Higher-order Galilean contractions

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Abstract

A Galilean contraction is a way to construct Galilean conformal algebras from a pair of infinite-dimensional conformal algebras, or equivalently, a method for contracting tensor products of vertex algebras. Here, we present a generalisation of the Galilean contraction prescription to allow for inputs of any number of conformal algebras, resulting in new classes of higher-order Galilean conformal algebras. We provide several detailed examples, including infinite hierarchies of higher-order Galilean Virasoro algebras, affine Kac-Moody algebras and the associated Sugawara constructions, and $W_3$ algebras.
1 Introduction

The Galilean Virasoro algebra appears in studies of asymptotically flat three-dimensional spacetimes, see [1] and references therein. It can be constructed [2, 3, 4, 5, 6] as an Inönü-Wigner contraction [7, 8, 9, 10] of a commuting pair of Virasoro algebras. The Galilean W3 algebra [11, 12, 13, 14] likewise follows by contracting a pair of W3 algebras [15]. Many other Galilean conformal algebras with extended symmetries have been worked out [16, 13, 14], including contractions of higher-rank WN algebras [17, 18, 19, 20, 21]. These constructions are all based on contractions of pairs of symmetry algebras, or equivalently, contractions of tensor products of two vertex algebras. In this note, we present a generalisation to allow for inputs of any number of symmetry algebras. This solidifies ideas put forward in [14] and gives rise to new infinite hierarchies of higher-order Galilean conformal algebras.

In Section 2, we outline the generalised contraction prescription and illustrate it by working out the higher-order Galilean Virasoro and affine Kac-Moody algebras. In Section 3, we construct a Sugawara operator [22] for each Galilean Kac-Moody algebra; its central charge is given by the product of the contraction order and the dimension of the underlying Lie algebra. We also show that the Sugawara construction commutes with the Galilean contraction procedure. In Section 4, we apply the Galilean contractions to the W3 algebra and thereby obtain an infinite hierarchy of higher-order W3 algebras. Section 5 contains some concluding remarks.

2 Galilean contractions

2.1 Operator-product algebras and star relations

It is often convenient to combine the generators of the symmetry algebra of a conformal field theory into generating fields of the form

\[ A(z) = \sum_{n \in -\Delta_A + \mathbb{Z}} A_n z^{-n-\Delta_A}, \]  

(2.1)

where \( \Delta_A \) is the conformal weight of \( A \). We are interested in the corresponding operator-product algebra (OPA) \( \mathcal{A} \), where the operator-product expansion (OPE) of the two fields \( A, B \in \mathcal{A} \) is given by

\[ A(z)B(w) = \sum_{n=-\infty}^{\Delta_A+\Delta_B} [AB]_n(w) \frac{z^{-n-\Delta_A}}{(z-w)^n}. \]  

(2.2)

Here, if nonzero, \([AB]_n\) is a field of conformal weight \( \Delta_A + \Delta_B - n \). As the nontrivial information of an OPE is stored in the singular terms, one often ignores the non-singular terms, writing

\[ A(z)B(w) \sim \sum_{n=1}^{\Delta_A+\Delta_B} [AB]_n(w) \frac{z^{-n-\Delta_A}}{(z-w)^n}. \]  

(2.3)

The normal ordering of \( A, B \in \mathcal{A} \) is given by \( (AB) = [AB]_0 \). We use \( \mathbb{I} \) to denote the identity field.

An OPA \( \mathcal{A} \) is said to be conformal if it contains a distinct field \( T \) generating a Virasoro subalgebra. In that case, a field \( A \in \mathcal{A} \) is called a scaling field if

\[ [TA]_2 = \Delta_A A, \quad [TA]_1 = \partial A. \]  

(2.4)

Such a field is quasi-primary if \( [TA]_3 = 0 \), and primary if \( [TA]_n = 0 \) for all \( n \geq 3 \). Let \( \mathcal{B} \) denote a basis for the linear span of the quasi-primary fields in \( \mathcal{A} \). Relative to this, the OPE (2.3) reads

\[ A(z)B(w) \sim \sum_{Q \in \mathcal{B}} \sum_{n=0}^{\Delta_A+\Delta_B-\Delta_Q} C^Q_{A,B} \frac{\beta^{\Delta_Q,n} \partial^n Q(w)}{(z-w)^{\Delta_A+\Delta_B-\Delta_Q-n}}, \]  

(2.5)
with structure constants $C_{A,B}^Q$ and

$$
\beta_{\Delta_A,\Delta_B}^{\Delta Q;n} = \frac{(\Delta_A - \Delta_B + \Delta Q)_n}{n!(2\Delta Q)_n}, \quad (x)_n = \prod_{j=0}^{n-1} (x + j).
$$

(2.6)

Compactly, we may represent the OPE (2.5) by the so-called star relation

$$
A \ast B \simeq \sum_{Q \in B_A} C_{A,B}^Q \{Q\},
$$

(2.7)

where $\{Q\}$ represents the sum over $n$. We refer to [14, 23] for more details on the algebraic structure of an OPA.

### 2.2 Contraction prescription

For $N \in \mathbb{N}$, we consider the tensor-product algebra

$$
\mathcal{A}^\otimes N = \bigotimes_{i=0}^{N-1} \mathcal{A}(i),
$$

(2.8)

where, for simplicity, $\mathcal{A}(0), \ldots, \mathcal{A}(N-1)$ are copies of the same OPA $\mathcal{A}$, up to the value of their central parameters (such as central charges). For $\epsilon \in \mathbb{C}$, let

$$
A_{i,\epsilon} = \epsilon^i \sum_{j=0}^{N-1} \omega^{ij} A_{(j)}, \quad c_{i,\epsilon} = \epsilon^i \sum_{j=0}^{N-1} \omega^{ij} c_{(j)}, \quad i = 0, \ldots, N-1,
$$

(2.9)

where $A_{(j)}$ (respectively $c_{(j)}$) denotes the field $A \in \mathcal{A}_{(j)}$ (respectively the central parameter $c$), and $\omega$ is the principal $N$th root of unity: $\omega = e^{2\pi i/N}$. For $\epsilon \neq 0$, the map

$$
\mathcal{A}^\otimes N \to \mathcal{A}^\otimes N, \quad (A_{(0)}, \ldots, A_{(N-1)}) \mapsto (A_{0,\epsilon}, \ldots, A_{N-1,\epsilon}),
$$

(2.10)

(and similarly for the central parameters) is invertible, with

$$
A_{(i)} = \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-ij} \epsilon^{-j} A_{j,\epsilon}, \quad i = 0, \ldots, N-1.
$$

(2.11)

In the special case $N = 2$, we have $\omega = -1$ and

$$
A_{0,\epsilon} = A_{(0)} + A_{(1)}, \quad A_{1,\epsilon} = \epsilon(A_{(0)} - A_{(1)}),
$$

(2.12)

with inverses

$$
A_{(0)} = \frac{1}{2}(A_{0,\epsilon} + \frac{1}{\epsilon} A_{1,\epsilon}), \quad A_{(1)} = \frac{1}{2}(A_{0,\epsilon} - \frac{1}{\epsilon} A_{1,\epsilon}).
$$

(2.13)

In [14], these fields are denoted by

$$
A = A_{(0)}, \quad \bar{A} = A_{(1)}, \quad A_{+}^\epsilon = A_{0,\epsilon}, \quad A_{-}^\epsilon = A_{1,\epsilon}.
$$

(2.14)

For $\epsilon = 0$, the map (2.10) is singular (unless $N = 1$), indicating that a new algebraic structure emerges in the limit $\epsilon \to 0$, where

$$
A_{i,\epsilon} \to A_i, \quad c_{i,\epsilon} \to c_i.
$$

(2.15)

If the resulting algebra is a well-defined OPA, we refer to it as the $N$th-order Galilean OPA $\mathcal{A}_{\mathcal{G}}^N$. In particular, if $\mathcal{A}$ is an OPA of Lie type (that is, the underlying algebra of modes is a Lie algebra), then all the corresponding higher-order Galilean contractions are indeed well-defined and readily obtained. This is illustrated by the Virasoro and affine Kac-Moody algebras in Section 2.3.
2.3 Galilean Virasoro and affine Kac-Moody algebras

The Virasoro OPA $\mathfrak{Vir}$ of central charge $c$ is of Lie type and generated by $T$, with star relation

$$T \ast T \simeq \frac{c}{2} \mathbb{I} + 2\{T\}. \quad (2.16)$$

The Galilean Virasoro algebra of order $N$, $\mathfrak{Vir}_G^N$, is generated by the fields $T_0, \ldots, T_{N-1}$, with central parameters $c_0, \ldots, c_{N-1}$ and star relations

$$T_i \ast T_j \simeq \begin{cases} \frac{c_{i+j}}{2} \mathbb{I} + 2\{T_{i+j}\}, & i + j < N, \\ 0, & i + j \geq N. \end{cases} \quad (2.17)$$

This yields an infinite family of extended Virasoro algebras, $\{\mathfrak{Vir}_G^N | N \in \mathbb{N}\}$, where $\mathfrak{Vir}_G^1 \simeq \mathfrak{Vir}$ while $\mathfrak{Vir}_G^2$ is the familiar Galilean Virasoro algebra $[2, 3, 4, 5, 6, 13, 14]$. For small $N$, the Galilean Virasoro algebras $\mathfrak{Vir}_G^N$ have recently appeared in [24].

The OPE of two fields in an affine Kac-Moody (or current) algebra $\hat{\mathfrak{g}}$ (where the central element $K$ has been replaced by $k \mathbb{I}$, with $k$ the level) is given by

$$J^a(z)J^b(w) \sim \kappa_{ab} k \frac{1}{(z-w)^2} + f_{ab} \{J^c(w)\} \frac{1}{z-w}, \quad (2.18)$$

where $f_{ab}^c$ are structure constants and $\kappa$ the Killing form of the underlying finite-dimensional Lie algebra $\mathfrak{g}$. (As is customary, the summation over the basis label $c$ is not displayed.) The corresponding OPA is of Lie type, and we find that $\hat{\mathfrak{g}}_G^N$ is generated by $\{J_i^a | a = 1, \ldots, \dim \mathfrak{g}; i = 0, \ldots, N-1\}$, with nontrivial star relations

$$J_i^a \ast J_j^b \simeq \kappa_{ab} k_{i+j} \mathbb{I} + f_{ab}^c \{J_{i+j}^c\}, \quad i + j \in \{0, \ldots, N-1\}. \quad (2.19)$$

In the limit $N \to \infty$, we obtain the algebra $\hat{\mathfrak{g}}_G^\infty$; it is generated by $\{J_i^a | a = 1, \ldots, \dim \mathfrak{g}; i \in \mathbb{N}_0\}$, with nontrivial star relations

$$J_i^a \ast J_j^b \simeq \kappa_{ab} k_{i+j} \mathbb{I} + f_{ab}^c \{J_{i+j}^c\}. \quad (2.20)$$

It follows that

$$\hat{\mathfrak{g}}_G^\infty \cong \hat{\mathfrak{g}} \otimes \mathbb{C}[t] \quad (2.21)$$

and

$$\hat{\mathfrak{g}}_G^N \cong \hat{\mathfrak{g}} \otimes \mathbb{C}[t]/\langle t^N \rangle, \quad (2.22)$$

extending to general $N$ the construction of the Takiff algebras considered in [25, 26]. We similarly have

$$\mathfrak{Vir}_G^\infty \cong \mathfrak{Vir} \otimes \mathbb{C}[t], \quad \mathfrak{Vir}_G^N \cong \mathfrak{Vir} \otimes \mathbb{C}[t]/\langle t^N \rangle. \quad (2.23)$$

3 Generalised Sugawara constructions

In [14], we constructed a Sugawara operator for Galilean affine Kac-Moody algebras (of order 2), and showed that this process commutes with the Galilean contraction procedure. We find that a similar result holds for the higher-order Galilean affine Kac-Moody algebras, manifested by the commutativity of the diagram

$$\hat{\mathfrak{g}} \otimes N \xrightarrow{\text{Sug} \otimes N} \mathfrak{Vir} \otimes N \xrightarrow{\text{Gal}} \hat{\mathfrak{g}}_G^N \xrightarrow{\text{Gal Sug}} \mathfrak{Vir}_G^N$$

To verify this, separate analyses of the two branches are presented in the following two subsections: The lower branch is considered in Section 3.1, the upper one in Section 3.2.
3.1 Galilean Sugawara construction

For the generators of $\mathfrak{Vir}_G^N$, we make the ansatz

$$T_i = \sum_{r,s=0}^{N-1} \lambda^r_s \kappa_{ab}(J^a_r J^b_s), \quad i = 0, \ldots, N - 1,$$

(3.1)

where $\kappa_{ab}$ are elements of the inverse Killing form on $\mathfrak{g}$. The task is now to determine the coefficients $\lambda^r_s$ such that

$$T_i * J^a_j \simeq \begin{cases} \{J^a_i J^a_j\}, & i + j \in \{0, \ldots, N - 1\}, \\ 0, & i + j \geq N. \end{cases}$$

(3.2)

We show below that this is indeed possible. It subsequently follows that $\mathfrak{Vir}_G^N = (T_0, \ldots, T_{N-1})$, with central charges

$$c_0 = N \dim \mathfrak{g}, \quad c_1, \ldots, c_{N-1} = 0.$$

(3.3)

First, we compute the OPE

$$J^a_j(z)T_i(w) = \sum_{r,s=0}^{N-1} \frac{\lambda^r_s}{(z-w)^2} \left\{ k_{j+r}J^a_r(w) + k_{j+s}J^a_s(w) + 2h^\vee J^a_{j+r+s}(w) \right\}$$

$$+ \sum_{r,s=0}^{N-1} \frac{\lambda^r_s \kappa_{bc}}{z-w} \left\{ f^{ab}_{cd}(J^d_{j+r}J^c_s)(w) + f^{ac}_{bd}(J^d_{j+r}J^b_s)(w) \right\},$$

(3.4)

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, arising through the relation $\kappa_{bc} f^{ab}_{cd} f^{de} = 2h^\vee \delta^a_e$. To satisfy (3.2), the first sum must equal $J^a_i J^a_j(w)/(z-w)^2$ while the second sum must vanish. The second-sum constraint implies that

$$\lambda^r_s = \begin{cases} \lambda^r_s^{N-1}, & r + s = N - 1 + \ell \quad (\ell = 0, \ldots, N - 1), \\ 0, & r + s < N - 1. \end{cases}$$

(3.5)

This leaves $N$ coefficients, $\lambda^0_{i,0}, \ldots, \lambda^{N-1}_{i,N-1}$, for each $i \in \{0, \ldots, N - 1\}$. The first-sum constraint then requires that

$$2 \sum_{n=j}^{N-1} \sum_{\ell=0}^{n-j} \lambda^r_s^{i,N-1} k_{N-1-n+j+\ell} J^a_n + 2N h^\vee \lambda^0_{i,N-1} \delta_{j,0} J^a_{N-1} = \begin{cases} J^a_{i+j}, & i + j \leq N - 1, \\ 0, & i + j \geq N. \end{cases}$$

(3.6)

For each $i$, this translates into a lower-triangular system of linear equations:

$$2 \begin{pmatrix} k^i_{N-1} \\ k^i_{N-2} & k^i_{N-1} \\ \vdots & \ddots & \ddots \\ k_1 & \ddots & k_{N-1} \\ k'_0 & k_1 & \cdots & k_{N-2} & k_{N-1} \end{pmatrix} \begin{pmatrix} \lambda^0_{i,0} \\ \vdots \\ \vdots \\ \lambda^{N-1}_{i,N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix},$$

(3.7)

where $k'_{0} = k_0 + N h^\vee$, and where the only nonzero component on the right-hand side is a 1 in position $i + 1$. To solve these systems, we must assume that $k_{N-1} \neq 0$, in which case the problem reduces to
inverting the lower-triangular Toeplitz matrix

\[
A = \begin{pmatrix}
1 & 1 \\
1 & a_1 \\
1 & a_2 & 1 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & a_2 & a_1 & 1
\end{pmatrix},
\tag{3.8}
\]

where

\[
a_m = \frac{k_{N-1-m} + Nh^\vee \delta_{m,N-1}}{k_{N-1}}, \quad m = 1, \ldots, N-1.
\tag{3.9}
\]

The inverse is itself a lower-triangular Toeplitz matrix with 1’s on the diagonal,

\[
A^{-1} = \begin{pmatrix}
b_0 & b_0 \\
b_1 & b_0 & b_0 \\
b_2 & \vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & b_0 \\
b_{N-1} & \cdots & b_2 & b_1 & b_0
\end{pmatrix}, \quad b_0 = 1,
\tag{3.10}
\]

and we find that the nontrivial matrix elements are given by

\[
b_n = \sum_{p \in (N_0)^n} (-1)^{|p|} \frac{\delta_{|p|,n} |p|!}{p_1! \cdots p_n!} a_1^{p_1} \cdots a_n^{p_n},
\tag{3.11}
\]

where

\[
|p| = \sum_{i=1}^n p_i, \quad |p|! = \sum_{i=1}^n i^{|p|}, \quad p = (p_1, \ldots, p_n).
\tag{3.12}
\]

It follows that

\[
\chi_i^{\ell,N-1} = \begin{cases}
0, & \ell = 0, \ldots, i-1, \\
\frac{b_{k-i}}{2k_{N-1}}, & \ell = i, \ldots, N-1,
\end{cases}
\tag{3.13}
\]

so the unique expression for \(T_i\) of the form [3.1] is given by

\[
T_i = \sum_{n=0}^{N-1-i} \frac{b_n}{2k_{N-1}} \sum_{t=0}^{N-1-i-n} \kappa_{ab}(J_{i+n+1}^a J_{N-1-t}^b),
\tag{3.14}
\]

For \(N = 2\), we thus recover the Galilean Sugawara construction obtained in [14],

\[
T_0 = \frac{\kappa_{ab}}{2k_1} (J_0^a J_1^b + (J_1^a J_0^b)) - \frac{k_0 + 2h^\vee}{2(k_1)^2} \kappa_{ab}(J_1^a J_1^b), \quad T_1 = \frac{\kappa_{ab}}{2k_1} (J_1^a J_1^b),
\tag{3.15}
\]

whereas for \(N = 3\), we find the new expressions

\[
T_0 = \frac{\kappa_{ab}}{2k_2} ((J_0^a J_2^b + (J_1^a J_1^b) + (J_2^a J_0^b)) - \frac{k_1}{2(k_2)^2} (J_1^a J_2^b + (J_2^a J_1^b)) + \frac{(k_1)^2 - (k_0 + 3h^\vee)k_2}{2(k_2)^3} \kappa_{ab}(J_2^a J_2^b),
T_1 = \frac{\kappa_{ab}}{2k_2} ((J_1^a J_2^b + (J_2^a J_1^b)) - \frac{k_1}{2(k_2)^2} (J_2^a J_2^b), \quad T_2 = \frac{\kappa_{ab}}{2k_2} (J_2^a J_2^b).
\]
For each \( i = 0, \ldots, N - 1 \), the value of the central parameter \( c_i \) follows from the leading pole in the OPE \( T_0(z)T_i(w) \). Using (3.14), we compute

\[
T_0(z)T_i(w) \sim \sum_{n=0}^{N-1-i} \frac{b_n}{2k_{N-1}} \sum_{t=0}^{N-1-i-n} \frac{\kappa_{ab}^2 \kappa_{ab}^2 \kappa_{a\eta} k_{N-1+i+n} (z-w)^{\eta}}{(z-w)^4} + \cdots, \tag{3.16}
\]
suppressing all subleading poles. Since \( k_a = 0 \) for \( a \geq N \), this term is zero unless \( n + i = 0 \), that is, unless \( n = i = 0 \). From \( \kappa_{ab} \kappa_{ab} = \dim \mathfrak{g} \), we then obtain the announced result (3.3).

### 3.2 Sugawara before Galilean contraction

On the individual factors of \( \mathfrak{g}^{\otimes N} \), the Sugawara construction is given by

\[
T_i = \frac{\kappa_{ab}}{2(k_i + h^\vee)} (J_i^a J_i^b), \quad c_i = \frac{k_i \dim \mathfrak{g} - \kappa_i}{k_i + h^\vee}, \quad i = 0, \ldots, N - 1. \tag{3.17}
\]

Changing basis as in (2.9) introduces

\[
T_{i,\epsilon} = \epsilon^i \sum_{j=0}^{N-1} \omega^j T_j = \epsilon^i \sum_{j=0}^{N-1} \omega^j \sum_{\ell, \ell' = 0}^{N-1} \omega^{-j(\ell+\ell')} \epsilon^{-\ell-\ell'} \kappa_{ab} (J_{\ell, \epsilon}^a J_{\ell', \epsilon}^b) \frac{2N(\sum_{m=0}^{N-1} \omega^{-j \epsilon} \kappa_{m, \epsilon} + Nh^\vee)}{1 + \sum_{m=1}^{N-1} a_{m, \epsilon} \omega^j \epsilon^m}, \tag{3.18}
\]

where

\[
a_{m, \epsilon} = \frac{k_{N-1-m, \epsilon} + Nh^\vee \delta_{m, N-1}}{k_{N-1, \epsilon}}, \quad m = 1, \ldots, N - 1. \tag{3.19}
\]

Now, using that a lower-triangular \( N \times N \) Toeplitz matrix of the form (3.8) decomposes as

\[
A = I + a_1 \eta + \cdots + a_{N-1} \eta^{N-1}, \tag{3.20}
\]

where \( I \) is the identity matrix and \( \eta \) the \( N \times N \) matrix

\[
\eta = \begin{pmatrix}
0 & & & \\
1 & 0 & & \\
& \ddots & \ddots & \\
& & 0 & \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}, \tag{3.21}
\]

we can use the result for \( A^{-1} \) in (3.10)-(3.11) to expand the expression for \( T_{i,\epsilon} \) in powers of \( \epsilon \). We thus find that

\[
T_{i,\epsilon} = \frac{1}{2Nk_{N-1, \epsilon}} \sum_{j, \ell, \ell' = 0}^{N-1} (\omega^j \epsilon)^{N-1+i-\ell-\ell'} \kappa_{ab} (J_{\ell, \epsilon}^a J_{\ell', \epsilon}^b) \left( \sum_{n=0}^{N-1} b_{n, \epsilon} (\omega^j \epsilon)^{b} + \mathcal{O}(\epsilon^N) \right)
\]

\[
= \frac{1}{2Nk_{N-1, \epsilon}} \sum_{\ell, \ell', \epsilon, n=0}^{N-1} b_{n, \epsilon} \kappa_{ab} (J_{\ell, \epsilon}^a J_{\ell', \epsilon}^b) \sum_{j=0}^{N-1} (\omega^j \epsilon)^{N-1+i-\ell-\ell'+n} + \mathcal{O}(\epsilon^{i+1}), \tag{3.22}
\]

where

\[
b_{0, \epsilon} = 1, \quad b_{n, \epsilon} = \sum_{p \in \mathbb{N}_0^n} (-1)^{|p|} \frac{\delta_{p, i-n, |p|}}{p_1! \cdots p_n!} a_{p_1, \epsilon} \cdots a_{p_n, \epsilon}, \quad n = 1, \ldots, N - 1. \tag{3.23}
\]
The summation over \( j \) yields a factor of the form

\[
\sum_{j=0}^{N-1} \omega^j (N-1+i-\ell-\ell'+n) = \begin{cases} N, & N - 1 + i - \ell - \ell' + n \equiv 0 \pmod{N}, \\ 0, & N - 1 + i - \ell - \ell' + n \not\equiv 0 \pmod{N}, \end{cases}
\]

and since \( N - 1 + i - \ell - \ell' + n > -N \), it follows that the \( T_{i,\epsilon} \)-coefficients to \( \epsilon^m \) for \( m \) negative are 0. The limit \( \epsilon \to 0 \) is therefore well-defined, resulting in

\[
T_i = \frac{1}{2kN-1} \sum_{\ell,\ell',n=0}^{N-1} b_{n,k\ell,\epsilon} \delta_{N-1+i-\ell-\ell'+n,0},
\]

whose nonzero terms are seen to match the expression in (3.14). For the central parameters, we evaluate

\[
c_{i,\epsilon} = \epsilon^i \sum_{j=0}^{N-1} \omega^j c_{(j)} = \epsilon^i \sum_{j=0}^{N-1} \omega^j \sum_{\ell=0}^{N-1} \omega^{-j} \epsilon^{-\ell} k_{\ell,\epsilon} \dim g \sum_{\ell'=0}^{N-1} \omega^{-j} \epsilon^{-\ell'} k_{\ell',\epsilon} + Nh^\vee
\]

from which it follows that

\[
c_i = \frac{\dim g}{k_{N-1,\epsilon}} \sum_{\ell,n=0}^{N-1} b_{n,k\ell,\epsilon} \delta_{N-1+i-\ell+n,0} = N \dim g \delta_{i,0},
\]

again confirming (3.3).

### 4 Galilean \( W_3 \) algebras

Higher-order Galilean contractions can also be applied to \( W \)-algebras. Below, we present the results for the \( W_3 \) algebra.

#### 4.1 \( W_3 \) algebra

The \( W_3 \) algebra [15] of central charge \( c \) is generated by a Virasoro field \( T \) and a primary field \( W \) of conformal weight 3, with star relations

\[
T \ast T \simeq \frac{3}{2} \{1\} + 2\{T\}, \quad T \ast W \simeq 3\{W\}, \quad W \ast W \simeq \frac{3}{8} \{1\} + 2\{T\} + \frac{39}{22+\sqrt{5}} \Lambda^{2,2},
\]

where

\[
\Lambda^{2,2} = (TT) - \frac{3}{10} g^2 T,
\]

is quasi-primary.

#### 4.2 Galilean \( W_3 \) algebra of order 2

Following [13][14], we now recall the structure of the second-order Galilean \( W_3 \) algebra [11][12][29]. It is generated by the four fields \( T_0, T_1, W_0, W_1 \), with central parameters \( c_0 \) and \( c_1 \), and nontrivial star relations

\[
T_i \ast T_j \simeq \frac{c_{i+j}}{2} \{1\} + 2\{T_{i+j}\}, \quad T_i \ast W_j \simeq 3\{W_{i+j}\}, \quad i \ast j \in \{0,1\},
\]
and

\[ W_0 \ast W_0 \simeq \frac{2}{3} \{I\} + 2\{T_0\} + \frac{64}{5c_1} \{\Lambda_{0,1}^{2,2}\} - \frac{32(44+5c_0)}{25c_1^2} \{\Lambda_{1,1}^{2,2}\}, \quad W_0 \ast W_1 \simeq \frac{2}{3} \{I\} + 2\{T_1\} + \frac{32}{5c_1} \{\Lambda_{1,1}^{2,2}\}, \]  

(4.4)

where

\[ \Lambda_{0,1}^{2,2} = (T_0 T_1) - \frac{3}{10} \partial^2 T_1, \quad \Lambda_{1,1}^{2,2} = (T_1 T_1) \]  

(4.5)

are quasi-primary. We note that a nonzero \( c_1 \) can be scaled away by renormalising as \( T_1, W_1 \rightarrow \frac{T_1}{c_1}, W_1 = \frac{W_1}{c_1} \).

### 4.3 Infinite hierarchy

For any \( N \in \mathbb{N} \), the algebra \( W_3^{\otimes N} \) is generated by the \( 2N \) fields \( \{T_{(i)}, W_{(i)} \mid i = 0, \ldots, N - 1\} \), and has central charges \( \{c_{(i)} \mid i = 0, \ldots, N - 1\} \). As outlined in the following, the corresponding Galilean algebra is well-defined. In tune with the general prescription in Section 2.2, we thus confirm that the \( N \)th-order Galilean \( W_3 \) algebra (\( W_3 \)\(^N\)) is generated by the fields \( \{T_i, W_i \mid i = 0, \ldots, N - 1\} \) and has central parameters \( \{c_i \mid i = 0, \ldots, N - 1\} \).

First, it straightforwardly follows that

\[ T_i \ast T_j \simeq \frac{\epsilon_{ij}}{2} \{I\} + 2\{T_{i+j}\}, \quad T_i \ast W_j \simeq 3\{W_{i+j}\}, \quad i + j \in \{0, \ldots, N - 1\}, \]  

(4.6)

while

\[ T_i \ast T_j \simeq T_i \ast W_j \simeq W_i \ast W_j \simeq 0, \quad i + j \geq N. \]  

(4.7)

To determine \( W_i \ast W_j \) in \( (W_3)^N \) for \( i + j = 0, \ldots, N - 1 \), we compute the corresponding star relation

\[ W_{i,\epsilon} \ast W_{j,\epsilon} = \epsilon^{i+j} \sum_{r,s=0}^{N-1} \frac{c_{(r)}}{3} \{I\} + 2\{T_{(r)}\} + \frac{32}{22+5c_{(r)}} \{\Lambda_{(r)}^{2,2}\}, \]  

(4.8)

Recycling the expansion techniques of Section 3, we find that

\[ \sum_{r=0}^{N-1} \frac{32}{22+5c_{(r)}} (\omega^r \epsilon)^{i+j} \Lambda_{(r)}^{2,2} = \frac{32}{5Nc_{N-1,\epsilon}} \sum_{n,\ell,\ell'=0}^{N-1} b_{n,\epsilon} \sum_{r=0}^{N-1} (\omega^r \epsilon)^{N-1+i+j-\ell-\ell'+n} (T_{\ell,\epsilon} T_{\ell',\epsilon}) \]  

\[ - \frac{48}{25c_{N-1,\epsilon}} \sum_{n,\ell=0}^{N-1} b_{n,\epsilon} \sum_{r=0}^{N-1} (\omega^r \epsilon)^{N-1+i+j-\ell-n} \partial^2 T_{\ell,\epsilon} + O(\epsilon^{i+j+1}), \]  

(4.9)

where \( b_{n,\epsilon} \) (and \( b_n \) appearing in (4.11) below) are given as in (3.23) (respectively (3.11)), but now based on

\[ a_{m,\epsilon} = \frac{c_{N-1-m,\epsilon} + \frac{22N}{3} \delta_{m,N-1}}{c_{N-1,\epsilon}}, \quad a_m = \frac{c_{N-1-m} + \frac{22N}{3} \delta_{m,N-1}}{c_{N-1}}, \quad m = 1, \ldots, N - 1. \]  

(4.10)

In the limit \( \epsilon \rightarrow 0 \), this yields

\[ \sum_{r=0}^{N-1} \frac{32}{22+5c_{(r)}} (\omega^r \epsilon)^{i+j} \Lambda_{(r)}^{2,2} \rightarrow \sum_{n=0}^{N-1-i-j} \frac{32b_n}{5c_{N-1}} \sum_{\ell=0}^{N-1-i-j-n} (T_{i+j+n} T_{N-1-i}) - \frac{48N}{25c_{N-1}} \partial^2 T_{N-1} \delta_{i,0} \delta_{j,0}. \]  

(4.11)
Observing that, for every pair \( r, s \in \{0, \ldots, N - 1\} \) such that \( r + s \in \{N - 1, \ldots, 2N - 2\} \),

\[
\Lambda_{r,s}^{2,2} = (T_r T_s) - \frac{3}{10} \partial^2 T_{N-1} \delta_{r+s,N-1}
\] (4.12)
is a quasi-primary field with respect to \( T_0 \), we then conclude that, for \( i + j \in \{0, \ldots, N - 1\} \),

\[
W_i * W_j \simeq \frac{c_{i+j}}{3} \{I\} + 2\{T_{i+j}\} + \sum_{n=0}^{N-1-i-j} \frac{32b_n}{5c_{N-1}} \sum_{t=0}^{N-1-i-j-n} \{\Lambda_{i+j+n+t,N-1-t}^{2,2}\}.
\] (4.13)

Using that \( \Lambda_{r,s}^{2,2} = \Lambda_{s,r}^{2,2} \), this can be written as

\[
W_i * W_j \simeq \frac{c_{i+j}}{3} \{I\} + 2\{T_{i+j}\}
+ \sum_{n=0}^{N-1-i-j} \frac{32b_n}{5c_{N-1}} \left( \frac{N-1-i-j-n}{2} \sum_{t=0}^{N-1-i-j-n} \{\Lambda_{i+j+n+t,N-1-t}^{2,2}\} + \{\Lambda_{N-1-i+j+n,N-1-i+j+n}^{2,2}\} \right),
\] (4.14)
where the last term is present only if \( \frac{N-1-i+j+n}{2} \) is integer.

Let us illustrate our findings by summarising the nontrivial star relations for the third-order Galilean algebra \((W_3)^3_G\): The six generating fields \( T_0, T_1, T_2, W_0, W_1, W_2 \) satisfy (4.6)–(4.7) with \( N = 3 \) as well as

\[
\begin{align*}
W_0 * W_0 & \simeq \frac{c_4}{3} \{I\} + 2\{T_0\} + \frac{64}{3c_2} \{\Lambda_{0,2}^{2,2}\} + \frac{32}{5c_2} \{\Lambda_{1,1}^{2,2}\} - \frac{64c_1}{3c_2} \{\Lambda_{1,2}^{2,2}\} - \frac{32(66+5c_0)c_2-5(c_1)^2}{25(c_2)^4} \{\Lambda_{2,2}^{2,2}\}, \\
W_0 * W_1 & \simeq \frac{c_4}{3} \{I\} + 2\{T_1\} + \frac{64}{3c_2} \{\Lambda_{0,2}^{2,2}\} - \frac{32c_1}{5c_2} \{\Lambda_{2,2}^{2,2}\}, \\
W_0 * W_2 & \simeq W_1 * W_0 \simeq \frac{c_4}{3} \{I\} + 2\{T_2\} + \frac{32}{5c_2} \{\Lambda_{2,2}^{2,2}\},
\end{align*}
\] (4.15–4.17)
where

\[
\begin{align*}
\Lambda_{0,2}^{2,2} = (T_0 T_2) - \frac{3}{10} \partial^2 T_2, & \quad \Lambda_{1,1}^{2,2} = (T_1 T_1) - \frac{3}{10} \partial^2 T_2, \quad \Lambda_{1,2}^{2,2} = (T_1 T_2), \quad \Lambda_{2,2}^{2,2} = (T_2 T_2)
\end{align*}
\] (4.18)
are quasi-primary.

### 4.4 Renormalisation

We now consider \((W_3)^N_G\) in the special case where

\[
c_i = c^i, \quad i = 1, \ldots, N - 1,
\] (4.19)
for some \( c \in \mathbb{C} \), leaving only two independent central parameters: the central charge \( c_0 \) and \( c \). The \( a_m \) coefficients in (4.10) then simplify to

\[
a_m = c^{-m} \left( 1 + \left[ c_0 + \frac{22N}{5} \right] - 1 \right) \delta_{m,N-1}, \quad m = 1, \ldots, N - 1.
\] (4.20)

Correspondingly, the inverse of the matrix \( A \) in (3.20) is given by

\[
A^{-1} = I - c^{-1} \eta + \left[ 1 - c_0 - \frac{22N}{5} \right] (c^{-1} \eta)^{N-1},
\] (4.21)
so (for \( N > 2 \))

\[
b_0 = 1, \quad b_1 = -c^{-1}, \quad b_n = 0 \ (1 < n < N - 1), \quad b_{N-1} = \left[ 1 - c_0 - \frac{22N}{5} \right] c^{-(N-1)}.
\] (4.22)
Let us also introduce the renormalised generators
\[
\hat{T}_i = c^{-i} T_i, \quad \hat{W}_i = c^{-i} W_i, \quad i = 0, \ldots, N - 1,
\] (4.23)
and ditto quasi-primary fields
\[
\hat{\Lambda}_{r,s}^{2,2} = c^{-r-s} \Lambda_{r,s}^{2,2}.
\] (4.24)
In terms of these, the nontrivial star relations are given by \((i + j \in \{0, \ldots, N - 1\})\)
\[
\hat{T}_i \ast \hat{T}_j \simeq \frac{\delta_{i+j,0}}{2} \{\mathbb{I}\} + 2\{\hat{T}_{i+j}\}, \quad \hat{T}_i \ast \hat{W}_j \simeq 3\{\hat{W}_{i+j}\}, \tag{4.25}
\]
and
\[
\hat{W}_i \ast \hat{W}_j \simeq \frac{\delta_{i+j,0}}{3} \{\mathbb{I}\} + 2\{\hat{T}_{i+j}\} + \frac{32}{3} [1 - c_0 - \frac{22N}{3}] \{\hat{\Lambda}_{N-1,N-1}^{2,2}\} \delta_{i+j,0}
\]
\[+ \frac{32}{3} \sum_{n=0,1}^{N-1-i-j-n} \sum_{t=0}^{N-1-t} (-1)^n \{\hat{\Lambda}_{i+j+n+t,N-1-t}^{2,2}\}. \tag{4.26}
\]
The central parameter \(c\) has thus been absorbed by a renormalisation of the algebra generators.

A similar absorption is also possible in the Galilean Sugawara construction of Section 31 with
\[
\hat{J}_i^a = k^{-i} J_i^a, \quad \hat{T}_i = k^{-i} T_i, \quad i = 0, \ldots, N - 1,
\] (4.27)
where \(k_i = k^i, i = 1, \ldots, N - 1\), for some \(k \in \mathbb{C}^\times\). The renormalised Galilean Virasoro generators are then given by
\[
\hat{T}_i = 1 \frac{1}{2} \sum_{n=0,1}^{N-1-i-n} \sum_{t=0}^{N-1-t} \kappa_{ab} (\hat{J}_{i+n+t}^a \hat{J}_{N-1-t}^b) + \frac{1}{2} [1 - k_0 - Nh] \kappa_{ab} (\hat{J}_{N-1-i}^a \hat{J}_{N-1}^b) \delta_{i,0}, \tag{4.28}
\]
while the nontrivial star relations read \((i + j \in \{0, \ldots, N - 1\})\)
\[
\hat{J}_i^a \ast \hat{J}_j^b \simeq \kappa_{ab} k_0^{\delta_{i+j,0}} \{\mathbb{I}\} + f^{abc} \{\hat{J}_{i+j}^c\}, \quad \hat{T}_i \ast \hat{J}_j^b \simeq \{\hat{J}_{i+j}^a\}, \quad \hat{T}_i \ast \hat{T}_j \simeq \frac{N \dim \mathfrak{g}}{2} \{\mathbb{I}\} \delta_{i+j,0} + 2\{\hat{T}_{i+j}\}. \tag{4.29}
\]

5 Discussion

In our continued exploration \[13,14\] of Galilean contractions, we have presented a generalisation of the contraction prescription to allow for inputs of any number of OPAs or vertex algebras. This has resulted in hierarchies of higher-order Galilean conformal algebras, including Virasoro, affine Kac-Moody and \(W_3\) algebras.

Asymmetric Galilean \(N = 1\) superconformal algebras, corresponding to an \(N = (1, 0)\) supersymmetry, can be obtained \[27,28,29,30\] from a Galilean contraction of the tensor product, \(S\mathfrak{Vir} \otimes \mathfrak{Vir}\), of an \(N = 1\) superconformal algebra, \(S\mathfrak{Vir}\), and the Virasoro algebra. As we hope to discuss in detail elsewhere, this extends to contractions of a conformal symmetry algebra with any subalgebra thereof. For example, one readily generalises our contraction prescription to the asymmetric tensor product \(W_3 \otimes \mathfrak{Vir}\), where one contracts the Virasoro subalgebra of \(W_3\) with a separate Virasoro algebra. This yields an OPA generated by fields \(T_0, T_1, W\), with nonzero star relations \((i + j \in \{0, 1\})\)
\[
T_i \ast T_j \simeq \frac{\delta_{i+j}}{2} \{\mathbb{I}\} + 2\{T_{i+j}\}, \quad T_0 \ast W \simeq 3\{W\}, \quad W \ast W \simeq \frac{c_0}{2} \{\mathbb{I}\} + 2\{T_1\} + \frac{32}{3} \{2^2 \Lambda_{1,1,1}^{2,2}\}. \tag{5.1}
\]
There is significant freedom in such contractions, leading to a variety of inequivalent Galilean algebras.

Other avenues for future research include representation theory and free-field realisations. The representation theory of the Galilean Virasoro algebra, also known as the \(W(2,2)\) algebra, has already
been studied in some detail \[31, 32, 33, 34, 35, 36\]. In general, though, the representation theory of Galilean algebras remains largely undeveloped and is entirely unexplored in the case of the higher-order algebras introduced in the present note.

Free-field realisations \[37, 38, 39, 40, 41, 42, 43, 44, 45\] have been central to many developments in and applications of conformal field theory, and it seems natural to expect that free fields will play a similar role when Galilean conformal symmetries are present. This includes the representation theory of the Galilean algebras alluded to above. Although realisations of the Galilean Virasoro algebra and some of its superconformal extensions have been considered \[46, 36, 30\], a systematic approach and general results are still lacking.

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