COMPACTIFICATION FOR ESSENTIALLY FINITE-TYPE MAPS

SURESH NAYAK

Abstract. We show that any separated essentially finite-type map $f$ of noetherian schemes globally factors as $f = hi$ where $i$ is an injective localization map and $h$ a separated finite-type map. In particular, via Nagata’s compactification theorem, $h$ can be chosen to be proper. We apply these results to Grothendieck duality. We also obtain other factorization results and provide essentialized versions of many general results such as Zariski’s Main Theorem, Chow’s Lemma, and blow-up descriptions of birational maps.

1. Introduction

Let us denote by $S_f$ the category of separated finite-type morphisms of noetherian schemes. Nagata’s compactification theorem ([Ng], [C2], [Lu]) states that any $S_f$-map $f: X \to S$ factors as $X \xrightarrow{i} \overline{X} \xrightarrow{g} S$ where $i$ is an open immersion and $g$ is a proper map. One of its applications lies in Grothendieck duality where the only known approach to defining the twisted-inverse-image pseudofunctor $(−)^!$ over all of $S_f$ relies on the compactification theorem. This approach, pioneered by Deligne and Verdier ([D], [V]), and developed further by Lipman, Neeman, Sastry and others ([Lt], [AJL], [Ng], [Si]) is also general at the level of complexes for it automatically permits working with those that have quasi-coherent homology and not just coherent ones. Thus $(−)^!$ is realized as a $D_{qc}^+$-valued functor and its characterizing properties are all expressed in $D_{qc}^+$ terms. (Here $D_{qc}^+$ has the usual meaning, namely the derived category of complexes $E$ such that $H^n(E)$ is quasi-coherent for all $n$ and vanishes for $n \ll 0$.)

In this paper we generalize Nagata’s theorem to the category $S_e$ of separated essentially finite-type maps of noetherian schemes and in the process we also extend a few other general results from the finite-type setup to the essentially finite-type one. A particular consequence, which is one of the main motivations behind this paper, is that for ordinary noetherian schemes, $(−)^!$ can be defined over all of $S_e$.

By definition, a map in $S_e$, over sufficiently small affine open subsets of the base and source, corresponds to ring homomorphisms that are essentially of finite-type, see 2.1(a). Keeping aside considerations from duality, it seems...
reasonable to clarify at the outset, what the right notion of compactifiability in \( S_e \) should be. Specifically, let us address the question: how should the notions of proper maps and open immersions be generalized in \( S_e \)? The answers are perhaps not so surprising but are also not so obvious at first glance. We show below that the universally closed morphisms in \( S_e \) are automatically proper (Remark 4.2). Moreover, we define the class of localizing immersions—these are injective localizing morphisms, i.e., morphisms that are one-to-one and on sufficiently small open sets on the base and source correspond to localization of rings, (see [2.7, 2.8.2])—and we show that they give a natural generalization of open immersions. (In particular, finite-type + localizing immersion = open immersion, see [2.8.3]). This is convenient from the perspective of duality theory because the candidates for \( f^! \) in both the cases are already available, namely, the right adjoint \( f^\times \) to the derived direct image \( Rf_* \) if \( f \) is proper and the inverse image \( f^* \) if \( f \) is a localizing map, and the required compatibilities between \((-)^\times \) and \((-)^* \) are generally known.

Our “essential” compactification theorem states that any map in \( S_e \) factors as a localizing immersion followed by a proper map (Theorem 4.1). We show this by putting together Nagata’s theorem from the finite-type case and the key global factorization theorem namely, that any \( S_e \)-map factors as a localizing immersion followed by a finite-type map in \( S_e \) (Theorem 3.6). Such a factorization is available locally by definition, but it is not clear why one can find it globally. We should also mention here that any attempt to mimic the known proofs of Nagata’s compactification theorem seems difficult to carry out because working with localizing immersions as opposed to open immersions has certain limitations, see [2.9].

Our methods also apply to produce extensions of other general results from the finite-type case to the essentially finite-type case. These include Chow’s lemma, Zariski’s main theorem, elimination of indeterminacies of a rational map via blow-ups and some results concerning birational maps, see [4]. These extensions were inspired by Conrad’s exposition ([C2]) of Deligne’s notes on Nagata’s theorem. We also prove other global factorization results which show that an \( S_e \)-map with a given property factors globally as a localizing immersion followed by a finite-type map with the same property (see 5.11).

Finally, in our applications to duality, apart from the basic one, namely that \((-)^! \) is defined over all of \( S_e \) (Theorem 5.3), there is also a characterization of essentially perfect maps (\( S_e \)-maps of finite tor dimension) in terms of relative dualizing complexes, such as, \( f : X \to Y \) is essentially perfect iff the natural map is an isomorphism \( f^! O_Y \otimes Lf^*(-) \to f^! (-) \) on \( D^{qc}_c(Y) \), see Theorem 5.9.

Prior to this paper, the most general results on duality over \( S_e \) would come by applying the main results of [N7] or by assuming additional hypothesis that the underlying schemes admit dualizing complexes and working with \( D_c^+ \) instead of \( D^{qc}_c \), see [S2 9.2,9.3] for instance. Recent work of
Yekutieli and Zhang [YZ] on rigid dualizing complexes also develops duality over essentially finite-type maps. But their emphasis is different and the scope of their work is limited compared to ours.

2. Preliminaries on localizing immersions

Localizing immersions are defined in 2.7 below. This basic notion is used throughout the paper. In 2.8 we give its elementary properties which show that in many respects localizing immersions behave like open immersions. However, also see 2.9.

Let \( \phi : A \to B \) be a ring homomorphism. We will call \( \phi \) a localizing homomorphism if \( B \) is a localization of \( A \) with \( \phi \) as the canonical map.

Definition 2.1.

(a) A map \( f : X \to Y \) of noetherian schemes is said to be essentially of finite type if every point \( y \in Y \) has an affine open neighborhood \( V = \text{Spec}(A) \) such that \( f^{-1}V \) is covered by finitely many affine open \( U_i = \text{Spec}(B_i) \) for which the corresponding ring homomorphisms \( \phi_i : A \to B_i \) are essentially of finite type.

(b) If, moreover, in (a), each \( \phi_i \) is a localizing homomorphism, then we say that \( f \) is localizing.

2.2. The defining properties in (a) and (b) behave well with respect to composition and base change. Thus if \( f : X \to Y \) and \( g : Y \to Z \) are maps of noetherian schemes and if both \( f \) and \( g \) are essentially of finite type, then so is the composition \( gf \); if \( gf \) and \( g \) are essentially of finite type then so is \( f \); and if \( Y' \to Y \) is any scheme-map with \( Y' \) noetherian, then \( X' := Y' \times_Y X \) is noetherian and the natural projection \( f' : X' \to Y' \) is essentially of finite type. Substituting “localizing” in place of “essentially of finite-type” everywhere gives similar valid statements.

2.3. If \( f : X \to Y \) is essentially of finite type with \( Y \) affine, say \( Y = \text{Spec}(A) \), then \( X \) is covered by finitely many affine open \( V_j = \text{Spec}(B_j) \) such that the associated ring homomorphism \( \phi_j : A \to B_j \) is essentially of finite type. However it is not clear and may not be true in general that for every affine open \( V = \text{Spec}(B) \) in \( X \), the corresponding ring homomorphism \( \phi : A \to B \) is essentially of finite type. As before, substituting “localizing” in place of “essentially finite-type” everywhere gives analogous statements.

2.4. Let us consider some examples of localizing maps. For any scheme \( Y \) and any point \( y \in Y \), the natural map \( \text{Spec}(\mathcal{O}_{Y,y}) \to Y \) is localizing. In the finite-type case, open immersions are localizing maps but not conversely. Here are some general examples. We will start with a base scheme \( Y \).

- Let \( U, V \) be open subschemes of \( Y \). Let \( X \) be the gluing of \( U \) and \( V \) along a nonempty open subset of \( U \cap V \). Then the natural map \( X \to Y \) is localizing but not separated in general.
Let \( \{U_i\} \) be a finite collection of open subsets of \( Y \). Let \( X \) be the disjoint union \( \bigsqcup_i U_i \). Then the natural map \( X \to Y \) is localizing and separated but is not an open immersion in general.

In both these examples \( X \to Y \) is not one-to-one. It is easy to show that a finite-type localizing map that is also set-theoretically injective, is an open immersion, \(^{(2.5.3)}\). This motivates our choice of considering the notion of injective localizing morphisms as a natural extension of that of open immersions when going from the finite-type to the essentially finite-type case. This choice will be further vindicated by the properties and theorems that will follow.

2.5. Let \( X \) be a noetherian scheme. The subset \( G_X \) consisting of all the generic points of \( X \) is finite and discrete and equals the intersection of all the dense open subsets of \( X \). Via restriction from \( X \) we may equip \( G_X \) with the sheaf of rings \( \mathcal{O}_X|_{G_X} \). Clearly, \( G_X \cong \bigsqcup_{\gamma \in G_X} \text{Spec}(\mathcal{O}_{X,\gamma}) \). In particular, \( G_X \) forms an Artinian affine scheme. The natural induced scheme-map \( G_X \to X \) is localizing because it is evidently so on each component \( \text{Spec}(\mathcal{O}_{X,\gamma}) \).

For any noetherian scheme \( X \), the associated Artinian scheme \( G_X \) will be called as the generic subscheme of \( X \). If \( f: X \to Y \) is localizing, then the generic points of \( X \) map to those of \( Y \) and each point (component) of \( G_X \) maps isomorphically to its image in \( G_Y \).

**Lemma 2.6.** Let \( f: X \to Y \) be localizing. Then the following are equivalent.

(i) The map \( f \) is set-theoretically injective.

(ii) The map \( f \) is separated and sends \( G_X \) injectively inside \( G_Y \).

**Proof.** (i) \( \implies \) (ii). We use the valuative criterion to check that \( f \) is separated. Let \( V \) be a valuation ring with quotient field \( K \). Suppose there exist maps \( \alpha: \text{Spec}(K) \to X \) and \( \beta: \text{Spec}(V) \to Y \) such that \( \beta \) restricts to \( f \alpha \) on \( \text{Spec}(K) \). For \( i = 1, 2 \), let \( \alpha_i: \text{Spec}(V) \to X \) be maps lifting \( \alpha \) that agree with \( \beta \) on \( \text{Spec}(K) \). Since \( f \) is injective, the \( \alpha_i \)'s agree with each other set-theoretically. Let \( x \in X \) be the image of the closed point of \( V \) under \( \alpha \), so that \( y = f(x) \) is the image of the closed point under \( \alpha \). Since \( V \) is local, \( \alpha_i \)'s factors through the natural map \( \text{Spec}(\mathcal{O}_{X,x}) \to X \) while \( \alpha \) factors through \( \text{Spec}(\mathcal{O}_{Y,y}) \to Y \). Since \( f \) is localizing, hence the natural map \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \) is an isomorphism, whence \( \alpha_1 = \alpha_2 \). By the valuative criterion, \( f \) is separated.

(ii) \( \implies \) (i). Suppose \( y \in Y \) has two distinct points, say \( x_1, x_2 \), in its preimage. Since \( f \) is localizing, there exist open neighborhoods \( V_i \) of \( x_i \) respectively such that \( f|_{V_i} \) is injective. Since \( G_X \) injects into \( G_Y \), hence neither \( y \) nor the \( x_i \)'s are generic and moreover \( V := V_1 \cap V_2 \) is a nonempty open set whose image in \( Y \) contains all the generic points that specialize to \( y \). Since \( V_1 \cup V_2 \) is separated over \( Y \) hence the natural map \( \phi: V \to V_1 \times_Y V_2 \) is a closed immersion. By applying a base-change \( Y' = \text{Spec}(\mathcal{O}_{Y,y}) \to Y \), we obtain a closed immersion \( \phi': V' \to V'_1 \times_Y V'_2 \) which is also an open
immersion since \( V'_i \sim Y' \). Since \( Y' \) is connected, we obtain \( V' = V'_i \), thus forcing \( x_1 = x_2 \), a contradiction. \( \square \)

**Definition 2.7.** We define a map \( f: X \to Y \) of noetherian schemes to be a localizing immersion, if \( f \) satisfies the two equivalent conditions of Lemma 2.6 above.

2.8. Here are some basic properties of localizing immersions. In what follows, all schemes shall be tacitly assumed to be noetherian.

2.8.1. The property of being a localizing immersion behaves well under compositions and base-change. Thus the assertions in 2.2 hold with “localizing immersion” in place of “essentially of finite type”. The assertions about compositions are obvious. Regarding base-change, note that if \( f: X \to Y \) is a localizing immersion, then for any \( y \in Y \), the fiber-map \( X_y \to \text{Spec}(k(y)) \) is an isomorphism. This property on fibers carries over to \( f': X' \to Y' \), whence \( f' \) is injective.

2.8.2. If \( f: X \to Y \) is a localizing immersion, then \( f(X) \) is stable under generalization in \( Y \) and the natural map of ringed-spaces induces an isomorphism \( \psi: (X, O_X) \sim (f(X), O_Y|_{f(X)}) \). All the assertions can be checked locally on \( Y \), hence we assume that \( Y \) is affine, say \( \text{Spec}(A) \). Let us write \( X \) as a finite union of open subschemes \( U_i \), each of the form \( U_i = \text{Spec}(S_i^{-1}A) \). Since each \( f(U_i) \) is stable under generalization, so is the union \( f(X) \). The natural topological map \( |\psi|: X \to f(X) \) is a continuous bijection which is a homeomorphism when restricted to each \( U_i \) and hence \(|\psi|\) is a homeomorphism. Since \( \psi \) is an isomorphism over each \( U_i \), it so globally.

2.8.3. A localizing immersion that is of finite-type is an open immersion. For a ring \( A \) and a multiplicative subset \( S \subset A \), if \( S^{-1}A \) is finitely generated over \( A \), then there exists an element \( s \in S \) such that the natural map \( A[1/s] \to S^{-1}A \) is an isomorphism. Thus if \( f: X \to Y \) is of finite type and localizing, then over sufficiently small open subsets of \( X \), \( f \) is an open immersion. Hence \( f(X) \) is an open subset of \( Y \). If \( f \) is also injective, then by 2.8.2 it is an open immersion.

2.8.4. In the situation of 2.8.2 if \( Y' \) is a noetherian scheme over \( Y \), then the fiber-product \( X \times_Y Y' \) maps homeomorphically to the inverse image of \( f(X) \) in \( Y' \). This follows easily from 2.8.2.

2.8.5. A surjective (and hence bijective) localizing immersion is an isomorphism. More generally, if \( f: X \to Y \) is a localizing immersion and \( f(X) \) is a closed set, then \( f \) is an isomorphism of \( X \) onto a union of connected components of \( Y \). The first assertion follows immediately from 2.8.2. For the second one it suffices to show, keeping in mind \( Z = f(X) \), that any closed subset \( Z \subset Y \) that is stable under generalization equals a union of connected components of \( Y \). Since \( Z^c := Y \setminus Z \) is stable under specialization, it contains the closure of each of the generic points of \( Y \) lying in it. Every
point of $Z^c$ lies in one of these closures because $Z^c$, being open, is stable under generization. Thus $Z^c$ is also closed.

2.8.6. If $f : X \to Y$ is a localizing immersion, then any coherent ideal $\mathcal{I}$ in $\mathcal{O}_X$ extends to one in $\mathcal{O}_Y$, i.e., there exists a coherent ideal $\mathcal{J}$ in $\mathcal{O}_Y$ such that $\mathcal{J}\mathcal{O}_X = \mathcal{I}$. Indeed, in view of 2.8.2 let us first think of $X$ as a subset of $Y$. Then the required ideal $\mathcal{J}$ is the kernel of the composition $\phi$ of the natural maps $\mathcal{O}_Y \to f_*\mathcal{O}_X \to f_*(\mathcal{O}_X/\mathcal{I})$, because applying the exact restriction functor $f^{-1}$ to $\phi$ results in the composition of

$$\mathcal{O}_X = f^{-1}\mathcal{O}_Y \to f^{-1}f_*(\mathcal{O}_X/\mathcal{I}) = \mathcal{O}_X/\mathcal{I},$$

which also identifies with the canonical map $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$.

2.8.7. A localizing immersion is a localizing monomorphism and vice-versa. Suppose $f : X \to Y$ is a localizing immersion and for $i = 1,2$ there are scheme maps $g_i : Z \to X$ such that $fg_1 = fg_2$. Since $f$ is set-theoretically injective, $g_1$ and $g_2$ agree set-theoretically. In particular, for any open subset $V \subset X$, we have $g_1^{-1}V = g_2^{-1}V$. It suffices to check that as $V$ varies over sufficiently small open subsets of $X$, we have $g_1 = g_2$ on $g_i^{-1}V$. Therefore we may assume that $X,Y$ are affine and $f$ corresponds to a localizing homomorphism $A \to S^{-1}A$. The universal property of localization now shows that $g_1 = g_2$. Thus $f$ is a monomorphism.

Conversely, suppose $f : X \to Y$ is a localizing monomorphism and there are points $x_1, x_2 \in X$, $y \in Y$ such that $y = f(x_i)$. Since $f$ is localizing, the induced maps on residue fields $k(y) \to k(x_i)$ are isomorphisms. Let $g_i$ be the composition of the natural maps $\text{Spec}(k(y)) \to \text{Spec}(k(x_i)) \to X$. Then $fg_1 = fg_2$ forces $g_1 = g_2$. Thus $x_1 = x_2$.

2.8.8. A scheme map $f : X \to Y$ is a localizing immersion if and only if every $y \in f(X)$ admits an affine open neighborhood $V = \text{Spec}(A)$ in $Y$ such that $U = f^{-1}V$ is affine, say $U = \text{Spec}(B)$, and the corresponding ring homomorphism $A \to B$ is localizing. In particular, $f$ is quasi-affine. It suffices to prove the “only if” part as the remaining assertions follow easily.

Let $f : X \to Y$ be a localizing immersion. As the assertion is local on $Y$, we may assume that $Y$ is affine, say $Y = \text{Spec}(A)$, to begin with. Let $y$ be a point in $f(X)$ and $x$ its unique pre-image. Then $x$ has a neighborhood of the form $U = \text{Spec}(S^{-1}A)$. In view of 2.8.2 there exists an open subset $V \subset Y$ such that $f^{-1}V = U$. Upon shrinking $V$ to a basic affine open neighborhood $\text{Spec}(A[1/f])$ of $y$ and $U$ to $\text{Spec}(S^{-1}A[1/f])$, the assertion follows.

2.9. Some aspects of handling open immersions such as gluing of schemes do not carry over to localizing immersions. Here we give an example of pathological behavior that illustrates this.

Let us call a subset $Z$ of a scheme $Y$ a localized subset if there exists a localizing immersion $X \to Y$ whose set-theoretic image equals $Z$. By 2.8.2 a localizing immersion over $Y$ is determined up to a unique isomorphism by its set-theoretic
image, which is necessarily stable under generization. However, not every subset of \( Y \) stable under generization is a localized one as the following example shows.

Let \( k \) be a field. Let \( A = k[T_1, T_2] \). Set \( Y = \mathbb{A}^2 = \text{Spec}(A) \). Let \( C \) be an irreducible curve in \( \mathbb{A}^2 \) and \( p \) a closed point on \( C \). Let \( Z \subset \mathbb{A}^2 \) be the complement of the set \( C' \) consisting of all the closed points of \( C \) except \( p \). Clearly \( Z \) is stable under generization. But \( Z \) is not localized. Indeed, if it were so, then \( p \) admits an affine open neighborhood \( U = \text{Spec}(B) \) in \( Z \) where \( B \) is a localization of \( A \). Since \( U \) is the restriction of an open subset of \( \mathbb{A}^2 \) to \( Z \), therefore \( U^c = \mathbb{A}^2 \setminus U \) is the union of \( C' \) and a closed subset of \( \mathbb{A}^2 \) away from \( p \). Since \( U^c \) contains finitely many points of codimension one and all but one of the closed points in \( C \), it cannot be the union of hypersurfaces in \( \mathbb{A}^2 \). This contradicts the fact that \( B \) is a localization of \( A \).

Curiously enough, \( Z \) is the union of two localized subsets of \( \mathbb{A}^2 \) and this demonstrates the problems in trying to glue along localizing immersions and also of identifying localized subsets. Let \( X_1 \) be the open subset in \( \mathbb{A}^2 \) whose complement is \( C \) and let \( X_2 = \text{Spec}(\mathcal{O}_{\mathbb{A}^2, p}) \). Since \( X_1 \cup X_2 = Z \), it means the obvious naive attempt at glueing the \( X_i \)'s along their intersection cannot produce a localizing immersion into \( \mathbb{A}^2 \).

3. A Global Factorization

The final result of this section, Theorem 3.6, is one of the basic main results of this paper.

As usual, all schemes will be tacitly assumed to be noetherian.

3.1. Recall that for any scheme-map \( f : X \to Y \) the schematic image of \( f \) refers to the closed subscheme of \( Y \) defined by the kernel of the natural map \( \phi : \mathcal{O}_Y \to f_*\mathcal{O}_X \). We say that \( f \) has schematically dense image (or \( X \) has schematically dense image in \( Y \)) if \( \phi \) is injective, i.e, the schematic image is \( Y \). If \( X \) has schematically dense image in \( Y \) then it also has topologically dense image (i.e., \( \overline{f(X)} = Y \)) because the kernel of \( \phi \) is always supported on any open subset away from \( f(X) \). Thus, for any \( f : X \to Y \), the natural factorization \( X \xrightarrow{g} \overline{X} \xrightarrow{i} Y \), with \( \overline{X} \) as the schematic image of \( X \), is one where \( g \) is schematically dense and \( i \) is a closed immersion. The property of having schematically dense image is preserved under flat base change, and in particular, base change by a localizing morphism.

**Lemma 3.2.** Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be scheme-maps such that \( gf \) is a localizing immersion and \( g \) is separated. If \( f \) has schematically dense image, then it is a localizing immersion, while if \( f \) is proper, it is a closed immersion.

**Proof.** Consider the following diagram of natural maps where the square is cartesian and \( i \) satisfies \( gi = 1_X \) and \( hi = f \).

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y' & \xrightarrow{g'} & X \\
\downarrow{h} & & \downarrow{gf} & & \\
Y & \xrightarrow{g} & Z
\end{array}
\]
Here \( h \) is a localizing immersion and since \( g' \) is separated, \( i \) is a closed immersion. By 2.8.6 we can write \( f = i_1 h_1 \) where \( i_1 \) is a closed immersion and \( h_1 \) is a localizing immersion. If \( f \) has schematically dense image, then \( i_1 \) is an isomorphism while if \( f \) is proper then \( h_1 \) is a closed immersion by 2.8.5.

In view of 2.8.2 if a localizing immersion \( f \colon X \to Y \) has schematically dense image, then we also abbreviate and say that \( f \) is \textit{schematically dense} (or \( X \) is schematically dense in \( Y \)). Schematic denseness of \( f \) is equivalent to every open neighborhood \( U \) of \( f(X) \) being schematically dense in \( Y \). If \( X \) is schematically dense in \( Y \), then it is also so in every open neighborhood \( U \) in \( Y \).

Let \( f \colon X \to Y \) be a localizing immersion. If \( X \) is topologically dense in \( Y \), then the induced map of generic subschemes \( G_X \to G_Y \) is an isomorphism. In general, we can always find an open neighborhood \( U \) of \( f(X) \) such that \( X \) is schematically dense in \( U \). Indeed, if \( \mathcal{I} \) is the kernel of the natural map \( \mathcal{O}_Y \to f_* \mathcal{O}_X \), then taking \( U \) to be the complement of the support of \( \mathcal{I} \) works.

For any such \( U \), it holds that \( G_X \overset{\sim}{\to} G_U \subset G_Y \).

**Proposition 3.3.** Let

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & Y_1 \\
\downarrow & & \downarrow g \\
X & \xrightarrow{f_2} & Y_2
\end{array}
\]

be a commutative diagram of separated scheme-maps where each \( f_i \) is a localizing immersion and \( g \) is essentially of finite type. Then there exists an open subscheme \( U \subset Y_1 \) containing \( f_1(X) \) such that \( g|_U : U \to Y_2 \) is a localizing immersion. In particular, if \( g \) is of finite type, then \( g|_U \) is an open immersion.

**Proof.** The last statement follows from 2.8.3. Replacing \( Y_i \) by open subsets \( V_i \) containing \( f_i(X) \) and satisfying \( g(V_1) \subset V_2 \) does not affect the assertion in the proposition. Hence by shrinking \( Y_i \)'s if necessary, we may assume without loss of generality that \( X \) is schematically dense in both, \( Y_1 \) and \( Y_2 \). Thus we have \( G_{Y_1} \cong G_X \cong G_{Y_2} \). By Lemma 2.6 it suffices to find \( U \) containing \( f_1(X) \) such that \( g|_U \) is localizing. This problem can verified locally on \( Y_2, Y_1 \) and \( X \) and so we shall now assume that all three are affine.

Specifically, let \( X = \text{Spec}(B) \), \( Y_1 = \text{Spec}(A_1) \), assume that for each \( i \), there is an isomorphism \( B \cong S_i^{-1} A_i \) for a suitable multiplicative subset \( S_i \) in \( A_i \) and assume that \( A_1 \) is essentially of finite type over \( A_2 \).

Since \( f_1 \) is schematically dense, the natural map \( A_i \to B \) is injective. For convenience, let us identify each \( A_i \) with its image in \( B \). Thus there is a containment of rings \( A_2 \subset A'_1 \subset A_1 \subset B \) where \( A'_1 \) is finitely generated over \( A_2 \) and \( A_1 = T^{-1} A'_1 \) for some multiplicative subset \( T \subset A'_1 \). Let \( a_1, \ldots, a_n \) be in \( A'_1 \) and \( x_1, \ldots, x_n \) in \( S_2 \) such that \( A'_1 = A_2[a_1/x_1, \ldots, a_n/x_n] \). Then for
It holds that $A_1'[1/x] = A_2[1/x]$. Thus $A_1[1/x]$ is a localization of $A_2[1/x]$, so that $U = \text{Spec}(A_1[1/x])$ gives the desired open set.

**Proposition 3.4.** Consider the following commutative diagram of separated scheme-maps

\[
x \xrightarrow{f_1} Y_1 \xrightarrow{\pi_1} S
\]

where $\pi_i$ are of finite type and $f_i$ are localizing immersions. Then there exists a localizing immersion $h: X \to W$ for some scheme $W$ of finite type over $S$ such that $W \to Y_i$ satisfying $g_i h = f_i$.

**Proof.** Set $P := Y_1 \times_S Y_2$. Let $Z$ be the schematic image of $X$ in $P$ so that the obvious natural map from $X$ to $P$ factors as $X \xrightarrow{j} Z \xrightarrow{k} P$ where $k$ is a closed immersion and $j$ has schematically dense image. Set $g_i := p_i k: Z \to Y_i$ where $p_i$ is the canonical projection $P \to Y_i$. Since $f_i = g_i j$, is a localizing immersion and $j$ has schematically dense image, hence by Lemma 3.2, $j$ is a localizing immersion. By Proposition 3.3 there are open subschemes $U_i$ of $Z$ containing $j(X)$ such that $g_i|_{U_i}$ is an open immersion. Then $W := U_1 \cap U_2$ gives the desired scheme satisfying the proposition.

**Proposition 3.5.** Let $f: X \to S$ be a separated scheme map. Suppose there are two open subsets $U_1, U_2$ covering $X$ such that for each $i$, $f|_{U_i}$ factors as

\[
U_i \xrightarrow{k_i} Y_i \xrightarrow{p_i} S
\]

where $k_i$ is a localizing immersion and $p_i$ is separated and of finite type. Then $f$ admits a factorization $X \xrightarrow{k} Y \xrightarrow{p} S$ where $k$ is a localizing immersion and $p$ is separated and of finite type.

**Proof.** Set $U_{12} := U_1 \cap U_2$. By Proposition 3.3 there is scheme $Y_{12}$, a localizing immersion $h: U_{12} \to Y_{12}$ and open immersions $g_i: Y_{12} \to Y_i$ such that $g_i h = k_i|_{U_i}$. Since each $U_i$ is homeomorphic to its image in $Y_i$ (2.8.2), there are open subsets $W_i \subset Y_i$ such that $k_i^{-1} W_i = U_{12}$. Replacing $Y_{12}$ by $g_1^{-1} W_1 \cap g_2^{-1} W_2$, we may assume that $Y_{12} \times_{Y_i} U_i = U_{12}$ for each $i$. To summarize, we now have the following commutative diagram where the two parallelograms on the left are cartesian, $h, k_i$ are localizing immersions, $g_i$ are open immersions and $p_i$ are of finite type.

```
```

**Diagram:**

```
```
The above-mentioned properties of this diagram are not affected by any further shrinking of $Y_{12}$ to an open subset $Y'_{12}$ containing $h(U_{12})$ and replacing $g_1$ by $g_1|_{Y'_{12}}$. Likewise, shrinking $Y_1$ to an open subset $Y'_1$, $Y_{12}$ to $g_1^{-1}Y'_1$ and correspondingly modifying $p_1, g_1$ has no effect. A similar statement holds for shrinking $Y_2$.

Since $X$ is separated over $S$, the natural immersion $U_{12} \to U_1 \times_S U_2$ has a closed image. However $Y_{12} \to Y_1 \times_S Y_2$ need not be a closed immersion and so the gluing of $Y_i$ along $Y_{12}$ need not be separated over $S$. Our aim now is to shrink $Y_i$'s and $Y_{12}$ suitably so that separation is achieved.

Consider the following commutative diagram where all the squares are cartesian and where $ba$ in the bottom row is the obvious natural map, displayed as factoring through the schematic image $\overline{U}_{12}$ of $U_{12}$ in $U_1 \times_S Y_2$.

$$
\begin{array}{ccc}
U'_{12} & \xrightarrow{a'} & \overline{U}_{12} & \xrightarrow{b'} & U_1 \times_S U_2 & \xrightarrow{\text{projection}} & U_2 \\
\downarrow{k'_2} & \downarrow{\tau_2} & \downarrow{k_2} & \downarrow{\text{projection}} & \downarrow{k_2} \\
U_{12} & \xrightarrow{a} & \overline{U}_{12} & \xrightarrow{b} & U_1 \times_S Y_2 & \xrightarrow{\text{projection}} & Y_2
\end{array}
$$

Thus $b, b'$ are closed immersions, while the vertical arrows are all localizing immersions. Now note that $ba$ is an immersion, so that $a$ is an open immersion. Indeed, the natural immersion $U_{12} \subset U_1 \to U_1 \times_S Y_1$ also factors naturally as $U_{12} \xleftarrow{e} U_1 \times_S Y_1 \xrightarrow{\text{open}} U_1 \times_S Y_1$ whence $e$ is an immersion. As $U_1 \times_S Y_1$ is open inside $U_1 \times_S Y_2$ therefore $ba$ is an immersion. By definition of $\overline{U}_{12}$ it follows that $a$ is a schematically dense open immersion. Thus $a'$ too is a schematically dense open immersion.

In view of Proposition 2.8.4 and 2.8.5 we see that $k'_2$ is an isomorphism. Separatedness of $X/S$ implies that $U'_{12} \cong U_{12}$ is closed in $U_1 \times_S U_2$, whence $a'$ is a closed immersion. Since $a'$ is schematically dense, it is an isomorphism. In view of Proposition 2.8.4 we see that the natural projection of $b(\overline{U}_{12} \setminus U_{12}) \subset U_1 \times_S Y_2$ to $Y_2$ is a set $Z$ disjoint from $k_2(U_2)$. As $k_2(U_2)$ is stable under generalization 2.8.2, it is also disjoint from the closure $\overline{Z}$ of $Z$ in $Y_2$. Therefore, performing an open base-change by $Y'_2 := Y_2 \setminus \overline{Z} \to Y_2$ to the bottom row of the above diagram transforms $a$ into an isomorphism. Let us replace $Y_2$ by $Y'_2$, $Y_{12}$ by $g_2^{-1}Y'_2$, so that we may assume henceforth that $a$ is an isomorphism, i.e., $U_{12}$ is a closed subscheme of $U_1 \times_S Y_2$.

Now consider the following diagram where too the squares are cartesian and $dc$ is the obvious natural immersion displayed as factoring through the schematic closure $\overline{Y}_{12}$ of $Y_{12}$ in $Y_1 \times_S Y_2$.

$$
\begin{array}{ccc}
Y'_{12} & \xrightarrow{c'} & \overline{Y}_{12} & \xrightarrow{d'} & U_1 \times_S Y_2 & \xrightarrow{\text{projection}} & U_1 \\
\downarrow{k'_i} & \downarrow{\tau_i} & \downarrow{k_i} \\
Y_{12} & \xrightarrow{c} & \overline{Y}_{12} & \xrightarrow{d} & Y_1 \times_S Y_2 & \xrightarrow{\text{projection}} & Y_1
\end{array}
$$
Thus the vertical arrows are localizing immersions, \( c, c' \) are schematically dense open immersions, while \( d, d' \) are closed immersions. Since \( U_{12} \) is isomorphic to \( Y_{12} \times_{Y_1} U_1 \), we may assume that \( Y'_{12} = U_{12} \) and \( k_1' = h \). Moreover we may identify \( c', d' \) with \( a, b \) of the earlier diagram respectively. Since \( a \) has been arranged to be an isomorphism, therefore \( c' \) is an isomorphism.

Arguing as before, we see that the projection of \( d(Y_{12} \setminus Y_{12}) \subset Y_1 \times_S Y_2 \) to \( Y_1 \) is a set \( W \) disjoint from \( k_1(U_1) \) and hence the closure \( W \) is also disjoint from \( k_1(U_1) \). Thus replacing \( Y_1 \) with the open subset \( Y_1 \setminus \overline{W} \) and correspondingly modifying \( Y_{12} \) ensures that \( c \) is an isomorphism.

Since we have found suitable \( Y_i \) and \( Y_{12} \) for which \( Y_{12} \to Y_1 \times_S Y_2 \) is a closed immersion, the gluing of \( Y_1 \) and \( Y_2 \) along \( Y_{12} \) yields a scheme \( Y \) that is separated and of finite-type over \( S \). The natural induced map \( k: X \to Y \) is localizing and injective by our choice of \( Y_{12} \) whence it is a localizing immersion as required.

**Theorem 3.6.** Let \( f: X \to S \) be a separated essentially finite-type map of noetherian schemes. Then \( f \) factors as \( X \xrightarrow{k} Y \xrightarrow{p} S \) where \( k \) is a localizing immersion and \( p \) is separated and of finite type.

**Proof.** By definition, there exists a finite open cover \( \{U_i\} \) of \( X \) such that \( f|_{U_i} \) factors as \( U_i \xrightarrow{k_i} Y_i \xrightarrow{p_i} S \) where \( k_i \) is a localizing immersion and \( p_i \) is separated and of finite type. By Proposition 3.5 and induction on the number of elements in the open cover, the theorem follows. \( \square \)

### 4. General Applications

Below, we give various applications of the results from the previous section.

As a consequence of Theorem 3.6, we can now extend Nagata’s compactification theorem [Ng], [C2], [Lu] to essentially finite-type maps.

**Theorem 4.1.** Let \( f: X \to S \) be a separated essentially finite-type map of noetherian schemes. Then \( f \) factors as \( X \xrightarrow{k} Y \xrightarrow{p} S \) where \( k \) is a localizing immersion and \( p \) is proper.

**Remark 4.2.** In the context of the above theorem we note that replacing “finite type” with “essentially finite type” in the definition of properness does not define a new condition, i.e., a separated essentially-finite-type map \( f: X \to S \) that is also universally closed, is necessarily proper. Indeed, by Theorem 3.6 \( f \) factors as \( X \xrightarrow{i} Y \xrightarrow{k} S \) where \( i \) is a localizing immersion and \( k \) is separated and of finite type. Separatedness of \( k \) implies that \( i \) is universally closed and hence by 2.8.5 \( i \) is a closed immersion. Thus \( f \) is of finite type and hence proper.

The remaining general applications in this section will play no role in the results of § 5.

Next let us look at Zariski’s Main Theorem. We define a map \( f: X \to Y \) of noetherian schemes to be *essentially quasi-finite*, if it is essentially of finite
type and for any $y \in Y$, the fiber $X_y$ is algebraically finite, i.e., $X_y \cong \text{Spec}(A)$ where $A$ has finite vector-space dimension over the residue field $k(y)$ at $y$.

**Theorem 4.3.** Let $f : X \to Y$ be a separated essentially quasi-finite map of noetherian schemes. Then $f$ factors as $X \xrightarrow{i} Z \xrightarrow{h} Y$ where $i$ is a localizing immersion and $g$ is finite.

**Proof.** By Theorem 3.6 we can factor $f$ as $X \xrightarrow{j} Z \xrightarrow{g} Y$ where $j$ is a localizing immersion and $g$ is separated and of finite type. Pick a point $x \in X$ and let $z = j(x)$, $y = f(x)$. Let $G = Z_y$ and $F = X_y$ be the corresponding fibers over $y$. Since $G$ is of finite type over $k(y)$ and $O_{G,z} \cong O_{F,x}$ is a finite $k(y)$-module, therefore $z$ is an isolated point of $G$. Thus the set $U$ of points in $Z$ that are isolated in their fiber over $Y$ contains $j(X)$ and by [EGAIV, 13.1.4], $U$ is open. Replacing $Z$ by $U$ (and $g$ by $g|_U$) we may therefore assume that the fibers of $g$ are discrete and hence that $g$ is quasi-finite. By Zariski’s Main Theorem, $g$ factors as $Z \xrightarrow{l} Z' \xrightarrow{h} Y$ where $l$ is open and $h$ finite. Using $Z'$ in place of $Z$ and setting $i = lj$ we deduce the theorem. □

The rest of this section concerns some general results on rational maps and blow-ups. These generalize statements in the finite-type case, all of which can all be found in [C2]. Let us fix some terminology and recall some general facts about blow-ups.

4.4. Let $f : X \to Y$ be a localizing immersion and $\mathcal{I}$ a coherent ideal in $O_Y$. Let $\tilde{Y} = \text{Bl}_\mathcal{I}(Y)$ be the blow-up of $Y$ with respect to $\mathcal{I}$. We say that $\tilde{Y}$ is an $X$-admissible blow-up of $Y$ (or the blow-up map $\pi: \tilde{Y} \to Y$ is $X$-admissible) if the closed subscheme of $Y$ defined by $\mathcal{I}$ is disjoint from $f(X)$. This is equivalent to the condition that for any $y \in f(X)$, we have $\mathcal{I}_y = O_{Y,y}$ and is also equivalent to requiring that there exist an open neighborhood $U$ of $f(X)$ such that $\mathcal{I}_U = O_U$. In such a situation, $f$ lifts to the blow-up $\tilde{Y}$, i.e., there is a localizing immersion $\tilde{f}: X \to \tilde{Y}$ such that $\pi \tilde{f} = f$.

For a scheme map $f: X \to Y$ and blow-ups $\pi_1: \tilde{X} \to X$, $\pi_2: \tilde{Y} \to Y$, we say that a map $f': \tilde{X} \to \tilde{Y}$ lifts $f$ if the following diagram commutes.

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f'} & \tilde{Y} \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
X & \xrightarrow{f} & Y
\end{array}
$$

If we fix the blow-up of $Y$, say $\tilde{Y} = \text{Bl}_\mathcal{I}(Y)$, then any lift $f'$ of $f$ factors through the canonical map $\tilde{f}: \text{Bl}_{\mathcal{I} O_X}(X) \to \text{Bl}_\mathcal{I}(Y)$. If $f$ is a localizing immersion, then with $f' = \tilde{f}$, the above diagram is cartesian as $f$ is flat. In this case, if $\pi_2$ is $X$-admissible, then $\pi_1$ is an isomorphism, so our notation of $\tilde{f}$ remains consistent with that of the previous paragraph.

If $U$ is an open subscheme of a noetherian scheme $X$ then $U$ is schematically dense in $\tilde{X} = \text{Bl}_J(X)$ where $J$ is any coherent ideal defining the closed
set $X\setminus U$. This follows from the fact that the complement of $U$ in $\tilde{X}$ is defined by an invertible ideal, namely $\mathcal{I}$. If $U$ is dense (resp. schematically dense) in $X$, then it remains so in any $U$-admissible blow-up $\tilde{X} = \text{Bl}_\mathcal{I}(X)$ as can be seen by looking at the open immersions $U \to X \setminus V(\mathcal{I}) = \tilde{X} \setminus V(\mathcal{I}) \to \tilde{X}$, all of which are dense (resp. schematically dense).

The statements in the previous paragraph can be generalized to localizing immersions because if $f: Z \to X$ is a localizing immersion, then there is an open subset $U \subset X$ containing $f(Z)$ such that $Z$ is schematically dense in $U$.

Thus there is a blow-up $\tilde{X}$ such that $Z$ is schematically dense in $\tilde{X}$ namely, the blow-up of $X$ along $X \setminus U$. Also, if $Z$ is dense (or schematically dense) in $X$, then it remains so in any $Z$-admissible blow-up of $X$.

A composition of blow-ups is again a blow-up. In fact, for an open inclusion $U \subset X$, let $X'' \xrightarrow{q} X' \xrightarrow{p} X$ be $U$-admissible blow-ups (for $q$, note that $p^{-1}U \cong U$). Then $pq$ is also a blow-up, which is necessarily $U$-admissible (see [C2, Lemma 1.2]).

The following theorem generalizes the result that a “birational” map can be transformed into an open immersion after suitably blowing up the base and source, ([C2, Corollary 4.4]). The finite-type version there actually seems to follow immediately from [ibid, Lemma 2.7] itself and also appears in [R, Proposition 3, p. 30].

**Theorem 4.5.** Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be separated essentially finite-type maps of noetherian schemes such that $f$ and $gf$ are localizing immersions. Then there are $X$-admissible blow-ups $\tilde{Y}$, $\tilde{Z}$ and a lift $g': \tilde{Y} \to \tilde{Z}$ of $g$ such that $g'$ is a localizing immersion.

**Remark 4.6.** (The proper case) Assume that $X$ is dense in $Z$. If $g$ is proper, then so is $g'$ so that by [2.8.5] $g'$ is an isomorphism.

**Proof.** By Theorem 3.6, $g$ factors as $Y \xrightarrow{k} Y_1 \xrightarrow{p} Z$ where $k$ is a localizing immersion and $p$ is separated and of finite type. By Proposition 3.4, $p$ is a birational map, i.e., there exists an open subset $U \subset Y_1$ containing $kf(X)$ such that $p|U$ is an open immersion. If $U$ is dense in $Y_1$ and $Z$, then by [C2, Corollary 4.4] there is a $U$-admissible blow-up of $Z$ such that the natural induced map $\tilde{p}: \tilde{Y}_1 \to \tilde{Z}$ is an open immersion. In the general case, keeping in mind that a composition of $U$-admissible blow-ups is again a ($U$-admissible) blow-up, we first blow-up $Y_1$ and $Z$ along the largest coherent ideal defining $Y_1 \setminus U$ and $Z \setminus U$ respectively, so that $U$ becomes schematically dense in $Y_1, Z$ and then use [C2, Corollary 4.4]. Thus we have an open immersion $p': \tilde{Y}_1 \to \tilde{Z}$ that lifts $p$. The $U$-admissible blow-up $\tilde{Y}_1$ induces an $X$-admissible blow-up $\tilde{Y}$ of $Y$ and composing the natural map $\tilde{k}: \tilde{Y} \to \tilde{Y}_1$ with $p'$ gives us a lifting $g'$ of $g$ as desired. $\square$

Next we consider the gluing of two or more schemes (of essentially finite-type over a base scheme $S$) along open immersions coming from a common scheme. Such a gluing need not result in a scheme separated over the base
but after suitable \( U \)-admissible blow-ups it does. The case of gluing two schemes can be thought of as an abstract version of Chow’s lemma.

**Proposition 4.7.** Let \( S \) be a noetherian scheme and let \( f_i: U \to Y_i \) be a finite collection of localizing \( S \)-immersions of essentially finite type \( S \)-schemes. Then there exist \( U \)-admissible blow-ups \( \tilde{Y}_i \to Y_i \), a separated finite-type \( S \)-scheme \( Z \) and dense localizing \( S \)-immersions \( g_i: \tilde{Y}_i \to Z \) such that for all \( i, j \) we have \( g_i f_i = g_j f_j \).

\textbf{Proof.} It suffices to prove the result in the case where there are only two \( Y_i \)'s since the general case follows by induction in view of the fact that composition of \( U \)-admissible blow-ups is again a \( U \)-admissible blow-up. By Theorem 3.6, the natural map \( Y_i \to S \) factors as \( Y_i \xrightarrow{k_i} Z_i \xrightarrow{p_i} S \) where \( k_i \) is a localizing immersion and \( p_i \) is separated and of finite type. By Proposition 3.4, there exists a localizing immersion \( h: X \to W \) for some scheme \( W \) of finite type over \( S \) and there are open \( S \)-immersions \( e_i: W \to Z_i \) such that \( e_i h = g_i f_i \). By \([C2, Corollary 2.10]\) there exist \( W \)-admissible blow-ups \( \tilde{Z}_i \to Z_i \) and a separated finite-type \( S \)-scheme \( Z \) together with open immersions \( \tilde{Z}_i \to Z \) that agree on \( W \). By construction, these open immersions can also be arranged to be dense. The blow-ups on \( Z_i \) induce corresponding ones on \( Y_i \) so setting \( g_i \) to be the composition of the natural maps \( \tilde{Y}_i \to \tilde{Z}_i \to Z \) proves the theorem. \( \Box \)

Let us generalize Chow’s Lemma. We call a map \( Z \to S \) of noetherian schemes \textit{essentially quasi-projective} if it factors as \( Z \xrightarrow{j} Y \xrightarrow{p} S \) where \( j \) is a localizing immersion and \( p \) is a projective morphism. Here we use the definition of quasi-projectivity and projectivity as in \([EGAII, \S 5.3, 5.5]\)

**Theorem 4.8.** Let \( \pi: Y \to S \) be a separated essentially-finite-type morphism of noetherian schemes. Let \( f: U \to Y \) be a dense localizing immersion such that the natural map \( \pi \circ f: U \to S \) is essentially quasi-projective. Then there exists an \( U \)-admissible blow-up \( \tilde{Y} \to Y \) such that the natural map \( \tilde{Y} \to S \) is essentially quasi-projective.

\textbf{Proof.} Let \( U \xrightarrow{f'} U^* \xrightarrow{\pi'} S \) be a factorization of \( \pi \circ f \) with \( f' \) a localizing immersion and \( \pi' \) a projective morphism. By Proposition 4.7 with \( Y_1 = Y \), \( Y_2 = U^* \) and \( f_1 = f \), \( f_2 = f' \), we find that for suitable \( U \)-admissible blow-ups of \( Y \) and \( U^* \), there are dense localizing immersions \( g_1: \tilde{Y} \to Z \), \( g_2: \tilde{U} \to Z \) that agree on \( U \) where \( Z \) is separated and of finite-type over \( S \). Since \( g_2 \) is dense and proper, by \([2.8.5]\) it is an isomorphism. Thus \( Z \) is projective over \( S \) and \( \tilde{Y} \) essentially quasi-projective. \( \Box \)

The finite-type version of Theorem 4.8 is closely related to a general result (\([C2, Theorem 2.4]\)) about eliminating indeterminacy of rational maps via blow-ups. This result is one of the important steps in the proof of Nagata’s compactification theorem. Here is the analogous result for essentially finite type maps.
Theorem 4.9. Let $S$ be a noetherian scheme and let $X,Y$ be noetherian schemes separated and of essentially finite type over $S$. Suppose there are $S$-morphisms $X \overset{j}{\rightarrow} Z \overset{f}{\rightarrow} Y$ with $j$ a localizing immersion. Then there exists a $Z$-admissible blow-up $\tilde{X}$, a localizing immersion $j': Z' \to \tilde{X}$ extending the natural inclusion $\tilde{j}: Z \to \tilde{X}$ and an $S$-morphism $f': Z' \to Y$ extending $f$, such that the natural map $Z' \to \tilde{X} \times_S Y$ is a closed immersion.

Remark 4.10. Suppose $Y \to S$ is proper and $Z$ is dense in $X$ so that $Z, Z'$ are dense in $\tilde{X}$. Properness of the composition $Z' \to \tilde{X} \times_S Y \to X$ implies that $Z' = \tilde{X}$. Thus the domain of the rational map $f$ extends to the whole of $\tilde{X}$ in this case.

One way of proving Theorem 4.9 would be to reduce it to the original result from the finite-type case via our main factorization theorem (3.6). Though this can be carried out, we use a somewhat different approach below and deduce it using Theorem 4.5. This illustrates the close relation between these results.

Proof. For any choice of $Z', \tilde{X}$, specifying an $S$-map $Z' \to Y$ is equivalent to specifying a map $Z' \to X \times_S Y$. Thus replacing $S$ by $X$ and $Y$ by $X \times_S Y$ does not change the problem. Hence we shall from now on assume that $S = X$. In particular, there is now a natural map $g: Y \to X$.

First we find a $Z$-admissible blow-up $\tilde{X} \to X$ such that $Z$ is schematically dense in $\tilde{X}$. Next note that replacing $X$ by $\tilde{X}$ and $Y$ by $\tilde{X} \times_X Y$ does not affect the problem. Hence we can and will now assume that $Z$ is schematically dense in $X$. This assumption is not affected by any further $Z$-admissible blow-up of $X$. Therefore for any choice of a further blow-up $\tilde{X}$ and any possible $f'$ extending $f$, the schematic image of $f'$ equals that of $f$. Hence, by replacing $Y$ with the schematic image of $f$, we may assume without loss of generality that $f$ has schematically dense image.

Since $j = gf$ is a localizing immersion, by Lemma 3.2, $f$ is a localizing immersion. Hence by Theorem 4.5, there exists a $Z$-admissible blow-up of $X$ such that the natural induced map $\tilde{g}: \tilde{Y} \to \tilde{X}$ is a localizing immersion. We choose $Z' = \tilde{Y}$. It remains to verify that the natural map $\tilde{Y} \overset{h}{\rightarrow} \tilde{X} \times_X Y$ is a closed immersion. Let $p$ denote the canonical projection $\tilde{X} \times_X Y \to \tilde{X}$. Since $h$ is proper and $ph = \tilde{g}$ is a localizing immersion, by Lemma 3.2, $h$ is a closed immersion. □

5. Duality for essentially-finite-type maps

The main applications to duality given here, Theorems 5.3 and 5.9, follow quickly from the existence of essential compactifications (Theorem 4.1) since the proofs are already there in literature. Towards the end we show that many properties of essentially finite-type maps can be approximated by finite-type ones by means of a global factorization result, see Proposition 5.11.
5.1. Let us recall some basic notation. For a scheme $X$, we use $\mathbf{D}(X)$ to denote the derived category of the category of $\mathcal{O}_X$-modules, $\mathbf{D}_{\mathit{QC}}(X)$ (resp. $\mathbf{D}_c(X)$) to denote the full subcategory whose objects are the complexes having quasi-coherent (resp. coherent) homology and $\mathbf{D}^+_\ast(X)$ (resp. $\mathbf{D}_+\ast(X)$), resp. $\mathbf{D}^\ast(X)$) to denote the full subcategory of complexes $\mathcal{F}$ such that $H^n\mathcal{F} = 0$ for $n \gg 0$ (resp. $n \ll 0$, resp. $|n| \gg 0$).

Recall that $\mathbf{S}_e$ is the category of essentially finite-type morphisms of noetherian schemes. An essentially étale map of noetherian schemes is a separated formally étale map that is essentially of finite type. Essentially étale maps form a larger subcategory of $\mathbf{S}_e$ than the one of localizing maps.

5.2. One of the main results in [Ny, Theorem 7.1.6] was that for ordinary (noetherian) schemes, a pseudofunctorial construction of $(-)^!$ satisfying a flat-base-change isomorphism is valid over the category $C$ of composites of proper maps and essentially étale maps. In particular, the theorem shows that the existence of essential compactifications is not needed for pseudofunctorially defining $(-)^!$ over $C \subset \mathbf{S}_e$, but it does not prove that $(-)^!$ is defined over all of $\mathbf{S}_e$. Theorem 4.1 completes the picture because now we know that $C = \mathbf{S}_e$. For convenience, let us briefly recall the basic defining properties of $(-)^!$.

$(-)^!$ is a contravariant $\mathbf{D}^\ast_{\mathit{QC}}$-valued pseudofunctor on $C$ such that:

- on proper maps, $(-)^!$ is pseudofunctorially isomorphic to the right adjoint of the right-derived direct image pseudofunctor $\mathbf{R}f_\ast$;
- on essentially étale maps, $(-)^!$ equals the inverse image pseudofunctor $(-)^\ast$;
- for any fibered square $s$ of morphisms of noetherian schemes as follows

$$
\begin{array}{ccc}
U & \xrightarrow{j} & X \\
\downarrow{g} & & \downarrow{f} \\
V & \xrightarrow{i} & Y
\end{array}
$$

where $f$ is proper (and hence is in $\mathbf{S}_e$) and $i$ is flat, there is a flat-base-change isomorphism $\beta_s: j^! f^! \sim \sim g^! i^!$, (see [Li3, 4.4.3]). Moreover, if $i, j$ are essentially étale, then $\beta_s$ agrees with the natural isomorphisms

$$
j^! f^! \sim j^! f^! \sim (f j)^! = (i g)^! \sim g^! i^! \sim g^! i^!.
$$

These properties together with the existence of essential compactifications extend the abstract theory of $(-)^!$ to all of $\mathbf{S}_e$:

**Theorem 5.3.** The Grothendieck duality pseudofunctor $(-)^!$ exists on the entire category $\mathbf{S}_e$ of separated essentially-finite-type maps of noetherian schemes and satisfies compatibility with flat base change.
5.4. For smooth or finite maps, \((-)^!\) has a concrete description. We only describe the functorial aspects and not the pseudofunctorial one over these subcategories.

Let \(f : X \to Y\) be a closed immersion. Since \(f\) is proper, \(f^! \cong f^\times\) the right adjoint of \(Rf_*\), so one can use \(f^!(\cdot) \cong f^{-1}\mathbf{R}\text{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \cdot)\), (see for instance, [H] p. 172, 6.8)). In particular, if \(f\) is a regular immersion induced locally by regular sequences of length \(d\), then we also have the fundamental local isomorphism \(f^!(\cdot) \cong \omega[-d] \otimes \mathbf{L}f^*(-)\) ([ibid. p. 180, 7.3]), where \(\omega\) is the \(d\)-th exterior power of the normal module for \(f\). (Also see [L] p. 111) for a more intrinsic description of this isomorphism.)

Recall that a scheme-map is called \textit{essentially smooth} if it is in \(\mathbf{S}_e\) and it is formally smooth. Any such map, say \(f\), has a locally constant relative dimension that corresponds to the rank of the module of relative differentials \(\Omega_f\), ([EGAIV] 16.10.2)). If \(f : X \to Y\) is essentially smooth of relative dimension \(d\), then Verdier’s argument in [V] Theorem 3 shows that there is a natural isomorphism \(f^!(\cdot) \cong f^*(-) \otimes_X \Omega^d[d]\) and in particular that \(f^!\mathcal{O}_Y \cong \Omega^d[d]\). (Verdier’s argument can be carried out quite formally and the main input is the fundamental local isomorphism above applied to the diagonal map \(X \to X \times_Y X\) which is locally given by a regular sequence of length \(d\).)

It follows that for any \(\mathbf{S}_e\)-map \(f : X \to Y\), we have \(f^!\mathcal{D}_c^+(Y) \subset \mathcal{D}_c^+(X)\). Indeed, since \(f\) factors locally on \(X\) as a closed immersion into a smooth map, and in these cases the above formulas for \(f^!\) preserve coherence of homology, the same holds for the general case.

5.5. Let \(X\) be a noetherian scheme. For any \(x \in X\) the natural map \(\lambda_x : \text{Spec}(\mathcal{O}_{X,x}) \to X\) is a localizing immersion. Via the exact global-sections functor \(\Gamma\) on \(\text{Spec}(\mathcal{O}_{X,x})\), and using \(\Gamma\) to denote \(R\Gamma\), for any \(\mathcal{F} \in \mathcal{D}^-_{\text{qct}}(X)\) we obtain natural isomorphisms

\[
\Gamma\lambda_x^!\mathcal{F} = \Gamma\lambda_x^*\mathcal{F} \cong \mathcal{F}_x.
\]

More generally, if \(f : X \to Y\) is a map in \(\mathbf{S}_e\) and \(f_x : \text{Spec}(\mathcal{O}_{X,x}) \to Y\) is the canonical map, then for any \(\mathcal{F} \in \mathcal{D}^+_{\text{qct}}(Y)\) we obtain natural isomorphisms

\[
\Gamma f_x^!\mathcal{F} = \Gamma(f\lambda_x)^!\mathcal{F} \cong \Gamma\lambda_x^!f_x^!\mathcal{F} \cong (f^!\mathcal{F})_x.
\]

Our next application concerns perfect complexes and perfect maps in \(\mathbf{S}_e\). Let us recall some basic facts about them. We follow mostly the treatment given in Lipman’s notes [L] 4.9).

5.6. Let \(X\) be a noetherian scheme. A complex \(\mathcal{F} \in \mathcal{D}(X)\) is called \textit{perfect} if it is locally \(\mathcal{D}\)-isomorphic to a strictly perfect complex, i.e., a bounded complex of finite-rank free \(\mathcal{O}_X\)-modules. In particular, \(\mathcal{F} \in \mathcal{D}^b_c(X)\). Let \(f : X \to Y\) be a scheme-map in \(\mathbf{S}_e\). A complex \(\mathcal{F} \in \mathcal{D}(X)\) is called \(f\)-\textit{perfect} if it has coherent homology and if it has finite flat \(f\)-amplitude, i.e., there exist integers \(m \leq n\) such that for any \(x \in X\), \(\mathcal{F}_x\) is isomorphic in \(\mathcal{D}(\mathcal{O}_{Y,f(x)})\) to a complex of flat \(\mathcal{O}_{Y,f(x)}\)-modules that lives between degrees \(m\) and \(n\). If
$X = Y$ and $f = 1_X$, then $\mathcal{F}$ is $f$-perfect iff it is perfect. We call $f$ essentially perfect if $\mathcal{O}_X$ is $f$-perfect.

5.7. We shall soon give a description of $f^!$ for an essentially perfect map $f$ in terms of the relative dualizing complex $f^!\mathcal{O}_Y$, namely a natural isomorphism $f^!(-) \cong f^!\mathcal{O}_Y \otimes_X Lf^*(-)$. Following Lipman’s notes, we first define the map underlying this isomorphism for any $f$ in $\text{Se}$.

Let $X \xrightarrow{i} \overline{X} \xrightarrow{h} Y$ be a factorization of a scheme-map $f: X \to Y$ in $\text{Se}$, with $i$ a localizing immersion and $h$ proper. Since $h$ is proper, $h^!$ is right adjoint to $Rh^*$. For any $\mathcal{F} \in D^+_{\text{qc}}(Y)$, there results a natural map

$$\chi^h_f: h^!\mathcal{O}_Y \otimes_{\overline{X}} Lh^*\mathcal{F} \to h^!\mathcal{F},$$

namely, $\chi^h_f$ is the map adjoint to the composition

$$Rh^*(h^!\mathcal{O}_Y \otimes_{\overline{X}} Lh^*\mathcal{F}) \xrightarrow{p} Rh_*h^!\mathcal{O}_Y \otimes_{\mathcal{F}} \mathcal{F} \xrightarrow{\tau} \mathcal{F},$$

where $p$ results from the projection isomorphism [Li3, Prop. 3.9.4] and $\tau$ is the trace map corresponding to the right-adjointness of $h^!$ to $Rh_*$. Thus for any factorization $f = hi$ as above and for any $\mathcal{F} \in D^+_{\text{qc}}(Y)$, we obtain a natural map

$$\chi^{[i,h]}_f: f^!\mathcal{O}_Y \otimes_X Lf^*\mathcal{F} \to f^!\mathcal{F}$$

via the composition of the following natural maps (where $i^* = Li^*$)

$$f^!\mathcal{O}_Y \otimes_X Lf^*\mathcal{F} \cong i^*h^!\mathcal{O}_Y \otimes_X i^*Lh^*\mathcal{F} \cong i^*(h^!\mathcal{O}_Y \otimes_{\overline{X}} Lh^*\mathcal{F}) \xrightarrow{\chi^h_f} i^*h^!\mathcal{F}.$$

**Proposition 5.8.** With notation as above, if $f = hi_1$ is another factorization with $i_1$ a localizing immersion and $h_1$ proper, then for any $\mathcal{F} \in D^+_{\text{qc}}(Y)$, it holds that $\chi^{[i,h]}_f = \chi^{[i_1,h_1]}_f$.

Henceforth, for $f, \mathcal{F}$ as above we shall denote $\chi^{[i,h]}_f$ by $\chi^f_f$.

**Proof.** One follows the same steps as in Lipman’s notes [Li3, 4.9.2.2]. For the two factorizations of $f$ as above, first one must find a “dominating” factorization. This can be done using the proof of Proposition 3.4 or one can use this Proposition to reduce to the finite-type case and then proceed as in loc. cit. The rest of the proof goes through using localizing immersions in place of open ones and using the properties in 2.8. □

**Theorem 5.9.** Let $f: X \to Y$ be a map in $\text{Se}$. Then the following conditions are equivalent.

(i) The map $f$ is essentially perfect, i.e., $\mathcal{O}_X$ is $f$-perfect.

(ii) For any open $U \subset X$ and any factorization of $f|_U$ as $U \xrightarrow{j} Z \xrightarrow{g} Y$ where $j$ is a closed immersion and $g$ essentially smooth, $j_*\mathcal{O}_U$ is a perfect $\mathcal{O}_Z$-complex.

(iii) The complex $f^!\mathcal{O}_Y$ is $f$-perfect.
(iv) \( f^! O_Y \in D^b_c(X) \), and for every \( F \in D^+_{qc}(Y) \), the map

\[
\chi^{f^! O_Y, F}_X : f^! O_Y \otimes_X Lf^* F \to f^! F
\]

is an isomorphism.

**Proof.** Here again, one argues as in Lipman’s notes [Li3, Theorem 4.9.4]. Locally on \( X \), the factorization of \( f \) as in (ii) always exists and so the proof of loc. cit. goes through without any difficulties.

\[\square\]

5.10. Recall that a map \( f : X \to Y \) is called Cohen-Macaulay (resp. Gorenstein) if \( f \) is flat and the local rings of the fibers of \( f \) are Cohen-Macaulay (resp. Gorenstein). Before proceeding further let us recall a well-known fact usually not stated in the generality that we need below: Let \( f : X \to Y \) be a flat map in \( S \). Then \( f \) is Cohen-Macaulay iff for any connected open subset \( U \subset X \), \( f^! O_Y \) has exactly one non-vanishing homology and this homology is \( f \)-flat. In particular, \( f \) is Gorenstein iff \( f^! O_Y \) is invertible, i.e., on every connected component of \( X \), the unique nonvanishing homology of \( f^! O_Y \) is an invertible \( O_X \)-module.

To prove this, first we recall that if \( f \) is Cohen-Macaulay, then on over connected components of \( X \), \( f \) has a constant relative dimension. Indeed, for any fiber, each of its connected components is equidimensional since it is Cohen-Macaulay. Hence by [EGAIV 15.4.3], for any integer \( r \), the set of points \( x \in X \) such that the fiber through \( x \) has dimension \( r \) forms an open set. Therefore, over any connected open subset of \( X \), all the fibers have the same dimension. Now we refer to [C1, Theorem 3.5.1] or [Li1, Lemma 1] to complete the proof keeping in mind that as the assertions of the previous paragraph are local on \( X \) and \( Y \), we may reduce to the case where \( f \) factors as a closed immersion into a formally smooth map. Note that the unique nonvanishing homology occurs in degree \(-n\) where \( n \) is the (local) relative dimension.

In the following proposition we shall use the notation used in 5.5 above, namely that for any \( x \in X \), \( \lambda_x \) denotes the natural map \( \text{Spec}(O_{X,x}) \to X \) while if \( f : X \to Y \) is a scheme map, then \( f_x \) denotes the natural map \( \text{Spec}(O_{X,x}) \to Y \).

**Proposition 5.11.** Let \( P \) be a property of scheme maps in \( S \) such that for any map \( f : X \to Y \) in \( S \) the following conditions hold.

(a) For any localizing immersion \( i : W \to X \), if \( f \) satisfies \( P \) then so does \( fi \). (We also say that \( P \) is stable under localization.)

(b) If \( f \) is of finite type, then we have the following:

(i) If \( f_x \) satisfies \( P \) for every \( x \in X \), then so does \( f \);

(ii) the set of all points \( x \in X \) such that \( f_x \) satisfies \( P \) is open.

Then (i), (ii) of (β) hold in general and for any \( f \) that satisfies \( P \), there exists a factorization \( X \xrightarrow{i} Z \xrightarrow{h} Y \) where \( i \) is a localizing immersion and \( h \) is a separated finite-type map satisfying \( P \).
Proof. By Theorem 3.6 any $S_e$-map $f: X \to Y$ factors as $X \xrightarrow{i} Z \xrightarrow{h} Y$ with $i$ a localizing immersion and $h$ a separated finite-type map. For any $x \in X$, with $z = i(x)$ we have $\mathcal{O}_{X,x} \cong \mathcal{O}_{Z,z}$. Therefore the $P$-locus of $f$, namely the set of points $x \in X$ such that $f_x$ satisfies $P$, equals $i^{-1}V$ where $V$ is the $P$-locus of $h$, whence it is open.

Suppose $f_x$ satisfies $P$ for each $x \in X$. Then $V$ contains $i(X)$ and therefore, using $V$ in place of $Z$ and replacing $i, h$ suitably we may assume that in the factorization $f = hi$ above, $h$ satisfies $P$. Thus $f$ satisfies $P$. Conversely if $f$ satisfies $P$, then so does $f_x$ and hence arguing as above yields the desired factorization of $f$. \qed

5.12. Examples

We give some examples of properties where Proposition 5.11 applies. A minor note here is that as a local condition, to have $f_x$ satisfy $P$ could a-priori be viewed as a strong requirement since the presence of the non-closed points in $\text{Spec}(\mathcal{O}_{X,x})$ means the behavior of $f$ at those points too is being considered. However in the presence of the finiteness hypothesis on $f$, it remains equivalent to more familiar local versions of $P$.

In what follows, each of the properties considered is stable under localization, so we only discuss how (i) and (ii) of Proposition 5.11 hold.

(a). $P = \text{Flat}$. Here $f_x$ being flat is equivalent to $f$ being flat at $x$ in the usual sense, i.e., $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module. Thus (i) holds and for (ii) we refer to [EGAIV, 11.1.1].

(b). $P = \text{Essentially Perfect}$. As in (a), $f_x$ being essentially perfect is equivalent to $\mathcal{O}_{X,x}$ being a perfect $\mathcal{O}_{Y,y}$-complex. Thus (i) holds while for (ii) we note that in the notation of Theorem 5.9, the perfect locus of $f|_U$ equals the inverse image under $j$ of the perfect locus of the coherent $\mathcal{O}_Z$-module $j_*\mathcal{O}_U$, which is open.

(c). $P = \text{Cohen-Macaulay (CM)}$. Since the fibers of $f_x$ are again localizations of the fibers of $f$ and (i) holds for flatness we see that (i) holds in this case too. For (ii) we argue using [EGAIV] 12.1.6. Indeed, by \textit{loc. cit} the set $U$ of points $x \in X$ such that the fiber through $x$ has a CM local ring at $x$ forms an open set. If $f_x$ is CM, then $x \in U$ and conversely if $x$ is in $U$, then the natural image of $\text{Spec}(\mathcal{O}_{X,x})$ in $X$ is contained in $U$, whence $f_x$ is CM. Thus (ii) holds.

(d). $P = \text{Gorenstein}$. For (i) we argue as in (c). For (ii), first, by (a) above, we may shrink $X$ if necessary and assume that $f$ is flat. If $f_x$ is Gorenstein, then using the isomorphism $(f^!\mathcal{O}_Y)_x \cong \Gamma f^!_x\mathcal{O}_Y$ (see 5.5) and exactness of $\Gamma$ we obtain from 5.10 that $(f^!\mathcal{O}_Y)_x$ has a unique homology which is a free $\mathcal{O}_{X,x}$-module of rank 1. Since $f^!\mathcal{O}_Y \in \mathbf{D}^b_c(X)$, it follows that $x$ has an open neighborhood $U$ such that $(f^!\mathcal{O}_Y)|_U$ has a unique invertible homology, whence by 5.10 (ii) holds.

(e). $P = \text{Essentially smooth}$. If $f_x$ is (essentially) smooth then it is also flat. Thus smoothness for each $f_x$ implies flatness for $f$ and moreover that every fiber of $f$, being locally smooth, is smooth. Since flatness + smooth
fibers is equivalent to smoothness ([EGAIV, 17.5.1]), (i) holds. For (ii), we argue as in (c).

(f) $P =$ Unramified. By [LNS, Prop 2.6.1], the module of relative differentials $\Omega_{f_x}$ on $\text{Spec}(\mathcal{O}_{X,x})$ is coherent. Moreover, there is a natural isomorphism $\Omega_{f_x} \cong \lambda_x^* \Omega_f$ as the natural map $\lambda_x : \text{Spec}(\mathcal{O}_{X,x}) \to X$ is (essentially) étale. As in [5.5], there result natural isomorphisms $\Gamma \Omega_{f_x} \cong \Gamma \lambda_x^* \Omega_f \cong (\Omega_f)_x$ where $\Gamma$ is the (exact and faithful) global-sections functor on $\text{Spec}(\mathcal{O}_{X,x})$. Thus (i) and (ii) can be verified by looking at stalks of $\Omega_f$.

(g) $P =$ Essentially Étale. One can use (e) while keeping track of the rank of module of differentials as in (f).

Acknowledgements. I thank Professor Joseph Lipman for suggesting the main compactification problem to me and for various discussions on it.

References

[AJL] L. Alonso-Tarrío, A. Jeremías-López, and J. Lipman, *Duality and flat base change on formal schemes*, Studies in Duality on Noetherian formal schemes and non-Noetherian ordinary schemes, Contemporary Mathematics, vol. 244, Amer. Math. Soc. Providence RI, 1999, pp. 1–87.

[C1] B. Conrad, *Grothendieck Duality and Base Change*, Lecture Notes in Math. 1750, Springer-Verlag, New York, 2000.

[C2] , *Deligne’s notes on Nagata compactifications*, J. Ramanujan Math. Soc. 22 (2007), 205–257.

[D] P. Deligne, Cohomologie à support propre et construction du foncteur $f^!$, in *Residues and Duality*, Lecture Notes in Math., 20, Springer-Verlag, New York, (1966), pp. 404–421.

[EGAII] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. III: Étude globale élémentaire de quelques classes de morphismes*, Publ. Math. I.H.E.S. 8 (1961).

[EGAIII] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. III: Étude cohomologique des faisceaux cohérents*, Publ. Math. I.H.E.S. 11, 17 (1961-3).

[EGAIV] , *Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas*, Publ. Math. I.H.E.S. 20, 24, 28, 32 (1964-7).

[EGAI] , *Éléments de géométrie algébrique. I*, Springer-Verlag, Heidelberg, 1971.

[H] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math. 20, Springer-Verlag, New York, 1966.

[I] L. Illusie, *Généralités sur les conditions de finitude dans les catégories dérivées, etc.*, Théorie des Intersections et Théorème de Riemann-Roch (SGA6), Lecture Notes in Math. 225, Springer-Verlag, New York, 1971, 78–296.

[Li1] J. Lipman, *Double-point resolutions of deformations of rational singularities*, Compositio Math. 38 (1979), 37–43.

[Li2] , *Dualizing sheaves, Differentials, and Residues on Algebraic Varieties*, Astérisque, 117, Soc. Math. de France, 1984.

[Li3] , *Notes on derived functors and Grothendieck duality*, Lecture Notes in Math. Springer-Verlag, (to appear); preprint, <http://www.math.purdue.edu/~lipman>.

[LNS] J. Lipman, S. Nayak, and P. Sastry, *Pseudofunctorial behavior of Cousin complexes on formal schemes*, Contemp. Math. 375, Amer. Math. Soc., Providence, RI, 2005, pp. 3–133.

[Lu] W. Lütkebohmert, *On compactification of schemes*, Manuscripta Math. 80 (1993), 95–111.
[Ne] A. Neeman, *The Grothendieck duality theorem via Bousfield’s techniques and Brown representability*, J. Amer. Math. Soc., 9 (1996), pp. 205–236.

[Ng] M. Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ., 2 (1962), 1–10.

[Ny] S. Nayak, *Pasting Pseudofunctors*, Contemporary Math. 375, Amer. Math. Soc., Providence, RI, 2005, pp. 195–271.

[R] M. Raynaud, *Flat modules in algebraic geometry*, Compositio Math. 24 (1972), 11–31.

[S1] P. Sastry, *Base change and Grothendieck duality for Cohen-Macaulay maps*, Compositio Math., 140 (2004), pp. 729–777.

[S2] P. Sastry, *Duality for Cousin complexes*, Contemporary Math., 375 Amer. Math. Soc., Providence, RI (2005), pp. 137–192.

[V] J. L. Verdier, *Base change for twisted inverse image of coherent sheaves*, Algebraic Geometry, Oxford Univ. press, 1969, pp. 393–408.

[YZ] A. Yekutieli, J. Zhang, *Rigid dualizing complexes on schemes*, (preliminary version of a series of papers), arXiv:math.AG/0405570 v3, 7 Apr 2005.

Chennai Mathematical Institute, Plot H1, SIPCOT IT Park, Siruseri-603103, INDIA

Current address: Avery Hall, University of Nebraska, Lincoln, NE-68588, USA

E-mail address: snayak@cmi.ac.in, snayak@math.purdue.edu