EXOTIC CLASSES FOR MEASURED FOLIATIONS
BY STEVEN HURDER

A measured foliation \((F, \mu)\) is a \(C^2\)-foliation \(F\) on a smooth manifold \(M\) and a transverse invariant measure \(\mu\) for \(F\) [14]. Inspired by the foliation index theorem of Connes [4, 5], we study the result of integrating normal data to \(F\) over the leaf space \(M/F\). This produces new secondary-type exotic classes for measured foliations [7]. These classes have applications to \(SL_q\)-foliations, to the study of groups of volume-preserving diffeomorphisms, and also are useful for relating the geometry of \(F\) to the values of the usual secondary classes [8, 9].

**Theorem 1.** Let \((F, \mu)\) be a measured foliation of codimension \(q\) on \(M\). If either \(M\) is closed and orientable, or \(\mu\) is absolutely continuous (so it is represented by a closed form \(d\mu\)), then there is a well-defined characteristic map

\[ \chi_\mu : H^*(\Omega_q, O_q) \rightarrow H^{*+q}(M). \]

We call the image of \(\chi_\mu\) the \(\mu\)-classes of \((F, \mu)\).

For \(M^m\) compact and \(y_I \in H^n(\Omega_q, O_q)\), the class \(\chi_\mu(y_I)\) is defined as the geometric current in \(H_m^n - n - q(M)\) obtained by integrating over the leaf space of \(F\), via \(\mu\), the leaf classes corresponding to \(y_I\). Duality then produces the invariant in \(H^{*+q}(M)\). If \(d\mu\) is a closed form representing \(\mu\), then a cocycle representing \(\chi_\mu(y_I)\) is \(\Delta(y_I) \cdot d\mu\), where \(\Delta : WO_q \rightarrow A'(M)\) is the secondary map for \(F\), [2, 10]. Complete details and properties of \(\chi_\mu\) are described in [7].

The values of the \(\mu\)-classes depend on the measure \(\mu\) and the dynamical behavior of \(F\) in a neighborhood of the support of \(\mu\). It is conjectured that sub-exponential growth of the leaves of \(F\) implies the \(\mu\)-classes vanish; this can be shown in some cases. Examples can be constructed for which all of the \(\mu\)-classes are nontrivial.

The canonical measure associated to an \(SL_q\)-foliation \((F, \omega)\)—where \(\omega\) is a transverse invariant volume form—defines a characteristic map \(\chi_\omega : H^*(sl_q, SO_q) \rightarrow H^{*+q}(M)\), and these come from universal classes for the Haefliger classifying space \(B\Gamma_{SL_q}\). There are additional \(\mu\)-classes for measured foliations with framed normal bundles, and corresponding universal classes for \(B\Gamma_{SL_q}\), the homotopy fiber of \(B\Gamma_{SL_q} \rightarrow BSL_q\).

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Theorem 2. The characteristic maps

\[ \chi: H^n(\mathfrak{sl}_q, SO_q) \rightarrow H^{n+q}(B\Gamma_{SL_q}), \]
\[ \chi^g: H^n(\mathfrak{sl}_q) \rightarrow H^{n+q}(B\Gamma_{SL_q}) \]

are injective for \( n < \lceil (q - 1)/4 \rceil \). Further, \( \chi^g \) is injective on the image \( H^*(so_q) \rightarrow H^*(sl_q) \).

To show nontriviality, we compute the values of the \( \mu \)-classes for the foliations obtained by suspending the action of \( SL_q \mathbb{Z} \) on \( T^q \). This type of example was suggested to us by W. Thurston. The corresponding characteristic map is related to the Van Est map of continuous cohomology \( H^*(sl_q, SO_q) \rightarrow H^*(SL_q; \mathbb{R}) \), which is injective in degrees less than \( \lceil (q - 1)/4 \rceil \) by Borel [1].

Corollary 1. There are inclusions of \( \mathbb{Q} \)-vector spaces

\[ H_n(so_q; \mathbb{R}) \subset H_n(B\text{Diff}_\omega^c \mathbb{R}^q; \mathbb{Q}) \quad \text{for } n < q, \text{ and } \mathbb{R} \subset H_q(B\text{Diff}_\omega^c \mathbb{R}^q; \mathbb{Q}) \quad \text{for all } q \geq 3. \]

It was shown that \( H_1(B\text{Diff}_\omega^c \mathbb{R}^q; \mathbb{Z}) = 0 \) for \( q > 2 \) by Thurston-Banyaga. Corollary 1 gives the first nonvanishing results for the group homology in degrees less than \( q + 1 \); in degrees \( \geq q + 1 \), the secondary classes of \( SL_q \)-foliations detect nontrivial homology of \( B\text{Diff}_\omega^c \mathbb{R}^q \). McDuff has investigated in [12] the geometrical significance of some of these new invariants for \( \text{Diff}_\omega^c \mathbb{R}^q \) and also defined further interesting classes.

The residuable secondary classes are the cocycles \( y_f c_f \) in \( H^*(WO_q) \) with degree \( c_f = 2q \) maximal. The "integration over the fiber" process is faithful on these classes, so a residue theory can be developed for them. Given a measured foliation \((F, \mu)\) with support \( \mu = M \), the residuable classes decompose into the measure class \( d\mu \) product with a leaf invariant. This observation can be used to relate the residuable secondary classes with the geometry of \( F \).

Theorem 3. Let \( F \) be a codimension \( q \) compact foliation (that is, each leaf of \( F \) is compact) on a closed manifold \( M \). Each residuable secondary class \( \Delta_\omega(y_f c_f) \in H^*(M) \) is then zero.
The idea of the proof is to integrate $\Delta(yTc_j)$ over $M$, decompose this integral over saturated sets—the Epstein filtration of the bad set—where each saturated set has a transverse invariant measure of maximal support. Each integral decomposes into a weighted sum of leaf classes, and then we show the leaf classes for a compact foliation uniformly vanish. Details appear in [8].

For codimension one foliations remarkable progress has been made in relating the geometry of a foliation with its Godbillon-Vey invariant [3, 13]. For higher codimensions, it is expected that a geometric interpretation of the residue secondary classes can be achieved by utilizing the techniques of the proof of Theorem 3, the residue theorem for foliations [6] and the properties of the $\mu$-classes. Some progress on this problem is given in [9].

ADDED IN PROOF. G. Duminy has recently proved that a codimension-one foliation on a compact manifold with nonvanishing Godbillon-Vey invariant must have a resilient leaf (L'Invariant de Godbillon-Vey d'un feuilletage se localise dans les feuilles ressort, preprint.)

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08544
