THE BIRCH–SWINNERTON-DYER EXACT FORMULA FOR QUADRATIC TWISTS OF ELLIPTIC CURVES

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Abstract. In the present short paper, we obtain a general lower bound for the 2-adic valuation of the algebraic part of the central value of the complex L-series for the quadratic twists of any elliptic curve $E$ over $\mathbb{Q}$. We also prove the existence of an explicit infinite family of quadratic twists with analytic rank 0 for a large family of elliptic curves.

1. Introduction

Let $E$ be any elliptic curve defined over $\mathbb{Q}$. We take any global minimal Weierstrass equation for $E$, and write $\Delta_E$ for its discriminant, $\omega_E$ for its Néron differential, and $\Omega_E^+$ for the least positive real period of $\omega_E$. Let $L(E, s)$ be the complex L-series of $E$. Since $E$ is modular by Wiles’ theorem, $L(E, 1)/\Omega_E^+$ is a rational number. Define $c_\infty(E) = \delta_E \Omega_E^+$, where $\delta_E = 1$ or 2 is the number of connected components of $E(\mathbb{R})$. For each finite prime $\ell$, let $c_\ell = [E(\mathbb{Q}_\ell) : E_0(\mathbb{Q}_\ell)]$, where $E_0(\mathbb{Q}_\ell)$ is the subgroup of $E(\mathbb{Q}_\ell)$ consisting of points with non-singular reduction modulo $\ell$. Let $\Sha(E)$ denote the Tate–Shafarevich group of $E$. If $L(E, 1) \neq 0$, Kolyvagin’s theorem tells us that both $E(\mathbb{Q})$ and $\Sha(E)$ are finite. In addition, the conjecture of Birch and Swinnerton-Dyer predicts that

\begin{equation}
\label{BSD}
\frac{L(E, 1)}{c_\infty(E)} = \prod_\ell c_\ell(E) \cdot |\Sha(E)| / |E(\mathbb{Q})_{\text{tor}}|^2,
\end{equation}

Remarkable progress has been made towards the proof this exact formula by the methods of Iwasawa theory. In particular, the $p$-part of the this formula is known to hold for all sufficiently large primes $p$ (see the nice survey article by Coates [6]). However, many of the most interesting classical problems involve the small primes $p$ where this formula still has not been established in general, and notably for the prime $p = 2$: see, for example, the remarkable work of Tian [16][17][18] on the congruent number problem, and Smith [15] on Goldfeld’s conjecture. In the present paper, we use elementary methods, which involve no Iwasawa theory, to prove both some weak forms of the 2-part of (1.1), and also some special cases of the 2-part of (1.1), when $E$ runs over a large family of quadratic twists of some fixed elliptic curve defined over $\mathbb{Q}$.

We write $ord_2$ for the order valuation of $\mathbb{Q}$ at the prime 2, normalized so that $ord_2(2) = 1$ and $ord_2(0) = \infty$. In all that follows, $M$ will always denote a square free positive or negative integer such that $M \equiv 1 \mod 4$ with $M \neq 1$, and we shall write $E(M)$ for the twist of $E$ by the extension $\mathbb{Q}(\sqrt{M})/\mathbb{Q}$. For any odd prime $q \mid M$, we define

$$t_E(M) := \sum_{q \mid M} t(q), \quad \text{where} \quad t(q) = \begin{cases} 1 & \text{if } 2 \mid c_q(E(M)); \\ 0 & \text{if } 2 \not\mid c_q(E(M)). \end{cases}$$

We recall that $E$ is said to be optimal if the map from the modular curve $X_0(C)$ to $E$ does not factor through any other elliptic curve defined over $\mathbb{Q}$.

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Theorem 1.1. Let $E$ be any optimal elliptic curve over $\mathbb{Q}$ having conductor $C$. Then, for all square free integers $M \equiv 1 \mod 4$ with $(M, C) = 1$ such that $L(E(M), 1) \neq 0$, we have
\begin{equation}
\text{ord}_2(L(E(M), 1)/c_\infty(E(M))) \geq t_E(M) - 1 - \text{ord}_2(\nu_E),
\end{equation}
where $\nu_E$ is the Manin constant of $E$.

Note that, if we assume $\nu_E$ is odd, the theorem gives the lower bound $t_E(M) - 1$. In fact, it has been shown that $\nu_E$ is odd for $4 \nmid C$ by the work of Mazur [13], Abbes–Ullmo [1], Agashe–Ribet–Stein [2] and Česnavičius [4]. In addition, Cremona [10] has shown numerically that $\nu_E$ is 1 for $C \leq 60000$. Previously, Zhao [23][24][25] proved several results like (1.2) for the congruent number family of elliptic curves, which were subsequently used by Tian in his induction arguments on the congruent number problem. Coates–Kim–Liang–Zhao [7] proved an analogue of (1.2) for a wide class of elliptic curves with complex multiplication, which was then applied in [8] to prove a generalisation of Birch’s lemma, and the 2-part of the Birch–Swinnerton-Dyer conjecture for quadratic twists of $X_0(49)$. These earlier methods work only for elliptic curves with complex multiplication, since they make use of the fact that the value at $s = 1$ of the complex $L$-series of such a curve is a sum of Eisenstein series. However, the methods used in this paper are valid for all elliptic curves defined over $\mathbb{Q}$.

We now give some non-vanishing results for quadratic twists of certain elliptic curves defined over $\mathbb{Q}$. The following conjecture is folklore.

Conjecture 1.2. For any elliptic curve $E$ over $\mathbb{Q}$, and any positive integer $r$, there are infinitely many square-free integers $M$, having exactly $r$ prime factors, such that $L(E(M), 1) \neq 0$.

We shall verify some special cases of this conjecture for certain elliptic curves in the following theorem.

Theorem 1.3. Let $E$ be an optimal elliptic curve over $\mathbb{Q}$ such that
\begin{enumerate}
\item $E(\mathbb{Q})[2] \subseteq \mathbb{Z}/2\mathbb{Z}$;
\item $\text{ord}_2(L(E, 1)/c_\infty(E)) = -\text{ord}_2(|E(\mathbb{Q})[2]|) \cdot \nu_E$.
\end{enumerate}
If $S$ is infinite, then for any positive integer $r$, there are infinitely many square-free integers $M$, having exactly $r$ prime factors in $S$, such that $L(E(M), 1) \neq 0$.

Proof. When $E(\mathbb{Q})[2] \cong \mathbb{Z}/2\mathbb{Z}$, the theorem follows from Theorem 5.2, and [3, Theorem 1.1] (in particular for $\Delta_E > 0$). When $E(\mathbb{Q})[2]$ is trivial, the theorem follows from [20, Theorem 1.1 & Theorem 1.3].
Remark 1.4. The conclusion of Theorem 1.3 also holds for the quadratic twists of elliptic curves lying in the same isogeny class as \( E \), since they have the same complex \( L \)-function as the corresponding quadratic twists of \( E \). We explain in the next section, using the Chebotarev density theorem, why \( S \) is usually non-empty, and how to construct an explicit infinite set primes with positive density lying in it in most cases. Note that when \( \Delta_E > 0 \), we assume that \( q \equiv 1 \mod 4 \), since our argument in the proof of Theorem 1.3 does not work for primes \( q \) with \( q \equiv 3 \mod 4 \). Indeed, when \( q \equiv 3 \mod 4 \), one has \( L(E(q), 1) = 0 \) for some \( E \) and some primes \( q \) satisfying all the other conditions in the definition of the set \( S \), for example, using the labelling of curves introduced by Cremona [9], 34a1 with \( q = 3 \), 99c1 with \( q = 7 \).

Remark 1.5. Theorem 1.3 can be applied to the family of quadratic twists of many elliptic curves \( E/Q \). In particular, it can be applied to the following \( E \) from Cremona’s Tables [9] with conductor \( C < 100 \): 11a1, 14a1, 19a1, 20a1, 26a1, 27a1, 34a1, 35a1, 36a1, 37b1, 38a1, 38b1, 44a1, 46a1, 49a1, 50a1, 50b1, 51a1, 52a1, 54a1, 54b1, 56b1, 66a1, 66c1, 67a1, 69a1, 73a1, 75a1, 75c1, 76a1, 77c1, 80b1, 84a1, 84b1, 89b1, 92a1, 94a1, 99c1, 99d1.

In a subsequent paper [14] with Jie Shu, we will prove parallel results such that the family of quadratic twists of elliptic curves has a rational point of infinite order and then verify the 2-part of the Birch and Swinnerton-Dyer conjecture for those curves by applying the results in this paper.

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2. The infinitude of \( S \)

We show in this section that, in most cases, the set of primes \( S \) has positive density in the set of all rational primes, by using the Chebotarev density theorem. If \( q \) is an odd prime of good reduction of \( E \), then multiplication by 2 is an automorphism of the formal group of \( E \) at \( q \), and so reduction modulo \( q \) gives an isomorphism from the 2-primary subgroup of \( E(Q) \) onto the 2-primary subgroup of \( E(F_q) \). Recall that when \( E(Q)[2] \cong \mathbb{Z}/2\mathbb{Z} \), write \( \phi : E \to E' \) for 2-isogeny defined over \( Q \).

Lemma 2.1. Let \( q \) be an odd prime of good reduction for \( E \). (i) If \( E(Q)[2] = 0 \), then \( E(Q)[2] = 0 \) if and only if \( [Q_q, E(2)] : Q_q = 3 \). (ii) If \( E(Q)[2] = \mathbb{Z}/2\mathbb{Z} \), then \( E(Q)[4] = \mathbb{Z}/2\mathbb{Z} \) if and only if both \( [Q_q, E(2)] : Q_q = 2 \) and also \( [Q_q, E'(2)] : Q_q = 2 \).

Proof. Assuming \( q \) is an odd prime of good reduction, then \( Q_q(E(2)) \) is an unramified, and therefore cyclic extension of \( Q_q \), whose Galois group is a quotient group of Gal(\( Q(E(2)) \)). But Gal(Q(E(2))/Q) is naturally isomorphic to a subgroup of GL(2, \( \mathbb{F}_2 \)) and Gal(Q_q(E(2))/Q_q) is therefore cyclic of order 1, 2, or 3. Now we can choose an equation for \( E \) of the form \( y^2 = f(x) \), where \( f(x) \) is a polynomial in \( Z[x] \) of degree 3, which has good reduction at \( q \). Then the non-zero points of order 2 on \( E \) are given by the points \((0, \alpha)\), where \( \alpha \) runs over the three roots of \( f(x) \). Thus it is clear that \( E(Q_q)[2] \neq 0 \) if and only if \( f(x) \) is a monic irreducible modulo \( q \), or equivalently \( [Q_q, E(2)] : Q_q = 3 \), proving (i). Assume now that \( E(Q)[2] = \mathbb{Z}/2\mathbb{Z} \). Suppose first that \( E(Q)[4] = \mathbb{Z}/2\mathbb{Z} \). Then \( E(Q_q)[2] = \mathbb{Z}/2\mathbb{Z} \), and so \( [Q_q, E(2)] : Q_q = 2 \). Moreover, since \( |E'(F_q)| = |E(F_q)| \), we must have \( E'(Q)[4] = E'(Q_q)[2] = \mathbb{Z}/2\mathbb{Z} \), and so \( [Q_q, E'(2)] : Q_q = 2 \). Conversely, assume that \( [Q_q, E(2)] : Q_q = 2 \) and \( [Q_q, E'(2)] : Q_q = 2 \). If, on the contrary, there exists a point \( P \) in \( E(Q_q) \) which is of exact order 4, then \( P' = \phi(P) \) would be a point in \( E'(Q_q) \) of order dividing 4. Moreover, if we write \( \phi' : E' \to E \) for the isogeny dual to \( \phi \), then \( \phi'(P') = 2P \neq 0 \). It follows that \( E'(Q_q)[4] \) contains an element which is not in the kernel.
of the isogeny $\phi'$, and so we must have $E'(\mathbb{Q}_q)[2]$ necessarily has order 4, contradicting our assumption. This completes the proof.

\begin{corollary}
Assume $E(\mathbb{Q})[2] = 0$. (i) There is a positive density of odd primes of good reduction $q$ such that $E(\mathbb{Q}_q)[2] = 0$. (ii) There is a positive density of odd primes of good reduction such that both $q \equiv 1 \mod 4$ and $E(\mathbb{Q}_q)[2] = 0$.
\end{corollary}

\begin{proof}
Let $H = \mathbb{Q}(E[2])$. Assertion (i) follows easily on applying the Chebotarev density theorem to $\text{Gal}(H/\mathbb{Q})$, recalling that this Galois group is either cyclic of order 3, or the symmetric group on 3 elements. Turning to assertion (ii), and in what follows we write $I = \mathbb{Q}(i)$. Denote $\mathfrak{H} = HI, G = \text{Gal}(\mathfrak{H}/\mathbb{Q})$. Suppose first that $\text{Gal}(H/\mathbb{Q})$ is cyclic of order 3, so that $G$ is a cyclic group of order 6. Let $\mathcal{Q}$ be the set of all odd primes of good reduction $q$ whose Frobenius element in $G$ is of exact order 3. Then $q \in \mathcal{Q}$ must split in the quadratic extension $I/\mathbb{Q}$, and the assertion follows immediately by the Chebotarev density theorem. When $\text{Gal}(H/\mathbb{Q})$ is the symmetric group on 3 elements, $G$ will have order 6 or 12. Again we simply take an element $g \in G$ of exact order 3, and let $\mathcal{Q}$ be the set of primes of odd good reduction whose Frobenius elements in $G$ lie in the conjugacy class of $g$. Since $g$ has order 3, the primes in $\mathcal{Q}$ must split in the quadratic extension $I/\mathbb{Q}$, and the assertion again follows from the Chebotarev density theorem. This completes the proof.
\end{proof}

When $E(\mathbb{Q})[2] \cong \mathbb{Z}/2\mathbb{Z}$, recall that $\Delta_E$ is the minimal discriminant of $E$, and we also write $\Delta_{E'}$ for the minimal discriminant of $E'$. Recall that $F = \mathbb{Q}(E(\mathbb{Q})[2]) = \mathbb{Q}(\sqrt{\Delta_E})$ and $F' = \mathbb{Q}(E'(\mathbb{Q})[2]) = \mathbb{Q}(\sqrt{\Delta_{E'}})$. Of course, the prime factors of both $\Delta_E$ and $\Delta_{E'}$ are precisely the set of primes of bad reduction, but the exponents to which these primes occur are usually different for $E$ and $E'$, and the signs of $\Delta_E$ and $\Delta_{E'}$ may also differ.

\begin{corollary}
Assume $E(\mathbb{Q})[2] = 0$. Then there is a positive density of odd primes of good reduction $q$ such that $E(\mathbb{Q}_q)[4] = \mathbb{Z}/2\mathbb{Z}$ if and only if $F' \neq \mathbb{Q}$.
\end{corollary}

\begin{proof}
Note that $[F : \mathbb{Q}] = 2$. If $F' = \mathbb{Q}$, Lemma 2.1 shows that there is no odd good prime $q$ such that $E(\mathbb{Q}_q)[4] = \mathbb{Z}/2\mathbb{Z}$. Suppose next that $[F' : \mathbb{Q}] = 2$. Hence we are seeking odd primes $q$ of good reduction which are inert in both $F$ and $F'$. If $F = F'$, there is clearly a set of positive density of such primes $q$, and so we can assume that $F \neq F'$. Then the compositum $J = FF'$ will be a biquadratic extension of $\mathbb{Q}$, and we let $K$ be the third quadratic extension of $\mathbb{Q}$ contained in $J$. Let $\mathcal{Q}$ be the set of all odd primes of good reduction $q$ which have a prime of $K$ lying above them which is inert in $J$. Since $J = FK = F'K$ is an extension of $\mathbb{Q}$ whose Galois group is isomorphic to the non-cyclic group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we see that every prime $q \in \mathcal{Q}$ must split completely in $K$. In turn, this implies that every prime $q \in \mathcal{Q}$ must be inert in both $F$ and $F'$. Now the set $\mathcal{Q}$ has positive density in the set of all primes by the Chebotarev theorem, and the proof of (ii) is now follows from (ii) of Lemma 2.1.
\end{proof}

\begin{corollary}
Assume that $E(\mathbb{Q})[2] = \mathbb{Z}/2\mathbb{Z}$ and we have both $F \neq \mathbb{Q}(i)$ and $F' \neq \mathbb{Q}, \mathbb{Q}(i)$. Then there is a set of positive density of odd primes $q$ of good reduction such that both $q \equiv 1 \mod 4$ and $E(\mathbb{Q}_q)[4] = \mathbb{Z}/2\mathbb{Z}$.
\end{corollary}

\begin{proof}
We use a similar argument to that used in the proof of Corollary 2.3. First suppose that $F = F'$. Then the compositum $\mathfrak{H} = FI$ is a biquadratic extension of $\mathbb{Q}$, and we take $\mathcal{Q}$ to be the set of odd primes $q$ of good reduction which have a prime of $I$ lying above them which is inert in $\mathfrak{H}$. Then all primes in $\mathcal{Q}$ must split completely in $I$, and must be inert in the field $F$, proving the corollary in this case. Hence we can assume that $F \neq F'$, so that $J = FF'$ is a biquadratic extension of $\mathbb{Q}$. Let $K$ be the third quadratic extension of $\mathbb{Q}$ contained in $J$. If $K = I$, exactly the same argument as in the proof of Corollary 2.3 proves the assertion here. Hence we can assume that $K \neq I$, whence the field $D = JF$ has Galois group over $\mathbb{Q}$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Then $D$ is a quadratic extension of the field $KI$. We now take $\mathcal{Q}$ to be the set of odd primes
Denote \(\{\alpha, \beta\} := \Gamma_0(\mathcal{H}) \cup \mathcal{H}\), and denote \(\langle \alpha, \beta \rangle\) to be an integral homology class in \(H_1(\mathcal{H}, \mathbb{Z})\) from \(\alpha\) to \(\beta\). Denote \(\langle \alpha, \beta, f \rangle := \int_\mathcal{H}^{\beta} 2\pi i f(z) dz\).

For any positive integer \(l \mid m\), we denote \(\chi_l^0\) to be the principal Dirichlet character modulo \(l\), and \(\chi_l\) to be any primitive Dirichlet character modulo \(l\). For \(\chi = \chi_l^0 : \chi_l\), we define
\[
\langle m \rangle_\chi := \sum_{k \in (\mathbb{Z}/m\mathbb{Z})^\times} \chi(k) \langle \{0, \frac{k}{m}\}, f \rangle.
\]

In particular, we have \(\langle m \rangle_{\chi_l} = \langle m \rangle_{\chi_l^0}\). For simplicity, we use \(\langle m \rangle_{\chi^0}\) instead of both of them if there is no danger of confusion.

**Proposition 3.1.** We have
\[
(\sum_{l \mid m} l - a_m)L(E, 1) = -\sum_{l \mid m \text{ and } \ell \mod l} \langle \{0, \frac{k}{l}\}, f \rangle = -\sum_{j=1}^{r(m)} 2^{r(m)-j} \sum_{d \mid m \text{ and } r(d)=j} \langle d \rangle_{\chi^0}^d,
\]
where \(l\) runs over all positive divisors of \(m\); and
\[
L(E, \chi_m, 1) = \frac{g(\overline{\chi_m})}{m} \sum_{k \mod m} \chi_m(k) \langle \{0, \frac{k}{m}\}, f \rangle = \frac{g(\overline{\chi_m})}{m} \cdot \langle m \rangle_{\chi_m},
\]
where
\[
g(\overline{\chi_m}) := \sum_{k \mod m} \overline{\chi_m(k)} e^{2\pi i \frac{k}{m}}.
\]

**Proof.** See Manin [12, Theorem 4.2], or Cremona [9, Chapter 3].

In what follows, we always denote \(N_q := |E(\mathbb{F}_q)| = 1 + q - a_q\).

**Proposition 3.2.** We have
\[
N_q N_{q^2} \cdots N_{q^{r(m)}} L(E, 1) = (-1)^{r(m)} \sum_{j=1}^{r(m)} \sum_{d \mid m \text{ and } r(d)=j} \prod_{q' \mid q} (1 - q) \cdot \langle d \rangle_{\chi^0}^d;
\]
\[
\langle m \rangle_{\chi_d} = (a_q - 2\chi_d(q)) \langle m/q \rangle_{\chi_d} = \prod_{q' \mid q} (a_q - 2\chi_d(q)) \cdot \langle d \rangle_{\chi_d},
\]
where \( r(m) \geq 2, \ d > 1 \) is a positive integer dividing \( m \), and \( q \) is a prime dividing \( \frac{m}{\pi} \).

**Proof.** See [3, Lemma 2.1] for the proof of (3.3), and [3, Lemma 4.1] for (3.4). \( \square \)

### 3.2. Period lattice

Let \( \omega_E \) be a Néron differential on a global minimal Weierstrass equation for \( E \), and \( \mathfrak{L}_E \) be the period lattice of \( \omega_E \). Denote \( \mathcal{O}_E^+ \ (i\mathcal{O}_E^-) \) to be the least positive real (purely imaginary) period of \( \omega_E \), and similarly \( \mathcal{O}_f^+ \ (i\mathcal{O}_f^-) \) to be the least positive real (purely imaginary) period of \( f \). Recall that when \( \Delta_E < 0 \) (resp. \( \Delta_E > 0 \)), the period lattice \( \mathfrak{L}_E \) has a \( \mathbb{Z} \)-basis of the form \( [\mathcal{O}_E^+, (\mathcal{O}_E^+ + i\mathcal{O}_E^-)/2] \) (resp. \( [\mathcal{O}_E^+, i\mathcal{O}_E^-] \)), where \( \mathcal{O}_E^+ \) and \( \mathcal{O}_E^- \) are both real, and the period lattice \( \Lambda_f \) of \( f \) has a \( \mathbb{Z} \)-basis of the form \( [\mathcal{O}_f^+, (\mathcal{O}_f^+ + i\mathcal{O}_f^-)/2] \) (resp. \( [\mathcal{O}_f^+, i\mathcal{O}_f^-] \)), where \( \mathcal{O}_f^+ \) and \( \mathcal{O}_f^- \) are also both real.

Recall that \( \nu_E \) denotes the Manin constant of \( E \), then we have \( c_{\infty}(E) := \delta_E \mathcal{O}_E^+ \), \( c_{\infty}(E) := \delta_E \mathcal{O}_E^- \), and \( c_f := \delta_E \mathcal{O}_f^+ \), \( c_f^- := \delta_E \mathcal{O}_f^- \), where \( \delta_E \) is the number of connected components of \( E(\mathbb{R}) \). Thus, we can write

\[
\langle \{0, \frac{k}{m}\}, f \rangle = \frac{(s_{k,m}c_f + it_{k,m}c_f^-)}{2},
\]

where \( s_{k,m}, t_{k,m} \) are integers depending on \( f \). In particular, \( s_{k,m}, t_{k,m} \) are of the same parity when \( \Delta_E < 0 \).

**Proposition 3.3.** We have

\[
\langle m \rangle_{\chi_m} / c_f = \sum_{(k,m)=1}^{(m-1)/2} s_{k,m};
\]

\[
\langle m \rangle_{\chi_m} / c_f = \sum_{(k,m)=1}^{(m-1)/2} \chi_m(k)\frac{s_{k,m}}{2} \quad (\text{if } m \equiv 1 \mod 4);
\]

\[
\langle m \rangle_{\chi_m} / ic_f^- = \sum_{(k,m)=1}^{(m-1)/2} \chi_m(k)\frac{t_{k,m}}{2} \quad (\text{if } m \equiv 3 \mod 4).
\]

**Proof.** When \( m \equiv 1 \mod 4 \), we have \( \chi_m(k) = \chi_m(m-k) \), and when \( m \equiv 3 \mod 4 \), we have \( \chi_m(k) = -\chi_m(m-k) \). Note that \( \langle \{0, \frac{k}{m}\}, f \rangle \) and \( \langle \{0, \frac{m-k}{m}\}, f \rangle \) are complex conjugate periods of \( f \), the assertions follow immediately in view of (3.5). \( \square \)

Moreover, combining (3.1) with (3.6), we have the following result.

**Corollary 3.4.** We have

\[
\text{ord}_2(N_qL(E, 1)/c_f) \geq 0,
\]

for any odd prime \( q \) with \( (q, C) = 1 \).
3.3. Integrality at 2. For any positive odd square-free integer \( m \) with \( (m, C) = 1 \), we write \( m = m^+m^- \), where all the prime factors of \( m^+ \) are congruent to 1 modulo 4, and all the prime factors of \( m^- \) are congruent to 3 modulo 4. We have the following result of integrality at 2, which plays an important role to carry out our induction arguments in the proof of our main theorems.

**Proposition 3.5.** Let \( E \) be an optimal elliptic curve over \( \mathbb{Q} \) with conductor \( C \). Let \( m \) be any integer of the form \( m = m^+m^- = q_1q_2\cdots q_{r(m)} \) with \( (m, C) = 1 \), and \( q_1, \ldots, q_{r(m)} \) arbitrary distinct odd primes. Then we have

\[
\sum_{d|\langle m \rangle} \langle m \rangle_{\chi_d/cf} = 2^{r(m)-1} \Psi_m; \quad \sum_{d|\langle m \rangle} \langle m \rangle_{\chi_d/\psi_f} = 2^{r(m)-1} \Psi'_m,
\]

where \( d = d^+d^- \), \( r(d^-) \) is the number of prime factors of \( d^- \), and \( \Psi_m, \Psi'_m \) are integers. Moreover, \( \Psi_m, \Psi'_m \) are integers of the same parity when \( \Delta_E < 0 \).

**Proof.** By (3.4), (3.6), (3.7) and (3.8), we have

\[
\sum_{d|\langle m \rangle} \langle m \rangle_{\chi_d} = \sum_{d|\langle m \rangle} \langle m \rangle_{\chi_0} + \sum_{d|\langle m \rangle} \prod_{1<d<m} (a_q - 2\chi_d(q)) \cdot \langle d \rangle_{\chi_d} + \frac{1 + (-1)^{r(m)}}{2} \langle m \rangle_{\chi_m} = s \cdot cf;
\]

\[
\sum_{d|\langle m \rangle} \langle m \rangle_{\chi_d} = \sum_{d|\langle m \rangle} \prod_{1<d<m} (a_q - 2\chi_d(q)) \cdot \langle d \rangle_{\chi_d} + \frac{1 - (-1)^{r(m)}}{2} \langle m \rangle_{\chi_m} = t \cdot \psi_f,
\]

where \( s, t \) are integers.

On the other hand, by the definition of \( \langle m \rangle_{\chi_d} \) and in view of (3.5), we have

\[
\sum_{d|\langle m \rangle} \langle m \rangle_{\chi_d} = 2^{r(m)} \sum_{k\in(\mathbb{Z}/m\mathbb{Z})^\times} \langle \{0, \frac{k}{m}\}, f \rangle = 2^{r(m)-1} \sum_{k\in(\mathbb{Z}/m\mathbb{Z})^\times} (s_{k,m} cf + it_{k,m} \psi_f),
\]

where \( \sum^* \) means that \( k \) runs over all the elements of \( (\mathbb{Z}/m\mathbb{Z})^\times \) such that \( \chi_{q_i}(k) = 1 \) for all \( 1 \leq i \leq r(m) \). Then combining (3.12) with (3.10) and (3.11), it follows that

\[
s = 2^{r(m)-1} \Psi_m := 2^{r(m)-1} \sum_{k\in(\mathbb{Z}/m\mathbb{Z})^\times} s_{k,m}; \quad t = 2^{r(m)-1} \Psi'_m := 2^{r(m)-1} \sum_{k\in(\mathbb{Z}/m\mathbb{Z})^\times} t_{k,m}.
\]

The last assertion of the proposition holds since \( s_{k,m}, t_{k,m} \) are integers of the same parity when \( \Delta_E < 0 \). This completes the proof of the proposition.

4. Lower Bound

In order to prove our main theorem, we should understand the behaviour of the Tamagawa factors under twisting, for example, one can see [5, §7]). Recall that \( c_q(E^{(M)}) \) denotes the Tamagawa factor at the prime \( q \mid M \), since \( (q, 2C) = 1 \), reduction modulo \( q \) on \( E \) gives an isomorphism \( E(\mathbb{Q}_q)[2] \cong E(\mathbb{F}_q)[2] \). Hence, we have

\[
ord_2(c_q(E^{(M)})) = ord_2(\#E^{(M)}(\mathbb{Q}_q)[2]) = ord_2(\#E(\mathbb{Q}_q)[2]) = ord_2(\#E(\mathbb{F}_q)[2])
\]

Recall that \( ord_2(|E(\mathbb{F}_q)|) = ord_2(N_q) \), it follows that \( 2 \mid c_q(E^{(M)}) \) if and only if \( 2 \mid N_q \). Hence, we have

\[
t(m) := t_E(M) = \sum_{2|c_q(E^{(M)})} \frac{q|m}{2|N_q} = 1 = \sum_{2|c_q(E^{(M)})} \frac{q|m}{2|N_q}
\]
where $m = |M|$. It is plainly that $t(m) = t(m_1) + t(m_2)$ for $m = m_1m_2$, and $t(m) = r(m)$ if $|E(Q)_{\text{tor}}|$ is even.

**Lemma 4.1.** Let $E$ be the optimal elliptic curve over $Q$ attached to $f$. Let $m = q_1q_2 \cdots q_{r(m)}$ be a product of $r(m)$ distinct odd primes with $(m, C) = 1$, $r(m) \geq 1$. We have

$$\text{ord}_2(\langle m \rangle_{\chi_m}/c_f) \geq \begin{cases} \text{ord}_2(L(E^{(M)}, 1)/c_{\infty}(E^{(M)})) \geq t_E(M) - 1 - \text{ord}_2(\nu_E). \\
\text{ord}_2(\langle m \rangle_{\chi_m}/c_f) \geq \begin{cases} r(m) - 1 & \text{if } |E(Q)_{\text{tor}}| \text{ is even}; \\
t(m) & \text{if } |E(Q)_{\text{tor}}| \text{ is odd}. 
\end{cases}
\end{cases}$$

**Proof.** When $r(m) = 1$, in view of (3.9), the lemma holds obviously if $|E(Q)_{\text{tor}}|$ is even. If $|E(Q)_{\text{tor}}|$ is odd, there exists a prime $p$ coprime to $2C$ such that $N_p$ is odd, so $\text{ord}_2(L(E, 1)/c_f) \geq 0$. It follows that

$$\text{ord}_2(\langle q \rangle_{\chi_q}/c_f) \geq \text{ord}_2(N_q) \geq t(q).$$

We now prove the lemma by induction on $r(m)$. Assume $r(m) > 1$, and that the lemma is true for all divisors $n > 1$ of $m$ with $n \neq m$. For $m = q_1q_2 \cdots q_{r(m)}$, in view of (3.3), we have that

$$\langle m \rangle_{\chi_m} = (-1)^r(m)N_q, \cdots N_q, L(E, 1) - \sum_{j=1}^{r(m)-1} \sum_{n|m} \prod_{q \mid n} (1 - q)(\langle n \rangle_{\chi_n}).$$

By our assumption, we see that

$$\text{ord}_2(\prod_{q \mid n} (1 - q)(\langle n \rangle_{\chi_n})/c_f) \geq \begin{cases} r(\frac{m}{n}) + r(n) - 1 = r(m) - 1 & \text{if } |E(Q)_{\text{tor}}| \text{ is even}; \\
t(\frac{m}{n}) + t(n) = t(m) & \text{if } |E(Q)_{\text{tor}}| \text{ is odd}. 
\end{cases}$$

Same result holds for $\text{ord}_2(N_q, N_q, \cdots N_q, L(E, 1)/c_f)$. Hence, the assertion for $m = q_1q_2 \cdots q_{r(m)}$ follows. This completes the proof of the lemma. □

**Theorem 4.2 (Theorem 1.1).** Let $E$ be an optimal elliptic curve over $Q$ with conductor $C$. Let $M = e q_1 q_2 \cdots q_r$ be a product of $r$ odd distinct primes with $(M, C) = 1$, where the sign $e = \pm 1$ is chosen so that $M \equiv 1 \mod 4$. We then have

$$\text{ord}_2(L(E^{(M)}, 1)/c_{\infty}(E^{(M)})) \geq t_E(M) - 1 - \text{ord}_2(\nu_E).$$

**Proof.** We denote $d = m^+ m^- = q_1 q_2 \cdots q_{r(m)}$, and

$$c_{\infty}^+ = \begin{cases} c_{\infty}(E) & \text{if } m \equiv 1 \mod 4; \\
ic_{\infty}(E) & \text{if } m \equiv 3 \mod 4,
\end{cases} \quad \text{and} \quad c_{f}^+ = \begin{cases} c_f & \text{if } m \equiv 1 \mod 4; \\
ic_f & \text{if } m \equiv 3 \mod 4.
\end{cases}$$

Since

$$\text{ord}_2(L(E^{(M)}, 1)/c_{\infty}(E^{(M)})) = \text{ord}_2(\sqrt{ML(E^{(M)}, 1)/c_{\infty}^+(E)}) = \text{ord}_2(\langle m \rangle_{\chi_m}/c_{f}^+) - \text{ord}_2(\nu_E),$$

we only need to work on $\langle m \rangle_{\chi_m}/c_{f}^+$. We shall prove this lemma by induction on $r(m)$.

When $r(m) = 1$, say $m = q$, Proposition 3.5 shows that

$$\langle q \rangle_{\chi_q}/c_{f} = \Psi_q (q)$$

and

$$\langle q \rangle_{\chi_q}/c_{f} = \Psi_q (q)$$

then the lemma follows immediately by Lemma 4.1.

Now assume that $r(M) > 1$, and that the theorem is true for all divisors $n \equiv 1 \mod 4$ of $M$ with $n \neq M$, so we have that $\text{ord}_2((d)_{\chi_d}/c_{f}^+) \geq t(d) - 1$, where $d \mid M$ and $1 < d < q_1 q_2 \cdots q_r := m$. Again by Proposition 3.5, we have

$$\langle m \rangle_{\chi_m} = \sum_{d | m, 2 \nmid d} \prod_{1 < d < m} (a_q - 2 \chi(d)g) \cdot \langle q \rangle_{\chi_q} + \langle m \rangle_{\chi_m} = 2^{r-1} \Psi_m \cdot c_f (q)$$

and
\[
\sum_{d \mid m, 2 \nmid (d - 1)} \prod_{q \mid d} (a_q - 2\chi_d(q)) \cdot (d)_{\chi_d} + \langle m \rangle_{\chi_m} = 2^{r-1} \Psi_m' \cdot i\epsilon_f^- \quad \text{(if } 2 \nmid r(m^-)) \text{.}
\]

Note that
\[
\text{ord}_2(\prod_{q \mid d} (a_q - 2\chi_d(q)) \cdot (d)_{\chi_d} / c_f^2) \geq \frac{t(m)}{d} + t(d) - 1 = t(m) - 1,
\]
and in view of the above two equations, it follows that \(\text{ord}_2(\langle m \rangle_{\chi_m} / c_f^2) \geq t(m) - 1\) again by Lemma 4.1. This completes the proof of the theorem. \(\square\)

5. NON-VANISHING RESULT

In this section, we shall verify Conjecture 1.2 for elliptic curves with only one non-trivial rational 2-torsion point. In particular, we will show that in some sub-families of quadratic twists of certain elliptic curves, there exists an explicit infinite family of quadratic twists with analytic rank 0. Of course, we can use the precise \(L\)-values to verify the 2-part of the Birch and Swinnerton-Dyer conjecture, combining with results given by the classical 2-descents. We first prove the following lemma, which admits stronger lower bound than the conclusion in Lemma 4.1.

**Lemma 5.1.** Let \(E\) be the optimal elliptic curve over \(\mathbb{Q}\) with analytic rank zero attached to \(f\). Let \(m\) be any integer of the form \(m = m^+m^- = q_1q_2 \cdots q_{r(m)}\), with \((m, C) = 1\), \(r(m^+) \geq 1\), \(r(m^-) \geq 1\), and \(q_1, \ldots, q_{r(m)}\) arbitrary distinct odd primes. If \(\text{ord}_2(N_{q_i}) = 1\) for any \(1 \leq i \leq r(m)\), then we have
\[
\text{ord}_2(\langle m \rangle_{\chi_m^o} / c_f) > r(m) - 1.
\]

**Proof.** We first prove the case when \(r(m) = 2\), say \(m = m^+m^- = q_1q_2\) with \(q_1 \equiv 1 \mod 4\), \(q_2 \equiv 3 \mod 4\). By (3.3), we have that
\[
N_{q_1}N_{q_2}L(E, 1) = (1 - q_2)\langle q_1 \rangle_{\chi^o} + (1 - q_1)\langle q_2 \rangle_{\chi^o} + \langle q_1q_2 \rangle_{\chi^o}.
\]
In view of (3.1), we have \(\text{ord}_2(S_{q_i} / c_f) = \text{ord}_2(N_{q_i}L(E, 1) / c_f)\) \((i = 1, 2)\), it follows that
\[
\text{ord}_2(N_{q_1}N_{q_2}L(E, 1) / c_f) = \text{ord}_2((1 - q_2)\langle q_1 \rangle_{\chi^o} / c_f) < \text{ord}_2((1 - q_1)\langle q_2 \rangle_{\chi^o} / c_f).
\]
Thus, we must have \(\text{ord}_2(q_1q_2)_{\chi^o} / c_f) > \text{ord}_2(N_{q_1}N_{q_2}L(E, 1) / c_f)\).

We now prove this lemma by induction on \(r(m)\). Assuming that \(r(m) > 2\), and that the lemma is true for all divisors \(n = n^+n^- > 1\) of \(m\) with \(r(n^+) \geq 1\), \(r(n^-) \geq 1\), and \(n \neq m\).

Then, for \(m = q_1q_2 \cdots q_{r(m)}\), we have
\[
(5.1) \prod_{q \mid m} N_q \cdot L(E, 1) = (-1)^{r(m)} \prod_{q \mid m^-} (1 - q) \cdot \langle m^+ \rangle_{\chi^o} + \sum_{j=1}^{r(m)-1} \sum_{n \neq m^+ \atop n \neq m^- \atop r(n) = j} \prod_{q \mid m} (1 - q) \cdot \langle n \rangle_{\chi^o} + \langle m \rangle_{\chi^o}
\]
again by (3.3). Since
\[
\text{ord}_2(\langle m^+ \rangle_{\chi^o} / c_f) = \text{ord}_2(\prod_{q \mid m^+} N_q \cdot L(E, 1) / c_f))
\]
by [3, Lemma 2.2] and \(\text{ord}_2(1 - q) = \text{ord}_2(N_q)\) for \(q \mid m^-\), it follows that
\[
(5.2) \quad \text{ord}_2(\prod_{q \mid m^-} (1 - q) \cdot \langle m^+ \rangle_{\chi^o} / c_f) = \text{ord}_2(\prod_{q \mid m} N_q \cdot L(E, 1) / c_f)).
\]
However, when \( n \mid m \) and \( n \neq m^+ \), we claim
\[
\text{ord}_2\left( \prod_{q \mid m} (1 - q) \cdot \langle n \rangle_{\chi^m/c_f} \right) > \text{ord}_2\left( \prod_{q \mid m} N_q \cdot L(E, 1)/c_f \right).
\]

Indeed, when \( m^+ \mid n \neq m^+ \), we have
\[
\text{ord}_2\langle n \rangle_{\chi^m/c_f} > \text{ord}_2\left( \prod_{q \mid m} N_q \cdot L(E, 1)/c_f \right)
\]
by our assumption, the claim then follows immediately since \( \text{ord}_2(1 - q) = \text{ord}_2(N_q) \) for \( q \mid \frac{m}{n} \).

When \( m^+ \nmid n \), we have
\[
\text{ord}_2\langle n \rangle_{\chi^m/c_f} \geq \text{ord}_2\left( \prod_{q \mid m} N_q \cdot L(E, 1)/c_f \right)
\]
by our assumption and by [3, Lemma 2.2] and [22, Lemma 2.2], where the equality holds when \( n \mid m^+ \) or \( n \nmid m^+ \). Since \( m^+ \nmid n \), there must exist at least one prime factor of \( \frac{m}{n} \) which is congruent to 1 modulo 4, it follows that \( \text{ord}_2\left( \prod_{q \mid \frac{m}{n}} (1 - q) \right) > \text{ord}_2\left( \prod_{q \mid \frac{m}{n}} N_q \right) \). Thus, the claim is true for both cases. In view of (5.1), and combining with (5.2) and (5.3), it easily follows that
\[
\text{ord}_2\langle n \rangle_{\chi^m/c_f} > \text{ord}_2\left( \prod_{q \mid m} N_q \cdot L(E, 1)/c_f \right) \geq r(m) - 1.
\]

This completes the proof of the lemma.

When \( E \) has only one rational 2-torsion point and no rational cyclic 4-isogeny, recall that, if \( \Delta_E < 0 \), \( S \) is an infinite set of odd primes
\[
S = \{ q \mid \text{ord}_2(\vert E(\mathbb{F}_q)\vert) = \text{ord}_2(\vert E(\mathbb{Q})[2]\vert) \},
\]
whence we have \( \text{ord}_2(\vert E(\mathbb{F}_q)\vert) = \text{ord}_2(N_q) = 1. \)

**Theorem 5.2.** Let \( E \) be an optimal elliptic curve over \( \mathbb{Q} \) with conductor \( C \). Assume that
\begin{enumerate}
  \item \( \Delta_E < 0; \)
  \item \( E(\mathbb{Q})[2] \cong \mathbb{Z}/2\mathbb{Z}; \)
  \item \( E \) has odd Manin constant;
  \item \( \text{ord}_2(L(E, 1)/c_\infty(E)) = -1. \)
\end{enumerate}
Let \( M = q_1q_2 \cdots q_r \) be a product of \( r \) distinct primes in \( S \), where the sign \( \epsilon = \pm 1 \) is chosen so that \( M \equiv 1 \mod 4 \). Then \( L(E^{(M)}, 1) \neq 0 \), and we have
\[
\text{ord}_2(L(E^{(M)}, 1)/c_\infty(E^{(M)})) = r - 1.
\]

In particular, \( E^{(M)}(\mathbb{Q}) \) and \( \text{III}(E^{(M)}) \) are both finite.

**Proof.** As usual, we write \( M = \epsilon M^+ M^- \), where all the prime factors of \( M^+ \) are congruent to 1 modulo 4, and all the prime factors of \( M^- \) are congruent to 3 modulo 4. The theorem has been proved when \( M = M^+ \) (see [3, Theorem 1.1]), and \( M = M^- \) (see [22, Theorem 1.2]). We only need to prove the case when \( r(M^+) \geq 1 \) and \( r(M^-) \geq 1 \). We next start with the case \( r(M^+) = r(M^-) = 1 \), and prove the theorem by induction on \( r(M) \).

We first work on the case \( r(M) = 2 \), say \( M = -q_1q_2 \) with \( q_1 \equiv 1 \mod 4 \) and \( q_2 \equiv 3 \mod 4 \). By Proposition 3.5 we have that
\[
\langle (q_1q_2)_{\chi^m/c_f} (a_{q_2} - 2\chi_{q_1}(q_2)) \rangle_{\chi^m/c_f} = \Psi_{q_1q_2}; \quad \langle (q_1 - 2\chi_{q_1}(q_2)) \rangle_{\chi^m/c_f} = \Psi'_{q_1q_2}.
\]
Note that we have proved \( \text{ord}_2\langle (q_1)_{\chi^m} \rangle = \text{ord}_2\langle (q_2)_{\chi^m} \rangle = 0 \) in [20, Theorem 1.5], then \( \text{ord}_2\langle (a_{q_2} - 2\chi_{q_1}(q_2)) \rangle_{\chi^m/c_f} > 0 \), and \( \text{ord}_2\langle (q_1q_2)_{\chi^m/c_f} \rangle > 1 \) by Lemma 5.1, it follows that \( \Psi_{q_1q_2} \) must be
even, so is \( \Psi_{q_1q_2}' \) by the last assertion of Lemma 5.1. But \( \text{ord}_2((a_{q_1} - 2q_{q_2}(q_1))\langle q_2 \rangle_{\chi_{q_2}}/ic_f^{-1}) = 1 \), hence \( \text{ord}_2((\langle q_1, q_2 \rangle_{\chi_{q_1q_2}}/ic_f^{-1})) = 1 \). This proves the case \( r(M) = 2 \).

Now assume that \( r(M) > 2 \), and that the theorem is true for \( r(M) < r \). We shall prove the theorem in two cases. When \( r(M^-) \) is even, by Proposition 3.5, we have

\[
\langle M^+M^- \rangle \chi_0 + \sum_{d\mid M^+M^-, 2\mid r(d^-)} \prod_{q \mid M^+M^-} (a_q - 2\chi_d(q)) \cdot \langle d \rangle_{\chi_d} + \langle M^+M^- \rangle \chi_{M^+M^-} = 2^{r(M)} \Psi_{M^+M^-} \cdot c_f^{-1};
\]

\[
\text{ord}_2\left( \prod_{q \mid M^+M^-} (a_q - 2\chi_d(q)) \cdot \langle d \rangle_{\chi_d}/c_f^{-1} \right) = r(M) - 1 \quad \text{if } M^- \mid d;
\]

\[
\text{ord}_2\left( \prod_{q \mid M^+M^-} (a_q - 2\chi_d(q)) \cdot \langle d \rangle_{\chi_d}/c_f^{-1} \right) > r(M) - 1 \quad \text{if } M^- \nmid d,
\]

by our assumption. If \( 2 \nmid r(d^-) \), but \( r(M^-) \) is even, we must have \( M^- \nmid d \). Combining with (5.5) and (5.6), it follows that \( \Psi_{M^+M^-}' \) must be even, so is \( \Psi_{M^+M^-} \) as \( \Delta_E < 0 \). We now investigate the middle terms of the left-hand side of (5.4) divided by \( c_f \), there are exactly \( 2^{r(M)} - 1 \) terms which have 2-adic valuation \( \text{ord}_2((M^+M^-) \chi_0/c_f^{-1}) = 1 \). This proves the case \( \Psi_{M^+M^-} \).

When \( r(M^-) \) is odd, by Proposition 3.5, we have

\[
\langle M^+M^- \rangle \chi_0 + \sum_{d\mid M^+M^-, 2\mid r(d^-)} \prod_{q \mid M^+M^-} (a_q - 2\chi_d(q)) \cdot \langle d \rangle_{\chi_d} = 2^{r(M)} \Psi_{M^+M^-} \cdot c_f^{-1};
\]

\[
\text{ord}_2\left( \prod_{q \mid M^+M^-} (a_q - 2\chi_d(q)) \cdot \langle d \rangle_{\chi_d}/c_f^{-1} \right) = r(M) - 1 \quad \text{if } M^- \mid d;
\]

\[
\text{ord}_2\left( \prod_{q \mid M^+M^-} (a_q - 2\chi_d(q)) \cdot \langle d \rangle_{\chi_d}/c_f^{-1} \right) > r(M) - 1 \quad \text{if } M^- \nmid d,
\]

by our assumption. If \( 2 \nmid r(d^-) \), but \( r(M^-) \) is odd, we must have \( M^- \nmid d \). Combining with (5.5) and (5.6), it follows that \( \Psi_{M^+M^-}' \) must be even, so is \( \Psi_{M^+M^-} \) as \( \Delta_E < 0 \). We now investigate the middle terms of the left-hand side of (5.4) divided by \( c_f \), there are exactly \( 2^{r(M)} - 1 \) terms which have 2-adic valuation \( \text{ord}_2((M^+M^-) \chi_0/c_f^{-1}) = 1 \). This proves the case \( \Psi_{M^+M^-} \).

If \( 2 \mid r(d^-) \), but \( r(M^-) \) is odd, we must have \( M^- \mid d \). In view of (5.6) and (5.7), we conclude that \( \Psi_{M^+M^-}' \) must be even by Lemma 5.1, so is \( \Psi_{M^+M^-} \) again as \( \Delta_E < 0 \). Similar to the case \( 2 \mid r(M^-) \), same argument shows that

\[
\text{ord}_2\left( \prod_{q \mid M^+M^-} (a_q - 2\chi_d(q)) \cdot \langle d \rangle_{\chi_d}/c_f^{-1} \right) = r(M) - 1.
\]

Therefore, we have \( \text{ord}_2((M^+M^-) \chi_{M^+M^-}/ic_f^{-1}) = r(M) - 1 \). This completes the proof for both cases in the induction argument. Since \( \nu_E \) is odd, it follows that

\[
\text{ord}_2(L(E(M), 1)/c_{\infty}(E(M))) = \text{ord}_2((M^+M^-) \chi_{M^+M^-}/c_f^{\pm}) = r - 1.
\]

This completes the proof of the theorem combining the celebrated theorems of Gross–Zagier and Kolyvagin. \( \square \)
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