Spinning BTZ Black Hole versus Kerr Black Hole:

A Closer Look

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Abstract

By applying Newman’s algorithm, the AdS$_3$ rotating black hole solution is “derived” from the nonrotating black hole solution of Bañados, Teitelboim, and Zanelli (BTZ). The rotating BTZ solution derived in this fashion is given in “Boyer-Lindquist-type” coordinates whereas the form of the solution originally given by BTZ is given in a kind of an “unfamiliar” coordinates which are related to each other by a transformation of time coordinate alone. The relative physical meaning between these two time coordinates is carefully studied. Since the Kerr-type and Boyer-Lindquist-type coordinates for rotating BTZ solution are newly found via Newman’s algorithm, next, the transformation to Kerr-Schild-type coordinates is looked for. Indeed, such transformation is found to exist. And in this Kerr-Schild-type coordinates, truely maximal extension of its global structure by analytically continuing to “antigravity universe” region is carried out.

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I. Introduction

It had long been thought that black hole solutions cannot exist in 3-dim. since there is no local gravitational attraction and hence no mechanism to confine large densities of matter. It was, therefore, quite a surprise when Bañados, Teitelboim, and Zanelli (BTZ) \cite{1} have recently constructed the Anti-de Sitter (AdS$_3$) spacetime solution to the Einstein equations in 3-dim. that can be interpreted as a black hole solution. They included the negative cosmological constant in the 3-dim. vacuum Einstein theory and then found both the rotating and nonrotating black hole solutions. In the mean time, a curious relationship between the nonrotating and the rotating spacetime solutions of Einstein theory in 4-dim. also has been long known. Newman et al. \cite{4} discovered long ago that one can “derive” Kerr solution from the Schwarzschild solution in vacuum Einstein theory and Kerr-Newman solution from the Reissner-Nordström solution in Einstein-Maxwell theory via the “complex coordinate transformation” scheme acting on metrics written in terms of null tetrad of basis vectors. 

In the present work, we attempt the same derivation but in this time in 3-dim. spacetime. Namely, we see if the rotating version of BTZ black hole solution can indeed be “derived” from its nonrotating counterpart via Newman’s method. And in doing so, our philosophy is that the 3-dim. situation can be thought of as, say, the $\theta = \pi/2$ - slice of the 4-dim. one (where $\theta$ denotes the polar angle). Interestingly enough, we do end up with the rotating version of BTZ black hole solution but in a different coordinate system from the one originally employed in BTZ’s solution ansatz. And as we shall see shortly, it turns out that the rotating BTZ solution derived here in this work is given in “Boyer-Lindquist-type” \cite{5} coordinates whereas the original form of the solution given by BTZ is given in a kind of an “unusual” coordinates which are related to the “familiar” Boyer-Lidquist-type coordinates by a transformation of time coordinate alone. We can easily understand the reason for this result as follows ; much as the Kerr solution “derived” from the Schwarzschild solution by Newman’s complex coordinate transformation method naturally comes in Kerr coordinates \cite{2,4} from which one can transform to Boyer-Lindquist coordinates \cite{5}, the rotating BTZ black hole solution derived from its nonrotating counterpart in this manner comes in Kerr-
type coordinates again from which one can make a transformation to Boyer-Lindquist-type coordinates as well. And then one can realize that by performing a transformation from the Boyer-Lindquist-type time coordinate $t$ to a “new” time coordinate $\tilde{t}$ following the transformation law, $\tilde{t} = t - a\phi$ (where $a$ is proportional to the angular momentum per unit mass and $\phi$ denotes the azimuthal angle), our “derived” rotating BTZ black hole solution can indeed be put in the form originally given by BTZ. As expected, the rotating BTZ black hole solution given in Boyer-Lindquist-type coordinates takes on the structure which resembles that of Kerr solution more closely than it is in BTZ’s rather unusual time coordinate. However, one can readily realize that it is the BTZ’s time coordinate $\tilde{t}$ that is the usual Killing time coordinate, not the Boyer-Lindquist-type one. The careful analysis of the relative physical meaning between the two different choices of coordinates will be presented later on in the discussion. Then this successful derivation of rotating BTZ black hole solution from its non-rotating counterpart via Newman’s algorithm plus the fact that the resultant metric comes in Kerr-type and Boyer-Lindquist-type coordinates which are familiar ones well-known in the study of Kerr black hole solution in 4-dim. lead us to attempt to complete the study of rotating BTZ solution in parallel with that of Kerr solution. Recall that in addition to Kerr and Boyer-Lindquist coordinates, there is one more coordinate system of central importance, i.e., the Kerr-Schild coordinates. And in this coordinates, one can envisage peculiar features of Kerr spacetime such as the “ring” structure of curvature singularity and the analytic continuation of (Boyer-Lindquist) $r$-coordinate from positive to negative values. Therefore, we similarly look for further transformation to Kerr-Schild-type coordinates for the rotating BTZ metric. It turns out that such Kerr-Schild-type coordinates indeed exists and in terms of which the rotating BTZ black hole metric takes on precisely the same form as that of Kerr metric sliced along the $\theta = \pi/2$ equatorial plane. As a result, this one-to-one correspondence between the metric (or coordinates) of rotating BTZ spacetime and that of Kerr spacetime allows us to envisage the structure of rotating BTZ solution in different direction from that in which the previous study of rotating BTZ solution has been done. For example, the representation of the rotating BTZ black hole metric in Kerr-Schild-type coordinates allows
us to realize that the global structure of the rotating hole can be maximally extended by analytically continuing the (Boyer-Lindquist-type) $r$-coordinate from positive (“our universe” region) to negative values (“antigravity universe” region) through $r = 0$ in a similar manner to the case of Kerr spacetime in 4-dim.

II. Derivation of the $\text{AdS}_3$ rotating black hole solution via Newman’s algorithm

As mentioned earlier, Newman et al. [4] discovered curious “derivations” of stationary, axisymmetric metric solutions from static, spherically-symmetric solutions in 4-dim. Einstein theory. To attempt the similar work in 3-dim. spacetime, we start with the nonrotating version of BTZ black hole solution in 3-dim. as a “seed” solution to construct its rotating counterpart. The nonrotating BTZ black hole solution written in Schwarzschild-type coordinates $(t, r, \phi)$ is given by

$$ds^2 = (-M + \frac{r^2}{l^2})dt^2 - (-M + \frac{r^2}{l^2})^{-1}dr^2 - r^2d\phi^2$$

(1)

where $l$ is related to the negative cosmological constant by $l^{-2} = -\Lambda$ and $M$ is an integration constant that can be identified with the ADM mass of the black hole. Now in order to “derive” a rotating black hole solution applying the complex coordinate transformation scheme of Newman et al., we begin by assuming that this 3-dim. black hole geometry is a $\theta = \pi/2$ - slice of a static, spherically-symmetric 4-dim. geometry given by

$$ds_4^2 = \lambda^2(r)dt^2 - \lambda^{-2}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(2)

with $\lambda^2(r) = (-M + r^2/l^2)$. The essential reason for this “dimensional continuation” is to introduce the null tetrad system of vectors on which Newman’s complex coordinate transformation method is technically based. We do not, however, ask nor demand that this 4-dim. geometry be an explicit solution of Einstein equation in 4-dim. as well. We just demand that only its $\theta = \pi/2$ - slice be a solution of Einstein equation in 3-dim. Remarkably, then, upon the series of operations ; dimensional continuation $\rightarrow$ Newman’s derivation method $\rightarrow$ dimensional reduction by setting $\theta = \pi/2$, we end up with a legitimate rotating black hole solution to 3-dim. Einstein equation as we shall see shortly.
Consider now the transformation to the Eddington-Finkelstein-type retarded null coordinates \((u, r, \phi)\) defined by 
\[ u = t - r_*, \quad \text{with} \quad r_* = \int dr (g_{rr} / -g_{tt})^{1/2}. \]
In terms of these null coordinates, the nonrotating BTZ black hole metric takes the form
\[ ds^2_1 = \lambda^2(r)du^2 + 2dudr - r^2(d\theta^2 + \sin^2 \theta d\phi^2). \] (3)

Into this 4-dim. Riemannian space we next introduce a tetrad system of vectors \(l_{\mu}, n_{\mu}, m_{\mu}\) and \(\bar{m}_{\mu}\) (with \(\bar{m}_{\mu}\) being the complex conjugate of \(m_{\mu}\)) satisfying the following orthogonality property 
\[ l_{\mu}n^{\mu} = -m_{\mu}\bar{m}^{\mu} = 1 \]
with all other scalar products vanishing. In terms of this null tetrad of basis vectors, the spacetime metric is written as 
\[ g^{\mu\nu} = l_{\mu}n_{\nu} + n_{\mu}l_{\nu} - m_{\mu}\bar{m}_{\nu} - \bar{m}_{\mu}m_{\nu}. \]

Then now one can obtain the contravariant components of the metric and the null tetrad vectors from the covariant components of the metric given in eq.(3) as
\[ g^{00} = 0, \quad g^{11} = -\lambda^2(r), \quad g^{10} = 1, \]
\[ g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2 \sin^2 \theta} \] (4)
and
\[ l^{\mu} = \delta_{1}^{\mu}, \quad n^{\mu} = \delta_{0}^{\mu} - \frac{1}{2} \lambda^2(r)\delta_{1}^{\mu}, \]
\[ m^{\mu} = \frac{1}{\sqrt{2r}} (\delta_{2}^{\mu} + \frac{i}{\sin \theta} \delta_{3}^{\mu}), \]
\[ \bar{m}^{\mu} = \frac{1}{\sqrt{2r}} (\delta_{2}^{\mu} - \frac{i}{\sin \theta} \delta_{3}^{\mu}) \] (5)
respectively. Now the radial coordinate \(r\) is allowed to take complex values and the tetrad is rewritten in the form
\[ l^{\mu} = \delta_{1}^{\mu}, \quad n^{\mu} = \delta_{0}^{\mu} - \frac{1}{2} (-M + \frac{r\bar{r}}{l^2})\delta_{1}^{\mu}, \]
\[ m^{\mu} = \frac{1}{\sqrt{2\bar{r}}} (\delta_{2}^{\mu} + \frac{i}{\sin \theta} \delta_{3}^{\mu}), \]
\[ \bar{m}^{\mu} = \frac{1}{\sqrt{2\bar{r}}} (\delta_{2}^{\mu} - \frac{i}{\sin \theta} \delta_{3}^{\mu}) \]
(6)
with \(\bar{r}\) being the complex conjugate of \(r\) (note that part of the algorithm is to keep \(l^{\mu}\) and \(n^{\mu}\) real and \(m^{\mu}\) and \(\bar{m}^{\mu}\) the complex conjugate of each other). We now formally perform the “complex coordinate transformation”
\[ r' = r + ia \cos \theta, \quad \theta' = \theta, \]
\[ u' = u - ia \cos \theta, \quad \phi' = \phi \]

on tetrad vectors \( l^\mu, n^\mu \) and \( m^\mu \) (\( \bar{m}^\mu \) is, as stated, defined as the complex conjugate of \( m^\mu \)).

If one now allows \( r' \) and \( u' \) to be real, we obtain the following tetrad

\[ l^\mu = \delta^\mu_1, \]
\[ n^\mu = \delta^\mu_0 - \frac{1}{2} [-M + \frac{r'^2 + a^2 \cos^2 \theta}{l^2}] \delta^\mu_1, \]
\[ m^\mu = \frac{1}{\sqrt{2(r' + ia \cos \theta)}} [ia \sin \theta (\delta^\mu_0 - \delta^\mu_1) + \delta^\mu_2 + \frac{i}{\sin \theta} \delta^\mu_3] \]

from which one can readily read off the contravariant components of the metric and then obtain the covariant components by inversion as (henceforth we drop the “prime”)

\[ ds^2_4 = (-M + \Sigma) du^2 + 2a \sin^2 \theta [1 - (-M + \Sigma)] dud\phi + 2dudr - 2a \sin^2 \theta drd\phi - \Sigma d\theta^2 - [r^2 + a^2 + a^2 \sin^2 \theta (1 - (-M + \Sigma))] \sin^2 \theta d\phi^2 \]

where \( \Sigma \equiv (r^2 + a^2 \cos^2 \theta) \). Now at this stage, considering that the geometry in 3-dim. can be thought of as the \( \theta = \pi/2 \)-slice of the full, 4-dim. one, we simply set \( \theta = \pi/2 \) in the metric above to arrive at the rotating black hole metric in 3-dim. given by

\[ ds^2 = (-M + \frac{r^2}{l^2}) (du - ad\phi)^2 + 2(du - ad\phi)(dr + ad\phi) - r^2 d\phi^2. \]

Also note that the metric above we “derived” via Newman’s complex coordinate transformation method is given in terms of Kerr-type coordinates \([2,4]\) \((u,r,\phi)\) which can be thought of as the generalization of the retarded null coordinates. Thus one might want to further transform it into the one written in Boyer-Lindquist-type coordinates \([5]\) \((t,r,\hat{\phi})\) that can be viewed as the generalization of the Schwarzschild coordinates. This can be achieved via the transformation

\[ dt = du + \frac{(r^2 + a^2)}{\Delta} dr, \quad d\hat{\phi} = d\phi + \frac{a}{\Delta} dr \]

where \( \Delta \equiv r^2(-M + r^2/l^2) + a^2 \). Finally, the rotating AdS\(_3\) black hole solution given in Boyer-Lindquist-type coordinates is given by (henceforth we drop “hat” on \( \phi \) coordinate)
\[ ds^2 = (-M + \frac{r^2}{l^2})dt^2 + 2a[1 - (-M + \frac{r^2}{l^2})]dtd\phi \]

\[ - [r^2 + a^2 + a^2(1 - (-M + \frac{r^2}{l^2}))]d\phi^2 - \frac{r^2}{\Delta}dr^2. \]  

(12)

And it is straightforward to check that the “derived” rotating black hole solution given in eq.(12) does satisfy the AdS$_3$ Einstein equation, \( R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - l^{-2}g_{\mu\nu} = 0. \)

It is a little puzzling, however, that the derived rotating black hole solution of ours in eq.(12) above does not appear exactly the same as the one originally constructed by BTZ. Nonetheless, both our solution above and the one originally obtained by BTZ (written in our notation convention in which \( a = J/2 \) with \( J \) appearing in original BTZ’s work [1])

\[ ds^2 = (-M + \frac{r^2}{l^2})d\tilde{t}^2 + 2a\tilde{t}d\phi - r^2d\phi^2 - \frac{r^2}{\Delta}dr^2 \]  

(13)

correctly reduce, in the vanishing angular momentum limit \( a \to 0, \) to the static, spherically-symmetric black hole solution which was the starting point of our solution construction. Therefore, presumably the two rotating black hole solutions must be related to each other by a coordinate transformation. Indeed, one can check straightforwardly that the two are related by the transformation of the time coordinate alone

\[ \tilde{t} = t - a\phi. \]  

(14)

Therefore the two metrics given in eqs.(12) and (13) represent one and the same AdS$_3$ black hole solution modulo gauge transformation.

III. Boyer-Lindquist-type coordinates versus BTZ coordinates

It is interesting to note that the rotating BTZ black hole solution given in the Boyer-Lindquist-type time coordinate in eq.(12) resembles the structure of Kerr solution in 4-dim. more closely than that given in the BTZ time coordinate in eq.(13). Therefore, it seems that now we are left with the question; between \( t \) and \( \tilde{t} \), which one is the usual Killing time coordinate? It appears that it is \( \tilde{t} \), the BTZ time coordinate that is the usual Killing time coordinate, not \( t \), the Boyer-Lindquist-type time coordinate. And this conclusion is based on the following observation. Note that both in Boyer-Lindquist-type coordinates \((t, r, \phi)\)
and in BTZ coordinates \((\tilde{t}, r, \phi)\), generally \(|g_{t\phi}| \) represents \textit{angular momentum per unit mass} as observed by an accelerating observer at some fixed \(r\) and \(\phi\). (We stress here that the quantity we are about to introduce is the angular momentum \textit{per unit mass} at some point \(r\). Generally, it should be distinguished from the total angular momentum of the (4-dim.) spacetime measured in the asymptotic region, \(\tilde{J} = (16\pi)^{-1} \int_S \epsilon_{\mu\nu\alpha\beta} \nabla^\alpha \psi^\beta\) in the notation convention of ref.[6] with \(S\) being a large sphere in the asymptotic region and \(\psi^\mu = (\partial/\partial \phi)^\mu\) being the rotational Killing field. Thus in these definitions, angular momentum \textit{per unit mass} may change under a coordinate transformation although the total angular momentum \(\tilde{J}\) remains coordinate independent.) To see this quickly, one only needs to write the rotating black hole metric in ADM’s (2+1)-split form

\[
d s^2 = N^2(r) dt^2 - f^{-2}(r) dr^2 - R^2(r)[N^\phi(r) dt + d\phi]^2. \tag{15}
\]

Then in Boyer-Lindquist-type time coordinate \(t\), metric components correspond to

\[
\begin{align*}
f^2(r) &= (-M + \frac{r^2}{l^2} + \frac{a^2}{r^2}), \\
R^2(r) &= [r^2 + a^2 + a^2\{1 - (-M + \frac{r^2}{l^2})]\}, \\
N^2(r) &= (-M + \frac{r^2}{l^2}) + R^{-2}(r)a^2\{1 - (-M + \frac{r^2}{l^2})\}^2, \\
N^\phi(r) &= -R^{-2}(r)a\{1 - (-M + \frac{r^2}{l^2})\}
\end{align*}
\] (16)

whereas in BTZ time coordinate \(\tilde{t}\), they correspond to

\[
\begin{align*}
N^2(r) &= f^2(r) = (-M + \frac{r^2}{l^2} + \frac{a^2}{r^2}), \\
N^\phi(r) &= -\frac{a}{r^2}, \quad R^2(r) = r^2.
\end{align*}
\]

Now it is apparent in this ADM’s space-plus-time split form of the metric given in eq.(15) that the shift function \(N^\phi(r)\) corresponds to the angular velocity \(|N^\phi(r)| = \Omega(r)\) and obviously \(g_{\phi\phi} = R^2(r)\) represents (radius associated with the proper circumference)\(^2\). That is, \(\int_0^{2\pi}(g_{\phi\phi})^{1/2} d\phi = 2\pi R(r)\) is the proper circumference of a circular orbit around the axis of rotation. Therefore the quantity \(R^2(r) |N^\phi(r)| = J\) can be identified with the \textit{angular
momentum per unit mass at some point (with fixed $r$ and $\phi$) from the axis of rotation.

Thus from $-R^2(r)N^\phi(r) = g_{t\phi}$, the angular momentum per unit mass in Boyer-Lindquist-type time coordinate and in BTZ time coordinate are given respectively by

$$J = |g_{t\phi}| = a[1 - (-M + \frac{r^2}{l^2})],$$

$$J^{BTZ} = |g_{t\phi}| = a.$$  \hfill (17)

Now from eq.(17), it is manifest that the angular momentum per unit mass given in BTZ time coordinate $\tilde{t}$ is finite and constant all over the hypersurface whereas that given in Boyer-Lindquist-type time coordinate $t$ grows indefinitely as $r \to \infty$. This can be attributed to the fact that the coordinate $\tilde{t}$ BTZ used is defined asymptotically using the asymptotic symmetries and hence approaches the AdS time at spatial infinity [1]. Therefore we can conclude that it is the BTZ time coordinate $\tilde{t}$ which is the usual Killing time. This is certainly in contrast to what happens in the familiar Kerr black hole geometry in 4-dim. where the usual Boyer-Lindquist time coordinate is the Killing time coordinate. And it seems that this discrepancy comes from the fact that the 3-dim. BTZ black hole is not asymptotically flat but asymptotically anti-de Sitter whereas the 4-dim. Kerr black hole is asymptotically flat. The next question one might want to ask and answer could then be ; what is the relative physical meaning of these two time coordinates $t$ and $\tilde{t}$ ? To get a quick answer to this question, we go back and look at the coordinate transformation law given in eq.(14) relating the two time coordinates $t$ and $\tilde{t}$. Namely, taking the dual of the transformation law $\delta \tilde{t} = \delta t - a\delta \phi$, we get

$$\left(\frac{\partial}{\partial \tilde{t}}\right)^\mu = \left(\frac{\partial}{\partial t}\right)^\mu - \frac{1}{a}\left(\frac{\partial}{\partial \phi}\right)^\mu$$ \hfill (18)

or

$$\tilde{\xi}^\mu = \xi^\mu - \frac{1}{a} \psi^\mu$$

where $\xi^\mu = (\partial/\partial t)^\mu$ and $\psi^\mu = (\partial/\partial \phi)^\mu$ denote Killing fields corresponding to the time translational and the rotational isometries of the spinning black hole spacetime respectively and $\tilde{\xi}^\mu = (\partial/\partial \tilde{t})^\mu$ denotes the Killing field associated with the isometry of the hole’s metric.
under the BTZ time translation. Now this expression for the BTZ time translational Killing field $\tilde{\xi}^{\mu}$ implies that in BTZ time $\tilde{t}$, the time translational generator is given by the linear combination of the Boyer-Lindquist-type time translational generator and the rotational generator. In plain English, this means that in BTZ time coordinate, the action of time translation consists of the action of Boyer-Lindquist-type time translation and the action of rotation in opposite direction to $a$, i.e., to the rotation direction of the hole. Thus the BTZ time coordinate $\tilde{t}$ can be interpreted as the coordinate, say, of a frame which rotates around the axis of the spinning BTZ black hole in opposite direction to that of the hole outside its static limit.

Finally, we come to another peculiar point worthy of note. The rotating BTZ black hole metric, when represented in Boyer-Lindquist-type coordinates as given in eq.(12), exhibits the following exotic behavior. Notice that in the Boyer-Lindquist-type coordinates, the rotational Killing field $\psi^{\mu} = (\partial/\partial \phi)^\mu$ fails to remain strictly spacelike while it does remain everywhere spacelike in BTZ coordinates. That is, from

$$\psi^{\mu} \psi_{\mu} = g_{\phi \phi} = r^2 (1 - a^2/l^2) + a^2 (M + 2)$$

where we used $l^{-2} = -\Lambda = |\Lambda| > 0$, we can realize that;

(i) for $a < 1/\sqrt{|\Lambda|}$ (i.e., slowly spinning hole), $g_{\phi \phi} > 0$, namely, $\psi^{\mu}$ is everywhere spacelike and hence the “proper circumference” of a circular orbit around the axis of rotation, $\int_{0}^{2\pi} (g_{\phi \phi})^{1/2} d\phi = 2\pi (g_{\phi \phi})^{1/2}$, is positive-definite and increases with $r$, and

(ii) for $a = 1/\sqrt{|\Lambda|}$, $g_{\phi \phi} = (M + 2)a^2 = \text{const.}$, namely, $\psi^{\mu}$ is again everywhere spacelike but the proper circumference $2\pi (g_{\phi \phi})^{1/2}$ remains constant at all distances from the axis of rotation, and lastly

(iii) for $a > 1/\sqrt{|\Lambda|}$ (i.e., rapidly spinning hole), $g_{\phi \phi} = -(|\Lambda|a^2 - 1)r^2 + (M + 2)a^2$ could become negative-definite for $r > a\sqrt{\frac{M + 2}{|\Lambda|a^2 - 1}}$ or for $r < -a\sqrt{\frac{M + 2}{|\Lambda|a^2 - 1}}$ (r may be extended to “negative” values as we shall see in the next section). This implies that in these “far” regions, the $\phi$-coordinate could become timelike which, in turn, leads to the occurrence of the
closed timelike curves signaling the possible violation of causality. This is a reminiscence of what happens in the case of Kerr black hole spacetime. Namely, as was first pointed out by Carter [5], the closed timelike curves can occur (i.e., $\psi^\mu$ can go timelike) as one approaches from negative-$r$ region toward the ring singularity on the equatorial plane of Kerr spacetime. And this could be another evidence that it is the BTZ time coordinate, not that of Boyer-Lindquist-type, which bears better physical relevance.

IV. Kerr-Schild-type coordinates and the maximal extension of global structure

Thus far, we have attempted the analysis of the AdS$_3$ black hole solution by BTZ in a fashion which is parallel with that of Kerr black hole in 4-dim. Namely, starting from the nonrotating BTZ black hole solution, we derived the rotating solution via Newman’s complex coordinate transformation method. Then, similarly to what happens in the case of Kerr black hole in 4-dim., we obtained, as a result of this Newman’s algorithm, the rotating BTZ black hole solution first in Kerr-type coordinates $(u, r, \phi)$ given in eq.(10). By performing the transformation in eq.(11), next we represented the black hole metric in Boyer-Lindquist-type coordinates $(t, r, \hat{\phi})$ given in eq.(12). Therefore in order to complete our study of rotating BTZ black hole solution in parallel with that of Kerr solution, we are naturally led to the remaining final step. That is, we look for further transformation to Kerr-Schild-type coordinates just as we did in the study of Kerr solution in 4-dim. Thus at this point, it seems appropriate to briefly review the physical roles played by these three alternative coordinate systems in which the rotating Kerr metric was represented. Historically, the Kerr solution was originally found in Kerr coordinates [2] and then the Kerr-Schild coordinates [3] was introduced and finally the Boyer-Lindquist coordinates was discovered [5]. Firstly, the Kerr coordinates $(u, r, \theta, \phi)$ can be thought of as the generalization of Eddington-Finkelstein null coordinates and hence is free of coordinate singularities. Next, the Boyer-Lindquist coordinates $(t, r, \theta, \hat{\phi})$ can be viewed as the generalization of (accelerated) Schwarzschild coordinates. In this system, it is straightforward to see how the charged Kerr metric reduces to the familiar Schwarzschild ($a = e = 0$) or Reissner-Nordstrom ($a = 0$) metrics (where $a$ and $e$ denote the angular momentum per unit mass and the electric charge respectively).
is also clear that the spacetime is asymptotically-flat in the limit of large positive or negative values of $r$. Lastly, the Kerr-Schild coordinates $(\tilde{t}, x, y, z)$ is a quasi-Cartesian coordinates and the well-known “ring” structure of curvature singularity can only be uncovered in this coordinate system. Besides, the true maximal extension by analytically continuing the $r$-coordinate from positive to negative values can be performed in this system.

The transformation law from Kerr to Boyer-Lindquist coordinates for Kerr solution is the same as that given in eq.(11) except that now $\Delta = r^2 - 2Mr + a^2$. Next, the transformation law from Boyer-Lindquist to Kerr-Schild coordinates is given by

$$x + iy = (r + ia) \sin \theta e^{i\tilde{\phi}},$$
$$z = r \cos \theta, \quad \tilde{t} = v - r$$

where

$$\tilde{\phi} = \tilde{\phi} + \int dr \frac{a}{\Delta}, \quad v = t + \int dr \frac{r^2 + a^2}{\Delta}$$

and $r$ is determined implicitly by $r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2 z^2 = 0$. Then the Kerr metric given in this Kerr-Schild coordinates reads

$$ds^2 = (-d\tilde{t}^2 + dx^2 + dy^2 + dz^2)$$
$$+ \frac{r^2(r^2 + a^2 - \Delta)}{r^4 + a^2 z^2} \left[ \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r} + d\tilde{t} \right]^2.$$ 

Now we come back to the representation of the rotating BTZ black hole solution in Kerr-Schild-type coordinates. When we derived the rotating BTZ black hole solution from its nonrotating counterpart via Newman’s algorithm, our philosophy was that the 3-dim. spacetime of interest can be thought of as, say, the $\theta = \pi/2$- slice (i.e., the equatorial plane) of the 4-dim. one. Thus in accordance with this philosophy, we take the coordinate transformation law from Boyer-Lindquist-type $(t, r, \tilde{\phi})$ to Kerr-Schild-type $(\tilde{t}, x, y)$ coordinates to be the special case of that for Kerr solution in 4-dim. given above when $\theta = \pi/2$. Then

$$x + iy = (r + ia)e^{i\tilde{\phi}}, \quad \tilde{t} = v - r$$

(22)
where

\[ \tilde{\phi} = \hat{\phi} + \int dr \frac{a}{\Delta}, \quad v = t + \int dr \frac{r^2 + a^2}{\Delta} \]

with \( \Delta = r^2(-M + r^2/l^2) + a^2 \) now and \( r \) is determined implicitly by \( x^2 + y^2 = r^2 + a^2 \).

Then the rotating BTZ black hole metric given in this Kerr-Schild-type coordinates reads

\[ ds^2 = (-d\tilde{t}^2 + dx^2 + dy^2) + \left( \frac{r^2 + a^2 - \Delta}{r^2} \left[ \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + dt \right] \right)^2. \] (23)

Note first that the relation

\[ r^2 = x^2 + y^2 - a^2 \] (24)

implies that each set of values \((x, y)\) corresponds to two different points distinguished by positive and negative values of \( r \). This indicates that \( r \) can take negative values as well as positive values. In fact the analytic continuation of \( r \)-coordinate from positive to negative values through \( r = 0 \) is permissible as the rotating BTZ black hole metric remains regular at \( r = 0 \). It is also interesting to note that this form of the rotating BTZ black hole metric written in Kerr-Schild-type coordinates corresponds precisely to that of the Kerr black hole metric in 4-dim. again written in Kerr-Schild coordinates if one sets \( z = r \cos \theta = 0 \). This point seems to indicate that our philosophy, i.e., viewing the 3-dim. black hole spacetime as the \( \theta = \pi/2 \)-slice (equatorial plane) of a 4-dim. spacetime was not so naive, after all. Besides, it further supports our earlier statement that it is the Boyer-Lindquist-type coordinates we discovered in the present work, not the BTZ time coordinate, in which the form of the spinning BTZ black hole metric resembles more closely that of the Kerr black hole metric in 4-dim. Lastly, we now turn to the truly maximal analytic extension of the global structure of spinning BTZ black hole spacetime that can be completed only when one works in this Kerr-Schild-type quasi-Cartesian coordinates. Namely, when the spinning BTZ black hole metric is written in this Kerr-Schild-type coordinates, it becomes natural to consider the analytic continuation of \( r \)-coordinate from positive to negative values through \( r = 0 \) in a similar
manner to the case of Kerr black hole in 4-dim. Even after this “truly” maximal analytic extension (involving the extension to negative-\(r\)), however, the Carter-Penrose conformal diagram of the spinning BTZ black hole spacetime remains almost the same except that the spatial infinity in the negative-\(r\) side, \(r = -\infty\) now should be incorporated. Just like \(r = +\infty\), \(r = -\infty\) is also represented by a timelike line and hence the slightly modified conformal diagram showing this new ingredient is given in Fig.1.

**V. Discussions**

To summarize, here in this work we explored the structure of rotating BTZ spacetime in parallel with that of Kerr spacetime in 4-dim. And such a study was initiated from our attempt to apply Newman’s method of generating a spinning black hole solution from a static black hole solution in 4-dim. Einstein theory to the 3-dim. situation. More concretely, we employed the algorithm \(\rightarrow\) dimensional continuation \(\rightarrow\) Newman’s derivation method \(\rightarrow\) dimensional reduction by setting \(\theta = \pi/2\) to successfully obtain a legitimate rotating AdS\(_3\) black hole solution. And as a consequence of this study, we discovered that there are two alternative time coordinates in describing the rotating AdS\(_3\) black hole solution one can select from to investigate its various physical contents. Thus let us elaborate on the complementary roles played by the two alternative time coordinates. First, note that the two time coordinates, that of Boyer-Lindquist-type, \(t\) and that of BTZ, \(\tilde{t}\) coincide for nonrotating case \(a = 0\) and become different only for rotating case \(a \neq 0\) as one can see in their relation, \(\tilde{t} = t - a\phi\). Next, the Boyer-Lindquist-type time coordinate \(t\) is the “familiar” time coordinate in which the rotating BTZ black hole spacetime metric closely resembles that of the Kerr (or Kerr-de Sitter [7,8]) spacetime metric in 4-dim. Thus for direct and parallel comparison between the structure of 3-dim. rotating BTZ black hole and that of 4-dim. Kerr (or Kerr-de Sitter) black hole, it seems appropriate to work in the Boyer-Lindquist-type time coordinate. The BTZ time coordinate \(\tilde{t}\), on the other hand, may look “unusual” in that it can be identified with the time coordinate of a non-static observer who rotates in opposite direction to that of the spinning hole. It is, however, this BTZ time coordinate, not the familiar Boyer-Lindquist-type time coordinate, which is the right Killing time coordinate in
terms of which the total mass and particularly the angular momentum can be well-defined in the asymptotic region of this asymptotically anti-de Sitter spacetime. Also it has been well-studied in the literature [1] that this BTZ time coordinate is particularly advantageous in exploring the global structure of the rotating BTZ black hole since it allows one to transform to Kruskal-type coordinates and eventually allows one to draw the Carter-Penrose conformal diagram much more easily than the case when one employs the usual Boyer-Lindquist-type time coordinate. Therefore when investigating various physical contents of the spinning BTZ black hole, the two time coordinates $t$ and $\tilde{t}$ appear to play mutually complementary roles. Lastly, the discovery of the subsequent transformation from Boyer-Lindquist-type to Kerr-Schild-type quasi-Cartesian coordinates allowed, among other things, us to further uncover the hidden rich global structure of the rotating BTZ black hole spacetime. Namely, in the Kerr-Schild-type coordinates, it became natural to maximally extend the global structure by analytically continuing to “antigravity universe” region (i.e., negative $r$-region) just as we did in the case of Kerr black hole spacetime in 4-dim.

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Figure Caption

FIG.1 Carter-Penrose conformal diagram of *truly* maximally-extended spinning BTZ black hole spacetime. Note that it is almost the same as that before the analytic continuation to negative-\( r \) except for the emergence of negative-\( r \) region with a new timelike infinity at \( r = -\infty \).
