Hardy–Littlewood–Sobolev inequality for $p = 1$

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Consider the Riesz potential $I_\alpha$, where $\alpha \in (0, d)$,

$$I_\alpha[f](x) = \int_{\mathbb{R}^d} |\xi|^{-\alpha} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi, \quad x \in \mathbb{R}^d, \; f \in L_1(\mathbb{R}^d).$$
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This operator appears frequently in the study of PDE (it might be though of as "inverse differentiation"), or in geometric measure theory (to measure dimension of sets), or in many other places.
Theorem (Hardy–Littlewood–Sobolev inequality)

The operator $I^{\alpha}$ is bounded as an $L^p$ to $L^q$ mapping if and only if $\frac{1}{p} - \frac{1}{q} = \alpha d$ and $1 < p < q < \infty$.

$$\|I^{\alpha}[f]\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$
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\[ \| I_\alpha[f] \|_{L_q(\mathbb{R}^d)} \lesssim \| f \|_{L_p(\mathbb{R}^d)}. \]
Invariance and sharpness

\[ \| I_\alpha[f] \|_{L^q(\mathbb{R}^d)} \lesssim \| f \|_{L^p(\mathbb{R}^d)}. \]

The HLS inequality is shift invariant and also dilation invariant. This means that the norms on the left hand side and the right hand side do not change if we shift the function \( f \) and are multiplied by the same constant if we dilate \( f \). In particular, the dilation invariance leads to the necessity of

\[ \frac{1}{p} - \frac{1}{q} = \alpha d. \]

In the endpoint case \( p = 1 \), that is

\[ \| I_\alpha[f] \|_{L^q} \lesssim \| f \|_{L^1}, \]

the inequality fails. Indeed, if we plug \( f = \delta_0 \) (which is formally illegal), we get

\[ I_\alpha[\delta_0](x) = c |x|^{\alpha - d}. \]

This function does not belong to \( L^{d-\alpha} \).

However, an interested analyst may prove the weak type bound in \( L^{d-\alpha}, \infty \).
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**Theorem (Classical Sobolev embedding)**

If \( \frac{1}{p} - \frac{1}{q} = \frac{1}{d} \) and \( 1 < p < q < \infty \), then

\[ \|f\|_{L_q} \lesssim \|\nabla f\|_{L_p}, \quad f \in C_0^\infty(\mathbb{R}^d). \]
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$$\|f\|_{L^{d-d-1}} \lesssim \|\nabla f\|_{L^1}, \quad f \in C_0^\infty (\mathbb{R}^d).$$

Using the Calderón–Zygmund theory (or rather the Mikhlin multiplier theorem), this may be restated as

$$\|I_1[g]\|_{L^{d-d-1}} \lesssim \|g\|_{L^1}, \quad g = \nabla f.$$

Recall that HLS fails for $p = 1$ when tested against a delta measure. In a sense, the Gagliardo–Nirenberg inequality says that the vectorial expression $\nabla f$ cannot concentrate as well as delta measures do.
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So it is natural to ask: "For what spaces (classes of functions) \( W \) does the inequality

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**Theorem (Hardy and Littlewood, 1927)**

The inequality

$$\| l_1^{1/2} f \|_{L^2(\mathbb{R})} \lesssim \| f \|_{L^1(\mathbb{R})}, \quad \hat{f}(\xi) = 0, \quad \xi < 0,$$

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In modern terms this may be stated as \( I_\alpha : H_1 \to L^d_{d-\alpha} \), where \( H_1 \) is either complex or the real Hardy class (Stein–Weiss inequality).
Yet another example was found about twenty years ago by Bourgain and Brezis:

\[ \| I_1[g] \|_{L^d} \lesssim \| g \|_{L^1}, \quad \text{div} g = 0. \]
More examples: Bourgain–Brezis inequalities

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So here $g$ is a vector field on $\mathbb{R}^d$ and vectorial behavior is important (as in the Gagliardo–Nirenberg inequality).
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So here $g$ is a vector field on $\mathbb{R}^d$ and vectorial behavior is important (as in the Gagliardo–Nirenberg inequality).

**Theorem (Van Schaftingen, 2011)**

The inequality

$$\| I_1[g] \|_{L^{d \over d-1}(\mathbb{R}^d)} \lesssim \| g \|_{L^1(\mathbb{R}^d)}, \quad Ag = 0,$$

where $A$ is a homogeneous elliptic differential operator holds true if and only if the space $^* - \text{clos}_{\text{Meas}}(\{Ag = 0 \mid g \in C^\infty_0(\mathbb{R}^d)\})$ does not contain vectorial delta measures.
Theorem

Let $\mathcal{W}$ be a closed linear subspace of $S'(\mathbb{R}^d, \mathbb{R}^\ell)$ that is invariant under translations and dilations, let $\alpha \in (0, d)$. The constant in the inequality

$$\| I_\alpha[f] \|_{L^\frac{d}{d-\alpha}} \lesssim \| f \|_{L^1}, \quad f \in \mathcal{W},$$

is uniform with respect to all $f \in \mathcal{W}$, for which the right hand side is finite, if and only if $\mathcal{W}$ does not contain the charges $a \otimes \delta_0$, $a \in \mathbb{R}^\ell \setminus \{0\}$. 
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For those analysts who are interested, we may replace the space $L_{\frac{d}{d-\alpha}}$ on the left with a finer space $L_{\frac{d}{d-\alpha}, 1}$; this solves several open problems going back to Van Schaftingen and even Bourgain–Brezis (here I am glad to mention that Hernandez and Spector solved B.-B. conjecture independently).
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$$\{\mu \geq 0 \mid \exists a \in \mathbb{R}^\ell \setminus \{0\} \quad a \otimes \mu \in \mathcal{W}\}$$

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has Hausdorff dimension larger than $\delta > 0$. The quantification of this qualitative statement is expressed via uniform estimates for the heat extension of $\mu$ (similar estimates were earlier used by Bennett, Carbery, and Tao to prove the multilinear restriction conjecture). Here it is:

$$\| H[\mu](\cdot, t) \|_{L_p(\mathbb{R}^d)} \leq t^{-\frac{d(p-1)}{2p} + \delta} \| H[\mu](\cdot, 1) \|_{L_p(\mathbb{R}^d)}.$$

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HLS for $p = 1$
One then has to decompose $f$ into parts where it resembles a measure in

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and parts where it is far from being a rank-one positive measure (those parts are even easier to handle).
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Thank you!