**A two-dimensional dividend problem for collaborating companies and an optimal stopping problem**

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**ABSTRACT**

We consider two insurance companies with wealth processes described by two independent Brownian motions with drift. The goal of the companies is to maximize their expected aggregated discounted dividend payments until ruin. The companies are allowed to help each other by means of transfer payments. But in contrast to Gu et al. [(2018). Optimal dividend strategies of two collaborating businesses in the diffusion approximation model. *Mathematics of Operations Research* 43(2), 377–398], they are not obliged to do so, if one company faces ruin. We show that the problem is equivalent to a mixture of a one-dimensional singular control problem and an optimal stopping problem. The value function is explicitly constructed and a verification result is proved. Moreover, the optimal strategy is provided as well.

**1. Introduction**

Since B. de Finetti has suggested in his seminal paper (de Finetti 1957) an alternative to the classical ruin probability approach to evaluate an insurance company, namely the maximization of expected discounted dividends paid by the company, there has been a lot of research in this direction. We just mention a few examples.

Gerber proved in Gerber (1969) that, if the underlying wealth process follows the classical Cramer-Lundberg model, it is optimal to use a so called band strategy, which degenerates in the case of exponential individual claims to a barrier strategy, meaning that you have to keep your endowment just below a certain barrier value by paying dividends (see also Schmidli 2008 for a different proof of these facts). Similarly, Asmussen and Taksar (1997) showed that a barrier strategy is optimal in the diffusion approximation case as well. The problem is discussed for more general diffusion processes in Marciniak and Palmowski (2016), and in the case of finite time horizon one can find results in Grandits (2013), Grandits (2014) and Grandits (2015). For a survey on the optimal dividend literature, see e.g. Avanzi (2009) and the monograph (Azcue and Muler 2014).

Although most of the literature concerns the one-dimensional case, there has been some interest in the two-dimensional case recently. For example, Gerber and Shiu (2006) give sufficient conditions on the model parameters for a two-dimensional Brownian motion with drift, under which it is optimal for the two companies to merge. In Albrecher et al. (2017), a two-dimensional Cramer-Lundberg model is considered and, among other things, the value function is characterized as the smallest viscosity supersolution of the corresponding Hamilton Jacobi Bellman (HJB) equation. The paper, which is closest to the present one, is Gu et al. (2018). There the wealth process of the two
companies is described by two independent Brownian motions with drift. Collaboration between the companies is allowed, which means that as soon as the wealth of one company hits zero, the other one can help by means of transfer payments (actually transfer payments are allowed at any time). It is also assumed that the solvent company has not only the possibility to help the insolvent one, but it is obliged to do so. Hence, in this model the possible ruin of both companies happens at the same time. It is shown that a barrier strategy, where the barrier value corresponds to the optimal value for the merged companies, is the optimal strategy.

This last assumption of the model in Gu et al. (2018) is the one, which we abandon. In our model the solvent company can help the insolvent one, but it is not obliged to do so. (Similarly as in (Gu et al. 2018), transfer payments are possible at any time, so it is allowed in our setting to ‘liquidate’ one company by transferring its wealth completely to the other one at an arbitrary time.)

Our paper can be considered as a singular stochastic control problem, subject to discretionary stopping. Some articles in this direction are Davis and Zervos (1994), Davis and Zariphopoulou (1995), Karatzas and Wang (2000), Karatzas et al. (2000), Karatzas and Zambiris (2006) and Zervos (2003).

The schedule of our paper is as follows. In Section 2, we introduce the model and prove some preliminary properties of the value function. Moreover, we show that our problem can be seen as a one-dimensional mixture of a singular control problem and an optimal stopping problem. The HJB equation has the form of a variational inequality, but with three operators instead of the usual two. In Section 3, we construct the value function explicitly. By this we mean that at most three transition points have to be calculated numerically, and in between these points the value function is the solution of ODE’s with constant coefficients. Let us note already here that – in contrast to Gu et al. (2018) – our value function is not $C^2$. In fact, it will turn out to be $C^1$, but with an a.e. bounded second derivative. In Section 4, we prove a verification theorem and the optimal strategy is also provided. A numerical example concludes the paper.

2. The model and preliminary results

The wealth of the two companies in question is described by the vector $Z_t := (X_t, Y_t)$, with

\[
\begin{align*}
    dX_t &= \mu_1 \, dt + \sigma_1 \, dW_t^{(1)} - dL_t^{(1)} + dC_t^{(2,1)} - dC_t^{(1,2)}, \\
    dY_t &= \mu_2 \, dt + \sigma_2 \, dW_t^{(2)} - dL_t^{(2)} + dC_t^{(1,2)} - dC_t^{(2,1)},
\end{align*}
\]

(1)

initial value $Z_0 = (X_0, Y_0) = z = (x, y)$, positive constants $\mu_1, \mu_2, \sigma_1, \sigma_2$ and $x, y \in \mathbb{R}_0^+$. The process $L_t := (L_t^{(1)}, L_t^{(2)})$ is caglad, monotone non-decreasing and $F$-progressively measurable, where $F$ is the completed natural filtration of the two independent Brownian motions $(W_t^{(1)}, W_t^{(2)})$. To include payments at time zero, we formally define $L_{t-} = 0$. These measurability requirements are standard in singular control theory, and can be found, e.g. in Fleming and Soner (2006, p. 296). The same holds for the process $C_t := (C_t^{(1,2)}, C_t^{(2,1)})$. The processes $L_t^{(1)}$ and $L_t^{(2)}$ describe the accumulated dividend payments of company one respectively company two until time $t$, whereas the process $C_t^{(1,2)}$ describes the accumulated transfer payments from company one to two, and similarly for $C_t^{(2,1)}$.

In Gu et al. (2018), the additional admissibility requirement for the control processes $(L_t, C_t), (L_t, C_t) \in \mathcal{A}$, with

\[
\mathcal{A} := \left\{ (L_t, C_t) \left| X_t \geq 0, Y_t \geq 0, \text{ for } t \leq \tau^{(ru)} \right. \right\}
\]

is imposed, where $\tau^{(ru)} := \inf \{ t \geq 0 | X_t < 0, Y_t < 0 \}$ denotes the moment when both companies are ruined. We replace it by

\[
\mathcal{A} := \left\{ (L_t, C_t) \left| X_{t\wedge \tau} \geq 0, Y_{t\wedge \tau} \geq 0, \text{ for } \tau \leq \tau^{(ru)} \right. \right\},
\]
with an arbitrary stopping time \( \tau \). This means that the companies help each other only until the stopping time \( \tau \leq \tau(\text{ru}) \). If \( \tau \) is chosen as the ruin time of one company, this company gets no help any more at this time. If it is not the ruin time a priori, one transfers all the endowment of one company to the 'better' one at this arbitrary stopping time. Naturally, we assume that the remaining company behaves optimally after time \( \tau \). This means that in our optimal control problem the control processes run actually only up to time \( t = \tau \), because the optimal solution after this time is well known, namely the optimal 'one-dimensional solution'.

This gives the following mixture of singular control and optimal stopping problem, with the objective functional

\[
J(z, L, C, \tau) := E_x \left[ \int_0^\tau e^{-\beta t} \left( dL_t^{(1)} + dL_t^{(2)} \right) + e^{-\beta \tau} g(X_\tau + Y_\tau) \right],
\]

and value function

\[
V(z) := \sup_{(L, C) \in A, \tau} J(z, L, C, \tau \leq \tau(\text{ru})).
\]

If not otherwise stated, the integral always includes possible jumps at the lower limit, but not at the upper one. \( \beta > 0 \) is a discounting factor and we define

\[
g(s := x + y) := \max \left( V^{(1)}(s), V^{(2)}(s) \right),
\]

where \( V^{(1)}(s), V^{(2)}(s) \) are the value functions of the one-dimensional dividend maximization problem for the parameters \((\mu_1, \sigma_1), (\mu_2, \sigma_2)\), respectively. These can be found, e.g. in Jeanblanc-Picqué and Shiryaev (1995) or Asmussen and Taksar (1997), and for latter use we give the one-dimensional result in

**Theorem 2.1**: The value function \( V^{(1)}(x) \) for the one-dimensional dividend maximization problem \( E_x [\int_0^{\tau(\text{ru})} e^{-\beta t} dL_t] \), where the endowment process is described by linear Brownian motion with drift \( \mu_1 \) and volatility \( \sigma_1 \), is given by the following \( C^2(\mathbb{R}_0^+) \) function.

\[
V^{(1)}(x) := \begin{cases} 
\frac{e^{\theta_+ x} - e^{\theta_- x}}{\theta_+} - \frac{\theta_- e^{\theta_- b_1}}{\theta_+ e^{\theta_+ b_1}}, & x \leq b_1, \\
\frac{\mu_1}{\beta} (x - b_1), & x \geq b_1.
\end{cases}
\]

\( \theta_+, \theta_- \) are the positive, respectively, negative roots of the corresponding characteristic polynomial, i.e.

\[
\theta_+ = -\frac{\mu_1}{\sigma_1^2} + \sqrt{\frac{\mu_1^2}{\sigma_1^4} + \frac{2\beta}{\sigma_1^2}} > 0,
\]

\[
\theta_- = -\frac{\mu_1}{\sigma_1^2} - \sqrt{\frac{\mu_1^2}{\sigma_1^4} + \frac{2\beta}{\sigma_1^2}} < 0.
\]

Finally, \( b_1 \) is a positive constant, representing the value of the barrier in the barrier dividend strategy, which is optimal here. One has

\[
b_1 = \frac{1}{\theta_+ - \theta_-} \ln \left( \frac{\theta_-^2}{\theta_+^2} \right).
\]

An analogous result holds clearly for \( V^{(2)}(y) \).
Our next Lemma shows that the value function $V(x, y)$ actually depends only on the sum $s = x + y$. By slight abuse of notation we do not introduce a new symbol for the value function, but write $V = V(s)$.

**Lemma 2.1**: The value function $V$ of problem (4) depends actually only on $s = x + y$, i.e. $V = V(s)$.

**Proof**: Let $(x, y) \neq (0, 0)$ be a given initial endowment. Let further $(L, C, \tau)$ be an $\epsilon$-optimal admissible strategy for $(x, y)$, i.e.

$$J(x, y, C, L, \tau) = \mathbb{E}_{x,y} \left[ \int_0^\tau e^{-\beta t} \left( dL_t^{(1)} + dL_t^{(2)} \right) + e^{-\beta \tau} g(X_\tau + Y_\tau) \right] > V(x, y) - \epsilon.$$ 

Consider $(x + \Delta, y - \Delta)$ with $\Delta \in [-x, y]$. We introduce now a new admissible strategy by transforming capital from one company to the other at time $t = 0$, s.t. we have again an endowment $(x, y)$. Then we proceed with the original strategy, i.e. we have, for $t > 0$ and $\Delta < 0$,

$$\hat{C}_t^{(2,1)} = -\Delta + C_t^{(2,1)},$$
$$\hat{C}_t^{(1,2)} = C_t^{(1,2)}.$$ 

For $\Delta > 0$ we simply exchange the indices one and two. Furthermore, we leave $(L, \tau)$ unchanged, i.e. $(\hat{L}, \hat{\tau}) = (L, \tau)$. This leads to

$$J(x + \Delta, y - \Delta, \hat{L}, \hat{C}, \hat{\tau}) = J(x, y, L, C, \tau) > V(x, y) - \epsilon,$$

and since $\epsilon$ was arbitrary small and the given strategy arbitrary, we conclude $V(x + \Delta, y - \Delta) \geq V(x, y)$. Starting with $(x + \Delta, y - \Delta)$, one gets analogously $V(x + \Delta, y - \Delta) \leq V(x, y)$, proving $V(x, y) = V(x + \Delta, y - \Delta)$. 

Our next result is more or less trivial and shows the monotonicity of the value function.

**Lemma 2.2**: We have

$$V(s + \Delta) \geq V(s),$$

for $\Delta \geq 0$.

**Proof**: Clearly, each strategy $(L, C, \tau)$, which is admissible for a total endowment $s$, is admissible for $s + \Delta$ (if we possibly shift capital from one company to the other at time $t = 0$, to ensure that both companies start with an endowment at least as good as the original one).

For later use we provide the following.

**Definition 2.1**: 

$$\hat{L}_t := L_t^{(1)} + L_t^{(2)},$$
$$\hat{S}_t := X_t + Y_t,$$
$$\bar{\mu} := \mu_1 + \mu_2,$$
$$\bar{\sigma}^2 := \sigma_1^2 + \sigma_2^2,$$

hence $S_t$ fulfills the SDE $dS_t = \bar{\mu} dt + \bar{\sigma} dW_t - d\hat{L}_t$, with a Brownian motion $W_t$.

In the next remark, we provide a useful property of the value function.
Remark 2.1: We give a lower bound for our value function. Indeed, the nonnegativity of the function \( g(s) \), the definition of the time horizon and the class of admissible controls implies that

\[
V(s) \geq \mathcal{V}(s) := \sup_{\mathcal{L}} E_s \int_0^{\tau^{(m)}} e^{-\beta t} \, dt.
\]

Hence,

\[
V(s) \geq \mathcal{V}(s) = \begin{cases} \frac{e^{\bar{\theta}+} - e^{-s}}{\bar{\theta} e^{\bar{\theta}+} - \bar{\theta} e^{-\bar{\theta}+}}, & s \leq \bar{b}, \\ \frac{\mu}{\beta} + (s - \bar{b}), & s \geq \bar{b}, \end{cases}
\]

with

\[
\bar{\theta}_+ = \frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2\beta}{\sigma^2}} > 0,
\]

\[
\bar{\theta}_- = \frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2\beta}{\sigma^2}} < 0,
\]

\[
\bar{b} = \frac{1}{\bar{\theta}_+ - \bar{\theta}_-} \ln \left( \frac{\hat{\theta}_+}{\hat{\theta}_-} \right).
\]

In an analogous fashion one can establish that

\[
V(s) \geq \sup_{\tau} E_s \left[ e^{-\beta \tau} (V_1(S_\tau) \lor V_2(S_\tau)) \right]
\]

demonstrating that the value function dominates also the value of an associated standard optimal stopping problem with payoff \( g(s) \).

In the last part of the present section we shall provide a property of the value function \( V(s) \), which will be useful in the next section, where we shall construct \( V \) and the optimal strategy rather explicitly.

We start with a rewriting of the HJB equation and introduce some new notation, i.e. the HJB equation can be written equivalently as

\[
\min \left\{ \mathcal{M} V, V - g \right\} = 0, \tag{7}
\]

with \( \mathcal{M} V := \min \{ -\mathcal{L} V + \beta V, V' - 1 \} \). Furthermore, we introduce the following sets

\[
\mathcal{S} := \left\{ s \in \mathbb{R}^+ \mid V(s) = g(s) \right\},
\]

\[
\mathcal{C} := \mathbb{R}^+ \setminus \mathcal{S},
\]

\[
\mathcal{D}^{(1)} := \left\{ s \in \mathbb{R}^+ \mid \mathcal{M} V^{(1)}(s) \geq 0 \right\},
\]

\[
\mathcal{D}^{(2)} := \left\{ s \in \mathbb{R}^+ \mid \mathcal{M} V^{(2)}(s) \geq 0 \right\}, \tag{8}
\]

where \( \mathcal{S} \) is called the stopping region, \( \mathcal{C} \) the continuation region, and the \( \mathcal{D} \)'s will also be used below to construct the value function.

As final preparatory result we give necessary conditions on the form of the sets \( \mathcal{D} \).
Lemma 2.3: The set \( D^{(1)} \) has necessarily one of the two following forms; either we have \( D^{(1)} = [s_1^{(1)}, \infty) \), with \( s_1^{(1)} := \mu_2 / \beta + b_1 \), or \( D^{(1)} = (0, s_0^{(1)}) \cup [s_1^{(1)}, \infty) \), with some \( s_0^{(1)} \in (0, b_1) \), holds. Analogously one finds: \( D^{(2)} = [s_1^{(2)}, \infty) \), with \( s_1^{(2)} := \mu_1 / \beta + b_2 \), or \( D^{(2)} = (0, s_0^{(2)}) \cup [s_1^{(2)}, \infty) \), with some \( s_0^{(2)} \in (0, b_2) \).

Note that \( b_1, b_2 \) correspond to the optimal barrier heights in the \((\mu_1, \sigma_1)\), resp. \((\mu_2, \sigma_2)\)-models.

Proof: We restrict our proof to the set \( D^{(1)} \).

By the definition of the set \( D^{(1)} \), see (8), one knows that on \( D^{(1)} \), \(-1 + V^{(1)}(s) \geq 0 \), as well as \( H(s) := -L V^{(1)}(s) + \beta V^{(1)}(s) \geq 0 \), hold. Since \( V^{(1)} \) is the solution of the \((\mu_1, \sigma_1)\)-problem, the first inequality holds on \( R^+ \). Therefore, we get the following characterization of \( D^{(1)} \)

\[
D^{(1)} = \left\{ s \in R^+ \mid H(s) = -L V^{(1)}(s) + \beta V^{(1)}(s) \geq 0 \right\}.
\]

We first consider the interval \( s \in [b_1, \infty) \). There one finds by the smooth fit condition at the point \( b_1 \), see e.g. Gerber and Shiu (2006), (2.10), the following explicit formula, \( V^{(1)}(s) = \mu_1 / \beta + (s - b_1) \). Plugging in this expression in the definition of \( H(s) \), provides

\[
H(s) = -\mu_2 + \beta(s - b_1),
\]

for \( s \geq b_1 \), i.e. a linear increasing function with zero at \( s_1^{(1)} \).

Let us now introduce the operator \( L := \mu_1 \partial / \partial s + (\sigma_1^2 / 2)(\partial^2 / \partial s^2) \), and consider the interval \((0, b_1)\). By construction we have \( L V^{(1)} - \beta V^{(1)} = 0 \) there, hence \( H(s) = -(\sigma_1^2 / 2) V^{(1)'}(s) - \mu_2 V^{(1)'}(s) \). By linearity we get the equation

\[
L H(s) - \beta H(s) = 0.
\]

We distinguish now several cases.

Case 1. \( H(0) \leq 0 \). We assert that in this case one has

\[
H(s) < 0, \quad \text{for all } s \in (0, b_1),
\]

giving \( D^{(1)} = [s_1^{(1)}, \infty) \). Assume first that \( H(0) < 0 \) holds. We then argue by contradiction. If (10) was false, then \( H(s) \) would have a strictly positive maximum, say \( s_{\text{max}} \in (0, b_1) \), contradicting (9); (note that \( H(s_{\text{max}}) = H'(s_{\text{max}}) = 0 \) is impossible, since this would imply \( H \equiv 0 \).)

If \( H(0) = 0 \) holds, then we first note that \( H'(0) = 0 \) is impossible by the same reasons as above for \( s_{\text{max}} \). Moreover, \( H'(0) > 0 \) would imply a strictly positive maximum, hence again a contradiction. If \( H'(0) < 0 \), then we have either \( H(s) < 0 \), for all \( s \in (0, b_1) \), giving \( D^{(1)} = [s_1^{(1)}, \infty) \) or the existence of a strictly positive maximum, which leads again to a contradiction. Altogether, we have in Case 1: \( D^{(1)} = [s_1^{(1)}, \infty) \).

Case 2. \( H(0) > 0 \). As above we cannot have any extrema with value zero. Moreover, strictly positive maxima or strictly negative minima for \( H(s) \) are again prohibited by Equation (9). The remaining case provides a monotone decreasing function \( H(s) \) on \([0, b_1)\), with some zero in \((0, b_1)\), which we call \( s_0^{(1)} \). Hence, \( D^{(1)} = (0, s_0^{(1)}) \cup [s_1^{(1)}, \infty) \) holds in this case.

3. Explicit construction of the value function

In this section, we shall give a rather explicit construction of the value function and the optimal strategy. By ‘rather explicit’ we mean the following. In a first step at most four points on the \( s \)-axis have to be determined numerically. Once these points are known, the value function can be found...
by integrating ODE’s with constant coefficients, the solution of which are of course well known. The optimal strategy uses some of these points as barrier values for barrier strategies, see below for details.

We start with the definition of the constant $\gamma$.

**Definition 3.1:** The constant $\gamma \geq 0$ is chosen in a way, s.t. we have $V^{(1)}(s) < V^{(2)}(s) = g(s)$, for $s \in (0, \gamma)$, as well as $V^{(2)}(s) < V^{(1)}(s) = g(s)$ on $(\gamma, \infty)$.

Note that by Proposition A.1, proven in the Appendix, there exists at most one such ‘sign change point’ (SCP) $\gamma$. Moreover, if $\gamma = 0$, we have $V^{(1)}(s) \geq V^{(2)}(s)$, for all $s \geq 0$. Finally, the order of the $V^{(i)}$’s is chosen w.l.o.g.

**Remark 3.1:** Note that at the point $\gamma$, if $\gamma > 0$, the function $g$ has an upwards jump in the first derivative, i.e. $g'(\gamma^+) - g'(\gamma^-) > 0$.

Indeed, by definition of $g$ we have certainly $g'(\gamma) = g'(\gamma^-) - g'(\gamma^+) \leq 0$. We now argue by contradiction. So assume $g'(\gamma^+) = g'(\gamma^-)$; we then have necessarily $g''(\gamma^+) = g''(\gamma^-)$, because otherwise $D(s) := V^{(1)}(s) - V^{(2)}(s)$ would not change the sign in $\gamma$. Hence, there would be a zero of third order of $D(s)$ at $\gamma$. But it can be shown with the same methods (basically Descartes rule), as are used in Proposition A.1, that this is impossible.

For the following procedure compare also Figure 1. We continue with the definition of the point $\bar{s}$ as the solution of $g(\bar{s}) = \mu/\beta$, and we assert that

$$\bar{s} > \gamma$$

(11)

holds. Since $g$ is monotone increasing, the assertion is equivalent to $\mu/\beta > g(\gamma)$. Now we have

$$g(\gamma) < \max \left( V^{(1)}(b_1), V^{(2)}(b_2) \right) = \max \left( \frac{\mu_1}{\beta}, \frac{\mu_2}{\beta} \right) < \frac{\mu}{\beta},$$

showing (11).

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**Figure 1.** Case 2.2: The functions $g(s)$ (solid line), vs. $V(s)$ (dotted), vs. $\bar{V}(s + \bar{s})$ (dash-dotted) on the interval $[0, \bar{s}]$. 

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Next we choose \( \bar{r} \in \mathbb{R} \), s.t. \( \nabla(\bar{s} + \bar{r}) = \mu/\beta \) holds, and moreover, let

\[
\Delta := \inf \left\{ r > \bar{r} \mid \nabla(s + r) \geq g(s), \quad \forall s \in [0, \bar{s}] \right\},
\]

i.e. we shift the solution \( \nabla \) horizontally, until it touches \( g(s) \) from above. Note that we have \( g'(s) < \nabla'(s + \bar{r}), \) for \( s \in [\bar{s} - \epsilon, \bar{s}] \), with \( \epsilon \) small, since \( g'(s) = 1 \) and \( \nabla'(s + \bar{r}) > 1 \) holds there, implying that \( \Delta \) is well defined.

Finally, define \( s_2 \) as follows

\[
s_2 := \sup \left\{ 0 \leq s \leq \bar{s} \mid g(s) = \nabla(s + \Delta) \right\}.
\]

By construction we have now

\[
g(s_2) = \nabla(s_2 + \Delta), \\
g'(s_2) = \nabla'(s_2 + \Delta), \\
g''(s_2) \leq \nabla''(s_2 + \Delta).
\]

Observe that we have necessarily \( s_2 \neq \gamma \), since we have an upwards kink of \( g \) at \( \gamma \), as noticed in Remark 3.1.

We start now with the construction of the value function and distinguish several cases, depending where the point \( s_2 \) lies.

**Case 0:** \( s_2 = 0 \).

This would mean \( \nabla(s + \Delta) \geq g(s) \), for all \( s \), and \( \nabla(0 + \Delta) = 0 \), hence \( \Delta = 0 \). We set in this case

\[
V(s) := \nabla(s), \quad s \geq 0,
\]

i.e. the case where we never stop the collaboration.

**Case 1:** \( s_2 < \gamma \).

Here we define the value function as

\[
V(s) := \begin{cases} 
  g(s) = V^{(2)}(s), & s \in [0, s_2], \\
  \nabla(s + \Delta), & s \in [s_2, \infty). 
\end{cases}
\]

We assert now

**Claim 1:** The point \( s_2 \) defined in (13) fulfills \( s_2 \leq b_2 \), and \( s_2 \in D^{(2)} \); hence, we are in the second case of Lemma 2.3, implying even \( s_2 \leq s_0^{(2)} < b_2 \), in the notation used there.

For \( s_2 \leq b_2 \), we notice that, for \( s \in [b_2, \bar{s}] \), we have \( V^{(2)}(s) = 1 \), as well as \( \nabla(s) > 1 \), for \( \nabla(s) \in [\mu_2/\beta, \mu/\beta] \).

For \( s_2 \in D^{(2)} \), we have to show \(-\mathcal{L}V^{(2)}(s_2) + \beta V^{(2)}(s_2) \geq 0 \). We know that \(-\mathcal{L}(\nabla(s_2 + \Delta)) + \beta \nabla(s_2 + \Delta) = 0 \) holds, hence we have to show \(-\frac{1}{(\sigma^2/2)}V^{(2)}(s_2) \geq \frac{1}{(\sigma^2/2)}\nabla''(s_2 + \Delta) \), which follows from the third equation of (14).

Lemma 2.3 provides finally \( s_2 \leq s_0^{(2)} \), since we have \( s_2 \leq b_2 \), proving Claim 1. Note also that we have in this case

\[
D^{(2)} = (0, s_0^{(2)}) \cup [s_1^{(2)}, \infty),
\]

with some \( s_0^{(2)} \in (0, b_2) \), and \( s_1^{(2)} := \mu_1/\beta + b_2 \).

**Case 2:** \( s_2 > \gamma \).

We distinguish two subcases.
Case 2.1: $\nabla(s + \Delta)$ touches (from above) $g(s)$ also left from $\gamma$ (i.e. it touches $V^{(2)}$) at a point, say at $\hat{s}_2 < \gamma$.

Here we are actually back in Case 1, by defining

$$V(s) := \begin{cases} 
  g(s) = V^{(2)}(s), & s \in [0, \hat{s}_2], \\
  \nabla(s + \Delta), & s \in [\hat{s}_2, \infty), 
\end{cases} \quad (17)$$

and Claim 1 holds as well for the point $\hat{s}_2$.

Case 2.2: $\nabla(s + \Delta) > g(s)$, for all $s \in [0, \gamma]$.

Let $s_3 \in (\gamma, s_2)$, solve the system

$$\frac{\sigma^2}{2} v'' + \bar{\mu} v' - \beta v = 0,$$

$$v(s_3) = V^{(1)}(s_3),$$

$$v'(s_3) = V^{(1)}'(s_3),$$

and call the solution function $v^{(s_3)}$. Furthermore, define

$$s_4 := \inf \left\{ s_3 > \gamma \left| v^{(s_3)}(s) \geq g(s), \forall s \leq s_3 \right. \right\},$$

$$s_5 := \sup \left\{ s \in [0, \gamma] \left| g(s) = v^{(s_4)} \right. \right\},$$

and finally the value function as

$$V(s) := \begin{cases} 
  V^{(2)}(s), & s \in [0, s_5], \\
  v^{(s_4)}(s), & s \in [s_5, s_4], \\
  V^{(1)}(s), & s \in [s_4, s_2], \\
  \nabla(s + \Delta), & s \in [s_2, \infty). 
\end{cases} \quad (18)$$

One can find a picture of this procedure in Figure 1.

Analogously to Claim 1 one proves for this case

Claim 2: One has $s_4 \leq s_2 < b_1$, $s_4, s_2 \in D^{(1)}$, and $s_4, s_2 \leq s_0^{(1)}$, with $D^{(1)} = (0, s_0^{(1)}) \cup [s_1^{(1)} \infty)$. Finally, $s_5 < b_2, s_5 \in D^{(2)}$ and $s_5 \leq s_0^{(2)}$, with $D^{(2)} = (0, s_0^{(2)}) \cup [s_1^{(2)} \infty) \text{ holds.}$

This concludes the construction of the function $V(s)$, and the next theorem asserts that this function solves the HJB equation in a certain sense.

Theorem 3.1: The function $V(s)$, defined by (15)–(18), solves the HJB equation

$$\min \left\{ -\mathcal{L}V + \beta V, V' - 1, V - g \right\} = 0$$

pointwise, except at points $s \in \partial S$, which are at most three points. Near these points $V''(s)$ is locally bounded.

Moreover, the boundary condition $V(0) = 0$ is fulfilled as well.

Proof: We restrict the proof to the case where $V$ is defined via Equation (18), because the other cases work analogously and are actually simpler.

We consider the set, where $V(s)$ is $C^2$, i.e. we consider $s \in (0, s_5) \cup (s_5, s_4) \cup (s_4, s_2) \cup (s_2, \infty)$, and start with
Case 1: $s \in (0, s_5)$.
By Claim 2, after equation (18), we have

$$s_5 \leq s_0^{(2)} \Rightarrow (0, s_5) \subset D^{(2)} \Rightarrow MV^{(2)}(s) \geq 0 \Rightarrow MV(s) \geq 0.$$ 

Moreover, since $s_5 \leq \gamma$ holds, we have $V(s) = V^{(2)}(s) = g(s)$ on $s \in (0, s_5)$, hence

$$\min \{MV, -g + V\} = 0,$$

finishing this case.

Case 2: $s \in (s_5, s_4)$.
By construction $\mathcal{L}V(s) + \beta V(s) = 0$, as well as $V(s) \geq g(s)$ hold on this interval. Hence, it remains to show

$$V'(s) \geq 1. \quad (19)$$

We know that $V'(s_4) \geq 1$ and $V'(s_5) \geq 1$ hold and claim that we have

$$V''(s_4-) < 0. \quad (20)$$

Indeed, $V'''(s_4+) < 0$, since $s_4 < b_1$, hence $-\mu_1 V''(s_4+) + \beta V'(s_4+) < 0$. As $V$ is $C^1$ at the point $s_4$ by construction, we find $-\mu_1 V''(s_4-) + \beta V'(s_4-) < 0$, hence $-\mu_1 V''(s_4-) + \beta V'(s_4-) < 0$. This provides in turn $(\bar{\sigma}^2/2) V''(s_4-) < 0$, proving (20). Analogously one shows $V'''(s_5+) = V''(s_5+) < 0$.

Now, in the interval $(s_5, s_4)$ $V(s)$ is smooth, and we consider therefore the equation for $W(s) := V'(s)$ there and get by a simple differentiation

$$\frac{\bar{\sigma}^2}{2} W''(s) + \bar{\mu} W'(s) - \beta W(s) = 0. \quad (21)$$

We argue by contradiction. If we assume the existence of a $s^* \in (s_5, s_4)$, with $V'(s^*) = W(s^*) < 1$, the boundary behavior of $W$ at $s_4$, which we have proved above, implies the existence of a local maximizer of $W$, say $\hat{s} \in (s^*, s_4)$, with $W(\hat{s}) > 1$, $W'(\hat{s}) = 0$, $W''(\hat{s}) \leq 0$, contradicting (21). This shows (19) and concludes Case 2.

Case 3: $s \in (s_4, s_2)$.
By Claim 2 above, we have

$$s_2 \leq s_0^{(1)} \Rightarrow (0, s_2) \subset D^{(1)} \Rightarrow MV^{(1)}(s) \geq 0, \quad s \in (0, s_2) \Rightarrow MV(s) \geq 0, \quad s \in (s_4, s_2).$$

Moreover, we have $s_4 > \gamma$, which implies $V(s) = V^{(1)}(s) = g(s)$ on $(s_4, s_2)$. Altogether, this gives $\min\{MV, V - g\} = 0$, completing Case 3.

Case 4: $s \in (s_2, \infty)$.
Here we have $V(s) = \nabla(s + \Delta)$, which implies the validity of

$$\min \left\{ \nabla(s + \Delta) - 1, -\mathcal{L}\nabla(s + \Delta) + \beta \nabla(s + \Delta) \right\} = 0,$$

since this is nothing else but the classical HJB equation for the one-dimensional $(\bar{\mu}, \bar{\sigma})$-problem, and $V(s)$ is just a shift of the classical solution. So it remains to show that $\nabla(s + \Delta) \geq g(s)$ holds. But this is true by construction, see the definition of $\Delta$, Equation (12), finishing Case 4.

Since local boundedness of $V'''(s)$ near the exceptional points is clear by construction, the proof is complete. \qed
4. The verification result and the optimal strategy

We show in this section that the function, defined in (15)–(18) in the previous section, provides our value function. Moreover, the optimal strategy is given as well. We have the following theorem, the proof of which is an adaption of the proof of Theorem 10.4.1 of Øksendal (2003). (Alternatively, one could use Peskir (2005), where an Ito formula under weak assumptions is proved.)

**Theorem 4.1:** Let \( G := \mathbb{R}^+ \).

(a) Assume we have a function \( V : \mathbb{R}_0^+ \rightarrow \mathbb{R} \) fulfilling

(i) \( V \in C^1(G) \cap C(\overline{G}) \)

(ii) \( V \geq g, \ V' \geq 1, \lim_{t \to \tau_G} V(S_t) = g(0) = 0 \), a.s., where \( \tau_G \) is the exit time of \( S_t \) for the set \( G \). Moreover, using the notation defined in (8), we assume

(iii) \( \mathbb{E}_s [\int_0^{\tau_G} \mathbb{1}_{\{\partial C\}}(S_t) \, dt] = 0 \)

(iv) \( V \in C^2(G \setminus \partial C) \), and the second derivatives of \( V(s) \) are locally bounded near \( \partial C \)

(v) \( -L V(s) + \beta V(s) \geq 0 \) on \( G \setminus \partial C \),

Then we have

\[
V(s) \geq \mathbb{E}_s \left[ \int_0^\tau e^{-\beta t} \, dL_t + e^{-\beta \tau} g(S_\tau) \right] = J(s, C, \overline{L}, \tau).
\]

for all admissible strategies \((C, \overline{L}, \tau)\), i.e. \( V \) is an upper bound for the value function.

(b) The function \( V(s) \) constructed in Section 3, see (15)–(18) there, fulfills all the assumptions demanded in point (a).

(c) The function \( V(s) \) constructed in Section 3, see (15)–(18) there, provides the value function of our problem.

Moreover, the optimal strategy can be described in the following way.

**Description of the optimal strategy.**

**Case A:** \( V(s) \geq g(s), \) for all \( s \geq 0 \).

Here it is optimal never to stop the sum process \( S_t \) prematurely, i.e. in terms of the original companies: the solvent company has to support the insolvent one until both companies are ruined. Moreover, for the sum process a dividend strategy with barrier \( \overline{b} \) has to be used.

Let us remark that, since the transfer processes between the two companies does not influence the sum process, this optimal strategy can be realized in many different ways. E.g. one possibility would be that the dividends are always paid out by company one i.e. Vis a upper bound for the value function.

**Case B:** \( \{V(s) < g(s)\} \neq \emptyset \), and w.l.o.g. \( V^{(1)}(s) \geq V^{(2)}(s), s \geq 0 \), i.e. \( \gamma = 0 \).

Here one should use the barrier strategy with barrier value \( \overline{b} \), until we reach the stopping time \( \tau := \inf\{t \geq 0 \mid S_t \leq s_2\} \) (s defined in (13)); at time \( t = \tau \) one should liquidate company 2 and run company 1 in the optimal way, i.e. with barrier strategy \( b_1 \).

In the following we mean by liquidating one company that one should keep the remaining one in the optimal way.

**Case C.1:** \( \{V(s) < g(s)\} \neq \emptyset \), \( \gamma > 0 \), \( s_2 < \gamma \).

Here one should run the barrier strategy with barrier value \( \overline{b} \), until we reach the stopping time \( \tau := \inf\{t \geq 0 \mid S_t \leq s_2\} \); at time \( t = \tau \) one should liquidate company 1.

**Case C.2.a:** \( \{V(s) < g(s)\} \neq \emptyset \), \( \gamma > 0 \), \( \overline{V}(s + \Delta) \) touches \( g(s) \) left of \( \gamma \) at \( \hat{s}_2 \) and right of \( \gamma \) at \( s_2 \).

The optimal strategy is to use a barrier strategy with level \( \hat{b} \) and to stop at \( \tau := \inf\{t \geq 0 \mid S_t \leq \hat{s}_2\} \); at time \( t = \tau \) one should liquidate company 1.

**Case C.2.b:** \( \{V(s) < g(s)\} \neq \emptyset \), \( \gamma > 0 \), \( \overline{V}(s + \Delta) \) touches \( g(s) \) only right of \( \gamma \) at \( s_2 \).

The optimal strategy is:
$s > s_2$: use a barrier strategy with level $\overline{b}$ until $\tau := \inf\{t \geq 0 \mid S_t \leq s_2\}$; at time $t = \tau$ liquidate company 2.

$s \in [s_4, s_2]$: liquidate company 2.

$s \in (s_5, s_4)$: wait until the stopping time $\tau := \inf\{t \geq 0 \mid S_t \leq s_5 \text{ or } S_t \geq s_4\}$; if $S_\tau = s_5$, liquidate company 1, otherwise company 2.

$s \in [0, s_5]$: liquidate company 1.

**Proof of (a).** We start the proof with the remark that, as in Øksendal (2003), Theorem 10.4.1, we have the following approximation result for our candidate function $V$.

We can find a sequence of functions $V_j \in C^2(G) \cap C(\overline{G}), j = 1, 2, \ldots$, s.t

1. $V_j \to V$ uniformly on compact subsets of $\overline{G},$ as $j \to \infty$,
2. $V_j' \to V'$ uniformly on compact subsets of $\overline{G},$ as $j \to \infty$,
3. $(\mathcal{L}V_j)_{j=1}^\infty$ is locally bounded on $G$.

(See Øksendal 2003, (D.17) for (β).)

Let now $\{G_R\}_{R=1}^\infty$ be a sequence of open bounded sets, s.t. $G = \bigcup_{R=1}^\infty G_R$ holds. Furthermore, let $T_R := \min\{R, \inf\{t > 0 \mid S_t \notin G_R\}\}$.

We get now for an arbitrary admissible strategy by Dynkin's formula

$$V_j(s) = \mathbb{E}_s \left[ e^{-\beta(\tau \wedge T_R)} (V_j(S_{\tau \wedge T_R}) - \mathbb{E}_s \left[ \int_0^{\tau \wedge T_R} (\mathcal{L} - \beta) V_j(S_t)e^{-\beta r} \, dr \right) \right] + \mathbb{E}_s \left[ \int_0^{\tau \wedge T_R} V_j'(S_t)e^{-\beta r} \, d\mathcal{L}_r \right]$$

Going to the limit $j \to \infty$ gives, employing (α)-(δ),

$$V(s) = \mathbb{E}_s \left[ e^{-\beta(\tau \wedge T_R)} V(S_{\tau \wedge T_R}) \right] - \mathbb{E}_s \left[ \int_0^{\tau \wedge T_R} (\mathcal{L} - \beta) V(S_t)e^{-\beta r} \, dr \right] + \mathbb{E}_s \left[ \int_0^{\tau \wedge T_R} V'(S_t)e^{-\beta r} \, d\mathcal{L}_r \right] \geq \mathbb{E}_s \left[ e^{-\beta(\tau \wedge T_R)} g(S_{\tau \wedge T_R}) \right] + \mathbb{E}_s \left[ \int_0^{\tau \wedge T_R} e^{-\beta r} \, d\mathcal{L}_r \right],$$

where we have used (ii) and (v) in the last inequality. Finally, letting $R \to \infty$ on the right-hand side, we get by Fatou's Lemma

$$V(s) \geq \mathbb{E}_s \left[ e^{-\beta \tau} g(S_{\tau}) \right] + \mathbb{E}_s \left[ \int_0^\tau e^{-\beta r} \, d\mathcal{L}_r \right] \geq J(s, C, \overline{\mathcal{L}}, \tau),$$

concluding the proof of (a).

**Proof of (b).** This is easy to check

**Proof of (c).** We confine here to the Case 2.2 (see formula (18)) in the construction of the value function and Case C.2.b for the optimal strategy, since this is the most complicated case, and the other cases work analogously.

We first note that by Theorem 3.1 our function $V(s)$ fulfills the HJB-equation pointwise except at three points.

Case 1: $s > s_2$.

Let $V_j$ be the approximating sequence for our candidate function $V$, assured above. Furthermore, let $T^*_j$ be the barrier strategy with barrier value $\overline{b}$, which fulfills $\overline{b} > s_2$ by construction, and let
Obviously \( \tau^* := \inf\{t > 0|S_t = s_2\} \) \((C_t^*\) is, as remarked above, one possible transfer process, leading to \((\overline{L}^*, \tau^*)\). Obviously \( \tau^* \) is a.s. finite, and let

\[
\tau_{k,R} := \inf\{t > 0|S_t = s_2 + 1/k\} \land R,
\]

for \( k \) large enough, s.t. \( s_2 + 1/k < s \) holds. Clearly, for \( k, R \to \infty, \tau_{k,R} \) tends to \( \tau^* \) a.s.

Again by Dynkin's formula we have

\[
E_s\left[e^{-\beta \tau_{k,R}} V_j(S_{\tau_{k,R}})\right] = V_j(s) + E_s\left[\int_0^{\tau_{k,R}} (L - \beta) V_j(S_r)e^{-\beta r} \, dr\right]
- E_s\left[\int_0^{\tau_{k,R}} V_j(S_r)e^{-\beta r} \, dL^*_r\right].
\]

Using our approximation result and letting \( j \to \infty \) provides

\[
E_s\left[e^{-\beta \tau_{k,R}} V(S_{\tau_{k,R}})\right] = V(s) + E_s\left[\int_0^{\tau_{k,R}} (L - \beta) V(S_r)e^{-\beta r} \, dr\right]
- E_s\left[\int_0^{\tau_{k,R}} V(S_r)e^{-\beta r} \, dL^*_r\right]
= V(s) - E_s\left[\int_0^{\tau_{k,R}} e^{-\beta r} \, dL^*_r\right],
\]

where in the last equality we have used the construction of the value function and the optimal strategy. Hence, we have

\[
V(s) = E_s\left[e^{-\beta \tau^*} V(S_{\tau^*})\right] + E_s\left[\int_0^{\tau^*} e^{-\beta r} \, dL^*_r\right],
\]

and after \( k, R \to \infty \) we end up using dominated, respectively monotone convergence, with

\[
V(s) = E_s\left[e^{-\beta \tau^*} V(S_{\tau^*})\right] + E_s\left[\int_0^{\tau^*} e^{-\beta r} \, dL^*_r\right] = E_s[g(s)] = g(s) = V(s).
\]

concluding case 1.

Case 2: \( s \in [s_4, s_2] \).
Here we use \( \tau^* = 0, \overline{L}^*_t \equiv 0 \), giving

\[
J(s, C^*, \overline{L}^*, \tau^*) = E_s\left[e^{-\beta \tau^*} V(S_{\tau^*})\right] = E_s[g(s)] = g(s) = V(s).
\]

Case 3: \( s \in [s_5, s_4] \).
We set \( \overline{L}^*_t \equiv 0 \) and \( \tau^* := \inf\{t > 0|S_t = s_5 \text{ or } S_t = s_4\} \). Again \( \tau^* \) is a.s. finite, and we use the approximating stopping times \( \tau_{k,R} := \inf\{t > 0|S_t = s_5 + 1/k \text{ or } S_t = s_4 - 1/k\} \land R, \) for \( k \) large enough.

In the same way as in Case 1 one gets

\[
V(s) = E_s\left[e^{-\beta \tau^*} V(s_5)\right] + E_s\left[e^{-\beta \tau^*} V(s_4)\right] = E_s\left[g(S_{\tau^*})\right] = E_s\left[e^{-\beta \tau^*} g(S_{\tau^*})\right].
\]

Case 4: \( s \in [0, s_5] \).
As in Case 2 we use \( \tau^* = 0, \overline{L}^*_t \equiv 0 \), and get

\[
J(s, C^*, \overline{L}^*, \tau^*) = E_s\left[e^{-\beta \tau^*} V(S_{\tau^*})\right] = E_s[g(s)] = g(s) = V(s),
\]

concluding our proof.
5. A numerical example

We chose here $\mu_1 = \sigma_1 = \beta = 1$, as well as $\mu_2 = 0.5, \sigma_2 = 0.295$. Performing the procedure described in Section 3, yields that we have here Case 1 in the construction of the value function. One gets $\gamma \approx 0.46$ and $s_2 \approx 0.04$, and a picture can be found in Figure 2. Concerning the optimal strategy, we have Case C.1 of Theorem 4.1c.

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Appendix

A.1 A sign change result

The aim of this section is to show that there is at most one point, where the function \( g(s) \) changes from being identical to \( V^{(1)}(s) \) to \( V^{(2)}(s) \) (or vice versa). We start with some definitions.

Let \( D(s) := V^{(1)}(s) - V^{(2)}(s) \), furthermore, let \( D^{(0)}(s) := D^{(1)}(s) - D^{(2)}(s) \), with \( D^{(1)}(s) := C_1(e^{\theta_1 s} - e^{\theta_2 s}) \), \( D^{(2)}(s) := C_2(e^{\theta_2 s} - e^{\theta_1 s}) \), where \( C_2, \gamma_+, \gamma_- \) correspond to the parameters \( (\mu_2, \sigma_2) \) in the same way as \( C_1, \theta_+, \theta_- \) to the parameters \( (\mu_1, \sigma_1) \), see Theorem 2.1. I.e. \( D^{(0)} \) describes the difference of the ‘exponential part’ of the individual value functions. We start with

Lemma A.1: \( D^{(0)}(s) \) changes sign at most at two positive points, i.e. it has at most two positive ‘sign change points’ (SCP). Moreover, if it has two SCP, then

\[ b_1 \land b_2 \leq \text{second smallest positive local extremum of } D^{(0)}(s) \]

holds.

Proof: We distinguish several cases.

Case 0 (trivial): \( \mu_1 = \mu_2, \sigma_1 = \sigma_2 \Rightarrow D^{(0)}(s) \equiv 0. \)

Case 1: \( \theta_+ = \gamma_+ \).

We first introduce a new independent variable \( \rho := e^s \) and note that we shall – by a slight abuse of notation – simultaneously use \( D^{(0)}(s) \) and \( D^{(0)}(\rho) \). Hence, one finds

\[ D^{(0)}(\rho) = (C_1 - C_2) \rho^{\theta_+} + C_2 \rho^{\gamma_-} - C_1 \rho^{\theta_-}. \]
We shall now employ Descartes’ sign rule and note that it is (by a shift in the exponent and a continuity argument) also valid for negative and/or irrational exponents. Consequently, \( D^{(0)}(\rho) \) has at most two positive zeroes, and since \( \rho = 1 \) is an obvious zero, we have at most one zero larger than one.

We note that the number of SCP is certainly less or equal than the number of zeroes, hence \( D^{(0)}(\rho) \) has at most one SCP larger than one, and therefore \( D^{(0)}(s) \) has at most one positive SCP, concluding Case 1.

For later use we make here the following

Remark A.1: In Case 1 we have either

(a) \( C_1 \neq C_2 \Rightarrow |D^{(0)}(\rho)| \to \infty, \) for \( \rho \to \infty, \) or
(b) \( C_1 = C_2 \Rightarrow \) starting from \( \rho = 1, D^{(0)}(\rho) \) is first strictly increasing and then strictly decreasing (or vice versa).

Case 2: w.l.o.g. \( \theta_+ > \gamma_+ \).

This assumption implies

\[
D^{(0)}(s) \to \infty, \quad \text{for} \quad s \to \infty.
\] (A1)

We divide now further and start with

Case 2.1: \( \theta_+ > \gamma_+, \theta_- \geq \gamma_- \).

We order the function \( D^{(0)}(\rho) \) with respect to the powers in the individual terms and get \( D^{(0)}(\rho) = C_1 \rho^{\theta_+} - C_2 \rho^{\gamma_+} - C_1 \rho^{\theta_-} + C_2 \rho^{\gamma_-} \). As in Case 1, we get by Descartes’ sign rule at most two positive zeroes for \( D^{(0)}(\rho) \), hence at most one positive SCP of \( D^{(0)}(s) \).

Case 2.2: \( \theta_+ > \gamma_+, \theta_- < \gamma_- \). The ordered function reads in this case

\[
D^{(0)}(\rho) = C_1 \rho^{\theta_+} - C_2 \rho^{\gamma_+} + C_2 \rho^{\gamma_-} - C_1 \rho^{\theta_-}.
\]

We note that the function \( D^{(0)}(\rho) \) can neither be identical zero near \( \rho = 1 \) nor can it oscillate. Hence, we can divide further.

Case 2.2.1: for \( \rho \in (1, 1 + \epsilon), \epsilon \) small, \( D^{(0)}(\rho) < 0 \).

Let now \( \rho_1 \) be the first SCP larger than 1. We argue by contradiction, so assume there is a second SCP, say \( \rho_2 \), larger than \( \rho_1 \). We note now that immediately right of \( \rho_2, D^{(0)}(\rho) \) is negative, hence, by (A.1), there has to be a third SCP larger than \( \rho_2 \), say \( \rho_3 \). Together with the trivial zero \( \rho = 1 \), this gives a total of four positive zeroes for \( D^{(0)}(\rho) \), contradicting Descartes’ rule.

Case 2.2.2: for \( \rho \in (1, 1 + \epsilon), \epsilon \) small, \( D^{(0)}(\rho) > 0, \) hence \( D^{(0)}(0) > 0 \).

We claim that in this case \( b_1 \wedge b_2 \) is less or equal to the second smallest local extremum of \( D^{(0)}(s) \), which is in this case the first local minimum.

We first assume \( b_2 \leq b_1 \) and note that \( D^{(1)'}(\rho) \) is a decreasing function in \([0, e^{b_1}]\) and increasing in \([e^{b_1}, \infty)\). An analogous assertion holds for \( D^{(2)'}(\rho) \) with \( b_2 \). Moreover, we have

\[
D^{(1)'}(\rho = e^{b_1}) = D^{(1)'}(s = b_1) e^{-b_1} = e^{-b_1} \leq D^{(2)'}(\rho = e^{b_2}) = D^{(2)'}(s = b_2) e^{-b_2} = e^{-b_2},
\]

\[
D^{(1)'}(0) > D^{(2)'}(0),
\]

\[
D^{(1)'}(\infty) > D^{(2)'}(\infty).
\] (A2)

We now argue by contradiction. So assume that the zeroes of \( D^{(0)'}(\rho) \), say \( \xi_1 < \xi_2 \), are both less than \( \exp(b_2) \). We clearly have

\[
D^{(1)'}(\xi_i) = D^{(2)'}(\xi_i), \quad i = 1, 2.
\]

A simple continuity argument (draw a picture!), using (A.2), shows that we have a third zero \( \xi_3 \in [e^{b_2}, e^{b_1}] \). But this is a contradiction to Descartes’ rule, since we have

\[
D^{(0)'}(\rho) = C_1 \theta_+ \rho^{\theta_+-1} - C_2 \gamma_+ \rho^{\gamma_+-1} + C_2 \gamma_- \rho^{\gamma_-} - C_1 \theta_- \rho^{\theta_- -1},
\]

which has only two sign changes in the sequence of its coefficients.

For the case \( b_2 > b_1 \), we just note that we have here \( D^{(0)'}(b_1) < 0 \), implying \( b_1 \) is less than the first local minimum of \( D^{(0)}(\rho) \) (otherwise we would have three zeroes of \( D^{(0)'}(\rho) \), again a contradiction to Descartes’ rule).

Finally, we have our last proposition.

**Proposition A.1:** \( D(s) \) has at most one positive SCP.
Proof: We first describe, how one can construct $D(s)$, starting with $D^{(0)}(s)$.

For $s \in [0, b_1 \land b_2]$, they are identical. In the interval $[b_1 \land b_2, b_1 \lor b_2]$, $D(s)$ is either strictly increasing or strictly decreasing. Finally, for $s \geq b_1 \lor b_2$, $D(s)$ is equal to a constant. Hence we can generate the function $D(s)$ by appending smoothly, i.e. $C^2$, a graph at the point $(b_1 \land b_2, D^{(0)}(b_1 \land b_2))$, which is first strictly monotone and then constant.

A simple geometric consideration proves now the assertion of the Proposition. One has just to perform the 'append- ing procedure', described in the paragraph above, in all subcases (thereby also using Remark A.1!), we have considered in the proof of Lemma A.1. This reveals that one can generate not more than one SCP in all for $D(s)$. ■