Analytic $SU(N)$ Skyrmions at finite Baryon density

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We construct analytic (3+1)-dimensional Skyrmions living at finite Baryon density in the $SU(N)$ Skyrme model that are not trivial embeddings of $SU(2)$ into $SU(N)$. We used Euler angles decomposition for arbitrary $N$ and the generalized hedgehog Ansatz at finite Baryon density. The Skyrmions of high topological charge that we find represent smooth Baryonic layers whose properties can be computed explicitly. In particular, we determine the energy to Baryon charge ratio for any $N$ showing the smoothness of the large $N$ limit. The closeness to the BPS bound of these configurations can also be analyzed. The energy density profiles of these finite density Skyrmions have lasagna-like shape in agreement with recent experimental findings. The shear modulus can be precisely estimated as well and our analytical result is close to recent numerical studies in the literature.

Keywords: nuclear pasta, large $N$, skyrmions, multi-solitons

I. INTRODUCTION

The characterization of the phase diagram of the low energy limit of QCD at finite Baryon density and low temperatures has motivated intense research in the last two decades, see [1] and references therein. Analytic models are scarce and new exact results are hard to obtain. A well known example is the (3+1)-dimensional Nambu-Jona-Lasinio (NJL) model that shares some of the analytical difficulties of the low energy limit of QCD (see [2] for a review). Together with the uselessness of perturbation theory at low energy, this means that the complicated phase diagram of low energy QCD cannot be easily analyzed with the available analytic techniques (see [3–5] and references therein).

A remarkable feature of low energy QCD at finite Baryon density is that at low temperature very complex structures appear. When the Baryon density is increased, a phase that is commonly defined as nuclear pasta appears. In [6–13], the presence of “baryonic layers” was disclosed, which will be the main focus of the present paper. Such a name arises from the fact that most of the baryonic charge and energy density is concentrated within lasagna-shaped regions in three dimensions$^1$. Many physical properties of these configurations are currently under investigation, such as the elasticity of nuclear pasta and their transport properties [10–13]. The high topological charge of nuclear pasta makes it hard to study analytically.

As powerful numerical techniques are available to analyze these configurations (see, for instance, [10–13] and references therein), why should one insist so much in finding analytic solutions? There are many reasons to strive for analytic solutions even when numerical techniques are available. Firstly, it could be enough to remind all the fundamental concepts that we have understood thanks to the availability of the Kerr solutions in General Relativity and of the non-Abelian monopoles and instantons in Yang-Mills-Higgs theory. Secondly, as in the present case, analytic solutions can disclose relevant physical properties of very complex structures which are difficult to analyze even numerically.

Until recently, these types of non-homogeneous condensates in the low energy limit of QCD in (3+1)-dimensions could not be properly understood analytically. A further problem is that, computationally, the large $N_f$ and large $N_c$ limits must be addressed carefully [14, 15]. One of the goals of the present paper is to shed light on the large $N_f$ behavior of these complex structures.

$^1$ Nuclear spaghetti and nuclear gnocchi phases are also known to appear: see the references quoted above.
A simplified version of the low energy limit of QCD that encodes many relevant features is the (1+1)-dimensional version of the NJL model, also known as chiral Gross-Neveu model [16–19]. Such a model possesses a crystalline phase at low temperature and finite Baryon density [20–27]. These results suggest that ordered structures must also appear in the low energy limit of QCD. At leading order in the ’t Hooft expansions [28–30], the low energy limit of QCD is described by the Skyrme theory [31] (see [32, 33] for reviews). Despite the Bosonic nature of the Skyrmion described by the Skyrme theory [31] (see [30, 34–37]).

Here, we will analyze the appearance of complex structures at finite Baryon density in the SU(N) Skyrme model in (3+1)-dimensions. We will focus on the analytic computations of relevant physical properties, such as the energy density, the energy per Baryon and the shear modulus of nuclear-lasagna like structures, such as the energy density, the energy per Baryon and its large N behavior. There-

while the range of coordinates is

\[ 0 \leq r \leq 2\pi , \quad 0 \leq \gamma \leq 2\pi , \quad 0 \leq \varphi \leq 2\pi . \]  

with the caveat that, despite the chosen values, they are not periodic! The parameters \( L_r, L_\gamma, \) and \( L_\varphi \) represent the size of the box within which the Skyrmion is confined.

A. Quantities of high physical interest.

Firstly, the main goal of the paper is to compute the energy per Baryon and its large N behavior. Therefore, only solutions with non-vanishing Baryon charge have been considered. The usual definition of Baryon charge in the Skyrme model (see [30, 31, 35, 36]) is

\[ W = B = \frac{1}{24\pi^2} \int_{\{t=\text{const}\}} \rho_B , \]  

so that a necessary condition in order to have non-trivial topological charge is

\[ \rho_B \neq 0 . \]  

From the geometrical point of view, the above condition can be interpreted as saying that the Skyrmion “fills a three-dimensional spatial volume”, at least locally. On the other hand, such a condition is not sufficient in general. One also has to require that the spatial integral of \( \rho_B \) be a non-vanishing integer:

\[ \frac{1}{24\pi^2} \int_{\{t=\text{const}\}} \rho_B \in \mathbb{Z} . \]  

Usually, this second requirement allows to fix some of the parameters and integration constants of the Ansatz, as we will see in the following. However, there are more global conditions to be satisfied, as it will be explained below. Hence, in the following we will only consider solutions satisfying both the condition in Eq. (II.7) and the one in Eq. (II.8).

Secondly, the energy density (the 0–0 component of the energy-momentum tensor) reads

\[ T_{00} = -\frac{K}{2} \text{tr} \left[ R_0 R_0 - \frac{1}{2} g_{00} R^\alpha R_\alpha \
+ \frac{\lambda}{4} \left( g^{\alpha\beta} F_{0\alpha} F_{0\beta} - \frac{g_{00}}{4} F_{\sigma\rho} F^{\sigma\rho} \right) \right] , \]  

2 Pioneering works on the Skyrmion model at finite density are [38–42] and references therein.

3 We remind the reader that the \( N \) of the SU(N) of the Skyrme model corresponds to \( N_f \).
where $F_{\mu\nu} = [R_{\mu}, R_{\nu}]$. Thus, the total energy $E$ of the Skyrmion is the spatial integral of the above quantity.

$$E = \int_{t=\text{const}} \sqrt{-g} T_{00}.$$  

We define a Skyrmion $U$ to be static if its energy density defined above is static. In other words, a Skyrmion is static, if it corresponds to a static distribution of energy density. It is worth to note that this definition is more general than the naive definition of a static Skyrmion as a static $SU(N)$-valued configuration $U$ which does not depend on time. In particular an elegant approach to avoid Derrick’s famous no-go theorem on the existence of solitons corresponds to search for a time-periodic Ansatz such that the energy density of the configuration is still static, as it happens for boson stars [55] (in the simpler case of $U(1)$-charged scalar field: see [56] and references therein). The Ansatz to be defined in the next sections will have exactly this property. Moreover, unlike what happens for the usual Bosons star Ansatz for $U(1)$-charged scalar fields, the present Ansatz for $SU(N)$-valued scalar field also possesses a non-trivial topological charge. Thus, we are interested in solutions in which the energy density has non-trivial local maxima, which could be identified with the position of the Skyrmions.

Given a solution of $SU(N)$ with Baryonic charge $B$ and energy $E$ living in the metric (II.3) we have already mentioned, it is very interesting to analyze the following quantity (which is nothing but the energy per Baryon of the configuration $g(N,a)$)

$$\frac{E}{B} \equiv \frac{E}{N} g(N,a), \quad \text{(II.10)}$$

where $a$ is any set of integration constants which characterizes the given solution. It is especially interesting to understand the behavior of $g(N,a)$ defined above when $N$ is large (the ’t Hooft limit). Here and in the following we will call the function $g(N,a)$ the “$g$-factor”. The very deep question is whether or not, in the given family of solutions one is considering, one can define

$$g^*(a) = \lim_{N \to \infty} g(N,a) \quad \text{(II.11)}$$

and if this limit is well defined. In particular, one would like to know whether or not “the closeness to the BPS bound” improves when $N$ is large. Indeed, it is worth to remind that in the $SU(2)$ case all the known solutions with non-vanishing topological charge exceed the bound by at least the 20%. Hence, one would like to know whether, in the ’t Hooft limit, the “closeness of Skyrmions to the BPS bound” is finite or whether it grows without bound. This issue is deeply related with the so-called Veneziano limit [29], which is a variant of the ’t Hooft limit in which also the flavour number $N_f$ goes to infinity in such a way that $N_c/N_f$ stays finite. The Veneziano limit allows to take into account the effects of quarks while keeping the advantages of the ’t Hooft topological expansion. Since, to arrive at the Skyrme model as an effective low energy limit of QCD, $N_c$ must be already large, the large $N$ limit which we are considering here (in which $N$ is the one of the $SU(N)$ Skyrme model), can be considered as a sort of Veneziano limit applied to the Skyrme model itself. The fact that such a limit is smooth is a very non-trivial result which would be very difficult to prove directly on the QCD Lagrangian.

The above discussion clearly shows that in order to declare a solution of the Skyrme field equations as ”physically interesting” two criteria must be satisfied:

1) The topological charge of the solution must be non-vanishing

2) The energy density $T_{00}$ as function of the coordinates must have an interesting pattern.

### III. LOCAL SOLUTIONS

Using the Euler angles for $SU(N)$ determined in [43, 44] together with the Ansatz for non-spherical Skyrmions living at finite Baryon density in [46–54], one arrives at the following Ansatz for the $SU(N)$ Skyrme:

$$U[t, r, \varphi, \gamma] = e^{i k \sqrt{m} k} e^{i m k}, \quad \Phi = \frac{t}{L^2} - \varphi \quad \text{(III.1)}$$

with a suitable choice of $k$ in $su(N)$ and $h(r)$ in the Cartan subalgebra $H$ to be specified below, $m$ a non-vanishing integer number, and where we recall that the metric is given by (II.3). When necessary to expand with respect to the basis of $su(N)$, we will also write

$$h(r) = y_1(r) J_1 + \ldots + y_{N-1}(r) J_{N-1}, \quad \text{(III.3)}$$

with (see app. A)

$$J_k = i(E_k,k - E_{k+1,k+1}) \quad k = 1, \ldots, N-1. \quad \text{(III.4)}$$

In general we will use the simplifying notations

$$h' = \frac{d}{dr} h(r), \quad h'' = \frac{d^2}{dr^2} h(r). \quad \text{(III.5)}$$
As for $k$, for $c_j$ arbitrary complex numbers, forming the components of the vector $\xi \in \mathbb{C}^{N-1}$, we choose
\[
k \equiv k_\xi = \sum_{j=1}^{N-1} (c_j \lambda_j - c_j^* \lambda_j^\dagger), \quad (III.6)
\]

$\lambda_j \equiv \lambda_\alpha_j$ being the eigenmatrices of the simple roots (App.A). We get

**Proposition 1.** From the Ansatz (III.1), (III.2), (II.3), the equations of motion reduce to
\[
h'' = \frac{\lambda m^2}{4L_1^2} (\{k, [k, h'']\} - [k, [h', [h', k]]]), \quad (III.7)
\]

\[
\sum_{j<k} (\alpha_j (h')^2 - \alpha_k (h')^2 - i (\alpha_j (h'') - \alpha_k (h''))) c_j c_k [\lambda_j, \lambda_k] \quad - h.c. = 0, \quad (III.9)
\]

where $h.c.$ stays for Hermitian conjugate, and $\alpha_j$ are a suitable choice of simple roots of $SU(N)$, defined in App. A1. Indeed, using (B.3) and (B.5), we can rewrite (III.7) as
\[
h'' = \frac{\lambda m^2}{4L_1^2} \left\{ \sum_{j<k} [i(\alpha_j (h'') - \alpha_k (h'')) - (\alpha_j (h')^2 - \alpha_k (h')^2)] c_j c_k [\lambda_j, \lambda_k] - h.c. - 2 \sum_{j=1}^{N-1} \alpha_j (h'') |c_j|^2 J_j \right\}. \quad (III.10)
\]

Now we use general properties of simple roots. Since $\lambda_j$ are eigenmatrices relative to simple roots, it happens that or $[\lambda_j, \lambda_k] = 0$ or it is an eigenmatrix relative to a positive root.\(^4\) Similar considerations follow for $\lambda_j^\dagger$ w.r.t. negative roots. It follows that none of these terms can lie in $H$, so that projecting equation (III.10) on $H$ we get ($h''$ belongs in $H$ by definition) we get (III.8), while projecting on the complement we get (III.9). These equations could be expressed even more explicitly in components, by exploiting (III.3) and using
\[
\text{where the prime indicates derivation w.r.t. } r.
\]

The proof is given in appendix B. Exploiting (III.3) and (III.6) we can further simplify the equations of motion, which can be put in the following form.
\[
h'' + \frac{\lambda m^2}{2L_1^2} \sum_{j=1}^{N-1} \alpha_j (h'') |c_j|^2 J_j = 0, \quad (III.8)
\]

\[
\sum_{j=1}^{N-2} (\alpha_j (h')^2 - \alpha_{j+1} (h')^2 - i (\alpha_j (h'') - \alpha_{j+1} (h''))) c_j c_{j+1} E_{j,j+2} - h.c. = 0. \quad (III.11)
\]

\(^4\) That is a linear combination of simple roots with non negative integer coefficients

We will assume $\xi$ to be generic, with this meaning that all the $c_j$ are non zero. Since $E_{j,j+2}$, including their
conjugates, are all linearly independent, this gives
\[ \alpha_j(h')^2 - \alpha_{j+1}(h')^2 - i(\alpha_j(h'') - \alpha_{j+1}(h'')) = 0, \]
for \( j = 1, \ldots, N - 2 \).

Since \( \alpha_j \) are real valued, we get also
\[ \alpha_j(h'') = \alpha_{j+1}(h''), \quad \alpha_j(h')^2 - \alpha_{j+1}(h')^2 = 0, \]
for \( j = 1, \ldots, N - 2 \). \hspace{1cm} (III.12)

The firsts of these give
\[ \alpha_j(h'') = \alpha_1(h''), \quad j = 2, \ldots, N - 1. \] \hspace{1cm} (III.13)

We have two possibilities. Or \( h'' = 0 \), or not. We will now show that the second case leads to a contradiction. First, notice that if \( h'' \neq 0 \) then it must be \( \alpha_j(h'') \neq 0 \) for at least one \( j \) (since the \( \alpha_j \) are linearly independent), so that all \( \alpha_j(h'') \) are equal and different from zero. From the second of (III.12) we have that there must exist signs \( \varepsilon_j \) such that
\[ \alpha_j(h') = \varepsilon_j \alpha_1(h'), \quad j = 2, \ldots, N - 1. \] \hspace{1cm} (III.14)

Deriving it w.r.t. \( r \) must give (III.13), so that \( \varepsilon_j = 1 \) for all \( j \), and we are left with the linear system of equations
\[ \alpha_j(h') = \alpha_1(h'), \quad j = 2, \ldots, N - 1. \] \hspace{1cm} (III.15)

Since the \( \alpha_j \) are linearly independent (of rank \( N - 1 \)) this is a set of \( N - 2 \) linearly independent equations for \( h' \in H \). Since \( H \) is \( N - 1 \) dimensional, the space of solutions is one dimensional and the general solution of it is
\[ h'(r) = f(r)v, \]
where \( f \) is an arbitrary function and \( v \in H \) is the unique matrix satisfying \( \alpha_j(v) = 1 \) for all \( j \) (which we will compute later, for now it is sufficient to know it exists). We now replace this solution in (III.8). We immediately get
\[ f'(r) \left( v + \frac{\lambda m^2}{2L^2} \sum_{j=1}^{N-1} |c_j|^2 J_j \right) = 0. \]

Since we have assumed \( h'' \neq 0 \), we have \( f' \neq 0 \) and, therefore,
\[ v = \frac{\lambda m^2}{2L^2} \sum_{j=1}^{N-1} |c_j|^2 J_j. \]

After applying \( \alpha_k \) to this equality, using that \( \alpha_k(v) = 1 \) and noticing that \( \alpha_k(J_j) = C_{A^{-1}k,j} \) are the components of the Cartan matrix, we get
\[ 1 = -\frac{\lambda m^2}{2L^2} \sum_{j=1}^{N-1} C_{A^{-1}k,j} |c_j|^2, \quad j = 1, \ldots, N - 1. \]

This relation can be inverted easily: if we consider 1 at varying \( j \) to be the components of a vector in \( \mathbb{R}^{N-1} \), we can apply the inverse Cartan matrix to both members, thus getting
\[ |c_j|^2 = \frac{2L^2}{\lambda m^2} \sum_{k=1}^{N-1} C_{A^{-1}k,j}. \]

Since \( \lambda \) is positive and the same is true for the elements of the inverse Cartan matrix (A.15), we see that this led us to a contradiction. Therefore, the only possibility is that \( f'(r) = 0 \), which is equivalent to \( h''(r) = 0 \).

Hence, we proceed in investigating the first possibility, \( h'' = 0 \). In this case (III.8) is automatically satisfied and (III.9) reduces to (III.14). Its solution is
\[ h'(r) = av \] \hspace{1cm} (III.16)
where \( a \) is a constant and \( v \in H \) is the unique matrix solving \( \alpha_j(v) = \varepsilon_j, \quad j = 1, \ldots, N - 1 \) where \( \varepsilon_j \in \{0,1\} \) (and \( \varepsilon_1 = 1 \)). Since \( \varepsilon_1 \) is fixed, this gives \( 2^{N-2} \) solutions for every choice of \( c_j \) in \( k \). As we will see in the explicit example of \( SU(4) \), however, not all of these are really distinct solutions. There is a convenient way to express \( v \) explicitly. Indeed, let us write\(^5\) \( h = av \), where \( a \) is a constant and \( v \in H \) is a matrix
\[ v = \text{diag}(v_1, \ldots, v_N) \] \hspace{1cm} (III.17)
such that \( \alpha_i(v_\varepsilon) = \varepsilon_i, \quad \varepsilon_i = \pm 1, \quad i = 1, \ldots, N - 1 \) and of course \( \sum_{i=1}^{N} v_i = 0 \). These equations are easily solved by writing \( v = \sum_{j=1}^{N-1} w_j J_j \) so that the equations are
\[ \varepsilon_k = \sum_{j=1}^{N-1} C_{A^{-1}k,j} w_j \]
and the solution is
\[ w_j = \sum_{k=1}^{N-1} C_{A^{-1}j,k} \varepsilon_j \] \hspace{1cm} (III.18)

\(^5\) We omit an irrelevant additive integration constant
and
\[ v_\varepsilon = \sum_{j,k} C_{AN-1,j,k}^{-1} \varepsilon_k J_j. \] (III.19)

We have thus proved

**Proposition 2.** All the solutions of the equations of motion (II.2) determined by the ansätze (III.1), (III.2), (II.3) are given by
\[ h(r) = arv_\varepsilon, \] (III.20)
\[ v_\varepsilon = \sum_{j,k} C_{AN-1,j,k}^{-1} \varepsilon_k J_j, \] (III.21)

where \( a \) is a real constant and \( \varepsilon_j \) are signs, with \( \varepsilon_1 = 1 \).

This solutions are only local solutions, which means that they solve the differential equations. They do not extend automatically to global solutions, that are solution with a well defined Baryon number. Looking for global solutions is the task of the next section.

**IV. GLOBAL SOLUTIONS**

Up to now we have found the most general solution of the differential Skyrme equation. Nevertheless, it is not sufficient to determine a Skyrmion, since global conditions have to be imposed in order to get a solution with a well defined topological charge. This condition is not simply equivalent to impose that the topological charge must be integer (this is just a consequence of the right topological condition) but that it has to wrap a homological cycle an entire number of times (mathematically, it has to cover a cycle, that means to be a surjective map with a well defined degree). We will normalize the parametrizations so to have all ranges in \([0, 2\pi]\).

**A. Statement of the problem**

The difficulty in passing from local solutions to global solutions is twofold. In order to illustrate it, let us consider the specific example of \( SU(4) \) when \( k \) is given by \( c_i = 1 \). For getting a well defined global solution, the function
\[ g(\gamma) = e^{m\gamma k} \] (IV.1)
is expected to provide a good coordinate of the image, of the solution. Since the target space of the map is compact, this requires that if we extend the range of \( \gamma \) to the whole \( \mathbb{R} \), \( g(\gamma) \) must result to be a periodic function. Now, a simple calculation show that the eigenvalues of \( k \) are \( \pm \mu_+, \pm \mu_- \), with
\[ \mu_\pm = \frac{i}{2}(\sqrt{5} \pm 1). \] (IV.2)

This means that, for a suitable unitary constant matrix \( U \), we have
\[ g(\gamma) = U \text{diag}(e^{m\gamma \mu_+}, e^{-m\gamma \mu_+}, e^{m\gamma \mu_-}, e^{-m\gamma \mu_-}) U^\dagger. \] (IV.3)

In particular, its elements have periodicities \( T_\pm \) with
\[ T_\pm = \frac{2\pi}{m|\mu_\pm|}. \] (IV.4)

But since
\[ \frac{T_+}{T_-} = \frac{1}{2}(3 + \sqrt{5}) \] (IV.5)
is not rational, they have not a common period and the orbit never close, so is not a periodic function but, rather, its orbit describes a curve which densely covers a bi-torus in \( SU(4) \). In particular, it is not possible to use \( g(\gamma) \) as a good factor to get a finite covering of a cycle, despite it gives a solution of the equations of motion. It doesn’t provide a solution with a well defined topological number and must be discarded. One has to tackle the problem of looking for acceptable matrices \( k \), that are matrices generating a well defined period.

Assuming we have solved the periodicity problem, there is a second subtlety to be tackled: how to determine the right range of the coordinates in order to correctly cover a cycle. First notice that \( \pi_3(SU(N)) = \mathbb{Z} \). This suggests that homotopically we have just one representative for any given topological (Baryonic) charge. Moreover, since \( \pi_2(SU(N)) = 0 \), we have also \( H_3(SU(N), \mathbb{Z}) = \mathbb{Z} \), so we have also a unique homological representative. Nevertheless, the solutions have not to be identified under deformation, but at most under gauge equivalence. But since the action is not gauge invariant, in our case all different representatives in a given equivalence class must to be considered as different solutions.

We will distinguish three different classes of solutions. The first two classes have canonical representatives: the ones of \( SU(2) \)-type, which belong in every class, and the ones of \( SO(3) \)-type, which belong in even classes only. They can be simply understood as follows. For any given \( N \) we can embed the representations of \( su(2) \) into \( su(N) \). Exponentiating, they will
give realisations of $SU(2)$ or $SO(3)$, depending on the specific representation. This give rise to pure $SU(2)$-type or $SO(3)$-type solutions. However, they can be continuously deformed, by varying the corresponding $z$ when allowed, giving rise to solutions that are not embeddings, so we can consider them as true $SU(N)$ solutions. But there exist a third class of solutions that cannot be obtained as continuous deformations of embeddings. Their existence is due to the fact that $SU(N)$ has a center isomorphic to $\mathbb{Z}_N$, which acts continuously on $SU(N)$, see appendix A. In particular, if $\Gamma$ is a normal subgroup of the center, then one can construct the group $SU(N)_\Gamma := SU(N)/\Gamma$. The new class of solutions are generated by cycles in $SU(N)$ that reduce to cycles of $SU(N)_\Gamma$ after the quotient. We will call them genuine $SU(N)$ solutions. We will consider them carefully in the explicit examples of $SU(3)$ and $SU(4)$, where everything is exactly computable, but now we shortly describe the $SU(2)$-type and $SO(3)$-type, where some details are a priory known, see App. C.

An $SU(2)$-type cycle has the form

$$U(\phi, \gamma, \theta) = e^{\phi h'r} e^{m\gamma k},$$

where $h'$ is constant and the coordinate must run as follows. The range of $r$ must be $T/4$, where $T$ is the period of $e^{h'r}$. The range of $\gamma$ must be $T_k$, the period of $e^{\gamma k}$ (with $m = 1$), and the range of $\phi$ must be $T_k/2$. Therefore, the convenient choice for the coordinates is

$$\varphi \in [0, T_k/2], \quad r \in [0, T/4], \quad \gamma \in [0, T_k],$$

corresponding to the Baryon number

$$B = mB_0,$$

where $B_0$ is the fundamental charge of the given Skyrminion. For $SO(3)$-type cycles the interval for $\phi$ must cover an integer period, so that the ranges must be

$$\varphi \in [0, T_k], \quad r \in [0, T/2], \quad \gamma \in [0, T_k],$$

and the corresponding Baryon number is

$$B = 2mB_0.$$

The $SO(3)$-type can be defined as “di-Baryon class” after the seminal works [35] [36]. These results were extended, keeping spherical symmetry, to the $SU(N)$ case in [57] [58] [59] [60] leading to numerical non-embedded configurations in the $SU(N)$ Skyrme model. In the present paper we will generalize those findings to the non-spherical case at finite Baryon density achieving, moreover, analytic solutions.

### B. $SU(3)$ Skyrmions

Let us apply the above formalism to the case $N = 3$. In this case we will see that the problem of periodicity will not arise.

#### 1. $SO(3)$-type solutions and genuine $SU(3)$ solutions

The matrix $k$ is

$$k_c = \left( \begin{array}{cc} 0 & c_1 \\ -c_1^* & 0 \\ 0 & -c_2^* \\ 0 & 0 \end{array} \right). \quad (IV.6)$$

We put $\|z\|^2 = |c_1|^2 + |c_2|^2$. Then, the characteristic equation is

$$\left( \lambda^2 + \|z\|^2 \right) \lambda = 0. \quad (IV.7)$$

The eigenvalues are $\lambda_0 = 0$ and $\lambda \pm = \pm i \|z\|$, so that

$$g(\gamma) = e^{\mp k_c} \quad (IV.8)$$

is periodic with period

$$T_k = \frac{2\pi}{\|z\|}. \quad (IV.9)$$

Now, we pass to determine the Cartan element. We have two possibilities according to the two possible choices for $z$:

$$z_{\pm} = \left( \begin{array}{c} 1 \\ \pm 1 \end{array} \right). \quad (IV.10)$$

The inverse Cartan matrix for $SU(3)$ is

$$C_{A_2}^{-1} = \frac{1}{3} \left( \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right). \quad (IV.11)$$

Thus, we find the two solutions

$$h_+(r) = ar(J_1 + J_2), \quad (IV.12)$$
$$h_-(r) = \frac{a}{3}(J_1 - J_2). \quad (IV.13)$$

The period of $\exp h_+(r)$ is

$$T_{h_+} = \frac{2\pi}{a}. \quad (IV.14)$$

while the one of $\exp h_+(r)$ is

$$T_{h_-} = \frac{6\pi}{a}. \quad (IV.15)$$
Now, we have to discuss the global properties in order to fix the ranges of the parameters. To this end, accordingly to appendix C, we have to look for the intersection between the orbit of \( \h \pm \) and the one of \( \gamma k \). Using the characteristic equation we immediately see that, see App. E1,

\[
e^{\gamma k} = I + \frac{\sin(||\gamma||)}{||\gamma||} k + 2 \frac{\sin^2(||\gamma||)}{||\gamma||^2} k \quad \text{(IV.16)}
\]

so that the intersection we are looking for is just the unit matrix \( I \). However, we can notice that the orbit of \( h_-(r) \) contains the elements

\[
\exp h-(2\pi/a) = e^{\frac{2\pi i}{a}} I, \quad \exp h-(4\pi/a) = e^{\frac{4\pi i}{a}} I, \quad \text{(IV.17)}
\]

which are both in the center of \( SU(3) \). Following App. C, we conclude that \( h_-(r) \) defines a genuine \( SU(3) \) solution, while only \( h_+(r) \) is of \( SO(3) \)-type.

In order to correctly define the solution we thus have to identify the ranges as follows. First, it is convenient to normalise \( c \) so that \( ||c|| = 1 \). This is just equivalent to re-scale the coordinates \( \Phi \) and \( \gamma \). Therefore, we fix once for all the metric to be

\[
d\text{s}^2 = -dt^2 + L_t^2 dr^2 + L_\gamma^2 d\gamma^2 + L_\varphi^2 d\varphi^2, \quad \text{(IV.18)}
\]

with range of coordinates

\[
0 \leq r \leq 2\pi, \quad 0 \leq \gamma \leq 2\pi, \quad 0 \leq \varphi \leq 2\pi, \quad \text{(IV.19)}
\]

with the caveat that, despite the chosen values, \emph{none of the coordinates is periodic!} Our Skyrmions are living in a rectangular box.

\textbf{SO(3) type solutions.} We already know that \( r \) must cover 1/2 of the period of the Cartan torus, which implies that we have to fix \( a = \frac{\pi}{2} \). Hence, our solutions are

\[
U_\pm(t, r, \varphi, \gamma) = e^{\Phi k} e^{a r (J_1 \pm J_2)} e^{m \gamma k}, \quad \text{(IV.20)}
\]

\[
\Phi = \frac{t}{L_\varphi} - \varphi, \quad \text{(IV.21)}
\]

\[
\varphi, \gamma, r \in [0, 2\pi], \quad \text{(IV.22)}
\]

\[
B = 2m. \quad \text{(IV.23)}
\]

More explicitly

\[
U_\pm(t, r, \varphi, \gamma) = \left( I + \sin(\Phi)k + 2 \sin^2 \left( \frac{\Phi}{2} \right) k \right) \text{diag}(e^{i\varphi}, 1, e^{-i\varphi}) \left( I + \sin(m\gamma)k + 2 \sin^2 \left( \frac{m\gamma}{2} \right) k \right).
\]

We can now compute the energy end the factor \( g_+ = \frac{E_m}{2m} \). We omit details here, since are particular cases of the general one for generic \( N \) considered below. We get

\[
g_+(m, c) = L_t L_\gamma L_\varphi \frac{K^3}{m} \left[ \frac{4}{L_\varphi^2} + \frac{1}{8L_r^2} + \frac{\lambda}{16L_\gamma^2 L_\varphi^2} + \frac{m^2}{L_\gamma^2} \left( 2 + \frac{\lambda}{32L_\gamma^2} + \frac{2\lambda}{L_\varphi^2} (1 - 3|c_1|^2|c_2|^2) \right) \right],
\]

where \(|c_1|^2 + |c_2|^2 = 1\). In particular, for each value of \( m, |g_+(m, c)| \) takes its minimum at \(|c_1| = |c_2|\), which is

\[
g_+(m, c) = L_t L_\gamma L_\varphi \frac{K^3}{m} \left[ \frac{4}{L_\varphi^2} + \frac{1}{8L_r^2} + \frac{\lambda}{16L_\gamma^2 L_\varphi^2} + \frac{m^2}{L_\gamma^2} \left( 2 + \frac{\lambda}{32L_\gamma^2} + \frac{\lambda}{2L_\varphi^2} \right) \right].
\]

Some comments are in order now. The reason for which the solution we have just described are of \( SO(3) \)-type can be understood remembering that we are working with \( 3 \times 3 \) matrices, which carry naturally a representation of spin 1 of the rotation group. Indeed, the minimum energy case just discussed, in which \(|c_j| = 1/\sqrt{2} \), corresponds exactly to the case when the matrices \( h_+ \) and \( k_\pm \) are the generators of the group \( SO(3) \) in the representation of spin 1. The other solution, for every fixed \( m \), are continuous deformat-
motions obtained varying \( \xi \), which does not changes their topological nature, and in particular the Baryon number, but it changes the energy. One can easily check that for generic \( \xi \) the matrices \( h_+ \) and \( k_\pm \) do not generate a subgroup. One may wander if this is related to the fact that their energy is not a minimum. The present remark suggests how to look for \( SU(2) \)-type solutions.

**Genuine \( SU(3) \) type solutions.** Since this case does not enter in the canonical classes, we have to manage separately the determination of the correct ranges (then normalised to \( 2\pi \) as specified above). As for \( r \), we will prove in proposition 3 that in order to have \( r \) ranging in \([0, 2\pi]\), one has always to fix \( a = \frac{1}{2} \). For what concerns the other coordinates, let us notice that \( h_-(r) \) does not commute with \( k_\pm \) but it commutes with \( k_\mp \). Therefore, for \( g(\gamma) = e^{\gamma k_\pm} \), we see that \( g(T_k/2) \) commutes with \( e^{h_-(r)} \). This means that we can write

\[
g(\Phi + T_k/2)e^{h_-(r)}g(\gamma) = g(\Phi)g(T_k/2)e^{h_-(r)}g(\gamma) = g(\Phi)e^{h_-(r)}g(T_k/2)g(\gamma) = g(\Phi)e^{h_-(r)}g(\gamma + T_k/2).
\]

If we assume that \( U^\pm_\gamma[\Phi, r, \gamma] = g(\Phi)e^{h_-(r)}g(\gamma) \) is covering a cycle, the relation \( U^\pm_\gamma[\Phi + T_k/2, r, \gamma] = U^\pm_\gamma[\Phi, r, \gamma + T_k/2] \) shows that we are covering it twice unless we restrict one of the two ranges, of \( \Phi \) and of \( \gamma \), to one half the period of \( g \). We choose to reduce \( \Phi \), so we replace \( \Phi \) with \( \Phi/2 \). So, our solution is

\[
U^\pm_\gamma[t, r, \varphi, \gamma] = e^{\frac{\Phi}{2}k_\pm e^{(J_1 \pm J_2)e^{\gamma \lambda k_\pm}}} 
\]

so we replace \( \Phi \) with \( \Phi/2 \). Explicitly,

\[
U^\pm_\gamma[t, r, \varphi, \gamma] = 
\]

For \( U_- \), \( g \) results to be independent from \( \xi \):

\[
g_-(m, \xi) = L_r L_\gamma L_\varphi \frac{K}{2m} \left[ \frac{4}{L_\varphi^2} + \frac{2}{3L_r^2} + \frac{\lambda}{4L_\varphi^2 L_r^2} + 8 \frac{m^2}{L_\gamma^2} \left( 1 + \frac{\lambda}{16L_r^2} + \frac{\lambda}{4L_\varphi^2} \right) \right] .
\]

2. \( SU(2) \)-type solutions

It is now clear that in order to find \( SU(2) \)-type solutions we have to consider deformations of spin \( \frac{1}{2} \) representations. This can be obtained by “reducing matrices” down to \( 2 \times 2 \), and can be achieved by choosing

\[
k \equiv k_c = \begin{pmatrix} 0 & c \\ -c^* & 0 \end{pmatrix}.
\]

where \( c \) is a phase. This is not the same thing as simply putting \( c_2 = 0 \) in \( k_\pm \) in the sense that we have to choose \( k = k_c \) before solving equation (III.9). Indeed, in (III.9) we assumed that all simple roots enter the game. This fixes the set of possible choices of \( h(r) \), and if in the above solutions we deform smoothly \( \xi \) to \((c, 0)\), we cannot move away from our topological classes. This is confirmed by the fact that if we put \( c_2 = 0 \), the matrix \( k \) reduces to a \( 2 \times 2 \) matrix, but the \( k_\pm \) do not allow to reduce the representation down to \( C^2 \). We have to make a discontinuous deformation. The point is that for \( c_2 = 0 \) the root \( a_2 \) does not enter into equation (III.9) that, indeed, for \( N = 3 \) becomes
just an identity. This means that when \( c_2 = 0 \) we can choose for \( h(r) \) any combination

\[
h(r) = arJ_1 + brJ_2,
\]

with the only caveat that \( e^{h(r)} \) must be periodic, so that \( a \) and \( b \) must be in rational ratio. We can set

\[
h_q(r) = arJ_1 + aqrJ_2, \quad q \in \mathbb{Q}.
\]

For \( q = \pm 1 \) we fall down to the previous \( SO(3) \)-solutions, while, for \( q = 0 \) provides a canonical embedding of \( SU(2) \) into \( SU(3) \), thus identifying an \( SU(2) \)-type solution. It is worth to mention that, since \( q \in \mathbb{Q} \) it cannot be deformed continuously among the three values, compatibly with the fact that the case \( q = 0 \) is not in the same topological class of the other ones and, indeed, we may wander what happens

\[
g_0(n,c) = \frac{K \pi^3}{n} \left[ \frac{2}{L_\phi^2} + \frac{1}{4L_r^2} + \frac{\lambda}{8L_\phi^2L_r^2} + \frac{n^2}{L_\gamma^2} \left( 4 + \frac{\lambda}{4L_r^2} + \frac{\lambda}{L_\phi^2} \right) \right].
\]

### C. \( SU(N) \) Skyrmions

We will now consider the class of Skyrmions associated to the matrix \( k \) given by

\[
k_n = \sum_{j=1}^{N-1} (c_j E_{j,j+1} - c_j^* E_{j+1,j}).
\]

We will limit ourselves to the case when all the \( c_j \) are different from zero. Here, we have to face the problem of establish for which choices of \( c_j \) the matrix \( e^{ck_n} \) is periodic. By now, let us assume to have solved it and write down the corresponding solution:

\[
U_{\sigma}^k[t,r,\varphi,\gamma] = e^{\sigma B r} e^{\frac{\Phi}{L_\phi}} e^{a \varphi r e^{i \varphi k_n}},
\]

\[
\Phi = \frac{t}{L_\phi} - \varphi,
\]

\[
\varphi, \gamma, r \in [0,2\pi],
\]

\[
B = \frac{\sigma}{2} n \|c\|^2,
\]

where \( \sigma = 1 \) for \( SO(3) \)-type solutions and \( \sigma = 1/2 \) for \( SU(2) \)-type solutions, and \( v_r \) is given by (III.21). For generic genuine solutions the value of \( \sigma \) must be computed case by case. For any admissible \( c \) these are \( 2^{N-2} \) solutions (since \( \varepsilon_1 = 1 \)). In principle \( \sigma \) could depend on \( N \) and \( c \). However, we will now show that this is not the case and the value of \( \sigma \) is completely fixed by requiring that the normalized interval \( [0,2\pi] \) for \( r \) must have the extension necessary to cover once a cycle:

**Proposition 3.** If \( \exp(\alpha v_r r) \) is such that \( r \in [0,2\pi] \), and the corresponding map \( U_{\sigma}^k[t,r,\varphi,\gamma] \) has not to cover a cycle more than once, then necessarily \( \sigma = \frac{1}{2} \).

**Proof.** The proof is simply based on the same strategy used for example in [43]: one first constructs the invariant measure restricted to the hypothetical cycle; the resulting measure will depend explicitly on some of the coordinates and will vanish at specific value of that coordinate. The good range for such a coordinate to cover just once a cycle is any range between two vanishing points. The nice fact is now that the Haar measure restricted to a cycle, a part from an eventual normalization constant, is just \( \rho_B \), which is computed in App.F. Since it results to depend on \( r \) via \( \sin(\sigma r) \), we see that a suitable good interval for \( r \) is \( [0,\pi/\sigma] \). Since we want it to be \( [0,2\pi] \), it must be \( \sigma = \frac{1}{2} \).

Therefore, we definitely have

\[
a = \frac{1}{2}
\]

in any case. Now, we can compute the \( g \) factor for our solutions. To this end, first note that
\[ T_{00} = -\frac{K}{2} \text{Tr} \left( \frac{1}{2} (R^\gamma R_\gamma + R^r R_r) + \frac{\lambda}{16} F_{\rho\sigma} F^{\rho\sigma} + R_t R_t + \frac{\lambda}{4} g^{\alpha\beta} F_{\alpha\beta} F_{\alpha\beta} \right) \]
\[ = -\frac{K}{4} \text{Tr} \left( \frac{R^2_r}{L^2_r} + \frac{R^2_\gamma}{L^2_\gamma} \right) - \frac{K \lambda}{16} \text{Tr}(F_{\gamma r})^2 - \frac{K}{2} \text{Tr} R^2_t - \frac{K \lambda}{8 L^2_\gamma} \text{Tr} \left( \frac{F^2_{\rho r}}{L^2_r} + \frac{F^2_{\rho \gamma}}{L^2_\gamma} \right), \] (IV.42)

according to Appendix B, and we used
\[ R_t = \frac{1}{L_\phi} R_\phi, \quad F_{\alpha\beta} = \frac{1}{L_\phi} F_{\alpha\beta}. \] (IV.43)

According to (B.6), (B.7), (B.8), with \( a = \frac{1}{2} \), we have
\[ \text{Tr} R^2_t = \frac{\sigma^2}{L^2_\phi} \text{Tr} \frac{v_\phi^2}{\| c \|^2}, \] (IV.44)
\[ \text{Tr} R^2_\gamma = m^2 \text{Tr} \frac{v_\phi^2}{\| c \|^2}, \] (IV.45)
\[ \text{Tr} R^2_t = \frac{1}{4} \text{Tr} \frac{v_\phi^2}{\| c \|^2} - \frac{1}{4} \sum_{j,k} C^{-1}_{A N - 1, j, k} \varepsilon_j \varepsilon_k = - \frac{1}{4} \| v_\phi \|^2, \] (IV.46)
\[ \text{Tr}(F_{\gamma r})^2 = m^2 \text{Tr}(v_\phi^2) \]
\[ \text{Tr}(F_{\rho r})^2 = \sigma^2 \text{Tr}(\| c \|^2), \] (IV.47)
\[ \text{Tr}(F_{\rho \gamma})^2 = \sigma^2 \text{Tr}(v_\phi^2) \]
\[ \text{Tr}(F_{\phi r}) = \sigma^2 \text{Tr}(v_\phi^2), \] (IV.48)
\[ \text{Tr}(F_{\phi \gamma}) = \sigma^2 \text{Tr}(v_\phi^2) \]
\[ \text{Tr}(F_{\phi \rho}) = \sigma^2 \text{Tr}(v_\phi^2) \]
\[ \text{Tr}(F_{\phi \gamma}) = \sigma^2 \text{Tr}(v_\phi^2) \]

Replacing in the expression for \( T_{00} \) and using that the energy is
\[ E = \int_0^{2\pi} dr \int_0^{2\pi} d\varphi \int_0^{2\pi} d\gamma L_r L_\phi L_\gamma T_{00}(r), \] (IV.50)
we get
\[ E = L_r L_\gamma L_\phi \| c \|^2 \frac{K}{2} \pi^3 \left[ 16 \frac{\sigma^2}{L^2_\phi} + \| v_\phi \|^2 \frac{\sigma^2}{L^2_\phi} \right. \]
\[ \left. + 8 \frac{m^2}{L^2_\gamma} \left( 1 + \frac{\lambda}{16 L^2_r} + \frac{\lambda^2}{L^2_\phi \| c \|^2} \right) \left( \sum_{j=1}^{N-1} |c_j|^4 + \sum_{j=1}^{N-1} |c_j|^2 |c_{j+1}|^2 \left( \frac{1}{2} - 3 \varepsilon_j \varepsilon_{j+1} \right) \right) \right]. \] (IV.51)

In a similar way one can compute the baryon number. This is done in appendix F, with the result
\[ B = 2m\sigma \| c \|^2. \] (IV.52)

From these results we immediately get the \( g \)-factor:
\[ g(N, m, \varepsilon) = L_r L_\gamma L_\phi \frac{K \pi^3}{4m} \left[ 16 \frac{\sigma^2}{L^2_\phi} + \| v_\phi \|^2 \frac{\sigma^2}{L^2_\phi} \right. \]
\[ \left. + 8 \frac{m^2}{L^2_\gamma} \left( 1 + \frac{\lambda}{16 L^2_r} + \frac{\lambda^2}{L^2_\phi \| c \|^2} \right) \left( \sum_{j=1}^{N-1} |c_j|^4 + \sum_{j=1}^{N-1} |c_j|^2 |c_{j+1}|^2 \left( \frac{1}{2} - 3 \varepsilon_j \varepsilon_{j+1} \right) \right) \right]. \] (IV.53)
Up to now, we have assumed $\zeta$ to be normalised so that $g(\gamma) = e^{\gamma k} \zeta$ has period $2\pi$. However, we will not have really found a solution until we will be able to specify for which $\zeta$ the function $g$ is periodic. Therefore, we cannot further postpone to tackle this problem.

However, before considering it in general, we want now concentrate on a very particular case, when $\varepsilon_j = 1$ for all $j$. In this case

$$g(N, m, \zeta) = L_x L_y L_0 \frac{K \pi^3}{4 \sigma m} \left[ \frac{16 \sigma^2}{L_y^2} \|v\|^2 + \frac{\sigma^2 \lambda}{L_x^2 L_y^2} + 8 \frac{m^2}{L_y^2} \left( 1 + \frac{\lambda}{16 L_y^2} + \frac{\lambda \sigma^2}{L_x^2 \|\zeta\|^2} \left( \sum_{j=1}^{N-1} |c_j|^4 - \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 \right) \right) \right].$$ \hfill (IV.54)

It is clear that, among all possible choices for $\varepsilon_j$, this minimises the energy, apart from possible effects due to $\|v\|$. We want also minimise with respect to the $c_j$, assuming the normalisation of $\|\zeta\|$ fixed. Introducing a Lagrange multiplier $\Lambda$, we have to extremize the function

$$f(\zeta) = \sum_{j=1}^{N-1} |c_j|^4 - \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 - \Lambda \|\zeta\|^2.$$ \hfill (IV.55)

Deriving with respect to $|c_j|^2$ we get the system

$$C_{AN-1} \|c\|^2 = \Lambda \textbf{1},$$ \hfill (IV.56)

$\textbf{1}$ being the vector in $\mathbb{R}^{N-1}$ having all elements equal to 1. This gives the solution

$$|c_j|^2 = \frac{\Lambda}{2} \gamma(N - j).$$ \hfill (IV.57)

Interestingly this also solves automatically the periodicity problem. It is easy to see (App. D) that

$$c_j = \zeta_j \sqrt{\frac{\Lambda}{2}} \gamma(N - j), \quad \Lambda = \begin{cases} \frac{1}{2} & \text{for odd } N \\ 1 & \text{for even } N \end{cases}$$ \hfill (IV.58)

where $\zeta_j$ are arbitrary phases, give a matrix $e^{\gamma k} \zeta$ that is periodic in $\gamma$ with period $2\pi$. For $v$ we find

$$v = \sum_{j,k} C^{-1}_{AN-1,j,k} \gamma_j.$$ \hfill (IV.59)

Moreover, we have

**Proposition 4.** If $c_j$ are given by (IV.58), and $v$ is as in (IV.59), then

$$\|\zeta\|^2 = \frac{\Lambda}{12} N(N^2 - 1),$$ \hfill (IV.60)

$$\|v\|^2 = \frac{1}{12} N(N^2 - 1),$$ \hfill (IV.61)

and

$$\sum_{j=1}^{N-1} |c_j|^4 - \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 = \frac{\Lambda^2}{24} N(N^2 - 1) = \frac{\Lambda}{2} \|\zeta\|^2.$$ \hfill (IV.62)

Proof. The first result follows immediately by the well known formulas

$$\sum_{j=1}^{N-1} j = \frac{N(N - 1)}{2},$$ \hfill (IV.63)

$$\sum_{j=1}^{N-1} j^2 = \frac{N(N - 1)(2N - 1)}{6}.$$ \hfill (IV.64)

For the second expression notice that, by using (A.15),

$$\|v\|^2 = \sum_{j,k} C^{-1}_{AN-1,j,k} \sum_{j,k} C_{AN-1,j,k},$$

$$= \frac{1}{N} \left[ \sum_{j<k} j(N - k) + \sum_{j \geq k} k(N - j) \right],$$

$$= - \frac{1}{N} \sum_{j<k} jk + \sum_{j \geq k} j + \sum_{j \geq k} k,$$

$$= - \frac{1}{N} \left( \sum_{j=1}^{N-1} j \right)^2 + \sum_{j=1}^{N-1} j(N - j - 1)$$

$$+ \sum_{k=1}^{N-1} k(N - k),$$ \hfill (IV.65)

and the final expression again follows after applying the above well known formulas.

For the last formula, notice that the $c_j$ are solutions
We can also notice that the minimum is reached at therein and the ones in the last proposition, we get done in general in appendix G. By using the formulas of (IV.69) with respects to Dynkin index \( I \). Notice that for \( N \) odd \( I \) is even, so \( B \) is always integer.

Finally, we are also interested in minimizing expression (IV.69) with respects to \( L_a, a = \varphi, r, \gamma \). This is done in general in appendix G. By using the formulas therein and the ones in the last proposition, we get that the minimum is reached at

\[
L_\varphi = \frac{\sqrt{3}}{2^2}, \quad L_r = \frac{\sqrt{3}}{4}, \quad L_\gamma = \frac{m \sqrt{3}}{\sigma 2^2}, \]

with corresponding minimal value

\[
g_{\text{min}} = K \sqrt{3} (1 + 2 \sqrt{2}). \quad (IV.72)
\]

Now, \( f \) is the sum of two homogeneous pieces, one of degree 4 end the other of degree 2. Therefore, we can use the Euler theorem\(^6\) to rewrite the last as

\[
0 = 4 \left( \sum_{j=1}^{N-1} |c_j|^4 - \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 \right) - 2 \lambda \| \vec{c} \|^2, \quad (IV.68)
\]

which completes the proof. \( \square \)

Using this results and noticing that \( \sigma^2 \Lambda = 1/2 \), we find for the energy per Baryon

\[
g_{\min, \text{stand}} = \pi \frac{1 + 2 \sqrt{2}}{6} \approx 2.00456. \quad (IV.73)
\]

Using normalised units (corresponding to \( \lambda = 1 \) and \( K = (6\pi^2)^{-1} \)) we get

\[
\frac{\rho_B}{U^{-1}dU} \neq 0.
\]

Notice that this is independent from \( N \) and it is expected to be the absolute minimum with respect to any choice of \( \varepsilon_j \). We will not try to prove this conjecture here, we will limit ourselves to check it for \( N = 4 \) here below. The comparison with [61] is very interesting. The present results are slightly above the bound in [61] due to the time-dependence in the Ansatz. Note however that the present time-dependence cannot be undone as the present solutions wrap in a topologically non-trivial way also around the time direction. To the best of our knowledge, this is the first analytic computations showing explicitly how the closeness to the BPS bound “evolves” with \( N \) in the \( SU(N) \) Skyrme model.

To be more specific, as it has been already emphasized, we are interested in topologically non-trivial solutions. In the present context this means that we only consider \( SU(N) \) Ansatz such that

\[
\rho_B = Tr \left( U^{-1}dU \right)^3 \neq 0.
\]

As it has been discussed in the previous sections, \( \rho_B \) represents the Baryon density when it is non-vanishing along three-dimensional space-like hypersurfaces \( \Sigma_{t=\text{const}} \).
In these cases, the integral of $\rho_B$ over $\Sigma_{t=const}$ represents the Baryon charge. While, mathematically, these integrals represent how many times the SU($N$)-valued Skyrmions wrap around $\Sigma_{t=const}$. On the other hand, $\rho_B$ can be topologically trivial also along time-like hypersurfaces. In this case, one can also consider the wrapping of the SU($N$)-valued configurations along three-dimensional time-like hypersurfaces. The configurations which have been constructed here are, as a direct check easily reveals, topologically non-trivial in two ways. Not only they possess non-vanishing Baryonic charge, they are also wrapped non-trivially along time-like hypersurfaces. Indeed, if one considers

$$U_{t}\{t, r, \varphi, \gamma\} = e^{i\Phi_{C}\xi_{C}^{a}a_{\alpha}r_{\epsilon}m_{\gamma}k_{C}^{\beta}},$$

$$\Phi = \frac{t}{L_{\varphi}} - \varphi,$$

then the corresponding topological density has one space-like component and one time-like component:

$$\rho_B \sim dr \wedge d\varphi \wedge d\gamma - dr \wedge d\left(\frac{t}{L_{\varphi}}\right) \wedge d\gamma.$$  

In particular, it implies that these SU($N$) Skyrmions wrap non-trivially around the three-dimensional time-like $\{\varphi = const\}$ hypersurfaces. The consequence of this fact is that the time-dependence of the present configuration “cannot be undone” otherwise the winding number corresponding to the $\{\varphi = const\}$ hypersurfaces would change.

**D. Solving the periodicity problem**

The solution of this problem is provided in App.E.7 We discuss here the main results. The vectors $\xi \in \mathbb{C}^{N-1}$ having all components different from zero and allowing for a periodic function $g(\gamma) = e^{i\gamma}\xi$, with period $2\pi$, form a family

$$\xi = g(m, \omega, l),$$  

where $m = (m_1, \ldots, m_n)$, is a finite strictly increasing sequence of strictly positive coprime integers, $n$ is the integer part of $N/2$, $m \in \{0, 2\pi\}^{N-1}$, and $l \in W \subset \mathbb{R}^{N-1}$ is a set of parameters parametrizing the strictly positive real solutions of the algebraic system

$$\sum_{j=1}^{N-1} \zeta_j = \sum_{a=1}^{n} m_a^2, \quad \text{(IV.74)}$$

$$\sum_{j_1 \ll \ldots \ll j_k \leq N-1} \zeta_{j_1} \ldots \zeta_{j_k} = \sum_{a_1 \ll \ldots \ll a_k \leq n} m_{a_1}^2 \ldots m_{a_k}^2, \quad \text{for } k = 2, \ldots, n, \quad \text{(IV.75)}$$

in the real variables $\zeta_j$, $j = 1, \ldots, N - 1$.

The parameters $\omega$ and $l$ form a moduli space $T^{N-1} \times W$. The relevant physical quantities depend only on $|c_j|$ so are independent on the components in $N - 1$ dimensional torus. Therefore, we can say that only $W$ represents the relevant moduli. As one could expect, in particular, the Baryon number associated to a solution constructed with $\xi(m, \omega, l)$ depends only on $m$ and not on the continuous moduli:

$$B = 2\pi m \sum_{a=1}^{n} m_a^2. \quad \text{(IV.77)}$$

The general form of $g(\gamma)$ is

$$e^{ik}\xi = f_0(\gamma, m)[\xi] + \sum_{j=1}^{N-1} f_j(\gamma, m)k^j, \quad \text{(IV.78)}$$

where the $f_0, \beta = 0, \ldots, N - 1$ are linear combinations of 1 and $\sin(m_a\gamma), \cos(m_a\gamma)$, with rational functions of $m$ as coefficients, and satisfying $f_0(0, m) = 1, f_j(0, m) = 0$ for $j > 0$. In particular, the dependence on the continuous moduli is only through the $k^j$.

**E. Back to $N = 4$**

Following App.E.3, for any two coprime positive integers $p$ and $q$ such that $p > q$, for $N = 4$ we can find four families of solutions, each one parametrized by three real phases $\alpha_1, \alpha_2, \alpha_3$ and a real modulus $\tau \in [q, p]$. Each of these families is specified by one of the four possible inequivalent choices for the discrete vector $\epsilon$. Recall that in this case the inverse Cartan matrix is

$$C_{A_3}^{-1} = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}. \quad \text{(IV.79)}$$

We also have, see App.E.3,
\[ g_4(x) \equiv e^{xk} = \left( \frac{p^2}{p^2 - q^2} \cos(qx) - \frac{q^2}{p^2 - q^2} \cos(px) \right) k_x + \left( \frac{1}{p^2 - q^2} (\cos(qx) - \cos(px)) \right) k_x^2 + \left( \frac{1}{p^2 - q^2} \left( \frac{\sin(qx)}{q} - \frac{\sin(px)}{p} \right) \right) k_x^3, \] (IV.80)

with

\[ k_x = \begin{pmatrix} 0 & e^{i\alpha_1} & 0 & 0 \\ -e^{-i\alpha_1} & 0 & e^{i\alpha_2} & 0 \\ 0 & -e^{-i\alpha_2} & 0 & \frac{p^2}{\tau} e^{i\alpha_3} \\ 0 & 0 & \frac{p^2}{\tau} e^{-i\alpha_3} & 0 \end{pmatrix}, \]

(IV.81)

\[ k_x^2 = \begin{pmatrix} -\tau^2 & 0 & -e^{-i(\alpha_1+\alpha_2)} \tau \psi & 0 \\ 0 & -e^{-i(\alpha_1+\alpha_2)} \tau \psi & -\tau^2 & 0 \\ 0 & e^{-i(\alpha_2+\alpha_3)} \frac{p^2}{\tau} \psi & 0 & -\frac{p^2}{\tau^2} e^{-i\alpha_3} \psi \\ e^{-i(\alpha_1+\alpha_2)} \tau \psi & 0 & e^{i(\alpha_2+\alpha_3)} \frac{p^2}{\tau} \psi & 0 \end{pmatrix}, \]

(IV.82)

\[ k_x^3 = \begin{pmatrix} 0 & -e^{i\alpha_1} \tau (p^2 + q^2 - \frac{p^2 q^2}{\tau^2}) & 0 & e^{i(\alpha_2+\alpha_3)} pq \psi \\ 0 & e^{i\alpha_2} (p^2 + q^2) \psi & 0 & 0 \\ e^{-i(\alpha_1+\alpha_2+\alpha_3)} pq \psi & 0 & -e^{i(\alpha_1+\alpha_2+\alpha_3)} pq \psi & 0 \\ e^{-i(\alpha_1+\alpha_2+\alpha_3)} pq \psi & 0 & 0 & e^{i(\alpha_1+\alpha_2+\alpha_3)} pq \psi \end{pmatrix}, \]

(IV.83)

and

\[ \psi = \sqrt{p^2 + q^2 - \tau^2 - \frac{p^2 q^2}{\tau^2}}, \]

(IV.84)

\[ \log(\psi) = \sqrt{p^2 + q^2 - \tau^2 - \frac{p^2 q^2}{\tau^2}}. \]

\[ \log(\psi) = \frac{1}{\sqrt{p^2 + q^2 - \tau^2 - \frac{p^2 q^2}{\tau^2}}}, \]

(IV.85)

1. The almost SU(2)-type solutions

The SU(2) solution is expected to be identified by \( \varepsilon_a = (1, 1, 1) \). Indeed, from (III.21) we have

\[ v_a = \frac{i}{2} \text{diag}(3, 1, -1, -3), \quad ||v_a||^2 = 5, \]

(IV.85)

which is exactly the matrix representing the diagonal generator of SU(2) in the spin 3/2 representation. However, this is not true in general and we will see that in this series only the one with \( (p, q) = (3, 1) \) is deformable to an SU(2) embedding. Let us first look at the coordinate ranges. Regarding the range of \( r \), it is completely fixed by proposition 3. As for the remaining ranges, they must correspond to the period of \( g_4 \), unless there are (finite discrete) subgroups of the \( U(1) \) group generated by \( g_4 \), which commute with \( v_a \). Since \( v_a \) does not commute with \( k_x^j \), \( j = 1, 2, 3 \)

\[ \text{s.l.c. is particularly grateful to Laurent Lafforgue for suggesting him how to tackle this problem in full generality.} \]
\( g_4(\pi) = -1 \). Therefore, we see that for \( p - q \) odd, there are no discrete symmetries, and the ranges of \( \Phi \) and \( \gamma \) must coincide with the whole period, so \( \sigma = 1 \). Instead, for \( p - q \) even we have

\[
g_4(\Phi + \pi)e^{\frac{i}{2} \nu a r} g_4(\gamma + \pi) = g_4(\Phi)(-\pi) e^{\frac{i}{2} \nu a r} (-\pi) g_4(\gamma) = g_4(\Phi)e^{\frac{i}{2} \nu a r} g_4(\gamma), \quad (IV.90)
\]

so we see that to any point on the image there correspond two different coordinates \((\Phi, \gamma)\) and \((\Phi + \pi, \gamma + \pi)\), unless we restrict one of the two ranges to half a period. We choose to do it with \( \Phi \), and, in order to keep its range to be \([0, 2\pi]\), we fix \( \sigma = 1/2 \).

The field is

\[
U_a = g_4(\sigma_{p-q} \Phi) \begin{pmatrix} e^{\frac{i}{2} v r} & 0 & 0 & 0 \\ 0 & e^{\frac{i}{2} v r} & 0 & 0 \\ 0 & 0 & e^{-\frac{i}{2} v r} & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{2} v r} \end{pmatrix} g_4(m \gamma),
\]

(IV.91)

\[
\Phi = \frac{t}{L_\varphi} - \varphi,
\]

(IV.92)

\[
\sigma_{p-q} = \begin{cases} \frac{1}{2} & \text{if } p - q \text{ is even} \\ 1 & \text{if } p - q \text{ is odd} \end{cases}.
\]

(IV.93)

The Baryon number is \( B_a = 2\sigma_{p-q} m(p^2 + q^2) \), while for the \( g \)-factor we get

\[
g_a(p, q, m, \tau) = L_\sigma L_\varphi \frac{K \pi^3}{4 \sigma_{p-q} m} \left[ \frac{16 \sigma_{p-q}^2}{L_\varphi^2} + \frac{5}{(p^2 + q^2)L_\varphi^2} + \frac{\lambda \sigma_{p-q}^2}{L_\varphi^2} + \frac{8 m^2}{L_\varphi^2} \right] \left( 1 + \frac{\lambda_0}{16 L_\varphi^2} \right).
\]

(IV.94)

The corresponding minimal energy, expressed in normalized units, is given by (G.16), which in this case becomes:

\[
g_a(p, q, \tau) = \frac{\pi}{3\sqrt{2}} \left[ 2 + \sqrt{5} \left( 1 - \frac{3\tau^2}{p^2 + q^2} + \frac{3\tau^4}{(p^2 + q^2)^2} + \frac{4p^2 q^2}{(p^2 + q^2)^2} + \frac{3p^4 q^4}{(p^2 + q^2)^2} - \frac{1}{(p^2 + q^2)^2} \right) \right].
\]

(IV.95)

We can further minimise w.r.t. \( \tau \). Setting \( x = \tau^2 \), we have to find the stationary points in

\[
q^2 < x < p^2. \quad (IV.96)
\]

Deriving the expression in the square root w.r.t. \( x \) and multiplying by \((p^2 + q^2)^2 x^3/6\), we get the equation

\[
0 = \left( x^2 - p^2 q^2 \right) \left( x^2 - \frac{x}{2} (p^2 + q^2) + p^2 q^2 \right). \quad (IV.97)
\]

This gives the admissible solutions \((x)\) is positive \( x_0 = pq \),

\[
x_0 = pq, \quad x_\pm = \frac{1}{4} \left( \frac{(p^2 + q^2)^2}{16} - p^2 q^2 \right) \pm \sqrt{\frac{(p^2 + q^2)^2}{16} - p^2 q^2}.
\]

(IV.98)

\( x_0 \) is always present, while \( x_\pm \) are stationary points only when the square root is real, that is when

\[
(p^2 + q^2)^2 - 16 p^2 q^2 > 0. \quad (IV.99)
\]

Setting \( z = p/q \) this means \( x^4 - 14 x^2 + 1 > 0 \) so (since \( p/q > 1 \))

\[
x^2 > 7 + \sqrt{48} = (2 + \sqrt{3})^2, \quad (IV.100)
\]

and, finally,

\[
\frac{p}{q} > 2 + \sqrt{3}. \quad (IV.101)
\]

Taking the second derivative of the above expression and evaluating it in \( x_0 \), we get that \( x_0 \) is the absolute minimum (at fixed \( p \) and \( q \)) if

\[
9 - \frac{p}{q} \frac{q}{p} > 0, \quad (IV.102)
\]

that is (recalling \( p \geq q \)), for

\[
1 \leq \frac{p}{q} < \frac{1}{2}(9 + \sqrt{77}), \quad (IV.103)
\]

otherwise the minimum is placed in \( x_\pm \). In conclusion
\[
g_{a,\text{min}}(p, q) = \frac{\pi}{3\sqrt{2}} \left[ 2 + \sqrt{5} \chi_a(p, q) \right],
\]
where

\[
\chi_a(p, q) = \begin{cases} 
1 + 10 - \frac{pq}{(p^2 + q^2)^2} & \text{if } 1 < \frac{p}{q} < \frac{1}{2}(9 + \sqrt{77}) \\
1 - 2\frac{pq}{(p^2 + q^2)^2} & \text{otherwise}.
\end{cases}
\]

The absolute minimum in the family is the minimum of the first row. Setting \( x = \frac{pq}{(p^2 + q^2)^2} \), we see that \( 1 + 10x^2 - 6x \) has a minimum for \( x = 3/10 \), which correspond to \( p = 3, q = 1 \). The corresponding absolute minimal energy is exactly (IV.73). This is not surprising at all, since the \( \epsilon \) corresponds to solution (IV.69) for \( N = 4 \) (use (IV.57) with \( \Lambda = 2 \) in (E.59)). This corresponds to the undeformed \( SU(2) \) embedding, as anticipated.

2. The case \( \epsilon_b = (1, 1, -1) \)

In this case we get

\[
v_b = i \text{diag}(1, 0, -1), \quad \|v_b\|^2 = 2.
\]

The Baryonic charge is

\[
B = 2\sigma_{p-q}m(p^2 + q^2).
\]

For the \( g \)-factor we get

\[
g_b(p, q, m, \tau) = L_r L_\gamma L_\phi \frac{K \pi^3}{4 \sigma_{p-q} m} \left[ \frac{16\tau_{p-q}^2}{L_r^2} + \frac{2}{(p^2 + q^2)L_r^2} + \frac{2\tau_{p-q}^2}{L_r^2 L_\gamma^2} + 8 \frac{m^2}{L_\gamma^2} \left( 1 + \frac{\lambda}{16L_r^2} \right) \right] + 8 \frac{m^2 \sigma_{p-q}^2}{L_\gamma^2} \left( 1 + \frac{p^2 q^2}{(p^2 + q^2)^2} + \frac{3\tau^4}{(p^2 + q^2)^2} - \frac{3\tau^2}{p^2 + q^2} \right),
\]

The corresponding minimal energy, given by (G.16), in this case becomes:

\[
g_b(p, q, \tau) = \frac{\pi}{3\sqrt{2}} \left[ 2 + \sqrt{2} \left( 1 + \frac{p^2 q^2}{(p^2 + q^2)^2} + \frac{3\tau^4}{(p^2 + q^2)^2} - \frac{3\tau^2}{p^2 + q^2} \right) \right].
\]

We can further minimise w.r.t. \( \tau \). Setting \( x = \tau^2 \), it is immediate to see that in this case the minimum is reached for

\[
x_0 = \frac{p^2 + q^2}{2},
\]

to which it corresponds the value

\[
g_{b,\text{min}}(p, q) = \frac{\pi}{3\sqrt{2}} \left[ 2 + \sqrt{2} \left( \frac{1}{4} + \frac{p^2 q^2}{(p^2 + q^2)^2} \right) \right].
\]
For fixed \( q \), this is a monotonic decreasing function of \( p \), so there is no an absolute minimum in this family. However, notice that the lower bound is

\[
g_{b, \text{bound}} = \frac{\pi}{6(1+2\sqrt{2})},
\]

which is (IV.73).

We finally notice that this kind of solutions are not deformations of an \( SU(2) \) or \( SO(3) \) embedding, despite one could suspect it. Indeed, \( v_\epsilon \), may at most belong to the representation\(^8\) \( 1 \oplus 0 \oplus 0 \) or \( 1 \oplus 0 \) embedded in \( SU(4) \). If so, there should exist a deformation of \( k_{\omega} \), a particular value of the moduli, such that \( k_{\omega} \) belongs into the same representation. But in both cases the particular solution would be embedded in \( SU(3) \) also and then it would require \( q = 0 \) or \( p = 0 \).

\[
g_c(p, q, m, \rho) = L_\gamma L_{\gamma'} L_\phi \frac{K \pi^3}{\sigma_{p-q} m} \left[ \frac{16 \sigma_{p-q}}{L_\rho^2} + \frac{1}{2(p^2 + q^2)L_\gamma^2} + \frac{\lambda \sigma_{p-q}}{L_\rho^2 L_\gamma^2} + \frac{8 m^2}{L_\gamma^2} \left( 1 + \frac{\lambda}{16L_\rho^2} \right) \right].
\]

The corresponding minimal energy, given by (G.16), in this case becomes:

\[
g_c(p, q, |\rho|) = \frac{\pi}{3\sqrt{2}} \left[ 2 + \left( 1 - 2\frac{p^2 q^2}{(p^2 + q^2)^2} \right)^{\frac{3}{2}} \right].
\]

This is independent on \( \tau \) and for fixed \( q \) it is a monotonic increasing function of \( p \). It follows that the lower bound is reached for \( p = q = 1 \) (the value 1 is enforced by the request that \( p \) and \( q \) are coprime, but the result depends only on \( p/q \))

\[
g_{c, \text{bound}} = g_c(1, 1) = \frac{\pi}{6}(1+2\sqrt{2}),
\]

which, again, is (IV.73). However, this is not allowed, since for \( p = q = 1 \) the functions \( f_j \) are not periodic and the solution of the equations does not yield a well defined global solution! In this particular family the absolute minimum is instead

\[
g_{c, \text{bound}} = g_c(2, 1) = \frac{\pi}{3\sqrt{2}}(2 + \frac{\sqrt{17}}{5}) \approx 2.0916,
\]

\[
3. \text{ The case } \varepsilon_c = (1, -1, 1)
\]

In this case we have

\[
v_c = \frac{i}{2}(1, -1, 1), \quad ||v_c||^2 = 1.
\]

Reasoning as before, we see that the field is now

\[
U_c = g_4(\sigma_{p-q} \Phi) \begin{pmatrix} e^{\frac{i}{4} \tau} & 0 & 0 \\ 0 & e^{-\frac{i}{4} \tau} & 0 \\ 0 & 0 & e^{\frac{i}{4} \tau} & 0 \end{pmatrix} g_4(m\gamma),
\]

\[
\Phi = \frac{t}{L_\rho} - \varphi,
\]

\[
\sigma_{p-q} = \begin{cases} 1 & \text{ for } p - q \text{ odd} \\
\frac{1}{2} & \text{ for } p - q \text{ even} \end{cases}.
\]

The Baryonic charge is

\[
B = 2\sigma_{p-q} m(p^2 + q^2).
\]

For the \( g \)-factor we get

\[
g_c(p, q, m, \rho) = L_\gamma L_{\gamma'} L_\phi \frac{K \pi^3}{\sigma_{p-q} m} \left[ \frac{16 \sigma_{p-q}}{L_\rho^2} + \frac{1}{2(p^2 + q^2)L_\gamma^2} + \frac{\lambda \sigma_{p-q}}{L_\rho^2 L_\gamma^2} + \frac{8 m^2}{L_\gamma^2} \left( 1 + \frac{\lambda}{16L_\rho^2} \right) \right].
\]

\[
4. \text{ The case } \varepsilon_d = (1, -1, -1)
\]

In this case

\[
v_d = i(0, -1, 0, 1), \quad ||v_d||^2 = 2.
\]

This case looks to be very similar to the case \( b \). Indeed, the reader can be easily check that the matrices

\[\text{We are using the convention that } s \text{ indicates the representation of spin } s.\]
\(v_b, k_\perp\) transform into \(v_d, k_\perp\) under the map
\[
\text{Mat}(N, \mathbb{C}) \rightarrow \text{Mat}(N, \mathbb{C}),
\]
\[
a_{j,k} \mapsto a_{N-j,N-k},
\]
\[
\mathbb{T}^3 \times W \rightarrow \mathbb{T}^3 \times W,
\]
\[
(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}, \tau) \mapsto (e^{i\alpha_3}, e^{i\alpha_2}, e^{i\alpha_1}, pq/\tau).
\]

Under this map the inverse Cartan matrix is invariant and \(\xi_k \mapsto -\xi_d \equiv \xi_{dl}\), where the last equivalence is by a global rescaling. This sort of duality makes the two families perfectly equivalent and giving the same minima.

**Remark:** We see that of the four predicted sequences of families the true inequivalent ones are the first three ones, while the \(d\) case is not really new. It is natural to expect that such duality extends to any \(N\), but this would require a deeper understanding of the global properties of the relevant moduli space \(W\). To this aim, it would be interesting to investigate the explicit cases \(N = 5\) and \(N = 6\). This, however, goes beyond the scope of the present work.

## V. SHEAR MODULUS FOR LASAGNA STATES

On the crust of ultra compact objects, like neutron stars, nucleons form large structures called pasta states. Knowing the elasticity properties of the crust may be very important to understand the structure of the gravitational waves emitted in a collision with a black hole. An important recent result has been found in [12] where, using numerical simulations based on the phenomenological nucleon-nucleon potential, the authors showed that the shear modulus for nuclear lasagna can have a much value larger than previous estimates. Here we give a first principle explanation of it as an application of the Skyrmionic model. To compute the shear modulus associated to lasagna states, our strategy will be to first compute it for the \(SU(2)\) case for the solutions determined in \([50, 53]\), by employing its relation with the \(1 + 1\) computations presented in \([62]\).

Let us begin with a review \([53]\). We will consider the symmetric case\(^9\) in Equations (13) and (16) of \([53]\), namely
\[
p = q, \quad l_2 = l_3 = \sqrt{A}.
\]

This means that we are considering configurations in which the \(SU(2)\) Skyrmions live in a box of volume \(V_{tot}\),
\[
V_{tot} = 16\pi^3 A l_1
\]
where \(l_1\) is the length along the \(r\) direction (which is the coordinate of the profile \(H\) in Eq. (13) of \([53]\)). The Baryonic charge corresponding to the Ansatz in Equations (12), (13) and (14) of \([53]\) is
\[
B = pq = p^2 \quad (V.1)
\]
(see below Eq. (24) page 5 of \([53]\)). Then, the \(SU(2)\) field equations for the Ansatz in Equations (12), (13), (14) and (16) of \([53]\) with a static profile \(H = H(r)\), reduce to
\[
-\frac{d^2 u}{dr^2} + \Gamma^2 \sin u = 0, \quad (V.2)
\]
where
\[
u(r) = 4H(r), \quad 0 \leq r \leq 2\pi, \quad (V.3)
\]
\[
\Gamma^2 = \left(\frac{B}{A}\right) \frac{\lambda^2}{4 + 2\lambda^2 A}, \quad (V.4)
\]
where \(\frac{B}{A}\) can be interpreted as the Baryon density per unit of area of the Lasagna configuration (up to \(\pi\) factors). In order to compare directly the present results with the ones in \([62]\), it is convenient to define the rescaled coordinate \(y\) as follows
\[
y = \Gamma r, \quad 0 \leq y \leq 2\pi\Gamma, \quad (V.5)
\]
so that the field equation (\(V.2\)) becomes
\[
-\frac{d^2 u}{dy^2} \sin u = 0 \Leftrightarrow \left(\frac{du}{dy}\right)^2 = 1 - \cos u + C, \quad (V.6)
\]
and the boundary conditions in order to have Baryonic charge \(B = pq = p^2\) are
\[
H(2\pi) = \frac{\pi}{2} \Leftrightarrow u(2\pi\Gamma) = 2\pi. \quad (V.7)
\]

Now, equations (\(V.5\)), (\(V.6\)) and (\(V.7\)) (which are equivalent to the results in \([53]\)) can be compared directly with equations (2.4), (2.7) and (2.9) of \([62]\). In particular, the dictionary between the results of \([62]\) and the present ones is
\[
\phi(x) \rightarrow u(y), \quad (V.8)
\]
\[
L \rightarrow 2\pi\Gamma, \quad (V.9)
\]
\[
k \rightarrow \sqrt{\frac{2}{C + 2}} \equiv \tau, \quad (V.10)
\]
\[
k' \rightarrow \sqrt{\frac{C}{C + 2}} \quad (V.11)
\]

\(\text{Notice that we are referring to the } p \text{ and } q \text{ in } [53], \text{ which have different meaning than the } p \text{ and } q \text{ used in the previous section}\)
where the left hand side (with respect to the “→”) is from [62] while the right hand side comes from the above equations. Equations (2.9) and (2.10) of [62] read

\[ L = 2\pi I_{-1/2}(\tau), \]
\[ I_{-1/2}(\tau) = \int_{0}^{\pi/2} dy \left( 1 - \tau^2 \sin^2 y \right)^{-1/2}, \]

that is

\[ \pi\Gamma = \sqrt{\frac{2}{C+2}} \int_{0}^{\pi/2} dy \left( 1 - \frac{2}{C+2} \sin^2 y \right)^{1/2}, \]

which fixes the integration constant \( C \) in Eq. (V.6) in terms of \( \Gamma \)

\[ C = C(\Gamma), \]

which depends on the Baryon charge as well as the size of the box in which the configuration lives. Now, with the above dictionary, we can write the speed of sound of the phonons using Eq. (3.15) of [62]:

\[ V_{\text{phonons}} = \sqrt{\frac{C}{2}} \frac{\pi\Gamma}{\int_{0}^{\pi/2} dy \left( 1 - \frac{2}{C+2} \sin^2 y \right)^{1/2}} = \sqrt{\frac{G_{SU(2)}}{T_{00}}}, \]

where the \( T_{00} \) is given in Eq. (28) of [53]. Thus we have the following expression for the shear modulus \( G_{SU(2)} \) in the \( SU(2) \) case

\[ G_{SU(2)} = \left( V_{\text{phonons}} \right)^2 T_{00}. \]

We can then estimate it as follows. In place of \( T_{00} \) we use its mean value computed as

\[ \bar{T}_{00} = \frac{E_{0 \text{min}}^{31}}{16\pi^3 L \gamma}, \]

where \( E_{0 \text{min}} \) is the minimal energy corresponding to \( B = 1 \). From Table 1 of [62] we see that \( B/A \) is independent from \( B \) for the minimal energy configuration. Using the values in the table\(^{10}\) we get

\[ \bar{T}_{00} \simeq 1.26 \times 10^{34} \text{erg/cm}^3. \]

With the same values, from (V.4) we obtain

\[ \Gamma \simeq 0.371, \quad \pi\Gamma \simeq 1.166. \]

Therefore condition (V.12), which is easily solved numerically after noticing that \( I_{-\frac{1}{2}}(\tau) = K(\tau^2) \), the first complete elliptic integral, gives

\[ C \simeq 2.73 \]

and

\[ V_{\text{phonons}} \simeq 0.1198. \]

Finally,

\[ G_{SU(2)} \simeq 1.8 \times 10^{32} \text{erg/cm}^3. \]

Notice that the present value is expected to an approximation from above, since we are using a Skyrmionic effective model. From the above analysis, taking into account (IV.73), we can infer that in any case the true value should be \( G_{SU(2)} \gtrsim 10^{31} \text{erg/cm}^3 \). The comparison with [12] is very good especially taking into account that we only used the Skyrme model.

At this point we can use the new solutions found in the present work to relate the shear modulus for \( SU(N) \) case to the one for \( SU(2) \).

Let us consider the minimal energy per nucleon (IV.72). After multiplying by \( B \) and dividing by the volume, which, because of (IV C) is proportional to \( \lambda^2 \), we get

\[ \bar{T}_{00} \propto \frac{K}{\lambda} N(N^2 - 1). \]

On the other hand, the Baryon density is

\[ n = \frac{B}{8\pi^2 L \gamma L \gamma} \propto \frac{N(N^2 - 1)}{\lambda^{3/2}}, \]

which solved for \( \lambda \) and replaced in \( \bar{T}_{00} \) gives

\[ \bar{T}_{00} \propto n^{2/3} \sqrt[3]{N(N^2 - 1)}. \]

Assuming the speed of sound to be essentially independent from \( N \), as suggested by the fact that all the component of \( T_{\mu\nu} \) scale in the same way with \( N \), we get that the dependence of the shear modulus from \( N \) is

\[ G_{SU(N)} \propto \sqrt[3]{N(N^2 - 1)}, \]

so that we get the final estimate for the value of the shear modulus \( G_{SU(N)} \) of the \( SU(N) \) Skyrme model as

\[ G_{SU(N)} = a(N) G_{SU(2)} \]

\[ a(N) = \sqrt[3]{\frac{N(N^2 - 1)}{6}}. \]

\(^{10}\) Notice that with these values the baryon density is \( n \simeq 0.0468 \text{fm}^{-3} \approx 0.05 \text{fm}^{-3} \), the same value used in the simulations of [12].
VI. CONCLUSION AND PERSPECTIVES

In conclusion, we constructed the first examples of analytic (3+1)-dimensional Skyrmions living at finite Baryon density in the SU(N) Skyrme model (which are not trivial embeddings of SU(2) into SU(N)) for any N. These results allow to compute explicitly the energy to Baryon charge ratio for any N and to discuss its smooth large N limit as well as the closeness to the BPS bound. The energy density profiles of these finite density Skyrmions have lasagna-like shape. A quite remarkable by-product of the present analysis is that we have been able to estimate analytically the shear modulus of lasagna-shaped configurations which appear at finite Baryon density. Our estimate agrees with recent results [12] based on many body simulations in nuclear physics using phenomenological nucleon-nucleon interaction potentials.

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Appendix A: General facts and conventions about SU(N)

In this section we collect some general facts we applied for getting the solutions. Let V, (,) the N-dimensional complex vector space isomorphic to \( \mathbb{C}^N \), endowed with the canonical hermitian product
\[
(z|w) = \sum_{j=1}^{n} z_j^* w_j, \quad z,w \in \mathbb{C}^n,
\]
\[(x+iy)^* = x - iy, \quad x,y \in \mathbb{R}.
\]

The unitary group U(N) \( \equiv U(V) \) is the group of unitary transformations of V. Looking U(V) as automorphisms of V determines the smallest fundamental representation, simply called V. The action of U(V) over V induces an action on the external products \( \wedge^k V \) of V, and the corresponding homomorphisms
\[
U(V) \rightarrow \text{Aut}(\wedge^k V), \quad k = 1, \ldots, N
\]
are all representations, also called \( \wedge^k V \). For \( k = 1, \ldots, N-1 \) are all faithful (i.e. the kernel of the map is the identity transformation) and are called the fundamental representations. Any other finite dimensional representation is obtained by their tensor products. \( \wedge^N V \) is not faithful. The corresponding kernel is a normal subgroup of U(N) called the special group SU(N) \( \equiv SU(V) \).

SU(N) is a compact simply connected simple Lie group of rank N−1. It essentially means that it contains a maximal abelian torus \( T_N \) of dimension \( N-1 \). On V, it is represented by the diagonal \( N \times N \) matrices T such that
\[
\prod_{j=1}^{N} T_{jj} = 1, \quad |T_{jj}| = 1, \quad j = 1, \ldots, N.
\]

The center \( Z_N \) of SU(N) is the subgroup of T consisting of the elements commuting with the whole SU(N) (equivalently, it is the kernel of the adjoint representation). It consists of matrices of the form \( \omega I \), where \( \omega^N = 1 \) and I is the identity matrix. Therefore, \( Z_N \simeq \mathbb{Z}_N \). All the other compact simple Lie groups locally isomorphic to SU(N) are the quotients \( SU(N)/\Gamma \), where \( \Gamma \) is any given subgroup of \( Z_N \). They are not simply connected, since their first homotopy group is \( \pi_1(SU(N)/\Gamma) \simeq \mathbb{Z}/\Gamma \). SU(N) is the universal covering of all of them. In particular, for \( N = 2 \) we have just two groups, which are SU(2) and \( SU(2)/\mathbb{Z}_2 \simeq SO(3) \).

To any Lie group G one associates the corresponding Lie algebra \( L(G) \), which is the algebra of left invariant vector fields\(^{11} \) over G, endowed with the Lie bracket product. In matrix representation it reduces to the commutator \([,]\). Since the groups SU(N)/\( \Gamma \) are locally isomorphic to SU(N), their Lie algebras are all isomorphic. One gets
\[
\mathfrak{su}(N) \equiv LIE(SU(N)) = \{ X \in \text{Mat}(N) | X^\dagger = -X, \quad \text{Tr}X = 0 \}\quad (A.1)
\]

i.e. the antihermitian traceless \( N \times N \) complex matrices.

In particular, \( H := \text{Lie}(T_N) \) is a maximal abelian subalgebra of \( \mathfrak{su}(N) \), having the property that, for any \( X \in H \), the linear map \( ad_X : \mathfrak{su}(N) \rightarrow \mathfrak{su}(N) \) defined by \(^{12} \) \( ad_X(Y) = [X,Y] \) for any \( Y \in \mathfrak{su}(N) \), is diagonalizable (on the complexification of the algebra).

We see from the definition that \( \mathfrak{su}(N) \) is a real vector

\(^{11}\) i.e. the vector fields invariant under the action of the left translation \( L_g : G \rightarrow G, L_g(h) = gh \), for any given \( g \in G \)

\(^{12}\) This is called the adjoint action and defines the adjoint representation of the algebra over itself
space of dimension \( N^2 - 1 \). A basis can be easily obtained as follows. For any \( j,k = 1, \ldots, N \) we define the matrix \( E_{j,k} \) with elements

\[
(E_{j,k})_{mn} = \delta_{jm} \delta_{kn}. \tag{A.2}
\]

They are called the elementary matrices. With these notations, a basis of \( \mathfrak{su}(N) \) is given by

\[
\begin{align*}
A_{j,k} &= (E_{j,k} - E_{k,j}), & S_{j,k} &= i(E_{j,k} + E_{k,j}), \\
1 &\leq j < k \leq N, \\
J_h &= i(E_{j,j} - E_{j+1,j+1}), & h &= 1, \ldots, N - 1. \tag{A.3}
\end{align*}
\]

In particular, the matrices \( J_h \) span the Cartan subalgebra \( H \).

1. Roots and simple roots

A concept that is particularly helpful for most of the calculations we need is the one of roots. These are related to the above observation regarding the diagonalizability of \( ad_X \) for any \( X \in H \). The diagonalizability must be checked on \( \mathfrak{su}(N) \otimes \mathbb{C} \), which is generated by the complex span of the basis given above, in place of the real span. Notice that the complex span contains the matrices \( E_{i,j}, i \neq j \). This is sufficient to determine all the eigenvectors and eigenvalues of \( ad_X \) for any given \( X \in H \).

To this aim, let us specify \( H \) as follows:

\[
H = \left\{ X = i \sum_{j=1}^{n} c_j E_{j,j} \left| \sum_{j} c_j = 0 \right\} \right\}_{\mathbb{R}}, \tag{A.5}
\]

where \( \{ \cdots \}_{\mathbb{R}} \) we mean the span over \( \mathbb{R} \) of \( \cdots \). Thus, we immediately see that

\[
\begin{align*}
[X, E_{j,j}] &= i(c_j - c_k)E_{j,k}, & \quad (A.6) \\
[X, J_h] &= 0, & \quad (A.7)
\end{align*}
\]

so that \( E_{j,k} \) and \( J_h \) are eigenmatrices of the adjoint action of \( X \), with eigenvalues \( i(c_j - c_k) \) and 0 respectively. The point is that the eigenvalues depend linearly on \( X \). Let us consider the linear operators \( L_j \), \( j = 1, \ldots, N \) defined by

\[
L_j : \text{Mat}(N) \to \mathbb{C}, \quad A \mapsto A_{j,j}.
\]

Then, we can write \( i c_j = L_j(X) \) so that

\[
\begin{align*}
\text{ad}_X(E_{j,k}) &= (L_j - L_k)(X) E_{j,k}. & \quad (A.7)
\end{align*}
\]

The linear operators

\[
\beta_{j,k} := L_j - L_k : H \to \mathbb{C} \quad \tag{A.8}
\]

are said the non vanishing roots of \( \mathfrak{su}(N) \). The corresponding eigenspaces are one dimensional. Beyond these, there is the vanishing root defining the 0 eigenvalue, which eigenspace is \( H \), so that has degeneration equal to the rank \( r = N - 1 \). In particular, the set of non vanishing roots contains a set of \( r \) linearly independent roots, having the property that all the remaining roots are combination of them with all non positive or all non negative integer coefficients. These are called the simple roots and are

\[
\beta_j := L_j - L_{j+1}, \quad j = 1, \ldots, N - 1. \tag{A.9}
\]

Finally, for convenience, we introduce the less conventional concept of real valued roots \( \alpha_{j,k} = -i \beta_{j,k} \), \( \alpha_j = -i \beta_j \), which we will simply call again roots and simple roots. With this convention, for the simple roots \( \alpha_j \), we can also write

\[
\alpha_j : H \to \mathbb{R}, \quad X \mapsto -\text{Tr}(J_jX), \tag{A.10}
\]

useful for practical purposes. This also shows that the \( \alpha_j \) are linearly independent. We name the corresponding eigenvectors \( \lambda_{\alpha_j} \equiv \lambda_j = E_{j,j+1} \), so that

\[
[X, \lambda_j] = i \alpha_j(X) \lambda_j, \quad \forall X \in H. \tag{A.11}
\]

Notice that \( \lambda_{-\alpha_j} = \lambda_{\alpha_j}^\dagger \), so

\[
[X, \lambda_j^\dagger] = -i \alpha_j(X) \lambda_j^\dagger. \tag{A.12}
\]

2. Some further technical facts

There is a canonical way to introduce a scalar product on the real space spanned by the simple roots. We however bypass the historical construction and employ (A.10) to define the scalar product

\[
(\alpha_j | \alpha_k) := -\text{Tr}(J_j J_k). \tag{A.13}
\]

On

\[
H^*_{\mathbb{R}} := \langle \alpha_1, \ldots, \alpha_{N-1} \rangle_{\mathbb{R}}
\]

it is an euclidean scalar product. One then defines the \( r \times r \) Cartan matrix\(^{13}\) \( C_{A_{N-1}} \) with components

\[
(C_{A_{N-1}})_{j,k} := 2 \frac{\left( \alpha_j | \alpha_k \right)}{\left( \alpha_j | \alpha_j \right)} = (\alpha_j | \alpha_k)
\]

\[
= 2 \delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1}.
\]

\(^{13}\) The name comes from the fact that in the Dynkin classification the algebra \( \mathfrak{su}(N) \) is called \( A_r \), where \( r \) is the rank
The Cartan matrix is strictly positive definite. Indeed, for any vector \((x^1, \ldots, x^r) \in \mathbb{R}^r\) we have
\[
\sum_{j,k} x^j x^k (C_{A_{N_1}}^{-1})_{j,k} = 2 \sum_{j=1}^r x_j^2 - \sum_{j=1}^{r-1} 2x_j x_{j+1} \\
= x_1^2 + x_r^2 + \sum_{j=1}^{r-1} (x_j - x_{j+1})^2,
\]
which is strictly positive and vanishes only for \(x_j = 0\) for all \(j\). In particular, the Cartan matrix is invertible and, indeed, one easily checks that
\[
(C_{A_{N_1}}^{-1})_{j,k} = \frac{1}{N} \min(j, k)(N - \max(j, k)).
\]
Another important fact to notice is that for \(j, k\) one has
\[
[\lambda_j, \lambda_k] = \delta_{j+1,k} E_{j+2}.
\]

**Appendix B: Proof of Proposition 1**

In order to prove the proposition, it is convenient to work with the coordinates \(\Phi\) and \(T = t + L_\gamma \Phi\). The metric takes the form \(ds^2 = -L_\Phi dt d\Phi + L_\gamma^2 dr^2 + L_\gamma^2 d\gamma^2\). With the given Ansatz, after replacing \(\Phi\) with \(\sigma\Phi\) for constant \(\sigma\) (for convenience), for the \(L_{\mu}\) we get
\[
R_T = 0, \quad R^T = -\frac{2}{L_\Phi} R_\Phi, \\
R_\Phi = \sigma e^{-m\gamma k} e^{-h(r)} k e^{h(r)} e^{m\gamma k}, \quad R^\Phi = 0, \\
R_r = e^{-m\gamma k} h'(r) e^{m\gamma k}, \quad R^r = \frac{1}{L_\gamma^2} R_\gamma, \\
R_\gamma = m k, \quad R^\gamma = \frac{1}{L_\gamma^2} R_\gamma.
\]
For \(F_{\mu\nu} = [R_\mu, R_\nu]\), with \(x = e^{h(r)} k e^{h(r)}\), we get the non-vanishing components
\[
F_{\Phi r} = -F_{r\Phi} = \sigma e^{-m\gamma k} [x, h'] e^{m\gamma k}, \\
F_{\Phi \gamma} = -F_{\gamma \Phi} = \sigma m e^{-m\gamma k} [x, k] e^{m\gamma k}, \\
F_{r \gamma} = -F_{\gamma r} = m e^{-m\gamma k} [h', k] e^{m\gamma k}.
\]
Setting \(L_{\mu} := [L^\nu, F_{\mu\nu}]\), the equations of motion are
\[
0 = \partial^\mu R_\mu + \frac{\lambda}{4} \partial^\mu L_{\mu}.
\]
Using that nothing depends on \(T\) and that there are no lower \(T\) components, these reduce to
\[
0 = \frac{1}{L_\gamma^2} \partial_\gamma (R_\gamma + \frac{\lambda}{4} L_\gamma) + \frac{1}{L_\gamma^2} \partial_\gamma (R_\gamma + \frac{\lambda}{4} L_\gamma).
\]
But
\[
\partial_\gamma r_\gamma = 0, \\
\partial_\gamma r_\gamma = e^{-m\gamma k} h'' e^{m\gamma k}, \\
\partial_\gamma L_\gamma = \partial_\gamma \left( \frac{m^2}{L_\gamma^2} e^{-m\gamma k} [k, [h', k]] e^{m\gamma k} \right) \\
= \frac{m^2}{L_\gamma^2} e^{-m\gamma k} [k, [h'', k]] e^{m\gamma k}, \\
\partial_\gamma L_\gamma = -\frac{m^2}{L_\gamma^2} \partial_\gamma \left( e^{-m\gamma k} [h', [h', k]] e^{m\gamma k} \right) \\
= \frac{m^2}{L_\gamma^2} e^{-m\gamma k} [k, [h', [h', k]]] e^{m\gamma k},
\]
so (B.1) becomes
\[
0 = \frac{1}{L_\gamma^2} e^{-m\gamma k} \left( h'' - \frac{\lambda}{4} \frac{m^2}{L_\gamma^2} ([k, [k, h'']] - [k, [h', [h', k]]]) \right) e^{m\gamma k},
\]
which proves the proposition.

1. Further details

Making use of (III.6) and (A.11) we can write
\[
[h', k] = \sum_{j=1}^{N-1} (c_j [h', \lambda_j] - c^*_j [h', \lambda_j]) = i \sum_{j=1}^{N-1} \alpha_j (h')(c_j \lambda_j + c^*_j \lambda_j).
\]
Repeating the same calculation:

\[ [h', [h', k]] = i \sum_{j=1}^{N-1} (\alpha_j(h')c_j [h', \lambda_j] + \alpha_j(h')c_j^* [h', \lambda_j^*]) = - \sum_{j=1}^{N-1} \alpha_j(h')^2 (c_j \lambda_j - c_j^* \lambda_j^*). \]

Finally,

\[ [k, [h', [h', k]]] = - \sum_{k=1}^{N-1} \sum_{j=1}^{N-1} \alpha_j(h')^2 \left\{ c_k c_j [\lambda_k, \lambda_j] + c_k^* c_j^* [\lambda_k^*, \lambda_j^*] - c_k c_j^* [\lambda_k, \lambda_j^*] - c_k^* c_j [\lambda_k^*, \lambda_j] \right\} = - \sum_{k=1}^{N-1} \sum_{j=1}^{N-1} \alpha_j(h')^2 \left\{ c_k c_j [\lambda_k, \lambda_j] + c_k^* c_j^* [\lambda_k^*, \lambda_j^*] - c_k c_j^* [\lambda_k, \lambda_j^*] + c_k^* c_j [\lambda_k^*, \lambda_j] \right\}. \]

The last two terms cancel after summation, while the first terms vanish for \( j = k \), so we get

\[ [k, [h', [h', k]]] = - \sum_{j<k} (\alpha_j(h')^2 - \alpha_k(h')^2) \left( c_j c_k [\lambda_j, \lambda_k] + c_j^* c_k^* [\lambda_j^*, \lambda_k^*] \right), \]

where we have changed the order of commutators in the first sum and exchanged the name of variable in the second sum. Therefore

\[ [k, [h', [h', k]]] = \sum_{j<k} (\alpha_j(h')^2 - \alpha_k(h')^2) \left( c_j c_k [\lambda_j, \lambda_k] + c_j^* c_k^* [\lambda_j^*, \lambda_k^*] \right). \tag{B.3} \]

Similarly,

\[ [k, h'']= -[h'', k] = -i \sum_{j=1}^{N-1} \alpha_j(h'') (c_j \lambda_j + c_j^* \lambda_j^*), \]

and

\[ [k, [k, h'']] = -i \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \alpha_j(h'') \left[ c_j c_k [\lambda_k, \lambda_j] - c_j^* c_k^* [\lambda_k^*, \lambda_j^*] - c_j c_k^* [\lambda_k, \lambda_j^*] + c_j^* c_k [\lambda_k^*, \lambda_j] \right]. \]

The first two terms can be treated as above, giving the contribution

\[ i \sum_{j<k} (\alpha_j(h'') - \alpha_k(h'')) \left( c_j c_k [\lambda_j, \lambda_k] - c_j^* c_k^* [\lambda_j^*, \lambda_k^*] \right), \]

while the last two terms, after renaming the labels in the first of the sums, give the contribution

\[ -i \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} (\alpha_j(h'') + \alpha_k(h'')) c_k c_j^* [\lambda_k, \lambda_j^*]. \]

Now,

\[ [\lambda_k, \lambda_j^*] = [E_{k,k+1}, E_{j+1,j}], \]

which in components is

\[ [E_{k,k+1}, E_{j+1,j}]_{m,r} = \delta_{j,k} (E_{j,j} - E_{j+1,j+1})_{m,r} \]

so that

\[ [\lambda_k, \lambda_j^*] = -i \delta_{j,k} J_j. \tag{B.4} \]

We finally get

\[ [k, [k, h'']] = i \sum_{j<k} (\alpha_j(h'') - \alpha_k(h'')) \cdot \left( c_j c_k [\lambda_j, \lambda_k] - c_j^* c_k^* [\lambda_j^*, \lambda_k^*] \right) - 2 \sum_{j=1}^{N-1} \alpha_j(h'') |c_j|^2 J_j. \tag{B.5} \]
2. A further proposition

We want now to state another technical proposition:

**Proposition 5.** Assume \( k_{\underline{3}} = \sum_{j=1}^{N-1} (c_j E_{j+1,j} - c_j^* E_{j-1,j}) \), \( h' \in H \) a matrix such that \( \alpha_j(h') = \varepsilon_j a \), where \( \varepsilon_j \) is a sign, \( j = 1, \ldots, N - 1 \), and \( x := e^{-h'r} k_{\underline{3}} e^{h'r} \). Then

\[
\text{Tr}(k_{\underline{3}}^2) = -2\|x\|^2, \tag{B.6}
\]

\[
\text{Tr}(h'[k_{\underline{3}}][h', k_{\underline{3}}]) = \text{Tr}(h'[x][h', x]) = -2a^2\|x\|^2, \tag{B.7}
\]

and

\[
\text{Tr}(x[k_{\underline{3}}][x, k_{\underline{3}}]) = -8\sin^2(ar) \left( \sum_{j=1}^{N-1} |c_j|^4 + \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 \frac{1}{2}(1 - 3\varepsilon_j \varepsilon_{j+1}) \right). \tag{B.8}
\]

Proof. First, we have

\[
\text{Tr}(k_{\underline{3}}^2) = \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} (c_j c_k \text{Tr}(\lambda_j \lambda_k^*) + c_j^* c_k^* \text{Tr}(\lambda_j^* \lambda_k^*))
\]

\[
- c_j^* c_k \text{Tr}(\lambda_j \lambda_k^*) - c_j c_k^* \text{Tr}(\lambda_j^* \lambda_k^*)
\]

(B.9)

where we used the notation \( \lambda_j = E_{j,j+1} \). Since \( \lambda_j \) is upper diagonal, so is \( \lambda_j \lambda_k \), hence \( \text{Tr}(\lambda_j \lambda_k) = 0 \).

Similarly, \( \text{Tr}(\lambda_j^* \lambda_k^*) = 0 \). On the other hand

\[
\text{Tr}(\lambda_j^* \lambda_k) = \sum_{n=1}^{N} \sum_{m=1}^{N} (E_{j+1,j})_{nm} (E_{k,k+1})_{mn}
\]

\[
= \sum_{n=1}^{N} \sum_{m=1}^{N} \delta_{j+1,n} \delta_{j+1,n} \delta_{k+1,m} \delta_{k,m}
\]

\[
= \delta_{kj} = \text{Tr}(\lambda_j \lambda_k). \tag{B.10}
\]

This proves (B.6).

Now, notice that

\[
[h', x] = [h', e^{-h'r} k_{\underline{3}} e^{h'r}] = e^{-h'r}[h', k_{\underline{3}}] e^{h'r} \tag{B.11}
\]

since \( h' \) commutes with \( e^{h'r} \). Therefore,

\[
\text{Tr}([h', x][h', x]) = \text{Tr}(e^{-h'r}[h', k_{\underline{3}}][h', k_{\underline{3}}] e^{h'r})
\]

\[
= \text{Tr}([h', k_{\underline{3}}][h', k_{\underline{3}}]), \tag{B.12}
\]

because of the cyclicity property of the trace. So we are left with the computation of \( \text{Tr}([h', k_{\underline{3}}][h', k_{\underline{3}}]) \). Using (B.2) and the fact that the only non vanishing traces are \( \text{Tr}(\lambda_j^* \lambda_k) = \delta_{j,k} \), we get

\[
\text{Tr}([h', k_{\underline{3}}][h', k_{\underline{3}}]) = \text{Tr} \left( \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} (i \alpha_j(h') \lambda_j c_j^* + i \alpha_j(h') c_j \lambda_j^*) (i \alpha_k(h') \lambda_k c_k^* + i \alpha_k(h') c_k \lambda_k^*) \right)
\]

\[
= -2 \sum_{j=1}^{N-1} \alpha_j(h')^2 c_j c_j^* = -2a^2\|x\|^2, \tag{B.13}
\]

where we used that \( \alpha_j(h')^2 = (\varepsilon_j a)^2 = a^2 \). This proves (B.7).

Let us write \( h = h'r \). Therefore,

\[
x = e^{-h} k_{\underline{3}} e^h = \sum_{j=1}^{N-1} (c_j e^{-h} \lambda_j e^h - h.c.). \tag{B.14}
\]

Using the notation \( ad_X(Y) = [X, Y] \) for any pair of
matrices $X, Y \in \mathfrak{su}(N)$, we first notice the identity
\[
e^{tX}Ye^{-tX} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n a^X_n(Y), \tag{B.15}
\]
where with $a^X_n$ we mean the iterated application of $ad_X$. Indeed,
\[
\frac{d}{dt} (e^{tX}Ye^{-tX}) = e^{tX}ad_X(Y)e^{-tX}. \tag{B.16}
\]
Hence
\[
\frac{d^n}{dt^n} \big|_{t=0} (e^{tX}Ye^{-tX}) = e^{tX}ad_X^n(Y)e^{-tX} \big|_{t=0} = a^X_n(Y), \tag{B.17}
\]
so that (B.15) is the Taylor expansion of $e^{tX}Ye^{-tX}$.

For $Y = k_x$, $X = h$ and $t = -1$, and using that $ad_h(\lambda_j) = i\alpha_j(h) = i\varepsilon_j at$ we then have
\[
e^{-h}\lambda_j e^h = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n a^h_n(\lambda_j)
= \sum_{n=0}^{\infty} \frac{1}{n!} (-i\varepsilon_j ar)^n\lambda_j
= e^{-i\varepsilon_j ar}\lambda_j. \tag{B.18}
\]

So
\[
x = \sum_{j=1}^{N-1} (c_j e^{-i\varepsilon_j ar}\lambda_j - c^*_j e^{i\varepsilon_j ar}\lambda^\dagger_j), \tag{B.19}
\]
and
\[
[x, k_x] = \sum_{j,k} \left( c_j c_k e^{-i\varepsilon_j ar} [\lambda_j, \lambda_k] + c_j^* c_k^* e^{i\varepsilon_j ar} [\lambda^\dagger_j, \lambda^\dagger_k] - c_j c_k e^{-i\varepsilon_j ar} [\lambda_j, \lambda^\dagger_k] - c_j^* c_k^* e^{i\varepsilon_j ar} [\lambda^\dagger_j, \lambda_k] \right). \tag{B.20}
\]

By using (B.4), we see that the last two terms sum up to
\[
- \sum_{j=1}^{N-1} |c_j|^2 J_j i(e^{-i\varepsilon_j ar} - e^{i\varepsilon_j ar})
= -2 \sum_{j=1}^{N-1} |c_j|^2 \sin(\varepsilon_j ar) J_j. \tag{B.21}
\]

On the other hand
\[
[\lambda_j, \lambda_k] = [E_{j,j+1}, E_{k,k+1}]
= \delta_{k+1,j+1} E_{j,j+2} - \delta_{k+1,j} E_{j,j+2}, \tag{B.22}
\]
so that
\[
\sum_{j,k} c_j c_k e^{-i\varepsilon_j ar} [\lambda_j, \lambda_k]
= \sum_{j=1}^{N-2} c_j c_{j+1} (e^{-i\varepsilon_j ar} - e^{-i\varepsilon_{j+1} ar}) E_{j,j+1}. \tag{B.23}
\]

and, similarly, by taking the hermitian conjugate,
\[
\sum_{j,k} c^*_j c^*_k e^{i\varepsilon_j ar} [\lambda^\dagger_j, \lambda^\dagger_k]
= - \sum_{j=1}^{N-2} c^*_j c^*_{j+1} (e^{i\varepsilon_j ar} - e^{i\varepsilon_{j+1} ar}) E_{j+1,j}. \tag{B.24}
\]
This leads to
\[
[x, k_x] = \sum_{j=1}^{N-2} \left( c_j c_{j+1} (e^{-i\varepsilon_j ar} - e^{-i\varepsilon_{j+1} ar}) E_{j,j+2} - c^*_j c^*_{j+1} (e^{i\varepsilon_j ar} - e^{i\varepsilon_{j+1} ar}) E_{j+2,j} \right)
- 2 \sum_{j=1}^{N-1} |c_j|^2 \sin(\varepsilon_j ar) J_j. \tag{B.25}
\]
Using that the only non vanishing traces are
\[
\text{Tr}(E_{j,j+2}E_{k+2,k}) = \text{Tr}(E_{j+2,j}E_{k,k+2}) = \delta_{jk}, \quad \text{Tr}(J_jJ_k) = -2\delta_{jk} + \delta_{j,k+1} + \delta_{j+1,k}.
\]
we get
\[
\text{Tr}(\omega_{x,k}[\omega_{x,k}]) = -2 \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 |e^{-i\varepsilon_j a r} - e^{-i\varepsilon_{j+1} a r}|^2 - 8 \sum_{j=1}^{N-1} |c_j|^4 \sin^2(\varepsilon_j a r)
\]
\[
+ \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 \sin(\varepsilon_j a r) \sin(\varepsilon_{j+1} a r).
\]

Now,
\[
|e^{-i\varepsilon_j a r} - e^{-i\varepsilon_{j+1} a r}|^2 = 2(1 - \cos(\ar(\varepsilon_j - \varepsilon_{j+1})))
\]
\[
= 4 \sin^2\left(\frac{\ar(\varepsilon_j - \varepsilon_{j+1})}{2}\right).
\]

Since \((\varepsilon_j - \varepsilon_{j+1})/2 = 0, \pm 1\), we can write
\[
\sin^2\left(\frac{\ar(\varepsilon_j - \varepsilon_{j+1})}{2}\right) = \frac{1}{2}(1 - \varepsilon_j \varepsilon_{j+1}) \sin^2(\ar). \tag{B.28}
\]

Also
\[
\sin(\varepsilon_j a r) \sin(\varepsilon_{j+1} a r) = \sin^2(\ar) \varepsilon_j \varepsilon_{j+1} \tag{B.29}
\]
and \(\sin^2(\varepsilon_j a r) = \sin^2(\ar)\), so that summing all up we get (B.8).

\[\text{Appendix C: SU}(2) \text{ versus } \text{SO}(3).
\]

Despite these being very well known facts, in this appendix we want to discuss the difference between \(\text{SU}(2)\) and \(\text{SO}(3)\), since it is crucial to identify our solutions. Locally, the two groups coincide, they have the same Lie algebra. However, \(\text{SU}(2)\) is simply connected, while \(\text{SO}(3)\) is not. Indeed, \(\text{SU}(2)\) is the universal covering of \(\text{SO}(3)\). It has a nontrivial center \(Z_{\text{SU}(2)} = \pm I\), \(I\) being the unit element, and there is a surjective projection
\[
\pi : \text{SU}(2) \rightarrow \text{SO}(3) \tag{C.1}
\]
having \(Z_{\text{SU}(2)}\) as kernel. \(\text{SO}(3)\) has trivial kernel and \(\pi_1(\text{SO}(3)) \cong Z_{\text{SU}(2)}\). We can also write
\[
\text{SO}(3) \cong \text{SU}(2)/Z_{\text{SU}(2)}\tag{C.2}
\]
Now, let us illustrate the crucial difference we are interested in. Let \(\tau_i, i = 1, 2, 3\) a canonical basis of Lie(G), \(G\) being one of the two groups. We can then realise the group by means of the Euler parametrisation. This means that the generic element \(g\) of the group has the form
\[
g(a, b, c) = e^{a\tau_3} e^{b\tau_2} e^{c\tau_3}. \tag{C.3}
\]
\(a, b, c\) are the Euler angles. Each of the exponentials has a period (depending on the normalisation of the matrices), say \(T_3\) for \(a\) and \(c\), and \(T_2\) for \(b\). The strategy to correctly cover \(G\) exactly one time is explained in [44] and works as follows. To be sure to cover \(G\) one integer number of times one first allow the coordinates to run each one in the respective period. This number, in general, is larger than one because of redundancies, due to two reasons. The first reason is that the central element, parametrised by \(b\), is chosen in the maximal torus (the exponential of the Cartan matrix). The redundancies correspond to the action of the Weyl group to the torus. This action is determined by the algebra and is the same for both \(\text{SU}(2)\) and \(\text{SO}(3)\). It shows that indeed moving \(b\) along a period quadruplicates the determination of the points for \(\text{SU}(2)\) and duplicates for \(\text{SO}(3)\), and one can reduce the range of \(b\) down to \(T_2/4\) or \(T_2/2\) respectively. At this point, the difference between \(\text{SU}(2)\) and \(\text{SO}(3)\) appears. Indeed, for \(\text{SO}(3)\) this is the end of the story, it is already covered just one time, while for \(\text{SU}(2)\) it remains a redundancy and we covered it twice. This redundancy is due to the fact that
\[
e^{b\tau_2} \cap e^{c\tau_3} = \begin{cases} I & \text{if } G = \text{SO}(3) \\ \Delta = e^{(T_3/2)\tau_3} & \text{if } G = \text{SU}(2) \end{cases} \tag{C.4}
\]
Therefore, since $\Delta^2 = I$

\[ g(a, b, c) = e^{aT_1}e^{bT_2}e^{cT_3} \]

\[ = e^{aT_3}e^{bT_2}e^{-aT_3} \]

\[ = e^{aT_3}e^{bT_2}e^{-aT_3} \]

\[ = e^{aT_3}e^{bT_2}e^{-aT_3} \]

\[ = g(a - T_3/2, b, c - T_3/2). \quad (C.5) \]

This redundancy is eliminated by reducing the range of $a$ down to $T_3/2$ for $SU(2)$. This is the way, relevant to our case, to distinguish the two kind of solutions: if the above intersection id $\Delta$, then the ranges of the variables $a, b, c$ are $T_3/2, T_2/4, T_3$ respectively, and the group is $SU(2)$, otherwise the ranges are $T_3, T_2/2, T_3$, and the group is $SO(3)$.

Finally, we want to add a final remark relevant for recognising genuine solutions: for $SO(3)$ generator $\tau$ it happens of course that the orbit $\exp(x\tau)$ never meets the center, while if $\tau$ is an $SU(2)$ generator, then $\exp(x/2\tau)$ is the only non trivial generator of the center of $SU(2)$. No other elements of the center of $SU(N)$ can be met these kind of orbits.

### Appendix D: Representations of $SU(2)$ and periodicity

It is well known from representation theory that spin $J$ representation of $SU(2)$ has generators $T_1, T_2, T_3$ given by the $N \times N$ matrices, with $N = 2J + 1$

\[
(T_1)_{m,n} = \frac{i}{2} \sqrt{m(N - m)} \delta_{m,n-1} \\
+ \frac{i}{2} \sqrt{m(N - m)} \delta_{m-1,n} , \quad (D.1)
\]

\[
(T_2)_{m,n} = \frac{1}{2} \sqrt{m(N - m)} \delta_{m,n-1} \\
- \frac{1}{2} \sqrt{m(N - m)} \delta_{m-1,n} , \quad (D.2)
\]

\[
(T_3)_{m,n} = i(J + 1 - m) \delta_{m,n} . \quad (D.3)
\]

Each of these matrices is diagonalizable with eigenvalues given by the ones of $T_3$. Since

\[ U^\dagger \exp(xT_j)U = \exp(xU^\dagger T_j U) \quad (D.4) \]

it follows that the periodicity of

\[ f_j(x) = \exp(xT_j) \quad (D.5) \]

depends only on the eigenvalues and so all $f_j$ have the same periodicity, which is obviously $2\pi$ for odd $N$ and $4\pi$ for even $N$.

On the other hand, let us consider the matrices $k_{\mathcal{Z}}$ and $g(x) = \exp(xk_{\mathcal{Z}})$. The possible periodicity of $g$

depends on the eigenvalues of $k_{\mathcal{Z}}$. It is easy to see that the coefficients of the characteristic polynomial of $k_{\mathcal{Z}}$ depend only on $|c_j|^2$, so the phases of $c_j$ are irrelevant for the periodicity. In particular, this means that the matrix $\exp(xT_2)$ with

\[
(T_2)_{m,n} = \frac{\zeta_m}{2} \sqrt{m(N - m)} \delta_{m,n-1} \\
- \frac{\zeta_m}{2} \sqrt{n(N - n)} \delta_{m-1,n} , \quad |\zeta_j| = 1 , (D.6)
\]

has the same periodicity of $f_2(x)$.

### Appendix E: Solving the periodicity problem

In section IV we have seen that for $N$ higher than 3 there is a further difficulty to overcome in order to find a global solution: generically the matrix $g(x) = e^{xk}$ is not periodic and its orbit densely fills a torus of dimension strictly higher than one. This phenomenon corresponds to the fact that the one parameter subgroup $g(x)$ is not indeed a Lie subgroup but only an imbedded subgroup. Therefore, for arbitrary choices of the coefficients $c_j$, the matrix

\[ k_{\mathcal{Z}} = \sum_{j=1}^{N-1} (c_j E_{j,j+1} - c_j^* E_{j+1,j}) . \quad (E.1) \]

cannot be used to generate global solutions unless the corresponding $g(x)$ is periodic. We will now tackle this problem in general. For the sake of completeness we will first show that no problems arise in the case $N = 3$.

#### 1. The case $N = 3$

In this simple case we have

\[ k_{\mathcal{Z}} = \begin{pmatrix} 0 & c_1 & 0 \\ -c_1^* & 0 & c_2 \\ 0 & -c_2^* & 0 \end{pmatrix} . \quad (E.2) \]

The corresponding characteristic polynomial is

\[ P_\lambda(x) := \det(\lambda I - k_{\mathcal{Z}}) = \lambda^2 + ||c||^2 . \quad (E.3) \]

The eigenvalues are therefore $0, \pm i ||c||$, which are in rational ratios so $g(\gamma) = \exp(\gamma k_{\mathcal{Z}})$ is periodic, in particular, with period $2\pi/||c||$. For other purposes, we compute explicitly $g(\gamma)$. To this aim, let us first notice that, by Cayley-Hamilton theorem, $k_{\mathcal{Z}}$ satisfies

\[ k_{\mathcal{Z}}(k_{\mathcal{Z}}^2 + ||c||^2 I) = 0 , \quad (E.4) \]
where \( I \) and \( \mathbb{O} \) are the identity and the null \( 3 \times 3 \) matrices. This implies \( k_\zeta^3 = -\|\zeta\|^2 k_\zeta \) so that any power of \( k_\zeta \) can be reduced to a power lower than three. Hence

\[ e^{\gamma k_\zeta} = g_1(\gamma)I + g_2(\gamma)k_\zeta + g_3(\gamma)k_\zeta^2, \quad (E.5) \]

for three functions satisfying \( g_1(0) = 1, \ g_2(0) = g_3(0) = 0, \) since \( e^0 = I \). Deriving (E.5) w.r.t. \( \gamma \) and using the characteristic equation, we get

\[ g_1'(\gamma)I + g_2'(\gamma)k_\zeta + g_3'(\gamma)k_\zeta^2 = k_\zeta e^{\gamma k_\zeta} \]

\[ = g_1(\gamma)k_\zeta + g_2(\gamma)k_\zeta^2 + g_3(\gamma)k_\zeta^3 \]

\[ = (g_1(\gamma) - \|\zeta\|^2 g_3(\gamma))k_\zeta + g_2(\gamma)k_\zeta^2, \quad (E.6) \]

so that

\[ g_1'(\gamma) = 0, \quad (E.7) \]
\[ g_2'(\gamma) = g_1(\gamma) - \|\zeta\|^2 g_3(\gamma), \quad (E.8) \]
\[ g_3'(\gamma) = g_2(\gamma), \quad (E.9) \]

with the Cauchy conditions \( g_1(0) = 1, g_2(0) = g_3(0) = 0 \) (so that \( g_2'(0) = 1 \)). From the first equation we immediately get \( g_1(\gamma) = 1 \), while deriving the second one and replacing from the third, we get

\[ g_2''(\gamma) = -\|\zeta\|^2 g_2(\gamma), \quad g_2(0) = 0, \quad g_2'(0) = 1, \quad (E.10) \]

which has solution

\[ g_2(\gamma) = \frac{\sin(\|\zeta\|\gamma)}{\|\zeta\|}. \quad (E.11) \]

Finally, from the third equation we get

\[ g_3(\gamma) = \int_0^\gamma dx \frac{\sin(\|\zeta\|\gamma)}{\|\zeta\|} = \frac{1 - \cos(\|\zeta\|\gamma)}{\|\zeta\|^2} \]

\[ = 2\frac{\sin^2\left(\frac{\|\zeta\|\gamma}{2}\right)}{\|\zeta\|^2}. \quad (E.12) \]

Therefore

\[ e^{\gamma k_\zeta} = I + \frac{\sin(\|\zeta\|\gamma)}{\|\zeta\|^2} k_\zeta + 2\frac{\sin^2\left(\frac{\|\zeta\|\gamma}{2}\right)}{\|\zeta\|^2} k_\zeta^2. \quad (E.13) \]

2. The general case

One can in principle solve this problem as follows. Since \( k_\zeta \) is anti hermitian, it can be diagonalized in \( \mathbb{C} \), with pure imaginary eigenvalues. Moreover, if \( \lambda \) is an eigenvalue, also \( -\lambda = \lambda^* \) is. Therefore, if \( S \) is the integer part of \( N/2 \) (so that \( N = 2S \) or \( N = 2S+1 \) for \( N \) even and odd respectively), generically we have \( S \) distinct non vanishing eigenvalues. Let \( U \) be a unitary matrix such that

\[ k = U^\dagger \sigma U, \quad (E.14) \]

where \( \sigma \) is the diagonal form of \( k \), say

\[ \sigma = \begin{cases} \text{diag}(i\lambda_1, -i\lambda_1, \ldots, i\lambda_S, -i\lambda_S), & \text{N even} \\ \text{diag}(i\lambda_1, -i\lambda_1, \ldots, i\lambda_S, -i\lambda_S, 0), & \text{N odd} \end{cases} \quad (E.15) \]

with \( \lambda_j > 0 \). Since

\[ e^{xk} = e^{xU^\dagger \sigma U} = U^\dagger e^{x\sigma} U, \quad (E.16) \]

\( e^{xk} \) is periodic if and only if \( e^{x\sigma} \) is. Now, \( e^{T\sigma} \) is the identity iff and only if

\[ e^{T\lambda_j} = 1 \quad (E.17) \]

for all \( j = 1, \ldots, S \), that means \( T\lambda_j = n_j 2\pi \), with \( n_j \) a positive integer (obviously, we assume \( T > 0 \)) for any \( j = 1, \ldots, S \). Therefore,

\[ \frac{\lambda_j}{\lambda_k} = \frac{n_j}{n_k} \quad (E.18) \]

so that all pairs of eigenvalues must have rational quotients. Of course, this condition is satisfied for \( N \leq 3 \), and any choice of \( \zeta \) is allowed. But for \( N \geq 4 \) we cannot choose the \( c_j \) arbitrarily: only those values, such that \( k \) admits eigenvalues with rational ratios are allowed. Notice that \( \zeta \) remains defined up to a real multiplicative constant: if \( t \in \mathbb{R}, \) then \( k_\zeta = tk_\zeta \). The eigenvalues are the solutions of the characteristic polynomial

\[ P_N(x) = \det(xI - k_\zeta), \quad (E.19) \]

of degree \( N \) in \( x \). Since \( k_\zeta \) is antihermitian, its eigenvalues are purely imaginary and, moreover, if \( \mu \) is a nonvanishing eigenvalue, then also \( -\mu = -\mu^* \) is an eigenvalue. So the non vanishing eigenvalues are in pairs and, if \( N \) is odd, there is at least one zero eigenvalue. Moreover, since in the factorization of the polynomial the nonvanishing eigenvalues \( \mu \) must appear in the factors \((x - \mu)(x + \mu) = x^2 - \mu^2\), we see that the general form of the polynomial must be
The coefficients $a_j$ are not the same for $N$ odd and for $N$ even, but it is convenient to keep the same name so that we can generically write the equation for the non vanishing eigenvalues as
\[
g^n + a_1 y^n - 1 + \ldots + a_{n-1} y + a_n = 0, \quad y = x^2. \tag{E.21}
\]
We can be more precise:

**Proposition 6.** Using the notation $j \ll k$ for $k - j \geq 2$, we have
\[
a_1 = ||z||^2, \tag{E.22}
a_k = \sum_{j_1 \ll j_2 \ll \ldots \ll j_k} |c_{j_1}|^2 |c_{j_2}|^2 \ldots |c_{j_k}|^2, \quad k = 2, \ldots, n. \tag{E.23}
\]

**Proof.** It can be easily proven by induction. We have already seen it for $N = 3$. A direct computation shows that it is true also for $N = 4$, since $P_4(x) = x^4 + x^2(|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 |c_1|^2 |c_3|^2).$ Now, assume it to be true for $N$ and $N - 1$. Let $k_n$ be the matrix $n \times n$ defined as $k_n$ with components $c_1, \ldots, c_n$. This way,

\[
P_6(x) = x^6 + ||z||^2 x^4 + x^2 (|c_1|^2 |c_1|^2 + |c_2|^2) + |c_1|^2 |c_3|^2 + |c_3|^2 (|c_1|^2 + |c_2|^2 + |c_3|^2)) + |c_1| c_3 c_5^2. \tag{E.28}
\]

Notice that, assuming that all $c_j$ are different from zero, we have always $a_n \neq 0$, so these are truly non zero eigenvalues. Now, condition (E.18) is equivalent to require that there exist a positive real number $z$ and $n$ positive integers $m_j$, $j = 1, \ldots, n$ such that the non vanishing eigenvalues of $k_n$ must have the form $\lambda_j^z = \pm im_j z$. This happen if the solutions of (E.21) are
\[
y_j = -z^2 m_j^2. \tag{E.29}
\]

At this point, we can notice that the coefficient of the above polynomial can be written in terms of the solutions as:
\[
a_1 = -\sum_{j=1}^{N} y_j, \tag{E.30}
a_2 = \sum_{j_1 < j_2} (-y_{j_1}) (-y_{j_2}), \tag{E.31}
\ldots
\]
\[
a_n = \sum_{j_1 < \ldots < j_n} (-y_{j_1}) \cdots (-y_{j_n}). \tag{E.33}
\]

We see $k_n$ as a submatrix of $k_{n+1}$ obtained erasing the last row and column. Let
\[
P_n(x) = \det(xI_{n \times n} - k_n). \tag{E.24}
\]

Developing the determinant with the Laplace rule applied to the last row, we easily find
\[
P_{N+1}(x) = xP_N(x) + |c_N|^2 P_{N-1}(x). \tag{E.25}
\]
The first addendum contains all the monomials of the stated form but the terms containing $|c_N|^2$. The second addendum contains all the terms of the stated form containing $|c_N|^2$. The proposition is proved. \qed
Comparing with the last proposition, we get the following set of equations for the \(|c_j^2| =: \zeta_j\):

\[
\sum_{j=1}^{N-1} \zeta_j = z^2 \sum_{a=1}^{n} m_a^2, \quad (E.34)
\]

\[
\sum_{j_1 \ll \cdots \ll j_k \leq N-1} \zeta_{j_1} \cdots \zeta_{j_k} = z^{2k} \sum_{a_1 < \cdots < a_k \leq n} m_{a_1}^2 \cdots m_{a_k}^2, \quad k = 2, \ldots, n. \quad (E.35)
\]

This is a set of \(n\) equations in \(N-1\) real positive variables. We will now show that it has generically an \((N-1-n)\)-dimensional space of solutions in the interesting region, which is for \(\zeta_j\) positive. To this end, we assume the generic situation where all \(m_a\) are different, and order them in an increasing sequence \(m_1 < m_2 < \cdots < m_n\). We will show later that the condition on the \(m_a\) cannot be weakened in order to get periodic solutions. Then, we show that there is a simple solution on the boundary of the region of interest, which is (if \(N\) is odd we assume the null eigenvalue to be the last one, \(\lambda_{2n+1} = 0\))

\[
\zeta_{2n} = 0, \quad \zeta_{2n-1} = z^2 m_{a_1}^2, \quad a = 1, \ldots, n. \quad (E.36)
\]

Next, we claim that starting from this point, we can find a smooth family of solutions \(\zeta_{2n-1}(\{\zeta_{2b}\})\) in a small open neighbourhood of \(\zeta_{2b} = 0\). In particular, it implies that there are positive (by continuity) \(\zeta_{2n-1}\)'s parametrized by small positive \(\zeta_j\)'s. This is sufficient to show that there is generically a moduli space of real dimension \(N-n-1\) for the solutions for the above system.

**Proof of the claim.** To prove the claim, let us consider the functions

\[
F_1(\zeta_1, \ldots, \zeta_{N-1}) = \sum_{j=1}^{N-1} \zeta_j, \quad (E.37)
\]

\[
F_k(\zeta_1, \ldots, \zeta_{N-1}) = \sum_{j_1 < \cdots < j_k \leq N-1} \zeta_{j_1} \cdots \zeta_{j_k}, \quad k = 2, \ldots, n, \quad (E.38)
\]

and the square submatrix \(M\) of its Jacobian defined by

\[
M_{a,b} = \frac{\partial F_a}{\partial \zeta_{2b-1}} |_{\zeta_j = z_j}, \quad (E.39)
\]

We want to compute the determinant of this matrix. It does not change if we subtract the first column to all the other ones. In doing this, the first line becomes \(\delta_{1,j}\), so that we can compute the determinant by applying the Laplace formula to the first line. So, the determinant is equal to the determinant of the new matrix with the first row and the first column canceled out. To understand how this matrix appears, let us notice that the second row is

\[
M_{2,1} = \sum_{c \neq b} z^2 m_c^2 - \sum_{c \neq 1} z^2 m_c^2 = z^2 (m_b^2 - m_1^2), \quad (E.47)
\]
and, more in general,
\[ M_{k,b} - M_{k,1} = \sum_{c_1 < \ldots < c_{k-1}} z^{2k-2} m_{c_1} \cdots m_{c_{k-1}} \]
\[ \quad \text{for } c_j \neq b \]
\[ \quad c_1 < \ldots < c_{k-1} \]

\[ = z^2 (m_b^2 - m_1^2) \sum_{c_1 < \ldots < c_{k-2}} m_{c_1} \cdots m_{c_{k-2}}. \]
\[ \quad 1 \neq c_j \neq b \]

Therefore, from the b-th column of the reduced matrix, b = 2, . . . , n, has a factor \( z^2 (m_b^2 - m_1^2) \) and since the determinant is multilinear on the columns, we get
\[ \det(M) = \prod_{b=2}^{n} z^2 (m_b^2 - m_1^2) \det(\tilde{M}), \quad \text{(E.49)} \]

where \( \tilde{M} \) is a \((n-1) \times (n-1)\) matrix whose first row has all elements equal to 1 and
\[ \tilde{M}_{k,b} = \sum_{c_1 < \ldots < c_{k-1}} z^{2k-2} m_{c_1} \cdots m_{c_{k-1}}. \quad \text{(E.50)} \]
\[ \quad c_1 < \ldots < c_{k-1} \quad 1 \neq c_j \neq b \]

In other words, we see that \( \tilde{M} \) is like \( M \) but in one lower dimension and where \( m_1 \) has disappeared. We can then repeat inductively the same construction, finally arriving to the conclusion
\[ \det(M) = \prod_{a < b} z^2 (m_b^2 - m_a^2). \quad \text{(E.51)} \]

Since \( m_a^2 < m_b^2 \) for \( a < b \), we see that this determinant is different from zero. The proof of the claim then is an immediate consequence of the implicit function theorem. \( \square \)

Going back to the \( c_j \), we then see that in general
\[ c_j = \xi_j \sqrt{\zeta_j (m_j, \ell)}, \quad \text{(E.52)} \]
for arbitrary phases \( \xi_j, j = 1, \ldots, N-1 \), with \( m \in \mathbb{N}_5 \), \( \ell \in W \subset \mathbb{R}^{N-n-1} \). The parameters \( t_j \) parametrize the above family of solutions. We can always assume that the integer \( m_j \) are coprime. Indeed, if \( m \) is a common divisor of \( m_j \) so that \( m_j = m s_j \), then we can write \( m = m_0 \) and \( m_0 \) can be reabsorbed in \( z \). Having assumed this, we can now fix \( z \) in such the way that \( e^{inx} \) has period \( 2\pi \). Indeed, since the non vanishing eigenvalues of \( k_- \) are \( \lambda_j^\pm = \pm iz m_j \), since the \( m_j \) are coprime, the common period of the associated exponential is \( 2\pi/z \). This fixes \( z = 1 \).

Notice in particular that in this case
\[ ||\xi||^2 = \sum_{j=1}^{n} m_j^2 = ||m||^2. \quad \text{(E.53)} \]

In particular, the associated baryon number is
\[ B = 2\sigma ||m||^2. \quad \text{(E.54)} \]

We have proved:

**Proposition 7.** For \( N = 2n \) or \( N = 2n + 1 \) and for any \( n \)-tuple of strictly increasing coprime positive integers \( m_a, a = 1, \ldots, n \), the matrices \( k_- \) such that \( e^{inx} \) has period \( 2\pi \) is a family of dimension \( 2N - 2 + n \), where \( n \) is the integer part of \( N/2 \). Beyond \( m \), this family is described by \( N - 1 \) phases and by \( N - n - 1 \) real parameters varying in a set \( W \), parametrizing the solutions of the system
\[ \sum_{j=1}^{N-1} \xi_j = \sum_{a=1}^{n} m_a^2, \quad \text{(E.55)} \]
\[ \sum_{a_1 < \ldots < a_k \leq N-1} \xi_{a_1} \cdots \xi_{a_k} = \sum_{a_1 < \ldots < a_k \leq n} m_{a_1}^2 \cdots m_{a_k}^2, \quad \text{for } k = 2, \ldots, n. \quad \text{(E.56)} \]

Correspondingly, the fundamental Baryon number is \( B_0 = 2\sigma ||m||^2. \)

One says that these matrices have a moduli space
\[ \mathcal{M} = T^{N-1} \times W, \quad \text{(E.57)} \]
where \( T^{N-1} \) is the torus generated \( y \) the phases and \( W \subset \mathbb{R}^{N-n-1} \) is the moduli space of the system. It is difficult to say something of general about the global properties of \( W \). We will study in general the case \( N = 4 \) where all computations are feasible explicitly.

**Remark:** for \( N = 3 \) we have \( n = 1 \) and, therefore, only one integer \( m \) that must be equal to 1 (to be “coprime”). So \( \xi \) must have norm 1 and the fundamental Baryon number is \( B = 2\sigma \).

### 3. The \( N = 4 \) case

Let us apply the above results to the case of \( SU(4) \). We have \( n = 2 \), so we expect the dimension of \( W \) to be 1. The eigenvalues equation for \( k \) is
\[ 0 = \lambda^4 + \lambda^2 ||\xi||^2 + |c_1|^2 |c_3|^2. \quad \text{(E.58)} \]
The four solutions are \( i\lambda_+, i\lambda_-, -i\lambda_+, -i\lambda_- \), with
\[
\lambda_{\pm} = \sqrt{\frac{|c_1|^2}{4} + \frac{|c_1||c_3|}{2}} \pm \sqrt{\frac{|c_1|^2}{4} - \frac{|c_1||c_3|}{2}}. \tag{E.59}
\]

Let \( q < p \) a pair of positive coprime integer numbers. Then, we have to solve the system
\[
\zeta_1 + \zeta_2 + \zeta_3 = p^2 + q^2 , \tag{E.60}
\]
\[
\zeta_1 \zeta_3 = p^2 q^2 . \tag{E.61}
\]

Notice that this gives
\[
\lambda_+ = p , \quad \lambda_- = q . \tag{E.62}
\]

Now, let us replace
\[
\zeta_3 = \frac{p^2 q^2}{\zeta_1} \tag{E.63}
\]
in the first equation, so that
\[
\zeta_1 + \frac{p^2 q^2}{\zeta_1} - (p^2 + q^2) = -\zeta_2. \tag{E.64}
\]

Since we have to require \( \zeta_2 > 0 \), we see that it must be
\[
\zeta_1^2 - (p^2 + q^2)\zeta_1 + p^2 q^2 < 0. \tag{E.65}
\]

This is equivalent to say
\[
q^2 < \zeta_1 < p^2 . \tag{E.66}
\]

So we can use \( \tau = \sqrt{\zeta_1} \) as a modulus to represent \( W \). The moduli space, including the boundary, is therefore
\[
\mathcal{M}_4 = \mathbb{T}^3 \times [q, p] . \tag{E.67}
\]

For
\[
(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}, \tau) \in \mathcal{M}_4 , \tag{E.68}
\]
we have
\[
\zeta = \left( e^{i\alpha_1} \tau; e^{i\alpha_2} \sqrt{p^2 + q^2 - \tau^2 - \frac{p^2 q^2}{\tau^2}}; \frac{pq}{\tau} e^{i\alpha_3} \right) . \tag{E.69}
\]

The corresponding period is of course
\[
T = 2\pi \tag{E.70}
\]
and the fundamental Baryon number is
\[
B_0 = 2\sigma(p^2 + q^2) . \tag{E.71}
\]

Finally, we can compute the exponential. Rewriting the characteristic polynomial as
\[
P(x) = x^4 + x^2(p^2 + q^2) + p^2 q^2 , \tag{E.72}
\]
we see that the matrix \( k \equiv k_\infty \) satisfies
\[
k^4 = -(p^2 + q^2)k^2 - p^2 q^2 I . \tag{E.73}
\]

This implies that there must exist four functions \( f_j(x) \), \( j = 0, 1, 2, 3 \) such that
\[
e^{\tau k} = f_0(x)I + f_1(x)k + f_2(x)k^2 + f_3(x)k^3 , \tag{E.74}
\]
with \( f_0(0) = 1, f_a(0) = 0, a = 1, 2, 3 \). From
\[
\frac{d}{dx}e^{\tau k} = ke^{\tau k} \tag{E.75}
\]
we get
\[
f_0'(x)I + f_1'(x)k + f_2'(x)k^2 + f_3'(x)k^3 \]
\[
= f_0(x)k + f_1(x)k^2 + f_2(x)k^3 + f_3(x)(- (p^2 + q^2)k^2 - p^2 q^2 I) , \tag{E.76}
\]
which gives the system of differential equations
\[
f_0' = -p^2 q^2 f_3 , \tag{E.77}
\]
\[
f_1' = f_0 , \tag{E.78}
\]
\[
f_2' = f_1 - (p^2 + q^2) f_3 , \tag{E.79}
\]
\[
f_3' = f_2 . \tag{E.80}
\]

with the Cauchy conditions \( f_j(0) = \delta_{j,0} \). Using the fourth equation in the third one we get
\[
f_3'' = f_1 - (p^2 + q^2) f_3 , \quad f_3''(0) = 0 . \tag{E.81}
\]

Deriving again and using the second equation:
\[
f_3''' = f_0 - (p^2 + q^2) f_3' , \quad f_3'''(0) = 1 . \tag{E.82}
\]

Deriving a last time and using the first equation, we finally get the Cauchy problem
\[
f_3''' + (p^2 + q^2) f_3'' + p^2 q^2 f_3 = 0 , \tag{E.83}
\]
\[
f_3(0) = 0 , \quad f_3'(0) = 0 , \quad f_3''(0) = 0 , \quad f_3'''(0) = 1 . \tag{E.84}
\]

This is easily solved and gives also \( f_2 = f_3' , \quad f_1 = f_2' + (p^2 + q^2) f_3 \), and finally \( f_0 = f_1' \). For \( p > q \), we get
\[
f_0(x) = \frac{p^2}{p^2 - q^2} \cos(qx) - \frac{q^2}{p^2 - q^2} \cos(px) , \tag{E.85}
\]
\[
f_1(x) = \frac{p^2}{q(p^2 - q^2)} \sin(qx) - \frac{q^2}{p(p^2 - q^2)} \sin(px) , \tag{E.86}
\]
\[
f_2(x) = \frac{1}{p^2 - q^2} (\cos(qx) - \cos(px)) , \tag{E.87}
\]
\[
f_3(x) = \frac{1}{p^2 - q^2} \left( \frac{\sin(qx)}{q} - \frac{\sin(px)}{p} \right) . \tag{E.88}
\]
In the case \( p = q = 1 \) we have
\[
 f_3(x) = -\frac{1}{2} x \cos x + \frac{1}{2} \sin x. \tag{E.89}
\]
This is sufficient to show that the case when \( p = q \) must be excluded, since the solution is no more periodic.

**Appendix F: The Baryonic number**

The Baryon number is defined by the integral
\[
 B = \frac{1}{24\pi^2} \int e^{ijk} \text{Tr}(R_i R_j R_k) \sqrt{g} dr \ d\phi \ d\gamma. \tag{F.1}
\]
Now,
\[
e^{ijk}(R_i R_j R_k) = \frac{3}{L_r L_\gamma L_\varphi} e^{\gamma\varphi} \text{Tr}(R_\gamma R_\varphi) = -\frac{3\sigma m}{L_r L_\gamma L_\varphi} \text{Tr}(h^\dagger k_c x),
\]
where we used the explicit expressions for the \( R_a \). After using (B.25), we get
\[
e^{ijk}(R_i R_j R_k) = -\frac{6\sigma m}{L_r L_\gamma L_\varphi} \sum_{j=1}^{N-1} |c_j|^2 \varepsilon_j \sin(ar) \text{Tr}(h^\dagger J_j),
\]
and using that
\[
 -\varepsilon_j \text{Tr}(h^\dagger J_j) = a,
\]
we finally get
\[
e^{ijk} \text{Tr}(R_i R_j R_k) = \frac{6\sigma m}{\sqrt{g}} |\xi|^2 a \sin(ar).
\]
Replacing in the integral and integrating we get
\[
 B = 2m\sigma |\xi|^2. \tag{F.2}
\]

**Remark:** the form
\[
 \omega = e^{ijk} \text{Tr}(R_i R_j R_k) \sqrt{g} dr \ d\phi \ d\gamma \tag{F.3}
\]
is nothing but the pull back on the rectangular box of the volume form \( \text{Tr}(R \wedge R \wedge R) \) over the cycle, see for example [43].

**Appendix G: Minimal energy per Baryon**

Let us minimise expression (IV.53) w.r.t. the \( L_a \), \( a = \phi, r, \gamma \). Let us rewrite it in the form
\[
g(L_\varphi, L_r, L_\gamma) = DL_\varphi L_r L_\gamma \left[ \frac{A^2}{L_\varphi^2} + \frac{B^2}{L_r^2} + \frac{C^2}{L_\gamma^2} + \frac{M^2}{L_\gamma^2} \left( 1 + \frac{\alpha^2}{L_\varphi^2} + \frac{\beta^2}{L_r^2} \right) \right], \tag{G.1}
\]
where
\[
 D = \frac{K \pi^3}{4\sigma m}, \quad A = 4\sigma, \quad B = \frac{\|v_c\|}{\|\xi\|},
\]
\[
 C = \sigma \sqrt{\lambda}, \quad M = 2\sqrt{2} m, \quad \beta = \frac{\sqrt{\lambda}}{4},
\]
\[
 \alpha = \sqrt{\frac{\sigma}{\|\xi\|}} \left( \sum_{j=1}^{N-1} |c_j|^4 + \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 \left( \frac{1}{2} - \frac{3}{2} \varepsilon_j \varepsilon_{j+1} \right) \right)^{\frac{1}{2}}. \tag{G.2}
\]

Deriving w.r.t. \( L_j \) and setting
\[
x = \frac{1}{L_\varphi^2}, \quad y = \frac{1}{L_r^2}, \quad z = \frac{M^2}{L_\gamma^2}, \tag{G.3}
\]
we get the equations for the stationary points:
\[
 A^2 x + B^2 y + C^2 x y - z(1 + \alpha^2 x + \beta^2 y) = 0, \tag{G.4}
\]
\[
 A^2 x - B^2 y + C^2 x y - z(1 - \alpha^2 x + \beta^2 y) = 0, \tag{G.5}
\]
\[
 -A^2 x + B^2 y + C^2 x y - z(1 + \alpha^2 x - \beta^2 y) = 0. \tag{G.6}
\]
Solving the first equation w.r.t. \( z \) and replacing in the remaining equations, we get:

\[
z = \frac{A^2 x + B^2 y + C^2 xy}{1 + \alpha^2 x + \beta^2 y}, \tag{G.7}
\]

\[
0 = B^2 y + B^2 \beta^2 y^2 - \alpha^2 x (A^2 + C^2 y), \tag{G.8}
\]

\[
0 = A^2 x (1 + \alpha^2 x) - \beta^2 y^2 (B^2 + C^2 x). \tag{G.9}
\]

From the third equation we get:

\[
y^2 = \frac{A^2 x}{\beta^2} \frac{1 + \alpha^2 x}{B^2 + C^2 x}. \tag{G.10}
\]

which replaced in the second term of the second equation gives

\[
(\alpha^2 x^2 C^2 - B^2)(y + \frac{A^2 x}{B^2 + C^2 x}) = 0. \tag{G.11}
\]

Since we are looking for positive \( x, y, z \), the second factor is strictly positive and the only allowed solution is \( x = \frac{B}{\alpha C} \). Replacing in (G.10) and then in (G.7), we get

\[
x = \frac{B}{\alpha C}, \quad y = \frac{A}{\beta C}, \quad z = \frac{AB}{\alpha \beta}. \tag{G.12}
\]

Therefore,

\[
\frac{1}{L_\varphi^2} = \frac{\|v_\varphi\|}{\lambda \sigma^2} \left( \sum_{j=1}^{N-1} |c_j|^4 + \right.
\]

\[
\left. \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 \left( \frac{1}{2} - \frac{3}{2} \varepsilon_j \varepsilon_{j+1} \right) \right)^{-\frac{1}{2}} \tag{G.13}
\]

\[
\frac{1}{L_\gamma^2} = \frac{16}{\lambda}, \tag{G.14}
\]

\[
\frac{1}{L_\sigma^2} = \frac{2\|v_\sigma\|}{\lambda m^2} \left( \sum_{j=1}^{N-1} |c_j|^4 + \right.
\]

\[
\left. \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 \left( \frac{1}{2} - \frac{3}{2} \varepsilon_j \varepsilon_{j+1} \right) \right)^{-\frac{1}{2}} \tag{G.15}
\]

and the corresponding energy per Baryon in standard units \((K = (6\pi^2)^{-1}, \lambda = 1)\) is

\[
g(\varphi, \varepsilon) = \frac{\pi}{\sqrt{2}} \left[ 2 + \frac{\|v_\varphi\|}{|\Delta|^2} \sum_{j=1}^{N-1} |c_j|^4 + \right.
\]

\[
\left. \sum_{j=1}^{N-2} |c_j|^2 |c_{j+1}|^2 \left( \frac{1}{2} - \frac{3}{2} \varepsilon_j \varepsilon_{j+1} \right) \right]^{\frac{1}{2}} \tag{G.16}
\]

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