Abstract

In this paper we compute classical Minkowsky spacetime solutions of pure SU(2) and SU(3) gauge theories, in Landau gauge. The solutions are regular everywhere except at the origin and/or infinity, are characterized by a four momentum $k$ such that $k^2 = 0$ and resemble QED configurations. The classical solutions suggest a particle-independent description of hadrons, similarly to the Atomic and Nuclear energy levels, which is able to reproduce the heavy quarkonium spectrum with a precision below 10%. Typical errors in the theoretical mass prediction relative to the measured mass being of the order of 2-4%.

1 Introduction and Motivation

For nonabelian gauge theories the classical field equations are nonlinear partial differential equations. Finding solutions of the classical field equations has proved to be quite a challenge and we still do not know its general solution. The usual approach to classical gauge configurations is to looks for ansatz that simplify the mathematical problem at hand. The ansatz should allow the computation of solutions of the partial differential equations and, simultaneously, should provide a proper description of the field’s dynamics.

For pure gauge SU(2) theory several classical configurations are known. They have been computed using different ansatz, see [1] for a review. For SU(3), classical configurations can be built from SU(2) solutions. In principle, the SU(3) gauge dynamics goes beyond SU(2) classical configurations and, in order to understand its dynamics, one should try to enlarge the set of known solutions as much as possible.

The interest on the classical gauge configurations goes beyond classical field theory. Indeed, the classical solutions of a field theory are a first step towards the
understanding of the associated quantum theory. For QCD, classical configurations built from SU(2) configurations, in particular the instanton [2, 3, 4, 5, 6], are used to address different aspects of hadronic phenomenology. The large number of studies involving instantons shows that classical gauge configurations have a role to play in the associated quantum gauge theory. Hopefully, it would be interesting if these configurations could provide hints about two of the main open problems of strong interactions, namely quark confinement in hadrons and/or chiral symmetry breaking mechanism.

In this paper we report on new Minkowsky spacetime solutions for pure gauge classical SU(2) and SU(3) theories. The Landau gauge solutions were obtained after using a technique inspired on Cho’s work [7, 8, 9, 10] to write the Yang-Mills fields. The generalized Cho-Faddeev-Niemeijer ansatz, combined with a spherical-like basis in color space, reduces the nonlinear field equations to linear abelian-like equations and replaces the Landau gauge condition, a linear gauge condition, by a set of coupled equations.

For SU(2), the gauge fields are functions of a vector field $\hat{A}_\mu$ and a scalar field $\theta_2$ and the ansatz reproduces the original Cho-Faddeev-Niemeijer ansatz. For SU(3), besides $\hat{A}_\mu$ and $\theta_2$ an extra vector field $C_\mu$ is required. The relation between $\hat{A}_\mu, \theta_2, C_\mu$ and the gluon fields is nonlinear. For the ansatz discussed in the paper, the classical field equations become Maxwell equations for $\hat{A}_\mu$ and $C_\mu$ but the Euler-Lagrange equations don’t provide information on $\theta_2$. For pure gauge theories, the classical action is independent of $\theta_2$. It is the gauge fixing condition which restricts $\theta_2$. This seems to suggest that $\theta_2$ is not a dynamical field and, therefore, that it can be chosen arbitrarily. However, if one considers matter fields the action is a functional of $\partial_\mu \theta_2$. Instead of fixing $\theta_2$ a priori, we choose to compute this scalar function solving simultaneously the classical equations of motion and the gauge fixing condition.

The classical field configurations considered in this paper have finite action and energy. They are regular everywhere, except at the origin and/or at infinity, and are characterized by a four-vector $k$ such that $k^2 = 0$, like in the solutions of the linearized field equations. Each four-vector $k$ identifies linear combinations of solutions of the type

$$a e^{\pm ikx} + b e^{\pm ikx}.$$ 

The first components are the usual free field solutions and are eigenfunctions of $i\partial_\mu$ with real eigenvalues. The second components are also eigenfunctions of $i\partial_\mu$ but with pure imaginary eigenvalues. In the general solution of the Euler-Lagrange equations, the vector fields $\hat{A}_\mu$ and $C_\mu$ are linear combinations of the pure complex exponential functions. The scalar $\theta_2$ is given by a combination of both type of exponential functions. The ansatz allows the computation of classical vacuum configurations, fields that share the same general properties as those just described.

After discussing classical solutions of the field equations, we investigate the properties of quarks in the background of the classical gauge configurations. We show that the quark problem can be reduced to three electromagnetic-type problems and, that from the point of view of these electromagnetic problems, not all the couplings are attractive couplings - the maximum number of attractive couplings being two. Moreover, if one look for eigenfunctions of $i\partial_\mu$, it turns out that they are the gauge invariant combinations which represent mesons $\bar{\psi} \gamma^a \psi$ and baryons $\epsilon_{abc} \bar{\psi}^a \psi^b \psi^c$ in the quark model. The third color component of $\psi$
and the product $\psi_1^* \psi_2$ of the color components are also eigenfunctions of $i\partial_0$, but are not gauge invariant wave functions. Motivated by this observation, we try to understand the heavy quarkonium meson spectrum, i.e. charmonium and bottomonium, from the point of view of quarks propagating in the background of classical configurations. Assuming that the spectrum is a combination of two hidrogenium like spectrum and identifying physical states as eigenstates of $i\partial_0$, we are able to reproduce the quarkonium meson spectrum below 10% error for all mesons, with typical errors below 4% for the meson masses. Note that this study of the particles spectrum is a first order approximation to the complete problem. The level of accuracy achieve with such a simple picture of an hadron suggests that it is a good starting point for a more detailed calculation.

The paper is organized has follows. In section 2 we discuss the construction of the ansatz for SU(2) and SU(3) pure gauge fields. In section 3 we compute Minkowsky space-time SU(2) solutions in Landau gauge. Section 4 repeats the previous section but for SU(3) gauge theory. In section 5 we discuss the coupling of quarks to the classical SU(3) configurations and in section 6 discuss the relevance of this configurations for the heavy quarkonium spectrum. In section 7 we give the conclusions. The appendix contains material used along the paper.

2 Gauge Fields for SU(2) and SU(3)

For $SU(N)$ gauge theories the lagrangian reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$$  \hspace{1cm} (1)

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf_{abc} A_\mu^b A_\nu^c,$$  \hspace{1cm} (2)

and $A_\mu^a$ are the gluon fields.

Let $n^a$ be a covariant constant real scalar field in the adjoint representation. From the definition it follows that

$$D_\mu n^a = \partial_\mu n^a + ig (F_{\mu\nu}^b)_{ac} A_\nu^b = 0;$$  \hspace{1cm} (3)

the generators of the adjoint representation are $(F_{\mu\nu}^b)_{ac} = -if_{bac}$. Given a gluon field it is always possible to solve the above equations for the scalar field $n$. The set of equations are linear partial differential equations for $n$. In general, the solution of (3) is not uniquely defined unless boundary conditions are provided. Solving (3) is similar to solve the Laplace equation, where the uniqueness theorem requires boundary conditions. From a formal point of view, the set of equations (3), together with appropriate boundary conditions, defines a mapping from the gluon field $A_\mu^a$ to $n^a$.

Can we say something about the inverse mapping? From the point of view of $A_\mu^a$, equation (3) is a linear equation. This observation may suggest that given a scalar field $n$, the solution of equations (3) is a unique gluon field $A_\mu^a$. However, we will see that, in general, $n$ does not uniquely determines the gluon field, although it helps in reducing the number of independent fields that should be considered when writing $A_\mu^a$. 

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Let us discuss the mapping $n \rightarrow A^a_\mu$. From (3) it follows

$$n \partial_\mu n = \frac{1}{2} \partial_\mu n^2 = 0,$$

and one can always choose $n^2 = 1$. Let us write the gluon field as

$$A^a_\mu = n^a \hat{A}_\mu + X^a_\mu,$$

where the field $X$ is orthogonal to $n$ in the sense

$$n \cdot X_\mu = \sum_a n^a X^a_\mu = 0.$$

To proceed we multiply (3) by $(F^{ed})_{ea} n^d$. For SU(2) theory one has the following relation

$$f_{abc} f_{dec} = \delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd}.$$

For SU(3) we use

$$f_{abc} f_{dec} = \frac{2}{3} (\delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd}) + (d_{ade} d_{bec} - d_{adb} d_{eca})$$

where

$$d_{abc} = \frac{1}{4} \text{Tr} \left( \lambda^a \left\{ \lambda^b, \lambda^c \right\} \right),$$

$\lambda^a$ being the Gell-Mann matrices for SU(3). Solving for the gauge fields we get, after some algebra,

$$f_{cda} n^d \partial_\mu n^a - \Lambda g X^a_\mu - g (d_{ebh} d_{dch} - d_{dbh} d_{ech}) X^b_\mu n^c n^d = 0.$$

In (10) $\Lambda$ is defined as follows

$$\Lambda = \begin{cases} \frac{2}{3}, & \text{SU}(3), \\ 1, & \text{SU}(2). \end{cases}$$

Equation (10) suggest the following form for $X^a_\mu$,

$$X^a_\mu = \frac{1}{\Lambda g} f_{abc} n^b \partial_\mu n^c + Y^a_\mu$$

with $Y^a_\mu$ verifying the constraint

$$n \cdot Y_\mu = 0$$

for SU(3) Yang-Mills theory and

$$Y_\mu = 0$$

for SU(2) gauge group.

In terms of $\hat{A}_\mu$, $n^a$ and $Y^a_\mu$ the gauge fields are given by

$$A^a_\mu = \hat{A}_\mu n^a + \frac{1}{\Lambda g} f_{abc} n^b \partial_\mu n^c + Y^a_\mu,$$
with \( n \) and \( Y \) verifying the constraints

\[
\begin{align*}
n \cdot Y_{\mu} &= 0, \quad (16) \\
D_\mu n^a &= 0. \quad (17)
\end{align*}
\]

Let us consider the gauge transformation properties of \( n \), \( Y \) and \( \hat{A} \). The field \( n \) is, by definition, covariant constant. It follows that \( -i (F^c)_{ab} n^c \partial_\mu n^b \) belongs to the adjoint representation of the gauge group. Demanding that \( Y \) is in the adjoint representation of the group, \( A_\mu \) transforms under the gauge group as follows

\[
\hat{A}_\mu \rightarrow \hat{A}_\mu + \frac{1}{g} n \cdot \partial_\mu \omega. \quad (18)
\]

Note that for constant \( n \), (18) mimics the transformation of an abelian field. Constraints (16) and (17) are scalars under gauge transformations and parameterization (15) provides a gauge invariant decomposition of the gluon field \( A_a^\mu \).

If (15), (16) and (17) define a complete parameterization of the gluon fields, the total number of independent fields on both sides of (15) should be the same. Certainly, the counting of field components is larger on the r.h.s of (15) compared to the l.h.s. However, the counting of the number of independent fields is not obvious, specially in what concerns \( n \), \( \hat{A} \) and \( Y \), and will not be discussed in this paper. Instead, we will proceed looking at solutions of the Euler-Lagrange equations by exploring the ansatz defined in this section.

### 3 Classical SU(2) Gauge Theory

In order to solve classical pure SU(2) gauge theory we consider the spherical basis in color space defined as

\[
\begin{align*}
e_1 &= \begin{pmatrix} s_1 c_2 \\ s_1 s_2 \\ c_1 \end{pmatrix}, & e_2 &= \begin{pmatrix} c_1 c_2 \\ c_1 s_2 \\ -s_1 \end{pmatrix}, & e_3 &= \begin{pmatrix} s_2 \\ -c_2 \\ 0 \end{pmatrix}, \quad (19)
\end{align*}
\]

where \( s_i = \sin \theta_i, \ c_i = \cos \theta_i \) and \( \theta_1 \) and \( \theta_2 \) are functions of spacetime. For SU(2), \( Y^a_{\mu} = 0 \) and after identifying \( n \) with \( e_1 \) the gauge fields become

\[
\begin{align*}
A_1^\mu &= s_1 c_2 \left( \hat{A}_\mu - \frac{1}{g} c_1 \partial_\mu \theta_2 \right) - \frac{1}{g} s_2 \partial_\mu \theta_1, \quad (20) \\
A_2^\mu &= s_1 s_2 \left( \hat{A}_\mu - \frac{1}{g} c_1 \partial_\mu \theta_2 \right) + \frac{1}{g} c_2 \partial_\mu \theta_1, \quad (21) \\
A_3^\mu &= \hat{A}_\mu c_1 + \frac{1}{g} s_2^2 \partial_\mu \theta_2 = c_1 \left( \hat{A}_\mu - \frac{1}{g} c_1 \partial_\mu \theta_2 \right) + \frac{1}{g} \partial_\mu \theta_2, \quad (22)
\end{align*}
\]

the gluon field tensor is given by

\[
F^{\mu\nu}_a = n^a F_{\mu\nu}, \quad (23)
\]

where

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad A_\mu = \hat{A}_\mu - \frac{1}{g} c_1 \partial_\mu \theta_2 \quad (24)
\]
and the classical equations of motion

\[ n^a \partial_\nu F^{\mu \nu} = 0. \] (25)

The structure of \( n \) reduces the last set of equations to

\[ \partial_\nu F^{\mu \nu} = 0, \] (26)
i.e. the ansatz makes classical pure SU(2) gauge theory equivalent to Maxwell theory. That the equivalence between classical SU(2) gauge theory and an abelian theory is not perfect can be seen by looking at the classical hamiltonian density

\[ H = F^{\alpha \beta} \partial^0 A^a_\beta - \mathcal{L} \]

\[ = \mathcal{F}^{\alpha 0} \partial^0 \hat{A}_\beta - \left( -\frac{1}{4} \mathcal{F}^{\mu \nu} \mathcal{F}_{\mu \nu} \right) + \text{additional terms} \] (27)

and at the classical spin tensor

\[ S^{\alpha \beta} = F^{\alpha \beta} \hat{A}^{\alpha \beta} - F^{\alpha \beta} \hat{A}^{\beta \alpha} \]

\[ = \mathcal{F}^{\beta 0} \hat{A}^{\alpha} - \mathcal{F}^{\alpha 0} \hat{A}^{\beta} + \frac{\cos \theta_1}{g} \left( \mathcal{F}^{\beta 0} \partial^\alpha \theta_2 - \mathcal{F}^{\alpha 0} \partial^\beta \theta_2 \right) \]

\[ = \mathcal{F}^{\beta 0} \hat{A}^{\alpha} - \mathcal{F}^{\alpha 0} \hat{A}^{\beta}. \] (28)

For the pure gauge theory, the field equations (26) don’t fix unambiguously the components of gluon field \( A^a_\mu \). In particular, the determination of \( \theta_1 \) and \( \theta_2 \) requires additional conditions on \( A^a_\mu \), i.e. one has to rely on a gauge fixing condition.

So far we have considered the field equations associated to the gluon field built after identifying \( n \) with \( \epsilon_1 \). Different choices for \( n \) will reproduce essentially the picture just described. The simplest gluon field is obtained with \( n = \epsilon_3 \). Then

\[ A^1_\mu = \sin \theta_2 \hat{A}_\mu, \] (29)

\[ A^2_\mu = -\cos \theta_2 \hat{A}_\mu, \] (30)

\[ A^3_\mu = \frac{1}{g} \partial_\mu \theta_2, \] (31)

the gluon field tensor being

\[ F^{a \mu \nu} = n^a F^{\mu \nu}, \] (32)

where

\[ F^{\mu \nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu \] (33)

and the classical equations of motion of motion are

\[ \partial^{\nu} F^{\mu \nu} = 0. \] (34)

For this gluon field the equivalence between classical SU(2) and an abelian theory is even more striking. The hamiltonian density is a hamiltonian density of the “abelian theory” associated to \( \hat{A} \),

\[ H = \mathcal{F}^{\beta 0} \partial^0 \hat{A}_\beta - \mathcal{L} \] (35)
and the spin tensor reproduces the spin tensor of the same "abelian theory"

$$S^{\alpha\beta} = \mathcal{F}^{\beta 0} \hat{A}^\alpha - \mathcal{F}^{\alpha 0} \hat{A}^\beta.$$  \hspace{1cm} (36)

As previously, the field equations don’t fix completely $A^\alpha_\mu$ and to compute the gluon fields one has to work on a particular gauge. The Landau gauge condition

$$\partial^\mu A^\alpha_\mu = 0,$$  \hspace{1cm} (37)

now reads

$$\partial^\mu \hat{A}_\mu = 0, \quad \partial^\mu \theta_2 \hat{A}_\mu = 0, \quad \partial_\mu \partial^\mu \theta_2 = 0,$$  \hspace{1cm} (38)

i.e. the ansatz reduces the nonlinear field equations to a set of linear equations (34) and a linear gauge condition to a set of coupled equations.

Let us discuss the properties of solutions of (34) and (38). Particular solutions of the gauge fixing condition are

1. Type I $\partial^\mu \theta_2 = 0$,
2. Type II $\hat{A}_\mu = 0$.  \hspace{1cm} (39)

For type I solutions, the field equations are the Maxwell equations and the classical finite action SU(2) configurations are the finite action QED-like configurations associated with the vector field $A^\mu_\mu$; $\theta_2$ is a constant field. The other family of solutions have null action, energy and spin. They are interesting because type II solutions includes a new class of configurations. The function $\theta_2$ is a solution of the Klein-Gordon equation. The requirement of finite action does not constraint $\theta_2$ and the general solution is

$$\theta_2(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \left[ a(k) e^{-ikx} + a^*(k) e^{ikx} \right] +$$

$$\int \frac{d^3k}{(2\pi)^3 2k_0} \left[ b(k) e^{ikx} + c(k) e^{-ikx} \right],$$  \hspace{1cm} (40)

where it was assumed that all integrations are well defined. $\theta_2$ is a linear combination of exponential functions characterized by a four-vector $k$ satisfying the condition $k^2 = 0$. The components proportional to $a$ are eigenfunctions of $-i\partial_\mu$ associated to real eigenvalues $\pm k_\mu$. These components can be identified with free gluons. The components proportional to $b$ or $c$ are eigenfunctions of the same operator but the associated eigenvalues are pure imaginary numbers $\pm ik_\mu$, i.e. they cannot describe free particles. Considering that free gluons have never been observed in nature, configurations of the last kind should not be disregarded \textit{a priori}. Actually, the same observation may suggest that the components that one should disregard are those proportional to $a$. The above reasoning is valid for a linear theory like QED. For SU(2), the identification of free field components with eigenfunctions of $-i\partial_\mu$ is arguable and this naive reasoning should be taken with care.

The general solution of the field equations with finite action in the Landau gauge is

$$\hat{A}_\mu = \sum_\lambda a(\vec{k}_\lambda, \lambda) \epsilon_\mu(\vec{k}, \lambda) e^{\pm ikx},$$  \hspace{1cm} (41)

$$\theta_2 = \left[ a(\vec{k}) e^{-ikx} + a^*(\vec{k}) e^{ikx} \right] +$$

$$\left[ b(\vec{k}) e^{ikx} + c(\vec{k}) e^{-ikx} \right],$$  \hspace{1cm} (42)
where $k^2 = 0$ and the three independent polarization vectors $\epsilon_{\mu}(k, \lambda)$ verify the conditions
\begin{equation}
    k^\mu \epsilon_{\mu}(k, \lambda) = 0, \quad \lambda = 1 \ldots 3 .
\end{equation}
For the special case of $k = (\omega, 0, 0, \omega)$, possible choices for the polarization vectors are
\begin{equation}
    \epsilon_{\mu}(k, 1) = k_{\mu} / \omega, \quad \epsilon_{\mu}(k, 2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_{\mu}(k, 3) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
\end{equation}

However, similarly to classical electrodynamics, one can redefine the vector field $A_\mu$ by adding up a gradient of a scalar function $\eta$ such that $\partial^2 \eta = 0$ and $\partial_\mu \theta_2 \partial^\mu \eta = 0$. The particular choice $\eta = \pm i a(k, 1) e^{\pm ikx} / \omega$ satisfies the above requirements and removes the longitudinal component from $A_\mu$. The vector field $A_\mu$ can be made transverse but the gluon field built from (41) - (42) has a longitudinal component associated to $\theta_2$. Remember that energy and spin are independent of the longitudinal component of the gluon field and that $\theta_2$ plays a role in the interaction with matter fields. In what concerns only pure gauge SU(2) theory, the finite action classical solutions are essentially QED-like solutions associated to $A_\mu$ field.

A special class of configurations are the classical vacuum solutions. They are solutions with a null gluon field tensor,
\begin{equation}
    F^a_{\mu\nu} = 0 .
\end{equation}
For the ansatz considered, this means
\begin{equation}
    \hat{A}_\mu = \partial_\mu \chi ,
\end{equation}
where $\chi$ is any differentiable function of spacetime. Vacumm configurations are determined by two scalar functions $\chi$ and $\theta_2$. The field equations don’t give us information about the nature of these functions. The Landau gauge condition requires that
\begin{equation}
    \partial_\mu \partial^\mu \chi = 0 , \quad \partial^\mu \theta_2 \partial_\mu \chi = 0 , \quad \partial_\mu \partial^\mu \theta_2 = 0 ,
\end{equation}
whose general solution is, again, characterized by a four-vector $k$ satisfying $k^2 = 0$.

\begin{align}
    \chi &= \left[ a_\chi(\vec{k}) e^{-ikx} + a_\chi^*(\vec{k}) e^{ikx} \right] +  \\
    &\quad \left[ b_\chi(\vec{k}) e^{ikx} + c_\chi(\vec{k}) e^{-ikx} \right] , \quad (48)
\end{align}
\begin{align}
    \theta_2 &= \left[ a_\theta(\vec{k}) e^{-ikx} + a_\theta^*(\vec{k}) e^{ikx} \right] +  \\
    &\quad \left[ b_\theta(\vec{k}) e^{ikx} + c_\theta(\vec{k}) e^{-ikx} \right] . \quad (49)
\end{align}
Solutions (48) and (49) are similar to the finite action solutions discussed previously. The main difference being that now the gluon field is pure longitudinal and, as before, $\chi$ can be chosen such that $\hat{A}_\mu = 0$. Then, the vacuum field is
\begin{equation}
    A^a_\mu = \delta^{a3} k_\mu \left\{ a(\vec{k}) e^{-ikx} + a^*(\vec{k}) e^{ikx} \right\} + \left[ b(\vec{k}) e^{ikx} + c(\vec{k}) e^{-ikx} \right] .
\end{equation}
Our study of SU(2) is a first flavor for the classical configurations of SU(3) pure gauge theory. As we will see in the next section, the “abelian projection” and the main characteristics of the SU(2) solutions are also present in the SU(3) classical configurations.

4 Classical SU(3) Gauge Theory

To solve the classical equations of motion for QCD, we choose a spherical like basis in the color space - see appendix for definitions. Setting $n = \vec{e}_3$, condition (17) becomes

\[
-\frac{1}{2} \partial_\mu n^a - g f_{abc} Y^b_\mu n^c = 0.
\]

This set of equations provides the following relations between the $Y^a_\mu$ fields

\[
\begin{aligned}
Y_\mu^2 &= -Y_\mu^1 \cot \theta_2, \\
Y_\mu^3 &= -\frac{1}{2g} \partial_\mu \theta_2, \\
Y_\mu^4 = Y_\mu^5 = Y_\mu^6 = Y_\mu^7 &= 0. 
\end{aligned}
\]

Taking into account (52) to (54) the gluon field is given by

\[
(A^a_\mu) = \begin{pmatrix}
-\sin \theta_2 \hat{A}_\mu + Y_\mu^1 \\
\cos \theta_2 \hat{A}_\mu - \cot \theta_2 Y_\mu^1 \\
\partial_\mu \theta_2 / g \\
0 \\
0 \\
0 \\
Y_\mu^8
\end{pmatrix}.
\]

Condition (16) simplifies the gluon field into

\[
A^a_\mu = n^a \hat{A}_\mu + \delta^a_8 \frac{1}{g} \partial_\mu \theta_2 + \delta^a_8 C_\mu;
\]

in the last equation $C_\mu = Y_\mu^8$. The corresponding gluon field tensor components are

\[
F^a_{\mu\nu} = n^a A_{\mu\nu} + \delta^a_8 C_{\mu\nu}
\]

where

\[
A_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu, \quad C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu.
\]

The classical action is a functional of the “photon like” fields $\hat{A}_\mu$ and $C_\mu$,

\[
\mathcal{L} = -\frac{1}{4} (A^2 + C^2),
\]

and the classical equations of motion

\[
D^\nu F^a_{\mu\nu} = \partial^\nu F^a_{\nu\mu} - g f_{abc} A^b_\mu F^c_{\nu\mu} = 0
\]

(55)

(56)

(57)

(58)

(59)

(60)
are reduced to
\[ \partial_\nu A^{\mu \nu} = 0, \quad \partial_\nu C^{\mu \nu} = 0. \]  
(61)
The hamiltonian density
\[ H = A^{\beta 0} \partial^0 \hat{A}_\beta + C^{\beta 0} \partial^0 C_\beta - \mathcal{L} \]  
(62)
and the spin tensor
\[ S = \left( A^{\beta 0} \hat{A}^\alpha - A^{\alpha 0} \hat{A}^\beta \right) + \left( C^{\beta 0} C^\alpha - C^{\alpha 0} C^\beta \right) \]  
(63)
are given by the sum of the contributions of two abelian-like theories associated with \( \hat{A} \) and \( C \). Similarly to what was observed for the SU(2) theory, the ansatz (56) maps SU(3) to a set of two linear theories, making classical pure SU(3) gauge theory formally equivalent to QED with two “photon fields”. Note that the “photon fields” are not coupled and that their coupling to the fermionic fields requires different Gell-Mann matrices.

The classical equations of motion (61) are independent of \( \theta^2 \). Therefore, to compute classical configurations we consider the Landau gauge. In terms of \( \hat{A}_\mu \), \( C_\mu \) and \( \theta^2 \) the gauge condition reads
\[ \partial_\mu \hat{A}_\mu = 0, \quad \partial_\mu C_\mu = 0, \quad \partial_\mu \theta^2 = 0. \]  
(64)
The solutions of (61) and (64) are similar to the classical configurations discussed for SU(2). The main difference comes from having now two “photon” fields instead of a single one. The general finite action solution is then given by
\[ \hat{A}_\mu = \sum_\lambda a_A(\vec{k}, \lambda) \epsilon_\mu(k, \lambda) e^{\pm ikx}, \]  
(65)
\[ \theta_2 = \left[ a(\vec{k}) e^{-ikx} + a^*(\vec{k}) e^{ikx} \right] + \left[ b(\vec{k}) e^{ikx} + c(\vec{k}) e^{-ikx} \right], \]  
(66)
\[ C_\mu = \sum_\lambda a_C(\vec{k}, \lambda) \epsilon_\mu(k, \lambda) e^{\pm ikx}, \]  
(67)
where \( k^2 = 0 \) and the sum over polarizationas runs only over the transverse polarizations.

Let us discuss the classical vacuum solutions of (61) and (64). Following the approach as for SU(2), one can set \( \hat{A}_\mu = C_\mu = 0 \), i.e.
\[ A^a_\mu = \delta^{a 3} \frac{1}{g} \partial_\mu \theta_2. \]  
(68)
Then, (64) reduces to
\[ \partial^2 \theta_2 = 0 \]  
(69)
and that \( \theta_2 \) does not verify any special kind of boundary condition. The above equation can be solved by separation of variables in the usual way. In spherical
coordinates, the solution is

\[ \theta_2 = \sum_{l=0}^{+\infty} \sum_{m=-l}^{l} \left\{ \begin{array}{l} a(\omega)e^{-i\omega t} + a^*(\omega)e^{+i\omega t} \end{array} \right. \times \]

\[ \left[ A_{lm}(\omega)j_l(\omega r) + B_{lm}(\omega)n_l(\omega r) \right] Y_{lm}(\theta, \phi) \]

\[ + \left[ b(\omega)e^{-\omega t} + c(\omega)e^{+\omega t} \right] \times \]

\[ \left[ C_{lm}(\omega)j_l(\omega r) + D_{lm}(\omega)n_l(\omega r) \right] Y_{lm}(\theta, \phi) \]

\[ + \left[ et + f \right] \left[ F_{lm} r^l + \frac{G_{lm}}{\mu + 1} \right] Y_{lm}(\theta, \phi) \} , \]

(70)

where \( \omega \) has dimensions of mass, \( j_l \) is the spherical Bessel function of order \( l \) and \( n_l \) is the spherical Neumann function of order \( l \). The parameter \( \omega \) is related to the four-vector \( k \) \((k_0 = |\vec{k}| = \omega)\) and comes into (70) when the separation between time and spatial parts of \( \theta_2 \) is performed. Note that, as in SU(2), the classical gluon field vacuum is a pure longitudinal field.

5 Quarks in the Background of Classical Configurations

Let us discuss now the properties of quark fields propagating in the background of the classical SU(3) configurations computed in the previous section. In terms of color components, the Dirac equation

\[ (i\slashed{\partial} - m) \psi = 0 \]  

(71)

reads

\[ (i\slashed{\partial} - m) \psi^1 + \frac{g}{2} e^{-i\theta_2} \slashed{A} \psi^2 - \frac{1}{2} \theta_2 \psi^1 - \frac{g}{2\sqrt{3}} \gamma^2 \psi^1 = 0 , \]  

(72)

\[ (i\slashed{\partial} - m) \psi^2 - \frac{g}{2} e^{i\theta_2} \slashed{A} \psi^1 + \frac{1}{2} \theta_2 \psi^2 - \frac{g}{2\sqrt{3}} \gamma^2 \psi^2 = 0 , \]  

(73)

\[ (i\slashed{\partial} - m) \psi^3 + \frac{g}{\sqrt{3}} \gamma^3 \psi^3 = 0 . \]  

(74)

Introducing the spinor fields \( \phi \) and \( \eta \) defined by

\[ \psi^1 = e^{-\frac{i}{2} \theta_2} \phi , \quad \psi^2 = i e^{\frac{i}{2} \theta_2} \eta , \]  

(75)

(72)-(74) can be written as a set of uncoupled equations

\[ \left[ i\slashed{\partial} - m - \frac{g}{2} \left( \frac{\gamma^2}{\sqrt{3}} - \slashed{A} \right) \right] \phi^{(+)} = 0 , \]  

(76)

\[ \left[ i\slashed{\partial} - m - \frac{g}{2} \left( \frac{\gamma^2}{\sqrt{3}} + \slashed{A} \right) \right] \phi^{(-)} = 0 , \]  

(77)

\[ (i\slashed{\partial} - m) \psi^3 + \frac{g}{\sqrt{3}} \gamma^3 \psi^3 = 0 , \]  

(78)
where
\[ \phi^{(\pm)} = \phi \pm \eta. \]  

The vector fields \( \hat{A}_\mu \) and \( C_\mu \) are electromagnetic-type fields and the problem of solving the Dirac equation is reduced to a set of three electromagnetic-like problems. Note that in (72)-(74) the electromagnetic couplings cannot all be attractive couplings. Indeed, the maximum number of attractive couplings is two. In terms of dynamics this means that, at least, for one the equations the lowest energy solution is the free particle solution of mass \( m \) at rest.

Let \( \{ \phi_n^{(+)}(E^{(+)}), \phi_n^{(-)}(E^{(-)}), \phi_n^{(3)}(E^{(3)}) \} \) be the Dirac spinor and associated energy for the solutions of equations (76), (77), (78) respectively. Then, the original Dirac field is given by
\[
\psi_1(x) = \frac{e^{-\frac{1}{2} \theta_2}}{2} \sum_n \left[ \phi_n^{(+)}(\vec{x}) e^{-iE^{(+)}_n t} + \phi_n^{(-)}(\vec{x}) e^{-iE^{(-)}_n t} \right],
\]
\[
\psi_2(x) = i \frac{e^{\frac{1}{2} \theta_2}}{2} \sum_n \left[ \phi_n^{(+)}(\vec{x}) e^{-iE^{(+)}_n t} - \phi_n^{(-)}(\vec{x}) e^{-iE^{(-)}_n t} \right],
\]
\[
\psi_3(x) = \sum_n \left[ \phi_n^{(3)}(\vec{x}) e^{-iE^{(3)}_n t} \right].
\]

If one identifies physical states with eigenstates of the operator \( i\partial_0 \), the candidates to be physical particles are
\[
\text{any product of } \psi_3, \quad \text{products of } \psi_2 \psi_1, \quad \sum_a \bar{\psi} \Gamma^a \psi^a, \quad \epsilon_{abc} \chi_{\alpha \beta \gamma} \psi^a_\alpha \psi^b_\beta \psi^c_\gamma;
\]

roman letters stand for color indices and greek letters for spin indices. In (83), the first and second wave functions are not gauge invariant, therefore the associated eigenvalue of \( i\partial_0 \) is not gauge invariant\(^1\) and they cannot be associated to physical particles. On the other hand, the second and third types of wave functions are gauge invariant and their energy spectrum can be identified with physical states. These wave functions are the usual quark model wave functions for mesons and baryons.

Let us assume that physical particles are described by the following wave functions
\[
M = \sum_a \bar{\psi}^a \Gamma^a \psi^a, \quad B = \epsilon_{abc} \chi_{\alpha \beta \gamma} \psi^a_\alpha \psi^b_\beta \psi^c_\gamma.
\]

Taking into account (80)-(82), the meson wave functions are given by
\[
M(x) = \frac{1}{4} \sum_{n,k} \left\{ \begin{array}{l}
\bar{\phi}_n^{(+)}(\vec{x}) \Gamma \phi_k^{(+)}(\vec{x}) e^{-i(E^{(+)}_k - E^{(+)}_n) t} + \\
\bar{\phi}_n^{(-)}(\vec{x}) \Gamma \phi_k^{(-)}(\vec{x}) e^{-i(E^{(-)}_k - E^{(-)}_n) t} + \\
\bar{\phi}_n^{(3)}(\vec{x}) \Gamma \phi_k^{(3)}(\vec{x}) e^{-i(E^{(3)}_k - E^{(3)}_n) t}
\end{array} \right\}
\]

\(^1\)by a gauge transformation the wave function can be made time independent. A state with zero \( i\partial_0 \) eigenvalue.
while the baryon wave functions are

\[
B \sim \sum_{n_1, n_2, n_3} \left\{ A_{n_1 n_2 n_3} e^{-i\left( E_{n_1}^{(+)} + E_{n_2}^{(+) - E_{n_3}^{(3)} \right) t}} + B_{n_1 n_2 n_3} e^{-i\left( E_{n_1}^{(-)} + E_{n_2}^{(-)} + E_{n_3}^{(3)} \right) t} + C_{n_1 n_2 n_3} e^{-i\left( E_{n_1}^{(-)} + E_{n_2}^{(-)} + E_{n_3}^{(3)} \right) t} \right\},
\]

where \( A_{n_1 n_2 n_3}, B_{n_1 n_2 n_3} \) and \( C_{n_1 n_2 n_3} \) are the baryon spatial wave function. In this picture, mesons and baryons are described as systems of non-interacting quarks in the background of the classical configurations considered in last section. Of course, this is not the true picture of a meson or a baryon but a first order approximation to hadrons. This way of viewing hadronic matter is identical to the description of the atomic and nuclear energy levels.

The energy spectrum described by \( E_{n_1}^{(+)}, E_{n_1}^{(-)} \) and \( E_{n_1}^{(3)} \) are electromagnetic spectra. Assuming that the quark-gluon interaction can be viewed as static interaction, described by a Coulombic potential, one can immediately compute the associated particles masses. Before starting to compare the theoretical spectrum with measured masses, one should first try to identify the family of hadrons that fits better such a theoretical description. The success of heavy quark effective field theory \([13, 14]\) and nonrelativistic potential studies of heavy mesons \([15]\) suggests that heavy mesons are good testing grounds for the above hypotheses. This idea is corroborated by quenched lattice QCD investigations, an approximation that solves exactly the bosonic sector of the theory and propagates quarks in the gluon fields. It is well known that such an approximation provides better results for heavy quark systems \([16]\). Furthermore, it is well known that charmonium and bottomonium spectrum are close to hydrogen-like spectrum. All this observations motivates the next section.

6 Classical Gluonic Configurations, Charmonium and Bottomonium Spectrum

According to our picture of hadrons, mesons and baryons are classified with hydrogen-like levels. The main difference to the hydrogen atom being that now there are two independent sequence of levels, i.e. the Coulombic potential is not an interquark potential but describes the interaction between a quark and a background gluonic field. According to (85), the two independent spectrum do not mix.

The solution of the Dirac equation in a Coulomb potential is well known [17]. The energy spectrum is given by

\[
E_{n, j} = \frac{m}{\sqrt{1 + \left( \frac{\alpha^2}{(n - \delta_j)^2} \right)}}, \quad \delta_j = j + \frac{1}{2} - \sqrt{\left( j + \frac{1}{2} \right)^2 - \alpha^2},
\]

with \( n = 1, 2, \ldots \) and \( j = \frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2} \). The spectrum has a twofold degeneracy except for the state \( j = n - \frac{1}{2} \). This degenerate states can be distinguished by their orbital angular momentum \( l = j \pm \frac{1}{2} \) (for \( j = n - \frac{1}{2} \), \( l = n - 1 \)). The quarkonium levels are given by (87) with the coupling constant
are written the allowed states are identified with the usual spectroscopic notation and in parentheses.

Table 1: Bottomonium and charmonium spectrum according to Particle Data Book [18].

\[
\begin{array}{|c|c|c|}
\hline
\text{Particle} & J^P & M (\text{MeV}) \\
\hline
\eta_b(1S) & 0^+ & 9300 \pm 20 \\
\Upsilon_b(1S) & 1^- & 9460.30 \pm 0.26 \\
\chi_{b0}(1P) & 0^+ & 9859.9 \pm 1.0 \\
\chi_{b1}(1P) & 1^+ & 9892.7 \pm 0.6 \\
\chi_{b2}(1P) & 2^+ & 9912.6 \pm 0.5 \\
\Upsilon_b(2S) & 1^- & 10023.26 \pm 0.31 \\
\chi_{b0}(2P) & 0^+ & 10232.1 \pm 0.6 \\
\chi_{b1}(2P) & 1^+ & 10255.2 \pm 0.5 \\
\chi_{b2}(2P) & 2^+ & 10268.5 \pm 0.4 \\
\Upsilon_b(3S) & 1^- & 10355.2 \pm 0.6 \\
\Upsilon_b(4S) & 1^- & 10580.0 \pm 3.5 \\
\Upsilon_b(10860) & 1^- & 10865 \pm 8 \\
\Upsilon_b(11020) & 1^- & 11019 \pm 8 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\eta_c(1S) & 0^+ & 2979.7 \pm 1.5 \\
J/\psi(1S) & 1^- & 3096.87 \pm 0.04 \\
\chi_{c0}(1P) & 0^+ & 3415.1 \pm 0.8 \\
\chi_{c1}(1P) & 1^+ & 3510.51 \pm 0.12 \\
\chi_{c2}(1P) & 2^+ & 3556.18 \pm 0.13 \\
h_c(1P) & ?^+ & 3526.14 \pm 0.24 \\
\eta_c(2S) & 0^+ & 3594 \pm 5 \\
\psi(2S) & 1^- & 3685.96 \pm 0.09 \\
\psi(3770) & 1^- & 3769.9 \pm 2.5 \\
\psi(3865) & 2^- & 3836 \pm 13 \\
\psi(4040) & 1^- & 4040 \pm 10 \\
\psi(4160) & 1^- & 4159 \pm 20 \\
\psi(4415) & 1^- & 4415 \pm 6 \\
\hline
\end{array}
\]

\(\alpha\) replaced by the strong coupling constant multiplied by a factor \(F^2\),

\[
F = \left\{ \frac{1}{2} \left( a - \frac{c}{\sqrt{3}} \right), \quad - \frac{1}{2} \frac{\sqrt{3}}{2} \left( a + \frac{c}{\sqrt{3}} \right), \quad \frac{c}{\sqrt{3}} \right\},
\]

where \(a\) and \(c\) are, respectively, the \(\hat{A}_\mu\) and \(C_\mu\) amplitudes; see (76)-(78). In the electromagnetic spectrum, the Lamb shift resolves the degeneracy between \(n s_j\) and \(n p_j\) states, the \(s\) states acquire an excess of energy. In our calculation of the particles masses we will not take into account such a correction but, certainly, a more precise computation should take into account the Lamb shift and corrections to the particle-independent approximation assumed in our description of hadrons.

As quark masses we take the central values of the 2002 edition of the Particle Data Book [18],

\[
m_c = 1200 \text{ MeV}, \quad m_b = 4850 \text{ MeV}.
\]

For the strong coupling constants we use the central values given in [19],

\[
\alpha_s(m_c) = 0.42124, \quad \alpha_s(m_b) = 0.21174.
\]

In the following it is assumed that the quantum numbers of charmonium and bottomonium given in the Particle Data Book are correct. Table 1 is a resumé of particles masses and quantum numbers.

For the double hidrogenium spectrum, the sequence of the lowest energy levels is

\[
1s_1^2(0^-), \quad \left( 1s_2, 2p_2 \right)(0^+, 1^+), \quad \left( 1s_2, 2s_2 \right)(0^-, 1^-), \quad \left( 1s_2, 2p_2 \right)(1^+, 2^+);
\]

states are identified with the usual spectroscopic notation and in parentheses are written the allowed \(J^P\) values. In each of the sequences in Table 1, the first
quarkonium states can be interpreted as same of the (91) states. Note that, for charmonium $\chi(1P)$ states, the difference in mass between the $0^+$ and $1^+$ is almost twice the difference between the $1^+$ and $2^+$ states. Then, one can take as the mass of the $(1s_{1/2}, 2p_{1/2}) (1^+, 2^+)$ state, the mean value of $\chi_{c1}$ and $\chi_{c2}$ masses and identify $\chi_{c0}$ as $(1s_{1/2}, 2p_{1/2}) (0^+)$. Then, the mass difference between these two states is given by

$$E_{2 \ 3/2} - E_{2 \ 1/2} = \Delta M(F^2).$$  \hspace{1cm} (92)$$

From (92) one can compute $F$. A similar reasoning applies to the bottomonium$^2$ $\chi_b(1P)$ particles. The $F$ values computed using table 1 masses are

$$F = \begin{cases} 
1.5259131 & \text{from } \chi_c(1P), \\
1.7781466 & \text{from } \chi_b(1P).
\end{cases}$$  \hspace{1cm} (93)$$

The corresponding effective strong coupling constant being

$$\alpha(m_c) = \alpha_s(m_c) F^2 = \begin{cases} 
0.98082 & \text{using the } \chi_c(1P) \text{ result}, \\
1.33188 & \text{using the } \chi_b(1P) \text{ result},
\end{cases}$$  \hspace{1cm} (94)$$

and

$$\alpha(m_b) = \alpha_s(m_b) F^2 = \begin{cases} 
0.49302 & \text{using the } \chi_c(1P) \text{ result}, \\
0.66948 & \text{using the } \chi_b(1P) \text{ result}.
\end{cases}$$  \hspace{1cm} (95)$$

Note that one of the $\alpha(m_c)$ values is larger than one. This implies that there states whose coulombic energy (87) is imaginary, meaning that they are not stable states. Then, according to our description, the number of available charmonium states is lower than the number of bottomonium states.

In order to reproduce quantitatively the particles masses, one has to compute the contribution from the gluonic field. A value for the gluonic energy, $E_{\text{glue}}$, can be obtain using the $F$ values (93) and assuming that

$$M \left[ (1s_{1/2}, 2p_{1/2}) (0^+) \right] = E_{\text{glue}} + E_1 + E_{2 \ 1/2}.$$  \hspace{1cm} (96)$$

Then, using the values of table 1

$$E_{\text{glue}} = \begin{cases} 
2253.66 \text{ MeV} & \text{from } \chi_c(1P), \\
1729.72 \text{ MeV} & \text{from } \chi_b(1P).
\end{cases}$$  \hspace{1cm} (97)$$

The gluonic energy (97) provides a large fraction of the particle masses. We do not have a justification for the above values but it is surprising that the values reported in (97) are compatible with quenched lattice QCD estimates for the ground and first excited states of $J^P = 0^+$ glueballs masses [20]. In this paper we will not discuss a possible origin of $E_{\text{glue}}$.

We are now in position of discussing the theoretical meson spectrum and compare it with the experimental data. For bottomonium the lightest particles are $1s_{1/2}^2 (0^-)$ states. Their mass values being

$$M \left[ (b \bar{b}; 1s_{1/2}^2(0^-)) \right] = \begin{cases} 
8935.20 \text{ MeV} & \text{from } \chi_c(1P), \\
10692.85 \text{ MeV} & \text{from } \chi_b(1P).
\end{cases}$$  \hspace{1cm} (98)$$

$^2$The observation also applies to $\chi_b(2P)$ states.
For charmonium the spectrum associated to $\chi_c(1P)$ has $1s^2_{1/2}(0^-)$,

$$M \left[ \bar{c}c; 1s^2_{1/2}(0^-) \right] = 2721.46 \text{ MeV} \quad \text{from } \chi_c(1P),$$

(99)

has the lowest energy eigenstate. The stable lowest energy eigenstate associated
to the spectrum computed using $\chi_b(1P)$ data is a $2p_{3/2}$ configuration. The
lightest meson mass associated to this spectrum is

$$M \left[ \bar{c}c; 2p^2_{3/2}(0^-, 2^-) \right] = 3520.13 \text{ MeV} \quad \text{from } \chi_b(1P).$$

(100)

According to these results, the lightest quarkonium meson should be a $J^P = 0^-$
meson. This result agrees with the measured data.

For $J^P = 0^-$ particles our predictions are resumed in the following tables

| Charmonium $0^-$ Spectrum | $\chi_b(1P)$ |
|---------------------------|--------------|
| $1s^2_{1/2}$ | 2721.46 | $\eta_c(1S)$ | 8.7% |
| $1s_{1/2}; 2s_{1/2}$ | 3415.10 | $\eta_c(2S)$ | 4.98% |
| $1s_{1/2}; 3s_{1/2}$ | 3583.15 | $\eta_c(3S)$ | 0.30% |
| $2p^2_{1/2}$ | 4108.85 |

| Bottomonium $0^-$ Spectrum ($\chi_b(1P)$) |
|---------------------|
| $1s^2_{1/2}$ | 8935.20 | $\eta_b(1S)$ | 3.9% |
| $1s_{1/2}; 2s_{1/2}$ | 9859.90 |

The tables read as follows (from left to right): configuration, theoretical particle mass in MeV, closest observed particle, absolute error in the theoretical prediction of the particle mass. For charmonium the table includes configurations built from the two independent spectrum. From now on, each particle will appear only in the line referring to the closest mass prediction in the respective table. The tables shows that the difference between the theoretical predictions and the observed masses is below 10%. The largest error is for $1s^2_{1/2}$ configuration, where one expects to see a significant Lamb shift correction. Moreover, our description of mesons predicts more states than those observed in experiments. As we will see below, this comment applies to all particles quantum number.

For $J^P = 0^+$ particles our predictions are resumed in the following tables

| Charmonium $0^+$ Spectrum ($\chi_c(1P)$) |
|---------------------|
| $1s_{1/2}; 2p_{1/2}$ | 3415.10 | $\chi_0(1P)$ | 0.0% |
| $1s_{1/2}; 3p_{1/2}$ | 3583.15 |

| Bottomonium $0^+$ Spectrum ($\chi_b(1P)$) |
|---------------------|
| $1s_{1/2}; 2p_{1/2}$ | 9859.90 | $\chi_0(1P)$ | 0.0% |
| $1s_{1/2}; 3p_{1/2}$ | 10044.12 | $\chi_0(2P)$ | 1.8% |
| $2s_{1/2}; 2p_{1/2}$ | 10784.63 |

Note that $\chi_0(1P)$ were used as inputs to compute $E_{glue}$ and $F$. For $0^+$ states, only the prediction for $\chi_0(2P)$ can be compared with the observed data.
theoretical value for the mass of $\chi_b(2P)$ has an error which is below 2%. Naively, one would expect $s$ states to have larger corrections than $p$ states coming from higher order processes. In particular, the Lamb shift correction is negligible small for a $3p$ state, significant for $1s$ state, quite small for $2s$ and smaller for a $3s$ state. This relative importance of the Lamb shift correction at least points in the right direction to explain the errors listed in the above tables. One should not forget that there are corrections to this particle-independent description of mesons. The nature of these corrections is substantially different from the Lamb shift correction and they are expected to be larger when the overlap between quarks is larger; for example, when quark and antiquark are in the same orbital.

For $J^P = 1^-$ particles our predictions are resumed in the following tables.

| Charmonium $1^-$ Spectrum ($\chi_c(1P)$) |
|----------------------------------------|
| $1s_{1/2}; 2s_{1/2}$ | 3415.10 | $J/\psi$ | 10.3% |
| $1s_{1/2}; 3s_{1/2}$ | 3583.15 |
| $1s_{1/2}; 3d_{3/2}$ | 3617.49 |
| $2p_{1/2}; 2p_{3/2}$ | 4227.00 | $\psi(4040)$ | 4.6% |
| $2s_{1/2}; 3s_{1/2}$ | 4276.79 | $\psi(4160)$ | 2.8% |
| $2p_{1/2}; 3p_{1/2}$ | 4276.79 |
| $2s_{1/2}; 3d_{3/2}$ | 4276.79 | $\psi(4160)$ | 2.8% |
| $2p_{1/2}; 3p_{3/2}$ | 4311.14 |
| $2p_{3/2}; 3p_{1/2}$ | 4395.04 |
| $2p_{3/2}; 3p_{3/2}$ | 4429.39 | $\psi(4415)$ | 0.3% |
| $3p_{1/2}; 3p_{3/2}$ | 4451.06 |

| Charmonium $1^-$ Spectrum ($\chi_b(1P)$) |
|----------------------------------------|
| $2p_{3/2}; 2p_{3/2}$ | 3683.26 | $\psi(2S)$ | 0.07% |
| $3d_{3/2}; 3d_{5/2}$ | 3863.30 | $\psi(3770)$ | 2.5% |
| $n \geq 4$ state | |

| Bottomonium $1^-$ Spectrum ($\chi_b(1P)$) |
|----------------------------------------|
| $1s_{1/2}; 2s_{1/2}$ | 9859.90 | $\Upsilon(1S)$ | 4.2% |
| $1s_{1/2}; 3s_{1/2}$ | 10044.12 | $\Upsilon(2S)$ | 0.2% |
| $2s_{1/2}; 3d_{3/2}$ | 10056.87 | $\Upsilon(3S)$ | 2.9% |
| $2p_{1/2}; 3p_{3/2}$ | 10056.87 | $\Upsilon(4S)$ | 2.3% |
| $2s_{1/2}; 3s_{1/2}$ | 10968.85 | $\Upsilon(10860)$ | 1.0% |
| $2p_{1/2}; 3p_{1/2}$ | 10968.85 |
| $2s_{1/2}; 3d_{3/2}$ | 10981.61 |
| $2p_{1/2}; 3p_{3/2}$ | 10981.61 |
| $2p_{3/2}; 3p_{1/2}$ | 11011.60 |
| $2p_{3/2}; 3p_{3/2}$ | 11024.36 | $\Upsilon(11020)$ | 0.05% |
| $3p_{1/2}; 3p_{3/2}$ | 11165.83 |
| $3s_{1/2}; 3d_{3/2}$ | 11165.83 |

Apart $J/\psi$, the errors in the theoretical predictions are under 5%. For $\psi(2S)$, $\psi(4415)$, $\Upsilon(2S)$, $\Upsilon(10860)$ and $\Upsilon(11020)$ the agreement between theory and experiment is below 1%. Such an excellent match between theory and experiment
seems to suggest that our picture of hadronic matter can provide a good starting point for more precise heavy hadron computations. In what concerns \( J/\psi \), the error in the theoretical mass is the largest in the calculation discussed in this paper. Again, one should not forget that \( J/\psi \) is described by a configuration where one expects to see large corrections. Note that the corresponding state in the bottomonium spectrum \( \Upsilon(1S) \) has the largest error in the bottomonium spectrum. For \( J/\psi \) and \( \Upsilon(1S) \), the error in the theoretical value for their mass is roughly a factor of two larger than the next largest error.

For \( J^P = 1^+ \) particles our predictions are resumed in the following tables.

| Charmonium 1\(^+\) Spectrum \((\chi_c(1P))\) |
|-----------------------------------------------|
| 1\(s_{1/2}:2p_{1/2}\) | 3510.51 |
| 1\(s_{1/2}:2p_{3/2}\) | 3533.35 |
| \(\chi_c(1P)\) | 0.7% |
| 1\(s_{1/2}:3p_{1/2}\) | 3617.49 |

| Bottomonium 1\(^+\) Spectrum \((\chi_b(1P))\) |
|-----------------------------------------------|
| 1\(s_{1/2}:2p_{1/2}\) | 9859.90 |
| 1\(s_{1/2}:2p_{3/2}\) | 9902.65 |
| \(\chi_b(1P)\) | 0.1% |
| \(1s_{1/2}:3p_{1/2}\) | 10044.12 |
| \(1s_{1/2}:3p_{3/2}\) | 10056.87 |
| \(\chi_b(2P)\) | 1.9% |
| 2\(p_{1/2}:2p_{3/2}\) | 10827.38 |

Remember that \(\chi_1(1P)\) was used as input to compute \( F \). Similarly as what was observed previously, the only predict mass has an error of \( \sim 2\% \).

For \( J^P = 2^+ \) particles our predictions are resumed in the following tables.

| Charmonium 2\(^+\) Spectrum \((\chi_c(1P))\) |
|-----------------------------------------------|
| 1\(s_{1/2}:2p_{3/2}\) | 3533.35 |
| \(\chi_c(2P)\) | 0.7% |
| 1\(s_{1/2}:3p_{1/2}\) | 3617.49 |

| Bottomonium 2\(^+\) Spectrum \((\chi_b(1P))\) |
|-----------------------------------------------|
| 1\(s_{1/2}:2p_{3/2}\) | 9902.65 |
| \(\chi_b(1P)\) | 0.1% |
| 1\(s_{1/2}:3p_{3/2}\) | 10056.87 |
| \(\chi_b(2P)\) | 2.1% |
| 2\(p_{1/2}:2p_{3/2}\) | 10827.38 |

Remember that \(\chi_2(1P)\) states where used to compute \( F \). For the particle \(\chi_b(2P)\), the situation is similar to \(\chi_b(2P)\).

For \( J^P = 2^- \) particles our predictions are resumed in the following tables.

| Charmonium 2\(^-\) Spectrum \((\chi_c(1P))\) |
|-----------------------------------------------|
| 1\(s_{1/2}:3d_{3/2}\) | 3617.49 |
| 1\(s_{1/2}:3d_{5/2}\) | 3621.61 |
| 1\(p_{1/2}:2p_{3/2}\) | 4227.00 |

| Charmonium 2\(^-\) Spectrum \((\chi_b(1P))\) |
|-----------------------------------------------|
| 2\(p_{3/2}\) | 3520.13 |
| 2\(p_{3/2}:3p_{3/2}\) | 3683.26 |
| 3\(p_{3/2}\) | 3846.38 |
| \(\psi(3836)\) | 0.3% |
| 3\(d_{3/2}\) | 3846.38 |
| 3\(d_{5/2}\) | 3863.30 |
| Bottomonium $2^-$ Spectrum ($\chi_b(1P)$) |
|------------------------------------------|
| $1s_{1/2}; 3d_{3/2}$ | 10056.87 |
| $1s_{1/2}; 3d_{5/2}$ | 10060.14 |
| $2p_{1/2}; 2p_{3/2}$ | 10827.38 |

Unfortunately, only one $2^-$ was observed and we can not say much about our predictions. For $\psi(3836)$ the theoretical prediction matches the experimental mass with high accuracy. Note that for the configuration $3p_{3/2}$ the Lamb shift correction is expected to be negligible.

In the charmonium spectrum there is a particle whose quantum numbers are not known, $h_c(1P)$. Following the same reasoning as before, we can look for the closest theoretical mass value. For $h_c(1P)$ we find two configurations which are very good candidates to represent $h_c(1P)$, namely

$$[\chi_c(1P)] 1s_{1/2}^2; 2p_{3/2}^2 (1^+, 2^+) \quad M = 3533.35 \text{ MeV} - \text{Error} = 0.2\% , (101)$$

$$[\chi_b(1P)] 2p_{1/2}^2 (0^–, 2^-) \quad M = 3520.13 \text{ MeV} - \text{Error} = 0.2\% . (102)$$

In what concerns the charmonium and bottomonium spectrum, one can conclude that the particle-independent description is able to reproduce the particle masses with a 10% precision. Indeed, for almost all particles, the level of precision is below 5% error and for same particles, the level of precision is under 1%. This result is encouraging, specially if one notes that for these last states no large corrections, from higher order processes, are expected to take place. Moreover, the particle-independent picture relates the two spectrum and predicts a number of new particles that can be tested experimentally. Another interesting observation being that what we call $E_{\text{glue}}$, see (97) for values, is compatible with the quenched lattice QCD estimates for the ground state and first excited state $0^+$ glueballs [20].

The particle data book [18] reports a bottom-charm meson with a mass of $6.4 \pm 0.39 \pm 0.13$ GeV. There is not to much experimental information on the properties of this meson. For example, the quantum numbers of this particle were never measured. Assuming, that the bottom-charm meson is the lowest mass state, our picture of a meson gives the following prediction

$$6.71 \text{ GeV}, \quad J^P = 0^–, 1^- \text{ from } \chi_c(1P) \text{ spectrum} \quad (103)$$

$$6.23 \text{ GeV}, \quad J^P = 1^+, 2^+ \text{ from } \chi_b(1P) \text{ spectrum}. \quad (104)$$

The best agreement between theory and experiment happens for (104) state (error is 2.7%).

For baryons the particle-independent model also makes predictions. Unfortunately, there is no experimental information on baryons made up of only $c$ or $b$ quarks. Instead of giving a detailed spectrum, we simply quote the prediction for the lowest $bbb$ and $ccc$ states

$$M (bbb) = 12.54 \text{ GeV} \quad J^P = \frac{3^+}{2}, \quad (105)$$

$$M (ccc) = 2.96 \text{ GeV} \quad J^P = \frac{3^+}{2}. \quad (106)$$

Surprisingly, the charm prediction is a rather light state.
7 Results and Conclusions

In this paper we report on Minkowsky space-time solutions of classical pure SU(2) and SU(3) gauge theories in Landau gauge. The solutions were obtained after writing the gluon field in a particular way (15) and choosing a spherical like basis in color space. The two steps seem to be crucial to map SU(2) and SU(3) to abelian like theories, replacing the nonlinear classical field equations by linear equations. If the construction of the ansatz remembers the “abelian projection” technique, in our case, it is only after the choice of the spherical like basis that the “abelian nature” of the theory shows up. On the other hand, the ansatz transforms the Landau gauge condition, a linear gauge fixing condition, in a set of coupled equations.

Despite same appealing properties of the ansatz (15), we do not know if in general our gluon field provides a good dynamical description of the gluonic degrees of freedom. If so, does it means that the dynamics of classical gauge theories is, essentially, the dynamics of various uncoupled photonic fields? The idea is interesting but, at present, we cannot answer this question.

The classical configurations discussed are regular everywhere except at the origin and/or infinity. Moreover, the SU(2) and SU(3) solutions of the Euler-Lagrange equations are similar to classical electrodynamic fields. The difference is a longitudinal component in the gauge fields, which does not contributes neither to the energy nor the spin of the theory but couples to the fermionic fields. Like in classical electrodynamics, the solutions are exponential functions and are identified by a light-type four-vector \( k_\mu, k^2 = 0 \). The longitudinal component of the gauge field is given by a combination of pure complex exponential functions and real exponential functions. These last components make the gluonic field to diverge for large space and time values.

In what concerns the coupling of the classical configurations to fermionic fields, the quark-gluon interaction is equivalent to three independent electromagnetic-type couplings. The solution of the Dirac equation for quarks in the background of the gluon fields computed in section 4, suggests that physical particles are the usual quark model mesons and baryons. The hadronic wave function is given by the slater determinant of single-quark wave functions for electromagnetic problems. In this picture, an hadron is described by a particle-independent model like in Atomic and Nuclear Physics.

The classical gluonic configurations of section 4 include, as particular solution, the Coulomb potential. The spectrum of the Coulomb problem is well known. If one assumes a particle-independent model, for this particular solution it is possible to compute the associated particle spectrum. For heavy quarkonium our calculation shows that such a picture reproduces the full quarkonium spectrum with an error of 10% or bellow. Indeed, typical errors in mass prediction are between 2% and 5% and for same states the error in the theoretical mass is bellow 1%. Note that, if the Coulomb gluonic configurations are the relevant configurations for heavy quarkonium, such an approach is similar to the quenched approximation used in lattice QCD. One should not forget that such a picture of what is an hadron is not the full story but, maybe, a good starting point for a more precise and detailed calculation.

The result for the charmonium and bottomonium spectrum seems to suggest that the classical gluonic configurations computed here can tell us something about hadronic properties. However, before giving a clear answer to such a
question a number of other questions have to be answered: can the coulomb picture described in this paper reproduce the meson properties (masses, decay widths, etc.) to a high precision? What is the meaning of $E_{\text{glue}}$? Should one think on mesons as a $q\bar{q} \otimes (0^+ \text{ glueball})$? What about light mesons and baryons?

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**Appendix**

The color space has dimension eight. For an eight dimension space, the spherical basis requires seven angles. Considering the following seven functions $\theta_i(x)$, $i = 1...7$ we define the basis as follows

\[
\vec{e}_1 = \begin{pmatrix}
\sin \theta_1 \cos \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \sin \theta_7 \\
\sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \sin \theta_7 \\
\cos \theta_1 \sin \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \sin \theta_7 \\
\cos \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \sin \theta_7 \\
\cos \theta_4 \sin \theta_5 \sin \theta_6 \sin \theta_7 \\
\cos \theta_5 \sin \theta_6 \sin \theta_7 \\
\cos \theta_6 \sin \theta_7 \\
\cos \theta_7
\end{pmatrix}
\] (107)

\[
\vec{e}_2 = \begin{pmatrix}
\cos \theta_1 \\
\cos \theta_1 \\
\cos \theta_1 \\
\cos \theta_1 \\
\cos \theta_1 \\
\cos \theta_1 \\
\cos \theta_1 \\
\cos \theta_1
\end{pmatrix}
\begin{pmatrix}
\cos \theta_2 \\
\sin \theta_2 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\] (108)

\[
\vec{e}_3 = \begin{pmatrix}
\sin \theta_2 \\
\cos \theta_2 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\] (109)

\[
\vec{e}_4 = \begin{pmatrix}
\sin \theta_1 \cos \theta_2 \cos \theta_3 \\
\sin \theta_1 \sin \theta_2 \cos \theta_3 \\
\cos \theta_1 \cos \theta_3 \\
\cos \theta_1 \cos \theta_3 \\
\cos \theta_1 \cos \theta_3 \\
\cos \theta_1 \cos \theta_3 \\
\cos \theta_1 \cos \theta_3 \\
\cos \theta_1 \cos \theta_3
\end{pmatrix}
\] (110)
\[ \vec{e}_5 = \begin{pmatrix} \sin \theta_1 \cos \theta_2 \sin \theta_3 \cos \theta_4 \\ \sin \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_4 \\ \cos \theta_1 \sin \theta_3 \cos \theta_4 \\ \cos \theta_3 \cos \theta_4 \\ -\sin \theta_4 \\ 0 \\ 0 \end{pmatrix} \] (111)

\[ \vec{e}_6 = \begin{pmatrix} \sin \theta_1 \cos \theta_2 \sin \theta_3 \sin \theta_4 \cos \theta_5 \\ \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \cos \theta_5 \\ \cos \theta_1 \sin \theta_3 \sin \theta_4 \cos \theta_5 \\ \cos \theta_3 \sin \theta_4 \cos \theta_5 \\ \cos \theta_4 \cos \theta_5 \\ -\sin \theta_5 \\ 0 \\ 0 \end{pmatrix} \] (112)

\[ \vec{e}_7 = \begin{pmatrix} \sin \theta_1 \cos \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \cos \theta_6 \\ \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \cos \theta_6 \\ \cos \theta_1 \sin \theta_3 \sin \theta_4 \sin \theta_5 \cos \theta_6 \\ \cos \theta_3 \sin \theta_4 \sin \theta_5 \cos \theta_6 \\ \cos \theta_4 \sin \theta_5 \cos \theta_6 \\ \cos \theta_5 \cos \theta_6 \\ -\sin \theta_6 \\ 0 \end{pmatrix} \] (113)

\[ \vec{e}_8 = \begin{pmatrix} \sin \theta_1 \cos \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \cos \theta_7 \\ \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \cos \theta_7 \\ \cos \theta_1 \sin \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \cos \theta_7 \\ \cos \theta_3 \sin \theta_4 \sin \theta_5 \sin \theta_6 \cos \theta_7 \\ \cos \theta_4 \sin \theta_5 \sin \theta_6 \cos \theta_7 \\ \cos \theta_5 \sin \theta_6 \cos \theta_7 \\ \cos \theta_6 \cos \theta_7 \\ -\sin \theta_7 \end{pmatrix} \] (114)

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