Spin Networks and Anyonic Topological Computing

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ABSTRACT

We review the $q$-deformed spin network approach to Topological Quantum Field Theory and apply these methods to produce unitary representations of the braid groups that are dense in the unitary groups.

Keywords: braiding, knotting, linking, spin network, Temperley–Lieb algebra, unitary representation.

1. INTRODUCTION

This paper describes the background for topological quantum computing in terms of Temperley–Lieb Recoupling Theory. This is a recoupling theory that generalizes standard angular momentum recoupling theory, generalizes the Penrose theory of spin networks and is inherently topological. Temperley–Lieb Recoupling Theory is based on the bracket polynomial model for the Jones polynomial. It is built in terms of diagrammatic combinatorial topology. The same structure can be explained in terms of the $SU(2)_q$ quantum group, and has relationships with functional integration and Witten’s approach to topological quantum field theory. Nevertheless, the approach given here will be unrelentingly elementary. Elementary, does not necessarily mean simple. In this case an architecture is built from simple beginnings and this architecture and its recoupling language can be applied to many things including: colored Jones polynomials, Witten–Reshetikhin–Turaev invariants of three manifolds, topological quantum field theory and quantum computing.

In quantum computing, the application is most interesting because the recoupling theory yields representations of the Artin Braid group into unitary groups $U(n)$. These representations are dense in the unitary group, and can be used to model quantum computation universally in terms of representations of the braid group. Hence the term: topological quantum computation.

In this paper, we outline the basics of the Temperley–Lieb Recoupling Theory, and show explicitly how unitary representations of the braid group arise from it. We will return to this subject in more detail in subsequent papers. In particular, we do not describe the context of anyonic models for quantum computation in this paper. Rather, we concentrate here on showing how naturally unitary representations of the braid group arise in the context of the Temperley–Lieb Theory. For the reader interested in the relevant background in anyonic topological quantum computing we recommend the following references {1–5, 10, 11, 13, 14}.

Here is a very condensed presentation of how unitary representations of the braid group are constructed via topological quantum field theoretic methods. For simplicity assume that one has a single (mathematical) particle with label $P$ that can interact with itself to produce either itself labeled $P$, or itself with the null label $\ast$. When $\ast$ interacts with $P$ the result is always $P$. When $\ast$ interacts with $\ast$ the result is always $\ast$. One considers process spaces where a row of particles labeled $P$ can successively interact, subject to the restriction that the end result is $P$. For example the space $V[(ab)c]$ denotes the space of interactions of three particles labeled $P$. The particles are placed in the positions $a, b, c$. Thus we begin with $(PP)P$. In a typical sequence of interactions, the first two $P$'s interact to produce a $\ast$, and the $\ast$ interacts with $P$ to produce $P$.

$(PP)P \rightarrow (\ast)P \rightarrow P$.

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In another possibility, the first two $P$'s interact to produce a $P$, and the $P$ interacts with $P$ to produce $P$.

$$(PP)P \rightarrow (P)P \rightarrow P.$$ 

It follows from this analysis that the space of linear combinations of processes $V[(ab)c]$ is two dimensional. The two processes we have just described can be taken to be the the qubit basis for this space. One obtains a representation of the three strand Artin braid group on $V[(ab)c]$ by assigning appropriate phase changes to each of the generating processes. One can think of these phases as corresponding to the interchange of the particles labeled $a$ and $b$ in the association $(ab)c$. The other operator for this representation corresponds to the interchange of $b$ and $c$. This interchange is accomplished by a unitary change of basis mapping

$$F : V[(ab)c] \rightarrow V[a(bc)].$$

If

$$A : V[(ab)c] \rightarrow V[(ba)c]$$

is the first braiding operator (corresponding to an interchange of the first two particles in the association) then the second operator

$$B : V[(ab)c] \rightarrow V[(ac)b]$$

is accomplished via the formula $B = F^{-1}AF$ where the $A$ in this formula acts in the second vector space $V[a(bc)]$ to apply the phases for the interchange of $b$ and $c$.

In this scheme, vector spaces corresponding to associated strings of particle interactions are interrelated by recoupling transformations that generalize the mapping $F$ indicated above. A full representation of the Artin braid group on each space is defined in terms of the local interchange phase gates and the recoupling transformations. These gates and transformations have to satisfy a number of identities in order to produce a well-defined representation of the braid group. These identities were discovered originally in relation to topological quantum field theory. In our approach the structure of phase gates and recoupling transformations arise naturally from the structure of the bracket model for the Jones polynomial. Thus we obtain a knot-theoretic basis for topological quantum computing.

2. SPIN NETWORKS AND TEMPERLEY – LIEB RECOUPLING THEORY

In this section we discuss a combinatorial construction for spin networks that generalizes the original construction of Roger Penrose. The result of this generalization is a structure that satisfies all the properties of a graphical $TQFT$ as described in our paper on braiding and universal quantum gates, and specializes to classical angular momentum recoupling theory in the limit of its basic variable. The construction is based on the properties of the bracket polynomial. A complete description of this theory can be found in the book “Temperley – Lieb Recoupling Theory and Invariants of Three-Manifolds” by Kauffman and Lins.

The “$q$-deformed” spin networks that we construct here are based on the bracket polynomial relation. View Figure 1 and Figure 2.
\[ n \text{ strands} = n \]
\[ n = \ldots \]
\[ (\sigma) \sim (1/\sigma!) \sum (A^{-3})^{(\sigma)} \]
\[ (\sigma) \varepsilon S_{n} \]
\[ n = (1/\sigma!) \sum (A^{-3})^{(\sigma)} \delta \]

Figure 1. Basic Projectors

\[ 2 = \ldots = -1/\delta \]
\[ n = \ldots = -\Delta /\Delta \]
\[ n = \ldots = -\Delta n /\Delta n+1 \]

\[ \Delta_{-1} = 0 \quad \Delta_{0} = 1 \]
\[ \Delta_{n+1} = \delta \Delta_{n} - \Delta_{n-1} \]

Figure 2. Two Strand Projector

\[ a + j = a \]
\[ j + k = b \]
\[ i + k = c \]

Figure 3. Trivalent Vertex
In Figure 1 we indicate how the basic projector (symmetrizer, Jones-Wenzl projector) is constructed on the basis of the bracket polynomial expansion. In this technology, a symmetrizer is a sum of tangles on $n$ strands (for a chosen integer $n$). The tangles are made by summing over braid lifts of permutations in the symmetric group on $n$ letters, as indicated in Figure 1. Each elementary braid is then expanded by the bracket polynomial relation, as indicated in Figure 1, so that the resulting sum consists of flat tangles without any crossings (these can be viewed as elements in the Temperley–Lieb algebra). The projectors have the property that the concatenation of a projector with itself is just that projector, and if you tie two lines on the top or the bottom of a projector together, then the evaluation is zero. This general definition of projectors is very useful for this theory. The two-strand projector is shown in Figure 2. Here the formula for that projector is particularly simple. It is the sum of two parallel arcs and two turn-around arcs (with coefficient $-1/d$, with $d = -A^2 - A^{-2}$ is the loop value for the bracket polynomial. Figure 2 also shows the recursion formula for the general projector. This recursion formula is due to Jones and Wenzl and the projector in this form, developed as a sum in the Temperley–Lieb algebra (see Section 5 of this paper), is usually known as the Jones–Wenzl projector.

The projectors are combinatorial analogs of irreducible representations of a group (the original spin nets were based on $SU(2)$ and these deformed nets are based on the quantum group corresponding to $SU(2)$). As such the reader can think of them as “particles”. The interactions of these particles are governed by how they can be tied together into three-vertices. See Figure 3. In Figure 3 we show how to tie three projectors, of $a, b, c$ strands respectively, together to form a three-vertex. In order to accomplish this interaction, we must share lines between them as shown in that Figure so that there are non-negative integers $i, j, k$ so that $a = i + j, b = j + k, c = i + k$. This is equivalent to the condition that $a + b + c$ is even and that the sum of any two of $a, b, c$ is greater than or equal to the third. For example $a + b \geq c$. One can think of the vertex as a possible particle interaction where $[a]$ and $[b]$ interact to produce $[c]$. That is, any two of the legs of the vertex can be regarded as interacting to produce the third leg.

There is a basic orthogonality of three vertices as shown in Figure 4. Here if we tie two three-vertices together so that they form a “bubble” in the middle, then the resulting network with labels $a$ and $b$ on its free ends is a multiple of an $a$-line (meaning a line with an $a$-projector on it) or zero (if $a$ is not equal to $b$). The multiple is compatible with the results of closing the diagram in the equation of Figure 4 so the the two free ends are identified with one another. On closure, as shown in the Figure, the left hand side of the equation becomes a Theta graph and the right hand side becomes a multiple of a “delta” where $\Delta_a$ denotes the bracket polynomial evaluation of the $a$-strand loop with a projector on it. The $\Theta(a, b, c)$ denotes the bracket evaluation of a theta graph made from three trivalent vertices and labeled with $a, b, c$ on its edges.

There is a recoupling formula in this theory in the form shown in Figure 5. Here there are “6-j symbols”, recoupling coefficients that can be expressed, as shown in Figure 7, in terms of tetrahedral graph evaluations and theta graph evaluations. The tetrahedral graph is shown in Figure 6. One derives the formulas for these coefficients directly from the orthogonality relations for the trivalent vertices by closing the left hand side of the recoupling formula and using orthogonality to evaluate the right hand side. This is illustrated in Figure 7.

Finally, there is the braiding relation, as illustrated in Figure 8.

With the braiding relation in place, this $q$-deformed spin network theory satisfies the pentagon, hexagon and braiding naturality identities needed for a topological quantum field theory. All these identities follow naturally from the basic underlying topological construction of the bracket polynomial. One can apply the theory to many different situations.
Figure 4. Orthogonality of Trivalent Vertices

Figure 5. Recoupling Formula

Figure 6. Tetrahedron Network
\[
\sum_j \{ a b i \ c d j \} = \left( \frac{\Theta(a,b,j) \ \Theta(c,d,j)}{\Delta_j} \right)^j \delta^k_j
\]

\[
= \sum_j \{ a b i \ c d j \} \frac{\Theta(a,b,k) \ \Theta(c,d,k)}{\Delta_k}
\]

\[
\{ a b i \ c d k \} = \frac{Tet \left[ \begin{array}{c} a b i \\ c d k \end{array} \right] \Delta_k}{\Theta(a,b,k) \ \Theta(c,d,k)}
\]

Figure 7. Tetrahedron Formula for Recoupling Coefficients

\[
\lambda^a b = \lambda_c^a b = (-1)^{(a+b-c)/2} \frac{(a'+b'-c')/2}{A} \quad x' = x(x+2)
\]

Figure 8. LocalBraidingFormula
2.1. Evaluations

In this section we discuss the structure of the evaluations for $\Delta_n$ and the theta and tetrahedral networks. We refer to\(^8\) for the details behind these formulas. Recall that $\Delta_n$ is the bracket evaluation of the closure of the $n$-strand projector, as illustrated in Figure 4. For the bracket variable $A$, one finds that

$$\Delta_n = (-1)^n \frac{A^{2n+2} - A^{-2n-2}}{A^2 - A^{-2}}.$$  

One sometimes writes the quantum integer

$$[n] = (-1)^{n-1} \Delta_{n-1} = \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}.$$  

If

$$A = e^{i\pi/2r},$$

where $r$ is a positive integer, then

$$\Delta_n = (-1)^n \frac{\sin((n + 1)\pi/r)}{\sin(\pi/r)}.$$  

Here the corresponding quantum integer is

$$[n] = \frac{\sin(n\pi/r)}{\sin(\pi/r)}.$$  

Note that $[n + 1]$ is a positive real number for $n = 0, 1, 2, \ldots r - 2$ and that $[r - 1] = 0$.

The evaluation of the theta net is expressed in terms of quantum integers by the formula

$$\Theta(a, b, c) = (-1)^{m+n+p} \frac{[m+n+p+1]![m]![n]![p]}{[m+n]![n+p]![p+m]}.$$  

where

$$a = m + p, b = m + n, c = n + p.$$  

Note that

$$(a + b + c)/2 = m + n + p.$$  

When $A = e^{i\pi/2r}$, the recoupling theory becomes finite with the restriction that only three-vertices (labeled with $a, b, c$) are admissible when $a + b + c \leq 2r - 4$. All the summations in the formulas for recoupling are restricted to admissible triples of this form.

2.2. Symmetry and Unitarity

The formula for the recoupling coefficients given in Figure 7 has less symmetry than is actually inherent in the structure of the situation. By multiplying all the vertices by an appropriate factor, we can reconfigure the formulas in this theory so that the revised recoupling transformation is orthogonal, in the sense that its transpose is equal to its inverse. This is a very useful fact. It means that when the resulting matrices are real, then the recoupling transformations are unitary.
Figure 9 illustrates this modification of the three-vertex. Let $\text{Vert}[a, b, c]$ denote the original 3-vertex of the Temperley–Lieb recoupling theory. Let $\text{ModVert}[a, b, c]$ denote the modified vertex. Then we have the formula

$$\text{ModVert}[a, b, c] = \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\Theta(a, b, c)} \text{Vert}[a, b, c].$$

**Lemma.** For the bracket evaluation at the root of unity $A = e^{i\pi/2r}$ the factor

$$f(a, b, c) = \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\Theta(a, b, c)}$$

is real, and can be taken to be a positive real number for $(a, b, c)$ admissible (i.e. with $a + b + c \leq 2r - 4$).

**Proof.** By the results from the previous subsection,

$$\Theta(a, b, c) = (-1)^{(a+b+c)/2} \hat{\Theta}(a, b, c)$$

where $\hat{\Theta}(a, b, c)$ is positive real, and

$$\Delta_a \Delta_b \Delta_c = (-1)^{(a+b+c)}[a + 1][b + 1][c + 1]$$

where the quantum integers in this formula can be taken to be positive real. It follows from this that

$$f(a, b, c) = \sqrt{\frac{[a + 1][b + 1][c + 1]}{\Theta(a, b, c)}},$$

showing that this factor can be taken to be positive real. This completes the proof.

In Figure 10 we show how this modification of the vertex affects the non-zero term of the orthogonality of trivalent vertices (compare with Figure 4). We refer to this as the “modified bubble identity.” The coefficient in the modified bubble identity is

$$\sqrt{\frac{\Delta_a \Delta_b \Delta_c}{\Delta_a}} = (-1)^{(b+c-a)/2} \sqrt{\frac{[b + 1][c + 1]}{[a + 1]}}$$

where $(a, b, c)$ form an admissible triple. In particular $b + c - a$ is even and hence this factor can be taken to be real.

We rewrite the recoupling formula in this new basis and emphasize that the recoupling coefficients can be seen (for fixed external labels $a, b, c, d$) as a matrix transforming the horizontal “double-Y” basis to a vertically disposed double-Y basis. In Figures 11, 12 and 13 we have shown the form of this transformation, using the matrix notation

$$M[a, b, c, d]_{ij}$$

for the modified recoupling coefficients. In Figure 11 we derive an explicit formula for these matrix elements. The proof of this formula follows directly from trivalent–vertex orthogonality (See Figures 4 and 7.), and is given in Figure 11. The result shown in Figure 11 and Figure 12 is the following formula for the recoupling matrix elements.

$$M[a, b, c, d]_{ij} = \text{ModTet} \left( \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right) \sqrt{\frac{\Delta_a \Delta_b \Delta_c \Delta_d}{\Delta_a \Delta_b \Delta_c \Delta_d}}$$

where $\sqrt{\Delta_a \Delta_b \Delta_c \Delta_d}$ is short-hand for the product

$$\sqrt{\frac{\Delta_a \Delta_b \Delta_c \Delta_d}{\Delta_a \Delta_b \Delta_c \Delta_d}}.$$
\[
\sqrt{\sqrt{\Delta_a \Delta_b \Delta_c}} = \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\sqrt{\Theta(a, b, c)}}
\]

Figure 9. Modified Three Vertex

\[
\begin{align*}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array} & = \frac{\Theta(a, b, c)}{\Delta_a} \\
\begin{array}{c}
\text{b} \\
\text{c}
\end{array} & = \frac{\sqrt{\Delta_a \Delta_b \Delta_c}}{\Theta(a, b, c)} \\
\begin{array}{c}
\text{c} \\
\text{a}
\end{array} & = \sqrt{\frac{\Delta_b \Delta_c}{\Delta_a}}
\end{align*}
\]

Figure 10. Modified Bubble Identity

\[
= (-1)^{(a+b-j)/2}(-1)^{(c+d-j)/2}(-1)^j \sqrt{\frac{[a+1][b+1]}{[j+1]}} \sqrt{\frac{[c+1][d+1]}{[j+1]}}[j+1]
\]

In this form, since \((a, b, j)\) and \((c, d, j)\) are admissible triples, we see that this coefficient can be taken to be real, and its value is independent of the choice of \(i\) and \(j\). The matrix \(M[a, b, c, d]\) is real-valued.

It follows from Figure 12 (turn the diagrams by ninety degrees) that

\[M[a, b, c, d]^{-1} = M[b, d, a, c].\]

In Figure 14 we illustrate the formula

\[M[a, b, c, d]^T = M[b, d, a, c].\]

It follows from this formula that

\[M[a, b, c, d]^T = M[a, b, c, d]^{-1}.\]

Hence \(M[a, b, c, d]\) is an orthogonal, real-valued matrix.
\[
\sum_k \frac{\Delta a \Delta b}{\Delta_j} \cdot \frac{\Delta c \Delta d}{\Delta_j} \cdot \delta_{j^k} = \sum_k \frac{\Delta a \Delta b}{\Delta_j} \cdot \frac{\Delta c \Delta d}{\Delta_j} \cdot \delta_{j^k} = \frac{\Delta a \Delta b}{\Delta_j} \cdot \frac{\Delta c \Delta d}{\Delta_j} \cdot \delta_{j^k}
\]

Figure 11. Derivation of Modified Recoupling Coefficients

\[
\sum_j \frac{\Delta a \Delta b}{\Delta_j} \cdot \frac{\Delta c \Delta d}{\Delta_j} \cdot \delta_{j^k} = \frac{\Delta a \Delta b}{\Delta_j} \cdot \frac{\Delta c \Delta d}{\Delta_j} \cdot \delta_{j^k}
\]

Figure 12. Modified Recoupling Formula

\[
\frac{\Delta a \Delta b}{\Delta_j} \cdot \frac{\Delta c \Delta d}{\Delta_j} \cdot \delta_{j^k} = \frac{\Delta a \Delta b}{\Delta_j} \cdot \frac{\Delta c \Delta d}{\Delta_j} \cdot \delta_{j^k}
\]

Figure 13. Modified Recoupling Matrix
Theorem. In the Temperley–Lieb theory we obtain unitary (in fact real orthogonal) recoupling transformations when the bracket variable $A$ has the form $A = e^{i\pi/2r}$. Thus we obtain families of unitary representations of the Artin braid group from the recoupling theory at these roots of unity.

Proof. The proof is given the discussion above.

Remark. The density of these representations will be taken up in a subsequent paper.

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