Generic pure quantum states as steady states of quasi-local dissipative dynamics

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Abstract
We investigate whether a generic pure state on a multipartite quantum system can be the unique asymptotic steady state of locality-constrained purely dissipative Markovian dynamics. In the tripartite setting, we show that the problem is equivalent to characterizing the solution space of a set of linear equations and establish that the set of pure states obeying the above property has either measure zero or measure one, solely depending on the subsystems’ dimension. A complete analytical characterization is given when the central subsystem is a qubit. In the $N$-partite case, we provide conditions on the subsystems’ size and the nature of the locality constraint, under which random pure states cannot be quasi-locally stabilized generically. Also, allowing for the possibility to approximately stabilize entangled pure states that cannot be exact steady states in settings where stabilizability is generic, our results offer insights into the extent to which random pure states may arise as unique ground states of frustration-free parent Hamiltonians. We further argue that, to a high probability, pure quantum states sampled from a $t$-design enjoy the same stabilizability properties of Haar-random ones as long as suitable dimension constraints are obeyed and $t$ is sufficiently large. Lastly, we demonstrate a connection between the tasks of quasi-local state stabilization and unique state reconstruction from local tomographic information, and provide a constructive procedure for determining a generic $N$-partite pure state based on knowledge of the support of any two of the reduced density matrices of about half the parties, improving over existing results.
Keywords: generic pure quantum states, uniquely determined quantum states, stability properties, Markovian quantum dynamics, engineered dissipation

(Some figures may appear in colour only in the online journal)

1. Introduction

Generic states of composite (bipartite or multipartite) quantum systems have long played an important role in the exploration of fundamental questions ranging from quantum information theory to the foundations of quantum and statistical mechanics [1, 2]. For instance, consideration of pure quantum states of \( N \) parties, drawn at random from a uniform distribution, has been instrumental in shedding light on the extent to which knowledge of the ‘parts’ may determine the ‘whole’ generically [3, 4]: that is, in the context of the quantum marginal (or ‘local consistency’) problem [5, 6], in understanding whether a given set of reduced density matrices (RDMs) is compatible with a global quantum state and, if so, whether the latter is uniquely determined by such ‘local’ information. Random pure states have also enabled significant progress in characterizing the nature of the correlations that a multipartite quantum state may exhibit generically [7] and in developing a simplified theory of entanglement in high-dimensional bipartite settings, by leveraging concentration of measure effects [1, 8]. In turn, similar mathematical tools have provided a fresh approach to the problem of thermalization and the emergence of statistical ensembles [9, 10].

Our interest in generic quantum states in this work is motivated by the problem of characterizing the conditions under which convergence to a stable equilibrium may be achieved in naturally occurring or engineered dissipative evolutions subject to realistic resource constraints. Specifically, we ask whether a generic quantum pure state \(|\psi\rangle\) on \( N d \)-dimensional subsystems can be asymptotically stable under purely dissipative Markovian dynamics subject to specified ‘quasi-locality’ constraints, namely, whether it is dissipatively quasi-locally stabilizable (DQLS) in the sense of [11, 12], as we will formalize later. Quantum stabilization problems have been extensively investigated in recent years in the context of dissipative quantum state preparation and quantum engineering, with special emphasis on the stabilization of entangled pure states of relevance to quantum information processing (QIP) [13] and condensed-matter physics—see also [14–21] and references therein for representative contributions.

All existing characterizations, however, have addressed stabilization of a quantum state of interest, without offering insight on properties that stabilizable sets may enjoy as a whole. For a fixed multipartite structure on the underlying Hilbert space \( \mathcal{H} \), are there features of a target state and a quasi-local (QL) constraint together, that make stabilization more likely than not? Can we possibly identify non-trivial settings where pure-state stabilizability is almost always feasible, so, in fact, it is typical? Besides adding to the conceptual understanding of the constrained stabilization problem, generic stabilizability would offer new venues for approximate stabilization of states that may otherwise be unattainable—so-called ‘practical’ stabilization in control-theoretic parlance [22]. Additional motivation for characterizing the stabilizability properties of generic pure states stems from the fact that any pure state that enjoys the DQLS properties can be equivalently thought of as a unique ground state of a frustration-free (FF) QL parent Hamiltonian [11]. Thus, from a many-body physics standpoint, such a characterization provides insight into the existence and structure of the corresponding QL parent Hamiltonians, thereby complementing recent studies of ground-state properties of generic QL Hamiltonians [23].
While, throughout our analysis, we will focus for concreteness on characterizing stabilizability of a generic $N$-partite state $|\psi\rangle$ under continuous-time QL Markovian dynamics described by a semigroup master equation [24], we stress that our approach and conclusions are equally applicable to QL discrete-time dynamics, as considered in [20, 25]. The content of the paper and our main results may be summarized as follows:

- **Section 2** introduces the relevant system-theoretic background and establishes some preliminary results that serve as a foundation for our subsequent analysis. In particular, after formalizing the QL constraints of interest in terms of a specified neighborhood structure $\mathcal{N}$ on $\mathcal{H}$, and recalling the existing characterization of DQLS pure states, section 2.2 provides a reformulation of the DQLS condition by directly constraining the action of suitable sets of neighborhood operators acting on $|\psi\rangle$ (theorem 2.4). As a by-product, this yields an improved understanding of the DQLS property itself: while the latter was known to be invariant under the action of local unitary transformations on the target state $|\psi\rangle$ [11], we further prove here invariance under the larger class of stochastic local operations and classical communication (SLOCC) transformations [26].

In the simplest case of a tripartite ($N = 3$) setting, the reformulation of the DQLS condition may be further recast in the form of a system of linear equations, the solution space of which determines whether stabilization of $|\psi\rangle$ may be achieved.

- **Section 3** is the core section of our paper, presenting a detailed analysis of the tripartite setting in, say, dimensions $d_a \times d_b \times d_c$ (where $d_a \leq d_c$ without loss of generality), relative to two-body QL constraints, $\mathcal{N}_{ab} \equiv \{N_{ab}, N_{bc}\}$. By utilizing the maximal Schmidt rank property of generic pure states along any bi-partition, a ‘no-go’ result is established (theorem 3.2), preventing QL stabilization whenever $d_a d_b \leq d_c$. While this immediately implies that the DQLS property is a measure zero property in such cases, our analysis makes it clear that, remarkably, DQLS states form either a measure zero or a measure one set in all other possible tripartite settings, when $d_a d_b > d_c$ (theorem 3.4). Section 3.3 provides a complete characterization of the behavior of generic tripartite pure states when $d_b = 2$, essentially identifying the combination of subsystem dimensions $d \times 2 \times (d + 1)$ as the only one for which the DQLS property holds generically (theorem 3.7). For $d_b > 2$, we are still able to analytically characterize a number of settings of interest (theorem 3.10), and arrive at a general conjecture for arbitrary $d_a, d_b, d_c$ based on numerical exploration of the remaining instances.

- **Section 4** collects a number of extensions and implications of the tripartite results of section 3. Notably, a no-go condition for the DQLS property is also established for generic $N$-partite pure states (theorem 4.1), making precise the intuition that scalable QL stabilization cannot be achieved generically: as $N$ increases, the set of DQLS states relative to a specified neighborhood structure $\mathcal{N}$ has measure zero if the sizes of the neighborhoods $\mathcal{N}_j \in \mathcal{N}$ (that is, physically, the range of the couplings) are kept fixed. When such a no-go theorem applies, we further show that no FF QL Hamiltonian can have a generic pure state in its ground space, as one may intuitively expect. Building on our analysis, an algorithmic procedure is presented, for determining whether the DQLS property holds for given inputs $(|\psi\rangle, \mathcal{N})$. Implications beyond exact stabilization and beyond generic pure states are addressed in sections 4.2 and 4.4, respectively. In particular, we show how maximally entangled pure states may be approximately stabilized to arbitrary accuracy in multipartite scenarios where DQLS holds generically, and argue that a number of conclusions on the DQLS property (or lack thereof) may be extended to quantum states whose RDMs (or ‘marginals’) do retain certain generic features—notably, states sampled from $t$-design [27, 28].
Section 5 illustrates connections between the task of pure state dissipative QL stabilization and the one of reconstructing the global state using information from its neighborhood RDMs, as relevant in practice to quantum state tomography protocols via local measurements [29]. Our main result, contained in proposition 5.2, improves over existing ones in several ways: we show that knowledge about the support of any two RDMs on no more than (about) half of the subsystems suffices to uniquely determine a generic $|\psi\rangle$ on $N$ qudits among all possible quantum states; further to that, we provide a constructive procedure for generic-state reconstruction, that is directly informed by the DQLS characterization we leverage in our approach.

We conclude in section 6 by highlighting additional related questions that may warrant future investigation. In the appendix, we collect a number of useful properties that generic matrices enjoy as well as complete proofs of two of our main theorems (theorems 3.7 and 3.10), which we deemed too lengthy for inclusion in the main text.

2. Background and preliminaries

2.1. Dissipative quasi-local stabilization: prior results

Consider a multipartite quantum system composed of $N$ distinguishable, finite-dimensional subsystems (or ‘qudits’). The corresponding Hilbert space, $\mathcal{H} \simeq \mathbb{C}^d$, has a tensor product structure given by

$$\mathcal{H} = \bigotimes_{a=1}^{N} \mathcal{H}_a, \quad a = 1, \ldots, N, \quad \dim(\mathcal{H}_a) = d_a, \quad \dim(\mathcal{H}) = d. \quad (1)$$

Let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators acting on $\mathcal{H}$. For $X \in \mathcal{B}(\mathcal{H})$, its adjoint operator is given by $X^\dagger$. When using a matrix representation of operators, $\mathbb{M}^{d_1 \times d_2}$ shall denote the space of complex matrices in $d_1 \times d_2$ dimensions. Physical states of the system are represented in general by density operators, which form a set of trace-one, positive semi-definite operators denoted by $\mathcal{D}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$, and whose extreme points are the pure states. Unless otherwise noted, we shall take a multipartite pure state $|\psi\rangle$ to be a normalized vector in the tensor-product space $\mathcal{H}$.

If $\rho_0$ represents the initial state of the system, say, at time $t = 0$, then for any $t > 0$ the evolved state is given by a completely-positive trace-preserving (CPTP) linear map (or quantum channel) $T_t$ acting on $\mathcal{B}(\mathcal{H})$, namely [13],

$$\rho(t) \equiv T_t(\rho_0) = \sum_k M_k \rho_0 M_k^\dagger, \quad \sum_k M_k^\dagger M_k = I, \quad M_k \in \mathcal{B}(\mathcal{H}),$$

where $I$ is the identity operator in $\mathcal{B}(\mathcal{H})$. The operator-sum representation of $T_t$ given above, although not unique, exists if and only if the map is CP, whereas the second equality enforces the TP property. The evolution of a large class of quantum systems of physical interest can be described using quantum dynamical semigroups (QDS). A QDS is a continuous, one-parameter family of CPTP maps $\{T_t\}_{t \geq 0}$, with $T_0 = I$, the identity map on $\mathcal{B}(\mathcal{H})$, and obeying the Markov property, $T_t \circ T_s = T_{t+s}$ for $t, s \geq 0$. These conditions ensure the existence of a linear map $\mathcal{L}$ (or Liouvillian) that generates the semigroup via $T_t = e^{t\mathcal{L}}$ and in terms of which the resulting dynamics may be represented in the canonical Gorini–Kossakowski–Sudarshan–Lindblad form ($\hbar \equiv 1$) [13, 24]:
\[ \dot{\rho}(t) \equiv \mathcal{L}[\rho(t)] = -i[H, \rho(t)] + \sum_k \left( L_k \rho(t) L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho(t)\} \right), \quad t \geq 0. \]  

Here, \( H = H^\dagger \) is the system’s effective Hamiltonian and \( \{ L_k \} \) are the Lindblad (noise) operators, that are associated to the non-unitary component of the dynamics. In this paper we consider only time-invariant dynamics, so that \( H, \{ L_k \} \) do not depend on time. A given choice of \( H \) and \( \{ L_k \} \) unambiguously identifies the QDS generator associated with it, which we denote by \( \mathcal{L}(H, \{ L_k \}) \). Vice-versa, the choice of \( H \) and \( \{ L_k \} \) that specify the action of a given \( \mathcal{L} \) on \( \mathcal{B}(\mathcal{H}) \) is not unique: any transformation of the form

\[ L_k \equiv L_k + c_k I, \quad \mathbf{T} \equiv H - \frac{i}{2} \sum_k (c_k^* L_k - c_k L_k^\dagger), \quad c_k \in \mathbb{C}, \]  

can be verified to leave the generator unchanged, that is, \( \mathcal{L}(H, \{ L_k \}) = \mathcal{L}(\mathbf{T}, \{ T_k \}) \).

The lattice geometry or coupling topology in the multipartite system under consideration typically impose constraints on the structure of \( \mathcal{L} \). Building on our previous work \([11, 12, 19, 20, 25]\), we introduce locality constraints by specifying subsets of subsystems that are allowed to be acted upon non-trivially by the dynamics:

**Definition 1.** For the multipartite system (1), a neighborhood \( \mathcal{N}_j \subseteq \{1, \ldots, N\} \) is a set of indexes that labels a group of subsystems. A neighborhood structure is a collection of neighborhoods \( \mathcal{N} = \{ \mathcal{N}_j \}_{j=1}^M \) for some finite \( M \).

The neighborhood complement of \( \mathcal{N}_j \), denoted by \( \overline{\mathcal{N}}_j \), consists of subsystems that are not included in \( \mathcal{N}_j \), such that \( \mathcal{N}_j \cup \overline{\mathcal{N}}_j = \{1, \ldots, N\} \). Given a quantum state \( \rho \in \mathcal{D}(\mathcal{H}) \), its reduced density matrix (RDM) on the neighborhood \( \mathcal{N}_j \) is given by \( \rho_{\mathcal{N}_j} = \text{Tr}_{\overline{\mathcal{N}}_j}(\rho) \), where \( \text{Tr}_{\overline{\mathcal{N}}_j} \) is the partial trace over the subsystems in \( \overline{\mathcal{N}}_j \). In this work we consider only neighborhood structures that are complete, that is, \( \bigcup_{j=1}^M \mathcal{N}_j = \{1, \ldots, N\} \). For fixed \( \mathcal{N}_j \), a neighborhood operator is an operator that acts non-trivially on only one neighborhood. For example, \( X = X_{\mathcal{N}_j} \otimes I_{\overline{\mathcal{N}}_j} \), with \( X_{\mathcal{N}_j} \in \mathcal{B}(\mathcal{H}_{\mathcal{N}_j}) \) and \( I_{\overline{\mathcal{N}}_j} = \otimes_{a \in \overline{\mathcal{N}}_j} I_a \), is a neighborhood operator on \( \mathcal{N}_j \).

**Definition 2.** Given a neighborhood structure \( \mathcal{N} \), the QDS generator \( \mathcal{L} \) is QL if it may be expressed as a sum of neighborhood generators, that is, \( \mathcal{L} = \sum_{j=1}^M \mathcal{L}_{\mathcal{N}_j} \otimes I_{\overline{\mathcal{N}}_j} \), where \( I_{\overline{\mathcal{N}}_j} \) is the identity map acting on \( \mathcal{B}(\mathcal{H}_{\overline{\mathcal{N}}_j}) \).

Quasi-locality of the generator \( \mathcal{L} \) implies \([12]\) that there exists a representation \( \mathcal{L} \equiv \mathcal{L}(H, \{ L_k \}) \), such that the associated \( H \) and \( \{ L_k \} \) both satisfy QL constraints, that is, \( H = \sum_{j=1}^M H_{\mathcal{N}_j} \otimes I_{\overline{\mathcal{N}}_j} \) and \( L_k = L_{\mathcal{N}_j} \otimes I_{\overline{\mathcal{N}}_j} \), for all \( k \).

As our main aim is to explore stabilizability properties by QL-constrained dynamics, we next recall the relevant stability notions:

**Definition 3.** A state \( \rho_0 \) is globally asymptotically stable (GAS) if it is an invariant state of the generator (2), that is, \( \mathcal{L}[\rho_0] = 0 \), and for any initial state the evolution approaches \( \rho_0 \) in the long time limit, that is, \( \lim_{t \to \infty} e^{\mathcal{L}t}[\rho_0] = \rho_0 \), for all \( \rho_0 \in \mathcal{D}(\mathcal{H}) \).

As shown in \([19]\) (corollary 1), if a generator \( \mathcal{L}(H, \{ L_k \}) \) makes a target pure state \( \rho_\psi = |\psi\rangle\langle\psi| \) GAS, then by a suitable transformation of \( H \) and \( \{ L_k \} \) as specified in equation (3), the same generator can be represented in a standard form \( \mathcal{L}(\overline{H}, \{ \overline{L}_k \}) \), such that \( \overline{H}|\psi\rangle = \overline{h}|\psi\rangle \) and \( \overline{L}_k|\psi\rangle = 0 \), for all \( k \). Notice that such a transformation preserves the QL structure of the Hamiltonian and the Lindblad operators. In view of this, the following definition may be given for pure-state stabilization:
Definition 4. A pure state \( \rho_d = |\psi\rangle\langle\psi| \in \mathcal{D}(\mathcal{H}) \) is DQLS relative to a neighborhood structure \( \mathcal{N} \) if there exists a QL generator in standard form \( \mathcal{L}(\overline{\mathcal{H}}, \{\overline{\mathcal{L}}_i\}) \), with \( \overline{\mathcal{H}} \equiv 0 \), for which \( \rho_d \) is GAS.

The above definition implies that if \( \rho_d \) is DQLS, then \( \rho_d \in \ker(\mathcal{L}(\overline{\mathcal{L}}_k)) \) for each \( k \), which means that \( \rho_d \) must be invariant under the dynamics relative to each neighborhood.

Finally, in order to characterize the set of DQLS states, we shall need the following linear-algebraic tools [12]:

Definition 5. Consider two inner product spaces \( V_A, V_B \). Given a vector \( \vec{v} \in V_A \otimes V_B \) with Schmidt decomposition \( \vec{v} = \sum \mu_i \vec{a}_i \otimes \vec{b}_i \), the Schmidt span of \( \vec{v} \) relative to \( V_A \) is the subspace \( \Sigma_A(\vec{v}) \equiv \text{span}\{\vec{a}_i \in V_A : \vec{v} = \sum \mu_i \vec{a}_i \otimes \vec{b}_i, \vec{b}_i \in V_B\} \). The extended Schmidt span of \( \vec{v} \), denoted by \( \Sigma_A(\vec{v}) \) as \( \Sigma_A(\vec{v}) \equiv \Sigma_A(\vec{v}) \otimes V_B \).

For notational simplicity, from now on we represent a pure state \( \psi \) by the corresponding normalized state vector \( |\psi\rangle \in \mathcal{H} \). With a slight abuse of terminology, \( \Sigma_{\mathcal{N}_j}(|\psi\rangle) \) will be denoted as the Schmidt span of \( |\psi\rangle \) relative to the neighborhood \( \mathcal{N}_j \) instead of the space \( \mathcal{H}_{\mathcal{N}_j} \). By construction, notice that

\[
\Sigma_{\mathcal{N}_j}(|\psi\rangle) = \text{supp}(\rho_{\mathcal{N}_j}), \quad \Sigma_{\mathcal{N}_j}(|\psi\rangle) = \text{supp}(\rho_{\mathcal{N}_j} \otimes I_{\mathcal{N}_j}).
\]

The first characterization of DQLS states has been provided in [11], and is reported below in its equivalent form in terms of the Schmidt span:

**Theorem 2.1. ([11, 12])** A multipartite pure state \( |\psi\rangle \in \mathcal{H} \) is DQLS relative to the neighborhood structure \( \mathcal{N} = \{\mathcal{N}_j\}_{j=1}^M \) if and only if

\[
\bigcap_{j=1}^M \Sigma_{\mathcal{N}_j}(|\psi\rangle) \equiv \mathcal{H}_0(|\psi\rangle) = \text{span}\{|\psi\rangle\}. \tag{4}
\]

The proof of this theorem in [11] makes it clear that the target-dependent subspace \( \mathcal{H}_0(|\psi\rangle) \) defined by (4) is the smallest subspace containing \( |\psi\rangle \), which can be made GAS using QL dissipation alone (that is, a QL generator that has \( \overline{\mathcal{H}} \equiv 0 \) in standard form). We shall refer to \( \mathcal{H}_0(|\psi\rangle) \) as the DQLS subspace associated to \( |\psi\rangle \), relative to \( \mathcal{N} \). Clearly, \( |\psi\rangle \) is DQLS if and only if its corresponding DQLS subspace is one-dimensional.

An equivalent characterization of DQLS states, which also provides them with a transparent physical interpretation, is also proved as a corollary to theorem 2.1 in [11]:

**Corollary 2.2 ([11]).** A multipartite pure state \( |\psi\rangle \in \mathcal{H} \) is DQLS relative to \( \mathcal{N} \) if and only if it is the unique ground state of a FF QL parent Hamiltonian respecting the same neighborhood structure.

Recall that a QL Hamiltonian \( H = \sum H_j \) is FF if any ground state of \( H \) is also a ground state of each \( H_j \). A canonical FF QL parent Hamiltonian \( H_{|\psi\rangle} \) may be constructed from the state \( |\psi\rangle \) itself by letting

\[
H_{|\psi\rangle} = \sum H_j, \quad H_j \equiv I - \Pi_{\Sigma_{\mathcal{N}_j}(|\psi\rangle)} \otimes I_{\mathcal{N}_j}, \quad \text{where } \Pi_{\Sigma_{\mathcal{N}_j}(|\psi\rangle)} \text{ is the (orthogonal) projector onto the Schmidt span } \Sigma_{\mathcal{N}_j}(|\psi\rangle).
\]

**Remark 2.3.** As mentioned in the introduction, we shall focus henceforth on continuous-time Markovian dynamics, described by equation (2). The relevant notions of quasi-locality and stabilization are also applicable, however, to multipartite systems that evolve according to discrete-time Markovian dynamics. In the general non-homogeneous (time-varying) case, the state evolution is then given by a series of CPTP maps \( \{T_t\} \), such that \( \rho_{t+1} = T_t(\rho_t) \).
\(\rho_t \in \mathcal{D}(\mathcal{H}), \ t = 0, 1, 2, \ldots,\) and each such map is constrained to be a neighborhood map, namely, for each \(t\) we may write \(T = \mathcal{T}_{\mathcal{N}_j} \otimes I_{\mathcal{N}_j}^{\prime}\) for some \(j\). It has been shown in [25] that the mathematical characterization of pure states that are stabilizable by QL discrete-time dynamics yields the same necessary and sufficient condition as that for the DQLS states in equation (4). Thus, it follows that our analysis is applicable to QL discrete-time stabilizing dynamics as well.

2.2. Reformulation of the DQLS condition for general pure states

Our main aim is to study the DQLS property of generic pure states. Given \(|\psi\rangle \in \mathcal{H}\) and its DQLS subspace \(\mathcal{H}_0(|\psi\rangle)\) for a specified neighborhood structure \(\mathcal{N}\), the key mathematical idea underlying our approach is to characterize \(\mathcal{H}_0(|\psi\rangle)\) in terms of sets of neighborhood operators acting on \(|\psi\rangle\). This characterization can be employed to reformulate the DQLS condition in equation (4) for an arbitrary number \(N\) of subsystems. For \(N = 3\), the restated DQLS condition can be further reformulated as a set of linear equations whose coefficients are determined by the entries of \(|\psi\rangle\) in the standard basis. The solution space to this set of equations is shown to be in correspondence with \(\mathcal{H}_0(|\psi\rangle)\) and hence directly determines the DQLS property of \(|\psi\rangle\).

In section 3, such a reformulation will be instrumental in leveraging the properties of generic pure states toward understanding their DQLS nature.

2.2.1. \(N\)-partite setting. The desired characterization of the DQLS subspace of a pure state \(|\psi\rangle\) in terms of sets of neighborhood operators stems from the following:

**Theorem 2.4.** Let \(|\psi\rangle \in \mathcal{H}\) be a multipartite pure state and \(\mathcal{H}_0(|\psi\rangle)\) the corresponding DQLS subspace relative to the neighborhood structure \(\mathcal{N} = \{\mathcal{N}_j\}_{j=1}^M\). Another pure state \(|\psi'\rangle\) belongs to \(\mathcal{H}_0(|\psi\rangle)\) if and only if there exists operators \(X_{\mathcal{N}_j} \in \mathcal{B}(\mathcal{H}_{\mathcal{N}_j})\) such that:

\[
|\psi'\rangle = (I_{\mathcal{N}_j} \otimes X_{\mathcal{N}_j})|\psi\rangle, \quad j = 1, \ldots, M.
\]

**Proof.** To prove the forward implication, let \(|\psi\rangle = \sum_j \mu_\xi \eta_j \otimes \gamma_j \subseteq \mathcal{H}_{\mathcal{N}_j}\) and \(|\psi'\rangle = \sum_j \sigma_k \xi_j \otimes \eta_j \otimes \gamma_j\) be the Schmidt decomposition of the two pure states relative to the \(\mathcal{N}_j\) bipartition. Since \(|\psi'\rangle \in \mathcal{H}_0(|\psi\rangle)\), then \(|\psi'\rangle \in \Sigma\mathcal{N}_j(|\psi\rangle), \forall j\). Hence, it follows that \(\text{span}\{|\xi_1\rangle \otimes \eta_j \rangle \subseteq \Sigma\mathcal{N}_j(|\psi\rangle) = \text{span}\{|\eta_j\rangle \otimes \xi_1\rangle\}, \text{for all } j\). Thus, we may express \(|\psi'\rangle = \sum_k \xi_j \otimes \eta_j \otimes \gamma_j\), with \(\xi_k, \eta_k \in \mathbb{C}\), for all \(k\). Choose the operator \(X_{\mathcal{N}_j}\) such that \(X_{\mathcal{N}_j} |\eta_j\rangle \otimes \gamma_j = \sum_k \xi_k (\sigma_k / \mu_k) |\xi_k\rangle \otimes \gamma_j\). This guarantees that \((I_{\mathcal{N}_j} \otimes X_{\mathcal{N}_j})|\psi\rangle = |\psi'\rangle\), for all \(j\), as claimed. Conversely, to prove that any \(|\psi'\rangle\) that is expressed by equation (5) belongs to \(\mathcal{H}_0(|\psi\rangle)\), note that since \(I_{\mathcal{N}_j} \otimes X_{\mathcal{N}_j}\) is a linear transformation acting non-trivially only on \(\mathcal{H}_{\mathcal{N}_j}\), \(\Sigma\mathcal{N}_j((I_{\mathcal{N}_j} \otimes X_{\mathcal{N}_j})|\psi\rangle) \subseteq \Sigma\mathcal{N}_j(|\psi\rangle)\). Hence, for any \(|\psi'\rangle\) satisfying equation (5),

\[
\text{span}\{|\psi'\rangle\} \subseteq \bigcap_j \Sigma\mathcal{N}_j(|\psi\rangle) = \mathcal{H}_0(|\psi\rangle),
\]

from which the stated result follows. \(\square\)

As a corollary, the anticipated restatement of the DQLS condition in equation (4) follows:

**Corollary 2.5.** A multipartite pure state \(|\psi\rangle \in \mathcal{H}\) is DQLS relative to the neighborhood structure \(\mathcal{N} = \{\mathcal{N}_j\}_{j=1}^M\) if and only if the only operators \(\{X_{\mathcal{N}_j}\} \subseteq \mathcal{B}(\mathcal{H}_{\mathcal{N}_j})\) that satisfy equation (5) act as a scalar multiple of the identity on their Schmidt spans \(\Sigma\mathcal{N}_j\), for all \(j\).
Proof. Assume that $|\psi\rangle$ is DQLS relative to $\mathcal{N}$. Thus, $\mathcal{H}_0(|\psi\rangle) = \text{span}\{|\psi\rangle\}$ and any $|\psi'\rangle \in \mathcal{H}_0(|\psi\rangle)$ is $|\psi\rangle$ itself, up to a constant scalar factor. Following equation (5), this is possible only if $X_{\mathcal{N}_j}$ act as the identity (up to a constant scalar factor) on the corresponding $\Sigma_{\mathcal{N}_j}(|\psi\rangle)$, for all $j$. To show the reverse implication, assume that the only choice of $\{X_{\mathcal{N}_j}\}$ satisfying equation (5) is indeed the identity (up to a constant scalar factor) on $\Sigma_{\mathcal{N}_j}(|\psi\rangle)$, for all $j$. This means that any $|\psi'\rangle \in \mathcal{H}_0$ is proportional to $|\psi\rangle$ and hence the latter is DQLS. $\square$

The above results are useful to establish an important feature of the DQLS property of pure states, namely, its invariance under the action of SLOCC transformations, through which a given quantum state may be converted into another with a non-zero probability of success [26]. Recall that two pure states are said to be in the same SLOCC class when they are related by an invertible local transformation of the form $\otimes_{a=1}^N X_a$, where $X_a \in \mathcal{B}(\mathcal{H}_a)$ for each $a$ [30].

The following result then holds:

**Proposition 2.6.** The dimension of the DQLS subspace $\mathcal{H}_0(|\psi\rangle)$ of a multipartite pure state $|\psi\rangle \in \mathcal{H}$, relative to any fixed neighborhood structure, is preserved under arbitrary SLOCC transformations.

Proof. Let the two pure states $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ belong to the same SLOCC class. This means they are related as $|\phi\rangle = (\otimes_{a=1}^N X_a)|\psi\rangle$, where each $X_a \in \mathcal{B}(\mathcal{H}_a)$ is an invertible operator. Thanks to the linearity of $X_a$, the Schmidt spans of these two states are related as $\Sigma_{\mathcal{N}_j}(|\phi\rangle) = (\otimes_{a=1}^N X_a)\Sigma_{\mathcal{N}_j}(|\psi\rangle)$ for any neighborhood $\mathcal{N}_j$ and, correspondingly, $\Sigma_{\mathcal{N}_j}(|\phi\rangle) = (\otimes_{a=1}^N X_a)\Sigma_{\mathcal{N}_j}(|\psi\rangle)$ for the extended Schmidt spans. Therefore,

$$\bigcap_{j=1}^M \Sigma_{\mathcal{N}_j}(|\phi\rangle) = \left( \bigotimes_{a=1}^N X_a \right) \bigcap_{j=1}^M \Sigma_{\mathcal{N}_j}(|\psi\rangle),$$

whereby it follows that the corresponding DQLS subspaces relative to $\mathcal{N} = \{\mathcal{N}_j\}_{j=1}^M$ are related as $\mathcal{H}_0(|\phi\rangle) = (\otimes_{a=1}^N X_a)\mathcal{H}_0(|\psi\rangle)$. Thanks to the invertible nature of all these transformations, $\dim(\mathcal{H}_0(|\phi\rangle)) = \dim(\mathcal{H}_0(|\psi\rangle))$. $\square$

As a corollary, if $|\psi\rangle$ is DQLS, then any $|\phi\rangle$ belonging to the same SLOCC class is also DQLS relative to the chosen $\mathcal{N}$. In the non-DQLS case, the proof establishes the relation between the DQLS subspaces of two pure states that are in the same SLOCC class through equation (6). Notice that the converse of the above proposition is clearly not true, as the example of a W-state and GHZ-state on 3 qubits illustrates: these two states have two-dimensional DQLS subspaces relative to any non-trivial neighborhood structure [11], yet they do not belong to the same SLOCC class [30].

2.2.2. Tripartite setting. We now specialize to tripartite quantum systems, with the corresponding Hilbert space $\mathcal{H} \equiv \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$ and dimensions $d_a \times d_b \times d_c$, where we assume $d_c \geq d_a$ without loss of generality. The relevant neighborhood structure is denoted by $\mathcal{N}_{\text{tri}} \equiv \{\mathcal{N}_{ab}, \mathcal{N}_{bc}\}$ (see also figure 1). Let $|\psi\rangle \in \mathcal{H}$. Then, by theorem 2.4, the operators $X_a \in \mathcal{B}(\mathcal{H}_a), X_c \in \mathcal{B}(\mathcal{H}_c)$ that satisfy the equation

$$X_a \otimes I_b \otimes I_c |\psi\rangle = (I_a \otimes I_b \otimes X_c)|\psi\rangle,$$

are such a transformation need not be norm-preserving. However, the DQLS behavior of a pure state can be verified to be independent of its norm. This is further discussed in remark 3.1.
As long as $|\psi\rangle$ denote the vector form of $X\otimes b = Xa\otimes |i\rangle$, we may rewrite equation (7) in terms of the matrices $d_{ij}$ denoting its transpose in the standard basis. It is important to verify that equation (7) is the singular value decomposition of $Ai$ with respect to the Hilbert spaces.

Next, recall equation (9). Thanks to the linear nature of $\mathcal{P}T((Xa \otimes I_c)\phi_i) = XaA_i$ and $\mathcal{P}T((I_a \otimes X_c)\phi_i) = A_iX_c^T$, we may rewrite equation (7) in terms of the matrices $\{A_i\}$ as
\[ X_a A_i = A_i X_i^T, \quad i = 0, \ldots, d_b - 1, \]  
(11)

with \{X_a, X_i\}, in matrix form, treated as unknowns. This makes it clear that the nature of the solution space of this linear system depends upon the DQLS property of the input state \(|\psi\rangle\).

However, the original DQLS condition given in equation (7) is based on the action of the operators \(X_a \in B(H_a), X_i \in B(H_i)\) restricted to the Schmidt spans \(\Sigma_a(\{|\psi\rangle\}), \Sigma_i(\{|\psi\rangle\})\), respectively, by virtue of theorem 2.4. At the same time, solutions to equation (11) describe the action of \(X_a X_i\), in matrix form, restricted to the combined range of \{\(A_i\)\} and \{\(A_i^T\)\], that is, \(\text{span}\{\text{range}(A_i)\}_{i=0}^{d_b-1}\) and \(\text{span}\{\text{range}(A_i^T)\}_{i=0}^{d_b-1}\), respectively. Hence, in order to formally show that equation (7) is equivalent to equation (11), we need to further show that the corresponding subspaces indeed coincide.

**Lemma 2.7.** With \(A_i\) defined as in equation (10), the following equalities hold:

\[ \Sigma_a(\{|\psi\rangle\}) = \text{span}\{\text{range}(A_i)\}_{i=0}^{d_b-1}, \quad \Sigma_i(\{|\psi\rangle\}) = \text{span}\{\text{range}(A_i^T)\}_{i=0}^{d_b-1}. \]

**Proof.** For notational simplicity, we denote subspaces like \(\text{span}\{\text{range}(\phi)\}_{ac}\) by their vector representation in the standard basis. Following equation (10), we already know that \(\text{range}(A_i) = \Sigma_a(\{|\psi\rangle\})\). Therefore, \(\text{span}\{\text{range}(A_i)\}_{i=0}^{d_b-1} = \Sigma_a(\{|\psi\rangle\})\). Similarly, \(\text{span}\{\text{range}(A_i^T)\}_{i=0}^{d_b-1} = \Sigma_i(\{|\psi\rangle\})\).

Following equation (8), \(|\psi\rangle_{ac} = b(|i\rangle_{bc})\). If \(|\psi\rangle = \sum_{i} \psi_{ia} |\gamma_{ia}\rangle \otimes |\eta_{ia}\rangle\) in the Schmidt-decomposed form for the \(a|bc\) bipartition, then \(|\phi\rangle_{ac} = \sum_{i} \psi_{ia} \gamma_{ia} \otimes |\eta_{ia}\rangle\) for all \(i\). Thus, it follows that

\[ \Sigma_a(\{|\psi\rangle_{ac}\}) = \text{span}\{\gamma_{ia} : b(|i\rangle_{bc}) \neq 0, i\} \subseteq \Sigma_a(\{|\psi\rangle\}), \quad i = 0, \ldots, d_b - 1, \]

where \(0\) denotes the null vector in \(H_c\). Therefore, \(\text{span}\{\Sigma_a(\{|\psi\rangle_{ac}\})\}_{i=0}^{d_b-1} \subseteq \Sigma_a(\{|\psi\rangle\})\). To show that the two subspaces indeed coincide, assume that a vector \(|\psi\rangle_{ac} \in \Sigma_a(\{|\psi\rangle\})\) exists such that \(|\psi\rangle_{ac} \in \Sigma_a(\{|\psi\rangle\})\). The latter implies that \(\text{span}\{\text{range}(A_i)\}_{i=0}^{d_b-1} = \Sigma_a(\{|\psi\rangle\})\). This further shows that \(\Sigma_a(\{|\psi\rangle\}) = \Sigma_i(\{|\psi\rangle\})\). In a similar way, it can be shown that \(\text{span}\{\text{range}(A_i^T)\}_{i=0}^{d_b-1} = \Sigma_c(\{|\psi\rangle\})\). \(\square\)

We can now combine the above lemma with corollary 2.5 to obtain the desired reformulation of the DQLS condition for general tripartite pure states:

**Proposition 2.8.** Consider a tripartite pure state \(|\psi\rangle\) in \(H_a \otimes H_b \otimes H_c\), and let the set of matrices \{\(A_i\)\} be constructed from \(|\psi\rangle\) according to equations (8) and (10). Then \(|\psi\rangle\) is DQLS relative to \(N_{\text{lin}} = \{N_{ab}, N_{bc}\}\) if and only if the only solutions \(\{X_a, X_i\}\) to the linear set of conditions \(X_a A_i = A_i X_i^T, i = 0, \ldots, d_b - 1\), act as a scalar multiple of the identity matrix on \(\text{span}\{\text{range}(A_i)\}_{i=0}^{d_b-1}\) and \(\text{span}\{\text{range}(A_i^T)\}_{i=0}^{d_b-1}\), respectively.

In this way, we have recast the problem of determining the DQLS property of a tripartite pure state \(|\psi\rangle\) into one of characterizing the set of matrices \(X_a\) and \(X_i\) that solves the linear system in equation (11). The latter is guaranteed to have at least one (non-zero) solution, which corresponds to \(X_a\) and \(X_i\) being proportional to the identity when restricted to the appropriate subspaces. The existence of any other non-trivial solution implies that \(|\psi\rangle\) is not DQLS. Thus, the structure of the solution space of equation (11) determines whether or not \(|\psi\rangle\) is DQLS. We next show that the dimension of the solution space is in fact the only relevant parameter in the case of generic tripartite states.
3. Stabilizability properties of generic tripartite pure states

In this section, we study the implications of the reformulated DQLS condition given in proposition 2.8 in the context of generic tripartite pure states.

3.1. Random and generic tripartite pure states: basic features

A random pure state in a finite-dimensional Hilbert space $\mathcal{H} \simeq \mathbb{C}^d$ is akin to a random variable that is sampled from the set of pure states according to the (unique) unitarily invariant Fubini–Study measure [1, 3]. Random states may be generated as the orbit of a reference pure state $|\phi\rangle \in \mathcal{H}$ under the action of a random, Haar-distributed unitary operator $U \in U(d)$, where $U(d)$ is the unitary group of degree $d$. The entries of a random (un-normalized) pure-state vector relative to any basis are i.i.d. complex Gaussian random variables [2]. As the Fubini–Study measure does not make any reference to the underlying factorization of $\mathcal{H}$, note that the sampling described above gives rise, in turn, to random pure states without a particular tensor structure. A generic pure state $|\psi\rangle \in \mathcal{H}$ is a typical element in the set of random pure states, in the sense that it possesses all the properties that hold for a measure one set of states with respect to the Fubini–Study measure. Thus, the collection of such states is itself a measure one subset of pure states with respect to the same measure. In view of this, to establish whether random pure states are DQLS (or not) with probability one, it suffices to prove that a generic $|\psi\rangle$ is DQLS (or not). In particular, with probability one a generic $N$-partite pure state has maximal Schmidt rank with respect to any bipartition of the subsystems, and the corresponding Schmidt coefficients are distinct [31]—both of which properties will be heavily used in the following.

A random tripartite pure state can be given a decomposition with respect to the central subsystem $b$ according to equation (8). Notice that the entries of the vector representation of each $|\phi_{ac}\rangle_{ac}$ are a subset of the entries of the random pure-state vector with respect to the standard basis in $\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$. For this reason, each $|\phi_{ac}\rangle_{ac}$, in this basis, is a random complex vector in $d_a \times d_c$ dimensions. Thus, $\{|\phi_{ac}\rangle_{ac}\}$ correspond to a set of random pure (un-normalized) vectors in $\mathcal{H}_a \otimes \mathcal{H}_c$ that are linearly independent on each other. Recall that each $A_i$ is a reshaped version of the $|\phi_{ac}\rangle_{ac}$ in its Schmidt basis, given by equation (10). Accordingly, $A_i$ is a random matrix of appropriate dimension. Further to that, if $|\psi\rangle$ is a generic state in $\mathcal{H}$, then, by construction, $\{A_i\}$ is a set of linearly independent, generic matrices in $\mathbb{M}^{d_a \times d_c}$ (see also appendix 7.1).

**Remark 3.1.** We have so far dealt with normalized states $|\psi\rangle$. For this reason, each state $|\phi_{ac}\rangle_{ac}$ that is derived from $|\psi\rangle$ using equation (8) has norm at most equal to one. For the corresponding matrices $\{A_i\}$, this translates into the sum of the squares of their singular values obeying $\sum_i \sigma_{ik}^2 \leq 1$, for all $i$. While a generic matrix need not satisfy this condition, this problem can be circumvented by noticing that the characterization of DQLS states given in equation (4) is independent upon normalization, as $\text{span}\{|\psi\rangle\} = \text{span}\{\alpha|\psi\rangle, \\forall \alpha \in \mathbb{C}\}$. Indeed, it can be verified that $H_0(\alpha|\psi\rangle) = H_0(|\psi\rangle)$. The reformulated DQLS condition for tripartite states given by proposition 2.8 reflects this property: $|\psi\rangle \mapsto \alpha|\psi\rangle$ implies that $\{A_i\} \mapsto \{\alpha A_i\}$ and the set $\{X_a, X_c\}$ that satisfies $X_a A_i = A_i X_c^T$ also satisfies $X_a (\alpha A_i) = (\alpha A_i) X_c^T$. Thus, each $A_i$ that is derived from a normalized generic $|\psi\rangle$ can be viewed as a representative of the family of generic matrices $\alpha A_i$ and the restriction on the sum of its singular values is irrelevant.
3.2. General results for arbitrary tripartite settings

3.2.1. No-go results and generic nature of the DQLS property. Our first result is a no-go theorem for the DQLS property of generic tripartite pure states, valid under certain combination of the subsystem dimensions.

**Theorem 3.2.** Consider a generic tripartite pure state $|\psi\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$, where $d_a \leq d$ without loss of generality, $|\psi\rangle$ is not DQLS relative to $N_m = \{N_{ab}, N_{bc}\}$ if $d_a d_b \leq d_c$. The dimension of the DQLS subspace $\mathcal{H}_0(|\psi\rangle)$ is $d_a^2$.

**Proof.** The Schmidt rank of $|\psi\rangle$ for the ab|c bipartition is $\min\{\dim(\mathcal{H}_{ab}), \dim(\mathcal{H}_c)\} = \min\{d_a d_b, d_c\}$, generically. If $d_a d_b \leq d_c$, then $\dim(\Sigma_{ab}(|\psi\rangle)) = d_a d_b$ and $\Sigma_{ab}(|\psi\rangle) = \mathcal{H}_a \otimes \mathcal{H}_b$. As a result, the extended Schmidt span $\Sigma_{abc}(|\psi\rangle) = \Sigma_{ab}(|\psi\rangle) \otimes \mathcal{H}_c$, the entire Hilbert space itself. The DQLS subspace is now $\mathcal{H}_0(|\psi\rangle) = \Sigma_{abc}(|\psi\rangle) \cap \Sigma_{bc}(|\psi\rangle) = \Sigma_{bc}(|\psi\rangle)$, the smallest of the two subspaces. On the other hand, the Schmidt rank of the a|bc bipartition is $d_a$, generically ($d_a \leq d_c$ assumed). Therefore, $\Sigma_{bc}(|\psi\rangle)$ is a $d_a^2$-dimensional subspace. Thus, $\dim(\mathcal{H}_0) = d_a^2 > 1$ which violates the necessary and sufficient condition for DQLS in theorem 2.1. \qed

The above result shows that as the difference between the outer subsystems’ dimensions $d_a$ and $d_c$ gets bigger, generic pure states tend to be non-DQLS. An immediate corollary is that DQLS pure states, for $d_a d_b \leq d_c$, form a measure zero set. While for tripartite systems that do not fall under the above no-go condition (that is, for which $d_a d_b > d_c$), the analysis is more involved, we next show that, remarkably, the DQLS behavior or its absence is, nonetheless, a generic feature. The result builds on a number of properties we collect in the following:

**Proposition 3.3.** Consider a generic tripartite pure state $|\psi\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$, $N_m = \{N_{ab}, N_{bc}\}$ and, without loss of generality, $d_a \leq d_c$. If the subsystem dimensions obey $d_a d_b > d_c$, then:

i. $\Sigma_{ab}(|\psi\rangle) = \mathcal{H}_a$ and $\Sigma_{bc}(|\psi\rangle) = \mathcal{H}_c$.

ii. $|\psi\rangle$ is DQLS relative to $N_m$ if and only if $\{X_a, X_c\}$ that satisfy equations (11), act as a scalar multiple of the identity on $\mathcal{H}_a$ and $\mathcal{H}_c$, respectively.

iii. The elements of the DQLS subspace $\mathcal{H}_0(|\psi\rangle)$ relative to $N_m$ are in 1–1 correspondence with pairs $\{X_a, X_c\}$ of solutions of equations (11), hence the resulting dimensions are equal.

**Proof.** We prove each statement separately.

i. The Schmidt rank of a generic $|\psi\rangle$ for the ab|c bipartition is $d_a$, since $d_a \leq d_c$. Thus, $\Sigma_{ab}(|\psi\rangle) = \mathcal{H}_a$. Due to the extra assumption $d_a d_b > d_c$, the Schmidt rank of the ab|c bipartition is equal to $d_c$, generically, and therefore, $\Sigma_{bc}(|\psi\rangle) = \mathcal{H}_c$.

ii. Combining lemma 2.7 with the result just proved, the combined range of $\{A_i\}$, namely, $\text{span}\{\text{range}(A_i)\}_{i=1}^{d_a-1}$, coincide with $\mathcal{H}_a$. Similarly, the combined range of $\{A_j^T\}$ coincides with $\mathcal{H}_c$. Hence, it follows from proposition 2.8 that $|\psi\rangle$ is DQLS relative to $N_m$ if and only if $X_a$ and $X_c$ act as the identity (up to a constant scalar factor) on $\mathcal{H}_a$ and $\mathcal{H}_c$, respectively.

iii. In corollary 2.5, we proved that $|\psi'\rangle \in \mathcal{H}_0(|\psi\rangle)$ if and only if it satisfies equation (5) for some choice of $X_{ac} \in \mathcal{B}(\mathcal{H}_a \mathcal{H}_c)$, for all $j$. This observation, applied to the tripartite case with $N_m = \{N_{ab}, N_{bc}\}$, implies that $|\psi'\rangle \in \mathcal{H}_0(|\psi\rangle)$ if and only if there exist some $X_a \in \mathcal{B}(\mathcal{H}_a)$ and $X_c \in \mathcal{B}(\mathcal{H}_c)$ such that...
\[ |\psi'\rangle = (X_a \otimes I_b \otimes I_c) |\psi\rangle = (I_a \otimes I_b \otimes X_c) |\psi\rangle. \] (12)

What remains to be shown is that any fixed choice of \( \{X_a, X_c\} \) that satisfies this equation can be identified with a unique \( |\psi'\rangle \in \mathcal{H}_0(|\psi\rangle) \), and vice-versa. The forward implication follows straightforwardly from linearity. To prove the reverse implication, assume that there exist two different choices, \( \{X_a, X_c\} \) and \( \{Y_a, Y_c\} \), such that they both satisfy equation (12) for the same \( |\psi'\rangle \). Restructuring this equation leads to the form given in equation (11). After eliminating \( |\psi'\rangle \), we get

\[ (X_a - Y_a)A_i = 0, \quad A_i(X_c - Y_c)^T = 0, \quad i = 0, \ldots, d_b - 1. \] (13)

The first equation results in \( X_a = Y_a \). This is because a generic matrix \( A_i \in M^{d_a \times d_c} \) with \( d_a \leq d_c \), has no non-trivial left kernel. Now, form another matrix

\[ \mathcal{A} = \begin{bmatrix} A_0 \\ \vdots \\ A_{d_b-1} \end{bmatrix}, \]

which is also generic, by construction. The second equation in equation (13) then implies \( \mathcal{A}(X_c - Y_c)^T = 0 \). Since \( d_a d_b \geq d_c \), \( \mathcal{A} \in M^{d_a d_b \times d_c} \) has no non-trivial right kernel, hence \( X_c = Y_c \). Therefore, for a given \( |\psi'\rangle \in \mathcal{H}_0(|\psi\rangle) \), there exists a unique choice of \( \{X_a, X_c\} \) that satisfies equation (12), whereby it also follows that the dimensions of these two spaces are equal. \( \Box \)

Notice that the assumption \( d_a d_b > d_c \) is key to show that the Schmidt spans of a generic \( |\psi\rangle \) on subsystems \( a \) and \( c \) coincide with the entire space. Had that not been the case, the search for any non-trivial solutions to equations (11) would have to be restricted to some subspace of the corresponding Hilbert spaces, making the analysis more involved. Although statement (ii) in proposition 3.3 is essentially a restatement of proposition 2.8 for generic tripartite pure states, combining statements (ii) and (iii) allows us to determine the dimension of the DQLS subspace of the target state, in addition to its DQLS behavior. Altogether, we are now in a position to prove the anticipated result on the generic nature of the DQLS property itself in those tripartite settings that do not meet the no-go condition:

**Theorem 3.4.** Consider a tripartite quantum system on \( \mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \), with neighborhood structure \( N_{\text{tri}} = \{N_{ab}, N_{bc}\} \) and \( d_a \leq d_c \), without loss of generality. When the subsystem dimensions obey the condition \( d_a d_b > d_c \), DQLS pure states relative to \( N_{\text{in}} \) form either a measure zero or a measure one set.

**Proof.** Consider a generic \( |\psi\rangle \in \mathcal{H} \). Since \( d_a d_b > d_c \) and \( d_a \leq d_c \), by proposition 3.3 we know that \( \Sigma_a(|\psi\rangle) = \mathcal{H}_a \), \( \Sigma_c(|\psi\rangle) = \mathcal{H}_c \), and \( |\psi\rangle \) is DQLS relative to \( N_{\text{in}} \) if and only if equation (12) is satisfied for \( X_a = \lambda A_a \) and \( X_c = \lambda C_a \), with \( \lambda \in \mathbb{C} \). Define \( T = \text{span}\{X_a \otimes I_b \otimes I_c, I_a \otimes I_b \otimes X_c, X_a \otimes B(H_a), X_c \in B(H_c)\} \), which is a \( (d_a^2 + d_c^2 - 1) \)-dimensional subspace of \( B(\mathcal{H}) \). In terms of \( T \), \( |\psi\rangle \) is DQLS relative to \( N_{\text{in}} \) if and only if \( Y|\psi\rangle = 0 \) for \( Y \in T \) implies that \( Y = 0 \). Consider the representation of \( |\psi\rangle \) and \( X_a, X_c \) in the standard basis for the respective spaces. If we vectorize \( Y \) by stacking up its columns into a single vector denoted \( \text{vec}(Y) \), and use the property \( \text{vec}(AXB) = (B^T \otimes A)\text{vec}(X) \) [32], the condition \( Y|\psi\rangle = 0 \) is equivalent to

\[ (B^T \otimes A)\text{vec}(X) = 0. \]
\[ M(\vec{\psi})^T \text{vec}(Y) = 0, \quad \text{where } M(\vec{\psi}) \equiv \vec{\psi} \otimes I_a \otimes I_b \otimes I_c. \quad (14) \]

Thus, the existence of any non-trivial \(\text{vec}(Y)\) in the right kernel of \(M^T(\vec{\psi})\) is equivalent to the non-DQLS nature of the generic \(|\psi\rangle\) which satisfies our assumption. To check whether this is the case, let us re-parametrize \(T\) by introducing an isometric embedding \(V : C^{n^2} \to \text{vec}(T)\), such that \(V(\vec{t}) = \text{vec}(t)\), where \(\{t_i\}\) denotes a fixed basis in \(T\) and \(\{\vec{t}_i\}\) is the standard basis in \(C^{n^2}\). Using the fact that \(V\) is full (column-) rank, we have that \(Y \neq 0\) can satisfy equation \((14)\), for a pure state \(|\psi\rangle\), if and only if \(A \equiv M(\vec{\psi})^T V\) has a non-trivial right kernel. This is equivalent to saying that \(A^TA\) is non-invertible. Therefore, a generic \(|\psi\rangle\) fails to be DQLS if and only if

\[ D(\psi) \equiv \det[(M(\vec{\psi})^T V)^T M(\vec{\psi})^T V] = 0. \quad (15) \]

The above may be viewed as a polynomial equation, whose coefficients are determined by \(V\) and whose unknowns are given by the entries of \(|\psi\rangle\langle\psi|\) in the standard basis. The roots of a polynomial in \(C^n\), for any \(n\), are a measure zero set whenever the latter is non-zero, or a measure one set for the trivial (zero) polynomial. In the first case, when equation \((15)\) implies that \(Y = 0\) is the only solution to equation \((14)\) for all but a zero measure set of instances, the DQLS property is measure one. If \(D(\psi) \equiv 0\), all \(V\)-dependent coefficients vanish identically, hence non-trivial solutions \(Y\) always exists and DQLS is, correspondingly, a measure zero property.

Since the proof of the above theorem hinges on the fact that, for \(d_a d_b > d_c\), the Schmidt spans of a generic \(|\psi\rangle\) with respect to the \(a|bc\rangle (ab|c)\) bi-partitions coincide with the full \(H_a\) (\(H_c\)), we can extend some of our conclusions to any general tripartite state \(|\phi\rangle\), whose Schmidt spans on the relevant bi-partitions are maximal. This is equivalent to saying that the corresponding one-body RDMs, \(\rho_a \equiv \rho_{\Sigma_a}\) and \(\rho_c \equiv \rho_{\Sigma_c}\), are full-rank. Observe that when DQLS is a measure zero property, the polynomial \(D(\phi)\) defined in equation \((15)\) from a state \(|\phi\rangle\) (generic or otherwise), whose RDMs \(\rho_a\) and \(\rho_c\) are full-rank, is also identically zero. We can thus give the following corollary:

**Corollary 3.5.** Consider a tripartite quantum system on \(\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c\) and neighborhood structure \(N_{\text{int}} = \{N_{ab}, N_{bc}\}\), such that the subsystem dimensions obey \(d_a d_b > d_c\) and \(d_a \leq d_c\). If a generic pure state \(|\psi\rangle\in \mathcal{H}\) is DQLS relative to \(N_{\text{int}}\) with probability zero, then an arbitrary pure state \(|\phi\rangle\in \mathcal{H}\), with full-rank RDMs \(\rho_a \in \mathcal{D}(\mathcal{H}_a)\) and \(\rho_c \in \mathcal{D}(\mathcal{H}_c)\), is also not DQLS relative to \(N_{\text{int}}\).

### 3.2.2. SLOCC canonical form of generic pure states.

While theorem 3.4 guarantees that the DQLS property is either measure zero or measure one in the set of pure states, further analysis is required to characterize the behavior in cases where \(d_a d_b > d_c\). Toward this, we leverage the SLOCC-invariance of the DQLS property to simplify our analysis, by transforming the given generic state into another state with useful properties. The following lemma will be needed:

**Lemma 3.6.** Any bipartite pure state \(|\phi\rangle\in \mathcal{H}_1 \otimes \mathcal{H}_2\) with maximal Schmidt rank may be transformed into the maximally entangled (un-normalized) state, namely \(|\Omega\rangle = \sum_k |e_k\rangle_1 \langle f_k|_2\), through the action of local invertible operators. Here, \([|e_i\rangle]_{i=1}^{d_a}\) and \([|f_m\rangle]_{m=1}^{d_2}\) represent the standard bases in \(\mathcal{H}_1\) and \(\mathcal{H}_2\), respectively.
Proof. The maximal Schmidt rank of a bipartite state is equal to the dimension of the smallest component Hilbert space in the bipartition. Assume \( d_1 \leq d_2 \), without loss of generality, and let the Schmidt decomposition of \( |\phi\rangle \) be given by \( |\phi\rangle = \sum_{k=1}^{d_2} \mu_k |x_k\rangle |y_k\rangle \). Construct the linear operator \( M_1 \in \mathcal{B}(\mathcal{H}_1) \) such that \( M_1 = \sum_{k=1}^{d_1} (1/\mu_k) e_k^\dagger x_k \). This is invertible since \( \{|x_k\rangle\}_{k=1}^{d_2} \) forms a basis of \( \mathcal{H}_1 \), thanks to the maximal Schmidt rank of \( |\phi\rangle \). Now, choose \( M_2 \in \mathcal{B}(\mathcal{H}_2) \) to be the unitary that transforms the orthogonal basis \( \{|y_m\rangle\}_{m=1}^{d_1} \) (by extending the Schmidt vectors appropriately) into \( \{|f_k\rangle\}_{k=1}^{d_2} \) in \( \mathcal{H}_2 \). \( \square \)

We make use of this fact to convert a generic state \( |\psi\rangle \) into another pure state \( (M_a \otimes I_b \otimes M_c)|\psi\rangle \), in such a way that

\[
(M_a \otimes M_c)|\phi_{0c}\rangle = |\Omega_{ac}\rangle,
\]

where \( |\phi_{0c}\rangle \) belongs to the decomposition of \( |\psi\rangle \) given by equation (8). Lemma 3.6 applies because \( |\phi_{0c}\rangle \) has maximal Schmidt rank, owing to its generic nature. Let \( \{\bar{A}_i\} \) denote the set of matrices associated to \( (M_a \otimes I_b \otimes M_c)|\psi\rangle \), derived using equation (8) and equation (10). With the modified \( |\phi_{ac}\rangle \) being maximally entangled, these matrices have the form:

\[
\bar{A}_0 = \begin{bmatrix} I_{d_0} & 0_{d_0d} \end{bmatrix}, \quad \bar{A}_i = M_a A_i M_c^\dagger, \quad i = 1, \ldots, d_b - 1,
\]

where the block \( I_{d_0} \) denotes the identity matrix in \( \mathbb{M}_{d_a,d_a} \), corresponding to the Schmidt coefficients of the maximally entangled state, and \( 0_{d_0d} \) denotes the zero matrix in \( \mathbb{M}_{d_0,d} \), with \( d = d_a - d_b \). Since the dimension of \( \mathcal{H}_0(|\psi\rangle) \) and \( \mathcal{H}_0(M_a \otimes I_b \otimes M_c|\psi\rangle) \) are the same thanks to proposition 2.6. Hence, in order to determine the DQLS nature of \( |\psi\rangle \) we can study the set of linear equations (11) associated to \( \{\bar{A}_i\} \), instead of \( \{A_i\} \).

The form of \( \bar{A}_0 \) in equation (17) considerably simplifies the problem of solving the set of equations (11) for \( \{\bar{A}_i\} \). For this reason, and for the sake of simplicity, we hereafter denote by \( |\psi\rangle \) an SLOCC-transformed generic tripartite pure state, such that the corresponding \( |\phi_{0c}\rangle = |\Omega_{ac}\rangle \) in equation (8), with \( |\Omega_{ac}\rangle \) being the maximally entangled state. We also drop the tilde in denoting \( \{\bar{A}_i\} \), the set of matrices constructed from the transformed \( |\psi\rangle \) following equations (8)–(10). In order to analyze the form of the solutions, we introduce a block decomposition of the matrices involved in equation (11). Since \( X_a \) is the smallest matrix, it is not further decomposed. The decomposition of the other matrices is summarized in table 1. Equation (11), with \( A_0 = \begin{bmatrix} I_{d_0} & 0_{d_0d} \end{bmatrix} \), simplifies the form of the matrix \( X_c \) as shown below:

\[
\bar{X}_{0c} = X_a, \quad \bar{X}_{0d} = 0_{d_0d}.
\]

3.3. Analytical characterization for a central qubit subsystem, \( d_b = 2 \)

We now provide analytical results that fully characterize the DQLS nature of generic tripartite pure states when the central subsystem is a qubit. In this case, the set of relevant equations (11) are associated to the matrices \( A_0 \) and \( A_1 \). Since we are considering the SLOCC-transformed generic pure state, \( A_0 \) is equal to identity padded with zeros for adjusting the dimensions. As a consequence, equations (18) hold for the different blocks of \( X \). Now apply equations (11) to \( A_1 \) to obtain the following conditions:

\[
X_a A_1 = A_{00} X_a + A_{0d} \bar{X}_{0d},
\]

\[
X_a A_0 = A_{00} X_a + A_{0d} \bar{X}_{0d}.
\]

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where, for notational simplicity, we have dropped the superscript 1 while denoting the blocks of $A_1$. The case $d \times d \times d (d = 0)$ is easier to analyze as the relevant matrices $X_a, X_c, A_0$ and $A_1$ have the same dimensionality. As a result, in the decomposition of table 1, all the blocks except the $d_a \times d_a$ ones vanish, and equations (19) and (20) reduce to

$$X_aA_{00} = A_{00}X_a,$$

with $X^T_c = X_c$ (notice that $A_{00}$ is the same as $A_1$, but we stick to the former notation for consistency). In the more general scenario $d_a \times 2 \times d_c$, where $d = d_c - d_a > 0$, we know from theorem 3.2 that generic pure states are non-DQLS when $d_a \leq d_c$, in which case the corresponding DQLS subspace $\mathcal{H}_0$ is $d_{a}^2$-dimensional. As we are going to show, the only situation in which the DQLS property is found to be generic corresponds to the combination of dimensions $d_a \times 2 \times (d_a + 1)$. More precisely, our analytical characterization is summarized in the following theorem, whose lengthy proof we postpone, for clarity, to appendix 7.2:

**Theorem 3.7.** Let $|\psi\rangle$ be a generic tripartite pure state in $\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$, with dimensions $d_a \times d_b \times d_c$ and neighborhood structure $\mathcal{N}_{tr} = \{\mathcal{N}_{ab}, \mathcal{N}_{bc}\}$. When $d_b = 2$,

i. $|\psi\rangle$ is not DQLS relative to $\mathcal{N}_{tr}$ when $d_a = d_c$. The dimension of $\mathcal{H}_0(|\psi\rangle)$ is $d_a^2$.

ii. $|\psi\rangle$ is not DQLS relative to $\mathcal{N}_{tr}$ when $d_c > d_a$, except for $d_c = d_a + 1$. The dimension of $\mathcal{H}_0(|\psi\rangle)$ is $\min\{d_a^2, (d_c - d_a)^2\}$.

### 3.4. Stabilizability properties beyond $d_b = 2$

Consider next a general tripartite system in $d_a \times d_b \times d_c$ dimensions, for $d_b \neq 2$. Numerical evidence shows that the dimension combinations which do not fall under the no-go condition of theorem 3.2 yield a positive result to the DQLS test—with some exceptions. It appears as if this is an instance of an over-constrained system of equations, with equation (11) having more constraints than in the $d_b = 2$ cases, whereas the number of unknowns remain the same. This observation would suggest to pursue a constraint-counting analysis similar to the one used in proving theorem 3.7. However, notwithstanding the fact that the number of constraints in equation (11) increases with $d_b$, they remain linearly dependent on each other to a large extent. This is verified by the fact that the system is always guaranteed to admit at least one non-trivial solution, namely, $X_a$ and $X_c$ proportional to the identity matrix. In view of this intricate general structure, we focus on providing analytic results for some specific cases of interest, and use numerical tools to investigate the remaining ones.
3.4.1. Analytical results. We begin with the following observation:

**Proposition 3.8.** If a generic tripartite pure state in $d_a \times d_b \times d_c$ dimensions is DQLS relative to $N_m = \{N_{ab}, N_{bc}\}$, the same property holds in any $d_a \times d_b' \times d_c$ dimensions, with $d_b' > d_b$.

**Proof.** It can be verified that none of the above dimension combinations fall under the no-go theorem 3.2. The set of equations $X_i A_i = A_i X_i^T$ for $i = 0, \ldots, (d_b' - 1)$, where $\{A_i\}$ denotes a set of generic matrices in $M^{d_a \times d_b'}$, determines the DQLS nature of generic pure states in $d_a \times d_b' \times d_c$ dimensions, following proposition 3.3. Since, by assumption, generic pure states in $d_a \times d_b \times d_c$ dimensions are DQLS, $\{X_a, X_c\}$ that simultaneously solve any $d_b$ of these equations are proportional to the identity in the corresponding space. Hence, this holds for the entire set too and thus the generic pure states in $d_a \times d_b' \times d_c$ dimensions are also DQLS, for $d_b' > d_b$. □

Note that the converse of the above proposition is generally false. A counter example is given by considering $2 \times 3 \times 2$ versus $2 \times 2 \times 2$: as we will soon prove (theorem 3.10), generic pure states are DQLS relative to $N_m$ in the former case, yet we know from theorem 3.7 that they are not in the latter.

For the remainder of our analysis, we consider equations (11) in terms of the block-decomposition of the relevant matrices (table 1). As we will be using the SLOCC-transformed version of the generic pure state, the matrices $\{A_i\}$ have $A_0 = \left[ I_{d_a} \mid 0_{d_a} \right]$. The consequence of this transformation is to simplify the block structure of $X_i$ by setting $X_{00} = X_a$ and $X_{0d} = 0$. By taking into account this simplification, we can now rewrite equations (11) with respect to the block decomposition of the relevant matrices as follows:

\[ X_a A_{0i} = A_{0i}^T \tilde{X}_{a0i}, \quad i = 1, \ldots, d_b - 1. \]  \( (22) \)

\[ X_a A_{0i}^T = A_{0i}^T X_a + A_{0d}^T \tilde{X}_{a0i}, \quad i = 1, \ldots, d_b - 1. \]  \( (23) \)

Notice that for $d \equiv d_a - d_a > 0$, equation (22) looks similar to the set of equations (11) with $\{A_i\}, \{X_a, X_c\}$ replaced by $\{A_{0i}^T\}, \{X_a, \tilde{X}_{a0i}\}$, respectively, and the system dimensions given by $d_a \times (d_b - 1) \times d$. Thus, we see that the DQLS property of a generic pure state in $d_a \times (d_b - 1) \times d$ dimensions influences the solution space of equation (22), according to proposition 3.3. The proposition given below builds further on this observation.

**Proposition 3.9.** If a generic tripartite pure state in $d_a \times (d_b - 1) \times d$ dimensions is DQLS relative to $N_m = \{N_{ab}, N_{bc}\}$, the same holds for a generic tripartite pure state in $d_a \times d_b \times (d_a + d)$ dimensions, for $0 < d < d_a(d_b - 1)$.

**Proof.** If the generic pure states in $d_a \times (d_b - 1) \times d$ dimensions are DQLS, then the only choice of $X_a$ and $X_c$ that satisfy equations (11) is proportional to the identity. In order to determine the DQLS property of generic pure states in $d_a \times d_b \times (d_a + d)$ dimensions, let us analyze the modified version of equations (11), given by equations (22) and (23). Since equation (22) is identical to equation (11) with $\{A_i\}, \{X_a, X_c\}$ replaced by $\{A_{0i}^T\}, \{X_a, \tilde{X}_{a0i}\}$, respectively, we see that the only choice of $\{X_a, \tilde{X}_{a0i}\}$ that satisfy equation (22) must be the identity, following our assumption. As a consequence, equation (23) is modified to $A_{0i}^T X_{a0i} = 0$, for all $i$. Since we are only considering the cases where $d_a d_b > d_a$ (or, $d < d_a(d_b - 1)$), which are not covered under the no-go theorem 3.2, the only choice of $X_{a0i}$ that fall in the right kernel of each generic matrix in $\{A_{0i}^T\}$ is the zero matrix (as established at the end of the proof.
of statement (iii) in proposition 3.3). Let us now collect together all the above observations, in order to determine the solution space to equations (22) and (23) in $d_a \times d_b \times (d_a + \bar{d})$ dimensions. We already showed that $X_a = I$. Recall that, in block form, $X^T_c = \begin{bmatrix} X_a & \vec{0} \\ \vec{0} & \bar{X}_{d\bar{d}} \end{bmatrix}$. If $\bar{X}_{d\bar{d}}$ and $\bar{X}_{d0}$ are the zero matrix and the identity matrix, respectively, it follows that $X_c = I$.

Therefore, invoking proposition 3.3, generic pure states in $d_a \times d_b \times (d_a + \bar{d})$ dimensions are also DQLS.

We are now in a position to characterize a number of special cases, which we collect in the following theorem. Since the proof is lengthy, we again defer it to appendix 7.2.

**Theorem 3.10.** Let $|\psi\rangle$ be a generic tripartite pure state in $H_a \otimes H_b \otimes H_c$, with neighborhood structure $N_{\text{tri}} = \{N_{ab}, N_{bc}\}$ and central subsystem dimension $d_b > 2$. Then, $|\psi\rangle$ is DQLS relative to $N_{\text{tri}}$ for the following combinations of subsystem dimensions:

i. $d_a \times d_b \times d_a$

ii. $d_a \times d_b \times (d_a + 1)$

iii. $d_a \times d_b \times nd_a$, with $1 < n < d_b$.

**3.4.2. Numerical results.** In order to investigate the cases where an analytical proof is lacking, we resort to numerical techniques, by generating random pure states in various dimensions\(^5\) and assessing their DQLS property relative to $N_{\text{tri}} = \{N_{ab}, N_{bc}\}$. In light of theorem 3.4, we know that the set of DQLS states has either measure zero or measure one, hence a randomly generated state with probability one will have the same property of almost all its peers. Table 2 summarizes our numerical results in $d_a \times d_b \times d_c$ dimensions when the central subsystem is a qutrit, $d_b = 3$. It can be seen that, for fixed values of $d_a$, increasing $\bar{d}$ results in a gradual shift from DQLS to non-DQLS behavior, as intuitively expected. This is not the case, however, when $d_b = 2$, whereby $d_a \times 2 \times (d_a + 1)$ is the only dimension combination for which random states are DQLS, generically. With the exception of $4 \times 3 \times 11$ and $5 \times 3 \times 14$ dimensions, the transition from DQLS to non-DQLS behavior occurs precisely when $\bar{d} = (d_b - 1)d_a$, that is, $d_ad_b = d_c$, which corresponds to the threshold for the applicability of the no-go theorem 3.2. The above two dimension combinations are the only cases in this table that are not DQLS and do not fall under the no-go category. Additional numerical investigation of such ‘exceptional cases’ suggest that (i) they do occur for $\bar{d}$ close to $(d_b - 1)d_a$

\(^5\)Specifically, we employed T Cubitt’s randPsi.m function in MATLAB, available online at www.dr-qubit.org/matlab/randPsi.m
(the no-go threshold) for \( d_a > d_b \); and (ii) the number of such occurrences is approximately given by \( \frac{d_a}{d_b} \). These trends are captured in table 3, where we have chosen \( d_b = 4 \).

Based on our combined analytical and numerical results, we conclude our analysis of the generic DQLS property of random tripartite pure states with the following:

**Conjecture 3.11.** Let \( |\psi\rangle \) be a generic pure state in \( d_a \times d_b \times d_c \) dimensions, with \( d_b > 2 \). Then:

i. When \( d_a \leq d_b \), \( |\psi\rangle \) is generically DQLS relative to \( N_{ab} \) if and only if \( d_c < \frac{d_a}{d_b} \).

ii. When \( d_a > d_b \), \( |\psi\rangle \) is generically DQLS relative to \( N_{ab} \) if \( d_c < \frac{d_a}{d_b} - \lfloor \frac{d_a}{d_b} \rfloor \).

### 4. Additional results and implications

#### 4.1. Stabilizability properties in the generic multipartite setting

The analytical tools developed in section 2.2 and the results in section 3 pertain to generic tripartite pure states relative to the bipartite neighborhood structure \( N_{ab}, N_{bc} \). Addressing multipartite quantum systems, with arbitrary neighborhood structures, is very hard. Nonetheless, building on our tripartite analysis, it is still possible to identify conditions that preclude generic DQLS behavior in multipartite settings. First, by extending theorem 3.2, a no-go condition for the DQLS property of generic multipartite pure states may be established, based on the sizes of the relevant neighborhoods in \( N \):

**Theorem 4.1.** Consider a generic multipartite pure state \( |\psi\rangle \in \mathcal{H} \) and a neighborhood structure \( N = \{N_j\}_{j=1}^M \). \( |\psi\rangle \) is not DQLS relative to \( N \) if \( \dim(\mathcal{H}_{N_j}) \leq \dim(\mathcal{H}_{N_i}) \), for all \( j \). Additionally, the DQLS subspace of \( |\psi\rangle \) coincides with \( \mathcal{H} \).

**Proof.** For a generic pure state \( |\psi\rangle \), the Schmidt rank with respect to any bipartition is maximal. Therefore, for any \( N_j \cup N_i \) partition, \( \dim(\Sigma_{N_j}(|\psi\rangle)) = \dim(\mathcal{H}_{N_i}) \), since \( \dim(\mathcal{H}_{N_j}) \leq \dim(\mathcal{H}_{N_i}) \) by assumption. Hence, \( \Sigma_{N_j}(|\psi\rangle) = \mathcal{H}_{N_j} \) and \( \Sigma_{N_i}(|\psi\rangle) = \mathcal{H} \). The DQLS subspace in this case is \( \mathcal{H}_0(|\psi\rangle) = \bigcap_{j=1}^M \Sigma_{N_j}(|\psi\rangle) = \mathcal{H} \), implying that \( |\psi\rangle \) is not DQLS according to theorem 2.1. □

As a consequence, it follows that in many systems of practical interest, where the number of subsystems is large but only finite-range (typically, nearest or next-to-nearest neighbor) couplings are considered, generic pure states are not DQLS, as one may intuitively expect. In fact, the above result also implies that, if one begins with an \( N \)-partite quantum system where generic states are DQLS relative to the chosen neighborhood structure, the sizes of the

| \( d_a \) | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | ✓ | × | × | × | × | × | × | × | × | × | × | × | × | × |
| 5 | ✓ | ✓ | ✓ | ∅ | × | × | × | × | × | × | × | × | × | × |
| 6 | ✓ | ✓ | ✓ | ✓ | ✓ | ∅ | × | × | × | × | × | × | × | × |
| 7 | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ∅ | × | × | × | × | × | × | × |
| 8 | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ∅ | × | × | × | × | × | × |

Table 3. DQLS behavior of random pure states for \( d_b = 4 \), with symbols having the same meaning as in table 2. In particular, circled symbols now correspond to the dimensions where the tested target states are not DQLS, but they do not fall under the no-go category.
neighborhoods have to be scaled appropriately as $N$ increases, in order for the DQLS property to be retained.

**Remark 4.2.** Similar to corollary 3.5, it is worth noting that the no-go theorem 4.1 is applicable to any pure state $|\phi\rangle$ which has full-rank RDMs in all the neighborhoods under consideration. Since a full-rank RDM on neighborhood $N_j$ is equivalent to saying that $\Sigma_{N_j}(|\phi\rangle) = \mathcal{H}$, such states always fail to be DQLS. For instance, one may verify that qubit graph states have maximally mixed RDMs on any two-qubit neighborhoods and hence they are non-DQLS under nearest-neighbor interactions, consistent with known results [11, 14] and the fact that they cannot be exact ground states of two-body FF parent Hamiltonians [34]. Another interesting class of states that are non-DQLS for the same reason comprises so-called $k$-uniform states, namely, $N$-party pure states such that their all $k$-body RDMs for $k \leq \lfloor N/2 \rfloor$ are maximally mixed [35].

To derive other useful tests for the DQLS property by leveraging our tripartite results, we need a way to relate multipartite and multi-neighborhood settings to the known tripartite scenario. The key observation is that DQLS properties are preserved when aggregating, or coarse-graining, these neighborhoods:

**Definition 6.** Let $|\psi\rangle \in \mathcal{H} = \otimes_{a=1}^{N} \mathcal{H}_a$, where each $\mathcal{H}_a$ is a finite-dimensional Hilbert space, with neighborhood structure $N = \{N_j\}_{j=1}^{M}$. A neighborhood structure $N' = \{N'_j\}_{j=1}^{M'}$ is a coarse-graining of $N$ if for each $N_j \in N$ there exists some $N'_j \in N'$, such that $N_j \subseteq N'_j$. Equivalently, $N'$ can be called a refinement of $N'$.

We next show that the DQLS property of a general pure state relative to a given $N$ is preserved if we move to any coarse-graining of the same, as expected:

**Proposition 4.3.** If the multipartite pure state $|\psi\rangle \in \mathcal{H}$ is DQLS relative to the neighborhood structure $N = \{N_j\}_{j=1}^{M}$, then $|\psi\rangle$ is DQLS relative to any coarse-graining of $N$ as well.

**Proof.** Let $N' = \{N'_j\}_{j=1}^{M'}$ denote the coarse-graining of $N$. By definition 4, $|\psi\rangle$ is DQLS relative to $N$ if and only if there exists a QL generator $\mathcal{L}(\mathcal{H}, \{T_a\})$ in standard form, with $\mathcal{H} = 0$, such that each term in $\mathcal{L}$ acts non-trivially only on a given neighborhood $N_j \in N$ and $\mathcal{L}$ makes $|\psi\rangle \langle \psi|$ GAS. Since each $N'_j \in N'$ is such that $N'_j \supseteq N_j$ for some $N_j \in N$, $\mathcal{L}$ is QL with respect to the coarse-grained neighborhood structure $N'$ as well. Therefore, the same QL generator renders $|\psi\rangle$ DQLS relative to $N'$.

This, in turn, implies that if $|\psi\rangle$ is not DQLS relative to a certain neighborhood structure, then $|\psi\rangle$ is not DQLS relative to any refinement as well. Given a multipartite generic pure state and an associated $N$, one can then coarse-grain $N$ into a two-neighborhood neighborhood structure $N'$, similar to the one described in figure 1, and analyze it in the light of the tripartite results given in the previous sections. If the generic pure state is not DQLS relative to $N'$, then it is not DQLS relative to $N$ also.

We may organize all the results on the DQLS property of generic pure states into a DQLS checker algorithm that can analytically decide the DQLS nature in many cases. The algorithm takes as input a generic target pure state $|\psi\rangle$ and a neighborhood structure $N = \{N_j\}_{j=1}^{M}$ on $\mathcal{H}$, and proceeds through the following steps:

(i) Check if there exists any neighborhood $N'_k \in N$ such that $\dim(\mathcal{H}_{N'_k}) \leq \dim(\mathcal{H}_{N_j})$. If so, form a modified neighborhood structure $\tilde{N}$ by ignoring all such neighborhoods. If there exists an $N'_k$ satisfying the above condition, its corresponding extended Schmidt span
\[ \Sigma_{\mathcal{N}}(|\psi\rangle) = \mathcal{H}, \] according to the reasoning given in the proof of theorem 4.1. As a result, \[ \Sigma_{\mathcal{N}}(|\psi\rangle) \] contributes trivially to the DQLS subspace \[ \mathcal{H}_0(|\psi\rangle) = \bigcap_{i=1}^{\ell} \Sigma_{\mathcal{N}_i}(|\psi\rangle) \] relative to \( \mathcal{N} \), and hence can be ignored in the analysis. In other words, for any neighborhood on which the Schmidt span of \( |\psi\rangle \) is full-rank, the invariance condition on the dissipative generator renders the corresponding neighborhood generator trivial, and therefore disposable (see definition 4). This also shows that the DQLS subspaces of \( |\psi\rangle \) relative to \( \mathcal{N} \) and \( \bar{\mathcal{N}} \) coincide with each other. If all the neighborhoods in \( \mathcal{N} \) can be eliminated in this way, then, as proved in theorem 4.1, \( |\psi\rangle \) is not DQLS relative to \( \mathcal{N} \).

(ii) Check if there exists any pair of neighborhoods \( \mathcal{N}_k, \mathcal{N}_l \in \bar{\mathcal{N}} \), such that \( \mathcal{N}_k \cup \mathcal{N}_l = \{1, \ldots, N\} \). Then if \( |\psi\rangle \) is DQLS relative to the neighborhood structure \( \{\mathcal{N}_k, \mathcal{N}_l\}, |\psi\rangle \) is also DQLS relative to \( \mathcal{N} \).

(iii) As a result of ignoring the smaller neighborhoods in step (i), if there exist subsystems that do not belong to any \( \mathcal{N}_j \in \bar{\mathcal{N}} \), then \( |\psi\rangle \) is not DQLS relative to \( \mathcal{N} \). In such cases, the extended Schmidt spans of any of the remaining neighborhoods may be written as
\[
\Sigma_{\mathcal{N}_j} = (\Sigma_{\mathcal{N}_j} \otimes_{a \in \bar{\mathcal{N}}(\mathcal{N}_j \cup r)} \mathcal{H}_a) \otimes_{a \in r} \mathcal{H}_a,
\]
where \( r \) collects all the subsystems that do not belong to any of the relevant neighborhoods. Then
\[
\mathcal{H}_0(|\psi\rangle) = \bigcap_{N_j \in \bar{\mathcal{N}}} \Sigma_{N_j} \otimes_{a \in \bar{\mathcal{N}}(N_j \cup r)} \mathcal{H}_a \otimes_{a \in r} \mathcal{H}_a,
\]
and \( \dim(\mathcal{H}_0(|\psi\rangle)) \) is at least equal to the product of the dimensions of the leftover subsystems. Since the latter exceeds one, we know that \( |\psi\rangle \) is not DQLS.

(iv) Coarse-grain \( \bar{\mathcal{N}} \) to form a two-neighborhood structure \( \bar{\mathcal{N}}' \), as in figure 1. If \( |\psi\rangle \) is not DQLS relative to \( \bar{\mathcal{N}}' \), then \( |\psi\rangle \) is non-DQLS relative to \( \bar{\mathcal{N}}' \) as well.

The utility of our DQLS checker algorithm may be illustrated through a simple example. Consider \( \mathcal{N} = 5 \) qudit subsystems \{a, \ldots, e\}, with neighborhood structure \( \mathcal{N} = \{N_j\}_{j=1}^5 = \{N_{ab}, N_{ace}, N_{abe}, N_{cde}\} \), as depicted in figure 2. Our aim is to determine whether a generic pure state \( |\psi\rangle \) in this setting is DQLS or not. First, following step (i), we eliminate \( N_{ab} \), as it contributes trivially to \( \mathcal{H}_0(|\psi\rangle) \). Next, we observe that \( N_{ace} \cup N_{bde} = \{a, \ldots, e\} \). Thus, by step (ii), if \( |\psi\rangle \) is DQLS relative to \( \mathcal{N}_{in} = \{N_{ace}, N_{bde}\} \), corresponding to effective subsystem dimensions \( d^2 \times d \times d^2 \), it is DQLS relative to \( \mathcal{N} \) as well. This will be the case if \( d \neq 2 \), following theorem 3.10. Thus, the algorithm terminates successfully at step (ii) for \( d \neq 2 \) and we conclude that \( |\psi\rangle \) is DQLS relative to \( \mathcal{N} \) in this case. For the qubit case instead (\( d = 2 \)), theorem 3.7 shows that \( |\psi\rangle \) is non-DQLS relative to \( \mathcal{N}_{in} \) and hence we proceed to the next steps of the algorithm. Step (iii) is not helpful here, because following the elimination of \( N_{ab} \) no subsystem is left uncovered by the remaining neighborhoods. In step (iv), a possible non-trivial coarse-graining of \( \mathcal{N} \) is given by \( \mathcal{N}' = \{N_{ace} \cup N_{cde}, N_{bde}\} \), with effective subsystem dimensions \( 2 \times 4 \times 4 \) when \( d = 2 \). However, \( |\psi\rangle \) is verified to be DQLS relative to \( \mathcal{N}' \) by theorem 3.10. Accordingly, for the 5-qubit system the algorithm is inconclusive and one has to resort to numerical techniques in order to explicitly determine whether \( \mathcal{H}_0(|\psi\rangle) \) is one-dimensional or not.

4.2. Practical stabilization under QL constraints

From a practical standpoint, a potential application of our results on the DQLS property of random pure states is to enable arbitrarily accurate approximate stabilization of target pure states that are otherwise non-DQLS. We illustrate the basic idea through a QIP-motivated example.
Consider the 4-qubit GHZ state, \( |\psi_{\text{GHZ}}\rangle = \frac{|0000\rangle + |1111\rangle}{\sqrt{2}} \), with respect to three-body neighborhoods \( \mathcal{N} = \{N_{123}, N_{234}\} \). The GHZ state belongs to the measure zero set of non-DQLS states in this setting [11], although, as we know from theorem 3.10, random pure states of 4-qubits are generically DQLS relative to \( \mathcal{N} \). We now show that one can always find a DQLS pure state \( |\psi_\varepsilon\rangle \), such that \( |\psi_{\text{GHZ}}\rangle \) can be approximately stabilized by QL dissipation with arbitrary accuracy \( \varepsilon > 0 \). To do so, pick a random unitary \( U \in U(16) \) with respect to the Haar measure and, for fixed \( \varepsilon \), define the integer \( n \equiv \lceil \frac{1}{\varepsilon} \rceil \). Correspondingly, let \( U_\varepsilon \equiv U^n \) and \( |\psi_\varepsilon\rangle \equiv U_\varepsilon |\psi_{\text{GHZ}}\rangle \). Since the eigenphases of \( U \) are uniformly distributed in \( [0, 2\pi) \), the eigenphases of \( U_\varepsilon \) are uniformly random within \( [0, 2\pi/n) \), with the associated eigenvectors being unchanged [33]. The set of pure states \( |\psi_\varepsilon\rangle \) that are generated from the unitaries following this modified distribution forms a finite measure subset of the pure states in \( \mathcal{H} \), and it thus follows that such states are generically DQLS. We now claim that, with probability one, the preparation fidelity

\[
F \equiv |\langle \psi_{\text{GHZ}} | (I - U_\varepsilon) |\psi_{\text{GHZ}}\rangle|^2 \geq 1 - \varepsilon \frac{\|H\|_{\text{op}}}{1 - \varepsilon \|H\|_{\text{op}}} \approx 1 - 2\varepsilon \|H\|_{\text{op}} + O(\varepsilon^2).
\]

Thus, \( \|I - U_\varepsilon\|_{\text{op}} \leq \varepsilon \|H\|_{\text{op}} + O(\varepsilon^2) \). By using standard properties of operator norm, the desired result follows:

\[
F = |1 - \langle \psi_{\text{GHZ}} | (I - U_\varepsilon) |\psi_{\text{GHZ}}\rangle|^2 \geq 1 - \varepsilon \frac{\|H\|_{\text{op}}}{1 - \varepsilon \|H\|_{\text{op}}} \approx 1 - 2\varepsilon \|H\|_{\text{op}} + O(\varepsilon^2).
\]

A similar analysis may be carried out for any pure state that belongs to the zero measure set of non-DQLS states in a setting where pure states are generically DQLS.
4.3. Generic pure states as ground states of frustration-free Hamiltonians

As we remarked (corollary 2.2), DQLS states may be physically characterized as unique ground states of FF QL parent Hamiltonians. Thus, our results may be equivalently interpreted as identifying some QL notions under which almost any pure state may arise as the unique ground state of a FF Hamiltonian—and, likewise, many other QL notions for which such a description is not possible. In the spirit of practical stabilization as discussed above, it also follows that non-DQLS states in those settings for which random pure states are generically DQLS may be written as approximate unique ground states of some FF QL parent Hamiltonian.

When generic states are not DQLS because of the no-go theorem 4.1, we can actually draw a stronger conclusion. As the proof of the above no-go shows, when neighborhoods are too small, in the sense that they all contain no more than half of the subsystems, the DQLS subspace coincides with the entire \( H \). We now show that the following property holds for all pure states:

**Proposition 4.4.** Let \( H = \sum_k H_k = \sum_k h_{N_k} \otimes I_{\overline{N_k}} \) be any FF Hamiltonian QL relative to \( N = \{N_k\} \). If \( |\psi\rangle \) is a ground state of \( H \), its DQLS subspace \( \mathcal{H}_0(|\psi\rangle) \) is contained in the ground-state space of \( H \).

**Proof.** Without loss of generality, we can scale the ground-state energy of each \( h_{N_k} \) such that it coincides with zero, the same holding for \( H \), thanks to its FF property. Since \( H|\psi\rangle = 0 \) implies that \( (h_{N_k} \otimes I_{\overline{N_k}})|\psi\rangle = 0 \), for all \( k \), we also have that \( \text{tr}(h_{N_k} \rho_{N_k}) = 0 \). Using the fact that both \( h_{N_k} \) and \( \rho_{N_k} \) are positive semi-definite, it also follows that

\[
\text{supp}(\rho_{N_k}) = \Sigma_{N_k}(|\psi\rangle) \subseteq \ker(h_{N_k}), \quad \forall k. \tag{24}
\]

Recall that the DQLS subspace of \( |\psi\rangle \) is given by \( \mathcal{H}_0(|\psi\rangle) = \bigcap_k (\Sigma_{N_k}(|\psi\rangle) \otimes \mathcal{H}_{\overline{N_k}}) \). For any \( |\phi\rangle \in \mathcal{H}_0(|\psi\rangle) \), \( |\phi\rangle \in \Sigma_{N_k}(|\psi\rangle) \otimes \mathcal{H}_{\overline{N_k}} \), for all \( k \). It can then be verified that \( \Sigma_{N_k}(|\phi\rangle) \subseteq \Sigma_{N_k}(|\psi\rangle) \), for all \( k \). Using equation (24), this shows that \( \Sigma_{N_k}(|\phi\rangle) \subseteq \ker(h_{N_k}) \), and hence \( H|\phi\rangle = 0 \), for any \( |\phi\rangle \in \mathcal{H}_0(|\psi\rangle) \). Accordingly, \( \mathcal{H}_0(|\psi\rangle) \subseteq \ker(H) \).

From the above observations, we conclude that for the multipartite settings where theorem 4.1 applies, no FF QL Hamiltonian other than a trivial (scalar) one can have a generic pure state in its ground-state space. This is to be expected in view of the fact that since generic states have full-rank neighborhood RDMs, each term \( h_{N_k} \) that has the corresponding RDM in its ground space can only be the identity, by equation (24).

**Remark 4.5.** We have already commented on the applicability of the extended no-go theorem 4.1 to special classes of pure states with full-rank RDMs in remark 4.2. As a consequence of proposition 4.4, these states are also not expressible as the exact ground states of any non-trivial FF QL Hamiltonians with appropriate neighborhood structure. Interestingly, the fact that qubit graph states are never exact ground states of non-trivial two-body FF Hamiltonians recovers a known result independently established in [34].

4.4. QL stabilization beyond generic pure states

While our main emphasis has been on generic pure states, it is worth stressing that the key mathematical features upon which our conclusions rest may hold for different sets of target states. In the tripartite setting, corollary 3.5 shows that when DQLS is a measure zero property
relative to $N_m$, arbitrary pure states with full-rank RDMs $\rho_{N_m}, \rho_{N_m}$ on subsystems $a$ and $c$ also behave the same way. Similarly, remark 4.2 demonstrates that quantum states with full-rank neighborhood RDMs $\rho_{N_{\lambda}}$ fall under the reach of the extended no-go theorem 4.1 and thus are non-DQLS.

In a similar spirit, since generating random pure states on $\mathcal{H} \simeq \mathbb{C}^d = (\mathbb{C}^2)^\otimes N$ is well known to entail resources that scale exponentially in $N$ [1], ‘pseudo-random’ quantum states have been extensively analyzed in QIP [27, 28, 36, 37], that share only some properties with the former yet suffice for relevant tasks. Loosely speaking, a quantum state $t$-design is a subset of states that approximates the Haar distribution in that a pure state sampled from the design is indistinguishable from a Haar-sampled random pure state, given access to at most $t$ copies of the state [28]. More convenient to our scope is an equivalent definition of a $t$-design in terms of polynomials on pure-state amplitudes [38]. Let $|\psi\rangle = \sum_{i=1}^d \alpha_i |i\rangle$ with respect to the standard basis for $\mathbb{C}^d$, and let $P_{(t)}(\psi) \equiv P(\alpha_1, \ldots, \alpha_d, \alpha_1^*, \ldots, \alpha_d^*)$ be a complex polynomial in $2d$ variables that is balanced in the sense that it has degree at most $t$ in both $\alpha_1, \ldots, \alpha_d$ and $\alpha_1^*, \ldots, \alpha_d^*$. A probability distribution $(p_i, |\phi_i\rangle)$ is called an exact quantum state $t$-design if the expected value of any $t$-order balanced polynomial $P_{(t)}(\psi)$ over the set coincides with the expectation over all (normalized) pure states in $\mathcal{H}$,

$$\int P_{(t)}(\psi) \, d\psi = \sum_i p_i P_{(t)}(\phi_i), \quad \sum_i p_i = 1,$$

where the integral is taken with respect to the Haar-induced (Fubini–Study) measure.

Our claim is that, for sufficiently large $t$ and suitable subsystem dimensions, pure target states sampled from a $t$-design share the DQLS properties of fully random pure states, to high probability. In essence, this follows from the fact that suitable concentration bounds hold for the statistical moments of $t$-designs [39]. Consider first a tripartite system on $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$, with neighborhood structure $N_m = \{N_{ab}, N_{bc}\}$, and let $|\phi_i\rangle \in \mathcal{H}$ be sampled from a $t$-design. The first relevant bound constraints the entropy across any bipartition: specifically, by using lemma 2.7 in [39] for an exact design ($\epsilon = 0$), one may show that the probability for the von Neumann entropy of subsystem $a$ across the tri-partition $a|bc$ (say, $S(\rho_a)$), to deviate from its maximum value by more than $\delta > 0$ is upper-bounded as

$$\mathbb{P}[*S(\rho_a) - S_{\max}(\rho_a)] \geq \delta \leq \left( \frac{d_a}{Cd_b d_c} t \right)^{1/8},$$

where $\delta = \delta + d_a/(d_b d_c \ln 2), \alpha > 0$, and the constant $C \approx (4/9 \pi^2)(2^\alpha - 1)^2$. The above bound implies that, with high probability, $|\phi_i\rangle$ has full-rank RDM on subsystem $a$, with a similar result holding across the bi-partition $c|ab$. Similarly, theorem 1.2 in [39] shows that, if $P_{(k,k)}(\cdot)$ is any balanced polynomial of degree at most $k$, its expectation is close to its Haar expectation value with high probability, provided that $t \geq 2k$.

Suppose now that the subsystem dimensions satisfy $d_c < d_a d_b$ (note that $d_a < d_b d_c$ is automatically satisfied in our setting), so that, by theorem 3.4, DQLS states form either a measure zero or a measure one set. In the former case, thanks to the bound in equation (25), corollary 3.5 is applicable to $|\phi_i\rangle$ and it thus follows that the state is not DQLS, with high probability. When instead the DQLS property is generic relative to $N_m$, the polynomial $D(\cdot)$ defined in equation (15) differs from zero. By construction, $D(\phi_i)$ is a balanced polynomial of degree at most $(d_a^2 + d_c^2 - 1)$ in terms of $|\phi_i\rangle$ in the standard basis, thus its expectation value over the $t$-design coincides with that of the Haar measure, as long as $t \geq (d_a^2 + d_c^2 - 1)$. Since this expectation value is strictly positive when Haar-random pure states are DQLS with probability one, $D(\phi_i)$ is non-zero, with high probability, due to theorem 1.2 in [39]. It then follows that
5. From quasi-local stabilization to uniquely determined quantum states

5.1. DQLS versus UDA pure states

As mentioned in the introduction, the task of stabilizing a globally specified quantum state in the presence of QL constraints is naturally related to aspects of the quantum marginal problem and the general theme of how ‘parts’ relate to the ‘whole’ in multipartite settings. Mathematically, a direct connection between a DQLS state and its set of neighborhood RDMs is established in theorem 2.1: the intersection of the extended Schmidt spans or, equivalently, the supports of the RDMs (extended to a global state by appending identities of the proper dimensions) must return the support of the target alone. Since the relevant set of RDMs in the DQLS problem is derived from the given target pure state $|\psi\rangle$, the ‘existence’ part of the quantum marginal problem is automatically satisfied. Notwithstanding, knowledge about the QL stabilizability properties of $|\psi\rangle$ may shed light onto whether and how the state may be uniquely reconstructed solely based on QL information.

Following established terminology [40], given a list of density operators $\{\rho_k\}$, we say that $|\psi\rangle$ is uniquely determined among all pure states (UDP) or uniquely determined among all states (UDA) if no other pure or, respectively, arbitrary quantum state has the same $\{\rho_k\}$ as its RDMs (or marginals); for example, k-body RDMs have the form $\rho_{i_1\cdots i_k} = \text{Tr}_{i_{k+1},\ldots,i_n}(|\psi\rangle\langle\psi|)$, $i_j \in \{1, \ldots, N\}$. While the UDA property clearly implies UDP, it was proved recently that the converse does not hold true in general [29]. It is also known that a DQLS state is always UDA by its neighborhood RDMs [12]; for later reference, we re-establish this result in the present context.

**Theorem 5.1.** [12] If a multipartite pure state $|\psi\rangle \in \mathcal{H}$ is DQLS relative to the neighborhood structure $\mathcal{N} = \{\mathcal{N}_j\}_{j=1}^M$, then $|\psi\rangle$ is UDA by the set of all its neighborhood RDMs $\{\rho_{\mathcal{N}_j}\}$.

**Proof.** We first show that any quantum state $\tau \in \mathcal{D}(\mathcal{H})$ with the same set of RDMs $\{\text{Tr}_{\mathcal{N}_j}(\tau) = \rho_{\mathcal{N}_j}\}$ as $|\psi\rangle\langle\psi|$ is such that $\text{supp}(\tau) \subseteq \mathcal{H}_0(|\psi\rangle)$. Express $\tau \equiv \sum_i \rho_i |\phi_i\rangle\langle\phi_i|$ in its eigenbasis, and let $|\phi\rangle \equiv \sum_i \sqrt{p_i} |\phi_i\rangle |i\rangle \in \mathcal{H} \otimes \mathcal{H}$ be a purifcation of $\tau$, with $\{|i\rangle\}$ forming an orthonormal basis in the ancilla space $\mathcal{H}$. Also, let $|\overline{\phi}\rangle \equiv |\psi\rangle |k\rangle$, with $|k\rangle \in \mathcal{H}$. For fixed $j$, consider the $\mathcal{N}_j$-bi-partition of $\mathcal{H}$. By assumption, $\text{Tr}_{\mathcal{N}_j,\overline{\mathcal{N}_j}}(|\phi\rangle \langle\phi|) = \text{Tr}_{\mathcal{N}_j,\overline{\mathcal{N}_j}}(|\overline{\phi}\rangle \langle\overline{\phi}|)$; it then follows that $(I_{\mathcal{N}_j} \otimes U_{\mathcal{N}_j,\overline{\mathcal{N}_j}})|\overline{\phi}\rangle = |\phi\rangle$, for all $j$, where $U_{\mathcal{N}_j,\overline{\mathcal{N}_j}} \in B(\mathcal{H}_{\mathcal{N}_j} \otimes \mathcal{H}_{\overline{\mathcal{N}_j}})$ is a unitary transformation. Furthermore, each eigenvector of $\tau$ with non-zero eigenvalue is expressible as $|\phi\rangle \equiv (I_{\mathcal{N}_j} \otimes X_{\overline{\mathcal{N}_j}})|\psi\rangle$, with $X_{\overline{\mathcal{N}_j}} = \sum_i U_{\mathcal{N}_j,\overline{\mathcal{N}_j}} |i\rangle \langle i|$, for all $p_i \neq 0$, and for all $\mathcal{N}_j \in \mathcal{N}$. As a result, $|\phi\rangle \in \mathcal{H}_0(|\psi\rangle)$ for all $i$, by theorem 2.4, which implies $\text{supp}(\tau) \subseteq \mathcal{H}_0(|\psi\rangle)$. Since $|\psi\rangle$ is DQLS relative to $\mathcal{N}$, $\mathcal{H}_0(|\psi\rangle)$ is one-dimensional and thus $\tau = |\psi\rangle\langle\psi|$, as claimed. 

In addition to showing that DQLS implies UDA, the above result can be useful in searching for UDA states relative to $\mathcal{N}$: as the proofs shows, if the given pure state $|\psi\rangle$ is not UDA, any state $\tau$, pure or mixed, that shares the same RDMs must be supported in the DQLS subspace of $|\psi\rangle$ relative to $\mathcal{N}$. Hence, characterizing the DQLS subspace of a non-UDA pure state $|\psi\rangle$ may
be useful in reducing the search space for other global states that are ‘joined’ or determined by the same RDMs as $|\psi\rangle$.

While DQLS implies UDA, the converse is not true in general: a counter-example is provided by the W-state on $N$-qubits, which is known to be UDA by the set of all its two-body RDMs [41], yet fails to be DQLS relative to any non-trivial neighborhood structure [11]. That the DQLS property is a strictly stronger requirement than UDA is to be expected, since DQLS implies that knowledge about the support of the neighborhood RDMs alone can fully specify the global state. Conversely, non-DQLS states that are UDA require complete information about the RDMs in order to uniquely specify the global state. Uniquely determined pure states are, in turn, a strict subset of all possible pure states: for instance, the GHZ-state on $N$-qubits, which is not DQLS relative to any non-trivial neighborhood structure [11], is non-UDA (hence non-UDP) with respect to the set of $k$-body RDMs for any $k < N$. A pictorial summary is given in figure 3.

5.2. Generic pure-state reconstruction in multipartite quantum systems

Studying a multipartite quantum state based on the properties of its RDMs yields important insight on how correlations are distributed between different parties in the system: in fact, if an $N$-party state can be uniquely reconstructed using a given set of $k$-party RDMs ($k < N$), then the correlations it contains are limited to subsets of these $k$ parties [42]. Early results in this direction are due to Linden et al [7], who established the UDA nature of generic pure states of 3-qubit systems with respect to their 2-party marginals. These studies were subsequently extended to arbitrary 3-qudit systems in [4, 43, 44] and to 4-qudit systems in [45]. In particular, Linden and Wootters showed that (i) a generic $|\psi\rangle$ in $d_a \times d_b \times d_c$ dimensions (with $d_a \leq d_c$, w.l.o.g.), is UDA by $\rho_{ab}$ and $\rho_{bc}$, provided that $d_b \geq d_a + d_c - 1$ and, building on this result, that (ii) a generic $|\psi\rangle$ on $N$ qudits, with $N$ sufficiently large, is UDA by only two out of its $\binom{N}{k}$ $k$-body RDMs, where it suffices that $k \leq \lfloor 2N/3 \rfloor$. For $N = 3$, it is worth noting that, assuming validity of conjecture 3.11, the inequality in (i) implies $d_ad_b > d_c$, so that $|\psi\rangle$ is indeed DQLS with respect to $N_{ab}$, hence consistently UDA. For the case of $N$ qudits, the bound in (ii) was further improved to $k \leq \lceil N/2 \rceil$ in [43], at the cost of considering about $N/2$ such RDMs.

Here, we strengthen the multipartite result by showing that a generic $N$-qudit pure state is UDA by only two of its $k$-body RDMs, with $k \approx N/2$, directly leveraging DQLS arguments. This affords two important advantages: not only does our approach avoid the need for full information about the RDMs, but it is constructive in nature, offering an explicit procedure for obtaining the global state via the DQLS condition in equation (4).

**Proposition 5.2.** A generic multipartite pure state on $N$ qudits ($N \geq 3$) is UDA by its RDMs on any two neighborhoods $N_{ab}$ and $N_{bc}$ such that $N_{ab} \cup N_{bc} = \{1, 2, \ldots, N\}$, where it suffices that each contains no more than $[\frac{N+1}{2}]$ subsystems ($d > 2$) or no more than $[\frac{N+3}{2}]$ subsystems ($d = 2$).

**Proof.** Let us organize the system of $N$ qudits into three subsystems $a, b$ and $c$, such that their dimensions are given by

- $d^{\frac{N-1}{2}} \times d^2 \times d^{\frac{N+1}{2}}$ for $N$ even, $d \geq 2$.
- $d^{\frac{N-1}{2}} \times d \times d^{\frac{N+1}{2}}$ for $N$ odd, $d > 2$.
- $d^{\frac{N-3}{2}} \times d^2 \times d^{\frac{N+1}{2}+1}$ for $N$ odd, $d = 2$. 

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From theorem 3.10, we know that a generic tripartite pure state in $d_a \times d_b \times d_c$ dimensions ($d_b > 2$) is DQLS relative to the neighborhood structure $N_{\text{tri}} = \{N_{ab}, N_{bc}\}$ when $d_a = d_c$ or $d_a = d_c/2$. Hence, by theorem 5.1, in order to uniquely reconstruct a generic pure state in these dimensions, knowledge of the RDMs $\{\rho_{ab}, \rho_{bc}\}$ is sufficient. The separate consideration of the $d = 2$ case (for $N$ odd) is due to the fact that generic pure states in $d_a \times 2 \times d_c$ dimensions are not DQLS when $d_a = d_c$, due to theorem 3.7.

It is important to appreciate that both the neighborhood structure and the order of the subsystems are, in general, important in determining the DQLS property of $|\psi\rangle$. For example, a generic tripartite pure state in $2 \times 2 \times 5$ dimensions is not DQLS relative to $N'_{\text{tri}} = \{N_{ab}, N_{bc}\}$, by theorem 3.2. But, $|\psi\rangle$ is DQLS relative to $N'_{\text{tri}}' = \{N_{ac}, N_{cb}\}$ (where $b$ and $c$ have been swapped), as generic states in $2 \times 5 \times 2$ dimensions are DQLS by theorem 3.10. While $N_{\text{tri}}$ and $N'_{\text{tri}}'$ are not equivalent from a control-theoretic standpoint as they represent different constraints, subsystems’ ordering need not be important for quantum state reconstruction in typical settings where all $k$-body RDMs are assumed to be available. Hence, when inferring the UDA property from DQLS results in such cases, it is allowable to permute subsystems as needed. With reference to the above example, choosing $N'_{\text{tri}}'$ allows one to conclude that $|\psi\rangle$ is UDA by all two-body RDMs—in particular, $\{\rho_{ac}, \rho_{bc}\}$ alone suffice.

One may verify that the results in proposition 5.2 extends those of [44] which, while also being constructive, are restricted to UDP. Further to that, as anticipated, equation (4) requires information only about the support of the relevant RDMs, as opposed to full information about these RDMs, as in previous works. We may thus summarize our result about generic-state reconstruction as follows: A generic pure state on $N$ qudits ($N > 3$) is UDA by the support of any of its two $k$-body RDMs, where $k \approx N/2$ and the reconstruction procedure is given by equation (4).

6. Conclusion

We have characterized the extent to which generic pure states on a finite-dimensional multipartite quantum system can be the unique asymptotic equilibrium state of purely dissipative Markovian dynamics, subject to specified locality constraints. Beside further contributing to the understanding of fundamental properties of both generic quantum states and quantum stabilization problems, we have shown how our results have implications for a number of
problems relevant to QIP. Specifically, on the one hand we have addressed QL stabilization of pure states drawn from a state $t$-design, as opposed to the full Haar measure, and approximate QL stabilization of pure states that do not stabilizable in exact form; on the other hand, we have revisited uniquely determined quantum states and quantum state tomography based on reduced density matrices in the light of the underlying stabilizability properties.

A number of related questions are prompted by the present analysis. First, it may be interesting to understand whether (and, if so, why) the fact that stabilizability occurs either with probability one or with probability zero is a feature exhibited by more general constrained stabilization settings than the QL tripartite one we have demonstrated here—both in regard to $N > 3$ and the possibility of different resource constraints (for instance, symmetry-constrained dissipative dynamics), as well as to in connection to ‘phase transition’ phenomena that are known to occur for entanglement sharing and quantum marginals [46]. Second, by making contact with the analysis of mixed-state QL stabilization in [12], it is natural to ask about stabilizability properties of random mixed states. In this case, one may consider the standard induced ensemble resulting from taking a partial trace over part of a random pure state or ensembles of random density matrices distributed according to other probability measures [1, 47], again beginning from the simplest setting of a tripartite target system.

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Note added in proof. After this paper was submitted for publication, UDA properties of generic $N$-qudit pure states have been independently analyzed in [48]. In particular, the authors present there a result that is equivalent to our proposition 5.2, except for odd $N$ and $d = 2$ (that is, odd number of qubits), in which case our proof technique yields two slightly larger RDMs ($\lfloor N+3/2 \rfloor$-body rather than $\lceil N+1/2 \rceil$-body as in [48]), while still only requiring knowledge of their support.

Appendix

A.1. Properties of generic matrices

Lemma A.1. Let $A \in \mathbb{M}^{d \times d}$ be a generic matrix. If $B$ is any matrix such that $[A,B] = 0$, then $B = P(A)$, where $P(A)$ is a polynomial in $A$ with degree at most $(d - 1)$.

Proof. The characteristic polynomial of the generic square matrix $A$ has arbitrary coefficients, hence its roots are all distinct and $A$ is diagonalizable. Let $\{\mu_i(A)\}_{i=1}^d$ denote the spectrum of $A$ with $\mu_i(A) \neq \mu_j(A)$ when $i \neq j$. Since $[A,B] = 0$, $A$ and $B$ are diagonal in the same basis. Let $\{\mu_i(B)\}_{i=1}^d$ be the spectrum of $B$ and $\vec{b} = [\mu_1(B), \ldots, \mu_d(B)]^T$. In a basis where both $A$ and $B$ are diagonal, their non-zero entries are related as $\mu_j(b) = P(\mu_j(A))$, for all $j$, if $B$ is a polynomial in $A$ where $P(\cdot)$ is of some finite degree. In order to show this, construct the Vandermonde matrix.
\[ V_A \equiv \begin{bmatrix}
1 & \mu_1(A) & \ldots & \mu_1^{d-1}(A) \\
1 & \mu_2(A) & \ldots & \mu_2^{d-1}(A) \\
\vdots & \vdots & \ddots & \vdots \\
1 & \mu_d(A) & \ldots & \mu_d^{d-1}(A) 
\end{bmatrix} \]

Then, \( \det(V_A) = \prod_{1 \leq i \leq d} (\mu_i(A) - \mu_j(A)) \), which is non-zero as all the \( \mu_j(A) \)’s are distinct. Accordingly, \( V_A \) is invertible. Thanks to the Cayley–Hamilton theorem, any polynomial in \( A \) is expressible as \( P(A) = \sum_{k=0}^d c_k A^k \), \( c_k \in \mathbb{C} \). In a basis where \( A \) (hence \( P(A) \)) is diagonal, the non-zero entries of \( P(A) \) are given by the entries of the vector \( V_A \tilde{c} \), where \( \tilde{c} = [c_0, \ldots, c_{d-1}]^T \). In order to verify that the same holds for \( B \), choose in particular \( \tilde{c} = V_A^{-1} \tilde{b} \), so that \( \mu_j(B) = P(\mu_j(A)) \) for all \( j \); thus, \( B = P(A) \), where \( P(\cdot) \) is a polynomial of degree at most \( (d - 1) \), as claimed.

**Lemma A.2.** Let \( A \in \mathbb{M}^{d \times d} \) be a generic matrix. \( \tilde{b} \in \mathbb{C}^d \) is a generic vector independent of \( A \). If \( P(A) \) is a polynomial in \( A \) satisfying \( P(A)\tilde{b} = \tilde{0} \), then \( P(A) \) is the zero polynomial.

**Proof.** Since \( A \) is a generic square matrix, it has \( d \) distinct eigenvalues and is diagonalizable with its set of left/right eigenvectors forming a basis for \( \mathbb{C}^d \). Let \( \{a_j\}_{j=1}^d \) denote the spectrum of \( A \) and \( \{\tilde{\alpha}_j\}_{j=1}^d \) the corresponding right-eigenvectors, respectively, such that \( A \tilde{\alpha}_j = a_j \tilde{\alpha}_j \), for all \( j \). Let \( P(A) = \sum_{k=0}^{d-1} p_k A^k \), for \( p_k \in \mathbb{C} \), for all \( k \). Owing to the generic nature of \( \tilde{b} \), \( \tilde{b} = \sum_{j=1}^d b_j \tilde{\alpha}_j \), for \( b_j \in \mathbb{C} \) and \( b_j \neq 0 \), for all \( j \). Following our assumption, \( \tilde{b} \) belongs to the right kernel of \( P(A) \). That is,

\[ P(A)\tilde{b} = \sum_{j=1}^d b_j \left( \sum_{k=0}^{d-1} p_k A^k \right) \tilde{\alpha}_j = \sum_{j=1}^d b_j \left( \sum_{k=0}^{d-1} p_k a_j^k \right) \tilde{\alpha}_j = \tilde{0}. \]

Since \( \{\tilde{\alpha}_j\} \) is a linearly independent set, it follows that \( \sum_{k=0}^{d-1} p_k a_j^k = 0 \) for all \( j \), which in turn shows that \( P(A) = \tilde{0} \).

**Lemma A.3.** Let \( A, B \in \mathbb{M}^{d \times d} \) be two generic matrices, independent of each other. The only non-trivial matrix that commutes with both of them simultaneously is the identity.

**Proof.** Let \( X \) be a matrix that commutes simultaneously with \( A \) and \( B \). Due to the generic nature of \( A \), it has all distinct eigenvalues and, following lemma A.1, \( X = P(A) \), which is a polynomial of degree \( (d - 1) \) or lower. Also, \([P(A), B] = \tilde{0}\). Let \( \{a_j\}_{j=1}^d \) denote the spectrum of \( A \) and \( \{\tilde{\alpha}_j\}_{j=1}^d \) the corresponding right-eigenvectors, such that \( A \tilde{\alpha}_j = a_j \tilde{\alpha}_j \), for all \( j \). The latter form a basis for \( \mathbb{C}^d \). Let \( P(A) = \sum_{k=0}^{d-1} p_k A^k \), for all \( k \). If \( \tilde{\beta} \) denotes a right-eigenvector of \( B \) with eigenvalue \( b \), \( BP(A)\tilde{\beta} = bP(A)\tilde{\beta} \). Due to the generic nature of \( B \), all its eigenvalues are distinct and therefore, \( P(A)\tilde{\beta} = \lambda \tilde{\beta} \) for some \( \lambda \in \mathbb{C} \). The eigenvectors of two independent generic matrices are also independent of each other, and hence \( \tilde{\beta} = \sum_{j=1}^d c_j \tilde{\alpha}_j \), for \( c_j \in \mathbb{C} \) and \( c_j \neq 0 \), for all \( j \). Hence,

\[ P(A)\tilde{\beta} - \lambda \tilde{\beta} = \sum_{j=1}^d c_j \left( \sum_{k=0}^{d-1} p_k a_j^k - \lambda \right) \tilde{\alpha}_j = \sum_{j=1}^d c_j \left( \sum_{k=0}^{d-1} p_k a_j^k - \lambda \right) \tilde{\alpha}_j = \tilde{0}. \]
This implies that $\sum_{k=1}^{d} p_k d_k^j = \lambda$ for all $j$, because $\{\alpha_j\}$ are linearly independent. This in turn shows that $P(A) = \lambda I$, that is, $X$ is proportional to the identity.

A.2. Technical proofs

We present here complete proofs that were not included in the main text for brevity.

**Theorem A.1.** Let $|\psi\rangle$ be a generic tripartite pure state in $\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$, with dimension $d_a \times d_b \times d_c$ and neighborhood structure $N_{\text{tri}} = \{N_{ab}, N_{bc}\}$. When $d_b = 2$,

i. $|\psi\rangle$ is not DQLS relative to $N_{\text{tri}}$ when $d_a = d_c$. The dimension of $\mathcal{H}_0(|\psi\rangle)$ is $d_a$.

ii. $|\psi\rangle$ is not DQLS relative to $N_{\text{tri}}$ when $d_c > d_a$, except for $d_c = d_a + 1$. The dimension of $\mathcal{H}_0(|\psi\rangle)$ is $\min\{d_a^2, (d_c - d_a)^2\}$.

**Proof.** We prove each statement separately.

i. Let $d_a = d_c \equiv d$. The DQLS nature of generic pure states in $d \times 2 \times d$ dimensions is determined by the solution space of equation (21). Since $A_{00}$ is a generic square matrix, it has all distinct eigenvalues. For this reason, any matrix that commutes with $A_{00}$ belongs to the space spanned by the first $(d - 1)$ powers of $A_{00}$, including the identity (see lemma A.1). Thus, $[X_a, X_c = X_d^T]$ that satisfy equation (11) belongs to this $d$-dimensional space. According to proposition 3.3, the DQLS subspace $\mathcal{H}_0(|\psi\rangle)$ is $d$-dimensional ($d > 1$), hence $|\psi\rangle$ is not DQLS.

ii. In this case, $d = d_c - d_a > 0$. We only need to focus on the case where $d < d_a$, $d \neq 0$, that is not covered under the no-go theorem 3.2. Let us reorganize equations (19) and (20) into a compact form by vectorizing the matrices $X_a, X_b$, and $X_{00}$, i.e. by stacking up their columns and using the property $\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X)$ [32]:

$$
C\vec{v} \equiv \begin{bmatrix}
A_{00}^T \otimes I_{00} & -I_{\bar{d}\bar{d}} \otimes A_{0\bar{d}} & 0 \\
A_{00}^T \otimes I_{00} - I_{00} \otimes A_{00}^T & 0 & -I_{00} \otimes A_{0\bar{d}}
\end{bmatrix}
\begin{bmatrix}
\text{vec}(X_a) \\
\text{vec}(X_{\bar{d}})
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

(A.1)

Here, $I_{00}, I_{\bar{d}\bar{d}}$ are the $d_a \times d_a$ and $\bar{d} \times \bar{d}$ identity matrices, respectively; $C$ is called the coefficient matrix, and $\vec{v}$ is formed by stacking the columns of $\text{vec}(X_a), \text{vec}(X_{\bar{d}})$, and $\text{vec}(X_{00})$. This homogeneous system of linear equations has $dd_a + d_a^2$ constraints and $dd_a + d_a^2 + d^2$ unknowns. Evidently, it is under-constrained and has at least one non-trivial solution (corresponding to $X_a, X_{\bar{d}}$ proportional to the identity and $X_{00}$ being the zero matrix). The total number of linearly independent constraints in equation (A.1), which is given by the row rank of $C$, determines the dimension of the solution space. To determine the row rank of $C$, observe that the first $d_a^2$ rows of $C$ (upper block) are linearly independent of the rest (lower block), by construction. For the upper block, linearly dependent rows, if any, imply that there exist some non-zero vector $\text{vec}(P)$ such that

$$
\text{vec}(P)^T[A_{0\bar{d}}^T \otimes I_{00} - I_{\bar{d}\bar{d}} \otimes A_{0\bar{d}}] = 0.
$$

This condition is equivalent to $P^TA_{0\bar{d}} = 0$ and $A_{0\bar{d}}P^T = 0$, where $P$ is the matrix form of $\text{vec}(P)$. That is, columns (rows) of $P$ belong to the left (right) kernel of $A_{0\bar{d}}$. Due to its generic nature, $A_{0\bar{d}}$ does not have a non-trivial right kernel when $\bar{d} < d_a$. Therefore, there exists no non-trivial $P$ that simultaneously satisfies the conditions given above and hence, the rows in the upper block of $C$ are linearly independent of each other. Similarly, linear dependencies of the lower block, if any, imply the existence of a vector $\text{vec}(Q)$ such that
\[
\text{vec}(Q)^T[A_{00}^T \otimes I_{00} - I_{00} \otimes A_{00}^T - I_{00} \otimes A_{10}] = 0,
\]

or, equivalently, the conditions: \([A_{00}, Q^T] = 0\) and \(Q^TA_{10} = 0\), satisfied for some matrix \(Q\). Any \(Q^T\) that commutes with the generic matrix \(A_{00}\) is such that \(Q^T = P(A_{00})\), the latter being some polynomial in \(A_{00}\) by lemma A.1. By invoking lemma A.2, it also follows that if \(P(A_{00})A_{10} = 0\) for another generic matrix \(A_{10}\), then \(P(A_{00})\) is the zero polynomial (trivial solution). This shows that the rows in the lower block of \(C\) are also linearly independent of each other. Thus, we conclude that the coefficient matrix \(C\) has maximal row rank. Hence, the homogeneous system in equation (A.1) has \(d_a d_b + d^2\) independent constraints versus \(d_a d_b + d^2\) variables, and the dimension of the solution space is \(d^2\).

When \(d = 1\), it is thus verified that the only non-trivial solution corresponds to \(X_a\) and \(X_c\) proportional to the identity in the respective spaces. Accordingly in \(d_a \times 2 \times (d_a + 1)\) dimensions, generic pure states are DQLS according to proposition 3.3. When \(d > 1\) instead (with \(d < d_a\)), proposition 3.3 shows that the dimension of \(H_d(\langle \psi |)\), is \(d^2 > 1\), therefore generic states are non-DQLS, as claimed.

\[\Box\]

**Theorem A.2.** Let \(|\psi\rangle\) be a generic tripartite pure state in \(H_a \otimes H_b \otimes H_c\), with neighborhood structure \(\mathcal{N}_{tr}\) = \(\{\mathcal{N}_{ab}, \mathcal{N}_{bc}\}\) and central subsystem dimension \(d_b \geq 2\). Then, \(|\psi\rangle\) is DQLS relative to \(\mathcal{N}_{tr}\) for the following combinations of subsystem dimensions:

i. \(d_a \times d_b \times d_a\)

ii. \(d_a \times d_b \times (d_a + 1)\)

iii. \(d_a \times d_b \times nd_a\), with \(1 < n < d_b\)

**Proof.** We prove each statement separately.

i. \(d_a \times d_b \times d_a\). This case is an extension to theorem 3.7(i), which shows that generic pure states in \(d_a \times 2 \times d_a\) dimensions are not DQLS relative to \(\mathcal{N}_{tr}\). Equation (21), which determined the DQLS nature of generic pure states when \(d_b = 2\), is now modified to \(X_a A_{10}^T = A_{10} X_a\), \(i = 1, \ldots , d_b - 1\). Due to lemma A.3, the only non-trivial matrix that commutes with two or more generic matrices simultaneously is the identity. Thus, the only choice of \(\{X_a, X_c = X_d\}\) that satisfy the set of equations (11) for \(d_b > 2\) is proportional to the identity, and by proposition 3.3, generic pure states in \(d_a \times d_b \times d_a\) dimensions are DQLS relative to \(\mathcal{N}_{tr}\), for any \(d_b > 2\).

ii. \(d_a \times d_b \times (d_a + 1)\). Following theorem 3.7(ii), generic pure states in \(d_a \times 2 \times (d_a + 1)\) dimensions are DQLS relative to \(\mathcal{N}_{tr}\). It then suffices to apply proposition 3.8 to show that generic pure states in \(d_a \times d_b \times (d_a + 1)\) dimensions are also DQLS relative to \(\mathcal{N}_{tr}\), for any \(d_b > 2\).

iii. \(d_a \times d_b \times nd_a\), \(1 < n < d_b\). When \(n \geq d_b\), these cases fall under the no-go theorem 3.2, and hence are not considered in this proof. We first prove that generic pure states in \(d_a \times 3 \times 2d_a\) dimensions are DQLS relative to \(\mathcal{N}_{tr}\). Consider the modified version of the set of equations (11) given by equations (22) and (23), for \(d_b = 3\). The block decomposition of the relevant matrices are given in table 1. In this case, \(A_{10}^T\) are generic square matrices since \(d = d_a\), and hence invertible. Eliminating \(X_{1d}\) in equation (22) for \(i = 1, 2\) leads to the condition \([A_{0d}^T(A_{10}^{-1})^{-1}, X_a] = 0\). Let \(Q \equiv A_{0d}^T(A_{10}^{-1})^{-1}\). Since \(Q\) is a generic matrix, lemma A.1 implies that \(X_a = P(Q)\), with \(P(\cdot)\) being a polynomial of degree at most \((d_a - 1)\). Equation (23) then rewrites...
\[ \hat{X}_{0d} = (A_{0d}^d)^{-1} (P(Q)A_{00}^d - A_{00}^d P(Q)), \]
\[ \hat{X}_{0d} = (A_{0d}^d)^{-1} (P(Q)A_{00}^d - A_{00}^d P(Q)). \]

We see that RHS of both these equations are equal to each other. Owing to the full-rank nature of the generic matrices that are involved in these equations, we now show that they are satisfied only by \( \hat{X}_{0d} = 0 \). For this purpose, observe that equating the LHS of the two equations we obtain \( Q [P(Q), A_{00}^d] [P(Q), A_{00}^d] = 0 \), where \( A_{00}^d = QA_{00}. \) Due to the generic, invertible nature of the matrices involved in this relationship, \( Q \) and \( (A_{00}^d - A_{00}^d) \) are independent of each other. Following lemma A.3, \( P(Q) = I \), otherwise this implies that a non-identity matrix \( P(Q) \) commutes with two independent, generic matrices \( Q \) and \( (A_{00}^d - A_{00}^d) \). Therefore, \( X_a = P(Q) = I \). This in turn shows that \( \hat{X}_{dd} = I \) and \( \hat{X}_{0d} = 0 \). Referring to table 1, these observations imply \( X_c = I \). Thus, by proposition 3.3, generic pure states in \( d_a \times 3 \times 2d_a \) dimensions are DQLS.

The above result is now extended with the help of propositions 3.8 and 3.9 to show that generic pure states in \( d_a \times d_b \times nd_a \) dimensions are DQLS relative to \( N_{q1} \), for any \( d_b \geq 3 \) and \( 1 < n < d_b \).

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