Yang-Mills thermodynamics

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Received October 4, 2007, in final form ????; Published online ????
Original article is available at http://www.emis.de/journals/SIGMA/2007/001/

Abstract. We present a quantitative analysis of Yang-Mills thermodynamics in 4D flat spacetime. The focus is on the gauge group SU(2). Results for SU(3) are mentioned in passing. Although all essential arguments and results were reported elsewhere we summarize them here in a concise way and offer a number of refinements and some additions.

Key words: holonomy; calorons; Polyakov loop; adjoint Higgs mechanism; spatial coarse-graining; deconfinement; renormalizability; Legendre transformation; unitary-Coulomb gauge; maximal resolution; loop expansion; irreducible bubble diagram; BPS monopole; monopole condensate; abelian Higgs mechanism; dual gauge field; preconfinement; center-vortex loop; spin-1/2 fermion; ’t Hooft loop; Hagedorn transition; condensate of paired center-vortex loops; total confinement; asymptotic series; $\lambda\phi^4$-theory in 1D; Borel summability and analytic continuation

2000 Mathematics Subject Classification: 70S15; 74A15; 82B10; 82B28

1 Introduction

It was Planck who first demonstrated the power of statistical methods in quantitatively understanding a gauge theory [1]. His important suggestion was to subject indeterministic phase and amplitude changes of a single resonator in the wall of a cavity – in thermal equilibrium with the contained electromagnetic radiation – to an averaging procedure dictated by the laws of (statistical) thermodynamics. Appealing to a known, classically derived result on gross features of the so-called black-body spectrum (Wien’s displacement law), postulating a partitioning of the total energy into multiples of a smallest unit, and appealing to Boltzmann’s statistical definition of entropy, Planck deduced his famous radiation law. As an aside, he discovered a universal quantum of action needed to relate the entropy (disorder) and mean energy of a single resonator to its frequency. The robustness of his result is demonstrated by the fact that even for physical objects sizably deviating from ideal black bodies Planck’s radiation law holds to a high degree of accuracy.

The purpose of the present article is to give a concise presentation of results, accumulated over the last four years, on generalizations of the thermal U(1) gauge theory studied by Planck: SU(2) and SU(3) Yang-Mills thermodynamics. It is possible that an SU(2) gauge symmetry, dynamically broken down to U(1) by a deconfining thermal ground state, underlies photon propagation [2] [3] [4] [5] [6]. We do not here consider SU(N) gauge theories with $\infty \geq N \geq 4$ the reason being nonunique phase diagrams [7] [8].

Yang-Mills thermodynamics strongly relates to the concept of emergent phenomena. On the most basic level, temperature itself is an emergent phenomenon depending on the fluctuating...
degrees of freedom defining it. Conversely, the mass of a magnetic monopole, which, as a short-lived field configuration contributes to the thermodynamics of the Yang-Mills ground state at high temperatures, is determined by temperature. That is, a single monopole owes its existence to the existence of all other fluctuating monopoles and antimonopoles in the ensemble.

In Yang-Mills thermodynamics various temperature-dependent emergent phenomena, facilitated by topologically nontrivial mappings from submanifolds of four dimensional spacetime into the (sub)manifold(s) of the gauge group, dominate the ground-state thermodynamics in three different phases. Albeit their microscopic dynamics is complex and not accessible to analytic treatment, a thermodynamically implied spatial coarse-graining down to a uniquely determined resolution, determined by the Yang-Mills scale and temperature, yields accurate and technically manageable representations of the (ultraviolet-regulated) partition function at any given temperature.

Here the term spatial coarse-graining refers to the process of eliminating short-wavelength gauge-field fluctuations in favor for effective fields and their couplings in a reformulation of the same partition function at lower and lower resolution and at a given temperature. This coarse-graining leads to the emergence of an effective action $S_{\text{eff}}$ being a functional of the effective fields which determines the weight for the functional integration over the latter in the reformulated partition function valid at a given maximal resolution.

Remarkably, starting out from exact BPS saturated solutions to the euclidean field equations in the deconfining phase, the derivation of the thermal ground state in the deconfining, high-temperature phase makes no reference to the way of how the continuum partition function of the theory is regularized in the ultraviolet. This is a consequence of the possibility to compute the expectation value of a uniquely determined operator, representing the phase $\hat{\phi}$ of an adjoint scalar field $\phi$, over suitable, topologically nontrivial, and BPS saturated configurations first and to subsequently investigate its average effect on integrated fluctuations and its direct effect on explicit field configurations in the topologically trivial sector. No reference is made to an ultraviolet regularization in the process. When decreasing the maximal resolution in the BPS saturated situation it is possible to deduce definite information about $\phi$’s modulus $|\phi|$ from its phase $\hat{\phi}$ since the process of spatial coarse-graining, which for $\hat{\phi}$ rapidly saturates the limit of vanishing maximal resolution at a finite maximal resolution, yields a spacetime homogeneous (and of course isotropic) value of $|\phi|$ starting from a maximal resolution determined by $|\phi|$.

That is, spatial coarse-graining over noninteracting, BPS saturated configurations in the (only admissible) sector with topological charge modulus $|Q| = 1$ dimensionally formally reduces this sector of the theory from $D = 4$ to $D = 1$ (quantum mechanics with periodic-in-time trajectories) at a finite maximal resolution. Thermodynamically, the gauge-invariant quantity $|\phi|$ must not carry energy (or momentum in 4D) because of its inherited BPS saturation. This is also the reason why no local vertices of the field $\phi$ with the topologically trivial, coarse-grained gauge field involving three or more external legs of the latter may exist: Such vertices would on the level of the effective theory convey energy-momentum transfer from the topologically trivial to the topologically nontrivial but BPS saturated sector of the theory which contradicts the very existence of the field $\phi$. Notice, however, that on the fundamental level topologically trivial fluctuations do interact with the topologically nontrivial sector exchanging energy-momentum associated with a resolution higher than $|\phi|$ thus having no visible effect on the field $\phi$. These interactions, introducing a temporary (anti)caloron holonomy in addition to short-range radiative corrections, are described by a pure-gauge configuration in the effective theory which in fact lifts the energy density of the thermal ground-state from zero to a finite positive value.

Despite the fact that the Stefan-Boltzmann limit is approached in a power-like and thus rapid

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2 An exception is the nonthermal behavior shortly below the Hagedorn transition.

3 Due to the nonfluctuating nature of the field $\phi$ or higher Lorentz-spin fields, potentially generated by a coarse-graining over the BPS sector of the fundamental theory, infinite-volume thermodynamics excludes the existence of the latter and demands homogeneity (constancy) of the former’s gauge invariant modulus.
way with increasing temperature the inherent resolution $|\phi|$ of the thermal Yang-Mills system decreases with temperature, and thus any memory of the short-distance regularization of the partition function is wiped out, see also [9] for the corresponding lattice observation (increasing delocalization of topological charge with increasing temperature). Conceptually, this is in accord with the situation known in zero-temperature perturbation theory where our ignorance about the values and UV regularization dependence of bare parameters is shown to be no obstacle to the predictivity of the quantum Yang-Mills theory since, upon their dressing at a finite resolution, only finitely many such parameters need to be fixed [10, 11, 12, 13].

Following the appreciated advice of a Referee, the presentation in this paper resorts to a hybrid style: A statement, whose validity is argued for in a more physical rather than rigorous mathematical way, and a number of definitions are highlighted by slanted script and are interspersed into the argumentation, a statement whose validity under stated assumptions is verified by direct calculation is presented according to custom within the mathematics literature, apologies to the irritated reader. But even though the subject presented (4D quantum field theory) still awaits its rigorous mathematical foundation the pragmatic line of pursuit followed in the present work unearths a number of unexpected, quantitatively very precisely representable facts which, as the author is convinced of, do not depend on a future, rigorous formulation of quantum field theory.

For maximal benefit it is recommended to read the present article in conjunction with [7]. That article contains extended discussions of the involved physics, explicit expressions for the relevant topological field configurations, and graphic displays of numerical results. Although the use of differential forms would simplify certain statements in the beginning of Sec.2 our presentation entirely resorts to the component notation.

The following conventions will be used: Einstein summation (summation over doubly occurring indices), solely lower-case indices for contractions in the euclidean formulation, lower-case and upper-case indices for contractions in the real-time formulation, and natural units ($\hbar = k_B = c = 1$).

The outline of this work is as follows: In Sec.2 we remind the reader of basic facts about thermal Yang-Mills gauge field theory. Sec.3 discusses the deconfining phase where a thermal ground state emerges upon a spatial coarse-graining over interacting, BPS saturated field configurations and where the Yang-Mills scale occurs as a purely nonperturbative constant of integration. We give tight estimates on the goodness of the finite-volume saturation of the coarse-graining process, and we account for the radiative corrections in the effective theory. The thermodynamics of the intermediate, preconfining phase is addressed in Sec.4. Here the unbroken abelian gauge symmetry of the deconfining phase is dynamically broken by monopole-antimonopole condensate(s). Emphasis is put on a discussion of supercooling which takes place because the switch from small to large caloron/anticaloron holonomy is energetically disfavored. Finally, in Sec.5 we elucidate the process of the decay of the monopole-antimonopole condensate of the preconfining phase giving rise to a zero-pressure and zero-energy density ground state in the confining phase. In that phase no propagating gauge modes exist, and the spectrum is represented by single or selfintersecting center-vortex loops which we interpret as spin-1/2 fermions. The naive series for the thermodynamic pressure represents an asymptotic expansion, and we show its Borel summability for complex values of the expansion parameter. Upon continuation to the physical regime a sign-definite imaginary part is encountered which for sufficiently small temperatures, however, is, smaller than the definite real part.
2 Thermal Yang-Mills theory

2.1 Euclidean formulation and symmetries

On a flat, four-dimensional euclidean spacetime with coordinates \(0 \leq \tau \leq \beta \equiv T^{-1}\) (time) and \(\bar{x}\) (infinite three-dimensional space) the partition function \(Z\) of a pure Yang-Mills gauge-field theory subject to the gauge group \(SU(N)\) is formally defined as

\[
Z \equiv \int A_\mu(\tau=0,\bar{x})=A_\mu(\tau=\beta,\bar{x}) \mathcal{D}A_\mu \exp[-S],
\]

where the gauge-field configuration \(A_\mu\) is Lie-algebra valued, \(A_\mu \equiv A^a_\mu t_a\), \((a = 1, \ldots, N^2 - 1)\), with the generators \(t_a\) in the fundamental representation normalized as \(\text{tr} t_a t_b = \frac{1}{2} \delta_{ab}\), and \(T\) is the temperature. The action \(S\) is defined as \(S = \frac{1}{2g^2} \text{tr} \int_0^\beta d\tau \int d^3x \, F_{\mu\nu} F^{\mu\nu}\) where \(g\) is a dimensionless coupling constant, \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu]\), \((\mu, \nu = 1, \ldots, 4)\), and the measure of (functional) integration is \(\mathcal{D}A_\mu \equiv \prod_{\tau,\bar{x},a} dA^a_\mu(\tau, \bar{x})\). Here the product is over the continuously varying values of the coordinates \(\tau, \bar{x}\), and over \(a = 1, \ldots, N^2 - 1\).

For the gauge group \(SU(2)\) we set \(t_a = \frac{1}{2} \lambda_a\) where \(\lambda_a\) are the Pauli matrices. The integration measure \(\mathcal{D}A_\mu \equiv \prod_{\tau,\bar{x}} dA^a_\mu(\tau, \bar{x})\) is ill-defined as it stands. If the field theory is endowed with an ultraviolet regularization then the infinite product is over a discrete index. We will argue that Yang-Mills thermodynamics, formally defined\(^4\) by Eq. (1), does not relate to the way how sense is made of the formal object in Eq. (1) by introducing a minimal length as long as the ultraviolet scale. As already discussed in the Introduction at a given temperature \(T\) a unique maximal resolution \(|\phi|\) appears to emerge. Since \(|\phi|\) decays like a power when increasing \(T\) it is guaranteed that the high-temperature physics is insensitive to any definite ultraviolet substantialization of Eq. (1). The action density \(\frac{1}{2g^2} \text{tr} F_{\mu\nu} F^{\mu\nu}\) in Eq. (1) is invariant under gauge transformations \(A_\mu \rightarrow \Omega A_\mu \Omega^\dagger + i \Omega \partial_\mu \Omega^\dagger\), where \(\Omega\) is an element of \(SU(N)\) in the fundamental representation, but the functional integration is carried out over gauge-in-equivalent, periodic-in-\(\tau\) gauge-field configurations.

In lattice definitions of the partition function in Eq. (1) one can show its invariance under temporally local center transformations, that is, under gauge transformations which are periodic up to a multiplication with a center element: \(\Omega(\tau = 0, \bar{x}) = Z \Omega(\tau = \beta, \bar{x})\) where \(Z \in \mathbb{Z}_N\) and thus the relevant gauge group actually is \(SU(N)/\mathbb{Z}_N\). If this (electric) center symmetry of the Yang-Mills action \(S\), subjected to the gauge group \(SU(N)\), is broken dynamically then a \(\mathbb{Z}_N\) degeneracy of the ground state must be present. Furthermore, the action \(S\) is invariant under continuous spatial rotations and translations. There is also an invariance w.r.t. time translations \(\tau \rightarrow \tau + \tau_0, 0 \leq \tau_0 \leq \beta\).

2.2 BPS saturated field configurations at finite temperature

The Euler-Lagrange equations, \(D_\mu F_{\mu\nu} = 0\) (stationarity of the action, \(\frac{\delta S}{\delta A_\mu} = 0\)) are solved by configurations \(A_\mu\) obeying the (anti)selfduality condition \(F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}\). Here \(D_\mu = \partial_\mu - i[A_\mu, \cdot]\) and \(\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\kappa\lambda} F_{\kappa\lambda}\) where \(\epsilon_{1234} = 1\). (Anti)selfdual configurations saturate the Bogomol’nyi bound on the action (BPS saturation):

\[
S = \frac{8\pi^2}{g^2} |Q|\]

where \(Q \equiv \frac{1}{32\pi^2} \int_0^\beta d\tau \int d^3x A^a_\mu \tilde{F}^a_{\mu\nu} \in \mathbb{Z}\) is the topological charge. \(Q\) is finite and quantized if according boundary conditions are imposed. At finite temperature we consider BPS saturated, finite-action configurations \(A_\mu\) which behave accordingly at spatial infinity and are

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\(^4\)This partition function implies Legendre transformations between formally defined thermodynamical quantities like pressure (minus free energy), energy density, and entropy density.
periodic in $\tau$. In this case $Q$ is an integer. As is usual, we refer to these configurations as calorons/anticalorons for a positive/negative sign of $Q$. (The presence of spatial boundaries, which are not at infinity, would explicitly break the translational invariance. We so far have no well backed up insight on how to treat Yang-Mills thermodynamics analytically in this case which, in a softened fashion, is also conveyed by external sources. A possibility for the treatment of mild distortions is to deform the undistorted situation adiabatically by letting $T \to T(\vec{x})$ and/or $\Lambda \to \Lambda(\vec{x})$ where $\Lambda$ denotes the Yang-Mills scale). Solutions to the Euler-Lagrange equations, which are not (anti)selfdual, have been constructed numerically, for recent work on axially symmetric finite-temperature and instanton/antiinstanton configurations and see [14] and [15], respectively.

**Proposition 1.** The euclidean energy-momentum tensor $\theta_{\mu\nu} \equiv -F_{\mu\lambda}^a F_{\nu \lambda}^a + \frac{1}{4} \delta_{\mu\nu} F_{\kappa\lambda}^a F_{\kappa \lambda}^a$ and every local, scalar composite of the form $\text{tr} t^a F_{\mu a} F_{\nu a}$, $\text{tr} t^a F_{\mu a} F_{\nu a}$, $\text{tr} t^a F_{\mu a} F_{\nu a} F_{\lambda a}$, $\text{tr} t^a F_{\mu a} F_{\nu a} F_{\lambda a}$, $\text{tr} t^a F_{\mu a} F_{\nu a} F_{\lambda a} F_{\mu a}$, \ldots vanish identically on a BPS saturated field configuration $A_\mu$.

**Proof.** Routine computation.

Notice that for $Q = 0$ we have $S = 0$ implying that calorons in this sector are pure gauges: $A_\mu = i\Omega \partial_\mu \Omega^\dagger$.

The Polyakov loop $P(\vec{x})[A]$ is defined as $P(\vec{x})[A] \equiv \mathcal{P} \exp[i \int_0^\beta A_4(\tau, \vec{x})]$ where the symbol $\mathcal{P}$ demands path-ordering.

A caloron/anticaloron is said to be of trivial holonomy if $P_\infty[A] \equiv \lim_{|\vec{x}| \to \infty} P(\vec{x})[A] \in$ center of the gauge group.

(Recall that for $SU(N)$: center $= \{\exp[2\pi i k/N] \mathbb{1}_N | k = 0, 1, \ldots, N - 1\} = \mathbb{Z}_N$.)

Notice that for the gauge group SU(N) this definition does not depend on the choice of gauge as long as $\Omega(\tau = 0, \vec{x}) = Z \Omega(\tau = \beta, \vec{x})$, where $Z \in \mathbb{Z}_N$, since $P_\infty[A] \xrightarrow{\Omega} \Omega(\tau = 0) P_\infty[A] \Omega^\dagger(\tau = \beta)$.

**Example 1.** For the gauge group SU(2) the following calorons/anticalorons (Harrington-Shepard (HS) [16]) are of trivial holonomy and of topological charge $Q = \pm 1$:

$$A_\mu^C(\tau, \vec{x}) = \eta_{\mu a}^0 t_a \partial_\mu \ln \Pi(\tau, r) \quad \text{(caloron, } Q = +1)$$

$$A_\mu^A(\tau, \vec{x}) = \eta_{\mu a}^0 t_a \partial_\mu \ln \Pi(\tau, r) \quad \text{(anticaloron, } Q = -1)$$

where the ’t Hooft symbols $\eta_{\mu a}$ and $\bar{\eta}_{\mu a}$ are defined as $\eta_{\mu a} = \epsilon_{\mu a}^\alpha + \delta_{\mu a}^\alpha \delta_{\nu 4} - \delta_{\nu a}^\alpha \delta_{\mu 4}$ and $\bar{\eta}_{\mu a} = \epsilon_{\mu a}^\alpha - \delta_{\mu a}^\alpha \delta_{\nu 4} + \delta_{\nu a}^\alpha \delta_{\mu 4}$, and the prepotential $\Pi$ is given as

$$\Pi(\tau, r) = 1 + \pi \rho^2 \frac{\sinh \left(\frac{2\pi r}{\beta}\right)}{\beta r}, (r \equiv |\vec{x}|).$$

The dimensionful modulus $\rho$ is inherited from the singular-gauge instanton configuration with prepotential $\Pi_0(\tau, r) = 1 + \frac{\beta^2}{\tau^2 + \tau^2}$ since $\Pi$ is obtained from $\Pi_0$ by superimposing its infinitely many images in mirrors placed at $\tau = 0$ and $\tau = \beta$ to generate periodicity in $\tau$. Additional moduli are the shifts $\tau \to \tau + \tau_z$, $\left(0 \leq \tau_z \leq \beta\right)$, and $\vec{x} \to \vec{x} + \vec{z}$ and, if one wishes, global gauge transformations.

**Example 2.** For the gauge group SU(2) there exist [14, 15, 16, 19, 20, 23] explicitly constructed [24, 25, 26, 27, 28] BPS calorons/anticalorons (Lee-Lu-Kraan-van-Baal (LLKvB)) of nontrivial holonomy and topological charge $Q = \pm 1$. For the trivial-holonomy case configurations with $|Q| > 1$ were constructed in [21, 22].
In contrast to their trivial-holonomy counterparts [29] these configurations are in isolation unstable under quantum deformation [42]. For a holonomy sufficiently close to trivial and for $\rho > 0$ the static BPS magnetic monopole and antimonopole constituents [39, 40, 41] attract under the influence of quantum fluctuations and thus eventually annihilate one another. This relaxes the LLKvB caloron or anticaloron back to the stable situation of a HS caloron or anticaloron. For a holonomy far from trivial and for $\rho > 0$ BPS magnetic monopole and antimonopole repulse [42] one another under the influence of quantum fluctuations. As a consequence, the large-holonomy LLKvB caloron or anticaloron dissociates into a pair of a screened BPS magnetic monopole and its antimonopole. Screening occurs due to the presence of short-lived magnetic dipoles that are provided by intermediary small-holonomy LLKvB calorons and anticalorons.

In the seminal work [29] nontrivial-holonomy calorons/anticalorons were argued to not contribute to the partition function based on the observation that quantum corrections produce a term in their effective action which diverges like the three-volume of the system. This argument is certainly correct if holonomy is considered a quantity that is externally sustained at a definite value. Hence no explicit integration over the holonomy must occur in any first-principle evaluation of the (ultraviolet regularized) partition function at sufficiently large temperature. However, viewed as a dynamical, short-lived quantity nontrivial holonomy does occur through the quantum induced deformation of the trivial-holonomy case. Partially based on theoretical work on fermionic zero modes, used as a diagnostics for lumps of topological charge [35, 36] to avoid the application of a cooling procedure to a given configuration, this is impressively demonstrated by many lattice investigations. A nonexhaustive list of references is [9, 30, 31, 32, 33, 34].

The construction of nontrivial-holonomy calorons/anticalorons of higher topological charge for a Yang-Mills theory subject to the gauge group SU(N) was investigated by Bruckmann and van Baal in [37] and the explicit form of the solution was given for $|Q| = 2$. The interesting result is that these calorons possess $n|Q|$ constituents monopoles whose sum of magnetic charges (with respect to $U(1)^{N-1}$) is nil. Recently, a nontrivial-holonomy caloron of $Q = 2$ and nonvanishing, overall magnetic charge was constructed [38].

**Remark 1.** On the level of the euclidean saddlepoint one has for the masses $m_1$ of a BPS magnetic monopole and $m_2$ of its antimonopole inside a LLKvB caloron: $m_1 + m_2 = 8\pi^2T$ [24]. Thus already on the classical level one observes the remarkable fact that the emergence of a particular monopole depends on the emergence of temperature or in other words on the existence of all other fluctuating monopoles and antimonopoles and propagating gauge fields in the ensemble.

### 2.3 Propagating fields at finite temperature: $Q=0$

In a given gauge and in the euclidean formulation the topologically trivial sector $\{\delta A_\mu\}$ is represented by a superposition of plane waves:

$$\delta A_\mu(\tau, \vec{x}) = \sum_{n=-\infty}^{n=\infty} \exp \left[ 2\pi in \frac{\tau}{\beta} \right] \delta \tilde{A}_{\mu,n}(\vec{x})$$

(3)

where $\delta \tilde{A}_{\mu,n}(\vec{x}) = \int d^3k \alpha_{\mu,n}(k) \exp[ik \cdot \vec{x}]$ and the function $\alpha_{\mu,n}(\vec{k})$ falls off sufficiently fast in $|\vec{k}|$ and in $n$ for the integrals and the sum in Eq. (3) to exist, respectively. The quantity $\frac{2\pi n}{\beta}$ is called $n$th Matsubara frequency.

**Remark 2.** Upon a Wick rotation $\tau \rightarrow it$, $(t$ real), one shows that the propagator of the field $\delta A_\mu$ decomposes into a quantum part (describing a particle of four-momentum $p$ possibly being off the mass shell, $p^2 \equiv p^\mu p_\mu \neq 0$) and a thermal part (describing thermalized on-shell propagation), see for example [43].
3 Deconfining phase

3.1 Thermal ground state: Interacting calorons/anticalorons

If not stated otherwise we consider the gauge group SU(2) from now on. We perform a spatial coarse-graining over the sector of nontrivially BPS saturated (nonpropagating) field configurations in singular gauge to arrive at a nonpropagating adjoint scalar field $\phi$ of spacetime independent modulus. Our strategy is to derive $\phi$’s second-order equation of motion and, by requiring compatibility with BPS saturation, to subsequently determine the field $\phi$ (modulus and phase) in terms of $T$ and a constant of integration $\Lambda$. The perturbative renormalizability of the sector with propagating gauge fields ($Q = 0$) \cite{10, 11, 12, 13} and the requirement of gauge invariance then yield a unique effective action which is associated with a maximal resolution given by $\phi$’s modulus.

If in the effective action a spatially homogeneous composite field emerges after spatial coarse-graining over the sector of nontrivially BPS saturated field configurations of trivial holonomy then this composite is a scalar under rotations (O(3) scalar) and transforms in the adjoint representation of the gauge group SU(2).

Note 1. The $A_4$-component and (nonlocal) products thereof are O(3) scalars only in covariant gauges.

As stated in the Introduction, the term ‘spatial coarse-graining’ refers to a lowering of the maximal resolution available in the system at a given temperature when keeping the partition function fixed. The term ‘effective action’ refers to minus the exponent in the weight according to which an average over configurations is performed in the partition function after spatial coarse-graining.

exclusion of explicit nontrivial holonomy: As shown in \cite{29}, calorons/anticalorons with explicit nontrivial holonomy induce a one-loop effective action which diverges with the three-volume of the system. Thus explicit holonomy must not enter the process of spatial coarse-graining in the caloron/anticaloron sector. (Short-lived implicit holonomy, however, emerges by quantum deformation of the trivial-holonomy case and is responsible for the generation of short-lived magnetic dipoles (large thermodynamic weight) or screened magnetic monopoles and antimonopoles (very small thermodynamic weight).

transformation property under O(3): Since nontrivially BPS saturated field configurations of trivial holonomy are nonpropagating field configurations their coarse-grained counterparts represent spatially homogeneous background fields in the effective theory. But the existence of a nontrivial O(3) tensor after coarse-graining would spontaneously break rotational invariance which is impossible in the absence of microscopic degrees of freedom that single out a direction in space at vanishing momentum.

gauge transformation property: The scalar $\phi$ must transform homogeneously under a change of gauge for otherwise the coarse-grained gauge field $\delta A_\mu$ would have to form a composite to couple to $\phi$ in a gauge-invariant way. The existence of such a composite on the level of the effective action would, however, contradict perturbative renormalizability \cite{10, 11, 12, 13} which states that all propagating degrees of freedom are represented by $\delta A_\mu$ itself. The only homogeneously

\footnote{This is the statement that the only effect of integrating out fluctuations in the sector with $Q = 0$ is a resolution dependence of the gauge coupling and the normalization of a plane wave in a given gauge. Modulo these radiatively generated effects the effective action describing fluctuations in the sector with $Q = 0$ has the same form as the fundamental action. In the absence of external sources probing the thermal system its inherent resolution $|\phi|$ is a function of temperature. As a consequence, the effective gauge coupling is a function of temperature and, in the (only) physical gauge, the wave function normalization is the characteristic function $\chi$: $\chi$ equals unity if the plane wave resolves its environment by less than $|\phi|$, and $\chi$ equals zero if the plane wave would resolve by more than $|\phi|$ since such a fluctuation already is integrated out.}
transforming, nontrivial quantities in the fundamental theory are (nonlocal) products of the field strength $F_{\mu\nu}$. Since

$$t_a t_b = \frac{1}{2} \{t_a, t_b\} + \frac{1}{2} [t_a, t_b] = \frac{1}{2} \left( \frac{1}{2} \delta_{ab} I_2 + i \epsilon_{abc} t_c \right)$$

we may without restriction of generality schematically write

$$\phi^{a_1 \ldots a_K} \sim \text{tr} \left( t^{a_1} \ldots t^{a_K} F \ldots F \right), \quad (K \geq 1)$$

with appropriate contractions of the Lorentz indices and parallel transports for the field strength $F \equiv F^a t_b$ implied. By virtue of Eq. (4) this can always be decomposed into spin-0 and spin-1 representations of SU(2). The case of the spin-0 representation (gauge-invariant composite) is irrelevant because it decouples and no energy-momentum is associated with it, see proposition \[\text{Proposition 1}\]. Thus we are left with the spin-1 representation which proves the claim.

**Notation:** We denote by $\{(\tau, 0), (\tau, \vec{x})\}$ the spacelike Wilson line $\mathcal{P} \exp \left[ i \int_{(\tau,0)}^{(\tau,\vec{x})} dz A_\mu (z) \right]$ where the path of integration is a straight line.

The following is the unique definition for the set $K$ of $\tau$-dependent algebra-valued functions which contains $\phi$’s phase $\hat{\phi} \equiv \frac{\phi}{|\phi|^2}$, $|\phi|^2 \equiv \text{tr} \frac{1}{2} \phi^2$:

$$K = \left\{ \sum_{\alpha=C,A} \int d^3 x \int d\rho \text{ tr } \bar{t} F_{\mu\nu}(\tau, \vec{0}) \alpha \left\{ (\tau, \vec{0}), (\tau, \vec{x}) \right\}_\alpha F_{\mu\nu}(\tau, \vec{x}) \alpha \left\{ (\tau, \vec{x}), (\tau, \vec{0}) \right\}_\alpha \right\}, \quad (5)$$

where the sum is over a HS caloron and anticaloron (singular gauge) and $\bar{t} \equiv (t^1, t^2, t^3)$. Explicitly, the set $K$ is parametrized by the coordinates of the caloron/anticaloron center $z_{C,A} = (\tau_{C,A}, \vec{z}_{C,A})$. The integrals are over infinite space and the entire range $0 \leq \rho \leq \infty$ of the scale parameter $\rho$, and, as we shall see, the set $K$ is implicitly parametrized by arbitrary rescalings and global gauge transformations.

**HS caloron/anticaloron:**

Again, explicit holonomy is excluded by the result of the semiclassical calculation in [29] but implicit, short-lived holonomy emerges by the quantum deformation of the trivial-holonomy case. **local definition:**

This is excluded by Proposition [1].

**curved path for evaluation of Wilson line $\{(\tau, 0), (\tau, \vec{x})\}$:**

Since the path is purely spacelike there exists no mass scale on the level of BPS saturated field configurations to parameterize curvature.

**higher $n$-point functions:**

Since $K$ contains the dimensionless phase $\hat{\phi}$ all its members must be dimensionless. Considering nonlocal, $n$-fold products of $F_{\mu\nu}$ with $n > 2$ together with the associated additional space integrations, a factor of $\beta^{2-n}$ would have to be introduced to make these contributions dimensionless. Since there are no explicit dependences on $\beta$ on the level of BPS saturated field configurations, see beginning of Sec. 2.2, this possibility does not exist.

**moduli-space average:**

(i) Integrations over the dimensionful moduli $\rho$ and $\tau_{C,A}, \vec{z}_{C,A}$ must have a flat measure since the members of $K$ make no reference to any scale on the level of BPS saturated field configurations.

(ii) The right-hand side of Eq. (5) transforms in the adjoint representation. Shifting the caloron/anticaloron spatial center from $\vec{0}$ to $\vec{z}_{C,A}$ and honoring spherical symmetry, an additional pair of Wilson lines would have to be introduced to parallel transport $F_{\mu\nu}$ from $(0, \vec{0})$ to $(0, \vec{z}_{C,A})$. Integrating then over $\vec{z}_{C,A}$ yields zero. (If this integral would not vanish then the scale $\beta$ would occur explicitly in the definition of the dimensionless members of $K$. But this is forbidden on
the level of BPS saturation, see beginning of Sec. 2.2.)

(iii) An integration over \( \tau_{C,A} \) yields zero. (The case of a constant term in the Fourier series associated with the integrand again would imply that the scale \( \beta \) occurs explicitly in the definition of the dimensionless members of \( K \).)

(iv) Since the members of \( K \) are gauge-variant objects integrations over the global gauge orientations of the caloron/anticaloron yield zero and thus are forbidden.

\[ \text{shift} \, \vec{0} \to \vec{y} \neq 0: \] Shifting \( \vec{0} \to \vec{y} \) in Eq. (5) but leaving the caloron/anticaloron center fixed at \( \vec{0} \), spherical symmetry would imply the need for an additional pair of Wilson lines to connect \( \vec{y} \) with \( \vec{0} \). But this is just a global gauge rotation of the unshifted situation and thus does not alter \( K \).

**caloron/anticaloron with \( |Q| > 1 \):**

Besides the translational moduli there are \( m > 1 \) dimensionful moduli in such a configuration. For example, a trivial-holonomy caloron with \( Q = 2 \) has three dimensionful moduli: two scale parameters and the distance between the two centers of its topological charge. Considering an \( n \)-point function (\( n \) nonlocal factors of the field strength \( F_{\mu\nu} \)) with \( n - 1 \) integrations over space and a flat-measure integration over the \( m \) dimensionful moduli (not counting the shift moduli) of the caloron, we arrive at a mass dimension \( 2n - 3(n - 1) - m = 3 - n - m \) of the object. To avoid the introduction of explicit powers of \( \beta \) (BPS saturation) in the definition of \( K \) this mass dimension needs to vanish. But for \( n \geq 2 \) and \( m > 1 \) we have \( 3 - n - m \neq 0 \).

**Proposition 2.** The Wilson line \( \{(\tau, \vec{0}), (\tau, \vec{x})\}_{C,A} \) evaluates to

\[ \{ (\tau, \vec{0}), (\tau, \vec{x}) \}_{C,A} = \cos g \pm \bar{2}i t_b \frac{x^b}{r} \sin g , \]

where \( g = g(\tau, r, \rho) = g(\beta \hat{\tau}, \beta \hat{r}, \beta \hat{\rho}) \equiv \hat{g}(\hat{r}, \hat{\rho}) \equiv \int_0^1 ds \frac{d}{d \tau} \log \Pi(\tau, \sigma r, \rho) \). The \( + \) or \(-\) sign relates to a caloron or an anticaloron, respectively. Explicitly, one has

\[ \hat{g} = -\pi^2 \rho^2 \sin(2\pi \hat{r}) \int_0^1 ds \left( \frac{\sinh(2\pi \hat{s})}{2 \pi \rho^2} \right) \frac{\sinh(2\pi \hat{s})}{[\cosh(2\pi \hat{s}) - \cos(2\pi \hat{r})][\cosh(2\pi \hat{s}) - \cos(2\pi \hat{r}) + \frac{\pi \rho^2}{2} \sinh(2\pi \hat{s})]} . \]

The function \( \hat{g} \) exists and approaches constancy in \( \hat{r} \) more than exponentially fast with increasing \( \hat{r} > 1 \).

**Note 2.** To point out essential properties only we have set the phases \( \tau_C \) and \( \tau_A \) of the \( \tau \) dependences of caloron and anticaloron equal to zero. They can easily be reinstated by letting \( \tau \to \tau + \tau_{C,A} \).

**Proof.**

**irrelevance of path-ordering:**

Observe that \( \int_{(\tau, \vec{0})}^{(\tau, \vec{x})} dz_\mu A_\mu(z) |_{C,A} = \pm t_b x^b \partial_r \int_0^1 ds \log \Pi(\tau, s r, \rho) \). That is, the integrand in the exponent of \( \{(\tau, \vec{0}), (\tau, \vec{x})\}_{C,A} \) varies along a fixed direction in the Lie algebra of SU(2). Path-ordering thus can be omitted.

**explicit form of \( \{(\tau, \vec{0}), (\tau, \vec{x})\}_{C,A} \):** Routine computation.

**existence of \( \hat{g} \) for all values of its arguments:**

The potentially problematic point in the domain of integration is \( s = 0 \) for \( \hat{r} = k \in \mathbb{Z} \). By Taylor expanding the sine function in front of the integral in Eq. (7) about \( \hat{r} = k \) and by Taylor expanding the cosine and the cosine hyperbolic functions in the denominator of the integrand about \( \hat{r} = k \) and \( s = 0 \), respectively, one easily checks that the limit \( \hat{r} \to k \) exists when \( \hat{\rho} \geq 0 \).
and $\hat{r} \geq 0$.

**saturation property for growing $\hat{r} > 0$**:

Split the integration in Eq. (7) as $I = \int_0^1 ds I_1 + I_2 \equiv \int_0^{\frac{\pi}{2}} ds + \int_{\frac{\pi}{2}}^1 ds = \int_0^1 dz + \int_{\frac{\pi}{2}}^1 dz$. $I_1$ does not depend on $\hat{r}$. The integrand $I$ is given as

$$I(z, \hat{r}, \hat{\tau}) \equiv \frac{\sinh z}{z[\cosh z - \cos(2\pi \hat{r})][\cosh z - \cos(2\pi \hat{r}) + \frac{2(\pi \hat{r})^2}{z} - \sinh z]}.$$ (8)

For the integration in $I_2$ the integrand $I$ is bounded from above as

$$I(z, \hat{r}, \hat{\tau}) \leq \frac{2e^z}{(e^z - 2 \cos(2\pi \hat{\tau}))^2}, \quad \forall \hat{r}, \hat{\tau}; z \geq 1.$$ (9)

Since $\int_{\frac{\pi}{2}}^1 dz I(z, \hat{r}, \hat{\tau}) = \int_0^{\infty} dz I(z, \hat{r}, \hat{\tau}) = \int_{\frac{\pi}{2}}^\infty dz I(z, \hat{r}, \hat{\tau})$ and since, by virtue of Eq. (9), the modulus of the second summand is bounded by $\frac{\pi e^{\pi \hat{r}}}{2(2\pi \hat{\tau})^2}$ we are assured a more than exponentially fast saturation in $\hat{r}$. Numerically, $\int_{\frac{\pi}{2}}^\infty dz I(z, \hat{r}, \hat{\tau}) < 10^{-5}$ for $\hat{r} > 2$ independently of $\hat{\rho}$. (Saturation in $\hat{\rho}$ is now trivial.) □

**Proposition 3.** The integrand in Eq. (5), when evaluated on a caloron, is as

$$-i\beta \frac{2^{32\pi^4}}{3} x^a \pi^2 \hat{\rho}^4 + \hat{\rho}^2 (2 + \cos(2\pi \hat{\tau})) \times F[\hat{g}, \Pi],$$ (10)

where the functional $F$ is given as

$$F[\hat{g}, \Pi] = 2\cos(2\hat{g}) \left( 2\left[ \frac{\partial_\tau \Pi}{\Pi^2} - \frac{\partial_\tau \partial_\tau \Pi}{\Pi} \right] + \sin(2\hat{g}) \left( 2\left[ \frac{\partial_\tau \Pi}{\Pi^2} - \frac{\partial_\tau \partial_\tau \Pi}{\Pi} \right] + \frac{\partial_\tau \Pi}{\Pi} - \frac{\partial_\tau \partial_\partial_\tau \Pi}{\Pi} \right) \right).$$ (11)

**Proof.** Lengthy routine computation, see [44, 45]. □

**Proposition 4.** The integrand in Eq. (5), when evaluated on an anticaloron, is obtained by a parity transformation, $\bar{x} \rightarrow -\bar{x}$, of the integrand evaluated on a caloron.

**Note 3.** For equal temporal phases, $\tau_C = \tau_A$, it then follows that the integrands cancel. But, as we will show, nontrivial BPS saturation of the field $\phi$ requires that $\tau_C - \tau_A = \pm \frac{\pi}{2}$.

**Proof.** It is easily checked that $F_{\mu\nu}(\tau, \bar{x}) = F_{\mu\nu}(\tau, -\bar{x})$ and that

$$\left\{ (\tau, \bar{0}), (\tau, \bar{x}) \right\}^C_A = \left\{ (\tau, \bar{x}), (\tau, \bar{0}) \right\}^C_A = \left\{ (\tau, -\bar{x}), (\tau, \bar{0}) \right\}^A_A = \left\{ (\tau, -\bar{x}), (\tau, \bar{0}) \right\}^A_A \right\}^A_A.$$ This proves the claim. □

**Remark 3.** Due to the appearance of the factor $\frac{a^a}{r}$ in the expression (10) the unconstrained angular integration in Eq. (5) yields zero. Thus for the final integration over $\hat{\rho}$ to possess a nonvanishing integrand the radial integral must diverge.

**Proposition 5.** The only term in $F[\hat{g}, \Pi]$, which gives rise to the divergence of the radial integral, is $-\sin(2\hat{g}) \frac{\partial^2 \Pi}{\Pi}$ This divergence is logarithmic.

**Note 4.** Since only spatial derivatives are involved this term arises from magnetic-magnetic correlations. But it is the magnetic sector whose insufficient screening gives rise to the poor convergence properties in thermal perturbation theory [47].
Proof. Obviously, no divergence arises when \( \hat{r} \to 0 \). We have
\[
\partial_\tau \Pi(\tau, r) = \beta^{-1} \partial_\tau \hat{\Pi}(\hat{\tau}, \hat{r}) \overset{\hat{r} \gg 1}{\sim} \frac{(2\pi \hat{\rho})^2}{\beta \hat{r}} \sin(2\pi \hat{\tau}) \exp(-2\pi \hat{r}),
\]
(12)
\[
\partial_\tau^2 \Pi(\tau, r) = \beta^{-2} \partial_\tau^2 \hat{\Pi}(\hat{\tau}, \hat{r}) \overset{\hat{r} \gg 1}{\sim} 2 \frac{(2\pi \hat{\rho})^2}{\beta \hat{r}} \left(4 \sin(2\pi \hat{\tau}) \exp(-4\pi \hat{r}) - \cos(2\pi \hat{\tau}) \exp(-2\pi \hat{r})\right).
\]
(13)
Thus all terms in \( F[\hat{g}, \Pi] \) containing \( \partial_\tau \Pi \) or \( \partial_\tau^2 \Pi \) give rise to finite contributions to the radial integral. (The measure is \( d\hat{r} \hat{r}^2 \).) The same holds true for the term with \( (\partial_\tau \Pi)^2 \) since
\[
\Pi(\tau, r) \equiv \hat{\Pi}(\hat{\tau}, \hat{r}) \overset{\hat{r} \gg 1}{\sim} 1 + \frac{\pi \hat{\rho}^2}{r} \Rightarrow (\partial_\tau \Pi(\tau, r))^2 \overset{\hat{r} \gg 1}{\sim} \beta^{-2} \frac{\pi^2 \hat{\rho}^4}{r^4}.
\]
(14)
But
\[
\partial_\tau^2 \Pi(\tau, r) \overset{\hat{r} \gg 1}{\sim} \beta^{-2} \frac{2\pi \hat{\rho}^2}{r^3}.
\]
(15)
Thus
\[
- \int_0^\infty dr \hat{r}^2 \sin(2\hat{g}) \frac{\partial_\tau^2 \Pi}{\Pi} \sim -\beta \left( \text{finite} + 2\pi \hat{\rho}^2 \left( \lim_{\hat{r} \to \infty} \sin(2\hat{g}) \right) \int_0^\infty \frac{d\hat{r}}{\hat{r}} \right),
\]
(16)
where the \( \sim \) sign indicates that the right-hand side approaches the left-hand side more than exponentially fast for increasing \( \hat{R} > 1 \), see Proposition 2. Obviously, the integral in Eq. (16) diverges logarithmically.

Remark 4. In summary, the quantity to be evaluated is
\[
i \frac{64\pi^5}{3} \int d\hat{\rho} \hat{\rho}^3 \frac{\pi^2 \hat{\rho}^4 + \hat{\rho}^2 (2 + \cos(2\hat{\tau}))}{(2\pi \hat{\rho}^2 + 1 - \cos(2\hat{\tau}))^2} \int d\Omega \frac{x^a}{r} \int_0^\infty \frac{d\hat{r}}{\hat{r}} \sin(2\hat{g}).
\]
(17)
Remark 5. gauge-invariant way The angular integration in the expression (17),
\[
\int d\Omega \frac{x^a}{r} = \int_{\alpha_C}^{\alpha_C + 2\pi} d(\cos \theta) \int_{-1}^{+1} d\varphi \frac{x^a}{r}, \quad (0 \leq \alpha_C \leq 2\pi),
\]
(18)
is regularized by introducing a defect/surplus angle \( \eta' \ll 1 \) for the azimuthal integration in \( \varphi \): \( \alpha_C \to \alpha_C \pm \eta' \) (lower integration limit) and \( \alpha_C \to \alpha_C \mp \eta' \) (upper integration limit). This singles out a unit vector \( \hat{n}_C \equiv (\cos \alpha_C, \sin \alpha_C, 0) \). Obviously, a rotation of \( \hat{n}_C \) is induced by a rotation of the cartesian coordinates in which the transition to polar coordinates is performed. But for \( \hat{\phi} \in \mathcal{K} \) this amounts to nothing but a global gauge rotation. Thus no breaking of rotational symmetry is introduced by the angular regularization.

Proposition 6. Without restriction of generality the contribution from the anticaloron is also regularized in the \( x_1 x_2 \)-plane with angle \( \alpha_A \) (global gauge choice). Then we arrive at
\[
\mathcal{K} = \Xi_C (\delta^{a_1} \cos \alpha_C + \delta^{a_2} \sin \alpha_C) A (2\pi (\hat{\tau} + \hat{\tau}_C)) + \Xi_A (\delta^{a_1} \cos \alpha_A + \delta^{a_2} \sin \alpha_A) A (2\pi (\hat{\tau} + \hat{\tau}_A)),
\]
(19)
where \( \Xi_C, \Xi_A \in \mathbb{R} \) (undetermined: \( 0 \) [angular integr.]\( \times \infty \) [radial integr. subject to dimensional smearing, see (44)]) and \( 0 \leq \hat{\tau}_C, \hat{\tau}_A \leq 1 \) (undetermined: modulus of caloron/anticaloron which cannot be averaged over, see argument (iii) above Prop. 2.). The function \( A(2\pi \hat{\tau}) \) in Eq. (14) is given as
\[
A(2\pi \hat{\tau}) = \frac{32\pi^7}{3} \int_0^\xi d\hat{\rho} \hat{\rho}^4 \left[ \lim_{\hat{r} \to \infty} \sin(2\hat{g}(\hat{\tau}, \hat{\rho})) \right] \frac{\pi^2 \hat{\rho}^2 + \cos(2\pi \hat{\tau}) + 2}{(2\pi^2 \hat{\rho}^2 - \cos(2\pi \hat{\tau}) + 1)^2}.
\]
(20)
The integral over \( \hat{\rho} \) in Eq. (20) diverges cubically for \( \xi \to \infty \).
Figure 1. The function $A(\frac{2\pi\tau}{\beta})$ plotted over two periods with different values of $\xi$. For comparison the function $272\xi^3\sin(\frac{2\pi\tau}{\beta})$ is plotted as a dashed line.

**Proof.** Routine computation using the fact that $\hat{g}$ saturates for $\hat{\rho} \to \infty$ (see Prop. 2).

**Theorem 1.** The function $A(2\pi\hat{\tau})$ rapidly approaches $\text{const}_{\infty} \times \xi^3\sin(2\pi\hat{\tau})$ where $\text{const}_{\infty} = 272.018$.

**Proof.** Since $\hat{g}$ saturates one may evaluate the integral numerically thus proving the claim.

**Remark 6.** Already for $\xi = 3$ one has $\text{const}_{\infty} - \text{const}_{\infty}$ = 0.025, and the functional dependence on $\hat{\tau}$ practically is a sine, see Fig. 1. Since there is such a fast saturation towards a sine function the prefactor $272.018 \times \xi^3$, which is computed numerically, can be absorbed into the undetermined, real number $\Xi_{C,A}$ in Eq. (19). In this sense the result for $K$ is independent of the cutoff $\xi$ for $\xi$ sufficiently large, see again Fig. 1.

**Theorem 2.** The set $K$ coincides with the kernel of the linear differential operator $D \equiv \partial^2_{\hat{\tau}} + \left(\frac{2\pi}{\beta}\right)^2$ acting on an adjoint scalar field $\hat{\phi}$ with two polarizations. Thus $D$ is uniquely determined by $K$.

**Proof.** There are two independent ‘polarizations’ contained in $K$ which are given by the unit vectors $\hat{n}_C$ and $\hat{n}_A$. For each polarization there is an undetermined phase shift $\hat{\tau}_{C,A}$ and an undetermined amplitude $|\Xi_{C,A}|$, and each polarization is annihilated by $D$. Modulo global gauge rotations there are thus two real parameters for each polarization of $\hat{\phi}$ which span the solution space of $D\hat{\phi} = 0$, and $D$ is determined uniquely.
\( D \) is linear. Under the ansatz that \( |\phi| \) is spacetime independent, Thm. 2 implies that the field \( \phi \) possesses a canonic kinetic term \( \text{tr} \left( (\partial_\tau \phi)^2 \right) \) in its effective, euclidean Lagrangian density.

**Theorem 3.** The adjoint scalar field \( \phi \) is subject to the euclidean Lagrangian density

\[
\mathcal{L}_\phi = \text{tr} \left( (\partial_\tau \phi)^2 + V(|\phi|^2) \right)
\]

with \( V(|\phi|^2) \equiv \Lambda^6 |\phi|^{-2} \) and \( \Lambda \) an arbitrary mass scale. Here \( |\phi|^{-1} \equiv \frac{\phi}{|\phi|^2} \).

**Proof.** Due to the BPS saturation of coarse-grained calorons/anticalorons no explicit temperature dependence may appear in \( \phi \)'s effective action. Together with the statement preceding Thm. 3 this implies an effective action of the form (21) with a yet unknown potential \( V(|\phi|^2) \).

The according Euler-Lagrange equations are

\[
\partial_\tau^2 \phi^a = \frac{\partial V(|\phi|^2)}{\partial |\phi|^2} \phi^a \quad \text{(in components)} \iff \partial_\tau^2 \phi = \frac{\partial V(|\phi|^2)}{\partial |\phi|^2} \phi \quad \text{(in matrix form)}.
\]

Since \( \phi \)'s motion is within a plane in the three-dimensional vector space of the SU(2) Lie algebra, since \( |\phi| \) is independent of spacetime, and since \( \phi \)'s phase \( \hat{\phi} \) is of period unity, see Thm. 2, one may, without restriction of generality (global gauge choice), write the solution to Eq. (22) as

\[
\phi = 2 |\phi| t_1 \exp(\pm \frac{4\pi i}{\beta} t_3 \tau). \tag{23}
\]

BPS saturation, or equivalently, the vanishing of the euclidean energy density and Eq. (23) imply

\[
|\phi|^2 \left( \frac{2\pi}{\beta} \right)^2 - V(|\phi|^2) = 0. \tag{24}
\]

On the other hand, comparing \( \partial_\tau^2 \phi + \left( \frac{2\pi}{\beta} \right)^2 \phi = 0 \), see Thm. 2 with Eq. (22), we have

\[
\left( \frac{2\pi}{\beta} \right)^2 = - \frac{\partial V(|\phi|^2)}{\partial |\phi|^2}. \tag{25}
\]

Together, Eqs. (24) and (25) yield

\[
\frac{\partial V(|\phi|^2)}{\partial |\phi|^2} = - \frac{V(|\phi|^2)}{|\phi|^2}. \tag{26}
\]

Eq. (26) is a first-order differential equation whose solution is

\[
V(|\phi|^2) = \Lambda^6 |\phi|^2, \tag{27}
\]

where \( \Lambda \) denotes an arbitrary mass scale (the Yang-Mills scale).

**Corollary 1.** The modulus of the field \( \phi \) is given as \( |\phi| = \sqrt{\frac{\Lambda^6}{2\pi}} \) and hence, modulo a global change of gauge,

\[
\phi = 2 \sqrt{\frac{\Lambda^6}{2\pi}} t_1 \exp(\pm \frac{4\pi i}{\beta} t_3 \tau). \tag{28}
\]

\(^6\)Compare with discussion in the Introduction: The search for configurations \( \phi \) with constant modulus and, up to global gauge transformations, pre-determined phase \( \hat{\phi} \) leads to the existence of a unique and consistent (no contradiction to saturation of \( \hat{\phi} \)) value of \( |\phi| \) and thus to the according maximal resolution.

\(^7\)Notice our notational convention: \( V \) is either a scalar-valued function of its scalar-valued argument or a matrix-valued function of its matrix-valued argument. In both cases the functional dependence is identical.
Proof. Substitute Eq. (27) into Eq. (24), solve for $|\phi|$ and substitute the result into Eq. (23). 

Remark 7. The field $\phi$ represents coarse-grained nonpropagating, noninteracting, BPS saturated field configurations of topological charge modulus $|Q| = 1$ and trivial holonomy. Thus it should itself not propagate. This is explicitly checked by computing the mass $M_{\delta \phi}$ of potential fluctuations $\delta \phi$ about the configuration in Eq. (28) as

$$M_{\delta \phi}^2 = 2 \frac{\partial^2 V}{\partial |\phi|^2} \bigg|_{|\phi| = \sqrt{\frac{8\pi^2}{2\bar{\kappa}}}} = 48\pi^2 T^2 = 12\lambda^3 |\phi|^2,$$

where $\lambda \equiv \frac{2\pi T}{\bar{\kappa}}$. Since $\lambda$ is considerably larger than unity in the deconfining phase, see below where it is derived that $\lambda \geq \lambda_c = 13.87$, and since the scale $|\phi|$ represents the maximal resolution (off-shellness) of any fluctuation after coarse-graining, we conclude that the field $\phi$ does not fluctuate: neither thermally nor quantum mechanically. The field $\phi$ thus represents a spatially homogeneous background for the dynamics of the coarse-grained, propagating gauge field (sector with $Q = 0$).

Notice that with $\lambda_c = 13.87$ one obtains $|\phi|^{-1} \geq 8.221 \times \left(\frac{\lambda_c}{\lambda}\right)^{3/2}$, $(\lambda \geq \lambda_c)$. But for $\hat{r} = 8.221 \times \left(\frac{\lambda_c}{\lambda}\right)^{3/2}$ the exponentially suppressed term below Eq. (9) is a correction of less than one in $10^{22}$! At the same time, setting $\xi = 8.221 \times \left(\frac{\lambda_c}{\lambda}\right)^{3/2}$ in Eq. (20), one is deep inside the saturation regime for the set $\mathcal{K}$, see Fig. IV. Thus, with a maximal resolution $|\phi|$ in the effective theory (corresponding to a length scale $|\phi|^{-1}$ up to which short-distance fluctuations in the fundamental fields are coarse-grained over to derive the effective theory) the infinite-volume limit used to derive $\mathcal{K}$ and in turn the differential operator $\mathcal{D}$ is extremely well approximated.

According to Rem. 7 the configuration in Eq. (28) is not altered by interactions with the gauge fields in the sector with $Q = 0$. Thus the nonperturbative emergence of the scale $\Lambda$ is not influenced by this sector. Compare this with the situation in perturbation theory at $T = 0$ where $\Lambda$ is the pole position for the evolution of the fundamental gauge coupling $g$ in the sector with $Q = 0$. There, the value of $\Lambda$ is dictated by the value of $g$ at a given resolution, and two assumptions enter. First, one assumes properties of the perturbative expansion that are sufficiently close to those of an asymptotic series to justify the low-order truncation of the beta function. Second, one assumes that close to the pole the perturbative prediction for the evolution of $g$ can smoothly be extrapolated to the regime where $g \gg 1$. For an extended discussion see [46].

For the case of SU(3) the field $\phi$ winds in each of the three (nearly independent) SU(2) subalgebras for a third of the period $\beta$, for details see [7].

After spatial coarse-graining the effective Lagrangian density, subject to a maximal resolution $|\phi|$ for propagating gauge fields, is given as

$$\mathcal{L}_{\text{eff}}(a_\mu) = \text{tr} \left( \frac{1}{2} G^{\mu\nu} G_{\mu\nu} + (D_\mu \phi)^2 + \frac{\Lambda^6}{\phi^2} \right),$$

(30)

where $G^{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - ie[a_\mu, a_\nu] \equiv C^{\mu\nu}_a t_a$, $a_\mu = a_\mu^a t_a$ is the coarse-grained, propagating gauge field in the sector with $Q = 0$, $D_\mu \phi = \partial_\mu \phi - ie[a_\mu, \phi]$, and $e$ is the effective gauge coupling.

Why is this statement true? The form of the term $\frac{1}{2} G^2$ is as in the fundamental theory due

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\(^{a}\)Notice that only the case of pure thermodynamics is discussed here. If an external probe is applied to the thermal system then an additional momentum scale enters, and $|\phi|$ no longer represents the only scale of resolution.

\(^{b}\)Notice that in contrast to Eq. (1) we have not absorbed the coupling into the gauge field. A gauge transformation acting on $\phi$ and $a_\mu$ thus reads: $\phi \rightarrow \Omega \phi \Omega^\dagger$ and $a_\mu \rightarrow \Omega a_\mu \Omega^\dagger + \frac{1}{2} \Omega \partial_\mu \Omega^\dagger$, $\Omega \in \text{fund}(SU(2))$ or $\text{fund}(SU(3))$. 
to perturbative renormalizability \cite{10,11,12,13}, and the only gauge-invariant way to couple the Lagrangian density of Eq. (21) to the coarse-grained sector with \( Q = 0 \), which itself cannot generate any composite of the field \( a_\mu \), is to do the replacement \( \partial_\tau \phi \rightarrow D_\tau \phi \). When we say ‘the only gauge-invariant way’ we only consider local effective vertices of \( \phi \) with \( a_\mu \) which do not involve three or more external legs of the latter. (Nonlocal interactions always can be expanded into a local series involving powers of covariant derivatives.) Such vertices would mediate energy-momentum exchange from the sector \( Q=0 \) to \( |Q| = 1 \) after coarse-graining. This, however, would contradict the BPS nature of the field \( \phi \) and thus its very existence. But \( \phi \)’s existence has just been established. Therefore no vertices involving the field \( \phi \) and \( a_\mu \) other than the mass operator contained in \( \text{tr} (D_\mu \phi)^2 \) (unitary gauge, no energy-momentum transfer but massiveness of off-Cartan modes after infinite resummation of mass insertion) exist.

The fluctuating field \( a_\mu \) is integrated out loop expanding the logarithm of the partition function about the free quasiparticle situation. This loop expansion is nontrivial due to the term \( ie[a_\mu, a_\nu] \) in \( G_{\mu
u} \), which leads to the occurrence of three-vertices and four-vertices. The momentum transfer in these vertices is subject to constraints imposed by the existence of the maximal resolution \( |\phi| \). The evolution of the effective coupling \( e \) is determined by the invariance of Legendre transformations between thermodynamic quantities under the applied coarse-graining up to a given loop order, see below.

Apart from (small) radiative corrections and modulo global gauge transformations the full ground state of the effective theory in the deconfining phase (taking into account the interactions between and fundamental radiative modifications of calorons and anticalorons) is given by the configuration in Eq. (28) and the pure-gauge configuration \( a_\mu^\text{gs} = \mp \delta_\mu A_\beta / \pi^3 t_3 \).

The following Euler-Lagrange equation for \( a_\mu \) is implied by \( L_{\text{eff}} \) in Eq. (30):

\[
D_\mu G^{\mu\nu} = ie[\phi, D^\nu \phi].
\]

But Eq. (31) is solved by \( \phi \) and \( a_\mu^\text{gs} \) by virtue of \( G_{\mu
u}[a_\mu^\text{gs}] = D^\nu[a_\mu^\text{gs}] \phi \equiv 0 \).

Due to \( G_{\mu
u}[a_\mu^\text{gs}] = D^\nu[a_\mu^\text{gs}] \phi \equiv 0 \) the ground-state associated Lagrangian density is given as \( L_{\text{eff}}[a_\mu^\text{gs}] = \text{tr} \Lambda_{\phi}^3 = 4\pi \Lambda^3 T, \quad (T \equiv \beta^{-1}) \). Thus interactions between and radiative corrections within calorons and anticalorons lift the energy density \( \rho^\text{gs} \) of the ground state from zero in the case of BPS saturation to \( \rho^\text{gs} = 4\pi \Lambda^3 T \). The fact that the ground-state pressure \( P^\text{gs} = -\rho^\text{gs} \), is microscopically explained by small-holonomy calorons and anticalorons having their BPS magnetic monopoles-antimonopole constituents \cite{17,18,24,25,26,27} attract one another under the influence of radiative corrections, for a detailed discussion which is based on the work in \cite{42}, see \cite{7,49}. The case of the ‘excitation’ of a large holonomy, which according to \cite{42} leads to the dissociation of the associated caloron/anticaloron and hence to the liberation of a screened magnetic monopole and its antimonopole, is extremely rare \cite{7}. Deviations from the equation of state \( P^\text{gs} = -\rho^\text{gs} \), which are due to those nonrelativistic, screened magnetic monopoles and antimonopoles, are described in part by the radiative corrections to the total pressure and energy density in the effective theory.

**Proposition 7.** In the effective theory the winding gauge, where \( \phi \) is as in Eq. (28) and \( a_\mu^\text{gs} \) is as above, and the unitary gauge, where \( \phi = 2 |\phi| t_3, \ a_\mu^\text{gs} = 0 \), are connected by a singular but admissible periodic gauge transformation. Under this gauge transformation the Polyakov loop \( P[a_\mu^\text{gs}] \) is transformed from \( P = -1 \) to \( P = 1 \) which points out the electric \( \mathbb{Z}_2 \) degeneracy of the ground state and thus deconfinement.

**Proof.** Since \( \phi \rightarrow \bar{\Omega}(\tau) \phi \bar{\Omega}(\tau) \) under the gauge transformation it is easily checked that \( \bar{\Omega}(\tau) \) is given as

\[
\bar{\Omega}(\tau) = \Omega_\phi Z(\tau) \Omega(\tau),
\]
where $\Omega(\tau) \equiv \exp[\pm 2\pi i \tau_3]$, $Z(\tau) = \left(2\Theta(\tau - \frac{\beta}{2}) - 1\right)1_2$, and $\Omega_{gl} = \exp[i\tau_3 t_2]$. The function $\Theta$ is defined as

$$\Theta(x) = \begin{cases} 0, & (x < 0), \\ \frac{1}{r}, & (x = 0), \\ 1, & (x > 0). \end{cases} \quad (33)$$

Thus $\tilde{\Omega}(\tau)$ is periodic but not smooth. The periodicity of fluctuations $\delta a_\mu$, however, is not affected by this gauge transformation. Namely, writing $a_\mu = a_{bg,\mu} + \delta a_\mu$, we have

$$a_\mu \to \tilde{\Omega}(a_{bg,\mu} + \delta a_\mu)\tilde{\Omega}^\dagger + \frac{i}{e} \tilde{\Omega} \partial_\mu \tilde{\Omega}^\dagger = \Omega_{gl} \left(\Omega(a_{bg,\mu} + \delta a_\mu)\Omega^\dagger + \frac{i}{e} \left(\Omega \partial_\mu \Omega^\dagger + Z \partial_\mu Z\right)\right)\Omega_{gl}^\dagger,$$

$$= \Omega_{gl} \left(\Omega \delta a_\mu \Omega^\dagger + 2\frac{i}{e} \delta \left(\tau - \frac{\beta}{2}\right) Z\right)\Omega_{gl}^\dagger = \Omega_{gl} \Omega \delta a_\mu (\Omega_{gl}^\dagger). \quad (34)$$

Now $\Omega_{gl}(\tau = 0) = -\Omega_{gl}(\tau = \beta)$. Thus the periodicity of the fluctuation $\delta a_\mu$ is unaffected by the gauge transformation induced by $\tilde{\Omega}(\tau)$ (admissibility of this change of gauge). To show the claimed transformation of the Polyakov loop on $a^{bg}_{\mu}$ is trivial.

One can easily show that for SU(3) the Polyakov loop $P[a^{bg}_{\mu}]$ forms a three-dimensional representation of the electric center symmetry $\mathbb{Z}_3$, see [7]. Thus also for SU(3) the deconfining property of the thermal ground state follows.

### 3.2 Thermal quasiparticle excitations

In this section we obtain the tree-level mass spectrum for emergent thermal quasiparticles in the effective theory, and we derive the evolution of the effective gauge coupling $e$. Next, we give analytic expressions for the temperature dependence of thermodynamic quantities on the level of free quasiparticle fluctuations. We also comment on the trace anomaly of the energy-momentum tensor.

We refer to an excitation, which possesses a temperature-dependent mass on tree-level in the effective theory, as a thermal quasiparticle.

**Proposition 8.** In the effective theory dynamical gauge symmetry breaking $SU(2)\to U(1)$ is manifested for the sector with $Q = 0$ by quasiparticle masses $m_a$. One has

$$m_a^2 = -2e^2 tr[\phi, t_3][\phi, t_3]. \quad (35)$$

Thus, $m^2 = m_1^2 = m_2^2 = 4e^2 \frac{A^3}{g^2 T}$ and $m_3 = 0$.

**Proof.** Since $a^{bg}_{\mu} = 0$ in unitary gauge formula (35) can be read off from the Lagrangian density $\mathcal{L}_{eff}$ in Eq. (30), and since $\phi = 2 |\phi| t_3$ in unitary gauge the explicit expression for the mass $m$ follows.

**Remark 8.** Imposing unitary gauge, with gauge condition $\phi = 2 |\phi| t_3$, $a^{bg}_{\mu} = 0$, and in addition Coulomb gauge for the unbroken U(1) subgroup, with gauge condition $\partial_\mu a^3_\mu = 0$, yields a completely fixed gauge if the real-valued gauge function $\theta$ in $\Omega_3 \equiv \exp(i\theta t_3)$ vanishes at spatial infinity. This gauge is physical because it exhibits the quasiparticle mass spectrum, the physical number of polarizations – three for $a = 1, 2$ and two for $a = 3$ –, and the transversality of the gauge field $a^3_\mu$ associated with the unbroken subgroup U(1).

**Remark 9.** For SU(3) the unbroken subgroup is U(1)$^2$ and six out of eight independent directions in the SU(3) Lie algebra acquire mass, for details see [7, 8].
Notice that the number of degrees of freedom before coarse-graining matches those after coarse graining. Namely, for SU(2) one has three species of propagating gauge fields \((Q=0\) sector) times two polarizations each plus two species of charge-one scalar magnetic monopoles \((|Q|=1\) sector) before coarse-graining and two species of massive gauge fields times three polarization each plus one species of massless gauge field times two polarizations each. In both cases one obtains eight degrees of freedom. For SU(3) one obtains 22 degrees of freedom before and after coarse-graining.

In unitary-Coulomb gauge and on the level of free quasiparticles the real-time propagators of the fields \(a^1_\mu, a^2_\mu\) and \(a^3_\mu\) are given as

\[
D^{1,2}_\mu(p) = -\tilde{D}_\mu(p^2 - m^2 + i0 + 2\pi\delta(p^2 - m^2)n_B(|p_0|/T)), \\
D^3_\mu(p) = -\left\{P^T_\mu(p^2 + i0 + 2\pi\delta(p^2)n_B(|p_0|/T)) - i\frac{u_\mu u_\nu}{p^2}\right\},
\]

where \(\tilde{D}_\mu = (g_\mu_\nu - \frac{p_\mu p_\nu}{m^2})p^0_\nu - P^0_\nu = P^0_\nu = 0\), \(P^ij = \delta^ij - \frac{p^i p^j}{p^2}\), \(u = (1, 0, 0, 0)\) represents the four-velocity of the heat bath, and \(n_B(x) = 1/(e^x - 1)\) denotes the Bose-Einstein distribution function.

Because of the existence of a maximal resolution scale \(|\phi|\) the deviation of the momentum \(p_\mu\) in Eqs. (36, 37) from its mass shell is constrained as

\[
|p^2| \leq |\phi|^2, \quad \text{for } a=3, \quad |p^2 - m^2| \leq |\phi|^2, \quad \text{for } a=1,2.
\]

Conditions (38) fix the momentum transfer in a three-vertex by momentum conservation.

The following conditions fix the momentum transfer in a four-vertex:

\[
|(p_1 + p_2)^2| \leq |\phi|^2, \quad \text{\(s\) channel}; \quad |(p_3 - p_1)^2| \leq |\phi|^2, \quad \text{\(t\) channel}; \quad |(p_2 - p_3)^2| \leq |\phi|^2, \quad \text{\(u\) channel}.
\]

These conditions follow from the fact that massless intermediate modes in the fundamental theory, which do not exist in the effective theory but dress the four-vertex such that the latter appears to be local, may not resolve distances smaller than \(|\phi|^{-1}\).

On the one-loop level in the effective theory (gas of noninteracting thermal quasiparticles and massless excitations) the contribution \(\Delta V\) of quantum fluctuations (arising from terms without the factor \(n_B\) in Eqs. (36, 37)) is negligibly small.

We estimate \(\Delta V\) by the contribution of the massless mode \(a^3_\mu\) appropriately weighted by the number of polarizations of all fields \(a^1_\mu, a^2_\mu, \) and \(a^3_\mu\):

\[
|\Delta V| \leq \frac{1}{\pi^2} \int_0^{|\phi|} dp p^3 \log \left(\frac{p}{|\phi|}\right) = \frac{\phi^4}{16\pi^2} = \frac{\lambda^{-3}}{32\pi^2} V.
\]

Since \(\lambda\) is considerably larger than unity \(\Delta V\) can safely be neglected.

---

10 According to the definition of the set \(\mathcal{K}\) in Sec. 3.1 topologically nontrivial field configurations with \(|Q|=1\) only contribute to the thermal ground state. As a consequence, only magnetic monopoles of magnetic charge modulus unity occur as their constituents.

11 An analytic continuation \(\tau \rightarrow -it, t \) and \(\tau\) real, is performed, see.

12 In calculating radiative corrections we have checked that under \(|\phi| \rightarrow \xi|\phi|\) in (38) and (39), where \(\xi\) is of order unity, the results are remarkably stable.
Theorem 4. On the one-loop level the evolution of the effective coupling $e$ is determined by the following first-order ordinary differential equation:

$$\partial_a \lambda = \frac{24 \lambda^4 a}{(2\pi)^6} \frac{D(2a)}{1 + \frac{24 \lambda^4 a^2}{(2\pi)^6} D(2a)},$$

(41)

where $a \equiv \frac{m}{T} = 2\pi e\lambda^{-3/2}$ and $D(y) \equiv \int_0^\infty dx \frac{x^2}{\sqrt{x^2+y^2}} \frac{1}{\exp(\sqrt{x^2+y^2})-1}$. The evolution governed by Eq. (41) possesses two fixed points: $a = 0$ and $a = \infty$. The latter is associated with a critical temperature $\lambda_c$ of value $\lambda_c = 13.87$. An attractor to the evolution exists. It is given as $a(\lambda) = 4\sqrt{2\pi^2}\lambda^{-3/2}$ for $\lambda \gg \lambda_c$ and $a(\lambda) \propto -\log(\lambda - \lambda_c)$ for $\lambda \lesssim \lambda_c$. The trace of the energy-momentum tensor $\theta_{\mu\nu}$ grows as $\theta_{\mu\nu} = \frac{e^{-3P}}{\lambda^3} = 12\lambda$ for $\lambda \gg \lambda_c$.

Proof. The Legendre transformation $\rho = T \frac{dP}{dT} - P$ between total energy density $\rho$ and total pressure $P$, which follows from the (ultraviolet regularized) partition function formulated in terms of fundamental fields, needs to be honored in the effective theory. Since there are implicit temperature dependences in the parameters $|\phi|$ and $e$ of the effective theory for the coarse-grained fluctuations $\delta a_\mu$, the derivatives w.r.t. temperature of these parameters ought to cancel one another. A necessary and sufficient condition for this to take place is $\partial_m P = 0$. Because one may neglect the quantum part, on the one-loop level $P$ and $\rho$ are determined by the thermal parts of the propagators in Eqs. (36) and (37) (terms with the factor $n_B$). They are given as follows:

$$P(\lambda) = -\Lambda^4 \left\{ \frac{2\lambda^4}{(2\pi)^6} \left[ \frac{1}{\lambda} \frac{d}{d\lambda} \right] \left[ 2\bar{P}(0) + 6\bar{P}(2a) \right] + 2\lambda \right\},$$

(42)

$$\rho(\lambda) = \Lambda^4 \left\{ \frac{2\lambda^4}{(2\pi)^6} \left[ \frac{1}{\lambda} \frac{d}{d\lambda} \right] \left[ 2\bar{P}(0) + 6\bar{P}(2a) \right] + 2\lambda \right\},$$

(43)

where $\bar{P}(y) \equiv \int_0^\infty dx x^2 \log \left[ 1 - \exp(-\sqrt{x^2+y^2}) \right]$ and $\bar{\rho}(y) \equiv \int_0^\infty dx x^2 \frac{\sqrt{x^2+y^2}}{\exp(\sqrt{x^2+y^2})-1}$. The evolution equation (41) follows by applying $\partial_{(aT)}P$ to the expression in Eq. (42) and setting the result equal to zero. The right-hand side of the evolution equation (41) indeed vanishes at $a = \infty$ (essential zero due to the exponential in the integrand for the function $D(2a)$) and at $a = 0$ (algebraic zero since $D(0)$ exists: $D(0) = \frac{2\pi^2}{6}$). Since the right-hand side of Eq. (41) is negative definite this equation is equivalent to

$$1 = -\frac{24\lambda^3}{(2\pi)^6} \left( \lambda \frac{d}{d\lambda} + a \right) a D(2a).$$

(44)

For $a \ll 1$ the Taylor expansion of the function $D(2a)$ can be truncated at zeroth order\textsuperscript{13}. This simplifies Eq. (44) as

$$1 = -\frac{\lambda^3}{(2\pi)^4} \left( \lambda \frac{d}{d\lambda} + a \right) a,$$

(45)

and the solution, subject to the initial condition $a(\lambda_i) = a_i \ll 1$ is

$$a(\lambda) = 4\sqrt{2\pi^2}\lambda^{-3/2} \left( 1 - \frac{\lambda}{\lambda_i} \left[ 1 - \frac{a_i\lambda^3}{32\pi^4} \right] \right)^{1/2}.$$ 

\textsuperscript{13}In [50] it was shown that albeit the coefficients in the Taylor expansion of $D(y)$ about $y = 0$ diverge for orders larger or equal than quadratic this formal series can be resummed to a smooth function in $y$. For this process the zeroth-order coefficient serves as a boundary condition and thus is relevant.
Thus for $\lambda \ll \lambda_i$ the function $a(\lambda)$ runs into the attractor $a(\lambda) = 4\sqrt{2}\pi^2\lambda^{-3/2}$. Since $a \equiv \frac{m}{\sqrt{T}} = 2\pi e\lambda^{-3/2}$ there is a plateau $e \equiv \sqrt{8}\pi$ in this regime. Because the attractor increases with decreasing $\lambda$ the condition $a \ll 1$ will be violated at small temperatures. The estimate $14.61 > \lambda_c$ is obtained by setting the attractor equal to unity. Since the true solution in this regime will continue to grow with decreasing $\lambda$ (negative definiteness of right-hand side of Eq. (41)) the right-hand side of Eq. (41) will be exponentially suppressed. This verifies the nontrivial thermal ground state, see Thm. 3. In a physics model the initial temperature perturbation theory), but it arises as a purely nonperturbative integration constant owing to the Yang-Mills scale $\Lambda$ is not determined by the initial value $a_i = 2\pi e_i\lambda_i^{-3/2}$ (as it would in perturbation theory), but it arises as a purely nonperturbative integration constant owing to the nontrivial thermal ground state, see Thm. 3. In a physics model the initial temperature $T_i = \frac{\lambda_i\Lambda}{2\pi}$ is naturally given by the scale where the assumption of a smooth spacetime manifold supporting the Yang-Mills theory breaks down. According to present consensus this scale is the Planck mass. The constancy of $\epsilon$ for $a \ll 1$ signals that the magnetic charge $g = \frac{4\sqrt{2}}{\pi}$ of a screened magnetic monopoles, liberated by a dissociating caloron/anticaloron of large holonomy, is conserved during most of the evolution. For $\lambda \ll \lambda_c$ the magnetic charge $g$ and the mass $M_{\text{mon}} \sim \frac{4\sqrt{2}}{\pi} \epsilon^2$ of a screened magnetic monopole vanish, see Rem. I.

In a completely analogous way one obtains for SU(3) the evolution equation

$$\frac{\partial a}{\partial \lambda} = -\frac{12\lambda^4 a}{(2\pi)^6} \frac{D(a) + 2D(2a)}{1 + \frac{12\lambda^4 a^2}{(2\pi)^6}(D(a) + 2D(2a))}. \tag{48}$$

The attractor for $a \ll 1$ reads $a(\lambda) = \frac{4}{\sqrt{3}}\pi^2\lambda^{-3/2}$, and the plateau value is $e \equiv \frac{4}{\sqrt{3}}\pi$. The numerical value for $\lambda_c$ is $\lambda_c = 9.475$, and one obtains $\theta_{\mu\mu} = 24\pi\Lambda^3 T = 12\lambda\Lambda^4$ for $\lambda \gg \lambda_c$.

### 3.3 Radiative corrections

Here we discuss the systematic computation of radiative corrections in terms of a respective loop expansion within the effective theory for the deconfining phase. This loop expansion has little resemblance with its perturbative counterpart. Each nonvanishing diagram is infrared and ultraviolet finite. While the former property is due to the nonperturbative emergence of (quasiparticle-)mass on tree level (adjoint Higgs mechanism) the ultraviolet finiteness follows from the existence of a scale $|\phi|$ of maximal resolution: In a physical gauge quantum fluctuations are constrained to maximal hardness $|\phi|^2$ by the thermal ground state, and their action is small as compared to that of thermal (on-shell) modes. Frankly speaking, this is the reason for the rapid converence of loop expansions. We also exhibit two calculational examples: The on-shell one-loop polarization tensor of the massless mode and the dominant two-loop correction to the pressure.
After a euclidean rotation $p_0 \rightarrow ip_4$, $(p_0$ and $p_4$ real) the second condition in (38) reads $|p^2 + 4e^2| |\phi|^2 \leq |\phi|^2$. Since this is never true in the attractor regime because $e \geq \sqrt{8\pi}$ for SU(2) and $e \equiv \frac{4}{\sqrt{3}}\pi$ for SU(3) one concludes that massive quasiparticles do propagate thermally only.

Alternatively, staying in Minkowskian signature, the four-momentum squared of a massive mode needs tuning to the mass squared of about one part in three-hundred for SU(2) and similarly for SU(3). Again, this implies that a massive quasiparticle propagates over long distances and thus thermalizes. Also massive quasiparticles cannot be created by quantum processes because this would invoke momentum transfers of at least twice their mass. But this is about 35 times larger than the maximally allowed resolution in the effective theory.

An irreducible bubble diagram is defined by the property that no one-particle reducible diagram for the polarization tensor is created by cutting in any possible way an internal line of the diagram.

It was argued in [49] that, when computing bubble diagrams contributing to the thermal pressure, the resummation of polarization tensors, subject to an insertion of an irreducible bubble diagram, avoids the occurrence of pinch singularities (powers of delta functions) due to a relaxation to finite-width spectral functions in the thermal parts of the so-obtained real-time propagators. Technically, the procedure for resumming polarization tensors is to compute them in real time subject to the constraints (38) and (39), perform an analytic continuation to imaginary time in the external momentum, carry out the resummation, and finally continue back to real time.

For a four momentum $p = (p_0, \vec{p})$ circulating in a loop we refer to $p_0$ and $|\vec{p}|$ as the two independent radial loop momenta. We denote by $\tilde{K}$ the number of independent radial loop momenta in a given irreducible bubble diagram.

Independent hypersurfaces $H_i$, $(i = 1, \ldots, h \leq \tilde{K})$, in the $\tilde{K}$-dimensional space $\mathbb{R}^{\tilde{K}}$ are defined by the property that in a whole environment $U$ of their intersection $\bigcap_{i=1}^h H_i$ the normal vectors $\hat{n}_i$ to $H_i$ (computed anywhere on $U \cap H_i$) are linearly independent.

Remark 10. If $h = \tilde{K}$ then it follows that $\bigcap_{i=1}^{\tilde{K}} H_i$ is a set of discrete points.

Proposition 9. For an irreducible bubble diagram containing only $V_4$-many four-vertices and no three-vertices the number $K$ of independent constraints on the loop momenta is estimated as $K \geq \frac{7}{2} V_4$.

Proof. The number $I$ of internal lines in such a diagram is $I = 2 V_4$ [52]. Because of (38) there are thus $2 V_4$ constraints on propagating momenta and according to (39) at least $\frac{3}{2} V_4$ constraints on momentum transfers in vertices. (For this estimate pair the four-vertices in the diagram.) Together this gives $K \geq \frac{7}{2} V_4$. ■

Proposition 10. For an irreducible bubble diagram containing only $V_3$-many three-vertices and no four-vertices the number $K$ of independent constraints on the loop momenta is given as $K = \frac{5}{2} V_3$.

Proof. The number $I$ of internal lines in such a diagram is $I = \frac{3}{2} V_3$ [52]. Because of (38) there are thus $\frac{3}{2} V_3$ constraints on propagating momenta. No additional constraints arise because the momentum transfer in the vertex, induced by two external legs, coincides by momentum conservation with the momentum of the third leg. Thus $K = \frac{5}{2} V_3$. ■

Proposition 11. For an irreducible bubble diagram containing only $V_4$-many four-vertices and no three-vertices the number $\tilde{K}$ of independent radial loop momenta is given as $\tilde{K} = 2 V_4 + 2$. 

Proof. The Euler characteristic for a spherical polyhedron reads $2 = V - I + F$ where $V$ is the number of vertices, $I$ the number of edges, and $F$ the number of faces. Since a connected bubble diagram is a spherical polyhedron with one face removed the identification of $F = L + 1$, where $L$ denotes the number of loops (or left-over faces), yields $L = I - V + 1$. Combining this with $I = 2V_4$ and using the fact that $\tilde{K}$ is twice the number $L$ proves the claim.

Proposition 12. For an irreducible bubble diagram containing only $V_3$-many three-vertices and no four-vertices the number $\tilde{K}$ of independent radial loop momenta is given as $\tilde{K} = V_3 + 2$.

Proof. Combining $L = I - V + 1$ with $I = 3\frac{V}{2}$ and using the fact that $\tilde{K}$ is twice the number $L$ proves the claim.

Corollary 2. From Props. 9, 10, 11, and 12 one concludes that

$$\frac{\tilde{K}}{K} \leq \frac{4}{7} \left(1 + \frac{1}{V_4}\right), \quad \text{(four-vertices only);} \quad \frac{\tilde{K}}{K} = \frac{2}{3} \left(1 + \frac{2}{V_3}\right), \quad \text{(three-vertices only).} \quad (49)$$

Since any subdiagram (obtained by cutting more than one internal line) of an irreducible bubble diagram is again irreducible it follows that at a fixed number $V$ of vertices with $V = V_4 + V_3 \geq 2$ the ratio $\frac{\tilde{K}}{K}$ is minimal for $V_3 = 0$ and maximal for $V_4 = 0$.

If the constraints (38) and (39) were equations and not inequalities one would conclude from Cor. 2 that the intersection of the independent hypersurfaces $H_i$ specified by them would be empty for a number $L$ of loops greater than a finite number $L_{\text{max}}$. From $\frac{\tilde{K}}{K} \leq \frac{2}{3} \left(1 + \frac{2}{V_3}\right)$ we have $V_{\text{max}} = 4$ which by virtue of the proof to Prop. 11 implies that $L_{\text{max}} \leq 5$.

For completeness let us investigate the generalization of the Euler characteristic for a spherical polyhedron without any handles to the situation of nonplanar bubble diagrams. Notice that the latter can be considered spherical polyhedra with one face removed and a nonvanishing number of handles added (genus $g > 0$): $V - I + L + 1 = 2 \rightarrow V - I + L + 1 = 2 - 2g$, (50)

where again $I$ is the number of internal lines, $L$ the number of loops, and $g$ represents the genus of the polyhedral surface (the number of handles). Notice that the right-hand side of the right-hand side equation is the full Euler-L’Huilliers characteristics. Reasoning as above but now based on the general situation of $g \geq 0$ expressed by the right-hand side equation in Eqs. (50), we arrive at

$$\frac{\tilde{K}}{K} \leq \frac{4}{7} \left(1 + \frac{1}{V_4} (1 - 2g)\right), \quad (V = V_4),$$

$$\frac{\tilde{K}}{K} \leq \frac{2}{3} \left(1 + \frac{2}{V_3} (1 - 2g)\right), \quad (V = V_3). \quad (51)$$

According to Eqs. (51) the demand $\frac{\tilde{K}}{K} \leq 1$ for a compact support of the loop integrations is always satisfied for $g \geq 1$ since the number of vertices needs to be positive: $V_4 \geq 0$ and $V_3 \geq 0$. Recall that at $g = 0$ this is true only for $V_4 \geq 2$ and $V_3 \geq 6$, respectively. We thus conclude that bubble diagrams of a topology deviating from planarity are much more severely constrained than their planar counterparts.

Because other thermodynamic quantities are related to the pressure by (successive) Legendre transformations this would imply their exact calculability as well. But since (38) and (39) are inequalities the associated hypersurfaces $H_i$ are fattened, and the situation is less clear cut. However, for large $V_3$ the ratio $\frac{\tilde{K}}{K}$ approaches the value $\frac{2}{3}$, which is considerably smaller than
Figure 2. \(\left| \frac{G}{T^2} \right|\) as a function of \(X \equiv \frac{\|\vec{p}\|}{T}\) for \(\lambda = 1.12 \lambda_c\) (black), \(\lambda = 2 \lambda_c\) (dark grey), \(\lambda = 3 \lambda_c\) (grey), \(\lambda = 4 \lambda_c\) (light grey), \(\lambda = 20 \lambda_c\) (very light grey). The dashed curve is a plot of the function \(f(X) = 2 \log_{10} X\). There is screening to the left \((G > 0)\) and antiscreening \((G < 0)\) to the right of the cusps. The massless mode is strongly screened at \(X\)-values for which \(\log_{10} \left| \frac{G}{T^2} \right| > f(X) \left(\frac{\sqrt{G}}{T} > X\right)\), that is, to the left of the dashed line. For \(\lambda = \lambda_c\) the function \(G\) vanishes identically.

unity, in a powerlike-way suggesting that \(\bigcap_{i=1}^{K} H_i = \emptyset\) for sufficiently large but finite \(K\) (or \(V_3\) or \(L\)). Here \(H_i\) now refers to a fattened hypersurface. We thus arrive at the following conjecture:

The loop expansion of the pressure in the effective theory for the deconfining phase terminates at a finite loop order.

Remark 11. The argument presented in favor of the truth of this conjecture apply to both the SU(2) and the SU(3) case. In \[53\] we have observed that for the SU(2) case the three-loop irreducible bubble diagram with \(V_4 = 2\) and \(V_3 = 0\), containing two internal lines with massless and two internal lines with massive particles, vanishes identically.

Example 3. Here we provide a one-loop example for a typical radiative correction: the polarization tensor \(\Pi_{\mu\nu}\) for the massless mode with \(p^2 = 0\) in the effective theory for deconfining SU(2) Yang-Mills thermodynamics \[4\]. Without restriction of generality one may assume that \(\vec{p}\) points into the 3-direction. Then the only nontrivial entries in \(\Pi_{\mu\nu}\) are \(\Pi_{11} = \Pi_{22} \equiv G(p_0 = |\vec{p}|, p, T, \Lambda)\). One easily checks that only the tadpole diagram with the massive modes \((a = 1, 2)\) circulating in the loop contributes for one-shell external momentum \((p^2 = 0)\). That is, for \(p^2 = 0\) and on the one-loop level there is no imaginary part in the screening function \[14\] \(G\). By a (lengthy) routine computation, which takes into account the constraints \[39\] and recalls that massive modes propagate thermally only, one obtains the following result:

\[
\frac{G}{T^2} = \left[ \int_{-\infty}^{\xi(X,\lambda)} d\xi \int_{\rho_m(X,\xi,\lambda)}^{\rho_M(X,\xi,\lambda)} d\rho + \int_{\xi_m(X,\lambda)}^{\xi(X,\lambda)} d\xi \int_{0}^{\rho_m(X,\xi,\lambda)} d\rho \right] \times \\
eq e^{2(\lambda)} \lambda^{-3} \left( -4 + \frac{\rho^2}{4e^{2(\lambda)}} \right) \rho \frac{n_B \left( 2\pi \lambda^{-3/2} \sqrt{\rho^2 + \xi^2 + 4e^{2(\lambda)}} \right)}{\sqrt{\rho^2 + \xi^2 + 4e^{2(\lambda)}}}, \tag{52}
\]

\[14\] On the two-loop level and at \(p^2 = 0\) \(G\) receives an imaginary contribution whose modulus, however, is strongly suppressed as compared to the modulus of the one-loop result, for the ratio of two-loop to one-loop contributions to the pressure see \[4\].

\[15\] The three constraints in \(39\) collapse onto one constraint for this diagram.
The ratio of the dominant two-loop correction to the pressure and the ground-state subtracted one-loop result as a function of $\lambda$.

where the dimensionless quantities $\xi_m, \xi_M, \rho_m,$ and $\rho_M$ are given as

$$\xi_M(X, \lambda) \equiv \frac{\pi}{2X} \frac{4e^2 \mp 1}{\lambda^{3/2}} - 2\frac{X}{\pi} \frac{\lambda^{3/2}}{4e^2 \mp 1},$$

$$\rho_M(X, \xi, \lambda) \equiv \sqrt{\left(\frac{\pi}{X}\right)^2 \left(\frac{4e^2 \mp 1}{\lambda^{3}}\right)^2 - 2\frac{2\pi}{X} \frac{4e^2 \mp 1}{\lambda^{3/2}} \xi - 4e^2},$$

the dimensionless quantity $X$ is defined as $X \equiv \frac{|\vec{p}|}{T}$, and $e(\lambda)$ follows from the solution $a(\lambda)$ of the evolution equation (44) by virtue of $e(\lambda) = a(\lambda) \frac{2\pi}{\lambda} \frac{\lambda^{3/2}}{2}$. A plot of $\log_{10} G_T$ in dependence of $X$ for various values of $\lambda$ is presented in Fig. 3. By virtue of the constraints expressed in (53) and (54) it is straightforward to show that the function $G$ possesses an essential zero at $X = 0$.

Example 4. In Fig. 3 the dependence on temperature of the ratio of the dominant two-loop diagram for the pressure (with $V_3 = 2$ and $V_4 = 0$) and the ground-state subtracted one-loop result is depicted. We refrain from quoting formulas and refer the reader to [4].

Remark 12. Combining the results of [7], [4], and [53], the ratio of two-loop corrections to the pressure to the ground-state subtracted one-loop result is, depending on temperature, at most $\sim 10^{-2}$ and the ratio of three-loop corrections to the ground-state subtracted one-loop result at most $\sim 2 \times 10^{-7}$.

4 Preconfining phase

4.1 Thermal ground state: Interacting magnetic monopoles/antimonopoles

The preconfining phase is the mediator between deconfinement at high temperatures and the low-temperature confining phase where (dual) gauge fields do not propagate. This phase occupies a very narrow region in the phase diagrams of SU(2) and SU(3) Yang-Mills thermodynamics. This and the fact that lattice simulations operate at a finite spatial volume and thus have difficulty to exhaustively capture the important long-range correlations inherent to the monopole-antimonopole condensing ground state are the reasons why the preconfining phase has escaped its detection in numerical experiments [54, 57, 58]. We know that at $\lambda_c$ one direction in the SU(2) algebra remains precisely massless and thus propagates. At the same time we know that
screened magnetic monopoles and antimonopoles become massless at $\lambda_c$ and thus are prone to condensation. Even though the monopole-antimonopole condensate induces a dynamical breaking of the (dual) $U(1)$ gauge symmetry this does not lead to the immediate decoupling of the dual gauge mode. Why and at what point this happens and how the deconfining ground state relates to the monopole-antimonopole condensate (tunneling) when lowering the temperature is subject to insightful analysis. On average, the process of gradually generating an extra polarization of the dual gauge mode by tunneling transitions between the deconfining ground state, characterized by a short-lived and small caloron or anticaloron holonomy, and the monopole-antimonopole condensate, characterized by a stable and large caloron or anticaloron holonomy \[42, 54, 55\] appears to be relevant when addressing the emergence of intergalactic magnetic fields in applying $SU(2)$ Yang-Mills thermodynamics to describe thermalized photon propagation, see \[3, 4, 5, 6, 7, 8\].

In this section a derivation of the thermal ground state in the preconfining phase is given where magnetic monopoles and their antimonopoles are pairwise condensed. The process of condensation is extremely subtle microscopically: As temperature approaches $\lambda_c$ from above, the screening of a given preexisting magnetic monopole and its antimonopole (liberated by the dissociation of a large-holonomy caloron or anticaloron) is suddenly enhanced (logarithmic pole in $e(\lambda)$). Although this rapidly suppresses their mass and magnetic charge it does not yet lead to the formation of a stable condensate. For $\lambda < \lambda_c$ the average caloron-anticaloron holonomy gradually increases with decreasing temperature (supercooling) giving rise to more frequent caloron and anticaloron dissociation processes. As a consequence, the magnetic-charge screening of a given monopole increasingly is due to alike monopoles and antimonopoles (and decreasingly due to annihilating monopoles and antimonopoles inside a small-holonomy caloron/anticaloron) in an ever more stable condensate. Although the derivation of a complex scalar field describing the monopole-antimonopole condensate only relies on the limit of total screening $e \to \infty$, which takes place at $\lambda_c$, the formation of a stable condensate is seen to occur at a slightly smaller temperature.

**Remark 13.** The condensate of monopoles and antimonopoles starts to form at $\lambda_c$, where $e = \infty$, and the Yang-Mills system ‘forgets’ about the existence of the mass scale $\Lambda$ since the mass of monopoles and antimonopoles vanishes by total screening and since there is no interaction between them because the magnetic coupling $g = \frac{e}{\sqrt{2}}$ vanishes. Also, since screened monopoles and antimonopoles are at rest w.r.t. the heat bath (previously created by dissociating large-holonomy calorons and anticalorons \[42\]) no kinetic energy of their motion exists. Thus, if, after an appropriate spatial coarse-graining, the condensate is described by a spatially homogeneous field $\varphi$, then this field by itself must be BPS saturated, that is, its energy-momentum tensor (or, after integrating out any dependence on space, its euclidean energy density) vanishes identically.

The potential for the formation of a monopole-antimonopole condensate opens up for $e \to \infty$. After an appropriate spatial coarse-graining and on the level of no interactions between monopoles and/or antimonopoles this condensate is, in the euclidean formulation, described by an inert, spatially homogeneous, and BPS saturated complex scalar field $\varphi = |\varphi| \exp \left[ \pm 2\pi i \frac{\beta}{\beta} \right]$ where $|\varphi| = \sqrt{\frac{\Lambda^3}{2\pi}}$ and $\Lambda$ is an arbitrary mass scale. Taking interactions between monopoles and antimonopoles into account, the thermal ground state is described by $\varphi$ and the pure-gauge configuration $a_{\mu}^{D,ws} = \mp \delta_{\mu4} \frac{2e}{\sqrt{2}}$ of the dual abelian gauge field $a_\mu^D$. Here $g$ is the magnetic coupling, and the ground-state energy density $\rho^{ws}$ and pressure $P^{ws}$ are given as $\rho^{ws} = -P^{ws} = \pi \Lambda^3 T$.

**complex scalar field $\varphi$:**

For the effective theory \[30\] in unitary gauge out of the three directions $a_{\mu}^{1,2,3}$ in the Lie algebra only $a_\mu^3$ propagates for $e \to \infty$, the other two gauge modes decouple because their quasiparticle
mass diverges. For \( \lambda \leq \lambda_c \) we define the dual gauge field \( a^D_\mu \) by the coarse-grained version of \( a^3_\mu \). Since \( a^3_\mu \) is a free field for \( \lambda \leq \lambda_c \) this coarse-graining is trivial. In particular no (local or nonlocal) composites of the field \( a^D_\mu \) may propagate in the effective theory for the preconfining phase, and \( a^D_\mu \rightarrow a^D_\mu + \frac{1}{\theta} \Omega \partial_\mu \Omega^\dagger \) under a gauge transformation with \( \Omega \in U(1) \). Because of the rotational symmetry of the thermal system the monopole-antimonopole condensate is described by a scalar field \( \varphi \). Since \( \varphi \) by itself is BPS saturated it would be irrelevant to the thermodynamics of the preconfining phase if it was a U(1) gauge singlet (real scalar). But then the only option for coupling \( \varphi \) to \( a^D_\mu \) in a gauge-invariant way is the transformation law \( \varphi \rightarrow \Omega^\dagger \varphi \) (complex scalar field).

\( \varphi \)'s phase:

The (dimensionless) phase \( \theta \) with \( \varphi = |\varphi| \exp[i\theta] \) is defined by the geometrically and thermally averaged magnetic flux \( F_{\pm,\text{th}} \) through a two-dimensional sphere, \( S_{2,R=\infty} \), of vanishing curvature (infinite radius \( R \)) induced by a monopole-antimonopole system \( \rho \) at zero-momentum and \( e \rightarrow \infty \). In accord with Rem. \( [13] \) this is the only possible definition of a dimensionless quantity which does not make any reference to a scale. Consider a system of a zero-momentum monopole and its zero-momentum antimonopole. In unitary gauge, where independently of position the adjoint Higgs field of the monopole or antimonopole configuration points into a fixed direction in the SU(2) algebra and Dirac strings compensate for the magnetic flux through a closed surface surrounding the monopole or antimonopole, we introduce unit vectors \( \hat{x}_m \) and \( \hat{x}_a \) for monopole and antimonopole, respectively. These vectors signal the direction of the Dirac strings (both pointing away from the respective center of charge). Let \( \delta \equiv \angle(\hat{x}_m, \hat{x}_a) \) and both monopole and antimonopole be placed on the same side of \( S_{2,R=\infty} \). (It can easily be checked below that this is no restriction of generality.) Now a single monopole or a single antimonopole, whose Dirac string does not pierce \( S_{2,R=\infty} \), would induce a magnetic flux \( F_{\pm} \) through \( S_{2,R=\infty} \) of \( F_{\pm} = \pm \frac{4\pi}{e} \). It is then easy to see \( [7] \) that, (geometrically) averaging over all directions of \( \hat{x}_m \) and \( \hat{x}_a \) at a given angle \( \delta \), the flux \( \bar{F}_{\pm}(\delta) \) of the monopole-antimonopole system is given as

\[
\bar{F}_{\pm}(\delta) = \pm \frac{\delta}{2\pi} \frac{2\pi}{e} = \frac{2\delta}{e}, \quad (0 \leq \delta \leq \pi).
\]  

(55)

After screening the mass \( M_{m+a} \) of the monopole-antimonopole system is given as \( [24] \) \( M_{m+a} = \frac{8\pi^2}{e^2} \). Thus, coupling this system to the heat bath, the thermally averaged flux \( \bar{F}_{\pm,\text{th}}(\delta) \) reads

\[
\bar{F}_{\pm,\text{th}}(\delta) = 4\pi \int d^2p \delta^{(3)}(p) n_B(\beta E(p)) \bar{F}_{\pm}(\delta),
\]  

(56)

where \( E(p) \equiv \sqrt{M^2_{m+a} + p^2} \), and \( n_B(x) \equiv \frac{1}{\exp[x]-1} \) denotes the Bose function. Since

\[
\lim_{\bar{p} \rightarrow 0} \left( \exp \left[ \beta \sqrt{M^2_{m+a} + \bar{p}^2} \right] - 1 \right) = \frac{8\pi^2}{e} \left( 1 + \frac{1}{2} \frac{8\pi^2}{e} + \frac{1}{6} \left( \frac{8\pi^2}{e} \right)^2 + \cdots \right)
\]  

(57)

one finally has

\[
\lim_{e \rightarrow \infty} \bar{F}_{\pm,\text{th}}(\delta) = \pm \frac{\delta}{\pi} \equiv \frac{\theta}{2\pi}, \quad (0 \leq \delta \leq \pi).
\]  

(58)

Since the angle \( \delta \), which, after eliminating any dependence on space by the inclusion of vanishing spatial momentum into the average thermal flux, see Eq. \( [56] \), has lost its original geometric meaning but still ought to parametrize a periodic situation, we may set \( \pm \frac{\delta}{\pi} = \pm \frac{\theta}{2\pi}, \quad (0 \leq \tau \leq \pi) \).

\( ^{16} \) A monopole is \textit{correlated} with its antimonopole in the sense that the former owes its existence to the latter and vice versa since their origin is the charge separation enabled by the strong deformation of a small-holonomy caloron/anticaloron.
That is, periodicity of the flux in dependence of an angle in the monopole-antimonopole condensate is, after spatial coarse-graining, promoted to the periodicity in euclidean time of the associated, spatially homogeneous, BPS saturated field $\varphi$. Therefore, we have

$$\varphi = |\varphi| \exp \left[ \pm 2\pi i \frac{\tau}{\beta} \right].$$

(59)

Since $|\varphi|$ is spatially homogeneous with the same justification as for the field $\phi$ in the deconfining phase the field $\varphi$ is annihilated by the linear differential operator:

$$\bar{D} \equiv \partial_\tau^2 + \left( \frac{2\pi}{\beta} \right)^2 :$$

(59)

where $\varphi^*$ denotes the complex conjugate of $\varphi$.

No explicit temperature dependence may appear in the euclidean action for the field $\varphi$ on the level of noninteracting monopoles and antimonopoles and $e \to \infty$. According to Eq. (60) and because of gauge invariance one may thus write

$$S_\varphi = \int_0^\beta d\tau \int d^3x \left( \frac{1}{2} \partial_\tau \varphi^* \partial_\tau \varphi + \frac{1}{2} V(|\varphi|^2) \right),$$

(61)

where $V(|\varphi|^2)$ is a to-be-determined gauge-invariant potential and $|\varphi|^2 = \varphi^* \varphi$. By virtue of Eq. (59) the Euler-Lagrange equation, which follows from the action (61), reads

$$\partial_\tau^2 \varphi = \frac{\partial V(|\varphi|^2)}{\partial |\varphi|^2} \varphi \quad \text{Eq. (59)}$$

$$\text{subject to } \varphi^* \varphi \neq 0 \Rightarrow \left( \frac{2\pi}{\beta} \right)^2 = - \frac{\partial V(|\varphi|^2)}{\partial |\varphi|^2}. \quad \text{(62)}$$

On the other hand, the field $\varphi$ is BPS saturated (vanishing of the euclidean energy density). Eq. (61) and Eq. (59) thus implies that

$$|\varphi|^2 \left( \frac{2\pi}{\beta} \right)^2 - V(|\varphi|^2) = 0. \quad \text{(63)}$$

Together, Eqs. (62) and (63) yield

$$\frac{\partial V(|\varphi|^2)}{\partial |\varphi|^2} = - \frac{V(|\varphi|^2)}{|\varphi|^2}. \quad \text{(64)}$$

The solution to the first-order equation (64) reads

$$V(|\varphi|^2) = \frac{\Lambda^6}{|\varphi|^2},$$

(65)

where $\Lambda$ is a mass scale which appears as a constant of integration. Substituting Eq. (65) into Eq. (63) yields

$$|\varphi| = \sqrt{\frac{\Lambda^3}{2\pi T}} = \sqrt{\frac{\Lambda^3 \beta}{2\pi}}. \quad \text{(66)}$$

The quantity $|\varphi|$ sets the scale of maximal resolution in the effective theory. An $S_{2,R=|\varphi|^{-1}}$ separating a monopole in the interior from its antimonopole in the exterior (or vice versa) experiences the same magnetic flux as an $S_{2,R=\infty}$ since in the condensate the monopole-antimonopole

---

17By a global U(1) gauge rotation a phase shift $\tau \to \tau + \tau_0$ can be introduced (global rotation of the Dirac strings), and $|\varphi|$ so far is an undetermined normalization. This freedom spans a two-dimensional vector space which coincides with the kernel $\mathcal{K}$ of $\mathcal{D}$ and thus determines $\mathcal{D}$ uniquely.
distance and their core-size is nil. Thus monopole and antimonopole cannot probe the finite curvature of $S_{2,R=|\varphi|^{-1}}$ and the infinite-surface limit is trivially saturated in the spatial coarse-graining.

\(\varphi\)'s inertness:

By virtue of Eqs. (65) and (66) one has

$$\partial_{|\varphi|} V(|\varphi|^2) = 6 \Lambda^3 |\varphi|^2 = 24 \pi^2 T^2,$$

where $\Lambda \equiv \frac{2\pi T}{g^2}$. We will show below that $\Lambda \geq 7.075$. Thus the field $\varphi$ neither fluctuates quantum mechanically nor thermally.

full action and $a_{\mu}^{D,gs}$:

Since the field $a_{\mu}^{3}$ does not interact with itself the coarse-grained field $a_{\mu}^{D}$ obeys the same form of the action. Also, local $U(1)$ gauge invariance dictates that $\partial_{\tau} \rightarrow D_{\mu} \equiv \partial_{\mu} + i g a_{\mu}^{D}$. The effective action for the preconfining phase thus reads

$$S = \int_{0}^{\beta} d\tau \int d^{3}x \left[ \frac{1}{4} G_{\mu \nu}^{D} G_{\mu \nu}^{D} + \frac{1}{2} (D_{\mu} \varphi)^{*} D_{\mu} \varphi + \frac{1}{2} |\varphi|^{2} \right],$$

where $G_{\mu \nu}^{D} \equiv \partial_{\mu} a_{\nu}^{D} - \partial_{\nu} a_{\mu}^{D}$. Making use of the inertness of the field $\varphi$, the Euler-Lagrange equations, which follow from the action (68), are given as

$$\partial_{\mu} G_{\mu \nu}^{D} = ig \left[ (D_{\nu} \varphi)^{*} \varphi - \varphi D_{\nu} \varphi^{*} \right].$$

By virtue of $D_{\nu} \varphi = 0$ the pure-gauge configuration $a_{\mu}^{D,gs} = \mp \frac{2\pi n}{g^{2}} \delta_{\mu 4}$ solves Eq. (69). Inserting $a_{\mu}^{D,gs}$ and $\varphi$ into (68) one reads off the ground-state energy density and pressure as $\rho^{gs} = -P^{gs} = \pi \Lambda^3 T$.

The Polyakov loop, evaluated on the dual gauge-field configuration $a_{\mu}^{D,gs}$, is unity independently of the choice of admissible gauge.

Let us show this. The field $\varphi$ remains periodic under $\varphi \rightarrow \Omega \varphi$ (admissible change of gauge) if and only if $\Omega = \exp \left[ i \left( 2\pi n \frac{\vec{x}}{g^{2}} + \alpha(\vec{x}) \right) \right]$ where $n \in \mathbb{Z}$, and $\alpha$ is a real function of space only.

Hence $a_{\mu}^{D} \rightarrow a_{\mu}^{D} + \frac{2\pi n}{g^{2}} \delta_{\mu 4} + \frac{\partial_{\mu} \alpha(\vec{x})}{g} \delta_{\mu j}$ under $\Omega$, and thus the periodicity of $a_{\mu}^{D}$ is (trivially) assured. (Here $j = 1, 2, 3$.) In particular,

$$a_{\mu}^{D,gs} \rightarrow (\mp \frac{2\pi}{g^{2}} + \frac{2\pi n}{g^{2}}) \delta_{\mu 4} + \frac{\partial_{\mu} \alpha(\vec{x})}{g} \delta_{\mu j} = \frac{2\pi (n \mp 1)}{g^{2}} \delta_{\mu 4} + \frac{\partial_{\mu} \alpha(\vec{x})}{g} \delta_{\mu j}.$$

Thus in any admissible gauge the Polyakov loop $P$ on $a_{\mu}^{D,gs}$ is unity:

$$P[a_{\mu}^{D,gs}] = \exp \left[ ig \int_{0}^{\beta} d\tau a_{\mu}^{D,gs} \right] = 1.$$

The electric $\mathbb{Z}_{2}$ degeneracy of the ground state, which occurred in the deconfining phase, no longer exists in the preconfining phase. Since the magnetic coupling $g$ remains finite inside this phase this does, however, not imply complete confinement since the dual gauge field, albeit massive, still propagates.

In a way completely analogous to SU(2) one derives for SU(3) the following effective action for the preconfining phase [7]:

$$S = \sum_{l=1}^{2} \int_{0}^{\beta} d\tau \int d^{3}x \left[ \frac{1}{4} G_{\mu \nu, l}^{D} G_{\mu \nu, l}^{D} + \frac{1}{2} (D_{\mu, l} \varphi_{l})^{*} D_{\mu, l} \varphi_{l} + \frac{1}{2} |\varphi_{l}|^{2} \right].$$

Since SU(3) → U(1)$^{2}$ in the deconfining phase there are now two independent species of magnetic monopoles, the dual gauge fields, $a_{\mu, 1}^{D}$, $a_{\mu, 2}^{D}$, and the monopole-antimonopole condensate,
represented by inert complex scalar fields \( \varphi_1, \varphi_2 \). The magnetic coupling \( g \) and the scale \( \Lambda \) are universal, \( a^{D_{gs}}_{\mu,1} = a^{D_{gs}}_{\mu,2} = \frac{2\sqrt{2}}{g^3} \delta \mu_4 \), and the ground-state energy density and pressure are given as \( \rho^{gs} = -P^{gs} = 2\pi \Lambda^3 T \). The Polyakov loop, evaluated on the ground-state configurations \( a^{D_{gs}}_{\mu,1}, a^{D_{gs}}_{\mu,2} \), is unity in any admissible gauge also for SU(3). This shows that the electric \( Z_3 \) degeneracy of the ground state of the deconfining phase no longer persists in the preconfining phase.

4.2 Thermal quasiparticle excitations of the dual gauge field

**Proposition 13.** In the effective theory for the preconfining phase the dynamical breaking of the residual gauge symmetry \( U(1) \) (for SU(2)) and \( U(1)^2 \) (for SU(3)) is manifested in terms of a quasiparticle mass \( m \) for the dual gauge field. One has \( m = g|\varphi| = g|\varphi_1| = g|\varphi_2| = aT \) where \( a = 2\pi g\Lambda^{-3/2} \).

**Proof.** In unitary gauge, \( \varphi = |\varphi| = \varphi_{1,2} \) and \( a^{D_{gs}}_{\mu,1} = a^{D_{gs}}_{\mu,2} = 0 \), the relation \( m = g|\varphi| = g|\varphi_{1,2}| \) for the mass of the fluctuations \( \delta a^{D_{gs}}_{\mu,1}, \delta a^{D_{gs}}_{\mu,2} \) can be read off from (65) and (71), respectively (abelian Higgs mechanism), and \( a = 2\pi g\Lambda^{-3/2} \) then follows from Eq. (66) and the definition \( \lambda \equiv 2\pi T / \Lambda \).

**Remark 14.** Thus, the excitations in the effective theory for the deconfining phase are free thermal quasiparticles.

The contribution of quantum fluctuations to the thermodynamic pressure \( \Delta V \) in the preconfining phase is negligible.

For both SU(2) and SU(3) one obtains in close analogy to the deconfining phase the following estimate for the ratio \( \Delta V / V \):

\[
\left| \frac{\Delta V}{V} \right| \leq \frac{\tilde{\lambda}^{-3}}{24\pi^2}.
\]  

(72)

As we shall see, \( \tilde{\lambda} \geq 7.075 \) (SU(2)) and \( \tilde{\lambda} \geq 6.467 \) (SU(3)). Thus \( \Delta V \) is a small correction to the (dominant) tree-level result \( V \).

**Proposition 14.** The scales \( \tilde{\Lambda} \) (preconfining phase) and \( \Lambda \) (deconfining phase) are related as

\[
\tilde{\Lambda} = \left( 4 + \frac{\lambda_c^3}{720\pi^2} \right)^{1/3} \Lambda, \quad \text{(for SU(2))} \quad \Lambda = \left( 2 + \frac{\lambda_c^3}{720\pi^2} \right)^{1/3} \Lambda, \quad \text{(for SU(3))}.
\]  

(73)

**Proof.** At \( \lambda_c \), where \( e = \infty \) and \( g = 0 \), the pressure \( P \) is continuous. Moreover, no higher loop corrections to the one-loop result exist in the deconfining phase since in unitary gauge the fluctuations \( a^{1,2}_{\mu,1} \) (SU(2)) and \( a^{1,2,4,5,6,7}_{\mu,2} \) (SU(3)) decouple at \( \lambda_c \). Equating at \( \lambda_c = \frac{4}{3} \tilde{\lambda} \), the right-hand side of Eq. (42) with the right-hand side of

\[
P(\tilde{\lambda}_c) = -\tilde{\Lambda}^4 \left[ \frac{6\lambda_c^4}{(2\pi)^6} \tilde{P}(0) + \frac{\tilde{\lambda}_c}{2} \right],
\]  

(74)

for the preconfining phase (negligible quantum part), yields the claim for SU(2). For SU(3) one needs to equate the right-hand sides of

\[
P(\lambda_c) = -\Lambda^4 \left\{ \frac{8\lambda_c^4}{(2\pi)^6} P(0) + 2\lambda_c \right\}
\]

(75)

\(^{18}\)The total pressure \( P \) is already negative at \( \lambda_c \), see [7].
and
\[
P(\lambda_c) = -\lambda^4 \left\{ \frac{12\lambda^4}{(2\pi)^6} \bar{P}(0) + \lambda_c \right\}.
\]  

(76)

**Theorem 5.** The evolution of the magnetic coupling \( g \) with temperature is described by the first-order differential equation
\[
\partial_a \lambda = \frac{12\lambda^4}{(2\pi)^6} \frac{a D(a)}{1 + \frac{12\lambda^4}{(2\pi)^6} D(a)},
\]
where \( a = 2\pi g \lambda^{-3/2} \), and the function \( D(y) \) is defined below Eq. (41).

**Proof.** Because quantum contribution to the pressure \( P \) can be neglected in the effective theory for the preconfining phase one has for SU(2)
\[
P(\lambda) = -\lambda^4 \left\{ \frac{6\lambda^4}{(2\pi)^6} \bar{P}(a) + \frac{\lambda}{2} \right\},
\]
where the function \( \bar{P}(y) \) is defined below Eq. (43). The SU(3) pressure is just twice the SU(2) pressure. As in the deconfining phase, the invariance of the Legendre transformations between thermodynamic quantities under the applied coarse-graining implies for the effective theory that \( \partial_{(\alpha T)} \bar{P} = 0 \), and for both SU(2) and SU(3) the same evolution equation (77) follows.

**Remark 15.** Numerically, the initial condition for the evolution described by Eq. (77) is \( g(\lambda_c) = 0 \) for \( \lambda_c = 8.478 \) (SU(2)) and \( \lambda_c = 7.376 \) (SU(3)). For decreasing \( \lambda < \lambda_c \) the magnetic coupling \( g \) rises rapidly and runs into a logarithmic pole at \( \lambda_c' \): \( g \propto -\log(\lambda - \lambda_c') \). Numerically, one has \( \lambda_c' = 7.075 \) (SU(2)) and \( \lambda_c' = 6.467 \). Taking the mass \( m \) of the dual gauge mode as an order parameter for the dynamical breaking of U(1) (SU(2)) and U(1)\(^2\) (SU(3)) and postulating that \( m = K(T_c - T)^\nu \) for \( T \lesssim T_c \), where \( K \) and \( \nu \) are constants, one extracts mean-field critical exponents: \( \nu = 1/2 \).

**Remark 16.** The energy density \( \rho \) divided by \( T^4 \) in the preconfining phase is given as
\[
\frac{\rho}{T^4}(\lambda) = \frac{(2\pi)^4}{\lambda^4} \left\{ \frac{6\lambda^4}{(2\pi)^6} \bar{P}(a) + \frac{\lambda}{2} \right\}, \quad \text{(for SU(2))}; \quad \frac{\rho}{T^4}(\lambda) = \frac{(2\pi)^4}{\lambda^4} \left\{ \frac{12\lambda^4}{(2\pi)^6} \bar{P}(a) + \lambda \right\}, \quad \text{(for SU(3))}.
\]

(79)

**Remark 17.** With a slight abuse of notation we refer to \( \rho(\lambda) \) as the functional dependence of the energy density on temperature \( \lambda \) in the preconfining phase and to \( \rho(\lambda) \) as the functional dependence of the energy density on temperature \( \lambda \) in the deconfining phase. Thus \( \rho(\lambda) \) and \( \rho(\lambda) \) are different functions of their arguments.

**Proposition 15.** At \( \lambda_c = \frac{\lambda}{\lambda_c} \lambda_c \) the energy density \( \rho \) exhibits a positive jump when decreasing the temperature. One has \( \Delta(\lambda_c) = \frac{\rho(\lambda_c) - \rho(\lambda_c - 0)}{T_c^4} = \frac{4}{3} \frac{\pi^2}{30} \) for SU(2) and \( \Delta(\lambda_c) = \frac{8}{3} \frac{\pi^2}{30} \) for SU(3).

**Proof.** Routine computation considering Eqs. (73).

**Remark 18.** The existence of the gap \( \Delta \) signals that the monopole-antimonopole condensate only builds up gradually as temperature falls below \( \lambda_c \). This is intuitively understandable because the condensation would require the influx of an infinite number of totally screened monopole-antimonopole pairs from infinity which costs energy. To facilitate the condensation of
additional monopole-antimonopole pairs (stable condensate) by total screening needs an increase of the average caloron/anticaloron holonomy from almost trivial to maximal. Then the pairs of liberated monopoles and antimonopoles screen one another, and no transport from infinity is needed. Although this process is hard to grasp microscopically, after spatial coarse-graining the critical temperature at which a stable condensate forms (defined by the property that the system is more likely to be preconfining than deconfining) can be determined exactly \[3\].

**Remark 19.** Notice that the number of degrees of freedom before coarse-graining matches those after coarse-graining. Namely, for SU(2) one has one species of propagating gauge field times two polarizations plus one species of center-vortex loop, see Sec.5 before coarse-graining and one species of massive, dual gauge field times three polarizations. Thus, one obtains three degrees of freedom before and three degrees of freedom after coarse-graining. For SU(3) one obtains six degrees of freedom before and after coarse-graining.

### 4.3 Supercooling

A stable condensate of monopoles and antimonopoles exists for temperatures \(\bar{\lambda}\) with \(\bar{\lambda}_c < \bar{\lambda} < \bar{\lambda}_s\) where \(\bar{\lambda}_{c'} < \bar{\lambda}_s < \bar{\lambda}_c\).

At \(\bar{\lambda}_{c'}\), where \(g = \infty\), one has \(\rho(\bar{\lambda}_{c'}) = 8\pi^4 \bar{\lambda}^{-3}\) (for SU(2)) and \(\rho(\bar{\lambda}_{c'}) = 16\pi^4 \bar{\lambda}^{-3}\) (for SU(3)).

On the other hand, continuing the energy density of the deconfining phase down to \(\lambda_{c'} = \frac{4}{\pi} \bar{\lambda}_{c'}\) and using Eqs. (73) yields

\[
\frac{\rho(\lambda_{c'})}{T_{c'}^4} = \frac{\pi^2}{15} + \frac{32\pi^4}{4 + \frac{\lambda_{c'}}{20\pi^2}} \bar{\lambda}_{c'}^{-3}, \quad \text{(for SU(2))} \quad \frac{\rho(\lambda_{c'})}{T_{c'}^4} = \frac{2\pi^2}{15} + \frac{32\pi^4}{2 + \frac{\lambda_{c'}}{20\pi^2}} \bar{\lambda}_{c'}^{-3}, \quad \text{(for SU(3))}.
\]

The second summands in Eqs. (80) practically coincide with the above expressions for \(\frac{\rho(\lambda_{c'})}{T_{c'}^4}\).

Thus we conclude that \(\frac{\rho(\lambda_{c'})}{T_{c'}^4} < \frac{\rho(\lambda_{c'})}{T_{c'}^4}\) for both SU(2) and SU(3). But according to Prop. (15) we have \(\rho(\lambda_{c'}) > \rho(\lambda_{c'})\) for both SU(2) and SU(3). Since both functions \(\rho(\lambda_{c'})\) and \(\rho(\lambda_{c'})\) are continuous in the ranges \(\bar{\lambda}_{c'} \leq \bar{\lambda} \leq \bar{\lambda}_c\) and \(\lambda_{c'} \leq \lambda \leq \lambda_c\), respectively, there is at least one intersection. Numerically, one shows that only a single intersection takes place, see Fig.4. For SU(2) one obtains the following values: \(\lambda_s = 12.15\) or \(\bar{\lambda}_s = 7.428\). Now, cooling the system, a stable condensate (system is more likely to be found in preconfining than in deconfining state) starts to take place at \(\bar{\lambda}_s\), and the claim follows.

**Remark 20.** The typical, maximal core-size \(R_{\text{core}}(\bar{\lambda})\) of an unstable center-vortex loop is given as \(R_{\text{core}}(\bar{\lambda}) \sim \frac{1}{m} = \frac{1}{|\varphi|}\). For \(\lambda_{c'} \leq \bar{\lambda} \leq \bar{\lambda}_c\) and \(\lambda_{c'} \leq \lambda \leq \lambda_c\), respectively, where a stable monopole-antimonopole condensate exists, we have \(g \geq 8.3\) according to Eq. (77). Thus \(\frac{R_{\text{core}}(\bar{\lambda})}{|\varphi|} \leq 0.12\). That is, collapsing center-vortex loops are not resolved and the monopole-antimonopole condensate appears to be spatially homogeneous.

**Remark 21.** To describe the average effect of tunneling between the two trajectories \(\frac{\rho(\lambda)}{T}\) and \(\frac{\rho(\lambda)}{T}\) for \(\bar{\lambda}_s \leq \bar{\lambda} \leq \bar{\lambda}_c\) and \(\lambda_s \leq \lambda \leq \lambda_c\), respectively, one may think of the following ‘droplet’ model. Let \(V \subset V_{\text{tot}}\) be two volumina. The thermal probability density \(P(V, V_{\text{tot}}, \bar{\lambda})\) for measuring a fraction \(\frac{V}{V_{\text{tot}}}\) of condensed magnetic monopoles and antimonopoles is given as

\[
P(V, V_{\text{tot}}, \bar{\lambda}) \equiv d(\bar{\lambda}) \lambda^3 \exp\left[\frac{d(\bar{\lambda})\lambda^3(V_{\text{tot}} - V)}{\exp[d(\bar{\lambda})\lambda^3 V_{\text{tot}}] - 1}\right].
\]

\(^{19}\)The resolution is given by \(|\varphi|\)
Figure 4. The quantities $\rho(\lambda)/T^4$ (dashed line) and $\rho(\bar{\lambda})/T^4$ (solid grey line) in the preconfining phase as functions of $\lambda$. The black solid line is associated with $\rho(\lambda)/T^4$ in the deconfining phase.

where $d(\bar{\lambda}) \equiv \Delta(\bar{\lambda}) \frac{\bar{\lambda}}{(2\pi)^3}$, and $\Delta$ is the temperature-dependent (positive) difference between trajectories $\rho(\bar{\lambda})/T^4$ and $\rho(\lambda)/T^4$. Notice that for $d(\bar{\lambda})\bar{\lambda}^3V_{\text{tot}} \gg 1$ the probability density $P(V,V_{\text{tot}},\bar{\lambda})$ ceases to depend on $V_{\text{tot}}$. In the model of Eq. (81) and for SU(2) the average polarization number $N_p$ of the U(1) gauge field calculates as

$$N_p(\bar{\lambda}) = \int_0^{V_{\text{tot}}} dV P(V,V_{\text{tot}},\bar{\lambda}) \left( 3 \frac{V}{V_{\text{tot}}} + 2 \frac{V_{\text{tot}} - V}{V_{\text{tot}}} \right) = 2 + \int_0^{V_{\text{tot}}} dV P(V,V_{\text{tot}},\bar{\lambda}) \frac{V}{V_{\text{tot}}}.$$  (82)

Keeping $V_{\text{tot}}$ fixed, this yields $\lim_{\bar{\lambda} \to \bar{\lambda}_c} N_p = \lim_{d \to 0} N_p = \frac{5}{2}$. A similar model for the regime $\bar{\lambda}_c' \leq \bar{\lambda} \leq \bar{\lambda}_s$ shows that $N_p$ increases towards $N_p = 3$ for $\bar{\lambda} \searrow \bar{\lambda}_c'$.

5 Confining phase

In this section we turn to the confining phase which starts to set in at the temperature $T_c'$ where formerly instable, untwisted center-vortex loops become stable and massless \cite{7} and the dual gauge field decouples because its mass diverges, see Rem. 15. At $T_c'$ the magnetic $Z_2$ (for SU(2)) and $Z_3$ (for SU(3)) symmetries start to be broken dynamically. Twisted or untwisted center-vortex loops, which are liberated during the subsequent decay of the monopole-antimonopole condensate, are interpreted as spin-1/2 fermions.

The transition from the preconfining to the confining phase is genuinely nonthermal: Relying on the results of \cite{59} for the number of connected bubble diagrams in a $\lambda\phi^4$ theory one proves by Borel summation and analytic continuation that the pressure increasingly develops sign-indefinite imaginary parts as temperature is increased from zero towards $T_c'$ \cite{60}. One also proves that at $T = 0$ the pressure is precisely zero, see below \cite{60}.

In the preconfining phase closed lines of magnetic flux form by the collective dissociation of large-holonomy calorons/anticalorons. Inside their cores magnetic monopoles travel oppositely directed to their antimonopoles along the direction of the flux. The magnetic flux $F_{\pm,0}$ through

\footnote{Since there are no isolated magnetic charges in the preconfining phase (monopoles and antimonopoles are condensed) these flux lines cannot end and thus are closed.}
Thus the expectation Φ of ˆΦ changes phase under a singular gauge transformations of
the contour C: mediated by a local magnetic center jump. Such a center jump is associated with
an extra quantum of center flux, induced by a center-vortex loop piercing MC in addition the
multitude of vortices which had generated the finite value of Φ to begin with. The process of
having an additional vortex pierce MC proceeds in real time, and it is clear that the smooth
dynamics of having Φ change its phase is described by a complex scalar field even though for
SU(2) it is sufficient for the equilibrium situation to assume Φ to be a Z2-charged real quantity.
(Imagine the adiabatic limit, where no kinetic energy is associated with the piercing center-
 vortex loop, no change in energy is conveyed to Φ by this process. If Φ were real and the process
were to be described by smooth dynamics then Φ would have to change its sign in a smooth
way (exhibiting a zero in the process) which is in contradiction to energy conservation.)

Consider now a spatial circle of infinite radius Sr=∞ centered at x̅. The thermally averaged
flux F±,0;th of a system of a center-vortex loop and its flux-reversed partner at rest through
Ar=∞ is in the limit of vanishing core-size and mass (λ → λc, g → ∞) given as

\[ F_{±,0} = \begin{cases} \pm \frac{2\pi}{g} i & 0 \end{cases} \]

depending on whether a center-vortex flux pierces A once (±\frac{2\pi}{g}) or whether it pierces AC not
at all or twice (0). Notice that F±,0 does not depend on the velocity of the train of monopoles
and antimonopoles travelling along the vortex line, for a discussion see [7].

The phase of the dual order-parameter for confinement, the ’t Hooft-loop expectation (a complex
field Φ), takes on discrete values. These are 0, iπ for SU(2) and 0, ±\frac{2\pi}{g}i for SU(3).

The ’t Hooft-loop operator Φ(⃗x, C) [62] is defined as the exponential of the magnetic flux of the
dual gauge field aDµ along the minimal surface MC spanned by an oriented and closed spatial
curve C centered at the point ⃗x:

\[ \hat{Φ}(⃗x, C) \propto \exp[i g \oint_C dz_i a_i^D] . \]

Thus the expectation Φ of Φ changes phase under a singular gauge transformations of aDµ along
the contour C mediated by a local magnetic center jump. Such a center jump is associated with
an extra quantum of center flux, induced by a center-vortex loop piercing MC in addition the
multitude of vortices which had generated the finite value of Φ to begin with. The process of
having an additional vortex pierce MC proceeds in real time, and it is clear that the smooth
dynamics of having Φ change its phase is described by a complex scalar field even though for
SU(2) it is sufficient for the equilibrium situation to assume Φ to be a Z2-charged real quantity.
(Imagine the adiabatic limit, where no kinetic energy is associated with the piercing center-
vortex loop, no change in energy is conveyed to Φ by this process. If Φ were real and the process
were to be described by smooth dynamics then Φ would have to change its sign in a smooth
way (exhibiting a zero in the process) which is in contradiction to energy conservation.)

Consider now a spatial circle of infinite radius Sr=∞ centered at x̅. The thermally averaged
flux F±,0;th of a system of a center-vortex loop and its flux-reversed partner at rest through
Ar=∞ is in the limit of vanishing core-size and mass (λ → λc, g → ∞) given as

\[ \lim_{g \to \infty} F_{±,0;th} = 4\pi \int d^3 p \delta^{(3)}(\vec{p}) n_B(\beta, 2 E_v(\vec{p}, \tilde{\lambda}_c)) \]  

\[ F_{±,0} = \begin{cases} 0 & \pm \frac{\chi^{3/2}}{\pi} \end{cases} \]

where \( E_v(0, \tilde{\lambda}) \sim \pi |\varphi(\tilde{\lambda})| \) is the typical mass of a single center-vortex loop at temperature \( \tilde{\lambda} \) [7].
Notice the use of the Bose function nB for the system of two center-vortex loops of opposite-flux:
Even though each vortex loop is interpreted as a spin-1/2 fermion (two polarizations also in the
case of selfintersections [7, 61]) the system is of spin zero. The value zero in Eq. (85) is realized
if Ar=∞ is pierced an even number of times by the center-vortex loops in the system, and the
values \( \pm \frac{\chi^{3/2}}{\pi} \) correspond to odd numbers of piercings. It is obvious that in the SU(2) case an
identification of \( \pm \frac{\chi^{3/2}}{\pi} \) takes place which is not true for SU(3). Properly normalized, the discrete
values in Eq. (85) are phase changes in Φ for the creation of a single center-vortex loop. In SU(2)
they are from 0 to iπ and from iπ to 0. For SU(3) the process 0 to ±i\frac{2\pi}{3} and ±i\frac{2\pi}{3} to 0 create
two distinct species of center-vortex loops. Since the spatial extent of a given center-vortex loop
is unresolvable (g / ∞), for discussion see [7]) and since in the condensate the distance between
a center-vortex loop and its flux-reversed partner is zero, the vanishing-curvature situation is
trivially saturated at finite curvature, Sr=∞.

The process of decay of the monopole-antimonopole condensate(s) and the formation of the
center-vortex condensate is described by real-time dynamics of the order-parameter $\Phi$ subject to the potentials

\[ V(\Phi) = \left( \tilde{\Lambda}^3 - \tilde{\Phi} \right) \left( \frac{\tilde{\Lambda}^3}{\Phi} - \tilde{\Lambda} \right), \quad \text{(for SU(2))}, \]

\[ V(\Phi) = \left( \tilde{\Lambda}^3 - \Phi^2 \right) \left( \frac{\tilde{\Lambda}^3}{\Phi^2} - \tilde{\Lambda} \right), \quad \text{(for SU(3))}, \tag{87} \]

where $\tilde{\Lambda} \sim 2^{1/3} \bar{\Lambda}$ (for SU(2)) and $\tilde{\Lambda} \sim \bar{\Lambda}$ (for SU(3)).

At the onset of center-vortex loop condensation thermal equilibrium is maintained at overall negative pressure. Periodic BPS saturated trajectories\(^{21}\) along euclidean time, describing the onset of vortex-loop condensation, exist for the potentials in Eqs. (86) and (87), see \([63]\). As in the other two phases, this periodicity is due to the pole-term in the ‘square-root’ of $V$ which endows the field $\Phi$ with a winding number. (The ‘superpotential’ $W$ has a branch cut, see \([63]\).) Furthermore, the potential $V$ needs to satisfy the following requirements: (i) invariance under (local) magnetic center transformation only (no larger symmetry), (ii) dynamical realization of the latter (flux creation, negative tangential curvature for jump-like behavior in real time), (iii) the only minima of $V$ are center-degenerate and at zero energy density (center-vortex loops in condensate do not interact and are massless), and (iv) as in the other two phases, a single mass scale $\tilde{\Lambda}$ enters $V$. It is easy to check, see also \([64]\), that modulo U(1) invariant rescalings and adding term of the form $\Delta V = \kappa \left( \tilde{\Lambda}^2 - \tilde{\Lambda}^{-2(n-1)}(\bar{\Phi}\Phi)^n \right)^{2k}$, $(\kappa > 0, k = 1, 2, 3, \ldots, n \in \mathbb{Z})$, which do increase the curvature of $V$ at its minima, the potentials in Eqs. (86) and (87) are unique. Demanding at the onset of the condensation of center-vortex loops that the (negative) pressure be continuous in the euclidean formulation, the above relation between the scales $\bar{\Lambda}$ and $\tilde{\Lambda}$ follows.

**Remark 22.** Writing $\Phi = |\Phi| \exp[i \frac{\Theta}{\tilde{\Lambda}}]$, one has

\[
\frac{\partial^2 V(\Phi)}{|\Phi|^2} |_{\Phi_{\text{min}}} = \frac{\partial^2 V(\Phi)}{|\Phi|^2} |_{\Phi_{\text{min}}} = \begin{cases} 8 & \text{(SU(2))} \\ 18 & \text{(SU(3))} \end{cases},
\]

where $\Phi_{\text{min}} = \pm \bar{\Lambda}$ (for SU(2)) and $\Phi_{\text{min}} = \tilde{\Lambda} \exp[i \frac{2\pi k}{3}]$, $(k = 0, 1, 2)$, (for SU(3)). Since $|\Phi_{\text{min}}|$ is the scale of maximal resolution once $\Phi$ has settled into one of its minima we conclude that the field $\Phi$ does no longer fluctuate. This, in turn, implies that no tunneling to another minimum (flux creation) takes place once $\Phi$ has settled into $\Phi_{\text{min}}$\(^{7}\).

Naively, that is, without taking into account contact interactions between and internal excitations within (twisted) center-vortex loops, the thermodynamic SU(2) pressure is estimated by the following asymptotic-series representation\(^{22}\)

\[
P_{\infty} \leq \frac{M^4}{2\pi^2} \hat{\beta}^{-4} \left( \frac{7\pi^4}{180} + \sqrt{2\pi} \hat{\beta}^{3/2} \sum_{l=0}^{L} a_l \sum_{n \geq 1} (32\lambda)^n n^{2+1} \right), \tag{90} \]

\(^{21}\)A canonic kinetic term in $\Phi$’s effective action

\[
S = \int d^4 x \left( \frac{1}{2} \left( \partial_\mu \Phi \right)^* \partial^\mu \Phi - \frac{1}{2} V \right) \tag{88} \]

is inherited from the effective action for the field $\varphi$: At the onset of vortex condensation thermal equilibrium prevails, and the vortex condensate coincides with the monopole-antimonopole condensate.

\(^{22}\)Apologies for introducing the variable $\lambda$ twice in this paper, here with a different meaning than in Secs.\(^3\) and
Figure 5. The core of a selfintersection in a center vortex loop.

Figure 6. Untwisted and twisted center-vortex loops up to $n = 3$

where $\tilde{\beta} \equiv \frac{M}{T}$, $M \sim \tilde{\Lambda}$, $\lambda \equiv e^{-\tilde{\beta}}$, $L < \infty$, and $a_l \in \mathbb{Z}$.

According to Rem. [22] no (naive) contribution to the pressure arises from the ground state: The fermionic gas has thermalized to a given temperature by the decay of the monopole-antimonopole condensate, and the field $\Phi$ is settled into one of the minima of the potential $V$. Because of its two polarization states (two directions of center flux for both SU(2) and SU(3) inherited from an untwisted progenitor center-vortex loop) any center-vortex loop with $n$ selfintersections ($n = 0, 1, 2, \cdots$) is interpreted as a spin-1/2 fermion. Its mass is $nM$ where $M$ corresponds to the mass of a single intersection point (a $\mathbb{Z}_2$ or a $\mathbb{Z}_3$ monopole or antimonopole is associated with the core of the flux eddy, see Fig. 5 marking the intersection – a plastic visualization of the concept of spin) which, in turn, is comparable to the scale $\tilde{\Lambda}$. Since there are two possible charges of the monopole singled out in the core of the intersection there are $C_n = 2^n$ many possible charge states of a center-vortex loop with $n$ selfintersections. Topologically, the multiplicity $N_n$ of these solitons is known exactly up to $n = 6$ [65], see Fig. 6. For $n \gg 1$ the form $N_n \sim \left( \sum_{l=0}^{L} a_l n^l \right) n! 16^n$ was obtained in [59] by an analysis of the ground-state energy of the
anharmonic quantum mechanical oscillator ($\lambda\phi^4$-theory in one dimension). Naturally, $a_l \in \mathbb{Z}$. Taking into account the spin degeneracy, the total multiplicity $M_n$ of a center-vortex loop with $n$ selfintersections is given as

$$M_n = 2 \times N_n \times C_n \equiv 2 \times 2^n \times \left( \sum_{l=0}^{L} a_l n^l \right) n! 16^n. \quad (91)$$

Separating off the massless sector (single center-vortex loops) and negelecting any interaction and internal excitability, one has

$$P_n = \frac{M^4}{2\pi^2} \beta^{-4} \left( \frac{7\pi^4}{180} + \beta^3 \sum_{n \geq 1} M_n \int_0^\infty dx x^2 \log \left( 1 + e^{-\beta \sqrt{n^2 + x^2}} \right) \right)$$

$$\leq \frac{M^4}{2\pi^2} \beta^{-4} \left( \frac{7\pi^4}{180} + \beta^3 \sum_{n \geq 1} M_n n^2 K_2(n\beta) \right)$$

$$\sim \frac{M^4}{2\pi^2} \beta^{-4} \left( \frac{7\pi^4}{180} + \sqrt{\frac{\pi}{2}} \beta^2 \sum_{n \geq 1} M_n \lambda^n n^2 \right)$$

$$\leq \frac{M^4}{2\pi^2} \beta^{-4} \left( \frac{7\pi^4}{180} + \frac{2}{\pi} \beta^2 \sum_{l=0}^{L} a_l \sum_{n \geq 1} (32\lambda)^n n^2 n^\frac{4}{2} + l \right). \quad (92)$$

Here $K_2$ denotes a modified Bessel function. In Eq. (92) the first $\leq$ sign holds strictly for the linear truncation of the expansion of the logarithm about unity, and the $\sim$ sign indicates that terms of order $(\beta n)^{-1}$ have been neglected in the asymptotic expression for the Bessel function. This is relevant for studying the analyticity structure of the Borel resummed series. The second $\leq$ sign holds because we have made use of the large-$n$ expression for $N_n$ of Eq. (91). Obviously, the expression in Eq. (90) represents an asymptotic series in $\lambda$ (zero radius of convergence). Upon sending $\lambda \rightarrow -\lambda$ in Eq. (92) notice the formal similarity to the expansion of the ground-state energy of an anharmonic quantum mechanical oscillator \[66, 67\] ($\lambda\phi^4$-theory in one dimension) for which Borel summability was proven, see \[65\] and refs. therein.

The fact that the (naive) partition function diverges because of an over-exponentially in energy rising density of states is known to be associated with a so-called Hagedorn transition \[68\].

The asymptotic estimate in Eq. (90) is Borel summable for $\lambda < 0$.

Notice that the case $\lambda < 0$ corresponds to an analytic continuation from positive-real values of $\beta = \beta_1 + i\beta_2$ to complex values: $\beta_2 = 0 \rightarrow \beta_2 = \pm \pi$.

Let us now show the above claimed Borel summability. Defining $\bar{P}_\text{mass}(\lambda) \equiv P_n - \frac{7\pi^4}{7200} \left( \frac{M}{\beta} \right)^4$ and $\bar{\lambda} \equiv 32\lambda$, the Borel transformation of $\bar{P}_\text{mass}(\bar{\lambda})$ is given as

$$\bar{P}_\text{mass}(\bar{\lambda}) \equiv \sum_{l=0}^{L} a_l \sum_{n \geq 1} \bar{\lambda}^n n^\frac{4}{2} + l \xrightarrow{\text{Borel}} B\bar{P}_\text{mass}(\bar{\lambda}) \equiv \sum_{l=0}^{L} a_l \sum_{n \geq 1} \bar{\lambda}^n n^\frac{4}{2} + l. \quad (93)$$

\footnote{Apologies for introducing the variable $\bar{\lambda}$ twice in this paper, here with a different meaning than in Sec.4}
Thus, $B_{\text{mass}}(\bar{\lambda})$ is a superposition of polylogarithms:

$$B_{\text{mass}}(\bar{\lambda}) = \sum_{l=0}^{L} a_l \text{Li}_{-\left(\frac{3}{2} + t\right)}(\bar{\lambda}). \quad (94)$$

The functions $\text{Li}_{-\left(\frac{3}{2} + t\right)}(\bar{\lambda})$ are real-analytic for $\bar{\lambda} < 1$. To perform the inverse Borel transformation

$$\hat{P}_{\text{mass}}(\bar{\lambda}) \equiv \sum_{l=0}^{L} a_l \hat{P}_l(\bar{\lambda}) \equiv \int_{0}^{\infty} dt \, e^{-t} B_{\text{mass}}(\bar{\lambda} t), \quad (95)$$

where

$$\hat{P}_l(\bar{\lambda}) \equiv \int_{0}^{\infty} dt \, e^{-t} \text{Li}_{-\left(\frac{3}{2} + t\right)}(\bar{\lambda} t), \quad (96)$$

we notice the following integral representation of $\text{Li}_s(z)$, valid for all $s, z \in \mathbb{C}$ [69]:

$$\text{Li}_s(z) = \frac{i z}{2} \int_{C} du \, \frac{(-z)^u}{(1 + u)^s \sin(\pi u)}, \quad (97)$$

where the path $C$ is along the imaginary axis from $-i\infty$ to $+i\infty$ with an indentation to the left of the origin. Inserting Eq. (97) into Eq. (96) for $\bar{\lambda} = -|\lambda| < 0$ and interchanging the order of integration, we have

$$\hat{P}_l(\bar{\lambda}) = -i \int_{C} du \, \frac{(1 + u)^{\frac{3}{2} + l}}{1 - e^{-2\pi i u}} \, e^{-\pi i u} e^{(1 + u) \log(-\bar{\lambda})} \Gamma(u + 2). \quad (98)$$

Since, by Stirling’s formula [24] the gamma function $\Gamma(u + 2)$ decays exponentially fast for $u \to \pm i\infty$, the integral over $u$ in Eq. (98) exists and defines the real-analytic [25] function $\hat{P}_l(\bar{\lambda})$, $(\bar{\lambda} < 0)$.

The function $\hat{P}_l(\bar{\lambda})$ is analytic for a much larger range $\bar{\lambda} \in \mathbb{C}$. Notice, however, that for $\arg(\bar{\lambda}) \to \pm \pi$ a branch cut is expected for $\hat{P}_l(\bar{\lambda})$. Also, one shows that $\hat{P}_l(0) = 0$ by using $e^{(1 + u) \log(-\bar{\lambda})} = -\lambda e^{u \log(-\bar{\lambda})}$ in Eq. (98).

The approximate behavior $\Phi_l(\bar{\lambda})$ of $\hat{P}_l(\bar{\lambda})$ is suggested [26] as follows:

$$\Phi_l(\bar{\lambda}) = \frac{\sum_{r=0}^{R_l} \alpha_{2r+1} \left(\log\left(-\gamma_{2r+1}\bar{\lambda}\right)\right)^{2r+1}}{\sum_{s=0}^{S_l} \beta_{2s} \left(\log\left(-\delta_{2s}\bar{\lambda}\right)\right)^{2s}}, \quad (99)$$

where $\gamma_{2r+1}, \delta_{2s} \in \mathbb{R}_+, \alpha_{2r+1}, \beta_{2s} \in \mathbb{R}$ and $S_l = R_l + 1$. Numerically, one has for example

$$\Phi_0(\bar{\lambda}) = 0.0570 \frac{\log(-0.154\bar{\lambda})}{1 + 0.220(\log(-0.494\bar{\lambda}))^2},$$

$$\Phi_1(\bar{\lambda}) = 0.0212 \frac{\log(-10.2\bar{\lambda}) + 0.00142(\log(-0.109\bar{\lambda}))^3}{1 + 0.128(\log(-1.09\bar{\lambda}))^2 + 0.0544(\log(-0.886\bar{\lambda}))^4}. \quad (100)$$

---

24 $\Gamma(z) = \sqrt{2\pi} \, e^{-z^2/2} \, e^{H(z)}$ where $H(z) \equiv \sum_{n \geq 0} \left(\frac{z + n + 1/2}{z + n}\right) \log\left(1 + \frac{1}{z + n}\right)$ converges for $z \in \mathbb{C}_-$ and $\lim_{z \to -\infty} H(z) = 0$, see [70].

25 This follows from Eq. (99) and the fact that $\text{Im} \left[\text{Li}_{-\left(\frac{3}{2} + t\right)}(z)\right] \equiv 0$ for $z \leq 0$.

26 One has:

$$\int_{0}^{\infty} dx \, e^{-ax} \sin(\log(-\bar{\lambda}) x) = \frac{a}{a^2 + (\log(-\bar{\lambda}))^2}, \quad \int_{0}^{\infty} dx \, e^{-ax} \cos(\log(-\bar{\lambda}) x) = \frac{\log(-\bar{\lambda})}{a^2 + (\log(-\bar{\lambda}))^2}, \quad (a > 0).$$
Since for $\Phi_l$ one has $\Phi_l(\bar{\lambda} = 0) = 0$ due to a higher power of the logarithmic singularity in the numerator than in the denominator it is clear that $|\text{Im}\Phi_l|$ grows slower than $|\text{Re}\Phi_l|$ for sufficiently small, real-positive values of $\lambda$ increasing from zero. Also, $\text{Re}\Phi_l$ is continuous across the branch cut. The growing importance of imaginary contaminations of the physical pressure with increasing temperature signals the growing deviation from genuine thermal equilibrium: A sign-indefinite imaginary part implies the existence of exponentially fast growing and decaying plasma modes and thus turbulences.

Since the only difference for SU(3) is the occurrence of two types of center-vortex loops one obtains the result for the latter by simply multiplying the SU(2) result by two.

6 Conclusion

A detailed discussion and partial analysis of the thermodynamics of SU(2) and SU(3) Yang-Mills theory has been given. As for the case of SU(2) there appears to be a wealth of applications in particle physics [7,8], cosmology [2,3,4,5], and plasma physics [60,71] (dark energy by virtue of the axial anomaly, leptons and their interactions). To enable future contact with experiment in the case of SU(3) (strong interactions) the dynamics of electric-magnetically dual gauge-group factors (fractional quantum Hall effect) needs to be explored in their confining phases.

Acknowledgments

The author would like to acknowledge joyful collaboration with Francesco Giacosa on some aspects of the here-presented material. I am also grateful for excellent work conducted by my students. Many thanks go to Nucu Stamatescu for interesting discussions, and to Markus Schwarz and Francesco Giacosa for their helpful comments on the manuscript. I am indebted to my family, in particular to my wife Karin Thier, for their persistent help and understanding over the last four years. I also would like to thank Frans Klinkhamer for suggesting the present paper and for having the far-sight and fairness to care about my institutional survival. Finally, I would like to thank the Referee for a very thorough assessment of the manuscript and various helpful suggestions for improvement.

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