INTERSECTION COHOMOLOGY COMPLEXES ON LOW RANK FLAG VARIETIES

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WITH AN APPENDIX BY TOM BRADEN

ABSTRACT. We present a combinatorial procedure, based on the $W$-graph of the Coxeter group, which shows that the graded dimension of the stalks of intersection cohomology complexes of certain Schubert varieties is independent of the characteristic of the coefficient field. Our procedure exploits the existence and uniqueness of parity sheaves. In particular we are able to show that the characters of all intersection cohomology complexes with coefficients in a field on the flag variety $G/B$ of type $A_n$ for $n < 7$ are given by Kazhdan-Lusztig basis elements. By results of Soergel, this implies a part of Lusztig’s conjecture for $SL(n)$ with $n ≤ 7$. We also give examples where our techniques fail.

In the appendix by Tom Braden examples are given of intersection cohomology complexes on the flag varieties of $SL(8)$ and $SO(8)$ whose stalks have different graded dimension in characteristic 2.

1. INTRODUCTION

Let $k$ be a field of characteristic $p ≥ 0$. Let $G$ be a reductive algebraic group over $\mathbb{C}$, $B ⊂ G$ a Borel subgroup and $(W, S)$ the Weyl group and its simple reflections. Denote by $D^b_\Lambda(G/B)$ the bounded derived category of sheaves of $k$-vector spaces on $G/B$ constructible along $B$-orbits. In $D^b_\Lambda(G/B)$ there exist the intersection cohomology sheaves $IC(w)$. These sheaves are a certain extension of the constant sheaf in degree $-\ell(w)$ on the Bruhat cell $BwB/B$ to its closure.

Let $H$ be the Hecke algebra of $(W, S)$ over $\mathbb{Z}[v, v^{-1}]$ normalised so as to satisfy

$$H_s H_w = \left\{ \begin{array}{ll}
H_{sw} & \text{if } sw > w \\
(v-v^{-1})H_w + H_{sw} & \text{if } sw < w
\end{array} \right.$$ 

and let $\{H_w \mid w ∈ W\}$ be the Kazhdan-Lusztig basis of $H$. It satisfies $H_w ∈ H_w + \bigoplus_{x < w} v^{-1}N[v^{-1}]H_x$. Given a finite dimensional graded vector space $V = \bigoplus V_i$ let $P(V) = \sum (\dim V_i)v^i$ be its Poincaré polynomial.
The character of a sheaf $F \in D_b^G(B)$ is the element of $\mathcal{H}$ given by

$$\operatorname{ch}(F) = \sum_{w \in W} P(H^*(F_w)) v^*(w) H_w.$$ 

If $k$ is of characteristic zero, a theorem of Kazhdan and Lusztig [KL80] says that $\operatorname{ch}(\mathcal{IC}(w)) = H_w$. Thus the Poincaré polynomials of the stalks of the intersection cohomology sheaves are given by Kazhdan-Lusztig polynomials. It then follows that the same is true in almost all characteristics, however for any given characteristic almost nothing is known.

It is a difficult question to determine over which fields one has $\operatorname{ch}(\mathcal{IC}(w)) = H_w$ and, if not, what these characters are. It has been known since the original papers of Kazhdan and Lusztig ([KL79] and [KL80]) that in non-simply laced cases the intersection cohomology complexes may have a different character in characteristic 2. (This happens, for example, in the only non-smooth Schubert variety in the flag variety of $\mathcal{G}(4)$.) In 2002 Braden discovered examples of Schubert varieties in simply laced types $A_7$ and $D_4$ where the character of the intersection cohomology sheaf in characteristic 2 is different to all other characteristics (see the appendix).

In this article we define combinatorially a certain subset $\sigma(W) \subset W$ of separated elements and show:

**Theorem 1.1.** Suppose that $x \in \sigma(W)$, then $\operatorname{ch}(\mathcal{IC}(w)) = H_w$ for any field $k$.

The determination of the characters of $\mathcal{IC}(w)$ is closely related to the decomposition theorem in positive characteristic. Given a simple reflection $s \in S$ let $P_s$ be the corresponding standard minimal parabolic subgroup and consider the quotient map

$$G/B \xrightarrow{\pi_s} G/P_s.$$ 

If $k$ is of characteristic zero, the decomposition theorem implies that $\pi_{ss}\mathcal{IC}(w)$ is a direct sum of shifts of intersection cohomology sheaves. This need not be true if $k$ is of positive characteristic. Given $w \in W$ and $s \in S$ let $\{w_1, \ldots, w_i\}$ be all the parameters of Kazhdan-Lusztig basis elements that appear in the product $H_w H_s$. Then we have:

**Theorem 1.2.** Suppose that $w$ and $w_1, \ldots, w_n$ lie in $\sigma(W)$. Then the decomposition theorem holds for $\pi_{ss}\mathcal{IC}(w)$.

Whilst being of considerable intrinsic interest, these questions are also important in representation theory. Assume that $k$ is algebraically closed, that $\operatorname{char} k = p$ is greater than the Coxeter number of $W$ and
let \( G^\vee \) be a split semi-simple and simply connected algebraic group over \( k \) whose root system is dual to that of \( G \), and let \( B^\vee \supset T^\vee \) be a Borel subgroup and maximal torus in \( G^\vee \). Fix positive roots \( R^+ \subset X(T^\vee) \) so that the roots corresponding to \( B \) are those lying in \( -R^+ \). To each weight \( \lambda \in X(T^\vee) \) one may associate a standard module \( H^0(\lambda) \) which is non-zero if and only if \( \lambda \) is dominant, in which case it contains a unique simple submodule \( L(\lambda) \).

A conjecture of Lusztig [Lus80] expresses the characters of the simple modules \( L(\lambda) \) in terms of the (known) characters of standard modules. A particular case of the conjecture is the following: let \( \rho \in X(T^\vee) \) denote the half-sum of the positive roots, and \( st = (p-1)\rho \) the Steinberg weight, then it is conjectured that, for all \( x, y \in W \),

\[
[H^0(st + x\rho) : L(st + y\rho)] = h_{x,y}(1)
\]

where \( h_{x,y} \in \mathbb{Z}[v^{-1}] \) is the Kazhdan-Lusztig polynomial indexed by \( x, y \in W \). A theorem of Soergel [Soe00] says that (1) is equivalent to the semi-simplicity of \( \pi_s^\ast IC(x) \) for all \( x \in W \) and \( s \in S \).

Of course, in order to apply Theorems 1.1 and 1.2 it is necessary to know the set \( \sigma(W) \). The essential ingredient in the calculation of \( \sigma(W) \) is the \( W \)-graph of the Coxeter system \((W, S)\). Unfortunately, even in simple situations the \( W \)-graph can be very complicated and no general description is known. However, using Fokko du Cloux’s program Coxeter [dC] it is possible to use a computer to determine the set \( \sigma(W) \) for low rank Weyl groups. The simplest situation is when \( \sigma(W) = W \). This only occurs in type A in low rank:

**Theorem 1.3.** If \( G \) is of type \( A_n \) for \( n \leq 6 \) then \( \sigma(W) = W \). Hence, in all characteristics the intersection cohomology complexes have characters given by Kazhdan-Lusztig basis elements and the decomposition theorem holds for \( \pi_s^\ast IC(x) \) for all \( s \in S \) and \( x \in W \).

It also follows that (1) holds for \( G^\vee = SL_n(k) \) with \( n \leq 7 \) if \( k \) has characteristic at least \( n + 1 \). Using results of Braden in the appendix, we are also able to extend this theorem to cover \( G^\vee \) of type \( D_4 \).

In other types and type \( A_n \) for \( n \geq 7 \) our techniques are not as effective. In most examples that we have computed \( \sigma(W) \) is not the entire Weyl group. However, we are able to confirm the characters and decomposition theorem for many intersection cohomology complexes in ranks \( \leq 6 \). It also seems that the elements \( x \notin \sigma(W) \) for which our methods fail will provide an interesting source of future research.

Indeed in the appendix Braden shows that, both in type \( D_4 \) and \( A_7 \), if \( k \) is of characteristic 2, then for minimal elements in \( W \setminus \sigma(W) \) one
has \( \text{ch}(\text{IC}(w)) \neq H_w \). These two examples, combined with the case of dihedral groups, leads one to suspect a close relationship between \( W \setminus \sigma(W) \) and those intersection cohomology complexes for which one has \( \text{ch}(\text{IC}(w)) \neq H_w \) for some field of coefficients \( k \).

Let us briefly mention that, in [BM01] Braden and MacPherson give an algorithm for the calculation of the stalks of the intersection cohomology complexes, using only the fixed points and one-dimensional orbits of a maximal torus on the flag variety. (This data is encoded in the so-called “moment graph” of the flag variety.) The forthcoming paper [FW] of Fiebig and the author extends this result, showing that the moment graph of the flag variety can be used to calculate the characters of parity sheaves, and hence determine those intersection cohomology complexes for which \( \text{ch}(\text{IC}(w)) \neq H_w \). Thus the results of this paper could (at least in principle) be deduced from the moment graph. In fact, the computations of torsion in the appendix translate easily into the moment graph language and give a proof that the moment graph sheaves corresponding to certain Bott-Samelson varieties do not split as expected unless \( 2 \) is invertible in the coefficient ring.

Lastly, in this article we only consider field coefficients. However versions of the intersection cohomology complex exist with coefficients in \( \mathbb{Z} \) and the statement that \( \text{ch}(\text{IC}(w)) = H_w \) for a field of characteristic \( p \) is equivalent to the absence of \( p \)-torsion in the stalks and costalks of an intersection cohomology complex over \( \mathbb{Z} \). The reader is referred to [Jut09] for more details.

The structure of the paper is as follows. In Section 2 we review the Hecke algebra and Kazhdan-Lusztig basis in more detail and recall the \( W \)-graph associated to \( (W, S) \). In Section 3 we discuss the parity sheaves, introduced in [JMW09], which are our main theoretical tool. In Section 4 we define the subset \( \sigma(W) \subset W \) and prove Theorems 1.1 and 1.2. In Section 5 we discuss the calculation of the sets \( \sigma(W) \) via computer and give some examples of the sets \( \sigma(W) \) for low rank Weyl groups.

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2. **The Hecke Algebra and \( W \)-graphs**

In this section we recall the Hecke algebra and Kazhdan-Lusztig basis in slightly more detail. Let \( (W, S) \) be a Coxeter system with
Bruhat order $\leq$ and length function $\ell : W \to \mathbb{N}$. Given $w \in W$ we define the left and right descent set to be

$$\mathcal{L}(w) = \{ s \in S \mid sw < w \} \quad \text{and} \quad \mathcal{R}(w) = \{ s \in S \mid ws < w \}.$$ 

Recall that the Hecke algebra is the free $\mathbb{Z}[v, v^{-1}]$-module with multiplication given by

$$H_s H_w = \begin{cases} H_{sw} & \text{if } sw > w, \\ (v - v^{-1})H_w + H_{sw} & \text{if } sw < w. \end{cases}$$

The elements $H_w$ are invertible and there is an involution $h \mapsto \overline{h}$ on $\mathcal{H}$ which sends $H_w$ to $H_{w^{-1}}$ and $v$ to $v^{-1}$. We will call elements fixed by this involution self-dual.

There exists a basis $\{ H_w \}$ of $\mathcal{H}$ called the Kazhdan-Lusztig basis which is uniquely determined by requiring:

1. The $H_w$ are self-dual;
2. $H_w = \sum_{x \leq w} h_{x,w} H_x$ where $h_{w,w} = 1$ and $h_{x,w} \in v^{-1} \mathbb{Z}[v^{-1}]$ for $x \neq w$.

The polynomials $h_{x,w}$ are (up to a renormalisation) the Kazhdan-Lusztig polynomials. One may check, for example, that $H_s = H_s + v^{-1} H_{id}$.

The action of $H_s$ for $s \in S$ on the Kazhdan-Lusztig basis has a particularly simple form. We denote by $\mu(x, w)$ the coefficient of $v^{-1}$ in $h_{x,w}$. Then:

$$H_w H_s = \begin{cases} (v + v^{-1})H_w & \text{if } ws < w, \\ \overline{H_{sw}} + \sum_{x < w; s \in \mathcal{R}(x)} \mu(x, w) H_x & \text{if } ws > w. \end{cases}$$

(A similar formula describes multiplication by $H_s$ on the left).

Thus all the information about the action of $H_s$ on the left and right on the Kazhdan-Lusztig basis may be encoded in a labelled graph, known as the $W$-graph. The vertices correspond to the elements of $W$ and are labelled with the left and right descent sets. There is a directed edge between $x$ and $y \in W$ if $\mu(x, y) \neq 0$, in which case the edge is labelled with the value of $\mu(x, y)$. For more details on the Kazhdan-Lusztig basis and $W$-graphs the reader is referred to [KL79], [Hum90] or [Soe97].

3. Parity Sheaves

In this section we briefly recall some basic properties of “parity sheaves” introduced in [JMW09] and motivated by [Soe00]. These are our main technical tool.

We recall briefly the setting of [JMW09]. We fix a field of coefficients $k$. All spaces will be complex algebraic $H$-varieties, for $H$ a complex linear algebraic group. Given an $H$-space $X$, we write...
for the bounded derived category of constructible $k$-sheaves on $X$ and $D^b_{H}(X)$ for the bounded $H$-equivariant derived category of constructible sheaves of $k$-vector spaces on $X$ (see [BL94]). By abuse of language, we call objects in $D^b_{H}(X)$ complexes. We denote by $\text{For} : D^b_{H}(X) \to D^b_{H}(X)$ the forgetful functor (see [BL94]). If $H$ has finitely many orbits on $X$ then the image of the forgetful functor is contained in $D^b_{G}(X)$, the full subcategory of $D^b_{H}(X)$ consisting of sheaves whose cohomology is locally constant along $H$-orbits. The endomorphism ring of any indecomposable object in $D^b_{H}(X)$ is local, and hence the Krull-Schmidt theorem holds in $D^b_{H}(X)$.

All maps will be equivariant morphisms of complex algebraic varieties. Given a map $f : X \to Y$ we have functors $f_*, f_!$ from $D^b_{H}(X)$ to $D^b_{H}(Y)$ and $f^*, f^!$ from $D^b_{H}(Y)$ to $D^b_{H}(X)$. Similar functors exist between $D^b_{c}(X)$ and $D^b_{c}(Y)$. On the categories $D^b_{H}(X), D^b_{H}(Y), D^b_{c}(X)$ and $D^b_{c}(Y)$ we have the Verdier duality functor, which we denote by $\mathbb{D}$. We have isomorphisms of functors $\mathbb{D}f_* \cong f_!\mathbb{D}$ and $\mathbb{D}f^* \cong f^!\mathbb{D}$. All functors $f^*, f_!, f^!, f_*$ commute with the forgetful functor.

Now let $G$ denote a reductive complex algebraic group and $B \supset T$ a Borel subgroup and maximal torus. Let $W$ denote the Weyl group and $S \subset W$ the set of simple reflections. Throughout $X = G/P$, where $P$ is either $B$ or a minimal standard parabolic subgroup $P_s$ corresponding to $s \in S$ (i.e. $P_s := \overline{B_sB}$). We regard $X$ as a $B$-variety. Each $B$-orbit is isomorphic to an affine space and the strata are classified by $W$ if $P = B$ and $W/\langle s \rangle$ if $P = P_s$. Given $w \in W$ (resp. $\overline{w} \in W/\langle s \rangle$) we denote by $X_w$ (resp. $X_{\overline{w}}$) the stratum $BwB/B$ (resp. $B\overline{w}P_s/P_s$), by $i_w : X_w \hookrightarrow G/B$ (resp. $i_{\overline{w}} : X_{\overline{w}} \hookrightarrow G/P_s$) its inclusion and by $k_w$ (resp. $k_{\overline{w}}$) in the $B$-equivariant constant sheaf on $X_w$ (resp. $X_{\overline{w}}$).

For brevity, $? \in \{*, !\}$. A complex $\mathcal{F} \in D^b_{B}(X)$ is $?\text{-even}$ if $i_w^*\mathcal{F}$ (resp. $i_{\overline{w}}^*\mathcal{F}$) is isomorphic to a direct sum of constant sheaves concentrated in even degrees, for all strata $X_w \subset G/B$ (resp. $X_{\overline{w}} \subset G/P_s$). A complex is $?\text{-even}$ if it is both $?*$ and $!\text{-even}$. A complex $\mathcal{F}$ is (?!)-odd if $\mathcal{F}[1]$ is (?!)-even. A complex $\mathcal{F} \in D^b_{B}(X)$ is (?!)-parity if we have an isomorphism $\mathcal{F} \cong \mathcal{F}_0 \oplus \mathcal{F}_1$ with $\mathcal{F}_0$ (?!)-even and $\mathcal{F}_1$ (?!)-odd. Note that direct sums and summands of (?!)-parity sheaves are (?!)-parity.

An indecomposable parity complex is called a parity sheaf. The following theorem shows that one may classify parity sheaves on the flag variety in a similar way to intersection cohomology complexes:

**Theorem 3.1 ([JMW09, 4.4]).** There exists (up to isomorphism) a unique parity sheaf $\mathcal{E}(w) \in D^b_{B}(X)$ with support contained in $\overline{X}_w$ and $i_w^*\mathcal{E}(w) \cong \mathcal{F}(w)$.
$k_w[\dim X_w]$. Each $\mathcal{E}(w)$ is self-dual and any indecomposable parity complex is isomorphic to $\mathcal{E}(w)[m]$ for some $w \in W$ and $m \in \mathbb{N}$.

Given a $\ast$-parity complex $\mathcal{E} \in D^b_B(G/B)$ and $w \in W$ we may write $i_w^*\mathcal{E} \cong V(w) \otimes k_w$ for some finite dimensional graded $k$-vector space $V(w) = \oplus V(w)_i$. We define the character of $\mathcal{E}$ in the Hecke algebra to be:

$$\text{ch}(\mathcal{E}) = \sum_{i \in \mathbb{Z}, w \in W} (\dim V(w)_i)u^iH_w.$$

**Remark 1.** Given a $\ast$-parity complex $\mathcal{E}$ it is easily seen that $\text{ch}(\mathcal{E})$ agrees with the character of $\text{For}(\mathcal{E}) \in D^b_B(G/B)$ as defined in the introduction.

A similar character map exists for $D^b_B(G/P_s)$ for a simple reflection $s \in S$. Let $\mathcal{H}_s$ denote the left ideal $\mathcal{H}H_s$ in $\mathcal{H}$. Then $\mathcal{H}_s$ is free with basis $H_w H_s$ for $w \in W^s$, where $W^s \subset W$ denotes the subset of elements $w \in W$ such that $s \notin R(w)$. Given a $\ast$-parity complex $\mathcal{E} \in D^b_B(G/P_s)$ and $w \in W^s$ we can write $i_w^*\mathcal{E} \cong V(w) \otimes k_w$ for some graded $k$-vector space $V(w) = \oplus V(w)_i$. We define the character of $\mathcal{E}$ to be:

$$\text{ch}(\mathcal{E}) = \sum_{i \in \mathbb{Z}, w \in W^s} (\dim V(w)_i)u^iH_wH_s.$$

For any $s \in S$ we have obvious maps given by inclusion and multiplication:

$$\mathcal{H} \xrightarrow{\text{inc}} \mathcal{H}$$

The quotient map $\pi_s : G/B \to G/P_s$ induces functors:

$$D^b_B(G/B) \xrightarrow{\pi_s} D^b_B(G/P_s)$$

The following lemma is well-known (see [Spr82], Lemme 2.6):

**Lemma 3.2.**

1. If $\mathcal{E} \in D^b_B(G/B)$ is $\ast$-parity, then so is $\pi_s^*\mathcal{E}$ and

$$\text{ch}(\pi_s^*\mathcal{E}) = \text{ch}(\mathcal{E})H_s.$$

2. If $\mathcal{E} \in D^b_B(G/P_s)$ is $\ast$-parity, then so is $\pi_s^*\mathcal{E}$ and

$$\text{ch}(\pi_s^*\mathcal{E}[1]) = \text{inc}\left(\text{ch}(\mathcal{E})\right).$$
(3) If \( \mathcal{E} \in D^b_B(G/B) \) or \( D^b_B(G/P_s) \) is \(*\)-parity then
\[
\text{ch}(\mathcal{E}[1]) \cong v^{-1} \text{ch}(\mathcal{E}).
\]

Proof. We first show the character relations. Statement (3) is a straightforward consequence of the definitions and (2) follows from the definitions and the fact that \( \pi_s^{-1}(X_\pi) = X_w \sqcup X_ws \) for \( w \in W \). It remains to show (1).

We prove (1) by induction on the number of \( w \in W \) for which \( i^*_w \mathcal{E} \neq 0 \). If this number is one, then (by definition of \(*\)-parity) \( \mathcal{E} \) is necessarily a direct sum of shifts of \( i^*_w \mathcal{K}_w [\ell(w)] \) for some \( w \in W \). We may assume that \( \mathcal{E} \cong i^*_w \mathcal{K}_w [\ell(w)] \). Let us write \( \pi_\omega \mathcal{E} \) for the image of \( w \) in \( W/\langle s \rangle \). If \( ws > w \) then \( \pi_s \) restricts to an isomorphism \( X_w \to X_\pi \). Hence
\[
\pi_{s*} \mathcal{E} \cong i^*_w \mathcal{K}_w [\ell(w)].
\]
If \( ws < w \) then the restriction of \( \pi_s \) to \( X_w \) induces a (trivial) \( C \)-bundle over \( X_w \), hence
\[
\pi_{s*} \mathcal{E} \cong i^*_w \mathcal{K}_w [\ell(w) - 2].
\]
A simple calculation in the Hecke algebra then shows that in both cases
\[
\text{ch}(\pi_{s*} \mathcal{E}) = \text{ch}(\mathcal{E}) H_s
\]
as claimed.

We now turn to the general case. We may assume without loss of generality that \( \mathcal{E} \) is \(*\)-even. Choose \( w \in W \) so that \( X_w \) is open in the support of \( \mathcal{E} \) and let \( i : \overline{\text{supp} \mathcal{E}} \setminus X_w \to G/B \) denote the inclusion. Then \( i^*_w \mathcal{E} \cong i^*_w \mathcal{E} \) and we have a distinguished triangle of \(*\)-even sheaves
\[
i_{i^*_w} \mathcal{E} \to \mathcal{E} \to i_{i^*_w} \mathcal{E} [1] \to
\]
By induction \( \pi_{s*} \) applied to the first or third terms is \(*\)-parity and (1) holds. It follows that the same is true of \( \mathcal{E} \) because \( \text{ch}(\mathcal{E}) = \text{ch}(i_{i^*_w} \mathcal{E}) + \text{ch}(i_{i^*_w} \mathcal{E}) \) and \( \text{ch}(\pi_{s*} \mathcal{E}) = \text{ch}(\pi_{s*} i_{i^*_w} \mathcal{E}) + \text{ch}(\pi_{s*} i_{i^*_w} \mathcal{E}) \).

It remains to see that \( \pi_s \) and \( \pi^* \) preserve the classes of parity sheaves. However this follows immediately because \( \pi_s \mathcal{D} \cong \mathcal{D} \pi_s \) (as \( \pi \) is proper) and \( (\pi^*[1]) \mathcal{D} \cong \mathcal{D} (\pi^*[1]) \) (because \( \pi \) is a smooth fibration with fibres of complex dimension 1).

Consider \( G \) as a \( B \times B \)-space via \( (b_1, b_2) \cdot g := b_1 gb_2^{-1} \). As the second copy of \( B \)-acts freely on \( G \), the quotient equivalence ([BL94, 2.6.2]) yields an equivalence of triangulated categories:
\[
Q^* : D^b_B(G/B) \to D^b_B(B \times B(G)).
\]
Consider the inversion map \( i : G \to G \). Then this is \( B \times B \)-equivariant with respect to the swap map \( B \times B \to B \times B : (b_1, b_2) \mapsto (b_2, b_1) \). This induces an equivalence

\[
i^* : D^b_{B \times B}(G) \to D^b_{B \times B}(G)
\]

Consider the functor \( \iota := (Q^*)^{-1}i^*Q^* : D^b_B(G/B) \to D^b_B(G/B) \). Then \( \iota \) commutes with \( D \). It is easy to see \( \iota \) preserves parity complex and that, for a parity complex \( E \in D^b_B(G/B) \),

\[
(3) \quad \chi (\iota (E)) = j (\chi (E))
\]

where \( j : \mathcal{H} \to \mathcal{H} \) is the anti-involution \( H_w \mapsto H_{w^{-1}} \).

Define endofunctors on \( D^b_B(G/B) \) by

\[
(-) \vartheta_s := \pi_s^* \varpi_{s*}(-)[1] \quad \text{and} \quad \vartheta_s(-) := \iota^* \pi_s^* \varpi_{s*} \iota^*(-)[1].
\]

Then the functors \( (-) \vartheta_s \) and \( \vartheta_s(-) \) preserve parity sheaves; the shift is chosen so that \( D (\vartheta \mathcal{E}) \cong \vartheta (D \mathcal{E}) \) and \( D (\vartheta_s \mathcal{E}) \cong (D \mathcal{E}) \vartheta_s \). By (3) and the above lemma,

\[
\chi (\vartheta \mathcal{E}) = \chi (\mathcal{E}) \mathcal{H}_s \quad \text{and} \quad \chi (\vartheta_s \mathcal{E}) = \mathcal{H}_s \chi (\mathcal{E})
\]

for parity sheaves \( \mathcal{E} \in D^b_B(G/B) \).

The first result about the characters of parity sheaves is the following:

**Proposition 3.3.** For all \( w \in W \), \( \chi (\mathcal{E}(w)) \in \mathcal{H} \) is self-dual.

**Proof.** We proceed via induction on \( \ell(w) \) with the base case being trivial. Let us fix \( w \) and choose \( s \in S \) with \( sw < w \). By Theorem 3.1 we may write

\[
\vartheta_s \mathcal{E}(sw) \cong \mathcal{E}(w) \oplus \mathcal{G}
\]

where

\[
\mathcal{G} \cong \bigoplus_{\substack{x < w \\eta \in \mathbb{Z}}} \mathcal{E}(x)[\eta]^{m_{x, \eta}}.
\]

The Verdier self-duality of \( \vartheta_s \mathcal{E}(sw) \) and each \( \mathcal{E}(x) \) for \( x < w \) together with Krull-Schmidt implies

\[
m_{x,-\eta} = m_{x,\eta}.
\]

By induction, the \( \chi (\mathcal{E}(x)) \) for \( x < w \) are self-dual. Hence both \( \chi (\mathcal{G}) \) and \( \chi (\vartheta_s \mathcal{E}(sw)) = \mathcal{H}_s \chi (\mathcal{E}(sw)) \) are self-dual. Thus so is \( \chi (\mathcal{E}(w)) \).

Let \( s \in S \) be a simple reflection. The next proposition relates parity sheaves on \( G/P_s \) to those on \( G/B \):
Proposition 3.4. Let $w \in W$ be such that $ws < w$ and denote by $\overline{w}$ the image of $w$ in $W/\langle s \rangle$. We have isomorphisms
\[ \pi_s^* \mathcal{E}(\overline{w})[1] \cong \mathcal{E}(w) \]
and
\[ \pi_{ss}^* \mathcal{E}(w) \cong \mathcal{E}(\overline{w})[-1] \oplus \mathcal{E}(\overline{w})[1]. \]

Proof. As $\mathcal{E}(w)$ is a direct summand of $\pi_s^* \mathcal{E}(\overline{w})[1]$ we have
\[ \text{ch}(\mathcal{E}(w)) = H_w + v^{-1}H_{ws} + \ldots. \]
It follows (by considering $i_{\overline{w}}^* \pi_{ss}^* \mathcal{E}(w)$) that
\[ \pi_{ss}^* \mathcal{E}(w) \cong \mathcal{E}(\overline{w})[1] \oplus \mathcal{E}(\overline{w})[-1] \oplus \mathcal{G} \]
for some parity complex $\mathcal{G}$. We may also decompose
\[ \pi_{ss}^* \mathcal{E}(\overline{w})[1] \cong \mathcal{E}(w) \oplus \mathcal{G}'. \]

Hence
\[ \pi_{ss}^* \pi_s^* \mathcal{E}(\overline{w})[1] \cong \mathcal{E}(w)[1] \oplus \mathcal{E}(\overline{w})[-1] \oplus \mathcal{G} \oplus \pi_{ss}^* \mathcal{G}'. \]

However, by Lemma 3.2,
\[ \text{ch}(\pi_{ss}^* \pi_s^* \mathcal{E}(\overline{w})[1]) = (v + v^{-1}) \text{ch}(\mathcal{E}(\overline{w})) \]
and so $\text{ch}(\mathcal{G}) = \text{ch}(\pi_{ss}^* \mathcal{G}') = 0$. Hence $\mathcal{G}$ and $\mathcal{G}'$ are zero. \qed

In $D_b^\Lambda(G/B)$ there exist the intersection cohomology sheaves $\mathcal{IC}(w)$ which may be defined as $(i_w)_! k_w[\ell(w)]$ (see [BBD82]). The intersection cohomology complexes admit equivariant lifts $\mathcal{IC}_B(w) \in D_b^k(G/B)$ which are uniquely determined (up to isomorphism) by requiring $\text{For}(\mathcal{IC}_B(w)) \cong \mathcal{IC}(w)$ (see [BL94, 5.2]).

We will need the following in the next section:

Proposition 3.5. For $w \in W$, $\mathcal{E}(w) \cong \mathcal{IC}_B(w)$ if and only if $\text{ch}(\mathcal{E}(w)) = H_w$. Hence if $\text{ch}(\mathcal{E}(w)) = H_w$ then $\text{For}(\mathcal{E}(w)) \cong \mathcal{IC}(w)$.

Proof. Recall that $\mathcal{IC}(w)$ is the unique complex in $D_b^\Lambda(G/B)$ satisfying the four conditions:

1. $\mathcal{IC}(w)$ is Verdier self-dual;
2. $i_y^! \mathcal{IC}(w) = 0$ for $y < w$;
3. $i_w^* \mathcal{IC}(w) \cong k_w[\ell(w)]$;
4. $i_w^* \mathcal{IC}(w)$ is concentrated in degrees $< -\ell(x)$ for $x < w$.

Note that 1), 2) and 3) are always satisfied by $\text{For}(\mathcal{E}(w))$.

Now, if $\text{ch}(\mathcal{E}(w)) = H_w$ then (4) is also satisfied and so $\text{For}(\mathcal{E}(w)) \cong \mathcal{IC}(w)$. On the other hand, if $\text{ch}(\mathcal{E}(w)) \neq H_w$ then $\text{ch}(\mathcal{E}(w))$ is a self-dual and positive combination of Kazhdan-Lusztig basis elements.
containing more than one term (again by Proposition 3.3). It is straightforward to see that if this is the case then $\text{For}(E(w))$ cannot satisfy (4). □

4. SEPARATED ELEMENTS

Let \{\(E(w) \mid w \in W\)\} be a set of representatives of isomorphism classes of indecomposable parity sheaves on \(G/B\). We would like to investigate when their characters are equal to the Kazhdan-Lusztig basis. The set \{\(\text{ch}(E(w)) \mid w \in W\)\} yields a self-dual basis of \(\mathcal{H}\) with certain positivity properties which are shared by the Kazhdan-Lusztig basis. In this section we investigate to what extent these properties already determine the basis.

Example 4.1. As motivation, let us consider some examples:

1. (Simplistic) Let \(x \in W\) and suppose that \(x_s > x\), \(H_x H_s = H_{x_s}\) and \(\text{ch}(E(x)) = H_x\). We know that \(E(x)\) is a direct summand of \(E(x) \vartheta_s = \pi_s^* \pi_s \ast E(x)[1]\) with self-dual character. Hence \(E(x) \vartheta_s \cong IC_B(x)\) and so \(\text{ch}(E(x)) = H_{x_s}\) by the uniqueness of the Kazhdan-Lusztig basis.

2. (More realistic) Fix \(x \in W\) and suppose that we can show that \(\text{ch}(E(x)) = H_{x_s}\) for all \(s \in \mathcal{R}(x)\). Suppose further that the only Kazhdan-Lusztig basis element that appears with non-zero coefficient in all expressions \(H_x H_s\) with \(s \in \mathcal{R}(x)\) is \(H_x\). Then, using the fact that \(E(x)\) occurs as a direct summand in \(E(x) \vartheta_s = \pi_s^* \pi_s \ast E(x)[1]\) for all \(s \in \mathcal{R}(x)\), it follows that \(\text{ch}(E(w)) = H_x\).

In the above examples it is essential to know how the elements \(H_s\) for \(s \in S\) act on the Kazhdan-Lusztig basis. This is precisely the information provided by the \(W\)-graph discussed in Section 2.

We start with some definitions. Given an element \(h = \sum a_x H_x \in \mathcal{H}\) we define the \(KL\)-support to be the set:

\[\text{supp}_{KL}(h) = \{x \mid a_x \neq 0\}\]

We say that \(h\) is \(KL\)-supported in degree 0 if, in addition, all \(a_x \in \mathbb{N}\).

Given \(h, h' \in \mathcal{H}\) we may write the difference \(h' - h\) in the standard basis as

\[h' - h = \sum p_x H_x\]

If all \(p_x \in \mathbb{N}[v, v^{-1}]\) we write \(h \leq h'\). Note that if \(E\) is a direct summand of a parity complex \(G\) then \(\text{ch}(E) \leq \text{ch}(G)\).
Lemma 4.2. Suppose $E$ is a direct summand of a parity complex $G$ whose character is KL-supported in degree 0. Then the character of $E$ is also KL-supported in degree zero and

$$\text{supp}_{KL}(\text{ch}(E)) \subset \text{supp}_{KL}(\text{ch}(G)).$$

Proof. Because $\text{ch}(G)$ is KL-supported in degree 0, it is isomorphic to a direct sum of indecomposable parity sheaves $E(w)$ without shifts. Moreover, each $E(w)$ that occurs is KL-supported in degree zero. Hence $E$ is also isomorphic to a direct sum of such indecomposable parity sheaves and the lemma follows. □

Remark 2. A more conceptual proof of the above lemma is provided by the following (which will not be needed below). The character of a self-dual parity complex $E$ is KL-supported in degree zero if and only if $E$ is perverse. Now if $G$ is as in the lemma it is perverse and splits into a direct sum

$$\bigoplus_{x \in W, \eta \in \mathbb{Z}} E(x)[\eta]^{\oplus m_{x, \eta}}.$$

However we must have $m_{x, \eta} = 0$ if $\eta \neq 0$ as each direct summand must be perverse. Hence any direct summand is self-dual and perverse.

Given a subset $Z \subset W$ we denote by

$${}^sZ = \{ x \in Z \mid sx > x \} \text{ and } Z^s = \{ x \in Z \mid xs > x \}.$$

We now define a function $f_W : Y \to \mathcal{P}(W)$ from some subset $Y \subset W$ (to be defined below) to the power set of $W$. This function and its domain are defined inductively as follows:

1. $f_W(id_W) = \{id_W\}$.
2. Suppose we have defined $f_W$ on all $y < x$. Then $x$ belongs to $Y$ if there exists $s \in \mathcal{L}(x)$ or $t \in \mathcal{R}(x)$ such that either

$${}^s f_W(sx) = f_W(sx) \text{ or } f_W(x)^t = f_W(x).$$

In this case we define $f_W(x)$ to be the set:

$$f_W(x) = \bigcap_{s \in \mathcal{L}(x), \ s f_W(sx) = f_W(sx)} \left( \bigcup_{w \in f_W(sx)} \text{supp}_{KL}(H_sH_w) \right) \cap \bigcap_{t \in \mathcal{R}(x), \ f_W(x)^t = f_W(x)} \left( \bigcup_{w \in f_W(x)} \text{supp}_{KL}(H_wH_t) \right).$$
Example 4.3. Let $W$ be a dihedral group:
$$D_n = \langle s, t \mid s^2 = t^2 = (st)^n = id \rangle.$$
If $(st)^m$ (resp. $(st)^ms$) is not the longest element then:
$$f_W((st)^m) = \{(st)^m, (st)^{m-1}, \ldots, st\}$$
$$f_W((st)^ms) = \{(st)^ms, (st)^{m-1}s, \ldots, s\}$$
For the longest element $w_0$ one has $f_W(w_0) = \{w_0\}$.

Definition 1. If $f_W$ is defined on $x \in W$ and $f_W(x) = \{x\}$ we say that $x$ is separated. The set of all separated elements will be denoted by $\sigma(W)$.

Example 4.4. If $W$ is a dihedral group $D_n$ then it follows from above that the separated elements are $\{id, s, t, st, ts, w_0\}$. In particular, $A_2$ and $A_1 \times A_1$ are the only rank two Weyl groups in which $\sigma(W) = W$.

Remark 3. One may show (for example using the results of [FW]) that if $G$ is a Kac-Moody group having Weyl group $W$ a dihedral group $D_n$, then the elements $w \in \sigma(W)$ are precisely those for which $\text{ch}(\text{IC}(w)) = H_w$ in all characteristics.

Theorem 1.1 in the introduction is an immediate consequence (using Proposition 3.5) of the following:

Proposition 4.5. If $f_W$ is defined on $x \in W$ then $\text{ch}(E(x))$ is KL-supported in degree 0 and
$$\text{supp}_{KL}(\text{ch}(E(x))) \subset f_W(x)$$
In particular, for all $x \in \sigma(W)$ we have $\text{ch}(E(x)) = H_x$.

Proof. Clearly $\text{ch}(E(id)) = H_{id}$ and so we may assume by induction that $\text{supp}_{KL}(\text{ch}(E(w))) \subset f_W(w)$ for all $w < x$, where $x \in W$ is some element on which $f_W$ is defined. Without loss of generality we may assume, by the inductive definition of $f_W$ above, that there exists some $s \in \mathcal{L}(x)$ so that $f_W(sx) = f_W(x)$. Hence $\text{ch}(\vartheta_s E(sw)) = H_s \text{ch}(E(sw))$ is KL-supported in degree zero.

We may now apply Lemma 4.2 to conclude that $\text{ch}(E(x))$ is KL-supported in degree zero and that
$$\text{supp}_{KL}(\text{ch}(E(x))) \subset \bigcup_{w \in f_W(x)} \text{supp}_{KL}(H_s H_w).$$
However, as $\text{ch}(E(x))$ is KL-supported in degree zero and is a direct summand of all $\vartheta_t E(tx)$ for all $t \in \mathcal{L}(x)$ we conclude that
$$\text{supp}_{KL}(\text{ch}(E(x))) \subset \bigcup_{w \in f_W(x)} \text{supp}_{KL}(H_s H_w).$$
An identical argument applies on the right. The proposition then follows by intersecting these conditions. □

The following remains from the introduction:

Proof of Theorem 1.2. Under the assumptions of the theorem, $E(w) \cong IC_B(w)$ and $E(w)d_w$ splits as a direct sum of parity sheaves, all of which are isomorphic to equivariant intersection cohomology complexes. We are then done by the above Proposition, combined with Proposition 3.5 and Proposition 3.4. □

5. Results of Computer Calculations

In this section we give some examples of the sets $\sigma(W) \subset W$ for low rank Weyl groups. As is clear from the definition of $f_W$, the only piece of information needed to calculate the $\sigma(W)$ and $f_W$ is the $W$-graph of $(W, S)$. However, no general description of the $W$-graph is known (for descriptions of some subgraphs see [LS81] and [Ker83] and for a description of the computational aspects of the problem see [dC02] and [OK95]).

Thus, in order to calculate $f_W$ and $\sigma(W)$ we have to restrict ourselves to examples. This involves two steps:

1) calculation of the $W$-graph of $(W, S)$, and
2) calculation of the function $f_W$ using the $W$-graph.

Step 1) is computationally quite difficult, especially when the Weyl group is large. Luckily there exists the program Coxeter written by Fokko du Cloux [dC], which calculates the $W$-graph very efficiently. Step 2) is then relatively straightforward. A crude implementation in Magma (whose routines for handling Coxeter groups proved very useful) as well as the $W$-graphs obtained from Coxeter are available at:

http://people.maths.ox.ac.uk/williamsong/torsion/

This site also contains a complete description of the sets $\sigma(W)$ and $f_W$ for all Weyl groups of ranks less than 6.

We will now describe examples of the sets $\sigma(W)$. The values of $f_W$ on values not in $\sigma(W)$ may be found on the above web page.

5.1. $A_n, n \leq 6$. Here $\sigma(W) = W$. Thus all intersection cohomology complexes with coefficients of any characteristic have the same characters as in characteristic zero and the decomposition theorem is always true. This is the statement of Theorem 1.3.
5.2. $A_7$. Let $W = A_7$ with Coxeter generators $s_i$ with $i \in \{1, \ldots, 7\}$ corresponding to the simple transpositions $(i, i+1)$. In $W$, 38 of the 40320 elements do not belong to $\sigma(W)$. The elements which do not lie in $\sigma(W)$ break up naturally into five groups, which we now describe.

Consider the following elements of $W$:

- \(w_1 = 46718235\)
- \(w_2 = 67823451\)
- \(w_3 = 84567123\)
- \(w_4 = 62845173\)
- \(w_5 = 84627351\)

The first group consists of

\[ K_1 = \{uw_1v \mid u, v \in \langle s_4 \rangle \} \]

The second group consists of

\[ K_2 = \{w_2, w_2s_5, w_2s_1, w_2s_1s_5, s_3w_2, s_7w_2, s_3s_7w_2\} \]

(note that $w_2$ is maximal in $K_2$). The third group $K_3$ is obtained from $K_2$ by inversion (or by applying the automorphism $s_i \mapsto s_{8-i}$). It contains $w_3$ as a maximal element. The fourth group consists of

\[ K_4 = \{uw_4v \mid u, v \in \langle s_2, s_6 \rangle \} \]

The fifth group consists of

\[ K_5 = \{uw_5v \mid u, v \in \langle s_4 \rangle \} \]

It would be interesting to investigate the intersection cohomology complexes corresponding to the minimal elements in $K_i$ for $i \neq 1$ directly.

Note that the set $K_1$ has already arisen in Kazhdan-Lusztig combinatorics; these are the so-called “hexagon permutations” of Billey and Warrington (see [BW01], the name refers to a characteristic hexagon shape appearing in their heap representation).

We say that a permutation $w \in S_n$ contains the pattern of a permutation $y \in S_m$ if there is a collection of indices $1 \leq i_1 < \cdots < i_m \leq n$ so that $w(i_1), \ldots, w(i_m)$ are in the same relative order as $y(1), \ldots, y(m)$. Otherwise, we say that $w$ avoids the pattern $y$. For a more general notion of pattern avoidance which works for general Coxeter groups, see [BP05]. A geometric interpretation of pattern avoidance is given in [BB03].

The significance of hexagon permutations is explained by the following result:
Theorem 5.1 ([BW01]). Let $w$ be a reduced word for an element $w \in S_n$. Then $\pi : \Sigma_w \to X_w$ is a small map if and only if the permutation $w$ avoids the pattern 321 and the four hexagon permutations.

The Bott-Samelson resolutions of the hexagon permutations are semi-small, but not small. In the appendix, Braden treats the hexagon permutations in detail and shows that $\text{ch}(E(w)) \neq H_w$ if $k$ is of characteristic 2.

5.3. $B_3$. Consider $W = B_3$ with generators

$$
\begin{array}{ccc}
  s & \rightarrow & t \\
  \downarrow & & \downarrow \\
  \rightarrow & & u \\
\end{array}
$$

This is the first group on which $f_W$ is not defined everywhere; it is not defined on the element $stsuts$. This means that there may be a characteristic in which the parity sheaf corresponding to $stsuts$ is not perverse. Of the 48 elements of $W$, 20 do not lie in $\sigma(W)$. They are:

- $utu, tut, utu, tuts, tsut, sutu, tsutu,$
- $(tsu)^2, tu, st, st, sut, sutu,$
- $tu, tsut, tsutut, tsututu, tsututut, stuts, stutsutu, stutsutut.$

5.4. $B_4$ and $B_5$. In $B_4$, which contains 384 elements, $f_W$ is not defined on 8 elements, and 180 elements do not lie in $\sigma(W)$. In $B_5$, which contains 3840 elements, $f_W$ is not defined on 26 elements and 1696 elements do not lie in $\sigma(W)$.

5.5. $D_4$. We label our generators $s, t, u$ and $v$ of $W$ as follows:

$$
\begin{array}{ccc}
  s & \rightarrow & t \\
  \downarrow & & \downarrow \\
  \rightarrow & & u \\
\end{array}
$$

Here, $f_W$ is defined on all of $W$ and 7 elements do not belong to $\sigma(W)$. Let $\tau$ be the automorphism of $W$ mapping $s \mapsto u \mapsto v \mapsto s$. The elements not in $\sigma(W)$ are $w_1 = tvtutsut$, $\tau(w_1)$ and $\tau^2(w_1)$ as well as $w_2 = suvtsu, tw_2, w_2t$ and $tw_2t$. In the appendix, Braden discusses the case of $w_2$ in more detail.

In fact, in this example one may extend the techniques of Section 4 to deduce combinatorially that one always has

$$
\text{ch } E(\tau^i(w_1)) = H_{\tau^i(w_1)} \text{ for } i \in \{0,1,2\}.
$$

Clearly it is enough to see this for $i = 0$. As $tvw_1$ is separated, $E(w_1)$ occurs as a direct summand of $\mathcal{E}(vw_1)$. But as $vw_1$ is separated and $\text{supp}_{KL}(H_{vw_1}) \cap f_W(w_1) = \{w_1\}$ we conclude that $\text{ch } E(w_1) = H_{w_1}$. 
5.6. $D_5$ and $D_6$. In $D_5$, $f_W$ is not defined on one element, and 176 of the 1920 elements in $W$ do not lie in $\sigma(W)$. In $D_6$, which contains 23040 elements, $f_W$ is not defined on 33 elements and 3259 elements of $W$ do not lie in $\sigma(W)$.

5.7. $F_4$. In $F_4$, $f_W$ is not defined on 23 elements and 621 of the 1152 do not lie in $\sigma(W)$.

5.8. $G_2$. In this case we have already calculated $\sigma(W)$ in Example 4.3. Here we obtain nothing new. If $W = \langle t, st \mid s^2 = t^2 = (st)^5 = 1 \rangle$ then $\sigma(W) = \{1, s, t, st, ts, ststs\}$. However these Schubert varieties are smooth, and so the $\text{ch}(\text{IC}(w, k)) = H_w$ in any characteristic if $w \in \sigma(W)$.

5.9. Further calculations. Let us briefly describe how the above algorithm may be taken further with some of the geometric input contained in the appendix.

For example, if $W$ is of type $A_7$ then, with notation as in 5.2, the calculations in the appendix allow one to conclude that $\text{ch}(\mathcal{E}(w_1)) = H_{w_1}$ if $\text{char} k \neq 2$. Then the above algorithm may be used to deduce that $\text{ch}(\mathcal{E}(w)) = H_w$ for all hexagon permutations $w$.

Similarly in $D_4$ if one knows, with notation as in 5.5, that $\text{ch}(\mathcal{E}(w_2)) = H_{w_2}$ then it follows that $\mathcal{E}(tw_2), \mathcal{E}(w_2t)$ and $\mathcal{E}(tw_2t)$ all have characters given by Kazhdan-Lusztig basis elements. By the calculations of the appendix this occurs if and only if $\text{char} k \neq 2$. It follows that Lusztig’s conjecture is true around the Steinberg weight in type $D_4$.

One can also turn these arguments around to deduce that one has $\text{ch}(\mathcal{E}(w)) \neq H_w$ (and hence $\text{ch}(\text{IC}(w)) \neq H_w$) for families of elements, once one knows that it occurs once. For example, if $W$ is of type $D_4$ and one knows that $\text{ch}(\mathcal{E}(w_2)) \neq H_{w_2}$ (as is the case in characteristic 2) one cannot have $\text{ch}(\mathcal{E}(w)) = H_w$ for $w \in \{tw_2, w_2t, tw_2t\}$. Indeed, assume that $\text{ch}(\mathcal{E}(tw_2)) = H_{tw_2}$. Then, as $stw_2$ is separated $\mathcal{E}(w_2)$ occurs as a direct summand of $\vartheta_s \mathcal{E}(tw_2)$. But $\text{supp}_{\text{KL}}(H_{tw_2}) \cap \{swv\} = \emptyset$ and so the calculations of 5.5 show that $\mathcal{E}(w) = H_w$. Similar arguments apply in the other cases.

Similarly one may show in type $A_7$ that if $\text{ch}(\mathcal{E}(w)) \neq H_w$ for one hexagon permutation then this is true for all of them. A geometric explanation for this is given in Remark 6 of the appendix.
Let $G$ be a semisimple complex group, and fix a choice of a Borel subgroup $B$, maximal torus $T$, and opposite Borel subgroup $B^-$. Let $\Phi^+ \subset \Phi \subset X(T)$ be the corresponding sets of roots and positive roots, chosen so that the weights of $\text{Ad} T$ acting on $\mathfrak{b}$ are $-\Phi^+ \cup \{0\}$. Let $W = N(T)/T$ denote the Weyl group and $S \subset W$ the set of simple reflections. Let $\ell : W \to \mathbb{N}$ be the length function, and $\leq$ the Bruhat-Chevalley order on $W$.

Consider the flag variety $X = G/B$; $G$ acts on $X$ by left multiplication. The set of $T$-fixed points is in bijection with $W$ by $w \mapsto \tilde{w}B/B$, where $\tilde{w}$ is any lift of $w$ to $G$. Using this bijection, we abuse notation and refer to points in $X$ and elements of $W$ by the same symbols.

The flag variety $X$ has two decompositions by Bruhat cells and dual Bruhat cells $X = \bigsqcup_{w \in W} X_w = \bigsqcup_{w \in W} S_w$, where $X_w = B \cdot w$ and $S_w = B^- \cdot w$. The notation $S_w$ is chosen to indicate that it should be thought of as a normal slice at $w$ to the stratification $\{X_w\}$.

It is also easy to describe the one-dimensional $T$-orbits in $X$. The closure of a one-dimensional orbit is a closed irreducible $T$-invariant curve; we will refer to such curves as “$T$-curves” for short. For any positive root $\mu \in \Phi^+$ (simple or not), let $s_\mu \in W$ denote the corresponding reflection.

**Proposition A.1.** For any $\mu \in \Phi^+$ and any $w \in W$, there is a unique $T$-curve $C$ which contains $w$ and $ws_\mu$, and all $T$-curves are of this form. The weight of the action of $T$ on the tangent space $T_w C$ is $w(\mu)$.

Since $s_{w(\mu)} = ws_\mu w^{-1}$, the above formula for the tangent weight can also be given, up to sign, by saying that if $C$ is the $T$-curve joining $w$ and $s_{\mu'}w$ for some $\mu' \in \Phi^+$, then the $T$-weight of $T_w C$ is $\pm \mu'$. The sign can then be specified by noting that the weight is in $\Phi^+$ if and only if $w \leq s_{\mu'}w$.

**The Bott-Samelson variety.** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ be a sequence of simple roots, not necessarily distinct, and let $w = (s_1, s_2, \ldots, s_l)$ be the corresponding sequence of simple reflections: $s_i = s_{\alpha_i}$. Put $w = s_1 s_2 \cdots s_l$. Then $w$ is a reduced word for $w$ if $\ell(w) = l$. 

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For each simple reflection $s_i$, let $P_i$ be the corresponding minimal parabolic containing $B$, whose Lie algebra is $b \oplus g_{\alpha_i}$. The Bott-Samelson variety $\Sigma = \Sigma_w$ associated to $w$ is defined to be the quotient

$$(P_1 \times P_2 \times \cdots \times P_l)/B^l,$$

where $B^l = B \times \cdots \times B$ acts on $P_1 \times P_2 \times \cdots \times P_l$ on the right by

$$(x_1, x_2, \ldots, x_l) \cdot (b_1, b_2, \ldots, b_l) = (x_1b_1b_1^{-1}x_2b_2, \ldots, b_{l-1}^{-1}x_l(b_l)).$$

Let $[x_1, \ldots, x_l]$ denote the point of $\Sigma$ corresponding to $(x_1, \ldots, x_l) \in P_1 \times \cdots \times P_l$.

If $1 \leq k \leq l$, define a map $\pi_k: \Sigma \to X$ by $\pi_k([x_1, \ldots, x_l]) = x_1 \cdots x_kB$. It is a $B$-equivariant map, where $B$ acts on $\Sigma$ via $b \cdot [x_1, \ldots, x_l] = [bx_1, x_2, \ldots, x_l]$. Put $\pi = \pi_l$.

We wish to describe the set $(\Sigma_w)^T$ of $T$-fixed points. Let $D_l = (\mathbb{Z}/2\mathbb{Z})^l = \{0, 1\}^l$. For each $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(l)) \in D_l$, define $p(\varepsilon) = p_w(\varepsilon) \in \Sigma_w$ and $w^\varepsilon \in W$ by

$$p(\varepsilon) = [s_1^{\varepsilon(1)}, \ldots, s_l^{\varepsilon(l)}], \quad w^\varepsilon = s_1^{\varepsilon(1)} \cdots s_l^{\varepsilon(l)}.$$

For any $\varepsilon \in D_l$ and any $1 \leq k \leq l$, define $\varepsilon[k] = (\varepsilon(1), \ldots, \varepsilon(k), 0, \ldots, 0) \in D_l$. As usual, we will refer to elements of $W$ and points of $X^T$ by the same symbols.

**Proposition A.2.** The map $\varepsilon \mapsto p(\varepsilon)$ is a bijection between $D_l$ and $(\Sigma_w)^T$. For any $\varepsilon \in D_l$, we have $\pi_k(p(\varepsilon)) = w^{\varepsilon[k]}$, and, in particular, $\pi(p(\varepsilon)) = w^\varepsilon$.

**Example A.3.** Let $G = SL(3, \mathbb{C})$. There are two simple roots, call them $\rho_1$ and $\rho_2$.

Let $w = (s_{\rho_1}, s_{\rho_2}, s_{\rho_1}, s_{\rho_2}, s_{\rho_1})$. Let $w_0 = s_{\rho_1}s_{\rho_2}s_{\rho_1}$ denote the longest element in $W$. Then there are five $T$-fixed points in $\pi^{-1}(w_0)$, namely $p(\varepsilon)$, where

$$\varepsilon \in \{11100, 01110, 00111, 10011, 11001\}.$$

Let us denote these five elements of $D_5$ by $\varepsilon_1, \ldots, \varepsilon_5$. There exists a $T$-curve containing $p(\varepsilon_i)$ and $p(\varepsilon_j)$ if and only if $i = j \pm 1 \mod 5$. The tangent weights of the curves joining $p(\varepsilon_i)$ to $p(\varepsilon_{i+1})$ for $i = 1, \ldots, 5$ are $-\rho_1, -\rho_2, \rho_1, \rho_2 + \rho_1$, and $\rho_2$.

The composition of $\pi_2: \Sigma_w \to X$ with the projection $X \to G/P_1 \cong \mathbb{CP}^2$ restricts to a birational map $\pi^{-1}(w_0) \to \mathbb{CP}^2$ which identifies $\pi^{-1}(w_0)$ with the blow-up of $\mathbb{CP}^2$ at two points. The exceptional fibers are the $T$-curves joining $p(\varepsilon_1)$ to $p(\varepsilon_2)$ and $p(\varepsilon_3)$ to $p(\varepsilon_4)$.
The one-dimensional $T$-orbits of $\Sigma_w$ are more difficult to classify than the fixed points. Unlike the flag variety $G/B$, Bott-Samelson varieties generally have infinitely many $T$-curves. We will describe a collection of $T$-curves which span the tangent space at each fixed point, but there are in general many other $T$-curves.

Denote the standard basis of $D_l$ by $\delta_i$, where $\delta_i(j) = \delta_{ij}$ is the Kronecker $\delta$-function. For any $\varepsilon \in D_l$ and $1 \leq i \leq k$, we have a $T$-curve joining $p(\varepsilon)$ and $p(\varepsilon + \delta_i)$, namely

$$\{[x'_1, \ldots, x'_l] \mid x'_j = x_j, j \neq i\},$$

where $p(\varepsilon) = [x_1, \ldots, x_l]$. This curve projects under $\pi$ to the $T$-curve in $G/B$ which joins $w^\varepsilon$ and $w^{\varepsilon + \delta_i}$, and so the tangent weight of this curve at $p(\varepsilon)$ is $w^\varepsilon[i-1](\alpha_i)$. Note that $T$-curves which project down to fixed points, such as the ones in Example A.3, are not of this type.

**Bialynicki-Birula cells.** Besides their definition as $B$-orbits, the Bruhat cells $\{X_w\}$ in the flag variety $X$ can also be described as Bialynicki-Birula cells for the action of a dominant cocharacter $\zeta : \mathbb{C}^* \to T$. For any $w \in W$, we have

$$X_w = B \cdot w = \{x \in X \mid \lim_{t \to \infty} \zeta(t) \cdot x = w\}.$$

The Bott-Samelson variety $\Sigma_w$ will not in general have finitely many $B$-orbits, but we can still consider its Bialynicki-Birula cells (for the same cocharacter $\rho$). Given $\varepsilon \in D_l$, we define

$$\Sigma_{w,\varepsilon} = \{x \in \Sigma_w \mid \lim_{t \to \infty} \zeta(t) \cdot x = p(\varepsilon)\}.$$

**Theorem A.4** ([Gau01, Här]). The dimension of the Bialynicki-Birula cells is given by

$$\dim_{\mathbb{C}} \Sigma_{w,\varepsilon} = \#\{1 \leq k \leq l \mid w^{\varepsilon[k]}(\mu_k) \in -\Phi^+\} = \#\{1 \leq k \leq l \mid \ell(w^{\varepsilon[k]}) > \ell(w^{\varepsilon[k]}s_k)\}.$$

The cell $\Sigma_{w,\varepsilon}$ fibers linearly over $X_w$, $w = w^\varepsilon$, with fiber of dimension

$$\dim_{\mathbb{C}} \Sigma_{w,\varepsilon} \cap \pi^{-1}(w) = \#\{1 \leq k \leq l \mid w^{\varepsilon[k-1]}(\mu_k) \in -\Phi^+\} = \#\{1 \leq k \leq l \mid \ell(w^{\varepsilon[k-1]}) > \ell(w^{\varepsilon[k-1]}s_k)\}.$$

These cells give a paving by affines of the fiber $\pi^{-1}(w)$. 
Obstruction to splitting the Bott-Samelson sheaf. Fix a word $w$ as above, and let $A_{w,k} = R\pi_*k\Sigma_w$, an object in $D^b(X)$. We want to understand when does the decomposition theorem hold for $A_{w,k}$, i.e. when does it split as a direct sum of (shifted) intersection cohomology sheaves.

Put $S_y = S_y \setminus \{ y \}$. We will be interested in the natural homomorphism

$$\phi_{y,w,k} : H^\bullet(\pi^{-1}(S_y), \pi^{-1}(S_y); k) \to H^\bullet(\pi^{-1}(S_y); k)$$

of cohomology groups. Because $\pi$ is proper, this is the same as the map obtained by applying hypercohomology to the adjunction morphism $(i_y)_!i_y^!(A_{w,k}|_{S_y}) \to A_{w,k}|_{S_y}$, where $i_y : \{ y \} \to S_y$ is the inclusion. On the other hand $(i_y)_!i_y^!(IC(X_w; k)|_{S_y}) \to IC(X_w; k)|_{S_y}$ is an isomorphism if $y = w$ and is zero if $y \neq w$, so if the decomposition theorem holds for the projection $\pi$ with $k$ coefficients, then for any $y \leq w$ the cokernel of $\phi_{y,w,k}$ will have no torsion.

Remark 4. When the resolution $\pi$ is semi-small, this map was previously considered by [dCM02, JMW09], in the guise of an intersection form on the top Borel-Moore homology of $\pi^{-1}(y)$ with $k$ coefficients. In the semi-small case $\phi_{y,w,k}$ is an isomorphism modulo torsion, and it is an isomorphism if and only if the decomposition theorem holds. When $\pi$ is not semi-small we do not know to what extent the lack of torsion in $\text{Coker} \phi_{y,w,k}$ implies the splitting of $A_{w,k}$.

The next two propositions allow us to calculate $\phi_{y,w,k}$ by computing the analogous map in $T$-equivariant cohomology with $\mathbb{Z}$ coefficients.

**Proposition A.5.** Both the source and target of the homomorphism $\phi_{y,w,k}$ are free $\mathbb{Z}$-modules. Consequently, we have $\phi_{y,w,k} = \phi_{y,w,\mathbb{Z}} \otimes \mathbb{Z} \otimes \mathbb{Z}$ for all rings $\mathbb{Z}$.

**Proof.** Because there is a cocharacter of $T$ which contracts $S_y$ onto $y$, we have an isomorphism between $H^\bullet(\pi^{-1}(S_y); k) = HH^\bullet(A_{w,k}|_{y})$. Using the properness of $\pi$, this is the same as the cohomology of the fiber $\pi^{-1}(y)$. Since this fiber has a paving by affines, its cohomology is torsion free. The torsion freeness of the domain of the map $\phi_{y,w,\mathbb{Z}}$ follows from Lefschetz duality and the following lemma. □

**Lemma A.6.** The space $\pi^{-1}(S_y)$ is a smooth manifold.

**Proof.** Let $N, N^{-}$ be the unipotent parts of $B, B^-$. Let $N_y = yN^{-}y^{-1}$. Then $U = N_y \cdot y$ is a $T$-invariant open neighborhood of $y$, and the map
\( N_y \to U, g \mapsto g \cdot y \) is an isomorphism. This map identifies \( N_y \cap N \) with the cell \( X_y \) and \( N_y \cap N^\perp \) with the slice \( S_y \). Since \( \pi: \Sigma_w \to X \) is \( N_y \)-equivariant and \( (N_y \cap N)(N_y \cap N^\perp) = N_y \), it follows that \( \pi^{-1}(U) \cong \pi^{-1}(S_y) \times X_y \). Since \( \pi^{-1}(U) \) is an open subvariety of a smooth variety, \( \pi^{-1}(S_y) \) must be smooth. \( \square \)

Remark 5. If the fiber \( Y := \pi^{-1}(y) \) is smooth, then we can replace the pair \( (\pi^{-1}(S_y), \pi^{-1}(S^0_y)) \) by \( (N, N^0) \), where \( N \) is the total space of the normal bundle \( \mathcal{N} \) to \( Y \) in \( \pi^{-1}(S_y) \), and \( N^0 \) is the complement of the zero section in \( N \). In this case, by the Thom isomorphism theorem we have \( H^\bullet(N, N^0; \mathbb{k}) \cong H^\bullet-2d(Y; \mathbb{k}), d = \text{rank} \mathcal{N}, \) and the map \( \phi_{y,w,\mathbb{k}} \) can be identified with multiplication by the Euler class \( e(\mathcal{N}) \).

The spaces involved are all \( T \)-invariant, so we can also consider the corresponding map in equivariant cohomology:

\[ \phi^T_{y,w,\mathbb{k}}: H^\bullet_T(\pi^{-1}(S_y), \pi^{-1}(S^0_y); \mathbb{k}) \to H^\bullet_T(\pi^{-1}(S_y); \mathbb{k}). \]

The same arguments as the non-equivariant case, together with standard arguments about equivariant formality and localization, prove the following.

**Proposition A.7.** The source and target of the equivariant homomorphism \( \phi^T_{y,w,\mathbb{k}} \) are free \( H^\bullet_T(pt; \mathbb{k}) \)-modules, and \( \phi_{y,w,\mathbb{k}} = \phi^T_{y,w,\mathbb{k}} \otimes H^\bullet_T(pt; \mathbb{k}) \mathbb{k} \).

The restriction map \( H^\bullet_T(Y) \to H^\bullet_T(Y^T) \) is an injection.

**The hexagon permutation.** We let \( G = GL(8, \mathbb{C}) \), so \( W \) is the symmetric group on the set \( \{1, \ldots, 8\} \). Taking the torus \( T \) to be the diagonal matrices, the lattice \( X(T) \) of characters is naturally identified with \( \mathbb{Z}^n \). Let \( \beta_i \) be the \( i \)th standard basis vector of \( \mathbb{Z}^n \). The roots of \( G \) are then the vectors \( \rho_{ij} = \beta_i - \beta_j \), for \( 1 \leq i, j \leq 8, i \neq j \).

Choose the Borel subgroup \( B \) to be the lower triangular matrices. With this choice, the positive roots are \( \rho_{ij} \) with \( i < j \), and the simple roots are \( \rho_i \overset{\text{def}}{=} \rho_{i,i+1}, i = 1, \ldots, 7 \). The simple reflection \( s_{\rho_i} \) corresponding to \( \rho_i \) is the transposition of \( i \) and \( i + 1 \).

We now fix \( w = w_1 \) where \( w_1 \) is the shortest of the “hexagon permutations” introduced in 5.2. The one-line notation of \( w \) is 46718235. A reduced word \( w \) for \( w \) can be given by the sequence of simple roots

\[ \alpha = (\alpha_1, \ldots, \alpha_{14}) = (\rho_3, \rho_2, \rho_1, \rho_5, \rho_4, \rho_3, \rho_2, \rho_6, \rho_5, \rho_4, \rho_3, \rho_7, \rho_6, \rho_5). \]

Reduced words for the other three hexagon permutations are obtained from this by appending \( \rho_4 \) at the beginning or the end, or both.

Let \( y = s_{\rho_2}s_{\rho_3}s_{\rho_2}s_{\rho_5}s_{\rho_6}s_{\rho_5} \); it is given in one-line notation as 14327658.
Theorem A.8. The source and target of the map
\[ \phi^8 = \phi^8_{y,w,z} : H^8(\pi^{-1}(S_w), \pi^{-1}(S'_w)) \to H^8(\pi^{-1}(S_w)) \]
(the degree eight part of \( \phi_{y,w,z} \)) are both isomorphic to \( \mathbb{Z} \); its cokernel is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \).

To prove this, we look more closely at the fiber \( Y = \pi^{-1}(y) \). We will compute the map \( \phi^8 \) by working with \( T \)-equivariant cohomology and localization. First we describe the \( T \)-fixed points in \( Y \).

Recall that we denote the standard basis of \( D_{14} \) by \( \{ \delta_i \} \). Define elements of \( D_{14} \) by
\[
\begin{align*}
\lambda_1 &= \delta_1 + \delta_2 + \delta_6 & \mu_1 &= \delta_4 + \delta_8 + \delta_9 \\
\lambda_2 &= \delta_2 + \delta_6 + \delta_7 & \mu_2 &= \delta_8 + \delta_9 + \delta_{13} \\
\lambda_3 &= \delta_6 + \delta_7 + \delta_{11} & \mu_3 &= \delta_9 + \delta_{13} + \delta_{14} \\
\lambda_4 &= \delta_7 + \delta_{11} + \delta_1 & \mu_4 &= \delta_{13} + \delta_{14} + \delta_4 \\
\lambda_5 &= \delta_{11} + \delta_1 + \delta_2 & \mu_5 &= \delta_{14} + \delta_4 + \delta_8 \\
\end{align*}
\]
and
\[ \nu = \delta_5 + \delta_{10}. \]

Proposition A.9. There are 29 \( T \)-fixed points in \( Y = \pi^{-1}(y) \). They are given by
\[
\begin{align*}
(\text{a}) & \quad p(\lambda_i + \mu_j), 1 \leq i, j \leq 5, \text{ and} \\
(\text{b}) & \quad p(\lambda_i + \mu_j + \nu), i, j \in \{4, 5\}.
\end{align*}
\]

\( Y \) has two irreducible components. The first, call it \( Y_1 \), is isomorphic to \( Z \times Z \), where \( Z \) is isomorphic to \( \mathbb{P}^2 \) blown up at two points. The \( T \)-fixed points in \( Y_1 \) are the ones given by (a) above.

The other component \( Y_2 \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \). Its \( T \)-fixed points are the four of type (b) and the four of type (a) where \( i, j \in \{4, 5\} \).

Proof. The enumeration of the points of \( Y^T \) is straightforward, using Proposition A.2.

Let \( w' \) be the (non-reduced) word corresponding to the sequence of simple roots
\[ (\rho_3, \rho_2, \rho_5, \rho_3, \rho_2, \rho_6, \rho_5, \rho_3, \rho_6, \rho_5). \]
It is a subword of \( w \); specifically, the roots \( \rho_1, \rho_4 \) and \( \rho_7 \) have been omitted. There is an embedding \( \Sigma_w \hookrightarrow \Sigma_w' \) which fills in the coordinates for the missing roots with the identity element of \( P_i \). The simple reflections \( \rho_2, \rho_3, \rho_5, \rho_6 \) that appear in \( w' \) generate the Weyl group of the group \( GL(3) \times GL(3) \), embedded into \( GL(8) \) as block diagonal matrices acting on the middle two factors in the decomposition \( \mathbb{C}^8 = \mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C} \). It follows that we have an isomorphism \( \Sigma_w' \cong \Sigma_1 \times \Sigma_2 \), where the factors are the Bott-Samelson varieties for
(\rho_3, \rho_2, \rho_3, \rho_2, \rho_3) \) and \( (\rho_5, \rho_6, \rho_5, \rho_6, \rho_5) \), respectively. Both \( \Sigma_1 \) and \( \Sigma_2 \) are isomorphic to the Bott-Samelson variety in Example A.3, and it is easy to see that \( Y \cap \Sigma_{w'} \) is a product of two copies of the fiber from that example.

To see that \( Y_1 := Y \cap \Sigma_{w'} \) is an irreducible component of \( Y \), note that a computation with Theorem A.4 shows that the paving by affines of \( Y \) given by intersecting with the Białynicki-Birula cells has only one cell of dimension four, namely \( \Sigma_{w, \lambda_1 + \mu_1} \cap Y \), and all other cells are of smaller dimension. So the closure of this cell must be a component of \( Y \), and it is the only four-dimensional component, so it is equal to \( Y_1 \).

To understand the other component, note that there is only one cell in \( Y \setminus Y_1 \) of dimension three, namely \( C := \Sigma_{w, \lambda_4 + \mu_4 + \nu} \cap Y \), and all other cells are of smaller dimension. If we let \( w'' \) be the smallest subword of \( w \) containing the nonzero entries of \( \lambda_4, \lambda_5, \mu_4, \mu_5, \) and \( \nu \), then the \( T \)-fixed points of \( Y_2 := Y \cap \Sigma_{w''} \) are the ones given in (b) in the statement of the theorem. It is easy to check that \( \pi_2, \pi_7, \) and \( \pi_{12} \) each map these fixed points onto a pair of fixed points in \( X \), and so they each map \( Y_2 \) onto a \( T \)-curve. The map \( \pi_2 \times \pi_7 \times \pi_{12} \) gives the required isomorphism of \( Y_2 \) with \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \). \( \square \)

The fiber \( Y \) is not smooth, but there is only one component of dimension four, so we can reduce the computation of \( \phi^8 \) to the smooth case: it is isomorphic to the restriction map \( H^8(N, N\setminus Y_1; k) \to H^8(N; k) = H^8(Y_1; k) \), where \( N \) is the total space of the normal bundle \( N \) to \( Y_1 \) in \( \pi^{-1}(S_w) \). As remarked earlier, \( \phi \) can be identified with multiplication by the Euler class \( e(N) \in H^8(Y_1; k) \), so the image of \( \phi^8 \) is spanned by \( e(N) \). We will compute this class by computing the equivariant Euler class \( e_T(N) \in H_T^8(Y_1; k) \) and then finding its image in ordinary cohomology.

To do this, we split the normal bundle \( N \) into line bundles. We have seen in the proof of Proposition A.9 that \( Y_1 = \Sigma_{w'} \cap \pi^{-1}(S_y) \); we claim that this intersection is transverse. Since \( \pi(\Sigma_{w'}) = X_y \) and \( \pi \) is \( B \)-equivariant, \( \pi \) is a submersion over the cell \( X_y \), and the required transversality follows from the transversality of \( S_y \) and \( X_y \) in \( X \). Let \( w_1, \ldots, w_4 \) be the subwords of \( w \) of length 11 which contain all of the simple reflections in \( w' \), together with one of the remaining simple reflections \( \rho_1, \rho_4, \rho_4, \rho_7 \). If \( \mathcal{L}_i \) is the restriction to \( Y_1 \) of the normal bundle to \( \Sigma_{w'} \) in \( \Sigma_{w, \nu} \), then we have \( N \cong \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \).

To compute the equivariant Euler classes \( e_T(\mathcal{L}_i) \), we compute their restrictions to the fixed point set \( Y_1^T = \{ p(\lambda_j + \mu_k) \mid 1 \leq j, k \leq 5 \} \). The restriction \( e_T(\mathcal{L}_i)|_{p(\lambda_j + \mu_k)} \in H_T^2(p(\lambda_j + \mu_k)) \cong X(T) \) is just
the $T$-weight of the tangent space to the unique $T$-curve containing $p(\lambda_j + \mu_k)$, contained in $\Sigma_w$, and not contained in $\Sigma_w'$. This curve is the curve joining $p(\lambda_j + \mu_k)$ and $p(\eta_1 + \lambda_j + \mu_k)$, where $\eta_1 = \delta_3$, $\eta_2 = \delta_5$, $\eta_3 = \delta_{10}$, $\eta_4 = \delta_{12}$. Using Propositions A.1 and A.2, we can compute that its $T$-weight is the sum of the entries under $\lambda_j$ and $\mu_k$ in the following tables:

|       | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
|-------|-------------|-------------|-------------|-------------|-------------|
| $e_T(L_1)$ | $\rho_1 + \rho_2 + \rho_3$ | $\rho_1 + \rho_2$ | $\rho_1$ | $\rho_1 + \rho_2 + \rho_3$ |           |
| $e_T(L_2)$ | $\rho_3 + \rho_4$ | $\rho_4$ | $\rho_4 + \rho_4$ |           |           |
| $e_T(L_3)$ | $\rho_2 + \rho_3 + \rho_4$ | $\rho_2 + \rho_3 + \rho_4$ | $\rho_3 + \rho_4$ |           |           |
| $e_T(L_4)$ | 0 | 0 | 0 | 0 | 0 |

|       | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\mu_5$ |
|-------|--------|--------|--------|--------|--------|
| $e_T(L_1)$ | 0 | 0 | 0 | 0 | 0 |
| $e_T(L_2)$ | $\rho_5$ | 0 | $\rho_5$ | $\rho_5$ | $\rho_5$ |
| $e_T(L_3)$ | $\rho_5 + \rho_6$ | $\rho_5 + \rho_6$ | $\rho_5$ | $\rho_5$ | $\rho_5$ |
| $e_T(L_4)$ | $\rho_5 + \rho_6 + \rho_7$ | $\rho_6 + \rho_7$ | $\rho_7$ | $\rho_5 + \rho_6 + \rho_7$ | |

The equivariant class $e_T(N) = e_T(L_1)e_T(L_2)e_T(L_3)e_T(L_4)$ induces the same class in $H^8(Y_1)$ as

$$(e_T(L_1) - (\rho_1 + \rho_2))(e_T(L_2) - (\rho_3 + \rho_4 + \rho_5))(e_T(L_3) - (\rho_3 + \rho_4 + \rho_5))(e_T(L_4) - (\rho_6 + \rho_7)),$$

where we abuse notation and write a weight $\rho_i \in X(T) = H^2_T(pt)$ instead of its pullback under the map $Y_1 \to pt$. After a little computation one sees that this class restricts to zero at every point of $(Y_1)^T$ except $p(\lambda_3 + \mu_1)$ and $p(\lambda_1 + \mu_3)$, where it has the same restriction as $e_T(T_{Y_1})$, the equivariant Euler class of the tangent bundle to $Y_1$. (To compute the localization of $e_T(T_{Y_1})$ to the fixed points, use the identification of the weights of $T$-curves in Example A.3. Note that the labeling of the fixed points $\varepsilon_1, \ldots, \varepsilon_5$ in that example corresponds to the labeling of the fixed points $\lambda_1, \ldots, \lambda_5$ and $\mu_1, \ldots, \mu_5$ of $\Sigma_1$ and $\Sigma_2$.) Then the Atiyah-Bott localization formula shows that $e(N)$ is twice a generator of $H^8(Y_1)$, as desired.

**Remark 6.** The other hexagon permutations can be shown to have 2-torsion by a similar computation; we give the main points. Let $\bar{w} = (s_4)^a w(s_4)^b$ and $\bar{y} = (s_4)^a y(s_4)^b$ for $a, b \in \{0, 1\}$. The fiber $\bar{Y} = \pi^{-1}(\bar{y})$ is still four-dimensional, but now the union of the components of maximal dimension is isomorphic to $Z_a \times Z_b$, where $Z_0 = Z$ and $Z_1 = Z \cup (\mathbb{P}^1 \times \mathbb{P}^1)$, the union taken so that $\{0\} \times \mathbb{P}^1$ is identified with a $T$-curve in $Z$ with trivial normal bundle. The excess intersection
formula then implies that matrix of $\phi^8$ is diagonal under the natural bases given by the components of $\tilde{Y}$ (in other words, the components are orthogonal under the intersection form). The normal bundle to the component $Z \times Z$ is the same as before, so we have $\det \phi^8 \in 2\mathbb{Z}$.

**Torsion example in D4.** Let $G = SO(8; \mathbb{C})$. We follow the notation of Section 5.5: the simple reflections in $W$ are $s, t, u, v$ where $s, u, v$ all commute with each other. Let $w$ be the word $(s, u, v, t, s, u, v)$, put $w = \pi(w)$, and let $y = suv$.

**Proposition A.10.** The $T$-fixed points in $Y := \pi^{-1}(y)$ are

$$\{ p(\varepsilon) \in D_7 \mid \varepsilon(4) = 0 \text{ and } \varepsilon(i) + \varepsilon(i+4) = 1 \text{ for } i = 1, \ldots, 3 \}.$$  

The fiber $Y$ is the transverse intersection of $\Sigma_{(s,u,v,s,u,v)} \subset \Sigma_w$ and $\pi^{-1}(S_y)$. It is $T$-equivariantly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, where the $T$-weights on the three factors are $\rho_s, \rho_u, \rho_v$, respectively.

Since $Y$ is smooth, we have $H^\bullet(\pi^{-1}(S_y), \pi^{-1}(S_y); \mathbb{k}) \cong H^{*-2}(Y)$, and the map $\phi = \phi_{y,w,Z}$ can be identified with multiplication by $e(\mathcal{L})$ on $H^\bullet(Y)$, where $\mathcal{L}$ is the normal bundle to $Y$ in $\pi^{-1}(S_y)$. As in the previous example we compute this by computing the localization of the equivariant class $e_T(\mathcal{L})$ to the fixed points $Y^T$. We have

$$e_T(\mathcal{L})|_{p(\varepsilon)} = \rho_t + \varepsilon(1)\rho_s + \varepsilon(2)\rho_u + \varepsilon(3)\rho_v,$$

so $e(\mathcal{L}) = \alpha + \beta + \gamma$, where $\alpha, \beta, \gamma \in H^2(Y)$ are the pullbacks of a generating class of $H^2(\mathbb{P}^1)$ by the three projection maps. Multiplication by this class from $H^2(Y)$ to $H^4(Y)$ is given by the matrix

$$\begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}$$

with respect to the natural monomial basis in $\alpha, \beta, \gamma$. This matrix has determinant $-2$, so $\text{Coker } \phi^4$ has 2-torsion.

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