Chern Classes and Lie-Rinehart Algebras

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Abstract. Let $A$ be a $F$-algebra where $F$ is a field, and let $W$ be an $A$-module of finite presentation. We use the linear Lie-Rinehart algebra $V_W$ of $W$ to define the first Chern-class $c_1(W)$ in $H^2(V_W|_U, O_U)$, where $U$ in Spec$(A)$ is the open subset where $W$ is locally free. We compute explicitly algebraic $V_W$-connections on maximal Cohen-Macaulay modules $W$ on the hypersurface-singularities $B_{mn} = x^m + y^n + z^2$, and show that these connections are integrable, hence the first Chern-class $c_1(W)$ vanishes. We also look at indecomposable maximal Cohen-Macaulay modules on quotient-singularities in dimension 2, and prove that their first Chern-class vanish.

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Introduction

Classically, the Chern-classes of a locally free coherent $A$-module $W$ are defined using the curvature $R_V$ of a connection $\nabla : W \to W \otimes \Omega^1_X$. A connection $\nabla$ gives rise to a covariant derivation $\nabla : \text{Der}_F(A) \to \text{End}_F(W)$. If we more generally consider a coherent $A$-module $W$, a connection $\nabla$ might not exist. In this paper we will consider the problem of defining Chern-classes for $A$-modules $W$ using a covariant derivation defined on a certain sub-Lie-algebra $V_W$ of $\text{Der}_F(A)$: For an arbitrary $A$-module $W$ there exists a sub-Lie-algebra $V_W$ of $\text{Der}_F(A)$ called the linear Lie-Rinehart algebra, and also the notion of a $V_W$-connection. There exists a complex, the Chevalley-Hochschild complex $C^\bullet(V_W, W)$ for the $A$-module $W$ with a flat $V_W$-connection, generalizing the classical deRham-complex. If $A$ is a regular ring, the derivation module $\text{Der}_F(A)$ is locally free, and it follows that the complex $C^\bullet(\text{Der}_F(A), A)$ is quasi-isomorphic to the complex $\Omega^\bullet_{A/F}$, hence the Chevalley-Hochschild complex of $\text{Der}_F(A)$ can be used to compute the algebraic...
deRham-cohomology of $A$. The complex $C^\bullet(V_W, A)$ generalizes simultaneously the algebraic deRham-complex and the Chevalley-Eilenberg complex. This is due to $\mathbb{22}$. A natural thing to do is to investigate possibilities of defining Chern-classes of $A$-modules equipped with a $V_W$-connection, generalizing the classical Chern-classes defined using the curvature of a connection. Invariants for Lie-Rinehart algebras have been considered by several authors (see $\mathbb{8}$, $\mathbb{10}$ and $\mathbb{14}$), however invariants for $A$-modules with a $V_W$-connection, where $V_W$ is the linear Lie-Rinehart algebra of $W$ does not appear to be treated in the litterature and that is the aim of this work. In this paper we also develop techniques to do explicit calculations of Chern-classes of maximal Cohen-Macaulay modules on hypersurface singularities and two-dimensional quotient singularities.

We define for any $F$-algebra $A$ where $F$ is any field, and any $A$-module $W$ which is locally free on an open subset $U$ of $\text{Spec}(A)$, the first Chern-class $c_1(W)$ in $H^2(V_W|_U, \mathcal{O}_U)$, where $H^2(V_W|_U, \mathcal{O}_U)$ is the Chevalley-Hochschild cohomology of the restricted linear Lie-Rinehart algebra $V_W$ of $W$, with values in the sheaf $\mathcal{O}_U$. This is Theorem $\mathbb{3.2}$. We also prove in Theorem $\mathbb{2.1}$ existence of explicit $V_W$-connections $\nabla^{\psi, \phi}$ on a class of maximal Cohen-Macaulay modules $W$ on the Brieskorn singularities $B_{m,n,2}$, which in fact are defined over any field $F$ of characteristic prime to $m$ and $n$. We prove in Theorem $\mathbb{3.3}$ that the $V_W$-connections defined in Theorem $\mathbb{2.1}$ are all regular, hence the first Chern-class is zero. Finally we prove in Theorem $\mathbb{4.2}$ that for any maximal Cohen-Macaulay module $W_\rho$ on any two-dimensional quotient-singularity $C^2/G$, the first Chern class $c_1(W_\rho)$ is zero.

1. **Kodaira-Spencer maps and linear Lie-Rinehart algebras**

Let $A$ be an $F$-algebra, where $F$ is any field, and let $W$ be an $A$-module. In this section we use the Kodaira-Spencer class and the Kodaira-Spencer map to define the linear Lie-Rinehart algebra $V_W$ of $W$, and the obstruction $l_c(W)$ in $\text{Ext}^1_A(V_W, \text{End}_A(W))$ for existence of a $V_W$-connection on $W$.

**Definition 1.1.** Let $P$ be an $A$-bimodule. The *Hochschild-complex* of $A$ with values in $P$ $C^\bullet(A, P)$ is defined as follows:

$$C^p(A, P) = \text{Hom}_F(A^p, P)$$

with differentials $d^p : C^p(A, P) \to C^{p+1}(A, P)$ defined by

$$d^p\phi(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}) = a_1\phi(a_2 \otimes \cdots \otimes a_{p+1}) + \sum_{1 \leq i \leq p} (-1)^i\phi(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{p+1}) + (-1)^{p+1}\phi(a_1 \otimes \cdots \otimes a_p)a_{p+1}.$$  

We adopt the convention that $C^0(A, P) = P$, and $d^0(p)(a) = pa - ap$ for all $p$ in $P$, and $a$ in $A$. The $i$'th cohomology $H^i(C^\bullet(A, P))$ is denoted $HH^i(A, P)$.

There exists an exact sequence

$$0 \to HH^0(A, P) \to P \to \text{Der}_F(A, P) \to HH^1(A, P) \to 0,$$

and it is well known that if we let $P = \text{Hom}_F(W, V)$ where $W$ and $V$ are $A$-modules, then $HH^i(A, P)$ equals $\text{Ext}^i_A(W, V)$. Recall the definition of a *connection* on $W$: it is an $F$-linear map $\nabla : W \to W \otimes \Omega^1_{A/F}$, with the property that $\nabla(aw) = a\nabla(w) + w \otimes da$, where $d : A \to \Omega^1_{A/F}$ is the *universal derivation*. Put $P = \text{Hom}_F(W, W \otimes \Omega^1_{A/F})$.
in 11.1 and construct an element \( C \) in \( \text{Der}_F(A, \text{Hom}_F(W, W \otimes \Omega^1_{A/F})) \) in the following way: \( C(a)(w) = w \otimes da \).

**Definition 1.2.** The class \( C = \text{ks}(W) \) in \( \text{Ext}^1_A(W, W \otimes \Omega^1_{A/F}) \) is the **Kodaira-Spencer class** of \( W \).

Note that \( \text{ks}(W) \) is also referred to as the **Atiyah-class** of \( W \).

**Proposition 1.3.** Let \( A \) be any \( F \)-algebra, and let \( W \) be any \( A \)-module, then \( \text{ks}(W) = 0 \) if and only if \( W \) has a connection.

**Proof.** We see that \( \text{ks}(W) = 0 \) if and only if there exists an element \( \nabla : W \to W \otimes \Omega^1_{A/F} \) with the property that \( d^W \nabla = C \). This is if and only if

\[
(\nabla a - a \nabla)(w) = \nabla(aw) - a \nabla(w) = C(a)(w) = w \otimes da,
\]

hence \( \nabla \) is a connection, and the claim follows. \( \square \)

**Definition 1.4.** Let \( A \) be an \( F \)-algebra where \( F \) is any field. A **Lie-Rinehart algebra** on \( A \) is a \( F \)-Lie-algebra and an \( A \)-module \( g \) with a map \( \alpha : g \to \text{Der}_F(A) \) satisfying the following properties:

\[
\begin{align*}
(1.4.1) & \quad \alpha(a\delta) = a\alpha(\delta) \\
(1.4.2) & \quad \alpha([\delta, \eta]) = [\alpha(\delta), \alpha(\eta)] \\
(1.4.3) & \quad [\delta, \alpha \eta] = a[\delta, \eta] + \alpha(\delta)(a)\eta
\end{align*}
\]

for all \( a \in A \) and \( \delta, \eta \in g \). Let \( W \) be an \( A \)-module. A **\( g \)-connection** \( \nabla \) on \( W \), is an \( A \)-linear map \( \nabla : g \to \text{End}_F(W) \) which satisfies the **Leibniz-property**, i.e.

\[
\nabla(\delta)(aw) = a\nabla(\delta)(w) + \alpha(\delta)(a)w
\]

for all \( a \in A \) and \( w \in W \). We say that \( (W, \nabla) \) is a **\( g \)-module** if \( \nabla \) is a homomorphism of Lie-algebras. The **curvature** of the \( g \)-connection, \( R_\nabla \) is defined as follows:

\[
R_\nabla(\delta \wedge \eta) = [\nabla \delta, \nabla \eta] - \nabla[\delta, \eta].
\]

**Example 1.5.** Any connection \( \nabla \) on \( W \), gives an action \( \nabla : \text{Der}_F(A) \to \text{End}_F(W) \) with the property that \( \nabla(\delta)(aw) = a\nabla(\delta)(w) + \delta(a)w \) for any \( \delta \) in \( \text{Der}_F(A) \), \( a \) in \( A \) and \( w \) in \( W \). The Lie-algebra \( g = \text{Der}_F(A) \) is trivially a Lie-Rinehart algebra, hence \( W \) has a \( g \)-connection.

Given any Lie-algebroid \( g \), and any \( A \)-module \( W \) with a \( g \)-connection, the set of \( g \)-connections on \( W \) is a **torsor** on the set \( \text{Hom}_A(g, \text{End}_A(W)) \). Put \( P = \text{Hom}_F(W, W) \) in 1.1.1 and define for all \( \delta \) in \( \text{Der}_F(A) \) the following element \( C(\delta) \) in \( \text{Der}_F(A, \text{Hom}_F(W, W)) \): \( C(\delta)(a)(m) = \delta(a)m \). We get an \( A \)-linear map

\[
C : \text{Der}_F(A) \to \text{Der}_F(A, \text{Hom}_F(W, W)).
\]

**Definition 1.6.** Let \( A \) be any \( F \)-algebra, and let \( W \) be any \( A \)-module. We define the **Kodaira-Spencer map**

\[
g : \text{Der}_F(A) \to \text{Ext}^1_A(W, W)
\]

as follows: \( g(\delta) = C(\delta) \) in sequence 1.1.1. We let \( \text{ker}(g) = V_W \) be the **linear Lie-Rinehart algebra** of \( W \).
One immediately checks that the $A$-sub module $V_W$ of $\text{Der}_F(A)$ satisfies the axioms of definition 1.4, hence $V_W$ is indeed a Lie-Rinehart algebra.

**Proposition 1.7.** Let $A$ be any $F$-algebra and $W$ any $A$-module. There exists an $F$-linear map

$$\rho : V_W \to \text{End}_F(W)$$

with the property that $\rho(\delta)(aw) = a\rho(\delta)(w) + \delta(a)w$ for all $\delta$ in $V_W$, $a$ in $A$ and $w$ in $W$.

**Proof.** Assume that $g(\delta) = 0$. Then there exists a map $\rho(\delta)$ in $\text{Hom}_F(W,W)$ with the property that $d^0 \rho(\delta) = C(\delta)$. This is if and only if $\rho(\delta)(aw) = a\rho(\delta)(w) + \delta(a)w$, hence for all $\delta$ in $V_W$ we get a map $\rho(\delta)$, and the assertion follows. $\square$

Given any $A$-module $W$, we now pick any map $\rho : V_W \to \text{End}_F(W)$ with the property that $\rho(\delta)(aw) = a\rho(\delta)(w) + \delta(a)w$, which exists by proposition 1.7. Put $P = \text{Hom}_F(V_W, \text{End}_A(W))$ in sequence 1.1.1 and consider the element $L$ in $\text{Der}_F(A,P)$ defined as follows: $L(a)(\delta)(w) = a\rho(\delta)(w) - \rho(\eta\delta)(w)$.

**Definition 1.8.** Let $lc(W) = \mathcal{L}$ in $\text{HH}^1(A,P) = \text{Ext}_A^1(V_W, \text{End}_A(W))$.

One verifies that the class $lc(W)$ is independent of choice of map $\rho$, hence it is an invariant of $W$.

**Theorem 1.9.** Let $A$ be any $F$-algebra, and $W$ any $A$-module, then $lc(W) = 0$ if and only if $W$ has a $V_W$-connection.

**Proof.** Assume $lc(W) = 0$. Then there exists a map $\eta$ in $\text{Hom}_F(V_W, \text{End}_A(W))$ with the property that $d^0 \rho(\delta) = L$. Then the map $\rho + \eta = \nabla : V_W \to \text{End}_A(W)$ is a $V_W$-connection, and the assertion follows. $\square$

From a groupoid in schemes (roughly speaking an algebraic stack, see [16]) one constructs a Lie-Rinehart algebra in a way similar to the way one constructs the Lie-algebra from a group-scheme. A natural problem is to find necessary and sufficient criteria for the linear Lie-Rinehart algebra to be integrable to a groupoid in schemes.

Note that for any $A$-submodule and $k$-sub-Lie algebra $g$ of $\text{Der}_k(A)$, there exists a generalized universal enveloping algebra $\text{U}(A,g)$ which is a sub-algebra of $\text{D}(A)$, where $\text{D}(A)$ is the ring of differential-operators of $A$. The algebra $\text{U}(A,g)$ has the property that there is a one-to-one correspondence between $A$-modules with a flat $g$-connection and left $\text{U}(A,g)$-modules. There exists a generalized PBW-theorem for the algebra $\text{U}(A,g)$ when $g$ is a projective $A$-module (see [23]). The dual algebra $\text{U}(A,g)^\ast$ is commutative and its spectrum $\text{Spec}(\text{U}(A,g)^\ast)$ is a formal equivalence-relation in schemes. Note also that all constructions in this section globalize.

2. **Explicit examples: Algebraic $V_W$-connections**

In this section we apply the theory developed in the previous section to compute explicitly algebraic $V_W$-connections on a class of maximal Cohen-Macaulay modules on isolated hypersurface singularities $B_{mn2} = x^m + y^n + z^2$. Let in the following $F$ be a field of characteristic zero, and $A = F[[x,y,z]]/x^m + y^n + z^2$. We are interested in maximal Cohen-Macaulay modules on $A$, and such modules have
a nice description, due to [6]: Consider the two matrices

\[
\phi = \begin{pmatrix}
x^{m-k} & y^{n-l} & 0 & z \\
y' & -x_k & z & 0 \\
z & 0 & -y^{n-l} & -x_k \\
0 & z & x^{m-k} & -y'
\end{pmatrix}
\]

and

\[
\phi = \begin{pmatrix}
x^k & y^{n-l} & z & 0 \\
y' & -x_{m-k} & 0 & z \\
z & 0 & -y' & x^k \\
0 & 0 & -x_{m-k} & -y^{n-l}
\end{pmatrix},
\]

where \(1 \leq k \leq m\) and \(1 \leq l \leq n\). Let \(f\) be the polynomial \(x^m + y^n + z^2\). The matrices \(\phi\) and \(\psi\) have the property that \(\phi\psi = \psi\phi = f I\) where \(I\) is the rank 4 identity matrix. Hence we get a complex of \(A\)-modules

\[
\cdots \rightarrow \psi P \rightarrow \phi P \rightarrow \psi P \rightarrow \phi P \rightarrow W(\phi, \psi) \rightarrow 0.
\]

Note that the sequence \((2.0.1)\) is a complex since \(\phi\psi = \psi\phi = f I = 0\). By [6], the module \(W = W(\phi, \psi)\) is a maximal Cohen-Macaulay module on \(A\). The ordered pair \((\phi, \psi)\) is a matrix-factorization of the polynomial \(f\). We want to compute explicitly algebraic \(V_W\)-connections on the modules \(W = W(\phi, \psi)\) for all \(1 \leq k \leq m\) and \(1 \leq l \leq n\), i.e. we want to give explicit formulas for \(A\)-linear maps \(\nabla^{\phi, \psi} = \nabla : V_W \rightarrow \text{End}_F(W)\) satisfying \(\nabla(\delta)(aw) = a\nabla(\delta)(w) + \delta(a)w\) for all \(a\) in \(A\), \(w\) in \(W\) and \(\delta\) in \(V_W\). Hence first we have to compute generators and syzygies of the derivation-modules \(\text{Der}_F(A)\). A straight-forward calculation shows that \(\text{Der}_F(A)\) is generated by the derivations

\[
\delta_0 = 2nx\partial_x + 2my\partial_y + mnz\partial_z
\]

\[
\delta_1 = mx^{m-1}\partial_y - ny^{n-1}\partial_z
\]

\[
\delta_2 = -2z\partial_x + mx^{m-1}\partial_z
\]

\[
\delta_3 = -2z\partial_y + ny^{n-1}\partial_z,
\]

hence we get a surjective map of \(A\)-modules \(\eta : A^4 \rightarrow \text{Der}_F(A) \rightarrow 0\). A calculation shows that the syzygy-matrix of \(\text{Der}_F(A)\) is the following matrix

\[
\rho = \begin{pmatrix}
y^{n-1} & z & 0 & x^{m-1} \\
2x & 0 & -2z & -2y \\
0 & nz & ny^{n-1} & -nz \\
-mz & ny & -mx^{m-1} & 0
\end{pmatrix},
\]

hence we get an exact sequence of \(A\)-modules

\[
\cdots \rightarrow A^4 \rightarrow^\rho A^4 \rightarrow^\eta \text{Der}_F(A) \rightarrow 0.
\]

A calculation shows that the Kodaira-Spencer map \(g : \text{Der}_F(A) \rightarrow \text{Ext}^1_A(W, W)\) is zero for the modules \(W\), hence \(V_W = \text{Der}_F(A)\), and the calculation also provides us with elements \(\nabla(\delta_i)\) in \(\text{End}_F(W)\) with the property that \(\nabla(\delta_i)(aw) = a\nabla(\delta_i)(w) + \delta_i(a)w\) for \(i = 0, \ldots, 3\). Hence we get an \(F\)-linear map \(\nabla : V_W \rightarrow \text{End}_F(W)\). Given any map \(e_i\) in \(\text{End}_A(W)\), it follows that the map \(\nabla(\delta_i) + e_i\) again is an element in \(\text{End}_F(W)\) with desired derivation-property, hence we seek endomorphisms \(e_0, \ldots, e_3\) in \(\text{End}_A(W)\) with the property that the adjusted map \(\nabla : V_W \rightarrow \text{End}_F(W)\) defined by \(\nabla(\delta_i) = \nabla(\delta_i) + e_i\) is \(A\)-linear. We see that we have to solve equations in the ring \(\text{End}_A(W)\). In the examples above, it turns out if one looks closely that one can
find elements in $\text{End}_A(W)$ by inspection so as to reduce the problem to solve linear equations in the field $F$. If one does this, one arrives at the following expressions:

\[(2.0.2)\quad \nabla_{\delta_0} = \delta_0 + A_0 =
\begin{pmatrix}
\frac{nk + ml}{2mn} & 0 & 0 & 0 \\
0 & \frac{2mn - ml - nk}{2mn} & 0 & 0 \\
0 & 0 & \frac{1}{2}mn + ml - nk & 0 \\
0 & 0 & 0 & \frac{1}{2}mn + nk - ml
\end{pmatrix}.
\]

(2.0.3) $\nabla_{\delta_1} = \delta_1 + \begin{pmatrix} 0 & b_2 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ 0 & 0 & b_4 & 0 \\ 0 & b_3 & 0 & 0 \end{pmatrix} = \delta_1 + A_1,$

with $b_1 = \frac{1}{4}(mn - 2nk - 2ml)x^{k-1}y^{l-1},$ $b_2 = \frac{1}{4}(3mn - 2ml - 2nk)x^{m-k-1}y^{n-l-1},$

$b_3 = \frac{1}{4}(2nk - mn - 2ml)x^{m-k-1}y^{l-1}$ and $b_4 = \frac{1}{4}(2nk - 2ml + mn)x^{k-1}y^{n-l-1}.$

(2.0.4) $\nabla_{\delta_2} = \delta_2 + \begin{pmatrix} 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \\ c_1 & 0 & 0 & 0 \\ c_2 & 0 & 0 & 0 \end{pmatrix} = \delta_2 + A_2,$

with $c_1 = \frac{1}{m}(\frac{1}{2}mn - ml - nk)x^{k-1},$ $c_2 = \frac{1}{n}(\frac{1}{2}mn - ml - nk)x^{m-k-1},$ $c_3 = \frac{1}{n}(\frac{1}{2}mn + ml - nk)x^{m-k-1}$ and $c_4 = \frac{1}{n}(ml - nk - \frac{1}{2}mn)x^{k-1}.$

(2.0.5) $\nabla_{\delta_3} = \delta_3 + \begin{pmatrix} 0 & 0 & 0 & d_4 \\ 0 & 0 & d_3 & 0 \\ 0 & d_2 & 0 & 0 \\ d_1 & 0 & 0 & 0 \end{pmatrix} = \delta_3 + A_3,$

where $d_1 = \frac{1}{m}(\frac{1}{2}mn - ml - nk)y^{l-1},$ $d_2 = \frac{1}{n}(ml + nk - \frac{3}{2}mn)y^{n-l-1},$ $d_3 = \frac{1}{m}(\frac{3}{2}mn + ml - nk)y^{l-1}$ and $d_4 = \frac{1}{m}(\frac{1}{2}mn - ml + nk)y^{n-l-1}.$

**Theorem 2.1.** For all $1 \leq k \leq m$ and $1 \leq l \leq n$ the equations (2.0.2)-(2.0.5) define algebraic $V_W$-connections $\nabla^{\phi,\psi} : V_W \to \text{End}_F(W)$ where $W = W(\phi, \psi).

**Proof.** The module $W$ is given by the exact sequence

$\cdots \to^{\psi} A^4 \to^{\phi} A^4 \to W \to 0,$

hence an element $w$ in $W$ is an equivalence class $\overline{a}$ of an element $a$ in $A^4.$ Let $a$ in $A^4$ be the element

\[a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}\]

and consider the class $w = \overline{a}$ in $W = A^4/im\phi.$ Define the $V_W$-connection $\nabla$ as follows:

$\nabla(\delta_i)(w) = (\delta_i + A_i) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$
Then one verifies that this definition gives a well-defined $A$-linear map $\nabla^{\phi,W} = \nabla: V_W \to \text{End}_F(W)$, and we have proved the assertion. \qed

Note that the connections $\nabla^{\phi,W}$ from Theorem 2.1 exist over any field $F$ of characteristic prime to $m$ and $n$.

In Theorem 2.1 we saw examples where $V_W$ was the whole module of derivations for a class of modules on some hypersurface singularities. In [19], Theorem 5.1 the splitting type of the principal parts $P^k(\mathcal{O}(d))$ is calculated on the projective line over a field of characteristic zero. When $d \geq 1$, he following formula is shown:

$$\mathcal{P}^1(\mathcal{O}(d)) \cong \mathcal{O}(d-1) \oplus \mathcal{O}(d-1).$$

It follows that the Atiyah sequence (see [2])

$$0 \to \Omega^1 \otimes \mathcal{O}(d) \to \mathcal{P}^1(\mathcal{O}(d)) \to \mathcal{O}(d) \to 0$$

does not split hence $\mathcal{O}(d)$ does not have a connection $\nabla: \mathcal{O}(d) \to \Omega^1 \otimes \mathcal{O}(d)$.

It follows that there does not exist an action

$$\rho: T_{\mathbb{P}^1} \to \text{End}(\mathcal{O}(d)),$$

hence we see that for $\mathcal{O}(d)$ on $\mathbb{P}^1$ over a field of characteristic zero, the linear Lie-Rinehart algebra $V_{\mathcal{O}(d)}$ is a strict sub-sheaf of the tangent sheaf $T_{\mathbb{P}^1}$.

3. Chern-classes

In this section we define for any $F$-algebra $A$ where $F$ is any field, and any $A$-module $W$ of finite presentation with a $V_W$-connection the first Chern-class $c_1(W)$ in $H^2(V_W|_U, \mathcal{O}_U)$, where $U$ in Spec($A$) is the open subset where $W$ is locally free, and $H^1(V_W|_U, \mathcal{O}_U)$ is the Chevalley-Hochschild cohomology of the restricted Lie-Rinehart algebra $V_W|_U$ with values in the sheaf $\mathcal{O}_U$.

**Definition 3.1.** Let $g$ be a Lie-Rinehart algebra and $W$ a $g$-module. The Chevalley-Hochschild complex $C^*(g, W)$ is defined as follows:

$$C^p(g, W) = \text{Hom}_A(g^{\wedge p}, W),$$

with differentials $d^p: C^p(g, W) \to C^{p+1}(g, W)$ defined by

$$d^p\phi(g_1 \wedge \cdots \wedge g_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} g_i \phi(g_1 \wedge \cdots \wedge \bar{g}_i \wedge \cdots \wedge g_{p+1}) +$$

$$\sum_{i<j} (-1)^{i+j} \phi([g_i, g_j] \wedge \cdots \wedge \bar{g}_i \wedge \cdots \wedge \bar{g}_j \wedge \cdots \wedge g_{p+1}).$$

Here $g \phi(g_1 \wedge \cdots \wedge g_p) = \nabla(g) \phi(g_1 \wedge \cdots \wedge g_p)$ and overlined elements should be excluded. The cohomology $H^i(C^*(g, W))$ is denoted $H^i(g, W)$.

Note that $C^*(g, W)$ is a complex if and only if the $g$-connection $\nabla$ is flat, i.e a morphism of Lie-algebras. The complex $C^*(g, W)$ is a complex generalizing simultaneously the algebraic deRham-complex and Chevalley-Eilenberg complex. Now given a $V_W$-connection $\nabla$ on $W$, we immediately get a $V_W$-connection on $\text{End}_A(W)$, denoted $ad\nabla$. We see that if $\nabla$ is flat, it follows that $ad\nabla$ is flat. Consider the open subset $U$ of Spec($A$) where $W$ is locally free. We restrict the $V_W$-connection to $U$, to get a connection $\nabla: V_W|_U \to \text{End}_{\mathcal{O}_U}(W|_U)$. Since $W|_U$ is
Define an affine scheme with an isolated singularity. Pick a representation $G$ of dimension two, and let $\ker c^2(\mathbb{V}_W|\mathcal{U}_i, \mathcal{O}_{\mathcal{U}_i})$. One checks that the element $c^i = \text{trace} \circ R\mathcal{C}_i|U_i$ is a cocycle of the complex $C^*(\mathbb{V}_W|U_i, \mathcal{O}_{U_i})$ for all $i$. We have constructed elements

$$c^i \in C^2(\mathbb{V}_W|U_i, \mathcal{O}_{U_i})$$

which coincide on intersections since trace is independent with respect to choice of basis, hence the sheaf-structure on $C^2(\mathbb{V}_W|U, \mathcal{O}_U)$ gives a uniquely defined element $c$ in $C^2(\mathbb{V}_W|U, \mathcal{O}_U)$, such that $c|U_i = c^i$ for all $i$. There exists a commutative diagram

$$
\begin{array}{ccc}
C^p(\mathbb{V}_W|U, \mathcal{O}_U) & \xrightarrow{\rho^p} & C^{p+1}(\mathbb{V}_W|U, \mathcal{O}_U) \\
|U_i & & |U_i \\
C^p(\mathbb{V}_W|U_i, \mathcal{O}_{U_i}) & \xrightarrow{\rho^p} & C^{p+1}(\mathbb{V}_W|U_i, \mathcal{O}_{U_i})
\end{array}
$$

which proves that the element $c$ is a cocycle in the complex $C^*(\mathbb{V}_W|U, \mathcal{O}_U)$.

**Theorem 3.2.** There exists a class $c_1(W)$ in $H^2(\mathbb{V}_W|U, \mathcal{O}_U)$ which is independent with respect to choice of $\mathbb{V}_W$-connection.

**Proof.** Existence of the class $c_1(W)$ follows from the argument above. Independence with respect to choice of connection is straightforward. 

Note that if the $\mathbb{V}_W$-connection is flat, the first Chern-class $c_1(W)$ is zero. Note also that the construction in this section can be done with any $A$-module $W$ of finite presentation with a $g$-connection, where $g$ is any Lie-Rinehart algebra.

**Theorem 3.3.** The $\mathbb{V}_W$-connections $\nabla^{\phi,\psi}$ calculated in Theorem 2.1 are flat hence $c_1(W(\phi, \psi)) = 0$ for all the modules $W(\phi, \psi)$ on the singularities $B_{mn}$.

**Proof.** Easy calculation.

Note that the flat $\mathbb{V}_W$-connections in Theorem 3.3 give rise to a class of left modules on the algebra of differential operators $D(A)$ where $A$ is the ring $k[x,y,z]/f$ and $f = x^n + y^n + z^2$. Note also that Kolmo has in [13] computed the Alexander-polynomial of an irreducible plane curve $C$ in $\mathbb{C}^2$ using a certain logarithmic deRham-complex $\Omega^*_{\mathbb{C}_C}(\ast C)$. It would be interesting to see if the Alexander-polynomial can be computed in terms of a $g$-connection.

4. **Surface Quotient-Singularities**

In this section we consider maximal Cohen-Macaulay modules on quotient singularities of dimension two, and their first Chern-class. Let now $F = C$ be the complex numbers, and let $G \subseteq GL(2, F)$ be a finite sub-group with no pseudo-reflections. Consider the natural action $G \times F^2 \to F^2$, and the quotient $X = F^2/G$. It is an affine scheme with an isolated singularity. Pick a representation $\rho : G \to GL(V)$ where $V$ is an $F$-vectorspace, and consider the $A$-module $V \otimes_F A$ where $A = F[x, y]$. Define a $G$-action as follows: $g(v \otimes a) = \rho(g)v \otimes ga$, and let $W_\rho = (V \otimes_F V)^G$ be the $G$-invariants of the $G$-action defined. Then by [9] $W_\rho$ is a maximal Cohen-Macaulay
module on $A^G$. If $\rho$ is indecomposable, $M_\rho$ is irreducible. The McKay correspondence in dimension 2 says that all maximal Cohen-Macaulay modules on $A^G$ arise this way. Define a $G$-action on $g = \text{Der}_F(A)$ as follows: $(g\delta)(a) = g\delta(g^{-1}a)$, and let $g^G$ be the $G$-invariant derivations. The module $V \otimes_F A$ is a free $A$-module, hence there exists trivially a regular $g$-connection on $V \otimes_F A$. This implies that we get an induced $g^G$-connection on $V \otimes_F A$.

**Proposition 4.1.** There exists an action of $g^G$ on $W_\rho$.

**Proof.** Straightforward. □

From [24] it follows that $g^G$ is isomorphic to $\text{Der}_F(A^G)$, hence we have proved that all maximal Cohen-Macaulay modules $W_\rho$ on $\text{Spec}(A^G)$ possess a $\text{Der}_F(A^G)$-connection, and these are all regular connections since they are induced by the trivial one on $V \otimes_F A$.

**Theorem 4.2.** Let $X = \text{Spec}(A^G)$ be a 2-dimensional quotient singularity and let $W_\rho$ be a maximal Cohen-Macaulay module on $X$ then $c_1(W_\rho) = 0$.

**Proof.** Follows from the argument above. □

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