Geometrical Quantization in Fock Space

V.P. Maslov and O.Yu. Shvedov

Sub-faculty of Quantum Statistics and Field Theory,
Department of Physics, Moscow State University,
Vorobievy gory, Moscow 119899, Russia

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We investigate an infinite dimensional analog of the theory of Lagrangian manifolds with complex germs. To such a manifold we assign a canonical operator that depends on creation and annihilation operators. This operator is by definition the geometrical quantization for these isotropic manifolds with complex germs. We prove that for secondary quantized equations this quantization is the asymptotics for the Cauchy problem. Results of Berezin are used thoroughly in the construction of the canonical operator and in proofs of the theorems.
1 Introduction

Construction of asymptotic solutions for multiparticle Schrödinger and Liouville equations as the number of particles tends to infinity was investigated in [15, 16]. The construction looks like follows. Both Schrödinger and Liouville multiparticle equations may be presented in a unified form through the creation and annihilation operators:

\[ i \frac{d \hat{\Phi}}{dt} = \left( T_{mn} \hat{\psi}_m^+ \hat{\psi}_n^- + \frac{\epsilon}{2} V_{k l r s} \hat{\psi}_k^+ \hat{\psi}_l^- \hat{\psi}_r^+ \hat{\psi}_s^- \right) \hat{\Phi} \]

Here \( \hat{\Phi} \) is a Fock space element, \( \hat{\psi}_j^\pm \) are creation and annihilation operators in this space [2, 12], and we sum over the repeating indices, \( m, n, k, l, r, s = \overline{1, \infty} \).

The coefficients \( T_{mn} \) and \( V_{k l r s} \) are

1. for the Schrödinger equation

\[
T_{mn} = \int dx f_m^*(x) \left( -\Delta/2 + U(x) \right) f_n(x),
\]

\[
V_{k l r s} = \int dx dy f_k^*(x) f_l^*(y) V(x, y) f_r(y) f_s(x),
\]

\[ x, y \in \mathbb{R}^\nu, \nu \in \mathbb{N}, \]

where \( \{ f_1, f_2, \ldots \} \) is an orthonormal basis in \( L^2(\mathbb{R}^\nu) \), \( \Delta \) is the Laplace operator in \( \mathbb{R}^\nu \), \( U \) is an external potential, \( V \) is the potential of the interparticle interaction. Form (1) of the multiparticle Schrödinger equation originates to papers [4, 5, 6, 9].

2. for the Liouville equation

\[
T_{mn} = i \int dp dq f_m^*(p, q) \left( \frac{\partial U(q)}{\partial p} - p \frac{\partial}{\partial q} \right) f_n(p, q),
\]

\[
V_{k l r s} = i \int dp_1 dp_2 dq_1 dq_2 f_k^*(p_1, q_1) f_l^*(p_2, q_2) \left( \frac{\partial V(q_1, q_2)}{\partial q_1} \frac{\partial}{\partial p_1} + \frac{\partial V(q_1, q_2)}{\partial q_2} \frac{\partial}{\partial p_2} \right) f_r(p_1, q_1) f_s(p_2, q_2),
\]

\[ p, q, p_1, q_1, p_2, q_2 \in \mathbb{R}^\nu, \nu \in \mathbb{N}, \]

where \( \{ f_1, f_2, \ldots \} \) is an orthonormal basis in \( L^2(\mathbb{R}^{2\nu}) \), \( U \) and \( V \) are as in the previous case an external potential and the interparticle interaction potential respectively. Schönberg [19, 20] was the first one who suggested the form (1) for the multiparticle Liouville equation, see also [18].

The asymptotics of the solution of equation (1) was constructed as follows. After the substitution \( \sqrt{\epsilon} \hat{\psi}_j^\pm = \hat{\phi}_j^\pm \) equation (1) becomes

\[
i \epsilon \frac{d \hat{\Phi}}{dt} = H(\hat{\phi}_j^+, \hat{\phi}_j^-) \hat{\Phi},
\]

(2)
for

\[ H(\hat{\phi}_j^+, \hat{\phi}_j^-) = T_{mn} \hat{\phi}_m^+ \hat{\phi}_n^- + \frac{1}{2} V_{klrs} \hat{\phi}_k^+ \hat{\phi}_l^- \hat{\phi}_r^+ \hat{\phi}_s^- \]

where

\[ [\hat{\phi}_j^-, \hat{\phi}_l^+] = \epsilon \delta_{jl} \quad (3) \]

Property (3) allows to apply the quasiclassical methods to equation (2) with $\epsilon$ being the parameter of the quasiclassical expansion.

Since the mean number of particles in the state $\hat{\Phi}$ is equal to

\[ N = \frac{\langle \hat{\Phi}, \hat{\phi}_l^+ \hat{\phi}_l^- \hat{\Phi} \rangle}{\langle \hat{\Phi}, \hat{\Phi} \rangle} = \frac{1}{\epsilon} \frac{\langle \hat{\phi}_l^+ \hat{\phi}_l^- \hat{\Phi} \rangle}{\langle \hat{\Phi}, \hat{\Phi} \rangle}, \quad l = 1, \infty \]

and the quantity $\langle \hat{\Phi}, \hat{\phi}_l^+ \hat{\phi}_l^- \hat{\Phi} \rangle / \langle \hat{\Phi}, \hat{\Phi} \rangle$ is of order $\epsilon^0$ the quasiclassical methods allow to construct approximate solutions of the equation (1) at $\epsilon \to 0, N \to \infty, \epsilon N \to \alpha = \text{const.}$

To the equation (2) there corresponds a classical infinite dimensional Hamiltonian system with the Hamiltonian

\[ H(\hat{Q}_j - i\hat{P}_j, \hat{Q}_j + i\hat{P}_j) / \sqrt{2} \]

The results relating to statistical physics may be rigorously justified. For a part of them such justification was given in [12].

An analogous parameter arises in quantum field theory. Considering formal asymptotic expansions over this small parameter does not allow however to justify this asymptotics since the quantum field theory itself does not have rigorous mathematical meaning. Some approaches to this problem have been developed only in partial cases, see [22, 11]. And without its solution a justification of heuristic asymptotics is of course impossible. Only the postulated perturbation theory series makes sense in the quantum field theory up to now. One can expect that a geometrical quantization coinciding with the "classical" equations in the case when all the commutators vanish should be postulated as well.

The concept of geometrical quantization has been essentially introduced in the works by Bohr, Sommerfeld, and de Broglie, even before the works by Schrödinger, Heisenberg, and Dirac. It is also a rapidly developing concept in modern mathematics [8, 10].

This paper is organized as follows. Section 2 contains definition of a canonical operator in Fock space. We define objects of geometric quantization – Lagrangian manifolds with complex germs, and assign elements of Fock sapce to these manifolds, justify a connection between our and traditional definition of a canonical operator. These definitions differ because of divergences in infinite dimensional case. Traditional definition is not applicable in this case. In section 3 we define canonical transformation of a Lagrangian manifold with complex germ and justify germ axioms. We show that a canonical operator approximately satisfies corresponding secondary - quantized equations. Theorem is formulated in section 4. Section 5 contains the construction of another asymptotics with the help of complex germ creation and annihilation operators. Many examples are cited in sections 4, 5. In section 6 we prove the theorem.
2 A canonical operator

We consider in the present paper finite dimensional isotropic manifolds with infinite dimensional complex germ in an infinite dimensional phase space (see the definition below). We assign a canonical operator to such a manifold. For the sake of simplicity we give a not absolutely invariant definition of the canonical operator. This definition is however sufficient for solving the Cauchy problem.

This canonical operator approximately satisfies the corresponding secondary quantized equations provided that the initial conditions for these equations correspond to isotropic finite dimensional manifolds (or, in the stationary case, there exist stable isotropic manifolds).

In specific examples such isotropic manifolds usually have dimension 1 or 0.

2.1 Objects of geometrical quantization

Define now the objects of geometrical quantization.

By $H^n$ denote the space of complex square summable symmetric functions of $n$ variables $i_1, \ldots, i_n \in \mathbb{N}$. Introduce in $H^n$ a scalar product by the following formula

$$(f; g) = \sum_{i_1, \ldots, i_n=1}^{\infty} f^*_{i_1 \ldots i_n} g_{i_1 \ldots i_n}; \quad f, g \in H^n.$$ 

Denote by $H$ the Fock space $\bigoplus_{n=0}^{\infty} H^n$, and by $\hat{\Phi}^{(k)} \in H^k$ the $k$-th component of $\hat{\Phi} \in H$. Consider creation and annihilation operators acting in $H$ in the following way:

$$(\hat{\psi}^+ \hat{\Phi})^{(k-1)}_{j_1 \ldots j_{k-1}} = k^{1/2} \hat{\Phi}^{(k)}_{j_1 \ldots j_{k-1} j},$$

$$(\hat{\psi}^- \hat{\Phi})^{(k)}_{j_1 \ldots j_k} = k^{-1/2} \sum_{i=1}^{k} \hat{\Phi}^{(k-1)}_{j_1 \ldots j_{i-1} j_{i+1} \ldots j_k} \delta_{jj},$$

$j,j_i \in \mathbb{N}$.

By $\hat{\Phi}_0$ denote the following element of the space $H$: $\hat{\Phi}_0^{(0)} = 1, \hat{\Phi}_0^{(i)} = 0, i \geq 1$. As usual we call elements of the space $H$ state vectors in the Fock presentation.

We will also use in the present paper the presentation of elements in $H$ as the Berezin generating functionals [4].

To each element $\hat{\Phi} \in H$ we assign the generating functional

$$\Phi(a_1^*, a_2^*, \ldots) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{i_1, \ldots, i_n=1}^{\infty} \hat{\Phi}^{(n)}_{i_1 \ldots i_n} a_{i_1}^* \cdots a_{i_n}^*,$$

$$a_j^* \in \mathbb{C}, \sum_{j=1}^{\infty} |a_j^*|^2 < \infty.$$

In this presentation the creation operators $\hat{\psi}_j^+$ are multiplications by $a_j^*$, and the annihilation operators $\hat{\psi}_j^-$ are derivation operators $\partial/\partial a_j^*$. 

We will also make use of the Schrödinger coordinate presentation (or the $Q$-presentation). In this presentation we assign to each element $\hat{\Phi} \in \mathcal{H}$ the functional

$$
\Phi_Q(q_1, q_2, \ldots) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{i_1, \ldots, i_n=1}^{\infty} \hat{\Phi}^{(n)}_{i_1 \ldots i_n} \cdot \prod_{i=1}^{n} \left[ (q_{i} - \epsilon \partial / \partial q_{i}) (2\epsilon)^{-1/2} \right] \exp \left( - \sum_{i=1}^{\infty} q_{i}^2 / 2\epsilon \right).
$$

In this presentation the creation and annihilation operators $\hat{\psi}^{\pm}_{j}$ have the form $\hat{\psi}^{\pm}_{j} = \left( q_{j} \mp \epsilon \partial / \partial q_{j} \right) / \sqrt{2\epsilon}$.

By $\mathcal{L}$ denote the set

$$
\mathcal{L} = \{(P_1, Q_1; P_2, Q_2; \ldots) : P_i \in \mathbb{R}, Q_i \in \mathbb{R}, \sum_{i=1}^{\infty} (P_i^2 + Q_i^2) < \infty \}.
$$

Let us define now finite dimensional isotropic manifolds and corresponding Lagrangian manifolds with complex germs.

Let $\Lambda^k$ be a $k$-dimensional surface in $\mathcal{L}$, and $\tau_1, \tau_2, \ldots, \tau_k$ local coordinates on $\Lambda^k$. For the sake of brevity denote the sequence $(P_1, P_2, P_3, \ldots)$ by $P$, the sequence $(Q_1, Q_2, Q_3, \ldots)$ by $Q$, and the sequence $(\tau_1, \tau_2, \ldots, \tau_k)$ by $\tau$.

**Definition 1** A manifold $\Lambda^k = \{P = P(\tau), Q = Q(\tau)\}$ is called isotropic if the following axioms hold

**m1)** For any $\lambda_1, \ldots, \lambda_k, \lambda_i \in \{0, 1, 2, \ldots\}, i = \overline{1,k}$ the derivatives

$$
\frac{\partial^{\lambda_1 + \ldots + \lambda_k}}{\partial \tau_{\lambda_1}^{\lambda_1} \ldots \partial \tau_{\lambda_k}^{\lambda_k}} P_j, \quad \frac{\partial^{\lambda_1 + \ldots + \lambda_k}}{\partial \tau_{\lambda_1}^{\lambda_1} \ldots \partial \tau_{\lambda_k}^{\lambda_k}} Q_j, \quad j = 1, 2, 3, \ldots
$$

exist and the series

$$
\sum_{j=1}^{\infty} \left[ \left( \frac{\partial^{\lambda_1 + \ldots + \lambda_k}}{\partial \tau_{\lambda_1}^{\lambda_1} \ldots \partial \tau_{\lambda_k}^{\lambda_k}} P_j \right)^2 + \left( \frac{\partial^{\lambda_1 + \ldots + \lambda_k}}{\partial \tau_{\lambda_1}^{\lambda_1} \ldots \partial \tau_{\lambda_k}^{\lambda_k}} Q_j \right)^2 \right]
$$

converges.

**m2)** If

$$
\sum_{j=1}^{\infty} \left[ \left( \sum_{m=1}^{k} \frac{\partial P_j}{\partial \tau_m} \xi_m \right)^2 + \left( \sum_{m=1}^{k} \frac{\partial Q_j}{\partial \tau_m} \xi_m \right)^2 \right] = 0,
$$

for a $\xi_m \in \mathbb{R}$ then $\xi_m = 0, m = \overline{1,k}$.

**m3)**

$$
\sum_{j=1}^{\infty} \left( \frac{\partial P_j}{\partial \tau_m} \frac{\partial Q_j}{\partial \tau_n} - \frac{\partial P_j}{\partial \tau_n} \frac{\partial Q_j}{\partial \tau_m} \right) = 0, m, n = \overline{1,k}.
$$

Denote by $M^k$ the universal covering space of the isotropic manifold $\Lambda^k$. 
Definition 2 A complex germ is a set of planes \( r(\tau), \tau \in M^k \) in the abstract space \( \mathbb{R}^\infty \times \mathbb{R}^\infty \). Each plane consists of the vectors

\[
\begin{pmatrix}
  w_1(\alpha, \tau) \\
  w_2(\alpha, \tau) \\
  \vdots \\
  z_1(\alpha, \tau) \\
  z_2(\alpha, \tau) \\
  \vdots
\end{pmatrix}
\]

\[
w_i(\alpha, \tau) = \sum_{j=1}^{\infty} B_{ij}(\tau)\alpha_j, \quad i = 1, \infty,
\]

\[
z_i(\alpha, \tau) = \sum_{j=1}^{\infty} C_{ij}(\tau)\alpha_j, \quad i = 1, \infty,
\]

where \( \alpha \) ranges over infinite sequences \( \alpha = (\alpha_1, \alpha_2, \ldots) (\alpha_i \in \mathbb{C}, i = 1, \infty, \sum_{i=1}^{\infty} |\alpha_i|^2 < \infty) \) and the following axioms hold:

r1) Let \( \tau', \tau'' \) be two points of the universal covering space \( M^k \) such that they project into one and the same point of \( \Lambda^k \). Then there exists an unitary operator \( A(\tau', \tau'' \rangle \) such that

\[
B(\tau'') = B(\tau')A(\tau', \tau''), C(\tau'') = C(\tau')A(\tau', \tau''),
\]

r2) For \( a = 1, k, i = 1, \infty \)

\[
B_{ia}(\tau) = \partial P_i(\tau)/\partial \tau_a, C_{ia}(\tau) = \partial Q_i(\tau)/\partial \tau_a.
\]

r3) \( B^T(\tau)C(\tau) - C^T(\tau)B(\tau) = 0. \)

(Here \( B^T \) and \( C^T \) are transpose matrices to \( B \) and \( C \) respectively.)

r4) \( C^+(\tau)B(\tau) - B^+(\tau)C(\tau) = iL, \)

where \( L \) is the diagonal matrix with first \( k \) diagonal elements equal to zero and all others to 1.

r5) For any \( \lambda_1, \ldots, \lambda_k; i = 1, k \) the derivatives

\[
F_{mn}^{(\lambda_1, \ldots, \lambda_k)} = \frac{\partial^\lambda_1 + \ldots + \lambda_k}{\partial \tau_1^{\lambda_1} \ldots \partial \tau_k^{\lambda_k}} \left( \frac{C_{mn}(\tau) + iB_{mn}(\tau)}{\sqrt{2}} \right),
\]

\[
G_{mn}^{(\lambda_1, \ldots, \lambda_k)} = \frac{\partial^\lambda_1 + \ldots + \lambda_k}{\partial \tau_1^{\lambda_1} \ldots \partial \tau_k^{\lambda_k}} \left( \frac{C_{mn}(\tau) - iB_{mn}(\tau)}{\sqrt{2}} \right),
\]

exist, operators \( G^{(\lambda_1, \ldots, \lambda_k)} \) are bounded, and operators \( F^{(\lambda_1, \ldots, \lambda_k)} \) are Hilbert-Schmidt operators.

r6) The operator \( (C(\tau) - iB(\tau))/\sqrt{2} \) has a bounded inverse operator.

A pair consisting of an isotropic manifold and a complex germ will be called a Lagrangian manifold with complex germ since they form together a germ of an infinite dimensional complex Lagrangian manifold.
Remark 1 Let \( r(\tau'), r(\tau'') \) be the planes in two points \( \tau', \tau'' \) of the universal covering space \( M^k \) that project into one and the same point of the isotropic manifold \( \Lambda^k \). It follows then from the axiom r1 that these two planes differ from each other only by a parametrization.

Remark 2 In the finite dimensional case axioms r2-r4 are equivalent to traditional axioms of the complex germ [13, 14, 1].

Remark 3 For a zero dimensional isotropic manifold the axioms of the complex germ mean that the matrix

\[
\begin{pmatrix}
G & \bar{F} \\
F & \bar{G}
\end{pmatrix}
\]

is the matrix of a Berezin proper canonical transformation (see [3]).

Assign the function \( \phi_l(\tau) = (Q_l(\tau) + iP_l(\tau))/\sqrt{2} \) to each isotropic manifold \( \Lambda^k = \{P = P(\tau), Q = Q(\tau)\} \).

2.2 Heuristic motivation for definition of a canonical operator

Consider a heuristic method to derive state vectors, which are approximate solutions to secondary-quantized equations (2) and corresponds to Lagrangian manifolds with complex germs.

Equation (2) in \( Q \)-presentation has the form of infinite-dimensional Schrödinger equation,

\[
i\epsilon \frac{\partial \Phi_Q}{\partial t} = H \left( \frac{\hat{Q}_j - i\hat{P}_j}{\sqrt{2}}, \frac{\hat{Q}_j + i\hat{P}_j}{\sqrt{2}} \right) \Phi_Q
\]

where operators \( \hat{Q}_j \) are multiplications by \( q_j \), operators \( \hat{P}_j \) are derivation operators \(-i\epsilon\partial/\partial q_j\). Thus, \( \hat{Q}_j \) and \( \hat{P}_j \) are infinite-dimensional analogs of coordinate and momentum operators, while \( \epsilon \) is the analog of Planck constant. As it was mentioned above, one can apply semiclassical technique to equation (4) as \( \epsilon \) tends to zero.

Semiclassical approximate solutions of the following type

\[
\phi(q) \exp \left( \frac{i}{\epsilon} S(q) \right),
\]

where the sequence \((q_1, q_2, \ldots)\) is denoted by \( q \), \( S \) is a real functional, are widely used in physics (see, for example, [12]). Functionals (5) are of order \( O(1) \) as \( \epsilon \to 0 \) at all \( q \).

In this paper we consider yet another type of asymptotic solutions to quantized equations. These solutions are not small and have rapidly oscillating form only if the distance between the point \( q \) and surface

\[
\{Q(\tau), \tau \in \Lambda^k\}
\]

is of order \( \epsilon^{1/2} \) as \( \epsilon \) tends to zero, i.e., the following quantity

\[
\min_{\tau \in \Lambda^k} \sum_{i=1}^{\infty} (q_i - Q_i(\tau))^2
\]
is of order $O(\epsilon)$. If quantity (7) is of order $O(\epsilon^{1-\delta})$, $\delta > 0$, then these solutions are exponentially small.

First of all, consider the case of zero-dimensional Lagrangian manifold, i.e. point $(P_1, Q_1, P_2, Q_2, ...) \in \mathcal{L}$. Complex germ method leads to the asymptotics of the following type

$$
\Phi_Q(q) = \mathcal{F} \left( \frac{q - Q}{\sqrt{\epsilon}} \right) \exp \left( i \sum_{\infty} P_i (q_i - Q_i) / \epsilon \right) \quad (8)
$$

Function $\mathcal{F}$ in formula (8) rapidly decays as its argument tends to infinity and can be expressed through the complex germ (see subsection 2.5 for more details).

Give Fock presentation for functional (8). First, consider for the sake of simplicity case $P = Q = 0$. If

$$
\mathcal{F} \left( \frac{q}{\sqrt{\epsilon}} \right) = \mathcal{F}_0 \left( \frac{q}{\sqrt{\epsilon}} \right) = \exp(-\sum_{\infty} q_i^2 / 2\epsilon) \quad (9)
$$

then functional (8) corresponds to the vacuum state vector $\hat{\Phi}_0$ in Fock presentation. If functional $\mathcal{F}$ is arbitrary, then we can extract factor (9) from functional (8) and present corresponding state vector in the form

$$
\mathcal{F}_1 \left( \frac{\hat{Q}}{\sqrt{\epsilon}} \right) \hat{\Phi}_0 \quad (10)
$$

where $\mathcal{F}_1 = \mathcal{F} / \mathcal{F}_0$. Operator $\hat{Q}/\sqrt{\epsilon}$ can be expressed as a function of creation and annihilation operators

$$
\frac{\hat{Q}_i}{\sqrt{\epsilon}} = \frac{\hat{\psi}_i^+ + \hat{\psi}_i^-}{\sqrt{2}} \quad (11)
$$

which does not depend on $\epsilon$. Any function of operator (11) transform vacuum vector to $\epsilon$-independent element of the Fock space, which corresponds to functional (8) if $P = Q = 0$.

Consider now the general case, which can be reduced to the considered case. Namely, functional (8) can be presented in the form

$$
\exp \left( i \sum_{\infty} P_l q_l / \epsilon \right) \exp \left( i Q_l \delta q_l / \delta q_l \right) \mathcal{F}_0 \left( q / \sqrt{\epsilon} \right) \quad (12)
$$

where $l = 1, \infty$. Expression (12) can be simplified by obtaining Baker-Hausdorff formula

$$
e^{A_1 + A_2} = e^{A_1} e^{A_2} e^{-[A_1, A_2]/2} \quad (13)
$$

for operators 

$$
A_1 = i P_l q_l / \epsilon, A_2 = -Q_l \delta q_l / \delta q_l, l = 1, \infty
$$

From

$$
A_1 + A_2 = \frac{1}{\sqrt{\epsilon}} \sum_{\infty} (\hat{\psi}_i^+ \phi_i - \hat{\psi}_i^- \phi_i^*)
$$

we obtain the following state vector in Fock presentation, corresponding to functional (12) :

$$
U_\phi \exp \left( -\frac{i}{2\epsilon} \sum_{\infty} P_l Q_l \right) \hat{Y} \quad (14)
$$
where

\[ U_\phi = \exp\left( \frac{1}{\sqrt{\epsilon}} \sum_{l=1}^{\infty} (\hat{\psi}_l^+ \phi_l - \hat{\psi}_l^- \phi_l^*) \right) \]

\( \hat{Y} \) is \( \epsilon \)-independent state vector, which can be expressed through creation operators [2]:

\[ \hat{Y} = \hat{Y}(\hat{\psi}^+) \hat{\Phi}_0 \]

\[ \hat{Y}(\hat{\psi}^+) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \hat{Y}^{(n)}_{i_1,\ldots,i_n} \hat{\psi}_{i_1}^+ \ldots \hat{\psi}_{i_n}^+ \]

From Baker-Hausdorff formula and the following commutation relations [2]

\[ U_\phi^{-1} \hat{\psi}_l^+ U_\phi = \hat{\psi}_l^+ + \phi_l^*/\sqrt{\epsilon}, \quad U_\phi^{-1} \hat{\psi}_l^- U_\phi = \hat{\psi}_l^- + \phi_l/\sqrt{\epsilon} \] (15)

which can be verified by using commutation relations between creation and annihilation operators, we obtain that vector (14) can be expressed in a form

\[ \Psi_{\phi,Y} = Y(\hat{\psi}^+ - \phi^*/\sqrt{\epsilon}) \exp \left( \frac{1}{\epsilon} \left[ g + \sum_{l=1}^{\infty} \phi_l (\sqrt{\epsilon} \hat{\psi}_l^+ - \phi_l^*) \right] \right) \hat{\Phi}_0 \] (16)

where

\[ g = \phi_l^* \phi_l / 2 + (\phi_l^* \phi_l^* - \phi_l \phi_l) / 4, l = 1, \infty \] (17)

We shall usually use Gaussian functionals \( Y \)

\[ Y(\hat{\psi}^+) = c \exp(\hat{\psi}_i^+ M_{ij} \hat{\psi}_j^+ / 2), i, j = 1, \infty \] (18)

where \( c \) is a constant, \( M \) is a Hilbert-Schmidt operator, \( \| M \| < 1 \).

Thus, we have obtained state vector (16), corresponding to zero-dimensional isotropic manifold. We shall assign matrix \( M \) to each complex germ in subsection 2.3. Discussion about connection between complex germ theories in \( Q \)-presentation and in Fock space in more details will be presented in subsection 2.5.

Consider now asymptotics in \( Q \)-presentation, which are peaked in the vicinity of the surface (1) if \( k > 0 \). In this subsection we discuss partial case of manifold \( \Lambda^k \), which satisfies the conditions

\[ Q_{k+1} = Q_{k+2} = \ldots = P_{k+1} = P_{k+2} = \ldots = 0 \] (19)

General case will be considered in subsection 2.5.

Complex germ asymptotics in a case (19) has the form

\[ \mathcal{F} \left( q_1, \ldots, q_k; \frac{q_{k+1}}{\sqrt{\epsilon}}, \frac{q_{k+2}}{\sqrt{\epsilon}}, \ldots \right) \exp \left( \frac{i}{\epsilon} S(q_1, \ldots, q_k) \right), \] (20)

where \( \mathcal{F} \) rapidly decays at infinity, \( S \) is a real function. Isotropic manifold, corresponding to asymptotics (20), can be parametrized by coordinates \( \tau_a = Q_a, a = 1, k \)

\[ Q_a(\tau) = \tau_a, P_a(\tau) = \partial S/\partial \tau_a \]
In order to give Fock presentation of functional (20) it is convenient to reduce it to functionals (8). Namely, consider the following superposition of expressions (19)

\[
\int \frac{d\tau}{\epsilon^{k/2}} f \left( \tau_1, \ldots, \tau_k; \frac{q_1 - \tau_1}{\sqrt{\epsilon}}, \ldots, \frac{q_k - \tau_k}{\sqrt{\epsilon}}, \frac{q_{k+1}}{\sqrt{\epsilon}}, \frac{q_{k+2}}{\sqrt{\epsilon}}, \ldots \right) 
\times \exp \left( \frac{i}{\epsilon} \left[ S(\tau) + \sum_{a=1}^{k} \frac{\partial S}{\partial \tau_a}(\tau)(q_a - \tau_a) \right] \right) 
\]  

(21)

and choose rapidly decaying at infinity function \( f \) in order to make functional (21) approximately equal to (20). Consider the following substitution in integral (21):

\[ \xi_a = (q_a - \tau_a)/\sqrt{\epsilon}, a = 1, k, \]

which transforms it to the following expression:

\[
\int d\xi_1 \ldots d\xi_k f \left( q_1 - \xi_1 \sqrt{\epsilon}, \ldots, q_k - \xi_k \sqrt{\epsilon}; \xi_1, \ldots, \xi_k; \frac{q_{k+1}}{\sqrt{\epsilon}}, \frac{q_{k+2}}{\sqrt{\epsilon}}, \ldots \right) 
\times \exp \left( \frac{i}{\epsilon} \left[ S(q - \xi \sqrt{\epsilon}) + \sum_{a=1}^{k} \frac{\partial S}{\partial q_a}(q - \xi \sqrt{\epsilon})\xi_a \sqrt{\epsilon} \right] \right) 
\]

(22)

If \( \epsilon \to 0, \xi = \text{const} \) then the exponent in formula (22) has the form

\[
\exp \left[ -\frac{i}{2} \xi_a \frac{\partial^2 S(q)}{\partial q_a \partial q_b} \xi_b + \frac{i}{\epsilon} S(q) \right] 
\]

Taking into account that \( f \) rapidly decays, we obtain that formulas (21) and (20) are approximately equal if

\[
F \left( q_1, \ldots, q_k; \frac{q_{k+1}}{\sqrt{\epsilon}}, \frac{q_{k+2}}{\sqrt{\epsilon}}, \ldots \right) 
= \int d^k \xi f \left( q_1, \ldots, q_k; \xi_1, \ldots, \xi_k; \frac{q_{k+1}}{\sqrt{\epsilon}}, \frac{q_{k+2}}{\sqrt{\epsilon}}, \ldots \right) \exp \left[ -\frac{i}{2} \xi_a \frac{\partial^2 S(q)}{\partial q_a \partial q_b} \xi_b \right] 
\]

(23)

Of course, there are many such functions \( f \).

Functional (21) has the following form in Fock presentation

\[
\int \frac{d\tau}{\epsilon^{k/2}} Y(\hat{\psi}^+ - \phi^*/\sqrt{\epsilon}) \exp \left( \frac{1}{\epsilon} [g(\tau) + iS(\tau) + \sum_{l=1}^{\infty} \phi_l(\tau)(\sqrt{\epsilon}\hat{\psi}_l^+ - \phi^*_l(\tau))] \right) \Phi_0 
\]

(24)

In formula (24) \( g(\tau) \) has the form (17). We shall usually use functionals \( Y \) of the form (18). Vector (24) will be multiplied by \( \epsilon^{k/4} \) for making its norm of order \( O(1) \). Notice also that

\[
g(\tau) + iS(\tau) = \int_{\tau(0)}^{\tau} \phi_l d\phi^*_l + g(\tau(0)) + iS(\tau(0)) 
\]

We shall define a canonical operator by formula, analogous to (24) in subsection 2.4; some auxiliary lemmas will be proved in subsection 2.3. Connection between the expression (24) and traditional canonical operator, corresponding to arbitrary Lagrangian manifold with complex germ, will be discussed in subsection 2.5.
2.3 Some auxiliary lemmas

We are going now to prove some lemmas.

**Lemma 1** Let $L$ be an operator in $l^2$ that maps a sequence $(\xi_1, \xi_2, \xi_3, \ldots)$ into the sequence $(0, 0, \ldots, 0, \xi_{k+1}, \xi_{k+2}, \ldots)$, and let $Y$ be a bounded operator in $l^2$ satisfying the following properties:

1. $(\xi, Y\xi) \geq 0, \xi \in l^2$;
2. if $L\xi = 0, (\xi, Y\xi) = 0$, then $\xi = 0$.

Then there exists a positive $\kappa$, such that $(\xi, (L + Y)\xi) \geq \kappa(\xi, \xi)$ for any $\xi$.

**Proof.** It follows from the assumptions of the lemma that $(\xi, (L + Y)\xi) \geq (L\xi, L\xi)$. (25)

On the other hand, the property 2 implies that for $L\xi = 0$

$$(\xi, Y\xi) \geq \sigma(\xi, \xi), \sigma > 0.$$ (26)

It follows then that

$$(\xi, (L + Y)\xi) = ((L - E)\xi, Y(E - L)\xi) + (L\xi, Y(E - L)\xi) + (E - L)\xi, YL\xi) + (L\xi, (YL + L)\xi) \geq \sigma(\xi, \xi) - 2\|Y\|\sqrt{(L\xi, L\xi)(\xi, \xi)} - (\|Y\| + \sigma)(L\xi, L\xi),$$

where $\|Y\| = \sup_{\|\xi\|=1} \|Y\xi\|$.

Suppose that for any $\delta > 0$ there exists a vector $\xi \in l^2$ such that $(\xi, (L + Y)\xi) < \delta(\xi, \xi)$. It follows then from (25) that $(\xi, Y\xi) < \delta(\xi, \xi)$, and it follows from (26) that

$$(\xi, (L + Y)\xi) \geq (\sigma - (\|Y\| + \sigma)\delta - 2\|Y\|\sqrt{\delta})(\xi, \xi).$$

This inequality for $\delta$ small enough contradicts the condition $(\xi, (L + Y)\xi) < \delta(\xi, \xi)$. The contradiction obtained proves Lemma 1.

Denote by $W_{ab}(\tau), \tau \in \Lambda^k, a, b = 1, k$ the inverse matrix to the matrix $\sum_{i=1}^{\infty} \frac{\partial \phi_{i}^*}{\partial \tau_{a}} \frac{\partial \phi_{i}}{\partial \tau_{b}}$, which is invertable by axiom m2. We set

$$M_{ij}(\tau) = ((C + iB)(\tau)(C - iB)^{-1}(\tau))_{ij} - \sum_{a,b=1}^{k} \frac{\partial \phi_{i}^*}{\partial \tau_{a}} W_{ab}(\tau) \frac{\partial \phi_{j}}{\partial \tau_{b}}$$ (27)

**Lemma 2** The operator $M$ has the following properties

1. $M$ is a Hilbert-Schmidt operator;
2. $\|M\| < 1$;
3. $\sum_{j=1}^{\infty} M_{ij} \partial \phi_{j}^{*}/\partial \tau_{a} = 0, \ a = 1, k.$
Proof. Since the operators $C + iB$ and $\sum_{a,b=1}^{k} \partial_{\phi_i a} W_{ab} \partial_{\phi_j b}$ are Hilbert-Schmidt operators, and the operator $(C - iB)^{-1}$ is a bounded operator, we have that $M$ is a Hilbert-Schmidt operator.

Axiom r2 and definition (27) of $M$ imply that

$$\sum_{j=1}^{\infty} M_{ij} \partial \phi_j^*/\partial \tau_a = 0, \quad a = 1, k.$$  

Expression (27) imply as well that

$$E - M^+ M = ((C - iB)^+)^{-1} \left( (C + iB)^+ \sum_{a,b=1}^{k} \partial_{\phi_i a} W_{ab} \partial_{\phi_j b} \right) (C + iB) + 2L (C - iB)^{-1}$$

Lemma 1 implies that for some $\rho > 0 (\xi, (E - M^+ M) \xi) > \rho (\xi, \xi)$. It follows then that $\|M\| < 1$. Lemma 3 is proved.

2.4 Definition of a canonical operator

Consider an aggregate consisting of a number $\epsilon > 0$, a sequence $\phi_j, \sum_{j=1}^{\infty} |\phi_j|^2 < \infty$, and a Hilbert-Schmidt operator $M : l^2 \to l^2, \|M\| < 1$. Assign the following element of $\mathcal{H}$ to such an aggregate:

$$\hat{\Phi}_{\phi,M} = \exp \left\{ \frac{1}{\epsilon} \phi_j (\sqrt{\epsilon} \hat{\psi}_j^+ - \phi_j^*) + \frac{1}{2\epsilon} (\hat{\psi}_j^+ \sqrt{\epsilon} - \phi_j^*) M_{ij} (\hat{\psi}_j^+ \sqrt{\epsilon} - \phi_j^*) \right\} \hat{\Phi}_0,$$

(28)

where we mean summing over repeating indices, $i, j = 1, \infty$.

Remark 4 Expanding the exponent in formula (28) into the power series over the variable $\epsilon$ we obtain the $l$-th component of $\hat{\Phi}_{\phi,M} \in \mathcal{H}$ in the following form:

$$(\hat{\Phi}_{\phi,M})^{(l)}_{i_1 \ldots i_l} = \sum_{k=0}^{[l/2]} \frac{c}{\sqrt{l!2^k k!}} a_{i_1} \ldots a_{i_l} \cdots \sum_{1 \leq j_1 \neq \ldots \neq j_{2k} \leq l} M_{ij_1 i_2} \ldots M_{ij_{2k-1} i_{2k}} a_{i_1} \ldots a_{i_{2k}}$$

(29)

where

$$a_m = (\phi_m - M_{mn} \phi_n^*)/\sqrt{\epsilon}, \quad m, n = 1, \infty$$

$$c = \exp \left\{ -\frac{1}{\epsilon} \phi_j^* \phi_j - \frac{1}{2\epsilon} \phi_j^* M_{ij} \phi_j^* \right\}, \quad i, j = 1, \infty$$
Consider a Lagrangian manifold with complex germ satisfying the quantization conditions
\[
\frac{1}{2\pi\epsilon} \int_l P_i dQ_i = n_i, \quad n_i \in \mathbb{Z}, i = 1, \infty
\] (30)
l is an arbitrary closed path on the isotropic manifold \(\Lambda^k\).

Let \(f\) be an infinitely differentiable function with compact support on the isotropic manifold \(\Lambda^k\), and let \(\tau^{(0)}\) be an arbitrary point on \(\Lambda^k\).

Denote
\[
g = \phi_i^*(\tau^{(0)})\phi_i(\tau^{(0)})/2 + (\phi_i^*(\tau^{(0)})\phi_i^*(\tau^{(0)}) - \phi_i(\tau^{(0)})\phi_i(\tau^{(0)}))/4, l = 1, \infty,
\]
\[
F(\tau) = \frac{C(\tau) + iB(\tau)}{\sqrt{2}}, G(\tau) = \frac{C(\tau) - iB(\tau)}{\sqrt{2}}, \tau \in \Lambda^k.
\]

We assign the following element of \(H\) to the Lagrangian manifold with complex germ \([\Lambda^k, r]\), the function \(f\), the point \(\tau^{(0)} \in \Lambda^k\), and the number \(\epsilon > 0\):
\[
\hat{K}^{\epsilon}_{[\Lambda^k, r], \tau^{(0)}} f = \frac{1}{\Lambda_k} \frac{d\sigma}{d\tau_1...d\tau_k} \frac{\exp\left(\frac{1}{4\epsilon^2} \int_{\tau^{(0)}}^{\tau} \phi_i(\tau')d\phi_i^*(\tau')\right)}{\sqrt{\det G(\tau)G(\tau)}} \frac{d\xi}{\phi(\tau), M(\tau)}
\] (31)
where \(M\) is defined by formula (27), \(d\sigma\) is the following measure on \(\Lambda^k\)
\[
d\sigma = \sqrt{\det \frac{\partial \phi_i^*}{\partial \tau_a}(\tau)\frac{\partial \phi_i}{\partial \tau_b}(\tau)} d\tau,
\]
d\(\tau \equiv d\tau_1 ... d\tau_k; a, b = 1, k, i = 1, \infty\). This measure does not depend on the choice of local coordinates \(\tau_1, ..., \tau_k\) on \(\Lambda^k\).

**Remark 5** Since the manifold \(\Lambda^k\) is isotropic the integral \(\int_{\tau^{(0)}}^{\tau} \phi_i(\tau')d\phi_i^*(\tau')\) does not depend locally on the path.

**Remark 6** The integrand in (31) is a one-valued function due to the quantization conditions (30).

**Remark 7** It follows from the axiom r4 that
\[
(C - iB)^+(C - iB) = 2L + (C + iB)^+(C + iB)
\] (32)
Formula (32) implies that the operator \(G^+(\tau)G(\tau) - E\) is an operator of trace class, and therefore Fredholm determinant for the operator \(G^+(\tau)G(\tau)\) is defined.

**Remark 8** By the axiom r6 \(\det[G^+(\tau)G(\tau)] > 0\)
2.5 Connection with the traditional definition of a canonical operator

Consider now the relation between definition [31] of a canonical operator and the traditional definition of a canonical operator [13, 1]. Write down the vectors (28) and (31) in $\mathcal{Q}$-presentation. Consider an auxiliary presentation with the wave function depending on case of focal points into consideration.

Let $I$ be a finite set of positive integers, $I = \{i_1, i_2, \ldots, i_l\}$. Denote by $\Lambda$ the diagonal matrix with $\Lambda_{jj} = 1$ for $j \not\in I$, $\Lambda_{jj} = -j$ for $j \in I$. We assign the following presentation ($I$-presentation) of the vectors of $\mathcal{H}$ to each set $I$.

To each element of $\mathcal{H}$ we assign the functional

$$
\Phi_I(q_1^I, q_2^I, \ldots) = \int \frac{dq_{i_1} \ldots dq_{i_l}}{(2\pi\epsilon)^{l/2}} \exp \left( -\frac{i}{\epsilon} \sum_{j \in I} q_j^I q_j^* \right) \cdot \Phi_Q(q_1^I, \ldots, q_i^I, q_{i+1}^I, \ldots, q_{i-1}^I, q_i^I, q_{i+1}^I, \ldots)
$$

The $Q$-presentation is a special case of the $I$-presentation for $I = \emptyset$.

In order to give $I$-presentations of the vectors (28) and (31) use the formula [2]

$$
\Phi(a^*) = \int K_z(a^*) \Phi(z^*) e^{-z^*z} \prod dz^* dz.
$$

Here $K_z(a^*)$ is the generating functional corresponding to the vector $K_z = \exp(\sum_{j=1}^{\infty} z_j \hat{\psi}_j^+ \hat{\Phi}_0$, the measure in the functional integral is defined in [4], and $\hat{\Phi}$ is an arbitrary element of $\mathcal{H}$.

Introduce following notation

$$
P_j^I(\tau) = -Q_j(\tau), j \in I, P_j^I(\tau) = P_j(\tau), j \not\in I;$$

$$Q_j^I(\tau) = P_j(\tau), j \in I, Q_j^I(\tau) = Q_j(\tau), j \not\in I;$$

$$\phi_j^I(\tau) = (Q_j^I(\tau) + i P_j^I(\tau)) / \sqrt{2} = \Lambda_{jm} \phi_m(\tau), j, m = 1, \infty,$$

$$z_j^I = \Lambda_{jm} z_m, z_j^I = \Lambda_{jm} z_m, j, m = 1, \infty.$$

The functional

$$
(K_z)_I(q_1^I, q_2^I, \ldots) = \exp \left\{ \frac{1}{2\sqrt{\epsilon}} z_m^I \left( q_m^I - \epsilon \frac{\partial}{\partial q_m^I} \right) \right\} \exp \left\{ -\frac{1}{2\epsilon} q_m^I q_m^I \right\}
$$

$$= \exp \left\{ -\frac{1}{2} z_m^I z_m^I + \sqrt{2} \int \frac{\partial}{\partial q_m^I} q_m^I q_m^I - \frac{1}{2\epsilon} q_m^I q_m^I \right\},
$$

$m = 1, \infty$

corresponds to the vector $\hat{K}_z$ in the $I$-presentation.

This expression follows from the Baker-Hausdorff formula

$$
e^{A_1 + A_2} = e^{A_1} e^{A_2} e^{-\frac{1}{2} [A_1, A_2]}, A_1 = \frac{1}{\sqrt{2\epsilon}} z_m^I q_m^I, A_2 = -\sqrt{\frac{\epsilon}{2}} z_m^I \frac{\partial}{\partial q_m^I}.$$
and the equality

\[ \exp \left\{ a_i \frac{\partial}{\partial q_i^l} \right\} f(q_1^l, q_2^l, \ldots) = f(q_1^l + a_1, q_2^l + a_2, \ldots). \]

It now follows that the functional corresponding to the vector \( \hat{\Phi}_{\phi, M} \) has the following form in \( I \)-presentation:

\[
(\Phi_{\phi, M})_I(q_1^l, q_2^l, \ldots) = \int \prod dz^* dz \exp \left( -z^*_l z_l^I - \frac{1}{2} z^*_l z_l^I + \sqrt{\epsilon} z^*_l q_l^I - \frac{1}{2\epsilon} q_l^I q_l^I \right) \\
+ \frac{1}{2\epsilon}(\epsilon z^*_l - \phi_l^I)M_l^I(z^*_l \epsilon - \phi^*_l) \\
- \frac{1}{\epsilon} \phi^*_l (\epsilon z_l^I - \phi^*_l), \quad l, m = 1, \infty,
\]

where \( M^I = \Lambda M \Lambda \).

By (2) this integral equals to

\[
(\Phi_{\phi, M})_I(q_1^l, q_2^l, \ldots) = \exp \left\{ \frac{1}{\epsilon} \phi_l^I - \phi^*_l \right\} \left( q_l^I - \phi^*_l \right) \\
+ \frac{i}{2\epsilon} \left( q_l^I - \phi^*_l \right) A^I_{mn} \left( q_n^I - \phi_l^I \right) - \frac{1}{2\epsilon} \phi_l^I \phi^*_l \\
A^I = i(E - M^I)(E + M^I)^{-1}
\]

\[(33)\]

**Remark 9** Not any element of \( \mathcal{H} \) of form (28) may be presented in form (33). Indeed, \( \det(E + M^I) \) exists only if the operator \( M^I \) is an operator of trace class, and the (28) may be defined in other cases as well. For example,

\[
\exp \left\{ \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n} (\phi^*_n)^2 \right\} \hat{\Phi}_0 \in \mathcal{H}
\]

but this vector may not be written down in the \( Q \)-presentation. It means that the definition of a canonical operator in Fock presentation \((71)\) is more general than the corresponding definition in \( Q \)-presentation.

Consider now the vector \((71)\) in the \( I \)-presentation.

By \((71)\) and \((33)\) we have

\[
(K^\epsilon_{[A^k, r], \tau(0)} f)_I(q_1^l, q_2^l, \ldots) = \int d\tau f(\tau) \sqrt{\frac{\det \frac{\partial \phi_l^I}{\partial r_a}(\tau) \frac{\partial \phi^*_l}{\partial r_b}(\tau)}{(2\pi)^{k/2} \epsilon^{k/4}} \\
\exp \left\{ \frac{i}{\epsilon} f^I_{(0)} P^I_l(\tau') + \frac{i}{\epsilon} E^I_l(\tau) (q_l^I - Q_l^I(\tau)) + \frac{i}{\epsilon} (q_l^I - Q_l^I(\tau)) A^I_{lm}(\tau) (q_m^I - Q_m^I(\tau)) \right\} \\
\sqrt{\det \left[ \frac{C(\tau) - iB(\tau)}{\sqrt{2}} \right] + \left[ \frac{C(\tau) - iB(\tau)}{\sqrt{2}} \right]} \sqrt{\det(E + M^I(\tau))}
\]
of domains $\Omega^\alpha$. We present now a heuristic justification of a relation between the definition of a canonical operator in $Q$-presentation (35) and the traditional definition of a canonical operator [13, 1].

Cover the support of the function $f$ on the isotropic manifold $\Lambda^k$ by a finite number of domains $\Omega^\alpha$, $\alpha = 1, \ldots, M$ such that each domain has a one-to-one projection on one of the coordinate planes of the form

\[
P_j = 0, \quad j \notin I^\alpha, \quad Q_j = 0, \quad j \in I^\alpha
\]

\[I^\alpha = \{i_1^\alpha, i_2^\alpha, \ldots, i_{M^\alpha}^\alpha\} \subset \mathbb{N}.
\]

Consider a partition of unity $1 = \sum_{\alpha=1}^M e^\alpha(\tau)$, where functions $e^\alpha(\tau)$ are infinitely differentiable with a support in $\Omega^\alpha$. Introduce a notation $f^\alpha(\tau) = f(\tau)e^\alpha(\tau)$.

The canonical operator in the $Q$-presentation has the form

\[
(K^\prime_{[\Lambda^k, r], \tau(0)}) f^\alpha = \sum_{a=1}^M \int \frac{d\tau q^\alpha_1 \cdots d\tau q^\alpha_{M^\alpha}}{(2\pi \epsilon)^{1/2}} \exp \left( -\frac{i}{\epsilon} \sum_{j \in I^\alpha} q^\alpha_j \right)
\]

\[\cdot \left( K^\prime_{[\Lambda^k, r], \tau(0)} f^\alpha \right)_{I^\alpha} \left( q^\alpha_1, \ldots, q^\alpha_{M^\alpha-1}, q^\alpha_{M^\alpha}, q^\alpha_{M^\alpha+1}, \ldots, q^\alpha_{M^\alpha-1}, q^\alpha_{M^\alpha}, q^\alpha_{M^\alpha+1}, \ldots \right) \quad (36)
\]

Calculate now the integral in formula (34) that expresses $(K^\prime_{[\Lambda^k, r], \tau(0)} f^\alpha)_{I^\alpha}$.

Introduce following notation:

\[
B^I_{mn} = -C^I_{mn}, \quad B^I_{mn} = B^I_{mn}, \quad m \in I
\]

\[B^I_{mn} = B^I_{mn}, \quad C^I_{mn} = C^I_{mn}, \quad m \notin I
\]
and $S_{ab}$ is the inverse matrix to the matrix

$$\frac{\partial Q^I_i}{\partial \tau_a} A^I_{ij} \frac{\partial Q^I_j}{\partial \tau_b} - \frac{\partial P^I_i}{\partial \tau_a} \frac{\partial Q^I_i}{\partial \tau_b}, \quad a, b, = \overline{1, k}, \quad i, j = \overline{1, \infty}.$$ 

The following statement holds.

**Statement 1**

1.

$$(B^I(C^I)^{-1})_{mn} = A^I_{mn} - \left( A^I_{mr} \frac{\partial Q^I_r}{\partial \tau_a} - \frac{\partial P^I_r}{\partial \tau_a} \right) S_{ab} \left( \frac{\partial Q^I_i}{\partial \tau_b} A^I_{jn} - \frac{\partial P^I_i}{\partial \tau_b} \right),$$

$$j, m, n, r = \overline{1, \infty}, \quad a, b = \overline{1, k} \quad (37)$$

2.

$$\frac{\det \left( \frac{\partial \phi^I_a}{\partial \tau_a} \frac{\partial \phi^I_a}{\partial \tau_b} \right)}{\det (E + M^I)} = \det \left[ \frac{1}{2} \left( E - iB^I(C^I)^{-1} \right) \right] \cdot \det \left[ \frac{1}{i} \left( \frac{\partial Q^I_m}{\partial \tau_a} A^I_{mn} \frac{\partial Q^I_n}{\partial \tau_b} - \frac{\partial P^I_m}{\partial \tau_a} \frac{\partial Q^I_m}{\partial \tau_b} \right) \right]$$

$$m, n = \overline{1, \infty}, \quad a, b = \overline{1, k} \quad (38)$$

**Proof.** Property (37) follows from the following expressions for $B^I(C^I)^{-1}$ and $A^I$:

$$B^I(C^I)^{-1} = i(E + N^I)^{-1}(E - N^I),$$

$$N^I_{mn} = M^I_{mn} + \frac{\partial \phi^I_m}{\partial \tau_a} W_{ab} \frac{\partial \phi^I_n}{\partial \tau_b}, \quad a, b = \overline{1, k}, \quad m, n = \overline{1, \infty},$$

where $W_{ab}$ is the inverse matrix to the matrix $\frac{\partial \phi^I_m}{\partial \tau_a} \frac{\partial \phi^I_m}{\partial \tau_b}, a, b = \overline{1, k}, m = \overline{1, \infty},$

$$A^I = i(E + M^I)^{-1}(E - M^I)$$

and from the property $M_{mn} \frac{\partial \phi^I_m}{\partial \tau_a} = 0$.

The proof of formula (38) bases on the following Lemma.

**Lemma 3** Let $y^a, z^a \in l^2, a = \overline{1, k},$ and let $R$ be the operator in $l^2$ of the form $R \kappa = \kappa - \sum_{c=1}^{k} y^c(z^c, \kappa), \kappa \in l^2.$

Then $\det R = \det (\delta_{ab} - (y^a, z^b)), a, b = \overline{1, k}.$

**Proof.** Choose an orthonormal basis $\{e_s, s = \overline{1, \infty}\}$ in $l^2$ such that only the first $k$ components of the vectors $y^a, a = \overline{1, k}$ differ from zero. Then

$$\det R = \det (\delta_{ij} - \sum_{c=1}^{\infty} (e_i, y^c)(z^c, e_j)), \quad i, j = \overline{1, k}.$$
Since the functions $\det(\delta_{ab} - \alpha(y^a, z^b))$ and $\det(\delta_{ij} - \alpha \sum_{c=1}^\infty (e_i, y^c)(z^c, e_j))$ are polynomials in $\alpha$, and Taylor-series expansions of their logarithms as $\alpha \to 0$ coincide these functions coin stimulate well.

This completes the proof of Lemma 3. Lemma 3 implies that

$$\det \left( \delta_{mn} + \frac{1}{m} \frac{\partial \phi_n}{\partial x_m} W_{ab} \frac{\partial \phi_a}{\partial x_b} \right) = \det \left( \delta_{ab} + W_{ac} \frac{\partial \phi_a}{\partial x_c} (E + M) \right), \quad a, b, c = 1, k, \quad s, m, n = 1, \infty$$

Formula (38) now follows. The statement is proved.

Therefore we have

$$\langle K^{\varepsilon}_{[\Lambda_k], \tau(0), f_\alpha} \rangle_{I_{\Lambda_k}} (q^I_{1\alpha}, q^I_{2\alpha}, \ldots) = \int_{\Lambda_k} \frac{d\gamma^I}{(2\pi)^{k/2}} \frac{1}{\sqrt{\det(C^{I_{\alpha}} \sqrt{2})}} \left( \frac{1}{\varepsilon} \frac{\partial \gamma^I_{\alpha}}{\partial x} \gamma^I_{\alpha} \right) \int_{\tau(0)} \frac{d\tau}{\sqrt{\det((C - iB)/\sqrt{2})}} \right) > 0$$

(this real part cannot vanish since

$$\Re \det(C^{I_{\alpha}} \sqrt{2})/(\varepsilon^{I_{\alpha}} \det((C - iB)/\sqrt{2})) > 0).$$

Find the asymptotics of integral (38). Since $\Im A^I > 0$ only the domain in $\mathbb{R}^k$ where $\langle q^I_{m\alpha} - Q^I_{m\alpha}(\tau) \rangle > 0$ provides a non exponentially small contribution. Denote by $\tau(q)$ the point of minimum of $\langle q^I_{m\alpha} - Q^I_{m\alpha}(\tau) \rangle$.

Expanding the exponent in a neighborhood of the point $\tau$ we obtain

$$f_{\tau(0)} (P^I_{m\alpha} (\tau') dQ^I_{m\alpha}(\tau') + P^I_{m\alpha}(\tau)(q^I_{m\alpha} - Q^I_{m\alpha}(\tau)) = \frac{1}{2} \varepsilon (q^I_{m\alpha} - Q^I_{m\alpha}(\tau))^2) \quad m, n = 1, \infty, \quad a, b = 1, k, \quad t_a = (\tau_a - \tau_0)/\varepsilon, \quad \xi_m = (q_m - Q_m(\tau))/\varepsilon$$

Integrating over $t_a, \xi_m$ we obtain the following asymptotics

$$\langle K^{\varepsilon}_{[\Lambda_k], \tau(0), f_\alpha} \rangle_{I_{\Lambda_k}} (q^I_{1\alpha}, q^I_{2\alpha}, \ldots) = e^{\text{const}} \int_{\tau(0)} \frac{d\tau}{\sqrt{\det(C^{I_{\alpha}} \sqrt{2})}} (40)$$
Remark 10 It follows therefore that the definition of a canonical operator in the present paper differs from the classical definition only by the factor \( \text{const} \exp\left(\frac{i}{2} \text{Arg} \det C \right) \). This factor changes the quantization conditions and the transport equation for function \( f \).

This heuristic will not however be used in the proof of the theorem.

3 Time evolution of a Lagrangian manifold with complex germ

Consider now a canonical transformation of a Lagrangian manifold with complex germ. Let

\[
H(\phi^*, \phi) = \sum_{m=1}^{s} \sum_{n=1}^{s} H_{i_1 \ldots i_m j_1 \ldots j_n}^{(m,n)} \phi_{i_1}^* \ldots \phi_{i_m}^* \phi_{j_1} \ldots \phi_{j_n}
\]  

(41)

\[
\bar{H}_{i_1 \ldots i_m j_1 \ldots j_n}^{(m,n)} = H_{j_1 \ldots j_n i_1 \ldots i_m}^{(n,m)}
\]

is symmetric separately over \( i_1, \ldots, i_m \) and over \( j_1, \ldots, j_n \).

Definition 3 We say that a canonical transformation \( D_t H, t \in [0, T] \) of a Lagrangian manifold with complex germ \([\Lambda^k_t, r_t]\) corresponds to the Hamiltonian \( H \), if

1. there exists on the segment \([0, T]\) a solution \( \phi_j(\tau, t) \) of the Cauchy problem

\[
i \dot{\phi}_j = \frac{\partial H}{\partial \phi_j^*}(\phi_j^*, \phi),
\]

\[
\phi_j(\tau, t)|_{t=0} = \phi_j^0(\tau) = (Q_j^0(\tau) + iP_j^0(\tau))/\sqrt{2},
\]

(42)

such that for any \( \lambda_1, \ldots, \lambda_k \), \( \lambda_i \in \{0, 1, 2, \ldots\}, \ i = 1, k \) the derivatives

\[
\frac{\partial^{\lambda_1+\ldots+\lambda_k}}{\partial \tau_1^{\lambda_1} \ldots \partial \tau_k^{\lambda_k}} \phi_j(\tau, t)
\]

exist, and the series \( \sum_{j=1}^{\infty} \left| \frac{\partial^{\lambda_1+\ldots+\lambda_k}}{\partial \tau_1^{\lambda_1} \ldots \partial \tau_k^{\lambda_k}} \phi_j(\tau, t) \right|^2 \) converges.

2. there exists on the segment \([0, T]\) a solution of the following Cauchy problem

\[
i \dot{\Pi}_{mn}(\tau, t) = -\frac{\partial^2 H}{\partial \phi_m \partial \phi_l^*} \Pi_{ln}(\tau, t) - \frac{\partial^2 H}{\partial \phi_m \partial \phi_l} \Omega_{ln}(\tau, t)
\]

\[
i \dot{\Omega}_{mn}(\tau, t) = \frac{\partial^2 H}{\partial \phi_m^* \partial \phi_l} \Pi_{ln}(\tau, t) + \frac{\partial^2 H}{\partial \phi_m^* \partial \phi_l} \Omega_{ln}(\tau, t)
\]

(43)

\[
\Pi_{mn}(\tau, 0) = \delta_{mn}, \quad \Omega_{mn}(\tau, 0) = 0 \quad m, n, l = 1, \infty
\]
(the arguments \(\phi^*(\tau, t), \phi(\tau, t)\) at the derivatives of \(H\) are omitted) such that for any
\(\lambda_1, \ldots, \lambda_k, \quad \lambda_i \in \{0, 1, 2, \ldots\}\), \(i = 1, k\) the derivatives

\[
\Pi^{(\lambda_1, \ldots, \lambda_k)}_{mn} = \frac{\partial^{\lambda_1+\ldots+\lambda_k}}{\partial \tau_1^{\lambda_1} \ldots \partial \tau_k^{\lambda_k}} \Pi_{mn}(\tau, t)
\]

\[
\Omega^{(\lambda_1, \ldots, \lambda_k)}_{mn} = \frac{\partial^{\lambda_1+\ldots+\lambda_k}}{\partial \tau_1^{\lambda_1} \ldots \partial \tau_k^{\lambda_k}} \Omega_{mn}(\tau, t)
\]

exist, the operators \(\Pi^{(\lambda_1, \ldots, \lambda_k)}\) are bounded, and the operators \(\Omega^{(\lambda_1, \ldots, \lambda_k)}\) are Hilbert-Schmidt operators.

A canonical transformation \(\mathcal{D}_H^{t}\) of a Lagrangian manifold with complex germ is a set of transformations mapping the manifold \(\Lambda^k_0\) in the manifold

\[
g^t_{H-k} = \Lambda^k_t = \left\{ P_j = \frac{\phi_j(\tau, t) - \phi_j^*(\tau, t)}{\sqrt{2i}}, Q_j = \frac{\phi_j(\tau, t) + \phi_j^*(\tau, t)}{\sqrt{2}} \right\}
\]

where \(\phi_j(\tau, t)\) is a solution of the system \((\Omega)\), and the matrices \(B(\tau, 0), C(\tau, 0)\) into the matrices \(B(\tau, t), C(\tau, t)\) such that

\[
\begin{pmatrix}
C - iB \\
C + iB
\end{pmatrix}(\tau, t) = \begin{pmatrix}
\Pi & \bar{\Omega} \\
\Omega & \bar{\Pi}
\end{pmatrix}(\tau, t) \begin{pmatrix}
C - iB \\
C + iB
\end{pmatrix}(\tau, 0)
\]

Remark 11 The substitution

\[
\phi_j = (Q_j + iP_j)/\sqrt{2}, \phi_j^* = (Q_j - iP_j)/\sqrt{2}
\]

makes the equations \((\Omega)\) Hamiltonian with the Hamiltonian function \(H((Q_j - iP_j)/\sqrt{2}, (Q_j + iP_j)/\sqrt{2})\).

Remark 12 The equations for the matrices \(B\) and \(C\) are equations in variations for this Hamiltonian system (see \([\Omega, \Omega]\)).

Lemma 4 The pair consisting of the manifold \(\Lambda^k_t\) and the complex germ corresponding to the matrices \(B(\tau, t), C(\tau, t)\) is a Lagrangian manifold with complex germ.

Proof. First of all, check that the following matrix

\[
\begin{pmatrix}
\Pi & \bar{\Omega} \\
\Omega & \bar{\Pi}
\end{pmatrix}
\]

is a matrix of a proper canonical transformation, i.e.

\[
\Pi^+\Pi - \Omega^+\Omega = E, \quad \Omega^T\Pi = \Pi^T\Omega, \Pi\Pi^+ - \bar{\Omega}\Omega^T = E, \Omega\Pi^+ = \bar{\Pi}\Omega^T.
\]
At the initial moment these properties obviously hold. Equations (43) imply that
\[ (\Pi^+\Pi - \Omega^+\Omega)^\cdot = 0, \quad (\Omega^T\Pi - \Pi^T\Omega)^\cdot = 0 \]
\[ (\Pi\Pi^+ - \bar{\Omega}\bar{\Omega}^T)^\cdot = 0, \quad (\Omega\Pi^+ - \bar{\Pi}\bar{\Omega}^T)^\cdot = 0. \]
Properties (46) hold therefore at any moment of time.

Check now the axioms of a Lagrangian manifold with complex germ. Properties (46) imply germ axioms r3, r4 and axiom m3. Axiom m2 follows from the invertibility of the matrix (45) Axioms m1, r5 follow immediately from the definition 3. Axiom r1 follows from linearity of the system (43). Axiom r2 follows from the equation (42).

Berezin have shown [2] that for matrix of a proper canonical transformation (45) \[ ||\Pi^+ - \bar{\Omega}|| < 1 \] and operator \( \Pi^+ - \bar{\Omega} \) is bounded. Formula (32) implies that
\[ ||(C + iB)(0,\tau)\bar{\Omega}(C + iB)^{-1}(0,\tau)|| \leq 1, \text{so} \]
\[ ||\Pi^+ - \bar{\Omega}(C + iB)(0,\tau)(C + iB)^{-1}(0,\tau)|| < 1. \]
Thus, the following operator
\[ (C + iB)^{-1}(0,\tau)(E + \Pi^+\bar{\Omega}(C + iB)(0,\tau)(C - iB)^{-1}(0,\tau))^{-1}\Pi^{-1}(t,\tau) = (C - iB)^{-1}(t,\tau) \]
is bounded, so axiom r6 is also satisfied. Lemma 3 is proved.

It is also easy to check that the quantization condition (30) holds at any moment of time whenever it holds at the initial moment.

**Remark 13** In the case of a germ in a point the preserving of the germ axioms has the following sense: the product of two matrices of proper canonical transformations \[ [2] \]
\[ \left( \begin{array}{cc} \Pi & \bar{\Omega} \\ \Omega & \Pi \end{array} \right) \quad \text{and} \quad \left( \begin{array}{cc} G & \bar{F} \\ F & G \end{array} \right) \]
is a matrix of a proper canonical transformation.

## 4 Connection between geometrical and canonical quantization and some examples

Suppose \( H(\sqrt{\epsilon}\hat{\psi}^+, \sqrt{\epsilon}\hat{\psi}^-) \) is a selfadjoint operator in \( \mathcal{H} \) of the form
\[ H(\sqrt{\epsilon}\hat{\psi}^+, \sqrt{\epsilon}\hat{\psi}^-) = \sum_{m,n=1}^s H_{\nu,\mu}(m,n) \epsilon^{\frac{m+n}{2}} \hat{\psi}^+_{\nu} \hat{\psi}^-_{\mu} \ldots \hat{\psi}^+_{\mu} \hat{\psi}^-_{\nu}, \quad (47) \]

Consider the Cauchy problem for the equation
\[ i\frac{\partial \hat{\Phi}(t)}{\partial t} = \frac{1}{\epsilon} H(\sqrt{\epsilon}\hat{\psi}^+, \sqrt{\epsilon}\hat{\psi}^-) \hat{\Phi}(t), \quad \hat{\Phi}(t) \in \mathcal{H}, \]
\[ \hat{\Phi}\big|_{t=0} = K^\epsilon_{[\Lambda_0,\tau_0]} \int_0^\tau f_0 \] \[ (48) \]

Introduce a notation
\[ \hat{\Psi}^*(t) = \int \frac{dt f(t)}{(2\pi)^{k/2}} t^{\frac{\alpha}{2}} \sqrt[\frac{1}{2}]{\frac{\delta [G^*(\tau,t)G(\tau,t)]}{\delta [G^*(\tau,t)G(\tau,t)]}} \]
\[ \cdot \exp \left\{ \frac{1}{\epsilon}[g + \int_{(\tau(0),0)}^{(\tau,\tau)} (\phi_0(\tau',t')d\phi^*_0(\tau',t') - i H(\phi^*_0(\tau',t'), \phi(\tau',t')dt'))] \right\} \hat{\Phi}_{\phi(\tau,t),M(\tau,t)} \]
\[ l = 1, \infty, \quad a, b = 1, k. \]
where
\[ g = \phi^*_I(\tau(0), 0) \phi_I(\tau(0), 0)/2 + (\phi^*_I(\tau(0), 0) \phi_I(\tau(0), 0) - \phi_I(\tau(0), 0) \phi^*_I(\tau(0), 0))/4, \]
\[ f(\tau, t) = \exp \left\{ -\frac{i}{4} \int_0^t dt' \frac{\partial^2 H}{\partial \phi_m \partial \phi_n} F G^{-1}_{mn} \right\} f_0(\tau) \]
\[ + \left( \frac{\partial^2 H}{\partial \phi_{m_n} \partial \phi_{n_k}} F G^{-1}_{mn} \right) f_0(\tau) \]
\[ (50) \]

**Theorem 1** Suppose \( H_{i_1...i_m,j_1...j_n}^{(m,n)} \) are number sets such that
\[ \| H(\sqrt{\epsilon} \hat{\psi}^+, \sqrt{\epsilon} \hat{\psi}^-) \hat{\Phi}(\phi(\tau,t), M(\tau,t)) \| \leq C_1, \quad \epsilon \in (0, \epsilon_0), C_1 > 0. \]
where \( M(\tau,t) \) is defined by formula (27). Let the canonical transformation \( D_H \) correspond to the Hamiltonian \( H \), and let the series \( \sum_{n,m=1}^{\infty} \left| \frac{\partial^2 H}{\partial \phi_m \partial \phi_n} \right|^2 \) converge.

Then the solution of the Cauchy problem (48) may be presented in the form
\[ \hat{\Phi}(t) = \hat{\Psi}(t) + \hat{\delta}^{(1)}(t) \]
\[ (\delta^{(1)}(t), \delta^{(1)}(t)) \rightarrow 0 \quad \epsilon \rightarrow 0 \]
\[ (\hat{\Phi}(t), \hat{\Phi}(t)) = O(1) \quad \epsilon \rightarrow 0. \]

A proof of this theorem will be given below as a corollary of a more general statement. Hence it will be proved that the canonical operator really gives the asymptotics of the Cauchy problem solution, i.e. that the geometrical quantization is compatible with the canonical one.

Consider now some examples.

**Example 1** The following approximate solution of the equation (48) coinciding up to a constant factor with the vector \( \hat{\Phi}(\phi(t), M(t)) \) corresponds to a zero dimensional isotropic manifold:
\[ \hat{\Psi}'(t) = \exp(g/\epsilon) \]
\[ f_0 \exp \left\{ -\frac{i}{4} \left( \frac{\partial^2 H}{\partial \phi_m \partial \phi_n} M_{mn}(t') + \frac{\partial^2 H}{\partial \phi_{m_n} \partial \phi_{n_k}} M_{mn}(t') \right) dt' \right\} \]
\[ \cdot \exp \left( \frac{1}{\epsilon} \int_0^t (\phi(t') \dot{\phi}_s(t') - iH) dt' \right) \hat{\Phi}(\phi(t), M(t)) \]
\[ m, n, l = 1, \ldots, \infty, \]
where we omit arguments \( \phi(t'), \phi(t) \) at \( H \).

The matrix \( M \) satisfies the equation
\[ i \dot{M}_{mn} = \frac{\partial^2 H}{\partial \phi_m \partial \phi_n} + \frac{\partial^2 H}{\partial \phi_{m_n} \partial \phi_{n_k}} M_{mn} + \frac{\partial^2 H}{\partial \phi_{m_n} \partial \phi_r} M_{rn}, \]
\[ m, n, r = 1, \ldots, \infty \]

In the Fock presentation this vector coincides with the vector (29) up to a constant factor. Note that all the components of the vector corresponding to a zero dimensional isotropic manifold differ from zero.
Example 2 \([10, 17]\)

Let \(H\) be of the form

\[
H(\phi^*, \phi) = \sum_{n=1}^{s} H_{n_i \ldots i_n}^{(n)} \phi_{n_i}^* \cdots \phi_{n_i}^* \phi_{i_1} \cdots \phi_{i_1}.
\]

The family \(g'_{H}\) maps a one dimensional isotropic manifold of the form

\[
\phi_j(\tau) = \tilde{\phi}_j^0 \exp(i\tau), \tau \in [0, 2\pi)
\]

into an isotropic manifold of the form

\[
\phi_j(\tau, t) = \tilde{\phi}_j(t) \exp(i\tau), \tau \in [0, 2\pi),
\]

where \(\tilde{\phi}_j\) is a solution of the equations (42).

Introduce as usual \(F = (C + iB)/\sqrt{2}, G = (C - iB)/\sqrt{2}\).

The equations in variations for \(F\) and \(G\)

\[
i\dot{F}_{mn} = \frac{\partial^2 H}{\partial \phi^*_m \partial \phi^*_n} G_{rn} + \frac{\partial^2 H}{\partial \phi^*_m \partial \phi_r} F_{rn},
\]

\[
i\dot{G}_{mn} = -\frac{\partial^2 H}{\partial \phi_m \partial \phi^*_r} G_{rn} - \frac{\partial^2 H}{\partial \phi_m \partial \phi_r} F_{rn}, \quad m, r, n = 1, \infty
\]

have a solution of the form

\[
F(\tau, t) = \tilde{F}(t) e^{i\tau}, \quad G(\tau, t) = \tilde{G}(t) e^{-i\tau}.
\]

The quantities

\[
\det G^+ G, \quad \det \frac{\partial \phi^*_l}{\partial \tau_a} \frac{\partial \phi_l}{\partial \tau_b} = \tilde{\phi}_l^* \tilde{\phi}_l, \quad l = 1, \infty
\]

do not depend on \(\tau\). If \(f_0\) does not depend on \(\tau\), the function \(f(\tau, t)\) does not depend on \(\tau\) as well.

In this case vector \([49]\) has therefore the following form

\[
\Psi^e(t) = \exp(g/\epsilon) \times \frac{f(t)}{\sqrt{\det \tilde{G}(t)}} \exp \left( \frac{1}{\epsilon} \int_0^t \left( \tilde{\phi}_l(t') \tilde{\phi}^*_l(t') - iH \right) dt' \right) 
\]

\[
\times \tilde{M}_{jl} \tilde{\phi}_l \left( \tilde{\phi}_j^* \sqrt{\epsilon e^{i\tau}} - \phi_j^* (t) \right) \tilde{\phi}_l \left( \tilde{\phi}_l^* \sqrt{\epsilon e^{i\tau}} - \phi_l^* (t) \right) \right) \tilde{\phi}_0, \quad j, l = 1, \infty
\]

The quantization condition is of the form

\[
\tilde{\phi}_l^* \tilde{\phi}_l = \epsilon N, \quad N \in \mathbb{Z}.
\]
It is easy to see that integrating over $\tau$ makes zero all the components of $\hat{\Psi}^\epsilon$ but the $N$-th. This component has the following form (up to a normalizing factor):

$$
(\Psi^\epsilon)_{i_1 \cdots i_N}^{(N)} = \tilde{\phi}_{i_1} \cdots \tilde{\phi}_{i_N} \sum_{k=0}^{[N/2]} \frac{1}{2^k k!} \sum_{1 \leq j_1 \neq \cdots \neq j_{2k} \leq N} M_{ij_1 j_2} \cdots M_{ij_{2k-1} i_{2k}} c(t),
$$

(51)

$$
c(t) = \frac{f(t)}{\sqrt{\det G(t) G^+(t)}} \exp \left\{ \frac{1}{\epsilon} \int_0^t \left( \tilde{\phi}(t') \tilde{\phi}^*(t') - iH \right) dt' \right\}
$$

$$
\tilde{M}_{ij} = (FG^{-1})_{ij} - \tilde{\phi}_l \tilde{\phi}_j / (\tilde{\phi}_m^* \tilde{\phi}_m), \quad l, j, m = 1, \infty
$$

Formula (51) coincides with that obtained in [13] for the stationary case. It may be also obtained by taking the $N$-th component of the state vector corresponding to a germ in a point (see [16, 17]).

The matrix $M$ that vanishes at the initial moment may differ from zero at other moments of time. It now follows that an $N$-particle wave function may not be decomposed into a product of one-particle wave functions (i.e. in form (51) with $M = 0$) as $\epsilon \to 0, N \to \infty, \epsilon N \to \text{const}$, even in the case it is decomposed into such a product at the initial moment. For the case of classical statistical mechanics this statement has been stated and proved in [12].

**Example 3** The approach developed here may be also spread formally to the case of quantum field theory, for scalar quantum electrodynamics, for example. Formulas, however, look more simple for the theory of a real scalar field with self-action. This theory describes an approximation of $\pi$-mesons interaction [3, 21]. The Lagrangian for the theory is of the form

$$
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2} \phi^2 - \frac{g}{4} \phi^4, \quad \mu = 0, 1, \ldots, d-1,
$$

(52)

where $d$ is the space-time dimension.

The following Hamiltonian

$$
\mathcal{H}(\hat{p}(\cdot), \hat{q}(\cdot)) = \int d^{d-1}x \left( \frac{1}{2} \hat{p}^2(x) + \frac{1}{2} (\nabla \hat{q}(x))^2 + m^2 \hat{q}^2(x) + \frac{g}{4} \hat{q}^4(x) \right),
$$

(53)

corresponds to the Lagrangian (52).

After the substitution $\sqrt{\hat{q}} \hat{p} = \hat{P}$, $\sqrt{\hat{g}} \hat{q} = \hat{Q}$ the Hamiltonian (53) becomes

$$
\mathcal{H}(\hat{P}(\cdot), \hat{Q}(\cdot)) = \frac{1}{g} \int d^{d-1}x \left( \frac{1}{2} \hat{P}^2(x) + \frac{1}{2} (\nabla \hat{Q}(x))^2 + m^2 \hat{Q}^2(x) + \frac{1}{4} \hat{Q}^4(x) \right),
$$

\begin{equation}
[\hat{Q}(x), \hat{P}(y)] = i g \delta(x - y)
\end{equation}
The theory may be interpreted in the terms of particles with the help of following creation
and annihilation operators:

\[ \hat{q}(x) = \frac{1}{(2\pi)^{d+1}} \int \frac{dl}{\sqrt{2\sqrt{l^2 + m^2}}} \left( \hat{\psi}^+(l)e^{-ilx} + \hat{\psi}^-(l)e^{ilx} \right) \]

\[ \hat{p}(x) = \frac{i}{(2\pi)^{d+1}} \int dl \sqrt{\frac{l^2 + m^2}{2}} \left( \hat{\psi}^+(l)e^{-ilx} - \hat{\psi}^-(l)e^{ilx} \right) \]

According to the outlined scheme one may assign solutions of secondary quantized equa-
tions of the sort (49) to each Lagrangian manifold with complex germ.

For constructing such an asymptotics one must solve a Hamiltonian system and a system
of equations in variations.

The Hamiltonian system has the form

\[ \dot{Q}(x, t) = P(x, t) \]

\[ \dot{P}(x, t) = \Delta Q(x, t) - m^2 Q(x, t) - Q^3(x, t) \]

and the equations in variations over the variables \( \delta P, \delta Q \) are

\[ \dot{B}(x, t, \tau) = \Delta C(x, t, \tau) - m^2 C(x, t, \tau) - 3C(x, t, \tau)Q^2(x, t, \tau) \]

\[ \dot{C}(x, t, \tau) = B(x, t, \tau) \]

The outlined conception may be applied without a preliminar regularization if

\[ \int dx(Q(x, t, \tau)\sqrt{m^2 - \Delta Q(x, t, \tau)} + P(x, t, \tau)\frac{1}{\sqrt{m^2 - \Delta}}P(x, t, \tau)) < \infty, \]

and the operator \( BC^{-1} - i\sqrt{m^2 - \Delta} \) is a Hilbert- Schmidt operator.

If these conditions are not satisfied, regularization must be treated more closely.

The conception of geometrical quantization may be applied as well to the scalar quantum
electrodynamics in the \( \alpha \)-gauge with the Lagrangian

\[ L = -\frac{1}{4}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) - \frac{1}{2\alpha}(\partial_{\mu}A^{\mu})^2 \]

\[ + (\partial_{\mu} + ieA_{\mu})\Phi^{*}(\partial_{\mu} - ieA_{\mu})\Phi - m^2\Phi^{*}\Phi - \frac{g}{4}(\Phi^{*}\Phi)^2. \]

5 Complex germ creation and annihilation operators

The theory of complex germs allows to construct not only asymptotic solutions of type (49)
of secondary quantized equations. Other asymptotic solutions of the equation (48) may be
obtained also with the help of so called germ creation and annihilation operators.

Suppose a basis on a complex germ is chosen so that the matrix \( A \) from the axiom r1
of a complex germ is diagonal. Consider following creation and annihilation operators on a
Lagrangian manifold with complex germ:

\[ A_{\alpha}(\tau, t) = C_{\alpha\alpha}(\tau, t) \left( \hat{\psi}^+_m - \frac{\phi_{\alpha}(\tau, t)}{\sqrt{\epsilon}} \right) - F_{\alpha\alpha}(\tau, t) \left( \hat{\psi}^-_m - \frac{\phi_{\alpha}(\tau, t)}{\sqrt{\epsilon}} \right), \]

\[ A_{\alpha}(\tau, t) = C_{\alpha\alpha}(\tau, t) \left( \hat{\psi}^-_m - \frac{\phi_{\alpha}(\tau, t)}{\sqrt{\epsilon}} \right) - F_{\alpha\alpha}(\tau, t) \left( \hat{\psi}^+_m - \frac{\phi_{\alpha}(\tau, t)}{\sqrt{\epsilon}} \right), \]

\[ \text{(54)} \]
\[
F = (C + iB)/\sqrt{2}, \quad G = (C - iB)/\sqrt{2}, \quad \alpha = k + 1, \infty, \quad m = 1, \infty
\]

Let \( \nu_\alpha \in \{0, 1, 2, \ldots\} \), \( \alpha = k + 1, \infty \), \( \sum_{\alpha=k+1}^{\infty} \nu_\alpha < \infty \). Let \( l(\tau', \tau'') \) be a closed path on \( \Lambda^k \), such that it is covered by a path on \( M^k \) with the beginning in point \( \tau' \) and the end in point \( \tau'' \). Suppose that the following quantization condition holds for any such path

\[
\frac{1}{2\pi \epsilon} \oint_{l(\tau', \tau'')} P_m dQ_m = \sum_{\alpha=k+1}^{\infty} \gamma_\alpha(\tau', \tau'') \nu_\alpha + n, \quad n \in \mathbb{Z} \tag{55}
\]

where \( \gamma_\alpha(\tau', \tau'') \) are determined by the condition \( A_{\alpha, \alpha}(\tau', \tau'') = e^{i\gamma_\alpha(\tau', \tau'')} \).

Consider the following element of \( \mathcal{H} \):

\[
\hat{\Psi}^\epsilon(t) = \int \frac{df(\tau,t)}{(2\pi \epsilon)^{2d/4}} \frac{\det \frac{\partial^2}{\partial \Phi^\epsilon(\tau,t) \partial \Phi^\epsilon(\tau,t)}}{\sqrt{\det G^\epsilon(\tau,t)G(\tau,t)}} \cdot \exp \left( i \int S(\tau, t) \right) \hat{A}_{k+1} \hat{A}_{k+2} \ldots \hat{F}_{\phi(\tau,t), M(\tau,t)}, \tag{56}
\]

where

\[
S(\tau, t) = \frac{1}{2\pi} \phi^*_\epsilon(\tau(0), 0) \phi(\tau(0), 0) + \frac{1}{4\pi} \left( \phi^*_\epsilon(\tau(0), 0) \phi(\tau(0), 0) - \phi(\tau(0), 0) \phi^*_\epsilon(\tau(0), 0) \right) + f(\tau, t) \frac{1}{4\pi} \phi(\tau', t') d\phi^*_\epsilon(\tau', t') - H(\phi^*_\epsilon(\tau', t'), \phi(\tau', t')) dt',
\]

and \( f(\tau, t), M(\tau, t) \) are determined from the formulas (50) and (27) respectively. Let \( \hat{\Phi}(t) \) be the solution of (18), satisfying the initial condition \( \hat{\Phi}(0) = \hat{\Psi}^\epsilon(0) \).

**Theorem 2** Suppose

\[
\|H(\sqrt{\epsilon \phi^*_\epsilon}, \sqrt{\epsilon \phi^-\epsilon}) \hat{A}_{k+1} \hat{A}_{k+2} \ldots \hat{F}_{\phi(\tau,t), M(\tau,t)} \| \leq C_2, \quad \epsilon \in (0, \epsilon_0), C_2 > 0.
\]

Let the canonical transformation \( \mathcal{D}_H^\epsilon \) correspond to the Hamiltonian \( H \), and the series

\[
\sum_{m,n=1}^{\infty} |\partial^2 H/\partial \phi_m \partial \phi_n|^2
\]

converge. Then

\[
\hat{\Phi}(t) = \hat{\Psi}^\epsilon(t) + \hat{\delta}_\epsilon(t),
\]

\[
(\hat{\delta}_\epsilon(t), \hat{\epsilon}_\epsilon(t)) \xrightarrow{\epsilon \to 0} 0, \quad (\hat{\Phi}(t), \hat{\Phi}(t)) \Rightarrow O(1), \quad \epsilon \to 0.
\]

**Remark 14** In the case of C-Lagrangian manifold \( \mathcal{L} \), when \( \gamma_\alpha = 0 \), a canonical operator can be generalized.

Let \( f_{\alpha_1, \ldots, \alpha_n}(\tau) \), where \( \alpha_i = k + 1, \infty \), \( n = 0, \infty \) be a set of infinitely differentiable functions with compact support on the Lagrangian manifold with complex germ \( [\Lambda^k, r] \), where the following series

\[
\sum_{n=1}^{\infty} \sum_{\alpha_1, \ldots, \alpha_n = k+1}^{\infty} |f_{\alpha_1, \ldots, \alpha_n}(\tau)|^2
\]
converges at all \( \tau \).

The following element of \( H \) is assigned to the Lagrangian manifold with complex germ \([\Lambda, r]\), the set of functions \( f^{(n)}_{\alpha_1, \ldots, \alpha_n}(\tau) \), the point \( \tau(0) \in \Lambda \), and the number \( \epsilon > 0 \):

\[
\hat{K}^{[\Lambda, r], \tau(0)}_\epsilon f = \int \frac{d\tau}{(2\pi)^k} \frac{\sqrt{\det G_\gamma(\tau)}}{\sqrt{\det G^+(\tau)G(\tau)}} \cdot \exp \left( \frac{i}{\epsilon} S(\tau) \right) \sum_{n=1}^{\infty} \sum_{\alpha_1, \ldots, \alpha_n = k+1} \frac{1}{\sqrt{\det \partial \phi_l \partial \tau a(\tau)}} \partial_{\phi_l} \partial_{\tau b} \left( \frac{2}{2^4} \sqrt{\det G + (\tau, G(\tau) \cdot \exp \left( i \epsilon \hat{\phi}(\tau, M(\tau), l = 1, \infty, a, b = 1, k \right) \right.}
\]

If \( f(0) = f \) and other \( f^{(n)} = 0 \) then canonical operator (57) is equal to canonical operator (57).

**Example 4** Consider the Hamiltonian from example 2, and suppose \( \phi_l(t) \) is a solution of the form \( \phi_l(t) = \tilde{\phi}_l \exp(-i\Omega t) \) of equation (42).

Suppose that there exist matrices \( \tilde{F} \) and \( \tilde{G} \) such that

\[
(\beta_l + \Omega) \tilde{G}_{ml} = \sum_{r=1}^{\infty} \frac{\partial^2 H}{\partial \phi_m \partial \phi_r} \tilde{G}_{rl} + \sum_{r=1}^{\infty} \frac{\partial^2 H}{\partial \phi_l \partial \phi_r} \tilde{F}_{rl},
\]

\[-(\beta_l - \Omega) \tilde{F}_{ml} = \sum_{r=1}^{\infty} \frac{\partial^2 H}{\partial \phi_m^* \partial \phi_r} \tilde{G}_{rl} + \sum_{r=1}^{\infty} \frac{\partial^2 H}{\partial \phi_l^* \partial \phi_r} \tilde{F}_{rl},\]

\[\beta_l \in \mathbb{R}, \quad \tilde{G}_{m1} = \tilde{\phi}_{m1}^*, \quad \tilde{F}_{m1} = -\tilde{\phi}_{m1}\]

\[\tilde{F}^T \tilde{G} = \tilde{G}^T \tilde{F}, \quad \tilde{G}_{m\alpha} \tilde{G}_{m\beta} - \tilde{F}_{m\alpha}^* \tilde{F}_{m\beta} = \delta_{\alpha\beta}, \quad m, l = 1, \infty, \alpha, \beta = 1, \infty\]

\( \tilde{F} \) is a Hilbert–Schmidt operator and \( \tilde{G} \) has a bounded inverse operator.

Consider a Lagrangian manifold with complex germ corresponding to the following function \( \phi_l \)

\[\phi_l(\tau, t) = \tilde{\phi}_l \exp(i(\tau - \Omega t))\]

with matrices \( F \) and \( G \)

\[F_{ml}(\tau, t) = \tilde{F}_{ml} \exp(-i(\beta_l/\Omega - 1)(\tau - \Omega t)),\]

\[G_{ml}(\tau, t) = \tilde{G}_{ml} \exp(-i(\beta_l/\Omega + 1)(\tau - \Omega t)).\]

The matrix \( A(0, 2\pi) \) is

\[A(0, 2\pi) = \text{diag}\{1, \exp(2\pi i \beta_2/\Omega), \exp(2\pi i \beta_3/\Omega), \ldots\}\]

And the quantization condition is

\[\tilde{\phi}_l^* \tilde{\phi}_l = \epsilon N + \epsilon \sum_{\lambda=2}^{\infty} \beta_\lambda \nu_\lambda/\Omega.\]
Vector (56) depends on time as

\[ \mathcal{E} = \frac{1}{\epsilon} H(\phi^*, \phi) + \frac{1}{4} \sum_{m,n=1}^{\infty} \left( \frac{\partial^2 H}{\partial \phi_m \partial \phi_n} (F G^{-1})_{mn} + \frac{\partial^2 H}{\partial \phi_m^* \partial \phi_n^*} (F G^{-1})_{mn}^* \right). \]

Hence it is an approximate solution of the stationary equation

\[ \mathcal{E} \hat{\Phi} = \frac{1}{\epsilon} H(\hat{\psi}^+ \sqrt{\epsilon}, \hat{\psi}^- \sqrt{\epsilon}) \hat{\Phi} \] (58)

**Example 5** Consider now the complex germ given by the following matrices:

\[ F_{ml}(\tau, t) = F_{ml} \exp(-i \Omega t + i \tau) \exp(i \beta t) \]

\[ G_{ml}(\tau, t) = G_{ml} \exp(i \Omega t - i \tau) \exp(i \beta t) \]

The matrix \( A(0, 2\pi) \) is the identity matrix, and the quantization condition has the form

\[ \hat{\phi}^*_l \hat{\phi}_l = \epsilon N. \]

In this example vector (56) is also an approximate solution of the equation (58) for

\[ \mathcal{E} = \frac{1}{\epsilon} H(\phi^*, \phi) + \frac{1}{4} \sum_{m,n=1}^{\infty} \beta_\lambda \nu_\lambda, \quad m, n = 1, \infty \]

This solution coincide with that given in [15].

Find now a relation between the results of examples 4 and 5. Let \( \phi^{(1)}_m, \phi^{(2)}_m \) be solutions of the equations

\[ \Omega_1 \phi^{(1)}_m = \partial H/\partial \phi^{(1)*}_m, \quad \Omega_2 \phi^{(2)}_m = \partial H/\partial \phi^{(2)*}_m \]

such that

\[ \phi^{(1)*}_m \phi^{(1)}_m = \epsilon N + \epsilon \sum_{\lambda=1}^{\infty} \beta_\lambda \nu_\lambda / \Omega_1, \quad \phi^{(2)*}_m \phi^{(2)}_m = \epsilon N, \]

\[ \phi^{(1)}_m = \phi^{(2)}_m + \chi_m, \quad \|\chi\| = O(\epsilon), \quad \Omega_2 - \Omega_1 = O(\epsilon). \]

We have

\[ H(\phi^{(1)*}_1, \phi^{(1)}_1) - H(\phi^{(2)*}_1, \phi^{(2)}_1) = \Omega_2 (\phi^{(2)}_m \chi_m^* + \phi^{(2)}_m \chi_m) + O(\epsilon^2), \]

\[ (\phi^{(1)}_1, \phi^{(1)}_1) - (\phi^{(2)}_1, \phi^{(2)}_1) = \phi^{(2)}_m \chi_m^* + \phi^{(2)}_m \chi_m + O(\epsilon^2), \quad m = 1, \infty \]

It now follows that the values of \( \mathcal{E} \) in two examples coincide up to \( O(\epsilon^1) \).

**Remark 15** We have shown for a specific example how different quantization conditions may be used with simultaneous change of the transport equation. Convyniency determines the choice of quantization conditions. In the finite dimensional case a quantization condition of the kind

\[ \frac{1}{2\pi \epsilon} \oint_{(\tau', \tau'')} P_m dQ_m = \sum_{\alpha=k+1}^{D} \gamma_{\alpha}(\tau', \tau'') (\nu_{\alpha} + \frac{1}{2}) + n, \quad n \in \mathbb{Z} \]

is often used ([1]).

In the infinite dimensional case this condition makes no sense even for the simplest Hamiltonian \( H(\phi^*, \phi) = \sum_{i=1}^{\infty} \phi_i^* \phi_i \) and for the isotropic manifold with complex germ as in example 4.1.
Example 6 Consider one more (heuristic) way to deduce a formula for $E$ (see [11]).

Note, that if a family $\mathcal{D}'_H$ acts on a two dimensional isotropic manifold with complex germ in the following way

\[
(g^t\phi)(\tau_1, \tau_2) = \phi(\tau_1 + \Omega_1 t, \tau_2 + \Omega_2 t)
\]

\[
F(\tau_1, \tau_2, t) = F(\tau_1 + \Omega_1 t, \tau_2 + \Omega_2 t)
\]

\[
G(\tau_1, \tau_2, t) = G(\tau_1 + \Omega_1 t, \tau_2 + \Omega_2 t)
\]

then multiplication of the function $f_{\tau_1, \tau_2, 0}$ by $\exp(i(n_1\tau_1 + n_2\tau_2))$ leads to multiplication of the vector $[a]$ by $\exp(-i(n_1\Omega_1 + n_2\Omega_2)t)$, $n_1, n_2 \in \mathbb{Z}$.

Consider now "almost invariant" two dimensional isotropic manifolds close to one dimensional manifolds in examples 4, 5:

\[
\phi_t(t, \tau_1, \tau_2) = e^{-i(\tau_1 - \Omega t)}(\phi_t + \delta(F_t e^{i(\tau_2 + \beta m t)} + \tilde{G}_t e^{-i(\tau_2 + \beta m t)})), \quad \delta \to 0.
\]

The values of $E$ corresponding to these almost invariant manifolds are $E^{(a)} = E^{(0)} + \beta_n \mu_m$, $\mu_m \in \mathbb{Z}$. Since these values must agree with the value for a neighborhood of a one dimensional manifold the last value is of the form $E^{(0)} + \sum_{m=1}^{\infty} \beta_m \nu_m$, $\nu_m \in \{0, 1, 2, \ldots\}$.

6 Proof of the theorem

Let us now prove theorem 2 on the base of the following lemma.

Introduce notation

\[
S(\tau) = \frac{1}{2} \phi_j^*(\tau(0)) \phi_j(\tau(0)) + \frac{1}{4} (\phi_j^*(\tau(0)) \phi_j(\tau(0)) - \phi_j(\tau(0)) \phi_j(\tau(0)))
\]

\[+ \frac{1}{4} \int_{\tau(0)}^{} \phi_j(\tau') d\phi_j^*(\tau'), \quad j = 1, \infty
\]

We assign to number sets $\mathcal{D}_{ij}(\tau)$, $i, j = 1, \infty$ and $(Y_a^{n})_{i_1 \ldots i_n}(\tau)$, $a = 1, 2$, $n = 1, s$, $i_p = 1, \infty$ the following operators in $\mathcal{H}$:

\[
Y_a(\tau, \hat{\psi}^+ - \frac{\phi_j^*(\tau)}{\sqrt{\epsilon}}) = \sum_{n=0}^{s} (Y_a^{n})_{i_1 \ldots i_n}(\tau)(\hat{\psi}^+_{i_1} - \frac{\phi_j^*(\tau)}{\sqrt{\epsilon}}) \ldots (\hat{\psi}^+_{i_n} - \frac{\phi_j^*(\tau)}{\sqrt{\epsilon}})
\]

\[\times \exp\{\frac{1}{2}(\hat{\psi}^+_{i_1} - \frac{\phi_j^*(\tau)}{\sqrt{\epsilon}})D_{ij}(\tau)(\hat{\psi}^+_{i_2} - \frac{\phi_j^*(\tau)}{\sqrt{\epsilon}})\}, \quad i, j, i_1, \ldots, i_n = 1, \infty, \quad a = 1, 2.
\]

Introduce also the following notation

\[
\hat{\Phi}^a = \int \frac{d\tau}{e^{k/4}} Y_a\left(\tau, \hat{\psi}^+ - \frac{\phi_j^*(\tau)}{\sqrt{\epsilon}}\right)
\]

\[\times \exp\{\frac{1}{\sqrt{\epsilon}} \phi_j(\tau)(\hat{\psi}^+_{j} - \phi_j^*(\tau))\} \hat{\phi}_0, \quad a = 1, 2, \quad j = 1, \infty.
\]

The commutation relations for the operators $\psi_j^\pm$ imply

\[
\hat{\Phi}^a = \int \frac{d\tau \exp(\frac{1}{2}(S(\tau) - \frac{1}{2} \phi_j(\tau) \phi_j^*(\tau)))}{e^{k/4}}
\]

\[\times e^{\frac{1}{\sqrt{\epsilon}} \phi_j(\tau)(\hat{\psi}^+_{j} - \phi_j^*(\tau)) \hat{\psi}^-_{j}} Y_a(\tau, \hat{\psi}^+) \hat{\Phi}_0, \quad j = 1, \infty, \quad a = 1, 2
\]
Lemma 5 Let the matrix \( D_{ij}(\tau) \) correspond to a Hilbert–Schmidt operator \( D \) in \( l^2 \), \( \| D \| < 1 \), \( D_{ij}(\tau) \) and \( (Y_a^n)_{i_1...i_n}(\tau) \) being smooth functions of \( \tau \), functions \( (Y_a^n)_{i_1...i_n}(\tau) \) having compact supports,

\[
\sum_{i_1...i_n} |(Y_a^n)_{i_1...i_n}(\tau)|^2 < c, \quad n = 1, s, \quad a = 1, 2.
\]

Then

\[
(\hat{\Phi}_1^\epsilon, \hat{\Phi}_2^\epsilon) \xrightarrow{\epsilon \to 0} \int d\tau \int d\xi (Y_1(\tau, \hat{\psi}^+)) \hat{\Phi}_0, \\
\exp(\xi_0(\partial_{\phi_j}^i(\tau)\hat{\psi}_j^+ - \partial_{\phi_j}^i(\tau)\hat{\psi}_j^-))Y_2(\tau, \hat{\psi}^+)} \hat{\Phi}_0), \quad j = 1, \infty.
\]

Proof. We have

\[
(\hat{\Phi}_1^\epsilon, \hat{\Phi}_2^\epsilon) = \int dx dx' \epsilon^{2(S(\tau') - S(\tau))} \epsilon^{k(\phi_j, \phi_j^\epsilon(\tau) + \phi_j^\epsilon(\tau'))} \\
\times e^{\epsilon(\phi_j^\epsilon(\tau) - \phi_j^\epsilon(\tau'))} (\hat{\Phi}_0, Y_1^\epsilon(\tau, \hat{\psi}^-)) \\
Y_2(\tau', \hat{\psi}^+)} \hat{\Phi}_0), \quad j = 1, \infty, b, c, d = 1, k, \\
\xi = (\tau - \tau')/\epsilon, \quad |R_{bcd}(\tau, \tau')| < \text{const}, \\
\sum_{j=1}^{\infty} |J_j(\tau, \tau')|^2 < \text{const}.
\]

Integrating over \( \tau \) and \( \xi \) as \( \epsilon \to 0 \) we obtain (59). Lemma 5 is proved. Lemma 3 implies

Corollary 1 \( \| \hat{\psi}(t) \| = O(1) \) as \( \epsilon \to 0 \).

Proof. It is sufficient to prove that only the integral

\[
\int d\xi \hat{\Phi}_0, e^{2 \psi_m^\epsilon - M_{mn} \psi_n^\epsilon - \alpha_{k+1}^\epsilon \alpha_{k+2}^\epsilon} \\
\times \exp(\xi_0(\partial_{\phi_j}^i(\psi_m^\epsilon) - \partial_{\phi_j}^i(\psi_n^\epsilon))) \hat{\alpha}^\epsilon \hat{\alpha}_{k+1}^\epsilon \hat{\alpha}_{k+2}^\epsilon \\
\hat{\alpha}^\epsilon = \hat{G}_{ma} \hat{\psi}_m^\epsilon - \hat{F}_{ma} \hat{\psi}_m^\epsilon, \quad \hat{\alpha} = G_{ma} \psi_m - F_{ma} \psi_m,
\]

differs from zero. Show first that

\[
\int d\xi (\hat{X}, \exp(\xi_0(\partial_{\phi_j}^i(\psi_m^\epsilon) - \partial_{\phi_j}^i(\psi_n^\epsilon))) \hat{\alpha}^\epsilon \hat{\alpha}_{k+1}^\epsilon \hat{\alpha}_{k+2}^\epsilon) \Phi_0 = 0, \\
\beta \in \{k+1, k+2, \ldots\}, \quad b = 1, k, \quad m, n = 1, \infty,
\]

\[
\hat{X} = \hat{\alpha}^\epsilon \ldots \hat{\alpha}_{\alpha_1}^\epsilon \hat{\alpha}_{\beta_1}^\epsilon \ldots \hat{\alpha}_{\beta_n}^\epsilon (\frac{1}{2} \hat{\psi}_m^\epsilon + M_{mn} \psi_n^\epsilon \Phi_0, \\
\alpha_1, \ldots, \alpha, \beta_1, \ldots, \beta_n \in \{k+1, k+2, \ldots\}.
\]
Indeed, make use of the presentation of the scalar product in the left hand side of (61) as a functional integral (see [2]):

\[
\int d^k \xi \int \prod dz^* dX(z^*)(G_{m\beta} \partial / \partial z_m - F_{m\beta} z_m) \\
\times \exp \{ \xi_b(\partial \phi_m z_m - \partial \phi_m \partial \phi_m z_m) \} \exp \left( \frac{1}{2} \left( \frac{1}{2} z_m M_{mn} z_n \right) e^{-z_m^* z_m} \right)
\]

Integrating over \( \xi \) and using the property \( M_{mn} \partial \phi_m / \partial \phi_n = 0 \) we conclude that the functional integral is equal to zero.

Check now that integral (60) differs from zero. The commutation relations

\[
[a_\alpha, a_\beta] = [a_\alpha, \partial \phi_m \hat{\psi}_m^+ - \partial \phi_m \hat{\psi}_m^-] = [a_\alpha, \hat{\psi}_m^+] - [a_\alpha, \hat{\psi}_m^-] = 0,
\]

make this integral equal to

\[
(\nu_{k+1})!(\nu_{k+2})! \ldots \int d^k \xi (\hat{\Phi}_0, \exp \left( \frac{1}{2} \psi^*_m M_{mn} \psi_n \right)) \\
\times \exp \left( \int \frac{1}{2} \psi^*_m M_{mn} \psi_n \right) \exp \left( \frac{1}{2} \hat{\psi}^*_m M_{mn} \hat{\psi}_n \right) \hat{\Phi}_0.
\]

It is easy to check that the last expression differs from zero. The corollary is proved.

Now we may give a proof of theorem 2. Check that

\[
(i \frac{\partial}{\partial t} - \frac{1}{\epsilon} H(\sqrt{\epsilon} \hat{\psi}^+, \sqrt{\epsilon} \hat{\psi}^-)) \hat{\Psi}^\epsilon (t) \longrightarrow 0 \quad \epsilon \to 0
\]

(62)

The vector \( \hat{\Psi}^\epsilon (t) \) may be presented as

\[
\hat{\Psi}^\epsilon (t) = \int \frac{d\tau \chi(\tau, t)}{e^{k/4}} \exp \left( \frac{i}{\epsilon} S(\tau, t) - \frac{1}{\epsilon} \phi_j^*(\tau, t) \phi_j(\tau, t) \right) \\
\times e^{\frac{1}{\sqrt{\epsilon}} \phi_j(\tau, t) \hat{\psi}_j^+} e^{-\frac{1}{\sqrt{\epsilon}} \phi_j^*(\tau, t) \hat{\psi}_j^-} (\hat{\phi}_{k+1})^{\nu_{k+1}}(\hat{\phi}_{k+2})^{\nu_{k+2}} \ldots \hat{\Phi}_{0,M(\tau, t)}, \quad j = 1, \infty
\]

\[
\chi(\tau, t) = \frac{f(\tau, t) \sqrt{\det \frac{\partial \phi_j}{\partial \tau_j}(\tau, t) \frac{\partial \phi_j^*}{\partial \tau_j^*}(\tau, t)}}{(2\pi)^{k/2} \sqrt{\det G^+(\tau, t) G(\tau, t)}}
\]

\[
a, b = 1, k, \quad j = 1, \infty.
\]

Since

\[
i \frac{\partial}{\partial \tau} \left( e^{-\frac{1}{\sqrt{\epsilon}} \phi_j^*(\tau, t) \hat{\psi}_j^-} e^{\frac{1}{\sqrt{\epsilon}} \phi_j^*(\tau, t) \hat{\psi}_j^+} \right) = e^{-\frac{1}{\sqrt{\epsilon}} \phi_j^*(\tau, t) \hat{\psi}_j^-} e^{\frac{1}{\sqrt{\epsilon}} \phi_j^*(\tau, t) \hat{\psi}_j^+} \\
\times (i \frac{\partial}{\partial \tau} - \frac{i}{\sqrt{\epsilon}} \phi_j^*(\tau, t) + \frac{i}{\sqrt{\epsilon}} (\hat{\phi}_j^* \hat{\psi}_j^- - \hat{\phi}_j \hat{\psi}_j^-))
\]
where \( j = 1, \infty \), arguments \( t, \tau \) at \( \phi_j^*, \phi_j \) are omitted, we obtain
\[
\langle i \frac{\partial}{\partial t} - \frac{1}{\epsilon} H(\sqrt{\epsilon} \hat{\psi}^+, \sqrt{\epsilon} \hat{\psi}^-) \rangle \hat{\psi}^+ (t) = \int \frac{d\tau}{\epsilon^{k/4}}
\]
\[
\times \exp\left( \frac{i}{\epsilon} S(\tau, t) - \frac{1}{2\epsilon} \phi_j^*(\tau, t) \phi_j(t, t) + \frac{1}{\sqrt{\epsilon}} (\phi_j(\tau, t) \hat{\psi}_j^+ - \phi_j^*(\tau, t) \hat{\psi}_j^-) \right)
\]
\[
\times \left( i \frac{\partial}{\partial t} + \frac{i}{\sqrt{\epsilon}} (\phi_j(\tau, t) \hat{\psi}_j^+ - \phi_j^*(\tau, t) \hat{\psi}_j^-) - \frac{1}{\epsilon} \frac{\partial S}{\partial t} \right)
\]
\[
- \frac{i}{\epsilon} \phi_j^*(\tau, t) \phi_j(t, t) - \frac{1}{\epsilon} H(\phi^* + \sqrt{\epsilon} \hat{\psi}^+, \phi + \sqrt{\epsilon} \hat{\psi}^-) \right)
\]
\[
\times \chi(\tau, t)(\hat{a}_{k+1}^+)^{\nu_k+1}(\hat{a}_{k+2}^+)^{\nu_k+2} \ldots \hat{\Phi}_{0, M}(\tau, t), \quad j = \overline{1, \infty}.
\]

The conditions of the theorem imply that the norm of the vector
\[
H(\phi^* + \sqrt{\epsilon} \hat{\psi}^+, \phi + \sqrt{\epsilon} \hat{\psi}^-)
\]
is uniformly bounded. Since the operator \( H(\phi^* + \sqrt{\epsilon} \hat{\psi}^+, \phi + \sqrt{\epsilon} \hat{\psi}^-) \) is a polynomial in \( \sqrt{\epsilon} \)
of the form
\[
H(\phi^* + \sqrt{\epsilon} \hat{\psi}^+, \phi + \sqrt{\epsilon} \hat{\psi}^-) = \sum_{n=0}^{s} H_{m+n}(\phi_{m+n}^*) \phi_{m+n}(\tau, t)
\]
the norms of the vectors
\[
\frac{\partial^{m+n} H}{\partial \phi_{i_1}^* \ldots \partial \phi_{i_m}^* \partial \phi_{j_1} \ldots \partial \phi_{j_n}} \hat{\psi}_{i_1}^* \ldots \hat{\psi}_{i_m}^* \hat{\psi}_{j_1} \ldots \hat{\psi}_{j_n} (\hat{a}_{k+1}^+)^{\nu_k+1}(\hat{a}_{k+2}^+)^{\nu_k+2} \ldots \hat{\Phi}_{0, M}(\tau, t)
\]
are also uniformly bounded. By Lemma 3, the right hand side of the formula (34) may be presented as
\[
\int \frac{d\tau}{\epsilon^{k/4}} \exp\left( \frac{i}{\epsilon} S(\tau, t) - \frac{1}{2\epsilon} \phi_j^*(\tau, t) \phi_j(t, t) \right)
\]
\[
+ \frac{1}{\sqrt{\epsilon}} (\phi_j(\tau, t) \hat{\psi}_j^+ - \phi_j^*(\tau, t) \hat{\psi}_j^-) \left( i \frac{\partial}{\partial t} - \frac{\partial^2 H}{2 \partial \phi_{m}^* \partial \phi_{n}^*} \hat{\psi}_{m}^* \hat{\psi}_{n}^+ \right)
\]
\[
- \frac{1}{2 \partial \phi_{m}^* \partial \phi_{n}^*} \hat{\psi}_{m}^* \hat{\psi}_{n}^+ = \frac{\partial^2 H}{\partial \phi_{m}^* \partial \phi_{n}^*} \hat{\psi}_{m}^* \hat{\psi}_{n}^+ \chi(\tau, t)
\]
\[
(\hat{a}_{k+1}^+)^{\nu_k+1}(\hat{a}_{k+2}^+)^{\nu_k+2} \ldots \hat{\Phi}_{0, M}(\tau, t) + O(\epsilon^{1/2})
\]
\[
j, m, n = \overline{1, \infty}.
\]

where we made use of the relations
\[
\frac{\partial S}{\partial t} = -i \phi_l \phi_l^* - H(\phi^*, \phi)
\]
by the definition of \( S \), and
\[
i \dot{\phi}_l = \frac{\partial H}{\partial \phi_l^*}, \quad -i \dot{\phi}_l^* = \frac{\partial H}{\partial \phi_l}, \quad l = \overline{1, \infty}
\]
by equations (12).

It is easy to check that
\[
[i \frac{\partial}{\partial t} - \hat{H}_2, \hat{a}_\alpha^+] = [i \frac{\partial}{\partial t} - \hat{H}_2, \hat{c}_\alpha] = [i \frac{\partial}{\partial t} - \hat{H}_2, \frac{\partial \phi_j}{\partial \tau_a} \hat{\psi}_j^+ - \frac{\partial \phi_j^*}{\partial \tau_a} \hat{\psi}_j^-] = 0, \tag{65}
\]
\[\alpha = k + 1, \infty, \quad a = 1, k, \quad j = 1, \infty, \quad \hat{H}_2 = \frac{1}{2} \frac{\partial^2 H}{\partial \phi_m \partial \phi_n} \hat{\psi}_m^+ \hat{\psi}_n^+ + \frac{1}{2} \frac{\partial^2 H}{\partial \phi_m \partial \phi_n} \hat{\psi}_m^- \hat{\psi}_n^- + \frac{\partial^2 H}{\partial \phi_m \partial \phi_n} \hat{\psi}_m^+ \hat{\psi}_n^-, \quad m, n = 1, \infty \]

Let us make use of Lemma 5, commutation relations (65) and the functional integral presentation for the scalar product (3). It is then easy to check that if for any \(z\)
\[
\int d^k \xi \exp(-\frac{1}{2} \xi \partial_j \phi \partial_{\tau_c} \phi) \chi e^{\frac{1}{2} \chi^2} = 0, \quad m, n, j = 1, \infty, \quad b, c = 1, k \tag{66}
\]
(where arguments \(t, \tau\) at the functions \(M, \chi, \phi^*, \phi\) and arguments \(\phi^*(\tau, t), \phi(\tau, t)\) at the derivatives of \(H\) are omitted) then the norm of the vector \((i \partial/\partial t - H) \hat{\tilde{\Psi}^\epsilon}(t)\) tends to zero at \(\epsilon \to 0\).

Formula (66) may be checked by calculating the Gaussian integral at \(\xi\) using presentations for \(\chi, M, F, G\).

The property (62) is therefore proved. Estimate now \(\hat{\delta}^\epsilon(t)\). Introduce notation
\[
\hat{H} = \frac{1}{\epsilon} H(\sqrt{\epsilon} \hat{\psi}^+, \sqrt{\epsilon} \hat{\psi}^-).
\]
\[
\hat{\kappa}^\epsilon(t) = \hat{\tilde{\Psi}^\epsilon}(t) - \hat{\Phi}(t), \quad \hat{s}^\epsilon(t) = (i \frac{\partial}{\partial t} - \hat{H}) \hat{\tilde{\Psi}^\epsilon}(t).
\]

The function \(\hat{\kappa}^\epsilon(t)\) is the solution of the Cauchy problem
\[
i \frac{\partial}{\partial t} \hat{\kappa}^\epsilon(t) - \hat{H} \hat{\kappa}^\epsilon(t) = \hat{s}^\epsilon(t), \quad \hat{\kappa}^\epsilon(0) = 0
\]
and it is
\[
\hat{\kappa}^\epsilon(t) = \int_0^t dt' \exp(-i \hat{H}(t - t')) \hat{s}^\epsilon(t').
\]

Hence
\[
\|\hat{\kappa}^\epsilon(t)\| \leq \int_0^t \|\hat{s}^\epsilon(t')\| dt' \to 0 \quad \epsilon \to 0
\]
that implies
\[
\|\hat{\delta}^\epsilon(t)\| \to 0 \quad \epsilon \to 0
\]

Theorem 2 is proved.

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