Topological Index for Free–Fermion Systems in Disordered Media

N. J. B. Aza  A. F. Reyes-Lega  L. A. Sequera M.

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Abstract

We use infinite dimensional self–dual CAR C∗–algebras to study the existence of a Z2–index, which classifies free–fermion systems embedded on Zd disordered lattices. Combes–Thomas estimates are pivotal to show that the Z2–index is uniform with respect to the size of the system. We additionally deal with the set of ground states to completely describe the mathematical structure of the underlying system. Furthermore, the weak∗–topology of the set of linear functionals is used to analyze paths connecting different sets of ground states.

Keywords: Operator Algebras, Disordered fermion systems, Z2–index, ground states.

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1 Introduction

A considerable number of mathematical results concerning gapped Hamiltonians of fermions has been achieved in recent years. Among the most important ones are topological protection under small perturbations and the persistence of the spectral gap for interacting fermions [Has19, DS19]. We study a Z2–projection index (Z2–PI) that generalizes the one introduced long ago by Araki–Evans in their work where the two possible thermodynamic phases of the classical two-dimensional Ising model
are characterized using operator algebras technologies [AE83]. Here we deal with disordered free–fermion systems on the lattice within the mathematical framework of self–dual CAR C∗–algebras. In particular, their structure contains the information on the symmetries of free fermions embedded in disordered systems, and is also useful to study interacting fermion systems, even with superconducting terms [ABPM20]. To be precise, the Z2–PI is defined in terms of well–defined basis projections related to a self–adjoint operator, which typically is the Hamiltonian of the system acting on a separable Hilbert space H. See Definition 2 below. Thus, the Z2–PI is introduced to discriminate nonequivalent representations [EK98, BVF01].

A very important problem in this context is the classification of topological matter in general. The current classification scheme can be traced back to Dyson’s [Dys62] classical work from 1962. Of course that work did not contemplate topological aspects for such systems, but it provided the setting on which more recent work has been based. Indeed, a completion of this early work was made by Altland and Zirnbauer [AZ97], leading to the identification of new symmetry classes. These ideas were generalized by Kitaev [Kit09] and led to a “periodic table” of topological insulators and superconductors. In that work, Kitaev showed how the classification can be achieved in terms of Bott periodicity and K–theory. More recently an exhaustive and complete version of the classification was made by Ryu et al. [RSFL10]. They explore arbitrary dimensions making use again of classifying spaces given by the Cartan symmetric spaces along with the Bott periodicity in a more strong way. This allows them to consider disordered systems and shows the explicit relation between gapped Hamiltonians and Anderson localization phenomena, a very important result for this kind of problem.

The first iconic example of a topological fermionic system is the quantum Hall effect. The observed quantization of the conductivity was explained by Thouless et al. [TKNdN82] and led to the recognition of the important role played by the Chern number. The restrictions on the validity of this result where eventually overcome by Bellissard [BvESB94] and collaborators, in what was to become one of the main examples of applications of noncommutative geometry to physics. This was a big step to deal with more realistic models that consider disordered media. In this line of ideas there are more recent works, due to Carey et al [CHM+06], [BCR16], where Bellissard’s techniques are generalized to deal with a wider class of systems.

For interacting systems rigorous proofs of quantization of conductivity were provided [GMP16, BDF18]. These studies rely on the study of families of gapped Hamiltonians, such that any two elements on these families can be continuously deformed into one another. The latter was demonstrated rigorously by Bachmann, Michalakis, Nachtergaele and Sims [BMNS12] by studying spectral flow of quantum spin system under a “quasi–adiabatic” evolution. They proved that such related systems verify the same Lieb–Robinson bounds and in its thermodynamic limit the spectral flow has a cocycle structure for the automorphism in the algebra of observables. By using the dual space of the underlying algebras considered they also studied the ground states associated.

From the point of view of physics, fundamental properties of such systems are deduced from the study of the set of ground states in the thermodynamic limit and zero temperature. Relevant examples include electronic conduction problems (e.g. quantum Hall effect), or the study of different phases of matter. Nevertheless, knowledge of ground states of concrete models is a huge challenge in general. This is due to the fact that there is no general procedure to find the full set of ground states for specific systems. As far as we know there are very few mathematical physics results about the existence of ground states, in contrast to the theoretical point of view, see [AT85, CNN18]. Instead, one generally verifies the existence of the ground state energy for specific physical systems.

In this paper we focus on the study of Z2–PI for non–interacting fermion systems. We specifically deal with unique ground states associated to families of gapped Hamiltonians. In a subsequent paper

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1Ground state energy can be understood as the states associated to the lowest energy of a physical system. For example, Giuliani and Jauslin use rigorous renormalization methods to prove the existence of the ground state energy for the bilayer graphene [GJ16].
[AR20], we will report on results about the $\mathbb{Z}_2$–PI for interacting fermions. Observe that the technical tools in that case differ from the current study and other technologies such that Lieb–Robinson bounds will be required.

To conclude, our main results are Theorems 1 and 2, as well as the set of Lemmata 1, 2 and 3. From the mathematical point of view, Theorem 1 is reminiscent of the interacting case, however, we additionally state the $\mathbb{Z}_2$–PI result, classifying the parity of the Bogoliubov $\star$–automorphism of the infinite self–dual CAR $C^\star$–algebras. On the other hand, Theorem 2 deals with subsets of the ground states set. In particular, open spectral gap ground states are considered. As a particular case of the general Theorem 2, we prove that in the weak$^\star$–topology, paths connecting states in different components of the $\mathbb{Z}_2$–PI implies the existence of a Hamiltonian having 0 as a eigenvalue.

The paper is organized as follows:

• Section 2 presents the mathematical framework of CAR $C^\star$–algebras. We introduce self–dual CAR $C^\star$–algebras, which were introduced long ago by Araki in his elegant study of non–interacting but non–gauge invariant fermion systems. We recall pivotal properties of general CAR $C^\star$–algebras.

• In Section 3 we state the main Theorems, as well as some relevant definitions concerning the $\mathbb{Z}_2$–PI and comment on the weak$^\star$–topology of the set of states. In particular we discuss the conditions for a system to have pure or mixed states.

• Section 4 is devoted to all technical proofs. We prove the existence of a spectral flow automorphism for self–dual Hilbert spaces, for families of differentiable Hamiltonians with unbounded one–site potentials. Then, the existence of strong limits for the dynamics, the spectral flow automorphism and the weak$^\star$–convergence of ground states are proven. Well–known Combes–Thomas estimates are invoked for families of gapped Hamiltonians, which will permit to analyze two–point correlation functions such that we obtain the trace class properties for relevant unitary operators.

• We finally include Appendix A, providing a general framework of graphs with special attention to disordered models.

Notation 1.

A norm on the generic vector space $\mathcal{X}$ is denoted by $\| \cdot \|_{\mathcal{X}}$ and the identity map of $\mathcal{X}$ by $1_{\mathcal{X}}$. The space of all bounded linear operators on $(\mathcal{X}, \| \cdot \|_{\mathcal{X}})$ is denoted by $\mathcal{B}(\mathcal{X})$. The unit element of any algebra $\mathcal{A}$ is always denoted by 1, provided it exists of course. The scalar product of any Hilbert space $\mathcal{X}$ is denoted by $\langle \cdot , \cdot \rangle_{\mathcal{X}}$ and $\text{Tr}_{\mathcal{X}}$ represents the usual trace on $\mathcal{B}(\mathcal{X})$.

2 Mathematical Framework and Physical Setting

We introduce the mathematical framework based on Araki’s self–dual formalism [Ara68, Ara71]. Our setting considers disorder effects, which come as is usual in physics, i.e., from impurities, crystal lattice defects, etc. Thus, disorder can modeled by (a) a random external potential, like in the celebrated Anderson model, (b) a random Laplacian, i.e., a self–adjoint operator defined by a next–nearest neighbor hopping term with random complex–valued amplitudes. In particular, random vector potentials can also be implemented. (c) Finally, one–site unbounded operators are in the scope of the current work.

2.1 Self–dual CAR Algebra

If not otherwise stated, $\mathcal{H}$ always stands for a (complex, separable) Hilbert space. If $\mathcal{H}$ is finite–dimensional, we will assume it is even–dimensional, i.e., $\dim \mathcal{H} \in 2\mathbb{N}$. Let $\Gamma : \mathcal{H} \to \mathcal{H}$ be an
antiunitary involution on \( \mathcal{H} \), i.e., an antilinear operator such that \( \Gamma^2 = 1_{\mathcal{H}} \) and\(^2\)

\[
\langle \Gamma \varphi_1, \Gamma \varphi_2 \rangle_{\mathcal{H}} = \langle \varphi_2, \varphi_1 \rangle_{\mathcal{H}}, \quad \varphi_1, \varphi_2 \in \mathcal{H}.
\]

The space \( \mathcal{H} \) endowed with the involution \( \Gamma \) is named a \textit{self–dual Hilbert space} and yields \textit{self–dual CAR algebra}:

**Definition 1 (Self–dual CAR algebra).**

A self–dual CAR algebra \( \text{sCAR}(\mathcal{H}, \Gamma) \equiv (\text{sCAR}(\mathcal{H}, \Gamma), +, \cdot, *) \) is a \( C^* \)–algebra generated by a unit \( 1 \) and a family \( \{ \text{B}(\varphi) \}_{\varphi \in \mathcal{H}} \) of elements satisfying Conditions 1.–3.:

1. The map \( \varphi \mapsto \text{B}(\varphi)^* \) is (complex) linear.
2. \( \text{B}(\varphi)^* = \text{B}(\Gamma(\varphi)) \) for any \( \varphi \in \mathcal{H} \).
3. The family \( \{ \text{B}(\varphi) \}_{\varphi \in \mathcal{H}} \) satisfies the CAR: For any \( \varphi_1, \varphi_2 \in \mathcal{H} \),
   \[
   \text{B}(\varphi_1)\text{B}(\varphi_2)^* + \text{B}(\varphi_2)^*\text{B}(\varphi_1) = \langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} \ 1. \]

For a historic verview on Self–dual CAR algebras and some of their basic properties see [Ara68, Ara71, Ara87, Ara88, EK98]. Note that by the CAR (1), the antilinear map \( \varphi \mapsto \text{B}(\varphi) \) is necessarily injective and contractive. Therefore, \( \mathcal{H} \) can be embedded in \( \text{sCAR}(\mathcal{H}, \Gamma) \).

Conditions 1.–3. of Definition 1 only define self–dual CAR algebras up to Bogoliubov \(*\)–automorphisms\(^3\) (see (5)). In [ABPM20] is done an explicit construction of a \(*\)–isomorphic self–dual CAR algebras from \( \mathcal{H} \) and \( \Gamma \). This is done via \textit{basis projections} [Ara68, Definition 3.5], whose highlights the relationship between CAR algebras and their self–dual counterparts.

**Definition 2 (Basis projections).**

A basis projection associated with \( \mathcal{H} \), \( \Gamma \) is an orthogonal projection \( P \in \mathcal{B}(\mathcal{H}) \) satisfying \( \Gamma P \Gamma = P^\perp \equiv 1_{\mathcal{H}^\perp} - P \). We denote by \( \mathfrak{h}_P \) the range \( \text{ran}(P) \) of the basis projection \( P \). The set of all basis projections associated with \( \mathcal{H} \), \( \Gamma \) will be denoted by \( \text{p}(\mathcal{H}, \Gamma) \).

For any \( P \in \text{p}(\mathcal{H}, \Gamma) \) a few remarks are in order.

\( \mathfrak{h}_P \) must satisfy the conditions

\[
\Gamma(\mathfrak{h}_P) = \mathfrak{h}_P^\perp \quad \text{and} \quad \Gamma(\mathfrak{h}_P^\perp) = \mathfrak{h}_P.
\]

By [Ara68, Lemma 3.3], an explicit \( P \in \text{p}(\mathcal{H}, \Gamma) \) can always be constructed because \( \dim \mathcal{H} = 2N \). Moreover, \( \varphi \mapsto (\Gamma \varphi)^* \) is a unitary map from \( \mathfrak{h}_P^\perp \) to the dual space \( \mathfrak{h}_P^* \). In this case we can identify \( \mathcal{H} \) with

\[
\mathcal{H} \equiv \mathfrak{h}_P \oplus \mathfrak{h}_P^*,
\]

and

\[
\text{B}(\varphi) \equiv \text{B}_P(\varphi) \equiv \text{B}(P, \varphi) + \text{B} \left( \Gamma P^\perp \varphi \right)^*.
\]

Therefore, there is a natural isomorphism of \( C^* \)–algebras from \( \text{sCAR}(\mathcal{H}, \Gamma) \) to the CAR algebra \( \text{CAR}(\mathfrak{h}_P) \) generated by the unit \( 1 \) and \( \{ \text{B}_P(\varphi) \}_{\varphi \in \mathfrak{h}_P} \). In other words, a basis projection \( P \) can be used to fix so–called \textit{annihilation} and \textit{creations} operators. Each basis projection \( P \) associated with \( \mathcal{H}, \Gamma \),

\(^2\)We will assume that the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} \) associated to some Hilbert space \( \mathcal{H} \) is a sesquilinear form on \( \mathcal{H} \) such that is antilinear in its first component while is linear in the second one.

\(^3\)An analogous result for CAR algebra is, for instance, given by [BR03b, Theorem 5.2.5].
by (3), $\mathfrak{h}_P$ can be seen as a one–particle Hilbert space. We shall see below, that for simplicity, $\mathfrak{h}$ is the so–called one–particle Hilbert space, is finite–dimensional.

As shown in [Ara68, Ara71], self–dual CAR algebras naturally arise in the diagonalization of quadratic fermionic Hamiltonians (Definition 3), via Bogoliubov transformations defined as follows:

For any unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $UT = \Gamma U$, the family of elements $B(U \varphi)_{\varphi \in \mathcal{H}}$ satisfies Conditions (a)–(c) of Definition 1 and, together with the unit $1$, generates sCAR($\mathcal{H}$, $\Gamma$). Like in [Ara71, Section 2], such a unitary operator $U \in \mathcal{B}(\mathcal{H})$ commuting with the antiunitary map $\Gamma$ is called a Bogoliubov transformation, and the unique $^*$–automorphism $\chi_U$ such that

$$
\chi_U (B(\varphi)) = B(U \varphi), \quad \varphi \in \mathcal{H},
$$

is called in this case a Bogoliubov $^*$–automorphism. Note that a Bogoliubov transformation $U \in \mathcal{B}(\mathcal{H})$ always satisfies

$$
\det (U) = \det (\Gamma U) = \overline{\det (U)} = \pm 1.
$$

If $\det (U) = 1$, we say that $U$ is in the positive connected set $\mathbb{U}_+$. Otherwise $U$ is said to be in the negative connected set $\mathbb{U}_-$. $\chi_U (B(\varphi))$ is said to be even (respectively odd) if and only if $U \in \mathbb{U}_+$ (respectively $U \in \mathbb{U}_-$).

Clearly, if $P \in p(\mathcal{H}, \Gamma)$, see Definition 2, and $U \in \mathcal{B}(\mathcal{H})$ is a Bogoliubov transformation, then $P_U = U^*PU$ is another basis projection. Conversely, for any pair $P_1, P_2 \in p(\mathcal{H}, \Gamma)$ there is a (generally not unique) Bogoliubov transformation $U$ such that $P_2 = U^*P_1U$. See [Ara68, Lemma 3.6]. In particular, Bogoliubov transformations map one–particle Hilbert spaces onto one another.

Considering the Bogoliubov $^*$–automorphism (5) with $U = -1_{\mathcal{H}}$, an element $A \in \text{sCAR}(\mathcal{H}, \Gamma)$, satisfying

$$
\chi_{-1_{\mathcal{H}}} (A) = \begin{cases} 
A & \text{is called even}, \\
-A & \text{is called odd},
\end{cases}
$$

Note that the subspace $\text{sCAR}(\mathcal{H}, \Gamma)^+$ of even elements is a sub–$C^*$–algebra of $\text{sCAR}(\mathcal{H}, \Gamma)$.

It is well–known that in quantum mechanics are taken even elements for the description of fermion systems. For example, are used self–adjoint (even) elements of the CAR algebra which are quadratic in the creation and annihilation operators, like for instance the Bogoliubov approximation of the celebrated (reduced) BCS model. In the context of self–dual CAR algebra, those elements are called bilinear Hamiltonians and are self–adjoint bilinear elements:

**Definition 3 (Bilinear elements of self–dual CAR algebra).**

Given an orthonormal basis $\{\psi_i\}_{i \in I}$ of $\mathcal{H}$, we define the bilinear element associated with $H \in \mathcal{B}(\mathcal{H})$ to be

$$
\langle B, HB \rangle \doteq \sum_{i,j \in I} \big< \psi_i, H \psi_j \big> \mathcal{H} B \big( \psi_j \big) B \big( \psi_i \big)^*.
$$

Note that $\langle B, HB \rangle$ does not depend on the particular choice of the orthonormal basis, but does depend on the choice of generators $\{B(\varphi)\}_{\varphi \in \mathcal{H}}$ of the self–dual CAR algebra $\text{sCAR}(\mathcal{H}, \Gamma)$, and by (1), bilinear elements of $\text{sCAR}(\mathcal{H}, \Gamma)$ have adjoints equal to

$$
\langle B, HB \rangle^* = \langle B, H^*B \rangle, \quad H \in \mathcal{B}(\mathcal{H}).
$$

**Bilinear Hamiltonians** are then defined as bilinear elements associated with self–adjoint operators $H = H^* \in \mathcal{B}(\mathcal{H})$. They include all second quantizations of one–particle Hamiltonians, but also models that are not gauge invariant. Important models in condensed matter physics, like in the BCS theory of superconductivity, are bilinear Hamiltonians that are not gauge invariant.

Without loss of generality (w.l.o.g.), our analysis of bilinear elements can be restricted to operators $H \in \mathcal{B}(\mathcal{H})$ satisfying $H^* = -TH\Gamma$, which, in particular, have zero trace, i.e., $\text{Tr}_{\mathcal{H}} (H) = 0$. We call such operators self–dual operators:
Definition 4 (Self–dual operators).
A self–dual operator on \((\mathcal{H}, \Gamma)\) is an operator \(H \in \mathcal{B}(\mathcal{H})\) satisfying the equality \(H^* = -\Gamma H \Gamma\). If, additionally, \(H\) is self–adjoint, then we say that it is a self–dual Hamiltonian on \((\mathcal{H}, \Gamma)\).

We say that the basis projection \(P\) (Definition 2) (block–) “diagonalizes” the self–dual operator \(H \in \mathcal{B}(\mathcal{H})\) whenever

\[
H = \frac{1}{2} \left( PH_P P - P^\perp \Gamma H_P \Gamma P^\perp \right), \quad \text{with} \quad H_P = 2PH_P \in \mathcal{B}(\mathfrak{h}_P).
\]

In this situation, we also say that the basis projection \(P\) diagonalizes \(\langle B, HB\rangle\), similarly to [Ara68, Definition 5.1].

By the spectral theorem, for any self-dual Hamiltonian \(H\) on \((\mathcal{H}, \Gamma)\), there is always a basis projection \(P\) diagonalizing \(H\). In quantum physics, as discussed in Section 2.1, \(\mathfrak{h}_P\) is in this case the one–particle Hilbert space and \(H_P\) the one–particle Hamiltonian.

2.2 Quasi–Free Dynamics

Bilinear Hamiltonians are used to define so-called quasi–free dynamics: For any \(H = H^* \in \mathcal{B}(\mathcal{H})\), we define the continuous group \(\{\tau_t\}_{t \in \mathbb{R}}\) of \(*\)–automorphisms of \(\text{sCAR}(\mathcal{H}, \Gamma)\) by

\[
\tau_t(A) \doteq e^{-it\langle B, HB\rangle} A e^{it\langle B, HB\rangle}, \quad A \in \text{sCAR}(\mathcal{H}, \Gamma), \ t \in \mathbb{R}.
\]

Provided \(H\) is a self–dual Hamiltonian on \((\mathcal{H}, \Gamma)\) (Definition 4), this group is a quasi–free dynamics, that is, a strongly continuous group of Bogoliubov \(*\)–automorphisms, as defined in Equation (5). Straightforward computations using Definitions 1 and 3, together with the properties of the antiunitary involution \(\Gamma\), lead to show that

\[
\exp \left( -\frac{z}{2} \langle B, HB\rangle \right) B (\varphi)^* \exp \left( \frac{z}{2} \langle B, HB\rangle \right) = B \left( e^{zH} \varphi \right)^*,
\]

even for any self–dual operator \(H\) on \((\mathcal{H}, \Gamma)\), all \(z \in \mathbb{C}\) and \(\varphi \in \mathcal{H}\).

Moreover, for \(\{\tau_t\}_{t \in \mathbb{R}}\), we define the linear subspace

\[
\mathcal{D}(\delta) \doteq \{ A \in \text{sCAR}(\mathcal{H}, \Gamma) : t \mapsto \tau_t(A) \text{ is differentiable at } t = 0 \} \subset \text{sCAR}(\mathcal{H}, \Gamma)
\]

and the linear operator (unique, generally unbounded) \(\delta : \mathcal{D} \to \text{sCAR}(\mathcal{H}, \Gamma)\) by

\[
\delta(A) \doteq \left. \frac{d\tau_t(A)}{dt} \right|_{t=0}.
\]

The operator \(\delta\) is called the generator of \(\tau\) and \(\mathcal{D}(\delta)\) is the (dense) domain of definition of \(\delta\). Here we will assume that \(\delta\) is a symmetric unbounded derivation, i.e., the domain \(\mathcal{D}(\delta)\) of \(\delta\) is a dense \(*\)–subalgebra of \(\mathfrak{A}\) and, for all \(A, B \in \mathcal{D}(\delta)\),

\[
\delta(A)^* = \delta(A^*), \quad \delta(AB) = \delta(A)B + A\delta(B).
\]

Note that the set of all symmetric derivation on \(\mathcal{D}(\delta)\) can be endowed with a real vector space structure. In fact, for any symmetric derivations \(\delta_1\) and \(\delta_2\) and all real numbers \(\alpha_1, \alpha_2\), expression

\[
(\alpha_1\delta_1 + \alpha_2\delta_2)(A) \doteq \alpha_1\delta_1(A) + \alpha_2\delta_2(A), \quad A \in \mathcal{D}(\delta),
\]

gives rise to another symmetric derivation \(\alpha_1\delta_1 + \alpha_2\delta_2\) on \(\mathcal{D}(\delta)\).
2.3 States

A linear functional $\omega \in \text{sCAR}(\mathcal{H}, \Gamma)^*$ is a “state” if it is positive and normalized, i.e., if for all $A \in \text{sCAR}(\mathcal{H}, \Gamma)$, $\omega(A^*A) \geq 0$ and $\omega(1) = 1$. In the sequel, $\mathcal{E} \subset \text{sCAR}(\mathcal{H}, \Gamma)^*$ will denote the set of all states on $\text{sCAR}(\mathcal{H}, \Gamma)$. Note that any $\omega \in \mathcal{E}$ is Hermitian, i.e., for all $A \in \text{sCAR}(\mathcal{H}, \Gamma)$, $\omega(A^*) = \omega(A)$. Since $\text{sCAR}(\mathcal{H}, \Gamma)$ is a unital $C^*$-algebra, $\mathcal{E}$ is a weak$^*$-compact convex set, such that its extremal points coincide with the pure states [BR03a, Theorem 2.3.15]. The latter, combined with the fact that $\text{sCAR}(\mathcal{H}, \Gamma)$ is separable allows to claim that the set of states $\mathcal{E}$ is metrizable in the weak$^*$-topology [Rud91, Theorem 3.16]. Note that the existence of extremal points is a consequence of the Krein–Milman Theorem. More specifically, if $\mathbb{E}(\mathcal{E})$ denotes the set of extremal points of $\mathcal{E}$,

$$\mathcal{E} = \text{cch} \left( \mathbb{E}(\mathcal{E}) \right),$$

where, for $\mathcal{X}$ a Topological Vector Space and $A \subset \mathcal{X}$, $\text{cch}(A)$ refers to the closed convex hull of $A$. Such extremal points $\mathbb{E}(\mathcal{E})$ or pure states cannot be written as a linear combination of any states. As an application of the extremal states is that these are used to write any “mixed state” $\omega$.

In particular, if the state $\omega \in \mathcal{E}$ is separable allows to claim that the set of states $\mathcal{E}$ allows to write any “mixed state” $\omega \in \mathcal{E}$. By a mixed state $\omega \in \mathcal{E}$, we mean that, there are states $\{\omega_j\}_{j=1}^m \in \mathbb{E}(\mathcal{E})$, $m \in \mathbb{N}$, and positive real numbers, $0 \leq \lambda_j \leq 1$ for $j \in \{1, \ldots, m\}$, with $\sum_{j=1}^m \lambda_j = 1$ satisfying

$$\omega = \sum_{j=1}^m \lambda_j \omega_j. \tag{14}$$

In particular, if the state $\omega \in \mathcal{E}$ is pure, $\omega = \sum_{j=1}^m \lambda_j \omega_j$ implies that $\omega = \omega_1 = \cdots = \omega_m$, and $\lambda_1 = \cdots = \lambda_m = \frac{1}{m}$.

As is usual, for the state $\omega \in \mathcal{E}$ on $\text{sCAR}(\mathcal{H}, \Gamma)$, $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ denotes its associated cyclic representation: $\mathcal{H}_\omega$ is the Hilbert space associated to $\omega$, and is given by the closure of (the linear span) of the set $\{\pi_\omega(A)\Omega_\omega: A \in \text{sCAR}(\mathcal{H}, \Gamma)\}$,

$$\mathcal{H}_\omega = \pi_\omega(\text{sCAR}(\mathcal{H}, \Gamma))\Omega_\omega,$$

i.e., $\mathcal{H}_\omega$ is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}_\omega}$, $\pi_\omega$ a faithful representation from $\text{sCAR}(\mathcal{H}, \Gamma)$ into $\mathcal{B}(\mathcal{H}_\omega)$, the set of bounded operators acting on $\mathcal{H}_\omega$, and $\Omega_\omega \in \mathcal{H}_\omega$ is a unit cyclic vector with respect to $\pi_\omega(\text{sCAR}(\mathcal{H}, \Gamma))$. More specifically, for all $A \in \text{sCAR}(\mathcal{H}, \Gamma)$ we write

$$\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega).$$

$(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is the so-called GNS construction, which is unique up to unitary equivalence. If the state $\omega \in \mathcal{E}$ is mixed, see Expression (14), its associated representation $(\mathcal{H}_\omega, \pi_\omega)$ is reducible, that is, it can be decomposed as a direct sum $\pi_\omega = \bigoplus_{j \in \mathcal{J}} \pi_{\omega_j}$ on $\mathcal{H}_\omega = \bigoplus_{j \in \mathcal{J}} \mathcal{H}_{\omega_j}$. Here, $\{\mathcal{H}_j\}_{j \in \mathcal{J}}$ is a countable family of orthogonal Hilbert spaces, by meaning that for two different Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\{\mathcal{H}_j\}_{j \in \mathcal{J}}$, $\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}_j} = 0$ for all $\varphi_1 \in \mathcal{H}_1$ and all $\varphi_2 \in \mathcal{H}_2$. The set $\{\pi_{\omega_j}\}_{j \in \mathcal{J}}$ are representations of $\text{sCAR}(\mathcal{H}, \Gamma)$ on proper subspaces of $\mathcal{H}_\omega$. In particular if $\omega$ is pure, its representation $(\mathcal{H}_\omega, \pi_\omega)$ is irreducible and as mentioned $\omega$ is an extremal point $\mathbb{E}(\mathcal{E})$ of the set of states on $\text{sCAR}(\mathcal{H}, \Gamma)$.

States $\omega \in \mathcal{E}$ are said to be quasi–free when, for all $N \in \mathbb{N}_0$ and $\varphi_0, \ldots, \varphi_{2N} \in \mathcal{H}$,

$$\omega(B(\varphi_0) \cdots B(\varphi_{2N})) = 0, \tag{15}$$

\footnote{For the Topological Vector Space $\mathcal{X}$, $\overline{\mathcal{X}}$ denotes its closure.}
The state \( \omega \) is called the quasi–free state with symbol \( S_\omega \), which is the symbol of a quasi–free state on \( \mathcal{B}(\mathcal{H}) \), that is, a positive operator \( S_\omega \in \mathcal{B}(\mathcal{H}) \) such that
\[
0 \leq S_\omega \leq 1_{\mathcal{H}} \quad \text{and} \quad S_\omega + \Gamma S_\omega \Gamma = 1_{\mathcal{H}},
\]
through the conditions
\[
\langle \varphi_1, S_\omega \varphi_2 \rangle_{\mathcal{H}} = \varphi_1 \rho \varphi_2, \quad \varphi_1, \varphi_2 \in \mathcal{H}.
\]
Conversely, any self–adjoint operator satisfying (18) uniquely defines a quasi–free state through Equation (19). In physics, \( S_\omega \) is called the one–particle density matrix of the system. Note that any basis projection associated with \( (\mathcal{H}, \Gamma) \) can be seen as a symbol of a quasi–free state on sCAR\((\mathcal{H}, \Gamma)\). Such state is pure and called a Fock state [Ara71, Lemma 4.3]. Araki shows in [Ara71, Lemmata 4.5–4.6] that any quasi–free state can be seen as the restriction of a quasi–free state on sCAR\((\mathcal{H} \oplus \mathcal{H}, \Gamma \oplus (-\Gamma))\), the symbol of which is a basis projection associated with \( (\mathcal{H} \oplus \mathcal{H}, \Gamma \oplus (-\Gamma)) \). This procedure is called purification of the quasi–free state.

Quasi–free states obviously depend on the choice of generators of the self–dual CAR algebra. Another example of a quasi–free state is provided by the tracial state:

**Definition 5 (Tracial state).**

The tracial state \( \operatorname{tr} \in \mathcal{E} \) is the quasi–free state with symbol \( S_{\operatorname{tr}} \triangleq \frac{1}{2} 1_{\mathcal{H}} \).\[\]

The tracial state can be used to highlight the relationship between quasi–free states and bilinear Hamiltonians. In fact, one can show, c.f. [ABPM20], that for any \( \beta \in (0, \infty) \) and any self–dual Hamiltonian \( H \) on \( (\mathcal{H}, \Gamma) \) the positive operator \( (1 + e^{-\beta H})^{-1} \) satisfies Condition (18) and is the symbol of a quasi–free state \( \omega_H^{(\beta)} \) satisfying
\[
\omega_H^{(\beta)}(A) = \frac{\operatorname{tr} \left( A \exp \left( \frac{\beta}{2} (B, HB) \right) \right)}{\operatorname{tr} \left( \exp \left( \frac{\beta}{2} (B, HB) \right) \right)}, \quad A \in \text{sCAR}(\mathcal{H}, \Gamma).
\]

The state \( \omega_H^{(\beta)} \in \mathcal{E} \) is named the (\( \tau_t, \beta \))–Gibbs state, thermal equilibrium state, or KMS–state, associated with the self–dual (one–particle) Hamiltonian \( H \) on \( (\mathcal{H}, \Gamma) \) at fixed \( \beta \in (0, \infty) \). As is usual, we call to the parameter \( \beta \in (0, \infty) \) the inverse (non–negative) temperature of a physical system. Note that, given \( H \in \mathcal{B}(\mathcal{H}) \), we also can define two particular quasi–free states \( \omega_H^{(0)} \) and \( \omega_H^{(\infty)} \).
which satisfy (20) for the convergent sequence \( \{ \beta_n \}_{n \in \mathbb{N}} \subset \mathbb{R}_0 \cup \{ \infty \} \) to a \( \beta \subset \mathbb{R}_0 \cup \{ \infty \} \). The former case is closely related with the tracial state in Definition 5, and corresponds to the infinite temperature. Namely, the state at \( \beta = \lim_{n \to \infty} \beta_n = 0 \) is known as trace state or chaotic state. This particular name comes from the fact that physically it corresponds to the state of maximal entropy which occurs at infinite temperature. Its uniqueness is a well–known property. On the other hand, states at \( \beta = \lim_{n \to \infty} \beta_n = \infty \) are also thermal equilibrium states. More generally, these are defined by:

**Definition 6 (Ground state).**

Let \( \omega \in \mathcal{E} \) be a state on \( \text{sCAR}(\mathcal{H}, \Gamma) \) and let \( H \in \mathcal{B}(\mathcal{H}) \) be a self–dual Hamiltonian on \( (\mathcal{H}, \Gamma) \). We say that \( \omega \equiv \omega_H^{(\infty)} \) is a ground state if it satisfies

\[
i \omega(A^* \delta(A)) \geq 0,
\]

for all \( A \in \mathcal{D}(\delta) \). Here \( \delta \) is the generator with domain \( \mathcal{D}(\delta) \), of the continuous group \( \{ \tau_t \}_{t \in \mathbb{R}} \) of \(^*\)–automorphisms of \( \text{sCAR}(\mathcal{H}, \Gamma) \) given by (10).

From now on, we will denote by \( \mathcal{E}^{(\beta)} \in \mathbb{C} \) the set of all KMS states at inverse temperature \( \beta \in \mathbb{R}_0^+ \cup \{ \infty \} \) associated to the self–dual Hamiltonian \( H \) on \( (\mathcal{H}, \Gamma) \). A few of remarks regarding \( \mathcal{E}^{(\beta)} \) are discussed:

To lighten the notations, in the sequel when we refer to the KMS state \( \omega_H^{(\beta)} \) we will omit any mention of the dependence on \( H \), i.e., \( \omega_H^{(\beta)} \equiv \omega^{(\beta)} \). For \( \beta \in \mathbb{R}_0^+ \cup \{ \infty \} \), \( \omega^{(\beta)} \in \mathcal{E}^{(\beta)} \) is \( \tau \) invariant or stationary, i.e., \( \omega^{(\beta)} \circ \tau = \omega^{(\beta)} \). See [BR03b, Propositions 5.3.3 and 5.3.19]. In contrast, the tracial case \( \beta = 0 \) not necessarily is. Then, for \( \beta \in \mathbb{R}_0^+ \cup \{ \infty \} \), \( \omega \equiv \omega^{(\beta)} \), there is a strongly continuous one–parameter unitary group \( \{ e^{it\omega} \}_t \in \mathbb{R} \) with generator \( \mathcal{L}_\omega = \mathcal{L}^{\omega} \) satisfying \( e^{it\mathcal{L}_\omega} \Omega_\omega = \Omega_\omega \) such that for any \( t \in \mathbb{R} \)

\[
\pi_\omega(\tau_t(A)) = e^{-it\mathcal{L}_\omega} \pi_\omega(A) e^{it\mathcal{L}_\omega} \quad \text{and} \quad e^{it\mathcal{L}_\omega} \in \pi_\omega(\text{sCAR}(\mathcal{H}, \Gamma))''.
\]

If any \( A \in \mathcal{D}(\delta) \subseteq \text{sCAR}(\mathcal{H}, \Gamma) \),

\[
\pi_\omega(A) \Omega_\omega \in \mathcal{D}(\mathcal{L}) \quad \text{and} \quad \mathcal{L}(\pi_\omega(A) \Omega_\omega) = \pi_\omega(\delta(A)) \Omega_\omega.
\]

If \( \omega \) is a ground state, then the generator satisfies \( \mathcal{L}_\omega \geq 0 \).

For \( \beta \in \mathbb{R}_0^+ \), the set \( \mathcal{E}^{(\beta)} \in \mathcal{E} \) form a weak\(^*\)–compact convex set that also is a simplex,\(^{5}\) while the set of ground states or KMS states at inverse temperature \( \infty \), \( \mathcal{E}^{(\infty)} \subseteq \mathcal{E} \), form a face \( \mathcal{F} \), i.e., a subset of a compact convex set \( \mathcal{K} \) such that if there are finite linear combination

\[
\omega = \sum_{j=1}^n \lambda_j \omega_j \quad \text{with} \quad \sum_{j=1}^n \lambda_j = 1
\]

of elements \( \{ \omega_j \}_{j=1}^n \in \mathcal{K} \) and \( \omega \in \mathcal{F} \) then \( \{ \omega_j \}_{j=1}^n \in \mathcal{F} \).

Let \( A \in \mathcal{B}(\mathcal{H}) \) be a bounded self–dual operator on \( (\mathcal{H}, \Gamma) \), such that \( E_\Sigma(A) = \chi_\Sigma(A) \) define the spectral projection of \( A \) on the Borel set \( \Sigma \subset \mathbb{C} \). Here, \( \chi_\Sigma : \Sigma \to \{ 0, 1 \} \) is the so–called characteristic function on \( \Sigma \subset \mathbb{R} \), with \( \chi_\Sigma^2 = \chi_\Sigma \). For \( H \), a self–adjoint Hamiltonian on \( (\mathcal{H}, \Gamma) \), i.e., \( H = -\Gamma H \Gamma \), we denote by \( E_0, E_- \) and \( E_+ \), the restrictions of the spectral projections of \( H \) on \( \{ 0 \} \), the negative real numbers \( \mathbb{R}^- \) and the positive real numbers \( \mathbb{R}^+ \), respectively. Using functional calculus we note that

\[
H = \int_{\text{spec}(H)} \lambda dE_\lambda = \int_{-\infty}^\infty \lambda dE_\lambda,
\]

\(^{5}\)This is true because one can show that the set of KMS \( \mathcal{E}^{(\beta)} \in \mathcal{E} \) form a base of the cone which is also a lattice [BR03a, Chapter 4].
where $\text{spec}(H)$ denotes the spectrum of $H$. Thus, one verify that

\[(21) \quad \Gamma E_\lambda \Gamma = E_{-\lambda} \quad \text{for all} \quad \lambda \in \mathbb{R} \quad \text{and} \quad E_0 + E_+ + E_- = 1_H.\]

In particular, we have $\Gamma E_0 \Gamma = E_0$. However, we strongly will assume throughout this paper that $E_0 = 0$ so that the ground state is unique. For details see [AT85][Theorems 3 and 4]. By (21), both $E_+$ and $E_-$ are basis projections in $p(\mathcal{H}, \Gamma)$: $E_\pm \Gamma = 1_H - E_\pm$, i.e., ground states can be uniquely characterized by their spectral projections $E_\pm$. In particular, the symbol $S_\omega$ at (19) can corresponds to the spectral projection $E_+$ on the positive real numbers, associated to the self–dual Hamiltonian $H$ on $(H, \Gamma)$ in such a way that ground states are uniquely determined by the two–point correlation function defined by:

\[(22) \quad \omega(B(\varphi_1)B(\Gamma \varphi_2)) = \langle \varphi_1, E_+ \varphi_2 \rangle_H, \quad \varphi_1, \varphi_2 \in \mathcal{H}.\]

Thus, for a quasi–free system associated to some self–dual Hamiltonian $H$, the set of all ground states $\mathcal{E}_H^{(\infty)} \equiv \mathcal{E}^{(\infty)}$, is studied via (positive) spectral projections of $H$. Additionally, straightforward calculations show the uniqueness of ground states, even under small perturbations. See [BR03b, Chapter 5] and [Has19] for recent results on the stability of free fermion systems. More generally, for a unital $C^*$–algebra the quasi–free state for $\beta \in (0, \infty]$ is unique. We are able to define:

**Definition 7 (Quasi–free ground states).**

The state $\omega \in \text{sCAR}(\mathcal{H}, \Gamma)^*$ satisfying (18), (19) and (22) it will be called quasi–free ground state. The set of all quasi–free ground states it will denoted by $q\mathcal{E}^{(\infty)} \subset \mathcal{E}^{(\infty)}$. □

### 2.4 Gapped Systems

We consider the (possibly unbounded) operator $h \in \mathcal{H}$, for some (finite–dimensional) Hilbert space $\mathcal{H}$, whose spectrum is denoted by $\text{spec}(h) \subset \mathbb{R} \cup \{\infty\}$. If $h = h^*$, we say that system described by $\mathcal{H}$ have a gap if when we measure the spectrum of the associated Hamiltonian exists a strictly positive distance $\gamma \in \mathbb{R}^+$ between the two lowest eigenvalues $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{R}$ such that $\mathcal{E}_2 - \mathcal{E}_1 > \gamma$, with $\mathcal{E}_1 \equiv \inf \text{spec}(h)$. Hence, $\gamma$ also named spectral gap, it is known to be the difference between the lower energy of the system and the energy of its first excited physical state. In Definition 8 below, we formally express this. On the other hand, in the scope of fermion systems, Definition 9 is suitable for our interests. Then, by denoting by $\mathcal{d}(X, Y)$, the distance between the sets $X, Y \in \mathbb{R}$:

\[\mathcal{d}(X, Y) \equiv \inf \{d(x, y) : x \in X, y \in Y\},\]

with $d(x, y) \equiv |x - y|$ for $x, y \in \mathbb{R}$, we define:

**Definition 8 (Gapped Hamiltonians).**

Let $\mathcal{H}$ be a (one–particle) Hilbert space and consider $h \in \mathcal{H}$ the (one–particle) Hamiltonian, that is, a self–adjoint operator $h = h^*$, whose spectrum is denoted by $\text{spec}(h) \subset \mathbb{R}$. We will say that $h$ is a gapped Hamiltonian if there are $\Sigma$ and $\Sigma$, nonempty and disjoint subsets of $\text{spec}(h)$, such that $\Sigma \cup \Sigma = \text{spec}(h)$ and exists $\gamma \equiv \inf \mathcal{d}(\Sigma, \Sigma) > 0$.

**Remark 1.** In the latter definition $\Sigma$ can be though as the Borel set in $\mathbb{R}$ that contain the isolated eigenvalue $\mathcal{E}_1$, which carry the information of the lowest energy associated to the physical system to consider. Note that if we define the family of elements of $\Sigma$ with indices on $\mathbb{N} \setminus \{1\}$ as the map $\mathcal{E} : \mathbb{N} \setminus \{1\} \to \Sigma$, such that $\mathcal{E} \equiv \{\mathcal{E}_n\}_{n \in \mathbb{N} \setminus \{1\}}$, the rest of eigenvalues of $\mathcal{H}$, given $\mathcal{E}$, belongs to $\Sigma$.

Definition 8 is completely general and usually is used to study spectrum related to physical systems. Nevertheless, our primary interest is the fermionic case and then we need to consider an alternative expression. In order to find such an expression recall Definition 4 of a self–dual operator $H \in \mathcal{B}(\mathcal{H})$, where one consider a self–dual Hilbert space $(\mathcal{H}, \Gamma)$, with $\mathcal{H}$ a finite–dimensional Hilbert space with orthonormal basis given by $\{\psi_i\}_{i \in \mathcal{I}}$. Hence, for any $H \in \mathcal{B}(\mathcal{H})$ satisfying $H^* = -\Gamma H \Gamma$ we have:
(i) \( \text{Tr}_{\mathcal{H}}(H) = 0. \)

(ii) \( \text{spec}(\Lambda 1_{\mathcal{H}} - H) = \lambda - \text{spec}(H) \) for \( \lambda \in \mathbb{C}. \)

Both (i) and (ii) are fundamental to study the underlying systems we are considering. On the other hand, the physical terms we are dealing are expressed by

\[
d\Gamma(h) + d\Upsilon(g) = -\langle [B, [\kappa(h) + \bar{\kappa}(g)] B \rangle + \frac{1}{2}\text{Tr}_h(h) \mathbf{1},
\]

that is, a bilinear element of a self–dual Hamiltonian (see again Definition 4) plus a constant term. This is the typical case of a free–fermion system with quasi–free dynamics provided by some bilinear Hamiltonian \( H \in \text{sCAR}(\mathcal{H}, \Gamma). \) Instead of considering \( H \), observe equivalently that

\[
- \langle [B, [F + G] B \rangle + \text{Tr}_{h_P}(PFP) \mathbf{1},
\]

(23)

give us the description of the systems, where \( F \) and \( G \) are self–dual Hamiltonians on \( \mathcal{H} \), and \( P \in \mathfrak{p}(\mathcal{H}, \Gamma) \) is a basis projection with range \( \text{ran}(P) = h_P. \). As already mentioned \( F_P \doteq 2PHP \) is the so–called one–particle Hamiltonian, then, w.l.o.g. we can remove the term \( \text{Tr}_{h_P}(PFP) \mathbf{1} \), by writting (23) as

\[
- \langle [B, \tilde{F} + G] B \rangle,
\]

(24)

for \( \tilde{F} \doteq F - \frac{1}{|I|}\text{Tr}_{h_P}(PFP)\kappa(1_{h_P}) \), with \( |I| \) the cardinality of the Hilbert space \( \mathcal{H} \), and the map \( \kappa \) being defined by

\[
\kappa(h) \doteq \frac{1}{2} (P_b h P_b - \Gamma P_b h^* P_b \Gamma), \quad h \in \mathcal{B}(h).
\]

See also (9). Since \( H \doteq \tilde{F} + G \) is a self–dual Hamiltonian, we use \( h \doteq 2P_b H P_b \) and \( g \doteq 2P_b H \Gamma P_b \), in order to describe any quadratic Fermionic Hamiltonian. In fact, given \( P \in \mathfrak{p}(\mathcal{H}, \Gamma) \) with \( \text{ran}(P) = h \) and the self–dual Hamiltonian \( H \in \mathcal{B}(\mathcal{H}) \), the bounded operators on \( h \)

\[
h \doteq 2PHP \quad \text{and} \quad g \doteq 2PHPG,
\]

provide all the possible free–fermion models. Further, we can add to Expression (23) or (24) a self–adjoint element \( W \in \text{sCAR}(\mathcal{H}, \Gamma) \) which could carried the interparticle interaction terms, but for simplicity we will omit in the following. The latter will be considered in a subsequent paper [AR20]. Finally, based in Definition 8, items (i) and (ii), and above comments we can well–define the following:

**Definition 9 (Fermionic Gapped Hamiltonians).**

Let \( (\mathcal{H}, \Gamma) \) be a self–dual Hilbert space and consider \( H \in \mathcal{B}(\mathcal{H}) \) be a self–dual Hamiltonian with spectrum denoted by \( \text{spec}(H) \subset \mathbb{R} \). We will say that \( H \) is a **gapped Hamiltonian** if exists \( g \in \mathbb{R}^+ \) satisfying the gap assumption

\[
g \doteq \inf \{ \epsilon > 0: [-\epsilon, \epsilon] \cap \text{spec}(H) \neq \emptyset \}. \]

Observe that for fermionic systems, Definitions 8 and 9 are equivalent. In fact, in Definition 9, \( \Sigma \in \mathbb{R} \) is a finite interval with \( a \doteq \inf \{ \Sigma \} \) and \( b \doteq \sup \{ \Sigma \} \), \( \Sigma \) is nothing but \( -\Sigma \), so that \( -a \doteq \sup \{ \Sigma \} \) and \( -b \doteq \inf \{ \Sigma \} \). Then, self–dual formalism permits consider a symmetric decomposition of the spectrum. Therefore \( \Sigma \) can be understood as a Borel set on \( \mathbb{R}^+ \) related to the positive part of the energy while \( \Sigma \doteq -\Sigma \) its symmetric negative part: the gap \( g \) centered at zero separates these. We
finally stress following Definition 9 that denoting by $\Sigma_0$ and $-\Sigma_0$ the remaining two open sets, their closures respectively are $\Sigma$ and $-\Sigma$.

Due to above reasons, from now on, we will only consider fermion systems. Thus, let now consider the family of self–dual Hamiltonians $\{H_s\}_{s \in C} \in \mathcal{B}(\mathcal{H})$ on $(\mathcal{H}, \Gamma)$, where $C$ is the compact set $[0, 1]$. In particular, $\{H_s\}_{s \in C}$ will define a differentiable family of self–adjoint operators on $\mathcal{B}(\mathcal{H})$. More specifically, for any $s \in C$ we will consider that the map $s \mapsto H_s$ is strongly differentiable so that $\partial_s H_s \in \mathcal{B}(\mathcal{H})$. Thus, among the models we are taking into account, Anderson model is a particular case, as discussed in Appendix A. See [BPH14] and [ABPR19]. Following Definition 9 we now define:

**Definition 10 (Phase of the Matter).**

Let $C \equiv [0, 1]$ and $\{H_s\}_{s \in C} \in \mathcal{B}(\mathcal{H})$ be a family of self–dual Hamiltonians on $(\mathcal{H}, \Gamma)$. We will say that $H_s$ is a $s$–gapped Hamiltonian if the gap assumption in Definition 9 is satisfied for any $s \in C$. $\{H_s\}_{s \in C}$ describes the same phase of the matter if there is $g \in \mathbb{R}^+$, independent of $s$, such that for any $s \in C$ there is a uniform lower bond, i.e., $\inf_{s \in C} g \geq g > 0$. In this situation we will say that $\{H_s\}_{s \in C}$ is in the $g$–phase.

Observe that a difference between ground states associated to family of Hamiltonians $\{H_s\}_{s \in C}$ in the $g$–phase and the general definition of ground states (Definition 6) is necessary. In fact, one can prove that if the family of Hamiltonians is gapped, then its associated ground states $\{\omega_s\}_{s \in C}$ satisfy:

\begin{equation}
\omega_s(A^*\delta(A)) \geq g_s(\omega_s(A^*A) - |\omega_s(A)|^2), \quad \text{for any } s \in C \text{ and } A \in \mathcal{B}(\delta),
\end{equation}

with $g_s \in \mathbb{R}^+, s \in C$, and $\inf_{s \in C} g_s \geq g > 0$. For details see [Mat13]. In the sequel we will say that states satisfying above inequality are gapped ground states.

### 3 Main Results

We study gapped Hamiltonians satisfying the following Assumption:

**Assumption 1.**

Take $C \equiv [0, 1]$. (a) $H_\emptyset = \{H_s\}_{s \in C} \in \mathcal{B}(\mathcal{H})$ is a differentiable family of self–dual Hamiltonians on the $g$–phase such that $\partial H_\emptyset = \{\partial_s H_s\}_{s \in C} \in \mathcal{B}(\mathcal{H})$. (b) At the infinite volume we assume that the sequences of self–dual Hamiltonians $H_{s,L}: C \to \mathcal{B}(\mathcal{H}_\infty)$ and $\partial_s H_{s,L}: C \to \mathcal{B}(\mathcal{H}_\infty)$ are strongly and pointwise convergent, that is, $\lim_{L \to \infty} H_{s,L} = H_{s,\infty}$ and $\lim_{L \to \infty} \partial_s H_{s,L} = \partial_s H_{s,\infty}$ in the strong sense.

Now, for any self–dual Hilbert space $(\mathcal{H}, \Gamma)$, take $P_1 \in \mathcal{P}(\mathcal{H}, \Gamma)$ and $P_2 \in \mathcal{P}(\mathcal{H}, \Gamma)$ basis projections, the “$\mathbb{Z}_2$–projection index” ($\mathbb{Z}_2$–PI) $\sigma: \mathcal{P}(\mathcal{H}, \Gamma) \times \mathcal{P}(\mathcal{H}, \Gamma) \to \mathbb{Z}_2$ is the map defined by:

\begin{equation}
\sigma(P_1, P_2) \equiv (-1)^{\dim(P_1 \wedge P_2^+)}.
\end{equation}

Here, $\wedge$ symbolizes the lower bound or intersection of the basis projections $P_1$ and $P_2$ in $\mathcal{B}(\mathcal{H})$. Note that the $\mathbb{Z}_2$–PI defines a topological group with two components. Then, we analyze the class of Hamiltonians described by last assumption and their connection with topological indexes. We formally state one of the main results of the paper:

**Theorem 1 ($\mathbb{Z}_2$–projection Index):**

Take $C \equiv [0, 1]$ and let $H_{g_s} = \{H_{s,\infty}\}_{s \in C} \in \mathcal{B}(\mathcal{H}_\infty)$ be a differentiable family of self–dual Hamiltonians on $(\mathcal{H}_\infty, \Gamma_\infty)$ at the $g_\infty$–phase, with $\partial H_{g_s} = \{\partial_s H_{s,\infty}\}_{s \in C} \in \mathcal{B}(\mathcal{H}_\infty)$, see Definition 10 and Assumption 1 (b). For any $s \in C$, $\mathcal{E}_{+s,\infty}$ denotes the spectral projection associated to the positive part of $\text{spec}(H_{s,\infty})$ and consider the $\mathbb{Z}_2$–PI given by (26). Then:
(1) For any \( s \in \mathcal{C} \), \( H_{0,\infty} \) is unitarily equivalent to \( H_{s,\infty} \) via the unitary operator \( V_s \in \mathcal{B}(\mathcal{H}_\infty) \) satisfying the differential equation (29) below.

(2) For \( s \in \mathcal{C} \), the \( \mathbb{Z}_2\)-PI \( \sigma(H_{0,\infty}, H_{s,\infty}) \equiv \sigma(E_{+,0,\infty}, E_{+,s,\infty}) \) satisfies

\[
\sigma(H_{0,\infty}, H_{s,\infty}) = \begin{cases} 
1, & \text{if } V_s^{(\infty)} \in \mathcal{U}_+ \\
-1, & \text{if } V_s^{(\infty)} \in \mathcal{U}_- .
\end{cases}
\]

Thus, the Bogoliubov \( {}^* \)-automorphism \( \chi_{V_s}^{*} \) is inner and maintain its parity, even or odd, over the family \( \mathbf{H}_{0,\infty} \).

\textbf{Proof.} (1) For any \( A \in \mathcal{B}(\mathcal{H}_\infty) \) and all \( s \in \mathcal{C} \) one define the spectral flow automorphism \( \kappa_s : \mathcal{B}(\mathcal{H}_\infty) \to \mathcal{B}(\mathcal{H}_\infty) \) by

\[
\kappa_s(A) = \left(V_s^{(\infty)}\right)^* AV_s^{(\infty)},
\]

where \( V_s \in \mathcal{B}(\mathcal{H}_\infty) \) is the unitary operator satisfying \( V_0 = 1_{\mathcal{H}_\infty} \), and the differential equation (29). See Lemmata 1–2 and Corollary 4. In particular, since any Hamiltonian \( H_{s,\infty} \) in \( \mathbf{H}_{0,\infty} \) can be written as

\[
H_{s,\infty} = \int_{-\infty}^{\infty} \lambda dE_{\lambda,s,\infty},
\]

with \( \Gamma E_{\lambda,s,\infty} = E_{-\lambda,s,\infty} \) for all \( \lambda \in \mathbb{R} \) and \( E_{-\lambda,s,\infty} + E_{+,s,\infty} = 1_{\mathcal{H}_\infty} \), by Lemmata 1–2, (1) follows.

(2) Concerning the \( \mathbb{Z}_2\)-PI \( \sigma(P_1, P_2) \) we only need to invoke [EK98, Theo. 6.30 and Lemma 7.17]:

(a) \( \sigma(P_1, P_2) = \sigma(P_2, P_1) \), (b) If \( P_1, P_2 \) is a Hilbert–Schmidt class operator, then \( \sigma(P_1, P_2) \) is continuous in \( P_1 \) and \( P_2 \) with respect to the norm topology in \( \mathcal{P}(\mathcal{H}, \Gamma) \) (c) If \( U \in \mathcal{B}(\mathcal{H}) \) is a unitary operator such that \( U \Gamma = \Gamma U \) and \( 1_{\mathcal{H}} - U \) is a trace class operator, then \( \sigma(P, UPU^*) = \det U \). Then we to verify these statements to the family of positive spectral projections \( \{E_{+,s,\infty}\}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}_\infty) \).

By (21)–(22) and comments around it, any positive spectral projection in \( \{E_{+,s,\infty}\}_{s \in \mathcal{C}} \) is a basis projection and thus \( \{E_{+,s,\infty}\}_{s \in \mathcal{C}} \subseteq \mathcal{P}(\mathcal{H}_\infty, \Gamma_\infty) \). W.l.o.g. take \( E_{+,0,\infty} \) and \( E_{+,s,\infty} \) with \( s \in \mathcal{C} \). We split the proof in three steps:

(i) Part (a) follows from

\[
E_{+,0,\infty} \wedge E_{+,s,\infty} = \Gamma_{\infty} \left(E_{+,0,\infty} \wedge E_{+,s,\infty}\right) \Gamma_{\infty}.
\]

(ii) We need to verify that \( E_{+,0,\infty} - E_{+,s,\infty} \) is Hilbert–Schmidt class. By Lemma 1, there is a unitary bounded operator \( V_s^{(\infty)} \in \mathcal{B}(\mathcal{H}_\infty) \) such that

\[
\left(V_s^{(\infty)}\right)^* E_{+,s,\infty} V_s^{(\infty)} = E_{+,0,\infty},
\]

with \( V_0^{(\infty)} = 1_{\mathcal{H}_\infty} \). On the other hand, for any separable Hilbert space \( \mathcal{H} \), \( A \in \mathcal{B}(\mathcal{H}) \) and any orthonormal basis \( \{ \psi_i \}_{i \in I} \) of \( \mathcal{H} \) the trace of \( A \)

\[
\text{tr}_{\mathcal{H}}(A) = \sum_{i \in I} \langle \psi_i, A \psi_i \rangle,
\]

does not depend of the choice of the orthonormal basis. Thus, since \( E_{+,s,\infty} \) and \( E_{+,0,\infty} \) are self–adjoint operators on \( \mathcal{B}(\mathcal{H}_\infty) \) we get

\[
\text{tr} \left( (E_{+,0,\infty} - E_{+,s,\infty})^2 \right) = 2 \sum_{i \in I} \left\{ \langle \psi_i, E_{+,0,\infty} \psi_i \rangle - \langle E_{+,0,\infty} V_s^{(\infty)} \psi_i, V_s^{(\infty)} E_{+,0,\infty} \psi_i \rangle \right\}.
\]

Denote by \( \mathfrak{h}_{E_{+,0,\infty}} \), the range \( \text{ran}(E_{+,0,\infty}) \) of \( E_{+,0,\infty} \) (see Definition 2), and assume that \( \{ \psi_i \}_{i \in I} \) is an orthonormal basis of \( \mathfrak{h}_{E_{+,0,\infty}} \). Because of the invariance on the choice of the orthonormal basis of the trace, it follows that last expression equals zero, and then \( E_{+,0,\infty} - E_{+,s,\infty} \) is Hilbert–Schmidt class, the desired.
(iii) By Corollary 2, for any \( s \in \mathcal{C} \) and \( L \in \mathbb{R}_0^+ \cup \{ \infty \} \) the unitary operator \( V_s^{(L)} \in \mathcal{B}(\mathcal{H}_L) \) commutes with \( \Gamma_L \) and by Lemma 3, for \( L \in \mathbb{R}_0^+ \cup \{ \infty \} \), \( 1_{\mathcal{H}_L} - V_s^{(L)} \) is a trace class operator, where in particular \( 1_{\mathcal{H}_L} - V_s^{(\infty)} \) is a trace class operator per unit volume. Then

\[
\sigma (E_{+,0,\infty}, E_{+,s,\infty}) = \det \left( V_s^{(L)} \right), \quad L \in \mathbb{R}_0^+ \cup \{ \infty \}.
\]

By Lemma 3 the Bogoliubov \(^*\)-automorphism \( \Upsilon_s^{(L)} \) on \( \mathfrak{A}_L \) given Expression (36) below is inner. Additionally, if \( V_s^{(L)} \in \mathcal{B}(\mathcal{H}_L) \) has some parity, we say \( V_s^{(L)} \in \mathfrak{H}_L \) (see (6)), then by [EK98, Theorem 6.15], \( \Upsilon_s^{(L)} \) is even or odd and this parity holds for the family of Hamiltonians \( H_\mathfrak{g}_L \).

This complete the proof of the Theorem.

\[\text{End}\]

Remark 2. Note that there is an alternative form of the \( \mathbb{Z}_2 \)-PI (26) in terms of orthogonal complex structures [BVF01, EK98]. There, the index appears naturally in the proof of the Shale–Stinespring Theorem and is related to the parity of the ground states. In [CGRL18], this approach to the \( \mathbb{Z}_2 \)-index was used to study ground states for finite Kitaev chains with different boundary conditions. More recently, for infinite translationally invariant fermionic chains, Bourne and Schulz-Baldes classify ground states using orthogonal complex structures [BSB20]. Observe that Theorem 1 above generalizes the mentioned results in the sense that we do not require translational invariant conditions neither one-dimensional systems only. Finally, by Lemma 3 one notes that the \( \mathbb{Z}_2 \)-PI is uniform with respect to the size of the systems.

Hitherto in this paper we have been interested in physical systems with open gap, which is the case of systems of last Theorem. In fact, Theorem 1 claims that two self–dual Hamiltonians, \( H_{0,\infty}, H_{1,\infty} \in \mathbf{H}_{g_{\infty}} \) acting on \( \mathcal{H}_{\infty} \), can be connected by a path described by the spectral flow automorphism \( \kappa_s: \mathcal{B}(\mathcal{H}_{\infty}) \to \mathcal{B}(\mathcal{H}_{\infty}) \). As is usual, a path is nothing but a continuous application \( \kappa: [0,1] \to \mathcal{B}(\mathcal{H}_{\infty}) \) connecting the initial point \( \kappa_0 = H_{0,\infty} \) and the terminal point \( \kappa_1 = H_{1,\infty} \). That is the same, \( H_{0,\infty} \) and \( H_{1,\infty} \) are the extremal points of the path. Observe that for any \( s, r \in \mathcal{C} \) with \( H_{s,\infty}, H_{r,\infty} \in \mathbf{H}_{g_{\infty}} \), we can write

\[
H_{s,\infty} = \kappa_{s,r}(H_{r,\infty}), \quad \text{with} \quad \kappa_{s,r} = \kappa_s^{-1} \circ \kappa_r,
\]

and then there is a path such that \( H_{s,\infty} \) and \( H_{r,\infty} \) are their extremal points, and then the family \( \mathbf{H}_{g_{\infty}} \) is arcwise connected at the strong operator topology of \( \mathcal{H}_{\infty} \), and as a consequence it is also a connected set at the same topology.

Nevertheless, we also are worried about the possibility of having two self–dual Hamiltonians, \( H_{0,\infty}, H_{2,\infty} \in \mathbf{H}_{g_{\infty}} \) acting on \( \mathcal{H}_{\infty} \), but both not respect conditions of Definition 10. For example, if \( H_{0,\infty} \in \mathbf{H}_{g_{\infty}} \) while \( H_{2,\infty} \notin \mathbf{H}_{g_{\infty}} \), Theorem 2 (see Corollary 1 too) below shows that the path \( \tilde{\kappa} \) connecting both Hamiltonians close the gap, by meaning that there is a Hamiltonian \( \tilde{H} \in \mathcal{B}(\mathcal{H}_{\infty}) \) on \( \tilde{s} \) such that 0 is an eigenvale of \( \tilde{H} \). Concerning the latter, observe that one can study the gap closure in terms of the self–dual CAR, \( C^* \)-algebra \( \mathfrak{A}_{\infty} \cong s\mathcal{CAR}(\mathcal{H}_{\infty}, \Gamma) \), in such a way that we associate to \( H_{0,\infty} \) and \( H_{2,\infty} \) the bilinear elements \( \langle B, H_{0,\infty}B \rangle \) and \( \langle B, H_{2,\infty}B \rangle \) on \( \mathfrak{A}_{\infty} \). See Expression (44) below. Instead, we can equivalently use the set of states \( \mathcal{E}(\infty) \in \mathfrak{A}_{\infty}^* \). Since in the current work we are dealing with the set of quasi–free ground states \( q\mathcal{E}(\infty) \subset \mathcal{E}(\infty) \) of Definition 7, we will analyze the gap closure using \( q\mathcal{E}(\infty) \).

First of all, because of the properties of \( \mathcal{E}(\infty) \) provided at Section 2.3 and Theorem 4, \( q\mathcal{E}(\infty) \) is a metrizable weak\(^*\)-compact set. In the scope of gapped systems, for gapped quasi–free ground states \( \Omega_{g_{\infty}} \equiv \{ \omega_s \}_{s \in \mathcal{C}} \) on the \( g_{\infty} \)-phase, following Theorems 1 and 5, \( \Omega_{g_{\infty}} \) is arcwise connected, and hence it is a connected set in the weak\(^*\)-topology. Here, the Bogoliubov \(^*\)-automorphisms \( \Upsilon_s \) in Theorem 5 plays the role of implementing the path, namely, following Corollary 3 we are able to write

\[
\omega_s = \omega_0 \circ \Upsilon_s, \quad \text{for any} \quad \omega_s \in \Omega_{g_{\infty}} \quad \text{and} \quad s \in \mathcal{C},
\]
and hence we get for any \( r, s \in \mathcal{C} \) that

\[
(27) \quad \omega_r = \omega_s \circ \Upsilon_{s,r}, \quad \text{with} \quad \omega_r, \omega_s \in \Omega_{g,\infty},
\]

with \( \Upsilon_{s,r} = \Upsilon_{s}^{-1} \circ \Tilde{\Upsilon}_r \), which satisfies a cocycle \(^{\ast}\)-automorphism condition.

We now invoke the following result of metric spaces:

**Proposition 1 (Alfândega’s Theorem).**

Let \( \mathcal{M} = (\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \) be a non-empty metric space. Consider \( \mathcal{C}, \mathcal{X} \subset \mathcal{M} \), where \( \mathcal{C} \) is a connected set having common points with \( \mathcal{X} \) and \( \mathcal{M} \setminus \mathcal{X} \). Then, \( \mathcal{C} \) has a point on \( \partial \mathcal{X} \), the boundary of \( \mathcal{X} \).

**Proof.** We proceed following [Lim77, Prop. 9–Chap. 4]. We claim that \( x \in \mathcal{M} \) constructed as follows satisfies the assumptions of the Theorem: On the one hand, note that exists \( x \in \partial(\mathcal{C} \cap \mathcal{X}) \subset \mathcal{C} \). On the other hand, for any \( \epsilon > 0 \) there are \( y \in \mathcal{C} \cap \mathcal{X} \subset \mathcal{X} \) with \( d_M(x, y) < \epsilon \) and \( z \in \mathcal{C} \cap \mathcal{X} \subset \mathcal{M} \setminus \mathcal{X} \) with \( d_M(x, z) < \epsilon \).

In the context of metric spaces Alfândega Theorem’s is known as a generalization of the Intermediate value theorem [Lim77]. Observe that, by definition of boundary of \( \mathcal{X} \), any open ball with radius \( r \in \mathbb{R}^+ \) and center in \( p \in \partial \mathcal{X} \), \( B(p, r) \), has at least one point on \( \mathcal{X} \) and one point on \( \mathcal{M} \setminus \mathcal{X} \). To fix ideas we desire to apply Alfândega Theorem’s for subsets of the metrizable weak\(^{\ast}\)-topology set space \( \mathcal{C} \) associated to the self–dual CAR \( C^{\ast}\)-algebra \( \Omega (\infty) \).

More precisely, consider the quasi–free ground states \( qe(\infty) \subset \mathcal{E}(\infty) \), as well as the family of gapped quasi–free ground states \( \Omega_{g,\infty} \subset qe(\infty) \) above defined. For \( s \in \mathcal{C} \), take \( \omega_s \in \Omega_{g,\infty} \subset \mathcal{E}(\infty) \) and \( \omega_2 \in \mathcal{E}(\infty) \setminus \Omega_{g,\infty} \), and suppose that there is a path \( \gamma \) connecting \( \omega_s \) and \( \omega_2 \). Note that by Expressions (18), (19) and (22), there are positive spectral (basis) projections \( E_{+,\omega_s,\infty}, E_{+,\omega_2,\infty} \in p(\mathcal{H}_\infty, \Gamma_\infty) \) on \( (\mathcal{H}_\infty, \Gamma_\infty) \). We claim the second main result of the current paper:

**Theorem 2:**

Let \( H_{g,\infty} \) be the family of self–dual Hamiltonians of Theorem 1 associated to the family of gapped quasi–free ground states \( \Omega_{g,\infty} \). Let \( \omega_2 \in \mathcal{E}(\infty) \setminus \Omega_{g,\infty} \) be a quasi–free ground state with associated self–dual Hamiltonian \( H_{2,\infty} \in \mathcal{B}(\mathcal{H}_\infty) \) constructed from the positive spectral projection \( E_{+,\omega_2,\infty} \). For some \( s \in \mathcal{C} \) fix \( \omega_s \in \Omega_{g,\infty} \), and suppose that there is a path \( \gamma : [0, 1] \rightarrow qe(\infty) \) such that \( \gamma(0) = \omega_s \) and \( \gamma(1) = \omega_2 \) are the extremal points of \( \gamma \). Then, there is a self–dual Hamiltonian \( \tilde{H}_\infty \in \mathcal{B}(\mathcal{H}_\infty) \) with associated ground state \( \tilde{\omega} \) on \( \gamma \) so that \( 0 \in \text{spec}(\tilde{H}_\infty) \).

**Proof.** Let \( \{ A_n \}_{n \in \mathbb{N}} \) be a countable set of operators on \( \mathcal{A}(\infty) \) so that \( \| A_n \|_{\mathcal{A}_\infty} \leq 1 \) for all \( n \in \mathbb{N} \). It is a well–know fact that the metric

\[
\mathcal{D}_{\mathcal{E}(\infty)}(\omega_1, \omega_2) \doteq \sum_{n \in \mathbb{N}} \frac{1}{2^n} |\omega_1(A_n) - \omega_2(A_n)|, \quad \omega_1, \omega_2 \in \mathcal{E}(\infty),
\]

induces the weak\(^{\ast}\)-topology on the set of states \( \mathcal{E}(\infty) \). In particular, the open ball with center on \( \omega \in \mathcal{E}(\infty) \) and radius \( \epsilon \in \mathbb{R}^+ \) is defined by

\[
\mathcal{B}(\omega, \epsilon) \doteq \{ \omega' \in \mathcal{E}(\infty); \mathcal{D}_{\mathcal{E}(\infty)}(\omega, \omega') < \epsilon \} \subset \mathcal{E}(\infty).
\]

By the hypothesis of the Theorem, we can use Alfândega Theorem’s, Proposition 1, such that we know that there is a ground state \( \tilde{\omega} \) on \( \gamma \) so that \( \tilde{\omega} \in \partial \Omega_{g,\infty} \). Thus, for an open ball with center in \( \tilde{\omega} \in \partial \Omega_{g,\infty} \) and radius \( \epsilon \in \mathbb{R}^+ \) note that there are \( \omega_{\text{in}} \in \Omega_{g,\infty} \) and \( \omega_{\text{out}} \in qe(\infty) \setminus \Omega_{g,\infty} \) so that \( \mathcal{D}_{\mathcal{E}(\infty)}(\tilde{\omega}, \omega_{\text{in}}) < \epsilon \) and \( \mathcal{D}_{\mathcal{E}(\infty)}(\tilde{\omega}, \omega_{\text{out}}) < \epsilon \). By defining \( \omega' \doteq \frac{1}{2}(\omega_{\text{in}} + \omega_{\text{out}}) \), we can use the triangle inequality in order to obtain: \( \mathcal{D}_{\mathcal{E}(\infty)}(\tilde{\omega}, \omega') < \epsilon \). Because \( \epsilon \) is arbitrary it follows that \( \tilde{\omega} \) is a mixed ground state, that is, a convex combination of pure ground states. Finally, by [EK98, Propos. 6.37] the self–dual Hamiltonian \( \tilde{H}_\infty \in \mathcal{B}(\mathcal{H}_\infty) \) associated to the ground state \( \tilde{\omega} \) has 0 as an eigenvalue, and the proof concludes.
As a straightforward consequence we have the following Corollary:

**Corollary 1.**

Let \( \mathbf{H}_{g_{\infty,1}}, \mathbf{H}_{g_{\infty,2}} \) be two family of self–dual Hamiltonians satisfying Theorem 1, for \( g_{\infty,1}, g_{\infty,2} \in \mathbb{R}^+ \) as in Definition 10. Consider the \( \mathbb{Z}_2 – \text{PI} \) given by (26) such that \( \sigma_1 \neq \sigma_2 \), with \( \sigma_1 \), the \( \mathbb{Z}_2 – \text{PI} \) associated to the family \( \mathbf{H}_{g_{\infty,i}} \) following Theorem 1, for \( i \in \{1, 2\} \). For some \( s, r \in \mathcal{C} \) fix \( H_{s,1}, H_{r,2} \in \mathcal{B}(\mathcal{H}_{\infty}) \), and suppose that there is a path \( \tilde{\kappa} : [0,1] \to \mathcal{B}(\mathcal{H}_{\infty}) \) such that \( \tilde{\kappa}(0) = H_{s,1} \) and \( \tilde{\kappa}(1) = H_{r,2} \) are the extremal points of \( \tilde{\kappa} \). Then, there is a self–dual Hamiltonian \( H_{\infty} \in \mathcal{B}(\mathcal{H}_{\infty}) \) on \( \tilde{\kappa} \) so that \( 0 \in \text{spec}(\tilde{H}_{\infty}) \).

**Proof.** Combine Theorems 1 and 2.

### 4 Technical Proofs

#### 4.1 Existence of the spectral flow automorphism

**Lemma 1.**

Take \( \mathcal{C} \equiv [0,1] \) and let \( H_s \) as in Assumption 1. For any \( s \in \mathcal{C} \), \( E_{+,s} \) will denote the spectral projection associated to the positive part of \( \text{spec}(H_s) \). Then, for the family of spectral projections \( \{E_{+,s}\}_{s \in \mathcal{C}} \), there exists a family of automorphisms \( \{\kappa_s\}_{s \in \mathcal{C}} \) on \( \mathcal{B}(\mathcal{H}) \) satisfying

\[
\kappa_s(E_{+,s}) = E_{+,0}.
\]

**Proof.** The arguments of the proof are completely standard and we state these for the sake of completeness, c.f. [Kat13, BMNS12, NSY18b]. Take \( \mathcal{C} \equiv [0,1] \) and consider \( \{H_s\}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}) \) be a differentiable family of self–dual Hamiltonians on the \( g \)–phase. Fix \( s \in \mathcal{C} \) and let \( E_{+,s} \) be the spectral projection of \( H_s \) on \( \Sigma_s \). Note that if the automorphism \( \kappa_s : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) satisfying \( \kappa_s(E_{+,s}) = E_{+,0} \) exists, implies that is it unitarily implemented by a differentiable unitary operator \( V_s \in \mathcal{B}(\mathcal{H}) \) defined by

\[
\kappa_s(E_{+,s}) = V_s^* E_{+,s} V_s, \quad \text{with} \quad V_0 = 1_{\mathcal{H}},
\]

and satisfying the differential equation

\[
\partial_s V_s = -i \mathcal{D}_{g,s} V_s,
\]

where for the gap \( g \), \( \mathcal{D}_{g,s} : \mathcal{C} \to \mathcal{B}(\mathcal{H}) \) is a pointwise self–adjoint bounded operator. Here, \( \partial_s \) indicates the derivative with respect to \( s \in \mathcal{C} \). Now, for any \( H_s \), we write its spectral projection on \( \Sigma_s \) by

\[
E_{+,s} = \frac{1}{2\pi i} \int_{\Gamma_s} R_\zeta(H_s) d\zeta,
\]

where, for any \( s \in \mathcal{C} \), \( R_\zeta(H_s) \in \mathcal{B}(\mathcal{H}) \) is the resolvent set of \( H_s \). In (30), for any \( s \in \mathcal{C} \), \( \Gamma_s \) is a chain, that is, \( \Gamma_s \) is a finite collection of closed rectifiable curves \( \gamma_s \) in \( \mathbb{C} \). In particular, \( \Gamma_s \) surrounds \( \Sigma_s \) and is in the complement of \( \overline{\Sigma}_s \). By using the second resolvent equation, i.e.,

\[
R_\zeta(A) - R_\zeta(B) = R_\zeta(A)(A - B)R_\zeta(B),
\]

for any operators \( A, B \in \mathcal{B}(\mathcal{H}) \) and any \( \zeta \in \text{spec}(A) \cap \text{spec}(B) \), one can show that

\[
\partial_s E_{+,s} = -\frac{1}{2\pi i} \int_{\Gamma_s} R_\zeta(H_s) (\partial_s H_s) R_\zeta(H_s) d\zeta,
\]
and follows that the derivative $\partial_s E_{+,s}$ is well–defined on $\mathcal{C}$. A combination of (28)–(29) and $\kappa_s (E_{+,s}) = E_{+,0}$ yield us to

$$\partial_s E_{+,s} = -i[\mathfrak{D}_g, E_{+,s}].$$

Additionally, since for any $s \in \mathcal{C}$, $E_{+,s}$ is an orthogonal projection then

$$E_{+,s} (\partial_s E_{+,s}) E_{+,s} = E_{+,s} (\partial_s E_{+,s}) E_{+,s} = 0,$$

where for any $s \in \mathcal{C}$, $E_{+,s}$ denotes the orthogonal complement of $E_{+,s}$, i.e., $E_{+,s} \perp 1 - E_{+,s}$. From the latter identity we get the following one

$$\partial_s E_{+,s} = E_{+,s} (\partial_s E_{+,s}) E_{+,s} + E_{+,s} (\partial_s E_{+,s}) E_{+,s},$$

and together with (31) and the fact that $E_{+,s}, E_{+,s}$ are basis projections, see (21), we arrive to

$$\partial_s E_{+,s} = -\frac{1}{\pi i} \text{Re} \left( \oint_{\Sigma_s} (E_{+,s} R \Gamma (H_s) (\partial_s H_s) R \Gamma (H_s) E_{-,s}) \, d\zeta \right).$$

Here, the self–adjoint operator $\text{Re} (A) \in \mathcal{B} (\mathcal{H})$ is the real part of $A \in \mathcal{B} (\mathcal{H})$, given by $\text{Re} (A) = \frac{1}{2} (A + A^*)$. Similarly, $\text{Im} (A) \in \mathcal{B} (\mathcal{H})$, the imaginary part of $A$, is the self–adjoint operator is usually defined by $\text{Im} (A) = \frac{1}{2i} (A - A^*)$.

Then, the existence of the automorphism $\kappa_s$ is equivalent to find the operator $\mathfrak{D}_g$ such that (32) and (33) are satisfied. This is precisely that is done at [BMNS12], and in the present context we explicitly write $\partial_s E_{+,s}$ as

$$\partial_s E_{+,s} = \int_{\Sigma_s} \int_{\Sigma_s} \frac{2}{\mu + \lambda} \text{Re} \left( dE_{\mu,+,s} (\partial_s H_s) dE_{-\lambda,+,s} \right),$$

where for any $s \in \mathcal{C}$, $E_{\mu,+,s}$ is a resolution of the identity supported on the positive (negative) part of $\text{spec}(H_s)$, i.e.,

$$E_{+,s} \perp \int_{\pm \Sigma_s} dE_{\lambda,+,s}.$$  

In the next, we verify that the self–adjoint bounded operator

$$\mathfrak{D}_g (\partial_s H_s) \mathfrak{M}_g (t) dt, \quad \text{for any} \quad s \in \mathcal{C},$$

satisfies (32) and (33). Here, $\mathfrak{M}_g (t) : \mathbb{R} \to \mathbb{R}$ is an odd function on $L^1 (\mathbb{R})$ such that its Fourier transform, $\hat{\mathfrak{M}}_g : \mathbb{R} \to \mathbb{R}$ is given for $\mu \neq 0$ by

$$\hat{\mathfrak{M}}_g (\mu) \equiv -\frac{1}{\sqrt{2\pi \mu}}.$$

For a complete description of the properties of $\mathfrak{M}_g$ see [BMNS12, MZ13, NSY18b]. We now note that for any operator $B \in \mathcal{B} (\mathcal{H})$ and any orthogonal projection $P \in \mathcal{B} (\mathcal{H})$ we get

$$-i[B, P] = i (PB (1-P) - (1-P)BP).$$

In particular, note that by taking $B$ as $\mathfrak{D}_g, P$ as the spectral projection of $H_s$ on $\Sigma_s$, i.e., $E_{+,s}$, we have

$$-i[\mathfrak{D}_g, E_{+,s}] = i \int_{\Sigma_s} \int_{\Sigma_s} e^{i(\mu - \lambda)} dE_{\mu,+,s} (\partial_s H_s) dE_{-\lambda,+,s} \mathfrak{M}_g (t) d\lambda d\mu dt$$

$$= i \sqrt{2\pi} \int_{\Sigma_s} \int_{\Sigma_s} dE_{\mu,+,s} (\partial_s H_s) dE_{-\lambda,+,s} e^{i(\lambda - \mu)} (\lambda - \mu) d\lambda d\mu$$

$$= \int_{\Sigma_s} \int_{\Sigma_s} \frac{2}{\mu + \lambda} \text{Re} \left( dE_{\mu,+,s} (\partial_s H_s) dE_{-\lambda,+,s} \right)$$

$$= \int_{\Sigma_s} \int_{\Sigma_s} \frac{2}{\mu + \lambda} \text{Re} \left( dE_{\mu,+,s} (\partial_s H_s) dE_{-\lambda,+,s} \right).$$
where we have used (34) and that $\widehat{\mathcal{M}}_{\varphi}$ is an odd function.

In particular, the unitary operator $V_s$ satisfying the differential equation (29) commutes with the involution $\Gamma$, i.e., $\Gamma V_s = V_s \Gamma$. In fact, for any $s \in \mathcal{C}$, let $C_s \in \mathcal{B}(\mathcal{H})$ defined by $C_s \triangleq [\Gamma, V_s]$, such that $C_s^* = [V_s^*, \Gamma]$, and we would like to show that $C_s = 0$. To do this, observe that the self–adjoint bounded operator $\mathcal{D}_{b,s}$ given by Expression (35), commutes with $\Gamma$. Using (29), after some calculations we have

$$\partial_s C_s = i \mathcal{D}_{b,s} C_s \quad \text{and} \quad \partial_s C_s^* = i C_s^* \mathcal{D}_{b,s}.$$ 

From the left hand side equation one has $\partial_s C_s^* = -i C_s^* \mathcal{D}_{b,s}$, which comparing with the right hand side equation, we obtain $C_s = 0$. We had proven:

**Corollary 2 (Bogoliubov Transformation).**

For any $s \in \mathcal{C} \equiv [0, 1]$, the unitary operator $V_s$ satisfying the differential equation (29) commutes with the involution $\Gamma$, i.e., $\Gamma V_s = V_s \Gamma$, then $V_s$ is a Bogoliubov transformation, see (5).}

One primary consequence of Lemma 1 and Corollary 2 is the existence of a strongly continuous family of one–parameter (Bogoliubov) group $\Upsilon_s \equiv \{\Upsilon_s\}_{s \in \mathcal{E}}$ of $^*\mathrm{a}$–automorphisms of sCAR$(\mathcal{H}, \Gamma)$, implemented by the Bogoliubov automorphism $V_s$. To be precise, for the one–parameter unitary group $\{V_s\}_{s \in \mathcal{E}}$ implementing the family of automorphism $\{\kappa_s\}_{s \in \mathcal{C}}$ of Lemma 1 over the family of spectral projections $\{E_{+,s}\}_{s \in \mathcal{C}}$ we are able to show that, for any $s$, the (Bogoliubov) $^*\mathrm{a}$–automorphism

$$\Upsilon_s(B(\varphi)) \equiv \chi_{V_s^*} (B(\varphi)) = B(V_s^* \varphi),$$

exists, for any $\varphi \in \mathcal{H}$. The latter can be easily verified using bilinear elements, which are described by Definition 3. More generally, for any family of self–dual Hamiltonians $\{H_s\}_{s \in \mathcal{E}} \in \mathcal{B}(\mathcal{H})$ as in Assumption 1, they have associated the family of bilinear elements $\{\langle B, H_s B \rangle\}_{s \in \mathcal{C}} \in \mathrm{sCAR}(\mathcal{H}, \Gamma)$ given by,

$$\Upsilon_s(\langle B, H_s B \rangle) = \langle B, H_0 B \rangle, \quad \text{for any} \quad s \in \mathcal{C},$$

(see Definition 3).

As stressed in comments around Expression (26) for any pair $P_1, P_2 \in \mathfrak{p}(\mathcal{H}, \Gamma)$ exists a Bogoliubov transformation $U$ relating both, i.e., a unitary operator $U \in \mathcal{B}(\mathcal{H})$ so that $P_2 = U^* P_1 U$, with $U \Gamma = \Gamma U$. Thus, if $\det(U) = 1$ ($\det(U) = -1$) we say that $U$ is in the positive (negative) connected component. Following [EK98, Theo. 6.30 and Lemma 7.17] for the special class of $U$ satisfying (i) $U \Gamma = \Gamma U$ and (ii) $1 - U$ trace class, the topological index $\sigma(P, U^* PU)$ coincides with $\det(U)$. Note that Lemma 1 tell us about the existence of a family of unitary operators $\{V_s\}_{s \in \mathcal{E}}$ which implements the family of automorphisms $\{\kappa_s\}_{s \in \mathcal{E}}$ on $\mathcal{B}(\mathcal{H})$. However, we need to specify with which kind of Hamiltonians we are dealing. A wide class of fermion systems are those satisfying Lemma 2 and Proposition 3 below. More concretely, our results will permit to consider disordered fermions systems in which the spectral gap does not close. Note that a suitable control of the properties of $\{V_s\}_{s \in \mathcal{E}}$ is closely related to the recently results found by Hastings in [Has19]. Then, as already mentioned we invoke [EK98, Theo. 6.30 and Lemma 7.17] in order to distinguish different physical systems on the same $\varphi$–phase for some positive $\varphi \in \mathbb{R}^+$ (see Definition 10) and these are classified by two components even in the interacting setting, see [NSY18a]. Additionally, Theorem 1 holds for a wide family of (possibly unbounded) random Hamiltonians $H_\varphi$ on the $\varphi$–phase.

From now on, we will expose some issues about quasi–free ground states for $\varphi \in \mathbb{R}^+$ and $\varphi = 0$. The former are called *gapped quasi–free ground states* (see Expression (25) and comments around it) and *per se* any information about the number of these is unknown, however, as already mentioned at the quasi–free setting, their uniqueness is guaranteed. Because of the set $\mathcal{C}$ of ground states is metrizable in the weak$^*$–topology, we denote by $\mathcal{C}_0 \equiv (\mathcal{C}_\varphi, \mathcal{D}_\varphi)$ and $\mathcal{C}_0 \equiv (\mathcal{C}_0, \mathcal{D}_0)$ the metric spaces in the weak$^*$–topology related to the quasi–free ground states for $\varphi \in \mathbb{R}^+$ and $\varphi = 0$ respectively.
particular, one notes that $\mathfrak{E}_g$ and $\mathfrak{E}_0$ are not homeomorphic since, as we will see on Corollary 3, the representations associated to $\mathfrak{E}_g$ are reducible whereas those associated to $\mathfrak{E}_0$ not. This is clear from the fact that there is not homeomorphism between one connected metric space and another one that is disconnected\(^6\). Then the representations associated to $\mathfrak{E}_g$ and $\mathfrak{E}_0$ are not physically equivalent as the intuition says. Instead, Corollary 3 below claims that any two gapped quasi–free ground states associated to the quasi–free dynamics of two gapped Hamiltonians on the same g–phase are unitarily equivalent, thus, their irreducible representations also are.

In order to prove last statement, recall Expressions (18), (19) and (22), where for any $s \in \mathcal{C} \equiv [0, 1]$ one knows that the positive (on $\Sigma_s$) spectral projection $E_{+,s} \in \mathcal{B}(\mathcal{H})$ associated to the self–dual Hamiltonian $H_s$ on $(\mathcal{H}, \Gamma)$ there is a unique quasi–free ground state $\omega_s \in \mathfrak{E}_g$ such that

$$\omega_s (B(\varphi_1)B(\Gamma\varphi_2)) = \langle \varphi_1, E_{+,s}\varphi_2 \rangle_{\mathcal{H}}, \quad \varphi_1, \varphi_2 \in \mathcal{H}.$$ 

One more time, $H_g = \{H_s\}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H})$ is a family of self–dual Hamiltonians at the g–phase satisfying Assumption 1, and in this case, for any $s \in \mathcal{C}$, $\omega_s$ is also a gapped quasi–free ground states. By using the family of automorphism $\{\kappa_s\}_{s \in \mathcal{C}}$ on $\mathcal{B}(\mathcal{H})$ of Lemma 1, with $V_s \in \mathcal{B}(\mathcal{H})$ the unitary operator implementing $\kappa_s$, we note that

$$\omega_s = \omega_0 \circ \Upsilon_s, \quad s \in \mathcal{C}. \quad (37)$$

Here, $\Upsilon_s$ is the one–parameter (Bogoliubov) *–automorphism of sCAR($\mathcal{H}, \Gamma$) given by Expression (36). Additionally, let $\omega_g = \{\omega_s\}_{s \in \mathcal{C}}$ be a family of gapped quasi–free ground states associated to self–dual Hamiltonians $H_g$ on some self–dual Hilbert space $(\mathcal{H}, \Gamma)$ at the g–phase, with the same assumptions of Lemma 1. What Expression (37) means in terms of representations is that the associated (irreducible) GNS representation $(\mathcal{H}_{\omega_g}, \pi_{\omega_g}, \Omega_{\omega_g})$ is unique (up to unitary equivalence): for all $A \in \text{sCAR}(\mathcal{H}, \Gamma)$ one write

$$\omega_g(A) = (\Omega_{\omega_g}, \pi_{\omega_g}(A)\Omega_{\omega_g}),$$

where the latter notation means that for any two states $\omega_{s_1}, \omega_{s_2} \in \mathfrak{E}_g$ exists an isomorphism $\mathcal{J}_{s_1,s_2}$ from $\mathcal{H}_{\omega_{s_1}}$ to $\mathcal{H}_{\omega_{s_2}}$ satisfying

$$\pi_{\omega_{s_2}} (A) = \mathcal{J}_{s_1,s_2}^{-1} \pi_{\omega_{s_1}} (A) \mathcal{J}_{s_1,s_2}, \quad i.e., \pi_{\omega_{s_2}} \text{ and } \pi_{\omega_{s_2}} \text{ are unitarily equivalent as well as their associated cyclic vectors } \Omega_{\omega_{s_1}} \text{ and } \Omega_{\omega_{s_2}}.$$

Additionally, following Definition 6 and comments around it, there is a strongly continuous one–parameter unitary group $\left( e^{it\mathcal{L}_{\omega_g}} \right)_{t \in \mathbb{R}}$ with generator $\mathcal{L}_{\omega_g} = \mathcal{L}^*_{\omega_g} \geq 0$ satisfying $e^{it\mathcal{L}_{\omega_g}} \Omega_{\omega_g} = \Omega_{\omega_g}$ such that for $t \in \mathbb{R}$, $e^{it\mathcal{L}_{\omega_g}} \in \pi_{\omega_g}(\text{sCAR}(\mathcal{H}, \Gamma))''$ and any $A \in \text{sCAR}(\mathcal{H}, \Gamma)$

$$e^{it\mathcal{L}_{\omega_g}} \pi_{\omega_g}(A)\Omega_{\omega_g} = \pi_{\omega_g} \left( \tau_t(A) \right) \Omega_{\omega_g}.$$ 

We summarize the latter at the following Corollary:

**Corollary 3.**

Consider a family of self–dual Hamiltonians $H_g \in \mathcal{B}(\mathcal{H})$ satisfying Assumption 1 at the g–phase. Let $\omega_g \in \mathfrak{E}_g$ be a family of gapped quasi–free ground states associated to $H_g$. Thus, the associated (irreducible) GNS representation $(\mathcal{H}_{\omega_g}, \pi_{\omega_g}, \Omega_{\omega_g})$ is unique (up to unitary equivalence). In particular, any state $\omega_s \in \mathfrak{E}_g$, $s \in \mathcal{C}$, is related with $\omega_0 \in \mathfrak{E}_g$ by Expression (37), namely, $\omega_s = \omega_0 \circ \Upsilon_s$. \(\blacksquare\)

\(^6\)In particular $\mathfrak{E}_g$ is a weak∗–compact convex set metrizable in the weak∗–topology that can be written as $\mathfrak{E}_g = \mathfrak{E}_{g–} \cup \mathfrak{E}_{g+}$, for $\mathfrak{E}_{g–}$ and $\mathfrak{E}_{g+}$ nonempty and disjoint sets metrizable in the weak∗–topology. Here, $\mathfrak{E}_{g–}$ and $\mathfrak{E}_{g+}$ are associated to the negative and positive components of the unitary operators respectively.
4.2 Dynamics, ground states and spectral flow automorphism at the Thermodynamic limit

For $d \in \mathbb{N}$, let $\mathbb{Z}^d$ be the Cayley graph as defined in Appendix A, see Expression (58), and let the spin set $\mathcal{S}$, such that $\mathcal{E} \cong \mathbb{Z}^d \times \mathcal{S}$. Since we are dealing with fermions, w.l.o.g., these can be treated as negatively charged particles. The cases of particles positively charged can be treated by exactly the same methods. Then, in order to take the thermodynamic limit we define the Hilbert spaces $\mathcal{H}_\mathcal{E} \cong \ell^2(\mathcal{S}) \oplus \ell^2(\mathcal{S}^*)$ and $\mathcal{H}_L \cong \ell^2(\Lambda_L; \mathcal{H}_\mathcal{E})$ for all $L \in \mathbb{R}_0^+ \cup \{\infty\}$, where $\Lambda_L$ for $L \in \mathbb{R}_0^+ \cup \{\infty\}$ is defined by the increasing sequence of cubic boxes

$$
\Lambda_L \doteq \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : |x_1|, \ldots, |x_d| \leq L\} \in \mathcal{P}_I(\mathbb{Z}^d),
$$

of side length $\mathcal{O}(L)$. Note that, such a sequence is a “Van Hove net”, i.e., the volume of the boundaries $\partial\Lambda_L \subset \Lambda_L \subset \mathcal{P}_I(\mathbb{Z}^d)$ is negligible w.r.t. the volume of $\Lambda_L$ for $L$ large enough: $\lim_{L \to \infty} \{\partial\Lambda_L/|\Lambda_L|\} = 0$.

We now fix any antiunitary involution $\Gamma_\mathcal{E}$ on $\mathcal{H}_\mathcal{E}$. For any $L \in \mathbb{R}_0^+ \cup \{\infty\}$, we define an antiunitary involution $\Gamma_L$ on $\mathcal{H}_L$ by

$$
(\Gamma_L \varphi)(x) \doteq \Gamma_\mathcal{E}(\varphi(x)), \quad x \in \Lambda_L, \quad \varphi \in \mathcal{H}_L.
$$

Then, $(\mathcal{H}_L, \Gamma_L)$ is a local self–dual Hilbert space for any $L \in \mathbb{R}_0^+ \cup \{\infty\}$. Note that $\mathcal{H}_\mathcal{E}$ and $\mathcal{H}_L$ are finite–dimensional, with even dimension, whenever $L < \infty$: Let

$$
\mathcal{X}_L \doteq \Lambda_L \times \mathcal{E} \times \{+, -\}, \quad L \in \mathbb{R}_0^+ \cup \{\infty\}.
$$

The canonical orthonormal basis $\{\epsilon_x\}_{x \in \mathcal{X}_L}$ of $\mathcal{H}_L$, $L \in \mathbb{R}_0^+ \cup \{\infty\}$, now is defined by

$$
\epsilon_x(y) = \delta_{x,y} f_{x,v}, \quad x = (x, s, v) \in \mathcal{X}_L, \quad y \in \Lambda_L,
$$

where $f_{s,t} = \Gamma_\mathcal{E} f_{s,-} \in \mathcal{H}_\mathcal{E}$ and $f_{s,-}(t) = \delta_{s,t}$ for any $s, t \in \mathcal{S}$.

In the self–dual formalism, a lattice fermion system in infinite volume is defined by a self–dual Hamiltonian $H_\infty \in \mathcal{B}(\mathcal{H}_\infty)$ on $(\mathcal{H}_\infty, \Gamma_\mathcal{E})$, that is, $H_\infty = H_\infty^* = -\Gamma_\mathcal{E} H_\infty \Gamma_\mathcal{E}$. See Definition 4 which is here extended to the infinite–dimensional case. For a fixed basis projection $P_\infty$ diagonalizing $H_\infty$, the operator $P_\infty H_\infty P_\infty$ is the so–called one–particle Hamiltonian associated with the system. To obtain the corresponding self–dual Hamiltonians in finite volume we use the orthogonal projector $P_{\mathcal{H}_L} \in \mathcal{B}(\mathcal{H}_L)$ on $\mathcal{H}_L$ and define

$$
H_L \doteq P_{\mathcal{H}_L} H_\infty P_{\mathcal{H}_L}, \quad L \in \mathbb{R}_0^+.
$$

By construction, if $H_\infty$ is a self–dual Hamiltonian on $(\mathcal{H}_\infty, \Gamma_\mathcal{E})$, then, for any $L \in \mathbb{R}_0^+$, $H_L$ is a self–dual Hamiltonian on $(\mathcal{H}_L, \Gamma_L)$. Note that $P_{\mathcal{H}_L}$ strongly converges to $1_{\mathcal{H}_\infty}$ as $L \to \infty$.

For the self–dual Hilbert space $(\mathcal{H}_\infty, \Gamma_\mathcal{E})$, the self–dual CAR algebra associated is denoted by $\mathfrak{A}_\infty \doteq \text{sCAR}(\mathcal{H}_\infty, \Gamma_\mathcal{E})$, with generator elements $\xi$ and $\{B(\epsilon_x)\}_{x \in \mathcal{X}_L}$ satisfying CAR Expressions of Definition 4. The subalgebra of even elements of $\mathfrak{A}_\infty$ (see (7)) will be denoted by $\mathfrak{A}_\infty^+$ in the sequel. For $\Lambda \in \mathcal{P}_I(\mathbb{Z}^d)$ and the finite–dimensional (one–particle) Hilbert space $\mathcal{H}_\Lambda \doteq \ell^2(\mathcal{H}_\mathcal{E}; \Lambda)$ with involution given by (39), we identify the finite dimensional CAR $C^*$–algebra

$$
\mathfrak{A}_\Lambda \doteq \text{sCAR}(\mathcal{H}_\Lambda, \Gamma_\Lambda), \quad \Lambda \in \mathcal{P}_I(\mathbb{Z}^d),
$$

By fixing $m \geq 1$, the boundary $\partial\Lambda$ of any $\Lambda \subset \mathbb{Z}^d$ is defined by $\partial\Lambda \doteq \{x \in \Lambda : \exists y \in \mathbb{Z}^d \setminus \Lambda \text{ with } d_m(x, y) \leq m\}$, where for $\epsilon \in (0, 1]$, $d_\epsilon(x, y) : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ is a well–defined pseudometric related to the distance between $x, y$ in the lattice $\mathbb{Z}^d$ [BP13]. W.l.o.g. we will take the $\epsilon$–euclidian distance $d_\epsilon(x, y) \doteq |x - y|^\epsilon$. 

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1By fixing $m \geq 1$, the boundary $\partial\Lambda$ of any $\Lambda \subset \mathbb{Z}^d$ is defined by $\partial\Lambda \doteq \{x \in \Lambda : \exists y \in \mathbb{Z}^d \setminus \Lambda \text{ with } d_m(x, y) \leq m\}$, where for $\epsilon \in (0, 1]$, $d_\epsilon(x, y) : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ is a well–defined pseudometric related to the distance between $x, y$ in the lattice $\mathbb{Z}^d$ [BP13]. W.l.o.g. we will take the $\epsilon$–euclidian distance $d_\epsilon(x, y) \doteq |x - y|^\epsilon$. 

---
with the $C^*$–subalgebra generated by the unit 1 and $\{B(\epsilon_x)\}_{x \in \mathbb{X}_\Lambda}$. Then, we define by
\begin{equation}
\mathfrak{A}^{(0)}_\infty = \bigcup_{\Lambda \in \mathcal{P}(\mathbb{Z}^d)} \mathfrak{A}_\Lambda \subset \mathfrak{A}_\infty,
\end{equation}
the normed $*$–algebra of local elements, which is dense in $\mathfrak{A}_\infty$.

From Definition 6 one notes that existence of ground states strongly relies on the existence of the dynamics at the thermodynamical limit. The latter means that the sequence $\{\Lambda_L\}_{L \in \mathbb{R}_0^+ \cup \{\infty\}}$, defined by (38), eventually will contain all the finite subsets, $\mathcal{P}_f(\mathbb{Z}^d)$ of $\mathbb{Z}^d$ as $L \to \infty$. In fact, for any $H_L = H_L^* \in \mathcal{B}(\mathcal{H}_L)$ one can associate a quasi–free dynamics (10) defining a continuous group $\{\tau_t^{(L)}\}_{t \in \mathbb{R}}$, of finite volume $*$–automorphisms of $\mathfrak{A}_L \equiv \mathfrak{A}_{AL}$ by
\begin{equation}
\tau_t^{(L)}(A) = e^{-it(B, H_L B)} A e^{it(B, H_L B)}, \quad A \in \mathfrak{A}_\infty, \quad t \in \mathbb{R}.
\end{equation}
See (42) and (44). The associated finite volume generator or finite symmetric derivation is given by (13), namely,
\begin{equation}
\delta^{(L)}(A) = -i[\langle B, H_L B \rangle, A], \quad A \in \mathfrak{A}^{(0)}_\infty,
\end{equation}
while the infinite volume generator or symmetric derivation is given by
\begin{equation}
\delta(A) = -i\langle B, H_\infty B \rangle, A \in \mathfrak{A}^{(0)}_\infty.
\end{equation}

For $L \in \mathbb{R}_0^+$ and $\Lambda_L \in \mathcal{P}_f(\mathbb{Z}^d)$, denote by $\Lambda_L^c \equiv \mathbb{Z}^d \setminus \Lambda_L$ the complement of $\Lambda_L$. Then, $\mathfrak{A}_{\Lambda^c} \equiv \text{scAR}(\mathcal{H}_\Lambda, \Gamma_{\Lambda^c})$, will be the $C^*$–subalgebra generated by the unit 1 and $\{B(\epsilon_x)\}_{x \in \mathbb{X}_{\Lambda^c}}$. The bilinear elements associated to the (border) terms on $\Lambda_L$ and $\Lambda^c_L$ are (cf. Definition 3):
\begin{equation}
\langle B, \partial H_L B \rangle = \sum_{x_1, x_2 \in \mathbb{X}_\infty} \langle \epsilon_{x_2}, \partial H_L \epsilon_{x_1} \rangle_{\mathcal{H}_\infty} B(\epsilon_{x_1}) B(\epsilon_{x_2})^*,
\end{equation}
with $\mathcal{H}^c_L \equiv \mathcal{H}_{\Lambda^c}$ and
\begin{equation}
\partial H_L = P_{\mathcal{H}_L} H_\infty P_{\mathcal{H}_L} + P_{\mathcal{H}^c_L} H_\infty P_{\mathcal{H}^c_L},
\end{equation}
where for any $\Lambda_L \in \mathcal{P}_f(\mathbb{Z}^d)$, $P_{\mathcal{H}_L} \in \mathcal{B}(\mathcal{H}_L)$ is the orthogonal projector on $\mathcal{H}_L$, see Expression (42).

Theorem 3 (Infinite volume dynamics):
\begin{quote}
Assume that the sequence $\{H_L\}_{L \in \mathbb{R}_0^+}$ of self–dual Hamiltonians $H_L \in \mathcal{B}(\mathcal{H}_L)$ strongly converges to $H_\infty \in \mathcal{B}(\mathcal{H}_\infty)$. For $L \in \mathbb{R}_0^+$, the continuous group $\{\tau_t^{(L)}\}_{t \in \mathbb{R}}$ with generator $\delta^{(L)}$ converges strongly to a continuous group $\{\tau_t\}_{t \in \mathbb{R}}$ with generator $\delta$ as $L \to \infty$.
\end{quote}

\textbf{Proof.} The proof of the statements are completely standard, and we present here for the sake of completeness. We can combine Expressions (10) and (11) such that for any self–dual Hamiltonian $H_\infty \in \mathcal{B}(\mathcal{H}_\infty)$ we have
\begin{equation}
\tau_t^{(L)}(B(\varphi)) = B\left(\left(U_t^{(L)}\right)^\varphi\right) \quad \text{and} \quad \tau_t(B(\varphi)) = B\left(U_t^\varphi\right).
\end{equation}
Here, for $L \in \mathbb{R}_0^+$, $\tau_t^{(L)} = \chi_{e^{itH_L}}$ and $\tau_t = \chi_{e^{itH_\infty}}$ so that
\begin{equation}
\left\{U_t^{(L)} = e^{itH_L}\right\}_{t \in \mathbb{R}} \quad \text{and} \quad \left\{U_t = e^{itH_\infty}\right\}_{t \in \mathbb{R}}
\end{equation}
are the one–parameter unitary groups on $(\mathcal{H}_\infty, \Gamma_\infty)$ associated to the finite and infinite dynamical systems, respectively. Note that for any $\varphi \in \mathcal{H}_\infty$, $B(\varphi)$ is bounded (see Definition 1). Then, using that $\|B(\varphi)\|_\mathfrak{A}_\infty \leq \|\varphi\|_\mathcal{H}_\infty$, for any $L_1, L_2 \in \mathbb{R}_0^+$, with $L_2 \geq L_1$, we have
\begin{equation}
\left\|\tau_t^{(L_2)}(B(\varphi)) - \tau_t^{(L_1)}(B(\varphi))\right\|_\mathfrak{A}_\infty \leq \left\|U_t^{(L_2)} - U_t^{(L_1)}\right\|_{\mathcal{B}(\mathcal{H}_\infty)} \|\varphi\|_\mathcal{H}_\infty.
\end{equation}
We can write
\[ U_t^{(L_2)} - U_t^{(L_1)} = i \int_0^t \partial_s \left( U_{t-s}^{(L_1)} U_s^{(L_2)} \right) ds = i \int_0^t U_{t-s}^{(L_1)} (H_{L_2} - H_{L_1}) U_s^{(L_2)} ds, \]
so that
\[ \| \tau_t^{(L_2)} (B(\varphi)) - \tau_t^{(L_1)} (B(\varphi)) \|_{\mathcal{A}_\infty} \leq |t| \| H_{L_2} - H_{L_1} \|_{\mathcal{B}(\mathcal{H}_\infty)} \| \varphi \|_{\mathcal{H}_\infty}. \]

Since the sequence \( \{ H_{\alpha} \}_{\alpha \in \mathbb{R}_+^0} \) strongly converges to \( H_\infty \) as \( L \to \infty \), last expression shows that it is a Cauchy sequence of self-adjoint operators. Therefore, the continuous group of \(*\)-automorphisms \( \{ \tau_t^{(L)} \}_{t \in \mathbb{R}} \), \( L \in \mathbb{R}_+^0 \), strongly converges to \( \{ \tau_t \}_{t \in \mathbb{R}} \) for all \( t \in \mathbb{R} \).

To show the existence of the generator, we restrict our study to bounded self–dual Hamiltonians. An extension to unbounded self–dual Hamiltonians can be found using similar arguments that in [BP16, Theorem 4.8]. With the same notation of above, note that the difference between finite volume generators is (see (45))
\[ \delta^{(L_2)}(A) - \delta^{(L_1)}(A) = -i[\langle B, (H_{L_2} - H_{L_1}) B \rangle, A], \quad A \in \mathfrak{A}_L. \]

We can write for \( L_1, L_2 \in \mathbb{R}_+^0 \), with \( L_2 \geq L_1 \),
\[ H_{L_2} = H_{L_1} + P_{\mathcal{H}_{L_2}} H_\infty P_{\mathcal{H}_{L_2}} \| \mathcal{H}_{L_1} \| + P_{\mathcal{H}_{L_2}} H_\infty P_{\mathcal{H}_{L_2}} \| \mathcal{H}_{L_1} \| + P_{\mathcal{H}_{L_2}} \| \mathcal{H}_{L_1} \| H_\infty P_{\mathcal{H}_{L_2}} \| \mathcal{H}_{L_1} \|, \]
where \( P_{\mathcal{H}_{L_2}} H_\infty P_{\mathcal{H}_{L_1}} \equiv P_{\mathcal{H}_{L_2}} - P_{\mathcal{H}_{L_1}} \in \mathcal{B}(\mathcal{H}_{L_2}) \) is the orthogonal projector on \( \mathcal{H}_{L_2} \setminus \mathcal{H}_{L_1} \). Then, for any fix \( \Lambda \subseteq \Lambda_{L_1} \supsetneq \Lambda_{L_2} \)
\[ \| \delta^{(L_2)}(A) - \delta^{(L_1)}(A) \|_{\mathcal{A}_\infty} \leq 2 \| A \|_{\mathcal{A}_\infty} \left( \sum_{x_1, x_2 \in \mathcal{X}_{L_2} \setminus \mathcal{X}_{L_1}} | \langle \mathbf{e}_{x_2}, H_\infty \mathbf{e}_{x_1} \rangle_{\mathcal{H}_\infty} | + 2 \mathcal{X}_{L_1, L_2}(\Lambda) \right), \]
with
\[ \mathcal{X}_{L_1, L_2}(\Lambda) = \max \left\{ \sum_{x_1 \in \mathcal{X}_{L_2} \setminus \mathcal{X}_{L_1}} \sum_{x_2 \in \mathcal{X}_{L_2}} \langle \mathbf{e}_{x_2}, H_\infty \mathbf{e}_{x_1} \rangle_{\mathcal{H}_\infty}, \sum_{x_2 \in \mathcal{X}_{L_2} \setminus \mathcal{X}_{L_1}} \sum_{x_1 \in \mathcal{X}_{L_1}} \langle \mathbf{e}_{x_2}, H_\infty \mathbf{e}_{x_1} \rangle_{\mathcal{H}_\infty} \right\}. \]

We finally note that for \( L_1, L_2 \to \infty \), \( \| \delta^{(L_2)}(A) - \delta^{(L_1)}(A) \|_{\mathcal{A}_\infty} \) goes to zero and the sequence \( \{ \delta^{(L)} \}_{L \in \mathbb{R}_+^0} \) is Cauchy. In fact, it is (absolutely) convergent for any \( A \in \mathfrak{A}_\infty : \delta(A) = \lim_{L \to \infty} \delta^{(L)}(A), \quad A \in \mathfrak{A}_\infty. \) In particular, note that for the local element \( A \in \mathfrak{A}_L \)
\[ \| \langle B, H_L B \rangle, A \|_{\mathcal{A}_\infty} \leq 2 \| A_L \|_{\mathcal{A}_\infty} \max \left\{ \sup_{x_1 \in \mathcal{X}_L} \sum_{x_2 \in \mathcal{X}_L} | \langle \mathbf{e}_{x_2}, H_L \mathbf{e}_{x_1} \rangle_{\mathcal{H}_\infty} |, \right\} \]
\[ \quad \sup_{x_2 \in \mathcal{X}_L} \sum_{x_1 \in \mathcal{X}_L} | \langle \mathbf{e}_{x_2}, H_L \mathbf{e}_{x_1} \rangle_{\mathcal{H}_\infty} |. \]

By finally remark, that the second Trotter–Kato Approximation Theorem [EBN+06, Chap. III, Sect. 4.9] assures that \( \delta : \mathfrak{A}_\infty \to \mathfrak{A}_\infty \) is the generator of \( \{ \tau_t \}_{t \in \mathbb{R}} \). For complete details see [BP16].

**Remark 3.** Observe that for any \( t \in \mathbb{R}, \Lambda \in \mathcal{P}(\mathbb{Z}^d), A \in \mathfrak{A}_\Lambda \) and \( L_1, L_2 \in \mathbb{R}_0^+ \) with \( \Lambda \subseteq \Lambda_{L_1} \subsetneq \Lambda_{L_2} \) we have:
\[ \tau_t^{(L_2)}(A) - \tau_t^{(L_1)}(A) = \int_0^t \frac{d}{ds} \left( \tau_s^{(L_2)} \left( \tau_{t-s}^{(L_1)}(A) \right) \right) ds \]
\[ = -i \int_0^t \tau_s^{(L_2)} \left( \langle B, (H_{L_2} - H_{L_1}) B \rangle, \tau_{t-s}^{(L_1)}(A) \right) ds \]

Because of Expression (11), the boundedness of the generators $1$ and $\{B(\epsilon_{x})\}_{x \in \mathcal{X}_{L}}$, and due to $\Lambda \subset \Lambda_{1}$, one has:

$$\left\| \tau^{(L_{2})}_{t}(A) - \tau^{(L_{1})}_{t}(A) \right\|_{\mathcal{A}_{\infty}} \leq 2 \left\| A \right\|_{\mathcal{A}_{\infty}} \left( \sum_{x_{1}, x_{2} \in \mathcal{X}_{L_{2}} \setminus \mathcal{X}_{L_{1}}} \left| \langle \epsilon_{x_{2}}, H_{\infty} \epsilon_{x_{1}} \rangle_{\mathcal{H}_{\infty}} \right| + 2 \mathcal{X}_{L_{1}, L_{2}}(\Lambda) \right),$$

where $\mathcal{X}_{L_{1}, L_{2}}(\Lambda)$ is given by (49). Last inequality is reminiscent to Lieb–Robinson bounds (LRB) used to show the existence of dynamics for the interacting short-range case [BP16].

In order to study quasi–free ground states at infinite volume we use:

**Proposition 2.**

Let $\{H_{L}\}_{L \in \mathbb{R}_{0}^{+}} \in \mathcal{B}(\mathcal{H}_{\infty})$ be a sequence of self–dual Hamiltonians on $(\mathcal{H}_{\infty}, \Gamma_{\infty})$ strongly convergent to $H_{\infty} \in \mathcal{B}(\mathcal{H}_{\infty})$. For any $L \in \mathbb{R}_{0}^{+} \cup \{\infty\}$, $E_{+, L}$ will denotes the spectral projection on $\mathbb{R}^{+}$ associated to the self–dual Hamiltonian $H_{L}$. If zero is not eigenvalue of $H_{\infty}$, then $E_{+}$ will be the strongly limit of the sequence $\{E_{+, L}\}_{L \in \mathbb{R}_{0}^{+}}$, i.e., $\lim_{L \to \infty} E_{+, L} = E_{+}$.

**Proof.** The proof is found in [AE83, Lemma 3.3.]

For any $L \in \mathbb{R}_{0}^{+}$ let us define the set of local quasi–free ground states by $q\mathcal{E}^{(L)} \subset q\mathcal{E}^{(\infty)}$ on $\mathfrak{A}_{L} \subset \mathfrak{A}_{\infty}$. See Definition 7. To be explicit, for any self–dual Hamiltonian $H_{\infty} \in \mathcal{B}(\mathcal{H}_{\infty})$ on $(\mathcal{H}_{\infty}, \Gamma_{\infty})$ and any orthogonal projection $P_{\mathcal{H}_{L}} \in \mathcal{B}(\mathcal{H}_{\infty})$ on $\mathcal{H}_{L}$ the local Hamiltonian is given by (42), namely,

$$H_{L} = P_{\mathcal{H}_{L}} H_{\infty} P_{\mathcal{H}_{L}},$$

which has associated the local Gibbs state defined by

$$\varrho_{\Lambda_{L}} \left( B(\varphi_{1, \Lambda_{L}}) B(\varphi_{2, \Lambda_{L}}) \right) = \frac{\langle \varphi_{1, \Lambda_{L}}, E_{+, L} \varphi_{2, \Lambda_{L}} \rangle_{\mathcal{H}_{L}}}{\langle \varphi_{1, \Lambda_{L}}, \varphi_{2, \Lambda_{L}} \rangle_{\mathcal{H}_{L}}},$$

for $\varphi_{j, \Lambda_{L}} \in \mathcal{H}_{L}, j = \{1, 2\}$, where $E_{+, L}$ denotes the sequence of spectral projections of Proposition 2. Then, using Expressions (16)–(17) the local quasi–free ground state $\omega_{\Lambda_{L}} \in q\mathcal{E}^{(L)}$ is determined to be

$$\omega_{\Lambda_{L}} \left( B(\varphi_{1, \Lambda_{L}}) B(\varphi_{2, \Lambda_{L}}) B(\varphi_{3, \Lambda_{L}}) B(\varphi_{4, \Lambda_{L}}) \right) = 2 \varrho_{\Lambda_{L}} \left( B(\varphi_{1, \Lambda_{L}}) B(\varphi_{2, \Lambda_{L}}) \right) \text{tr} \left( B(\varphi_{3, \Lambda_{L}}) B(\varphi_{4, \Lambda_{L}}) \right)$$

where $\varphi_{j, \Lambda_{L}} \in \mathfrak{A}_{\Lambda_{L}} \equiv \mathfrak{A}_{L}, j = \{3, 4\}$, and $\text{tr} \in \mathcal{E}$ is the tracial state of Definition 5, cf. [AM03, Section 4.2]. In particular, if $\varphi_{3, \Lambda_{L}} = \varphi_{4, \Lambda_{L}} = 1_{\mathcal{H}_{\infty}}$ one has

$$\omega_{\Lambda_{L}} \left( B(\varphi_{1, \Lambda_{L}}) B(\varphi_{2, \Lambda_{L}}) \right) = \varrho_{\Lambda_{L}} \left( B(\varphi_{1, \Lambda_{L}}) B(\varphi_{2, \Lambda_{L}}) \right).$$

Additionally, by linearity, for any two even elements $A \in \mathfrak{A}_{L}^{+}$ and $B \in \mathfrak{A}_{L}^{c}$, see (7), we get from (50):

$$\omega_{\Lambda_{L}} \left( AB \right) = 2 \varrho_{\Lambda_{L}} \left( A \right) \text{tr} \left( B \right),$$

see again [AM03, Section 4.2]. We now state:

**Theorem 4 (Quasi–free ground states):**

The local quasi–free ground state $\omega_{\Lambda_{L}}$ converges to

$$\omega \left( B(\varphi_{1}) B(\varphi_{2}) \right) = \langle \varphi_{1}, E_{+} \varphi_{2} \rangle_{\mathcal{H}_{\infty}},$$

in the weak*–topology, where $E_{+} \in \mathcal{B}(\mathcal{H}_{\infty})$ is the spectral projection on $\mathbb{R}^{+}$ associated to the self–dual Hamiltonian $H_{\infty} \in \mathcal{B}(\mathcal{H}_{\infty})$, and $\varphi_{1}, \varphi_{2} \in \mathcal{H}_{\infty}$. ☐

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Proof. Take $L_1, L_2 \in \mathbb{R}_0^+$, with $L_2 \geq L_1$ such that $\Lambda_{L_2} \supseteq \Lambda_{L_1}$. Thus, we analyze the following difference:

$$D\omega_{L_2},\omega_{L_1} = \omega_{L_2} \left( B(\varphi_{1, L_2})B(\varphi_{2, L_2})^* \right) - \omega_{L_1} \left( B(\varphi_{1, L_1})B(\varphi_{2, L_1})^* \right).$$

Here, in the way that was defined the set of boxes $\Lambda_L$, given by (38), for $j = \{1, 2\}$ we canonically identify $\varphi_{j, L_1} \in \mathcal{H}_{L_1}$ with the element $\varphi_{j, L_1} \oplus 0_{\Lambda_{L_2} \setminus \Lambda_{L_1}} \in \mathcal{H}_{L_2}$. The spectral projections on $\mathbb{R}^+$ are related by $E_{L_2} = E_{L_1} \oplus 1_{\Lambda_{L_2} \setminus \Lambda_{L_1}} \in \mathcal{B}(\mathcal{H}_{L_2})$. Straightforward calculations yield us to note that

$$\lim_{L_1 \to \infty} \lim_{L_2 \to \infty} D\omega_{L_2},\omega_{L_1}$$

equals zero.

We are now in a position to prove the properties of the family of automorphisms $\kappa_s : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, for any $s \in \mathcal{C}$, given by Assumption 1 and Lemma 1, which are associated to a differentiable family of self–dual Hamiltonians $H_\beta \in \mathcal{B}(\mathcal{H}_\infty)$ in the $\varphi$–phase, see Definition 10. Observe that the existence of such $\kappa_s$ is strongly related to the existence of a differentiable unitary operator $V_s \in \mathcal{B}(\mathcal{H})$ satisfying the non–autonomous differential equation, Expression (29):

$$\partial_s V_s = -i\mathcal{D}_{_0, s} V_s, \quad \text{with} \quad V_0 = 1_{\mathcal{M}},$$

where $\{\mathcal{D}_{_0, s}\}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}_\infty)$ is a family of self–adjoint operators that is found to be

$$\mathcal{D}_{_0, s} = \int_\mathbb{R} e^{iH_s} (\partial_s H_s) e^{-iH_s} 2\mathcal{M}_0(t)dt,$$

with $\mathcal{M}_0 : \mathbb{R} \to \mathbb{R}$ an integrable odd function such that its properties are summarized in [BMNS12, MZ13] and references therein. In the sequel, for any $s \in \mathcal{C}$, $V_s, H_s$ and $\partial_s H_s$ have to be understood as the strong limit of the sequences $\{V_{s, L}\}_{L \in \mathbb{R}_0^+}, \{H_{s, L}\}_{L \in \mathbb{R}_0^+}$ and $\{\partial_s H_{s, L}\}_{L \in \mathbb{R}_0^+}$ respectively. We formulate:

**Lemma 2.**

Take $\mathcal{C} \equiv [0, 1]$, fix $s \in \mathcal{C}$ and consider the family of operators satisfying Assumption 1. Then, the sequence of automorphisms $\{\kappa_{s, L}\}_{L \in \mathbb{R}_0^+} : \mathcal{B}(\mathcal{H}_L) \to \mathcal{B}(\mathcal{H}_L)$ of Lemma 1 on the local self–dual Hilbert space $(\mathcal{H}_L, \Gamma_L)$ strongly converges uniformly on $\mathcal{C}$ to $\kappa_s : \mathcal{B}(\mathcal{H}_\infty) \to \mathcal{B}(\mathcal{H}_\infty)$. More precisely, for any $L \in \mathcal{P}([0, 1])$, $B \in \mathcal{B}(\mathcal{H}_\Lambda)$ and $L \in \mathbb{R}_0^+$ such that $\Lambda \subset \Lambda_L$ we have

$$\lim_{L \to \infty} \left\| \kappa_{s, L}(B) - \kappa_s(B) \right\|_{\mathcal{B}(\mathcal{H}_\infty)} = 0, \quad \text{for any} \quad s \in \mathcal{C}. \quad \diamondsuit$$

**Proof.** Fix $L \in \mathcal{P}([0, 1])$ and take $L_1, L_2 \in \mathbb{R}_0^+$, with $L_2 \geq L_1$ such that $\Lambda_{L_2} \supseteq \Lambda_{L_1} \supseteq \Lambda$. We proceed in a similar way that in [BP16, Lemma 4.4]. Note that with a few modifications of the proof we can arrive that the result works even in the interparticle case [AR20].

For any $L \in \mathbb{R}_0^+$, let $V_s(L) \in \mathcal{B}(\mathcal{H}_L)$ be the unitary operator satisfying the differential equation (29). For $s, r \in \mathcal{C}$, one defines the unitary element

$$U_L(s, r) = V_s(L) \left( V_r(L)^* \right)^*,$$

which satisfies $U_L(s, s) = 1_{\mathcal{M}}$ for all $s \in \mathcal{C}$ while

$$\partial_s U_L(s, r) = -i\mathcal{D}_{_0, s} U_L(s, r) \quad \text{and} \quad \partial_r U_L(s, r) = iU_L(s, r)\mathcal{D}_{_0, r}. $$

Note that for $B \in \mathcal{B}(\mathcal{H}_\Lambda)$ one can write

$$\kappa_{s, L_2}(B) - \kappa_{s, L_1}(B) = \int_0^1 \partial_r(U_{L_2}(0, r)U_{L_1}(r, s)BU_{L_1}(s, r)U_{L_2}(r, 0))dr.$$
Straightforward calculations show us that the derivative inside the integral is

$$iU_{L_2}(0, r) \left[ \left( \mathcal{D}_{g,r}^{(L_2)} - \mathcal{D}_{g,r}^{(L_1)} \right), U_{L_1}(r, s)BU_{L_1}(s, r) \right] U_{L_2}(r, 0)$$

with $s, r \in \mathcal{C}$, and for $L \in \mathbb{R}_0^+$, and $A \in \mathcal{P}_1(\mathbb{Z}^d), A \subset A_L$.

On the other hand, for any $s \in \mathcal{C}, t \in \mathbb{R}^+, A \in \mathcal{P}_1(\mathbb{Z}^d)$ and $L \in \mathbb{R}_0^+$ such that $A_L \supset A$, define the $s$–automorphism $\tilde{\tau}_{s,t}^{(L)} : \mathcal{B}(\mathcal{H}_L) \to \mathcal{B}(\mathcal{H}_L)$ by

$$\tilde{\tau}_{s,t}^{(L)}(B) = e^{iH_{s,L}t}BE^{-itH_{s,L}},$$

with $H_{s,L}$ a self–dual Hamiltonian on $(\mathcal{H}_L, \Gamma_L)$. Then, for $L_1, L_2 \in \mathbb{R}_0^+$, one can write the following

$$\tilde{\tau}_{s,t}^{(L_2)}(B) - \tilde{\tau}_{s,t}^{(L_1)}(B) = \int_0^t \partial_u \left( \tilde{\tau}_{s,t}^{(L_2)} \circ \tilde{\tau}_{s,t}^{(L_1)}(B) \right) du$$

$$= i \int_0^t \tilde{\tau}_{s,t}^{(L_u)} \left( \left[ H_{s,L_2} - H_{s,L_1}, \tilde{\tau}_{s,t}^{(L_1)}(B) \right] \right) du,$$

where, for a fix $s \in \mathcal{C}$, the difference $H_{s,L_2} - H_{s,L_1} \in \mathcal{B}(\mathcal{H}_\infty)$ is given by Expression (48), namely,

$$H_{s,L_2} - H_{s,L_1} = P_{\mathcal{H}_{L_2}}H_{s,\infty}P_{\mathcal{H}_{L_1}} + P_{\mathcal{H}_{L_2}}\mathcal{H}_{L_1}H_{s,\infty}P_{\mathcal{H}_{L_2}} + P_{\mathcal{H}_{L_2}}\mathcal{H}_{L_1}H_{s,\infty}P_{\mathcal{H}_{L_2}}\mathcal{H}_{L_1},$$

where $P_{\mathcal{H}_{L_2}}\mathcal{H}_{L_1} = P_{\mathcal{H}_{L_2}} - P_{\mathcal{H}_{L_1}} \in \mathcal{B}(\mathcal{H}_\infty)$ is the orthogonal projector on $\mathcal{H}_{L_2} \setminus \mathcal{H}_{L_1}$. Here, $H_{s,\infty}$ is the self–dual Hamiltonian on $(\mathcal{H}_\infty, \Gamma_\infty)$ at infinite volume. It follows that,

$$\| \tilde{\tau}_{s,t}^{(L_2)}(B) - \tilde{\tau}_{s,t}^{(L_1)}(B) \|_{\mathcal{B}(\mathcal{H}_\infty)} \leq \int_0^t \| \left[ H_{s,L_2} - H_{s,L_1}, \tilde{\tau}_{s,t}^{(L_1)}(B) \right] \| du$$

$$\leq 2|t| \| H_{s,L_2} - H_{s,L_1} \|_{\mathcal{B}(\mathcal{H}_\infty)} \| B \|_{\mathcal{B}(\mathcal{H}_\infty)}.$$  (54)

By Assumption 1, $\{ H_{s,L} \}_{L \in \mathbb{R}_0^+}$ is a sequence of operators which strongly converges to $H_{s,\infty}$ as $L \to \infty$, then last expression is a Cauchy sequence of self–adjoint operators. Hence, for all $t \in \mathbb{R}$, $\tilde{\tau}_{s,t}^{(L)}$ converges strongly on $\mathcal{B}(\mathcal{H}_L)$ to $\tilde{\tau}_{s,t}$, as $L \to \infty$.

That is important to stress is that the difference $\mathcal{D}_{g,r}^{(L_2)} - \mathcal{D}_{g,r}^{(L_1)}$ in Expression (53) can be written as follows

$$\mathcal{D}_{g,r}^{(L_2)} - \mathcal{D}_{g,r}^{(L_1)} = \int_\mathbb{R} \left( \tilde{\tau}_{r,t}^{(L_2)}(\partial_r \{ H_{r,L_2} \}) - \tilde{\tau}_{r,t}^{(L_1)}(\partial_r \{ H_{r,L_1} \}) \right) \mathcal{W}_g(t) dt + \int_\mathbb{R} \left( \tilde{\tau}_{r,t}^{(L_2)}(\partial_r \{ H_{r,L_1} \}) - \tilde{\tau}_{r,t}^{(L_1)}(\partial_r \{ H_{r,L_1} \}) \right) \mathcal{W}_g(t) dt.$$

From which one has

$$\| \kappa_{s}^{(L_2)}(B) - \kappa_{s}^{(L_1)}(B) \|_{\mathcal{B}(\mathcal{H}_\infty)} \leq 2 \| B \|_{\mathcal{B}(\mathcal{H}_\infty)} |s|$$

$$\sup_{r \in \mathcal{C}} \left( \int_\mathbb{R} \| \partial_r \{ H_{r,L_2} \} - \partial_r \{ H_{r,L_1} \} \|_{\mathcal{B}(\mathcal{H}_\infty)} |\mathcal{W}_g(t)| dt \right)$$

$$+ \int_\mathbb{R} \left( \| \tilde{\tau}_{r,t}^{(L_2)}(B) - \tilde{\tau}_{r,t}^{(L_1)} \|_{\mathcal{B}(\mathcal{H}_\infty)} |\mathcal{W}_g(t)| dt \right)$$

Hence, for a fix $s \in \mathcal{C}$, by Assumption 1 and Inequality (54) one notes that the right hand side of the last inequality vanishes as $L_2$ and $L_1$ go to $\infty$. Thus, $\kappa_{s}^{(L)}$ is a pointwise Cauchy sequence as $L \to \infty$ and hence the family of automorphism $\kappa_{s}^{(L)}$ converges strongly on $\mathcal{B}(\mathcal{H}_L)$ to $\kappa_{s}$ as $L \to \infty$.  

As a consequence we have:

End
Corollary 4.
Take same assumptions of Lemma 2. Then, for any $s \in \mathcal{C}$, the sequence of unitary operators $V_s^{(L)} \in \mathbb{B}(\mathcal{H}_\infty)$ strongly converges to some $V_s \in \mathbb{B}(\mathcal{H}_\infty)$.

Proof. As is usual, it is enough to show that the sequence $V_s^{(L)} \in \mathbb{B}(\mathcal{H}_\infty)$ is a Cauchy’s sequence. Note that for any $s \in \mathcal{C}$ and $L_1, L_2$ with $L_2 \geq L_1$, we can write:

$$
\left( V_s^{(L_2)} \right)^* - \left( V_s^{(L_1)} \right)^* = \int_0^1 \partial_r \left( U_{L_2}(0, r)U_{L_1}(r, s) \right) dr,
$$

where for any $s, r \in \mathcal{C}$, $U_L(s, r)$ is the unitary element defined by (51)–(52). Straightforward calculation yield to

$$
\left( V_s^{(L_2)} \right)^* - \left( V_s^{(L_1)} \right)^* = i \int_0^1 U_{L_2}(0, r) \left( \mathcal{D}_{0,r}^{(L_2)} - \mathcal{D}_{0,r}^{(L_1)} \right) U_{L_1}(r, s) dr.
$$

By using a similar argument that in (55) we arrive to the desired result. We omit the details.

End

4.3 Decay estimates of correlations and gapped quasi–free ground states
Fix $\varepsilon \in (0, 1]$ and let $(\mathcal{H}_\infty, \Gamma_\infty)$ be the self–dual Hilbert space as defined in subsection 4.2. Moreover, consider the family of self–adjoint operators $\{A_s\}_{s \in \mathcal{C}} \in \mathbb{B}(\mathcal{H}_\infty)$. Thus, for any $s \in \mathcal{C}$ we define the constants

$$
S(A_s, \mu) = \sup_{x_1 \in \mathbb{X}_\infty \atop x_2 \in \mathbb{X}_\infty} \sum \left( e^{\|x_1 - x_2\|^2} - 1 \right) |\langle \epsilon_{x_1}, A_s \epsilon_{x_2} \rangle| \in \mathbb{R}_0^+ \cup \{\infty\},
$$

for $\mu \in \mathbb{R}_0^+$ and

$$
\Delta(A_s, z) = \inf \{|z - \lambda| : \lambda \in \text{spec}(A_s)\}, \quad z \in \mathbb{C},
$$

as the distance from the point $z$ to the spectrum of $A_s$. $\mathbb{X}_\infty$ is defined by (40). Here, $\mu$ is not necessarily the same for two different operators $A_{s_1}, A_{s_2} \in \{A_s\}_{s \in \mathcal{C}}$, but in the sequel w.l.o.g. we will assume this. Since the function $x \mapsto (e^{\varepsilon x} - 1)/x$ is increasing on $\mathbb{R}^+$ for any fixed $r \geq 0$, it follows that

$$
(56) \quad S(A_s, \mu_1) \leq \frac{\mu_1}{\mu_2} S(A_s, \mu_2), \quad \mu_2 \geq \mu_1 \geq 0.
$$

We have the following Combès–Thomas estimates:

Proposition 3 (Combès–Thomas).
Let $\varepsilon \in (0, 1], \mathcal{C} = [0, 1]$, the family of self–adjoint operators $\{A_s\}_{s \in \mathcal{C}} \in \mathbb{B}(\mathcal{H}_\infty), \mu \in \mathbb{R}_0^+$ and $z \in \mathbb{C}$. If $\Delta(A_s, z) > S(A_s, \mu)$ then for any $s \in \mathcal{C}$ and $x = (x, s, v)$, $y = (y, t, w) \in \mathbb{X}_\infty$

$$
|\langle \epsilon_x, (z - A_s)^{-1} \epsilon_y \rangle| \leq \sup_{s \in \mathcal{C}} \left\{ \frac{e^{-\mu \|x - y\|^2}}{\Delta(A_s, z) - S(A_s, \mu)} \right\}.
$$

For a proof see [AW15, Theorem 10.5]. Some immediate consequences are summarized as follows:

Corollary 5.
Let $\varepsilon \in (0, 1], \mathcal{C} = [0, 1]$, the family of self–adjoint operators $\{A_s\}_{s \in \mathcal{C}} \in \mathbb{B}(\mathcal{H}_\infty), \mu \in \mathbb{R}_0^+$ and all $x = (x, s, v), y = (y, t, w) \in \mathbb{X}_\infty$. Then,
We are able to state the following:

At this point is useful to introduce the normalized 3D \( \in \mathbb{E} \) Cauchy–Schwarz inequality, the result follows. For further details see [Proof. (a) is proven like (b) (Gapped Case) For \( \eta \) and uniformly bounded by \( \|G\|_{\infty} \) we have

\[
\langle \epsilon_x, G(A_s) \epsilon_y \rangle \leq D_{5,(a)} \|G\|_{\infty} e^{-\mu \min \{ 1, \inf_{s \in \mathbb{E}} \{ \frac{\eta}{4S(A_s, \mu)} \} \} |x-y|^e}.
\]

Moreover, for any function \( G(z) : \mathbb{C} \to \mathbb{C} \) analytic on \( |\Re(z)| \leq \eta \) and uniformly bounded by \( \|G\|_{\infty} \) we have

\[
\langle \epsilon_x, G(A_s) \epsilon_y \rangle \leq D_{5,(b)} \|G\|_{\infty} e^{-\mu \min \{ 1, \inf_{s \in \mathbb{E}} \{ \frac{\eta}{4S(A_s, \mu)} \} \} |x-y|^e}.
\]

(b) (Gapped Case) For \( z \in \mathbb{C} \) such that \( \inf_{s \in \mathbb{E}} \Delta(A_s, z) \geq g/2 > 0 \), with \( g \) as in Definition 10:

\[
|\langle \epsilon_x, (z - A_s)^{-1} \epsilon_y \rangle| \leq 4g^{-1} \exp \left( -\mu \min \{ 1, \inf_{s \in \mathbb{E}} \{ \frac{g}{4S(A_s, \mu)} \} \} |x-y|^e \right).
\]

Moreover, for \( \eta \in (0, g/2] \), and any function \( G(z) : \mathbb{C} \to \mathbb{C} \) analytic on \( z \in \mathbb{R}^+ \times \eta + i \eta [-1, 1] \) and uniformly bounded by \( \|G\|_{\infty} \) we have

\[
\langle \epsilon_x, E_+ G(A_s) E_+ \epsilon_y \rangle \leq D_{5,(c)} \|G\|_{\infty} e^{-\mu \min \{ 1, \inf_{s \in \mathbb{E}} \{ \frac{\eta}{4S(A_s, \mu)} \} \} |x-y|^e}.
\]

In all inequalities, the numbers \( D_{5,(a)}, D_{5,(b)}, D_{5,(c)} \in \mathbb{R}^+ \) are suitable constants. \( \square \)

**Proof.** (a) is proven like [AG98, Theorem 3 and Lemma 3]. (b) First part is a consequence of Theorem 3 together Inequality (56). On the other hand, we use Cauchy’s integral formula to write, for all real \( E \in \mathbb{R} \setminus \{ \eta \} \),

\[
\chi_{(\eta, \infty)} G(E) = \frac{1}{2\pi i} \int_{\eta}^{\infty} \left( \frac{G(u - i\eta)}{u - E - i\eta} - \frac{G(u + i\eta)}{u - E + i\eta} \right) du - \frac{1}{2\pi} \int_{-\eta}^{\eta} G(\eta - E + iu) du,
\]

which yields

\[
\chi_{(\eta, \infty)} G(E) = \frac{\eta}{\pi} \int_{\eta}^{\infty} \left( \frac{G(u - i\eta)}{u - E} + \frac{G(u + i\eta)}{u - E} \right) du - \frac{2\eta}{\pi} \int_{\eta}^{\infty} \frac{G(u)}{(u - E)^2 + \eta^2} du
\]

\[
+ \frac{1}{2\pi} \int_{0}^{\eta} \frac{G(\eta - iu)}{\eta - iu - E + 2\eta} du + \frac{1}{2\pi} \int_{0}^{\eta} \frac{G(\eta + iu)}{\eta + iu - E - 2\eta} du - \frac{1}{2\pi} \int_{-\eta}^{\eta} \frac{G(\eta + iu)}{\eta - E + iu} du.
\]

By spectral calculus, together with the last equality, part (a) of this Lemma, Inequality (57) and the Cauchy–Schwarz inequality, the result follows. For further details see [ABPM20, Lemma 5.12]. \( \text{[End]} \)

At this point is useful to introduce the normalized *trace per unit volume* as

\[
\text{Tr}(\cdot) \doteq \lim_{L \to \infty} \frac{1}{\dim(H_L)} \text{tr}_{H_L}(\cdot).
\]

We are able to state the following:
Lemma 3.

Take $\mathcal{C} \equiv [0, 1]$ and consider the family of operators satisfying assumptions of Corollary 5 (a) for \( \{\partial_{\psi} F_s^{(L)}\}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}_L) \), \( L \in \mathbb{R}_0^+ \). Then, for the pointwise sequence \( V^{(L)} : \mathcal{C} \to \mathcal{B}(\mathcal{H}) \), \( L \in \mathbb{R}_0^+ \), of unitary operators satisfying (29), \( 1 - V^{(L)} \) is trace class. Thus, for \( L \in \mathbb{R}_0^+ \), the family of one–parameter (Bogoliubov) group \( \{ \Upsilon^{(L)} \}_{s \in \mathcal{C} \in \mathbb{R}} \) of \(^*\)-automorphisms on \( \mathfrak{A}_L \) (see (43)), given by (36), is inner. Moreover, \( 1_{\mathcal{H}_\infty} - V^{(\infty)} \) is trace class per unit volume, where \( V^{(\infty)} \in \mathcal{B}(\mathcal{H}) \) is that given by Corollary 4.

Proof. For \( s \in \mathcal{C} \) and \( L \in \mathbb{R}_0^+ \), let \( W_s^{(L)} \in \mathcal{B}(\mathcal{H}_L) \) be the partial isometry arising from the polar decomposition of \( 1_{\mathcal{H}_L} - V^{(L)} \)

\[
1_{\mathcal{H}_L} - V^{(L)} = W_s^{(L)} |1_{\mathcal{H}_L} - V^{(L)}|.
\]

From which one can calculate the trace of \( 1 - V^{(L)} \) as follows

\[
\text{tr}_{\mathcal{H}_L} \left| 1_{\mathcal{H}_L} - V^{(L)} \right| = \sum_{\infty} \left( \psi_i, \left( W_s^{(L)} \right)^* \left( 1_{\mathcal{H}_L} - V^{(L)} \right) \psi_i \right).
\]

Note that for the unitary bounded operator \( V^{(L)} \) on \( \mathcal{H}_L \) we can write \( 1 - V^{(L)} = i \int_0^s \mathcal{D}_{s,r}^{(L)} V^{(L)} \, dr \). Then, by combining the explicit form of \( D_{s,r}^{(L)} \) given by (35), Cauchy–Schwarz inequality, Corollary 5 (a), and other simple arguments we arrive to

\[
\left| \text{tr}_{\mathcal{H}_L} \left| 1_{\mathcal{H}_L} - V^{(L)} \right| \right| \leq D_{\text{lem. 3}} |s| |A_L| |G| \int_{\mathbb{R}} \left| \mathcal{M}_g(t) \right| \, dt \sum_{x \in \mathcal{A}_L} e^{-\mu \inf \left\{ 1, \inf \left\{ \frac{\|x\|}{\mathcal{O}(\mathcal{H}, \Gamma)} \right\} \right\} \} |x|^s.
\]

Then, \( 1_{\mathcal{H}_L} - V^{(L)} \) is trace class, and \( V^{(L)} \in \mathcal{B}(\mathcal{H}_L) \) is a unitary operator, it follows from [Ara87, Theorem 4.1] that the \(^*\)-automorphism \( \Upsilon^{(L)} \) on \( \mathfrak{A}_L \) is inner. Because there is a volume factor in last inequality we state that \( 1_{\mathcal{H}_\infty} - V^{(\infty)} \) is trace class per unit volume. 

A combination of Corollary 4 and Lemma 3 yields to:

Corollary 6.

Take same assumptions of Lemma 2. Then, the one–parameter (Bogoliubov) group \( \Upsilon^{(L)} \) on \( \mathfrak{A}^{(0)} \) converges uniformly for \( s \in \mathcal{C} \) as \( L \to \infty \) to the one–parameter (Bogoliubov) group \( \Upsilon^s \) on \( \mathfrak{A}_\infty \) thus defining a strongly continuous group on \( \mathfrak{A}_\infty \). Moreover, \( \left( \Upsilon^{(L)} \right)^{-1} \) exists and strongly converges to \( \Upsilon^{-1}. \)

Proof. Note that the sequence of one–parameter (Bogoliubov) group \( \Upsilon^{(L)} \) on \( \mathfrak{A}^{(0)} \) is Cauchy for any \( B \in \mathfrak{A}^{(0)} \). We omit the details. Existence of \( \left( \Upsilon^{(L)} \right)^{-1} \) is a straight conclusion from Corollary 2, its convergence is immediate. We also omit the details.

For any \( L \in \mathbb{R}_0^+ \), \( q^{\infty}(L) \subset q^{\infty} \) denotes the local quasi–free ground states on \( \mathfrak{A}_L \subset \mathfrak{A}_\infty \). We postulate:

Theorem 5 (Gapped quasi–free ground states):

Take \( \mathcal{C} \equiv [0, 1] \) and consider the family of self–dual Hamiltonians satisfying Assumption 1 (b). Fix \( L \in \mathbb{R}_0^+ \), and let \( \{ \omega^{(L)} \}_{s \in \mathcal{C}} \subset q^{\infty} \) be the family of gapped quasi–free ground states associated to the family of Hamiltonians \( \{ H_s^{(L)} \}_{s \in \mathcal{C}} \in \mathcal{B}(\mathcal{H}_L) \). Then,

\[\text{For } U \in \mathcal{B}(\mathcal{H}), \text{ a Bogoliubov transformation, the Bogoliubov } \ast-\text{automorphism } \chi \text{ on } s\text{CAR}(\mathcal{H}_\Gamma) \text{ is inner if and only if } 1_{\mathcal{H}_\infty} - 1 \text{ is trace class and } \text{det}(U) = \pm 1, \text{ see } [\text{Ara87, Theorem 4.1}]. \]

\[\text{Recall that } \mathfrak{A}_\infty \text{ is the completeness of the normed } \ast-\text{algebra } \mathfrak{A}^{(0)} \text{ given by } (44). \]
(1) \( \omega_s^{(L)} = \omega_0^{(L)} \circ \Upsilon_s^{(L)}, \) for all \( s \in \mathcal{C}, \) where \( \Upsilon_s^{(L)} \) is the finite–volume Bogoliubov *–automorphism on \( \mathfrak{A}_L \) of Corollary 6.

(2) Let \( \omega_s \in \mathfrak{qE}^{(\infty)} \) be the weak*–limit of \( \omega_s^{(L)} \in \mathfrak{qE}^{(L)} \) and \( \Upsilon_s \) the infinite–volume Bogoliubov *–automorphism on \( \mathfrak{A}_\infty \) associated to the sequence \( \Upsilon_s^{(L)} \) of Corollary 6. With respect to the weak*–topology, the following three statements are equivalent: (a) \( \lim_{L \to \infty} \omega_s^{(L)} = \omega_s. \) (b) \( \lim_{L \to \infty} \omega_s^{(L)} \circ \Upsilon_s = \omega_s \circ \Upsilon_s. \) (c) \( \lim_{L \to \infty} \omega_s^{(L)} \circ \Upsilon_s^{(L)} = \omega_s \circ \Upsilon_s. \) ♦

Proof. (1) follows from Corollary 3 and Lemma 3. (2) Fix \( s \in \mathcal{C}. \) Note that the existence of the weak*–limit \( \omega_s \) is consequence of Theorem 4 while the existence of the Bogoliubov *–automorphism \( \Upsilon_s \) is a consequence of Corollary 6. Now, take any \( A \in \mathfrak{A}_\infty \) and note that (a) \( \Rightarrow \) (b) because

\[
\left| \omega_s^{(L)} \circ \Upsilon_s(A) - \omega_s \circ \Upsilon_s(A) \right| \leq \left| \omega_s^{(L)} - \omega_s \right| \| A \|_{\mathfrak{A}_\infty},
\]

(b) \( \Rightarrow \) (c) follows by recognizing \( \omega_s^{(L)} \) and \( \omega_s \) as states and writing

\[
\left| \omega_s^{(L)} \circ \Upsilon_s^{(L)}(A) - \omega_s \circ \Upsilon_s(A) \right| \leq \left| \omega_s^{(L)} - \omega_s \right| \| A \|_{\mathfrak{A}_\infty} + \left| \omega_s^{(L)} \right| \left| \Upsilon_s^{(L)}(A) - \Upsilon_s(A) \right|_{\mathfrak{A}_\infty},
\]

and we have that the left hand side of last inequality is zero. Finally, we note that

\[
\left| \omega_s^{(L)}(A) - \omega_s(A) \right| \leq \left| \omega_s^{(L)} \circ \Upsilon_s^{(L)} - \omega_s \circ \Upsilon_s \right| \| A \| + \left| \omega_s \circ \Upsilon_s \right| \left| \Upsilon_s^{(L)}(A) - \Upsilon_s(A) \right|_{\mathfrak{A}_\infty},
\]

and from Corollary 6, the right hand side of last inequality is zero, thus (c) \( \Rightarrow \) (a).

A Disordered models on general graphs

Consider the graph \( \mathfrak{G}, \) that is, \( \mathfrak{G} = \mathfrak{V} \times \mathfrak{E}, \) where \( \mathfrak{V} \) is the so called set of vertices and \( \mathfrak{E} \) is called set of edges. For any, \( v, w \in \mathfrak{V}, \) and \( (v, w) \in \mathfrak{V} \times \mathfrak{V}, \) \( v \) and \( w \) are called the endpoints of \( (v, w) \in \mathfrak{E}. \) Further, for \( v, w \in \mathfrak{V}, \) \( \mathfrak{E} \) not contain element of the form \( (v, v) \) and \( (v, w) \in \mathfrak{E} \) iff \( (w, v) \in \mathfrak{E}. \) Thus, unless otherwise indicated, we will assume that the edges are not–oriented. Note that the element \( g \in \mathfrak{G} \) can also be written as \( g = (v, e) \) for some \( v \in \mathfrak{V} \) and \( e \in \mathfrak{E}. \) Additionally, for \( e \in (0, 1] \) and any \( v, w \in \mathfrak{V}, \) one can endow to \( \mathfrak{G} \) of a pseudometric \( \delta : \mathfrak{G} \times \mathfrak{G} \to \mathbb{R}^+_0 \cup \{ \infty \}, \) that is, an equivalence relation satisfying the metric properties on \( \mathfrak{G}, \) except that \( \delta(v, w) = 0 \) does not implies that \( v = w. \) Then, \( \delta \) is closely related to the size of the path with the minimum number of edges joining vertices \( v \) and \( w. \) For \( \mathfrak{G}, \) \( \mathcal{P}(\mathfrak{G}) \subset 2^\mathfrak{G} \) will denote the set of all finite subsets of \( \mathfrak{G}. \) We refer the reader to [LP17] for a complete discussion about graphs.

Among the graphs that physicists consider is included the \( d–\)dimensional cubic lattice or crystal \( \mathbb{Z}^d, \) for \( d \in \mathbb{N}, \) taken as a subset of \( \mathbb{R}^d \) in the following way:

\[
\mathbb{Z}^d = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_j \in \mathbb{Z} \text{ for any } 1 \leq j \leq d\}.
\]

Usually, \( \mathbb{Z}^d \) is used because of its symmetric properties, as translations ... However, one can take more general assumptions by considering Cayley graphs. These are defined via the group \( \mathfrak{V} \equiv (\mathfrak{V}, \cdot) \) generated by the subset \( \mathfrak{V} \equiv (v, \cdot). \) Then, we associate to any element of \( \mathfrak{V} \) a vertex of the Cayley graph \( \mathfrak{G} \) and the set of edges is defined by

\[
\mathfrak{E} = \{(v, w) \in \mathfrak{V}^2 : v^{-1}w \in \mathfrak{V}\}.
\]

In the \( \mathbb{Z}^d \) case, the group \( \mathfrak{V} \equiv (\mathfrak{V}, +) \) is the so–called translation group.

From the physical point of view, mobility or confinement of particles embedded in a graph \( \mathfrak{G} \equiv \mathfrak{V} \times \mathfrak{E} \)
will rely on the *impurities* of the material, crystal lattice defects (as in the $\mathbb{Z}^d$ case), etc., which usually are modeled (in the simplest case) by random (one–site) external potentials on the set of vertices $\mathcal{V}$ as follows: We take the probability space $(\Omega, \mathcal{A}_\Omega, \alpha_\Omega)$, where $\Omega = [-1, 1]^{|\mathcal{V}|}$. For any $v \in \mathcal{V}$, $\Omega_v$ is an arbitrary element of the Borel $\sigma$–algebra $\mathcal{A}_v$ of the Borel set $[-1, 1]$ w.r.t. the usual metric topology. Then, $\mathcal{A}_\Omega$ is the $\sigma$–algebra generated by the cylinder sets $\prod_{v \in \mathcal{V}} \Omega_v$, where $\Omega_v = [-1, 1]$ for all but finitely many $v \in \mathcal{V}$. Additionally, we assume that the distribution $\alpha_\Omega$ is an arbitrary ergodic probability measure on the measurable space $(\Omega, \mathcal{A}_\Omega)$. I.e., it is invariant under the action

$$
\rho \mapsto \chi^{(\Omega)}_v(\rho) \doteq \chi^{(\Omega)}_v(\rho), \quad v \in \mathcal{V},
$$

of the group $\mathcal{G} \equiv (\mathcal{V}, \cdot)$ on $\Omega$ and $\alpha_\Omega(\emptyset) \in \{0, 1\}$ whenever $\emptyset \in \mathcal{A}_\Omega$ satisfies $\chi^{(\Omega)}_v(\emptyset) = \emptyset$ for all $v \in \mathcal{V}$. Here, for any $\rho \in \Omega$, $v \in \mathcal{V}$ and $w \in \mathcal{V}$

$$
\chi^{(\mathcal{G})}_v(\rho)(w) \doteq \rho \left( v^{-1} w \right).
$$

As is usual, $\mathbb{E}[\cdot]$ denotes the expectation value associated with $\alpha_\Omega$.

For the Cayley graph $\mathcal{G} = \mathcal{V} \times \mathcal{E}$, $\mathcal{h} = L^2(\mathcal{G}, \mathbb{C})$ will denote a separable Hilbert space associated to $\mathcal{G}$ with scalar product $\langle \cdot, \cdot \rangle_\mathcal{h}$ and canonical orthonormal basis denoted by $\{e_v\}_{v \in \mathcal{V}}$, which is defined by $e_v(w) = \delta_{v-1} w$, $\forall v$, with $v$ the generator set of $\mathcal{V}$ and $\mathcal{h}_{\mathcal{G}}$ the unit on $\mathcal{V}$. For any, $\rho \in \Omega$, one introduces the external potential $V_\rho \in \mathcal{B}(\mathcal{h})$ as the self–adjoint multiplication operator operator $V_\rho : \mathcal{G} \rightarrow [-1, 1]$. On the other hand, one define for the compact set $\mathcal{C} \equiv [0, 1]$, the family of graph Laplacians $\{\Delta_{\mathcal{G}, s}\}_{s \in \mathcal{E}}$ defined for any $s \in \mathcal{E}$ by

$$
[\Delta_{\mathcal{G}, s}(\psi)](v) = \text{deg}_{\mathcal{G}}(v) \psi(v) - s \sum_{d(v,w)=1} \psi(v^{-1}w), \quad v \in \mathcal{V}, \: \psi \in \mathcal{h}
$$

where for any $\epsilon \in [0, 1]$, $d_\epsilon : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}^+ \cup \{\infty\}$ is a pseudometric on $\mathcal{G}$. In (59), on the right hand side, $\text{deg}_{\mathcal{G}}(v)$ is the number of nearest neighbors to vertex $v$, or degree of $v$. If $\{\text{deg}_{\mathcal{G}}(v)\}_{v \in \mathcal{V}} \in \mathbb{N}$ is the same for all $v \in \mathcal{V}$, we say that the graph is regular.

The random tight–binding (Anderson) model is the one–particle Hamiltonian defined by

$$
h^{(\rho)}_{\mathcal{G}, s} \doteq \Delta_{\mathcal{G}, s} + \lambda V_\rho, \quad \rho \in \Omega, \: \lambda \in \mathbb{R}^+_0.
$$

See [AW15] for further details. In [ABPR19], we consider a more general setting such that hopping disorder is present. I.e., we associate to particles a hop probability on the non–oriented edges $\mathcal{E}$. In this case, one deals with *hopping amplitudes* and the probability space $(\Omega, \mathcal{A}_\Omega, \alpha_\Omega)$ is properly implemented.

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