Research Article

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Range-kernel weak orthogonality of some elementary operators

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Abstract: We study the range-kernel weak orthogonality of certain elementary operators induced by non-normal operators, with respect to usual operator norm and the Von Neumann-Schatten $p$-norm ($1 \leq p < \infty$).

Keywords: range-kernel orthogonality, elementary operator, Schatten $p$-classes, quasinormal operator, subnormal operator, $k$-quasihyponormal operator

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1 Introduction

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators acting on a complex separable Hilbert space $\mathcal{H}$. Given $A, B \in B(\mathcal{H})$, we define the generalized derivation $\delta_{A,B} : B(\mathcal{H}) \to B(\mathcal{H})$ by $\delta_{A,B}(X) = AX - XB$.

Let $X \in B(\mathcal{H})$ be a compact operator, and let $s_1(X) \geq s_2(X) \geq \ldots \geq 0$ denote the eigenvalues of $|X| = (X^*X)^{1/2}$ arranged in their decreasing order. The operator $X$ is said to belong to the Schatten $p$-class $C_p(\mathcal{H})$, if

$$\|X\|_p = \left( \sum_{i=1}^{\infty} s_i(X)^p \right)^{1/p} = \text{tr}(|X|^p)^{1/p} < +\infty,$$

where $\text{tr}$ denotes the trace functional. In case $p = \infty$, we denote by $C_\infty(\mathcal{H})$, the ideal of compact operators equipped with the norm $\|X\|_\infty = s_1(X)$. For $p = 1$, $C_1(\mathcal{H})$ is called the trace class, and for $p = 2$, $C_2(\mathcal{H})$ is called the Hilbert-Schmidt class and the case $p = \infty$ corresponds to the class. For more details, the reader is referred to [1]. In the sequel, we will use the following further notations and definitions. The closure of the range of an operator $T \in B(\mathcal{H})$ will be denoted by $\text{ran}_T$ and $\ker T$ denotes the kernel of $T$. The restriction of $T$ to an invariant subspace $\mathcal{M}$ will be denoted by $|T|_{\mathcal{M}}$, and the commutator $AB - BA$ of the operators $A, B$ will be denoted by $[A, B]$. We recall the definition of Birkhoff-James’s orthogonality in Banach spaces [2,3].

Definition 1. If $X$ is a complex Banach space, then for any elements $x, y \in \mathcal{X}$, we say that $x$ is orthogonal to $y$, noted by $x \perp y$, iff for all $\alpha, \beta \in \mathbb{C}$ there holds

$$\|\alpha y + \beta x\| \geq \|\beta x\|,$$

for all $\alpha, \beta \in \mathbb{C}$ or $\mathbb{R}$.

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If $M$ and $N$ are linear subspaces in $X$, we say that $M$ is orthogonal to $N$ if $x \perp_{B,J} y$ for all $x \in M$ and all $y \in N$. The orthogonality in this sense is asymmetric.

Let $* : \mathcal{B} \to \mathcal{B}(X)$ be an involution defined on a linear subspace $\mathcal{B}$ of $\mathcal{B}(X)$ onto the algebra of all bounded linear operators acting on the Banach space $X$ and $\mathcal{B}^* = \mathcal{B}$. According to the definition given by Harte [4], if $E$ is called the Fuglede operator if $\ker E \subseteq \ker E^*$.

The elementary operator is an operator $E$ defined on Banach $(\mathcal{A}, \mathcal{B})$-bimodule $M$ with its representation $E(x) = \sum_{i=1}^n a_i x b_i$, where $a = (a_i) \in \mathcal{A}^n$, $b = (b_i) \in \mathcal{B}^n$ are $n$-tuples of algebra elements. The length of $E$ is defined to be the smallest number of multiplication terms required for any representation $\sum a_i x b_i$ for $E$.

In this note, we consider $\mathcal{A} = \mathcal{B} = B(\mathcal{H})$ and $\mathcal{B} = B(\mathcal{H})$ or $\mathcal{B} = C_p : (1 \leq p < \infty)$ and the length of $E$ will be less or equal to 2, i.e., if $A = (A_1, A_2), B = (B_1, B_2)$ are 2-tuples of operators in $B(\mathcal{H})^2$, then the elementary operator induced by $A$ and $B$ is defined by $E(x) = A_1 x B_1 - A_2 x B_2$ for all $x \in \mathcal{B}$. We will denote by $E$ the formal adjoint of $E$ defined by $E(x) = \sum_{i=1}^n A_i^* x B_i^*$ for all $x \in \mathcal{B}$. Note also that $E \in C_p$ and $E^*(X) = \sum_{i=1}^n B_i X A_i$ on any separable ideal of compact operator, where $E^*$ is the operator adjoint of $E$ in the sense of duality.

J. Anderson [5] proved that if $A$ and $B$ are normal operators, then
\[
\text{for all } X, S \in B(\mathcal{H}) : S \in \ker \delta_{A,B} \Rightarrow \|\delta_{A,B}(X) + S\| \geq \|S\|.
\]
This means that the kernel of $\delta_{A,B}$ is orthogonal to its range.

F. Kittaneh [6] extended this result to an u.i. ideal norm $\mathcal{J}$ in $B(\mathcal{H})$, by proving that the range of $\delta_{A,B}^\mathcal{J}$ is orthogonal to $\ker \delta_{A,B} \cap \mathcal{J}$.

A detailed study of range-kernel orthogonality for generalized derivation $\delta_{A,B}$ has received much attention in recent years and has been carried out in a large number of studies [3,5,7–13] and are based on the following result.

**Theorem 2.** Let $A, B$ be operators in $B(\mathcal{H})$. If $\delta_{A,B}$ is Fuglede, then the range of $\delta_{A,B}$ (resp. the range of $\delta_{A,B}^\mathcal{J}$) is orthogonal to the kernel of $\delta_{A,B}$ (resp. the kernel of $\delta_{A,B}^\mathcal{J}$) for all $1 \leq p \leq \infty$.

D. Keckic [14] and A. Turnšek [15] extended Theorem 2 to the elementary operator $E$ defined by $E(x) = AXB - CXD$, where $(A, C)$ and $(B, D)$ are 2-tuples of commuting normal operators. Duggal [16] generalized the famous theorem to the case $(A, C)$ and $(B, D)$ are 2-tuples of commuting operators, where $A, B$ are normal and $C, D^*$ are hyponormal.

In this paper, our goal is to extend the previous theorem to non-normal operators including quasi-normal, subnormal, and $k$-quasihyponormal operators.

In the following, we recall some definitions about the range-kernel weak orthogonality.

**Definition 3.** [4] If $E : X \to \mathcal{Y}$ and $T : \mathcal{Y} \to \mathcal{Z}$ are bounded linear operators between Banach spaces and $0 < k \leq 1$,
\[
[s \in \ker T \Rightarrow \text{dist}(s, \text{ran} E) \geq \|s\|] \Rightarrow T \perp_k E \iff \ker T \perp_k \text{ran} E.
\]
We say that $T$ is weakly orthogonal to $E$, written $T \perp E$, or equivalently
\[
\ker T \perp \text{ran} E \iff \exists k : 0 < k \leq 1 : T \perp_k E.
\]
For $0 < k \leq 1$, we say that $(E, T)$ has a $\frac{1}{k}$ gap if $T \perp_k E$.

If $T = E$ and $k \neq 1$, we shall call $E$ w-orthogonal $(E \perp E)$, consequently we get a $\frac{1}{k}$-gap between the subspaces $\ker E$ and $\text{ran} E$, which corresponds to the “range-kernel weak-orthogonality” for an operator $E$. If $k = 1$, we shall say that $T$ is orthogonal to $E$, written $T \perp E$, also if $X = (Y) = \mathcal{Z}$ and $T = E$ we get a 1-gap between the subspaces $\ker E$ and $\text{ran} E$. 
$T$ is said to be a quasi-normal if $[T, T^* T] = 0$, subnormal if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a normal operator $N \in B(\mathcal{K})$ such that $N|_{\mathcal{H}} = T$. Also, $T$ is called hyponormal if $[T^*, T]$ is a positive operator. Furthermore, we have the following proper inclusion

$$T \text{ normal } \Rightarrow T \text{ quasi-normal } \Rightarrow T \text{ subnormal } \Rightarrow T \text{ hyponormal}.$$ 

$A \in B(\mathcal{X})$ is said the Fuglede operator [13] if $\ker A \subseteq \ker A^\dagger$.

Recall that an $n$-tuple $A = (A_1, A_2, \ldots, A_n) \in B(\mathcal{H})^n$ is said commuting (resp. doubly commuting) if $[A_i, A_j] = 0$ (resp. $[A_i, A_j] = 0$ and $[A_i^*, A_j^*] = 0$) for all $i, j = 1, \ldots, n, i \neq j$. The $n$-tuple $A$ is said to be normal if $A$ is commuting and each $A_i$ $(i = 1, \ldots, n)$ is normal, and $A$ is subnormal if $A$ is the restriction of a normal $n$-tuple to a common invariant subspace. Clearly, every normal $n$-tuple is subnormal $n$-tuple. Any other notation or definition will be explained as and when required.

## 2 Preliminaries

We summarize the results given by D. Keckic [14], A. Turnšek [15], and B. P. Duggal [16] in the following theorem.

**Theorem 4.** [14–16] Let $A, B$ be normal operators, $C, D^*$ be hyponormal operators in $B(\mathcal{H})$ such that $[A, C] = [B, D] = 0$ and $\mathcal{J} = B(\mathcal{H})$ or $\mathcal{J} = C_p : 1 \leq p \leq \infty$, then

(i) If $\ker A \cap \ker C = \{0\} = \ker B^* \cap \ker D^*$, then for all $X \in B(\mathcal{H})$ such that $E(X), E(X) \in \mathcal{J}$

$$S \in \ker E \cap \mathcal{J} \Rightarrow \min\|E(X) + S\|_{\mathcal{J}}, \|E(X) + S\|_{\mathcal{J}} \geq \|S\|_{\mathcal{J}}.$$ 

(ii) If $\ker A \cap \ker C \neq \{0\}$ or $\ker B^* \cap \ker D^* \neq \{0\}$, then there exists $k$ verifying $0 < k < 1$ such that

$$\forall X \in B(\mathcal{H}), \quad \forall S \in \ker E \cap \mathcal{J} : \|E(X) + S\|_{\mathcal{J}} \geq k\|S\|_{\mathcal{J}}, \quad \text{where } E(X) \in \mathcal{J}.$$ 

(iii) If $\mathcal{J} = C_2(\mathcal{H})$ with its inner product $(X, Y) = \text{tr}(Y^* X)$ and $E$ is defined on $C_2(\mathcal{H})$, then for all $S \in \ker E$, we get

$$\|E(X) + S\|_2 = \|E(X)\|_2 + \|S\|_2; \quad \|\tilde{E}(X) + S\|_2 = \|\tilde{E}(X)\|_2 + \|S\|_2.$$ 

We recall some useful results which are important in the sequel.

**Lemma 5.** [13] Let $A, B$ be commuting operators in $B(\mathcal{H})$ with no trivial kernel.

1. Let $\xi$ be the elementary Fuglede operator defined by $\xi(X) = AX^* - BX^*$, then

   (i) if $\ker A \cap \ker B = \{0\}$, then $\ker A$ reduces $A$ and $\ker B$ reduces $B$.

   (ii) if $[A, B^*] = 0$ and $\ker A \cap \ker B \neq \{0\}$, then $\ker A \cap \ker B$ reduces $A$ and $B$.

2. Let $A \in B(\mathcal{H}), B \in B(\mathcal{K})$, and $E(X) = AXB; X \in B(\mathcal{K}, \mathcal{H})$, then $E$ is the Fuglede operator if and only if $\ker A$ reduces $A$ and $\ker B^*$ reduces $B^*$.

**Lemma 6.** [9] Let $T$ be an operator represented by block matrix as $T = (T_{i,j})_{i,j=1}^n$.

(i) If $T \in B(\mathcal{H})$, then $\frac{1}{n^2} \sum_{i,j} \|T_{i,j}\|^2 \leq \|T\|^2 \leq \sum_{i,j} \|T_{i,j}\|^2$

(ii) If $T \in C_p(T); 1 \leq p \leq \infty$, then

$$\frac{1}{n^{p-2}} \|T\|_p^p \leq \sum_{i,j} \|T_{i,j}\|_p^p \leq \|T\|_p^p \quad \text{for all } 2 \leq p < \infty,$$

$$\|T\|_p^p \leq \sum_{i,j} \|T_{i,j}\|_p^p \leq \frac{1}{n^{p-2}} \|T\|_p^p \quad \text{for all } 1 \leq p < 2.$$
3 Main results

Proposition 7. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $A \in B(\mathcal{H}), B \in B(\mathcal{K})$, and $E \in B(B(\mathcal{K}, \mathcal{H}))$ such that $E(X) = AXB$. If $E$ is the Fuglede operator, then $E$ is $w$-orthogonal, $E \perp \tilde{E}$, and the inequality
\[
\min\|E(X) + S\|_{\mathcal{F}}, \|\tilde{E}(X) + S\|_{\mathcal{F}} \geq \|S\|_{\mathcal{F}}
\]
holds if
- $\mathcal{F} = B(\mathcal{K}, \mathcal{H})$ with $k = 1/2$ or
- $\mathcal{F} = C_p(\mathcal{K}, \mathcal{H})$ with $k = \frac{1}{2 - p}$, if $1 \leq p \leq 2$ and $k = \frac{1}{2 + p}$, if $2 \leq p < \infty$.

Proof. If $A$ and $B^*$ are injective operators, then $E$ is injective. So there is nothing to prove.

Suppose that $A$ or $B^*$ is the non-injective operator. From Lemma 5(2) and with respect to the decompositions:
\[
\mathcal{H} = (\ker A)^\perp \oplus \ker A, \quad \mathcal{K} = (\ker B^*)^\perp \oplus \ker B^*
\]
we obtain $A = A_i \oplus A; B = B_1 \oplus 0$. Let $X = (X_{ij})_{i,j=1,2} : \mathcal{K} \to \mathcal{H}$. Then \(E(X) = A_i X_{11} B_1 \oplus 0\).

From the injectivity of $A_i$ and $B_1^*$ it yields that any $S = (S_{ij})_{i,j=1,2}$ in ker $E$ can be written as
\[
S = \begin{bmatrix} 0 & S_{12} \\ S_{21} & S_{22} \end{bmatrix},
\]
where the operators $S_{12}, S_{21},$ and $S_{22}$ are arbitrary.

Hence, for all $S \in \ker E$ and all $X \in B(\mathcal{K}, \mathcal{H})$, by Lemma 5(2), we have
\[
\min\|E(X) + S\|_{\mathcal{F}}, \|\tilde{E}(X) + S\|_{\mathcal{F}} \geq \frac{1}{2}(\|S_{12}\|^2 + \|S_{21}\|^2 + \|S_{22}\|^2)^{1/2} \geq \frac{1}{2}\|S\|.
\]

Also, for all $S \in \ker E \cap C_p(\mathcal{K}, \mathcal{H})$ and all $X \in B(\mathcal{K}, \mathcal{H})$ such that $E(X) \in C_p(\mathcal{K}, \mathcal{H})$, we get
(i) if $1 \leq p \leq 2$, then
\[
\min\|E(X) + S\|_{\mathcal{F}}, \|\tilde{E}(X) + S\|_{\mathcal{F}} \geq \frac{1}{2}\|S_{12}\|^p + \|S_{21}\|^p + \|S_{22}\|^p)^{1/p} \geq \frac{1}{2^{1/p}}\|S\|_{\mathcal{F}}.
\]

(ii) if $2 \leq p < \infty$, then
\[
\min\|E(X) + S\|_{\mathcal{F}}, \|\tilde{E}(X) + S\|_{\mathcal{F}} \geq \frac{1}{2^{1/p}}(\|S_{12}\|^p + \|S_{21}\|^p + \|S_{22}\|^p)^{1/p} \geq \frac{1}{2^{1/p}}\|S\|_{\mathcal{F}}.
\]

Using Lemma 5(2), we get a simple form of the previous result as follows.

Corollary 8. If $\ker A$ reduces $A$ and $\ker B^*$ reduces $B^*$, then $E$ is $w$-orthogonal, $E \perp \tilde{E}$ and satisfies the relation (6).

In the sequel $\xi$ denotes the elementary operator defined by
\[
\xi(X) = AXA^* - BXB^*
\]
from $B(\mathcal{H})$ to $\mathcal{F}$, where $A$ and $B$ are operators in $B(\mathcal{H})$ and $\mathcal{F} = B(\mathcal{H})$ or $\mathcal{F} = C_p(\mathcal{H}); 1 \leq p < \infty$.

Lemma 9. Let $\Delta$ be the elementary operator defined on $B(\mathcal{H})$ by $\Delta(X) = AXB - X$, where $A, B \in B(\mathcal{H})$. If $\Delta$ is a Fuglede operator, then $\Delta$ is orthogonal and $\Delta \perp \Delta$.

Proof. The proof is the same as the one in Theorem 4. □
Proposition 10. Let $A, B$ be doubly commuting operators in $B(\mathcal{H})$ and

$$\xi : B(\mathcal{H}) \to C_p(\mathcal{H}); \quad (1 \leq p \leq \infty, \quad p \neq 2).$$

If $\ker A$ reduces $A$, $\ker B$ reduces $B$ and $\xi$ orthogonal, then $\ker A \cap \ker B = \{0\}$.

Proof. We consider the following three cases:

(i) Let us suppose that $N = \ker A \cap \ker B \neq \{0\}$.

If $\ker A \neq \ker B$ and $\ker B \notin \ker A$, then with respect to the decomposition

$$\mathcal{H} = (\ker B)^\perp \oplus (\ker B \oplus N) \oplus N$$

and from the hypothesis it yields

$$A = A_1 \oplus A_2 \oplus 0, \quad B = B_1 \oplus 0, \quad \text{and} \quad S = (S_{ij})_{i,j=1,...,3},$$

where $A_2$ is an injective operator. Hence,

$$S \in \ker \xi \Rightarrow A_1 S_{11} A_1^* = B_1 S_{11} B_1^*; \quad A_1 S_{12} A_1^* = A_2 S_{21} A_1^* = 0; \quad S_{22} = 0$$

and the other entries are arbitrary. Choosing $X$ and $S$ as follows:

$$X = 0 \oplus (e \otimes e) \oplus 0, \quad S = 0 \oplus \begin{bmatrix} 0 & R \\ R^* & C \end{bmatrix},$$

where $e$ is a non-zero vector in $\mathcal{H}$, $R$ is an operator of rank one, and $C$ is a self-adjoint operator of rank one. Then

$$\xi(X) + S = 0 \oplus \begin{bmatrix} A_2 e \oplus A_2 e & R \\ R^* & C \end{bmatrix}.$$ 

Applying Lemma (2.4) [15], we get

$$\|\xi(X) + S\|_p < \left\| \begin{bmatrix} 0 & R \\ R^* & C \end{bmatrix} \right\|_p = \|S\|_p \quad (p \neq 2).$$

(ii) If $\ker A \neq \ker B$ and $\ker A \notin \ker B$, we proceed similarly as in the first case, it suffices to replace $A$ by $B$ and $B$ by $A$ in the preceding argument.

(iii) If $\ker A = \ker B$, then with respect to the decomposition

$$\mathcal{H} = (\ker B)^\perp \oplus \ker B,$$

we get

$$A = A_1 \oplus 0, \quad B = B_1 \oplus 0.$$

Letting $S = (S_{ij})_{i,j=1,...,3}$. Then

$$S \in \ker \xi \Rightarrow A_1 S_{11} A_1^* = B_1 S_{11} B_1^*,$$

and the other entries are arbitrary. Choosing $X$ and $S$ as

$$X = (e \otimes e) \oplus 0, \quad S = \begin{bmatrix} 0 & R \\ R^* & C \end{bmatrix},$$

where $e$ and $R$ are as in (i). Then

$$\xi(X) + S = \begin{bmatrix} A_1 e \otimes A_1 e - B_1 e \otimes B_1 e & R \\ R^* & C \end{bmatrix}.$$ 

If $A_1 e \otimes A_1 e = B_1 e \otimes B_1 e$ for all $e \in \mathcal{H}$, then it follows from this fact and the injectivity of $A_1$ and $B_1$ that $A_1 = a B_1$ with $|a| = 1$ and hence $\xi = 0$, which is a contradiction with the assumption $\xi \neq 0$. We use Lemma (2.4) [15] to complete the proof for $p \notin \{1, 2\}$. 

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In the case $p = 1$, let us assume that $A_1 e \otimes A_1 e - B_1 e \otimes B_1 e$ is an operator of rank 2 and has eigenvalues $\lambda_1, \lambda_2$ with $|\lambda_1| = |\lambda_2|$ for all $e \in \mathcal{H}$, then we can check that

$$|\lambda_i| = |\lambda_2| \Rightarrow \|A_1 e\| = \|B_1 e\| \quad \text{and} \quad \langle A_1 e, B_1 e \rangle = 0 \quad \forall e \in \mathcal{H}.$$ 

If $\ker B^* \subseteq \ker B$, then by the injectivity of $B_1^*$, it follows that $A_1 = 0$, which is a contradiction since $A \neq 0$.

If $\ker B^* \not\subseteq \ker B$, with respect to the decomposition

$$\mathcal{H} = (\ker B_1^*)^\perp \oplus \ker B_1^* \oplus \ker B$$

we get

$$A = A_1 \oplus A_2 \oplus 0, \quad B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \oplus 0, \quad \text{and} \quad S = (S_{i,j})_{i=1,...,3}.$$ 

Since $A_i$ is injective and $S \in \ker \xi$, we obtain

$$A_1 S_{11} A_1^* = B_1 S_{11} B_1^*; \quad S_{12} = S_{21} = S_{22} = 0$$

and the other entries are arbitrary.

We rewrite $S$ on the following decomposition

$$\mathcal{H} = (\ker B_1^*)^\perp \oplus \ker B \oplus \ker B_1^*$$

and choose $S_{11} = S_{23} = S_{32} = 0$, $S_{13} = R$, $S_{31} = R^*$, $S_{33} = C$, and $X = e \otimes e \oplus 0$ ($R, C, e$ are as in (i)). Then

$$\xi(X) + S = \begin{bmatrix} A_1 e \otimes A_1 e - B_1 e \otimes B_1 e & R^* \\ R & C \end{bmatrix} \oplus 0.$$ 

If $A_1 e \otimes A_1 e - B_1 e \otimes B_1 e$ is an operator of rank two and has eigenvalues $\lambda_1, \lambda_2$ with $|\lambda_1| = |\lambda_2|$ for all $e \in \mathcal{H}$, then from the previous argument we conclude that $A_1 = 0$. On the other hand, if $B_2 \neq 0$, then from $[A, B] = 0$ it follows that $A_2 = 0$, also a contradiction with the fact that $A \neq 0$.

If $B_2 = 0$ (ker $B^*$ reduces $B^*$), then $B = B_1 \oplus 0, A = 0 \oplus A_2 \oplus 0$ ($A_2$ is injective), and $S = (S_{i,j})_{i,j=1,...,3}$. Hence, it follows from $ASA^* = BSB^*$ that

$$S = \begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{21} & 0 & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$ 

and the other entries are arbitrary.

To conclude the proof we can argue similarly as in the first case (i). \qed

**Corollary 11.** Let $(A_1, A_2), (B_1, B_2)$ be 2-tuples of doubly commuting operators in $\mathcal{B}(\mathcal{H})$ and $E : \mathcal{B}(\mathcal{H}) \to \mathcal{C}_p(\mathcal{H});$ $(1 \leq p \leq \infty, \ p \neq 2)$ be the elementary operator defined by $E(X) = A_1 X B_1 - A_2 X B_2$ such that $\ker A_1, \ker A_2, \ker B_1^*, \text{and ker} \ker B_2^*$ reduce $A_1, A_2, B_1^*$, and $B_2^*$, respectively. If $E$ is orthogonal, then

$$\ker A_1 \cap \ker A_2 = \{0\} = \ker B_1^* \cap \ker B_2^*.$$ 

**Proof.** Consider the space $\mathcal{H} \oplus \mathcal{H}$ and the following decompositions

$$A = A_1 \oplus B_1^*, \quad B = A_2 \oplus B_2^*, \quad \text{and} \quad Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$ 

Then, for all $X \in \mathcal{B}(\mathcal{H})$, we have $AY = BYB^* = A_1 X B_1 - A_2 X B_2$, ker $A = \ker A_1 \cap \ker B_1^*$, and ker $B = \ker A_2 \cap \ker B_2^*.$ Hence,

$$\ker A_1 \cap \ker A_2 = \{0\} = \ker B_1^* \cap \ker B_2^* \iff \ker A \cap \ker B = \{0\}.$$ 

So to achieve the proof, it suffices to apply the previous proposition. \qed
Theorem 12. Let $A_1, A_2, N_1$, and $N_2$ be operators in $B(H)$ such that $(N_1, N_2)$ is 2-tuple normal and $[A_1, N_1] = [A_2, N_2] = 0$. Let $E(X) = A_1XA_2 - N_1XN_2$ such that $E(X), \tilde{E}(X) \in \mathcal{J}$. If $E$ is the Fuglede operator, then $E$ is $w$-orthogonal and $E \perp \tilde{E}$. Furthermore,

(i) If $\mathcal{J} = C_p(H)$: $(1 \leq p < \infty, p \neq 2)$, then

\[ E \text{ is orthogonal if and only if } \ker A_1 \cap \ker N_1 = \{0\} = \ker A_2^* \cap \ker N_2^*; \]

(ii) If $\ker A_1 \cap \ker N_1 = \{0\} = \ker A_2^* \cap \ker N_2^*$, then $E$ is orthogonal and $E \perp \tilde{E}$;

(iii) If $\ker A_1 \cap \ker N_1 \neq \{0\}$ or $\ker A_2^* \cap \ker N_2^* \neq 0$, then for all $X \in B(H)$ and all $S \in \ker E \cap \mathcal{J}$,

\[ \min \{\|E(X) + S\|_{\mathcal{J}}, \|\tilde{E}(X) + S\|_{\mathcal{J}}\} \geq k\|S\|_{\mathcal{J}}, \]

where

\[ -\mathcal{J} = B(H) : k = \frac{1}{2^p}; \]

\[ -\mathcal{J} = C_p(H) : k = \frac{1}{2^p - 2p}, \text{ if } 1 \leq p \leq 2 \text{ and } k = -\frac{1}{2^p - 4p}, \text{ if } 2 < p < \infty. \]

Proof.

(i) The implication $(\Rightarrow)$ follows from Corollary 8.

Let $\xi : B(H) \to \mathcal{J}$ be the elementary operator defined by $\xi(X) = AXA^* - NXX^*$, where $A, N \in B(H)$, $N$ is normal with $[A, N] = 0$. Assume that $\xi$ is the Fuglede operator.

(ii) Suppose that $N$ is invertible and set $D = N^{-1}A$. $\xi$ is Fuglede implies that $\xi_0$ is Fuglede, where $\xi_0$ is the elementary operator induced by $D$. By Lemma 2.4 [16], we have

\[ \|\xi(X) + S\|_{\mathcal{J}} = \|AXA^* - NXX^* + S\|_{\mathcal{J}} = \|D(NX^*)D^* - NXX^* + S\|_{\mathcal{J}} \geq \|S\|_{\mathcal{J}}. \]

Similarly, it can be shown that $\|\xi_0(X) + S\|_{\mathcal{J}} \geq \|S\|_{\mathcal{J}}$ for any operator $S \in (\ker \xi) \cap \mathcal{J}$.

(iii) Suppose that $N$ is injective, set $\Delta_n = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{n}; n \in \mathbb{N}\}$ and $\mu(\Delta_n)$ denotes the corresponding spectral projection, where $P_n = I - \mu(\Delta_n)$ converges strongly to $I$.

From $[A, N] = 0$ and by Fuglede-Putnam’s theorem it follows that $[A, N^\tau] = 0$ and therefore $P_n^\tau H$ reduces both $A$ and $N$. Let

\[ \mathcal{H} = P_n^\tau H \oplus (I - P_n^\tau)H, \]

then $A = A_{1n} \oplus A_{2n}; N = N_{1n} \oplus N_{2n}$ and $P_n = I \oplus 0$, where $N_{1n}$ is invertible. Hence,

\[ P_n(\xi(X) + S)P_n = [A_{1n}X_{1n}A_{1n}^* - N_{1n}X_{1n}N_{1n} + S_{1n}] \oplus 0 \]

for all $X = (X_{ij})_{i,j=1,2} \in B(H)$ and $S = (S_{ij})_{i,j=1,2} \in \ker \xi$ implying

\[ \|\xi(X) + S\|_{\mathcal{J}} \geq \|P_n(\xi(X) + S)P_n\|_{\mathcal{J}} = \|A_{1n}X_{1n}A_{1n}^* - N_{1n}X_{1n}N_{1n} + S_{1n}\|_{\mathcal{J}} \geq \|S_{1n}\|_{\mathcal{J}}. \]

Since $P_n(\xi^*(X) + S)P_n = [A_{1n}^*X_{1n}A_{1n} - N_{1n}X_{1n}N_{1n} + S_{1n}] \oplus 0$, then

\[ \|\xi^*(X) + S\|_{\mathcal{J}} \geq \|P_n(\xi^*(X) + S)P_n\|_{\mathcal{J}} = \|A_{1n}^*X_{1n}A_{1n} - N_{1n}X_{1n}N_{1n} + S_{1n}\|_{\mathcal{J}} \geq \|S_{1n}\|_{\mathcal{J}}. \]

On the other hand, for all positive integer $n$, we have

\[ \|S_{1n}\|_{\mathcal{J}} = \|P_nS_{1n}\| \leq \min \{\|\xi(X) + S\|_{\mathcal{J}}, \|\xi^*(X) + S\|_{\mathcal{J}}\} < \infty \]

and thus $\sup \|P_nS_{1n}\| < \infty$ for all $S \in \ker \xi \cap \mathcal{J}$ and $X \in B(H)$. It follows by Lemma 3 [14] that

\[ \min \{\|\xi(X) + S\|_{\mathcal{J}}, \|\xi^*(X) + S\|_{\mathcal{J}}\} \geq \|S\|_{\mathcal{J}}. \]

(iii) Suppose that $N$ is an arbitrary normal operator.

If $\ker A \cap \ker N = \{0\}$, then $\mathcal{H}$ may be decomposed as

\[ \mathcal{H} = (\ker N)^\perp \oplus \ker A \oplus \ker A \oplus \ker N. \]
Since $\xi$ is Fuglede and by Lemma 5(i), we have $\ker A$ reduces $A$, $A = A_{11} \oplus 0 \oplus A_{22}$ and $N = N_{11} \oplus N_{22} \oplus 0$. For $X = (X_{ij})_{i,j=1,2,3} \in B(\mathcal{H})$, set $\xi_i(X_{ij}) = A_{11}X_{ij}A_{11}^* - N_{11}X_{ij}N_{11}^*$ and $S = (S_{ij})_{i,j=1,2,3} \in \ker \xi$, then

$$\xi_i(S_{ij}) = 0 \quad \text{and} \quad A_{11}S_{13}A_{12}^* = A_{22}S_{31}A_{11}^* = A_{22}S_{31}A_{12}^* = 0,$$

$$N_{11}S_{12}N_{22}^* = N_{22}S_{21}N_{11}^* = N_{22}S_{22}N_{22}^* = 0.$$  

Since $\xi$ is Fuglede, then $S \in \ker \xi$ and

$$\xi_i(S_{ij}) = 0 \quad \text{and} \quad A_{11}^*S_{13}A_{22} = A_{22}^*S_{31}A_{11} = A_{22}^*S_{31}A_{22} = 0,$$

$$N_{11}^*S_{12}N_{22} = N_{22}^*S_{21}N_{11} = N_{22}^*S_{22}N_{22} = 0.$$  

(7)

Since $N_{11}, N_{22}, A_{11},$ and $A_{22}$ are injective, we get from (ii) that $\xi_i$ is orthogonal, $\xi_i \perp \tilde{\xi}_i$ and $S_{13} = S_{31} = S_{12} = S_{22} = S_{21} = 0$ and any operator $S \in \ker \xi$ has the form

$$S = \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & 0 & S_{23} \\ 0 & S_{32} & 0 \end{bmatrix},$$

where $S_{23}$ and $S_{32}$ are arbitrary.

Let $S_{23} = U_{23}S_{23}$, $S_{22} = U_{22}S_{22}$ be the polar decompositions of $S_{23}$ and $S_{32}$, respectively, let $V$ be the operator defined by

$$V = I \oplus \begin{bmatrix} 0 & U_{22} \\ U_{23} & 0 \end{bmatrix}.$$  

Then

$$\|\xi(X) + S\|_F \geq \|V(\xi(X) + S)\|_F = \left\| \begin{bmatrix} \xi_i(X_{11}) + S_{11} & * & * \\ * & |S_{21}| & * \\ * & * & |S_{23}| \end{bmatrix} \right\|_F.$$

Applying Lemma 6, we get

- if $F = B(\mathcal{H})$:

$$\|\xi(X) + S\| \geq \max\{\|\xi_i(X_{11}) + S_{11}\|, \|S_{22}\|, \|S_{23}\|\} \geq \max\{\|S_{11}\|, \|S_{22}\|, \|S_{23}\|\} = \|S\|._F.$$

- if $F = C_p(\mathcal{H}); (1 \leq p < \infty)$:

$$\|\xi(X) + S\|_p \geq (\|\xi_i(X_{11}) + S_{11}\|_p^p + \|S_{22}\|_p^p + \|S_{23}\|_p^p)^{1/p} \geq (\|S_{11}\|_p^p + \|S_{22}\|_p^p + \|S_{23}\|_p^p)^{1/p} = \|S\|_p.$$  

By the same method, we have that $\xi \perp \tilde{\xi}$. If $M = \ker A \cap \ker N \neq \{0\}$ and $\mathcal{H}$ is decomposed as $\mathcal{H} = (\ker N)^\perp \oplus [\ker N \cap M] \oplus M,$ then by Lemma 5(ii) and the fact that $\xi$ is Fuglede, it follows that $M$ reduces $A$, and hence $A = A_{11} \oplus A_{22} \oplus 0, \quad N = N_{11} \oplus 0.$

For $X = (X_{ij})_{i,j=1,2,3}$, we set $\xi_i(X) = A_{11}X_{ij}A_{11}^* - N_{11}X_{ij}N_{11}^*$ and let $S = (S_{ij})_{i,j=1,2,3} \in \ker \xi$. From the injectivity of $A_{11}$ and $A_{22}$, we obtain

$$\xi_i(S_{ij}) = 0 \quad \text{and} \quad S_{12} = S_{21} = S_{22} = 0.$$  

By simple computation, we get

$$\|\xi(X) + S\|_F = \left\| \begin{bmatrix} \xi_i(X_{11}) + S_{11} & S_{13} \\ S_{31} & S_{32} + S_{33} \end{bmatrix} \right\|_F.$$
Applying Lemma 6 yields

- for \( \mathcal{F} = B(\mathcal{H}) \):
  \[
  \|\xi(X) + S\|^2 \geq \frac{1}{2^2} (\|\xi(X_{11} + S_{11})\|^2 + \|S_{22}\|^2 + \|S_{33}\|^2) \geq \frac{1}{2^2} \|S\|^2.
  \]

- for \( \mathcal{F} = C_p(\mathcal{H}) ; 1 \leq p \leq 2 \):
  \[
  \|\xi(X) + S\|^p \geq \frac{1}{2^{2-p}} (\|\xi(X_{11} + S_{11})\|^p + \|S_{22}\|^p + \|S_{33}\|^p) \geq \frac{1}{2^{2-p}} \|S\|^p.
  \]

- for \( \mathcal{F} = C_p(\mathcal{H}) ; 2 \leq p < \infty \):
  \[
  \|\xi(X) + S\|^p \geq \frac{1}{2^{2-p}} (\|\xi(X_{11} + S_{11})\|^p + \|S_{22}\|^p + \|S_{33}\|^p) \geq \frac{1}{2^{2-p}} \|S\|^p.
  \]

Similarly,

- for \( \mathcal{F} = B(\mathcal{H}) \):
  \[
  \|\xi(X) + S\| \geq \frac{1}{2} \|S\|.
  \]

- for \( \mathcal{F} = C_p(\mathcal{H}) ; 1 \leq p \leq 2 \):
  \[
  \|\xi(X) + S\|_p \geq \frac{1}{2^{\frac{2}{-p}}} \|S\|_p.
  \]

- for \( \mathcal{F} = C_p(\mathcal{H}) ; 2 \leq p < \infty \):
  \[
  \|\xi(X) + S\|_p \geq \frac{1}{2^{\frac{2}{-p}}} \|S\|_p.
  \]

Let us now finish the proof for the elementary operator \( E(X) = A_1 X A_2 - N_1 X N_2 \). Consider the space \( \mathcal{H} \oplus \mathcal{H} \) and define the following operators on \( B(\mathcal{H} \oplus \mathcal{H}) \) as

\[
N = N_1 \oplus N_2^*; \quad A = A_1 \oplus A_2^*; \quad \text{and} \quad Z = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}
\]

It is clear that \( N \) is normal operator and \([A_1, N_1] = [A_2, N_2] = 0 \) imply \([A, N] = 0 \) and \( E(Z) = AZA^* - NZN^* \).

Applying the preceding result, the proof is complete.

Let \( A, B \in B(\mathcal{H}) \) and \( \Omega \) be a set in \( B(\mathcal{H})^2 \) defined by \((A, B) \in \Omega \) if and only if \( \Delta(X) = ANXBM - X \) is the Fuglede operator for any normal \( N, M \) satisfying \([N, A] = [M, B] = 0 \).

It follows from the definition that \( \Delta \) is Fuglede and \( \Omega \neq \emptyset \) since \((I, I) \in \Omega \).

It is shown in [13] that if \((A, B) \in \Omega \), then the elementary operator \( E \) defined by \( E(X) = NXM - AXB \) is Fuglede for any normal operators \( N \) and \( M \) in \( B(\mathcal{H}) \). So as a consequence of the previous theorem, we get the following corollaries.

**Corollary 13.** Let \( A, B \in B(\mathcal{H}) \) and \( N, M \) be arbitrary normal operators in \( B(\mathcal{H}) \) such that \([N, A] = [M, B] = 0 \) and \( E \) be the elementary operator defined by \( E(X) = NXM - AXB \) for all \( X \in B(\mathcal{H}) \).

If \((A, B) \in \Omega \), then \( E \) is \( w \)-orthogonal and \( E \in \tilde{E} \). Furthermore, \( E \) and \( \tilde{E} \) verify assertions (i), (ii), and (iii) in Theorem 12.

**Lemma 14.** [13] Let \( A, B \in B(\mathcal{H}) \) and \( N, M \) be normal operators in \( B(\mathcal{H}) \) such that \([N, A] = [M, B] = 0 \), then \((A, B) \in \Omega \) in each of the following cases:

(i) \( A \) and \( B^* \) are hyponormal operators;

(ii) \( A \) is \( k \)-quasihyponormal and \( B^* \) is injective \( k \)-quasihyponormal operator.
Corollary 15. If \((A_1, A_2), (B_1, B_2)\) are 2-tuples of commuting operators in \(B(\mathcal{H})\) such that \(A_1, B_1\) are normal operators and \(E(X) = A_1XB_1 - A_2XB_2\), then \(E\) is w-orthogonal and \(E \not\perp \bar{E}\). Furthermore, \(E\) and \(\bar{E}\) satisfy assertions (i), (ii), and (iii) cited in Theorem 12, in each of the following cases:

(i) \(A_i\) and \(B_i^*\) are hyponormal operators;
(ii) \(A_i\) is \(k\)-quasihyponormal and \(B_i^*\) is injective \(k\)-quasihyponormal operator.

In the next theorem, we give a positive answer to a question raised by P. B. Duggal [16]: Is Theorem 2 still true if the hypothesis is related to \(A\) and \(B^*\) being subnormal?

Theorem 16. If \(A\) and \(B^*\) are 2-tuples of commuting subnormal operators in \(B(\mathcal{H})\) such that \(A = (A_1, A_2), B = (B_1, B_2), \) and \(E(X) = A_1XB_1 - A_2XB_2\), then \(E\) is w-orthogonal and \(E \not\perp \bar{E}\).

Proof. From the definition of subnormality of 2-tuple operator, we have \(A\) is the restriction of a 2-tuple normal \(N = (N_1, N_2)\) to a common invariant subspace \(\mathcal{H}\) and \(B^*\) is the restriction of a 2-tuple normal \(M = (M_1, M_2)\) to a common invariant subspace \(\mathcal{H}\) equivalent to \(A_i = N_i|\mathcal{H}; B_i^* = M_i|\mathcal{H}\), \(i = 1, 2\) with \(N_i, M_i\) are normal operators on a Hilbert space \(\mathcal{K} \supseteq \mathcal{H}\) and \([N_i, N_2] = [M_i, M_2] = 0\). If \(S \in \ker E\), then for all \(X \in B(\mathcal{H})\), we have

\[N_1\tilde{X}M_1^* - N_2\tilde{X}M_2^* + \tilde{S} = \begin{bmatrix} E(X) + S & 0 \\ 0 & 0 \end{bmatrix},\]

where \(\tilde{X} = X \oplus 0\) and \(\tilde{S} = S \oplus 0\). Hence,

\[\|N_1\tilde{X}M_1^* - N_2\tilde{X}M_2^* + \tilde{S}\|_f = \|E(X) + S\|_f.\]

Since \(N_i, M_i : i = 1, 2\) are normal, we get the w-orthogonality of \(E\). With similar argument, \(E \not\perp \bar{E}\) follows. 

Corollary 17. If \((A_1, A_2)\) and \(B = (B_1, B_2)\) are 2-tuples of commuting operators in \(B(\mathcal{H})\) such that \([A_1, A_1^*A_1 + A_2^*A_2] = [B_1^*, B_1B_1^* + B_2B_2^*] = 0; i = 1, 2\) and \(E(X) = A_1XB_1 - A_2XB_2\), then \(E\) is w-orthogonal and \(E \not\perp \bar{E}\).

Proof. By assumption, \(A\) and \(B^*\) are spherically quasi-normal commuting 2-tuples (see definition [17]) and also by [17], \(A\) and \(B^*\) are subnormal 2-tuples. So the desired result follows from Theorem 16. 

Theorem 18. Let \((A_1, A_2), (B_1, B_2)\) be 2-tuples of doubly commuting operators in \(B(\mathcal{H})\) and \(E\) be the elementary operator defined by \(E(X) = A_1XB_1 - A_2XB_2\) such that \(E(X), \bar{E}(X) \in \mathcal{J}\), \(A_i, B_i^*\) are quasi-normal operators and \(A_2, B_2^*\) are \(k\)-quasihyponormal operators \((k \geq 1)\) with \(\ker A_2 \subseteq \ker A_2^*\) and \(\ker B_2^* \subseteq \ker B_2\). Then \(E\) is w-orthogonal and \(E \not\perp \bar{E}\). Furthermore,

1. If \(\mathcal{J} = C_p(\mathcal{H}); 1 \leq p < \infty, p \neq 2\), then

   \(E\) is orthogonal if and only if \(\ker A_1 \cap \ker N_1 = \ker A_1^* \cap \ker N_1^* = \{0\}\).

2. If \(\ker A_1 \cap \ker N_1 = \{0\} = \ker A_1^* \cap \ker N_1^*\), then \(E\) is orthogonal and \(E \perp \bar{E}\);

3. If \(\ker A_1 \cap \ker A_2 \neq 0 \text{ or } \ker B_1^* \cap \ker B_2^* \neq 0\), then for all \(X \in B(\mathcal{H})\) and all \(S \in \ker E \cap \mathcal{J}\),

   \[
   \min\|E(X) + S\|_f, \|\bar{E}(X) + S\|_f \geq k\|S\|_f.
   \]

   - For \(\mathcal{J} = B(\mathcal{H}), k = \frac{1}{6}\);

   - For \(\mathcal{J} = C_p(\mathcal{H}), k = \frac{1}{6-1/p}, \text{ if } 1 \leq p \leq 2\);

   - For \(\mathcal{J} = C_p(\mathcal{H}), k = \frac{1}{6-1/p}, \text{ if } 2 \leq p < \infty\).
Proof. Consider the following decompositions
\[ \mathcal{H} = \mathcal{H}_1 = (\ker A_2)^\perp \oplus \ker A_2, \quad \mathcal{H} = \mathcal{H}_2 = (\ker B_2)^\perp \oplus \ker B_2^* \]
Then
\[ A_2 = C_1 \oplus 0, \quad B_2 = C_2 \oplus 0, \]
where \( C_1, C_2 \) are injective \( k \)-quasihyponormal and by hypothesis, we get
\[ A_1 = T_1 \oplus T_2, \quad B_1 = R_1 \oplus R_2 \]
with \( T_1, T_2, R_1^*, \) and \( R_2^* \) are quasinormal operators and \( (T_1, C_1), (R_1, C_2) \) are 2-tuples of doubly commuting operators.
Since \( \ker T_1 \) reduces \( T_1 \) (resp. \( \ker R_1^* \) reduces \( R_1^* \)) and by commutativity, we have that
\[ T_1 = A_{11} \oplus 0, \quad R_1 = B_{11} \oplus 0, \quad C_1 = A_{21} \oplus A_{22}, \quad C_2 = B_{21} \oplus B_{22} \]
with respect to the following decompositions
\[ (\ker A_2)^\perp = (\ker T_1)^\perp \oplus \ker T_1, \quad (\ker B_2)^\perp = (\ker R_1^*)^\perp \oplus \ker R_1^*, \]
where \( A_{11} \) and \( B_{11}^* \) are injective quasinormal operators, \( A_{21}, A_{22}, B_{21}^*, \) and \( B_{22}^* \) are injective \( k \)-quasihyponormal. We set \( A_{11} = U|A_{11}| \) and \( B_{11} = V|B_{11}| = |B_{11}^*| V \). Then it follows from the injectivity of \( A_{11} \) and \( B_{11}^* \) that \( U \) and \( V^* \) are isometry operators and by \( |T_1, C_1| = |T_1^*, C_1^*| = 0 \), we get that
\[ |A_{11}, A_{21}| = |A_{11}^*, A_{21}| = 0. \] (8)
Then \( [A_{11}^*, A_{21}] = [A_{11}, A_{21}] = 0 \) and \( \|A_{11}, A_{21}\| = \|A_{11}^*, A_{21}\| = 0 \). Hence, \( \|U, A_{21}\| = \|U^*, A_{21}\| = 0 \).
Similarly, we obtain that \( \|V, B_{21}\| = \|V^*, B_{21}\| = 0 \), and \( (UA_{21}^*)^\perp \) is \( k \)-quasihyponormal. Indeed,
\[ (A_{21}^* U)^k [U^* A_{21}^* U] A_{21}^* U^k = (A_{21}^*)^k U^k [U^*, U] A_{21}^* U^k + U^k A_{21}^* U \]
\[ = U^k A_{21}^* U + U^k A_{21}^* U \leq 0. \]
Similarly, we get \((V B_{21})^*\) is an injective \( k \)-quasihyponormal.
Let \( X, S \in B(\mathcal{H}) : X = (X_{ij})_{i,j=1,2,3} \) and \( S = (S_{ij})_{i,j=1,2,3} \). If \( S \in \ker E \), then
\[ A_{11} S_{11} B_{11} = A_{21} S_{21} B_{21}, \] (9)
\[ A_{11} S_{13} R_3 = T_3 S_{13} B_{11} = T_2 S_{13} R_2 = 0, \] (10)
\[ A_{31} S_{23} B_{22} = A_{22} S_{23} B_{22} = A_{23} S_{22} B_{22} = 0. \] (11)
And
\[ U^*(A_{21} X_{21} B_{21} - A_{21} X_{31} B_{21} + S_{11}) V^* = |A_{11}| X_{11} B_{11}^* - U^* A_{21} X_{11} B_{21} + U^* S_{11} V^*. \]
We derive from 8 that
\[ |A_{11}| U S_{11} V^* B_{11} = U^* A_{21} U^* S_{11} V^* B_{21} V^*. \]
Applying Corollary 8,
\[ \|A_{11} X_{11} B_{11} - A_{21} X_{11} B_{21} + S_{11}\| \geq \|U^*(A_{11} X_{11} B_{11} - A_{21} X_{11} B_{21} + S_{11}) V^*\| \geq \|U^* S_{11} V^*\|. \]
from the injectivity of \( A_{11} \) and its polar decomposition, we have
\[ (\ker T_1)^\perp = (\ker A_{11})^\perp = (\ker U)^\perp; \quad \ker A_{11} = \text{ran } U \]
and \( U : (\ker U)^\perp \rightarrow \text{ran } U \) is unitary. Taking the following decompositions yields
\[ (\ker U)^\perp = \text{ran } U \oplus (\text{ran } U)^\perp; \quad (\ker V^*)^\perp = \overline{\text{ran } V^*} \oplus (\text{ran } V^*)^\perp. \]
Then

\[
A_{11} = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} B_{11} & 0 \\ \zeta & 0 \end{bmatrix}, \quad S_{11} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},
\]

\[
U = \begin{bmatrix} U_1 \\ 0 \end{bmatrix} : (\ker U)^+ \to \text{ran } U \oplus (\text{ran } U)^+,
\]

\[
V = [V_1 \ 0] : \text{ran } V^* \oplus (\text{ran } V^*)^+ \to (\ker V^*)^+.
\]

From the commutativity, we obtain

\[
A_{21} = A_{12}^1 \oplus A_{21}^2; \quad B_{21} = B_{12}^1 \oplus B_{21}^2.
\]

By simple computation, we get

\[
U^* S_{11} V^* = U_1^* S_{11}^1 V_1^*, \quad A_{12}^1 S_{11}^2 B_{12}^2 = A_{21}^2 S_{11}^1 B_{21}^2 = A_{21} S_{11}^2 B_{21}^2 = 0.
\]

From the injectivity of \(A_{1i}\) and \(B_{2i}^2\), \(i = 1, 2\), we derive that

\[
S_{11}^2 = S_{11}^1 = S_{11} = 0.
\]

The injectivity of \(A_{31}, B_{22}, A_{22}, B_{21}, A_{11}\), and \(B_{11}^*\) in the equalities 10 and 11 implies that

\[
S_{12} = S_{21} = S_{32} = 0, \quad S_{13} R_2 = T_2 S_{31} = T_2 S_{32} R_2 = 0.
\]

Setting \(E_{11}(X_{11}) = A_{11} X_{11} B_{11} - A_{21} X_{11} B_{21}\),

(i) If \(\ker A_1 \cap \ker A_2 = \{0\} = \ker B_{11}^* \cap \ker B_{21}^*\), then \(S_{13} = S_{33} = S_{31} = 0\) and therefore any operator \(S \in \ker E\) has the form

\[
S = \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & 0 & S_{32} \\ 0 & S_{23} & 0 \end{bmatrix},
\]

where \(S_{23}\) and \(S_{32}\) are arbitrary with

\[
\|S_{11}\|_\mathcal{F} = \|S_{11}^{1}\|_\mathcal{F} = \|U_1^* S_{11}^1 V_1^*\|_\mathcal{F} = \|U^* S_{11} V^*\|_\mathcal{F}
\]

and

\[
\|E(X) + S\|_\mathcal{F} = \begin{bmatrix} E_{11}(X_{11}) + S_{11} & * & * \\ * & * & S_{32} \\ * & S_{23} & * \end{bmatrix}_\mathcal{F}.
\]

Let \(S_{23} = U_{23} S_{23}\), \(S_{32} = S_{32}^* S_{32}\) be the polar decomposition of \(S_{23}\) and \(S_{32}\), respectively, and set the operator

\[
V = I \oplus \begin{bmatrix} 0 & U_{12} \\ U_{23}^* & 0 \end{bmatrix}. \quad \text{Then}
\]

\[
\|E(X) + S\|_\mathcal{F} \geq \|V(E(X) + S)\|_\mathcal{F} \geq \begin{bmatrix} E_{11}(X_{11}) + S_{11} & * & * \\ * & |S_{32}| & * \\ * & * & |S_{23}| \end{bmatrix}_\mathcal{F}.
\]

Applying Lemma 5, we get

\[
- \mathcal{J} = B(\mathcal{H}) : \quad \|E(X) + S\| \geq \max\{\|E(X_{11}) + S_{11}\|, \|S_{32}\|, \|S_{23}\|\} \geq \max\{\|S_{11}\|, \|S_{32}\|, \|S_{23}\|\} = \|S\|.
\]

\[
- \mathcal{J} = C_p(\mathcal{H}) : (1 \leq p < \infty)
\]

\[
\|E(X) + S\|_p \geq (\|E(X_{11}) + S_{11}\|_p + \|S_{32}\|_p + \|S_{23}\|_p^{1/p})^{1/p} \geq (\|S_{11}\|_p + \|S_{32}\|_p + \|S_{23}\|_p^{1/p})^{1/p} = \|S\|_p^{1/p}.
\]
(ii) If \( \ker A_1 \cap \ker A_2 \neq \{0\} \) or \( \ker B_1^* \cap \ker B_2^* \neq \{0\} \), then any operator \( S \in \ker E \) has the form

\[
S = \begin{bmatrix}
S_{11} & 0 & S_{13} \\
0 & 0 & S_{23} \\
S_{13} & S_{32} & S_{33}
\end{bmatrix},
\]

where \( S_{23} \) and \( S_{32} \) are arbitrary. By simple calculation, we have

\[
\|E(X) + S\|_{\mathcal{J}} = \left\| \begin{bmatrix}
E_{11}(X_{11}) + S_{11} & + & A_{11}X_{13} + S_{13} \\
* & + & * \\
T_{2}X_{31}B_{11} + S_{31} & + & T_{2}X_{33}R_{2} + S_{33}
\end{bmatrix} \right\|_{\mathcal{J}}.
\]

It is well known that the kernel of a quasinormal operator is a reduced subspace, then by application of Corollary 8, we obtain

\[
\|A_{11}X_{13} + S_{13}\|_{\mathcal{J}} \geq k\|S_{13}\|_{\mathcal{J}} \\
\|T_{2}X_{31}B_{11} + S_{31}\|_{\mathcal{J}} \geq k\|S_{31}\|_{\mathcal{J}} \\
\|T_{2}X_{33}R_{2} + S_{33}\|_{\mathcal{J}} \geq k\|S_{33}\|_{\mathcal{J}}.
\]

Therefore, by Lemma 6, we get

\[-\mathcal{J} = B(\mathcal{H});
\]

\[
\|E(X) + S\|^2 \geq \frac{1}{3^2} \left( \|E_{11}(X_{11}) + S_{11}\|^2 + \frac{1}{2^2} \|S_{23}\|^2 + \|S_{31}\|^2 + \|S_{33}\|^2 \right) \geq \frac{1}{6^2}\|S\|^2.
\]

\[-\mathcal{J} = C_p(\mathcal{H}); (2 \leq p < \infty)
\]

\[
\|E(X) + S\|_p \geq \frac{1}{3^{p-2}}\|S_{11}\|_p + \frac{1}{2^{p-2}}\|S_{23}\|_p + \|S_{31}\|_p + \|S_{33}\|_p \geq \frac{1}{2^{p-2}} \frac{1}{3^{p-2}} \|S\|_p = \frac{1}{6^{p-2}}\|S\|_p.
\]

\[-\mathcal{J} = C_p(\mathcal{H}); (1 \leq p \leq 2)
\]

\[
\|E(X) + S\|_p \geq \frac{1}{3^{1-p}}\|S_{11}\|_p + \frac{1}{2^{1-p}}\|S_{23}\|_p + \|S_{31}\|_p + \|S_{33}\|_p \geq \frac{1}{2^{1-p}} \frac{1}{3^{1-p}} \|S\|_p = \frac{1}{6^{1-p}}\|S\|_p.
\]

\[\square\]

4 Conclusion

D. Keckic [14] and A. Turnšek [15] extended Theorem 2 to the elementary operator \( E \) defined by \( E(X) = AXB - CXD \), where \( A, C \) and \( B, D \) are 2-tuples of commuting normal operators. Duggal [16] generalized the famous theorem to the case \( A, C \) and \( B, D \) are 2-tuples of commuting operators, where \( A, B \) are normal and \( C, D^* \) are hyponormal.

In this paper, Theorem 2 was extended to non-normal operators including quasinormal, subnormal, and \( k \)-quasi-hyponormal operators. The main results are Theorems 12 and 18, both of considerable value in the relevant area of research, also the paper includes new ideas, along with a few new tools and techniques, and likely to attract considerable attention from researchers in operator theory and Banach space theory.

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