Global weak solutions and long time behavior for 1D compressible MHD equations without resistivity

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Abstract: We study the initial-boundary value problem for 1D compressible MHD equations of viscous non-resistive fluids in the Lagrangian mass coordinates. Based on the estimates of upper and lower bounds of the density, weak solutions are constructed by approximation of global regular solutions, the existence of which has recently been obtained by Jiang and Zhang in [17]. Uniqueness of weak solutions is also proved as a consequence of Lipschitz continuous dependence on the initial data. Furthermore, long time behavior for global solutions is investigated. Specifically, based on the uniform-in-time bounds of the density from above and below away from zero, together with the structure of the equations, we show the exponential decay rate in $L^2$- and $H^1$-norm respectively, with initial data of arbitrarily large.

Keywords: non-resistive MHD equations, global weak solutions, long time behavior, initial-boundary value problem

1 Introduction

The motion of conducting fluids is described by the system of magnetohydrodynamics (MHD). In Eulerian coordinates, a typical model for 3D compressible MHD fluids assumes the following form (see [5]):

\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0, \]
\[ (\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \nu \Delta \mathbf{u} + (\nu + \eta) \text{div}\mathbf{u} + (\nabla \times \mathbf{b}) \times \mathbf{b}, \]
\[ \mathbf{b}_t = \nabla \times (\mathbf{u} \times \mathbf{b}) - \lambda \nabla \times (\nabla \times \mathbf{b}), \]
\[ \text{div}\mathbf{b} = 0. \]
Here the unknown functions $\rho, \mathbf{u} \in \mathbb{R}^3$, $p$ and $\mathbf{b} \in \mathbb{R}^3$ denote the density of fluid, the velocity, the pressure and the magnetic field, respectively. The viscosity coefficients $\nu$ and $\eta$ satisfy

$$\nu > 0, \ 3\eta + 2\nu \geq 0.$$ 

Moreover, $\lambda \geq 0$ is the resistivity coefficient which represents the magnetic diffusion of the field $\mathbf{b}$. The compressible fluid is assumed to be isentropic, which means the pressure $p$ is prescribed through the following constitutive relation:

$$p(\rho) = A\rho^\gamma, \quad (1.5)$$

where $A$ is a positive constant and the adiabatic exponent $\gamma > 1$.

Assuming that the resistivity coefficient $\lambda$ is positive, based on the pioneering work of P. L. Lions [24], E. Feireisl et al. [11], Hu and Wang [14] obtained the global existence and large time behavior of global weak solutions to 3D equations of compressible MHD flows. B. Ducomet and E. Feireisl [7] proved the global weak solutions to the Navier-Stokes-Fourier system, coupled with the Maxwell equations with finite energy initial data.

In fact, the resistivity coefficient $\lambda$ is extremely small in practical models, and the fluid is often referred to as perfect conductor if $\lambda = 0$. Thus, it is reasonable to consider the compressible isentropic MHD equations without resistivity, where (1.4) reads as

$$\mathbf{b}_t = \nabla \times (\mathbf{u} \times \mathbf{b}).$$

Compared with the case of positive resistivity, mathematical investigations to (1.1)-(1.4) with $\lambda = 0$ are relatively few. Obviously, zero resistivity introduces extra difficulty to build global solutions. The only known results on the multi-dimensional case is the recent work of Wu and Wu [29], where the authors have established global well-posedness for the initial value problem of 2D compressible non-resistive MHD system with initial data close to the stationary solution $\mathbf{u}_s = (0, 0), \mathbf{b}_s = (1, 0)$. This is an extension of early results of Lin et al. [23, 30] for incompressible MHD to the compressible case. As to the incompressible MHD without resistivity, see also [26, 33].

In this article, we focus on the MHD equations without resistivity and restrict ourselves to the simplest one-dimensional case. By assuming

$$\rho = \rho(x, t), \mathbf{u} = (u(x, t), 0, 0), \mathbf{b} = (0, 0, b(x, t)),$$

where $x \in \mathbb{R}$ is the spatial variable, (1.1)-(1.4) (with $\lambda = 0$) are reduced to (see [12])

$$\rho_t + (\rho u)_x = 0, \quad (1.6)$$

$$(\rho u)_t + (\rho u^2 + p(\rho) + \frac{1}{2} b^2)_x = \mu u_{xx}, \quad (1.7)$$

$$b_t + (bu)_x = 0, \quad (1.8)$$

where the pressure $p$ satisfies (1.5) and $\mu = 2\nu + \eta > 0$.

Recently, Jiang and Zhang in [17] obtained the global well-posedness of strong solutions to the initial-boundary value problem for (1.6)-(1.8) with initial data of arbitrary size, by making a full use of the effective viscous flux, the material derivative and the structure of the equations.
See also Yu [31] for a similar result concerning the appearance of vacuum, but with more restriction on the initial magnetic field. We refer to [8, 32] for more results on 1D compressible heat-conductive MHD equations with vanishing resistivity.

It should be noted that if the resistivity coefficient $\lambda$ is included above, (1.8) becomes

$$b_t + (bu)_x = \lambda b_{xx}.$$  \hspace{1cm} (1.9)

There are many investigations for system (1.6)-(1.7), (1.9). Kazhikhov and Smagulov in [21] announced the global well-posedness of strong solutions to the one-dimensional compressible, heat-conductive, viscous fluids with resistivity. Fan, Jiang and G. Nakamura in [9] obtained the existence, uniqueness and Lipschitz continuous dependence on the initial data of global weak solutions to a similar system.

As it is well-known, the classical method to handle one-dimensional models in fluid mechanics is the use of Lagrangian mass coordinates. To this end, we assume the fluid occupies the interval $[0, 1]$. Let

$$y = \int_0^x \varrho(\xi, t) d\xi, \quad s = t.$$  \hspace{1cm} (1.10)

Then (1.6)-(1.8) are reformulated as

$$\tau_t = u_x,$$  \hspace{1cm} (1.11)
$$u_t = \sigma_x,$$  \hspace{1cm} (1.12)
$$(b\tau)_t = 0,$$  \hspace{1cm} (1.13)

with

$$\tau := \varrho^{-1}$$

the specific volume of the flow and the effective viscous flux

$$\sigma := \frac{\mu u_x}{\tau} - A\tau^{-\gamma} - \frac{1}{2} b^2.$$  \hspace{1cm} (1.14)

Here for convenience, we still use $(x, t)$ instead of $(y, s)$ to denote the spatial and temporal variables. Without loss of generality, we assume the conserved total mass on $[0, 1]$ is one unit. We then supplement system (1.11)-(1.14) with the following initial and boundary conditions:

$$\tau(x, 0) = \tau_0(x), \quad u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x), \quad x \in [0, 1],$$  \hspace{1cm} (1.15)
$$u(0, t) = u(1, t) = 0, \quad t \in (0, \infty).$$  \hspace{1cm} (1.16)

The present paper is dedicated to the study of global weak solutions to the initial-boundary value problem (1.11)-(1.16). Based on estimates of upper and lower bounds of the density, we first construct weak solutions by approximation of global regular solutions, the existence of which is guaranteed by Jiang and Zhang [17]. Then we show the stability of weak solutions, that is, the Lipschitz continuous dependence on the initial data. The uniqueness of global weak solutions follows as a consequence of stability. In particular, similar to the results for one-dimensional Navier-Stokes(-Fourier) system, see [11, 13, 16, 35] among others, our results show that neither vacuum nor concentration can form in finite time for weak solutions. Furthermore, based on the uniform-in-time bounds of the density from above and below away from zero, the exponential decay estimates of solutions are obtained in $L^2$- and $H^1$-norm respectively.
It should be noted that the stabilization for 1D compressible barotropic Navier-Stokes equations has been well-established since the work of Kanel [18] and Kazhikhov [20]. Extensions to more general barotropic case or the inclusion of external forces can be found in [3, 4, 25, 27, 28, 34]. The reader may consult [2] for the stability of 1D Navier-Stokes-Fourier system in bounded domain. Also see [15, 19, 22] for the case of unbounded domains.

Before giving the main results of this paper, we introduce the notations and functional spaces used throughout this paper. Denote \( \Omega := (0, 1) \), \( \Omega_t := \Omega \times (0, t) \). Let \( p \in [1, \infty] \), \( k \) be a positive integer. We denote the usual Lebesgue space \( L^p(\Omega) \) by \( L^p \), with its norm \( \| \cdot \|_{L^p} \); \( H^k \) denotes the usual Sobolev space \( H^k(\Omega) \), with its norm \( \| \cdot \|_{H^k} \); \( L^p(0, T; X) \) is the space of all strongly measurable, \( p \)-th power integrable functions from \( (0, T) \) to \( X \), with \( X \) being some Banach space and its corresponding norm \( \| \cdot \|_{L^p(0, T; X)} \). The Sobolev space \( W^{1,p}(0, T; X) \) consists of all functions \( v \in L^p(0, T; X) \) such that \( v_t \) exists in the weak sense and belongs to \( L^p(0, T; X) \). The Banach space \( C([0, T]; X) \) stands for all continuous functions from \([0, T]\) to \( X \).

Concerning with the initial-boundary value problem (1.11)-(1.16) for an isentropic, viscous and compressible flow, the first result of this paper is the existence of global weak solutions.

**Theorem 1.1** Assume

\[
\inf_{x \in (0,1)} \tau_0(x) > 0, \quad \tau_0 \in L^\infty, \quad u_0 \in L^2, \quad b_0 \in L^\infty. \tag{1.17}
\]

Then there exists a weak solution \((\tau, u, b)\) to (1.11)-(1.16) in the time interval \([0, T]\) for any fixed \( T \in (0, \infty) \). Moreover, there exists a constant \( C > 0 \), such that

\[
C^{-1} \leq \tau(x, t) \leq C, \quad |b(x, t)| \leq C, \quad \text{for a.e.} \quad (x, t) \in \Omega_T,
\]

\[
\|u\|_{L^\infty(0, T; L^2)} + \|u_t\|_{L^2(0, T; L^2)} + \|\tau_t\|_{L^2(0, T; L^2)} + \|b_t\|_{L^2(0, T; L^2)} \leq C. \tag{1.19}
\]

Here and in the next theorem, the letter \( C \) denotes a generic positive constant depending only on the parameters \( A, \gamma, \mu \), the fixed time \( T \) and the initial data. The definition of weak solutions will be given in the next section.

The next theorem concerns the stability of weak solutions obtained in Theorem 1.1.

**Theorem 1.2** Let \((\tau, u, b)\) and \((\hat{\tau}, \hat{u}, \hat{b})\) be two weak solutions on \([0, 1] \times [0, T]\) corresponding to the initial data \((\tau_0, u_0, b_0)\) and \((\hat{\tau}_0, \hat{u}_0, \hat{b}_0)\). Then there exists a constant \( C > 0 \) such that

\[
\|\tau - \hat{\tau}\|_{L^\infty(\Omega_T)} + \|u - \hat{u}\|_{L^\infty(0, T; L^2)} + \|b - \hat{b}\|_{L^\infty(\Omega_T)} + \|(u - \hat{u})_x\|_{L^2(0, T; L^2)} \leq C(\|\tau_0 - \hat{\tau}_0\|_{L^\infty} + \|b_0 - \hat{b}_0\|_{L^\infty} + \|u_0 - \hat{u}_0\|_{L^2}). \tag{1.20}
\]

Obviously, Theorem 1.2 in particular implies the uniqueness of weak solutions.

**Corollary 1.1** Under the assumptions of Theorem 1.1 there exists a unique global weak solution to the initial-boundary value problem (1.11)-(1.16).

The subsequent two theorems are associated with the long time behavior for global solutions to (1.11)-(1.16).
Theorem 1.3 Let the assumption (1.17) be satisfied and \((\tau, u, b)\) be the unique weak solution to (1.11)-(1.16). Then there exist two positive constants \(C_1, C_2\) which are independent of time, such that

\[
\|\tau - \tau_s\|_{L^2} + \|u(t)\|_{L^2} + \|b - b_s\|_{L^2} \leq C_1 \exp(-C_2t), \quad \text{for any } t \geq 0. \tag{1.21}
\]

Here, \((\tau_s, u_s, b_s)\) are the stationary solution to (1.11)-(1.16) which will be introduced in Section 5. Here and in the next theorem, we denote \(C, C_i\) by generic positive constants depending only on the parameters of the system, the initial data and the stationary solution.

Given more regular initial data, we are able to strengthen the exponential decay rate of solutions in \(H^1\)-norm. To be more precise, we have

Theorem 1.4 Assume

\[
\inf_{x \in (0,1)} \tau_0(x) > 0, \quad \tau_0 \in W^{1,\infty}, \quad u_0 \in H^1_0, \quad b_0 \in W^{1,\infty}. \tag{1.22}
\]

Let \((\tau, u, b)\) be the unique strong solution to (1.11)-(1.16). Then there exist two positive constants \(C_3, C_4\), such that

\[
\|\tau - \tau_s\|_{H^1} + \|u(t)\|_{H^1} + \|b - b_s\|_{H^1} \leq C_3 \exp(-C_4t), \quad \text{for any } t \geq 0. \tag{1.23}
\]

The key point to obtain these results, especially Theorem 1.1 and 1.3 on the existence of global weak solution and its long time behavior, is the observation that under the Lagrangian formulation, the magnetic field \(b\) is solved out as \(b = b_0 \tau_0 \tau^{-1}\). This observation results in the momentum equation a non-standard pressure law \(p = p(x, \tau)\). The dependence of \(p\) on the spatial variable \(x\) makes it difficult to apply the traditional approaches for 1D isentropic Navier-Stokes equations such as in [20], especially for uniform pointwise estimates for the density. To overcome this difficulty, we have to modify the methods developed in [2] to handle the full Navier-Stokes-Fourier system as well as in [34] to treat a wider class of pressure laws. Moreover, it is also new for the large time behavior of the specific volume as well as the magnetic field in the sense that they approach to the nontrivial stationary solution \((\tau_s, b_s)\) determined by (5.20) and (5.21).

The rest of this paper is organized as follows. In Section 2 we recall the existence of global strong solution due to Jiang and Zhang [17] under the framework of Eulerian coordinates. In Section 3 we prove Theorem 1.1 by approximation of strong solutions. The proof of Theorem 1.2 is completed in Section 4 by modifying the ideas used in [16]. The proof of Theorem 1.3 and Theorem 1.4 are finished in Section 5 by means of establishing the necessary uniform-in-time estimates.

2 Preliminary Results

To establish the existence of weak solution, we use approximation of strong solutions, the existence of which has been obtained in [17] in the framework of Eulerian coordinates. It should be pointed out that for the initial-boundary value problem (1.11)-(1.16), the global existence (and uniqueness) of strong solutions still holds in our case of Lagrangian formulation. Here, for completeness and later use, we just state this result and give a sketch of the proof. Throughout the present and the next two sections, the letter \(C\) denotes a generic positive constant which is described after the statement of Theorem 1.1 in the introduction.
Proposition 2.1 Assume that the initial data \((\tau_0, u_0, b_0)\) given in (1.15) satisfy
\[
\min_{0 \leq x \leq 1} \tau_0(x) > 0, \quad (\tau_0, b_0) \in H^1, \quad u_0 \in H^1_0.
\] (2.1)

Then there exists a unique strong solution \((\tau, u, b)\) in the time interval \([0, \infty)\) to the initial-boundary value problem (1.11)-(1.16) such that
\[
(\tau, b) \in L^\infty_{loc}(0, \infty; H^1), \quad (\tau_t, b_t) \in L^2_{loc}(0, \infty; L^2), \quad u \in L^\infty_{loc}(0, \infty; H^1_0), \quad u_t \in L^2_{loc}(0, \infty; H^2).
\] (2.2)

Furthermore, for any fixed \(0 < T < \infty\), there exists a positive constant \(C\) such that
\[
C^{-1} \leq \tau(x, t) \leq C, \quad \text{for any } (x, t) \in [0, 1] \times [0, T],
\] (2.4)
\[
\|\tau, b, u\|_{L^\infty(0, T; H^1)} + \|\tau_t, b_t, u_{xx}, u_t\|_{L^2(\Omega_T)} \leq C.
\] (2.5)

The proof of this proposition is essentially based on global a priori estimates. We first give the standard energy estimates without proof.

Lemma 2.1 Let \((\tau, u, b)\) be a smooth solution to the initial-boundary value problem (1.11)-(1.16) on \([0, 1] \times [0, T]\). Then
\[
\int_0^1 \tau(x, t) dx = \int_0^1 \tau_0(x) dx = 1, \quad \text{for any } t \in [0, T],
\] (2.6)
and
\[
\sup_{0 \leq t \leq T} \int_0^1 \left( \frac{1}{2} u^2 + \frac{A}{\gamma - 1} \tau^{1-\gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right) dx + \mu \int_0^T \int_0^1 \frac{u^2}{\tau} dx \, ds \leq C.
\] (2.7)

The next lemma gives the upper and lower bounds of the specific volume, which is essential for the proof of Proposition 2.1. Here we modify the argument of Antontsev et. al., see [2].

Lemma 2.2 Let \((\tau, u, b)\) be a smooth solution to the initial-boundary value problem (1.11)-(1.16) on \([0, 1] \times [0, T]\). Then
\[
C^{-1} \leq \tau(x, t) \leq C, \quad \text{for any } (x, t) \in [0, 1] \times [0, T].
\] (2.8)

Proof. Note that (1.12) can be rewritten, using (1.11), (1.13) and (1.14), as
\[
u_t = \left( \frac{u_x}{\tau} - A \tau^{-\gamma} - \frac{1}{2} b_0^2 \right)_x
= \left( \mu (\log \tau)_t - A \tau^{-\gamma} - \frac{1}{2} b_0^2 \tau^{-2} \right)_x.
\]
By (2.6) and the mean value theorem, for each \(t \in [0, T]\), there exists \(a(t) \in [0, 1]\), such that \(\tau(a(t), t) = 1\). Integrating the above equation first over \((0, t)\) with respect to \(t\), then over \((a(t), x)\) \((x\) is an arbitrarily fixed point in \([0, 1]\)) with respect to \(x\), and then taking exponential
on both sides of the resulting equation, we obtain the following representation of the specific volume $\tau$ as follows.

$$Y(t)\tau(x,t) = \tau_0(x)B(x,t)\exp\left\{\frac{1}{\mu} \int_0^t \left( A\tau^{-\gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right) (x,s)ds \right\}, \tag{2.9}$$

where

$$B(x,t) := \exp\left( \frac{1}{\mu} \int_{a(t)}^x u(\xi,t) - u_0(\xi) d\xi \right);$$

$$Y(t) := \tau_0(a(t))\exp\left\{\frac{1}{\mu} \int_0^t \left( A\tau^{-\gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right)(a(t),s)ds \right\}.$$

By Lemma 2.1 and Cauchy-Schwarz’s inequality,

$$C^{-1} \leq B(x,t) \leq C, \quad C^{-1} \leq Y(t), \quad \text{for any } (x,t) \in [0,1] \times [0,T]. \tag{2.10}$$

We compute

$$\frac{\partial}{\partial t} \exp\left\{\frac{1}{\mu} \int_0^t \left( A\tau^{-\gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right) (x,s)ds \right\}$$

$$= \frac{1}{\mu} \left( A\tau^{-\gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right) \exp\left\{\frac{1}{\mu} \int_0^t \left( A\tau^{-\gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right) (x,s)ds \right\}$$

$$= \frac{1}{\mu} \left( A\tau^{-\gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right) Y(t)\tau(x,t)\tau_0^{-1}(x)B^{-1}(x,t).$$

Integrating the above equation over $(0,t)$ with respect to $t$ gives

$$\exp\left\{\frac{1}{\mu} \int_0^t \left( A\tau^{-\gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right) (x,s)ds \right\}$$

$$= 1 + \frac{1}{\mu} \int_0^t \left( A\tau^{-\gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right) (x,s)Y(s)\tau_0^{-1}(x)B^{-1}(x,s)ds.$$

By substituting the above identity into (2.9), we find

$$Y(t)\tau(x,t) = \tau_0(x)B(x,t)$$

$$\times \left\{ 1 + \frac{1}{\mu} \int_0^t \left( A\tau^{-\gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right) (x,s)Y(s)\tau_0^{-1}(x)B^{-1}(x,s)ds \right\}. \tag{2.11}$$

Integrating (2.11) over $(0,1)$ with respect to $x$ and by virtue of (2.6), (2.7) and (2.10),

$$Y(t) \leq C \left( 1 + \int_0^t Y(s)ds \right),$$

which together with Gronwall’s inequality yields

$$Y(t) \leq C, \quad \text{for any } t \in [0,T]. \tag{2.12}$$

Therefore, (2.9), (2.10) and (2.12) imply

$$C^{-1} \leq \tau(x,t), \quad \text{for any } (x,t) \in [0,1] \times [0,T]. \tag{2.13}$$

The upper bound of the specific volume $\tau$ follows immediately from (2.10)-(2.13). This completes the proof of Lemma 2.2.
Remark 2.1 We note that in [17], to deal with the vanishing resistivity problem in Eulerian coordinates, the authors have to use a different approach to show the boundedness of the density from above and below by making a full use of the effective viscous flux, the material derivative and the structure of the equations. In particular, in their proof the lower boundedness of the density follows from the boundedness of the magnetic field, while in our case the boundedness of the magnetic field follows directly from that of the specific volume obtained in Lemma 2.2.

Once we have Lemma 2.1 and Lemma 2.2 at hand, it remains to derive the higher order energy estimates for the specific volume \( \tau \), the magnetic field \( b \), and the velocity field \( u \). We list the higher order energy estimates with detailed proof omitted here, see [2, 17].

**Lemma 2.3** Let \((\tau, u, b)\) be a smooth solution to the initial-boundary value problem (1.11)-(1.16) on \([0, 1] \times [0, T]\). Then there exists a constant \( C > 0 \) such that

\[
\|\tau_t, \tau_x, b_t, b_x, u_x\|_{L^\infty(0, T; L^2)} \leq C, \tag{2.14}
\]

\[
\|u_{xx}, u_t, \tau_{xx}, b_{tx}\|_{L^2(\Omega_T)} \leq C. \tag{2.15}
\]

Based on these a priori estimates, the global existence of strong solutions to the initial-boundary value problem (1.11)-(1.16) can be proved in a standard way.

Finally we introduce the definition of weak solution to the MHD system (1.11)-(1.13).

**Definition 2.1** We say that \((\tau, u, b)\) is a weak solution to MHD system (1.11)-(1.13) on \([0, 1] \times [0, T]\) with boundary condition (1.16) and initial data \((\tau_0, u_0, b_0)\) satisfying

\[
\inf_{x \in (0, 1)} \tau_0(x) > 0, \tau_0, b_0 \in L^\infty, u_0 \in L^2,
\]

provided that

\[
\tau \in W^{1,2}(0, T; L^2), u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1_0), b \in W^{1,2}(0, T; L^2),
\]

\[
\inf_{(x,t) \in \Omega_T} \tau(x,t) > 0, \tau, b \in L^\infty(\Omega_T),
\]

\[
\tau_t = u_x, b = b_0 \tau_0^{-1} \text{ a.e. in } \Omega_T,
\]

\[
\tau(x, 0) = \tau_0(x) \text{ for a.e. } x \in \Omega,
\]

and that for any test function \( \psi \in C_c^\infty(\Omega \times [0, T]) \), the following integral identity holds:

\[
\int_0^T \int_\Omega u \psi_t - \left( \frac{ux}{\tau} - A \tau^{-\gamma} - \frac{1}{2} b^2 \right) \psi_x \, dx \, dt + \int_\Omega u_0 \psi(x, 0) \, dx = 0.
\]

3 Existence of weak solutions

In this section, to prove Theorem 1.1, we first obtain a strong solution to the initial-boundary value problem (1.11)-(1.16) by regularizing the initial data and then show the existence of global weak solutions.
Under the assumptions of initial data in (1.17), we construct a sequence \((\tau_0^\epsilon, u_0^\epsilon, b_0^\epsilon)\), by regularizing the initial data, such that
\[b_0^\epsilon, \tau_0^\epsilon \in C^2([0,1]), u_0^\epsilon \in C_\epsilon^2((0,1)),\]
\[C^{-1} \leq \tau_0^\epsilon \leq C, (\tau_0^\epsilon, u_0^\epsilon, b_0^\epsilon) \to (\tau_0, u_0, b_0) \text{ strongly in } L^2 \text{ as } \epsilon \to 0^+,\]
\[\|b_0^\epsilon\|_{L^\infty} \leq \|b_0\|_{L^\infty}.\]

Now we consider the initial-boundary value problem (1.11)-(1.16) with \((\tau_0, u_0, b_0)\) replaced by the approximate initial data \((\tau_0^\epsilon, u_0^\epsilon, b_0^\epsilon)\). It follows from Proposition 2.1 that there exists a unique global strong solution \((\tau^\epsilon, u^\epsilon, b^\epsilon)\) such that
\[0 < \tau^\epsilon < \infty \text{ in } [0,1] \times [0,T],\]
\[u^\epsilon \in L^\infty(0,T;L^2), u_x^\epsilon \in L^2(0,T;L^2),\]
\[\tau_t^\epsilon \in L^2(0,T;L^2), b^\epsilon_t \in L^2(0,T;L^2).\]

It should be pointed out that a careful review of Lemmas 2.1-2.2 shows that the approximate solutions \((\tau_0^\epsilon, u_0^\epsilon, b_0^\epsilon)\) have the following uniform-in-\(\epsilon\) bounds:
\[C^{-1} \leq \tau^\epsilon(x,t) \leq C, \text{ for any } (x,t) \in [0,1] \times [0,T],\]
\[\|u^\epsilon\|_{L^\infty(0,T;L^2)} + \|u_x^\epsilon\|_{L^2(0,T;L^2)} \leq C,\]
\[\|\tau_t^\epsilon\|_{L^2(0,T;L^2)} + \|b^\epsilon_t\|_{L^2(0,T;L^2)} \leq C.\]

In order to pass to the limits to obtain the existence of weak solutions to (1.11)-(1.16), we have to show that the specific volume \(\tau^\epsilon\) exists as a strong limit of \(\tau_0^\epsilon\), due to the nonlinearity of the system. For this purpose, we give the following crucial lemma. Let \(\Delta_h w(x) := w(x+h) - w(x)\), which is the difference of \(w\) with respect to \(x\).

**Lemma 3.1** For any \(0 < h < 1\), there holds
\[\|\Delta_h \tau^\epsilon\|_{L^\infty(0,T;L^2)} \leq C(\|\Delta_h \tau_0\|_{L^2} + \|\Delta_h b_0\|_{L^2} + h).\]

**Proof.** Set
\[\sigma^\epsilon := \mu \frac{u_x^\epsilon}{\tau^\epsilon} - A(\tau^\epsilon)^{1-\gamma} - \frac{1}{2}(b^\epsilon)^2, a_0^\epsilon := b_0^\epsilon \tau_0^\epsilon.\]
Thus \((\tau^\epsilon, u^\epsilon, b^\epsilon)\) satisfies the following system:
\[\tau_t^\epsilon = u_x^\epsilon,\]
\[u_t^\epsilon = \sigma_x^\epsilon,\]
\[(b^\epsilon \tau^\epsilon)_t = 0.\]

Note that (3.4)-(3.6) together give us
\[\tau_t^\epsilon = \frac{\sigma^\epsilon}{\mu} \mu \tau^\epsilon + \frac{A}{\mu}(\tau^\epsilon)^{1-\gamma} + \frac{(a_0^\epsilon)^2}{2\mu \tau^\epsilon}.\]
Multiplying (3.7) by \( \exp \left( -\frac{1}{\mu} \int_0^t \sigma^\varepsilon(x,s)ds \right) \) and integrating the resulting equation over \((0,t)\) with respect to \(t\) yields
\[
\exp \left( -\frac{1}{\mu} \int_0^t \sigma^\varepsilon(x,s)ds \right) \tau^\varepsilon = \tau_0^\varepsilon + \frac{1}{\mu} \int_0^t \exp \left( -\frac{1}{\mu} \int_0^\xi \sigma^\varepsilon(x,s)ds \right) \left( A(\tau^\varepsilon)^{1-\gamma} + \frac{(a_0^\varepsilon)^2}{2\tau^\varepsilon} \right) (x,\xi)d\xi.
\]

Hence we have
\[
\tau^\varepsilon = \left[ \tau_0^\varepsilon + \frac{1}{\mu} \int_0^t \exp \left( -\frac{1}{\mu} \int_0^\xi \sigma^\varepsilon(x,s)ds \right) \left( A(\tau^\varepsilon)^{1-\gamma} + \frac{(a_0^\varepsilon)^2}{2\tau^\varepsilon} \right) (x,\xi)d\xi \right] \times \exp \left( \frac{1}{\mu} \int_0^t \sigma^\varepsilon(x,s)ds \right), \tag{3.8}
\]

By defining
\[
B^\varepsilon(x,t) := \exp \left( \frac{1}{\mu} \int_0^t \sigma^\varepsilon(x,s)ds \right),
\]
and recalling (3.1), one sees
\[
C^{-1} \leq B^\varepsilon(x,t) \leq C, \text{ for any } (x,t) \in [0,1] \times [0,T]. \tag{3.9}
\]
Consequently, (3.8) reads
\[
\tau^\varepsilon = B^\varepsilon \left\{ \tau_0^\varepsilon + \frac{1}{\mu} \int_0^t (B^\varepsilon)^{-1}(x,\xi) \left( A(\tau^\varepsilon)^{1-\gamma} + \frac{(a_0^\varepsilon)^2}{2\tau^\varepsilon} \right) (x,\xi)d\xi \right\},
\]

and direct computation shows that
\[
\Delta_h \tau^\varepsilon(x,t) = \Delta_h B^\varepsilon(x,t) \left\{ \tau_0^\varepsilon(x+h) + \frac{1}{\mu} \int_0^t (B^\varepsilon)^{-1} \left( A(\tau^\varepsilon)^{1-\gamma} + \frac{(a_0^\varepsilon)^2}{2\tau^\varepsilon} \right) (x+h,\xi)d\xi \right\} + B^\varepsilon(x,t) \left[ \Delta_h \tau_0^\varepsilon + \frac{1}{\mu} \int_0^t (B^\varepsilon)^{-1}(x+h,\xi) \Delta_h \left( A(\tau^\varepsilon)^{1-\gamma} + \frac{(a_0^\varepsilon)^2}{2\tau^\varepsilon} \right) (x,\xi)d\xi \right] - B^\varepsilon(x,t) \left[ \frac{1}{\mu} \int_0^t (B^\varepsilon)^{-1}(x+h,\xi)(B^\varepsilon)^{-1}(x,\xi) \left( A(\tau^\varepsilon)^{1-\gamma} + \frac{(a_0^\varepsilon)^2}{2\tau^\varepsilon} \right) (x,\xi)d\xi \right],
\]

which, by (3.1) and (3.9), implies
\[
\| \Delta_h \tau^\varepsilon(x,t) \|_{L^2} \leq C(\| \Delta_h B^\varepsilon(x,t) \|_{L^2} + \| \Delta_h \tau_0^\varepsilon \|_{L^2}) \tag{3.10}
\]
\[
+ C \int_0^t \| \Delta_h B^\varepsilon(x,\xi) \|_{L^2} + \| \Delta_h \tau^\varepsilon(x,\xi) \|_{L^2} + \| \Delta_h a_0^\varepsilon \|_{L^2} d\xi \tag{3.11}
\]
\[
\leq C \left( h \| u^\varepsilon - u_0^\varepsilon \|_{L^\infty(0,T;L^2)} + \| \Delta_h \tau_0^\varepsilon \|_{L^2} + \| \Delta_h b_0^\varepsilon \|_{L^2} + \int_0^t \| \Delta_h \tau^\varepsilon(x,\xi) \|_{L^2} d\xi \right) \]

10
\[
C \left( h + \|\Delta_h \tau_0\|_{L^2} + \|\Delta_h b_0\|_{L^2} + \int_0^t \|\Delta_h \tau^\epsilon(x, \xi)\|_{L^2} d\xi \right). \tag{3.12}
\]

An application of Gronwall’s inequality to (3.12) yields

\[
\|\Delta_h \tau^\epsilon\|_{L^\infty((0, T); L^2)} \leq C \left( \|\Delta_h \tau_0\|_{L^2} + \|\Delta_h b_0\|_{L^2} + h \right),
\]

thus completing the proof of Lemma 3.1.

Note that (3.1)-(3.3) allow us to extract a subsequence of \((\tau^\epsilon, u^\epsilon, b^\epsilon)\), still denoted by \((\tau^\epsilon, u^\epsilon, b^\epsilon)\), such that as \(\epsilon \to 0^+\), the following weakly or weakly-\(\star\) convergences hold:

\[
\tau^\epsilon \to \tau \text{ weakly in } L^\infty(0, T; L^\infty), \tag{3.13}
\]

\[
u^\epsilon \to u \text{ weakly-\(\star\) in } L^\infty(0, T; L^2), \tag{3.14}
\]

\[
(\tau^\epsilon_t, u^\epsilon_x) \to (\tau_t, u_x) \text{ weakly in } L^2(0, T; L^2). \tag{3.15}
\]

In addition, for the limit functions \((\tau, u)\), we have

\[
C^{-1} \leq \tau(x, t) \leq C, \text{ for a.e. } (x, t) \in \Omega_T, \tag{3.16}
\]

\[
\|u\|_{L^\infty(0, T; L^2)} + \|u_x\|_{L^2(0, T; L^2)} + \|\tau_t\|_{L^2(0, T; L^2)} \leq C. \tag{3.17}
\]

By (3.3) and Lemma 3.1, we deduce that for any \(0 < h < 1, 0 < s < T\), there holds

\[
\|\tau^\epsilon(\cdot + h, \cdot + s) - \tau^\epsilon\|_{L^\infty(0, T-s; L^2)} \leq C \left( \|\Delta_h \tau_0\|_{L^2} + \|\Delta_h b_0\|_{L^2} + h + s^{1/2} \right). \tag{3.18}
\]

Recalling the criterion of compactness of sets in \(L^2(0, T; L^2)\) and invoking (3.13), (3.18) implies

\[
\tau^\epsilon \to \tau \text{ strongly in } L^2(0, T; L^2) \text{ as } \epsilon \to 0^+. \tag{3.19}
\]

By means of defining

\[
b := b_0 \tau_0 \tau^{-1},
\]

one checks easily, by virtue of (3.11), (3.15), (3.19) and (3.19), that

\[
b^\epsilon \to b \text{ strongly in } L^2(0, T; L^2) \text{ as } \epsilon \to 0^+, \tag{3.20}
\]

\[
b^\epsilon_t \to b_t \text{ weakly in } L^2(0, T; L^2) \text{ as } \epsilon \to 0^+, \tag{3.21}
\]

\[
\|b_t\|_{L^2(0, T; L^2)} \leq C. \tag{3.22}
\]

Based on Lemma 3.1 and the analysis of weak convergence given above, we are now ready to give the proof of Theorem 1.1.

We multiply (3.5) by any \(\phi \in C_c^\infty((0, 1) \times [0, T])\), then integrate over \(\Omega_T\), and perform an integration by parts. Letting \(\epsilon \to 0^+\), taking (3.13), (3.19), (3.19)–(3.21) into account, we find that \((\tau, u, b)\) obtained is a global weak solution to the initial-boundary value problem (1.11)-(1.16), by gathering the results for \((\tau, u, b)\) derived above. Moreover, the estimates (1.18) and (1.19) follow from (3.16), (3.17) and (3.22). The proof of Theorem 1.1 is therefore complete.
4 Uniqueness of weak solutions

In this section, we prove Theorem 1.2 by modifying the arguments used in [1, 16]. The proof is based on the following three lemmas.

Lemma 4.1 Let the assumptions of Theorem 1.2 be satisfied. Then the following representations are valid in $\Omega_T$:

$$\tau(x,t) = \exp\left(\frac{1}{\mu} \int_0^t \sigma(x,s) \, ds\right) \times \left[\tau_0 + \int_0^t \exp\left(-\frac{1}{\mu} \int_0^\xi \sigma(x,s) \, ds\right) \left(\frac{A}{\mu} \tau^{1-\gamma} + \frac{1}{2\mu} b_0^2 \tau^{-1} - 1\right) \frac{(A \mu \tau_0 - 1)^2}{2}\right],$$

(4.1)

and

$$\int_0^t \sigma(x,s) \, ds = (J_{\Omega}(u-u_0))(x,t) + \int_0^t <\sigma(\cdot, s) > \, ds,$$

(4.2)

where the linear operator $J_{\Omega}$ is defined by

$$J_{\Omega}w(x) := \int_0^x w(\xi) d\xi - <\int_0^x w(\xi) d\xi >, <w> := \int_0^1 w(x) dx.$$

Proof. Obviously, (1.11)-(1.14), and Theorem 1.1 imply the following relations:

$$\tau_t = \frac{\sigma}{\mu} \tau + A \left(\frac{1}{\mu} \tau^{1-\gamma} + \frac{1}{2\mu} b_0^2 \tau^{-1}\right),$$

(4.3)

$$\left(\int_0^t \sigma(x,s) \, ds\right)_x = u - u_0.$$  

(4.4)

Multiplying (4.3) by $\exp\left(-\frac{1}{\mu} \int_0^t \sigma(x,s) \, ds\right)$ and integrating the resulting equation over $(0,t)$ with respect to $t$ gives (4.1). In addition, applying the operator $J_{\Omega}$ to (4.1) yields (4.2) immediately.

Before stating the next lemma, for simplicity, we introduce the notations below.

$$(\Delta \sigma, \Delta u, \Delta b) := (\sigma - \bar{\sigma}, u - \bar{u}, b - \bar{b}),$$

$$(\Delta \tau_0, \Delta u_0, \Delta b_0) := (\tau_0 - \bar{\tau}_0, u_0 - \bar{u}_0, b_0 - \bar{b}_0),$$

$$\Delta \sigma := \sigma - \bar{\sigma}, \quad \bar{\sigma} := \frac{\bar{u}}{\bar{\tau}} - A(\bar{\tau})^{-\gamma} - \frac{1}{2}(\bar{b_0})^2(\bar{\tau})^{-2},$$

$$g := \exp\left(\frac{1}{\mu} \int_0^t \sigma(x,s) \, ds\right), \quad \bar{g} := \exp\left(\frac{1}{\mu} \int_0^t \bar{\sigma}(x,s) \, ds\right),$$

$$K := \frac{A}{\mu} \tau^{1-\gamma} + \frac{1}{2\mu} b_0^2 \tau^{-1}, \quad \bar{K} := \frac{A}{\mu}(\bar{\tau})^{1-\gamma} + \frac{1}{2\mu}(\bar{b_0})^2(\bar{\tau})^{-1},$$

$$\bar{\sigma} := (\bar{\tau})^{-1}, \Delta \phi := \phi - \bar{\phi}.$$  

Then our essential lemma with respect to the supremum norm of $\Delta \tau$ reads as follows.
Lemma 4.2 Let the assumptions of Theorem 1.2 be fulfilled. Then for any $t \in (0, T)$,
\[
\|\Delta \tau\|_{L^\infty(\Omega_t)} \leq C (\|\Delta \tau_0\|_{L^\infty} + \|\Delta b_0\|_{L^\infty} + \|\Delta u_0\|_{L^2}) \\
+ \|\Delta u\|_{L^\infty(0,t;L^2)} + \|(\Delta u)_x\|_{L^2(0,t;L^2)}. 
\]  
(4.5)

**Proof.** It follows from (1.18) that
\[
C^{-1} \leq g, \widetilde{g} \leq C. 
\]  
(4.6)

Direct computation, by (4.1), shows that
\[
\Delta \tau = g \left[ \Delta \tau_0 + \int_0^t K \left( \frac{1}{g} - \frac{1}{\widetilde{g}} \right) + \frac{K - \widetilde{K}}{\widetilde{g}} d\xi + (g - \widetilde{g}) \left( \tau_0 + \int_0^t \frac{\widetilde{K}}{g} d\xi \right) \right]. 
\]  
(4.7)

Using (4.6) and (1.18), we estimate
\[
|\Delta \tau| \leq C \left( |\Delta \tau_0| + \int_0^t \left| \int_0^\xi \Delta \sigma d\xi \right| + |\Delta \tau| + |\Delta b_0| + |\Delta \tau_0| d\xi \right) + C \left| \int_0^t \Delta \sigma d\xi \right|, 
\]  
which means
\[
|\Delta \tau| \leq C \left( |\Delta \tau_0| + |\Delta b_0| + \int_0^t \left| \int_0^\xi \Delta \sigma d\xi \right| + |\Delta \tau| d\xi \right) + C \left| \int_0^t \Delta \sigma d\xi \right|. 
\]  
(4.8)

Obviously, (4.8) yields the bound
\[
\|\Delta \tau(\cdot, t)\|_{L^\infty} \leq C \left( \|\Delta \tau_0\|_{L^\infty} + \|\Delta b_0\|_{L^\infty} + \left\| \int_0^\xi \Delta \sigma d\xi \right\|_{L^\infty(\Omega_t)} + \int_0^t \|\Delta \tau(\cdot, \xi)\|_{L^\infty} d\xi \right). 
\]  
(4.9)

By virtue of (4.2), we estimate the third term on the right hand side of (4.9) in the following manner.
\[
\left\| \int_0^\xi \Delta \sigma d\xi \right\|_{L^\infty(\Omega_t)} \leq \|J_\Omega \Delta u_0\|_{L^\infty} + \|J_\Omega \Delta u\|_{L^\infty(\Omega_t)} + \|\Delta \sigma\|_{L^1(\Omega_t)}. 
\]  
(4.10)

It is easy to see
\[
\Delta \sigma = \mu \tau^{-1}(\Delta u)_x + \varphi, 
\]  
(4.11)

where
\[
\varphi := \mu(\Delta \varrho) \bar{u}_x - \frac{K}{\tau} + \frac{\widetilde{K}}{\tau}. 
\]

Thus, by invoking (1.18), we arrive at
\[
|\Delta \sigma| \leq C |(\Delta u)_x| + |\varphi|, 
\]  
\[
|\varphi| \leq C |\Delta \tau| |(\bar{u}_x)| + 1 + C |(\Delta b_0) + |\Delta \tau_0)|. 
\]  
(4.12)
In terms of multiplying (4.17) by $\Delta u$

Here $\zeta$ after integration by parts that

This completes the proof of Lemma 4.2.

Consequently,

$$\|\Delta \sigma\|_{L^1(\Omega_t)} \leq C \left( \|(\Delta u)_x\|_{L^1(\Omega_t)} + \|\Delta b_0\|_{L^\infty} + \|\Delta \tau_0\|_{L^\infty} + \int_0^t \zeta(s) \|\Delta \tau(\cdot, s)\|_{L^\infty} \, ds \right),$$

(4.13)

where

$$\zeta(t) := \|\widetilde{u}_x(\cdot, t)\|_{L^2} + 1.$$

In accordance with (4.19), valid is

$$\|\zeta\|_{L^2(0, T)} \leq C.$$  

(4.14)

By (4.10) and (4.13), (4.9) is further estimated as follows.

$$\|\Delta \tau(\cdot, t)\|_{L^\infty} \leq C(\|\Delta \tau_0\|_{L^\infty} + \|\Delta b_0\|_{L^\infty} + \|J_{\Omega}\Delta u_0\|_{L^\infty} + \|J_{\Omega}\Delta u\|_{L^\infty(\Omega_t)})$$

$$+ C \left( \|(\Delta u)_x\|_{L^1(\Omega_t)} + \int_0^t \zeta(s) \|\Delta \tau(\cdot, s)\|_{L^\infty} \, ds \right),$$

(4.15)

where

$$\zeta(t) := \zeta(t) + 1.$$

Applying again Gronwall’s inequality to (4.15) yields

$$\|\Delta \tau\|_{L^\infty(\Omega_t)} \leq C(\|\Delta \tau_0\|_{L^\infty} + \|\Delta b_0\|_{L^\infty} + \|J_{\Omega}\Delta u_0\|_{L^\infty}$$

$$+ \|J_{\Omega}\Delta u\|_{L^\infty(\Omega_t)} + \|(\Delta u)_x\|_{L^1(\Omega_t)}),$$

which combined with (4.12), Cauchy-Schwarz’s inequality implies that

$$\|\Delta \tau\|_{L^\infty(\Omega_t)} \leq C(\|\Delta \tau_0\|_{L^\infty} + \|\Delta b_0\|_{L^\infty} + \|\Delta u_0\|_{L^2}$$

$$+ \|\Delta u\|_{L^\infty(0, t; L^2)} + \|(\Delta u)_x\|_{L^2(0, t; L^2)}).$$

This completes the proof of Lemma 4.2

The next lemma concerns the energy estimate of $\Delta u$.

**Lemma 4.3** Let the hypotheses of Theorem 1.2 be satisfied. Then for any $t \in (0, T]$,

$$\|\Delta u\|_{L^\infty(0, t; L^2)} + \|(\Delta u)_x\|_{L^2(0, t; L^2)} \leq C(\|\Delta u_0\|_{L^2} + \|\Delta b_0\|_{L^\infty} + \|\Delta \tau_0\|_{L^\infty}$$

$$+ \|\zeta\|_{L^\infty(\Omega_t)} \|\Delta \tau(\cdot, s)\|_{L^\infty} \|_{L^2(0, t)}).$$

(4.16)

Here $\zeta(t) = \|\widetilde{u}_x(\cdot, t)\|_{L^2} + 1.$

**Proof.** By (4.12) and (4.11), we have

$$(\Delta u)_t = [\mu \tau^{-1}(\Delta u)_x + \varphi]_x.$$  

(4.17)

In terms of multiplying (4.17) by $\Delta u$ and integrating the resulting equation over $\Omega_t$, we obtain after integration by parts that

$$\|\Delta u\|_{L^\infty(0, t; L^2)} + \|(\Delta u)_x\|_{L^2(0, t; L^2)} \leq C(\|\Delta u_0\|_{L^2} + \|\varphi\|_{L^2(\Omega_t)}),$$

(4.18)
where (1.18) and Cauchy-Schwarz’s inequality have been invoked. We conclude readily, by virtue of (1.22), that

$$
\| \varphi \|_{L^2(\Omega)} \leq C(\| \zeta \|_{L^\infty} + \| \Delta b_0 \|_{L^\infty} + \| \Delta \tau_0 \|_{L^\infty}).
$$

(4.19)

Thus, Lemma 4.3 is proved by substituting (4.19) into (1.18).

Based on the previous lemmas, we now give the proof of Theorem 1.2.

To this end, we first notice that the energy virtue of (4.12), that

$$
\| \Delta \tau(\cdot, t) \|_{L^\infty} + \| \Delta u \|_{L^\infty(0, t; L^2)} + \| (\Delta u)_x \|_{L^2(0, t; L^2)}
$$

which implies

$$
\| \Delta \tau(\cdot, t) \|_{L^\infty} + \| \Delta u \|_{L^\infty(0, t; L^2)} + \| (\Delta u)_x \|_{L^2(0, t; L^2)}
$$

$$
\leq C \left( \| \Delta b_0 \|_{L^\infty} + \| \Delta \tau_0 \|_{L^\infty} + \| \Delta u_0 \|_{L^2} \right)^2 + \int_0^t \zeta^2(s) \| \Delta \tau(\cdot, s) \|_{L^\infty} \, ds.
$$

(4.20)

An application of Gronwall’s inequality to (4.20) gives

$$
\int_0^t \zeta^2(s) \| \Delta \tau(\cdot, s) \|_{L^\infty} \, ds \leq C \left( \| \Delta b_0 \|_{L^\infty} + \| \Delta \tau_0 \|_{L^\infty} + \| \Delta u_0 \|_{L^2} \right)^2.
$$

As a consequence, we obtain

$$
\| \Delta \tau \|_{L^\infty(\Omega_t)} + \| \Delta u \|_{L^\infty(0, t; L^2)} + \| (\Delta u)_x \|_{L^2(0, t; L^2)}
$$

$$
\leq C \left( \| \Delta b_0 \|_{L^\infty} + \| \Delta \tau_0 \|_{L^\infty} + \| \Delta u_0 \|_{L^2} \right).
$$

(4.21)

In addition, it follows from Definition 2.1 and (1.18) that

$$
\| \Delta b \|_{L^\infty(\Omega_t)} \leq C \left( \| \Delta \tau_0 \|_{L^\infty} + \| \Delta b_0 \|_{L^\infty} + \| \Delta \tau \|_{L^\infty(\Omega_t)} \right).
$$

(4.22)

Thus (1.20) is verified if we multiply (1.22) by $\frac{1}{2C}$ and add the resulting inequality to (4.21).

**Remark 4.1** In fact, as the classical results on one-dimensional compressible Navier-Stokes-Fourier system, the weak solution $(\tau, u, b)$ obtained in Theorem 1.1 satisfies

$$
\tau \in C([0, T]; L^\infty), \quad u \in C([0, T]; L^2), \quad b \in C([0, T]; L^\infty),
$$

for any fixed $0 < T < \infty$, see [1, 35].

### 5 Large time behavior

The crucial step to the proof of Theorem 1.3 lies in obtaining the uniform-in-time bounds of the density from above and below away from zero. To this end, we first notice that the energy estimates given in Lemma 2.1 are uniform with respect to time. For the sake of convenience, we rewrite it as follows.
Lemma 5.1 Let \((\tau, u, b)\) be the unique weak solution to (1.11)-(1.16) under the assumption (1.17). Then
\[
\int_0^1 \tau(x, t) dx = \int_0^1 \tau_0(x) dx = 1, \text{ for any } t \in [0, \infty),
\]  
and
\[
\sup_{0 \leq t < \infty} \int_0^1 \left( \frac{1}{2} u^2 + \frac{A}{\gamma - 1} \tau^{1 - \gamma} + \frac{1}{2} b_0^2 \tau^{-2} \right) dx + \mu \int_0^\infty \int_0^1 \frac{u^2}{\tau} dx ds \leq C.
\]
During this section, the letter \(C, C_i\) denote generic positive constants independent of the time. Following Zlotnik [34], we first consider the boundary value problem with a parameter \(t \geq 0\) as follows.
\[
(\rho w)_x = f, \quad x \in (0, 1), \quad w|_{x=0,1} = 0.
\]
Here \(\rho = \rho(t, x) > 0\) and \(f\) are given functions in \(\Omega \times (0, \infty)\) and \(w\) is the unknown function. Let \(\eta = \rho^{-1}\) and \(v\) satisfy \(\eta_x = v_x\). Denote \(\Lambda f := \rho w_x\). We report the following results on \(\Lambda f\) from [34].

Lemma 5.2 There holds
\[
(\Lambda f^1)(x, t) = - \int_x^1 f^1(\xi, t) d\xi + \int_0^1 \eta(y, t) \int_y^1 f^1(\xi, t) d\xi dy,
\]
\[
(\Lambda f^2)(x, t) = f^2(x, t) - \int_0^1 \eta(y, t) f^2(y, t) dy,
\]
\[
(\Lambda f^3)_t(x, t) = (\Lambda f^3)(x, t) + \int_0^1 v(y, t) f^3(y, t) dy,
\]
\[
\|\Lambda f^4\|_{L^\infty} \leq 2 \|f^4\|_{L^1},
\]
where \(f^1(\cdot, t), f^2(\cdot, t), f^4(\cdot, t), (\eta f^2)(\cdot, t) \in L^1\) for any \(t \geq 0\) and \(f^3, f^3_t, v f^3 \in L^1(\Omega \times (0, T))\) for any \(T \in (0, \infty)\).

Based on Lemmas 5.1, 5.2 we can obtain the uniform-in-time bounds of the density from above and below away from zero, which plays a crucial role in deriving exponential decay estimates.

Lemma 5.3 Let \((\tau, u, b)\) be the unique weak solution to (1.11)-(1.16) under the assumption (1.17). Then
\[
C^{-1} \leq \tau(x, t) \leq C, \text{ for any } (x, t) \in [0, 1] \times [0, \infty).
\]

Proof. By setting
\[
P(x, \tau) := A \tau^{-\gamma} + \frac{1}{2} b_0^2 \tau^{-2},
\]
we rewrite (1.12) as
\[
(\rho u_x)_x = \frac{1}{\mu} (u_t + P(x, \tau)_x).
\]
In view of Lemma 5.2 and (1.16),
\[ \varrho u_x = \frac{1}{\mu} \left( (Au)_t - \int_0^1 u^2 dx + P(x, \tau) - \int_0^1 \tau P(x, \tau) dx \right). \]

Thus, by (1.11), we see
\[ (\log \tau)_t = \frac{1}{\mu} \left( (Au)_t - \int_0^1 u^2 dx + P(x, \tau) - \int_0^1 \tau P(x, \tau) dx \right). \tag{5.10} \]

On the one hand, using (5.2), we get
\[ \int_0^1 \tau P(x, \tau) dx \leq C_1, \]
and there exists \( C_2 > 0 \) such that
\[ P(x, \tau) > C_1, \text{ if } 0 < \tau < C_2. \tag{5.11} \]

Now we fix \( x \in [0, 1] \) and set
\[ \tau_0(x) := \min \{ \tau_0(x), C_2 \}. \]

If there exists \( t_2 \in (0, \infty) \) such that
\[ \tau(x, t_2) < \tau_0(x), \]
then, due to the continuity of \( \tau(x, t) \) with respect to \( t \) (see Remark 1.11), there exists \( t_1 \in [0, t_2) \) such that
\[ \tau(x, t_1) = \tau_0(x), \tau(x, t) < \tau_0(x), \text{ for } t \in (t_1, t_2]. \tag{5.12} \]

Integrating (5.10) both sides over \( (t_1, t_2) \) with respect to \( t \) yields
\[ \mu \log \tau(x, t_2) - \mu \log \tau(x, t_1) = (Au)(x, t_2) - (Au)(x, t_1) \]
\[ - \int_{t_1}^{t_2} \int_0^1 u^2 dx ds + \int_{t_1}^{t_2} \left( P(x, \tau) - \int_0^1 \tau P(x, \tau) dx \right) ds. \tag{5.13} \]

Taking advantage of (5.2) and (5.11), one easily finds
\[ \|(Au)(t)\|_{L^\infty} \leq C, \text{ for any } t \in [0, \infty); \tag{5.14} \]

while the third term on the right-hand side of (5.13) can be estimated by
\[ \int_{t_1}^{t_2} \int_0^1 u^2 dx ds \leq \int_{t_1}^{t_2} \|u\|_{L^\infty}^2 ds \leq \int_{t_1}^{t_2} \|u_x\|_{L^1}^2 ds \]
\[ \leq \int_{t_1}^{t_2} \left( \int_0^1 \frac{u_x^2}{\tau} dx \right) \int_0^1 \tau dx ds \leq C, \tag{5.15} \]

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where Hölder’s inequality and Lemma 5.1 have been used. As a consequence, by gathering (5.11), (5.12), (5.14) and (5.15), we deduce from (5.13) that

$$\tau(x, t_2) \geq \tau_0(x) \exp \left( -\frac{C}{\mu} \right).$$

(5.16)

On the other hand, by employing (5.1) and Jensen’s inequality, there holds

$$\int_0^1 \tau P(x, \tau) \, dx \geq A \int_0^1 \tau^{1-\gamma} \, dx \geq A \left( \int_0^1 \tau \, dx \right)^{1-\gamma} \geq A,$$

and there exists $C_3 > 0$ such that

$$P(x, \tau) < A, \text{ if } \tau > C_3.$$

(5.17)

Now we fix $x \in [0, 1]$ and set

$$\bar{\tau}_0(x) := \max \{ \tau_0(x), C_3 \}.$$

If there exists $t_2 \in (0, \infty)$ such that

$$\tau(x, t_2) > \bar{\tau}_0(x),$$

then, due to the continuity of $\tau(x, t)$ with respect to $t$, there exists $t_1 \in [0, t_2)$ such that

$$\tau(x, t_1) = \tau_0(x), \quad \tau(x, t) > \bar{\tau}_0(x), \text{ for } t \in (t_1, t_2].$$

(5.18)

Therefore, similar to the derivation of (5.16), it follows from (5.14), (5.15), (5.17) and (5.18) that

$$\tau(x, t_2) \leq \bar{\tau}_0(x) \exp \left( \frac{C}{\mu} \right).$$

(5.19)

This completes the proof of Lemma 5.3 by combining (5.16) with (5.19).

Before turning to the proof of Theorem 1.3, we give the unique stationary solution of (1.11)-(1.16) denoted by $(\tau_s, u_s, b_s)$, obeying

$$\int_0^1 \tau_s(x) \, dx = \int_0^1 \tau_0(x) = 1, \quad u_s = 0,$$

(5.20)

$$A \tau_s^{-\gamma} + \frac{1}{2} b_s^2 = C_0, \quad b_s = b_0 \tau_0 \tau_s^{-1},$$

(5.21)

where the constant $C_0$ is determined by the normalized condition $\int_0^1 \tau_s(x) \, dx = 1$. It is obvious that $\tau_s$ is upper and lower bounded, i.e., there exists a constant $C > 0$ such that

$$C^{-1} \leq \tau_s \leq C.$$

(5.22)

Furthermore, with the regularity class $L^\infty$ imposed on the initial data, we have

$$\| (\tau_s)_x \|_{L^\infty} \leq C.$$

(5.23)
5.1 Exponential decay in $L^2$-norm

With Lemmas 5.1-5.3 at hand, we are now in a position to give the proof of Theorem 1.3. The proof is essentially based on the energy method by modifying the idea used in [25, 28, 34].

Firstly, owing to (5.21), we rewrite (1.12) as

$$u_t + (P(x, \tau) - P(x, \tau_s))_x = (\mu \varphi u_x)_x. \quad (5.24)$$

Multiplying both sides by $u$ and integrating the resulting equation over $(0, 1)$ with respect to $x$,

$$\frac{d}{dt} \int_0^1 \left( \frac{1}{2} u^2 + A \frac{\tau - \tau_{s}}{\gamma - 1} \tau_{s}^{\gamma + 1} + A \tau_{s}^{\gamma 2} \tau + \frac{1}{2} b_0^2 \tau_{s}^{2} \tau - 1 + \frac{1}{2} b_0^2 \tau_{s}^{2} \tau - 2 \tau_{s}^{-2} \right) dx + \int_0^1 \mu \varphi u_x^2 dx = 0, \quad (5.25)$$

where (1.11) is used. Denote

$$\Phi_1(\tau, \tau_s) := \frac{A}{\gamma - 1} \tau_{s}^{\gamma + 1} + A \tau_{s}^{\gamma 2} \tau - \frac{A \gamma}{\gamma - 1} \tau_{s}^{\gamma + 1};$$

$$\Phi_2(x, \tau, \tau_s) := \frac{1}{2} b_0^2 \tau_{s}^{2} \tau - 1 + \tau_{s}^{2} \tau - 2 \tau_{s}^{-2} \tau_{s}^{-1}. (5.28)$$

Then (5.25) is equivalent to

$$\frac{d}{dt} \int_0^1 \left( \frac{1}{2} u^2 + \Phi_1(\tau, \tau_s) + \Phi_2(x, \tau, \tau_s) \right) dx + \int_0^1 \mu \varphi u_x^2 dx = 0, \quad (5.26)$$

Note that $\Phi_1(\tau, \tau_s)$ can be written as

$$\Phi_1(\tau, \tau_s) = A \tau_{s}^{\gamma + 1} G \left( \frac{\tau}{\tau_s} \right),$$

where

$$G(z) = z - \frac{\gamma}{\gamma - 1} + \frac{1}{\gamma - 1} z^{-\gamma + 1}. \quad (5.27)$$

It follows that that

$$G(1) = G'(1) = 0, \quad G''(z) > 0, \quad \text{if } z > 0.$$

As a consequence, by invoking (5.8) and (5.22), we conclude that

$$C^{-1}(\tau - \tau_s)^2 \leq \Phi_1(\tau, \tau_s) \leq C(\tau - \tau_s)^2. \quad (5.27)$$

Similarly it holds that

$$0 \leq \Phi_2(x, \tau, \tau_s) \leq C(\tau - \tau_s)^2. \quad (5.28)$$

For a positive parameter $\varepsilon$, we multiply both sides by $\varepsilon \int_0^1 (\tau - \tau_s) d\xi$ and integrate the resulting equation over $(0, 1)$ with respect to $x$ to find

$$\frac{d}{dt} \int_0^1 \varepsilon u \mathcal{J}(\tau - \tau_s) dx - \varepsilon \int_0^1 (P(x, \tau) - P(x, \tau_s)) (\tau - \tau_s) dx$$

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\[ -\varepsilon \int_0^1 u^2 \, dx + \varepsilon \int_0^1 \mu g u_x (\tau - \tau_s) \, dx = 0, \quad (5.29) \]

where, for simplicity, we have set
\[ J(\tau - \tau_s) := \int_0^x (\tau - \tau_s) \, d\xi. \]

Adding (5.29) to (5.26) yields
\[ \frac{d}{dt} \int_0^1 \left( \frac{1}{2} u^2 + \Phi_1(\tau, \tau_s) + \Phi_2(x, \tau, \tau_s) + \varepsilon u J(\tau - \tau_s) \right) \, dx 
+ \int_0^1 \mu g u_x^2 \, dx - \varepsilon \int_0^1 (P(x, \tau) - P(x, \tau_s)) (\tau - \tau_s) \, dx 
= \varepsilon \int_0^1 u^2 \, dx - \varepsilon \int_0^1 \mu g u_x (\tau - \tau_s) \, dx. \quad (5.30) \]

Obviously, (5.8) and (5.22) imply
\[ -\varepsilon \int_0^1 (P(x, \tau) - P(x, \tau_s)) (\tau - \tau_s) \, dx \geq C_1 \varepsilon \int_0^1 (\tau - \tau_s)^2 \, dx. \quad (5.31) \]

Similarly, by Cauchy-Schwarz’s inequality and (5.8), we see
\[ \left| \varepsilon \int_0^1 \mu g u_x (\tau - \tau_s) \, dx \right| \leq C_2 \varepsilon \int_0^1 \mu g u_x^2 \, dx + \frac{C_1 \varepsilon}{2} \int_0^1 (\tau - \tau_s)^2 \, dx; \quad (5.32) \]
\[ C_3 \int_0^1 \mu g u_x^2 \, dx \geq \int_0^1 u^2 \, dx. \quad (5.33) \]

In view of (5.31)–(5.33), it follows from (5.30) that
\[ \frac{d}{dt} \int_0^1 \left( \frac{1}{2} u^2 + \Phi_1(\tau, \tau_s) + \Phi_2(x, \tau, \tau_s) + \varepsilon u J(\tau - \tau_s) \right) \, dx 
+ \frac{C_1 \varepsilon}{2} \int_0^1 (\tau - \tau_s)^2 \, dx + \left[ \left( 1 - \frac{C_2 \varepsilon}{2C_1} \right) - C_3 \varepsilon \right] \int_0^1 \mu g u_x^2 \, dx \leq 0. \quad (5.34) \]

An application of Cauchy-Schwarz’s inequality again shows
\[ \left| \int_0^1 \varepsilon u J(\tau - \tau_s) \, dx \right| \leq \frac{\varepsilon}{2} \int_0^1 u^2 \, dx + \frac{\varepsilon}{2} \int_0^1 (\tau - \tau_s)^2 \, dx. \quad (5.35) \]

Therefore, defining
\[ \mathcal{E} := \int_0^1 \left( \frac{1}{2} u^2 + \Phi_1(\tau, \tau_s) + \Phi_2(x, \tau, \tau_s) + \varepsilon u J(\tau - \tau_s) \right) \, dx, \]
and gathering (5.27), (5.28) and (5.35), after choosing $\varepsilon$ to be a sufficiently small constant, we arrive at

$$C_4^{-1} \left( \int_0^1 (\tau - \tau_s)^2 dx + \int_0^1 u^2 dx \right) \leq \mathcal{E} \leq C_4 \left( \int_0^1 (\tau - \tau_s)^2 dx + \int_0^1 u^2 dx \right).$$

(5.36)

Finally, combining (5.34) with (5.36) gives

$$\frac{d}{dt} \mathcal{E} + C_5 \left( \int_0^1 (\tau - \tau_s)^2 dx + \int_0^1 u^2 dx \right) \leq 0,$$

from which one obtains the decay estimate after using (5.36) and integration

$$\| (\tau - \tau_s)(t) \|_{L^2} + \| u(t) \|_{L^2} \leq C_6 \exp(-C_7 t), \text{ for any } t \geq 0.$$ 

(5.37)

Due to (5.8) and (5.22), there holds

$$\| (b - b_s)(t) \|_{L^2} \leq C \| (\tau - \tau_s)(t) \|_{L^2}.$$ 

(5.38)

This completes the proof of Theorem 1.3 by adding (5.38) to (5.37).

At this stage, we intend to give an interesting remark concerning a special case of Theorem 1.3.

**Remark 5.1** Suppose, in addition to (1.17), if the absolute value of the ratio between the initial magnetic field and density is a positive constant, then the stationary magnetic field will be a piecewise constant. As a simple example, assume there exists a positive constant $\theta$ such that

$$b_0(x)\tau_0(x) = \begin{cases} \theta, & \text{if } x \in \left(0, \frac{1}{2}\right), \\ -\theta, & \text{if } x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Then, in accordance with (5.34)-(5.24), the stationary solution exactly takes

$$(\tau_s, u_s) = (1, 0), \quad b_s(x) = \begin{cases} \theta, & \text{if } x \in \left(0, \frac{1}{2}\right), \\ -\theta, & \text{if } x \in \left(\frac{1}{2}, 1\right). \end{cases}$$

### 5.2 Exponential decay in $H^1$-norm

Inspired by the method introduced in [27, 28, 34], we give the proof of Theorem 1.4 in this section. To this end, we need the uniform-in-time bound of the density and the velocity in $H^1$-norm. Specifically, we have

**Lemma 5.4** Let $(\tau, u, b)$ be the unique strong solution to (1.11)-(1.16), with the hypotheses of Theorem 1.4 be satisfied. Then

$$\| \tau_x \|_{L^2} + \| u_x \|_{L^2} \leq C.$$ 

(5.39)
**Proof.** Denoting

\[ F := u - \mu \left( \log \tau \right)_x, \quad a(x) := b_0^2(x) \tau_0^2(x), \]

we rewrite (1.12), by means of (1.11), as

\[ F_t + (P(x, \tau))_x = 0. \quad (5.40) \]

By using the fact that

\[ (P(x, \tau))_x = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial \tau} \tau, \]

we multiply (5.40) both sides by \( F \) and integrate the resulting equation over \((0, 1)\) to find

\[ \frac{d}{dt} \int_0^1 \left( \frac{1}{2} F^2 \right) dx + \frac{1}{\mu} \int_0^1 \left( A \gamma \tau - \gamma + a(x) \tau^{-2} \right) F^2 dx \]

\[ = - \int_0^1 \left( \frac{1}{2} a'(x) \tau^{-2} F \right) dx + \frac{1}{\mu} \int_0^1 \left( A \gamma \tau - \gamma + a(x) \tau^{-2} \right) F u dx, \]

the right-hand side of which can be estimated by

\[ \left| \int_0^1 \left( \frac{1}{2} a'(x) \tau^{-2} F \right) dx \right| \leq C \|a'\|_{L^2} \|F\|_{L^2} \leq \delta_1 \|F\|_{L^2}^2 + C_\delta_1 \|a'\|_{L^2}^2; \]

\[ \left| \frac{1}{\mu} \int_0^1 \left( A \gamma \tau - \gamma + a(x) \tau^{-2} \right) F u dx \right| \leq C \|u\|_{L^2} \|F\|_{L^2} \leq \delta_2 \|F\|_{L^2}^2 + C_\delta_2 \|u\|_{L^2}^2, \]

where (5.8) and Cauchy-Schwarz’s inequality have been used. Hence, by choosing \( \delta_1, \delta_2 \) to be sufficiently small and invoking (5.2), we obtain

\[ \frac{d}{dt} \int_0^1 F^2 dx + C_1 \int_0^1 F^2 dx \leq C_2. \quad (5.41) \]

It follows that

\[ \int_0^1 F^2 dx \leq C, \]

which particularly implies, recalling (5.2) and (5.8), that

\[ \int_0^1 \tau_x^2 dx \leq C. \quad (5.42) \]

To proceed, we write (1.12) as

\[ u_t + (P(x, \tau))_x = \mu u_x u_x + \mu u_{xx}, \]

followed by multiplying both sides by \( u_{xx} \), integrating over \((0, 1)\) with respect to \( x \). Then after integration by parts we see

\[ \frac{1}{2} \frac{d}{dt} \int_0^1 u_x^2 dx + \int_0^1 \mu u_{xx}^2 dx = \int_0^1 (P(x, \tau))_x u_{xx} dx - \int_0^1 \mu g_x u_x u_{xx} dx \]

\[ 22 \]
\[ \leq \|u_{xx}\|_{L^2} \| (P(x, \tau))_x \|_{L^2} + C \| \tau_x \|_{L^2} \| u_{xx} \|_{L^2} \| u_x \|_{L^\infty}. \] (5.43)

Furthermore, owing to Gagliardo-Nirenberg inequality, one has
\[ \|u_x\|_{L^\infty} \leq \delta_3 \|u_{xx}\|_{L^2} + C \delta_3 \|u\|_{L^2}. \] (5.44)

As a consequence, due to (5.8), (5.42), (5.44) and Cauchy-Schwarz’s inequality, after choosing \( \delta_3 \) to be sufficiently small, we conclude from (5.43) that
\[ \frac{d}{dt} \int_0^1 u_x^2 dx + C_3 \int_0^1 u_{xx}^2 dx \leq C_4 \left( \int_0^1 u^2 dx + \int_0^1 (P(x, \tau))^2_x dx \right). \] (5.45)

Obviously, (5.2), (5.8) and (5.42) together lead to
\[ C_4 \left( \int_0^1 u^2 dx + \int_0^1 (P(x, \tau))^2_x dx \right) \leq C_5. \]

In addition, since \( \int_0^1 u_x dx = 0 \),
\[ \int_0^1 u_{xx}^2 dx \leq \int_0^1 u_x^2 dx. \]

Thus, we strengthen (5.45) as
\[ \frac{d}{dt} \int_0^1 u_x^2 dx + C_0 \int_0^1 u_{xx}^2 dx \leq C_7. \]

Hence
\[ \int_0^1 u_x^2 dx \leq C. \] (5.46)

This completes the proof of Lemma 5.4 by adding (5.46) to (5.42).

Based on the previous lemmas, we are now in a position to prove Theorem 1.4. Recall that
(1.12) is equivalent to
\[ u_t + (P(x, \tau) - P(x, \tau_s))_x = \mu \left( \log \left( \frac{\tau}{\tau_s} \right) \right)_x. \] (5.47)

We multiply (5.47) both sides by \( \left( \log \left( \frac{\tau}{\tau_s} \right) \right)_x \) and integrate the resulting equation over \((0, 1)\) with respect to \(x\), to infer that
\[ \frac{d}{dt} \left\{ \frac{\mu}{2} \int_0^1 \left( \log \left( \frac{\tau}{\tau_s} \right) \right)_x^2 dx - \int_0^1 u \left( \log \left( \frac{\tau}{\tau_s} \right) \right)_x dx \right\} 
- \int_0^1 (P(x, \tau) - P(x, \tau_s))_x \left( \log \left( \frac{\tau}{\tau_s} \right) \right)_x dx = \int_0^1 \theta u_x^2 dx. \] (5.48)

To proceed, we write the second term on the left-hand side of (5.48) as follows.
\[ - \int_0^1 (P(x, \tau) - P(x, \tau_s))_x \left( \log \left( \frac{\tau}{\tau_s} \right) \right)_x dx = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3, \]

where
\[ \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \] are terms that need to be estimated.
where
\[ R_1 := A \gamma \int_0^1 \left[ \tau^{-\gamma - 1} \tau_x - \tau_s^{-\gamma - 1} (\tau_s)_x \right] \left[ \tau^{-1} \tau_x - \tau_s^{-1} (\tau_s)_x \right] dx; \]
\[ R_2 := \int_0^1 \left[ a \tau^{-3} \tau_x - a \tau_s^{-3} (\tau_s)_x \right] \left[ \tau^{-1} \tau_x - \tau_s^{-1} (\tau_s)_x \right] dx; \]
\[ R_3 := -\frac{1}{2} \int_0^1 a' (\tau^{-2} - \tau_s^{-2}) \left[ \tau^{-1} \tau_x - \tau_s^{-1} (\tau_s)_x \right] dx. \]

Notice that \( R_1 \) can be reformulated as
\[ R_1 = A \gamma \int_0^1 \left[ \tau^{-\gamma - 1} (\tau_x - (\tau_s)_x) + (\tau_s)_x (\tau^{-\gamma - 1} - \tau_s^{-\gamma - 1}) \right] \times \left[ \tau^{-1} (\tau_x - (\tau_s)_x) + (\tau_s)_x (\tau^{-1} - \tau_s^{-1}) \right] dx. \]

Consequently, using (5.8), (5.22), (5.23) and Cauchy-Schwarz’s inequality, we get the estimate
\[ R_1 \geq C_1 \int_0^1 (\tau_x - (\tau_s)_x)^2 dx - C_2 \int_0^1 (\tau - \tau_s)^2 dx. \] (5.49)

In a similar manner, we have
\[ R_2 \geq -\delta_1 \int_0^1 (\tau_x - (\tau_s)_x)^2 dx - C\delta_1 \int_0^1 (\tau - \tau_s)^2 dx; \] (5.50)
\[ R_3 \geq -\delta_2 \int_0^1 (\tau_x - (\tau_s)_x)^2 dx - C\delta_2 \int_0^1 (\tau - \tau_s)^2 dx. \] (5.51)

In view of (5.49) - (5.51), we choose \( \delta_1, \delta_2 \) to be sufficiently small and arrive at
\[ -\int_0^1 (P(x, \tau) - P(x, \tau_s))_x \left( \log \left( \frac{\tau}{\tau_s} \right) \right)_x dx \geq C_3 \int_0^1 (\tau_x - (\tau_s)_x)^2 dx - C_4 \int_0^1 (\tau - \tau_s)^2 dx. \] (5.52)

It follows from (5.48), by (5.52), that
\[ \frac{dH}{dt} + C_3 \int_0^1 (\tau_x - (\tau_s)_x)^2 dx \leq C_4 \int_0^1 (\tau - \tau_s)^2 dx + \int_0^1 \varrho u_x^2 dx, \] (5.53)
where we have set
\[ H := \frac{\mu}{2} \int_0^1 \left( \log \left( \frac{\tau}{\tau_s} \right) \right)_x^2 dx - \int_0^1 u \left( \log \left( \frac{\tau}{\tau_s} \right) \right)_x dx. \]

In addition, by invoking (5.8), (5.22), (5.23) and Cauchy-Schwarz’s inequality, we see
\[ C_5 \int_0^1 (\tau_x - (\tau_s)_x)^2 dx - C_6 \left( \int_0^1 (\tau - \tau_s)^2 dx + \int_0^1 u^2 dx \right) \leq H \leq C_7 \left( \int_0^1 (\tau_x - (\tau_s)_x)^2 dx + \int_0^1 (\tau - \tau_s)^2 dx + \int_0^1 u^2 dx \right). \] (5.54)
We multiply (5.53) both sides by \( \delta_3 \) and add the resulting inequality to (5.34), after choosing \( \delta_3 \) to be sufficiently small, to conclude that
\[
\frac{d}{dt} (E + \delta_3 \mathcal{H}) + C_8 (E + \delta_3 \mathcal{H}) \leq 0.
\]
Then one checks easily, with the help of (5.36) and (5.54), that
\[
C_9^{-1} \left( \int_0^1 (\tau_x - (\tau_s)_x)^2 \, dx + \int_0^1 (\tau - \tau_s)^2 \, dx + \int_0^1 u^2 \, dx \right)
\leq E + \delta_3 \mathcal{H} \leq C_9 \left( \int_0^1 (\tau_x - (\tau_s)_x)^2 \, dx + \int_0^1 (\tau - \tau_s)^2 \, dx + \int_0^1 u^2 \, dx \right).
\]
Moreover, the right-hand side of (5.45) is estimated as follows:
\[
\left( \int_0^1 u^2 \, dx + \int_0^1 (P(x, \tau))^2 \, dx \right) = \left( \int_0^1 u^2 \, dx + \int_0^1 (P(x, \tau) - P(x, \tau_s))^2 \, dx \right)
\leq C_{10} \left( \int_0^1 (\tau_x - (\tau_s)_x)^2 \, dx + \int_0^1 (\tau - \tau_s)^2 \, dx \right),
\]
due to (5.38), (5.42) and (5.43). Therefore, (5.45) implies
\[
\frac{d}{dt} \int_0^1 u_x^2 \, dx + C_3 \int_0^1 u_{xx}^2 \, dx \leq C_{11} \left( \int_0^1 (\tau_x - (\tau_s)_x)^2 \, dx + \int_0^1 (\tau - \tau_s)^2 \, dx \right). \tag{5.57}
\]
Again as we have done previously, multiplying (5.57) both sides by \( \delta_4 \) and adding the resulting inequality to (5.44), after choosing \( \delta_4 \) to be sufficiently small, gives
\[
\frac{d}{dt} \left( E + \delta_3 \mathcal{H} + \delta_4 \int_0^1 u_x^2 \, dx \right) + C_{12} \left( E + \delta_3 \mathcal{H} + \delta_4 \int_0^1 u_x^2 \, dx \right) \leq 0. \tag{5.58}
\]
This establishes the decay estimate
\[
\| (\tau - \tau_s)(t) \|_{H^1} + \| u(t) \|_{H^1} \leq C_{13} \exp(-C_{14} t), \quad \text{for any } t \geq 0, \tag{5.59}
\]
by integrating (5.58) and noting the simple fact that
\[
C_{15}^{-1} \left( \int_0^1 (\tau_x - (\tau_s)_x)^2 \, dx + \int_0^1 (\tau - \tau_s)^2 \, dx + \int_0^1 u^2 \, dx + \int_0^1 u_x^2 \, dx \right)
\leq \left( E + \delta_3 \mathcal{H} + \delta_4 \int_0^1 u_x^2 \, dx \right)
\leq C_{15} \left( \int_0^1 (\tau_x - (\tau_s)_x)^2 \, dx + \int_0^1 (\tau - \tau_s)^2 \, dx + \int_0^1 u^2 \, dx + \int_0^1 u_x^2 \, dx \right).
\]
Finally, the decay estimate
\[
\| (\bar{b} - \bar{b}_s)(t) \|_{H^1} \leq C_{16} \exp(-C_{17} t), \quad \text{for any } t \geq 0, \tag{5.60}
\]
is a direct consequence of (5.8), (5.22), (5.42), (5.59) and Sobolev’s inequality. The proof of Theorem 1.4 is thus finished by adding (5.60) to (5.59).

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