AN EXAMPLE OF A NEW SIMPLE THEORY

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Abstract. We construct a countable simple theory which, in Keisler’s order, is strictly above the random graph (but “barely so”) and also in some sense orthogonal to the building blocks of the recently discovered infinite descending chain. As a result we prove in ZFC that there are incomparable classes in Keisler’s order.

Recent work on the structure of Keisler’s order is changing our understanding of the so-called simple unstable theories, a class which includes the random graph and pseudofinite fields. Indeed, when we discovered recently [12] that Keisler’s order has infinitely many classes – overturning a long-standing idea that it had five or six – these infinitely many classes were within the simple theories. We are starting to see that the simple unstable theories may have very interesting layers of complexity above that of the random graph, arising from the interaction of randomness with underlying constraints, with no obvious analogue in the stable case.

Basic questions remain open about the structure of Keisler’s order on the simple theories, in part because of a lack of examples. For instance, among the unstable theories, the Keisler-minimum class is the class of the random graph. The infinitely many classes arise from disjoint unions of certain random hypergraphs with forbidden substructures (higher analogues of the triangle-free random graph [4]) forming a descending chain above the random graph. It was not known whether there were theories strictly between the random graph and this infinite descending chain.

In the present paper, we build a new simple theory, really a family of theories, illustrating how the randomness may interact with an underlying structure with some forbidden configurations, even with only a graph edge (no hyperedges required). We analyze its saturation and non-saturation in regular ultrapowers, concluding it is strictly above the random graph but in a precise sense orthogonal to the higher analogues of the triangle-free random graph. As a consequence of our strategy, we prove there are incomparable classes in Keisler’s order just in ZFC; this was known to be true under the existence of a supercompact cardinal, as noticed independently by Ulrich [15] and the authors [13]. Along the way we give a gentle introduction to some key methods of [9], [11], [12], the papers which have been the foundation of our current work on simple theories.

We conclude with some comments on open problems and on work in progress.

Thanks: Malliaris was partially supported by DMS-1553653 and by a Minerva Research Foundation membership at the IAS. Shelah was partially supported by European Research Council grant 338821. Both authors thank NSF grant 1362974 (Rutgers) and ERC 338821. This is paper 1140 in Shelah’s list.
1. Basic definitions

Keisler’s order compares complete countable theories via the difficulty of saturating their regular ultrapowers.

**Definition 1.1** (Keisler’s order, 1967 [5]). Let $T_1, T_2$ be complete countable theories. We say $T_1 \preceq T_2$ if for every infinite $\lambda$, every regular ultrafilter $\mathcal{D}$ on $\lambda$, every model $M_1 \models T_1$, every model $M_2 \models T_2$, if $(M_2)^\lambda/\mathcal{D}$ is $\lambda^+$-saturated, then $(M_1)^\lambda/\mathcal{D}$ is $\lambda^+$-saturated.

We remind the reader of Keisler’s result that if $\mathcal{D}$ is a regular ultrafilter on $\lambda$ and $M \equiv N$ in a countable language, then $M^\lambda/\mathcal{D}$ is $\lambda^+$-saturated iff $N^\lambda/\mathcal{D}$ is $\lambda^+$-saturated. Thus, the choice of $M_1$ and $M_2$ in Definition 1.1 is only important up to elementary equivalence. A further introduction to Keisler’s order can be found in the recent lecture notes [1] sections 2-3 and in sections 1-2 of [11].

**Convention 1.2.** In what follows:

1. All theories are complete and countable, unless otherwise stated.
2. When $\mathcal{D}$ is a regular ultrafilter on $I$ and $T$ is a theory, we will say “$\mathcal{D}$ is good for $T$” to mean that for some (equivalently every) model $M \models T$, the ultrapower $M^I/\mathcal{D}$ is $|I|^+\text{-saturated}$.

Our recent work on simple theories started with an idea in [9] to increase the range of ultrafilter construction. The idea is to build regular ultrafilters in two stages: by building a regular filter $\mathcal{D}_0$ on $I$ so that the quotient Boolean algebra $\mathcal{P}(I)/\mathcal{D}_0$ is isomorphic to some specific Boolean algebra $\mathcal{B}$, and then building an ultrafilter $\mathcal{D}_*$ on $\mathcal{B}$ (which, a priori, need not be regular), finally combining $\mathcal{D}_0$ and $\mathcal{D}_*$ in the natural way to obtain an ultrafilter $\mathcal{D}$ on $I$.

A key lemma, “separation of variables,” says that in this setup there is a natural translation between realizing types in the ultrapower $M^I/\mathcal{D}$ and showing that certain related patterns, called “possibility patterns,” have multiplicative refinements in $\mathcal{D}_*$ (see below). As a result, in many subsequent saturation-of-ultrapowers arguments it is most convenient to work in the Boolean algebra $\mathcal{B}$, a completion of a free boolean algebra.

**Definition 1.3.** $\mathcal{B}_{2^\lambda, \mu, \theta}$ is the free Boolean algebra generated by $2^\lambda$ independent partitions each of size $\mu$, where intersections of $< \theta$ elements of distinct partitions are nonempty. For a boolean algebra $\mathcal{B}$, let $\mathcal{B}^1$ denote its completion.
In the context of our free Boolean algebras, the following notation will be helpful in referring to basis elements (“choose \( g \in \text{FI}_{\mu,\theta}(2^\lambda) \) so that \( x_g \leq \varepsilon \)).

**Definition 1.4.** Let

\[
\text{FI}_{\mu,\theta}(2^\lambda) = \{ h : h \text{ is a function, } \text{dom}(h) \subseteq 2^\lambda, \text{range}(h) \subseteq \mu, |\text{dom}(h)| < \theta \}.
\]

For \( g \in \text{FI}_{\mu,\theta}(2^\lambda) \), let \( x_g \) denote the corresponding nonzero element of \( \mathfrak{B} \).

**Convention 1.5.** We will assume that giving \( \mathfrak{B} = \mathfrak{B}_{\alpha,\mu,\theta}^1 \) determines \( \alpha, \mu, \theta \) and a set of generators \( \langle x_f : f \in \text{FI}_{\mu,\theta}(\alpha) \rangle \).

**Definition 1.6** (Regular ultrafilters built from tuples, from \([9]\) Theorem 6.13). Suppose \( \mathcal{D} \) is a regular ultrafilter on \( I, |I| = \lambda \). We say that \( \mathcal{D} \) is built from \((\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)\) when:

1. \( \mathcal{D}_0 \) is a regular, \(|I|^{+}\)-excellent filter on \( I \)
   (for our purposes here, it is sufficient to use regular and good)
2. \( \mathfrak{B} \) is a Boolean algebra
3. \( \mathcal{D}_* \) is an ultrafilter on \( \mathfrak{B} \)
4. there exists a surjective homomorphism \( j : \mathcal{P}(I) \to \mathfrak{B} \) such that:
   a. \( \mathcal{D}_0 = j^{-1}(\{1_{\mathfrak{B}}\}) \)
   b. \( \mathcal{D} = \{ A \subseteq I : j(A) \in \mathcal{D}_* \} \).

**Theorem 1.8** (“Separation of variables”, \([9]\) Theorem 6.13). Suppose that \( \mathcal{D} \) is a regular ultrafilter on \( I \) built from \((\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)\). Then the following are equivalent:

1. \( \mathcal{D}_* \) is \((|I|, \mathfrak{B}, T)\)-moral, see \([9]\) Definition 6.3.
2. \( \mathcal{D} \) is good for \( T \).

The definition of “moral” is a bit long to quote, but can be easily summarized by saying that the way we will use Theorem 1.8 is the following. Suppose \( \mathcal{D} \) is a regular ultrafilter on \( I, |I| = \lambda \), \( \mathcal{D} \) is built from \((\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)\), \( M \models T \), and \( N = M^I/\mathcal{D} \) is the ultrapower. Suppose \( p = \{ \varphi_\alpha(x, \bar{a}_\alpha) : \alpha < \lambda \} \) is a type in the ultrapower. Let \( \bar{B} = \langle B_\alpha : \alpha \in [\lambda]^{<\aleph_0} \rangle \) be a sequence of elements of \( \mathcal{D} \) such that for each finite \( u \),

\[
B_\alpha = \{ t \in I : M \models \exists x \left( \bigwedge_{\alpha \in u} \varphi_\alpha(x, \bar{a}_\alpha[t]) \right) \}.
\]

Let \( \bar{b} = \langle b_\alpha : \alpha \in [\lambda]^{<\aleph_0} \rangle \) be a sequence of elements of \( \mathcal{D}_* \) such that \( j(B_\alpha) = b_\alpha \) for each \( u \). Then to show \( p \) is realized, it will suffice to show that \( \bar{b} \) has a multiplicative refinement in \( \mathfrak{B} \), i.e. there exists a sequence \( \langle b'_\alpha : \alpha \in [\lambda]^{<\aleph_0} \rangle \) of elements of \( \mathcal{D}_* \) such that for each finite \( u, b'_\alpha \leq b_\alpha \) and such that for each finite \( u \), \( v, b'_\alpha \cap b'_v = b'_{\alpha \cup v} \).

**Notation 1.7.** Let \( \mathcal{F} = \{ f : f \text{ an increasing function from } \mathbb{N} \text{ to } N \setminus \{0\} \} \).

**Claim 1.8.** Let \( \mathfrak{B} = \mathfrak{B}_{2^\lambda,\mu,\theta}^1 \). Suppose \( \langle g_\alpha : \alpha \in U \rangle \) is a subset of \( \text{FI}_{\mu,\theta}(2^\lambda) \), where \( U \) is of cardinality > \( \mu \), and \( m < \omega \). Then there is \( u \subseteq U, |u| = m \) such that \( \bigcup_{\alpha \in u} g_\alpha \) is a function.

**Proof.** By induction on \( n \leq m \) we will choose \( \beta_n, U_n, \) and \( \langle f^n_\alpha : \alpha \in U_n \rangle \) such that:

1. \( \beta_n \in U_{n-1}, \) where \( U_{-1} = U \)
2. \( U_n \subseteq U \) is of cardinality > \( \mu \)
3. \( f^n_\alpha \in \text{FI}_{\mu,\theta}(2^\lambda) \) and \( f^n_\alpha \supseteq g_\alpha, \) for all \( \alpha \in U \)
4. \( n < n' \) implies \( f^n_\alpha \subseteq f^n_{\alpha'} \)
5. \( n < n' \) implies \( U_{n'} \subseteq U_n \)
We will repeatedly use the fact that no set of size $> \mu \leq M$. M. MALLIARIS AND S. SHELAH say $\langle t | _U$ the construction could not continue), and since a subset $U$ parameter an increasing function $\alpha$ element $x$. Let $\beta \alpha \in U$ for every $\alpha \in U_{n-1}$, the element $x_{\alpha_{n-1}}$ must be compatible with one of the elements of our antichain (since the construction could not continue), and since $U_{n-1}$ is of cardinality $> \mu$, there is a subset $U_n \subseteq U_{n-1}$ of cardinality $\mu$ whose elements are all $\beta_n$ compatible with a single element of the antichain. Let $\beta_n$ be the subscript of this single element.

For all $\alpha \in U_n$, let $f^\alpha_n = f^{\alpha_{n-1}}_n \cup f^{\alpha_{n-1}}_{\beta_n}$. (For $\alpha \in U \setminus U_n$, let $f^\alpha_n = f^{\alpha_{n-1}}_{\beta_n}$.) So we can carry the construction.

When we arrive to $m$, we have defined $f^{m}_{\beta_0}, \ldots, f^{m}_{\beta_m}$ and by construction,

$$0 < x_{f^m_{\beta_m}} \leq \cdots \leq x_{f^m_{\beta_0}}.$$  

Recalling (3) and (4) of the inductive hypothesis, we conclude that

$$x_{f^m_{\beta_0}} \cap \cdots \cap x_{f^m_{\beta_m}} > 0$$

i.e. when $u = \{\beta_0, \ldots, \beta_m\}$, $\bigcup_{\beta \in u} g_\beta$ is a function, as desired. (This proof is also easy by the $\Delta$-system lemma.)

\[\square\]

2. Construction

In this section we build our example, really a family of examples taking as a parameter an increasing function $f \in \mathcal{F}$ (see [17]). We begin with the definition, and continue with a leisurely discussion of the types in models of this theory.

**Notation 2.1.** Let $f \in \mathcal{F}$.

(a) For each $k < \omega$, let $T_k = T_{f,k} = \{ \eta : \eta \in k^\omega \text{ and } \ell < k \to \eta(\ell) \leq f(\ell) \}$.

(b) Let $T_{[k]} = T_{f,[k]} = \bigcup_{j \leq k} T_j$.

(c) Let $T = T_f = \bigcup_{k < \omega} T_k$.

(d) Let $\lim(T) = \{ \eta \in \omega^\omega : \eta | k \in T_k \text{ for all } k < \omega \}$.  

(e) We may write $T, T_k, \text{ etc.}$ omitting $f$ when it is clear from context.

Definition 2.2 describes a subtree of $T_{[k]}$ which is maximal subject to containing no full splitting.

**Definition 2.2.** We say $s \subseteq T_{[k]}$ is $k$-maximal if either $k = 0$, or $k > 0$ and $s$ satisfies: it is nonempty and closed under initial segment, every maximal node belongs to $T_k$, for no $\rho \in T_{[k]}$ do we have $\{ \rho^- | \ell : \ell \leq f(|\rho|) \} \subseteq s$, and $s$ is maximal subject to these conditions.

**Notation 2.3.** Let $f \in \mathcal{F}$.

(f) Let $S_k = S_{f,k} = \{ s : s \subseteq T_{[k]} \text{ is } k\text{-maximal} \}$.

(g) Let $S = S_f = \bigcup_k S_k$.

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(6) for every $\alpha \in U_n$, $x_{f^\alpha_n} \leq x_{f^\beta_n}$
(h) Let \( \lim(S) = \{ \nu : \nu = (s_k : k < \omega) \text{ is such that } s_k \in S_k \text{ for each } k < \omega \text{ and } s_k \subseteq s_{k'} \text{ for all } k < k' < \omega \} \).

(i) We may write \( S, S_k, \text{ etc. } \) omitting \( f \) when it is clear from context.

To emphasize that \( \lim(S) \) and \( \lim(T) \) are parallel, we might think of elements of \( \lim(T) \) as sequences \( \langle \eta_k : k < \omega \rangle \) where each \( \eta_k \in T_k \) and \( k < k' < \omega \) implies \( \eta_k \sqsubseteq \eta_{k'} \). Elements of \( \lim(T) \) are branches of \( T \) under the ordering \( \sqsubseteq \), and elements of \( \lim(S) \) are branches of \( S \) under the ordering \( \subseteq \). In particular, since all of the \( T_k \)'s and \( S_k \)'s are finite and \( f \) is increasing, both \( T \) and \( S \) are countable, and both \( \lim(T) \) and \( \lim(S) \) have cardinality \( \leq 2^{\aleph_0} \).

We will define our theory \( T_f \) as the model completion of the following universal theory \( T_{0,f} \).

**Definition 2.4.** Let \( f \in F \). We define \( T_{0,f} \) as the following universal theory.

1. \( \tau(T_{0,f}) = \{ P, Q, R, P_{\eta}, Q_s : \eta \in T_k, s \in S_k, k < \omega \} \)
   where \( R \) is a binary predicate and the \( Q_s \)'s, \( P_\eta \)'s, \( P, Q \) are unary predicates.

2. For a \( \tau \)-model \( M \), \( M \models T_{0,f} \) iff:
   (a) \( P^M, Q^M \) is a partition of \( M \)
   (b) for each \( k < \omega \), \( (P^M_\eta : \eta \in T_k) \) is a partition of \( P^M \)
   (c) for each \( k < \omega \), \( (Q^M_s : s \in S_k) \) is a partition of \( Q^M \)
   (d) \( \eta \sqsubseteq \eta' \in T \) implies \( P^M_\eta \subseteq P^M_{\eta'} \) and \( s \subseteq s' \in S \) implies \( Q^M_s \subseteq Q^M_{s'} \)
   (e) \( R^M \subseteq Q^M \times P^M \) satisfies:
      - if \( \eta \in T_k \), and \( b, a \in P^M_{\eta' \ell \cap \langle f(k) \rangle} \) for \( \ell \leq f(k) \), then
        \[ M \models \neg(\exists x)(Q(x) \land \bigwedge_{\ell \leq f(k)} R(x, b)) \]
      - if \( (b, a) \in R^M \) and \( b \in Q_s, s \in S_k, a \in P_\eta, \eta \in T_k \) then \( \eta \in s \).

3. Let \( T_f \) be the model completion of \( T_{0,f} \).

Informal description:

\( Q \) and \( P \) partition the domain. If we think of the elements of \( P \) as being indexed by the leaves of some infinite, finitely branching tree whose branching at height \( k \) is \( f(k) \), then for each \( \eta \in T_k \), \( P_\eta \) names the elements in the cone above \( \eta \). In particular, for each finite \( k \) and \( \eta \in T_k \), \( P^M_{\eta' \ell \cap \langle f(k) \rangle} \) for \( \ell \leq f(k) \) partition \( P^M_{\eta'} \).

\( R \), the edge relation, relates elements of \( Q \) to elements of \( P \). The only restriction on \( R \) is that for each element \( b \in Q \) and each \( k < \omega \) and \( \eta \in T_k \), \( b \) cannot connect to some element in each piece of the partition \( \{ P^M_{\eta' \ell \cap \langle f(k) \rangle} : \ell \leq f(k) \} \). So for each \( b \in Q^M \), each \( k < \omega \) and each \( \eta \in T_k \), either \( R(b, x) \) misses the set \( P^M_\eta \) entirely, or else it misses one of \( P^M_{\eta' \ell \cap \langle f(k) \rangle} \) for \( \ell \leq f(k) \).

The example could end here, except that we would like quantifier elimination, which is where the \( Q_s \)'s will help. These predicates name already definable sets. \( Q^M_s(b) \) tells us that \( \{ \eta \in T_k : M \models (\exists x)(R(b, x) \land P_\eta(x)) \} = s \in S_k \). [Of course we can’t formally quantify over \( \eta \in T_k \), but for each \( k \), there are only finitely many \( P_\eta \)'s to consider, so the set named by \( Q_s \) is indeed a definable set.] Condition (2)(c) uses that the \( s \) are maximal in the sense of \( \subseteq \).

Note that in models of \( T_f \), the model completion, whenever \( R(b, x) \) is consistent with \( P_\eta(x) \), exactly one of the formulas \( \{ P_{\eta' \ell \cap \langle f(k) \rangle}(x) : \ell \leq f(k) \} \) will be inconsistent with \( R(b, x) \). So the \( Q_s \)'s with \( s \) maximal are indeed the relevant ones.
Claim 2.5. \( T_{0,f} \) has a model completion \( T_f \). In fact \( T_{0,f} \) is relational, has amalgamation and the joint embedding property. Moreover \( T_f \) is simple, has elimination of quantifiers (well, for formulas in at least one free variable, as there are no individual constants), has no algebraicity, and has trivial forking.

Proof Sketch. We’ll briefly describe the two key cases for quantifier elimination, ignoring the atomic formulas \( x = a \).

(a) given \( P(y_0),\ldots,P(y_n) \), does there exist \( x \) s.t. \( Q(x) \land \bigwedge_{i \leq n} R(x,y_i) ? \)

Informally, the answer is no if and only if the \( y_i \)’s fall in all pieces of some successor partition, that is, for some \( k < \omega \) and \( \eta \in T_k \), for each \( \ell \leq f(k+1) \) there is at least one \( i \leq n \) such that \( P^M_{\eta^-(\ell)}(y_i) \). Since \( f \) is increasing \([\underline{1.7}]\), we can choose \( k_\ast \) such that \( f(k_\ast) > n \), and it is then sufficient to ensure that this doesn’t happen for any \( \eta \in T_\ast \). Since the number of pieces in each successor partition is finite and \( n \) is finite, this can be expressed by a disjunction of all legal possibilities.

If we now modify (a) by adding finitely many conditions on the \( y_i \)’s, of the form \( P_\eta(y_i) \) or \( \neg P_\eta(y_i) \) for some \( \eta \in T \), and by adding finitely many conditions on \( x \) of the form \( Q_s(x) \) or \( \neg Q_s(x) \) for some \( s \in S \), these conditions simply affect which of all the possible “legal possibilities” remain legal.

(b) given \( Q(y_0),\ldots,Q(y_n) \), does there exist \( x \) s.t. \( P(x) \land \bigwedge_{i \leq n} R(y_i,x) ? \)

Let \( k_\ast \) be minimal so that \( f(k_\ast) > n \). Informally, the answer to (b) is yes if and only if for some \( \ell \leq f(k_\ast+1) \), and some \( \eta \in T_{k_\ast} \), \( \bigwedge_{i \leq n} (R(y_i,x) \land P^M_{\eta^-(\ell)}(x)) \) is consistent \([\underline{1.7}]\) (Why? We may verify by induction on \( k \geq k_\ast \) that there is \( \eta_k \in T_k \) such that \( \bigwedge_{i \leq n} (R(y_i,x) \land P^M_{\eta_k}(x)) \) is consistent. Let \( \eta_\ast = \eta_k \). At \( k > k_\ast \), for each \( i \leq n \), \( R(y_i,x) \) will be inconsistent with precisely one of the formulas \( \{P^M_{\eta^-(\ell)} : \ell \leq f(k+1)\} \), so \( n \) is not large enough to rule out one which works for all. Conversely, if the answer to (b) is no, we can bound the height of an inconsistency by noting that each \( y_i \) which is consistent with \( P_\nu \) can only miss one piece of the successor partition \( \{P^M_{\eta^-(\ell)} : \ell \leq f(\lg(\nu))\} \).

Since the number of pieces in each successor partition is finite and \( n \) is finite, (b) can be expressed by a disjunction over all legal possibilities, using the possible types for the \( y_i \) as expressed by the \( Q_s \)’s. As before, if we modify (b) by adding conditions on the \( y_i \)’s expressing which pieces of which successor partitions they may miss (expressible by the \( Q_s \)’s) and by adding conditions on \( x \) expressing in which pieces of which partitions it may be (expressible by the \( P_\nu \)’s), these conditions simply affect which of all the possible “legal possibilities” remain legal. \( \square \)

Remark 2.6. The properties of \( T_{0,f} \) listed above in \([\underline{2.6}]\) are not themselves sufficient to imply \( T_{0,f} \) has a model completion, as \( \tau(T_{0,f}) \) is a compact, but still it is easy to prove there is one by considering \( \tau_n = \{P_\eta,Q_\nu,Q,P,R : \eta \in \bigcup_{k < n} T_k, \nu \in \bigcup_{k < n} S_k\} \) as explained above.

Before proceeding to an analysis of saturation and non-saturation in ultrapowers of models of \( T_f \), let us briefly describe the types we will be dealing with. Recall that \( T_{\operatorname{rg}} \) is the theory of the random graph.

\[1\text{Note this is equivalent to saying: for every } k \leq k_\ast, \text{ for some } \ell \leq f(k+1), \text{ and some } \eta \in T_k, \text{ we have that } \bigwedge_{i \leq n} (R(y_i,x) \land P^M_{\eta^-(\ell)}(x)) \text{ is consistent.}\]
Claim 2.7. Let $M \models T_I$. Let $I$ be any infinite set, $|I| = \lambda$, and $D$ a regular ultrafilter on $I$ which is good for $T_{rg}$. Then to show $N = M^I/D$ is $\lambda^+$-saturated, it suffices to show that $N$ realizes all partial types of the following form:

(a) $\{Q(x)\} \cup \{R(x,a) : a \in A\}$ for $A \subseteq P^N$, $|A| \leq \lambda$.
(b) $\{P(x)\} \cup \{R(b,x) : b \in B\}$ for $B \subseteq Q^N$, $|B| \leq \lambda$.

Proof. Fix a model $N$. To analyze saturation of $N$, it suffices to consider 1-types, so there are two cases: the type contains $P(x)$, or it contains $Q(x)$. Since the ultrafilter is good for the random graph, necessarily $\mu(D) \geq \lambda^+$ (i.e. any pseudofinite set has size at least $\lambda^+$), so we may safely ignore formulas of the form “$x \neq a$.” Recall that for models with countable vocabularies, which is always our case here, saturation of ultrapowers reduces to saturation of $\varphi$-types ([7], Theorem 12). [So we should expect to not need to consider types with infinitely many distinct $Q_s$’s or $P_q$’s.]

Let $p \in S(C,N)$, $C \subseteq N$, $|C| \leq \lambda$, $p$ not algebraic.
Without loss of generality,

(i) $N \models C \subseteq N$.
(ii) if $\eta \in T$, $Q(x) \in p(x)$ and $(\exists z)(P_\eta(z) \land R(x,z)) \in p$, then for some $a_\eta \in P^N_\eta$ we have $R(x,a_\eta) \in p$.
(iii) if $s \in S$, $P(x) \in p(x)$ and $(\exists z)(Q_s(z) \land R(z,x)) \in p$, then for some $b_s \in Q^N_s$ we have $R(b_s,x) \in p$.

[We can assume this because for all $\eta \in T$ and all $s \in S$, $|P^N_\eta|$, $|Q^N_s|$ are $> \lambda$ because $D$ is regular and $N = M^I/D$.]

Now let:

(a) $A_1 = \{a \in C : P(x) \land R(a,x) \in p \text{ or } Q(x) \land R(x,a) \in p\}$. 
(b) $A_0 = \{a \in C : P(x) \land \neg R(a,x) \in p \text{ or } Q(x) \land \neg R(x,a) \in p\}$. 
(c) if $P(x) \in p(x)$ then let:
  • $\eta = \eta_p \in \text{lim}(T)$ be such that $p_2(x) := \{P_\eta|x(\ell) : \ell < \omega\} \subseteq p(x)$,
  • $p_1(x) = \{R(a,x) : a \in A_1\}$, and
  • $p_0(x) = \{\neg R(a,x) : a \in A_0\}$.
(d) if $Q(x) \in p(x)$ then let:
  • $\nu = \nu_p \in \text{lim}(S)$ be such that $p_2(x) := \{Q_\nu|x(\ell) : \ell < \omega\} \subseteq p(x)$,
  • $p_1(x) = \{R(x,a) : a \in A_1\}$, and
  • $p_0(x) = \{\neg R(x,a) : a \in A_0\}$.

The $p$ is equivalent to $p_1(x) \cup p_0(x) \cup p_2(x)$, by elimination of quantifiers and our assumptions (i),(ii),(iii).

Now if $\eta \in T$, $P_\eta(x) \in p$ then $p_1(x) \vdash P_\eta(x)$. [Why? Let $k = \lg(\eta)$ and let $S^* = \{s \in S_k : \eta \in s\} \subseteq S_k$. So for each $s \in S^*$, there is $b_s \in Q^N_s$ such that $R(b_s,x) \in p$ by (ii), so it is easy to see that $\{R(b_s,x) : s \in S^*\} \cup P_\eta(x)$.

Similarly, if $Q_s(x) \in p(x)$ for $s \in S$ then $p_1(x) \vdash Q_s(x)$. So in both cases,

$$p_1(x) \vdash p_2(x).$$

It remains to handle $p_0$. Suppose, then, that our ultrapower realizes $p_1(x)$. The sets $A_1,A_0 \subseteq N$ defined in (a), (b) are disjoint and both of size $\leq \lambda$. As $D$ is regular, there are disjoint pseudo-finite internal sets $X_0, X_1$ such that $A_\ell \subseteq X_\ell$ for $\ell = 0, 1$ and $X_0 \cap X_1 = \emptyset$. Suppose $c \in N$ realizes $p_1(x)$. There are two cases depending on whether $Q(x) \in p$ or $P(x) \in p$. If $Q(x) \in p$, there is $c' \in N$ such that:

• $(\forall y \in X_1)(R(c',y) \iff R(c,y))$
• $(\forall y \in X_0)(\neg R(c',y))$. 
So \( \ell \) realizes \( p_1(x), p_2(x) \) and \( p_0(x) \), so we are done. If \( P(x) \in p \), replace \( R(\ell', y) \) by \( R(y, \ell') \) and \( R(\ell, y) \) by \( R(y, \ell) \) in the above quotation. This completes the proof. \( \square \)

3. A NON-SATURATION RESULT FOR \( T_f \)

**Notation 3.1.** When \( \langle a_\alpha : \alpha < \kappa \rangle \), and \( w \subseteq \kappa \), write \( \bar{a}_w \) to mean \( \langle a_\alpha : \alpha \in w \rangle \).

Recall also our notation for \( T \), \( T_{[k]} \), etc. from 2.1.

**Lemma 3.2.** Let \( f \in F \). Let \( \lambda \) be any infinite cardinal and \( \mu < \kappa \leq 2^{\aleph_0} \). Then no ultrafilter \( D \) of \( \mathcal{B} = \mathcal{B}^1_{\lambda, \mu, \kappa} \) is \( (\kappa^+, T_f) \)-moral.

**Proof.** We use the setup of separation of variables, so we have in mind the background set \( I = \lambda \), a homomorphism \( j : \mathcal{P}(\lambda) \to \mathcal{B} \), an ultrafilter \( D_* \) on \( \mathcal{B} \), and also a model \( M \models T_f \) and an ultrapower \( M^f/D \). We will build a possibility pattern in the variables \( x \) and \( \langle x_\alpha : \alpha < \kappa \rangle \) and show it doesn’t have a multiplicative refinement.

Recall that \( \mathcal{B}^+ = \mathcal{B} \setminus \{0\} \).

(3.A) For each \( \rho \in T_k \) and \( \alpha < \kappa \), define \( a[P_\rho(x_\alpha)] \in \mathcal{B}^+ \) by induction on \( k < \omega \):

\((a)\) if \( k = 0 \), i.e. \( \rho \upharpoonright k = \epsilon \), \( a[P_\rho(x_\alpha)] = \mathbf{1}_{\mathcal{B}} \).

\((b)\) for \( k \) and \( i \leq f(k) \), let \( g_{\alpha, k, i} \) be the function with domain \( \{\omega \alpha + k\} \) such that \( g_{\alpha, k, i}(\omega \alpha + k) = i \). Then for \( \rho \in T_k \), \( \ell \leq f(k) \) define

\[ a[P_\rho \hat{\cdot}(x_\alpha)] = a[P_\rho(x_\alpha)] \cap x_{g_{\alpha, k, i}}. \]

\((c)\) for \( \alpha \neq \beta < \kappa \),

\[ a[x_\alpha \neq x_\beta] = 1_{\mathcal{B}}. \]

Without loss of generality we assume \( x_{g_{\alpha, k, 0}} \in D_* \) for \( \alpha < \kappa \).

By (3.A),

(3.C)

\[ a[x_\alpha \neq x_\beta] \in D_* \]

and by quantifier elimination, \( a[\varphi(\bar{x}, \bar{v})] \) is determined for any finite \( v \subseteq \kappa \) and any \( \varphi(\bar{x}, \bar{v}) \) in the language of \( T \). It follows that for each \( \alpha < \kappa \) and each \( k < \omega \), \( a[P_{\langle 0, k \rangle}(x_\alpha)] \in D_* \), where \( \langle 0, k \rangle \) is \( (0, \ldots, 0) \) \( (k \) times). By (3.A), (3.B), (3.C), the sequence

\[ \bar{b} = \langle b_u : u \in [\kappa]^{<\kappa_0} \rangle \quad \text{where} \quad b_u = a[\exists x \bigwedge_{\alpha \in u} R(x, x_\alpha)] \]

is a possibility pattern. [That is, in the related ultrapower we are considering the type \( \{R(x, a_\alpha) : \alpha < \kappa\} \) where the \( a_\alpha \)'s are pairwise distinct realizing the type \( \{P_{\rho, \langle k \rangle}(x) : k < \omega\} \) where \( \rho = (0, 0, \ldots) \).]

2The proof will take place entirely in \( \mathcal{B} \), but it is simply asserting that we could build a type \( \{R(x, a_\alpha) : \alpha < \kappa\} \) in the ultrapower \( M^f/D \), i.e. we could choose our parameters \( a_\alpha \), in such a way that each \( a[\varphi(\bar{x}, \bar{u})] \) in the proof below is really the image of \( \{t \in I : M \models \varphi[\bar{a}_u[t]]\} \) under \( j \), and likewise for each finite \( u \subseteq \kappa \), \( b_u \) is the image of \( \{t \in I : M \models \exists x \bigwedge_{\alpha \in u} R(x, a_\alpha[t])\} \) under \( j \).

3The intent is that the elements \( a_\alpha[t] \) all satisfy the “constant zero” branch in the ultrapower, so \( p(x) \) is clearly a type. The potential problem in realizing the type is that in each index model, by our construction (3.A), the elements \( a_\alpha[t] \) may project to different branches “\( D \)-rarely”, possibly violating the condition on splitting.
Towards contradiction assume

\[(3.E) \quad \bar{b}' = (b'_u : u \in [\kappa]^{<\aleph_0})\]

is a multiplicative refinement of \( \bar{b} \).

For \( \alpha < \kappa \) choose \( g_\alpha \in \text{Fl}_{\aleph_0}(2^\lambda) \) such that \( x_{g_\alpha} \leq b'_{(\alpha)} \). We will repeatedly use the fact that no set of size \( \mu \) (e.g. no set of size \( \kappa \)) is covered by a countable union of sets of cardinality \( \leq \mu \). Given an ordinal \( \beta \), let us say “the remainder of \( \beta \) mod \( \omega \) is \( k \)” to mean that \( \beta = \omega \gamma + k \) for some ordinal \( \gamma \) and integer \( k \).

Since each \( g_\alpha \) has finite domain, there is a smallest integer \( k_\alpha \) such that for every \( \beta \in \text{dom}(g_\alpha) \), the remainder of \( \beta \) mod \( \omega \) is \( \leq k_\alpha \). As \( \mathcal{T}_\alpha \) has cardinality \( > \mu \), there is some finite \( k_* \) such that \( \mathcal{U}_1 = \{ \alpha < \kappa : k_\alpha = k_* \} \) is of cardinality \( > \mu \).

For each \( \alpha \in \mathcal{U}_1 \), the elements

\[ \{ \text{a}[P_\nu(x_\alpha)] : \nu \in \mathcal{T}_\alpha, k \leq k_* \} \]

form a maximal antichain of \( b_\alpha \). For each \( \alpha \in \mathcal{U}_1 \), choose \( \nu_\alpha \in \mathcal{T}_{k_*} \) such that

\[ x_{g_\alpha} \cap \text{a}[P_{\nu_\alpha}(x_\alpha)] > 0. \]

By the construction in (3.A) above, we can translate as follows: letting

\[ g_\alpha^* = g_\alpha \cup \bigcup_{k \leq k_*} \{ (\omega \alpha + k, \nu_\alpha(k)) \} \]

we have that

\[(3.F) \quad x_{g_\alpha^*} = x_{g_\alpha} \cap \text{a}[P_{\nu_\alpha}(x_\alpha)] > 0. \]

Moreover, for each \( \alpha \in \mathcal{U}_1 \), it is still the case that for every \( \beta \in \text{dom}(g_\alpha^*) \), the remainder of \( \beta \) mod \( \omega \) is still \( \leq k_* \). Since \( \mathcal{T}_{k_*} \) is finite, for some \( \nu_* \in \mathcal{T}_{k_*} \),

\[ \mathcal{U}_2 = \{ \alpha \in \mathcal{U}_1 : \nu_\alpha = \nu_* \} \]

is of cardinality \( > \mu \).

Choose some very large finite \( m \) (at least \( f(k_* + 1) \) suffices, see below). Apply Claim 1.8 to \( (g_\alpha^* : \alpha \in \mathcal{U}_2) \) and \( m \), so the Claim says: Suppose \( (g_\alpha^* : \alpha \in \mathcal{U}_2) \) is a subset of \( \text{Fl}_{\mu, \kappa_0}(2^\lambda) \), where \( \mathcal{U}_2 \) is of cardinality \( > \mu \), and \( m < \omega \). Then there is \( u \subseteq \mathcal{U}_2 \), \( |u| = m \) such that \( \bigcup_{\alpha \in u} g_\alpha^* \) is a function.

Let \( u \subseteq \mathcal{U}_2 \) be as returned by Subclaim 1.8. Then \( g^* = \bigcup_{\alpha \in u} g_\alpha^* \) is a function, and it is still the case that for all \( \beta \in \text{dom}(g^*) \), the remainder of \( \beta \) mod \( \omega \) is \( \leq k_* \).

Since we assumed (for a contradiction) that \( b' \) is multiplicative and refines \( b \), and each \( x_{g_\alpha} \leq b_\alpha \), we have that for every \( v \subseteq u \),

\[(3.G) \quad 0 < x_{g^*} = \bigcap_{\alpha \in v} x_{g_\alpha} \leq \bigcap_{\alpha \in v} b_\alpha = b'_v \leq b_v. \]

Thus, for every \( \ell \leq f(k_* + 1) \) and every \( \alpha \in u \),

\[ \text{dom}(g^*) \cap \{ \omega \alpha + k_* + 1 \} = \emptyset \]

and clearly for \( \alpha \neq \alpha' \in u \),

\[ \{ \omega \alpha + k_* + 1 \} \cap \{ \omega \alpha' + k_* + 1 \} = \emptyset. \]

So for any \( f(k_* + 1) \) distinct \( \alpha \)'s in \( u \), say we enumerate them as \( (\alpha_\ell : \ell \leq f(k_* + 1)) \),

\[ x_{g_*} \cap \bigcap_{\ell \leq f(k_* + 1)} \text{a}[P_{\nu_*} \circ (\ell)(x_{\alpha_{\ell}})] > 0 \]
in other words, recalling (3.F),
\[
\bigcap_{\ell \leq f(k_*)} x_{g_{k_*}^{\alpha}} \cap \bigcap_{\ell \leq f(k_*)} a[P_{\nu^*}(x_{\alpha_{i_\ell}})] \cap \bigcap_{\ell \leq f(k_*+1)} a[P_{\nu^* \cdot \langle \ell \rangle}(x_{\alpha_{i_\ell}})] > 0
\]
Rewriting \( v = \{ \alpha_{i_\ell} : \ell \leq f(k_*) \} \subseteq u, \) this contradicts the fact from (3.G) that \( \bigcap_{\alpha \in v} x_{g_{k_*}^{\alpha}} \subseteq \bigcap_{\alpha \in v} x_{g_{\alpha}^\theta} \leq b_v. \) So there can be no such \( \bar{b}', \) which completes the proof. □

4. \( T_f \) is explicitly simple

This section proves Theorem 4.2, but its aim is equally or even primarily pedagogical, to exposit a way of measuring simple theories. In [11] we defined “explicit simplicity,” a way of stratifying the complexity of simple theories using cardinals \((\lambda, \mu, \theta, \sigma)\) satisfying Definition 4.1 below. This is motivated in the introduction to [11], §3. The definition of “\((\lambda, \mu, \theta, \sigma)\)-explicitly simple” implies that \( T \) is simple, and it follows from the definitions that this becomes weaker as \( \mu \) increases. [11] Theorem 4.10 had shown that this definition holds for any simple theory \( T \) when |\( T \)| < \( \sigma \) and we use the largest nontrivial number of “colors,” \( \mu^+ = \lambda \).

This new characterization of simplicity suggested a program of stratifying simple theories according to the necessary value of \( \mu \). For the random graph one color is enough, but it turned out in [12] that the case, of, say, the tetrahedron-free three-hypergraph, needs either \( \mu^+ = \lambda \) or \( \mu^{++} = \lambda \). Moreover, the idea of the infinite descending chain of [12] is essentially to look for theories whose \( \mu \) must satisfy \( \mu < \lambda \leq \mu^{+n} \) for larger and larger finite \( n \) inspired by ideas on free sets in set mappings and the Kuratowski-Sierpinski characterization of the \( \aleph_n \)'s (for more on set mappings, see [2], [6]).

We left as an open question, Question 10.1 of [11], whether it was possible to build a simple theory where \( \mu \), the range of the coloring function, is truly uncountable but does not depend on \( \lambda \).

In this section we show that \( T_f \) is not such an example. In light of §3 Theorem 4.2 tells us that when \( \sigma > |T| \), uncountable \( \mu \) is not necessary to be in a class strictly above the random graph. This is a delicate point: it highlights that the definition of explicit simplicity requires \( \sigma > |T| \), with the closure of a finite set (in the relevant algebra) giving rise to an elementary submodel. So \( \sigma > |T| \) is an assumption in Theorem 4.2 even though \( \theta > |T| \) would seem more natural for our case. For, as we will see, \( \sigma > |T| \) and knowing a type of \( T_f \) over an elementary submodel is already enough to control consistency of its automorphic images. Indeed, if \( \sigma \) were finite, the story would be different: we will see at the end of the proof below a suggested strategy for saturation of ultrapowers in §5.

We now review the setup.

**Definition 4.1 ([11] Definition 1.1)**. Call \((\lambda, \mu, \theta, \sigma)\) suitable when:

1. \( \sigma \leq \theta \leq \mu < \lambda \)

---

4This is motivation, not a mathematical statement: [12] isn’t carried out in the language of explicit simplicity, since there we had a concrete family of theories with trivial forking and so could simplify the definitions somewhat, e.g. dropping the requirement that closures be submodels.

5It is natural to give a general definition of explicit simplicity for \( \sigma = \aleph_0 \) which allows the closure of finite sets to be finite, and applies to theories with trivial forking, see after the proof.
Theorem 4.2. is suitable and $\sigma > \theta$ quoted for ease of reading. We are aiming for Definition 4.3; all the terms mentioned in the definition will be defined subsequently.

Proof. We will simply follow [11], Section 3, but all relevant definitions have been quoted for ease of reading. We are aiming for Definition 4.3 all the terms mentioned in the definition will be defined subsequently.

Definition 4.3 (Explicitly simple, [11] Definition 3.2). Assume $\lambda, \mu, \theta, \sigma$ are suitable. We say $T$ is $(\lambda, \mu, \theta, \sigma)$-explicitly simple if $T$ is simple and for every $N \models T$, $||N|| = \lambda$, $p \in \text{S}(N)$ nonalgebraic,

(a) there exists a presentation $m$ of $p$. 
(b) for every presentation $m$ of $p$, there is a presentation $n$ of $p$ refining $m$ and a function $G : R_n \to \mu$ such that $G$ is an intrinsic coloring of $R_n$.

$T_f$ is indeed simple. So suppose we are given $N \models T$, $||N|| = \lambda$, $p \in \text{S}(N)$ nonalgebraic. We will have two cases: the case where $P(x) \in p$, and the case where $Q(x) \in p$. The only difference will come at the end.

Definition 4.4 (Presentation, [11] Definition 3.3). Suppose we are given $N \models T$, $||N|| = \lambda$, and $p \in \text{S}(N)$. A $(\lambda, \theta, \sigma)$-presentation for $p$ is the data of an enumeration and an algebra,

$$m = ((\varphi_\alpha(x, a_\alpha^*) : \alpha < \lambda) : \mathcal{M})$$

where these objects satisfy:

(1) $p = (\varphi_\alpha(x, a_\alpha^*) : \alpha < \lambda)$ is an enumeration of $p$, which induces an enumeration $\langle a_\alpha^* : \alpha < \lambda \rangle$ of $\text{dom}(N)$, possibly with repetitions, and with the $a_\alpha^*$ possibly imaginary.

(2) $\mathcal{M}$ is an algebra on $\lambda$ with $< \theta$ functions.

(3) For any finite $u \subseteq \lambda$, $|\text{cl}_{\mathcal{M}}(u)| < \sigma$. Thus, for any $u \subseteq \lambda$, if $|u| < \sigma$ then $|\text{cl}_{\mathcal{M}}(u)| < \sigma$, and if $|u| < \theta$ then $|\text{cl}_{\mathcal{M}}(u)| < \theta$.

(4) $\text{cl}_{\mathcal{M}}(\emptyset)$ is an infinite cardinal $\leq |\mathcal{T}|$, so an initial segment of $\lambda$.

$M_* := N \upharpoonright \{ a_\alpha^* : \alpha < \text{cl}_{\mathcal{M}}(\emptyset) \}$ is a distinguished elementary submodel of $N$, and we require that $p$ does not fork over $M_*$. Moreover, for each $u \in [\lambda]^{<\sigma}$, $N_u := N \upharpoonright \{ a_\alpha^* : \alpha \in \text{cl}_{\mathcal{M}}(u) \}$ is an elementary submodel of $N$, and $\langle \varphi_\alpha(x, a_\alpha^*) : \alpha \in \text{cl}_{\mathcal{M}}(u) \rangle$ is a complete type over this submodel which dnf over $M_*$. (In particular, $\{ \varphi_\alpha(x, a_\alpha^*) : \alpha \in \text{cl}_{\mathcal{M}}(\emptyset) \}$ is a complete type over $M_*$.)

(6) If $\alpha \in \text{cl}_{\mathcal{M}}(u)$, $\beta \leq \alpha$, writing $A_\beta = \{ a_\gamma^* : \gamma < \beta \}$, we have that $\text{tp}(a_\alpha^*, A_\beta \cup M_*)$ does not fork over $\{ a_\gamma^* : \gamma \in \text{cl}_{\mathcal{M}}(u) \cap \beta \} \cup M_*$. In a context where $(\lambda, \theta, \sigma)$ are given, “presentation” means “$(\lambda, \theta, \sigma)$-presentation.”

To show $p$ has a presentation, first fix a countable elementary submodel $M_* \subseteq N$. Choose an enumeration $\langle \varphi_\alpha(x, a_\alpha^*) : \alpha < \lambda \rangle$ of $p$ so that:

- $\{ a_\alpha^* : \alpha < \omega \} = \text{dom}(M_*)$,
- $\{ \varphi(x, a_\alpha^*) : \alpha < \omega \} = p \upharpoonright M_*$,
- and $\{ a_\alpha^* : \alpha < \lambda \} = \text{dom}(N)$,
noting the sequence \((\alpha^*_\alpha : \alpha < \lambda)\) may have repetitions. This is easily done as \(x \neq a\) belongs to \(p\) for every \(a \in N\). For the algebra \(M\), we add three kinds of functions.

- First choose countably many unary functions \(\{g_n : n < \omega\}\), where \(g_n\) is the constant function \(n\), to ensure that the “closure of the empty set” is \(\omega\).
- Second, choose countably many functions which are analogues of Skolem functions for \(T_f\). That is, for each formula \(\varphi(x, \bar{y})\) of \(\mathcal{L}(\tau(T_f))\), let \(f_{\varphi(\bar{y})}\) be a new function symbol, and interpret these countably many new function symbols as Skolem functions for \(T_f\) in \(N\), in each case choosing the witness \(a^*_\alpha\) of smallest index \(\alpha < \lambda\). Then for each \(\varphi(x, \bar{y})\) add to the algebra a new \(\ell(\bar{y})\)-place function symbol \(gf_{\varphi(x, \bar{y})}\) which mirrors the action of the Skolem function on the indices \(\lambda\): \(gf_{\varphi}(\bar{a^*_\lambda}) = v\) only if \(f_{\varphi}(\bar{a}^*_\lambda) = a^*_\lambda\).
- Third, we want to ensure that the type restricted to closed sets is complete. For each \(\psi(x, y) \in \{R(x, y), x = y, Q(x) \land y = y, P(y)\} \cup \{Q_s(x) : s \in S\} \cup \{P_s(y^\prime) : \eta \in T\}\), let \(h_\psi(y)\) be defined so that \(h_\psi(\alpha) = \beta\) if \(\beta < \lambda\) is the least ordinal such that \(\varphi_\beta(x, a^*_\alpha)\) is equivalent mod \(T\) to either \(\psi(x, a^*_\alpha)\) or its negation\(^7\). Since there is no nontrivial forking, this suffices, and \((\varphi_\alpha(x,a^*_\alpha) : \alpha < \lambda), M\) is a presentation.

**Definition 4.5** (Refinements of presentations, Def. 3.6) Suppose we are given \(N \models T\), \(\|N\| = \lambda\), and \(p \in S(N)\). Let \(m = (\varphi_m, M_m)\), \(n = (\varphi_n, M_n)\) be presentations of \(p\). We say that \(n\) refines \(m\) when:

1. \(\varphi_m = \varphi_n\).
2. \(\text{cl}_{M_m}(\emptyset) = \text{cl}_{M_n}(\emptyset)\).
3. \(M_m \subseteq M_n\), i.e. the algebra of \(n\) extends that of \(m\).

In a refinement, the enumeration stays the same, the distinguished elementary submodel stays the same, but we may add a few more functions to the algebra if we wish. In our case it isn’t necessary; we’ll just show directly that every presentation has a coloring. For the rest of the proof, then, assume we have been given some fixed presentation \(m\).

**Definition 4.6.** (The set of quadruples \(R_m\), Def. 3.9) Let \(m\) be a presentation of a given type \(p = p_m\). Then \(R = R_m\) is the set of \(r = (u, w, q, r)\) such that:

1. \(u \in [\lambda]^{< \omega}, w \in [\lambda]^{< \omega}\) and \(w = \text{cl}_{M}(u)\).
2. \(u \subseteq \text{cl}_{M}(u) \subseteq w\).
3. \(q = q(\bar{w})\) is a complete type in the variables \(\bar{w}\) such that:
   - (a) for any finite \(v \subseteq \text{cl}_{M}(\emptyset)\), if \(M_* \models \psi(\bar{w})\) then \(\psi(\bar{w}) \in q\).
   - (b) for any finite \(\{\alpha_0, \ldots, \alpha_n\} \subseteq u\), \(\exists x \bigwedge_{0 \leq \alpha \leq n} \varphi_\alpha(x, a^*_\alpha) \in q\).
4. \(r = r(x, \bar{w})\) is a complete type in the variables \(x, \bar{w}\), extending \(q(\bar{w}) \cup \{\varphi_\alpha(x, a^*_\alpha) : \alpha \in u\}\).

\(^6\)we never consider \(\emptyset\) as a base set, so this effectively is the set which is contained in every closure of every nontrivial set; or if you prefer, consider an algebra to be a structure on \(\lambda\) with functions and no relations and a constant (interpreted as any element of \(\omega\)).

\(^7\)It’s sufficient if the restriction of \(p\) to a closed set generates a complete type. The reason to ask that \(\{\varphi(x, a^*_\alpha) : \alpha < \omega\} = p \upharpoonright M_*\) above with equality instead of \(\upharpoonright\) was just to ensure the closure of the empty set didn’t grow.
\(\text{(5) if } \bar{b}_w^* \text{ realizes } q(\bar{w}_w) \text{ in } \mathcal{E}_T \text{ and } \alpha < \text{cl}_{M}(\emptyset) \implies b_\alpha^* = a_\alpha^*, \text{ then}
\)
\begin{enumerate}
\item \(r(x, \bar{b}_w) \text{ is a type which does not fork over } M_* \text{ and extends } p \mid M_*\).
\item\(\text{if } w' \subseteq w \text{ is } M_\lambda \text{-closed, } \mathcal{E}_T \mid \{b_\alpha^* : \alpha \in w'\} \preceq \mathcal{E}_T \text{ and } r(x, \bar{b}_w) \mid \bar{b}_w \text{ is a complete type over this elementary submodel.}\)
\item\(\text{if } w' \subseteq w \text{ is } M_\lambda \text{-closed and } \alpha \in w' \text{ then } \text{tp}(b_\alpha^*, \{b_\beta^* : \beta \in w \cap \alpha\}) \text{ dnf over } \{b_\beta^* : \beta \in w' \cap \alpha\}.\)
\end{enumerate}

Note that \(u\) need not be closed. So in our case, \(r\) will describe a type in the variables \(x, \bar{x}_w\) which agrees with \(p \mid M_*\) on \(\{x_\alpha : \alpha < \omega\}\); it will then contain new conditions stating that \(x\) connects to additional elements \(x_\alpha\) and stating in which “leaves” of the tree these \(x_\alpha\)’s fall. [For example, if \(\{\varphi_\alpha(x, x_\alpha) : \alpha < \omega\} = \{R(x, x_\alpha) : \alpha < \omega\}\), a priori \(x_\alpha\) need not be in the same leaf as \(a_\alpha^*\) for \(\alpha \geq \omega\).]

**Definition 4.7** (A non-triviality condition, \[11\] Definition 3.10). Suppose we are given \(\bar{r} = (r_t = (u_t, w_t, q_t, r_t) : t < t_* < \sigma)\) from \(\mathcal{R}_m\). Say that \(\bar{b} = (b_\alpha^* : \alpha \in \bigcup_1 w_1)\), with each \(b_\alpha^* \in \mathcal{E}\) (possibly imaginary), is a good instantiation of \(\bar{r}\) when the following conditions hold.

1. \(\alpha \in \text{cl}_{M}(\emptyset) \implies b_\alpha^* = a_\alpha^*.\)
2. For each \(t < t_*\), \(\bar{b} \mid w_t\) realizes \(q_t(\bar{w}_t).\)
3. For each \(t < t_*\), if \(v \subseteq w_t \cap w_{t'}\) is finite, then:
   (a) For each formula \(\psi(\bar{r}_t)\), \(\psi(\bar{b}_t) \in q_t \iff \psi(\bar{b}_{t'}) \in q_{t'}\).
   (b) For each formula \(\psi(x, \bar{r}_t)\), \(\psi(x, \bar{b}_t) \in r_t \iff \psi(x, \bar{b}_{t'}) \in r_{t'}\).
4. If \(\beta \in w_{t'}\) for some \(t < t_*\) then
   \[\text{tp}(b_\beta^*, \{b_\gamma^* : \gamma \in \bigcup_{s \leq t} w_s \text{ and } \gamma < \beta\}) \text{ dnf over } \{b_\beta^* : \beta \in w_{t' \cap \beta}\}.\]
5. For each \(t < t_*\), if \(w' \subseteq w\) and \(\text{cl}_{M}(w') = w'\) then \(\mathcal{E}_T \mid \{b_\alpha^* : \alpha \in w'\} \preceq \mathcal{E}_T\) and \(r_t(\bar{b}_w)\) is a complete type over this elementary submodel which does not fork over \(M_*\) (noting that the domain of \(M_*\) is \(\{b_\alpha^* : \alpha \in \text{cl}_{M}(\emptyset)\}\) by the first item).

Definition 4.7 is simply to make the definition of coloring meaningful by ruling out trivial inconsistency, as will be clear from the next definition.

**Definition 4.8** (Coloring, \[11\] Definition 3.11). Let \(m\) be a \((\lambda, \theta, \sigma)\)-presentation and \(\mathcal{R} = \mathcal{R}_m\) be from \[4.6\]. Call \(G : \mathcal{R}_m \rightarrow \mu\) an intrinsic coloring of \(\mathcal{R}_m\) if whenever
\[\bar{r} = (r_t = (u_t, w_t, q_t, r_t) : t < t_* < \sigma)\]
is a sequence of elements of \(\mathcal{R}_m\) and \(\bar{b} = (b_\alpha^* : \alpha \in \bigcup_{t \leq t_*} w_t)\) is a good instantiation of \(\bar{r}\), if \(G \mid \{r_t : t < t_*\}\) is constant, then the set of formulas
\[\{\varphi_\alpha(x, b_\alpha^*) : \alpha \in u_t, \varphi_\alpha \in r_t, t < t_*\}\]
is a consistent partial type which does not fork over \(M_*\).

**End of quotations**
It remains to find a coloring given our fixed $m$, and therefore $R$.

**Case 1:** $Q(x) \in p$. The surprise in this case is that since $p \upharpoonright M_x$ is determined, we know whether $Q_x(x)$ for all $s \in S_k$ and all $k < \omega$. This means the set $\Lambda$ of possible leaves $\eta$ such that $p \upharpoonright M_x \cup \{R(x,a) \cup \{P_{\eta \upharpoonright k}(a) : k < \omega\} \}$ is fixed by $p \upharpoonright M_x$ and inherited by any $r$ from some $r \in R$. So whenever

$$\bar{\eta} = (t_\ell = (u_\ell, w_\ell, q_\ell, r_\ell) : t < t_\ast < \sigma)$$

is a sequence of elements of $R_m$ and $\bar{b}^* = (b^*_\alpha : \alpha \in \bigcup_{t < t_\ell} w_t)$ is a good instantiation of $\bar{\eta}$, it must be the case that for each $t < t_\ast$ and each $\alpha \in w_t$, $r_\ell$ determines that the leaf of $b^*_\alpha$ must be $\eta$ for some $\eta \in \Lambda$. It follows that

$$\{\varphi_\alpha(x, b^*_\alpha) : \alpha \in u_t, \, \varphi_\alpha \in r_\ell, \, t < t_\ast\}$$

is a consistent partial type which does not fork over $M_x$.

**Case 2:** $P(x) \in p$. Since $p$ is a type, there will already be $\eta_\ast \in \text{lim}(T)$ such that $p \vdash P_{\eta_\ast \upharpoonright k}(x)$ for all $k < \omega$. This information will be part of $p \upharpoonright M_x$. So only a single color is needed.

That is, suppose we are given a sequence $\bar{r}$ of elements of $R$, on which $G$ is constant and equal to $\beta$, and $\bar{b}^* = (b^*_\alpha : \alpha \in \bigcup_{t < t_\ell} w_t)$ which is a good instantiation of $\bar{r}$. As we have ruled out trivial inconsistency, by our observation,

$$\{\varphi_\alpha(x, b^*_\alpha) : \alpha \in u_t, \, \varphi_\alpha \in r_\ell, \, t < t_\ast\} \cup \{P_{\eta_\ast \upharpoonright k}(x) : k < \omega\}$$

is a consistent partial type which does not fork over $M_x$, and this suffices. \qed

**Discussion 4.9.** It is still interesting to ask what would happen if we had available only finitely much information from $p \upharpoonright M_x$. Would some coloring work, which does not rely on having already determined the predicates $Q_S$ or $P_\nu$? In the remainder of this section we consider this, which will give the key idea for dealing with ultrapowers in the next section. We handle just the case of $Q(x) \in p$ as an illustration, since both cases are worked out in detail in the next section.

For each $a \in P^{\mathcal{C}_{T_f}}$, let leaf$(a)$ denote the “leaf of $a$,” i.e., the unique $\eta \in \text{lim}(T)$ such that $\models P_{\eta \upharpoonright k}(a)$ for all $k < \omega$.

**Definition 4.10.** Call $B \subseteq \text{lim}(T)$ a blocking set when: for every $A \subseteq \mathcal{C}_{T_f}$,

if $\{\text{leaf}(a) : a \in A\} \cap B = \emptyset$ then $\{R(x, a) : a \in A\}$ is a partial type.

[We can also give a direct definition, using $2^{2^{\aleph_0}}$.] Let $B = \langle B_\alpha : \alpha < 2^{2^{\aleph_0}} \rangle$ enumerate them, possibly with repetition. Now for each $r = (u, w, q, r) \in R$, and each $\alpha \in w$, as $q$ is a type, each $x_\alpha$ is either determined to belong to $P$ or to $Q$. If $x_\alpha$ belongs to $P$, then (again since $q$ is a type) there is $\eta_\alpha \in \text{lim}(T)$ such that $q \vdash P_{\eta_\alpha \upharpoonright k}(x_\alpha)$ for $k < \omega$. Moreover, as $r$ is a type, there is at least one blocking set $B$ such that

$$\{\text{leaf}_r(x_\alpha) : \alpha \in w, \, R(x, x_\alpha) \in q\} \cap B = \emptyset.$$

Let $\beta_\ast$ be an index for this $B$ in the enumeration $B$, say for definiteness a minimal index. Choose the coloring function $G$ so that $G(r) = \beta_\ast$ for all $r \in R$.

\footnote{But see Comment 4.11}
Let’s verify that this works. Suppose we are given a sequence \( \mathfrak{t} \) of elements of \( R \), on which \( G \) is constant and equal to \( \beta \), and \( \mathfrak{b} = \langle b^*_\alpha : \alpha \in \bigcup_{i<\lambda} u_i \rangle \) which is a good instantiation of \( \mathfrak{t} \). Since we have ruled out trivial inconsistency with \( 4.2 \), inconsistency cannot come from direct disagreement in the sense that, say, \( R(x,b^*_\alpha) \) appears in one instance and \( \neg(R(x,b^*_\alpha)) \) appears in another. It will suffice to show that the type restricted to positive instances of \( R \) is consistent. By definition of \( G \), \( \{ \text{leaf}(b^*_\alpha) : \alpha \in u_t, t < t_*, \forall \alpha \in q_t \} \cap B_\beta = \emptyset \), hence the set of formulas
\[
\{ \varphi_\alpha(x,b^*_\alpha) : \alpha \in u_t, \varphi_\alpha \in r_t, t < t_*, \forall \alpha(x,y) = R(x,y) \}
\]
is a consistent partial type which does not fork over \( M_* \), and this suffices.

**Comment 4.11.** In fact, as the theory has trivial forking, we may use \( \sigma = \aleph_0 \), \( \theta = \aleph_1 \) and various natural changes to the definition to accommodate this, such as having the closure of a set be itself; see Observation 3.5 of [11] and the paragraph before it. With these modifications, we can use co-finite blocking sets only, hence we can replace \( 2^{\aleph_0} \) above by \( \aleph^\theta_0 \). We plan to address this in future work (but see also the proof of Theorem 5.7).

5. A saturation result for \( T_f \)

**Observation 5.1** (see e.g. Jech Theorem I.5.20). To satisfy Definition 11 we may take e.g. \( \sigma = \theta = \aleph_1 \), \( \mu = 2^{\aleph_0} \) and \( \lambda = \mu + \ell \) for any finite \( \ell > 0 \).

Perfect ultrafilters were defined and shown to exist in [11] §9, for the case of suitable \( \langle \lambda, \mu, \aleph_0, \aleph_0 \rangle \). [These were called \( \langle \lambda, \mu \rangle \)-perfect, with \( \theta, \sigma \) omitted when countable.] Here we make the essentially cosmetic changes to extend this definition to allow for possibly uncountable \( \theta \), starting with the definition.

**Definition 5.2** (Support of a sequence, [11] Definition 5.6.1). Let \( \mathfrak{b} = \langle b_u : u \in [\lambda]^{<\aleph_0} \rangle \) be a sequence of elements of \( \mathfrak{B} = \mathfrak{B}^{\lambda,\mu,\theta}_{\aleph_0} \). We say \( X \) is a support of \( \mathfrak{b} \) in \( \mathfrak{B} \) when \( X \subseteq \{ x_f : f \in \Pi_{\lambda,\theta}(\alpha) \} \) and for each \( u \in [\lambda]^{<\aleph_0} \) there is a maximal antichain of \( \mathfrak{B} \) consisting of elements of \( X \) all of which are either \( \leq b_u \) or \( \leq 1 - b_u \). Though there is no canonical choice of support we will write \( \text{supp}(\mathfrak{b}) \) to mean some support.

Definition 5.3 extends [11], Definition 9.1 to possibly uncountable \( \theta \).

**Definition 5.3** (Perfect ultrafilters, for suitable \( \langle \lambda, \mu, \theta, \aleph_0 \rangle \)). Let \( \langle \lambda, \mu, \theta, \aleph_0 \rangle \) be suitable. We say that an ultrafilter \( \mathcal{D}_* \) on \( \mathfrak{B} = \mathfrak{B}^{\lambda,\mu,\theta}_{\aleph_0} \) is \( \langle \lambda, \mu, \theta, \aleph_0 \rangle \)-perfect when (A) implies (B):
\[
\langle b_u : u \in [\lambda]^{<\aleph_0} \rangle \text{ is a monotonic sequence of elements of } \mathcal{D}_*
\]
and \( \text{supp}(b) \) is a support for \( \mathfrak{b} \) of cardinality \( \leq \lambda \), see 5.2 such that for every \( \alpha < 2^\lambda \) with \( \bigcup \{ \text{dom}(f) : x_f \in \text{supp}(\mathfrak{b}) \} \subseteq \alpha \), there exists a multiplicative sequence
\[
\langle b'_u : u \in [\lambda]^{<\aleph_0} \rangle
\]
of elements of \( \mathfrak{B}^+ \) such that
(a) \( b'_u \leq b_u \) for all \( u \in [\lambda]^{<\aleph_0} \),
(b) for every \( c \in \mathfrak{B}^+_{\aleph_0,\theta} \cap \mathcal{D}_* \), no intersection of finitely many members of
\[
\{ b'_{\{i\}} \cup (1 - b'_{\{i\}}) : i < \lambda \}
\]
is disjoint to \( c \).
Definition 5.4. Suppose \( \text{Definition 5.4 extends [12], Definition 3.11 to possibly uncountable } \theta \).

\[ \text{Theorem} \]

Let \( \theta \geq \aleph \) for an uncountable but constant value of \( \theta \). The main result of this section is that perfect ultrafilters are able to saturate \( T_f \) for an uncountable but constant value of \( \mu \).

\[ \text{Theorem 5.5.} \]

Let \( \lambda, \mu, \theta, \aleph \) be suitable, and in addition suppose \( \mu \geq \aleph \) and \( \theta \geq \aleph \) (e.g. let \( \theta = \aleph \), \( \mu = \aleph \), and \( \lambda = \mu^+ \) for any finite \( n \)). Let \( \mathcal{D} \) be a \( (\lambda, \mu, \theta, \aleph) \)-perfect ultrafilter on \( I, |I| = \lambda \). Then \( \mathcal{D} \) is good for \( T_f \), i.e. for any \( M \models T_f \), the ultrapower \( M^I / \mathcal{D} \) is \( \lambda^+ \)-saturated.

\[ \text{Proof.} \]

We begin with the usual setup. We fix \( \mathcal{D}_0, \mathcal{B} = \mathcal{B}_{2,\mu,\theta}^1 \) and a \( (\lambda, \mu, \theta, \aleph) \)-perfect ultrafilter \( \mathcal{D}_+ \) on \( \mathcal{B} \) such that \( \mathcal{D} \) is built from \( \mathcal{D}_0, \mathcal{B}, \mathcal{D}_+ \) via \( \mathcal{D} \). We choose \( M \models T_f \) as the index model, without loss of generality \( \lambda^+ \)-saturated (by regularity of \( \mathcal{D} \) the choice of \( M \) will not matter). We fix some lifting from \( M^I / \mathcal{D} \) and each index \( t \in I \) the projection \( a[t] \) is well defined. If \( c = \langle c_i : i < m \rangle \in \mathcal{B}(M^I / \mathcal{D}) \) we then use \( c[t] \) to denote \( \langle c_i[t] : i < m \rangle \).

Following Claim [2\text{.3}] it suffices to consider partial types of the following form. (Moreover, since \( R \) is not symmetric, \( Q(x) \) and \( P(x) \) are implied by the rest of the partial type in each case, so we may omit them.)

\[ (1) \text{ } \{ Q(x) \} \cup \{ R(x, a) : a \in A \} \text{ for } A \subseteq \mathcal{P}N, |A| \leq \lambda. \]

\[ (2) \text{ } \{ P(x) \} \cup \{ R(b, x) : b \in B \} \text{ for } B \subseteq \mathcal{Q}N, |B| \leq \lambda. \]

Fix a partial type \( p = p(x) \) which is either of type (1) or type (2). Depending on which there will be some minor choices to make in the proof below. Recall two useful facts from the proof of Claim [2\text{.3}] above: in models of \( T_f \), for each finite \( n, \)

\[ \text{Fact A} \]

For elements \( a_0, \ldots, a_n \in N, N \models (\exists x) \bigwedge_{i \leq n} R(x, a_i) \) if and only if there exist \( \eta_0, \ldots, \eta_n \in T_{k_n} \) such that \( N \models (\exists x) \bigwedge_{i \leq n} ( R(x, a_i) \land P_\eta (a_i) ) \), where \( k_n \) is minimal such that \( f(k_n) > n \).

\[ \text{Fact B} \]

For elements \( b_0, \ldots, b_n \in N, N \models (\exists x) \bigwedge_{i \leq n} R(b_i, x) \) if and only if there exists \( \eta \in T_{k} \) such that \( N \models (\exists x) \bigwedge_{i \leq n} ( R(b_i, x) \land P_\eta (x) ) \), where \( k_n \) is minimal such that \( f(k_n) > n \).

We'll follow the strategy of [12], Theorem 4.1.

We begin with the case where \( p \) is of type (1).

Without loss of generality (possibly \( |A| < \lambda \), but \( ||N|| \geq 2^\lambda \) so this is no problem),

\[ (5.A) \quad \text{let } \langle a_i : i < \lambda \rangle \text{ list the elements of } A \text{ without repetition.} \]
This induces an enumeration of $p$ as
\[ (5.B) \quad \langle R(x, a_i) : i < \lambda \rangle. \]
As usual, for each finite $u \subseteq \lambda$, let
\[ (5.C) \quad B_u := \{ t \in I : M \models \exists x \bigwedge_{i \in u} R(x, a_i[t]) \} \quad \text{and} \quad b_u = j(B_u). \]
and let
\[ (5.D) \quad \mathbf{b} = \langle b_u : u \in [\lambda]^{<\aleph_0} \rangle. \]
First we build an appropriate support for $\mathbf{b}$. This will require handling equality and leaves. For equality, for each $i, j \in \lambda$ let
\[ (5.E) \quad A_{a_i = a_j} := \{ t \in I : a_i[t] = a_j[t] \} \quad \text{and} \quad a_{i,j} := j(A_{a_i = a_j}). \]
For leaves, for each $a \in \lambda$, let
\[ (5.F) \quad a_{P_{\eta}(a_i)} = j(\{ t \in I : M \models P_{\eta}(a_i[t]) \}). \]
Remembering that $\theta > \aleph_0$, for each $\eta \in \text{lim}(\mathcal{T})$, define
\[ (5.G) \quad a_{\text{leaf}(a_i) = \eta} := \bigcap_{k < \omega} a_{P_{\eta+k}(a_i)}. \]
Then $\mathbf{c}$ will be nonzero for some, but not necessarily all, $\eta$, however, for each $i$, $\mathbf{d}$
\[ (5.H) \quad \langle a_{\text{leaf}(a_i) = \eta} : \eta \in \text{lim}(\mathcal{T}) \rangle \]
is a maximal antichain of $\mathcal{B}$.

For each $i < \lambda$ let $\mathcal{F}_{i,j}$ be the set of all $f \in \text{FI}_{\mu, \theta}(2^\lambda)$ such that for some $j \leq i$, the three conditions $\mathbf{g}$, $\mathbf{h}$, $\mathbf{i}$ hold:
\[ (5.I) \quad x_f \leq a_{a_i = a_j}. \]
\[ (5.J) \quad \text{for all } k < j, \ x_f \cap a_{a_i = a_k} = 0. \]
\[ (5.K) \quad \text{for some } \eta, \ x_f \leq a_{\text{leaf}(a_i) = \eta}. \]
For each finite $u \subseteq \lambda$, define $\mathcal{F}_u$ to be $\bigcap \{ \mathcal{F}_{i,j} : i \in u \}$. Each $\mathcal{F}_u$ is upward closed, i.e. $f \in \mathcal{F}_u$ and $g \in \text{FI}_{\mu, \theta}(2^\lambda)$ and $g \supseteq f$ implies $g \in \mathcal{F}_u$.

For each $u \in [\lambda]^{<\aleph_0}$, by induction on $\zeta < \lambda$, choose a maximal antichain $(x_f, \epsilon < \epsilon_*$) of elements of $\mathcal{B}$ such that (i) each $x_f$ is either $\leq b_u$ or $\leq 1 - b_u$, and (ii) each $f \in \mathcal{F}_u$ and $0 \in \text{dom}(f)$. Necessarily the construction will stop at an ordinal $< \mu^+$, but $\geq \mu$ because $0 \in \text{dom}(f)$. Re-index this antichain as
\[ (5.L) \quad \langle x_{f_{u,\zeta}} : \zeta < \mu \rangle. \]
Then
\[ (5.M) \quad \{ x_{f_{u,\zeta}} : \zeta < \mu, u \in [\lambda]^{<\aleph_0} \} \]
is a support of $\mathbf{b}$ in the sense of Definition $5.2$. When $u = \{i\}$, we will often write $f_{i,\zeta}$ to mean $f_{\{i\},\zeta}$.

Second, we build a multiplicative refinement for $\mathbf{b}$.
\[ (5.N) \quad \text{Fix } \alpha < 2^\lambda \text{ such that } \bigcup \{ \text{dom}(f_{u,\zeta}) : \zeta < \mu, u \in [\lambda]^{<\aleph_0} \} \subseteq \alpha. \]
As before, let \( \text{leaf}(a) \) denote the unique \( \eta \in \lim(T) \) such that \( \models P_{\eta|k}(a) \) for all \( k < \omega \), and call \( X \subseteq \lim(T) \) a blocking set when: for every \( A \subseteq P^{cT} \),

\[
\text{if } \{ \text{leaf}(a) : a \in A \} \cap X = \emptyset \text{ then } \{ R(x,a) : a \in A \} \text{ is a partial type.}
\]

As \( \mu \geq 2^{\aleph_0} \), let

\[
\langle X_\epsilon : \epsilon < \mu \rangle
\]

be an enumeration, possibly with repetitions, of all co-finite blocking sets.\(^9\) Let \( H \) be the function from \( \{ f_{i,\zeta} : i < \lambda, \zeta < \mu \} \times \mu \) to \( \{ 0, 1 \} \) given by

\[
H(f_{i,\zeta}, \epsilon) = 1 \text{ iff } \eta \not\in X_\epsilon
\]

where \( \eta \) is the unique element of \( \lim(T) \) such that \( x_{f_{i,\zeta}} \leq a_{\text{leaf}(a_i)=\eta} \). (Very informally, \( H \) returns 1 if a type avoiding the blocking set \( B_\epsilon \) may contain \( a_i \) as it appears on \( x_{f_{i,\zeta}} \).

We’ll need a new antichain to help us divide the work:

\[
\text{(5.Q)} \quad \text{let } \bar{c} = \langle c_\epsilon : \epsilon < \mu \rangle \text{ be given by } c_\epsilon = x_{\{a_{\langle a,\epsilon \rangle}\}}.
\]

Any element of this antichain will have nonzero intersection with any of the elements from \( \mathcal{B}_{\alpha,\mu,\theta}^+ \), our protagonists so far.

Finally, for each \( i < \lambda \), define

\[
\text{(5.R)} \quad b'_{\{i\}} = \left( \bigcup \{ x_{f_{i,\zeta}} \cap c_\epsilon : \zeta < \mu \text{ and } H(f_{i,\zeta}, \epsilon) = 1 \} \right) \cap b_{\{i\}}.
\]

Define

\[
\text{(5.S)} \quad b'_u = \bigcap_{i \in u} b'_{\{i\}}, \text{ and let } \bar{b}' = \langle b'_u : u \in [\lambda]^{<\aleph_0} \rangle.
\]

By definition, \( \bar{b}' \) is multiplicative. Our final task is to make sure the hypotheses of Definition 5.3 are satisfied, i.e. that for our multiplicative sequence \( \bar{b}' \),

\( a \) \( b'_u \leq b_u \) for all \( u \in [\lambda]^{<\aleph_0} \),

\( b \) for every \( c \in \mathcal{B}_{\alpha,\mu,\theta}^+ \cap \mathcal{D}_\lambda \), no intersection of finitely many members of \( \{ b'_{\{i\}} \cup (1 - b_{\{i\}}) : i < \lambda \} \) is disjoint to \( c \).

For (a), suppose for a contradiction that for some \( u \in [\lambda]^{<\aleph_0} \) there were a nonzero \( c \leq b'_u \setminus b_u \).

Without loss of generality,

\[
\text{(5.T)} \quad c \leq c_\epsilon \text{ for some } \epsilon < \mu,
\]

and also, since \( u \) is finite,

\[
\text{(5.U)} \quad c \text{ is either below or disjoint to all elements in } \{ x_{f_{i,\zeta}} : i \in u, \zeta < \mu \}.
\]

So for each \( i \in u \) there is \( \zeta_i < \mu \) with \( c \leq x_{f_{i,\zeta_i}} \). By (5.T) and the definition (5.R),

\[
\text{(5.V)} \quad c \leq \bigcap_{i \in u} x_{f_{i,\zeta_i}} \text{ and } \bigwedge_{i \in u} H(f_{i,\zeta_i}, \epsilon) = 1.
\]

Now by our hypothesis, \( c \leq b_{\{i\}} \), meaning

\[
\text{(5.W)} \quad c \leq a_{\exists x R(x,a_i)} \text{ for } i \in u.
\]

\(^9\)A priori we could use all blocking sets and \( \mu = 2^{\aleph_0} \), but the nice point is that in our present setup the co-finite blocking sets suffice. Note that here \( p(x) \) being a set of positive instances of \( R(x,y) \) helps. With negation, we’d need \( f_{\mu,\zeta} \)’s deciding all cases of \( \exists x R(x,a_i) \), \( \neg R(x,a_i) \), \( a_i = a_j \).
Let $k_*$ be minimal so that $f(k_*) > |u|$. Then as $H(f_{i,\xi}, \epsilon) = 1$ for $i \in u$,
\begin{equation}
\tag{5.X}
\exists \nu \in X, \nu \leq k_* \{a[i] \sim P_{\rho}[k_{i}] : i \in u, \rho \in X, k \leq k_*\}.
\end{equation}

Informally, on $c$ none of the parameters $a_i$ fall into the predicates forbidden by the blocking set, at least up to level $k_*$ (this suffices for our contradiction, recalling Fact A from the beginning of the proof). And $c \cap b_n = \emptyset$ means
\begin{equation}
\tag{5.Y}
\exists \nu \in X, \nu \leq k_* \{a[i] \sim P_{\rho}[k_{i}] : i \in u, \rho \in X, k \leq k_*\}.
\end{equation}

Since $\bar{b}$ is a possibility pattern and $c > 0$, this means we should be able to find values for $a_i$ in $\mathcal{C}_T$, which would make this combination of formulas true, but this is impossible because $X_i$ is a blocking set (so avoiding it gives a type). This contradiction shows that (a) holds, so $\bar{b}$ is a multiplicative refinement of $\bar{b}$.

For (b), it will suffice to show that for any $a \in D_\lambda$ such that $\text{supp}(a) \subseteq \alpha$, and any finite $u \subseteq \lambda$,
\[ a \cap \bigcap \{b'_1[i] \cup (1 - b'_1[i]) : i \in u\} > 0. \]
Without loss of generality, we can write $u = v \cup w$ where for each $i \in v$, $a \leq 1 - b'_1[i]$ and for each $i \in w$, $a \leq b'_1[i]$. If $w$ is empty we are done, so suppose not, and it will suffice to show that
\[ a \cap \bigcap \{b'_1[i] : i \in w\} > 0. \]
As $b_w \in D_\lambda$, without loss of generality $a \leq b_w$, and we may choose $g \in \text{Fl}_{\mu, \theta}(\lambda)$ such that $x_f \leq a$. Moreover, for each $i \in w$, we may assume that there is some $i < \mu$ such that $x_g \leq x_{f,i}$. So we have that
\[ 0 < x_g \leq \bigcap \{x_{f,i} : i \in w\} \leq b_w. \]
Recall that by our choice of partitions, for each $f_{i,\xi}$, there is a unique $\eta = \eta_i \in \text{lim}(T)$ such that $x_{f_i,\xi} \leq a_{\text{leaf}(a_i)} = \eta_i$. Because this intersection is $\leq b_w$, there is some blocking set $X_i$ such that $X \cap \{\eta_i : i \in w\} = \emptyset$. Then
\[ 0 < c_r \cap \bigcap \{x_{f,i} \cap x_g \leq \bigcap \{b'_1[i] : i \in w\} \]
which completes the proof of (b). This completes the proof of Case 1.

For case (2), the strategy is similar, with a few changes to reflect the dual type. For clarity we give the entire argument, renaming the parameter set as $B$.

Without loss of generality,
\begin{equation}
\tag{6.A}
\{b_i : i < \lambda\} \text{ list the elements of } B \text{ without repetition.}
\end{equation}
This induces an enumeration of $p$ as
\begin{equation}
\tag{6.B}
\langle R(b_i, x) : i < \lambda\rangle.
\end{equation}
For each finite $u \subseteq \lambda$, let
\begin{equation}
\tag{6.C}
B_u := \{t \in I : M \models \exists x \bigwedge_{i \in u} R(b_i[t], x)\} \quad \text{and } b_u = \mathfrak{j}(B_u).
\end{equation}
and let
\begin{equation}
\tag{6.D}
\mathfrak{b} = \langle b_u : u \in [\lambda]^{<\aleph_0}\rangle.
\end{equation}
First we build an appropriate support for $\bar{b}$. As before, for each $i, j \in \lambda$ let
$$(6.E) \quad A_{b_i=b_j} := \{ t \in I : b_j[t] = b_j[t] \} \quad \text{and let} \quad a_{b_i=b_j} := j(A_{b_i=b_j}).$$

Now for each $\eta \in T$,
$$(6.F) \quad a_{\exists x (R(x) \land P_\eta(x))} = j( \{ t \in I : M \models \exists x (R(x) \land P_\eta(x)) \} ).$$

As $\theta > \aleph_0$, for each $i < \lambda$ and $\eta \in \text{lim}(T)$, define
$$(6.G) \quad a^{\alpha}_{\exists x (R(x) \land \text{leaf}(x) = \eta)} := \bigcap_{k<\omega} a_{\exists x (R(x) \land \text{leaf}(x) = \eta)}.$$  

For each $s \in S$,
$$(6.H) \quad a_{Q_s(b_i)} = j( \{ t \in I : M \models Q_s(b_i[t]) \} ).$$

For each $\nu \in \text{lim}(S)$, letting $\nu = \langle s_k : k < \omega \rangle$ (recalling 2.3) so $\nu(k)$ denotes $s_k$, define
$$(6.I) \quad a^{\alpha}_{Q_{\nu}(b_i)} := \bigcap_{k<\omega} a_{Q_{\nu(k)}(b_i)}.$$  

Then for each $i < \lambda$,
$$(6.J) \quad \langle a^{\alpha}_{Q_{\nu}(b_i)} : \nu \in \text{lim}(S) \rangle$$

is a partition of $b_\langle i \rangle$. For each finite $u \subseteq \lambda$ let $F_u$ be the set of all $f \in F_{\mu, \theta}(2^\lambda)$ such that the conditions (6.1), (6.3) hold:
$$(6.K) \quad f \in F_u \text{ and } g \supseteq f \text{ implies } g \in F_u.$$  

Each $F_u$ is upward closed, i.e. $f \in F_u$ and $g \supseteq f$ implies $g \in F_u$.  

For each $u \in [\lambda]^{<\aleph_0}$, by induction on $|\zeta| < \lambda$, choose a maximal antichain $\langle x_{f_\langle \epsilon \rangle} : \epsilon < \epsilon_\zeta \rangle$ of elements of $2^\mathfrak{a}$ such that (i) each $x_{f_\langle \epsilon \rangle}$ is either $\leq b_\langle u \rangle$ or $\leq 1 - b_\langle u \rangle$ and (ii) each $f_\langle \epsilon \rangle \in F_u$. Necessarily the construction will stop at an ordinal $< \mu^+$. Renumber this antichain as
$$\langle x_{f_{u, \zeta}} : \zeta < \mu \rangle.$$  

Then
$$(6.L) \quad \{ x_{f_{u, \zeta}} : \zeta < \mu, u \in [\lambda]^{<\aleph_0} \}$$

is a support of $\bar{b}$ in the sense of Definition 5.2.  

When $u = \{i \}$, we will again write
$$f_{i, \zeta}$$

to mean $f_{\langle i \rangle, \zeta}$.  

Second, we build a multiplicative refinement for $\bar{b}$.  

Fix $\alpha < 2^\lambda$ such that $\bigcup \{ \text{dom}(f_{u, \zeta}) : \zeta < \mu, u \in [\lambda]^{<\aleph_0} \} \subseteq \alpha$.  

As $\mu \geq 2^{\aleph_0}$, let
$$(6.M) \quad \langle \eta : \epsilon < \mu \rangle$$
be an enumeration, possibly with repetitions, of all leaves \(\eta \in \lim(\mathcal{T})\). Let \(G\) be the function from \(\{f_{i,\zeta} : i < \lambda, \zeta < \mu\} \times \mu\) to \(\{0, 1\}\) given by
\[(6.N)\quad G(f_{i,\zeta}, \epsilon) = 1 \iff x_{f_{i,\zeta}} \leq a^{\langle Q_r(b_i) \rangle} \text{ and } \eta_\zeta \upharpoonright k \in \nu(k) \text{ for all } k < \omega.\]

We’ll need a new antichain to help us divide the work:
\[(6.O)\quad \text{let } \bar{c} = \langle c_\epsilon : \epsilon < \mu \rangle \text{ be given by } c_\epsilon = x_{\{(\alpha + 1, \epsilon)\}}.\]

Any element of this antichain will have nonzero intersection with any of the elements from \(\mathcal{B}_{\alpha, \theta}^+.\)

For each \(i < \lambda\), define
\[(6.P)\quad b'_{\{i\}} = \left( \bigcup \{x_{f_{i,\zeta}} \cap c_\zeta : \zeta < \mu \text{ and } G(f_{i,\zeta}, \epsilon) = 1 \} \right) \cap b_{\{i\}}.\]

Define
\[b'_u = \bigcap_{i \in u} b'_{\{i\}}, \text{ and let } b' = \langle b'_u : u \in [\lambda]^{<\aleph_0} \rangle.\]

By definition, \(b'\) is multiplicative. Again we address the hypotheses of Definition 5.3. For (a), suppose for a contradiction that for some \(u \in [\lambda]^{<\aleph_0}\) there were a nonzero
\[c \leq b'_u \setminus b_u.\]

Without loss of generality,
\[(6.Q)\quad c \leq c_\epsilon \text{ for some } \epsilon < \mu,\]
and also
\[(6.R)\quad c \text{ is either below or disjoint to all elements in } \{x_{f_{i,\zeta}} : i \in u, \zeta < \mu\}.\]

So for each \(i \in u\) there is some \(\zeta_i < \mu\) with \(c \leq x_{f_{i,\zeta_i}}\), and then by (6.Q) and the definition (6.P),
\[(6.S)\quad c \leq \bigcap_{i \in u} x_{f_{i,\zeta_i}} \text{ and } \bigwedge_{i \in u} G(f_{i,\zeta_i}, \epsilon) = 1.\]

Now by our hypothesis, \(c \leq b_{\{i\}}\), meaning
\[(6.T)\quad c \leq a[\exists x R(b_i, x)] \text{ for } i \in u.\]
And \(c \cap b_u = \emptyset\) means
\[(6.U)\quad c \leq a[\neg \exists x \bigwedge_{i \in u} R(b_i, x)].\]

Recalling the definition of \(G\) and (6.Q), for every \(i \in u\),
\[(6.V)\quad c \leq x_{f_{i,\zeta_i}} \leq a^{\langle (\exists x)(R(b_i, x) \land \text{leaf}(x) = \eta_\zeta) \rangle}.\]

But (6.T), (6.U), and (6.V) together are a contradiction. [Why? (6.U) must be witnessed by full splitting at some finite stage, but (6.V) guarantees that at every finite stage there is a specific piece of the successor partition which is reserved for \(x\).] This contradiction shows that condition (a) of the definition of perfect holds, so \(b'\) is indeed a multiplicative refinement of \(b\).

For (b), it will suffice to show that for any \(c \in \mathcal{D}_*\) such that \(\text{supp}(a) \subseteq \alpha\), and any finite \(u \subseteq \lambda\),
\[c \cap \bigcap_{i \in u} \{b'_{\{i\}} \cup (1 - b_{\{i\}}) : i \in u\} > 0.\]
Without loss of generality, we can write \( u = v \cup w \) where for each \( i \in v \), \( c \leq 1 - b_{i(i)} \) and for each \( i \in w \), \( c \leq b_{i(i)} \). If \( w \) is empty we are done, so suppose not, and it will suffice to show that
\[
c \cap \bigcap \{ b_{i(i)} : i \in w \} > 0.
\]
As \( b_w \in D_\ast \), without loss of generality
\[
c \leq b_w.
\]
Recalling Fact B from the beginning of the proof, let \( k \ast \) be minimal so that \( f(k_\ast) > |w| \), and then
\[
\bigcup \{ a[(\exists x) \bigwedge_{i \in w} R(b_i, x) \wedge P_{\eta}(x)] : \eta \in T_{k_\ast} \} = b_w
\]
so, after possibly shrinking \( c \) by taking intersections, we may assume there is \( \eta_\ast \in T_{k_\ast} \) such that
\[
c \leq a[(\exists x) \bigwedge_{i \in w} R(b_i, x) \wedge P_{\eta}(x)].
\]
Again by Fact B, this implies there is some \( \eta \in \lim(T) \) such that \( \eta_\ast \leq \eta \) and for all finite \( k \),
\[
c \cap a[(\exists x) \bigwedge_{i \in w} R(b_i, x) \wedge P_{\eta \upharpoonright k}(x)] > 0
\]
so as \( \theta > 0 \), without loss of generality
\[
c \leq a^w(\exists x)(R(b_i, x) \wedge \text{leaf}(x) = \eta)^\prime\prime.
\]
Recalling the support (6.L), after possibly shrinking \( c \) by taking intersections, there are \( \zeta_i \) for each \( i \in w \) such that
\[
c \leq x_{f_i, \zeta_i}
\]
i.e. for each \( i \in w \),
\[
0 < x_{f_i, \zeta_i} \cap c \leq a^w(\exists x)(R(b_i, x) \wedge \text{leaf}(x) = \eta)^\prime\prime.
\]
Our choice of partitions in 6.J and 6.L means that for each \( i \in w \) there is \( \nu_i \in \lim(S) \) such that
\[
x_{f_i, \zeta_i} \leq a^wQ_{\nu_i}(b_i)^\prime\prime.
\]
Recalling 6.K, we conclude from these two equations that \( \eta \upharpoonright k \in \nu_i(k) \) for all \( k < \omega \), for each \( i \in w \). So letting \( \epsilon \) be such that \( \eta = \eta_\epsilon \) in the enumeration 6.M, \( G(f_{i, \zeta_i}, \epsilon) = 1 \) for all \( i \in w \). We have shown that
\[
c \cap \bigcap_{i \in w} x_{f_i, \zeta_i} \cap c_\epsilon > 0
\]
and this suffices.

This completes Case 2, and so completes the proof of the Theorem. \( \square \)
7. Consequences for Keisler’s order

**Theorem 7.1.** Let \( f \in \mathcal{F} \). In Keisler’s order, \( T_f \) is strictly above the theory of the random graph.

**Proof.** Recall that \( T_{\text{rg}} \) is minimum in Keisler’s order among the unstable theories (4, Conclusion 5.3). So as \( T_f \) is unstable, \( T_{\text{rg}} \leq T_f \). By Lemma 3.2, if \( D \) is a regular ultrafilter on \( \lambda \) built from \((D_0, B = B_{2}^\lambda, \aleph_0, \aleph_0, D_*)\) where \( D_0 \) is any regular good [or so-called excellent] filter on \( \lambda \) and \( D_* \) is any ultrafilter on \( B \), then \( D \) is not good for \( T_f \). On the other hand, by (10) Theorem 3.2 in the case \( \mu = \aleph_0 \), there is such an ultrafilter which is good for \( T_{\text{rg}} \). This shows that \( T_{\text{rg}} \triangleright T_f \). □

We recall the higher analogues of the triangle-free random graph, studied by Hrushovski (4). In particular, he proved that each \( T_{n,k} \) is simple unstable with trivial forking for \( n > k > 2 \).

**Definition 7.2.** Recall that \( T_{n,k} \) denotes the generic \((n+1)\)-free \((k+1)\)-hypergraph, i.e. the model completion of the theory (in a language with a single \((k+1)\)-place relation, interpreted as a hyperedge, so symmetric and irreflexive) stating that there do not exist \((n+1)\) distinct elements of which every \((k+1)\) are an edge.

The infinite descending chain in Keisler’s order obtained in (12) was given by \( \cdots \triangleright T_{m} \triangleright T_{n-1} \triangleright \cdots \triangleright T_1 \triangleright T_0 \) where \( T_n \) is the disjoint union of the theories \( T_{k+1,k} \) for \( k > 2n + 2 \).

In the Appendix, Theorem 10.9 below we update (12) Claim 5.1 to allow for the possibility of uncountable \( \theta \).

**Theorem (12), Claim 5.1 for possibly uncountable \( \theta \), Theorem 10.9 below.** Suppose that:

1. for integers \( 2 \leq k < \ell \), and e.g. \( \theta = \aleph_1, \mu = 2^{\aleph_0}, \lambda = \mu^{+\ell} \), or just: \((\lambda, k, k+1) \rightarrow k+1 \) in the sense of (12) Notation 1.2
2. \( B = B_{2}^{\lambda, \mu, \theta} \)
3. \( D_* \) is an ultrafilter on \( B \)
4. \( T = T_{k+1,k} \)

Then \( D_* \) is not \((\lambda, T)\)-moral.

**Theorem 7.3.** For any finite \( k \geq 2 \), \( T_f \) and \( T_{k+1,k} \) are incomparable in Keisler’s order. More precisely:

1. let \( D \) be a \((\lambda, \mu, \aleph_0, \aleph_0)\)-perfected ultrafilter on \( \lambda \) where
   \( \lambda = \aleph_{k-1}, \ \mu = \aleph_0, \quad \theta = \aleph_0 \).
   Then \( D \) is not good for \( T_f \), but it is good for \( T_{k+1,k} \).
2. let \( D_* \) be a \((\lambda, \mu, \aleph_1, \aleph_0)\)-perfected ultrafilter on \( \lambda \) where
   \( \mu \geq 2^{\aleph_0} \) and \( \lambda = \mu^{+n} \) for \( n > k + 1 \).
   Then \( D \) is good for \( T_f \), but it is not good for \( T_{k+1,k} \).

**Proof.** (1) The non-saturation is Lemma 3.2 via separation of variables, and the saturation is (12) Theorem 4.1.
(2) The saturation is Theorem 5.5 and the non-saturation is Theorem 10.9 via separation of variables. □
8. Baseline saturation

In this section we give a mathematical sense in which $T_f$ is very close to the random graph in Keisler's order.

**Lemma 8.1.** Let $f \in F$. Let $M \models T_f$. Suppose $D$ is a regular ultrafilter on $\lambda$, and let $N = M^f / D$. Suppose:

1. $D$ is $(2^{|\lambda|^+})^+$-good
2. $D$ is good for $T_{rg}$.

Then $N$ is $\lambda^+$-saturated.

We will use:

**Fact 8.2** ([7] Lemma 9). Let $D$ be a regular ultrafilter on $\lambda$ such that $\text{lcf}(\omega, D) \geq \lambda^+$. Suppose $M$ is countable, $N = M^\lambda / D$ its regular ultrapower, and $A \subseteq N$, $|A| \leq \lambda$. Let $\langle P_n : n < \omega \rangle$ be a sequence of induced predicates such that $P_n \supseteq P_{n+1} \supseteq A$ for all $n < \omega$. Then there exists an induced predicate $Y$ such that $P_n \supseteq Y$ for all $n < \omega$ and $Y \supseteq A$.

**Proof of Lemma 8.1.** Since $D$ is good for $T_{rg}$, it follows that $\text{lcf}(\omega, D) \geq \lambda^+$, so the hypotheses of Fact 8.2 are satisfied.

It suffices to consider the two kinds of positive $\varphi$-types from Claim 8.7.

First, consider $p(x) = \{R(x, a) : a \in A\}$ where $A \subseteq P^N$. Consider $\Lambda = \{\eta \in \text{lim}(\mathcal{T}) : \eta = \text{leaf}(a) \text{ for some } a \in A\}$. For each $\eta \in \Lambda$, let $A_\eta = \{a \in A : \text{leaf}(a) = \eta\}$, so $\{A_\eta : \eta \in \Lambda\}$ is a partition of $A$, and note that because $p(x)$ is a partial type, the complement of $\Lambda$ is necessarily a blocking set. Recall that a subset $C$ of $N$ is called induced (or internal) if we could expand the language by a new predicate $X$ and interpret $X$ in each index model $M_i$ to obtain $M_i^+, \frac{\text{in such a way that in the}}{\text{resulting ultraproduct } N^+, X \text{ names } C.}$

For each $\eta \in \Lambda$, apply Fact 8.2 with $A_\eta$ and $\langle P_{\eta|n} : n < \omega \rangle$ here for $A$ and $\langle P_n : n < \omega \rangle$ there, to obtain an internal pseudo-finite set $Y_\eta \supseteq A_\eta$ such that $Y_\eta \subseteq \bigcap_{n<\omega} P_{\eta|n}$. Let $N^+$ refer to the ultraproduct in the expanded language, where we have predicates for all of the $Y_\eta$’s. Then for each $\eta$, and each $n < \omega$, by Los’ theorem, $N^+ \models (\forall x)(Y_\eta(x) \implies P_{\eta|n}(x))$. Since our original $p$ was a type, it follows that

$$p_1(x) = \{R(x, a) : \eta \in \Lambda, a \in Y_\eta\}$$

is a partial type extending $p$. Also $p_2(x) \vdash p_1(x)$ in $N^+$ where

$$p_2(x) = \{(\forall y)(y \in Y_\eta \implies R(y, x)) : \eta \in \Lambda\}.$$  

Note that $p_2(x)$ is a type in $N^+$ because the $Y_\eta$’s are pseudo-finite. Moreover, $p_2(x)$ is a type over the empty set, but in a language of size continuum. If $D$ is $(2^{|\lambda|^+})^+$-good, $p_1$ is realized, so $p$ is realized, which completes this case.

Second, consider $q(x) = \{R(b, x) : b \in B\}$ where $B \subseteq Q^N$. For each finite $k$, $\langle Q_k : s \in S_k \rangle$ is a partition of $Q^N$. Hence for each $b \in Q^N$, there is a unique sequence $\nu_b = \langle s_n, b : n < \omega \rangle$ which is a branch of $\bigcup_k S_k$ in the $\subseteq$-order, and the proof in this case is just the dual of the proof in case 1. That is, let $\Lambda = \{\nu_b : b \in B\} \subseteq \text{lim}(S)$, and just as before, use Lemma 8.2 to choose some pseudo-finite internal $Y_\nu$ for each $\nu \in \Lambda$, such that $b \in Y_\nu$, and $Y_\nu \subseteq \bigcap_{n<\omega} Q_{s_n, b}$. (Note that $\Lambda$ has size no more than continuum, as each $S_k$ is finite.) As before, working in the expanded model $N^+$ with predicates for the $Y$’s,

$$q_1(x) = \{R(b, x) : \nu \in \Lambda, b \in Y_\nu\}$$
is a partial type extending \( q \). Moreover \( q_2(x) \vdash q_1(x) \) in \( N^+ \) where
\[
q_2(x) = \{(\forall y)(y \in Y_\nu \implies R(y,x)) : \nu \in \Lambda\}.
\]
Again, \( p_2(x) \) is a type in \( N^+ \) because the \( Y_\nu \)'s are pseudo-finite. Moreover, \( p_2(x) \) is a type over the empty set in a language of size continuum. As \( D \) is \((2^{\aleph_0})^+\)-good, \( p_1 \) is realized, so \( p \) is realized, which completes the proof. \( \square \)

**Discussion 8.3.** We did not use much about an ultrapower in Lemma 8.1. With more bookkeeping, the proof could be carried out in a model \( N^+ \) expanding a model of a theory of \( T_f \), provided that the hypotheses of the Lemma are translated. For example, we could ask that:

i) \( T^+ = Th(N^+) \) has sufficient set theory,

ii) there is a countable model of \( T^+ \) which expands \((\mathbb{N},<)\) (so the domains of any model of \( T^+ \) are linearly ordered, and behave like pseudofinite sets),

iii) \( T^+ \) expands the theory of the random graph,

iv) \( N^+ \) is \((2^{\aleph_0})^+\)-saturated,

v) the reduct of \( N^+ \) to the language of the random graph is \( \lambda^+ \)-saturated,

vi) \( N^+ \) is \( \lambda \)-regular, meaning that every set of size \( \leq \lambda \) is contained in a pseudofinite set [i.e. is not unbounded in the reduct to linear order].

It follows from condition vi) of Discussion 8.3 that in \( N^+ \mid \{<\} \), the set above any countable strictly increasing sequence in the underlying linear order has co-initiality at least \( \lambda^+ \), and the set below any countable strictly decreasing sequence in the underlying linear order has co-finality at least \( \lambda^+ \). [More precisely, it follows from the fact that the reduct to some theory with the order property is \( \lambda^+ \)-saturated; see e.g. the coding argument in [14], Theorem 4.8 p. 379.] This is essentially what is needed to prove Fact 8.2.

Lemma 8.1 suggests it may be productive to look at a coarser picture of Keisler’s order for some large constant \( \mu \), or perhaps even \( \mu^+ = \lambda \).

**Definition 8.4.**

1. \( T_1 \preceq_{\lambda,\mu} T_2 \) is defined as usual but the ultrafilter has to be \( \mu^+ \)-good.

2. Similarly for \( T_1 \preceq_{\lambda,\mu}^* T_2 \), defined as usual but the \( T_1 \)-model is \( \mu^+ \)-saturated.

Recall that the random graph is minimum among the unstable theories in Keisler’s order, and that any regular ultrapower of any complete countable theory is \( \aleph_1 \)-saturated. In this notation, for any infinite \( \lambda \) the random graph is \( \preceq_{\lambda,\aleph_0} \)-minimal among countable unstable theories.

**Conclusion 8.5.** For any \( f \in F \) and any infinite \( \lambda \), the theory \( T_f \) is \( \preceq_{\lambda,2^{\aleph_0}} \)-minimal among countable unstable theories.

**Appendix: perfect ultrafilters for uncountable \( \theta \)**

In [11], we considered suitable tuples of cardinals \( (\lambda, \mu, \theta, \sigma) \), see Definition 4.1. We defined

\[D \text{ is a } (\lambda, \mu, \theta, \sigma)\text{-ultrafilter on the Boolean algebra } \mathfrak{M}_{2^{\lambda},\mu,\theta}\]

in the case where \( \theta = \sigma = \aleph_0 \), and we proved that such ultrafilters did indeed exist.

In this Appendix, we upgrade that definition and existence proof to include the case of uncountable \( \theta \). The proof is almost word-for-word the same as that of [11].
§9, but to eliminate doubt, we have reproduced that proof here with the minor changes. We defined “support” in §2 above and “perfect” in §3 above.

**Convention 9.6.** Throughout this section we assume:

\[ \lambda \geq \mu^{<\theta} \geq \theta = \text{cof}(\theta) \geq \sigma = \aleph_0. \]

Without loss of generality we may assume \( \theta > \sigma \), as the case \( \theta = \sigma = \aleph_0 \) was the case of \( [11] \) §9.

**Observation 9.7.** Suppose \( \alpha < 2^\lambda \) is fixed, \( D_\alpha \) is an ultrafilter on \( \mathcal{B}^{1}_{\alpha, \mu, \theta} \subseteq \mathcal{B} = \mathcal{B}^{2}_{2^\alpha, \mu, \theta} \), and \( (b_u : u \in [\lambda]^{<\aleph_0}) \) is a sequence of elements of \( D_\alpha \). Suppose that there exists a multiplicative sequence \( (b'_u : u \in [\lambda]^{<\aleph_0}) \) of elements of \( \mathcal{B}^+ \) such that

(a) \( b'_u \leq b_u \) for all \( u \in [\lambda]^{<\aleph_0} \),

(b) for every \( c \in \mathcal{B}^{2+}_{\alpha, \mu, \theta} \cap D_\alpha \), no intersection of finitely many members of

\[ \{b'_i \cup (1 - b_i) : i < \lambda\} \]

is disjoint to \( c \).

Then there is a multiplicative sequence \( (b''_u : u \in [\lambda]^{<\aleph_0}) \) such that (a), (b) hold with \( b'_u, b'_i \) replaced by \( b''_u, b''_i \) respectively, and such that some support of \( b'' \) is contained in \( \mathcal{B}^{\alpha+\lambda, \mu, \theta} \).

**Proof.** Without loss of generality there is \( \mathcal{V} \) of cardinality \( \lambda \) such that some support of \( b' \) is contained in \( \{x_f : f \in F_{\mu, \theta}(\mathcal{V})\} \). Let \( \pi \) be a permutation of \( 2^\lambda \) which is the identity on \( \alpha \) and takes \( \mathcal{U} \) into \( \alpha + \lambda \). This induces an automorphism \( \rho \) of \( \mathcal{B} \) which is the identity on \( \mathcal{B}^{\alpha, \mu, \theta} \), so in particular is the identity on \( D_\alpha \) and thus on \( b \). For each \( u \in [\lambda]^{<\aleph_0} \), let \( b''_u = \rho(b'_u) \). Then clearly \( b'' \) fits the bill. \( \square \)

**Theorem 9.8 (Existence).** Let \( (\lambda, \mu, \theta, \aleph_0) \) be suitable. Let \( \mathcal{B} = \mathcal{B}^{1}_{2^\alpha, \mu, \theta} \). Then there exists a \( (\lambda, \mu, \theta, \aleph_0) \)-perfect ultrafilter on \( \mathcal{B} \).

**Proof.** Begin by letting \( (D_\delta = (b_{\delta, u} : u \in [\lambda]^{<\aleph_0}) : \delta < 2^\lambda) \) be an enumeration of the monotonic sequences of elements of \( \mathcal{B}^+ \), each occurring cofinally often. Let \( z : 2^\lambda \rightarrow 2^\lambda \) be an increasing continuous function which satisfies: \( z(0) \geq 0 \) and for all \( \beta < 2^\lambda \), \( z(\beta) + \lambda = z(\beta + 1) \). By induction on \( \delta < 2^\lambda \) we will construct \( (D_\delta : \delta < 2^\lambda) \), an increasing continuous sequence of filters with each \( D_\delta \) an ultrafilter on \( \mathcal{B}^{z(\delta), \mu, \theta} \), to satisfy:

\[ (*) \text{ if } \delta = \beta + 1, \text{ if it is the case that } \]

\( (b_{\beta, u} : u \in [\lambda]^{<\aleph_0}) \) is a monotonic sequence of elements of \( D_\beta \) and there exists a choice of \( \text{supp}(\delta) \cup \{\text{dom}(f) : x_f \in \text{supp}(\delta)\} \subseteq \beta \) and there exists a multiplicative sequence

\( (b'_u : u \in [\lambda]^{<\aleph_0}) \)

of elements of \( \mathcal{B}^+ \) such that

(a) \( b'_u \leq b_{\beta, u} \) for all \( u \in [\lambda]^{<\aleph_0} \),

(b) for every \( c \in \mathcal{B}^{z(\delta), \mu, \theta}(\mathcal{B}) \), no intersection of finitely many members of

\[ \{b'_i \cup (1 - b_i) : i < \lambda\} \]

is disjoint to \( c \).

then there is a sequence \( b'' = (b''_u : u \in [\lambda]^{<\aleph_0}) \) of elements of \( \mathcal{B}^+ \) such that:

(i) \( b''_u \leq b_{\beta, u} \) for all \( u \in [\lambda]^{<\aleph_0} \),
(ii) for every $c \in B^+_{\mathcal{L}(\beta)\mu\theta} \cap D_\beta$, no intersection of finitely many members of $\{b^\prime_{\{i\}} \cup (1 - b_{\beta\{i\}}) : i < \lambda\}$ is disjoint to $c$.

(iii) some support of $b''$ is contained in $B^+_{\mathcal{L}(\delta)\mu\theta}$, and

(iv) $D_\delta$ is an ultrafilter on $B^+_{\mathcal{L}(\delta)\mu\theta}$ which extends $D_\beta \cup \{b^\prime_u : u \in [\lambda]^{<\aleph_0}\}$.

The induction may be carried out at limit stages because all of the $D_\delta$ are ultrafilters. Suppose $\delta = \beta + 1$. If $b$ satisfies the quoted condition, then let $b''$ be given by Observation 9.7, using $z(\beta)$ here for $\alpha$ there. Then (i), (ii), (iii) are satisfied, so we need to prove that

$$D_\beta \cup \{b''_u : u \in [\lambda]^{<\aleph_0}\}$$

has the finite intersection property. As $D_\beta$ is an ultrafilter on $B^+_{\mathcal{L}(\beta)\mu\theta}$, and $b'$ is a multiplicative sequence, it suffices to prove that for any $c \in D_\beta$ and any finite $u \subseteq \lambda$,

$$c \cap \bigcap \{b^\prime_{\{i\}} : i \in u\} > 0.$$  

As $b_{\{i\}} \in D_\beta$ for each $i \in u$, we may assume that $c \cap (1 - b_{\{i\}}) = 0$ for each $i \in u$. Then we are finished by assumption (ii). This completes the induction. Let $D_* = \bigcup_{\beta < 2} D_\beta$.

Let us check that $D_*$ is indeed a perfect ultrafilter. If $b$ satisfies condition (A), let $U$ be as there, and let $\delta = \beta + 1$ be an ordinal $< 2^\lambda$ such that $B_\beta = b$ and $U \subseteq B_{\beta,\mu,\theta}$, which is possible as we listed each sequence cofinally often. Then since $D_\beta$ was an ultrafilter, $D_* \upharpoonright B_{\beta,\mu,\theta} = D_\beta$ so at stage $\delta$ condition (*) of the inductive hypothesis will be activated and we will have ensured that $b$ has a multiplicative refinement in $D_*$. \[\square\]

**APPENDIX: NON-SATURATION FOR $T_{k+1,k}$ AND UNCOUNTABLE $\theta$**

In this Appendix we update the non-saturation result from [12] to allow for possibly uncountable $\theta$. The proof is the same. It has just been slightly rewritten for readability, since it seems a good occasion to call attention to Question 10.11.

**Theorem 10.9** ([12], Claim 5.1 for possibly uncountable $\theta$). *Suppose that:*

1. For integers $2 \leq k < \ell$, $\theta \geq \aleph_0$, and e.g. $\mu = 2^{\aleph_0}$, $\lambda = \mu^+$, or just: $(\lambda, k, \mu^+) \rightarrow k + 1$ in the sense of [12] Notation 1.2

2. $\mathcal{B} = B^+_{\lambda\mu\theta}$ [for $\theta$ possibly uncountable]

3. $D_\star$ is an ultrafilter on $\mathcal{B}$

4. $T = T_{k+1,k}$

*Then $D_\star$ is not $(\lambda, T)$-moral.*

**Proof.** Fix the objects given in the statement of the theorem. We’ll often use the notation $T_{n,k}$ instead of $T_{k+1,k}$, but $n = k + 1$ seems necessary in this proof, as we will point out. We will use [12] Claim 1.6, which when applied to $(\lambda, k, \mu^+)$ gives:

(10.A) there is a model $M \models T_{n,k}$ such that:

- $M$ has size $\geq \lambda$, and there are $\lambda$ elements of its domain $\{b_\alpha : \alpha < \lambda\}$ such that if we let $P = \{w \in [\lambda]^n : (\forall u \in [w]^{k+1})(M \models R(b_u))\}$ denote the indices for near-forbidden configurations, then for any

[10] recall that in $T_{n,k}$ the forbidden configuration is a set of $n + 1$ vertices of which every $k + 1$ form an edge. So if $\{b_\alpha : \alpha \in U\}$ is near-forbidden, then $\{R(x, b_\alpha) : v \in [w]^k\}$ is not a type.
\[ F : [\lambda]^k \to [\lambda]^{\leq \mu} \] such that \( u \subseteq F(u) \) for all \( u \in \text{dom}(F) \), there exists \( w \in \mathcal{P} \) such that \((\forall v \in [w]^k)(w \not\subseteq F(v))\).

Informally, the conclusion is that for any such \( F \), some near-forbidden configuration escapes the control of its \( k \)-element subsets.

The strategy will be to build a possibility pattern that has no multiplicative refinement. Fix a sequence of ordinals \( (\alpha_w : w \in \mathcal{P}) \), each \( < \lambda^+ \), with no repetitions. For each \( w \in \mathcal{P} \), fix a function \( g_w \in \text{FI}_{\mu,0}(2^\lambda) \) such that \( \text{dom}(g_w) = \{ \alpha_w \} \) and \( x_{g_w} = 0 \mod \alpha_w \).

The strategy will be to build a possibility pattern that has no multiplicative refinement. Fix a sequence of ordinals \( (\alpha_w : w \in \mathcal{P}) \), each \( < \lambda^+ \), with no repetitions. For each \( w \in \mathcal{P} \), fix a function \( g_w \in \text{FI}_{\mu,0}(2^\lambda) \) such that \( \text{dom}(g_w) = \{ \alpha_w \} \) and \( x_{g_w} = 0 \mod \alpha_w \).

Let
\[
(10.B) \quad \langle v_\alpha : \alpha < \lambda \rangle
\]
list \([\lambda]^k\) without repetition.\(^{11}\) Let \( \Omega = [\lambda]^{<\kappa_0} \) and for each \( s \in \Omega \), let\(^{12}\)
\[
(10.C) \quad b_s = 1_{\mathcal{B}} - \bigcup\{x_{g_w} : w \in \mathcal{P} \text{ and } [w]^k \subseteq \{v_\beta : \beta \in s\}\}.
\]

In order to check that \( \bar{b} = \langle b_s : s \in \Omega \rangle \) is really a representation of some type in the ultrapower, it will suffice by compactness to argue as follows (since we may always choose the index model to be a \( \lambda^+ \)-saturated elementary extension of \( M \)).

First note that for each finite \( s \subseteq \lambda \), each nonzero \( c \in \mathcal{B} \), \( c \) induces a partition of \( \{b_{s'} : s' \subseteq s \} \) according to whether \( c \cap b_{s'} = 0 \) or \( c \cap b_{s'} > 0 \). By shrinking \( c \) if necessary, we may assume \( c \) induces a partition of \( \{b_{s'} : s' \subseteq s \} \) according to whether \( c \cap b_{s'} = 0 \) or \( c \leq b_{s'} \). Let \( \text{vert}(s) := \bigcup\{v_\beta : \beta \in s\} \) be the set of indices for all elements mentioned in formulas in \( s \). Finally, by shrinking \( c \) if necessary, we may also assume that either \( c \leq x_{g_w} \) or \( c \leq 1 - x_{g_w} \) for every \( w \in \mathcal{P} \) such that \( w \subseteq \text{vert}(s) \). It suffices to show that for each such finite \( s \) and nonzero \( c \), we may choose elements \( \{b_{s_\beta} : \beta \in \text{vert}(s)\} \) in \( M \) such that
\[
(10.D) \quad M \models (\exists x)(\bigwedge_{\beta \in s'} R(x, b_{s_\beta}')) \text{ when } c \leq b_{s'},
\]
and
\[
(10.E) \quad M \models \neg(\exists x)(\bigwedge_{\beta \in s'} R(x, b_{s_\beta}')) \text{ when } c \cap b_{s'} = 0.
\]

Consider the elements \( \{b_{s_\beta} : \beta \in \text{vert}(s)\} \). In \( M \), this set may have some edges on it. Informally, what we will do is for each \( w \subseteq \text{vert}(s) \) such that \( w \in \mathcal{P} \), we remove the edge on \( \{b_{s_\beta} : \beta \in w\} \) if and only if \( c \leq 1 - x_{g_w} \). Formally, we choose a set of distinct elements \( \{b_{s_\beta} : \beta \in \text{vert}(s)\} \) such that \( R(b_{s_\beta_0}, \ldots, b_{s_\beta_k}) \) if and only if \( \{\beta_0, \ldots, \beta_k\} \in \mathcal{P} \) and \( c \leq x_{g_w} \).

Let’s check that \( (10.D) \) and \( (10.E) \) are satisfied.

If \( c \cap b_{s'} = 0 \), then there is some \( w \) such that \( w \subseteq \text{vert}(s) \), \( w \in \mathcal{P} \), \( [w]^k \subseteq \{v_\beta : \beta \in s\} \) and \( c \leq x_{g_w} \). [Suppose not. If there were no \( w \subseteq \text{vert}(s') \) such that \( w \in \mathcal{P} \) and \( [w]^k \subseteq \{v_\beta : \beta \in s'\} \), then by definition in \( (10.C) \), \( b_w = 1_{\mathcal{B}} \), so we contradict \( c > 0 \). So there must be some such \( w \). Let \( w_0, \ldots, w_i \) be a list of all such \( w_i \). Then again by \( (10.C) \), \( \bigcup_{j \leq i} x_{g_{w_j}} = 1 - b_{s'} \), so it must be that \( c \leq x_{g_{w_j}} \) for some \( j \leq i \).]

\(^{11}\)We are aiming at a type in the ultrapower of the form \( \{R(x, \bar{a}_{v_\alpha}) : \alpha < \lambda \} \). Since \( u, v, w \) are used for sets of indices, we use \( s \in \Omega \) for a finite set of formulas in the type.

\(^{12}\)Note that the condition in \( (10.C) \) is set to avoid \( x_{g_w} \) only if all \( k \)-element subsets of \( w \) occur as \( v_\beta \) for some \( \beta \in s \). It’s not enough that each element of \( w \) occurs in some \( v_\beta \). For example, in the tetrahedron-free three-hypergraph, if \( w = \{1, 2, 3\} \) and \( \{v_\beta : \beta \in s\} = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\} \), then \( b_s \cap x_{g_w} = 0 \), but not if \( \{v_\beta : \beta \in s\} = \{\{1, 2\}, \{1, 3\}\} \).
By construction, $M \models R(\bar{b}^\prime_w)$, so $\{R(x, \bar{b}_{v_\beta}) : \beta \in s'\} \supseteq \{R(x, \bar{b}_{v_\beta}) : \beta \in [w]^k\}$ is indeed inconsistent.

If $c \leq b_x$, then for any $w$ such that $w \subseteq \text{vert}(s)$, $w \in \mathcal{P}$, and $[w]^k \subseteq \{v_\beta : \beta \in s\}$, we must also have (by definition of $b_x$ in (10.C) that $c \leq x_{g_w}$. But in this case we removed the edge on $\{b_{v_\beta} : \beta \in w\}$. Since this is true for all relevant $w$, $\{R(x, \bar{b}_{v_\beta}) : \beta \in s'\}$ is indeed consistent.

This shows that $\bar{b} = (b_x : s \in \Omega)$ is indeed a possibility pattern, and it remains to show it has no multiplicative refinement. Suppose for a contradiction that $\bar{b}' = (b'_x : s \in \Omega)$ were a multiplicative refinement of $\bar{b}$, i.e. $\bar{b}'$ is a sequence of elements of $D$, such that $s_1, s_2 \in \Omega$ implies $b'_{s_1} \cap b'_{s_2} = b'_{s_1 \cap s_2}$ and for each $s \in \Omega$, $b'_s \leq a_s$. As each $b'_{i(\beta)}$ in $2^\beta$, we may write $b'_{i(\beta)} = \bigcup\{x_{\beta, i} : i < i(\beta) \leq \mu\}$ where $\langle h_{\beta, i} : i < i(\beta) \rangle$ is a set of pairwise inconsistent functions from $\text{Fl}_{\mu, \alpha}(2^\lambda)$. Let $S_\beta = \bigcup\{\text{dom}(h_{\beta, i}) : i < i(\beta)\}$, so $S_\beta \subseteq 2^\lambda$ has cardinality $\leq \mu : \theta = \mu$.

**Subclaim 10.10.** Let $n = k + 1$. If $w \in \mathcal{P}$ then $\alpha_w \in \bigcup\{S_\beta : v_\beta \in [w]^k\}$.

**Proof.** Let $x = \{\beta : v_\beta \in [w]^k\}$, which is a finite set since (10.B) was without repetitions. As $\Bbb{B}$ is multiplicative, $b'_x = \bigcap\{b'_\beta : \beta \in x\}$. Let $f \in \text{Fl}_{\mu, \alpha}(2^\lambda)$ be such that $x_f \leq b'_x$. Thus $x_f \leq b'_{i(\beta)}$ for each $\beta \in x$. Let $g = f \setminus \bigcup\{S_\beta : \beta \in x\}$, noting that $g$ must be nonempty [indeed, if $\text{dom}(f) \cap S_\beta = \emptyset$ for some $\beta \in x$, then necessarily $x_f \cap (1 - b'_{\beta}) > 0$]. Then $x_g \leq b'_{i(\beta)}$ for all $\beta \in x$. This implies that $x_g \leq b'_x \leq b_x$ because $b'$ refines $b$. By definition in (10.C),

$$b_x = 1_{\mathcal{B}} - \bigcup\{x_{g_w} : w \in \mathcal{P} \text{ and } [w]^k \subseteq \{v_\beta : \beta \in x\}\}.$$ 

So as $[w]^k \subseteq \{v_\beta : \beta \in x\}$, necessarily $x_g \cap x_{g_w} = 0_{\mathcal{B}}$. Since our Boolean algebra $\mathcal{B}$ was generated freely, it must be that $\text{dom}(g_w) \cap \text{dom}(g) \neq \emptyset$, but $\text{dom}(g_w) = \{\alpha_w\}$. This shows that $\alpha_w \in \bigcup\{S_\beta : v_\beta \in [w]^k\}$ as desired. 

This proves Subclaim 10.10.  

Finally, define $F : [\lambda]^k \rightarrow [\lambda]^{\leq \mu}$ by: if $v \in [\lambda]^k$ let $\beta$ be such that $v = v_\beta$, and let

$$F(v) = \bigcup\{w \in [\lambda]^n : w \in \mathcal{P} \text{ and } \alpha_w \in S_\beta\} \cup v.$$ 

Then $F(v)$ is well defined, $F(v) \subseteq \lambda$, and $|F(v)| \leq \mu$ for $v \in [\lambda]^k$, since (10.B) is without repetition and $|S_\beta| \leq \mu$. Now for all $w \in \mathcal{P}$, there is $v = v_\beta \in [w]^k$ such that $\alpha_w \in S_\beta$. Thus $w \subseteq F(v)$. This shows that for all $w \in \mathcal{P}$,

$$(\exists w \in [w]^k)(w \subseteq F(v)).$$

This is a contradiction to (10.A), so $\bar{b}$ does not have a multiplicative refinement. Thus, $D$, cannot be moral for $T_{k+1,k}$. This completes the proof.  

**Question 10.11.** Does Theorem 10.9 hold for $n > k + 1$?

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