REGULARIZED TRACE OF STURM-LIOUVILLE EQUATION
WITH SINGULARITY ON A BOUNDED SEGMENT

ILYAS HAŞIMOĞLU

Abstract. Let \( \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \ldots \) and \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \) be the eigenvalues of the operators \( L_0 \) and \( L \) which are formed by differential expressions

\[
L_0[y] = -y'' + \frac{\nu^2 - 1/4}{x^2} y, \quad L[y] = -y'' + \frac{\nu^2 - 1/4}{x^2} y + q(x)y, \quad \nu \geq 1/2
\]

respectively and with the same boundary conditions Eq.(2.4)

\[ y(0) = y(1) = 0 \]

We prove that under some conditions on \( q(x) \), the following formula for traces

\[
\sum_{n=1}^{\infty} (\lambda_n - \mu_n) = -\frac{2q(0) + q(1)}{4}
\]

with Eq.(2.4) holds.

1. Introduction

In literature there are numerous papers devoted to the calculation of regularized trace of scalar differential operators which is the generalization of concept of matrix trace. First work in this direction belongs to I.M.Gelfand and B.M.Levitan [1], where the formula for the sum of differences of eigenvalues of two regular Sturm-Liouville operators on the \([0, \pi]\), was obtained. This work has numerous continuations. In [2-16] the regularized traces are calculated in several cases. In [2] the formula was obtained for the sum of differences of eigenvalues of two singular self-adjoint Sturm-Liouville operators which differ by finite potentials. In [4-8] by using zeta function and teta function V.A.Sadovnichiy has obtained formulae for regularized traces for wide class of differential operators. In [7] the author considered the following operator

\[
L y = -y'' + \frac{\nu^2 - 1/4}{x^2} y + q(x)y = \lambda^2 y
\]
on \((0, \pi]\), for the case when the potential \(q(x)\) is finite on the neighborhood of zero and is a sufficiently smooth function. In this case the series \(\sum_{n=1}^{\infty} (\lambda_n - (n + \frac{\nu}{2} - \frac{1}{2})^2)\) is calculated.

The aim of this article is to calculate the regularized trace with the peculiarity on the bounded segment. Namely we consider the following problem.

Let \(L_0\) and \(L\) be operators acting in \(L_2(0, 1)\) and formed by differential expressions

\[
\ell_0[y] = -y'' + \frac{v^2 - 1/4}{x^2} y, \quad \ell[y] = -y'' + \frac{v^2 - 1/4}{x^2} y + q(x)y, \quad \nu \geq 1/2
\]

respectively and with same boundary conditions Eq. (2.4)

\[y(0) = y(1) = 0,\]

where potential \(q(x)\) is a bounded function and satisfies the following conditions:

a) \(q(x)\) satisfies the Hölder condition with order \(\alpha > 0\) on the neigbour of zero, i.e. there is \(\varepsilon > 0\) such that for any \(x \in [0, \varepsilon]\) the following inequality holds

\[|q(x) - q(0)| < \text{const. } x^\alpha\]

b) \(\int_0^1 q(x) dx = 0\)

Our aim is to find the regularized trace of these operators.

2. MAIN RESULTS

Let \(\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \ldots\) and \(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots\) be the eigenvalues of the operators \(L_0\) and \(L\), respectively. Then the following theorem holds

**Theorem.** Let the function \(q(x)\) satisfy conditions a) and b).

Then \(\sum_{n=1}^{\infty} (\lambda_n - \mu_n) = -\frac{2q(0) + q(2.4)}{4}\)

It is shown in [11] that \(\sum_{n=1}^{\infty} (\lambda_n - \mu_n - (q\ell_n, \ell_n)) = 0\), where \(\ell_n\) are eigenfunctions of \(L_0\). If \(\sum_{n=1}^{\infty} (q\ell_n, \ell_n)\) is convergent then \(\sum_{n=1}^{\infty} (\lambda_n - \mu_n) = \sum_{n=1}^{\infty} (q\ell_n, \ell_n)\).

Eigenfunctions of the operator \(L_0\) have the form [2]

\[\ell_n = \sqrt{2x} \frac{J_v(j_n x)}{J_{v+1}(j_n)}\]

where \(J_v(x)\) is the Bessel function, \(j_1 \leq j_2 \leq j_3 \leq \ldots\) are positive roots of \(J_v(z)\).
\[
\sum_{n=1}^{\infty} (q\ell_n, \ell_n) = \lim_{N \to \infty} \sum_{n=1}^{N} (q\ell_n, \ell_n)
\] (2.1)

\[
= \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{J_{v+1}(j_n)} \int_{0}^{x} 2x J_{v}^2(j_nx) q(x) dx
\] (2.2)

\[
= \lim_{N \to \infty} \int_{0}^{1} \left( \sum_{n=1}^{N} \frac{2x J_{v}^2(j_nx)}{J_{v+1}^2(j_n)} \right) q(x) dx
\] (2.3)

To find this limit we will study the asymptotical behavior of the function

\[ T_N(x) = \sum_{n=1}^{N} 2x J_{v}^2(j_nx) J_{v+1}(j_n) \] with and \( \varepsilon > 0 \)

For later use we note the following lemma

**Lemma.** For the function \( T_N(x) \),

\[ T_N(x) = \frac{A_N}{\pi} - \frac{\cos(2x A_N - v\pi)}{2 \sin x} + \frac{\psi(A_N x)}{x} \]

where \( A_N = (N + \frac{v}{2} + \frac{1}{4})\pi \), when \( 0 < x < 1 \),

\( \psi(A_N x) \to 0 \) as \( N \to \infty \).

**Proof of Lemma.** To get formula for \( T_N(x) \) we express the \( m \)th term of the sum \( T_N(x) \) in the form of residue at point \( j_m \) of some function of complex variable \( z \), which has poles at the points \( j_1, j_2, ..., j_N \).

Consider the following complex function.

\[ \frac{zz \{ J_{v}^2(zz) - J_{v-1}(zz)J_{v+1}(zz) \}}{J_{v}^2(z)} \]

First, prove that this function has a residue \( 2x J_{v}^2(j_mx) / J_{v+1}(j_m) \) at \( z = j_m \).

If \( z = j_m + \theta \), where \( \theta \) is small, then

\[ J_{v}(z) = \theta J'_{v}(j_m) + \frac{1}{2} \theta^2 J''_{v}(j_m) + ... \]

and hence

\[ zJ_{v}^2(z) = \theta^2 j_m J_{v}^2(j_m) + \theta^3 (j_m) \{ j_m J''_{v}(j_m) + J'_{v}(j_m) \} + ... \]

Using Bessel differential equation it is easy to prove that the coefficient of \( \theta^3 \) on the right hand side of this expression is zero.

Therefore the residue of function

\[ \frac{g(z)}{zJ_{v}^2(z)} = \frac{z^2 x \{ J_{v}^2(zz) - J_{v-1}(zz)J_{v+1}(zz) \}}{zJ_{v}^2(z)} \] (2.4)
at $j_m$ is

$$\frac{g'(j_m)}{j_m J^2_{\nu}(j_m)} = 2x \frac{J_{\nu}^2(j_mx)}{J_{\nu+1}^2(j_m)}$$

As a contour of integration we take the rectangle with vertex at $\pm iB, A_N \pm iB$. Here $B \to \infty$ and $j_N < A_N < j_{N+1}$. For $A_N$ we take the value $(N + \frac{1}{2} + \frac{1}{4}) \pi$, in the case when $N$ is sufficiently large this value is between $j_N$ and $j_{N+1}$.

It is easy to prove that the function (2.4) is an odd function of $z$, therefore the integral on the left sides of rectangle is zero. If $z = u + iw$, then for large $|w|$ and for $u \geq 0$ integrand will have an order $O(e^{-2(1-x)|w|})$ and consequently for given value of $A_N$, integrals on upper and lower sides converge to zero as $B \to \infty$, when $0 < x < 1$.

Thus we obtain

$$T_N(x) = \frac{1}{2\pi i} \int_{A_N-iB}^{A_N+iB} \frac{zx\{J_{\nu}^2(xz) - J_{\nu+1}(xz)J_{\nu-1}(xz)\}}{J_{\nu}^2(z)} \; dz$$

When $j^{-1+\varepsilon}_N \leq x < 1$, where $0 < \varepsilon < 1/2, \; |xz| \to \infty$, the Bessel functions in the integrant can be replaced by the corresponding asymptotes with large arguments

$$J_{\nu}^2(xz) - J_{\nu-1}(xz)J_{\nu+1}(xz) = \frac{2}{\pi xz} \left[ 1 - \frac{1}{2xz} \cos(2xz - \nu\pi) \right] \left( 1 + O\left( \frac{1}{(xz)^2} \right) \right)$$

$$J_{\nu}^2(z) = \frac{2}{\pi z} \left[ \frac{1}{2} + \frac{\sin(2z - \nu\pi)}{2} + \frac{\cos(2z - \nu\pi)(2\nu - 1)(2\nu + 1)}{8z} \right] \left( 1 + O\left( \frac{1}{z^2} \right) \right)$$
Then if \( N \to \infty \)

\[
T_N(x) = \frac{1}{2\pi i} \lim_{B \to \infty} \int_{A_N - iB}^{A_N + iB} \frac{xz\{J_v^2(xz) - J_{v+1}(xz)J_{v-1}(xz)\}}{J_v^2(xz)} dz
\]

\[
= \frac{1}{2\pi i} \lim_{B \to \infty} \int_{A_N - iB}^{A_N + iB} \frac{z[1 - \frac{1}{2\pi xz} \cos(2xz - v\pi)](1 + 0 \left(\frac{1}{(xz)^2}\right))}{1 + \sin(2z - v\pi)} dz
\]

\[
\approx \frac{1}{\pi i} \int_{A_N - i\infty}^{A_N + i\infty} \frac{z}{1 + \sin(2z - v\pi)} dz
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{(A_N + iw) dv}{1 + \cos(2iw)} - \frac{1}{2\pi x} \int_{-\infty}^{+\infty} \frac{\cos(2x A_N + 2ixw - v\pi)}{1 + \cos(2iv)} dw
\]

\[
= \frac{A_N}{\pi} - \frac{\cos(2\pi x A_N - v\pi)}{2\pi x} \int_{-\infty}^{+\infty} \frac{\cosh(2xw)}{1 + \cosh(2w)} dw
\]

\[
= \frac{A_N - \cos(2\pi x A_N - v\pi)}{2 \sin \pi x}
\]

That is we obtain

\[
T_N(x) = \frac{A_N}{\pi} - \frac{\cos(2\pi x A_N - v\pi)}{2 \sin \pi x} + \frac{\psi(A_N x)}{x},
\]

where \( A_N = (N + \frac{v}{2} + \frac{1}{4})\pi \) and \( \psi(A_N x) = O(\lim_{B \to \infty} \int_{A_N - i\infty}^{A_N + i\infty} \frac{\sin(2xz - v\pi)}{xz(1 + \sin(2z - v\pi))} dz) \)

when \( 0 < x < 1 \), \( \psi(A_N x) \to 0 \) as \( N \to \infty \).

Really, if we put \( z = A_N + iw \), the integral

\[
\int_{A_N - iB}^{A_N + iB} \frac{\sin(2z - v\pi)}{xz(1 + \sin(2z - v\pi))} dz
\]

(2.5)
can be rewritten as

\[
2A_N \sin(2xA_N - v\pi) \int_0^\infty \frac{\cosh(2xw)}{x(A_N^2 + w^2)(1 + \cosh(2w))} dw
\]
\[
+ 2 \cos(2xA_N - v\pi) \int_0^\infty \frac{w \sinh(2xw)}{x(A_N^2 + w^2)(1 + \cosh(2w))} dw
\]

When \(j W^{1+\varepsilon} < x < 1\), absolute value of (2.5) doesn’t exceed

\[
\frac{1}{xA_N} \int_0^\infty \frac{\cosh(2xv)}{1 + \cosh(2v)} dv + \frac{2}{xA_N^2} \int_0^\infty \frac{v \sinh(2xv)}{1 + \cosh(2v)} dv
\]

This shows, that \(\psi(A_Nx) \to 0\) as \(N \to \infty\). If \(x = 1\)

\[
\lim_{B \to \infty} \int_{A_N-iB}^{A_N+iB} \frac{\sin(2z-v\pi)}{z(1 + \sin(2z-v\pi))} dz = \lim_{B \to \infty} \int_{-B}^{B} \frac{\cosh(2w) dw}{(A_N + iw)(1 + \cosh(2w))}
\]

\[
= \lim_{B \to \infty} \left[ 2A_N \int_0^B \frac{\cosh(2w)}{(A_N^2 + w^2)(1 + \cosh(2w))} dw - i \int_{-B}^{B} \frac{w \cosh(2w) dw}{(A_N^2 + w^2)(1 + \cosh(2w))} \right]
\]

\[
= \lim_{B \to \infty} 2A_N \int_0^B \frac{\cosh(2w)}{(A_N^2 + w^2)(1 + \cosh(2w))} dw
\]

Absolute value of this integral doesn’t exceed \(2A_N \int_0^\infty \frac{dw}{A_N^2 + w^2} = \pi\).

We obtain that \(\psi(A_Nx) \to 0\) as \(N \to \infty\) when \(j N^{1+\varepsilon} < x < 1\) and \(|\psi(A_Nx)| < c\) when \(x = 1\).

**Proof of Theorem.** Now using the asympotes of the function

\[T_N(x) = \sum_{n=1}^N 2x \frac{j_n^2(j_n x)}{j_{n+1}^2(j_n)},\] we get
\[
\sum_{n=1}^{\infty} (\lambda_n - \mu_n) = \lim_{N \to \infty} \sum_{n=1}^{N} \int_{0}^{ \frac{2\pi}{A_N(x+\nu)}} q(x)dx = \lim_{N \to \infty} \int_{0}^{A_N^{1+\varepsilon}} T_N(x)q(x)dx
\]

\[
= \lim_{N \to \infty} \left[ \int_{0}^{ A_N^{1+\varepsilon}} T_N(x)(q(x) - q(0))dx + q(0)N \right]
\]

\[
= \lim_{N \to \infty} \left[ \int_{0}^{ A_N^{1+\varepsilon}} T_N(x)(q(x) - q(0))dx - \int_{0}^{ A_N^{1+\varepsilon}} (\frac{A_N}{\pi} - \frac{\cos(2A_N x - \nu \pi)}{2 \sin \pi x})q(x) - q(0))dx + q(0)N \right]
\]

\[
= \lim_{N \to \infty} \left[ \int_{0}^{ A_N^{1+\varepsilon}} T_N(x)(q(x) - q(0))dx + \int_{0}^{ A_N^{1+\varepsilon}} \frac{\psi(A_N x)}{x}(q(x) - q(0))dx \right]
\]

\[
= \lim_{N \to \infty} \int_{0}^{ A_N^{1+\varepsilon}} T_N(x)(q(x) - q(0))dx + \frac{1}{A_N^{1+\varepsilon}} \int_{0}^{ A_N^{1+\varepsilon}} \frac{\psi(A_N x)}{x}(q(x) - q(0))dx
\]

when \(0 < \varepsilon < \frac{\alpha}{\alpha + 1}\).

Using the inequality \(|\sqrt{x}J_\nu(x)| < \text{const}\), for any \(x [17]\), it is easy to prove that \(T_N(x) \leq \text{const} N(3)\).

Take into account (3) and condition a), one can easily see that

\[
\lim_{N \to \infty} \left| \int_{0}^{ A_N^{1+\varepsilon}} T_N(x)(q(x) - q(0))dx \right| \leq \lim_{N \to \infty} \int_{0}^{ A_N^{1+\varepsilon}} T_N(x)|q(x) - q(0)|dx
\]

\[
\leq \lim_{N \to \infty} \text{const} \int_{0}^{ A_N} N x^\alpha dx
\]

\[
= \text{const} \lim_{N \to \infty} N A_N^{1-\alpha+\varepsilon(\alpha+1)} = 0
\]

for \(0 < \varepsilon < \frac{\alpha}{\alpha + 1}\).

Moreover it is easy to prove that

\[
\lim_{N \to \infty} \int_{0}^{ A_N^{1+\varepsilon}} \left( \frac{A_N}{\pi} - \frac{\cos(2A_N x - \nu \pi)}{2 \sin \pi x} \right)(q(x) - q(0))dx = 0 \quad (2.6)
\]
\[ \lim_{N \to \infty} \int_{A_N}^{1+x} \frac{\psi(A_Nx)}{x} (q(x) - q(0)) \, dx = 0 \]  

(2.7)

Taking into account (3) – (2.7) and condition b), we obtain

\[ \lim_{N \to \infty} \int_0^\infty \frac{1}{\pi} \left( \frac{A_N}{2} - \cos(2A_Nx-\pi) \right) (q(x) - q(0)) \, dx + q(0)N \]

\[ = \lim_{N \to \infty} \frac{A_N}{\pi} \int_0^1 q(x) \, dx - \frac{A_N}{\pi} q(0) + Nq(0) \]

\[ - \frac{1}{4} \int_0^1 \frac{\cos(2N+\frac{1}{2})}{\sin \frac{\pi}{2} x} (q(x) - q(0)) \, dx \]

\[ = - \left( \frac{1}{2} + \frac{1}{4} \right) q(0) - \frac{q(1)-q(0)}{4} = - \frac{2q(0)+q(1)}{4} \]

That is,

\[ \sum_{n=1}^{\infty} (\lambda_n - \mu_n) = - \frac{2q(0)+q(1)}{4} \]

This completes the proof of the theorem.

ÖZET:

\[ \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \ldots \text{ ve } \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \text{sayıları} \]

\[ \ell_n[y] = -y'' + \frac{\nu^2-1/4}{x^2} y, \quad \ell[y] = -y'' + \frac{\nu^2-1/4}{x^2} y + q(x)y, \quad \nu \geq 1/2 \]

diferansiyel ifadeleri ve \( y(0) = y(1) = 0 \) sınır şartları ile tamlanmış sırasıyla \( L_0 \) ve \( L_1 \) operatörlerinin özeğeleri olun. Makalede \( q(x) \) fonksiyonun bazı şartları sağlandığında

\[ \sum_{n=1}^{\infty} (\lambda_n - \mu_n) = - \frac{2q(0)+q(1)}{4} \text{ formülü ispatlanmıştır.} \]

REFERENCES

[1] I.M.Gelfand, B.M.Levitan, On the simple identity for the eigenvalues of second order differential operator. Doklady Akademii Nauk SSSR, 88(1953) 593-596
[2] M.G.Gasimov , The sum of difference of eigenvalues of two singular Sturm-Liouville operators. Doklady Akademii Nauk SSSR, 151(1963) 1014-1017
[3] L.A.Dikiy, On the Gelfand-Levitan formula. Uspekhi Mat. Nauk, 7(1953) 119-123
[4] L.A.Dikiy, Zeta function of ordinary differential equation on the segment. Izvestiya Akademii Nauk SSSR, ser. Matematika, 19(1955) 187-200
[5] V.A.Sadovnichiy, The trace of ordinary differential operators. Mat. Zametki 1(1967) 179-188
[6] V.A.Sadovnichiy, The identity of eigenvalues of Dirac system and some other systems of high order. Vestnik MGU 3(1967) 37-47
[7] V.A.Sadovnichiy, Some identities for the eigenvalues of singular differential operators. Relations for the zeros of the Bessel function. Vestnik MGU , ser. Matem. Mechan. 3(1971) 77-86
[8] V.A.Sadovnichiy, Zeta function and eigenvalues of differential operators. Differentsialniye uravneniya 10(1974) 1276-1285
[9] V.A.Sadovnichiy, The trace of high order differential operators. Differentsialniye uravneniya 28(1996) 1611-1624
[10] F.G.Maksudov, M.Bairamoglu, A.A.Adigozelov, On the regularized trace on the segment of Sturm-Liouville operator with unbounded operator coefficients. Doklady Akademii Nauk SSSR, 277(1984) 795-799
[11] V.A.Lyubishkin, Formulae of Gelfand-Levitan and Crayn. Math. Sbornik, 182(1991) 1786-1795
[12] V.A.Lyubishkin, V.E. Podolskiy, On the summarizing of regularized traces of differential operators. Mat. Zametki, 54(1993), 33-38
[13] V.G. Papanicolaou, Trace formulas and the behavior of large eigenvalues. SIAM J. Math.Anal. 26(1995) 218-237
[14] F.Gesztesy, H.Helden, B.Simon and Z.Zhao, Trace formulae and inverse spectral theory for Schroedinger operators. Bull. Amer. Math. Soc. 29(1993) 250-255
[15] V.A.Vinokurov, V.A.Sadovnichiy, Asymptote of eigenvalues and eigenfunctions and trace formula for the potential contains $\delta$—function. Doklady Akademii Nauk SSSR, 376(2001) , 445-448
[16] A.M.Savchuk, A.A.Shkalikov, Trace formula for the Sturm-Liouville operator with singular potential. Mat. Zametki, 69(2001) 427-442
[17] G.N.Watson, A tretise on the theory of Bessel functions. Cambridge University Press, Cambridge, 1992

Current address: Ilyas Ha¸ simo¼glu; Karabuk University, Karabuk, TURKEY and Institute of Mathematics, Academy of Sciences of Azerbaijan, Baku, Azerbaijan
E-mail address: ilyas_hashimov@yahoo.com
URL: http://communications.science.ankara.edu.tr/index.php?series=A1