Uniform Estimates for Oscillatory Integral Operators with Polynomial Phases

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Abstract

In this paper, we shall prove the uniform sharp $L^p$ decay estimates for a class of oscillatory integral operators with polynomial phases. By this one-dimensional result, we can use the rotation method to obtain uniform sharp $L^p$ estimates of certain higher-dimensional oscillatory integral operators.

Keywords Oscillatory integral operator, van der Corput’s Lemma, Uniform estimate

Mathematics Subject Classification 47G10, 44A05.

1 Introduction

In this paper, we mainly consider the stability of certain oscillatory integral operators. The issue of stability for oscillatory integrals includes two major cases: (i) stable estimates under a small perturbation of a given function; see Karpushkin [14], Phong-Stein-Sturm [24], Phong-Sturm [25] and Greenblatt [8]; (ii) uniform estimates over a large class of phases satisfying certain nondegeneracy conditions; see Carbery-Christ-Wright [1], Carbery-Wright [2], Christ-Li-Tao-Thiele [3], Phong-Stein-Sturm [26] and Gressman [12]. The second case of stability will be investigated for certain oscillatory integral operators with polynomial phases.

Consider oscillatory integral operators of form

$$ T_\lambda f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \varphi(x,y) f(y) dy, \quad (1.1) $$

where $\lambda$ is a real parameter and $\varphi$ is a smooth cut-off function. For nondegenerate phases $S$, Hörmander [13] established the optimal $L^2$ decay estimate; see also Phong-Stein [21, 22] and Phong-Stein-Sturm [26] for uniform $L^2$ decay estimates. For general degenerate real-analytic phases $S$, Phong-Stein [22] established the relation between the decay rate of the $L^2$ operator norm of $T_\lambda$ and the Newton distance of the phase $S$; see Phong-Stein [20, 21] for earlier related results. Rychkov [28] extended this result to most smooth phases and full generalizations to smooth phases were established by Greenblatt [3]. For higher dimensional analogues, Tang [34] and Greenleaf-Pramanik-Tang [9] obtained sharp $L^2$ decay estimates under certain genericity assumptions. Recently, Xiao [35] proved the full range of sharp $L^p$ decay estimate for $T_\lambda$; see also [36, 37, 31] for some related and earlier work. We also refer the reader to Greenleaf-Seeger [11] for a survey on degenerate oscillatory and Fourier integral operators.

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By imposing uniform positive lower bounds on certain mixed derivatives of the phase $S$, Carbery-Christ-Wright \cite{1} established uniform sharp growth estimates for sublevel set operators associated with $S$. In \cite{1}, the authors also obtained uniform decay estimates for oscillatory integral operators with non-sharp decay exponents except for some special cases; see also \cite{2}. Up to logarithmic terms, Phong-Stein-Sturm \cite{26} obtained uniform sharp $L^p$ estimates for a class of multilinear sublevel set operators and oscillator integral operators. In this paper, we shall remove the logarithmic terms in these estimates of \cite{26} for $T_\lambda$. The phase $S$ is a polynomial in $\mathbb{R}^2$ and takes the following form

$$S(x, y) = \sum_{k\eta + l = d} a_{k,l} x^k y^l$$

for two numbers $\eta, d > 0$. It is clear that $S$ has nonvanishing partial derivatives $\partial_x^k \partial_y^l S$ for positive integers $k, l$ with $a_{k,l} \neq 0$.

Now we state our main result in this paper.

**Theorem 1.1** Assume $S$ is a real-valued polynomial of form (1.2) in $\mathbb{R}^2$. Let $T_\lambda$ be an oscillatory integral operator as in (1.1). Then there exists a constant $C$, depending only on the cut-off $\varphi$ and the degree $\deg(S)$ of the phase $S$, such that for all positive integers $k, l$ satisfying $k\eta + l = d$,

$$\|T_\lambda f\|_{L^p} \leq C|a_{k,l}|^{-\frac{1}{k+\eta}} |\lambda|^{-\frac{1}{k+\eta+d}} \|f\|_{L^p}, \quad p = \frac{k + l}{k}. \quad (1.3)$$

Our proof of the theorem relies on some uniform damping estimates related to $T_\lambda$. Roughly speaking, one of the damping estimates is a uniform $L^2$ decay rate for oscillatory integral operators with damping factors related to the Hessian of the phase. Another damping estimate consists of uniform $L^1 \rightarrow L^{1,\infty}$, $H^1_E \rightarrow L^1$ and $L^1 \rightarrow L^1$ boundedness for certain damped oscillatory integral operators. Here $H^1_E$ is a variant of the Hardy space $H^1$ associated with the phase $S$; see Phong-Stein \cite{19}, Pan \cite{16}, Greenleaf-Seeger \cite{10} and Shi-Yan \cite{31}.

In this paper, the damped oscillatory integral operators with polynomial phases are of form

$$W_z f(x) = \int_{\mathbb{R}} e^{i z S(x,y)} |D(x,y)|^2 \varphi(x,y) f(y) dy, \quad (1.4)$$

where $D$ is the damping factor and $z$ is the damping exponent. When the damping factor $D$ is taken as the Hessian $S''_{xy}$, Phong-Stein \cite{21} \cite{22} proved the sharp $L^2$ decay estimate for $W_\lambda$ with the operator norm depending on upper bounds for $S''_{xy}$ together with its higher derivatives; see Seeger \cite{30} for decay $O(|\lambda|^{-1/2})$ with $\text{Re}(z) > 1/2$. As a consequence of our uniform $L^2$ damping estimates, we are able to establish the stability of the result in Phong-Stein \cite{22} for $W_z$ with polynomial phases of form (1.2). More precisely, we have the following theorem.

**Theorem 1.2** Assume $S$ is a real-valued polynomial of form (1.2). Let $W_z$ be defined as in (1.4) with $D(x,y) = S''_{xy}(x,y)$. Then there exists a constant $C = C(\deg(S))$, depending only on the degree of $S$, such that

$$\|W_z f\|_{L^2} \leq C(1 + |z|)^2 M |\lambda|^{-1/2} \|f\|_{L^2}$$

for all $z \in \mathbb{C}$ with $\text{Re}(z) = \frac{1}{2}$ and all $\varphi \in C_0^\infty(\mathbb{R}^2)$ satisfying

$$\sup_{\Omega} \sum_{k=0}^2 \left( (\delta_{\Omega,k}(x))^k |\partial_x^k \varphi(x,y)| + (\delta_{\Omega,v}(y))^k |\partial_y^k \varphi(x,y)| \right) \leq M,$$
where $M$ is a positive number and $\Omega$ is a horizontally and vertically convex domain such that the cut-off $\varphi$ is supported in $\Omega$. Here $\delta_{\Omega,h}(x)$ and $\delta_{\Omega,v}(y)$ denote the length of the cross sections $\{y : (x,y) \in \Omega\}$ and $\{x : (x,y) \in \Omega\}$, respectively.

With the rotation method and a variant of Stein-Weiss interpolation with change of measures, we can apply Theorem 1.1 to obtain some uniform $L^p$ estimates for higher dimensional oscillatory integral operators. In Section 6, we shall also discuss some examples related to a conjecture raised by Greenleaf-Pramanik-Tang in [9].

## 2 Preliminaries

In this section, we shall first present some basic notions concerning horizontally and vertically convex domains. With these useful notions, uniform $L^2$ estimates will be established for non-degenerate oscillatory integral operators which are supported in horizontally (vertically) convex domains. These decay estimates are often known as the operator version of van der Corput lemma; see [21, 22, 26]. Finally, we also include an interpolation result with change of power weights.

**Definition 2.1** Let $\Omega \subseteq \mathbb{R}^2$. We say that $\Omega$ is horizontally convex if $(x,z),(y,z) \in \Omega$ imply $(u,z) \in \Omega$ for all $x \leq u \leq y$. Similarly, $\Omega$ is said to be vertically convex if $(x,y),(x,z) \in \Omega$ imply $(x,v) \in \Omega$ for all $y \leq v \leq z$.

We shall give a simple relation between horizontal convexity and the following curved trapezoid.

**Definition 2.2** Assume $g$ and $h$ are two monotone functions on $[a,b]$ with $g \leq h$. Then we call

$$
\Omega = \{(x,y) \mid a \leq x \leq b, \; g(x) \leq y \leq h(x)\}
$$

a curved trapezoid.

It is clear that a curved trapezoid is vertically convex. By the monotonicity of $g,h$, one can verify that a curved trapezoid is also horizontally convex. If the monotonicity assumption on $g,h$ is dropped, the horizontal convexity for $\Omega$ is not generally true. But we have the following lemma.

**Lemma 2.1** Assume $g,h$ are two continuous functions satisfying $g \leq h$ on $[a,b]$. If $\Omega = \{(x,y) \mid a \leq x \leq b, \; g(x) \leq y \leq h(x)\}$ is horizontally convex, then $[a,b]$ can be divided into three subintervals $I_1,I_2,I_3$ with disjoint interiors such that each domain

$$
\Omega_i = \{(x,y) \mid x \in I_i, \; g(x) \leq y \leq h(x)\}
$$

is a curved trapezoid. In other words, both $g$ and $h$ are monotone on each $I_i$.

**Proof.** Choose a point $c \in [a,b]$ such that $h$ achieves its maximum at $c$. Then $h$ must be increasing on $[a,c]$ and decreasing on $[c,d]$. Otherwise, we can choose two points $a \leq x_1 < x_2 \leq c$ such that $h(x_1) > h(x_2)$. By the continuity of $h$, we can choose $x_3 \in [x_2,c]$ with $h(x_3) = h(x_1)$. Since $\Omega$ is horizontally convex, we find that $(u,z) \in \Omega$ for all $x_1 \leq u \leq x_3$ if $z = h(x_1)$. But this contradicts the fact $(x_2,z) \notin \Omega$. Thus $h$ does not decrease on $[a,c]$. By a similar argument, we can prove that $h$ is decreasing on $[c,b]$. 

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Assume $g$ attains its minimum at some point $d \in [a, b]$. By the above argument, it follows that $g$ is decreasing on $[a, d]$ and is increasing on $[d, b]$.

Without loss of generality, assume $c \leq d$. Then $I_i$ are obtained by taking $a, c, d, b$ as the endpoints of these intervals. The proof is therefore complete. $\square$

For horizontally and vertically convex domains, some notations will be used frequently.

- $\delta_{\Omega,h}(x)$: the length of the cross section $\{y : (x, y) \in \Omega \}$. Here the subscript $h$ means that $\delta_{\Omega,h}$ is a function of the horizontal variable.
- $\delta_{\Omega,v}(y)$: the length of the cross section $\{x : (x, y) \in \Omega \}$. The subscript $v$ suggests that $y$ is the vertical component.
- $\Omega^*_h$: a horizontally expanded domain of $\Omega$; see Definition 2.3
- $\Omega^*_v$: a vertically expanded domain of $\Omega$; see Definition 2.3
- $\Omega^*$: an expanded domain of $\Omega$ of form $\Omega^* = \Omega^*_h \cup \Omega^*_v$.
- $I_{\Omega,h}(x)$: the cross section $\{y : (x, y) \in \Omega \}$. The subscript $h$ means that $I_{\Omega,h}$ is taken with respect to the horizontal component.
- $I_{\Omega,v}(y)$: the cross section $\{x : (x, y) \in \Omega \}$. The subscript $v$ means that $I_{\Omega,v}$ is taken with respect to the vertical component.
- $a \wedge b$: the minimum of two real numbers $a$ and $b$.
- $a \vee b$: the maximum of $a$ and $b$.

Now we state the operator van der Corput lemma for nondegenerate oscillatory integral operators. The following lemma, with varying width of the “curve box”, was first established by Phong-Stein-Sturm in [26]. For convenience of readers, we also include a proof here.

**Lemma 2.2** Assume $S$ is a real-valued polynomial in $\mathbb{R}^2$. Let $T_\lambda$ be defined as in (1.7) with supp $(\varphi)$ contained in a curved trapezoid $\Omega = \{(x, y) \mid a \leq x \leq b, \ g(x) \leq y \leq h(x) \}$. Suppose $S$ satisfies the following conditions:

(i) For some $\mu, A_1 > 0$,
\[
\mu \leq \left| S_{xy}(x, y) \right| \leq A_1 \mu
\]
for all $(x, y) \in \Omega$.

(ii) We use $\delta_{\Omega,h}(x)$ to denote the length of the cross-section $I_{\Omega,h}(x) = \{y \mid (x, y) \in \Omega \}$. There exists a constant $A_2 > 0$ such that
\[
\sum_{k=0}^{2} \sup_{(x,y) \in \Omega} \left( \delta_{\Omega,h}(x) \right)^k |\partial_y^k \varphi(x, y)| \leq A_2.
\]

Then there exists a constant $C = C(\deg(S), A_1)$ such that
\[
\|T_\lambda f\|_{L^2} \leq C A_2 \|\lambda \mu|^{-1/2}\|f\|_{L^2}.
\]

**Proof.** We first assume that $g, h$ are increasing on $[a, b]$. We shall decompose $\Omega$ into a sequence of domains $\{\Omega_i\}$ such that these domains satisfy the almost orthogonality property. Then it suffices to treat one $T_i$ supported in $\Omega_i$, where $T_i$ is defined as $T_\lambda$ with insertion of the characteristic function $\chi_{\Omega_i}$ into the cut-off of $T_\lambda$. More precisely, the decomposition procedure is described as follows.

Let $x_0 = a$. Find a point $a \leq x_1 \leq b$ such that $g(x_1) = h(x_0)$. Then we choose $x_2 > x_1$ such that $g(x_2) = h(x_1)$. Generally, if $x_k$ is chosen, then select one $x_{k+1} > x_k$ such that
Then its transpose $D$. For $satisfies $g$, $h$ satisfying $g(b) = h(b)$ and $g(x) < h(x)$ for all $x \in [a, b]$, this decomposition process consists of infinite steps.

Let $\Omega_i = \{(x, y) \mid x_i \leq x \leq x_{i+1}, g(x) \leq y \leq h(x)\}$. Define $T_i$ as $T_\lambda$ in (1.1), but with the cut-off multiplied by $\chi_{\Omega_i}$. Then it is clear that $T = \sum T_i$. For $i, j$ satisfying $|i - j| \geq 2$, $T_i T_j^* = T_j^* T_i = 0$. The reason is that if $|i - j| \geq 2$ then the projections of $\Omega_i$ and $\Omega_j$ into the $x$–axis (also $y$–axis) are disjoint up to a set of measure zero. Write $T = \sum_{i \text{ even}} T_i + \sum_{i \text{ odd}} T_i$.

Observe that $\|T_i\| \leq \|T_i\|_*$ and $\|T_i\|_* \leq \sup_i \|T_i\|_*$. This observation implies $\|T\| \leq 2\sup_i \|T_i\|$. Thus it suffices to estimate one $T_i$. For simplicity, we may assume $a = x_i$ and $b = x_{i+1}$. Under this assumption, we have $h(a) \geq g(b)$. With some abuse of notation, we also write $T_\lambda$ instead of $T_i$.

Now we use the classical $TT^*$ method to prove our desired estimate. First, the kernel of $T_\lambda T_\lambda^*$ is given by

$$K(x, y) = \int e^{i\lambda[S(x, z) - S(y, z)]} \varphi(x, z) \overline{\varphi(y, z)} dz.$$ 

For $z$ satisfying $\varphi(x, z) \overline{\varphi(y, z)} \neq 0$, it follows from Assumption (i) that

$$\Phi(z) := \partial_z [S(x, z) - S(y, z)] = \int_y^x \partial_u \partial_z S(u, z) du$$

satisfies

$$|\Phi(z)| \geq \mu |x - y|.$$ 

For fixed $x, y$, define a differential operator $D$ as

$$Df(z) = (i\lambda)^{-1}[\partial_z S(x, z) - \partial_z S(y, z)]^{-1} f'(z).$$

Then its transpose $D^t$ is

$$D^t f(z) = \frac{\partial}{\partial z} \left[ -\frac{f(z)}{i\lambda(\partial_z S(x, z) - \partial_z S(y, z))} \right].$$

By integration by parts,

$$K(x, y) = \int_R D^2 \left( e^{i\lambda[S(x, z) - S(y, z)]} \right) \varphi(x, z) \overline{\varphi(y, z)} dz = \int_R e^{i\lambda[S(x, z) - S(y, z)]} (D^t)^2 (\varphi(x, z) \overline{\varphi(y, z)}) dz.$$

Let $\Delta_1$ and $\Delta_2$ be two curved trapezoids defined by

$$\Delta_1 = \{ (x, y) \mid a \leq x \leq b, h(a) \leq y \leq h(x) \},$$

$$\Delta_2 = \{ (x, y) \mid a \leq x \leq b, g(x) \leq y \leq h(a) \}.$$ 

For $z$ satisfying $\varphi(x, z) \overline{\varphi(y, z)} \neq 0$, we see that $z$ must belong to $[g(x), h(x)] \cap [g(y), h(y)]$ and that

$$\left| \frac{d}{dz} \left( \frac{1}{\Phi(z)} \right) \right| = \left| \frac{\Phi'(z)}{\Phi(z)^2} \right| \leq C \frac{\delta^{-1} \mu |x - y|}{(\mu |x - y|)^2} \leq C (\delta \mu |x - y|)^{-1},$$

where

$$\frac{d}{dz} \left( \frac{1}{\Phi(z)} \right) = \frac{\Phi'(z)}{\Phi(z)^2}.$$ 

\[ \text{5} \]
where \( \delta = (\delta_{\Delta_1, h}(x) \wedge \delta_{\Delta_1, h}(y)) \vee (\delta_{\Delta_2, h}(x) \wedge \delta_{\Delta_2, h}(y)) \) and \( C = C(deg(S), A_1) \). Similarly, we also have
\[
\left| \frac{d^2}{dz^2} \left( \frac{1}{\Phi(z)} \right) \right| = \left| \frac{\Phi''(z)}{\Phi(z)} - 2 \frac{\Phi'(z)^2}{\Phi(z)^2} \right| \leq C\delta^{-2}(|x-y|)^{-1}
\]
with the positive number \( \delta \) defined as above. Observe that \((D^t)^2(\varphi(x, z)\overline{\varphi}(y, z))\) is a linear combination of the following terms
\[
\frac{1}{(i\lambda)^2} \frac{d^{m_1}}{dz^{m_1}} \left( \frac{1}{\Phi(z)} \right) \frac{d^{m_2}}{dz^{m_2}} \left( \frac{1}{\Phi(z)} \right) \frac{\partial^{m_3} \varphi}{\partial z^{m_3}} \frac{\partial^{m_4} \overline{\varphi}}{\partial z^{m_4}}
\]
with nonnegative integers \( m_1 \) satisfying \( m_1 + m_2 + m_3 + m_4 = 2 \).

Combining above results together with Assumption (ii), we have
\[
|K(x, y)| \leq CA_2^2 \left( |\lambda| \overline{\mu} |x - y| \right)^{-2} \left[ (\delta_{\Delta_1, h}(x) \wedge \delta_{\Delta_1, h}(y)) \vee (\delta_{\Delta_2, h}(x) \wedge \delta_{\Delta_2, h}(y)) \right]^{-1}.
\]
Taking absolute value into the integral of the kernel \( K \), we see that
\[
|K(x, y)| \leq A_2^2 \left( \delta_{\Delta_1, h}(x) \wedge \delta_{\Delta_1, h}(y) + \delta_{\Delta_2, h}(x) \wedge \delta_{\Delta_2, h}(y) \right).
\]
These two inequalities imply
\[
|K(x, y)| \leq CA_2^2 \left[ \frac{\delta_{\Delta_1, h}(x)}{(1 + |\lambda| \overline{\mu} |x - y|)^2} + \frac{\delta_{\Delta_2, h}(y)}{(1 + |\lambda| \overline{\mu} \delta_{\Delta_2, h}(y)|x - y|)^2} \right].
\]
with \( \delta = (\delta_{\Delta_1, h}(x) \wedge \delta_{\Delta_1, h}(y)) \vee (\delta_{\Delta_2, h}(x) \wedge \delta_{\Delta_2, h}(y)) \).

Since \( g, h \) are increasing, it is clear that
\[
\delta_{\Delta_1, h}(x) \wedge \delta_{\Delta_1, h}(y) = \begin{cases} \delta_{\Delta_1, h}(x), & \text{if } x \leq y; \\ \delta_{\Delta_1, h}(y), & \text{if } y \leq x; \end{cases}
\]
and
\[
\delta_{\Delta_2, h}(x) \wedge \delta_{\Delta_2, h}(y) = \begin{cases} \delta_{\Delta_2, h}(y), & \text{if } x \leq y; \\ \delta_{\Delta_2, h}(x), & \text{if } y \leq x. \end{cases}
\]
If \( x \leq y \), then
\[
|K(x, y)| \leq CA_2^2 \left[ \frac{\delta_{\Delta_1, h}(x)}{(1 + |\lambda| \overline{\mu} \delta_{\Delta_1, h}(x)|x - y|)^2} + \frac{\delta_{\Delta_2, h}(y)}{(1 + |\lambda| \overline{\mu} \delta_{\Delta_2, h}(y)|x - y|)^2} \right].
\]
This implies
\[
\int_{x \leq y} |K(x, y)||f(y)g(x)|dxdy \leq CA_2^2 |\lambda\mu|^{-1} \left[ \int Mf(x)|g(x)|dx + \int |f(y)|Mg(y)dy \right] \leq CA_2^2 |\lambda\mu|^{-1}\|f\|_2\|g\|_2,
\]
where \( M \) is the Hardy-Littlewood maximal operator. In the case \( y \leq x \), the same estimate is true for the kernel \( K \). It follows immediately that
\[
\|T_\lambda\| \leq CA_2 |\lambda\mu|^{-1/2},
\]
where \( T_\lambda \) is the Hardy-Littlewood maximal operator.
where the constant $C$ depends only on $\deg(S)$ and $A_1$.

By the same argument as above, we can prove the desired estimate if $f, g$ are decreasing. There are two remaining cases: (i) $f$ is increasing and $g$ is decreasing; (ii) $f$ is decreasing and $g$ is increasing. In these two remaining cases, the proof is more direct since we need not decompose $\Omega$ into small pieces as above. In both cases (i) and (ii), the kernel $K$ is bounded by

$$|K(x, y)| \leq CA_2^2 \left[ \frac{\delta_{\Omega,x}(x)}{(1 + |\lambda| \mu \delta_{\Omega,x}(x)|x - y|)^2} + \frac{\delta_{\Omega,y}(y)}{(1 + |\lambda| \mu \delta_{\Omega,y}(y)|x - y|)^2} \right].$$

Invoking the maximal operator as above, we can also prove the desired estimate. The proof is therefore complete. \qed

Let $\Omega$ be a horizontally and vertically convex domain in $\mathbb{R}^2$. If its horizontal and vertical cross-sections are closed intervals, then there exist two numbers $a, b$ and functions $g, h$ such that $\Omega$ can be written as

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \ g(x) \leq y \leq h(x) \}. \quad (2.5)$$

Similarly, for some $c, d \in \mathbb{R}$ and functions $u, v$, the region $\Omega$ can also be given by

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, \ u(y) \leq x \leq v(y) \}. \quad (2.6)$$

Now we shall define an expanded domain $\Omega^*$ for which $\delta_{\Omega^*,x}$ and $\delta_{\Omega^*,y}$ are suitably larger than $\delta_{\Omega,x}$ and $\delta_{\Omega,y}$. Throughout the rest of this section, we always assume $\Omega$ is a horizontally and vertically convex domain in $\mathbb{R}^2$.

**Definition 2.3** We say that $\Omega^*_h$ is a horizontally expanded domain of $\Omega$, if there exists a positive number $\epsilon > 0$ and a nonnegative function $\eta$ on $[c, d]$ such that

$$\Omega^*_h = \{(x, y) \mid c \leq y \leq d, \ u(y) - \eta(y) \leq x \leq v(y) + \eta(y) \}$$

and for each $y$

$$\delta_{\Omega^*_h,y}(y) \geq (1 + 2\epsilon)\delta_{\Omega,y}(y).$$

Similarly, $\Omega^*_v$ is said to be a vertically expanded domain of $\Omega$ if there exists a number $\epsilon > 0$ and a nonnegative function $\gamma$ on $[a, b]$ such that

$$\Omega^*_v = \{(x, y) \mid a \leq x \leq b, \ g(x) - \gamma(x) \leq y \leq g(x) + \gamma(x) \}$$

and there holds

$$\delta_{\Omega^*_v,x}(x) \geq (1 + 2\epsilon)\delta_{\Omega,x}(x), \ x \in [a, b].$$

**Definition 2.4** The set $\Omega^*$ is said to be an expanded domain of $\Omega$, if there exist horizontally and vertically expanded domains $\Omega^*_h$ and $\Omega^*_v$, defined as in Definition 2.3, such that $\Omega^* = \Omega^*_h \cup \Omega^*_v$.

**Definition 2.5** Let $I$ be a bounded interval in $\mathbb{R}$. We use $I^*(B)$ to denote the concentric interval with length expanded by the factor $B > 0$.

We now state a useful lemma concerning expanded intervals.

**Lemma 2.3** Assume $I_1$ and $I_2$ are two bounded intervals in $\mathbb{R}$. Then if $I_1 \cap I_2 \neq \emptyset$, then we have $|I_1^*(B) \cap I_2^*(B)| \geq (B - 1)|I_1| \wedge |I_2|$ for $B > 1$ with $|I|$ being the length of the interval $I$. 
Proof. Assume $|I_1| \leq |I_2|$ and $I_1 \cap I_2 = (a,b)$. For simplicity, write $I_1$ as $I_1 = (c_1 - \delta_1, c_1 + \delta_1)$. By Definition 2.5, we see that

$$\left(a - \frac{B-1}{2} \delta_1, b + \frac{B-1}{2} \delta_1\right) \subseteq I_1^*(B) \cap I_2^*(B)$$

for $B > 1$. Hence $|I_1^*(B) \cap I_2^*(B)| \geq (B-1)\delta_1 = (B-1)|I_1|$. \hfill \qed

With the above preliminaries, we can now present the oscillation estimate to treat the almost orthogonality between two oscillatory integral operators. It should be pointed out that operators considered here are supported on horizontally (vertically) convex domains. For operators supported on curved trapezoids, the corresponding oscillation estimate was obtained by Phong and Stein in \[23\].

**Lemma 2.4** Let $S$ be a real-valued polynomial in $\mathbb{R}^2$ with degree $\deg(S)$. Assume $T_{\lambda}^{(1)}$ and $T_{\lambda}^{(2)}$ are defined as $T_{\lambda}$ in (1.22), but with the cut-off $\varphi$ replaced by $\varphi_1$ and $\varphi_2$ respectively. Suppose that $\supp(\varphi_i) \subseteq \Omega_i$ for two horizontally and vertically convex domains $\Omega_1$ and $\Omega_2$. Assume that all of the following conditions are true.

(i) For some $\mu$, $A > 0$, there exist expanded domains $\Omega_1^*$ and $\Omega_2^*$, defined as in Definition 2.4, such that

$$\mu \leq |S_{xy}'(x,y)| \leq A\mu, \quad (x,y) \in \Omega_1^*.$$  

(ii) For any horizontal line segment $L$ joining one point in $\Omega_1^*$ and another one in $\Omega_2^*$, the Hessian $S_{xy}''(x,y)$ does not change sign on $L$ and $\sup_{L} |S_{xy}''(x,y)| \leq A\mu$.

(iii) There exists a positive number $B$ such that for each $y$, we have

$$I_{\Omega_2^*,v}(y) \subseteq I_{\Omega_1^*,v}(y,B),$$

where $I_{\Omega_i^*,v}(y)$ is the horizontal cross-section of $\Omega_i^*$ at height $y$, i.e., $I_{\Omega_i^*,v}(y) = \{x : (x,y) \in \Omega_i^*\}$, and the interval $I_{\Omega_i^*,v}(y,B)$ has the same center as $I_{\Omega_i^*,v}(y)$, but its length is $B$ times as long as that of $I_{\Omega_i^*,v}(y)$.

(iv) For two positive numbers $M_1$ and $M_2$, it is true that

$$\sum_{k=0}^{2} \sup_{\Omega_i} (\delta_{\Omega_i,h}(x))^k |\partial^k_y \varphi_1(x,y)| \leq M_i, \quad i = 1, 2.$$  

Then there exists a constant $C$, depending only on $\deg(S)$, $A$ and the expanded factors $\epsilon$ appearing in the definition of $\Omega_1^*$ and $\Omega_2^*$ such that

$$\|T_{\lambda}^{(1)} T_{\lambda}^{(2)*}\|_{L^2 \to L^2} \leq CM_1 M_2 |\lambda \mu|^{-1}.$$  

**Proof.** Let $K$ be the kernel associated with $T_{\lambda}^{(1)} T_{\lambda}^{(2)*}$. Then $K$ can be written as

$$K(x,y) = \int_{\mathbb{R}} e^{i\lambda[S(x,z) - S(y,z)]} \varphi_1(x,z) \overline{\varphi_2(y,z)} dz.$$  

For those $z$ such that $\varphi_1(x,z) \overline{\varphi_2}(y,z) \neq 0$, we deduce form the assumptions (i), (ii) and (iii) that

$$\Phi(z) = \partial_z S(x,z) - \partial_z S(y,z)$$

\[2.7\]
satisfies

$$|\Phi(z)| = \left| \int_y^x \partial_u \partial_z S(u, z) du \right| \geq C \mu |x - y|,$$

where the constant $C$ depends only on the factor $B$ in the assumption (iii) and $\epsilon_1, \epsilon_2$ appearing in the definition of $\Omega_1^*, \Omega_2^*$. By integration by parts, we have

$$K(x, y) = \int_{\mathbb{R}} e^{i \lambda (S(x, z) - S(y, z))} \varphi_1(x, z) \overline{\varphi_2}(y, z) dz$$

$$= \int_{\mathbb{R}} D^t \left( e^{i \lambda (S(x, z) - S(y, z))} \right) \varphi_1(x, z) \overline{\varphi_2}(y, z) dz$$

$$= \int_{\mathbb{R}} e^{i \lambda (S(x, z) - S(y, z))} (D^t)^2 \left( \varphi_1(x, z) \overline{\varphi_2}(y, z) \right) dz,$$

where $D$ is the differential operator $Df(z) = [i \lambda \Phi(z)]^{-1} f'(z)$ and $D^t$ is its transpose, i.e., $D^t f(z) = -\partial_z [i \lambda \Phi(z)]^{-1} f(z)$. On the other hand, by the assumption (ii), we see that $|\Phi(z)| \leq A \mu$ for $z \in I_{\Omega_1, h}(x) \cap I_{\Omega_2, h}(y)$. If $I_{\Omega_1, h}(x) \cap I_{\Omega_2, h}(y) = \emptyset$, then $K(x, y) = 0$. Thus we need only consider those $x, y$ for which $I_{\Omega_1, h}(x) \cap I_{\Omega_2, h}(y) \neq \emptyset$. By Lemma 2.3, we have

$$|I_{\Omega_1, h}(x) \cap I_{\Omega_2, h}(y)| \geq |I_{\Omega_1, h}(x) \cap I_{\Omega_2, h}(y)|$$

$$\geq \epsilon |I_{\Omega_1, h}(x) \cap I_{\Omega_2, h}(y)|$$

with $\epsilon = 2 \epsilon_1 \wedge \epsilon_2$. Here $\epsilon_1, \epsilon_2$ are the expanded factors for $\Omega_1^*$ and $\Omega_2^*$. It follows from the polynomial property that

$$\left| \frac{d}{dz} \left( \frac{1}{\Phi(z)} \right) \right| = \left| -\frac{\Phi'(z)}{\Phi^2(z)} \right| \leq \frac{C \mu |x - y| |I_{\Omega_1, h}(x) \cap I_{\Omega_2, h}(y)|^{-1}}{(\mu |x - y|)^2}$$

$$\leq \frac{C \mu |x - y| (\delta_{\Omega_1, h}(x) \wedge \delta_{\Omega_2, h}(y))^{-1}}{(\mu |x - y|)^2}$$

for some constant $C = C(deg(S), \epsilon_1, \epsilon_2)$. Here the factor $\epsilon_i$ appears in the definition of $\Omega_i^*$. Likewise, it is also true that

$$\left| \frac{d^2}{dz^2} \left( \frac{1}{\Phi(z)} \right) \right| \leq \frac{|\Phi''(z)|}{(\Phi(z))^2} + 2 \frac{|\Phi'(z)|^2}{(\Phi(z))^3} \leq C \mu |x - y|^{-1} (\delta_{\Omega_1, h}(x) \wedge \delta_{\Omega_2, h}(y))^{-2}.$$
with \( a = \delta_{\Omega_1,h}(x) \) and \( b = \delta_{\Omega_2,h}(y) \). Thus for any \( f, g \in L^2 \),
\[
\int_{\mathbb{R}^2} |K(x, y)||f(y)g(x)|dxdy \leq CM_1M_2|\lambda\mu|^{-1} \left( \int g(x)|Mf(x)|dy + \int f(y)|Mg(y)|dy \right)
\]
\[
\leq CM_1M_2|\lambda\mu|^{-1}\|f\|_{L^2}\|g\|_{L^2},
\]
where \( M \) is the Hardy-Littlewood maximal operator, and the constant \( C \) depends on \( \text{deg}(S) \), \( \epsilon_1, \epsilon_2 \) and \( A \), but not on \( \lambda, \mu \) and \( f, g \). Thus we complete the proof of the lemma.

\[\square\]

**Remark 2.1** If we change the role of \( x \) and \( y \) in the lemma, we also have a similar estimate for the \( L^2 \) operator norm of \( T_\lambda^{(1)} T_\lambda^{(2)} \). Now we shall point out that the bounds \( M_1, M_2 \) appearing in the assumption (iv) can be slightly improved under certain additional conditions.

Assume all assumptions in the lemma are true. Suppose \( \phi_1 \) and \( \phi_2 \) are also supported in another two horizontally and vertically convex domains \( \Omega_i \) and \( \tilde{\Omega}_i \), respectively. Then we can improve the bound \( M_i \) by replacing \( \Omega_i \) by \( \Omega_i \cap \tilde{\Omega}_i \) in (iv) for \( i = 1, 2 \). For the proof of this improved result, we can follow the above argument line by line, but with \( \delta_{\Omega_i,h} \) replaced by \( \delta_{\tilde{\Omega}_i,r,h} \).

Now we shall present a uniform decay estimate for \( T_\lambda \) with non-sharp decay exponent. This result will be used in our proof of \( H^1_E \rightarrow L^1 \) boundedness for damped oscillatory integral operators.

**Lemma 2.5** Let \( T_\lambda \) be defined as in (1.7) with \( S \) being a real-valued polynomial in \( \mathbb{R}^2 \). For two positive integers \( j, k \), assume \( |\partial_x^j \partial_y^k S(x,y)| \geq 1 \) on the unit square \( U := \{|x| < 1/2, |y| < 1/2\} \). Then there exists a decay exponent \( \delta > 0 \), depending only on the degree of \( S \), such that we have for all \( \phi \in C_0^\infty(U) \)
\[
\|T_\lambda \phi\|_{L^2} \leq C|\lambda|^{-\delta}\|\phi\|_{L^2},
\]
where the constant \( C \) depends only on \( \text{deg}(S) \) and \( \phi \).

For the proof of this lemma with non-sharp \( \delta \), we refer the reader to Ricci-Stein [27] for the above estimate with \( \delta < \frac{1}{2\text{deg}(S)} \) and to Carbery-Christ-Wright [11] for a more general estimate (for smooth functions) with \( \delta = \frac{1}{2(j+k+1)} \). The maximal decay \( |\lambda|^{-\frac{1}{2(j+k)}} \) was proved by Phong-Stein-Sturm [26]. For further related topics, we refer the reader to see Christ-Li-Tao-Thiele [3] and Greenblatt [7] for uniform estimates under more general non-degeneracy concepts.

The following lemma is useful in the interpolation with change of power weights. Its formulation and proof are in many ways like the Stein-Weiss interpolation with change of measures; see Stein-Weiss [33], Pan-Sampson-Szeptycki [17] and Shi-Yan [31].

**Lemma 2.6** Assume \( T \) is a sublinear operator defined for simple functions in \( \mathbb{R} \) with Lebesgue measure. Suppose that there exist two constants \( A, B > 0 \) such that

(i) \( \|Tf\|_{L^\infty} \leq A\|f\|_{L^1} \) for all simple functions \( f \);

(ii) \( \|x^aTf\|_{L^{p_0}} \leq B\|f\|_{L^{p_0}} \), where \( a \in \mathbb{R} \), \( 1 < p_0 < \infty \) and \( a \neq -1/p_0 \).
Then for all $0 < \theta < 1$, there exists a constant $C = C(a, p_0, \theta)$ such that
\[
\left\|\left|x\right|^{\gamma}Tf\right\|_{L^p} \leq CA^{1-\theta}B^{\theta}\|f\|_{L^p}, \quad \gamma = -(1-\theta) + \theta a, \quad \frac{1}{p} = (1-\theta) + \frac{\theta}{p_0},
\]
where $f$ is an arbitrary simple function.

Proof. By the assumption (i), it is easy to see that $|x|^{-1}T$ is bounded from $L^1(dx)$ into $L^{1,\infty}(dx)$. Since $|x|^{\alpha}T$ is bounded on $L^{p_0}(dx)$, we can define an operator $Sf = |x|^bTf$ and a measure $d\mu = |x|^c dx$ such that $S$ is bounded from $L^1(dx)$ into $L^{1,\infty}(d\mu)$ and maps $L^{p_0}(dx)$ continuously into $L^{p_0}(d\mu)$.

We shall now need a simple fact that $|x|^\lambda$ belongs to $L^{1,\infty}(|x|^{-1-\lambda}dx)$ for any nonzero real number $\lambda$. By this fact, we see that $S$ is bounded from $L^1(dx)$ into $L^{1,\infty}(d\mu)$ provided that $b+c = -1$ and $b \neq 0$. On the other hand, using assumption (ii), we see that $S$ is bounded from $L^{p_0}(dx)$ into $L^{p_0}(d\mu)$ if $p_0b+c = p_0a$. We solve this two equations for $b$ and $c$, and then obtain
\[
b = \frac{p_0a + 1}{p_0 - 1}, \quad c = -1 - \frac{p_0a + 1}{p_0 - 1}.
\]

Recall that we have assumed $b \neq 0$, i.e., $a \neq -1/p_0$. By the Marcinkiewicz interpolation theorem, we have a constant $C = C(a, p_0, \theta)$ such that
\[
\left\|\left|x\right|^bTf\right\|_{L^p(|x|^c dx)} \leq CA^{1-\theta}B^{\theta}\|f\|_{L^p(dx)}
\]
with $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_0}$. Note that
\[
b + \frac{c}{p} = b + c \left(1 - \theta + \frac{\theta}{p_0}\right) = (1 - \theta)(b + c) + \theta \left(b + \frac{c}{p_0}\right) = -(1 - \theta) + \theta a.
\]
Then the desired inequality in the lemma follows immediately. \hfill \Box

3 Damped Oscillatory Integral Operators on $L^2$

Assume $S$ is a real-valued polynomial of the form (1.2). Then its Hessian $S''_{xy}$ can be written as
\[
S''_{xy}(x, y) = c_0x^\alpha y^\eta \prod_{i=1}^{N}(x - \alpha_i y^\eta)
\]
with $c_0 \neq 0$, $\alpha_1, \alpha_2, \cdots, \alpha_N \in \mathbb{C} \setminus \{0\}$ and $\eta > 0$. Since $\lambda$ and $S$ can be replaced by $\lambda c_0$ and $S/c_0$ in (1.1), we may assume $c_0 = 1$. By choosing $s$ indices $i_1 < i_2 < \cdots < i_s$, we define the corresponding damping factor $D$ of the form
\[
D(x, y) = x^\alpha(x - \alpha_{i_1} y^\eta)(x - \alpha_{i_2} y^\eta) \cdots (x - \alpha_{i_s} y^\eta).
\]
Let $W_z$ be the damped oscillatory integral operator
\[
W_zf(x) = \int_{\mathbb{R}} e^{i\lambda S(x, y)} |D(x, y)|^{\frac{1}{2}} \varphi(x, y)f(y)dy,
\]
where $z \in \mathbb{C}$ lies in an appropriate strip.

To estimate the $L^2$ norm of $W_z$ by Lemma 2.2 and Lemma 2.4, we also require that the damping function $|D(x, y)|^{\frac{1}{2}}$ should behave like a polynomial in $x$ and $y$. However both $D$ and $|D|^{\frac{1}{2}}$ are not polynomials generally. For this reason, we shall introduce the concept of polynomial type functions; see Phong-Stein [22, 23].
Definition 3.1 Assume a function $F$ is of class $C^N$ on an interval $J$. We say that $F$ is of polynomial type with order $N$ if there exists a constant $C$ such that

$$\sum_{k=0}^{N} |I|^k \sup_{x \in I} |F^{(k)}(x)| \leq C \sup_{x \in I} |F(x)|$$

holds for all intervals $I \subseteq J$.

A polynomial $P$ in $\mathbb{R}$ of degree $N$ is of polynomial type with order $N$ on any bounded interval $I$. We also need the following two facts related to the concept of polynomial type functions.

- Let $F$ be a polynomial type function of order $N$ on $J$. If $0 < \mu \leq |F(x)| \leq A \mu$ for some $\mu, A > 0$ and all $x \in J$, then $|F|^z$ is also of polynomial type with order $N$ on $J$ for any $z \in \mathbb{C}$.

- If $F$ is of polynomial type with order $N$ on some dyadic interval $J \subseteq \mathbb{R}^+$, then, for any $\eta > 0$, the function $G(x) = F(x^n)$ is also of polynomial type with order $N$ on $J^{1/\eta}$, where the interval $J^{1/\eta}$ is defined by $J^{1/\eta} = \{x^{1/\eta} : x \in J\}$.

Theorem 3.1 Let $S$ and $D$ be defined as above with $\eta \geq 1$. If $W_z$ is defined as in (3.10), then there exists a constant $C$, depending only on $\deg(S)$ and the cut-off $\varphi$, such that

$$\|W_z f\|_{L^2} \leq C(1 + |z|^2) \left( |\lambda| \prod_{k \notin \{i_1, \ldots, i_s\}} |\alpha_k| \right)^{-\gamma} \|f\|_{L^2}, \quad \gamma = \frac{1}{2(n + (N - s)\eta + 1)}$$

(3.11)

for $z \in \mathbb{C}$ with real part

$$\Re(z) = \frac{m + s - n - (N - s)\eta}{2(n + (N - s)\eta + 1)} \cdot \frac{1}{m + s}.$$  \hspace{1cm} (3.12)

More precisely, the constant $C$ can take the following form:

$$C(\deg(S)) \sup_{\Omega} \sum_{k=0}^{2} \left( (\delta_{\Omega,h}(x))^k |\partial_y^k \varphi(x,y)| + (\delta_{\Omega,v}(y))^k |\partial_x^k \varphi(x,y)| \right)$$

(3.13)

for all $\varphi$ supported in a horizontally and vertically convex domain $\Omega$. Here $C(\deg(S))$ is a constant depending only on the degree of $S$.

Remark 3.1 The assumption $\eta \geq 1$ is crucial in our proof for technical reasons. If $\eta < 1$, we shall instead change the role of $x$ and $y$.

Proof. Choose a smooth bump function $\Phi$ such that $\supp(\Phi) \subseteq [1/2, 2]$ and $\sum_j \Phi(x/2^j) = 1$ for $x > 0$. Define $W_{j,k}$ as $W_z$ by insertion of $\Phi(x/2^j)\Phi(y/2^k)$ into the cut-off. In other words,

$$W_{j,k} f(x) = \int_{\mathbb{R}} e^{i\lambda S(x,y)} |D(x,y)|^2 \Phi(x/2^j)\Phi(y/2^k) \varphi(x,y) f(y) dy.$$  \hspace{1cm} (3.14)

Here we only consider the operator $W_z$ in the first quadrant. Estimates in other quadrants can be treated similarly. We assume $z$ has real part in (3.12) and $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_N|$
throughout this proof.

**Step 1.** All \(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_k}\) are real numbers.

We first prove the theorem under the additional assumption \(\alpha_{i_l} \in \mathbb{R}\). One can see that our arguments are also applicable without essential change in presence of complex \(\alpha_{i_l}\); see Step 2.

**Case (i)** \(|\alpha_1|2^{(k-1)\eta} \geq 2^{j+2}\).

Assume first \(m = 0\). For fixed \(k\), we define \(W_k = \sum_j W_{j,k}\) with the summation taken over all \(j\) satisfying Case (i). Then \(W_k\) is supported in the rectangle \(R_k\): \(|x| \leq |\alpha_1|2^{(k-1)\eta-1}, 2^{k-1} \leq y \leq 2^{k+1}\). For \(|k-l| \geq 2, W_k W_l^* = 0\). Thus we may assume \(k \geq l\). Observe that the Hessian of \(S\) has the uniform upper and lower bounds on an expanded rectangle \(R_k^\ast\) of \(R_k\). More precisely, we have

\[
|S''(x,y)| \approx 2^{kn} \left[ \prod_{i=1}^N |\alpha_i| \right] 2^{Nk\eta}
\]

on the expanded rectangle \(R_k^\ast\): \(|x| \leq |\alpha_1|2^{(k-1)\eta-1}, 2^{k-1-\varepsilon} \leq y \leq 2^{k+1+\varepsilon}\) with \(\varepsilon > 0\) satisfying \(\varepsilon\eta \leq \frac{1}{4}\). It is easily verified that \(|x-\alpha_i y^\eta| \approx |\alpha_i|2^{\eta n}\) with bounds depending only on \(\eta\). Moreover \(S''(x,y)\) does not change sign on all vertical line segments joining two points in \(R_k^\ast\) and \(R_l^\ast\). We can apply Lemma 2.4 to get

\[
\|W_k^* W_l\| \leq C \left| \lambda \right| 2^{kn} \left[ \prod_{i=1}^N |\alpha_i| \right] 2^{Nk\eta}\]

By the size estimate of for each \(W_k\), we deduce from \(\|W_k^* W_l\| \leq \|W_k^\ast\| \|W_l\|\) that

\[
\|W_k^* W_l\| \leq C \left( 2^{kn} \prod_{t=1}^s |\alpha_{i_t}| \right) Re(z) \left| |\alpha_1|2^{\eta n}\right\}^{1/2} 2^{k/2} \left( 2^{kn} \prod_{t=1}^s |\alpha_{i_t}| \right) Re(z) \left( |\alpha_1|2^{\eta n}\right\}^{1/2} 2^{l/2}.
\]

Taking a convex combination of the above two estimates, we obtain

\[
\|W_k^* W_l\| \leq C \left[ \left( \prod_{i=1}^N |\alpha_i| \right)^{-\theta} \left( \prod_{t=1}^s |\alpha_{i_t}| \right)^{1-\theta} \right] \left( |\alpha_1|2^{\eta n}\right\}^{(1-\theta)/2} 2^{k(1-\theta)/2} \left( |\alpha_1|2^{\eta n}\right\}^{(1-\theta)/2} 2^{l(1-\theta)/2}.
\]

Take \(\theta = 2\gamma\). We collect terms involving \(\alpha_i\) in the above inequality and obtain

\[
\left( \prod_{i=1}^N |\alpha_i| \right)^{-\theta} \left( \prod_{t=1}^s |\alpha_{i_t}| \right)^{1-\theta} \leq \left( \prod_{i=1}^N |\alpha_i| \right)^{-\theta} \left( \prod_{t=1}^s |\alpha_{i_t}| \right)^{2 \Re(z) + (1-\theta)/s}.
\]
where we have used the assumption $|\alpha_1| \leq |\alpha_2| \leq \cdots |\alpha_N|$ and the fact
\[
2 \text{Re}(z) + \frac{1 - \theta}{s} = s - n - (N - s)\eta \cdot \frac{1}{s} + \frac{n + (N - s)\eta}{n + (N - s)\eta + 1} \cdot \frac{1}{s} = \theta.
\]

On the right side of (3.15), we add all exponents of $2^k$ and obtain
\[
-(n + N\eta)\theta + s\eta \text{Re}(z) + \frac{1 - \theta}{2} - \frac{1 - \theta}{2}
= -\frac{n + N\eta}{n + (N - s)\eta + 1} + s\eta \cdot \frac{1}{2[n + (N - s)\eta + 1]} \cdot \frac{1}{s} + \frac{n + (N - s)\eta}{2(n + (N - s)\eta + 1)} \cdot \frac{1}{s} \cdot \frac{n + (N - s)\eta}{n + (N - s)\eta + 1}
+ \frac{1}{2} \cdot \frac{n + (N - s)\eta}{n + (N - s)\eta + 1}
= -\frac{n + N\eta}{n + (N - s)\eta + 1} + \frac{1}{2} \cdot \frac{s\eta}{n + (N - s)\eta + 1} + \frac{1}{2} \cdot \frac{n + (N - s)\eta}{n + (N - s)\eta + 1}
= \frac{1}{2} \cdot \frac{n + N\eta}{n + (N - s)\eta + 1}.
\]
The exponent of $2^l$ in (3.15) equals
\[
s\eta \text{Re}(z) + \frac{1 - \theta}{2} - \frac{1 - \theta}{2}
= \frac{n + N\eta}{n + (N - s)\eta + 1} + \frac{n + (N - s)\eta}{2(n + (N - s)\eta + 1)} \cdot \frac{1}{s} \cdot \frac{n + (N - s)\eta}{n + (N - s)\eta + 1}
= \frac{1}{2} \cdot \frac{n + N\eta}{n + (N - s)\eta + 1}.
\]
Then the inequality (3.15) becomes
\[
\|W_k^*W_l\| \leq C(1 + |z|^2)|\lambda|^{\theta} \left( \prod_{k \notin \{i_1, \ldots, i_s\}} |\alpha_k| \right)^{-\theta} 2^{-|k - l|\delta}
\]
with $\theta = 2\gamma$ and $\delta = (n + N\eta)/[2(n + (N - s)\eta + 1)] > 0$.

Now we address Case (i) with $m > 0$. Observe that $W_{j,k}$ is supported in the rectangle $R_{j,k}$: $x \in [2^j - 1, 2^j + 1]$, $y \in [2^{k-1}, 2^{k+1}]$. For some $\epsilon > 0$ satisfying $\epsilon \leq \frac{1}{4N}$, we define the expanded rectangle
\[
R_{j,k}^*: 2^{j-3/2} \leq x \leq 2^{j+3/2}, \quad 2^{k-1-\epsilon} \leq y \leq 2^{k+1+\epsilon}.
\]
Then we can verify that all assumptions in Lemma 2.4 are true. Since $W_{j,k}^*W_{j',k'}^* = 0$ for $|k - k'| \geq 2$, we shall view $2^k$ and $2^{k'}$ as the same number. By Lemma 2.4, $W_{j,k}^*W_{j',k'}^*$ satisfies, assuming $j \geq j'$,
\[
\|W_{j,k}^*W_{j',k'}^*\| \leq C \left[ |\lambda|2^{jm}2^{kn} \left( \prod_{i=1}^{N} |\alpha_i| \right) 2^{kN\eta} \right]^{-1} 2^{jm} \left( \prod_{u=1}^{s} |\alpha_{i_u}| \right) 2^{ks\eta} \text{Re}(z) \frac{1}{z} \left[ 2^{j'm} \left( \prod_{u=1}^{s} |\alpha_{i_u}| \right) 2^{k's\eta} \right] \text{Re}(z)
\]
Since $\|W_{j,k}W_{j',k'}^*\| \leq \|W_{j,k}\| \|W_{j',k'}^*\|$, it follows from size estimates for each operator that

$$
\|W_{j,k}W_{j',k'}^*\| \lesssim \left[ 2^{jm} \left( \prod_{i=1}^{s} |\alpha_{i_u}| \right) \right]^{2kn} \left[ 2^{j'm} \left( \prod_{i=1}^{s} |\alpha_{i_u}| \right) \right]^{2k'sn} \Re(z) 2^{j/2} 2^{k/2} 2^{j'/2} 2^{k'/2}.
$$

We take a convex combination of the above two inequalities and obtain

$$
\|W_{j,k}W_{j',k'}^*\| \lesssim \left[ \lambda |2^{jm} 2^{kn} \left( \prod_{i=1}^{N} |\alpha_i| \right) \right]^{-\theta} \left[ 2^{jm} \left( \prod_{i=1}^{s} |\alpha_{i_u}| \right) \right]^{2k'sn} \Re(z) 2^{(1-\theta)/2} \times
$$

$$
2^{k(1-\theta)/2} \left[ 2^{j'm} \left( \prod_{i=1}^{s} |\alpha_{i_u}| \right) \right]^{2j(1-\theta)/2} 2^{k(1-\theta)/2} 2^{j(1-\theta)/2} 2^{k(1-\theta)/2}.
$$

Put $\theta = 2\gamma$. Since $|k-k'| \leq 1$, we can identify $k'$ with $k$ in this estimate. Then the exponent of $2^k$ is equal to

$$
\tau := -(n+N\eta)\theta + 2s\eta \Re(z) + 1 - \theta
$$

$$
= \frac{n + N\eta}{n + (N-s)\eta + 1} + \frac{2s\eta m + s - n - (N-s)\eta}{n + (N-s)\eta + 1} \cdot \frac{1}{2(m+s)} + \frac{n + (N-s)\eta}{n + (N-s)\eta + 1}.
$$

Recall that $|\alpha_{1}|2^{(k-1)\eta} \geq 2^{j+2}$ in Case (i). Inserting this estimate into (3.16) and collecting terms concerning $\alpha_i$, we obtain

$$
\left( \prod_{i=1}^{N} |\alpha_i| \right)^{-\theta} \left( \prod_{u=1}^{s} |\alpha_{i_u}| \right) \left| \alpha_1 \right|^{-\tau/\eta} \leq \left( \prod_{i=1}^{N} |\alpha_i| \right)^{-\theta} \left( \prod_{u=1}^{s} |\alpha_{i_u}| \right) \left( \prod_{u=1}^{s} |\alpha_{i_u}| \right)^{\frac{\tau}{s\eta}}
$$

$$
\leq \left( \prod_{i=1}^{N} |\alpha_i| \right)^{-\theta} \left( \prod_{u=1}^{s} |\alpha_{i_u}| \right)^{\theta}
$$

$$
\leq \left( \prod_{k \notin \{i_1, i_2, ..., i_s\}} |\alpha_k| \right)^{-\theta},
$$

where we have used the equality

$$
2 \Re(z) - \frac{\tau}{s\eta} = 2 \cdot \frac{m + s - n - (N-s)\eta}{2[n + (N-s)\eta + 1]} \cdot \frac{1}{m + s} + \frac{n + (N-s)\eta}{n + (N-s)\eta + 1} \cdot \frac{m}{m + s}
$$

$$
= \frac{1}{n + (N-s)\eta + 1} = \theta.
$$

At the same time, the exponent of $2^j$ becomes

$$
\frac{\tau}{\eta} - m\theta + m \Re(z) + 1 - \theta + \frac{2}{2}
$$

$$
= -\frac{1}{\eta} \cdot \frac{n + (N-s)\eta}{n + (N-s)\eta + 1} \cdot \frac{s\eta}{m + s} - \frac{m}{m + s} - \frac{n + (N-s)\eta + 1}{n + (N-s)\eta + 1} +
$$

$$
m \cdot \frac{m + s - n - (N-s)\eta}{2[n + (N-s)\eta + 1]} \cdot \frac{1}{m + s} + \frac{n + (N-s)\eta}{n + (N-s)\eta + 1}
$$

$$
= -\frac{s}{2(m+s)} \cdot (1 - \theta) - \frac{m}{2} \theta.
$$

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By direct calculation, we see that the exponent of $2^j$ is

$$m \Re(z) + \frac{1 - \theta}{2} = m \cdot \frac{m + s - n - (N - s)\eta}{2[n + (N - s)\eta + 1]} \cdot \frac{1}{m + s} + \frac{n + (N - s)\eta}{2[n + (N - s)\eta + 1]} \cdot \frac{m}{2}.$$ 

Combining these estimates, we get

$$\|W_{j,k} W^*_j, k'\| \lesssim (1 + |z|^2)|\lambda|^{-\theta} \left( \prod_{k \notin \{i_1, i_2, \ldots, i_s\}} |\alpha_k| \right)^{-\theta} 2^{-|j - j'|\delta}$$

with $\delta$ defined by

$$\delta = \frac{s}{2(m + s)} \cdot (1 - \theta) + \frac{m}{2} \cdot \theta > 0.$$

By the same argument, we can show that $W^*_j, k' = W_j^* W^*_j$ satisfies a similar estimate. We omit the details here.

**Case (ii)** $2^j \geq |\alpha_N|2^{(k+1)\eta} + 2$.

As in **Case (i)**, the argument is slightly different depending on whether $n = 0$. First consider the case $n = 0$. For each $j$, define $W_j = \sum_k W_{j,k}$ with the summation taken over all $k$ satisfying the condition in **Case (ii)**. Then $W_j$ is supported in the rectangle $R_j : 2^{j-1} \leq x \leq 2^{j+1}$, $0 \leq y \leq (|\alpha_N|^{-1}2^{j-2})^{1/\eta}$. Since $W_j^* W_{j'} = 0$ for $|j - j'| \geq 2$, it is enough to estimate $W_j^* W_{j'}$. Observe that $\|W_j W^*_j\| = \|W_j^* W_{j'}\|$. We may assume $j \geq j'$ in the following proof.

It should be pointed out that the almost orthogonality estimate in Lemma 2.4 is not applicable here since $D(x, y)$ is not a polynomial type function in $y$ on $0 \leq y \leq (|\alpha_N|^{-1}2^{j-2})^{1/\eta}$. However, if $s = N$ then $D$ is a polynomial in $\mathbb{R}^2$. Assume first $s < N$. Since both $D(x, y)$ and $|D(x, y)|^2$ are polynomial functions of order 2 with respect to $x$, we can apply Lemma 2.2 to obtain the following oscillation estimate

$$\|W_j\| \leq C(1 + |z|^2) \left( |\lambda|2^m2^{iN} \right)^{-1/2} \left( 2^m2^{is} \right)^{\Re(z)}.$$ 

By the Schur test, we obtain the size estimate

$$\|W_j\| \leq C \left( 2^{jm2^{is}} \right)^{\Re(z)} 2^{j/2} \left( |\alpha_N|^{-1}2^{j-2} \right)^{1/\eta}.$$ 

In the oscillation estimate, the exponent of $2^j$ is

$$-\frac{m + N}{2} + (m + s) \Re(z) = \frac{-m + N}{2} + (m + s)\left( \frac{\theta}{2} - \frac{1 - \theta}{2(m + s)} \right)$$

$$= \frac{-m + N}{2} - \frac{1 - \theta}{2} + (m + s)\frac{\theta}{2} < 0$$

because of $s < N$. On the other hand, the exponent of $2^j$ in the size estimate is

$$(m + s) \Re(z) + \frac{1}{2} \frac{1}{2\eta} = (m + s) \left( \frac{\theta}{2} - \frac{1 - \theta}{2(m + s)} \right) + \frac{1}{2} + \frac{1}{2\eta}$$

$$= (m + s)\frac{\theta}{2} + \frac{\theta}{2} + \frac{1}{2\eta} > 0.$$
Balancing the oscillation and size estimates, we will obtain
\[
\left\| \sum_j W_j \right\| \leq \sum_j \left\| W_j \right\| \leq C(1 + |z|^2) \left( |\lambda| \prod_{k \notin \{i, i_2, \ldots, i_s\}} |\alpha_k| \right)^{-\gamma} \|f\|_{L^2}.
\]

Now we turn to the case when \( D(x, y) \) is a polynomial type function in \( y \) on \([0, (|\alpha_N|^{-1}2^{j-1})^{1/\eta}]\); for example \( s = N \). One can also verify the assumptions in Lemma 2.4. Hence \( W_j W_j^* \) satisfies
\[
\left\| W_j W_j^* \right\| \leq C \left[ |\lambda|2^{jm2jN} \right]^{-1} \left( 2^{jm2j} \right) \text{Re} (z) \left( 2^{j'} m 2^{j'} \right) \text{Re} (z).
\]

By the Schur test, \( \|W_j W_j^*\| \) is bounded by
\[
\left\| W_j \right\| \left\| W_j^* \right\| \leq \left[ \left( 2^{jm2j} \right) \text{Re} (z) \right]^{2j/2} \left( |\alpha_N|^{-1}2^{j-2} \right)^{1/\eta} \left[ \left( 2^{jm2j} \right) \text{Re} (z) \right]^{2j'/2} \left( |\alpha_N|^{-1}2^{j'-2} \right)^{1/\eta}.
\]

A convex combination of these two estimates yields
\[
\left\| W_j W_j^* \right\| \leq \left[ \left( 2^{jm2jN} \right) \text{Re} (z) \right]^{-\theta} \left( 2^{jm2j} \right) \text{Re} (z) \times \left( 2^{j'} m 2^{j'} \right) \text{Re} (z) \times \left( 2^{j'(1-\theta)/2} \left( |\alpha_N|^{-1}2^{j'-2} \right)^{1/\eta} \right).
\]

In the above estimate, the exponent of \( 2^{j'} \) equals
\[
-(m + N)\theta + (m + s) \text{Re} (z) + \frac{1 - \theta}{2} + \frac{1 - \theta}{2\eta} \tag{3.17}
\]
\[
= -\frac{(m + N)}{(N - s)\eta + 1} + (m + s) \cdot \frac{(m + s - (N - s)\eta)}{2[(N - s)\eta + 1]} \cdot \frac{1}{m + s} + \frac{(N - s)}{2[(N - s)\eta + 1]} \cdot \frac{1}{m + s} + \frac{(N - s)}{2[(N - s)\eta + 1]}
\]
and the exponent of \( 2^{j'} \) is given by
\[
(m + s) \text{Re} (z) + \frac{1 - \theta}{2\eta} + \frac{1 - \theta}{2\eta} \tag{3.17}
\]
\[
= (m + s) \cdot \frac{m + s - (N - s)\eta}{2[(N - s)\eta + 1]} \cdot \frac{1}{m + s} + \frac{(N - s)\eta}{2[(N - s)\eta + 1]} + \frac{(N - s)}{2[(N - s)\eta + 1]}
\]
\[
= \frac{(m + N)}{2[(N - s)\eta + 1]}.
\]

On the other hand, we see that \((1 - \theta)/2\eta = (N - s)\theta\) and
\[
|\alpha_N|^{-\frac{1 - \theta}{2\eta}} \leq \left( \prod_{k \notin \{i, \ldots, i_s\}} |\alpha_k| \right)^{-\theta}.
\]
Almost orthogonality principle.

Assume we take a convex combination with \( R \). The size estimate is, using \( \lambda \) is a consequence of Plancherel theorem. Here we need only consider the case \( \delta > 0 \).

To obtain the almost orthogonality, we shall verify the exponent of 2 with \( k \).

The operator \( S'_{\alpha} \) is comparable to a fixed value on an expanded region \( R'_{j,k} \) which can be defined as in Case (i). Other assumptions in Lemma 2.4 are also true.

Assume \( j \geq j' \). By Lemma 2.3 there holds

\[
\|W_{j,k}W_{j',k'}^*\| \leq C \left|\lambda|2jm\theta|2jN\right|^{-1} \left[2jm\theta\right]^{\text{Re}(z)} \left[2jm\theta\right]^{\text{Re}(z)}.
\]

The size estimate is, using \( \|W_{j,k}W_{j',k'}^*\| \leq \|W_{j,k}\|\|W_{j',k'}^*\| \)

\[
\|W_{j,k}W_{j',k'}^*\| \leq C \left[2jm\theta\right]^{\text{Re}(z)} \left[2jm\theta\right]^{\text{Re}(z)}\left[2jm\theta\right]^{\text{Re}(z)}
\]

We take a convex combination with \( \theta = 2 \gamma \) and obtain

\[
\|W_{j,k}W_{j',k'}^*\| \leq \left[\lambda|2jm\theta|2jN\right]^{-\theta} \left[2jm\theta\right]^{\text{Re}(z)} \left[2jm\theta\right]^{\text{Re}(z)}\left[2jm\theta\right]^{\text{Re}(z)}
\]

All terms and their exponents are given as follows:

\[
2^j : \quad -(m + N)\theta + (m + s) \text{Re}(z) + \frac{1 - \theta}{2}; \\
2^{j'} : \quad (m + s) \text{Re}(z) + \frac{1 - \theta}{2}; \\
2^k : \quad -n\theta + \frac{1 - \theta}{2}, \\
2^{k'} : \quad \frac{1 - \theta}{2}.
\]

Since \( \left|k - k'\right| \leq 1 \) and \( 2^{k'} \geq (\left|\alpha N\right|^{-1}2^j)^{1/\eta} \), the product of two terms concerning \( 2^k \) and \( 2^{k'} \) is bounded by \( C(\left|\alpha N\right|^{-2})^{-n(1 - \theta)/\eta} \). Then the new exponent of \( 2^j \) becomes

\[
-(m + N)\theta + (m + s) \text{Re}(z) + \frac{1 - \theta}{2} + \frac{1 - \theta}{2} - n\theta + \frac{n\theta}{\eta} + (N - s)\theta
\]

\[
= \left(N + \frac{m}{2} - \frac{s}{2}\right)\theta.
\]

To obtain the almost orthogonality, we shall verify the exponent of \( 2^j \) equals \(-(N + \frac{m}{2} - s)\theta \).

In fact, this is true. By direct computation,

\[
-(m + N)\theta + (m + s) \text{Re}(z) + \frac{1 - \theta}{2}
\]

\[
= -(m + N)\theta + (m + s) \frac{\theta}{2} - \frac{1 - \theta}{2} + \frac{1 - \theta}{2}
\]

\[
= -\left(N + \frac{m}{2} - \frac{s}{2}\right)\theta.
\]
On the other hand, the bound concerning $\alpha_i$ is equal to $\langle |\alpha_N|^{-1} \rangle (-n\theta + 1 - \theta) / \eta$. Observe that $(-n\theta + 1 - \theta) / \eta = (N - s)\theta$. By the assumption $|\alpha_1| \leq \cdots \leq |\alpha_N|$, we see that $\langle |\alpha_N|^{-1} \rangle (-n\theta + 1 - \theta) / \eta$ is less than or equal to $\langle \prod_{k \notin \{i_1, \ldots, i_s\}} |\alpha_k| \rangle^{-\theta}$. Combining this fact with above results, we obtain

$$\|W_{j,k}W_{j',k'}^*\| \leq C \left( |\lambda| \prod_{k \notin \{i_1, \ldots, i_s\}} |\alpha_k| \right)^{-\theta} 2^{-|k-k'|/\delta}$$

with $\delta$ given by

$$\delta = \left( N + \frac{m}{2} - \frac{s}{2} \right) \theta.$$

A similar argument also shows that $W_{j,k}W_{j',k'}^*$ satisfies the same estimate.

Case (iii) $|\alpha_1| 2^{(k-1)\eta - 2} < 2^j < \alpha_N 2^{(k+1)\eta + 2}$.

For some large number $N_0 = N_0(\eta) > 0$, consider the intervals $I_i = (|\alpha_i| 2^{-2N_0}, |\alpha_i| 2^{2N_0})$ with $1 \leq i \leq N$. Since $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_N|$, we see that $I_i \cap I_{i+1} = \emptyset$ if and only if $|\alpha_{i+1}| / |\alpha_i| \geq 2^{4N_0}$.

Let $\{t_1, t_2, \cdots, t_a\} \subseteq \{1, 2, \cdots, N\}$ consist of all indices $i$ such that $|\alpha_{i+1}| / |\alpha_i| \geq 2^{4N_0}$. Of course, the set may very well be empty. We shall divide our proof into two subcases.

Subcase (a) $|\alpha_{t_r} 2^{(k+1)\eta + 2} \leq 2^j \leq |\alpha_{t_{r+1}} 2^{(k-1)\eta - 2}$ for some $1 \leq r \leq a$.

Define $\Xi_1 = \{1, 2, \cdots, t_r\}$ and $\Xi_2 = \{t_r + 1, t_r + 2, \cdots, N\}$. Then on the support of $W_{j,k}$, $|S_{x,y}''(x,y)|$ is comparable to

$$|S_{x,y}''(x,y)| \approx 2^{jm} 2^{kn} \left( \prod_{t \in \Xi_1} 2^j \right) \left( \prod_{t \in \Xi_2} |\alpha_t| 2^{kn} \right)$$

and the damping factor $D(x,y)$ has size

$$|D(x,y)| \approx 2^{jm} \left( \prod_{t \in \Xi_1 \cap \Theta} 2^j \right) \left( \prod_{t \in \Xi_2 \cap \Theta} |\alpha_t| 2^{kn} \right)$$

where $\Theta = \{i_1, i_2, \cdots, i_s\}$. Since $W_{j,k}W_{j',k'}^* = 0$ for $|k-k'| \geq 2$, we assume now $|k-k'| \leq 1$ and $j \geq j'$ without loss of generality. By Lemma 2.4, there holds

$$\|W_{j,k}W_{j',k'}^*\| \lesssim \left[ |\lambda| 2^{jm} 2^{kn} 2^j |\Xi_1| \left( \prod_{t \in \Xi_2} |\alpha_t| 2^{kn} \right) \right]^{-1} \left[ 2^{jm} 2^j |\Xi_1 \cap \Theta| \left( \prod_{t \in \Xi_2 \cap \Theta} |\alpha_t| 2^{kn} \right) \right]^{\Re(z)} \times \left[ 2^{jm} 2^j |\Xi_1 \cap \Theta| \left( \prod_{t \in \Xi_2 \cap \Theta} |\alpha_t| 2^{kn} \right) \right]^{\Re(z)}.$$
By the size estimate of each $W_{j,k}$, we have, using $\|W_{j,k}W_{j',k'}^*\| \leq \|W_{j,k}\|||W_{j',k'}^*||_2$,$$
abla W_{j,k}W_{j',k'}^*\| \lesssim \left[ 2^{jm/2} |\Xi_1\cap\Theta| \left( \prod_{t \in \Xi_2 \cap \Theta} |\alpha_t|2^k \right)^{\text{Re}(z)} \right] 2^{j/2} 2^{k/2} \times \left[ 2^{jm/2} |\Xi_1\cap\Theta| \left( \prod_{t \in \Xi_2 \cap \Theta} |\alpha_t|2^k \right)^{\text{Re}(z)} \right] 2^{j/2} 2^{k'/2}.$$With $\theta = 2\gamma$, we take a convex combination of the oscillation and size estimates. This yields$$\|W_{j,k}W_{j',k'}^*\| \leq C |\lambda|^{-\theta} 2^{a_j} 2^{k'} 2^{j'} 2^{k'_{j'}}^2,$$
abla(3.18)$$\text{where the above exponents are given as follows:}$$a_j = -(m + |\Xi_1|)|\theta| + (m + |\Xi_1 \cap \Theta|) \text{Re}(z) + \frac{1 - \theta}{2},$$d'_j = (m + |\Xi_1 \cap \Theta|) \text{Re}(z) + \frac{1 - \theta}{2},$$b_k = -n\theta - |\Xi_2| |\theta| + |\Xi_2 \cap \Theta| \text{Re}(z) \eta + \frac{1 - \theta}{2},$$b'_{k'} = |\Xi_2 \cap \Theta| \text{Re}(z) \eta + \frac{1 - \theta}{2}.$$By the assumption $|k - k'| \leq 1$, we see that$$2^{k} 2^{k'} \leq C 2^{k[2|\Theta \cap \Xi_2| \text{Re}(z) \eta + 1 - \theta - |\Xi_2| |\eta| - n\theta]}.$$Define $s_1 = |\Xi_1 \cap \Theta|$ and $s_2 = |\Xi_2 \cap \Theta|$. Then $s = s_1 + s_2$. Using $1 - \theta = [n + (N - s)\eta] \theta$, $$2|\Theta \cap \Xi_2| \text{Re}(z) \eta + 1 - \theta - |\Xi_2| |\eta| - n\theta = 2s_2 \eta \text{Re}(z) + (1 - \theta) - (N - t_r) |\eta| - n\theta = 2s_2 \eta \text{Re}(z) + (1 - \theta) - (N - s) |\eta| - (s - t_r) |\eta| - n\theta = 2s_2 \eta \text{Re}(z) + (t_r - s) |\eta| = -(s - t_r) |\eta|.$$Since $|\alpha_t|2^{2\eta} \leq 2^j, 2^{j'} \leq |\alpha_{t'+1}|2^{2\eta}$ and $|k - k'| \leq 1$, the right side of (3.19) can be divided into two terms, and each term can treated as follows:$$2^{k[t_r - s_1]\eta} \leq \left( \frac{1}{|\alpha_{t_r}2^{2j'}(t_r - s_1)\eta} \right)^{\text{Re}(z)} \text{and} \quad 2^{k\eta(\theta - 2 \text{Re}(z))} s_2 \leq \left( \frac{1}{|\alpha_{t_r+1}|2^{2j}} \right)^{-(\theta - 2 \text{Re}(z))s_2}.$$Substituting this estimate into (3.18), we obtain the new exponent of $2^j$:$$-(\theta - 2 \text{Re}(z)) s_2 + a_j = -(\theta - 2 \text{Re}(z)) s_2 - (m + t_r) \theta + (m + s_1) \text{Re}(z) + \frac{1 - \theta}{2} = -s_2 \theta + 2s_2 \text{Re}(z) - (m + s_1) \theta - (t_r - s_1) \theta + (m + s) \text{Re}(z) - s_2 \theta + \frac{1 - \theta}{2} = -(m + s) \theta + s_2 \text{Re}(z) - (t_r - s_1) \theta + (m + s) \theta/2.$$
Note that \(- (m + s) \theta / 2 + s_2 \Re (z) = -(m + s) \Re (z) - \frac{1 - \theta}{2} + s_2 \Re (z)\). It follows from the above equality that
\[
-(\theta - 2 \Re (z)) s_2 + a_j = -(m + s) \Re (z) - \frac{1 - \theta}{2} + s_2 \Re (z) - (t_r - s_1) \theta \\
= - (m + s_1) \Re (z) - \frac{1 - \theta}{2} - (t_r - s_1) \theta .
\]
It is easy to see that the new exponent of \(2^{j'}\) is
\[
(t_r - s_1) \theta + a_j = (t_r - s_1) \theta + (m + |\Xi_1 \cap \Theta|) \Re (z) + \frac{1 - \theta}{2} \\
= (t_r - s_1) \theta + (m + s_1) \Re (z) + \frac{1 - \theta}{2}.
\]
Combining the above estimates, we obtain
\[
\|W_{j,k} W_{j',k'}^*\| \leq C (1 + |z|^2) \lambda^{-\delta - |j-j'|^\delta}
\]
where
\[
\delta = (t_r - s_1) \theta + (m + s_1) \Re (z) + \frac{1 - \theta}{2} \\
= (t_r - s_1) \theta + \frac{m + s_1}{2} \cdot \theta - \frac{m + s_1}{2(m + s)} \cdot (1 - \theta) + \frac{1 - \theta}{2} \\
= (t_r - s_1) \theta + \frac{m + s_1}{2} \cdot \theta + \frac{s_2}{2(m + s)} (1 - \theta).
\]
Note that if \(\{t_1, \ldots, t_a\} = \emptyset\) then we need only consider Subcase (b) below. Hence we can assume \(t_r \geq 1\). It follows immediately that \(\delta > 0\). The constant \(C\) is bounded by a constant multiple of \(\left(\prod_{k \in \mathcal{E}} |\alpha_k|^\theta\right)^{-\delta}\). Its dependence on \(\alpha_i\) comes from (3.18) and (3.20), and we can verify this claim as follows,
\[
\left(\prod_{\Xi_2} |\alpha_t|\right)^{-\theta} \left(\prod_{\Xi_2 \cap \Theta} |\alpha_t|\right)^{2 \Re (z)} \left(\frac{1}{|\alpha_{t_r}|}\right)^{(t_r - s_1) \theta} \left(\frac{1}{|\alpha_{t+1}|}\right)^{(\theta - 2 \Re (z)) s_2} \leq \left(\prod_{t \in \Theta} |\alpha_t|\right)^{-\theta}
\]
where we have used the following two inequalities:
\[
|\alpha_{t_r}|^{-(t_r - s_1) \theta} \leq \left(\prod_{t \in \Xi_2 \cap \Theta} |\alpha_t|\right)^{-\theta} \quad \text{and} \quad |\alpha_{t+1}|^{(\theta - 2 \Re (z)) s_2} \leq \left(\prod_{t \in \Xi_2 \cap \Theta} |\alpha_t|\right)^{\theta - 2 \Re (z)}.
\]
Therefore we have obtained the desired estimate in Subcase (a).

**Subcase (b)** \(|\alpha_{t+1}|2^{(k-1)\eta - 2} < 2^l < |\alpha_{t+1}|2^{(k+1)\eta + 2}\) for some \(0 \leq r \leq a\).

We use \(t_0\) to denote \(t_0 = 0\). Let \(G_0 = \{t_r + 1, t_r + 2, \ldots, t_{r+1}\}\) and \(\Theta = \{i_1, i_2, \ldots, i_b\}\). Choose \(e_0 \in G_0\) arbitrarily. For clarity, set \(e_0 = t_r + 1\), the least number in \(G_0\). Define \(W_{j,k}^{G_0, t_0}\) as \(W_{j,k}\), but with the cut-off multiplied by \(\Phi\left(\sigma_0 \frac{x - \alpha_{t_0} y^\nu}{2\sigma_0}\right)\). Here \(\sigma_0\) takes either + or −.
Note that $|\alpha_{t+1}|/|\alpha_t| \leq 2^{4N_0}$. By the almost orthogonality principle, there exists a constant $C$, depending on $N_0$, such that
\[
\left\| \sum_{j,k} W_{j,k} \right\| \leq C \sup_{j,k} \|W_{j,k}\|,
\]
where both the summation and the supremum are taken over all $(j, k)$ satisfying Subcase (b). Associated with fixed $j, k, l_0$, we can decompose $G_0$ into the following three subsets:
\[
\begin{align*}
G_{1,1} &= \{i \in G_0 \mid |\alpha_i - \alpha_{e_0}|2^{kn} \geq 2^{l_0+N_0}\} \\
G_{1,2} &= \{i \in G_0 \mid |\alpha_i - \alpha_{e_0}|2^{kn} \leq 2^{l_0-N_0}\} \\
G_{1,3} &= \{i \in G_0 \mid 2^{l_0-N_0} < |\alpha_i - \alpha_{e_0}|2^{kn} < 2^{l_0+N_0}\}.
\end{align*}
\]
If $G_{1,3}$ is empty, then our decomposition is finished. Otherwise, choose the least number $e_1$ in $G_{1,3}$, and define $W_{j,k,l_0}^{\sigma_1}$ as $W_{j,k,l_0}^{\sigma}$ by inserting $\Phi \left( \frac{x-\alpha_{e_i}y}{2^l} \right)$ into the cut-off of $W_{j,k,l_0}^{\sigma}$, where $\sigma = +$ or $-$. Since the number of $l_0$ satisfying $|\alpha_{e_1} - \alpha_{e_0}|2^{kn} \approx 2^{l_0}$ is bounded by a constant $C(N_0)$, there exists a constant $C$, depending only on $N_0$, such that
\[
\left\| \sum_{l_0} W_{j,k,l_0}^{\sigma} \right\| \leq C \sup_{l_0} \|W_{j,k,l_0}^{\sigma}\|,
\]
where the summation and superemum are taken over all $l_0$ satisfying $|\alpha_{e_1} - \alpha_{e_0}|2^{kn} \approx 2^{l_0}$. Further decompose $G_{1,3}$ as follows:
\[
\begin{align*}
G_{2,1} &= \{i \in G_{1,3} \mid |\alpha_i - \alpha_{e_1}|2^{kn} \geq 2^{l_1+N_0}\} \\
G_{2,2} &= \{i \in G_{1,3} \mid |\alpha_i - \alpha_{e_1}|2^{kn} \leq 2^{l_1-N_0}\} \\
G_{2,3} &= \{i \in G_{1,3} \mid 2^{l_1-N_0} < |\alpha_i - \alpha_{e_1}|2^{kn} < 2^{l_1+N_0}\}.
\end{align*}
\]
If $G_{2,3}$ is nonempty, we continue this decomposition procedure and obtain three subsets $G_{3,1}$, $G_{3,2}$, $G_{3,3}$ of $G_{2,3}$.

Generally, if $G_{i,1}, G_{i,2}, G_{i,3}$ are given with $G_{i,3} \neq \emptyset$, then we shall choose the least element $e_i$ of $G_{i,3}$ and decompose $G_{i,3}$, for each fixed $l_i$, as follows:
\[
\begin{align*}
G_{i+1,1} &= \{t \in G_{i,3} \mid |\alpha_t - \alpha_{e_i}|2^{kn} \geq 2^{l_i+N_0}\} \\
G_{i+1,2} &= \{t \in G_{i,3} \mid |\alpha_t - \alpha_{e_i}|2^{kn} \leq 2^{l_i-N_0}\} \\
G_{i+1,3} &= \{t \in G_{i,3} \mid 2^{l_i-N_0} < |\alpha_t - \alpha_{e_i}|2^{kn} < 2^{l_i+N_0}\}.
\end{align*}
\]
Then we obtain three disjoint subsets $G_{i+1,1}, G_{i+1,2}, G_{i+1,3}$. This process will continue until $G_{w,3} = \emptyset$ for some $w$. We shall prove later that this process will terminate in finite steps.

Although our construction of $G_{i,1}, G_{i,2}, G_{i,3}$ depends on $l_{i-1}$, we may regard $\{G_{i,1}, G_{i,2}, G_{i,3}\}$ as a three-tuple satisfying three properties:

(i) $G_{i,u} \cap G_{i,v} = \emptyset$ for $u \neq v$;

(ii) if $G_{i,3}$ is nonempty then so is $G_{i+1,2}$;

(iii) $G_{i-1,3} = G_{i,1} \cup G_{i,2} \cup G_{i,3}$ with $G_{0,3} = G_0$. 

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These properties imply $|G_{i+1,3}| \leq |G_{i,3}| - 1$ provided that $G_{i,3}$ is nonempty. Thus the above decomposition process stops in finite steps.

Since $G_0$ is a finite set, $|G_0| = t_{r+1} - t_r \leq N$, the number of all three tuples $(G_{i,1}, G_{i,2}, G_{i,3})$, for each $i \geq 1$, is bounded by a constant depending only on $N$. Therefore, in each step, we can divide the summation over $l_i$ into a finite summation by restricting $l_i$ in $(G_{i,1}, G_{i,2}, G_{i,3})$. In other words, $l_i$ satisfies inequalities associated with $G_{i,u}$; for example, inequalities associated with $G_{i,1}$ are $|\alpha_t - \alpha_{e_i}|2^{kn} \geq 2^{l_i + N_0}$, where $t \in G_{i,1}$ and $e_i$ is the least member in $G_{i,2}$. This observation will simplify our proof of the decay estimate of $W_{j,k,l,\ldots,l_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}}$.

Assume $W_{j,k,l,\ldots,l_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}}$ and $W_{j,k,l,\ldots,l'_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}}$ are two operators such that $l_{w-1}$ and $l'_{w-1}$ lie in the same three tuple $(G_{w-1,i})^3_{i=1}$. Now we shall verify the assumptions in Lemma 2.4. First observe that $W_{j,k,l,\ldots,l_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}}$ is supported in the intersection of $\Omega \supseteq \text{supp}(\varphi)$ and the following horizontally (also vertically) convex domain

$$\Omega_{j,k,l,\ldots,l_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}} = \text{Closure of} \left\{ (x,y) : \Phi \left( \frac{x}{2^t} \right) \Phi \left( \frac{y}{2^k} \right) \prod_{t=0}^{w-1} \Phi \left( \frac{\sigma_t x - \alpha_{e_1} y^\eta}{2^t} \right) = 0 \right\}.$$

Define an expanded region $\Omega_{j,k,l,\ldots,l_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}}$ by

$$2^{j-1} - \epsilon 2^j \leq x \leq 2^{j+1} + \epsilon 2^j, \quad 2^{k-1} - \epsilon 2^k \leq y \leq 2^{k+1} + \epsilon 2^k,$$

$$2^{l-1} - \epsilon 2^l \leq \sigma_t (x - \alpha_{e_1} y^\eta) \leq 2^{l+1} + \epsilon 2^l, \quad 0 \leq t \leq w - 1$$

for some sufficiently small $\epsilon = \epsilon(\eta) > 0$. Observe that $W_{j,k,l,\ldots,l_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}} = 0$ unless $l_{w-1} \leq \min\{j, l_0, \ldots, l_{w-1}\} + C(\eta)$. One can verify that all conditions in Lemma 2.4 are true for $W_{j,k,l,\ldots,l_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}}$ and $W_{j,k,l,\ldots,l'_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}}$.

Assume $l_{w-1} \geq l'_{w-1}$. By Lemma 2.4

$$\| W_{j,k,l,\ldots,l_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}} \| \leq \| W_{j,k,l,\ldots,l'_{w-1}}^{\sigma_0,\ldots,\sigma_{w-1}} \|$$
In this convex combination, the bound involving $k\gamma$ where the last two terms are upper bounds for the measure of $u$ with

\[ \prod_{t \in G_{w,1} \cap \Theta} |\alpha_t - \alpha_{e_{w-1}}| 2^{k\eta} \prod_{t \in G_{w,2} \cap \Theta} 2^{l_{w-1}} \times \]

\[ 2^{l_{w-1}/2} \left( \frac{2^{l_{w-1}}}{|\alpha_{e_{w-1}}| 2^{k(\eta-1)}} \right)^{1/2} \]

where the last two terms are upper bounds for the measure of

\[ \left\{ y \mid \Phi \left( \frac{y}{2^k} \right) \Phi \left( \frac{x - \alpha_{e_{w-1}} y^\eta}{2u} \right) \neq 0 \right\} \]

with $u \in \{l_{w-1}, l'_{w-1}\}$. A convex combination of the above oscillation and size estimates with $\theta = 2\gamma$ yields the following terms and their exponents:

- $2^j$: $2(m + |\Xi_1 \cap \Theta|) \Re(z) - (m + |\Xi_1|)\theta$,
- $2^k$: $-n\theta - (|G_{1,1}| + |G_{2,1}| + \cdots + |G_{w,1}| + |\Xi_2|)\eta\theta + 2 (|G_{1,1} \cap \Theta| + |G_{2,1} \cap \Theta| + \cdots + |G_{w,1} \cap \Theta| + |\Xi_2 \cap \Theta|) \eta \Re(z) - (\eta - 1)(1 - \theta)$,
- $2^{l_0}$: $2|G_{1,2} \cap \Theta| \Re(z) - |G_{1,2}|\theta$,
- $2^{l_{w-2}}$: $2|G_{w-1,2} \cap \Theta| \Re(z) - |G_{w-1,2}|\theta$,
- $2^{l_{w-1}}$: $-|G_{w,2}|\theta + |G_{w,2} \cap \Theta| \Re(z) + 1 - \theta$,
- $2^{l'_{w-1}}$: $|G_{w,2} \cap \Theta| \Re(z) + 1 - \theta$.

In this convex combination, the bound involving $\alpha_1$ is

\[ \prod_{t \in G_{1,1} \cap \Theta} |\alpha_t - \alpha_{e_0}| \prod_{t \in G_{1,1} \cap \Theta} |\alpha_t - \alpha_{e_1}| \cdots \prod_{t \in G_{1,1} \cap \Theta} |\alpha_t - \alpha_{e_{w-1}}| \prod_{t \in \Xi_2} |\alpha_t| \]

\[ |\alpha_{e_{w-1}}|^{-(1-\theta)} \times \]

\[ \prod_{t \in G_{1,1} \cap \Theta} |\alpha_t - \alpha_{e_0}| \prod_{t \in G_{1,1} \cap \Theta} |\alpha_t - \alpha_{e_1}| \cdots \prod_{t \in G_{1,1} \cap \Theta} |\alpha_t - \alpha_{e_{w-1}}| \prod_{t \in \Xi_2 \cap \Theta} |\alpha_t| \]

Define $s_1 = |\Xi_1 \cap \Theta|$, $s_2 = \{|t_r + 1, \ldots, t_{r+1}\} \cap \Theta|$ and $s_3 = |\Xi_2 \cap \Theta|$. It is clear that $s = s_1 + s_2 + s_3$. Now we shall deal with the term $2^{k\gamma_k}$ with $\gamma_k$ given as above. The definition of $G_{i,1}$ implies that, for $1 \leq t \leq w$,}

\[ 2^{k(-|G_{1,1}|\theta + 2|G_{1,1} \cap \Theta| \Re(z)\eta)} \lesssim \left( \frac{2^{l_{t-1}}}{\min_{u \in G_{1,1}} |\alpha_u - \alpha_{e_{t-1}}|} \right)^{|G_{1,1}|\theta + 2|G_{1,1} \cap \Theta| \Re(z)\eta}, \]

since $-|G_{1,1}|\theta + 2|G_{1,1} \cap \Theta| \Re(z)\eta \leq 0$. Our decomposition of $G_0$ implies that there exists an absolute constant $C$ such that $l_i \leq j + C N_0$ and $l_{i+1} \leq l_i + C N_0$ for each $i$. With the restriction
of \(j, k\) in **Subcase (b)**, it is true that

\[
2^k(-\Theta_2\Theta + 2) < \left( \frac{2^{1/2}}{\alpha_{r+1}} \right) ^{ -2\Theta_2\Theta + 2).
\]

For the same reason as well as the assumption \(\eta \geq 1\), there also holds

\[
2^{k[-n\theta-(\eta-1)(1-\theta)]} < \left( \frac{2^{1/2}}{\alpha_{r+1}} \right) ^{ -\frac{n\theta-\frac{2}{\eta}(1-\theta)}{\gamma}}.
\]

Since \(2|G_{i-2}\cap \Theta|Re(z) - |G_{i-2}| \leq 0\), for \(0 \leq i \leq w-2\), we also have

\[
2^{\ell_2 \Theta} < \left( \frac{2^{1/2}}{\alpha_{r+1}} \right) ^{ -\frac{n\theta-\frac{2}{\eta}(1-\theta)}{\gamma}}.
\]

Inserting these inequalities into the above estimate for \(\|W_{\sigma_0,\ldots,\sigma_{w-1}}^{w_{1,0,\ldots,1}} \cdot W_{j,k,l_{w-1}}^{j_{1,0,\ldots,1}}\|\), we see that the resulting exponent of \(2^{\ell_2 \Theta}\) equals

\[
2(m + |\Xi_1\cap \Theta|)Re(z) - (m + |\Xi_1|\theta - \frac{n\theta}{\eta} - \sum_{i=1}^{w} |G_{i,1}| + |\Xi_2|)\theta + \\
2 \left( \sum_{i=1}^{w} (G_{i,1} \cap \Theta) + |\Xi_2| \cap \Theta| \right) Re(z) - \frac{(\eta - 1)}{\eta} (1-\theta) + \\
\sum_{i=1}^{w-1} (2|G_{i,2} \cap \Theta| Re(z) - |G_{i,2}| \theta - |G_{w,2}| \theta + |G_{w,2} \cap \Theta| Re(z) + 1 - \theta) = (2m + 2s - |G_{w,2} \cap \Theta|)Re(z) - (m + s)\theta
\]

\[
= (m + s)\theta - (1 - \theta) - |G_{w,2} \cap \Theta| Re(z) - (m + s)\theta = -|G_{w,2} \cap \Theta| Re(z) - (1 - \theta).
\]

The new bound involving \(\alpha_i\) is bounded by a constant multiple of

\[
\left( \prod_{t \in \Xi_2} |\alpha_t| \right)^{ -\theta} \left( \prod_{t \in \Xi_2 \cap \Theta} |\alpha_t| \right)^{2Re(z) \alpha_{r+1}|^b},
\]

where

\[
b = -(1-\theta) + |\Xi_2|\theta - 2|\Xi_2 \cap \Theta| Re(z) + \frac{n\theta}{\eta} + \frac{\eta - 1}{\eta} (1-\theta)
\]

\[
= -(1-\theta) + |\Xi_2 \cap \Theta|\theta + |\Xi_2 \setminus \Theta|\theta - 2|\Xi_2 \cap \Theta| Re(z) + \frac{n\theta}{\eta} + (1-\theta) - \frac{n + (N-s)\eta}{\eta} \theta
\]

\[
= |\Xi_2 \cap \Theta|\theta + |\Xi_2 \setminus \Theta|\theta - 2|\Xi_2 \cap \Theta| Re(z) - (N-s)\theta, \text{ using } N-s = |\Xi_1 \setminus \Theta| + |G_0 \setminus \Theta| + |\Xi_2 \setminus \Theta|,
\]

\[
= |\Xi_2 \cap \Theta|\theta - 2|\Xi_2 \cap \Theta| Re(z) - |\Xi_1 \setminus \Theta|\theta - |G_0 \setminus \Theta|\theta.
\]

This implies

\[
\left( \prod_{t \in \Xi_2} |\alpha_t| \right)^{ -\theta} \left( \prod_{t \in \Xi_2 \cap \Theta} |\alpha_t| \right)^{2Re(z) \alpha_{r+1}|^b} \leq \left( \prod_{t \in \Theta} |\alpha_t| \right)^{ -\theta}.
\]
Now we address the case \( \eta < d \) with we first choose indices \( 1 \leq i_1, \ldots, i_s \leq N \). Here \( C \) as in the theorem. Indeed, we can deduce from Remark 2.1 that all above constants \( C \) can obtain a similar estimate for \( W^{\sigma_{i_1}, \ldots, \sigma_{i_s}}_{j,k,l_0,i_1,\ldots,i_s} \). By further decomposition as in Step 1, the proof applies without change and the desired estimate follows.

**Step 2. Existence of complex \( \alpha_i \)**

Assume \( \alpha_{i_0} \) has nonzero imaginary part for some \( 1 \leq t_0 \leq s \). For the two cases \( 2^j+2 \leq |\alpha_{i_0}|2^{(k+1)\eta} \) and \( 2^j \geq |\alpha_{i_0}|2^{(k+1)\eta}+2 \), the earlier arguments in Step 1 can apply without any change. So we need only consider the range \( 2^j \approx |\alpha_{i_0}|2^{kn} \). Note that

\[
|x - \alpha_{i_0}y| \approx |x - \text{Re} (\alpha_{i_0})y| + |\text{Im} (\alpha_{i_0})|y|^n.
\]

If \( |\text{Re} (\alpha_{i_0})| \leq |\text{Im} (\alpha_{i_0})| \), then \( |x - \alpha_{i_0}y| \approx |\alpha_{i_0}|2^{kn} \). The earlier arguments also produce the desired estimate. Assume now \( |\text{Re} (\alpha_{i_0})| > |\text{Im} (\alpha_{i_0})| \). By balancing the oscillation and size estimates, we see that only **Subcase (b) of Case (iii)** requires a separate treatment. Indeed, if a further decomposition, near the root \( x = \alpha_{i_0}y^\eta \) of the Hessian \( S''_{xy} \), is required, we shall insert \( \Phi (\sigma \text{Re}(\alpha_{i_0})y^\eta) \) into \( W_{j,k} \). With insertion of this cut-off, we have

\[
|x - \alpha_{i_0}y| \approx \max \{2^j, |\text{Im} (\alpha_{i_0})|2^{kn}\}.
\]

By further decomposition as in **Step 1**, with \( \text{Re} (\alpha_{i_0}) \) in place of \( \alpha_{i_0} \), the Hessian is bounded from both below and above by the same bound up to a multiplicative constant. The previous proof applies without change and the desired estimate follows.

It remains to show that all constants \( C \) appearing in above estimates have more precise form as in the theorem. Indeed, we can deduce from Remark 2.4 that all above constants \( C \) can take the following form

\[
C(\text{deg}(S)) \sup_{\Omega} \sum_{k=0}^2 \left( (\delta_{\Omega,h}(x))^k |\partial_x^k \varphi(x,y)| + (\delta_{\Omega,v}(y))^k |\partial_x^k \varphi(x,y)| \right).
\]

Here \( C(\text{deg}(S)) \) is a constant depending only on the degree of \( S \).

Combining all above results, we have completed the proof of the theorem. \( \square \)

As mentioned in Remark 3.1 the assumption \( \eta \geq 1 \) is necessary in the proof of Theorem 3.1. Now we address the case \( \eta < 1 \). Let \( \nu = \eta^{-1} > 1 \). Since \( S \) is a polynomial, (3.8) can be rewritten as

\[
S''_{xy}(x,y) = d_0 x^n y^n \prod_{i=1}^M (x^\nu - \beta_i y)
\]

with \( d_0, \beta_i \in \mathbb{C}\setminus\{0\} \). Without loss of generality, we may assume \( c_0 = 1 \). For some \( 0 \leq s \leq M \), we first choose indices \( 1 \leq i_1 < \cdots < i_s \leq N \). Then define the damping factor \( D \) as

\[
D(x,y) = x^m \prod_{t=1}^s (x^\nu - \beta_{i_t} y).
\]
Here we may take \( s = 0 \) and then define \( D(x, y) = x^m \). The following theorem can be regarded as a variant of Theorem 3.1.

**Theorem 3.2** Assume \( S \) is a real-valued polynomial such that its Hessian is given by (3.21) with \( \nu \geq 1 \). Let \( W_z \) be defined as in (3.10). Then there exists a constant \( C \), depending only on \( \deg(S) \) and \( \varphi \), such that

\[
\| W_z f \|_{L^2} \leq C(1 + |z|^2) \left( |\lambda| \prod_{k \notin \{i_1, \ldots, i_s\}} |\beta_k| \right)^{-\gamma} \| f \|_{L^2}, \quad \gamma = \frac{1}{2(n + M - s + 1)},
\]

where \( z \in \mathbb{C} \) has real part

\[
\text{Re}(z) = \frac{m + s\nu - n - (M - s)}{2(n + M - s + 1)} \cdot \frac{1}{m + s\nu}.
\]

**Remark 3.2** The proof of this theorem is the same as that of Theorem 3.1. For this reason, we omit the details here. It should be pointed out that the constant \( C \) in the above theorem can also take the form in Theorem 3.1. By the same argument as above, we can also prove Theorem 3.2 with uniformity on both the phases and the cut-off functions.

We conclude this section with a \( L^2 \) damping estimate for \( W_{j,k} \) in (3.11). Assume

\[
\Theta = \{ i_1, i_2, \ldots, i_s \} = \{ t_r + 1, t_r + 2, \ldots, t_{r+1} \}
\]

where \( t_1, t_2, \ldots, t_a \) are defined as in Case (iii) of the proof of Theorem 3.1. In the case \( 2^i \approx |\alpha_{i_a}|2^{k_n} \), we define the damping factor

\[
D(x, y) = \left( |\lambda| |\alpha_{i_a}|^{-1} 2^{-k(\eta-1)} A \right)^{-\frac{1}{2\eta}} + \prod_{t=1}^{s} |x - \alpha_{i_t} y|^\eta,
\]

where \( A \) is a positive number given by

\[
A = 2^{mj} 2^{kn} 2^{j_t} \left( \prod_{t=t_{r+1}+1}^{N} |\alpha_t|^{2k_n} \right).
\]

**Theorem 3.3** Assume \( m = 0 \) in (3.8) and \( \{ i_1, i_2, \ldots, i_s \} = \{ t_r + 1, t_r + 2, \ldots, t_{r+1} \} \) for some \( 0 \leq r \leq a - 1 \) and \( \|\alpha_{i_a}\|^2 (k-1)^{\eta-2} \leq 2^j \leq \|\alpha_{i_a}\|^{2(k+1)^{\eta+2}} \). Let \( W_{j,k} \) be the damped oscillatory integral operator (3.14) with \( D(x, y) \) defined by (3.21). Then there exists a constant \( C \) of form (3.13) such that the decay estimate (3.11) is still true for \( W_{j,k} \) with the damping exponent \( z \in \mathbb{C} \) having real part (3.12).

**Proof.** With \( 2^j \approx |\alpha_{i_a}|2^{k_n} \), \( S_{xy}'' \) behaves like \( A \prod_{t=1}^{s} (x - \alpha_{i_t} y) \) on the support of \( W_{j,k} \). As in the proof of Theorem 3.1 we can decompose \( W_{j,k} \) as

\[
W_{j,k} = \sum_{\sigma \in \Sigma} W_{j,k,l_0,l_1,\ldots,l_{w-1}}^{\sigma_0,\sigma_1,\ldots,\sigma_{w-1}},
\]

On the support of each operator \( W_{j,k,l_0,l_1,\ldots,l_{w-1}}^{\sigma_0,\sigma_1,\ldots,\sigma_{w-1}} \), the Hessian \( S_{xy}'' \) (also \( D \)) is bounded from both below and above by the same bound up to a multiplicative constant.
If $D$ has size comparable to $\prod_{t=1}^s |x - \alpha_i y^t|$ on the support of $W_{r,\sigma_1,\cdots,\sigma_{w-1}}^{\sigma_0,\sigma_1,\cdots,\sigma_{w-1}}$, then all its partial derivatives also have the same upper bounds as $\prod_{t=1}^s (x - \alpha_i y^t)$. With this observation, the desired estimate can be proved by the same argument as in our proof of Theorem 3.1.

It remains to consider these operators $W_{r,\sigma_1,\cdots,\sigma_{w-1}}^{\sigma_0,\sigma_1,\cdots,\sigma_{w-1}}$ for which $D$ has size

$$|D(x, y)| \approx \left( |\lambda||\alpha_i|^{-1} 2^{-k(\eta-1)} A \right)^{-\frac{1}{2}}. $$

In other words, it suffices to prove that the desired estimate holds if

$$\prod_{t=1}^s |x - \alpha_i y^t| \leq \left( |\lambda||\alpha_i|^{-1} 2^{-k(\eta-1)} A \right)^{-\frac{1}{s+2}}. $$

(3.26)

Since $\alpha_1, \cdots, \alpha_s$ have equivalent sizes and $y$ is restricted in the dyadic interval $[2^{k-1}, 2^{k+1}]$, we use the Schur test to obtain, by estimating the length of horizontal and vertical cross-sections of the support of $W_{r,\sigma_1,\cdots,\sigma_{w-1}}^{\sigma_0,\sigma_1,\cdots,\sigma_{w-1}}$, the following two size estimates:

$$\left\| \sum_{l_w=1}^{s} W_{r,\sigma_1,\cdots,\sigma_{w-1}}^{\sigma_0,\sigma_1,\cdots,\sigma_{w-1}} f \right\|_{L^2} \leq C \left( |\lambda||\alpha_i|^{-1} 2^{-k(\eta-1)} A \right)^{-\frac{1}{s+2}} \left( |\alpha_i| 2^k \right)^{-\frac{1}{2}} \|f\|_{L^2}$$

and

$$\left\| \sum_{l_w=1}^{s} W_{r,\sigma_1,\cdots,\sigma_{w-1}}^{\sigma_0,\sigma_1,\cdots,\sigma_{w-1}} f \right\|_{L^2} \leq C \left( |\lambda||\alpha_i|^{-1} 2^{-k(\eta-1)} A \right)^{-\frac{1}{s+2}} \text{Re}(z) 2^{j/2} \|f\|_{L^2},$$

where the above summations are taken over all $l_w=1$ such that the inequality (3.26) is true on the support of $W_{r,\sigma_1,\cdots,\sigma_{w-1}}^{\sigma_0,\sigma_1,\cdots,\sigma_{w-1}}$. Let $\gamma$ be given by (3.11) and define $\theta$ as follows.

$$\theta = \frac{n + (N - s)\eta + 2}{2[n + (N - s)\eta + 1]} = \frac{1}{2} + \gamma.$$  

Note that $\gamma = \frac{s}{s+2} \text{Re}(z) + \frac{\theta}{s+2}$. A convex combination of the above estimates yields

$$\left\| \sum_{l_w=1}^{s} W_{r,\sigma_1,\cdots,\sigma_{w-1}}^{\sigma_0,\sigma_1,\cdots,\sigma_{w-1}} f \right\|_{L^2} \leq C \left( |\lambda||\alpha_i|^{-1} 2^{-k(\eta-1)} A \right)^{-\gamma} \left( |\alpha_i| 2^k \right)^{-\eta/2} \left( 2^{j/2} \right)^{1-\gamma} \|f\|_{L^2},$$

where the exponents of $2^j$ and $2^k$ are given by

$$2^j:\ a_j = -t_r \gamma + \frac{1}{2}(1-\theta),$$

$$2^k:\ b_k = [(\eta - 1) - n - (N - t_{r+1})\eta] \gamma - (\eta - 1)\theta/2 + (1 - \theta)/2.$$  

By direct calculation, we have $\eta a_j + b_k = 0$. In fact, using $\theta = \frac{1}{2} + \gamma$, $1 - \theta = \frac{1}{2} - \gamma$ and $s = t_{r+1} - t_r$, one can see that $\eta a_j + b_k$ equals

$$-t_r \gamma + \frac{1}{2}(1-\theta)\eta + [(\eta - 1) - n - (N - t_{r+1})\eta] \gamma - (\eta - 1)\theta/2 + (1 - \theta)/2$$

$$= [(\eta - 1) - n - (N - t_{r+1})\eta - t_r\gamma] \gamma + \frac{1}{2} \cdot \left( \frac{1}{2} - \gamma \right) \cdot \eta - (\eta - 1) \cdot \frac{1}{2} + \frac{1}{2} \cdot \left( \frac{1}{2} - \gamma \right) \cdot \frac{1}{2}$$

$$= [(\eta - 1) - n - (N - s)\eta] \gamma + \frac{1}{2} \cdot \left( \frac{1}{2} - \gamma \right) \cdot \eta - \frac{1}{2} \cdot \left( \frac{1}{2} + \gamma \right) \cdot \eta + \frac{1}{2} \cdot \left( \frac{1}{2} + \gamma \right) + \frac{1}{2} \cdot \left( \frac{1}{2} - \gamma \right)$$

$$= \eta \gamma - \frac{1}{2} - \gamma + \frac{1}{2}$$

$$= 0.$$  

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Since sizes of $\alpha_i$ are equivalent, the bound concerning $\alpha_i$ in the resulting estimate is given by

$$|\alpha_i|^{-t_r \gamma} \left( \prod_{t=t_r+1}^{N} |\alpha_t| \right)^{-\gamma} |\alpha_i|^{-\theta/2} |\alpha_i|^{(1-\theta)/2} L$$

$$\leq |\alpha_i|^{-t_r \gamma} \left( \prod_{t=t_r+1}^{N} |\alpha_t| \right)^{-\gamma} |\alpha_i|^{1/2-\theta} \text{ with } \theta = \frac{1}{2} + \gamma$$

$$\leq \left( \prod_{t \notin \Theta} |\alpha_t| \right)^{-\gamma} \text{ since } \Theta = \{i_1, i_2, \ldots, i_s\} = \{t_r + 1, t_r + 2, \ldots, t_r + 1\}.$$ Combining above results, we have completed the proof of the theorem. \(\square\)

4 Damped Oscillatory Integral Operators on $H^1_E$

In this section, we shall establish uniform $H^1_E \rightarrow L^1$ estimates for damped oscillatory integral operators considered in Section 3. Generally, these operators are not bounded from $H^1_E$ into $L^1$. However, we can decompose them into three parts such that each operator has desired properties.

Assume $S$ is a real-valued polynomial in $\mathbb{R}^2$ and its Hessian $S''_{xy}$ can be written as

$$S''_{xy}(x, y) = x^m y^n \prod_{i=1}^{N} (x - \alpha_i y^{\eta})$$

with $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{R} \setminus \{0\}$ and $\eta > 0$. If $N = 0$, then $S$ is a monomial up to pure $x, y$ terms and the uniform estimate in this case is known; see for example [17] and [31]. For convenience, we also include a simple proof here. Without of loss of generality, we may assume $m \geq n$. If $m = 0$, then $T_\lambda$ is nondegenerate and the desired result follows immediately. Assume $m > 0$ and $D(x, y) = x^m$. It is clear that $W_z$, defined by (3.10), is bounded from $L^1$ into $L^{1,\infty}$ for $\Re(z) = -1/m$. Combining this fact with Theorem 3.1 we can apply Lemma 2.6 to find that $T_\lambda$ is bounded on $L^p$ with $p = (m + n + 2)/(m + 1)$. Moreover, the operator norm of $T_\lambda$ on $L^p$ satisfies

$$\|T_\lambda\|_{L^p \rightarrow L^p} \leq C|\lambda|^{-\frac{1}{m+n+2}}$$

with the constant $C$ depending only on the cut-off $\varphi$ and $\deg(S)$. Therefore we can assume $N \geq 1$ throughout this section.

Consider the following damped oscillatory integral operator

$$W_z f(x) = \int_{\mathbb{R}} e^{i\lambda S(x, y)} |D(x, y)|^z \varphi(x, y) f(y) dy,$$

where $z \in \mathbb{C}$ has fixed real part $\Re(z) = b$. In this section, we shall further assume that the cut-off $\varphi$ is supported in the unit cube $Q : |x|, |y| \leq 1/2$. Choose a bump function $\Phi \in C_0^\infty(\mathbb{R})$ such that (i) $\text{supp}(\Phi) \subseteq [1/2, 2]$; (ii) $\sum_{j \in \mathbb{Z}} \Phi(x/2^j) = 1$ for all $x > 0$. Let $W_{z,k}$ be defined as $W_z$, but with insertion of $\Phi(x/2^j)\Phi(y/2^k)$ into the cut-off of $W_z$. First choose $s$ indices
\( i_1 < i_2 < \cdots < i_s \). Now we shall define the damping factor \( D \) as follows.

- If either \( m > 0 \) or \( \max_{1 \leq t \leq s} |\alpha_{i_t} - \alpha_{i_s}| \geq |\alpha_{i_s}|/4 \) is true, we take \( z \in \mathbb{C} \) with \( \text{Re}(z) = -\frac{1}{m+s} \)

\[
D(x,y) = x^m \prod_{t=1}^s (x - \alpha_{i_t} y^m).
\]

- If \( m = 0 \) and \( \max_{1 \leq t \leq s} |\alpha_{i_t} - \alpha_{i_s}| < |\alpha_{i_s}|/4 \), we first choose a fixed large number \( N_0 = N_0(\eta) \). Let \( \{t_1, t_2, \cdots, t_o\} \) consist of all indices \( 1 \leq i \leq N \) such that \( |\alpha_{i+1}|/|\alpha| \geq 2^{4N_0} \); see Case (iii) in our proof of Theorem 3.1. Since the sizes of \( \alpha_{i} \) are equivalent, for large \( N_0 \), we must have \( \{i_1, i_2, \cdots, i_s\} \subset \{t_r + 1, t_r + 2, \cdots, t_r+1\} \) for some \( 0 \leq r \leq o \). Here we use the notation \( t_0 = 0 \).

In this case, it is more convenient to replace the set \( \{i_1, i_2, \cdots, i_s\} \) by \( \{t_r + 1, t_r + 2, \cdots, t_{r+1}\} \).

In other words, we shall set

\[
\Theta := \{i_1, i_2, \cdots, i_s\} = \{t_r + 1, t_r + 2, \cdots, t_{r+1}\}.
\]

With this revision, it should be pointed out that \( \max_{1 \leq t \leq s} |\alpha_{i_t} - \alpha_{i_s}| < |\alpha_{i_s}|/4 \) may not still hold. But this does not affect our final results. Now we shall define damping factors \( D_{j,k} \) for each \( W_{j,k} \) as follows.

\[
D_{j,k}(x,y) = \begin{cases} 
\prod_{t=1}^s (x - \alpha_{i_t} y^m), & \text{if } 2j \geq |\alpha_{i_s}|2^{\eta(k+1)+2} \text{ or } 2j \leq |\alpha_{i_s}|2^{\eta(k-1)-2} \\
|\lambda| |\alpha_{i_s}|^{-1}2^{-k(\eta-1)}A^{\frac{1}{m+s}} + \prod_{r=1}^s |x - \alpha_{i_r} y^m|, & \text{otherwise};
\end{cases}
\]

where \( A \) is defined as in (3.25).

As a variant of the classical Hardy space \( H^1 \), we now shall define the space \( H^1_E \) associated with the phase \( \lambda S \) and the set \( \Theta = \{i_1, i_2, \cdots, i_s\} \) considered above; see Phong-Stein [19], Pan [16], Greenleaf-Seeger [10], Shi-Yan [31] and Xiao [35] for earlier work related to this space.

**Definition 4.1** Let \( I_k \) be the dyadic interval \( I_k = [2^{k-1}, 2^{k+1}] \). Associated with \( \lambda S \) and the index set \( \Theta \), we say that a Lebesgue measurable function \( a \) is an atom in \( H^1_E(I_k) \) if there exists an interval \( I \subset I_k \) with the following three properties:

1. \( \text{supp}(a) \subset I_k \);
2. \( |a(x)| \leq |I|^{-1}, \text{ a.e. } x \in I; \)
3. \( \int e^{i\lambda S(\alpha_s \xi_0 \cdot y)} a(y) dy = 0 \) with \( c_1 \) the center of \( I \).

The space \( H^1_E(I_k) \) consists of all \( L^1 \) functions \( f \) which can be written as \( f = \sum_j \lambda_j a_j \), where \( \{a_j\} \) is a sequence of \( H^1_E(I_k) \) atoms and \( \lambda_j \) are complex numbers with \( \sum |\lambda_j| < \infty \). The norm of \( f \) in \( H^1_E(I_k) \) is defined by

\[
\|f\|_{H^1_E} = \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j, \text{ a}_j \text{ atoms in } H^1_E(I_k) \right\}.
\]

Now we state our main result in this section.
Theorem 4.1 Assume $S, W_z$ and the cut-off $\Phi$ are given as above. Let $W_{j,k}$ be defined as $W_z$ with the cut-off multiplied by $\Phi(x/2^j)\Phi(y/2^k)$. If $\eta \geq 1$, then we can decompose $\mathbb{Z}^2$ into three disjoint subsets $\Delta_1, \Delta_2$ and $\Delta_3$ such that each $W_i = \sum_{(j,k) \in \Delta_i} W_{j,k}$, $1 \leq i \leq 3$, satisfies

$$\|W_1f\|_{L^1,\infty} \leq C\|f\|_{L^1}, \quad \|W_3f\|_{L^1} \leq C\|f\|_{L^1}$$

and, for $f \in H^1(I_k)$,

$$\|W_{j,k}f\|_{L^1} \leq C(1+|z|)\|f\|_{H^1_k} \quad \text{with} \quad (j,k) \in \Delta_2,$$

where the above constants $C$ depend only on $\eta$ and $\varphi$.

Remark 4.1 As in Section 3, we can treat the case $\eta < 1$ by writing $S''_{xy}$ as in (3.21). Similarly, we shall define the damping factor $D$ in the first case as $D(x,y) = x^m \prod_{i=1}^{s} (x^\nu - \beta_i y)$ with $\text{Re}(z) = -1/(m + s\nu)$. While for the second case, we take $z$ with $\text{Re}(z) = -\frac{1}{s\nu}$ and define $D_{j,k}$ by

$$D_{j,k}(x,y) = \prod_{i=1}^{s} (x^\nu - \beta_i y).$$

We shall point out that this definition is slightly different from the case $\eta \geq 1$. In fact, the following integral of Hilbert type is finite:

$$\int_{|x|^\eta \approx |\beta_i y|} |x^\nu - \beta_i y|^{-1/\nu} \, dx \leq C(\nu)$$

provided that $|\beta_i - \beta_i y| < \frac{1}{4}|\beta_i|$. Hence, by Fubini’s theorem and Hölder’s inequality, we see that $W_{j,k}$ is bounded on $L^1$ with a bound $C(\nu)$ for each $(j,k) \in \Delta_2$. Other statements in the above theorem are also true in this case.

Proof. According to the definition of $D(x,y)$, we shall prove the theorem in two steps.

**Step 1.** $m > 0$ or $|\alpha_t - \alpha_k| \geq |\alpha_t|/4$ for some $1 \leq t \leq s$.

Recall that each $W_{j,k}$ is defined as $W_z$, but with the cut-off multiplied by $\Phi(x/2^j)\Phi(y/2^k)$.

Now we divide the argument into several cases.

**Case (i) $2^j > |\alpha_t|2^{\eta(k+1)+2}$**.

The set of all these $(j,k)$ is denoted by $\Delta_1$. Put $W_1 = \sum_{(j,k) \in \Delta_1} W_{j,k}$. Then $W_1$ satisfies the trivial estimate

$$\|W_1f\|_{L^1,\infty} \leq C\left(\sup |\varphi|\right)\|f\|_{L^1}.$$

**Case (ii) $|\alpha_t|2^{\eta(k-1)-2} \leq 2^j \leq |\alpha_t|2^{\eta(k+1)+2}$**.

Let $\Delta_2$ be the set of these $(j,k)$ and $W_2 = \sum_{\Delta_2} W_{j,k}$. Then we have

$$\|W_2f\|_{L^1} \leq C\left(\sup |\varphi|\right)\|f\|_{L^1}.$$
for some constant $C$. This can be verified by Fubini’s theorem.

In fact, there holds

\[
\int_{\mathbb{R}} |W_2 f(x)| \, dx \leq C \int_{\mathbb{R}} \int_{|\alpha_i| \approx |x|} |D(x, y)|^{-1/(m+s)} |\varphi(x, y)||f(y)| \, dy \, dx
\]

\[
\leq C \sup |\varphi| \int_{|y'| < 1} \left( \int_{|x'| \approx |\alpha_i| / |y'|} |D(x, y)|^{-1/(m+s)} \, dx \right) |f(y)| \, dy
\]

\[
\leq C \sup |\varphi| \int_{|y'| < 1} \left( \int_{|x'| \approx 1} |D(x, |\alpha_i|^{-1/\eta})|^{-1/(m+s)} \, dx \right) |f(y)| \, dy
\]

\[
\leq C \sup |\varphi| \|f\|_{L^1},
\]

where we have used the assumption that either $m > 0$ or $|\alpha_i_t - \alpha_i_s| > |\alpha_i_s|/4$ for some $t$. We shall point out that the above constants $C$ depend only on $m, s, \eta$.

**Case (iii) $2^j < |\alpha_{i_s}|2^{(k-1)-2}$.**

Let $\Delta_3$ be the set of these pairs $(j, k)$. Define $W_3 = \sum_{(j, k) \in \Delta_3} W_{j,k}$. By Fubini’s theorem, we can prove that $W_3$ is bounded on $L^1$ with the operator norm less than a constant multiple of $\sup |\varphi|$. By use of Fubini’s theorem and change of variables, we have

\[
\int |W_3 f(x)| \, dx \leq \int \int_{|\alpha_{i_s} y'| > 2|x|} |D(x, y)|^{-1/(m+s)} |\varphi(x, y)||f(y)| \, dy \, dx
\]

\[
\leq C \sup |\varphi| \int_{|y'| < 1} \left( \int_{|x'| < 1}|x' - |\alpha_i_s||^{-1/\eta} \, dx' \right) |f(y)| \, dy
\]

\[
\leq C \sup |\varphi| \|f\|_{L^1}.
\]

Combining all results in the above three cases, we have completed the proof of the theorem for Step 1.

**Step 2.** $m = 0$ and $|\alpha_i_t - \alpha_i_s| \leq |\alpha_i_s|/4$ for all $1 \leq t \leq s$.

The crux of our proof lies in this case. Define $W_1, W_2$ and $W_3$ as above. With the same argument, we can show that $W_1$ maps $L^1$ into $L^{1,\infty}$, and that $W_3$ are bounded on $L^1$. Moreover, their operator norms are bounded by a constant multiple of $\sup |\varphi|$. Generally, $W_2$ does not have the mapping properties of $W_1$ and $W_2$. Now we present a special case in which $W_2$ is still bounded on $L^1$. In fact, if there exists a complex $\alpha_{it_0}$ such that $|\Re (\alpha_{it_0})| \leq |\Im (\alpha_{it_0})|$, then $W_2$ is bounded from $L^1$ into itself. For $|x| \approx |\alpha_i| |y| \approx |\alpha_i| |y|$, we have

\[
|x - \alpha_i y| \approx |x - \Re (\alpha_i) y| + |x - \Im (\alpha_i) y| \approx x.
\]

On the support of $W_{j,k}$, this implies

\[
|D(x, y)| \approx |x| \prod_{t \neq t_0} |x - \alpha_i y|
\]
up to constants depending only on \( \eta \). It follows that

\[
\int_{x \approx |\alpha_i| y^q} |D(x,y)|^{-1/s} \, dx \leq C \int_{x \approx |\alpha_i| y^q} \left( x \prod_{t \neq t_0} |x - \alpha_t y^q| \right)^{-1/s} \, dx
\]

\[
\leq C(\eta).
\]

By Fubini’s theorem, one can see that \( W_2 \) is bounded on \( L^1 \).

**Case (i). All \( \alpha_{i_1}, \cdots, \alpha_{i_w} \) are real.**

Though \( W_2 \) is not bounded on \( L^1 \) generally, there exists a constant \( C \) such that for all \((j, k) \in \Delta_2, \)

\[
\|W_{j,k}f\|_{L^1} \leq C\|f\|_{H^1_k}, \quad f \in H^1_k(I_k),
\]

where \( I_k = [2^{k-1}, 2^{k+1}] \). Before our proof of this endpoint estimate, we shall first establish the following \( L^2 \) estimate:

\[
\|W_{j,k}f\|_{L^2} \leq C(\|\alpha_i\|^{-1} 2^{-k(\eta - 1)})^{1/2}\|f\|_2
\]

with \( C = C(deg(S), \varphi) \). By the decomposition method in **Case (iii)** of our proof of Theorem 3.1, we can decompose \( W_{j,k} \) as

\[
W_{j,k} = \sum W_{j,k,l_0,\cdots,l_{w-1}}^{s_{0},\cdots,s_{w-1}}
\]

where the summation ranges over all possible operators appearing in our decomposition. As in Section 3, we shall classify these operators by a sequence of **three-tuples** \( \{G_{i_1,1}, G_{i_2,2}, G_{i_3,3}\}_{i=0}^w \) with the following properties:

(i) \( G_{i_1,1}, G_{i_2,2} \) and \( G_{i_3,3} \) are disjoint and \( G_{i_3,3} = G_{i+1,1} \cup G_{i+1,2} \cup G_{i+1,3} \).

(ii) \( G_{i_2,2} \neq \emptyset \) for \( 1 \leq i \leq w \), and \( G_{j,3} \neq \emptyset \) for \( 1 \leq j \leq w - 1 \).

(iii) \( G_{w,3} = \emptyset \).

Here we set \( G_{0,1} = G_{0,2} = \emptyset \) and \( G_{0,3} = \{t_r + 1, \cdots, t_{r+1}\} \). Given a sequence of these three-tuples \( \{G_{i_1,1}, G_{i_2,2}, G_{i_3,3}\}_{i=0}^w \), we say that \( W_{j,k,l_0,\cdots,l_{w-1}}^{s_{0},\cdots,s_{w-1}} \) is of class \( \{G_{i_1,1}, G_{i_2,2}, G_{i_3,3}\}_{i=0}^w \) if, for \( 1 \leq i \leq w, \)

\[
G_{i_1,1} = \{t \in G_{i-1,3} | |\alpha_t - \alpha_{e_{i-1}}| 2^{k} n \geq 2^{l_{i-1}+N_0} \}
\]

\[
G_{i_2,2} = \{t \in G_{i-1,3} | |\alpha_t - \alpha_{e_{i-1}}| 2^{k} n \leq 2^{l_{i-1}-N_0} \}
\]

\[
G_{i_3,3} = \{t \in G_{i-1,3} | 2^{l_{i-1}-N_0} < |\alpha_t - \alpha_{e_{i-1}}| 2^{k} n < 2^{l_{i-1}+N_0} \},
\]

where \( G_{0,3} = \emptyset = \{t_r + 1, \cdots, t_{r+1}\} \), and \( e_{i-1} \) is the least integer in \( G_{i-1,2} \).

Since the number of sequences of all three-tuples \( \{G_{i_1,1}, G_{i_2,2}, G_{i_3,3}\}_{i=0}^w \), with the above properties, is bounded by a constant depending only on the Cardinality of \( G_0 \), it is enough to establish the \( H^1_k \rightarrow L^1 \) estimate for \( \sum W_{j,k,l_0,\cdots,l_{w-1}}^{s_{0},\cdots,s_{w-1}} \) with the summation taken over all operators of each class \( \{G_{i_1,1}, G_{i_2,2}, G_{i_3,3}\}_{i=0}^w \).
Now assume that $\mathcal{G} = \{G_{i,1}, G_{i,2}, G_{i,3}\}_{i=1}^w$ is fixed. Define

$$W_{\mathcal{G}} = \sum_{j,k,l} W_{j,k,l}^{\sigma_0, \cdots, \sigma_{w-1}}$$

with the summands taken over operators of class $\mathcal{G}$. In this summation, the number of integers $l_0, l_1, \cdots, l_{w-2}$ is finite. To prove that $W_{\mathcal{G}}$ is bounded from $H_k^1$ into $L^1$, it suffices to show

$$\left\| \sum_{l_{w-1}} W_{j,k,l_0, \cdots, l_{w-1}} f \right\|_{L^1} \leq C \|f\|_{H_k^1(I_k)}$$

for a constant $C$ independent of $l_0, \cdots, l_{w-2}$. Now we can reduce (4.29) to the following $L^2$ estimate:

$$\left\| \sum_{l_{w-1}} W_{j,k,l_0, \cdots, l_{w-1}} f \right\|_{L^2} \leq C \left( |\alpha_i|^{-1} 2^{-k(\eta-1)} \right)^{1/2} \|f\|_{L^2}$$

with $C = C(\varphi, \eta)$. Observe that on the support of $W_{j,k,l_0, \cdots, l_{w-1}}$, we have

$$\prod_{i=1}^{l_{w-1}+1} |x - \alpha_i y| \approx B := \prod_{i=1}^{w} \left( \prod_{t \in G_{i,1}} |\alpha_t - \alpha_{e_{i-1}}| 2^{k \eta} \right) \left( \prod_{t \in G_{i,2}} 2^{2^{l-1}} \right)$$

with $G_{w,2} \neq \emptyset$ and $G_{w,3} = \emptyset$. For those $l_{w-1} \in \mathbb{Z}$ such that

$$B < \left( |\alpha_i|^{-1} 2^{-k(\eta-1)} A \right)^{-s/(s+2)}$$

we see that the length $\Delta x$ of horizontal cross-sections satisfies, for each fixed $y$,

$$\Delta x \leq C \left( |\alpha_i|^{-1} 2^{-k(\eta-1)} A \right)^{-1/(s+2)}$$

and for each fixed $x$, the length of vertical cross-sections is bounded by

$$\Delta y \leq C \left( |\alpha_i|^{-1} 2^{-k(\eta-1)} A \right)^{-1/(s+2)} \left( |\alpha_i|^{-1} 2^{-k(\eta-1)} \right)^{1/2}.$$ 

Hence, by the Schur test, we see that the estimate (4.30) is true, where the summation is taken over all $l_{w-1}$ satisfying (4.32). For other integers $l_{w-1}$, the damping factor $D$ has size $|D(x, y)| \approx B$ on the support of $W_{j,k,l_0, \cdots, l_{w-1}}$. Here $B > 0$ is defined as in (4.31).

By the operator van der Corput lemma, we have

$$\|W_{j,k,l_0, \cdots, l_{w-1}}^{\sigma_0, \cdots, \sigma_{w-1}}\| \leq C \left( |\alpha| AB \right)^{-1/2} B^{-1/s}.$$ 

On the right side of this inequality, the exponent of $2^{l_{w-1}}$ is $(-\frac{1}{2} - \frac{1}{s})|G_{w,2}|$. Recall that $l_{w-1}$ satisfies the reverse of the inequality (4.32). Thus we have

$$\sum_{l_{w-1}} \|W_{j,k,l_0, \cdots, l_{w-1}}^{\sigma_0, \cdots, \sigma_{w-1}}\| \leq C \left( |\alpha_i|^{-1} 2^{-k(\eta-1)} \right)^{1/2}.$$ 

Indeed, if we define $b > 0$ by

$$b := \prod_{i=1}^{w} \left( \prod_{t \in G_{i,1}} |\alpha_t - \alpha_{e_{i-1}}| 2^{k \eta} \right) \prod_{i=1}^{w-1} \left( \prod_{t \in G_{i,2}} 2^{2^{l-1}} \right),$$

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Then $B = b \ 2^{l_w-1} |G_{w,2}|$. It follows immediately that
\[
\sum_{l_w=1}^{\infty} \|W_{j,k,l,w-1}^\alpha\| \leq C \sum_{l_w=1}^{\infty} \left( |\alpha| A b 2^{l_w-1} |G_{w,2}| \right)^{1/2} \left( b 2^{l_w-1} |G_{w,2}| \right)^{1/s} \\
\leq C \sum_{l_w=1}^{\infty} \left( |\alpha| A b^{1/2} b^{-1/s} 2^{-l_w-1} |G_{w,2}| (1/2+1/s) \right) \\
\leq C \left( |\alpha_i w|^{-1} 2^{-k(q-1)} \right)^{1/2},
\]
where we have used the fact that $l_{w-1}$ satisfies
\[
\left( |\alpha| A b \right)^{s/(s+2)} < b 2^{l_w-1} |G_{w,2}|.
\]
This proves our claim $\text{(4.30)}$.

Now we turn to address the $H^1_E \rightarrow L^1$ estimate for $W_{j,k}$. Recall that we have assumed that the cut-off $\varphi$, appearing in the definition of $T_\lambda$, is supported in the unit rectangle $|x| \leq 1/2$, $|y| \leq 1/2$.

Define a set $G$ by
\[
G = \bigcup_{t=1}^{s} \{ x : |x - \alpha_i c_j^\alpha| \leq C(\eta) |\alpha_i| 2^{k(q-1)} |I| \}.
\]
Then, by Hölder’s inequality, we can apply the $L^2$ estimate $\text{(4.29)}$ to obtain
\[
\|W_{j,k} a\|_{L^1(G)} \leq |G|^{1/2} \|W_{j,k} a\|_{L^2} \\
\leq C |G|^{1/2} \left( |\alpha_i| 2^{k(q-1)} \right)^{-1/2} \|a\|_{L^2} \\
\leq C \left( |\alpha_i| 2^{k(q-1)} |I| \right)^{1/2} \left( |\alpha_i| 2^{k(q-1)} \right)^{-1/2} |I|^{-1/2} \\
\leq C.
\]
Since $G$ is the union of $s$ intervals, it may not be an interval generally. However, we shall see that the interval $(\alpha_1 c_1^\eta, \alpha_s c_s^\eta)$ can be included in $G$ such that $\|W_{j,k} a\|_{L^1(G)} \leq C$ is still true. Let $F$ be the set
\[
F = \{ x : |x - \alpha_i c_j^\eta| \leq 2|\alpha_i - \alpha_j| c_j^\eta \}. \quad \text{(4.33)}
\]
As just claimed, there exists a constant $C = C(\eta, \varphi)$ such that $\|W_{j,k} a\|_{L^1(F)} \leq C$. For $x \in G^c$,
\[
|x - \alpha_i y^\eta| \approx |x - \alpha_i c_j^\eta|, \ y \in I, \quad \text{(4.34)}
\]
for all $1 \leq t \leq s$. By this fact, we have
\[
\int_{F \cap G} |W_{j,k} a(x)| dx \leq C \int_{F \setminus G} \prod_{t=1}^{s} |x - \alpha_i c_j^\eta|^{-1/s} dx \\
\leq C \prod_{t=1}^{s} |x - \alpha_i c_j^\eta|^{-1/s} dx \\
\leq C,
\]
and
where the last inequality is true since the singularities \(\alpha_i c_i^n, \ldots, \alpha_i c_i^j\) in the above integral are suitably separated in \(F\). More precisely, for \(x \in F\), we have
\[
\max\{|x - \alpha_i c_i^n|, |x - \alpha_i c_i^j|\} \geq C|\alpha_i - \alpha_i|c_i^n. \tag{4.35}
\]

By the definition (4.33) of \(F\), (4.34) and (4.35), we have, for \(x \in F^c \cap G^c\),
\[
\min_{1 \leq t \leq s} |x - \alpha_i y^n| \approx \min_{1 \leq t \leq s} |x - \alpha_i c_i^j| \approx \max_{1 \leq t \leq s} |x - \alpha_i y^n| \approx \max_{1 \leq t \leq s} |x - \alpha_i y|^n \tag{4.36}
\]
and all these terms are not less than a constant multiple of \(|\alpha_i - \alpha_i|c_i^n\). Now we use the decomposition method in Section 3 and obtain \(W_{j,k} = \sum W_{j,k,l_0,\ldots,l_{w-1}}\). Since \(W_{j,k,l_0,\ldots,l_{w-1}}\) is supported in the domain
\[
2^{j-1} \leq x \leq 2^{j+1}, \ 2^{k-1} \leq y \leq 2^{k+1}, 2^{t-1-1} \leq \sigma_t-1(x - \alpha_{e_t-1}y^n) \leq 2^{t-1+1}, \ 1 \leq t \leq w,
\]
it follows from (4.36) that, for \(x \in F^c \cap G^c\), we may assume
\[
2^{t-1} \geq C \max\{|\alpha_i|2^{(k-1)|I|}, |\alpha_i - \alpha_i||2^{kn}\}, \ 1 \leq t \leq w.
\]
On the other hand, the sizes of \(|x - \alpha_i c_i^j|\) are equivalent up to constants depending only on \(\eta\). In other words, we can assume \(l_0 \approx l_1 \approx \cdots \approx l_{w-1} \) now.

With the above observations, we are now going to prove
\[
\left\| \sum_{l_{w-1}} W_{j,k,l_0,\ldots,l_{w-1}} a \right\|_{L^1(F^c \cap G^c)} \leq C \tag{4.37}
\]
for fixed \(l_0, \ldots, l_{w-2}\), where \(a\) is an atom in \(H^1(E)(I_k)\). First we treat the simplest case \(w \geq 2\). For \(x \in F^c \cap G^c\), it follows from (4.36) that
\[
D_{j,k}(x, y) \geq \prod_{t=1}^s |x - \alpha_i y^n| \approx |x - \alpha_{e_0} c_i^n|^s.
\]

On the other hand, the cut-off function of \(W_{j,k,l_0,\ldots,l_{w-1}}\) has the factor \(\Phi\left(\sigma_0 \frac{x - \alpha_{e_0} y^n}{2^{l_0}}\right)\). This implies
\[
\int_{F^c \cap G^c} \left| \sum_{l_{w-1}} W_{j,k,l_0,\ldots,l_{w-1}} a(x) \right| dx \leq C \int_{|x - \alpha_{e_0} c_i^n| \approx 2^{l_0}} |x - \alpha_{e_0} c_i^n|^{-1} dx \leq C.
\]
This completes the proof of (4.37) for \(w \geq 2\).

Now we turn to address the desired estimate (4.37) for \(w = 1\). Before our proof of the estimate (4.37), we shall need the following inequality, which is reminiscent of the Hörmander condition for singular integrals (see Stein [32]),
\[
\sup_{y \in I} \int_{F^c \cap G^c} |K(x, y) - K(x, c_1)| dx \leq C(\varphi, \eta)(1 + |z|), \quad \Re(z) = \frac{1}{s}, \tag{4.38}
\]
where the kernel \(K\) is defined by
\[
K(x, y) = \varphi(x, y)\Phi\left(\frac{x}{2^r}\right)\Phi\left(\frac{y}{2^r}\right)|D(x, y)|^2. \tag{4.39}
\]
Let $M$ be defined by

$$M(x, y) = \varphi(x, y) \Phi \left( \frac{x}{2^j} \right) \Phi \left( \frac{y}{2^k} \right).$$

It is clear that $K(x, y) = M(x, y)|D(x, y)|^2$. By the mean value theorem,

$$|M(x, y) - M(x, c_I)| \leq \|\partial_{y}\varphi\|_{\infty} \|\Phi\|_{\infty}^2 |I| + \|\varphi\|_{\infty} \|\Phi\|_{\infty} \|\Phi'\|_{\infty} \left| \frac{|I|}{2^k} \right| \leq C |I|$$

for $y \in I$ and $k \leq 0$. Recall that $\varphi$ is supported in the unit rectangle $|x| \leq 1/2$, $|y| \leq 1/2$. Thus we need only consider $j, k \leq 0$.

For $x \in F^c \cap G^c$ and $y \in I$, it follows from (4.36) that

$$\left| |D(x, y)|^2 - |D(x, c_I)|^2 \right| \leq C \left| \frac{\sum_{u=1}^s \prod_{t=1}^s \left| x - \alpha_i c_I^t \right|^{-\delta_t}}{2^j} \right| \left| \alpha_i y - \alpha_i c_I^t \right|\right| \leq C \left| \frac{\sum_{u=1}^s \prod_{t=1}^s \left| x - \alpha_i c_I^t \right|^{-\delta_t}}{2^j} \right| \left| \alpha_i y - \alpha_i c_I^t \right|$$

where $\delta_t^u$ is the Kronecker symbol. Note also that the integral in (4.38) is taken over all $x \in F^c \cap G^c$ and $\Phi(x/2^j) \neq 0$. For this reason and $2^j \approx |\alpha_i|^{2k\eta}$, we can restrict the integration over $C_1(\eta)|\alpha_i|^{2k(\eta - 1)} |I| \leq |x - \alpha_i c_I^t| \leq C_2(\eta)|\alpha_i|^{2k\eta}$. With these observations, we see that the integral in (4.38) is bounded by a constant multiple of $2^j \log(C_2^{-1}) + 1$. By the assumption $I \subseteq [2^{k-1}, 2^{k+1}]$, this quantity is bounded above by a constant $C$ independent of $j, k, y, c_I$. Hence we have completed the proof of (4.38).

With the Hömander condition (4.38), to obtain (4.37), it is enough to show that there exists a constant $C$ such that

$$\int_{F^c \cap G^c} \left| D(x, c_I) \right|^2 \Phi(x/2^j) \Phi(c_I/2^k) \varphi(x, c_I) \left| \int_I e^{i\lambda S(x, y)} a(y) dy \right| dx \leq C, \quad \text{Re}(z) = -\frac{1}{s}. \quad (4.40)$$

Recall that $\text{supp}(a) \subseteq I \subseteq I_k = [2^{k-1}, 2^{k+1}]$. Using the cancellation property of $a$, we see that for $x \in F^c \cap G^c \cap I_j$,

$$\left| \int_I e^{i\lambda S(x, y)} a(y) dy \right| = \left| \frac{\pi}{4} \int_I \int_x^y 2 \partial_u \partial_v S(u, v) du dv \right| a(y) dy \right| \leq C |\lambda| \int x_{c_I}^y \int c_I^s \partial_u \partial_v S(u, v) du dv \leq C |\lambda| A \left| x - \alpha_i c_I^t \right|^{s+1} |I|,$$

where $A > 0$ is defined by (5.25). Choose $\mu > 0$ such that $|\lambda| A \mu^{s+1} |I| = 1$. It is easily verified that (4.40) is true if integration is taken over the set of all $x$ satisfying $|x - \alpha_i c_I^t| \leq \mu$ and $x \in F^c \cap G^c$. Define $\gamma_0 = \max\{C(\eta)|\alpha_i|^{2k(\eta - 1)} |I|, C(\eta)|\alpha_i - \alpha_{i-s}^{2k\eta}, \mu\}$. Write

$$\int_{|x - \alpha_i c_I^t| > \gamma_0} |K(x, c_I)| \left| \int_I e^{i\lambda S(x, y)} a(y) dy \right| dx \leq C_1 \int_{|x - \alpha_i c_I^t| \leq C_2 \mu} \left| x - \alpha_i c_I^t \right|^{-1} \left| \int_I e^{i\lambda S(x, y)} a(y) dy \right| dx \leq C_1 \sum_{l=0}^{\log(C_2/\gamma_0)} \int_{|x| < 2 l} \left| e^{i\lambda S(2^l x + \alpha_i c_I^t |I| y + c_I)} \right| \left| a(|I| y + c_I) dy \right| dx,$$
where $K(x,y)$ is given by (1.39). $C_1 = C_1(\eta, \varphi)$ and $C_2 = C_2(\eta)$. To apply the uniform decay estimate in Lemma 2.5, we shall verify that the above phases are uniformly non-degenerate. In fact, if $N_0 = N_0(\eta)$ is sufficiently large, then we have

$$
|\lambda| \left| \partial_x \partial_y [S(2^l \gamma_0 x + \alpha_i s_1^n, |y|)] \right| = |\lambda| |2^l \gamma_0| I \left| \partial_x \partial_y S(2^l \gamma_0 x + \alpha_i s_1^n, |y|) \right|
\geq \frac{C}{\lambda} |2^l \gamma_0| I \left| (A 2^l \gamma_0)^s \right|
\geq C 2^{(s+1)l}
$$

for $1 \leq |x| \leq 2$, $y \in I$ and $0 \leq l \leq \log(C 2^j / \gamma_0)$. By Lemma 2.5, there exists a positive number $\delta = \delta(deg(S))$ such that

$$
\int_{|x-\alpha_i| > \gamma_0} |K(x, c_l)| \int_I e^{i \lambda S(x,y)} a(y) dy \leq C \sum_{l=0}^{\infty} 2^{-s+1)l\delta} \leq C.
$$

Case (ii). At least one of $\alpha_1, \ldots, \alpha_s$ is complex.

The treatment of complex roots is similar to that of Step 2 in our proof of Theorem 3.1. Without loss of generality, we may assume that $\alpha_i$ has nonzero imaginary part. If $\Re(\alpha_i) \leq \Im(\alpha_i)$, as shown at the beginning of our proof in this step, then $W_{j,k}$ is bounded on $L^1$. So we need only take care of the case $\Re(\alpha_i) > \Im(\alpha_i)$ in which the above arguments apply without essential change.

\section{5 Proof of the main result}

In this section, we shall prove Theorem 1.1 by exploiting damping estimates established in Section 3 and Section 4.

\textbf{Proof.} Throughout this section, we choose $s$ indices $\{i_1, i_2, \ldots, i_s\} = \{1, 2, \ldots, s\}$. Then the damping factor $D$ in Theorem 3.1 is equal to

$$
D(x, y) = x^m \prod_{t=1}^{s} (x - \alpha_t y^n).
$$

We first assume $\eta \geq 1$ and $m + s \geq n + (N - s)\eta$. As in the proof of Theorem 3.1 we shall divide our proof here into two steps.

\textbf{Step 1.} Either $m > 0$ or $|\alpha_{i_t} - \alpha_{i_s}| \geq \frac{|\alpha_{i_t}|}{4}$ is true for some $1 \leq t \leq s$.

Choose a bump function $\Phi$ as in the previous two sections. In other words, $\Phi \in \mathcal{C}^\infty(\mathbb{R})$ satisfies (i) $\text{supp}(\Phi) \subseteq [1/2, 2]$ and (ii) $\sum_{j \in \mathbb{Z}} \Phi(x/2^j) = 1$ for $x > 0$. Define $W_{j,k}$ by

$$
W_{j,k} f(x) = \int_{\mathbb{R}} e^{i \lambda S(x,y)} |\tilde{D}(x,y)|^2 \varphi(x,y) \Phi \left( \frac{2^j}{x} \right) \Phi \left( \frac{y}{2^k} \right) f(y) dy
$$

(5.42)

where the damping factor $\tilde{D}$ is slightly different from $D$ and $z \in \mathbb{C}$ lies in the following strip:

$$
-\frac{1}{m + s} \leq \Re(z) \leq \gamma := \frac{m + s - n - (N - s)\eta}{2[n + (N - s)\eta + 1]} \cdot \frac{1}{m + s}.
$$

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Theorem 3.1, we can establish the above proof of Theorem 3.1, as the damping factor $D$ and its partial derivatives. Hence following the proof of Theorem 3.1, we can establish the above $L^2$ decay estimate. The $L^1 \to L^{1,\infty}$ estimate for $W_1$ is obvious for $z \in \mathbb{C}$ with $\text{Re}(z) = -1/(m+s)$; see also Section 4.

By Lemma 2.6 with $T$ defined by $W_1 f = |x|^{(m+s)z} T \lambda f$, $a = (m+s)\gamma$ and $p_0 = 2$, we obtain

$$\| W_1 f \|_{L^p} \leq C(1 + |z|^2) \left( |\lambda| \prod_{k \in \{i_1, \ldots, i_s\}} |\alpha_k| \right)^{-\gamma} \| f \|_{L^2}$$

with $\text{Re}(z) = 0$, \quad (5.43)

where $\delta$ and $p$ are given by

$$\delta = \frac{1}{m+s+n+(N-s)\eta+2}, \quad p = \frac{m+s+n+(N-s)\eta+2}{m+s+1}. \quad (5.44)$$

Case (ii) $|\alpha_k|2^{(k-1)\eta-2} < 2^l < |\alpha_k|2^{(k+1)\eta+2}$.

The set of all $(j,k)$ satisfying Case (ii) is denoted by $\Delta_2$. As in Section 4, we define $W_2 = \sum_{\Delta_2} W_{j,k}$. In this case, put $\tilde{D}(x,y) = D(x,y)$. Then it is also true that

$$\| W_2 f \|_{L^2} \leq C(1 + |z|^2) \left( |\lambda| \prod_{k \notin \{i_1, \ldots, i_s\}} |\alpha_k| \right)^{-\gamma} \| f \|_{L^2}, \quad \gamma = \frac{1}{2[n+(N-s)\eta+1]}$$

for $z \in \mathbb{C}$ with real part as in Theorem 3.1. On the other hand, as in Section 4, it is easy to prove $W_2$ satisfies

$$\| W_2 f \|_{L^1} \leq C\| f \|_{L^1}, \quad \text{Re}(z) = -\frac{1}{m+s}.$$ 

By the Stein complex interpolation theorem, for $z$ with real part $\text{Re}(z) = 0$, we see that $W_2$ also satisfies the estimate (5.43) for $W_1$.

Case (iii) $2^l \leq |\alpha_k|2^{(k-1)\eta-2}$.

We use the notation $\Delta_3$ to denote the set of all $(j,k)$ satisfying Case (iii). Define $W_3 = \sum_{\Delta_3} W_{j,k}$ with $\tilde{D}(x,y) = D(x,y)$. As in the previous two cases, we can prove that $W_3$ satisfies
the same estimates as \( W_2 \) in Case (ii). An interpolation yields the desired \( L^p \) decay estimate \(^\text{[5.33]}\) with \( W_3 \) in place of \( W_1 \).

**Step 2.** \( m = 0 \) and \(|\alpha_t - \alpha_k| < \frac{|t_0|}{4} \) for all \( 1 \leq t \leq s \).

We can divide our proof into three cases as in Step 1. The arguments for Case (i) and Case (iii) are the same as that of Step 1. Now we turn to Case (ii). For each fixed \((j, k) \in \Delta_2\), the number of all \((j', k') \in \Delta_2\) such that the horizontal (also vertical) projections of the supports of \( W_{j,k} \) and \( W_{j',k'} \) have nonempty intersections is bounded by a constant \( C = C(\eta) \). By the almost orthogonality, there exists a constant \( C = C(\eta) \) such that

\[
\|W_2 f\|_{L^p} \leq C \sup_{(j,k) \in \Delta_2} \|W_{j,k} f\|_{L^p}, \quad 1 \leq p \leq \infty.
\]

Thus it suffices to prove that \( W_{j,k} \) satisfies \(^\text{[5.33]}\) for each \((j, k) \in \Delta_2 \) with bounds independent of \((j, k)\).

The proof of the estimate for \( W_2 \) is somewhat different from Step 1. As in Section 3 and Section 4, the set \( \{t_1, t_2, \cdots , t_n\} \) consists of all \( 1 \leq i \leq N \) such that \(|\alpha_{i+1}|/|\alpha_i| \geq 2^4N_0 \) for some large \( N_0 = N_0(\eta) \). If this set is empty, we shall define \( t_1 = N \). With our assumptions, it is easy to see that sizes of \( \alpha_i, \alpha_{i+1}, \cdots , \alpha_{i+s} \) are equivalent up to absolute constants. At the beginning of this section, we have chosen \( \{i_1, i_2, \cdots , i_s\} = \{1, 2, \cdots , s\} \). This implies that \( \{1, 2, \cdots , s\} \) is a subset of \( \{1, 2, \cdots , t_1\} \) provide that \( N_0 \) is sufficiently large. Now we shall first prove the corresponding \( L^p \) estimate for the special case \( s = t_1 \). Then by interpolation we are able to show the \( L^p \) estimate for general \( s \).

Since \( m + s \geq n + (N - s)\eta \) and \( s \leq t_1 \), it is clear that \( m + t_1 \geq n + (N - t_1)\eta \). The damping factor \( D_{j,k} \) for \( W_{j,k} \) is defined by Theorem \(^\text{[4.1]}\) i.e.,

\[
D_{j,k}(x, y) = \left( |\lambda|/|\alpha_{i_1}| \right)^{-1} 2^{-k(\eta - 1)} A \prod_{t=1}^{t_1} |x - \alpha_{i_t} y|^{i_t} + \prod_{t=1}^{t_1} |x - \alpha_{i_t} y|
\]

with \( A > 0 \) given by

\[
A = 2^{kn} \prod_{t=1}^{N} |\alpha_t|^{2k\eta}.
\]

By Theorem \(^\text{[3.3]}\) we have for each \((j, k) \in \Delta_2\)

\[
\|W_{j,k} f\|_{L^2} \leq C(1 + |z|^2) \left( |\lambda| \prod_{t=t_1+1}^{N} |\alpha_t| \right)^{-\gamma} \|f\|_{L^2}, \quad \gamma = \frac{1}{2(n + (N - t_1)\eta + 1)}, \quad \Re(z) = -\frac{1}{t_1},
\]

with \( z \) having real part in Theorem \(^\text{[3.3]}\). Here the constant \( C \) depends only on \( \deg(S) \) and the cut-off \( \varphi \) as in \(^\text{[3.3]}\). On the other hand, it follows from Theorem \(^\text{[4.1]}\) that for each \((j, k) \in \Delta_2\)

\[
\|W_{j,k} f\|_{L^1} \leq C(1 + |z|) \|f\|_{H^1_{\alpha_0}(\Delta_2)}, \quad \Re(z) = -\frac{1}{t_1},
\]

where the constant \( C \) depends only \( \deg(S) \) and the cut-off \( \varphi \).

By interpolation between the above \( L^2 \to L^2 \) and \( H^1_{\alpha_0} \to L^1 \) estimates, we see that there exists a constant \( C = C(\deg(S), \varphi) \) such that

\[
\sup_{\Delta_2} \|W_{j,k} f\|_{L^p} \leq C(1 + |\Im(z)|^2) \left( |\lambda| \prod_{t=t_1+1}^{N} |\alpha_t| \right)^{-\gamma} \|f\|_{L^p}, \quad \Re(z) = 0, \quad (5.45)
\]
with the supremum taken over all \((j, k)\) in \(\Delta_2\) and \(\gamma, p\) being given by
\[
\gamma = \frac{1}{t_1 + n + (N - t_1)\eta + 2}, \quad p = \frac{t_1 + n + (N - t_1)\eta + 2}{t_1 + 1}.
\]

With the above estimate \((5.45)\), we are able to prove the corresponding \(L^p\) estimates for \(1 \leq s \leq t_1\) satisfying \(m + s \geq n + (N - s)\eta\). However, we also need another \(L^p\) decay estimate corresponding to \(s = N\) for which \(\{i_1, \ldots, i_s\} = \{1, \ldots, N\}\). The above arguments in Step 1 and this step are applicable without change. Indeed, if either \(m > 0\) or \(\sup_{1 \leq t \leq N} |\alpha_t - \alpha_N| \geq |\alpha_N|\) is true, we can follow the argument in Step 1 to obtain
\[
\|W_z f\|_{L^p} \leq C(1 + |z|^2)|\lambda|^{-\gamma}\|f\|_{L^p}, \quad \gamma = \frac{1}{N+2}, \quad p = N+2,
\]
where \(C = C(\deg(S), \varphi)\). If \(m = 0\) and \(\sup_{1 \leq t \leq N} |\alpha_t - \alpha_N| < |\alpha_N|/4\), we can use the same argument in this step to obtain the above estimate since \(t_1 = N\) in this case. By a duality argument, with the role of \(x\) and \(y\) changed, we have
\[
\|W_z f\|_{L^p} \leq C(1 + |z|^2) \left(\prod_{t=s+1}^{N} |\alpha_t|\right)^{-\gamma}\|f\|_{L^p}, \quad \gamma = \frac{1}{n + N\eta + 2}, \quad p = n + N\eta + 2. \tag{5.46}
\]

Invoking the above estimates \((5.45)\) and \((5.46)\) with \(m = 0\), we claim that for a constant \(C = C(\deg(S), \eta)\) we have
\[
\|W_z f\|_{L^p} \leq C(1 + |z|^2) \left(\prod_{t=s+1}^{N} |\alpha_t|\right)^{-\gamma}\|f\|_{L^p} \tag{5.47}
\]
with \(1 \leq s \leq t_1\) satisfying \(s \geq n + (N - s)\eta\) and \(\gamma, p\) given by
\[
\gamma = \frac{1}{s + n + (N - s) + 2}, \quad p = \frac{s + n + (N - s) + 2}{s + 1}.
\]

Now we are going to prove this claim. Choose \(\theta \in [0, 1]\) such that
\[
\frac{s + 1}{s + n + (N - s)\eta + 2} = \frac{t_1 + 1}{t_1 + n + (N - t_1)\eta + 2} \cdot \theta + \frac{1}{n + N\eta + 2} \cdot (1 - \theta).
\]
Set \(A = n + N\eta + 2\) and \(B = \eta - 1\). Subtracting \(-\frac{1}{B} = -\frac{1}{\eta - 1}\) from both sides of the above equality, we obtain
\[
\frac{A + B}{(A - Bs)B} = \frac{A + B}{(A - Bt_1)B} \cdot \theta + \frac{A + B}{AB} \cdot (1 - \theta).
\]
Hence we have
\[
\frac{1}{(A - Bs)} = \frac{1}{(A - Bt_1)} \cdot \theta + \frac{1}{A} \cdot (1 - \theta).
\]
It follows that
\[
\frac{1}{(A - Bs)} - \frac{1}{A - Bt_1} = \left(\frac{1}{A} - \frac{1}{(A - Bt_1)}\right) \cdot (1 - \theta).
\]
which implies
\[ \frac{t_1 - s}{A - Bs} = \frac{t_1}{A} \cdot (1 - \theta). \]  
(5.48)

By interpolation between the two \( L^p \) estimates (5.45) and (5.46), we get
\[ \|W_z f\|_{L^p} \leq C (1 + |\text{Im}(z)|^2) \left( \prod_{t=t_1+1}^{N} |\alpha_t|^{1 - \gamma_1 \theta} \right) \left( \prod_{t=1}^{N} |\alpha_t|^{1 - \gamma_2 (1 - \theta)} \right) \|f\|_{L^p}, \text{ Re}(z) = 0, \]
where \( p \) is given by
\[ p = \frac{s + n + (N - s)\eta + 2}{s + 1} \]
and we use \( \gamma_1 \) and \( \gamma_2 \) to denote the decay exponents in (5.45) and (5.46), respectively. With the \( \theta \) considered above, we see that \( \gamma_1 \theta + \gamma_2 (1 - \theta) = 1/(A - Bs) \) and
\[
\left( \prod_{t=t_1+1}^{N} |\alpha_t|^{-\gamma_1 \theta} \right) \left( \prod_{t=1}^{N} |\alpha_t|^{-\gamma_2 (1-\theta)} \right) = \left( \prod_{t=t_1+1}^{N} |\alpha_t|^{-1/(A - Bs)} \right) \left( \prod_{t=1}^{N} |\alpha_t|^{-\gamma_2 (1-\theta)} \right)
\approx \left( \prod_{t=t_1+1}^{N} |\alpha_t|^{-1/(A - Bs)} \right) \left| \alpha_{t_1} \right|^{-t_1 \gamma_2 (1-\theta)}
\approx \left( \prod_{t=s+1}^{N} |\alpha_t|^{-1/(A - Bs)} \right),
\]
where we have used the equality (5.48) and the assumption that \( \alpha_1, \ldots, \alpha_{t_1} \) have equivalent sizes up to constants depending only on \( \eta \). This proves our claim (5.47).

Combining all above results, we have established the desired \( L^p \) estimate for \( T_\lambda \) in (1.1) with \( \eta \geq 1 \). Similarly, we can deal with the case \( \eta < 1 \) without essential change. Using the above interpolation methods, we are able to prove this decay estimate by invoking Theorem 5.2 and Remark 4.1.

Now we turn to prove Theorem 1.1. First assume \( k \geq l \). We can choose an integer \( s \) such that one of the following two statements is true:

(i) \( m + s + 1 = k \) and \( n + (N - s)\eta + 1 = l \) if \( \eta \geq 1 \);

(ii) \( m + s \nu + 1 = k \) and \( n + M - s + 1 = l \) with \( \nu = \eta^{-1} \) if \( \eta < 1 \).

Here the notations \( M \) and \( \nu \) are defined as in Theorem 5.2. Choose \( \{i_1, i_2, \ldots, i_s\} = \{1, 2, \ldots, s\} \). Without loss of generality, we assume that the coefficient \( c_0 \), appearing in (3.8) and (3.21), is equal to one. The reason is that \( c_0 \) can be incorporated into the real parameter \( \lambda \). Since \( |\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_N| \) and
\[ a_{k,l} = (-1)^\sigma \cdot \frac{1}{k} \cdot \frac{1}{l} \cdot \sum_{1 \leq j_1 < j_2 < \cdots < j_{N-s} \leq N} \alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_{N-s}} \]
with \( \sigma = N - s \) if \( \eta \geq 1 \) and \( \sigma = M - s \) if \( \eta < 1 \), we see that \( |a_{k,l}| \) is not greater than a constant multiple of the absolute value of the product of \( \alpha_{s+1}, \alpha_{s+2}, \ldots, \alpha_N \). By these results, the operator \( T_\lambda \) satisfies the \( L^p \) decay estimate in Theorem 1.1 for \( k \geq l \). While for \( k < l \), the desired estimate can be proved by a duality argument.
\[ \Box \]
6 Higher dimensional oscillatory integral operators

In this section, we shall establish uniform sharp \( L^p \) estimates for higher dimensional oscillatory integral operators. Our main tools are the one dimensional result in Theorem 1.1 and a variant of Stein-Weiss interpolation with change of measures. As a consequence of the \( L^p \) estimates, we can extend the \( L^2 \) decay estimate in Tang [34] for \((2 + 1)\)-dimensional oscillatory integral operators. At the end of this section, some counterexamples will be presented to clarify the necessity of our assumptions.

In this section, we consider the higher dimensional oscillatory integral operator of form:

\[
T_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x,y)} \varphi(x,y) f(y) dy, \quad x \in \mathbb{R}^n,
\]

where \( \lambda \in \mathbb{R} \) is a parameter and \( \varphi \) is a smooth cut-off. To establish uniform estimates for \( T_\lambda \), we need \( L^p \) decay estimates with change of cut-off functions. We state this as follows.

**Proposition 6.1** Let \( T_\lambda \) be an oscillatory integral operator defined as in \((6.49)\) and \( 1 \leq p \leq \infty \). Then the following two statements are true.

(i) Let \( n_X = n_Y = 1 \). Assume that there exists a nonnegative number \( \sigma \) such that

\[
\|T_\lambda f\|_{L^p} \leq C|\lambda|^{-\sigma} \|f\|_{L^p}
\]

for all smooth cut-off \( \varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}) \) with the constant \( C \) independent of \( \lambda \). Then this decay estimate is still true with the cut-off \( \varphi(x,y) = \psi(x^{\gamma_1}, y^{\gamma_2}), \chi_{\mathbb{R}^+(x)} \chi_{\mathbb{R}^+}(y) \), where \( \psi \in C_0^\infty(\mathbb{R}^2) \), \( \gamma_1, \gamma_2 > 0 \) and \( \chi_{\mathbb{R}^+} \) denotes the characteristic function of \( \mathbb{R}^+ \).

(ii) Assume \( n_X, n_Y \geq 1 \). Then the decay estimate \((6.50)\) holds for all \( \varphi \in C_0^\infty(\mathbb{R}^{n_X} \times \mathbb{R}^{n_Y}) \) if and only if it is true for all \( x \)-radial and \( y \)-radial \( \varphi \in C_0^\infty(\mathbb{R}^{n_X} \times \mathbb{R}^{n_Y}) \).

**Remark 6.1** This proposition turns out to be very useful in the treatment of uniform estimates for oscillatory integral operators. Its proof is very simple and invoke the Fourier expansion for smooth periodic functions.

**Proof.** We first prove (i). Choose a radial function \( \omega \in C_0^\infty(\mathbb{R}^2) \) such that \( \omega(x,y) = 1 \) for \((x,y) \in [-1,1]^2 \) and \( \omega = 0 \) outside the square \([-2,2]^2 \). Since \( \psi \) is a smooth cut-off, we can choose the smallest positive number \( a > 0 \) such that \( \text{supp}({\psi}) \subseteq [-a,a]^2 \). For convenience, we shall extend \( \psi \) periodically outside the larger rectangle \([-2a,2a]^2 \); we use \( \tilde{\psi} \) to denote this periodic extension. Since \( \tilde{\psi} \) is smooth, we have the following Fourier expansion.

\[
\tilde{\psi}(x,y) = \sum_{k,l \in \mathbb{Z}} a_{k,l} e^{ik\frac{x}{2a} + il\frac{y}{2a}},
\]

where \( a_{k,l} \) are the Fourier coefficients given by

\[
a_{k,l} = \frac{1}{(4a)^2} \int_{[-2a,2a]^2} \tilde{\psi}(x,y) e^{-i(k\frac{x}{2a} + l\frac{y}{2a})} dx dy.
\]

By our choice of \( \omega \), we have \( \psi(x,y) = \omega(x/a, y/a) \tilde{\psi}(x,y) \). It follows immediately from the above expansion that

\[
\psi(x,y) = \omega(\frac{x}{a}, \frac{y}{a}) \left( \sum_{k,l \in \mathbb{Z}} a_{k,l} e^{i(k\frac{x}{2a} + l\frac{y}{2a})} \right).
\]
Since $\omega$ is radial, we may write $\omega(x, y) = \omega(\sqrt{x^2 + y^2})$ with some abuse of notation. For clarity, we write the operator $T_\lambda$ as $T_\lambda(S, \varphi)$ to stress its dependence on the phase $S$ and the cut-off $\varphi$. Then it follows from the above equality that

$$T_\lambda(S, \psi(x^{\gamma_1}, y^{\gamma_2})) = \sum_{k,l \in \mathbb{Z}} a_{k,l} T_\lambda \left( S + \frac{k \pi}{\lambda} x^{\gamma_1} + \frac{l \pi}{\lambda} y^{\gamma_2}, \omega \left( \frac{\sqrt{x^{\gamma_1} + y^{\gamma_2}}}{\lambda} \right) \right) f$$

for $f \in L^p$ satisfying $\operatorname{supp}(f) \subseteq \mathbb{R}^+$. On the other hand, the pure $x, y$ terms in the phase of $T_\lambda$ do not affect the operator norm of $T_\lambda$. Since $\omega = 1$ near the origin, $\omega \left( \frac{\sqrt{x^{\gamma_1} + y^{\gamma_2}}}{\lambda} \right)$ is a smooth cut-off. The rapid decay of $a_{k,l}$ implies $\sum |a_{k,l}| < \infty$. By our assumption, the $L^p$ operator norms of operators in the above summation have a uniform power decay bound $C|\lambda|^{-\sigma}$. This implies that

$$\left\| T_\lambda(S, \psi(x^{\gamma_1}, y^{\gamma_2}) \chi_{\mathbb{R}^+}(x) \chi_{\mathbb{R}^+}(y) \right\|_{L^p \to L^p} \leq C|\lambda|^{-\sigma}$$

with the constant $C$ independent of $\lambda$.

The second statement can be proved as above. Indeed, following the above argument, we can show that every $\varphi \in C_0^\infty(\mathbb{R}^{n_x} \times \mathbb{R}^{n_y})$ can be written as the product of an absolutely convergent Fourier series and an $x$-radial and $y$-radial smooth cut-off in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$. On the other hand, it is easy to see that insertion of each term $e^{i\theta p \cdot \hat{S}(x,y)}$ in the cut-off of $T_\lambda$ does not affect the $L^p$ operator norm. Thus the statement (ii) follows immediately.

Assume $S$ is a real-valued polynomial in $\mathbb{R}^2$ as in Theorem 6.1. For arbitrary two $\eta_1, \eta_2 \geq 1$, define

$$T_\lambda^{\eta_1, \eta_2} f(x) = \int_{\mathbb{R}} e^{i \lambda S(x^{\eta_1}, y^{\eta_2})} \varphi(x, y) \chi_{\mathbb{R}^+}(x) \chi_{\mathbb{R}^+}(y) f(y) dy. \quad (6.51)$$

**Theorem 6.2** Assume $S$ is a real-valued polynomial described as in Theorem 6.1. Let $T_\lambda^{\eta_1, \eta_2}$ be defined by (6.51) with $\varphi \in C_0^\infty$ and $\eta_1, \eta_2 \geq 1$. Then there exists a constant $C = C(\deg(S), \eta_1, \eta_2, \varphi)$ such that

$$\left\| T_\lambda^{\eta_1, \eta_2} f \right\|_{L^p} \leq C \left( |a_{k,l}| |\lambda| \right)^{-\gamma} \| f \|_{L^p}, \quad \gamma = \frac{1}{k \eta_1 + l \eta_2}, \quad p = \frac{k \eta_1 + l \eta_2}{k \eta_1}.$$

**Proof.** First assume $\eta_1 \geq 1, \eta_2 = 1$. By Theorem 6.1, we see that $T_\lambda(S, \varphi)$ satisfies the estimate in the theorem, but with $\gamma = 1/(k + l)$ and $p = (k + l)/k$. By Proposition 6.4, this estimate is also true if the cut-off $\varphi(x, y)$ is replaced by $\varphi(x^{1/\eta_1}, y) \chi_{\mathbb{R}^+}(x) \chi_{\mathbb{R}^+}(y)$. Now we can apply Lemma 2.6 with $a = 0$ and $p_0 = (k + l)/k$ to obtain

$$\int_0^\infty \left| \int_0^\infty \exp(i \lambda S(x, y)) \varphi(x^{1/\eta_1}, y) f(y) dy \right|^p x^{-\theta p_0} dx \leq C \left( |a_{k,l}| |\lambda| \right)^{-\frac{p}{k \eta_1} (1 - \theta)} \int_0^\infty |f(x)|^p dx$$

for $1 < p \leq p_0$, where $\theta$ satisfies $1/p = \theta + (1 - \theta)/p_0$. Put $p = \frac{k \eta_1 + l}{k \eta_1}$. It is easy to see that $\theta = \frac{p_0}{p} - \frac{1}{p_0 - 1}$ and hence $1 - \theta p = \frac{p_0 - 1}{p_0 - 1} = \frac{1}{\eta_1}$. It follows that

$$\int_0^\infty \left| \int_0^\infty \exp(i \lambda S(x^{\eta_1}, y)) \varphi(x, y) f(y) dy \right|^p dx \leq C \left( |a_{k,l}| |\lambda| \right)^{-\frac{1}{k \eta_1}} \int_0^\infty |f(x)|^p dx, \quad p = \frac{k \eta_1 + l}{k \eta_1}.$$
By a duality argument, an application of Lemma 2.6 again yields
\[
\int_0^\infty \left| \int_0^\infty \exp(i\lambda S(x^n, y^n)) \varphi(x, y)f(y)dy \right|^p dx \\
\leq C \left( |a_{k,l}| |\lambda| \right)^{-\frac{1}{2n}} \int_0^\infty |f(x)|^p dx, \quad p = \frac{k\eta_1 + l\eta_2}{k\eta_1}.
\]
This completes the proof of the theorem. \(\square\)

Now consider the following class of real-valued polynomials in \(\mathbb{R}^{n_X} \times \mathbb{R}^{n_Y}\) with \(n_X, n_Y \geq 1\):
\[
S(x, y) = \sum_{i=1}^N P_i(x)Q_i(y),
\]
where \(P_i\) and \(Q_i\) are real-valued homogeneous polynomials in \(\mathbb{R}^{n_X}\) and \(\mathbb{R}^{n_Y}\), respectively. Then we can state our main result for higher dimensional oscillatory integral operators as follows.

**Theorem 6.3** Assume \(S\) is a real-valued polynomial defined as above. Suppose the following conditions hold.

(i) There exist two integers \(m \geq n_X\) and \(n \geq n_Y\) such that \(\deg(P_i) = k_i m\) and \(\deg(Q_i) = l_i n\) for two positive integers \(k_i, l_i\).

(ii) For two numbers \(c > 0\) and \(d > 0\), \(S\) can be written as
\[
S(x, y) = \sum_{k_i + l_i = d} P_i(x)Q_i(y)
\]
with \(k_i, l_i\) having the same meaning as in (i). Then there exists a constant \(C = C(\deg(S), \varphi)\) such that \(\|T_\lambda f\|_{L^p(\mathbb{R}^{n_X})}\) is bounded by
\[
C|\lambda|^{-\gamma} \left( \int_{S_X^{n_X}} |P_i(x')|^{-\frac{n_X}{k_i m}} dx' \right)^{1/p} \left( \int_{S_Y^{n_Y}} |Q_i(y')|^{-\frac{n_Y}{l_i n}} dy' \right)^{1/p'} \|f\|_{L^p(\mathbb{R}^{n_Y})},
\]
where \(p\) and \(\gamma\) are given as follows:
\[
p = \frac{k_i m/n_X + l_i n/n_Y}{k_i m/n_X}, \quad \gamma = \frac{1}{k_i m/n_X + l_i n/n_Y}.
\]

**Proof.** We shall apply Theorem 1.1 to prove this theorem with the rotation method. By polar coordinates, we write \(x = \rho x'\) and \(y = ry'\) with \(\rho = |x|\) and \(r = |y|\). Thus the phase \(S\) can be written as
\[
S(x, y) = \sum_{k_i + l_i = d} P_i(x')Q_i(y')\rho^{k_i m}r^{l_i n}.
\]
We first calculate the \(L^p\) norm of \(T_\lambda f\) in the radial direction.
\[
\int_0^\infty |Tf(x)|^p \rho^{n_X-1} d\rho \\
= \int_0^\infty \left| \int_{S_Y^{n_Y}} \int_0^\infty e^{i\lambda S(\rho x', ry')} f(ry')\varphi(\rho x', ry')r^{n_Y-1} dr d\sigma(y') \right|^p d\rho \\
= C \int_0^\infty \left| \int_{S_Y^{n_Y}} \int_0^\infty e^{i\lambda S(\rho^{n_X} x', \frac{1}{r^{n_Y}} y')} \varphi(\rho^{n_X} x', \frac{1}{r^{n_Y}} y') f(\frac{1}{r^{n_Y}} y') dr d\sigma(y') \right|^p d\rho.
\]
By Minkowski’s inequality and Hölder’s inequality, we can apply Theorem 6.2 to obtain
\[
\left( \int_0^\infty |Tf(px')|^p \rho^{nX-1} \, dp \right)^{1/p} \leq C \int_{S^{nY-1}} \left( \int_0^\infty \left| e^{i\lambda S(\rho^{1/nX} x', x^{1/nY} y')} \varphi(\rho^{1/nX} x', r^{1/nY} y') f(r^{1/nY} y') \right|^p \, dp \right)^{1/p} \, d\sigma(y')
\]
\[
\leq C \int_{S^{nY-1}} |\lambda|^{-\delta_1} |P_i(x')|^{-\delta_i} |Q_i(y')|^{-\delta_1} \|f(\cdot, y')\|_{L^p(R^{nY-1} \, dr)} \, d\sigma(y')
\]
\[
\leq C |\lambda|^{-\delta_1} |P_i(x')|^{-\delta_i} \left( \int_{S^{nY-1}} |Q_i(y')|^{-\frac{nY}{m}} \, d\sigma(y') \right)^{1/p'} \|f\|_{L^p(R^{nY})}, \quad \delta_i = \frac{1}{k_i m/nX + l_i n/nY}.
\]

In the second inequality, it should be pointed out that our application of Theorem 6.2 produces a constant \(C\) independent of \(x', y'\). In fact, by Fourier expansion, this can be verified by Proposition 6.1. Taking the \(L^p\) norm of the above integrals over \(S^{nX-1}\), we obtain the desired result. \(\square\)

As a consequence of the above theorem, we obtain the following \(L^p\) boundedness of higher dimensional oscillatory integral operators without cut-off function.

**Theorem 6.4** Assume \(S\) is a real-valued polynomial satisfying all assumptions in Theorem 6.3. Let \(T(S, \phi \equiv 1)\) be defined as \(T_\lambda\) in (6.49), but with \(\lambda S\) replaced by \(S\). If, for some \(i\), \(P_i\) and \(Q_i\) satisfy the integrability conditions:
\[
\int_{S^{nX-1}} |P_i(x')|^{-\frac{nX}{m}} \, d\sigma(x') < \infty \quad \text{and} \quad \int_{S^{nY-1}} |Q_i(y')|^{-\frac{nY}{m}} \, d\sigma(y') < \infty,
\]
then \(T\) is bounded from \(L^p(\mathbb{R}^{nY})\) into \(L^p(\mathbb{R}^{nX})\) with \(p\) defined as in Theorem 6.3.

Now we discuss the necessity of degree gaps between polynomials \(P_i\) (also \(Q_i\)) in Theorem 6.3. Some examples show that the assumption (i) is necessary. On the other hand, these examples are related to a conjecture raised by A. Greenleaf, M. Pramanik and W. Tang in [9].

We first present an example in \((4+4)\) dimensions. For \(x, y \in \mathbb{R}^4\), let \(\rho = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}\) and \(r = |y| = \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}\). Consider the phase \(S(x, y) = (\rho^4 - r^2)^{36}\). Then the decay estimates in Theorem 6.3 are not true for \(T_\lambda\) with this phase. Since Theorem 6.4 is a corollary of Theorem 6.3, it suffices to show that \(T\) is unbounded on certain Lebesgue spaces. In fact, we shall see that \(T\), associated with phase \(S(x, y) = (\rho^4 - r^2)^{36}\), is not bounded from \(L^p(\mathbb{R}^4)\) into itself for all \(1 \leq p \leq \infty\). Assume the converse. With change of variables in polar coordinates, we would see that the corresponding \(L^p\) boundedness is true in \((1+1)\) dimension with phase \(S(u, v) = (u - \sqrt{v})^{36}\). In other words, we would obtain a constant \(C < \infty\) such that
\[
\left( \int_0^\infty \left| \int_0^\infty e^{i(u - \sqrt{v})^{36}} f(v) \, dv \right|^p \, du \right)^{1/p} \leq C \left( \int_0^\infty |f(v)|^p \, dv \right)^{1/p}.
\]

It is easy to see that this inequality is not true for \(p = 1, \infty\). For \(1 < p < \infty\), let \(f\) be the characteristic function of the interval \(I_M = (M^2, M^2 + \epsilon_0 M)\) with \(M \geq 1\) and \(0 < \epsilon_0 \ll 1\). For \(u \in (M, M + \epsilon_0)\) and \(v \in I_M\), we have \(|u - \sqrt{v}| \leq C \epsilon_0\). For sufficiently small \(\epsilon_0 > 0\), the left side of (6.53) is not less than a constant multiple of \(M\). But the right side equals \((\epsilon_0 M)^{1/p}\). Therefore the above inequality is not true for large \(M\). Write \(S(x, y) = (\rho^4 - r^2)^{36}\) as the form in Theorem 6.3 i.e., \(S(x, y) = \sum P_i(x)Q_i(y)\). Then the degree gap between \(Q_i\)
and $Q_{i+1}$ is strictly less than $n_Y = 4$. This explains why we shall impose the assumptions $|\text{deg}(P_i) - \text{deg}(P_{i+1})| \geq n_X$ and $|\text{deg}(Q_i) - \text{deg}(Q_{i+1})| \geq n_Y$ on the phase $S$. More generally, consider the phase $S(x, y) = (\rho^{2m} - r^{2n})^N$ with $m, n \geq 1$ and $N \geq 2$. Here $\rho = |x|$ and $r = |y|$. If $2m = n_X$ and $2n < n_Y$, then $T_\lambda$ does not satisfy the decay estimate in Theorem 6.5.3.

For a real analytic function $S \in C^\omega(\Omega)$ with $\Omega$ being a neighborhood of the origin in $\mathbb{R}^{n_X+n_Y}$, we write it as the Taylor expansion $S(x, y) = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\alpha y^\beta$. For our purpose, we may assume all pure $x$ and $y$ terms in this series vanish. Then the Newton polyhedron of $S$ is invariant under linear transformations on $\mathbb{R}^n$. The modified Newton distance $R = \inf_{x \in \mathbb{R}^n} \frac{1}{\|\nabla S(x)\|}$ is strictly less than $4$. This explains why we shall impose the assumptions $|\delta, \cdots, \delta| \in \mathbb{R}^{n_X+n_Y}$ belongs to $N(S)$. Since the decay rate of $\|T_\lambda\|$ is invariant under linear transformations on $\mathbb{R}^{n_X}$ and $\mathbb{R}^{n_Y}$, Greenleaf, Pramanik and Tang \cite{9} introduced the following modified Newton distance

$$\delta_{\text{mod}}(S) = \sup \{ \delta(S(Ax, By)) : A \in \text{GL}(n_X), \ B \in \text{GL}(n_Y) \}.$$ 

It was conjectured in \cite{9} that $T_\lambda$, with phase $S$, should satisfy the following decay estimate

$$\|T_\lambda f\|_{L^2} \leq C|\lambda|^{-\frac{1}{2\delta_{\text{mod}}(S)}} \left( \log|\lambda| \right)^p \|f\|_{L^2} \quad (6.54)$$

for some $p \geq 0$ and a constant $C$ independent of $\lambda$. For homogeneous polynomial phases satisfying various genericity assumptions, for example homogeneous polynomial phases with either full rank or rank one Hessian away from the origin, Greenleaf, Pramanik and Tang \cite{9} proved (6.54). Especially, their result is optimal in $(2+2)$-dimensions under genericity assumptions.

For the phases considered above, the decay estimate (6.54) is not true. For example, $S(x, y) = (\rho^4 - r^2)^{36}$, both the Newton distance and the modified one are equal to 12. Now we prove $\delta_{\text{mod}}(S) = 12$. First observe that all terms in $S(x, y)$ take the form $x^\alpha y^\beta$ with

$$\frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4} + \frac{\beta_1 + \beta_2 + \beta_3 + \beta_4}{2} = 36.$$

This implies that $\delta(S), \delta_{\text{mod}}(S) \geq 12$. On the other hand, $S(x, y)$ contains the following term

$$\left( x_1^2 + x_2^2 + x_3^2 + x_4^2 \right)^{24} \left( y_1^2 + y_2^2 + y_3^2 + y_4^2 \right)^{24}$$

which contains the monomial $\prod_{i=1}^4 (x_i^{12} y_i^{12})$. This shows that $\delta(S) \leq 12$. Hence $\delta(S) = 12$. Moreover, we can prove that $S(Ax, By)$ contains also the monomial $(x_1 x_2 x_3 x_4)^{12}(y_1 y_2 y_3 y_4)^{12}$ for arbitrary $A, B \in \text{GL}(4)$. Now we show $\delta_{\text{mod}}(S) = 12$ with a different method. One can verify that $S(Ax, By)$ has the term $(x^T A^T Ax)^{24}(y^T B^T By)^{24}$ which of course contains monomials of form $x_i^{18} y_j^{18}$ for $1 \leq i, j \leq 4$. Then $(48 e_i, 48 e_j)$ belongs to $N(S(Ax, By))$. By a convex combination, we see that $12(e_1 + e_2 + e_3 + e_4, e_1 + e_2 + e_3 + e_4)$ belongs to the Newton polyhedron of $S(Ax, By)$. Here $e_i$ is the $i$-th coordinate unit vector in $\mathbb{R}^4$. So $\delta_{\text{mod}}(S) = 12$. If (6.54) were true, we would obtain

$$\|T_\lambda f\|_{L^2} \leq C|\lambda|^{-1/24} \left( \log |\lambda| \right)^p \|f\|_{L^2} \quad (6.55)$$

for some $p \geq 0$. Assume the smooth cut-off $\varphi$ is radial in both variables $x$ and $y$. For radial functions $f$, it follows from the above estimate that

$$\left( \int_0^\infty \left( \int_0^\infty e^{i\lambda(\rho - \sqrt{\tau})^{36}} \varphi(\rho^{1/4}, r^{1/2}) f(r) dr \right)^2 d\rho \right)^{1/2} \leq C|\lambda|^{-1/24} \left( \log |\lambda| \right)^p \|f\|_{L^2}$$

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for all $f \in L^2(\mathbb{R}^+)$. Indeed, the optimal decay rate of the left side is $\frac{1}{36}(< \frac{1}{24})$ for general $f \in L^2$. Choose a smooth cut-off $\varphi$ which is positive near the origin. Then for some small $0 < \epsilon_0 < 1$, we have $\varphi(x, y) > 0$ for $|x|, |y| \leq \epsilon_0$. For sufficiently large $\lambda$, we define $f$ to be the characteristic function of $(\frac{\epsilon_0}{2}, \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2}|\lambda|^{-1/36})$. If we take $\rho$ in the interval $(\sqrt{\epsilon_0/2}, \sqrt{\epsilon_0/2} + \sqrt{\epsilon_0/2}|\lambda|^{-\frac{1}{36}})$, then we see that the left side of the above inequality is not less than a constant multiple of $|\lambda|^{-1/24}$.

By change of variables, one has $\sigma = \frac{1}{36}$; see [21, 22]. Hence the above inequality does not hold.

Generally, we may consider the phase $S(x, y) = (\rho^{2m} - r^{2n})^N$ with $m, n \geq 1$ and $N \geq 2$. Here $\rho$ and $r$ denote the length of $x$ and $y$, respectively. If $2m = nX$ and $2n < nY$, then we have $\delta_{mod}(S) = 2N (\frac{nX}{m} + \frac{nY}{n})^{-1}$ which is less than $N/2$. Thus the decay rate $1/(2\delta_{mod}(S)) > 1/N$. As above, we can prove the optimal $L^2$ decay rate for $T_\lambda(S, \varphi)$ is $1/N$. This shows that the decay estimate (6.54) is not generally true.

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