Schur Function Expansions and the Rational Shuffle Theorem

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Abstract

Gorsky and Negut introduced operators $Q_{m,n}$ on symmetric functions and conjectured that, in the case where $m$ and $n$ are relatively prime, the expression $Q_{m,n}(1)$ is given by the Hikita polynomial $H_{m,n}[X;q,t]$. Later, Bergeron-Garsia-Leven-Xin extended and refined the conjectures of $Q_{m,n}(1)$ for arbitrary $m$ and $n$ which we call the Extended Rational Shuffle Conjecture. In the special case $Q_{n+1,n}(1)$, the Rational Shuffle Conjecture becomes the Shuffle Conjecture of Haglund-Haiman-Loehr-Remmel-Ulyanov, which was proved in 2015 by Carlsson and Mellit as the Shuffle Theorem. The Extended Rational Shuffle Conjecture was later proved by Mellit as the Extended Rational Shuffle Theorem. The main goal of this paper is to study the combinatorics of the coefficients that arise in the Schur function expansion of $Q_{m,n}(1)$ in certain special cases.

Leven gave a combinatorial proof of the Schur function expansion of $Q_{2,2n+1}(1)$ and $Q_{2n+1,2}(1)$. In this paper, we explore several symmetries in the combinatorics of the coefficients that arise in the Schur function expansion of $Q_{m,n}(1)$. Especially, we study the hook-shaped Schur function coefficients, and the Schur function expansion of $Q_{m,n}(1)$ in the case where $m$ or $n$ equals 3.

Keywords: Macdonald polynomials, parking functions, Dyck paths, Rational Shuffle Theorem

1 Introduction

The Rational Shuffle Theorem, as a rational generalization of the Shuffle Theorem, comes from the study of the ring of diagonal harmonics. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two sets of $n$ variables. The ring of diagonal harmonics consists of those polynomials in $\mathbb{Q}[X,Y]$ which satisfy the following system of differential equations

$$\partial_{x_1}^a \partial_{y_1}^b f(X,Y) + \partial_{x_2}^a \partial_{y_2}^b f(X,Y) + \ldots + \partial_{x_n}^a \partial_{y_n}^b f(X,Y) = 0$$

for each pair of integers $a$ and $b$ such that $a + b > 0$. Haiman in [15] proved that the ring of diagonal harmonics has dimension $(n+1)^{n-1}$, and the bigraded Frobenius characteristic of the $S_n$-module of diagonal harmonics, $DH_n(X;q,t)$, is given by

$$DH_n(X;q,t) = \nabla e_n,$$

where $\nabla$ (nabla) is the symmetric function operator defined by Bergeron and Garsia [4], and $e_n$ is the elementary symmetric function of degree $n$.

Let $n$ be a positive integer. An $(n,n)$-Dyck path $P$ is a lattice path from $(0,0)$ to $(n,n)$ which always remains weakly above the main diagonal $y = x$. Given a Dyck path $P$, we can get an
(n, n)-parking function \( \pi \) by labeling the cells east of and adjacent to the north steps of \( P \) with integers \( \{1, \ldots, n\} \) such that the labels are strictly increasing in each column. The set of parking functions of size \( n \) is denoted by \( \mathcal{P}_n \). Figure 1 (a) gives an example of a (5, 5)-parking function.

Let \( F_\alpha[X] \) denote the fundamental quasi-symmetric function of Gessel [10] associated to the composition \( \alpha \), and let area, dinv, pides be statistics of parking functions. The Classical Shuffle Conjecture proposed by Haglund, Haiman, Loehr, Remmel and Ulyanov [14] gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of the ring of diagonal harmonics. The Shuffle Conjecture has been proved by Carlsson and Mellit [6] as the Shuffle Theorem that for all \( n \geq 0 \),

\[
\nabla e_n = \sum_{\pi \in \mathcal{P}_n} t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} F_{\text{pides}(\pi)}[X].
\]  

Let \( m \) and \( n \) be positive integers. An \((m, n)\)-Dyck path is a lattice path from \((0, 0)\) to \((m, n)\) which always remains weakly above the main diagonal \( y = \frac{m}{n}x \). An \((m, n)\)-parking function \( \pi \) is obtained by labeling the north steps of an \((m, n)\)-Dyck path in a similar way to the \((n, n)\) case. Figure 1 (b) gives an example of a (3, 5)-parking function. Let area, dinv, pides and ret be statistics of rational parking functions (will be defined later), then Bergeron, Garsia, Leven and Xin [5] extended the combinatorial side of the Shuffle Theorem to the extended Hikita polynomial

\[
H_{m,n}[X; q, t] := \sum_{\pi \in \mathcal{P}_{m,n}} [\text{ret}(\pi)] t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} F_{\text{pides}(\pi)}[X].
\]  

Gorsky and Negut introduced the symmetric function operator \( Q_{m,n} \) for \( m \) and \( n \) coprime in [11] (where they call the operator \( \tilde{Q}_{m,n} \)), and Bergeron, Garsia, Leven and Xin [5] generalized the operator \( Q_{m,n} \) so that the coprimality condition is removed, extending the algebraic side of the Shuffle Theorem from \( \nabla e_n \) to \( Q_{m,n}(1) \). The Extended Rational Shuffle Theorem is that, for any pair of positive integers \((m, n)\),

\[
Q_{m,n}(1) = H_{m,n}[X; q, t],
\]  

which was proved by Mellit [21].

A more important goal is to find the Schur function expansion of \( \nabla e_n \) since that would allow us to find the bigraded \( S_n \)-isomorphism type of the ring of diagonal harmonics, see [15]. More generally, we would like to find a combinatorial interpretation of the coefficients that arise in the Schur function expansion of \( Q_{m,n}(1) \). The main goal of this paper is to find such Schur function expansions in the case where \( m \) or \( n \) equals 3. The Schur function expansion of \( Q_{m,n}(1) \) in the case...
where \( m \) and \( n \) are coprime and either \( m \) or \( n \) equals \( 2 \) was given by Leven [19]. That is, let \([n]_{q,t}\) be the \( q,t \)-analogue of the integer \( n \) that

\[
[n]_{q,t} := \frac{q^n - t^n}{q - t} = q^{n-1} + q^{n-2}t + \cdots + t^{n-1},
\]

then Leven [19] gave a proof of the following theorem.

**Theorem 1** (Leven). For any integer \( k \geq 0 \),

\[
Q_{2k+1,2}(1) = H_{2k+1,2}[X; q, t] = [k]_{q,t}s_2 + [k + 1]_{q,t}s_{11},
\]

and

\[
Q_{2,2k+1}(1) = H_{2,2k+1}[X; q, t] = \sum_{r=0}^{k} [k + 1 - r]_{q,t}s_{2r+12k+1-2r}.
\]

We want to write the coefficients of Schur functions in \( Q_{m,n}(1) \) as symmetric functions in variables \((q,t)\). A \((q,t)\)-Schur function \( s_\lambda(q,t) \) is non-zero only if the partition \( \lambda \) has no more than two parts, and

\[
s_{(a,b)}(q,t) = (qt)^b[a-b+1]_{q,t}.
\]

Thus, the right hand side of Equations \((6)\) and \((7)\) can be written as \( s_{k-1}(q,t)s_2 + s_k(q,t)s_{11} \) and \( \sum_{r=0}^{k} s_{k-r}(q,t)s_{2r+12k+1-2r} \) respectively.

In fact, the coefficients of Schur functions in \( Q_{m,n}(1) \) as \((q,t)\)-Schur functions are Schur positive due to the work of Bergeron [2] that the ring of diagonal harmonics forms both an \( S_n \)-module and a \( GL_2 \)-module which commute with each other.

By the Extended Rational Shuffle Theorem formulated in [5], we can extend Leven’s theorem to compute the Schur function expansion of \( Q_{m,n}(1) \) where either \( m \) or \( n \) is equal to \( 2 \), but \( m \) and \( n \) are not coprime. We give a proof of the following theorem in Section 2.2.

**Theorem 2.** For any integer \( k > 0 \),

\[
Q_{2k,2}(1) = H_{2k,2}[X; q, t] = (s_{k-1}(q,t) + s_{k-2}(q,t))s_2 + (s_k(q,t) + s_{k-1}(q,t))s_{11},
\]

and

\[
Q_{2,2k}(1) = H_{2,2k}[X; q, t] = \sum_{r=0}^{k} (s_{k-r}(q,t) + s_{k-r-1}(q,t))s_{2r+12k+1-2r}.
\]

The coefficient at \( s_{1^n} \) in \( Q_{m,n}(1) \) is known as the rational \( q,t \)-Catalan number, computed by Gorsky and Mazin [12] for the case \( n = 3 \) and studied by Lee, Li and Loehr [18] for the case \( n = 4 \). The coefficients at hook-shaped Schur functions were discussed by Armstrong, Loehr and Warrington [1].

In this paper, we explore the combinatorics of the Schur function expansion of \( Q_{m,n}(1) \) in several special cases.

In Section 2, we provide backgrounds of the problem in both combinatorial side (parking function side) and algebraic side (symmetric function side). Then in Section 3, we prove several symmetries of the coefficients of Schur functions in the Extended Rational Shuffle Theorem. Let \([s_\lambda]_{m,n}\) be the coefficient of the Schur function \( s_\lambda \) in both \( Q_{m,n}(1) \) and \( H_{m,n}[X; q, t] \), then we can combinatorially prove

**Theorem 3.** For all \( m, n > 0 \) and \( \lambda' \vdash (n - am) \),
(a) \([s_1^n]_{m,n} = [s_n]_{m+n,n}\),
(b) \([s_{m,n}^\lambda]_{m,n} = [s_\lambda]_{m,n-am}\),
(c) \([s_{k1n-k}]_{m,n} = [s_{k1m-k}]_{n,m}\).

Nakagane [22] obtained a similar result to Theorem 3 independently.

In Section 4, we prove the following theorem to give explicit formulas for the Schur function expansion of \(Q_{m,3}(1)\) from both symmetric function side and combinatorial side.

**Theorem 4.** For any integer \(k \geq 0\),

\[
Q_{3k+1,3}(1) = H_{3k+1,3}[X; q, t] = \left(\sum_{i=0}^{k-1} s_{(k+2i-1, k-1)}(q, t)\right) s_3 + \left(\sum_{i=0}^{k-1} (s_{(k+2i,k-1)}(q, t) + s_{(k+2i+1,k-1)}(q, t))\right) s_{21} + \left(\sum_{i=0}^{k} s_{(k+2i,k-1)}(q, t)\right) s_{111},
\]

(11)

\[
Q_{3k+2,3}(1) = H_{3k+2,3}[X; q, t] = \left(\sum_{i=0}^{k-1} s_{(k+2i,k-1)}(q, t)\right) s_3 + \left(\sum_{i=0}^{k-1} (s_{(k+2i+1,k-1)}(q, t) + s_{(k+2i+2,k-1)}(q, t))\right) s_{21} + \left(\sum_{i=0}^{k} s_{(k+2i+1,k-1)}(q, t)\right) s_{111},
\]

(12)

\[
Q_{3k,3}(1) = H_{3k,3}[X; q, t] = \left(\sum_{i=0}^{k-1} (s_{(k+2i-3,k-1)}(q, t) + s_{(k+2i-2,k-1)}(q, t))\right) s_3 + \left(s_{(k+1,k-1)}(q, t) + 2s_{(k,k-1)}(q, t) + s_{(k-1,k-1)}(q, t)\right) s_{21} + \left(\sum_{i=1}^{k-1} (s_{k+2i-2,k-1}(q, t) + 2s_{k+2i-1,k-1}(q, t) + 2s_{k+2i,k-1}(q, t) + s_{k+2i+1,k-1}(q, t))\right) s_{111}.
\]

(13)

Note that this independently proves the Shuffle Theorem and the Extended Rational Shuffle Theorem when \(n \leq 3\).

In Section 5, we prove several Schur function coefficient formulas and symmetries in \(Q_{3,n}(1)\) (some of which are consequences of Theorem 3), and conjecture a concise recursive formula for Schur function coefficients \([s_\lambda]_{3,n}\) generally for any \(\lambda \vdash n\). In particular, we study a new symmetry that

\[
[s_{2^a 1^b}]_{3,n} = [s_{2^a 1^b}]_{3,3(a+b)-n},
\]

(14)

and a combinatorial action on parking functions called the switch map \(S\).
2 Background

To state our results, we shall first introduce details about the Extended Rational Shuffle Theorem. This will require a series of definitions. We omit the word “Extended” as long as it will not cause any ambiguity.

2.1 Combinatorial side

Let $m$ and $n$ be positive integers. The set of $(m, n)$-Dyck paths is denoted by $D_{m,n}$. For an $(m, n)$-Dyck path, the cells that are cut through by the main diagonal will be called diagonal cells. Figure 2 (a) gives an example of a $(5,7)$-Dyck path, and Figure 2 (b) gives an example of a $(4,6)$-Dyck path, where the diagonal cells are shaded.

![Figure 2: A $(5,7)$-Dyck path, a $(4,6)$-Dyck path, a $(5,7)$-parking function and its car ranks.](image)

For an $(m, n)$-Dyck path, we have the statistic area defined as follows.

**Definition 1 (area).** The number of full cells between an $(m, n)$-Dyck path $P$ and the main diagonal is denoted by $\text{area}(P)$.

The cells above a Dyck path $P$ are called coarea cells of $P$, and they form a Ferrers diagram (in English notation) of a partition $\lambda(P)$. In the example in Figure 2 (a), $\lambda(P) = (3, 3, 1, 1)$ or $\begin{array}{ccc} & & 3 \\ & 3 & \\ 1 & & 1 \end{array}$.

For any partition $\mu$ and any cell $c$ in the Ferrers diagram of $\mu$, we let $\text{arm}(c)$ be the number of cells to the right of $c$ in $\mu$ and $\text{leg}(c)$ be the number of cells below $c$ in $\mu$. Let $\chi(x)$ denote the function that takes value 1 if its argument $x$ is true, and 0 otherwise, then we can define the path $\text{dinv}$ ($\text{pdinv}$) statistic of an $(m, n)$-Dyck path.

**Definition 2 (pdinv).** The $\text{pdinv}$ of an $(m, n)$-Dyck path $P$ is given by

$$\text{pdinv}(P) := \sum_{c \in \lambda(P)} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c)} + 1 \leq \frac{m}{n} < \frac{\text{arm}(c) + 1}{\text{leg}(c)} \right).$$

We can get an $(m, n)$-parking function $\pi$ by labeling the north steps of an $(m, n)$-Dyck path with the integers $\{1, \ldots, n\}$ such that the numbers increase in each column from bottom to top, and we will refer to these labels as cars. The underlying Dyck path is denoted by $\Pi(\pi)$, and the partition formed by the collection of cells above the path $\Pi(\pi)$ is denoted by $\lambda(\pi)$. The set of $(m, n)$-parking functions is denoted by $P_{m,n}$. Figure 2 (c) pictures a $(5,7)$-parking function based on the $(5,7)$-Dyck path in Figure 2 (a).

Next we define statistics ides and pides for rational parking functions. For any pair of coprime positive integers $m$ and $n$, we define the rank of a cell $(x, y)$ in the $(m, n)$-grid to be $\text{rank}(x, y) :=$
my − nx. If m and n are not coprime, we shall generalize the rank to be $\text{rank}(x, y) := my − nx + \left\lfloor \frac{x\gcd(m,n)}{m} \right\rfloor$. Figure 2 (d) shows the ranks of cars in Figure 2 (c). The word (or diagonal word), $\sigma(\pi)$ (or word($\pi$)), of $\pi$, is obtained by reading cars from highest to lowest ranks. In our example in Figure 2 (c), $\sigma(\pi) = 7563412$. We define $\text{ides}(\pi)$ to be the descent set of $\sigma(\pi)^{-1}$. In other words, we have

**Definition 3 (ides).** Let $\pi$ be any parking function, then

$$\text{ides}(\pi) := \{i \in \sigma(\pi) : i + 1 \text{ is to the left of } i \text{ in } \sigma(\pi)\} = \{i : \text{rank}(i) < \text{rank}(i + 1)\}.$$ 

Then we define $\text{pides}(\pi)$ to be the composition set of $\text{ides}(\pi)$.

**Definition 4 (pides).** For any $(m, n)$-parking function $\pi$, if $\text{ides}(\pi) = \{i_1 < i_2 < \cdots < i_d\}$, then

$$\text{pides}(\pi) := \{i_1, i_2 - i_1, \ldots, n - i_d\}.$$ 

In Figure 2 (c), we have $\text{ides}(\pi) = \text{ides}(7563412) = \{2, 4, 6\}$, and $\text{pides}(\pi) = \{2, 2, 2, 1\}$.

We have the following two remarks about the statistics word, ides and pides.

**Remark 1.** Let $i < j$ be two cars in the parking function $\pi$. If $i$ is to the left of $j$ in $\sigma(\pi)$, then the cars $i, j$ must be in different columns.

**Proof.** In $\sigma(\pi)$, the allocation of $i$ and $j$ implies that $\text{rank}(i) > \text{rank}(j)$. If $i$ and $j$ are in the same column, then it must be the case that $j$ lies on top of $i$, which leads to a contradiction with $\text{rank}(i) > \text{rank}(j)$. Thus, $i$ and $j$ must be in different columns.

**Remark 2.** For $\pi \in \mathcal{P}_{m,n}$, the parts in the composition set $\text{pides}(\pi)$ are less than or equal to $m$.

**Proof.** Suppose to the contrary. If $M \in \text{pides}(\pi)$ where $M > m$, then there exist $M$ cars $k, k + 1, \ldots, k + M - 1$ with decreasing ranks. By Remark 1 the $M$ cars are in different columns, which is impossible, thus the assumption that $M \in \text{pides}(\pi)$ is not true.

In many papers (e.g. [8, 19]), the statistic $\text{dinv}$ of a parking function is defined by 3 components — path $\text{dinv}$ ($\text{pdinv}$), max $\text{dinv}$ ($\text{maxdinv}$) and temporary $\text{dinv}$ ($\text{tdinv}$).

**Definition 5 (tdinv).** Let $\pi$ be any $(m, n)$-parking function, then

$$\text{tdinv}(\pi) := \sum_{\text{cars } i < j} \chi(\text{rank}(i) < \text{rank}(j) < \text{rank}(i) + m).$$

In Figure 2 (c), $\text{tdinv}(\pi) = 7$ since the pairs of cars contributing to $\text{tdinv}$ are (1, 3), (1, 4), (3, 5), (3, 6), (4, 6), (5, 7) and (6, 7). Then, the statistic max $\text{dinv}$ of a path is defined as the maximum of temporary dinvs of parking functions on the path.

**Definition 6 (maxdinv).** For any parking function $\pi$,

$$\text{maxdinv}(\pi) := \max\{\text{tdinv}(\pi') : \Pi(\pi') = \Pi(\pi)\}.$$ 

Finally, the statistic $\text{dinv}$ is defined as follows.

**Definition 7 (dinv).** For any parking function $\pi$,

$$\text{dinv}(\pi) := \text{tdinv}(\pi) + \text{pdinv}(\Pi(\pi)) − \text{maxdinv}(\pi).$$
We shall apply this definition of dinv in several combinatorial proofs in Section 3.

Notice that the statistics pdinv and maxdinv of a parking function \( \pi \) are determined by the underlying Dyck path \( \Pi(\pi) \). We also write \( \text{pdinv}(\pi) \) and \( \text{maxdinv}(\Pi(\pi)) \) for path dinv of parking function \( \pi \) and max dinv of its path.

Further, the component \( (\text{pdinv}(\Pi(\pi)) - \text{maxdinv}(\Pi(\pi))) \) in the definition of dinv(\( \pi \)) combines to a statistic of rational Dyck paths. Our definition of dinv(\( \pi \)) in Section 4 will follow the formulation by Leven and Hicks [16], who gave a simplified formula for dinv(\( \pi \)) by defining the statistic \( \text{dinv correction}(\text{dinvcorr}) \) that satisfies dinvcorr(\( \Pi(\pi) \)) = \( \text{pdinv}(\Pi(\pi)) - \text{maxdinv}(\Pi(\pi)) \).

**Definition 8** (dinvcorr). Let \( P \) be any \((m,n)\)-Dyck path and set \( \frac{0}{0} = 0 \) and \( \frac{x}{0} = \infty \) for all \( x \neq 0 \), then

\[
\text{dinvcorr}(P) := \sum_{c \in \lambda(P)} \chi \left( \frac{\text{arm}(c)+1}{\text{leg}(c)+1} \leq \frac{m}{n} < \frac{\text{arm}(c)}{\text{leg}(c)} \right) - \sum_{c \in \lambda(P)} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c)} \leq \frac{m}{n} < \frac{\text{arm}(c)+1}{\text{leg}(c)+1} \right).
\]

An alternative definition of dinv is

**Definition 9** (dinv, alt.). Let \( \pi \) be any \((m,n)\)-parking function, then

\[
\text{dinv}(\pi) := \text{tdinv}(\pi) + \text{dinvcorr}(\Pi(\pi)).
\]

Note that the statistic dinvcorr only depends on the path \( P \), and it is the difference of two sums \( \sum_{c \in \lambda(P)} \chi \left( \frac{\text{arm}(c)+1}{\text{leg}(c)+1} \leq \frac{m}{n} < \frac{\text{arm}(c)}{\text{leg}(c)} \right) \) and \( \sum_{c \in \lambda(P)} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c)} \leq \frac{m}{n} < \frac{\text{arm}(c)+1}{\text{leg}(c)+1} \right) \), of which at most one is nonzero. If \( m = n \), then there is no dinvcorr. If \( m \neq n \), we count dinvcorr by all the cells in \( \lambda(P) \). Given a cell \( c \in \lambda(P) \), we highlight the vertical line segment \( N \) which is a north step of the path \( P \) to the east of \( c \), and the horizontal line segment \( E \) which is a east step of \( P \) to the south of \( c \). We draw two lines with slope \( \frac{n}{m} \) from the north end and south end of \( N \).

1. If \( n > m \), the cells of type (a) and (b) in Figure 3 contribute \(-1\) to dinvcorr,
2. If \( m > n \), the cells of type (c) and (d) in Figure 3 contribute \(1\) to dinvcorr.

![Figure 3: Types of cells that contribute to dinvcorr.](image)

A composition \( \alpha \) of \( n \) is a sequence of positive integers summing up to \( n \), denoted by \( \alpha \vdash n \). Suppose \( \alpha = (\alpha_1, \ldots, \alpha_k) \) is a composition of \( n \) with \( k \) parts. We associate a subset \( S(\alpha) \) of \( \{1, \ldots, n-1\} \) to \( \alpha \) by setting

\[
S(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}.
\]

We let \( F_\alpha[X] \) denote the fundamental quasi-symmetric function of Gessel [10] associated to \( \alpha \) where \( X = \{x_1, \ldots, x_n\} \):

\[
F_\alpha[X] := \sum_{1 \leq \alpha_1 \leq a_2 \leq \cdots \leq a_n \leq n \atop i \in S(\alpha) \Rightarrow a_i < a_{i+1}} x_{a_1}x_{a_2}\cdots x_{a_n}.
\]
Then following Hikita \cite{17}, the \textit{Hikita polynomial} \( H_{m,n}[X; q, t] \) where \( m \) and \( n \) are coprime is defined by

\[
H_{m,n}[X; q, t] := \sum_{\pi \in P_{m,n}} t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} F_{\text{ides}(\pi)}[X].
\] (16)

Due to the work of Haglund, Haiman, Loehr, Remmel and Ulyanov \cite{14}, the refinement of Hikita polynomial on each Dyck path is symmetric over the variables in \( \{x_1, \ldots, x_n\} \), thus Hikita polynomials are symmetric functions.

Hikita did not define non-coprime Hikita polynomials, thus we generalize Hikita polynomials to non-coprime case as follows. Given \( m, n \) coprime and \( k \geq 1 \), we defined the return, \( \text{ret}(\pi) \), of a \((km, kn)\)-parking function \( \pi \) to be the smallest positive integer \( i \) such that the supporting path of \( \pi \) goes through the point \((im, in)\). Then following the formulation of Bergeron, Garsia, Leven and Xin \cite{5}, the \textit{extended Hikita polynomial} is defined to be

\[
H_{km,kn}[X; q, t] := \sum_{\pi \in P_{km,kn}} [\text{ret}(\pi)] t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} F_{\text{ides}(\pi)}[X].
\] (17)

For the same reason, extended Hikita polynomials are also symmetric functions in \( X \).

### 2.2 Algebraic side

For any partition \( \mu \) of \( n \), let \( \tilde{H}_\mu \) be the modified Macdonald symmetric function \cite{20} associated to \( \mu \), and let \( \nabla \) be the linear operator defined in terms of the modified Macdonald symmetric functions \( \tilde{H}_\mu(X; q, t) \) by

\[
\nabla \tilde{H}_\mu := t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu,
\] (18)

where \( \mu' \) is the conjugate of \( \mu \), and \( n(\mu) = \sum_i (i - 1)\mu_i \).

The Shuffle Conjecture proposed by Haglund, Haiman, Loehr, Remmel and Ulyanov \cite{14} which was proved by Carlsson and Mellit \cite{6} as the Shuffle Theorem can be stated as follows.

**Theorem 5 (Carlsson-Mellit).** For all \( n \geq 0 \),

\[
\nabla e_n = H_{n+1,n}[X; q, t].
\] (19)

Gorsky and Negut\cite{11} introduced operators \( Q_{m,n} \) on symmetric functions in the case where \( m \) and \( n \) are coprime. The \( Q_{m,n} \) operators of the Gorsky-Negut can be defined in terms of the operators \( D_k \) which were introduced by Bergeron and Garsia \cite{4}. In the plethystic notation, the action of \( D_k \) on a symmetric function \( F[X] \) is defined by

\[
D_k F[X] = F \left[ X + \frac{M}{z} \sum_{i \geq 0} (-z)^i e_i[X] \right]_{z^k},
\] (20)

where \( M = (1 - t)(1 - q) \).

Then one can construct a family of symmetric function operators \( Q_{m,n} \) for any pair of coprime positive integers \((m, n)\) as follows. First for any \( n \geq 0 \), set \( Q_{1,n} = (-1)^n D_n \). Next, one can recursively define \( Q_{m,n} \) for \( m > 1 \) as follows. Consider the \( m \times n \) lattice with diagonal \( y = \frac{n}{m} x \). We choose \((a, b)\) such that \((a, b)\) is the lattice point which is closest to the diagonal, and

\[
\begin{vmatrix}
c & d \\
a & b \\
\end{vmatrix} > 0
\]
where \((c, d) = (m - a, n - b)\). In such a case, we will write
\[
\text{Split}(m, n) = (a, b) + (c, d).
\] (21)

Note that the pairs \((a, b)\) and \((c, d)\) are coprime since any point of the form \((kx, ky)\) is further from the diagonal than the point \((x, y)\). Then we have the following recursive definition of the \(Q_{m, n}\) operators:
\[
Q_{m, n} = \frac{1}{M}[Q_{c, d}, Q_{a, b}] = \frac{1}{M}(Q_{c, d}Q_{a, b} - Q_{a, b}Q_{c, d}).
\] (22)

Figure 4 gives an example of \(\text{Split}(3, 5)\). \(\text{Split}(3, 5) = (1, 2) + (2, 3)\), so that
\[
Q_{3, 5} = \frac{1}{M}[Q_{2, 3}, Q_{1, 2}] = \frac{1}{M}[Q_{2, 3}, D_{2}].
\] (23)

![Figure 4: The geometry of Split(3, 5).](image)

The same procedure gives \(Q_{2, 3} = \frac{1}{M}[Q_{1, 1}, Q_{1, 2}] = \frac{1}{M}[-D_{1}, D_{2}]\). Therefore,
\[
Q_{3, 5} = \frac{1}{M^2}[-D_{1}, D_{2}, D_{2}] = \frac{1}{M^2}(-D_{2}D_{2}D_{1} + 2D_{2}D_{1}D_{2} - D_{1}D_{2}D_{2}).
\] (24)

For the non-coprime case, we can define the \(Q_{km,kn}\) operator as follows. We choose one of the lattice points, \((a, b)\), in the \(km \times kn\) lattice satisfying \(b(km - a) - a(km - b) > 0\) that are not on the diagonal and closest to the diagonal, then we set
\[
Q_{km,kn} = \frac{1}{M}[Q_{km-a,kn-b}, Q_{a,b}].
\] (25)

This recursive definition is well-defined as it is proved in [5] that any choice of such point \((a, b)\) defines the same operation.

Gorsky and Negut [11] have the following lemma about the operators \(\nabla\) and \(Q_{m,n}\) for the coprime case.

**Lemma 6** (Gorsky-Negut). For any coprime positive integers \(m, n\),
\[
\nabla Q_{m,n} \nabla^{-1} = Q_{m+n,n}.
\] (26)

We can generalize Lemma 6 to the case \((kn, n)\).
Lemma 7. For any positive integers $k, n$,
\[ \nabla Q_{kn} \nabla^{-1} = Q_{(k+1)n,n}. \] (27)

Proof. By definition of the operator $Q_{m,n}$, we have
\[ Q_{kn,n} = \frac{1}{M}[Q_{kn-1,n}Q_{1,0}] = \frac{1}{M}(Q_{kn-1,n}Q_{1,0} - Q_{1,0}Q_{kn-1,n}). \]
Using Lemma 6, we have
\[ \nabla Q_{kn,n} \nabla^{-1} = \frac{1}{M}(\nabla Q_{kn-1,n} \nabla^{-1} - \nabla Q_{1,0} \nabla^{-1} \nabla Q_{kn-1,n} \nabla^{-1}) = \frac{1}{M}(Q_{kn+n-1,n}Q_{1,0} - Q_{1,0}Q_{kn+n-1,n}) = Q_{(k+1)n,n}. \]

Now we are ready to prove Theorem 2.

Proof of Theorem 2. One can verify the base case for both equations:
\[ Q_{2,2}(1) = s_2 + (s_1(q,t) + 1)s_{11}. \] (28)

Note that by Lemma 7, $Q_{2,2} = \nabla^{-k-1}Q_{2,2} \nabla^{k+1}$, so we have
\[ Q_{2k,2}(1) = \nabla^{-k-1}Q_{2,2}(1) = \nabla^{k+1}(s_2 + (s_1(q,t) + 1)s_{11}). \] (29)

It can be proved by induction that
\[ \nabla^n s_2 = -qt[n-1]_{q,t}s_2 - qt[n]_{q,t}s_{11}, \] (30)
\[ \nabla^n s_{11} = [n]_{q,t}s_2 + qt[n+1]_{q,t}s_{11}, \] (31)
thus,
\[ Q_{2k,2}(1) = \nabla^{-k-1}s_2 + (q + t + 1)\nabla^{-k-1}s_{11} = ([k]_{q,t} + [k-1]_{q,t})s_2 + ([k]_{q,t} + [k]_{q,t})s_{11}. \] (32)

For Equation (10), we use the result of Leven (Equation (23) of [19]) that
\[ D_aD_b(1) = (-1)^{a+b}e_a e_b + (-1)^{a+b-1}M \sum_{i=1}^{b} [i]_{q,t}e_{a+i} e_{b-i}. \] (33)

Expanding the operator $Q_{2,2k}$ gives
\[ Q_{2,2k}(1) = \frac{1}{M}(D_{k+1}D_{k-1}(1) - D_{k-1}D_{k+1}(1)) = \sum_{i=1}^{k+1} [i]_{q,t}e_{k+1-i}e_{k-1+i} - \sum_{i=1}^{k-1} [i]_{q,t}e_{k-1-i}e_{k+1+i} = \sum_{r=0}^{k} ([k - r + 1]_{q,t} + [k - r]_{q,t})s_{2r+1}2^{k-2r+1}, \] (34)
which proves Equation (10).

We shall use Lemma 6 and Lemma 7 to prove Theorem 4 algebraically in Section 4.1.
2.3 The Extended Rational Shuffle Theorem

For the rational case, we consider pairs of positive integers \((km, kn)\), where \(m\) and \(n\) are coprime and \(k\) is a positive integer. The Extended Rational Shuffle Conjecture of Bergeron-Garsia-Leven-Xin [5], which generalizes a previous conjecture by Gorsky and Negut [11], has been shown to hold by Mellit [21]. So we have the Extended Rational Shuffle Theorem as follows.

**Theorem 8** (Mellit). For all pairs of coprime positive integers \((m, n)\) and all \(k \in \mathbb{Z}^+\), we have

\[
Q_{km,kn}(1) = H_{km,kn}[X; q, t].
\]  

(35)

The original Rational Shuffle Theorem proposed by Gorsky and Negut [11] and proved by Mellit [21] in the case where \(m\) and \(n\) are relatively prime is the special case when \(k = 1\) in Theorem 8. Further details and the complete picture about the Rational Shuffle Theorem is outlined in [3]. Notice that \(\nabla \epsilon_n = Q_{n+1,n}(1)\), and \(\nabla \epsilon_n\) is not the same as \(Q_{n,n}(1)\).

The main goal of this paper is to study the combinatorics of the Schur function expansion of \(Q_{m,n}(1)\). Given that Mellit has proved the Rational Shuffle Theorem, we can find the Schur function expansion in one of two ways. That is, we can use the properties of \(Q_{m,n}\) to find the Schur function expansion of \(Q_{m,n}(1)\) which we will refer to as working on the symmetric function side of the Rational Shuffle Theorem. Second, one could start with the Hikita polynomial \(H_{m,n}[X; q, t]\) and expand that polynomial into Schur functions which we will call working on the combinatorial side of the Rational Shuffle Theorem.

Since it is proved that \(Q_{m,n}(1) = H_{m,n}[X; q, t]\), we let \(s_\lambda_{m,n}\) denote the coefficient of Schur function \(s_\lambda\) in both polynomials \(Q_{m,n}(1)\) and \(H_{m,n}[X; q, t]\).

2.4 An alternative expression for the combinatorial side

We shall introduce an alternative expression for Hikita polynomials due to the fact that the Hikita polynomials \(H_{m,n}[X; q, t]\) are symmetric in \(\{x_1, \ldots, x_n\}\).

Suppose that \(\alpha = (\alpha_1, \ldots, \alpha_k)\) is a composition of \(n\) into \(k\) parts \((k \leq n)\), then we set \(\alpha_j = 0\) for \(j > k\). We let \(X = \{x_1, \ldots, x_n\}\) be the set of \(n\) variables and

\[
\Delta_\alpha[X] := \det ||x_i^{\alpha_j+n-j}|| = \sum_{\sigma \in S_n} \text{sgn}(\sigma)\sigma(x_1^{\alpha_1+n-1} \cdots x_n^{\alpha_n+n-n}).
\]

We let \(\Delta[X] := \det ||x_i^{n-j}||\) be the Vandermonde determinant. The Schur symmetric function \(s_\alpha[X]\) associated to \(\alpha\) can be defined by \(s_\alpha[X] := \frac{\Delta_\alpha[X]}{\Delta[X]}\).

It is well-known that for any such composition \(\alpha\), either we have \(s_\alpha[X] = 0\) or there is a partition \(\lambda \vdash n\) such that \(s_\alpha[X] = \pm s_\lambda[X]\). In fact, there is a straightening relation which allows us to prove that fact. Namely, if \(\alpha_i+1 > \alpha_i\), then

\[
s(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_k)[X] = -s(\alpha_1, \ldots, \alpha_{i+1}-1, \alpha_i+1, \ldots, \alpha_k)[X].
\]  

(36)

In a remarkable and important paper, Egge, Loehr and Warrington [7] gave a combinatorial description of how to start with a quasi-symmetric function expansion of a homogeneous symmetric
function $P[X]$ of degree $n$, and transform it into an expansion in terms of Schur functions. The following theorem due to Garsia and Remmel [9] is implicit in the work of [7], but is not explicitly stated and it allows one to find the Schur function expansion by using the straightening laws.

**Theorem 9** (Garsia-Remmel). *Suppose that $P[X]$ is a symmetric function which is homogeneous of degree $n$ and*

$$P[X] = \sum_{\alpha \vdash n} a_\alpha F_\alpha[X].$$  \hspace{1cm} (37)

*Then*

$$P[X] = \sum_{\alpha \vdash n} a_\alpha s_\alpha[X].$$  \hspace{1cm} (38)

Recall that $\text{pides}(\sigma)$ is the composition set of $\text{ides}(\sigma)$, then Theorem 9 and the straightening action allow us to transform $H_{m,n}[X; q, t]$ into Schur function expansion that

$$H_{m,n}[X; q, t] = \sum_{\pi \in \mathcal{P}_{m,n}} [\text{ret}(\pi)] t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} F_{\text{pides}(\pi)}$$

$$= \sum_{\pi \in \mathcal{P}_{m,n}} [\text{ret}(\pi)] t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} s_{\text{pides}(\pi)}.$$

From Section 3, we shall use the expression (39) for $H_{m,n}[X; q, t]$ to prove several facts about the coefficients in the Schur function expansions of the Rational Shuffle Theorem.

### 3 Combinatorial results about Schur function expansions of the $(m, n)$ case

We shall work on the combinatorial side by studying the Hikita polynomials in this section, and we use the expression of Hikita polynomials in Equation (39).

In the rational $(m, n)$ case, we have $n$ cars, i.e. the word of an $(m, n)$-parking function is a permutation of $[n] = \{1, \ldots, n\}$. Recall that $[s_\lambda]_{m,n}$ is the coefficient of $s_\lambda$ in $H_{m,n}[X; q, t]$. By Remark 2 in Section 2.1, $[s_\lambda]_{m,n} \neq 0$ implies that $\lambda$ must be of the form $m^\alpha \cdots 1^\alpha_1$ with $\sum_{i=1}^m i \alpha_i = n$, i.e. $[s_\lambda]_{m,n} \neq 0$ only if the partition $\lambda$ has parts of size less than or equal to $m$. In this section, we shall prove the 3 symmetries about $[s_\lambda]_{m,n}$ described in Theorem 3 stated as the following three lemmas.

**Lemma 10.** $[s_{1^n}]_{m,n} = [s_n]_{m+n,n}.$

Note that a parking function with pides $n$ must have word $12\cdots n$, and a parking function with pides $1^n$ must have word $n \cdots 21$.

![Figure 5: Bijection between $\mathcal{P}_{m,3}$ with word 123 and $\mathcal{P}_{m+3,3}$ with word 321.](image)
A parking function in $P_{m,n}$ with word $n \cdots 21$ corresponds to a unique $(m,n)$-Dyck path, and a parking function in $P_{m+n,n}$ with word $12 \cdots n$ corresponds to a unique $(m+n,n)$-Dyck path with no consecutive north steps. As shown in Figure 5, we can obtain a parking function in $P_{m+n,n}$ with word $12 \cdots n$ by pushing a staircase into a parking function $\pi \in P_{m,n}$ with word $n \cdots 21$. Given $\pi \in P_{m,n}$ with word $n \cdots 21$, let $\lambda = \lambda(\pi)$, we define $\text{hstr}(\pi) \in P_{m+n,n}$, the horizontal stretch of $\pi$, to be the parking function with word $12 \cdots n$ and $\lambda(\text{hstr}(\pi)) = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_{n-1} + 1)$, then

**Theorem 11.**

$$\text{hstr} : \{ \pi \in P_{m,n} : \text{word}(\pi) = n \cdots 21 \} \to \{ \pi \in P_{m+n,n} : \text{word}(\pi) = 12 \cdots n \},$$

$$\pi \mapsto \text{hstr}(\pi)$$

is a bijection, and

$$\text{area}(\text{hstr}(\pi)) = \text{area}(\pi),$$

(40)

$$\text{dinv}(\text{hstr}(\pi)) = \text{dinv}(\pi),$$

(41)

$$\text{ret}(\text{hstr}(\pi)) = \text{ret}(\pi).$$

(42)

**Proof.** The bijectivity of the map $\text{hstr}$ is clear since the map is invertible. Comparing the coarea of both parking functions immediately proves Equations (40) and (42). To prove Equation (41), recall that $\text{dinv}(\pi) = \text{tdinv}(\pi) + \text{dinvcorr}(\pi)$, we shall compare the two components of $\text{dinv}$, i.e. $\text{tdinv}$ and $\text{dinvcorr}$.

For a parking function $\pi \in P_{m,n}$ with word $\text{word}(\pi) = n \cdots 21$, its temporary $\text{dinv}$ reaches the maximum possible value of its path $\Pi(\pi)$, i.e. any two north steps with rank difference less than $m$ will contribute 1 to $\text{tdinv}$. For any two north steps, we fire two lines parallel to the diagonal from the two end points of the upper north step, then rank difference less than $m$ means that either the upper line or the lower line intersects the lower north step. The two cases are pictured in Figure 6.

On the other hand, the parking function $\text{hstr}(\pi) \in P_{m+n,n}$ always has no $\text{tdinv}$ since $\text{word}(\text{hstr}(\pi)) = 12 \cdots n$. We shall show that the increment of $\text{dinvcorr}$ makes up for the missing $\text{tdinv}$.

For a parking function $\pi \in P_{m,n}$, suppose that there are $j$ cells in row $r$ of $\pi$ (counting from bottom to top) with leg $i$ in the English partition $\lambda(\pi)$, and their arms are $a, a + 1, \ldots, a + j - 1$, pictured in Figure 7 (a). We fire two lines with slope $\frac{n}{m}$ from the two end points of the north step (called $N_1$) in row $r$, then they intersect the east steps (called $EEs$) below the $j$ cells at points $A, B$ which have horizontal distances $\frac{m}{n}$ and $\frac{m(i+1)}{n}$ to $N_1$.

Now consider the parking function $\text{hstr}(\pi) \in P_{m+n,n}$. By definition of $\text{hstr}$, there are $j + 1$ cells in row $r$ with leg $i$ in the partition $\lambda(\text{hstr}(\pi))$, and their arms are $a + i, a + i + 1, \ldots, a + i + j.$

![Figure 6: Pairs of north steps contributing to tdinv.](image-url)
pictured in Figure 7(b). We again fire two lines with slope \( \frac{n}{m+n} \) from the two end points of the north step \( N_1 \) in row \( r \), then they intersect the east steps below the \( j+1 \) cells at points \( A, B \) which have horizontal distances \( \frac{mi}{n} + i \) and \( \frac{m(i+1)}{n} + i + 1 \) to \( N_1 \).

![Figure 7: Cells in row \( r \) with leg \( i \).](image)

Now recall the definition of the dinv correction. The dinvcorr contribution of \( N_1 \) in each picture is equal to the whole east steps contained in line segment \( AB \). The line segment \( AB \) in \( hstr(\pi) \) contains one more east step than \( AB \) in \( \pi \) in the following 2 cases:

1. In \( \pi \), A is not on EEs but B is on EEs.
2. In \( \pi \), A is on EEs.

In case (1), the car in row \( r \) of \( \pi \) produces a tdinv with the car in the row immediately below EEs; in case (2), the car in row \( r \) of \( \pi \) produces a tdinv with the car in the row of the next north step that the upper line fired from \( N_1 \) intersects. Thus, the new dinvcorr in case (1) and case (2) matches the tdinv in the two cases in Figure 6, and the increment of dinv correction is equal to \( tdinv(\pi) \), which proves the theorem.

Since \( hstr \) is an \((\text{area,dinv,ret})\)-preserving bijection, \([s_{1^n}]_{m,n} = [s_n]_{m+n,n} \) follows immediately.

**Lemma 12.** \([s_{m^{\alpha_m} \cdots 1^{\alpha_1}}]_{m,n} = [s_{m^{\alpha_m+1} \cdots 1^{\alpha_1}}]_{m,n+m}\)

This is a rewording of Theorem 3(b). For a parking function \( \pi \in \mathcal{P}_{m,n} \), we define a map \( vstr \), vertical stretch, that we push a staircase down to \( \pi \), then replace the car \( i \) in \( \pi \) by \( i + m \), and fill the bottom of the \( m \) columns of the new parking function with cars \( 1, \ldots, m \) in a rank decreasing way to get \( vstr(\pi) \), as shown in Figure 8.

Similar to Theorem 11, we have the following theorem about the vertical stretch action.

**Theorem 13.**

\[
vstr : \{ \pi \in \mathcal{P}_{m,n} : \text{pides}(\pi) = m^{\alpha_m} \cdots 1^{\alpha_1} \} \rightarrow \{ \pi \in \mathcal{P}_{m,n+m} : \text{pides}(\pi) = m^{\alpha_m+1} \cdots 1^{\alpha_1} \}, \]

\[
\pi \mapsto vstr(\pi)
\]
Figure 8: Bijection between $P_{3,n}$ with pides $3^a2^b1^c$ and $P_{3,n+3}$ with pides $3^a+1^b2^b1^c$.

is a bijection, and

$$\text{area}(vstr(\pi)) = \text{area}(\pi),$$  \hspace{1cm} (43)

$$\text{dinv}(vstr(\pi)) = \text{dinv}(\pi),$$  \hspace{1cm} (44)

$$\text{ret}(vstr(\pi)) = \text{ret}(\pi).$$  \hspace{1cm} (45)

Proof. The bijectivity follows from the invertibility of the map. Equations (43) and (45) are true for the same reason as Equations (40) and (42). The proof of Equation (44) is based on a similar idea to the proof of (41): the action of $vstr$ changes each car $i$ in $\pi$ into $i + m$, and the rank is also increased by $m$, thus the temporary dinv of $\pi$ is equal to the temporary dinv of the cars $m+1, \ldots, m+n$ in $vstr(\pi)$. Since the dinv correction is negative, we can match each tdinv between cars $1, 2, \ldots, m$ and $m+1, \ldots, m+n$ with a new negative dinv correction, showing that the change of dinv is zero.

Lemma 14. $[s_{k1}^n]_{m,n} = [s_{k1}^m]_{n,m}$. We shall prove the special case when $k = 1$ first. That is, we first show $[s_{1}^n]_{m,n} = [s_{1}^m]_{n,m}$. The bijection for this identity is that we can transpose the path of $\pi \in P_{m,n}$ and fill the word $(n, n-1, \ldots, 1)$ to get $\pi' \in P_{n,m}$.

It is obvious that $\pi'$ has the same area as $\pi$ since their underlying Dyck paths are transposes of each other. For the statistic dinv, recall that the tdinv of a parking function with word $(n, n-1, \ldots, 1)$ is equal to the maxdinv of the path, thus

$$\text{dinv}(\pi) = \text{tdinv}(\pi) + \text{pdinv}(\Pi(\pi)) - \text{maxdinv}(\pi)$$

$$= \text{pdinv}(\Pi(\pi))$$

$$= \sum_{c \in \lambda(\Pi(\pi))} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c) + 1} \leq \frac{m}{n} < \frac{\text{arm}(c) + 1}{\text{leg}(c)} \right).$$  \hspace{1cm} (46)

From the Equation (46), we see that dinv is symmetric about $m$ and $n$, and preserved by the transpose action. Thus Figure shows an example of this bijection.

Then we consider the identity $[s_{k1}^{n-k}]_{m,n} = [s_{k1}^{m-k}]_{n,m}$. This bijective proof is similar to that of $[s_{1}^{n}]_{m,n} = [s_{1}^{m}]_{n,m}$.

That is, given a parking function $\pi \in P_{m,n}$ with pides $k1^{n-k}$, one transposes the path and labels the path to produce pides $k1^{m-k}$. If there are only $k$ peaks (which means $k$ different columns)
in the Dyck paths, then the filling of cars in both \((m, n)\) and \((n, m)\) cases are unique since the cars \(1, \ldots, k\) must be filled in a rank-decreasing way at bottom of each column in the two parking functions, while the remaining cars should be filled in a rank-increasing way in the remaining north steps. One can check that they have the same area and \(\text{dinv}\) values.

Figure 10: A \((5, 8)\)-Dyck path with 3 peaks.

Otherwise, in any rational \((m, n)\)-Dyck path \(P\) with \(j > k\) peaks, the car \(k\) must be in the first row since it has the smallest rank, and there are \(\binom{j-1}{k-1}\) ways to choose columns for cars \(1, \ldots, k - 1\) in the north steps of both \(P\) and its transpose \(P'\), while the remaining cars should be filled in a rank-increasing way in the remaining north steps. We want to match the \(\binom{j-1}{k-1}\) possible positions of cars \(1, \ldots, k - 1\) in both \((m, n)\) and \((n, m)\) cases by a similar idea.

We still use Definition \(\text{dinv}\) as the definition of \(\text{dinv}\), and the fact that a path \(P\) and its transpose \(P'\) have the same \(\text{pdinv}\). For a parking function \(\pi \in \mathcal{P}_{m,n}\) with pides \(k^{1^n-k}\) and a parking function \(\pi' \in \mathcal{P}_{n,m}\) with pides \(k^{1^m-k}\), the component \((\text{tdinv}(\pi) - \text{maxdinv}(\pi))\) counts the missing \(\text{dinv}\) created by the first \(k\) cars (since the cars greater than \(k\) are filled in a way to generate maximum possible \(\text{dinv}\)).

Taking the \((5, 8)\)-Dyck path \(P\) in Figure 10 for an example. It has three peaks, and the three circles are the positions that the first \(k\) cars can be filled in. Next we consider its transpose \(P'\). We can use the same picture for the underlying Dyck path, and fill cars in its east steps. In this way, the three crosses are the positions that the first \(k\) cars can be filled in. The rank of each cross is determined by the lattice point to the northeast of itself, and the rank of each circle is determined by the lattice point to the southwest of itself, and these lattice points are exactly the three valley points (including the start point) of the Dyck path, marked in the picture. Thus, each cross is paired with a circle by a certain valley point.
For any parking function on $P$ with pides $k1^{n-k}$, we find the $k$ circled positions that contain the first $k$ cars, then we choose the corresponding $k$ crossed positions in the path $P'$. In this way of matching, the dinv component $(tdinv(\pi) - \text{maxdinv}(\pi))$ in two parking functions are the same, thus $[s_{k1^{n-k}}]_{m,n} = [s_{k1^{m-k}}]_{n,m}$ is proved.

Note that Theorem 3 (c) is a result about hook-shaped Schur functions. As we proved within this result, Theorem 3 (c) implies the following corollary.

**Corollary 15.** For all $m, n > 0$,

$$[s_1^n]_{m,n} = [s_1^m]_{n,m}.$$

### 4 Schur function expansions of the $(m, 3)$ Case

The Rational Shuffle Theorem when $n = 3$ has a nice Schur function expansion, summarized in Theorem 4. For example, one can compute the Schur function expansion of $Q_{3k+1,3}(1)$ by Maple to get Table 4.

In this section, we give two proofs of Theorem 4 by both working on the symmetric function side and the combinatorial side of the Rational Shuffle Theorem. Our proofs independently prove the Rational Shuffle Theorem and the Shuffle Theorem when $n \leq 3$.

**Table 1:** Coefficients of $s_{\lambda}$ in $Q_{3k+1,3}(1)$.

| $k$ | $Q_{3k+1,3}(1)$ | $[s_3]_{3k+1,3}$ | $[s_{21}]_{3k+1,3}$ | $[s_{111}]_{3k+1,3}$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| 0   | $Q_{1,3}(1)$    | 0               | 0               | $s_0(q,t)$      |
| 1   | $Q_{4,3}(1)$    | $s_0(q,t)$      | $s_1(q,t) + s_2(q,t)$ | $s_3(q,t)$ + $s_{11}(q,t)$ |
| 2   | $Q_{7,3}(1)$    | $s_3(q,t)$      | $s_4(q,t) + s_5(q,t)$ | $s_6(q,t)$ + $s_{41}(q,t)$ + $s_{22}(q,t)$ |
| 3   | $Q_{10,3}(1)$   | $s_6(q,t)$      | $s_7(q,t) + s_8(q,t)$ | $s_9(q,t)$ + $s_{71}(q,t)$ + $s_{52}(q,t)$ + $s_{33}(q,t)$ |
| 4   | $Q_{13,3}(1)$   | $s_9(q,t)$      | $s_{10}(q,t) + s_{11}(q,t)$ | $s_{12}(q,t)$ + $s_{101}(q,t)$ + $s_{82}(q,t)$ + $s_{63}(q,t)$ + $s_{44}(q,t)$ |
4.1 Algebraic proof — $Q_{m,3}(1)$

We shall use Leven’s method in [19] to prove the theorem by induction. We use the following lemma about $(q,t)$-Schur functions to simplify our computation.

**Lemma 16.** Let $n, k \geq 0$ be two non-negative integers, we have

$$s_{n-1}(q,t)s_{k-1}(q,t) = s_{n+k-2}(q,t) + s_{k-1,1}(q,t)s_{n-2}(q,t). \quad (47)$$

**Proof.**

$$s_{n-1}(q,t)s_{k-1}(q,t) - (s_{n+k-2}(q,t) + s_{k-1,1}(q,t)s_{n-2}(q,t)) \min(k-2,n-3)$$

$$= \sum_{i=0} s_{n+k-i-4,i+1,1}(q,t) = 0. \quad \square$$

Since $\nabla a = a$ for any constant $a$, Lemma 6 and Lemma 7 allow us to write a recursion for $Q_{m,3}$ operator that

$$Q_{m+3,3}(1) = \nabla Q_{m,3}(1) = \nabla Q_{m,3}(1). \quad (48)$$

Using the recursion, we can prove Theorem 4 by inducting on $m$. We shall give the complete algebraic proof of Equation (11) in Theorem 4, and omit the algebraic proof of Equations (12) and (13), only listing the base cases that

$$Q_{2,3}(1) = s_{21} + s_{1}(q,t)s_{11}, \quad (49)$$
$$Q_{3,3}(1) = s_{3} + (s_{2}(q,t) + 2s_{1}(q,t) + 1)s_{21} + (s_{11}(q,t) + s_{1}(q,t) + s_{2}(q,t) + s_{3}(q,t)). \quad (50)$$

**Proof of Equation (11).** When $k = 0$, we can obtain by direct computation that

$$Q_{1,3}(1) = s_{111}, \quad (51)$$

which satisfies Equation (11). Then we induct on $k$ to prove Equation (11) that suppose the Schur function coefficients of $Q_{3k+1,3}(1)$ are the following:

$$[s_{3}]_{3k+1,3} = \sum_{i=0}^{k-1} s_{(k+2i-1,k-i-1)}(q,t), \quad (52)$$
$$[s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1} (s_{(k+2i,k-i-1)}(q,t) + s_{(k+2i+1,k-i-1)}(q,t)), \quad (53)$$
$$[s_{111}]_{3k+1,3} = \sum_{i=0}^{k} s_{(k+2i,k-i)}(q,t), \quad (54)$$

we want to show that

$$[s_{3}]_{3k+1,3} = \sum_{i=0}^{k} s_{(k+2i-1,k-i-1)}(q,t), \quad (55)$$
$$[s_{21}]_{3k+1,3} = \sum_{i=0}^{k} (s_{(k+2i,k-i-1)}(q,t) + s_{(k+2i+1,k-i-1)}(q,t)), \quad (56)$$
$$[s_{111}]_{3k+1,3} = \sum_{i=0}^{k+1} s_{(k+2i,k-i)}(q,t), \quad (57)$$

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One can directly compute that
\[
\begin{align*}
\nabla s_3 &= s_{2,2}(q,t)s_{21} + s_{3,2}(q,t)s_{111}, \\
\nabla s_{21} &= s_{2,1}(q,t)s_{21} - s_{3,1}(q,t)s_{111}, \\
\nabla s_{111} &= s_3 + (s_1(q,t) + s_2(q,t))s_{21} + (s_{11}(q,t) + s_3(q,t))s_{111}.
\end{align*}
\]

By Equation (48), we have
\[
Q_{3(k+1)+1,3}(1) = [s_3][3k+4,3]s_3 + [s_{21}][3k+4,3]s_{21} + [s_{111}][3k+4,3]s_{111} \\
= \nabla Q_{3k+1,3}(1) \\
= \nabla ([s_3][3k+1,3]s_3 + [s_{21}][3k+1,3]s_{21} + [s_{111}][3k+1,3]s_{111}) \\
= [s_3][3k+1,3] \nabla s_3 + [s_{21}][3k+1,3] \nabla s_{21} + [s_{111}][3k+1,3] \nabla s_{111} \\
= [s_{111}][3k+1,3]s_3 \\
+ (s_{22}(q,t)[s_3][3k+1,3] - s_21(q,t)[s_{21}][3k+1,3] + (s_1(q,t) + s_2(q,t))[s_{111}][3k+1,3])s_{21} \\
+ (s_{32}(q,t)[s_3][3k+1,3] - s_{31}(q,t)[s_{21}][3k+1,3] + (s_{11}(q,t) + s_3(q,t))[s_{111}][3k+1,3])s_{111},
\]
which implies that
\[
\begin{align*}
[s_3][3k+4,3] &= [s_{111}][3k+1,3], \\
[s_{21}][3k+4,3] &= s_{2,2}(q,t)[s_3][3k+1,3] - s_21(q,t)[s_{21}][3k+1,3] + (s_1(q,t) + s_2(q,t))[s_{111}][3k+1,3], \\
[s_{111}][3k+4,3] &= s_{3,2}(q,t)[s_3][3k+1,3] - s_{31}(q,t)[s_{21}][3k+1,3] + (s_{11}(q,t) + s_3(q,t))[s_{111}][3k+1,3].
\end{align*}
\]

By the recursions above, one can verify Equations (55), (56) and (57) using Lemma 16.

\section*{4.2 Combinatorial side — $H_{m,3}[X; q, t]$}

Now we consider the Hikita polynomial defined by Equation (59). Any parking function $\pi \in P_{m,3}$ has 3 rows, thus only has 3 cars: \{1, 2, 3\}, and the word $\sigma(\pi)$ can be any permutation $\sigma \in S_3$. Table 2 shows the $s_{\text{pides}}$ contribution of the 6 permutations in $S_3$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$\sigma \in S_3$ & 123 & 132 & 213 & 231 & 312 & 321 \\
\hline
$s_{\text{pides}}$ & $s_3$ & $s_{21}$ & $s_{12} = 0$ & $s_{21}$ & $s_{12} = 0$ & $s_{111}$ \\
\hline
\end{tabular}
\caption{$s_{\text{pides}}$ contribution of permutations in $S_3$.}
\end{table}

By our notation, \[H_{m,3}[X; q, t] = [s_3][m,3]s_3 + [s_{21}][m,3]s_{21} + [s_{111}][m,3]s_{111}.\] We can work out the combinatorial side of the Rational Shuffle Theorem in the case where $n = 3$ using (59).

\subsection*{4.2.1 Combinatorics of $H_{3k+1,3}[X; q, t]$}

We show the combinatorics of $H_{3k+1,3}[X; q, t]$ by enumerating the parking functions on $(3k + 1) \times 3$ lattice to prove the following formulas for the coefficients of Schur functions in $H_{3k+1,3}[X; q, t]$ (in
Given a parking function $\pi \in \mathcal{P}_{3k+1,3}$, we let $\Pi = \Pi(\pi)$ be the path of $\pi$. Its dinv correction is non-negative since $3k + 1 > 3$ for $k \geq 1$, and

$$\text{dinvcorr}(\pi) = \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c) + 1}{\text{leg}(c) + 1} \leq \frac{m}{n} < \frac{\text{arm}(c)}{\text{leg}(c)} \right).$$

The partition corresponding to the Dyck path $\Pi$ has at most 2 parts, so $\text{leg}(c)$ of a cell $c \in \lambda(\Pi)$ is either 0 or 1. Taking Figure 11 for reference, we have

(a) $c \in \lambda(\Pi)$ with $\text{leg}(c) = 0$ and $1 \leq \text{arm}(c) < k$ contributes 1 to dinv correction, marked $\bigcirc$ in Figure 11

(b) $c \in \lambda(\Pi)$ with $\text{leg}(c) = 1$ and $k < \text{arm}(c) \leq 2k - 1$ contributes 1 to dinv correction, marked $\bigtriangleup$ in Figure 11.

Further, we can directly count the statistics area and dinv correction (dinvcorr) from the partition $\lambda(\Pi)$. We write $\lambda = (\lambda_1, \lambda_2) = \lambda(\Pi)$, then $\lambda \subseteq \lambda_0 = (2k, k)$, i.e. $\lambda_1 \leq 2k$ and $\lambda_2 \leq k$. Clearly, the area of $\Pi$ is counted by $|\lambda_0| - |\lambda|$, i.e.

$$\text{area}(\Pi) = 3k - \lambda_1 - \lambda_2.$$  

We can also write the formula for dinv correction according to the partition $\lambda$:

$$\text{dinvcorr}(\Pi) = \begin{cases} 
\lambda_1 - 1 & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 \leq k, \\
\lambda_1 - 1 & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 > k, \\
k - 1 & \text{if } \lambda_2 = \lambda_1 \geq 1, \\
\lambda_1 - 2 & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \leq k, \\
2\lambda_1 - k - 3 & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \geq k + 1, \\
2\lambda_2 + k - 2 & \text{if } \lambda_2 \geq 1 \text{ and } \lambda_1 - \lambda_2 \geq k + 1.
\end{cases}$$

(69)
Note that the return statistic is always 1 since $3k + 1$ and 3 are coprime. We shall compute $[s_3]_{3k+1,3}$ first.

From Table 2 we see that only the parking functions in $P_{3k+1,3}$ with word 123 contribute to the coefficient of $s_3$. We also notice that the 3 cars should be in different columns, otherwise there are cars $i < j$ with $\text{rank}(i) < \text{rank}(j)$, contradicting with the restriction that the word of the parking function is 123. Thus we have one $\pi \in P_{3k+1,3}$ with word 123 on each $(3k + 1, 3)$ Dyck path which has no consecutive north steps.

Let $\lambda(\pi) = (\lambda_1, \lambda_2)$ be the partition associated to the Dyck path $\Pi(\pi)$ (see Figure 12), then $\text{area}(\pi)$ is counted by Equation (68). Since the ranks of cars 1, 2, 3 are decreasing, there is always no $\text{tdinv}$, thus $\text{dinv}(\pi) = \text{dinvcorr}(\Pi)$, which is counted by the latter 3 cases (since $\lambda_1 > \lambda_2 > 0$) of Equation (69).

$$\lambda(\pi)$$

1

2

3

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{parking_function_example.png}
\caption{Example: a parking function $\pi \in P_{7,3}$ with word 123.}
\end{figure}

For $[s_3]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+1]q,t$, we construct each term $(qt)^{k-1-i}[3i+1]q,t$ as a sequence of parking functions. Since each parking function corresponds to a unique partition $\lambda \subset (2k,k)$ with 2 distinct parts, we shall use partitions to represent parking functions in $P_{3k+1,3}$ with diagonal word 123. For each $i$, we define the following 3 branches of partitions (parking functions with word 123):  

\begin{align*}
\Lambda_1 &= \{(k + i + 1, k), (k + i, k - 1), \ldots, (k + 2, k - i + 1)\}, \\
\Lambda_2 &= \{(2k, i), (2k - 1, i - 1), \ldots, (2k + 1 - i, 1)\}, \\
\Lambda_3 &= \{(k + 1, k - i), (k, i + 1), \ldots, (k - i + 1, k - i)\}.
\end{align*}

The branch $\Lambda_1$ contains all the partitions $\lambda$ such that $\lambda_1 - \lambda_2 = i + 1 \leq k$ with $\lambda_2 > k - i$, the branch $\Lambda_2$ contains all the partitions $\lambda$ such that $\lambda_1 - \lambda_2 = 2k - i > k$, and the branch $\Lambda_3$ contains all the partitions $\lambda$ such that $\lambda_2 = i + 1$ and $\lambda_1 - \lambda_2 \leq k - i$. Notice that $|\Lambda_1| = |\Lambda_2|$. As shown in Figure 13 the construction begins with alternatively taking partitions from $\Lambda_1$ and $\Lambda_2$, ending with the last partition of $\Lambda_2$. Then continue the chain by taking partitions in $\Lambda_3$ and end the chain with the last partition $(k - i + 1, k - i)$ in $\Lambda_3$. The weights of the parking functions are $(qt)^{k-1-i}q^{3i}, (qt)^{k-1-i}q^{3i-1}t, \ldots, (qt)^{k-1-i}t^{3i}$ following the order of the chain.

To be more precise, it is not difficult to check that each parking function with diagonal word
which sum up to \((qt)^{k-1-i}[3i + 1]_{q,t}\). This proves that \([s_{3}]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i + 1]_{q,t}\). Figure 14 shows an example of the combinatorial construction of the coefficient \([s_{3}]_{10,3}\).

We can combinatorially prove \([s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i + 2]_{q,t} + [3i + 3]_{q,t})\) in a similar way. In this case, we have 2 possible diagonal words: 132 and 312. In both cases, the car 2 has the smallest rank, which means the label of the first (lowest) row must be 2. Thus, the pair of cars (1,2) does not produce a \(tdinv\). If we let \(\lambda = (\lambda_1, \lambda_2)\) be the partition corresponding to the path.
\( \Pi \), and let the labels of row 1, row 2, row 3 (counting from bottom to top) be \( \ell_1, \ell_2, \ell_3 \), then we have the following formula for temporary dinv:

\[
\text{tdinv}(\pi) = \begin{cases} 
\chi(\ell_3 > \ell_2) & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 \leq k, \\
\chi(\ell_3 > \ell_1) + \chi(\ell_2 > \ell_3) & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 > k, \\
\chi(\ell_2 > \ell_1) & \text{if } \lambda_2 = \lambda_1 \geq 1, \\
\chi(\ell_2 > \ell_1) + \chi(\ell_3 > \ell_2) & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \lambda_1 \leq k, \\
\chi(\ell_2 > \ell_1) + \chi(\ell_3 > \ell_1) + \chi(\ell_3 > \ell_2) & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \lambda_1 \geq k + 1, \\
\chi(\ell_2 > \ell_1) + \chi(\ell_3 > \ell_1) + \chi(\ell_2 > \ell_3) & \text{if } \lambda_2 \geq 1 \text{ and } \lambda_1 - \lambda_2 \geq k + 1.
\end{cases}
\]  

(73)

In the construction of the coefficient \( [s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}([3i + 2]_{q,t} + [3i + 3]_{q,t}) \), we construct each term \((qt)^{k-1-i}[3i + 2]_{q,t}\) or \((qt)^{k-1-i}[3i + 3]_{q,t}\) as a sequence of parking functions. First, we define the following 3 branches of parking functions to obtain the term \((qt)^{k-1-i}[3i + 3]_{q,t}\):

\[
\Lambda_1 = \{ \pi : \lambda(\pi) \in \{(2k, i + 1), (2k - 1, i), \ldots, (2k - i, 1)\}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3) \}, \\
\Lambda_2 = \{ \pi : \lambda(\pi) \in \{(2k, i), (2k - 1, i - 1), \ldots, (2k - i, 0)\}, (\ell_1, \ell_2, \ell_3) = (2, 3, 1) \}, \\
\Lambda_3 = \{ \pi : \lambda(\pi) \in \{(k, k - i - 1), \ldots, (k - i, k - i - 1)\}, (\ell_1, \ell_2, \ell_3) = (2, 3, 1) \}.
\]

With the 3 branches defined, the construction is similar to that of \((qt)^{k-1-i}[3i + 1]_{q,t}\) as a component of \([s_{3}]_{3k+1,3}\). We alternatively take the chain by taking partitions in \(\Lambda_3\) ending with the last partition of \(\Lambda_2\). Then we continue the chain by taking partitions in \(\Lambda_4\) ending with the last parking function corresponding to the partition \((k - i, k - i - 1)\) with labels \((\ell_1, \ell_2, \ell_3) = (2, 3, 1)\) in \(\Lambda_3\). The weights of the parking functions are \((qt)^{k-1-i}q^{3i+2}, \ldots, (qt)^{k-1-i}q^{3i+2}\), which sum up to \((qt)^{k-1-i}[3i + 3]_{q,t}\).

Second, we define another three branches of parking functions for \((qt)^{k-1-i}[3i + 2]_{q,t}\):

\[
\Lambda_4 = \{ \pi : \lambda(\pi) \in \{(k + i + 1, k), (k + i, k - 1), \ldots, (k + 1, k - i)\}, (\ell_1, \ell_2, \ell_3) = (2, 3, 1) \}, \\
\Lambda_5 = \{ \pi : \lambda(\pi) \in \{(k + i, k), (k + i - 1, k - 1), \ldots, (k - i)\}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3) \}, \\
\Lambda_6 = \{ \pi : \lambda(\pi) \in \{(k - 1, k - i), \ldots, (k - i, k - i)\}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3) \}.
\]

The construction is the same as that of \((qt)^{k-1-i}[3i + 3]_{q,t}\), and the weights of the parking functions are \((qt)^{k-1-i}q^{3i+1}, \ldots, (qt)^{k-1-i}q^{3i+1}\) which sum up to \((qt)^{k-1-i}[3i + 2]_{q,t}\).

Thus we have proved that \([s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}([3i + 2]_{q,t} + [3i + 3]_{q,t})\). Figure 15 shows an example of the combinatorial construction of the coefficient \([s_{21}]_{7,3}\).

The identity that \([s_{111}]_{3k+1,3} = \sum_{i=0}^{k}(qt)^{k-1-i}[3i + 1]_{q,t} = [s_{3}]_{3k+4,3}\) is a consequence of the following corollary of Theorem 3(a):

Corollary 17. For any \(m > 0\), \([s_{111}]_{m,3} = [s_{3}]_{m+3,3}\).

4.2.2 Combinatorics of \(X_{3k+2,3}[X; q, t]\)

We study the combinatorics of \(H_{3k+2,3}[X; q, t]\) in a similar manner by enumerating the parking functions on the \((3k + 2) \times 3\) lattice to prove the following formulas for the coefficients of Schur
functions in $H_{3k+2,3}[X; q, t]$ (in $q, t$-analogue notation):

\[
[s_3]_{3k+2,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i + 2]_{q,t},
\]

(74)

\[
[s_{21}]_{3k+2,3} = \sum_{i=-1}^{k-1} (qt)^{k-1-i}([3i + 3]_{q,t} + [3i + 4]_{q,t}), \quad \text{and}
\]

(75)

\[
[s_{111}]_{3k+2,3} = \sum_{i=0}^{k} (qt)^{k-i}[3i + 2]_{q,t}.
\]

(76)

Given a parking function $\pi \in \mathcal{P}_{3k+2,3}$ with $\Pi(\pi) = \Pi$, we can compute the dinv correction of $\pi$ by examining the cells $c \in \lambda(\Pi)$. Taking Figure 16 for reference,

(a) $c \in \lambda(\Pi)$ with leg($c$) = 0 and $1 \leq \text{arm}(c) < k$ contributes 1 to dinv correction, marked $\circ$ in Figure 16.

(b) $c \in \lambda(\Pi)$ with leg($c$) = 1 and $k < \text{arm}(c) \leq 2k$ contributes 1 to dinv correction, marked $\triangle$ in Figure 16.

Figure 16: The dinv correction of a $(3k + 2, 3)$-Dyck path when $k = 4$.

Further, we can directly count the statistics area and dinv correction (dinvcorr) from the partition $\lambda(\Pi) = (\lambda_1, \lambda_2) \subseteq (2k + 1, k)$ of a $(3k + 2, 3)$-Dyck path $\Pi$. Similar to Equation (68), we have
area(Π) = 3k + 1 - \lambda_1 - \lambda_2. \quad (77)

The dinv correction formula is the same as Equation (69), and the return is still always 1.

To prove \([s_{3k+2,3}] = \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i + 2]_{q,t},\) we shall construct the following 3 branches of partitions (parking functions with word 123) for each term \((qt)^{k-1-i}[3i + 2]_{q,t}:\)

\[
\begin{aligned}
\Lambda_1 &= \{(2k + 1, i + 1), (2k, i), \ldots, (2k + 1 - i, 1)\}, \\
\Lambda_2 &= \{(k + i + 1, k), (k + i, k - 1), \ldots, (k + 1, k - i)\}, \\
\Lambda_3 &= \{(k, k - i), (k - 1, i + 1), \ldots, (k - i + 1, k - i)\}.
\end{aligned}
\]

Then, we can follow the same construction as the \((3k + 1, 3)\) case to obtain all parking functions with word 123 and their weights \((qt)^{k-1-i}[3i + 2]_{q,t} = (qt)^{k-1-i}q^{3i+1}, \ldots, (qt)^{k-1-i}q^{3i+1}.\)

Similarly, to prove \([s_{21}]_{3k+2,3} = \sum_{i=1}^{k-1} (qt)^{k-1-i}([3i + 3]_{q,t} + [3i + 4]_{q,t}),\) we have 6 branches of parking functions as follows:

\[
\begin{aligned}
\Lambda_1 &= \{\pi : \lambda(\pi) \in \{(k + 2, k), \ldots, (k + 2, k - i)\}, (\ell_1, \ell_2, \ell_3) = (2, 3, 1)\}, \\
\Lambda_2 &= \{\pi : \lambda(\pi) \in \{(k + 1, k), \ldots, (k + 1, k - i)\}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3)\}, \\
\Lambda_3 &= \{\pi : \lambda(\pi) \in \{(k, k - i), \ldots, (k, k - i)\}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3)\}, \\
\Lambda_4 &= \{\pi : \lambda(\pi) \in \{(2k + 2, i + 1), \ldots, (2k - i + 1, 1)\}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3)\}, \\
\Lambda_5 &= \{\pi : \lambda(\pi) \in \{(2k + 1, i), \ldots, (2k - i + 1, 0)\}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3)\}, \\
\Lambda_6 &= \{\pi : \lambda(\pi) \in \{(k, k - i), \ldots, (k, k - i)\}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3)\}.
\end{aligned}
\]

Then, the total weight of parking functions in the first 3 branches is \((qt)^{k-1-i}[3i + 4]_{q,t}\), and the total weight of parking functions in the last 3 branches is \((qt)^{k-1-i}[3i + 3]_{q,t}\).

The proof of \([s_{11}]_{3k+2,3} = [s_{3}]_{3(k+1)+2,3} = \sum_{i=0}^{k} (qt)^{k-i}[3i + 2]_{q,t}\) follows from Corollary [17].

4.2.3 Combinatorics of \(H_{3k,3}[X; q, t]\)

Notice that the area and dinv of parking functions in \(P_{3k,3}\) are equal to those of the parking functions in \(P_{3k+1,3}\). Given a parking function \(\pi \in P_{3k+1,3}\) where \(\lambda(\pi) = \lambda = (\lambda_1, \lambda_2)\), the return statistic of \(\pi\) is formulated as

\[
\text{ret}(\pi) = 2\chi(\lambda_1 = 2k) + \chi(\lambda_2 = k) - 2\chi(\lambda_1 = 2k) \cdot \chi(\lambda_2 = k).
\]

By the Extended Rational Shuffle Theorem in the non-coprime case,

\[
H_{3k,3}[X; q, t] = \sum_{\pi \in P_{3k,3}} \left[\text{ret}(\pi)\right]^1q^{\text{area}(\pi)}s_{\text{pides}(\pi)}F_{\text{pides}(\pi)}[X] = \sum_{\pi \in P_{3k+1,3}} \left[\text{ret}(\pi)\right]^1q^{\text{area}(\pi)}s_{\text{pides}(\pi)}. \quad (78)
\]
To prove the following formulas for the coefficients of Schur functions in $H_{3k+1,3}[X; q, t]$,

\begin{align}
[s_{3}]_{3k,3} &= \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i-1]_{q,t} + [3i]_{q,t} + [3i+1]_{q,t}), \tag{79} \\
[s_{21}]_{3k,3} &= (qt)^{k-1}([3]_{q,t} + 2[2]_{q,t} + [1]_{q,t}) \\
&\quad + \sum_{i=1}^{k-1} (qt)^{k-1-i}([3i]_{q,t} + 2[3i+1]_{q,t} + 2[3i+2]_{q,t} + [3i+3]_{q,t}), \tag{80} \\
[s_{111}]_{3k,3} &= \sum_{i=0}^{k} (qt)^{k-i}([3i-1]_{q,t} + [3i]_{q,t} + [3i+1]_{q,t}), \tag{81}
\end{align}

we use the constructions of $[s_{3}]_{3k+1,3}, [s_{21}]_{3k+1,3}, [s_{111}]_{3k+1,3}$ and modify the weight of parking functions with nonzero returns. We use the same sets of partitions $\Lambda_{1}, \Lambda_{2}, \Lambda_{3}, \Lambda_{4}, \Lambda_{5}, \Lambda_{6}$ as Section 4.2.1.

For $[s_{3}]_{3k,3}$, the first parking function in each set $\Lambda_{1}$ and $\Lambda_{2}$ has return statistic 1, except that the first parking function in $\Lambda_{1}$ when $i = k - 1$ has return 2. All the remaining parking functions have return 3. Then we prove Equation (79) by summing up the parking function weights.

For $[s_{21}]_{3k,3}$, the first parking function in each of the sets $\Lambda_{1}, \Lambda_{2}, \Lambda_{4}, \Lambda_{5}$ has return statistic 1 since the second parts of these partitions are $k$, except that the first parking function in $\Lambda_{1}$ or $\Lambda_{4}$ when $i = k - 1$ has return 2. All the remaining parking functions have return 3. Then again we obtain Equation (80) by direct computation.

The proof of $[s_{111}]_{3k,3} = [s_{3}]_{3(k+1),3}$ follows from Corollary 17.

5 Combinatorial results about Schur function expansions of the $(3, n)$ case

5.1 Recursive formula for $[s_{\lambda}]_{3,n}$

In $(3, n)$ case, we have $n$ cars, i.e. the word of a $(3, n)$ parking function is a permutation of $[n]$. By Remark 2, $[s_{\lambda}]_{3,n} \neq 0$ implies that $\lambda$ must be of the form $3^a 2^b 1^c$ with $3a + 2b + c = n$, i.e. $[s_{\lambda}]_{3,n} \neq 0$ only if the partition $\lambda$ only has parts of sizes less than or equal to 3.

We have the following corollary of Theorem 3 summarizing several symmetries about $[s_{\lambda}]_{3,n}$.

Corollary 18. For all $n > 0$ and $a, b, c \geq 0$,

(a) $[s_{3^a 2^b 1^c}]_{3,n} = [s_{2^b 1^c}]_{3,n-3a}$,

(b) $[s_{1^n}]_{3,n} = [s_{111}]_{n,3}$,

(c) $[s_{21^n-2}]_{3,n} = [s_{21}]_{n,3}$.

Further, we have found the straightening action in parking functions combinatorially from parking functions with pides $\{\cdots, 1, 3, \cdots\}$ to parking functions with pides $\{\cdots, 2, 2, \cdots\}$, which is an involution $\Phi$ whose fixed points are the coefficients of $[s_{2^1 1^0}]_{3,n}$. We call the fixed points of $\Phi$ the fixed parking functions. The details of the involution $\Phi$ will be given in Section 5.2.1.
Let $a, b$ be positive integers. We have conjectured a bijection $\mathcal{S}$ between the fixed parking functions with pides $2^a1^b$ and the fixed parking functions with pides $2^b1^a$ in the coprime case, mapping the 2 cars (or 1 car) causing part 2 (or 1) in pides $2^a1^b$ to 1 car (or 2 cars) causing part 1 (or 2) in pides $2^b1^a$. The map $\mathcal{S}$ has nice properties summarized in Theorem 24, Theorem 25 and Conjecture 26 in Section 5.2.2, and they imply the coprime case of another important symmetry: Conjecture 19. For all $a, b, n \geq 0$,

$$[s_{2^a1^b}]_{3,n} = [s_{2^b1^a}]_{3,3(a+b)-n}.$$  

The results above show that the problem of computing the Schur function expansion of $Q_{3,n}(1)$ can be reduced to the problem of finding the coefficients of Schur functions of the form $s_{2^a1^b}$ where $a < b$. Finally, we conjecture a recursive formula for such coefficients $[s_{2^a1^b}]_{3,n}$ where $a < b$.

Conjecture 20. Let $a < b$ then

$$[s_{2^a1^b}]_{3,n} = (qt)[s_{2^b1^a-3}]_{3,n-3} + \sum_{i=0}^{a} [b+i]_{q,t}.$$  

We have verified this formula by Maple for $n < 27$. If the conjectures are true, then we have solved the Schur function expansion in the $(3, n)$ case.

5.2 The involution $\Phi$ and the map $\mathcal{S}$

We shall first introduce an involution on $(3, n)$-parking functions whose pideses contain 1, 3 or 2, 2. The fixed points of the involution is a subset of parking functions whose pideses do not contain 1, 3. Then, we conjecture a bijection between the fixed parking functions with pides $2^a1^b$ and the fixed parking functions with pides $2^b1^a$ for the coprime case.

5.2.1 The involution $\Phi$

An involution $f$ of a set $S$ is a bijection from $S$ to itself, such that $f^2 = \text{id}$ is the identity map. An element $s \in S$ such that $f(s) = s$ is called a fixed point of the involution.

Suppose that there is a weight function $w(s)$ for each elements $s \in S$. A sign-reversing involution $f$ of the set $S$ (with respect to the weight $w$) is an involution such that for all $s \in S$, if $f(s) \neq s$, then $w(f(s)) = -w(s)$. As a consequence, we have

$$\sum_{s \in S} w(s) = \sum_{s \in S} w(f(s)) = \sum_{s \in S, f(s) = s} w(s),$$  

i.e. we only need to consider the fixed points of $f$ when computing the total weight of the set $S$.

By definition,

$$Q_{3,n}(1) = H_{3,n}[X; q, t] = \sum_{\pi \in \mathcal{P}_{3,n}} [\text{ret}(\pi)]_{1} t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} s_{\text{pides}(\pi)}$$  

is a sum of weights of parking functions in $\mathcal{P}_{3,n}$, where the weight of a parking function $\pi$ is $[\text{ret}(\pi)]_{1} t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} s_{\text{pides}(\pi)}$. In this section, we build a sign-reversing involution $\Phi$ for the set $\mathcal{P}_{3,n}$ in order to simplify the computation of the polynomial $Q_{3,n}(1)$.  

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Note that by the straightening action on Schur functions, we have

\[ s_{\lambda 13 \mu} = -s_{\lambda 22 \mu}, \]  

(84)

where \( \lambda \) and \( \mu \) are two compositions, and \( \lambda 13 \mu \) (or \( \lambda 22 \mu \)) is the composition obtained by first listing all the parts in \( \lambda \), then two parts of sizes 1 and 3 (or 2 and 2), finally all the parts in \( \mu \).

Let \( P_{3,n|\lambda 13 \mu} \) be the set of parking functions in \( P_{3,n} \) with pides \( \lambda 13 \mu \) and \( P_{3,n|\lambda 22 \mu} \) be the set of parking functions in \( P_{3,n} \) with pides \( \lambda 22 \mu \), then we can give an involution \( \Phi \) of the set \( P_{3,n|\lambda 13 \mu} \cup P_{3,n|\lambda 22 \mu} \), such that

- all the fixed points of \( \Phi \) are in the set \( P_{3,n|\lambda 22 \mu} \), and
- the set of non-fixed points in \( P_{3,n|\lambda 22 \mu} \) are in bijection with the set \( P_{3,n|\lambda 13 \mu} \).

Let \( \pi \) be a parking function in \( P_{3,n} \). If \( \text{pides}(\pi) = \lambda 13 \mu \), then without loss of generality, we suppose that the cars causing pides 13 are 1, 2, 3, 4, which means that \( \text{rank}(1) < \text{rank}(2) > \text{rank}(3) > \text{rank}(4) \). There are 3 possible subwords (subsequences of the words of \( \pi \)) formed by the 4 cars, which are

\[ 2341, \ 2314, \ 2134. \]

On the other hand, the cars 2, 3, 4 are in different columns since \( \text{rank}(2) > \text{rank}(3) > \text{rank}(4) \). Given that there are only 3 columns, we have three possible placements of the four cars:

(I) Cars 1 and 4 are in the same column.

(II) Cars 1 and 2 are in the same column.

(III) Cars 1 and 3 are in the same column.

If the four cars form a word 2341, then (I), (II), (III) are all possible; if the four cars form a word 2314, then only (II), (III) are possible; if the four cars form a word 2134, then only (II) is possible.

Next, we consider the case when \( \text{pides}(\pi) = \lambda 13 \mu \), i.e. the cars 1, 2, 3, 4 cause pides 22, and \( \text{rank}(1) > \text{rank}(2) < \text{rank}(3) > \text{rank}(4) \). The possible words are

\[ 3412, \ 3142, \ 3124, \ 1324, \ 1342. \]

The cars 1, 2 and the cars 3, 4 have to be in different columns since \( \text{rank}(1) > \text{rank}(2) \) and \( \text{rank}(3) > \text{rank}(4) \), thus we have the following five possible placements of the four cars:

(i) Both cars 1, 3 and 2, 4 are in the same column.

(ii) Only cars 1 and 4 are in the same column.

(iii) Only cars 2 and 4 are in the same column.

(iv) Only cars 1 and 3 are in the same column.

(v) Only cars 2 and 3 are in the same column.

If the four cars form a word 3412, then (i), (ii), (iii), (iv) and (v) are all possible; if the four cars form a word 3142, then only (i), (iii), (iv) and (v) are possible; if the four cars form a word 3124, then only (iv) and (v) are possible; if the four cars form a word 1324, then only (v) is possible; if the four cars form a word 1342, then only (iii) and (v) are possible.
For any permutation \( \sigma \in S_n \) and any \( \pi \in \mathcal{P}_{3,n} \), we let \( \sigma \cdot \pi \) be the parking function obtained by permuting the cars of \( \pi \) by the permutation \( \sigma \). We also let \( \text{word}(\pi) \) be the word of the cars 1, 2, 3, 4. Then we can define the map \( \Phi \) on the set \( \mathcal{P}_{3,n|\lambda_1\lambda_3\mu} \) that

\[
\Phi|_{\mathcal{P}_{3,n|\lambda_1\lambda_3\mu}} : \mathcal{P}_{3,n|\lambda_1\lambda_3\mu} \to \mathcal{P}_{3,n|\lambda_2\mu}.
\]

Following is the detailed definition, while the words and the placements of the images are recorded in each case:

\[
\Phi(\pi) = \begin{cases} 
(1, 2)\pi & \text{if } \text{word}(\pi) = 2341 \text{ and placement is (I). } \Phi(\pi) \text{ has word 1342 (iii).} \\
(1, 2, 3)\pi & \text{if } \text{word}(\pi) = 2341 \text{ and placement is (II). } \Phi(\pi) \text{ has word 3142 (v).} \\
(1, 2)\pi & \text{if } \text{word}(\pi) = 2314 \text{ and placement is (III). } \Phi(\pi) \text{ has word 1324 (v).} \\
(1, 2, 3)\pi & \text{if } \text{word}(\pi) = 2341 \text{ and placement is (IV). } \Phi(\pi) \text{ has word 3124 (v).} \\
(2, 3)\pi & \text{if } \text{word}(\pi) = 2314 \text{ and placement is (II). } \Phi(\pi) \text{ has word 2341 (III).} \\
(1, 3, 2)\pi & \text{if } \text{word}(\pi) = 3124 \text{ and placement is (v). } \Phi(\pi) \text{ has word 3142 (I).} \\
(1, 2)\pi & \text{if } \text{word}(\pi) = 3142 \text{ and placement is (I). } \Phi(\pi) \text{ has word 2341 (II).} \\
(1, 2)\pi & \text{if } \text{word}(\pi) = 1324 \text{ and placement is (II). } \Phi(\pi) \text{ has word 2314 (III).} \\
(1, 2)\pi & \text{if } \text{word}(\pi) = 3142 \text{ and placement is (III). } \Phi(\pi) \text{ has word 3124 (IV).} \\
(1, 2)\pi & \text{if } \text{word}(\pi) = 3142 \text{ and placement is (IV). } \Phi(\pi) \text{ has word 3124 (IV).} \end{cases}
\]

Then we shall define the map \( \Phi \) on the set \( \mathcal{P}_{3,n|\lambda_2\mu} \) that

\[
\Phi|_{\mathcal{P}_{3,n|\lambda_2\mu}} : \mathcal{P}_{3,n|\lambda_2\mu} \to \mathcal{P}_{3,n|\lambda_1\lambda_3\mu} \cup \mathcal{P}_{3,n|\lambda_2\mu}.
\]

Notice that the non-fixed points in \( \mathcal{P}_{3,n|\lambda_2\mu} \) are mapped into the set \( \mathcal{P}_{3,n|\lambda_1\lambda_3\mu} \). We have

\[
\Phi(\pi) = \begin{cases} 
\pi & \text{if } \text{word}(\pi) = 3412, \text{ or word}(\pi) = 3142 \text{ and placement is (I), (iii), (iv).} \\
(1, 3, 2)\pi & \text{if } \text{word}(\pi) = 3142 \text{ and placement is (v). } \Phi(\pi) \text{ has word 2341 (II).} \\
(2, 3)\pi & \text{if } \text{word}(\pi) = 3124 \text{ and placement is (iv). } \Phi(\pi) \text{ has word 2134 (II).} \\
(1, 3, 2)\pi & \text{if } \text{word}(\pi) = 3124 \text{ and placement is (v). } \Phi(\pi) \text{ has word 2314 (II).} \\
(1, 2)\pi & \text{if } \text{word}(\pi) = 1324 \text{ and placement is (II). } \Phi(\pi) \text{ has word 2341 (III).} \\
(1, 2)\pi & \text{if } \text{word}(\pi) = 1324 \text{ and placement is (III). } \Phi(\pi) \text{ has word 3142 (I).} \\
(1, 2)\pi & \text{if } \text{word}(\pi) = 1342 \text{ and placement is (v). } \Phi(\pi) \text{ has word 3124 (IV).} \end{cases}
\]

The first case above gives the fixed points of \( \Phi \).

It is easy to check that the map \( \Phi \) preserves area and the dinv since \( \Phi \) does not change the Dyck path of \( \pi \), and it also preserves the cars other than \( \{1, 2, 3, 4\} \). Since \( \Phi \) changes the sign of the weight of each non-fixed point, it follows immediately that \( \Phi \) forms a sign-reversing involution of the set of parking functions in \( \mathcal{P}_{3,n} \) with pides of either \( \lambda_1\lambda_3\mu \) or \( \lambda_2\mu \). As we mentioned, the set of fixed points of this involution is

\[
\text{fp}(\Phi) = \{ \pi \in \mathcal{P}_{3,n|\lambda_2\mu} : \text{word}(\pi) = 3412, \text{ or word}(\pi) = 3142 \text{ (i), (iii), (iv)}. \}
\]

If we apply the involution \( \Phi \) to all the parking functions \( \pi \in \mathcal{P}_{3,n} \) that we compute pides(\( \pi \)) and scan from left to right to find the first occurrence of either (1, 3) or non-fixed (2, 2) and apply \( \Phi \) at that position. Then clearly, we have

**Theorem 21.** \( \Phi \) is a sign-reversing involution of the set of parking functions in \( \mathcal{P}_{3,n} \).

The fixed points in \( \mathcal{P}_{3,n} \) have weakly decreasing pides in the form \( 3^{a_1}2^{a_2+b} \), and these parking functions contribute to the coefficients of the Schur functions. Thus, we have the Schur positivity of the \( m = 3 \) case:

**Corollary 22.** The polynomial \( Q_{3,n}(1) \) is Schur positive, and

\[
[s_{2^a1^b}]_{3,n} = \sum_{\pi \in \mathcal{P}_{3,n}, \text{pides}(\pi) = 2^a1^b, \pi \text{ fixed by } \Phi} \text{ret}(\pi)\binom{\text{area}(\pi)}{1} q^{\text{dinv}(\pi)}. \tag{87}
\]
5.2.2 The map $S$ and the symmetry $[s_{2a+1}]_{3,n} = [s_{2b+1}]_{3,3(a+b)−n}$

For any parking function $\pi \in \mathcal{P}_{3,n}$ with $\text{pides}(\pi) = 2^a1^b$, we study the placement of the $a$ pairs of numbers
\[
\{(1, 2), (3, 4), \ldots, (2a-1, 2a)\}
\]
and the $b$ singletons
\[
\{2a + 1, \ldots, 2a + b\}.
\]
Note that the two cars in each pair cannot be placed in the same column since the rank of the smaller car is larger than the rank of the bigger car.

Since there are 3 columns, we have $\binom{3}{2}$ ways to choose columns for each pair $(2i - 1, 2i)$. We name the 3 columns from left to right by $\ell, c, r$. Once we determine 2 columns for the pair, the filling of the two cars in the pair is fixed by their ranks since $\text{rank}(2i - 1) > \text{rank}(2i)$. Now, we define the notation for the placement of a pair $(2i - 1, 2i)$:

1. $L$ means $(2i - 1, 2i)$ are in the left 2 columns $\ell, c$,
2. $R$ means $(2i - 1, 2i)$ are in the right 2 columns $c, r$,
3. $C$ means $(2i - 1, 2i)$ are in columns $\ell, r$.

Similarly, we have $\binom{3}{1}$ ways to choose a column for each singleton. For a singleton $j$, we define the notation for the placement:

1. $L$ means $j$ is in the left column $\ell$,
2. $R$ means $j$ is in the right column $r$,
3. $C$ means $j$ is in column $c$.

Now we are ready to describe the map $S$. Given a parking function $\pi \in \mathcal{P}_{3,n}$ with $\text{pides}(\pi) = 2^a1^b$, we track the placements of the $a$ pairs of cars $\{(1, 2), \ldots, (2a - 1, 2a)\}$ and the $b$ singleton cars $\{2a + 1, \ldots, 2a + b\}$. Let the $a + b$ placements of these $a + b$ objects be $p_1, \ldots, p_a, p_{a+1}, \ldots, p_{a+b}$ (here $p_i$ is one of $L, R$ or $C$).

Then we consider $b$ pairs of cars $\{(1, 2), \ldots, (2b - 1, 2b)\}$ and a singleton cars $\{2b + 1, \ldots, 2b + a\}$. We build a new parking function $S(\pi)$ by first counting how many cars in each column if we assign the $b + a$ placements $p_{b+a}, \ldots, p_1$ to the $b + a$ objects in this reversed order, then constructing the path according to the number of cars of each column. Finally, we fill from the first pair $(1, 2)$ to the last singleton $2b + a$ based on the rule that $\text{rank}(2i - 1) < \text{rank}(2i)$ for $i \leq b$ for the $b$ pairs of cars and the column placement choice $p_{b+a}, \ldots, p_1$. We call this map the switch map $S$. Figure [17] shows an example that we can construct a parking function in $\mathcal{P}_{3,5}$ with pides $21^3$ from a parking function in $\mathcal{P}_{3,7}$ with pides $2^31$.

We shall prove several properties of the map $S$. It is even not obvious that the image of a parking function is still above the diagonal, thus we shall show that

**Theorem 23.** If $\pi$ is a $(3, n)$-parking function with pides $2^a1^b$, then $S(\pi)$ is also a parking function.

**Proof.** We still consider the $a$ pairs of cars $\{(1, 2), \ldots, (2a - 1, 2a)\}$ and the $b$ singleton cars $\{2a + 1, \ldots, 2a + b\}$ of $\pi$. Suppose that there are $\ell_1, c_1, r_1$ placements of the first $a$ pairs of cars which
are $L$, $R$ and $C$ respectively, and $\ell_2, c_2, r_2$ placements of the last $b$ singleton cars which are $L$, $R$ and $C$ respectively. Without loss of generality, we suppose that $n = 3k + 1$. Then we have

\begin{align*}
\ell_1 + c_1 + r_1 &= a, \\
\ell_2 + c_2 + r_2 &= b, \\
2a + b &= 3k + 1.
\end{align*}

(88) (89) (90)

Since $\pi$ is a parking function, the path of the parking function should be above the diagonal, thus the number of cars in the left column is at least $k + 1$ and the number of cars in the left two columns is at least $2k + 1$.

Note that an $L$ placement of a pair contribute 1 left car and 1 center car, a $C$ placement of a pair contribute 1 left car and 1 right car, and an $R$ placement of a pair contribute 1 right car and 1 center car. The contribution of the singleton cars are obvious. Thus the number of cars in the left column is $\ell_1 + c_1 + \ell_2$, and the number of cars in the left 2 columns is $2\ell_1 + r_1 + c_1 + \ell_2 + c_2 = a + \ell_1 + \ell_2 + c_2$, and we have that

\begin{align*}
\ell_1 + c_1 + \ell_2 &\geq k + 1, \\
a + \ell_1 + \ell_2 + c_2 &\geq 2k + 1.
\end{align*}

(91) (92)

Next, for $S(\pi)$, it has $\ell_2, c_2, r_2$ placements of the first $b$ pairs of cars which are $L$, $R$ and $C$ respectively, and $\ell_1, c_1, r_1$ placements of the last $a$ singleton cars which are $L$, $R$ and $C$ respectively. The total number of cars is equal to $2a + b = 3(a + b) - (2b + a) = 3(a + b) - 3k - 1 = 3(a + b - k - 1) + 2$, the number of cars in the left column should be at least $a + b - k = (2a + b) - a - k = 3k + 1 - a - k = 2k + 1 - a$, and the number of cars in the left two columns should be at least $2a + 2b - 2k = b + (3k + 1) - 2k = b + k + 1$. $S(\pi)$ is a parking function if the following equations hold:

\begin{align*}
\ell_1 + c_2 + \ell_2 &\geq 2k + 1 - a, \\
b + \ell_1 + \ell_2 + c_1 &\geq b + k + 1.
\end{align*}

(93) (94)

Clearly, (91) implies (94), (92) implies (93).

Next, we have the formula for area.

**Theorem 24.** Let $\pi$ be a $(3,n)$-parking function with pides $2^a1^b$. Using the definitions of $\ell_1, c_1, r_1, \ell_2, c_2, r_2$ in the proof of Theorem 23. Let $L = \ell_1 + \ell_2, R = r_1 + r_2, C = c_1 + c_2$, then

\begin{align*}
\text{area}(\pi) &= L - R - 1.
\end{align*}

(95)
Proof. We want to compute the area of a parking function as the difference of its coarea and the maximum coarea of a \((3, 2a + b)\)-parking function. The maximum coarea of a \((3, 2a + b)\)-parking function is 
\[
\frac{(2a+b-1)(3-1)}{2} = 2a + b - 1.
\]

Notice that the cars in the right column contribute 2 to coarea, and the cars in the center column contribute 1 to coarea, thus the coarea of \(\pi\) is
\[
\ell_1 + 3r_1 + 2c_1 + 2r_2 + c_2 = a + 2(r_1 + r_2) + (c_1 + c_2) = a + 2R + C.
\]
Then,
\[
\text{area}(\pi) = 2a + b - 1 - (a + 2R + C) = a + (L + R + C) - 1 - (a + 2R + C) = L - R - 1. \tag{97}
\]

It follows immediately from Theorem 24 that

**Theorem 25.** For any \(\pi \in \mathcal{P}_{3,n}\) with \(\text{pides}(\pi) = 2^a 1^b\),
\[
\text{area}(\pi) = \text{area}(S(\pi)). \tag{98}
\]

We have not yet proved, but verified all parking functions with less than or equal to 10 rows for the following conjecture:

**Conjecture 26.** For any \(\pi \in \mathcal{P}_{3,n}\) with \(\text{pides}(\pi) = 2^a 1^b\),

(a) \(\text{dinv}(\pi) = \text{dinv}(S(\pi))\).

(b) When \(n\) and 3 are coprime, if \(\pi\) is a fixed point of \(\Phi\), then so is \(S(\pi)\), and \(\text{pides}(S(\pi)) = 2^b 1^a\).

Notice that \(S\) does not preserve the “return” statistic, and the return of any parking function in the coprime case is 1 which is trivial. In the non-coprime case when \(n\) is a multiple of 3, Conjecture 26 (b) will fail. Further, we have the following results:

**Theorem 27.** In the case when 3 and \(n\) are coprime, Conjecture 26 (b) implies that the map \(S\) is a bijection between the fixed parking functions with \(\text{pides} 2^a 1^b\) and the fixed parking functions with \(\text{pides} 2^b 1^a\), and
\[
[s_{2^a 1^b}]_{3,n} = [s_{2^b 1^a}]_{3,3(a+b)-n}. \tag{99}
\]

Proof. The bijectivity follows from the fact that the map \(S\) is invertible. Equation (99) follows from the bijectivity and Equation (87).

5.2.3 The switch map \(S\) in the \(m\) column case

We haven’t completely understood how to use straightening to compute the coefficients of \(s_\lambda\) for general \((m, n)\) case, but computations in Maple have led us to conjecture the following:

**Conjecture 28.** For all \(m, n > 0\) and \(\alpha_i \geq 0\),
\[
[s_{(m-1)\alpha_{m-1}(m-2)\alpha_{m-2}...1\alpha_1}]_{m,n} = [s_{(m-1)\alpha_1(m-2)\alpha_2...1\alpha_{m-1}}]_{m,m \sum_{i=1}^{m-1} \alpha_i-n}. \tag{100}
\]

On the other hand, the switch map \(S\) that we have defined for the three column case can be naturally generalized to the \(m\) column case, which conjecturally has many nice properties and is considered to be useful in proving Conjecture 28. The definition of an \(m\) columns switch map will need some new definitions.
Given any parking function \( \pi \in \mathcal{P}_{m,n} \), we suppose that \( s, s + 1, \ldots, s + r - 1 \) form an increasing subsequence of the word \( \sigma(\pi) \), then by Remark \[ \text{Theorem 30.} \] the cars \( s, s + 1, \ldots, s + r - 1 \) must be placed in \( r \) different columns in a rank decreasing way.

There are \( \binom{m}{r} \) possible choices to pick \( r \) columns for such cars \( s, s + 1, \ldots, s + r - 1 \). Let \( p = \{s_1, s_2, \ldots, s_r\} \subset \{1, \ldots, m\} \) be a possible placement (i.e. the choice of columns), then we define the reverse complement of \( p \) to be \( p^\text{rc} := \{1, \ldots, m\} \setminus \{m + 1 - s_r, \ldots, m + 1 - s_1\} \), which is a placement for \( m - r \) cars.

Given \( \mu = \mu_1 \cdots \mu_k \models n \). Suppose that \( \sigma(\pi) \), the word of \( \pi \), is a shuffle of the increasing sequences \( (1, \ldots, \mu_1), (\mu_1 + 1, \ldots, \mu_1 + \mu_2), \ldots, (n - \mu_k + 1, \ldots, n) \), and the placement of the sequence \((\mu_1 + \ldots + \mu_{i-1} + 1, \ldots, \mu_1 + \ldots + \mu_i)\) is \( p_i \), then we construct \( S(\pi) \) as follows.

Let \( \mu^\gamma := \mu_1^\gamma \cdots \mu_k^\gamma \), where \( \mu_i^\gamma := m - \mu_{k+1-i} \). We build the word of \( S(\pi) \) to be a shuffle of \((1, \ldots, \mu_1^\gamma), (\mu_1^\gamma + 1, \ldots, \mu_1^\gamma + \mu_2^\gamma), \ldots, (n - \mu_k^\gamma + 1, \ldots, n)\), and choose \( p^\text{rc} \) as the placement of \((\mu_1^\gamma + \ldots + \mu_{i-1}^\gamma + 1, \ldots, \mu_1^\gamma + \ldots + \mu_i^\gamma)\). This construction is well defined, and we can invert the map when the composition \( \mu \) is given.

For example, suppose that there are \( m = 4 \) columns. Take \( \mu = (2, 2, 3) \models n \) where \( n = 7 \). For a \((4, 7)\)-parking function \( \pi \) whose word \( \sigma(\pi) = 5613472 \) is a shuffle of \((1, 2), (3, 4), (5, 6, 7)\) with placements \( \{1, 3\}, \{1, 2\}, \{1, 2, 4\} \), we construct \( S(\pi) \) such that its word is a shuffle of \((1, 2, 3), (4, 5)\) and the placements are \( \{2\}, \{2, 1\}, \{1, 3\} \), shown in Figure 18.

Figure 18: An example of \( \pi \) and \( S(\pi) \).

Using the same technique as the 3 column case, we have

**Theorem 29.** \( \pi \) is an \((m, n)\)-parking function if and only if \( S(\pi) \) is a parking function. Further, \( \text{area}(\pi) = \text{area}(S(\pi)) \).

For a composition \( \mu = \mu_1 \cdots \mu_k \models n \), we say a permutation \( \sigma \) is a shuffle of \( \mu \) if \( \sigma \) is a shuffle of the increasing sequences \( (1, \ldots, \mu_1), (\mu_1 + 1, \ldots, \mu_1 + \mu_2), \ldots, (n - \mu_k + 1, \ldots, n) \). Then we have:

**Theorem 30.** The switch map \( S \) is a bijection between \((m, n)\)-parking functions whose words are shuffle of \( \mu = \mu_1 \cdots \mu_k \) and \((m, mk - n)\)-parking functions whose words are shuffle of \( \mu^\gamma = (m - \mu_k) \cdots (m - \mu_1) \).

The switch map of \( m \) column case still preserves the dinv statistic experimentally (summarized in the following conjecture), which we are not able to prove.

**Conjecture 31.** For any \( \pi \in \mathcal{P}_{m,n} \) where \( \sigma(\pi) \) is a shuffle of \( \mu \models n \),

\[ \text{dinv}(\pi) = \text{dinv}(S(\pi)) \].

33
Thus conjecturally, the switch map $S$ is an area, dinv-preserving bijective map between $(m, n)$-parking functions whose words are shuffle of $\mu$ and $(m, mk - n)$-parking functions whose words are shuffle of $\mu'$. In the end, we shall discuss some results related to Theorem 29, Theorem 30 and Conjecture 31.

Referring to Haglund’s work in [13], for any parking function whose word is a shuffle of $\mu = \mu_1 \cdots \mu_k \models n$, we can replace the cars $\mu_1 + \ldots + \mu_i - 1, \ldots, \mu_1 + \ldots + \mu_i$ with number $i$ to obtain a word parking function with cars $1^{\mu_1} \cdots k^{\mu_k}$ with the same area and dinv statistics. Further when $m$ and $n$ are coprime, we have

$$Q_{m,n}(1) = \sum_{\pi \in \mathcal{P}_{m,n}, \, \sigma(\pi) \text{ is a shuffle of } \mu} q^{\text{area}(\pi)} q^{\text{dinv}(\pi)}.$$  \hfill (101)

By the definition of Hall scalar product, for any symmetric function $f$, we have

$$\langle f, h_{\mu} \rangle = f \big|_{m_{\mu}}.$$  \hfill (102)

Thus, the properties of the switch map $S$ (Theorem 29, Theorem 30 and Conjecture 31) imply the coprime case of the following conjecture:

**Conjecture 32.** For $m, n > 0$, $\mu = \mu_1 \cdots \mu_k \models n$ and $\mu' = (m - \mu_k) \cdots (m - \mu_1)$,

$$\langle Q_{m,n}(1), h_{\mu} \rangle = \langle Q_{m,mk-n}(1), h_{\mu'} \rangle.$$  \hfill (103)

Since the area-preserving property of $S$ is proved in Theorem 29 we have the following theorem which is a special case of Conjecture 32.

**Theorem 33.** For coprime integers $(m, n)$, $\mu = \mu_1 \cdots \mu_k \models n$ and $\mu' = (m - \mu_k) \cdots (m - \mu_1)$,

$$\langle Q_{m,n}(1) \big|_{q=1}, h_{\mu} \rangle = \langle Q_{m,mk-n}(1) \big|_{q=1}, h_{\mu'} \rangle.$$  \hfill (104)

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