Holographic Screening Length on Parallel Motion of Quark-Antiquark Pair in Four Dimensional Strongly Coupled $\mathcal{N} = 4$ super-Yang-Mills plasma

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Abstract. We study the screening length of a quark-antiquark pair moving in a strongly coupled hot plasma of $\mathcal{N} = 4$ super-Yang-Mills using AdS/CFT correspondence where the background metric is five dimensional AdS black hole. We take the string solution as such the separation length $L$ of quark-antiquark pair is parallel to the string velocity $v$. The screening length and the bound energy are computed numerically using Mathematica. We find that the plots are bounded from below by some functions that are related to the momentum flow of the drag force configuration $P_c$. We compare the result by computing the screening length in the quark-antiquark reference frame by boosting the AdS black hole.

1. Introduction

The existence of Quark Gluon Plasma produced in heavy ion collision’s experiment at RHIC and the new LHC can be indicated by the suppression of $J/\psi$ meson production, $c\bar{c}$ pair. This phenomena is understood qualitatively when temperature of the plasma is larger than the Hagedorn temperature where the potential interaction of $c\bar{c}$ pair would not able to hold them anymore so that $J/\psi$ meson will dissociate and be screened in the Quark Gluon Plasma[1]. The screening potential of $c\bar{c}$ pair depends on a separation length between $c$ and $\bar{c}$, $L$ which could have a value up to some maximum length $L_{\text{max}}$, called screening length, where beyond this length the screening potential becomes flat.

From the string theory point of view, a heavy quarks pair described by a Wilson loop is defined as an open string where both ends attached on the probe brane. Evaluating the Wilson loop will tell us information about the screening length dependent of the quark-antiquark potential. The procedure to evaluate the Wilson loop goes by extremizing the action of corresponding gravity dual theory [2, 3, 4]. A rigorous calculation on the screening length of a moving quark-antiquark pair at finite temperature four dimensional $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory from AdS/CFT correspondence was pioneered by Liu, Rajagopal, and Wiedemann [5]. It was then generalized to arbitrary dimension for conformal and non-conformal gauge theories by Cáceres, Natsuume, and Okamura [6]. The calculations there were done by going to reference frame of the moving quark-antiquark pair or explicitly by boosting the background metric to the direction of quark-antiquark’s velocity. A different approach was done in the reference frame of the plasma by Chernicoff, Garcia, and Guijosa [7] and furthermore they also calculated the energy from both
approaches. One interesting result of these calculations is the present of boost factor, \((1 - v^2)\), in the screening length. In Ref. [6], it was shown that the scale of this factor depends on the dimension of black hole backgrounds. However, this scaling factor is valid in ultra-relativistic limit and disagrees with numerical fitting that was found in Ref. [7].

In this article, we are interested in computing the screening length of a heavy quark-antiquark, \(q\bar{q}\), pair moving in the strongly coupled \( \mathcal{N} = 4 \) super-Yang-Mills plasma where the corresponding background metric is five dimensional AdS black hole in Poincare coordinate. Here, we consider the separation length to be parallel with the velocity. We further would like to make comparison with the computation in the reference frame of the \(q\bar{q}\) pair where the plasma is moving hence the background metric is a boosting black hole which can be obtained simply by boosting procedure as in Ref. [6].

2. Screening Length
For our purpose here the corresponding background metric is the five dimensional AdS black hole \([8]\)

\[
ds^2 = \frac{r^2}{R^2} \left( -h(r)dt^2 + \frac{R^2}{r^2} \frac{1}{h(r)} dr^2 \right) + R^2 \frac{1}{r^2} d\vec{x}^2,
\]

\[
h(r) = 1 - \frac{r_H^2}{r^4}, \quad T_H = \frac{r_H}{\pi R^2},
\]

where \(R\) is the radius of curvature of AdS space, \(T_H\) is the Hawking temperature, and \(r_H\) is the even horizon. We ignore the compact part of the space, \(S^5\), as it can be effectively removed in this case. This background metric in AdS/CFT dictionary corresponds to the strongly coupled \( \mathcal{N} = 4 \) super-Yang-Mills through the following relation \([10]\)

\[
\frac{R^4}{\alpha'^2} = \lambda = \frac{g_{YM}^2 N_c}{\pi}, \quad \lambda \gg 1,
\]

where \(\sqrt{\alpha'}\) is the fundamental string length scale, \(\lambda\) is the \(^{'}\)Hooft coupling, \(g_{YM}\) is the Yang-Mills coupling, and \(N_c\) is the number of color in \(SU(N_c)\) gauge symmetry.

The dynamic of the string is classically given by the Nambu-Goto action under the background (1),

\[
S = -\frac{1}{2\pi\alpha'} \int d\sigma^2 \sqrt{-|g_{\alpha\beta}|} g_{\alpha\beta} \equiv G_{\mu\nu} \partial_{\alpha} X^\mu \partial_\beta X^\nu,
\]

where \(\sigma^\alpha\) is the worldsheet coordinate, with \(\sigma^\alpha \equiv (\tau, \sigma)\), \(X^\mu(\sigma^\alpha)\) is the space-time coordinate where the string worldsheet is embedded, and \(G_{\mu\nu}\) is the space-time metric (1). From there, we obtain the equation of motion \([9]\)

\[
\nabla_{\alpha} P_{\mu}^\alpha = 0,
\]

\[
P_{\mu}^\alpha = \frac{\pi_{\mu}^\alpha}{\sqrt{-g}}, \quad \pi_{\mu}^\alpha = \frac{\delta S}{\delta \partial_{\alpha} X^\mu},
\]

where \(\pi_{\mu}^\alpha\) is the canonical worldsheet momentum with \(g = \det g_{\alpha\beta}\).

Using the usual gauge \(\tau = t\), while for \(\sigma\) will be determined later for convenient, we choose the string to be moving in the direction of \(x_3\)-axis with the following ansatz

\[
x_3(\sigma^\alpha) \equiv v \tau + x_3(\sigma),
\]

\[
r(\sigma^\alpha) \equiv r(\sigma),
\]

(6) (7)
thus the equations of motion are given by

\[ x'^2_3 \left( r^4 - r^4_H \right)' + r'^2 \left( \frac{r^4(1-v^2) - r^4_H}{r^4 - r^4_H} \right)' - 2r'\sqrt{-g} \frac{\partial}{\partial \sigma} \left[ \frac{r'^4 \left( r^4(1-v^2) - r^4_H \right)}{\sqrt{-g} r^4 - r^4_H} \right] = 0, \tag{8} \]

\[ \sigma \pi^\sigma_{x_3} = \frac{\partial}{\partial \sigma} \left[ \frac{r^4 - r^4_H}{\sqrt{-g} x'^3_3} \right] = 0, \tag{9} \]

with \( \pi^\sigma_{x_3} \) is constant and \( ' = \frac{\partial}{\partial \sigma} \). For quark-antiquark system, the solution for \( r \) must satisfy conditions

\[ r(-L/2) = r(L/2) = +\infty, \quad r'(\sigma_p) = 0, \tag{10} \]

where the length of string is \( L \) and \(-L/2 < \sigma_p < L/2\) is the turning point where \( r \) takes the minimum value \( r(\sigma_p) = r_p \geq r_H \). Using the symmetry of the ends of the string, we can set \( \sigma_p = 0 \). Another gauge that we might use is \( x_3(\sigma) = \sigma \) where both equations of motion are similar. We can simply choose a gauge \( x_3 = v\tau + \sigma \) in the first place which gives us the equation (8) that has more complicated form than equation (9).

As both equations are similar we prefer to use equation (9) which is simpler to find the separation length, now is precisely \( L \), of quark-antiquark pair. The solutions for \( r \) are given by

\[ r' = \pm \frac{r^4 - r^4_H}{P} \sqrt{\frac{r^4 - (P^2 + r^4_H)}{r^4(1-v^2) - r^4_H}}, \tag{11} \]

where \( P \equiv \pi^\sigma_{x_3} \) is a constant denoting amount of \( x_3 \)-component of momentum flow or transfer on the string. The force on the components of string must correspond to the direction of string motion which is in \( x_3 \)-coordinate thus we interpret the force proportional to \( \pi^\sigma_{x_3} \) instead of \( \pi^\sigma \). For quark-antiquark pair \( P > 0 \), where \( P = 0 \) would correspond to a straight string configuration, and the positive sign in (11) is proportional to the energy flows from the boundary down to \( r = r_p \) opposed to the negative sign. The quark-antiquark pair can be built out of two configurations (a) and (b) as shown in Figure 1 with

\[
\begin{aligned}
\frac{dr}{d\sigma} &= \begin{cases} 
\frac{r^4 - r^4_H}{P} \sqrt{\frac{r^4 - (P^2 + r^4_H)}{r^4(1-v^2) - r^4_H}}, & 0 < \sigma \leq L/2 \\
0, & \sigma = 0 \\
-\frac{r^4 - r^4_H}{P} \sqrt{\frac{r^4 - (P^2 + r^4_H)}{r^4(1-v^2) - r^4_H}}, & -L/2 \leq \sigma < 0 
\end{cases}.
\end{aligned}
\tag{12}
\]

Both solutions (11) only differ by a sign \( P \to -P \) therefore we might want to solve only one of them. At \( r(0) = r_p \), we must have \( r^4_p - (P^2 + r^4_H) = 0 \) which implies that \( r_p > r_H \) for \( P > 0 \). There is a critical radius \( r^4_c = \frac{r^4_H}{1-v^2} \) and it is also \( r_c > r_H \) for \( v \neq 0 \). If \( r_p > r_c \) then the geometry is cut-off below at \( r = r_p \) and string configuration for the quark-antiquark bound is formed. If \( r_p < r_c \) the geometry is also cut-off at \( r = r_c \) however this solution is not physical because the string is singular in the middle at \( r = r_c \) so we will ignore this solution. If \( r_p = r_c \) then the condition (10) could not be satisfied and the string configuration physically would be similar to the drag force configuration thus for quark-antiquark pair we must have \( r_p > r_c \) or \( P^2 > v^2 r^4_H \). So for some fixed velocity \( v \), or \( r_c \), the string configuration is characterized by amount of momentum transfer \( P \) on the string with \( P^2 \geq v^2 r^4_c \). At \( P^2 = P^2_c = v^2 r^4_c \), the string tends to represent a single quark and if we increase the momentum transfer \( P \) the quark-antiquark pair is formed.

From equation (11), we obtain an integral formula for computing separation length as below

\[
\int_0^{L/2} d\sigma = \frac{L}{2} = \int_{r_p}^\infty \frac{P}{r^4 - r^4_H} \sqrt{\frac{r^4(1-v^2) - r^4_H}{r^4 - (P^2 + r^4_H)}}. \tag{13}
\]
We cannot solve the integral above analytically so we are going to plot the integral numerically. The numerical solution of the separation length for various value of $v$ is shown in Figure 2. Our calculation here is in the same foot as in Ref. [6] for parallel case, $\theta = 0$, where they only discuss the ultra-relativistic limit and analytically compute the scaling factor of the screening length.

As one can see, we produce similar profile of screening length $L_s$ as in the perpendicular case, Ref. [7], except that the momentum transfer is bounded from below at $P^2 = P_c^2 \equiv v^2 r_c^4$, for fixed $v$, where below this value the integral (13) becomes complex. Therefore our plot do not start from $P/R^2 = 0$, which is different from the plot produced in Ref. [7], instead we start from $P = P_c$. Similar to the perpendicular case, the separation length in Figure 2 consist of two regions separated by the maximum $L_{\text{max}}$ called the screening length at the point $P = P_{\text{max}}$. For
$P_c < P < P_{\text{max}}$, the string configuration is metastable while for $P \geq P_{\text{max}}$ is stable, for detail discussion see Ref. [7].

Unlike the perpendicular case, this quark-antiquark pair of string configuration requires an external force to keep it moves with a constant velocity. The amount of required external force is proportional to $P_c$ suggested from the drag force computation[10]. The momentum flow $P$ on the string is bigger than the external force and so it is natural to think that the difference is related to the potential to keep quark-antiquark pair bounded. In this sense, the quark-antiquark pair still feels a drag force by the present of this external force. One can see from the Figure 2 the larger momentum flow $P$, for $P > P_{\text{max}}$, the smaller separation length thus one can consider the quark-antiquark pair effectively as a particle. In addition, since the tip of the string at $r_p$ is farther from the horizon and is closer to the boundary, $r_p \rightarrow \infty$, the quark-antiquark pair is insensitive to the presence of the plasma. The string has a very short length in a very large $P$ hence it might correspond to light quark-antiquark pair. This light quark-antiquark pair could be the constituent of other plasma located at the UV-brane, where the geometry is cut-off by the presence of UV-brane in order to remove the UV divergence and to make the quark mass, or correspondingly the string length, to be large but finite.

3. Total energy of the string

We are now interested to compute the total energy of the string. Energy density for the string in quark-antiquark pair setup above is given by [10]

$$\mathcal{H}_{\text{bound}} = 2T_0 \frac{(r^4 - r_H^4)^2 - r^4 P^2 v^2}{(r^4 - r_H^4) \sqrt{(r^4(1 - v^2) - r_H^4)(r^4 - r_H^4 - P^2)}}, \quad P^2 > v^2 r_c^4. \quad (14)$$

As usual the bound energy density is divergent at the boundary, $\sigma = \pm L/2$, therefore it must be substracted with the unbound energy density of two quark and antiquark moving with constant velocity $v$ where the energy density is

$$\mathcal{H}_{\text{unbound}} = 2T_0 \frac{(r^4 - r_H^4)^2 - r^4 P_c^2 v^2}{(r^4 - r_H^4) \sqrt{(r^4(1 - v^2) - r_H^4)(r^4 - r_H^4 - P_c^2)}}, \quad P_c^2 = v^2 r_c^4. \quad (15)$$

This unbound energy density will cancel the boundary divergence of the bound energy density with the cost of a new divergence at the horizon. As explained in Ref. [7], and also Ref. [10], this source of divergence coming from the unbound energy density because of infinite amount of energy flowing down from boundary, supplied by external force, to the horizon to keep the quark and antiquark move with constant velocity. Furthermore this also produce some ambiguity in removing this horizon divergence.

As resolution we need to compute the energy density in the rest frame of the moving string and this can be done by the following coordinate transformation [11]:

$$t \rightarrow \gamma (t - \nu x_3), \quad x_3 \rightarrow \gamma (x_3 - \nu t), \quad \gamma = (1 - v^2)^{-1/2}$$

with $\gamma$ is the boost factor. The resulting five dimensional “Boost-AdS” black hole is

$$ds^2 = -\frac{r^4 - r_H^4}{r^2 R^2} \gamma^2 (dt - \nu dx_3)^2 + \frac{r^2 R^2}{r^4 - r_H^4} dr^2 + \frac{r^2}{R^2} \gamma^2 (dx_3 - \nu dt)^2 + \frac{r^2}{R^2} (dx_1^2 + dx_2^2). \quad (17)$$

Taking the same gauge, $t = \tau$ and $x_3 = \sigma$, the static solution for radial coordinates, $r \equiv r(\sigma)$, is given by

$$r' = \frac{r^4 - r_H^4}{P \gamma} \sqrt{\frac{r^4 - (P^2 + r_H^4)}{r^4 \gamma^2 - r_H^4}}, \quad (18)$$
Figure 3: Plots of normalized energy for quark-antiquark pair $H/(2T_0R)$ as function of separation length $L/R$ for various velocity $v$ with fixed $r_H/R = 1$. The metastable configuration is colored with red while the stable configuration is colored with blue. The dashed black line is the saturated energy $H_{sat}$ where all possible energies in this line are forbidden for the quark-antiquark pair configuration.

where $P$ is again the same $x_3$-component of momentum transfer as in the corresponding static black hole case. One can immediately see from (18) that separation length will be different from the static case by a boost factor $\gamma$ thus the screening length as well, $L_{Boost} = \gamma L_{AdS}$. In this “Boost-AdS” black hole the bound energy density turns out to be

$$H_{bound} = 2T_0 \sqrt{\frac{r^4 - r_H^4 \gamma^2}{r^4 - r_H^4 - P^2}}, \quad P^2 > v^2 r_c^4.$$  

with the integral range $r_p \leq r < \infty$. The corresponding unbound energy density is simply written as

$$H_{unbound} = 2T_0 \sqrt{\frac{r^4 - r_H^4 \gamma^2}{r^4 - r_H^4 - P_c^2}}, \quad P_c^2 = v^2 r_c^4.$$  

(20)

As we can see the boundary divergence of the bound energy is removed by the unbound energy without introducing the horizon divergence. So, we can define a normalized energy for quark-antiquark pair

$$H = \int_{r_H}^{\infty} dr \ H_{bound} - \int_{r_H}^{\infty} dr \ H_{unbound}.$$  

(21)
Rewriting the normalized energy $H$ as a function of separation length $L$ instead of $P$ using formula (13), we plot numerically the normalized energy for various value of velocity $v$, see Figure 3. The characteristic behaviour of the normalized energy is similar as in Ref. [7] except that the normalized energy for fixed velocity $v$ has a minimum length in the metastable configuration, red line of Figure 3, given by the dashed black line. This dashed black line is produced by the saturated energy of the following function

$$H_{\text{sat}}(P) = -2T_0 (r_p - r_H).$$

(22)

4. Conclusion and Discussion

In section 2, we have computed the separation length as a function of momentum transfer $P$ and plotted it for various value of velocity $v$. In the plot of Figure 2, for fixed velocity $v$, the separation length has minimum momentum transfer $P_c > 0$ for $v \neq 0$. This is because $P$ is bounded from below where its value has to be bigger than the drag force momentum transfer, $P_c = vr_c^2$. Below this value there is no physical solution for string configuration of quark-antiquark pair. Therefore right after the momentum transfer $P$ passes this bound the quark-antiquark pair is formed at the boundary with non-zero separation length but this quark-antiquark pair is unstable or at least metastable. The reason is because the tip of the string in this configuration, located at $r_p$, is very close to $r_c$ where at $v = 0$, $r_c = r_H$. However, there is another configuration of string, which is much more stable than the previous one, with the same separation length at high momentum transfer $P$ where the tip of string, $r_p$, is far away from $r_c$. Both configurations separated by the maximum separation length called the screening length as discussed lengthy in Ref. [7]. An interesting line shown in Figure 2 is the line of bounded momentum transfer $P = P_c$ colored with black dashed. This line’s profile is mimicking the line’s profile of the separation length, especially for $v = 0$, where there is a maximum length. We also computed the bound energy density of the quark-antiquark pair. Similarly the normalized bound energy of quark-antiquark pair also has minimum length on its metastable configuration(red) as shown by dashed line in Figure 3.

Acknowledgments

This work is supported by Kompetitif-LIPI 2013 grant with project’s title: “Dinamika Fluida pada Energi Tinggi”.

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