Hidden symmetries and the fermionic sector of eleven-dimensional supergravity

Thibault Damour\textsuperscript{1}, Axel Kleinschmidt\textsuperscript{2} and Hermann Nicolai\textsuperscript{2}

\textsuperscript{1}Institut des Hautes Etudes Scientifiques  
35, Route de Chartres, F-91440 Bures-sur-Yvette, France  

\textsuperscript{2}Max-Planck-Institut für Gravitationsphysik  
Albert-Einstein-Institut  
Mühlenberg 1, D-14476 Potsdam, Germany

Abstract: We study the hidden symmetries of the fermionic sector of $D = 11$ supergravity, and the rôle of $K(E_{10})$ as a generalised ‘R-symmetry’. We find a consistent model of a massless spinning particle on an $E_{10}/K(E_{10})$ coset manifold whose dynamics can be mapped onto the fermionic and bosonic dynamics of $D = 11$ supergravity in the near space-like singularity limit. This $E_{10}$-invariant superparticle dynamics might provide the basis of a new definition of M-theory, and might describe the ‘de-emergence’ of space-time near a cosmological singularity.

Eleven-dimensional supergravity (SUGRA$_{11}$) [1] is believed to be the low-energy limit of the elusive ‘M-theory’, which is, hopefully, a unified framework encompassing the various known string theories. Understanding the symmetries of SUGRA$_{11}$ is therefore important for reaching a satisfactory formulation of M-theory. Many years ago it was found that the toroidal dimensional reduction of SUGRA$_{11}$ to lower dimensions leads to the emergence of unexpected (‘hidden’) symmetry groups, notably $E_7$ in the reduction to four non-compactified spacetime dimensions [2], $E_8$ in the reduction to $D = 3$ [2, 3, 4, 5], and the affine Kac–Moody group $E_9$ in the reduction to $D = 2$ [6, 7]. It was also conjectured [8] that the hyperbolic Kac–Moody group $E_{10}$
might appear when reducing SUGRA\textsubscript{11} to only one (time-like) dimension.

Recently, the consideration, à la Belinskii, Khalatnikov and Lifshitz [9], of the near space-like singularity limit\textsuperscript{1} of generic inhomogeneous bosonic eleven-dimensional supergravity solutions has uncovered some striking evidence for the hidden rôle of $E\textsubscript{10}$ [10, 11]. Ref. [11] related the gradient expansion ($\partial_x \ll \partial_t$), which organises the near space-like singularity limit [12], to an algebraic expansion in the height of positive roots of $E\textsubscript{10}$. A main conjecture of [11] was the existence of a correspondence between the time evolution, around any given spatial point $x$, of the supergravity bosonic fields $g^{(11)}_{MN}(t, x)$, $A^{(11)}_{MNP}(t, x)$, together with their infinite towers of spatial gradients, on the one hand, and the dynamics of a structureless massless particle on the infinite-dimensional coset space $E\textsubscript{10}/K(E\textsubscript{10})$ on the other hand. Here, $K(E\textsubscript{10})$ is the maximal compact subgroup of $E\textsubscript{10}$. Further evidence for the rôle of the one-dimensional non-linear sigma model $E\textsubscript{10}/K(E\textsubscript{10})$ in M-theory was provided in [13, 14, 15, 16].

An earlier and conceptually different proposal aiming at capturing hidden symmetries of M-theory, and based on the very-extended Kac–Moody group $E\textsubscript{11}$, was made in [17, 18] and further developed in [19, 20, 21]. A proposal combining the ideas of [18] and [11] was put forward in [22, 23, 24].

In this letter, we extend the bosonic coset construction of [11] to the full supergravity theory by including fermionic variables; more specifically, we provide evidence for the existence of a correspondence between the time evolution of the coupled supergravity fields $g^{(11)}_{MN}(t, x)$, $A^{(11)}_{MNP}(t, x)$, $\psi^{(11)}_M(t, x)$ and the dynamics of a spinning massless particle on $E\textsubscript{10}/K(E\textsubscript{10})$. Previous work on $E\textsubscript{10}$ which included fermions can be found in [13, 25].\textsuperscript{2}

To motivate our construction of a fermionic extension of the bosonic one-dimensional $E\textsubscript{10}/K(E\textsubscript{10})$ coset model we consider the equation of motion of the gravitino in $D = 11$ supergravity [1].\textsuperscript{3} Projecting all coordinate indices on

\textsuperscript{1}This limit can also be viewed as a small tension limit, $\alpha' \to \infty$.

\textsuperscript{2}Results similar to some of the ones reported here have been obtained in [26].

\textsuperscript{3}We use the mostly plus signature; $M, N, \ldots = 0, \ldots, 10$ denote spacetime coordinate (world) indices; $m, n, p, \ldots = 1, \ldots, 10$ denote spatial coordinate indices, and the indices $i, j, k, l = 1, \ldots, 10$ label the non-orthonormal frame components $\theta^i_m dx^m$. Spacetime
an elfbein $E^A_{(1)} = E^A_{(11)} dx^M$, the equation of motion for $\psi^{(11)} = E^M_{(11)} \psi^{(11)}_M$ is (neglecting quartic fermion terms)

$$0 = \hat{\mathcal{E}}_A := \Gamma^B \left[ (D_A(\omega) + \mathcal{F}_A) \psi^{(11)}_B - (D_B(\omega) + \mathcal{F}_B) \psi^{(11)}_A \right],$$  

where $D_A(\omega) = E^M_{(11)} A_M$ denotes the moving-frame covariant derivative. $D_A(\omega) \psi^{(11)}_B = \partial_A \psi^{(11)}_B + \omega^{(11)}_{ABC} \psi^{(11)}_C + \frac{1}{2} \Gamma^{(11)}_{MNP} \Gamma^{(11)}_{CD} \psi^{(11)}_B$, and where $\mathcal{F}_A := + \frac{1}{144} \left( \Gamma^{(11)}_{BCDE} - 8 \delta_A^{(11)} \Gamma^{(11)}_{CDE} \right) F_{BCDE}$ denotes the terms depending on the 4-form field strength $F_{MNPQ}^{(11)} = 4 \partial_M A_{NPQ}^{(11)}$. Here $\omega^{(11)}_{ABC} = - \omega^{(11)}_{ACB} = E^M_{(11)} \omega^{(11)}_{MBC}$ denotes the moving frame components of the spin connection, with $\omega^{(11)}_{ABC} = \frac{1}{2} (\Omega_{ABC}^{(11)} + \Omega_{CAB}^{(11)} - \Omega_{BAC}^{(11)})$, where $\Omega_{ABC}^{(11)}$ are the coefficients of anholonomicity. Following [11, 15] we use a pseudo-Gaussian (zero-shift) coordinate system $t, x^m$ and we accordingly decompose the elfbein $E^A_{(11)}$ in separate time and space parts as $E^a_{(11)} = N dt$, $E^a_{(11)} = e^a_{(10)m} dx^m$. We note that the zehnbein $E^a_{(11)} = e^a_{(10)}$ is related to the non-orthogonal, time-independent spatial frame $\theta^i(x) = \theta^i_m(x) dx^m$ used in [11] via $e^a_{(10)} = S^a_i \theta^i [15]$.

Using the $D = 11$ local supersymmetry to impose the relation $\psi^{(11)}_0 = \Gamma^a_0 \psi^{(11)}_a$, and defining $\mathcal{E}_a := N g^{1/4} \Gamma^a \hat{\mathcal{E}}_a$ (with $g^{1/2} = |e^a_{(10)m}|$), we find that the spatial components of the gravitino equation of motion (1), when expressed in terms of a rescaled $\psi^{(10)}_a := g^{1/4} \psi^{(11)}_a$, take the following form

$$\mathcal{E}_a = \partial_t \psi^{(10)}_a + \omega^{(11)}_{ab} \psi^{(10)}_b + \frac{1}{4} \psi^{(10)}_a \Gamma^{cd} \psi^{(10)}_{cd}$$

$$- \frac{1}{12} \Gamma^{(11)}_{abcd} \Gamma^{(10)d} \psi^{(10)}_a - \frac{1}{3} \Gamma^{(11)}_{tabc} \Gamma^{(10)c} + \frac{1}{6} \Gamma^{(11)}_{bcde} \Gamma^{(10)c}$$

$$+ \frac{N}{144} \Gamma^{(11)}_{abcd} \Gamma^{(10)d}$$

$$+ \frac{N}{9} \Gamma^{(11)}_{abcd} \Gamma^{(10)c} + \frac{N}{2} \Gamma^{(11)}_{abc} \Gamma^{(10)d} \psi^{(10)}_e$$

$$+ \frac{N}{4} \psi^{(11)}_{a} \Gamma^{(10)c} \Gamma^{(10)_d} - \frac{N}{4} \omega^{(11)}_{abc} \Gamma^{(10)c} \Gamma^{(10)_d}$$

$$+ N g^{1/4} \Gamma^{(10,d)} \left( 2 \partial_t \psi^{(11)}_b - \partial_t \psi^{(11)}_a - \frac{1}{2} \partial^{(11)}_a \psi^{(11)}_b + \omega^{(11)}_{00a} \psi^{(11)}_b + \frac{1}{2} \omega^{(11)}_{00a} \psi^{(11)}_a \right).$$

Lorentz (flat) indices are denoted $A, B, C, \ldots, F = 0, \ldots, 10$, while $a, b, \ldots, f = 1, \ldots, 10$ denote purely spatial Lorentz indices. We use the conventions of [1, 2] except for the replacement $\Gamma^{M}_{CJS} = +i \Gamma^{M}_{here}$ (linked to the mostly plus signature) which allows us to use real gamma matrices and real (Majorana) spinors. The definition of the Dirac conjugate is $\bar{\psi} := \psi^{T} \Gamma^{0}_{here}$, and thus differs from [1] by a factor of $i$. The field strength $F_{MNPQ}^{here}$ used in this letter is equal to $+1/2$ the one used in [11].
Refs. [11, 15] defined a dictionary between the temporal-gauge bosonic supergravity fields $g_{mn}^{(11)}(t, x), A_{mn}^{(11)}(t, x)$ (and their first spatial gradients: spatial connection and magnetic 4-form) and the four lowest levels $h_i^a(t), A_{ijk}(t), A_{i1...i_a}(t), A_{i0|i_1...i_8}(t)$ of the infinite tower of coordinates parametrising the coset manifold $E_{10}/K(E_{10})$. Here, we extend this dictionary to fermionic variables by showing that the rescaled, SUSY gauge-fixed gravitino field $\psi_{a}^{(10)}$ can be identified with the first rung of a ‘vector-spinor-type’ representation of $K(E_{10})$, whose Grassmann-valued representation vector will be denoted by $\Psi = (\psi_a, \psi_{...}, ...)^4$. We envisage $\Psi$ to be an infinite-dimensional representation of $K(E_{10})$ which is decomposed into a tower of $SO(10)$ representations, starting with a vector-spinor one $\psi_a$. Our labelling convention is that coset quantities, such as $A_{ijk}$ or $\Psi$ do not carry sub- or superscripts, whereas supergravity quantities carry an explicit dimension label.

We shall give several pieces of evidence in favour of this identification and of the consistency of this $K(E_{10})$ representation. As in the bosonic case, the correspondence $\psi_{a}^{(10)}(t, x) \leftrightarrow \psi_{a}^{\text{coset}}(t) \equiv \psi_a(t)$ is defined at a fixed, but arbitrary, spatial point $x$. A dynamical system governing a ‘massless spinning particle’ on $E_{10}/K(E_{10})$ will be presented as an extension of the coset dynamics of [11] and we will demonstrate the consistency of this dynamical system with the supergravity model under this correspondence. More precisely, we will first show how to consistently identify the Rarita–Schwinger equation (2) with a $K(E_{10})$-covariant equation

$$0 = \mathcal{D}^\text{vs} \Psi(t) := \left( \partial_t - \mathcal{Q}(t) \right) \Psi(t).$$

This equation expresses the parallel propagation of the vector-spinor-type ‘$K(E_{10})$ polarisation’ $\Psi(t)$ along the $E_{10}/K(E_{10})$ worldline of the coset particle. Our notation here is as follows. A one-parameter dependent generic group element of $E_{10}$ is denoted by $\mathcal{V}(t)$. The Lie algebra valued ‘velocity’ of $\mathcal{V}(t)$, namely $v(t) = \partial_t \mathcal{V} \mathcal{V}^{-1} \in \mathfrak{e}_{10} \equiv \text{Lie}(E_{10})$ is decomposed into its ‘symmetric’ and ‘antisymmetric’ parts according to $\mathcal{P}(t) := v_{\text{sym}}(t) := \frac{1}{2}(v(t) + v^T(t)), \mathcal{Q}(t) := v_{\text{anti}}(t) := \frac{1}{2}(v(t) - v^T(t))$, where the transposition

$^4$By contrast, [25] considered ‘Dirac-spinor-type’ representations of $K(E_{10})$.  

4
$(\cdot)^T$ is the generalised transpose of an $\mathfrak{e}_{10}$ Lie algebra element $x^T := -\omega(x)$ defined by the Chevalley involution $\omega$ [27]. $K(E_{10})$ is defined as the set of ‘orthogonal elements’ $k^{-1} = k^T$. Its Lie algebra $\mathfrak{e}_{10} = \text{Lie}(K(E_{10}))$ is made of all the antisymmetric elements of $\mathfrak{e}_{10}$, such as $Q$.

The bosonic coset model of [11] is invariant under a global $E_{10}$ right action and a local $K(E_{10})$ left action $\mathcal{V}(t) \rightarrow k(t)\mathcal{V}(t)g_0$. Under the local $K(E_{10})$ action, $\mathcal{P}$ varies covariantly as $\mathcal{P} \rightarrow kPk^{-1}$, while $\mathcal{Q}$ varies as a $K(E_{10})$ connection $\mathcal{Q} \rightarrow k\mathcal{Q}k^{-1} + \partial_t k k^{-1}$, with $\partial_t k k^{-1} \in \mathfrak{e}_{10}$ following from the orthogonality condition. The coset equation (3) will therefore be $K(E_{10})$ covariant if $\Psi$ varies, under a local $K(E_{10})$ left action, as a certain (‘vector-spinor’) linear representation

$$\Psi \rightarrow v^\mathfrak{v}_\Psi(k) \cdot \Psi$$

and if $\Psi$ in (3) is the value of $\mathcal{Q} \in \mathfrak{e}_{10}$ in the same representation $v^\mathfrak{v}_\Psi$. In order to determine the concrete form of $\Psi$ in the vector-spinor representation we need an explicit parametrisation of the coset manifold $E_{10}/K(E_{10})$.

Following [11, 15] we decompose the $E_{10}$ group w.r.t. its $\text{GL}(10)$ subgroup. Then the $\ell = 0$ generators of $\mathfrak{e}_{10}$ are $\mathfrak{gl}(10)$ generators $K^a_b$ satisfying the standard commutation relations $[K^a_b, K^c_d] = \delta^a_c K^a_d - \delta^a_d K^c_b$. The $\mathfrak{e}_{10}$ generators at levels $\ell = 1, 2, 3$ as $\text{GL}(10)$ tensors are, respectively, $E_{a_1a_2a_3}^{a_4a_5a_6} = E^{a_1a_2a_3}_{a_4a_5a_6}$, $E_{a_1...a_8}^{a_9} = E_{a_1...a_8}^{a_9}$, and $E_{a_0|a_1...a_8}^{a_9} = E_{a_0|a_1...a_8}^{a_9}$, where the $\ell = 3$ generator is also subject to $E_{a_0|a_1...a_8}^{a_9} = 0$. In a suitable (Borel) gauge, a generic coset element $\mathcal{V} \in E_{10}/K(E_{10})$ can be written as $\mathcal{V} = \exp(X_h) \exp(X_A)$ with

$$X_h = h^b_a K^a_b, \quad X_A = \frac{1}{3!} A_{a_1a_2a_3} E^{a_1a_2a_3} + \frac{1}{6!} A_{a_1...a_6} E^{a_1...a_6} + \frac{1}{9!} A_{a_0|a_1...a_8} E^{a_0|a_1...a_8} + \ldots.$$  

Defining $e^i_a := (\exp h)^{i}_a = \delta^i_a + h^i_a + \frac{1}{2!} h^i_a h^a + \ldots$ and $\bar{e}^a_{\ i} := (e^{-1})^{a\ i}$ one finds that the velocity $v \in \mathfrak{e}_{10}$ reads, expanded up to $\ell = 3$,

$$v = \bar{e}^b_i \partial_t e^i_a K^a_b + \frac{1}{3!} e^{i_1}_a e^{i_2}_a e^{i_3}_a A_{i_1i_2i_3} E^{a_1a_2a_3} + \frac{1}{6!} e^{i_1}_a e^{i_2}_a ... e^{i_6}_a A_{i_1...i_6} E^{a_1...a_6} + \frac{1}{9!} e^{i_1}_a e^{i_2}_a ... e^{i_8}_a A_{i_1...i_8} E^{a_0|a_1...a_8}.$$
Here, $DA_{i_1 i_2 i_3} = \partial_t A_{i_1 i_2 i_3}$, and the more complicated expressions for $DA_{i_1 \ldots i_6}$ and $DA_{i_0 | i_1 \ldots i_8}$ were given in [11]. In the expansion (6) of $v$ one can think of the indices on the generators $K^a_b$ etc. as flat (Euclidean) indices. As for the indices on $DA_{i_1 i_2 i_3}$ etc. the dictionary of [11, 15] shows that they correspond to a time-independent non-orthonormal frame $\theta^i = \theta^i_m dx^m$. The object $e^i_a = (\exp h)^i_a$ (which is the ‘square root’ of the contravariant ‘coset metric’ $g^{ij} = \sum_a e^i_a e^j_a$) relates the two types of indices, and corresponds to the inverse of the matrix $S_{ai}$ mentioned above. The parametrization (5) corresponds to a special choice of coordinates on the coset manifold $E_{10}/K(E_{10})$.

We introduce the $\mathfrak{k}_{10}$ generators through

$$\begin{align*}
J^{ab} &= K^a_b - K^b_a, \\
J^{a_1 \ldots a_6} &= E^{a_1 \ldots a_6} - F_{a_1 \ldots a_6}, \\
J^{a_0 | a_1 \ldots a_8} &= E^{a_0 | a_1 \ldots a_8} - F_{a_0 | a_1 \ldots a_8},
\end{align*}$$

where $F_{a_1 a_2 a_3} = (E^{a_1 a_2 a_3})^T$ etc., that is, with the general normalization $J = E - F$. Henceforth, we shall refer to $J^{ab}$, $J^{a_1 a_2 a_3}$, $J^{a_1 \ldots a_6}$, and $J^{a_0 | a_1 \ldots a_8}$ as being of ‘levels’ $\ell = 0, 1, 2, 3$, respectively. However, this ‘level’ is not a grading of $\mathfrak{k}_{10}$; rather one finds for commutators that $[\mathfrak{k}^{(\ell)}, \mathfrak{k}^{(\ell')}] \subset \mathfrak{k}^{(\ell + \ell')} \oplus \mathfrak{k}^{(|\ell - \ell|)}$ (in fact, $\mathfrak{k}_{10}$ is neither a graded nor a Kac–Moody algebra). Computing the antisymmetric piece $Q$ of the velocity $v$ we conclude that the explicit form of the fermionic equation of motion (3) is

$$\left( \partial_t - \frac{1}{2} e^{ib} \partial_t e^i_a \mathcal{V}^{ab} - \frac{1}{2} \frac{1}{3!} e^{i_1 a_1} \ldots e^{i_3 a_3} DA_{i_1 \ldots i_3} \mathcal{V}^{a_1 a_2 a_3} \\
- \frac{1}{2} \frac{1}{6!} e^{i_1 a_1} \ldots e^{i_6 a_6} DA_{i_1 \ldots i_6} \mathcal{V}^{a_1 \ldots a_6} \\
- \frac{1}{2} \frac{1}{9!} e^{i_0 a_0} \ldots e^{i_8 a_8} DA_{i_0 | i_1 \ldots i_8} \mathcal{V}^{a_0 | a_1 \ldots a_8} + \ldots \right) \Psi = 0.$$ 

Here, $\mathcal{V}^{ab} := R(J^{ab})$ etc. are the form the $\mathfrak{k}_{10}$ generators take in the sought-for vector-spinor representation $\Psi$. The crucial consistency condition for $\Psi$ to be a linear representation is that the generators $\mathcal{V}^{ab}$ etc. (to be deduced
below) should satisfy the abstract $\mathfrak{K}_{10}$ commutation relations

\[
\begin{align*}
[J^{ab}, J^{cd}] & = \delta^{bc} J^{ad} + \delta^{ad} J^{bc} - \delta^{ac} J^{bd} - \delta^{bd} J^{ac} \equiv 4\delta^{bc} J^{ad} \\
[J^{a_1a_2a_3}, J^{b_1b_2b_3}] & = J^{a_1a_2a_3b_1b_2b_3} - 18\delta^{a_1b_1}\delta^{a_2b_2} J^{a_3b_3} \\
[J^{a_1a_2a_3}, J^{b_1...b_6}] & = J^{[a_1a_2a_3]b_1...b_6} - 5! \delta^{a_1b_1} \delta^{a_2b_2} \delta^{a_3b_3} J^{b_4b_5b_6} \\
[J^{a_1...a_6}, J^{b_1...b_6}] & = -6 \cdot 6! \delta^{a_1b_1} \cdots \delta^{a_5b_5} J^{a_6b_6} + \ldots \\
[J^{a_1a_2a_3}, J^{b_0|b_1...b_8}] & = -336 (\delta^{b_0b_1b_2} J^{b_3...b_8} - \delta^{b_2b_1b_3} J^{b_4...b_9}) + \ldots \\
[J^{a_1...a_6}, J^{b_0|b_1...b_8}] & = -8! (\delta^{b_0b_1b_2b_3} J^{b_4b_5b_6} - \delta^{b_1b_2b_3} J^{b_4b_5b_6}) + \ldots \\
[J^{a_0|a_1...a_8}, J^{b_0|b_1...b_8}] & = -8 \cdot 8! (\delta^{a_0b_1...b_8} J^{a_2...a_8} - \delta^{a_1b_1...b_8} J^{a_0b_0} - \delta^{a_2b_1...b_8} J^{a_0b_0} \\
&\quad+ 8 \delta^{a_0b_0} \delta^{a_1b_1...b_7} J^{a_2b_6} + 7 \delta^{a_0b_0} \delta^{a_2b_2} J^{a_0a_7} J^{a_1b_8}) + \ldots
\end{align*}
\]

computed up to $\ell = 3$ in the basis for $\mathfrak{e}_{10}$ used in [15]. We use the flat Euclidean $\delta^{ab}$ of $SO(10)$ to raise and lower indices. As $SO(10)$ representation the generator $J^{a_0|a_1...a_8}$ is reducible with irreducible components $\tilde{J}$ and $\hat{J}$ defined by $\tilde{J}^{a_1|a_2...a_9} = J^{a_1|a_2...a_9} - \frac{8}{3} \delta^{a_1a_2} J^{a_3...a_9}$ and $\hat{J}^{a_3...a_9} = \delta^{a_1a_2} J^{a_3...a_9}$. Neglecting $\tilde{J}^{a_0|a_1...a_8}$, the corresponding commutators for $K(E_{11})$ were already computed in [21]. In eq. (9) we have used a shorthand notation where the terms on the r.h.s. should be antisymmetrised (with weight one) according to the antisymmetries on the l.h.s., as written out for the $SO(10)$ generators $J^{ab}$ in the first line. For the mixed symmetry generator $J^{a_0|a_1...a_8}$ this includes only antisymmetrisation over $[a_1 ... a_8]$. Under $SO(10)$ the tensors on the higher levels rotate in the standard fashion.

To compare eqs. (2) and (8) we now use the bosonic dictionary obtained in [11, 15]. In terms of our present conventions, and in terms of ‘flat’ indices on both sides this dictionary consists of asserting the correspondences

\[
\begin{align*}
e^i_a & \leftrightarrow \theta^i_m e^{m}_{(10) a}, & DA_{a_1a_2a_3} & \leftrightarrow 2F^{(11)}_{a_1a_2a_3} = 2NF^{(11)}_{0a_1a_2a_3}, \\
DA_{a_1...a_6} & \leftrightarrow -\frac{2}{4!} N\epsilon_{a_1...a_6b_1...b_4} F^{(11)}_{b_1...b_4}, & DA_{a_0|a_1...a_8} & \leftrightarrow \frac{3}{2} N\epsilon_{a_0a_1...a_8b_1b_2} \tilde{F}^{(10)}_{b_1b_2a_0}.
\end{align*}
\]

\footnote{To convert ‘frame’ indices $i, j, k, \ldots$ into ‘flat’ ones $a, b, c, \ldots$, one uses $e^i_a$ on the coset side, and $e^{(10) a} := \theta^i_m e^{m}_{(10) a} \equiv (S^{-1})^i_a$ on the SUGRA side.}
Here, as in [15], \( \tilde{\Omega}_{abc}^{(10)} = \Omega_{abc}^{(10)} - \frac{2}{9} \delta_{c[a} \Omega_{b]d}^{(10)} \) denotes the tracefree part of the spatial anholonomy coefficient \( \Omega_{abc}^{(10)} = 2 \epsilon_{(10)a}^m \epsilon_{(10)b}^n \theta_m e_c^{(10)n} \).

Using the correspondences (10), as well as their consequence \(- \frac{1}{2} (\epsilon^b \partial e^i_a - \epsilon^a \partial e^i_b) \leftrightarrow + \omega_{(11)}^{(1)} = N \omega_{0ab}^{(11)} \), we can tentatively re-interpret most terms in the supergravity equation (2) as terms in the putatively \( K(E_{10}) \) covariant equation (8). Using, as is always locally possible, a spatial frame such that the trace \( \omega_{b0c}^{(11)} = 0 \) (and therefore \( \tilde{\Omega}_{0bc}^{(10)} = \Omega_{0bc}^{(10)} \)), and neglecting, as in the bosonic case [11], the frame spatial derivatives \( \partial_a \psi_b^{(10)} \) and \( \partial_a N = -N \omega_{0ab}^{(11)} \), we can identify eq. (2) with eq. (8) if we define the action of \( K(E_{10}) \) generators in the vector-spinor representation by

\[
\begin{align*}
(J_{\Lambda}^{(0)} \psi)_a & := \Lambda_{ab} \psi^b + \frac{1}{4} \Lambda_{bc} \Gamma^{bc} \psi_a, \\
(J_{\Lambda}^{(1)} \psi)_a & := \frac{1}{12} \Lambda_{bcd} \Gamma^{bcd} \psi_a + \frac{2}{3} \Lambda_{abc} \Gamma^{b} \psi^c - \frac{1}{6} \Lambda_{bcd} \Gamma^{abc} \psi^d, \\
(J_{\Lambda}^{(2)} \psi)_a & := \frac{1}{1440} \Lambda_{bcdefg} \Gamma^{bcdefg} \psi_a + \frac{1}{180} \Lambda_{bcdefg} \Gamma_{abcdef} \psi^g - \frac{1}{72} \Lambda_{abcdef} \Gamma^{abcdef} \psi^f, \\
(J_{\Lambda}^{(3)} \psi)_a & := \frac{2}{3} \cdot \frac{1}{8!} \left( \Lambda_{bc1 \ldots 8} \Gamma_a^{c1 \ldots c8} \psi_b^c + 8 \Lambda_{a[1 \ldots 8} \Gamma_a^{c1 \ldots c7} \psi_{c8} \\
& + 2 \Lambda_{b[bc1 \ldots 7} \Gamma_a^{c1 \ldots c7} \psi_{c8} - 28 \Lambda_{b[bc1 \ldots 7} \Gamma_a^{c1 \ldots c6} \psi_{c7} \right),
\end{align*}
\]
algebra. The result of this computation is

\[
\begin{align*}
\left( \left[ J^v_J(1), J^v_J(1) \right] \Psi \right)_a &= 20 \left( J^v_J(2) \Psi \right)_a - \left( J^v_J(0) \Psi \right)_a, \\
\left( \left[ J^v_J(1), J^v_J(2) \right] \Psi \right)_a &= 56 \left( J^v_J(3) \Psi \right)_a - \frac{1}{6} \left( J^v_J(1) \Psi \right)_a,
\end{align*}
\]

where the \( J^v_J(\ell) \) are defined as above, but now with new parameters given by

\[
\Sigma_{ab}^{(0)} = \Lambda_{d_1d_2[a} \Lambda'_{b]} d_1d_2, \quad \Sigma_{b_1...b_6}^{(2)} = \Lambda_{[b_1b_2b_3} \Lambda'_{b_4b_5b_6]}, \quad \Sigma_{a_1a_2a_3}^{(1)} = \Lambda_{b_1b_2b_3} \Lambda'_{b_4b_5b_6} a_1a_2a_3,
\]

and \( \Sigma_{a_0|a_1...a_8}^{(3)} = \Lambda_{a_0[a_1a_2} \Lambda'_{a_3...a_8]} - \Lambda_{a_0[a_1a_2} \Lambda'_{a_3...a_8]a_0} \). One can now check that the relations (12) are consistent with the \( K(E_{10}) \) commutators (9). All other commutators have to produce terms on the r.h.s. which have contributions of ‘level’ \( \ell > 3 \) and therefore cannot be checked fully. However, we have verified, where possible, that the expected contributions of the lower levels appear with the correct normalisation required by the structure constants of (9). Therefore we find that the vector-spinor representation \( J^v_J(\ell) \) of \( K(E_{10}) \) which we deduced from comparing (2) and (8) is a good linear representation up to the level we have supergravity data to define it.

Using arguments from the the general representation theory of Lie algebras one can actually show that the checks we have carried out are sufficient to guarantee the existence of an extension of the vector-spinor representation \( J^v_J(\ell) \) to ‘levels’ \( \ell > 3 \) on the same components \( \psi_a \). That is, we can define on \( \psi_a \) alone an unfaithful, irreducible 320-dimensional representation of \( K(E_{10}) \) on which infinitely many \( K(E_{10}) \) generators are realised non-trivially. For this definition it is sufficient to define the action of \( J^v_J(0) \) and \( J^v_J(1) \) on \( \psi_a \) and check Serre-type compatibility conditions [28]. We view the fact that the \( J^v_J(2) \) and \( J^v_J(3) \) transformations deduced from the supergravity correspondence above agree with this general construction as strong evidence for the relevance of the vector-spinor component of the infinite-dimensional \( K(E_{10}) \) spinor \( \Psi = (\psi_a, \ldots) \) we have in mind. If one repeats the same analysis for the Dirac spinor, where the representation matrices on this 32-dimensional space are given in terms of anti-symmetric \( \Gamma \)-matrices (see (16) below), one finds that one can consistently realise \( K(E_{10}) \) on a 32-component spinor of \( SO(10) \). The fact that the anti-symmetric \( \Gamma \)-matrices together with \( \Gamma^0 \) span
the fundamental representation of $SO(32)$ has led a number of authors to propose $SO(32)$ as a ‘generalised holonomy’ for M-theory [29, 30]. That this group, like the larger group $SL(32)$ proposed in [31], cannot be realised as a *bona fide* symmetry was subsequently pointed out in [32] where it was shown that no suitable spinor (i.e. double valued) representation with the correct number of components of these generalised holonomy groups exist. Our approach is radically different, since we have an action not of $SO(32)$ but of $K(E_{10})$, with infinitely many generators acting in a non-trivial manner, on a *bona fide* spinor representation of $SO(10)$. We therefore evade the conclusions of [32].\(^6\) The appearance of an unfaithful representation for the fermions was already noted and studied in the affine case for $K(E_9)$, which shows very similar features consistent with our present findings [34]. One possibility to construct a faithful representation of $K(E_{10})$ already pointed out there might be to consider the tensor product of such unfaithful representations with a faithful representation, like the adjoint $\mathfrak{t}_{10}$ or the coset $\mathfrak{e}_{10} \ominus \mathfrak{k}_{10}$.\(^7\) More details on these aspects will be given in a future publication [35].

A deeper confirmation of the hidden $K(E_{10})$ symmetry of SUGRA\(_{11}\) is obtained by writing down a $K(E_{10})$ invariant action functional describing a massless spinning particle on $E_{10}/K(E_{10})$. We will be brief and defer the details to [35]. The bosonic part of the action is the one of [11]

$$S_{\text{bos}} = \int dt \frac{1}{4n} \langle P(t)|P(t)\rangle$$

(13)

where $\langle \cdot | \cdot \rangle$ is the standard invariant bilinear form on $\mathfrak{e}_{10}$ [27] and where the coset ‘lapse’ function $n$ can be identified with the rescaled supergravity lapse

\(^6\)The transition from $SO(10)$ to $SO(32)$ (or $SO(1,10)$ to $SL(32)$) requires $\Gamma^{abc}$ which is associated with the rank three gauge field. The importance of the rank three generator in the context of M5-brane dynamics was already stressed in [33] and also features in [21] where it is seen as part of $K(E_{11})$. However, it is an open question whether there exists a vector-spinor-type representation of $K(E_{11})$, which would be analogous to (11) and thus also compatible with [32].

\(^7\)Let us also note that the 320-dimensional representation of $K(E_{10})$ is compatible with the fermionic representations studied in [13].
\( N g^{-1/2} \) (denoted \( \tilde{N} \) in [12]).

The fermionic term we add to this action reads

\[
S_{\text{term}} = -\frac{i}{2} \int dt \left( \Psi(t) \mid vs \mathcal{D} \Psi(t) \right)_{\text{vs}},
\]

where \((\cdot|\cdot)_{\text{vs}}\) is a \( K(E_{10}) \) invariant symmetric form on the vector-spinor representation space. Observe that this symmetric form is actually \textit{anti}-symmetric when evaluated on \textit{anti}-commuting (Grassmann valued) fermionic variables \( \Psi(t) \), such that e.g. \( (\Psi(t)|\Psi(t))_{\text{vs}} = 0 \). On the lowest component of \( \Psi = (\psi_a, \ldots) \) it is explicitly given by \( (\Psi|\Phi)_{\text{vs}} = \psi_a^T \Gamma^{ab} \phi_b \). The invariance of this form under the generators \( J^{(\ell)} \) defined in (11) is a quite restrictive condition. We have verified that invariance holds, but only since we are working over a \textit{ten-dimensional} Clifford algebra. By using induction arguments we find that \( (\Psi|\Phi)_{\text{vs}} \) is invariant not only under (11) but under the (unfaithful) extension to the \textit{full} \( K(E_{10}) \) transformations mentioned above. We expect that the form \( (\Psi|\Phi)_{\text{vs}} \) will extend to an invariant symmetric form on a \textit{faithful} representation \( \Psi = (\psi_a, \ldots) \).

Further important hints of a hidden \( K(E_{10}) \) symmetry come from considering the local SUSY constraint \( S^{(11)} = 0 \) which is proportional to the time component of the Rarita Schwinger equation (1). First, we find that, under the dictionary of [11, 15], \( S^{(11)} \) is mapped into a \( K(E_{10}) \) covariant constraint of the form \( P \odot \Psi = 0 \), when neglecting frame gradients \( \partial_a \psi_b \) as we have done in the derivation of (11). The product \( \odot \) symbolises a map from the tensor product of \( e_{10} \odot e_{10} \) with \( \Psi \) onto a Dirac-spinor-type representation space of \( e_{10} \). The coset constraint \( P \odot \Psi = 0 \) suggests to augment the action \( S_{\text{bos}} + S_{\text{term}} \) by a ‘Noether’ term of the form

\[
S_{\text{Noether}} = \int dt \left( \chi(t) | P(t) \odot \Psi(t) \right)_s,
\]

with a local Dirac-spinor \( \chi(t) \) Lagrange multiplier (that is, a one-dimensional ‘gravitino’). The total action \( S_{\text{bos}} + S_{\text{term}} + S_{\text{Noether}} \) is expected to be not only invariant under \( K(E_{10}) \), but also (disregarding \( \Psi^4 \) terms) under time-dependent supersymmetry transformations which involve a Dirac-spinor-type \( K(E_{10}) \) representation \( \epsilon(t) \). In this case the \( \chi = 0 \) gauge fixed action will
be invariant under residual quasi-rigid supersymmetry transformations constrained to satisfy \( \bar{s} \mathcal{D} \epsilon(t) \equiv (\partial_t - \bar{Q}) \epsilon = 0 \). This equation is formally the same as (3) and (8) but now the generators are found to be (cf. [25])

\[
\begin{align*}
\bar{s} J^{ab} &= \frac{1}{2} \Gamma^{ab}, \\
\bar{s} J^{a_1 a_2 a_3} &= \frac{1}{2} \Gamma^{a_1 a_2 a_3}, \\
\bar{s} J^{a_1 \ldots a_6} &= \frac{1}{2} \Gamma^{a_1 \ldots a_6}, \\
\bar{s} J^{a_0 | a_1 \ldots a_8} &= 12 \delta^{a_1 \ldots a_8}_{a_0 b_1 \ldots b_7} \gamma^{b_1 \ldots b_7}. 
\end{align*}
\] (16)

The particular form of the Dirac-spinor representation on \( \ell = 3 \) implies that the irreducible component \( \bar{s} J^{a_0 | a_1 \ldots a_8} \) is mapped to zero under this correspondence: indeed, there is no way to represent a non-trivial Young tableau purely in terms of gamma matrices. This is in contrast to the vector-spinor representation (11).

In summary, we have given evidence for the following generalisation of the correspondence conjectured in [11]: The time evolution of the eleven-dimensional supergravity fields \( g_{MN}^{(11)}(t, \mathbf{x}), A_{MNP}^{(11)}(t, \mathbf{x}), \psi_M^{(11)}(t, \mathbf{x}) \) and their spatial gradients (considered around any given spatial point \( \mathbf{x} \), in temporal gauge and with fixed SUSY gauge) can be mapped onto the dynamics of a (supersymmetric) spinning massless particle \((\mathcal{V}(t), \Psi(t))\) on \( E_{10}/K(E_{10}) \). The \( E_{10} \)-invariant quantum dynamics of this superparticle might provide the basis of a new definition of M-theory. Much work remains to be done to extend the evidence indicated here, for instance by proving the existence of irreducible faithful (and hence infinite-dimensional) ‘vector-spinor-type’ and ‘Dirac-spinor-type’ representations of \( K(E_{10}) \).

Let us finally note on the physical side, that we deem it probable that the proposed correspondence between M-theory and the coset model is such that the two sides do not have a common range of physical validity: Indeed, the coset model description emerges in the near space-like singularity limit \( T \to 0 \), where \( T \) denotes the proper time\(^8\), which indicates that the coset description might be well defined only when \( T \ll T_{\text{Planck}} \), i.e. in a strong curvature regime where the spacetime description ‘de-emerges’.

\(^8\)The coordinate and ‘coset time’ \( t \) used above is (in the gauge \( n = 1 \)) roughly proportional to \(-\log T\), and actually goes to \(+\infty\) near the space-like singularity.
Acknowledgements
We thank Ofer Gabber and Pierre Vanhove for informative discussions and Bernard Julia for clarifying the conventions of [1, 2]. AK and HN gratefully acknowledge the hospitality of IHES during several visits. This work was partly supported by the European Research and Training Networks ‘Superstrings’ (contract number MRTN-CT-2004-512194) and ‘Forces Universe’ (contract number MRTN-CT-2004-005104).

References

[1] E. Cremmer, B. Julia and J. Scherk, Supergravity theory in 11 dimensions, Phys. Lett. B 76 (1978) 409–412

[2] E. Cremmer and B. Julia, The SO(8) Supergravity, Nucl. Phys. B 159 (1979) 141

[3] B. Julia, Application of supergravity to gravitation theories, in: Unified Field Theories of more than 4 Dimensions, eds. V. De Sabbata and E. Schmutzer, World Scientific (Singapore, 1983)

[4] N. Marcus and J. H. Schwarz, Three-Dimensional Supergravity Theories, Nucl. Phys. B 228 (1983) 145

[5] H. Nicolai, D = 11 Supergravity With Local SO(16) Invariance, Phys. Lett. B 187 (1987) 316

[6] B. Julia, Group Disintegrations, in: S. W. Hawking and M. Roček (Eds.), Superspace and Supergravity, Proceedings of the Nuffield Workshop, Cambridge, Eng., Jun 22 – Jul 12, 1980, Cambridge University Press (Cambridge, 1981)

[7] H. Nicolai, The Integrability Of N = 16 Supergravity, Phys. Lett. B 194 (1987) 402

[8] B. Julia, in: Lectures in Applied Mathematics, Vol. 21 (1985), AMS-SIAM, p. 335; preprint LPTENS 80/16

[9] V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, Oscillatory Approach To A Singular Point In The Relativistic Cosmology, Adv. Phys. 19 (1970) 525
[10] T. Damour and M. Henneaux, $E_{10}$, $B E_{10}$ and arithmetical chaos in superstring cosmology, Phys. Rev. Lett. 86 (2001) 4749, hep-th/0012172

[11] T. Damour, M. Henneaux and H. Nicolai, $E_{10}$ and a "small tension expansion" of M-theory, Phys. Rev. Lett. 89 (2002) 221601, hep-th/0207267

[12] T. Damour, M. Henneaux and H. Nicolai, Cosmological Billiards, Class. Quant. Grav. 20 (2003) R145–R200, hep-th/0212256

[13] A. Kleinschmidt and H. Nicolai, $E_{10}$ and $SO(9,9)$ invariant supergravity, JHEP 0407 (2004) 041, hep-th/0407101

[14] A. Kleinschmidt and H. Nicolai, IIB supergravity and $E_{10}$, Phys. Lett. B 606 (2005) 391, hep-th/0411225

[15] T. Damour and H. Nicolai, Eleven dimensional supergravity and the $E_{10}/K(E_{10})$ $\sigma$-model at low $A_9$ levels, in: Group Theoretical Methods in Physics, Institute of Physics Conference Series No. 185, IoP Publishing, 2005, hep-th/0410245

[16] T. Damour and H. Nicolai, Higher order M theory corrections and the Kac-Moody algebra $E_{10}$, Class. Quant. Grav. 22 (2005) 2849, hep-th/0504153

[17] P. C. West, Hidden superconformal symmetry in M theory, JHEP 0008 (2000) 007, hep-th/0005270

[18] P. C. West, $E_{11}$ and M theory, Class. Quant. Grav. 18 (2001) 4443–4460, hep-th/0104081

[19] I. Schnakenburg and P. C. West, Kac-Moody symmetries of IIB supergravity, Phys. Lett. B 517 (2001) 421, hep-th/0107181

[20] I. Schnakenburg and P. C. West, ‘Massive IIA supergravity as a non-linear realisation, Phys. Lett. B 540 (2002) 137, hep-th/0204207

[21] P. C. West, $E_{11}$, $SL(32)$ and central charges, Phys. Lett. B 575 (2003) 333–342, hep-th/0307098

[22] F. Englert and L. Houart, $G^{+++}$ invariant formulation of gravity and M-theories: exact BPS solutions, JHEP 0401 (2004) 002, hep-th/0311255
[23] F. Englert and L. Houart, $G^{+++}$ invariant formulation of gravity and M-theories: Exact intersecting brane solutions, JHEP 0405 (2004) 059, hep-th/0405082

[24] F. Englert, M. Henneaux and L. Houart, From very-extended to overextended gravity and M-theories, JHEP 0502 (2005) 070, hep-th/0412184

[25] S. de Buyl, M. Henneaux and L. Paulot, Hidden symmetries and Dirac fermions, Class. Quant. Grav. 22 (2005) 3595, hep-th/0506009

[26] S. de Buyl, M. Henneaux and L. Paulot, Extended $E_8$ invariance of 11-dimensional supergravity, hep-th/0512292

[27] V. G. Kac, Infinite dimensional Lie algebras, 3rd edition, Cambridge University Press (Cambridge, 1990)

[28] S. Berman, On generators and relations for certain involutory subalgebras of Kac-Moody Lie Algebras, Commun. Alg. 17 (1989) 3165–3185

[29] M. J. Duff and K. S. Stelle, Multimembrane solutions of $D=11$ supergravity, Phys. Lett. B 253 (1991) 113–118

[30] M. Duff and J. T. Liu, Hidden space-time symmetries and generalized holonomy in M theory, Nucl. Phys. B 674 (2003) 217–230, hep-th/030140

[31] C. Hull, Holonomy and symmetry in M theory, hep-th/0305039

[32] A. Keurentjes, The topology of U Duality (sub)groups, Class. Quantum. Grav. 21 (2004) 1695-1708, hep-th/0309106

[33] O. Bärwald and P. C. West, Brane rotating symmetries and the five-brane equations of motion, Phys. Lett. B 476 (2000) 157–164, hep-th/9912226

[34] H. Nicolai and H. Samtleben, On $K(E_9)$, Q. J. Pure Appl. Math. 1 (2005) 180, hep-th/0407055

[35] T. Damour, A. Kleinschmidt and H. Nicolai, in preparation.