CONTROLLABILITY OF SOBOLEV TYPE FUZZY DIFFERENTIAL EQUATION WITH NON-INSTANTANEOUS IMPULSIVE CONDITION

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Abstract. In this manuscript, we investigate the existence, uniqueness and controllability results of a Sobolev type fuzzy differential equation with non-instantaneous impulsive conditions. Non-linear functional analysis, Banach fixed point theorem and fuzzy theory are the main techniques used to establish these results. In support, an example is given to validate the obtained analytical findings.

1. Introduction. The concept of fuzzy set theory was introduced by L. A. Zadeh in 1965 [37], which is primarily based on the fact that “all things happening in real world are unstable and unpredictable”. This idea was put forward and successfully applied in several research areas. Recently this theory has been further developed and a large number of applications of this theory have been considered. Taking into consideration that differential equations play vital role in the modelling of a wide range of physical problems for example, bio medical problems, population models and hydraulic process etc [18, 14, 12]. Knowing that incomplete and imprecision information is a part of experimental observation and calculation, so authors used a fuzzy environment instead of a fixed value and hence turning the differential equations into fuzzy differential equations that help us to overcome the uncertainty. For the initial studies on fuzzy differential equations, we refer [37, 18, 14, 12, 20].

Sobolev type equation appears in many physical problems like thermodynamics, flow of fluid through rocks, shear in second order fluid and propagation of long waves of small amplitude and so on. For more, one can see [8, 1, 7, 35] and reference therein. Recently, few works are reported on the controllability of Sobolev type differential equations. In [28], Meraj et al. discussed the approximate controllability of a class of non-autonomous Sobolev type integro-differential equations having non-instantaneous impulses with nonlocal initial condition. In [4], Arora et al. studied the sufficient conditions for the approximate controllability of a semilinear Sobolev type evolution system in Banach spaces. To study more about Sobolev type differential equations, one can see [27, 13, 6].

Also, in literature, many physical problems are classified by abrupt changes in their states. These abrupt changes are recognized as impulsive effects in the

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system. Many physical issues can be modelled as impulsive differential equations. The concept of impulsive differential equations provided beneficial results in the field of science and engineering. In the literature, the impulsive system is broadly classified into two categories; one is the instantaneous impulsive system in which the length of these abrupt changes is very small in comparison with the whole evolution process. For instance in heart pulsates, natural disasters and shocks etc. Second one is the non-instantaneous impulsive system in which the length of these abrupt changes continues over a very finite time interval. For more of differential equations with non-instantaneous impulses, we refer to [5, 3, 36].

In 1960, Kalman was the first person who introduced the concept of controllability that formed the backbone of modern control theory. The problem of controllability has been extensively studied by many authors, see [32, 30, 31, 26, 21, 22, 34] and references therein. Moreover, in addition there are only few results available on the controllability of Sobolev type fuzzy differential systems. Balasubramaniam et al. [9] studied the existence and uniqueness results of fuzzy solution for the neutral fuzzy functional differential equation. In [19] Park et al. discussed the exact controllability for fuzzy differential equations. For more study about controllability of fuzzy differential equations, one can see [10, 2, 23, 25, 24, 33].

Therefore, motivated from the above discussion, here we are interested to find the existence, uniqueness and controllability of the following Sobolev type fuzzy differential equation with non-instantaneous impulses

\[
\frac{d}{d\sigma} [z(\sigma) + \mathfrak{N}(\sigma, z(\sigma))] = A z(\sigma) + \psi(\sigma, z(\sigma)) + u(\sigma), \quad \sigma \in (k_j, \sigma_{j+1}],
\]

\[j = 0, 1, 2, \cdots, n, \quad (1)\]

\[z(\sigma) = z_j(\sigma, z(\sigma_j^-)), \quad \sigma \in (\sigma_j, k_j], \quad j = 1, 2, 3, \cdots, n, \]

\[z(\sigma_0) = z_0,\]

where \(A : J \to \mathbb{K}^n\) denotes a fuzzy coefficient and \(J = [0, T], \mathbb{K}^n\) is the collection of all upper semi continuous convex normal fuzzy numbers with bounded \(\beta\)-level intervals. Also, we have that points \(k_j\) and \(\sigma_j\) satisfy the sequence \(0 = k_0 = \sigma_0 < \sigma_1 < k_1 < \sigma_2 < \cdots < \sigma_n < k_n < \sigma_{n+1} = T < \infty\). \(z(\sigma_j^-)\) represent the left and right limit of the state function \(z\) at \(\sigma_j\), respectively. The non-linear function \(\psi, \mathfrak{N}\) is define from \(J \times \mathbb{K}^n\) into \(\mathbb{K}^n\) and \(u : J \to \mathbb{K}^n\) is a control function. The functions \(z_j(\sigma, z(\sigma_j^-))\) represent non-instantaneous impulses during the interval \((\sigma_j, k_j]), j = 1, 2, \cdots, n\).

The primary contribution and advantage of this paper are as follows. In this work, we have considered a Sobolev type fuzzy differential equations with non-instantaneous impulses (1) and analyse the existence and uniqueness of the solution. Further, some sufficient conditions are established to guarantee that the system (1) is exact controllable. Also, an example is given to verify the proposed theoretical results. Note that the problem (1), considered in this manuscript is new on fuzzy set theory. We trust that the acquired outcomes will give a significant contribution to the existing literature on the topic. The rest of the manuscript is organized as follows. In Section 2, we give some basic definitions, preliminaries, and important lemmas. In Sections 3, we give the existence and uniqueness results of fuzzy solution. Section 4 is devoted to the controllability results of given system and in the last Section 5, an example is given to validate the obtained analytical results.
2. Preliminaries and assumptions. We define $PC(J;\mathbb{K}^n) = \{ z : J \to \mathbb{K}^n : z \in C((\sigma_j, \sigma_{j+1}); \mathbb{K}^n), j = 0, 1, \ldots, n \text{ and there exists } z(\sigma_j) \text{ and } z(\sigma_{j+1}) \text{, } j = 1, 2, \ldots, n, \text{ with } z(\sigma_j^+) = z(\sigma_j) \}$ for the space of piecewise continuous functions.

Next, we define some important definitions and lemmas with suitable references.

Definition 2.1. [38] A membership function of a fuzzy set $D$ on the universal set $\Xi$ is a function $\theta_D : \Xi \to [0, 1]$ which satisfy the following properties:

1. If $\theta_D(\rho) = 1$, then $\rho$ is totally in fuzzy set $D$, 
2. If $0 < \theta_D(\rho) < 1$, then $\rho$ is partially in fuzzy set $D$, 
3. If $\theta_D(\rho) = 0$, then $\rho \notin D$.

Definition 2.2. [38] We consider a fuzzy set $w$ of $\mathbb{R}^n$ and $\theta_w : \mathbb{R}^n \to [0, 1]$ is a membership function then the fuzzy set $w$ is said to be fuzzy number if it satisfy the following properties:

1. $w$ is normal, i.e., $\exists \rho_0 \in \mathbb{R}$ with $\theta_w(\rho_0) = 1$.
2. $w$ is fuzzy convex, i.e., $\theta_w(\sigma \rho + (1 - \sigma)\rho) \geq \min\{\theta_w(\rho), \theta_w(\rho)\}$, $\forall \sigma \in [0, 1], \rho, \rho' \in \mathbb{R}^n$.
3. $w$ is upper semi continuous on $\mathbb{R}^n$, i.e., $\forall \epsilon > 0, \exists \tau > 0$ such that $\theta_w(\rho) - \theta_w(\rho_0) < \epsilon, |\rho - \rho_0| < \tau$. 
4. $w$ is compactly supported, i.e., $cl\{\rho \in \mathbb{R}^n; \theta_w(\rho) > 0\}$ is compact.

We denote $\mathbb{K}^n$ for the space of all fuzzy sets in $\mathbb{R}^n$.

Definition 2.3. [38] We define 

$$d_H([\mu]^\beta, [\gamma]^\beta) = \max\{d([\mu]^\beta, [\gamma]^\beta), d([\mu]^\beta, [\mu]^\beta) : \beta \in (0, 1]\}, \mu, \gamma, \in \mathbb{K}^n.$$ 

($\mathbb{K}^n, d_H$) becomes a complete metric space, see [15]. Also, $\forall \mu, \gamma, \kappa, x \in \mathbb{K}^n$ satisfies the following properties :

(i) $d(\mu + \kappa, \gamma + \kappa) = d(\mu, \gamma)$; (ii) $d(\mu + \kappa, 0) = d(\mu, 0) + d(\kappa, 0)$;
(iii) $d(\mu + \gamma, \kappa + \mu) = d(\gamma, \kappa)$; (iv) $d(\mu + \gamma, x + \kappa) \leq d(\mu, \gamma) + d(\gamma, x)$;
(v) $d(\mu - \gamma, \kappa - x) \leq d(\mu, \kappa) + d(\gamma, x), \mu, \gamma, \kappa, x \in \mathbb{K}^n$ exist;
(vi) $d(\mu\kappa, \gamma\kappa) = \|d(\mu, \gamma)\|, \kappa \in \mathbb{R}$.

Definition 2.4. [15] We define the metric $d_\infty$ on $\mathbb{K}^n$ by

$$d_\infty(\mu, \gamma) = \sup\{d_H([\mu]^\beta, [\gamma]^\beta) : \beta \in (0, 1], \forall \mu, \gamma \in \mathbb{K}^n\}.$$ 

Clearly, one can see that $d_\infty$ is a metric in $\mathbb{K}^n$ and $(\mathbb{K}^n, d_\infty)$ forms a metric space which is complete.

Now, We define the metric $H_1$ on $PC(J;\mathbb{K}^n)$ by

$$H_1(\mu, \gamma) = \sup\{d_\infty(\mu(\sigma), \gamma(\sigma)) : \sigma \in J, \forall \mu, \gamma \in PC(J;\mathbb{K}^n)\}.$$ 

Clearly, $(PC(J;\mathbb{K}^n), H_1)$ is a complete metric space.

Definition 2.5. [29] Let $D$ over $\Xi$ be a fuzzy set and $\theta_D$ be the membership function on $D$. The support of a fuzzy set $D$ is the subset of $\Xi$ at which the grade of membership $\theta_D(\rho)$ is positive, i.e.,

$$\text{supp}(D) = \{\rho | \rho \in \Xi, \theta_D(\rho) > 0\},$$

we always assume that $\text{supp}(D) \neq \emptyset$.

Definition 2.6. [29] For $\beta \in (0, 1]$, the $\beta$ - level set of $D$ is defined by

$$[D]^\beta = \{\rho | \rho \in \Xi, \theta_D(\rho) \geq \beta\}$$
and for $\beta = 0$, we define the closure of the support

$$[D]^0 = cl\{\rho \in \Xi, \theta_D(\rho) \geq 0\}.$$  

**Definition 2.7.** [29] The core of $D$ is defined as

$$core(D) = \{\rho | \rho \in \Xi, \theta_D(\rho) = 1\}$$

and we say that $D$ is normal if $core(D) \neq \emptyset$.

**Lemma 2.8.** [11] We called two fuzzy numbers $g$ and $h$ are equal, i.e., $g = h$ if $\theta_g(w) = \theta_h(w)$ for all $w \in \mathbb{R}$. If

$$g = h \iff [g]^{\beta} = [h]^{\beta}, \quad \forall \beta \in (0, 1].$$

**Lemma 2.9.** [11] If $g, h \in \mathbb{K}^n$, then for $\beta \in (0, 1],$

$$[g + h]^{\beta} = [g_a^{\beta} + h_a^{\beta}, g_b^{\beta} + h_b^{\beta}],$$

$$[g \times h]^{\beta} = [\min\{h_i^{\beta} j_i^{\beta}\}, \max\{h_i^{\beta} j_i^{\beta}\}], \quad i, j = a, b.$$  

$$[g - h]^{\beta} = [g_a^{\beta} - h_a^{\beta}, g_b^{\beta} - h_b^{\beta}].$$

**Lemma 2.10.** [16, 17] For $0 < \beta \leq 1$, let $[g_a^{\beta}, h_b^{\beta}]$ be a collection of non-empty intervals. If

$$[g_a^{\beta}, g_b^{\beta}] \subset [g_a^{\beta}, g_b^{\beta}], \quad 0 < \beta \leq \alpha$$

and

$$\left[\lim_{k \to \infty} g_a^{\beta_k}, \lim_{k \to \infty} g_b^{\beta_k}\right] = [g_a^{\beta}, g_b^{\beta}],$$

$(\beta_k)$ is a non decreasing sequence converging to $\beta \in (0, 1]$, then the family $[g_a^{\beta}, g_b^{\beta}]$ are called the $\beta$-level set of a fuzzy number $g \in \mathbb{K}^n$.

Conversely, if $[g_a^{\beta}, g_b^{\beta}]$ are the $\beta$-level set of a fuzzy number $g \in \mathbb{K}^n$, then the above two conditions hold.

For any real interval $I$, a map $z : I \to \mathbb{K}^n$ is said to be a fuzzy process if $z$ can be written as

$$[z(\sigma)]^{\beta} = [z_p^{\beta}(\sigma), z_q^{\beta}(\sigma)], \quad \sigma \in I, \quad 0 < \beta \leq 1.$$  

$z'(\sigma) \in \mathbb{K}^n$ of a fuzzy process is defined by

$$[z'(\sigma)]^{\beta} = [(z_p^{\beta})'(\sigma), (z_q^{\beta})'(\sigma)], \quad 0 < \beta \leq 1.$$  

We define the fuzzy integral as follows

$$\left[\int_a^b [z(\sigma)]^{\beta} \right]^{\beta} = \left[\int_a^b z_p^{\beta}(\sigma)d\sigma, \int_a^b z_q^{\beta}(\sigma)d\sigma\right], \quad \forall a, b \in I,$$

provided that the right side Lebesgue integrals in the above equation are exists.

We need the following assumptions, in order to prove our results

**(A1)** Functions $\psi, \mathcal{M} : J \times \mathbb{K}^n \to \mathbb{K}^n$ are continuous and satisfies a global Lipschitz condition i.e.;

$$d_H([\mathcal{M}(s, \xi(s))]^{\beta}, [\mathcal{M}(s, \zeta(s))]^{\beta}) \leq M_{\mathcal{M}} d_H([\xi(s)]^{\beta}, [\zeta(s)]^{\beta}),$$

$$d_H([\psi(s, \xi(s))]^{\beta}, [\psi(s, \zeta(s))]^{\beta}) \leq M_{\psi} d_H([\xi(s)]^{\beta}, [\zeta(s)]^{\beta}),$$

for all $\xi(\cdot), \zeta(\cdot) \in \mathbb{K}^n$, and finite positive constants $M_{\mathcal{M}}, M_{\psi}$. 

The functions $\mathfrak{f}_j : [s_j, k_j] \times \mathbb{K}^n \to \mathbb{K}^n$, $j = 1, 2, \cdots, n$, satisfies the global Lipschitz condition i.e.;

$$d_H([s_j(s), \xi(s))]^{[\mathfrak{f}_j(s) \xi(s)]^{[\mathfrak{f}_j(s)]}} \leq M_2 d_H([\xi(s)]^{[\mathfrak{f}_j(s)]}, [\xi(s)]^{[\mathfrak{f}_j(s)]}),$$

for all $(\xi(\cdot), \xi(\cdot)) \in \mathbb{K}^n$ and finite positive constant $M_2$, s.t. $j = 1, 2, \cdots, n$.

**Definition 2.11.** A function $z \in PC(J; \mathbb{K}^n)$ is said to be a solution of system (1) if $z$ satisfies the following integral equations

\[
z(\sigma_0) = z_0,
\]

\[
z(\sigma) = S(\sigma)[z_0 + M(0, z_0) + \int_0^\sigma S(\sigma - s)[-A\mathcal{M}(s, z(s)) + \psi(s, z(s))]ds
\]

\[
- M(\sigma, z(\sigma)), \quad \forall \quad \sigma \in (0, \sigma_1],
\]

\[
z(\sigma) = \mathfrak{f}_j(\sigma, z(\sigma^-)), \quad \forall \quad \sigma \in (s_j, k_j], \quad j = 0, 1, 2, \cdots, n,
\]

\[
z(\sigma) = S(\sigma - k_j)[\mathfrak{f}_j(k_j, z(\sigma^-)) + M(0, z_0)]
\]

\[
+ \int_{k_j}^\sigma S(\sigma - s)[-A\mathcal{M}(s, z(s)) + \psi(s, z(s))]ds - M(\sigma, z(\sigma)),
\]

\[
\forall \quad \sigma \in (k_j, \sigma_{j+1}].
\]

3. **Existence and uniqueness of solution.** Here, we state and prove existence and uniqueness theorem for the solution of system (1) by using Banach fixed point theorem.

**Theorem 3.1.** If all the assumptions (A1)-(A4) are satisfied, then the system (1) has a unique solution on $J$.

**Proof.** We define an operator $\Lambda : PC(J; \mathbb{K}^n) \to PC(J; \mathbb{K}^n)$ such that

\[
(\Lambda z)(\sigma) = S(\sigma)[z(\sigma) + M(0, z(\sigma)) + \int_0^\sigma S(\sigma - s)[-A\mathcal{M}(s, z(s)) + \psi(s, z(s))]ds
\]

\[
- M(\sigma, z(\sigma)), \forall \quad \sigma \in (0, \sigma_1],
\]

\[
(\Lambda z)(\sigma) = \mathfrak{f}_j(\sigma, z(\sigma^-)), \quad \forall \quad \sigma \in (s_j, k_j], \quad j = 0, 1, 2, \cdots, n,
\]

\[
(\Lambda z)(\sigma) = S(\sigma - k_j)[\mathfrak{f}_j(k_j, z(\sigma^-)) + M(0, z_0)]
\]

\[
+ \int_{k_j}^\sigma S(\sigma - s)[-A\mathcal{M}(s, z(s)) + \psi(s, z(s))]ds - M(\sigma, z(\sigma)),
\]

\[
\forall \quad \sigma \in (k_j, \sigma_{j+1}].
\]

Clearly, the fixed point of the operator $\Lambda$ is the solution of considered system (1). Therefore, we need to show the fixed point of the operator $\Lambda$. The proof of this theorem is divided into three cases as follows:
Case I: For any $\sigma \in (0, \sigma_1]$, $\xi, \zeta \in \mathbb{K}^n$, we have

$$d_H([\Lambda \xi(\sigma)]^\beta, [\Lambda \zeta(\sigma)]^\beta)$$

$$= d_H([S(\sigma)[\xi_0 + M(0, \xi_0)] - M(\sigma, \xi(\sigma))] - \int_0^\sigma A s(\sigma - s) M(s, \xi(s))$$

$$+ \int_0^\sigma S(\sigma - s) \psi(s, \xi(s))ds)^\beta, [S(\sigma)[\zeta_0 + M(0, \zeta_0)] - M(\sigma, \zeta(\sigma))]$$

$$- \int_0^\sigma A s(\sigma - s) M(s, \zeta(s)) + \int_0^\sigma S(\sigma - s) \psi(s, \zeta(s))ds)^\beta)$$

$$\leq MLA^* \int_0^\sigma d_H([\xi(s)]^\beta, [\zeta(s)]^\beta)ds.$$

For detail of the above inequality, please see the Appendix 1.

Now,

$$d_\infty([\Lambda \xi(\sigma)], [\Lambda \zeta(\sigma)]) = \sup_{\beta \in (0, 1)} d_H([\Lambda \xi(\sigma)]^\beta, [\Lambda \zeta(\sigma)]^\beta)$$

$$= MLA^* \int_0^\sigma \sup_{\beta \in (0, 1)} d_H([\xi(s)]^\beta, [\zeta(s)]^\beta)ds$$

$$= MLA^* \int_0^\sigma d_\infty(\xi(s), \zeta(s))ds.$$

Hence,

$$H_1(\Lambda \xi, \Lambda \zeta) = \sup_{\sigma \in (0, \sigma_1]} d_\infty([\Lambda \xi(\sigma)], [\Lambda \zeta(\sigma)])$$

$$\leq MLA^* \int_0^\sigma \sup_{\sigma \in (0, \sigma_1]} d_\infty(\xi(s), \zeta(s))ds$$

$$= MLA^* \sigma_1 H_1(\xi, \zeta).$$

Set, $r_0 = MLA^* \sigma_1$. Thus, for sufficiently small $\sigma_1$, $MLA^* \sigma_1 < 1$.

Case II: For any $\sigma \in (\sigma_j, k_j]$, we have

$$d_H([\Lambda \xi(\sigma)]^\beta, [\Lambda \zeta(\sigma)]^\beta) \leq M_3, d_H([\xi(\sigma)]^\beta, [\zeta(\sigma)]^\beta).$$

Therefore,

$$d_\infty([\Lambda \xi(\sigma)], [\Lambda \zeta(\sigma)]) = \sup_{\beta \in (0, 1]} d_H([\Lambda \xi(\sigma)]^\beta, [\Lambda \zeta(\sigma)]^\beta)$$

$$\leq M_3, d_H(\xi(\sigma), \zeta(\sigma)).$$

Thus,

$$H_1(\Lambda \xi, \Lambda \zeta) = \sup_{\sigma \in (\sigma_j, k_j]} d_\infty([\Lambda \xi(\sigma)], [\Lambda \zeta(\sigma)])$$

$$\leq M_3, H_1(\xi, \zeta).$$
Case III: For any $\sigma \in (k_j, \sigma_{j+1}]$, $\xi, \zeta \in \mathbb{K}^n$, we have

$$d_H([(\Lambda \xi)(\sigma)]^\beta, [(\Lambda \zeta)(\sigma)]^\beta)$$

$$= d_H\left( [S(\sigma - k_j)(\mathcal{J}_j(k_j, \xi(\sigma^-)) + \mathcal{M}(0, \xi_0)) - \mathcal{M}(\sigma, \xi(\sigma)) - \int_{k_j}^\sigma A S(\sigma - s) \mathcal{M}(s, \xi(s)) + \int_{k_j}^\sigma S(\sigma - s) \psi(s, \xi(s))ds]^{\beta}, \right.$$

$$\left. [S(\sigma - k_j)(\mathcal{J}_j(k_j, \zeta(\sigma^-)) + \mathcal{M}(0, \zeta_0)) - \mathcal{M}(\sigma, \zeta(\sigma)) - \int_{k_j}^\sigma A S(\sigma - s) \mathcal{M}(s, \zeta(s)) + \int_{k_j}^\sigma S(\sigma - s) \psi(s, \zeta(s))ds]^{\beta}\right)$$

$$\leq M M_3 d_H([(\xi(\sigma_j^-)]^\beta, [(\zeta(\sigma_j^-)]^\beta) + \dot{M} \Lambda^* \int_{k_j}^\sigma d_H([(\xi(s)]^\beta, [\zeta(s)]^\beta)ds.$$

For detail of the above inequality, please see the Appendix 2.

Thus,

$$d_\infty([(\Lambda \xi)(\sigma)], [(\Lambda \zeta)(\sigma)]) = \sup_{\beta \in (0,1]} d_H([(\Lambda \xi)(\sigma)]^\beta, [(\Lambda \zeta)(\sigma)]^\beta)$$

$$= M \sup_{\beta \in (0,1]} M_3 d_H([(\xi(\sigma_j^-)]^\beta, [\zeta(\sigma_j^-)]^\beta)$$

$$+ \dot{M} \Lambda^* \int_{k_j}^\sigma \sup_{\beta \in (0,1]} d_H([(\xi(s)]^\beta, [\zeta(s)]^\beta)ds$$

$$= M (M_3 d_\infty(\xi(\sigma_j^-), \zeta(\sigma_j^-)) + \dot{M} \Lambda^* \int_{k_j}^\sigma d_\infty(\xi(s), \zeta(s))ds.$$

Hence,

$$\mathcal{H}_1((\Lambda \xi), (\Lambda \zeta)) = \sup_{\sigma \in (k_j, \sigma_{j+1}] \ d_\infty([(\Lambda \xi)(\sigma)], [(\Lambda \zeta)(\sigma)])$$

$$\leq M \sup_{\sigma \in (k_j, \sigma_{j+1}] \ d_\infty(\xi(\sigma_j^-), \zeta(\sigma_j^-))$$

$$+ \dot{M} \Lambda^* \int_{k_j}^\sigma \sup_{\sigma \in (k_j, \sigma_{j+1}] \ d_\infty(\xi(s), \zeta(s))ds$$

$$= M (M_3 \mathcal{H}_1(\xi, \zeta) + \dot{M} \Lambda^*(\sigma_{j+1} - k_j)\mathcal{H}_1(\xi, \zeta))$$

$$\leq M (M_3 + \dot{M} \Lambda^*(\sigma_{j+1} - k_j))\mathcal{H}_1(\xi, \zeta)$$

$$\leq r_j \mathcal{H}_1(\xi, \zeta),$$

where $r_j = M (M_3 + \dot{M} \Lambda^*(\sigma_{j+1} - k_j))$.

From the above three cases, we conclude that

$$\mathcal{H}_1((\Lambda \xi), (\Lambda \zeta)) = \sup_{\sigma \in J} d_\infty([(\Lambda \xi)(\sigma)], [(\Lambda \zeta)(\sigma)]) \leq r \mathcal{H}_1(\xi, \zeta).$$

By the assumption (A4), $\Lambda$ is a strict contraction mapping. Therefore, by using Banach fixed point theorem, $\Lambda$ has a unique fixed point, which is the unique solution of system (1).
4. Exact controllability. Steering of dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls is called exactly controllable system. In this section, we investigate exact controllability of the system (1), for this we define the following fuzzy mapping $G: \hat{P}(R) \to \mathbb{K}^n$ such that

$$G^\beta(u) = \begin{cases} \int_0^T S^\beta(T - s)u(s)ds, & \text{if } u \in \Gamma_u, \\ 0, & \text{otherwise}, \end{cases}$$

where $\Gamma_u$ is the closure support of $u$ and $\hat{P}(R)$ is defined as the closed compact control functions in $R$. Then, there exists $G_j^\beta (j = p, q)$ such that

$$G_p^\beta(u_p) = \int_0^T S_p^\beta(T - s)u_p(s)ds, \quad u_p(s) \in [u_p^0(s), u^1(s)],$$

$$G_q^\beta(u_q) = \int_0^T S_q^\beta(T - s)u_q(s)ds, \quad u_q(s) \in [u^1(s), u_q^0(s)].$$

Assuming that $G_p^\beta, G_q^\beta$ are the bijective mappings. Hence, the $\beta$-level set of control term $u(\sigma)$ are defined as follow

**Case I:** For $\sigma \in (0, \sigma_1)$, we have

$$[u(\sigma)]^\beta = [u_p^\beta(\sigma), u_q^\beta(\sigma)]$$

$$= ([G_p^{-1})^\beta((z^{\sigma_1})_p^\beta - S_p^\beta(\sigma_1)(z_0)_p^\beta + M_p^\beta(0, (z_0)_p^\beta)]$$

$$+ \int_0^{z_1} S_p^\beta(\sigma_1 - s)[A_p^\beta M_p^\beta(s, z(s)) - \psi_p^\beta(s, z(s))] + M_p^\beta(\sigma_1, z(\sigma_1))),$$

$$([G_q^{-1})^\beta((z^{\sigma_1})_q^\beta - S_q^\beta(\sigma_1)(z_0)_q^\beta + M_q^\beta(0, (z_0)_q^\beta)]$$

$$+ \int_0^{z_1} S_q^\beta(\sigma_1 - s)[A_q^\beta M_q^\beta(s, z(s)) - \psi_q^\beta(s, z(s))] + M_q^\beta(\sigma_1, z(\sigma_1))]. \quad (2)$$

**Case II:** For $\sigma \in (k_j, \sigma_{j+1})$, we have

$$[u(\sigma)]^\beta$$

$$= [u_p^\beta(\sigma), u_q^\beta(\sigma)]$$

$$= ([G_p^{-1})^\beta((z^{\sigma_{j+1}})_p^\beta - S_p^\beta(\sigma_{j+1} - k_j)]\mathcal{F}_p^\beta(k_j, z(\sigma_{j+1})) + M_p^\beta(0, (z_0)_p^\beta)]$$

$$+ \int_{k_j}^{\sigma_{j+1}} S_p^\beta(\sigma_{j+1} - s)[A_p^\beta M_p^\beta(s, z(s)) - \psi_p^\beta(s, z(s))]ds + M_p^\beta(\sigma_{j+1}, z(\sigma_{j+1}))),$$

$$([G_q^{-1})^\beta((z^{\sigma_{j+1}})_q^\beta - S_q^\beta(\sigma_{j+1} - k_j)]\mathcal{F}_q^\beta(k_j, z(\sigma_{j+1})) + M_q^\beta(0, (z_0)_q^\beta)]$$

$$+ \int_{k_j}^{\sigma_{j+1}} S_q^\beta(\sigma_{j+1} - s)[A_q^\beta M_q^\beta(s, z(s)) - \psi_q^\beta(s, z(s))]ds + M_q^\beta(\sigma_{j+1}, z(\sigma_{j+1}))). \quad (3)$$

**Definition 4.1.** The non-instantaneous fuzzy control system (1) is said to be controllable on $J$, if for every $z_0, z^T \in \mathbb{K}^n$, there exists a control function $u : J \to \mathbb{K}^n$ such that the solution of (1) satisfies $z(\sigma_0) = z_0$ and $[z(T)]^\beta = [z^T]^\beta$.

**Definition 4.2.** A function $z(\sigma) \in PC(J, \mathbb{K}^n)$ is called the solution of the fuzzy system (1), if $z$ is the solution of the following integral equation...
Proof. Put the value of $u(\sigma)$ from equation (2) and $\sigma = \sigma_1$ in equation (4), we get

$$z(\sigma) = S(\sigma)[z_0 + M(0, z_0)] + \int_0^\sigma S(\sigma - s)[-AM(s, z(s)) + \psi(s, z(s)) + u(s)]ds$$

$$- M(\sigma, z(\sigma)), \quad \forall \quad \sigma \in (0, \sigma_1],$$

(4)

$$z(\sigma) = \mathcal{S}_j(\sigma, z(\sigma_j^\pm)) \quad \forall \quad \sigma \in (\sigma_j, k_j], \quad j = 0, 1, 2, \cdots, n,$$

(5)

$$z(\sigma) = S(\sigma - k_j)(\mathcal{S}_j(k_j, z(\sigma_j^+)) + M(0, z(\sigma))) + \int_{k_j}^\sigma S(\sigma - s)[-AM(s, z(s))$$

$$+ \psi(s, z(s)) + u(s)]ds - M(\sigma, z(\sigma)), \quad \forall \quad \sigma \in (k_j, \sigma_{j+1}].$$

(7)

**Lemma 4.3.** If the assumptions (A1)-(A3) are satisfied and $z^{\sigma_1} \in \mathbb{K}^n$ be an arbitrary point. Then, the control input $u(\sigma)$ transfer the system from given initial state to desire final state $z^{\sigma_1}$.

Proof. Put the value of $u(\sigma)$ from equation (2) and $\sigma = \sigma_1$ in equation (4), we get

$$[z(\sigma_1)]^\beta$$

$$= [S_p^\beta(\sigma_1)(z_0)^\beta + M_p^\beta(0, z_0)] - \int_0^{\sigma_1} S_p^\beta(\sigma_1 - s)A_p^\beta M_p^\beta(s, z(s))ds$$

$$+ \int_0^{\sigma_1} S_p^\beta(\sigma_1 - s)\psi_p^\beta(s, z(s))ds + G_p^\beta(G_p^\beta)^{-1}((z^{\sigma_1})^\beta_p - S_p^\beta(\sigma_1)(z_0)^\beta_p$$

$$+ M_p^\beta(0, z_0)^\beta_p] + \int_0^{\sigma_1} S_q^\beta(\sigma_1 - s)[A_q^\beta M_q^\beta(s, z(s)) - \psi_q^\beta(s, z(s))]ds$$

$$+ M_q^\beta(0, z_0)^\beta_q] - \int_0^{\sigma_1} S_q^\beta(\sigma_1 - s)A_q^\beta M_q^\beta(s, z(s))ds + \int_0^{\sigma_1} S_q^\beta(\sigma_1 - s)\psi_q^\beta(s, z(s))ds$$

$$+ G_q^\beta(G_q^\beta)^{-1}((z^{\sigma_1})^\beta_q - S_q^\beta(\sigma_1)(z_0)^\beta_q + M_q^\beta(0, z_0)^\beta_q]$$

$$+ \int_0^{\sigma_1} S_q^\beta(\sigma_1 - s)[A_q^\beta M_q^\beta(s, z(s)) - \psi_q^\beta(s, z(s))]ds + M_q^\beta(\sigma_1, z(\sigma_1))]$$

$$- M_q^\beta(\sigma_1, z(\sigma))$$

$$= [(z^{\sigma_1})^\beta_p, (z^{\sigma_1})^\beta_q]$$

$$= [z^{\sigma_1}]^\beta.$$
This completes the proof of Lemma 4.4.

In order to prove the controllability results, we need the following additional assumptions:

(A5) For \( R < 1 \), where \( R = \max\{R_0, R_j, M_{j_3}\} \) for \( j = 1, 2, \cdots, p \), \( R_0 = MLA^* (1 + MT) \sigma_1 \) and \( R_j = (M M_{j_3} + MLA^* (\sigma_j + 1 - k_j) + M^2 M_j (\sigma_j + 1 - k_j) + M^2 L A^* (\sigma_j + 1 - k_j) (T - k_j)) \).

(A6) Linear system corresponding to (1) is controllable.

**Theorem 4.5.** Assume that all the assumptions (A1)-(A6) are satisfied. Then, the fuzzy system (1) is exactly controllable on \( J \).

**Proof.** We define an operator \( \Lambda : PC(J; \mathbb{K}^n) \rightarrow PC(J; \mathbb{K}^n) \) such that

\[
(\Lambda \xi)(\sigma) = S(\sigma) [\xi_0 + M(0, \xi_0) + \int_0^\sigma S(\sigma - s)[-AM(s, \xi(s)) + \psi(s, \xi(s)) + u(s)] ds - M(\sigma, \xi(\sigma))], \forall \ \sigma \in (0, \sigma_1],
\]

\[
(\Lambda \xi)(\sigma) = \mathcal{A}_j(\sigma, \xi(\sigma^-)), \ \forall \ \sigma \in (s_j, k_j], \ j = 0, 1, 2, \cdots, n,
\]

\[
(\Lambda \xi)(\sigma) = S(\sigma - k_j) [\mathcal{A}_j(k_j, \xi(\sigma^-)) + M(0, \xi_0)] + \int_k^\sigma S(\sigma - s)[-AM(s, \xi(s)) + \psi(s, \xi(s)) + u(s)] ds - M(\sigma, \xi(\sigma)), \forall \ \sigma \in (k_j, \sigma_{j+1}],
\]

where \( u(\sigma) \) is given by equations (2) and (3) in the intervals \( (0, \sigma_1) \) and \( (k_j, \sigma_{j+1}) \), \( j = 1, 2, \cdots, n \), respectively. Now, from Lemma 4.3 and 4.4, we have \([z(\sigma_1)]^\beta = [z(\sigma_1)^\beta\] and \([z(\sigma_{j+1})]^\beta = [z(\sigma_{j+1})]^\beta\] for \( j = 1, 2, \cdots, n \), respectively. To prove this theorem, we consider the following three cases:

**Case 1:** For any \( \sigma \in (0, \sigma_1) \) and \( \xi, \zeta \in \mathbb{K}^n \), we have

\[
d_H([[(\Lambda \xi)(\sigma)]^\beta, (\Lambda \zeta)(\sigma)]^\beta) \leq MLA^* \int_0^\sigma d_H ([\xi(s)]^\beta, [\zeta(s)]^\beta) ds + M^2 L A^* \int_0^\sigma \int_0^T d_H ([\xi(s)]^\beta, [\zeta(s)]^\beta) ds ds.
\]
Thus,
\[
d_{\infty}([\Lambda \xi](\sigma), [\Lambda \zeta](\sigma)) = \sup_{\beta \in (0,1)} d_H([\Lambda \xi](\sigma)]^\beta, [\Lambda \zeta](\sigma)]^\beta) \\
= M L A^* \int_0^\sigma \sup_{\beta \in (0,1)} d_H([\xi(s)]^\beta, [\zeta(s)]^\beta) \, ds \\
+ M^2 L A^* \int_0^\sigma \int_0^T \sup_{\beta \in (0,1)} d_H([\xi(s)]^\beta, [\zeta(s)]^\beta) \, ds \, ds \\
= M L A^* \int_0^\sigma d_{\infty}(\xi(s), \zeta(s)) \, ds \\
+ 2 M^2 L A^* \int_0^\sigma \int_0^T d_{\infty}(\xi(s), \zeta(s)) \, ds.
\]

Hence,
\[
H_1(\Lambda(\xi), \Lambda(\zeta)) = \sup_{\sigma \in (0, \sigma_1]} d_{\infty}([\Lambda \xi](\sigma), [\Lambda \zeta](\sigma)) \\
\leq M L A^* \int_0^\sigma \sup_{\sigma \in (0, \sigma_1]} d_{\infty}(\xi(s), \zeta(s)) \, ds \\
+ M^2 L A^* \int_0^\sigma \int_0^T \sup_{\sigma \in (0, \sigma_1]} d_H([\xi(s)]^\beta, [\zeta(s)]^\beta) \, ds \, ds \\
= M L A^* \sigma_1 H_1(\xi, \zeta) + M^2 L A^* \sigma_1 \sigma_1 T H_1(\xi, \zeta) \\
= M L A^* (1 + MT) \sigma_1 H_1(\xi, \zeta) \\
= R_0 H_1(\xi, \zeta).
\]

**Case II:** For any \( \sigma \in (\sigma_j, k_j) \), we have
\[
d_H([\Lambda \xi](\sigma)]^\beta, [\Lambda \zeta](\sigma)]^\beta) \leq M_3 d_H([\xi(\sigma)]^\beta, [\zeta(\sigma)]^\beta).
\]

Thus,
\[
d_{\infty}([\Lambda \xi](\sigma), [\Lambda \zeta](\sigma)) = \sup_{\beta \in (0,1)} d_H([\Lambda \xi](\sigma)]^\beta, [\Lambda \zeta](\sigma)]^\beta) \\
\leq M_3 d_H(\xi(\sigma), \zeta(\sigma)).
\]

Hence,
\[
H_1((\Lambda \xi), (\Lambda \zeta)) = \sup_{\sigma \in (k_j, \sigma_{j+1})} d_{\infty}([\Lambda \xi](\sigma), [\Lambda \zeta](\sigma)) \\
\leq M_3 H_1(\xi, \zeta).
\]

**Case III:** For any \( \sigma \in (k_j, \sigma_{j+1}) \), we have
\[
d_H([\Lambda \xi](\sigma)]^\beta, [\Lambda \zeta](\sigma)]^\beta) \leq M M_3 d_H([\xi(\sigma_j^-)]^\beta, [\zeta(\sigma_j^-)]^\beta) \\
+ M L A^* \int_{k_j}^\sigma d_H([\xi(s)]^\beta, [\zeta(s)]^\beta) \, ds \\
+ M^2 M_3 \int_{k_j}^\sigma d_H([\xi(\sigma_j^-)]^\beta, [\zeta(\sigma_j^-)]^\beta) \, ds \\
+ M^2 L A^* \int_{k_j}^\sigma \int_{k_j}^{\sigma_j} d_H([\xi(s)]^\beta, [\zeta(s)]^\beta) \, ds \, ds.
\]
Hence, 
\[
d_{\infty}([\Lambda\xi](\sigma), [\Lambda\zeta](\sigma)) = \sup_{\beta \in (0, 1)} d_{H}([\Lambda\xi](\sigma)]^\beta, [\Lambda\zeta](\sigma)]^\beta) \\
= M \sup_{\beta \in (0, 1)} M_3 d_{H}([\xi(\sigma^-)]^\beta, [\zeta(\sigma^-)]^\beta) \\
+ MLA^* \int_{k_j}^\sigma d_{H}([\xi(s)]^\beta, [\zeta(s)]^\beta)ds \\
+ M^2 M_3 \int_{k_j}^\sigma \sup_{\beta \in (0, 1)} d_{H}([\xi(\sigma^-)]^\beta, [\zeta(\sigma^-)]^\beta)ds \\
+ M^2 L A^* \int_{k_j}^\sigma \int_{k_j}^T \sup_{\beta \in (0, 1)} d_{H}([\xi(s)]^\beta, [\zeta(s)]^\beta)dsds \\
= MM_3 d_{\infty}(\xi(\sigma^-), \zeta(\sigma^-)) + MLA^* \int_{k_j}^\sigma d_{\infty}(\xi(\sigma^-), \zeta(\sigma^-))ds \\
+ M^2 M_3 \int_{k_j}^\sigma d_{\infty}(\xi(\sigma^-), \zeta(\sigma^-))ds \\
+ M^2 L A^* \int_{k_j}^\sigma \int_{k_j}^T d_{\infty}(\xi(s), \zeta(s))dsds.
\]

Hence, 
\[
H_1((\Lambda\xi), (\Lambda\zeta)) = \sup_{\sigma \in (k_j, \sigma_{j+1})} d_{\infty}([\Lambda\xi](\sigma), [\Lambda\zeta](\sigma)) \\
\leq M \sup_{\sigma \in (k_j, \sigma_{j+1})} M_3 d_{\infty}(\xi(\sigma^-), \zeta(\sigma^-)) \\
+ MLA^* \int_{k_j}^\sigma d_{\infty}(\xi(s), \zeta(s))ds \\
+ M^2 M_3 \int_{k_j}^\sigma \sup_{\sigma \in (k_j, \sigma_{j+1})} d_{\infty}(\xi(\sigma^-), \zeta(\sigma^-))ds \\
+ M^2 L A^* \int_{k_j}^\sigma \int_{k_j}^T \sup_{\sigma \in (k_j, \sigma_{j+1})} d_{\infty}(\xi(s), \zeta(s))dsds \\
= MM_3 H_1(\xi, \zeta) + MLA^*(\sigma_{j+1} - k_j) H_1(\xi, \zeta) \\
+ M^2 M_3 (\sigma_{j+1} - k_j) H_1(\xi, \zeta) \\
+ M^2 L A^*(\sigma_{j+1} - k_j) (T - k_j) H_1(\xi, \zeta) \\
= (MM_3 + MLA^*(\sigma_{j+1} - k_j) + M^2 M_3 (\sigma_{j+1} - k_j) \\
+ M^2 L A^*(\sigma_{j+1} - k_j) (T - k_j) ) H_1(\xi, \zeta) \\
= R_j H_1(\xi, \zeta).
\]

From the above three cases, we conclude that 
\[
H_1((\Lambda\xi), (\Lambda\zeta)) = \sup_{\sigma \in J} d_{\infty}([\Lambda\xi](\sigma), [\Lambda\zeta](\sigma)) \leq RH_1(\xi, \zeta).
\]

By the hypothesis (A5), \(A\) is a strict contraction mapping. Therefore, by using Banach fixed point theorem, system (1) has a unique fixed point. Thus, system (1) is exact controllable on \(J\). \(\square\)
Remark 1. We can extend the results of this manuscript to the nonlocal problem given below:

\[
\frac{d}{d\sigma}[z(\sigma) + \mathcal{M}(\sigma, z(\sigma))] = \mathcal{A}z(\sigma) + \psi(\sigma, z(\sigma)) + u(\sigma), \quad \sigma \in (k_j, \sigma_{j+1}],
\]

\[
j = 0, 1, 2, \cdots, n,
\]

\[
z(\sigma) = \mathcal{F}_j(\sigma, z(\sigma^-)), \quad \sigma \in (\sigma_j, k_j], \quad j = 1, 2, 3, \cdots, n,
\]

\[
z(\sigma_0) + g(., z(.)) = z_0,
\]

where all the symbols are same as defined in (1) and \( g(., z(.)) \) is a nonlinear continuous function which satisfy some suitable conditions. For this system (8), we can establish the existence, uniqueness and controllability results by using the fuzzy set theory, Banach fixed point theorem and some non-linear functional analysis results.

5. Example. Consider the fuzzy differential system with non-instantaneous impulsive condition

\[
\frac{d}{d\sigma}[z(\sigma) + 2\sigma z^2(\sigma)] = 3z(\sigma) + 3\sigma z^2(\sigma) + u(\sigma), \quad \sigma \in (0, 0.5) \cup (0.7, 1],
\]

\[
z(\sigma) = \frac{\cos(\sigma)}{e^{2\sigma}}z(\sigma^-), \quad \sigma \in (0.5, 0.7],
\]

\[
z(\sigma_0) = 0.
\]

Here, we set \( \sigma_0 = k_0 = 0, \sigma_1 = 0.5, k_1 = 0.7, \sigma_2 = T = 1, \psi(\sigma, z(\sigma)) = \dot{3}\sigma z^2(\sigma), \mathcal{M}(\sigma, z(\sigma)) = 2\sigma z^2(\sigma) \) and \( \mathcal{F}_1(\sigma, z(\sigma^-)) = \frac{\cos(\sigma)}{e^{2\sigma}}z(\sigma^-) \).

The \( \beta \)-level of fuzzy number 0, 1, 2 and 3 are

\[
[0]^\beta = [\beta - 1, 1 - \beta], \quad \beta \in [0, 1],
\]

\[
[1]^\beta = [\beta, 2 - \beta], \quad \beta \in [0, 1],
\]

\[
[2]^\beta = [\beta + 1, 3 - \beta], \quad \beta \in [0, 1],
\]

\[
[3]^\beta = [\beta + 2, 4 - \beta], \quad \beta \in [0, 1].
\]

Now, the \( \beta \)-level set of \( \mathcal{M}(\sigma, z(\sigma)) = 2\sigma z^2(\sigma) \) is

\[
[\mathcal{M}(\sigma, z(\sigma))]^\beta = [2\sigma z^2(\sigma)]^\beta
\]

\[
= [\sigma^2(\beta)(z^2(\sigma))]^\beta
\]

\[
= \sigma([\beta + 1]z^2(\sigma))^2, (3 - \beta)(z^2(\sigma))^2].
\]

Therefore,

\[
d_H([\mathcal{M}(\sigma, \xi(\sigma))]^\beta, [\mathcal{M}(\sigma, \zeta(\sigma))]^\beta)
\]

\[
= d_H([\beta + 1](\xi_p^\beta(\sigma))^2, (3 - \beta)(\xi_p^\beta(\sigma))^2], [\beta + 1](\zeta_p^\beta(\sigma))^2, (3 - \beta)(\zeta_p^\beta(\sigma))^2])
\]

\[
\leq \sigma \max\{[(\beta + 1)](\xi_p^\beta(\sigma))^2 - (\zeta_p^\beta(\sigma))^2], (3 - \beta)(\xi_p^\beta(\sigma))^2 - (\zeta_p^\beta(\sigma))^2]\}
\]

\[
\leq 3\sigma \max\{|\xi_p^\beta(\sigma) + \zeta_p^\beta(\sigma)|, |\xi_p^\beta(\sigma) - \zeta_p^\beta(\sigma)|, |\xi_p^\beta(\sigma) + \zeta_p^\beta(\sigma)|, |\xi_p^\beta(\sigma) - \zeta_p^\beta(\sigma)|\}
\]

\[
\leq 3\sigma|\xi_p^\beta(\sigma) + \zeta_p^\beta(\sigma)| d_H([\xi(\sigma)]^\beta, [\zeta(\sigma)]^\beta)
\]

\[
\leq l_1 d_H([\xi(\sigma)]^\beta, [\zeta(\sigma)]^\beta),
\]

where \( l_1 = 3\sigma|\xi_p^\beta(\sigma) + \zeta_p^\beta(\sigma)| \) satisfy the condition (A1).
Now, the $\beta$-level set of $\psi(\sigma, z(\sigma)) = 3\sigma z^2(\sigma)$ is
\[
[\psi(\sigma, z(\sigma))]^\beta = [3\sigma z^2(\sigma)]^\beta
= \sigma [3\sigma z^2(\sigma)]^\beta
= \sigma [(\beta + 2)(z_p^\beta(\sigma))^2, (4 - \beta)(z_q^\beta(\sigma))^2].
\]
Therefore,
\[
d_H([\psi(\sigma, \xi(\sigma))]^\beta, [\psi(\sigma, z(\sigma))]^\beta)
= d_H(\sigma[(\beta + 2)(z_p^\beta(\sigma))^2, (4 - \beta)(z_q^\beta(\sigma))^2],\sigma[(\beta + 2)(z_p^\beta(\sigma))^2, (4 - \beta)(z_q^\beta(\sigma))^2])
\leq \sigma\max\{(\beta + 2)|z_p^\beta(\sigma)|^2 - (z_q^\beta(\sigma))^2], (4 - \beta)|z_q^\beta(\sigma)|^2 - (z_p^\beta(\sigma))^2]\}
\leq 4\sigma\max\{|z_p^\beta(\sigma) + z_q^\beta(\sigma)|\max\{|z_p^\beta(\sigma) - z_q^\beta(\sigma)|, |z_q^\beta(\sigma) - z_p^\beta(\sigma)|\}
\leq 4\sigma[z_p^\beta(\sigma) + z_q^\beta(\sigma)]d_H([\xi(\sigma)]^\beta, [\xi(\sigma)]^\beta)
\leq l_2 d_H([\xi(\sigma)]^\beta, [\xi(\sigma)]^\beta),
\]
where $l_2 = 4\sigma|z_p^\beta(\sigma) + z_q^\beta(\sigma)|$ also satisfy the condition (A1).

Now, the $\beta$-level set of $3_1(\sigma, z(\sigma_1^-))$ is
\[
[3_1(\sigma, z(\sigma_1^-))]^\beta = \left[\frac{\cos(\sigma)}{e^\sigma} z(\sigma_1^-)\right]^\beta
= \frac{\cos(\sigma)}{e^\sigma}[(\beta, 2 - \beta]z(\sigma_1^-)]^\beta]
= \frac{\cos(\sigma)}{e^\sigma}[\beta z_p^\beta(\sigma_1^-), (2 - \beta)z_q^\beta(\sigma_1^-)].
\]
Therefore,
\[
d_H([3_1(\sigma, \xi(\sigma_1^-))]^\beta, [3_1(\sigma, z(\sigma_1^-))]^\beta)
= d_H(\frac{\cos(\sigma)}{e^\sigma}[\beta z_p^\beta(\sigma_1^-), (2 - \beta)z_q^\beta(\sigma_1^-)], \frac{\cos(\sigma)}{e^\sigma}[\beta z_p^\beta(\sigma_1^-), (2 - \beta)z_q^\beta(\sigma_1^-)])
\leq \max\{\beta \frac{\cos(\sigma)}{e^\sigma}|z_p^\beta(\sigma_1^-) - z_q^\beta(\sigma_1^-)|, (2 - \beta) \frac{\cos(\sigma)}{e^\sigma}|z_q^\beta(\sigma_1^-) - z_p^\beta(\sigma_1^-)|
\leq (2 - \beta) \frac{\cos(T)}{e^T} \max\{|z_p^\beta(\sigma_1^-) - z_q^\beta(\sigma_1^-)|, |z_q^\beta(\sigma_1^-) - z_p^\beta(\sigma_1^-)|
\leq (2 - \beta) \frac{\cos(T)}{e^T} d_H([\xi(\sigma_1^-)]^\beta, [\xi(\sigma_1^-)]^\beta)
\leq l_3 d_H([\xi(\sigma_1^-)]^\beta, [\xi(\sigma_1^-)]^\beta),
\]
where $l_3 = (2 - \beta) \frac{\cos(T)}{e^T}$ satisfies the condition (A2).

Thus, all the conditions of Theorem 3.1 are fulfilled. Hence, system (9) has a unique fuzzy solution.

Let target state $z(\sigma_1) = z(\sigma_{j+1}) = \tilde{z}$. Now, we can find the $\beta$-level set of control term $u(\sigma)$ as
Case 1: When $\sigma \in (0, \sigma_1]$, we have

$$[u(\sigma)]^\beta = [u_p^\beta(\sigma), u_q^\beta(\sigma)]$$

$$= [(G_p^\beta)^{-1}((\beta + 1) - S_p^\beta(\sigma_1)(\beta - 1)$$

$$+ \int_0^{\sigma_1} S_p^\beta(\sigma_1 - s)|\sigma(\beta + 1)(z_p^\beta(\sigma))^2|ds$$

$$- \int_0^{\sigma_1} S_p^\beta(\sigma_1 - s)|\sigma(\beta + 2)(z_p^\beta(\sigma))^2|ds + \sigma(\beta + 1)(z_p^\beta(\sigma))^2),$$

$$(G_q^\beta)^{-1}((3 - \beta) - S_q^\beta(\sigma_1)(\beta - 1)$$

$$+ \int_0^{\sigma_1} S_q^\beta(\sigma_1 - s)|\sigma(3 - \beta)(z_q^\beta(\sigma))^2|ds$$

$$- \int_0^{\sigma_1} S_q^\beta(\sigma_1 - s)|\sigma(4 - \beta)(z_q^\beta(\sigma))^2|ds + \sigma(3 - \beta)(z_q^\beta(\sigma))^2].$$

Case 2: When $\sigma \in (k_j, \sigma_{j+1}]$, $j = 1, 2, \cdots, n$, we have

$$[u(\sigma)]^\beta = [u_p^\beta(\sigma), u_q^\beta(\sigma)]$$

$$= [(G_p^\beta)^{-1}((\beta + 1) - S_p^\beta(\sigma_{j+1} - k_j)\frac{\cos(\sigma)}{e^{2\sigma}}\beta z_p^\beta(\sigma))$$

$$+ \int_{k_j}^{\sigma_{j+1}} S_p^\beta(\sigma_{j+1} - s)((\beta + 2)\sigma(\beta + 1)(z_p^\beta(\sigma))^2|ds$$

$$- \int_{k_j}^{\sigma_{j+1}} S_p^\beta(\sigma_{j+1} - k_j)((\beta + 2)\sigma(z_p^\beta(\sigma))^2|ds + \sigma(\beta + 1)(z_p^\beta(\sigma))^2),$$

$$(G_q^\beta)^{-1}((3 - \beta) - S_q^\beta(\sigma_{j+1} - k_j)\frac{\cos(\sigma)}{e^{2\sigma}}(2 - \beta)z_q^\beta(\sigma))$$

$$+ \int_{k_j}^{\sigma_{j+1}} S_q^\beta(\sigma_{j+1} - s)((4 - \beta)\sigma(3 - \beta)(z_q^\beta(\sigma))^2|ds$$

$$- \int_{k_j}^{\sigma_{j+1}} S_q^\beta(\sigma_{j+1} - k_j)((4 - \beta)(z_q^\beta(\sigma))^2|ds + \sigma(3 - \beta)(z_q^\beta(\sigma))^2].$$

Now, for $\sigma \in (0, \sigma_1]$, we have

$$[z(\sigma)]^\beta = [S_p^\beta(\sigma_1)(\beta - 1) - \int_0^{\sigma_1} S_p^\beta(\sigma_1 - s)|\sigma(\beta + 2)(\beta + 1)(z_p^\beta(\sigma))^2|ds$$

$$+ \int_0^{\sigma_1} S_p^\beta(\sigma_1 - s)|\sigma(\beta + 2)(z_p^\beta(\sigma))^2|ds$$

$$+ (G_p^\beta)(G_p^\beta)^{-1}((\beta + 1) - S_p^\beta(\sigma_1)(\beta - 1)$$

$$+ \int_0^{\sigma_1} S_p^\beta(\sigma_1 - s)|\sigma(\beta + 2)(z_p^\beta(\sigma))^2|ds$$

$$- \int_0^{\sigma_1} S_p^\beta(\sigma_1 - s)|\sigma(\beta + 2)(z_p^\beta(\sigma))^2|ds + \sigma(\beta + 1)(z_p^\beta(\sigma))^2$$

$$- \sigma(\beta + 1)(z_p^\beta(\sigma))^2, S_p^\beta(\sigma_1)(\beta - 1$$

$$- \int_0^{\sigma_1} S_q^\beta(\sigma_1 - s)|\sigma(4 - \beta)(3 - \beta)(z_q^\beta(\sigma))^2|ds$$

$$+ \int_0^{\sigma_1} S_q^\beta(\sigma_1 - s)|\sigma(4 - \beta)(z_q^\beta(\sigma))^2|ds.$$
\[ + \left( G_{q}^{\beta} \right)^{-1} (3 - \beta) - S_{q}^{\beta} (\sigma_1) (\beta - 1) \]
\[ + \int_{0}^{\sigma_1} S_{q}^{\beta} (\sigma_1 - s) \sigma (4 - \beta) (3 - \beta) (z_{q}^{\beta} (\sigma))^2 ds \]
\[ - \int_{0}^{\sigma_1} S_{q}^{\beta} (\sigma_1 - s) \sigma (4 - \beta) (z_{q}^{\beta} (\sigma))^2 ds + \sigma (3 - \beta) (z_{q}^{\beta} (\sigma))^2 - \sigma (3 - \beta) (z_{q}^{\beta} (\sigma))^2 \]
\[ = ((\beta + 1), (3 - \beta)) \]
\[ = (\tilde{2})^{\beta}. \]

Similarly, for \( \sigma \in (k_1, \sigma_{j+1}], j = 1, 2, \ldots, n \), we can find that \( z(\sigma_{j+1})^{\beta} = (\tilde{2})^{\beta} \).
Thus, all the condition of Theorem 4.5 are satisfied. Hence, system (9) is exact controllable.

6. Conclusion. In this manuscript, by using the concept of fuzzy number, Banach fixed point theorem and Non-linear functional analysis, we established the existence of a unique solution of Sobolev type fuzzy differential system (1). In addition, we examine exact controllability results of system (1). In the last, we have given an example to validate the obtained analytical results. In future, we can study the existence, uniqueness and controllability results for fractional fuzzy differential equation, stochastic fuzzy differential equations, Intuitionistic fuzzy differential equations, etc. Also, to study the dynamic behaviour of physical system with uncertainty, fuzzy differential equations is useful.

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Appendix 1. For any $\sigma \in (0, \sigma_1]$, $\xi, \zeta \in \mathbb{K}^n$, we have
\[
d_H([[(\Lambda_1)(\sigma)]^{\beta}, [[\Lambda_2](\sigma)]^{\beta})
= d_H([S(\sigma)[\xi_0 + \mathcal{M}(0, \xi_0)] - \mathcal{M}(\sigma, \xi(\sigma)) - \int_0^\sigma A S(\sigma - s) \mathcal{M}(s, \xi(s))
+ \int_0^\sigma S(\sigma - s) \psi(s, \xi(s)) ds \right]^\beta, [S(\sigma)[\xi_0 + \mathcal{M}(0, \xi_0)] - \mathcal{M}(\sigma, \xi(\sigma)) - \int_0^\sigma A S(\sigma - s) \mathcal{M}(s, \xi(s))
+ \int_0^\sigma S(\sigma - s) \psi(s, \xi(s)) ds \right]^\beta])
\leq d_H([-\int_0^\sigma A S(\sigma - s) \mathcal{M}(s, \xi(s)) + \int_0^\sigma S(\sigma - s) \psi(s, \xi(s)) ds \right]^\beta, [-\int_0^\sigma A S(\sigma - s) \mathcal{M}(s, \xi(s)) + \int_0^\sigma S(\sigma - s) \psi(s, \xi(s)) ds \right]^\beta)
\leq \int_0^\sigma d_H([-\int_0^\sigma A S(\sigma - s) \mathcal{M}(s, \xi(s)) + S(\sigma - s) \psi(s, \xi(s)) ds \right]^\beta ds,
\leq \int_0^\sigma d_H([-\int_0^\sigma A S(\sigma - s) \mathcal{M}(s, \xi(s)) + S(\sigma - s) \psi(s, \xi(s)) ds \right]^\beta ds,
\leq \int_0^\sigma d_H([-\int_0^\sigma A S(\sigma - s) \mathcal{M}(s, \xi(s)) + S(\sigma - s) \psi(s, \xi(s)) ds \right]^\beta ds,
\leq \int_0^\sigma \max \left([(-\int_0^\sigma A S(\sigma - s) \mathcal{M}(s, \xi(s)) + S(\sigma - s) \psi(s, \xi(s)) ds \right]^\beta ds,
- \int_0^\sigma \max \left([(-\int_0^\sigma A S(\sigma - s) \mathcal{M}(s, \xi(s)) + S(\sigma - s) \psi(s, \xi(s)) ds \right]^\beta ds,
\right)\right),
\]
Thus,

\[
\left| \left[ (-\mathcal{A}S^\beta_q (\sigma - s) \mathcal{H}^\beta_q (s, \xi(s)) + S^\beta_q (\sigma - s) \psi^\beta_q (s, \xi(s))) \right]
\right|
\leq \left| \left[ (-\mathcal{A} + I) S^\beta_p (\sigma - s) \left( [\mathcal{H}^\beta_p (s, \xi(s)) + \psi^\beta_p (s, \xi(s))] - [\mathcal{H}^\beta_q (s, \xi(s)) + \psi^\beta_q (s, \xi(s))] \right) \right| \right|
\]

\[
\leq \int_0^\sigma \max \left( \left| \left[ (-\mathcal{A} + I) S^\beta_p (\sigma - s) \left( [\mathcal{H}^\beta_p (s, \xi(s)) + \psi^\beta_p (s, \xi(s))] - [\mathcal{H}^\beta_q (s, \xi(s)) + \psi^\beta_q (s, \xi(s))] \right) \right| \right| \right) \]

\[
= \left| -\mathcal{A} + I \right| \left| \mathcal{A} + I \right| \int_0^\sigma d_\mathcal{H} \left( [\mathcal{H}(s, \xi(s))]^\beta, [\mathcal{H}(s, \zeta(s))]^\beta \right) ds
\]

\[
+ \left| -\mathcal{A} + I \right| \left| \mathcal{A} + I \right| \int_0^\sigma d_\mathcal{H} \left( [\psi(s, \xi(s))]^\beta, [\psi(s, \zeta(s))]^\beta \right) ds
\]

\[
\leq \left| -\mathcal{A} + I \right| \left| \mathcal{A} + I \right| ML \int_0^\sigma d_\mathcal{H} \left( [\mathcal{H}(s)]^\beta, [\mathcal{H}(s)]^\beta \right) ds
\]

\[
= \left| -\mathcal{A} + I \right| \left| \mathcal{A} + I \right| \int_0^\sigma d_\mathcal{H} \left( [\psi(s)]^\beta, [\psi(s)]^\beta \right) ds
\]

\[
\leq M L A^* \int_0^\sigma d_\mathcal{H} \left( [\mathcal{H}(s)]^\beta, [\mathcal{H}(s)]^\beta \right) ds,
\]

where, \( A^* = \left| -\mathcal{A} + I \right| \).

Thus,

\[
d_\mathcal{H} \left( [(\mathcal{A}\zeta)(\sigma)]^\beta, [(\mathcal{A}\zeta)(\sigma)]^\beta \right) \leq M L A^* \int_0^\sigma d_\mathcal{H} \left( [\mathcal{H}(s)]^\beta, [\mathcal{H}(s)]^\beta \right) ds.
\]

**Appendix 2.** For any \( \sigma \in (k_j, \sigma_{j+1}], \xi, \zeta \in \mathbb{K}^n \), we have

\[
(\mathcal{A}\zeta)(\sigma) = S(\sigma - k_j) (\mathcal{Y}(k_j, \xi((\sigma^-))) + \mathcal{Y}(0, \xi_0)) - \mathcal{Y}(\sigma, \xi(\sigma))
\]

\[
- \int_{k_j}^\sigma A S(\sigma - s) \mathcal{H}(s, \xi(s)) + \int_{k_j}^\sigma S(\sigma - s) \psi(s, \xi(s)) ds
\]

and

\[
(\mathcal{A}\zeta)(\sigma) = S(\sigma - k_j) (\mathcal{Y}(k_j, \xi((\sigma^-))) + \mathcal{Y}(0, \xi_0)) - \mathcal{Y}(\sigma, \zeta(\sigma))
\]

\[
- \int_{k_j}^\sigma A S(\sigma - s) \mathcal{H}(s, \zeta(s)) + \int_{k_j}^\sigma S(\sigma - s) \psi(s, \zeta(s)) ds.
\]

Therefore,

\[
d_\mathcal{H} \left( [(\mathcal{A}\zeta)(\sigma)]^\beta, [(\mathcal{A}\zeta)(\sigma)]^\beta \right)
\]

\[
= d_\mathcal{H} \left( [S(\sigma - k_j) (\mathcal{Y}(k_j, \xi((\sigma^-))) + \mathcal{Y}(0, \xi_0)) - \mathcal{Y}(\sigma, \xi(\sigma))
\]

\[
- \int_{k_j}^\sigma A S(\sigma - s) \mathcal{H}(s, \xi(s)) + \int_{k_j}^\sigma S(\sigma - s) \psi(s, \xi(s)) ds] \right)^\beta,
\]
\begin{equation*}
\begin{aligned}
&\left[S(\sigma - k_j)\{\mathcal{H}_j(k_j, \zeta(\sigma_j^-)) + \mathcal{M}(0, \zeta_0)\} - \mathcal{M}(\sigma, \zeta(\sigma))
\right.
\left. - \int_{k_j}^{\sigma} \mathcal{A}S(\sigma - s)\mathcal{M}(s, \zeta(s)) + \int_{k_j}^{\sigma} S(\sigma - s)\psi(s, \zeta(s))ds\right]^\beta
\end{aligned}
\end{equation*}
\begin{equation*}
\begin{aligned}
&= d_\mathcal{H}\left(\left[S(\sigma - k_j)\{\mathcal{H}_j(k_j, \zeta(\sigma_j^-)) + \mathcal{M}(0, \zeta_0)\}\right]^\beta
\left. - \left[\mathcal{M}(\sigma, \zeta(\sigma))\right]^\beta\right.
\left. + \left[- \int_{k_j}^{\sigma} \mathcal{A}S(\sigma - s)\mathcal{M}(s, \zeta(s)) + \int_{k_j}^{\sigma} S(\sigma - s)\psi(s, \zeta(s))ds\right]^\beta,
\right.
\left. \left[S(\sigma - k_j)\{\mathcal{H}_j(k_j, \zeta(\sigma_j^-)) + \mathcal{M}(0, \zeta_0)\}\right]^\beta
\left. + \left[- \int_{k_j}^{\sigma} \mathcal{A}S(\sigma - s)\mathcal{M}(s, \zeta(s)) + \int_{k_j}^{\sigma} S(\sigma - s)\psi(s, \zeta(s))ds\right]^\beta\right)'
\end{aligned}
\end{equation*}
\begin{equation*}
\begin{aligned}
&\leq d_\mathcal{H}\left(\left[S(\sigma - k_j)\{\mathcal{H}_j(k_j, \zeta(\sigma_j^-))\}, S_\mathcal{E}(\sigma - k_j)\{\mathcal{E}_j(k_j, \zeta(\sigma_j^-))\}\right]
\end{aligned}
\end{equation*}
\begin{equation*}
\begin{aligned}
&\left[\mathcal{S}_p^\beta(\sigma - k_j)\{\mathcal{H}_j^\beta(p_j, \zeta(\sigma_j^-))\}, \mathcal{S}_q^\beta(\sigma - k_j)\{\mathcal{E}_j^\beta(q_j, \zeta(\sigma_j^-))\}\right]
\end{aligned}
\end{equation*}
\begin{equation*}
\begin{aligned}
&+ \int_{k_j}^{\sigma} d_\mathcal{H}\left(\left[- \mathcal{A}S(\sigma - s)\mathcal{M}(s, \zeta(s)) + S(\sigma - s)\psi(s, \zeta(s))ds\right]^\beta,\right.
\left. + \left[- \mathcal{A}S(\sigma - s)\mathcal{M}(s, \zeta(s)) + S(\sigma - s)\psi(s, \zeta(s))ds\right]^\beta\right)'
\end{aligned}
\end{equation*}
\begin{equation*}
\begin{aligned}
&\leq d_\mathcal{H}\left(\left[S_\mathcal{E}(\sigma - k_j)\{\mathcal{H}_j^\beta(p_j, \zeta(\sigma_j^-))\}, S_\mathcal{E}(\sigma - k_j)\{\mathcal{E}_j^\beta(q_j, \zeta(\sigma_j^-))\}\right]
\end{aligned}
\end{equation*}
\begin{equation*}
\begin{aligned}
&\left[\mathcal{S}_p^\beta(\sigma - k_j)\{\mathcal{H}_j^\beta(p_j, \zeta(\sigma_j^-))\}, \mathcal{S}_q^\beta(\sigma - k_j)\{\mathcal{E}_j^\beta(q_j, \zeta(\sigma_j^-))\}\right]
\end{aligned}
\end{equation*}
\begin{equation*}
\begin{aligned}
&+ \int_{k_j}^{\sigma} d_\mathcal{H}\left(\left[- \mathcal{A}S_\mathcal{E}(\sigma - s)\mathcal{M}_\mathcal{E}(s, \zeta(s)) + \mathcal{S}_\mathcal{E}(\sigma - s)\psi_\mathcal{E}(s, \zeta(s)),\right.ight.
\left. \mathcal{S}_\mathcal{E}(\sigma - s)\mathcal{M}_\mathcal{E}(s, \zeta(s)) + \mathcal{S}_\mathcal{E}(\sigma - s)\psi_\mathcal{E}(s, \zeta(s))\right)
\end{aligned}
\end{equation*}
Thus,
\[
\max \left( \left| S_p^\beta (\sigma - k_j) (3_{j^p} \delta_k (k_j, \xi(\sigma^-)) - S_p^\beta (\sigma - k_j) (3_{j^p} \delta_k (k_j, \zeta(\sigma^-))) \right|, \right.
\]
\[
\left. \left| S_q^\beta (\sigma - k_j) (3_{j^q} \delta_k (k_j, \xi(\sigma^-)) - S_q^\beta (\sigma - k_j) (3_{j^q} \delta_k (k_j, \zeta(\sigma^-))) \right| \right)
\]
\[
+ \int_{k_j}^{\sigma} \max \left( \left| (\mathcal{M}_p^\beta (s, \xi(s)) + \psi_p^\beta (s, \xi(s))) \right|, \right.
\]
\[
\left. \left| \mathcal{A} + I \right| S_q^\beta (\sigma - s) (\mathcal{M}_q^\beta (s, \xi(s)) + \psi_q^\beta (s, \xi(s))) \right| \right)
\]
\[
\leq \max \left( \left| S_p^\beta (\sigma - k_j) (3_{j^p} \delta_k (k_j, \xi(\sigma^-)) - (3_{j^p} \delta_k (k_j, \zeta(\sigma^-))) \right|, \right.
\]
\[
\left. \left| S_q^\beta (\sigma - k_j) (3_{j^q} \delta_k (k_j, \xi(\sigma^-)) - (3_{j^q} \delta_k (k_j, \zeta(\sigma^-))) \right| \right)
\]
\[
+ \int_{k_j}^{\sigma} \max \left( \left| (\mathcal{M}_p^\beta (s, \xi(s)) + \psi_p^\beta (s, \xi(s))) \right|, \right.
\]
\[
\left. \left| \mathcal{A} + I \right| S_q^\beta (\sigma - s) (\mathcal{M}_q^\beta (s, \xi(s)) + \psi_q^\beta (s, \xi(s))) \right| \right)
\]
\[
\leq MD_H \left( (3_{j^p} \delta_k (k_j, \xi(\sigma^-)) - (3_{j^p} \delta_k (k_j, \zeta(\sigma^-))), (3_{j^q} \delta_k (k_j, \xi(\sigma^-)) - (3_{j^q} \delta_k (k_j, \zeta(\sigma^-))) \right)
\]
\[
+ | - \mathcal{A} + I | \mathcal{M} \int_{k_j}^{\sigma} d_H ([\mathcal{M}(s, \xi(s))]^\delta, [\mathcal{M}(s, \zeta(s))]^\delta) \right) ds
\]
\[
+ | - \mathcal{A} + I | \mathcal{M} \int_{k_j}^{\sigma} d_H ([\psi(s, \xi(s))]^\delta, [\psi(s, \zeta(s))]^\delta) \right) ds
\]
\[
\leq MD_H (\mathcal{M}(s, \xi(\sigma^-))]^\delta, [\mathcal{M}(s, \zeta(\sigma^-))]^\delta) + | - \mathcal{A} + I | ML \int_{k_j}^{\sigma} d_H ([\xi(s)]^\delta, [\zeta(s)]^\delta) \right) ds
\]
\[
+ | - \mathcal{A} + I | ML \int_{k_j}^{\sigma} d_H ([\xi(s)]^\delta, [\zeta(s)]^\delta) \right) ds
\]
\[
\leq M_M d_H ([\xi(\sigma^-)]^\delta, [\zeta(\sigma^-)]^\delta) + MLA^* \int_{k_j}^{\sigma} d_H ([\xi(s)]^\delta, [\zeta(s)]^\delta) \right) ds.
\]
Thus,
\[
d_H ([\mathcal{L}(\sigma)]^\delta, [(\mathcal{L}(\sigma)]^\delta) \leq M_M d_H ([\xi(\sigma^-)]^\delta [\zeta(\sigma^-)]^\delta)
\]
\[
+ MLA^* \int_{k_j}^{\sigma} d_H ([\xi(s)]^\delta, [\zeta(s)]^\delta) \right) ds.
\]