The relative tensor product and a minimal fiber product in the setting of $C^*$-algebras

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Abstract
We introduce a relative tensor product of $C^*$-modules and a spatial fiber product of $C^*$-algebras that are analogues of Connes’ fusion of correspondences and the fiber product of von Neumann algebras introduced by Sauvageot, respectively, and study their categorical properties. These constructions form the basis for our approach to quantum groupoids in the setting of $C^*$-algebras that is published separately.

1 Introduction and Preliminaries
We introduce a relative tensor product of $C^*$-modules and a fiber product of $C^*$-algebras, study their categorical properties, and give some examples. These constructions are fundamental to our approach to quantum groupoids in the setting of $C^*$-algebras. Roughly, a quantum groupoid consists of an algebra $B$, thought of as the functions on the unit space, an algebra $A$, thought of as functions on the total space, a homomorphism $r: B \to A$ and an antihomomorphism $s: B \to A$ corresponding to the range and the source map, and a comultiplication $\Delta: B \to A \ast_r A$ corresponding to the multiplication of the quantum groupoid. Here, the algebra $A \ast_r A$ is a fiber product whose precise definition depends on the class of the algebras involved. In the purely algebraic context, this fiber product is the $\times_R$-product of Takeuchi [20], whereas in the setting of von Neumann algebras, one represents $A$ on a Hilbert space $H$ that arises from some GNS-construction for the Haar weights of the quantum groupoid, and constructs the fiber product $A \ast_r A$ as an operator algebra on a relative tensor product $H_s \otimes_r H$. In the setting of von Neumann algebras, the relative tensor product $H_s \otimes_r H$ was defined by Connes and is often called Connes’ fusion, see [3] [9] [10], the fiber product $A \ast_r A$ was defined by Sauvageot [17], and the data $(B, A, r, s, \Delta)$ forms a Hopf-von Neumann bimodule in the sense of Vallin [27]. We propose the corresponding definitions for the setting of $C^*$-algebras and use
them in [21] to define and study compact $C^*$-quantum groupoids. Let us add that in the setting of von Neumann algebras, a comprehensive theory of measurable quantum groupoids was developed by Lesieur and Enock [6, 7, 8, 14]. Similarly as in the theory of quantum groups, an important role in operator-algebraic approaches to quantum groupoids is played by fundamental unitaries that generalize the multiplicative unitaries of Baaj and Skandalis [1]. In the setting of von Neumann algebras, these unitaries were introduced by Vallin [28]. Using the theory presented in this article, we introduced and studied fundamental unitaries for quantum groupoids in the setting of $C^*$-algebras [23, 24], generalizing large parts of [1]. For space reasons, these applications had to be kept separate; an article is in preparation.

An earlier approach to quantum groupoids and their fundamental unitaries in the setting of $C^*$-algebras was developed in [22, 26]. The theory presented here and in [21, 23, 24] overcomes serious restrictions of this previous approach; see [25] for a comparison.

Although our approach is based on the theory of Hilbert $C^*$-modules, it only involves elementary algebraic constructions and can easily be adapted to a variety of settings like the purely algebraic one or the setting of von Neumann algebras, where one recovers the constructions mentioned above.

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Plan Let us describe the plan of this paper in more detail.

In the first part, we introduce a new class of modules over $C^*$-algebras and a relative tensor product for such modules that is closely modeled after Connes’ fusion of correspondences. The relative tensor product has nice functorial properties and can be considered as the composition in a bicategory of modules. We carefully motivate our definitions and explain the close relation between the setting of $C^*$-algebras and $W^*$-algebras which appears both on the formal level and in form of several functors.

In the second part, we consider $C^*$-algebras and $W^*$-algebras represented on modules and construct a fiber product of such algebras. We start with a reformulation of Sauvageot’s construction [16] for $W^*$-algebras and then introduce a spatial fiber product of $C^*$-algebras. As in the setting of $W^*$-algebras, our fiber product is functorial, but unfortunately it fails to be associative. This deficiency is shared by Takeuchi’s $\times_R$-product; in all of our applications [21, 23, 24], non-associativity will be compensated by some form of coassociativity. In the final subsection, we study fiber products of commutative $C^*$-algebras and give some examples.

In the appendix, we construct a minimal fiber product of $C^*$-algebras that is independent of chosen representations as a Kan extension.

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Terminology and notation

Given a subset \( Y \) of a normed space \( X \), we denote by \( [Y] \) \( X \) the closed linear span of \( Y \).

All sesquilinear maps like inner products of Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one. Let \( H, K \) be Hilbert spaces. We canonically identify \( \mathcal{L}(H, K) \) with a subspace of \( \mathcal{L}(H \oplus K) \).

Given subsets \( X \subseteq \mathcal{L}(H) \) and \( Y \subseteq \mathcal{L}(H, K) \), we denote by \( X' \) the commutant of \( X \) and by \([Y]\) the \( \sigma \)-weak closure of \( Y \).

Given a \( C^* \)-subalgebra \( A \subseteq \mathcal{L}(H) \) and a \( * \)-homomorphism \( \pi : A \to \mathcal{L}(K) \), let

\[
\mathcal{L}^*(H, K) := \{ T \in \mathcal{L}(H, K) \mid Ta = \pi(a)T \text{ for all } a \in A \}.
\]

We use the ket-bra notation and define for each \( \xi \in H \) operators \( |\xi\rangle : C \to H \), \( \lambda \mapsto \lambda \xi \), and \( \langle \xi | : H \to C \), \( \xi' \mapsto \langle \xi | \xi' \rangle \).

**Hilbert \( C^* \)-modules**

We shall extensively use (right) Hilbert \( C^* \)-modules \([12]\).

Let \( A \) and \( B \) be \( C^* \)-algebras. Given Hilbert \( C^* \)-modules \( E \) and \( F \) over \( B \), we denote the space of all adjointable operators \( E \to F \) by \( \mathcal{L}_B(E, F) \). Let \( E \) and \( F \) be \( C^* \)-modules over \( A \) and \( B \), respectively, and let \( \pi : A \to \mathcal{L}_B(F) \) be a \( * \)-homomorphism.

Then the internal tensor product \( E \otimes F \) is the Hilbert \( C^* \)-module over \( B \) \([12] \S 4\] which is the closed linear span of elements \( \eta \otimes \pi \xi \), where \( \eta \in E \) and \( \xi \in F \) are arbitrary, and

\[
\langle \eta \otimes \pi \xi | \eta' \otimes \pi \xi' \rangle = \langle \xi | \pi((\eta \eta')) \xi' \rangle
\]

and \( \eta \otimes \pi b = \eta \otimes \pi b \) for all \( \eta, \eta' \in E \), \( \xi, \xi' \in F \), and \( b \in B \). We denote the internal tensor product by “\( \otimes \)” and drop the index \( \pi \) if the representation is understood; thus, for example, \( E \otimes F = E \otimes _\pi F = E \otimes _F F \).

We also define a flipped internal tensor product \( F \otimes \rho E \) as follows. We equip the algebraic tensor product \( F \otimes E \) with the structure maps \( (\xi \otimes \eta)\xi' \otimes \eta' := \langle \xi | \pi((\eta \eta')) \xi' \rangle \), \( (\xi \otimes \eta)\xi' \otimes \eta' := \xi b \otimes \eta \), form the separated completion, and obtain a Hilbert \( C^* \)-B-module \( F \otimes E \) which is the closed linear span of elements \( \eta \otimes \xi \), where \( \eta \in E \) and \( \xi \in F \) are arbitrary, and

\[
\langle (\eta \otimes \xi) | (\eta' \otimes \xi') \rangle = \langle \eta | \pi((\eta \eta')) \xi' \rangle
\]

and \( (\eta \otimes \xi) b = \eta b \otimes \xi \) for all \( \eta, \eta' \in E \), \( \xi, \xi' \in F \), and \( b \in B \). As above, we drop the index \( \pi \) and simply write “\( \otimes \)” instead of “\( \pi \otimes \)” if the representation \( \pi \) is understood. Evidently, the usual and the flipped internal tensor product are related by a unitary map \( \Sigma : F \otimes E \overset{\sim}{\to} E \otimes F, \eta \otimes \xi \mapsto \xi \otimes \eta \).

For each \( \xi \in E \), the maps

\[
l_\pi^F(\xi) : F \to E \otimes F, \eta \mapsto \xi \otimes \eta, \quad r_\pi^F(\xi) : F \to F \otimes E, \eta \mapsto \eta \otimes \xi,
\]

are adjointable operators, and for all \( \eta \in F \), \( \xi' \in E \),

\[
l_\pi^F(\xi)^* (\xi' \otimes \eta) = \pi((\eta | \xi')) \eta = r_\pi^F(\xi)^* (\eta \otimes \xi').
\]

Again, we drop the superscript \( \pi \) in \( l_\pi^F(\xi) \) and \( r_\pi^F(\xi) \) if this representation is understood.

Finally, let \( E_1, E_2 \) be \( C^* \)-modules over \( A \), let \( F_1, F_2 \) be \( C^* \)-modules over \( B \) with \( * \)-homomorphisms \( \pi_i : A \to \mathcal{L}_B(F_i) (i = 1, 2) \), and let \( S \in \mathcal{L}_A(E_1, E_2) \), \( T \in \mathcal{L}_B(F_1, F_2) \) such that \( T \pi_1(a) = \pi_2(a)T \) for all \( a \in A \). Then there exists a unique operator \( S \otimes T \in \mathcal{L}_B(E_1 \otimes F_1, E_2 \otimes F_2) \) such that \( (S \otimes T)(\eta \otimes \xi) = S \eta \otimes T \xi \) for all \( \eta \in E_1, \xi \in F_1 \), and \( (S \otimes T)^* = S^* \otimes T^* \) \([5] \) Proposition 1.34].
Weights We shall use the theory of proper KMS-weights on $C^*$-algebras [10] and adopt the following conventions. Let $\mu$ be a faithful proper KMS-weight on a $C^*$-algebra $B$. We denote by $\mathcal{R}_\mu = \{ b \in B \mid \mu(b^*b) < \infty \}$ the space of square-integrable elements, by $\sigma^\mu$ the modular automorphism group, by $H_\mu$ the GNS-space, by $\Lambda_\mu: \mathcal{R}_\mu \to H_\mu$ the GNS-map, and by $J_\mu: H_\mu \to H_\mu$ the modular conjugation associated to $\mu$. We denote by $B^{\text{op}}$ the opposite $C^*$-algebra of $B$, which coincides with $B$ as a Banach space with involution but has the reversed multiplication, and by $\mu^{\text{op}}: B^{\text{op}} \to \mathbb{C}$ the opposite state of $\mu$, given by $\mu^{\text{op}}(b^{\text{op}}) := \mu(b)$ for all $b \in B$. One easily verifies that $\mu^{\text{op}}$ is a faithful proper KMS-weight, that the modular automorphism group $\sigma^{\mu^{\text{op}}}$ is given by $\sigma^{\mu^{\text{op}}}_t(b^{\text{op}}) = \sigma^\mu_{-t}(b)^{\text{op}}$ for all $b \in B$, $t \in \mathbb{R}$, and that $\mathcal{R}_{\mu^{\text{op}}} = (\mathcal{R}_\mu)^{\text{op}}$. Moreover, one can always choose the GNS-space and GNS-map for $\mu^{\text{op}}$ such that $H_{\mu^{\text{op}}} = H_\mu$ and $\Lambda_{\mu^{\text{op}}}(b^{\text{op}}) = J_\mu \Lambda_\mu(b)^*$ for all $b \in \mathcal{R}_{\mu^{\text{op}}}$, and then $J_{\mu^{\text{op}}} = J_\mu$ and $\pi_{\mu^{\text{op}}}(b^{\text{op}}) = J_\mu \pi_\mu(b)^* J_\mu$ for all $b \in B$. We shall also use the theory of normal semifinite faithful weights on $W^*$-algebras and von Neumann algebras, standard results and notation of Tomita–Takesaki theory [19], and the fact that every faithful proper KMS-weight $\mu$ on a $C^*$-algebra $B$ extends uniquely to a normal semifinite faithful weight $\bar{\mu}$ on $\pi_\mu(B)'$. Usually, we will identify $B$ and $B^{\text{op}}$ with operators on $H_\mu$ without explicitly mentioning $\pi_\mu$ or $\pi_{\mu^{\text{op}}}$.

2 The relative tensor product

In this section, we introduce a framework of bimodules over operator algebras and a relative tensor product for such modules that is closely modeled after Connes’ fusion of correspondences [3]. The main definitions and constructions are of a purely algebraic nature; therefore, we can treat the case of $C^*$-algebras and of $W^*$-algebras simultaneously.

2.1 Motivation

The main problem in the construction of a relative tensor product for modules over $C^*$-algebras is to find the appropriate notion of a module. In this subsection, we motivate our approach.

Let us first recall the construction of the relative tensor product in the setting of von Neumann algebras [3] [10]. The starting point is a $W^*$-algebra $B$ with a normal semifinite faithful weight $\mu$, a representation of the opposite $W^*$-algebra $B^{\text{op}}$ on a Hilbert $H$, and a representation of $B$ on a Hilbert space $K$. The original construction of Connes involves the concept. An element $\eta \in H_B$ is called $\mu^{\text{op}}$-bounded if there exists a (necessarily unique) operator $R_{\mu^{\text{op}}}(\eta): H_\mu \to H$ such that $R_{\mu^{\text{op}}}(\eta) \Lambda_{\mu^{\text{op}}}(b^{\text{op}}) = b^{\text{op}} \eta$ for all $b^{\text{op}} \in \mathcal{R}_{\mu^{\text{op}}}$. The space of all $\mu^{\text{op}}$-bounded elements is denoted by $D(H, \mu^{\text{op}})$.

Let us identify $B$ and $B^{\text{op}}$ with operators on $H_\mu$ as in Section [10]. Since each $R_{\mu^{\text{op}}}(\eta)$ commutes with left multiplication by elements of $B^{\text{op}}$, we have that $R_{\mu^{\text{op}}}(\eta)^* R_{\mu^{\text{op}}}(\eta') \in (B^{\text{op}})' = B \subseteq \mathcal{L}(H_\mu)$ for all $\eta, \eta' \in D(H, \mu^{\text{op}})$. Therefore,
we can define a $B$-valued sesquilinear product $\langle \cdot | \cdot \rangle_\mu$ on $D(H; \mu^{op})$ by
\[
\langle \eta | \eta' \rangle_\mu = R_{\mu^{op}}(\eta) ^* R_{\mu^{op}}(\eta') \quad \text{for all } \eta, \eta' \in D(H; \mu^{op})
\]
and a sesquilinear form $\langle \cdot | \cdot \rangle$ on the algebraic tensor product $D(H; \mu^{op}) \otimes K$ by
\[
\langle \eta \otimes \xi | \eta' \otimes \xi' \rangle := \langle \xi | (\eta \otimes \eta') \mu \xi' \rangle \quad \text{for all } \eta, \eta' \in H, \xi, \xi' \in K.
\]
The relative tensor product of $H$ and $K$ with respect to $\mu$ is the Hilbert space obtained from $D(H; \mu^{op}) \otimes K$ by forming the separated completion with respect to this sesquilinear form.

Equivalently, one can define the relative tensor product without reference to bounded elements. Let
\[
I := \{ T \in \mathcal{L}(H_\mu, H) \mid Tb^{op} = b^{op}T \text{ for all } b^{op} \in B^{op} \}.
\]
Then $I^* I \subseteq (B^{op})' = B$, and we can define a sesquilinear form $\langle \cdot | \cdot \rangle$ on the algebraic tensor product $I \otimes K$ by
\[
\langle T \otimes \xi | T' \otimes \xi' \rangle := \langle \xi | (T^* T') \xi' \rangle \quad \text{for all } T, T' \in I, \xi, \xi' \in K.
\]
Evidently, the map $D(H; \mu^{op}) \otimes K \rightarrow I \otimes K$ given by $\eta \otimes \xi \mapsto R_{\mu^{op}}(\eta) \otimes \xi$ extends to an isometry between the Hilbert space completions for the respective sesquilinear forms introduced above, and one can show that this isometry is an isomorphism.

Let us note that in the setting of von Neumann algebras, the representation of $B^{op}$ on $H$ is completely determined by the space $I$ and conversely [15]; this correspondence is an easy consequence of von Neumann’s bicommutant theorem.

We want to adapt the construction outlined above to the setting of $\mathcal{C}^*$-algebras and face the following problem. If $B$ is a $\mathcal{C}^*$-algebra, then the commutant $(B^{op})' \subseteq \mathcal{L}(H_\mu)$ is $B''$ and in general not $B$, and we cannot use formula (2) to define a sesquilinear form as in (3).

Apparently, we need to replace the space $D(H; \mu^{op})$ or, equivalently, the space $I$ defined in (4) by a subspace $\tilde{I} \subseteq I$ that is small enough in the sense that $I^* \tilde{I} \subseteq B$ and large enough in the sense that $IH_\mu$ is dense in $H$. In general, such a subspace $\tilde{I}$ is not uniquely determined by a representation of $B^{op}$ on $H$ alone.

But conversely, each such subspace $\tilde{I}$ determines a representation of $B^{op}$ on $H$ by the formula $b^{op}T \tilde{\xi} = Tb^{op}\tilde{\xi}$ for all $T \in \tilde{I}$ and $\tilde{\xi} \in H_\mu$. Therefore, we shall make such a subspace $\tilde{I}$ part of the module structure on $H$.

For illustration, let us consider the following special case. Assume that $B = C(X)$ for some compact space $X$. Then the representation of $B^{op} = B$ on $H$ allows us to write $H$ as a direct integral of a measurable field of Hilbert spaces on $X$, and $I$ corresponds the space of all measurable essentially bounded sections of this field. Now, to choose a subspace $\tilde{I} \subseteq I$ as above is essentially equivalent to choosing a continuous structure on the measurable field. Usually, such a continuous structure is not uniquely determined by the measurable structure.

### 2.2 Background on modules of operators

Before we introduce the classes of modules involved in the relative tensor product construction, we fix some terminology and recall several results on modules of
operator algebras that are concrete in the sense that they are represented as
operators between Hilbert spaces. We denote by \([X]\) the norm-closed and by
\([X]\) the \(\sigma\)-weakly closed linear span of a set \(X\) of operators.

**Definition 2.1.** A (nondegenerate) concrete \(C^*\)-algebra \(A_H = (H, A)\) consists
of a Hilbert space \(H\) and a (nondegenerate) norm-closed \(*\)-subalgebra \(A \subseteq \mathcal{L}(H)\).

A (nondegenerate) concrete \(C^*\)-module \(E^K_H = (H, K, E)\) consists of Hilbert
spaces \(H, K\) and a norm-closed subspace \(E \subseteq \mathcal{L}(H, K)\) satisfying \(EE^*E \subseteq E\)
(and \([EH] = K, [E^*K] = H\)). Replacing norm-closures by the \(\sigma\)-weak closures,
we similarly define (nondegenerate) concrete \(W^*\)-algebras and \(W^*\)-modules.

Given a concrete \(C^*\)-algebra \(A_H\), we identify \(M(A)\) with a \(C^*\)-subalgebra of
\(\mathcal{L}([AH]) \subseteq \mathcal{L}(H)\); then also \(M(A)_H\) is a concrete \(C^*\)-algebra. Clearly, each
conge concrete \(W^*\)-module is a concrete \(C^*\)-module, the weak closure of each concrete
\(C^*\)-module is a concrete \(W^*\)-module, and likewise for algebras. Finally, a
nondegenerate concrete \(W^*\)-algebra is the same as a von Neumann algebra.

**Proposition 2.2.** Let \(E^K_H\) be a concrete \(C^*\)-module.

1. \(A := [E^*E]\) and \(B := [EE^*]\) are \(C^*\)-algebras, \(E\) is a Hilbert \(C^*\)-\(B\)-\(A\)-bimodule with respect to the obvious structure maps, \([EE^*E] = E\), and
\(E^K_H\) is nondegenerate if and only if \(A_H\) and \(B_K\) are.

2. There exist isometries \(m_E : E \otimes H \to K, \xi \otimes \zeta \mapsto \xi \zeta\), and \(n_E : H \otimes E \to K, \zeta \otimes \xi \mapsto \xi \zeta\), which are unitary if and only if \([EH] = K\).

3. There exists a normal \(*\)-homomorphism \(\rho_E : A' \to B'\), \(x \mapsto m_E((\text{id}_E \otimes x)m_E^*\),
and \(\rho_E(x||_{EH}|) = 0\) and \(\rho_E(x)\zeta = \xi x\zeta\) for all \(x \in A'\), \(\xi \in E\), \(\zeta \in H\).

4. The embedding \(E \otimes H \hookrightarrow [E] \otimes H\) is equal to \(m_E^*[E]m_E^*\) and an isomorphism,
and \(\rho[E] = \rho_E\).

**Proof.** i–iii) Straightforward; for the relation \([EE^*E] = E\), see [12] p. 5. \(\square\)

**Proposition 2.3.** There exist categories

- **\(C^*\)-mod**, whose objects are all concrete \(C^*\)-algebras and whose morphisms
  between objects \(A_H\) and \(B_K\) are all concrete \(C^*\)-modules \(E^K_H\) satisfying
  \(E^*E \subseteq A\), \(EA \subseteq E\), \(EE^* \subseteq B\), \(BE \subseteq E\); the composition of morphisms
  \(E^K_H\) and \(F^K_K\) being \([FE]_H^*\).

- **\(W^*\)-mod**, which is defined similarly like \(C^*\)-mod, but \(C^*\)-algebras and
  \(C^*\)-modules are replaced by \(W^*\)-algebras and \(W^*\)-modules, and the
  composition of morphisms \(E^K_H\) and \(F^K_K\) is \([FE]_H^*\).

- **vN**, whose objects are all von Neumann algebras and whose morphisms
  are all normal \(*\)-homomorphisms.

There exist functors

- \([\cdot] : C^*\)-mod \(\to W^*\)-mod, given by \(A_H \mapsto [A]_H\), \(E^K_H \mapsto [E]_H^K\).
• $W^*\text{-mod} \to \mathfrak{vN}$, given by $A_H \mapsto A'_H, E^K_H \mapsto \rho_E$;

• $\mathfrak{vN} \to W^*\text{-mod}$, given by $A_H \mapsto A'_H, \rho \mapsto \mathcal{L}^0(H, K)$ for all $A_H, B_K \in \text{ob}\, \mathfrak{vN}, \rho \in \mathfrak{vN}(A_H, B_K)$.

The last two functors restrict to inverse isomorphisms of $\mathfrak{vN}$ with the full subcategory of $W^*\text{-mod}$ formed by all von Neumann algebras, that is, nondegenerate concrete $W^*$-algebras.

Proof. The existence of the categories and functors is evident. The last assertion above is proved in [13].

The preceding proposition implies the following easy result:

Proposition 2.4. Let $E^K_H$ be a concrete $C^*$-module and $A = [E^*E], B = [EE^*]$.

i) In $C^*\text{-mod}$, $E^K_H$ is an isomorphism from $A_H$ to $B_K$ with inverse $(E^*)_K^H$.

ii) For each nondegenerate concrete $C^*$-algebra $C_L$, we have $\rho_{C} = \text{id}_{C^*}$.\[\]

iii) $\rho_{E^*}$ is left/right inverse to $\rho_{E}$ if $A$ is nondegenerate/if $B$ is nondegenerate.

Corresponding assertions hold for concrete $W^*$-modules and $W^*$-algebras.

2.3 Modules and bimodules over algebras and bases

We shall define a relative tensor product for the following classes of modules:

Definition 2.5. Let $\mathfrak{A}_\alpha$ be a nondegenerate concrete $C^*$-algebra. A $C^*\text{-}\mathfrak{A}_\alpha$-module $H_\alpha = (H, \alpha)$ is a Hilbert space $H$ with a nondegenerate concrete $C^*$-module $\alpha^H_\alpha$ satisfying $[\alpha^*\alpha] = \mathfrak{A}$. We denote by $C^*\text{-mod}_{\mathfrak{A}_\alpha}$ the category of all $C^*\text{-}\mathfrak{A}_\alpha$-modules, where the set of morphisms between objects $H_\alpha$ and $K_\beta$ is $\mathcal{L}(H_\alpha, K_\beta) := \{T \in \mathcal{L}(H, K) \mid T_\alpha \subseteq \beta, T^*\beta \subseteq \alpha\}$.

Let $\mathfrak{A}_\alpha$ be a von Neumann algebra. We define the category $W^*\text{-mod}_{\mathfrak{A}_\alpha}$ of $W^*$-$\mathfrak{A}_\alpha$-modules similarly as $C^*\text{-mod}_{\mathfrak{A}_\alpha}$, replacing the norm-closure by the $\sigma$-weak closure.

The following properties of morphisms are easily verified:

Lemma 2.6. Let $\mathfrak{A}_\alpha$ be a nondegenerate concrete $C^*$-algebra (or $W^*$-algebra), $H_\alpha, K_\beta$ $C^*\text{-}\mathfrak{A}_\alpha$-modules (or $W^*\text{-}\mathfrak{A}_\alpha$-modules), and $S \in \mathcal{L}(H_\alpha, K_\beta)$. Then the map $S^* \alpha \mapsto \beta$ given by $\xi \mapsto S\xi$ belongs to $\mathfrak{L}_H(\alpha, \beta)$, and $(S^*)^* = (S^*)^*$, $S = m_{\beta}S \alpha \otimes \text{id}m^*_{\alpha}$ and $S\rho_\alpha(a) = \rho_{\beta}(a)S$ for all $a \in \mathfrak{A}'$.

The categories of modules introduced above are related to the following categories of representations and of Hilbert $C^*$-modules. For each $C^*$-algebra $\mathfrak{A}$, we denote by $C^*\text{-mod}_{\mathfrak{A}}$ the category of all Hilbert $C^*$-modules over $\mathfrak{A}$ with all adjointable operators as morphisms. Let $\mathfrak{A}_\alpha$ be a von Neumann algebra. A representation of $\mathfrak{A}$ is a Hilbert space $H$ with a faithful normal nondegenerate $^*$-homomorphism $\rho: \mathfrak{A} \to \mathcal{L}(H)$, briefly written $H^\rho = (H, \rho)$. We denote by
The functors \( \text{mr} \) and \( \text{rm} \) are inverse equivalences of categories.

**Proposition 2.7.** For each nondegenerate concrete \( C^* \)-algebra \( \mathfrak{A}_S \), there exist functors

1. \( \text{C}^*\text{-mod}_{\mathfrak{A}_S} \to \text{C}^*\text{-mod}_{\mathfrak{A}_S} \), given by \( E \mapsto (l_\beta(E))_{\beta \in \mathcal{B}_S} \) and \( T \mapsto T \otimes \text{id}_{\mathcal{B}_S} \);
2. \( \text{C}^*\text{-mod}_{\mathfrak{A}_S} \to \text{C}^*\text{-mod}_{\mathfrak{A}_S} \), given by \( H_\alpha \mapsto \alpha \) and \( T \mapsto T_* \);
3. \( [\cdot] : \text{C}^*\text{-mod}_{\mathfrak{A}_S} \to \text{W}^*\text{-mod}_{\mathfrak{A}_S} \), given by \( H_\alpha \mapsto H_{\{\alpha\}} \) and \( T \mapsto T \);
4. \( \text{mr} : \text{W}^*\text{-rep}_{\mathfrak{A}_S} \to \text{W}^*\text{-rep}_{\mathfrak{A}_S} \), given by \( (H, \rho) \mapsto (H, \mathcal{L}(S, H)) \) and \( T \mapsto T \).

The functors \( \text{mr} \) and \( \text{rm} \) are inverse equivalences of categories.

**Proof.** Immediate from Proposition 2.8.

Next, we introduce bimodules. To motivate the definition given below, let us first consider \( W^* \)-bimodules over von Neumann algebras, \( \mathfrak{A}_S \) and \( \mathfrak{B}_R \). Clearly, a \( W^* \)-\( \mathfrak{A}_S \)-\( \mathfrak{B}_R \)-bimodule should be a triple \( (H, \alpha, \beta) \), where \( H_\alpha \) is a \( W^* \)-\( \mathfrak{A}_S \)-module, \( H_\beta \) a \( W^* \)-\( \mathfrak{B}_R \)-module, and \( (H, \rho_\alpha, \rho_\beta) \) a correspondence between \( \mathfrak{A}_S \) and \( \mathfrak{B}_R \), that is, we should have \( [\rho_\alpha(\mathfrak{A}'), \rho_\beta(\mathfrak{B}')] = 0 \). Note that by Proposition 2.8 and Lemma 2.6

\[
[r_\alpha(\mathfrak{A}'), r_\beta(\mathfrak{B}')] = 0 \iff r_\alpha(\mathfrak{A}') \beta = \beta \iff r_\beta(\mathfrak{B}') \alpha = \alpha. \tag{5}
\]

If we pass from von Neumann algebras to concrete \( C^* \)-algebras, the equivalence breaks down — the condition on the left hand side is too weak, whereas the other two conditions are too strong. The solution is to replace the commutants \( \mathfrak{A}^1 \) and \( \mathfrak{B}^1 \) by suitable \( C^* \)-subalgebras.

**Definition 2.8.** A \( C^* \)-base or \( W^* \)-base \( (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^1) \) consists of a Hilbert space \( \mathfrak{H} \) and commuting nondegenerate \( C^* \)-algebras or von Neumann algebras \( \mathfrak{A}, \mathfrak{A}^1 \subseteq \mathcal{L}(\mathfrak{H}) \), respectively. Two \( C^* \)-bases or \( W^* \)-bases \( (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^1) \) and \((\mathfrak{H}, \mathfrak{B}, \mathfrak{B}^1)\) are equivalent if \( \text{Adv}(\mathfrak{A}) = \mathfrak{B} \) and \( \text{Adv}(\mathfrak{A}^1) = \mathfrak{B}^1 \) for some unitary \( V \in \mathcal{L}(\mathfrak{H}, \mathfrak{H}) \).

**Examples 2.9.**

i) We denote by \( t = (\mathbb{C}, \mathbb{C}, \mathbb{C}) \) the trivial \( C^* \)-base, given by the Hilbert space \( \mathbb{C} \) and twice the \( C^* \)-subalgebra \( \mathbb{C} = \mathcal{L}(\mathbb{C}) \).

ii) Let \( a = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^1) \) be a \( C^* \)- or \( W^* \)-base. Then \( a^1 := (\mathfrak{H}, \mathfrak{A}^1, \mathfrak{A}) \) is a \( C^* \)- or \( W^* \)-base, called the opposite of \( a \). If \( a \) is a \( C^* \)-base, then \( \{a\} := (\mathfrak{H}, [\mathfrak{A}], [\mathfrak{A}^1]) \) is a \( W^* \)-base.

iii) If \( \mathfrak{A}_S \) is a Neumann algebra, then \( (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^1) \) is a \( W^* \)-base. We call a \( C^* \)- or \( W^* \)-base \( (\mathfrak{A}, \mathfrak{B}, \mathfrak{B}^1) \) reduced if \( [\mathfrak{B}', (\mathfrak{B}^1)'] = 0 \), or, equivalently, if \( \mathfrak{B}'' = (\mathfrak{B}^1)' \).
iv) Let $\mu$ be a proper KMS-weight on a $C^*$-algebra $A$ with GNS-space $H_\mu$, GNS-representation $\pi_\mu: A \rightarrow \mathcal{L}(H_\mu)$, and opposite GNS-representation $\pi_{\mu^op}: A^{op} \rightarrow \mathcal{L}(H_\mu)$, $a \mapsto J_\mu \pi_\mu(a^*) J_\mu$ (see Section 7). Then the tuple $(H_\mu, \pi_\mu(A), \pi_{\mu^op}(A^{op}))$ is a $C^*$-base. Its opposite $(H_\mu, \pi_{\mu^op}(A^{op}), \pi_\mu(A))$ is equivalent to the $C^*$-base associated to the opposite weight $\mu^op$ on $A^{op}$. Indeed, $H_\mu$ can be considered as the GNS-space for $\mu^op$ via the opposite GNS-map $\Lambda_{\mu^op}: \mathcal{M}_{\mu^op} \rightarrow H_\mu$, $a^{op} \mapsto J_\mu \Lambda_\mu(a^*)$, and then $J_\mu \Lambda_\mu(a^*) J_\mu = \pi_\mu(A)$; see Section 7.

v) Let $A$ be a $W^*$-algebra. Similarly as in iv), we associate to every normal semifinite faithful weight $\mu$ on $A$ a $W^*$-base $(H_\mu, \pi_\mu(A), \pi_{\mu^op}(A^{op}))$, and the opposite weight $\mu^op$ on $A^{op}$ gives rise to the opposite $W^*$-base. In contrast to the setting of $C^*$-algebras, the $W^*$-bases associated to any two normal semifinite faithful weights $\mu$ and $\nu$ on $A$ are equivalent, as one can deduce from [11, §IX.1].

Let $\mu$ be a proper KMS-weight on a $C^*$-algebra $A$ and $\alpha$ the $C^*$-base associated to it as in iv). Then $\mu$ extends to a normal semifinite faithful weight $\mu$ on $[\pi_\mu(A)]$, and the $W^*$-base associated to $\mu$ above can be identified with $[a]$, as one can deduce from [11, §1.7].

From now on, we always denote $C^*$- and $W^*$-bases as follows:

$$a = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^1), \quad a_j = (\mathfrak{H}_j, \mathfrak{A}_j, \mathfrak{A}_j^1), \quad b = (\mathfrak{A}, \mathfrak{B}, \mathfrak{B}^1), \quad c = (\mathfrak{L}, \mathfrak{C}, \mathfrak{C}^1), \quad \ldots \quad (6)$$

**Definition 2.10.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be $W^*$-algebras. A $W^*$-$(\mathfrak{A}, \mathfrak{B})$-representation is a triple $\rho H = (H, \rho, \sigma)$, where $H^\rho$ and $H^\sigma$ are representations of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, and $[\rho(\mathfrak{A}), \sigma(\mathfrak{B})] = 0$. We denote by $W^\ast$-$\text{rep}_{\mathfrak{A}, \mathfrak{B}}$ the category of all $W^*$-$(\mathfrak{A}, \mathfrak{B})$-representations, where the set of morphisms between objects $\rho H^\sigma$ and $\rho' H^\sigma$ is $L^\rho(H^\sigma, H^\sigma) \cap L(H^\rho, H^\rho)$. Let $a$ and $b$ be $C^*$-bases. A $C^*$-$(a, b)$-module is a triple $a H_\beta = (H, \alpha, \beta)$, where $H_\alpha$ is a $C^*$-$\mathfrak{A}$-$\mathfrak{B}$-module, $H_\beta$ a $C^*$-$\mathfrak{B}$-$\mathfrak{A}$-module, and $[\rho_\alpha(\mathfrak{A}^1)] \beta = \beta, [\rho_\beta(\mathfrak{B}^1) \alpha] = \alpha$. We denote by $C^*$-$\text{mod}_{a, b}$ the category of all $C^*$-$(a, b)$-modules, where the set of morphisms between objects $a H_\beta$ and $\gamma K_\delta = L(a H_\beta, \gamma K_\delta) \cap L(H_\alpha, K_\delta)$. If $a$ and $b$ are $W^*$-bases, we similarly define the category $W^\ast$-$\text{mod}_{a, b}$.

Till the end of this subsection, let $a = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^1)$ and $b = (\mathfrak{A}, \mathfrak{B}, \mathfrak{B}^1)$ be $C^*$-bases. Proposition 2.7 and relation (5) imply:

**Proposition 2.11.** There exists a functor $[-, -]: C^*$-$\text{mod}_{a, b} \rightarrow W^\ast$-$\text{mod}_{[a], [b]}$ given by $a H_\beta \mapsto [a] H_\beta$ and $T \mapsto T$. If $a$, $b$ are reduced, there exist functors

- $\text{mr}: W^\ast$-$\text{mod}_{[a], [b]} \rightarrow W^\ast$-$\text{rep}_{\mathfrak{A}, \mathfrak{B}}$, given by $a H_\beta \mapsto (H, \rho_\alpha, \rho_\beta)$ and $T \mapsto T$;

- $\text{rm}: W^\ast$-$\text{rep}_{\mathfrak{A}^1, \mathfrak{B}} \rightarrow W^\ast$-$\text{mod}_{[a], [b]}$, given by the assignments $\rho H^\sigma \mapsto (H, L^\rho(\mathfrak{H}, H), L^\sigma(\mathfrak{A}, H))$ and $T \mapsto T$;

and then the functors $\text{mr}$ and $\text{rm}$ are inverse isomorphisms of categories.
Examples 2.12.  
i) \( a^\dagger \mathfrak{A} \) is a \( C^\ast \t(A^1, a) \)-module since \( \rho_{a^\dagger} (\mathfrak{A}) \mathfrak{A} \) = \( \mathfrak{A} \mathfrak{A} \) = \( \mathfrak{A} \) and \( \rho_{a^\dagger} (\mathfrak{A}) \mathfrak{A}^1 = [\mathfrak{A}]^1 \mathfrak{A} \) = \( \mathfrak{A}^1 \) by Proposition 2.4 ii).

ii) Let \( H \) be a Hilbert space and \( |H| = \{ |\xi| : |\xi| \in H \} \subseteq \mathcal{L}(C, H) \). Then \( (H, |H|) \) is a \( C^\ast \t\)-module, where \( \tau \) is the trivial \( C^\ast \)-base (Example 2.9 i)). If \( H_\alpha \) is a \( C^\ast \mathfrak{A}_\beta \)-module, then \( (H, \alpha, |H|) \) is a \( C^\ast \t\)-module.

iii) If \( \mathfrak{A} = \mathfrak{A}^1 \), then \( a^\dagger = a \), and if additionally \( H_\alpha \) is a \( C^\ast \alpha \)-module, then \( aH_\alpha \) is a \( C^\ast \t(a^1, a) \)-module because then \( \rho_{a^\dagger} (\mathfrak{A}^1) \alpha \) = \( [\alpha \mathfrak{A}^1] = \alpha \).

iv) The category \( C^\ast \mathfrak{A}_\mathfrak{B} \) has direct sums, which can be constructed as follows. Let \( (\mathfrak{H}_i)_i \) be a family of \( C^\ast \t(a, b) \)-modules, where \( \mathfrak{H}_i = (H_i, \alpha_i, \beta_i) \) for each \( i \). Denote by \( \oplus_i \alpha_i \subseteq \mathcal{L}(\mathfrak{H}_i, \oplus H_i) \) the norm-closed linear span of all operators of the form \( \zeta \mapsto (\zeta \xi)_i \), where \( (\xi)_i \) is contained in the algebraic direct sum \( \bigoplus_i \alpha_i \xi \), and similarly define \( \bigoplus_i \beta_i \subseteq \mathcal{L}(\mathfrak{H}^i, \oplus H_i) \). One easily verifies that the triple \( \bigoplus_i \mathfrak{H}_i := \bigoplus (H_i, \alpha_i, \bigoplus \beta_i) \) is a \( C^\ast \t(a^1, b^1) \)-module, that for each \( j \), the canonical inclusions \( i_j: H_j \rightarrow \bigoplus \mathfrak{H}_i \) and projection \( p_j: \bigoplus \mathfrak{H}_i \rightarrow H_j \) are morphisms \( \mathfrak{H}_j \rightarrow \bigoplus \mathfrak{H}_i \) and \( \bigoplus \mathfrak{H}_i \rightarrow \mathfrak{H}_j \), and that with respect to these maps, \( \bigoplus \mathfrak{H}_i \) is the direct sum of the family \((\mathfrak{H}_i)_i\).

Another important source of \( C^\ast \t(b, b^1) \)-modules is discussed in Example 2.13.

Of course, each of the preceding examples has an analogue in the setting of \( W^\ast \)-algebras.

### 2.4 The relative tensor product in the setting of \( C^\ast \)-algebras

The relative tensor product of modules introduced below is a simple algebraic reformulation of Connes’ fusion and was motivated already in Subsection 2.1. We first consider the setting of \( C^\ast \)-algebras. Throughout this subsection, let \( a, b, c, A \in C^\ast \)-bases as in \( \mathfrak{A}_\mathfrak{B} \). Let \( H_\beta \) be a \( C^\ast \mathfrak{B}_\mathfrak{B} \)-module and \( K_\gamma \) a \( C^\ast \mathfrak{B}_\mathfrak{B} \)-module. Then the \textit{relative tensor product} of \( H_\beta \) and \( K_\gamma \) is the Hilbert space

\[
H_\beta \otimes_{b,c} K := \beta \oplus \mathfrak{B} \otimes \gamma.
\]

It is spanned by elements \( \xi \otimes \zeta \otimes \eta \), where \( \xi \in \beta, \zeta \in \mathfrak{B}, \eta \in \gamma \), and the inner product is given by \( \langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \xi | \xi' \rangle \langle \zeta | \zeta' \rangle \langle \eta | \eta' \rangle \). By definition and by Proposition 2.4 ii), there exist isomorphisms

\[
\Sigma: H_\beta \otimes_{b,c} K \rightarrow K_\gamma \otimes_{b,c} H, \quad \xi \otimes \zeta \otimes \eta \mapsto \eta \otimes \zeta \otimes \xi,
\]

\[
n_{\beta, \gamma} := \text{id} \otimes n_{\gamma}: H_\beta \otimes_{b,c} K \rightarrow \beta \otimes_{b,c} K, \quad \xi \otimes \zeta \otimes \eta \mapsto \xi \otimes \eta \zeta,
\]

\[
m_{\beta, \gamma} := m_{\beta} \otimes \text{id}: H_\beta \otimes_{b,c} K \rightarrow H_{\rho_\beta} \otimes_{b,c} K, \quad \xi \otimes \zeta \otimes \eta \mapsto \xi \otimes \zeta \otimes \eta.
\]

Example 2.13. If \( b \) is the trivial \( C^\ast \)-base \( t \) (Example 2.4 i)), then \( \beta = |H| \), \( \gamma = |K| \), and \( H_{|H|} \otimes |K| \mathfrak{K} \cong H \otimes K \) via \( |\xi| \otimes \lambda \otimes |\eta| \mapsto |\xi| \otimes \eta \).
The proof of the following result is straightforward:

**Lemma 2.14.** i) For each \( \xi \in \beta \) and \( \eta \in \gamma \), there exist operators \( |\xi|_1 \in L(K, H_\beta \otimes \gamma K) \) and \( |\eta|_2 \in L(H_\beta, H_\beta \otimes \gamma K) \) with adjoints \( \langle \xi |_1 := |\xi|_1^* \) and \( \langle \eta |_2 := |\eta|_2^* \) such that for all \( \xi' \in \beta \), \( \zeta \in \mathfrak{K} \), \( \eta' \in \gamma \),

\[
|\xi|_1(\eta' \zeta) = \xi \otimes \zeta \otimes \eta', \quad \langle \xi |_1(\xi' \otimes \zeta \otimes \eta') = \rho_\gamma(\xi^* \xi')\eta' \zeta = \eta' \xi^* \xi' \zeta,
\]

\[
|\eta|_2(\xi' \zeta) = \xi' \otimes \zeta \otimes \eta, \quad \langle \eta |_2(\xi' \otimes \zeta \otimes \eta') = \rho_\beta(\eta^* \eta')\xi' \zeta = \xi^* \eta' \eta' \zeta.
\]

ii) \( |\beta|_1 := \{|\xi|_1 \xi \in \beta \} \) is a concrete \( C^*-\rho_\gamma(\mathfrak{B})_K \)-module and \( |\gamma|_2 := \{|\eta|_2 \eta \in \gamma \} \) a concrete \( C^*-\rho_\beta(\mathfrak{B}^t)_K \)-module.

We put \( |\beta|_1 := |\beta|_1^* \), \( |\gamma|_2 := |\gamma|_2^* \). Note that \( ||\beta|_1 \gamma || = ||\gamma|_2 \beta || \subseteq L(\mathfrak{K}, H_\beta \otimes \gamma K) \).

By Proposition 2.2 and 2.4 we have isomorphisms

\[
\rho_{|\beta|_1} : \rho_\gamma(\mathfrak{B}^t) \to |\beta|_1(\rho_\gamma(\mathfrak{B}))^t \subseteq L(\beta_\otimes \gamma K), \quad T \mapsto \id \otimes \eta T \eta, =: \id \otimes T,
\]

\[
\rho_{|\gamma|_2} : \rho_\beta(\mathfrak{B}^t)^t \to |\gamma|_2|\gamma|_2^t \subseteq L(\beta_\otimes \gamma K), \quad S \mapsto m_\beta \text{Sm}_\beta \otimes \id =: S \otimes \id.
\]

**Lemma 2.15.** Let \( T \in \rho_{|\beta|_1}\mathfrak{B}^t \), \( S \in \rho_{|\beta|_2}\mathfrak{B}^t \). If one of the following conditions holds, then \( \id \otimes T, S \otimes \id = 0 \):

i) \( T \in L(K_\beta) \), ii) \( S \in L(K_\beta) \), iii) \( \beta \) is reduced.

**Proof.** Let \( \omega = \xi \otimes \zeta \otimes \eta, \omega' = \xi' \otimes \zeta' \otimes \eta' \in H_\beta \otimes \gamma K. \) Then \( \eta^* T \eta' \in \mathfrak{B}^t \), \( \xi^* S \xi' \in \mathfrak{B}^t \). In case i), \( \eta^* T \eta' \in \mathfrak{B}^t \); in case ii), \( \xi^* S \xi' \in \mathfrak{B}^t \). In all cases, \( \eta^* T \eta' \) and \( \xi^* S \xi' \) commute and \( \langle \omega|\id \otimes T)(S \otimes \id)\omega' = \langle \xi|\eta^* T \eta'\rangle\langle \xi^* S \xi'\rangle = \langle \zeta|\eta^* S \xi'\rangle\langle \xi^* S \xi'\rangle = \langle \omega|\id \otimes T\rangle\omega' \).

If \( T \in \rho_{|\beta|_1}\mathfrak{B}^t \) and \( S \in \rho_{|\beta|_2}\mathfrak{B}^t \) satisfy one of the conditions i)-iii) above, we put

\[
S \otimes T := (\id \otimes T)(S \otimes \id) = (S \otimes \id)(\id \otimes T) \in L(H_\beta \otimes \gamma K).
\]

Given \( C^* \)- or \( W^* \)-algebras \( A \subseteq L(H_\beta) \) and \( B \subseteq L(K_\gamma) \), we denote by

\[
A \otimes B \subseteq L(H_\beta \otimes \gamma K) \quad / \quad A \bar{\otimes} B \subseteq L(H_\beta \otimes \gamma K)
\]

the \( C^* \)- or \( W^* \)-algebra, respectively, generated by all operators \( a \otimes b \), where \( a \in A, b \in B \).

**Proposition 2.16.** Let \( \mathcal{H} = a_\beta H_\beta \) be a \( C^*-(a_1, b) \)-module and \( K = \gamma K_\delta \) a \( C^* -(b^t, c) \)-module. Put

\[
\alpha \triangleleft \gamma := \|\gamma|_2 \alpha \| \subseteq L(\mathfrak{K}, H_\beta \otimes \gamma K), \quad \beta \triangleright \delta := \|\beta|_1 \delta \| \subseteq L(\mathfrak{K}, H_\beta \otimes \gamma K).
\]

Then \( \mathcal{H} \otimes K := (\alpha \triangleright \gamma)(H_\beta \otimes \gamma K)(\beta \triangleright \delta) \) is a \( C^* -(a_1, c) \)-module.
In the situation above, we call \( H \otimes K \) the relative tensor product of \( H \) and \( K \).

Note the following commutative diagram, where the morphisms are concrete \( C^* \)-modules and composition is defined as in \( C^*\)-mod:

The relative tensor product has the following categorical properties:

**Proposition 2.17.** Let \( H = \alpha H_\beta, H^1 = \alpha_1 H^1_\beta, H^2 = \alpha_2 H^2_\beta \) be \( C^* \)-\((a^1, b)\)-modules, \( K = \gamma K_\delta, K^1 = \gamma_1 K^1_\delta, K^2 = \gamma_2 K^2_\delta \) \( C^* \)-\((b^1, c)\)-modules, and \( L = \phi L_\delta \) a \( C^* \)-\((t^1, d)\)-module.

1) \( S \otimes T \in L(H^1 \otimes K^1, H^2 \otimes K^2) \) for all \( S \in L(H^1, H^2), T \in L(K^1, K^2) \).

2) The composition of the maps

\[
(H_\beta \otimes K^i)_{(\beta, \delta)} \otimes \epsilon L \xrightarrow{m(b^i, \epsilon)} (H_\beta \otimes (\gamma K)\rho_\delta) \otimes \epsilon \xrightarrow{n_\delta, \gamma \otimes \epsilon} \beta \otimes \rho_\delta \otimes \epsilon, \\
\beta \otimes \rho_\delta \otimes \epsilon \xrightarrow{id \circ m_\epsilon} \beta \otimes \rho_\delta \otimes \epsilon, \\
\beta \otimes \rho_\delta \otimes \epsilon \xrightarrow{n_\delta, \gamma \otimes \epsilon} (K_\delta \otimes \epsilon L) \xrightarrow{n_\delta, \gamma \otimes \epsilon} (\gamma K) \otimes \epsilon L \xrightarrow{m(b^i, \epsilon)} H_\beta \otimes (K^1 \otimes \epsilon L) \xrightarrow{(\rho_\delta, \gamma \otimes \epsilon)} H_\beta \otimes (\gamma L_\delta) \\
\] is an isomorphism \( a_{a^i, b^i, c^i, d^i}(L, K, \gamma) : (H \otimes K) \otimes L \rightarrow H \otimes (K \otimes L) \).

3) Put \( U := a_{a^i, b^i, c^i, d^i} \). Then there exist isomorphisms

\[
r_{a^i, b^i}(H) := m_\beta \circ n_{b^i, \beta} : H \otimes U \rightarrow H, \xi \otimes \zeta \otimes b^i \rightarrow \xi b^i \zeta = \rho_\beta(b^i) \zeta, \\
l_{b^i, c^i}(K) := n_{\gamma} \circ m_{b^i, \gamma} : U \otimes K \rightarrow K, b \otimes \zeta \otimes \eta \rightarrow \eta b \zeta = \rho_\gamma(b) \zeta.
\]

4) Let \( (\mathcal{H}_i) \) be a family of \( C^* \)-\((a^i, b)\)-modules and \( (K^j) \) a family of \( C^* \)-\((b^i, c)\)-modules. For each \( i, j \), denote by \( \iota^i_j : \mathcal{H}_i \rightarrow \mathbb{I}_j^i \mathcal{H}_i, \iota^j_k : K_j \rightarrow \mathbb{I}_k^j K_j \)
\[ \pi^{i}_{K^j}, \pi^{j}_{K^j} \text{ and } \pi^{a}_{M} : \mathcal{H}^{i} \to \mathcal{H}, \pi^{j}_{K^j} : \mathcal{K}^{j} \to \mathcal{K} \text{ the canonical inclusions and projections, respectively. Then there exist inverse isomorphisms} \]
\[ \mathbb{B}_{i,j}[^{b}_{b}](\mathcal{H}^{i} \otimes \mathcal{K}^{j}) \cong \mathbb{B}_{i,j}[^{b}_{b}](\mathcal{H}^{i} \otimes (\mathbb{B}_{j} \mathcal{K}^{j})) \text{ given by } (\omega_{i,j})_{i,j} \mapsto \sum_{i,j}(\pi^{i}_{K^j} \otimes \pi^{j}_{K^j})(\omega_{i,j}) \]
\[ \text{and } ((\pi^{i}_{K^j} \otimes \pi^{j}_{K^j})(\omega))_{i,j} \mapsto \omega, \text{ respectively.} \]

**Proof.** i) If \( S, T \) are as above and \( \mathcal{H}^{i} = \alpha, H_{ij}^{b}, \mathcal{K}^{j} = \gamma_{ij}^{b} \) for \( i, j = 1, 2 \), then \( (S \otimes T)^{b} |\gamma_1|_{2} \alpha_1 = |T \gamma_1|_{2} \alpha_2 \) and similarly \( (S \otimes T)^{b} |\beta_1|_{2} \delta_1 \subseteq |T \beta_1|_{2} \delta_2, \)
\( (S \otimes T)^{b} |\gamma_2|_{2} \alpha_2 \subseteq |T \gamma_2|_{2} \alpha_1, (S \otimes T)^{b} |\beta_2|_{2} \delta_2 \subseteq |T \beta_2|_{2} \delta_1. \)

ii) Straightforward.

iii) \( r_{\alpha,b}(\mathcal{H}) \cdot (\alpha \otimes \mathcal{B}^{b}_1) = [\rho_{\beta}(\mathcal{B}^{b}_1) |\alpha] = \alpha \) and \( r_{\alpha,b}(\mathcal{H}) \cdot (\beta \otimes \mathcal{B}) = [\beta \mathcal{B}] = \beta. \)

For \( l_{b,c}(\mathcal{K}) \), the arguments are similar.

iv) Straightforward. \( \square \)

Recall that a bicategory \( \mathcal{B} \) consists of a class of objects \( \text{ob} \mathcal{B} \), a category \( \mathcal{B}(A, B) \) for each \( A, B \in \text{ob} \mathcal{B} \) whose objects and morphisms are called \( 1 \)-cells and \( 2 \)-cells, respectively, a functor \( c_{A,B,C} : \mathcal{B}(B,C) \times \mathcal{B}(A,B) \to \mathcal{B}(A,C) \) ("composition") for each \( A, B, C \in \text{ob} \mathcal{B} \), an object \( 1_{A} \in \mathcal{B}(A, A) \) ("identity") for each \( A \in \text{ob} \mathcal{B} \), an isomorphism \( a_{A,B,C,D}(f,g,h) : c_{A,B,C} \circ c_{B,C,D}(f,g,h) \to c_{A,C,D}(h,c_{A,B,C}(g,f)) \) in \( \mathcal{B}(A,D) \) ("associativity") for each triple of \( 1 \)-cells \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \) in \( \mathcal{B} \), and isomorphisms \( l_{A}(f) : c_{A,B,A}(f,1_{A}) \to f \) and \( r_{B}(f) : c_{A,B,B}(1_{B}, f) \to f \) in \( \mathcal{B}(A,B) \) for each \( 1 \)-cell \( A \xrightarrow{f} B \) in \( \mathcal{B} \), subject to several axioms \[ 13. \]

T eros but straightforward calculations show:

**Theorem 2.18.** There exists a bicategory \( \mathcal{C}^{*} \text{-bimod} \) such that

- the objects are all \( \mathcal{C}^{*} \)-bases and \( \mathcal{C}^{*} \text{-bimod}(a,b) = \mathcal{C}^{*} \text{-mod}_{a,b} \) for all \( \mathcal{C}^{*} \)-bases \( a, b; \)
- the functor \( c_{a,b,c} \) is given by \( (\gamma_{A}, \delta_{A}, b_{A}) \mapsto \alpha H_{\delta_{b}} \otimes \gamma_{A}, K_{b} \) and \( (T, S) \mapsto S \otimes T, \)
- respectively, and the identity \( 1_{a} \) is \( \mathbb{B}_{b}^{a} \mathcal{H}_{a} \mathcal{K}_{a} \) for all \( \mathcal{C}^{*} \)-bases \( a, b, c, d; \)
- \( a, r, l \) are as in Proposition 2.17.

### 2.5 The relative tensor product in the setting of \( W^{*} \)-algebras

We carry over the definitions and results of Subsection 2.4 to the setting of \( W^{*} \)-algebras, replacing the norm closures in \[ 8. \] by the \( \sigma \)-weak closures. Moreover, we carry over the definition of the bicategory \( \mathcal{C}^{*} \text{-bimod} \) and obtain a bicategory \( \mathcal{W}^{*} \text{-bimod} \). These two bicategories are related by a functor as follows.

Let \( \mathcal{B}, \mathcal{B}' \) be bicategories. Recall that a functor \( \mathcal{F} : \mathcal{B} \to \mathcal{B}' \) consists of an assignment \( \text{ob} \mathcal{B} \to \text{ob} \mathcal{B}', A \mapsto FA, A \mapsto FA, A \mapsto FA, \) a functor \( F_{A,B} : \mathcal{B}(A,B) \to \mathcal{B}'(FA,FB) \) defined for each \( A, B \in \text{ob} \mathcal{B} \), a 2-cell \( \phi_{A,B,C}(g,f) : c_{A,B,F,B}(F_{B,C}g,F_{A,B}f) \to c_{A,C,F,D}(g,f) \) for each pair of \( 1 \)-cells \( A \xrightarrow{f} B \xrightarrow{g} C \) in \( \mathcal{B} \), and a 2-cell \( \phi_{A} : 1_{A} \to F_{A,A}1_{A} \) for each \( A \in \text{ob} \mathcal{B} \), subject to several axioms \[ 13. \]

Again, tedious but straightforward calculations show:
Theorem 2.19. i) Let $H = \alpha H_\beta$ be a $C^\ast$-$(\alpha, b)$-module and $K = \gamma K_\delta$ a $C^\ast$-$\delta$-module. Then the embedding $\beta \otimes \mathfrak{K} \otimes \gamma \hookrightarrow [\beta] \otimes \mathfrak{K} \otimes [\gamma]$ is an isomorphism of $W^*\otimes([\alpha], [\epsilon])$-modules

$$[\cdot \otimes (\mathcal{H} \otimes K): \mathcal{H} \otimes K \to \mathcal{H} \otimes \mathcal{K}].$$

ii) There exists a functor $[\cdot]: W^\ast\text{-bimod} \to W^\ast\text{-bimod}$ given by $a \mapsto [a]$ on objects, $\alpha H_\beta \mapsto [\alpha H_\beta] = [\alpha] H[\beta]$ on 1-cells, the identity on 2-cells, $\phi_{a,b,\alpha}(\gamma K_\delta, \alpha H_\beta) = \mathcal{I}_{\mathfrak{K}}^{-1} \circ (H_{\beta \otimes \gamma} K_\delta)$ for each pair of 1-cells $a_\alpha \beta_\gamma \rightarrow b_\delta$, $\gamma$ in $C^\ast\text{-bimod}$, and $\phi_a = id_H$ for each object $a$.

The preceding constructions are related to Connes’ fusion (see Subsection 2.1) as follows.

Proposition 2.20. Let $B$ be a $W^\ast$-algebra, $H^\rho$ a $W^\ast\cdot B^\text{op}$-representation, $K^\sigma$ a $W^\ast\cdot B$-representation, and $\mu$ a normal semifinite faithful weight on $B$ with associated $W^\ast$-base $\mathfrak{B}$ (Example 2.9 v)). Put

$$\tilde{\rho} := \rho \circ \pi^{-1}_a: \mathfrak{B} \overset{\pi^{-1}_a}{\rightarrow} B^\text{op} \overset{L}{\rightarrow} \mathcal{L}(H), \quad \tilde{\sigma} := \sigma \circ \pi^{-1}_a: \mathfrak{B} \overset{\pi^{-1}_a}{\rightarrow} B \overset{\sigma}{\rightarrow} \mathcal{L}(K),$$

$$\beta := \mathcal{L}^\delta(\mathfrak{B}, H), \quad \gamma := \mathcal{L}^\delta(\mathfrak{B}, K).$$

Then there exists a unitary $V_{H^\rho, K^\sigma}: H^\rho \otimes_\beta K \to H\alpha \otimes_\gamma K$ given by $\eta \otimes \omega \mapsto R_{\mu \rho}(\eta \otimes n^*_\gamma(\omega))$ for all $\eta \in D(H; \mu \rho)$, $\omega \in K$.

Proof. Let $\eta, \eta' \in D(H; \mu \rho)$. Then $\sigma(\langle \eta | \eta' \rangle_{\mu \rho}) = \tilde{\sigma}(R_{\mu \rho}(\eta \ast R_{\mu \rho}(\eta'))) \ast \tilde{\sigma}(R_{\mu \rho}(\eta'))$ and $\tilde{\sigma} = \rho_{\gamma}|_{\mathfrak{B}}$ by Proposition 2.23. Now, there exists an isometry $V_{H^\rho, K^\sigma}$ as claimed because for all $\omega, \omega' \in K$,

$$\langle \eta \otimes \omega | \eta' \otimes \omega' \rangle = \langle \omega | \rho_{\gamma}(R_{\mu \rho}(\eta) \ast R_{\mu \rho}(\eta')) \omega' \rangle = \langle R_{\mu \rho}(\eta) \otimes n^*_\gamma(\omega) | R_{\mu \rho}(\eta') \otimes n^*_\gamma(\omega') \rangle.$$  

This isometry is surjective because the relation $[R_{\mu \rho}(D(H; \mu \rho)) H_{\mu}] = H$ (proof of) Lemma IX.3.3 implies

$$(m_\beta \otimes id)(V_{H^\rho, K^\sigma}(H_\beta \otimes _\sigma K)) = [R_{\mu \rho}(D(H; \mu \rho)) H_{\mu}] \otimes \gamma = (m_\beta \otimes id)(H_\beta \otimes _\gamma K).$$

Propositions 2.23 and 2.20 imply:

Corollary 2.21. Let $\mathfrak{B}$ be the $C^\ast$-base associated to a proper KMS-weight $\mu$ on a $C^\ast$-algebra $B$ (Example 2.9 iv)), let $H_\beta$ be a $C^\ast\cdot \mathfrak{B}_R$-module, and let $K_\gamma$ be a $C^\ast\cdot \mathfrak{B}_R^1$-module. Denote by $\hat{\mu}$ the normal semifinite faithful extension of
μ to \([\mathcal{B}]\), identify \([\mathcal{B}]^{op}\) with \([\mathcal{B}]'\), and put \(\rho := \rho_\beta : [\mathcal{B}]^{op} \rightarrow \mathcal{L}(H)\), \(\sigma := \rho_\gamma : [\mathcal{B}] \rightarrow \mathcal{L}(K)\). Then \([\beta]\) = \(\mathcal{L}\rho(H_\mu, H)\), \([\gamma]\) = \(\mathcal{L}\sigma(H_\mu, K)\), and we have isomorphisms

\[
H_\rho \otimes_\mu K \xrightarrow{V_{H_\rho,K\sigma}} H_{[\beta]} \otimes_{[\rho]} L_{[\gamma]} K \xleftarrow{\cong} H_\beta \otimes_{[\gamma]} K.
\]

There exists a bicategory such that the objects are \(W^*\)-algebras, the category of morphisms between objects \((A,\mu)\) and \((B,\nu)\) is \(W^*\text{-rep}_{A,B}\), and the composition is given by Connes’ fusion [2] Proposition II.5]. Straightforward but tedious verifications show that the unitaries \(V_{H_\rho,K\sigma}\) yield a functorial embedding of this bicategory into \(W^*\text{-bimod}\).

3 The spatial fiber product

In this section, we use the relative tensor product to define a spatial fiber product of \(C^*\)-algebras. This fiber product is an analogue of the fiber product of von Neumann algebras [17] but lacks several desirable properties like associativity. Nevertheless, it is functorial with respect to a natural class of morphisms and suits our applications in [21 23 24].

3.1 \(C^*\)-algebras and \(W^*\)-algebras represented on modules

We consider \(C^*\)- and \(W^*\)-algebras which are represented on modules of the type introduced in Section 2. Again, we write \(C^*\)- and \(W^*\)-bases as in [17].

**Definition 3.1.** Let \(\mathfrak{a}\) be a \(C^*\)-base. A (nondegenerate) \(C^*\text{-}\mathfrak{a}\)-algebra is a pair \(A_\mathfrak{a}^H = (H_\mathfrak{a}, A)\), where \(H_\mathfrak{a}\) is a \(C^*\)-\(\mathfrak{A}\)-module, \(A_H\) a (nondegenerate) concrete \(C^*\)-algebra, and \(\rho_\mathfrak{a}(\mathfrak{A}')A \subseteq A\). We denote by \(C^*\text{-alg}_\mathfrak{a}^{(nd)}\) the category of all (nondegenerate) \(C^*\text{-}\mathfrak{a}\)-algebras, where the morphisms between objects \(A_\mathfrak{a}^H\), \(B_\mathfrak{a}^H\) are all \(*\)-homomorphisms \(\pi : A \rightarrow B\) satisfying \(\beta = [\mathcal{L}\sigma(H_\alpha, K_\beta)\alpha]\), where \(\mathcal{L}\sigma(H_\alpha, K_\beta) := \mathcal{L}(H_\alpha, K_\beta) \cap \mathcal{L}(H, K)\) (see [11]). Given a \(W^*\text{-base}\) \(\mathfrak{a}\), we similarly define the category \(W^*\text{-alg}_\mathfrak{a}^{(nd)}\) of all (nondegenerate) \(W^*\text{-}\mathfrak{a}\)-algebras, where the morphisms are assumed to be normal.

Let \(\mathfrak{A}\) be a \(W^*\)-algebra. A von Neumann-\(\mathfrak{A}\)-algebra is a pair \(A_\mathfrak{A}^H = (A_H, \rho)\), where \(A_H\) is a von Neumann algebra and \(\rho : \mathfrak{A} \rightarrow A\) a unital normal \(*\)-homomorphism. The class of all von Neumann-\(\mathfrak{A}\)-algebras forms a category \(v\mathcal{N}\text{-alg}_\mathfrak{A}\), where the morphisms between objects \((A_H, \rho), (B_K, \sigma)\) are all \(\pi \in v\mathcal{N}(A_H, B_K)\) satisfying \(\pi \circ \rho = \sigma\).

**Remark 3.2.** Let \(\mathfrak{a}\) be a reduced \(W^*\)-base and \(A_\mathfrak{a}^H\) a nondegenerate \(W^*\text{-}\mathfrak{a}\)-algebra. Then \(A' \subseteq \rho_\mathfrak{a}(\mathfrak{A}') = \mathcal{L}(H_\alpha)\) by Proposition 2.7.

In our applications to quantum groupoids [21], \(C^*\text{-}\mathfrak{a}\)-algebras arise as follows:

**Example 3.3.** Let \(\mathfrak{b}\) be the \(C^*\)-base associated to a KMS-state \(\mu\) on a unital \(C^*\)-algebra \(B\) (Example 2.4 iv)), let \(A\) be a unital \(C^*\)-algebra such that \(1_A \in \ldots\)
\(B \subseteq A\), and let \(\phi\colon A \to B\) be a conditional expectation such that \(\nu := \mu \circ \phi\) is a KMS-state and \(\sigma_t^B|_B = \sigma_t^A\), \(\phi \circ \sigma_t^B = \sigma_t^A \circ \phi\) for all \(t \in \mathbb{R}\). Denote by \(\zeta_\phi\colon H_\nu \to H_\nu\) the isometry given by \(\Lambda_\mu(b) \mapsto \Lambda_\nu(b)\) and put \(H := H_\nu\), \(\alpha := [\pi_\nu(A)\zeta_\phi], \beta := [\pi_\nu^p(A^{op})\zeta_\phi]\). Then \(\alpha H_\beta\) is a \(C^*\)-\((b,b')\)-module, \(\rho_\alpha \circ \pi_\mu = \pi_\nu^p\), \(\rho_\beta \circ \pi_\mu = \pi_\nu\), and \(\pi_\nu(A) + \pi_\nu^p((A \cap B')^{op}) \subseteq \mathcal{L}(H_\alpha)\), \(\pi_\nu^p(A^{op}) + \pi_\nu(A \cap B') \subseteq \mathcal{L}(H_\beta)\) [21, Lemma 3.7]. Moreover, \(\pi_\nu(A)_\beta\) is a nondegenerate \(C^*\)-\(b\)-algebra because \(\rho_\beta(3)\pi_\nu(A) = \pi_\nu(B)\pi_\nu(A) \subseteq \pi_\nu(A)\), and similarly, \((\pi_\nu^p(A^{op}))_\beta\) is a nondegenerate \(C^*\)-\(b^1\)-algebra.

**Example 3.4.** Let \(a\) be a \(C^*\)-base and \(A_H^a\) a \(C^*\)-\(a\)-algebra. If we identify \(M(A)\) with a \(C^*\)-subalgebra of \(L([AH])\) in the canonical way, \(M(A)_H^a\) becomes a \(C^*\)-\(a\)-algebra.

Let us collect some easy properties of morphisms:

**Proposition 3.5.** Let \(a\) be a \(C^*\)-/\(W^*\)-base, \(\pi\) a morphism of \(C^*\)-/\(W^*\)-\(a\)-algebras \(A_H^a\) and \(B_K^a\), and \(\gamma = L^*(H_\alpha, K_\beta)\). Then:

i) \([\gamma H] = K\) and \(\gamma \in C^*/W^*-\text{mod}(L(H_\alpha) \cap A', L(K_\beta) \cap \pi(A'))\).

ii) \(\pi = \rho_\gamma|A\) is normal and \(\pi(\rho_\alpha(x)a) = \rho_\beta(x)\pi(a)\) for all \(x \in \mathfrak{A}\), \(a \in A\).

iii) If \(A_H\) is nondegenerate, then \(\pi(A)_K\) is nondegenerate.

iv) If \(a\) is a reduced \(W^*\)-base and \(A_H, B_K\) are nondegenerate \(W^*\)-algebras, then \(L^*(H_\alpha, K_\beta) = L^*(H, K)\) and the maps \(\pi \mapsto L^*(H, K)\) and \(\rho_\gamma \mapsto \gamma\) are inverse bijections \(W^*-\text{alg}_a(A_H^a, B_K^a) \cong \{\gamma \in W^*-\text{mod}(A_H^a, C_K^a) \mid [\gamma \alpha] = \beta, C \subseteq B\}\) a von-Neumann algebra).

**Proof.** We only prove iv), the other statements follow immediately from the definitions. Assume that \(a\) is a reduced \(W^*\)-base and \(A_H, B_K\) are nondegenerate \(W^*\)-algebras. By Proposition 2.3, \(\rho_\beta|_{\mathfrak{N}} = \rho_\gamma|_{\mathfrak{N}} = \rho_\gamma \circ \rho_\alpha \circ \rho_\alpha|_{\mathfrak{N}} = \pi \circ \rho_\alpha|_{\mathfrak{N}}\), so \(L^*(H, K) \subseteq L(H_{\rho_\alpha}, K_{\rho_\beta})\). Let \(C \subseteq B\) be a von-Neumann algebra and \(\delta \in W^*-\text{mod}(A_H^a, C_K^a)\) satisfy \([\delta \alpha] = \beta\), and let \(\pi := \rho_\beta\colon A_H^a \to C_K^a\). Then by Proposition 2.3, \(\pi \circ \rho_\alpha = \rho_\beta\) and therefore \(L^*(H_\alpha, K_\beta) = L^*(H, K) = \delta\). The claim follows.

A combination of this result with Proposition 2.7 shows:

**Proposition 3.6.** Let \(a\) be a \(C^*\)-base.

i) There exist functors \([\cdot]_H: C^*-\text{alg}_{\text{nd}} \to W^*-\text{alg}_{\text{nd}}\) given by \(A_H^a \mapsto [A]_H^{\alpha}\) and \(\pi \mapsto \bar{\pi}\), where \(\bar{\pi}\) denotes the normal extension.

ii) If \(a\) is reduced, then there exist isomorphisms \(\nu N_{\text{alg}}_{\text{nd}} \cong W^*-\text{alg}_{\text{nd}}\) given by \((H, A) \mapsto (\text{rm} H, A), \pi \mapsto \pi\) and \((H, A) \mapsto (\text{mr} H, A), \pi \mapsto \pi\).
Definition 3.7. Let $a, b$ be $C^*$-bases. A (nondegenerate) $C^*$-$(a, b)$-algebra is a pair $A_H^{a,b} = (\rho_H, A)$, where $\rho_H$ is a $C^*$-$(a, b)$-module, $A_H^{a,b}$ a (nondegenerate) $C^*$-$a$-algebra, and $A_H^{b}$ a $C^*$-$b$-algebra. We denote by $C^*$-alg$_{a,b}$ the category of all (nondegenerate) $C^*$-$(a, b)$-algebras, where $C^*$-alg$_{a,b}$ $(A_H^{a,b}, B_K^{a,b}) = C^*$-alg$_a (A_H^{\alpha}, B_K^{\alpha}) \cap C^*$-alg$_b (A_H^{\beta}, B_K^{\beta})$ for all $A_H^{\alpha}, B_K^{\beta}$. Given $W^*$-bases $a, b$, we similarly define the category $W^*$-alg$_{a,b}$ of all (nondegenerate) $W^*$-$(a, b)$-algebras.

Let $\mathfrak{A}, \mathfrak{B}$ be $W^*$-algebras. A von Neumann-$(\mathfrak{A}, \mathfrak{B})$-algebra is a pair $A_H^{\rho,\sigma} = (\rho H^\sigma, A)$, where $\rho H^\sigma$ is a $W^*$-$(\mathfrak{A}, \mathfrak{B})$-representation, $A_H$ a von Neumann algebra, and $\rho(\mathfrak{A}), \sigma(\mathfrak{B}) \subseteq A$. The class of all von Neumann-$(\mathfrak{A}, \mathfrak{B})$-algebras forms a category $\text{vN-}\text{alg}_{\mathfrak{A}, \mathfrak{B}}$, where $\text{vN-}\text{alg}_{\mathfrak{A}, \mathfrak{B}} (A_H^{\rho,\sigma}, B_K^{\phi,\psi}) = \text{vN-}\text{alg}_{\mathfrak{A}} (A_H^{\rho}, B_K^{\phi}) \cap \text{vN-}\text{alg}_{\mathfrak{B}} (A_H^{\sigma}, B_K^{\psi})$.

Of course, we have an analogue of Proposition 5.6 for the categories introduced above. For later use, we note the following direct sum and direct product construction.

Example 3.8. Let $a, b$ be $C^*$-bases and $(A_i)_i$ a family of $C^*$-$(a, b)$-algebras, where $A_i = ((H_i, \alpha_i, \beta_i), A_i)$ for each $i$. Then the $c_0$-sum of $C^*$-algebras $\bigoplus_i A_i$ and the $l^\infty$-product of $C^*$-algebras $\prod_i A_i$ are naturally represented on the Hilbert space $\bigoplus_i H_i$, and the pairs $\bigoplus_i A_i := \bigoplus_i (H_i, \alpha_i, \beta_i, \bigoplus_i A_i)$ and $\prod_i A_i := \prod_i (H_i, \alpha_i, \beta_i, \prod_i A_i)$ are $C^*$-$(a, b)$-algebras, called the direct sum and the direct product of the family $(A_i)_i$, respectively. One easily verifies that for each $j$, the canonical maps $A_j \to \bigoplus_i A_i \to \prod_i A_i \to A_j$ are morphisms of $C^*$-$(a, b)$-algebras $A_j \to \bigoplus_i A_i \to \prod_i A_i \to A_j$, and that $\bigoplus_i A_i$ and $\prod_i A_i$ are nondegenerate if all of the $(A_i)_i$ are nondegenerate.

Similarly, one defines the direct sum in the setting of $W^*$-algebras.

3.2 The spatial fiber product of $W^*$-algebras

Before we introduce a spatial fiber product for $C^*$-algebras, we reformulate the well-known fiber product construction for von Neumann algebras 17 in the language of $W^*$-algebras and $W^*$-modules. Throughout this subsection, let $a, b, c$ be $W^*$-bases as in 10. We start with the following lemma:

Lemma 3.9. Let $E_H^K$ be a concrete nondegenerate $W^*$-module and $A_H$ a concrete $W^*$-algebra such that $E^*EA \subseteq A$.

i) For each $T \in \mathcal{L}(K)$, the following conditions are equivalent: (a) $E^*TE \subseteq A$; (b) $T \in [EAE^*]$; (c) $TE + T^*E \subseteq [EA]$. If $A_H$ is nondegenerate, these conditions are equivalent to (d) $T \in \rho_E (A')$.

ii) The set of all $T \in \mathcal{L}(K)$ satisfying conditions (a)–(c) is a $W^*$-algebra.

Proof. i) Using the relation $id_K \in [E^*E]$, we find $E^*TE \subseteq A \Rightarrow T \in [E^*TEE^*] \subseteq [EAE^*]$ $\Rightarrow TE + T^*E \subseteq [EAE^*] \subseteq [EA] \Rightarrow E^*TE \subseteq [E^*EA] \subseteq A$. 

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If \( A = A'' \), then the relations \( \xi^* T \xi' b = \xi^* T \rho_E(b) \xi' \) and \( b \xi^* T \xi' = \xi^* T \rho_E(b) T \xi' \), valid for all \( \xi, \xi' \in E, b \in A' \), imply the equivalence of (a) and (d).

ii) Obvious.

**Proposition 3.10.** Let \( A^b_H \) be a \( W^* - b \)-algebra and \( B^b_K \) a \( W^* - b^1 \)-algebra.

i) The following subspaces of \( \mathcal{L}(H_\beta \otimes \gamma K) \) coincide and are a \( W^* \)-algebra:

(a) \( \{ T \in \mathcal{L}(H_\beta \otimes \gamma K) \mid \langle \beta_1 T | \beta_1 \rangle \subseteq B, \langle \gamma_2 T | \gamma_2 \rangle \subseteq A \} \);

(b) \( \llbracket \gamma \rrbracket_2 A \langle \beta | \beta \rangle \cap \| \beta \|_1 B \langle \beta | \beta \rangle_1 \} \);

(c) \( \{ T \in \mathcal{L}(H_\beta \otimes \gamma K) \mid T | \beta_1 \rangle_1 + T^* | \beta_1 \rangle_1 \subseteq \| \beta \|_1 B, T \langle \gamma_2 \rangle_2 + T^* \langle \gamma_2 \rangle_2 \subseteq \| \gamma \|_2 A \} \} \).

We denote the \( W^* \)-algebra in (a)–(c) by \( A^b_\beta \beta, B \).

ii) If \( A_H, B_K \) are nondegenerate, then \( A' \subseteq \rho_\beta(\mathcal{B}^1)' \), \( B' \subseteq \rho_\gamma(\mathcal{B}^1)' \), and \( A^b_\beta \beta, B = \rho_\gamma \langle \beta | \beta \rangle \cap \rho_\beta \langle \beta | \beta \rangle (B^1)' = \langle (A^1 \otimes \text{id}) \cap (\text{id} \otimes B^1)' \rangle \).

iii) If \( A^\alpha_\beta \beta \) is a \( W^* -(a^1, b) \)-algebra and \( B^\gamma_\delta K \) a \( W^* -(b^1, c) \)-algebra, then \( (a) H_\beta \otimes \gamma K \delta, A^b_\beta \delta \beta, B \) is a \( W^* -(a^1, c) \)-algebra.

**Proof.** Assertions i) and ii) follow immediately from Lemma 3.9 and assertion iii) follows easily from equation \( \theta \).

**Definition 3.11.** We denote the \( W^* \)-algebra in 3.10 (i)–(iii) by \( A^b_\beta \gamma, B \) and call it the \( W^* \)-fiber product of \( A^b_H \) and \( B^b_K \). If \( A^\alpha_\beta \beta \) is a \( W^* -(a^1, b) \)-algebra and \( B^\gamma_\delta K \) a \( W^* -(b^1, c) \)-algebra, then we call the \( W^* -(a^1, c) \)-algebra \( A^\alpha_\beta \beta \otimes \gamma K \delta, A^b_\beta \delta \beta, B \) the \( W^* \)-fiber product of \( A^\alpha_\beta \beta \) and \( B^\gamma_\delta K \).

To prove that the \( W^* \)-fiber product is functorial, we use the following lemma:

**Lemma 3.12.** Let \( \phi \) be a morphism of \( W^* - b \)-algebras \( A^b_H, C^b_L \) and let \( B^b_K \) be a \( W^* - b^1 \)-algebra. Then \( I := \mathcal{L}(H, L) \otimes \text{id} \) is a concrete \( W^* \)-module, \( \rho_1(A^b_\beta \gamma, B) \subseteq C^b_\beta \gamma, B \), and \( \langle \eta_x | \rho_1(x) | \eta \rangle_2 = \phi(\langle \eta | x \rangle_2 | \eta \rangle_2) \) for all \( x \in A^b_\beta \gamma, B \) and \( \eta, \eta' \in \gamma \).

**Proof.** The first assertion is evident. Let \( x \in A^b_\beta \gamma, B \). Then

\[
\rho_1(x) | \lambda_1 \rangle_1 = \rho_1(x) | Ix | \lambda_1 \rangle_1 \subseteq \| Ix | \beta_1 \rangle_1 \| \| I | \beta_1 \rangle_1 \| \subseteq \| I \rangle_1 B \]

and \( \langle \eta_x | \rho_1(x) | \eta \rangle_2 = \phi(\langle \eta_x | x \rangle_2 | \eta \rangle_2) \) for all \( \eta, \eta' \in \gamma \) because

\[
\langle \eta | \rho_1(x) | x \rangle_2 S = \langle \eta' | x \rangle \rho_1(x) | S \otimes \text{id} \rangle | \eta \rangle_2 = \langle \eta' | x \rangle \rho_1(x) | S \otimes \text{id} \rangle | x \rangle_2 = \rho_1(x) | x \rangle_2 = \phi(\langle \eta | x \rangle_2) S
\]

for all \( S \in \mathcal{L}(H, L) \). By Lemma 3.9 we can conclude \( \rho_1(x) \in C^b_\beta \gamma, B \).
Proposition 3.13. Let $\phi$ be a morphism of $W^\ast$-($a^1, b$)-algebras $A_H^{a, b}$, $C_L^{a, \lambda}$ and $\psi$ a morphism of $W^\ast$-($b^1, c$)-algebras $B_K^{\gamma, \delta}$, $D_M^{\mu, \nu}$. Then there exists a unique morphism $\phi \circ \psi$ of $W^\ast$-($a^1, c$)-algebras $A_H^{a, c} B_K^{\gamma, \delta}$ and $C_L^{a, \lambda} D_M^{\mu, \nu}$ such that

$$((\phi \circ \psi)(x))R = Rx \quad \text{for all } x \in A_H^{a, b}B \text{ and } R \in I_MJ_H + J_LI_K, \quad (10)$$

where $I_X = L^\phi(H, L) \otimes id_X$, $J_Y = id_Y \otimes L^\psi(K, M)$ for $X \in \{K, M\}$, $Y \in \{H, L\}$.

Proof. The space $E := L^\phi(H, L) \otimes L^\psi(K, M)$ is a concrete $W^\ast$-module, $E \subseteq [I_MJ_H] \cap [J_LI_K]$, and $A_H^{a, b}B \subseteq (E^E)'$. Therefore, every morphism $\phi \circ \psi$ satisfying (10) is equal to the restriction of $\rho_E$. Conversely, we may choose $\phi \circ \psi$ to be this restriction. Indeed, Proposition 2.3 shows that then $\phi \circ \psi$ is equal to the compositions

$$A_H^{a, b} B \overset{\rho_L}{\to} C_L^{a, \lambda} B \overset{\rho_M}{\to} C_L^{a, \mu} D, \quad A_H^{a, b} B \overset{\rho_L}{\to} A_H^{a, \mu} D \overset{\rho_M}{\to} C_L^{a, \mu} D,$$

whence (10) holds, $[L^\phi(H, L) \otimes L^\psi(K, M)](\alpha \otimes \gamma) \subseteq [(L^\phi(K, M) \gamma) \otimes L^\psi(H, L)\alpha]\subseteq [\otimes L^\psi(K, M)](\alpha \otimes \gamma)$ contains

$$[(L^\phi(H, L) \otimes L^\psi(K, M))\gamma] = \|L^\phi(K, M)\gamma\|_2 \|L^\psi(H, L)\alpha\|_2 = \|\gamma\|_2 \|\alpha\|_2 \subseteq \lambda \otimes \nu,$$

and similarly $[L^\phi(H, L) \otimes L^\psi(K, M)\gamma] = \lambda \otimes \nu$. \qed

Theorem 3.14. There exists a bicategory $W^\ast$-$\text{alg}_{(nd)}$ such that

- the objects are all $W^\ast$-bases and $W^\ast$-$\text{alg}_{(nd)}(a, b) = W^\ast$-$\text{alg}_{(nd)}(a^1, b)$ for all $W^\ast$-bases $a, b$;
- $c_{a, b, c}$ is given by $(B_K^{\gamma, \delta}, A_H^{a, b}) \mapsto A_H^{a, \mu} B_K^{\gamma, \delta}$ and $(\psi, \phi) \mapsto \phi \circ \psi$, and the identity $1_b$ is $L(V)^{B^1, \mu}$;
- the isomorphisms $a_{a, b, c}(\phi, \psi, \phi, \psi), r_{a, b}(A_H^{a, b}), l_{a, b}(B_K^{\gamma, \delta})$ are given by conjugation by $a_{a, b, c}(L_{\phi, \psi, \phi}, \gamma, K_{\delta, \alpha}H_{\beta})$, $r_{a, b}(\alpha H_{\beta}), l_{a, b}(\gamma, K_{\delta})$, respectively, for all $A_H^{a, b}, B_K^{\gamma, \delta}, C_L^{a, \lambda}$ and all $W^\ast$-bases $a, b, c, \delta$.

Proof. Tediou but straightforward. \qed

### 3.3 A spatial fiber product of $C^\ast$-algebras

To define a fiber product of $C^\ast$-algebras, we start from the characterization of the fiber product of $W^\ast$-algebras given in Proposition 3.10 and formulate
a $C^*$-algebraic analogue of Lemma 3.9. Let $E^K_H$ be a concrete nondegenerate $C^*$-module and $A_H$ a concrete $C^*$-algebra satisfying $E^*E_A \subseteq A$. Put

$$\text{Ind}_E(A) := \{T \in \mathcal{L}(K) \mid TE + T^*E \subseteq [EA]\} \subseteq \mathcal{L}(K).$$

**Definition 3.15.** Let $E^K_H$ be a nondegenerate concrete $C^*$-module. The $E$-strong-$*$, $E$-strong, and $E$-weak topology on $\mathcal{L}(K)$ are the topologies induced by the families of seminorms $T \mapsto \|T\xi\| + \|T^*\xi\| (\xi \in E)$, $T \mapsto \|T\xi\| (\xi \in E)$, and $T \mapsto \|\xi^*T\xi\| (\xi, \xi' \in E)$, respectively. Given a subset $X \subseteq \mathcal{L}(K)$, denote by $[X]_E$ the closure of span $X$ with respect to the $E$-strong-$*$ topology.

**Remarks 3.16.**

i) If $H$ is a Hilbert space, then $|H\rangle^H_E$ is a nondegenerate concrete $C^*$-module, and the associated topologies on $\mathcal{L}(H)$ introduced above coincide with the strong-$*$, the strong and the weak operator topology, respectively.

ii) Evidently, the multiplication in $\mathcal{L}(K)$ is separately continuous with respect to the topologies introduced above, and the involution $T \mapsto T^*$ is continuous with respect to the $E$-strong-$*$ and the $E$-weak topology.

**Proposition 3.17.** Let $E^K_H$ be a concrete nondegenerate $C^*$-module, $A_H$ a concrete $C^*$-algebra satisfying $E^*E_A \subseteq A$. Then $\text{Ind}_E(A)$ is a $C^*$-algebra and

$$[E^* \text{Ind}_E(A)E] \subseteq A, \quad \text{Ind}_E(A) = [EAE^*]_E, \quad \text{Ind}_E(M(A)) \subseteq M(\text{Ind}_E(A)).$$

$\text{Ind}_E(A)_K$ is nondegenerate if and only if $A_H$ is nondegenerate; in that case, $\text{Ind}_E(A) \subseteq \mathcal{L}(K[\text{Ind}_E(A)])$.

**Proof.** The proof of the first two assertions is straightforward.

We have $[EAE^*]_E \subseteq \text{Ind}_E(A)$ because $[EAE^*]_E \subseteq [EAE^*]_E \subseteq [EA]$. To see that $[EAE^*]_E \supseteq \text{Ind}_E(A)$, choose a bounded approximate unit $(u_\nu)_\nu$ for the $C^*$-algebra $[EE^*]$ and observe that for each $T \in \text{Ind}_E(A)$, the net $(u_\nu Tu_\nu)_\nu$ lies in $[EE^* \text{Ind}_E(A)E] \subseteq [EAE^*]$ and converges to $T$ in the $E$-strong-$*$ topology.

If $S \in \text{Ind}_E(M(A))$, $T \in \text{Ind}_E(A)$, then $ST \in \text{Ind}_E(A)$ because $ST \subseteq [SAE] \subseteq [EM(A)] \subseteq [EA]$ and $T^*S^*E \subseteq [TEM(A)] \subseteq [EAM(A)] = [EA]$.

If $\text{Ind}_E(A)_K$ is nondegenerate, then $[AH] \supseteq [E^* \text{Ind}_E(A)E] = [E^* \text{Ind}_E(A)] = [E^*K] = H$. Conversely, if $A_H$ is nondegenerate, then $[EAE^*]_K$ and hence also $\text{Ind}_E(A)_K$ is nondegenerate.

Throughout this subsection, let $a, b, c$ be $C^*$-bases as in (3). Moreover, let $A^\beta_H$ be a $C^*$-b-algebra and $B^\gamma_K$ a $C^*$-b*-algebra. We apply the induction procedure to $A$, $B$ and $|\gamma\rangle_2 \subseteq \mathcal{L}(H, H_B \otimes \gamma K)$, $|\beta\rangle_1 \subseteq \mathcal{L}(K, H_B \otimes \gamma K)$ (see Subsection 2.4), respectively, and let

$$A^\beta_{b^\gamma}B := \text{Ind}_{|\gamma\rangle_2}(A) \cap \text{Ind}_{|\beta\rangle_1}(B) \subseteq \mathcal{L}(H_B \otimes \gamma K).$$

(11)

Thus, $A^\beta_{b^\gamma}B$ consists of all $T \in \mathcal{L}(H_B \otimes \gamma K)$ satisfying $T|\gamma\rangle_2 + T^*|\gamma\rangle_2 \subseteq [|\gamma\rangle_2A]$ and $T|\beta\rangle_1 + T^*|\beta\rangle_1 \subseteq [|\beta\rangle_1B]$, and we have the following commutative in the
category where the objects are Hilbert spaces, morphisms are concrete $C^*$-modules, and composition is defined as in $C^*$-mod:

\[ H \xrightarrow{\gamma_2} H_{\beta} \otimes_\gamma K \xrightarrow{\beta_1} K \]

\[ A \]

\[ A \xrightarrow{\gamma_2} H_{\beta} \otimes_\gamma K \xrightarrow{\beta_1} K \]

Put $A^{(\beta)} := A \cap \mathcal{L}(H_{\beta})$, $B^{(\gamma)} := B \cap \mathcal{L}(K_{\gamma})$, $X := (A^{(\beta)} \otimes \text{id}) + (\text{id} \otimes B^{(\gamma)})$, and

\[ M_s(A^{(\beta)} \otimes B^{(\gamma)}) := \{ T \in \mathcal{L}(H_{\beta} \otimes_\gamma K) \mid TX, XT \subseteq A^{(\beta)} \otimes B^{(\gamma)} \}. \]

**Proposition 3.17.** Let $A^{(\beta)}_H$ be a $C^*_b$-algebra and $B^{(\gamma)}_K$ a $C^*_b$-algebra.

i) $\langle \beta \rangle_1 (A^{(\beta)} \otimes B^{(\gamma)}) | \beta \rangle_1 \subseteq B$ and $\langle \gamma \rangle_2 (A^{(\beta)} \otimes B^{(\gamma)}) | \gamma \rangle_2 \subseteq A$.

ii) $A^{(\beta)} \otimes B^{(\gamma)} \subseteq A^{(\beta)} \otimes B$ and $M(A^{(\beta)} \otimes B^{(\gamma)}) \subseteq M(A^{(\beta)} \otimes B)$.

iii) If $[A^{(\beta)}] = \beta$ and $[B^{(\gamma)}] = \gamma$, then $A^{(\beta)} \otimes B^{(\gamma)}$ is nondegenerate and we have

\[ M_s(A^{(\beta)} \otimes B^{(\gamma)}) \subseteq A^{(\beta)} \otimes B^{(\gamma)}. \]

iv) $A^{(\beta)} \otimes B^{(\gamma)}$ contains $\text{id}_{(H_{\beta} \otimes_\gamma K)}$ if and only if $\rho_{\beta}(B^1) \subseteq A$ and $\rho_{\gamma}(B) \subseteq B$.

v) If $A^{(\beta)} \otimes B^{(\gamma)}$ is nondegenerate, then the $C^*$-algebra $[\beta^* A \beta] \cap [\gamma^* B \gamma] \subseteq \mathcal{L}(\mathbb{C})$ is nondegenerate.

vi) Assume that $A^{(\beta)}_{H^{(\gamma)}}$ is a $C^*_b$-algebra and $B^{(\gamma)}_{K^{(\delta)}}$ a $C^*_b$-algebra.

Then $(A^{(\beta)}_{H^{(\gamma)}} \otimes B^{(\gamma)}_{K^{(\delta)}}) \subseteq A^{(\beta)}_{H^{(\gamma)}} \otimes B^{(\gamma)}_{K^{(\delta)}}$ is a $C^*_b$-algebra.

**Proof.**

i) Immediate from Proposition 3.17.

ii) This follows from the relations $(A^{(\beta)} \otimes B^{(\gamma)}) | \beta \rangle_1 \subseteq [(A^{(\beta)} \otimes B^{(\gamma)}) | \beta \rangle_1] \subseteq [\beta_1 B], (A^{(\beta)} \otimes B^{(\gamma)}) | \gamma \rangle_2 \subseteq [(A^{(\beta)} \otimes B^{(\gamma)}) | \gamma \rangle_2] \subseteq [\gamma_2 A]$ and Proposition 3.17.

iii) Assume that $[A^{(\beta)}] = \beta$ and $[B^{(\gamma)}] = \gamma$. Then $A^{(\beta)} \otimes B^{(\gamma)}$ and hence also $A^{(\beta)} \otimes B^{(\gamma)}$ is nondegenerate. If $T \in M_s(A^{(\beta)} \otimes B^{(\gamma)})$, then $T | \beta \rangle_1 \subseteq T (A^{(\beta)} \otimes \text{id}) | \beta \rangle_1 \subseteq [(A^{(\beta)} \otimes B^{(\gamma)}) | \beta \rangle_1] \subseteq [\beta_1 B]$ and similarly $T | \gamma \rangle_2 \subseteq [\gamma_2 A]$. Conversely, if the last two inclusions hold, then $| \gamma \rangle_2 = [\gamma_2 B_1] \subseteq [\gamma_2 A]$ and similarly $| \beta \rangle_1 \subseteq [\beta_1 B]$, whence $\text{id}_{(H_{\beta} \otimes_\gamma K)} \subseteq A^{(\beta)} \otimes B^{(\gamma)}$. 

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Example 3.20. Let \( A \) be an infinite-dimensional Hilbert space and identify \( \ell^2(\mathbb{N}) \) with \( A \). Then \( A \) is not contained in \( \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \) because the flip \( (A_1,A_2) \mapsto (A_2,A_1) \) is nondegenerate, we therefore have \( [\mathbb{R}] \supseteq \left\{ \gamma \eta_1 \Omega_1 | \gamma \Omega_1 \right\} \).}



vi) The product \( X := \rho_{(\omega,\gamma_1)}(\mathbb{A}^i_{\lambda})(\gamma_{\lambda,\gamma_1}) \) is contained in \( \beta \lambda \gamma \lambda, \beta \gamma \lambda \). Even in special situations, it seems to be difficult to describe the fiber product more precisely.

**Definition 3.19.** The fiber product of a \( C^*-(\mathbb{A},\mathbb{B}) \)-algebra \( A_{\lambda \gamma} \) and a \( C^*-(\mathbb{B},\mathbb{C}) \)-algebra \( B_{\lambda} \) is the \( C^*-(\mathbb{A},\mathbb{C}) \)-algebra \( A_{\lambda \gamma} \gamma \lambda := \rho_{(\lambda \gamma,\lambda \gamma)}(\mathbb{C})(\gamma_{\lambda,\gamma_1}) \subseteq A_{\lambda \gamma,\gamma_1} \).

Example 3.21. Let \( H \) be an infinite-dimensional Hilbert space and identify \( H \) with \( \ell^2(\mathbb{N}) \) as in Example 3.12. We show that the flip \( \Sigma : H \times H \rightarrow H \times H, \xi \times \eta \mapsto \eta \times \xi \), is not contained in \( \mathcal{L}(H) \times H \mathcal{L}(H) \). Indeed, let \( (\xi_\nu) \) be an orthonormal basis for \( H \) and let \( \eta \in H \) be non-zero. Then \( \langle \xi_\nu \rangle_1 \Sigma \eta_1 = |\eta| \xi_\nu \) and hence \( \sum_\nu \langle |\xi_\nu| \Sigma \eta_1 \rangle_1^2 = \infty \). On the other hand, one easily verifies that \( \sum_\nu \langle |\xi_\nu| S \rangle_1^2 < \infty \) for each \( S \in [H]_1 \mathcal{L}(H) \), and hence \( \Sigma \eta \not\in [H]_1 \mathcal{L}(H) \).

Further concrete examples will be given in Example 3.26 and Subsection 3.6.

### 3.4 Categorical properties of the spatial fiber product of \( C^*-\)algebras

We shall see that the fiber product of \( C^*-\)algebras introduced above is functorial, nonassociative, unital only in a restricted sense, contained in the fiber product of the associated \( W^*-\)algebras, and distributive with respect to finite direct sums. To prove functoriality, we use the following lemma.
Lemma 3.22. Let \( \phi \) be a morphism of \( C^*\)-\( b \)-algebras \( A_H^{\beta} \) and \( C_L^{\lambda} \), and let \( B_K^{\gamma} \) be a \( C^*\)-\( b^\perp \)-algebra. Then \( I := \mathcal{L}^\phi(H, L) \otimes \text{id} \) is a concrete \( C^* \)-module, \( \rho_I(A_{\beta, b} \gamma)B \subseteq C_{\lambda, b} \gamma B \), there exists a bounded linear map \( j_\phi : [\gamma]_{2}A \to [\gamma]_{2}C \) given by \( [\gamma]_{2}a \mapsto [\gamma]_{2}\phi(a) \), and \( \rho_I(x)[\gamma]_{2} = j_\phi(x)[\gamma]_{2} \) for all \( x \in A_{\beta, b} \gamma B \), \( \eta \in \gamma \).

Proof. The first assertion is evident. The existence of \( j_\phi \) follows from the fact that for all \( \eta, \eta' \in \gamma \), \( a', a' \in A \), \(([\gamma]_{2}\phi(a'))^*([\gamma]_{2}\phi(a')) = \phi(a') \rho_\lambda(\eta^* \eta') \phi(a') = \phi(([\gamma]_{2}a)^*([\gamma]_{2}a')) \). For all \( x \in A_{\beta, b} \gamma B \), \( \eta \in \gamma \), \( \rho_I(x)[\gamma]_{2} = j_\phi(x)[\gamma]_{2} \) in \([\gamma]_{2}C \) because \( \rho_I(x)[\gamma]_{2} = \rho_I(x)(S \otimes \text{id})[\gamma]_{2} = (S \otimes \text{id})x[\gamma]_{2} = j_\phi(x)[\gamma]_{2} \) for all \( S \in \mathcal{L}^\phi(H, L) \). Therefore, \( \rho_I(A_{\beta, b} \gamma B)[\gamma]_{2} \subseteq [\gamma]_{2}C \). Finally, \( \rho_I(A_{\beta, b} \gamma B)[\lambda]_{1} = \rho_I(A_{\beta, b} \gamma B)[\lambda]_{1} \subseteq [I(A_{\beta, b} \gamma B)][\lambda]_{1} \subseteq [I][\lambda]_{1}B = [\lambda]_{1}B \).

\[ ((\phi \ast \psi)(x))R = Rx \quad \text{for all } x \in A_{\beta, b} \gamma B \text{ and } R \in I_MJ_H + J_LI_K. \quad (12) \]

where \( I_X = \mathcal{L}^\phi(H, L) \otimes \text{id}_X \), \( J_Y = \text{id}_Y \otimes \mathcal{L}^\phi(K, M) \) for \( X \in \{K, M\}, Y \in \{H, L\} \).

Proof. Completely analogous to the proof of Proposition 3.13 §.

The fiber product is not functorial with respect to nondegenerate morphisms into multiplier algebras:

Remark 3.24. Let \( A_H^{\beta}, C_L^{\lambda} \) be \( C^*\)-\( b \)-algebras, let \( B_K^{\gamma}, D_M^{\nu} \) be \( C^*\)-\( b^\perp \)-algebras, and let \( \phi \) and \( \psi \) be morphisms of \( A_H^{\beta}, M(C) \), and \( B_K^{\gamma}, M(D) \), respectively. Then there exists a \( * \)-homomorphism \( \phi \ast \psi : A_{\beta, b} \gamma B \to M(C) \ast \mu M(D) \to M(C) \ast \mu D \). However, we do not know whether the relations \([\phi(A)C] = C\) and \([\psi(B)D] = D\) imply \([((\phi \ast \psi)(A_{\beta, b} \gamma B))(C \ast \mu D)] = C \ast \mu D \).

The fiber products of \( C^* \)-algebras and \( W^* \)-algebras are related as follows:

Theorem 3.25. i) \([A_{\beta, b} \gamma B] \subseteq [A][a]_{[b]}[\gamma][B] \) for each \( C^*\)-\( b \)-algebra \( A_H^{\beta} \) and \( C^*\)-\( b^\perp \)-algebra \( B_K^{\gamma} \).

ii) \([\phi \ast \psi] = [\phi][\psi][A_{\beta, b} \gamma B] \) whenever \( \phi \) is a morphism of \( C^*\)-\( (a^\perp, b) \)-algebras \( A_H^{\alpha, \beta}, C_L^{\lambda, \gamma} \) and \( \psi \) is a morphism of \( C^*\)-\( (b^\perp, c) \)-algebras \( B_K^{\gamma, \delta}, D_M^{\nu, \mu} \).
Proof. i) $\langle \beta \mid 1_A^* A_{\beta \gamma}^* B \rangle \langle \beta \rangle_1 \subseteq \langle \beta \mid 1_A^* A_{\beta \gamma}^* B \rangle \subseteq \langle \beta \rangle_1$ and similarly $\langle \gamma \mid 2_A^* A_{\beta \gamma}^* B \rangle \langle \gamma \rangle_2 \subseteq \langle \gamma \rangle_2$ by Proposition 3.13 (i). Now, i) follows from Proposition 3.10. ii) Since the multiplication is separately $\sigma$-weakly continuous, $\mathcal{L}_\beta(H_{\beta}, L_H) \subseteq \mathcal{L}^\phi(H_{\beta}, L_{M_H})$ and $\mathcal{L}_\gamma(K_{\gamma}, M_H) \subseteq \mathcal{L}^\psi(K_{\gamma}, M_{[\psi]})$. Now, ii) follows from equations (10) and (12).

Unfortunately, the fiber product of $C^*$-algebras is not associative. More precisely, let $A_{b_c}^\alpha$ be a $C^*(a, b)$-algebra, $B_{K_{\beta}^\delta}^\gamma$ a $C^*(b^1, c)$-algebra, and $C_{L_{\beta}}^\phi$ a $C^*(c^1, d)$-algebra. Then we can form the fiber products $(A_{b_c}^\alpha B_{K_{\beta}^\delta}^\gamma)_b \otimes_C C_{L_{\beta}}^\phi$ and $A_{b_c}^\alpha \otimes_{\alpha} B_{K_{\beta}^\delta}^\gamma \otimes_{\gamma} C_{L_{\beta}}^\phi$. The following example shows that these $C^*$-algebras need not be identified by the canonical isomorphism $a_{a, b, c, d}(L_{\beta}, \gamma K_{\delta}, \alpha H_{\beta})$ of Proposition 2.17. However, identifying the underlying Hilbert spaces via this isomorphism, we can define a “minimal” $C^*$-fiber product $A_{b_c}^\alpha B_{K_{\beta}^\delta}^\gamma \otimes_{\epsilon} C_{L_{\beta}}^\phi := ((A_{b_c}^\alpha B_{K_{\beta}^\delta}^\gamma)_b \otimes_{\beta} C_{L_{\beta}}^\phi) \cap (A_{b_c}^\alpha \otimes_{\alpha} B_{K_{\beta}^\delta}^\gamma \otimes_{\gamma} C_{L_{\beta}}^\phi)$, which is functorial in each component.

Example 3.26. Let $a = b = c = d = t$ (see Example 2.9 i)) and $A_{b_c}^\beta B_{K_{\beta}^\delta}^\gamma = C_{L_{\beta}}^\phi = \mathcal{L}(H)^{[H], [H]}_H$, where $H$ is a separable Hilbert space. Then we can identify $H_{\beta}^\otimes \otimes \gamma K_{\delta}^\otimes \otimes \epsilon \subseteq H \otimes H \otimes H \cong H^{\otimes 3}$. We construct an element $T$ of $X := \mathcal{L}(H)^{[H], [H]}_H \mathcal{L}(H)^{[H], [H]}_H \mathcal{L}(H)$ that does not belong to $Y := \mathcal{L}(H)^{[H], [H]}_H \mathcal{L}(H)^{[H], [H]}_H \mathcal{L}(H)$. Choose an orthonormal basis $(\epsilon_n)_{n \in \mathbb{N}}$ of $H$ and define $T \in \mathcal{L}(H^{\otimes 3})$ by

$$T(\epsilon_k \otimes \epsilon_l \otimes \epsilon_m) = \begin{cases} \epsilon_k \otimes \epsilon_l \otimes \epsilon_m & \text{for all } k, l, m \in \mathbb{N} \text{ s.t. } m \leq k + l, \\ \epsilon_l \otimes \epsilon_k \otimes \epsilon_m & \text{for all } k, l, m \in \mathbb{N} \text{ s.t. } m > k + l. \end{cases}$$

For each $\xi \in H$ and $\omega \in H^{\otimes 2}$, define $|\xi\rangle_1, |\xi\rangle_3 \in \mathcal{L}(H^{\otimes 2}, H^{\otimes 3})$ and $|\omega\rangle_1, |\omega\rangle_2 \in \mathcal{L}(H, H^{\otimes 3})$ by $v \mapsto \langle \xi | v \rangle \xi$ and $v \mapsto \langle \omega | v \rangle \xi$, respectively. Then

$$T|\epsilon_k \otimes \epsilon_l\rangle_1 = |\epsilon_k \otimes \epsilon_l\rangle_1 P_{l+k} + |\epsilon_l \otimes \epsilon_k\rangle_1 (id - P_{l+k}), P_{l+k} := \sum_{m \leq k+l} |\epsilon_m\rangle\langle \epsilon_m|,$$

whence $T|H^{\otimes 2}\rangle_1 \in |H^{\otimes 2}\rangle_1 \mathcal{L}(H)$, and

$$T|\epsilon_m\rangle_3 = |\epsilon_m\rangle_3 (id + \Sigma_m), \text{ where } \Sigma_m := \sum_{k,l \in \mathbb{N}} |\epsilon_l \otimes \epsilon_k - \epsilon_k \otimes \epsilon_l\rangle \langle \epsilon_l \otimes \epsilon_k|,$$

whence $T|H\rangle_3 \in [|H]\rangle_3 (id + \mathcal{K}(H) \otimes \mathcal{K}(H)) = [|H]\rangle_3 (|H\rangle \mathcal{L}(H))$ by Proposition 3.18 (ii), iv. Since $T = T^*$, we can conclude $T \in X$.

However, $T|\epsilon_1\rangle_1 \notin [|H]\rangle_1 \mathcal{L}(H^{\otimes 2})$ and therefore $T \notin Y$. Indeed, one easily verifies that $\sum_l \|\langle \epsilon_l |_S\rangle\|^2 < \infty$ for each $S \in [|H]\rangle \mathcal{L}(H^{\otimes 2})$ (where $\langle \epsilon_1 |_1 = |\epsilon_1\rangle_1$), but $T|\epsilon_1\rangle_1 |\epsilon_1 \otimes \epsilon_1\rangle_2 = |\epsilon_1 \otimes \epsilon_1\rangle_2$ and hence $\|\langle \epsilon_1 |_T |\epsilon_1\rangle\|^2 = 1$ for each $l \in \mathbb{N}$.

Let us now investigate whether the fiber product construction relative to a $C^*$-base $b$ admits a categorical unit. Such a unit should be a $C^*-(b^1, b)$-algebra.
such that for all $C^*-(a^1, b)$-algebras $A_{H}^{\alpha, \beta}$ and all $C^*-(b^1, c)$-algebras $B_{K}^{\gamma, \delta}$, we have $A_{H}^{\alpha, \beta} = \text{Ad}_r(A_{H}^{\alpha, \beta} \ast \mathfrak{U}_R^{\beta, \beta})$ and $B_{K}^{\gamma, \delta} = \text{Ad}_l(\mathfrak{U}_R^{\beta, \beta} \ast B_{K}^{\beta, \beta})$, where $r = l_{B_{K}^{\beta, \beta}}$ and $l = l_{B_{K}^{\beta, \beta}}$ (see Proposition 2.17). The relations $r|\beta) = \beta$, $l|\beta) = \rho_{\beta}(\mathfrak{B})$, imply

$$\text{Ad}_r(A_{\beta} \star \mathfrak{M}) = \text{Ind}_{\beta}(\mathfrak{M}) \cap \text{Ind}_{\rho_{\beta}(\mathfrak{M})}(A),$$

$$\text{Ad}_l(\mathfrak{M} \ast \gamma B) = \text{Ind}_{\rho_{\beta}(\mathfrak{M})}(B) \cap \text{Ind}_{\gamma}(\mathfrak{M}).$$

We deduce that if $\mathfrak{B}$ is a unital, then $\text{Ind}_{\rho_{\beta}(\mathfrak{M})}(A) = A$ and $\text{Ind}_{\rho_{\beta}(\mathfrak{M})}(B) = B$, and then the $C^*-(b^1, b)$-algebra $\mathcal{L}(\mathfrak{R}) \mathfrak{U}_R^{\beta, \beta}$ is a unit for the fiber product on the full subcategories of all $A_{H}^{\alpha, \beta}$ and $B_{K}^{\gamma, \delta}$ satisfying $A \subseteq \text{Ind}_{\beta}(\mathcal{L}(\mathfrak{R}))$ and $B \subseteq \text{Ind}_{\gamma}(\mathcal{L}(\mathfrak{R})).$

**Remarks 3.27.**

i) If $A \subseteq \text{Ind}_{\alpha}(\mathcal{L}(\mathfrak{F}))$ and $B \subseteq \text{Ind}_{\gamma}(\mathcal{L}(\mathfrak{C}))$, then $A_{\beta} \star \gamma B \subseteq \text{Ind}_{(\alpha \times \gamma)}(\mathcal{L}(\mathfrak{F})) \cap \text{Ind}_{(\beta \times \delta)}(\mathcal{L}(\mathfrak{R})).$

ii) If $\text{Ind}_{\beta}(\mathfrak{B}) = \mathcal{L}(\mathfrak{H})$, and if $\mathfrak{B}$ is unital, then $\text{Ad}_r(A_{\beta} \star \mathfrak{M} \mathfrak{B}) = A \cap \mathcal{L}(\mathfrak{H}) = A_{\beta}.$

iii) $\text{Ad}_r(\mathfrak{M} \ast \mathfrak{B}) = \mathcal{L}(\mathfrak{R} \mathfrak{M}) \cap \mathcal{L}(\mathfrak{K} \mathfrak{M}) = M(\mathfrak{B}) \cap M(\mathfrak{B}).$

The fiber product is compatible with finite sums, but neither with infinite c₀-sums nor with infinite $l^\infty$-sums.

**Proposition 3.28.** Let $(A_i)$ be a finite family of $C^*-(a^1, b)$-algebras and $(B_j)$ a finite family of $C^*-(b^1, c)$-algebras. For each $i, j,$ denote by $e_i: A_i \rightarrow \mathbb{1}, A^i$, $e_j: B_j \rightarrow \mathbb{1}, B^j,$ and $\pi_{A_i}: A_i \rightarrow A^i$, $\pi_{B_j}: B_j \rightarrow B^j$ the canonical inclusions and projections, respectively. Then there exist inverse isomorphisms $\mathbb{1} = \mathbb{1}$ and $\mathbb{1} = \mathbb{1}$, given by $(x, y) \mapsto \sum_{i, j}(e_i \ast e_j)(x, y)$ and $((\pi_{A_i} \ast \pi_{B_j})(y))_{i, j} \mapsto y,$ respectively.

**Examples 3.29.**

i) For each $i, j \in \mathbb{N}$, let $A^i$ and $B^j$ be the $C^*$-algebra $\mathbb{C}$ and identify $\bigoplus_{i, j} \mathbb{C} \otimes \mathbb{C}$ with $l^2(\mathbb{N} \times \mathbb{N})$ in the canonical way. Then $\bigoplus_{i, j} A^i \ast B^j$ corresponds to $\mathcal{C}(\mathbb{N} \times \mathbb{N})$, represented on $l^2(\mathbb{N} \times \mathbb{N})$ by multiplication operators. We shall see in Example 3.30(ii) that $(\bigoplus_{i} A^i) \ast (\bigoplus_{j} B^j) \cong \mathcal{C}(\mathbb{N} \times \mathbb{N})$ is strictly larger and contains, for example, the characteristic function of the diagonal $\{(x, x) \mid x \in \mathbb{N}\}.$

ii) Let $A$ and $B$ be the $C^*$-algebra $\mathcal{K}(\mathcal{H})_{H}$ for all $j$, where $H = l^2(\mathbb{N})$, and identify $H_{H} \otimes H$ with $H \otimes H$ as in Example 2.15. Choose an orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of $H$ and put $y_j := (e_j \otimes e_0) < e_0 \otimes e_0 > \in \mathcal{K}(H \otimes H)$ for each $j \in \mathbb{N}$. Then $y := (y_j)_{j \in \mathbb{N}} \in \bigoplus_{j} A^i \ast B^j$ because $y_j \in \mathcal{K}(H) \otimes \mathcal{K}(H)$, and...
\[ A_j^* B_j \] for all \( j \in \mathbb{N} \), but with respect to the canonical identification \( \bigoplus_j H \otimes H \equiv H \otimes \left( \bigoplus_j H \otimes H \right) \), we have \( y \notin A_\ast(\prod_j B_j) \) because \( y|_{\langle e_0 \rangle} \) corresponds to the family \( \{(e_j)_{1|\langle e_0 \rangle}\}_{j} \in \prod_j \mathcal{L}(H, H \otimes H) \subseteq \mathcal{L}(\bigoplus_j H, \bigoplus_j H \otimes H) \) which is not contained in the space \( \|H\|_{\mathcal{L}(\bigoplus_j H)} \).

### 3.5 Slice maps on the spatial \( C^* \)-fiber product

We discuss two classes of slice maps on the spatial fiber product of \( C^* \)-algebras. The first class arises from a simple induction procedure for maps that have the following nice presentation. Let \( E_\mathbb{N} \) be a concrete \( C^* \)-module, let \( E^{(n)} \) be the set of all sequences \( \xi = (\xi_k)_{k \in \mathbb{N}} \) in \( E \) for which the sum \( \sum_k \xi_k^* \xi_k \) converges in norm, and put \( \|\xi\| := \left\| \sum_k \xi_k^* \xi_k \right\|^{1/2} \) for each \( \xi \in E^{(n)} \). Then standard arguments show that for all \( \xi, \xi' \in E^{(n)} \), there exists a bounded linear map

\[ \omega_{\xi, \xi'} : \mathcal{L}(K) \to \mathcal{L}(H), \quad T \mapsto \sum_k \xi_k^* T \xi_k', \]

where the sum converges in norm, and \( \|\omega_{\xi, \xi'}\| \leq \|\xi\| \|\xi'\| \). We put \( \Omega_E := \{ \omega_{\xi, \xi'} | \xi, \xi' \in E^{(n)} \} \).

**Proposition 3.30.** Let \( A_\beta^{(n)} \) be a \( C^* \)-\( b \)-algebra and \( B_\gamma^{(n)} \) a \( C^* \)-\( b^1 \)-algebra.

i) If \( \xi \in \beta^{(n)} \) and \( \eta \in \gamma^{(n)} \), then \( |\xi\rangle_1 := (|\xi_k\rangle_1)_k \in |\beta\rangle^{(n)}_1 \) and \( |\eta\rangle_2 := (|\eta_k\rangle_2)_k \in |\gamma\rangle^{(n)}_2 \).

ii) Let \( \phi \in \Omega_{\beta\gamma} \), \( \psi \in \Omega_{\gamma\eta} \) and choose \( \xi, \xi' \in \beta^{(n)} \), \( \eta, \eta' \in \gamma^{(n)} \) such that \( \phi = \omega_{\xi, \xi'}, \psi = \omega_{\eta, \eta'} \). Then the maps \( \phi \ast \text{id} := \omega_{|\xi\rangle_1, |\xi'\rangle_1} \) and \( \text{id} \ast \psi := \omega_{|\eta\rangle_2, |\eta'\rangle_2} \) do not depend on the choice of \( \xi, \xi', \eta, \eta' \) and satisfy \( \phi \circ (\text{id} \ast \psi) = \psi \circ (\phi \ast \text{id}) \), \( (\phi \ast \text{id})(A_{\beta\gamma} \ast \gamma, B) \subseteq B \), \( (\text{id} \ast \psi)(A_{\beta\gamma} \ast \beta, B) \subseteq A \).

**Proof.**

i) Immediate from the relation \( \sum_{k=1}^n |\xi_k\rangle_1^* |\xi_k\rangle_1 = \rho_\gamma \left( \sum_{k=1}^n |\xi_k\rangle_1^* |\xi_k\rangle_1 \right), \) which holds for all \( \xi_1, \ldots, \xi_n \in \beta, \) and from the corresponding formula for elements \( \eta_1, \ldots, \eta_n \in \gamma. \)

ii) We have \( \phi \circ (\text{id} \ast \psi) = \psi \circ (\phi \ast \text{id}) \) because \( |\xi_k\rangle_1 \eta_l = |\eta_l\rangle_2 \xi_k \) and \( |\xi_k\rangle_1 \eta_l' \) = \( |\eta_l'\rangle_2 \xi_k \) for all \( k, l \). Varying \( \phi \), we conclude that \( \text{id} \ast \psi \) does not depend on the choice of \( \eta, \eta' \). Similarly, \( \phi \ast \text{id} \) is independent of the choice of \( \xi, \xi' \). Finally, \( (\phi \ast \text{id})(A_{\beta\gamma} \ast \gamma, B) \subseteq [\langle \beta | (A_{\beta\gamma} \ast \gamma, B) | \beta \rangle_1] \subseteq [\langle \beta | \beta \rangle_{1} B] = [\rho_\beta(B) B] \subseteq B \) and similarly \( (\text{id} \ast \psi)(A_{\beta\gamma} \ast \beta, B) \subseteq A. \)

The second class of slice maps arises from a completely positive map on one factor of a spatial fiber product. The range of such a slice map are operators on certain KSGNS-constructions, that is, internal tensor products with respect to completely positive linear maps (see [12, 84–95]). We abbreviate “completely positive linear” by “c.p.”. Let \( A, C \) be \( C^* \)-algebras and \( \phi : A \to C \) a c.p. map. Recall that \( \phi \) is strict if for some bounded approximate unit \( (u_\nu)_\nu \) of \( A \), the net \( (\phi(u_\nu))_\nu \) converges strictly in \( M(C) \), and that in this case, \( \phi \) extends uniquely.
to a c.p. map \( \tilde{\phi} : M(A) \to M(C) \) that is strictly continuous on bounded subsets of \( \mathcal{B}(\mathcal{H}_{\beta} \otimes \gamma) \).

**Lemma 3.31.** Let \( A_{\beta}^{b} \) be a C*-b-algebra, \( K \) a C*-b\(^{1}\)-module, \( L \) a Hilbert space, and \( \phi : A \to \mathcal{L}(L) \) a strict c.p. map. Put \( X := \{ x \in \mathcal{L}(H_{\beta} \otimes \gamma, K) | \langle \gamma|x|\gamma \rangle_{2} \subseteq A \} \) and \( \theta := \tilde{\phi} \circ \rho_{\beta} \). Then there exists a unique c.p. map \( \phi \circ \text{id} : X \to \mathcal{L}(L_{\theta} \otimes \gamma) \) such that for all \( \zeta, \zeta' \in L, \eta, \eta' \in \gamma \), and \( x \in X \),

\[
\langle \zeta \otimes \eta | (\phi \circ \text{id})(x)(\zeta' \otimes \eta') \rangle = \langle \zeta | \phi((|\eta|_{2}x|\eta\rangle_{2})\zeta' \rangle. \tag{14}
\]

**Proof.** Let \( x = (x_{ij})_{i,j} \in M_{n}(X) \) be positive and \( \zeta_{1}, \ldots, \zeta_{n} \in L, \eta_{1}, \ldots, \eta_{n} \in \gamma \), where \( n \in \mathbb{N} \). Put \( d := \text{diag}(\eta_{1}, \ldots, \eta_{n})_{2} \). Then \( 0 \leq \langle \eta_{1}|x_{ij}|\eta_{j}\rangle_{2} \leq ||x||d^{*}d \), so \( 0 \leq (\phi((|\eta|_{2}x_{ij}|\eta_{j}\rangle_{2})) \leq \|x\|\|\tilde{\phi}(d^{*}d)\| \) and

\[
0 \leq \sum_{i,j} \langle \zeta_{i} | \phi((|\eta|_{2}x_{ij}|\eta_{j}\rangle_{2}) \zeta_{j} \rangle \leq \|x\| \sum_{i,j} \langle \zeta_{i} \otimes \eta_{i} | \zeta_{j} \otimes \eta_{j} \rangle.
\]

Consequently, there exists a c.p. map \( \phi \circ \text{id} : X \to \mathcal{L}(L_{\theta} \otimes \gamma) \) satisfying (14). \( \square \)

**Lemma 3.31** and routine calculations yield the following result:

**Proposition 3.32.** Let \( A_{\beta}^{b} \) be a C*-b-algebra, \( B_{\beta}^{b} \) a C*-b\(^{1}\)-algebra, \( L \) a Hilbert space, and \( \phi : A \to \mathcal{L}(L) \) a strict c.p. map. Put \( \theta := \tilde{\phi} \circ \rho_{\beta} \). Then \( A_{\beta} \otimes B \) belongs to the domain of definition of \( \phi \circ \text{id} \) and \( (\phi \circ \text{id})(A_{\beta} \otimes B) \subseteq (\text{id} \otimes (B_{\beta} \cap \mathcal{L}(K_{\gamma})))^{\prime} \subseteq \mathcal{L}(L_{\theta} \otimes \gamma) \).

**Remarks 3.33.**

1. Assume that \( \phi((\rho_{\beta}(b^{1})a) = \phi((\rho_{\beta}(b^{1}))a) \) for all \( b^{1} \in \mathcal{B}, a \in A \). Then \( \theta \) is a *-homomorphism, there exists a bounded linear map \( \Phi : [\gamma|_{2}A] \to \mathcal{L}(L, L_{\theta} \otimes \gamma) \), \( |\eta|_{2} \mapsto |\eta|_{2} \phi(a) \), and \( ((\phi \circ \text{id})(x))(\zeta \otimes \eta) = \Phi(x|\eta|_{2})\zeta \) for all \( x \in A_{\beta} \otimes B, \zeta \in L, \eta \in \gamma \).

2. Assume that \( L = \mathbb{R} \) and \( \phi \in \Omega_{\beta} \). Then we can identify \( \mathbb{R} \otimes \gamma \) with \( K \) via \( n_{\gamma} \), and the map \( \phi \circ \text{id} \) constructed above coincides with the one constructed in Proposition 3.30.

3. Assume that \( C_{L}^{\lambda} \) is a C*-b-algebra and that there are \( S_{1}, \ldots, S_{n}, T_{1}, \ldots, T_{n} \in \mathcal{L}(H_{\beta}, L_{\lambda}) \) such that \( \phi(a) = \sum_{i} S_{i}aT_{i}^{*} \in C \) for all \( a \in A \). Then \( \theta = \rho_{\lambda} \) by Lemma 2.6, we can identify \( L_{\rho_{\lambda}} \otimes \gamma \) with \( L_{\rho_{\lambda}} \otimes \gamma, K \), and one easily verifies \( (\phi \circ \text{id})(x) = \sum_{i}(S_{i} \otimes \text{id})x(T_{i}^{*} \otimes \text{id}) \in C_{L_{\lambda}} \otimes B \) for all \( x \in A_{\beta} \otimes B \).

Similarly, one can construct slice maps for completely positive maps defined on the second factor in a fiber product.
3.6 The commutative case

We finally discuss fiber products of $C^*$-algebras, where one or both of the factors involved are commutative, and give some examples. First, we need to recall some preliminaries on Hilbert $C^*$-modules and Hilbert bundles, and fix some notation. A standard reference is [4].

Let $B$ be a commutative $C^*$-algebra. We denote by $\hat{B}$ the spectrum of $B$, that is, the set of nonzero *-homomorphisms $B \to \mathbb{C}$. Let $E$ be a Hilbert $C^*$-$B$-module. Then $E$ corresponds to a continuous bundle of Hilbert spaces on $\hat{B}$ as follows. Put $E_\chi := E \otimes_\chi \mathbb{C}$ for each $\chi \in \hat{B}$ and $\text{Tot}(E) := \bigsqcup_{\chi \in \hat{B}} E_\chi$, and denote by $p_E : \text{Tot}(E) \to \hat{B}$ the natural projection. Each $\xi \in E$ yields a section $\sigma_\xi$ of $p_E$, given by $\chi \mapsto \xi_\chi := \chi \otimes_\chi 1 \in E_\chi$. Equipped with the weakest topology that makes $p_E$ and the map $\text{Tot}(E) \to \mathbb{C}$ given by $E_\chi \ni \xi \mapsto \|\xi_\chi - \xi\|$ continuous for each $\chi \in E$, the space $\text{Tot}(E)$ becomes a continuous bundle of Hilbert spaces.

Denote by $\Gamma_0(\text{Tot}(E))$ the space of continuous sections $\sigma$ of $p_E$ for which the function $\xi \mapsto \|\sigma(\xi)\|$ vanishes at infinity. Then $\Gamma_0(\text{Tot}(E))$ carries a natural structure of a Hilbert $C^*$-$B$-module such that the map $E \to \Gamma_0(\text{Tot}(E))$ given by $\xi \mapsto \sigma_\xi$ is an isomorphism of Hilbert $C^*$-$B$-modules.

Put $\text{Tot}\mathcal{L}(E) := \bigsqcup_{\chi \in \hat{B}} \mathcal{L}(E\chi)$. Let $A$ be a commutative $C^*$-algebra and $\rho : B \to M(A)$ a nondegenerate $*$-homomorphism. Then $E \otimes_\rho A$ is a Hilbert $C^*$-$A$-module. For each $\chi \in \hat{A}$, we identify $(E \otimes_\rho A)_\chi = E \otimes_\rho A \otimes_\chi \mathbb{C}$ with $E \otimes_{(\chi \circ \rho)} \mathbb{C} = E_{(\chi \circ \rho)}$. We call a map $f : \hat{A} \to \text{Tot}\mathcal{L}(E)$ weakly continuous and vanishing at infinity if $p_E \circ f$ is continuous and for all $\xi, \xi' \in E$, the map $\chi \mapsto \langle \xi_{p_E(f(\chi))}, f(\chi)\xi'_{p_E(f(\chi))} \rangle$ lies in $C_0(A) \cong A$. Similarly, we call a map $f : \hat{A} \to \text{Tot}\mathcal{L}(E)$ strong-$*$ continuous and vanishing at infinity if $p_E \circ f$ is continuous and for all $\xi, \xi' \in E$, the map $\chi \mapsto f(\chi)\xi_{p_E(f(\chi))}$ lies in $\Gamma_0(E \otimes_\rho A)$.

In the following lemma, we apply the preceding discussion to $B = \mathfrak{B}^1$, $E = \gamma$.

Lemma 3.34. Let $A^\beta_H$ be a $C^*$-$b$-algebra, $K, \gamma$ a $C^*$-$b^1$-module, $x \in \mathcal{L}(H, b, K)$. Assume that $\mathfrak{B}^1, A$ are commutative and $[\rho_\beta(\mathfrak{B}^1)A] = A$ and $[\gamma]_{2x}[\gamma]_2 \subseteq A$. Define $F_x : \hat{A} \to \text{Tot}\mathcal{L}(\gamma)$ by $\chi \mapsto (\chi \ast \text{id})(x)$. Then:

i) $F_x$ is weakly continuous vanishing at infinity.

ii) $x \in \text{Ind}_1(\gamma)_2(A)$ if and only if $F_x$ is strong-$*$ continuous vanishing at infinity.

Proof. First, note that for all $\xi, \eta \in \gamma$ and $\chi \in \hat{A}$,

$$\chi(\langle \eta|2x|\xi\rangle_2) = \langle 1_{(\chi \otimes \rho_\beta)} \otimes \eta(\chi \ast \text{id})(x)(1_{(\chi \otimes \rho_\beta)} \otimes \xi) \rangle = \langle \eta(\chi \otimes \rho_\beta)|F_x(\chi)\xi_{(\chi \otimes \rho_\beta)} \rangle.$$ 

i) For each $\xi, \eta \in \gamma$, the map $\chi \mapsto \langle \eta(\chi \otimes \rho_\beta)|F_x(\chi)\xi_{(\chi \otimes \rho_\beta)} \rangle$ equals $\langle \eta(\gamma)|2x|\xi\rangle_2 \in A$.

ii) Assume that $F_x$ is strong-$*$ continuous vanishing at infinity, and let $\xi \in \gamma$. Then the map $\chi \mapsto F_x(\chi)\xi_{(\chi \otimes \rho_\beta)}$ lies in $\Gamma_0(\gamma \otimes \rho_\beta A)$ and has the form $\sigma_\omega$ for some $\omega \in \gamma \otimes \rho_\beta A$. We identify $\gamma \otimes \rho_\beta A$ with $[\gamma]_{2A} \subseteq \mathcal{L}(H, b, \gamma, K)$ in the canonical manner and find that $x|\xi\rangle_2 = \omega$ because $\chi(\langle \eta|2x|\xi\rangle_2) = \langle \eta(\chi \otimes \rho_\beta)|\omega_{(\chi \otimes \rho_\beta)} \rangle = \langle \eta(\gamma)|2x|\xi\rangle_2$.
\[ \chi(\eta|2\omega) \text{ for all } \chi \in \hat{A}, \eta \in \gamma. \text{ Since } \xi \in \gamma \text{ was arbitrary, we can conclude } x|\gamma \subseteq [\gamma]_2 A. \text{ A similar argument, applied to } x^* \text{ instead of } x, \text{ shows that } x^*|\gamma \subseteq [\gamma]_2 A, \text{ and therefore } x \in \text{Ind}([\gamma]_2(A)). \text{ Reversing the arguments, we obtain the reverse implication.} \]

We now consider a special situation. Let \( X, Y, Z \) be locally compact Hausdorff spaces, \( p: X \to Z \) and \( q: Y \to Z \) continuous maps, \( \tau \) a Radon measure on \( Z \) with full support, and \( \phi = (\phi_z)_{z \in Z}, \psi = (\psi_z)_{z \in Z} \) families of measures such that

i) \( \phi_z \) and \( \psi_z \) are Radon measures with full support on \( X_z := p^{-1}(z) \) and \( Y_z := q^{-1}(z) \), respectively, for each \( z \in Z \),

ii) for each \( f \in C_c(X) \) and \( g \in C_c(Y) \), the maps \( z \mapsto \int_{X_z} f \, d\phi_z \) and \( z \mapsto \int_{Y_z} g \, d\psi_z \) are continuous.

Then we can define unique Radon measures \( \nu_X, \nu_Y, \nu \) on \( X, Y, X_p \times_Z q Y \), respectively, such that for all \( f \in C_c(X), g \in C_c(Y), h \in C_c(X_p \times_Z q Y) \),

\[
\int_X f \, d\nu_X = \int_Z \int_{X_z} f \, d\phi_z \, d\mu(z), \quad \int_Y g \, d\nu_Y = \int_Z \int_{Y_z} g \, d\psi_z \, d\mu(z), \quad \int_{X_p \times_Z q Y} h \, d\nu = \int_Z \int_{X_z} \int_{Y_z} h(x, y) \, d\psi_z(y) \, d\phi_z(x) \, d\mu(z).
\]

We equip \( C_c(X) \) with the structure of a pre-Hilbert-module over \( C_0(Z) \) such that \( (f|g)(z) = \int_X f(x) g(z(x)) \, d\mu \) and \( fh = f p^*(h) \) for all \( f, g \in C_c(X), h \in C_0(Z), z \in Z \), and denote the completion by \( L^2(X, \phi) \). Similarly, we define the Hilbert \( C^* \)-module \( L^2(Y, \psi) \). Then each \( f \in L^2(X, \phi) \) and \( g \in L^2(Y, \psi) \) yields an operator

\[
j_X(f): L^2(Z, \mu) \to L^2(X, \nu_X), \ h \mapsto f p^*(h), \quad j_Y(g): L^2(Z, \mu) \to L^2(Y, \nu_Y), \ h \mapsto g q^*(h).
\]

Let \( \mathfrak{b} \) be the \( C^* \)-base associated to the weight on \( C_0(Z) \) corresponding to \( \mu \) and

\[
A := C_0(X), \quad H := L^2(X, \nu_X), \quad \beta := j_X(L^2(X, \phi)) = j_Y(L^2(X, \phi)), \quad B := C_0(Y), \quad K := L^2(Y, \nu_Y), \quad \gamma := j_Y(L^2(Y, \psi)).
\]

Evidently, \( A \beta H \) is a \( C^* \)-\( \mathfrak{b} \)-algebra and \( B \gamma K \) a \( C^* \)-\( \mathfrak{b}^\perp \)-algebra. For each \( z \in Z \), we identify \( L^\infty(X_z, \phi_z) \) and \( L^\infty(Y_z, \psi_z) \) with \( C^* \)-subalgebras of \( L(L^2(X_z, \phi_z)) \cong \mathcal{L}(\beta_z) \) and \( L(L^2(Y_z, \psi_z)) \cong \mathcal{L}(\gamma_z) \), respectively.

**Proposition 3.35.** i) There exists a unitary \( U: H_\beta \otimes K \to L^2(X_p \times_Z q Y, \nu) \) such that \( \Phi(j_X(f) \otimes h \otimes j_Y(g))(x, y) = f(x) h(p(x)) g(y) \) for all \( f \in C_c(X), g \in C_c(Y), h \in C_c(Z), (x, y) \in X_p \times_Z q Y \).
ii) \( \text{Ad}_\gamma(A\beta^{\gamma}B) \) is the \( C^* \)-algebra of all \( f \in L^\infty(X^\beta_q Y, \nu) \) that have representatives \( f_X, f_Y \) such that the maps \( X \to \text{Tot}L(\gamma) \) and \( Y \to \text{Tot}L(\beta) \) given by \( x \mapsto f_X(x, \cdot) \in L^\infty(Y_{x\cdot}, \psi_{p(x)}) \) and \( y \mapsto f_Y(\cdot, y) \in L^\infty(X_{q(y)}, \phi_{q(y)}) \) respectively, are strong-
\( \ast \) continuous vanishing at infinity.

Proof. The proof of assertion i) is straightforward, and assertion ii) follows immediately from Theorem 3.35i) and Lemma 3.34ii).

Examples 3.36. i) Let \( X, Y \) be discrete, \( Z = \{0\} \), and let \( \phi_0, \psi_0 \) be the counting measures on \( X, Y \), respectively. Then

\[
C_0(X)_{\beta^*}C_0(Y) \cong \{ f \in C_0(X \times Y) \mid f(x, \cdot) \in C_0(Y) \text{ for all } x \in X, \]

\[
f(\cdot, y) \in C_0(X) \text{ for all } y \in Y.\]

This follows from Theorem 3.32 and the fact that for each \( f \in C_0(X \times Y) \), the maps \( X \to \mathcal{L}(Y^0) \), \( x \mapsto f(x, \cdot) \), and \( Y \to \mathcal{L}(X^0) \), \( y \mapsto f(\cdot, y) \), are strong-
\( \ast \) continuous vanishing at infinity if and only if \( f(\cdot, y) \in C_0(X) \) and \( f(x, \cdot) \in C_0(Y) \) for each \( y \in Y \) and \( x \in X \).

ii) Let \( X = \mathbb{N} \), \( Z = \{0\} \), and let \( \phi_0 \) be the counting measure. Then

\[
C_0(\mathbb{N})_{\beta^\ast}C_0(Y) \cong \{ f \in C_0(\mathbb{N} \times Y) \mid (f(x, \cdot))_x \text{ is a sequence in } C_0(Y) \}
\]

that converges strongly to \( 0 \in \mathcal{L}(L^2(Y, \psi_0)) \}

because for each \( f \in L^\infty(\mathbb{N} \times Y) \), the map \( Y \to \mathcal{L}(L^2(\mathbb{N})) \), \( y \mapsto f(\cdot, y) \), is strong-
\( \ast \) continuous vanishing at infinity if and only if \( f(x, \cdot) \in C_0(Y) \) for all \( x \in \mathbb{N} \).

iii) Let \( X = Y = [0, 1] \), \( Z = \{0\} \), and let \( \phi_0 = \psi_0 \) be the Lebesgue measure. For each subset \( I \subseteq [0, 1] \), denote by \( \chi_I \) its characteristic function. Then the function \( f \in L^\infty([0, 1] \times [0, 1]) \) given by \( f(x, y) = 1 \) if \( y \leq x \) and \( f(x, y) = 0 \) otherwise belongs to \( C([0, 1])_{\beta^\ast}C([0, 1]) \) because the functions \( [0, 1] \to L^\infty([0, 1]) \subseteq \mathcal{L}(L^2([0, 1])) \) given by \( x \mapsto f(x, \cdot) = \chi_{[0,x]} \) and \( y \mapsto f(\cdot, y) = \chi_{[y, 1]} \) are strong-
\( \ast \) continuous. In particular, we see that \( C([0, 1])_{\beta^\ast}C([0, 1]) \not\subseteq C([0, 1] \times [0, 1]) = C([0, 1]) \otimes C([0, 1]) \).

A Minimal fiber products in the setting of \( C^* \)-algebras

In this appendix, we use the spatial fiber product to construct a minimal fiber product of \( C^* \)-algebras that is independent of a chosen representation in the following sense. Let \( b = (\beta, \mathfrak{B}, \mathfrak{B}^1) \) be a \( C^* \)-base. Given \( C^* \)-algebras \( A, B \) with nondegenerate \( \ast \)-homomorphisms \( \sigma : \mathfrak{B} \to M(A) \) and \( \rho : \mathfrak{B}^1 \to M(B) \), we define a minimal fiber product of \( A \) and \( B \) with respect to \( \sigma, \rho, b \) as follows. We represent \( A \) as a \( C^* \)-\( \mathfrak{B}^1 \)-algebra and \( B \) as a \( C^* \)-\( b \)-algebra, form the spatial fiber.
product, and then take the limit of these spatial fiber products for all admissible representations. In categorical terms, this construction is a right Kan extension \[15\]. For background on category theory, see \[15\].

We fix some terminology. Let \( C \) be a \( C^* \)-algebra. A \( C^* \)-\( C \)-algebra \((A, \rho)\), briefly written \( A_\rho \), is a \( C^* \)-algebra \( A \) with a nondegenerate \(*\)-homomorphism \( \rho: C \to M(A) \). A morphism of \( C^* \)-\( C \)-algebras \( A_\rho \) and \( B_\sigma \) is a \(*\)-homomorphism \( \pi: A \to B \) satisfying \( \sigma(c)\pi(a) = \pi(\rho(c)a) \) for all \( c \in C, a \in A \). We denote the category of all \( C^* \)-\( C \)-algebras by \( \mathbf{C}^*\mathbf{alg}_C \) and write \( \mathbf{C}^*\mathbf{alg} = \mathbf{C}^*\mathbf{alg}_C \).

Similarly as in Example \[3.8\] we define finite sums in the categories \( \mathbf{C}^*\mathbf{alg}_b \) and \( \mathbf{C}^*\mathbf{alg}_{b^!} \), respectively. Let \( R_b \) and \( R_{b^!} \) be full subcategories of \( \mathbf{C}^*\mathbf{alg}_b \) and \( \mathbf{C}^*\mathbf{alg}_{b^!} \), respectively, which are closed under

- i) \( l^\infty \)-sums and
- ii) formation of subalgebras, that is, if \( c \in \{b, b^!\} \) and \( A_H^c \in R_c, B \subseteq A, \)
  \( \phi_H^c, \psi_H^c \) then \( \phi_H^c, \psi_H^c \in R_c \).

We construct a fiber product \((\cdot \otimes \cdot): \mathbf{C}^*\mathbf{alg}_{b^!} \times \mathbf{C}^*\mathbf{alg}_b \to \mathbf{C}^*\mathbf{alg}\) as the right Kan extension of the spatial fiber product \((\cdot \otimes \cdot): R_b \times R_{b^!} \to \mathbf{C}^*\mathbf{alg}\) constructed in Subsection \[3.3\] along the forgetful functor

\[
U: R_b \times R_{b^!} \to \mathbf{C}^*\mathbf{alg}_{b^!} \times \mathbf{C}^*\mathbf{alg}_b
\]

given by \((A_H^c, B_K^c) \mapsto (A_H^c, B_K^c)\) on objects and \((\phi, \psi) \mapsto (\phi, \psi)\) on morphisms.

Given a \( C^* \)-\( B \)-algebra \( A_\rho \) and a \( C^* \)-\( B \)-algebra \( B_\sigma \), we define the category \( R(A_\rho, B_\sigma) := (\{(A_\rho, B_\sigma) \mid U\}) \) of all admissible representations of \( A_\rho \) and \( B_\sigma \) as follows. The objects are tuples \((C, D, \phi, \psi)\), where \((C, D) \in R_b \times R_{b^!}\) and \((\phi, \psi)\) is a morphism from \((A_\rho, B_\sigma)\) to \( U(C, D)\) in \( \mathbf{C}^*\mathbf{alg}_{b^!} \times \mathbf{C}^*\mathbf{alg}_b \). Morphisms between two objects \((C_1, D_1, \phi_1, \psi_1)\) and \((C_2, D_2, \phi_2, \psi_2)\) are all morphisms \((\pi_C, \pi_D)\) from \((C_1, D_1)\) to \((C_2, D_2)\) in \( R_b \times R_{b^!}\) satisfying \( \phi_2 = \pi_C \circ \phi_1 \) and \( \psi_2 = \pi_D \circ \psi_1 \).

We denote by \( P_{(A_\rho, B_\sigma)}: R(A_\rho, B_\sigma) \to R_b \times R_{b^!}\) the projection, given by \((C, D, \phi, \psi) \mapsto (C, D)\) on objects and \((\pi_C, \pi_D) \mapsto (\pi_C, \pi_D)\) on morphisms.

**Proposition A.1.** Let \( A_\rho \) be a \( C^* \)-\( B \)-algebra and \( B_\sigma \) a \( C^* \)-\( B \)-algebra. Then the functor \( F = F_{(A_\rho, B_\sigma)} := (\cdot \otimes \cdot) \circ P_{(A_\rho, B_\sigma)}: R(A_\rho, B_\sigma) \to \mathbf{C}^*\mathbf{alg}\), given on objects and morphisms by \( F((C, D, \phi, \psi)) = C \otimes D \) and \( F(\pi_C, \pi_D) = \pi_C \otimes \pi_D \), has a limit consisting of a \( C^* \)-algebra \( A_\rho \star_\sigma B \) and a family of morphisms \((j_S: A_\rho \star_\sigma B \to FS)_{S \in R(A_\rho, B_\sigma)}\).

The proof requires some preparation. Given \( R_i = (C_i, D_i, \phi_i, \psi_i) \in R(A_\rho, B_\sigma) \) for \( i = 1, 2 \), we define a sum \( R_1 \boxplus R_2 = (C_1 \boxplus C_2, D_1 \boxplus D_2, \phi, \psi) \in R(A_\rho, B_\sigma) \), where \( \phi(a) = (\phi_1(a), \phi_2(a)) \) and \( \psi(b) = (\psi_1(b), \psi_2(b)) \) for all \( a \in A, b \in B \). Denote by \( G: R(A_\rho, B_\sigma) \to R(A_\rho, B_\sigma) \) the functor given by \((C, D, \phi, \psi) \mapsto (\phi(A), \psi(B), \phi, \psi)\) on objects and restriction on morphisms, and by \( \nu: G \to id\) the natural transformation given by the inclusions \( \phi(A) \mapsto C, \psi(B) \mapsto D \).
Lemma A.2. Let $A_{\rho}, B_{\sigma}$ be as above, $R_i = (C_i, D_i, \phi_i, \psi_i) \in \mathbf{R}(A_{\rho}, B_{\sigma})$ for $i = 1, 2,$ and let $\pi: R_1 \boxplus R_2 \to R_1$ be the projection. If $\ker \phi_1 \subseteq \ker \phi_2$ and $\ker \psi_1 \subseteq \ker \psi_2,$ then the map $\pi^{R_2}_{R_1}: \mathbf{F}G(R_1 \boxplus R_2) \xrightarrow{\mathbf{F}G(\pi)_{R_1 \boxplus R_2}} \mathbf{F}G(R_1 \boxplus R_2) \xrightarrow{\mathbf{F}G(\pi)^{-1}_{R_1}} \mathbf{F}G(R_1 \boxplus R_2)$ is injective.

Proof. Write $R_1 \boxplus R_2 = (C, D, \phi, \psi)$ and denote by $\pi_C: C \to C_1$ and $\pi_D: D \to D_1$ the projections. Then $\mathbf{F}G(R_1 \boxplus R_2) = \phi(A) \otimes \psi(B), \mathbf{F}G(R_1 \boxplus R_2) = C \otimes D,$ $\mathbf{F}R_1 = C_1 \otimes D_1,$ and $\mathbf{F}\pi = \pi_C \otimes \pi_D.$ We first show that the map $\pi_C \otimes \mathbf{id}: C \otimes D \to C_1 \otimes D$ is injective on $\phi(A) \otimes \psi(B).$ Write $D = D_{K_b}^\beta.$ On $\phi(A),$ the map $\pi_C$ is given by $(\phi_1(a), \phi_2(a)) \mapsto \phi_1(a)$ for all $a \in A.$ Since $\ker \phi_1 \subseteq \ker \phi_2,$ this map is injective. Now, assume that $x \in \phi(A) \otimes \psi(B)$ and $(\pi_C \otimes \mathbf{id})(x) = 0.$ Then $0 = \langle \beta_2(x\pi_C \otimes \mathbf{id})(x) \rangle_{\beta_2} = \pi_C(\langle \beta_2(x) \rangle_{\beta_2}),$ and since $\langle \beta_2(x) \rangle_{\beta_2} \subseteq \phi(A)$ and $\pi_C$ is injective on $\phi(A),$ we must have $\langle \beta_2(x) \rangle_{\beta_2} = 0$ and hence $x = 0.$ A similar argument shows that the map $\mathbf{id} \otimes \pi_D: C_1 \otimes D \to C_1 \otimes D_1$ is injective on $\phi_1(A) \otimes \psi(B),$ and hence $\pi_C \otimes \pi_D = (\mathbf{id} \otimes \pi_D) \circ (\pi_C \otimes \mathbf{id})$ is injective as well. \hfill \Box

of Proposition A.7. Using the assumptions on $\mathbf{R},$ we choose $R = (C, D, \phi, \psi) \in \mathbf{R}(A_{\rho}, B_{\sigma})$ such that $\ker \phi \subseteq \ker \phi_2,$ $\ker \psi \subseteq \ker \psi_2$ for every $(C_2, D_2, \phi_2, \psi_2) \in \mathbf{R}(A_{\rho}, B_{\sigma})$ and $R = GR.$ Then for each $S \in \mathbf{R}(A_{\rho}, B_{\sigma}),$ the map $\pi^S_R: \mathbf{F}G(R \boxplus S) \to \operatorname{Im} \pi^S_R \subseteq \mathbf{F}R$ is injective by Lemma A.2 and the map $j_S: \pi^S_R \cap (\pi^S_T)^{-1}: \operatorname{Im} \pi^S_R \to \mathbf{F}S$ makes the following diagram commute:

\[
\begin{array}{ccc}
\mathbf{F}G(R \boxplus S) & \xrightarrow{\pi^R_B} & \mathbf{F}S, \\
\pi^R_S & \searrow & \\
\operatorname{Im} \pi^S_R & \xrightarrow{j_S} & \operatorname{Im} \pi^S_T
\end{array}
\]

We show that the $C^*$-algebra $A_{\rho} \otimes^\sigma B := \bigcap_S \operatorname{Im} \pi^S_R \subseteq \mathbf{F}R$ and the collection $(j_S)_S$ form a limit for the functor $\mathbf{F},$ where the intersection is taken over all $S \in \mathbf{R}(A_{\rho}, B_{\sigma}).$ Note that this intersection exists even if $\mathbf{R}(A_{\rho}, B_{\sigma})$ is not small. Let $\pi$ be a morphism between objects $S, T$ in $\mathbf{R}(A_{\rho}, B_{\sigma}).$ In the diagram

\[
\begin{array}{ccc}
\mathbf{F}G(R \boxplus S) & \xrightarrow{\mathbf{F}G(\mathbf{id} \boxplus \pi)} & \mathbf{F}G(R \boxplus T) \\
\pi^R_S & \searrow & \pi^R_T \\
\mathbf{F}S & \xrightarrow{j_S} & \mathbf{F}T,
\end{array}
\]

the square and the upper, the left and the right triangle commute, and hence also the lower triangle commutes. Therefore, $A_{\rho} \otimes^\sigma B$ and $(j_S)_S$ form a cone on
If $C$ and $(k_S)_S$ form another cone on $F$, then $k_R(C) \subseteq \text{Im} \pi_R^S$ and $k_S = j_S \circ k_R$ for each $S \in R(A_\rho, B_\sigma)$ because the following diagram commutes:

$$
\begin{array}{ccc}
F_R & \xrightarrow{k_R} & C \\
\downarrow{\pi_R^S} & & \downarrow{\pi_S^S} \\
F_G(R \oplus S) & \xrightarrow{k_{G(R \oplus S)}} & F_S.
\end{array}
$$

Consequently, $A_\rho \ast_\sigma B$ and $(j_S)_S$ form a limit for $F$. \qed

The construction of the limit in the preceding proposition is functorial. Indeed, let $\pi_A$ be a morphism of $C^\ast$-$\mathfrak{B}$-algebras $A_\rho$, $\tilde{A}_\rho$, and let $\pi_B$ be a morphism of $C^\ast$-$\mathfrak{B}$-algebras $B_\sigma$, $\tilde{B}_\sigma$. Then we have a functor $(\pi_A, \pi_B) : R(\tilde{A}_\rho, \tilde{B}_\sigma) \to R(A_\rho, B_\sigma)$ given by $(C, D, \phi, \psi) \mapsto (C, D, \phi \circ \pi_A, \psi \circ \pi_B)$ on objects and the identity on morphisms, and for each $S \in R(\tilde{A}_\rho, \tilde{B}_\sigma)$, we have a map

$$
j(\pi_A, \pi_B) : A_\rho \ast_\sigma B \to F_{(A_\rho, B_\sigma)}(\pi_A, \pi_B)^S = F_{(\tilde{A}_\rho, \tilde{B}_\sigma)}^S.
$$

The family of these maps forms a cone on the functor $F_{(A_\rho, B_\sigma)}$ and therefore induces a map $\pi_A \ast_\sigma \pi_B : A_\rho \ast_\sigma B \to \tilde{A}_\rho \ast_\sigma \tilde{B}$.

**Theorem A.3.** There exists a unique functor $(\ast_\sigma) : C^\ast$-$\mathfrak{B}$-$\mathfrak{alg} \times C^\ast$-$\mathfrak{alg} \to C^\ast$-$\mathfrak{alg}$ given on objects by $(A_\rho, B_\sigma) \mapsto A_\rho \ast_\sigma B$ and on morphisms by $(\pi_A, \pi_B) \mapsto \pi_A \ast_\sigma \pi_B$.

**Proof.** This functor is the right Kan extension of $(\ast_\sigma)$ along $U$, and the result follows from Proposition A.1; see [15, X.3 Theorem 1]. \qed

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