Convergence of Finite Integral Method for Poisson Equation and Corresponding Eigenproblems

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March 12, 2014

Abstract

In this paper, we analyze the convergence of Finite Integral method (FIM) for Poisson equation with Neumann boundary condition on submanifolds isometrically embedded in Euclidean spaces. We also show the convergence of FIM for solving Helmholtz equation (aka the eigensystem of Laplace-Beltrami operator) with Neumann boundary. Finite Integral method is a numerical method to solve Poisson equation which was introduced in our previous paper [16]. Being a meshless method, FIM avoids the process of meshing the underlying domain, which can be very difficult for curved submanifolds. Therefore it is a good alternative to finite element method when the mesh structure of good quality is not available, which indeed occurs in many applications. Moreover, comparing to other meshless methods, FIM does not need to construct basis functions and thus is much easier to implement and more efficient.

The main theoretical contribution of this paper consists of two parts: (I) we prove that the numerical solution given by FIM will converge to the exact solution as the desity of the sample points tends to infinity; (II) the eigenvalues and corresponding eigenfunctions of the eigensystem also converge.

We extend the previous work of the convergence of the eigensystem of the integral Laplace operator and its discrete version to that of Laplace-Beltrami operator in two ways. First we show the convergence of the eigensystem in the presence of Neumann boundary where the integral Laplace operator is in fact dominated by the first order derivative. Second, we generalize standard Gaussian kernel to a large class of integral kernels, including ones with compact support. Unlike the previous approach, our approach avoids resorting to the heat kernel of the manifold, which make it possible to be extended to analyze the integral Laplace operator on more complicated spaces where the heat kernel is not defined.

The section of numerical results (Section 11) by Zhen Li shows the performance of finite integral method and its comparison with finite element method.

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1 Introduction

The Laplace-Beltrami operator of a Riemannian manifold encodes all intrinsic geometry of the manifold, which has many applications in computer graphics and vision, geometric processing and modeling, and machine learning, among others. The solutions to Helmholtz equation, also known as the eigensystem of the Laplace-Beltrami operator, have been used for representing shapes in computer graphics and vision for their analysis [22, 20], and representing data in machine learning for dimensionality reduction [1].

Finite element method has been widely used to solve Poisson equation on surfaces. Dziuk [14] showed finite element method converges quadratically in $L^2$ and linearly in $H^1$. Recently, Wardetzky [27] showed the same convergence rate holds even with approximating sequences of meshes. The eigensystem of Laplace-Beltrami operator also converges for finite element method linearly [25, 13, 27]. Boffi [6] showed the convergence of FEM for general compact operators. In FEM, the solution is approximated over a set of basis functions and the basis functions can be derived from the mesh elements. However, the quality of the basis functions depends on the quality of the mesh. A mesh with bad shaped elements may increase the condition number of involved linear systems and hence reduce the accuracy of the solution [23]. It is not an easy task to construct a mesh of good quality, especially for an irregular domain in $\mathbb{R}^d$ with $d \geq 3$ [9], and maintain the mesh quality when the domain is deforming over the time. For a general submanifold, the meshing task becomes even harder. It may be still manageable for meshing a surface (2-manifold) in $\mathbb{R}^3$ [10, 28]. However, it is extremely difficult to build a globally consistent mesh for a general submanifold in high dimensional spaces $\mathbb{R}^d$ [7].

In recent years, many efforts are devoted to develope alternative numerical method bypass the difficulty of constructing global mesh. Liang and Zhao [17] and Lai et al. [21] consider the problem of solving PDE on a submanifold with local meshes. However, they did not provide any theoretical analysis for their methods. In fact, the method that Lai et al. proposed based on local meshes is essentially FEM but over a non-consistent mesh. Belkin and Niyoyi derived the integral equation [2, 3] in the presence of no boundary through heat diffusion process on the manifold. They proved the pointwise convergence and the convergence of the eigensystem for the graph Laplacian with Gaussian weights in the probabilistic setting where the data points are uniformly sampled from the manifold in iid fashion. Belkin et al. [5] and Dey et al. [11] showed the same results hold in the geometric setting where the data points densely sample the manifold and the volume weights are estimated from the data points. In the presence of boundary, Lafon [15] and Belkin et al [4] observed that the weighted graph Laplacian is dominated by the first order derivative and thus fails to be true laplacian near the boundary. Recently, Singer and Wu [24] independently showed the convergence of the eigensystem of the discrete Laplace operator with Gaussian kernel in the presence of Neumann boundary. Their approach which resorts to heat kernel is different from ours.

The Finite Integral method is a kind of meshless method as this method does not require a global mesh with good quality. FIM only requires two volume vectors ($V$ and $A$ in Section 2) which can be well estimated from a mesh with possibly bad shaped elements as they are only used for approximating integrals on the manifold. Moreover, the volume vectors can be estimated from the data points by locally approximating tangent planes of the manifold, which is much easier than building a globally consistent mesh [18]. In the previous paper [16], many numerical examples are shown to demonstrated that FIM has good performance in solving Poisson equation
and corresponding eigensystem on manifold.

In this paper, we focus on the convergence of Finite Integral method which was not given in [16]. We consider the Poisson equation and its eigenproblem with Neumann boundary condition on a submanifold isometrically embedded in a Euclidean space $\mathbb{R}^d$ and have proved that the solution given by FIM does converge to the exact solution as the density of sample points tends to infinity both for Poisson equation and its eigenproblem. For Poisson equation, the convergence rate is obtained. For the eigensystem, we only prove the convergence. Our convergence result of the eigensystem (Theorem 3.2) extends the above result in two ways. First, we prove the convergence of the eigensystem in the presence of Neumann boundary. Second, we generalize Gaussian kernel to a large class of integral kernels, including ones with compact support. In fact, unlike the previous approaches [3], our approach of proving the convergence result does not involve heat kernel and heat operator of the manifold. It is known that the integral Laplace operator ($L_t$ defined in Equation (4.1)) and its discrete version ($L$ defined in Equation (2.6)) can be defined on more complicated spaces, like multiple manifolds with self intersections [4] or even general stratified spaces. One can infer the structures of the underlying complicated spaces from the behavior of this integral Laplace operator [3]. Our approach of avoiding using heat kernel may be extended to analyze the integral Laplace operator on more complicated spaces where the heat kernel is not defined.

The remaining of this paper is organized as following. In Section 2, we give the problems we consider and a brief introduction of the Finite Integral method. The main results are given in Section 3. The structure of the proof is shown in Section 4. Section 5 is the discussion and future work. In section 6, we give several basis estimates related to the property of the manifold which will be used often in the proof. The main body of the proof is given in Section 7, Section 8, Section 9 and Section 10. At last, a simple numerical example is shown in section 11.

### 2 Problems and Finite Integral Method

We consider the Poisson equation on a compact $k$-dimensional submanifold $\mathcal{M}$ in $\mathbb{R}^d$ with Neumann boundary condition

$$
\begin{align*}
-\Delta_{\mathcal{M}} u(x) &= f(x), \quad x \in \mathcal{M} \\
\frac{\partial u}{\partial n}(x) &= g(x), \quad x \in \partial \mathcal{M}
\end{align*}
$$

(P1.a)

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$. Let $g$ be the Riemannian metric tensor of $\mathcal{M}$. Given a local coordinate system $(x^1, x^2, \ldots, x^k)$, the metric tensor $g$ can be represented by a matrix $[g_{ij}]_{k \times k},$

$$
g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle, \quad i, j = 1, \ldots, k. \tag{2.1}
$$

Let $[g^{ij}]_{k \times k}$ is the inverse matrix of $[g_{ij}]_{k \times k}$, then it is well known that the Laplace-Beltrami operator is

$$
\Delta_{\mathcal{M}} = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} (g^{ij} \sqrt{\det g} \frac{\partial}{\partial x^j}) \tag{2.2}
$$

In this paper, the metric tensor $g$ is assumed to be inherited from the embedding space $\mathbb{R}^d$, that is, $\mathcal{M}$ isometrically embedded in $\mathbb{R}^d$ with standard Euclidean metric. If $\mathcal{M}$ is an open set in $\mathbb{R}^d$, then $\Delta_{\mathcal{M}}$ becomes standard Laplace operator, i.e., $\Delta_{\mathcal{M}} = \sum_{i=1}^{d} \frac{\partial^2}{\partial x^i x^i}.$
We can also write the above linear system in the matrix form. Assemble the following three eigenvalues listed in the increasing order and paper.

\[ \begin{cases} -\Delta_M u(x) = \lambda u(x), & x \in M \\ \frac{\partial u}{\partial n}(x) = 0, & x \in \partial M. \end{cases} \] (P1.b)

A pair \((\lambda, u)\) solving the above equation is called an eigenvalue and the corresponding eigenfunction of the Laplace-Beltrami operator \(\Delta_M\). It is well known that the spectrum of Laplace-Beltrami operator is discrete and all eigenvalues are nonnegative. Suppose \(0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \cdots\) are all eigenvalues listed in the increasing order and \(\phi_0, \phi_1, \phi_2, \cdots\) are corresponding eigenfunctions.

**Input data:** To numerically solve the problem \((\text{P1.a})\), and the problem \((\text{P1.b})\), assume we are given a set of sample points \(P\) and a subset \(S \subset P\) sampling the boundary of \(M\). List the points in \(P\) respectively \(S\) in a fixed order \(P = (p_1, \cdots, p_n)\) where \(p_i \in \mathbb{R}^d, 1 \leq i \leq n\), respectively \(S = (s_1, \cdots, s_m)\) where \(s_i \in P\).

In addition, assume we are given two vectors \(V = (V_1, \cdots, V_n)^t\) where \(V_i\) is an volume weight of \(p_i\) in \(M\), and \(A = (A_1, \cdots, A_m)^t\) where \(A_i\) is an area weight of \(s_i\) in \(\partial M\), so that for any Lipschitz function \(f\) on \(M\) respectively \(\partial M\), \(\int_M f(x) d\mu_x\) respectively \(\int_{\partial M} f(x) d\tau_x\) can be approximated by \(\sum_{i=1}^n f(p_i)V_i\) respectively \(\sum_{i=1}^m f(s_i)A_i\) \((\text{P1.a})\). Here \(d\mu_x\) and \(d\tau_x\) are the volume form of \(M\) and \(\partial M\), respectively.

Finally, the function \(f\) and the function \(g\) in the problem \((\text{P1.a})\) are specified by their evaluation at the sample points. Specifically, \(f\) respectively \(g\) is given as the vector \(f = (f_1, \cdots, f_n)^t\) with \(f_i = f(p_i)\) respectively \(g = (g_1, \cdots, g_m)^t\) with \(g_i = g(s_i)\).

**Finite integral method:** We are ready to describe our finite integral method for solving the above problems. Let \(R: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) to be a positive \(C^2\) function which is integrable over \([0, +\infty)\). Given a parameter \(t > 0\), for any \(x, y \in \mathbb{R}^d\), denote

\[ R_t(x, y) = C_t R\left(\frac{|x - y|^2}{4t}\right) \] (2.3)

where \(C_t = \frac{1}{(4\pi t)^{d/2}}\) is the normalizing factor. The function \(R_t(x, y)\) serves as the integral kernel. Let

\[ \tilde{R}(r) = \int_r^{+\infty} R(s)ds, \quad \tilde{R}_t(x, y) = C_t \tilde{R}\left(\frac{|x - y|^2}{4t}\right) \] (2.4)

Then \(\frac{\partial \tilde{R}}{\partial r} = -R\). When \(R(r) = e^{-r}\), \(\tilde{R}_t(x, y) = R_t(x, y) = C_t \exp\left(\frac{|x - y|^2}{4t}\right)\) are well-known Gaussian.

We consider the following linear system of \(u = (u_1, \cdots, u_n)^t\). For any point \(p_i \in P\)

\[-\frac{1}{t} \sum_{p_j \in P} R_t(p_i, p_j)(u_i - u_j)V_j = -2 \sum_{s_j \in S} \tilde{R}_t(p_i, s_j)A_j g_{ij} + \sum_{p_j \in P} \tilde{R}_t(p_i, p_j)f_{ij}V_j. \] (2.5)

We can also write the above linear system in the matrix form. Assemble the following three matrices. The first matrix, denoted \(\mathcal{L}\), is an \(n \times n\) matrix defined as for any \(p_i, p_j \in P\)

\[ \mathcal{L}_{ij} = \begin{cases} -\frac{1}{t} R_t(p_i, p_j)V_j & \text{if } i \neq j \\ -\sum_{i \neq j} L_{ij} & \text{if } i = j. \end{cases} \] (2.6)
The second matrix, denoted $I$, is also an $n \times n$ matrix defined as for any $p_i, p_j \in P$
\[ I_{ij} = \tilde{R}_t(p_i, p_j)V_j. \] (2.7)

The third matrix, denoted $B$, is an $n \times m$ matrix defined as for any $p_i \in P$ and any $s_j \in S$
\[ B_{ij} = \tilde{R}_t(p_i, s_j)A_j. \] (2.8)

With the above three matrices, the linear system in Equation 2.5 can be rewritten as
\[ -Lu = -2Bg + If \tag{P2.a} \]

We use the solution $u = (u_1, \cdots, u_n)^t$ to the above linear system to approximate the solution to
the problem [P1.a].

In addition, consider the following generalized eigenproblem
\[ -Lu = \lambda Iu. \tag{P2.b} \]

We use the eigenvalues and the eigenvectors of the above generalized eigenproblem to approximate
the eigensystem of Laplace-Beltrami operator with Neumann boundary condition, i.e., the solutions to the problem [P1.b]. The main theoretical contribution of this paper is to establish the
convergence results for the above finite integral method for solving the problem [P1.a] and the
problem [P1.b].

3 Main Results

Our main theoretical contribution is to establish the convergence results for our finite integral
method for solving the problem [P1.a] and the problem [P1.b]. To state the convergence result,
we need to describe the assumptions on the integral kernel $R$, the regularity of the submanifold $M$
and how well the input data approximate the submanifold $M$.

First we assume the function $R: \mathbb{R}^+ \to \mathbb{R}^+$ is $C^2$ and satisfies the following conditions:
(i) $R(r) = 0$ for $\forall r > 1$.
(ii) There exists a constant $\delta_0$ so that $R(r) > \delta_0$ for $\forall r < \frac{1}{2}$.

Note it is for the sake of simplicity of exposition that $R$ is assumed to be compactly supported.
Our convergence results also hold for any integral kernel that decays exponentially, like Gaussian
kernel $G_t(x, y) = C_t \exp \left( -\frac{|x-y|^2}{4t} \right)$. In fact, for any $s \geq 1$ and any $\epsilon > 0$, the $H^s$ mass of Gaussian
kernel over the domain $\Omega = \{ y \in M||x - y|^2 \geq t^{1+\epsilon} \}$ decays faster than any polynomial in $t$ as $t$
goes to 0, i.e., $\lim_{t \to 0} \frac{\|G_t(x, y)\|_{H^s(\Omega)}}{t^\alpha} = 0$ for any $\alpha$. Note since $R$ is $C^2$ and compactly supported,
$R, R', R''$ and $\tilde{R}$ are all bounded.

Second, we assume both $M$ and $\partial M$ are compact and $C^\infty$ smooth. Consequently, it is well
known that both $M$ and $\partial M$ have positive reaches.

Finally, we state the assumption on the input data. Our requirement to the input data is that
the integral of any Lipschitz function on $M$ or $\partial M$ is able to be well approximated from the input
data $P, S, V$ and $A$. We say a quadruple $(P, S, V, A)$ $h$-integral approximation of $(M, \partial M)$ if the
following two conditions hold. For any function $f$ in $C^1(M)$ or $C^1(\partial M)$, let $\|f\|_{C^1} = \|f\|_\infty + \|\nabla f\|_\infty$
and $\text{supp}(f)$ is the support of $f$. For a set $X$, denote $|X|$ to be its volume.
(i) For any function \( f \in C^1(M) \), there is a constant \( C \) independent of \( h \) and \( f \) so that

\[
\left| \int_M f(y) d\mu_y - \sum_{i=1}^n f(p_i)V_i \right| < Ch|\text{supp}(f)||f|_{C^1}, \quad \text{and} \quad (3.1)
\]

(ii) For any function \( f \in \partial M \), there is a constant \( C \) independent of \( h \) and \( f \) so that

\[
\left| \int_{\partial M} f(y) d\tau_y - \sum_{i=1}^n f(s_i)A_i \right| < Ch|\text{supp}(f)||f|_{C^1}. \quad (3.2)
\]

The solution of finite integral method is a vector \( u \) in \( \mathbb{R}^n \) while the solution of the problem (P1.a) is a function defined on \( M \). To make them comparable, for any solution \( u = (u_1, \ldots, u_n)^t \) to the problem (P1.a), we construct a function on \( M \)

\[
I_{f,g}(u)(x) = \frac{\sum_{p_j \in P} R_t(x, p_j)u_j V_j + 2t \sum_{s_j \in S} \frac{\bar{R}_t(x, s_j)g_j A_j - t \sum_{p_j \in P} \bar{R}_t(x, p_j)f_j V_j}{\sum_{p_j \in P} \bar{R}_t(x, p_j)V_j}}.
\]

It is easy to verify that \( I_{f,g}(u) \) interpolates \( u \) at the sample points \( P \), i.e., \( I_{f,g}(u)(p_j) = u_j \) for any \( j \) (see Proposition 4.1). Since \( I_{f,g}(u) \) is smooth, it is well-controlled by the vector \( u \) provided that \( P \) densely samples \( M \).

The following theorem guarantees the convergence of our finite integral method in solving the problem (P1.a).

**Theorem 3.1.** Assume both the submanifolds \( M \) and \( \partial M \) are \( C^\infty \) smooth and the input data \( (P, S, V, A) \) is an \( h \)-integral approximation of \((M, \partial M)\), \( u \) is the solution to Problem (P1.a) with \( f \in C^\infty(M) \) and \( g \in C^\infty(\partial M) \). Set \( f = (f(p_1), \ldots, f(p_n)) \), and \( g = (g(s_1), \ldots, g(s_m)) \). If the vector \( u \) is the solution to the problem (P2.a). Then there exists a constant \( C \) so that for any sufficient small \( t \)

\[
\|u - I_{f,g}(u)\|_{H^1(M)} \leq Ct^{1/4}(\|f\|_{H^1(M)} + \|g\|_{H^1(\partial M)}) + \frac{Ch}{t^2} \left( \|f\|_{\infty} + t^{-1/4}\|g\|_{\infty} \right). \quad (3.3)
\]

where \( s = k/2 + 3 \).

Similarly for any eigenvector vector \( u = (u_1, \ldots, u_n) \) of the eigenprolem (P2.b) with eigenvalue \( \lambda \), we construct a function on \( M \)

\[
I_{\lambda}(u)(x) = \frac{\lambda \sum_{p_j \in P} \bar{R}_t(x, p_j)u_j V_j - \sum_{p_j \in P} R_t(x, p_j)u_j V_j}{\sum_{p_j \in P} \bar{R}_t(x, p_j)V_j}.
\]

It is also easy to verify that \( I_{\lambda}(u) \) interpolates \( u \) at the sample points \( P \) (see Proposition 4.1). The following theorem guarantees the convergence of our finite integral method for solving the eigensystem problem (P1.b) of the Laplace-Beltrami operator with Neumann boundary condition.

**Theorem 3.2.** Assume the submanifold \( M \) and \( \partial M \) are \( C^\infty \) smooth and the input data \( (P, S, V, A) \) is an \( h \)-integral approximation of \((M, \partial M)\), Let \( \lambda_i \) be the \( i \)th smallest eigenvalue of \( \Delta_M \) and \( \phi_i \) is the corresponding function. Then there exist two sequences \( h_n \to 0 \) and \( t_n \to 0 \) as \( n \) going to \( \infty \), if
let \( \lambda_{i}^{h,n,t} \) and \( u_{i}^{h,n,t} \) be the \( i \)th eigenvalue and the corresponding vector of the eigenproblem \( \text{P2.b} \), then
\[
\lim_{n \to \infty} |\lambda_{i}^{h,n,t} - \lambda_i| = 0,
\]
and if the multiplicity of \( \lambda_i \) is 1,
\[
\lim_{n \to \infty} \| I_{\lambda_{i}^{h,n,t}}(u_{i}^{h,n,t}) - \phi_i \|_{L^2(M)} = 0.
\]

4 Intermediate Operators and Structure of the Proof

To connect the Poisson problem \( \text{P1.a} \) and the eigenproblem \( \text{P1.b} \) to their discrete version \( \text{P2.a} \) and \( \text{P2.b} \) respectively, we introduce two intermediate operators \( L_t \) and \( L_{t,h} \) as follows. For any function \( u \) on \( M \), define
\[
L_t u(x) = \frac{1}{t} \int_M R_t(x,y)(u(x) - u(y))d\mu_y, \quad \text{and} \quad (4.1)
\]
\[
L_{t,h} u(x) = \frac{1}{t} \sum_{p_j \in P} R_t(x,p_j)(u(x) - u(p_j))V_j. \quad (4.2)
\]
We call \( L_t \) the integral Laplace operator. Similar operators can also be found in [2], [26]. Consider the corresponding intermediate problems as follows. For the operator \( L_t \), consider the problem
\[
- L_t u(x) = -2 \int_{\partial M} \bar{R}_t(x,y)g(y)d\tau_y + \int_M \bar{R}_t(x,y)f(y)d\mu_y, \quad (P3.a)
\]
and the corresponding eigenproblem
\[
- L_{t,h} u(x) = \lambda \int_M \bar{R}_t(x,y)u(y)d\mu_y. \quad (P3.b)
\]
We call Equation \( \text{P3.a} \) the integral equation for Poisson problem. For the operator \( L_{t,h} \), consider the problem
\[
- L_{t,h} u(x) = -2 \sum_{s_j \in S} \bar{R}_t(x,s_j)g(s_j)A_j + \sum_{p_j \in P} \bar{R}_t(x,p_j)f(p_j)V_j, \quad (P4.a)
\]
and the corresponding eigenproblem
\[
- L_{t,h} u(x) = \lambda \sum_{p_j \in P} \bar{R}_t(x,p_j)u(p_j)V_j. \quad (P4.b)
\]
Using the standard theory of integral equation, we know that the solution of \( \text{P3.a} \) exists in \( L^2(M) \). Our focus is on how well this solution approximates that of Poisson equation.

We follow [26] and use the following proposition to show that there is a one to one correspondence between the solution to the problem \( \text{P2.a} \) and the solution to Equation \( \text{P4.a} \), and a one to one correspondence between each eigen-pair of the problem \( \text{P2.b} \) and that of the problem \( \text{P4.b} \). For any function \( u \) on \( M \), let \( \rho(u) \) denote the restriction of \( u \) to the sample points, i.e., \( \rho(u) = (u(p_1), \ldots, u(p_n))^t \).
Theorem 4.1. 1. If a function \( u \) on \( \mathcal{M} \) is the solution to the problem \([P4.a]\) if and only if the vector \( \rho(u) \) is a solution to the problem \([P2.a]\) and \( u = I_{\mathcal{L}}(\rho(u)) \).

2. If a function \( u \) is an eigenfunction of the eigenproblem \([P4.b]\) with the eigenvalue \( \lambda \), then the vector \( \rho(u) \) is an eigenvector of the eigenproblem \([P2.b]\) with the same eigenvalue \( \lambda \).

3. If the function \( u \) is an eigenfunction of the eigenproblem \([P4.b]\) with the eigenvalue \( \lambda \), then \( u \) has to be of the form \( u = I_{\lambda}(\rho(u)) \).

Now we relate the solution of the problem \([P4.a]\) to the solution of the problem \([P1.a]\). We do so in two steps. In the first step, we show that the operator \( L_{t,h} \) approximates the Laplace-Beltrami operator \( \Delta_{\mathcal{M}} \) via the operator \( L_t \). In the second step, we show that as is \( \Delta_{\mathcal{M}} \), both \( L_{t,h} \) and \( L_t \) are elliptic.

The following theorem says that the operator \( L_t \) well approximates the Laplace-Beltrami operator \( \Delta_{\mathcal{M}} \).

**Theorem 4.1.** Assume \( \mathcal{M} \) and \( \partial \mathcal{M} \) are \( C^\infty \). Let \( u(x) \) be the solution of the problem \([P1.a]\) and \( u_t(x) \) be the solution of corresponding problem \([P3.a]\). If \( f \in C^\infty(\mathcal{M}), g \in C^\infty(\partial \mathcal{M}) \) in both problems, then there exists a constant \( C \) depending only on \( \mathcal{M} \) and \( \partial \mathcal{M} \), so that for any sufficient small \( t \)

\[
\|L_t u - L_t u_t\|_{L^2(\mathcal{M})} \leq C t^{1/4} \|u\|_{C^2(\mathcal{M})} + C t^{1/2} \|u\|_{C^3(\mathcal{M})},
\]

\[
\|\nabla (L_t u - L_t u_t)\|_{L^2(\mathcal{M})} \leq C t^{-1/4} \|u\|_{C^2(\mathcal{M})} + C \|u\|_{C^3(\mathcal{M})} + C t^{1/2} \|u\|_{C^4(\mathcal{M})}.
\]

The following theorem says that the operator \( L_{t,h} \) well approximates \( L_t \).

**Theorem 4.2.** Assume \( \mathcal{M} \) and \( \partial \mathcal{M} \) are \( C^\infty \) and the input data \((P, S, V, A)\) is an \( h \)-integral approximation of \((\mathcal{M}, \partial \mathcal{M})\). Let \( u_t(x) \) be the solution of the problem \([P3.a]\) and \( u_{t,h} \) be the solution of the problem \([P4.a]\). If \( f \in C(\mathcal{M}), g \in C(\partial \mathcal{M}) \) in both problems, then there exists a constant \( C \) depending only on \( \mathcal{M} \) and \( \partial \mathcal{M} \) so that for any sufficient small \( t \) and \( \frac{h}{\sqrt{t}} \)

\[
\|L_{t,h} u - L_t u_t\|_{L^2(\mathcal{M})} \leq \frac{Ch}{t^{3/2}} \left( \|f\|_\infty + t^{-1/4} \|g\|_\infty \right),
\]

\[
\|\nabla (L_{t,h} u - L_t u_t)\|_{L^2(\mathcal{M})} \leq \frac{Ch}{t^2} \left( \|f\|_\infty + t^{-1/4} \|g\|_\infty \right).
\]

**Theorem 4.3.** Assume \( \mathcal{M} \) and \( \partial \mathcal{M} \) are \( C^\infty \). Let \( u(x) \) solves the integral equation

\[
L_t u = r(x)
\]

and \( r \in H^1(\mathcal{M}) \). Then, there exist a constant \( C > 0 \) independent on \( t \), such that

\[
\|u\|_{H^1(\mathcal{M})} \leq C \left( \|r\|_{L^2(\mathcal{M})} + t \|\nabla r\|_{L^2(\mathcal{M})} \right)
\]

as long as \( t \) are small enough.
Theorem 3.1 is a direct application of Theorem 4.1, 4.2, and 4.3.

Proof. of Theorem 3.1

Using Theorem Theorem 4.1 and 4.2, we have

\[ \| L_t (u - u_{t,k}) \|_{L^2(M)} \leq C t^{1/4} \| u \|_{C^3(M)} + \frac{Ch}{t^{3/2}} \left( \| f \|_\infty + t^{-1/4} \| g \|_\infty \right), \quad (4.9) \]

\[ \| \nabla L_t (u - u_{t,k}) \|_{L^2(M)} \leq C t^{-1/4} \| u \|_{C^4(M)} + \frac{Ch}{t} \left( \| f \|_\infty + t^{-1/4} \| g \|_\infty \right). \quad (4.10) \]

Then Theorem 4.3 gives that

\[ \| u - u_{t,k} \|_{H^1(M)} \leq C t^{1/4} \| u \|_{C^4(M)} + \frac{Ch}{t^{3/2}} \left( \| f \|_\infty + t^{-1/4} \| g \|_\infty \right) \] (4.11)

Then, using the well-known results of the regularity of the solution of Poisson equation,

\[ \| u \|_{H^{s+2}(M)} \leq C (\| f \|_{H^s(M)} + \| g \|_{H^s(\partial M)}) \quad (4.12) \]

and the Sobolev imbedding theorem saying the inclusion map

\[ H^s(M) \to C^4(M), \quad \forall s \geq k/2 + 3 \] (4.13)

is an embedding. This proves the theorem. \( \square \)

Next we prove the convergence of our finite integral method for solving the eigenproblem [P1,b]. We rely on the following result from perturbation theory [8] showing the convergence of the eigen-system.

Definition 4.1. (Convergences of operators)

Let \( (E, \| \cdot \|_E) \) be an arbitrary Banach space. \( B \) is the unit ball of \( E \). \( \{T_n\} \) is a sequence of bounded linear operator on \( E \).

1. \( \{T_n\} \) converges pointwisely, denoted by \( T_n \xrightarrow{p} T \), if \( \| T_n x - T x \|_E \to 0 \), for all \( x \in E \).

2. \( \{T_n\} \) converges compactly, denoted by \( T_n \xrightarrow{c} T \), if it converges pointwisely and for every sequence \( \{x_n\} \) in \( B \), the sequence \( (T - T_n)x_n \) is relatively compact in \( E \).

Theorem 4.4. (Perturbation results for compact convergence) Let \( (E, \| \cdot \|_E) \) be an arbitrary Banach space and \( T_n, T \) be bounded linear operator on \( E \) with \( T_n \xrightarrow{c} T \). Let \( \lambda \in \sigma(T) \) be an isolated eigenvalue with finite multiplicity \( m \) and \( A \subset C \) be an open neighborhood of \( \lambda \) such that \( \sigma(T) \cap A = \{\lambda\} \). Then

1. (convergence of eigenvalues) \( \exists \ N \in \mathbb{N} \) such that \( \forall n > N \), \( \sigma(T_n) \cap A \) is an isolated part of \( \sigma(T_n) \) consists of at most \( m \) different eigenvalues and their multiplicities sum up to \( m \). Moreover the sequence of sets \( \sigma(T_n) \cap A \) converges to the set \( \{\lambda\} \) in the sense that every sequence \( \{\lambda_n\} \) with \( \lambda_n \in \sigma(T_n) \cap A \) satisfies \( \lim_{n \to \infty} \lambda_n = \lambda \).

2. (convergence of eigenfunctions) If \( \lambda \) is a simple eigenvalue, \( \exists \ N \in \mathbb{N} \) such that \( \forall n > N \), \( \sigma(T_n) \cap A \) consists of a simple eigenvalue \( \lambda_n \). The corresponding eigenvector \( v_n \) converges up to a change of sign.
We consider the solution operators to the problems [P1.a], [P3.a], [P4.a] and prove their convergence results. Denote the operator $T : L^2(\mathcal{M}) \to H^2(\mathcal{M})$ to be the solution operator of the problem [P1.a] with $g = 0$, i.e., for any $f \in L^2(\mathcal{M})$, $T(f)$ with $\int_M T(f) = 0$ is the solution of the following problem:

$$
\begin{aligned}
-\Delta_{\mathcal{M}} u(x) &= f(x), \; x \in \mathcal{M}, \\
\frac{\partial u}{\partial n}(x) &= 0, \; x \in \partial \mathcal{M}.
\end{aligned}
$$

(4.14)

where $n$ is the outward normal vector of $\mathcal{M}$.

Denote $T_t : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ to be the solution operator of the problem [P3.a] with $g = 0$, i.e. $u = T_t(f)$ with $\int_M u = 0$ solves the following integral equation

$$
-\frac{1}{t} \int_M R_t(x,y) (u(x) - u(y)) dy = \int_M \bar{R}_t(x,y) f(y) dy.
$$

(4.15)

The last solution operator is $T_{t,h} : C(\mathcal{M}) \to C(\mathcal{M})$ which solves the problem [P4.a] with $g = 0$. By Proposition 4.1, we can write $T_{t,h}$ explicitly as follows.

$$
T_{t,h}(f)(x) = \frac{1}{w_{t,h}(x)} \sum_j R_t(x,x_j) u_j V_j - \frac{t}{w_{t,h}(x)} \sum_j \bar{R}_t(x,x_j) f(x_j) V_j
$$

(4.16)

where $w_{t,h}(x) = \sum_j R_t(x,x_j) V_j$ and $u = (u_1, \ldots, u_n)^t$ with $\sum_i u_i V_i = 0$ solves the problem [P2.a] with $g = 0$, i.e.,

$$
-\frac{1}{t} \sum_j R_t(x_i,x_j)(u_i - u_j)V_j = \sum_j \bar{R}_t(x_i,x_j) f(x_j)V_j
$$

(4.17)

We have the following bounds on the operators $T_t$ and $T_{t,h}$.

**Lemma 4.1.** Assume $\mathcal{M}$ and $\partial \mathcal{M}$ are $C^\infty$. Then

(i) For any fixed small enough $t$, there is a constant $C$ independent of $t$, so that for any $f \in C(\mathcal{M})$, $\|T_t f\|_{C^1(\mathcal{M})} \leq C \frac{t^{1/2}}{t^{1/2}} \|f\|_\infty$.

(ii) If $(P,S,V,A)$ is an $h$-integral approximation of $(\mathcal{M},\partial \mathcal{M})$, then for any fixed small enough $t$, there is a constant $C$ independent of $t$, so that for any $f \in C(\mathcal{M})$ and any sufficiently small $h$, $\|T_{t,h} f\|_{C^1(\mathcal{M})} \leq C \frac{t^{1/2}}{t^{1/2}} \|f\|_\infty$.

Finally, we need the following theorem which upper bounds the $H^1$ norm of the solution to the problem [P3.a] using the $L^2$ norm of $f$.

**Theorem 4.5.** Assume $\mathcal{M}$ and $\partial \mathcal{M}$ are $C^\infty$. For any $f \in L^2(\mathcal{M})$, there exist a constant $C > 0$ independent on $t$ and $f$, such that

$$
\|T_t(f)\|_{H^1(\mathcal{M})} \leq C \|f\|_{L^2(\mathcal{M})}
$$

(4.18)
Proof. Using Theorem 4.3, we have
\[ \|T_t(f)\|_{H^1(M)} \leq C \left( \|r\|_{L^2(M)} + t\|\nabla r\|_{L^2(M)} \right), \]  
(4.19)
where
\[ r(x) = \int_M \tilde{R}_t(x,y)f(y)d\mu_y. \]  
(4.20)
It is easy to show that
\[ \|r\|_{L^2(M)} \leq C\|f\|_{L^2(M)}, \quad \|\nabla r\|_{L^2(M)} \leq Ct^{-1/2}\|f\|_{L^2(M)}. \]
Then the theorem is proved.

With the above set up, we now prove the other main result, i.e., Theorem 3.2.

Proof. of Theorem 3.2
The proof consists of two parts. We will show that
\[ T_t \rightharpoonup T \quad \text{in} \quad L^2(M), \quad \text{as} \quad t \to 0, \quad \text{and} \]
\[ T_{t,h} \rightharpoonup T_t \quad \text{in} \quad C(M), \quad \text{as} \quad h \to 0, \quad t \ \text{fixed}. \]
Then, by Theorem 4.4, for any \( n \in \mathbb{N} \), there exists a small \( t_n \) so that
\[ |\lambda_{t_n}^i - \lambda_i| < 1/n \]
where \( \lambda_{t_n}^i \) is the \( i \)th eigenvalue of \( T_{t_n} \). Now fix \( t_n \). There exists \( h_n \) depending on \( t_n \) so that \( |\lambda_{h_n,t_n}^i - \lambda_{t_n}^i| < 1/n \).
This proves the theorem.

It remains to prove the above two sequences converge compactly. From Theorem 4.1 and Theorem 4.3, we have
\[ \|T_t(f) - T(f)\|_{L^2(M)} \leq Ct^{1/4}\|f\|_{H^2(M)} \]  
(4.21)
Using the fact that \( H^2(M) \) is dense in \( L^2(M) \) and that \( \|T_t\|_{L^2(M)} \) is bounded independent of \( t \) from Theorem 4.5, we can have for any \( f \in L^2(M) \),
\[ \|T_t(f) - T(f)\|_{L^2(M)} \to 0, \quad \text{as} \quad t \to 0. \]  
(4.22)
which implies that as \( t \to 0 \), \( T_t \) converges to \( T \) pointwisely in \( L^2(M) \). Moreover, we have
\[ \|T_t(f)\|_{H^1(M)} \leq C\|f\|_{L^2(M)}, \]  
(4.23)
from Theorem 4.5 and it is well known that
\[ \|T(f)\|_{H^1(M)} \leq C\|f\|_{L^2(M)}. \]  
(4.24)
This means that
\[ \left( \bigcup_{t>0} T_t(B) \right) \bigcup T(B) \text{ is bounded in } H^1(M) \]  
(4.25)
where \( B \) is the unit ball in \( L^2(M) \). Using the well known compact embedding theorem of \( H^1(M) \) being compactly embedded into \( L^2(M) \), we have that \( \left( \bigcup_{t>0} T_t(B) \right) \bigcup T(B) \) is relatively compact in \( L^2(M) \). This proves that \( T_t \rightharpoonup T \quad \text{in} \quad L^2(M) \) as \( t \to 0 \).
Next, we turn to prove that for any \( t > 0 \) fixed, \( T_{t,h} \xrightarrow{h \to 0} T_t \) in \( C(M) \) as \( h \to 0 \). Using Theorem 4.2 and Theorem 4.3, we have
\[
\|T_{t,h}(f) - T_t(f)\|_{L^2(M)} \leq \frac{C h}{t^2} \|f\|_{\infty}.
\] (4.26)

Since both \( T_{t,h}(f) \) and \( T_t(f) \) are \( C^\infty \) for any \( f \in C(M) \),
\[
\|T_{t,h}(f) - T_t(f)\|_{C(M)} \to 0, \quad \text{as} \quad h \to 0.
\] (4.27)

which means that as \( h \) goes to 0, \( T_{t,h} \) converges to \( T_t \) pointwisely in \( C(M) \). Furthermore, from Lemma 4.1, we have both \( T_t \) and \( T_{t,h} \) are bounded in \( C^1(M) \) for a fixed \( t \). Then by the well known Azela-Ascoli theorem, we have that for any fixed \( t > 0 \), \( \{h T_{t,h}(B) \} \bigcup T_t(B) \) is relatively compact in \( C(M) \).

5 Discussion and Future Work

We have presented a novel finite integral method for numerically solving PDE’s on submanifolds isometrically embedded in Euclidean spaces. We show the convergence of FIM for solving Poisson equation and the eigensystem of Laplace-Beltrami operator. In solving Poisson equation, our result shows that the convergence order of FIM is \( h^{1/9} \) in \( L^2 \). However, our experimental results [16] show the empirical convergence is linear in \( h \). Indeed, there are places in our analysis where we believe the error bounds can be improved. In solving the eigensystem of Laplace-Beltrami operator, our result does not show any convergence rate. This is one of the future directions that we would like to work on.

We have shown that both the integral Laplace operator \( L_t \) and its discrete version \( L \) are elliptic as is the Laplace(-Beltrami) operator \( \Delta_M \). It is well-known that the elliptic property is essential to many properties of \( \Delta_M \). As we know, \( L_t \) is in fact different from \( \Delta_M \), especially near the boundary. It is interesting to see how much \( L_t \) resembles \( \Delta_M \). For example, what is the regularity of the solution to the integral equation (P3.a).

Finally, we make a few of comments on the assumption of \( h \)-integral approximation. Note that it is much easier to meet the requirement of the input data \( (P, S, V, A) \) being an \( h \)-integral approximation of \( (M, \partial M) \) than to mesh consistently the underlying submanifold with good shaped elements. If the data points are sampled according to some known probability measure as assumed in [2, 3, 15], the volume weights \( V \) and \( A \) can be estimated from this probability measure. For instance, if the data points are uniformly sampled in iid fashion, then both \( V \) and \( A \) are constant vectors and the integrals of functions on the submanifold can be estimated up to a constant (the volume of the submanifold). In a more geometric setting, Luo et al. [18] showed that the volume weights \( V \) and \( A \) can be estimated by locally approximating the tangent space at each data points. The resulting quadruple \( (P, S, V, A) \) is an \( h \)-integral approximation of \( (M, \partial M) \) if \( P \) and \( S \) are \( h \)-dense and \( h \)-sparse sampling of the manifolds \( M \) and \( \partial M \) respectively.

Finite integral method is simple and easy to implement, and imposes less requirement on the input data. We believe that FIM has great potentials in many applications. Our preliminary results show that finite integral method can be extended for solving Poisson equation with Dirichlet
boundary [16], and even more complicated PDE’s. One may also mix FIM with FEM to take advantages of both methods. Finally, it is interesting to analyze the behavior of the integral Laplace operator on other spaces in addition to smooth manifolds.

6 Basic Estimates

In this section, we show some basic estimates which will be used frequently in the proof later on. Since we are interested in the convergence behavior of the operators as $t \to 0$, the involved estimates need to hold only for sufficiently small $t$. Thus we only need to consider the submanifolds $\mathcal{M}$ and $\partial \mathcal{M}$ at local scales at which they look like Euclidean space. Let $r$ be the minimum of the reaches of $\mathcal{M}$ and $\partial \mathcal{M}$. We have $r > 0$ if both $\mathcal{M}$ and $\partial \mathcal{M}$ are $C^2$ smooth.

Proposition 6.1. Assume both $\mathcal{M}$ and $\partial \mathcal{M}$ are $C^2$ smooth and $r$ is the minimum of the reaches of $\mathcal{M}$ and $\partial \mathcal{M}$. For any point $x \in \mathcal{M}$, there is a neighborhood $U \subset \mathcal{M}$ of $x$, so that there is a convex set $\Omega \subset \mathbb{R}^k$ with a parametrization $\Phi : \Omega \to U$ satisfying the following conditions. For any $\rho \leq 0$,

(i) $\Omega$ contains at least half of the ball $B_{\Phi^{-1}(x)}(\frac{\rho}{5}r)$, i.e., $\text{vol}(\Omega \cap B_{\Phi^{-1}(x)}(\frac{\rho}{5}r)) > \frac{1}{2}(\frac{\rho}{5}r)^kw_k$ where $w_k$ is the volume of unit ball in $\mathbb{R}^k$;

(ii) $B_x(\frac{\rho}{10}r) \cap \mathcal{M} \subset U$.

(iii) The Jacobian determinant of $\Phi$ is bounded: $|D\Phi| \leq (1 + 2|\Phi^{-1}(x) - \Phi^{-1}(z)|)(1 + 2\rho)^k$ over $\Omega$.

(iv) For any points $y, z \in U$, $1 - 2\rho \leq |\Phi^{-1}(y) - \Phi^{-1}(z)| \leq 1 + 3\rho$.

To prove the above proposition, we first cite a few results from Riemannian geometry on isometric embedding. For a submanifold $\mathcal{M}$ embedded in $\mathbb{R}^d$, let $d_\mathcal{M} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be the geodesic distance on $\mathcal{M}$, and $T_p\mathcal{M}$ and $N_p\mathcal{M}$ be the tangent space and the normal space of $\mathcal{M}$ at point $p \in \mathcal{M}$ respectively.

Lemma 6.1. (eg [12]) Assume $\mathcal{M}$ is a submanifold isometrically embedded in $\mathbb{R}^d$ with reach $r$. For any two $x, y$ on $\mathcal{M}$ with $|x - y| \leq r/2$,

$$|x - y| \leq d_\mathcal{M}(x, y) \leq |x - y|(1 + \frac{2|x - y|^2}{r^2}).$$

Lemma 6.2. (eg [5]) Assume $\mathcal{M}$ is a submanifold isometrically embedded in $\mathbb{R}^d$ with reach $r$. For any two $x, y$ on $\mathcal{M}$ with $|x - y| \leq r/2$,

$$\cos \angle T_x\mathcal{M}, T_y\mathcal{M} \leq 1 - \frac{2|x - y|^2}{r^2}.$$
Lemma 6.3. (eg [12]) Assume $M$ is a submanifold isometrically embedded in $\mathbb{R}^d$ with reach $r$. Let $N$ be any local normal vector field around a point $x \in M$. Then for any tangent vector $Y \in T_xM$

$$\frac{<Y, D_Y N>}{<Y,Y>} \leq \frac{1}{r}$$

where $D$ and $<\cdot,\cdot>$ are standard connection and standard inner product in $\mathbb{R}^d$.

In what follows, assume the hypotheses on $M$ and $\partial M$ in Proposition 6.1 hold. We prove the following two lemmas which bound the distortion of certain parametrization, which are used to build the parameterization in Proposition 6.1.

For a point $x \in M$, let $U_\rho = B_x(\rho r) \cap M$ with $\rho \leq 0.2$. We define the following projection map $\Psi : U_\rho \to T_xM = \mathbb{R}^k$ as the restriction to $U_\rho$ of the projection of $\mathbb{R}^d$ onto $T_xM$. It is easy to verify that $\Psi$ is one-to-one. Then $\Phi = \Psi^{-1} : \Psi(U_\rho) \to U_\rho$ is a parametrization of $U_\rho$. See Figure 1. We have the following lemma which bounds the distortion of this parametrization.

Lemma 6.4. For any point $y \in \Psi(U_\rho)$ and any $Y \in T_y(T_xM)$ for any $\rho \leq 0.2$,

$$|Y| \leq |D\Phi(y)| \leq \frac{1}{1-2\rho^2} |Y|.$$  

Proof. We have $\Phi(y) = y - l_T(y)N_T(y)$ where $N_T(y) \perp T_xM$ for any $y$ and $l_T(y) = |y - \Phi(y)|$. So $D\Phi(y) N_T(y) \perp T_xM$ for any $y$ and any $Y \in T_y(T_xM)$. Since $D\Phi(y) = Y - N_T(D\Phi(y)l_T) - l_T(D\Phi(y)N_T)$, the projection of $D\Phi(y)l_T$ to $T_xM$ is $Y$. At the same time, $D\Phi(y)$ is on $T_{\Phi(y)}M$. Since $|x - \Phi(y)| \leq \rho r$, from Lemma 6.2, $\cos \angle T_xM,T_yM = 1 - 2\rho^2$. This proves the lemma.

To ensure the convexity of the parameter domain $\Omega$ in Proposition 6.1, we need a different parameterization for the points near the boundary. For a point $x \in \partial M$, let $U_\rho = B_x(\rho r) \cap M$ with $\rho \leq 0.1$. We construct a map $\tilde{\Psi} : U_\rho \to T_x\partial M \times \mathbb{R} = \mathbb{R}^k$ as follows. For any point $z \in U_\rho$, let $\bar{z}$ be the closest point on $\partial M$ to $z$. Such $\bar{z}$ is unique. Let $n$ be the outward normal of $\partial M$ at $\bar{z}$. The projection $P$ of $\mathbb{R}^d$ onto $T_xM$ maps $z$ to a point on the line $\ell$ passing through $\bar{z}$ with the direction $n$. In fact, $P$ projects $N_\bar{z}\partial M$ onto the line $\ell$. If we let $V_{\rho_1} = N_{\bar{z}}\partial M \cap B_{\bar{z}}(\rho_1 r) \cap M$ with $\rho_1 \leq 0.2$, $P$ maps $V_{\rho_1}$ to the line $\ell$ in the one-to-one manner. Let $y^k = -(P(z) - \bar{z}) \cdot n$. Think of $\partial M$ as a submanifold. It is isometrically embedded in $\mathbb{R}^d$ as is $M$. As $|z - x| \leq 2|x - z| \leq 2\rho r$, we apply Lemma 6.4 by replacing $M$ with $\partial M$ and obtain the map $\tilde{\Psi}$ that maps $\bar{z}$ onto $T_x\partial M$. Define $\tilde{\Phi}(z) = (\tilde{\Phi}(\bar{z}), y^k)$. Since both $P|_{\rho_1}$ and $\tilde{\Psi}$ are one-to-one, so is $\tilde{\Psi}$. Then $\tilde{\Phi} = \tilde{\Psi}^{-1} : \tilde{\Psi}(U_\rho) \to U_\rho$ is a parametrization of $U_\rho$. See Figure 2. We have the following lemma which bounds the distortion of this parametrization $\tilde{\Phi}$.
Lemma 6.5. For any point \((y, y^k) \in \Psi(U_\rho)\) with \(\rho \leq 0.1\) and any tangent vector \(Y\) at \((y, y^k)\),

\[
(1 - 2\rho)|Y| \leq |D_Y \Phi(y, y^k)| \leq (1 + 2\rho)|Y|.
\]

Proof. Let \(\tilde{y} = \Phi(y) - y^k n\). We have \(\tilde{\Phi}(y, y^k) = \Phi(y) - y^k n(\Phi(y)) - l_T(\tilde{y})N_T(\tilde{y})\) where \(N_T(\tilde{y}) \perp T_{\tilde{\Phi}(\tilde{y})}M\). See Figure 2. We have

\[
D_Y \tilde{\Phi}(y, y^k) = D_Y \Phi - y^k D_Y n - nD_Y y^k - N_T(D_Y l_T - l_T D_Y N_T).
\]

Using the similar strategy of proving Lemma 6.4, we consider the projection of \(D_Y \tilde{\Phi}(y, y^k)\) to the space \(T_{\tilde{\Phi}(\tilde{y})}M\) to which it is almost parallel. Denote \(P\) this projection map. We bound \(P(D_Y \tilde{\Phi}(y, y^k))\). Let \(Y = (Y^1, \cdots, Y^k)\), \(Y_1 = (Y^1, \cdots, Y^{k-1}, 0)\) and \(Y_2 = (0, \cdots, 0, Y^k)\). We have \(D_Y \tilde{\Phi}(y, y^k) = D_{Y_1} \Phi(y, y^k) + D_{Y_2} \Phi(y, y^k)\). First consider each term involved in \(D_{Y_1} \Phi(y, y^k)\).

(i) \(D_{Y_1} \Phi(y)\) is a vector in \(T_{\Phi(y)} \partial M\), thus \(P(D_{Y_1} \Phi(y)) = D_{Y_1} \Phi(y)\). In addition, from Lemma 6.4, 

\[
|Y_1| \leq |D_{Y_1} \Phi(y)| \leq \frac{1}{1-2\rho^2}|Y_1|.
\]

(ii) \(D_{Y_1} n(y, y^k) = D_{Y_1} n(\Phi(y))\). First note that \(n \cdot D_{Y_1} n = 0\). Second, from Lemma 6.3, we have that the projection of \(D_{Y_1} n\) to the space \(T_{\Phi(y)} \partial M\) is upper bounded by \(\frac{1}{r}|D_{Y_1} \Phi|\). Since \(|y^k| < \rho r\), \(|P(y^k D_{Y_1} n)| \leq \frac{\rho^2}{2r^2}|Y_1|\).

(iii) Consider \(D_{Y_1} N_T(y, y^k)\). We have \(N_T \perp T_{\Phi(y)} M\). Let \(e_1, \cdots, e_k\) be the orthonormal basis of \(T_{\Phi(y)} M\) so that \(e_i N_T \cdot e_j = 0\) for \(i \neq j\). Locally extend \(e_1, \cdots, e_k\) to be an orthonormal basis of \(TM\) in a neighborhood of \(\Phi(y)\). We have for any \(e_i\)

\[
|D_{Y_1} N_T(y, y^k) \cdot e_i| = |D_{Y_1} n N_T(\tilde{y}) \cdot e_i| = |D_{(D_{Y_1} n) e_i} n T(\tilde{y}) \cdot e_i| = |D_{(D_{Y_1} n) e_i} N_T(\tilde{y}) \cdot e_i| \leq \frac{1}{r}|D_{Y_1} \Phi \cdot e_i|.
\]
where the last inequality is due to Lemma 6.3. Moreover, one can verify that $l_T(y) \leq \frac{\rho r}{2}$, which leads to

$$|P(l_T D_{Y_1} N_I)| \leq \frac{\rho^2}{2} |D_{Y_1} \Phi| \leq \frac{\rho^2}{2(1 - 2\rho^2)} |Y_1|$$

(iv) It is obvious that $n D_{Y_1} y^k = P(N_T D_{Y_1} l_T) = 0$.

Next consider each term involved in $D_{Y_2} \Phi(y, y^k)$.

(i) $n D_{Y_2} y^k = Y^k n$, which lies on $T_{\Phi(y)} M$. Moreover $n \perp D_{Y_1} \Phi(y)$.

(ii) As $N_T(y, y^k)$ remains perpendicular to $T_{\Phi(y)} M$ if we only vary $y^k$, we have

$$P(D_{Y_2} N_I(y, y^k)) = 0.$$

(iii) For the remaining terms, we have $D_{Y_2} \Phi(y) = y^k D_{Y_2} n = P(N_T D_{Y_2} l_T) = 0$.

On the other hand, we hand $D_{Y} \Phi(y, y^k)$ lie in the tangent space $T_{\Phi(y)} M$, and

$$\cos \angle T_{\Phi(y, y^k)} M, T_{\Phi(y)} M \leq 1 - 2\rho^2.$$ 

Putting everything together, we have

$$|Y| - \frac{\rho^2 + 2\rho}{2(1 - 2\rho^2)} |Y_1| \leq D_{Y} \Phi(y, y^k) \leq \frac{1}{(1 - 2\rho^2)^2} |Y| + \frac{\rho^2 + 2\rho}{2(1 - 2\rho^2)^2} |Y_1|.$$ 

This proves the lemma. \qed

Now we are ready to prove Proposition 6.1

Proof. of Proposition 6.1

First consider the case where $d(x, \partial M) > \frac{\rho}{2} r$. Set $U' = B_x(\frac{\rho}{2} r) \cap M$, and parametrize $U'$ using map $\Phi : \Psi(U') \to U'$. Since for any $y \in \partial U'$, $|x - y| = \frac{\rho}{2} r$, from Lemma 6.4, we have that $B_{\Phi^{-1}(x)}(\frac{\rho}{2(1 + \rho)} r)$ is contained in $\Psi(U')$. Set $\Omega = B_{\Phi^{-1}(x)}(\frac{\rho}{2(1 + \rho)} r)$ and $U = \Phi(\Omega)$. This shows the parametrization $\Phi : \Omega \to U$ satisfies the condition (i). By Lemma 6.4 and Lemma 6.1, it is easy to verify that $\Phi$ satisfies the other three conditions.

Next consider the case where $d(x, \partial M) \leq \frac{\rho}{2} r$. Let $\bar{x}$ be the closest point on $\partial M$ to $x$. Set $U' = B_x(\rho r) \cap M$ and parametrize $U'$ using map $\Phi : \Psi(U') \to U'$. By Lemma 6.5, $\Psi(U')$ contains half of the ball $B_{\Phi^{-1}(x)}(\frac{\rho}{1 + 2\rho} r)$. Let $\Omega$ be that half ball and $U = \Phi(\Omega)$. It is easy to verify that the parametrization $\Phi : \Omega \to U$ satisfies the condition (ii) and (iv). To see (ii), note that $|x - \bar{x}| \leq \frac{\rho}{2} r$. From Lemma 6.5 and Lemma 6.1

$$|\Psi(x) - \Psi(\bar{x})| \leq (1 + 2\rho)(1 + 2\rho^2)|x - \bar{x}|.$$ 

We have that $\Omega$ contains at least half of the ball centered at $\Phi^{-1}(x)$ with radius $(\frac{\rho}{1 + 2\rho} - \frac{\rho(1 + 2\rho)(1 + 2\rho^2)}{2}) r \geq \frac{\rho}{2} r$. This shows that $\Phi$ satisfies the condition (i). Similarly, the condition (ii) follows from (i) as $\Phi$ has bounded distortion (Lemma 6.5) and geodesic distance is bounded by Euclidean distance (Lemma 6.1). \qed

The above proposition have the following corollaries which are important in our proof. Their proof is straight forward once we change the variables to those in the parameter domain.
Corollary 6.1. Assume both $\mathcal{M}$ and $\partial \mathcal{M}$ are $C^2$ smooth. For sufficiently small $t$ and any $x \in \mathcal{M}$, there are constants $w_{\text{min}}, w_{\text{max}}$ depending only on the geometry of $\mathcal{M}, \partial \mathcal{M}$, so that

$$w_{\text{min}} \leq \int_{\mathcal{M}} R_t(x,y) d\mu_y \leq w_{\text{max}}.$$ 

Corollary 6.2. Assume both $\mathcal{M}$ and $\partial \mathcal{M}$ are $C^2$ smooth. For sufficiently small $t$ and any $x \in \mathbb{R}^d$, there is a constant $C$ depending only on the geometry of $\mathcal{M}, \partial \mathcal{M}$, so that for $i = 0, 1, 2$ and any integer $s \geq 0$,

$$\int_{\mathcal{M}} R_t^{(i)}(x,y) |x - y|^s d\mu_y \leq C t^{s/2},$$

where $R_t^{(i)}$ denote the $i$th derivative of $R$.

The above two corollaries lead to the bounds on the convolution using the integral kernel $R_t, R_t'$ and $\bar{R}$.

Corollary 6.3. Assume both $\mathcal{M}$ and $\partial \mathcal{M}$ are $C^2$ smooth. For any function $f \in L^2(\mathcal{M})$, let $u(x) = \int_{\mathcal{M}} K_t(x,y) f(y) d\mu_y$ where $K_t$ can be $R_t, R_t'$ and $\bar{R}$. For sufficiently small $t$ and any $x \in \mathcal{M}$, there is a constant $C$ depending only on the geometry of $\mathcal{M}, \partial \mathcal{M}$, so that

$$\|u\|_{L^2(\mathcal{M})} \leq C \|f\|_{L^2(\mathcal{M})}, \quad \text{and} \quad \|\nabla u\|_{L^2(\mathcal{M})} \leq \frac{C}{t^{1/2}} \|f\|_{L^2(\mathcal{M})},$$

(6.1)

and moreover, if $f \in C(\mathcal{M})$,

$$\|u\|_{C^1(\mathcal{M})} \leq \frac{C}{t^{1/2}} \|f\|_{\infty}.$$ 

(6.2)

Proof. We prove the corollary for $K_t = R_t$. The other cases can be done in the exact same way. Using Hölder inequality, we have

$$\|u\|_{L^2(\mathcal{M})}^2 = \int_{\mathcal{M}} \left( \int_{\mathcal{M}} C_t R_t \left( \frac{|x - y|^2}{4t} \right) f(y) d\mu_y \right)^2 d\mu_x$$

$$\leq \int_{\mathcal{M}} \left( \int_{\mathcal{M}} C_t R_t \left( \frac{|x - y|^2}{4t} \right) f^2(y) d\mu_y \right)^{1/2} \left( \int_{\mathcal{M}} C_t R_t \left( \frac{|x - y|^2}{4t} \right) d\mu_y \right)^{1/2}$$

$$\leq C \|f\|_{L^2(\mathcal{M})}^2$$

(6.3)

where the last inequality is due to Corollary 6.2.

The gradient of $u(x)$ on $\mathcal{M}$ is the projection of the gradient of $u(x)$ in $\mathbb{R}^d$, which means

$$|\nabla u(x)|^2 \leq \sum_{i=1}^d \left( \frac{\partial u}{\partial x^i} \right)^2 = \sum_{i=1}^d \left( \int_{\mathcal{M}} C_t R_t' \left( \frac{|x - y|^2}{4t} \right) \frac{x^i - y^i}{2t} f(y) d\mu_y \right)^2$$

$$\leq \left( \int_{\mathcal{M}} C_t R_t' \left( \frac{|x - y|^2}{4t} \right) f^2(y) d\mu_y \right)^{1/2} \left( \int_{\mathcal{M}} C_t R_t' \left( \frac{|x - y|^2}{4t} \right) f^2(y) d\mu_y \right)^{1/2}$$

$$\leq \frac{C}{t} \int_{\mathcal{M}} C_t R_t' \left( \frac{|x - y|^2}{4t} \right) f^2(y) d\mu_y.$$ 

(6.4)
Then we have \( \|\nabla u\|_{L^2(M)} \leq \frac{C}{t^{1/2}} \|f\|_{L^2(M)} \) and if \( f \) is continuous then \( \|u\|_\infty \leq \frac{C}{t^{1/2}} \|f\|_\infty \). It is obvious that \( \|u\|_\infty \leq C \|f\|_\infty \). This proves the corollary. 

Together with the assumption of input data, we also have the corollary saying the convolution with the integral kernel can be approximated from the data. Its proof is also straightforward once we change the variables to those in the parameter domain.

**Corollary 6.4.** Assume both \( M \) and \( \partial M \) are \( C^2 \) smooth and the input data \((P, S, V, A)\) is an \( h\)-integral approximation of \((M, \partial M)\). For sufficiently small \( t \) and \( \frac{h}{t^{1/2}} \), we have

(i) for any function \( f \in C^1(M) \), there is a constant \( C \) independent of \( t, h \) and \( f \) so that

\[
\left| \int_M K_t(x, y) f(y) d\sigma_y - \sum_{i=1}^n K_t(x, p_i) f(p_i) V_i \right| < \frac{Ch}{t^{1/2}} \|f\|_{C^1},
\]

(ii) for any function \( f \in \partial M \), there is a constant \( C \) independent of \( t, h \) and \( f \) so that

\[
\left| \int_{\partial M} K_t(x, y) f(y) d\tau_y - \sum_{i=1}^n K_t(x, s_i) f(s_i) A_i \right| < \frac{Ch}{t} \|f\|_{C^1},
\]

where \( K_t \) can be any of \( R_t, R'_t \) and \( \bar{R}_t \).

### 7 Proof of Theorem 4.3

We need some preparations to prove the Theorem 4.3. In the subsequent subsection, we will prove several results first and then give the proof of Theorem 4.3.

#### 7.1 Ellipticity of \( L_t \)

In this subsection, we prove that the operator \( L_t \) keeps one important property of the original Laplace-Beltrami operator, ellipticity. More precisely, we will prove following two theorems,

**Theorem 7.1.** For any function \( u \in L^2(M) \), there exist a constant \( C > 0 \) independent on \( t \) and \( u \), such that

\[
\langle u, L_t u \rangle_M \geq C \int_M |\nabla v|^2 d\mu_x
\]

where \( \langle f, g \rangle_M = \int_M f(x) g(x) d\mu_x \) for any \( f, g \in L^2(M) \) and

\[
v(x) = \frac{C_t}{w_t(x)} \int_M R \left( \frac{|x-y|^2}{4t} \right) u(y) d\mu_y,
\]

and \( w_t(x) = C_t \int_M R \left( \frac{|x-y|^2}{4t} \right) d\mu_y \).

**Theorem 7.2.** Assume \( M \) and \( \partial M \) are \( C^\infty \). There exist a constant \( C > 0 \) independent on \( t \) so that for any function \( u \in L^2(M) \) with \( \int_M u = 0 \) and for any sufficient small \( t \)

\[
\langle L_t u, u \rangle_M \geq C \|u\|^2_{L^2(M)}
\]
Theorem 7.1 can be proved by following two lemmas.

Lemma 7.1. For any function $u \in L^2(M)$, there exist a constant $C > 0$ independent on $t$ and $u$, such that

$$\frac{Ct}{t} \int_M \int_M R \left( \frac{|x-y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \geq C \int_M |\nabla v|^2 d\mu_x \quad (7.4)$$

where $v$ is defined as in (7.2).

Lemma 7.2. If $t$ is small enough, for any function $u \in L^2(M)$, there exist a constant $C > 0$ independent on $t$ and $u$, such that

$$\int_M \int_M R \left( \frac{|x-y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \leq C \int_M \int_M R \left( \frac{|x-y|^2}{4t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y. \quad (7.5)$$

To make the paper concise and easy to read, the details of the proof of above two lemmas are put in appendix.

Using Lemma 7.1 and Lemma 7.2, Theorem 7.1 is easy to proved by noticing following equality:

$$\langle u, L_t u \rangle = \int_M \int_M R \left( \frac{|x-y|^2}{4t} \right) u(x)(u(x) - u(y))d\mu_y d\mu_x$$

$$= -\int_M \int_M R \left( \frac{|x-y|^2}{4t} \right) u(y)(u(x) - u(y))d\mu_y d\mu_x$$

$$= \frac{1}{2} \int_M \int_M R \left( \frac{|x-y|^2}{4t} \right) (u(x) - u(y))^2 d\mu_y d\mu_x.$$

Then, Theorem 7.2 can be proved as following by utilizing Lemma 7.1

Proof. of Theorem 7.2

By Lemma 7.1 and Poincare inequality, there exist constant $C > 0$, such that

$$\int_M (v(x) - \bar{v})^2 d\mu_x \leq \frac{CCt}{t} \int_M \int_M R \left( \frac{|x-y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \quad (7.5)$$
where \( \bar{v} = \frac{1}{|M|} \int_M v(x) d\mu_x \). We have
\[
|\mathcal{M}| \bar{v} = \int_M v(x) d\mu_x
\]
\[
= \left| \int_M \int_M \frac{c_t}{w_t(x)} R \left( \frac{|x-y|^2}{4t} \right) (u(y) - u(x)) d\mu_y d\mu_x \right|
\]
\[
\leq \left( \int_M \int_M \frac{c_t}{w_t(x)} R \left( \frac{|x-y|^2}{4t} \right) d\mu_y d\mu_x \right)^{1/2}
\]
\[
\leq C \left( C_t \int_M \int_M R \left( \frac{|x-y|^2}{4t} \right) (u(y) - u(x))^2 d\mu_y d\mu_x \right)^{1/2}
\]
\[
\leq C \left( \frac{C_t}{\delta_0} \int_M \int_M R \left( \frac{|x-y|^2}{32t} \right) R \left( \frac{|x-y|^2}{32t} \right) (u(y) - u(x))^2 d\mu_y d\mu_x \right)^{1/2}
\]
\[
\leq C \left( \frac{C_t}{\delta_0} \int_M \int_M R \left( \frac{|x-y|^2}{32t} \right) (u(y) - u(x))^2 d\mu_y d\mu_x \right)^{1/2},
\]
where the second equality is due to \( \int_M u = 0 \). This enables one to upper bound the \( L_2 \) norm of \( v \) as follows. For \( t \) sufficiently small,
\[
\int_M (v(x))^2 d\mu_x \leq 2 \int_M (v(x) - \bar{v})^2 d\mu_x + 2 \int_M \bar{v}^2 d\mu_x
\]
\[
\leq C C_t \int_M \int_M R \left( \frac{|x-y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y
\]
\( \quad \text{ (7.6)} \)

Let \( \delta = \frac{u_{\min}}{2u_{\max} + u_{\min}} \). If \( u \) is smooth and close to its smoothed version \( v \), in particular,
\[
\int_M v^2(x) d\mu_x \geq \delta^2 \int_M u^2(x) d\mu_x,
\]
\( \quad \text{ (7.7)} \)
then we have shown \( < L_{\delta L} u, u >_M \geq C \|u\|_{L_2(M)}^2 \) for some constant \( C \), which proves the elliptic inequality of \( L_t \).

Now consider the case where \( \text{(7.7)} \) does not hold. In this case, \( u \) is not so smooth. Since \( L_t \) is sort of differential operator, \( L_t u \) is large, which in fact favors the elliptic inequality. Note that we now have
\[
\|u - v\|_{L^2(M)} \geq \|u\|_{L^2(M)} - \|v\|_{L^2(M)} > (1 - \delta)\|u\|_{L^2(M)}
\]
\[
> \frac{1 - \delta}{\delta} \|v\|_{L^2(M)} = \frac{2u_{\max}}{u_{\min}} \|v\|_{L^2(M)}.
\]
Then we have
\[
\frac{C_t}{t} \int_{\mathcal{M}} \int_{\mathcal{M}} R \left( \frac{|x - y|^2}{4t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y
\]
\[
= 2C_t \int_{\mathcal{M}} u(x) \int_{\mathcal{M}} R \left( \frac{|x - y|^2}{4t} \right) (u(x) - u(y)) d\mu_y d\mu_x
\]
\[
= 2 \int_{\mathcal{M}} u^2(x) w(x) d\mu_x - \int_{\mathcal{M}} u(x) v(x) w(x) d\mu_x
\]
\[
= 2 \int_{\mathcal{M}} (u(x) - v(x))^2 w(x) d\mu_x + \int_{\mathcal{M}} (u(x) - v(x)) v(x) w(x) d\mu_x
\]
\[
\geq 2 \int_{\mathcal{M}} (u(x) - v(x))^2 w(x) d\mu_x - \frac{2}{t} \left( \int_{\mathcal{M}} v^2(x) w(x) d\mu_x \right)^{1/2} \left( \int_{\mathcal{M}} (u(x) - v(x))^2 w(x) d\mu_x \right)^{1/2}
\]
\[
\geq \frac{w_{\text{min}}}{t} \int_{\mathcal{M}} (u(x) - v(x))^2 d\mu_x
\]
\[
\geq \frac{w_{\text{min}}}{t} (1 - \delta)^2 \int_{\mathcal{M}} u^2(x) d\mu_x.
\] (7.8)

Finally, notice that
\[
\int_{\mathcal{M}} \int_{\mathcal{M}} R \left( \frac{|x - y|^2}{4t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \leq \frac{1}{\delta_0} \int_{\mathcal{M}} \int_{\mathcal{M}} R \left( \frac{|x - y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y.
\]
which shows that the elliptic inequality holds for small \( t \). This completes the proof for the theorem. 

\[ \Box \]

### 7.2 Main proof of Theorem 4.3

Now, we are ready to prove Theorem 4.3.

**Proof.** Using Theorem 7.2, we have
\[
\|u\|^2_{L^2(\mathcal{M})} \leq C \langle u, L_t u \rangle = C \int_{\mathcal{M}} u(x) r(x) d\mu_x \leq C \|u\|_{L^2(\mathcal{M})} \|r\|_{L^2(\mathcal{M})}
\] (7.9)

which implies that
\[
\|u\|_{L^2(\mathcal{M})} \leq C \|r\|_{L^2(\mathcal{M})}.
\] (7.10)

Now we turn to estimate \( \|\nabla u\|_{L^2(\mathcal{M})} \).

Notice that we have following expression of \( u \),
\[
u(x) = v(x) + \frac{t}{w_t(x)} r(x)
\] (7.11)

where
\[
v(x) = \frac{1}{w_t(x)} \int_{\mathcal{M}} R_t(x, y) u(y) d\mu_y, \quad w_t(x) = \int_{\mathcal{M}} R_t(x, y) d\mu_y
\] (7.12)
then by Theorem\ref{thm:gradient_estimation}, we have

\[ \| \nabla u \|^2_{L^2(M)} \leq 2 \| \nabla v \|^2_{L^2(M)} + 2t^2 \left\| \nabla \left( \frac{r(x)}{w_t(x)} \right) \right\|^2_{L^2(M)} \]

\[ \leq C \| u \|_{L^2(M)}^2 + Ct \| r \|_{L^2(M)} + Ct^2 \| \nabla r \|^2_{L^2(M)} \]

\[ \leq C \| u \|^2_{L^2(M)} + Ct \| r \|^2_{L^2(M)} + Ct^2 \| \nabla r \|^2_{L^2(M)} \]

\[ \leq C \left( \| r \|^2_{L^2(M)} + t \| \nabla r \|_{L^2(M)} \right)^2. \] (7.13)

\[ \Box \]

8 Proof of Theorem 4.1

In this section, we prove that the integral equation in (P3.a) well approximates Poisson equation, by showing that the intermediate operator $L_t$ well approximates the Laplace-Beltrami operator $\Delta_M$ (Theorem 4.1).

As stated in Proposition 6.1, $M$ can be locally parametrized as follows.

\[ x = \Phi(\gamma) : \Omega \subset \mathbb{R}^k \to M \subset \mathbb{R}^d \] (8.1)

where $\gamma = (\gamma^1, \cdots, \gamma^k)^t \in \mathbb{R}^k$ and $x = (x^1, \cdots, x^d)^t \in M$. In what follows, we use the index with prime (e.g., $i'$), respectively without prime (e.g., $i$) to represent the coordinate component of $\gamma$ in the parameter domain, respectively of $x$ in the embedding space. We also use Einstein convention for the notational brevity.

Let $\partial_i' = \frac{\partial}{\partial \gamma^i'}$ be the tangent vector along the direction $\gamma^i'$. Since $M$ is a submanifold in $\mathbb{R}^d$ with induced metric, $\partial_i' = (\partial_i' \Phi^1, \cdots, \partial_i' \Phi^d)$ and the metric tensor

\[ g_{i'j'} = \langle \partial_i', \partial_j' \rangle = \partial_i' \Phi^l \partial_j' \Phi^l. \]

Let $g^{i'j'}$ denote the inverse of $g_{i'j'}$, i.e.,

\[ g^{i'j'} = g_{i'j'}^{-1} = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \]

For any function $f$ on $M$, $\nabla f = g^{i'j'} \partial_j' f \partial_i'$ denotes the gradient of $f$. For convenience, let $\nabla^j f$ denote the $x^j$ component of the gradient $\nabla f$, i.e.,

\[ \nabla^j f = \partial_j' \Phi^i g^{i'j'} \partial_i' f \quad \text{and} \quad \partial_j f = \partial_j' \Phi^i \nabla^j f. \] (8.2)
8.1 Proof of Theorem 4.1

Proof. Let \( r(x) = L_t u - L_t u_t \) be the residual, then we have

\[
\begin{align*}
    r(x) &= -\frac{1}{t} \int_{\mathcal{M}} R_t(x, y)(u(x) - u(y))d\mu_y + \frac{1}{t} \int_{\partial \mathcal{M}} \tilde{R}_t(x, y)g(y)d\tau_y - \int_{\mathcal{M}} \tilde{R}_t(x, y)f(y)d\mu_y \\
    &= -\frac{1}{t} \int_{\mathcal{M}} R_t(x, y)(u(x) - u(y))d\mu_y + \int_{\mathcal{M}} \tilde{R}_t(x, y)\Delta_M u(y)d\mu_y \\
    &\quad + \frac{1}{t} \int_{\mathcal{M}} (x - y) \cdot \nabla u(y)\tilde{R}_t(x, y)d\mu_y \\
    &= -\frac{1}{t} \int_{\mathcal{M}} R_t(x, y)(u(x) - u(y) - (x - y) \cdot \nabla u(y))d\mu_y + \int_{\mathcal{M}} \tilde{R}_t(x, y)\Delta_M u(y)d\mu_y
\end{align*}
\]

(8.3)

Here we use that fact that \( u \) is the solution of the Poisson equation with Neumann boundary condition \([\text{P1.a}]\), such that

\[
\int_{\partial \mathcal{M}} \tilde{R}_t(x, y)g(y)d\tau_y = \int_{\partial \mathcal{M}} \tilde{R}_t(x, y)\frac{\partial u}{\partial n}(y)d\tau_y
\]

\[
= \int_{\mathcal{M}} \tilde{R}_t(x, y)\Delta_M u(y)d\mu_y + \int_{\mathcal{M}} \nabla_{\mathcal{M}} \tilde{R}_t(x, y)\nabla u(y)d\mu_y
\]

\[
= \int_{\mathcal{M}} \tilde{R}_t(x, y)\Delta_M u(y)d\mu_y + \frac{1}{t} \int_{\mathcal{M}} (x - y) \cdot \nabla u(y)\tilde{R}_t(x, y)d\mu_y,
\]

and

\[
\int_{\mathcal{M}} \tilde{R}_t(x, y)f(y)d\mu_y = \int_{\mathcal{M}} \tilde{R}_t(x, y)\Delta_M u(y)d\mu_y.
\]

Fixing \( y \), the function \( R_t(x, y) \) has a support of \( B_y(2\sqrt{t}) \cap \mathcal{M} \). Thus when \( t \) is sufficiently small, we only need to consider the function \( u \) at a local support. As stated in Proposition 6.1, in that local support, there exists a nice local parametrization \( \Phi : \Omega \subset \mathbb{R}^k \to U \subset \mathcal{M} \), such that

1. \( (B_y(2\sqrt{t}) \cap \mathcal{M}) \subset U \) and \( \Omega \) is convex.

2. \( \Phi \in C^2(\Omega) \);

3. For any points \( x, y \in \Omega, \frac{1}{2} |x - y| \leq \|\Phi(x) - \Phi(y)\| \leq 2 |x - y| \).

Moreover, we denote \( \Phi(\beta) = x, \Phi(\alpha) = y, \xi = \beta - \alpha, \eta = \xi^i \partial_i \Phi(\alpha) \). and

\[
\begin{align*}
    r_1(x) &= -\frac{1}{t} \int_{\mathcal{M}} R_t(x, y) \left( u(x) - u(y) - (x - y) \cdot \nabla u(y) - \frac{1}{2} \eta^i \eta^j (\nabla^i \nabla^j u(y)) \right) d\mu_y \\
    r_2(x) &= \frac{1}{2t} \int_{\mathcal{M}} R_t(x, y) \eta^i \eta^j (\nabla^i \nabla^j u(y)) d\mu_y - \int_{\mathcal{M}} \eta^i (\nabla^i \nabla^j u(y)) \nabla^j \tilde{R}_t(x, y) d\mu_y \\
    r_3(x) &= \int_{\mathcal{M}} \eta^i (\nabla^i \nabla^j u(y)) \nabla^j \tilde{R}_t(x, y) d\mu_y + \int_{\mathcal{M}} \nabla (\eta^i (\nabla^i \nabla^j u(y))) \tilde{R}_t(x, y) d\mu_y \\
    r_4(x) &= \int_{\mathcal{M}} \nabla (\eta^i (\nabla^i \nabla^j u(y))) \tilde{R}_t(x, y) d\mu_y + \int_{\mathcal{M}} \tilde{R}_t(x, y)\Delta_M u(y)d\mu_y
\end{align*}
\]

(8.4)
then
\[ r(x) = r_1(x) - r_1(x) - r_3(x) + r_4(x). \] (8.5)

Next, we will prove the theorem by estimating above four terms one by one.

First, let us consider \( r_1. \) Using the fact that \( \Omega \) is convex and the Newton-Leibniz formula, we can get

\[
\begin{align*}
& u(x) - u(y) - (x - y) \cdot \nabla u(y) - \frac{1}{2} \eta^i \eta^j (\nabla^i \nabla^j u(y)) \\
& = \xi^i \xi^j \int_0^1 \int_0^1 \int_0^1 ds_1 ds_2 ds_3 \left( \partial_i \Phi^j (\alpha + s_2 s_1 \xi) \partial_j \Phi^i (\alpha + s_3 s_2 s_1 \xi) \nabla^j \nabla^i u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \right) ds_3 ds_2 ds_1 \\
& = \xi^i \xi^j \xi^m \xi^n \int_0^1 \int_0^1 \int_0^1 \int_0^1 ds_1 ds_2 ds_3 ds_4 \left( \partial_i \Phi^j (\alpha + s_2 s_1 \xi) \partial_j \Phi^i (\alpha + s_3 s_2 s_1 \xi) \nabla^j \nabla^i u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \right) ds_4 ds_3 ds_2 ds_1 \\
& + \xi^i \xi^j \xi^m \xi^n \int_0^1 \int_0^1 \int_0^1 \int_0^1 ds_1 ds_2 ds_3 ds_4 \left( \partial_i \Phi^j (\alpha + s_2 s_1 \xi) \partial_j \Phi^i (\alpha + s_3 s_2 s_1 \xi) \partial_m \Phi^j (\alpha + s_3 s_2 s_1 \xi) \partial_n \Phi^i (\alpha + s_3 s_2 s_1 \xi) \nabla^j \nabla^i u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \right) ds_4 ds_3 ds_2 ds_1 \\
& \nabla^j \nabla^i u(\Phi(\alpha + s_3 s_2 s_1 \xi)) ds_3 ds_2 ds_1
\end{align*}
\]

Using this equality and \( \Phi \in C^2(\Omega), \) it is easy to show that

\[
\begin{align*}
\left| u(x) - u(y) - (x - y) \cdot \nabla u(y) - \frac{1}{2} \eta^i \eta^j (\nabla^i \nabla^j u(y)) \right| & \leq C|\xi|^3 \|u\|_{C^3(\mathcal{M})} \\
\left| \nabla_x \left( u(x) - u(y) - (x - y) \cdot \nabla u(y) - \frac{1}{2} \eta^i \eta^j (\nabla^i \nabla^j u(y)) \right) \right| & \leq C|\xi|^2 \|u\|_{C^3(\mathcal{M})} + C|\xi|^3 \|u\|_{C^4(\mathcal{M})}
\end{align*}
\]

Then we have
\[
|r_1(x)| \leq \frac{C}{t} \|u\|_{C^3(\mathcal{M})} \int_{\mathcal{M}} R_t(x, y) |\xi|^3 d\mu_y \leq Ct^{1/2} \|u\|_{C^3(\mathcal{M})} \tag{8.6}
\]

and
\[
\begin{align*}
\left| \nabla r_1(x) \right| & \leq \frac{C}{t} \|u\|_{C^3(\mathcal{M})} \int_{\mathcal{M}} |\xi|^2 (R_t(x, y) + \nabla R_t(x, y) |\xi|) d\mu_y + \frac{C}{t} \|u\|_{C^4(\mathcal{M})} \int_{\mathcal{M}} R_t(x, y) |\xi|^3 d\mu_y \\
& \leq C \|u\|_{C^3(\mathcal{M})} + Ct^{1/2} \|u\|_{C^4(\mathcal{M})} \tag{8.7}
\end{align*}
\]

Now, we try to bound \( r_2. \) Notice that
\[
\begin{align*}
\nabla^j \tilde{R}_t(x, y) &= \frac{1}{2t} \partial_{m'} \Phi^j (\alpha) g^{m'n'} \partial_n \Phi^i (\alpha) (x^i - y^i) R_t(x, y), \\
\frac{\eta^j}{2t} R_t(x, y) &= \frac{1}{2t} \partial_{m'} \Phi^j (\alpha) g^{m'n'} \partial_n \Phi^i (\alpha) \xi^i \partial_i \Phi^j R_t(x, y)
\end{align*}
\]

Then, we have
\[
\begin{align*}
\nabla^j \tilde{R}_t(x, y) - \frac{\eta^j}{2t} R_t(x, y) \\
& = \frac{1}{2t} \partial_{m'} \Phi^j (\alpha) g^{m'n'} \partial_n \Phi^i (\alpha) (x^i - y^i) - \frac{\eta^j}{2t} R_t(x, y) \\
& = \frac{1}{2t} \partial_{m'} \Phi^j (\alpha) g^{m'n'} \partial_n \Phi^i (\alpha) \left( x^i - y^i - \xi^i \partial_i \Phi^j \right) R_t(x, y) \\
& = \frac{1}{2t} \xi^i \xi^j \partial_{m'} \Phi^j (\alpha) g^{m'n'} \partial_n \Phi^i \left( \int_0^1 \int_0^1 s \partial_j \partial_i \Phi^j (\alpha + \tau s \xi) d\tau ds \right) R_t(x, y)
\end{align*}
\]
Thus, we get
\[ \left| \nabla^j \bar{R}_t(x, y) - \frac{\eta^j}{2t} R_t(x, y) \right| \leq \frac{C|\xi|^2}{t} R_t(x, y) \]
\[ \left| \nabla_x \left( \nabla^j \bar{R}_t(x, y) - \frac{\eta^j}{2t} R_t(x, y) \right) \right| \leq \frac{C|\xi|}{t} R_t(x, y) + \frac{C|\xi|^3}{t^2} |R'_t(x, y)| \]

Then, we have following bound for \( r_2 \),
\[ |r_2(x)| \leq C\|u\|_{C^2(M)} \int_M \frac{|\xi|^3}{t} R_t(x, y) d\mu_y \leq Ct^{1/2}\|u\|_{C^2(M)}. \] (8.8)
\[ |\nabla r_2(x)| \leq C\|u\|_{C^2(M)} \int_M \frac{|\xi|^2}{t} \left( R_t(x, y) + \frac{|\xi|^2}{t} |R'_t(x, y)| \right) d\mu_y \leq C\|u\|_{C^2(M)}. \] (8.9)

\( r_3 \) is relatively easy to bound by using the well known Gauss formula. First we have
\[ r_3(x) = \int_{\partial M} \nabla^j (\nabla^i \nabla^j u) \bar{R}_t(x, y) d\tau_x \] (8.10)

Then, by direct calculation, we have
\[ \|r_3\|_{L^2(M)}^2 = \int_M \left( \int_{\partial M} \nabla^j (\nabla^i \nabla^j u) \bar{R}_t(x, y) d\tau_y \right)^2 d\mu_x \]
\[ \leq C\|u\|_{C^2(M)}^2 \int_M \left( \int_{\partial M} |\xi| \bar{R}_t(x, y) d\tau_y \right)^2 d\mu_x \]
\[ \leq C\|u\|_{C^2(M)}^2 \int_M \left( \int_{\partial M} |\xi|^2 \bar{R}_t(x, y) d\tau_y \right) \left( \int_{\partial M} \bar{R}_t(x, y) d\tau_y \right) d\mu_x \]
\[ \leq Ct^{1/2}\|u\|_{C^2(M)} \int_{\partial M} \left( \int_M \bar{R}_t(x, y) d\mu_x \right) d\tau_y \]
\[ \leq Ct^{1/2}\|u\|_{C^2(M)}^2. \] (8.11)

and
\[ \|\nabla r_3(x)\|_{L^2(M)}^2 = \int_M \left| \nabla \left( \int_{\partial M} \nabla^j (\nabla^i \nabla^j u) \bar{R}_t(x, y) d\tau_y \right) \right|^2 d\mu_x \]
\[ = \int_M \left| \int_{\partial M} \nabla^j (\nabla^i \nabla^j u) \left( \bar{R}_t(x, y) + \eta^i \nabla \bar{R}_t(x, y) \right) d\tau_y \right|^2 d\mu_x \]
\[ \leq C\|u\|_{C^2(M)}^2 \left( \int_M \left| \int_{\partial M} \bar{R}_t(x, y) d\tau_y \right|^2 d\mu_x + \int_M \left| \int_{\partial M} \frac{|\xi|^2}{t} \bar{R}_t(x, y) d\tau_y \right|^2 d\mu_x \right) \]
\[ \leq Ct^{-1/2}\|u\|_{C^2(M)} \] (8.12)

Now, we turn to bound the last term \( r_4 \). Notice that
\[ \nabla^j (\nabla^j u) = (\partial_{k'} \Phi^j) g^{k'\ell'} \partial_{\ell'} \left( \partial_{m'} \Phi^j \right) g^{m'\ell'} \left( \partial_{n'} u \right) \]
\[ = (\partial_{k'} \Phi^j) g^{k'\ell'} \partial_{\ell'} \left( \partial_{m'} \Phi^j \right) g^{m'\ell'} \left( \partial_{n'} u \right) + (\partial_{k'} \Phi^j) g^{k'\ell'} \partial_{\ell'} \left( \partial_{m'} \Phi^j \right) \partial_{\ell'} \left( g^{m'\ell'} \left( \partial_{n'} u \right) \right) \]
\[ = \frac{1}{\sqrt{g}} \partial_{m'} \left( \sqrt{g} g^{m'\ell'} \left( \partial_{n'} u \right) \right) \]
\[ = \Delta_M u. \] (8.13)
Here we use the fact that
\[(\partial_{k'}\Phi^j)g^{k'l'}(\partial_{l'}(\partial_{m'}\Phi^j)) = (\partial_{k'}\Phi^j)g^{k'l'}(\partial_{m'}(\partial_{l'}\Phi^j)) = (\partial_{m'}(\partial_{k'}\Phi^j))g^{k'l'}(\partial_{l'}\Phi^j) = \frac{1}{2}g^{k'l'}\partial_{m'}(g_{k'l'}) = \frac{1}{\sqrt{g}}(\partial_{m'}\sqrt{g}) \] (8.14)
Moreover, we have
\[g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{l'}\xi^l)(\partial_l\Phi^i)(\nabla^i\nabla^{j'}u) = -g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{l'}\Phi^i)(\nabla^i\nabla^{j'}u) = -g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{l'}\Phi^i)(\partial_{m'}\Phi^j)g^{m'n'}\partial_{n'}(\nabla^{j'}u) = -g^{i'j'}(\partial_{j'}\Phi^j)\partial_{l'}(\nabla^{j'}u) = -\nabla^{j'}(\nabla^{j'}u). \] (8.15)
where the first equalities are due to that \(\partial_{\ell'}\xi^\ell = -\delta^\ell_{\ell'}.\) Then we have
\[
\text{div} \left( \eta^l(\nabla^i\nabla^{j'}u(y)) \right) + \Delta_M u = \frac{1}{\sqrt{|g|}} \partial_{l'} \left( \sqrt{|g|} g^{j'j}(\partial_{j'}\Phi^j)(\nabla^i\nabla^{j'}u(y)) \right) - g^{j'j}(\partial_{j'}\Phi^j)(\partial_{l'}\xi^l)(\nabla^i\nabla^{j'}u) \]
\[= \frac{\xi^l}{\sqrt{|g|}} \partial_{l'} \left( \sqrt{|g|} g^{j'j}(\partial_{j'}\Phi^j)(\partial_{l'}\Phi^i)(\nabla^i\nabla^{j'}u(y)) \right) \] (8.16)
Here we use the equalities \([8.13],[8.15],\) \(\eta^l = \xi^l\partial_l\Phi^i\) and the definition of div,
\[
\text{div} X = \frac{1}{\sqrt{|g|}} \partial_{l'}(\sqrt{|g|} g^{j'j}(\partial_{j'}\Phi^j X^l)) = \frac{1}{\sqrt{|g|}} \partial_{l'}(\sqrt{|g|} g^{j'j}(\partial_{j'}\Phi^j X^l)) \] (8.17)
where \(X\) is a smooth tangent vector field on \(M\) and \((X^1, \ldots, X^d)^t\) is its representation in embedding coordinates.

Hence,
\[
r_4(x) = \int_M \frac{\xi^l}{\sqrt{|g|}} \partial_{l'} \left( \sqrt{|g|} g^{j'j}(\partial_{j'}\Phi^j)(\partial_l\Phi^i)(\nabla^i\nabla^{j'}u(y)) \right) R_{\ell'}(x, y) d\mu_y \]
Then it is easy to get that
\[
|r_4(x)| \leq C l^{1/2} \|u\|_{C^2(M)}, \] (8.18)
\[
|\nabla r_4(x)| \leq C \|u\|_{C^3(M)}. \] (8.19)
The proof is complete by combining \([8.6],[8.7],[8.8],[8.9],[8.11],[8.12],[8.18],[8.19].\)

### 9 Proof of Theorem 4.2
In this section, we estimate the discretization error introduced by approximating the integrals in \((P3.a)\) from the discrete data. We will show that the discrete laplace operator \(L\) well approximates the integral operator \(L_t\) (Theorem 4.2). First, we need to get a prior estimate of \(u\) which is the solution of the discrete integral equation \((2.5)\). This is obtained by showing that the discrete operator \(L\) also has the property of ellipticity.
9.1 Ellipticity of \( \mathcal{L} \)

First, we introduce a smooth function \( u \) that approximates \( u \) at the samples \( P \).

\[
u(x) = \frac{C_t}{w_{t,h}(x)} \sum_i R \left( \frac{|x - p_i|^2}{4t} \right) u_i V_i,
\]

(9.1) where \( w_{t,h}(x) = C_t \sum_i R \left( \frac{|x - p_i|^2}{4t} \right) V_i \). We have the following lemma about the function \( u \) and \( w_{t,h} \).

**Lemma 9.1.** Assume the submanifold \( M \) and \( \partial M \) are \( C^2 \) smooth and the input data \((P, S, V, A)\) is an \( h \)-integral approximation of \( M \). For both \( t \) and \( \frac{h}{t^{1/2}} \) sufficiently small, there exists a constant \( C_1, C_2 \) and \( C \) independent of \( t, h, x \), so that

\[
C_1 \leq w_{t,h}(x) \leq C_2, \quad \text{and} \quad |\nabla w_{t,h}(x)| \leq \frac{C}{t^{1/2}}
\]

Proof. Since the input data \((P, S, V, A)\) is an \( h \)-integral approximation of \( M \),

\[
|w_{t,h}(x) - C_t \int_M R \left( \frac{|x - y|^2}{4t} \right) d\mu_y| \leq \frac{C h}{t^{1/2}},
\]

which, together with Corollary 6.1, shows the bounds on \( w_{t,h}(x) \). Next, we show the bound on the gradient.

\[
|\nabla w_{t,h}(x)|^2 \leq \sum_{i=1}^{d} \left( \frac{\partial w_{t,h}}{\partial x^i} \right)^2 = \sum_{i=1}^{d} \left( \sum_j C_t R' \left( \frac{|x - p_j|^2}{4t} \right) \frac{x^i - p^i_j}{2t} V_j \right)^2
\]

\[
\leq \left( \sum_j C_t R' \left( \frac{|x - p_j|^2}{4t} \right) V_j \right) \left( \sum_j C_t R' \left( \frac{|x - p_j|^2}{4t} \right) V_j \right)
\]

\[
\leq \frac{C}{t}.
\]

(9.2)

Now, we can get following theorem on the ellipticity of \( \mathcal{L} \).

**Theorem 9.1.** If \( M \) and \( \partial M \) are \( C^\infty \) and the input data \((P, S, V, A)\) is an \( h \)-integral approximation of \((M, \partial M)\), then there exists a constant \( C > 0, C_0 > 0 \) independent on \( t \) so that for any \( u = (u_1, \cdots, u_n)^t \in \mathbb{R}^d \) with \( \sum_{i=1}^{n} u_i V_i = 0 \) and for any sufficient small \( t \) and \( \frac{h}{\sqrt{t}} \),

\[
\langle u, \mathcal{L}u \rangle_V \geq C \left( 1 - \frac{C_0 h}{\sqrt{t}} \right) < u, u > V
\]

(9.3)

where \( \langle u, v \rangle_V = \sum_i u_i v_i V_i \) for any \( u = (u_1, \cdots, u_n)^t, v = (v_1, \cdots, v_n)^t \in \mathbb{R}^d \).

The proof of above theorem is defered to Appendix C.

As an easy corollay of the elliptic property of \( \mathcal{L} \), we can get a priori estimate of \( u = (u_1, \cdots, u_n)^t \) which is the solution of problem (2.5).
Lemma 9.2. Suppose \( u = (u_1, \ldots, u_n)^t \) with \( \sum_i u_i V_i = 0 \) solves the problem (P2.a) with \( f = (f(p_1), \ldots, f(p_n))^t \) and \( g = (g(s_1), \ldots, g(s_m)) \) for \( f \in C^\infty(M) \) and \( g \in C^\infty(\partial M) \). Then

\[
\left( \sum_i u_i^2 V_i \right)^{1/4} \leq C(t^{-1/2}\|g\|_\infty + \|f\|_\infty). \quad (9.4)
\]

Proof. From the elliptic property of \( \mathcal{L} \), we have

\[
\sum_i u_i^2 V_i \leq C < \mathcal{L}u, u >_V
\]

\[
= \sum_i \left( -2 \sum_{s_j \in S} \bar{R}_t(p_i, s_j) g_j A_j + \sum_{p_j \in P} \bar{R}_t(p_i, p_j) f_j V_j \right) u_i V_i
\]

\[
\leq \left( \sum_i u_i^2 V_i \right)^{1/2} \left( \sum_i \left( 2\|g\|_\infty \sum_{s_j \in S} \bar{R}_t(p_i, s_j) A_j + \|f\|_\infty \sum_{p_j \in P} \bar{R}_t(p_i, p_j) V_j \right) V_i \right)^{1/2}
\]

\[
\leq C \left( \sum_i u_i^2 V_i \right)^{1/2} (t^{-1/4}\|g\|_\infty + \|f\|_\infty). \quad (9.5)
\]

This proves the lemma.

\[ \square \]

9.2 Main proof of Theorem 4.2

We are now ready to prove Theorem 4.2.

Proof. of Theorem 4.2

Since \( u_{t,h} \) solves the problem (P4.a), we have

\[
u_{t,h}(x) = \frac{1}{w_{t,h}(x)} \left( \sum_j R_t(x, p_j) u_j V_j + 2t \sum_j \bar{R}_t(x, s_j) g_j A_j - t \sum_j \bar{R}_t(x, p_j) f_j V_j \right) \quad (9.6)
\]

where \( u_j = u_{t,h}(p_j) \), \( g_j = g(s_j) \), \( f_j = f(p_j) \) and \( w_{t,h}(x) = \sum_j R_t(x, p_j) V_j \). For convenience, we set

\[
a_{t,h}(x) = \frac{1}{w_{t,h}(x)} \sum_j R_t(x, p_j) u_j V_j, \quad \text{and} \quad (9.7)
\]

\[
b_{t,h}(x) = \frac{2t}{w_{t,h}(x)} \sum_j \bar{R}_t(x, s_j) g_j A_j, \quad \text{and} \quad (9.8)
\]

\[
c_{t,h}(x) = -\frac{t}{w_{t,h}(x)} \sum_j \bar{R}_t(x, p_j) f(p_j) V_j. \quad (9.9)
\]

Next we upper bound the approximation error \( L_t(u_{t,h}) - L_{t,h}(u_{t,h}) \). Since \( u_{t,h} = a_{t,h} + b_{t,h} + c_{t,h} \),
we first upper bound the approximation error for each part as follows.

\[
|(L_t b_{t,h} - L_{t,h} b_{t,h})(x)|
= \frac{1}{t} \left| \int_M R_t(x, y) (b_{t,h}(x) - b_{t,h}(y)) d\mu_y - \sum_j R_t(x, p_j) (b_{t,h}(x) - b_{t,h}(p_j)) V_j \right|
\leq \frac{1}{t} |b_{t,h}(x)| \left| \int_M R_t(x, y) d\mu_y - \sum_j R_t(x, p_j) V_j \right|
+ \frac{1}{t} \left| \int_M R_t(x, y) b_{t,h}(y) d\mu_y - \sum_j R_t(x, p_j) b_{t,h}(p_j) V_j \right|
\leq \frac{Ch}{t^{3/2}} |b_{t,h}(x)| + \frac{Ch}{t^{3/2}} \|b_{t,h}\|_{C^1(M)}
\leq \frac{Ch}{t^{3/2}} \|g\|_{\infty} + \frac{Ch}{t^{3/2}} (t^{1/2} \|g\|_{\infty} + \|g\|_{\infty})
\leq \frac{Ch}{t^{3/2}} \|g\|_{\infty}.
\]  
(9.10)

And

\[
|(L_t c_{t,h} - L_{t,h} c_{t,h})(x)|
= \frac{1}{t} \left| \int_M R_t(x, y) (c_{t,h}(x) - c_{t,h}(y)) d\mu_y - \sum_j R_t(x, p_j) (c_{t,h}(x) - c_{t,h}(p_j)) V_j \right|
\leq \frac{1}{t} |c_{t,h}(x)| \left| \int_M R_t(x, y) d\mu_y - \sum_j R_t(x, p_j) V_j \right|
+ \frac{1}{t} \left| \int_M R_t(x, y) c_{t,h}(y) d\mu_y - \sum_j R_t(x, p_j) c_{t,h}(p_j) V_j \right|
\leq \frac{Ch}{t^{3/2}} |c_{t,h}(x)| + \frac{Ch}{t^{3/2}} \|c_{t,h}\|_{C^1(M)}
\leq \frac{Ch}{t^{3/2}} \|f\|_{\infty} + \frac{Ch}{t^{3/2}} (t \|f\|_{\infty} + t^{1/2} \|f\|_{\infty})
\leq \frac{Ch}{t} \|f\|_{\infty}.
\]  
(9.11)
Now we upper bound $\|L_t a_{t,h} - L_{t,h} a_{t,h}\|_{L_2}$. We have

$$
\int_{\mathcal{M}} (a_{t,h}(x))^2 \left| \int_{\mathcal{M}} R_t(x,y)d\mu_y - \sum_j R_t(x,p_j)V_j \right|^2 d\mu_x
\leq \frac{Ch^2}{t} \int_{\mathcal{M}} (a_{t,h}(x))^2 d\mu_x
\leq \frac{Ch^2}{t} \int_{\mathcal{M}} \left( \frac{1}{w_{t,h}(x)} \sum_j R_t(x,p_j)u_j V_j \right)^2 d\mu_x
\leq \frac{Ch^2}{t} \int_{\mathcal{M}} \left( \sum_j R_t(x,p_j)u_j^2 V_j \right) \left( \sum_j R_t(x,p_j)V_j \right) d\mu_x
\leq \frac{Ch^2}{t} \left( \sum_j u_j^2 V_j \int_{\mathcal{M}} R_t(x,p_j)d\mu_x \right)
\leq \frac{Ch^2}{t} \sum_j u_j^2 V_j.
$$

(9.12)

Let

$$
A = Ct \int_{\mathcal{M}} \frac{1}{w_{t,h}(y)} R \left( \frac{|x - y|^2}{4t} \right) R \left( \frac{|p_i - y|^2}{4t} \right) d\mu_y
- Ct \sum_j \frac{1}{w_{t,h}(p_j)} R \left( \frac{|x - p_j|^2}{4t} \right) R \left( \frac{|p_i - p_j|^2}{4t} \right) V_j.
$$

(9.13)

We have $|A| < \frac{Ch}{t^{1/2}}$ for some constant $C$ independent of $t$. In addition, notice that only when $|x - x_j|^2 \leq 16t$ is $A \neq 0$, which implies

$$
|A| \leq \frac{1}{\delta_0} |A| R \left( \frac{|x - p_j|^2}{4t} \right).
$$

(9.14)
Then we have
\[
\int_{\mathcal{M}} \left( \sum_i C_i u_i V_i A \right) d\mu_x
\]
\[
= \frac{Ch^2}{t} \int_{\mathcal{M}} \left( \sum_i C_i |u_i| V_i R \left( \frac{|x - p_i|^2}{4t} \right) \right)^2 d\mu_x
\]
\[
\leq \frac{Ch^2}{t} \int_{\mathcal{M}} \left( \sum_i C_i R \left( \frac{|x - p_i|^2}{4t} \right) \right) \left( \sum_i C_i R \left( \frac{|x - p_i|^2}{4t} \right) V_i \right) d\mu_x
\]
\[
\leq \frac{Ch^2}{t} \left( \sum_i \left( \int_{\mathcal{M}} C_i R \left( \frac{|x - p_i|^2}{4t} \right) d\mu_x (u_i^2 V_i) \right) \right)
\]
\[
\leq \frac{Ch^2}{t} \left( \sum_i u_i^2 V_i \right) .
\]
(9.15)

Combining Equation (9.12), (9.15) and Lemma 9.2,
\[
\|L_t a_{t,h} - L_t u_{t,h}\|_{L^2(\mathcal{M})}
\]
\[
= \left( \int_{\mathcal{M}} |(L_t(a_{t,h}) - L_t u_{t,h})(x)|^2 d\mu_x \right)^{1/2}
\]
\[
\leq \frac{1}{t} \left( \int_{\mathcal{M}} (a_{t,h}(x))^2 \left( \int_{\mathcal{M}} R_t(x, y) d\mu_y - \sum_j R_t(x, p_j)V_j \right)^2 d\mu_x \right)^{1/2}
\]
\[
+ \frac{1}{t} \left( \int_{\mathcal{M}} \left( \int_{\mathcal{M}} R_t(x, y) d\mu_y - \sum_j R_t(x, p_j) a_{t,h}(p_j)V_j \right)^2 d\mu_x \right)^{1/2}
\]
\[
\leq \frac{Ch}{t^{3/2}} \left( \sum_i u_i^2 V_i \right)^{1/2} \leq \frac{Ch}{t^{3/2}} \left( t^{-1/2} \|g\|_\infty + \|f\|_\infty \right) .
\]
(9.16)

Now assembling the parts together, we have the following upper bound.
\[
\|L_t u_{t,h} - L_t a_{t,h}\|_{L^2(\mathcal{M})}
\]
\[
\leq \|L_t a_{t,h} - L_t u_{t,h}\|_{L^2(\mathcal{M})} + \|L_t b_{t,h} - L_t b_{t,h}\|_{L^2(\mathcal{M})} + \|L_t c_{t,h} - L_t c_{t,h}\|_{L^2(\mathcal{M})}
\]
\[
\leq \frac{Ch}{t^{3/2}} (t^{-1/2} \|g\|_\infty + \|f\|_\infty) + \frac{Ch}{t^{3/2}} \|g\|_\infty + \frac{Ch}{t} \|f\|_\infty
\]
\[
\leq \frac{Ch}{t^2} (\|g\|_\infty + \|f\|_\infty) .
\]
(9.17)

At the same time, since \(u_t\) respectively \(u_{t,h}\) solves Problem \(\text{[P3.a]}\) respectively Problem \(\text{[P4.a]}\), we
have

\[ \|L_t(u_t) - L_{t,h}(u_{t,h})\|_{L^2(M)} \]
\[ = \left( \int_M ((L_tu_t - L_{t,h}u_{t,h})(x))^2 \, d\mu_x \right)^{1/2} \]
\[ \leq 2 \left( \int_M \left( \int_{\partial M} \tilde{R}_t(x,y)g(y) \, d\tau_y - \sum_j \tilde{R}_t(x,s_j)g(s_j)A_j \right)^2 \, d\mu_x \right)^{1/2} \]
\[ + \left( \int_M \left( \int_M \tilde{R}_t(x,y)f(y) - \sum_j \tilde{R}_t(x,p_j)f(p_j)V_j \right)^2 \, d\mu_x \right)^{1/2} \]
\[ \leq \frac{C h}{t} \|g\|_\infty + \frac{C h}{t^{1/2}} \|f\|_\infty. \] (9.18)

The complete $L^2$ estimate follows from Equation (9.17) and (9.18).

Next, we turn to prove the second part of the theorem, upper bound $\|\nabla L_t(u_t - u_{t,h})\|_{L^2(M)}$.

The techniques is similar those used in $L^2$ estimate.

First, let us consider the first term, $\|\nabla (L_t a_{t,h} - L_{t,h}a_{t,h})\|_{L^2}$. 

\[ \int_M |\nabla a_{t,h}(x)|^2 \left| \int_M R_t(x,y) \, d\mu_y - \sum_j R_t(x,p_j)V_j \right|^2 \, d\mu_x \]
\[ \leq \frac{C h^2}{t} \int_M |\nabla a_{t,h}(x)|^2 \, d\mu_x \]
\[ \leq \frac{C h^2}{t} \left( \int_M \left| \frac{1}{w_{t,h}(x)} \sum_j \nabla R_t(x,p_j)u_j V_j \right|^2 \, d\mu_x \right) \]
\[ + \int_M \left| \frac{\nabla w_{t,h}(x)}{w_{t,h}(x)} \sum_j R_t(x,p_j)u_j V_j \right|^2 \, d\mu_x \]
\[ \leq \frac{C h^2}{t^2} \int_M \left| \sum_j R_{2t}(x,p_j)u_j V_j \right|^2 \, d\mu_x \]
\[ \leq \frac{C h^2}{t^2} \int_M \left( \sum_j R_{2t}(x,p_j)u_j^2 V_j \right) \left( \sum_j R_{2t}(x,p_j)V_j \right) \, d\mu_x \]
\[ \leq \frac{C h^2}{t^2} \left( \sum_j u_j^2 V_j \int_M R_{2t}(x,p_j) \, d\mu_x \right) \]
\[ \leq \frac{C h^2}{t^2} \sum_j u_j^2 V_j. \] (9.19)
\[
\int_M |a_{t,h}(x)|^2 \left| \int_M \nabla R_t(x, y) d\mu_y - \sum_j \nabla R_t(x, p_j) V_j \right|^2 d\mu_x \\
\leq \frac{C h^2}{t^2} \int_M |a_{t,h}(x)|^2 d\mu_x \\
\leq \frac{C h^2}{t^2} \sum_j u_j^2 V_j. \quad (9.20)
\]

Let
\[
B = C_t \int_M \frac{1}{w_{t,h}(y)} \nabla R \left( \frac{|x - y|^2}{4t} \right) R \left( \frac{|p_i - y|^2}{4t} \right) d\mu_y \\
- C_t \sum_j \frac{1}{w_{t,h}(p_j)} \nabla R \left( \frac{|x - p_j|^2}{4t} \right) R \left( \frac{|p_i - p_j|^2}{4t} \right) V_j. \quad (9.21)
\]

We have $|B| < \frac{C h}{t^{1/2}}$ for some constant $C$ independent of $t$. In addition, notice that only when $|x - x_i|^2 \leq 16t$ is $B \neq 0$, which implies
\[
|B| \leq \frac{1}{\delta_0} |B| R \left( \frac{|x - p_i|^2}{4t} \right). \quad (9.22)
\]

Then we have
\[
\int_M \left| \int_M \nabla R_t(x, y)a_{t,h}(y) d\mu_y - \sum_j \nabla R_t(x, p_j)a_{t,h}(p_j) V_j \right|^2 d\mu_x \\
= \int_M \left( \sum_i C_t \delta_i V_i B \right)^2 d\mu_x \\
\leq \frac{C h^2}{t^2} \int_M \left( \sum_i C_t |\delta_i| V_i R \left( \frac{|x - p_i|^2}{32t} \right) \right)^2 d\mu_x \\
\leq \frac{C h^2}{t^2} \left( \sum_i u_i^2 V_i \right). \quad (9.23)
\]
Combining Equation (9.19), (9.20) and (9.23), we have

\[ \| \nabla (L_t a_{t,h} - L_{t,h}a_{t,h}) \|_{L^2(M)} \]
\[ = \left( \int_M \| (L_t(a_{t,h}) - L_{t,h}(a_{t,h})) (x) \|^2 \, d\mu_x \right)^{1/2} \]
\[ \leq \frac{1}{t} \left( \int_M (\nabla a_{t,h}(x))^2 \left| \int_M R_t(x,y) \, d\mu_y - \sum_j R_{t}(x,p_j) V_j \right|^2 \, d\mu_x \right)^{1/2} \]
\[ = \frac{1}{t} \left( \int_M (a_{t,h}(x))^2 \left| \int_M \nabla_x R_{t}(x,y) \, d\mu_y - \sum_j \nabla_x R_{t}(x,p_j) V_j \right|^2 \, d\mu_x \right)^{1/2} \]
\[ + \frac{1}{t} \left( \int_M \left| \int_M \nabla_x R_{t}(x,y) a_{t,h}(y) \, d\mu_y - \sum_j \nabla_x R_{t}(x,p_j) a_{t,h}(p_j) V_j \right|^2 \, d\mu_x \right)^{1/2} \]
\[ \leq \frac{Ch}{t^2} \left( \sum_i u_i^2 V_i \right)^{1/2} \leq \frac{Ch}{t^2} \left( \| f \|_\infty + t^{-1/4} \| g \|_\infty \right) \quad (9.24) \]

Using the similar method, we can get

\[ \| \nabla (L_t b_{t,h} - L_{t,h}b_{t,h}) \|_{L^2(M)} \leq \frac{Ch}{t^2} \| g \|_\infty, \]
\[ \| \nabla (L_t c_{t,h} - L_{t,h}c_{t,h}) \|_{L^2(M)} \leq \frac{Ch}{t^{3/2}} \| f \|_\infty, \]

Then we have

\[ \| \nabla (L_t u_{t,h} - L_{t,h}u_{t,h}) \|_{L^2(M)} \leq \frac{Ch}{t^2} \left( \| f \|_\infty + t^{-1/4} \| g \|_\infty \right). \]

At last, we can complete the proof by using following fact,

\[ \| \nabla (L_t(u_t) - L_{t,h}(u_{t,h})) \|_{L^2(M)} \]
\[ \leq \frac{2}{t} \left( \int_M \left( \int_{\partial M} \nabla_x \tilde{R}_{t}(x,y) g(y) \, d\tau_y - \sum_j \nabla_x \tilde{R}_{t}(x,s_j) g(s_j) A_j \right)^2 \, d\mu_x \right)^{1/2} \]
\[ + \left( \int_M \left( \int_M \nabla_x \tilde{R}_{t}(x,y) f(y) - \sum_j \nabla_x \tilde{R}_{t}(x,p_j) f(p_j) V_j \right)^2 \, d\mu_x \right)^{1/2} \]
\[ \leq \frac{Ch}{t^{3/2}} \| g \|_\infty + \frac{Ch}{t} \| f \|_\infty. \quad (9.29) \]

\[ \square \]

10 Bounds on Solution Operators

In this section, we prove the bounds on the solution operators \( T_t \) and \( T_{t,h} \) which are used to prove the convergence of the eigenproblem of Laplace-Beltrami operator.
Proof of Lemma 4.1

Let \( u = T_t(f) \). From the elliptic property of \( L_t \), there exist a constant \( C \) independent of \( t \)

\[
\|u\|_{L_2(M)} \leq C \left( \int_M \left( \frac{|x-y|^2}{4t} \right) f(y) d\mu_y \right) u(x) d\mu_x
\]

\[
\leq C \|u\|_{L_2(M)} \|f\|_{L_2(M)},
\]

which means \( \|u\|_{L_2(M)} \leq C \|f\|_{L_2(M)} \). At the same time, we can write

\[
u(x) = \frac{1}{w_t(x)} \int_M R_t(x, y) u(y) d\mu_y - \frac{t}{w_t(x)} \int_M \bar{R}_t(x, y) f(y) d\mu_y.
\]

This means \( u \) is \( C^\infty(M) \).

\[
|u(x)|^2 \leq C \int_M R_t(x, y) u^2(y) d\mu_y \int_M R_t(x, y) d\mu_y + C t^2 \|f\|_\infty^2
\]

\[
\leq C C_t \|u\|_{L_2(M)}^2 + C t^2 \|f\|_\infty^2 \leq C C_t \|f\|_\infty^2
\]

From Corollary 6.3, we have

\[
\|\nabla u\|_\infty \leq \frac{C}{t^{1/2}} (\|u\|_\infty + t \|f\|_\infty) \leq \frac{C C_t^{1/2}}{t^{1/2}} \|f\|_\infty.
\]

Now consider to bound \( T_{t,h} \). We can write

\[
T_{t,h}(f)(x) = \frac{1}{w_{t,h}(x)} \sum_j R_t(x, p_j) u_j V_j - \frac{t}{w_{t,h}(x)} \sum_j \bar{R}_t(x, p_j) f(p_j) V_j
\]

where \( \mathbf{u} = (u_1, \ldots, u_n) \) with \( \sum_i u_i V_i = 0 \) solves the problem (P2.a) with \( \mathbf{g} = 0 \). From Lemma 9.2, \( \left( \sum_j u_j^2 V_j \right)^{1/2} \leq C \|f\|_\infty \) for some constant \( C \) independent on \( t \). It is obvious that for some constant \( C \) independent on \( t \),

\[
|T_{t,h}(f)(x)| \leq C C_t^{1/2} \left( \sum_j u_j^2 V_j \right)^{1/2} + C t \|f\|_\infty \leq C C_t^{1/2} \|f\|_\infty,
\]

and

\[
|\nabla T_{t,h}(f)(x)| \leq C C_t^{1/2} \left( \sum_j u_j^2 V_j \right)^{1/2} + C t^{1/2} \|f\|_\infty \leq C C_t^{1/2} \|f\|_\infty.
\]

This proves the lemma.

\[
\square
\]

11 Numerical Results

In this section, we present the numerical results of finite integral method and compare it to that of finite element method.

We test both methods on unit disk where we know the ground truth. We discretize unit disk using a Delaunay mesh with 684 vertices shown in Figure 3(a). We refine the mesh to the meshes
with 2610, 10191 and 40269 vertices by respectively subdividing it once, twice and three times. In each subdivision, each triangle in the mesh is split into four smaller ones using the midpoints of the edges. Figure 3(b) shows the mesh after one subdivision. We run finite element method over these meshes. For our finite integral method, we remove the mesh topology and only retain the vertices as the input point set $P$. Those vertices on the boundary of the mesh are taken as the input point set $S$. The volume weight vector $V$ and $A$ are estimated using the method proposed in [18] as follows. Locally approximate the tangent space at a point and then project the nearby points onto the tangent space over which a Delaunay triangulation is computed in the tangent space. The volume weight of that point is computed as one third of the sum of the area of the triangles incident to that point in the Delaunay triangulation.

Set the Neumann boundary condition as that of the function $u_{gt} = \cos 2\pi r$ with $r = \sqrt{x^2 + y^2}$ and see how accurate the numerical methods can recover this function. The approximation error is computed in $L^2$: $err = \|u - u_{gt}\|_{L^2}$ where $u$ is the numerical solution obtained by either FEM or FIM. Table 1 shows the approximation errors of different methods over the above input data. As we can see, FEM has a quadratic convergence rate in $L^2$, which coincides with the classical theory of FEM. On the other hand, FIM has a linear convergence rate in $L^2$ numerically, which is better than what our theory predicts.

| $|V|$ | 684  | 2610 | 10191 | 40269 |
|-----|------|------|-------|-------|
| FEM | 0.0266 | 0.0070 | 0.0018 | 0.00044 |
| FIM | 0.2437 | 0.1306 | 0.0643 | 0.0312 |

Table 1: Convergence of Solving Poisson Equation.
Next, we consider computing Neumann eigensystems. Table 2 shows the convergence of the sixth eigenvalue and its corresponding eigenfunction. The sixth eigenvalue is in fact the smallest simple eigenvalue. We note that FEM is about 10 times more accurate than FIM in solving both Poisson Equation and its eigensystem when a good mesh of unit disk is available.

| \( V \) | 684 | 2610 | 10191 | 40296 |
|---|---|---|---|---|
| Eigenvalue | | | | |
| FEM | 0.1164 | 0.0371 | 0.0168 | 0.0116 |
| FIM | 0.8244 | 0.2570 | 0.0555 | 0.0212 |
| Eigenfunction | | | | |
| FEM | 0.0017 | 0.00064 | 0.00056 | 0.00054 |
| FIM | 0.0298 | 0.0158 | 0.0094 | 0.0066 |

Table 2: Convergence of Neumann Eigensystem. The approximation error of the eigenvalues is computed as \( |\lambda_i - \sigma_i| \) where \( \sigma_i \) is the ground truth and \( \lambda_i \) is the numerical solution.

Finally, we present an example which shows FEM is very sensitive to the quality of the mesh. Figure 4 shows the a triangle mesh of 10000 vertices of unit disk. Again, we attempt to numerically recover the harmonic \( u_{gt} = \cos 2\pi r \) from its Neumann boundary. Although it is a Delaunay mesh, the condition number of the stiff matrix of FEM reaches \( 1e + 20 \). Consequently, the solution of FEM has no accuracy. On the other hand, our FIM still gives a reasonable approximation of the solution. See Table 3.
Table 3: Approximation error over the triangle mesh shown in Figure 4.

### Acknowledgments
This research was partially supported by NSFC Grant (11201257 to Z.S., 11371220 to Z.S. and J.S. and 11271011 to J.S.), and National Basic Research Program of China (973 Program 2012CB825500 to J.S.).

### Appendix A: Proof of Lemma 7.1

**Proof.** We start with the evaluation of the $x^i$ component of $\nabla v$.

\[
\nabla^i v(x) = \frac{C_i^2}{2tw_i^2(x)} \int_M \int_M \nabla^i x^j \delta^j - R'^i \left( \frac{|x - y|^2}{4t} \right) R \left( \frac{|x - y'|^2}{4t} \right) u(y) d\mu_y d\mu_y
\]

\[
= \frac{C_i^2}{4tw_i^2(x)} \int_M \int_M K^i(x, y, y'; t)(u(y) - u(y')) d\mu_y d\mu_y
\]

(11.1)

where we set

\[
K^i(x, y, y'; t) = \nabla^i x^j \delta^j - R'^i \left( \frac{|x - y|^2}{4t} \right) R \left( \frac{|x - y'|^2}{4t} \right)
\]

(11.2)

Think of $\nabla^i x^j$ as the $i,j$ entry of the matrix $[\nabla^i x^j]$ and we have

\[
\nabla^i x^j \nabla^l x^i = (\partial_{j'} x^i)(\partial_{j'} x^j)(\partial_{i} x^l)(\partial_{i} x^j)
\]

\[
= g_{i}^{j'} g_{j'}^{i} (\partial_{j'} x^j)(\partial_{i} x^l)
\]

\[
= \delta_{j'}^{i'} (\partial_{j'} x^j)(\partial_{i} x^l)
\]

\[
= (\partial_{j'} x^j)(\partial_{i} x^l)
\]

\[
= \nabla^l x^j.
\]

(11.3)

This shows that the matrix $[\nabla^i x^j]$ is idempotent. At the same time, $[\nabla^i x^j]$ is symmetric, which implies that the eigenvalues of $\nabla x$ are either 1 or 0. Then we have the following upper bounds.
There exists a constant $C$ depending only on the maximum of $R$ and $R'$ so that

\[
\sum_{i=1}^{d} K^i(x, y, y'; t)^2 \leq 2 \left( R' \left( \frac{|x - y|^2}{4t} \right) R \left( \frac{|x - y'|^2}{4t} \right) \right)^2 \| \nabla v \| \| (x - y) \|^2
\]

\[
+ 2 \left( R' \left( \frac{|x - y|^2}{4t} \right) R \left( \frac{|x - y'|^2}{4t} \right) \right)^2 \| \nabla v \| \| (x - y') \|^2
\]

\[
\leq CR' \left( \frac{|x - y|^2}{4t} \right) R \left( \frac{|x - y'|^2}{4t} \right) \| x - y \|^2
\]

\[
+ CR' \left( \frac{|x - y|^2}{4t} \right) R \left( \frac{|x - y'|^2}{4t} \right) \| x - y' \|^2
\]

(11.4)

Thus by Corollary 6.2, there exists a constant $C$ independent of $t$ so that

\[
C_t^2 \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{\sum_{i=1}^{d} K^i(x, y, y'; t)^2}{t} \, d\mu_y \, d\mu_y' \leq C \int_{\mathcal{M}} \int_{\mathcal{M}} C_t R' \left( \frac{|x - y|^2}{4t} \right) \| x - y \|^2 \, d\mu_y \int_{\mathcal{M}} C_t R \left( \frac{|x - y'|^2}{4t} \right) \, d\mu_y'
\]

\[
+ C \int_{\mathcal{M}} \int_{\mathcal{M}} C_t R' \left( \frac{|x - y'|^2}{4t} \right) \| x - y' \|^2 \, d\mu_y' \int_{\mathcal{M}} C_t R \left( \frac{|x - y|^2}{4t} \right) \, d\mu_y
\]

\[
\leq C
\]

(11.5)

Since $R$ has a compact support, only when $|y - y'|^2 < 16t$ and $|x - \frac{y + y'}{2}|^2 < 4t$ is $K^i(x, y, y'; t) \neq 0$. Thus from the assumption on $R$, we have

\[
K^i(x, y, y'; t)^2 \leq \frac{1}{\delta_o^2} \sum_{i=1}^{d} K^i(x, y, y'; t)^2 R \left( \frac{|y - y'|^2}{32t} \right) R \left( \frac{|x - \frac{y + y'}{2}|^2}{8t} \right).
\]

We can upper bound the norm of $\nabla v$ as follows

\[
\| \nabla v \| = \frac{C_t}{16t^2 w_1^2(x)} \sum_{i=1}^{d} \int_{\mathcal{M}} \int_{\mathcal{M}} K^i(x, y, y'; t) (u(y) - u(y')) \, dy' \, dy
\]

\[
\leq \frac{C_t^4}{16t^2 w_1^4(x)} \sum_{i=1}^{k} \int_{\mathcal{M}} \int_{\mathcal{M}} K_t^2(x, y, y'; t) \left( R \left( \frac{|y - y'|^2}{32t} \right) R \left( \frac{|x - \frac{y + y'}{2}|^2}{8t} \right) \right)^{-1} \, d\mu_y' \, d\mu_y
\]

\[
= \frac{C_t}{16t^2 \delta_0^2 w_1^4(x)} \int_{\mathcal{M}} \int_{\mathcal{M}} \sum_{i=1}^{d} K^i(x, y, y'; t)^2 \, d\mu_y' \, d\mu_y
\]

\[
\int_{\mathcal{M}} \int_{\mathcal{M}} R \left( \frac{|x - \frac{y + y'}{2}|^2}{8t} \right) R \left( \frac{|y - y'|^2}{32t} \right) (u(y) - u(y'))^2 \, d\mu_y' \, d\mu_y
\]

\[
\leq \frac{CC_t^2}{t} \int_{\mathcal{M}} \int_{\mathcal{M}} R \left( \frac{|x - \frac{y + y'}{2}|^2}{8t} \right) R \left( \frac{|y - y'|^2}{32t} \right) (u(y) - u(y'))^2 \, d\mu_y' \, d\mu_y.
\]

(11.6)
Finally, we have

\[
\int_{\mathcal{M}} |\nabla v(x)|^2 d\mu_x \\
\leq \frac{CC^2}{t} \int_{\mathcal{M}} \left( \int_{\mathcal{M}} \int_{\mathcal{M}} R \left( \frac{|x-y'|^2}{8t} \right) R \left( \frac{|y-y'|^2}{32t} \right) (u(y) - u(y'))^2 d\mu_y d\mu_y \right) d\mu_x \\
= \frac{CC^2}{t} \int_{\mathcal{M}} \left( \int_{\mathcal{M}} \int_{\mathcal{M}} R \left( \frac{|x-y'|^2}{8t} \right) d\mu_x \right) R \left( \frac{|y-y'|^2}{32t} \right) (u(y) - u(y'))^2 d\mu_y d\mu_y \\
\leq \frac{CC_t}{t} \int_{\mathcal{M}} \int_{\mathcal{M}} R \left( \frac{|y-y'|^2}{32t} \right) (u(y) - u(y'))^2 d\mu_y d\mu_y. \tag{11.7}
\]

This proves the Lemma.

**Appendix B: Proof of Lemma [7.2]**

**Proof.** First, denote

\[
\mathcal{B}_{x_i}^r = \{ y \in \mathcal{M} : |x - y| \leq r \}, \quad \mathcal{M}_x^t = \{ y \in \mathcal{M} : |x - y|^2 \leq 32t \} \tag{11.8}
\]

Since the manifold $\mathcal{M}$ is compact, there exists $x_i \in \mathcal{M}$, $i = 1, \cdots, N$ such that

\[
\mathcal{M} \subset \bigcup_{i=1}^{N} \mathcal{B}_{x_i}^r \tag{11.9}
\]

Then, we have

\[
\int_{\mathcal{M}} \int_{\mathcal{M}} R \left( \frac{|x-y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \\
\leq \sum_{i=1}^{N} \int_{\mathcal{M}} \int_{\mathcal{B}_{x_i}^r} R \left( \frac{|x-y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \\
= \sum_{i=1}^{N} \int_{\mathcal{B}_{x_i}^r} \int_{\mathcal{B}_{x_i}^r} R \left( \frac{|x-y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \tag{11.10}
\]

In the last equality, we use the fact that $R$ is compactly supported and $t$ is small enough.

By Proposition [6.1], we have there exist a parametrization $\Phi_i : \Omega_i \subset \mathbb{R}^k \to U_i \subset \mathcal{M}, i = 1, \cdots, N$, such that

1. $\mathcal{B}_{x_i}^{2r} \subset U_i$ and $\Omega_i$ is convex.
2. $\Phi \in C^3(\Omega)$;
3. For any points $x, y \in \Omega$, $\frac{1}{2} |x - y| \leq \|\Phi_i(x) - \Phi_i(y)\| \leq 2 |x - y|$.

For any $x \in \mathcal{B}_x^r$ and $y \in \mathcal{B}_x^{2r}$, let

\[
z_j = \Phi \left( \left( \frac{j}{16} \right) \Phi^{-1}(x) + \left( 1 - \frac{j}{16} \right) \Phi^{-1}(y) \right), \quad j = 0, \cdots, 16. \tag{11.11}
\]
Apparently, \( z_0 = x, z_{16} = y \). Since \( \Omega_i \) is convex, we have \( \Phi^{-1}(z_j) \in \Omega_i, \ i = 0, \cdots, 16 \). Then utilizing property 3 of \( \Phi_i \), we can get

\[
\|z_j - z_{j+1}\| \leq 2\|\Phi^{-1}(z_j) - \Phi^{-1}(z_{j+1})\|
\leq \frac{1}{8}\|\Phi^{-1}(x) - \Phi^{-1}(y)\|
\leq \frac{1}{4}\|x - y\|
\]

(11.12)

Now, we are ready to estimate the integrals in (11.10).

\[
\int_{B_{x_i}} \int_{B_{x_i}} R \left( \frac{|x - y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y
\leq 16 \sum_{j=0}^{15} \int_{B_{x_i}} \int_{B_{x_i}} R \left( \frac{|x - y|^2}{32t} \right) (u(z_j) - u(z_{j+1}))^2 d\mu_x d\mu_y
\]

(11.13)

The support of \( R \) is \([0, 1]\) is used to get the last equality.

For any \( y \in M_{i_k} \),

\[
\|z_j - z_{j+1}\|^2 \leq \frac{1}{16}\|x - y\|^2 \leq 2t, \quad j = 0, \cdots, 15
\]

(11.14)

which implies that

\[
R \left( \frac{|z_j - z_{j+1}|^2}{4t} \right) \geq \delta_0, \quad j = 0, \cdots, 15.
\]

(11.15)

Now, we have

\[
\int_{B_{x_i}} \int_{M_{x_i}} R \left( \frac{|x - y|^2}{32t} \right) (u(z_j) - u(z_{j+1}))^2 d\mu_y d\mu_x
\]

\[
= \int_{B_{x_i}} \int_{M_{x_i}} R \left( \frac{|x - y|^2}{32t} \right) \left( R \left( \frac{|z_j - z_{j+1}|^2}{4t} \right) \right)^{-1} R \left( \frac{|z_j - z_{j+1}|^2}{4t} \right) (u(z_j) - u(z_{j+1}))^2 d\mu_y d\mu_x
\]

\[
\leq \frac{1}{\delta_0} \int_{B_{x_i}} \int_{M_{x_i}} R \left( \frac{|z_j - z_{j+1}|^2}{4t} \right) (u(z_j) - u(z_{j+1}))^2 d\mu_y d\mu_x
\]

\[
= \frac{1}{\delta_0} \int_{\Phi^{-1}_i(B_{x_i})} \int_{\Phi^{-1}_i(M_{x_i})} R \left( \frac{|z_j - z_{j+1}|^2}{4t} \right) (u(z_j) - u(z_{j+1}))^2 \left| \nabla \Phi \left( \xi_y \right) \right| d\xi_y d\xi_x
\]

(11.16)

\[
\leq \frac{4}{\delta_0} \int_{\Phi^{-1}_i(B_{x_i})} \int_{\Phi^{-1}_i(M_{x_i})} R \left( \frac{|z_j - z_{j+1}|^2}{4t} \right) (u(z_j) - u(z_{j+1}))^2 d\xi_y d\xi_x
\]

where \( \xi_x = \Phi^{-1}_i(x) \), \( \xi_y = \Phi^{-1}_i(y) \).

Let

\[
\xi_j = \Phi^{-1}_i(z_j) = \frac{j}{16} \xi_x + \left( 1 - \frac{j}{16} \right) \xi_y, \quad j = 0, \cdots, 16.
\]

(11.17)
It is easy to show that $\Phi_i(\xi_{z_j}) = z_j \in B_{2r_{\xi_i}}^r$, $j = 0, \cdots, 16$ by using the facts that for any $y \in M_x^t$

$$\|z_j - x\| \leq \sum_{l=1}^{j} \|z_l - z_{l-1}\| \leq \frac{j}{4} \|x - y\| \leq 15\sqrt{2t}, \quad j = 1, \cdots, 15 \quad (11.18)$$

and $x \in B_r x_i$ and $15\sqrt{2t} \leq r$. Then we have

$$\xi_{z_j} \in \Phi_i^{-1}(B_{2r_{\xi_i}}^r), \quad j = 0, \cdots, 16. \quad (11.19)$$

By changing variable, we get

$$\int_{\Phi_i^{-1}(B_{2r_{\xi_i}}^r)} \int_{\Phi_i^{-1}(M_x^t)} R \left( \frac{|z_j - z_{j+1}|^2}{4t} \right) (u(z_j) - u(z_{j+1}))^2 d\xi_x d\xi_{z_j} d\xi_{z_{j+1}} \leq 4 \cdot 8^k \int_{\Phi_i^{-1}(B_{2r_{\xi_i}}^r)} \int_{\Phi_i^{-1}(B_{2r_{\xi_i}}^r)} R \left( \frac{|z_j - z_{j+1}|^2}{4t} \right) (u(z_j) - u(z_{j+1}))^2 \left| \nabla \Phi(\xi_{z_j}) \right| \left| \nabla \Phi(\xi_{z_{j+1}}) \right| d\xi_x d\xi_{z_j} d\xi_{z_{j+1}} \leq 4 \cdot 8^k \int_{B_{2r_{\xi_i}}^r} \int_{B_{2r_{\xi_i}}^r} R \left( \frac{|z_j - z_{j+1}|^2}{4t} \right) (u(z_j) - u(z_{j+1}))^2 d\mu_{z_j} d\mu_{z_{j+1}} \leq C \int_{M_x^t} \int_{M_x^t} R \left( \frac{|x - y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \quad (11.20)$$

Finally, we can prove the lemma,

$$\int_{M_x^t} \int_{M_x^t} R \left( \frac{|x - y|^2}{32t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \leq C \sum_{i=1}^{N} \int_B \int_{M_{2r_{\xi_i}}^t} R \left( \frac{|x - y|^2}{4t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \leq CN \int_{M_x^t} \int_{M_x^t} R \left( \frac{|x - y|^2}{4t} \right) (u(x) - u(y))^2 d\mu_x d\mu_y \quad (11.21)$$

\[\square\]

Appendix C: Proof of Theorem 9.1
Proof. In the definition of \( u \) and \( w_{t,h}, \{ 0.1 \} \), replace \( t \) with \( t' = t/18 \). We have

\[
\begin{align*}
\int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(x, y) (u(x) - u(y))^2 \, d\mu_x d\mu_y \\
= \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(x, y) \left( \frac{1}{w_{t,h}(x)} \sum_i R_{t'}(x, p_i) u_i V_i - \frac{1}{w_{t,h}(y)} \sum_j R_{t'}(p_j, y) u_j V_j \right)^2 \, d\mu_x d\mu_y \\
= \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(x, y) \left( \frac{1}{w_{t,h}(x) w_{t,h}(y)} \sum_{i,j} R_{t'}(x, p_i) R_{t'}(p_j, y) V_i V_j(u_i - u_j) \right)^2 \, d\mu_x d\mu_y \\
\leq \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(x, y) \frac{1}{w_{t,h}(x) w_{t,h}(y)} \sum_{i,j} R_{t'}(x, p_i) R_{t'}(p_j, y) V_i V_j(u_i - u_j)^2 \, d\mu_x d\mu_y \\
= \sum_{i,j} \left( \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{1}{w_{t,h}(x) w_{t,h}(y)} R_{t'}(x, p_i) R_{t'}(p_j, y) R_{t'}(x, y) d\mu_x d\mu_y \right) V_i V_j(u_i - u_j)^2 \quad (11.22)
\end{align*}
\]

Denote

\[
A = \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{1}{w_{t,h}(x) w_{t,h}(y)} R_{t'}(x, p_i) R_{t'}(p_j, y) R_{t'}(x, y) d\mu_x d\mu_y
\]

and then notice only when \( |p_i - p_j|^2 \leq 36t' \) is \( A \neq 0 \). For \( |p_i - p_j|^2 \leq 36t' \), we have

\[
A \leq \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(x, p_i) R_{t'}(p_j, y) R_{t'}(x, y) R \left( \frac{|p_i - p_j|^2}{72t'} \right)^{-1} \frac{R \left( |p_i - p_j|^2 \right)}{72t'} \, d\mu_x d\mu_y \\
\leq \frac{CC_t}{\delta_0} \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(x, p_i) R_{t'}(p_j, y) R \left( \frac{|p_i - p_j|^2}{72t'} \right) \, d\mu_x d\mu_y \\
\leq CC_t \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(x, p_i) R_{t'}(p_j, y) R \left( \frac{|p_i - p_j|^2}{72t'} \right) \, d\mu_x d\mu_y \\
\leq CC_t R \left( \frac{|p_i - p_j|^2}{4t} \right). \quad (11.23)
\]

Combining Equation \((11.22), (11.23)\) and Lemma \(7.2\) we obtain

\[
C \frac{C_t}{t} \sum_{i,j} R \left( \frac{|p_i - p_j|^2}{4t} \right) (u_i - u_j)^2 V_i V_j \geq \int_{\mathcal{M}} (u(x) - \bar{u})^2 d\mu_x \quad (11.24)
\]

We now lower bound the RHS of the above equation using \( < \mathbf{u}, \mathbf{u} > \mathbf{v} \).

\[
|\mathcal{M}| |\bar{u}| = \left| \int_{\mathcal{M}} u(x) d\mu_x \right| = \left| \sum_i u_i \int_{\mathcal{M}} \frac{C_t}{\sqrt{w_{t,h}(x)}} R \left( \frac{|x - p_i|^2}{4t} \right) d\mu_x \right|. \quad (11.25)
\]

Let \( q(x) = \frac{C_t}{\sqrt{w_{t,h}(x)}} R \left( \frac{|x - p_i|^2}{4t} \right) \). There exists a constant \( C \) so that \( |q(x)| \leq CC_t \) and

\[
|\nabla q(x)| \leq \frac{C_t}{\sqrt{w_{t,h}(x)}} \left| \nabla R \left( \frac{|x - p_i|^2}{4t} \right) \right| + \frac{C_t |\nabla w_{t,h}(x)|}{w_{t,h}(x)} R \left( \frac{|x - p_i|^2}{4t} \right) \leq CC_t \frac{|\nabla w_{t,h}(x)|}{t^{1/2}}. \quad (11.26)
\]

Then, for sufficiently small \( t', h/t^{1/2} \), there exists a constant \( C \)

\[
\left| \int_{\mathcal{M}} \frac{C_t}{\sqrt{w_{t,h}(x)}} R \left( \frac{|x - p_i|^2}{4t} \right) d\mu_x - \sum_i \frac{C_t}{\sqrt{w_{t,h}(p_i)}} R \left( \frac{|p_i - p_j|^2}{4t} \right) V_i \right| \leq \frac{Ch}{t^{1/2}}. \quad (11.26)
\]

43
Thus we have
\[
|M||\bar{u}| \leq \sum_{i,j} C_l \frac{w_{l,t}(\mathbf{p}_i)}{w_{l,t}(\mathbf{p}_j)} R \left( \frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t'} \right) u_j V_i V_j + \frac{Ch}{t^{1/2}} (\sum_i |u_i V_i|)
\]
\[
\leq \sum_i u_i V_i + \frac{Ch}{t^{1/2}} (\sum_i |u_i V_i|)
\]
\[
= \frac{1}{|M|} \sum_{i,j} C_l \frac{w_{l,t}(\mathbf{p}_i)}{w_{l,t}(\mathbf{p}_j)} R \left( \frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t'} \right) (u_j - u_i) V_i V_j + \frac{Ch}{t^{1/2}} (\sum_i u_i^2 V_i)^{1/2}
\]
\[
\leq \frac{CC_l^{1/2}}{\sqrt{|M|}} \left( \sum_{i,j} R \left( \frac{|\mathbf{x} - \mathbf{p}_j|^2}{4t'} \right) (u_j - u_j)^2 V_i V_j \right)^{1/2} + \frac{Ch}{t^{1/2}} (\sum_i u_i^2 V_i)^{1/2} \tag{11.27}
\]
where the first equality is due to \( \sum_i u_i V_i = 0 \). Denote
\[
A = \int_M C_l \frac{w_{l,t}(\mathbf{x})}{w_{l,t}(\mathbf{p}_i)} R \left( \frac{|\mathbf{x} - \mathbf{p}_i|^2}{4t'} \right) \left( \frac{|\mathbf{x} - \mathbf{p}_j|^2}{4t'} \right) d\mu_{\mathbf{x}} - \sum_j C_l \frac{w_{l,t}(\mathbf{p}_j)}{w_{l,t}(\mathbf{p}_i)} R \left( \frac{|\mathbf{p}_j - \mathbf{p}_i|^2}{4t'} \right) \left( \frac{|\mathbf{p}_j - \mathbf{p}_j|^2}{4t'} \right) V_j \tag{11.28}
\]
and then \( |A| \leq \frac{Ch}{t^{1/2}} \). At the same time, notice that only when \( |\mathbf{p}_i - \mathbf{p}_j|^2 < 16t' \) is \( A \neq 0 \). Thus we have
\[
|A| \leq \frac{1}{\delta_0} |A|R \left( \frac{|\mathbf{p}_i - \mathbf{p}_i|^2}{32t'} \right), \tag{11.29}
\]
and
\[
\left| \int_M u^2(\mathbf{x}) d\mu_{\mathbf{x}} - \sum_j u^2(\mathbf{p}_j) V_j \right|
\]
\[
\leq \sum_{i,j} |C_l u_i u_j V_i V_j||A|
\]
\[
\leq \frac{Ch}{t^{1/2}} \sum_{i,j} |C_l R \left( \frac{|\mathbf{p}_i - \mathbf{p}_i|^2}{32t'} \right) u_i u_j V_i V_j|
\]
\[
\leq \frac{Ch}{t^{1/2}} \sum_{i,j} |C_l R \left( \frac{|\mathbf{p}_i - \mathbf{p}_i|^2}{32t'} \right) u_i^2 V_i V_i|
\]
\[
\leq \frac{Ch}{t^{1/2}} \sum_{i,j} u_i^2 V_i. \tag{11.30}
\]
Now combining Equation \((11.24)\), \((11.27)\) and \((11.30)\), we have for small \( t \)
\[
\sum_i u^2(\mathbf{p}_i) V_i = \int_M u^2(\mathbf{x}) d\mu_{\mathbf{x}} + \frac{Ch}{t^{1/2}} \sum_i u_i^2 V_i
\]
\[
\leq 2 \int_M (u(\mathbf{x}) - \bar{u})^2 d\mu_{\mathbf{x}} + 2\bar{u}^2 |M| + \frac{Ch}{t^{1/2}} \sum_i u_i^2 V_i
\]
\[
\leq \frac{CC_l}{t} \sum_{i,j} R \left( \frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t} \right) (u_i - u_j)^2 V_i V_j + \frac{Ch}{t} \sum_i u_i^2 V_i. \tag{11.31}
\]
Here we use the fact that for $t = 18t'$

$$R \left( \frac{|p_i - p_j|^2}{4t'} \right) \leq \frac{1}{\delta_0} R \left( \frac{|p_i - p_j|^2}{4t} \right).$$

Let $\delta = \frac{w_{\min}}{2w_{\max} + w_{\min}}$. If $\sum_i u^2(p_i)V_i \geq \delta^2 \sum_i u_i^2 V_i$, then we have completed the proof. Otherwise, we have

$$\sum_i (u_i - u(p_i))^2 V_i = \sum_i u_i^2 V_i + \sum_i u(p_i)^2 V_i - 2 \sum_i u_i u(p_i) V_i \geq (1 - \delta)^2 \sum_i u_i^2 V_i. \quad (11.32)$$

This enables us to prove ellipticity of $\mathcal{L}$ in the case of $\sum_i u^2(p_i)V_i < \delta^2 \sum_i u_i^2 V_i$ as follows.

$$C_t \sum_{i,j} R \left( \frac{|p_i - p_j|^2}{4t'} \right) (u_i - u_j)^2 V_i V_j$$

$$= 2C_t \sum_{i,j} R \left( \frac{|p_i - p_j|^2}{4t'} \right) u_i(u_i - u_j)V_i V_j$$

$$= 2 \sum_i u_i(u_i - u(p_i)) w_{t,h}(p_i) V_i$$

$$= 2 \sum_i (u_i - u(p_i))^2 w_{t,h}(p_i) V_i + 2 \sum_i u(p_i)(u_i - u(p_i)) w_{t,h}(p_i) V_i$$

$$\geq 2 \sum_i (u_i - u(p_i))^2 w_{t,h}(p_i) V_i - 2 \left( \sum_i u^2(p_i) w_{t,h}(p_i) V_i \right)^{1/2} \left( \sum_i (u_i - u(p_i))^2 w_{t,h}(p_i) V_i \right)^{1/2}$$

$$\geq 2 w_{\min} \sum_i (u_i - u(p_i))^2 V_i - 2 w_{\max} \left( \sum_i u^2(p_i) V_i \right)^{1/2} \left( \sum_i (u_i - u(p_i))^2 V_i \right)^{1/2}$$

$$\geq 2 (w_{\min}(1 - \delta)^2 - w_{\max}(1 - \delta)) \sum_i u_i^2 V_i$$

$$\geq w_{\min}(1 - \delta)^2 \sum_i u_i^2 V_i. \quad (11.33)$$

\[\square\]

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