OPTIMAL REGULARITY AND THE FREE BOUNDARY IN THE PARABOLIC SIGNORINI PROBLEM

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Abstract. We give a comprehensive treatment of the parabolic Signorini problem based on a generalization of Almgren's monotonicity of the frequency. This includes the proof of the optimal regularity of solutions, classification of free boundary points, the regularity of the regular set and the structure of the singular set.

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1. Introduction

Given a domain $\Omega$ in $\mathbb{R}^n$, $n \geq 2$, with a sufficiently regular boundary $\partial \Omega$, let $\mathcal{M}$ be a relatively open subset of $\partial \Omega$ (in its relative topology), and set $S = \partial \Omega \setminus \mathcal{M}$. We consider the solution of the problem

\begin{align}
\text{(1.1)} & \quad \Delta v - \partial_t v = 0 \quad \text{in } \Omega_T := \Omega \times (0, T], \\
\text{(1.2)} & \quad v \geq \varphi, \quad \partial_r v \geq 0, \quad (v - \varphi) \partial_r v = 0 \quad \text{on } \mathcal{M}_T := \mathcal{M} \times (0, T], \\
\text{(1.3)} & \quad v = g \quad \text{on } S_T := S \times (0, T], \\
\text{(1.4)} & \quad v(\cdot, 0) = \varphi_0 \quad \text{on } \Omega_0 := \Omega \times \{0\},
\end{align}

where $\partial_r$ is the outer normal derivative on $\partial \Omega$, and $\varphi : \mathcal{M}_T \to \mathbb{R}$, $\varphi_0 : \Omega_0 \to \mathbb{R}$, and $g : \mathcal{S}_T \to \mathbb{R}$ are prescribed functions satisfying the compatibility conditions: $\varphi_0 \geq \varphi$ on $\mathcal{M} \times \{0\}$, $g \geq \varphi$ on $\partial \mathcal{S} \times (0, T]$, and $g = \varphi$ on $\mathcal{S} \times \{0\}$, see Fig. 1. The condition (1.2) is known as the Signorini boundary condition and the problem (1.1)–(1.4) as the (parabolic) Signorini problem for the heat equation. The function $\varphi$ is called the thin obstacle, since $v$ is restricted to stay above $\varphi$ on $\mathcal{M}_T$. Classical examples where Signorini-type boundary conditions appear are the problems with unilateral constraints in elastostatics (including the original Signorini problem [Sig59,Fic63]), problems with semipermeable membranes in fluid mechanics (including the phenomenon of osmosis and osmotic pressure in biochemistry), and the problems on the temperature control on the boundary in thermics. We refer to the book of Duvaut and Lions [DL76], where many such applications are discussed and the mathematical models are derived.

Another historical importance of the parabolic Signorini problem is that it serves as one of the prototypical examples of evolutionary variational inequalities, going back to the foundational paper by Lions and Stampacchia [LS67], where the existence and uniqueness of certain weak solutions were established. In this paper, we work with a stronger notion of solution. Thus, we say that a function $v \in W^{1,0}_2(\Omega_T)$ solves (1.1)–(1.4) if

$$\int_{\Omega_T} \nabla \nabla (w - v) + \partial_t (w - v) \geq 0 \quad \text{for every } w \in \mathcal{K},$$

$$v \in \mathcal{K}, \quad \partial_t v \in L^2(\Omega_T), \quad v(\cdot, 0) = \varphi_0,$$

where $\mathcal{K} = \{w \in W^{1,0}_2(\Omega_T) \mid w \geq \varphi \text{ on } \mathcal{M}_T, \ w = g \text{ on } \mathcal{S}_T\}$. The reader should see Section 2.2 for the definitions of the relevant parabolic functional classes. The existence and uniqueness of such $v$, under some natural assumptions on $\varphi$, $\varphi_0$, and $g$ can be found in [Bre72,DL76,AUS86,AU96]. See also Section 3 for more details.

Two major questions arise in the study of the problem (1.1)–(1.4):

- the regularity properties of $v$;
- the structure and regularity of the free boundary

$$\Gamma(v) = \partial_{\mathcal{M}_T} \{ (x, t) \in \mathcal{M}_T \mid v(x, t) > \varphi(x, t) \},$$

where $\partial_{\mathcal{M}_T}$ indicates the boundary in the relative topology of $\mathcal{M}_T$.

Concerning the regularity of $v$, it has long been known that the spatial derivatives $\partial_{x_i} v$, $i = 1, \ldots, n$, are $\alpha$-Hölder continuous on compact subsets of $\Omega_T \cup \mathcal{M}_T$, for some unspecified $\alpha \in (0, 1)$. In the time-independent case, such regularity was first proved by Richardson [Ric78] in dimension $n = 2$, and by Caffarelli [Caf79] for
n ≥ 3. In the parabolic case, this was first proved by Athanasopoulos [Ath82], and subsequently by Uraltseva [Ura85] (see also [AU88]), under certain regularity assumptions on the boundary data, which were further relaxed by Arkhipova and Uraltseva [AU96].

We note that the Hölder continuity of $\partial_x v$ is the best regularity one should expect for the solution of (1.1)–(1.4). This can be seen from the example

$$v = \text{Re}(x_1 + ix_n)^{3/2}, \quad x_n ≥ 0,$$

which is a harmonic function in $\mathbb{R}^n$, and satisfies the Signorini boundary conditions on $M = \mathbb{R}^{n-1} \times \{0\}$, with thin obstacle $\varphi ≡ 0$. This example also suggests that the optimal Hölder exponent for $\partial_x v$ should be $1/2$, at least when $M$ is flat (contained in a hyperplane). Indeed, in the time-independent case, such optimal $C^{1,1/2}$ regularity was proved in dimension $n = 2$ in the cited paper by Richardson [Ric78]. The case of arbitrary space dimension (still time-independent), however, had to wait for the breakthrough work of Athanasopoulos and Caffarelli [AC04]. Very recently, the proof of the optimal regularity for the original Signorini problem in elastostatics was announced by Andersson [And12].

One of the main objectives of this paper is to establish, in the parabolic Signorini problem, and for a flat thin manifold $M$, that $\nabla v ∈ H^{1/2, 1/4}_{\text{loc}}(\Omega_T \cup M_T)$, or more precisely that $v ∈ H^{3/2, 3/4}_{\text{loc}}(\Omega_T \cup M_T)$, see Theorem 9.1 below. Our approach is inspired by the works of Athanasopoulos, Caffarelli, and Salsa [ACS08] and Caffarelli, Salsa, and Silvestre [CSS08] on the time-independent problem. In such papers a generalization of the celebrated Almgren’s frequency formula established in [Alm00] was used, not only to give an alternative proof of the optimal $C^{1,1/2}$ regularity of solutions, but also to study the so-called regular set $\mathcal{R}(v)$ of the free boundary $\Gamma(v)$. This approach was subsequently refined in [GP09] by the second and third named authors with the objective of classifying the free boundary points according to their separation rate from the thin obstacle $\varphi$. In [GP09] the authors also introduced generalizations of Weiss’s and Monneau’s monotonicity formulas, originally developed in [Wei99a] and [Mon03], respectively, for the classical obstacle problem. Such generalized Weiss’s and Monneau’s monotonicity formulas allowed to prove a structural theorem on the so-called singular set $\Sigma(v)$ of the free boundary, see
For an exposition of these results in the case when the thin obstacle $\phi \equiv 0$ we also refer to the book by the third named author, Shahgholian, and Uraltseva [PSU12, Chapter 9].

In closing we mention that, as far as we are aware of, the only result presently available concerning the free boundary in the parabolic setting is that of Athanassopoulos [Ath84], under assumptions on the boundary data that guarantee boundedness and nonnegativity of $\partial_t v$. In that paper it is shown that the free boundary is locally given as a graph

$$ t = h(x_1, \ldots, x_{n-1}), $$

for a Lipschitz continuous function $h$.

1.1. Overview of the main results. In this paper we extend all the above mentioned results from the elliptic to the parabolic case. We focus on the situation when the principal part of the diffusion operator is the Laplacian and that the thin manifold $M$ is flat and contained in $\mathbb{R}^{n-1} \times \{0\}$.

One of our central results is a generalization of Almgren’s frequency formula [Alm00], see Theorem 6.3 below. As it is well known, the parabolic counterpart of Almgren’s formula was established by Poon [Poo96], for functions which are caloric in an infinite strip $S_\rho = \mathbb{R}^n \times (-\rho^2, 0]$. In Section 6 we establish a truncated version of that formula for the solutions of the parabolic Signorini problem, similar to the ones in [CSS08] and [GP09]. The time dependent case presents, however, substantial novel challenges with respect to the elliptic setting. These are mainly due to the lack of regularity of the solution in the $t$-variable, a fact which makes the justification of differentiation formulas and the control of error terms quite difficult. To overcome these obstructions, we have introduced (Steklov-type) averaged versions of the quantities involved in our main monotonicity formulas. This basic idea has enabled us to successfully control the error terms.

Similarly to what was done in [GP09], we then undertake a systematic classification of the free boundary points based on the limit at the point in question of the generalized frequency function. When the thin obstacle $\phi$ is in the class $H^{\ell, \ell/2}$, $\ell \geq 2$, this classification translates into assigning to each free boundary point in $\Gamma(v)$ (or more generally to every point on the extended free boundary $\Gamma_*(v)$, see Section 4) a certain frequency $\kappa \leq \ell$, see Sections 7 and 10. At the points for which $\kappa < \ell$, the separation rate of the solution $v$ from the thin obstacle can be “detected”, in a sense that it will exceed the truncation term in the generalized frequency formula. At those points we are then able to consider the so-called blowups, which will be parabolically $\kappa$-homogeneous solutions of the Signorini problem, see Section 7.

Next, we show that, similarly to what happens in the elliptic case, the smallest possible value of the frequency at a free boundary point is $\kappa = 3/2$, see Section 8. We emphasize that our proof of this fact does not rely on the semiconvexity estimates, as in the elliptic case (see [AC04] or [CSS08]). Rather, we use the monotonicity formula of Caffarelli [Caf93] to reduce the problem to the spatial dimension $n = 2$, and then study the eigenvalues of the Ornstein-Uhlenbeck operator in domains with slits (see Fig. 2 in Section 8). The elliptic version of this argument has appeared earlier in the book [PSU12, Chapter 9]. The bound $\kappa \geq 3/2$ ultimately implies the optimal $R^{3/2, 3/4}_{\text{loc}}$ regularity of solutions, see Section 9.
We next turn to studying the regularity properties of the free boundary. We start with the so-called regular set $\mathcal{R}(v)$, which corresponds to free boundary points of minimal frequency $\kappa = 3/2$. Similarly to the elliptic case, studied in [ACS08, CSS08], the Lipschitz regularity of $\mathcal{R}(v)$ with respect to the space variables follows by showing that there is a cone of spatial directions in which $v - \varphi$ is monotone. The $1/2$-Hölder regularity in $t$ is then a consequence of the fact that the blowups at regular points are $t$-independent, see Theorem 11.3. Thus, after possibly rotating the coordinate axes in $\mathbb{R}^{n-1}$, we obtain that $\mathcal{R}(v)$ is given locally as a graph

$$x_{n-1} = g(x_1, \ldots, x_{n-2}, t),$$

where $g$ is parabolically Lipschitz (or $\text{Lip}(1, \frac{1}{2})$ in alternative terminology). To prove the Hölder $H^{\alpha, \alpha/2}$ regularity of $\partial_x g$, $i = 1, \ldots, n-2$, we then use an idea that goes back to the paper of Athanasopoulos and Caffarelli [AC85] based on an application of the boundary Harnack principle (forward and backward) for so-called domains with thin Lipschitz complement, i.e., domains of the type

$$Q_1 \setminus \{(x', t) \in Q_1' \mid x_{n-1} \leq g(x'', t)\},$$

see Lemma 11.9. This result was recently established in the work of the third named author and Shi [PS13]. We emphasize that, unlike the elliptic case, the boundary Harnack principle for such domains cannot be reduced to the other known results in the parabolic setting (see e.g. [Kem72, FGS84] for parabolically Lipschitz domains, or [HLN04] for parabolically NTA domains with Reifenberg flat boundary).

Another type of free boundary points that we study are the so-called singular points, where the coincidence set $\{v = \varphi\}$ has zero $\mathcal{H}^n$-density in the thin manifold with respect to the thin parabolic cylinders. This corresponds to free boundary points with frequency $\kappa = 2m$, $m \in \mathbb{N}$. The blowups at those points are parabolically $\kappa$-homogeneous polynomials, see Section 12.

Following the approach in [GP09], in Section 13 we establish appropriate parabolic versions of monotonicity formulas of Weiss and Monneau type. Using such formulas we are able to prove the uniqueness of the blowups at singular free boundary points $(x_0, t_0)$, and consequently obtain a Taylor expansion of the type

$$v(x, t) - \varphi(x', t) = q_n(x - x_0, t - t_0) + o(\|x - x_0, t - t_0\|^\kappa), \quad t \leq t_0,$$

where $q_n$ is a polynomial of parabolic degree $\kappa$ that depends continuously on the singular point $(x_0, t_0)$ with frequency $\kappa$. We note that such expansion holds only for $t \leq t_0$ and may fail for $t > t_0$ (see Remark 12.8). Nevertheless, we show that this expansion essentially holds when restricted to singular points $(x, t)$, even for $t \geq t_0$. This is necessary in order to verify the compatibility condition in a parabolic version of the Whitney’s extension theorem (given in Appendix B). Using the latter we are then able to prove a structural theorem for the singular set. For the elliptic counterpart of this result see [GP09]. It should be mentioned at this moment that one difference between the parabolic case treated in the present paper and its elliptic counterpart is the presence of new types of singular points, which we call time-like. At such points the blowup may become independent of the space variables $x'$. We show that such singular points are contained in a countable union of graphs of the type

$$t = g(x_1, \ldots, x_{n-1}),$$

where $g$ is a $C^1$ function. The other singular points, which we call space-like, are contained in countable union of $d$-dimensional $C^{1,0}$ manifolds ($d < n-1$). After a
possible rotation of coordinates in $\mathbb{R}^{n-1}$, such manifolds are locally representable as graphs of the type

$$(x_{d+1}, \ldots, x_{n-1}) = g(x_1, \ldots, x_d, t),$$

with $g$ and $\partial g$, $i = 1, \ldots, d$, continuous.

1.2. Related problems. The time-independent version of the Signorini problem is closely related to the obstacle problem for the half-Laplacian in $\mathbb{R}^{n-1}$

$$u \geq \varphi, \quad (-\Delta_x)^{1/2} u \geq 0, \quad (u - \varphi)(-\Delta_x)^{1/2} u = 0 \quad \text{in } \mathbb{R}^{n-1}.$$ 

More precisely, if we consider a harmonic extension of $u$ from $\mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\}$ to $\mathbb{R}^n_+$ (by means of a convolution with the Poisson kernel), we will have that

$$(-\Delta_x)^{1/2} u = -c_n \partial_{x_n} u \quad \text{on } \mathbb{R}^{n-1} \times \{0\}.$$ 

Thus, the extension $u$ will solve the Signorini problem in $\mathbb{R}^n$. More generally, in the above problem instead of the half-Laplacian one can consider an arbitrary fractional power of the Laplacian $(-\Delta_x)^s$, $0 < s < 1$, see e.g. the thesis of Silvestre [Sil07]. Problems of this kind appear for instance in mathematical finance, in the valuation of American options, when the asset prices are modelled by jump processes. The time-independent problem corresponds to the so-called perpetual options, with infinite maturity time. In such framework, with the aid of an extension theorem of Caffarelli and Silvestre [CS07], many of the results known for $s = 1/2$ can be proved also for all powers $0 < s < 1$, see [CSS08].

The evolution version of the problem above is driven by the fractional diffusion and can be written as

$$u(x', t) - \varphi(x') \geq 0,$$

$$( (-\Delta_x)^s + \partial_t ) u \geq 0,$$

$$(u(x', t) - \varphi(x')) ( (-\Delta_x)^s + \partial_t ) u = 0$$

in $\mathbb{R}^{n-1} \times (0, T)$ with the initial condition

$$u(x', 0) = \varphi(x').$$

This problem has been recently studied by Caffarelli and Figalli [CF12]. We emphasize that, although their time-independent versions are locally equivalent, the problem studied in [CF12] is very different from the one considered in the present paper.

In relation to temperature control problems on the boundary, described in [DL76], we would like to mention the two recent papers by Athanasopoulos and Caffarelli [AC10], and by Allen and Shi [AS13]. Both papers deal with two-phase problems that can be viewed as generalizations of the one-phase problem (with $\varphi = 0$) considered in this paper. The paper [AS13] establishes the phenomenon of separation of phases, thereby locally reducing the study of the two-phase problem to that of one-phase. A similar phenomenon was shown earlier in the elliptic case by Allen, Lindgren, and the third named author [ALP12].

1.3. Structure of the paper. In what follows we provide a brief description of the structure of the paper.

- In Section 2 we introduce the notations used throughout the paper, and describe the relevant parabolic functional classes.
• In Section 3 we overview some of the known basic regularity properties of the solution \( v \) of the parabolic Signorini problem. The main ones are:
\[ v \in W^{2,1}_{2,\text{loc}} \cap L^\infty_{\text{loc}}, \quad \nabla v \in H^{\alpha,\alpha/2}_{\text{loc}} \quad \text{for some} \quad \alpha > 0. \]
Such results will be extensively used in our paper.

• In Section 4 we introduce the classes of solutions \( \mathcal{S}_{\varphi}(Q^+_1) \) of the parabolic Signorini problem with a thin obstacle \( \varphi \), and show how to effectively “subtract” the obstacle by maximally using its regularity. In this process we convert the problem to one with zero thin obstacle, but with a non-homogeneous right hand side \( f \) in the equation. In order to apply our main monotonicity formulas we also need to to extend the resulting functions from \( Q^+_1 \) to the entire strip \( S^+_1 \). We achieve this by multiplication by a a cutoff function, and denote the resulting class of functions by \( \mathcal{S}^f(S^+_1) \).

• Section 5 contains generalizations of \( W^{2,1}_2 \) estimates to the weighted spaced with Gaussian measure. These estimates will be instrumental in the proof of the generalized frequency formula in Section 6 and in the study of the blowups in Section 7. In order not to distract the reader from the main content, we have deferred the proof of these estimates to the Appendix A.

• Section 6 is the most technical part of the paper. There, we generalize Almgren’s (Poon’s) frequency formula to solutions of the parabolic Signorini problem.

• In Section 7 we prove the existence and homogeneity of the blowups at free boundary points where the separation rate of the solution from the thin obstacle dominates the error (truncation) terms in the generalized frequency formula.

• In Section 8 we prove that the minimal homogeneity of homogeneous solution of the parabolic Signorini problem is \( 3/2 \).

• Section 9 contains the proof of the optimal \( H^{3/2,3/4}_{\text{loc}} \) regularity of the solutions of the parabolic Signorini problem.

• In the remaining part of the paper we study the free boundary. We start in Section 10 by classifying the free boundary points according to the homogeneity of the blowups at the point in question. We also use the assumed regularity of the thin obstacle in the most optimal way.

• In Section 11 we study the so-called regular set \( \mathcal{R}(v) \) and show that it can be locally represented as a graph with \( H^{\alpha,\alpha/2} \) regular gradient.

• In Section 12 we give a characterization of the so-called singular points.

• Section 13 contains some new Weiss and Monneau type monotonicity formulas for the parabolic problem. These results generalize the ones in [GP09] for the elliptic case.

• In Section 14 we prove the uniqueness of blowups at singular points and the continuous dependence of blowups on compact subsets of the singular set. We then invoke a parabolic version of Whitney’s extension theorem (given in Appendix B) to prove a structural theorem on the singular set.

2. Notation and preliminaries
2.1. Notation. To proceed, we fix the notations that we are going to use throughout the paper.

\[ \mathbb{N} = \{1, 2, \ldots\} \quad \text{(natural numbers)} \]
\[ \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \quad \text{(integers)} \]
\[ \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \quad \text{(nonnegative integers)} \]
\[ \mathbb{R} = (-\infty, \infty) \quad \text{(real numbers)} \]
\[ s^\pm = \max\{\pm s, 0\}, \quad s \in \mathbb{R} \quad \text{(positive/negative part of s)} \]
\[ \mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\} \quad \text{(Euclidean space)} \]
\[ \mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_n > 0\} \quad \text{(positive half-space)} \]
\[ \mathbb{R}_-^n = \{x \in \mathbb{R}^n \mid x_n < 0\} \quad \text{(negative half-space)} \]
\[ \mathbb{R}^{n-1} \quad \text{(thin space)} \]
\[ x' = (x_1, x_2, \ldots, x_{n-1}) \quad \text{for} \ x \in \mathbb{R}^n \]
\[ x'' = (x_1, x_2, \ldots, x_{n-2}) \quad \text{we also identify} \ x' \text{ with} \ (x', 0) \]
\[ |x| = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}, \quad x \in \mathbb{R}^n \quad \text{(Euclidean norm)} \]
\[ ||(x, t)|| = |x|^2 + |t|^{1/2}, \quad x \in \mathbb{R}^n, t \in \mathbb{R} \quad \text{(parabolic norm)} \]
\[ x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \quad x \in \mathbb{R}^n, \alpha \in \mathbb{Z}_+^n \quad \text{(Euclidean coordinates)} \]
\[ \overline{E}, E^c, \partial E \quad \text{closure, interior, boundary of the set} \ E \]
\[ \partial_X E \quad \text{boundary in the relative topology of} \ X \]
\[ \partial_p E \quad \text{parabolic boundary of} \ E \]
\[ E^c \quad \text{complement of the set} \ E \]
\[ \mathcal{H}^s(E) \quad s\text{-dimensional Hausdorff measure} \]
\[ \text{of} \text{a} \text{Borel} \text{set} \ E \]

For \( x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}, \) and \( r > 0 \) we let

\[ B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\} \quad \text{(Euclidean ball)} \]
\[ B_r^\pm(x_0) = B_r(x_0) \cap \mathbb{R}_+^n, \quad x_0 \in \mathbb{R}^{n-1} \quad \text{(Euclidean half-ball)} \]
\[ B'_r(x_0) = B_r(x_0) \cap \mathbb{R}_+^{n-1}, \quad x_0 \in \mathbb{R}^{n-1} \quad \text{(ball)} \]
\[ B''_r(x_0) = B'_r(x_0) \cap \mathbb{R}_+^{n-2}, \quad x_0 \in \mathbb{R}^{n-2} \quad \text{(thin ball)} \]
\[ \mathcal{E}_{r_0}^\eta = \{x' \in \mathbb{R}^{n-1} \mid x_{n-1} \geq \eta|x''|\}, \quad \eta > 0 \quad \text{(thin cone)} \]
\[ Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0] \quad \text{(parabolic cylinder)} \]
\[ Q_r^\pm(x_0, t_0) = B_r^\pm(x_0) \times (t_0 - r^2, t_0] \quad \text{(parabolic half-cylinders)} \]
\[ \dot{Q}_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r^2) \quad \text{(full parabolic cylinder)} \]
\[ Q'_r(x_0, t_0) = B'_r(x_0) \times (t_0 - r^2, t_0] \quad \text{(thin parabolic cylinder)} \]
\[ Q''_r(x_0, t_0) = B''_r(x_0) \times (t_0 - r^2, t_0] \quad \text{(thin parabolic cylinder)} \]
\[ S_r = \mathbb{R}^n \times (-r^2, 0] \quad \text{(parabolic strip)} \]
\[ S^+ = \mathbb{R}^n_+ \times (-r^2, 0) \] (parabolic half-strip)

\[ S' = \mathbb{R}^{n-1} \times (-r^2, 0) \] (thin parabolic strip)

When \( x_0 = 0 \) and \( t_0 = 0 \), we routinely omit indicating the centers \( x_0 \) and \((x_0, t_0)\) in the above notations.

\[
\partial e u, \partial e u \partial x_i u, \partial x_i = \partial e_i u,
\]

for standard coordinate vectors \( e_i, i = 1, \ldots, n \)

\[
\partial t u, \partial t u \partial x_i 1 \cdots \partial x_i k u = \partial x_i 1 \cdots \partial x_i k u
\]

\[
\partial \alpha x u, \partial \alpha 1 x_1 \cdots \partial \alpha n x_n u,
\]

for \( \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbb{Z}_+ \)

\[
\nabla u, \nabla u = (\partial x_i u, \ldots, \partial x_n u)
\]

(gradient)

\[
\nabla' u, \nabla' x u = (\partial x_i u, \ldots, \partial x_n u - 1)
\]

(tangential or thin gradient)

\[
\nabla'' u, \nabla'' x u = (\partial x_i u, \ldots, \partial x_n u - 2)
\]

where \( |\alpha| = \alpha_1 + \cdots + \alpha_n \)

\[
\Delta u, \Delta x u = \sum_{i=1}^{n} \partial x_i u
\]

(Laplacian)

\[
\Delta' u, \Delta' x u = \sum_{i=1}^{n-1} \partial x_i u
\]

(tangential or thin Laplacian)

We denote by \( G \) the \textit{backward heat kernel} on \( \mathbb{R}^n \times \mathbb{R} \)

\[
G(x, t) = \begin{cases} (-4\pi t)^{-\frac{n}{2}} e^{x^2/4t}, & t < 0, \\ 0, & t \geq 0. \end{cases}
\]

We will often use the following properties of \( G \):

\[(2.1) \quad \Delta G + \partial_t G = 0, \quad G(\lambda x, \lambda^2 t) = \lambda^{-n} G(x, t), \quad \nabla G = \frac{x}{2t} G. \]

Besides, to simplify our calculations, we define the differential operator:

\[(2.2) \quad Z u = x \nabla u + 2t \partial_t u, \]

which is the generator of the parabolic scaling in the sense that

\[(2.3) \quad Z u(x, t) = \frac{d}{d\lambda} \bigg|_{\lambda=1} u(\lambda x, \lambda^2 t). \]

Using (2.1), the operator \( Z \) can also be defined through the following identity:

\[(2.4) \quad Z u = 2t \left( \nabla u \frac{\nabla G}{G} + \partial_t u \right). \]
2.2. Parabolic functional classes. For the parabolic functional classes, we have opted to use notations similar to those in the classical book of Ladyzhenskaya, Solonnikov, and Uraltseva \([LSU67]\).

Let \( \Omega \subseteq \mathbb{R}^n \) be an open subset in \( \mathbb{R}^n \) and \( \Omega_T = \Omega \times (0, T) \) for \( T > 0 \). The class \( C(\Omega_T) = C^{0,0}(\Omega_T) \) is the class of functions continuous in \( \Omega_T \) with respect to parabolic (or Euclidean) distance. Further, given for \( m \in \mathbb{Z}_+ \) we say \( u \in C^{2m,m}(\Omega_T) \) if for \( |\alpha| + 2j \leq 2m \) \( \partial_x^\alpha \partial_t^j u \in C^{0,0}(\Omega_T) \), and define the norm

\[
\|u\|_{C^{2m,m}(\Omega_T)} = \sum_{|\alpha| + 2j \leq 2m} \sup_{(x,t) \in \Omega_T} |\partial_x^\alpha \partial_t^j u(x,y)|.
\]

The parabolic Hölder classes \( H^{\ell,\ell/2}(\Omega_T) \), for \( \ell = m + \gamma, m \in \mathbb{Z}_+, 0 < \gamma \leq 1 \) are defined as follows. First, we let

\[
\begin{align*}
\langle u \rangle^{(0)}_{\Omega_T} &= |u|_{\Omega_T} = \sup_{(x,t) \in \Omega_T} |u(x,t)|, \\
\langle u \rangle^{(m)}_{\Omega_T} &= \sum_{|\alpha| + 2j = m} |\partial_x^\alpha \partial_t^j u|^{(0)}_{\Omega_T}, \\
\langle u \rangle^{(\beta)}_{x,\Omega_T} &= \sup_{(x,t),(y,t) \in \Omega_T, 0 < |x-y| \leq \delta_0} \frac{|u(x,t) - u(y,t)|}{|x-y|^\beta}, \quad 0 < \beta \leq 1, \\
\langle u \rangle^{(\beta)}_{t,\Omega_T} &= \sup_{(x,t),(x,s) \in \Omega_T, 0 < |t-s| < \delta_0} \frac{|u(x,t) - u(x,s)|}{|t-s|^\beta}, \quad 0 < \beta \leq 1, \\
\langle u \rangle^{(\ell)}_{x,\Omega_T} &= \sum_{|\alpha| + 2j = m} \langle \partial_x^\alpha \partial_t^j u \rangle^{(\gamma)}_{x,\Omega_T}, \\
\langle u \rangle^{(\ell/2)}_{t,\Omega_T} &= \sum_{m-\gamma \leq |\alpha| + 2j \leq m} \langle \partial_x^\alpha \partial_t^j u \rangle^{((\ell-|\alpha|-2j)/2)}_{t,\Omega_T}, \\
\langle u \rangle^{(\ell)}_{\Omega_T} &= \langle u \rangle^{(\ell)}_{x,\Omega_T} + \langle u \rangle^{(\ell/2)}_{t,\Omega_T}.
\end{align*}
\]

Then, we define \( H^{\ell,\ell/2}(\Omega_T) \) as the space of functions \( u \) for which the following norm is finite:

\[
\|u\|_{H^{\ell,\ell/2}(\Omega_T)} = \sum_{k=0}^m \langle u \rangle^{(k)}_{\Omega_T} + \langle u \rangle^{(\ell)}_{\Omega_T}.
\]

The parabolic Lebesgue space \( L_q(\Omega_T) \) indicates the Banach space of those measurable functions on \( \Omega_T \) for which the norm

\[
\|u\|_{L_q(\Omega_T)} = \left( \int_{\Omega_T} |u(x,t)|^q dx dt \right)^{1/q}
\]

is finite. The parabolic Sobolev spaces \( W^{2m,m}_q(\Omega_T) \), \( m \in \mathbb{Z}_+ \), denote the spaces of those functions in \( L_q(\Omega_T) \), whose distributional derivative \( \partial_x^\alpha \partial_t^j u \) belongs to \( \in L_q(\Omega_T) \), for \( |\alpha| + 2j \leq 2m \). Endowed with the norm

\[
\|u\|_{W^{2m,m}_q(\Omega_T)} = \sum_{|\alpha| + 2j \leq 2m} \|\partial_x^\alpha \partial_t^j u\|_{L_q(\Omega_T)},
\]

\( W^{2m,m}_q(\Omega_T) \) becomes a Banach space.
We also denote by $W^{1,0}_q(\Omega_T)$, $W^{1,1}_q(\Omega_T)$ the Banach subspaces of $L^q(\Omega_T)$ generated by the norms
\[
\|u\|_{W^{1,0}_q(\Omega_T)} = \|u\|_{L^q(\Omega_T)} + \|\nabla u\|_{L^q(\Omega_T)},
\]
\[
\|u\|_{W^{1,1}_q(\Omega_T)} = \|u\|_{L^q(\Omega_T)} + \|\nabla u\|_{L^q(\Omega_T)} + \|\partial_t u\|_{L^q(\Omega_T)}.
\]
Let $E \subset S_R$ for some $R > 0$. The weighted Lebesgue space $L_p(E, G)$, $p > 1$, with Gaussian weight $G(x,t)$, will appear naturally in our proofs. The norm in this space is defined by
\[
\|u\|_{L_p(E,G)}^p = \int_E |u(x,t)|^p G(x,t) dx dt.
\]
When $E$ is a relatively open subset of $S_R$, one may also define the respective weighted Sobolev spaces. We will also consider weighted spatial Lebesgue and Sobolev spaces $L_p(\Omega, G(\cdot, s))$, and $W^m_p(\Omega, G(\cdot, s))$ with Gaussian weights $G(\cdot, s)$ on $\mathbb{R}^n$ for some fixed $s < 0$. For instance, the norm in the space $L_p(\Omega, G(\cdot, s))$ is given by
\[
\|u\|_{L_p(\Omega,G(\cdot, s))}^p = \int_\Omega |u(x)|^p G(x, s) dx.
\]

3. KNOWN EXISTENCE AND REGULARITY RESULTS

In this section we recall some known results about the existence and the regularity of the solution of the parabolic Signorini problem that we are going to take as the starting point of our analysis. For detailed proofs we refer the reader to the works of Arkhipova and Uraltseva [AU88, AU96]. For simplicity we state the relevant results only in the case of the unit parabolic half-cylinder $Q^+_1$.

Suppose we are given functions $f \in L_\infty(Q^+_1)$, $\varphi \in W^{2,1}_\infty(Q^+_1)$, $g \in W^{2,1}_\infty((\partial B_1)^+ \times (-1,0))$, and $\varphi_0 \in W^{2,0}_\infty(B^+_1)$ obeying the compatibility conditions
\[
\varphi_0 = g(\cdot, -1) \quad \text{a.e. on } (\partial B_1)^+,
\]
\[
\varphi_0 \geq \varphi(\cdot, -1) \quad \text{a.e. on } B^+_1,
\]
\[
g \geq \varphi \quad \text{a.e. on } \partial B^+_1 \times (-1,0].
\]

Given $\varphi$ and $g$ as above, we introduce the following closed subset of $W^{1,0}_2(Q^+_1)$
\[
\mathcal{R} = \{v \in W^{1,0}_2(Q^+_1) \mid v \geq \varphi \text{ a.e. on } Q^+_1, \ v = g \text{ a.e. on } (\partial B_1)^+ \times (-1,0)\}.
\]

We say that $u \in W^{1,0}_2(Q^+_1)$ satisfies
\[
(3.1) \quad \Delta u - \partial_t u = f(x,t) \quad \text{in } Q^+_1,
\]
\[
(3.2) \quad u \geq \varphi, \quad -\partial_{x_n} u \geq 0, \quad (u - \varphi)\partial_{x_n} u = 0 \quad \text{on } Q^+_1,
\]
\[
(3.3) \quad u = g \quad \text{on } (\partial B_1)^+ \times (-1,0],
\]
\[
(3.4) \quad u(\cdot, -1) = \varphi_0 \quad \text{on } B^+_1.
\]
if \( u \) solves the variational inequality

\[
\int_{Q_1^+} \left[ \nabla u (v - u) + \partial_t u (v - u) + f(v - u) \right] \geq 0 \quad \text{for any } v \in \mathcal{R},
\]

\[
u \in \mathcal{R}, \quad \partial_t u \in L_2(Q_1^+), \quad u(\cdot, -1) = \varphi_0 \quad \text{on } B_1^+.
\]

Under the assumptions above there exists a unique solution to the problem \( (3.1) - (3.4) \). Moreover, the solution will have Hölder continuous spatial gradient: \( \nabla u \in H^{\alpha, \beta} (Q_1^+ \cup Q_2^+) \) for any \( 0 < r < 1 \) with the Hölder exponent \( \alpha > 0 \) depending only on the dimension, see [AU88, AU96]. Below, we sketch the details in the case \( (3.5) \). Thus, the family \( \{ \varphi \} \) has uniform bounds for the family \( \{ \varphi \} \) in the case \( \varepsilon \to 0 \) weak limit as \( \varepsilon = 0 \) and \( g = 0 \). Besides, by choosing the test functions \( \eta = \beta_\varepsilon (w^r - w^s) | \beta_\varepsilon (w^r - w^s)^{p-2} | \), where \( w \) solves the boundary value problem

\[
\Delta w - \partial_t w = f^\varepsilon \quad \text{in } Q_1^+,
\]

\[
w = 0 \quad \text{on } \partial_\beta Q_1^+.
\]
one can show the global uniform bound
\[ \sup_{Q^+_i} |g^r(u^r)| \leq C_n \left( \|\varphi_0\|_{W^2(B^+_i)} + \|f\|_{L_\infty(Q^+_i)} \right). \]

For complete details, see the proofs of Lemmas 4 and 5 in [AU88].

Next we have a series of local estimates. With \( \zeta \in C_0^\infty(B_1) \), we take the function \( \eta = \partial_{x_i}(\partial_{x_i} u^r)\zeta^2(x) \), \( i = 1, \ldots, n \) in (3.6). Integrating by parts, we obtain the following local uniform, second order estimate
\[
\|D^2 u^r\|_{L_2(Q^+_i)} \leq C_{n,r} \left( \|\nabla u\|_{L_2(Q^+_i)}^2 + \|f\|_{L_2(Q^+_i)} \right) \\
\leq C_{n,r} \left( \|\varphi_0\|_{L_2(B^+_i)} + \|f\|_{L_2(Q^+_i)} \right), \quad 0 < r < 1.
\]

One should compare with our proof of Lemma 5.1 in Appendix A which is the weighted version of this estimate. Furthermore, with more work one can establish the following locally uniform spatial Lipschitz bound
\[
\|u^r\|_{W^2_\infty(Q^+_i)} \leq C_{n,r} \left( \|\varphi_0\|_{W^2(B^+_i)} + \|f\|_{L_\infty(Q^+_i)} \right), \quad 0 < r < 1,
\]
see Lemma 6 in [AU88].

Finally, one can show that there exists a dimensional constant \( \alpha > 0 \) such that \( \nabla u^r \in H^{\alpha,\alpha/2}(Q^+_r \cup Q^+_r) \) for any \( 0 < r < 1 \), with the estimate
\[
\|\nabla u^r\|_{H^{\alpha,\alpha/2}(Q^+_i \cup Q^+_i)} \leq C_{n,r} \left( \|\nabla u^r\|_{L_\infty(Q^+_i)} + \|f\|_{L_\infty(Q^+_i)} \right), \quad 0 < r < \rho < 1 \\
\leq C_{n,r} \left( \|\varphi_0\|_{W^2(B^+_i)} + \|f\|_{L_\infty(Q^+_i)} \right),
\]
see Theorem 2.1 in [AU96].

We summarize the estimates above in the following two lemmas.

**Lemma 3.1.** Let \( u \in W^{2,1}(Q^+_i) \) be a solution of the Signorini problem (3.1)–(3.4) with \( f \in L_2(Q^+_i) \), \( \varphi_0 \in W^{2,1}(B^+_i) \), \( \varphi = 0 \), and \( g = 0 \). Then, \( u \in W^{2,1}(Q^+_i) \) for any \( 0 < r < 1 \) and
\[
\|u\|_{W^2_1(Q^+_i)} \leq C_{n,r} \left( \|\varphi_0\|_{W^2(B^+_i)} + \|f\|_{L_2(Q^+_i)} \right).
\]

**Lemma 3.2.** Let \( u \in W^{2,1}(Q^+_i) \) be a solution of the Signorini problem (3.1)–(3.4) with \( f \in L_\infty(Q^+_i) \), \( \varphi_0 \in W^{2,1}(B^+_i) \), \( \varphi = 0 \), and \( g = 0 \). Then, for any \( 0 < r < 1 \), \( u \in L_\infty(Q^+_i) \), \( \nabla u \in H^{\alpha,\alpha/2}(Q^+_r \cup Q^+_r) \) with a dimensional constant \( \alpha > 0 \) and
\[
\|u\|_{L_\infty(Q^+_i)} + \|\nabla u\|_{H^{\alpha,\alpha/2}(Q^+_r \cup Q^+_r)} \leq C_{n,r} \left( \|\varphi_0\|_{W^2(B^+_i)} + \|f\|_{L_\infty(Q^+_i)} \right).
\]

We will also need the following variant of Lemma 3.2 that does not impose any boundary data \( g \) and \( \varphi_0 \).

**Lemma 3.3.** Let \( v \in W^{2,1}(Q^+_i) \cap W^{1,0}_\infty(Q^+_i) \) be a solution of the Signorini problem (3.1)–(3.4) with \( f \in L_\infty(Q^+_i) \), and \( \varphi \in H^{2,1}(Q^+_r) \). Then, for any \( 0 < r < 1 \), \( \nabla v \in H^{\alpha,\alpha/2}(Q^+_r \cup Q^+_r) \) with a universal \( \alpha > 0 \), and
\[
\|\nabla v\|_{H^{\alpha,\alpha/2}(Q^+_r \cup Q^+_i)} \leq C_{n,r} \left( \|v\|_{W^{1,0}_\infty(Q^+_i)} + \|f\|_{L_\infty(Q^+_i)} + \|\varphi\|_{H^{2,1}(Q^+_r)} \right).
\]

**Proof.** Consider the function
\[
u(x,t) = \left[ v(x,t) - \varphi(x',t) \right] \eta(x,t)
\]
with \( \eta \in C_0^\infty(Q^+_1) \), such that
\[
\eta = 1 \quad \text{on } Q_r, \quad \eta(x',x_n,t) = \eta(x',x_n,t).
\]
In particular, \( \partial_x \eta = 0 \) on \( Q'_1 \). Then \( v \) satisfies the conditions of Lemma 3.2 with \( \varphi = 0, \) \( g = 0, \) \( \varphi_0 = 0 \) and with \( f \) replaced by

\[
[f - (\Delta' - \partial_t)\varphi]\eta + (v - \varphi)(\Delta - \partial_t)\eta + 2(\nabla v - \nabla \varphi) \nabla \eta.
\]

The assumptions on \( v \) and \( \varphi \) now imply the required estimate from that in Lemma 3.2.

In Section 5 we generalize the estimates in Lemma 3.1 for the appropriate weighted Gaussian norms. The proof of these estimates are given in Appendix A. One of our main results in this paper is the optimal value of the Hölder exponent in Lemma 3.3. We show that \( \nabla u \in H^{1/2,1/4}_0 \), or slightly stronger, that \( u \in H^{3/2,3/4}_0 \), when \( f \) is bounded, see Theorem 9.1.

4. Classes of solutions

In this paper we are mostly interested in local properties of the solution \( v \) of the parabolic Signorini problem and of its free boundary. In view of this, we focus our attention on solutions in parabolic (half-)cylinders. Furthermore, thanks to the results in Section 3, we can, and will assume that such solutions possess the regularity provided by Lemmas 3.1 and 3.2.

**Definition 4.1** (Solutions in cylinders). Given \( \varphi \in H^{2,1}(Q'_1) \), we say that \( v \in S_{\varphi}(Q'_1) \) if \( v \in W^{2,1}_2(Q'_1) \cap L_\infty(Q'_1) \), \( \nabla v \in H^{\alpha/2}(Q'_1 \cup Q'_1) \) for some \( 0 < \alpha < 1 \), and \( v \) satisfies

\[
\begin{align*}
\Delta v - \partial_t v &= 0 \quad \text{in } Q'_1, \\
v - \varphi &\geq 0, \quad -\partial_{x_n} v \geq 0, \quad (v - \varphi)\partial_{x_n} v = 0 \quad \text{on } Q'_1, \\
(0,0) &\in \Gamma_s(v) := \partial_{Q'_1} \{(x',t) \in Q'_1 \mid v(x',0,t) = \varphi(x',t), \ \partial_{x_n} v(x',0,t) = 0\},
\end{align*}
\]

and

\[
(0,0) \in \Gamma_s(v) := \partial_{Q'_1} \{(x',t) \in Q'_1 \mid v(x',0,t) > \varphi(x',t)\}.
\]

Note that by definition \( \Gamma_s(v) \supset \Gamma(v) \). The reason for considering this extension is that parabolic cylinders do not contain information on “future times”. This fact may create a problem when restricting solutions to smaller subcylinders. The notion of extended free boundary removes this problem. Namely, if \((x_0,t_0) \in \Gamma_s(v)\) and \( r > 0 \) is such that \( Q^+_1(x_0,t_0) \subset Q^{+1}_1 \), then \((x_0,t_0) \in \Gamma(v|_{Q^+_1(x_0,t_0)}) \). Sometimes, we will abuse the terminology and call \( \Gamma_s(v) \) the free boundary.

Replacing \( Q^+_1 \) and \( Q'_1 \) by \( Q^+_R \) and \( Q'_R \) respectively in the definition above, we will obtain the class \( S_{\varphi}(Q^+_R) \). Note that if \( v \in S_{\varphi}(Q^+_R) \) then the parabolic rescaling

\[
v_R(x,t) = \frac{1}{C_R}v(Rx,R^2t),
\]

where \( C_R > 0 \) can be arbitrary (but typically chosen to normalize a certain quantity), belongs to the class \( S_{\varphi/R}(Q^+_1) \). Having that in mind, we will state most of the results only for the case \( R = 1 \).
The function \( v \in \mathbb{S}_\varphi(Q_1^+) \) allows a natural extension to the entire parabolic cylinder \( Q_1 \) by the even reflection in \( x_n \) coordinate:
\[
v(x', -x_n, t) := v(x', x_n, t).
\]
Then \( v \) will satisfy
\[
\Delta v - \partial_t v = 0 \quad \text{in } Q_1 \setminus \Lambda(v),
\]
where
\[
\Lambda(v) := \{(x', t) \in Q'_1 \mid v(x', 0, t) = \varphi(x', t)\},
\]
is the so-called coincidence set. More generally,
\[
\Delta v - \partial_t v \leq 0 \quad \text{in } Q_1,
\]
\[
\Delta v - \partial_t v = 2(\partial^n v_\ast \mathcal{H}^n)_{|\Lambda(v)} \quad \text{in } Q_1,
\]
in the sense of distributions, where \( \mathcal{H}^n \) is the \( n \)-dimensional Hausdorff measure and by \( \partial^n v \) we understand the limit from the right \( \partial_{x_n} v(x', 0+, t) \) on \( Q'_1 \).

We next show how to reduce the study of the solutions with nonzero obstacle \( \varphi \) to the ones with zero obstacle. As the simplest such reduction we consider the difference \( v(x, t) - \varphi(x', t) \), which will satisfy the Signorini conditions on \( Q'_1 \) with zero obstacle, but at an expense of solving nonhomogeneous heat equation instead of the homogeneous one. One may further extend this difference to the strip \( S_1^+ = \mathbb{R}_x^+ \times (-1, 0) \) by multiplying with a cutoff function in \( x \) variables. More specifically, let \( \psi \in C_0^\infty(\mathbb{R}^n) \) be such that
\[
0 \leq \psi \leq 1, \quad \psi = 1 \quad \text{on } B_{1/2}, \quad \text{supp} \psi \subset B_{3/4},
\]
\[
\psi(x', -x_n) = \psi(x', x_n), \quad x \in \mathbb{R}^n,
\]
and consider the function
\[
u(x, t) = [v(x, t) - \varphi(x', t)]\psi(x) \quad \text{for } (x, t) \in S_1^+.
\]
It is easy to see that \( u \) satisfies the nonhomogeneous heat equation in \( S_1^+ \)
\[
\Delta u - \partial_t u = f(x, t) \quad \text{in } S_1^+,
\]
\[
f(x, t) = -\psi(x)[\Delta \varphi - \partial_t \varphi] + [v(x, t) - \varphi(x', t)]\Delta \psi + 2\nabla v \nabla \psi,
\]
and the Signorini boundary conditions on \( S_1' \)
\[
u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u \partial_{x_n} u = 0 \quad \text{on } S_1'.
\]
Moreover, it is easy to see that \( f \) is uniformly bounded.

**Definition 4.2** (Solutions in strips). We say that \( u \in \mathbb{S}_\mathcal{f}(S_1^+) \), for \( f \in L_\infty(S_1^+) \)
if \( u \in W^{2,1}_2(S_1^+) \cap L_\infty(S_1^+) \), \( \nabla u \in H^{\alpha, \alpha/2}(S_1^+ \cup S_1') \), \( u \) has a bounded support and solves
\[
\Delta u - \partial_t u = f \quad \text{in } S_1^+,
\]
\[
u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u \partial_{x_n} u = 0 \quad \text{on } S_1',
\]
and
\[
(0, 0) \in \Gamma(u) = \partial S'_1 \{ (x', t) \in S_1^+ \mid u(x', 0, t) = 0, \quad \partial_{x_n} u(x', 0, t) = 0 \}.\]
If we only assume \( \varphi \in H^{2,1}(Q_1^n) \), then in the construction above we can only say that the function \( f \in L_\infty(S_1^n) \). For some of the results that we are going to prove (such as the optimal regularity in Theorem 9.1), this will be sufficient. However, if we want to study a more refined behavior of \( u \) near the origin, we need to assume more regularity on \( \varphi \).

Thus, if we assume \( \varphi \in H^{k,\gamma}(Q_1^n) \) with \( k = k + \gamma \geq 2 \), \( k \in \mathbb{N} \), \( 0 < \gamma \leq 1 \), then for its parabolic Taylor polynomial \( q_k(x',t) \) of parabolic degree \( k \) at the origin, we have

\[
|\varphi(x',t) - q_k(x',t)| \leq M \| (x',t) \|^k,
\]

for a certain \( M > 0 \), and more generally

\[
|\partial_{x'}^{\alpha'} \partial_t \varphi(x',t) - \partial_{x'}^{\alpha'} \partial_t q_k(x',t)| \leq M \| (x',t) \|^{k-|\alpha'|-2j},
\]

for \( |\alpha'| + 2 j \leq k \).

To proceed, we caloriically extend the polynomial \( q_k(x',t) \) in the following sense.

**Lemma 4.3** (Caloric extension of polynomials). For a given polynomial \( q(x',t) \) on \( \mathbb{R}^{n-1} \times \mathbb{R} \), there exists a caloric extension polynomial \( \tilde{q}(x,t) \) in \( \mathbb{R}^n \times \mathbb{R} \), symmetric in \( x_n \), i.e.,

\[
(\Delta - \partial_t)\tilde{q}(x,t) = 0, \quad \tilde{q}(x',0,t) = q(x',t), \quad \tilde{q}(x',-x_n,t) = \tilde{q}(x',x_n,t).
\]

Moreover, if \( q(x',t) \) is parabolically homogeneous of order \( \kappa \), then one can find \( \tilde{q}(x,t) \) as above with the same homogeneity.

**Proof.** It is easily checked that the polynomial

\[
\tilde{q}(x',x_n,t) = \sum_{j=0}^{N} (\partial_t - \Delta_{x'})^j q(x',t) \frac{x_n^{2j}}{2j!}
\]

is the desired extension. Here \( N \) is taken so that the parabolic degree of the polynomial \( q(x',t) \) does not exceed \( 2N \).

Let now \( \tilde{q}_k \) be the extension of the parabolic Taylor polynomial \( q_k \) of \( \varphi \) at the origin and consider

\[
v_k(x,t) := v(x,t) - \tilde{q}_k(x,t), \quad \varphi_k(x',t) := \varphi(x',t) - q_k(x',t).
\]

It is easy to see that \( v_k \) solves the Signorini problem with the thin obstacle \( \varphi_k \), i.e. \( v_k \in \mathcal{S}_{\varphi_k}(Q_1^n) \), now with an additional property

\[
|\partial_{x'}^{\alpha'} \partial_t \varphi_k(x',t)| \leq M \| (x',t) \|^{k-|\alpha'|-2j}, \quad \text{for } |\alpha'| + 2 j \leq k.
\]

Then if we proceed as above and define

\[
u_k(x,t) = |v_k(x,t) - \varphi_k(x',t)| \psi(x)
\]

\[
= |v(x,t) - \varphi(x,t) - (\tilde{q}_k(x,t) - q_k(x',t))| \psi(x)
\]

then \( u_k \) will satisfy (4.7)–(4.8) with the right-hand side

\[
f_k = -\psi(x) [\Delta' \varphi_k - \partial_t \varphi_k] + |v_k(x,t) - \varphi_k(x',t)| \Delta \psi + 2\nabla v_k \nabla \psi,
\]

which additionally satisfies

\[
|f_k(x,t)| \leq M \| (x,t) \|^{\ell-2}, \quad \text{for } (x,t) \in S_1^n,
\]
for $M$ depending only on $\psi$, $\|u\|_{W^{\frac{3}{2},0}(Q^+_1)}$, and $\|\varphi\|_{H^{\ell,\ell/2}(Q^+_1)}$. Moreover, for $|\alpha| + 2 j \leq k - 2$ one will also have

$$|\partial_x^\alpha \partial_t^j f_k(x,t)| \leq M_{\alpha,j} \|\varphi\|^{\ell - 2 - |\alpha|-2j} \text{ for } (x,t) \in Q^+_{1/2}.$$

We record this construction in the following proposition.

**Proposition 4.4.** Let $v \in \mathcal{S}_v(Q^+_1)$ with $\varphi \in H^{\ell,\ell/2}(Q^+_1)$, $\ell = k + \gamma \geq 2$, $k \in \mathbb{N}$, $0 < \gamma \leq 1$. If $q_k$ is the parabolic Taylor polynomial of order $k$ of $\varphi$ at the origin, $\tilde{q}_k$ is its extension given by Lemma 4.3, and $\psi$ is a cutoff function as in (4.4)–(4.5), then

$$u_k(x,t) = [v(x,t) - \tilde{q}_k(x,t) - (\varphi(x',t) - q_k(x',t))]\psi(x)$$

belongs to the class $\mathcal{S}(S^+_1)$ with

$$|f_k(x,t)| \leq M \|\varphi\|^{\ell-2} \text{ for } (x,t) \in S^+_1$$

and more generally, for $|\alpha| + 2j \leq k - 2$,

$$|\partial_x^\alpha \partial_t^j f_k(x,t)| \leq M_{\alpha,j} \|\varphi\|^{\ell - 2 - |\alpha| - 2j} \text{ for } (x,t) \in Q^+_{1/2}.$$ 

Furthermore, $u_k(x',0,t) = v(x',0,t) - \varphi(x',t)$, $\partial_{x_n} u_k(x',0,t) = \partial_{x_n} v(x',0,t)$ in $Q^+_{1/2}$ and therefore

$$\Gamma(u_k) \cap Q^+_{1/2} = \Gamma(v) \cap Q^+_{1/2},$$

$$\Gamma_* (u_k) \cap Q^+_{1/2} = \Gamma_* (v) \cap Q^+_{1/2}.$$ 

### 5. Estimates in Gaussian spaces

In this section we state $W^{2,1}_2$-estimates with respect to the Gaussian measure $G(x,t)dxdt$ in the half-strips $S^+_\rho$. The estimates involves the quantities that appear in the generalized frequency formula that we prove in the next section. Since the computations are rather long and technical, to help with the readability of the paper, we have moved the proofs to Appendix A.

**Lemma 5.1.** Let $u \in \mathcal{S}(S^+_1)$ with $f \in L_\infty(S^+_1)$. Then, for any $0 < \rho < 1$ we have the estimates

$$\int_{S^+_\rho} |t|^2 |\nabla u|^2 G \leq C_{n,\rho} \int_{S^+_1} (u^2 + |t|^2 f^2) G,$$

$$\int_{S^+_\rho} |t|^2 (|D^2 u|^2 + u_t^2) G \leq C_{n,\rho} \int_{S^+_1} (u^2 + |t|^2 f^2) G.$$ 

**Remark 5.2.** Even though the estimates above are most natural for our further purposes, we would like to note that slightly modifying the proof one may show that

$$\int_{S^+_\rho} |\nabla u|^2 G \leq C_{n,\rho} \int_{S^+_1} (u^2 + f^2) G,$$

i.e., without the weights $|t|$ and $|t|^2$ in the integrals for $|\nabla u|^2$ and $f^2$. More generally, the same estimate can be proved if $u(\Delta - \partial_t)u \geq 0$ in $S_1$, with the half-strips $S^+_\rho$ and $S^+_1$ replaced by the full strips $S_\rho$ and $S_1$. 

Lemma 5.3. Suppose $u_i \in \mathcal{S}^f(S^+_i)$, $i = 1, 2$, with $f_i \in L_\infty(S^+_i)$. Then for any $0 < \rho < 1$ we have the estimate
\[
\int_{S^+_\rho} |t| |\nabla (u_1 - u_2)|^2 G \leq C_{n, \rho} \int_{S^+_1} [(u_1 - u_2)^2 + |t|^2 (f_1 - f_2)^2] G.
\]

6. The generalized frequency function

In this section we will establish a monotonicity formula, which will be a key tool for our study. The origins of this formula go back to Almgren’s Big Regularity Paper [Alm00], where he proved that for (multiple-valued) harmonic functions in the unit ball, the frequency function
\[
N_u(r) = r \int_{B_r} |\nabla u|^2 \frac{G}{\int_{\partial B_r} u^2}
\]
is monotonically increasing in $r \in (0, 1)$. Versions of this formula have been used in different contexts, most notably in unique continuation [GL86, GL87] and more recently in the thin obstacle problem [ACS08, GP09]. Almgren’s monotonicity formula has been generalized by Poon to solutions of the heat equation in the unit strip $S_1$. More precisely, he proved in [Poo96] that if $\Delta u - u_t = 0$ in $S_1$, then its caloric frequency, defined as
\[
N_v(r) = \frac{r^2 \int_{\mathbb{R}^n} |\nabla u|^2 G(x,-r^2) dx}{\int_{\mathbb{R}^n} u(x,-r^2)^2 G(x,-r^2) dx},
\]
is monotone non-decreasing in $r \in (0, 1)$. This quantity, which to a large extent plays the same role as Almgren’s frequency function, differs from the latter in that it requires $u$ to be defined in an entire strip. Thereby, it is not directly applicable to caloric functions which are only locally defined, for instance when $u$ is only defined in the unit cylinder $Q_1$. One possible remedy to this obstruction is to consider an extension of a caloric function in $Q_1$ to the entire strip $S_1$ by multiplying it with a spatial cutoff function $\psi$, supported in $B_1$:
\[
v(x,t) = u(x,t) \psi(x).
\]
Such extension, however, will no longer be caloric in $S_1$, and consequently the parabolic frequency function $N_v$ is no longer going to be monotone. However, there is a reasonable hope that $N_v$ is going to exhibit properties close to monotonicity. In fact, to be able to control the error terms in the computations, we will need to consider a “truncated” version of $N$. Moreover, we will be able to extend this result to functions $u \in \mathcal{S}^f(S^+_1)$, and to functions $v \in \mathcal{S}_\infty(Q^+_1)$, via the constructions in Proposition 4.4.

To proceed, we define the following quantities:
\[
h_u(t) = \int_{\mathbb{R}^n_+} u(x,t)^2 G(x,t) dx,
\]
\[
i_u(t) = -t \int_{\mathbb{R}^n_+} |\nabla u(x,t)|^2 G(x,t) dx.
\]
for any function \( u \) in the parabolic half-strip \( S_1^+ \) for which the integrals involved are finite. Note that, if \( u \) is an even function in \( x_n \), then Poon’s parabolic frequency function is given by

\[
N_u(r) = \frac{i_u(-r^2)}{h_u(-r^2)}.
\]

There are many substantial technical difficulties involved in working with this function directly. To overcome such difficulties, we consider the following averaged versions of \( h_u \) and \( i_u \):

\[
H_u(r) = \frac{1}{r^2} \int_{-r^2}^{0} h_u(t)dt = \frac{1}{r^2} \int_{S_1^+} u(x, t)^2 G(x, t)dxdt,
\]

\[
I_u(r) = \frac{1}{r^2} \int_{-r^2}^{0} i_u(t)dt = \frac{1}{r^2} \int_{S_1^+} |t||\nabla u(x, t)||^2 G(x, t)dxdt.
\]

One further obstruction is represented by the fact that the above integrals may become unbounded near the endpoint \( t = 0 \), where \( G \) becomes singular. To remedy this problem we introduce the following truncated versions of \( H_u \) and \( I_u \). For a constant \( 0 < \delta < 1 \), let

\[
H^\delta_u(r) = \frac{1}{r^2} \int_{-\delta^2 r^2}^{0} h_u(t)dt = \frac{1}{r^2} \int_{S_1^+ \setminus S_1^+ \delta r} u(x, t)^2 G(x, t)dxdt,
\]

\[
I^\delta_u(r) = \frac{1}{r^2} \int_{-\delta^2 r^2}^{0} i_u(t)dt = \frac{1}{r^2} \int_{S_1^+ \setminus S_1^+ \delta r} |t||\nabla u(x, t)||^2 G(x, t)dxdt.
\]

The following lemma plays a crucial role in what follows.

**Lemma 6.1.** Assume that \( v \in C^{4,2}_0 (S_1^+ \cup S_1^-) \) satisfies

\[
\Delta v - \partial_t v = g(x, t) \quad \text{in } S_1^+.
\]

Then, for any \( 0 < \delta < 1 \) we have the following differentiation formulas

\[
(H^\delta_u)'(r) = \frac{4}{r} I^\delta_v(r) - \frac{4}{r^3} \int_{S_1^+ \setminus S_1^- \delta r} tvG - \frac{4}{r} \int_{S_1^+ \setminus S_1^- \delta r} tv_{x_n} G
\]

\[
(I^\delta_u)'(r) = \frac{1}{r^3} \int_{S_1^+ \setminus S_1^- \delta r} (Zv)^2 G + \frac{2}{r^3} \int_{S_1^+ \setminus S_1^- \delta r} t(Zv)gG + \frac{2}{r^3} \int_{S_1^- \setminus S_1^+ \delta r} tv_{x_n} (Zv)G,
\]

where the vector field \( Z \) is as in (2.2).

**Proof.** The main step in the proof consists in establishing the following differentiation formulas for \(-1 < t < 0\):

\[
h'_u(t) = \frac{2}{t} i_u(t) - 2 \int_{R^{n-1}} vG - 2 \int_{R^n} v_{x_n} G,
\]
and

\[ i'_\nu(t) = \frac{1}{2t} \int_{\mathbb{R}^n_+} (Zv)^2 G + \int_{\mathbb{R}^n_+} (Zv)gG + \int_{\mathbb{R}^{n-1}} v x_n (Zv) G. \]

Once this is done, then noting that

\[ H^{\delta}_{u'}(r) = \int_{-1}^{-\delta^2} h_u(r^2 s) ds, \quad I^{\delta}_{u'}(r) = \int_{-1}^{-\delta^2} i_u(r^2 s) ds, \]

we find

\[ (H^{\delta}_{u'})'(r) = 2r \int_{-1}^{-\delta^2} s h_u'(r^2 s) ds, \quad (I^{\delta}_{u'})'(r) = 2r \int_{-1}^{-\delta^2} s i_u'(r^2 s) ds. \]

Using (6.1) we thus obtain

\[ (H^{\delta}_{\nu'})'(r) = \frac{2}{r^3} \int_{-\delta^2}^{-\delta^2 r^2} th'_\nu(t) dt = \frac{2}{r^3} \int_{-\delta^2}^{-\delta^2 r^2} \left( 2i_\nu(t) - 2 \int_{\mathbb{R}^n_+} tvG(v, t) dx - 2 \int_{\mathbb{R}^{n-1}} vv_{x_n} G(v, t) dx \right) dt \]
\[ = \frac{4}{r} I^\delta_{\nu'}(r) - \frac{4}{r^3} \int_{S^+_r \setminus S^+_r} t vG - \frac{4}{r^3} \int_{S^+_r \setminus S^+_r} t vv_{x_n} G \]

The formula for \((I^{\delta}_{\nu'})'(r)\) is computed similarly. We are thus left with proving (6.1) and (6.2).

(1°) We start with claiming that

\[ i_\nu(t) = \frac{1}{2} \int_{\mathbb{R}^n_+} v Z v G + t \int_{\mathbb{R}^n_+} v g G + \int_{\mathbb{R}^{n-1}} v v x_n G. \]

Indeed, noting that \(\Delta(v^2/2) = v \Delta v + |\nabla v|^2\) in \(S^+_1\), and keeping in mind that the outer unit normal to \(\mathbb{R}^n_+\) on \(\mathbb{R}^{n-1}\) is given by \(\nu = -e_n = (0, \ldots, 0, -1)\), we integrate by parts to obtain

\[ i_\nu(t) = -t \int_{\mathbb{R}^n_+} \left( \Delta(v^2/2) - v \Delta v \right) G \]
\[ = t \int_{\mathbb{R}^n_+} v \nabla v G + t \int_{\mathbb{R}^{n-1}} v v x_n G + t \int_{\mathbb{R}^n_+} v v G + t \int_{\mathbb{R}^{n-1}} v v x_n G \]
\[ = t \int_{\mathbb{R}^n_+} \left( \nabla v \frac{\nabla G}{G} + v \right) v G + t \int_{\mathbb{R}^n_+} v v G + t \int_{\mathbb{R}^{n-1}} v v x_n G \]
\[ = \frac{1}{2} \int_{\mathbb{R}^n_+} v (Zv) G + t \int_{\mathbb{R}^n_+} v v G + t \int_{\mathbb{R}^{n-1}} v v x_n G, \]
where in the first integral of the last equality we have used (2.4). This proves the
claim.

(2◦) We now prove the formula (6.1) for $h'_v$. Note that for $\lambda > 0$ we have

$$h_v(\lambda^2 t) = \int_{\mathbb{R}^n_+} v(x, \lambda^2 t)^2 G(x, \lambda^2 t) dx$$

$$= \int_{\mathbb{R}^n_+} v(\lambda y, \lambda^2 t)^2 G(\lambda y, \lambda^2 t) \lambda^n dy = \int_{\mathbb{R}^n_+} v(\lambda y, \lambda^2 t)^2 G(y, t) dy.$$  

Here, we have used the identity $G(\lambda y, \lambda^2 t) \lambda^n = G(y, t)$. Differentiating with respect to $\lambda$ at $\lambda = 1$, and using (2.3), we therefore obtain

$$2th'_v(t) = 2 \int_{\mathbb{R}^n_+} vZvG,$$

or equivalently

$$h'_v(t) = \frac{1}{t} \int_{\mathbb{R}^n_+} vZvG.$$  

Using now the formula for $i_v$ in (1◦), and the fact that $\Delta v - v_t = g$, we obtain

$$h'_v(t) = \frac{2}{t} i_v(t) - 2 \int_{\mathbb{R}^n_+} v g G - 2 \int_{\mathbb{R}^{n-1}} vv_{x_n} G.$$  

(3◦) To obtain the differentiation formula (6.2) for $i_v$, note that using the scaling properties of $G$, similarly to what was done for $h_v$, we have

$$i_v(\lambda^2 t) = -\lambda^2 t \int_{\mathbb{R}^n_+} |\nabla v(\lambda y, \lambda^2 t)|^2 G(y, t) dy.$$  

Differentiating with respect to $\lambda$ at $\lambda = 1$, we obtain

$$2t i'_v(t) = -t \int_{\mathbb{R}^n_+} Z(|\nabla v|^2) G - 2t \int_{\mathbb{R}^n_+} |\nabla v|^2 G = -t \int_{\mathbb{R}^n_+} (Z(|\nabla v|^2 + 2|\nabla v|^2)) G.$$  

We now use the following easily verifiable identity

$$Z(|\nabla v|^2) + 2|\nabla v|^2 = 2\nabla v \cdot \nabla (Zv),$$
which, after substitution in the latter equation and integration by parts, yields
\[
2t i_\epsilon'(t) = -2t \int_{\mathbb{R}^n_+} \nabla v \cdot \nabla (Zv) G
\]
\[
= 2t \int_{\mathbb{R}^n_+} \Delta v (Zv) G + 2t \int_{\mathbb{R}^n_+} (Zv) \nabla v \nabla G + 2t \int_{\mathbb{R}^{n-1}} v_{x_n} (Zv) G
\]
\[
= 2t \int_{\mathbb{R}^n_+} g(Zv) G + 2t \int_{\mathbb{R}^n_+} (Zv) (\nabla v \nabla G + v_t) G + 2t \int_{\mathbb{R}^{n-1}} v_{x_n} (Zv) G
\]
\[
= 2t \int_{\mathbb{R}^n_+} g(Zv) G + \int_{\mathbb{R}^n_+} (Zv)^2 G + 2t \int_{\mathbb{R}^{n-1}} v_{x_n} (Zv) G.
\]
Hence, we obtain
\[
i_\epsilon'(t) = \frac{1}{2t} \int_{\mathbb{R}^n_+} (Zv)^2 G + \int_{\mathbb{R}^n_+} g(Zv) G + \int_{\mathbb{R}^{n-1}} v_{x_n} (Zv) G,
\]
which establishes (6.2). \qed

With Lemma 6.1 in hand, we turn to establishing the essential ingredient in the proof of our main monotonicity result.

**Proposition 6.2** (Differentiation formulas for $H_u$ and $I_u$). Let $u \in \mathcal{S}_f(S^+_1)$. Then, $H_u$ and $I_u$ are absolutely continuous functions on $(0,1)$ and for a.e. $r \in (0,1)$ we have
\[
H_u'(r) = \frac{4}{r} I_u(r) - \frac{4}{r^3} \int_{S^+_1} t u f G,
\]
\[
I_u'(r) \geq \frac{1}{r^3} \int_{S^+_1} (Zu)^2 G + \frac{2}{r^3} \int_{S^+_1} t(Zu) f G.
\]

**Proof.** We note that, thanks to the estimates in Lemma 5.1 above, all integrals in the above formulas are finite. The idea of the proof of the proposition is to approximate $u$ with smooth solutions $u^\epsilon$ to the Signorini problem, apply Lemma 6.1, and then pass to the limit in $\epsilon$. The limit process is in fact more involved than one may expect. One complication is that although we have the estimates in Lemma 5.1 for the solution $u$, we do not have similar estimates, uniform in $\epsilon$, for the approximating $u^\epsilon$. This is in fact the main reason for which we have to consider truncated quantities $H_u^\delta$ and $I_u^\delta$, let $\epsilon \to 0$ first, and then $\delta \to 0$. However, the main difficulty is to show that the integrals over $S^+_1$ vanish. This is relatively easy to do for $H_u$, since $uu_{x_n} = 0$ on $S^+_1$. On the other hand, proving the formula for $I_u'$ is considerably more difficult, since one has to justify that $ux_n Zu = 0$ on $S^+_1$. Furthermore, the vanishing of this term should be interpreted in a proper sense, since we generally only know that $Zu \in L^2(S^+_1)$, and thus its trace may not even be well defined on $S^+_1$.

With this being said, in the sequel we justify only the formula for $I_u'$, the one for $H_u'$ being analogous, but much simpler.
(1°) Assume that \( u \) is supported in \( B^+_R \times (-1, 0] \), \( R \geq 3 \). Multiplying \( u \) with a cutoff function \( \eta(t) \) such that \( \eta = 1 \) on \([-\tilde{r}^2, 0] \) and \( \eta = 0 \) on \((-1, -\tilde{r}^2) \) for \( 0 < r < \tilde{r} < 1 \), without loss of generality we may assume that \( u(\cdot, -1) = 0 \). We then approximate \( u \) in \( B^+_R \times (-1, 0] \) with the solutions of the penalized problem
\[
\Delta u^\varepsilon - \partial_t u^\varepsilon = f^\varepsilon \quad \text{in} \quad B^+_R \times (-1, 0],
\]
\[
\partial_{x_n} u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{on} \quad B^+_R \times (-1, 0],
\]
\[
u^\varepsilon = 0 \quad \text{on} \quad (\partial B_R)^+ \times (-1, 0],
\]
\[
u^\varepsilon (\cdot, -1) = 0 \quad \text{on} \quad B^+_R,
\]
where \( f^\varepsilon \) is a mollification of \( f \). For any \( \rho \in (0, 1) \), let
\[
Q^+_{R-1, \rho} = B^+_{R-1} \times (-\rho^2, 0], \quad Q^+_{R-1, \rho} = B^+_{R-1} \times (-\rho^2, 0].
\]
From the estimates in Section 3 we have
\[
\|u^\varepsilon\|_{W^{2,1}_2(Q^+_{R-1, \rho})}, \|u^\varepsilon\|_{L^\infty(Q^+_{R-1, \rho})}, \|\nabla u^\varepsilon\|_{L^\infty(Q^+_{R-1, \rho})} \leq C(\rho, u), \quad \max_{Q^+_{R-1, \rho}} |\beta_\varepsilon(u^\varepsilon)| \leq C(\rho, u),
\]
uniformly in \( \varepsilon \in (0, 1) \).

(2°) Now, in order to extend \( u^\varepsilon \) to \( S^+_1 \), pick a cutoff function \( \zeta \in C^\infty_0(\mathbb{R}^n) \) such that
\[
0 \leq \zeta \leq 1, \quad \zeta = 1 \quad \text{on} \quad B_{R-2}, \quad \text{supp} \zeta \subset B_{R-1}, \quad \zeta(x', -x_n) = \zeta(x', x_n),
\]
and define
\[
v^\varepsilon(x, t) = u^\varepsilon(x, t)\zeta(x).
\]
Note that since \( u \) is supported in \( B^+_R \times (-1, 0] \), \( v^\varepsilon \) will converge to \( u \) and we will have the uniform estimates
\[
\|v^\varepsilon\|_{W^{2,1}_2(S^+_1)}, \|v^\varepsilon\|_{L^\infty(S^+_1)}, \|\nabla v^\varepsilon\|_{L^\infty(S^+_1)} \leq C(\rho, u) < \infty.
\]
For \( v^\varepsilon \) we have that
\[
\Delta v^\varepsilon - \partial_t v^\varepsilon = f^\varepsilon \zeta + u^\varepsilon \Delta \zeta + 2\nabla v^\varepsilon \nabla \zeta =: g^\varepsilon \quad \text{in} \quad S^+_1.
\]
It is easy to see that \( g^\varepsilon \) converges strongly to \( f \) in \( L_2(S^+_\delta \setminus S^+_\delta \rho, G) \) for \( 0 < \delta < 1 \). Besides, we also have
\[
v^\varepsilon_{x_n} = u^\varepsilon_{x_n} \zeta, \quad Z v^\varepsilon = \zeta(Z u^\varepsilon) + (Z \zeta) u^\varepsilon \quad \text{on} \quad S^+_1,
\]
and therefore
\[
v^\varepsilon_{x_n} Z v^\varepsilon = \zeta(Z \zeta) u^\varepsilon \beta_\varepsilon(u^\varepsilon) + \zeta^2 u^\varepsilon_{x_n} (Z u^\varepsilon) \quad \text{on} \quad S^+_1.
\]

(3°) We now fix a small \( \delta > 0 \), apply the differentiation formulas in Lemma 6.1 to \( v^\varepsilon \) and pass to the limit. We have
\[
(I^+_{\nu^\varepsilon}(r)') = \frac{1}{\nu^\varepsilon} \int_{S^+_1 \setminus S^+_\delta} (Z v^\varepsilon)^2 G + \frac{2}{\nu^\varepsilon} \int_{S^+_1 \setminus S^+_\delta} t(Z v^\varepsilon)^2 G + \frac{4}{\nu^\varepsilon} \int_{S^+_1 \setminus S^+_\delta} t v^\varepsilon_{x_n} (Z v^\varepsilon) G
\]
\[
= J_1 + J_2 + J_3.
\]

(3.i°) To pass to the limit in \( J_1 \), we note that \( Z v^\varepsilon \) converges to \( Z u \) weakly in
Indeed, this follows from the uniform $W^{2,1}_2$ estimates on $v^\varepsilon$ in $2^+_r$ and the boundedness of $G$ in $S^+_r \setminus S^+_br$. Thus, in the limit we obtain
\[
\frac{1}{r^3} \int_{S^+_r \setminus S^+_br} (Zu)^2 G \leq \liminf_{\varepsilon \to 0} \frac{1}{r^3} \int_{S^+_r \setminus S^+_br} (Zv^\varepsilon)^2 G.
\]
Note that here we cannot claim equality, as we do not have a strong convergence of $Zv^\varepsilon$ to $Zu$.

(3.ii\textsuperscript{o}) In $J_2$, the weak convergence of $Zv^\varepsilon$ to $Zu$, combined with the strong convergence of $g^\varepsilon$ to $f$ in $L^2(S^+_r \setminus S^+_br, G)$ is enough to conclude that
\[
\frac{2}{r^3} \int_{S^+_r \setminus S^+_br} t(Zu)f G = \lim_{\varepsilon \to 0} \frac{2}{r^3} \int_{S^+_r \setminus S^+_br} t(Zv^\varepsilon)g^\varepsilon G.
\]
Moreover, the convergence will be uniform in $r \in [r_1, r_2] \subset (0, 1)$.

(3.iii\textsuperscript{o}) Finally, we claim that $J_3 \to 0$ as $\varepsilon \to 0$, i.e.,
\[
\lim_{\varepsilon \to 0^+} \frac{4}{r^3} \int_{S^+_r \setminus S^+_br} tv^\varepsilon (Zv^\varepsilon) G = 0.
\]
Indeed, we have
\[
\int_{S^+_r \setminus S^+_br} tv^\varepsilon (Zv^\varepsilon) G = \int_{S^+_r \setminus S^+_br} t\zeta(Z\zeta)u^\varepsilon \beta_\varepsilon(u^\varepsilon) G + \int_{S^+_r \setminus S^+_br} t\zeta^2 \beta_\varepsilon(u^\varepsilon)(Zu^\varepsilon) G =: E_1 + E_2.
\]
We then estimate the integrals $E_1$ and $E_2$ separately.

(3.iii.a\textsuperscript{o}) We start with $E_1$. Recall that $|\beta_\varepsilon(u^\varepsilon)| \leq C(\rho)$ in $Q'_{R-1,\rho}$. By (3.5), this implies that $u^\varepsilon \geq -C(\rho)\varepsilon$ in $Q'_{R-1,\rho}$ and therefore
\[
|E_1| \leq C(\rho)r^3\varepsilon, \quad 0 < r \leq \rho < 1.
\]

(3.iii.b\textsuperscript{o}) A similar estimate holds also for $E_2$, but the proof is a little more involved. To this end consider
\[
\mathcal{B}_\varepsilon(t) = \int_0^t \beta_\varepsilon(s)ds, \quad t \in \mathbb{R}.
\]
From (3.5), it is easy to see that
\[
\mathcal{B}_\varepsilon(t) = 0 \quad \text{for } t > 0, \quad \mathcal{B}_\varepsilon(t) \geq 0 \quad \text{for all } t, \quad \mathcal{B}_\varepsilon(t) = C_\varepsilon + \varepsilon t + \frac{t^2}{2\varepsilon} \quad \text{for } t \leq -2\varepsilon^2,
\]
with $C_\varepsilon = \mathcal{B}_\varepsilon(-2\varepsilon^2) \leq 2\varepsilon^3$. Note that the uniform bound $u^\varepsilon \geq -C(\rho)\varepsilon$ in $Q'_{R-1,\rho}$ implies that
\[
\max_{Q'_{R-1,\rho}} \mathcal{B}_\varepsilon(u^\varepsilon) \leq C(\rho)\varepsilon.
\]
To use this fact, note that
\[
E_2 = \int_{S_t \setminus S_{t_r}} t \zeta^2 \beta_r(u^\varepsilon)(Zu^\varepsilon)G = \int_{S_t \setminus S_{t_r}} t \zeta^2 [ZB_c(u^\varepsilon)]G
\]
\[
= \int_{S_t \setminus S_{t_r}} Z[t \zeta^2 B_c(u^\varepsilon)]G - \int_{S_t \setminus S_{t_r}} Z(t \zeta^2)B_c(u^\varepsilon)G
\]
\[
=: E_{22} - E_{21}.
\]

(3.iii.b.α°) The estimate for $E_{21}$ is straightforward:
\[
|E_{21}| \leq C(\rho)r^3 \varepsilon, \quad 0 < r \leq \rho.
\]

(3.iii.b.β°) To estimate $E_{22}$, denote $U^\varepsilon = t \zeta^2 B_c(u^\varepsilon)$. Then, substituting $t = -\lambda^2$, $x' = \lambda y'$, we have
\[
E_{22} = \int_{S_t} ZU^\varepsilon G \, dx' dt = \int_{\delta r} \int_{\mathbb{R}^{n-1}} (ZU^\varepsilon)(\lambda y', -\lambda^2)G(\lambda y', -\lambda^2)2\lambda^n dy' d\lambda
\]
\[
= 2 \int_{\delta r} \int_{\mathbb{R}^{n-1}} \lambda \frac{d}{d\lambda} U^\varepsilon(\lambda y', -\lambda^2)G(y', -1) dy' d\lambda
\]
\[
= 2 \int_{\mathbb{R}^{n-1}} [rU^\varepsilon(ry', -r^2) - \delta rU^\varepsilon(\delta ry', -\delta^2 r^2)]G(y', -1) dy'
\]
\[
- 2 \int_{\delta r} \int_{\mathbb{R}^{n-1}} U^\varepsilon(\lambda y', -\lambda^2)G(y', -1) dy' d\lambda
\]
and consequently
\[
|E_{22}| \leq C(\rho)r^3 \varepsilon, \quad 0 < r \leq \rho.
\]

Combining the estimates in (3.iii.a°)–(3.iii.b°) above, we obtain
\[
|J_3| = \frac{4}{r^3} \int_{S_t \setminus S_{t_r}} te^\frac{\varepsilon}{x_n}(Zv^\varepsilon)G \leq C(\rho)\varepsilon \to 0, \quad 0 < r \leq \rho.
\]

(4°) Now, writing for any $0 < r_1 < r_2 < 1$
\[
I^0_{u^\varepsilon}(r_2) - I^0_{u^\varepsilon}(r_1) = \int_{r_1}^{r_2} (I^0_{u^\varepsilon})'(r) dr,
\]
collecting the facts proved in (3.iii.a°)–(3.iii.f°), and passing to the limit as $\varepsilon \to 0$, we obtain
\[
I^0_u(r_2) - I^0_u(r_1) \geq \int_{r_1}^{r_2} \left( \frac{1}{r^3} \int_{S^+_t \setminus S^+_{t_r}} (Zu)^2 G + \frac{2}{r^3} \int_{S^+_t \setminus S^+_{t_r}} t(Zu)fG \right) dr.
\]
Next, note that by Lemma 5.1 the integrands are uniformly bounded with respect to $\delta$ (for fixed $r_1$ and $r_2$). Therefore, we can let $\delta \to 0$ in the latter inequality, to obtain
\[
I_u(r_2) - I_u(r_1) \geq \int_{r_1}^{r_2} \left( \frac{1}{r^{\lambda}} \int_S (Zu)^2 G + \frac{2}{r^3} \int t(Zu) f G \right) dr.
\]
This is equivalent to the sought for conclusion for $I'_u$.

To state the main result of this section, the generalized frequency formula, we need the following notion. We say that a positive function $\mu(r)$ is log-convex if $\log \mu(r)$ is a convex function of $\log r$. This simply means that
\[
\mu(e^{(1-\lambda)s + \lambda t}) \leq \mu(e^s)^{1-\lambda} \mu(e^t)^{\lambda}, \quad 0 \leq \lambda \leq 1.
\]
This is equivalent to saying that $\mu$ is locally absolutely continuous on $\mathbb{R}_+$ and $r \mu'(r)/\mu(r)$ is nondecreasing. For instance, $\mu(r) = r^\kappa$ is a log-convex function of $\log r$ for any $\kappa$. The importance of this notion in our context is that Almgren’s and Poon’s frequency formulas can be regarded as log-convexity statements in $\log r$ for the appropriately defined quantities $H_u(r)$.

**Theorem 6.3** (Generalized frequency formula). Let $u \in \mathcal{F}(S_1^+)$ with $f$ satisfying the following condition: there is a positive monotone nondecreasing log-convex function $\mu(r)$ of $\log r$, and constants $\sigma > 0$ and $C_\mu > 0$, such that
\[
\mu(r) \geq C_\mu r^{4 - 2\sigma} \int_{\mathbb{R}^n} f^2(\cdot, -r^2) G(\cdot, -r^2) \, dx.
\]
Then, there exists $C > 0$, depending only on $\sigma, C_\mu$ and $n$, such that the function
\[
\Phi_u(r) = \frac{1}{2} r e^{C r^\sigma} \frac{d}{dr} \log \max\{H_u(r), \mu(r)\} + 2(e^{C r^\sigma} - 1)
\]
is nondecreasing for $r \in (0, 1)$.

Note that on the open set where $H_u(r) > \mu(r)$ we have $\Phi_u(r) \sim \frac{1}{2} r H'_u(r)/H_u(r)$ which coincides with $2N_u$, when $f = 0$. The purpose of the “truncation” of $H_u(r)$ with $\mu(r)$ is to control the error terms in computations that appear from the right-hand-side $f$.

**Proof of Theorem 6.3**. First, we want to make a remark on the definition of $r \mapsto \Phi_u(r)$, for $r \in (0, 1)$. The functions $H_u(r)$ and $\mu(r)$ are absolutely continuous and therefore so is max{$H_u(r), \mu(r)$}. It follows that $\Phi_u$ is uniquely identified only up to a set of measure zero. The monotonicity of $\Phi_u$ should be understood in the sense that there exists a monotone increasing function which equals $\Phi_u$ almost everywhere. Therefore, without loss of generality we may assume that
\[
\Phi_u(r) = \frac{1}{2} r e^{C r^\sigma} \frac{\mu'(r)}{\mu(r)} + 2(e^{C r^\sigma} - 1)
\]
on $\mathcal{F} = \{H_u(r) \leq \mu(r)\}$ and
\[
\Phi_u(r) = \frac{1}{2} r e^{C r^\sigma} \frac{H'_u(r)}{H_u(r)} + 2(e^{C r^\sigma} - 1)
\]
in $\mathcal{O} = \{H_u(r) > \mu(r)\}$. Following an idea introduced in [GL80, GL87] we now note that it will be enough to check that $\Phi'_u(r) > 0$ in $\mathcal{O}$. Indeed, from the assumption
on \( \mu \), it is clear that that \( \Phi_u \) is monotone on \( \mathcal{F} \). Next, if \((r_0, r_1)\) is a maximal open interval in \( \mathcal{O} \), then \( H_u(r_0) = \mu(r_0) \) and \( H_u(r_1) = \mu(r_1) \) unless \( r_1 = 1 \). Besides, if \( \Phi_u \) is monotone in \((r_0, r_1)\), it is easy to see that the limits \( H_u'(r_0+) \) and \( H_u'(r_1-) \) will exist and satisfy

\[
\mu'(r_0^+) \leq H_u'(r_0^+), \quad H_u'(r_1^-) \leq \mu'(r_1^-) \quad \text{(unless } r_1 = 1) \]

and therefore we will have

\[
\Phi_u(r_0^-) \leq \Phi_u(r_0^+) \leq \Phi_u(r_1^-) \leq \Phi_u(r_1^+),
\]

with the latter inequality holding when \( r_1 < 1 \). This will imply the monotonicity of \( \Phi_u \) in \((0, 1)\).

Therefore, we will concentrate only on the set \( \mathcal{O} = \{ H_u(r) > \mu(r) \} \), where the monotonicity of \( \Phi_u(r) \) is equivalent to that of

\[
\left( \frac{r H_u'(r)}{H_u(r)} + 4 \right) e^{C r^\sigma} = 2 \Phi_u(r) + 4.
\]

The latter will follow, once we show that

\[
\frac{d}{dr} \left( \frac{r H_u'(r)}{H_u(r)} \right) \geq -C \left( \frac{r H_u'(r)}{H_u(r)} + 4 \right) r^{-1+\sigma}
\]

in \( \mathcal{O} \). Now, from Proposition 6.2 we have

\[
r H_u'(r) = \frac{I_u(r)}{H_u(r)} - \frac{4}{r^2} \frac{\int_{S^+_1} t u f G}{H_u(r)} := 4 E_1(r) + 4 E_2(r).
\]

We then estimate the derivatives of each of the quantities \( E_i(r) \), \( i = 1, 2 \).

(1°) Using the differentiation formulas in Proposition 6.2, we compute

\[
r^5 H_u^2(r) E_1'(r) = r^5 H_u^2(r) \frac{d}{dr} \left( \frac{I_u(r)}{H_u(r)} \right)
\]

\[
= r^5 (I_u'(r) H_u(r) - I_u(r) H_u'(r))
\]

\[
\geq r^2 H_u(r) \left( \frac{\int (Z u)^2 G + 2 \int t (Z u) f G}{S^+_1} \right)
\]

\[
- r^2 I_u(r) r^3 H_u'(r)
\]

\[
= r^2 H_u(r) \left( \frac{\int (Z u + t f)^2 G - \int t^2 f^2 G}{S^+_1} \right)
\]

\[
- \left( \frac{\int t u f G}{S^+_1} \right) r^3 H_u'(r)
\]

\[
= \frac{\int u^2 G \left( \frac{\int (Z u + t f)^2 G - \int t^2 f^2 G}{S^+_1} \right)}{S^+_1}
\]

\[
- \left( \frac{r^3}{2} H_u'(r) + \frac{\int t u f G}{S^+_1} \right) + \left( \frac{\int t u f G}{S^+_1} \right)^2
\]

\[
= \left[ \frac{\int u^2 G \left( \frac{\int (Z u + t f)^2 G - \int t^2 f^2 G}{S^+_1} \right)}{S^+_1} \right]
\]

\[
- \left( \frac{r^3}{2} H_u'(r) + \frac{\int t u f G}{S^+_1} \right)^2 + \left( \frac{\int t u f G}{S^+_1} \right)^2
\]

\[
= \left[ \frac{\int u^2 G \left( \frac{\int (Z u + t f)^2 G - \int t^2 f^2 G}{S^+_1} \right)}{S^+_1} \right]
\]

\[
- \left( \frac{r^3}{2} H_u'(r) + \frac{\int t u f G}{S^+_1} \right)^2 + \left( \frac{\int t u f G}{S^+_1} \right)^2
\]
Applying the Cauchy-Schwarz inequality we obtain that the term in square brackets above is nonnegative. Therefore,

\[ r^5 H_u^2(r) E'_1(r) \geq - \int_{S^+_r} u^2 G \int_{S^+_r} t^2 f^2 G \]

or equivalently,

\[ E'_1(r) \geq - \frac{1}{r^3 H_u(r)} \int_{S^+_r} t^2 f^2 G. \]

(2°) We next estimate the derivative of \( E_2(r) \).

\[
E'_2(r) = \frac{d}{dr} \left( - \frac{1}{r^2} \frac{\int_{S^+_r} tu f G}{H_u(r)} \right) \\
= \frac{2}{r^3} \frac{\int_{S^+_r} tu f G}{H_u(r)} - \frac{2}{r} \frac{\int_{R^n_+} (-r^2) u(\cdot, -r^2) f(\cdot, -r^2) G(\cdot, -r^2)}{H_u(r)} \\
+ \frac{1}{r^2} \frac{H'_u(r) \int_{S^+_r} tu f G}{H_u(r)^2} \\
\geq - \frac{2}{r^3 H_u(r)} \left( \int_{S^+_r} u^2 G \right)^{1/2} \left( \int_{S^+_r} t^2 f^2 G \right)^{1/2} \\
- \frac{2}{r H_u(r)} \left( \int_{R^n_+} u^2(\cdot, -r^2) G(\cdot, -r^2) \right)^{1/2} \left( r^4 \int_{R^n_+} f^2(\cdot, -r^2) G(\cdot, -r^2) \right)^{1/2} \\
- \frac{r H'_u(r)}{H_u(r)} \frac{1}{r^3 H_u(r)} \left( \int_{S^+_r} u^2 G \right)^{1/2} \left( \int_{S^+_r} t^2 f^2 G \right)^{1/2} \\
= - \frac{2}{r^2 H_u(r)^{1/2}} \left( \int_{S^+_r} t^2 f^2 G \right)^{1/2} \\
- \frac{2}{r H_u(r)} \left( \int_{R^n_+} u^2(\cdot, -r^2) G(\cdot, -r^2) \right)^{1/2} \left( r^4 \int_{R^n_+} f^2(\cdot, -r^2) G(\cdot, -r^2) \right)^{1/2} \\
- r H'_u(r) \frac{1}{H_u(r)^{3/2} H_u(r)^{1/2}} \left( \int_{S^+_r} t^2 f^2 G \right)^{1/2}. \]
(3°) Combining together the estimates for $E'_1(r)$ and $E'_2(r)$, we have
\[
\frac{d}{dr} \left( r \frac{H'_u(r)}{H_u(r)} \right) \geq - \frac{4}{r^3 H_u(r)} \int_{S_u^+} t^2 f^2 G - \frac{8}{r^2 H_u(r)^{1/2}} \left( \int_{S_u^+} t^2 f^2 G \right)^{1/2} \bigg( 1 + \frac{r H'_u(r)}{2 H_u(r)} \bigg) \leq 0.
\]
We estimate the third term separately. First, from
\[
\frac{d}{dr} \int_{S_u^+} u^2 G = 2r \int_{\mathbb{R}_+^n} u^2(\cdot, -r^2) G(\cdot, -r^2) \, dx
\]
we have
\[
\int_{\mathbb{R}_+^n} u^2(\cdot, -r^2) G(\cdot, -r^2) = \frac{1}{2r} \frac{d}{dr} (r^2 H_u(r)) = H_u(r) + \frac{r}{2} H'_u(r).
\]
From here we see that
\[
1 + \frac{r H'_u(r)}{2 H_u(r)} \geq 0
\]
and therefore we also have
\[
2 \left( \int_{\mathbb{R}_+^n} u^2(\cdot, -r^2) G(\cdot, -r^2) \right)^{1/2} = 2H_u(r)^{1/2} \left( 1 + \frac{r H'_u(r)}{2 H_u(r)} \right) \leq H_u(r) \left( 2 + \frac{r H'_u(r)}{2 H_u(r)} \right).
\]
Substituting into the inequality above, we then have
\[
\frac{d}{dr} \left( r \frac{H'_u(r)}{H_u(r)} \right) \geq - \frac{4}{r^3 H_u(r)} \int_{S_u^+} t^2 f^2 G - \frac{8}{r^2 H_u(r)^{1/2}} \left( \int_{S_u^+} t^2 f^2 G \right)^{1/2} \bigg( 2 + \frac{r H'_u(r)}{2 H_u(r)} \bigg) \leq 0.
\]
On the set \{ $H_u(r) > \mu(r)$ \}, we easily have
\[
H_u(r) \geq C \mu r^{4-2\sigma} \int_{\mathbb{R}_+^n} f^2(\cdot, -r^2) G(\cdot, -r^2),
\]
\[
H_u(r) \geq C \mu r^{-2-2\sigma} \int_{S_u^+} t^2 f^2 G,
\]
and consequently,
\[
\frac{d}{dr} \left( \frac{r H_u'(r)}{H_u(r)} \right) \geq -C r^{-1+2\sigma} - C r^{-1+\sigma} - \left( 2 + \frac{r H_u'(r)}{2 H_u(r)} \right) C r^{-1+\sigma} - C r \frac{H_u'(r)}{H_u(r)} r^{-1+\sigma}
\]
\[
\geq -C \left( \frac{r H_u'(r)}{H_u(r)} + 4 \right) r^{-1+\sigma}.
\]
Note that in the last step we have again used the fact that \(1 + \frac{r H_u'(r)}{H_u(r)} \geq 0\). The desired conclusion follows readily. \(\square\)

7. Existence and homogeneity of blowups

In this section, we show how the generalized frequency formula in Theorem 6.3 can be used to study the behavior of the solution \(u\) near the origin. The central idea is to consider some appropriately normalized rescalings of \(u\), indicated with \(u_r\) (see Definition 7.2), and then pass to the limit as \(r \to 0^+\) (see Theorem 7.3). The resulting limiting functions (over sequences \(r = r_j \to 0^+\)) are known as blowups. However, because of the truncation term \(\mu(r)\) in the generalized frequency function \(\Phi_u(r)\), we can show the existence of blowups only when the growth rate of \(u\) can be "detected," in a certain proper sense to be made precise below. Finally, as a consequence of the monotonicity of \(\Phi_u(r)\), we obtain that the blowups must be parabolically homogeneous solutions of the Signorini problem in \(S_\infty = \mathbb{R}^n \times (-\infty, 0]\).

Henceforth, we assume that \(u \in \mathcal{G}^f(S_1^+)\), and that \(\mu(r)\) be such that the conditions of Theorem 6.3 are satisfied. In particular, we assume that
\[
\left. r^4 \int_{\mathbb{R}^n} f^2(\cdot, -r^2)G(\cdot, -r^2) \, dx \right|_{r=0} \leq \frac{r^{2\sigma} \mu(r)}{C_\mu}.
\]
Consequently, Theorem 6.3 implies that the function
\[
\Phi_u(r) = \frac{1}{2} r e^{Cr^\sigma} \frac{d}{dr} \log \max\{H_u(r), \mu(r)\} + 2(e^{Cr^\sigma} - 1)
\]
is nondecreasing for \(r \in (0, 1)\). Hence, there exists the limit
\[
(7.1) \quad \kappa := \Phi_u(0+) = \lim_{r \to 0^+} \Phi_u(r).
\]
Since we assume that \(r \mu'(r)/\mu(r)\) is nondecreasing, the limit
\[
(7.2) \quad \kappa_\mu := \frac{1}{2} \lim_{r \to 0^+} \frac{r \mu'(r)}{\mu(r)}
\]
also exists. We then have the following basic proposition concerning the values of \(\kappa\) and \(\kappa_\mu\).

**Lemma 7.1.** Let \(u \in \mathcal{G}^f(S_1^+)\) and \(\mu\) satisfy the conditions of Theorem 6.3. With \(\kappa, \kappa_\mu\) as above, we have
\[
\kappa \leq \kappa_\mu.
\]
Moreover, if \(\kappa < \kappa_\mu\), then there exists \(r_u > 0\) such that \(H_u(r) \geq \mu(r)\) for \(0 < r \leq r_u\). In particular,
\[
\kappa = \frac{1}{2} \lim_{r \to 0^+} \frac{r H_u'(r)}{H_u(r)} = 2 \lim_{r \to 0^+} \frac{I_u(r)}{H_u(r)}.
\]
Proof: As a first step we show that

\((7.3)\) \(\kappa \neq \kappa_\mu \quad \Rightarrow \quad \) there exists \(r_u > 0\) such that \(H_u(r) \geq \mu(r)\) for \(0 < r \leq r_u\).

Indeed, if the implication claimed in \((7.3)\) fails, then for a sequence \(r_j \to 0^+\) we have \(H_u(r_j) < \mu(r_j)\). This implies that

\[\Phi_u(r_j) = \frac{1}{2} r_j e^{Cr_j} \frac{\mu'(r_j)}{\mu(r_j)} + 2(\epsilon Cr_j - 1),\]

and therefore

\[\kappa = \lim_{j \to \infty} \Phi_u(r_j) = \frac{1}{2} \lim_{j \to \infty} r_j \frac{\mu'(r_j)}{\mu(r_j)} = \kappa_\mu,\]

which contradicts \(\kappa \neq \kappa_\mu\). We have thus proved \((7.3)\).

Now, \((7.3)\) implies that, if \(\kappa \neq \kappa_\mu\), then

\[\Phi_u(r) = \frac{1}{2} r e^{Cr} \frac{H_u'(r)}{H_u(r)} + 2(\epsilon Cr - 1), \quad 0 < r < r_u.\]

Passing to the limit, we conclude that, if \(\kappa \neq \kappa_\mu\), then

\[(7.4)\] \[\kappa = \lim_{r \to 0^+} \Phi_u(r) = \frac{1}{2} \lim_{r \to 0^+} r \frac{H_u'(r)}{H_u(r)}.\]

However, in this case we also have

\[(7.5)\] \[\frac{1}{2} \lim_{r \to 0^+} r \frac{H_u'(r)}{H_u(r)} = 2 \lim_{r \to 0^+} \frac{I_u(r)}{H_u(r)}.\]

Indeed, recall that (see Proposition \(6.2\))

\[r \frac{H_u'(r)}{H_u(r)} = 4 \frac{I_u(r)}{H_u(r)} - \frac{4 \int_{S^+} tufG}{r^2 H_u(r)},\]

and \((7.5)\) will follow once we show that

\[\lim_{r \to 0^+} \frac{1}{r^2} \frac{4 \int_{S^+} tufG}{H_u(r)} = 0.\]

Using the Cauchy-Schwarz inequality, we have

\[\int_{S^+} tufG \leq \frac{1}{r^2 H_u(r)} \leq \frac{\left(\int_{S^+} t^2 f^2 G\right)^{1/2} \left(\int_{S^+} u^2 G\right)^{1/2}}{r H_u(r)^{1/2}} = \frac{\left(\int_{S^+} t^2 f^2 G\right)^{1/2}}{r H_u(r)^{1/2}} \leq \frac{\mu(r)}{C_\mu H_u(r)} \leq C_\mu^{-1} r^{\sigma} \to 0,\]

where in the last inequality before the limit we have used \((7.3)\). Summarizing, the assumption \(\kappa \neq \kappa_\mu\) implies \((7.4)\)–\((7.5)\) above. Therefore, the proof will be completed if we show that the case \(\kappa > \kappa_\mu\) is impossible.

So, assume towards a contradiction that \(\kappa > \kappa_\mu\), and fix \(0 < \varepsilon < \kappa - \kappa_\mu\). For such \(\varepsilon\) choose \(r_\varepsilon > 0\) so that

\[r \frac{H_u'(r)}{H_u(r)} > 2\kappa - \varepsilon, \quad r \frac{\mu'(r)}{\mu(r)} < 2\kappa_\mu + \varepsilon, \quad 0 < r < r_\varepsilon.\]

Integrating these inequalities from \(r\) to \(r_\varepsilon\), we obtain

\[H_u(r) \leq \frac{H_u(r_\varepsilon)}{r^{2\kappa - \varepsilon}} r^{2\kappa - \varepsilon}, \quad \mu(r) \geq \frac{\mu(r_\varepsilon)}{r^{2\kappa_\mu + \varepsilon}} r^{2\kappa_\mu + \varepsilon}.\]
Since by our choice of \( \varepsilon > 0 \) we have \( 2\kappa - \varepsilon > 2\kappa_\mu + \varepsilon \), the above inequalities imply that \( H_u(r) < \mu(r) \) for small enough \( r \), contrary to the established conclusion of (7.3) above. Hence, the case \( \kappa > \kappa_\mu \) is impossible, which implies that we always have \( \kappa \leq \kappa_\mu \). \( \Box \)

To proceed, we define the appropriate notion of rescalings that works well with the generalized frequency formula.

**Definition 7.2 (Rescalings).** For \( u \in \mathcal{S}_f(S_1^+) \) and \( r > 0 \) define the rescalings

\[
u_r(x,t) := \frac{u(rx, r^2t)}{H_u(r)^{1/2}}, \quad (x,t) \in S_{1/r}^+ = \mathbb{R}_+^n \times \left(-1/r^2, 0\right] .\]

It is easy to see that the function \( \nu_r \) solves the nonhomogeneous Signorini problem

\[
\Delta \nu_r - \partial_t \nu_r = f_r(x,t) \quad \text{in} \quad S_{1/r}^+, \\
\nu_r \geq 0, \quad -\partial_{x_n} \nu_r \geq 0, \quad \nu_r \partial_{x_n} \nu_r = 0 \quad \text{on} \quad S_{1/r}',
\]

with

\[
f_r(x,t) = \frac{r^2 f(rx, r^2t)}{H_u(r)^{1/2}} .
\]

In other words, \( \nu_r \in \mathcal{S}_f(S_{1/r}^+) \). Further, note that \( \nu_r \) is normalized by the condition

\[
H_{\nu_r}(1) = 1 ,
\]

and that, more generally, we have

\[
H_{\nu_r}(\rho) = \frac{H_u(pr)}{H_u(r)} .
\]

We next show that, unless we are in the borderline case \( \kappa = \kappa_\mu \), we will be able to study the so-called blowups of \( u \) at the origin. The condition \( \kappa < \kappa_\mu \) below can be understood, in a sense, that we can “detect” the growth of \( u \) near the origin.

**Theorem 7.3 (Existence and homogeneity of blowups).** Let \( u \in \mathcal{S}_f(S_1^+) \), \( \mu \) satisfy the conditions of Theorem 6.3 and

\[
\kappa := \Phi_u(0+) < \kappa_\mu = \frac{1}{2} \lim_{r \to 0+} \frac{\mu'(r)}{\mu(r)} .
\]

Then, we have:

i) For any \( R > 0 \), there is \( r_{R,u} > 0 \) such that

\[
\int_{S_R^+} (u_r^2 + |t||\nabla u_r|^2 + |t|^2|D^2 u_r|^2 + |t|^2(\partial_t u_r)^2)G \leq C(R), \quad 0 < r < r_{R,u} .
\]

ii) There is a sequence \( r_j \to 0+ \), and a function \( u_0 \) in \( S_{\infty}^+ = \mathbb{R}_+^n \times (-\infty, 0] \), such that

\[
\int_{S_R^+} (|u_{r_j} - u_0|^2 + |t||\nabla (u_{r_j} - u_0)|^2)G \to 0 .
\]

We call any such \( u_0 \) a blowup of \( u \) at the origin.
iii) $u_0$ is a nonzero global solution of Signorini problem:

$$
\Delta u_0 - \partial_t u_0 = 0 \quad \text{in } S^+_\infty
$$

$$
u_0 \geq 0, \quad -\partial_{x_n} u_0 \geq 0, \quad u_0 \partial_{x_n} u_0 = 0 \quad \text{on } S'_\infty,
$$

in the sense that it solves the Signorini problem in every $Q^+_R$.

iv) $u_0$ is parabolically homogeneous of degree $\kappa$:

$$
u_0(\lambda x, \lambda^2 t) = \lambda^\kappa u_0(x,t), \quad (x,t) \in S^+_\infty, \quad \lambda > 0
$$

The proof of Theorem 7.3 is based on the following lemmas.

**Lemma 7.4.** If $\kappa < \kappa' < \kappa_{\mu}$ and $r_u > 0$ is such that $\Phi_u(r) < \kappa'$ and $H_u(r) \geq \mu(r)$ for $0 < r < r_u$, then

$$
H_u(r) \geq \rho^{2\kappa'} \quad \text{for any } 0 < \rho \leq 1, \quad 0 < r < r_u,
$$

$$
H_u(R) \leq R^{2\kappa'} \quad \text{for any } R \geq 1, \quad 0 < r < r_u/R.
$$

**Proof.** From the assumptions we have

$$
\Phi_u(r) = \frac{1}{2} r e^{Cr}\frac{H_u'(r)}{H_u(r)} + 2(e^{Cr} - 1) \leq \kappa',
$$

for $0 < r < r_u$, which implies that

$$
\frac{H_u'(r)}{H_u(r)} \leq \frac{2\kappa'}{r}.
$$

Integrating from $\rho r$ to $r$ and exponentiating, we find

$$
\frac{H_u(r)}{H_u(\rho r)} \leq \rho^{-2\kappa'},
$$

which implies that

$$
H_u(\rho) = \frac{H_u(\rho r)}{H_u(r)} \geq \rho^{2\kappa'}.
$$

Similarly, integrating from $r$ to $Rr$ (under the assumption that $Rr \leq r_u$) we find

$$
H_u(R) = \frac{H_u(Rr)}{H_u(r)} \leq R^{2\kappa'}.
$$

□

**Lemma 7.5.** Under the notations of the previous lemma, for any $R \geq 1$ and $0 < r < r_u/R$, we have

$$
\int_{S^+_R} t^2 f^2 G \leq c_{\mu} R^{2+2\sigma+2\kappa'} r^{2\sigma}.
$$

**Proof.** Note that from the assumptions we have

$$
\mu(r) \geq C_{\mu} r^{4-2\sigma} \int_{R^+_\infty} f^2(-r^2) G(-r^2),
$$

$$
\mu(r) \geq C_{\mu} r^{-2-2\sigma} \int_{S^+_R} t^2 f^2 G.
$$
Now take $R \geq 1$. Then, making the change of variables and using the inequalities above, we have
\[
\int_{S_R^+}^2 f^2 G = \frac{r^4}{H_u(r)} \int_{S_R^+}^2 f(rx, r^2t)^2 G(x, t) dx dt \\
= \frac{1}{r^2 H_u(r)} \int_{S_R^+}^2 f^2 G \leq R^{2+2\sigma} r^{2\sigma} \frac{\mu(Rr)}{C \mu H_u(r)}.
\]

Thus, if $0 < r < u/R$, then $H_u(Rr) \geq \mu(Rr)$ and therefore
\[
\int_{S_R^+}^2 f^2 G \leq c \mu R^{2+2\sigma} r^{2\sigma} \frac{H_u(Rr)}{H_u(r)} \leq c \mu R^{2+2\sigma} r^{2\sigma}.
\]

This completes the proof.

We will also need the following well-known inequality (see [Gro75]) and one of its corollaries.

**Lemma 7.6** (log-Sobolev inequality). For any $f \in W^1_2(\mathbb{R}^n, G(\cdot, s))$ one has
\[
\int_{\mathbb{R}^n} f^2 \log(f^2) G(\cdot, s) \leq \left( \int_{\mathbb{R}^n} f^2 G(\cdot, s) \right) \log \left( \int_{\mathbb{R}^n} f^2 G(\cdot, s) \right) + 4|s| \int_{\mathbb{R}^n} |\nabla f|^2 G(\cdot, s).
\]

**Lemma 7.7.** For any $f \in W^1_2(\mathbb{R}^n, G(\cdot, s))$, let $\omega = \{|f| > 0\}$. Then,
\[
\log \frac{1}{|\omega|}\int_{\mathbb{R}^n} f^2 G(\cdot, s) \leq 2|s| \int_{\mathbb{R}^n} |\nabla f|^2 G(\cdot, s),
\]
where
\[
|\omega| = \int_{\omega} G(\cdot, s).
\]

**Proof.** Let $\psi(y) = y \log y$ for $y > 0$ and $\psi(0) = 0$. Then, the log-Sobolev inequality can be rewritten as
\[
\int_{\mathbb{R}^n} \psi(f^2) G(\cdot, s) \leq \psi \left( \int_{\mathbb{R}^n} f^2 G(\cdot, s) \right) + 2|s| \int_{\mathbb{R}^n} |\nabla f|^2 G(\cdot, s).
\]

On the other hand, since $\psi$ is convex on $[0, \infty)$, by Jensen’s inequality we have
\[
\frac{1}{|\omega|}\int_{\mathbb{R}^n} \psi(f^2) G(\cdot, s) \geq \psi \left( \frac{1}{|\omega|}\int_{\mathbb{R}^n} f^2 G(\cdot, s) \right).
\]

Combining these inequalities and using the identity $\lambda \psi \left( \frac{a}{\lambda} \right) - \psi(a) = a \log \frac{1}{\lambda}$, we arrive at the claimed inequality. \[\square\]

**Proof of Theorem 7.3** i) From Lemmas 7.4 and 7.5 as well as Lemma 5.1 for $R \geq 1$, $0 < r < r_{R,u}$ we have
\[
\int_{S_{R/2}^+} (u_r^2 + |t|\nabla u_r|^2 + |t|^2 |D^2 u_r|^2 + |t|^2 |\partial_t u_r|^2) G \leq C_R(1 + c \mu r^{2\sigma}).
\]

Since $R \geq 1$ is arbitrary, this implies the claim of part i).
ii) Note that, in view of Lemma 7.3, it will be enough to show the existence of \( u_0 \), and the convergence
\[
\int_{S_R^+} |u_{r_j} - u_0|^2 G \to 0.
\]
From Lemma 7.1 it follows that \( \kappa \geq 0 \) and therefore we obtain
\[
\frac{r H_u(r)}{H_u(r)} \geq -1, \quad 0 < r < r_0.
\]
Integrating, we obtain that for small \( \delta > 0 \)
\[
H_u(r \delta) \leq H_u(r) \delta^{-1},
\]
which gives
\[
H_u(\delta) \leq \delta^{-1}
\]
and consequently
\[
\int_{S_R^+} u^2 G < \delta, \quad 0 < r < r_0.
\]
Next, let \( \zeta_A \in C_0^\infty(\mathbb{R}^n) \) be a cutoff function, such that
\[
0 \leq \zeta \leq 1, \quad \zeta_A = 1 \quad \text{on } B_{A-1}, \quad \text{supp } \zeta \subset B_A.
\]
We may take \( A \) so large that \( \int_{\mathbb{R}^n \setminus B_{A-1}} G(x,t)dx < e^{-1/\delta} \) for \(-R^2 < t < 0\). Then from Lemma 7.7 we have that
\[
\int_{S_R^+ \cap \{ |x| \geq A \}} u^2 G \leq \int_{S_R^+} u^2(1 - \zeta_A)^2 G \leq \delta \int_{S_R^+} (u^2 + |t||\nabla u_r|^2)G \leq \delta C(R),
\]
for small enough \( r \), where in the last step we have used the uniform estimate from part i).

Next, notice that on \( E = E_{R,\delta,A} = (S_R^+ \setminus S_\delta^+) \cap \{ |x| \leq A \} = B_A \times (-R^2, -\delta^2) \) the function \( G \) is bounded below and above and therefore the estimates in i) imply that the family \( \{u_r\}_{0 < r < r_{R,n}} \) is uniformly bounded in \( W^{1,1}_2(E^0) \) and thus we can extract a subsequence \( u_{r_j} \) converging strongly in \( L_2(E) \) and consequently in \( L_2(E,G) \).

Letting \( \delta \to 0 \) and \( A \to \infty \), combined with the estimates above, by means of the Cantor diagonal method, we complete the proof of this part.

iii) We first start with the Signorini boundary conditions. From the estimates in ii) we have that \( \{u_r\} \) is uniformly bounded in \( W^{2,1}_2(B_R^+ \times (-R^2, -\delta^2)) \) for any \( 0 < \delta < R \). We thus obtain that \( u_{r_j} \to u_0 \) strongly, and \( \partial_x u_{r_j} \to \partial_x u_0 \) weakly in \( L_2(B_R^+ \times (-R^2, -\delta^2)) \). This is enough to pass to the limit in the Signorini boundary conditions and to conclude that
\[
u_0 \geq 0, \quad -\partial_x u_0 \geq 0, \quad u_0 \partial_x u_0 = 0 \quad \text{on } \mathbb{R}^{n-1} \times (-\infty, 0).
\]
Besides, arguing similarly, and using Lemma 7.5 we obtain that
\[
\Delta u_0 - \partial_t u_0 = 0 \quad \text{in } \mathbb{R}^n_+ \times (-\infty, 0).
\]
Thus, to finish the proof of this part it remains to show that \( u_0 \) is in the unweighted Sobolev class \( W^{1,1}_2(Q^+_R) \) for any \( R > 0 \). Because of the scaling properties, it is sufficient to prove it only for \( R = 1/8 \). We argue as follows. First, extend \( u_0 \) by even symmetry in \( x_n \) to \( \mathbb{R}^n \times (-\infty, 0) \). We then claim that \( u_0^\pm \) are subcaloric.

*We will prove later that \( \kappa \geq 3/2 \), but the information \( \kappa \geq 0 \) will suffice in this proof.
functions in $\mathbb{R}^n \times (-\infty, 0)$. Indeed, this would follow immediately, if we knew the continuity of $u_0$, since $u_0$ is caloric where nonzero. But since we do not know the continuity of $u_0$ at this stage, we argue as follows. By continuity of $u_r$ we easily obtain

$$(\Delta - \partial_t)u_r^\pm \geq -f_r^\pm \text{ in } B_R \times (-R^2, -\delta^2).$$

Then, passing to the limit as $r = r_j \to 0$, we conclude that $u_0^\pm$ are subcaloric, since $|f_r| \to 0$ in $L_2(B_R \times (-R^2, -\delta^2))$ by Lemma 7.5. Further, we claim that $u_0^\pm$ satisfy the sub mean-value property

$$u_0^\pm(x, t) \leq \int_{\mathbb{R}^n} u_0^\pm(y, -1)G(x - y, -t - 1)dy,$$

for any $(x, t) \in \mathbb{R}^n \times (-1, 0)$. The proof of this fact is fairly standard, since, by the estimates in part i), $u_0$ satisfies an integral Tychonoff-type condition in the strips $S_1 \setminus S_\delta$, $\delta > 0$. Nevertheless, for completeness we give the details below. For large $R > 0$ let $\zeta_R \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function such that $0 \leq \zeta_R \leq 1$, $\zeta_R = 1$ on $B_R$, supp $\zeta_R \subset B_{R+1}$, $|\nabla \zeta_R| \leq 1$. Let now $w = u_0^\pm \zeta_R$ in $\mathbb{R}^n \times (-1, 0)$. From the fact that $u_0^\pm$ are subcaloric, we have that

$$(\Delta - \partial_t)w \geq 2\nabla u_0^\pm \nabla \zeta_R.$$

The advantage of $w$ now is that it has a bounded support, and therefore we can write

$$u_0(x, t)^\pm \zeta_R(x) = w(x, t) \leq \int_{\mathbb{R}^n} u_0^\pm(y, -1)\zeta_R(y)G(x - y, -t - 1)$$

$$+ 2 \int_{-1}^t \int_{\mathbb{R}^n} |\nabla u_0(y, s)||\nabla \zeta_R(y, s)|G(x - y, s - t)dyds.$$

To proceed, fix $A > 0$ large and $a > 0$ small and consider $|x| \leq A$ and $-1 < t < -a$. We want to show that the second integral above will vanish as we let $R \to \infty$. This will be done with suitable estimates on the kernel $G$.

**Claim 7.8.** Let $|x| \leq A$, $-1 < s < -a < 0$, and $s < t < 0$. Then

$$G(x - y, s - t) \leq \begin{cases} 
CG(y, s), & \text{if } t - s < -s/8, \ |y| \geq R \\
CG(y, s)e^{C|y|}, & \text{if } t - s \geq -s/8
\end{cases}$$

with $C = C_{n, a, A}$, $R = R_{n, a, A}$.

**Proof.** (1°) $t - s < -s/8$. Choose $R = 2A + 1$ and let $|y| \geq R$. Then

$$|x - y|^2 \geq \frac{|x - y|^2}{2} + \frac{|x - y|^2}{2} \geq \frac{|R - A|^2}{2} + \frac{(|y|/2)^2}{2} \geq \frac{1}{2} + \frac{|y|^2}{8},$$

and therefore

$$G(x - y, s - t) = \frac{1}{(4\pi(t - s))^{n/2}}e^{-\frac{|x - y|^2}{4(t - s)}} \leq \frac{1}{(4\pi(t - s))^{n/2}}e^{-\frac{1}{8s}|x - y|^2}$$

$$\leq \frac{1}{(4\pi(t - s))^{n/2}}e^{-s/8}e^{\frac{|y|^2}{4s}} \leq C_n e^{\frac{|y|^2}{4s}}$$

$$\leq \frac{C_{n, a}}{(4\pi(-s))^{n/2}}e^{\frac{|y|^2}{4s}} \leq C_{n, a} G(y, s),$$
where we have used that the function $r \mapsto 1/(4\pi r)^{n/2}e^{-1/4r}$ is uniformly bounded on $(0, \infty)$.

$(2^o)$ Suppose now $t - s \geq -s/8$. Then

$$G(x - y, s - t) = \frac{1}{(4\pi(t - s))^{n/2}}e^{-\frac{|x - y|^2}{4(t - s)}} \leq \frac{8^{n/2}}{(4\pi(-s))^{n/2}}e^{-\frac{|x - y|^2}{4s}} \leq \frac{C_n}{(4\pi(-s))^{n/2}}e^{-\frac{|y|^2}{4s}e^{C_n|\rho|}} \leq C_n G(y, s)e^{C_n|\rho|}.$$

Hence, the claim follows. $\square$

Then, using the second estimate in Claim 7.8, for $(x,t)$ and by applying Cauchy-Schwarz we will have

$$\int_{\mathbb{R}^n} u_0^+(x, t)G(x - y, s - t)dy < \rho < 0.$$

Then, using the second estimate in Claim 7.8 for $(x,t) \in Q_{1/2}$ we obtain

$$|u_0(x, t)| \leq C_n \int_{\mathbb{R}^n} |u_0(y, -1)|G(y, -1)e^{C_n|\rho|}dy.$$

More generally, changing the initial point $s = -1$ to arbitrary point $s \in (-1, -1/2]$ we will have

$$|u_0(x, t)| \leq C_n \int_{S_1 \setminus S_{1/2}} |u_0(y, s)|G(y, s)e^{C_n|\rho|}dyds$$

and by applying Cauchy-Schwarz

$$\|u_0\|_{L^\infty(Q_{1/2})}^2 \leq C_n \int_{S_1} u_0^2 G = C_n H_{u_0}(1) < \infty.$$

The energy inequality applied to $u_0^\pm$ then yields $u_0 \in W^{1,0}(Q_{1/4})$. Further, applying the estimate in Lemma 3.1 for $(x,t) = u_0(x,t)\zeta(x,t)$, where $\zeta$ is a smooth cutoff function in $Q_{1/4}$, equal to 1 on $Q_{1/8}$, with $\varphi_0 = 0$ and $f = 2\nabla u_0 \nabla \zeta + u_0(\Delta \zeta - \partial_t \zeta)$, we obtain that $u_0 \in W^{2,1}_2(Q^+_{1/8})$. As remarked earlier, the scaling properties imply that $u_0 \in W^{2,1}_2(Q^+_R)$ for any $R > 0$.

iv) Finally, we show that $u_0$ is parabolically homogeneous of degree $\kappa$. Let $r_j \to 0+$ be such that $u_{r_j} \to u_0$ as in ii). Then by part ii) again we have for any $0 < \rho < 1$

$$H_{u_{r_j}}(\rho) \to H_{u_0}(\rho), \quad I_{u_{r_j}}(\rho) \to I_{u_0}(\rho).$$

Moreover, since by Lemma 7.4 $H_{u_0}(\rho) \geq \rho^{2\kappa}$ for sufficiently small $r$, we also have

$$H_{u_0}(\rho) \geq \rho^{2\kappa}, \quad 0 < \rho < 1.$$

Hence, we obtain that for any $0 < \rho < 1$

$$2 \frac{I_{u_0}(\rho)}{H_{u_0}(\rho)} = 2 \lim_{j \to \infty} \frac{I_{u_{r_j}}(\rho)}{H_{u_{r_j}}(\rho)} = 2 \lim_{j \to \infty} \frac{I_{u}(r_j \rho)}{H_{u}(r_j \rho)} = \kappa.$$
by Lemma 7.1. Thus the ratio $2I_{u_0}(\rho)/H_{u_0}(\rho)$ is constant in the interval $(0,1)$. Further, notice that passing to the limit in the differentiation formulas in Proposition 6.2, we will obtain the similar formulas hold for $u_0$ for any $0 < r < \infty$. Thus, from computations in step $(1^\circ)$ in Theorem 6.3, before the application of the Cauchy-Schwarz inequality, we have

$$
\frac{d}{dr}\left( \frac{I_{u_0}(r)}{H_{u_0}(r)} \right) \geq \frac{1}{r^5H_{u_0}(r)} \left[ \int_{S^+} u_0^2G \int_{S^+} (Zu_0)^2G - \left( \int_{S^+} u_0(Zu_0)G \right)^2 \right].
$$

Note here that $H_{u_0}(r)$ is never zero, since $H_{u_0}(r) \geq r^{2\kappa}$ for $r \leq 1$ and $H_{u_0}(r) \geq r^{-2}H_{u_0}(1) \geq r^{-2}$. And since we know that the above derivative must be zero it implies that we have equality in Cauchy-Schwarz inequality

$$
\int_{S^+} u_0^2G \int_{S^+} (Zu_0)^2G = \left( \int_{S^+} u_0(Zu_0)G \right)^2,
$$

which can happen only if for some constant $\kappa_0$ we have

$$
Zu_0 = \kappa_0u_0 \quad \text{in} \quad S_\infty^+,
$$
or that $u_0$ is parabolically homogeneous of degree $\kappa_0$. But then, in this case it is straightforward to show that

$$
H_{u_0}(r) = Cr^{2\kappa_0},
$$

$$
H'_{u_0}(r) = \frac{4}{r}I_{u_0}(r) = 2\kappa_0Cr^{2\kappa_0-1},
$$

and therefore

$$
2 \frac{I_{u_0}(r)}{H_{u_0}(r)} = \kappa_0.
$$

This implies that $\kappa_0 = \kappa$ and completes the proof of the theorem. \hfill \Box

8. Homogeneous global solutions

In this section we study the homogeneous global solutions of the parabolic Signorini problem, which appear as the result of the blowup process described in Theorem 7.3. One of the conclusions of this section is that the homogeneity $\kappa$ of the blowup is

either $\kappa = \frac{3}{2}$ or $\kappa \geq 2$,

see Theorem 8.6 below. This will have two important consequences: (i) the fact that $\kappa \geq 3/2$ will imply the optimal $H^{3/2,3/4}_{\text{loc}}$ regularity of solutions (see Theorem 9.1) and (ii) the “gap” $(3/2,2)$ between possible values of $\kappa$ will imply the relative openness of the so-called regular set (see Proposition 11.2).

We start by noticing that $\kappa > 1$.

**Proposition 8.1.** Let $u \in \mathcal{G}^1(S_1^+)$ be as in Theorem 7.3. Then, $\kappa \geq 1 + \alpha$, where $\alpha$ is the Hölder exponent of $\nabla u$ in Definition 4.2.

For the proof we will need the following fact.

**Lemma 8.2.** Let $u \in \mathcal{G}^1(S_1^+)$. Then,

$$
H_u(r) \leq C_u r^{2(1+\alpha)}, \quad 0 < r < 1,
$$

where $\alpha$ is the Hölder exponent of $\nabla u$ in Definition 4.2.
Proof. Since \((0, 0) \in \Gamma_\ast(u)\), we must have \(|\nabla u(0, 0)| = 0\). Recalling also that \(u\) has a bounded support, we obtain that
\[
|\nabla u| \leq C_0 \|(x, t)\|^\alpha, \quad (x, t) \in S_1^+.
\]
Let us show that for \(C > 0\)
\[
|u| \leq C \|(x, t)\|^{1+\alpha}, \quad (x, t) \in S_1^+.
\]
Because of the gradient estimate above, it will be enough to show that
\[
|u| \leq C r^{1+\alpha} \quad \text{in} \quad Q_r^+.\]
First, observe that since \(u \geq 0\) on \(Q'_r\), we readily have
\[
u \geq -C_0 r^{1+\alpha} \quad \text{in} \quad Q_r^+.\]
To show the estimate from above, it will be enough to establish that
\[
u(0, -r^2) \leq C_1 r^{1+\alpha}.
\]
Note that since \(u\) is bounded, it is enough to show this bound for \(0 < r < 1/2\).
Assuming the contrary, let \(r \in (0, 1/2)\) be such that \(u(0, -r^2) \geq C_1 r^{1+\alpha}\) with large enough \(C_1\). Then, from the bound on the gradient, we have \(u \geq (C_1 - C_0) r^{1+\alpha}\) on \(B_r \times \{-r^2\}\). In particular,
\[
u(\frac{1}{2}re_n, -r^2) \geq (C_1 - C_0) r^{1+\alpha}.
\]
Also, let \(M\) be such that \(|f(x, t)| \leq M\) in \(S_1^+\). Then, consider the function
\[
\tilde{u}(x, t) = u(x, t) + C_0 \langle 2r \rangle^{1+\alpha}.
\]
We will have
\[
\tilde{u} \geq 0, \quad |(\Delta - \partial_t)\tilde{u}| \leq M \quad \text{in} \quad Q_{2r}^+.
\]
Besides,
\[
\tilde{u}(\frac{1}{2}re_n, -r^2) \geq C_1 r^{1+\alpha}.
\]
Then, from the parabolic Harnack inequality (see e.g. [Lie96, Theorems 6.17–6.18])
\[
\tilde{u}(\frac{1}{2}e_n, 0) \geq C_n C_1 r^{1+\alpha} - Mr^2,
\]
or equivalently,
\[
u((\frac{1}{2}e_n, 0) \geq (C_n C_1 - C_0 2^{1+\alpha}) r^{1+\alpha} - Mr^2.
\]
But then from the bound on the gradient we will have
\[
u(0, 0) \geq (C_n C_1 - C_0 2^{1+\alpha} - C_0) r^{1+\alpha} - Mr^2 > 0,
\]
if \(C_1\) is sufficiently large, a contradiction. This implies the claimed estimate
\[
|u(x, t)| \leq C \|(x, t)\|^{1+\alpha}.
\]
The estimate for $H_u(r)$ is then a simple corollary:

$$H_u(r) \leq C r^{2(1+\alpha)}$$

□

The proof of Proposition 8.1 now follows easily.

Proof of Proposition 8.1. Let $\kappa' \in (\kappa, \kappa_{\mu})$ be arbitrary. Then, by Lemma 7.4 we have

$$H_u(r) \geq c_u r^{2\kappa'}, \quad 0 < r < r_u.$$  

On the other hand, by Lemma 8.2 (proved below) we have the estimate

$$H_u(r) \leq C r^{2(1+\alpha)}, \quad 0 < r < 1.$$  

Hence, $\kappa' \geq 1 + \alpha$. Since this is true for any $\kappa' \in (\kappa, \kappa_{\mu})$, we obtain that $\kappa \geq 1 + \alpha$, which is the sought for conclusion.

□

We will also need the following technical fact.

Lemma 8.3. Let $u_0$ be a nonzero $\kappa$-parabolically homogeneous solution of the Signorini problem in $S^{\infty}_+$, as in Theorem 7.3(iii). Then, $\nabla u_0 \in H^{\alpha, \alpha/2}_{\text{loc}}(\mathbb{R}^n_+ \cup \mathbb{R}^n_-)$ for some $0 < \alpha < 1$.

Remark 8.4. Note that this does not follow from Lemmas 3.2 or 3.3 directly, since they rely on $W^{2}_{\infty}$-regularity of $\varphi_0$ (which is given by the function $u_0$ itself), or $W^{1,0}_{\infty}$-regularity of $u_0$, which has to be properly justified.

Proof. Note that because of the homogeneity, it is enough to show that $g(x) := u_0(x, -1)$ is in $H^{\alpha, \alpha/2}_{\text{loc}}((\mathbb{R}^n_+ \cup \mathbb{R}^n_-) \times (-\infty, 0))$ for some $0 < \alpha < 1$.

Proposition 8.5 (Homogeneous global solutions of homogeneity $1 \leq \kappa < 2$). Let $u_0$ be a nonzero $\kappa$-parabolically homogeneous solution of the Signorini problem in $S^{\infty}_+ = \mathbb{R}^n_+ \times (-\infty, 0]$ with $1 < \kappa < 2$. Then, $\kappa = 3/2$ and

$$u_0(x, t) = C \Re(x' \cdot e + ix_n)^{3/2} \quad \text{in } S^{\infty}_+$$

for some tangential direction $e \in \partial B'_1$. □
Proof. Extend $u_0$ by even symmetry in $x_n$ to the strip $S_\infty$, i.e., by putting
$$u_0(x', x_n, t) = u_0(x', -x_n, t).$$
Take any $e \in \partial B'_1$, and consider the positive and negative parts of the directional derivative $\partial_e u_0$
$$v^+_e = \max\{\pm \partial_e u_0, 0\}.$$
We claim that they satisfy the following conditions
$$(\Delta - t) v^+_e \geq 0, \quad v^+_e \geq 0, \quad v^+_e \cdot v^-_e = 0 \quad \text{in} \quad S_\infty.$$ The last two conditions are obvious. The first one follows from the fact that $v^+_e$ are continuous in $\mathbb{R}^n \times (-\infty, 0)$ (by Lemma 8.2) and caloric where positive. Hence, we can apply Caffarelli’s monotonicity formula to the pair $v^\pm_e$, see [Caf93]. Namely, the functional
$$\varphi(r) = \frac{1}{r^4} \int_{S_r} |\nabla v^+_e|^2 G \int_{S_r} |\nabla v^-_e|^2 G,$$
is monotone nondecreasing in $r$. On the other hand, from the homogeneity of $u$, it is easy to see that
$$\varphi(r) = r^{4(\kappa-2)} \varphi(1), \quad r > 0.$$ Since $\kappa < 2$, $\varphi(r)$ can be monotone increasing if and only if $\varphi(1) = 0$ and consequently $\varphi(r) = 0$ for all $r > 0$. If fact, one has to exclude the possibility that $\varphi(1) = \infty$ as well. This can be seen in two different ways. First, by Remark 5.2 one has
$$\int_{S_1} |\nabla v^+_e|^2 G \leq C_n \int_{S_2} (v^+_e)^2 G \leq \int_{S_4} u_0^2 G.$$ Alternatively, from Theorem 7.3 (i) and (iv) it follows that
$$\int_{\mathbb{R}^n} |\nabla v^+_e|^2 G(\cdot, -1) dx = j^+ < \infty.$$ Then,
$$\int_{S_1} |\nabla v^+_e|^2 G(x, t) dx dt = \int_{S_1} |\nabla v^+_e(|t|^{1/2} y, t)|^2 G(|t|^{1/2} y, t)|t|^{n/2} dy dt$$
$$= j^+ \int_{-1}^0 |t|^{(\kappa-2)} dt < \infty,$$
since $\kappa > 1$.
From here it follows that one of the functions $v^\pm_e$ is identically zero, which is equivalent to $\partial_e u_0$ being either nonnegative or nonpositive on the entire $\mathbb{R}^n \times (-\infty, 0]$. Since this is true for any tangential direction $e \in \partial B'_1$, it thus follows that $u_0$ depends only on one tangential direction, and is monotone in that direction. Therefore, without loss of generality we may assume that $n = 2$ and that the coincidence set at $t = -1$ is an infinite interval
$$\Lambda_{-1} = \{(x', 0) \in \mathbb{R}^2 \mid u_0(x', 0, -1) = 0\} = (-\infty, a] \times \{0\} =: \Sigma_a^-.$$
On the other hand, let

\[ a \quad \Sigma^a = [a, \infty) \times \{0\} \]

Observe now that

\[ \lambda(\Sigma^a) = \lambda(\Sigma^-_a) \]

is the ground state for the Ornstein-Uhlenbeck operator in \( \mathbb{R}^2 \setminus \Sigma^-_a \), and therefore from the above equations we have

\[ \lambda(\Sigma^-_a) = \lambda(\Sigma^-_0) \]

On the other hand, it is easy to see that the function \( a \mapsto \lambda(\Sigma^-_a) \) is strictly monotone and therefore the above equality can hold only if \( a = 0 \). In particular,

\[ \kappa = 1 + 2\lambda(\Sigma^-_0) \]
We now claim that $\lambda(\Sigma^-_0) = 1/4$. Indeed, consider the function
\[ v(x) = \Re(x_1 + i|x_2|^{1/2}), \]
which is harmonic and homogeneous of degree $1/2$:
\[ \Delta v = 0, \quad x \nabla v - \frac{1}{2} v = 0. \]
Therefore,
\[ v = 0 \quad \text{on} \quad \Sigma^-_0, \quad -\Delta v + \frac{1}{2} x \nabla v = \frac{1}{4} v \quad \text{in} \quad \mathbb{R}^2 \setminus \Sigma^-_0. \]
Also, since $v$ is nonnegative, we obtain that $v$ is the ground state of the Ornstein-Uhlenbeck operator in $\mathbb{R}^2 \setminus \Sigma^-_0$. This implies in particular that $\lambda(\Sigma^-_0) = 1/4$, and consequently
\[ \kappa = 3/2. \]
Moreover, $g_1(x) = \partial_{x_1} u_0(x, -1)$ must be a multiple of the function $v$ above and from homogeneity we obtain that
\[ \begin{aligned}
\partial_{x_1} u_0(x, t) &= (-t)^{1/4} g_1(x/(-t)^{1/2}) = C(-t)^{1/4} \Re((x_1 + i|x_2|)/(-t)^{1/2})^{1/2} \\
&= C \Re(x_1 + i|x_2|)^{1/2}.
\end{aligned} \]
From here it is now easy to see that necessarily
\[ u_0(x, t) = C \Re(x_1 + i|x_2|)^{3/2}. \]
\[ \square \]
Combining Propositions 8.1 and 8.5 we obtain the following result.

**Theorem 8.6 (Minimal homogeneity).** Let $u \in \mathcal{S}(S^+_1)$ and $\mu$ satisfies the conditions of Theorem 6.3. Assume also $\kappa_\mu = \frac{3}{2}$ and
\[ \lim_{r \to 0^+} r \mu'(r)/\mu(r) \geq 3/2. \]
Then
\[ \kappa := \Phi_u(0+) \geq 3/2. \]
More precisely, we must have
\[ \text{either} \quad \kappa = 3/2 \quad \text{or} \quad \kappa \geq 2. \]
\[ \square \]

**Remark 8.7.** Very little is known about the possible values of $\kappa$. However, we can say that the following values of $\kappa$ do occur:
\[ \kappa = 2m - 1/2, \quad 2m, \quad 2m + 1, \quad m \in \mathbb{N}. \]
This can be seen from the following explicit examples of homogeneous solutions (that are actually $t$-independent)
\[ \Re(x_1 + ix_n)^{2m-1/2}, \quad \Re(x_1 + ix_n)^{2m}, \quad \Im(x_1 + ix_n)^{2m+1}. \]
It is known (and easily proved) that in $t$-independent case and dimension $n = 2$, the above listed values of $\kappa$ are the only ones possible. In all other cases, finding the set of possible values of $\kappa$ is, to the best of our knowledge, an open problem.
9. Optimal regularity of solutions

Using the tools developed in the previous sections we are now ready to prove the optimal regularity of solutions of the parabolic Signorini problem with sufficiently smooth obstacles. In fact, we will establish our result for a slightly more general class of functions solving the Signorini problem with nonzero obstacle and nonzero right-hand side.

**Theorem 9.1** (Optimal regularity in the parabolic Signorini problem). Let \( \varphi \in H^{2,1}(Q_1^+) \), \( f \in L_\infty(Q_1^+) \). Assume that \( v \in W^{2,1}_2(Q_1^+) \) be such that \( \nabla v \in H^{\alpha,\alpha/2}(Q_1^+ \cup Q_1') \) for some \( 0 < \alpha < 1 \), and satisfy

\[
\Delta v - \partial_t v = f \quad \text{in} \ Q_1^+,
\]

\[
v - \varphi \geq 0, \quad -\partial_n v \geq 0, \quad (v - \varphi)\partial_n v = 0 \quad \text{on} \ Q_1'.
\]

Then, \( v \in H^{3/2,3/4}(Q_{1/2}^+ \cup Q_{1/2}') \) with

\[
\|v\|_{H^{3/2,3/4}(Q_{1/2}^+ \cup Q_{1/2}')} \leq C_n \left( \|v\|_{W^{2,1}_2(Q_1^+)} + \|f\|_{L_\infty(Q_1^+)} + \|\varphi\|_{H^{2,1}(Q_1^+)} \right).
\]

The proof of Theorem 9.1 will follow from the interior parabolic estimates and the growth bound of \( u \) away from the free boundary \( \Gamma(v) \).

**Lemma 9.2.** Let \( u \in \mathcal{G}(S_1^+) \) with \( \|u\|_{L_\infty(S_1^+)} \), \( \|f\|_{L_\infty(S_1^+)} \leq M \). Then,

\[
H_r(u) \leq C_nM^2r^3.
\]

**Proof.** The \( L_\infty \) bound on \( f \) allows us to apply Theorem 6.3 with the following specific choice of \( \mu \) in the generalized frequency function. Indeed, fix \( \sigma = 1/4 \) and let \( \mu(r) = M^2r^{4-2\sigma} \). Then,

\[
r^{4-2\sigma} \int_{S_1^+} f^2(-,-r^2)G(-,-r^2) \leq \mu(r)
\]

and therefore

\[
\Phi_u(r) = \frac{1}{2} re^{Cr^\sigma} \frac{d}{dr} \log \max \{H_u(r), M^2r^{4-2\sigma}\} + 2(e^{Cr^\sigma} - 1)
\]

is monotone for \( C = C_n \). By Theorem 8.6 we have that \( \Phi_u(0+) \geq 3/2 \).

Now, for \( H_u(r) \) we have two alternatives: either \( H_u(r) \leq \mu(r) = M^2r^{4-2\sigma} \) or \( H_u(r) > \mu(r) \). In the first case the desired estimate is readily satisfied, so we concentrate on the latter case. Let \( (r_0, r_1) \) be a maximal interval in the open set \( \emptyset \ = \{ r \in (0,1) \mid H_u(r) > \mu(r) \} \). Then, for \( r \in (r_0, r_1) \) we have

\[
\Phi_u(r) = \frac{1}{2} re^{Cr^\sigma} \frac{H_u'(r)}{H_u(r)} + 2(e^{Cr^\sigma} - 1) \geq \Phi_u(0+) \geq \frac{3}{2}.
\]

We thus have,

\[
\frac{H_u'(r)}{H_u(r)} \geq \frac{3}{r}(1-Cr^\sigma), \quad r \in (r_0, r_1),
\]

which, after integration, implies

\[
\log \frac{H_u(r_1)}{H_u(r)} \geq \log \frac{r_1^3}{r^3} - Cr_1^\sigma,
\]

and therefore

\[
H_u(r) \leq Cr^3 \frac{H_u(r_1)}{r_1^3}.
\]
Now, for \( r_1 \) we either have \( r_1 = 1 \) or \( H_u(r_1) = \mu(r_1) \). Note that \( H_u(1) \leq M^2 \) from the \( L_{\infty} \) bound on \( u \), and thus in both cases we have \( H_u(r_1) \leq M^2 r_1^3 \). We thus have the desired conclusion
\[
H_u(r) \leq CM^2 r^3.
\]

To apply the results of the previous sections, we will need the following \( L_{\infty} - L_2 \) type estimates.

**Lemma 9.3.** Let \( w \) be a nonnegative function with at most polynomial growth at infinity in the strip \( S_R \), and such that for some \( \gamma > 0 \)
\[
\Delta w - \partial_t w \geq -M \| (x,t) \|^{\gamma-2} \quad \text{in } S_R.
\]
Then,
\[
\sup_{Q_{r/2}} w \leq C_n H_w(r)^{1/2} + C_{n,\gamma} M r^\gamma, \quad 0 < r < R.
\]

**Proof.** Choosing a constant \( C_{n,\gamma} > 0 \) we can guarantee that \( \tilde{w}(x,t) = w(x,t) + C_{n,\gamma} M (|x|^2 - t)^{\gamma/2} \) is still nonnegative, has a polynomial growth at infinity and satisfies
\[
\Delta \tilde{w} - \partial_t \tilde{w} \geq 0 \quad \text{in } S_R.
\]
Moreover,
\[
H_{\tilde{w}}(r)^{1/2} \leq H_w(r)^{1/2} + C_{n,\gamma} M \left( \frac{1}{r^2} \int_{S_r} (|x|^2 + |t|)^\gamma G(x,t) dx dt \right)^{1/2}.
\]

From the scaling properties of \( G \) it is easily seen that
\[
\frac{1}{r^2} \int_{S_r} (|x|^2 + |t|)^\gamma G(x,t) dx dt = C_n r^{2\gamma}
\]
and therefore we may assume that \( M = 0 \) from the beginning.

The rest of the proof is now similar to that of Theorem 7.3(iii).

Indeed, for \((x,t) \in Q_{r/2}\) and \( s \in (-r^2, -r^2/2] \) we have the sub mean-value property
\[
w(x,t) \leq \int_{\mathbb{R}^n} w(y,s) G(x-y, t-s) dy.
\]
This can be proved as in Theorem 7.3(iii) by combining the fact that \( \int_{S_\delta \setminus S_\rho} |\nabla w|^2 G < \infty \) for any \( 0 < \delta < \rho < R \), with the polynomial growth assumption on \( w \), and the energy inequality. On the other hand, for our choice of \( t \) and \( s \), we have \( |s|/4 < t-s < |s| \). Thus, arguing as in Claim 7.8 we have
\[
G(x-y, t-s) = \frac{1}{(4\pi(t-s))^{n/2}} e^{-|x-y|^2/4(t-s)} \leq \frac{C_n}{(4\pi|s|)^{n/2}} e^{-|x-y|^2/4|s|} \leq C_n G(y,s) e^{(x-y)/2|s|}.
\]
Hence, we obtain that
\[ w(x, t) \leq C_n \int_{\mathbb{R}^n} w(y, s)e^{(x-y)/|s|}G(y, s)dy. \]

Now, applying the Cauchy-Schwarz inequality, we will have
\[ w(x, t) \leq C_n \left( \int_{\mathbb{R}^n} w(y, s)^2G(y, s)dy \right)^{1/2} \left( \int_{\mathbb{R}^n} e^{(x-y)/|s|}G(y, s)dy \right)^{1/2} \leq C_n \left( \int_{\mathbb{R}^n} w(y, s)^2G(y, s)dy \right)^{1/2}, \]
where in the last step we have used that \(|x| \leq r\) and \(|s| \geq r^2/2\). Integrating over \(s \in [-r^2, -r^2/4]\), we obtain
\[ w(x, t) \leq C_n \frac{1}{r^2} \int_{-r^2}^{-r^2/2} \left( \int_{\mathbb{R}^n} w(y, s)^2G(y, s)dy \right)^{1/2} ds \leq C_n H_w(r)^{1/2}. \]
This completes the proof.

Lemma 9.4. Let \(w\) be a nonnegative bounded function in \(S_R\) satisfying
\[ 0 \leq w \leq M, \quad \Delta w - \partial_t w \geq -M \quad \text{in} \quad S_R, \]
and let \((x_0, t_0) \in S'_R, \ r_0 > 0\) be such that for some constant \(A\)
\[ \sup_{Q_r(x_0, t_0)} w \leq Ar^{3/2}, \quad 0 < r < r_0. \]
Then,
\[ w(x, t) \leq C \|(x-x_0, t-t_0)\|^{3/2}, \quad (x, t) \in S_R, \]
with \(C = C_n, r_0, R(A + M)\).

Proof. Without loss of generality we may assume that \(x_0 = 0\). Then, from the assumptions on \(w\) we have that
\[ w(x, t_0) \leq C|x|^{3/2}, \quad x \in \mathbb{R}^n, \]
and more generally
\[ w(x, t) \leq C \|(x, t-t_0)\|^{3/2}, \quad x \in \mathbb{R}^n, \ t \in (-R^2, t_0], \]
where \( C = C_{n,r_0,t}(A + M) \). To propagate the estimate to \( t \in (t_0,0) \) we use the sub-mean value property. Namely, for \( x \in \mathbb{R}^n \) and \( t_0 < t < 0 \) we have

\[
w(x,t) \leq \int_{\mathbb{R}^n} w(y,t_0) G(x - y,t - t_0) dy + M(t - t_0)
\]

\[
\leq C \int_{\mathbb{R}^n} |y|^{3/2} G(x - y,t - t_0) dy + M(t - t_0).
\]

To obtain the desired conclusion, we notice that the integral

\[
V(x,s) = \int_{\mathbb{R}^n} |y|^{3/2} G(x - y,s) dy, \quad x \in \mathbb{R}^n, s > 0
\]

is parabolically homogeneous of degree 3/2 in the sense that

\[
V(\lambda x, \lambda^2 s) = \lambda^{3/2} V(x,s), \quad \lambda > 0
\]

and therefore we immediately obtain that

\[
V(x,s) \leq C_n \| (x,s) \|^{3/2}, \quad x \in \mathbb{R}^n, s > 0.
\]

Consequently, this yields

\[
w(x,t) \leq C_n C \| (x,t - t_0) \|^{3/2} + M(t - t_0), \quad x \in \mathbb{R}^n, t \in (t_0,0],
\]

which implies the statement of the lemma.

**Lemma 9.5.** Let \( u \in \mathcal{E}^f(S^+_1) \) and \( (x_0,t_0) \in \Gamma_+(u) \cap Q_{3/4}^1 \). Then,

\[
|u(x,t)| \leq C \| (x - x_0, t - t_0) \|^{3/2}, \quad (x,t) \in S^+_1,
\]

with

\[
C = C_n (\| u \|_{L^\infty(S^+_1)} + \| f \|_{L^\infty(S^+_1)}).
\]

**Proof.** This is simply a combination of Lemma 9.2 for \( u \) and Lemmas 9.3–9.4 applied for \( w = u^\pm \).

**Proof of Theorem 9.1.** As in Section 4 let \( \psi \in C_0^\infty(\mathbb{R}^n) \) be a cutoff function satisfying (4.4)–(4.5), and consider

\[
u(x,t) = [v(x,t) - \varphi(x',t)]\psi(x).
\]

We may also assume that \( |\nabla \psi| \leq C_n \). We will thus have \( u \in \mathcal{E}^d(S^+_1) \) with

\[
\| g \|_{L^\infty(S^+_1)} \leq C_n \left( \|v\|_{W^{1,\infty}_d(Q^+_1)} + \| f \|_{L^\infty(Q^+_1)} + \| \varphi \|_{H^{2,1}(Q^+_1)} \right).
\]

In the remaining part of the proof, \( C \) will denote a generic constant that has the same form as the right-hand side in the above inequality. For \( (x,t) \in Q_{1/2} \), and

\[
Q^*_r(x,t) := \hat{Q}_r(x,t) \cap S_{\infty} = B_r(x) \times ((t - r^2, t + r^2) \cap (-\infty,0]),
\]

let

\[
d = d(x,t) = \sup \{ r > 0 \mid Q^*_r(x,t) \subset Q_1 \setminus \Gamma_+(u) \}.
\]

We first claim that

\[
|u| \leq Cd^{1/2} \quad \text{in} \ Q^*_d(x,t). \tag{9.3}
\]
Indeed, if \( d > 1/4 \), this follows from boundedness of \( u \). If instead \( d \leq 1/4 \), then there exist \((x_0, t_0) \in Q_{d/4}^* \cap \Gamma_+(u) \cap \partial Q_d^*(x, t)\) and by Lemma 9.5 we have the desired estimate. Next, we claim that

\[
|\nabla u| \leq Cd^{1/2} \quad \text{in } Q_{d/2}^*(x, t)
\]

This will follow from the interior gradient estimates, applied to the even or odd extension of \( u \) in \( x_n \) variable. More specifically, consider the intersection \( Q_d^*(x, t) \cap Q_1^* \). Since there are no points of \( \Gamma_+(u) \) in this set, we have a dichotomy: either (i) \( u > 0 \) on \( Q_d^*(x, t) \cap Q_1^* \), or (ii) \( u = 0 \) on \( Q_d^*(x, t) \cap Q_1^* \). Accordingly, we define

\[
\tilde{u}(x', x_n, t) = \begin{cases} 
  u(x', x_n, t), & x_n \geq 0 \\
  u(x', -x_n, t), & x_n \leq 0
\end{cases}
\]

in case (i),

\[
\tilde{u}(x', x_n, t) = \begin{cases} 
  u(x', x_n, t), & x_n \geq 0 \\
  -u(x', -x_n, t), & x_n \leq 0
\end{cases}
\]

in case (ii).

In either case \( \tilde{u} \) satisfies a nonhomogeneous heat equation

\[(\Delta - \partial_t)\tilde{u} = \tilde{g} \quad \text{in } Q_d^*(x, t),\]

for an appropriately defined \( \tilde{g} \). The claimed estimate for \( |\nabla u| = |\nabla \tilde{u}| \) now follows from parabolic interior gradient estimates, see e.g. [LSU67, Chapter III, Theorem 11.1]. Moreover, by [LSU67, Chapter IV, Theorem 9.1], one also has that \( \tilde{u} \in W^{2,1}_q(Q_{d/2}^*(x, t)) \) for any \( 3/2 < q < \infty \). To be more precise, we apply the latter theorem to \( \tilde{u}\zeta \), where \( \zeta \) is a cutoff function supported in \( Q_d^*(x, t) \), \( 0 \leq \zeta \leq 1 \), \( |\nabla \zeta| \leq C_n/d \), \( |\partial_t \zeta| \leq C_n/d^2 \). From the estimates on \( \tilde{u} \) and \( |\nabla \tilde{u}| \), we thus have

\[|(\Delta - \partial_t)(\tilde{u}\zeta)| \leq Cd^{-1/2},\]

which provides the estimate

\[\|D^2\tilde{u}\|_{L^q(Q_{d/2}^*(x, t))} + \|\partial_t \tilde{u}\|_{L^q(Q_{d/2}^*(x, t))} \leq Cd^{-1/2}d(n+2)/q.\]

Then, from the Sobolev embedding of \( W^{2,1}_q \) into \( H^{2-\frac{n+2}{q},1-\frac{n+2}{2r}} \) when \( q > n + 2 \), we obtain the estimates for H"older seminorms

\[\langle \nabla \tilde{u} \rangle_{Q_{d/2}^*(x, t)}^{(\alpha)} + \langle \tilde{u} \rangle_{t,Q_{d/2}^*(x, t)}^{((1+\alpha)/2)} \leq Cd^{1/2-\alpha},\]

for any \( 0 < \alpha < 1 \), see [LSU67, Chapter II, Lemma 3.3]. In particular, we have

\[
\langle \nabla \tilde{u} \rangle_{Q_{d/2}^*(x, t)}^{(1/2)} + \langle \tilde{u} \rangle_{t,Q_{d/2}^*(x, t)}^{(3/4)} \leq C.
\]

Now take two points \((x^i, t^i) \in Q_1^* \), \( i = 1, 2 \), and let \( d^i = d(x^i, t^i) \). Without loss of generality we may assume \( d_1 \geq d_2 \). Let also \( \delta = (|x^1| - |x^2|)^2 + |t^1 - t^2|^2)^{1/2} \).

Consider two cases:

1) \( \delta > \frac{1}{2} d_1 \). In this case, we have by (9.4)

\[
|\nabla u(x^1, t^1) - \nabla u(x^2, t^2)| \leq |\nabla u(x^1, t^1)| + |\nabla u(x^2, t^2)| \leq C(d^1)^{1/2} + C(d^2)^{1/2} \leq C\delta^{1/2}.
\]

2) \( \delta < \frac{1}{2} d_1 \). In this case, both \((x^i, t^i) \in Q_{d_1/2}^*(x^1, t^1)\), and therefore by (9.5)

\[
|\nabla u(x^1, t^1) - \nabla u(x^2, t^2)| \leq C\delta^{1/2}.
\]
This gives the desired estimate for the seminorm $\langle \nabla u \rangle_{Q_{1/2}}^{1/2}$. Arguing analogously, we can also prove a similar estimate for $\langle u \rangle_{Q_{1/2}}^{(3/4)}$, thus completing the proof of the theorem. □

10. Classification of free boundary points

After establishing the optimal regularity of the solutions, we are now able to undertake the study of the free boundary

$$\Gamma(v) = \partial \{ (x', t) \mid v(x', 0, t) > \varphi(x', t) \}.$$ 

We start with classifying the free boundary points and more generally points in

$$\Gamma_+(v) = \{ (x', t) \mid v(x', 0, t) = \varphi(x', t), \partial_{x_n} v(x', 0, t) = 0 \}.$$ 

As we will see, the higher is the regularity of $\varphi$, the finer is going to be the classification.

Let $v \in \mathcal{S}_{\varphi}(Q_1^+)$ with $\varphi \in H^{k+\gamma}(Q_1')$, $k \in \mathbb{N}$, $0 < \gamma \leq 1$ and $u_k \in \mathcal{S}_{f_k}(S_1^+)$ be as constructed in Proposition 4.4. In particular, $f_k$ satisfies

$$|f_k(x, t)| \leq M \|(x, t)\|^{\ell - 2}, \quad (x, t) \in S_1^+.$$ 

This implies that

$$\int_{\mathbb{R}^n_+} f_k(x, -r^2)^2 G(x, -r^2) dx \leq M^2 \int_{\mathbb{R}^n_+} (|x|^2 + r^2)^{\ell - 2} G(x, -r^2) dx$$

$$= M^2 r^{2\ell - 4} \int_{\mathbb{R}^n_+} (|y|^2 + 1)^{\ell} G(y, -1) dy$$

$$= C \ell M^2 r^{2\ell - 4}.$$ 

Thus, if we choose

$$\mu(r) = r^{2\ell_0}, \quad \text{with} \ k \leq \ell_0 < \ell \text{ and } \sigma \leq \ell - \ell_0,$$

then $\mu, f_k$ and $u_k$ will satisfy the conditions of Theorem 6.3. In particular, we will have that

$$\Phi^{(\ell_0)}_{u_k} (r) := \frac{1}{2} r e^{Cr\sigma} \frac{d}{dr} \log \max \{H_{u_k}(r), r^{2\ell_0}\} + 2(e^{Cr\sigma} - 1)$$

is monotone increasing in $r \in (0, 1)$ and consequently there exists the limit (see (7.1) above)

$$\kappa = \Phi^{(\ell_0)}_{u_k}(0+).$$

Recalling the definition (7.2) of $\kappa_\mu$, we note that in the present case we have

$$\kappa_\mu = \ell_0.$$ 

Therefore, by Lemma 7.1 we infer that

$$\kappa \leq \ell_0 < \ell.$$ 

Generally speaking, the value of $\kappa$ may depend on the cutoff function that we have chosen to construct $u_k$. However, as the next result proves, it is relatively straightforward to check that this is not the case.

**Lemma 10.1.** The limit $\kappa = \Phi^{(\ell_0)}_{u_k}(0+)$ does not depend on the choice of the cutoff function $\psi$ in the definition of $u_k$. 

Proof. Indeed, if we choose a different cutoff function $\psi'$, satisfying (4.4)–(4.5) and denote by $u_k'$ the function corresponding to $u_k$ in the construction above, and by $\kappa'$ the corresponding value as in (7.1), then by simply using the fact that $u_k = u_k'$ on $B^+_{1/2} \times (-1,0]$ and that $|G(x,t)| \leq e^{-c_n/r^2}$ for $|x| \geq 1/2$ and $-r^2 < t \leq 0$, we have

$$|H_{u_k}(r) - H_{u_k'}(r)| \leq C e^{-c_n/r^2}.$$ 

To show now that $\kappa = \kappa'$, we consider several cases.

1) If $\kappa = \kappa' = \ell_0$, then we are done.
2) If $\kappa < \ell_0$, then Lemma 7.1 implies that

$$2\kappa - \varepsilon \leq r H_{u_k}(r) \leq 2\kappa + \varepsilon, \quad 0 < r < r_\varepsilon.$$

Integrating these inequalities we obtain

$$c_\epsilon e^{2\kappa + \varepsilon} \leq H_{u_k}(r) \leq C_\epsilon r^{2\kappa - \varepsilon}, \quad 0 < r < r_\varepsilon,$$

for some (generic) positive constants $c_\epsilon$, $C_\epsilon$. This will also imply

$$c_\epsilon r^{2\kappa + \varepsilon} \leq H_{u_k'}(r) \leq C_\epsilon r^{2\kappa - \varepsilon}, \quad 0 < r < r_\varepsilon.$$

Now, if $\varepsilon$ is so small that $2\kappa + \varepsilon < \ell_0$, we will have that $H_{u_k'}(r) > \mu(r) = r^{\ell_0}$ for $0 < r < r_\varepsilon$. But then, we also have $\kappa' = \lim_{r \to 0} r H_{u_k'}(r)/H_{u_k'}(r)$, and therefore

$$c_\epsilon r^{2\kappa + \varepsilon} \leq H_{u_k'}(r) \leq C_\epsilon r^{2\kappa - \varepsilon}, \quad 0 < r < r_\varepsilon,$$

for arbitrarily small $\varepsilon > 0$. Obviously, the above estimates imply that $\kappa' = \kappa$.

3) If $\kappa' < \ell_0$, we argue as in 2) above. $\square$

Definition 10.2 (Truncated homogeneity). To stress in the above construction the dependence only of the function $v$, we will denote the quantity $\kappa = \Phi_{u_k}(0^+)$ by

$$\kappa_v(\ell_0)(0,0).$$

More generally, for $(x_0,t_0) \in \Gamma_+(v)$ we let

$$v(x_0,t_0)(x,t) := v(x_0 + x, t_0 + t),$$

which translates $(x_0,t_0)$ to the origin. Then, $v(x_0,t_0) \in \mathcal{I}^{(x_0,t_0)}(Q^+_r)$ for some small $r > 0$. The construction above has been carried out in $Q^+_1$, rather than $Q^+_r$. However, a simple rescaling argument generalizes it to any $r > 0$. Thus, we can define

$$\kappa_v(\ell_0)(x_0,t_0) = \kappa_v(\ell_0)(0,0),$$

which we will call the truncated homogeneity of $v$ at an extended free boundary point $(x_0,t_0)$.

Suppose now for a moment that the thin obstacle $\varphi$, that was assumed to belong to the class $H^{\ell,\ell/2}(Q^*_1)$, has a higher regularity. To fix the ideas, suppose $\varphi \in H^{\ell,\ell/2}(Q^*_1)$, for some $\ell \geq \ell_0 \geq 2$, with $\ell = \tilde{\ell} + \tilde{\gamma}$, $\tilde{\ell} \in \mathbb{N}$, $\tilde{\gamma} \leq 1$. We may thus define

$$\kappa_v(\ell_0)(x_0,t_0),$$

for any $\tilde{\ell} \leq \tilde{\ell}_0 < \tilde{\ell}$. It is natural to ask about the relation between $\kappa_v(\ell_0)(x_0,t_0)$ and $\kappa_v(\ell_0)(x_0,t_0)$.
Proposition 10.3 (Consistency of truncated homogeneities). If $\ell \leq \tilde{\ell}$, $\ell_0 \leq \tilde{\ell}_0$ are as above, then
\[
\kappa_v^{(\ell_0)}(x_0, t_0) = \min\{\kappa_v^{(\tilde{\ell}_0)}(x_0, t_0), \ell_0\}.
\]

This proposition essentially says that $\kappa_v^{(\ell_0)}(x_0, t_0)$ is the truncation of $\kappa_v^{(\tilde{\ell}_0)}(x_0, t_0)$ by the value $\ell_0$.

Proof. It will be sufficient to prove the statement for $(x_0, t_0) = (0, 0)$. To simplify the notation in the proof, we are going to denote
\[
\kappa = \kappa_v^{(\ell_0)}(0, 0), \quad \tilde{\kappa} = \kappa_v^{(\tilde{\ell}_0)}(0, 0),
\]
so we will need to show that
\[
\kappa = \min\{\tilde{\kappa}, \ell_0\}.
\]
First, we fix a cutoff function $\psi$ is the definition of the functions $u_k$ and $u_{\tilde{k}}$, and note that
\[
|u_k(x, t) - u_{\tilde{k}}(x, t)| \leq C\|\langle x, t \rangle \|^\ell.
\]
This implies that
\[
|H_{u_k}(r) - H_{u_{\tilde{k}}}(r)| \leq Cr^{2\ell}.
\]
Arguing as in the proof of Lemma 10.1, we obtain that, in fact,
\[
\kappa = \Phi^{(\ell_0)}(0+) = \Phi_{u_k}^{(\ell_0)}(0+).
\]
Using this information, from now on in this proof we will abbreviate $H_{u_k}(r)$ with $H(r)$.

We consider two cases.

(1°) Assume first that $\tilde{\kappa} < \ell_0$. In this case we need to show that $\kappa = \tilde{\kappa}$.

From the assumption we will have that $\tilde{\kappa} < \tilde{\ell}_0$ and by Lemma 7.1
\[
\tilde{\kappa} = \frac{1}{2} \lim_{r \to 0^+} \frac{rH'(r)}{H(r)}.
\]
Therefore, for any $\varepsilon > 0$ we obtain
\[
r \frac{H'(r)}{H(r)} \leq 2\tilde{\kappa} + \varepsilon, \quad 0 < r \leq r_\varepsilon.
\]
Integrating, we find
\[
H(r) \geq \frac{H(r_\varepsilon)}{r_\varepsilon^{2\kappa+\varepsilon}}, \quad 0 < r \leq r_\varepsilon.
\]
In particular, if $\varepsilon > 0$ is so small that $2\tilde{\kappa} + \varepsilon < 2\ell_0$, then $H(r) > r^{2\ell_0}$ and therefore
\[
\kappa = \Phi^{(\ell_0)}(0+) = \frac{1}{2} \lim_{r \to 0^+} \frac{rH'(r)}{H(r)} = \tilde{\kappa}.
\]

(2°) Assume now that $\tilde{\kappa} \geq \ell_0$. We need to show in this case that $\kappa = \ell_0$. In general, we know that $\kappa \leq \ell_0$, so arguing by contradiction, assume $\kappa < \ell_0$. We thus know by Lemma 7.1 that $H(r) \geq r^{2\ell_0}$ for $0 < r < r_0$, and
\[
\kappa = \frac{1}{2} \lim_{r \to 0^+} \frac{rH'(r)}{H(r)}.
\]
But then, we also have $H(r) \geq r^{2\ell_0}$ for $0 < r < r_0$ and therefore

$$\tilde{\kappa} = \Phi^{(\ell_0)}(0+) = \frac{1}{2} \lim_{r \to 0^+} \frac{rH'(r)}{H(r)} = \kappa < \ell_0,$$

contrary to the assumption.

□

**Definition 10.4** (Truncated homogeneity, part II). In view of Proposition 10.3, if $\varphi \in H^{\ell,\ell/2}(Q^+_1)$ we can push $\ell_0$ in the definition of the truncated homogeneity up to $\ell$ by setting

$$\kappa_v^{(\ell)}(x_0, t_0) = \sup_{\ell_0 < \ell} \kappa_v^{(\ell_0)}(x_0, t_0).$$

Indeed, Proposition 10.3 guarantees that $\ell_0 \mapsto \kappa_v^{(\ell_0)}$ is monotone increasing. Moreover, we have

$$\kappa_v^{(\ell_0)} = \min\{\kappa_v^{(\ell)}, \ell_0\}.$$

**Lemma 10.5.** The function $(x, t) \mapsto \kappa_v^{(\ell)}(x, t)$ is upper semicontinuous on $\Gamma_+(v)$ (with respect to Euclidean or, equivalently, parabolic distance), i.e., for any $(x_0, t_0) \in \Gamma_+(v)$ one has

$$\lim_{\delta \to 0} \sup_{Q^\delta(x_0, t_0) \cap \Gamma_+(v)} \kappa_v^{(\ell)}(x, t) \leq \kappa_v^{(\ell)}(x_0, t_0).$$

**Proof.** Suppose first $\kappa = \kappa_v^{(\ell)}(x_0, t_0) < \ell$ and fix $\ell_0 \in (\kappa, \ell)$. Then, for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that $u^{(\ell_0)}(r_\varepsilon) < \kappa + \varepsilon < \ell_0$. This implies that $H_u(r) \geq Cr^{2(\kappa + \varepsilon)}$ for $0 < r < r_\varepsilon$. Since the mapping $(x, t) \mapsto H_u^{(\sigma, \ell)}(r)$ is continuous on $\Gamma_+(v)$, we will have

$$H_u^{(\sigma, \ell)}(r_\varepsilon) \geq (C/2)r^{2(\kappa + \varepsilon)} > r^{2\ell_0},$$

if $|x - x_0|^2 + |t - t_0| < \eta_{\varepsilon}^2$, $(x, t) \in \Gamma_+(v)$, provided $r_\varepsilon$ and $\eta_{\varepsilon} > 0$ are small enough. In particular, this implies the explicit formula

$$\Phi^{(\ell_0)}_{u^{(\ell_0)}}(r_\varepsilon) = \frac{1}{2} r_\varepsilon e^{Cr_\varepsilon^\sigma} \frac{H_u^{(\sigma, \ell_0)}(r_\varepsilon)}{H_u^{(\sigma, \ell_0)}(r_\varepsilon)} + 2(e^{Cr_\varepsilon^\sigma} - 1).$$

Therefore, taking $\eta_{\varepsilon} > 0$ small, we can guarantee

$$|\Phi^{(\ell_0)}_{u^{(\ell_0)}}(r_\varepsilon) - \Phi^{(\ell_0)}_{u^{(\ell_0)}}(r_\varepsilon)| \leq \varepsilon,$$

if $|x - x_0|^2 + |t - t_0| < \eta_{\varepsilon}^2$, $(x, t) \in \Gamma_+(v)$. It follows that, for such $(x, t)$, one has

$$\kappa_v^{(\ell_0)}(x, t) \leq \Phi^{(\ell_0)}_{u^{(\ell_0)}}(r_\varepsilon) \leq \kappa + 2\varepsilon,$$

which implies the upper semicontinuity of $\kappa_v^{(\ell_0)}$ and $\kappa_v^{(\ell)}$ at $(x_0, t_0)$.

If $\kappa_v^{(\ell_0)}(x_0, t_0) = \ell$, the upper continuity follows immediately since $\kappa_v^{(\ell_0)} \leq \ell$. □

The truncated homogeneity $\kappa_v^{(\ell)}$ gives a natural classification of extended free boundary points.

**Definition 10.6** (Classification of free boundary points). Let $v \in \mathfrak{S}_\varphi(Q^+_1)$, with $\varphi \in H^{\ell,\ell/2}(Q^+_1)$. For $\kappa \in [3/2, \ell]$, we define

$$\Gamma_\kappa^{(\ell)}(v) := \{(x, t) \in \Gamma_+(v) \mid \kappa_v^{(\ell)}(x, t) = \kappa\}.$$
As a direct corollary of Proposition \[10.3\], we have the following consistency for the above definition.

**Proposition 10.7** (Consistency of classification). If \( \varphi \in H^{l,l/2}(Q'_1) \) with \( l \geq 2 \), then

\[
\Gamma^{(l)}_k(v) = \Gamma^{(\bar{l})}_k(v), \quad \text{if } k < l,
\]
\[
\Gamma^{(l)}_l(v) = \bigcup_{l \leq k \leq \bar{l}} \Gamma^{(\bar{l})}_k(v).
\]

The latter identity essentially means that, if \( \varphi \) is more regular than \( H^{l,l/2} \), then \( \Gamma^{(l)}_l(v) \) is an “aggregate” of points with higher homogeneities \( k \).

We conclude this section with the following description of the free boundary, based on the fact that the function \( \kappa^{(l)}_l(v) \) never takes certain values. We also characterize the points that are in the extended free boundary \( \Gamma_\ast(v) \), but not in the free boundary \( \Gamma(v) \).

**Proposition 10.8.** If \( v \in \mathcal{S}_\varphi(Q^+_1) \) with \( \varphi \in H^{l,l/2}(Q'_1) \), \( l \geq 2 \), then for any \((x_0,t_0) \in \Gamma_\ast(v)\), either we have

\[
\kappa^{(l)}_v(x_0,t_0) = \frac{3}{2}, \quad \text{or} \quad 2 \leq \kappa^{(l)}_v(x_0,t_0) \leq l.
\]

As a consequence,

\[
\Gamma_\ast(v) = \Gamma^{(l/2)}_3(v) \cup \bigcup_{2 \leq k \leq \bar{l}} \Gamma^{(l)}_k(v).
\]

Moreover,

\[
\Gamma_\ast(v) \setminus \Gamma(v) \subset \bigcup_{m \in \mathbb{N}} \Gamma^{(l)}_{2m+1}(v).
\]

**Proof.** The first part is nothing but Theorem \[8.6\]

Suppose now \((x_0,t_0) \in \Gamma_\ast(v) \setminus \Gamma(v)\) and that \( k = \kappa^{(l)}_v(x_0,t_0) < \ell \). Then, there exists a small \( \delta > 0 \) such that \( v = \varphi \) on \( Q'_\delta(x_0,t_0) \). Next, consider the translate \( v^{(x_0,t_0)} = v(x_0 + \cdot, t_0 + \cdot) \) and let \( u = u_k^{(x_0,t_0)} \) be obtained from \( v^{(x_0,t_0)} \) as in Proposition \[4.4\]. Since \( \kappa < \ell \), by Theorem \[7.3\] there exists a blowup \( u_0 \) of \( u \) over some sequence \( \tau \to 0^+ \). Since \( u = 0 \) on \( Q'_{\tau} \), \( u_0 \) will vanish on \( S'_{\infty} \).

Hence, extending it as an odd function \( \tilde{u}_0 \) of \( x_n \) from \( S'_{\infty} \) to \( S_{\infty} \), we will obtain a homogeneous caloric function in \( S_{\infty} \). Then, by the Liouville theorem, \( \tilde{u}_0 \) must be a caloric polynomial of degree \( k \). Thus, \( k \) is an integer. We further claim that \( k \) is odd. Indeed, \( \tilde{u}_0 \) solves the Signorini problem in \( S_{\infty}^+ \) and therefore we must have that \( -\partial_{x_n} u_0(x',0,t) \) is a nonnegative polynomial on \( S^\ast_{\infty} \) of homogeneity \( k - 1 \). The latter is possible when either \( k - 1 \) is even or if \( -\partial_{x_n} u_0 \) vanishes on \( S^\ast_{\infty} \). However, the latter case is impossible, since otherwise Holmgren’s uniqueness theorem would imply that \( u_0 \) is identically zero, contrary to Theorem \[7.3\]. Thus, the only possibility is that \( k - 1 \) is even, or equivalently, \( k \) is odd. Since we also have \( k \geq 3/2 > 1 \), we obtain that \( k \in \{2m + 1 \mid m \in \mathbb{N}\} \). \( \square \)

**Remark 10.9.** It is easy to construct \( v \in \mathcal{S}_\varphi(Q^+_1) \) such that \( \Gamma(v) = \emptyset \) and \( \Gamma_\ast(v) = \Gamma^{(l)}_{2m+1}(v) \neq \emptyset \). The simplest example is perhaps

\[
v(x,t) = -\text{Im}(x_1 + ix_n)^{2m+1}.
\]
It is easy to verify that \( v \in \mathcal{G}_0(Q^+_{1}) \), and \( v = 0 \) on \( Q^1_1 \). Thus, \( \Gamma(v) = \emptyset \). However, \( \Gamma_*(v) = \{0\} \times \{0\} \times (-1,0) \), and because of the \((2m+1)\)-homogeneity of \( v \) with respect to any point on \( \Gamma_*(v) \), if we choose \( \ell > 2m + 1 \), we have that \( \Gamma_*(v) = \Gamma^{(\ell)}_{2m+1}(v) \).

11. Free boundary: Regular set

In this section we study a special subset \( \mathcal{R}(v) \) of the extended free boundary. Namely, the collection of those points having minimal frequency \( \kappa = 3/2 \).

**Definition 11.1** (Regular set). Let \( v \in \mathcal{G}_\varphi(Q^+_{1}) \) with \( \varphi \in H^{\ell,\ell/2}(Q^1_1) \), \( \ell \geq 2 \). We say that \((x_0,t_0) \in \Gamma_*(v)\) is a regular free boundary point if it has a minimal homogeneity \( \kappa = 3/2 \), or equivalently \( \kappa_\varphi(x_0,t_0) = 3/2 \). The set
\[
\mathcal{R}(v) := \Gamma^{(\ell)}_{3/2}(v)
\]
will be called the regular set of \( v \).

We have the following basic fact about \( \mathcal{R}(v) \).

**Proposition 11.2.** The regular set \( \mathcal{R}(v) \) is a relatively open subset of \( \Gamma(v) \). In particular, for any \((x_0,t_0) \in \mathcal{R}(v)\) there exists \( \delta_0 > 0 \) such that
\[
\Gamma(v) \cap Q^\delta_0(x_0,t_0) = \mathcal{R}(v) \cap Q^\delta_0(x_0,t_0).
\]

**Proof.** First note that, by Proposition 10.8 we have \( \mathcal{R}(v) \subset \Gamma(v) \). The relative openness of \( \mathcal{R}(v) \) follows from the upper semicontinuity of the function \( \kappa_\varphi(\ell) \) and form the fact that it does not take any values between 3/2 and 2. \( \square \)

We will show in this section that, if the thin obstacle \( \varphi \) is sufficiently smooth, then the regular set can be represented locally as a \((n-2)\)-dimensional graph of a parabolically Lipschitz function. Further, such function can be shown to have Hölder continuous spatial derivatives. We begin with the following basic result.

**Theorem 11.3** (Lipschitz regularity of \( \mathcal{R}(v) \)). Let \( v \in \mathcal{G}_\varphi(Q^+_{1}) \) with \( \varphi \in H^{\ell,\ell/2}(Q^1_1) \), \( \ell \geq 3 \) and that \((0,0) \in \mathcal{R}(v)\). Then, there exist \( \delta = \delta_\varphi > 0 \), and \( g \in H^{1,1/2}(Q^\delta_0) \) (i.e., \( g \) is a parabolically Lipschitz function), such that possibly after a rotation in \( \mathbb{R}^{n-1} \), one has
\[
\Gamma(v) \cap Q^\delta_{\varphi} = \mathcal{R}(v) \cap Q^\delta_{\varphi} = \{(x',t) \in Q^\delta_{\varphi} \mid x_{n-1} = g(x'',t)\},
\]
\[
\Lambda(v) \cap Q^\delta_{\varphi} = \{(x',t) \in Q^\delta_{\varphi} \mid x_{n-1} \leq g(x'',t)\},
\]

For an illustration, see Fig. [3]. Following the well-known approach in the classical obstacle problem, see e.g. [PSU12 Chapter 4], the idea of the proof is to show that there is a cone of directions in the thin space, along which \( v - \varphi \) is increasing. This approach was successfully used in the elliptic Signorini problem in [ACS08], [CSS08], see also [PSU12 Chapter 9], and in the arguments below we generalize the constructions in these papers to the parabolic case. This will establish the Lipschitz regularity in the space variables. To show the \( 1/2 \)-Hölder regularity in \( t \) (actually better than that), we will use the fact that the \( 3/2 \)-homogeneous solutions of the parabolic Signorini problem are \( t \)-independent (see Proposition 8.5).

However, in order to carry out the program outlined above, in addition to (i) and (ii) in Theorem 7.3 above, we will need a stronger convergence of the rescalings \( u_\tau \) to the blowups \( u_0 \). This will be achieved by assuming a slight increase in the
regularity assumptions on the thin obstacle \( \varphi \), and, consequently, on the regularity of the right-hand side \( f \) in the construction of Proposition 4.4.

Lemma 11.4. Let \( u \in \mathcal{S}^f(S_1^+) \), and suppose that for some \( \ell_0 \geq 2 \)

\[
|f(x,t)| \leq M \| (x,t) \|^{\ell_0-2} \quad \text{in } S_1^+,
\]

\[
|\nabla f(x,t)| \leq L \| (x,t) \|^{(\ell_0-3)^+} \quad \text{in } Q_1^{+2},
\]

and

\[
H_u(r) \geq r^{2\ell_0}, \quad \text{for } 0 < r < r_0.
\]

Then, for the family of rescalings \( \{u_r\}_{0 < r < r_0} \) we have the uniform bounds

\[
\|u_r\|_{H^{3/2,3/4}(Q_1^+ \cup Q_R^+)} \leq C_u, \quad 0 < r < r_{u,R}.
\]

In particular, if the sequence of rescalings \( u_{r_j} \) converges to \( u_0 \) as in Theorem 7.3, then over a subsequence

\[
u_{r_j} \to u_0, \quad \nabla u_{r_j} \to \nabla u_0 \quad \text{in } H^{\alpha,\alpha/2}(Q_R^+ \cup Q_R^+),
\]

for any \( 0 < \alpha < 1/2 \) and \( R > 0 \).

Proof. Because of Theorem 9.1, it is enough to show that \( u_r, |\nabla u_r|, \) and \( f_r \) are bounded in \( Q_R^+ \). We have

\[
|f_r(x,t)| = \frac{r^2 |f(rx, r^2t)|}{H_u(r)^{1/2}} \leq \frac{M r^\ell_0 \| (x,t) \|^{\ell_0-2}}{H_u(r)^{1/2}} \leq M \| (x,t) \|^{\ell_0-2}, \quad (x,t) \in S_R^+.
\]

Besides, we have that

\[
|\nabla f_r(x,t)| = \frac{r^3 |\nabla f(rx, r^2t)|}{H_u(r)^{1/2}} \leq \frac{L r^{\max\{\ell_0,3\} \| (x,t) \|^{(\ell_0-3)^+}}{H_u(r)^{1/2}} \leq L \| (x,t) \|^{(\ell_0-3)^+}, \quad (x,t) \in Q_R^+.
\]
Then, the functions
\[ w_\pm = (u_\varepsilon)_\pm \ (\text{evenly reflected to } S^-_R) \]
satisfy
\[ \Delta w_\pm - \partial_t w_\pm \geq -M\| (x,t) \|^\ell_0 - 2 \quad \text{in } S_R. \]

By Lemma 9.3 we thus obtain
\[ \sup_{Q_{R/2}} |u_\varepsilon| \leq C(H_{u_\varepsilon}(R)^{1/2} + MR^{\ell_0}) \leq CR^{\ell_0}(1 + M), \]
for small \( r \). Then, by the energy inequality for \( w_\pm \) in \( Q_{R/2} \), we have
\[ \frac{1}{R^{n+2}} \int_{Q_{R/4}} |\nabla u_\varepsilon|^2 \leq CR^{-2}R^{2\ell_0}(1 + M)^2 + C R^{2}R^{2\ell_0(1 + M)^2} \leq CR^{2\ell_0-2}(1 + M)^2. \]

On the other hand, using that for \( i = 1, \ldots, n \),
\[ (w_i)_\pm = (\partial_x u_\varepsilon)_\pm \ (\text{evenly reflected to } S^-_R) \]
satisfy
\[ \Delta (w_i)_\pm - \partial_t (w_i)_\pm \geq -LR^{(\ell_0-3)}_+ \quad \text{in } Q_R, \]
then from \( L_\infty - L_2 \) estimate for subcaloric functions, we obtain
\[ \sup_{Q_{R/8}} |\nabla u_\varepsilon| \leq C_n R^{\max\{\ell_0-1,2\}}(1 + M + L). \]
Thus, \( u_\varepsilon, |\nabla u_\varepsilon| \) and \( f_\varepsilon \) are uniformly bounded in \( Q_{R/8}^+ \) for small \( r < r_{R,u} \), and this completes the proof of the lemma.

The next lemma will allow to deduce the monotonicity of the solution \( u \) in a cone of directions in the thin space, from that of the blowup. It is the parabolic counterpart of [ACS08, Lemma 4] and [CSS08, Lemma 7.2].

**Lemma 11.5.** Let \( \Lambda \) be a closed subset of \( \mathbb{R}^{n-1} \times (-\infty,0] \), and \( h(x,t) \) a continuous function in \( Q_1 \). For any \( \delta_0 > 0 \) there exists \( \varepsilon_0 > 0 \), depending only on \( \delta_0 \) and \( n \), such that if

i) \( h \geq 0 \) on \( Q_1 \cap \Lambda \),

ii) \( (\Delta - \partial_t)h \leq \varepsilon_0 \) in \( Q_1 \setminus \Lambda \),

iii) \( h \geq -\varepsilon_0 \) in \( Q_1 \),

iv) \( h \geq \delta_0 \) in \( Q_1 \setminus \{|x_n| \geq c_n\}, \ c_n = 1/(32\sqrt{n-1}) \),

then \( h \geq 0 \) on \( Q_1/2 \).

**Proof.** It is enough to show that \( h \geq 0 \) on \( Q_1/2 \setminus \{|x_n| \leq c_n\} \). Arguing by contradiction, let \( (x_0,t_0) \in Q_1/2 \setminus \{|x_n| \leq c_n\} \) be such that \( h(x_0,t_0) < 0 \). Consider the auxiliary function
\[ w(x,t) = h(x,t) + \frac{\alpha_0}{2(n-1)}|x' - x'_0|^2 + \alpha_0(t_0 - t) - \left( \alpha_0 + \frac{\varepsilon_0}{2} \right)x_n^2, \]
where \( \alpha_0 = \delta_0/2c_n^2 \). It is immediate to check that
\[ w(x_0,t_0) < 0, \quad (\Delta - \partial_t)w \leq 0 \quad \text{in } Q_1 \setminus \Lambda. \]

Now, consider the function \( w \) in the set \( U = (Q_{3/4} \cap \{|x_n| \leq c_n, t \leq t_0\}) \setminus \Lambda \). By the maximum principle, we must have
\[ \inf_{\partial U} w < 0. \]
Analyzing the different parts of \( \partial_p U \) we show that this inequality cannot hold:

1) On \( \Lambda \cap \partial_p U \) we have \( w \geq 0 \).
2) On \( \{ |x_n| = c_n \} \cap \partial_p U \) we have
   \[
   w(x, t) \geq h(x, t) - 2\alpha_0 x_n^2 \geq \delta_0 - 2\alpha_0 c_n^2 \geq 0,
   \]
   if \( \varepsilon_0 \leq 2\alpha_0 \).
3) On \( \{ |x_n| < c_n \} \cap \partial_p U \) we have
   \[
   w(x, t) \geq -\varepsilon_0 + \frac{\alpha_0}{2(n - 1)} |x' - x_0'|^2 - 2\alpha_0 x_n^2
   \geq -\varepsilon_0 + \alpha_0 \varepsilon_n,
   \]
   with
   \[
   \varepsilon_n = \frac{1}{128(n - 1)} - 2c_n^2 = \frac{3}{512(n - 1)} > 0.
   \]
   If we choose \( \varepsilon_0 < \alpha_0 \varepsilon_n \), we conclude that \( w \geq 0 \) on this portion of \( \partial_p U \).
4) On \( t = -9/16 \) we have
   \[
   w(x, t) \geq -\varepsilon_0 + \alpha_0 \frac{5}{16} - 2\alpha_0 c_n^2 \geq -\varepsilon_0 + \alpha_0 \left( \frac{5}{16} - 2c_n^2 \right) \geq 0,
   \]
   for \( \varepsilon_0 < \alpha_0 / 4 \).

In conclusion, if \( \varepsilon_0 \) is sufficiently small, we see that we must have \( \inf_{\partial_p U} w \geq 0 \), thus arriving at a contradiction with the assumption that \( h(x_0, t_0) < 0 \). This completes the proof. \( \square \)

**Proof of Theorem 11.3.** Let \( u = u_k \) and \( f = f_k \) be as in Proposition 4.4. From the assumption \( \ell \geq 3 \), we have that \( |f| \leq M \) in \( S_1^+ \) and \( |\nabla f| \leq L \) in \( Q_1^{1/2} \). We also choose \( \ell_0 = 2 \). We thus conclude that \( H_u(r) \geq r^{2\varepsilon_0} \) for \( 0 < r < r_u \). In view of Theorem 7.3, the rescalings \( u_{r_j} \) converge (over a sequence \( r = r_j \to 0^+ \)) to a homogeneous global solution \( u_0 \) of degree 3/2. Furthermore, we note that Lemma 11.4 is also applicable here. In view of Proposition 8.5, after a possible rotation in \( \mathbb{R}^{n-1} \), we may assume that
\[
u_0(x, t) = C_n \Re(x_{n-1} + ix_n)^{3/2}.
\]
It can be directly calculated that for any \( e \in \partial B_1' \)
\[
\partial_n u_0(x, t) = \frac{3}{2} C_n(e \cdot e_{n-1}) \Re(x_{n-1} + ix_n)^{1/2}
= \frac{3}{2} \sqrt{2} C_n(e \cdot e_{n-1}) \sqrt{x_{n-1}^2 + x_n^2 + x_{n-1}}.
\]
Thus, if for any given \( \eta > 0 \) we consider the thin cone around \( e_{n-1} \)
\[
C_\eta := \{ x' = (x'', x_{n-1}) \in \mathbb{R}^{n-1} | x_{n-1} \geq \eta |x''| \},
\]
then it is immediate to conclude that for any \( e \in C_\eta, |e| = 1, \)
\[
\partial_n u_0 \geq 0, \quad \text{in } Q_1^+, \quad \partial_{n,e} u_0 \geq \delta_{n,\varepsilon} > 0, \quad \text{in } Q_1^+ \cap \{ x_n \geq c_n \},
\]
where \( c_n = 1/(32\sqrt{n - 1}) \) is the dimensional constant in Lemma 11.5. We next observe that, by Lemma 11.4 for any given \( \varepsilon > 0 \) we will have for all directions \( e \in \partial B_1' \)
\[
|\partial_{n,e} u_{r_j} - \partial_{n,e} u_0 | < \varepsilon \quad \text{on } Q_1^+,
\]
provided \( j \) is sufficiently large. Moreover, note that in view of Proposition \ref{prop:4.4} we can estimate
\[
|\langle \Delta - \partial_t \rangle \partial_e u_{r_j} | = \frac{C r_j^\ell \| x(t) \|^{\ell - 3}}{H_u(r_j)^{1/2}} \leq C r_j^{\ell - \ell_0} \to 0 \quad \text{uniformly in } Q_1^+.
\]

Thus, the function \( h = \partial_e u_{r_j} \) (evenly reflected to \( Q_1 \)) will satisfy the conditions of Lemma \ref{lem:11.5} and therefore we conclude that
\[
\partial_e u_{r_j} \geq 0 \quad \text{in } Q_{1/2}^+, \quad \text{for any } e \in \mathcal{E}_{r_j}, \ |e| = 1,
\]
for \( j \geq j_\eta \). Scaling back, we obtain that
\[
\partial_e u \geq 0 \quad \text{in } Q_{r_j}^+, \quad \text{for any } e \in \mathcal{E}_{r_j}, \ |e| = 1,
\]
where \( r_\eta = r_{j_\eta}/2 \). Now a standard argument (see \cite[Chapter 4, Exercise 4.1]{PSU12}) implies that
\[
\{ u(x', 0, t) > 0 \} \cap Q_{r_\eta} = \{ (x', t) \in Q_{r_\eta} | x_{n-1} > g(x''', t) \},
\]
where, for every fixed \( t \in (-r_\eta^2, 0] \), \( x''' \mapsto g(x''', t) \) is a Lipschitz continuous function with
\[
|\nabla'' g| \leq \eta.
\]
We are now left with showing that \( g \) is \((1/2)\)-Hölder continuous in \( t \). In fact, we are going to show that \( |g(x, t) - g(x, s)| = \alpha |t - s|^{1/2} \), uniformly in \( Q''_{r_\eta/2} \).

Suppose, towards a contradiction, that for \( x''_j \in B_{r_\eta/2} \), \( -r_\eta^2/4 \leq s_j < t_j \leq 0 \), \( t_j - s_j \to 0 \), we have for some \( C > 0 \)
\begin{equation}
|g(x''_j, t_j) - g(x''_j, s_j)| \geq C |t_j - s_j|^{1/2}.
\end{equation}

Let
\[
x'_j = (x''_j, g(x''_j, t_j)), \quad y'_j = (x''_j, g(x''_j, s_j))
\]
and
\[
\delta_j = \max \{|g(x''_j, t_j) - g(x''_j, s_j)|, |t_j - s_j|^{1/2}\}.
\]

Let also
\[
\xi'_j = \frac{y'_j - x'_j}{\delta_j}, \quad \tau_j = \frac{s_j - t_j}{\delta_j^2}.
\]

Note that
\[
\xi'_j = |\xi'_j| e_{n-1}, \quad (\xi'_j, \tau_j) \in \partial_p Q'_1.
\]
Moreover, we claim that \( \delta_j \to 0 \). Indeed, we may assume that the sequences \( x'_j \), \( y'_j \), \( t_j \), \( \delta_j \) converge to some \( x', y', t \), \( \delta \) respectively. If \( \delta > 0 \) then we obtain that \( (x', t), (y', t) \in \Gamma(v) \). But \( y' - x' = |y' - x'| e_{n-1} \), which cannot happen since \( \Gamma(v) \) is given as a graph \( \{ x_{n-1} = g(x''', t) \} \) in \( Q'_{r_\eta} \). Thus, \( \delta_j \to 0 \).

Consider now the rescalings of \( u \) at \( (x_j, t_j) \) by the factor of \( \delta_j \):

\begin{equation}
w_j(x, t) = \frac{u(x_j + \delta_j x, t_j + \delta_j^2 t)}{H_u(x_j, t_j) (\delta_j)^{1/2}}.
\end{equation}

We want to show that the sequence \( w_j \) converges to a homogeneous global solution in \( S_\infty \), of homogeneity \( 3/2 \). For that purpose, we first assume that \( r_\eta \) is so small that
\[
\Gamma(u) \cap Q''_{r_\eta} = \Gamma_{3/2}(u) \cap Q''_{r_\eta}.
\]
This is possible by the upper semicontinuity of the mapping \((x, t) \mapsto \kappa_{u}(x, t) = \Phi^{(2)}_{u(x, t)}(0+)\) on \(\Gamma(u)\) as in Lemma 10.5, the equality \(\kappa_{u}(0, 0) = \kappa_{w}(0, 0) = 3/2,\) and Theorem 8.6. Moreover, arguing as in the proof of Lemma 10.5, we may assume that
\[
\Phi^{(2)}_{u(x, t)}(r) < 7/4, \quad \text{if } r < r_0, \quad (x, t) \in \Gamma(u) \cap Q_{r_0}^{'}.
\]
This assumption implies
\[
H_{u(x, t)}(r) \geq r^4, \quad \text{if } r < r_0, \quad (x, t) \in \Gamma(u) \cap Q_{r_0}'.
\]
Otherwise, we would have \(\Phi^{(2)}_{u(x, t)}(r) \geq 2,\) a contradiction. As a consequence, the functions
\[
\varphi_{r}(x, t) = \Phi^{(2)}_{u(x, t)}(r), \quad (x, t) \in \Gamma(u) \cap Q_{r_0}^{'}
\]
will have an explicit representation through \(H_{u(x, t)}(r)\) and its derivatives, and therefore will be continuous. We thus have a monotone family of continuous functions \(\{\varphi_{r}\}\) on a compact set \(K = \Gamma(u) \cap \overline{Q_{r_0}/2}\) such that
\[
\varphi_{r} \searrow 3/2 \quad \text{on } K \quad \text{as } r \searrow 0.
\]
By the theorem of Dini the convergence \(\varphi_{r} \to 3/2\) is uniform on \(K\). This implies that
\[
\varphi_{r_{j}}(x_{j}, t_{j}) \to 3/2 \quad \text{for any } (x_{j}, t_{j}) \in \Gamma(u) \cap \overline{Q_{r_{j}}/2}, \quad r_{j} \to 0.
\]
For the functions \(w_{j}\) defined in (11.2) above this implies
\[
\Phi^{(2)}_{w_{j}}(r) \to 3/2 \quad \text{as } j \to \infty,
\]
for any \(r > 0.\) Now, analyzing the proof of Theorem 7.3, we realize that the same conclusions can be drawn about the sequence \(w_{j}\) as for the sequence of rescalings \(w_{r_{j}}.\) In particular, over a subsequence, we have \(w_{j} \to w_{0}\) in \(L_{2, \text{loc}}(S_{\infty})\), where \(w_{0}\) is a \(3/2\)-homogeneous global solution of the Signorini problem. By Proposition 8.5 we conclude that for some direction \(e_{0} \in \mathbb{R}^{n-1}\) it must be
\[
w_{0}(x, t) = C_{n} \Re(x' \cdot e_{0} + ix_{n})^{3/2}.
\]
Further, since \(\partial_{e} w_{0} \geq 0\) for unit \(e \in \mathcal{C}^{'}_{\eta}\), we must have
\[
\partial_{e} w_{0} \geq 0, \quad \text{in } S_{\infty}^{+}, \quad e \in \mathcal{C}^{'}_{\eta}.
\]
Therefore,
\[
e_{0} \cdot e \geq 0 \quad \text{for any } e \in \mathcal{C}^{'}_{\eta} \Rightarrow e_{0} \cdot e_{n-1} > 0.
\]
Further, note that since \(u \in \mathcal{C}^{f}(S_{1}^{+})\) with \(|f| \leq M\) in \(S_{1}^{+},\) and \(|\nabla f| \leq L\) in \(Q_{1/2}^{+},\) we can repeat the arguments in the proof of Lemma 11.4 (with \(\ell_{0} = 2\)) to obtain for any \(R > 0\)
\[
(11.3) \quad w_{j} \to w_{0}, \quad \nabla w_{j} \to \nabla w_{0} \quad \text{in } H^{\alpha, \alpha/2}(Q_{R}^{+} \cup Q_{R}').
\]
Going back to the construction of the functions \(w_{j},\) note that \((\xi_{j}', \tau_{j}) \in \Gamma(w_{j}),\) in addition to \((\xi_{j}', \tau_{j}) \in \partial_{p}Q_{1}^{+}.\) Without loss of generality we may assume that \((\xi_{j}', \tau_{j}) \to (\xi_{0}', \tau_{0}) \in \partial_{p}Q_{1}^{+}.\) But then the convergence (11.3) implies that \(w_{0}(\xi_{0}', \tau_{0}) = 0\) and \(\nabla w_{0}(\xi_{0}', \tau_{0}) = 0.\) From the explicit formula for \(w_{0}\) it follows that
\[
(\xi_{0}', \tau_{0}) \in \{(x', t) \in S_{\infty}^{'} | x' \cdot e_{0} = 0\},
\]
or equivalently, \( \xi_0' \cdot e_0 = |\xi_0'|e_{n-1} \cdot e_0 = 0 \). Since \( e_{n-1} \cdot e_0 > 0 \), we must have \( \xi_0' = 0 \). Thus, we have proved that

\[
|\xi'_j| = \frac{|g(x''_j, s_j) - g(x''_j, t_j)|}{\max\{|g(x''_j, s_j) - g(x''_j, t_j)|, |t_j - s_j|^{1/2}\}} \to 0,
\]

which is equivalent to

\[
\frac{|g(x''_j, s_j) - g(x''_j, t_j)|}{|t_j - s_j|^{1/2}} \to 0,
\]

c contrary to our assumption (11.1).

We next show that, following an idea in [AC85], the regularity of the function \( g \) can be improved with an application of a boundary Harnack principle.

**Theorem 11.6** (Hölder regularity of \( \nabla'' g \)). In the conclusion of Theorem 11.3, one can take \( \delta > 0 \) so that \( \nabla'' g \in H^{\alpha, \frac{\alpha}{2}}(Q_\delta') \) for some \( \alpha > 0 \).

To prove this theorem we first show the following nondegeneracy property of \( \partial_x u \).

**Proposition 11.7** (Nondegeneracy of \( \partial_x u \)). Let \( v \in \mathcal{S}_p(Q_1^+) \) and \( u = u_k \) be as Theorem 11.3. Then, for any \( \eta > 0 \) there exist \( \delta > 0 \) and \( c > 0 \) such that

\[
\partial_x u \geq c \cdot d(x, t) \quad \text{in } Q_{1}^+, \quad \text{for any } e \in \mathcal{E}_n, \quad |e| = 1,
\]

where

\[
d(x, t) = \sup \{ r | Q_r(x, t) \cap Q_\delta \subset Q_\delta \setminus \Lambda(v) \}
\]

is the parabolic distance from the point \( (x, t) \) to the coincidence set \( \Lambda(v) \cap Q_\delta' \).

The proof is based on the following improvement on Lemma 11.5, which is the parabolic counterpart of [CSS08, Lemma 7.3].

**Lemma 11.8.** For any \( \delta_0 > 0 \) there exist \( \varepsilon_0 > 0 \) and \( c_0 > 0 \), depending only on \( \delta_0 \) and \( n \), such that if \( h \) is a continuous function on \( Q_1 \cap \{0 \leq x_n \leq c_n\}, \)

\( c_n = 1/(32\sqrt{n-1}) \), satisfying

i) \((\Delta - \partial_t)h \leq \varepsilon_0 \) in \( Q_1 \cap \{0 < x_n < c_n\} \),

ii) \( h \geq 0 \) in \( Q_1 \cap \{0 < x_n < c_n\} \),

iii) \( h \geq \delta_0 \), on \( Q_1 \cap \{x_n = c_n\} \),

then

\[
h(x, t) \geq c_0 x_n \quad \text{in } Q_{1/2} \cap \{0 < x_n < c_n\}.
\]

**Proof.** The proof is very similar to that of Lemma 11.5. Let \( (x_0, t_0) \in Q_{1/2} \cap \{0 < x_n < c_n\} \), and consider the auxiliary function

\[
w(x, t) = h(x, t) + \frac{\alpha_0}{2(n-1)}|x' - x'_0|^2 + \alpha_0(t_0 - t) - \left( \alpha_0 + \frac{\varepsilon_0}{2} \right)x_n^2 - c_0 x_n,
\]

with \( \alpha_0 = \delta_0/2c_n^2 \).

As before, we have \((\Delta - \partial_t)w \leq 0 \) in \( Q_1 \cap \{0 < x_n < c_n\} \). We now claim that \( w \geq 0 \) on \( U^+ = Q_{3/4} \cap \{0 < x_n < c_n, t < t_0\} \). This will follow once we verify that \( w \geq 0 \) on \( \partial_p U^+ \).

We consider several cases:

1) On \( \{x_n = 0\} \cap \partial_p U^+ \), we clearly have \( w \geq 0 \).
2) On \( \{ x_n = c_n \} \cap \partial_p U^+ \), one has
\[
w \geq \delta_0 - \frac{3}{2} \alpha_0 c_n^2 - c_0 c_n = \frac{1}{2} \alpha_0 c_n^2 - c_0 c_n \geq 0,
\]
provided \( \varepsilon_0 \leq \alpha_0 \) and \( c_0 \leq c_n \alpha_0 / 2 \).

3) On \( \{ 0 < x_n < c_n \} \cap \partial_p U^+ \) we have
\[
w \geq \frac{\alpha_0}{128(n-1)} - 2 \alpha_0 c_n^2 - c_0 c_n \geq \frac{3 \alpha_0}{512(n-1)} - c_0 c_n \geq 0,
\]
provided \( c_0 < 3 \alpha_0 /(512(n-1) c_n) \).

4) On \( \{ t = -9/16 \} \cap \partial_p U^+ \) we have
\[
w \geq \frac{5 \alpha_0}{16} - 2 \alpha_0 c_n^2 - c_0 c_n \geq \frac{\alpha_0}{4} - c_0 c_n \geq 0,
\]
provided \( c_0 < \alpha_0 / (4c_n) \).

In conclusion, for small enough \( \varepsilon_0 \) and \( c_0 \), we have \( w \geq 0 \) in \( U \), and in particular \( w(x_0, t_0) \geq 0 \). This implies that \( h(x_0, t_0) > c_0(x_0)_n \), as claimed. \( \square \)

**Proof of Proposition 11.7.** Considering the rescalings \( u_r \) as in the proof of Theorem 11.3 and applying Lemma 11.8, we obtain
\[
\partial_i u_r \geq c_n |x_n| \text{ in } Q_{1/2}, \quad e \in \mathcal{E}_n,
\]
for \( 0 < r = r_\eta \) small. (Here, we assume that \( u \) has been extended by even symmetry in \( x_n \) variable to \( Q_1 \).) Besides, by choosing \( r \) small, we can also make
\[
|\nabla - \partial_t| \partial_t u_r | \leq Cr^{t-t_0} \leq c_\eta \varepsilon_n \text{ in } Q_{1/2} \setminus \Lambda(u_r),
\]
for a dimensional constant \( \varepsilon_n > 0 \) to be specified below. Let now \( (x, t) \in Q_{1/4}^r \) and \( d = d_r(x, t) \) be the parabolic distance from \( (x, t) \) to \( \Lambda(u_r) \cap Q_1 \). Consider the lowest rightmost point on the boundary \( \partial Q_{d}(x, t) \)
\[
(x_*, t_*) = (x + c_n d, t - d^2).
\]
We have
\[
\partial_i u_r(x_*, t_*) \geq c_0 d.
\]
By the parabolic Harnack inequality (see, e.g., [Lie96, Theorems 6.17–6.18])
\[
\partial_i u_r(x, t) \geq c_n c_0 d - C_n c_0 \varepsilon_n d^2 \geq c d,
\]
if we take \( \varepsilon_n \) sufficiently small. Scaling back to \( u \), we complete the proof of the proposition. \( \square \)

A key ingredient in the proof of Theorem 11.6 is the following version of the parabolic boundary Harnack principle for domains with thin Lipschitz complements established in [PS13, Section 7]. To state the result, we will need the following notations. For a given \( L \geq 1 \) and \( r > 0 \) denote
\[
\Theta_r = \{ (x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} | (x', t) \in \Theta_{r'}', |x_{n-1}| < 4n Lr \},
\]
\[
\Theta_r = \{ (x, t) \in \mathbb{R}^n \times \mathbb{R} | (x', t) \in \Theta_{r'}', |x_n| < r \}.
\]
Lemma 11.9 (Boundary Harnack principle). Let
\[ \Lambda = \{(x',t) \in \Theta_1' \mid x_{n-1} \leq g(x'',t)\} \]
for a parabolically Lipschitz function \( g \) in \( \Theta_1'' \) with Lipschitz constant \( L \geq 1 \) such that \( g(0,0) = 0 \). Let \( u_1, u_2 \) be two continuous nonnegative functions in \( \Theta_1 \) such that for some positive constants \( c_0, C_0, M, \) and \( i = 1, 2, \)

i) \( 0 \leq u_i \leq M \) in \( \Theta_1 \) and \( u_i = 0 \) on \( \Lambda \),

ii) \( |(\Delta - \partial_t)u_i| \leq C_0 \) in \( \Theta_1 \setminus \Lambda \),

iii) \( u_i(x,t) \geq c_0 d(x,t) \) in \( \Theta_1 \setminus \Lambda \), where \( d(x,t) = \sup \{ r \mid \tilde{\Theta}_r(x,t) \cap \Lambda = \emptyset \} \).

Assume additionally that \( u_1 \) and \( u_2 \) are symmetric in \( x_n \). Then, there exists \( \alpha \in (0,1) \) such that
\[ \frac{u_1}{u_2} \in H^{\alpha,\alpha/2}(\Theta_1/2). \]
Furthermore, \( \alpha \) and the bound on the corresponding norm \( \|u_1/u_2\|_{H^{\alpha,\alpha/2}(\Theta_1/2)} \) depend only on \( n, L, c_0, C_0, \) and \( M \).

Remark 11.10. We note that, unlike the elliptic case, Lemma 11.9 above cannot be reduced to the other known results in the parabolic setting (see, e.g., [Kem72], [FGS84] for parabolically Lipschitz domains, or [HLN04] for parabolically NTA domains with Reifenberg flat boundary). We also note that this version of the boundary Harnack is for functions with nonzero right-hand side and therefore the nondegeneracy condition as in iii) is necessary. The elliptic version of this result has been established in [CSS08].

We are now ready to prove Theorem 11.6.

Proof of Theorem 11.6. Fix \( \eta > 0 \) and let \( \theta = \theta_\eta \) be such that \( e = (\cos \theta)e_{n-1} + (\sin \theta)e_j \in C' \) for \( j = 1, \ldots, n-2 \). Consider two functions
\[ u_1 = \partial_e u \quad \text{and} \quad u_2 = \partial_{e_{n-1}} u. \]

Then, by Proposition 11.7, the conditions of Lemma 11.9 are satisfied for some rescalings of \( u_1 \) and \( u_2 \). Hence, applying Lemma 11.9 and scaling back, we obtain that for a small \( \delta > 0 \), and \( \alpha \in (0,1) \),
\[ \frac{\partial_e u}{\partial_{e_{n-1}} u} \in H^{\alpha,\alpha/2}(\Theta_\delta). \]
This gives
\[ \frac{\partial_{e_j} u}{\partial_{e_{n-1}} u} \in H^{\alpha,\alpha/2}(\Theta_\delta), \quad j = 1, \ldots, n-2. \]
Hence, the level surfaces \( \{u = \varepsilon\} \cap \Theta'_\delta \) are given as graphs
\[ x_{n-1} = g_\varepsilon(x'',t), \quad x'' \in \Theta''_\delta, \]
with uniform in \( \varepsilon \) estimates on \( (\nabla'' g_\varepsilon)^{(\alpha)}_{\Theta''_\delta} \) for small \( \varepsilon > 0 \). Consequently, we obtain
\[ \nabla'' g \in H^{\alpha,\alpha/2}(\Theta''_\delta), \]
and this completes the proof of the theorem. \( \square \)
12. Free boundary: Singular set

We now turn to the study of a special class of free boundary points, called singular points, that are characterized by the property that the coincidence set has a zero density at those points with respect to the $\mathcal{H}^n$ measure in the thin space. In the time-independent Signorini problem the analysis of the singular set was carried in the paper [GP09].

**Definition 12.1** (Singular points). Let $v \in \mathfrak{S}_\varphi(Q^+_1)$ with $\varphi \in H^{\ell,\ell/2}(Q^+_1)$, $\ell \geq 2$. We say that $(x_0, t_0) \in \Gamma_*(v)$ is singular if

$$
\lim_{r \to 0+} \frac{\mathcal{H}^n(\Lambda(v) \cap Q^+_r(x_0, t_0))}{\mathcal{H}^n(Q^+_r)} = 0.
$$

We will denote the set of singular points by $\Sigma(u)$ and call it the singular set. We can further classify singular points according to the homogeneity of their blowup, by defining

$$
\Sigma_\ell(v) := \Sigma(v) \cap \Gamma^{(\ell)}_n(v), \quad \kappa \leq \ell.
$$

Since we are going to work with the blowups, we will write $\ell = k + \gamma$, for $k \in \mathbb{N}$, $0 < \gamma \leq 1$, and construct the functions $u_k \in \mathcal{S}(S^+_1)$ as in Proposition 4.4. By abusing the notation, we will write $0 \in \Sigma_\ell(u_k)$ whenever $0 \in \Sigma(v)$. Also, for technical reasons, similarly to what we did in the study of the regular set, we will assume that $\ell \geq 3$ for most of the results in this section.

The following proposition gives a complete characterization of the singular points in terms of the blowups and the generalized frequency. In particular, it establishes that

$$
\Sigma_\ell(v) = \Gamma_\ell(v) \quad \text{for} \quad \kappa = 2m, \quad m \in \mathbb{N}.
$$

**Proposition 12.2** (Characterization of singular points). Let $u \in \mathfrak{S}(S^+_1)$ with $|f(x, t)| \leq M ||(x, t)||^{\ell-2}$ in $S^+_1$, $|\nabla f(x, t)| \leq L ||(x, t)||^{\ell-3}$ in $Q^+_{1/2}$, $\ell \geq 3$ and $0 \in \Gamma^{(\ell)}_\kappa(u)$ with $\kappa < \ell$. Then, the following statements are equivalent:

1. $0 \in \Sigma_\kappa(u)$.
2. any blowup of $u$ at the origin is a nonzero parabolically $\kappa$-homogeneous polynomial $p_\kappa$ in $S^+_1$ satisfying

$$
\Delta p_\kappa - \partial_t p_\kappa = 0, \quad p_\kappa(x', 0, t) \geq 0, \quad p_\kappa(x', -x_n, t) = p_\kappa(x', x_n, t).
$$

We denote this class by $\mathfrak{P}_\kappa$.
3. $\kappa = 2m, \quad m \in \mathbb{N}$.

**Proof.** (i) $\Rightarrow$ (ii) Recall that the rescalings $u_r$ satisfy

$$
\Delta u_r - \partial_t u_r = f_r + 2(\partial_{x_n}^+ u_r) \mathcal{H}^n|_{\Lambda(u_r)} \quad \text{in} \quad S^+_1,r,
$$

in the sense of distributions, after an even reflection in $x_n$ variable. Since $u_r$ are uniformly bounded in $W^{2,1}_2(Q_{2r}^+)$ for small $r$ by Theorem 7.3, $\partial_{x_n}^+ u_r$ are uniformly bounded in $L_2(Q_{2r}^+)$. On the other hand, if $0 \in \Sigma(u)$, then

$$
\frac{\mathcal{H}^n(\Lambda(u_r) \cap Q_{2r}^+)}{R^n} = \frac{\mathcal{H}^n(\Lambda(u) \cap Q_{2r})}{(Rr)^n} \to 0 \quad \text{as} \quad r \to 0,
$$

and therefore

$$
(\partial_{x_n}^+ u_r) \mathcal{H}^n|_{\Lambda(u_r)} \to 0 \quad \text{in} \quad Q_r.
$$
in the sense of distributions. Further, the bound \(|f(x,t)| \leq M\|y(x,t)\|^{-2}\) implies that

\[
|f_r(x,t)| = \frac{r^2|f(rx,r^2t)|}{H_u(r)^{1/2}} \leq \frac{Mr^\ell}{H_u(r)^{1/2}}\|y(x,t)\|^{-2} \\
\leq Cr^{\ell-\ell_0}R^{\ell-2} \to 0 \quad \text{in} \quad Q_R,
\]

where \(\ell_0 \in (\kappa, \ell)\) and we have used the fact that \(H_\infty(r) \geq r^{2\ell_0}\) for \(0 < r < r_\infty\). Hence, any blowup \(u_0\) is caloric in \(Q_R\) for any \(R > 0\), meaning that it is caloric in the entire \(S_\infty = \mathbb{R}^n \times (-\infty, 0]\). On the other hand, by Proposition 7.3(iv), the blowup \(u_0\) is homogeneous in \(S_\infty\) and therefore has a polynomial growth at infinity. Then by the Liouville theorem we can conclude that \(u_0\) must be a homogeneous caloric polynomial \(p_\kappa\) of a certain integer degree \(\kappa\). Note that \(p_\kappa = u_0 \neq 0\) by construction. The properties of \(u\) also imply that \(p_\kappa(x',0,t) \geq 0\) for all \((x',t) \in S'_\infty\) and \(p_\kappa(x',-x_n,t) = p_\kappa(x',x_n,t)\) for all \((x',x_n,t) \in S'_\infty\). (ii) \(\Rightarrow\) (iii) Let \(p_\kappa\) be a blowup of \(u\) at the origin. Since \(p_\kappa\) is a polynomial, clearly \(\kappa \in \mathbb{N}\). If \(\kappa\) is odd, the nonnegativity of \(p_\kappa\) on \(\mathbb{R}^n - \{0\} \times \{-1\}\) implies that \(p_\kappa\) vanishes there identically, implying that \(p_\kappa \equiv 0\) on \(S'_\infty\). On the other hand, from the even symmetry in \(x_n\) we also have that \(\partial_{x_n} p_\kappa = 0\) on \(S'_\infty\). Since \(p_\kappa\) is caloric in \(\mathbb{R}^n\) and \(S'_\infty\) is not characteristic for the heat operator, Holmgren’s uniqueness theorem implies that \(p_\kappa \equiv 0\) in \(\mathbb{R}^n\), contrary to the assumption. Thus, \(\kappa \in \{2m \mid m \in \mathbb{N}\}\). 

(iii) \(\Rightarrow\) (ii) The proof of this implication is stated as a separate Liouville-type result in Lemma 12.3 below.

(ii) \(\Rightarrow\) (i) Suppose that 0 is not a singular point and that over some sequence \(r = r_j \to 0+\) we have \(\mathcal{H}^n(\Lambda(u_{r_j}) \cap Q^*_j) \geq \delta > 0\). By Lemma 11.4, taking a subsequence if necessary, we may assume that \(u_{r_j}\) converges locally uniformly to a blowup \(u_0\). We claim that

\[
\mathcal{H}^n(\Lambda(u_0) \cap Q^*_j) \geq \delta > 0.
\]

Indeed, otherwise there exists an open set \(U\) in \(S'_\infty\) with \(\mathcal{H}^n(U) < \delta\) such that \(\Lambda(u_0) \cap Q^*_j \subset U\). Then for large \(j\) we must have \(\Lambda(u_{r_j}) \cap Q^*_j \subset U\), which is a contradiction, since \(\mathcal{H}^n(\Lambda(u_{r_j}) \cap Q^*_j) \geq \delta > \mathcal{H}^n(U)\). Since \(u_0 = p_\kappa\) is a polynomial, vanishing on a set of positive \(\mathcal{H}^n\)-measure on \(S'_\infty\), it follows that \(u_0\) vanishes identically on \(S'_\infty\). But then, applying Holmgren’s uniqueness theorem one more time, we conclude that \(u_0\) must vanish on \(S_\infty\), which is a contradiction. This completes the proof of the theorem. \(\square\)

The implication (iii) \(\Rightarrow\) (ii) in Proposition 12.2 is equivalent to the following Liouville-type result, which is the parabolic counterpart of Lemma 1.3.3 in [GP09].

**Lemma 12.3.** Let \(v\) be a parabolically \(\kappa\)-homogeneous solution of the parabolic Signorini problem in \(S'_\infty\) with \(\kappa = 2m\) for \(m \in \mathbb{N}\). Then \(v\) is a caloric polynomial.

This, in turn, is a particular case of the following lemma, analogous to Lemma 1.3.4 in [GP09] in the elliptic case, which stems from Lemma 7.6 in [Mon09].

**Lemma 12.4.** Let \(v \in W^{1,1}_{2,\text{loc}}(S'_\infty)\) be such that \(\Delta v - \partial_t v \leq 0\) in \(S_\infty\) and \(\Delta v - \partial_t v = 0\) in \(S_\infty \setminus S'_\infty\). If \(v\) is parabolically \(2m\)-homogeneous, \(m \in \mathbb{N}\), and has a polynomial growth at infinity, then \(\Delta v - \partial_t v = 0\) in \(S_\infty\).
Proof. Consider \( \mu := \Delta v - \partial_t v \) in \( \mathbb{R}^n \times (-\infty, 0) \). By the assumptions, \( \mu \) is a nonpositive measure, supported on \( \{ x_n = 0 \} \times (-\infty, 0) \). We are going to show that in fact \( \mu = 0 \). To this end, let \( P(x, t) \) be a parabolically \( 2m \)-homogeneous caloric polynomial, which is positive on \( \{ x_n = 0 \} \times (-\infty, 0) \). For instance, one can take

\[
P(x, t) = \sum_{j=1}^{n-1} \text{Re}(x_j + ix_n)^{2m} + (-1)^m \sum_{k=0}^{m} \frac{m!}{(m-k)!(2k)!} x_n^{2k} t^{m-k}.\]

It is straightforward to check that \( P \) is caloric. Moreover on \( \{ x_n = 0 \} \times (-\infty, 0) \) we have

\[
P = \sum_{j=1}^{n-1} x_j^{2m} + (-t)^m,
\]

so it is positive on \( \{ x_n = 0 \} \times (-\infty, 0) \). Further, let \( \eta \in C_0^\infty((0, \infty)) \), with \( \eta \geq 0 \), and define

\[
\Psi(x, t) = \eta(t) G(x, t) = \eta(t) \left( -\frac{4}{\pi t} \right)^{n/2} e^{-|x|^2/4t}.
\]

Note that we have the following identity (similar to that of \( G(x, t) \))

\[
\nabla \Psi = \frac{x}{2t}.\]

We have

\[
\langle \Delta v, \Psi P \rangle = -\int_0^0 \int_{\mathbb{R}^n} \nabla v \cdot \nabla (\Psi P) \, dx \, dt
\]

\[
= -\int_0^0 \int_{\mathbb{R}^n} [\Psi \nabla v \cdot \nabla P + P \nabla v \cdot \nabla \Psi] \, dx \, dt
\]

\[
= \int_{\mathbb{R}^n} \int_0^0 [\Psi \Delta P + \nabla \Psi \cdot \nabla P - P \nabla v \cdot \nabla \Psi] \, dx \, dt
\]

\[
= \int_{\mathbb{R}^n} \int_0^0 \left[ v \Delta P + \frac{1}{2t} v(x \cdot \nabla P) - \frac{1}{2t} P(x \cdot \nabla v) \right] \Psi \, dx \, dt.
\]

We now use the identities \( \Delta P - \partial_t P = 0 \), \( x \cdot \nabla P + 2t \partial_t P = 2mP \), \( x \cdot \nabla v + 2t \partial_t v = 2mv \) to arrive at

\[
\langle \Delta v, \Psi P \rangle = \int_{\mathbb{R}^n} \int_0^0 [2mPv - P(x \cdot \nabla v)] \frac{\Psi}{2t} \, dx \, dt
\]

\[
= \int_{\mathbb{R}^n} \int_0^0 \partial_t v \Psi P \, dx \, dt
\]

\[
= \langle \partial_t v, \Psi P \rangle.
\]

Therefore, \( \langle \mu, \Psi P \rangle = 0 \). Since \( \mu \) is a nonpositive measure, this implies that actually \( \mu = 0 \) and the proof is complete. \( \square \)
Definition 12.5. Throughout the rest of the paper we denote by $\mathfrak{P}_\kappa$, $\kappa = 2m$, $m \in \mathbb{N}$, the class of $\kappa$-homogeneous harmonic polynomials described in Proposition 12.2(ii).

In the rest of this section we state our main results concerning the singular set: $\kappa$-differentiability at singular points (Theorem 12.6) and a structural theorem on the singular set (Theorem 12.12). The proofs will require additional technical tools (two families of monotonicity formulas) that we develop in the next section. The proofs themselves will be given in Section 14.

Theorem 12.6 ($\kappa$-differentiability at singular points). Let $u \in \mathcal{S}(S^+_1)$ with $|f(x,t)| \leq M\|(x,t)\|^{\kappa-2}$ in $S^+_1$, $\|f(x,t)\| \leq L\|(x,t)\|^{\kappa-3}$ in $Q^+_1$, $\ell \geq 3$, and $0 \in \Sigma_n(u)$ for $\kappa = 2m < \ell$, $m \in \mathbb{N}$. Then, there exists a nonzero $p_{x,t} \in \mathfrak{P}_\kappa$ such that

$$u(x,t) = p_{x,t}(x,t) + o(\|(x,t)\|^{\kappa}), \quad t \leq 0.$$  

Moreover, if $v \in \mathcal{S}(Q^+_1)$ with $v \in H^{\ell/2}(Q^+_1)$, $(x_0,t_0) \in \Sigma_n(v)$ and $u^{(x_0,t_0)}_k$ is obtained as in Proposition 4.4 for $v^{(x_0,t_0)} = v(x_0 + \cdot, t_0 + \cdot)$, then in the Taylor expansion

$$u_k^{(x_0,t_0)}(x,t) = p_{x,t}^{(x_0,t_0)}(x,t) + o(\|(x,t)\|^{\kappa}), \quad t \leq 0,$$

the mapping $(x_0,t_0) \mapsto p_{x,t}^{(x_0,t_0)}$ from $\Sigma_n(v)$ to $\mathfrak{P}_\kappa$ is continuous.

Remark 12.7. Note that since $\mathfrak{P}_\kappa$ is a convex subset of a finite-dimensional vector space, namely the space of all $\kappa$-homogeneous polynomials, its topology can be induced from any norm on that space. For instance, the topology can be induced from the embedding of $\mathfrak{P}_\kappa$ into $L_2(S^+_1)$ or even into $L_2(Q^*_1)$, since the elements of $\mathfrak{P}_\kappa$ can be uniquely recovered from their restriction to the thin space.

Remark 12.8. We want to emphasize here that the asymptotic development, as stated in Theorem 12.6, does not generally hold for $t > 0$. Indeed, consider the following example. Let $u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that

- $u(x,t) = -t - x_n^2/2$ for $x \in \mathbb{R}^n$ and $t \leq 0$.
- In $\{x_n \geq 0, t \geq 0\}$, $u$ solves the Dirichlet problem

$$\Delta u - \partial_t u = 0, \quad x_n > 0, t > 0,$$

$$u(x,0) = -x_n^2, \quad x_n \geq 0,$$

$$u(x,0,t) = 0 \quad t \geq 0.$$

- In $\{x_n \leq 0, t \geq 0\}$, we extend the function by even symmetry in $x_n$: $u(x',x_n,t) = u(x',-x_n,t)$.

It is easy to see that $u$ solves the parabolic Signorini problem with zero obstacle and zero right-hand side in all of $\mathbb{R}^n \times \mathbb{R}$. Moreover, $u$ is homogeneous of degree two and clearly $0 \in \Sigma_2(u)$. Now, if $p(x,t) = -t - x_n^2/2$, then $p \in \mathfrak{P}_2$ and we have the following equalities:

$$u(x,t) = p(x,t), \quad \text{for } t \leq 0,$$

$$u(x',0,t) = 0, \quad p(x',0,t) = -t \quad \text{for } t \geq 0.$$

So for $t \geq 0$ the difference $u(x,t) - p(x,t)$ is not $o(\|(x,t)\|^2)$, despite being zero for $t \leq 0$.  

Lemma 12.10 (Time-like singular points). Then 

Then an integer between 0 and \( n \), which we call the spatial dimension of the singular set. 

Definition 12.9 (Spatial dimension of the singular set). For a singular point \((x_0, t_0) \in \Sigma_\kappa(v)\) we define

\[
d_\kappa^{(x_0, t_0)} = \dim \{ \xi \in \mathbb{R}^{n-1} \mid \xi \cdot \nabla_x \partial_{x'} \partial_t^j p_\kappa^{(x_0, t_0)} = 0 \}
\]

for any \( \alpha' = (\alpha_1, \ldots, \alpha_{n-1}) \) and \( j \geq 0 \) such that \( |\alpha'| + 2j = \kappa - 1 \), which we call the spatial dimension of \( \Sigma_\kappa(v) \) at \((x_0, t_0)\). Clearly, \( d_\kappa^{(x_0, t_0)} \) is an integer between 0 and \( n - 1 \). Then, for any \( d = 0, 1, \ldots, n - 1 \) define

\[
\Sigma_\kappa^d(v) := \{(x_0, t_0) \in \Sigma_\kappa(v) \mid d_\kappa^{(x_0, t_0)} = d\}.
\]

The case \( d = n - 1 \) deserves a special attention.

Lemma 12.10 (Time-like singular points). Let \((x_0, t_0) \in \Sigma_\kappa^{n-1}(v), \kappa = 2m < \ell.\) Then

\[
p_\kappa^{(x_0, t_0)}(x, t) = C(-1)^m \sum_{k=0}^{m} \frac{t^{m-k} x_0^{2k}}{(m-k)! 2^k},
\]

for some positive constant \( C \). In other words, \( p_\kappa^{(x_0, t_0)} \) depends only on \( x_n \) and \( t \) and is unique up to a multiplicative factor. We call such singular points time-like.

Proof. The condition \( d_\kappa^{(x_0, t_0)} = n - 1 \) is equivalent to the following property of \( p_\kappa = p_\kappa^{(x_0, t_0)} \):

\[
\nabla_x \partial_{x'} \partial_t^j p_\kappa = 0,
\]

for any multi-index \( \alpha' = (\alpha_1, \ldots, \alpha_{n-1}) \) and \( j \) such that \( |\alpha'| + 2j = \kappa - 1 \). It is easy to see that this is equivalent to vanishing of \( \partial_{x_i} p_\kappa \) on \( S_\infty \) for \( i = 1, \ldots, n - 1 \). On the other hand, \( \partial_{x_i} p_\kappa \) is caloric in \( S_\infty \) and is also even symmetric in \( x_n \) variable, implying that \( \partial_{x_n} \partial_{x_i} p_\kappa = 0 \) on \( S_\infty \). Then, by Holmgren’s uniqueness theorem \( \partial_{x_i} p_\kappa \) is identically 0 in \( S_\infty \), implying that \( p_\kappa(x, t) \) depends only on \( x_n \) and \( t \). The homogeneity of \( p_\kappa \) implies that we can write it in the form

\[
p_\kappa(x, t) = \sum_{k=0}^{m} a_k \frac{t^{m-k} x_0^{2k}}{(m-k)! 2^k}.
\]

The rest of the proof is then elementary. \( \square \)

Definition 12.11 (Space-like and time-like manifolds). We say that a \((d + 1)\)-dimensional manifold \( S \subset \mathbb{R}^{n-1} \times \mathbb{R}, d = 0, \ldots, n - 2 \), is space-like of class \( C^{1,0} \), if locally, after a rotation of coordinate axes in \( \mathbb{R}^{n-1} \) one can represent it as a graph

\[
(x_d+1, \ldots, x_{n-1}) = g(x_1, \ldots, x_d, t),
\]

where \( g \) is of class \( C^{1,0} \), i.e., \( g \) and \( \partial_{x_i} g, i = 1, \ldots, d \) are continuous.

We say that \((n - 1)\)-dimensional manifold \( S \subset \mathbb{R}^{n-1} \times \mathbb{R} \) is time-like of class \( C^1 \) if it can be represented locally as

\[
t = g(x_1, \ldots, x_{n-1}),
\]

where \( g \) is of class \( C^1 \).
Figure 4. Structure of the singular set $\Sigma(v) \subset \mathbb{R}^2 \times (-\infty, 0]$ for the solution $v$ with $v(x_1, x_2, 0, t) = -t(t + x_1^2)^4$, $t \leq 0$ with zero thin obstacle. Note that the points on $\Sigma_{11}^1$ and $\Sigma_{10}^1$ are space-like, and the points on $\Sigma_2^2$ are time-like.

**Theorem 12.12** (Structure of the singular set). Let $v \in \mathcal{S}_\varphi(Q^+_\ell)$ with $\varphi \in H^{\ell,\ell/2}(Q^\ell)$, $\ell \geq 3$. Then, for any $\kappa = 2m < \ell$, $m \in \mathbb{N}$, we have $\Gamma_u(v) = \Sigma_\kappa(v)$. Moreover, for every $d = 0, 1, \ldots, n-2$, the set $\Sigma_d^\kappa(v)$ is contained in a countable union of $(d+1)$-dimensional space-like $C^1$ manifolds and $\Sigma_{n-1}^\kappa(v)$ is contained in a countable union of $(n-1)$-dimensional time-like $C^1$ manifolds.

For a small illustration, see Fig. 4.

13. Weiss and Monneau type monotonicity formulas

In this section we construct two families of monotonicity formulas that will play a crucial role in the study of the singular set. They generalize the corresponding formulas in [GP09] in the study of the elliptic thin obstacle problem.

The first family of monotonicity formulas goes back to the work of Weiss [Wei99a] in the elliptic case and [Wei99b] in the parabolic case; see also [CPS04].

**Theorem 13.1** (Weiss-type monotonicity formula). Let $u \in \mathcal{S}^f(S^+_\ell)$ with $|f(x,t)| \leq M\|\nu\|^{\ell-2}$ in $S^+_\ell$, $\ell \geq 2$. For any $\kappa \in (0, \ell)$, define the Weiss energy functional

$$W_u^\kappa(r) := \frac{1}{r^{2\kappa+2}} \int_{S^+_\ell} \left( |t||\nabla u|^2 - \frac{\kappa}{2} u^2 \right) G$$

$$= \frac{1}{r^{2\kappa}} \left( f_u(r) - \frac{\kappa}{2} H_u(r) \right), \quad 0 < r < 1.$$ 

Then, for any $\sigma < \ell - \kappa$ there exists $C > 0$ depending only on $\sigma$, $\ell$, $M$, and $n$, such that

$$\frac{d}{dr} W_u^\kappa(r) \geq -C r^{2\sigma-1}, \quad \text{for a.e. } r \in (0,1).$$

In particular, the function

$$r \mapsto W_u^\kappa(r) + Cr^{2\sigma}$$

is monotonically nondecreasing for $r \in (0,1)$.
Proof. The proof is by direct computation using Proposition 6.2. We have
\[ r^{2\kappa+1} \frac{d}{dr} W_{\kappa}^u(r) = rI'_u(r) - 2\kappa I_u(r) - \frac{\kappa}{2} r H'_u(r) + \kappa^2 H_u(r) \]
\[ \geq r \left( \frac{1}{r^3} \int_{S^+} (Zu)^2 G \, dx \, dt + \frac{2}{r^3} \int_{S^+} t(Zu)fG \, dx \, dt \right) - 2\kappa I_u(r) \]
\[ - \frac{\kappa}{2} r \left( \frac{4}{r} I_u(r) - \frac{4}{r^3} \int_{S^+} tuG \, dx \, dt \right) + \kappa^2 H_u(r) \]
\[ = \frac{1}{r^2} \int_{S^+} (Zu)^2 G \, dx \, dt + \frac{2}{r^2} \int_{S^+} t(Zu)fG \, dx \, dt \]
\[ + \frac{2\kappa}{r^2} \int_{S^+} tuG \, dx \, dt - 4\kappa I_u(r) + \kappa^2 H_u(r) \]
\[ = \frac{1}{r^2} \int_{S^+} (Zu)^2 G \, dx \, dt + \frac{2}{r^2} \int_{S^+} t(Zu)fG \, dx \, dt + \frac{2\kappa}{r^2} \int_{S^+} tuG \, dx \, dt \]
\[ - 4\kappa \left( \frac{1}{2r^2} \int_{S^+} (Zu)G \, dx \, dt + \frac{1}{r^2} \int_{S^+} tufG \, dx \, dt \right) + \kappa^2 \frac{1}{r^2} \int_{S^+} u^2 \, dx \, dt \]
\[ = \frac{1}{r^2} \int_{S^+} (Zu + tf - \kappa u)^2 G \, dx \, dt - \frac{1}{r^2} \int_{S^+} t^2 f^2 G \, dx \, dt. \]
Hence, using the integral estimate on \( f \) as at the beginning of Section 10 we obtain
\[ \frac{d}{dr} W_{\kappa}^u(r) \geq - \frac{1}{r^{2\kappa+3}} \int_{S^+} t^2 f^2 G \, dx \, dt \]
\[ \geq - C \frac{r^{2\kappa+2}}{r^{2\kappa+3}} \]
\[ \geq - C r^{2\sigma-1}, \]
which yields the desired conclusion. \( \square \)

Note that in Theorem 13.1 we do not require \( 0 \in \Gamma^{(\ell)}(u) \). However, if we do so, then we will have the following fact.

Lemma 13.2. Let \( u \) be as in Theorem 13.1 and assume additionally that \( 0 \in \Gamma^{(\ell)}(u), \kappa < \ell \). Then,
\[ W_{\kappa}^u(0+) = 0. \]

The proof will require the following growth estimate, which we will use a few more times in the remaining part of the paper.

Lemma 13.3 (Growth estimate). Let \( u \in \mathcal{S}_1^+(S_1^+) \) with \( |f(x,t)| \leq M|||(x,t)|||^{\ell-2}, \ell \geq 2, \) and \( 0 \in \Gamma^{(\ell)}(u) \) with \( \kappa < \ell \) and let \( \sigma < \ell - \kappa \). Then
\[ H_r(u) \leq C \left( \|u\|_{L^2(S_1^+,G)}^2 + M^2 \right) r^{2\sigma}, \quad 0 < r < 1, \]
with \( C \) depending only on \( \sigma, \ell, n. \)
Proof. Take \( \mu(r) = (\|u\|_{L^2(S^+_{1},G)}^2 + M^2)r^{2\ell-2\sigma} \) and proceed as in the proof of Lemma 9.2. We omit the details. \( \square \)

We can now prove Lemma 13.2.

Proof of Lemma 13.2. Since \( \kappa < \ell \), by Lemma 7.1 we have
\[
\lim_{r \to 0^+} \frac{I_u(r)}{H_u(r)} = \frac{\kappa}{2}.
\]
Further, by Lemma 13.3 above, we will have
\[
H_u(r) \leq Cr^{2\kappa}.
\]
Hence, we obtain
\[
\lim_{r \to 0^+} W_{u}^{\kappa}(r) = \lim_{r \to 0} \frac{H_u(r)}{H_u(r)} \left( \frac{I_u(r)}{H_u(r)} - \frac{\kappa}{2} \right) = 0. \quad \square
\]

The next monotonicity formula is specifically tailored for singular points. It goes back to the paper of Monneau [Mon03] for the classical obstacle problem. The theorem below is the parabolic counterpart of the Monneau-type monotonicity formula in [GP09].

**Theorem 13.4** (Monneau-type monotonicity formula). Let \( u \in \mathcal{G}^f(S^+_1) \) with \( |f(x,t)| \leq M\|x,t\|^{\ell-2} \) in \( S^+_1 \), \( |\nabla f(x,t)| \leq L\|x,t\|^{\ell-3} \) in \( Q^1_{1/2} \), \( \ell \geq 3 \). Suppose that \( 0 \in \Sigma(u) \) with \( \kappa = 2m < \ell \), \( m \in \mathbb{N} \). Further, let \( p_\kappa \) be any parabolically \( \kappa \)-homogeneous caloric polynomial from class \( \mathcal{P}_\kappa \) as in Definition 12.5. For any such \( p_\kappa \), define Monneau’s functional as
\[
M_{u,p_\kappa}^\kappa(r) := \frac{1}{r^{2\kappa+2}} \int_{S^+_r} (u - p_\kappa)^2 G, \quad 0 < r < 1,
\]
where \( u = u - p_\kappa \).

Then, for any \( \sigma < \ell - \kappa \) there exists a constant \( C \), depending only on \( \sigma \), \( \ell \), and \( n \), such that
\[
\frac{d}{dr} M_{u,p_\kappa}^\kappa(r) \geq -C \left( 1 + \|u\|_{L^2(S^+_r,G)} + \|p_\kappa\|_{L^2(S^+_r,G)} \right) r^{\sigma-1}.
\]
In particular, the function
\[
r \mapsto M_{u,p_\kappa}^\kappa(r) + Cr^\sigma
\]
is monotonically nondecreasing for \( r \in (0,1) \) for a constant \( C \) depending \( \sigma \), \( \ell \), and \( n \), \( \|u\|_{L^2(S^+_1,G)} \), and \( \|p_\kappa\|_{L^2(S^+_1,G)} \).

Proof. First note that \( W_{u}^{\kappa}(r) \) is constant in \( r \), which follows easily from the homogeneity of \( p_\kappa \). Then, since also \( 0 \in \Gamma_\kappa^{(t)}(p_\kappa) \), by Lemma 13.2 we have \( W_{p_\kappa}^{\kappa}(0+) = 0 \), implying that
\[
W_{p_\kappa}^{\kappa}(r) \equiv 0.
\]
Therefore, integrating by parts, we have

\[
W^\kappa_u(r) = W^\kappa_w(r) - W^\kappa_{p\kappa}(r)
\]

\[
= W^\kappa_w(r) - \frac{2}{r^{2\kappa+2}} \int_{S_r^+} t \nabla w \nabla p\kappa G - \frac{\kappa}{r^{2\kappa+2}} \int_{S_r^+} w p\kappa G
\]

\[
= W^\kappa_w(r) + \frac{2}{r^{2\kappa+2}} \int_{S_r^+} t w (\Delta p\kappa G + \nabla p\kappa \nabla G) - \frac{\kappa}{r^{2\kappa+2}} \int_{S_r^+} w p\kappa G
\]

\[
= W^\kappa_w(r) + \frac{1}{r^{2\kappa+2}} \int_{S_r^+} 2tw \left( \partial_r p\kappa + \nabla p\kappa x \frac{x}{2t} \right) G - \frac{\kappa}{r^{2\kappa+2}} \int_{S_r^+} w p\kappa G
\]

\[
= W^\kappa_w(r) + \frac{1}{r^{2\kappa+2}} \int_{S_r^+} w (Z p\kappa - \kappa p\kappa) G
\]

\[
= W^\kappa_w(r).
\]

Next, we want to compute the derivative of \( M^\kappa_{u,p\kappa} \). With this objective in mind, we remark that we can compute the derivative of \( H^\kappa_w \) by a formula similar to that in Lemma 6.1:

\[
H'_w(r) = \frac{4}{r} I_w(r) - \frac{4}{r^3} \int_{S_r^+} tw f G - \frac{4}{r^3} \int_{S_r^+} tw_n w G,
\]

for a.e. \( r \in (0,1) \). Indeed, we first note that Lemma 6.1 holds for smooth functions with a polynomial growth at infinity, since the same spatial integration by parts used to derive Lemma 6.1 is still valid. Then, as in Proposition 6.2 approximate \( u \) by the solutions \( u^\epsilon \) of the penalized problem, apply Lemma 6.1 for \((H^\delta_{w^\epsilon})'(r)\), where \( w^\epsilon = u^\epsilon \zeta - p\kappa \), and then pass to the limit as \( \epsilon \to 0 \) and \( \delta \to 0 \) as in Proposition 6.2.

We thus arrive at the formula for \( H'_w(r) \) given above.

We therefore can write

\[
\frac{d}{dr} M^\kappa_{u,p\kappa}(r) = \frac{H'_w(r)}{r^{2\kappa}} - \frac{2\kappa}{r^{2\kappa+1}} H_w(r)
\]

\[
= \frac{4}{r^{2\kappa+1}} I_w(r) - \frac{4}{r^{2\kappa+3}} \int_{S_r^+} tw f G - \frac{4}{r^{2\kappa+3}} \int_{S_r^+} tw_n w G - \frac{2\kappa}{r^{2\kappa+1}} H_w(r)
\]

\[
= \frac{4}{r} W^\kappa_w(r) - \frac{4}{r^{2\kappa+3}} \int_{S_r^+} tw f G + \frac{4}{r^{2\kappa+3}} \int_{S_r^+} tw_n p\kappa G.
\]

We now estimate each term on the right hand side.

To estimate the first term we use that, in view of Theorem 13.1 for an appropriately chosen \( C \), the function \( W^\kappa_u(r) + C r^{2\sigma} \) is nondecreasing, and therefore

\[
W^\kappa_u(r) = W^\kappa_u(r) \geq W^\kappa_u(0+) - C r^{2\sigma} = -C r^{2\sigma},
\]

where in the last equality we have used Lemma 13.2.
The second term can be estimated by the Cauchy-Schwarz inequality, the integral estimate on $f$ at the beginning of Section 10, and Lemma 13.3:

$$\begin{align*}
\frac{1}{r^{2\kappa+3}} \int_{S_r^+} twfG & \leq \frac{1}{r^{2\kappa+3}} \left( \int_{S_r^+} w^2G \right)^{1/2} \left( \int_{S_r^+} t^2f^2G \right)^{1/2} \\
& \leq \frac{Cr}{r^{2\kappa+3}} \left( Hu(r)^{1/2} + H_{p_\kappa}(r)^{1/2} \right) Cr^\ell+1 \\
& \leq C \left( 1 + \|u\|_{L_2(S_1^+,G)} + \|p_\kappa\|_{L_2(S_1^+,G)} \right) r^{\ell-\kappa-1} \\
& \leq C \left( 1 + \|u\|_{L_2(S_1^+,G)} + \|p_\kappa\|_{L_2(S_1^+,G)} \right) r^{\sigma-1}.
\end{align*}$$

For the last term, just notice that on $S'_1$,

$$t \leq 0, \quad u_{x_n} \leq 0, \quad p_\kappa \geq 0$$

and thus,

$$\int_{S_r} tu_{x_n}p_\kappa G \geq 0.$$

Combining all these estimates, we obtain,

$$\frac{d}{dr} M^\kappa_{u_\kappa p_\kappa}(r) \geq -Cr^{2\sigma-1} - C \left( 1 + \|u\|_{L_2(S_1^+,G)} + \|p_\kappa\|_{L_2(S_1^+,G)} \right) r^{\sigma-1} \geq -C \left( 1 + \|u\|_{L_2(S_1^+,G)} + \|p_\kappa\|_{L_2(S_1^+,G)} \right) r^{\sigma-1},$$

which is the desired conclusion. $\Box$

14. Structure of the singular set

In this section, we prove our main results on the singular set, stated at the end of Section 14 as Theorems 12.6 and 12.12.

We start by remarking that in the following proofs it will be more convenient to work with a slightly different type of rescalings and blowups, than the ones used up to now. Namely, we will work with the following $\kappa$-homogeneous rescalings

$$u^{(\kappa)}(x,t) := \frac{u(rx,r^2t)}{r^\kappa},$$

and their limits as $r \to 0+$. The next lemma shows the viability of this approach.

**Lemma 14.1** (Nondegeneracy at singular points). Let $u \in \mathcal{G}^i(S_1^+)$ with $|f(x,t)| \leq M\|f(x,t)\|^{\ell-3} \leq L\|f(x,t)\|^{\ell-3} \leq Q_{1/2}^+$, $\ell \geq 3$, and $0 \in \Sigma_\kappa(u)$ for $\kappa < \ell$. Then, there exists $c = c_u > 0$ such that

$$H_u(r) \geq cr^{2\kappa}, \quad \text{for any } 0 < r < 1.$$

**Proof.** Assume the contrary. Then for a sequence $r = r_j \to 0$ one has

$$H_u(r) = \frac{1}{r^2} \int_{S_r^+} u^2G = o(r^{2\kappa}).$$

Since $0$ is a singular point, by Proposition 12.2 we have that, over a subsequence,

$$u_r(x,t) = \frac{u(rx,r^2t)}{H_u(r)^{1/2}} \to q_\kappa(x,t),$$
as described in Theorem 7.3 for some nonzero \( q_\kappa \in \mathcal{P}_\kappa \). Now, for such \( q_\kappa \) we apply Theorem 13.4 to \( M_{\kappa, q_\kappa}^u(r) \). From the assumption on the growth of \( u \) it is easy to recognize that

\[
M_{\kappa, q_\kappa}^u(0+) = \int_{S^+_1} q_\kappa^2 G = \frac{1}{r^{2\kappa+2}} \int_{S^+_1} q_\kappa^2 G.
\]

Therefore, using the monotonicity of \( M_{\kappa, q_\kappa}^u(r) \) (see Theorem 13.4) for an appropriately chosen \( C > 0 \), we will have that

\[
C r^\sigma + \frac{1}{r^{2\kappa+2}} \int_{S^+_1} (u - q_\kappa)^2 G \geq \frac{1}{r^{2\kappa+2}} \int_{S^+_1} q_\kappa^2 G,
\]

or equivalently

\[
\frac{1}{r^{2\kappa+2}} \int_{S^+_1} (u^2 - 2uq_\kappa) G \geq -Cr^\sigma.
\]

After rescaling, we obtain

\[
\frac{1}{r^{2\kappa}} \int_{S^+_1} (H_u(r)u_r^2 - 2H_u(r)^{1/2} q_\kappa u_r q_\kappa) G \geq -Cr^\sigma,
\]

which can be rewritten as

\[
\int_{S^+_1} \left( \frac{H_u(r)^{1/2}}{r^\kappa} u_r^2 - 2u_r q_\kappa \right) G \geq -C \frac{r^{\kappa+\sigma}}{H_u(r)^{1/2}}.
\]

Now from the arguments in the proof of Lemma 7.4, we have \( H_u(r) > cr^{2\kappa'} \), for any \( \kappa' > \kappa \), for sufficiently small \( r \). Hence, choosing \( \kappa' < \kappa + \sigma \), we will have that \( r^{\kappa+\sigma}/H_u(r)^{1/2} \to 0 \). Thus, passing to the limit over \( r = r_j \to 0 \), we will arrive at

\[
-\int_{S^+_1} q_\kappa^2 \geq 0,
\]

which is a contradiction, since \( q_\kappa \not\equiv 0 \).

One consequence of the nondegeneracy at singular points is that the singular set has a topological type \( F_\sigma \); this will be important in our application of Whitney’s extension theorem in the proof of Theorem 12.12.

**Lemma 14.2** (\( \Sigma_\kappa(v) \) is \( F_\sigma \)). For any \( v \in \mathcal{G}_\sigma(Q_1^1) \) with \( \varphi \in H^{\ell, \ell/2}(Q_1^1) \), the set \( \Sigma_\kappa(v) \) with \( \kappa = 2m < \ell, m \in \mathbb{N} \), is of topological type \( F_\sigma \), i.e., it is a union of countably many closed sets.

**Proof.** For \( j \in \mathbb{N}, j \geq 2 \), let \( E_j \) be the set of points \((x_0, t_0) \in \Sigma_\kappa(v) \cap Q_{1-1/j} \) satisfying

\[
\frac{1}{j} r^{2\kappa} \leq H_{u_{\kappa}(x_0, t_0)}(r) \leq j r^{2\kappa}
\]

for every \( 0 < r < \min\{1 - |x_0|, \sqrt{1+t_0}\} \).

By Lemmas 13.3 and 14.1 we have that

\[
\Sigma_\kappa(v) \subset \bigcup_{j=2}^{\infty} E_j.
\]
So to complete the proof, it is enough to show that $E_j$ are closed. Indeed, if $(x_0, t_0) \in E_j$, then it readily satisfies (14.1). That, in turn, implies that $(x_0, t_0) \in \Gamma_k^{(f)}(v)$ and by Proposition 12.2 that $(x_0, t_0) \in \Sigma_k(v)$. Hence $(x_0, t_0) \in E_j$. This completes the proof.

We next show the existence and uniqueness of blowups with respect to $\kappa$-homogeneous rescalings.

**Theorem 14.3** (Uniqueness of $\kappa$-homogeneous blowups at singular points). Let $u \in S^f(S^+)$ with $|f(x, t)| \leq M\|\nu\|_{L^\infty}^{-2}$ in $S^+$, $|\nabla f(x, t)| \leq L\|\nu\|_{L^\infty}^{-3}$ in $Q^+_{1/2}$, $\ell \geq 3$, and $0 \in \Sigma_k(u)$ with $\kappa < \ell$. Then there exists a unique nonzero $p_\kappa \in \mathfrak{P}_\kappa$ such that

$$u^{(\kappa)}_r(x, t) := \frac{u(rx, r^{2\ell}t)}{r^\kappa} \to p_\kappa(x, t).$$

**Proof.** By Lemmas 13.3 and 14.1, there are positive constants $c$ and $C$ such that

$$cr^\kappa \leq H_u(r)^{1/2} \leq Cr^\kappa, \quad 0 < r < 1.$$ 

This implies that any limit $u_0$ of the $\kappa$-homogeneous rescalings $u^{(\kappa)}_r$ over any sequence $r = r_j \to 0+$ is just a positive multiple of a limit of regular rescalings $u_r(x, t) = u(rx, r^{2\ell}t)/H_u(r)^{1/2}$, since over a subsequence $H_u(r)^{1/2}/r^\kappa$ converges to a positive number. Since the limits of rescalings $u_r$ are polynomials of class $\mathfrak{P}_\kappa$, we obtain also that $u_0 \in \mathfrak{P}_\kappa$.

To show the uniqueness of $u_0$, we apply Theorem 13.4 with $p_\kappa = u_0$. This result implies that the limit $M^\kappa_{u, u_0}(0+)$ exists and can be computed by

$$M^\kappa_{u, u_0}(0+) = \lim_{r_j \to 0+} M^\kappa_{u, u_0}(r_j) = \lim_{j \to \infty} \int_{S^+_1} (u^{(\kappa)}_{r_j} - u_0)^2 G = 0.$$ 

The latter equality is a consequence of Theorem 7.3 and of the argument at the beginning of this proof. In particular, we obtain that

$$\int_{S^+_1} (u^{(\kappa)}_r - u_0)^2 G = M^\kappa_{u, u_0}(r) \to 0$$

as $r \to 0+$ (not just over $r = r_j \to 0+$!). Thus, if $u'_0$ is a limit of $u^{(\kappa)}_r$ over another sequence $r = r'_j \to 0$, we conclude that

$$\int_{S^+_1} (u'_0 - u_0)^2 G = 0.$$ 

This implies that $u'_0 = u_0$ and completes the proof of the theorem.

**Theorem 14.4** (Continuous dependence of blowup). Let $v \in S_f(Q^+_{1})$ with $\varphi \in H^{\ell, \ell/2}(Q^+_{1})$, $\ell \geq 3$, and $\kappa = 2m < \ell$, $m \in \mathbb{N}$. For $(x_0, t_0) \in \Sigma_k(v)$, let $u^{(x_0, t_0)}_k$ be as constructed in Proposition 4.4 for the function $v^{(x_0, t_0)} = v(x_0 + \cdot, t_0 + \cdot)$, and denote by $p^{(x_0, t_0)}_\kappa$ the $\kappa$-homogeneous blowup of $u^{(x_0, t_0)}_k$ at $(x_0, t_0)$ as in Theorem 14.3, so that

$$u^{(x_0, t_0)}_k(x, t) = p^{(x_0, t_0)}_\kappa(x, t) + o(\|\nu\|_\kappa^\kappa).$$
Then, the mapping \((x_0, t_0) \mapsto p_k(x_0, t_0)\) from \(\Sigma_\kappa(v)\) to \(\mathcal{P}_K\) is continuous. Moreover, for any compact subset \(K\) of \(\Sigma_\kappa(v)\) there exists a modulus of continuity \(\sigma = \sigma^K\), \(\sigma(0+) = 0\) such that

\[
|u_k^{x_0, t_0}(x, t) - p_k^{x_0, t_0}(x, t)| \leq \sigma(\|(x, t)\|)\|(x, t)\|^\alpha, \quad t \leq 0.
\]

for any \((x_0, t_0) \in K\).

**Proof.** Given \((x_0, t_0) \in \Sigma_\kappa(v)\) and \(\varepsilon > 0\) fix \(r_\varepsilon = r_\varepsilon(x_0, t_0) > 0\) such that

\[
M_{u_k(x_0, t_0), p_k(x_0, t_0)}(r_\varepsilon) = \frac{1}{r_\varepsilon^{2\kappa + 2}} \int_{S_{r_\varepsilon}^+} (u_k^{x_0, t_0} - p_k^{x_0, t_0})^2 G < \varepsilon.
\]

Then, there exists \(\delta_\varepsilon = \delta_\varepsilon(x_0, t_0)\) such that if \((x'_0, t'_0) \in \Sigma_\kappa(u)\) and \(\|(x'_0 - x_0, t'_0 - t_0)\| < \delta_\varepsilon\) one has

\[
M_{u_k(x'_0, t'_0), p_k(x_0, t_0)}(r_\varepsilon) = \frac{1}{r_\varepsilon^{2\kappa + 2}} \int_{S_{r_\varepsilon}^+} (u_k^{x'_0, t'_0} - p_k^{x_0, t_0})^2 G < 2\varepsilon.
\]

This follows from the continuous dependence of \(u_k^{x_0, t_0}\) on \((x_0, t_0) \in \Gamma(v)\) in \(L_2(S_{r_\varepsilon}^+, G)\) norm, which is a consequence of \(H^{\ell, \ell/2}\) regularity of the thin obstacle \(\varphi\). Then, from Theorem [13.4] we will have that

\[
M_{u_k(x'_0, t'_0), p_k(x_0, t_0)}(r_\varepsilon) = \frac{1}{r_\varepsilon^{2\kappa + 2}} \int_{S_{r_\varepsilon}^+} (u_k^{x'_0, t'_0} - p_k^{x_0, t_0})^2 G < 2\varepsilon + C r_\varepsilon^\sigma, \quad 0 < r < r_\varepsilon,
\]

for a constant \(C = C(x_0, t_0)\) depending on \(L_2\) norms of \(u_k^{x'_0, t'_0}\) and \(p_k^{x_0, t_0}\), which can be made uniform for \((x'_0, t'_0)\) in a small neighborhood of \((x_0, t_0)\). Letting \(r \to 0\) we will therefore obtain

\[
M_{u_k(x'_0, t'_0), p_k(x_0, t_0)}(0+) = \int_{S_{r_\varepsilon}^+} (p_k^{x'_0, t'_0} - p_k^{x_0, t_0})^2 G \leq 2\varepsilon + C r_\varepsilon^\sigma.
\]

This shows the first part of the theorem (see Remark [12.7].

To show the second part, we notice that we have

\[
\|u_k^{x'_0, t'_0} - p_k^{x'_0, t'_0}\|_{L_2(S_{r_\varepsilon}^+, G)} \leq \|u_k^{x'_0, t'_0} - p_k^{x_0, t_0}\|_{L_2(S_{r_\varepsilon}^+, G)} + \|p_k^{x_0, t_0}\|_{L_2(S_{r_\varepsilon}^+, G)} \leq 2(2\varepsilon + C r_\varepsilon^\sigma)^{1/2} r^{\kappa + 1},
\]

for \(\|(x'_0 - x_0, t'_0 - t_0)\| < \delta_\varepsilon, 0 < r < r_\varepsilon\), or equivalently

\[
\|u_r^{x'_0, t'_0} - p_k^{x_0, t_0}\|_{L_2(S_{r_\varepsilon}^+, G)} \leq 2(2\varepsilon + C r_\varepsilon^\sigma)^{1/2},
\]

where

\[
u_r^{x'_0, t'_0}(x, t) := \frac{u_k^{x'_0, t'_0}(rx, r^2 t)}{r^\kappa}
\]

is the homogeneous rescaling of \(u_k^{x'_0, t'_0}\). Making a finite cover of the compact \(K\) with full parabolic cylinders \(\overline{Q}_\delta(x_0, t_0)(x'_0, t'_0)\) for some \((x'_0, t'_0) \in K, i = 1, \ldots, N\), we see that (14.2) is satisfied for all \((x'_0, t'_0) \in K, r < r^K := \min\{r_\varepsilon(x'_0, t'_0) \mid i = 1, \ldots, N\}\) and \(C = C^K := \max\{C(x_0, t_0) \mid i = 1, \ldots, N\}\).
Now note that \( w_{r}(x', t') \) solves the parabolic Signorini problem in \( Q_{1}^{+} \) with zero thin obstacle and the right-hand side
\[
|\gamma_\rho(x', t') (x, t)| = \left| \frac{f(x', t')}{r^{\rho-2}} \right| \leq Mr^{L-\kappa} \text{ in } Q_{1}^{+}.
\]

Besides, \( p_{\kappa}(x', t') \) also solves the parabolic Signorini problem with zero thin obstacle, and zero right-hand side. This implies
\[
(\Delta - \partial_t) \left( w_{\rho}(x', t') - p_{\kappa}(x', t') \right) \geq -Mr^{L-\kappa} \to 0 \text{ in } Q_{1},
\]
and therefore using the \( L_{\infty} - L_{2} \) bounds as in the proof of Theorem 7.3(iii), we obtain that
\[
\|w_{\rho}(x', t') - p_{\kappa}(x', t')\|_{L_{\infty}(Q_{1/2})} \leq C_{\varepsilon},
\]
for all \((x', t') \in K, r < r_{\varepsilon}^{K} \) and \( C_{\varepsilon} \to 0 \text{ as } \varepsilon \to 0 \). It is now easy to see that this implies the second part of the theorem. \( \square \)

We are now ready to prove Theorems 12.6 and 12.12.

**Proof of Theorem 12.6** Simply combine Theorems 14.3 and 14.4 \( \square \)

**Proof of Theorem 12.12**

**Step 1:** Parabolic Whitney’s extension. For any \((x_{0}, t_{0}) \in \Sigma_{\kappa}(v)\) let the polynomial \( p_{\kappa}(x_{0}, t_{0}) \in \mathcal{P}_{\kappa} \) be as in Theorem 14.4. Write it in the expanded form
\[
p_{\kappa}(x_{0}, t_{0})(x) = \sum_{|\alpha|+2j=\kappa} a_{\alpha,j}(x_{0}, t_{0}) x^{\alpha}t^{j}.
\]

Then, the coefficients \( a_{\alpha,j}(x, t) \) are continuous on \( \Sigma_{\kappa}(v) \). Next, for any multi-index \( \alpha = (\alpha_{1}, \ldots, \alpha_{n}) \) and integer \( j = 0, \ldots, m \), let
\[
f_{\alpha,j}(x, t) = \begin{cases} 
0, & |\alpha| + 2j < \kappa \\
a_{\alpha,j}(x, t), & |\alpha| + 2j = \kappa,
\end{cases} \quad (x, t) \in \Sigma_{\kappa}(v).
\]

Then, we have the following compatibility lemma.

**Lemma 14.5.** Let \( K = E_{j} \) for some \( j \in \mathbb{N} \), as in Lemma 14.2. Then for any \((x_{0}, t_{0}), (x, t) \in K\)
\[
f_{\alpha,j}(x, t) = \sum_{|\beta| + 2k \leq |\alpha| - 2j} \frac{f_{\alpha+\beta,j+k}(x_{0}, t_{0})}{\beta!k!} (x - x_{0})^{\beta}(t - t_{0})^{k} + R_{\alpha,j}(x, t; x_{0}, t_{0})
\]
with
\[
|R_{\alpha,j}(x, t; x_{0}, t_{0})| \leq \sigma_{\alpha,j}(\|x - x_{0}, t - t_{0}\|)(x - x_{0}, t - t_{0})^{\kappa - |\alpha| - 2j},
\]
where \( \sigma_{\alpha,j} = \sigma_{\alpha,j}^{K} \) is a certain modulus of continuity.

**Proof.** 1) In the case \(|\alpha| + 2j = \kappa \) we have
\[
R_{\alpha,j}(x, t; x_{0}, t_{0}) = a_{\alpha,j}(x, t) - a_{\alpha,j}(x_{0}, t_{0})
\]
and the statement follows from the continuity of \( a_{\alpha,j}(x, t) \) on \( \Sigma_{\kappa}(v) \).

2) For \( 0 \leq |\alpha| + 2j < \kappa \) we have
\[ R_\alpha(x, t, x_0, t_0) = - \sum_{(\gamma, k) \geq (\alpha, j)} \frac{a_{\gamma, k}(x_0, t_0)}{(\gamma - \alpha)! (\gamma - k)!} (x - x_0)^{\gamma - \alpha} (t - t_0)^{k - j} \]

\[ = - \partial_x^\alpha \partial_t^j p_{\kappa}(x_0, t_0) (x - x_0, t - t_0). \]

Suppose now that there exists no modulus of continuity \( \sigma_{\alpha, j} \) such that (14.4) is satisfied for all \((x_0, t_0), (x, t) \in K\). Then, there exists \( \eta > 0 \) and a sequence \((x_0^i, t_0^i), (x^i, t^i) \in K\), with

\[
\max\{|x^i - x_0^i|, |t^i - t_0^i|^{1/2}\} =: \delta_i \to 0,
\]

such that

\[
(14.5) \quad \sum_{(\gamma, k) \geq (\alpha, j)} \frac{a_{\gamma, k}(x_0^i)}{(\gamma - \alpha)! (\gamma - k)!} (x^i - x_0^i)^{\gamma - \alpha} (t^i - t_0^i)^{k - j} \geq \eta \| (x^i - x_0^i, t^i - t_0^i) \|^{\kappa - |\alpha| - 2j}.
\]

Consider the rescalings

\[
w^i(x, t) = \frac{u_{x_0^i, t_0^i}(x, \delta_i^2)}{\delta_i^2}, \quad (\xi^i, \tau^i) = \left( \frac{x^i - x_0^i}{\delta_i}, \frac{t^i - t_0^i}{\delta_i^2} \right).
\]

Without loss of generality we may assume that \((x_0^i, t_0^i) \to (x_0, t_0) \in K\) and \((\xi^i, \tau^i) \to (\xi_0, \tau_0) \in \partial Q_1\). Then, by Theorem 14.4 for any \( R > 0 \) and large \( i \) we have for a modulus of continuity \( \sigma = \sigma^K \)

\[
|w^i(x, t) - p_{\kappa}(x_0^i, t_0^i)(x, t)| \leq \sigma(\delta_i \| (x, t) \|)(x, t) \|^{\kappa}, \quad (x, t) \in S_R,
\]

and therefore

\[
(14.6) \quad w^i(x, t) \to p_{\kappa}(x_0, t_0)(x, t) \quad \text{in} \quad L_\infty(Q_R).
\]

Note that we do not necessarily have the above convergence in the full parabolic cylinder \( Q_R \). Next, consider the rescalings at \((x^i, t^i)\) instead of \((x_0^i, t_0^i)\)

\[
\tilde{w}^i(x, t) = \frac{u_{x_0^i, t_0^i}(x, \delta_i^2)}{\delta_i^2}.
\]

Then, by the same argument as above

\[
(14.7) \quad \tilde{w}^i(x, t) \to p_{\kappa}(x_0, t_0)(x, t) \quad \text{in} \quad L_\infty(Q_R).
\]

We then claim that the \( H^{\ell, \ell/2} \) regularity of the thin obstacle \( \varphi \) implies that

\[
(14.8) \quad w^i(x + \xi^i, t + \tau^i) - \tilde{w}^i(x, t) \to 0 \quad \text{in} \quad L_\infty(Q_R)
\]

for any \( R > 0 \), or equivalently

\[
(14.9) \quad w^i(x, t) - \tilde{w}^i(x - \xi^i, t - \tau^i) \to 0 \quad \text{in} \quad L_\infty(Q_R).
\]
Indeed, if \(q_k^{(x_0,t_0)}(x', t)\) (as usual) denotes the \(k\)-th parabolic Taylor polynomial of the thin obstacle \(\varphi(x')\) at \((x_0, t_0)\), then

\[
q_k^2(x_0,t_0)(\delta_i(x' + \xi^i), \delta^2_i(t + \tau^i)) - q_k^1(x', \delta^2_i(t + \tau^i))
\]

\[
= \frac{\varphi(x_i^0 + \delta_i(x' + \xi^i), t_i^0 + \delta^2_i(t + \tau^i))}{\delta^2_i}
+ O(\delta^2_i \|x' + \xi^i, t + \tau^i\|^\ell)
\]

\[
= O(\delta^2_i - \kappa) \to 0
\]

and this implies the convergence \((14.8)\), if we write the explicit definition of \(w^i\) using the construction in Proposition \(4.4\).

To proceed further, we consider two different cases:

1) There are infinitely many indexes \(i\) such that \(\tau^i \geq 0\).

2) There are infinitely many indexes \(i\) such that \(\tau^i \leq 0\).

In both cases, passing to subsequences we may assume that \(\tau^i \geq 0\) \((\leq 0)\) for all indexes \(i\).

In case 1) we proceed as follows. If we take any \((x, t) \in Q_1\), because of the nonpositivity of \(\tau_i\) we have \((x - \xi^i, t - \tau^i) \in Q_2\). Passing to the limit in \((14.6)\), \((14.7)\), and \((14.9)\), we thus obtain

\[
(14.10) \quad p_k^{(x_0,t_0)}(x, t) = p_k^{(x_0,t_0)}(x - \xi_0, t - \tau_0), \quad \text{for any } (x, t) \in Q_1.
\]

Because of the real analyticity of polynomials, it follows that \((14.10)\) holds in fact for all \((x, t) \in \mathbb{R}^n \times \mathbb{R}\). But then, we also obtain

\[
(14.11) \quad p_k^{(x_0,t_0)}(x + \xi_0, t + \tau_0) = p_k^{(x_0,t_0)}(x, t), \quad \text{for any } (x, t) \in \mathbb{R}^n \times \mathbb{R}.
\]

In particular, this implies that

\[
\partial^2_x \partial^j_t p_k^{(x_0,t_0)}(\xi_0, \tau_0) = \partial^2_x \partial^j_t p_k^{(x_0,t_0)}(0, 0) = 0, \quad |\alpha| + 2j < \kappa.
\]

On the other hand, dividing both sides of \((14.5)\) by \(\delta^\kappa - |\alpha| - 2j\) and passing to the limit, we obtain

\[
|\partial^\alpha_x \partial^j_t p_k^{(x_0,t_0)}(\xi_0, \tau_0)| = \sum_{(\gamma, k) \geq (\alpha, j)} a_{\gamma, k}(x_0) \frac{\delta^{\gamma - \alpha}_i \delta^k}{|\gamma| + 2k = \kappa} \geq \eta > 0,
\]

a contradiction.

In case 2), when there are infinitely many indexes \(i\) so that \(\tau^i \leq 0\), passing to the limit in \((14.6)\), \((14.7)\), and \((14.8)\), we obtain \((14.11)\) for \((x, t) \in Q_1\). Again by real analyticity, we have the same conclusion for all \((x, t) \in \mathbb{R}^n \times \mathbb{R}\). Then, we complete the proof arguing as in case 1).

So in all cases, the compatibility conditions \((14.3)\)–\((14.4)\) are satisfied and we can apply the parabolic Whitney's extension theorem that we have proved in Appendix \(B\), see Theorem \(B.1\). Thus, there exists a function \(F \in C^{2m,m}(\mathbb{R}^n \times \mathbb{R})\) such that

\[
\partial^\alpha_x \partial^j_t F = f_{\alpha,j} \quad \text{on } K,
\]

for any \(|\alpha| + 2j \leq \kappa\).

\(^1\)just note that the arguments inside \(\varphi\) are the same
Step 2: Implicit function theorem. Suppose now \((x_0, t_0) \in \Sigma^n_d(v) \cap K\). We will consider two subcases: \(d \leq n - 2\) and \(d = n - 1\).

1) \(d \in \{0, 1, \ldots, n - 2\}\). Then, there are multi-indexes \((\beta'_i, k_i)\) with \(|\beta'_i| + 2k_i = \kappa - 1\), for \(i = 1, \ldots, n - 1 - d\) such that
\[
v_i = \nabla \delta_{x_i}^{\beta'_i} \partial_{t_i}^{k_i} F(x_0, t_0) = \nabla \delta_{x_i}^{\beta'_i} \partial_{t_i}^{k_i} p_\kappa(x_0, t_0)
\]
are linearly independent.

On the other hand,
\[
\Sigma^n_d(v) \cap K \subset \bigcap_{i=1}^{n-1-d} \{\delta_{x_i}^{\beta'_i} \partial_{t_i}^{k_i} F = 0\}.
\]

Therefore, in view of the implicit function theorem, the linear independence of \((\alpha, \beta)\) without loss of generality we may assume that \(u = 1\), for \(i = 1, \ldots, n - 1 - d\) such that
\[
\Sigma^n_d(v) \cap K \subset \{\partial_{t_i}^{m_1} F = 0\}.
\]

On the other hand, \(\Sigma^n_d(v) \cap K \subset \{\partial_{t_i}^{m_1} F = 0\}\).

Thus, by the implicit function theorem we obtain that \(\Sigma^n_d(v) \cap K\) in a neighborhood of \((x_0, t_0)\) is contained in a time-like \((n-1)\)-dimensional \(C^1\) manifold, as required. This completes the proof of the theorem. \(\square\)

**Appendix A. Estimates in Gaussian spaces: Proofs**

In this section we give the proofs of the estimates stated in Section 5.

**Proof of Lemma 5.1** (1°) For the given \(\rho > 0\), choose \(\rho < \rho_0 < 1\). Then, note that without loss of generality we may assume that \(u(\cdot, -1) = 0\), by multiplying \(u\) with a smooth cutoff function \(\eta(t)\) such that \(\eta = 1\) on \([-\rho^2, 0]\) and \(\eta = 0\) near \(t = -1\).

Next, fix a cutoff function \(\zeta_0 \in C^\infty_0(\mathbb{R}^n)\) and let \(R\) be so large that \(\zeta_0\) vanishes outside \(B_{R-1}\) and \(u\) vanishes outside \(B_{R-1} \times (-1, 0]\). Then, approximate \(u\) with the solutions of the penalized problem
\[
\begin{align*}
\Delta u_\epsilon - \partial_t u_\epsilon &= f_\epsilon &\text{in } B_R^+ \times (-1, 0],
\partial_{x_n} u_\epsilon = \beta_\epsilon(u_\epsilon) &\text{on } B_R^+ \times (-1, 0],
\epsilon u_\epsilon &= 0 &\text{on } (\partial B_R^+) \times (-1, 0],
u_\epsilon(\cdot, -1) = 0 &\text{on } B_R^+,
\end{align*}
\]
where \(f_\epsilon\) is a mollification of \(f\).

Let now \(r \in [\rho, \rho_0]\) be arbitrary. Then, for any small \(\delta > 0\) and \(\eta \in W^{1,0}_2(B_{R}^+ \times (-r^2, -\delta^2))\), vanishing on \((B_{R}^+ \setminus B_{R-1}^+) \times (-r^2, -\delta^2)\), we will have
\[
\int_{S^+ \setminus S^+_1} \nabla u_\epsilon \nabla \eta + u_\epsilon \eta + f_\epsilon \eta \, dx \, dt = - \int_{S^+ \setminus S^+_1} \beta_\epsilon(u_\epsilon) \eta \, dx \, dt.
\]
Now, for the cutoff function $\zeta_0$ as above, define the family of homogeneous functions in $S_1$ by letting
\[ \zeta_k(x, t) = |t|^{k/2}\zeta_0(x/|t|). \]
Then, choosing $\eta = u^\varepsilon\zeta_1^2 G$, we will have
\[
\int_{S_1^+ \setminus S_0^+} |\nabla u^\varepsilon|^2 \zeta_1^2 G + (u^\varepsilon \nabla u^\varepsilon \nabla G \zeta_1^2 + u^\varepsilon u_t^\varepsilon \zeta_1^2 G) + 2u^\varepsilon \zeta_1 \nabla u^\varepsilon \nabla \zeta_1 G + f^\varepsilon u^\varepsilon \zeta_1^2 G
\]
\[ = - \int_{S_1^+ \setminus S_0^+} \beta_\varepsilon(u^\varepsilon)u^\varepsilon \zeta_1^2 G dx' dt \leq 0, \]
where we have used that $s_\beta_\varepsilon(s) \geq 0$. Next, recalling that $\nabla G = x^2 t G$ and that $Z((u^\varepsilon)^2) = 2\varepsilon^2 (Zu^\varepsilon) = 2\varepsilon^2 (x\nabla u^\varepsilon + 2t\partial_t u^\varepsilon)$, we can rewrite the above inequality as
\[
\int_{S_1^+ \setminus S_0^+} |\nabla u^\varepsilon|^2 \zeta_1^2 G + \frac{1}{4\varepsilon} Z((u^\varepsilon)^2) \zeta_1^2 G + 2u^\varepsilon \zeta_1 \nabla u^\varepsilon \nabla \zeta_1 G + f^\varepsilon u^\varepsilon \zeta_1^2 G \leq 0. \]
We then can use the standard arguments in the proof of energy inequalities, except that we need to handle the term involving $Z((u^\varepsilon)^2)$. Making the change of variables $t = -\lambda^2$, $x = \lambda y$ and using the identities
\[ G(\lambda x, -\lambda^2) = \lambda^n G(x, -1), \quad \zeta_1(\lambda y, -\lambda^2) = \lambda \zeta_0(y), \]
we obtain
\[
\int_{S_1^+ \setminus S_0^+} \frac{1}{2\varepsilon} Z((u^\varepsilon)^2) \zeta_1^2 G dx dt
\]
\[ = - \int_{\mathbb{R}^n} \int_{\delta} r \lambda Z((u^\varepsilon)^2)(\lambda y, -\lambda^2) \zeta_0^2(y) G(y, -1) dy d\lambda \]
\[ = - \int_{\mathbb{R}^n} \int_{\delta} \lambda^2 \left[ \frac{d}{d\lambda} (u^\varepsilon(\lambda y, -\lambda^2)^2) \right] \zeta_0^2(y) G(y, -1) dy d\lambda \]
\[ = - \int_{\mathbb{R}^n} \int_{\delta} [\varepsilon^2 u^\varepsilon(r y, -r^2)^2 - \delta^2 u^\varepsilon(\delta y, -\delta^2)^2] \zeta_0^2(y) G(y, -1) dy \]
\[ + \int_{\mathbb{R}^n} \int_{\delta} 2\varepsilon u^\varepsilon(\lambda y, -\lambda^2)^2 \zeta_0^2 G(y, -1) dy d\lambda \]
\[ \geq -r^2 \int_{\mathbb{R}^n} u^\varepsilon(r y, -r^2)^2 \zeta_0^2(y) G(y, -1) dy \]
\[ = -r^2 \int_{\mathbb{R}^n} u^\varepsilon(\cdot, -r^2) \zeta_0^2 G(\cdot, -r^2) dy, \]
where we have used integration by parts is $\lambda$ variable. Thus, using Young’s inequality, we conclude that

$$\int_{S_r^+ \setminus S_\delta^+} |\nabla u|^2 \zeta_2^2 G \leq C_{n, \rho} \left( \int_{\mathbb{R}^n_+} u^\varepsilon(\cdot, -r^2)^2 \zeta_0^2 G(\cdot, -r^2) + \int_{S_r^+ \setminus S_\delta^+} [(u^\varepsilon)^2((\nabla \zeta_1)^2 + \zeta_0^2) + (f^\varepsilon)^2 \zeta_2^2] G \right).$$

Now, integrating over $r \in [\rho, \tilde{\rho}]$, we obtain

$$\int_{S_\rho^+} |\nabla u|^2 \zeta_2^2 G \leq C_{n, \rho} \int_{S_\tilde{\rho}^+} (u^2 + |t|^2 f^2) G.$$
Arguing as in step (1) above, we have
\[
\int_{S^+ \setminus S^+_t} \frac{1}{2t} Z \left( (u_x^e)^2 \right) \zeta_2^2 G = - \int_{\mathbb{R}^n_r} \left[ r^4 u_x^e (ry, -r^2) - \delta^4 u_x^e (\delta y, -\delta^2) \right] \zeta_0 (y)^2 G (y, -1) \\
+ 4 \int_{\mathbb{R}^n_t} \lambda \left( \frac{1}{2} u_x^e \right) \zeta_2^2 G \\
\geq -r^4 \int_{\mathbb{R}^n_t} u_x^e (ry, -r^2) \zeta_0 (y)^2 G (y, -1) dy \\
= -r^2 \int_{\mathbb{R}^n_t} u_x^e (\cdot, -r^2) \zeta_2^2 G (\cdot, -r^2).
\]

We further estimate, by the appropriate Young inequalities,
\[
\int_{S^+ \setminus S^+_t} \frac{1}{2t} Z \left( (u_x^e)^2 \right) \zeta_2^2 G \leq C_{n, \rho} \int_{S^+ \setminus S^+_t} \left( f^e \right)^2 \zeta_2^2 G + c_{n, \rho} \int_{S^+ \setminus S^+_t} \left| \nabla u_x^e \right|^2 \zeta_1^2 \frac{|x|^2}{t} G \\
\leq C_{n, \rho} \int_{S^+ \setminus S^+_t} \left( f^e \right)^2 \zeta_2^2 G \\
+ c_{n, \rho} \int_{S^+ \setminus S^+_t} \left[ \left| \nabla u_x^e \right|^2 (\zeta_1^2 + \left| \nabla \zeta_2 \right|) + \left| D^2 u_x^e \right|^2 \zeta_2^2 \right] G,
\]

with a small constant \( c_{n, \rho} > 0 \), where in the last step we have used that the following claim.

**Claim A.1.** For any \( v \in W^1_2 (\mathbb{R}^n, G) \) and \( t < 0 \) we have
\[
\int_{\mathbb{R}^n} \frac{v^2 |x|^2}{|t|} G (x, t) \leq C_n \int_{\mathbb{R}^n} (v^2 + |t||\nabla v|^2) G.
\]

**Proof.** Using that \( \nabla G = \frac{x}{|x|^2} G \), and then integrating by parts, we obtain
\[
\int_{\mathbb{R}^n} \frac{v^2 |x|^2}{|t|} G = - \int_{\mathbb{R}^n} v^2 x \cdot \nabla G = \int_{\mathbb{R}^n} \text{div}(xv^2) G \\
= n \int_{\mathbb{R}^n} v^2 G + \int_{\mathbb{R}^n} 2v (x \nabla v) G \\
\leq n \int_{\mathbb{R}^n} v^2 G + \int_{\mathbb{R}^n} v^2 \frac{|x|^2}{4|t|} G + \int_{\mathbb{R}^n} 4|t||\nabla v|^2 G,
\]

which implies the desired estimate. \( \square \)
Combining the estimates above, we obtain

\[
\int_{S^+_n \setminus S^+_{B^+}} |\nabla u^\varepsilon|^2 \zeta_2^2 G \\
\leq C_{n,p} \left( \int_{\mathbb{R}^+_n} u^\varepsilon_{x_n} (\cdot, -\varepsilon^2)^2 \zeta_1^2 G(\cdot, -\varepsilon^2) + \int_{S^+_n \setminus S^+_{B^+}} \|\nabla u^\varepsilon\|^2 (\zeta_1^2 + |\nabla \zeta_2|^2) + (f^\varepsilon)^2 \zeta_2^2 G \right) \\
+ c_{n,p} \int_{S^+_n \setminus S^+_{B^+}} |D^2 u^\varepsilon|^2 \zeta_2^2 G.
\]

(3°) Using the notations of the previous step, taking a test function \(\eta_{x_n}\) and integrating by parts, we will obtain

\[
\int_{S^+_n \setminus S^+_{B^+}} [\nabla u^\varepsilon \cdot \nabla \eta + u^\varepsilon_{x_n} \cdot \eta + f^\varepsilon \eta_{x_n}] dx dt = - \int_{S^+_n \setminus S^+_{B^+}} [u^\varepsilon \eta + \nabla' u^\varepsilon \cdot \nabla \eta] dx' dt.
\]

Plugging \(\eta = u^\varepsilon \zeta_2^2 G\), we will have

\[
\int_{S^+_n \setminus S^+_{B^+}} |\nabla u^\varepsilon_{x_n}|^2 \zeta_2^2 G + (u^\varepsilon_{x_n} \cdot \nabla u^\varepsilon_{x_n} \cdot \nabla G \zeta_2^2 + u^\varepsilon_{x_n} \cdot u^\varepsilon_{x_n} \zeta_2^2 G) + 2u^\varepsilon_{x_n} \zeta_2 \nabla u^\varepsilon_{x_n} \cdot \nabla \zeta_2 G \\
+ \int_{S^+_n \setminus S^+_{B^+}} f^\varepsilon u^\varepsilon_{x_n} \cdot \zeta_2^2 G + 2f^\varepsilon u^\varepsilon_{x_n} \cdot \zeta_2 (\zeta_2)_{x_n} G + f^\varepsilon u^\varepsilon_{x_n} \zeta_2^2 G_{x_n} \\
= - \int_{S^+_n \setminus S^+_{B^+}} [\nabla' u^\varepsilon \cdot \nabla' G \beta_{x_n}(u^\varepsilon) \zeta_2^2 + u^\varepsilon_{x_n} \cdot \beta_{x_n}(u^\varepsilon) \zeta_2^2 G] + |\nabla' u^\varepsilon_{x_n}|^2 \beta_{x_n}(u^\varepsilon) \zeta_2^2 G \\
- \int_{S^+_n \setminus S^+_{B^+}} 2u^\varepsilon_{x_n} \cdot \nabla' u^\varepsilon \cdot \nabla' \zeta_2 \zeta_2 G.
\]

We therefore have

\[
\int_{S^+_n \setminus S^+_{B^+}} |\nabla u^\varepsilon_{x_n}|^2 \zeta_2^2 G + \frac{1}{4\ell} \left( \frac{1}{(u^\varepsilon_{x_n})^2} \right) \zeta_2^2 G + 2u^\varepsilon_{x_n} \zeta_2 \nabla u^\varepsilon_{x_n} \cdot \nabla \zeta_2 G \\
+ \int_{S^+_n \setminus S^+_{B^+}} f^\varepsilon u^\varepsilon_{x_n} \cdot \zeta_2^2 G + 2f^\varepsilon u^\varepsilon_{x_n} \cdot \zeta_2 (\zeta_2)_{x_n} G + f^\varepsilon u^\varepsilon_{x_n} \zeta_2^2 G_{x_n} \\
\leq - \int_{S^+_n \setminus S^+_{B^+}} \frac{1}{4\ell} \left( B_{x_n}(u^\varepsilon) \right) \zeta_2^2 G + 2u^\varepsilon_{x_n} \cdot \nabla' u^\varepsilon \cdot \nabla' \zeta_2 \zeta_2 G = J_1 + J_2.
\]
To estimate $J_1$ we argue as before, however we now take into account that the spatial dimension is less by one:

$$J_1 = - \int_{S_t^+ \setminus S_{t^+}} \frac{1}{4t} Z(B_t(u^\varepsilon))(\zeta_2^2 G)$$

$$= \frac{1}{2} \int_{\mathbb{R}^{n-1}} [r^3 B_t(u^\varepsilon(ry^', - r^2)) - \delta^3 B_t(u^\varepsilon(\delta y^', - \delta^2))] \tilde{\zeta}_0(y')^2 G(y', -1) dy'$$

$$- \frac{3}{2} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \lambda^2 B_t(u^\varepsilon(\lambda y^', - \lambda^2)) \tilde{\zeta}_0(y')^2 G(y' - 1) dy' d\lambda$$

$$\leq C_{n, \rho \varepsilon},$$

since $B_t(s) \geq 0$ for any $s \in \mathbb{R}$ and we have used that $B_t(u^\varepsilon) \leq C_{n, \rho \varepsilon}$ on $B_{R-1}^t \times [\rho, \tilde{\rho}]$.

For more details see the proof of Proposition 6.2, step (3.iii.b).

In order to estimate $J_2$ we write it as a solid integral

$$J_2 = \int_{S_t^+ \setminus S_{t^+}} \partial x_n(u^\varepsilon_{x_n} u_{x_n}(\zeta_2^2 x_i)G)$$

$$= \int_{S_t^+ \setminus S_{t^+}} u^\varepsilon_{x_n} u^\varepsilon_{x_n}(\zeta_2^2 x_i)G + u^\varepsilon_{x_n} u^\varepsilon_{x_n}(\zeta_2^2 x_i)G + u^\varepsilon_{x_n} u^\varepsilon_{x_n}(\zeta_2^2 x_i)G + u^\varepsilon_{x_n} u^\varepsilon_{x_n}(\zeta_2^2 x_i)G + u^\varepsilon_{x_n} u^\varepsilon_{x_n}(\zeta_2^2 x_i)G = J_{21} + J_{22}.$$

The terms in $J_{21}$ are estimated in a standard way by the appropriate Young inequalities. The integral $J_{22}$ is estimated by using Claim A.1

$$|J_{22}| \leq \int_{S_t^+ \setminus S_{t^+}} |\nabla u^\varepsilon|^2 |\nabla \zeta_2| \frac{x_n}{t} G dx dt$$

$$\leq C_{n, \rho} \int_{S_t^+ \setminus S_{t^+}} |\nabla u^\varepsilon|^2 |\nabla \zeta_2|^2 G + c_{n, \rho} \int_{S_t^+ \setminus S_{t^+}} |\nabla u^\varepsilon|^2 |\zeta_1|^2 \frac{|x|^2}{|t|} G$$

$$\leq C_{n, \rho} \int_{S_t^+ \setminus S_{t^+}} |\nabla u^\varepsilon|^2 |\nabla \zeta_2|^2 G + c_{n, \rho} \int_{S_t^+ \setminus S_{t^+}} |\nabla u^\varepsilon|^2 (\zeta_1^2 + |\nabla \zeta_2|^2) G + |D^2 u|^2 \zeta_2^2 G],$$

for a small constant $c_{n, \rho} > 0$.

We further treat the term

$$\int_{S_t^+ \setminus S_{t^+}} \frac{1}{4t} Z((u^\varepsilon_{x_n})^2)\zeta_2^2 G$$
analogously to the similar term with \( u_{\varepsilon} \) in \((2)\). Collecting all estimates in this step, combined with appropriate Young inequalities, we obtain

\[
\int_{S^+_{\rho} \setminus S^+_{\delta}} |\nabla u_{\varepsilon}|^2 \zeta_2^2 G \leq C_{n,\rho} \int_{\mathbb{R}^n} u_{\varepsilon} (r, -r^2)^2 \zeta_1^2 G(r, -r^2) + C_{n,\rho} \int_{S^+_{\rho} \setminus S^+_{\delta}} |\nabla u_{\varepsilon}|^2 (\zeta_1^2 + |\nabla \zeta_2|^2 + |D^2 \zeta_3|^2) + (f_{\varepsilon})^2 \zeta_2^2 G
\]

\[
+c_{n,\rho} \int_{S^+_{\rho} \setminus S^+_{\delta}} |D^2 u_{\varepsilon}|^2 \zeta_2^2 G + C_{n,\rho} \varepsilon.
\]

\((4^o)\) Now combining the estimates in \((2)\) and \((3)\) above and integrating over \( r \in [\rho, \bar{\rho}] \) we obtain

\[
\int_{S^+_{\rho} \setminus S^+_{\delta}} |D^2 u_{\varepsilon}|^2 \zeta_2^2 G \leq C_{n,\rho} \int_{\mathbb{R}^n} u_{\varepsilon} (r, -r^2) \cdot ((\cdot, -r^2))^2 \zeta_2^2 G + C_{n,\rho} \varepsilon.
\]

As before, passing to the limit as \( \varepsilon \to 0 \), increasing the support of \( \hat{\zeta}_0 \), and then letting \( \delta \to 0 \), we conclude that

\[
\int_{S^+_{\rho}} |t|^2 |D^2 u|^2 G \leq C_{n,\rho} \int_{S^+_{\rho}} (|t| |\nabla u|^2 + |t|^2 f^2) G.
\]

Finally noticing that \( u_t = \Delta u - f \), we obtain the desired integral estimate for \( u_t \) as well. The proof is complete. \( \square \)

**Proof of Lemma 5.3**. The proof is very similar to part \((1)\) of the proof of Lemma 5.1. Indeed, for approximations \( u_{\varepsilon}^i \), \( i = 1, 2 \), we have the integral identities

\[
\int_{S^+_{\rho} \setminus S^+_{\delta}} (\nabla u_{\varepsilon}^i) \nabla \eta + \partial_t u_{\varepsilon}^i \eta + f_{\varepsilon}^i \eta) dx \, dt = -\int_{S^+_{\rho} \setminus S^+_{\delta}} \beta_{\varepsilon}(u_{\varepsilon}^i) \eta dx \, dt.
\]

Taking the difference, choosing \( \eta = (u_{\varepsilon}^1 - u_{\varepsilon}^2) \zeta_2^2 G \), and using the inequality

\[
|\beta_{\varepsilon}(u_{\varepsilon}^1) - \beta_{\varepsilon}(u_{\varepsilon}^2)| (u_{\varepsilon}^1 - u_{\varepsilon}^2) \geq 0,
\]

we complete the proof as in step \((1)\) of the proof of Lemma 5.1. \( \square \)

**Appendix B. Parabolic Whitney’s extension theorem**

Let \( E \) be a compact subset of \( \mathbb{R}^n \times \mathbb{R} \) and \( f : E \to \mathbb{R} \) a certain continuous function. Here we want to establish a theorem of Whitney type (see [Whi34]) that will allow the extension of the function \( f \) to a function of class \( C^{2m, m}(\mathbb{R}^n \times \mathbb{R}) \), \( m \in \mathbb{N} \). In fact, for that we need to have a family of functions \( \{f_{\alpha,j}\}_{\alpha+2j \leq m} \), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( j \) is a nonnegative integer.
Theorem B.1 (Parabolic Whitney’s extension). Let \( \{f_{\alpha,j}\}_{|\alpha|+2j \leq m} \) be a family of functions on \( E \), with \( f_{0,0} \equiv f \), satisfying the following compatibility conditions: there exists a family of moduli of continuity \( \{\omega_{\alpha,j}\}_{|\alpha|+2j \leq 2m} \) such that
\[
f_{\alpha,j}(x,t) = \sum_{|\beta|+2k \leq 2m-|\alpha|-2j} \frac{f_{\alpha+\beta,j+k}(x_0,t_0)}{\beta!k!} (x-x_0)^{\beta} (t-t_0)^{k} + R_{\alpha,j}(x,t;x_0,t_0)
\]
and
\[
|R_{\alpha,j}(x,t;x_0,t_0)| \leq \omega_{\alpha,j}((x-x_0,t-t_0)) (x-x_0,t-t_0)^{2m-|\alpha|-2j}.
\]
Then, there exists a function \( F \in C^{2m,m}(\mathbb{R}^n \times \mathbb{R}) \) such that \( F = f \) on \( E \) and moreover \( \partial^x \partial^t_F = f_{\alpha,j} \) on \( E \), for \( |\alpha|+2j \leq 2m \).

The proof of the following lemma is very similar to its Euclidean counterpart and is therefore omitted. (A slightly different version of this lemma can be found in [FS82] Lemma 1.67, for more general homogeneous spaces.)

Lemma B.2 (Parabolic Whitney cube decomposition). For any closed set \( E \) there exists a family \( \mathcal{W} = \{Q_i\} \) of parabolic dyadic cubes with the following properties:

(i) \( \bigcup_i Q_i = \Omega = E^c \)

(ii) \( Q_i \cap Q_j = \emptyset \) for \( i \neq j \)

(iii) \( c_n \ell(Q_i)^{\alpha} \leq \text{dist}_p(Q_i,E) \leq C_n \ell(Q_i)^{\beta} \) for some positive constants \( c_n, C_n \) depending only on the dimension \( n \).

For every \( Q_i \), let \( (x_i,t_i) \) be the center and \( \ell_i \) the size of the parabolic cube \( Q_i \). Then let \( Q^*_i = \delta_{1+\varepsilon}[-\lambda/2,\lambda/2] \times \cdots \times [-\lambda/2,\lambda/2] \times [-\lambda^2/2,\lambda^2/2] \), where \( \delta_{1+\varepsilon} : (x,t) \mapsto (\lambda x, \lambda^2 t) \) is the parabolic dilation. Clearly, the family of \( \{Q^*_i\} \) is no longer disjoint, however, we every point in \( E^c \) has a small neighborhood that intersects at most \( N = N_n \) cubes \( Q^*_i \), provided \( 0 < \varepsilon < \varepsilon_n \) is small. Then define
\[
\varphi_i(x,t) = \varphi \left( \frac{x-x_i}{\ell_i}, \frac{t-t_i}{\ell_i^2} \right),
\]
where \( \varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}) \) such that \( \varphi \geq 0, \varphi > 0 \) on \( I_1 \), \( \text{supp} \varphi \subset I_{1+\varepsilon} \), where
\[
I_\lambda = [-\lambda/2,\lambda/2] \times \cdots \times [-\lambda/2,\lambda/2] \times [-\lambda^2/2,\lambda^2/2].
\]
We also observe that
\[
|\partial^x \partial^t \varphi_i(x,t)| \leq A_{\alpha,j} \ell_i^{-|\alpha|-2j},
\]
for some constants \( A_{\alpha,j} \). Next, we define a partition of unity \( \{\varphi^*_i\} \) subordinate to \( \{Q^*_i\} \) as follows. Let
\[
\varphi^*_i(x,t) = \frac{\varphi_i(x,t)}{\Phi(x,t)}, \quad \Phi(x,t) = \sum_k \varphi_k(x,t), \quad (x,t) \in E^c.
\]
We will prove the latter formula. It is enough to check that the partial
and more generally
\[ \sum_{i} \varphi_{i}^{*} = 1 \quad \text{in } E^c, \]
where again the sum is locally finite. Moreover, it is easy to see that, similarly to
\( \varphi_{i} \), we have the estimates
\[ |\partial_{\alpha}^{2} \partial_{\beta}^{j} \varphi_{i}^{*}(x, t)| \leq A_{\alpha,j}^{*} e^{-|\alpha|-2j}. \]  

**Step 2:** For every \((x_0, t_0) \in E\) let
\[ P(x, t; x_0, t_0) = \sum_{|\alpha|+2j \leq 2m} \frac{f_{\alpha,j}(x_0, t_0)}{\alpha! j!} (x - x_0)^{\alpha}(t - t_0)^{j}. \]
In addition to \( P \), it is convenient to introduce
\[ P_{\alpha,j}(x; t_0) = \sum_{|\beta|+k \leq 2m - |\alpha| - 2j} \frac{f_{\alpha+\beta,j+k}(x_0, t_0)}{\beta! k!} (x - x_0)^{\beta}(t - t_0)^{k} \]
for \(|\alpha| + 2j \leq 2m\). Note that in fact \( P_{\alpha,j}(x; t, x_0, t_0) = \partial_{\alpha}^{2} \partial_{\beta}^{j} P(x, t; x_0, t_0) \). Then by definition
\[ f_{\alpha,j}(x, t) = P_{\alpha,j}(x, t; x_0, t_0) + R_{\alpha,j}(x, t; x_0, t_0), \]
for any \((x, t), (x_0, t_0) \in E\).

**Lemma B.3.** For any \((x_0, t_0), (x_1, t_1) \in E\) and \((x, t) \in \mathbb{R}^n \times \mathbb{R}, \) we have
\[ P(x, t; x_1, t_1) - P(x, t; x_0, t_0) = \sum_{|\beta|+k \leq 2m} R_{\beta,k}(x_1, t_1; x_0, t_0) \frac{(x - x_1)^{\beta}(t - t_1)^{k}}{\beta! k!} \]
and more generally
\[ P_{\alpha,j}(x, t; x_1, t_1) - P_{\alpha,j}(x, t; x_0, t_0) = \sum_{|\beta|+k \leq 2m - |\alpha| - 2j} R_{\alpha+\beta,j+k}(x_1, t_1; x_0, t_0) \frac{(x - x_1)^{\beta}(t - t_1)^{k}}{\beta! k!}. \]

**Proof.** We will prove the latter formula. It is enough to check that the partial
derivatives \( \partial_{\alpha}^{2} \partial_{\beta}^{j} \) of both sides equal to each other for \(|\beta| + 2k \leq 2m - |\alpha| - 2j\), as \both sides are polynomials of parabolic degree \( 2m - |\alpha| - 2j \). We have
\[ \partial_{\alpha}^{2} \partial_{\beta}^{j} P_{\alpha,j}(x, t; x_1, t_1)|_{(x, t) = (x_1, t_1)} = f_{\alpha+\beta,j+k}(x_1, t_1) \]
\[ \partial_{\alpha}^{2} \partial_{\beta}^{j} P_{\alpha,j}(x, t; x_0, t_0)|_{(x, t) = (x_1, t_1)} = P_{\alpha+\beta,j+k}(x_1, t_1; x_0, t_0), \]
which implies the desired equality.

**Step 3:** We are now ready to define the extension function \( F \). For every \( Q_i \) let \((y_i, s_i) \in E \) be such that \( \text{dist}_p(Q_i, E) = \text{dist}_p(Q_i, (y_i, s_i)) \). Note that \((y_i, s_i)\) is not necessarily unique. Then define
\[ F(x, t) = \begin{cases} f(x, t) = f_{0,0}(x, t), & (x, t) \in E \\ \sum_{i} P(x, t; y_i, s_i) \varphi_i^*(x, t), & (x, t) \in E^c. \end{cases} \]
From the local finiteness of the partition of unity, it is clear that $F$ is $C^\infty$ in $E^c$. Then we can define

$$F_{\alpha,j}(x,t) = \begin{cases} f_{\alpha,j}(x,t), & (x,t) \in E \\ \partial^\alpha_x \partial^j_t F(x,t), & (x,t) \in E^c, \end{cases}$$

for $|\alpha| + 2j \leq 2m$.

**Lemma B.4.** There exist moduli of continuity $\bar{\omega} = \bar{\omega}_{0,0}$ and $\tilde{\omega}_{\alpha,j}$, $|\alpha| + 2j \leq 2m$, such that for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $(x_0,t_0) \in E$ we have

$$|F(x,t) - P(x,t; x_0,t_0)| \leq \bar{\omega}((|x-x_0,t-t_0|)|(|x-x_0,t-t_0|^2)^m,$$

and more generally

$$|F_{\alpha,j}(x,t) - P_{\alpha,j}(x,t; x_0,t_0)| \leq \bar{\omega}_{\alpha,j}((|x-x_0,t-t_0|)^{|2m-|\alpha|-2j|, for $|\alpha| + 2j \leq 2m$.

**Proof.** Note that for $(x,t) \in E$, the estimates follows from the compatibility assumptions. For $(x,t) \in E^c$, we have

$$|F(x,t) - P(x,t; x_0,t_0)| = \left| \sum_i |P(x,t; y_i, s_i) - P(x,t; x_0,t_0)|\tilde{\varphi}_i^*(x,t) \right|$$

$$\leq \sum_i \sum_{|\alpha|+2j \leq 2m} |R_{\alpha,j}(y_i, s_i; x_0,t_0)| \frac{|x-y_i|^\alpha |t-s_i|^j}{\alpha!} \tilde{\varphi}_i^*(x,t)$$

$$\leq \sum_i \omega_{\alpha,j}(C_n(|x-y_i,t_0-s_i|) ||(x-y_i,t-s_i)||^2|m-|\alpha|-2j|+|\alpha|+2j\varphi_i^*(x,t)$$

$$\leq \bar{\omega}((|x-x_0,t-t_0|)||(|x-x_0,t-t_0|^2)^m,$$

using that

$$\|(x_0-y_i,t_0-s_i)\| \leq C_n\|(x-y_i,t-s_i)\| \leq C^2_n\|(x-x_0,t-t_0)\|$$

for $(x,t) \in Q_i^c$.

The second estimate in the lemma is obtained in a similar way. Indeed, we can write

$$|F_{\alpha,j}(x,t) - P_{\alpha,j}(x,t)| = |\partial^\alpha_x \partial^j_t [F(x,t) - P(x,t; x_0,t_0)]|$$

$$= |\partial^\alpha_x \partial^j_t \sum_i |P(x,t; y_i, s_i) - P(x,t; x_0,t_0)|\tilde{\varphi}_i^*(x,t)|$$

$$= \sum_{i, \beta \leq \alpha, k \leq j} C_{\beta,k}^{\alpha,j}|P_{\beta,k}(x,t; y_i, s_i) - P_{\beta,k}(x,t; x_0,t_0)|\partial^\beta_x \partial^{j-k}_{t} \tilde{\varphi}_i^*(x,t)$$

and then we argue as above by using Lemma B.3 and the estimates B.1. 

**Proof of Theorem B.4.** Note that Lemma B.4 implies that

$$\partial^\beta_x F_{\alpha,j}(x,t) = F_{\alpha+\beta,j}(x,t), \text{ for } |\beta| = 1, \text{ if } |\alpha| + 2j \leq 2m - 1$$

$$\partial_t F_{\alpha,j}(x,t) = F_{\alpha,j+1}(x,t), \text{ if } |\alpha| + 2j \leq 2m - 2$$

at every $(x,t) \in E$. The same equalities hold also in $E^c$, by the definition of $F_{\alpha,j}$. Thus, arguing by induction in the order of the derivative $|\alpha| + 2j \leq 2m$ and using Lemma B.4 we prove that everywhere in $\mathbb{R}^n \times \mathbb{R}$

$$\partial^\alpha_x \partial^j_t F = F_{\alpha,j}, \text{ for } |\alpha| + 2j \leq 2m.$$
We also note that $F_{\alpha,j}$ are continuous by Lemma 5.4. The proof is complete. □

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