On Nilcompactifications of Prime Spectra of Commutative Rings

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Abstract: Given a ring $R$ and $S$ one of its proper ideals, we obtain a compactification of the prime spectrum of $S$ through a mainly algebraic process. We name it the $R$–nilcompactification of $\text{Spec}S$. We study some categorical properties of this construction.

Keywords: Prime ideal, prime spectrum, spectral compactness, compactification.

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1 Introduction.

Compactification of a topological space is an important topic considered in a wide range of branches in mathematics as it guarantees a useful property for the space, see for example [4], [9], [10], [11], [12], [13], [14]. In this paper we study, from a categorical point of view, the compactification method of prime spectra which is presented in [3], named nilcompactification. Nilcompactification is a topological method obtained mainly through an algebraic process. The categorical point of view of nilcompactification, with some of its possible variations, offers an interesting wealth for this process. For the categorical concepts see, for example, [8]. This method of nilcompactification is functorial in a simple way in that it has interesting properties and the involved constructions provide us with different natural transformations. We can take into account that some compactification processes need to consider suitable subcategories to obtain a functorial behavior, as studied in [2] for the Alexandroff compactifications.

In this paper the word ring means commutative ring, not necessarily with identity. A homomorphism is a function between rings that respects addition and product. We suppose that prime ideals are proper ideals by definition. The prime spectrum of a ring $S$ is denoted by $\text{Spec}S$ and it is the set of

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prime ideals of $S$ endowed with the Zariski topology. In this topology the sets
\[ D_S(a) = \{ P \in \text{Spec}S : a \notin P \}, \]
where $a \in S$, provide a basis. The closed sets are
\[ V_S(I) = \{ P \in \text{Spec}S : P \supseteq I \}, \]
where $I$ is an ideal of $S$. It is known that the basic open sets are compact and for each ideal $I$ of $S$ the function $V_S(I) \to \text{Spec}(S/I) : P \mapsto P/I$ is a homeomorphism (see [5]). The Zariski topology is also called the hull-kernel topology because the closure of a subset $\mathcal{B}$ of $\text{Spec}S$ is the set
\[ \left\{ P \in \text{Spec}S : P \supseteq \bigcap_{J \in \mathcal{B}} J \right\} \]
and $\bigcap_{J \in \mathcal{B}} J$ is called the kernel of $\mathcal{B}$.

The nilradical of $S$, denoted $N(S)$, is precisely the kernel of $\text{Spec}S$.

It is known that $S$ is semiprime or reduced if $N(S) = \{0\}$.

A ring whose spectrum is compact is a spectrally compact ring. In particular, every unitary ring $R$ is spectrally compact because $D_R(1) = \text{Spec}R$.

We use the following notations:
- $\mathcal{CR}$: Category of commutative rings and homomorphisms of rings.
- $\mathcal{CR}_1$: Category of unitary commutative rings and homomorphisms of unitary rings.
- $\mathcal{CR}^s$: Category of commutative rings and surjective homomorphisms.
- Top: Category of topological spaces and continuous functions.
- $\mathcal{S}$: Category of spectral spaces and strongly continuous functions. A spectral space is a topological space that is homeomorphic to the prime spectrum of a unitary commutative ring. It is known that a topological space is spectral if and only if it is sober, compact and coherent (see [6]). A function is strongly continuous if it sends compact open sets in compact open sets by reciprocal image.

2 The mechanism of nilcompactification

The material of this section is taken from [3].

In this section $S$ is a fixed ring and $R$ is an i-extension of $S$, that is, a ring containing $S$ as ideal.
Given an ideal $I$ of $S$ it is clear that the set $\psi(I) = \{ x \in R : xS \subseteq I \}$ is an ideal of $R$. So, $\psi$ is a function from the set $\mathcal{J}(S)$ of the ideals of $S$, to the set $\mathcal{J}(R)$ of the ideals of $R$.

**Lemma 1.** The function $\psi : \mathcal{J}(S) \to \mathcal{J}(R)$ has the following properties:

1. If $P$ is a prime ideal of $S$ then $\psi(P)$ is a prime ideal of $R$ not containing $S$.
2. If $P$ and $Q$ are prime ideals of $S$ such that $\psi(P) = \psi(Q)$ then $P = Q$.
3. If $Q$ is a prime ideal of $R$ not containing $S$ then $Q \cap S$ is a prime ideal of $S$ and $\psi(Q \cap S) = Q$.
4. $\psi \left( \bigcap_{P \in \text{Spec} S} P \right) = \bigcap_{P \in \text{Spec} S} \psi(P)$.

**Proposition 2.** The function $\psi : \text{Spec} S \to \text{Spec} R$ is injective, continuous and open onto its image.

**Proof.** By (iii) of the previous lemma we have that $\psi$ is injective.

For continuity, it is enough to observe that if $r \in R$ then $\psi^{-1}(D_{R}(r)) = \bigcup_{s \in S} D_{S}(rs)$.

On the other hand, if $s \in S$ then it is easy to see that $\psi(D_{S}(s)) = \psi(\text{Spec} S) \cap D_{R}(s)$, then $\psi$ is open onto its image. 

Hereinafter we denote with $\text{Spec}_{S} R$ the image of the function $\psi$ restricted to $\text{Spec} S$. In other words, $\text{Spec}_{S} R$ is the set $\{ Q \in \text{Spec} R : Q \nsubseteq S \}$. Thus, $\text{Spec}_{S} R$ is homeomorphic to $\text{Spec}_{S} R$, seen as subspace of $\text{Spec} R$.

**Proposition 3.** $\text{Spec}_{S} R$ is an open of $\text{Spec} R$ and its closure is a subspace of $\text{Spec} R$ homeomorphic to $\text{Spec} (R/\psi(N(S)))$.

**Proof.** The first assertion follows from the equality $\text{Spec}_{S} R = \bigcup_{s \in S} D_{R}(s)$. On the other hand:

$$
\overline{\text{Spec}_{S} R} = \left\{ Q \in \text{Spec} R : Q \supseteq \bigcap_{P \in \text{Spec} S} \psi(P) \right\} \\
= \left\{ Q \in \text{Spec} R : Q \supseteq \psi \left( \bigcap_{P \in \text{Spec} S} P \right) \right\} \\
= \{ Q \in \text{Spec} R : Q \supseteq \psi(N(S)) \} \\
= V_{R}(\psi(N(S))) \\
\approx \text{Spec} (R/\psi(N(S))).
$$
Remark 4. Notice that the inclusion of \( \text{Spec}S \) in \( \text{Spec}(R/\psi(N(S))) \) is given by the function

\[
\lambda : \text{Spec}S \to \text{Spec}(R/\psi(N(S))) : P \mapsto \psi(P)/\psi(N(S)).
\]

Corollary 5. If \( R \) is spectrally compact (in particular if \( R \) has identity) then \( \text{Spec}(R/\psi(N(S))) \) is a compactification of \( \text{Spec}S \) in which \( \text{Spec}S \) is open.

Definition 6. If \( R \) is spectrally compact, the space \( \text{Spec}(R/\psi(N(S))) \) is called the \( R \)-nilcompactification of \( \text{Spec}S \).

It is well known that every ring is an ideal of a unitary ring (see [7]), therefore we obtain the following result:

Theorem 7. The spectrum of every ring has a spectral compactification.

Proof. It is enough to observe that if \( S \) is a ring, we can choose \( R \) unitary and hence \( \text{Spec}(R/\psi(N(S))) \) is a spectral space. \( \square \)

3 Functorial behavior of the mechanism of nilcompaction

Consider the category \( \mathcal{E} \) whose objects are the pairs \((S, R)\) with \( R \) a unitary \( i \)-extension of \( S \) and where the morphisms from \((S_1, R_1)\) to \((S_2, R_2)\) are the homomorphisms of unitary rings from \( R_1 \) to \( R_2 \) such that \( h(S_1) = S_2 \).

As \( S \) and \( R \) are variables, in this context, the functions \( \psi \) and \( \lambda \) defined in the previous section will be denoted \( \psi_{(S,R)} \) and \( \lambda_{(S,R)} \) respectively.

For each object \((S, R)\) of \( \mathcal{E} \), we define \( Q(S, R) = R/\psi_{(S,R)}(N(S)) \).

The following proposition can be proved without difficulty.

Proposition 8. If \( h : (S, R) \to (T, M) \) is a morphism of \( \mathcal{E} \), then

\[
Q(h) : Q(S, R) \to Q(T, M) : r + \psi_{(S,R)}(N(S)) \mapsto h(r) + \psi_{(T,M)}(N(T))
\]

is well defined and is a homomorphism of unitary rings.

Thus, \( Q \) is a functor from the category \( \mathcal{E} \) to the category \( \text{CR}_1 \) of unitary commutative rings.

We denote \( \text{NC} \) the contravariant functor \( \text{Spec} \circ Q : \mathcal{E} \to \mathcal{S} \) where \( \mathcal{S} \) is the category of the spectral spaces and the strongly continuous functions. So, \( \text{NC}(S, R) \) is the \( R \)-nilcompactification of \( \text{Spec}S \).
Some natural transformations:
Consider the functors $V : \mathcal{E} \to \mathcal{CR}_1$ and $W : \mathcal{E} \to \mathcal{CR}^*$ defined as follows:

\[
V(h : (S, R) \to (T, M)) = h : R \to M \quad \text{and} \quad W(h : (S, R) \to (T, M)) = h \mid_S : S \to T.
\]

The following proposition is a direct consequence of the Proposition 8.

**Proposition 9.** If for each object $(S, R)$ of $\mathcal{E}$, we denote by $\theta_{(S,R)} : R \to Q(S, R)$ the canonical function to the quotient, then $\theta = \left(\theta_{(S,R)}\right)_{(S,R) \in \text{Ob}\mathcal{E}}$ is a natural transformation from the functor $V$ to the functor $Q$.

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{V} & \mathcal{CR}_1 \\
(S, R) & \xrightarrow{V(h)} & Q(S, R) = R/\psi_{(S,R)}(N(S)) \\
\downarrow h & & \downarrow Q(h) \\
(T, M) & \xrightarrow{V(T, M)} & Q(T, M) = M/\psi_{(T,M)}(N(T))
\end{array}
\]

**Proposition 10.** $\psi = \left(\psi_{(S,R)}\right)_{(S,R) \in \text{Ob}\mathcal{E}}$ is a natural transformation from the functor $\text{Spec} \circ W$ to the functor $\text{Spec} \circ V$.

**Proof.** It is enough to observe that if $h : (S, R) \to (T, M)$ is a morphism of $\mathcal{E}$ then for each prime ideal $P$ of $T$ we have that $h^{-1}\left(\psi_{(T,M)}(P)\right) = \psi_{(S,R)}\left(h^{-1}(P)\right)$. \qed

**Proposition 11.** $\lambda = \left(\lambda_{(S,R)}\right)_{(S,R) \in \text{Ob}\mathcal{E}}$ is a natural transformation from the functor $\text{Spec} \circ W$ to the functor $\text{NC}$.

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\text{Spec}} & \text{Top} \\
(S, R) & \xrightarrow{\text{Spec}(S)} & \text{NC}(S, R) \\
\downarrow h & & \downarrow \text{NC}(h) \\
(T, M) & \xrightarrow{\text{Spec}(T)} & \text{NC}(T, M)
\end{array}
\]
Proof. Let $h : (S, R) \to (T, M)$ be a morphism of $\mathcal{E}$, $P \in \text{Spec} T$ and $r \in R$.

\[
\begin{align*}
r + \psi_{(S,R)}(N(S)) &\in Q(h)^{-1}(\psi_{(T,M)}(P) / \psi_{(T,M)}(N(S))) \\
\iff h(r) &\in \psi_{(T,M)}(P) \\
\iff h(r)T &\subseteq P \\
\iff h(rs) &\subseteq P \\
\iff rS &\subseteq h^{-1}(P) \\
\iff r &\in \psi_{(S,R)}(h^{-1}(P)) \\
\iff r + \psi_{(S,R)}(N(S)) &\in \psi_{(S,R)}(h^{-1}(P) / \psi_{(S,R)}(N(S)))
\end{align*}
\]

Thus, $NC(h)(\lambda_{(T,M)}(P)) = \lambda_{(S,R)}((\text{Spec} \circ W)(h)(P))$. ☐

**Proposition 12.** Let $h : (S, R) \to (T, M)$ be a morphism of the category $\mathcal{E}$. The function $h$ can be restricted to a function from $\psi_{(S,R)}(N(S))$ to $\psi_{(T,M)}(N(T))$.

Proof. Consider $r \in \psi_{(S,R)}(N(S))$, that is, $rs \in N(S)$ for all $s \in S$. Given $t \in T$, there exists $s \in S$ such that $h(s) = t$. Thus, $h(r)t = h(r)h(s) = h(rs) \in h(N(S)) \subseteq N(T)$ so that $h(r) \in \psi_{(T,M)}(N(T))$. ☐

Let $\chi : \mathcal{E} \to \mathcal{CR}$ be the functor defined by:

$$\chi(h : (S, R) \to (T, M)) = h |_{\psi_{(S,R)}(N(S))} : \psi_{(S,R)}(N(S)) \to \psi_{(T,M)}(N(T))$$

and for the object $(S, R)$ of $\mathcal{E}$, denote $j_{(S,R)}$ the natural inclusion of $\psi_{(S,R)}(N(S))$ into the ring $R$.

Notice that for each $h : (S, R) \to (T, M)$, morphism of the category $\mathcal{E}$, the square in the following figure is commutative; this allows us to state the following result.

| $\mathcal{E}$ | $\mathcal{CR}$ |
|---------------|----------------|
| $(S,R)$       | $\psi_{(S,R)}(N(S)) \xrightarrow{j_{(S,R)}} V(S,R) = R$ |
| $h$           | $\chi(h)$      | $V(h)$ |
| $(T,M)$       | $\psi_{(T,M)}(N(T)) \xrightarrow{j_{(T,M)}} V(T,M) = M$ |

**Proposition 13.** $j = (j_{(S,R)})_{(S,R) \in \text{Ob}\mathcal{E}}$ is a natural transformation from the functor $\chi$ to the functor $V$. 6
4 First variation

Fix a ring $S$. Let $\mathcal{E}(S)$ be the subcategory of $\mathcal{CR}_1$ whose objects are the i-extensions of $S$ and whose morphisms are those that can be restricted to the identity of $S$. It is clear that $\mathcal{E}(S)$ can be identified with a subcategory of $\mathcal{E}$ and therefore the functor $Q$ can be restricted to a functor $Q_S : \mathcal{E}(S) \rightarrow \mathcal{CR}_1$ and the functor $\text{NC}$ can be restricted to a functor $\text{NC}_S : \mathcal{E}(S) \rightarrow S$.

**Proposition 14.** If $h : R \rightarrow T$ is a morphism of $\mathcal{E}(S)$ then $Q_S(h) : Q(S,R) \rightarrow Q(S,T)$ is injective.

**Proof.**

$$r + \psi_{(S,R)}(N(S)) \in \ker Q_S(h) \iff h(r) \in \psi_{(S,T)}(N(S))$$

$$\iff h(r)s \in N(S) \text{ for all } s \in S$$

$$\iff h(rs) \in N(S) \text{ for all } s \in S$$

$$\iff rs \in N(S) \text{ for all } s \in S$$

$$\iff r \in \psi_{(S,R)}(N(S))$$

$$\iff r + \psi_{(S,R)}(N(S)) = 0$$

**Corollary 15.** If $h : R \rightarrow T$ is a surjective morphism of $\mathcal{E}(S)$ then $Q_S(h) : Q(S,R) \rightarrow Q(S,T)$ is an isomorphism and therefore $\text{NC}_S(h) : \text{NC}_S (T) \rightarrow \text{NC}_S (R)$ is a homeomorphism.

**Corollary 16.** If $h : R \rightarrow T$ is a morphism of $\mathcal{E}(S)$ then $\text{NC}_S(h)(\text{NC}_S(T))$ is dense in $\text{NC}_S (R)$.

**Proof.** It is consequence of the injectivity of $Q_S(h) : Q_S (R) \rightarrow Q_S(T)$ (see [5]).

**Proposition 17.** If $h : R \rightarrow T$ is a morphism of $\mathcal{E}(S)$, then $\text{NC}_S(h) \circ \lambda_{(S,T)} = \lambda_{(S,R)}$. 

\[ \begin{array}{c}
\text{Spec}(S) \\
\downarrow \lambda_{(S,R)} \\
\text{NC}_S(R) \\
\downarrow \lambda_{(S,T)} \\
\text{NC}_S(T) \\
\end{array} \]

\[ \begin{array}{c}
\text{NC}_S(h) \\
\end{array} \]
Proof. It is a direct consequence of the Proposition □

Corollary 18. If \( h : R \to T \) is a morphism of \( \mathcal{E}(S) \) then Spec\( S \) is a subspace of NC\( S(h) \) (NC\( S(T) \)).

5 Second variation

Given a ring \( S \), we denote \( U_0(S) \) the set \( S \times \mathbb{Z} \) endowed with the operations:

\[
(s, \alpha) + (t, \beta) = (s + t, \alpha + \beta) \\
(s, \alpha)(t, \beta) = (st + \beta s + \alpha t, \alpha \beta).
\]

It is well known that \( U_0(S) \) is a unitary ring and that, if we identify \( S \) with \( S_0 = S \times \{0\} \), \( S \) is an ideal of \( U_0(S) \). Besides we have the following universal property:

Theorem 19. Let \( S \) be a ring and consider \( \iota_S : S \to U_0(S) : s \mapsto (s, 0) \). If \( R \) is a unitary ring and \( g : S \to R \) is a homomorphism, then there exists a unique homomorphism of unitary rings \( \tilde{g} : U_0(S) \to R \) such that \( \tilde{g} \circ \iota_S = g \).

This property allows us to extend \( U_0 \) to a functor from the category \( \mathcal{CR} \) of commutative rings to the category \( \mathcal{CR}_1 \) of unitary rings, defining \( U_0(h) = \tilde{g_T} \circ h \), for a homomorphism \( h : S \to T \). We have also that \( \iota = (\iota_S)_{S \in \mathcal{CR}} \) is a natural transformation from the identity functor of the category \( \mathcal{CR} \) to the functor \( U_0 \) considered as endo-functor of \( \mathcal{CR} \).

Notice that if \( h : S \to T \) is a surjective homomorphism then \( U_0(h) \) is a morphism in the category \( \mathcal{E} \) from the object \( (S, U_0(S)) \) to the object \( (T, U_0(T)) \). Thus, \( U_0 \) can be seen as a functor from the category \( \mathcal{CR}^s \) of commutative rings and surjective homomorphisms to the category \( \mathcal{E} \).

The universal property of \( U_0 \) allows us to conclude immediately the following theorem:

Theorem 20. For each ring \( S \), \( U_0(S) \) is an initial object of the category \( \mathcal{E}(S) \).
\textbf{Proof.} Let $R$ be an object of $\mathcal{E}(S)$ and let $u : S \to R$ be the inclusion homomorphism. The universal property guarantees that there is a unique homomorphism of unitary rings $u_R : U_0(S) \to R$ such that $u_R \circ \iota_S = u$, that is, $u_R$ restricted to $S$ is the identity. \hfill \Box

As the functor $\text{NC}_S$ is injective in objects, it is clear that the image of $\text{NC}_S$ is a subcategory of $\mathcal{S}$, that we denote $\mathcal{NC}(S)$ (category of nilcompactifications of $S$). We denote $\text{NC}_0$ the functor $\text{NC} \circ U_0 : \mathcal{CR}^s \to \mathcal{S}$. Thus, we have the following result:

\textbf{Corollary 21.} For each ring $S$, $\text{NC}_0(S)$ is a final object of the category $\mathcal{NC}(S)$.

\textbf{Remark 22.} As a consequence of Corollary 15 we have that $\text{NC}_0(S)$ is homeomorphic to $\text{NC}_0(S/N(S))$. Therefore we can reduce the study of nilcompactifications to semi-prime or reduced rings.

\section{Third variation}

In this section we are working again with a fixed ring $S$. Let $R$ be an object of $\mathcal{E}(S)$ and consider the unique morphism $u_R : U_0(S) \to R$ of $\mathcal{E}(S)$. Then $\text{NC}_S(u_R) : \text{NC}_S(R) \to \text{NC}_0(S)$ is a (strongly) continuous function. We denote $\eta(R)$ the image of the function $\text{NC}_S(u_R)$.

\textbf{Theorem 23.} If $R$ is an object of $\mathcal{E}(S)$ then $\eta(R)$ is an $A$ class compactification of $\text{Spec}S$ and besides it is dense in $\text{NC}_0(S)$.

\textbf{Proof.} It is enough to observe that $\text{Spec}S \subseteq \eta(R) \subseteq \text{NC}_0(S)$ and that $\text{Spec}S$ is an open dense subspace of $\text{NC}_0(S)$. \hfill \Box

We consider the pre-order between compactifications given in [10]: let $(X', \tau')$ and $(X'', \tau'')$ be compactifications of $(X, \tau)$, with immersions $f' : (X, \tau) \to (X', \tau')$ and $f'' : (X, \tau) \to (X'', \tau'')$. It is said that $(X', \tau') \preceq (X'', \tau'')$ if there is an immersion $g : (X'', \tau'') \to (X', \tau')$ such that $f' \circ g = f''$. We denote this relation by $\preceq$. If $(X', \tau') \preceq (X'', \tau'')$ then $\eta(X', \tau') \subseteq \eta(X'', \tau'')$. If $(X', \tau') \preceq (X'', \tau'')$ and $(X'', \tau'') \preceq (X', \tau')$ then $(X', \tau') \equiv (X'', \tau'')$.
(X'', τ'') if there exists h : (X'', τ'') → (X', τ') continuous and surjective, such that f' = h ∘ f''. We obtain directly the following result:

**Proposition 24.** If R is an object of 𝒟(S) then η(R) is a compactification of SpecS smaller than NC_S(R).

**Corollary 25.** If NC_0(S) is a Hausdorff space then, it is the smallest nilcompactification of SpecS.

**Proof.** If R is an object of 𝒟(S) then η(R) is a compact subset of NC_0(S), then it is closed. Besides, η(R) is dense in NC_0(S), therefore NC_S(u_R) is surjective.

The proof of the following proposition is a simple routinary exercise:

**Proposition 26.** If h : R → T is a morphism of the category 𝒟(S) then η(T) ⊆ η(R).

If for each morphism h : R → T of the category 𝒟(S) we define η(h) : η(T) → η(R) : x ↦ x then, η is a functor from the category 𝒟(S) to the category Top of topological spaces and continuous functions.

We have then the following natural question: if R is an object of 𝒟(S), is η(R) a spectral space?

## 7 Two examples

In this section we present two examples which illustrate some results of this paper.

**Example 27.** Let B be a Boolean ring without identity. As B is semiprime, N(B) = 0. We are going to find ψ_{B,U_0(B)}(0) :

\[(a, α) ∈ ψ_{B,U_0(B)}(0) \text{ if and only if } ax + αx = 0 \text{ for each } x ∈ B, \text{ that is, } ax = αx \text{ for each } x ∈ B.\]

If α is even then ax = 0 for all x ∈ B, from which it follows that a = 0.

If α is odd then ax = x for all x ∈ B and therefore, a is the identity of B, which is absurd.

Hence, ψ_{B,U_0(B)}(0) = \{0\} × 2Z and Q(B, U_0(B)) = U_0(B) / \{0\} × 2Z.

In this case, NC(B, U_0(B)) is precisely the Alexandroff compactification of SpecB (see [1]).

The following example shows that different i-extensions of a ring can produce different nilcompactifications:
Example 28. Consider the ring without identity $S = x\mathbb{R} [x]$. Two different unitary $i$-extensions of $S$ are $U_0 (S)$ and $\mathbb{R} [x]$. Notice that the unique homomorphism from $U_0 (S)$ to $\mathbb{R} [x]$ is not surjective and there not exists a homomorphism from $\mathbb{R} [x]$ to $U_0 (S)$ . It is easy to see that $\psi_{(S,U_0 (S))} (0) = 0$ and $\psi_{(S,\mathbb{R} [x])} (0) = 0$. Therefore, $Q (S, U_0 (S)) = U_0 (S)$ and $Q (S, \mathbb{R} [x]) = \mathbb{R} [x]$, thus $NC (S, U_0 (S)) = \text{Spec} (U_0 (S))$ and $NC (S, \mathbb{R} [x]) = \text{Spec} \mathbb{R} [x]$. If we consider the surjective homomorphisms

$$
\pi : U_0 (S) \to \mathbb{Z} : (s, z) \mapsto z, \\
\beta : \mathbb{R} [x] \to \mathbb{R} : p (x) \mapsto p (0),
$$

we conclude, by the Correspondence Theorem, that $NC (S, U_0 (S))$ is a compactification of $\text{Spec} S$ by enumerable points, while $NC (S, \mathbb{R} [x])$ is a compactification of $\text{Spec} S$ by one point.

The following diagram summarizes the ideas presented in this paper.

[Diagram]

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The diagram includes arrows and labels indicating the relationships between various mathematical objects, such as $\mathcal{E}(S)$, $\text{Spec}$, $\text{NC}_S$, and $\text{CR}_1$. It illustrates the mappings and transformations described in the text, including natural transformations and adjoint functors. The diagram visually represents the compactifications and homomorphisms discussed in the example.
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