An engineering model for yield inception in slip-stick elastic contacts

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Abstract. The failure of the mechanical contact due to plastic yielding is generally predicted employing stress analysis coupled with the von Mises yield criterion, which uses the maximum of the second deviatoric stress invariant as a threshold value. This paper aims to establish the relation between the frictional regime and the normal and tangential loading components which lead to yield inception in the slip-stick spherical contact between similarly elastic materials. The Boussinesq and Cerruti fundamental solutions for the elastic half-space are used in a robust semi-analytical method based on the superposition principle applicable in the frame of linear elasticity, and enhanced with an acceleration technique derived from the convolution theorem. A rapid algorithm for accurate computation of elastic stresses induced in subsurface by a known but arbitrary distribution of surface tractions, normal or shear, is advanced. The obtained data is normalized to allow model extension to any elastic constants or contact curvature, and curve fitting is employed to derive simple empirical formulas pertinent to practical engineering applications.

1. Introduction
Stress analysis in the elastic contact between two spheres under combined normal and tangential loading is a fundamental problem in engineering, applicable in the modeling of particle–flow simulations or in study of contact between rough surfaces. While the surface tractions distribution in a normally loaded frictionless contact problem is known from the Hertz theory [1], the tangential contact problem, involving the establishment of a slip annular region surrounding the central stick contact zone, was solved separately by Cattaneo [2] and by Mindlin [3].

The potential contact failure due to plastic yielding is generally predicted using the von Mises yield criterion, which uses the maximum of the second deviatoric stress invariant to locate the plastic yield inception in the elastic body. The problem of assessing the stress field throughout an elastic and isotropic body, loaded by surface shear stresses acting simultaneously and proportional to normal Hertz-type stresses was solved by Hamilton and Goodman [4] for a circular point contact and by Bryant and Keer [5] for an elliptical contact. The latter work advances an explicit solution for the surface and subsurface stresses and displacements in a full slip contact, and indicates that the solution of the slip-stick problem could be obtained by superimposing three full-slip solutions. Due to problem complexity, the study of the influence of the stick region on the stress field is restricted to a single instance. The problem is revisited by Hills and Sackfield [6], who obtained an approximate solution to the stress field induced by the mutual compression of two spheres with dissimilar elastic properties. In
the approach advanced in this paper, the normal and tangential contact problems are decoupled, as the influence of shear tractions on the normal displacements is neglected and, consequently, the contact pressure is assumed to conserve a hertzian profile. A numerical solution to the surface stresses arising in the fully coupled slip-stick contact problem was achieved numerically by Spinu and Glovnea [7] and could be used to extend the work reported herein.

Considering the limited information existing in the literature, the goal of this paper is to establish the influence of the frictional regime and of the normal and tangential loading components on plastic yield inception in a slip-stick spherical contact between similarly elastic materials. The study is based on computation of the stress field using a robust semi-analytical method coupled with an acceleration technique based on the convolution theorem.

2. Semi-analytical model

The method applied herein in assessment of subsurface stresses induced in an elastic solid by a known distribution of surface tractions, normal and/or shear, is based on the superposition principle applicable in the frame of linear elasticity, and on the fundamental solutions derived for the elastic half-space by Boussinesq [8] and by Cerruti [9], expressing the stress and displacement field induced in an elastic and isotropic half-space by a normal or tangential concentrated force acting on the half-space boundary. The use of these fundamental solutions is authorized by the so-called half-space approximation employed in the theory of contact mechanics, assuming bodies of arbitrary (yet smooth) surface as semi-infinite solids confined by a plane boundary when computing the corresponding stress and displacement fields. This assumption is reasonable in case of concentrated contacts, when the dimensions of the contact area are small compared to the relevant dimensions of contacting bodies.

The main advantage of this semi-analytical method consists in its ability to handle arbitrary distributions of surface tractions, as long as the effect of the stress gradients can be simulated with adequate accuracy using a piecewise constant digitized counterpart. Stress risers are not to be expected in the contact between bodies assumed smooth in the contact region, and therefore relative errors of less than 1% can be obtained by imposing even coarser grids of $10^5$ points in the contact region and its vicinity.

For the contact scenario simulated in this paper, the surface stresses developed during contact evolution are known from the works of Hertz [1] for the normal tractions (i.e. contact pressure), and of Cattaneo [2] and of Mindlin [3] for the shear tractions. It should be noted that the contacting spheres are assumed to have identical elastic constants, which decouples the normal and the tangential contact problems. Consequently, shear tractions developed during application of the tangential load do not affect contact pressure, which preserve a Hertz profile specific to contacts undergoing normal loading only.

The contact model is obtained in a Cartesian coordinate system $x_1 x_2 x_3$ with the $x_1$ and $x_2$-axes contained in the common plane of contact, i.e. the plane tangent to the contacting surfaces in the initial point of contact. The normal loading consist in a force $W$ generating a contact pressure $p$, while the tangential T force, acting along direction of $\bar{x}_1$, induces the shear tractions $q_1$ and $q_2$. The semi-analytical model can readily handle tangential load components along both directions, but confining the direction of the tangential force to that of $\bar{x}_1$ limits the number of independent input parameters without loss of generality (the coordinate system could be conveniently rotated to accommodate any direction of the tangential force).

Let $\sigma_{ik}(x_1, x_2, x_3)$ denote the $ij$ stress tensor components induced by the generic surface traction $t_k(x_1, x_2)$ ($t_k \equiv p$ for $k = 3$ and $t_k \equiv q_k$ for $k = 1, 2$) acting on the contact area $A_c$. By applying the superposition principle in the frame of linear theory of elasticity, the following relation holds:
\[ \sigma_{ijk}(x_1, x_2, x_3) = \int \int_{\mathcal{A}} G_{ijk}(x_1 - x'_1, x_2 - x'_2, x_3) t_k(x'_1, x'_2) dx'_1 dx'_2 \]

where \( G_{ijk}(x_1, x_2, x_3), \; i, j, k = 1, 2, 3, \) denotes the appropriate Green function for the elastic half-space, expressing the \( ij \) stress tensor component induced at coordinates \( (x_1, x_2, x_3) \) by a unity surface traction acting in origin along direction of \( \hat{x}_k \). Closed-form expressions were derived by Boussinesq [8] for \( k = 3 \) and by Cerruti [9] for \( k = 1, 2 \). Analytical integration in equation 1 is difficult and leads to lengthy expressions even for the simplest case of Hertz contact, so it won’t be performed explicitly in this paper. A robust semi-analytical technique is employed instead, which can be generalized to allow for any known but otherwise arbitrary distribution of surface tractions.

In the semi-analytical model of stress analysis, it is convenient to divide the volume domain from the contact proximity into a collection of non-intersecting cuboids using a rectangular three-dimensional mesh having \( N_1 \times N_2 \times N_3 \) elementary domains, of side lengths \( d_1, d_2 \) and \( d_3 \), respectively. In each elementary cuboid, subsurface stresses are assumed constant and equal to the value computed using the coordinates of a representative point, usually the center of the cuboid. Surface stresses are handled in the same manner, but the set of upper peripheral cuboids is chosen so that the plane containing the centers coincides with the half-space boundary, as shown in figure 1. This arrangement guarantees accurate stress estimation on the boundary and immediately under the surface, which is important considering that the maximum von Mises equivalent stress is located on the free surface in high friction regimes.

In this manner, all otherwise continuous problem parameter are substituted by piecewise constant distributions, which brings the advantage of convenient numerical treatment. The stress in every subsurface cuboid is approximated by superposing the individual contributions of each elementary cuboid that touches the half-space boundary. Giving the discrete nature of the semi-analytical model, it is also convenient to introduce a simplified notation, in which continuous coordinates are substituted by integers indexing the position of the referred cuboid in the grid, e.g. the value of any parameter \( \tau \) in the center of the cuboid \( (i, j, k), \; i = 1..N_1, \; j = 1..N_2, \; k = 1..N_3 \), will be denoted by \( \tau(i, j, k) \). As the stress is assumed constant within each boundary patch, it can be factored outside the integral operator, yielding the semi-analytical counterpart of equation 1, in which integration with infinity as limits is substituted by a combination of summation and integration over rectangular boundary patches.
\[ \sigma_{ik}(n, p, q) = \sum_{(m, \ell) \in A_C} \left( t_k(m, \ell) \cdot \int_{x_2 \in [d_2/2]} \int_{x_1 \in [d_1/2]} G_{ijk} \left(x_i(n) - x_i', x_2(p) - x_2', x_3(q)\right) dx_1' dx_2' \right) \]  

(2)

where \( i, j, k = 1, 2, 3 \), \( m, n = 1..N_1 \), \( p, \ell = 1..N_2 \) and \( q = 1..N_3 \). The contribution of each elementary patch loaded by a unity surface traction can be expressed in closed-form, yielding the so-called influence coefficients \( IC_{ijk} \) derived by double integration of the corresponding Green functions \( G_{ijk} \) from equation (2). The semi-analytical counterpart of equation (1) can be further expressed as:

\[ \sigma_{ik}(n, p, q) = \sum_{(m, \ell) \in A_C} IC_{ijk}(n-m, p-\ell, q) \cdot t_k(m, \ell) \]  

(3)

in which the influence coefficient \( IC_{ijk}(n-m, p-\ell, q) \) expresses the contribution of a unity surface traction along direction of \( \hat{x}_i \), acting on the upper face the peripheral cuboid \( (m, \ell, 1) \), to the \( ij \) stress tensor component in the cuboid \( (n, p, q) \). These influence coefficients can be easily obtained in closed-form. The derivation of the influence coefficients for contacts involving thin elastic layers has already been discussed in [10]. It can be seen that equation 3, which is the semi-analytical counterpart of the analytical integration in equation (1), is a discrete double convolution with respect to directions of \( \hat{x}_i \) and \( \hat{x}_2 \). Specialized methods for increasing the algorithmic efficiency of convolution computation can be employed, as discussed in the following section.

3. Improvement of algorithmic efficiency

The double convolution in equation (3) is computationally intensive, having an order of computation of \( O(N^2) \) for a grid with \( N \) elements, thus constraining dramatically the resolution that can be imposed. The computational complexity can be reduced by employing spectral methods based on the convolution theorem, which transfer the convolution calculation from the space domain (SD) to the frequency domain (FD), where its computational complexity is reduced to an improved order of \( O(N \log N) \). The source of this reduction is the convolution theorem [11], which states that convolution in the SD can be computed as an element-by-element product in the FD, in \( O(N) \) operations. The operations resulting from applying the direct (FFT) and inverse (IFFT) fast Fourier transform to achieve transfer back and forth between the SD and the FD claim additional computational effort. This technique has already been successfully applied [12] to other types of convolution-related problems in the field of mechanics.

The term in the right-hand side of equation (3) is a two-dimensional discrete convolution with respect to directions of \( \hat{x}_i \) and \( \hat{x}_2 \), and can be treated efficiently by adapting the DCFFT technique advanced in [13]. The steps of the proposed algorithm are described herein.

Firstly, compute the three-dimensional arrays of influence coefficients. It should be noted that the same operations apply to all terms in equation (3), regardless of the referred traction or stress tensor component, i.e. regardless of the instance of indices \( i, j, k = 1, 2, 3 \). Therefore, the subscripts denoting the tensor components or the traction will be omitted in this section for brevity, and classical matrix notation will be employed instead, i.e. \( IC_{i,j,m} \), \( 1 \leq i \leq 2N_1 \), \( 1 \leq j \leq 2N_2 \), \( 1 \leq m \leq N_3 \), is the \( (i, j, m) \) component of any of the 18 three-dimensional arrays of influence coefficients entering equation 3, relating any stress tensor component to any surface traction. For further brevity, as the algorithm steps match along any direction, algorithm description will be limited to the one-dimensional case. The notation of the influence coefficients simplifies accordingly to \( IC_{i, \ell} \), \( 1 \leq \ell \leq 2N_k \), where \( k \) can be any
of the two directions \( k = 1, 2 \), and \( N_k \) is the number of grids in the direction of \( x_k \). The advanced algorithm can also be applied to any multidimensional configurations [12], when \( k > 2 \). The influence coefficient \( IC_1 \) corresponds to the negative greatest distance between two grid nodes taken along the \( x_k \)-axis, i.e. when the observation mark is indexed with 1 and the source mark with \( N_k \). The situation when the observation and the source points are interchanged corresponds to \( IC_{2N_k} \).

The next step consists in reordering of \( IC_1 \) terms to avoid the so-called periodicity error related to transfer to and from the FD, and its base is detailed in [14]. The main idea is that, whenever FFT is applied to a series, the latter is assumed periodical, which is not the case in the discussed contact scenario (but could be true for rough contacts problems for which a representative single asperity is considered). A new vector \( K_1 \) is generated by zero-padding and rearrangement in wrap-around order, i.e. \( K_1 = IC_{N_k+1} \) for \( 1 \leq \ell \leq N_k \); \( K_{N_k+1} = 0 \) and \( K_1 = IC_{N_k-N_k} \) for \( N_k + 2 \leq \ell \leq 2N_k \). The same notation convention is applied to the digitized tractions. The generic one-dimensional array of tractions \( t_\ell \), with \( 1 \leq \ell \leq N_k \), \( k = 1 \) or \( 2 \), is extended by zero-padding to match the size of the \( K_1 \) vector: \( P_\ell = t_\ell \) for \( 1 \leq \ell \leq N_k \) and \( P_\ell = 0 \) for \( N_k + 1 \leq \ell \leq 2N_k \).

In the following step, the \( K \) and \( P \) vectors are transferred to FD by means of FFT: \( \hat{K} = \text{FFT}(K) \), \( \hat{P} = \text{FFT}(P) \). The spectral array of elastic stresses if further obtained simply as element-wise product in the FD: \( \hat{\sigma}_\ell = \hat{K}_\ell \cdot \hat{P}_\ell \) for \( 1 \leq \ell \leq 2N_k \). Its SD counterpart is retrieved by means of inverse Fourier transform, \( \sigma = \text{IFFT}(\hat{\sigma}) \), and only the \( 1 \leq \ell \leq N_k \) terms are kept as algorithm output. This algorithm must be applied for any of the \( N^3 \) layers of constant depth, for any of the six stress tensor components and for any of the three surface tractions.

4. Results and discussions

The simulation of the slip-stick elastic contact was conducted by defining a normalized tangential force \( \bar{T} = T/(\mu W) \) which acts simultaneously with the initial normal loading. It should be noted that the contact is in a slip-stick regime for \( \bar{T} < 1 \) and in full sliding for \( \bar{T} = 1 \).

The frictional coefficient \( \mu \) is assumed constant in time and uniform on the contact area, which is a necessary assumption for traction computation according to [1, 2, and 3]. This model can handle
mapped frictional coefficients as well by employing the contact model advanced in [7] for surface traction assessment, but the number of independent input parameters increases impeding results generalization. Variation of the maximum von Mises equivalent stress $\sigma_{\text{vM}}$ with the frictional coefficient and with the level of the tangential loading is depicted in figure 2. This surface, denoted by $f(\mu, \bar{T})$, was obtained numerically by varying the input parameters with a step of 0.01 for both $\mu$ and $\bar{T}$, by computing the corresponding stress state and by extracting the magnitude of the maximum von Mises stress. The data points for specific fixed frictional coefficients were fitted to a polynomial model $f(\bar{T}) = \sum_{i=1}^{n} a_i \bar{T}^i$ and the dimensionless best-fit parameters are given in Table 1. It should be noted that, because of the applied normalization involving Hertz contact parameters, these results hold true for any contact load, elastic constants or contact curvature. This model allows rapid estimation of the relation between the load, the frictional regime and the yield inception in the elastic bodies. For a given frictional coefficient, the magnitude of the tangential load that lead to plastic flow can be obtained, according to the von Mises criterion, by solving the equation $f(\bar{T}) = Y/p_\mu$, where $Y$ is the yield strength of the elastic material. Conversely, the maximum allowed frictional coefficient for a given tangential loading level can be found by solving the equation $f(\mu) = Y/p_\mu$. Alternatively, any combination of frictional coefficient and tangential loading can be checked for consistency with the elastic limit by verifying that the inequality $f(\mu, \bar{T}) \leq Y/p_\mu$ is satisfied. This model provides simple, yet effective means of assessing the strength of the mechanical contact subjected to simultaneous normal and tangential loading.

Table 1. Best-fit parameters for assessing the maximum tangential load that can be accommodated for a fixed $\mu$.

| $\mu$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.1  | 0.62  | -0.00306 | 0.0691 | -0.5557 | 2.362 | -5.487 | 7.091 | -4.781 | 1.311 |
| 0.2  | 0.6201 | -0.01374 | 0.2948 | -2.435 | 10.31 | -23.87 | 30.77 | -20.69 | 5.658 |
| 0.3  | 0.6202 | -0.02549 | 0.5507 | -4.553 | 19.42 | -45.42 | 59.22 | -40.32 | 11.17 |
| 0.4  | 0.6193 | 0.07853 | -1.706 | 15.2 | -66.42 | 157.8 | -206.6 | 139.4 | -37.7 |
| 0.5  | 0.6177 | 0.2014 | -3.548 | 24.29 | -75.88 | 113.4 | -70.7 | 4.17 | 8.379 |
| 0.6  | 0.6207 | 0.00978 | -1.661 | 26.21 | -151.8 | 416.6 | -575.4 | 390 | -103.5 |
| 0.7  | 0.628 | -0.8206 | 16.66 | -128.2 | 471.7 | -910 | 961.4 | -529.7 | 119.5 |
| 0.8  | 0.6242 | -0.6052 | 15.62 | -146 | 631.3 | -1389 | 1641 | -997.5 | 245.4 |
| 0.9  | 0.6153 | 0.2342 | 0.4083 | -44.56 | 319.4 | -876.5 | 1178 | -780.4 | 204.6 |

5. Conclusions

This paper reports the derivation of a robust and efficient algorithm for stress analysis in frictional elastic contacts undergoing combined normal and tangential loading that lead to a slip-stick regime. The semi-analytical method employs digitization of a continuous double convolution by assuming piecewise constant distributions of involved parameters. The computation of the resulting discrete double convolution is accelerated by transferring the convolution computation to the spectral domain, where its complexity is reduced to that of an element-wise operation. Although additional treatment of convolution members is required to avoid the periodicity error, the advancement in computational efficiency is remarkable and allows for repeated computation of elastic stresses while varying the input parameters. The relation between the frictional and the loading parameters can be expressed by model fitting the results obtained in this manner, leading to simple, yet effective, equations that can be
easily incorporated in engineering models. Moreover, the semi-analytical method advanced herein can be used as a generic tool for stress analysis in the frame of theory of elasticity.

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