FACTORIAL MOMENTS
OF CONTINUOUS ORDER

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Abstract

The normalized factorial moments $F_q$ are continued to noninteger values of the order $q$, satisfying the condition that the statistical fluctuations remain filtered out. That is, for Poisson distribution $F_q = 1$ for all $q$. The continuation procedure is designed with phenomenology and data analysis in mind. Examples are given to show how $F_q$ can be obtained for positive and negative values of $q$. With $q$ being continuous, multifractal analysis is made possible for multiplicity distributions that arise from self-similar dynamics. A step-by-step procedure of the method is summarized in the conclusion.

1 Introduction

Continuation of the normalized factorial moments $F_q$ to arbitrary, noninteger values of $q$ is not just a mathematical problem. It has high phenomenological significance, and provides a powerful method to analyze experimental data that can reveal aspects about multiplicity fluctuations hitherto unexplored.

A historical review of the problem is in order. Bia/las and Peschanski [1] first introduced $F_q$ as a means of studying scaling behavior of multiplicity fluctuations as a function of the resolution scale $\delta$, a subject usually referred to as intermittency. Although a large part of the effect seen in the data has recently been found to be due to Bose-Einstein correlation among like-charge particles [2, 3], intermittency at a weaker level is still present for the unlike-charge particles and its origin remains to be clarified [4], especially for $q > 2$. There is far more dynamical information about
high-energy collisions than can be uncovered by studying two-particle correlation only. With that point of view forming the basis of our discussion here, we now outline the problems associated with the use of $F_q$. 

Let us first recall the definition of $F_q$:

$$F_q = \frac{\langle n (n - 1) \cdots (n - q + 1) \rangle}{\langle n \rangle^q}$$  \hspace{1cm} (1)$$

where $\langle \cdots \rangle$ denotes an average weighted by the multiplicity distribution $P_n$. The most outstanding property of $F_q$ discovered in Ref. \[1\] is that it filters out the statistical fluctuations, so any nontrivial behavior of $F_q$ is a direct indication of some features about the dynamics of particle production. A quick review of that will be given in the beginning of the next section. Another significant aspect about $F_q$ is that an event can contribute to (1) only if $n \geq q$. Thus for small $\delta$ where $\langle n \rangle$ is small in a bin, only rare events with high spikes ($n \geq q$) contribute. That is why it is sometimes said that intermittency measures spiky events. There are, however, disadvantages in the use of $F_q$. A corollary to the ability to select spiky events is its inability to extract any dynamical information about dips. It is by now generally recognized that rapidity gaps, like voids in galactic structure, are important to study. Those are, of course, large dips. In nuclear collisions where multiplicity per bin is large, unusual dips, which can be small but deep, are as significant as unusual spikes. For such fluctuations it is necessary to study $F_q$ for $q < 1$, especially negative $q$. Furthermore, for multifractal analysis of multiparticle production the continuation of $F_q$ to noninteger values of $q$ is necessary in order to allow differentiation with respective to $q$. These studies cannot be done, if $F_q$ is defined as in (1).

A method to investigate moments of arbitrary order was suggested several years ago in terms of the $G$ moments \[5\]. It was later modified to achieve better power-law behavior \[6\]. However, in overcoming the defects of $F_q$, the $G$ moments fail to retain the principal attribute of $F_q$, i.e. the screening of statistical fluctuations. Subtraction of the statistical component has to be done by hand \[4\]. Since that can be achieved only by simulation, the method is not elegant. But it has provided the first glimpses into the multifractal structure of particle production.

In this paper we describe a method that retains both attributes: it eliminates statistical fluctuations and is defined for continuous $q$. Although the mathematical technicalities involved may at first sight appear to be of theoretical interest only, the method is developed with phenomenology in mind. The purpose of the program is to extract quantitative information about dynamical fluctuations from the experimental data and to present it in a form suitable for comparison with theoretical predictions. Thus the problem of data analysis has not been overlooked in favor of mathematical expediency in the hope that the procedure can be readily accessible to direct experimental application.
2 The Problem

We first review the virtue of factorial moments. To say that $F_q$ filters out statistical fluctuation, one first assumes that the latter enters the multiplicity distribution as a convolution with the dynamical component

$$P_n = S \otimes D$$

(2)

where $S$ represents the statistical component, which we take to be the Poisson distribution $P^{(0)}_n$, and $D$ is the dynamical distribution. More specifically, (2) implies

$$P_n = \int_0^{\infty} dt \frac{t^n}{n!} e^{-t} D(t),$$

(3)

Let the numerator of (1) be denoted by $f_q$, i.e.

$$f_q = \sum_{n=q}^{\infty} \frac{n!}{(n-q)!} P_n.$$

(4)

which is well defined for $q$ being a positive integer. Substituting (3) in (4) and performing the summation yield

$$f_q = \int_0^{\infty} dt \ t^q D(t).$$

(5)

This is the $q$th moment of the dynamical $D(t)$ and is free of statistical contamination. Since $f_1 = \langle n \rangle$, we have

$$F_q = \frac{f_q}{f_1^q}.$$  \hspace{1cm} (6)

If $P_n$ is a Poisson distribution, then $D(t) = \delta(t - \langle n \rangle)$ and $f_q = \langle n \rangle^q$ so

$$F_q = 1, \quad \text{for integer} \quad q \geq 1.$$  \hspace{1cm} (7)

Because of this trivial result, one can state that any nontrivial $F_q$ reveals the nontrivial properties of $D(t)$.

To generalize $F_q$ to noninteger $q$, it must first be recognized that there is no unique continuation to complex $q$. Since $P_n$ must vanish as $n \to \infty$ (in fact, it must vanish for $n > N$ for some finite $N$ at any finite collision energy), there are only a finite number of the $F_q$ moments defined at integer $q$ values. Without an accumulation of $F_q$ at infinite $q$, unique continuation to noninteger $q$ is not possible. Put differently, we can add to $F_q$ any arbitrary function that vanishes at the finite range of integer $q$ where $F_q$ is specified and generate another function $\tilde{F}_q$ at noninteger $q$.

A simple way of continuing (4) to arbitrary $q$ is to replace the factorials by gamma functions, i.e.

$$f_q = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-q+1)} P_n.$$  \hspace{1cm} (8)
Procedures similar to (8), such as continuing $d^q G(z)/dz^q$ to fractional $q \mathbb{R}$, have been considered previously [3]-[11]. Since $\Gamma(z)$ has poles and oscillates rapidly among those poles when $z \leq 0$, $F_q$ as defined in (8) oscillates between large positive integers of $q$ and is highly suppressed at large negative $q$. The question is whether one wants that kind of behavior at noninteger values of $q$. If not, what are the guidelines by which one makes alternative choices of the continuation schemes?

In our view the only guideline is the primary rationale for considering factorial moments in the first place. And that is the elimination of statistical fluctuation at all $q$, not just at integer values of $q$. If one substitutes Poisson distribution into (8), one will find that $F_q \neq \langle n \rangle^q$. Fig. 1 (a) and (b) show the results for $F_q$ when

$$P_n = \mathcal{P}_n^{(0)} = \frac{\langle n \rangle^n e^{-\langle n \rangle}}{n!},$$

for $\langle n \rangle = 6$. Although (5) remains true for positive integers of $q$, $F_q$ is by no means equal to one for all $q$. The oscillations have larger amplitudes at high $q$, as revealed in Fig. 1(b). The situation is worse at smaller $\langle n \rangle$. In Fig. 2 we show the result for $\langle n \rangle = 1$ in (8), for which (9) and (8) are again used. Notice that there is no connected region of $q$ in which $F_q = 1$, not even between $q = 1$ and 2. Between $q = 9$ and 10 the peak of $|F_q|$ is greater than $10^4$. From these results it is therefore not possible to claim that (8) contains no statistical contribution at noninteger $q$. The reason for studying $F_q$ at noninteger $q$ is consequently lost. The $G_q$ moments [3, 9] would be better.

3 The Solution

To achieve our aim of retaining only the dynamical fluctuation in our continuation procedure, we demand that (3) defines $f(q)$ for all $q$. Hereafter, we use the notation that when $q$ appears as an argument of a function, instead of as a subscript, it is to be regarded as a continuous (complex) variable. A consequence of that condition is that the normalized factorial moment function satisfies

$$F(q) = 1$$

for all $q$ in the case of Poisson distribution. Equation (10) should be used as a test of the continuation procedure.

The burden of this procedure is to determine $D(t)$. If a theory specifies the dynamical distribution $D(t)$ completely, then (5) prescribes a unique continuation of $f_q$ to the complex function $f(q)$ at any $q$. But how is that to be checked by experiments where only $P_n$ is measured? Thus the procedure must supplement (5) with a way of determining $D(t)$ from $P_n$. This deconvolution process also cannot be made unique. The discrepancies show up as deviations of $F(q)$ from 1. The region where (10) fails significantly can fortunately be controlled and pushed to large $|q|$. 

To deconvolute (3) one could consider making the inverse Laplace transform of the generating function $G(z)$. However, there are difficulties connected with the fact that $G(z)$ determined from the experimental $P_n$ is a polynomial having no singularities in the finite $z$ plane.

Our proposal is to expand $P_n$ in terms of negative binomial distributions (NBD) $P_{n}^{NB}(j)$. One of the attributes of NBD is that it can also be expressed as a Poisson transform [12], as in (3). They do not form a complete set of orthogonal functions, so in general they cannot be the basis functions for the expansion of an arbitrary function. However, we do not have an arbitrary function. The experimental $P_n$ (ignoring errors for the moment) is a set of $N+1$ numbers for $n = 0, 1, \cdots, N$. Thus the expansion

$$P_n = \sum_{j=0}^{N} a_j P_{n}^{NB}(j)$$  \hspace{1cm} (11)

is well defined with $N+1$ coefficients $a_j$, provided we specify $P_{n}^{NB}(j)$ appropriately. One could consider other distributions instead of NBD, but for factorial moments that we shall eventually calculate NBD is most convenient.

Now $P_{n}^{NB}(j)$ is defined by [12]

$$P_{n}^{NB}(k_j, x_j) = \frac{\Gamma(n+k_j)}{\Gamma(n+1)\Gamma(k_j)} \left( \frac{k_j}{k_j+x_j} \right)^{k_j} \left( \frac{x_j}{k_j+x_j} \right)^n,$$ \hspace{1cm} (12)

where $x_j$ and $k_j$ specify the mean and inverse width of $P_{n}^{NB}(j)$. How they depend on $j$ will be discussed in the next section. We remark that $x_j$ is equal to $\bar{n}(j) = \sum_{n=0}^{\infty} n P_{n}^{NB}(j)$ only if the sum extends to $\infty$. Since our method is designed with phenomenological analysis in mind, where $P_n$ is given only for $n = 0, \cdots, N$, all sums over $n$ will be from 0 to $N$, whether the summand involves $P_n$ or $P_{n}^{NB}$. Consequently, $x_j$ is not exactly $\bar{n}(j)$. Although the discrepancy is small for the $x_j$ and $k_j$ to be chosen, accuracy will be important, as we shall see. Extending the sum in (11) to a larger upper limit $N'$ with $P_n = 0$ for $N+1 \leq n \leq N'$ would cause $a_j$ to be very large and highly sensitive to the accuracy of the calculation; it is a procedure that should be avoided. Hereafter $N$ will always be the maximum value of $n$ for which $P_n$ is measured to be nonzero, and $x_j$ and $k_j$ should only be regarded as real parameters of $P_{n}^{NB}(j)$ that will be varied in the expansion in (11). It should be recognized that because $P_{n}^{NB}(j)$ are all positive (unlike a harmonic function) and small on the wings, $a_j$ will have alternating signs and can have large absolute values if $P_n$ is small for a range of large $n$ values. Thus accuracy in the ensuing calculations will be essential. With $P_{n}^{NB}(j)$ specified, (11) is a set of $N+1$ simultaneous algebraic equations that can be solved for $a_j$ in terms of the experimental $P_n$. 

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Define $D^{NB}(t)$ by the negative-binomial versions of (3), i.e.,

$$P_n^{NB}(j) = \int_0^\infty dt \frac{t^n}{n!} e^{-t} D^{NB}(t, j). \quad (14)$$

Then it is known that [12]

$$D^{NB}(t, j) = \left( \frac{k_j}{x_j} \right)^{k_j} \frac{t^{k_j-1}}{\Gamma(k_j)} e^{-k_j t/x_j}. \quad (15)$$

The substitution of (3) and (14) to the two sides of (11) results in

$$D(t) = \sum_{j=0}^{N} a_j D^{NB}(t, j). \quad (16)$$

Thus we have inverted (3) and extracted the dynamical distribution $D(t)$ from the experimental data on $P_n$. In principle, this $D(t)$ can be compared directly with the theoretical distribution. However, the more familiar arena for comparison involves the factorial moments, which are more closely related to the data.

An interesting side remark that can be made here is that the feasibility of determining $D(t)$ from the data makes possible an experimental look at such theoretical quantity as the Landau free energy if the data on hadronic multiplicity distribution correspond to quark-hadron phase transition [13], or if the data are on photon distribution at the threshold of lasing in quantum optics [14]. That is because in such problems $D(t)$ in (3) is $e^{-F[t]}$, where $F[t]$ is the free energy of the system.

Returning to the problem on the factorial moments, we substitute (14) into our basic equation, (5), for continuation to complex $q$, and get

$$f(q) = \sum_{j=0}^{N} a_j f^{NB}(q, j). \quad (17)$$

where

$$f^{NB}(q, j) = \left( \frac{x_j}{k_j} \right)^q \frac{\Gamma(q + k_j)}{\Gamma(k_j)}. \quad (18)$$

This expression is obtained by integration over $t$ in (5) and is valid only for

$$\text{Re } q > -k_j. \quad (19)$$

Thus the domain of $q$ that can be continued into before encountering the first singularity is governed by the smallest value of $k_j$.

It should be noted that whereas $f_q$ as defined in (4) is obtained directly from the input $P_n$, $f(q)$ is determined from (17) after the inversion is done and the continuation procedure followed. Even at positive integer values of $q$, $f(q)$ is not exactly equal $f_q$.
because of finite accuracy in the computation. Following (8), we define the continued normalized factorial moments by

\[ F(q) = \frac{f(q)}{f(1)^q}, \] (20)

instead of \( f(q)/\langle n \rangle^q \). From (17) and (18) we then have

\[ F(q) = \sum_{j=0}^{N} a_j \left[ \frac{x_j}{f(1)} \right]^q F^{NB}(q, j) \] (21)

where

\[ F^{NB}(q, j) = \frac{\Gamma(q + k_j)}{\Gamma(k_j)k_j^q}. \] (22)

Equation (21) is our result, which is explicit once the values of \( a_j \) are determined.

4 The Range of \( x_j \) and \( k_j \)

To complete the description of our continuation procedure, it is necessary to specify \( x_j \) and \( k_j \), which represent the average and inverse width of \( P^{NB}_n(j) \). They should be chosen to be not too far from the values of \( x \) and \( k \) of the input \( P_n \), where

\[ x \equiv \langle n \rangle = \sum_{n=0}^{N} n P_n, \] (23)

\[ k \equiv (F_2 - 1)^{-1}, \quad F_2 = \langle n(n - 1) \rangle / x^2. \] (24)

We therefore define first

\[ \Delta_j = \Delta \left( -\frac{1}{2} + \frac{j}{N} \right), \] (25)

which ranges from \(-\Delta/2\) to \(+\Delta/2\) in equal steps, as \( j \) varies from 0 to \( N \). For \( x_j \) and \( k_j \) to vary from the lower to higher sides of \( x \) and \( k \), we set

\[ x_j = x \left( 1 + \Delta_j \right), \] (26)

\[ k_j = k \left( 1 + \Delta_j \right). \] (27)

Thus \( P^{NB}_n(j) \) with lower \( x_j \) has wider width (lower \( k_j \)); while that with higher \( x_j \) has narrower width. If \( \Delta \) is not too large, all components \( P^{NB}_n(j) \) will have comparable magnitudes at large \( n \), which should be small where \( P_n \) is small. Otherwise, \( a_j \) would be hypersensitive to the accuracy of the computation. With \( \Delta \) being free to
choose, the procedure is obviously not unique. As we have remarked earlier, no unique continuation should be expected from a finite set of numbers, $P_n$. However, there are guidelines for an optimal choice of $\Delta$.

We have mentioned in connection with (19) that the domain of continuation of $q$ is limited by the smallest value of $k_j$, which is $k(1 - \Delta/2)$. Thus to increase that domain we would want to have a small value for $\Delta$. However, with a small range of $x_j$ and $k_j$ it will be necessary to have highly accurate expansion coefficients $a_j$ in (11) to well represent $P_n$. So a larger value of $\Delta$ is preferred. This point needs to be demonstrated quantitatively. A way to do this is to calculate $f(1)$ and examine its dependence on $\Delta$. To that end we consider specific examples in the following.

Consider a sample $P_n$ given by

$$P_n^{(1)} = (n + 0.3)^3 e^{-0.5n}/Z, \quad n = 0, \ldots, N$$

where $N = 30$ and $Z$ is the normalization factor so that $\sum_{n=0}^{N} P_n^{(1)} = 1$. For this $P_n^{(1)}$ we have $x = \langle n \rangle = 7.6966$ and $k = 7.2193$. For $\Delta$ chosen to be in the range $0.1 \leq \Delta \leq 0.6$, we follow the procedure described in Sec. 3 and calculate $f(1)$. The result is shown in Fig. 3(a) as a function of $\Delta$. Evidently, there are fluctuations at small values of $\Delta$, but quite stable for $\Delta \geq 0.25$. However, when the vertical scale is expanded as shown in Fig. 3(b), we see that small fluctuations at a level of $< 0.3\%$ are still present until $\Delta \geq 0.4$, where $f(1) = 7.7016$. A discrepancy between $f(1)$ and $x$ is anticipated because of the finiteness of $N$ but at $0.06\%$ level it is unimportant. What is important is that $\Delta$ should not be too small. This example demonstrates the limitation of the method due to (19), if one attempts to continue $q$ to large negative values. From Fig. 3 an appropriate value for $\Delta$ can be set at $0.5$, for which $\min\{k_j\} = 0.75 k = 5.4145$. Thus $F(q)$ can be continued to $q \simeq -5.4$ before encountering divergence. In practice this range of negative $q$ is quite enough to exhibit the low $n$ behavior of $P_n$. Increasing $\Delta$ would improve the stability of the solution, but decrease the range of continued $q$. The choice of $\Delta = 0.5$ seems like a good compromise between the two opposite preferences.

Consider next another example where $\langle n \rangle$ is much smaller, corresponding to the situation where the phase-space cell size $\delta$ is small. Assume

$$P_n^{(2)} = (n + 1)^{0.5} e^{-n}/Z, \quad N = 20,$$

for which $x = 0.8352$ and $k = 1.395$. Fig. 4 shows the result of $f(1)$ vs $\Delta$, which is rather free of fluctuation for all $\Delta$ except near 0.1. The value of $f(1) = 0.8352$ is equal to $x$ to 5 significant figures. Choosing $\Delta = 0.5$ gives $\min\{k_j\} = 1.046$ which does not allow $q$ to go much beyond $-1$. Decreasing $\Delta$ would not improve the situation due to the limitation of small $k$ for $P_n^{(2)}$. Since $\Delta$ should not be changed in the analysis of data with varying $\delta$, we suggest that $\Delta$ be fixed at 0.5.
Continued Factorial Moments

The continuation procedure having been completely specified in Secs. 3 and 4, we can now proceed to the study of the normalized factorial moments $F(q)$. We continue to use the three distributions $P_n^{(i)}$, $i = 0, 1, 2$, as our sample inputs for $P_n$. Unless otherwise stated, $\Delta = 0$.5 is used.

For the Poisson distribution $P_n^{(0)}$, let us consider the same two cases: (a) $\langle n \rangle = 6$ and (b) $\langle n \rangle = 1$, already examined in Sec. 2, where the simple continuation scheme $n! \to \Gamma(n + 1)$ was used. Now, we assume $N = 30$ for (a) and $N = 10$ for (b) as the upper limits of $n$ in the experimental $P_n$. If $N = \infty$, $P_n^{(0)}$ would give $F_2 = 1$, so according to (24) $k$ would be infinite. For the finite $N$ chosen for the two cases, $F_2$ are still very close to 1, so $k$ would be extremely large. For our calculation it is sufficient to set $k = 10^4$. Using our procedure of continuation the results are shown in Fig. 5(a) and (b), for the two cases (a) and (b), respectively. Note the high resolution of the vertical scale. For $\langle n \rangle = 6$ in case (a) $F(q)$ is essentially 1 for $-10 < q < 30$. In case (b) where $\langle n \rangle = 1$, $F(q)$ is almost 1 for $-10 < q < 20$ except near the edges of that range. This case is more difficult to continue accurately because there are fewer values of nonvanishing $P_n^{(0)}$. Theoretically, $P_n^{(0)}$ is nonzero for any finite $n$, but we cut off at $N = 10$ to simulate a realistic situation where $\langle n \rangle$ is only 1. With only 11 values of $a_j$ the continuation to large $|q|$ cannot be expected to have extremely high accuracy. That is why $F(q)$ deviates from 1 near the two ends in Fig. 5(b). Nevertheless, the deviation is only of order 0.1%. Upon comparing Fig. 5 to Figs. 1 and 2, the advantage of this method over the simple scheme of Sec. 2 is self-evident.

For $P_n^{(1)}$ given in (28) with $N = 30$ the result of our calculation for $F(q)$ is shown in Fig. 6 (a) and (b). The dependence on $q$ is evidently very smooth. It grows rapidly at negative $q$, even though the first singularity is located at $q < -5$. We have also calculated $F(q)$ for $\Delta = 0.3$, the result of which is plotted in dashed lines, lying very close to the solid lines for $\Delta = 0.5$. For $q > 0$ the two cases are indistinguishable. For $q < -2$ the difference is actually not negligible in absolute value but because of the rapid rise of $F(q)$ it is not significant in terms of percentage discrepancy. In Fig. 6(b) we see that for the range of $q$ shown the difference between the two cases is totally insignificant. This is the most important range for the continuous $q$ problem, and we have found a reliable continuation of $F(q)$.

It should be pointed out that $F(0) = 1$ is not accidental. From (21) and (22) we see that $F(0) = \Sigma_j a_j$, which is 1 by virtue of the normalization of $P_n$ and $P_n^{NB}(j)$ in (11). Of course, $f_0$ is also 1 if the continuation scheme of (8) is followed.

Finally, let us come to the third example where $P_n^{(2)}$ is as given in (29) with $N = 20$. Now, the values of $x$ and $k$ are small. The calculated result for $F(q)$ is shown in Fig. 7 (a) and (b). There is a fast rise at large $q$ because of the smaller $x$ (compared to the case above). The continuation to negative $q$ encounters irregularity due to the small value of $k$. For $q > -0.5$, there is essentially no difference between the use of $\Delta = 0.5$ (solid line) and $\Delta = 0.3$ (dashed line). Only the solid line is plotted.
in Fig. 7(a); both are plotted in Fig. 7(b). The difference between the two \( \Delta \) cases becomes noticable and quantitatively significant only for \( q < -0.5 \), a region very close to the singularities. The reliability of our continuation should therefore be restricted to the domain to the right of the sudden downturn of \( F(q) \) around \( q = -0.5 \). In that domain our result is smooth and insensitive to \( \Delta \).

6 Multifractal Analysis

With \( F(q) \) continuable to noninteger \( q \), it is now possible to consider multifractal analysis, assuming that \( F(q) \) has a power-law dependence on the resolution scale \( \delta \) for a range of \( q \) covering both positive and negative values. Such an analysis was suggested previously using \( G \) moments, which are defined for all \( q \) by

\[
G(q) = \sum_i \left( \frac{n_i}{n_t} \right)^q
\]

where \( n_i \) is the multiplicity in bin \( i \), \( n_t = \sum_i n_i \), and the sum in \( i \) is over all nonempty bins. \( G(q) \) shows scaling behavior

\[
G(q) \sim \delta^{\tau(q)}
\]

for \( q > 1 \) in both experimental data and model simulation, when \( \theta(n_i - q) \) is included in the summand in \( \text{(30)} \). However, for \( q < 1 \) \( \text{(31)} \) is not valid for any extended range of \( \delta \), so multifractal analysis cannot be made for that range of \( q \). The problem is rooted in the empty-bin effect and the fact that \( G(q) \) contains statistical fluctuations. Since \( F(q) \) is now defined for noninteger \( q \), its scaling behavior

\[
F(q) \propto \delta^{-\varphi(q)}
\]

especially for negative \( q \), should be checked for a variety of existing data. If \( \text{(32)} \) is valid for a range of \( q \) around 1, multifractal analysis can then proceed without the necessity of subtracting out the statistical component, as was done for \( G(q) \).

We can relate \( G(q) \) and \( F(q) \), if we assume that \( \text{(30)} \) is defined for the dynamical distribution only without the statistical fluctuation. The sum over all bins in \( \text{(30)} \) can then be related to averaging over the dynamical distribution in \( \text{(5)} \). Using \( \text{(3)}, \text{(31)} \) and \( \text{(32)} \), we get

\[
\tau(q) = q - 1 - \varphi(q),
\]

where the \(-1\) comes from the fact that \( \sum_i \) by itself gives the total number of bins, which varies as \( \delta^{-1} \). Multifractal spectrum is then obtained by the Legendre transform

\[
f(\alpha) = q\alpha - \tau(q)
\]
with
\[ \alpha = d\tau(q)/dq \]  \hspace{1cm} (35)

Thus the verification of the power law (32) and the capability of calculating \( \alpha \) by differentiation with respect to \( q \), which we now have, make possible the presentation of the scaling properties of dynamical fluctuations in terms of the multifractal spectrum \( f(\alpha) \).

It is not our purpose here to examine specific dynamical models or experimental data and to extract their multiplicity fluctuation behaviors. However, we can demonstrate the nature of \( f(\alpha) \) if we assume that the scaling behavior (32) is true for a set of \( P_n(\delta) \) for a range of \( \delta \). Let us further assume that among those \( P_n(\delta) \), a specific one at some \( \delta_0 \) is exactly \( P_n^{(1)} \) given in (28). Then we have
\[ \varphi(q) = c \ln F(q) + c_1(q), \]  \hspace{1cm} (36)

where \( c = (-\ln \delta_0)^{-1} \) and \( c_1(q) \) is some function of \( q \) independent of \( \delta_0 \) arising from the proportionality factor in (22). Scaling behavior means that \( \varphi(q) \) is unchanged, as \( \delta \) is varied from \( \delta_0 \). The major part of the \( q \) dependence of \( \varphi(q) \) derives from that of \( F(q) \), which we know from Fig. 6. Apart from the unknown constant \( c \) and the unknown function \( c_1(q) \) in this example, we can determine \( f(\alpha) \) from \( F(q) \) by varying \( q \) parametrically. For illustrative purpose, let us assume that \( c_1(q) = 0 \) so that using (23)-(36) and the result of our calculated \( F(q) \) pertaining to \( P_n^{(1)} \), we can determine \( f(\alpha) \). In Fig. 8 we show the result for four possible values of \( c \). We stress that in a model or data analysis \( c \) is not a variable; \( \varphi(q) \) is determined from the log-log plots when there is scaling. Fig. 8 merely illustrates the possible form of \( f(\alpha) \), if \( \varphi(q) \) happens to coincide with the result of analyzing a particular \( P_n = P_n^{(1)} \) with a specific \( c \) in (36) and with \( c_1(q) = 0 \). The dashed line indicates where \( f(\alpha) = \alpha \), and is tangent to each of the \( f(\alpha) \) curves at \( q = 1 \). The range of \( q \) covered by \( f(\alpha) \) in Fig. 8 is, depending on \( c \), roughly \(-2 \leq q \leq 4 \), with the \( q = 0 \) point always occurring at the peak of the \( f(\alpha) \) curve. The multifractal dimension \( D_q \) is
\[ D_q = \tau(q)/(q - 1), \]  \hspace{1cm} (37)

which is related to \( \alpha \) by
\[ D_0 = f(\alpha_0), \quad D_1 = \alpha_1, \]  \hspace{1cm} (38)

where \( \alpha_q \) is the value of \( \alpha \) at \( q \). Thus \( \alpha_0 \) is where \( f(\alpha) \) is maximum, and \( \alpha_1 \) is where \( f(\alpha_1) = \alpha_1 \). The multifractal spectrum \( f(\alpha) \) is the most elegant way of displaying the scaling properties of dynamical fluctuation. Reduced to the bare minimum, two parameters can be used to characterize \( f(\alpha) \), viz. \( \alpha_0 \) and \( \alpha_1 \), the location of the peak and a measure of the width, respectively.

In many multiparticle production processes the scaling behavior (32) is not valid over an extended range of \( \delta \). The multifractal analysis described above cannot be...
applied then. However, it has been found phenomenologically \[17, 18, 19\] as well as theoretically \[13, 20\], not only in hadronic and nuclear collisions, but also in quantum optics \[14\], that $F_q$ satisfies a different scaling law

$$F_q \propto F_q^{\beta_q}.$$  \hspace{1cm} (39)

Let us assume that this behavior can be established for continuous $q$ so that the function $\beta(q)$ can be determined for a range of $q$ values both positive and negative. Then it is possible to define formally another spectrum, call it $g(\alpha)$, in exact analogy to (33)-(35), but without the geometrical implication of multifractality. Thus we define

$$\sigma(q) = q - 1 - \beta(q)$$  \hspace{1cm} (40)

$$g(\alpha) = q\alpha - \sigma(q)$$  \hspace{1cm} (41)

$$\alpha = d\sigma(q)/dq$$  \hspace{1cm} (42)

The only input into this scheme of description is $\beta(q)$, which is $\beta(q) = \ln F(q)/\ln F(2) + b(q)$, \hspace{1cm} (43)

where $b(q)$ is the log of the proportionality factor in (39). Mathematically, $\sigma(q)$ and $g(\alpha)$ correspond to $\tau(q)$ and $f(\alpha)$ if we set $c = 1/\ln F(2)$, but physically (32) need not be true, rendering $\varphi(q)$ meaningless, while (33) can well be true (no exception having been found so far). Ref. \[13\] gives an explicit example of (39) being valid for a problem that does not have (32).

In the example where $P_n^{(1)}$ is considered, let us assume that it belongs to the type of physical problems for which (39) is valid. Then the function $F(q)$ obtained is sufficient to determine $g(\alpha)$, assuming $b(q) = 0$. The result is shown in Fig. 9(a). Since $F(2) = 1.14$, $g(\alpha)$ corresponds to $f(\alpha)$ in Fig. 8 with $c = 7.64$. In Fig. 9(a) the peak occurs at $\alpha_0 = 1.45$ while the tangent point is at $\alpha_1 = 0.52$. If, on the other hand, $P_n^{(2)}$ is used as an example that has a scaling behavior (39), totally unrelated to $P_n^{(1)}$, then the corresponding spectrum $g(\alpha)$ is as shown in Fig. 9(b). Note that in this case the maximum $\alpha$ is 2.15 corresponding to $q = -0.24$. Our continuation to lower value of $q$ has led to a sudden downturn of $F(q)$ around $q = -0.4$, exhibited in Fig. 7(b). That causes a drastic change in the derivative in (12) at around $q = -0.24$ which in turn gives rise to an irregular behavior in $g(\alpha)$ at $\alpha = 2.15$. Thus the nature of $P_n^{(2)}$ prevents the use of $F(q)$ for $q < -0.24$, thereby setting an upper bound to how far to the right the spectrum $g(\alpha)$ can be developed. In that figure $\alpha_0 = 1.76$ and $\alpha_1 = 0.41$, quite different from the corresponding values in Fig. 9(a). These two figures are sufficient to indicate that $P_n^{(1)}$ and $P_n^{(2)}$ cannot belong to the same class of $F$-scaling factorial moments.

In summary, $g(\alpha)$ is a representation of the characteristics of the $F$-scaling behavior, (39), of $F(q)$, and may be a generally useful description of all multiparticle production processes.
7 Conclusion

We have presented a way to determine $F(q)$ for continuous $q$ such that it is 1 for all $q$ if the input distribution is Poissonian, i.e. the statistical fluctuation is filtered out. The range of $q$ into which $F(q)$ can be continued depends on the nature of $P_n$. Generally speaking, the low $n$ part of $P_n$ is characterized by the negative $q$ region of $F(q)$. Thus the study of the scaling behavior of multiplicity fluctuations can now be extended to dips, gaps and voids. All existing data that have been put to intermittency analysis at positive integer $q$ should be reanalyzed for continuous $q$. Similarly, such reanalysis should be done for all models and MC codes. Thus the confrontation between theory and experiment can now be extended to a significant portion of the real line of $q$, as compared to the previous situation where it has been done for only a few isolated points at the integer values. For comparison, the study of Bose-Einstein correlation is focused on only one point: $q = 2$.

If the dynamics of particle production is self-similar so that $F(q)$ exhibits a power-law dependence on the resolution scale $\delta$, then the intermittency index $\varphi(q)$ can be determined as a continuous function of $q$. It then follows that the multifractal spectrum $f(\alpha)$ can be derived without any ambiguity or need for correction to eliminate statistical contamination. On the other hand, if there is no power-law dependence on $\delta$, but there is $F$-scaling, i.e., $F_q \propto F_2^{\beta_q}$, which is more commonly observed, then the knowledge of $\beta(q)$ is sufficient to determine the spectrum $g(\alpha)$ that gives an excellent description of the self-similar behavior of the dynamics of fluctuations.

For the purpose of providing a convenient outline of the procedure to determine $F(q)$, $f(\alpha)$ and $g(\alpha)$, we summarize here the steps needed to do the analysis:

1. Starting with the input $P_n$, $n = 0, \cdots, N$, determine $x$ and $k$ from (23) and (24), and then $x_j$ and $k_j$ from (25)-(27) with $\Delta = 0.5$.

2. Using (12), set up $N + 1$ linear algebraic equations, (11), with $P_n^{NB}(j)$ as the matrix and $P_n$ as the input vector. Solve for $a_j$, $j = 0, \cdots, N$. [The use of MATHEMATICA has been found to be convenient.]

3. Use (21) to calculate $F(q)$ for a range of real $q$.

4. If the input $P_n(\delta)$ is known for a range of $\delta$, determine $F(q)$ as a function of $\delta$ and examine the validities of (32) and (39) over that range of $\delta$.

5. If (32) is valid for a subrange of $\delta$, then from $\varphi(q)$ and (33)-(35) determine $f(\alpha)$.

6. If (32) is invalid but (39) is valid, then from $\beta(q)$ and (40)-(42) determine $g(\alpha)$.

7. Compare theory and experiment at any of the three levels: $F(q)$, $f(\alpha)$ or $g(\alpha)$.
ACKNOWLEDGMENTS

I am grateful to I. M. Dremin and J.-L. Meunier for helpful discussions. This work was supported, in part, by the U. S. Department of Energy under Grant No. DE-FG06-91ER40637.

References

[1] A. Bialas and R. Peschanski, Nucl. Phys. B273, 703 (1986); 308, 867 (1988).

[2] *Soft Physics and Fluctuations*, edited by A. Bia{l}as, K. Fia{l}kowski, K. Zalewski and R. C. Hwa, (World Scientific, Singapore, 1994).

[3] N. Agababyan et al. (NA22), Z. Phys. C59, 405 (1993); N. Neumeister, et al. (UA1) Phys. Lett. B275, 186 (1992). P. Abreu et al. (DELPHI), Z. Phys. C63, 17 (1994).

[4] N. Agababyan et al. (NA22), Phys. Lett. B332, 458 (1994); W. Kittel, Proceedings of the 23rd International Symposium on Multiparticle Dynamics, Aspen, 1993, edited by M. M. Block and A. R. White (World Scientific, Singapore, 1994), p. 251.

[5] R. C. Hwa, Phys. Rev. D41, 1456 (1990); C.B. Chiu and R.C. Hwa, *ibid.* 43, 100 (1991); R. C. Hwa, in *Quark-Gluon Plasma*, edited by R. C. Hwa (World Scientific, Singapore, 1990).

[6] R. C. Hwa and J. Pan, Phys. Rev. D45, 1476 (1992); I. Derado, R. C. Hwa, G. Jancso and N. Schmitz, Phys. Lett. B283, 151 (1992).

[7] C. B. Chiu, K. Fialkowski and R. C. Hwa, Mod. Phys. Lett. A5, 2651 (1990).

[8] K. B. Oldham and J. Spanier, *The Fractional Calculus* (Academic Press, New York, 1974).

[9] M. Blazek, Phys. Lett. B247, 576 (1990).

[10] P. Duclos and J.-L. Meunier, Nice preprint INLN# 94.07.

[11] I. M. Dremin, Pisma v. ZhETF59 561 (1994); Uspekhi Fiz. Nauk 164, 785 (1994); JETP Lett. 59, 585 (1994).

[12] P. Carruthers and C. C. Shih, Int. J. Mod. Phys. A2, 1447 (1987).

[13] R. C. Hwa and M. T. Nazirov, Phys. Rev. Lett. 69, 741 (1992).

[14] M. R. Young, Y. Qu, S. Singh and R. C. Hwa, Optics Comm. 105, 325 (1994).
[15] J. Feder, *Fractals* (Plenum Press, New York, 1988).

[16] H. G. E. Hentschel and I. Procaccia, Physica **8D**, 435 (1983).

[17] W. Ochs, Phys. Lett. B**247**, 101 (1990); Z. Phys. C **50**, 339 (1991).

[18] M. Charlet, Ph. D. Thesis, University of Nijmegen (1994); P. Côté, Univ. of Ottawa preprint (1994).

[19] J. Pan and R. C. Hwa, Phys. Rev. D**48**, 168 (1993);

[20] R. C. Hwa, Phys. Rev. D**47**, 2773 (1993); Phys. Rev. C**50**, 383 (1994).
Figure Captions

Fig. 1 Normalized factorial moments $F_q$ in the simple continuation procedure using (8) with Poisson distribution (9) as input and with $\langle n \rangle = 6$. (a) $-10 \leq q \leq 25$; (b) $25 \leq q \leq 30$.

Fig. 2 Same as Fig. 1 but for $\langle n \rangle = 1$.

Fig. 3 Dependence of factorial moment $f(q = 1)$ on $\Delta$ when the input distribution is $P_n^{(1)}$. The resolution in (b) is higher than that in (a).

Fig. 4 Dependence of $f(q = 1)$ on $\Delta$ when the input distribution is $P_n^{(2)}$.

Fig. 5 Normalized factorial moments $F(q)$ with high vertical resolution for Poisson distribution with (a) $\langle n \rangle = 6$, and (b) $\langle n \rangle = 1$.

Fig. 6 $F(q)$ with $P_n^{(1)}$ as input distribution. Solid line is for $\Delta = 0.5$ dashed line for $\Delta = 0.3$. (a) $-3 \leq q \leq 10$; (b) $-1.5 \leq q \leq 2.5$.

Fig. 7 $F(q)$ with $P_n^{(2)}$ as input distribution. Solid line is for $\Delta = 0.5$ dashed line for $\Delta = 0.3$. (a) $-1 \leq q \leq 6$; (b) $-1 \leq q \leq 2$.

Fig. 8 Multifractal spectra $f(\alpha)$ for four values of $c$ (see text), if the input $P_n^{(1)}$ belongs to a class of scaling distributions. Dashed line is for $f(\alpha) = \alpha$.

Fig. 9 Spectrum function $g(\alpha)$ if (a) $P_n^{(1)}$ and (b) $P_n^{(2)}$ belong to separate classes of self-similar distributions satisfying $F$-scaling (3.9).
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