A Voronovskaja type formula for Soardi’s operators

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Abstract
In 1991 Soardi introduced a sequence of positive linear operators \( \beta_n \) associating to each function \( f \in C[0, 1] \) a polynomial function which is closely related to the Bernstein polynomials on \([-1, +1]\). One of the authors already studied the operators \( \beta_n \) in several papers. This paper is devoted to other properties of Soardi’s operators. We introduce a version \( \tilde{\beta}_n \) which can be expressed in terms of the classical Bernstein operators and present the relations between \( \beta_n \) and \( \tilde{\beta}_n \). We derive Voronovskaja-type results for both \( \beta_n \) and \( \tilde{\beta}_n \). Furthermore, rates of convergence for \( \tilde{\beta}_n \), respectively \( \beta_n \), are estimated. Finally, we study the first and second moments of \( \beta_n \).

Keywords Approximation by positive operators · Rate of convergence · Degree of approximation

Mathematics Subject Classification 41A36 · 41A25

1 Introduction
In 1991 Soardi [8] introduced the sequence of positive linear operators \( \beta_n \) associating to each function \( f \in C[0, 1] \) the polynomial function

\[
(\beta_n f)(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} f \left( \frac{n - 2k}{n} \right) \tilde{w}_{n,k}(x),
\]

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where
\[ \tilde{\omega}_{n,k}(x) = \frac{n+1-2k}{(n+1)2^{n+1}x} \binom{n+1}{k} \left[ (1-x)^k (1+x)^{n+1-k} - (1-x)^{n+1-k} (1+x)^k \right]. \]

Usually, the operators \( \beta_n \) are given in the form
\[ (\beta_n f)(x) = \sum_{k=0}^{m} f \left( \frac{n-2m+2k}{n} \right) w_{n,k}(x), \]
where \( m = \lfloor n/2 \rfloor \) and \( w_{n,k}(x) = \tilde{\omega}_{n,m-k}(x) \) are the fundamental polynomials. The definition and the proofs in [8] are based on properties of random walks on hypergroups. Soardi proved that, for each \( f \in C[0,1] \), the sequence \( (\beta_n f) \) is uniformly convergent to \( f \). Furthermore, by an intensive use of probabilistic tools, Soardi [8, Theorem 2] estimated the rate of convergence of \( (\beta_n f) \) in terms of the usual modulus of continuity:
\[ \| \beta_n f - f \| \leq \left( 55 + \frac{32}{n} \right) \omega \left( f; \frac{1}{\sqrt{n}} \right), \text{ for } f \in C[0,1]. \]

Shape preserving properties of the operators \( \beta_n \) were investigated in [5–7]. In particular, if \( f \in C[0,1] \) is increasing, then \( \beta_n f \) is increasing (see [6, Th. 2.1]; this fact will be used in Sect. 3). Moreover, if \( f \in C[0,1] \) is increasing and convex, then \( \beta_n f \geq f \) (see [6, Th. 3.1]; this inequality will be instrumental in Sect. 5).

For \( x \in (0,1) \) and bounded functions \( f \) on \([0,1]\), Rașa [6, Theorem 4.1] proved the Voronovskaja-type formula
\[ (\beta_n f)(x) = f(x) + \frac{1}{n} \left[ \left( \frac{1}{x} - 1 \right) f'(x) + \frac{1-x^2}{2} f''(x) \right] + o(1/n) \]
as \( n \to \infty \), provided that \( f''(x) \) exists.

This paper is devoted to other properties of Soardi’s operators. In Sect. 2 we introduce a version \( \tilde{\beta}_n \) which can be expressed in terms of the classical Bernstein operators. The relations between \( \beta_n \) and \( \tilde{\beta}_n \) are presented in Sect. 3. Section 4 contains Voronovskaja-type results for both \( \beta_n \) and \( \tilde{\beta}_n \). Rates of convergence for \( \tilde{\beta}_n \), respectively \( \beta_n \), are estimated in Sects. 5 and 2. The last two sections are devoted to the first and second moments of \( \beta_n \).

### 2 The variant \( \tilde{\beta}_n \) and its relation to Bernstein polynomials

In this section we introduce a variant \( \tilde{\beta}_n \) of Soardi’s operator which seems to be more natural. Replacing \( f \left( \frac{n-2k}{n} \right) \) with \( f \left( \frac{n+1-2k}{n+1} \right) \) leads to the definition
\[
\left( \tilde{\beta}_n f \right)(x) = \sum_{k=0}^{m} f \left( \frac{n + 1 - 2k}{n + 1} \right) \tilde{w}_{n,k}(x),
\]

where \( m = \lfloor n/2 \rfloor \). The index manipulation \( k \to n + 1 - k \) yields

\[
\left( \tilde{\beta}_n f \right)(x) = \sum_{k=n+1-m}^{n+1} f \left( \frac{n + 1 - 2k}{n + 1} \right) \tilde{w}_{n,k}(x).
\]

For even values of \( n \) we have

\[
2 \left( \tilde{\beta}_n f \right)(x) = \sum_{k=0}^{n+1} f \left( \frac{n + 1 - 2k}{n + 1} \right) \tilde{w}_{n,k}(x).
\]

This representation is valid also in the case of odd integers \( n \) since the term \( f(0) \tilde{w}_{n, \frac{n+1}{2}}(x) \) with \( k = \frac{n+1}{2} \) is vanishing. Hence, for all \( n \geq 0 \),

\[
\left( \tilde{\beta}_n f \right)(x) = \frac{1}{2} \sum_{k=0}^{n+1} f \left( \frac{n + 1 - 2k}{n + 1} \right) \tilde{w}_{n,k}(x).
\]

Writing

\[
\left( \tilde{\beta}_n f \right)(x) = \frac{1}{2x} \sum_{k=0}^{n+1} \frac{n + 1 - 2k}{n + 1} f \left( \left| \frac{1 - 2k}{n + 1} \right| \right)
\times \binom{n+1}{k} \left[ \left( \frac{1-x}{2} \right)^k \left( \frac{1+x}{2} \right)^{n+1-k} - \left( \frac{1-x}{2} \right)^{n+1-k} \left( \frac{1+x}{2} \right)^k \right]
\]

we obtain the following relation to the classical Bernstein polynomials.

**Lemma 1** For a function \( f \) on \([0, 1]\), we have the relation

\[
\left( \tilde{\beta}_n f \right)(x) = \frac{1}{2x} \left[ \left( B_{n+1}g \right) \left( \frac{1-x}{2} \right) - \left( B_{n+1}g \right) \left( \frac{1+x}{2} \right) \right],
\]

where

\[ g(t) = (1 - 2t) f(|1 - 2t|) \]

and \( B_n g \) denotes the classical Bernstein polynomial on \([0, 1]\).
3 Relations among the operators \( \beta_n \) and \( \tilde{\beta}_n \)

Consider the operators \( \beta_n : C [0, 1] \to C [0, 1] \) and \( \tilde{\beta}_n : C \left[ \frac{1}{n+1}, 1 \right] \to C [0, 1] \). Let

\[
\begin{align*}
    u_n : \left[ \frac{1}{n+1}, 1 \right] &\to [0, 1], u_n (t) = \frac{(n + 1) t - 1}{n} \\
v_n : [0, 1] &\to \left[ \frac{1}{n+1}, 1 \right], v_n (t) = \frac{nt + 1}{n+1}.
\end{align*}
\]

Then, for \( n = 1, 2, 3, \ldots \), \( v_n = u_n^{-1} \). We have \( \beta_n f = \tilde{\beta}_n (f \circ u_n) \), for \( f \in C [0, 1] \) and \( \tilde{\beta}_n g = \beta_n (g \circ v_n) \), for \( g \in C \left[ \frac{1}{n+1}, 1 \right] \). The shape preserving properties of \( \beta_n \) can be translated to \( \tilde{\beta}_n \). In particular, let \( h \in C^1 [0, 1] \). Then, the functions \( \| h' \| e_1 \pm h \) are monotonically increasing, hence \( \| h' \| \beta_n e_1 \pm \beta_n h \) are monotonically increasing. This implies \( \| h' \| (\beta_n e_1)' \pm (\beta_n h)' \geq 0 \), i.e.,

\[
-\| h' \| (\beta_n e_1)' \leq (\beta_n h)' \leq \| h' \| (\beta_n e_1)'.
\]

Since \( 0 \leq (\beta_n e_1)' \leq \frac{n-1}{n} \) (see [6,Theorem 2.1(i) and Rem. 2.3]) we obtain

\[
\| (\beta_n h)' \| \leq \frac{n-1}{n} \| h' \|, \text{ for all } h \in C^1 [0, 1] \quad (1)
\]

(see also [4,Ex. 4.1]).

Now let \( g \in C^1 \left[ \frac{1}{n+1}, 1 \right] \). Then

\[
\left\| (\tilde{\beta}_n g)' \right\| = \left\| \beta_n (g \circ v_n)' \right\| \leq \frac{n-1}{n} \left\| (g \circ v_n)' \right\|
\]

\[= \frac{n-1}{n} \left\| g'(v_n) v_n' \right\| \leq \frac{n-1}{n} \| g' \| \cdot \frac{n}{n+1},
\]

i.e.,

\[
\left\| (\tilde{\beta}_n g)' \right\| \leq \frac{n-1}{n+1} \| g' \|, \text{ for all } g \in C^1 \left[ \frac{1}{n+1}, 1 \right]. \quad (2)
\]

The inequalities (1) and (2) are instrumental in investigating the asymptotic behaviour of the iterates of \( \beta_n \) and \( \tilde{\beta}_n \); see [4].

Let \( f \in C [0, 1] \). Then, with \( \delta = \sqrt{\frac{3n+1}{(n+1)^2}} (1-x^2) \), we obtain from Theorem 2 below

\[
| (\beta_n f) (x) - f (x) | = \left| \left( \tilde{\beta}_n (f \circ u_n) \right) (x) - f (x) \right|
\]

\[\leq \left| \left( \tilde{\beta}_n (f \circ u_n) \right) (x) - (f \circ u_n) (x) \right| + \left| (f \circ u_n) (x) - f (x) \right|
\]

\[\leq 2\omega (f \circ u_n; \delta) + | f (u_n (x)) - f (x) | ,
\]

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where
\[ \omega (f \circ u_n; \delta) = \sup \left\{ |(f \circ u_n)(t_1) - (f \circ u_n)(t_2)| : \frac{1}{n+1} \leq t_1, t_2 \leq 1, |t_1 - t_2| \leq \delta \right\} \]
\[ = \sup \left\{ |f(u_n(t_1)) - f(u_n(t_2))| : \frac{1}{n+1} \leq t_1, t_2 \leq 1, |t_1 - t_2| \leq \delta \right\} \]
\[ = \sup \left\{ |f(s_1) - f(s_2)| : 0 \leq s_1, s_2 \leq 1, |s_1 - s_2| \leq \frac{n+1}{n} \right\} \]
\[ = \omega \left( f; \frac{n+1}{n} \delta \right). \]

Thus
\[ |(\beta_n f)(x) - f(x)| \leq 2\omega \left( f; \frac{n+1}{n} \frac{1}{n} \right) + \omega \left( f; \frac{|1-x|}{n} \right). \]

Consequently,
\[ |(\beta_n f)(x) - f(x)| \leq 2\omega \left( f; \frac{1}{n} \sqrt{(3n+1)(1-x^2)} \right) \]
\[ + \omega \left( f; \frac{1-x}{n} \right), \text{ for } f \in C [0, 1]. \] (3)

In particular,
\[ |(\beta_n f)(x) - f(x)| \leq 2\omega \left( f; \frac{1}{n} \sqrt{3n+1} \right) + \omega \left( f; \frac{1}{n} \right), \text{ for } f \in C [0, 1]. \]

See also Soardi’s estimate [8, Theorem 2]
\[ \|\beta_n f - f\| \leq \left( 55 + \frac{32}{n} \right) \omega \left( f; \frac{1}{\sqrt{n}} \right), \text{ for } f \in C [0, 1]. \]

### 4 Voronovskaja-type results for the operators $\beta_n$ and $\tilde{\beta}_n$

In 2000, Raša [6, Theorem 4.1] proved the following Voronovskaja-type formula for the operators $\beta_n$.

**Theorem 1** Let $x \in (0, 1)$ and $f$ be a bounded function on $[0, 1]$. If $f''(x)$ exists, then
\[ (\beta_n f)(x) = f(x) + \frac{1}{n} \left[ \left( \frac{1}{x} - 1 \right) f'(x) + \frac{1-x^2}{2} f''(x) \right] + o(1/n) \]
as $n \to \infty$. 

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If $x \neq 0$, i.e., $t \neq 1/2$, you can insert the well-known asymptotic formulas for $B_n$. One obtains

$$\left( \tilde{\beta}_n f \right) (x) = f (x) + \frac{1 - x^2}{n} \left[ \frac{1}{x} f' (x) + \frac{1}{2} f'' (x) \right] + o \left( \frac{1}{n} \right)$$

as $n \to \infty$. In the special case $x = 0$, we can use

$$\lim_{x \to 0} \frac{(1-x)^k (1+x)^{n+1-k} - (1-x)^{n+1-k} (1+x)^k}{x} = 2 (n+1-2k)$$

in order to obtain

$$\left( \tilde{\beta}_n f \right) (0) = (n+1) \left( B_{n+1} \hat{g} \right) \left( \frac{1}{2} \right),$$

where

$$\hat{g} (t) = (1-2t) g (t) = (1-2t)^2 f \left( |1-2t| \right).$$

The asymptotic behaviour can easily be derived if $f$ is an even function which is smooth in $x = 0$. If $f$ is not an even function, $\left( B_{n+1} \hat{g} \right) \left( \frac{1}{2} \right)$ is an unpleasant expression.

The link to Soardi’s original operator is given by

$$\beta_n f = \tilde{\beta}_n (f \circ u_n) \quad (4)$$

with $u_n (x) = ((n+1) t - 1) / n$. Therefore,

$$(\beta_n f) (x) = \left( \tilde{\beta}_n f_n \right) (x) = \left( \tilde{\beta}_n f \right) (x) + \frac{x - 1}{n} \left( \tilde{\beta}_n f' \right) (x) + o \left( \frac{1}{n} \right)$$

as $n \to \infty$. A look into the proof of asymptotic formulas for Bernstein polynomials reveals that the latter formula is valid if $f$ is only locally smooth.

We have

$$(B_n f) (x) \sim f (x) + \sum_{k=1}^{\infty} \frac{c_k (f, x)}{n^k} \quad (n \to \infty)$$

with

$$c_k (f, x) = \sum_{j=k}^{2k} a_{k,j} (x) f^{(j)} (x),$$
where \( a_{k,j}(x) \) are certain polynomials involving Stirling numbers of the first and the second kind. More precisely, we have

\[
(B_n f)(x) = f(x) + \sum_{k=1}^{q} \frac{1}{n^{k}} \sum_{s=k}^{2k} \frac{1}{s!} f^{(s)}(x) \sum_{\nu=0}^{s} a(k, s, \nu) x^{s-\nu} + o\left(\frac{1}{n^q}\right)
\]

as \( n \to \infty \), where

\[
a(k, s, \nu) = \sum_{r=\max\{\nu, k\}}^{s} (-1)^{s-r} \binom{s}{r} S(r - \nu, r - k) \sigma(r, r - \nu),
\]

provided that \( f \) is bounded on \([0, 1]\) and admits a derivative of order \( 2q \) at \( x \in [0, 1] \) (see [1, Remark 2]).

Let \( f \in C[0, 1] \). We define \( f \) on \([-1, +1]\) such that \( f \) becomes an even function, i.e., \( f(-x) = f(x) \). Put \( \varphi(t) = 1 - 2t \). If \( x \neq 0 \), i.e., \( t \neq 1/2 \), we have

\[
g(t) = \varphi(t) f(\varphi(t))
\]

and

\[
g^{(j)}(t) = (-2)^j \left[(1 - 2t) f^{(j)}(\varphi(t)) + j f^{(j-1)}(\varphi(t))\right],
\]

\[
g^{(j)}\left(\frac{1-x}{2}\right) = (-2)^j \left[x f^{(j)}(x) + j f^{(j-1)}(x)\right],
\]

\[
g^{(j)}\left(\frac{1+x}{2}\right) = (-2)^j \left[-x f^{(j)}(-x) + j f^{(j-1)}(-x)\right]
\]

\[
= -2^j \left[x f^{(j)}(x) + j f^{(j-1)}(x)\right].
\]

Then

\[
\left(\tilde{\beta}_{n-1}f\right)(x) = \frac{1}{x} (B_n g) \left(\frac{1-x}{2}\right) \sim f(x) + \sum_{k=1}^{\infty} \frac{x^{-1} c_k \left(g, \frac{1-x}{2}\right)}{n^k} \quad (n \to \infty).
\]

5 An estimate of the rate of convergence for the operators \( \tilde{\beta}_n \)

In this section we derive an estimate for the rate of convergence for the operators \( \tilde{\beta}_n \) in terms of the ordinary modulus of continuity \( \omega(f, \delta) \).

Put \( \tilde{g}(x) = g(1-x) \). Then \( (B_n g) \left(\frac{1+x}{2}\right) = (B_n g) \left(\frac{1-x}{2}\right) \) and

\[
\left(\tilde{\beta}_{n-1} f\right)(x) = \frac{1}{2x} (B_n (g - \tilde{g})) \left(\frac{1-x}{2}\right).
\]
For functions of the form 

\[ g(t) = (1 - 2t) f(|1 - 2t|) \]

we have \( \tilde{g} = -g \). Hence,

\[
\left( \tilde{\beta}_{n-1} f \right)(x) = \frac{1}{x} \left( B_n g \right) \left( \frac{1-x}{2} \right).
\]  

(5)

**Lemma 2** For all \( n \in \mathbb{N} \),

\[
\left( \tilde{\beta}_n e_1 \right)(x) \geq x \ (x \in [0, 1]).
\]

**Proof** With the notations of Sect. 3 we have

\[
\tilde{\beta}_n e_1 = \beta_n (e_1 \circ v_n) = \beta_n \left( \frac{n}{n+1} e_1 + \frac{1}{n+1} e_0 \right)
= \frac{n}{n+1} \beta_n e_1 + \frac{1}{n+1} \beta_n e_0.
\]

Since \( \beta_n \) preserves constant functions and \( \beta_n f \geq f \), for all increasing and convex functions \( f \in C [0, 1] \), we obtain

\[
\tilde{\beta}_n e_1 \geq \frac{n}{n+1} e_1 + \frac{1}{n+1} e_0 = e_1 + \frac{e_0 - e_1}{n+1} \geq e_1.
\]

\( \square \)

For reals \( t, x \), put \( \psi_x(t) = t - x \).

**Lemma 3** For all \( n \in \mathbb{N} \), the second central moment of \( \tilde{\beta}_n \) satisfies the estimate

\[
\left( \tilde{\beta}_n \psi_x^2 \right)(x) \leq \frac{3n+1}{(n+1)^2} \left( 1 - x^2 \right) \ (x \in [0, 1]).
\]

**Remark** The constant on the right-hand side is best possible on \([0, 1]\) because, for \( x = 0 \), we have \( \left( \tilde{\beta}_n \psi_0^2 \right)(0) = \left( \tilde{\beta}_n e_2 \right)(0) = (3n+1) / (n+1)^2 \).

**Proof** We have

\[
\left( \tilde{\beta}_n \psi_x^2 \right)(x) = \left( \tilde{\beta}_n e_2 \right)(x) - 2x \left( \tilde{\beta}_n e_1 \right)(x) + x^2 \left( \tilde{\beta}_n e_0 \right)(x) \leq \left( \tilde{\beta}_n e_2 \right)(x) - x^2
\]

on \([0, 1]\), where we used the inequality of Lemma 2. The desired estimate now follows from

\[
\left( \tilde{\beta}_{n-1} e_2 \right)(x) = x^2 + \left( 1 - x^2 \right) \left( 3n - 2 \right) / n^2.
\]

\( \square \)
Theorem 2 Let \( f : C [0, 1] \). For all \( n \in \mathbb{N} \), and \( \delta > 0 \),
\[
\left| \tilde{\beta}_n f(x) - f(x) \right| \leq \left( 1 + \frac{1}{\delta} \sqrt{\frac{3n + 1}{(n + 1)^2}} (1 - x^2) \right) \omega(f, \delta) \quad (x \in [0, 1]).
\]

Proof of Theorem 2 The estimate follows from Lemma 3 by standard arguments (see, e.g., [2, Theorem 5.1.2]).

Putting \( \delta = \sqrt{\frac{3}{n + 1}} \) immediately yields the following consequence.

Corollary 1 For all \( n \in \mathbb{N} \),
\[
\left| \tilde{\beta}_n f(x) - f(x) \right| \leq \left( 1 + \sqrt{1 - x^2} \right) \omega\left( f, \sqrt{\frac{3}{n + 1}} \right) \quad (x \in [0, 1]).
\]

6 An estimate of rate of convergence for the Soardi operator

As already mentioned in the introduction Soardi [8, Theorem 2] estimated the rate of convergence of the operators \( \beta_n \) in terms of the ordinary modulus of continuity:
\[
\| \beta_n f - f \| \leq \left( 55 + \frac{32}{n} \right) \omega\left( f, \frac{1}{\sqrt{n}} \right), \text{ for } f \in C [0, 1].
\]

In this section we improve this estimate considerably by diminishing the absolute constant.

Theorem 3 Let \( f : C [0, 1] \). For all \( n \in \mathbb{N} \), Soardi’s operator \( \beta_n \) satisfies the estimate
\[
\| \beta_n f - f \| \leq \left( 1 + \sqrt{3 + \frac{2}{n}} \right) \omega\left( f, \frac{1}{\sqrt{n}} \right) \quad (n \in \mathbb{N}).
\]

Remark 2 In particular, we have
\[
\| \beta_n f - f \| \leq c \cdot \omega\left( f, \frac{1}{\sqrt{n}} \right) \quad (n \in \mathbb{N}),
\]
where \( c = \left( 1 + \sqrt{5} \right) \approx 3.236. \)

The essential ingredient of the proof is the following estimate of the second central moment of the operators \( \beta_n \).

Lemma 4 For all \( n \in \mathbb{N} \), the second central moment of \( \beta_n \) satisfies the estimate
\[
\left( \beta_n \psi_x^2 \right)(x) \leq \frac{3}{n} \left( 1 - x^2 \right) + \frac{2}{n^2} (1 - x) \quad (x \in [0, 1]).
\]
 Remark 3 Since $1 - x \leq 1 - x^2$ on $[0, 1]$, we have
\[
\left( \beta_n \psi_x^2 \right) (x) \leq \frac{5}{n} \left( 1 - x^2 \right) \quad (x \in [0, 1]).
\]
Furthermore, for each $\varepsilon > 0$, there is an index $n_0$ such that for each $n > n_0$,
\[
\left( \beta_n \psi_x^2 \right) (x) \leq \frac{3 + \varepsilon}{n} \left( 1 - x^2 \right) \quad (x \in [0, 1]).
\]

Proof of Lemma 4 Using the relation $\beta_n f = \bar{\beta}_n (f \circ u_n)$ from Sect. 3 with $u_n (x) = ((n + 1) t - 1) / n$ we obtain
\[
\beta_n \psi_x^2 = \left( \frac{n + 1}{n} \right)^2 \bar{\beta}_n \left( e_1 - \frac{n x + 1}{n + 1} e_0 \right)^2
\]
\[
= \left( \frac{n + 1}{n} \right)^2 \left[ \bar{\beta}_n e_2 - 2 \frac{n x + 1}{n + 1} \bar{\beta}_n e_1 + \left( \frac{n x + 1}{n + 1} \right)^2 e_0 \right].
\]
By Lemma 2,
\[
\left( \beta_n \psi_x^2 \right) (x) \leq \left( \frac{n + 1}{n} \right)^2 \left[ x^2 + \left( 1 - x^2 \right) \frac{3n + 1}{(n + 1)^2} \frac{n x + 1}{n + 1} x + \left( \frac{n x + 1}{n + 1} \right)^2 \right]
\]
\[
= \frac{3n + 1}{n^2} (1 - x^2) + \left( \frac{n + 1}{n} \right)^2 \left( x - \frac{n x + 1}{n + 1} \right)^2
\]
\[
= \frac{3n + 1}{n^2} (1 - x^2) + \left( \frac{x - 1}{n} \right)^2
\]
\[
= \frac{3}{n} (1 - x^2) + \frac{1 - x^2 + (x - 1)}{n^2}
\]
\[
= \frac{3}{n} (1 - x^2) + \frac{2}{n^2} (1 - x),
\]
which is the desired estimate. \(\square\)

Proof of Theorem 3 By Lemma 4, it holds
\[
\left( \beta_n \psi_x^2 \right) (x) \leq \frac{3}{n} + \frac{2}{n^2} \quad (x \in [0, 1]).
\]
Using [2,(5.1.5)], we obtain
\[
|(\beta_n f) (x) - f (x)| \leq \left( 1 + \sqrt{n \left( \beta_n \psi_x^2 \right) (x)} \right) \omega \left( f; \frac{1}{\sqrt{n}} \right)
\]
\[
\leq \left( 1 + \sqrt{3 + \frac{2}{n}} \right) \omega \left( f; \frac{1}{\sqrt{n}} \right).
\]
This completes the proof. □

7 The second moment of $\beta_n$

We have

$$(\beta_ne_2)(x) = (\tilde{\beta}_ne_n^2)(x) = \left(\tilde{\beta}_n \left( \frac{n+1}{n} e_1 - \frac{1}{n} e_0 \right)^2 \right)(x)$$

$$= \left( \frac{n+1}{n} \right)^2 (\tilde{\beta}e_2)(x) + \frac{1}{n^2} - 2 \frac{n+1}{n^2} (\tilde{\beta}_ne_1)(x).$$

Since

$$(\tilde{\beta}e_2)(x) = x^2 + \frac{3n+1}{(n+1)^2} \left(1 - x^2\right)$$

we obtain

$$(\beta_ne_2)(x) \leq \left( \frac{n+1}{n} \right)^2 \left( x^2 + \frac{3n+1}{(n+1)^2} \left(1 - x^2\right) \right) + \frac{1}{n^2} - 2 \frac{n+1}{n^2} x$$

$$\leq \left( \frac{n+1}{n} \right)^2 x^2 + \frac{3n+1}{n^2} - 3 \frac{n+1}{n^2} x^2 + \frac{1}{n^2} - 2 \frac{n+2}{n^2} x$$

$$= x^2 + \frac{1}{n^2} \left( -n x^2 - (2n+2)x + 3n + 2 \right)$$

$$= x^2 + \frac{1 - x}{n^2} (nx + 3n + 2).$$

It follows

$$0 \leq (\beta_ne_2)(x) - x^2 \leq \frac{nx + 3n + 2}{n^2} (1 - x).$$

8 The value $(\beta_ne_1)(0)$ of the first moment

The operator $\beta_n$ does not reproduce the function $e_1(x) = x$, $x \in [0, 1]$. But $\beta_ne_1$ is increasing and convex ([6, Th. 2.1]), $\beta_ne_1 \geq e_1$ ([6, Th. 3.1]), and $\beta_ne_1(1) = 1$. Consequently,

$$0 \leq \beta_ne_1(x) - x \leq \beta_ne_1(0) (1 - x), \text{ for } x \in [0, 1].$$

So, we need a good control on $\beta_ne_1(0)$. This is our aim in what follows.
By Eq. (3), we infer that
\[ x \leq (\beta_ne_1)(x) \leq x + \frac{2}{n} \sqrt{(3n + 1)(1 - x^2)} + \frac{1 - x}{n}, \text{ for } x \in [0, 1]. \]

In particular, it follows that
\[ 0 \leq (\beta_ne_1)(0) \leq \frac{1 + 2\sqrt{3n + 1}}{n} \sim 2\sqrt{\frac{3}{n}} \quad (n \to \infty). \]

In the next section we derive closed expressions for \((\beta_ne_1)(0)\) and study its asymptotic behaviour as \(n\) tends to infinity. We prove that the exact asymptotic rate of convergence is
\[ (\beta_ne_1)(0) \sim \frac{2\sqrt{2}}{\sqrt{\pi n}} \quad (n \to \infty). \]

Note that \(2\sqrt{3} \approx 3.4641\) and \(2\sqrt{2/\pi} \approx 1.59577.\)

**Theorem 4** At \(x = 0\), the first moment of Soardi’s operator has the explicit representation
\[
(\beta_{2n}e_1)(0) = \frac{1}{2^{2n}} \left( 2 + \frac{1}{2n} \right) \binom{2n}{n} - \frac{1}{2n}, \quad (\beta_{2n-1}e_1)(0) = \frac{1}{2^{2n-1}} \left( 1 + \frac{1}{2n - 1} \right) \binom{2n}{n} - \frac{1}{2n - 1}
\]
and satisfies the asymptotic relation
\[ (\beta_ne_1)(0) = \frac{2\sqrt{2}}{\sqrt{\pi n}} - \frac{1}{n} + O \left( n^{-3/2} \right) \quad (n \to \infty). \]

**Proof** Since
\[
\lim_{x \to 0} \frac{1}{x} \left[ (1 - x)^k (1 + x)^{n+1-k} - (1 - x)^{n+1-k} (1 + x)^k \right] = 2(n + 1 - 2k),
\]
we have
\[ \tilde{w}_{n,k}(0) = \binom{n+1}{k} \frac{2(n + 1 - 2k)^2}{(n + 1) 2^{n+1}} \]
and
\[ (\beta_ne_r)(0) = \frac{1}{2^n(n + 1)n^r} \sum_{k=0}^{m} \binom{n+1}{k} (n + 1 - 2k)^2 (n - 2k)^r. \]
Although one can calculate it for arbitrary \( r \in \mathbb{N} \), we restrict ourselves to \( r = 1 \). Let us first consider the case of even parameters \( 2n \):

\[
(\beta_{2n} e_1)(0) = \frac{1}{2^{2n}(2n+1)n} \sum_{k=0}^{n} \binom{2n+1}{k} (2n+1-2k)^2 (n-k).
\]

Writing

\[
(2n+1-2k)^2 (n-k) = -4k^3 + 4(3n-2)k^2 + \left(-12n^2 + 4n - 1\right)k + n(2n+1)^2
\]

we obtain

\[
\sum_{k=0}^{n} \binom{2n+1}{k} (2n+1-2k)^2 (n-k) = -4(2n+1)^3 \sum_{k=3}^{n} \binom{2n-2}{k-3} + 4(3n-2)(2n+1)^2 \sum_{k=2}^{n} \binom{2n-1}{k-2} + \left(-12n^2 + 4n - 1\right)(2n+1) \sum_{k=1}^{n} \binom{2n}{k-1} + n(2n+1)^2 \sum_{k=0}^{n} \binom{2n+1}{k} = A + B + C + D.
\]

Now

\[
A = -4(2n+1)^3 \frac{2^{2n-2} - \binom{2n-2}{n-2} - \binom{2n-2}{n-1} - \binom{2n-2}{n}}{2}
\]

\[
= -2^{2n-1} (2n+1)^3 + 4(2n+1) \left[ n(n-1) \binom{2n}{n} + \frac{n^2}{2} \binom{2n}{n} \right]
\]

\[
= -2^{2n-1} (2n+1)^3 + 2n(2n+1)(3n-2) \binom{2n}{n},
\]

\[
B = 4(3n-2)(2n+1)^2 \frac{2^{2n-1} - \binom{2n-1}{n-1} - \binom{2n-1}{n}}{2}
\]

\[
= 2^{2n} (3n-2)(2n+1)^2 - 2(3n-2)(2n+1)^2 \binom{2n}{n},
\]

\[
C = \left(-12n^2 + 4n - 1\right)(2n+1) \frac{2^{2n} - \binom{2n}{n}}{2},
\]

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\[ D = 2^{2n} n (2n + 1)^2. \]

Finally,
\[
(\beta_{2n} e_1)(0) = \frac{1}{2^{2n} (2n + 1)} (A + B + C + D)
\]
\[
= - (2n - 1) + (6n - 4) + \left( -6n + 2 - \frac{1}{2n} \right) + (2n + 1)
\]
\[
+ \frac{1}{2^{2n}} \binom{2n}{n} \left[ 2 (3n - 2) - 4 (3n - 2) + \left( 6n - 2 + \frac{1}{2n} \right) \right]
\]
\[
= - \frac{1}{2n} + \frac{1}{2^{2n}} \binom{2n}{n} \left( 2 + \frac{1}{2n} \right).
\]

The well-known asymptotic behaviour of the central binomial coefficient (cf. Catalan constant \( \frac{1}{n+1} \binom{2n}{n} \))
\[
\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}} \left( 1 + O \left( \frac{1}{n} \right) \right) \quad (n \to \infty)
\]
leads to the asymptotic formula
\[
(\beta_{2n} e_1)(0) = \frac{2}{\sqrt{\pi n}} - \frac{1}{2n} + O \left( \frac{n^{-3/2}}{n} \right) \quad (n \to \infty).
\]

Now we consider the case of odd parameters \( 2n - 1 \):
\[
(\beta_{2n-1} e_1)(0) = \frac{1}{2^{2n-2} (2n - 1)} \sum_{k=0}^{n-1} \binom{2n}{k} (n - k)^2 (2n - 1 - 2k).
\]

Writing
\[
(n - k)^2 (2n - 1 - 2k) = -2k^3 + (6n - 7) k^2 + \left( -6n^2 + 8n - 3 \right) k + n^2 (2n - 1)
\]
we obtain
\[
\sum_{k=0}^{n-1} \binom{2n}{k} (n - k)^2 (2n - 1 - 2k)
\]
\[
= -2 (2n)^2 \sum_{k=3}^{n-1} \binom{2n - 3}{k - 3} + (6n - 7) (2n)^2 \sum_{k=2}^{n-1} \binom{2n - 2}{k - 2}
\]
\[
+ \left( -6n^2 + 8n - 3 \right) (2n) \sum_{k=1}^{n-1} \binom{2n - 1}{k - 1} + n^2 (2n - 1) \sum_{k=0}^{n-1} \binom{2n}{k}
\]
\[
=: A + B + C + D.
\]
Now

\[ A = -2 (2n)^3 \sum_{k=0}^{n-4} \binom{2n-3}{k} = -2 (2n)^3 \frac{2^{2n-3} - 2 \binom{2n-3}{n-3} - 2 \binom{2n-3}{n-2}}{2} \]

\[ = - (2n)^3 2^{2n-3} + 2 \binom{2n}{n} n (n-1) (n-2) + 2 \binom{2n}{n} n^2 (n-1) \]

\[ = - (2n)^3 2^{2n-3} + 4n (n-1)^2 \binom{2n}{n}, \]

\[ B = (6n - 7) (2n)^2 \sum_{k=0}^{n-3} \binom{2n-2}{k} = (6n - 7) (2n)^2 \frac{2^{2n-2} - 2 \binom{2n-2}{n-2} - (2n-2)}{2} \]

\[ = (6n - 7) \left[ (2n)^2 2^{2n-3} - \binom{2n}{n} n (n-1) - \frac{1}{2} \binom{2n}{n} n^2 \right] \]

\[ = (6n - 7) (2n)^2 2^{2n-3} - \frac{1}{2} (6n - 7) n (3n - 2) \binom{2n}{n}. \]

\[ C = \left( -6n^2 + 8n - 3 \right) (2n) \sum_{k=0}^{n-2} \binom{2n-1}{k} \]

\[ = \left( -6n^2 + 8n - 3 \right) (2n) \frac{2^{2n-1} - 2 \binom{2n-1}{n-1}}{2} \]

\[ = \left( -6n^2 + 8n - 3 \right) n \left( 2^{2n-1} - \binom{2n}{n} \right), \]

\[ D = n^2 (2n - 1) \frac{1}{2} \left( 2^n - \binom{2n}{n} \right). \]

Finally,

\[ (\beta_{2n-1} e_1) (0) \]

\[ = \frac{1}{2^{2n-2} n (2n - 1)} (A + B + C + D) \]

\[ = - (2n - 2) + (6n - 7) + \frac{2 \left( -6n^2 + 8n - 3 \right)}{2n - 1} + \frac{2n (2n - 1)}{2n - 1} \]

\[ + \frac{1}{2^{2n-2} (2n - 1)} \binom{2n}{n} \]

\[ \left[ 4 (n - 1)^2 - \frac{1}{2} (6n - 7) (3n - 2) + (6n^2 - 8n + 3) - \frac{1}{2} n (2n - 1) \right] \]

\[ = 4n - 5 + \frac{-8n^2 + 14n - 6}{2n - 1} + \frac{n}{2^{2n-2} (2n - 1)} \binom{2n}{n} \]

\[ = -\frac{1}{2n - 1} + \frac{2 (2n - 1) + 2 \binom{2n}{n}}{2^{2n} (2n - 1)} \binom{2n}{n} = \frac{-1}{2n - 1} + \frac{1}{2^{2n}} \binom{2n}{n} \left( 2 + \frac{2}{2n - 1} \right). \]
This proves the explicit representation for odd values of the parameter. As above we obtain the asymptotic formula

\[
(\beta_{2n-1}e_1)(0) = \frac{2}{\sqrt{\pi n}} - \frac{1}{2n-1} + O\left(\frac{n^{-3/2}}{}\right) \quad (n \to \infty).
\]

This completes the proof. \(\square\)

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**References**

1. Abel, U., Ivan, M.: Asymptotic expansion of the multivariate Bernstein polynomials on a simplex. Approx. Theory Appl. **16**, 85–93 (2000)
2. Altomare, F., Campiti, M.: Korovkin-type Approximation Theory and its Applications. De Gruyter Studies in Mathematics 17. Walter de Gruyter, Berlin, New York (1994)
3. Bustamante, J.: Bernstein Operators and their Properties. Birkhäuser/Springer, Cham (2017)
4. Gonska, H., Raşa, I., Rusu, M.-D.: Applications of an Ostrowski type inequality. J. Comput. Anal. Appl. **14**, 19–31 (2012)
5. Inoan, D., Raşa, I.: A recursive algorithm for Bernstein operators of second kind. Numer. Algor. **64**(4), 699–706 (2013)
6. Raşa, I.: On Soardi’s Bernstein operators of second kind. Rev. Anal. Numér. Théor. Approx. **29**(2), 191–199 (2000)
7. Raşa, I.: Classes of convex functions associated with Bernstein operators of second kind. Math. Ineq. Appl. **9**(4), 599–605 (2006)
8. Soardi, P.: Bernstein polynomials and random walks on hypergroups. Probability measures on groups X, Oberwolfach (1990), Plenum, New York, 387–393 (1991)

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