Hydrodynamic scaling from the dynamics of relativistic quantum field theory

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Hydrodynamic behavior is a general feature of interacting systems with many degrees of freedom constrained by conservation laws. To date hydrodynamic scaling in relativistic quantum systems has been observed in many high energy settings, from cosmic ray detections to accelerators, with large particle multiplicity final states. Here we show first evidence for the emergence of hydrodynamic scaling in the dynamics of a relativistic quantum field theory. We consider a simple scalar $\lambda\phi^4$ model in 1+1 dimensions in the Hartree approximation and study the dynamics of two colliding kinks at relativistic speeds as well as the decay of a localized high energy density region. The evolution of the energy-momentum tensor determines the dynamical local equation of state and allows the measurement of the speed of sound. Hydrodynamic scaling emerges at high local energy densities.

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Hydrodynamics has been used since the work of Landau [1] to describe the properties of high multiplicity final states in high energy particle collisions. While Landau’s motivations dealt with high energy cosmic rays there has been since ample evidence from accelerator experiments that hydrodynamic scaling (the longitudinal velocity $v_L = x/t$), which implies flat rapidity distributions, is the correct approximate kinematical constraint for the dynamics of high energy particle collisions.

These findings imply that hydrodynamic scaling must emerge from the dynamics of quantum field theories, if the latter are to be correct descriptions of collective behavior in particle physics models. While the applicability of quantum field theory in these regimes is not in doubt, it has not been demonstrated that hydrodynamic scaling, which implies that the energy density isosurfaces are surfaces of constant $\tau^2 = (t^2 - x^2)$, is achieved at sufficiently high center of mass collision energies.

Present and future experimental prospects for the study of hydrodynamic scaling in high energy experiments are tremendous. The relativistic heavy ion collider (RHIC) is presently producing the highest energy, highest multiplicity hadronic final states ever accessible in a controlled environment [2]. The Large Hadron Collider (LHC) will later produce even more spectacular events. The detailed understanding of hydrodynamic flows in these experiments constitutes the most promising way for the determination of the thermodynamic properties of nuclear matter at high temperatures [3], viz. its equation of state, and the nature of the confinement and chiral symmetry breaking transition. The connection between hydrodynamic scaling and flat rapidity distributions in the context of Landau’s hydrodynamical model was first discussed in [4]. In the context of “boost invariance”, the same scaling law was developed in detail by Bjorken [5].

Direct field theoretical methods, although still in their adolescence, offer much promise for the understanding of hydrodynamic scaling and the limits of its applicability. Moreover they make accessible regimes where particle coherence is important, which escape Boltzmann particle methods. In this letter we show, for the first time, how hydrodynamic scaling emerges from the dynamics of a simple 1+1 dimensional scalar field theory in the Hartree approximation. Our results allow us to map the equation of state as a function of space and time, and, under well known assumptions, determine the speed of sound $c_0$.

To exhibit the ubiquity of hydrodynamic scaling we study two different situations: one in which a hot region is formed in the wake of the collision of two leading particles (kinks) at relativistic velocities, and another simpler one where we construct a local energy overdensity which is allowed to relax under its own self-consistent evolution. To be definite we will be concerned with a scalar $\lambda\phi^4$ quantum field theory with Lagrangian density

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \mu_\Lambda^2 \phi^2 - \frac{1}{4} \lambda \phi^4.$$  \hspace{1cm} (1)

The well known Hartree approximation is the simplest nontrivial truncation of the coupled equations for the field’s correlation functions, which assumes that all connected correlation functions beyond the second are negligible [6]. This leads to a dynamical equation for the mean field $\phi \equiv \langle \phi \rangle$ and the connected two-point function. We write the quantum field $\phi = \varphi + \hat{\varphi}$, where $\hat{\varphi}$ are fluctuations, $\langle \hat{\varphi} \rangle = 0$. The equations of motion for $\varphi$ and the 2-point function $G(x,y) = \langle \hat{\varphi}(x) \hat{\varphi}(y) \rangle$ then are

$$[\Box - \mu_\Lambda^2 + \lambda \varphi^2(x) + 3\lambda G(x,x)] \varphi(x) = 0,$$

$$[\Box - \mu_\Lambda^2 + 3\lambda (\varphi^2(x) + G(x,x))] G(x,y) = 0.$$  \hspace{1cm} (2)

To solve the equation for the Green’s functions we will rely on a complete orthogonal mode basis $\psi_k(x)$
\[ \dot{\psi}(x) = \sum_k \left[ a_k^\dagger \psi_k^*(x) + a_k \psi_k(x) \right], \]  

where \( a_k^\dagger, a_k \) are creation and annihilation operators obeying canonical commutation relations. In terms of the mode fields \( \psi_k \), at zero temperature

\[ G(x, y) = \sum_k \psi_k(x) \psi_k^*(y). \]  

The effective mass squared of the propagator, \( \chi(x, t) \) must be finite which tells us how to choose the bare mass \( \mu_0^2 \). In 1+1 dimensions the self-energy has only a logarithmic divergence, which is eliminated by a simple mass renormalization. We choose

\[ -\mu_0^2 = m^2 - 3\lambda \int \frac{dk}{2\pi} \frac{1}{2\sqrt{k^2 + \chi}} = m^2 - 3\lambda G_0, \]  

leading to the existence of two homogeneous stable phases, corresponding to \( \chi_0 = m^2, \phi = 0 \) and \( \chi_0 = 2m^2, \phi^2 = m^2/\lambda \), i.e. a symmetric and broken symmetry phase, respectively. The renormalized equations are

\[ \left[ \Box + \chi(x) - 2\lambda \phi^2(x) \right] \varphi(x) = 0, \]
\[ \left[ \Box + \chi(x) \right] \psi_k(x, t) = 0 \quad \forall k, \]
\[ \chi(x) = m^2 + 3\lambda \phi^2(x) + 3\lambda G_R(x, x), \]

where the renormalized \( G_R(x, x) = G(x, x) - G_0 \). The challenge posed by Eqs. (6-7) in spatially inhomogeneous cases is that we need to solve many partial differential equations simultaneously. In a spatial lattice of linear size \( L \), the computational effort is of order \( L^2D \), per time step, where \( D \) is the number of space dimensions. Because of this demanding scaling we focus on \( D = 1 \).

We consider two classes of initial conditions: 1) Colliding kinks, in the broken phase where, at \( t = 0 \) we have

\[ \varphi(x, t = 0) = \frac{m}{\sqrt{\lambda}} \varphi_{\text{kink}}(x - x_0) \varphi_{\text{kink}}(-x - x_0), \]
\[ \varphi_{\text{kink}}(x) = \tanh \left( \frac{m x}{\sqrt{2}} \right), \]

with the kinks initially boosted towards each other at velocity \( v \), and 2) a Gaussian shape in the unbroken phase

\[ \varphi(x, 0) = \varphi_0 \exp \left[ -\frac{x^2}{2A} \right], \quad \partial_t \varphi(x, 0) = 0. \]

In both cases we adopt at \( t = 0 \) a Fourier plane-wave mode basis, characteristic of the unperturbed vacuum

\[ \psi_k(x, t) = \sqrt{\frac{\hbar}{2\omega_k}} e^{i(kx + \omega_k t)}, \quad \omega_k = \sqrt{k^2 + \chi_0}. \]

The orthonormality of the basis is preserved by the evolution, Eqs. (8-9).

To study the hydrodynamic behavior we need to specify the operator energy momentum tensor \( T^{\mu\nu} \)

\[ T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} L. \]

Its expectation value, in terms of \( \varphi \) and \( \psi_k \), is

\[ \langle T_{00} \rangle = \frac{1}{2} \langle \varphi^2 \rangle + \frac{1}{2} \langle \varphi^2 \rangle + \frac{1}{2} \sum_k [\psi_k^* \psi_k^2 + \psi_k^2 \psi_k^2] + V_H, \]
\[ \langle T_{11} \rangle = \frac{1}{2} \langle \varphi^2 \rangle + \frac{1}{2} \langle \varphi^2 \rangle + \frac{1}{2} \sum_k [\psi_k^* \psi_k^2 + \psi_k^2 \psi_k^2] - V_H, \]
\[ V_H = \frac{\lambda^2}{2\lambda} - \frac{\phi^2}{2}, \]
\[ \langle T_{01} \rangle = \langle T_{10} \rangle = \varphi \varphi_x + \frac{1}{2} \sum_k [\psi_k \psi_k^* + \psi_k^* \psi_k^*], \]

where all arguments are at \( x \). The subscripts \( x, t \) are shorthand for spatial and time derivatives, respectively. \( T_{00} \) and \( T_{11} \) contain two different types of ultraviolet divergent contributions. The first arises from the 1-loop integral \( G(x, x) \). This logarithmic divergence is removed by mass renormalization Eq. (5). The second divergence appears in the kinetic and spatial derivative fluctuation terms. This divergence is purely quadratic, and is already present in the free field theory in the vacuum sector. By comparing the mode sum with a covariant dimensional regularization scheme for the free field theory, one deduces that the correct subtraction in the mode sum scheme is given by

\[ \frac{1}{2} \sum_k [\psi_k^* \psi_k^2 + \psi_k^2 \psi_k^2] - \frac{1}{2} \sum_k [\psi_k^* \psi_k^2 + \psi_k^2 \psi_k^2 - |k|]. \]

In practice we discretize the fields \( \varphi(x) \) and the set \{ \psi_j \} on a spatial lattice with size \( N \) and spacing \( dx \) and use periodic boundary conditions in space. We choose \( dx = 0.125, N = 1024, \) and \( m^2 = 1, \lambda = 1, \hbar = 1 \). The dynamical equations are solved using a symplectic fourth order integrator (with a timestep \( dt = 0.025 \)). With these choices, in the finite volume \( L = N dx \), the momentum \( k \) takes a finite number of discrete values \( k_n = \frac{2\pi n}{L} \), with \( n = \{-N, \ldots, N - 1\} \) and continuum k-integrals become sums

\[ \int \frac{dk}{(2\pi)} \rightarrow L^{-1} \sum_n. \]

The frequency \( \omega_k \) now satisfies a lattice form of the dispersion relation, with

\[ \omega_k^2 = \hat{k}^2 + \chi^2, \quad \hat{k}^2 = \frac{2}{dx^2} (1 - \cos dx k_n). \]

These forms also require that the renormalization of \( T^{\mu\nu} \) be achieved using appropriate lattice choices. In particular we adopt \( |k_n| = \sqrt{k^2} \) in \( (4) \).

We are now ready to address the hydrodynamics of our field theory. Landau’s simplifying assumption, that pervades hydrodynamic simulations of multi-particle flows in high energy experiments, is that they behave collectively as a perfect fluid, corresponding to

\[ \langle T_{\mu\nu} \rangle = (\varepsilon + p) u^\mu u^\nu - g^{\mu\nu} p, \quad \partial^\mu \langle T_{\mu\nu} \rangle = 0, \]

where \( u^\mu = \gamma(1, v), (\gamma = 1/\sqrt{1 - v^2}) \) is the collective fluid velocity and \( \varepsilon \) and \( p \) are the comoving energy and
pressure densities. The latter are the eigenvalues of the energy momentum tensor, and can be obtained from the invariance of its trace and determinant $\varepsilon - p = T^\mu_\mu$, $\varepsilon = \text{Det}[T]$. The fluid velocity can be obtained from the form

$$T^{01} = (\varepsilon + p) \frac{v}{1 - v^2}. \quad (17)$$

A perfect fluid is the limiting hydrodynamic behavior of a collisionless plasma, where transport coefficients, such as viscosities are vanishing. In a mean-field approximation explicit collisions are neglected, so that the form (17) is compatible with our dynamical approximation.

Eq. (18) implies that energy density isosurfaces lie on hyperboloids $t^2 - x^2 = \text{const}$, a property that we can easily check in our results, see Figs. 1 and 3 corresponding to initial conditions 1) and 2), respectively. Fig. 2 shows the pressure in space-time for the same situation as in Fig. 1.

![FIG. 1. Contours of equal energy density in space (horizontal) and time (vertical), near the collision point of two kinks (at the origin of the plot), initially boosted towards each other at $v = 0.8$. The collision is symmetric under spatial reflection. We show the region after and to the right of the collision point. Red denotes the highest energy density, navy blue the lowest. The energy isosurfaces show signs of hydrodynamic scaling, following approximate hyperboloids, which are distorted because of the presence of the emerging kinks.](image1)

The attraction of scaling lies in the fact that the relation $x = \pm vt$ allows for significant simplifications of the hydrodynamic equations (14), which can then be expressed in terms of a single variable and thus become ordinary differential equations. These can then be solved analytically (4), generating predictions for the spatio-temporal behavior of hydrodynamic quantities such as energy density $\varepsilon$, temperature or entropy. For example, for equations of state where $dp/d\varepsilon = c_0^2$, where $c_0$ is the (constant) speed of sound, one easily derives in 1D (4)

$$\varepsilon(x,t)/\varepsilon_0 = (\tau/x_0)^{(1+c_0^2)}; \quad \tau = \sqrt{t^2 - x^2}, \quad (18)$$

where $\varepsilon_0, x_0$ are integration constants. The equation of state $p = c_0^2 \varepsilon$ can be obtained using simple assumptions about the (hadronic) excitation spectrum (4).

![FIG. 2. Pressure isosurfaces in space-time for the situation shown in Fig. 1. Asymptotically far from the kink trajectories the equation of state mimics that of a gas with $p = c_0^2 \varepsilon$. Close to the kinks the pressure gradients indicate space-time regions where strong energy flows are imminent, such as the area around the kink collision point, at the origin of the plot.](image2)

The red region carries away most of the energy in the form of a wave-packet traveling close to the speed of light. The energy isosurfaces follow exquisite hyperboloids, characteristic of hydrodynamic scaling.

![FIG. 3. $\varepsilon$ isosurfaces in space-time, for the decay of an initial Gaussian shape (10), with $A = 1$ and $\varphi_0 = 5$. The energy contained in the initial hot region is about two orders of magnitude larger than that deposited by the kink collision, shown in Figs. 1-2. The red region carries away most of the energy in the form of a wave-packet traveling close to the speed of light. The energy isosurfaces follow exquisite hyperboloids, characteristic of hydrodynamic scaling.](image3)
Besides the shape of energy isosurfaces hydrodynamic scaling also predicts that the single particle distribution functions are flat in the particle rapidity variable \( y = \frac{1}{2} \ln [(E + p_\parallel)/(E - p_\parallel)] \), where \( E \) and \( p_\parallel \) refer to the energy and momentum in the direction of the collision of an outgoing particle. This result is seen experimentally in the central rapidity region \( \Delta y \), and can be interpreted as arising either from the approximate boost invariance of two highly Lorentz contracted colliding nuclei at high energies \( \Delta y \), or from the fact that, in the center of mass frame, the Lorentz contracted source for particle production has a negligible longitudinal size, when compared to the asymptotic particle’s (pion) Compton wavelength \( \Delta y \). At sufficiently high deposited energies the two approaches lead to the same results.

Scaling solutions do not preserve global energy conservation. Thus energy isosurfaces must eventually deviate from the scaling hyperbola and join neighboring isosurfaces, creating characteristic horn like shapes. This behavior, which can be extracted directly from hydrodynamic equations, is also observed in the quantum field solutions, see Figs. 1 and 3.

Eq. (13) also allows us a measurement of the speed of sound \( c_0 \). Fig. 4 shows the decay in time of the energy density of the Gaussian mean-field profile already discussed in the context of Fig. 3. The resulting fit suggests a value of \( 1 \geq c_0 \geq 0.77 \), compatible with an ultrarelativistic equation of state \( c_0 = 1 \) in 1D.

In conclusion we have demonstrated for the first time, in a variety of settings, that hydrodynamic scaling emerges from the dynamics of quantum field theory at sufficiently high energy densities. We analyzed situations both with leading particles, which constitute the asymptotic states both before and after the collision, and following the evolution of a simple local energy overdensity. The extension of this type of calculation to 3D, where hydrodynamics is richer, and to include scattering, necessary for the description of real fluids, remain necessary steps to make real time studies of quantum fields predictive experimentally in the context of heavy ion collisions. Nevertheless studying the real time dynamics of quantum field theories demonstrates the applicability of models of particle physics in the largely unexplored limits of very large time and spatial scales, where they acquire fascinating macroscopic and non-perturbative properties.

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