Adding boundary terms to Anderson localized Hamiltonians leads to unbounded growth of entanglement

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Abstract

It is well known that in Anderson localized systems, starting from a random product state the entanglement entropy remains bounded at all times. However, we show that adding a single boundary term to an Anderson localized Hamiltonian leads to unbounded growth of entanglement. Our results imply that Anderson localization is not a local property. One cannot conclude that a subsystem has Anderson localized behavior without looking at the whole system, as a term that is arbitrarily far from the subsystem can affect the dynamics of the subsystem in such a way that the features of Anderson localization are lost.

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1 Introduction

In the presence of quenched disorder, the phenomenon of localization can occur not only in single-particle systems, but also in interacting many-body systems. The former is known as Anderson localization (AL) [1], and the latter is called many-body localization (MBL) [2–7]. In the past decade, significant progress has been made towards understanding AL and especially MBL.

A characteristic feature that distinguishes MBL from AL lies in the dynamics of entanglement. Initialized in a random product state, the entanglement entropy remains bounded at all times in AL systems [8], but grows logarithmically with time in MBL systems [9–12]. The logarithmic growth of entanglement can be understood heuristically [13–15] from the strong-disorder renormalization group [16–21] or a phenomenological model of MBL [22–23].

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Recently, it was rigorously proved that in MBL systems, the entanglement entropy obeys a volume law at long times \[24\].

Consider the random-field $XXZ$ chain with open boundary conditions

$$H_{XXZ} = \sum_{j=1}^{N-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \sum_{j=1}^{N} h_j \sigma_j^z,$$

where $\sigma_j^x, \sigma_j^y, \sigma_j^z$ are the Pauli matrices at site $j$, and $h_j$'s are independent and identically distributed uniform random variables on the interval $[-h, h]$. For $\Delta = 0$, this model reduces to the random-field $XX$ chain

$$H_{XX} = \sum_{j=1}^{N-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \sum_{j=1}^{N} h_j \sigma_j^z.$$

Using the Jordan–Wigner transformation, $H_{XX}$ is equivalent to a model of free fermions hopping in a random potential. It is AL for any $h > 0$. The $\Delta$ term in Eq. (1) introduces interactions between fermions. $H_{XXZ}$ is MBL for any $\Delta \neq 0$ and sufficiently large $h$ \[25–27\].

In $H_{XXZ}$, the $\Delta$ term representing interactions between fermions is extensive in that it is the sum of $N - 1$ local terms between adjacent qubits. Let

$$H_{XXb} = H_{XX} + \Delta \sigma_{N-1}^z \sigma_N^z = \sum_{j=1}^{N-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \sum_{j=1}^{N} h_j \sigma_j^z + \Delta \sigma_{N-1}^z \sigma_N^z.$$

Without the last term, $H_{XXb}$ is AL. In this paper, we show that in the dynamics generated by $H_{XXb}$, the effect of this boundary term invades into the bulk: Starting from a random product state the entanglement entropy obeys a volume law at long times. For large $h$, the coefficient of the volume law is almost the same as that in the dynamics generated by $H_{XXZ}$.

Our results imply that AL is not a local property. One cannot conclude that a subsystem has AL behavior without looking at the whole system, as a term that is arbitrarily far from the subsystem can affect the dynamics of the subsystem in such a way that the features of AL are lost.

We briefly discuss related works. Khemani et al. \[28\] showed nonlocal response to local manipulations in localized systems. Lezama and Bar Lev \[29\] studied the dynamics of an AL system with local noise. These works consider time-dependent Hamiltonians, and are thus different from ours. Vasseur et al. \[30\] studied the revival of a qubit coupled to one end of an AL system, but the coupling is chosen such that the whole system (including the additional qubit) is a model of free fermions. This is in contrast to $H_{XXb}$.

### 2 Results

**Definition 1** (entanglement entropy). The entanglement entropy of a bipartite pure state $\rho_{AB}$ is defined as the von Neumann entropy

$$S(\rho_A) := -\text{tr}(\rho_A \ln \rho_A)$$

of the reduced density matrix $\rho_A = \text{tr}_B \rho_{AB}$.
Figure 1: Dynamics of the half-chain entanglement entropy for \( H_{XXb} \) (blue circle, random-field \( XX \) chain with an additional boundary term), \( H_{XXZ} \) (red plus, random-field \( XXZ \) chain), and \( H_{XX} \) (green x-mark, random-field \( XX \) chain).

We initialize the system in a Haar-random product state \([31–34\).]

Definition 2 (Haar-random product state). In a system of \( N \) qubits, let \( |\Psi\rangle = \bigotimes_{j=1}^{N} |\Psi_j\rangle \) be a Haar-random product state, where each \( |\Psi_j\rangle \) is chosen independently and uniformly at random with respect to the Haar measure.

For all numerical results in the main text, we choose \( h = 16 \) and \( \Delta = 0.1 \), and average over 1000 samples (a sample consists of a random Hamiltonian and a random initial state). We choose \( N = 10 \) in Figure 1 and in the left panel of Figure 2. We use the Multiprecision Computing Toolbox for MATLAB (https://www.advanpix.com).

Figure 1 shows the dynamics of the entanglement entropy between the left and right halves of the system for \( H_{XXb} \) and \( H_{XXZ} \). We clearly see that the last term in Eq. (3) leads to slow entanglement growth.

Figure 2 shows that the entanglement entropy at long times obeys a volume law for \( H_{XXb} \) and \( H_{XXZ} \), and the coefficient of the volume law is very close to \( 1/2 \). This is consistent with the analytical prediction of Ref. [24]. Specifically, Theorem 3 in Ref. [24] states that the coefficient is upper bounded by \( 1/2 \) in the limit \( h \to +\infty \). In our numerical study, \( h = 16 \) is finite but very large (so that the models are deep in the localized regime). Therefore, we expect \( 1/2 \) to be an approximate upper bound. On the other hand, Theorem 1 in Ref. [24] states that the coefficient is lower bounded by \( 1/2 \) if the spectrum of the Hamiltonian has non-degenerate gaps.

Definition 3 (non-degenerate gap). The spectrum \( \{E_j\} \) of a Hamiltonian has non-degenerate
Figure 2: Left panel: The entanglement entropy between the first \( j \) and the last \( N - j \) qubits at long times for \( H_{XXb} \) (blue circle, random-field \( XX \) chain with an additional boundary term), \( H_{XXZ} \) (red plus, random-field \( XXZ \) chain), and \( H_{XX} \) (green x-mark, random-field \( XX \) chain). The black lines are \( S = \min\{j, N - j\}/2 \). Right panel: Finite-size scaling of the half-chain entanglement entropy at long times for \( H_{XXb} \) (blue), \( H_{XXZ} \) (red), and \( H_{XX} \) (green). The black line is \( S = 0.4709(N/2) - 0.4087 \). Both panels show that the entanglement entropy at long times obeys a volume law for \( H_{XXb} \) and \( H_{XXZ} \).

Gaps if the differences \( \{E_j - E_k\}_{j \neq k} \) are all distinct, i.e., for any \( j \neq k \),

\[
E_j - E_k = E_{j'} - E_{k'} \implies (j = j') \text{ and } (k = k').
\] (5)

Indeed, we have numerically verified (up to \( N = 12 \)) that the spectra of both \( H_{XXb} \) and \( H_{XXZ} \) almost surely have non-degenerate gaps.

In the right panel of Figure 2, we observe a constant correction to the volume law. This is expected, for such corrections also exist in many other contexts [35–45].

3 Summary and outlook

We have numerically shown that adding a single boundary term to an AL Hamiltonian leads to entanglement growth. Starting from a random product state the entanglement entropy obeys a volume law at long times, and the coefficient of the volume law is consistent with the analytical prediction of Ref. [24]. Our results imply that AL is not a local property. One cannot conclude that a subsystem has AL behavior without looking at the whole system, as a term that is arbitrarily far from the subsystem can affect the dynamics of the subsystem in such a way that the features of AL are lost.

Here are some interesting problems that deserve further study.

- Can we prove that the spectrum of \( H_{XXb} \) almost surely has non-degenerate gaps? A positive answer to this question would allow us to rigorously prove the title of this paper using Theorem 1 in Ref. [24].
• Can we develop an analytical understanding of how the entanglement entropy grows with time for $H_{XXb}$ by adapting the phenomenological model of MBL [22, 23]?

• How does $H_{XXb}$ scramble local information as measured by the out-of-time-ordered correlator [46–52]?

• It was argued that MBL is less stable in two and higher spatial dimensions [53, 54]. To what extent does an additional boundary term delocalize an AL system in higher dimensions?

Notes
The MIT-CTP preprint number of this paper was assigned on 9 Sep 2021 at 14:16 ET. Less than 10 minutes before submitting the arXiv version 1 of this paper, I became aware of a related work [55]. It studies the dynamics of a model that is different from but arguably conceptually similar to $H_{XXb}$ (3).

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Data availability statement
All data that support the findings of this study are included within the article.

A Additional numerical results
All numerical results in the main text are for $h = 16$ and $\Delta = 0.1$. Figure A.1 shows the numerical results for a different set of $(h, \Delta)$.

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Figure A.1: Numerical results for $h = 10$ and $\Delta = 1$. The blue circles and red pluses have the same meaning as in Figure 2. The black lines in the left and right panels are $S = \min\{j, N - j\}/2$ and $S = 0.4854(N/2) - 0.3886$, respectively.

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