On Thermometer Operation of Ultrasmall Tunnel Junctions

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Abstract

The temperature dependence of the $I-V$ characteristics of many single-electron tunneling devices enable thermometer operation of these systems. We investigate two normal conducting kinds of them, (a) a single junction in a high-impedance environment, and (b) a double junction. The characteristics of both devices show a crossover from Coulomb blockade at low temperatures to ohmic behavior at high temperatures. The related differential conductivity dip allows the determination of the junctions temperature. Both configurations (a) and (b) are expected to work at least within the range $0.5 \leq \beta E_C \leq 10$, where $E_C$ is the Coulomb energy of the system under investigation. We present an analytical solution for both low- and high-temperature case of (a) and (b) as well as numerical results and their fit, including the effect of co-tunneling in case of a double junction.

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I. INTRODUCTION

Sensoring turns out to be one of the possible applications of single–electron tunneling (SET) devices. After the establishment of the electrometer [1], that is one of the basic devices of SET, radiation detectors [2] and resistor comparators [3] were studied. Nowadays, SET–devices as thermometer are brought into the scope of investigation [4].

For this operation the known temperature dependence of the current through ultrasmall tunnel junctions is exploited. As the single junction without any environment, i.e. the voltage biased tunnel junction, does not show a reasonable influence of the temperature on its characteristic, the basic thermometer devices are (a) the normal conducting single junction in a high–impedance environment and (b) the normal conducting double junction. Studies on both (a) [5] and (b) [4] are done. This paper is restricted to (a) the infinite high–impedant ohmic environment ($R_E \rightarrow \infty$) and (b) the symmetric double junction ($R_1 = R_2$, $C_1 = C_2$). On the other hand, it extends [5] because of spotting the thermometer operation as well as [4] due to a more rigid theoretical treatment.

The suppression of thermal fluctuations in SET junctions is expressed by the condition $E_C \gg k_B T$, where $E_C$ is the Coulomb energy of the considered system. If this condition holds both (a) and (b) show a Coulomb blockade [3], [7]. This is the low–temperature case of our consideration. At high temperatures, $E_C \ll k_B T$, Coulomb effects are covered by thermal fluctuations and an ohmic current–voltage characteristic remains.

The thermal dependence is rather discussed in terms of the differential conductivity $G(V)$ of the junction(s) than the current. According to the Coulomb blockade $G(V)$ shows a dip at $V = 0$ (Fig. 1, 3). This dip is well–pronounced at low temperatures and is smeared out with increasing temperatures. Therefore, the shape of the dip allows the determination of the temperature of the junctions. This is the thermometer operation.

There are different measures of the mentioned dip that may be used for evaluation. In this paper the depth of the dip, i.e. $G(V = 0)$, is used only. It is preferred to the width of the dip because a precise measurement of the latter at higher temperatures would require
improper high voltages across the junction(s). Additionally, the depth of the dip is easier accessible in theory.

II. SINGLE JUNCTION

A single junction possesses the Coulomb energy $E_C = e^2/(2C)$, where $C$ is the junction capacitance. The current via this junction in a high–impedance ohmic environment ($R_E \to \infty$) is given by

$$I(V) = \frac{1}{eR} \left(1 - e^{-\beta eV}\right) \sqrt{\frac{\beta}{4\pi E_C}} \int \frac{d\varepsilon}{1 - e^{-\beta\varepsilon}} \exp \left[-\frac{\beta}{4E_C}(eV - E_C - \varepsilon)^2\right]$$

(1)

in terms of the junction resistance $R$ and the temperature $\beta = 1/(k_B T)$. To calculate this integral the first term of it is represented as

$$\frac{\varepsilon}{1 - e^{-\beta\varepsilon}} = \begin{cases} \varepsilon \sum_{n=0}^{\infty} e^{-n\beta\varepsilon} & \text{if } \varepsilon > 0 \\ -\varepsilon \sum_{n=0}^{\infty} e^{(n+1)\beta\varepsilon} & \text{if } \varepsilon < 0. \end{cases}$$

(2)

This representation breaks down at $\beta = 0$. Furthermore, the convergence of the sum is poor in the vicinity of this value. Therefore, this domain is excluded now but studied in detail later. Due to the sum, (1) simplifies to two integrals of the type

$$\int d\varepsilon x e^{-(ax^2+bx+c)} = \frac{1}{2a} e^{-c} - \frac{b}{2a} \sqrt{\frac{\pi}{a}} \exp \left(\frac{b^2}{a} - c\right) \text{erfc}\left(\frac{b}{\sqrt{a}}\right),$$

(3)

with $\text{erfc}(x) = 1 - \text{erf}(x)$. The corresponding calculation yields

$$I(V) = \frac{E_C}{2eR} \sum_{n=0}^{\infty} \left[f\left(\frac{eV}{E_C}, n\right) - f\left(-\frac{eV}{E_C}, n\right)\right]$$

(4)

with

$$f(x, n) = e^{-\beta E_C(x-1)^2/4} \left[\frac{4}{\sqrt{\pi} \beta E_C} \right.$$

$$-\left(2n - x + 1\right) \exp[\beta E_C(2n - x + 1)^2/4] \text{erfc}\left((2n - x + 1)\sqrt{\frac{\beta E_C}{4}}\right)$$

$$-\left(2n + x + 1\right) \exp[\beta E_C(2n + x + 1)^2/4] \text{erfc}\left((2n + x + 1)\sqrt{\frac{\beta E_C}{4}}\right) \right].$$
Obviously, the current is the difference of two contributions depending on each other by the substitution \( V \leftrightarrow -V \). The determination of the minimal differential conductivity yields in terms of

\[
\frac{d}{dy} \left[ f(y, n) - f(-y, n) \right] \bigg|_{y=0} = 4\sqrt{\frac{\beta E_C}{\pi}} e^{-\beta E_C/4} \]

\[
-2\beta E_C(2n + 1) e^{\beta E_C(n+1)} \text{erfc}\left((2n + 1)\sqrt{\frac{\beta E_C}{4}}\right)
\]

for \( G(V) \)

\[
G(V = 0) = \frac{dI}{dV} \bigg|_{V=0} = 2 R e^{-\beta E_C/4} \sum_{n=0}^{\infty} \left[ \sqrt{\frac{\beta E_C}{\pi}} - \beta E_C(n + \frac{1}{2}) e^{\beta E_C(n+1/2)} \text{erfc}\left((n + \frac{1}{2})\sqrt{\beta E_C}\right) \right].
\]

If \( \beta E_C > 1 \) holds the asymptotic expansion

\[
\sqrt{\pi \varepsilon} e^{\varepsilon^2} \text{erfc}(z) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \cdots (2m - 1)}{(2z^2)^m}
\]

and

\[
\sum_{n=0}^{\infty} (2n + 1)^{-2m} = \frac{(2m - 1) \pi^{2m}}{2(2m)!} |B_{2m}|
\]

can be used to transform (7) into

\[
G(V = 0) = -\frac{1}{R} \sqrt{\frac{\beta E_C}{\pi}} e^{-\beta E_C/4} \sum_{m=1}^{\infty} (-1)^m \frac{(2m - 1) \pi^{2m}}{(\beta E_C)^m m!} |B_{2m}|
\]

using the Bernoulli numbers \( B_{2m} \). The term corresponding to \( m = 1 \) yields the behavior of (10) for \( \beta E_C \gg 1 \). This is the short–dashed curve in Fig. 2.

In case of \( \beta E_C \ll 1 \) the evaluation of (7) uses

\[
\frac{\varepsilon}{1 - e^{-\beta \varepsilon}} = \frac{1}{\beta} e^{\beta \varepsilon/2} \frac{\beta \varepsilon/2}{\sinh \beta \varepsilon/2}.
\]

Therefore, the integral in (7) includes besides the Gaussian an other peak of the shape \( x/\sinh x \). Because the width of the Gaussian and the \( x/\sinh x \)–peak are proportional to
\((\beta E_C)^{-1/2}\) and \((\beta E_C)^{-1}\), respectively. Hence, the latter peak becomes broad and flat compared to the Gaussian in the limit \(\beta \to 0\). Therefore, it turns out to be sufficient to use the approximation

\[
\frac{\beta \varepsilon/2}{\sinh \beta \varepsilon/2} \approx 1
\]

in this limit and an integral of the Gaussian remains of (11). It results in

\[
I(V) = \frac{1}{\beta e R} \left(1 - e^{-\beta e V}\right) \exp \left[-\frac{\beta}{2}(E_C - 2eV)\right]
\]

for the current via the junction and gives

\[
G(V = 0) = \left. \frac{dI}{dV} \right|_{V=0} = \frac{1}{R} e^{-\beta E_C/4}
\]

for the minimal differential conductivity of the junction. This formula is represented in Fig. 2 by means of a long–dashed curve.

According to Fig. 2 an evaluation of (11) at \(\beta E_C \approx 1\) is desired, but hardly to achieve analytically. Therefore, a simple fit to the numerical data was performed, resulting in

\[
G(V = 0) = \frac{1}{R} \left(1 - \frac{2}{\pi} \arctan(0.5\beta E_C)\right).
\]

This formula corresponds to the dotted line in Fig. 2. The fit may be enhanced by introducing a polynomial into the argument of \(\arctan\). In the given simple form (15) enables an inversion to

\[
\beta E_C = \frac{1}{0.5} \tan \left[\frac{\pi}{2} \left(1 - G(V = 0)R\right)\right],
\]

a simple thermometer function.

III. DOUBLE JUNCTION

In case of the symmetric double junction the Coulomb energy is given by \(E_C = e^2/(4C)\), where \(C\) is the junction capacitance of each junction. Due to several states of the central
electrode of the double junction, that are determined by the number $n$ of extra charges on it, the current $I(V)$ turns out to be a sum over contributions of each state:

$$I(V) = e \sum_n \left[ r_1(n) - l_1(n) \right] \sigma(n) = e \sum_n \left[ r_2(n) - l_2(n) \right] \sigma(n) = \sum_n i(n) \sigma(n).$$  

(17)

In general, both the rates $r_{1,2}(n)$, $l_{1,2}(n)$ for both directions of the first and second junction and the occupation probability $\sigma(n)$ are voltage dependent. The voltage dependence of the rates is given in terms of the energy gain per tunnel event [10]

$$E_{r_{1,2}}(n) = 2 E_C \left( \frac{CV}{e} \mp n - \frac{1}{2} \right); \quad E_{l_{1,2}}(n) = 2 E_C \left( - \frac{CV}{e} \mp n + \frac{1}{2} \right)$$  

(18)

by

$$r_{1,2}(n) = \frac{1}{e^2 R} \frac{E_{r_{1,2}}(n)}{1 - e^{-\beta E_{r_{1,2}}(n)}}; \quad l_{1,2}(n) = \frac{1}{e^2 R} \frac{E_{l_{1,2}}(n)}{1 - e^{-\beta E_{l_{1,2}}(n)}}.$$  

(19)

where $R$ is the junction resistance.

According to the Likharev–Averin equation [10] the stationary occupation probabilities $\sigma(n)$ are governed by

$$\frac{\sigma(n+1)}{\sigma(n)} = \frac{r_1(n) + l_2(n)}{r_2(n+1) + l_1(n+1)}.$$  

(20)

In case of $n \gg 1$ this simplifies to

$$\frac{\sigma(n+1)}{\sigma(n)} \approx e^{-2n\beta E_C} \cosh 2\beta E_C \left( \frac{CV}{e} - \frac{1}{2} \right).$$  

(21)

Comparing with a Gaussian normal probability function with mean value 0 and width $s$

$$\sigma(n) = \frac{1}{s\sqrt{\pi}} \exp \left[ - \frac{1}{2} \left( \frac{n}{s} \right)^2 \right]$$  

(22)

$$\frac{\sigma(n+1)}{\sigma(n)} = \exp \left[ - \frac{n}{s^2} - \frac{1}{2s^2} \right],$$

the occupation probability can be approximated by this Gaussian, if

$$s = \frac{1}{\sqrt{2\beta E_C}}.$$  

(23)

is chosen. This approximation is in the limit $\beta \to 0$ exact. However, according to Fig. it seems to be reasonable, if $\beta E_C < 10$ holds. The main result of this consideration is the
weak voltage dependence of the occupation probabilities \( \sigma(n) \). Neglecting this dependence, the differential conductivity of the studied double junction might be expressed as

\[
G(V) = \frac{dI(V)}{dV} = \sum_n g(n)\sigma(n),
\]

where

\[
g(n) = \frac{di(n)}{dV}
\]

is used with regard to (17). Investigating the minimal differential conductivity of the double junction, i.e. \( V = 0 \), the calculation yields

\[
g(n)|_{V=0} = \frac{1}{2R} \left\{ 1 - \frac{\sinh \beta E_C + \beta E_C \cosh \beta E_C}{\cosh \beta E_C - \cosh 2n\beta E_C} \right\}
\]

In case of \( n = 0 \) this result simplifies to

\[
g(0)|_{V=0} = \frac{1}{R} \left( 1 - e^{\beta E_C} + \frac{\beta E_C e^{\beta E_C}}{(e^{\beta E_C} - 1)^2} \right).
\]

Further simplification is achieved in the case of \( \beta E_C \ll 1 \), and \( \beta E_C \gg 1 \) where

\[
\beta E_C \ll 1 : g(0) \approx \frac{1}{2R} \left( 1 - \frac{\beta E_C}{3} \right)
\]

\[
\beta E_C \gg 1 : g(0) \approx \frac{\beta E_C}{R} e^{-\beta E_C}
\]

holds, respectively. In both cases further evaluation is possible owing to (24) and the normalization of \( \sigma(n) \),

\[
\sum_n \sigma(n) = 1.
\]

As \( g(n) \) turns out to be independent of \( n \) up to second order in \( \beta E_C \) the approximation \( g(n) \approx g(0) \) can be exploited in the first case (\( \beta E_C \ll 1 \)). This results in

\[
G(V = 0) \approx g(0) \sum_n \sigma(n) = g(0) = \frac{1}{2R} \left( 1 - \frac{\beta E_C}{3} \right).
\]
This formula is represented by the long–dashed curve in Fig. 4. In the opposite limit, i.e. \( \beta E_C \gg 1 \), \( \sigma(n) \) is sharply peaked at \( n = 0 \) (see Fig. 5(c)) and the evaluation yields

\[
G(V = 0) \approx g(0)\sigma(0) = g(0) = \frac{\beta E_C}{R} e^{-\beta E_C}.
\]  

(31)

This result is shown by the short–dashed curve in Fig. 4.

Again, the missing analytical approximation at \( \beta E_C \) is substituted by the following fit:

\[
G(V = 0) = \frac{1}{2R}[1 - \tanh(0.3\beta E_C)]
\]  

(32)

\[
\beta E_C = \frac{1}{0.3} \text{artanh}[1 - 2G(V = 0)R]
\]

This fit is given by the dotted curve in Fig. 4. The second line of (32) enables the determination of the temperature out of the minimal differential conductivity \( G(V = 0) \).

IV. CO–TUNNELING

According to [11] the correlated transfer of electrons across both junctions of a double junction results in current contributions due to two different processes known as coherent and incoherent co–tunneling. The influence of the coherent co–tunneling can be neglected in this consideration because it depends on the size of the central electrode which is a design parameter. Hence, it should not be a problem to suppress this influence by means of a larger island between the junctions.

On the other hand, the current contribution of incoherent co–tunneling is independent of this size. It is given by [11]

\[
I_{in}(V) = \frac{\hbar}{12\pi e^2 R_1 R_2} \left( \frac{1}{\Delta E_1} + \frac{1}{\Delta E_2} \right)^2 \left[ (eV)^2 + (2\pi k_B T)^2 \right] V.
\]

(33)

The evaluation in case of the symmetric double junction \( R_1 = R_2 = R \), \( \Delta E_1|_{V=0} = \Delta E_2|_{V=0} = E_C \) at \( V = 0 \) yields for the contribution to the differential conductance

\[
G_{in}(V = 0) = \frac{2}{3R} \frac{R_Q}{R} \frac{1}{\beta E_C}
\]

(34)
using the abbreviation $R_Q = h/e^2$. In order to calculate the total differential conductance at $V = 0$, this term has to be added to the results of Sec. III. Hence, the thermometric use of the double junction at higher temperatures ($\beta E_C \ll 1$) requires high tunnel resistances. Fig. 6 shows the high–temperature approximation of the conductance $G(V = 0)$ for different tunnel resistances. As seen from Eq. (34) co–tunneling becomes negligible for low temperatures ($\beta E_C \gg 1$). In case of a realistic value $R \approx 10R_Q$ incoherent co–tunneling restricts the application to $\beta E_C \geq 0.5$. If co–tunneling reaches the order of the one–particle process, the calculation which bases on a perturbation theory becomes insecure.

V. DISCUSSION

There are obvious similarities in the behavior of the devices of Sec. II and Sec. III to build a simple thermometer on base of SET junctions as far as co–tunneling is excluded. In this case the characteristics $G(\beta E_C)$ are comparable, but due to co–tunneling that rises strongly with growing temperature the working field of the double junction is restricted to $0.5 \leq \beta E_C \leq 10$ approximately, whereas the single junction is expected to work reasonable within $0.1 \leq \beta E_C \leq 10$. Therefore, the single junction is preferred from a theoretician’s point of view.

In experimental realization, however, it is much harder to prepare a current biased single junction than a voltage biased double junction. Hence, an experimenter would prefer the second kind of SET–thermometer. According to these points the cons and pros of both configurations are balanced. It could be expected that multi junction arrays are better suited at high temperatures ($\beta E_C \leq 1$) because of suppressed co–tunneling.

Practical exploitation of SET thermometers requires the knowledge of the junction parameters, i.e. the junction capacitance and resistance. These parameters might be determined by measuring the $I(V)$ characteristic of the system at low temperatures $\beta E_C \gg 1$. After calibration the thermometer is ready to be operated at $\beta E_C \approx 1$. Therefore, the thermometers might be used in the mK–region and their field of application is rather nar-
row. Owing to quantum fluctuations the effect will be disturbed for very low temperatures if SET–conditions hold \((R \gg h/e^2)\). In case of a single junction a finite environment resistance will cause changes.

One possible kind of usage is as on–chip thermometer of other SET experiments. The advantage of it is the accurate determination of the junction temperature on the chip. Due to \(\beta E_C \approx 1\) the technical restrictions of the thermometer junctions are less rigid than those of the other SET junctions on chip, where \(\beta E_C \gg 1\) must be fulfilled. The disadvantage is the above mentioned calibration that requires even lower temperatures than those of the following experiment.

The presented theoretical calculations allow more detailed treatment of the thermal behavior of SET junctions. The search for an analytical approximation at \(\beta E_C \approx 1\) might be successful. An interesting point of the thermal behavior of small tunnel junctions that may become important within a more detailed treatment is the electron heating by the current \([12]\). However, in the standard framework applied here this effect is neglected. Eventually it must be taken into account for lower temperatures \((\beta E_C \gg 1)\). In this case the theoretical approach basing on thermodynamic equilibrium is questionable. It should be noted that in this case the electron temperature exceeds the substrate temperature \([12]\). In Sec. \([1]\) the study of a non–ohmic or non–infinite environment might make sense. The investigation of an asymmetry might be of interest in case of the double junction. Therefore, this paper gives a first theoretical approach only.

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FIGURES

FIG. 1. Differential conductivity of a single junction within a high–impedance environment for varying temperature showing the change of the differential conductivity dip at $V = 0$. The depth of the dip is estimated. The curves belong to $\beta E_C = 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1.0, 2.0, 5.0, 10, 20, 50, \text{ and } 100$.

FIG. 2. Minimal differential conductivity $G(V = 0)$ of a single junction in a high–impedance environment. The solid line links the values taken from Fig. 1, the dotted line corresponds to a fit formula given in the text. The long and short–dashed curves show analytical approximations for $\beta E_C \ll 1$ and $\beta E_C \gg 1$.

FIG. 3. Differential conductivity of a symmetric double junction in dependence on the applied voltage for temperatures $\beta E_C = 0.1, 0.2, 0.5, 1.0, 2.0, 5.0, 10, 20, 50, \text{ and } 100$. The peaks at low temperatures arise from the Coulomb staircase.

FIG. 4. Minimal differential conductivity $G(V = 0)$ of a symmetric double junction. Again, the solid line belongs to the data of Fig. 3 and the dotted line to a fit. The long and short–dashed curves show analytical approximations for $\beta E_C \ll 1$ and $\beta E_C \gg 1$.

FIG. 5. Actual occupation probability $\sigma(n)$ of $n$ charges on the central electrode of a symmetric double junction and a corresponding gaussian probability distribution for varying temperatures ($a$) $\beta E_C = 0.1$, ($b$) 1.0, ($c$) 10. For $\beta \to 0$ the Gaussian becomes exact.

FIG. 6. Influence of the incoherent co–tunneling on the differential conductance $G(V = 0)$ for different values of $R/R_Q$. As in Fig. 4, the dotted line corresponds to the Fit to guide the eye. The other curves show the total differential conductance in high–temperature approximation at $V = 0$ for the ratios $R/R_Q = 1, 10, 100, 1000$ from up to down.