A similarity degree characterization of nuclear $C^*$-algebras

by

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Abstract

We show that a $C^*$-algebra $A$ is nuclear iff there is a number $\alpha < 3$ and a constant $K$ such that, for any bounded homomorphism $u: A \to B(H)$, there is an isomorphism $\xi: H \to H$ satisfying $\|\xi^{-1}\| \|\xi\| \leq K\|u\|^\alpha$ and such that $\xi^{-1}u(\cdot)\xi$ is a $*$-homomorphism. In other words, an infinite dimensional $A$ is nuclear iff its length (in the sense of our previous work on the Kadison similarity problem) is equal to 2.

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In 1955, Kadison [14] formulated the following conjecture: any bounded homomorphism $u: A \to B(H)$, from a $C^*$-algebra into the algebra $B(H)$ of all bounded operators on a Hilbert space $H$, is similar to a $*$-homomorphism, i.e. there is an invertible operator $\xi: H \to H$ such that $x \to \xi u(x)\xi^{-1}$ satisfies $\xi u(x^*)\xi^{-1} = (\xi u(x)\xi^{-1})^*$ for all $x$ in $A$. This conjecture remains unproved, although many partial results are known (see [4, 10]). In particular, by [10], we know that $u$ is similar to a $*$-homomorphism iff it is completely bounded (c.b. in short) in the sense of e.g. [17] or [20] (to which we refer for background on c.b. maps). Moreover, we have

$$\|u\|_{cb} = \inf\{\|\xi\|\|\xi^{-1}\|\}$$

where the infimum runs over all invertible $\xi$ such that $\xi u(\cdot)\xi^{-1}$ is a $*$-homomorphism. Recall that, by definition, $\|u\|_{cb} = \sup\|u_n\|$ where $u_n: M_n(A) \to M_n(B(H))$ is the mapping taking $[a_{ij}]$ to $[u(a_{ij})]$. Thus Kadison’s conjecture is equivalent to the validity of the implication $\|u\| < \infty \Rightarrow \|u\|_{cb} < \infty$. In [18], the author proved that if a $C^*$-algebra $A$ verifies Kadison’s conjecture, then there is a number $\alpha$ for which there exists a constant $K$ so that any bounded homomorphism $u: A \to B(H)$ satisfies $\|u\|_{cb} \leq K\|u\|^\alpha$. Moreover, the smallest number $\alpha$ with this property is an integer denoted by $d(A)$ (and $\alpha = d(A)$ itself satisfies the property).

An analogous parameter can be defined for a discrete group $G$ and it is proved in [18] that $G$ is amenable iff $d(G) \leq 2$. The main result of this note is the analogous equivalence for $C^*$-algebras: a $C^*$-algebra $A$ is nuclear (or equivalently amenable, see below) iff $d(A) \leq 2$. In [18], we could only prove a partial result in this direction.

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Let $A, B$ be $C^*$-algebras. Let $\| \|_\alpha$ be a $C^*$-norm on their algebraic tensor product, denoted by $A \otimes B$; as usual, $A \otimes_\alpha B$ then denotes the $C^*$-algebra obtained by completing $A \otimes B$ with respect to $\| \|_\alpha$. By classical results (see [24]) the set of $C^*$-norms admits a minimal and a maximal element denoted respectively by $\| \cdot \|_{\min}$ and $\| \cdot \|_{\max}$. Then $A$ is called nuclear if for any $B$ we have $A \otimes_{\min} B = A \otimes_{\max} B$, or equivalently $\|x\|_{\min} = \|x\|_{\max}$ for any $x$ in $A \otimes B$. We refer the reader to [24] for more information on nuclear $C^*$-algebras. We note in particular that by results due to Connes and Haagerup ([7, 8]), a $C^*$-algebra is nuclear iff it is amenable as a Banach algebra (in B.E. Johnson’s sense).

The main result of this note is as follows.

**Theorem 1.** The following properties of $C^*$-algebra $A$ are equivalent:

(i) $A$ is nuclear.

(ii) There are $\alpha < 3$ and a constant $K$ such that any bounded homomorphism $u \colon A \to B(H)$ satisfies $\|u\|_{cb} \leq K\|u\|^\alpha$.

(iii) Same as (ii) with $K = 1$ and $\alpha = 2$.

The implication (i) $\Rightarrow$ (iii) is well known (see [2, 4]). In the terminology of [18], the similarity degree $d(A)$ is the smallest $\alpha$ for which the property considered in (ii) above is satisfied. It is proved in [18] that $d(A)$ is always an integer identified as the smallest length of a specific kind of factorization for matrices with entries in $A$.

With this terminology, the preceding theorem means that $A$ is nuclear iff $d(A) \leq 2$. In the infinite dimensional case, $d(A) > 1$ hence $A$ is nuclear iff $d(A) = 2$.

In his work on derivations (see [4] and [5]) Erik Christensen isolated the following property $D_k$ for a $C^*$-algebra. Here $k$ is any number $\geq 1/2$. A $C^*$-algebra $A$ has property $D_k$ if for any $H$, any representation $\pi \colon A \to B(H)$, and any $T$ in $B(H)$ the derivation $\delta_T : A \to B(H)$ defined by $\delta_T(a) = \pi(a)T - T\pi(a)$ satisfies

$$\|\delta_T\|_{cb} \leq 2k\|\delta_T\|.$$

With this terminology, Theorem 1 implies the following:

**Corollary 2.** Let $A$ be a $C^*$-algebra. The following assertions are equivalent.

(i) $A$ is nuclear.

(ii) $A$ satisfies property $D_k$ for some $k < 3/2$.

(iii) $A$ satisfies property $D_1$.

**Proof.** Here again the fact that (i) $\Rightarrow$ (iii) is well known (see [2, 4]). The equivalence between the similarity problem and the derivation problem was established by Kirchberg in [16]. Refining Kirchberg’s estimates, the author proved in [18] (see also [20, p. 139]) that property $D_k$ implies that the similarity degree $d(A)$ is at most $2k$. Thus (ii) $\Rightarrow$ (i) follows from the corresponding implication in Theorem 1. \hfill $\square$

The main point in Theorem 1 is (ii) $\Rightarrow$ (i). In our previous work, we could only prove that (ii) implies that $A$ is “semi-nuclear,” i.e. that whenever a representation $\pi : A \to B(H)$ generates a semifinite von Neumann algebra, the latter is injective. In this note, we show that the semifiniteness assumption is not needed. We use the same starting point as in [18], but we feel the idea of the present proof is more transparent than the one in [18]. In particular, we will use the following result which is part of Th 2.9 in [19] (obtained independently in [3]), but the latter is inspired by and closely related to Haagerup’s Th. 2.1 in [9].

**Theorem 3.** Let $N \subset B(H)$ be a von Neumann algebra. Then $N$ is injective iff there is a constant $C$ such that, for all $n$, if elements $x_i$ in $N$ ($i = 1, \ldots, n$) admit a decomposition $x_i = \alpha_i + \beta_i$ with $\alpha_i, \beta_i \in B(H)$ such that $\|\sum \alpha_i^* \alpha_i\| \leq 1$ and $\|\sum \beta_i^* \beta_i\| \leq 1$, then there is a decomposition $x_i = a_i + b_i$ with $a_i, b_i \in N$ such that $\|\sum a_i^* a_i\| \leq C^2$ and $\|\sum b_i b_i^*\| \leq C^2$.\hfill $\square$
Theorem 4. Let $M \subset B(H)$ be a von Neumann algebra with a cyclic vector. Let $y_1, \ldots, y_n$ in $M'$ be such that for any $x_1, \ldots, x_n$ in $M$ we have

\begin{equation}
\| \sum x_i y_i \| \leq \max \left\{ \| \sum x_i^* x_i \|^{1/2}, \| \sum x_i x_i^* \|^{1/2} \right\}.
\end{equation}

Then there is a decomposition

\begin{equation}
y_i = a_i + b_i
\end{equation}

with $a_i, b_i$ in $M'$ such that

\begin{align*}
\| \sum a_i a_i^* \| &\leq 1 \quad \text{and} \quad \| \sum b_i^* b_i \| \leq 1.
\end{align*}

Proof. We follow a well known kind of argument with roots in [9]; see also [23] and the proof of a theorem due to Kirchberg as presented in [20, §14] that we will follow closely below.

Recall that the “row and column” operator spaces $R_n \subset M_n$ and $C_n \subset M_n$ are defined by:

\begin{equation}
R_n = \text{span} [e_{1i} \mid 1 \leq i \leq n] \quad C_n = \text{span} [e_{i1} \mid 1 \leq i \leq n].
\end{equation}

Let $\Delta_n \subset C_n \oplus R_n$ be the operator space spanned by $\delta_i = e_{i1} \oplus e_{i1}$ $(i = 1, 2, \ldots, n)$. Our assumption means that the linear map

\begin{equation}
v: \Delta_n \otimes_{\min} M \to B(H)
\end{equation}

defined by

\begin{equation}
v \left( \sum \delta_i \otimes x_i \right) = \sum x_i y_i
\end{equation}

satisfies $\|v\| \leq 1$. (Indeed, it is easy to check that the majorant in (1) is equal to $\| \sum \delta_i \otimes x_i \|_{\min}$.)

Since $v$ is clearly a two-sided $M$-module map and $M$ has a cyclic vector, it follows by [22] (and unpublished work of Haagerup) that $\|v\|_{cb} = \|v\| \leq 1$. Let us assume $C_n \oplus R_n \subset B(K)$. Consider now the operator system $S \subset M_2(B(K) \otimes_{\min} M)$ formed of all elements of the form

\begin{equation}
\begin{pmatrix}
a & b \\
c^* & d
\end{pmatrix}
\end{equation}

with $a, d \in 1 \otimes M$, and $b, c \in \Delta_n \otimes M$. By [20] Lemma 14.5], the mapping

\begin{equation}
vhat: S \to M_2(B(H))
\end{equation}

defined by

\begin{equation}
\begin{pmatrix}
1 \otimes m_1 \\
c^* \\
1 \otimes m_2
\end{pmatrix}
\begin{pmatrix}
b \\
v(c)^* \\
m_2
\end{pmatrix}
\end{equation}

is a unital c.p. map (or satisfies $\|v\|_{cb} = 1$, which is the same for a unital map).

By Arveson’s extension theorem (see [24] p. 154 or [17]), $\hat{v}$ admits a unital c.p. extension $\hat{v}: M_2(B(K) \otimes_{\min} M) \to M_2(B(H))$. Then since $\hat{v}$ is a $*$-homomorphism when restricted to

\begin{equation}
\left\{ \begin{pmatrix} m_1 & 0 \\
0 & m_2 \end{pmatrix} \mid m_i \in M \right\} \simeq M \oplus M,
\end{equation}

it follows by multiplicative domain arguments (see e.g. [20] Lemma 14.3]) that $\hat{v}$ must be a bimodule map with respect to the natural actions of $M \oplus M$. Let $w: B(K) \otimes_{\min} M \to B(H)$ be the map defined by

\begin{equation}
w(\beta) = \begin{pmatrix} 0 & \beta \\
0 & 0 \end{pmatrix}.
\end{equation}
Since \( \hat{v} \) extends \( \tilde{v} \) we have
\[
w \left( \sum \delta_i \otimes x_i \right) = \sum x_i y_i \quad (x_i \in M).
\]

Moreover, since \( \hat{v} \) is bimodular, for any \( z_i \) in \( B(K) \) (since \( z_i \otimes y_i = (z_i \otimes 1)(1 \otimes y_i) = (1 \otimes y_i)(z_i \otimes 1) \)) we find
\[
w(z_i \otimes x_i) = w((z_i \otimes 1))x_i = x_i w(z_i \otimes 1)
\]
and in particular \( w(z_i \otimes 1) \in M' \). Thus if we set
\[
a_i = w((e_i 1 \oplus 0) \otimes 1) \quad \text{and} \quad b_i = w((0 \oplus e_i 1) \otimes 1)
\]
then \( a_i, b_i \in M' \), \( a_i + b_i = w(\delta_i 1) = x_i \). Finally
\[
\left\| \sum a_i a_i^* \right\|^{1/2} = \left\| \sum a_i \otimes e_i 1 \right\| \leq \|w\|_{cb} \left\| \sum e_i 1 \otimes e_i 1 \right\| \leq 1,
\]
and similarly
\[
\left\| \sum b_i b_i^* \right\|^{1/2} \leq 1. \quad \Box
\]

**Remark 1.** It is easy to see that the preceding result fails without the cyclicity assumption: Just consider the case \( M = \mathbb{C} \) and \( M' = B(H) \) with \( \dim(H) = \infty \).

**Remark 2.** The same proof gives a criterion for a map \( u: E \to M' \) defined on a subspace \( E \subset A \) of a general C*-algebra \( A \) to admit an extension \( \tilde{u}: A \to M' \) with \( \|\tilde{u}\|_{dec} \leq 1 \). This is essentially the same as Kirchberg’s [20, Th. 14.6].

**Remark 3.** The above Theorem 4.5 may be viewed as an analogue for the operator space \( R_n + C_n \) of Haagerup’s [9] Lemma 3.5] devoted to the operator space \( \ell_1^n \) equipped with its maximal structure, in the Blecher-Paulsen sense (see e.g. [20, §3]). Note that while he decomposes into products, we decompose into sums.

**Remark 4.** Let \( (E_0, E_1) \) be a compatible pair of operator spaces in the sense of [20, §2.7]. Then Remark 2 gives a sufficient criterion for a map \( u: E_0 + E_1 \to M' \) to admit a decomposition \( u = u_0 + u_1 \) with \( u_0: E_0 \to M' \) and \( u_1: E_1 \to M' \) satisfying \( \|u_0\|_{cb} \leq 1 \) and \( \|u_1\|_{cb} \leq 1 \). Assume that \( E_0 \subset A_0 \) and \( E_1 \subset A_1 \), where \( A_0, A_1 \) are C*-algebras, then this criterion actually ensures that there are extensions
\[
\tilde{u}_0: A_0 \to M' \quad \text{and} \quad \tilde{u}_1: A_1 \to M'
\]
with \( \|\tilde{u}_0\|_{dec} \leq 1 \) and \( \|\tilde{u}_1\|_{dec} \leq 1 \). In that formulation, the converse also holds up to a numerical factor 2. Note that, in the special case of interest to us, when \( E_0 = C \) and \( E_1 = R \), then we can take \( A_0, A_1 \) equal to \( K(\ell_2) \) (hence nuclear) so that the min and max norms are identical on \( (A_0 \oplus A_1) \otimes M \).

**Notation.** Let \( A \subset B(H) \) be any C*-subalgebra. For any \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) in \( A \), we denote
\[
\| (x_j) \|_{R \otimes C} = \max \left\{ \left\| \sum x_j^* x_j \right\|^{1/2}, \left\| \sum x_j x_j^* \right\|^{1/2} \right\},
\]
\[
\| (y_j) \|_{R + C} = \inf \left\{ \left\| \sum \alpha_j^* \alpha_j \right\|^{1/2} + \left\| \sum \beta_j \beta_j^* \right\|^{1/2} \right\},
\]
where the infimum runs over all \( \alpha_j, \beta_j \) in \( B(H) \) such that \( y_j = \alpha_j + \beta_j \). Note that, by the injectivity of \( B(H) \), the definition of \( \| (y_j) \|_{R + C} \) does not really depend on the choice of \( H \) or of the embedding \( A \subset B(H) \). The corresponding fact for \( \| (x_j) \|_{R \otimes C} \) is obvious.

**Corollary 5.** Let \( M \subset B(H) \) be a von Neumann algebra. Then \( M \) is injective iff there is a constant \( C \) such that, for all \( n \), all \( x_1, \ldots, x_n \) in \( M \) and \( y_1, \ldots, y_n \) in \( M' \), we have
\[
\left\| \sum x_i y_i \right\| \leq C \| (x_i) \|_{R \otimes C} \| (y_i) \|_{R + C}.
\]
Proof. If $M$ has a cyclic vector, then this follows immediately from Theorems 8 and 4 and the well known fact that $M'$ is injective iff $M$ is injective (see [99] p. 174). Now assume that $M$ has a finite cyclic set, i.e. there are $\xi_1, \ldots, \xi_N$ in $H$ such that $M\xi_1 + \cdots + M\xi_N$ is dense in $H$. Then the vector $(\xi_1, \ldots, \xi_N)$ in $H^N$ is cyclic for $M_N(M) \subset M_N(B(H))$. Moreover, it is easy to check that (14) remains true for $M_N(M)$ but with $C$ replaced by a constant $C(N)$ (possibly unbounded when $N$ grows). Nevertheless, by the first part of the proof it follows that $M_N(M)$ is injective and hence, a fortiori, $M$ is injective.

In the general case, let $\{\xi_i \mid i \in I\}$ be a dense subset of $H$. For any finite subset $J \subset I$, let $H_J$ be the closure of$$\{\sum_{j \in J} a_j(\xi_j) \mid a_j \in M\}.$$Note that $H_J$ is an invariant subspace for $M$, so that (since $M$ is self-adjoint) the orthogonal projection $P_J : H \to H_J$ belongs to $M'$. Let $\pi_J(a)$ denote the restriction of $a$ to $H_J$. Then $\pi_J : M \to B(H_J)$ is a normal representation, $\pi_J(M)$ admits a finite cyclic set (namely $\{\xi_i \mid i \in J\}$), and it is easy to check that our assumption (4) is still verified by $\pi_J(M)$ on $H_J$.

Thus, by the first part of the proof, $\pi_J(M)$ is injective. This clearly implies that the von Neumann algebra $M_J \subset B(H)$ generated by $P_J M$ and $I - P_J$ also is injective. Finally, since $M_J$ is the weak-* closure of the directed union of the $M_J$’s, we conclude that $M$ itself is injective.

Conversely, if $M$ injective then, by Remark 5 below, (14) holds with $C = 1$. \hfill \Box

Remark 5. Let $M \subset B(H)$ be an injective von Neumann algebra, so that there is a projection $P : B(H) \to M'$ with $\|P\|_{cb} = 1$. Then $M$ satisfies (14) with $C = 1$. To see this, assume $y_i \in M'$ and $\|y_i\|_{R+C} < 1$, so that $y_i = a_i + \beta_i$ with $\|\Sigma \alpha_i^* a_i\|^{1/2} + \|\Sigma \beta_i \beta_i^*\|^{1/2} < 1$. Then $y_i = a_i + b_i$ with $a_i, b_i \in M$ satisfying$$\|\sum a_i^* a_i\|^{1/2} + \|\sum b_i^* b_i\|^{1/2} \leq \|P\|_{cb} = 1.$$Indeed, $a_i = P\alpha_i$ and $b_i = P\beta_i$ clearly verify this.

Then for any $x_1, \ldots, x_n$ in $M$ we have by Cauchy-Schwarz$$\|\sum x_i a_i\| \leq \|\sum x_i x_i^*\|^{1/2} \|\sum a_i^* a_i\|^{1/2}$$and$$\|\sum b_i x_i\| \leq \|\sum x_i x_i^*\|^{1/2} \|\sum b_i^* b_i\|^{1/2},$$therefore, since$$\|\sum x_i y_i\| \leq \|\sum x_i a_i\| + \|\sum b_i x_i\|,$$we obtain finally$$\|\sum x_i y_i\| \leq \|(x_i)\|_{R+C} \|(y_i)\|_{R+C}.$$\hfill \Box

We will also use:

**Theorem 6 (18)**. A unital operator algebra $A$ satisfies property (ii) in Theorem 4 if and only if we have:

(iiv) There is a constant $K'$ satisfying the following: for any linear map $u : A \to B(H)$ for which there are a Hilbert space $K$, bounded linear maps $v_1, w_1$ from $A$ to $B(K, H)$ and $v_2, w_2$ from $A$ to $B(H, K)$ such that

$$\forall a, b \in A \quad u(ab) = v_1(a)v_2(b) + w_1(a)w_2(b)$$

(5)
we have
\[ \|u\|_{cb} \leq K'(\|v_1\| \|v_2\| + \|w_1\| \|w_2\|). \]

Remark. Note that (5) implies that the bilinear map \((a, b) \to u(ab)\) is c.b. on \(max(A) \otimes_h max(A)\) with c.b. norm \(\leq K'(\|v_1\| \|v_2\| + \|w_1\| \|w_2\|).\) Thus, Theorem 5 follows from the case \(d = 2\) of [13 Th. 4.2].

Another ingredient is the following Lemma which can be derived from [13] or from the more recent paper [21].

**Lemma 7.** Let \(E\) be a finite dimensional operator space and let \(A\) be a \(C^*\)-algebra. Assume that \(E\) is a “maximal” operator space (equivalently that \(E^*\) is a minimal one). Then for any c.b. map \(u: A \to E\) we have
\[ \forall n \; \forall a_1, \ldots, a_n \in A \; \forall \xi_i \in E^* \]
\[ \left| \sum (u(a_j), \xi_j) \right| \leq C\|u\|_{cb} \left( \left\| \sum a_j^* a_j \right\|^{1/2} + \left\| \sum a_j a_j^* \right\|^{1/2} \right) \cdot \sup_{x \in E} \left( \sum |\xi_j(x)|^2 \right)^{1/2} \]
where \(C\) is a numerical constant.

Proof. We may apply [13] Th. 1.4], arguing as in [13] Lemma 6.3 (using [19] Th. 17.13 to remove the exactness assumption) this yields (6) with \(C = 2\). Or we may invoke [21] Th. 0.3] taking into account [21] Lemma 2.3] (to remove the exactness assumption) and then we again obtain (6) with \(C = 2\).

For the convenience of the reader, we reproduce here the elementary Lemma 8 from [13].

**Lemma 8.** Let \((e_i)\) be the canonical basis of the operator space \(\max(\ell_2)\). Let \(H\) be any Hilbert space and let \(X\) be either \(B(\mathbb{C}, H)\) or \(B(H^*, \mathbb{C})\), or equivalently let \(X\) be either the column Hilbert space or the row Hilbert space. Then for all \(x_1, \ldots, x_n\) in \(X\) we have
\[ \left\| \sum_{i=1}^{n} x_i \otimes e_i \right\|_{X \otimes_{min} \max(\ell_2)} \leq \left( \sum_{i} \|x_i\|^2 \right)^{1/2}. \]

Proof. Assume \(X = B(\mathbb{C}, H)\) or \(B(H^*, \mathbb{C})\). We identify \(X\) with \(H\) as a vector space. Let \((\delta_m)\) be an orthonormal basis in \(H\). Observe that for any finite sequence \(a_m\) in \(B(\ell_2)\) we have in both cases
\[ \left\| \sum_{m} \delta_m \otimes a_m \right\|_{\min} \leq \left( \sum_{m} \|a_m\|^2 \right)^{1/2}. \]
whence we have, for any \(x_1, \ldots, x_n\) in \(X\),
\[ \left\| \sum_{i} x_i \otimes e_i \right\| = \left\| \sum_{m} \delta_m \otimes \sum_{i} \langle x_i, \delta_m \rangle e_i \right\| \]
\[ \leq \left( \sum_{m} \left\| \sum_{i} \langle x_i, \delta_m \rangle e_i \right\|^2 \right)^{1/2} \]
\[ = \left( \sum_{m,i} |\langle x_i, \delta_m \rangle|^2 \right)^{1/2} = \left( \sum_{i} \|x_i\|^2 \right)^{1/2}. \]
We now claim that we can write for all \(a, b\) the following:

\[
\|\sum x_j y_j\| \leq 4K'C\|\langle x_j\rangle\|_{R'\cap C}\|\langle y_j\rangle\|_{R+C}.
\]

Proof of Theorem 1. As we already observed, it suffices to show that (ii) implies that \(A\) is nuclear. Let \(\pi: A \to B(H)\) be a representation and let \(M = \pi(A)'\). Using Theorem 6 and Corollary 5, we will show that (ii) implies that \(M\) is injective. By the well known results of Choi–Effros and Connes (see \[3\]), this implies that \(A\) is nuclear. Since \(\pi(A) \simeq A/\ker(\pi)\) is a quotient of \(A\), it obviously inherits the property (ii). Thus we may as well replace \(\pi(A)\) by \(A\); we assume \(A \subset B(H)\) and let \(M = A''\). It suffices to show that \(M\) is injective.

Claim. We claim that for any \(x_1, \ldots, x_n\) in \(M\) and \(y_1, \ldots, y_n\) in \(M'\) we have

\[
(7) \quad \|\sum x_j y_j\| \leq 4K'C\|\langle x_j\rangle\|_{R'\cap C}\|\langle y_j\rangle\|_{R+C}.
\]

Note: It may be worthwhile for the reader to note that \(\|\langle y_j\rangle\|_{R+C}\) is (up to a factor 2) in operator space duality with \(\|\langle x_j\rangle\|_{R'\cap C}\); namely if we set

\[
\|\|\langle y_j\rangle\|\| = \sup \left\{ \left\| \sum x_j \otimes y_j \right\|_{\min} \right\}
\]

where the sup runs over all \((x_j)\) in \(B(\ell_2)\) such that \(\|\langle x_j\rangle\|_{R'\cap C} \leq 1\), then we have (see e.g. \[12\])

\[
\|\|\langle y_j\rangle\|\| \leq \|\langle y_j\rangle\|_{R+C} \leq 2\|\|\langle y_j\rangle\||.\]

To prove (7) we introduce the operator space \(E = \max(\ell_2^n)\), that is \(n\)-dimensional Hilbert space equipped with its "maximal" operator space structure in the Blecher-Paulsen sense (see \[20\] \S3). Let us now fix an \(n\)-tuple \((y_j)\) in \(M'\) such that \(\|\langle y_j\rangle\|_{R+C} < 1\). In addition, we fix \(\xi, \eta\) in the unit sphere of \(H\). Then we define a linear map \(u: M \to E\) as follows:

\[
u(x) = \sum_j \langle xy_j \xi, \eta \rangle e_j\]

where \(e_j\) is the canonical basis of \(\ell_2^n\). We will assume that \(E \subset B(K)\) completely isometrically. The reader may prefer to consider instead of \(u\), the bilinear form \((x, \xi) \to \langle u(x), \xi \rangle\) defined on \(M \times E'\) where \(E'\) is now \(\ell_2^n\) equipped with its "minimal" (or commutative) operator space structure obtained by embedding it isometrically into a commutative C*-algebra. We will now apply Theorem 6 to \(u\).

Since we assume \(\|\langle y_j\rangle\|_{R+C} < 1\), we can write

\[
y_j = \alpha_j + \beta_j\]

with \(\|\sum \alpha_j \alpha_j^*\| < 1\) and \(\|\sum \beta_j \beta_j^*\| < 1\). Note that, since \(y_j \in M'\), for all \(a, b\) in \(M\) we have

\[
aby_j = ay_j b
\]

and hence

\[
u(ab) = V(a, b) + W(a, b)
\]

where

\[
V(a, b) = \sum \langle a\alpha_j b\xi, \eta \rangle e_j,
\]

\[
W(a, b) = \sum \langle a\beta_j b\xi, \eta \rangle e_j.
\]

We now claim that we can write for all \(a, b\) in \(M\)

\[
(8) \quad V(a, b) = v_1(a)v_2(b) \quad \text{and} \quad W(a, b) = w_1(a)w_2(b)
\]

where

\[
v_1: M \to B(H \otimes K, K), \quad w_1: M \to B(H \otimes K, K)
\]

\[
v_2: M \to B(K, H \otimes K), \quad w_2: M \to B(K, H \otimes K)
\]
are linear maps all with norm $\leq 1$.

Indeed, let us set for $h \in H$, $k \in K$

$$v_1(a)(h \otimes k) = \sum_j (a \beta_j h, \eta) e_j k$$

$$w_1(a)(h \otimes k) = \langle ah, \eta \rangle k$$

$$v_2(b)(k) = b \xi \otimes k$$

$$w_2(b)(k) = \sum_j \alpha_j b \xi \otimes e_j k.$$ 

Then, it is easy to check (5). Also, we have trivially

$$\|w_1(a)\| = \|a^* \eta\| \leq \|a\|$$

$$\|v_2(b)\| = \|b \xi\| \leq \|b\|.$$ 

Moreover, by Lemma we have

$$\|v_1(a)\|^2 \leq \sum_j \|\beta_j^* a^* \eta\|^2 = \left\langle \sum \beta_j^* \beta_j^* a^* \eta, a^* \eta \right\rangle$$

$$\leq \|a^* \eta\|^2 \leq \|a\|^2$$

$$\|w_2(b)\|^2 \leq \sum_j \|\alpha_j b \xi\|^2 = \left\langle \sum \alpha_j^* \alpha_j b \xi, b \xi \right\rangle$$

$$\leq \|b \xi\|^2 \leq \|b\|^2.$$ 

By Theorem it follows that

$$\|u_{|A}\|_{cb} \leq 2K'.$$

Since $u: M \to B(K)$ is clearly normal (i.e. $\sigma(M, M_*)$ continuous) and since $A$ is $\sigma(M, M_*)$ dense in $M$, we clearly have (by the Kaplansky density theorem)

$$\|u\|_{cb} = \|u_{|A}\|_{cb} \leq 2K'.$$

Then by Lemma applied with $\xi_j$ biorthogonal to $e_j$, we have

$$\forall n \forall x_1, \ldots, x_n \in M \quad \left| \left\langle \sum x_j y_j \xi_j, \eta \right\rangle \right| \leq 4K' C \|x_j\|_{R \otimes C}.$$ 

Hence, taking the supremum over $\xi, \eta$ and using homogeneity, we obtain the claimed inequality (7). Then, by Corollary $M$ is injective.

**Remark.** Since Lemma actually holds whenever $A$ is an exact operator space (with $C$ replaced by twice the exactness constant 13, 21), the proof of Theorem 1 shows that any unital, exact (non selfadjoint) operator algebra $A \subset B(H)$ with $d(A) \leq 2$ in the sense of 18 satisfies (4) for some $C$.

The preceding arguments establish the following result of independent interest.

**Theorem 9.** A $C^*$-algebra $A$ is nuclear iff for any $C^*$-algebra $B$ there is a constant $C$ such that, for all $n$, all $x_1, \ldots, x_n$ in $A$ and all $y_1, \ldots, y_n$ in $B$ we have

$$(9) \quad \left\| \sum x_i \otimes y_i \right\|_{\max} \leq C \|(x_i)\|_{R \otimes C} \|(y_i)\|_{R \otimes C}.$$ 

**Proof.** Let $\pi: A \to \pi(A)'$ be a representation. Taking $B = \pi(A)'$ (and using the fact that the set of $n$-tuples $(x_i)$ in $A^{*n}$ with $\|(x_i)\|_{R \otimes C} \leq 1$ is the weak-* closure of its intersection with $A^n$, see e.g. 20, p. 303) we see that (9) implies (4) for $M = \pi(A)'$. Since this holds for any $\pi$, we may argue as in the preceding proof (replacing $\pi$ by $\pi_j$) to conclude that $\pi(A)'$ is injective, and hence that $A$ is nuclear. Conversely, if $A$ is nuclear it is easy to show (see Remark 15 that (9) holds with $C = 1$.  

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Theorem 10. A $C^*$-algebra $A$ is nuclear iff for any $C^*$-algebra $B$ there is a constant $C$ such that for all $n$, all $x_1, \ldots, x_n$ in $A$ and all $y_1, \ldots, y_n$ in $B$ we have

$$\|\sum x_i \otimes y_i\|_{\text{max}} \leq C \|\sum x_i \otimes \bar{x}_i\|_{\text{min}}^{1/2} \|\sum y_i \otimes \bar{y}_i\|_{\text{min}}^{1/2},$$

where the min norms are relative to $A \otimes \bar{A}$ and $B \otimes \bar{B}$.

Proof. It is known (see [19, (2.12)]) that $\|\Sigma x_i \otimes \bar{x}_i\|_{\text{min}}^{1/2} \leq \|(x_i)\|_{RC}$. Thus, arguing as above, we find that for any representation $\pi: A \to B(H)$ the von Neumann algebra $M = \pi(A)'$ satisfies the following: if $y_1, \ldots, y_n$ in $M'$ are such that $\|\Sigma y_i \otimes \bar{y}_i\|_{\text{min}} < 1$, then there are $a_i, b_i$ in $M'$ with $y_i = a_i + b_i$ such that $\|\Sigma a_i^* a_i\|^{1/2} < C$ and $\|\Sigma b_i^* b_i\|^{1/2} < C$. By [19, Th. 2.9], this ensures that $M'$ is injective, and hence $A$ is nuclear. \[\square\]

Remark 6. Note however that by [11] the inequality

$$\|\sum x_i \otimes \bar{x}_i\|_{\text{max}}^{1/2} \leq C \|\sum x_i \otimes \bar{x}_i\|_{\text{min}}^{1/2}$$

characterizes the weak expectation property, which is strictly more general than nuclearity.

Remark. It would be nice to know exactly which families of pairs of operator spaces in duality $(F_n, F_n')$ can be used instead of $F_n = R_n \cap C_n$ or $F_n = OH_n$ to characterize nuclearity (or injectivity) analogously to the above Theorems [2] and [11] (note that $F_n = R_n$ or $F_n = C_n$ obviously do not work).

We will say that a function $f: \mathbb{N} \to \mathbb{R}_+$ is “slowly growing” if, for any $\varepsilon > 0$, there is a constant $C_\varepsilon$ such that $f(n) \leq C_\varepsilon n^\varepsilon$ for all $n \geq 1$.

The rest of the paper is devoted to a technical refinement, based on the following observation: assume that in Theorem 3 the constant $C$ depends on $n$, i.e. $C = C(n)$ but that it is “slowly growing”. Then $N$ is injective.

Indeed, as for Theorem 3 this observation follows from the same argument as for [19, Th. 2.9], itself based on [9]. Recall Haagerup’s characterization of finite injective von Neumann algebras ([19, Lemma 2.2]): $N$ is finite and injective iff for any $n$-tuple $(u_i)$ of unitaries and any central projection $p$ in $N$ we have

$$n = \|\sum p u_i \otimes \bar{u}_i\|.$$

Actually, for this to hold it suffices that there exists a slowly growing function $C(n)$ such that for any $n$-tuple $(u_i)$ of unitaries and any central projection $p$ in $N$ we have

$$n \leq C(n) \|\sum p u_i \otimes \bar{u}_i\|.$$

Indeed, if we set $t = \sum p u_i \otimes \bar{u}_i$ and if we apply the preceding inequality to $(t^* t)^m$, take the $m$-th root and let $m$ go to infinity, then we find that [11] implies [10] (a similar trick appears in [9, Lemma 2.2]). Given that this is true, the above observation can be deduced, first in the case when $N$ is semifinite, and then in the general case, from the finite case by the same basic reasoning as in [9].

The following theorems are then easy to obtain in the same way as above.

Theorem 11. The following properties of a $C^*$-algebra are equivalent.

(i) $A$ is nuclear.

(ii) There is a slowly growing function $C: \mathbb{N} \to \mathbb{R}_+$ such that for any $n$ and any $C^*$-algebra $B$ we have:

$$\forall (x_i) \in A^n \quad \forall (y_i) \in B^n \quad \|\sum x_i \otimes y_i\|_{\text{max}} \leq C(n) \|(x_i)\|_{R^+ C} \|(y_i)\|_{R^+ C}.$$
(iii) There is a slowly growing function $C: \mathbb{N} \to \mathbb{R}_+$ such that for any $n$ and any $C^*$-algebra $B$, we have:

$$\forall (x_i) \in A^n \quad \forall (y_i) \in B^n \quad \left\| \sum x_i \otimes y_i \right\|_{\max} \leq C(n) \left\| \sum x_i \otimes \bar{x}_i \right\|_{\min}^{1/2} \left\| \sum y_i \otimes \bar{y}_i \right\|_{\min}^{1/2}.$$  

**Corollary 12.** A von Neumann algebra $M$ is injective iff there is a slowly growing function $C: \mathbb{N} \to \mathbb{R}_+$ such that, for any $n$, any mapping $u: \Delta_n \to M$ admits an extension $\tilde{u}: M_n \oplus M_n \to M$ such that

$$\| \tilde{u} \|_{cb} \leq C(n)\| u \|_{cb}.$$  

Remark. Consider a map $u: E \to F$ between operator spaces. Let $\gamma(u) = \inf \{\|v\|_{cb} \|w\|_{cb} \}$ where the infimum runs over all Hilbert spaces $H$ and all possible factorizations $u = vw$ of $u$ through $B(H)$ (here $v: B(H) \to F$ and $w: E \to B(H)$). Let $M$ be a von Neumann algebra. Assume that there is a constant $C$ so that, for any $n$, any $u: R_n \cap C_n \to M$ satisfies $\gamma(u) \leq C\|u\|_{cb}$. Then, by the preceding Corollary, $M$ is injective. Actually, even if $C = C(n)$ depends on $n$, but grows slowly when $n \to \infty$, we conclude that $M$ is injective, and hence, a posteriori, we can factor through $B(H)$ any $u$ that takes values in $M$, regardless of its domain. It seems interesting to investigate which (sequences of) operator spaces have the property that they “force” injectivity like $\{R_n \cap C_n\}$. One can show that $\{OH_n\}$ has that property too, but obviously not $\{R_n\}$ or $\{C_n\}$, since these are themselves injective!

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