Scalar 1-loop Feynman integrals as meromorphic functions in space-time dimension $d$, II: special kinematics

Khiem Hong Phan$^{1,2,\text{a}}$

$^1$ University of Science, Ho Chi Minh City, 227 Nguyen Van Cu, District 5, Ho Chi Minh City, Vietnam
$^2$ Vietnam National University, Ho Chi Minh City, Linh Trung Ward, Thu Duc District, Ho Chi Minh City, Vietnam

Received: 10 February 2020 / Accepted: 29 April 2020 / Published online: 12 May 2020

© The Author(s) 2020

Abstract Based on the method developed in Phan and Riemann (Phys Lett B 791:257, 2019), detailed analytic results for scalar one-loop two-, three-, four-point integrals in general $d$-dimension are presented in this paper. The calculations are considered all external kinematic configurations and internal mass assignments. Analytic formulas are expressed in terms of generalized hypergeometric series such as Gauss $2\text{F}_1$, Appell $F_1$ and Lauricella $F_S$ functions.

1 Introduction

Scalar one-loop integrals in general $d$ are important for several reasons. In general framework for computing two-loop and higher-loop corrections, higher-terms in the $\epsilon$-expansion ($\epsilon = 2 - d/2$) for one-loop integrals are necessary for building blocks. For example, they are used for building counter-terms. Furthermore, in the evaluations for multi-loop Feynman integrals, we may combine several methods in [1,2] to optimize the master integrals. As a result, the resulting integrals may include of one-loop functions in arbitrary space-time dimensions. Last but not least, scalar one-loop functions at $d = 4 + 2n \pm 2\epsilon$ with $n \in \mathbb{N}$ may be taken into account in the reduction for tensor one-loop Feynman integrals [3].

One-loop functions in space-time dimensions $d$ have been performed in [4–9]. However, not all of the calculations have covered at general configurations of external momenta and internal masses. Recently, scalar one-loop three-point functions have been performed by applying multiple unitarity cuts for Feynman diagrams [10]. In Ref. [10], analytic results have been presented in terms of hypergeometric functions in special cases of external invariants and internal masses. In more general cases, the results have only presented $\epsilon^0$-terms in space-time $d = 4 - 2\epsilon$. The algebraic structure of cut Feynman integrals, the diagrammatic coaction and its applications have proposed in [11–13]. However, the detailed analytic results for one-loop Feynman integrals have not been shown in the mentioned papers. More recently, detailed analytic results for one-loop three-point functions which are expressed in terms of Appell $F_1$ hypergeometric functions have been reported in [15].

A recurrence relation in $d$ for Feynman loop integrals has proposed by Tarasov [2,14,16]. By solving the differential equation in $d$, analytic results for scalar one-loop integrals up to four-points have been expressed in terms of generalized hypergeometric series such as Gauss $2\text{F}_1$, Appell $F_1$, Lauricella–Saran $F_S$ functions. In [16], boundary terms have obtained by applying asymptotic theory of complex Laplace-type integrals. These terms are only valid in sub-domain of external momentum and mass configurations which the theory are applicable. Hence, the general solutions for one-loop integrals in arbitrary kinematics have not been found [16], as pointed out in [17,18]. General solutions for this problem have been derived in [19] which proposed a different kind of recursion relation for one-loop integrals in comparison with [2,14,16]. In the scope of this paper, based on the method in Ref. [19], detailed analytic results for scalar one-loop two-, three- and four-point functions in general $d$-dimensions are presented. The calculations are considered all external kinematic configurations and internal mass assignments. Thus, we go beyond the material presented in [19].

The layout of the paper is as follows: In Sect. 2, we discuss briefly the method for evaluating one-loop integrals. We then apply this method for computing scalar one-loop two-, three- and four-point functions in Sects. 3, 4, and 5. Conclusions and plans for future work are presented in Sect. 6.

2 Methods

In this section, we describe briefly the method for evaluating scalar one-loop $N$-point Feynman integrals. Detailed
description for this method can be found in Ref. [19]. A
general recursion relation between scalar one-loop \( N \)-point
and \((N-1)\)-point Feynman integrals is shown in this section.
From the representation, analytic formulas for scalar one-
loop \( N \)-point functions can be constructed from basic inte-
grals which are scalar one-point integrals. For illustrating,
analytic expressions for scalar one-loop two-, three-, four-
point functions are derived in detail in next sections.

The scalar one-loop \( N \)-point Feynman integrals are defined:

\[
J_N(d; \{ p_i, p_j \}, \{ m^2_i \}) = \int \frac{d^d k}{i \pi^{d/2}} \frac{1}{P_1 P_2 \ldots P_N}.
\]

The inverse Feynman propagators are given:

\[
P_j = (k + q_j)^2 - m_j^2 + i \rho, \quad \text{for } j = 1, 2, \ldots, N.
\]

where \( p_j \) (\( m_j \)) for \( j = 1, 2, \ldots, N \) are external momenta (internal masses) respectively. The momenta \( q_j \) are given:

\[
q_1 = p_1, q_2 = p_1 + p_2, \ldots, q_j = \sum_{i=1}^{j-1} p_i \text{ and } q_N = \sum_{j=1}^{N} p_j = 0 \text{ thanks to momentum conservation.}
\]

They are inward as described in Fig. 1. The term \( i \rho \) is Feynman’s prescription and \( d \) is space-time dimension. Several cases of physical interests for \( d \) are \( d = 4 + 2n \pm 2e \) with \( n \in \mathbb{N} \).

In the complex-mass scheme [20], the internal masses take the form of \( m^2_j = m^2_{0j} - i m_{0j} \Gamma_j \) where \( \Gamma_j \geq 0 \) are decay widths of unstable particles.

The Cayley and Gram determinants [16] related to one-
loop Feynman \( N \)-point topologies are defined as follows:

\[
Y_N \equiv Y_{12 \ldots N} = \begin{vmatrix}
Y_{11} & Y_{12} & \ldots & Y_{1N} \\
Y_{12} & Y_{22} & \ldots & Y_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{1N} & Y_{2N} & \ldots & Y_{NN}
\end{vmatrix}
\]

with \( Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2 \).

In this report, analytic solutions for one-loop integrals are expressed in terms of generalized hypergeometric with arguments given by ratios of the above determinants. Hence, it is worth to introduce the following kinematic variables

\[
R_N \equiv R_{12 \ldots N} = -\frac{Y_N}{G_{N-1}} \quad \text{for } G_{N-1} \neq 0.
\]

The kinematics \( R_N \) play a role of the squared internal masses. In fact, when we shift \( m_j^2 \to m_j^2 - i \rho \), one verifies easily that \( R_N \to R_N - i \rho \) [16].

The recursion relation for \( J_N \) [19] is given (master equation):

\[
J_N(d; \{ p_i, p_j \}, \{ m^2_i \}) = \frac{1}{2 \pi i} \int_{-i \infty}^{+i \infty} ds \times \frac{\Gamma(-s)}{\Gamma\left(\frac{d-N+1}{2} + s\right) \Gamma(s + 1)} \left(\frac{1}{R_N}\right)^s \times \sum_{k=1}^{N} \left(\frac{\partial_k R_N}{R_N}\right) k^- J_N(d + 2s; \{ p_i, p_j \}, \{ m^2_i \}),
\]

for \( i, j = 1, 2, \ldots, N \) and \( \partial_k = \partial / \partial m_k^2 \). Here the operator \( k^- \) is defined as [19]

\[
k^- J_N(d; \{ p_i, p_j \}, \{ m^2_i \}) = \int \frac{d^d k}{i \pi^{d/2}} \frac{1}{P_1 P_2 \ldots P_{k-1} P_{k+1} \ldots P_{N-1} P_N}
\]

The relation (5) indicates that the integral \( J_N \) can be con-
bstructed by taking one-fold Mellin–Barnes (MB) integration
over \( J_{N-1} \) in \( d + 2s \). This representation has several ad-vantages. First, analytic formulas for \( J_N \) can be derived from basic functions which are scalar one-loop one-point func-
tions. Second, \( J_N \) is expressed as functions of kinematic variables such as \( m_j^2 \) for \( j = 1, 2, \ldots, N \) and \( R_N \). As a con-sequence of this fact, analytic expressions for \( J_N \) reflect the symmetry as well as threshold behavior of the corresponding Feynman topologies. Two special cases of (5) are also men-tioned as follows:

1. \( Y_N \to 0 \) and \( G_{N-1} \neq 0 \): In this case, \( R_N \to 0 \) and we have [16] (deriving this equation for \( N = 4 \) is shown in the Appendix D)

\[
J_N(d; \{ p_i, p_j \}, \{ m^2_i \}) = \frac{1}{d - N - 1}
\]
\begin{equation}
\sum_{k=1}^{N} \left( \frac{\partial Y_{N}}{G_{N-1}} \right) k^{-J_{N}(d-2; \{ p_{i} p_{j} \}, \{ m_{1}^{2} \})}.
\end{equation}

2. \( G_{N-1} \to 0 \) and \( Y_{N} \neq 0 \): In this case, \( R_{N} \to \infty \). We close the integration contour in (5) to the right. Taking residue contributions from poles of \( \Gamma (\ldots - s) \). In the limit \( R_{N} \to \infty \), we find only the term with \( s = 0 \) is non-zero. The result then reads

\begin{equation}
J_{N}(d; \{ p_{i} p_{j} \}, \{ m_{1}^{2} \}) = -\frac{1}{2} \sum_{k=1}^{N} \left( \frac{\partial Y_{N}}{Y_{N}} \right) k^{-J_{N}(d; \{ p_{i} p_{j} \}, \{ m_{1}^{2} \})}.
\end{equation}

This equation is equivalent to (65) in [21] and (3) in [16].

We turn our attention to apply the method for evaluating scalar one-loop Feynman integrals. The detailed evaluations for scalar one-loop two-, three- and four-point functions are presented in next sections. As we pointed out in this section, the prescription \( i \rho \) always follows with \( R_{N} = R_{N} - i \rho \). In order to simplify the notation, we omit \( i \rho \) in \( R_{N} \) in the next calculations. This term puts back into the final results when it is necessary.

3 One-loop two-point functions

The master equation for \( J_{2} \) is obtained by setting \( N = 2 \) in (5). MB representation for \( J_{2} \) then reads

\begin{equation}
J_{2}(d; p_{i}^{2}, m_{1}^{2}, m_{2}^{2}) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} ds \left( \Gamma(-s) \Gamma \left( \frac{d-2}{2} - s \right) \Gamma \left( \frac{d-1}{2} + s \right) \Gamma (s + 1) \right)
\end{equation}

\begin{equation}
\times \left( \frac{R_{2}}{R_{2}} \right)^{2} \left( \frac{d}{2} \right)^{d/2 + s} + (1 \leftrightarrow 2) \right),
\end{equation}

Note that we used the analytic formula for \( J_{1} \) in [22] with \( d \) shifted to \( d \to d + 2s \). We write \( J_{1} \) in \( d + 2s \) explicitly as follows:

\begin{equation}
J_{1}(d + 2s; m^{2}) = -\Gamma \left( \frac{2 - d}{2} - s \right) \left( m^{2} \right)^{d/2 + s}.
\end{equation}

In order to evaluate the MB integrals in (9), we close the integration contour to the right. The residue contributions to \( J_{2} \) at the sequence poles of \( \Gamma (-s) \) and \( \Gamma \left( \frac{d-2}{2} - s \right) \) are taken into account.

First, we calculate the residue at the poles of \( \Gamma (-s) \). In this case, \( s = m \) for \( m = 0, 1, \ldots, \mathbb{N} \). Subsequently, we can apply the reflect formula for gamma functions (103) in the

Appendix B. In detail, it is implied that

\begin{equation}
\Gamma \left( \frac{2 - d}{2} - s \right) \Gamma \left( \frac{d}{2} + s \right) = -(-1)^{s} \Gamma \left( \frac{4 - d}{2} \right) \Gamma \left( \frac{d - 2}{2} \right).
\end{equation}

With the help of (11), the MB representation in (9) is casted into the form of

\begin{equation}
J_{2} \left( \Gamma \left( \frac{4-d}{2} \right) \right) \bigg|_{s=m} = -\frac{\Gamma \left( \frac{d+1}{2} \right)}{2 \Gamma \left( \frac{d-1}{2} \right)} \times \frac{1}{2\pi i} \int_{-\infty}^{+\infty} ds \left( \Gamma(-s) \Gamma \left( \frac{d-1}{2} + s \right) \Gamma (s + 1) \right)
\end{equation}

\begin{equation}
\times \left( \frac{\partial R_{2}}{R_{2}} \right)^{2} (m_{1}^{2})^{d/2} + \left( 1 \leftrightarrow 2 \right).
\end{equation}

Using (1.6.1.6) in Ref. [25], these MB integrals are expressed in terms of Gauss hypergeometric series as follows:

\begin{equation}
J_{2} \left( \Gamma \left( \frac{4-d}{2} \right) \right) \bigg|_{s=m} = -\frac{\Gamma \left( \frac{d+1}{2} \right)}{2 \Gamma \left( \frac{d-1}{2} \right)} \times \left( \frac{\partial R_{2}}{R_{2}} \right)^{2} (m_{1}^{2})^{d/2} + \left( 1 \leftrightarrow 2 \right)
\end{equation}

provided that \( |m_{1}^{2}/R_{2}| < 1, |m_{2}^{2}/R_{2}| < 1 \) and \( \Re (d-2) > 0 \). Using (1.3.15) in [25], we arrive at another representation for (13)

\begin{equation}
J_{2} \left( \Gamma \left( \frac{4-d}{2} \right) \right) \bigg|_{s=m} = -\frac{\Gamma \left( \frac{d+1}{2} \right)}{2 \Gamma \left( \frac{d-1}{2} \right)} \times \left( \frac{\partial R_{2}}{R_{2}} \right)^{2} (m_{1}^{2})^{d/2} \sqrt{1 - m_{1}^{2}/R_{2}}
\end{equation}

\begin{equation}
\times \left( \frac{d}{2} \right)^{d/2} + \left( 1 \leftrightarrow 2 \right)
\end{equation}

provided that \( |m_{1}^{2}/R_{2}| < 1, |m_{2}^{2}/R_{2}| < 1 \) and \( \Re (d-2) > 0 \).

We next consider the residue at the second sequence poles of \( \Gamma \left( \frac{2-d}{2} - s \right) \). In this case, \( s = \frac{d-2}{2} + m \) for \( m \in \mathbb{N} \). These contributions read

\begin{equation}
J_{2} \bigg|_{s=\frac{d-2}{2}+m} = \sum_{m=0}^{\infty} \frac{(-1)^{m} \Gamma \left( \frac{d-2}{2} - m \right) \Gamma \left( \frac{d-1}{2} + m \right) \Gamma \left( m + \frac{1}{2} \right)}{m!} \times \left( \frac{\partial R_{2}}{R_{2}} \right)^{2} \left( \frac{m_{1}^{2}}{R_{2}} \right)^{m}
\end{equation}
\[
\frac{J_2}{\Gamma \left( \frac{4-d}{2} \right)} = \frac{1}{d-3} \left[ \frac{(m_1^2)^{d/2}}{(m_1^2 \pm m_2^2)^{d/2}} \right] + (1 \leftrightarrow 2) \tag{21}
\]

In the limit of \(m_1 \to m_2 = m\) and \(p^2 = 4m^2\), the result reads

\[
J_2 = \frac{\Gamma \left( \frac{4-d}{2} \right)}{2(d-3) (m^2)^{d/2}}. \tag{22}
\]

Following (8) for \(N = 2\), we arrive at

\[
J_2 = \Gamma \left( \frac{4-d}{2} \right) (m_1^2)^{d/2} \ _2F_1 \left[ \frac{4-d}{2}, \frac{1}{2} \ ; \ 1 - \frac{m_1^2}{m_2^2} \right] = \Gamma \left( \frac{2-d}{2} \right) \ _2F_1 \left[ \frac{2-d}{2}, \frac{1}{2} \ ; \ 1 - \frac{m_1^2}{m_2^2} \right]. \tag{23}
\]

If \(m_1^2 = m_2^2\), one presents \(J_2\) as

\[
J_2 = \Gamma \left( \frac{4-d}{2} \right) (m^2)^{d/2}. \tag{24}
\]

3.3 \(R_2 = m_1^2\) or \(R_2 = m_2^2\)

For the case of \(R_2 = m_1^2\) or \(R_2 = m_2^2\), one relies on (20). If \(R_2 = m_1^2 = m_2^2 = m^2\), the result in (20) simplifies to

\[
J_2 = \Gamma \left( \frac{4-d}{2} \right) (m^2)^{d/2}. \tag{25}
\]

Under the condition \(\Re(d-4) > 0\), when \(m^2 \to 0\), one then gets \(J_2 = 0\).
3.4 $m_1^2 = m_2^2 = 0$

If one of $m_i^2 = 0$, for $i = 1, 2$, we rely on (18). In the case of $m_1^2 = m_2^2 = 0$, from (18) we get

$$J_2 = \frac{\sqrt{\pi}}{2} \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4},$$

(26)

provided that $\Re e(d - 2) > 0$. We note that $p^2$ means $p^2 + i\rho$. Therefore, if $p^2 > 0$ the term $\left( - p^2/4 \right) \frac{d-4}{4}$ is well-defined.

4 One-loop three-point functions

Setting $N = 3$ in (5), master equation for $J_3$ reads

$$J_3 = J_3(d; \{p_i^2\}, \{m_i^2\}) = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \Gamma(-s) \Gamma \left( \frac{d-2}{2} + s \right) \Gamma(s + 1) \left( \frac{1}{R_5} \right)^s \times \sum_{k=1}^{3} \left( \frac{\partial_i R_3}{R_3} \right) k^3 J_3(d + 2s; \{p_i^2\}, \{m_i^2\}),$$

(27)

for $i = 1, 2, 3$. The term $k^3 J_3(d + 2s; \{p_i p_j\}, \{m_i^2\})$ becomes scalar one-loop two-point functions by shrinking an propagator $k$-th in the integrand of $J_3$. In the next steps, we take the contour integrals in (27). In order to understand how to take the contour integrals, we chose the term with $k = 3$ in (27) for illustrating. This term is written explicitly as follows:

$$J_{3,(123)} = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \left[ \Gamma(-s) \Gamma(s + 1) \Gamma \left( \frac{d-2}{2} + s \right) \frac{1}{2 \Gamma \left( \frac{d-2}{2} \right)} \times \left( \frac{\partial_3 R_3}{R_3} \right)^s J_2(d + 2s; \{p_i^2\}, \{m_i^2\}) \right] \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4},$$

(28)

$$= -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \sqrt{\pi} \Gamma(-s) \Gamma(s + 1) \Gamma \left( \frac{d-2}{2} + s \right) \Gamma \left( \frac{d-2}{2} + s \right) \Gamma \left( \frac{d-4}{2} + s \right) \times \left( \frac{\partial_3 R_3}{R_3} \right)^s \int_{-i\infty}^{+i\infty} ds \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4} \times \left( \frac{\partial_2 R_12}{R_3} \right)^s \frac{\partial_2 R_12}{\sqrt{1 - m_1^2/R_12}} + \frac{\partial_1 R_12}{\sqrt{1 - m_2^2/R_12}} \times (R_{12}) \frac{d-4}{4} \times \left( \frac{\partial_3 R_3}{R_3} \right) \frac{\partial_3 R_3}{R_3} + \frac{\partial_1 R_12}{\sqrt{1 - m_2^2/R_12}} \times (R_{12}) \frac{d-4}{4} \times \left( \frac{\partial_3 R_3}{R_3} \right)^s \int_{-i\infty}^{+i\infty} ds \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4} \times \left( \frac{\partial_3 R_3}{R_3} \right)^s \int_{-i\infty}^{+i\infty} ds \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4},$$

(29)

For taking the MB integrals in (29), one closes the integration contour to the right. The residue contributions at the poles of the $\Gamma(-s) \Gamma(s + 1) \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right)$ are taken into account.

First, the contributions of residua at the poles $s = m$ with $m \in \mathbb{N}$. In this case, one first applies the reflect formula (103) for gamma function:

$$\Gamma \left( \frac{4 - d}{2} - s \right) \Gamma \left( \frac{4 - d}{2} + s \right) = (-1)^s \Gamma \left( \frac{4 - d}{2} \right) \Gamma \left( \frac{d-2}{2} \right).$$

(30)

Using this identity, the first MB integration reads

$$J_{3,(123)}^{1-\text{term}} \left( \Gamma \left( \frac{4-d}{2} \right) \right) \bigg|_{s=m} = -\frac{\sqrt{\pi}}{4} \left( \frac{\partial_3 R_3}{R_3} \right) \left[ \frac{\partial_2 R_12}{\sqrt{1 - m_1^2/R_12}} + \frac{\partial_1 R_12}{\sqrt{1 - m_2^2/R_12}} \right] \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4} \times \left( \frac{\partial_3 R_3}{R_3} \right)^s \int_{-i\infty}^{+i\infty} ds \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4} \times \left( \frac{\partial_3 R_3}{R_3} \right)^s \int_{-i\infty}^{+i\infty} ds \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4},$$

(31)

This MB integral is then expressed in terms of hypergeometric $\frac{d}{2}$ as follows:

$$J_{3,(123)}^{1-\text{term}} \left( \Gamma \left( \frac{4-d}{2} \right) \right) \bigg|_{s=m} = -\frac{\sqrt{\pi}}{4} \left( \frac{\partial_3 R_3}{R_3} \right) \left[ \frac{\partial_2 R_12}{\sqrt{1 - m_1^2/R_12}} + \frac{\partial_1 R_12}{\sqrt{1 - m_2^2/R_12}} \right] \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4} \times \left( \frac{\partial_3 R_3}{R_3} \right)^s \int_{-i\infty}^{+i\infty} ds \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4} \times \left( \frac{\partial_3 R_3}{R_3} \right)^s \int_{-i\infty}^{+i\infty} ds \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4},$$

(32)

provided that $|R_{12}/R_3| < 1$ and $\Re e(d - 3) > 0$. The second MB integral reads

$$J_{3,(123)}^{2-\text{term}} \left( \Gamma \left( \frac{4-d}{2} \right) \right) \bigg|_{s=m} = -\frac{\sqrt{\pi}}{4} \left( \frac{\partial_3 R_3}{R_3} \right) \left[ \frac{\partial_2 R_12}{\sqrt{1 - m_1^2/R_12}} + \frac{\partial_1 R_12}{\sqrt{1 - m_2^2/R_12}} \right] \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4} \times \left( \frac{\partial_3 R_3}{R_3} \right)^s \int_{-i\infty}^{+i\infty} ds \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4} \times \left( \frac{\partial_3 R_3}{R_3} \right)^s \int_{-i\infty}^{+i\infty} ds \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \left( - p^2 \right) \frac{d-4}{4},$$

(33)
\[ J_{123}^{(d)} = \frac{\sqrt{\pi} \Gamma \left( \frac{d-2}{2} \right)}{4 \Gamma \left( \frac{d-1}{2} \right)} \left[ \frac{\partial_2 R_{12}}{\sqrt{1 - m_1^2/R_{12}}} + (1 \leftrightarrow 2) \right] \]

\[ \times \left( \frac{\partial_3 R_3}{R_3} \right) \left[ \frac{\partial_1 R_{12}}{\sqrt{1 - m_1^2/R_{12}}} + \frac{\partial_1 R_3}{\sqrt{1 - m_1^2/R_{12}}} \right] \left( R_{12} \right)^{d-4} \]

\[ \times F_1 \left( \frac{1}{2}; \frac{1}{2}; 1; \frac{m_1^2}{R_3 \sqrt{1 - m_1^2/R_{12}}} \right) + (1 \leftrightarrow 2) \right] \]

provided that \(|m_1^2/R_3| < 1, |m_1^2/R_{12}| < 1 \) and \( |R_{12}/R_3| < 1 \).

Summing all the above contributions, the final result for \( J_3 \) is written as a compact form

\[ J_3 \left( \frac{d}{2} \right) = -J_{123}^{(d=4)} \left( R_3 \right)^{d-4} + J_{123}^{(d)} \]

\[ + \left\{ (1, 2, 3) \leftrightarrow (2, 3, 1) \right\} \]

\[ + \left\{ (1, 2, 3) \leftrightarrow (3, 1, 2) \right\} \]

where \( J_{123}^{(d)} \) is obtained from (32) and (33). It is given by

\[ J_{123}^{(d=4)} = \frac{\sqrt{\pi} \Gamma \left( \frac{d-2}{2} \right)}{4 \Gamma \left( \frac{d-1}{2} \right)} \left[ \frac{\partial_2 R_{12}}{\sqrt{1 - m_1^2/R_{12}}} + (1 \leftrightarrow 2) \right] \]

\[ \times \left( \frac{\partial_3 R_3}{R_3} \right) \left[ \frac{\partial_1 R_{12}}{\sqrt{1 - m_1^2/R_{12}}} + \frac{\partial_1 R_3}{\sqrt{1 - m_1^2/R_{12}}} \right] \left( R_{12} \right)^{d-4} \]

\[ \times F_1 \left( \frac{1}{2}; \frac{1}{2}; 1; \frac{m_1^2}{R_3 \sqrt{1 - m_1^2/R_{12}}} \right) + (1 \leftrightarrow 2) \right] \]

(39)

provided that \(|m_1^2/R_3| < 1, |m_1^2/R_{12}| < 1 \) and \( |R_{12}/R_3| < 1 \).
One-fold integral and all transformations for $F_1$ can be found in Appendix B. Applying the relation (122) for $F_1$ in Appendix C, we arrive at another representation for (39, 40):

$$J_{123}^{(d-4)} = \frac{(\partial_3 R_3) (\partial_2 R_{12})}{2(m_1^2 - R_3)} \frac{\text{d}^d}{\text{d}^d x} F_1 \left(1; \frac{4-d}{2}, 1; \frac{3}{2}; 1 - \frac{R_{12} - m_1^2}{m_1^2}, \frac{R_{12} - m_1^2}{m_1^2} \right)$$

$$+ (1 \leftrightarrow 2), \quad (41)$$

$$J_{123}^{(d-4)} = \frac{(\partial_3 R_3) (\partial_2 R_{12})}{2(m_1^2 - R_3)} \times 2 F_1 \left[1, \frac{3}{2}; \frac{R_{12} - m_1^2}{R_3 - m_1^2} \right]$$

provided that the absolute value of the arguments of $2 F_1$ and the Appell functions $F_1$ in this presentation are less than 1.

4.1 Massless internal lines

For the massless case, under the condition $\Re (d - 2) > 0$, all terms related to Appell $F_1$ functions in (38) vanish, the result then reads

$$\frac{J_3}{\Gamma \left(\frac{4-d}{2}\right)} = - \frac{(3 R_3)}{2 R_3} \left(\frac{\partial_3 R_3}{R_3} \right) \frac{\text{d}^d}{\text{d}^d x} F_1 \left[1, \frac{3}{2}; \frac{R_{12}}{R_3} \right]$$

$$+ \frac{\sqrt{\pi}}{4} \frac{\Gamma \left(\frac{d-2}{2}\right)}{\Gamma \left(\frac{d-1}{2}\right)} \left(\frac{\partial_3 R_3}{R_3} \right) \frac{\text{d}^d}{\text{d}^d x} F_1 \left[1, \frac{3}{2}; \frac{R_{12}}{R_3} \right]$$

$$\times (R_3) \frac{\text{d}^d}{\text{d}^d x} F_1 \left[1, \frac{3}{2}; \frac{R_{12}}{R_3} \right]$$

$$+ \left\{1, 2, 3 \rightarrow (2, 3, 1) \right\}$$

$$+ \left\{1, 2, 3 \rightarrow (3, 1, 2) \right\}, \quad (43)$$

In order to cross check with the result in [6], we write $J_3$ as a function of $p_1^2, p_2^2, p_3^2$ explicitly

$$\frac{J_3}{\Gamma \left(\frac{4-d}{2}\right)} = - \left(\frac{p_1^2 + p_2^2 - p_3^2}{2 p_2^2 p_3^2} \right) \left(\frac{p_1^2 p_2^2 p_3^2}{\lambda(p_1^2, p_2^2, p_3^2)} \right) \frac{\text{d}^d}{\text{d}^d x} F_1 \left[1, \frac{3}{2}; \frac{-\lambda(p_1^2, p_2^2, p_3^2)}{4 p_2^2 p_3^2} \right]$$

$$+ \frac{\sqrt{\pi}}{4} \frac{\Gamma \left(\frac{d-2}{2}\right)}{\Gamma \left(\frac{d-1}{2}\right)} \left(\frac{p_1^2 + p_2^2 - p_3^2}{p_2^2 p_3^2} \right) \left(\frac{-p_1^2}{4} \right) \frac{\text{d}^d}{\text{d}^d x} F_1 \left[1, \frac{3}{2}; \frac{-\lambda(p_1^2, p_2^2, p_3^2)}{4 p_2^2 p_3^2} \right]$$

$$+ \left\{1, 2, 3 \rightarrow (2, 3, 1) \right\}$$

$$+ \left\{1, 2, 3 \rightarrow (3, 1, 2) \right\}.$$

4.2 $R_{ij} = 0$

We consider the terms in $J_3$ with $R_{12} = 0$ as an example. In this case, the terms $J_{123}^{(d)}$ and $J_{123}^{(d)}$ are given in the same form of (39) or (41). While the term $J_{123}^{(d)}$ is obtained by performing analytic continuation the result in (41). In detail, one takes the limit of $R_{12} \rightarrow 0$ in (41), we arrive at

$$J_{123}^{(d)} = \frac{(\partial_3 R_3) (\partial_2 R_{12})}{2(m_1^2 - R_3)} \frac{\text{d}^d}{\text{d}^d x} F_1 \left[1; \frac{4-d}{2}, 1; \frac{3}{2}; \frac{1}{m_1^2}, \frac{m_1^2}{m_1^2 - R_3} \right]$$

$$+ (1 \leftrightarrow 2). \quad (46)$$

Using (36) in Ref. [30], the term $J_{123}^{(d)}$ simplifies to

$$J_{123}^{(d)} = \frac{\Gamma \left(\frac{d-3}{2}\right)}{\Gamma \left(\frac{d-2}{2}\right)} \left(\frac{\partial_3 R_3}{4 R_3} \right) \frac{\text{d}^d}{\text{d}^d x} F_1 \left[1, \frac{3}{2}; \frac{m_1^2}{R_3} \right]$$

$$+ \left\{1, 2, 3 \rightarrow (2, 3, 1) \right\}$$

$$+ \left\{1, 2, 3 \rightarrow (3, 1, 2) \right\}.$$

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

is the Källén function. We remark that $p_i^2 \rightarrow p_i^2 + ip$ in this formula. With applying (1.3.3.5) in Ref. [25], one can present $J_3$ as

$$\frac{J_3}{\Gamma \left(\frac{4-d}{2}\right)} = \frac{2}{(p_1^2 - p_2^2 - p_3^2)} \left(\frac{p_1^2 p_2^2 p_3^2}{\lambda(p_1^2, p_2^2, p_3^2)} \right) \frac{\text{d}^d}{\text{d}^d x} F_1 \left[1, \frac{3}{2}; \frac{\lambda(p_1^2, p_2^2, p_3^2)}{(p_2^2 + p_3^2 - p_1^2)^2} \right]$$

$$+ \left\{1, 2, 3 \rightarrow (2, 3, 1) \right\}$$

$$+ \left\{1, 2, 3 \rightarrow (3, 1, 2) \right\}, \quad (45)$$

provided that $\frac{\lambda(p_1^2, p_2^2, p_3^2)}{(p_2^2 + p_3^2 - p_1^2)^2} < 1$ and $\Re (d - 2) > 0$.

This equation is equivalent to (10) in [6]. We note that we can arrive to this result by inserting $J_3$ at $d + 2$ in (26) into (27) and taking the corresponding MB integrals.
4.3 $R_{ij} = m_{\ell(j)}^2$ for $i, j = 1, 2, 3$

Next we consider $R_{12} = m_1^2$ as an example. In this case, the terms $J_{231}^{(d)}$, $J_{312}^{(d)}$ are given by (39). Beside that, one verifies
\[ \partial_2 R_{12} = 0. \]  
(49)

As a result, we obtain
\[ J_{123}^{(d=4)} = \frac{(\partial_3 R_3) (\partial_1 R_{12})}{2(m_2^2 - R_3)} {\left. 2 F_1 \left( 1, \frac{3}{2}; \frac{R_3 - m_2^2}{R_3 - m_2^2} \right) \right|}_{m_2^2}^{m_2^2} \]
(50)
\[ J_{123}^{(d)} = \frac{(\partial_3 R_3) (\partial_1 R_{12})}{2(m_2^2 - R_3)} \times F_1 \left( 1; \frac{4 - d}{2}, 1; \frac{3}{2}; \frac{1 - m_1^2}{m_2^2}, \frac{R_3 - m_2^2}{R_3 - m_2^2} \right) \]
(51)
provided that the amplitude of arguments of hypergeometric functions appearing in this formula are less than 1. For $R_{12} = m_1^2 = m_2^2$, the function $F_1$ in (50) is equal 1. The result reads
\[ J_{123}^{(d)} = \frac{(\partial_3 R_3) (m_2^2 - R_3)^{\frac{d-4}{2}}}{2(m_2^2 - R_3)}. \]
(52)

4.4 $R_3 = m_k^2$ for $k = 1, 2, 3$

As an example, consider the terms of $J_3$ in (38) with $R_3 = m_1^2$. One verifies that
\[ \partial_1 R_3 = 1, \quad \partial_i R_3 = 0, \quad \text{for} \quad i = 2, 3. \]
(53)

As a result, $J_3$ is casted into the form of
\[ \frac{J_3}{G_2} = -J_{231}^{(d=4)} (R_3)^{\frac{d-4}{2}} + J_{231}^{(d)}, \]
(54)
with $J_{231}^{(d)}$ taking the same form of (39) or (41). We take (41) as example for $J_{231}^{(d)}$. In detail, it takes
\[ J_{231}^{(d)} = \frac{(\partial_1 R_3) (\partial_3 R_{23})}{2(m_2^2 - R_3)} m_2^2 \times F_1 \left( 1; \frac{4 - d}{2}, 1; \frac{3}{2}; 1 - \frac{R_{23}^2}{m_2^2}, \frac{R_{23}^2 - m_2^2}{R_3 - m_2^2} \right) \]
(55)

Taking $d \rightarrow 4$, the result reads
\[ J_{231}^{(d=4)} = \frac{(\partial_1 R_3) (\partial_3 R_{23})}{2(m_2^2 - R_3)} m_2^2 \times 2 F_1 \left( 1, \frac{3}{2}; \frac{R_{23}^2 - m_2^2}{R_3 - m_2^2} \right) + (2 \leftrightarrow 3). \]
(56)

4.5 $G_2 \neq 0$ and $R_3 = 0$

By setting $N = 3$ in (7), one obtains
\[ J_3 = \frac{1}{(d - 4)} \sum_{k=1}^{3} \left( \frac{\partial_k Y_3}{G_2} \right) k^{-3} J_3(d - 2; \{ p_k^2 \}, \{ m_k^2 \}) . \]
(57)
The resulting of $k^{-3} J_3(d - 2; \{ p_k^2 \}, \{ m_k^2 \})$ is $J_2$ in (18) or in (20) with $d \rightarrow d - 2$. As an example, we take $J_2$ in (20) at $d - 2$. The result reads
\[ \frac{J_3}{(d-3)^2} = \frac{1}{(4 - d)} \left( \frac{\partial_3 Y_3}{G_2} \right) \left( \left( \frac{\partial_1 R_{12}}{R_{12}} \right) (R_{12})^{\frac{d-4}{2}} \right) \times 2 F_1 \left( \frac{3 - d}{2}, 1; 1 - \frac{m_1^2}{R_{12}} \right) + (1 \leftrightarrow 2) \]
\[ + \{(1, 2, 3) \leftrightarrow (2, 3, 1)\} \]
\[ + \{(1, 2, 3) \leftrightarrow (3, 1, 2)\}. \]
(58)

We can confirm (57) again by using analytic continuation result of $J_3$ in (41). In fact, when $R_3 \rightarrow 0$ the Eq. (41) becomes
\[ J_{123}^{(d)} = \frac{-\partial_1 R_3}{2} \left( \frac{\partial_3 R_{12}}{G_2} \right) \left( \left( \frac{\partial_1 R_{12}}{R_{12}} \right) (R_{12})^{\frac{d-4}{2}} \right) \]
\[ \times 2 F_1 \left( 1, \frac{4 - d}{2}, 1; \frac{3}{2}; 1 - \frac{R_{12}}{m_1^2}, 1 - \frac{R_{12}}{m_1^2} \right) + (1 \leftrightarrow 2) \]
(59)
\[ = \frac{1}{2} \left( \frac{\partial_1 R_3}{G_2} \right) \left( \frac{\partial_3 R_{12}}{G_2} \right) \left( \left( \frac{\partial_1 R_{12}}{R_{12}} \right) (R_{12})^{\frac{d-4}{2}} \right) \times 2 F_1 \left( \frac{3 - d}{2}, 1; 1 - \frac{m_1^2}{R_{12}} \right) + (1 \leftrightarrow 2) \]
(60)
\[ = \frac{-1}{2} \left( \frac{\partial_1 R_3}{G_2} \right) \left( \frac{\partial_3 R_{12}}{G_2} \right) \left( \left( \frac{\partial_1 R_{12}}{R_{12}} \right) (R_{12})^{\frac{d-4}{2}} \right) \times 2 F_1 \left( \frac{3 - d}{2}, 1; 1 - \frac{m_1^2}{R_{12}} \right) + (1 \leftrightarrow 2) \] .
(61)

Plugging (61) into (38), we arrive at (58).

4.6 $R_3 = R_{ij}$ for $i, j = 1, 2, 3$

We consider the case of $R_3 = R_{ij}$ as an example. In this case, one verifies that
\[ \partial_3 R_3 = 0. \]
(62)

As a result, $J_3$ is
\[ \frac{J_3}{\Gamma \left( \frac{4-d}{2} \right)} = -J_{231}^{(d=4)} (R_3)^{\frac{d-4}{2}} + J_{231}^{(d)} \]
\[ + \{(2, 3, 1) \leftrightarrow (3, 1, 2)\}. \]
(63)
Here \( J_{231}^{(d)} \) takes the same form as (39) or (41). We take (41) as example for \( J_{231}^{(d)} \). In detail, it takes

\[
J_{231}^{(d)} = \frac{\partial_1 R_3}{2(m_2^2 - R_3)} \left( \frac{d-d}{2} \right) \left( m_2^2 \right) \mathcal{F} \left( \begin{array}{c}
\frac{1}{2} \times F_1 \left( 1, \frac{3}{2} ; \frac{3}{2} - 1 \right) R_3 - m_2^2
\end{array} \right) + (2 \leftrightarrow 3),
\]

(64)

4.7 \( G_2 = 0 \)

Setting \( N = 3 \) in (8), the result reads

\[
J_3 = -\frac{1}{2} \sum_{k=1}^{3} \left( \partial_2 Y_3 \right) Y_3 J_3(d; \{ p_1^2 \}, \{ m_i^2 \}).
\]

(65)

This equation is equivalent to (46) in Ref. [21]. The term \( K = J_2(d; \{ p_i^2 \}, \{ m_i^2 \}) \) corresponds to \( J_2 \) in (20) as an example. We obtain

\[
J_3 = -\frac{1}{2} \sum_{k=1}^{3} \left( \partial_2 Y_3 \right) Y_3 J_3(d; \{ p_1^2 \}, \{ m_i^2 \}) \left( \frac{d-d}{2} \right) \left( m_2^2 \right) \mathcal{F} \left( \begin{array}{c}
\frac{1}{2} \times F_1 \left( 1, \frac{3}{2} ; \frac{3}{2} - 1 \right) R_3 - m_2^2
\end{array} \right) + (1 \leftrightarrow 2) + \left( 1, 2, 3 \right) \leftrightarrow (2, 3, 1)
\]

(66)

4.8 \( G_{1,i,j} = 0 \) for \( i, j = 1, 2, 3 \)

\( G_{1,i,j} \) are the Gram determinants of two-point functions which are obtained by shrinking a propagator \( k \neq i, j \) in the three-point ones. Taking \( G_{1,12} = 0 \) as an example, the term \( J_{123}^{(d)} \) is evaluated as follows. We put \( J_2 \) in (23) into (27).

Taking the corresponding MB integrals, the results read as form of (38) with

\[
J_{123}^{(d)} = -\frac{\partial_1 R_3}{2R_3} \left( \frac{2}{d} \right) \mathcal{F} \left( \begin{array}{c}
\frac{1}{2} \times F_1 \left( 1, \frac{3}{2} ; \frac{3}{2} - 1 \right) R_3 - m_2^2
\end{array} \right) + (1 \leftrightarrow 2),
\]

(67)

It is valid under the conditions that the arguments of the hypergeometric functions appearing in this formula are less that 1 and \( \text{Re}(d-2) > 0 \).

4.9 Cross check with other papers

We consider \( p_1^2 = p_3^2 = 0; p_1^2 \neq 0, m_1 = m_3 = 0 \) and \( m_2 \neq 0 \) as an example [10,15]. We confirm that

\[
R_3 = \frac{m_2^2 (p_1^2 + m_3^2)}{p_1^2}, \quad \partial_1 R_3 = \partial_3 R_3
\]

(68)

\[
= -\frac{1}{p_1^2 + m_3^2}, \quad R_3
\]

(69)

\[
= \frac{2m_2^2 + p_1^2}{m_2^2 (p_1^2 + m_3^2)}.
\]

(70)

The \( J_3 \) in (27) becomes

\[
J_3 = -\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d\gamma}{\Gamma(-\gamma)} \Gamma(s+1) \Gamma \left( \frac{d-2}{2} + s \right) \frac{1}{\gamma} \Gamma \left( \frac{d-2}{2} \right)
\]

(71)

\[
\times \left( \frac{1}{R_3} \right)^\gamma \frac{\sqrt{\pi}}{2} \frac{2m_2^2 + p_1^2}{m_2^2 (m_3^2 + p_1^2)} \Gamma \left( \frac{d-2}{2} + s \right) \frac{1}{\gamma} \Gamma \left( \frac{d-2}{2} \right)
\]

(72)

\[
\times \left( \frac{-p_1^2}{4} \right)^\frac{d-4+s}{2} - \frac{2}{\Gamma \left( \frac{d-2}{2} + s \right)} \frac{1}{\gamma} \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-4}{2} + s \right) \frac{1}{\gamma} \Gamma \left( \frac{d-4}{2} \right)
\]

(73)

\[
\times \frac{\frac{1}{\gamma} \Gamma \left( \frac{d-2}{2} + s \right) \Gamma \left( \frac{d-4}{2} + s \right) m_2^2}{2 \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-4}{2} \right)}.
\]

(74)
Taking into account the residue at the poles, we arrive at

\[ J_3^{(1,b)} = -\frac{\Gamma \left( \frac{d}{2} - 2 \right) \Gamma \left( 3 - \frac{d}{2} \right)}{2\Gamma \left( \frac{d}{2} - 1 \right)} \left( R_3 \right)^{4-z-2} \]

\[ \times 2F1 \left[ \frac{1}{2}; \frac{p_1^2 - \Gamma_1^1}{m_2^2} \right] \]  

(75)

Second type of MB integral is considered

\[ J_3^{(2)} = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\sqrt{\pi}}{2} \frac{\Gamma(\gamma-s) \Gamma(\gamma+\delta-s) \Gamma(\gamma+\epsilon-s)}{\Gamma(\gamma+\delta) \Gamma(\gamma+\epsilon)} \left( \frac{s}{d-2} \right)^{2} \frac{1}{\Gamma(d-s)} \]

\[ \times \left( \frac{1}{R_3} \right)^{2-s} \left( -\frac{p_1^2}{4} \right)^{2-s} \]  

(77)

For the first sequence poles of \( \Gamma(s) \), the result is

\[ J_3^{(2,a)} = -\frac{\sqrt{\pi}}{2} \frac{\Gamma \left( \frac{d}{2} - 1 \right) \Gamma \left( 2 - \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - \frac{1}{2} \right)} \frac{-p_1^2}{4} \left( \frac{d}{2} - 2 \right) \]

\[ \times 2F1 \left[ 1, \frac{d-2}{d-1}; -\frac{p_1^2}{4m_2^2} \right] \]  

(79)

Applying transforms for Gauss hypergeometric function which are (see (1.8.10) in [25] for the first relations and page 49, [26] for the later case)

\[ 2F1 \left[ a, b; c; z \right] = (1-z)^{-a+b} 2F1 \left[ c-a, c-b; z \right] \]  

(78)

\[ 2F1 \left[ a, b; 2b; z \right] = (1-z)^{-a/2} \left( \frac{z}{2b} \right)^{2} 2F1 \left[ \frac{a}{2}, b-a; \frac{z^2}{4(z-1)} \right] \]  

(79)

one obtains

\[ J_3^{(2,a)} = -\frac{\sqrt{\pi}}{2} \frac{\Gamma \left( \frac{d}{2} - 1 \right) \Gamma \left( 2 - \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - \frac{1}{2} \right)} \frac{-p_1^2}{4} \left( \frac{d}{2} - 2 \right) \]

\[ \times \frac{(p_1^2 + m_2^2)}{(p_1^2 + 2m_2^2)} 2F1 \left[ 1, \frac{d-2}{d-1}; -\frac{p_1^2}{4m_2^2} \right] \]  

(80)

Taking into account the residue at the poles \( \Gamma \left( \frac{4-d}{4} \right) - s \), we get

\[ J_3^{(2,b)} = -\frac{\Gamma \left( \frac{d}{2} - 2 \right) \Gamma \left( 3 - \frac{d}{2} \right)}{2\Gamma \left( \frac{d}{2} - 1 \right)} \frac{4m_2^2(p_1^2 + m_2^2)}{(2m_2^2 + p_1^2)^2} \]

\[ R_3^{d-2} 2F1 \left[ \frac{1}{2}, \frac{p_1^2}{(2m_2^2 + p_1^2)} \right] \]  

(81)

Using the relation (see Eq. (3.1.7) in [27])

\[ 2F1 \left[ a, b; z \right] = \left( 1 - z \right)^{-a} \times 2F1 \left[ \frac{a}{2}, b + \frac{1}{2}; \left( \frac{z}{2} \right)^2 \right] \]  

(82)

one gets

\[ J_3^{(2,b)} = -\frac{\Gamma \left( \frac{d}{2} - 2 \right) \Gamma \left( 3 - \frac{d}{2} \right)}{2\Gamma \left( \frac{d}{2} - 1 \right)} \frac{(p_1^2 + m_2^2)}{2m_2^2 + p_1^2} R_3^{d-2} \]

\[ \times 2F1 \left[ 1, \frac{1}{2}; -\frac{p_1^2}{m_2^2} \right] \]  

(83)

Combining all the terms, \( J_3 \) reads

\[ J_3 = \frac{\Gamma \left( \frac{2 - d}{2} \right) \Gamma \left( \frac{d}{2} - 1 \right) \Gamma \left( \frac{d}{2} - \frac{3}{2} \right)}{\Gamma \left( \frac{d}{2} - 1 \right) \Gamma \left( \frac{d}{2} - \frac{1}{2} \right) \Gamma \left( \frac{d}{2} - 2 \right)} \]

\[ \times \Gamma(\gamma-s) \Gamma(\gamma+1) \left( \frac{1}{\gamma} \right)^{s} \]

\[ \times \sum_{k=1}^{4} \frac{\partial_k R_4}{R_4} \]  

(84)

provided that \( |p_1^2/m_2^2| < 1 \) and \( \text{Re}(d-2) > 0 \). It agrees with Eq. (B2) in [10] and (22) in [15].

5 One-loop four-point functions

The master equation for \( J_4 \) is obtained from (5) with \( N = 4 \),

\[ J_4 = J_4(d; \{p_i^2, s, t, \{m_j^2\}) = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \]

\[ \times \frac{\Gamma(\gamma-s) \Gamma(\gamma+1) \left( \frac{1}{\gamma} \right)^{s}}{2\Gamma \left( \frac{d}{2} \right)} \]

\[ \times \sum_{k=1}^{4} \frac{\partial_k R_4}{R_4} \]  

(85)

We substitute the analytic solution for \( J_3(d+2s; \{p_i^2, \{m_j^2\}) \) in (38) into (85) and take the contour integrals in (85). With the help of MB integrations in (131, 134) in Appendix C, a compact expression for \( J_4 \) can be derived and expressed as follows:

\[ J_4 \]  

\[ \frac{\Gamma(\gamma-s) \Gamma(\gamma+1) \left( \frac{1}{\gamma} \right)^{s}}{2\Gamma \left( \frac{d}{2} \right)} \]

\[ + \sum_{k=1}^{4} \frac{\partial_k R_4}{R_4} \]

\[ \{1, 2, 3, 4 \} \leftrightarrow \{2, 3, 4, 1 \}\}

\[ + \sum_{k=1}^{4} \frac{\partial_k R_4}{R_4} \]

\[ \{1, 2, 3, 4 \} \leftrightarrow \{3, 4, 1, 2 \}\} \]

\[ \frac{\Gamma(\gamma-s) \Gamma(\gamma+1) \left( \frac{1}{\gamma} \right)^{s}}{2\Gamma \left( \frac{d}{2} \right)} \]

\[ + \sum_{k=1}^{4} \frac{\partial_k R_4}{R_4} \]

\[ \{1, 2, 3, 4 \} \leftrightarrow \{3, 4, 1, 2 \}\} \]
with
\[
J_{1234}^{(d)} = \left( -\frac{\partial_4 R_4}{2 R_4} \right) J_{123}^{(d=4)} (R_{123}) \frac{d-4}{2} F_1 \left[ \frac{d-3}{2} - \frac{1}{2} ; \frac{1}{2} \right] \frac{\partial_3 R_{123}}{2 R_{123}} \frac{R_{123}}{R_4} \right]
\]
\[
+ \frac{\sqrt{\Gamma} \left( \frac{d-2}{2} \right)}{\Gamma \left( \frac{d-1}{2} \right)} \left( \frac{\partial_3 R_{123}}{R_{123}} \right) \left( \frac{\partial_3 R_{123}}{R_{123}} \right) \partial_1 R_{123}
\]
\[
\times \left[ \frac{\partial_2 R_{123}}{\sqrt{1 - m_2^2/R_{123}}} + \frac{\partial_1 R_{123}}{\sqrt{1 - m_1^2/R_{123}}} \right]
\]
\[
\times \frac{(R_{123})^{d-4}}{\sqrt{1 - R_{123}/R_{12}}} F_1 \left( \frac{d - 3}{2} - \frac{1}{2} ; \frac{d - 1}{2} - \frac{1}{2} ; \frac{R_{12}}{R_4}, \frac{R_{12}}{R_{123}} \right)
\]
\[
- \frac{\Gamma \left( \frac{d-2}{2} \right)}{8 \Gamma \left( \frac{d-1}{2} \right)} \left( \frac{\partial_3 R_{123}}{R_{123}} \right) \left[ \frac{\partial_3 R_{123}}{R_{123} - m_1^2} + \frac{\partial_2 R_{123}}{R_{123} - m_2^2} \right]
\]
\[
x F_5 \left( \frac{d - 3}{2} - \frac{1}{2} ; \frac{d - 1}{2} - \frac{1}{2} ; \frac{R_{12}}{R_4}, \frac{R_{12}}{R_{123}} \right)
\]
\[
\times \left[ \frac{m_1^2}{m_1^2 - R_{123}} + \frac{m_2^2}{m_2^2 - R_{123}} \right] (1 \leftrightarrow 2)
\]
\[
+ \left\{ (1, 2, 3) \leftrightarrow (2, 3, 1) \right\}
\]
\[
+ \left\{ (1, 2, 3) \leftrightarrow (3, 1, 2) \right\}.
\]
where \( J_{123}^{(d=4)} , \ldots \) are given by (40). It is important that this representation is valid under the conditions that \( \Re(e (d - 3) > 0 \) and the absolute values of arguments of hypergeometric functions are smaller than one. If the absolute value of these arguments are larger than one, we have to perform analytic continuations for the Gauss hypergeometric and Appell \( F_1 \) functions, cf. [25, 31]. Further, the Saran function \( F_5 \) may be expressed by a Mellin–Barnes representation, or Euler integrals in this case. The result for \( J_4 \) has been shown in [19]. There are two important points we would like to emphasize in this paper as follows. (1) Ref. [16] have not shown conditions for the boundary term in (100). (2) \( J_4 \) is constructed from \( J_3 \) for arbitrary kinematics. However, the boundary term for \( J_3 \) for general kinematics have not been provided in [16], as mentioned in the previous section and in [18]. Subsequently, the first term in (99) of [16] is only valid in special kinematic regions. Therefore, the solution in (98) of Ref. [16] may not be considered as a complete solution for \( J_4 \). In contrast to [16], we provide a complete solution for \( J_4 \) in this article.

5.1 Massless internal lines

We are going to take \( m_i^2 \to 0 \) for \( i = 1, 2, 3, 4 \). The terms related to \( F_5 \) vanish. Therefore, in the massless case the result reads

\[
J_{1234}^{(d)} \left( \frac{\partial_4 R_4}{2 R_4} \right) \left( \frac{\partial_3 R_{123}}{2 R_{123}} \right) \left( \frac{\partial_1 R_{123}}{2 R_{123}} \right) \left( \frac{\partial_2 R_{123}}{2 R_{123}} \right)
\]
\[
\times \frac{\sqrt{\Gamma} \left( \frac{d-2}{2} \right)}{\Gamma \left( \frac{d-1}{2} \right)} \left( \frac{\partial_3 R_{123}}{R_{123}} \right) \left( \frac{\partial_3 R_{123}}{R_{123}} \right) \partial_1 R_{123}
\]
\[
\times \left[ \frac{\partial_2 R_{123}}{\sqrt{1 - m_2^2/R_{123}}} + \frac{\partial_1 R_{123}}{\sqrt{1 - m_1^2/R_{123}}} \right]
\]
\[
\times \frac{(R_{123})^{d-4}}{\sqrt{1 - R_{123}/R_{12}}} F_1 \left( \frac{d - 3}{2} - \frac{1}{2} ; \frac{d - 1}{2} - \frac{1}{2} ; \frac{R_{12}}{R_4}, \frac{R_{12}}{R_{123}} \right)
\]
\[
- \frac{\Gamma \left( \frac{d-2}{2} \right)}{8 \Gamma \left( \frac{d-1}{2} \right)} \left( \frac{\partial_3 R_{123}}{R_{123}} \right) \left[ \frac{\partial_3 R_{123}}{R_{123} - m_1^2} + \frac{\partial_2 R_{123}}{R_{123} - m_2^2} \right]
\]
\[
x F_5 \left( \frac{d - 3}{2} - \frac{1}{2} ; \frac{d - 1}{2} - \frac{1}{2} ; \frac{R_{12}}{R_4}, \frac{R_{12}}{R_{123}} \right)
\]
\[
\times \left[ \frac{m_1^2}{m_1^2 - R_{123}} + \frac{m_2^2}{m_2^2 - R_{123}} \right] (1 \leftrightarrow 2)
\]
\[
+ \left\{ (1, 2, 3) \leftrightarrow (2, 3, 1) \right\}
\]
\[
+ \left\{ (1, 2, 3) \leftrightarrow (3, 1, 2) \right\}.
\]

This is a new result for \( J_4 \) in the massless case at general \( d \). We are going to consider the special cases for \( J_4 \) in the following subsections.

5.2 \( R_4 = 0 \)

From (7), we set \( N = 4 \) and get

\[
J_4 = \frac{1}{d - 5} \sum_{k=1}^{4} \left( \frac{\partial_k Y_4}{G_3} \right) \kappa J_4 (d - 2; \{ p_i^2, s, t \}, \{ m_i^2 \}).
\]

The term \( \kappa J_4 (d - 2; \{ p_i^2, s, t \}, \{ m_i^2 \}) \) is given by \( J_3 \) in (38) with \( d \to d - 2 \).

5.3 \( R_4 = R_{ijk} \) for \( i, j, k = 1, 2, 3, 4 \)

As an example, we consider the case \( R_4 = R_{123} \). In this case, we verify that

\[
\left( \partial_1 R_4 \right) = 0.
\]

As a result, the terms \( J_{1234}^{(d)} \) vanish, other terms in (86) are of the same form in (87).
5.4 $R_4 = R_{ij}$ for $i, j = 1, 2, 3, 4$

For example, the terms of $J_4$ in (86) meet the condition $R_4 = R_{12}$. Because that $R_2$ depends only the internal masses $m_1^2, m_2^2$, one has

$$\partial_i R_4 = \partial_i R_2 = 0, \text{ for } i = 3, 4. \tag{92}$$

As a matter of this fact, only two terms $(2, 3, 4, 1)$ and $(3, 4, 1, 2)$ in (86) contribute to $J_4$.

5.5 $R_4 = m_k^2$ for $k = 1, 2, 3, 4$

For example, one considers the terms of $J_4$ in (86) having $R_4 = m_1^2$. One verifies that

$$\partial_i R_4 = 0 \text{ for } i = 2, 3, 4. \tag{93}$$

As a result, only the term $(2, 3, 4, 1)$ in (86) contributes to $J_4$.

5.6 $R_{ijk} = 0$ for $i, j, k = 1, 2, 3, 4$

We assume that $J_4$ in (86) contains $R_{123} = 0$ as an example. The term $(1, 2, 3, 4)$ in (87) with $R_{123} = 0$ is evaluated by applying the same previous procedure. The result reads

$$J_{1234}^{(d)} = -\frac{\sqrt{\pi}}{\Gamma \left(\frac{d-3}{2}\right) \Gamma \left(\frac{d-2}{2}\right)} \left(\frac{\partial_4 R_4}{R_4}\right) \left(\frac{\partial_3 Y_{123}}{R_{12}}\right) \times \left[\frac{\partial_2 R_1}{\sqrt{1 - m_1^2/R_{12}}} + \frac{\partial_1 R_2}{\sqrt{1 - m_2^2/R_{12}}}\right] \times (R_{12})^{d-6} \times 2F_1 \left[1, \frac{d-1}{2}; \frac{d-1}{2}; \frac{1}{2}; \frac{m_1^2}{R_4}, \frac{m_2^2}{R_12}\right] + \frac{1}{2} \Gamma \left(\frac{d-2}{2}\right) \left(\frac{\partial_4 R_4}{R_4}\right) \left(\frac{\partial Y_{123}}{R_{123}}\right) \times \left[\frac{\partial_2 R_1}{\sqrt{1 - m_1^2/R_{12}}}\right] \times F_1^{1; 2; 1}_{1; 1; 1} \left(\frac{d-3}{2}; \frac{d-3}{2}; \frac{1}{2}; \frac{m_1^2}{R_4}, \frac{m_2^2}{R_12}\right) + (1 \leftrightarrow 2) + \{(1, 2, 3) \leftrightarrow (2, 3, 1)\} + \{(1, 2, 3) \leftrightarrow (3, 1, 2)\}. \tag{94}$$

Where $F_1^{1; 2; 1}_{1; 1; 1}$ is Kampé de Fériet [33] (see Appendix B for more detail). We also refer to [34] which analytic continuations for a class of the Kampé de Fériet functions have been studied. This representation is valid if the amplitude of arguments of these hypergeometric functions are less than 1 and $\Re(d - 4) > 0$. In the massless case, one has

$$J_{1234}^{(d)} = \frac{\sqrt{\pi}}{\Gamma \left(\frac{d-3}{2}\right) \Gamma \left(\frac{d-2}{2}\right)} \left(\frac{\partial_4 R_4}{R_4}\right) \left(\frac{\partial_3 Y_{123}}{R_{12}}\right) (R_{12})^{d-6} \times 2F_1 \left[1, \frac{d-1}{2}; \frac{d-1}{2}; \frac{1}{2}; \frac{m_1^2}{R_4}, \frac{m_2^2}{R_12}\right] + \{(1, 2, 3) \leftrightarrow (2, 3, 1)\}.$$
\[
\frac{(R_{23})^{d-4}}{\Gamma\left(\frac{d-2}{2}\right)} \left[ \frac{\partial_2 R_{12}}{2} \right] \\
\times F_3 \left( \frac{d-3}{2}; 1, 1, 1; \frac{d-1}{2}; \frac{R_{23}}{R_4}, \frac{R_{23}}{R_{231}} \right) \\
+ \left( \frac{d-2}{2} \right) \left( \frac{\partial_2 R_{12}}{2} \right) \\
+ \left( 1 \leftrightarrow 2 \right) \left( \frac{d-3}{2} \right)
\]

This representation is valid if the amplitude of arguments of these hypergeometric functions are less than 1 and \( \Re(e^{-d-3}) > 0 \).

5.10 \( R_{ij} = m_i^2 \) or \( m_i^2 \) for \( i, j = 1, 2, 3, 4 \)

Taking \( R_{12} = m_1^2 \) as an example, one confirms that

\[
\partial_2 R_{12} = 0.
\]

As a result, the terms \( 1, 2, 3 \) of \( J_{1234}^{(d)} \) that are multiplied by \( \frac{\partial R_{23}}{\partial m_2^2} \) vanish. Other terms of \( J_{1234}^{(d)} \) are given by (87). The terms \( J_{2341}^{(d)}, J_{3412}^{(d)} \) and \( J_{4123}^{(d)} \) of \( J_4 \) are given by (87).

5.11 \( G_3 = 0 \)

In this case, one has

\[
J_4 = \frac{1}{2} \sum_{k=1}^{4} \left( \frac{\partial_3 Y_4}{Y_4} \right) k^{-} J_4(d; \{ p_i^2, s, t \}, \{ m_i^2 \}).
\]

This equation is equivalent with (65) in Ref. [21]. The term \( k^{-} J_4(d; \{ p_i^2, s, t \}, \{ m_i^2 \}) \) is given by \( J_3 \) in (38).

5.12 \( G_{2(ijk)} = 0 \) for \( i, j, k = 1, 2, 3, 4 \)

In the same notation, \( G_{2(ijk)} \) are the Gram determinants of \( J_3 \) that are obtained by shrinking \( k \)th propagator in \( J_4 \). We take \( |G_{2(123)}| = 0 \) as an example. By using (66) for \( J_{123}^{(d)} \), we then evaluate \( J_{1234}^{(d)} \) again, the result is

\[
J_{1234}^{(d)} = -\frac{\sqrt{\pi}}{8 \Gamma(d-2)} \left( \frac{\partial_3 R_{12}}{2} \right) \left( \partial_3 Y_{123} \right) \\
\times \left[ \frac{\partial_2 R_{12}}{\sqrt{1 - m_2^2/R_{12}}} + \frac{\partial_1 R_{12}}{\sqrt{1 - m_2^2/R_{12}}} \right] \\
\times \left( R_{12} \right)^{d-4} \frac{d-3}{2} F_2 \left( \frac{d-2}{2}; \frac{R_{12}}{R_4} \right) \\
+ \left( \frac{d-2}{2} \right) \left( \frac{\partial_3 Y_{123}}{2} \right) \\
+ \left( 1 \leftrightarrow 2 \right) \left( \frac{d-3}{2} \right)
\]

This representation is valid if the amplitude of arguments of these hypergeometric functions are less than 1 and \( \Re(e^{-d-3}) > 0 \).

5.13 \( G_{1(ij)} = 0 \) for \( i, j = 1, 2, 3, 4 \)

One assumes that the term \( J_{1234}^{(d)} \) has \( G_{1(12)} = 0 \). Recalculating this term, the result reads in term of Gauss and Appell \( F_3 \) functions

\[
J_{1234}^{(d)} = -\left( \frac{\partial_3 R_{12}}{2} \right) J_{123}^{(d=4)} \left( \frac{R_{12}}{d-2} \right) \frac{d-3}{2} F_2 \left( \frac{d-2}{2}; \frac{R_{12}}{R_4} \right) \\
\times \left[ \frac{\partial_3 R_{12}}{2 \Gamma(d-2)} \right] \left( \frac{\partial_3 R_{12}}{2} \right) \\
\times \left( \frac{d-2}{2} \right) \left( \frac{\partial_3 Y_{123}}{2} \right) \\
\times \left( \frac{d-2}{2} \right) \left( \frac{\partial_3 Y_{123}}{2} \right)
\]

where the terms \( J_{123}^{(d=4)} \) and \( J_{1234}^{(d=4)} \) are given in (40). This representation is valid if the amplitude of arguments of these hypergeometric functions are less than 1 and \( \Re(e^{-d-3}) > 0 \).

The Appell \( F_3 \) functions are described in detail in Appendix B (see (123) in more detail).
For future prospect of this work, a package which provides a general $\epsilon$-expansion and numerical evaluations for one-loop functions at general $d$ is planned. To achieve this purpose, many related works are worth mentioning in this paragraph. First, automatized analytic continuation of Mellin–Barnes integrals have been presented in [38]. The construction of Mellin–Barnes representations for Feynman integrals has been performed in [39,40]. Recent development for treating numerically Mellin–Barnes integrals in physical regions has been proposed in [41–43]. The hypergeometric functions in this work can be expressed as the multi-fold MB integrals and they may be evaluated numerically by following the above works. Furthermore, the $\epsilon$-expansion of the hypergeometric functions appearing in our analytic results may be also performed by using the packages Sigma, EvaluateMultiSums and Harmonic Sums [44–50]. Numerical $\epsilon$-expansion of hypergeometric functions may be done by using NumEXP [51]. Besides that, analytic $\epsilon$-expansion for the hypergeometric functions has been carried out in [52–59]. Differential reduction of generalized hypergeometric functions has been also reported in [60–63].

In the context of dimensional recurrence relations, the tensor reductions for one-loop up to five-point functions have been worked out in [64] and for higher-point functions have been developed in [65]. In practice, one encounters integrals with denominator powers higher than one and their reduction needs to be considered, see e.g. [3] for the scalar case. IBP reduction can be combined with dimensional recurrence relations to reduce them to master integrals of higher space-time dimensions.

6 Conclusions

In this article, we have been presented the analytic results for scalar one-loop two-, three- and four-point functions in detail. The results have been expressed in terms of Gauss $\, _2F_1$, Appell $\, _ F_1$ and $\, _ F_3$ hypergeometric functions. All cases of external momentum and internal mass assignments have considered in detail in this work. The higher-terms in the $\epsilon$-expansion for one-loop integrals can be performed directly from analytic expressions in this work. These terms are necessary building blocks in computing two-loop and higher-loop corrections. Moreover, one-loop functions in arbitrary $d$ in this work may be taken account in the evaluations for higher-loop Feynman integrals. The one-loop functions with $d \geq 4$ can also used in the reduction for tensor one-loop Feynman integrals. For future works, a package for numerical evaluations for one-loop integrals at general $d$ and general $\epsilon$-expansion for these integrals is planned. Additionally, the method can extend to evaluate two- and higher-loop Feynman integrals.

Acknowledgements This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under the Grant number 103.01-2019.346. The author would like to thank T. Riemann for helpful discussions and comments.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This is theoretical work, we present analytic results in this work. Therefore, we have not used any data.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Funded by SCOAP3.

Appendix A: Useful formulas

In this appendix, we show some useful formulas used in this paper. We applied the reflect formula for gamma functions [24]:

$$
\Gamma(1 - z - n) = (-1)^n \frac{\Gamma(z) \Gamma(1 - z)}{\Gamma(z + n)},
$$

provided that $z \in \mathbb{C}$ and $n \in \mathbb{N}$. We have mentioned the duplication formula for gamma functions [24]:

$$
\Gamma(2z) = \frac{2^{2z - 1} \Gamma(z) \Gamma(z + \frac{1}{2})}{\sqrt{\pi}},
$$

provided that $z \in \mathbb{C}$.

Appendix B: Generalized hypergeometric series

Generalized hypergeometric functions are presented in this appendix.

Gauss hypergeometric series

Gauss hypergeometric series are given [25]:

$$
\,_2F_1 \left[ a, b ; c ; z \right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},
$$

provided that $|z| < 1$. The pochhammer symbol $(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}$ is used.
The integral representation for Gauss hypergeometric functions [25] reads
\[
_{2}F_{1}\left[a, b ; c ; z\right] = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \int_{0}^{1} du \, u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a},
\]
provided that \(|z| < 1\) and \(\text{Re}(c) > \text{Re}(a) > 0\).

Basic linear transformation formulas for Gauss \(_{2}F_{1}\) hypergeometric functions which are collected from Refs. [25, 28], are listed as follows:

\[
_{2}F_{1}\left[a, b ; c ; z\right] = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \int_{0}^{1} du \, u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a},
\]

provided that \(|z| < 1\) and \(\text{Re}(c) > \text{Re}(b) > 0\).

The single integral representation for \(_{2}F_{1}\) reads [25]
\[
F_{1}(a; b, b'; c; x, y) = \frac{\Gamma(c) \Gamma(c-a) \Gamma(a)}{\Gamma(c-a-b) \Gamma(c-b) \Gamma(c)} \int_{0}^{1} du \, u^{a-1} \times (1-u)^{c-a-1}(1-xu)^{b}(1-yu)^{-b'},
\]
provided that \(\text{Re}(c) > \text{Re}(a) > 0\) and \(|x| < 1, |y| < 1\).

We collect all transformations for Appell \(_{1}\) functions from Refs. [25, 30]. The first relation for \(_{1}\) is mentioned,
\[
F_{1}(a; b, b'; c; x, y) = (1-x)^{-b}(1-y)^{-b'} \times F_{1}\left(c-a; b, b'; \frac{x}{x-1}, \frac{y-x}{1-y}\right).
\]

If \(b' = 0\), we arrive at the well-known Pfaff–Kummer transformation for \(_{2}F_{1}\). Further, we have
\[
F_{1}(a; b, b'; b + b'; x, y) = (1-x)^{-a} F_{1}\left(a, b' ; y-x \right) \times \left(x - b - b' + c, b'; \frac{x}{x-1}, \frac{y-x}{1-y}\right).
\]

Furthermore, if \(c = b + b'\), one then obtains
\[
F_{1}(a; b, b'; b + b'; x, y) = (1-x)^{-a} F_{1}\left(a, b' ; \frac{y-x}{1-y}\right).
\]

Similarly,
\[
F_{1}(a; b, b'; c, x, y) = (1-y)^{-a} \times F_{1}\left(a, b, c - b - b' ; \frac{y}{1-y}, \frac{y}{y-1}\right),
\]

and
\[
F_{1}(a; b, b'; c, x, y) = (1-x)^{c-a-b}(1-y)^{-b'} \times F_{1}\left(c-a; c - b - b', b'; x, \frac{x}{1-y}\right).
\]

A further relation for \(_{1}\) is given
\[
F_{1}(a + 1, 1, 1/2, a + 2; x, y) = \frac{\sqrt{\pi} \Gamma(a + 2)}{\Gamma(a + 3/2)} \times \frac{2 \Gamma(a + 2)}{\Gamma(a + 1)} \times F_{1}\left(1, a + 1; \frac{x}{a + 3/2}, \frac{1}{y(1-x)}\right).
\]

Appell series

Appell \(_{1}\) series are defined [25, 32]:
\[
F_{1}(a; b, b'; c, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)m(b')_{n} x^{m} y^{n}}{(c)_{m+n} m! n!},
\]
provided that \(|x| < 1\) and \(|y| < 1\).
Appell $F_3$ series

Appell $F_3$ series are written [25, 32]

$$F_3(a, a'; b, b'; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{m! n! (c)_{m+n}} x^m y^n,$$

provided that $\text{Re}(c) > \text{Re}(a) > 0$ and $|x| < 1, |y| < 1$.

J. Kampé de Fériet series

In addition, J. Kampé de Fériet series [32, 33] with two variables are shown:

$$F_{p,q}^{r,s} \left( \begin{array} { c c } { a_1, \ldots, a_p } & { b_1, b'_1; \ldots; b_q, b'_q } \\ { c_1, \ldots, c_r } & { d_1, d'_1; \ldots; d_s, d'_s } \\ x, y \end{array} \right)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1)_m \ldots (a_p)_m (b_1)_m (b'_1)_n \ldots (b_q)_m (b'_q)_n x^m y^n}{(c_1)_m \ldots (c_r)_m (d_1)_m (d'_1)_n \ldots (d_s)_m (d'_s)_n m! n!},$$

(124)

Lauricella–Saran function $F_S$

Lauricella–Saran function $F_S$ is defined in terms of a triple hypergeometric series [35]

$$F_S(a_1, a_2, a_3, b_1, b_2, b_3, \gamma_1, \gamma_1; x, y, z) = \sum_{r,m,n=0}^{\infty} \frac{(a_1)_r (a_2)_m (b_1)_r (b_2)_m (b_3)_n (\gamma_1)_r (\gamma_1)_m (\gamma_1)_n}{(c)_r (c)_m (c)_n} x^r y^m z^n.$$ (125)

The integral representation of $F_S$ is defined [35]:

$$F_S(a_1, a_2, a_3, b_1, b_2, b_3, \gamma_1, \gamma_1; x, y, z) = \frac{\Gamma(\gamma_1)}{\Gamma(a_1) \Gamma(\gamma_1 - a_1)} \int_0^1 dt \frac{r^\gamma \alpha^{a_1 - 1} (1 - t)^{a_1 - 1}}{(1 - \alpha + \alpha t)^{b_1}} F_1(a_2, b_2, b_3; \gamma_1 - a_1; \gamma_1, t, c, z),$$

(126)

provided that $\text{Re}(\gamma_1 - a_1 - a_2) > 0, \text{Re}(a_1) > 0, \text{Re}(a_2) > 0$, and $|x|, |y|, |z| < 1$.

Appendix C: The contour integrations

Type 1

Mellin–Barnes relation [29] is given:

$$\frac{1}{2\pi i} \int_{-\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(\lambda + s)}{\Gamma(\lambda)} (z)^s = \frac{1}{(1 + z)^\lambda}$$

(127)

provided that $|\text{Arg}(z)| < \pi$. The integration contour is chosen in such a way that the poles of $\Gamma(-s)$ and $\Gamma(\lambda + s)$ are well-separated.

Type 2

The Barnes-type integral for Gauss hypergeometric [25] reads

$$\frac{1}{2\pi i} \int_{-\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a + s) \Gamma(b + s)}{\Gamma(c + s)} (-z)^s$$

$$= \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} \times \frac{\Gamma(2)}{\Gamma(2)} F_1(a, b; c; z)$$

(128)

provided that $|\text{Arg}(z)| < \pi$ and $|z| < 1$.

Type 3

The next Barnes-type integral applied in this paper is

$$\frac{1}{2\pi i} \int_{-\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a + s) \Gamma(b + s)}{\Gamma(c + s)} (-x)^s$$

$$\times \frac{\Gamma(2)}{\Gamma(2)} F_1(a + m, b'; c + m; y)$$

$$= \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} F_1(a; b, b'; c; x, y)$$

(129)

with $|\text{Arg}(z)| < \pi$, $|\text{Arg}(z)| < \pi$ and $|x| < 1$ and $|y| < 1$. Under these conditions, one could close the contour of integration to the right. Subsequently, we have to take into account the residua of the sequence poles of $\Gamma(-s)$. The result is expressed as the summation of Gauss hypergeometric function. The summation is then identical as a series of Appell $F_1$ functions [30] as follows:

$$\frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} \sum_{m=0}^{\infty} \frac{(a)_m (b)_m x^m}{m!}$$

$$\times \frac{\Gamma(2)}{\Gamma(2)} F_1(a + m, b'; c + m; y)$$

(130)

Type 4

Furthermore, in this paper we evaluate the following integral:

$$\frac{1}{2\pi i} \int_{-\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a + s) \Gamma(b_1 + s)}{\Gamma(c + s)} (-x)^s$$

$$\times F_1(a' + s; b_2, b_3; c + s; y, z)$$

$$= \frac{\Gamma(a) \Gamma(b_1)}{\Gamma(c)} \left[ (1 - y)^{-b_2} (1 - z)^{-b_3} \right]$$

(131)

We close the contour of integration to the right. By taking into account the residua of the sequence poles of $\Gamma(-s)$, the
result reads
\[
\frac{\Gamma(a) \Gamma(b_1)}{\Gamma(c)} \sum_{m=0}^{\infty} \frac{(a)_m (b_1)_m}{(c)_m} \frac{x^m}{m!} F_1(a' + m; b_2, b_3; c + m; y, z) \quad (132)
\]
One applies the relation (115) for Appell functions in (132) as follows
\[
F_1(a' + m; b_2, b_3; c + m; y, z) = (1 - y)^{-b_2} (1 - z)^{-b_3} \times F_1(c - a'; b_2, b_3; c + m; \frac{y}{y - 1}, \frac{z}{z - 1}) . \quad (133)
\]
Equation (132) is then presented as a series of Lauricella functions \(F_{3}\) \([35]\)
\[
\frac{\Gamma(a) \Gamma(b_1)}{\Gamma(c)} (1 - y)^{-b_1} (1 - z)^{-b_3} \times \sum_{m,n,l=0}^{\infty} \frac{(a)_m (b_2)_n (b_3)_l}{(c)_m+n+l} \frac{x^m}{m!} y^n z^l \quad \frac{1}{n!}
\times \left( \frac{y}{y - 1} \right)^n \left( \frac{z}{z - 1} \right)^l \frac{\Gamma(a) \Gamma(b_1) \Gamma(c)}{\Gamma(c - a') \Gamma(c - a) \Gamma(c - b)} F_3(a, c - a'; c - a'; b_1, b_2, b_3; c, c, c; x, \frac{y}{y - 1}, \frac{z}{z - 1}) . \quad (134)
\]
provided that \(|x| < 1, \left| \frac{y}{y - 1} \right| < 1\) and \(\left| \frac{z}{z - 1} \right| < 1\).

The last formula is mentioned in this paper relates to a transformation of Mellin–Barnes integrals (see page 156 of Ref. [28], item 14.53 in page 290 of [29]) which is
\[
\int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(a + s) \Gamma(b + s)}{\Gamma(c + s)} (-z)^s
= \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(a + b - c - s) \Gamma(c - a + s) \Gamma(c - b + s)}{\Gamma(c - a) \Gamma(c - b)}
\times (1 - z)^{c - a - b + s} , \quad (135)
\]
provided that \(|\text{Arg}(-z)| < 2\pi\).

**Appendix D: Master equation for \(J_N\)**

General relation for \(J_N\) have been proved in [18]. In this appendix, we consider \(N = 4\) as an example. Performing Feynman parameterization for \(J_4\), one arrives at
\[
\frac{J_4}{\Gamma(4 - \frac{d}{2})} = \int dS_3 \left[ ax_1^2 + bx_2^2 + cx_3^2 \right.
+ 2 e x_1x_2 + 2 f x_1x_3 + 2 f x_2x_3
+ gx_1 + hx_2 + kx_3 + j - i\rho \frac{d}{2} \right] . \quad (136)
\]

Above coefficients \(a, b, c, \ldots, j\) are shown
\[
\begin{align*}
{a} & = p_1^2, & {b} & = p_2^2, & {c} & = (p_2 + p_3)^2, \\
{d} & = -p_1 p_2, & {e} & = p_1 p_2 - p_1 p_3, & {f} & = p_1^2 + p_2 p_3, \\
{g} & = m_1^2 - m_2^2 - p_1^2, & {h} & = -m_1^2, & {k} & = -m_2^2 + m_3^2 - (p_2 + p_3)^2, \\
{j} & = m_3^2.
\end{align*}
\]

We used the following notation
\[
\begin{align*}
\int dS_3 & = \int dx_1 \int dx_2 \int dx_3 \\
& = \int dx_3 \int dx_3 \int dx_3 \\
& \int dx_3 \int dx_3 \int dx_3 . \quad (137)
\end{align*}
\]

The integrand of \(J_4\) is
\[
\mathcal{M}_4(x_1, x_2, x_3) = \left[ x_1^2 + bx_2^2 + cx_3^2 \right. + 2 e x_1x_2 + 2 f x_1x_3 + 2 f x_2x_3
\quad + gx_1 + hx_2 + kx_3 + j.
\]
\[
= (x_1, x_2, x_3) \mathcal{G}_3 (x_2) (x_3) \mathcal{K}_4 . \quad (138)
\]

These matrices are given
\[
\mathcal{G}_3 = \left( \begin{array}{ccc}
\a & \d & \e \\
\b & \f & \g \\
\c & \h & \k \\
\end{array} \right) , \quad \mathcal{K}_4 = \left( \begin{array}{c}
g \\
h \\
k \\
\end{array} \right) , \quad \mathcal{K}_4 = j .
\]

The \(\Lambda_4(x_1, x_2, x_3)\) reads
\[
\Lambda_4(x_1, x_2, x_3) = a(x_1 - y_1)^2 + b(x_2 - y_2)^2
\quad + c(x_3 - y_3)^2 + 2 d (x_1 - y_1)(x_2 - y_2)
\quad + 2 e (x_1 - y_1)(x_3 - y_3)
\quad + 2 f (x_2 - y_2)(x_3 - y_3) . \quad (141)
\]
The vector $\vec{y}$ is defined as: $\vec{y} = (y_1, y_2, y_3) = -G^{-1}H^T$.

We write explicitly $y_1, y_2, y_3$ and $y_4$ as follows:

$$
y_1 = \frac{-bch + bek + cdg - dfk - efg + f^2h}{G_3} = \frac{\partial R_4}{\partial m_1^2},
$$

$$
y_2 = \frac{afg + beh - d(eg + fh) - abk + d^2k}{G_3} = \frac{\partial R_4}{\partial m_2^2},
$$

$$
y_3 = \frac{1 - y_1 - y_2 - y_3}{\partial R_4} = \frac{\partial R_4}{\partial m_3^2},
$$

$$
y_4 = 1 - 
$$

First, we consider $G_3 \neq 0$ and $R_4 \neq 0$. In this case, Mellin–Barnes relation is applied to decompose $A_4$'s integrand as follows:

$$
\frac{1}{\left[\Lambda_4(x_1, x_2, x_3) + R_4 - i\rho\right]^{4 - \frac{1}{2}}} = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma(4 - \frac{1}{2} + s)}{\Gamma(4 - \frac{1}{2})} \left(\frac{1}{R_4 - i\rho}\right)^{4 - \frac{1}{2}} \left[\Lambda_4(x_1, x_2, x_3)\right]^s,
$$

provided that $|\text{Arg} \left(\frac{\Lambda_4(x_1, x_2, x_3)}{R_4 - i\rho}\right)| < \pi$. With the help of the Mellin–Barnes relation, this brings the Feynman parameters integration to the simpler form:

$$
\mathcal{F}_4 = \int dS_3 \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4 - i\rho}\right]^s.
$$

In order to carry out this integral, we consider the following differential operator (see theorem of Bernstein [36], or [37])

$$
\mathcal{O}_4 = \frac{1}{2}(x_1 - y_1) \frac{\partial}{\partial x_1} + \frac{1}{2}(x_2 - y_2) \frac{\partial}{\partial x_2} + \frac{1}{2}(x_3 - y_3) \frac{\partial}{\partial x_3}.
$$

It is easy to check that

$$
\mathcal{O}_4 \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4}\right]^s = s \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4}\right]^s.
$$

As a matter of this fact, we can rewrite Feynman parameter integral as

$$
\mathcal{F}_4 = \frac{1}{s} \int dS_3 \mathcal{O}_4 \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4}\right]^s = \frac{1}{s} \left\{ \int_0^1 dx_1 \int_0^{1 - x_1} dx_2 \int_0^{1 - x_1 - x_2} dx_3 \ (x_3 - y_3) \frac{\partial}{\partial x_3} \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4}\right]^s + \int_0^1 dx_1 \int_0^{1 - x_1} dx_3 \ \right\}.
$$

\begin{table}[h]
\centering
\caption{The coefficients $A_i$, $B_i$, $C_i$, $D_i$, $E_i$, $F_i$}
\begin{tabular}{|c|c|c|c|c|}
\hline
\textit{i} & \textit{1} & \textit{2} & \textit{3} & \textit{4} \\
\hline
$A_i$ & $b$ & $a$ & $a$ & $a + c - 2e$ \\
$B_i$ & $c$ & $c$ & $b$ & $a + b - 2d$ \\
$C_i$ & $f$ & $e$ & $d$ & $a - d - e + f$ \\
$D_i$ & $h$ & $g$ & $g$ & $-2a + 2e - g + k$ \\
$E_i$ & $k$ & $k$ & $h$ & $-2a + 2d - g + h$ \\
$F_i$ & $j$ & $j$ & $j$ & $a + g + j$ \\
\hline
\end{tabular}
\end{table}

The last term in this equation is proportional to $\mathcal{F}_4$. It is then combined with $\mathcal{F}_4$ on the left side of Eq. (148). As a result, Eq. (148) is then casted into the form:

$$
\mathcal{F}_4 = \frac{\Gamma \left(s + \frac{3}{2}\right)}{2 \Gamma \left(s + \frac{1}{2}\right)} \left\{ \int_0^1 dx_1 \int_0^{1 - x_1} dx_2 \int_0^{1 - x_1 - x_2} dx_3 \ \right\}
$$

$$
\times \int dx_2 \int dx_3 \int dx_1 \ (x_3 - y_3) \frac{\partial}{\partial x_3} \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4}\right]^s + \int dx_2 \int dx_3 \int dx_1 \frac{\partial}{\partial x_2} \left[\frac{\Lambda_4(x_1, x_2, x_3)}{R_4}\right]^s.
$$

\[\Box\] Springer
Table 2 The coefficients $A_i, B_i, C_i, D_i, E_i, F_i$ in terms of external momenta and internal masses

| $i$ | 1   | 2   | 3   | 4   |
|-----|-----|-----|-----|-----|
| $A_i$ | $p_i^2$ | $p_1^2$ | $p_2^2$ | $p_3^2$ |
| $B_i$ | $p_i(p_2 + p_3)$ | $p_1(p_2 + p_3)$ | $p_1p_2$ | $p_4(p_1 + p_2)$ |
| $C_i$ | $- (p_i^2 + m_i^2 - m_1^2)$ | $- (p_1^2 + m_1^2 - m_i^2)$ | $- (p_1^2 - m_1^2 + m_i^2)$ | $- (p_2^2 + m_2^2 - m_i^2)$ |
| $D_i$ | $- (t + m_2^2 - m_i^2)$ | $- (t + m_2^2 - m_i^2)$ | $- (t - m_2^2 + m_i^2)$ | $- (s + m_1^2 - m_i^2)$ |
| $E_i$ | $m_2^2$ | $m_2^2$ | $m_2^2$ | $m_2^2$ |
| $F_i$ | | | | |

Taking over a Feynman parameter integration in Eq. (149), the result reads

$$
\mathcal{F}_4 = \frac{\Gamma(s + \frac{3}{2})}{2 \Gamma(s + \frac{3}{2})} \sum_{i=1}^{4} y_i \left. \int dS_2 \right|_{0}^{1} \left. \int dS_3 \right|_{0}^{1-x_3} \frac{\partial}{\partial x_1}\left\{ (x_1 - y_1) \right\}
$$

$$
= \frac{\Gamma(s + \frac{3}{2})}{2 \Gamma(s + \frac{3}{2})} \sum_{i=1}^{4} y_i \left. \int dS_2 \right|_{0}^{1} \left. \int dS_3 \right|_{0}^{1-x_3} \frac{\partial}{\partial x_1}\left\{ \mathcal{M}_3(A_i, B_i, C_i, D_i, E_i, F_i) - 1 \right\}.
$$

(150)

The coefficients $A_i, B_i, C_i, D_i, E_i, F_i$ are given in Table 1.

We then write the coefficients in terms of external momenta and internal masses which are given in Table 2.

Now we can write the Mellin–Barnes integral for $J_4$ as

$$
J_4(d; \{p_1^2, s, t, \{m_i^2\})
$$

$$
= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma\left(\frac{d-3}{2} + s\right) \Gamma(s + 1)}{2\Gamma\left(\frac{d-3}{2}\right)}
$$

$$
\times \left( \frac{1}{R_4} \right)^{d-4} \sum_{i=1}^{4} \left( \frac{\partial R_4}{\partial m_i^2} \right) \int dS_2
$$

$$
\times \left[ \mathcal{M}_3(A_i, B_i, C_i, D_i, E_i, F_i) \right]^{d-3+s}.
$$

(152)

This equation is then written as follows:

$$
J_4(d; \{p_1^2, s, t, \{m_i^2\})
$$

$$
= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{\Gamma(-s) \Gamma\left(\frac{d-3}{2} + s\right) \Gamma(s + 1)}{2\Gamma\left(\frac{d-3}{2}\right)}
$$

$$
\times \left( \frac{1}{R_4} \right)^{d-4} \sum_{i=1}^{4} \left( \frac{\partial R_4}{\partial m_i^2} \right) \int dS_2
$$

$$
\times k^{-d} \quad J_4(d + 2s; \{p_1^2, s, t, \{m_i^2\}).
$$

(153)

In the case of $R_4 = 0$, there is no Mellin–Barnes integral for $J_4$. We only apply the ring operator $\hat{O}_4$, the result arrives

$$
J_4(d; \{p_1^2, s, t, \{m_i^2\}) = \frac{1}{d-5}
$$

$$
\times \sum_{k=1}^{4} \left( \frac{\partial Y_4}{G_3} \right) k^{-d} \quad J_4(d + 2s; \{p_1^2, s, t, \{m_i^2\}).
$$

(154)

References

1. S. Laporta, Int. J. Mod. Phys. A 15, 5087 (2000)
2. O.V. Tarasov, Phys. Rev. D 54, 6479 (1996)
3. A.I. Davydychev, J. Math. Phys. 33, 358 (1992)
4. E.E. Boos, A.I. Davydychev, Theor. Math. Phys. 89, 1052 (1991), [Teor. Mat. Fiz. 89 (1991) 56]
5. A.I. Davydychev, R. Delbourgo, J. Math. Phys. 39, 4299 (1998)
6. A.I. Davydychev, Phys. Rev. D 61, 087701 (2000)
7. A.I. Davydychev, M.Y. Kalmykov, Nucl. Phys. B 699, 3 (2004)
8. A.I. Davydychev, Nucl. Instrum. Meth. A 559, 293 (2006)
9. C. Anastasiou, E.W.N. Glover, C. Oleari, Nucl. Phys. B 572, 307 (2000)
