I. INTRODUCTION

Recent experiments on dilute-gas Bose-Einstein condensates (BEC’s) have generated great interest both from theoretical and experimental points of view \cite{1}. At ultra-low temperatures the mean-field description for the macroscopic BEC wave-function is constructed using Hartree-Fock approximation and results in the Gross-Pitaevskii (GP) equation \cite{1}. The latter one reduces to the one-dimensional (1D) nonlinear Schrödinger (NLS) equation with an external potential, in particular, when the transverse dimensions of the condensate are on the order of its healing length and its longitudinal dimension is much longer than its transverse ones (see e.g. \cite{2,3}). This is termed the quasi-one dimensional (quasi-1D) regime of the GP equation. In this regime BECs remain phase-coherent, and the governing equations are one-dimensional.

Several families of stationary solutions for the cubic NLS with an elliptic function potential have been recently presented in Refs \cite{4,5} and their stability has been examined using analytic and numerical methods \cite{2,3,6,7,8,9}. In the quasi-1D regime, the GP equations for two interacting BEC’s reduce to coupled nonlinear Schrödinger (CNLS) equations with an external potential \cite{10,11} (see also Sec. 2 below). When the scattering lengths of the two components, which characterize inter-particle interactions, are close to each other, the CNLS equations reduce to the Manakov system with an external potential.

In the present paper we study the stationary two-component solutions of the CNLS with an external potential. Several cases of explicit solutions in terms of elliptic functions are analyzed and their stability properties are studied numerically. In particular, we derive a set of stationary solutions with trivial and non-trivial phases. We remark that some of the solutions presented in this paper were also analyzed independently in Ref. \cite{12}. In this work, however, all components of the CNLS were assumed to be proportional to the same elliptic function. We extend their results in the sense that we derive solutions of CNLS whose components are expressed through different elliptic functions. We also investigate the role played by these solutions as possible initial states from which localized matter waves (solitons) can be generated.

The paper is organized as follows. In section II we show how to derive one dimensional equations for coupled BEC starting from the original three dimensional problem using a multiple scale expansion in the small amplitude limit. In section III we present exact solution of the CNLS system with non trivial phases, while in section IV we analyze their limits (trigonometric and hyperbolic). In section V we derive stationary solutions with trivial phases for both proportional and non-proportional components. In Section VI we discuss stationary solutions with trivial phases of coupled BCS model a quasi one–dimensional interacting two-component Bose-Einstein condensate trapped in a standing light wave. New families of stationary solutions of the CNLS with a periodic potential are presented and their stability studied by direct numerical simulations. Some of these solutions allow reduction to Manakov system. From a physical point of view these solutions can be interpreted as exact Bloch states at the edge of the Brillouin zone. Some of them are stable while others are found to be unstable against modulations of long wavelength. The solutions which are modulationally unstable are shown to lead to the formation of localized ground states of the coupled BEC system.

PACS numbers: 03.75.Lm, 03.75.-b, 05.45.Yv, 42.65.-k
II. BASIC EQUATIONS

At very low temperatures, when the mean field approximation is applicable, the evolution of two interacting BECs can be described by two coupled GP equations \((j = 1, 2)\) (see e.g. \[10, 11\])

\[
i\hbar \frac{\partial \Psi_j}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_j(r) + \frac{4\pi\hbar^2}{m} \sum_{l=1,2} a_{jl}|\Psi_l|^2 \right] \Psi_j
\]

(2.1)

where atomic masses of the both components are assumed to be equal, \(V_j(r)\) is an external trap potential, and \(a_{ij}\) are the scattering lengths of the respective atomic interactions (other notations are standard). In the case when it consists of superposition of a magnetic trap providing cigar shape of the condensate (elongated, say, along the \(x\)-axis) and an optical trap inducing a lattice potential (which is assumed to be periodic along the \(x\)-axes) one has \((j = 1, 2)\)

\[
V_j(r) = \frac{m}{2}\omega_j^2(\lambda^2 x^2 + y^2 + z^2) + U(\kappa x), \quad U(\kappa x) = U(\kappa(x + L))
\]

(2.2)

Although in the last expression we have imposed equality of the optical potential for the both components, in a general case one has to distinguish the linear oscillator frequencies, \(\omega_1\) and \(\omega_2\), when considering the two components corresponding to the different magnetic moments. For example, in the experimental settings of \[13\] with \(^{87}\)Rb atoms \(\Omega = \frac{\omega_2}{\omega_1} = \sqrt{2}\). This fact has natural implication on the resulting form of the effective system of coupled 1D NLS equations. Indeed, different oscillator frequencies means that two components are located in two different parabolic potentials, and thus their effective densities are different when the number of atoms is equal. As a consequence, even at approximately equal s-wave scattering lengths, and thus for \(a_{11} \approx a_{22}\), the two components will experience different nonlinearities (the latter being proportional to the atomic densities).

Another important issue to be mentioned here is that a cigar-shaped BEC can be viewed as a waveguide for matter waves. As such it is characterized by its mode structure. As it is well known (c.f. with the nonlinear optical waveguides \[14\]) the intrinsic nonlinearity of a BEC results in the mode interaction (and thus energy distribution among modes). If however the nonlinearity is weak enough, the main state of the condensate can be considered as a weakly modulated ground state of the underline linear system. As it is clear that for a two-component BEC the respective small parameter is the ratio between the density energy of two-body interactions, \(\frac{4\pi\hbar^2 N_{a_{11}}}{m_0 \lambda}\) (hereafter \(a_j = \sqrt{\frac{\hbar}{m_0\omega_j}}\) is the linear oscillator length and \(N_j = \int |\Psi_j|^2 dr\) is the number of atoms of the \(j\)th component) to the density of the recoil energy \(\frac{\hbar^2}{2m_0\lambda^2}\).

In other words the small parameter of the problem can be identified as \(\epsilon = \frac{4\pi N_{a_{11}}}{\lambda}\) \(\ll 1\) \((\lambda = \bar{\lambda}_{11}\) is the total number of atoms). In this situation a self-consistent reduction of the original 3D system \(2.1\) to the effective 1D system of the coupled equations can be provided by means of the multiple-scale technique. Since the details of such a reduction have already published elsewhere \[3, 13\] for a single component BEC, here we only outline the main steps.

Let us first introduce dimensionless variables

\[
r' = (x', y', z'), \quad t' = \frac{1}{2}\omega_1 t, \quad \psi_j = \sqrt{\frac{2a_{1j}}{N_j}} \psi_j,
\]

and rewrite Eqs. 2.1 in the form

\[
i\psi_1 = [-\Delta' + T(r') + U(\kappa' x') + g_{11}|\psi_1|^2 + g_{21}|\psi_2|^2] \psi_1,
\]

(2.4)

\[
i\psi_2 = [-\Delta' + \Omega^2 T(r') + U(\kappa' x') + g_{12}|\psi_1|^2 + g_{22}|\psi_2|^2] \psi_2,
\]

(2.5)

where \(\kappa' = \kappa_1, \Omega = a_1^2/a_2^2\), and \(g_{ij} = 4\pi Na_{ij}/a_1\). Next consideration depends on the magnitude of \(\kappa\) (it is assumed that \(U(x)/U(x) = O(1)\). One can distinguish three main cases:

(i) \(\kappa' \ll \epsilon\) (say \(\kappa \sim \epsilon^2\)). In this case the periodic potential can be considered as smoothly varying, and somehow can be viewed as a limit of the case considered below.

(ii) \(\kappa' \approx \epsilon\) where \(\alpha \approx 1\). This is the case when the potential periodicity is of the order of the effective length of the nonlinearity. Below we concentrate on this last case.

To this end we consider two eigenvalue problems

\[
(-\Delta' + \lambda x'^2 + r_{11}^2)\varphi_1 = E_1\varphi_1,
\]

\[
(-\Delta' + \Omega(x'^2 + r_{11}^2))\varphi_2 = E_2\varphi_2
\]

(2.6)

whose normalized ground states are well known:

\[
\varphi_1 = \frac{\lambda^{1/4}}{\sqrt{\pi}^{1/4}} e^{-\frac{1}{4}(\lambda x'^2 + r_{11}^2)},
\]

\[
\varphi_2 = \frac{\Omega^{1/4}\lambda^{1/4}}{\sqrt{\pi}^{1/4}} e^{-\frac{1}{4}(\lambda x'^2 + r_{22}^2)}
\]

(2.7)

and \(E_j = \Omega^{j-1}(j + 2)\).

The next steps are conventional for the multiple scale expansion (see e.g. \[3\]). Namely we introduce scaled variables \(x_n = \epsilon^n x', r_{n} = \epsilon^n r_{11}'\) and \(t_n = \epsilon^n t' (n = 0, 1, 2, ...\) which are considered as independent and look for the solution of 2.4 in the form

\[
\psi_j = \sqrt{\frac{1}{g_{11}\lambda^{1/2}}} \left( e^{\psi_j^{(1)}} + e^{\psi_j^{(2)}} + \cdots \right)
\]

(2.8)
with
\[ \psi^{(1)}_j = Q_j(x_1, t_2)\varphi_j(x_0, r_0)e^{-iE_jt_0}, \quad j = 1, 2. \tag{2.9} \]
Here \( Q_j(x_1, t_2) \) describes slow modulation of the background state due to the nonlinearity.

Substituting (2.8) in (2.9), equating all terms at each of the \( \epsilon \) orders, and excluding secular terms, in the order \( \epsilon^3 \) we obtain
\[
\begin{align*}
&i \frac{\partial Q_1}{\partial t_2} = -\frac{\partial^2 Q_1}{\partial x_1^2} + V(\alpha x_1)Q_1 + \chi_1|Q_1|^2Q_1 + \chi|Q_2|^2Q_1, \\
&i \frac{\partial Q_2}{\partial t_2} = -\frac{\partial^2 Q_2}{\partial x_1^2} + V(\alpha x_1)Q_2 + \chi|Q_1|^2Q_2 + \chi_2|Q_2|^2Q_2,
\end{align*}
\tag{2.10, 2.11}
\]
where
\[
\chi_1 = \text{sign}(g_{11}) \int |\varphi_1|^4 \, dx = \frac{\text{sign}(g_{11})}{2^{3/2} \pi^{3/2}}, \\
\chi = \frac{g_{12}}{g_{11}} \int |\varphi_1|^2|\varphi_2|^2 \, dx = \frac{1}{\pi^{3/2}} \left( \frac{\Omega}{\Omega + 1} \right)^{3/2} \left| \frac{a_{12}}{|a_{11}|} \right|, \\
\chi_2 = \frac{g_{22}}{g_{11}} \int |\varphi_1|^4 \, dx = \frac{\Omega^{3/2}}{2^{3/2} \pi^{3/2}} \left| \frac{a_{22}}{|a_{11}|} \right|
\]
and its is taken into account that \( a_{12} = a_{21}. \) The system (2.10), (2.11) is a subject of our main interest.

III. STATIONARY SOLUTIONS WITH NON-TRIVIAL PHASES

After the change of notations: \( t_2 \to t, \ x_1 \to x, \beta = -\chi, \) and \( b_{1,2} = -\chi_{1,2} \) the system (2.10), (2.11) takes the well known form:
\[
\begin{align*}
&i \frac{\partial Q_1}{\partial t} + \frac{\partial^2 Q_1}{\partial x^2} + (b_1|Q_1|^2 + \beta|Q_2|^2)Q_1 - V_0\text{sn}^2(\alpha x, k)Q_1 = 0, \tag{3.1} \\
&B_j = -\beta_j A_j, \quad C_j^2 = \alpha^2 A_j^2 \beta_j (\beta_j - 1)(1 - \beta_j k^2), \tag{3.10} \\
&\omega_j = (1 + k^2)\alpha^2 + \frac{W}{\Delta} \left[ -\beta b_1 (b_2 - \beta) - \beta_2 (\beta - b_1) \right] - k^2 \alpha^2 \beta_j, \tag{3.11}
\end{align*}
\]

We restrict our attention to stationary solutions of these CNLS:
\[ Q_j(x, t) = q_j(x) \exp(-i\omega_j t + i\Theta_j(x) + ik_{0,j}), \tag{3.3} \]
where \( j = 1, 2, \) \( k_{0,j} \) are constant phases, \( q_j \) and \( \Theta_j(x) \) are real-valued functions and
\[ \Theta_j(x) = C_j \int_0^x \frac{dx'}{q_j'(x')}, \quad j = 1, 2. \tag{3.4} \]
where \( C_j, j = 1, 2 \) are constants of integration.

Following [3] we refer to solutions in the cases \( C_j = 0 \) and \( C_j \neq 0 \) as to trivial and nontrivial phase solutions, respectively. We notice that nontrivial phase solutions imply nonzero current of the matter – it is proportional to \( |q_j(x)|^2 \Theta_{jx} = C_j \), for each of the components – along \( x \)-axis, and hence seem to have no direct relation to present experimental setting for BECs (remember that the condensate is confined to a parabolic trap). Meantime, a system of coupled NLS equations appears to be a general model, having, for example, applications in nonlinear optics (see e.g. [14]). Bearing this in mind we consider both types of solutions.

An appropriate class of periodic potentials to model the quasi-1D confinement produced by a standing light wave is given by
\[ V(\alpha x) = V_0\text{sn}^2(\alpha x, k), \tag{3.5} \]
where \( \text{sn}(\alpha x, k) \) denotes the Jacobian elliptic sine function with elliptic modulus \( 0 \leq k \leq 1 \). Then, substituting the ansatz (3.3) in Eqs. (2.10, 2.11) and separating the real and imaginary part we get
\[
\begin{align*}
&\chi_1^3 q_{1xx} + (b_1 q_1^2 + \beta q_2^2)q_1^4 - V_0\text{sn}^2(\alpha x, k)q_1^4 + \omega_1 q_1^4 = C_1^2, \tag{3.6} \\
&\chi_2^3 q_{2xx} + (\beta q_1^2 + b_2 q_2^2)q_2^4 - V_0\text{sn}^2(\alpha x, k)q_2^4 + \omega_2 q_2^4 = C_2^2. \tag{3.7}
\end{align*}
\]
We seek solutions for \( q_j^2, j = 1, 2 \) as a quadratic function of \( \text{sn}(\alpha x, k) \):
\[ q_j^2 = A_j\text{sn}^2(\alpha x, k) + B_j, \quad j = 1, 2. \tag{3.8} \]
Inserting (3.8) in (3.6), (3.7) and equating the coefficients of equal powers of \( \text{sn}(\alpha x, k) \) results in the following relations among the solution parameters \( \omega_j, C_j, A_j \) and \( B_j \) and the characteristic of the optical lattice \( V_0, \alpha \) and \( k \):
\[ A_1 = \frac{(b_2 - \beta)W}{\Delta}, \quad A_2 = \frac{(b_1 - \beta)W}{\Delta}, \tag{3.9} \]
\[ B_j = -\beta_j A_j, \quad C_j^2 = \alpha^2 A_j^2 \beta_j (\beta_j - 1)(1 - \beta_j k^2) \tag{3.10} \]
\[ \omega_j = (1 + k^2)\alpha^2 + \frac{W}{\Delta} \left[ -\beta b_1 (b_2 - \beta) - \beta_2 (\beta - b_1) \right] - k^2 \alpha^2 \beta_j, \tag{3.11} \]
where \( j = 1, 2 \) and
\[ W = V_0 - 2\alpha^2 k^2, \quad \Delta = \chi_1^2 - \chi_2^2 = b_1 b_2 - \beta^2. \tag{3.12} \]
In order that our results (3.8), (3.11) are consistent with the parametrization (3.3), (3.4) we must ensure that both \( q_j^2(x) \) and \( \Theta_j(x) \) are real-valued; this means that \( C_j^2 \geq 0 \) and \( q_j^2(x) \geq 0 \). An elementary analysis shows that this is true provided one of the following pairs of conditions are satisfied:
\[
\begin{align*}
&\text{a)} \quad A_j \geq 0, \quad \beta_j \leq 0, \quad j = 1, 2; \tag{3.13} \\
&\text{b)} \quad A_j \leq 0, \quad 1 \leq \beta_j \leq \frac{1}{k^2}, \quad j = 1, 2. \tag{3.14}
\end{align*}
\]
Although our main interest is to analyze periodic solutions note that the solutions $Q_j$ in (3.3) are not always periodic in $x$. Indeed, let us first calculate explicitly $\Theta_j(x)$ by using the well known formula, see e.g. \[10: \]
\[
\int_0^x \frac{du}{\wp'(au) - \wp'(av)} = \frac{1}{\wp'(av)} \left[ 2x \zeta(av) + \frac{1}{\alpha} \ln \frac{\sigma(au - av)}{\sigma(au + av)} \right]
\]
where $\wp, \zeta, \sigma$ are standard Weierstrass functions.

In the case a) we replace $v$ by $iv_j$, set $\text{sn}^2(iav_j;k) = \beta_j < 0$ and
\[
e_1 = \frac{1}{3}(2-k^2), \quad e_2 = \frac{1}{3}(2k^2 - 1), \quad e_3 = -\frac{1}{3}(1+k^2),
\]
and rewrite the l.h.s in terms of Jacobi elliptic functions:
\[
\int_0^x \frac{du}{\text{sn}^2(iav;k) - \text{sn}^2(\omega;k)} = -\beta_j x - \beta_j^2 \int_0^x \frac{du}{\text{sn}^2(\omega,k) - \beta_j}
\]
Skipping the details we find the explicit form of $\Theta_j(x)$:
\[
\Theta_j(x) = C_j \int_0^x \frac{du}{A_j(\text{sn}^2(\omega;k) - \beta_j)}
= -\tau_j x + i \frac{\alpha}{2} \ln \frac{\sigma(ax + iav_j)}{\sigma(ax - iav_j)},
\]
\[
\tau_j = i \alpha \zeta(iav_j) + \frac{\alpha}{\beta_j} \sqrt{-\beta_j(1 - \beta_j)(1 - k^2 \beta_j)}.
\]

These formulae provide an explicit expression for the solutions $Q_j(x,t)$ with nontrivial phases; note that for real values of $v_j$ $\Theta_j(x)$ are also real. Now we can find the conditions under which $Q_j(x,t)$ are periodic. Indeed, from (3.15) we can calculate the quantities $T_j$ satisfying:
\[
\Theta_j(x + T_j) - \Theta_j(x) = 2\pi p_j.
\]

Then $Q_j(x,t)$ will be periodic in $x$ with periods $T_j = m_j \omega$ if there exist pairs of integers $m_j, p_j$, such that:
\[
m_j \omega = -\pi \left[ \text{sn}^2(iav_j\omega + \omega \tau_j) \right]^{-1}, \quad j = 1, 2.
\]
where $\omega$ (and $\omega'$) are the half-periods of the Weierstrass functions.

Of course the trivial phase solutions considered in the next sections are always periodic functions of $x$.

We will list also solutions for two particular choices of $b_1, b_2$ and $\beta$ which can be viewed as singular limits of the generic case considered above. The first one is
\[
\beta^2 = b_1 b_2, \quad b_1 \neq b_2
\]
which corresponds to the case (3.13). Then the solution is given by:
\[
A_2 = -\frac{b_1}{\beta} A_1, \quad V_0 = 2k^2 \alpha^2,
\]
\[
\omega_1 = (\beta_1 - \beta_2) b_1 A_1 + (1 + k^2) \alpha^2 - \alpha^2 k^2 \beta_1, (3.19)
\]
\[
\omega_2 = (\beta_1 - \beta_2) b_2 A_1 + (1 + k^2) \alpha^2 - \alpha^2 k^2 \beta_2,
\]
\[
C_j = \alpha^2 A_j^2 \beta_j (1 - 1)(1 - \beta_j k^2),
B_j = -\beta_j A_j, \quad j = 1, 2.
\]

The second particular case is the Manakov system; it corresponds to the choice $b_1 = b_2 = b$. The result is
\[
\omega_j = b(\beta_1 A_1 + \beta_2 A_2) + (1 + k^2) \alpha^2 - \alpha^2 k^2 \beta_j, \quad (3.20)
\]
\[
C_j = \alpha^2 A_j^2 \beta_j (1 - 1)(1 - \beta_j k^2),
B_j = -\beta_j A_j, \quad V_0 = b(A_1 + A_2) + 2k^2 \alpha^2, \quad j = 1, 2.
\]

We remark that in the two-component CNLS eqs. (3.6), (3.7) the constants $b_1, b_2$ and $\beta$ are assumed to be positive. However in our considerations we do not need this restrictions and our formulae are valid also for negative values of $b_1, b_2$ and $\beta$.

**IV. LIMITS OF THE NON-TRIVIAL PHASE SOLUTIONS**

**A. The limit $k \to 1$**

In this limit the elliptic functions reduce to hyperbolic functions. Specifically, $\text{sn}(x,k) = \tanh(x)$. Hence in this limit and for repulsive BECs the solutions have the form
\[
q_j^2 = A_j(\tanh^2(\alpha x) - \beta_j), \quad j = 1, 2,
\]
\[
C_j = -\alpha^2 A_j^2 \beta_j (1 - \beta_j)^2,
\]
\[
\omega_j = \beta_1 b_1 A_1 + \beta_2 b_2 A_2 + (2 - \beta_j) \alpha^2. \quad (4.1)
\]

The potential has only a single well or a single peak $V(x) = -V_0 \tanh^2(\alpha x)$. The consistency condition in this case takes the form:
\[
a) \quad A_j > 0, \quad \beta_j < 0, \quad j = 1, 2; \quad (4.2)
\]
while the second one (3.11) degenerates and dissappears.

The same limit combined with the condition $\beta^2 = b_1 b_2$ leads to:
\[
q_j^2(x) = A_j(\tanh^2(\alpha x) - \beta_j),
\]
\[
\omega_1 = (\beta_1 - \beta_2) b_1 A_1 + (2 - \beta_1) \alpha^2, \quad (4.3)
\]
\[
\omega_2 = (\beta_1 - \beta_2) b_2 A_1 + (2 - \beta_2) \alpha^2,
\]
\[
C_j = -\alpha^2 A_j^2 \beta_j (1 - \beta_j)^2,
A_2 = -\frac{b_1}{\beta} A_1, \quad V_0 = 2\alpha^2,
\]
and for the Manakov case \( b_1 = b_2 = \beta = b \) we have:

\[
q_j^2(x) = A_j(\tanh^2(\alpha x) - \beta_j), \\
\omega_j = b(\beta_1 A_1 + \beta_2 A_2) + (2 - \beta_j)\alpha^2, \\
C_j^2 = -\alpha^2 A_j^2 \beta_j(1 - \beta_j)^2, \\
V_0 = b(A_1 + A_2) + 2\alpha^2,
\]

Then the nontrivial phases are equal to \((\beta_j < 0)\):

\[
\Theta_j(x) = \alpha \sqrt{-\beta_j} x + \text{arctanh} \left( \frac{\tanh \alpha x}{\sqrt{-\beta_j}} \right). \tag{4.5}
\]

### B. The trigonometric limit

In the limit \( k \to 0 \), the elliptic functions reduce to trigonometric functions and \( V(x) = -V_0 \sin^2(\alpha x) \). Then

\[
q_j^2 = A_j(\sin^2(\alpha x) - \beta_j), \quad j = 1, 2, \\
\omega_1 = \alpha^2 + \beta_1 b_1 A_1 + \beta_2 b_2 A_2, \\
\omega_2 = \alpha^2 + \beta_1 \beta_2 A_1 + \beta_2 b_2 A_2, \\
C_j^2 = \alpha^2 A_j^2 \beta_j(1 - \beta_j - 1), \\
A_1 = \frac{b_2 - \beta}{b_1 b_2 - \beta^2}, \quad A_2 = \frac{b_1 - \beta}{b_1 b_2 - \beta^2},
\]

i.e., \( V_0 = b_1 A_1 + b_2 A_2 \). The consistency conditions \(3.13\) and \(3.14\) then take the form:

\begin{align*}
a) & \quad A_j > 0, \quad \beta_j < 0, \quad j = 1, 2; \tag{4.7} \\
b) & \quad A_j < 0, \quad \beta_j \geq 1, \quad j = 1, 2; \tag{4.8}
\end{align*}

If we assume \( \beta^2 = b_1 b_2 \) then

\[
q_j^2 = A_j(\sin^2(\alpha x) - \beta_j), \quad j = 1, 2, \\
\omega_j = \alpha^2 + (-1)^{j+1}(\beta_1 - \beta_2) b_1 A_1, \\
C_j^2 = \alpha^2 A_j^2 \beta_j(1 - \beta_j), \\
A_2 = -\frac{b_1}{\beta} A_1, \quad V_0 = 0,
\]

and in the Manakov case \( b_1 = b_2 = \beta = b \) we have:

\[
q_j^2 = A_j(\sin^2(\alpha x) - \beta_j), \quad j = 1, 2, \\
\omega_j = \alpha^2 + b(\beta_1 A_1 + \beta_2 A_2), \\
C_j^2 = \alpha^2 A_j^2 \beta_j(1 - \beta_j), \\
V_0 = b(A_1 + A_2).
\]

Therefore the phase integral \(4.14\) equals \((\beta_j < 0)\):

\[
\Theta_j = -\arctan \left( \frac{1 - \beta_j}{-\beta_j} \tan(\alpha x) \right). \tag{4.11}
\]

### V. Trivial phase solutions

In this section we consider solutions of \(3.1, 3.2\) with trivial phase, i.e. \( C_1 = C_2 = 0 \):

\[
Q_j(x, t) = e^{-i\omega_j t + i\kappa a_j} q_j(x), \quad j = 1, 2, \tag{5.1}
\]

and we will look for different possible choices for the functions \( q_1(x) \) and \( q_2(x) \). This type of solutions are more flexible and in certain cases survive reductions of the constants \( \beta^2 = b_1 b_2 \) or the limit to the Manakov case: \( b_1 = b_2 = \beta \). They are also relevant for processes in BEC and nonlinear optics \(14\).

In the following we shall consider the \( q_i(x) \) to be expressed in terms of Jacobi elliptic functions, i.e. we assume the following ansatz: \( q_i(x) = \gamma_i J_i(x) \), with \( J_i(x), i = 1, 2 \) being one of the Jacobi elliptic function \( \text{sn}(\alpha x, k) \), \( \text{cn}(\alpha x, k) \) or \( \text{dn}(\alpha x, k) \) and \( \gamma_i \) specifying both the real amplitudes and the constant phases in \(3.1\). Note that the CNLS \(3.1, 3.2\) possesses the gauge invariance \( Q_j \to Q_j e^{-i\kappa a_j} \). This allows one to fix up conveniently the initial phases of both \( Q_j(x) \). In most of the following examples we have made this choice by requiring that \( \gamma_j > 0 \). Direct substitution of the above ansatz into Eqs. \(5.1, 5.2\) provides a set of algebraic equations for the parameters whose solutions furnish exact ground states of the coupled BEC system.

**Case 1.** We start with

\[
q_1(x) = \gamma_1 \text{sn}(\alpha x, k), \quad q_2(x) = \gamma_2 \text{cn}(\alpha x, k), \tag{5.2}
\]

The functions in \(5.1\) are solutions of \(3.1\) provided the constants satisfy the relations:

\[
b_1 \gamma_1^2 - \beta \gamma_2^2 - W = 0, \\
\beta \gamma_1^2 - b_2 \gamma_2^2 - W = 0, \\
\beta \gamma_2^2 + \omega_1 - \alpha^2(k^2 + 1) = 0, \\
b_2 \gamma_2^2 + \omega_2 - \alpha^2 = 0.
\]

where for convenience we have introduced

\[
W = V_0 - 2k^2\alpha^2. \tag{5.4}
\]

From this system we can determine 4 of the constants in terms of the others. Let us split these constants into two groups. The first one:

\[
G_1 \simeq \{ b_1, \ b_2, \ \beta, \ W, \ \alpha, \ k, \}
\]

consists of constants determining the equations and the potential and we assume they are fixed. The second group of constants

\[
G_2 \simeq \{ \omega_1, \ \omega_2, \ \gamma_1, \ \gamma_2, \}
\]

characterize the corresponding soliton solution. Next we solve \(5.3\) and express the constants \( G_2 \) in terms of \( G_1 \).
If $\beta^2 \neq b_1 b_2$ we get the result:

$$\omega_1 = -\frac{\beta(b_1 - \beta)}{\beta^2 - b_1 b_2} W + \alpha^2 (k^2 + 1),$$

$$\omega_2 = -\frac{b_2(b_1 - \beta)}{(\beta^2 - b_1 b_2)} W + \alpha^2,$$  \hspace{1cm} (5.5)

$$\gamma_1^2 = \frac{(\beta - b_2)}{\beta^2 - b_1 b_2} W, \quad \gamma_2^2 = \frac{(b_1 - \beta)}{\beta^2 - b_1 b_2} W.$$  \hspace{1cm} (5.6)

The constraints on the constants $\gamma_1^2 > 0$ and $\gamma_2^2 > 0$ can be ensured in two ways:

$$\begin{cases} W < 0, & b_1 > \beta > b_2, \\ W > 0, & b_1 < \beta < b_2, \end{cases} \hspace{1cm} (5.7)$$

The case when $\beta^2 = b_1 b_2$ fixes up $W$ by

$$W = 0,$$  \hspace{1cm} (5.8)

and then

$$\gamma_2^2 = \frac{b_1}{b_2}, \quad \omega_1 = \alpha^2 (k^2 + 1) - b_1 \gamma_1^2, \quad \omega_2 = \alpha^2 - b_2 \gamma_2^2.$$  \hspace{1cm} (5.9)

Case 2. Here

$$q_1(x) = \gamma_1 \text{sn} (\alpha x, k), \quad q_2(x) = \gamma_2 \text{dn} (\alpha x, k), \hspace{1cm} (5.10)$$

The functions in (5.9) are solutions of (3.1) provided the constants satisfy the relations:

$$b_1 \gamma_1^2 + k^2 \beta \gamma_2^2 + W = 0,$$

$$\beta \gamma_1^2 + k^2 b_2 \gamma_2^2 + W = 0,$$

$$b_1 \gamma_1^2 + \omega_1 + \beta \gamma_2^2 - \alpha^2 = 0,$$

$$\beta \gamma_1^2 + b_2 \gamma_2^2 + \omega_2 - \alpha^2 k^2 = 0.$$  \hspace{1cm} (5.11)

The solution of eq. (5.10) gives:

$$\gamma_1^2 = \frac{b_1 - \beta}{\beta^2 - b_1 b_2} W, \quad \gamma_2^2 = \frac{b_1 - \beta}{k^2(\beta^2 - b_1 b_2)} W,$$  \hspace{1cm} (5.12)

$$\omega_1 = \alpha^2 (k^2 + 1) - b_1 \gamma_1^2, \quad \omega_2 = \alpha^2 b_2 \gamma_2^2.$$  \hspace{1cm} (5.13)

Case 3. Here

$$q_1(x) = \gamma_1 \text{cn} (\alpha x, k), \quad q_2(x) = \gamma_2 \text{dn} (\alpha x, k).$$  \hspace{1cm} (5.14)

The functions in (5.13) are solutions of (3.1) provided the constants satisfy the relations:

$$b_1 \gamma_1^2 + k^2 \beta \gamma_2^2 + W = 0,$$

$$\beta \gamma_1^2 + k^2 b_2 \gamma_2^2 + W = 0,$$

$$b_1 \gamma_1^2 + \omega_1 + \beta \gamma_2^2 - \alpha^2 = 0,$$

$$\beta \gamma_1^2 + b_2 \gamma_2^2 + \omega_2 - \alpha^2 k^2 = 0.$$  \hspace{1cm} (5.15)

The solution of eq. (5.14) gives:

$$\gamma_1^2 = \frac{b_1 - \beta}{\beta^2 - b_1 b_2} W, \quad \gamma_2^2 = \frac{b_1 - \beta}{k^2(\beta^2 - b_1 b_2)} W,$$  \hspace{1cm} (5.16)

$$\omega_1 = \alpha^2 \gamma_1^2 + \frac{b_1 (\beta - b_1)}{k^2(\beta^2 - b_1 b_2)} W + \frac{b_1 (\beta - b_2)}{\beta^2 - b_1 b_2} W,$$  \hspace{1cm} (5.17)

This solution allows the possibility to have $b_1 = b_2 = b$. It reduces to

$$\gamma_1^2 = \frac{W}{\beta + b}, \quad \gamma_2^2 = \frac{W}{k^2(\beta + b)}, \hspace{1cm} (5.18)$$

$$\omega_1 = \alpha^2 - \gamma_1^2 (b + \frac{\beta}{k^2}), \quad \omega_2 = \alpha^2 k^2 - \gamma_1^2 \left(\beta + \frac{b}{k^2}\right),$$  \hspace{1cm} (5.19)

Obviously to have $\gamma_2^2 > 0$ we need to require that

$$W < 0.$$  \hspace{1cm} (5.20)

Case 4. We put:

$$q_1(x) = \gamma_1 \text{dn} (\alpha x, k), \quad q_2(x) = \gamma_2 \text{sn} (\alpha x, k).$$  \hspace{1cm} (5.21)

Then the corresponding sets of parameters satisfy:

$$\gamma_1^2 = \frac{(\beta - b_2) W}{k^2(\beta^2 - b_1 b_2)}, \quad \gamma_2^2 = \frac{(\beta - b_1) W}{\beta^2 - b_1 b_2},$$  \hspace{1cm} (5.22)

$$\omega_1 = \alpha^2 k^2 - \frac{b_1 (b_2 - \beta) W}{k^2(\beta^2 - b_1 b_2)}, \quad \omega_2 = \alpha^2 (k^2 + 1) - \frac{\beta (b_2 - \beta) W}{k^2(\beta^2 - b_1 b_2)}.$$  \hspace{1cm} (5.23)

The subcase $b_1 = b_2 = b$ is impossible since from Eq. (5.22) there follows $k^2 \gamma_1^2 + \gamma_2^2 = 0$. It is possible however to put $\beta^2 = b_1 b_2$ in which case:

$$\gamma_2^2 = k^2 \frac{b_1}{b_2} \gamma_1^2, \hspace{1cm} (5.24)$$

$$\omega_1 = \alpha^2 k^2 - b_2 \gamma_1^2, \hspace{1cm} (5.25)$$

$$\omega_2 = \alpha^2 (k^2 + 1) - \frac{b_1}{b_2} \gamma_2^2.$$  \hspace{1cm} (5.25)
Case 5. Let now:

\[ q_1(x) = \gamma_1 \text{dn}(\alpha x, k), \quad q_2(x) = \gamma_2 \text{cn}(\alpha x, k). \] (5.26)

Then the corresponding sets of parameters satisfy:

\[ \gamma_1^2 = \frac{(\beta - b_2)W}{k^2(\beta^2 - b_1b_2)}, \quad \gamma_2^2 = \frac{(\beta - b_1)W}{\beta^2 - b_1b_2} \] (5.27)

\[ \omega_1 = \alpha^2 k^2 + \frac{\beta(\beta - b_1)W}{\beta^2 - b_1b_2} + b_1(\beta - b_2)W k^2(\beta^2 - b_1b_2), \] (5.28)

\[ \omega_2 = \alpha^2 + \frac{b_2(\beta - b_1)W}{\beta^2 - b_1b_2} + \frac{\beta(\beta - b_2)W}{k^2(\beta^2 - b_1b_2)}, \] (5.29)

The subcase \( b_1 = b_2 = b \) simplifies further (5.20) to:

\[ \gamma_1^2 = \frac{W}{k^2(\beta + b)}, \quad \gamma_2^2 = \frac{W}{\beta + b}, \] (5.30)

\[ \omega_1 = \alpha^2 k^2 + \left( \frac{b}{k^2(\beta + b)} \right) \frac{W}{\beta + b}, \] (5.31)

\[ \omega_1 = \alpha^2 + \left( \frac{b + \beta}{k^2(\beta + b)} \right) \frac{W}{\beta + b}. \] (5.32)

Here again it is natural to consider \( W < 0 \).

Let us finally consider three more cases in which the two components are proportional: \( q_1(x) = \gamma q_2(x) \) and \( q_1(x) \) is one of the three functions \( \text{sn}(\alpha x, k) \), \( \text{cn}(\alpha x, k) \) or \( \text{dn}(\alpha x, k) \). Such an ansatz imposes on the system 3.31, 3.32 the compatibility condition

\[ \gamma_1^2(\beta - b_2) + b_1 - \beta + \omega_1 - \omega_2 = 0 \] (5.33)

If (5.33) is fulfilled the system 5.1, 5.2 reduces effectively to the one-component case, which has been already studied; see also Section VII below.

Case 6. We choose:

\[ q_1(x) = \gamma_1 \text{sn}(\alpha x, k), \quad q_2(x) = \gamma_2 \text{sn}(\alpha x, k). \] (5.34)

Then the corresponding sets of parameters satisfy:

\[ \gamma_1^2 = \frac{(\beta - b_2)W}{\beta^2 - b_1b_2}, \quad \gamma_2^2 = \frac{(\beta - b_1)W}{\beta^2 - b_1b_2}, \] (5.35)

\[ \omega_1 = \omega_2 = \alpha^2(k^2 + 1). \]

The subcase \( b_1 = b_2 = b \) simplifies further (5.35) to:

\[ \gamma_1^2 = \gamma_2^2 = \frac{W}{\beta + b}, \quad \omega_1 = \omega_2 = \alpha^2(k^2 + 1). \] (5.36)

Here it is natural to consider \( W > 0 \).

Case 7. Assume:

\[ q_1(x) = \gamma_1 \text{cn}(\alpha x, k), \quad q_2(x) = \gamma_2 \text{cn}(\alpha x, k). \] (5.37)

Then the corresponding sets of parameters satisfy:

\[ \gamma_1^2 = \frac{(\beta - b_2)W}{\beta^2 - b_1b_2}, \quad \gamma_2^2 = \frac{(\beta - b_1)W}{\beta^2 - b_1b_2}, \] (5.38)

\[ \omega_1 = \omega_2 = \alpha^2 + W. \]

The subcase \( b_1 = b_2 = b \) simplifies further (5.38) to:

\[ \gamma_1^2 = \gamma_2^2 = -\frac{W}{\beta + b}, \quad \omega_1 = \omega_2 = \alpha^2 + W. \] (5.39)

Unlike case 6, here it is natural to consider \( W < 0 \).

Case 8. Let here:

\[ q_1(x) = \gamma_1 \text{dn}(\alpha x, k), \quad q_2(x) = \gamma_2 \text{dn}(\alpha x, k). \] (5.40)

Then the corresponding sets of parameters satisfy:

\[ \gamma_1^2 = \frac{(\beta - b_2)W}{k^2(\beta^2 - b_1b_2)}, \quad \gamma_2^2 = \frac{(\beta - b_1)W}{k^2(\beta^2 - b_1b_2)}, \] (5.41)

\[ \omega_1 = \omega_2 = \alpha^2 k^2 + \frac{W}{k^2}. \]

The subcase \( b_1 = b_2 = b \) simplifies further (5.41) to:

\[ \gamma_1^2 = \gamma_2^2 = -\frac{W}{k^2(\beta + b)}, \quad \omega_1 = \omega_2 = \alpha^2 k^2 + \frac{W}{k^2}. \] (5.42)

Here again it is natural to consider \( W < 0 \).

VI. MODULATIONAL INSTABILITY OF THE TRIVIAL PHASE SOLUTIONS AND LOCALIZED MATTER WAVES GENERATION

To discuss the stability of the above solutions we shall adopt a physical point. To this end we remark that all the trivial phase solutions, are periodic functions of period twice the period of the lattice (recall that the period \( a \) of potential in Eq. 3.5 is \( a = 2K(k^2)/\alpha \), where \( K(k^2) \) is the complete elliptic integral of the first kind). The corresponding wave-number of these solutions is \( K = \pi/\alpha \) which is just the boundary of the Brillouin zone of the uncoupled periodic linear system. Moreover, one can easily check that each component \( q_i(x), i = 1, 2 \), satisfy the Bloch condition

\[ q_i(x + R_n) = e^{i\alpha R_n} q_i(x), \quad R_n = na, \quad n \in \mathbb{N}, \] (6.1)

i.e. the trivial phase solutions are exact nonlinear Bloch states (note that also in the nonlinear case the concept of a Bloch state is well defined by Eq. 6.1). Although nonlinearity does not compromise Bloch property (this being a direct consequence of the translation invariance of the lattice), it can drastically influence the stability of
FIG. 1: Initial profile of stable cn-cn solution plotted against the potential profile (thick curve). The continuous and dotted thin curves denote, respectively, the modulo square of $q_1$ and $q_2$. The parameters are fixed as: $k^2 = 0.8$, $V_0 = 1$, $\alpha = 1$, $\beta = .5$, $b_1 = 1.0$, $b_2 = 0.6$. The amplitudes of the two components are $\gamma_1 = -0.414039$, $\gamma_2 = 0.92582$.

FIG. 2: Time evolution of the first component of the cn-cn solution reported in Fig. 1. To check stability the solution was slightly modulated in space with a profile of the form $0.001 \cos(x)$. A similar plot is obtained for the second component in Fig. 1. Parameters are fixed as in Fig. 1 and the modulational initial profile is taken as in Fig. 2. Notice the emergence of coupled soliton components out of the instability.

FIG. 3: Time evolution of the unstable sn-sn solution (notice that both components of the solution are depicted at each time). The initial amplitudes are taken $\gamma_1 = -0.41404$, and $\gamma_2 = 0.92582$. Parameters are fixed as in Fig. 1 and the modulational initial profile is taken as in Fig. 2. Notice the emergence of coupled soliton components out of the instability.

The possibility that localized states of soliton type can be generated from modulational instability of Bloch states at the edge of the Brillouin zone, was analytically and numerically proved for a single component BEC in optical lattice, in the cases of one [2], two and three spatial dimensions [15]. In order to explore the same possibility to occur also in the present periodic two-components system we recourse to numerical simulations. To this regard we have used an operator splitting method using fast Fourier transform to integrate Eqs. (3.1), (3.2) with initial condition taken as one of the exact solutions derived above modulated by a long wavelength $L (2\pi/k \ll \pi/L)$ and small amplitude sinusoidal profile.

In Fig. 1 we depict the initial profiles of the two components cn-cn solution plotted against the potential profile, while in Fig. 2 we show the time evolution of the first component of this solution in presence of a small modulation of the type $0.001 \cos(0.01x)$.

We see that the profile remains stable for long time (the same is true also for the other component) indicating that the cn-cn solution is stable against small modulations. We find that, except for this, all the other solutions display modulational instabilities out of which localized states emerge. This is clearly seen in Fig. 3 where the time evolution of the unstable sn-sn solution is reported. Notice that in contrast with Fig. 2 instability develops very quickly (already at time $t=10$), out of which two components bright soliton states emerge, as clearly seen at time $t = 20$ (notice that the bright soliton consists of two coupled solitons (one for each component) one bigger than the other.

In Fig. 4 and Fig. 5 we show the time evolution of unstable sn-cn solutions with different amplitude ratio of the sn and cn components. In Fig. 4 the stable cn com-
FIG. 4: Same as in Fig. 3 but for the unstable sn-cn solution. The initial amplitudes are taken as $\gamma_1 = -0.41404$, and $\gamma_2 = 0.92582$. Parameters are fixed as in Fig. 1. Notice that the cn component is larger and more stable. At $t=100$ a bright-dark soliton is formed in the center.

FIG. 5: Same as in Fig. 3 but for the unstable sn-cn solution. The initial amplitudes are taken as $\gamma_1 = 0.92582$, and $\gamma_2 = -0.41404$. Parameters are fixed as in Fig. 1. Notice that the unstable sn component dominates and soliton generation is more effective.

Component is larger than the unstable sn one, while in Fig. 5 we have the opposite. We see that, although in both cases we have instability, the case with larger stable component is obviously much more stable and less effective in creating localized states than the other. Also notice from Fig. 4 that a small amplitude localized state is formed from the sn component at time $t=100$ in the middle of the line which seems to have a character opposite to the one of the other component. We see indeed that when matter density is higher in one component it is lower in the other, this suggesting a sort of bright-dark coupling.

A better characterization of all possible states arising from the mixing of stable and unstable components requires a more accurate analysis. It is interesting to investigate also solutions involving dn components since these, in contrast with sn and cn components, have non zero spatial average, i.e. they are periodic waves on top of a constant background.

FIG. 6: Same as in Fig. 3 but for the unstable dn-dn solution. Initial amplitudes are $\gamma_1 = -0.46291$, and $\gamma_2 = 1.035099$. Other parameters are fixed as in Fig. 1.

In Fig. 6 we depict the time evolution of a dn-dn solution from which we see that it is modulationally unstable, leading to the formation of bright solitons of the same type observed for the sn-sn case. In Figs. 7-8 we depict similar evolutions for the cases dn-sn and dn-cn. Also in this case we observe that the mixing with the unstable sn component is more effective than the one with the stable cn component in creating localized excitations of soliton type (the three bright solitons formed in Fig. 7 at time $t \approx 10$ remain equally spaced and well localized also on longer times). By increasing the cn component of the dn-cn solution of Fig. 8, we also find that the time evolution becomes more stable, as discussed for the sn-cn case.

This analysis shows that the exact trivial phase solutions of the previous section are very useful to create localized excitations of two components BEC in optical lattice. The fact that these solutions are Bloch states at the edge of the Brillouin also suggests a way to create them in a real experiment. One could indeed start from a uniform density distribution of the matter in the potential wells, and accelerate the lattice until the state
amplitudes are \( \gamma \).

VII. DISCUSSIONS

In this section we briefly discuss the above results in comparison with those of Ref. \[12\] in which an \( n \)-component NLS-type equation with external potential, whose strength can be different for each component, was also considered. Changing somewhat notations to avoid confusion with ours we write it down in the form:

\[
\begin{align*}
 \frac{i}{\hbar} \psi_{j}(x) &= -\frac{1}{2\mu_{j}} \frac{\partial^{2}\psi_{j}}{\partial x^{2}} + V_{j}(x)\psi_{j} + \sum_{p=1}^{n} a_{jp}|\psi_{p}|^{2}\psi_{j} \\
 V_{j}(x) &= -V_{0j}\text{sn}^{2}(\alpha x, k), \quad j = 1, \ldots, n.
\end{align*}
\]

In Ref. \[12\] three types of trivial phase solutions are analyzed in more details for a rather special ansatz for \( \Psi \); in fact it is required there that all \( \Psi_{j} \) up to a standard phase factor are proportional to the same function \( \psi(x, k) \). This means that the systems of \( n \) equations will reduce to just one equation for \( \psi(x, k) \); the remaining \( n - 1 \) equations must follow as a consequence of the first one and a set of constraints on the coefficients \( a_{jt}, N_{j}, \mu_{j}, \omega_{j} \) in the notations of \[12\]. The same argument holds true also for three of our solutions, cases 6, 7 and 8 which we added just for the sake of completeness.

The Hamiltonian of the \( n \)-component NLS (7.1) is:

\[
H = \int dx \left[ \sum_{j=1}^{n} \frac{1}{2\mu_{j}} |\partial_{x}\psi_{j}|^{2} + \frac{1}{2} \sum_{j,p=1}^{n} a_{jp}|\psi_{j}|^{2}|\psi_{p}|^{2} + \sum_{j=1}^{n} V_{j}(x)|\psi_{j}|^{2} \right].
\]

where the integration goes over one period. Let us now assume that our solution is of the form:

\[
\psi_{j}(x) = n_{j}(x, t)\psi(x, t), \quad n_{j}(x, t) = e^{-i\omega_{j}t+i\Theta_{j}(x)}\sqrt{N_{j}}
\]

where \( N_{j} \geq 0 \) and \( \Theta_{j}(x) \) appears only in the non-trivial phase case and is determined by:

\[
\frac{d\Theta_{j}}{dx} = \frac{C_{j}}{N_{j}|\psi(x)|^{2}}.
\]

Inserting (7.4) into the Hamiltonian we easily get the following reduced Hamiltonian:

\[
H_{\text{red}} = \int dx \left[ \frac{1}{2} M_{0} |\partial_{x}\psi|^{2} + \frac{1}{2} M_{-1} \frac{1}{|\psi|^{2}} + V(x)|\psi|^{2} \right] + \frac{1}{2} W_{0}|\psi|^{4},
\]

\[
M_{0} = \sum_{j=1}^{n} \frac{1}{N_{j}h_{j}}, \quad M_{-1} = \sum_{j=1}^{n} \frac{N_{j}}{h_{j}},
\]

\[
V(x) = v_{0}\text{sn}^{2}(\alpha x, k), \quad W_{0} = \sum_{j,p=1}^{n} a_{jp}N_{j}N_{p}.
\]

which describes the dynamics of the effective field \( \psi(x, t) \). The result for the trivial phase solution case leads to \( H_{\text{red}} \) with \( M_{-1} = 0 \). That is why we believe that the multi-component effects should be analyzed by using ansatz more general than (7.4).
At the same time an important result of [12] is the detailed, both analytical and numerical, analysis of type B solutions to (7.1) though again using (7.4).

In the present paper we considered intrinsic two-component solutions i.e. solutions with different amplitudes. These solutions seems to have stability property which are not trivial consequence of the theorems proved in [12] and deserve additional studies. Besides enlarging the set of solutions, we have also shown the role played by these solutions as initial states from which localized matter waves (solitons) can be generated through the modulational instability mechanism.

VIII. CONCLUSIONS

In conclusion, we have considered the two-component CNLS with an elliptic potential as a model for trapped, quasi-one-dimensional two-component BECs. Classes of elliptic, solitary wave and trigonometric solutions have been presented.

From a physical point of view the solutions discussed in this paper are exact nonlinear Bloch states with wave number at the edges of the Brillouin zone. These solutions, except for the \( \text{cn}-\text{cn} \) one, are unstable under small amplitude and large wavelength modulations. Two component matter solitons arise from these unstable solution via a modulational instability mechanism which resemble the one described in Refs. [3, 17] for single component BEC in optical lattices.

Further perspectives of finding stable periodic solutions to the 2-component problem could be linked to investigations of finite-gap solutions of Manakov system given in terms of multi-dimensional \( \theta \)-functions [17] and [15] and to reduction of finite-gap solutions to elliptic functions [19]. Interesting classes of periodic solutions can be also obtained as the result of reduction of the Manakov system to completely integrable two-particle system interacting with fourth order potential [20, 21] and [1].

Acknowledgments

V.V.K. acknowledges Prof. Deconinck for discussing results in Ref. [12] prior their publication. V.S.G. and V.Z.E. wish to thank the Department of Physics “E.R.Caianiello” for the hospitality received, and the University of Salerno for providing an eight months research grant during which most of this work was done. V.V.K acknowledges support from the European grant, COSYC n.o HPRN-CT-2000-00158. M.S. acknowledge partial financial support from the MIUR, trough the inter-university project PRIN-2000, and from the European grant LOCNET grant no. HPRN-CT-1999-00163.