AREA OF THE COMPLEMENT OF THE FAST ESCAPING SETS OF A FAMILY OF ENTIRE FUNCTIONS

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Abstract. Let $f$ be an entire function with the form $f(z) = P(e^z)/e^z$, where $P$ is a polynomial with $\deg(P) \geq 2$ and $P(0) \neq 0$. We prove that the area of the complement of the fast escaping set (hence the Fatou set) of $f$ in a horizontal strip of width $2\pi$ is finite. In particular, the corresponding result can be applied to the sine family $\alpha \sin(z + \beta)$, where $\alpha \neq 0$ and $\beta \in \mathbb{C}$.

1. Introduction

Let $f: \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. Denote by $f^n$ the $n$-th iterate of $f$. The Fatou set $F(f)$ of $f$ is defined as the maximal open set in which the family of iterates \( \{f^n : n \in \mathbb{N}\} \) is normal in the sense of Montel. The complement of $F(f)$ is called the Julia set $J(f)$, which is denoted by $J(f)$. It is well known that $J(f)$ is a perfect completely invariant set which is either nowhere dense or coincides with $\mathbb{C}$. For more details about these sets, one can refer [Bea91], [CG93] and [Mil06] for rational maps, and [Ber93] and [EL92] for meromorphic functions.

Already in 1920s, Fatou considered the iteration of transcendental entire functions [Fat26] and one of his study object was $f(z) = \alpha \sin(z) + \beta$, where $0 < \alpha < 1$ and $\beta \in \mathbb{R}$. After Misiurewicz showed that the Fatou set of $f(z) = e^z$ is empty in 1981 [Mis81], the dynamics of exponential maps and trigonometric functions attracted many interests from then on. See [DK84], [DT86] and [DG87] for example. In particular, in 1987 McMullen [McM87] proved a remarkable result which states that the Julia set of $\sin(\alpha z + \beta)$, $\alpha \neq 0$ always has positive Lebesgue area and the Hausdorff dimension of the Julia set of $\lambda e^z$, $\lambda \neq 0$ is always 2. From then on a series of papers considered the area and the Hausdorff dimension of the dynamical objects of the transcendental entire functions, not only for the Julia sets in dynamical planes (see [Sta91], [Kar99a], [Kar99b], [Tan03], [Sch07], [Bar08], [RS10], [AB12], [Rem13], [Six15a] and the references therein for example), but also the bifurcation loci in the parameter spaces (see [Qiu94] and [ZL12]).

Unlike the polynomials, the Julia set of a transcendental entire function $f$ is always unbounded. Since the Fatou set of $f$ is dense in the complex plane (if $F(f) \neq \emptyset$), it is interesting to ask when the Fatou set of $f$ has finite area. For the sine function $f(z) = \sin z$, Milnor conjectured that the area of the Fatou set of $f$ is finite in a vertical strip of width $2\pi$. By applying the tools in [McM87], Schubert proved this conjecture in 2008 [Sch08].

For a transcendental entire function $f$, the escaping set $I(f)$ was studied firstly by Eremenko in [Ere89]. A subset of the escaping set, called the fast escaping set $A(f)$, was introduced by Bergweiler and Hinkkanen in [BH99]. These sets have received quite a lot of attention recently. Especially for the fast escaping set, see [Six11], [RS12], [Six13], [Six15b], [Evd16] and the references therein. In this paper,
we consider the area of the complement of the fast escaping sets of a family of entire functions and try to extend the result of Schubert to this class. Our main result is the following.

**Theorem 1.1.** Let $P$ be a polynomial with $\deg(P) \geq 2$ and $P(0) \neq 0$. Then the area of the complement of the fast escaping set of any function with the form $f(z) = P(e^z)/e^z$ is finite in any horizontal strip of width $2\pi$.

The method in this paper is strongly inspired by the work of McMullen and Schubert ([McM87] and [Sch08]). It is worth to mention that we give also a specific method can be adopted also to the type of entire functions with the form $f(z) = P(e^z)/e^z$. Since the fast escaping set of $f(z) = P(e^z)/e^z$ is the complement of the fast escaping set of $f$. In fact, we believe that our method can be adopted also to the type of entire functions with the form

$$f(z) = \frac{P(w)}{w^m} \circ \exp(z)$$

completely similarly, where $m \geq 1$ is a positive integer, $P$ is a polynomial with degree $\deg(P) \geq m + 1$ and $P(0) \neq 0$.

As a consequence of Theorem 1.1 and Theorem 3.1, we have the following result on the area of the complement of the fast escaping set of the sine family.

**Theorem 1.2.** Let $S$ be any vertical strip of width $2\pi$. Then the area of the complement of the fast escaping set of $f(z) = a \sin(z + \beta)$ with $\alpha \neq 0$ satisfies

$$\text{Area}(S \cap A(f)^c) \leq (4\pi + 4r) \left( x^* + r + 8c e^{1-x^*/2} \frac{r}{1 - e^{-r/2}} \right),$$

where

$$r = \frac{1}{8}, \quad c = \frac{536\sqrt{2}}{|\alpha|} + \frac{1}{|\alpha|^2}$$

and

$$x^* = \max \left\{ \log \left( 1 + \frac{18K}{|\alpha|} \right), \log \left( \frac{8(K + 1)}{|\alpha|} \right), 6 \log 2, 12 + 2 \log e \right\}$$

with $K = \max\{|\alpha|/2, |\beta|\}$. In particular, if $f(z) = \sin z$ or $\cos z$, then

$$\text{Area}(S \cap A(f)^c) < 361.$$
2. Distortion lemmas and some basic settings

2.1. Distortion quantities. As in [McM87] and [Sch08], we introduce some quantities of distortion in this subsection. Let $D$ be a bounded set in the complex plane $\mathbb{C}$ and let $f$ be a holomorphic function defined in a neighbourhood of $D$. We say that $f$ has bounded distortion on $D$ if there are positive constants $c$ and $C$, such that for all distinct $x$ and $y$ in $D$, one has

$$c < \frac{|f(x) - f(y)|}{|x - y|} < C.$$  

The quantity

$$L(f|_D) := \inf \{ C/c : c \text{ and } C \text{ satisfy } (1) \}$$

is the distortion of $f$ on $D$. By (1), we have

$$\sup_{z \in D} |f'(z)| \leq C \quad \text{and} \quad \inf_{z \in D} |f'(z)| \geq c.$$

Therefore, $L(f|_D)$ has a lower bound satisfying

$$L(f|_D) \geq \sup_{z \in D} \frac{|f'(z)|}{\inf_{z \in D} |f'(z)|}.$$  

The equality holds in this inequality if $D$ is a convex domain.

Let $\text{Area}(E)$ be the Lebesgue area of the measurable set $E \subset \mathbb{C}$. If $X$ and $D$ are two measurable subsets of the complex plane with $\text{Area}(D) > 0$, we use

$$\text{density}(X, D) := \frac{\text{Area}(X \cap D)}{\text{Area}(D)}$$

to denote the density of $X$ in $D$. If $c$ and $C$ satisfy (1), then $c^2 \text{Area}(X) \leq \text{Area}(f(X)) \leq C^2 \text{Area}(X)$. This means that

$$\text{density}(f(X), f(D)) \leq L(f|_D)^2 \text{density}(X, D).$$

The nonlinearity of $f$ on $D$ is defined as

$$N(f|_D) := \sup \left\{ \frac{|f''(z)|}{|f'(z)|} : z \in D \right\} \cdot \text{diam}(D),$$

provided the right-hand side is finite. In the following by square we mean a closed square whose sides are parallel to the coordinate axes. We will use the following relation between the distortion and nonlinearity on squares.
Lemma 2.1. Let $Q$ be a compact and convex domain in $\mathbb{C}$ (in particular if $Q$ is a square) and let $f$ be a conformal map defined in a neighbourhood of $Q$ with $N(f|_Q) < 1$. Then

$$L(f|_Q) \leq 1 + 2N(f|_Q).$$

Proof. Since $f$ is conformal, let $z_0$ be a point in $Q$ such that

$$|f'(z_0)| = \sup_{z \in Q} |f'(z)| > 0.$$ 

Since $Q$ is convex, for any $z \in Q$ we have

$$\frac{|f'(z) - f'(z_0)|}{|f'(z_0)|} = \left| \int_{z_0}^{z} \frac{f''(\zeta) d\zeta}{f'(z_0)} \right| \leq \frac{\sup_{z \in Q} |f''(z)|}{|f'(z_0)|} \cdot |z - z_0|$$

$$\leq \sup_{z \in Q} \left\{ \left| \frac{f''(z)}{f'(z)} \right| \right\} \cdot \text{diam}(Q) = N(f|_Q) < 1.$$ 

Therefore, the image of $Q$ under $f'(z)$ is contained in the disk $D(f'(z_0), |f'(z_0)|)$ and hence $\log f'(z)$ is well-defined on $Q$.

Since $Q$ is compact, let $z_1 \in Q$ such that

$$|f'(z_1)| = \inf_{z \in Q} |f'(z)| > 0.$$ 

Since $Q$ is convex and $\log f'(z)$ is well-defined, we have

$$\log L(f|_Q) = \log \left| \frac{f'(z_0)}{f'(z_1)} \right| \leq |\log f'(z_1) - \log f'(z_0)|$$

$$= \left| \int_{z_0}^{z_1} (\log f'(z))' dz \right| = \left| \int_{z_0}^{z_1} \frac{f''(z)}{f'(z)} dz \right|$$

$$\leq \sup_{z \in Q} \left\{ \left| \frac{f''(z)}{f'(z)} \right| \right\} \cdot \text{diam}(Q) = N(f|_Q).$$ 

Since $e^x \leq 1 + 2x$ for $x \in [0, 1)$, we have

$$L(f|_Q) \leq \exp(N(f|_Q)) \leq 1 + 2N(f|_Q). \quad \square$$

Remark. McMullen notes in [McM87] that $L(f|_Q)$ is bounded above by $1 + O(N(f|_Q))$ if $N(f|_Q)$ is small. After that Schubert states in [Sch08] that $L(f|_Q) \leq 1 + 8N(f|_Q)$ if $N(f|_Q) < 1/4$ but without a proof.

Let $n$ be a positive integer. For each $1 \leq i \leq n$, let $D_i \subset \mathbb{C}$ be an open set and $f_i : D_i \to \mathbb{C}$ a conformal map. Let $\sigma$ and $M$ be two positive constants satisfying

$$|f_i'(z)| > \sigma > 1 \quad \text{and} \quad \left| \frac{f_i''(z)}{f_i'(z)} \right| < M,$$ 

where $z \in D_i$ and $1 \leq i \leq n$.

Furthermore, let $Q_i \subset D_i$, $1 \leq i \leq n$ be squares with sides of length $r > 0$ satisfying $Q_{i+1} \subset f_i(Q_i)$ for all $1 \leq i \leq n - 1$. Define $V := f_n(Q_n)$ and

$$F := (f_n \circ \cdots \circ f_1)^{-1} : V \to Q_1.$$ 

Then $F$ is a conformal map. McMullen proved that the distortion of $F$ on $V$ is bounded above by a constant depending only on $\sigma$, $M$, and $r$, but not on $f_i$ and $n$ ([McM87]). Actually, this upper bound can be formulated in the following lemma.

Lemma 2.2. If the sides of length $r$ of $Q_i$ is chosen such that $r \leq 1/(4M)$ for all $1 \leq i \leq n$, then the distortion of $F$ on $V$ satisfies

$$L(F|_V) \leq \exp \left( \frac{\sigma}{\sigma - 1} \right).$$
Area of the Complement of the Fast Escaping Sets

Proof. Let $g_i$ be the inverse of $f_i$, which maps $f_i(Q_i)$ to $Q_i$ for $1 \leq i \leq n$. Recall that $V = f_n(Q_n)$. Define $V_i := g_1 \circ \cdots \circ g_n(V')$, where $1 \leq i \leq n$. In particular, $V_n = g_n(V') = Q_n$. Since $|f_i'(z)| > \sigma > 1$ for all $1 \leq i \leq n$, we have

$$\text{diam}(V_i) \leq \sqrt{2r}/\sigma^{n-i}, \quad \text{for all } 1 \leq i \leq n.$$  

Note that $V_1 \subset Q_1 \subset D_i$ for $1 \leq i \leq n$ since $Q_{i+1} \subset f_i(Q_i)$ for all $1 \leq i \leq n-1$. This means that there exists a square $Q'_i \subset Q_i$ such that $V_i \subset Q'_i$ and the length of the sides of $Q'_i$ is at most $\sqrt{2r}/\sigma^{n-i}$. Hence by [4], the nonlinearity of $f_i$ on $Q'_i$ satisfies

$$N(f_i|Q'_i) = \left( \sup_{z \in Q'_i} \frac{|f_i''(z)|}{|f_i'(z)|} \right) \cdot \text{diam}(Q'_i) \leq \frac{2Mr}{\sigma^{n-i}} \leq \frac{1}{2}.$$

By Lemma 2.1, we have

$$L(f_i|Q'_i) \leq 1 + \frac{4Mr}{\sigma^{n-i}}, \quad \text{for all } 1 \leq i \leq n.$$

For any holomorphic functions $f$ and $g$, it is straightforward to verify that the distortion of $f$ and $g$ satisfies

$$L(f|V) = L(f^{-1}|f(V)) \quad \text{and} \quad L((g \circ f)|V) \leq L(f|V)L(g|f(V)).$$

Hence, we have

$$L(F|V) = L((f_n \circ \cdots \circ f_1)|V_1) \leq L((f_1|V_1)\cdots L((f_n|V_n) \leq L(f_1|Q'_1)\cdots L(f_n|Q'_n) \leq \prod_{i=0}^{n-1} \left( 1 + \frac{4Mr}{\sigma^{n-i}} \right) \leq \prod_{i=0}^{n-1} \left( 1 + \frac{1}{\sigma^i} \right).$$

Since $\log(1+x) \leq x$ for all $x > 0$, we have

$$L(F|V) \leq \exp \left( \sum_{i=0}^{n-1} \frac{1}{\sigma^i} \right) < \exp \left( \sum_{i=0}^{\infty} \frac{1}{\sigma^i} \right) = \exp \left( \frac{\sigma}{\sigma - 1} \right). \quad \square$$

2.2. Nesting conditions, density and area. In his proof of the existence of Julia sets of entire functions having positive area, McMullen introduced a system of compact sets which satisfies the nesting conditions [McM87]. We now recall the precise definition.

Definition (Nesting conditions). For $k \in \mathbb{N}$, let $\mathcal{E}_k$ be a finite collection of measurable subsets of $C$, i.e. $\mathcal{E}_k := \{ E_{k,i} : 1 \leq i \leq d_k \}$, where each $E_{k,i}$ is a measurable subset of $C$ and $d_k := \# \mathcal{E}_k < +\infty$. We say that $\{ \mathcal{E}_k \}_{k=0}^\infty$ satisfies the nesting conditions if $\mathcal{E}_0 = \{ E_{0,1} \}$, where $E_{0,1}$ is a compact connected measurable set and for all $k \in \mathbb{N}$,

(a) every $E_{k+1,i} \in \mathcal{E}_{k+1}$ is contained in a $E_{k,j} \in \mathcal{E}_k$, where $1 \leq i \leq d_{k+1}$ and $1 \leq j \leq d_k$;

(b) every $E_{k,i} \in \mathcal{E}_k$ contains a $E_{k+1,j} \in \mathcal{E}_{k+1}$, where $1 \leq i \leq d_k$ and $1 \leq j \leq d_{k+1}$;

(c) $\text{Area}(E_{k,i} \cap E_{k,j}) = 0$ for all $1 \leq i, j \leq d_k$ with $i \neq j$; and

(d) there is $\rho_k > 0$ such that for all $1 \leq i \leq d_k$ and $E_{k,i} \in \mathcal{E}_k$, we have\footnote{We suppose that the inverse of $f$ exists in the first equality.}

$$\text{density}(E_{k+1}, E_{k,i}) := \text{density} \left( \bigcup_{j=1}^{d_{k+1}} E_{k+1,j}, E_{k,i} \right) \geq \rho_k.$$
Let \( \{E_k\}_{k=0}^\infty \) be a sequence satisfying the nesting conditions. Define \( E := \cap_{k=0}^\infty E_k \).

The following lemma was established in [McM87, Proposition 2.1].

**Lemma 2.3.** The density of \( E \) in \( E_{0,1} \) satisfies

\[
\text{density}(E, E_{0,1}) \geq \prod_{k=0}^\infty \rho_k.
\]

Now we give the definition of some regions which are needed in the following.

For \( x > 0 \), we define

\[
\Lambda(x) := \{z \in \mathbb{C} : |\text{Re} z| > x\}.
\]

For any given \( m, n \in \mathbb{Z} \) and \( r > 0 \), we define the closed square by

\[
Q_{r}^{m,n} := \{z \in \mathbb{C} : mr \leq \text{Re} z \leq (m+1)r \text{ and } nr \leq \text{Im} z \leq (n+1)r\}.
\]

Let

\[
Q_r := \{Q_{r}^{m,n} : m, n \in \mathbb{Z}\}
\]

be a partition of \( \mathbb{C} \) by the grids with sides of length \( r > 0 \). Sometimes we write \( Q_{r}^{m,n} \in Q_r \) as \( Q_r \), if we don’t want to emphasize the superscript of \( Q_{r}^{m,n} \).

**Lemma 2.4.** Let \( Q \subset \mathbb{C} \) be a square with sides of length \( r > 0 \) and suppose that \( f \) is conformal in a neighbourhood of \( Q \) with distortion \( L(f|_Q) < \infty \). For any \( x > 0 \) and \( z_0 \in Q \), we have

\[
\text{Area}(\cup\{Q_r \in Q_r : Q_r \cap (\partial f(Q) \cup (\partial \Lambda(x) \cap f(Q))) \neq \emptyset\}) \leq cr^2,
\]

where \( c = 16 + 12\sqrt{2}L(f|_Q)|f'(z_0)| \).

This lemma was established in [Sch08, Lemma 2.3] with a different coefficient \( c \). For completeness we include a proof here and the argument is slightly different.

**Proof.** If \( \gamma \subset \mathbb{C} \) is a vertical line with length \( l_1 > 0 \), it is clear that

\[
\#\{Q_r \in Q_r : Q_r \cap \gamma \neq \emptyset\} \leq 4 + \frac{2l_1}{r}.
\]

Let \( \gamma \subset \mathbb{C} \) be a continuous curve with length \( l_2 = 2\sqrt{2}kr > 0 \), where \( k \) is a positive integer. We claim that

\[
k' := \#\{Q_r \in Q_r : Q_r \cap \gamma \neq \emptyset\} \leq 4 + 8k.
\]

Indeed, if \( k = 1 \), then it is easy to see \( k' \leq 12 \). Assume that \( k = n + 1 \) and in this case \( k' \leq 4 + 8n \). If \( k = 0 \), let \( \gamma(t) : [0,1] \rightarrow \mathbb{C} \) be a parameterization of \( \gamma \) such that the length of \( \gamma([t_0,1]) \) is \( 2\sqrt{2}mr \) while the length of \( \gamma([t_0,1]) \) is \( 2\sqrt{2}r \), where \( 0 < t_0 < 1 \). Since \( \gamma([t_0,1]) \) can intersect at most 8 squares while \( \gamma([t_0,t_0]) \) can intersect at most 4 + 8 by the assumption, it follows that \( k' \leq 4 + 8(n+1) \) if \( k = n + 1 \). Hence the claim \([\text{3}]\) is proved.

For the general case, we assume that \( \gamma \subset \mathbb{C} \) is a continuous curve with length \( l_3 > 0 \). Let \( [x] \) be the integer part of \( x > 0 \). By \([\text{3}]\), we have

\[
\#\{Q_r \in Q_r : Q_r \cap \gamma \neq \emptyset\} \leq 4 + 8 \left[ \frac{l_3}{2\sqrt{2}r} \right] + 8 \leq 12 + \frac{2\sqrt{2}l_3}{r}.
\]

Since \( f \) is a conformal map in a neighbourhood of \( Q \), we conclude that \( \partial f(Q) = f(\partial Q) \). From \([\text{2}]\), the length of \( \partial f(Q) \) satisfies

\[
l_4 := \int_{\partial f(Q)} |d\xi| = \int_{\partial Q} |f'(z)||dz| \leq \sup_{z \in Q} |f'(z)| \cdot 4r \leq 4 L(f|_Q)|f'(z_0)| r.
\]
Similarly, the length of $\partial \Lambda(x) \cap f(Q)$ satisfies
\begin{equation}
\begin{aligned}
|\partial \Lambda(x) \cap f(Q)| &= \frac{2 \sup_{z \in Q} |f'(z)| \cdot \operatorname{diam}(Q)}{r} \\
&\leq \frac{2 \sqrt{2} L(f|Q)|f'(z_0)|}{r}.
\end{aligned}
\end{equation}
By (7), (9), (10) and (11), we have
\begin{align*}
\# \{Q_r \in Q_r : Q_r \cap (\partial f(Q) \cup (\partial \Lambda(x) \cap f(Q))) \neq \emptyset\} \\
&\leq \left( 4 + \frac{2 l_5}{r} \right) + \left( 12 + \frac{2 \sqrt{2} l_4}{r} \right) = 16 + \frac{2 l_5 + 2 \sqrt{2} l_4}{r} \\
&\leq 16 + 12 \sqrt{2} L(f|Q)|f'(z_0)|.
\end{align*}
The proof is finished if we notice that the area of each $Q_r$ is $r^2$. \hfill \Box

2.3. Basic properties of the polynomial and entire function. For $N \geq 2$, let $P$ be a polynomial with degree at least 2 which has the form
\begin{equation}
P(z) = a_0 + a_1 z + \cdots + a_N z^N,
\end{equation}
where $a_i \in \mathbb{C}$ for $0 \leq i \leq N$ and $a_0 a_N \neq 0$. In the rest of this article, the polynomial $P$ will be fixed. We denote
\begin{equation}
K := \max \{|a_0|, |a_1|, \ldots, |a_N|\} > 0.
\end{equation}

Lemma 2.5. Let $\varepsilon > 0$ be any given constant. The following statements hold:
\begin{enumerate}[(a)]
\item If $|z| \geq 1 + \frac{K}{\varepsilon |a_N|} > 1$, then
\begin{equation}
|P(z) - a_N z^N| \leq \varepsilon |a_N| |z|^N;
\end{equation}
\item If $|z| \leq \frac{\varepsilon |a_0|}{K + \varepsilon |a_0|} < 1$, then
\begin{equation}
|P(z) - a_0| \leq \varepsilon |a_0|.
\end{equation}
\end{enumerate}

Proof. By the definition of $K$ in (12), if $|z| \geq 1 + \frac{K}{\varepsilon |a_N|} > 1$, then
\begin{equation}
|P(z) - a_N z^N| \leq K(1 + |z| + \cdots + |z|^{N-1}) < K \frac{|z|^N}{|z| - 1} \leq \varepsilon |a_N| |z|^N.
\end{equation}
On the other hand, if $|z| \leq \frac{\varepsilon |a_0|}{K + \varepsilon |a_0|} < 1$, then
\begin{equation}
|P(z) - a_0| \leq K(|z| + \cdots + |z|^N) < K \frac{|z|}{1 - |z|} \leq \varepsilon |a_0|.
\end{equation}
\hfill \Box

Note that
\begin{equation}
P(z)/z = a_0 z^{-1} + a_1 + \cdots + a_N z^{N-1}
\end{equation}
is a rational function. Let $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ be the open disk centered at $a \in \mathbb{C}$ with radius $r > 0$. For each $R > 0$ and $\theta, \xi \in [0, 2\pi)$, we denote a closed domain
\begin{equation}
\mathbb{U}(R, \theta, \xi) := \{z \in \mathbb{C} : |z| \geq R \text{ and } \theta - \frac{\xi}{2} \leq \arg(z) \leq \theta + \frac{\xi}{2}\}.
\end{equation}

Lemma 2.6. For every $\theta \in [0, 2\pi)$, the rational function $P(z)/z$ is univalent in a neighborhood of $\mathbb{U}(2R_1, \theta, \frac{\pi}{N-1})$ and $\mathbb{D}(0, R_2/2)$, where
\begin{equation}
R_1 = 1 + \frac{4K}{|a_N|} \quad \text{and} \quad R_2 = \frac{|a_0|}{4K + |a_0|}.
\end{equation}
Proof. (a) If $|z| \geq R_1$, by Lemma 2.5(a) we have
\[
\left| \frac{P(z)}{z} - a_N z^{N-1} \right| \leq \frac{1}{4} |a_N| |z|^{N-1}.
\]
Then one can write $P(z)/z$ as
\[
P_1(z) = \frac{P(z)}{z} = a_N z^{N-1}(1 + \varphi(z)),
\]
where $\varphi(z)$ is holomorphic in $\mathbb{C} \setminus \{0\}$ and $|\varphi(z)| \leq 1/4$ if $|z| \geq R_1$.

Let $w_0 \in \mathbb{C} \setminus \{0\}$. For any $w \in \partial U(|w_0|/2, \arg(w_0), \pi)$, we have
\[
|w - w_0| > \frac{1}{4}(|w| + |w_0|).
\]
Let $g(z) := z^{N-1}$. For each $z_0 \in \mathbb{C}$ such that $|z_0| \geq 2R_1$, we define $w_0 := g(z_0) = z_0^{N-1}$. Note that $g^{-1}(U(|w_0|/2, \arg(w_0), \pi))$ consists of $N - 1$ disjoint closed domains:
\[
D_k := U\left(2^{-1/(N-1)}|z_0|, \arg(z_0) + \frac{2k\pi}{N-1} \frac{\pi}{N-1}\right),
\]
where $0 \leq k \leq N - 2$. Then for $0 \leq k \leq N - 2$, $z_k := z_0 e^{2k\pi i/(N-1)}$ is contained in the interior of $D_k$.

For any $z \in \partial D_k$ with $0 \leq k \leq N - 2$, we have $z^{N-1} \in \partial U(|w_0|/2, \arg(w_0), \pi)$. Combining (13) and (14), we have
\[
|z^{N-1} - z_0^{N-1}| > \frac{1}{4}(|z^{N-1} + |z_0|^{N-1}) \geq |z^{N-1} \varphi(z) - z_0^{N-1} \varphi(z_0)|.
\]
Define $\varphi_1(z) := a_N (z^{N-1} - z_0^{N-1})$ and $\varphi_2(z) := P_1(z) - P_1(z_0) = a_N z^{N-1}(1 + \varphi(z)) - a_N z_0^{N-1}(1 + \varphi(z_0))$. By Rouche's theorem, $\varphi_1(z) = 0$ and $\varphi_2(z) = 0$ have the same number of roots in each $D_k$, where $0 \leq k \leq N - 2$. Since $\varphi_1(z) = 0$ has exactly one root $z_k$ in each $D_k$, this means that $\varphi_2(z) = 0$ has exactly one root in each $D_k$, where $0 \leq k \leq N - 2$.

On the other hand, (14) holds also for $w \in \partial U(|w_0|/2, -\arg(w_0), \pi)$. By Rouche's theorem again, $\varphi_2(z) \equiv 0$ has no root in each $-D_k$, where $0 \leq k \leq N - 2$. By the arbitrariness of $z_0$, it means that $P_1(z) = P(z)/z$ is univalent in a neighborhood of $U(2R_1, \theta, \frac{\pi}{N-1})$, where $\theta \in [0, 2\pi)$.

(b) Similarly, by Lemma 2.5(b) one can write $P(z)/z$ as
\[
P_1(z) = \frac{P(z)}{z} = \frac{a_0}{z}(1 + \psi(z)),
\]
where $\psi(z)$ is holomorphic in $\mathbb{C}$ and $|\psi(z)| \leq 1/4$ if $|z| \leq R_2$. For each $z_0 \in \mathbb{D}(0, R_2/2) \setminus \{0\}$ and $z \in \partial \mathbb{D}(0, R_2)$, we have
\[
|z - z_0| > \frac{1}{4}(|z| + |z_0|).
\]
Hence
\[
\left| \frac{1}{z} \right| - \frac{1}{z_0} > \frac{1}{4} \frac{|z| + |z_0|}{|z_0|} \geq \left| \frac{\psi(z)}{z} - \frac{\psi(z_0)}{z_0} \right|.
\]
Define $\psi_1(z) := a_0(1/z - 1/z_0)$ and $\psi_2(z) := P_1(z) - P_1(z_0) = \frac{a_0}{z}(1 + \psi(z)) - \frac{a_0}{z_0}(1 + \psi(z_0))$. By Rouche's theorem, $\psi_1(z) = 0$ and $\psi_2(z) = 0$ have the same number of roots in $\mathbb{D}(0, R_2)$. Since $\psi_1(z) = 0$ has exactly one root $z_0$ in $\mathbb{D}(0, R_2)$, this means that $\psi_2(z) = 0$ has exactly one root in $\mathbb{D}(0, R_2)$. By the arbitrariness of $z_0$, it means that $P_1(z) = P(z)/z$ is univalent in a neighborhood of $\mathbb{D}(0, R_2/2)$. \qed
Since $P$ is a polynomial, it is easy to see that $P(e^z)/e^z$ is a transcendental entire function. We now give some quantitativeres on the mapping properties of $f(z) = P(e^z)/e^z$ by applying some properties of $P(z)/z$ obtained above. Recall that $\Lambda(x) = \{ z \in \mathbb{C} : |\Re z| > x \}$ for $x > 0$. We denote
\begin{equation}
K_0 := \min(|a_0|, |a_N|) > 0.
\end{equation}

**Corollary 2.7.** Let
\begin{equation}
r_0 := \frac{\pi}{N-1} \quad \text{and} \quad R_3 := \log \left(2 + \frac{8K}{K_0}\right).
\end{equation}
Then for any square $Q \subset \Lambda(R_3)$ with sides of length $r \leq r_0$, the restriction of $f(z) = P(e^z)/e^z$ on a neighbourhood of $Q$ is a conformal map.

**Proof.** We have $|e^z| \geq 2R_1$ if $\Re z \geq \log(2R_1)$ and $|e^z| \leq R_2/2$ if $\Re z \leq \log(R_2/2)$. Let $Q \subset \Lambda(R_3)$ be a square with sides of length $\pi/(N-1)$. It is easy to see that $\exp$ is injective in a neighbourhood of $Q$ and $\exp(Q)$ is contained in $\overline{\mathbb{D}(0, R_2/2)}$ or $U(2R_1, \theta, \pi/N)$ for some $\theta \in [0, 2\pi)$. This means that $f(z) = P(e^z)/e^z$ is conformal in a neighborhood of $Q$ by Lemma 2.6 \hfill □

We will use the following lemma to estimate $|f'(z)|$ and $|f''(z)/f'(z)|$ for $f(z) = P(e^z)/e^z$.

**Lemma 2.8.** Suppose that $|z| \geq R_4$ or $|z| \leq R_5$, where
\begin{equation}
R_4 = 1 + \max \left\{ \frac{2K+4}{|a_0|}, \frac{K}{|a_0|} \left( \frac{2N^2}{r_0^2} + 1 \right) \right\} \quad \text{and} \quad R_5 = \min \left\{ \frac{|a_0|}{2(N+2)}, \frac{1}{N} \sqrt{|a_0| R_0} \right\}.
\end{equation}
Then
\begin{equation}
\left| P'(z) - \frac{P(z)}{z} \right| > 2 \quad \text{and} \quad \left| \frac{z^2 P''(z)}{z P'(z) - P(z)} - 1 \right| < N.
\end{equation}

**Proof.** A direct calculation shows that
\begin{equation}
P'(z) = \sum_{k=1}^{N} k a_k z^{k-1} \quad \text{and} \quad P''(z) = \sum_{k=2}^{N} k(k-1) a_k z^{k-2}.
\end{equation}
This means that
\begin{equation}
P'(z) - \frac{P(z)}{z} = \sum_{k=1}^{N} k a_k z^{k-1} - \sum_{k=0}^{N} a_k z^{k-1} = \sum_{k=0}^{N} (k-1) a_k z^{k-1}
\end{equation}
and
\begin{equation}
\frac{z^2 P''(z)}{z P'(z) - P(z)} - 1 = \frac{\sum_{k=0}^{N} k(k-1) a_k z^{k-1}}{\sum_{k=0}^{N} (k-1) a_k z^{k-1}} - 1
= \frac{\sum_{k=0}^{N} k(k-1)^2 a_k z^{k-1}}{\sum_{k=0}^{N} (k-1) a_k z^{k-1}}.
\end{equation}
If $|z| \geq 1 + \frac{2K+4}{|a_0|} > 3$, by (17) we have
\begin{equation}
\left| P'(z) - \frac{P(z)}{z} \right| \geq |a_N| (N-1) |z|^{N-1} - K(N-1)(|z|^{N-2} + \cdots + |z| + 1)
\geq (N-1)|z|^{N-1} \left( |a_N| - \frac{K}{|z| - 1} \right)
\geq \frac{|a_N|}{2} |z|^{N-1} \geq \frac{|a_N|}{2} |z| > 2.
\end{equation}
If $|z| \leq \frac{|a_0|}{2(N+2)} < \frac{1}{2}$, we have
\begin{equation}
\left| P'(z) - \frac{P(z)}{z} \right| \geq \frac{|a_0|}{|z|} - K(N-1)(|z| + \cdots + |z|^{N-1})
\geq \frac{|a_0|}{|z|} - K(N-1) > \frac{|a_0|}{2} |z| \geq K N + 2 > 2.
\end{equation}
Lemma 2.10. The entire function $f(z) = e^z$.

Proof. Let $z \in \Lambda(R_0)$, such that $\gamma(t) \to \infty$ as $t \to \infty$ and $f(\gamma(t)) \to a$ as $t \to \infty$.

Corollary 2.9. Let $R_0 := \max \{ \log R_4, -\log R_0 \}$. Then for any $z \in \Lambda(R_0)$, the function $f(z) = P(e^z)/e^z$ satisfies

$$|f'(z)| > 2 \quad \text{and} \quad \frac{|f''(z)|}{|f'(z)|} < N.$$  

Proof. Denote $P_1(w) := P(w)/w$. Therefore, $f(z) = P(e^z)/e^z = P_1 \circ \exp(z)$. It is easy to check that

$$f'(z) = P'_1(e^z)e^z \quad \text{and} \quad f''(z) = P''_1(e^z)e^{2z} + P'_1(e^z)e^z.$$  

Let $w = e^z$. By a straightforward computation, we have

$$f'(z) = P'_1(w)w = P'(w) - \frac{P(w)}{w}$$

and

$$f''(z) = \frac{P''(w)w^2 + P'_1(w)w}{P'_1(w)w} = \frac{w^2P''(w)}{wP'(w) - P(w)} - 1.$$  

Then the result follows from Lemma 2.8 immediately.

2.4. Escaping and fast escaping sets. Let $f$ be a transcendental entire function. A point $a \in \mathbb{C}$ is called an asymptotic value of $f$ if there exists a continuous curve $\gamma(t) \subset \mathbb{C}$ with $0 < t < \infty$, such that $\gamma(t) \to \infty$ as $t \to \infty$ and $f(\gamma(t)) \to a$ as $t \to \infty$.

Lemma 2.10. The entire function $f(z) = P(e^z)/e^z$ does not have any finite asymptotic value.

Proof. Assume that $a \in \mathbb{C}$ is a finite asymptotic value of $f(z)$. Then by definition, there exists a continuous curve $\gamma(t) \subset \mathbb{C}$ with $0 < t < \infty$, such that $\gamma(t) \to \infty$ as $t \to \infty$ and $f(\gamma(t)) \to a$ as $t \to \infty$. This means that

$$\lim_{t \to \infty} \frac{P(w)}{w} \circ e^{\gamma(t)} = a.$$  

Denote $\gamma(t) = x(t) + iy(t)$ and let $w_1, w_2, \cdots, w_N$ be the $N$ roots of the equation $P(w) = aw$. We define the set $Y := \{ \arg w + 2k\pi : 1 \leq i \leq N, k \in \mathbb{Z} \}$. If $x(t)$ is unbounded as $t \to \infty$, then $f(\gamma(t))$ is also unbounded and this is a contradiction. Hence $|x(t)| \leq A$ for some constant $A > 0$ for all $t$. Since $\gamma(t) \to \infty$
as \( t \to \infty \), this implies that \( y(t) \to \infty \) as \( t \to \infty \). Therefore, for each \( y_0 \in \mathbb{R} \setminus Y \), there exists a sequence \( \{z_n\} \subset \gamma(t) \) such that \( \text{Im} z_n \to \infty \) as \( n \to \infty \) and \( \lim_{n \to \infty} e^{\text{Im} z_n} = e^{y_0} \). Since \( |x(t)| \leq A \), it follows that \( \lim_{n \to \infty} e^{x(t)} \neq 0 \). This implies that \( \lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} P(e^{z_n})/e^{z_n} \neq a \), which is a contradiction. \( \square \)

Let \( f \) be a transcendental entire function. The set

\[
I(f) := \{ z \in \mathbb{C} : f^{\circ n}(z) \to \infty \text{ as } n \to \infty \}
\]

is called the \textit{escaping set of} \( f \). We use \( \text{sing}(f^{-1}) \) to denote the set of \textit{singular values of} \( f \) which consists of all the critical values and asymptotic values of \( f \) and their accumulation points.

\textbf{Corollary 2.11.} The \textit{escaping set} \( I(f) \) of \( f(z) = P(e^z)/e^z \) is contained in the \textit{Julia set} \( J(f) \).

\textit{Proof.} It is clear that the set of the critical values of \( f(z) = P(e^z)/e^z \) is finite. From Lemma 2.10 it follows that \( \text{sing}(f^{-1}) \) is bounded. According to [BH99, Theorem 1], we have \( I(f) \subset J(f) \). \( \square \)

Actually, we will estimate the area of the complement of the fast escaping set in next section. Let \( f \) be a transcendental entire function. The \textit{maximal modulus function} is defined by

\[
M(r, f) := \max_{|z|=r} |f(z)|, \text{ where } r > 0.
\]

We use \( M^{n}(r, f) \) to denote the \( n \)-th iterate of \( M(r, f) \) with respect to the variable \( r > 0 \), where \( n \in \mathbb{N} \). The notation \( M(r, f) \) is written as \( M(r) \) if the function \( f \) is known clearly. A subset of the escaping set, called the \textit{fast escaping set} \( A(f) \) was introduced in [BH99] and can be defined [RS12] by

\[
A(f) := \{ z : \text{ there is } \ell \in \mathbb{N} \text{ such that } |f^{\circ (n+\ell)}(z)| \geq M^{n}(R) \text{ for } n \in \mathbb{N} \}.
\]

Here \( R > 0 \) is a constant such that \( M^{n}(R) \to \infty \) as \( n \to \infty \). It is proved in [RS12, Theorem 2.2(b)] that \( A(f) \) is independent of the choice of \( R \) such that \( M^{n}(R) \to \infty \) as \( n \to \infty \).

\textbf{Lemma 2.12.} Let \( R > 0 \) be a constant and define \( u_0 := R \). For \( n \geq 1 \), define \( u_n \) inductively by \( u_n := Re^{Ru_{n-1}} \). Let \( v_0 \in \mathbb{R} \) and define \( v_n \) inductively by \( v_n := e^{v_{n-1}} \) for \( n \geq 1 \). Then there is \( \ell \in \mathbb{N} \) such that \( v_{n+\ell} \geq 2Ru_n \) for all \( n \in \mathbb{N} \).

\textit{Proof.} For any \( v_0 \in \mathbb{R} \), there exists an integer \( \ell \in \mathbb{N} \) such that \( v_0 \geq 2R^2 \). Shifting the subscript of \( (v_n)_{n \in \mathbb{N}} \) if necessary, it is sufficient to prove that if \( v_0 \geq 2R^2 \), then \( v_n \geq 2Ru_n \) for all \( n \in \mathbb{N} \). Suppose that \( v_{n-1} \geq 2Ru_{n-1} \) for some \( n \geq 1 \) (note that \( v_0 \geq 2Ru_0 \)). We hope to obtain that \( v_n \geq 2Ru_n \). Note that \( v_n = e^{v_{n-1}} \geq e^{2Ru_{n-1}} \) and \( u_n = Re^{Ru_{n-1}} \). It is sufficient to obtain \( Ru_{n-1} \geq \log(2R^2) \). This is true since \( u_{n-1} \geq R \) and \( R^2 \geq \log(2R^2) \) for all \( R > 0 \). \( \square \)

\textbf{Corollary 2.13.} Let \( z_0 \in \mathbb{C} \) and suppose that \( z_n = f^{\circ n}(z_0) \) satisfies \( |z_n| \geq \xi_n \) for all \( n \in \mathbb{N} \), where \( \xi_n > 0 \) is defined inductively by

\[
\xi_n = 2\exp(\xi_{n-1}/2) \text{ with } \xi_0 > 0.
\]

Then \( z_0 \) is contained in the \textit{fast escaping set} of \( f(z) = P(e^z)/e^z \).

\textit{Proof.} Recall that \( N \geq 2 \) is the degree of the polynomial \( P \) and \( K > 0 \) is defined in [12]. According to Lemma 2.5, there exists \( \delta_0 \geq 1 \) such that if \( \delta \geq \delta_0 \), then the maximal modulus function of \( f \) satisfies

\[
M(\delta) = M(\delta, f) \leq 2Ke^{(N-1)\delta}.
\]
On the other hand, there exists $\delta_1 > 0$ such that for all $\delta \geq \delta_1$, then $M^{\infty}(\delta)$ is monotonically increasing as $n$ increases. Since the Julia set of $f$ is non-empty, this means that $M^{\infty}(\delta) \to \infty$ as $n \to \infty$ if $\delta \geq \delta_1$.

Define

$$R := \max\{2K_{\cdot}, (N - 1)\delta_0, \delta_1\} \geq 1.$$  

We denote $u_0 = R$ and for $n \geq 1$, define $u_n$ inductively by $u_n = R e^{Ru_{n-1}}$. Then we have $M^{\infty}(R) \leq u_n$ for all $n \in \mathbb{N}$. By the definition of $\xi_n$, we have $\xi_n = 2 \exp^{\infty}(\xi_0/2)$. Let $\nu_0 := \xi_0/2$ and define $\nu_n := e^{\nu_{n-1}}$ for $n \geq 1$. According to Lemma 2.12, there exists $\ell \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$|f^{n(\ell+1)}(z_0)| = |z_{n+\ell}| \geq \xi_{n+\ell} = 2\nu_{n+\ell} \geq 4Ru_n \geq u_n \geq M^{\infty}(R).$$

By the definition of $R$, we have $M^{\infty}(R) \to \infty$ as $n \to \infty$. This means that $z_0$ is contained in the fast escaping set of $f$. \hfill \Box

3. PROOF OF THE THEOREMS

3.1. Proof of Theorem 1.1. Recall that $N \geq 2$ is the degree of the polynomial $P$. Let $r > 0$ be fixed such that

$$r \leq \frac{1}{4N}.$$  

We define

$$x' := \max\{R_3, R_6, 6 \log 2\},$$

where $R_3$ and $R_6$ are constants introduced in Corollary 2.7 and Corollary 2.9 respectively.

Recall that $\Lambda(x) = \{z \in \mathbb{C} : |\text{Re} z| > x\}$ is the set defined in [5] for all $x > 0$. Let $Q_0$ be a square in $\Lambda(x)$ with sides of length $r$, where $x \geq x'$. Since $r < r_0 = \pi/(N - 1)$, from Corollary 2.7 we know that $f$ is conformal in a neighbourhood of $Q_0$. For $k \in \mathbb{N}$, define

$$x_k := 2 \exp^{\circ k}(x/2).$$

In particular, $x_0 = x \geq x'$ and we have $x_{k+1} := 2 \exp(x_{k}/2) > x_{k} \geq x'$ since $2e^{x/2} > x$ for all $x \in \mathbb{R}$.

Recall that $Q_r$ is a collection of grids with sides of length $r > 0$ defined in [6]. For any subset $E$ of $Q_0$ in $\Lambda(x_0)$ and $k \in \mathbb{N}$, define

$$\text{pack}(f^{\circ k}(E)) := \{Q_r \in Q_r : Q_r \subset f^{\circ k}(E) \cap \Lambda(x_k)\}.$$  

We now define a sequence of families of measurable sets satisfying the nesting conditions based on the square $Q_0$. Let $\mathcal{E}_0 := \{Q_0\}$ and for $k \geq 1$, define inductively

$$\mathcal{E}_k := \{F_k \subset Q_0 : F_k \subset E_{k-1} \in \mathcal{E}_{k-1} \text{ and } f^{\circ k}(F_k) \in \text{pack}(f^{\circ k}(E_{k-1}))\}.$$  

It is clear that $\mathcal{E}_k$ is a finite collection of measurable subsets of $\mathbb{C}$ for all $k \in \mathbb{N}$. Denote the elements of $\mathcal{E}_k$ by $E_{k,i}$, where $1 \leq i \leq d_k$.

By definition, for all $k \in \mathbb{N}$, we have $f^{\circ (k+1)}(E_{k,i}) = f(Q_{x_k}^i)$, where $Q_{x_k}^i$ is a square with sides of length $r$ and $Q_{x_k}^i \subset \Lambda(x_k)$. From [4], Corollary 2.9 and (25), we have

$$N(f|_{Q_{x_k}^i}) < N \sqrt{2r} \leq \frac{\sqrt{2}}{4}.$$  

By Lemma 2.1, the distortion of $f$ on $Q_{x_k}^i$ satisfies

$$L(f|_{Q_{x_k}^i}) \leq 1 + 2N(f|_{Q_{x_k}^i}) < 2.$$

3Note that $Q_{x_k}^i \subset \Lambda(x_k)$ is a square depending also on the subscript ‘$i$’ of $E_{k,i}$, where $k \in \mathbb{N}$ and $1 \leq i \leq d_k$. We omit this index here for simplicity.
For every $k \in \mathbb{N}$, let $z_k$ be any point in $Q^k \subset \Lambda(x_k)$. From (2) and (28) we have

$$\text{Area}(f(Q^k)) = \int_{Q^k} |f'(z)|^2 dx dy \geq \inf_{z \in Q^k} |f'(z)|^2 \cdot \text{Area}(Q^k)$$

$$\geq \frac{|f'(z_k)|^2}{(L(f|_{Q^k}))^2} \cdot r^2 > \frac{1}{4} |f'(z_k)|^2 r^2$$

and

$$\text{diam}(f(Q^k)) \leq \sup_{z \in Q^k} |f'(z)| \cdot \text{diam}(Q^k)$$

$$\leq L(f|_{Q^k})|f'(z_k)| \cdot \sqrt{2}r < 2\sqrt{2}|f'(z_k)|r.$$  

Recall that $K_0 = \min\{|a_o|, |a_N|\} > 0$ is the constant defined in (15). By (19), (20) and (22), we have

$$|f'(z_k)| > \frac{1}{2} K_0 e^{\text{Re} z_k} > \frac{1}{2} K_0 e^{x_k}.$$  

For $k \in \mathbb{N}$ and $1 \leq i \leq d_k$, we denote

$$B_1 := \bigcup \{Q_r \in Q_r : Q_r \subset f^{o(k+1)}(E_{k,i}) \cap (\mathbb{C} \setminus \Lambda(x_k+1))\}$$

and

$$B_2 := \bigcup \{Q_r \in Q_r : Q_r \cap (\partial f^{o(k+1)}(E_{k,i}) \cup (\partial \Lambda(x_k+1) \cap f^{o(k+1)}(E_{k,i})) \neq \emptyset\}.$$  

Recall that $f^{o(k+1)}(E_{k,i}) = f(Q^k)$ for some square $Q^k$ in $\Lambda(x_k)$ with sides of length $r$, where $k \in \mathbb{N}$ and $1 \leq i \leq d_k$. From (29), (30) and (31), we have

$$\frac{\text{Area}(B_1)}{\text{Area}(f^{o(k+1)}(E_{k,i}))} \leq \frac{2x_{k+1} \text{diam}(f^{o(k+1)}(E_{k,i}))}{\text{Area}(f^{o(k+1)}(E_{k,i}))}$$

$$= \frac{2x_{k+1} \text{diam}(f(Q^k))}{\text{Area}(f(Q^k))} \leq \frac{16\sqrt{2} x_{k+1}}{K_0 r} \cdot \frac{1}{K_0 e^{x_k}}.$$  

Note that $x_{k+1} \geq x_1 = 2e^{x/2}$ for all $k \in \mathbb{N}$ and $x \geq 6 \log 2$ by (26). By Lemma 2.4 (28), (29) and (31), we have

$$\frac{\text{Area}(B_2)}{\text{Area}(f^{o(k+1)}(E_{k,i}))} \leq \frac{(16 + 12\sqrt{2} L(f|_{Q^k}) |f'(z_k)|) r^2}{\text{Area}(f(Q^k))}$$

$$< \frac{32(2 + 3\sqrt{2} |f'(z_k)|)}{|f'(z_k)|^2} < \frac{256}{K_0^2 e^{2x_k}} + \frac{192\sqrt{2}}{K_0 e^{x_k}}$$

$$\leq \left(\frac{128}{K_0^2} \frac{1}{e^{2x/2}} + \frac{96\sqrt{2}}{K_0} \frac{1}{e^{x/2}}\right) x_{k+1} \leq \left(\frac{4K_0}{K_0} + \frac{12\sqrt{2}}{K_0}\right) x_{k+1}.$$  

For all $k \in \mathbb{N}$ and $1 \leq i \leq d_k$, by (32) and (33), we have

$$\text{density} \left(\bigcup \text{pack}(f^{o(k+1)}(E_{k,i})), f^{o(k+1)}(E_{k,i})\right)$$

$$\geq \frac{\text{Area}(B_1)}{\text{Area}(f^{o(k+1)}(E_{k,i}))} - \frac{\text{Area}(B_2)}{\text{Area}(f^{o(k+1)}(E_{k,i}))}$$

$$> 1 - c_0 \frac{x_{k+1}}{e^{x_k}} \geq 1 - c_1 \frac{x_{k+1}}{e^{x_k}},$$

where

$$c_1 \geq c_0 := \frac{32\sqrt{2}}{K_0 r} + \frac{1}{4K_0} + \frac{12\sqrt{2}}{K_0}.$$  

Comparing (26), we assume that $x^* > 0$ is a fixed constant such that

$$x^* \geq \max\{R_3, R_6, 6 \log 2, 12 + 2 \log c_1\}.$$
Moreover, we suppose that the sequence \( \{x_k\}_{k \in \mathbb{N}} \) in (27) is chosen such that the initial point satisfies \( x_0 = x \geq x^* \). Then, all the statements above are still true since \( x^* \geq x' \).

By a straightforward induction, one can show that for all \( k \in \mathbb{N} \) and \( x \in \mathbb{R} \),
\[
\exp^{(k+1)}(x) \geq \exp(k)(x).
\]
Since \( x_{k+1} = 2e^{x_{k}/2} \), we have
\[
\frac{x_{k+1}}{e^{x_{k}/2}} = \frac{2}{e^{x_{k}/2}} = \frac{2}{\exp(k+1)(x)/2} \leq \frac{2}{e^k} \cdot \frac{1}{e^{x/2}}.
\]
On the other hand, by (39), we have \( e^{x/2} \geq c_1 e^6 > 6c_1 e^4 \) since \( x \geq x_* \). Therefore,
\[
c_1 e^4 \frac{x_{k+1}}{e^{x_k}} \leq c_1 e^4 \cdot \frac{2}{e^k} \cdot \frac{1}{e^{x/2}} \leq c_1 e^4 \cdot \frac{2}{e^{x/2}} < \frac{1}{3}.
\]
Define \( V := f(Q^k) \) and let \( G := f^{-(k+1)} : V \to Q_0 \) be the inverse of \( f^{(k+1)}|_{E_k} \), where \( k \in \mathbb{N} \) and \( 1 \leq i \leq d_k \). By Lemma 2.2, Corollary 2.9 and (25), the distortion of \( G \) on \( V \) satisfies
\[
L(G|_V) < \exp\left(\frac{2^2}{c-1}\right) = e^2.
\]
From (3) and (39), we have
\[
density(E_{k+1}, E_{k,i}) = 1 - \density(E_{k,i} \setminus E_{k+1}, E_{k,i})
\]
\[
= 1 - \density(G(f^{(k+1)}(E_{k,i} \setminus E_{k+1})), E_k(E_{k+1}, E_{k,i}))
\]
\[
\geq 1 - L(G|_V)^2 \density\left(f^{(k+1)}(E_{k,i}) \setminus \bigcup \text{pack}\left(f^{(k+1)}(E_{k,i}), f^{(k+1)}(E_{k,i})\right)\right)
\]
\[
\geq 1 - e^4 \left(1 - \density\left(\bigcup \text{pack}\left(f^{(k+1)}(E_{k,i}), f^{(k+1)}(E_{k,i})\right)\right)\right).
\]
Therefore, by (34) and (38), we have
\[
density(E_{k+1}, E_{k,i}) \geq 1 - c_1 e^4 \frac{x_{k+1}}{e^{x_k}} \geq \frac{2}{3},
\]
where \( k \in \mathbb{N} \) and \( 1 \leq i \leq d_k \). For all \( k \in \mathbb{N} \), by setting
\[
\rho_k := 1 - c_1 e^4 \frac{x_{k+1}}{e^{x_k}},
\]
it is easy to see that \( \{E_k\}_{k=0}^\infty \) satisfies the nesting conditions.

Define \( E = \cap_{k=0}^\infty E_k \). Recall that \( A(f) \) is the fast escaping set of \( f \) defined in (24). Since every point \( z \in E_{k,i} \) satisfies \( f^j(z) \in \Lambda(x_j) \) for \( 0 \leq j \leq k \) and \( x_k \to +\infty \) as \( k \to \infty \), it means that \( E \) is contained in the fast escaping set \( A(f) \) by (27) and Corollary 2.13. According to Lemma 2.3, we have
\[
density(A(f), Q_0) \geq \density(E, Q_0) \geq \prod_{k=0}^\infty \rho_k.
\]
Note that \( \log(1 - t) > -2t \) for \( t \in (0, 1/2) \). By (38) and (41), we have
\[
\log \left( \prod_{k=0}^\infty \rho_k \right) = \sum_{k=0}^\infty \log \left( 1 - c_1 e^4 \frac{x_{k+1}}{e^{x_k}} \right) \geq -2 \sum_{k=0}^\infty c_1 e^4 \frac{x_{k+1}}{e^{x_k}}
\]
\[
\geq - \frac{4c_1 e^4}{e^{x/2}} \sum_{k=0}^\infty \frac{1}{e^k} > - \frac{8c_1 e^4}{e^{x/2}}.
\]
Since \( e^{-t} \geq 1 - t \) for all \( t \in \mathbb{R} \), we have
\[
density(A(f), Q_0) > \exp\left( - \frac{8c_1 e^4}{e^{x/2}} \right) \geq 1 - \frac{8c_1 e^4}{e^{x/2}}
\]
for all \( x \geq x^* \) and all square \( Q_0 \subset \Lambda(x) \) with sides of length \( r \).
Theorem 3.1. Let $S$ be any horizontal strip of width $2\pi$. Then the area of the complement of the fast escaping set of $f(z) = P(e^z)/e^z$ satisfies

$$\text{Area}(S \cap A(f)^c) \leq (4\pi + 4r) \left( x^* + r + 8c_1 e^{4-x^*/2} \frac{r}{1 - e^{-r/2}} \right) < \infty,$$

where $r$, $c_1$ and $x^*$ are any positive constants satisfying (25), (35) and (36) respectively.

Proof. Define the half strip $S_+$ by

$$S_+ := \{ z \in \mathbb{C} : 0 \leq \text{Im } z \leq 2\pi \text{ and } \text{Re } z \geq 0 \}.$$  

We take

$$m_0 = \lfloor x^*/r \rfloor + 1 \text{ and } n_0 = \lfloor 2\pi/r \rfloor + 1,$$  

where $\lfloor x \rfloor$ denotes the integer part of $x \geq 0$. Recall that $Q_r^{m,n}$ is defined as

$$Q_r^{m,n} := \{ z \in \mathbb{C} : mr \leq \text{Re } z \leq (m+1)r \text{ and } nr \leq \text{Im } z \leq (n+1)r \},$$

where $m,n \in \mathbb{Z}$. Since $Q_r^{m,n} \subset A(x^*)$ for all $m \geq m_0$, we get

$$\text{density}(A(f), Q_r^{m,n}) > 1 - \frac{8c_1 e^4}{\exp(mr/2)}$$

for all $m \geq m_0$ by (12). So

$$\text{Area}(S_+ \cap A(f)^c) \leq \text{Area} \left( \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{n_0} Q_r^{m,n} \setminus A(f) \right)$$

$$\leq \sum_{m=0}^{\infty} \sum_{n=0}^{n_0} \text{Area}(Q_r^{m,n} \setminus A(f))$$

$$\leq \sum_{m=0}^{\infty} \sum_{n=0}^{n_0} \left( 1 - \text{density}(A(f), Q_r^{m,n}) \right) \cdot \text{Area}(Q_r^{m,n}).$$

By (44) and (45), we obtain

$$\text{Area}(S_+ \cap A(f)^c) \leq r^2 \left( \sum_{m=0}^{m_0-1} \sum_{n=0}^{n_0} 1 + \sum_{m=m_0}^{\infty} \sum_{n=0}^{n_0} \frac{8c_1 e^4}{\exp(mr/2)} \right)$$

$$\leq (2\pi + 2r) \left( x^* + r + 8c_1 e^{4-x^*/2} \frac{r}{1 - e^{-r/2}} \right).$$

This means that $\text{Area}(S_+ \cap A(f)^c) < \infty$ for every fixed $r > 0$ satisfying (25). Similarly, one can obtain

$$\text{Area}(S_- \cap A(f)^c) \leq (2\pi + 2r) \left( x^* + r + 8c_1 e^{4-x^*/2} \frac{r}{1 - e^{-r/2}} \right),$$

where $S_- = \{ z \in \mathbb{C} : 0 \leq \text{Im } z \leq 2\pi \text{ and } \text{Re } z \leq 0 \}$. Since $f(z) = f(z + 2\pi i)$, for any horizontal strip $S$ of width $2\pi$, we have

$$\text{Area}(S \cap A(f)^c) \leq (4\pi + 4r) \left( x^* + r + 8c_1 e^{4-x^*/2} \frac{r}{1 - e^{-r/2}} \right).$$

This completes the proof of Theorem 3.1 and hence Theorem 1.1. \qed
3.2. **Proof of Theorem 1.2** Consider the quadratic polynomial
\[ P(z) = \frac{\alpha}{2} z^2 + i\beta z - \frac{\alpha}{2}, \]
where \( \alpha \neq 0 \) and \( \beta \in \mathbb{C} \).

We then have
\[ f(z) := P(e^z) e^z = \frac{\alpha}{2} e^{2z} + i\beta - \frac{\alpha}{2} e^{-z}. \]

Note that \( \alpha \sin(z + \beta) \) is conjugated by \( z \mapsto i(z + \beta) \) to \( f(z) \). In order to prove Theorem 1.2, it is sufficient to prove the corresponding statements on \( f \).

Now we collect all the needing constants in the proof. Note that the degree of \( P \) is \( \text{deg}(P) = N = 2 \). By (25) we fix the choice of \( r > 0 \) by setting \( r = 1/8 \).

By (15), we have \( K_0 = |\alpha|/2 \). From (35), we fix \( c_1 = c_0 = \frac{536\sqrt{2}}{|\alpha|} + \frac{1}{|\alpha|^2} \).

By (16), we have
\[ R_3 = \log \left( 2 + \frac{16K}{|\alpha|} \right), \]
where \( K = \max\{|\alpha|/2, |\beta|\} \).

According to Lemma 2.8, we have
\[ R_4 = \max \left\{ 1 + \frac{4(K + 2)}{|\alpha|}, 1 + \frac{18K}{|\alpha|} \right\} \quad \text{and} \quad R_5 = \min \left\{ \frac{|\alpha|}{8(K + 1)}, \frac{1}{4} \sqrt{\frac{|\alpha|}{2K}} \right\}. \]

Since \( K \geq |\alpha|/2 > 0 \), we have
\[ \frac{8(K + 1)}{|\alpha|} > \frac{8K}{|\alpha|} \geq 4 \sqrt{\frac{2K}{|\alpha|}} \quad \text{and} \quad \frac{8(K + 1)}{|\alpha|} = \frac{4K}{|\alpha|} + \frac{4(K + 2)}{|\alpha|} > 1 + \frac{4(K + 2)}{|\alpha|} \]
and
\[ 1 + \frac{18K}{|\alpha|} = 1 + \frac{16K}{|\alpha|} + \frac{K}{|\alpha|/2} \geq 2 + \frac{16K}{|\alpha|}. \]

Hence by (36), we can fix
\[ x^* = \max \left\{ \log \left( 1 + \frac{18K}{|\alpha|} \right), \log \left( \frac{8(K + 1)}{|\alpha|} \right), 6 \log 2, 12 + 2 \log c_1 \right\}. \]

By Theorem 3.1, the proof of Theorem 1.2 is finished modulo the statement on the sine and cosine functions.

Let \( S \) be a vertical strip with width \( 2\pi \). If \( \alpha = 1 \) and \( \beta = 0 \), then \( K = 1/2 \) and \( r = 1/8 \), \( c_1 = 536\sqrt{2} + 1 \) and \( x^* = 12 + 2 \log (536\sqrt{2} + 1) \).

From (43) we have
\[ \text{Area}(S \cap A(\sin z)^c) \leq \left( 4\pi + \frac{1}{2} \right) \left( \frac{97}{8} + 2 \log(536\sqrt{2} + 1) + \frac{1}{e^2 - e^{31/16}} \right) < 361. \]

If \( \alpha = 1 \) and \( \beta = \pi/2 \), then \( K = \pi/2 \) and we still have (46). Also from (43) we have
\[ \text{Area}(S \cap A(\cos z)^c) < 361. \]

This finishes the proof of Theorem 1.2.
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