A unified description of static and dynamic properties of Fermi liquids

N. Dupuis
Department of Physics, University of Maryland, College Park, MD 20742-4111, USA
Laboratoire de Physique des Solides, Université Paris-Sud, 91405 Orsay, France
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In Landau’s phenomenological Fermi-liquid theory (FLT), most physical quantities are derived from the knowledge of the energy variation $\delta E[\delta n]$ corresponding to a change $\delta n$ of the quasi-particle (QP) distribution function $n \equiv \{n_{k\sigma}\}$. We show that the internal energy $E[n]$ (or, more precisely, the thermodynamic potential $\Phi[n]$, expressed as a function of the QP distribution $n$, can be interpreted as an effective potential (in the sense of field theory), which is obtained from the free energy by a Legendre transformation. This allows to obtain explicitly $\delta \Phi$ (or $\delta E$) starting from a microscopic Hamiltonian and to relate the Landau $f$ function to the forward-scattering two-particle vertex without considering the collective modes as in the standard diagrammatic derivation of FLT. Out-of-equilibrium properties are obtained by extending the definition of the effective potential to space- and time-dependent configurations. $\Phi[n]$ is then a functional of the Wigner distribution function $n \equiv \{n_{k\sigma}(r,t)\}$. It contains information about both the static and dynamic properties of the Fermi liquid. In particular, it yields the quantum Boltzmann equation satisfied by $n_{k\sigma}(r,t)$. Finally, we show how $\delta \Phi[\delta n]$ can be derived (in the static case) using a finite-temperature renormalization-group approach. In agreement with previous results based on this technique, we find that the Landau $f$ function is defined by the fixed-point value of the $\Omega$-limit of the forward-scattering two-particle vertex.

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I. INTRODUCTION

Landau’s original approach to Fermi-liquid theory (FLT) is phenomenological. The main assumption is the existence of a one-to-one correspondence between the low-energy elementary excitations (quasi-particles (QP’s)) of the Fermi liquid and the elementary excitations (‘particles’ and ‘holes’) of the non-interacting fermion gas. Thermodynamic quantities are derived from the knowledge of the energy variation $\delta E[\delta n]$ corresponding to a change $\delta n$ of the QP distribution function $n \equiv \{n_{k\sigma}\}$. [$k$ and $\sigma$ denote the QP momentum and spin.] $\delta E[\delta n]$ is parameterized by the Landau $f$ function (or, equivalently, the Landau parameters) which describes the interaction between two QP’s. Non-equilibrium properties are obtained by generalizing the energy functional $\delta E[\delta n]$ to space- and time-dependent configurations, and by simultaneously considering the Boltzmann transport equation satisfied by the QP distribution.

Much effort has been dedicated to justify Landau’s phenomenological FLT from a microscopic Hamiltonian. The standard derivation consists in showing that the QP dynamics obtained from the microscopic two-particle vertex agrees with the conclusion of the phenomenological FLT. Although this demonstration provides a microscopic definition of the Landau $f$ function, it does not aim at calculating explicitly the energy functional $\delta E[\delta n]$.

The so-called statistical FLT provides an alternative approach which is in spirit much closer to the phenomenological theory of Landau. The main idea is to express the thermodynamic potential $\Phi[n]$ obtained from a microscopic Hamiltonian as a function of QP occupation numbers (see Refs. and references therein). The equilibrium QP distribution function is determined by minimizing the thermodynamic potential. The excitation energies are defined as functional derivatives of the energy with respect to the distribution function. They are usually referred to as statistical QP energies to distinguish them from the QP energies obtained from the poles of the single-particle Green’s function. Although this formulation provides a natural bridge between microscopic models and phenomenological FLT, few explicit calculations have been done in this framework. For instance, the Landau $f$ function (which can be obtained as the second functional derivative of the energy with respect to the QP distribution function) has not been related to the forward-scattering two-particle vertex as was done within the standard FLT. Besides, it is not clear how the collective modes of the Fermi liquid can be studied within this approach.

In this paper we propose a unified description of static and dynamic properties of Fermi liquids. The main idea is to interpret the thermodynamic potential $\Phi[n]$, expressed in terms of the QP distribution function $n$, as an effective potential (in the sense of field theory), which is obtained from the free energy by a Legendre transformation. A similar definition of the thermodynamic potential $\Phi[n]$ can be found in the statistical FLT and in particular in the work of Balian and De Dominicis. This allows to obtain explicitly $\delta \Phi$ (or $\delta E$) starting from a microscopic Hamiltonian and to relate the Landau $f$ function to the forward-scattering two-particle vertex without considering the collective modes as in the standard diagrammatic derivation of FLT. Dynamic properties are obtained by extending the definition of the effective potential to space- and time-dependent configurations. $\Phi[n]$
is then a functional of the Wigner distribution function \( n \equiv \{ n_{k\sigma}(r, t) \} \). We show how we can extract \( \Phi \) both the response functions of the Fermi liquid and the quantum Boltzmann equation satisfied by \( n_{k\sigma}(r, t) \).

Outline of the paper

In section II, we briefly review some basic results of the phenomenological FLT that are useful in subsequent sections. The microscopic definition of the thermodynamic potential \( \Phi[n] \) is given in section III. We first consider the free energy in the presence of an external source field \( h_{k\sigma} \), that couples to the QP number operator. The QP distribution function \( n \) is then obtained by taking the functional derivative of the free energy with respect to the source field. By performing a Legendre transformation, we obtain a functional \( \Phi[n] \) of the QP distribution function \( n \). In field theory, \( \Phi[n] \) is known as an effective potential.

In section III, we extend the definition of the effective potential \( \Phi[n] \) to space- and time-dependent configurations by considering the Wigner distribution function \( n \equiv \{ n_{k\sigma}(r, t) \} \). Restricting ourselves to small deviations \( \delta n_{k\sigma}(r, t) \) from equilibrium, we are able to compute the corresponding variation \( \delta \Phi[\delta n] \) of the effective potential. In the static and uniform limit \( \langle n_{k\sigma}(r, t) = n_{k\sigma} \rangle \), \( \delta \Phi \) reduces to the Landau functional studied in section III.

In section IV, we extend the definition of the effective potential \( \Phi[n] \) to space- and time-dependent configurations by considering the Wigner distribution function \( n \equiv \{ n_{k\sigma}(r, t) \} \). Restricting ourselves to small deviations \( \delta n_{k\sigma}(r, t) \) from equilibrium, we are able to compute the corresponding variation \( \delta \Phi[\delta n] \) of the effective potential. In the static and uniform limit \( \langle n_{k\sigma}(r, t) = n_{k\sigma} \rangle \), \( \delta \Phi \) reduces to the Landau functional studied in section III.

In section V, we use a finite-temperature renormalization-group (RG) approach to reproduce the results of section III. This approach provides an alternative method for computing the Landau’s functional \( \delta \Phi[\delta n] \) and confirms previous results based on this technique. We find that the \( f \) function is given by the fixed-point value of the \( \Omega \)-limit of the forward-scattering two-particle vertex \( (\Gamma^{\Omega}) \).

Unless otherwise specified, we consider a three-dimensional spin-\( \frac{1}{2} \) fermion gas (of volume \( \nu \)) with a spherical Fermi surface, and assume isotropy in spin space. Only short-range repulsive interactions are taken into account. We use a grand canonical formalism at finite temperature \( T \), the limit \( T \to 0 \) being taken at the end of the calculations. We take \( \hbar = k_B = 1 \) throughout the paper.

II. PHENOMENOLOGICAL FLT

In this section, we summarize the basic results of the phenomenological FLT which are useful for our purpose.

A. Landau’s functional \( \delta \Phi[\delta n] \)

In the ground state of a Fermi liquid, the QP distribution function corresponds to a filled Fermi sea: \( n^{(0)}_{k\sigma} = \Theta(k_F - k) \), where \( k_F \) is the Fermi momentum and \( \Theta(x) \) the step function. In the phenomenological FLT, one postulates that a change \( \delta n_{k\sigma} = n_{k\sigma} - n^{(0)}_{k\sigma} \) of the QP distribution induces the energy variation:

\[
\delta E[\delta n] = \sum_{k, \sigma} \epsilon_k \delta n_{k\sigma} + \frac{1}{2} \sum_{k, k', \sigma, \sigma'} f_{\sigma\sigma'}(k, k') \delta n_{k\sigma} \delta n_{k'\sigma'},
\]

neglecting terms of order \( O(\delta n^3) \). \( \epsilon_k \) is the energy of a QP in the absence of other excited QP’s (for simplicity, we consider only cases where \( \epsilon_k \) does not depend on spin). In the vicinity of the Fermi surface, it can be written as \( \epsilon_k \approx v_F^2 (k - k_F) + \mu \), where \( v_F^2 \) is the QP Fermi velocity and \( \mu \) the chemical potential. The term of order \( \delta n^2 \) in (2.1) comes from the interaction between quasi-particles. For states near the Fermi surface, \( k \approx k_F \) and \( k' \approx k_F \), the function \( f_{\sigma\sigma'}(k, k') = f_{\sigma\sigma'}(\theta) \) depends only on the angle \( \theta \) between \( k \) and \( k' \). The Landau parameters \( F_s^2 \) and \( F_s^0 \) are defined by expanding \( f \) on the basis of Legendre polynomials:

\[
2N(0) f_{\sigma\sigma'}(\theta) = \sum_{l=0}^{\infty} (F_s^l + \sigma' F_s^l) P_l(\cos \theta),
\]

where \( N(0) = k_F^3/2\pi^2 v_F^2 \) is the density of states per spin at the Fermi level.

In the grand-canonical ensemble at finite temperature, the physical quantity of interest is the thermodynamic potential \( \Phi[n] = E[n] - \mu N[n] - \beta^{-1} S[n] \). \( N[n] = \sum_{k, \sigma} n_{k\sigma} \) is the total QP number, \( S[n] \) the entropy, and \( \beta = 1/T \) the inverse temperature. Because of the correspondence between excitations of the non-interacting fermion gas and QP excitations, the entropy has the same expression as in a perfect Fermi gas:

\[
S[n] = - \sum_{k, \sigma} [n_{k\sigma} \ln n_{k\sigma} + (1 - n_{k\sigma}) \ln(1 - n_{k\sigma})].
\]
At equilibrium, the QP distribution function \( \bar{n} \) is obtained from the condition \( \delta \Phi / \delta n_{k\sigma} \big|_{\bar{n}} = 0 \):
\[
\bar{n}_{k\sigma} = n_F(\epsilon_k - \mu),
\] (2.4)
where \( n_F(x) = (e^{\beta x} + 1)^{-1} \) is the Fermi-Dirac factor, and
\[
\bar{\epsilon}_k = \frac{\delta E[n]}{\delta n_{k\sigma}} \bigg|_{\bar{n}} = \epsilon_k + \frac{1}{\nu} \sum_{k',\sigma'} \sigma' f_{\sigma'\sigma}(k, k') \left[ n_{k'\sigma'} - n_{k'\sigma'}^{(0)} \right],
\] (2.5)
is the QP energy corresponding to the equilibrium distribution \( \bar{n} \). Thus, if we expand \( \Phi[\bar{n} + \delta n] = \Phi[\bar{n}] + \delta \Phi[\delta n] \) around its equilibrium value, we obtain to lowest order in \( \delta n \)
\[
\delta \Phi[\delta n] = \frac{1}{2} \sum_{k, k', \sigma, \sigma'} \left[ -\frac{\delta \sigma_{\sigma'} \epsilon_{k, k'}}{n_F'(\epsilon_k - \mu)} \right] \delta n_{k\sigma} \delta n_{k'\sigma'},
\] (2.6)
where \( n_F'(x) = -\beta/4 \cosh^2(\beta x/2) \). There is no linear term in \( \delta n \) since \( \Phi[\bar{n}] \) is stationary at equilibrium.

### B. Deformation of the Fermi Surface

Because of the thermal factor \( 1/n_F'(\epsilon_k - \mu) \) in (2.6), small variations of the thermodynamic potential correspond to QP excitations lying in the thermal broadening of the Fermi surface \( (|\epsilon_k - \mu| \lesssim T) \). When \( T \to 0 \), these excitations have vanishing energies and can be viewed as resulting from a displacement \( u_\sigma(\mathbf{k}) \) of the Fermi surface. Here \( \mathbf{k} = k/\nu \) is a unit vector in the direction of \( \mathbf{k} \). It is sometimes convenient to express \( \delta \Phi \) directly in terms of \( u_\sigma(\mathbf{k}) \). Writing\[\]
\[
\delta n_{k\sigma} = v_F^\sigma u_\sigma(\mathbf{k}) n_F'(\epsilon_k - \mu),
\] (2.7)
we obtain
\[
\delta \Phi[u] = \frac{\nu v_F^\sigma N(0)}{2} \sum_{\sigma, \sigma'} \int \frac{d\Omega_k}{4\pi} u_\sigma^2(\mathbf{k})
\]
\[
+N(0) \int \frac{d\Omega_k}{4\pi} \frac{d\Omega_{k'}}{4\pi} f_{\sigma'\sigma}(\mathbf{k}, \mathbf{k}') u_\sigma(\mathbf{k}) u_{\sigma'}(\mathbf{k}') \}
\] (2.8)
in the limit \( T \to 0 \). We have used \( \epsilon_k \to \epsilon_k \) and \( n_F'(x) \to -\delta(x) \) when \( T \to 0 \). \( \Omega_k \) denotes the solid angle in the direction of \( \mathbf{k} \). Eq. (2.8) was first derived by Pomeranchuk considering the change in energy \( \delta E \) resulting from a Fermi surface displacement \( u_\sigma(\mathbf{k}) \) at \( T = 0 \). By requiring \( \Phi[u] \) to be minimum for \( u = 0 \), we easily deduce from (2.8) the stability conditions for a 3D Fermi liquid: \( F^i > 2l + 1 \) and \( F^a > -2l - 1 \).

### C. Dynamic properties

The Landau’s functional \( \delta \Phi[\delta n] \) allows to compute the thermodynamic properties of the Fermi liquid, but does not contain any information about the QP dynamics. To obtain the latter, one has to extend the definition of \( \delta E \) to out-of-equilibrium configurations:
\[
\delta E[\delta n^{cl}] = \sum_{k, \sigma} \int d^3r \epsilon_k \delta n^{cl}_{k\sigma}(r, t)
\]
\[
+ \frac{1}{2\nu} \sum_{k, \sigma, \sigma'} \int d^3r f_{\sigma'\sigma}(k, k') \times \delta n^{cl}_{k\sigma}(r, t) \delta n^{cl}_{k'\sigma'}(r, t).
\] (2.9)
\( n^{cl}_{k\sigma}(r, t) = n^{(0)}_{k\sigma} + \delta n^{cl}_{k\sigma}(r, t) \) gives the probability of finding a QP with momentum \( \mathbf{k} \) and spin \( \sigma \) at point \( r \) in space and time \( t \). We use the notation \( n^{cl} \) to emphasize that this approach is semiclassical, since it is assumed that one can simultaneously specify the momentum and position of the QP. For low-energy excitations, the interaction between QP’s (last term of Eq. (2.9)) can be assumed to be local in space. The time-dependent QP energy is defined by
\[
\epsilon_k(r, t) = \frac{\delta E[\delta n^{cl}]}{\delta n^{cl}_{k\sigma}(r, t)} \bigg|_{n^{cl}}
\]
\[
= \epsilon_k + \frac{1}{\nu} \sum_{k', \sigma'} f_{\sigma'\sigma}(k, k') \delta n^{cl}_{k'\sigma'}(r, t).
\] (2.10)
Eq. (2.9) is supplemented with the Boltzmann transport equation (2.11)
\[
\frac{\partial n^{cl}_{k\sigma}(r, t)}{\partial t} - \nabla_k n^{cl}_{k\sigma}(r, t) \cdot \nabla_r \epsilon_k(r, t)
\]
\[
+ \nabla_r n^{cl}_{k\sigma}(r, t) \cdot \nabla_r \epsilon_k(r, t) = 0,
\] (2.11)
where we have used the quasi-classical equations \( dx/dt = \nabla_k \epsilon_k(r, t) \) and \( d\mathbf{k}/dt = -\nabla_r \epsilon_k(r, t) \). To first order in \( \delta n^{cl} \), Eq. (2.11) reduces to
\[
\frac{\partial \delta n^{cl}_{k\sigma}(r, t)}{\partial t} + \nu \epsilon_k \delta n^{cl}_{k\sigma}(r, t)
\]
\[
+ \epsilon_k + \frac{1}{\nu} \sum_{k', \sigma'} f_{\sigma'\sigma}(k, k') \epsilon_k \delta n^{cl}_{k'\sigma'}(r, t) = 0.
\] (2.12)
The solution of this equation can be written as
\[
\delta n^{cl}_{k\sigma}(r, t) = v_F^\sigma u_\sigma(\mathbf{k}, q, \Omega) \delta(\epsilon_k - \mu)e^{i(q\cdot r - \Omega t)}
\] (2.13)
where \( u_\sigma(\mathbf{k}, q, \Omega) \) is a dynamic displacement of the Fermi surface. The solutions of Eq. (2.12) correspond to zero-sound \( (u_\uparrow = u_\downarrow) \) and spin-wave \( (u_\uparrow = -u_\downarrow) \) modes.
III. LANDAU’S FUNCTIONAL $\Phi$ AS AN EFFECTIVE POTENTIAL

In statistical mechanics and in the theory of phase transitions, it is natural to consider the Legendre transform of the free energy. In magnetism for instance, the Gibbs thermodynamic potential, which is a functional of the magnetization $M(\tau)$, is obtained from the free energy by performing a Legendre transformation.

In this section we show that the Landau’s functional $\Phi[n]$ can be naturally interpreted as an effective potential.

A. Effective potential $\Phi$

In order to define the effective potential $\Phi$, we first calculate the free energy $-\beta^{-1}\ln Z[h]$ in presence of a (real) external source field $h_{k\sigma}$ that couples to the QP occupation number operator $\hat{n}_{k\sigma}$. Then we obtain the Landau’s functional $\Phi[n]$ by performing a Legendre transformation.

We write the partition function in the Matsubara formalism as a functional integral over Grassmann variables $\psi^{(\tau)}(k,\tau)$ ($\tau$ is an imaginary time):

$$Z[h] = \int D\psi^* D\psi e^{-S[\psi^*, \psi] - S_h[\psi^*, \psi]},$$

(3.1)

where $S[\psi^*, \psi]$ is the action when $h_{k\sigma} = 0$. The source field contributes to the action a term

$$S_h[\psi^*, \psi] = \sum_{k,\sigma} h_{k\sigma} \int_0^\beta d\tau \hat{n}_{k\sigma}(\tau).$$

(3.2)

B. Non-interacting fermions

As a first example, we consider non-interacting fermions. Using $\hat{n}_{k\sigma}(\tau) = \psi^*_\sigma(k,\tau)\psi_\sigma(k,\tau)$, we obtain:

$$Z[h] = \int D\psi^* D\psi e^{\sum_{k,\sigma,\omega} \psi^*_\sigma(k,\omega)(i\omega - \epsilon_k - h_{k\sigma} + \mu)\psi_\sigma(k,\omega)} = \prod_{k,\sigma} \left(1 + e^{-\beta(h_{k\sigma} - \mu)}\right),$$

(3.6)

where $\epsilon_k$ is the energy of a fermion with momentum $k$. We have introduced the Fourier transformed field $\psi^{(\tau)}(k,\omega)$ where $\omega = \pi T(2m + 1)$ ($m$ integer) is a fermionic Matsubara frequency. Eq. (3.6) yields

$$n_{k\sigma} = n_F(\epsilon_k + h_{k\sigma} - \mu).$$

(3.7)

Inverting (3.7) and using (3.6), we eventually obtain

$$\Phi[n] = \sum_{k,\sigma} (\epsilon_k - \mu)n_{k\sigma} + \frac{1}{\beta} \sum_{k,\sigma} \left[ n_{k\sigma} \ln n_{k\sigma} - (1 - n_{k\sigma}) \ln(1 - n_{k\sigma}) \right].$$

(3.8)

Eq. (3.8) is the expected result for non-interacting fermions.

C. Interacting fermions

For interacting fermions, it is not possible to calculate exactly the thermodynamic potential $\Phi[n]$. However, we do not require the whole knowledge of $\Phi[n]$, but only its variation $\delta \Phi[\delta n]$ when the QP distribution function $n$ varies from its equilibrium value $\bar{n} = n|h=0$ by an amount $\delta n$. [For $T \to 0$, $\bar{n}_{k\sigma} = \Theta(k_F - k)$ corresponds to the ground-state distribution function.] This turns out to be a much easier task than obtaining $\Phi[n]$.

Expanding $\Phi[\bar{n} + \delta n]$ to second order in $\delta n$, we obtain

$$\delta \Phi[\delta n] = \frac{1}{2} \sum_{k,k',\sigma,\sigma'} \frac{\delta^2 \Phi[\bar{n}]}{\delta n_{k\sigma} \delta n_{k'\sigma'}} \delta n_{k\sigma} \delta n_{k'\sigma'}. $$

(3.9)

There is no linear term since $\Phi[\bar{n}]$ is stationary at equilibrium [Eq. (3.8)]. To proceed further, we use the following relation between the functional derivatives of the free energy $\beta^{-1}\ln Z[h]$ and those of the Legendre transform $\Phi[n]$:

$$\sum_{k_3,\sigma_3} \frac{\beta^{-1}\delta^2 \ln Z[h]}{\delta h_{k_3\sigma_3} \delta h_{k_3\sigma_3}} \bigg|_{h=0} = \delta_{\sigma_1, \sigma_2} \delta_{k_1, k_2}.$$

(3.10)

Introducing the matrix

$$\chi_{\sigma\sigma'}(k, k') = \beta^{-1}\delta^2 \ln Z[h] \bigg|_{h=0},$$

(3.11)

At equilibrium, i.e., in the absence of source field, $\Phi[n]$ is stationary with respect to small variations of the QP distribution function.
we thus obtain
\[ \delta \Phi[\delta n] = \frac{1}{\beta} \sum_{k,k',\sigma,\sigma'} \chi^{-1}_{\sigma\sigma'}(k,k') \delta n_{k\sigma} \delta n_{k'\sigma'}. \] (3.12)

Note that \( \chi_{\sigma\sigma'}(k,k') \) is nothing but the (linear) response function to the external source field \( h_{k\sigma} \). Comparing (2.3) and (3.12) (and assuming that the system is a Fermi liquid), we obtain a macroscopic definition of the Landau function:
\[ \frac{1}{\beta} f_{\sigma\sigma'}(k,k') = \frac{\delta_{\sigma,\sigma'} \delta_{k,k'}}{n_F(k) - \mu} + \chi^{-1}_{\sigma\sigma'}(k,k'). \] (3.13)

From now on, we ignore the difference between \( \tilde{c}_k \) and \( c_k \) since we are ultimately interested in the limit \( T \to 0 \). Thus, the calculation of the Landau function reduces to the calculation of the susceptibility \( \chi \). In the next section, we show how \( \chi \) can be calculated using the standard assumptions of FLT.

**D. Calculation of the Landau function**

In this section, we first express the Landau function in terms of the QP properties, and then relate it to the bare fermion properties.

1. **f in terms of QP properties**

Since the external field \( h_{k\sigma} \) couples to the QP’s, we have to distinguish between the QP field (that we denote by \( \psi(\tau) \)) and the field \( \psi(\tau) \) corresponding to the bare fermions. With these notations, we have \( \tilde{n}_{k\sigma}(\tau) = \tilde{\psi}_\sigma^*(k,\tau) \tilde{\psi}_\sigma(k,\tau) \), where the operator \( \tilde{n}_{k\sigma}(\tau) \) has been introduced in (3.2). From (3.2) and (3.11), we obtain
\[ \chi_{\sigma\sigma'}(k,k';\tilde{q}) = \frac{1}{\beta} \sum_{\omega,\omega'} \tilde{\chi}_{\sigma\sigma'}(k,k';\tilde{q}), \] (3.14)

where \( \tilde{\chi} \) is the two-particle Green’s function
\[ \tilde{\chi}_{\sigma\sigma'}(k,k';\tilde{q}) = \langle \tilde{\psi}_\sigma^*(k) \tilde{\psi}_\sigma(k+\tilde{q}) \tilde{\psi}_\sigma^*(k+\tilde{q}) \tilde{\psi}_\sigma(k+) \rangle_c \]
\[ = \langle \tilde{\psi}_\sigma^*(k) \tilde{\psi}_\sigma(k+\tilde{q}) \tilde{\psi}_\sigma^*(k+\tilde{q}) \tilde{\psi}_\sigma(k+) \rangle_c - \delta_{\tilde{q},0} \langle \tilde{\psi}_\sigma^*(k) \tilde{\psi}_\sigma(k+) \rangle \langle \tilde{\psi}_\sigma^*(k+\tilde{q}) \tilde{\psi}_\sigma(k) \rangle. \] (3.15)

The average values are evaluated in the absence of source field \( (h = 0) \), and the meaning of \( \langle \cdots \rangle_c \) is defined by the second line of (3.12). We use the notation \( \tilde{k} = (k,\tilde{q}) \), \( \tilde{q} = (q,\Omega) \), and \( \tilde{k}_\pm = \tilde{k} \pm \tilde{q}/2 \). \( \omega \) and \( \Omega \) denote fermionic and bosonic Matsubara frequencies, respectively. The Landau function is related to the limit \( \tilde{q} \to 0 \) of \( \chi_{\sigma\sigma'}(k,k';\tilde{q}) \) [Eq. (3.13)]. In a Fermi liquid, the forward-scattering limit \( \tilde{q} \to 0 \) is ill-defined and one should distinguish between the \( \Omega \) and \( \Omega \) limits, which correspond to \( \Omega/q \to 0 \) and \( q/\Omega \to 0 \), respectively. Since a static and uniform external field cannot create quasi-particle-quasiparticle excitations (and therefore modify the ground-state QP distribution function) in the \( \Omega \)-limit, \( \tilde{f} \) is obtained from the \( \Omega \)-limit of the function \( \chi \). It turns out to be more convenient to keep a finite \( \tilde{q} \) at intermediate stages, the limit \( \tilde{q} \to 0 \) (with \( \Omega/q \to 0 \)) being taken only at the end of the calculations. Besides, the function \( \chi \) (with \( \tilde{q} \) finite) will also have to be considered in the analysis of the dynamic properties of Fermi liquids (section V).

\[ \chi_{\sigma\sigma'}(\tilde{k},\tilde{k}';\tilde{q}) = \chi_{\sigma\sigma'}^{(0)}(\tilde{k},\tilde{k}';\tilde{q}) \]
\[ - \frac{1}{\beta \nu} \sum_{k_1,k_2,\sigma_1,\sigma_2} \tilde{\chi}_{\sigma\sigma'}^{(0)}(k_1,k_2;\tilde{q}) \tilde{I}_{\sigma_1\sigma_2}^{\text{rr}}(k_1,k_2;\tilde{q} \tilde{k}_1\tilde{k}_2), \] (3.16)

where
\[ \tilde{I}_{\sigma_1\sigma_2}^{\text{rr}}(k_1,k_2;\tilde{q} \tilde{k}_1\tilde{k}_2) \equiv \tilde{I}_{\sigma_1\sigma_2}^{\text{rr}}(k_1,k_2;\tilde{q} \tilde{k}_1\tilde{k}_2). \] (3.17)

is the irreducible QP vertex in the Landau channel. The non-interacting part
\[ \tilde{I}_{\sigma}^{(0)}(\tilde{k},\tilde{k}';\tilde{q}) = -\delta_{\sigma,\sigma'} \delta_{\tilde{k},\tilde{k}'} \tilde{G}(\tilde{k}+) \tilde{G}(\tilde{k}_-) \] (3.18)

is easily expressed in terms of the QP propagators
\[ \tilde{G}(\tilde{k}_+) = -\langle \tilde{\psi}_\sigma(\tilde{k}_+) \tilde{\psi}_\sigma^*(\tilde{k}_+) \rangle = \left[ i \omega + \frac{\Omega}{2} - c_\kappa \mp v_\kappa \cdot \frac{\tilde{q}}{2} + \mu \right]^{-1}, \] (3.19)

where \( c_\kappa \) is the QP energy and \( v_\kappa = \nabla_k \kappa \) the QP group velocity. Approximating \( c_\kappa \) by \( v_\kappa^2(k - k_F) + \mu \), we have \( v_\kappa^2 = v_F \cdot \kappa \) (with \( \hat{\kappa} = k/k \)). The quantity \( \tilde{I}^{(0)} \) becomes singular in the forward-scattering limit \( \tilde{q} \to 0 \) since the poles of the two QP propagators coalesce. In FLT, one assumes that \( \tilde{I}^{(0)} \) is the only singular quantity in this limit. This implies that the irreducible vertex \( \tilde{I}^{\text{irr}} \), which does not contain \( \tilde{I}^{(0)} \), is a non-singular quantity that has a well-defined limit when \( \tilde{q} \to 0 \). We therefore set \( \tilde{q} \to 0 \) in \( \tilde{I}^{\text{irr}} \). Furthermore, since the singularity of \( \tilde{I}^{(0)} \) is due to QP states in the vicinity of the Fermi surface, we can ignore the \( k \) and \( \omega \) dependence of \( \tilde{I}^{\text{irr}} \) (i.e., set \( k = k' = k_F \) and \( \omega = \omega' = 0 \)), which then becomes a function of \( \hat{k} \) and \( \hat{k}' \). The variables \( |k - k_F|, |k' - k_F| \) and \( \omega, \omega' \) are irrelevant in the RG sense (see section V). This allows to perform the frequency sums in (3.16) and obtain a Dyson equation for \( \chi \):
\[ \chi_{\sigma\sigma'}(k,k';q) = \chi_{\sigma\sigma'}^{(0)}(k,k';q) \]
\[ - \frac{1}{\nu} \sum_{k_1,k_2,\sigma_1,\sigma_2} \chi_{\sigma\sigma'}^{(0)}(k_1,k_1;\tilde{q}) \tilde{I}_{\sigma_1\sigma_2}^{\text{rr}}(k_1,k_2;\tilde{k}_2,\tilde{k}'_2). \] (3.20)

This can be rewritten as...
Comparing (3.13) and (3.21), and noting that Eqs. (3.23, 3.26) relate the Landau normalization factor. The QP field \( \overline{\Gamma}_1 \) in Eq. (3.21),\[ \chi^{(0)}(k, k'; \tilde{q}) = \delta_{\sigma,\sigma'}\delta_{k, k'}n^F_\nu(\epsilon_k - \mu) \]
\[ \left. \frac{v_k \cdot q}{\Omega_2 - v_k \cdot q} \right. \]
\[ \rightarrow -\delta_{\sigma,\sigma'}\delta_{k, k'}n^F_\nu(\epsilon_k - \mu) \] (3.22)
in the \( q \)-limit, we conclude that
\[ f_{\sigma\sigma'}(k, k') = \Gamma_{\sigma\sigma'}^{\text{irr}}(k, k'). \] (3.23)

We can proceed one step further by relating \( \overline{\Gamma}^{\text{irr}} \) to the total QP vertex \( \overline{\Gamma} \), using the equation
\[ \overline{\Gamma}_{\sigma\sigma'}(k, k'; \tilde{q}) = \Gamma_{\sigma\sigma'}^{\text{irr}}(k, k') - \frac{1}{\beta\nu} \sum_{k_1, k_2, \sigma_1, \sigma_2} \Gamma_{\sigma\sigma_1}(k, k_1) \]
\[ \times \chi^{(0)}_{\sigma\sigma_1}(k_1; k_2, k'; \tilde{q}) \Gamma_{\sigma_2\sigma'}(k_2, k'; \tilde{q}). \] (3.24)

Since \( \overline{\Gamma} \), as \( \Gamma^{\text{irr}} \), is independent of frequencies, we can perform the sum over \( \omega_1 \) and \( \omega_2 \) in (3.24), which yields
\[ \overline{\Gamma}_{\sigma\sigma'}(k, k'; \tilde{q}) = \Gamma_{\sigma\sigma'}^{\text{irr}}(k, k') - \frac{1}{\beta\nu} \sum_{k_1, k_2, \sigma_1, \sigma_2} \Gamma_{\sigma\sigma_1}(k, k_1) \]
\[ \times \chi^{(0)}_{\sigma\sigma_1}(k_1; k_2, k'; \tilde{q}) \Gamma_{\sigma_2\sigma'}(k_2, k'; \tilde{q}). \] (3.25)

\( \chi^{(0)} \) vanishing in the \( \Omega \)-limit [Eq. (3.22)], we obtain
\[ \Gamma_{\sigma\sigma'}^{\text{irr}}(k, k'; \tilde{q}) = \lim_{\Omega \to 0} \Gamma_{\sigma\sigma'}(k, k'; \tilde{q}) \]
\[ = \bar{\Gamma}_{\sigma\sigma'}^{\Omega}(k, k'). \] (3.26)

Eqs. (3.23, 3.26) relate the Landau \( f \) function to the \( \Omega \)-limit of the forward-scattering QP vertex.

2. \( f \) in terms of bare fermion properties

The last step of our derivation is to relate \( \bar{\Gamma}^{\text{irr}} \) to the vertex of the bare fermions. Consider first the single-particle propagator in a Fermi liquid. It can be written as
\[ G(\tilde{k}) = zG(\tilde{k}) + G_{\text{inc}}(\tilde{k}). \] (3.27)

We then deduce from (3.16) and (3.33)
\[ \bar{\Gamma}_{\sigma\sigma'}(k, k'; \tilde{q}) = z^2\Gamma_{\sigma\sigma'}(k, k'; \tilde{q}). \] (3.34)

Eq. (3.34) relates the QP vertex \( \bar{\Gamma} \) to the bare fermion vertex \( \Gamma \). From (3.23, 3.24, 3.34) we obtain the well-known expression of the Landau \( f \) function in a Fermi liquid:
\[ f_{\sigma\sigma'}(k, k') = z^2\bar{\Gamma}_{\sigma\sigma'}^{\Omega}(k, k'). \] (3.35)

IV. DYNAMIC PROPERTIES

The Landau's functional \( \Phi[n] \) does not contain any information about the dynamic properties of the Fermi liquid. In this section, we extend the definition of the effective potential \( \Phi \) to space- and time-dependent configurations by considering the Wigner distribution function.
(WDF) \( n \equiv \{ n_{k\sigma}(r, t) \} \). This allows to obtain the QP dynamics and the collective modes of the Fermi liquid without introducing the semiclassical distribution function \( n^cl_{k\sigma}(r, t) \) and the corresponding Boltzmann transport equation as in the phenomenological FLT (see section II C).

### A. Generalized effective potential \( \Phi \)

The best quantum analog to the semiclassical distribution function \( n^cl_{k\sigma}(r, t) \) is the WDF \( WDF \) [1]. The latter is not a true distribution function since it is not positive definite. However, as far as its moments are concerned, it behaves similarly to a distribution function [2,3]. For our purpose, we define the WDF in Matsubara time as

\[
WDF = \frac{1}{\beta v} \sum_{\omega} \langle \tilde{\psi}^*_\sigma(\tilde{k}_-, r') \tilde{\psi}^*_\sigma(\tilde{k}_+, r') \rangle, \tag{4.1}
\]

and its Fourier transform as

\[
n_{k\sigma}(\tilde{q}) = \frac{1}{\beta} \sum_{\omega} \langle \tilde{\psi}^*_\sigma(\tilde{k}_-, r') \tilde{\psi}^*_\sigma(\tilde{k}_+, r') \rangle. \tag{4.2}
\]

Note that these definitions involve the QP field \( \tilde{\psi}(\omega) \).

We are now in a position to define an effective potential \( \Phi[n] \), which is a functional of the WDF \( n \equiv \{ n_{k\sigma}(\tilde{q}) \} \). We shall proceed along the same lines as in section II. We consider the system in presence of a source field \( h_{k\sigma}(\tilde{q}) = h^s_{k\sigma}(\tilde{q}) \) that couples to the QP operator \( \hat{n}_{k\sigma}(\tilde{q}) = \beta^{-1} \sum_{\sigma} \tilde{\psi}^\dagger_\sigma(\tilde{k}_-) \tilde{\psi}^\dagger_\sigma(\tilde{k}_+) \). We write the partition function as in (3.1), with

\[
S_h = \beta \sum_{k, \sigma, \tilde{q}} h_{k\sigma}(\tilde{q}) \hat{n}_{k\sigma}(\tilde{q}). \tag{4.3}
\]

The WDF is obtained by taking the functional derivative of the free energy with respect to the source field:

\[
n_{k\sigma}(\tilde{q}) = \langle \hat{n}_{k\sigma}(\tilde{q}) \rangle = -\frac{1}{\beta} \frac{\delta \ln Z[h]}{\delta h_{k\sigma}(\tilde{q})}. \tag{4.4}
\]

The effective potential is then defined as

\[
\Phi[n] = -\frac{1}{\beta} \ln Z[h] - \sum_{k, \sigma, \tilde{q}} h_{k\sigma}(\tilde{q}) \hat{n}_{k\sigma}(\tilde{q}), \tag{4.5}
\]

and satisfies the ‘equation of state’

\[
\frac{\delta \Phi[n]}{\delta n_{k\sigma}(\tilde{q})} = -h_{k\sigma}(\tilde{q}). \tag{4.6}
\]

Even for non-interacting fermions, \( \Phi[n] \) cannot be calculated exactly. We shall therefore consider only small fluctuations \( \delta n_{k\sigma}(\tilde{q}) \) around the equilibrium state:

\[
n_{k\sigma}(\tilde{q}) = \delta n_{k\sigma}(\tilde{q}) + n_{k\sigma}(\tilde{q}). \tag{4.7}
\]

Eq. (3.10) can be easily generalized into

\[
\frac{\delta^{(2)} \Phi[n]}{\delta n_{k\sigma}(\tilde{q}) \delta n_{k'\sigma'}(\tilde{q})} = \delta_{\tilde{q}, \tilde{q}'} \chi_{\sigma\sigma'}(k, k'; \tilde{q}), \tag{4.8}
\]

where

\[
\chi_{\sigma\sigma'}(k, k'; \tilde{q}) = \beta^{-1} \delta^{(2)} \ln Z[h] \frac{\delta h_{k\sigma}(\tilde{q}) \delta h_{k'\sigma'}(\tilde{q})}{\delta h_{k\sigma}(\tilde{q})} \tag{4.9}
\]

is the susceptibility introduced in section II [Eqs. (3.14,3.13)]. The Kronecker symbol \( \delta_{\tilde{q}, \tilde{q}'} \) in (4.8) results from translational invariance. To second order in \( \delta n \), we thus have

\[
\delta \Phi[\delta n] = \frac{1}{2} \sum_{k, k', \sigma, \sigma'} \chi^{-1}_{\sigma\sigma'}(k, k'; \tilde{q}) \delta n_{k\sigma}(\tilde{q}) \delta n_{k'\sigma'}(\tilde{q}). \tag{4.10}
\]

Using Eqs. (3.21,3.22,3.23), we can write the effective potential in terms of the Landau \( f \) function:

\[
\delta \Phi[\delta n] = \frac{1}{2} \sum_{k, k', \sigma, \sigma'} \left\{ \frac{\delta \sigma, \sigma' \delta k, k' \cdot i\Omega - v_k \cdot q}{n'_{F}(\epsilon_k - \mu)} \frac{v_k \cdot q}{v_k \cdot q} + \frac{1}{\beta} f_{\sigma\sigma'}(k, k') \right\} \delta n_{k\sigma}(\tilde{q}) \delta n_{k'\sigma'}(\tilde{q}). \tag{4.11}
\]

Eq. (4.11) generalizes the Landau’s functional to space- and time-dependent configurations. As in the static and uniform case studied in section II, \( \delta \Phi[\delta n] \) is essentially parameterized by the Landau \( f \) function. We shall show in the next section that \( \delta \Phi \) contains all the information about the QP dynamics. Moreover, if we consider the \( q \)-limit, the WDF \( \{ n_{k\sigma}(r, t) \} \) reduces to the QP distribution function \( \{ n_{k\sigma} \} \), and \( \Phi[n] \) to the thermodynamic potential introduced in section II. Thus the effective potential \( \Phi \) describes static and dynamic properties of the Fermi liquid in a unified framework. This result should be contrasted with the phenomenological FLT which requires slightly different approaches to deal with static and dynamic properties (see section II), and strongly relies on semiclassical arguments for the latter.

### B. Quantum Boltzmann equation

In the absence of source field \( (h = 0) \), the stationarity condition of the effective potential [Eq. (4.11)] yields

\[
(v_k \cdot q - i\Omega) \delta n_{k\sigma}(\tilde{q}) - v_k \cdot q n'_{F}(\epsilon_k - \mu) \times \frac{1}{\beta} \sum_{k', \sigma'} f_{\sigma\sigma'}(k, k') \delta n_{k'\sigma'}(\tilde{q}) = 0. \tag{4.12}
\]
Eq. (4.12) is the quantum Boltzmann equation for the WDF. [Real-time quantities are obtained by the usual analytic continuation $i\Omega \rightarrow \Omega + i\theta^+$.] Note that in the simplest case we are considering here (no external electric field, small fluctuations around the equilibrium state, etc.), it is identical to the Boltzmann equation (2.12) satisfied by the semiclassical distribution function $n_{i\sigma}^{cl}(r,t)$.

The solution of (4.12) can be written as

$$\delta n_{\epsilon \sigma}(q) = v_F u_{\sigma}(\hat{k},\hat{q}) n'_{\epsilon}(\epsilon_k - \mu), \quad (4.13)$$

where $u_{\sigma}(\hat{k},\hat{q})$ is naturally interpreted as the dynamic Fermi surface displacement for spin $\sigma$ fermions. It satisfies the equation

$$(\nabla_k \cdot q - i\Omega) u_{\sigma}(\hat{k},\hat{q}) + v_k \cdot q N(0) \sum_{\sigma'} \int d\Omega \frac{k'}{4\pi} f_{\sigma\sigma'}(\hat{k},\hat{k}') u_{\sigma'}(\hat{k}',\hat{q}) = 0. \quad (4.14)$$

Here we have used $n'_{\epsilon}(\epsilon_k - \mu) = -\delta(\epsilon_k - \mu)$ when $T \rightarrow 0$. Eq. (4.14) yields the zero-sound (for $u_t = u_j$) and spin-wave (for $u_t = -u_j$) modes of the Fermi liquid.

It is also possible to express the effective potential directly in terms of the dynamic Fermi surface displacements:

$$\Phi[u] = \frac{\nu v_F^2 N(0)}{2} \sum_{\sigma,\sigma'} \int \frac{d\Omega k}{4\pi} \nabla_k \cdot q - i\Omega \int \frac{d\Omega k'}{4\pi} f_{\sigma\sigma'}(\hat{k},\hat{k}') u_{\sigma'}(\hat{k}',\hat{q})$$

$$\times u_{\sigma}(\hat{k},\hat{q})^2 + \sum_{\sigma} \int \frac{d\Omega k}{4\pi} f_{\sigma\sigma}(\hat{k},\hat{q})$$

$$\times u_{\sigma}(\hat{k},\hat{q}) u'_{\sigma}(\hat{k}',\hat{q}). \quad (4.15)$$

Here we have used $u_{\sigma}(\hat{k},\hat{q}) = u_{\sigma}^*(\hat{k},\hat{q})$. The equations of motion (4.14) are then directly obtained from the stationarity condition $\delta \Phi/\delta u_{\sigma}(\hat{k},\hat{q}) = 0$. Note also that (4.15) reduces to (2.3) in the $q$-limit.

C. Response functions

Since the effective potential is essentially determined by the susceptibility $\chi_{\sigma\sigma'}(\hat{k},\hat{k}',\hat{q})$, it also yields the response functions of the Fermi liquid. For instance, the charge and spin response functions are given by

$$\chi_{ch}(\hat{q}) = \frac{\nu}{2} \sum_{k,k',\sigma,\sigma'} \chi_{\sigma\sigma'}(k,k';\hat{q}),$$

$$\chi_{sp}(\hat{q}) = \frac{\nu}{2} \sum_{k,k',\sigma,\sigma'} \sigma' \chi_{\sigma\sigma'}(k,k';\hat{q}). \quad (4.16)$$

where $\chi$ is related to the effective potential by Eq. (4.10). Note that when calculating response functions, the difference between particles and QP’s can be ignored. This property is a consequence of the Ward identities that result from particle-number conservation (see Ref. [21] for a detailed discussion).

Eqs. (4.10, 4.11) determine $\chi^{-1}$. If, for simplicity, we consider the case where only the Landau parameters $F_0^0$ and $F_0^0$ are non-zero, the matrix $\chi^{-1}$ can easily be inverted, and we obtain the well-known expression:

$$\chi_{ch}(\hat{q}) = 2 N(0) \Omega_0(\Omega/\nu^* q) \quad (4.17)$$

$$\chi_{sp}(\hat{q}) = 2 N(0) \Omega_0(\Omega/\nu^* q). \quad (4.18)$$

V. RENORMALIZATION-GROUP APPROACH

Recently, RG techniques based on low-energy fermion effective actions have been applied to interacting fermions in dimension $d \geq 2$ by many authors (see Refs. [22, 23, 24] and references therein). The finite-temperature RG approach [25] was first applied to Fermi liquids in dimension $d \geq 2$ by Chitov and Sénéchal. Contrary to other works on the subject, it revealed that the effective interaction function in the Landau or zero-sound channel (particle-hole pairs at small total momentum and energy) does not stay marginal under the RG transformation, since its $\beta$-function is not identically zero. From the RG equations, the standard FLT results have been recovered [25]. It has been pointed out that the bare interaction function of the low-energy fermion effective action cannot be identified with the Landau $f$ function.

The latter, along with other observable parameters of a Fermi liquid, is obtained at the fixed-point of the RG equations, i.e., when all degrees of freedom have been integrated out [25]. The finite temperature RG approach has also given results that cannot be obtained within the standard derivation of FLT. Chitov and Sénéchal, taking into account the interferences between the zero-sound (ZS) and exchange (ZS') channels, have obtained RG equations that satisfy the Pauli principle contrary to the standard microscopic derivation of FLT. Performing a two-loop order calculation, the present author has obtained a non trivial expression of the wave-function renormalization factor [25], which was also obtained from 2D bosonization [17] and Ward Identities [21].

In this section, we show how the response function $\chi$ introduced in section [11] and therefore the Landau $f$ function, can be calculated using a finite-temperature (Kadanoff-Wilson) RG approach. We follow the procedure used in Ref. [15] to obtain the compressibility of a Fermi liquid. For simplicity we consider a 2D fermion gas.

We write the partition function $Z[h]$ as a functional integral over Grassmann variables [Eq. (3.1)], where, assuming that the high-energy degrees of freedom have
been integrated out (in a functional sense), the action describes fermionic degrees of freedom with $|k - k_F| < \Lambda_0 \ll k_F$. We write this low-energy effective action as $S^{\Lambda_0} + S^h_{\Lambda_0}$, where
\[
S^{\Lambda_0} = -\sum_{k, \sigma} \psi^*_\sigma(\hat{k}) (i\omega - \epsilon_k + \mu) \psi^*_\sigma(\hat{k}) + \frac{1}{4\beta v} \sum_{k_1, \ldots, k_4, \sigma_1, \ldots, \sigma_4} \Gamma^\Lambda_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}(\hat{k}_1, \hat{k}_2, \hat{k}_3, \hat{k}_4) \times \bar{\psi}^*_{\sigma_1}(\hat{k}_4) \psi^*_{\sigma_2}(\hat{k}_3) \psi_{\sigma_2}(\hat{k}_2) \bar{\psi}_{\sigma_1}(\hat{k}_1) \times \delta_{\hat{k}_1 + \hat{k}_2 + \hat{k}_3 + \hat{k}_4} \delta_{\omega_1 + \omega_2 + \omega_3 + \omega_4}. \tag{5.1}
\]

It will be shown below that the RG procedure ensures that the action is always expressed as a function of the QP field $\psi^*(\cdot)$. In (5.1), the wave-vectors $k$ satisfy $|k - k_F| < \Lambda_0$. The single-particle excitations are linearized around the Fermi surface: $\epsilon_k - \mu = v_F (k - k_F)$ where $v_F$ is the bare Fermi velocity. The bare (antisymmetrized) two-particle vertex $\Gamma^\Lambda_0$ is assumed to be a non-singular function of its arguments. The summation over wave vectors is defined by
\[
\frac{1}{\nu} \sum_k \equiv k_F \int_{k_F - \Lambda_0}^{k_F + \Lambda_0} dk \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi. \tag{5.2}
\]

ignoring irrelevant terms at tree-level.

As shown in section (11), the Landau’s functional $\delta F[\bar{n}]$ can also be expressed in terms of the Fermi surface displacements $n_{\bar{\sigma}}(\hat{k}) = n_{\bar{\sigma}}(\theta)$, or equivalently the ‘density’ variations $\delta n_{\bar{\sigma}}(\theta) = (k_F / 2\pi) n_{\bar{\sigma}}(\theta)$. Here and in the following, we denote by $\theta$ the direction of a given momentum $k$, i.e., $\hat{k} = (\cos \theta, \sin \theta)$. For purely technical convenience, we therefore consider an external field $h_{\bar{\sigma}}(\theta, \tau)$ that couples directly to the QP density operator $\bar{n}_{\bar{\sigma}}(\theta, \tau)$ and write the action $S^h_{\Lambda_0}$ as
\[
S^h_{\Lambda_0} = \sum_{\sigma} \int_0^{\beta} \frac{d\theta}{2\pi} h_{\bar{\sigma}}(\theta) \int_0^\theta d\tau \bar{n}_{\bar{\sigma}}(\theta, \tau), \tag{5.3}
\]

where
\[
\int_0^\theta d\tau \bar{n}_{\bar{\sigma}}(\theta, \tau) = vFN(0) \left( \int_{k_F - \Lambda_0}^{k_F + \Lambda_0} dk \right) \times \lim_{q \to 0} \left[ \psi^*_\sigma(\hat{k} + \hat{q}) \psi^*_\sigma(\hat{k}) \right]_{\Omega = 0}. \tag{5.4}
\]

We take the $q$-limit as discussed in section (11). $N(0) = k_F / 2\pi v_F$ is the bare 2D density of states per spin.

It is straightforward to show that the microscopic definition of the Landau $f$ function [Eqs. (5.11, 5.13)] becomes:
\[
\frac{\nu}{N(0)} [\delta_{\sigma, \sigma'} 2\pi \delta(\theta) + N(0) f_{\sigma, \sigma'}(\theta)] = \chi^{-1}_{\sigma, \sigma'}(\theta), \tag{5.5}
\]

\[
\chi^*_{\sigma, \sigma'}(\theta' - \theta') = 4\pi^2 \beta^{-1} \delta^{(2)} \ln Z[h] \frac{\delta h_{\bar{\sigma}}(\theta) \delta h_{\bar{\sigma}}(\theta')}{\partial h_{\bar{\sigma}}(\theta) h_{\bar{\sigma}}(\theta')} |_{h = 0}. \tag{5.5}
\]

The meaning of the notation $\chi^*$ is discussed below.

The constraint to have all momenta in the shell $|k - k_F| < \Lambda_0$ restricts the allowed scatterings to diffusion of particle-hole, or particle-particle, pairs with small total momentum $(q < \Lambda_0)$. Consequently, only two vertex functions have to be considered: the forward-scattering vertex function and the BCS vertex function. In the absence of BCS instability, we can neglect the latter. We denote by $\Gamma^\Lambda_{\sigma}(\hat{k}_1, \hat{k}_2; \hat{q})$ the forward-scattering vertex. [We use the notation $\Gamma^\Lambda_{\sigma}(\hat{k}_1, \hat{k}_2; \hat{q})$ or $\Gamma^\Lambda_{\sigma}(\hat{k}_1, \hat{k}_2; \hat{q})$]

The Kadanoff-Wilson RG procedure consists in successive partial integrations of the fermion field degrees of freedom in the infinitesimal momentum shell $\Lambda_0 e^{-dt} < |k - k_F| < \Lambda_0$ where $dt$ is the RG generator and $\Lambda(t) = \Lambda_0 e^{-t}$ the effective momentum cut-off at step $t$. Each partial integration is followed by a rescaling of radial momenta, frequencies and fields (i.e., $\omega' = \omega$, $k' - k_F = s(k - k_F)$ and $\psi' = \psi$ with $s = e^{dt}$) in order to let the quadratic part of the action (5.1) invariant and to restore the initial value of the cut-off. The partial integration modifies the parameters of the action which becomes functions of the flow parameter $t$. In the following, we note $\Gamma$ the running (i.e., cut-off dependent) vertex (we do not write its $t$-dependence explicitly). For the purpose of our calculation, it is sufficient to consider the $q$- and $\Omega$-limits of the forward-scattering vertex:
\[
\Gamma^\Lambda_{\sigma}(\theta_1 - \theta_2) = \lim_{q \to 0} \left[ \Gamma_{\sigma}(\theta_1, \theta_2; \hat{q}) \right]_{\Omega = 0}, \tag{5.6}
\]

\[
\Gamma^\Lambda_{\sigma}(\theta_1 - \theta_2) = \lim_{q \to 0} \left[ \Gamma_{\sigma}(\theta_1, \theta_2; \hat{q}) \right]_{\Omega = 0}. \tag{5.6}
\]

The RG process also generates corrections to the source field $h_{\bar{\sigma}}(\theta)$ along with higher-order terms in $h$. At step $t$, the source term in the action can be written as $\hat{h}^t$ (ignoring terms of order $h^3$)
\[
S^\Lambda = \sum_{\sigma, \sigma'} \int_0^{\beta} \frac{d\theta}{2\pi} h_{\bar{\sigma}}(\theta) \int_0^{\beta} \frac{d\theta'}{2\pi} z^{(h)}(\theta - \theta') \int_0^{\beta} d\tau \bar{n}_{\bar{\sigma}}(\theta', \tau) \times \lim_{q \to 0} \left[ \psi^*_{\sigma}(\hat{k} + \hat{q}) \psi^*_{\sigma}(\hat{k}) \right]_{\Omega = 0}. \tag{5.7}
\]

We do not write explicitly the dependence of $z^{(h)}$ and $\chi$ on the flow parameter $t$.

Note that $z^{(h)}$ and $\chi$ are not physical observables, since they do not result from the integration over all degrees of freedom. Only their fixed-point values $z^{(h)}$ and $\chi^*$, obtained when $\Lambda(t) = 0$, are physical (observable) quantities. The Landau $f$ function is therefore related to $\chi^*$.

[Note that $\chi^*$ corresponds to what we denoted by $\chi$ in the preceding sections.]
A. one-loop order

The integration of high-energy degrees of freedom \(|k - k_F| > \Lambda_0\) leading to the low-energy effective action Eq. (5.1) will in general generate a wave-function renormalization factor \(z_{\Lambda_0} < 1\) and an external field renormalization \(z^{(h)}|_{\Lambda_0}\). We ignore these complications which will be discussed in section V B. Thus, the initial conditions of the RG equations, besides \(\Gamma|_{\Lambda_0} = \Gamma^0\), are

\[
\begin{align*}
  z_{\sigma\sigma'}^{(h)}(\theta)|_{\Lambda_0} &= \delta_{\sigma,\sigma'} 2\pi \delta(\theta), \\
  \chi_{\sigma\sigma'}(\theta)|_{\Lambda_0} &= 0.
\end{align*}
\] (5.8)

The latter equation follows from the condition \(T < \Lambda_0\) and the fact that the external field \(h\) creates excitations only in the vicinity of the Fermi surface.

The external field renormalization at one-loop order is given by (see Fig. 2a in Ref. [10])

\[
dz^{(h)}_{\sigma\sigma'}(\theta - \theta') = \sum_{\alpha'} \int \frac{d\theta''}{2\pi} z_{\alpha''\sigma\sigma'}^{(h)}(\theta - \theta'') \Gamma_{\alpha''\sigma',\sigma'}^{(\alpha)}(\theta'' - \theta') \\
\times \frac{1}{\beta} \frac{k_F}{2} \int d\omega' \left[ (\tilde{G}(\omega'))^2 \right]^{1/2},
\] (5.9)

where \(\tilde{G}(\tilde{k}) = (i\omega - v_F(k - k_F))^{-1}\) is the QP propagator.

\(\int'\) indicates that the integration is restricted to the degrees of freedom that are in the infinitesimal momentum shell to be integrated out. Using

\[
\frac{1}{\beta} \sum_{\omega''} \int d\omega' (\tilde{G}(\omega'))^{1/2} = -\frac{\beta R}{v_F \cosh^2 \beta R} dt,
\] (5.10)

we obtain

\[
\frac{dz_{\sigma\sigma'}^{(h)}(l)}{dt} = -\frac{N(0) \beta R}{\cosh^2 \beta R} \sum_{\omega''} z_{\omega''\sigma\sigma'}^{(h)}(l) \Gamma_{\omega''\sigma',\sigma'}^{(\alpha)}(l).
\] (5.11)

We have introduced the dimensionless inverse temperature \(\beta R = v_F \beta A(t)/2\), and expanded in circular harmonics the quantities appearing in (5.9), \(z_{\sigma\sigma'}^{(h)}(\theta) = \sum_{t} z_{\sigma\sigma'}^{(h)}(l)e^{it\theta}\). Eq. (5.11) is solved by introducing

\[
\begin{align*}
  z^+_\alpha(l) &= z^{(h)}_{\uparrow\uparrow}(l), \\
  z^-_\alpha(l) &= z^{(h)}_{\downarrow\downarrow}(l),
\end{align*}
\] (5.12)

and the spin symmetric \((A^\sigma)\) and antisymmetric \((B^\sigma)\) parts of the two-particle vertex defined by

\[
2N(0) \Gamma^\sigma(l) = A^\sigma \delta_{\sigma_1,\sigma_2} \delta_{\sigma_3,\sigma} + B^\sigma \tau_{\sigma_1,\sigma_2} \tau_{\sigma_3,\sigma}.
\] (5.13)

where \(\tau\) denotes the Pauli matrices. Eq. (5.11) decouples into two independent equations:

\[
\frac{dln z^+_\alpha(l)}{dt} = -\frac{\beta R}{\cosh^2 \beta R} A^\alpha,
\]

\[
\frac{dln z^-_\alpha(l)}{dt} = -\frac{\beta R}{\cosh^2 \beta R} B^\alpha.
\] (5.14)

Eqs. (5.14) have to be supplemented with the one-loop RG equations for the vertex functions \(A^\sigma\) and \(B^\sigma\). As shown in Ref. [8], the latter can be written as

\[
\begin{align*}
  \frac{dA^\sigma}{dt} &= -\frac{\beta R}{\cosh^2 \beta R} A^\sigma + \frac{dA^\sigma_{\gamma}}{dt}, \\
  \frac{dB^\sigma}{dt} &= -\frac{\beta R}{\cosh^2 \beta R} B^\sigma + \frac{dB^\sigma_{\gamma}}{dt}.
\end{align*}
\] (5.15)

The first terms of the rhs of Eqs. (5.15) are the contribution of the ZS graph to the renormalization of \(\Gamma\). The contribution of the ZS' and BCS graphs is taken into account via the second terms on the rhs of (5.15) (see Fig. 1 in Ref. [1]). Because the thermal factor \(\beta R/\cosh^2 \beta R\) is a strongly peaked function of \(\Lambda(t)\) near \(\Lambda(0) = 0\) when \(T < 0\), these equations have the approximate solutions

\[
\begin{align*}
  A^\sigma_{\gamma}(\tau) &= \frac{A^\sigma_{\gamma}}{1 + (1 - \tau) A^\sigma_{\gamma}}, \\
  B^\sigma_{\gamma}(\tau) &= \frac{B^\sigma_{\gamma}}{1 + (1 - \tau) B^\sigma_{\gamma}}.
\end{align*}
\] (5.16)

for \(\Lambda(t) \lesssim T/v_F\). \(A^\sigma_{\gamma} = A^\sigma_{\gamma}|_{\Lambda(t)=0}\) and \(B^\sigma_{\gamma} = B^\sigma_{\gamma}|_{\Lambda(t)=0}\) are the fixed-point values of \(A^\sigma_{\gamma}\) and \(B^\sigma_{\gamma}\). We have introduced the parameter \(\tau = \tanh \beta R\) and used\(\tanh(\beta F \Lambda_0/2) \simeq 1\) for \(T < \Lambda_0/v_F\). Since Eqs. (5.16) hold beyond one-loop order, we postpone their detailed derivation to section V B. Eqs. (5.14) show that the RG flow of \(z^{(h)}\) becomes significant only at small energy when \(\Lambda(t) \lesssim T/v_F\). This allows to insert (5.16) into (5.14) and obtain

\[
\begin{align*}
  z^{(h)}_+(l) &= \frac{1}{1 + (1 - \tau) A^\sigma_{\gamma}}, \\
  z^{(h)}_-(l) &= \frac{1}{1 + (1 - \tau) B^\sigma_{\gamma}},
\end{align*}
\] (5.17)

using the initial conditions \(z^{(h)}_+(l)|_{\Lambda_0} = z^{(h)}_-(l)|_{\Lambda_0} = 1\).

The renormalization of \(\chi\) is given by (see Fig. 2b in Ref. [1])

\[
\frac{d\chi_{\sigma\sigma'}(\theta - \theta')}{dt} = -\sum_{\omega''} \frac{d\theta''}{2\pi} z^{(h)}_{\sigma\omega''}(\theta - \theta'') z^{(h)}_{\omega''\sigma'}(\theta' - \theta'') \\
\times \frac{k_F}{2\pi \beta R} \sum_{\omega'} \int d\omega' (\tilde{G}(\omega'))^{1/2}.
\] (5.18)

Expanding (5.18) in circular harmonics and using (5.10), we obtain

\[
\frac{d\chi_{\sigma\sigma'}(l)}{d\tau} = -\frac{N(0)}{\nu} \sum_{\omega''} z^{(h)}_{\omega''\sigma}(l) z^{(h)}_{\sigma\omega''}(l).
\] (5.19)

To derive (5.19) from (5.18), we have used the fact that \(z^{(h)}_{\sigma\sigma'}(\theta)\) is an even function of \(\theta\). Thus, we obtain
\[
\frac{d\chi_{\uparrow\uparrow}(l)}{dt} = -\frac{N(0)}{2\nu} [z_{+}^{(h)}(l)^2 + z_{-}^{(h)}(l)^2],
\]
\[
\frac{d\chi_{\uparrow\downarrow}(l)}{dt} = -\frac{N(0)}{2\nu} [z_{+}^{(h)}(l)^2 - z_{-}^{(h)}(l)^2].
\] (5.20)

The integration of (5.20) using (5.17) and the initial condition \(\chi_{\sigma\sigma}(l)\big|_{\Lambda_0} = 0\) yields the fixed-point matrix
\[
\chi_{\uparrow\uparrow}^*(l) = \frac{N(0)}{2\nu} \left[ \frac{1}{1 + A_{l}^{1\uparrow}} + \frac{1}{1 + B_{l}^{1\uparrow}} \right],
\]
\[
\chi_{\uparrow\downarrow}^*(l) = \frac{N(0)}{2\nu} \left[ \frac{1}{1 + A_{l}^{1\uparrow}} - \frac{1}{1 + B_{l}^{1\uparrow}} \right].
\] (5.21)

The inverse matrix is
\[
\chi_{\sigma\sigma}^{-1}(l) = \frac{\nu}{2N(0)} (2 + A_{l}^{1\uparrow} + B_{l}^{1\uparrow}),
\]
\[
\chi_{\uparrow\downarrow}^{-1}(l) = \frac{\nu}{2N(0)} (A_{l}^{1\uparrow} - B_{l}^{1\uparrow}).
\] (5.22)

From the microscopic definition of the Landau function \([\text{Eq. (5.3)}]\), we deduce
\[
f_{\sigma\sigma}(\theta) = \Gamma^{\Omega_{\sigma\sigma}, \sigma}_{\sigma}(\theta),
\]
(5.23)
a result first obtained in Ref. [8].

**B. Beyond one-loop**

In this section, we show that higher-order loop contributions do not change Eq. (5.23), except for a dependence of \(f\) on the wave-function renormalization factor. The RG equations can be solved exactly if we make the following assumptions (see also Ref. [10], section 5): we assume the existence of well-defined QP’s in the vicinity of the Fermi surface \(i\). Except the one-loop ZS graph, all the graphs are regular in the limit \(q \to 0\) \(i.e.\), give the same contribution in the \(q\)- and \(\Omega\)-limits \(ii\), and give a smooth contribution to the RG flow of various physical quantities \(iii\). [Because of the thermal factor \(\beta/4 \cosh^2 \beta_R\), which is a strongly peaked function of \(\Lambda(t)\) around \(\Lambda(t) = 0\) for \(T \to 0\), the ZS graph yields a singular contribution (with respect to \(\Lambda(t)\)) to the RG flow of \(\Gamma^q\).] Assumption \(ii\) can be explicitly verified at one-loop order. Assumption \(iii\) is only ‘approximate’ in a sense that is further discussed below.

Let us first consider the RG equations for the two-particle vertex. They can be written as
\[
\frac{dA_{l}^{\Omega}}{dt} = \frac{dA_{l}^{\Omega}}{dt}_{\text{ZS}} + \frac{dA_{l}^{\Omega}}{dt},
\] (5.24)
\[
\frac{dA_{l}^{\Omega}}{dt}_{\text{ZS}} = -\frac{\beta_R}{\cosh^2(\beta_R)} A_{l}^{\Omega},
\] (5.25)
and a similar equation for \(B_{l}^{\Omega}\). We have written explicitly the contribution of the one-loop ZS graph which distinguishes between the \(q\)- and \(\Omega\)-limits. The second term on the rhs of (5.24) includes the contribution of other one-loop graphs (ZS’ and BCS channels) as well as higher-order loop corrections. [Here we use assumption \(ii\) and the fact that the ZS graph vanishes in the \(\Omega\)-limit.]

It is tempting to neglect the term \(dA_{l}^{\Omega}/dt\) in (5.24) \(see\ for\ instance\ Ref. [23].\) Because of phase-space restrictions (which result from momentum conservation at the interaction vertices), the corresponding diagrams are suppressed by the small parameter \(\Lambda_0/k_F\) if one considers only low-energy states \(|k - k_F| \leq \Lambda_0\) (assuming that high-energy states \(|k - k_F| > \Lambda_0\) have already been integrated out). This property results from a frustration of the interferences between channels in dimension \(d \geq 2\). Solving (5.24) without the last term of the rhs is equivalent to an RPA calculation in the ZS channel \([24]\).

This RPA calculation can also be done using standard diagrammatic theory. Within this approximation, the Landau \(f\) function is naturally identified with the bare function \(\Gamma_{\Lambda_0}^{\Omega}\) \(\text{for instance Ref. [25].}\) We do not need to distinguish between the \(q\)- and the \(\Omega\)-limits, since \(\Gamma_{\Lambda_0}^{\Omega}\) is a regular function of its arguments.] A major drawback of this approximation is that the momentum scale \(\Lambda_0\), which is not a physical scale in the problem, enters the definition of the Landau parameters in an essential way. [For a further discussion of this RPA approximation, see Ref. [3]].

In Refs. [8,10] the present author and G. Chitov have proposed another approach to solve (5.24), which bears some similarities with Landau’s solution of the Bethe-Salpeter equation for the two-particle vertex. Besides the fact that it is not suppressed by the small parameter \(\Lambda_0/k_F\), the ZS graph presents another interesting feature. Its contribution to the RG flow of \(A^q\) and \(B^q\) becomes singular (with respect to \(\Lambda(t)\)) at low temperature since \(\beta_R/4 \cosh^2 \beta_R\) is exponentially suppressed for \(\Lambda(t) \gtrsim T\). To see how this allows to solve Eqs. (5.24,5.25), we integrate the latter (using \(A_{l}^{\Omega}\big|_{\Lambda_0} = A_{l}^{\Omega}_{\Lambda_0}\)):
\[
A_{l}^{\Omega}(t) = A_{l}^{\Omega}(t) - \int_0^t dt' \frac{\beta_R}{\cosh^2 \beta_R} A_{l}^{\Omega}(t')^2.
\] (5.26)

Iterating (5.26), we obtain
\[
A_{l}^{\Omega}(t) = A_{l}^{\Omega}(t) - \int_0^t dt' \frac{\beta_R}{\cosh^2 \beta_R} A_{l}^{\Omega}(t')^2 + \cdots
\] (5.27)

According to assumption \(iii\), \(A_{l}^{\Omega}(t')\) is a smooth function of the cut-off \(\Lambda(t')\) when \(\Lambda(t') \gtrsim T/v_F\). This implies \(A_{l}^{\Omega}(t)|_{\Lambda(t') \lesssim T/v_F} \simeq A_{l}^{\Omega}_{\Lambda_0}\) at low temperature. Since, on the other hand, \(\beta_R/4 \cosh^2 \beta_R\) is strongly peaked for \(\Lambda(t') \gtrsim T/v_F\), we can replace \(A_{l}^{\Omega}(t')\) in the rhs of (5.27) by its fixed-point value \(A_{l}^{\Omega}_{\Lambda_0}\). The RG equation for \(A^q\) then becomes (for \(\Lambda(t) \lesssim T/v_F\))
\[
A_{l}^{q}(t) = A_{l}^{\Omega}_{\Lambda_0} - \int_0^t dt' \frac{\beta_R}{\cosh^2 \beta_R} A_{l}^{q}(t')^2.
\] (5.28)

Eq. (5.28) corresponds to a decoupling of the ZS channel from the other channels: when \(\Lambda(t) \gtrsim T/v_F\), the ZS
graph does not contribute to the RG flow, and both $A^i$ and $A^\Omega$ evolve smoothly when the cut-off decreases (with $A^i \simeq A^\Omega$); when $\Lambda(t)$ varies from $\sim T/v_F$ to 0, only the ZS graph gives a significant contribution (assumption (iii)) and drives $A^i$ towards its fixed-point value $A^{\ast i}$ (but does not contribute to the renormalization of $A^\Omega$). The solution of Eq. (5.25) and the analog equation for $B^i$ is given by (5.16). We now obtain an RPA-like relation between two fixed-point quantities: $A^{\ast i} = A^\Omega_i/(1 + A^{\ast 3})$. Both quantities are physical in the sense that they are obtained by integrating all the degrees of freedom.

One should combine the RG equations of $A^i$ and $B^i$ with RG equations of the Fermi velocity $v_F(t)$ and the wave-function renormalization factor $z(t)$. Since the ZS loop does not appear in self-energy corrections, $v_F(t)$ and $z(t)$ are smooth functions of the cut-off $\Lambda(t)$. Thus, the argument leading to (5.28) and (5.16) also holds when solving (5.25), one should replace $v_F(t)$ and $z(t)$ by their fixed-point values $v_F^{\ast}$ and $z^{\ast i}$.

If assumption (iii) were to hold exactly, Eqs. (5.16) would be exact for $\Lambda(t) \lesssim T/v_F$ and $T \to 0$. It has been pointed out in Ref. 10 that the ZS graph also gives a singular contribution to the flow of $\Gamma^i(\theta - \theta')$ when the two incoming particles have parallel momenta $(\theta - \theta' \to 0)$. In this case, the ZS loop becomes identical to the ZS loop (see Fig. 1 of Ref. 10). Thus, assumption (iii) is not exact and Eqs. (5.16) hold only approximately. The singularity in the ZS channel is however restricted to very small angles $|\theta - \theta'| \approx T/v_F k_F$. Only for those angles do the ZS and ZS' channels interfere when $\Lambda(t) \lesssim T/v_F$. Consequently, only the components $\Gamma^i(l)$ with $l \gtrsim v_F k_F/T$ are affected by the small-angle singularity in the ZS channel. For most physical quantities (specific heat, effective mass, compressibility...), Eqs. (5.16) remain an excellent approximation. The singularity of the ZS graph becomes crucial if one is precisely interested in the value of $\Gamma^i(\theta - \theta')$ for $\theta - \theta' \to 0$. This conclusion has been checked by explicit calculation at one-loop order by Chitov and Sénéchal.

At each step of the RG transformation, the field is rescaled according to $\tilde{\psi}' = [z(dt)]^{-1/2} \tilde{\psi}$, where $z(dt)$ is the contribution to the wave-function renormalization factor $z(t)$ when the flow parameter $t$ increases by $dt$: $z(t + dt) = z(t)z(dt)$. This ensures that the propagator keeps the form $(i\omega - v_F(t)k)^{-1}$ Thus the field $\psi$ refers to QP's as anticipated.

Now we consider the renormalization of the source field. Although the action is expressed only in terms of the QP field, $h$ (as any other external field) couples a priori to the bare fermions. The coupling between the external field and the incoherent part of the single-particle spectral function appears indirectly through some renormalization of the external field (see for instance Ref. 23 for a discussion). Here we want to eliminate such renormalizations since $h$ couples directly to the QP's. To distinguish between the coherent (i.e., due to the QP's) and incoherent responses to the field $h$, we also consider the response to the field $h^\Omega$ that couples to the incoherent part of the density:

$$\int_0^\beta d\tau \rho^{\Omega}_{\sigma'}(\theta, \tau) = v_F N(0) \int_{k_F - \Lambda_0}^{k_F + \Lambda_0} dk \times \lim_{\Omega \to 0} [\tilde{\psi}'_n(\tilde{k} + \tilde{q}) \tilde{\psi}_n(\tilde{k})]_{q=0}, \quad (5.29)$$

When $T \to 0$, $h^\Omega$ cannot create coherent particle-hole pairs (i.e., quasi-particle-quasi-hole pairs) since the $\Omega$-limit is taken in (5.23). On the contrary, $h$ couples to both the coherent and incoherent parts. $n$-loop contributions ($n \geq 2$) to the field renormalization factor $z^{(b)}$ do not distinguish between $h$ and $h^\Omega$. The reason is that the singular ZS loop does not appear in $n$-loop ($n \geq 2$) diagrams. Therefore, the latter correspond to coupling of the external field to the incoherent part. On the contrary, the one-loop graph considered in the preceding section vanishes in the $\Omega$-limit and corresponds to coupling of the external field from the incoherent part. Since we want $h$ to couple only to the QP’s, only the one-loop diagram for $z^{(b)}$ has to be taken into account. QP’s are obtained not only by filtering out the incoherent part of the propagator, but also by rescaling the field according to $\psi' = [z(dt)]^{-1/2} \psi$. The latter implies a concomitant rescaling of the field $h' = h z(dt)$ to ensure that $h' \psi = h' \psi^* \psi'$. We conclude that the only renormalization of the external field comes from the one-loop contribution to $z^{(b)}$ considered in the preceding section.

We also note that the diagram shown in Fig. 2b of Ref. 10 is the only one contributing to the susceptibility in the Kadanoff-Wilson scheme, since it is the only diagram of order $O(h^2)$ (and $O(dt)$) generated by the RG procedure. Thus, we obtain the same RG equations as in the one-loop calculation, the bare Fermi velocity being replaced by its fixed-point value $v_F^{\ast}$. Eq. (5.25) holds at all order in a loop expansion (when the small-angle singularity in the ZS channel is neglected). Since the $\tilde{\psi}$’s have been rescaled at each step of the RG transformation, we eventually come to

$$f_{\sigma, \sigma'}(\theta) = z^2 q_{\sigma, \sigma'}(\theta), \quad (5.30)$$

where $q_{\sigma, \sigma'}^{\Omega}$ now refers to the bare fermions, and $z^*$ is the fixed-point value of $z(t)$. Eq. (5.30) agrees with the conclusion of Ref. 10.

VI. CONCLUSION

We have proposed a new microscopic description of Fermi liquids, which extends some early ideas of the statistical FLT. It is based on the introduction of an effective potential (in the sense of field theory) $\Phi[n]$, which is obtained from the free energy by a Legendre transformation.

In the more general case, the effective potential $\Phi[n]$ is a functional of the Wigner distribution function $n \equiv \cdots$
\{\eta_{k\sigma}(r,t)\}\). Small variations $\delta\Phi[\tilde{\eta}]$ around the equilibrium value are parameterized by the Landau $f$ function, which describes the interaction between QPs. $f$ has a precise microscopic definition in terms of the susceptibility $\chi$ introduced in section II. Using the standard assumptions of FLT, we have shown that this microscopic definition yields the usual identification between suppositions of FLT, we have shown that this microscopic definition in terms of the susceptibility value are parameterized by the Landau equation satisfied by $\eta_{k\sigma}(r,t)$. In the static and uniform limit, $\delta\Phi[\tilde{\eta}]$ is nothing but the variation of the thermodynamic potential corresponding to a change $\delta\eta$ of the QP distribution function. $\delta\Phi[\tilde{\eta}]$ was first introduced by Landau on phenomenological grounds to describe Fermi liquids. Thus, the effective potential describes both static and dynamic properties of Fermi liquids in a unified framework. It should be noted that this description does not rely on any semiclassical assumption.

The explicit calculation of $\delta\Phi[\tilde{\eta}]$ can be in principle extended to more complicated and/or realistic situations, for instance by taking into account the presence of impurities and the effect of an electric field.

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