Partial difference equations over compact Abelian groups, I: modules of solutions

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Abstract

Consider a compact Abelian group $Z$ and closed subgroups $U_1, \ldots, U_k \leq Z$. Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. This paper examines two kinds of functional equation for measurable functions $Z \rightarrow \mathbb{T}$.

First, given $f : Z \rightarrow \mathbb{T}$ and $w \in Z$, the resulting differenced function is

$$d_w f(z) := f(z - w) - f(z).$$

In this notation, we shall study solutions to the system of difference equations

$$d_{u_1} \cdots d_{u_k} f \equiv 0 \quad \forall u_1 \in U_1, u_2 \in U_2, \ldots u_k \in U_k.$$

These solutions form a subgroup of the group $\mathcal{F}(Z)$ of all measurable functions $Z \rightarrow \mathbb{T}$ (modulo Haar-a.e. equality). The subgroup of solutions is closed under convergence in probability and is globally invariant under rotations of $Z$, so it is a complete metrizable Abelian $Z$-module. We will give a recursive description of the structure of this $Z$-module in terms of the solution-modules of lower-degree equations of the same type.

Secondly, we study tuples of measurable functions $f_i : Z \rightarrow \mathbb{T}$ such that $f_i$ is invariant under translation by $U_i$ and also

$$f_1 + \cdots + f_k = 0.$$

Once again, we find that the space of such tuples comprises a complete metrizable $Z$-module admitting a simple recursive description in terms of the solutions to simpler such problems.

Loosely, these results are obtained from an abstract theory of a special class of $Z$-modules, assembled out of modules of functions on $Z$ and related

*Research supported by a fellowship from the Clay Mathematics Institute
groups. Most of our work will go into showing that this class of modules is closed under various natural operations. Knowing that, the above descriptions follow as easy consequences.

The motivation for this work is the problem of setting up a higher-dimensional analog of the inverse theory for the Gowers uniformity norms. In the setting of the higher-dimensional Szemerédi Theorem, one can generalize the definition of Gowers norms to a ‘directional’ notion which gives the right kind of control over counting patterns. However, it is not known how to extend to higher dimensions the ‘inverse theory’ that has now been developed in one dimension. The partial difference equations studied here can be naturally seen as the ‘extremal’ version of this problem, concerning functions whose directional Gowers norms take the maximum possible value. As such, their description provides a modest first step towards a general inverse theory. In addition to solving this strict, algebraic version of the inverse problem, our methods also give some information about the slightly relaxed version of the problem, which asks about functions whose Gowers norm is not strictly maximal, but very close to it.
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1 Introduction

Let $Z$ be a compact Abelian group, let $U_1, \ldots, U_k \leq Z$ be closed subgroups, and let $A$ be an Abelian Lie group. Let $m_Z$ be the Haar probability measure on $Z$. This paper will study two kinds of functional equation for measurable functions $Z \to A$, specified in terms of the subgroups $U_i$.

First, given $f : Z \to A$ and an element $w \in Z$, we define the associated **differenced function** to be

$$d_w f(z) := f(z - w) - f(z).$$

(1)

This is the obvious discrete analog of a directional derivative. Given a subgroup $W \leq Z$, we will sometimes write $d^W f$ for the function

$$W \times Z \to A : (w, z) \mapsto d_w f(z).$$

The two classes of equation to be studied are the following.
• The partial difference equation, or PD\textsuperscript{ce}E, associated to the tuple \((U_1, \ldots, U_k)\) is the system

\[
d_{u_1} \cdots d_{u_k} f = 0 \quad \forall u_1 \in U_1, u_2 \in U_2, \ldots, u_k \in U_k
\]  

(2)

(since we quotient by functions that vanish a.e., this means formally that for strictly every \(u_1, \ldots, u_k\), the left-hand side is a function \(Z \to A\) that vanishes at almost every \(z\)).

• Suppose now that \(f_i : Z \to A, i = 1, \ldots, k\), are measurable functions such that \(f_i\) is \(U_i\)-invariant (that is, \(f(z - u) = f(z)\) for all \(u \in U_i\) and \(z \in Z\)) and such that

\[
f_1(z) + \cdots + f_k(z) = 0 \quad \text{for } m_Z\text{-a.e. } z.
\]  

(3)

A tuple \((f_i)_{i=1}^k\) satisfying (3) will be called a zero-sum tuple of functions, and the problem of describing such tuples will be called a zero-sum problem.

Henceforth we refer to \(Z\) as the ambient group for either of these problems, and to \(U = (U_i)_{i=1}^k\) as the tuple of acting subgroups.

The target Lie group of greatest interest is \(A = T\) (or sometimes \(A = S^1\), when there is reason to write the equations multiplicatively). Most of our concrete examples will have target either \(T\) or \(Z\).

Zero-sum problems are closely related to PD\textsuperscript{ce}Es. The most obvious connection is the following. If \(f\) satisfies the PD\textsuperscript{ce}E associated to \(U = (U_1, \ldots, U_k)\), then we may write this as

\[
\sum_{e \subseteq [k]} (-1)^{|e|} f \circ q_e = 0,
\]

where

\[
q_e : U_1 \times \cdots \times U_k \times Z \to Z : (u_1, \ldots, u_k, z) \mapsto z - \sum_{i \in e} u_i.
\]

This is a zero-sum problem associated to the family of \(2^k\) subgroups \(\ker q_e, e \subseteq [k]\), of \(U_1 \times \cdots \times U_k \times Z\).

On the other hand, if \((f_1, \ldots, f_k)\) is a solution of (3), then for each \(i = 2, 3, \ldots, k\) one has \(d_{u_i} f_i = 0\) for all \(u \in U_i\), and so applying several of these operators to (3) gives

\[
d_{u_2} \cdots d_{u_k} f_1 = 0 \quad \forall u_2 \in U_2, \ldots, u_k \in U_k.
\]

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Another relation between the zero-sum problem and a $\text{PD}^{\text{ce}}$E is the following. Given $U = (U_i)_{i=1}^k$, the most obvious solutions to the associated $\text{PD}^{\text{ce}}$E are the sums

$$
\sum_{i=1}^k f_i
$$

in which each $f_i$ is $U_i$-invariant. Therefore there is a natural sum map from tuples of $U_i$-invariant functions to $\text{PD}^{\text{ce}}$E-solutions for the same subgroups $U_i$. The zero-sum tuples are precisely the elements of the kernel of this map: that is, they describe the non-uniqueness of the representation in (4). We will return to such issues of uniqueness at length later.

This paper will show that $\text{PD}^{\text{ce}}$Es and zero-sum problems are also related in that their modules of solutions all fall into a more general class of $Z$-modules having some special structure. Before formulating those results, it is worth collecting some motivating examples.

### 1.1 Some concrete examples

The above observations allow us to pass quickly between examples of zero-sum tuples and $\text{PD}^{\text{ce}}$E-solutions. Often examples are easier to find in the former setting, although we shall also discuss some $\text{PD}^{\text{ce}}$Es here.

Many examples will be written in the form

$$
f_1(M_1(\theta_1, \ldots, \theta_d)) + \cdots + f_k(M_k(\theta_1, \ldots, \theta_d)) \equiv 0,
$$

where

- $(\theta_1, \ldots, \theta_d)$ is an argument in $\mathbb{T}^d$;
- each $M_i$ is an $(r_i \times d)$-integer matrix for some $r_i < d$, interpreted as a homomorphism $\mathbb{T}^d \to \mathbb{T}^{r_i}$;
- the functions $f_i : \mathbb{T}^{r_i} \to A$ are measurable.

This is equivalent to asserting that $(f_i \circ M_i)_{i=1}^d$ is a zero-sum tuple on $\mathbb{T}^d$ in which the $i$th function is $(\ker M_i)$-invariant.

**Example 1.1.** Let us begin with the important special case

$$U_1 = U_2 = \ldots = U_k = Z.$$

When $k = 1$, the solutions are precisely the constant functions, and when $k = 2$ they are precisely the affine functions.
If $\chi_1$, $\chi_2$ and $\chi_3$ are any three characters on $Z$ which sum to zero, then they are a zero-sum triple for the subgroups $U_i := \ker \chi_i$, $i = 1, 2, 3$. For example, on $Z = \mathbb{T}^2$, the equation

$$\theta_1 + \theta_2 + (-\theta_1 - \theta_2) = 0$$

is an example of this kind using the three characters

$$\chi_1(\theta_1, \theta_2) = \theta_1, \quad \chi_2(\theta_1, \theta_2) = \theta_2 \quad \text{and} \quad \chi_3(\theta_1, \theta_2) = -\theta_1 - \theta_2.$$ 

In general, the solutions to this PDE with $k$ copies of $Z$ behave like polynomials $Z \rightarrow A$ of degree at most $k - 1$.

If we allow $Z = \mathbb{Z}^d$ (ignoring here that it is non-compact), then an easy exercise that the solutions to this PDE with $k$ copies of $Z$ are precisely the polynomials of degree at most $k - 1$. On the other hand, if $Z = \mathbb{T}^d$ and $A = \mathbb{T}$, then a simple exercise shows that the solution-module stabilizes at $k = 2$: if $f : Z \rightarrow A$ is a function such that for some $k$, all $k^{th}$ differenced functions of $f$ are zero, then $f$ is actually affine.

However, in case $Z = \mathbb{F}_q^d$ and $A = \mathbb{T}$, a vector space over the finite field $\mathbb{F}_q$, there are some subtle differences between functions satisfying this PDE and the classical notion of a polynomial as a sum of monomials. This phenomenon is the subject of a detailed analysis by Tao and Ziegler in their work [36] on the inverse problem for the Gowers norms over $\mathbb{F}_q^d$. We will return to the connection between our work and Gowers norms a little later.

**Example 1.2.** Now suppose that $Z_1 := U_1 + \ldots + U_k$ is a proper subgroup of $Z$. In this case the PDE and zero-sum problem effectively reduce to those on the subgroup $Z_1$. Knowing the possible solutions on $Z_1$, one may then obtain solutions on $Z$ by making a measurable, but otherwise completely independent, selection of solutions on each coset of $Z_1$, and all solutions on $Z$ are clearly of this kind.

For example if $Z_1 := U_1 + U_2 + U_3$ is a proper subgroup of $Z$, then one may let $(\chi_i, \bar{z})_{i=1}^3$ be a measurable selection of such a zero-sum triple of characters in $\tilde{Z}_1$ indexed by $\bar{z} \in Z/Z_1$, and now obtain a zero-sum triple on $Z$ by letting $\sigma : Z/Z_1 \rightarrow Z$ be a measurable section and setting

$$f_i(z) := \chi_i, z + Z_1(z - \sigma(z + Z_1)).$$

This phenomenon will be of great importance in the sequel. It can be simplified by working on each coset of $U_1 + \ldots + U_k$ separately, which effectively allows us to assume that $Z = U_1 + \ldots + U_k$. However, this does not evade the phenomenon completely, because our description of PDE-solutions for the tuple $(U_1, \ldots, U_k)$ will be relative to the solutions for the simpler tuples obtained from this one by omitting some entry, such as $(U_2, \ldots, U_k)$. Since one may have $U_2 + \ldots + U_k \subseteq \mathbb{T}$
$U_1 + \ldots + U_k$, we will need this ‘measurable-selection’ picture for describing the solutions to those simpler equations.

**Example 1.3.** The simplest nontrivial example in which the $U_i$s are distinct is the PD$^{ce}$E on $\mathbb{T}^2$ associated to $U_1 := \mathbb{T} \times \{0\}$ and $U_2 := \{0\} \times \mathbb{T}$. In this case $f$ is a solution if and only if

$$d_{(u,0)}d_{(0,v)}f(x, y) = f(x + u, y + v) - f(x, y + v) - f(x + u, y) + f(x, y) = 0$$

almost surely. Changing variables from $(z, y, u, v)$ to $(z, y', x', y')$ with $x' = x + u$ and $y' := y + v$, this becomes

$$f(x', y') - f(x, y') - f(x', y) + f(x, y) = 0$$

almost surely,

and this now re-arranges to

$$f(x, y) := (f(x, y') - f(x', y')) + f(x', y).$$

By Fubini’s Theorem, a.e. choice of $(x', y')$ is such that this re-arranged equation holds for a.e. $(x, y)$, so fixing such a choice of $(x', y')$, the right-hand side is manifestly a sum of a $U_1$-invariant function (i.e., depending only on $y$) and a $U_2$-invariant function (depending only on $x$).

For the PD$^{ce}$E associated to a general tuple $(U_i)_{i=1}^k$, the most obvious solutions are the generalization of the above: functions of the form $\sum_{i=1}^k f_i$ in which each $f_i$ is $U_i$-invariant.

Corresponding to this, the most obvious solutions to the zero-sum problem are those of the form

$$(0, \ldots, 0, f, 0, \ldots, 0, -f, 0, \ldots, 0),$$

where $f : Z \rightarrow A$ is invariant under $(U_i + U_j)$ for some $i < j$, and the non-zeros in this tuple are in the $i$th and $j$th positions, has zero sum. Further examples may then be obtained as sums of these for different pairs $(i, j)$.

The above change-of-variables trick may be performed whenever the subgroups $U_1, \ldots, U_k$ are **linearly independent**, meaning that for $(u_i)_{i=1}^k \in \prod_{i=1}^k U_i$ one has

$$\sum_{i} u_i = 0 \implies u_1 = u_2 = \ldots = u_k = 0.$$ 

In a sense, this is the extreme opposite of the case considered in Example 1.1. The change-of-variables leads to a simple solution of the PD$^{ce}$E in the linearly independent case: see Subsection 14.1 where it is shown that the ‘obvious’ solutions are the only ones.

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The above examples already give a large supply of \( \text{PD}^{\infty}\text{E}\)-solutions, and we can of course produce more examples by adding these together. They are all still rather simple, characterized by being ‘polynomial’ (perhaps actually invariant) on the cosets of some relevant subgroup. However, there is worse to come.

Let \( \lfloor \cdot \rfloor \) be the integer-part function on \( \mathbb{R} \), and for \( \theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z} \) let \( \{\theta\} \) be its unique representative in \([0, 1)\). A little abusively, we will call \( \{\cdot\} \) the ‘fractional-part map’.

**Example 1.4.** Define \( f : \mathbb{T} \times (\mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{T} \) by

\[
f(t, n) = \begin{cases} \frac{1}{2}\{t\} \mod 1 & \text{if } n = 0 \\ -\frac{1}{2}\{t\} \mod 1 & \text{if } n = 1 \end{cases}
\]

Let \( U_1 := U_2 := \mathbb{T} \times \{0\} \) and \( U_3 := \{0\} \times (\mathbb{Z}/2\mathbb{Z}) \). One computes easily that

\[
d_{(0,1)}f(t, n) = (-1)^{n-1}\{t\} \mod 1 = (-1)^{n-1}t,
\]

and hence that

\[d_{U_1}d_{U_2}d_{U_3}f = 0.\]

This \( f \) is a square-root of a character on \( \mathbb{T} \times (\mathbb{Z}/2\mathbb{Z}) \), but it is not a character, nor does its restriction agree with a character on any nonempty open subset of \( \mathbb{Z} \). It is also not invariant under any nontrivial subgroup of \( \mathbb{T} \times (\mathbb{Z}/2\mathbb{Z}) \).

**Example 1.5.** The simplest example lying between the ‘polynomial’ and linearly-independent cases is the \( \text{PD}^{\infty}\text{E} \)

\[d_{(u,0)}d_{(0,v)}d_{(w,w)}f(x, y) = 0 \tag{5}\]

for functions \( f : \mathbb{Z} \rightarrow A \). The relevant subgroups here are \( U_1 \) and \( U_2 \) as in the previous example and

\[U_3 = \{(w, w) \ | \ w \in \mathbb{T}\}.
\]

In this case, there is no simple change-of-variables from which one may read off the structure of \( f \), because the \( U_1 \)s are linearly dependent: of course, \( (w, w) = (w, 0) + (0, w) \).

When we return to this example in Subsection [4.2] it will illustrate several of the methods introduced on route. It corresponds to the first unresolved case of the higher-dimensional Gowers-norm inverse problem, to be described shortly. One indication of its delicacy is that the answer depends on the target group \( A \).

If \( A = \mathbb{T} \), then the only solutions to our \( \text{PD}^{\infty}\text{E} \) are sums of functions invariant under one of the \( U_1 \)s, as in the previous example.

However, if \( A = \mathbb{Z} \) then one finds a new solution:

\[f(x, y) := \{x\} + \{-y\}.
\]
To verify this, observe that among \( \mathbb{R} \)-valued functions we may write

\[
f(x, y) = \{x\} + \{-y\} - \{x - y\},
\]

which is a sum of pieces that are individually invariant under the subgroups \( U_1, U_2 \) and \( U_3 \). This illustrates that distinction between ‘trivial’ and ‘non-trivial’ solutions can change according to the choice of target group. (We will also see related, non-trivial \( \mathbb{T} \)-valued solutions later, which cannot be so easily trivialized.)

The function \( f \) not only solves the above PDCE, but it actually satisfies the equation

\[
f(x, y) - f(z, y + z) + f(z + y, z) - f(y, z) = 0,
\]

from which the above PDCE may be obtained by repeated differencing, as explained earlier. This equation has an important life of its own. It is the equation for a 2-cocycle in the inhomogeneous bar resolution of Moore’s measurable group cohomology. Moreover, standard calculations in that theory (as, for example, in [2]) show that this \( f \) is a generator for \( H^3_m(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z} \). Since, on the other hand, one can show that any solution to our simpler sub-equations would have to define a coboundary in the present setting, it follows that this function \( \sigma \) is not a sum of examples of those simpler kinds. It represents a new kind of solution that our theory must also be able to account for.

These facts will be proved in Section [14]. When \( A = \mathbb{T} \), even the absence of non-obvious solutions when \( A = \mathbb{T} \) seems to rely on the cohomological calculation \( H^2_m(\mathbb{T}, \mathbb{T}) = 0 \), which rules out a \( \mathbb{T} \)-valued cohomological example analogous to the above. This vanishing, in turn, requires some modestly heavy machinery: I do not know of an elementary proof. \( \triangleright \)

**Example 1.6.** Define \( \sigma : \mathbb{T}^3 \rightarrow \mathbb{T} \) by

\[
\sigma(\theta_1, \theta_2, \theta_3) := (\{\theta_1\} + \{\theta_2\}) \cdot \theta_3.
\]

Then one can verify directly that

\[
\sigma(\theta_1, \theta_2, \theta_3) - \sigma(\theta_1, \theta_2, \theta_3 + \theta_4) + \sigma(\theta_1, \theta_2 + \theta_3, \theta_4) - \sigma(\theta_1 + \theta_2, \theta_3, \theta_4) + \sigma(\theta_2, \theta_3, \theta_3) = 0 \quad \forall \theta_1, \theta_2, \theta_3, \theta_4. \quad (6)
\]

Now let \( Z := \mathbb{T}^4 \), and for \( i = 1, \ldots, 5 \) let \( \sigma_i(\theta_1, \theta_2, \theta_3, \theta_4) \) be the \( i \)th function appearing in the above alternating sum. Then the functions \( \sigma_i \) are respectively invariant under the following one-dimensional subgroups of \( Z \):

\[
\begin{align*}
U_1 &:= (0, 0, 0, 1) \cdot \mathbb{T}, & U_2 &:= (0, 0, -1, 1) \cdot \mathbb{T}, \\
U_3 &:= (0, -1, 1, 0) \cdot \mathbb{T}, & U_4 &:= (-1, 1, 0, 0) \cdot \mathbb{T}, \\
U_5 &:= (1, 0, 0, 0) \cdot \mathbb{T}.
\end{align*}
\]
Here we interpret each of these 4-vectors as a \((1 \times 4)\)-matrix, so the above notation means that
\[
U_1 = \{(0, 0, \theta) \mid \theta \in \mathbb{T}\},
\]
and similarly.

This is also an example of cohomological origin. This time, \(\sigma\) is a 3-cocycle in the inhomogeneous bar resolution for \(H^3_{\text{inh}}(\mathbb{T}, \mathbb{T}) \cong \mathbb{Z}\), and it turns out to be a generator for that cohomology group. As before, this will imply that it cannot be decomposed into a sum of solutions to simpler equations.

Thus, this kind of example provides a link to group cohomology. Similarly, any non-trivial generator element of \(H^p_{\text{inh}}(\mathbb{T}, \mathbb{T})\) (which is nonzero when \(p\) is odd \([2]\)) gives a zero-sum tuple with \(p + 2\) elements. In Subsection 14.2 we will fit these into a general class, and show that they never decompose into solutions of simpler equations.

A selection of more complicated examples will be offered in Subsection 14.3.

**Remark.** It is worth noting that the above examples, and also those to come in Subsection 14.3 all take the form of ‘step-polynomials’. In case \(Z = \mathbb{T}^d\), these are functions obtained by first imposing a ‘coordinate-system \(\mathbb{T}^d \to [0, 1]^d\) using the fractional-part map; then decomposing \(\mathbb{T}^d\) into regions according to various linear inequalities among these fractional parts; and finally taking a different polynomial in those fractional parts on each of the pieces.

This is not at all a coincidence. It turns out that solutions to PD\(ce\)s and zero-sum problems can always be decomposed, in a certain sense, into ‘basic solutions’ that are functions of this ‘semi-algebraic’ kind. This feature will be the subject of a future paper.

### 1.2 Modules of solutions

The examples above and in Subsection 14.3 exhibit considerable variety as individual functions. However, it will turn out that the global structure of the solution-modules admits a relatively simple ‘recursive’ description. In proving this, we will see that group cohomology is not just a source of examples: it will be the key tool for teasing this structure apart. The main structural results will be formulated next.

First recall that for any compact Abelian \(Z\), any Borel function from \(Z\) to a separable metric space must be lifted from some separable quotient of \(Z\). For our problems of interest, we will therefore lose no generality if we assume that \(Z\) itself is separable (or, equivalently, metrizable). This assumption is to be understood throughout the rest of the paper, and will usually not be remarked explicitly.

Let \(\mathcal{F}(Z, A)\) denote the space of measurable functions \(f : Z \to A\) modulo agreement \(m_Z\)-a.e., equipped with the topology of convergence in probability. This
becomes a topological \( Z \)-module when \( Z \) acts by translation: for \( w \in Z \), we denote the translation operator by

\[
R_w : \mathcal{F}(Z, A) \rightarrow \mathcal{F}(Z, A), \quad R_w f(z) := f(z - w).
\]

Because we are assuming \( Z \) is separable, the topology of \( \mathcal{F}(Z, A) \) is Polish: that is, it can be generated by a complete, separable, translation-invariant metric. This is the key point at which we have used the separability of \( Z \): without it, the function space \( \mathcal{F}(Z, A) \) would also not be separable, and so would fall outside the domain of some tools that we will need later.

We will usually abbreviate \( \mathcal{F}(Z, \mathbb{T}) = : \mathcal{F}(Z) \).

If \( d \) is a complete group metric on \( A \), than a suitable choice of metric on \( \mathcal{F}(Z, A) \) is offered by the conventional metric describing convergence in probability:

\[
d_0(f, g) := \inf \{ \varepsilon > 0 \mid m_Z \{ z \in Z \mid d(f(z), g(z)) > \varepsilon \} < \varepsilon \}.
\]

When we need an explicit metric on \( \mathcal{F}(Z, A) \) in the sequel, it will always be of this kind. In case \( A = \mathbb{T} \), we will usually use the metric \(| \cdot |\) inherited from the Euclidean distance on \( \mathbb{R} \).

Given \( k \in \mathbb{N} \), we will write \( [k] := \{1, 2, \ldots, k\} \).

Now let \( Z \) and its subgroups \( U_1, \ldots, U_k \) be as before. Given also a subset \( e \subseteq [k] \), we will always set

\[
U_e := \sum_{i \in e} U_i,
\]

where this is understood to be \( \{0\} \) in case \( e = \emptyset \).

Consider the PD\textsuperscript{ce} \( E \) associated to \( U \) for \( A \)-valued measurable functions. If one knows how to solve this PD\textsuperscript{ce} \( E \) in case the ambient group is \( U_{[k]} \), then in the general case one may simply make an independent (measurable) selection of solutions on every coset of \( U_{[k]} \) in \( Z \). Therefore the description of solutions in general reduces to the description of solutions in case \( Z = U_{[k]} \).

Now, for any \( e = \{i_1, \ldots, i_\ell\} \subseteq \{1, 2, \ldots, k\} \), let

\[
M_e := \{ f \in \mathcal{F}(Z, A) \mid d_{u_i} \cdots d_{u_{i_\ell}} f \equiv 0 \forall u_i \in U_{i_1}, u_{i_2} \in U_{i_2}, \ldots, u_{i_\ell} \in U_{i_\ell} \}.
\]

This is a family of closed \( Z \)-submodules of \( \mathcal{F}(Z, A) \), and they clearly satisfy

\[
a \subseteq e \quad \implies \quad M_a \subseteq M_e
\]

and \( M_\emptyset = \{0\} \). The largest module, \( M_{[k]} \), consists of the solutions to our PD\textsuperscript{ce} \( E \). Within it, we will sometimes refer to the elements of \( \sum_{e \subseteq [k]} M_e \) as the submodule
of degenerate solutions: these are sums of solutions to the different nontrivial simplifications of our PD\textsuperscript{ce}E. In many of the examples above, the function exhibited is interesting because it is not degenerate, as will be proved later.

However, it turns out that, in a certain sense, the full solution-module $M[k]$ cannot be too much larger than the submodule of degenerate solutions. In case $Z = U[k]$, we will find that $\sum_{e \in [k]} M_e$ is relatively open-and-closed inside $M[k]$. This will means that there are only countably many classes of solutions modulo the degenerate solutions, and that these classes are all separated by at least some positive $d_0$-distance $\varepsilon$. In case $Z \supseteq U[k]$, this picture still obtains on every coset of $U[k]$.

In order to prove this, we will also need a more complete structural picture of the module of degenerate solutions themselves. The module of degenerate solutions is the image of the homomorphism

$$\bigoplus_{i=1}^k M[k]_{\setminus i} \rightarrow M[k]$$

given by the obvious sum of inclusions. We will want to describe the module of degenerate solutions in terms of the individual modules $M[k]_{\setminus i}$, expecting that these have already been ‘understood’ in the course of an induction on $k$. However, to use this idea one must also describe the kernel of the above sum over inclusions: that is, describe the possible non-uniqueness in the representation of a degenerate solution as a sum of solutions to simpler equations.

Now one notices that this kernel has its own ‘degenerate’ elements. If $f \in M[k]_{\setminus \{i,j\}}$ for some $i < j$, then from this one obtains the zero-sum tuple

$$\begin{pmatrix} 0, 0, \ldots, 0, f, 0, \ldots, 0, -f, 0, \ldots, 0 \end{pmatrix} \in \bigoplus_{i=1}^k M[k]_{\setminus i},$$

where the nonzero entries are in positions $i$ and $j$ (similarly to Example 1.3 above). One may now add together such examples for different pairs $\{i, j\}$ to produce further examples.

Similarly to the situation with degenerate solutions to the original PD\textsuperscript{ce}E, we will find that sums of these degenerate examples are ‘most’ of the possible zero-sum tuples in $\bigoplus_{i=1}^k M[k]_{\setminus i}$.

These sums of degenerate examples are the image of a homomorphism

$$\bigoplus_{i<j} M[k]_{\setminus \{i,j\}} \rightarrow \bigoplus_{i=1}^k M[k]_{\setminus i},$$

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In order to analyze the degenerate examples of zero-sum tuples, we will now need to describe the kernel of this homomorphism.

Overall, one sees the emergence of the following structure. For each \( \ell \in \{1, \ldots, k-1\} \), there is a natural map

\[ \partial_{\ell+1} : \bigoplus_{a \in \binom{k}{\ell}} M_a \rightarrow \bigoplus_{e \in \binom{k}{\ell+1}} M_e \]

defined by requiring that

\[ (\partial_{\ell}((m_a)_a))_{\{i_1 \ldots < i_{\ell+1}\}} = \sum_{j=1}^{\ell+1} (-1)^{j-1} m_{\{i_1 \ldots < i_{\ell+1}\}\setminus \{i_j\}} \]

for each \( e = \{i_1 < \ldots < i_{\ell+1}\} \subseteq [k] \) and each

\[ (m_a)_a \in \bigoplus_{a \in \binom{k}{\ell}} M_a. \]

This is obviously a relative of the definition of boundary maps in simplicial cohomology, and just as in that theory one computes easily that \( \partial_{\ell+1} \partial_{\ell} = 0 \) for every \( \ell \). We have therefore constructed a complex of \( Z \)-modules

\[ 0 \rightarrow \bigoplus_{i=1}^k M_i \xrightarrow{\partial_2} \bigoplus_{i < j} M_{ij} \xrightarrow{\partial_3} \cdots \xrightarrow{\partial_{k-1}} \bigoplus_{|e|=k-1} M_e \xrightarrow{\partial_k} M_{[k]} \xrightarrow{\partial_{k+1}} 0 \]

(where of course \( \partial_1 \) and \( \partial_{k+1} \) are both the zero homomorphism).

In terms of this picture, we can finally formulate our main theorem relating \( M_{[k]} \) to the modules \( M_e \) for \( e \subseteq [k] \).

**Theorem A** Fix \( k \geq 2 \) and let \( (M_e)_{e \subseteq [k]} \) be as above. If \( Z = U_{[k]} \), then there is some \( \varepsilon > 0 \) such that for all \( \ell \in \{2, \ldots, k+1\} \), \( \text{img } \partial_{\ell-1} \) is relatively open-and-closed in \( \ker \partial_{\ell} \), with any distinct cosets of this submodule separated by at least \( \varepsilon \) in the metric \( d_0 \).

If \( Z \geq U_{[k]} \), the structure described above obtains upon restricting to any individual coset of \( U_{[k]} \).

The key to this theorem will be an abstract class of families of modules \( (M_e)_{e} \) for which the complex above admits such a structural description.

A similar structure obtains in the case of zero-sum tuples. To describe this, now for each \( e \subseteq [k] \) let

\[ N_e := \left\{ (f_i)_{i=1}^k \in \bigoplus_{i=1}^k F(Z, A)^{U_i} \left| \sum_{i=1}^k f_i = 0 \text{ and } f_i = 0 \ \forall i \in [k] \setminus e \right. \right\}. \]
Once again, \( N_a \leq N_e \) whenever \( a \subseteq e \), and this time \( N_a = \{0\} \) whenever \(|a| \leq 1\).

Now the same definition as above gives homomorphisms

\[
\partial_{\ell+1} : \bigoplus_{|a|=\ell} N_a \rightarrow \bigoplus_{|e|=\ell+1} N_e \text{ for each } \ell \in \{2, \ldots, k-1\}
\]

which fit into the complex

\[
0 \xrightarrow{\partial_2=0} \bigoplus_{i<j} N_{ij} \xrightarrow{\partial_3} \bigoplus_{|a|=3} N_a \xrightarrow{\partial_4} \cdots \xrightarrow{\partial_{k-1}} \bigoplus_{|e|=k-1} N_e \xrightarrow{\partial_k+N_{[k]}} \partial_{k+1}=0 \rightarrow 0.
\]

Zero-sum tuples enjoy the following analog of Theorem A.

**Theorem B** If \( Z = U_{[k]} \), then there is some \( \varepsilon > 0 \) such that in the above complex, for all \( \ell \in \{3, \ldots, k+1\} \), \( \text{im} \partial_{\ell-1} \) is relatively open-and-closed in \( \ker \partial_\ell \), with any distinct cosets of this submodule separated by at least \( \varepsilon \) in the metric \( d_0 \).

If \( Z \geq U_{[k]} \), the structure described above obtains upon restricting to any individual coset of \( U_{[k]} \).

In addition, we will prove the following.

**Theorem A'** (resp. B') **In Theorem A** (resp. B), there is a choice of \( \varepsilon > 0 \) that depends only on \( k \), not on \( Z \) or \( U \).

We shall first prove versions of Theorems A and B in which \( \varepsilon \) may also depend on the data \( Z \) and \( U \), but then show that this dependence can be removed by a separate compactness argument. Owing to this use of compactness, we will not obtain an explicit estimate for \( \varepsilon \) in terms of \( k \), although presumably one could be extracted by making the intermediate steps of the earlier proofs quantitative.

Let us next give an informal sketch of our approach to Theorem A. The approach to Theorem B will run along similar lines.

The key observation is that if \( f \in M_{[k]} \) then \( d_uf \in M_{[k-1]} \) for every \( u \in U_k \), and similarly under differencing by elements of the other \( U_i \). Thus, if one has already obtained enough information about the elements of \( M_{[k-1]} \) as part of some inductive hypothesis, one can try to obtain from this a description of the function

\[
U_k \rightarrow M_{[k-1]}: u \mapsto d_uf,
\]

and hence recover something of the structure of \( f \).

To execute this strategy, we will need some description of all maps \( U_k \rightarrow M_{[k-1]} \) that could arise in this way, before knowing the structure of \( M_{[k]} \). The key piece of structure that makes this possible is the relation

\[
d_{u+u'}f(z) = d_uf(z) + d_{u'}f(z + u).
\]
This follows immediately from the definition of \( d_u \). In the terminology of group cohomology, it asserts that the function

\[
    u \mapsto d_u f : U_k \rightarrow M_{[k-1]}
\]

is a 1-cocycle. The machinery of group cohomology (specifically, of the measurable version of group cohomology developed for locally compact groups and Polish modules by Calvin Moore \cite{Moore1977, Moore1976, Moore1975}) now makes it possible to describe the space of these 1-cocycles, provided one knows enough about the structure of the module \( M_{[k-1]} \). This will involve the whole of the complex appearing in Theorem A.

To bring this idea to fruition, we will define a quite abstract class of families of \( \mathbb{Z} \)-modules, and then show that it is closed under several natural operations, such as forming kernels, quotients, extensions and cohomology. The members of this class are ‘almost modest \( \Delta \)-modules’, and will be defined in Section 5 after several preparations have been made. Using these closure properties of this class, one can then show quite easily that the above modules \( M_e \) together constitute an example of an almost modest \( \Delta \)-module, and the conclusions of Theorems A are contained in this fact. A similar argument will give Theorem B.

### 1.3 Extremal inverse problems for Gowers norms

Although Theorem A and its proof are mostly algebraic in nature, it arises naturally as the extremal case of a much more analytic problem from arithmetic combinatorics: the inverse problem for directional Gowers norms.

The background to this problem begins with a famous result of Szemerédi.

**Theorem 1.7 (Szemerédi’s Theorem \cite{Szemeredi1983}).** For every \( \delta > 0 \) and \( k \in \mathbb{N} \) there is an \( N_0 \in \mathbb{N} \) such that the following holds: if \( N \geq N_0 \) is prime and \( E \subseteq \mathbb{Z}/N\mathbb{Z} \) has \( |E| \geq \delta N \), then

\[
    E \supseteq \{a, a + r, \ldots, a + (k - 1)r\}
\]

for some \( a \in \mathbb{Z}/N\mathbb{Z}, r \in (\mathbb{Z}/N\mathbb{Z})^* \).

This theorem has a long history, and a number of different proofs are now known. Starting with an ergodic-theoretic approach due to Furstenberg \cite{Furstenberg1977}, some of these approaches can be generalized to give the following higher-dimensional version:

**Theorem 1.8 (Furstenberg and Katznelson \cite{Furstenberg1985}).** For every \( \delta > 0 \) and \( d, k \in \mathbb{N} \) there is an \( N_0 \in \mathbb{N} \) such that the following holds: if \( N \geq N_0 \) is prime, if \( E \subseteq (\mathbb{Z}/N\mathbb{Z})^d \) has \( |E| \geq \delta N^d \), and if

\[
    \mathbf{v}_1, \ldots, \mathbf{v}_k \in (\mathbb{Z}/N\mathbb{Z})^d,
\]
then

\[ E \supseteq \{ a + r\mathbf{v}_1, \ldots, a + r\mathbf{v}_k \} \quad (7) \]

for some \( a \in (\mathbb{Z}/N\mathbb{Z})^d, r \in (\mathbb{Z}/N\mathbb{Z})^* \).

Of course, this implies Szemerédi’s Theorem by setting \( d = 1 \) and \( v_i = i - 1 \) for \( 1 \leq i \leq k \).

A much more complete introduction to these theorems can be found, for example, in the book [35] of Tao and Vu.

Our connection to these results is made by Gowers’ proof of Szemerédi’s Theorem from [13, 14]. First, let us introduce some notation. Let \( Z = (\mathbb{Z}/N\mathbb{Z})^d \) and \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) be as above, and suppose that that \( f_1, \ldots, f_k \) are bounded functions \( Z \rightarrow \mathbb{D} \), the closed unit disk in \( \mathbb{C} \). Then we define

\[
S(f_1, \ldots, f_k) := \frac{1}{N^{d+1}} \sum_{a \in Z, r \in \mathbb{Z}/N\mathbb{Z}} f_1(a + r\mathbf{v}_1) \cdots f_k(a + r\mathbf{v}_k).
\]

When \( f_1 = f_2 = \ldots = f_k = 1_E \), this is simply the fraction of those patterns of the kind in (7) that are contained in \( E \), except that we allow degenerate patterns for which \( r = 0 \). Most approaches to the Szemeredédi or Furstenberg-Katznelson Theorems effectively show that there is some constant \( c = c(k, d, \delta) > 0 \) such that

\[ S(1_E, 1_E, \ldots, 1_E) \geq c \quad \text{whenever} \quad |E| \geq \delta N^d. \]

Since, on the other hand, the total proportion of patterns with \( r = 0 \) is \( O(1/N) \), once \( N \) is large enough this implies that \( E \) must contain some non-degenerate patterns as well.

In order to prove this, Gowers (building on an older idea of Roth [26]) begins to develop a theory that distinguishes between functions \( Z \rightarrow D \) that are ‘structured’ and ‘random’ (although this terminology was introduced later by Tao). In case \( d = 1 \), he then shows that if one can find a large set \( E \) for which \( S(1_E, 1_E, \ldots, 1_E) \) is too small, then one can decompose \( 1_E \) as \( f + g \), a sum of a ‘structured’ function \( f \) and a ‘random’ function \( g \), so that \( g \) can effectively be ignored in the expression \( S \):

\[ S(1_E, 1_E, \ldots, 1_E) \approx S(f, f, \ldots, f), \]

and hence \( f \) also has a very small value \( S(f, f, \ldots, f) \). Using this, he is then able to extract another instance of the original problem with a smaller value of \( N \) and a subset having substantially larger density in the ambient group, but still having too few patterns inside it; iterating this procedure eventually leads to a contradiction.

A key part of this argument is setting up a suitable notion of ‘randomness’ for functions \( Z \rightarrow \mathbb{D} \). The new tools that one needs are the ‘directional Gowers
uniformity norms’. Let \( Z \) be any compact Abelian group (finite in the above examples) and let \( U = (U_1, U_2, \ldots, U_k) \) be a tuple of closed subgroups of \( Z \). If \( f : Z \rightarrow \mathbb{C} \) is a bounded measurable function, then the **directional Gowers uniformly norm of \( f \) over \( U \)** is the quantity

\[
\|f\|_{U(U)} := \left( \int_Z \int_{U_1} \cdots \int_{U_k} \nabla_{u_1} \nabla_{u_2} \cdots \nabla_{u_k} f(z) \, du_k \, du_{k-1} \cdots \, du_1 \, dz \right)^{2^{-k}},
\]

where

\[
\nabla_u f(z) := f(z - u) \cdot \overline{f(z)}
\]

(so this is a multiplicative analog of \( d_u \)).

Having introduced these norms, the technical key to their usefulness is a description of those functions \( f \) for which \( \|f\|_{U(U)} \) is not very small. Obtaining such a description is generally referred to as the **inverse problem** for such a Gowers norm. In one dimension a fairly complete answer is now known, starting from the work of Gowers, and now developed into a rich theory by Green, Tao and Ziegler [16, 17] and Szegedy [31, 32]. However, the analogous question for the general case remains mostly open (the papers [29, 30] of Shkredov make progress in some of the simplest cases in two dimensions).

Our work below bears on the most extreme form of this question. Clearly if \( f : Z \rightarrow \mathbb{D} \), then \( \|f\|_{U(U)} \leq 1 \). Since \( |f| \leq 1 \) everywhere, this norm is equal to 1 if and only if

\[
|\nabla_{u_1} \nabla_{u_2} \cdots \nabla_{u_k} f(z)| \equiv 1.
\]

Since the average over \( z, u_1, \ldots, u_k \) must be nonnegative and real, it must actually equal 1, so we seek to describe those \( f : Z \rightarrow S^1 \) for which

\[
\nabla_{u_1} \nabla_{u_2} \cdots \nabla_{u_k} f(z) = 1
\]
a.s.

If we now identify \( S^1 \) with \( T \) in the usual way, and write the above question using additive notation for \( T \), it becomes precisely the partial difference equation (2). Thus, our Theorem A contains the beginning of a description of those functions that would be relevant for a Gowers-like proof of the higher-dimensional Szemerédi Theorem.

Theorem A describes only those functions for which the directional Gowers norm is strictly maximal, and it seems likely that the general inverse problem for the Gowers norms will involve considerable complexity beyond those. However, the methods developed here do also provide a stability result for ‘almost-Gowers-maximal’ functions:

**Theorem C**  For all \( k \geq 1 \) and \( \varepsilon > 0 \) there is a \( \delta > 0 \) for which the following holds.
Let $U$ be a $k$-tuple of subgroups of a compact Abelian group $Z$, and let $M$ be the module of solutions to the associated PD$^cE$. If $f \in \mathcal{F}(Z)$ is such that
\[d_0(0, d^{U_1} \cdots d^{U_k} f) < \delta \quad \text{in} \quad \mathcal{F}(U_1 \times \cdots \times U_k \times Z),\]
then there is some $g \in M$ such that $d_0(f, g) < \varepsilon$ in $\mathcal{F}(Z)$.

Thus, approximate PD$^cE$-solutions lie close to exact solutions. In the setting of $S^1$-valued functions, this has the following simple corollary.

**Corollary C'** For all $k \geq 1$ and $\varepsilon > 0$ there is a $\delta > 0$ for which the following holds.

If $Z$ and $U$ are as before and $f : Z \to \mathbb{D}$ has the property that $\|f\|_{\mathcal{U}(U)} > 1 - \delta$, then there is an exact solution $g : Z \to S^1$ to the PD$^cE$ associated to $U$ such that $\|f - g\|_1 < \varepsilon$.

Our connection to the Gowers-norms inverse problem is well-illustrated in the cases that correspond to Examples 1.3 and 1.5 above. Those were formulated on $\mathbb{T}^2$, but the discussion carries over without much change to $(\mathbb{Z}/N\mathbb{Z})^2$.

Example 1.3 corresponds to the inverse problem for the directional Gowers norm
\[\frac{1}{N^4} \sum_{(z_1, z_2) \in (\mathbb{Z}/N\mathbb{Z})^2} \sum_{n_1, n_2 \in \mathbb{Z}/N\mathbb{Z}} \nabla_{(n_1, 0)} \nabla_{(0, n_2)} F(z_1, z_2)\]
over functions $F : Z \to S^1$. Like the PD$^cE$ itself, this inverse problem may be solved easily by a change-of-variables. This is the first (easy) step in Shkredov’s recent work [29, 30] in obtaining improved bounds in the problem of finding ‘corners’ in dense subsets of $(\mathbb{Z}/N\mathbb{Z})^2$, the simplest case of the two-dimensional Szemerédi Theorem. (The hard work is then in how he uses this structure; that will not be discussed here.)

On the other hand, Example 1.5 corresponds to maximizing the directional Gowers norm
\[\frac{1}{N^5} \sum_{(z_1, z_2) \in (\mathbb{Z}/N\mathbb{Z})^2} \sum_{n_1, n_2, n_3 \in \mathbb{Z}/N\mathbb{Z}} \nabla_{(n_1, 0)} \nabla_{(0, n_2)} \nabla_{(n_3, n_3)} F(z_1, z_2)\]
over functions $F : Z \to S^1$. This, in turn, is the directional Gowers norm that corresponds to the Szemerédi-type problem of finding a positive-density set of upright squares (that is, sets of the form
\[\{(z_1, z_2), (z_1 + h, z_2), (z_1, z_2 + h), (z_1 + h, z_2 + h)\}\]
for some \( z_1, z_2, h \in \mathbb{Z}/N\mathbb{Z} \) with \( h \neq 0 \) inside a positive-density subset of \((\mathbb{Z}/N\mathbb{Z})^2\).

A good inverse description is not known for this directional Gowers norm. As explained in Example [1,5] the only exact \( T \)-valued solutions to the PD\(^+\)E are functions of the form

\[
f(z_1, z_2) = f_1(z_1) + f_2(z_2) + f_3(z_1 - z_2)\]

that is, sums of solutions to the simpler sub-equations of (5). However, the proof we give for this depends on the cohomological vanishing \( H^2_m(\mathbb{Z}/N\mathbb{Z}, T) = 0 \). The result can be made a little robust in virtue of Corollary C, but I do not know what kind of weak quantitative analog of this cohomological result would be needed to give a solution of the full inverse problem.

**Remark.** In addition to Furstenberg and Katznelson’s ergodic-theoretic proof, purely combinatorial proofs of Theorem 1.8 are known. They use various versions of the hypergraph regularity lemma, due to Nagle, Rödl and Schacht [25], Gowers [15] and Tao [34]. In all of those combinatorial proofs, given the set \( E \) and the desired configuration \( \{v_1, \ldots, v_k\} \), one always assumes that the vectors \( v_i \) are linearly independent: if this is not so a priori, then \( E \) and the vectors \( v_i \) can be lifted to a set \( E' \) and some vectors \( v'_i \) in a higher-dimensional group such that the \( v'_i \) are linearly independent.

The linearly independent case of the Gowers inverse problem is not much harder to treat than its strict version, Example [1,3] above. This gives enough information to complete a proof using hypergraph regularity, but it leads to bounds on \( N \) that are better than tower-type, as does Gowers’ proof in one dimension. That improvement seems to require a much more detailed picture of the various functions involved, hence the need for the general inverse theory. The reason why one cannot focus only on the linearly independent case is discussed a little further in the closing remarks of [1].

\[\triangleright\]

## 2 Background on topological groups and modules

### 2.1 Compact Abelian groups

For any compact Abelian group \( Z \), we write \( \hat{Z} \) for its Pontryagin dual, and \( \mathcal{A}(Z) \) for its **affine group**, containing those members of \( \mathcal{F}(Z) \) that consist of a constant plus a character. As \( Z \)-modules, these ingredients fit into the short exact sequence

\[
0 \longrightarrow T \longrightarrow \mathcal{A}(Z) \longrightarrow \hat{Z} \longrightarrow 0,
\]

where the \( Z \)-actions on kernel and image are both trivial, but the action on \( \mathcal{A}(Z) \) is not: identifying \( \mathcal{A}(Z) \) with the Cartesian product \( T \times \hat{Z} \), translation by \( z \) corre-
sponds to the automorphism

\[(\theta, \chi) \mapsto (\theta + \chi(z), \chi)\]

of \(\mathbb{T} \times \hat{Z}\).

### 2.2 Polish modules

A topological Abelian group is **Polish** (some references use ‘polonais’) if its topology is generated by a complete, separable, translation-invariant metric.

If \(Z\) is a compact Abelian group, then a **Polish \(Z\)-module** is a Polish Abelian group \(M\) equipped with a jointly continuous action of \(Z\) by automorphisms. A **morphism** from one Polish \(Z\)-module \(M\) to another \(N\) is a continuous homomorphism \(\varphi : M \rightarrow N\) that intertwines the \(Z\)-actions. These modules and morphisms together define the **category** \(\text{PMod}(Z)\) of **Polish \(Z\)-modules**.

If \(U \leq Z\), then \((-)^U\) is the functor that selects the closed submodule of \(U\)-invariant elements of a given Polish \(Z\)-module. For example, \(\mathcal{F}(Z, A)^U\) consists of those members of \(\mathcal{F}(Z, A)\) that are lifts of members of \(\mathcal{F}(Z/U, A)\) through the quotient map.

If \(M\) is a Polish Abelian group, then we let \(\mathcal{F}(Z, M)\) be the Polish Abelian group of measurable functions \(Z \rightarrow M\) with the topology of convergence in probability. If \(M\) was a \(Z\)-module, then \(\mathcal{F}(Z, M)\) becomes a \(Z\)-module under the **diagonal \(Z\)-action**, which we still denote by \(R\):

\[R_w f(z) := w \cdot (f(z - w)).\]

Generalizing the above construction, given an inclusion of groups \(U \leq Z\) and a Polish \(U\)-module \(M\), the **co-induced \(Z\)-module of** \(M\) is the Polish group of \(U\)-equivariant measurable maps \(Z \rightarrow M\). It is denoted by \(\text{Coind}_Z^U M\), so in notation one has

\[\text{Coind}_Z^U M = \{ f \in \mathcal{F}(Z, M) \mid f(z - u) = u \cdot (f(z)) \ \forall z \in Z, \ u \in U\}.\]

It is given the action of \(Z\) by translation (not the same as the diagonal action).

One immediately has the relation

\[\text{Coind}_Z^U M = \text{Coind}_Z^V \text{Coind}_V^U M \quad \text{whenever} \ U \leq V \leq Z. \quad (8)\]

Similarly, if \(\varphi : M \rightarrow N\) is a morphism of Polish \(U\)-modules, then the **co-induced morphism of** \(\varphi\) is the morphism \(\text{Coind}_Z^U \varphi : \text{Coind}_Z^U M \rightarrow \text{Coind}_Z^U N\) defined by \((\text{Coind}_Z^U \varphi)(f) := \varphi \circ f\) for \(f \in \text{Coind}_Z^U M \leq \mathcal{F}(Z, M)\). With this construction for morphisms, \(\text{Coind}_Z^U\) becomes a functor \(\text{PMod}(U) \rightarrow \text{PMod}(Z)\). The obvious analog of (8) also holds for co-inductions of morphisms.
If a $Z$-module (resp. morphism) is the co-induced $Z$-module (resp. morphism) of some $U$-module (resp. morphism), then we will write simply that it is \textit{co-induced over} $U$. In view of (8), if a $Z$-module is co-induced over $U \leq Z$, then it is also co-induced over any $V$ such that $U \leq V \leq Z$, and similarly for morphisms.

General results from measure theory give an isomorphism

$$\mathcal{F}(Z, M) \cong \text{Coind}^Z_U \mathcal{F}(U, M)$$

for any inclusion $U \leq Z$ and any Polish Abelian group $M$; we will largely take this for granted. A more interesting example is the following.

\textit{Example 2.1.} Let $Z$ be an ambient group and $U$ a tuple of subgroups, let $M$ be the module of solutions in $\mathcal{F}(Z, A)$ to the PD$\omega$E associated to $U$, and let $M'$ be the module of solutions in $\mathcal{F}(U[k], A)$ to the same PD$\omega$E. Then

$$M = \text{Coind}^Z_{U[k]} M'.$$

Similarly, if $N$ is the module of zero-sum tuples in $\bigoplus_{i=1}^k \mathcal{F}(Z, A)^{U_i}$ and $N'$ is the module of zero-sum tuples in $\bigoplus_{i=1}^k \mathcal{F}(U[k], A)^{U_i}$, then

$$N = \text{Coind}^Z_{U[k]} N'.$$

For either problem, this is a formal expression of the fact that solutions on $Z$ are just arbitrary measurable selections of solutions on each coset of $U[k]$. $\triangle$

We shall always denote the identity in a Polish module by 0. A sequence $(m_n)_n$ in a Polish module is \textbf{null} if it converges to 0.

A morphism $\varphi : M \to N$ is \textbf{closed} if its image $\varphi(M)$ is a closed subgroup of $N$. This property may fail for some morphisms even if the modules $M$ and $N$ are Abelian Lie groups, so one must keep track of it separately.

A classical result of Banach asserts that a continuous, closed, bijective homomorphism from one Polish Abelian group to another is necessarily an isomorphism. In the setting of Banach spaces this is a well-known consequence of the Closed Graph Theorem, but it is more difficult to find a reference for the general-groups case (which needs a little more thought). One such is Section I.3 of Banach’s own book [3], and see also Remark (iv) in III.39.V of Kuratowski [21]. By factorizing an arbitrary closed continuous operator into a quotient, a bijection, and an inclusion, one easily deduces the following.

\textbf{Theorem 2.2.} For a morphism $\varphi : M \to N$ the following are equivalent:

1. $\varphi$ is closed;
2. If \( n_k \in \varphi(M) \) is a null sequence then there is a null sequence \( m_k \in M \) such that \( \varphi(m_k) = n_k \);

3. \( \varphi \) factorizes as

\[
M \xrightarrow{\text{quotient}} M / \ker \varphi \xrightarrow{\varphi_1} \varphi(M) \subseteq N,
\]

where \( \varphi_1 \) is a topological isomorphism from the quotient topology to the subspace topology.

Example 2.3. If \( \varphi : M \to N \) is a closed homomorphism, then it does not follow that its restriction to any closed submodule \( K \leq M \) is still closed, even if \( M \) and \( N \) are Abelian Lie groups. For example, if \( M = \mathbb{Z} \times \mathbb{R}, N = \mathbb{T} \) and \( \varphi \) is the coordinate projection to \( \mathbb{R} \) composed with the quotient homomorphism \( \mathbb{R} \to \mathbb{T} \), and if we let \( \alpha \in \mathbb{T} \) be irrational, then the subgroup \( K := \mathbb{Z} \cdot (1, \alpha) \leq M \) is closed, but its image under \( \varphi \) is the countable dense subgroup \( \mathbb{Z} \alpha \) of \( N \).

Following the conventions of Moore [23], a sequence of morphisms

\[ M \xrightarrow{\varphi} N \xrightarrow{\psi} P \]

is exact only if it is algebraically exact, in the sense that \( \varphi(M) = \ker \psi \): this of course requires that \( \varphi \) be closed. With this convention, a morphism \( \varphi \) is closed if and only if it can be inserted into an algebraically exact sequence

\[ M \xrightarrow{\varphi} N \xrightarrow{=} K \xrightarrow{} 0. \]

Relatedly, given a complex of Polish modules and continuous homomorphisms

\[ \ldots \to M_i \to M_{i+1} \to M_{i+2} \to \ldots, \]

we will call the complex closed if all of its homomorphisms are closed.

The module \( M \) is a quotient of modules \( P \) and \( Q \) if one has a short exact sequence

\[ 0 \to P \to Q \to M \to 0. \]

The following simple consequence of this theory will be useful later.

**Lemma 2.4.** A Polish module \( M \) is countable if and only if it is discrete.

**Proof.** If \( M \) is discrete, then second countability means it must have countably many points. On the other hand, if it is not discrete, then the complement of each point is an open dense set, so the intersection of any countable family of these complements is nonempty by the Baire Category Theorem.
Now suppose that \( A \rightarrow B \) is a surjection of Polish Abelian groups, and that \( Z \) is a compact Abelian group. Then the Measurable Selector Theorem implies that the resulting map \( \mathcal{F}(Z, A) \rightarrow \mathcal{F}(Z, B) \) is also surjective, and from this Theorem 2.2 gives the following. (It may also be proved directly from the Measurable Selector Theorem with a little more care.)

**Lemma 2.5.** Any null sequence in \( \mathcal{F}(Z, B) \) is the image of a null sequence in \( \mathcal{F}(Z, A) \).

\[ \square \]

### 3 Measurable cohomology for compact groups

#### 3.1 Overview

Group cohomology provides a powerful way to pick apart the structure of modules of a given group. In our setting (compact groups acting on modules of measurable functions) the appropriate theory is measurable cohomology for locally compact acting groups and Polish modules. This was developed by Calvin Moore in his important sequence of papers [22, 23, 24].

The basics of the theory can be found in those papers, and also in the more recent work [2], which resolves some outstanding issues from those earlier papers. This measurable theory largely parallels cohomology for discrete groups, which is nicely treated in many standard texts, such as Brown’s [4]. However, some standard techniques from the discrete world — most obviously, the construction of injective and projective resolutions — do not have straightforward generalizations.

We next offer a very terse summary of the foundations of the measurable theory. A more complete explanation, as well as proofs, can be found in the paper [23] of Moore’s sequence and the introduction to [2]. The reader with no familiarity with this theory may prefer to treat it entirely as a ‘black box’ on first reading. Our notation will largely follow [2].

If \( Z \) is a compact Abelian group and \( M \in \mathrm{PMod}(Z) \), then an \( M \)-valued cochain in degree \( p \) is an element of \( \mathcal{F}(Z^p, M) \), which space is regarded as a \( Z \)-module with the diagonal action. This module will sometimes be written \( C^p(Z, M) \). The inhomogeneous bar resolution of \( M \) is the following sequence of \( Z \)-modules and morphisms:

\[
M \xrightarrow{d} C^1(Z, M) \xrightarrow{d} C^2(Z, M) \xrightarrow{d} \ldots ,
\]
where for \( f \in C^p(Z, M) \) one defines
\[
    df(z_1, \ldots, z_{p+1}) := z_1 \cdot f(z_2, \ldots, z_{p+1}) + \sum_{i=1}^{p} (-1)^i f(z_1, \ldots, z_i + z_{i+1}, \ldots, z_p) + (-1)^{p+1} f(z_1, \ldots, z_p).
\]

Since we will need to work simultaneously with many different compact Abelian groups, we will sometimes write \( d^Z \) in place of \( d \) to record the acting group in question. When \( p = 0 \), this gives
\[
    d^Z f(w) := w \cdot f - f,
\]
which correctly generalizes \( \mathbb{1} \) when \( f \in F(Z, A) \) and \( Z \) acts on this module by rotations. (Beware that the argument \( w \) here appeared in \( \mathbb{1} \) in the subscript, and the argument \( z \) of \( \mathbb{1} \) is now hidden because we are treating \( f \) as an element of an abstract module.)

A routine calculation shows that \( d \circ d = 0 \). The \( p \)-\textbf{cocycles} are the elements of the subgroup
\[
    Z^p(Z, M) := \ker(d|C^p(Z, M)),
\]
the \( p \)-\textbf{coboundaries} are the elements of the further subgroup
\[
    B^p(Z, M) := \text{img}(d|C^{p-1}(Z, M)),
\]
and the \( p \)th \textbf{cohomology group} is
\[
    H^p_m(Z, M) := \frac{Z^p(Z, M)}{B^p(Z, M)}
\]
(the subscript ‘m’ reminds us that we work throughout with measurable cochains). Both \( Z^p \) and \( B^p \) inherit topologies as subspaces of the Polish space \( C^p \). The former is obviously closed, but the latter may not be. We sometimes consider \( H^p_m(Z, M) \) endowed with the quotient topology, with the warning that it may not be Hausdorff (in which case the topology is of little use). This quotient topology is Hausdorff if and only if \( B^p \) is closed, in which case the quotient topology is actually Polish.

This construction gives a sequence \( H^p_m(Z, -), p \geq 0, \) of functors from \( \text{PMod}(Z) \) to the category of all topological \( Z \)-modules. It is easy to check that if \( \varphi : M \to N \) is a morphism in \( \text{PMod}(Z) \), then the induced morphisms on cohomology
\[
    H^p_m(Z, \varphi) : H^p_m(Z, M) \to H^p_m(Z, N)
\]
are also continuous, even if the quotient topologies here are not Hausdorff.

As in the more classical setting of discrete groups, the sequence of functors \( H^p_m(Z, -) \) taken as a whole turns out to be a \textbf{universal cohomological} \( \delta \)-\textbf{functor}, meaning that it satisfies the following axioms:
• (interpretation in degree zero) \( H^0_m(Z, -) = (-)^Z \);

• (effaceability) for every \( M \in \text{PMod}(Z) \), there are \( N \in \text{PMod}(Z) \) and a closed injective morphism \( \varphi : M \to N \) such that the induced map on cohomology

\[
H^p_m(Z, \varphi) : H^p_m(Z, M) \to H^p_m(Z, N)
\]

is zero for all \( p \geq 1 \);

• (long exact sequence) if

\[
0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0
\]

is a short exact sequence in \( \text{PMod}(Z) \), then it gives rise to a long exact sequence

\[
0 \to H^0_m(Z, A) \xrightarrow{\alpha_0} H^0_m(Z, B) \xrightarrow{\beta_0} H^0_m(Z, C) \xrightarrow{s_1} H^1_m(Z, A) \xrightarrow{\alpha_1} \cdots
\]

\[
\cdots s_p H^p_m(Z, A) \xrightarrow{\alpha_p} H^p_m(Z, B) \xrightarrow{\beta_p} H^p_m(Z, C) \xrightarrow{s_{p+1}} \cdots,
\]

where \( \alpha_p := H^p_m(Z, \alpha) \), \( \beta_p := H^p_m(Z, \beta) \), and each \( s_p \) is a newly-constructed morphism called the ‘switchback morphism’ (or sometimes the ‘transgression map’).

A classical argument of Buchsbaum [5] shows that there can be only one sequence of functors on \( \text{PMod}(Z) \) satisfying all of these axioms. This is the basis for various results proving agreement between \( H^p_m(Z, -) \) and other cohomology theories, such as in Wigner’s work [38] and in [2].

In the setting of the effaceability axiom, one writes that \( \varphi \) ‘effaces’ \( M \). The standard construction of an effacing morphism for \( H^p_m(Z, -) \) relies on the following simple vanishing result.

**Lemma 3.1.** For any \( M \in \text{PMod}(Z) \) one has

\[
H^p_m(Z, F(Z, M)) = 0 \quad \forall p \geq 1.
\]

It follows at once that the embedding \( M \hookrightarrow F(Z, M) \) which identifies \( M \) with the constant functions is effacing. Given this lemma, the short exact sequence

\[
M \hookrightarrow F(Z, M) \to F(Z, M)/M
\]
gives rise to a cohomology long exact sequence that collapses at every third position starting from $H^1_m(Z, \mathcal{F}(Z, M))$. It therefore provides a sequence of isomorphisms

$$H^p_m(Z, \mathcal{F}(Z, M)/M) \cong H^{p+1}_m(Z, M) \quad \text{for } p \geq 1,$$

and also

$$\text{coker}(\mathcal{F}(Z, M)^Z \longrightarrow (\mathcal{F}(Z, M)/M)^Z) \cong H^1_m(Z, M).$$

By inspecting the proof of Lemma 3.1 at the level of individual cocycles, it is easy to prove that these are in fact topological isomorphisms, where all the cohomology groups and other quotients here are given their quotient topologies. We omit the details.

One can also show that all of the maps appearing in a long exact sequence are continuous for the quotient topologies (again, even if those topologies are not Hausdorff). The only nontrivial case is that of the switchback morphisms, which are covered by the following.

**Lemma 3.2.** For any short exact sequence of Polish $Z$-modules as above, the switchback isomorphisms

$$H^p_m(Z, C) \longrightarrow H^{p+1}_m(Z, A)$$

are continuous for the quotient topologies.

**Proof.** If $U \subseteq H^{p+1}_m(Z, A)$ is open in the quotient topology, then it is the image of an open set $\overline{U} \subseteq \mathcal{Z}^{p+1}(Z, A)$ which is a union of $\mathcal{B}^{p+1}(Z, A)$-cosets. This in turn is equal to $\overline{U} \cap \mathcal{Z}^{p+1}(Z, A)$ for some open set $\tilde{U} \subseteq \mathcal{Z}^{p+1}(Z, B)$, because $\mathcal{Z}^{p+1}(Z, A)$ is a closed subset of $\mathcal{Z}^{p+1}(Z, B)$. Since the boundary morphisms $d$ are continuous, it follows that $d^{-1}(\overline{U})$ is open in $C^p(Z, B)$, and therefore $\beta(d^{-1}(\overline{U}))$ is open in $C^p(Z, C)$ because $\beta$ is an open morphism. Finally, this implies that

$$\beta(d^{-1}(\overline{U})) = \beta(d^{-1}((\tilde{U}))) \cap \mathcal{Z}^p(Z, C)$$

is relatively open in $\mathcal{Z}^p(Z, C)$, as required.

The last piece of classical theory that we will need is the following property of cohomology for compact groups acting on Lie modules. It will be the building block for our later results on the structure of the (much larger) modules of PD$^E$-solutions and zero-sum tuples. It can also be found in [2]

**Proposition 3.3.** Suppose that $Z$ is a compact Abelian group and $A$ is a nilpotent Lie $Z$-module. If $p = 0$ then $H^p_m(Z, A) = A^Z$ is also a nilpotent Lie group, now with trivial $Z$-action. If $p \geq 1$ then $H^p_m(Z, A)$ is discrete.

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Thus, in degrees one and higher, ‘cohomology functors convert all locally compact modules into discrete modules’.

The results above are complemented by a selection of explicit calculations in Appendix A.

### 3.2 Cohomology groups as new modules

In case $\mathbb{Z}$ is a locally compact, second-countable Abelian group and $M \in \text{PMod}(\mathbb{Z})$, then the diagonal actions of $\mathbb{Z}$ on the cocycle modules $\mathbb{C}^p(\mathbb{Z}, M)$, $p \geq 0$, commute with $d : \mathbb{C}^p \rightarrow \mathbb{C}^{p+1}$. These diagonal actions therefore preserve the subgroups $\mathbb{Z}^p$ and $\mathbb{B}^p$, and so define an action of $\mathbb{Z}$ on the quotient groups $\mathbb{H}^p_{\text{in}}(\mathbb{Z}, M)$. It is a jointly continuous action if that quotient is Hausdorff.

**Remark.** This construction clearly requires that $\mathbb{Z}$ be Abelian. For a more general locally compact, second-countable group $G$, module $M \in \text{PMod}(G)$ and element $g \in G$, one finds that the diagonal transformations $R_g \curvearrowright \mathbb{C}^p(G, M)$ give isomorphisms

$$
\mathbb{Z}^p(G, M) \rightarrow \mathbb{Z}^p(G, M') \quad \text{and} \quad \mathbb{B}^p(G, M) \rightarrow \mathbb{B}^p(G, M'),
$$

and hence also $\mathbb{H}^p_{\text{in}}(G, M) \rightarrow \mathbb{H}^p(G, M')$, where $M'$ is the module defined by the underlying Polish group of $M$ with the conjugate action

$$
\pi^g(-) = \pi(g(-)g^{-1}) : G \curvearrowright M.
$$

I do not know of any applications for this rather more unwieldy picture, although it might be important if one attempted to extend the present work to non-Abelian groups.

In the Abelian setting, we will henceforth always assume that $\mathbb{H}^p_{\text{in}}(W, M)$ is endowed with this quotient action.

**Lemma 3.4.** If $W \leq \mathbb{Z}$ is an inclusion of locally compact and second-countable Abelian groups and $M \in \text{PMod}(W)$, then one has a canonical isomorphism of $\mathbb{Z}$-modules

$$
\mathbb{H}^p_{\text{in}}(W, \text{Coind}^Z_W M) \cong \text{Coind}^Z_W \mathbb{H}^p_{\text{in}}(W, M).
$$

**Proof.** Recalling that $\text{Coind}^Z_W (-) = \mathcal{F}(Z, -)^W$, this is most easily seen at the level of cocycles, where one has the obvious identifications of $\mathbb{Z}$-modules

$$
\mathbb{C}^p(W, \text{Coind}^Z_W M)) = \mathbb{C}^p(W, \mathcal{F}(Z, M)^W) = \mathcal{F}(W^p \times Z, M)^{W'} \\
\cong \mathcal{F}(Z, \mathcal{F}(W^p, M))^{W'} = \text{Coind}^Z_W \mathbb{C}^p(W, M),
$$

where $W'$ is a copy of $W$ that acts by rotation of the variable in $Z$, and on $M$, but not by rotation of the variable in $W^p$. \qed

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3.3 The Shapiro Isomorphism

An important tool in the sequel will be the measurable analog of the classical Shapiro Lemma, which gives a simplifying description of the cohomology of a co-induced module. We state it only for compact Abelian groups, although it holds for general locally compact, second-countable groups.

**Lemma 3.5.** If $W \leq Z$ is an inclusion of compact Abelian groups and $M \in \text{PMod}(W)$, then there are isomorphisms of functors

$$
H^p_m(Z, \text{Coind}^Z_W(-)) \cong H^p_m(W, -)
$$

for every $p \geq 0$.

For each $M \in \text{PMod}(Z)$, these all give isomorphisms of topological groups when the cohomology groups are given their quotient topologies.

**Proof.** The construction of these isomorphisms is given in Moore [23], and its basic properties are established there. The only missing detail is that they are topological; to sketch the proof of this we quickly recall the construction.

When $p = 0$, the desired isomorphism is simply

$$(\text{Coind}^Z_W M)^Z = (\mathcal{F}(Z, M)^W)^Z = M^W \quad \text{for each } M \in \text{PMod}(Z),$$

which is manifestly an isomorphism of Polish groups.

Now the general isomorphism of functors is obtained recursively in $p$: if it has already been constructed for some degree $p \geq 0$, then one uses the isomorphism (9) to give

$$H^{p+1}_m(Z, \text{Coind}^Z_W (-)) \cong H^p_m(Z, \mathcal{F}(Z, \text{Coind}^Z_W (-))/\text{Coind}^Z_W (-))$$

$$\cong H^p_m(Z, \text{Coind}^Z_W(\mathcal{F}(W, M)/M)) \overset{\text{induction}}{\cong} H^p_m(W, \mathcal{F}(W, M)/M) \cong H^{p+1}_m(W, M)$$

(or the obvious analog using (10) when $p = 0$). Since all of the individual isomorphisms here are already known to be topological, so is the isomorphism in degree $p + 1$.

**Definition 3.6.** The isomorphism above is the **Shapiro Isomorphism** from $H^*_m(Z, \text{Coind}^Z_W M)$ to $H^*_m(W, M)$.

We will often use a Shapiro Isomorphism by way of the following simple application.

Suppose that $Y, W \leq Z$ are two closed subgroups of a compact Abelian group and that

$$P_0 \leq Q_0$$

is an inclusion of Polish $Y$-modules such that $Q_0/P_0$ is discrete. Let

$$P := \text{Coind}^Z_Y P_0 \quad \text{and} \quad Q := \text{Coind}^Z_Y Q_0.$$
Lemma 3.7. Assume that $W + Y = Z$. In the situation above, the cokernel of the homomorphism

$$H^p_m(W, P) \longrightarrow H^p_m(W, Q)$$

is discrete for every $p \geq 0$.

Proof. Because $W + Y = Z$, a simple appeal to the Measurable Selector Theorem shows that as Polish Abelian groups one has

$$\text{Coind}^Z_Y P_0 = \{ f \in \mathcal{F}(Z, P_0) \mid f(z - y) = y \cdot f(z) \forall z \in Z, y \in Y \}$$

$$\cong \mathcal{F}(Z/Y, P_0) \cong \mathcal{F}(W/(W \cap Y), P_0) \cong \text{Coind}^W_{W \cap Y} P_0,$$

and similarly for $Q_0$. These Polish-group isomorphisms may not respect the action of $Z$, but they do respect the action of the subgroup $W$.

Therefore the Shapiro Isomorphism gives

$$H^p_m(W, P) \cong H^p_m(W, Q) \cong H^p_m(W \cap Y, P_0) \cong H^p_m(W \cap Y, Q_0).$$

Since the vertical isomorphisms here are topological isomorphisms (Lemma 3.5), it suffices to prove the desired conclusions for the bottom row instead. Equivalently, this reduces our task to the case $Y = Z$, and hence $P = P_0$ and $Q = Q_0$.

Assuming this, let $A := Q/P$, which is assumed discrete. The long exact sequence arising from this quotient is

$$(0) \longrightarrow P^W \longrightarrow Q^W \longrightarrow A^W$$

$$\longrightarrow H^1_m(W, P) \longrightarrow H^1_m(W, Q) \longrightarrow H^1_m(W, A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{p+1}_m(W, P) \longrightarrow H^{p+1}_m(W, Q) \longrightarrow H^{p+1}_m(W, A) \longrightarrow \cdots,$$

where all the maps are continuous.

Therefore, for each $p \geq 0$, one has a continuous injection

$$\text{coker}(H^p_m(W, P) \longrightarrow H^p_m(W, Q)) \hookrightarrow H^p_m(W, A).$$

Since the right-hand side is a discrete group by Proposition 3.3, so is the left-hand side. \qed
4 Polish complexes

If $Z$ is a compact Abelian group, a **Polish complex of $Z$-modules** is a (finite or infinite) sequence of Polish $Z$-modules and morphisms, say

$$\cdots \xrightarrow{\alpha_{\ell}} A_\ell \xrightarrow{\alpha_{\ell+1}} A_{\ell+1} \xrightarrow{\alpha_{\ell+2}} A_{\ell+2} \xrightarrow{\alpha_{\ell+3}} \cdots,$$

with the property that

$$\alpha_{\ell+1} \circ \alpha_{\ell} = 0 \quad \forall \ell.$$

This is the obvious adaptation of the usual notion from homological algebra. Often we will write simply ‘Polish complex’ if the group $Z$ is understood.

Given a Polish complex indexed as above, its **homology in position $\ell$** is the quotient

$$\ker \alpha_{\ell+1} / \text{img} \alpha_{\ell},$$

regarded as a topological group with the quotient topology. This quotient topology is Hausdorff if and only if $\alpha_{\ell}$ is a closed morphism, and in this case the quotient is also Polish.

The Polish complex is **exact in position $\ell$** if its homology is trivial in that position: that is, if $\ker \alpha_{\ell+1} = \text{img} \alpha_{\ell}$. It is **exact** if it is exact in all positions.

A Polish complex is **bounded** if it is finite and it starts and ends with the module 0. Then it may be written

$$0 \longrightarrow A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{\ell}} A_\ell \xrightarrow{\alpha_{\ell+1}} A_{\ell+1} \xrightarrow{\alpha_{\ell+2}} A_{\ell+2} \xrightarrow{\alpha_{\ell+3}} \cdots \xrightarrow{\alpha_k} A_k \longrightarrow 0$$

for some $k$.

4.1 Split complexes

A stronger notion than exactness is the following, which will be important in the sequel. It is also a standard idea from homological algebra.

**Definition 4.1.** Let $Z$ be a compact Abelian group, and let

$$0 = M_0 \xrightarrow{\alpha_1} M_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} M_k \xrightarrow{\alpha_{k+1}} M_{k+1} = 0$$

be any complex of topological $Z$-modules with continuous homomorphisms. Then this complex is **split** if there are topological $Z$-module homomorphisms

$$\beta_i : M_{i+1} \longrightarrow M_i, \quad i = 0, \ldots, k$$

such that $\beta_i \beta_{i+1} = 0$ for all $i$ and

$$\alpha_i \beta_{i-1} + \beta_i \alpha_{i+1} = \text{id}_{M_i} \quad \forall i \in \{1, 2, \ldots, k\}.$$
A routine exercise shows that if the above complex is split, then it is isomorphic to the complex

\[ 0 \rightarrow A_1 \rightarrow A_1 \oplus A_2 \rightarrow A_2 \oplus A_3 \rightarrow \ldots \rightarrow A_{k-1} \oplus A_k \rightarrow A_k \rightarrow 0, \]

where the maps are the obvious coordinate projections and where

\[ A_i \cong \ker \alpha_{i+1} \cong \ker \beta_{i-1}. \]

In the terminology of homological algebra, \((\beta_i)_i\) is a chain homotopy from the identity morphisms of this chain complex to the zero morphisms.

### 4.2 Almost discrete homology

In our later work, a special part will be played by complexes whose homology is controlled in the following specific sense.

**Definition 4.2.** Let

\[ 0 \rightarrow A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_k \rightarrow 0 \]

be a bounded complex in \(PMod(Z)\). Then for \(\ell_0 \in \{0, 1, \ldots, k\}\), it has \(\ell_0\)-almost discrete homology, if

- \(A_i = 0\) for \(i < \ell_0\),
- \(\ker(A_{\ell_0} \rightarrow A_{\ell_0+1})\) is locally compact, and
- the homology of the sequence is discrete at all positions \(> \ell_0\).

If, in addition, it actually has discrete homology at all positions, then it has \(\ell_0\)-discrete homology.

To emphasize the difference, \(\ell_0\)-discrete homology will sometimes be called strictly \(\ell_0\)-discrete.

This class owes its importance to some topological addenda to the Snake Lemma. It will be the workhorse of several later proofs, in which we must show how the properties of two complexes in a short exact sequence dictate those of the third.

For the rest of this subsection, fix a compact Abelian group \(Z\), and suppose that
is a commutative diagram in $\text{PMod}(\mathbb{Z})$ in which all columns are exact and each row is a complex. Naturally, this is referred to as a short exact sequence of complexes.

In the above diagram, let $I_\ell^A := \text{img}(\partial_\ell^A)$, $K_\ell^A := \ker(\partial_{\ell+1}^A)$, and $H_\ell^A := K_\ell^A / I_\ell^A$,

and similarly for the other rows. Then one may also apply the classical Snake Lemma to produce a long exact sequence tying all of these homology groups together:

$$0 \rightarrow H_1^A \rightarrow H_1^B \rightarrow H_1^C \rightarrow H_2^A \rightarrow H_2^B \rightarrow H_2^C \rightarrow \cdots \rightarrow H_k^A \rightarrow H_k^B \rightarrow H_k^C \rightarrow 0.$$ 

In practice this can give algebraic information about structure complexes, which may be useful for certain calculations. In general this long exact sequence does not give much information about the topologies of these homology groups. However, for complexes with almost discrete homology one can relate the topologies of the homologies of the three sequences.

We formulate these relatives of the Snake Lemma in three separate parts. In each case, the proof will amount to following the usual diagram chase of the Snake Lemma and finding where discreteness of homology may be applied.

**Lemma 4.3.** In the above short exact sequence of complexes, suppose that the first and second rows have $\ell_0$-almost (resp. strictly) discrete homology for some $\ell_0 \in \{1, 2, \ldots, k\}$. Then the third row also has $\ell_0$-almost (resp. strictly) discrete homology.

**Proof.** We give the proof for almost discrete rows, since the strictly discrete case follows in just the same way.
First note that, by the exactness of each column, if two of the modules in a column are 0, then so is the third. In light of this, \( \ell_0 \)-almost discreteness implies that \( A_i = B_i = C_i = 0 \) whenever \( i < \ell_0 \). Therefore we may simply truncate the above diagram to the left of position \( \ell_0 - 1 \), and so assume that \( \ell_0 = 1 \).

Now suppose that \( (c_n)_n \) is a null sequence in \( \ker \partial^C_{\ell+1} \) for some \( \ell \geq 1 \). Since \( \beta_\ell \) is closed, there is a null sequence \( (b_n)_n \) in \( B_\ell \) with \( \beta_\ell(b_n) = c_n \). It follows that \( b'_n := \partial^{B}_{\ell+1}(b_n) \) is a null sequence in ker \( \beta_{\ell+1} \), and hence that \( b'_n = \alpha_{\ell+1}(a_n) \) for some null sequence \( (a_n)_n \) in \( A_{\ell+1} \).

Now \( \alpha_{\ell+2}\partial^{A}_{\ell+2}(a_n) = \partial^{B}_{\ell+2}(b'_n) = \partial^{B}_{\ell+2}\partial^{B}_{\ell+1}(b_n) = 0 \), so \( a_n \in \ker \partial^{A}_{\ell+2} \) since \( \alpha_{\ell+2} \) is injective.

Therefore, since \( \text{img} \partial^{A}_{\ell} \) is closed and relatively open inside \( \ker \partial^{A}_{\ell+1} \), we have \( a_n = \partial^{A}_{\ell+1}(a_n^0) \) for some null sequence \( (a_n^0)_n \) in \( A_{\ell} \). Replacing each \( b_n \) with \( b_n - \alpha_\ell(a_n^0) \), it follows that this is still gives null sequence with image \( (c_n)_n \), and these now lie in \( \ker \partial^{B}_{\ell+1} \).

This has shown that \( \ker \partial^{C}_{\ell+1} \) contains \( \beta_\ell(\ker \partial^{B}_{\ell+1}) \) as a relatively open-and-closed subgroup. Since \( \partial^{C}_{\ell+1} \) is continuous, we also know that its kernel is closed, and hence that \( \beta_\ell(\ker \partial^{B}_{\ell+1}) \) is a closed subgroup, and \( \beta_\ell | \ker \partial^{B}_{\ell+1} \) is a closed morphism.

In case \( \ell = 1 \), we also have that \( \ker \partial^{B}_{1+1} \) is locally compact. Therefore its image \( \beta_\ell(\ker \partial^{B}_{1+1}) \) is locally compact, and since \( \ker \partial^{C}_{1+1} \) is a discrete extension of this, it is also locally compact.

On the other hand, if \( \ell > 1 \) then we can apply the discreteness of \( \ker \partial^{B}_{\ell+1} / \text{img} \partial^{B}_{\ell} \) to deduce that, after the adjustment above, we have \( b_n = \partial^{B}_{\ell}(b'_n) \) for some null sequence \( (b'_n)_n \) in \( B_{\ell-1} \). Applying \( \beta_\ell \), this gives

\[
c_n = \beta_\ell(b_n) = \beta_\ell(\partial^{B}_{\ell}(b'_n)) = \partial^{C}_{\ell}(\beta_{\ell-1}(b'_n)) \in \text{img} \partial^{C}_{\ell}
\]

for all sufficiently large \( n \), showing that the homology \( \ker \partial^{C}_{\ell+1} / \text{img} \partial^{C}_{\ell} \) is discrete if \( \ell > 1 \).

**Lemma 4.4.** In the above short exact sequence of complexes, suppose that the first and third rows have \( \ell_0 \)-almost (resp. strictly) discrete homology for some \( \ell_0 \in \{1, 2, \ldots, k\} \). Then the second row also has \( \ell_0 \)-almost (resp. strictly) discrete homology.

**Proof:** As in the previous lemma, we may assume \( \ell_0 = 1 \), and describe only the almost-discrete case.

First suppose that \( \ell = 1 \), and consider a null sequence \( (c_n)_n \) in \( \ker \partial^{C}_{2} \). We can now repeat the first few steps used in Case 1 above, which needed only the 1-almost discrete homology of the top row. This gives that \( \ker \partial^{C}_{2} \) contains \( \beta_1(\ker \partial^{B}_{2}) \) as a
relatively-open-and-closed subgroup. Since we assume that \( \ker \partial^C_2 \) is locally compact, it follows that \( \beta_1 \) restricts to a closed epimorphism \( \ker \partial^B_2 \to \beta_1(\ker \partial^B_2) \). On the other hand, \( (\ker \partial^B_2) \cap (\ker \beta_1) = \alpha_1(\ker \partial^A_3) \), so since this is closed it is also locally compact. Putting these facts together gives a presentation

\[
\ker \partial^A_2 \to \ker \partial^B_2 \to \beta_1(\ker \partial^B_2)
\]

of \( \ker \partial^B_2 \) as an extension of locally compact modules. Since we assume a priori that it is Polish, it is also locally compact.

Now suppose that \( \ell > 1 \), and that \( (b_n)_n \) is a null sequence in \( \ker \partial^B_{\ell+1} \). Then \( (\beta_\ell(b_n))_n \) is a null sequence in \( \ker \partial^C_{\ell+1} \), so for \( n \) sufficiently large we have \( \beta_\ell(b_n) = \partial^C_n(c_n) \) for some sequence \( (c_n)_n \) in \( C_{\ell-1} \). Since \( \partial^C_n \) is closed, we may assume that \( (c_n)_n \) is also null, and now since \( \beta_{\ell-1} \) is closed and surjective we may write \( c_n = \beta_\ell(b^o_n) \) for some null sequence \( (b^o_n)_n \) in \( B_{\ell-1} \).

It follows that \( b_n - \partial^B_\ell(b^o_n) \) forms a null sequence in \( \ker \partial^B_{\ell+1} \cap \ker \beta_\ell \). It is therefore equal to \( \alpha_\ell(a_n) \) for some null sequence \( (a_n)_n \) in \( A_\ell \). This sequence now satisfies

\[
\alpha_{\ell+1}(a_n) = \partial^B_{\ell+1}(b_n - \partial^B_\ell(b^o_n)) = 0,
\]

so since \( \alpha_{\ell+1} \) is injective we see that \( (a_n)_n \) is a null sequence in \( \ker \partial^A_{\ell+1} \). It therefore eventually equals \( \partial^A_\ell(a^o_n) \) for some sequence \( (a^o_n)_n \) in \( A_{\ell-1} \), giving that

\[
b_n = \partial^B_\ell(b^o_n) + \alpha_\ell(a_n) = \partial^B_\ell(b^o_n + \alpha_{\ell-1}(a^o_n))
\]

eventually lies in \( \text{img} \partial^B_\ell \), as required. \( \square \)

**Lemma 4.5.** In the above short exact sequence of complexes, suppose that the second and third rows have \( \ell_0 \)-almost (resp. strictly) discrete homology for some \( \ell_0 \in \{1, 2, \ldots, k\} \), and that \( A_i = 0 \) for all \( i < \ell_0 + 1 \). Then the first row has \( (\ell_0 + 1) \)-almost (resp. strictly) discrete homology.

**Proof.** Again assume \( \ell_0 = 1 \) and consider the almost-discrete case.

If \( \ell = 2 \), then \( \alpha_2 \) is an injective morphism identifying \( \ker \partial^A_3 \) with the closed subgroup

\[
(\ker \partial^B_2) \cap (\ker \beta_2) \leq \ker \partial^B_3.
\]

Since we assume \( \ker \partial^B_3 \) is locally compact, so is \( \ker \partial^A_3 \), which is the homology in position 2 of the top complex because our assumptions give \( \text{img} \partial^A_2 = 0 \).

Now suppose \( \ell > 2 \) and that \( (a_n)_n \) is a null sequence in \( \ker \partial^A_{\ell+1} \). Then \( (\alpha_\ell(a_n))_n \) is a null sequence in \( \ker \partial^B_{\ell+1} \), so for \( n \) sufficiently large it equals \( \partial^B_\ell(b_n) \) for some null sequence \( (b_n)_n \) in \( B_{\ell-1} \). 

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This now gives that \((\beta_{\ell-1}(b_n))_n\) is a null sequence in \(\ker \partial^C_\ell\), so for \(n\) sufficiently large it equals \(\partial^C_{\ell-1}(c_n)_n\) for some null sequence \((c_n)_n\) in \(C_{\ell-2}\). Letting \((b_n^\circ)_n\) be a lift of \((c_n)_n\) to a null sequence in \(B_{\ell-2}\), we obtain that

\[
\alpha_\ell(a_n) = \partial^B_\ell(b_n) = \partial^B_\ell(b_n - \partial^B_{\ell-1}(b_n^\circ))
\]

and \(b_n - \partial^B_{\ell-1}(b_n^\circ) \in \ker \beta_{\ell-1}\). Therefore this latter sequence equals \(\alpha_\ell(a_n^\circ)\) for some null sequence \((a_n^\circ)_n\) in \(A_{\ell-1}\), giving that

\[
\alpha_\ell(a_n) = \alpha_\ell(\partial^A_\ell(a_n^\circ))
\]

for all \(n\) sufficiently large. Since \(\alpha_\ell\) is injective and this right-hand side lies in \(\alpha_\ell(\image \partial^A_\ell)\) this completes the proof.

\[
\square
\]

5 \(\Delta\)-modules

Theorems A and B will be deduced from a significantly more abstract result, asserting that certain classes of Polish \(Z\)-module are preserved under some basic operations, such as cohomology functors and short exact sequences.

This section introduces these new classes of module.

5.1 \(\Delta\)-modules

The following definition will begin to single out the class of modules to which our methods will apply. More restricted sub-classes will be identified later.

**Definition 5.1 (Pre-\(\Delta\)-module).** Let \(Z\) a compact Abelian group and \(S\) a finite set. A pre-\(\Delta_S\)-module over \(Z\) is a family of topological \(Z\)-modules \((M_e)_{e \subseteq S}\) indexed by subsets of \(S\), together with a family of continuous homomorphisms \(\varphi_{a,e} : M_a \to M_e\) indexed by pairs \(a \subseteq e \subseteq S\), such that

\[
\Delta 1) \quad \varphi_{e,e} = \text{id}_{M_e} \text{ for all } e \subseteq S;
\]

\[
\Delta 2) \quad \varphi_{a,e} \circ \varphi_{b,a} = \varphi_{b,e} \text{ whenever } b \subseteq a \subseteq e.
\]

The homomorphisms \(\varphi_{a,e}\) are the structure morphisms of the pre-\(\Delta\)-module, and the individual modules \(M_e\) are the constituent modules of the pre-\(\Delta\)-module.

A pre-\(\Delta\)-module over \(Z\) is **Polish** if all of its constituents are Polish \(Z\)-modules.

In categorical terms, a pre-\(\Delta_S\)-module is a covariant functor into \(\text{PMod}(Z)\) from the category \(\Delta_S\) of subsets of \(S\) and order-preserving monomorphisms. This puts it into a well-known and very general framework of simplicial objects in homological algebra (see, for instance, Chapter 8 of Weibel’s textbook [37], and that
alg. top. book). However, it does not seem to offer a particularly efficient language for our purposes, and misses a great deal of additional structure that we will impose shortly, hence our adopting the bespoke term ‘pre-\(\Delta\)-module’.

One may visualize a pre-\(\Delta\)-module as an assignment of \(\mathbb{Z}\)-modules \(M_e\) to the vertices of the \(|S|\)-dimensional discrete cube, by identifying subsets of \(S\) with elements of \(\{0, 1\}^{|S|}\). For example, if \(S = [3]\) the picture is as follows (where only a few of the morphisms are labelled):

\[
\begin{array}{c}
M_{\{2,3\}} \\ M_{\{3\}} \\ M_{\{0\}}
\end{array}
\begin{array}{c}
M_{\{1,3\}} \\ M_{\{2\}} \\ M_{\{1\}}
\end{array}
\]

\(\varphi_{\emptyset, (3)}\)

\(\varphi_{\emptyset, (2)}\)

\(\varphi_{\emptyset, (1)}\)

This picture also ignores all of the extra structure that will be imposed next, but may offer some helpful intuition.

An important part of that extra structure depends on the following notion.

**Definition 5.2.** Suppose that \(M, N \in \text{PMod}(\mathbb{Z})\) and that \(U \leq \mathbb{Z}\). Then a derivation-action of \(U\) from \(N\) to \(M\) is a map

\[
U \rightarrow \text{Hom}_\mathbb{Z}(N, M) : u \mapsto \tilde{\nabla}_u
\]

which is jointly continuous regarded as a map \(U \times N \rightarrow M\) and which satisfies the relations

\[
\tilde{\nabla}_{u+u'} = \tilde{\nabla}_u + \tilde{\nabla}_{u'} \circ R_u \quad \forall u, u' \in U.
\]

If, in addition, \(\varphi : M \rightarrow N\) is a homomorphism of \(\mathbb{Z}\)-modules, then \(\tilde{\nabla}\) is a derivation-lift of \(U\) through \(\varphi\) if it is a derivation-action and if

\[
\varphi \circ \tilde{\nabla}_u = d_u \quad \forall u \in U.
\]

An alternative, more abstract definition of a derivation-action from \(N\) to \(M\) is that it is an element of the 1-cocycle group

\[
\mathcal{Z}^1(U, \text{Hom}_\mathbb{Z}(N, M)),
\]

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where $Z$ (and so also $U \leq Z$) acts on $\text{Hom}_Z(N,M)$ by pre-composition (this gives a well-defined action because $Z$ is Abelian). In these terms, the defining relation (11) is just the usual equation for a $1$-cocycle.

**Definition 5.3 ($\Delta$-module).** Let $Z$ be as before, let $Y \leq Z$ be a closed subgroup and $U = (U_i)_{i \in S}$ be a tuple of closed subgroups of $Z$. Then a $\Delta_S$-module directed by $(Z, Y, U)$, or $(Z, Y, U)$-$\Delta$-module is a pre-$\Delta_S$-module $(M_e)_{e, \varphi_{a,e}}$ over $Z$ together with a family of derivation-lifts

$$\tilde{\nabla}_{e,e\setminus i} : U_i \to \text{Hom}_Z(M_e, M_{e\setminus i}) \quad \text{for } i \in [k], \ e \subseteq [k]$$

satisfying the following additional axioms:

\begin{itemize}
  \item[\Delta 3)] each $M_e$ is of the form $\text{Coind}^Z_{Y + U_e} A_e$, where $U_e := \sum_{i \in e} U_i$ and $A_e$ is a topological $(Y + U_e)$-module;
  \item[\Delta 4)] whenever $a \subseteq e$, the homomorphism $\varphi_{a,e}$ is of the form $\text{Coind}^Z_{Y + U_e} \varphi'_{a,e}$, where
    $$\varphi'_{a,e} : \text{Coind}^Y_{Y + U_e} \varphi'_{a,e} \to A_e$$
    is a continuous homomorphism of $(Y + U_e)$-modules.
  \item[\Delta 5)] if $i \notin e$ then $\tilde{\nabla}_{e,e} = d^{U_i}$,
  \item[\Delta 6)] if $a \subseteq e \subseteq S$ and $i \in S$ then
    $$\varphi_{a \setminus i,e\setminus i} \circ \tilde{\nabla}_{a,e\setminus i} = \tilde{\nabla}_{e,e\setminus i} \circ \varphi_{a,e} \quad \forall a \subseteq e, \ \forall i,$$
  \item[\Delta 7)] and if $e \subseteq S$ and $i, j \in S$ then
    $$\tilde{\nabla}_{u_i,e\setminus \{i,j\}} \circ \tilde{\nabla}_{u_j,e\setminus \{i,j\}} = \tilde{\nabla}_{u_j,e\setminus \{i,j\}} \circ \tilde{\nabla}_{u_i,e\setminus i} \quad \forall (u_i, u_j) \in U_i \times U_j.$$
\end{itemize}

In the following, we will very often write $\mathcal{M}^Z_{Y,U}$ as a short-hand to indicate that $\mathcal{M}$ is a $(Z, Y, U)$-$\Delta$-module, somewhat similarly to the differential geometer’s notation ‘$M^n$’ for an $n$-dimensional manifold.

We will usually denote a $\Delta$-module by a script symbol such as $\mathcal{M}$, or simply by $(M_e)_{e}$, and write its associated structure morphisms and derivation-lifts as $\varphi_{a,e}$ and $\tilde{\nabla}_{a,e,e\setminus i}$ when necessary. In many instances $S$ will be $[k]$. Also, $M_{(i)}$ will usually be abbreviated to $M_i$ for $i \in S$.

The following simple lemma will not be used in the sequel, but begins to give some idea of how the extra axioms of a $\Delta$-module dictate its structure.
Lemma 5.4. Let $U = (U_1, \ldots, U_k)$, let $M = (M_e)_e$ be a non-zero $(Z, Y, U)$-$\Delta$-module, and suppose that $M$ is $U_i$-invariant (that is, every element of every $M_e$ is $U_i$-invariant). Then for each $e \subseteq [k]$, $M_e = (0)$ unless $Y + U_e \geq U_i$.

Proof. By axiom $(\Delta 3)$, each $M_e$ is of the form $\operatorname{Coind}^Z_{Y + U_e} A_e$ for some $(Y + U_e)$-module $A_e$. Such a module can be $U_i$-invariant only if

- either $U_i$ acts trivially on the set of cosets of $Y + U_e$, hence $Y + U_e \geq U_i$,
- or $A_e = (0)$, hence $M_e = (0)$.

Remark. The imposition of derivation-lifts in Definition 5.3 adds considerable weight to the definition of a $\Delta$-module. I do not know how much of the theory below could be developed without them: they are quite essential to the approach we take to our main theorems later. However, it might suffice to impose a weaker structure in their place, and build a corresponding theory around that, although the work required might be much longer. Some more detailed suggestions in this direction will be offered among the closing remarks.

Properties $(\Delta 1 - 7)$ are rather easier to understand in a special subclass which will include our first important examples.

Definition 5.5 (Inner $\Delta$-module). A $(Z, Y, U)$-$\Delta$-module is inner if all its structure morphisms $\varphi_{a,e}$ are closed and injective. In this case, by the Theorem 2.2, we may simply identify $M_a$ with its image $\varphi_{a,S}(M_a) \subseteq M_S$, and so regard our $\Delta$-module as a distinguished family of submodules of the module $M_S$.

In this case each $\varphi_{a,e}$ is actually an isomorphism $M_a \xrightarrow{\varphi_{a,e}} M_e$, and the defining equation of a derivation-lift shows that we must simply have

$$\varphi_{e \setminus i,e} \circ \tilde{\nabla}^e_{i,e} = d_u \quad \forall e, i, u \in U_i.$$ 

This implies that the homomorphisms $\tilde{\nabla}$ are simply given by ordinary differencing, following by the isomorphism $\varphi_{e \setminus i,e}^{-1} : \varphi_{e \setminus i,e}(M_{e \setminus i}) \rightarrow M_{e \setminus i}$. Indeed, given module and homomorphism data $(M_e)_e, (\varphi_{a,e})_{a \subseteq e}$ satisfying axioms $(\Delta 1 - 4)$ and with each $\varphi_{a,e}$ closed and injective, this specification gives the unique derivation-lifts that satisfy axioms $(\Delta 5 - 7)$ of a $\Delta$-module, if such exist.

Unfortunately, we will see that, even to study $\text{PD}^\infty$Es, we cannot limit ourselves to inner $\Delta$-modules. This is because innerness is not preserved by the operation of forming cohomology groups, which will be a crucial tool in our analysis. However, we will still need a sense in which applying differencing operators
moves one down to lower modules in the family. This feature is retained by forming cohomology, and it is this that motivates the more abstract notion of our ‘derivation-lifts’. Some non-inner examples will become available after cohomology $\Delta$-modules have been introduced formally in Section 9.

Remark. If $M$ is any $\Delta_S$-module in which all the structure morphisms $\varphi_{a,e}$ are closed, but not necessarily injective, then the family of images $\varphi_{a,S}(M_a)$ may nevertheless not define an inner $\Delta$-module, because the maps $\varphi_{a,S}$ may not be co-induced over the smaller subgroup $Y + U_a$. Only in the case of injective structure morphisms is this problem avoided.

Given a $(Z,Y,U)\Delta_S$-module $(M_e)_e$, any subcollection $(M_a)_{a \subseteq e}$ for $e \subseteq S$ inherits the structure of a $(Z,Y,U \rvert e)\Delta_e$-module in the obvious way, where $U \rvert e := (U_i)_{i \in e}$.

It will also be useful to note the following, a trivial consequence of Definition 5.3 and the relation (8).

Lemma 5.6. A $\Delta_S$-module directed by $(Z,Y,U)$ is also directed by $(Z,Y',U)$ whenever $Y \leq Y' \leq Z$.

The importance of a particular choice of $Y$ will appear later when we come to place some extra demands on the $(Y + U_e)$-modules $A_e$ in axioms ($\Delta 3$) and ($\Delta 4$).

The next definitions now almost write themselves.

Definition 5.7 ($\Delta$-submodules). Suppose that $\mathcal{M} = (M_e)_e$ is a $(Z,Y,U)\Delta_S$-module with structure morphisms $\varphi_{a,e}$ and derivation-lifts $\natural_{e,e}^a$. A collection of $Z$-submodules $N_e \leq M_e$ is compatible with these morphisms and derivation-lifts if each $N_e$ is of the form $\text{Coind}^Z_{Y + U_e} B_e$ for some $(Y + U_e)$-submodule $B_e \leq A_e$, where $A_e$ is as in axiom ($\Delta 3$), and also

$$\varphi_{a,e}(N_a) \subseteq N_e \quad \forall a \subseteq e \subseteq S$$

and

$$\natural^a_{e,e} (N_e) \subseteq N_{e \setminus i} \quad \forall e \subseteq S, \ i \in S, \ u \in U_i.$$

In this case, the associated $\Delta_S$-submodule of $\mathcal{M}$ is the family of submodules $(N_e)_e$ together with the restricted structure morphisms and derivation-lifts

$$\varphi_{a,e} \rvert N_a \quad \text{and} \quad u \mapsto \natural^a_{u,e} \rvert N_e.$$

Definition 5.8. Suppose that $\mathcal{M} = (M_e)_e$ is a $(Z,Y,U)\Delta$-module and $\mathcal{N} = (N_e)_e \leq (M_e)_e$ is a $(Z,Y,U)\Delta$-submodules. Then its quotient is the resulting $\Delta$-module $\mathcal{M} / \mathcal{N} = (N_e/M_e)_e$. It is immediate that it is also a $\Delta$-module directed by $(Z,Y,U)$.  

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Another definition that generalizes easily from modules to $\Delta$-modules is direct sum.

**Definition 5.9.** If $\mathcal{M}_1$ and $\mathcal{M}_2$ are both $(Z, Y, U)$-$\Delta_S$-modules, then their **direct sum** $\mathcal{M}_1 \oplus \mathcal{M}_2$ is the $\Delta_S$-module with constituent modules $M_{1,e} \oplus M_{2,e}$ for $e \subseteq S$, and similarly direct sums of all structure morphisms and derivation-lifts. It is immediate that this is also directed by $(Z, Y, U)$.

### 5.2 $\Delta$-morphisms

**Definition 5.10 ($\Delta$-morphisms).** Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $(Z, Y, U)$-$\Delta_S$-modules. Then a **$\Delta$-morphism** from $\mathcal{M}$ to $\mathcal{N}$ is a family $\Psi = (\Psi_e)_e$ of continuous $Z$-module homomorphisms $\Psi_e : M_e \to N_e$ such that:

- **(Consistency with structure morphisms)** $\Psi_e \circ \varphi_{a,e} = \varphi_{a,e} \circ \Psi_a$ whenever $a \subseteq e \subseteq S$;

- **(Consistency with derivation-lifts)** $\Psi_e \circ \nabla e : M_e \to N_e$ whenever $e \subseteq S$, and $i \in S$.

- **(Coinduced)** for every $e \subseteq S$, the homomorphism $\Psi_e$ is coinduced from a homomorphism of $(Y + U_e)$-modules.

A sequence of $\Delta$-morphisms $(M_e)_e \xrightarrow{\Psi} (N_e)_e \xrightarrow{\Phi} (P_e)_e$ is **exact** if each sequence of continuous homomorphisms $M_e \xrightarrow{\Psi_e} N_e \xrightarrow{\Phi_e} P_e$ is algebraically exact.

This above situation may be denoted by $\Psi : (M_e)_e \to (N_e)_e$, or just $\Psi : \mathcal{M} \to \mathcal{N}$, if no confusion can arise.

Given such a $\Delta$-morphism, its intertwining properties for the structure morphisms and derivation-lifts give immediately that the family of kernels $(\ker \Psi_e)_e$ is compatible with the structure morphisms and derivation lifts of $\mathcal{M}$. These therefore comprise a $\Delta$-submodule of $\mathcal{M}$, called the **kernel** $\Delta$-module of $\Psi$ and denoted $\ker \Phi$.

Similarly, the intertwining properties of $\Psi$ also imply that the family of images $(\img \Psi_e)_e$ comprise a $\Delta$-submodule of $\mathcal{N}$, called the **image** $\Delta$-module and denote $\img \Psi$.

Many of the nontrivial examples of $\Delta$-submodules that we will meet later arise as kernel or image $\Delta$-modules. Indeed, all $\Delta$-submodules may be represented as kernel $\Delta$-modules: if $\mathcal{N} \leq \mathcal{M}$ are a $(Z, Y, U)$-$\Delta_S$-module and $\Delta_S$-submodule respectively, then the quotient homomorphisms $M_e \to M_e/N_e$ for $e \subseteq S$ are easily seen to define a $\Delta$-morphism, and $\mathcal{N}$ is their kernel $\Delta$-module.
5.3 Structure complexes and structural properties

We now revisit the complexes introduced in Subsection 1.2. To do this, it is easiest to focus on indexing sets \( S \) that are totally ordered, so here we will simply assume that \( S = [k] \) for some \( k \).

It will help to have some more bespoke notation. Suppose that \( a \subseteq e \) are finite subsets of \( \mathbb{N} \) with \( |e| = |a| + 1 \), and let these sets be enumerated as \( e = \{ i_1 < i_2 < \ldots < i_s \} \) and \( a = e \setminus i_j \) for some \( j \in \{ 1, 2, \ldots, s \} \). Then the quantity \( \text{sgn}(e : a) \) is defined to be \( (-1)^{j-1} \). (This is where we make use of the chosen order on \( [k] \).)

Now suppose that \((M_e)_{e} \) is a \((Z,Y,U)\)-\( \Delta \)-module with structure morphisms \( (\varphi_{a,e})_{a \subseteq e} \), and that \( e \subseteq [k] \) is nonempty. Then we may construct from it the following sequence of modules and homomorphisms:

\[
0 \xrightarrow{\partial_0} M_\emptyset \xrightarrow{\partial_1} \bigoplus_{i \in e} M_i \xrightarrow{\partial_2} \bigoplus_{a \in \binom{e}{2}} M_a \xrightarrow{\partial_3} \cdots \xrightarrow{\partial_{|e|}} M_e,
\]

(12)

where \( \partial_\ell \) is defined by

\[
(\partial_\ell m)_b = \sum_{a \in \binom{b}{\ell-1}} \text{sgn}(b : a)\varphi_{a,b}(m_a) \quad \text{for } b \in \binom{e}{\ell}, \ m = (m_a)_{a \in \binom{e}{\ell-1}}.
\]

(13)

This construction is one of the central ideas of this paper. It is clearly motivated by simplicial cohomology: if every \( M_a \) is equal to some fixed Abelian group, then (12) is the simplicial cohomology complex of the \( k \)-simplex with coefficients in that group.

A quick check verifies the classical equality

\[
\text{sgn}(a \cup \{ s \} : a)\text{sgn}(a \cup \{ t \} : a) + \text{sgn}(a \cup \{ s, t \} : a \cup \{ s \})\text{sgn}(a \cup \{ s, t \} : a \cup \{ t \}) = 0
\]

(14)

whenever \( s \neq t \) and \( a \cap \{ s, t \} = \emptyset \), and as usual this implies the following:

**Lemma 5.11.** The sequence (12) is a complex: that is, \( \partial_{\ell+1} \partial_\ell = 0 \) for every \( \ell \).

The complex (12) is called the **structure complex of** \( M \) at \( e \), and \( \partial_\ell \) is the **boundary operator at position** \( \ell \) of that structure complex. The structure complex at \([k]\) will usually be called the **top** structure complex.

In case the \( M_a \) are all the same, the complex (12) is exact if \( e \neq \emptyset \). In general, the homology of the structure complex (12) is very reminiscent of the cohomology of a presheaf with respect to a given cover of a topological space. (We will return to this connection in the closing remarks of the paper.) In general, different modules
$M_a$ appear in the direct summands at each position of (12), so it need not be exact. However, a crucial feature of the examples of interest will be that the failure of exactness in (12) – that is, its homology – has some special structure.

**Definition 5.12 (Structural closure).** Let $\mathcal{M}$ be a $\Delta$-module, $e \subseteq [k]$ and $\ell \leq |e|$. Then $\mathcal{M}$ is **structurally closed at** $(e, \ell)$ if the boundary operator at position $\ell$ of its structure complex at $e$ is closed. It is **structurally closed** (‘$S$-closed’) if it is structurally closed at every $(e, \ell)$.

**Definition 5.13 (Modesty).** A $(Z, Y, U)\Delta$-module is **$\ell_0$-modest** if it is Polish and for each $e \subseteq [k]$, its structure complex at $e$ is co-induced over $Y + U_e$ from a complex of $(Y + U_e)$-modules that has $\ell_0$-discrete homology.

Our applications for this theory will also require that we consider $\Delta$-modules in which some of the lower-level structure complexes have a special status.

**Definition 5.14 (Almost modesty).** Suppose that $0 \leq \ell_0 \leq k$. If $\mathcal{M}$ is a $\Delta_k$-module, then it is **$\ell_0$-almost modest** if it is Polish and for each $e$, its structure complex at $e$ is co-induced over $Y + U_e$ from a complex of $(Y + U_e)$-modules that has $\ell_0$-almost discrete homology.

Clearly an $\ell_0$-modest $\Delta$-module is also $\ell_0$-almost modest. To emphasize the difference, a modest $\Delta$-module will sometimes be called **strictly** modest.

If $|e| < \ell_0$, then $\ell_0$-almost modesty actually implies that the structure complex at $e$ is trivial, and hence that $M_e = 0$.

If $|e| = \ell_0$, then the structure complex of an $\ell_0$-almost modest $\Delta$-module $\mathcal{M}$ at $e$ is simply

$$0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow M_e \rightarrow 0,$$

since all modules indexed by sets smaller than $\ell_0$ vanish. So an $\ell_0$-almost modest $\Delta$-module $(M_e)_e$ is allowed to have a layer of modules $M_e$ with $|e| = \ell_0$ whose structural homology is not co-induced-of-discrete.

The intuition behind the almost-modest case is that the constituents $M_e$ with $|e| = \ell_0$ are the ‘main ingredients’ of $\mathcal{M}$, and the constituents indexed by larger subsets $a \subseteq [k]$ are the result of applying various ‘corrections’ to those at level $\ell_0$. Those corrections are described by the structural homology.

**Remark.** This definition should also remind one of Proposition 3.3: if $Z$ is a compact Abelian group and $A$ is a locally compact Polish $Z$-module, then $H^p_m(Z, A)$ is discrete in its quotient topology for $p \geq 1$, and locally compact, but not necessarily discrete, for $p = 0$. This cohomological phenomenon will be key to several proofs of almost modesty later.

$\triangleright$
The notion of modesty gives more firmness to the rôle of $Y$ in the definition of a $\Delta$-module. On the one hand, $Y$ must be large enough that the relevant modules and morphisms may always be co-induced over $Y + U_e$ for the relevant $e$. On the other hand, given a modest $\Delta$-module, we cannot freely make $Y$ larger as in Lemma 5.6 without disrupting the definition of modesty. This is because the structural homology of $M$ at $e$ must be co-induced over $Y + U_e$ for a discrete (hence, ‘small’) module, and replacing this with co-induction over $Y' + U_e$ for some larger $Y'$ may disrupt this.

Nevertheless, there is sometimes still a little flexibility in the choice of $Y$, as given by the following, which again follows trivially from the definitions.

**Lemma 5.15.** An $\ell_0$-almost (resp. strictly) modest $(Z,Y,U)\Delta$-module $(M_e)_e$ is also an $\ell_0$-almost (resp. strictly) modest $(Z,Y',U)\Delta$-module provided

- $Y \leq Y' \leq Z$, and
- one has $Y + U_e = Y' + U_e$ for every $e$ such that $M_e \neq 0$.

Our main technical theorems later will assert that the class of (almost) modest $\Delta$-modules is closed under certain natural operations. The first instance of this closure, however, requires only a trivial check, which we omit.

**Lemma 5.16.** If $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\ell_0$-almost (resp. strictly) modest $(Z,Y,U)\Delta$-modules then so is $\mathcal{M}_1 \oplus \mathcal{M}_2$. □

### 5.4 First examples

Several examples of inner $\Delta$-modules can be introduced immediately, including those that underly Theorems A and B.

**Example 5.17.** For any Polish Abelian group $A$, the $Z$-module $\mathcal{F}(Z,A)$ is a $(Z,\{0\},\ast)\Delta$-module, where $\ast$ denotes the empty subgroup-tuple. □

**Example 5.18.** Given $Z$ and also $U = \{U_i\}_{i=1}^k$, one may obtain a $(Z,0,U)\Delta$-module $\mathcal{G}$ simply by setting $C_e := \mathcal{F}(Z,A)$ for every $e$, setting $\varphi_{a,e}^e := \text{id}$ whenever $a \subseteq e$, and setting $\hat{\nabla}^{e,e,e\setminus i} := d$ for all $e$ and $i$. All of the axioms $(\Delta 1 - 7)$ are trivial verifications in this case. In particular, the structure complex at every nonempty $e$ is exact, because it amounts to computing the higher simplicial cohomology of the $k$-simplex with coefficients in the fixed group $\mathcal{F}(Z,A)$. By Lemma 5.6 this is also a $(Z,Y,U)\Delta$-module for any other $Y \leq Z$. Such a $\Delta$-module will be called a **constant** $\Delta$-module. □

**Example 5.19.** A close relative $\mathcal{L}$ of the constant $\Delta$-module $\mathcal{G}$ is obtained as follows: set $L_0 := 0$ and $L_e := C_e$ for every other $e$, and let the structure morphisms and derivation-lifts be either zero or the same as for $\mathcal{G}$, as appropriate.
Once again, axioms (Δ1 − 7) are easily verified, and Lemma 5.6 lets us interpret this as a \((Z, Y, U)\)-Δ-module for any other \(Y \leq Z\).

In this case, the structure complex at \(\emptyset\) is trivial, so there is no homology there. On the other hand, the structure complex at any nonempty \(e\) is the same as for \(\mathcal{C}\), except that the entry in position 0 is now \(L_0 = 0\), rather than \(C_0 = \mathcal{F}(Z, A)\). This has the effect of removing all homology at position \((e, 0)\), and replacing it with a homology group equal to \(\mathcal{F}(Z, A)\) in position \((e, 1)\).

This example can easily be generalized to truncating \(\mathcal{C}\) below any other fixed level \(\ell\); the details are left to the reader.

Example 5.20. Now let \(U_1, \ldots, U_k \leq Z\) be as in the Introduction, and let \((M_e)_e\) be the family of modules of solutions to the associated heirarchy of PD\(^{\text{ce}}\)Es. These are all closed submodules of \(\mathcal{F}(Z, A)\), and \(M_a \leq M_e\) whenever \(a \leq e\), so we may let \(\varphi_{a,e} : M_a \to M_e\) be the inclusion. Also, if \(f \in M_e\) and \(i \in e\), then the definition implies that \(d^{U_i} f \in M_{e \setminus i}\), so this uniquely defines a derivation-lift \(\tilde{\nabla}_{e,e \setminus i}\) from \(M_e\) to \(M_{e \setminus i}\). So these solution-modules together define an inner \((Z, 0, U)\)-Δ-module.

This will be called the \textbf{solution Δ-module} of the PD\(^{\text{ce}}\)E associated to \(U\). Note that \(M_{[k]}\) is precisely the module of solutions to that PD\(^{\text{ce}}\)E. The solution Δ-module will be the centre of attention in Section 11.

Example 5.21. Let \(U_1, \ldots, U_k \leq Z\), and for \(e \subseteq [k]\) define

\[
P_e := \bigoplus_{i \in e} \mathcal{F}(Z, A)^{U_i}.
\]

There is an obvious inclusion morphism \(\varphi_{a,e}^{\mathcal{P}} : P_a \to P_e\) whenever \(a \subseteq e\), since \(P_a\) is a direct summand of \(P_e\). On the other hand, if \(f = (f_i)_{i \in e} \in P_e\), \(j \in e\) and \(u \in U_j\), then

\[
d^{U_j} f = (d^{U_j} f_i)_{i \in e},
\]

which is identically zero in coordinate \(j\). We may therefore canonically identify this with its projection to \(P_{e \setminus j}\), and this defines the derivation-lift \(\tilde{\nabla}_{u, e \setminus j}\). This gives another inner \((Z, Y, U)\)-Δ-module \(\mathcal{P}\).

Example 5.22. We may now combine Examples 5.19 (denoted \(\mathcal{L}\)) and 5.21 (denoted \(\mathcal{P}\)) as follows. Regard both \(\mathcal{L}\) and \(\mathcal{P}\) as being directed by \((Z, 0, U)\). For each \(e \subseteq [k]\), there is a homomorphism

\[
\Psi_e : P_e = \bigoplus_{i \in e} \mathcal{F}(Z, A)^{U_i} \to L_e = \mathcal{F}(Z, A) : (f_i)_{i \in e} \mapsto \sum_{i \in e} f_i.
\]

It is easy to check that the family \((\Psi_e)_e\) verifies the axioms of a \((Z, 0, U)\)-Δ-morphism from \(\mathcal{L}\) to \(\mathcal{P}\), and so \(N := \ker \Psi\) is a \((Z, 0, U)\)-Δ-submodule of \(\mathcal{P}\). Each \(N_e\) is precisely the module of zero-sum tuples introduced in Subsection 1.2.
This kernel $\Delta$-module will be called the zero-sum $\Delta$-module associated to $U$, and it will be the focus of Section 12.

One could also study the image of this $\Delta$-morphism as another example of a $(Z, 0, U)$-$\Delta$-module, but we will leave this aside. ⊳

6 Aggrandizement, restriction and reduction

6.1 Aggrandizement

Suppose that $c \subseteq e$ is an inclusion of finite sets, that $Y \leq Z$ is an inclusion of compact Abelian groups, and that $U = (U_i)_{i \in e}$ is a family of closed subgroups of $Z$, and let $U |_c$ be the subfamily $(U_i)_{i \in c}$.

Let $\mathcal{M}$ be a $(Z, Y, U |_c)$-$\Delta$-module

**Definition 6.1 (Aggrandizement).** The aggrandizement of $\mathcal{M}$ to $U$, say $\mathcal{M}^\wedge$, is the $(Z, Y, U)$-$\Delta$-module consisting of the following:

- **(modules)** $M^\wedge_a := M_{a \cap c}$, $\forall a \subseteq e$.

- **(structure morphisms)** $\phi^\wedge_{a,b} := \phi_{a \cap c, b \cap c}$, $\forall a \subseteq b \subseteq e$.

- **(derivation-lifts)** $\tilde{\phi}^\wedge_{a,a} := \tilde{\phi}_{a \cap c, (a \cap c) \backslash i}$, $\forall a \subseteq e, i \in e$.

This aggrandizement is denoted $Ag_{\mathcal{M}}^U$, or $Ag_{\mathcal{M}}^e$ if the tuple $U$ is clear. If $\mathcal{N}$ is a $(Z, Y, U)$-$\Delta$-module of the form $Ag_{\mathcal{M}}^e$ for some $(Z, Y, U |_c)$-$\Delta$-module $\mathcal{M}$, then $\mathcal{N}$ is said to be aggrandized from $c$.

It is easy to check that $Ag_{\mathcal{M}}^e$ satisfies the axioms of a $\Delta$-module. Axioms $(\Delta 1 - 2)$ are immediate. For $(\Delta 3)$ and $(\Delta 4)$, observe that if $a \subseteq e$ then

$$M_{a \cap c} = Coind_{Y + U_{a \cap c}}^Z A_{a \cap c} = Coind_{Y + U_e}^Z A'$$

and

$$\phi^\wedge_{a \cap c, e \cap c} = Coind_{Y + U_{a \cap c}}^Z \phi'_{a \cap c, e \cap c} = Coind_{Y + U_e}^Z \phi'',$$

where

$$A' = Coind_{Y + U_e}^Z A_{e \cap c}$$

is a Polish $(Y + U_e)$-module and

$$\phi'' = Coind_{Y + U_e}^Z \phi'_{a \cap c, e \cap c}$$

is a continuous homomorphism of $(Y + U_e)$-modules. The remaining relations of $(\Delta 5 - 7)$ just require re-writing those for $\mathcal{M}$ itself.
The structure complexes of an aggrandizement of $\mathcal{M}$ bear a simple relation to those of $\mathcal{M}$ itself. This fact will be crucially important in the sequel. Its proof is closely related to the homotopy invariance of classical homology (or rather, the special case that every complete simplex has the same homology as a point).

**Lemma 6.2 (Homotopical Lemma).** Suppose that $c \subseteq [k]$ and that $\mathcal{M} = (M_e)_e$ is a $(Z, Y, U|c)-\Delta$-module, and let $\mathcal{N} := \text{Ag}_{U|c} \mathcal{M}$. For each $\ell \in \{0, 1, \ldots, k\}$, let

$$N^{(\ell)} := \bigoplus_{|e| = \ell} N_e,$$

and let

$$0 \xrightarrow{\partial_0} N_0 \xrightarrow{\partial_1} N^{(1)} \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_k} N^{[k]} \xrightarrow{0}$$

be the top structure complex of $\mathcal{N}$. Then this complex is split.

**Proof.** Pick some $s \in [k] \setminus c$, and now define

$$\xi_\ell : N^{(\ell+1)} \rightarrow N^{(\ell)}$$

by

$$\xi_\ell((n_b|b|=\ell+1)_e) := \begin{cases} 0 & \text{if } s \in e \\ \text{sgn}(e \cup s : e)n_{e\cup s} & \text{if } s \not\in e \end{cases}$$

Since $s \not\in c$, for any $e$ one has $(e \cup s) \cap c = e \cap c$. Therefore if $(n_b)_b \in N^{(\ell+1)}$ and $|e| = \ell$, then

$$n_{e\cup s} \in N_{e\cup s} = M_{(e\cup s)\cap c} = M_{e\cap c},$$

so $\xi_\ell$ does indeed take values in $N^{(\ell)}$.

It remains to verify that these maps have the required properties.

**Step 1.** Suppose that $n = (n_a)|a|=\ell+2 \in N^{(\ell+2)}$, and let $|e| = \ell$. If $s \in e$ then $(\xi_\ell n)_e = 0$ directly from the definition. On the other hand, if $s \not\in e$, then

$$(\xi_\ell n)_e = \text{sgn}(e \cup \{s\} : e)\xi_{\ell+1} n_{e\cup \{s\}} = 0,$$

where this vanishing is because $s \in e \cup \{s\}$.

**Step 2.** Now let $m = (m_e)_e \in N^{(\ell)}$. If $s \in e$, then one has

$$(\partial_\ell \xi_{\ell-1} m)_e = \sum_{a \in (e^c)_{\ell-1}} \text{sgn}(e : a)\varphi_a^{\ell}((\xi_{\ell-1} m)_a)$$

$$= \sum_{a \in (e^c)_{\ell-1}, a \not\in s} \text{sgn}(e : a)\varphi_a^{\ell}(\text{sgn}(a \cup \{s\} : a)m_{a\cup\{s\}})$$

$$= \text{sgn}(e : e \setminus s)\varphi_{e\setminus s, e}(\text{sgn}(e : e \setminus s)m_e) = m_e,$$

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whereas
\[(\xi \partial^\ell_{+1} m)_e = 0\]
directly from the definition of \(\xi\).

On the other hand, if \(s \not\in e\) then
\[
(\partial^\ell_{+1} m)_e = \sum_{a \in \binom{e}{\ell-1}} \text{sgn}(e : a) \varphi^N_{a,e}(\xi_{\ell-1} m)_a
\]
\[
= \sum_{a \in \binom{e}{\ell-1}} \text{sgn}(e : a) \varphi^N_{a,e}(\text{sgn}(a \cup \{s\} : a)m_{a \cup \{s\}}),
\]
whereas
\[
(\xi \partial^\ell_{+1} m)_e
= \text{sgn}(e \cup \{s\} : e)(\partial_{+1} m)_{e \cup \{s\}}
\]
\[
= \text{sgn}(e \cup \{s\} : e) \sum_{b \in \binom{e \cup \{s\}}{\ell}} \text{sgn}(e \cup \{s\} : b) \varphi^N_{b,e \cup \{s\}}(m_b)
\]
\[
= \text{sgn}(e \cup \{s\} : e) \varphi^N_{e \cup \{s\}}(\text{sgn}(e \cup \{s\} : e)m_e)
\]
\[
+ \text{sgn}(e \cup \{s\} : e) \sum_{a \in \binom{e}{\ell-1}} \text{sgn}(e \cup \{s\} : a \cup \{s\}) \varphi^N_{a,e}(m_{a \cup \{s\}}).
\]

Adding these last two expressions and recalling the identity (14), one sees that all
the terms cancel except for \(\varphi^N_{e \cup \{s\}}(m_e) = m_e\). This completes the proof.

**Corollary 6.3.** In the situation of the previous lemma, the structure complex of \(\mathcal{N}\)
at \(e\) is:

- **the same as for \(\mathcal{M}\) itself if \(e \subseteq c\), and**

- **split if \(e \not\subseteq c\).**

**Proof.** If \(e \subseteq c\), then the structure complex for \(\mathcal{N}\) at \(e\) involves only the modules
\(M_{a \cap e} = M_a\) for \(a \subseteq e \subseteq c\), so the first case is clear. For the second, observe that
if \(e \setminus c \neq \emptyset\), then the structure complex of \(\mathcal{N}\) at \(e\) is
\[
0 \rightarrow M_0 \xrightarrow{\partial_1} \bigoplus_{i \in e} M_{c \cap (i)} \xrightarrow{\partial_2} \bigoplus_{a \in \binom{e}{2}} M_{c \cap a} \xrightarrow{\partial_3} \cdots \xrightarrow{\partial_k} M_{c \cap e},
\]
so this case follows from Lemma 6.2.\(\square\)
6.2 Restriction and reduction

Now let $\mathcal{M} = (M_e)_e$ be a $(Z, Y, U)\Delta_{[k]}$-module, and let $c \subseteq [k]$.

**Definition 6.4** (Restriction). The **restriction of** $\mathcal{M}$ **to** $c$, denoted $\mathcal{M} \downarrow c$, is the $(Z, Y, U \downarrow c)\Delta_c$-module whose constituent modules, structure morphisms and derivation lifts are precisely those of $\mathcal{M}$ that are indexed by subsets of $c$:

- **(modules:)** $M_a$ for $a \subseteq c$.
- **(structure morphisms:)** $\varphi_{a,b}^\mathcal{M}$ for $a \subseteq b \subseteq c$.
- **(derivation-lifts:)** $\tilde{\nabla}_{a,a \setminus i}^\mathcal{M}$ for $a \subseteq c$, $i \in c$.

It is obvious that these data do form a $(Z, Y, U \downarrow c)\Delta_c$-module, and that it inherits any of the properties of $S$-closed, modesty or almost-modesty if they are possessed by $\mathcal{M}$. It is also obvious that the structure complexes of $\mathcal{M} \downarrow c$ are simply those of $\mathcal{M}$ at subsets of $c$.

We can now combine restriction with aggrandizement.

**Definition 6.5** (Reduction). The **reduction of** $\mathcal{M}$ **at** $c$ is the $\Delta$-module $\mathcal{M} \downarrow Q = Ag^k_c(M \downarrow c)$.

Like $\mathcal{M}$ itself, all of its reductions are $(Z, Y, U)\Delta$-modules. Intuitively, it is simply the restriction $\mathcal{M} \downarrow c$ re-interpreted so that it is still directed by $(Z, Y, U)$.

**Lemma 6.6.** If $c \subseteq [k]$ and $e \subseteq [k]$, then the structure complex of $\mathcal{M} \downarrow Q$ at $e$ is

- the same as for $\mathcal{M}$ itself if $e \subseteq c$, and
- split if $e \nsubseteq c$.

**Proof.** This follows directly from Corollary 6.3.

**Corollary 6.7.** If $\mathcal{M}$ is a $(Z, Y, U)\Delta$-module and $c \subseteq [k]$, then $\mathcal{M} \downarrow Q$ is $\ell_0$-almost (resp. strictly) modest if and only if $\mathcal{M} \downarrow c$ is $\ell_0$-almost (resp. strictly) modest, and both are implied if $\mathcal{M}$ itself is $\ell_0$-almost (resp. strictly) modest.

**Proof.** This follows at once from Lemma 6.6 since the structure complexes of $\mathcal{M} \downarrow Q$ are the same as for $\mathcal{M} \downarrow c$, and they are among the structure complexes of $\mathcal{M}$ itself.

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7 A partial order on subgroup tuples

We will soon turn to some more results promising that (almost) modesty is preserved by some operations on $\Delta$-modules. These more difficult theorems will be proved by induction the data $(Z, Y, U)$, so we must first introduce the partial well-order that will direct this induction.

First, let us write $D$ for the class of all triples $(Z, Y, U)$ in which $Z$ is a compact separable Abelian group, $Y \leq Z$ is a closed subgroup and $U$ is a finite (possibly empty) tuple of closed subgroups of $Z$. (This $D$ is not formally a set, but the worried reader can simply restrict attention to only those $Z$ that are themselves closed subgroups of $T^\mathbb{N}$.)

**Definition 7.1** (Complexity order on subgroup data). Suppose that

$$(Z, Y, (U_1, \ldots, U_k)), (Z', Y', (U'_1, \ldots, U'_{k'})) \in D.$$ 

Then the tuple of data $(Z, Y, U)$ strictly precedes $(Z', Y', U')$, written $(Z, Y, U) \prec (Z', Y', U')$, if

- either $k < k'$,
- or $k = k'$, but

$$|\{i \leq k \mid U_i \leq Y\}| > |\{i' \leq k \mid U'_{i'} \leq Y'\}|.$$ 

The data $(Z, Y, U)$ are pure if $U_i \leq Y$ for every $i$.

This is easily seen to be a partial well-ordering. Its importance is that some of our deeper theorems about $\Delta$-modules — most notably, Theorem 10.1 in Section 10 — will be proved by induction on this partial order.

The Introduction discussed the idea of applying a difference operator to a zero-sum tuple of functions to obtain a 1-cocycle into the space of zero-sum tuples for a smaller tuple of subgroups. A more abstract version of this idea will be the basis for the induction on the partial order $\preceq$ that proves our main theorems. However, in some cases this reduction will work only for non-pure modules, and a separate proof will have to be given in the non-pure case. That case will usually depend upon the following simple structure theorem.

Let $(Z, Y, U) \in D$ be pure, and let $\mathcal{M} = (M_e)_e$ be a modest $(Z, Y, U)$-$\Delta$-module, and for $e \subseteq [k]$ and $\ell \leq |e|$ let $\partial_{e, \ell}$ be the boundary morphism in position $\ell$ of the structure complex of $\mathcal{M}$ at $e$. Let $K_{e, \ell} := \ker \partial_{e, \ell}$ and $I_{e, \ell} := \img \partial_{e, \ell}$.

**Proposition 7.2** (Structure of pure $\Delta$-modules). In the situation above, all the $M_e$ and also all the $I_{e, \ell}$ and $K_{e, \ell}$ are co-induced over $Y$ from discrete $Y$-module.
Proof. This is a simple induction on $|e|$.

When $e = \emptyset$, this follows directly from the modesty of the structure complex

$$0 \rightarrow M_\emptyset \rightarrow 0,$$

whose homology is $M_\emptyset$ itself.

Now given nonempty $e \subseteq [k]$, and assuming the conclusion is already known for all $a \subseteqneq e$, we need only inspect the structure complex at $e$. To lighten notation, we explain this in case $e = [k]$. With all kernels and images inserted, the structure complex reads

$$0 \rightarrow K_0 \hookrightarrow M_\emptyset \rightarrow I_0 \hookrightarrow K_1 \hookrightarrow M^{(1)} \rightarrow I_1 \hookrightarrow K_2 \hookrightarrow M^{(2)} \rightarrow I_2 \hookrightarrow \cdots \rightarrow I_{k-1} \hookrightarrow K_k = M_{[k]},$$

where all the morphisms here are co-induced over $Y$.

Here $K_0$ is a submodule of $M_\emptyset$ that is still co-induced over $Y$, so it is also co-induced-of-discrete over $Y$. The result now follows for all the remaining modules $I_\ell$ and $K_\ell$, including $K_k = M_{[k]}$, by induction on $\ell$. Given the desired structure for $K_\ell$ and $M^{(\ell)}$ with $\ell \leq k - 1$, it follows for $I_\ell$ in view of the presentation

$$K_\ell \hookrightarrow M^{(\ell)} \twoheadrightarrow I_\ell.$$  

On the other hand, each $K_\ell/I_{\ell-1}$ is assumed to be co-induced-of-discrete over $Y$, so given the desired structure for $I_{\ell-1}$, it follows also for $K_\ell$ owing to the presentation

$$I_{\ell-1} \hookrightarrow K_\ell \twoheadrightarrow K_\ell/I_{\ell-1}.$$

$\square$

8 Short exact sequences

The next natural notion for modules that should be extended to $\Delta$-modules is that of short exact sequences. The basic definition is obvious.

Definition 8.1. A short exact sequence of $(Z,Y,U)$-$\Delta$-modules is a sequence of $\Delta$-morphisms

$$0 \rightarrow (M_e)_e \rightarrow (N_e)_e \rightarrow (L_e)_e \rightarrow 0$$

which is exact in the sense of Definition 5.10.

Our interest here will be in how the properties of two of the $\Delta$-modules in a short exact sequence control the properties of the third.
Proposition 8.2. Suppose that

\[ 0 \rightarrow \mathcal{M} \xrightarrow{\Phi} \mathcal{N} \xrightarrow{\Psi} \mathcal{P} \rightarrow 0 \]

is a short exact sequence of $\Delta$-modules in which

\[ 0 \rightarrow M_e \xrightarrow{\Phi_e} N_e \xrightarrow{\Psi_e} P_e \rightarrow 0 \]

is an algebraically exact sequence of Polish modules for every $e$.

(1) If $\mathcal{M}$ and $\mathcal{N}$ are $\ell_0$-almost (resp. strictly) modest, then so is $\mathcal{P}$.

(2) If $\mathcal{M}$ and $\mathcal{P}$ are $\ell_0$-almost (resp. strictly) modest, then so is $\mathcal{N}$.

(3) If $\mathcal{N}$ and $\mathcal{P}$ are $\ell_0$-almost (resp. strictly) modest, and if $M_e = 0$ whenever $|e| \leq \ell_0$, then $\mathcal{M}$ is $(\ell_0 + 1)$-almost (resp. strictly) modest.

It seems to be necessary that we assume that all of the individual modules $M_e$, $N_e$ and $P_e$ are Polish a priori. I do not believe that knowing this for only two of the sequences is enough to imply it for the third.

**Proof.** We give the proof in the case of $\ell_0$-almost modesty, which effectively contains the strictly modest case. For each $e \subseteq [k]$, the structure complexes of $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ fit together into the vertical short exact sequence

\[ 0 \rightarrow M^{(\ell - 1)} \rightarrow M^{(\ell)} \rightarrow M^{(\ell + 1)} \rightarrow \cdots \rightarrow M^{[k]} \rightarrow 0 \]

\[ 0 \rightarrow N^{(\ell - 1)} \rightarrow N^{(\ell)} \rightarrow N^{(\ell + 1)} \rightarrow \cdots \rightarrow N^{[k]} \rightarrow 0 \]

\[ 0 \rightarrow P^{(\ell - 1)} \rightarrow P^{(\ell)} \rightarrow P^{(\ell + 1)} \rightarrow \cdots \rightarrow P^{[k]} \rightarrow 0 \]

The three conclusions result from applying Lemma 4.3, 4.4 or 4.5 to this picture, as appropriate. \qed
Remark. Proposition \( \text{8.2} \) and even Lemmas \( \text{4.3, 4.5} \) do not generalize to general \( \Delta \)-modules whose structural homology is locally compact.

The problem here is apparent already for pure \( \Delta \)-modules, even if we assume closure of all constituents and all structure morphisms. Put another way, one can construct short exact sequences of complexes consisting entirely of locally compact groups and closed homomorphisms, such that two of those complexes are closed, but the third is not.

For instance, let \( \alpha \in \mathbb{R} \) be irrational, let \( \varphi, \psi : \mathbb{Z} \rightarrow \mathbb{R} \) be respectively the obvious inclusion and the map \( n \mapsto \alpha n \), and let \( \overline{\psi} : \mathbb{Z} \rightarrow \mathbb{T} \) be the composition of \( \psi \) with the quotient \( \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} = \mathbb{T} \). Then the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \varphi \\
\mathbb{Z} & \rightarrow & \mathbb{R} \\
\downarrow \psi & & \downarrow \text{id} \\
\mathbb{Z} & \rightarrow & \mathbb{T} \\
\end{array}
\]

gives an example with closed morphisms in the first and second rows, but not the third; the diagram

\[
\begin{array}{ccc}
\mathbb{Z} & \rightarrow & \mathbb{T} \\
\downarrow \overline{\psi} & & \downarrow \text{id} \\
\mathbb{R} & \rightarrow & \mathbb{T} \\
\downarrow \psi & & \downarrow \text{mod 1} \\
\mathbb{T} & \rightarrow & 0 \\
\end{array}
\]

gives an example with closed morphisms in the second and third rows, but not the first; and the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \mathbb{T} \\
\downarrow & & \downarrow \text{id} \\
\mathbb{Z} & \rightarrow & \mathbb{T} \\
\downarrow \overline{\psi} & & \downarrow \text{id} \\
\mathbb{Z} & \rightarrow & 0 \\
\end{array}
\]

has closed morphisms on the first and last rows, but not the second. Each of these may be interpreted as a short exact sequence of, say, \((\mathbb{Z}, \mathbb{Z}, (U))\)-\( \Delta \)-modules with Polish constituents for any \( U \leq \mathbb{Z} \), if one desires. \( \triangleright \)
9 Cohomology

Suppose that $\mathcal{M} = (M_e)_e$ is a $(Z, Y, U)$-$\Delta$-module in which every $M_e$ is Polish, and that $W \leq Z$ is a closed subgroup. Then for each $p \geq 0$ we may form a new $\Delta$-module by applying the functor $\Pi^p_m(W, -)$ to all the modules, morphisms and derivation-lifts of $\mathcal{M}$.

To be explicit, the new data are the following.

- (Modules:) $\Pi^p_m(W, M_e)$ for $e \subseteq [k]$.

- (Structure morphisms:) These are obtained by composing cocycles with the structure morphisms $\varphi_{a,e} : M_a \rightarrow M_e$. If $d_W f \in B^p(W, M_a)$, then of course
  \[ \varphi_{a,e} \circ (d_W f) = d_W (\varphi_{a,e} f) \in B^p(W, M_e), \]
  so this composition sends coboundaries to coboundaries, and hence descends to a suitable structure morphism $\Pi^p_m(W, M_a) \rightarrow \Pi^p_m(W, M_e)$.

- (Derivation-lifts:) These are obtained similarly, as compositions with the derivation-lifts of $\mathcal{M}$ itself. This requires just a little more care. Certainly if $\sigma \in Z^p(W, M_e)$, then one easily verifies that $u \mapsto \nabla^{e,e\setminus i} u \circ \sigma$ satisfies (11), and it is jointly continuous as a function of $u$ and $\sigma$. If $\sigma = d_W f$ is a coboundary, then
  \[ \nabla^{e,e\setminus i} (d_W f) = d_W (\nabla^{e,e\setminus i} f), \]
  so this image takes values among coboundaries, and hence these also descend to derivation-lifts in $Z^1(U_i, \text{Hom}(\Pi^p_m(W, M_e) \rightarrow \Pi^p_m(W, M_{e\setminus i}))$.

Each $\Pi^p_m(W, M_a)$ has a natural quotient topology, and we consider them equipped as such, but these quotient topologies need not be Polish (because they may not even be Hausdorff).

The functoriality of $\Pi^p_m(W, -)$ means that it preserves all the relations among the structure morphisms and derivation lifts of $\mathcal{M}$. Lemma 3.4 together with the relation (8) show that if a module morphism is co-induced over some subgroup $Y \leq Z$, then its image under $\Pi^p_m(W, -)$ is still co-induced over $Y + W$. Lastly, it is easy to check that the new derivation-lifts still define continuous maps

\[ U_i \times \Pi^p_m(W, M_e) \rightarrow \Pi^p_m(W, M_{e\setminus i}). \]

Altogether this proves the following.
Lemma 9.1. The above construction defines a \((Z, Y + W, U)\)-\(\Delta\)-module. □

Definition 9.2. The new \(\Delta\)-module constructed above is the \(p\)th cohomology \(\Delta\)-module of \(\mathcal{M}\), and is denoted by \(H^p_m(W, \mathcal{M})\).

Remarks. (1) If \((M_e)_e\) is an inner \(\Delta\)-module, then nevertheless the resulting cohomology \(\Delta\)-modules \((H^p_m(W, M_e))_e\) need not be inner: an injection \(\varphi_{a,e} : M_a \to M_e\) need not give rise to an injection on cohomology \(H^p_m(W, M_a) \to H^p_m(W, M_e)\). This wrinkle is the main reason for not restricting our attention to inner modules altogether.

(2) The construction of cohomology \(\Delta\)-modules suggests a more general definition of functors between categories of \(\Delta\)-modules for different directing tuples of groups. Cohomology \(H^p_m(W, -)\) should then be an example of such a functor, and another should be given by applying \(\text{Coind}_Y^Z : \text{PMod}(Y) \to \text{PMod}(Z)\) to any \(\Delta\)-module with constituents from \(\text{PMod}(Y)\). I doubt there are any surprises in setting up such functors, but we will not use any other examples in the sequel, so we leave them aside.

We will prove later (Theorem 9.5) that modesty and almost modesty are preserved under forming cohomology, but that will require considerably more work. Here we record a few simpler relations.

Lemma 9.3. Suppose that \((Z, Y, U)\) is a tuple of subgroup data with \(U = (U_i)_{i \in c}\), that \(c \subseteq [k]\) and that \(\mathcal{M}\) is a \((Z, Y, U \upharpoonright c)\)-\(\Delta\)-module. Suppose also that \(W \leq Z\) and \(p \geq 0\). Then
\[
H^p_m(W, \text{Ag}_{c[k]} \mathcal{M}) = \text{Ag}_{c[k]} H^p_m(W, \mathcal{M}).
\]

Proof. This is an immediate consequence of Definitions 6.1 and 9.2. For instance, the left-hand derivation lifts are defined at the level of relative cocycles by
\[
\nabla^p(W, \text{Ag}_{c[k]} \mathcal{M}, a, a) \upharpoonright c : \sigma \mapsto \nabla^p(W, \text{Ag}_{c[k]} \mathcal{M}, a, a) \upharpoonright c \circ \sigma = \nabla^p(W, \mathcal{M}, a \cap c, a \cap c) \circ \sigma,
\]
which agrees with the derivation-lift acting on \(Z^p(W, Q_{a \cap c}, P_{a \cap c})\). □

Lemma 9.4. For any pre-semi-functional \(\Delta\)-module \(\mathcal{M}\), any \(W \leq Z\), any \(p \geq 0\) and any \(c \subseteq [k]\) one has
\[
H^p_m(W, \mathcal{M} \upharpoonright c) = (H^p_m(W, \mathcal{M})) \upharpoonright c
\]
and
\[
H^p_m(W, \mathcal{M} \downharpoonright c) = (H^p_m(W, \mathcal{M})) \downharpoonright c.
\]

Proof. This follows at once from the commutativity of all the relevant diagrams. □
9.1 Modesty of cohomology $\Delta$-modules

**Theorem 9.5.** Let $\mathcal{M}$ be an $\ell_0$-almost modest $(Z,Y,U)$-$\Delta$-module. Then the $(Z,W+Y,U)$-$\Delta$-module $\tilde{\Pi}_m^p(W,\mathcal{M})$ is $\ell_0$-almost modest, and it is modest in case $p \geq 1$ or $\mathcal{M}$ is modest.

**Remark.** A crucial feature of this theorem is that even if $\mathcal{M}$ is only almost modest, the cohomology $\Delta$-modules $H^p_m(W,M)$ are strictly modest for all $p \geq 1$: that is ‘higher degree cohomology converts almost modesty into strict modesty’. This will result from the fact that if $A$ is any locally compact Polish $W$-module then $H^p_m(W,A)$ is discrete for $p \geq 1$ (Proposition 3.3).

Let us now fix a $(Z,Y,U)$-$\Delta_{[k]}$-module $\mathcal{M} = (M_e)_e$ and another subgroup $W \leq Z$, and set up some notation. First, having fixed $W$, we will lighten notation in the rest of this section by abbreviating $H^p_m(W,\mathcal{M}) := H^p_m(W,\mathcal{M})$.

Since all modules and homomorphisms are co-induced over $W + Y + U_{[k]}$, and the desired structure is preserved by this co-induction, we may also assume without loss of generality that $W + Y + U_{[k]} = Z$. This will simplify the work. In the first place, it will frequently allow us to write ‘discrete’ rather than ‘co-induced-of-discrete’. More importantly, given a null sequence in a module of the form $\text{Coind}^Z_W(W,Y+U_{[k]},(\text{discrete}))$, we know only that it eventually equals zero above each coset of $W + Y + U_{[k]}$ separately, not that it eventually equals zero everywhere. Working ‘coset-wise’ hides the problem of keeping track of such ‘local’ convergence.

For any $e \subseteq [k]$ and $0 \leq \ell \leq |e|$, we set

$$M_e^{(\ell)} := \bigoplus_{a \in (\ell)} M_a,$$

so this is the $\ell$th entry in the structure complex of $\mathcal{M}$ at $e$.

Now consider the top structure complex of $(M_e)_e$, and insert all the images and kernels:

$$0 \rightarrow K_0 \hookrightarrow M_0 \twoheadrightarrow I_0 \hookrightarrow K_1 \hookrightarrow M^{(1)} \twoheadrightarrow I_1 \hookrightarrow K_2 \hookrightarrow M^{(2)} \twoheadrightarrow I_2 \hookrightarrow \cdots \hookrightarrow I_{k-1} \hookrightarrow K_k = M_{[k]}. \quad (16)$$

If $\ell < 0$ or $\ell > k + 1$, then we always interpret $M^{(\ell)} = I_\ell = K_\ell = 0$.

The structure complex for $H^p(M_{[k]})$ is obtained from (16) by omitting the kernels and images and applying the functor $H^p(-)$ to each of the modules $M^{(\ell)}$. In terms of the modules $B^p(W,M^{(\ell)})$ and $Z^p(W,M^{(\ell)})$, the top structure complex of $H^p(\mathcal{M})$ may be presented as follows:
where we use the notation $\partial_i$ and $\partial_{i*} := H^p_m(W,\partial_i)$ for the obvious structure morphisms.

Theorem 9.5 will be proved by an induction on the position of $(Z,Y,U)$ in the partial wellorder $\preceq$. It involves the following conclusions:

- That each $H^p(W_e)$ is Polish. As usual, since $M_e$ is Polish, this is equivalent to the closure of the submodule $B^p(W,M_e)$ in $Z^p(W,M_e)$.

- That $H^p(M)$ is $S$-closed, meaning that
  $\partial_{e,\ell}(Z^p(W,M_{e\{\ell-1\}})) + B^p(W,M_{e\{\ell\}})$
  is closed in $C^p(W,M_{e\{\ell\}})$ for each nonempty $e \subseteq [k]$ and each $\ell \in \{1, 2, \ldots, |e|\}$.

- That the homology of the structure complex of $H^p(M)$ at $e$ is co-induced over $W + Y + U_e$ from discrete $(W + Y + U_e)$-modules, or possibly co-induced-of-locally-compact in case $p = 0$ and $\ell_0 = k$.

First observe that if $e \subseteq [k]$, then the structure complex of $H^p(M)$ at $e$ is simply the structure complex of $H^p(M_{|e})$. Since $M_{|e}$ is a $(Z,Y,U_{|e})$-$\Delta$-module satisfying the same structural assumptions as $M$, and since

$$ (Z,Y,U_{|e}) \preceq (Z,Y+W,U) $$

for any $W$ (because $|e| < k$), the $\preceq$-induction hypothesis will already give all of the desired conclusions for the modules or structure complex of $H^p(M)$ at $e$. We will therefore focus on $e = [k]$ in the rest of the proof.

The required conclusions above with $e = [k]$ will be obtained by an inner induction on $\ell$, the position in the top structure complex of $H^p(M)$.

### 9.2 Some auxiliary results

For several of the properties that we need to establish, completing one cycle of the induction on $\ell$ will need different arguments in the ‘internal’ stages $\ell \leq k - 1$ from the final stage $\ell = k$.  

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For the internal stages, we now prove two auxiliary propositions assuming that Theorem 9.5 has already been shown for some previous cases in the order \( \preceq \). We then apply them to \( H^p(\mathcal{M}) \).

**Proposition 9.6.** Fix \((Z', Y', U')\), and suppose that Theorem 9.5 is known for any strictly modest \((Z_1, Y_1, U_1)\)-\(\Delta\)-module for which \((Z_1, Y_1, U_1) \preceq (Z', Y', U')\).

Suppose also that \(Y' + U'_1 = Z'\).

Let \(0 \leq \ell \leq k - 1\), and let \(\mathcal{P}\) be a \((Z', Y', U')\)-\(\Delta\)-module all of whose nontrivial restrictions are strictly modest. Then the image \(\partial_k(P^{(\ell-1)})\) is a closed submodule of \(P^{(\ell)}\).

**Proof.** This will be our first proof by \(\preceq\)-induction.

**Pure case** In case \((Z', Y', U')\) is pure, the module \(P^{(\ell)}\) is itself discrete by Proposition 7.2, so the result is obvious.

**Non-pure case** Now suppose that \(i \in [k]\) is such that \(Y' \not\supset U'_i\), and that \((f_n)_n\) is sequence in \(P^{(\ell-1)}\) such that \(\partial_k(f_n) \to 0\). Let \(\mathcal{P}^\circ := \mathcal{P}_{\ell,k \setminus i}\), and let \(\partial_\ell\), \(0 \leq \ell \leq k\), be boundary morphisms of the top structure complex of \(\mathcal{P}^\circ\).

Then \(\nabla U'_i f_n\) is a sequence in \(Z^1(U'_i, P^{(\ell-1)})\) such that
\[
\nabla U'_i \partial_\ell(f_n) \in Z^1(U'_i, P^{(\ell)})
\]
is null. Applying the continuous splitting homomorphism \(P^{(\ell-1)} \to P^{(\ell)}\) given by the second alternative in Lemma 6.6 it follows that
\[
\nabla U'_i f_n = \sigma_n + \partial_{\ell-1} \tau_n
\]
for some null sequence \((\sigma_n)_n\) in \(Z^1(U'_i, P^{(\ell-1)})\) and some sequence \((\tau_n)_n\) in \(Z^1(U'_i, P^{\epsilon(\ell-2)})\).

Now let
\[
\varphi_\ell := \bigoplus_{|e| = \ell-1} \varphi_{e \setminus i, e} : P^{\epsilon(\ell-1)} \to P^{\ell(\ell-1)},
\]
and let \(\sigma'_n = \varphi_\ell \sigma_n\) and \(\tau'_n = \varphi_{\ell-1} \tau_n\). Then the above becomes
\[
d^U_i f_n = \sigma'_n + \partial_{\ell-1} \tau'_n. \tag{17}
\]

Moving \(\sigma'_n\) to the left, this implies that \((\tau'_n)_n\) defines a sequence of cohomology classes in \(H^1_{\text{in}}(U'_i, P^{\ell(\ell-2)})\) which is sent to a null sequence in the quotient topologies by \(\partial_{\ell-1}\). Our assumption concerning Theorem 9.5 gives that \(\mathcal{Q} := H^1_{\text{in}}(U'_i, \mathcal{P})\) is a \((Z', Y' + U'_i, U')\)-\(\Delta\)-module whose nontrivial restrictions are all modest, and our choice of \(i\) gives that \((Z', Y' + U'_i, U') \preceq (Z', Y', U')\),
so by the hypothesis of our \( \preceq \)-induction applied to the sequence of cohomology classes \( ([\tau'_n])_n \), we may choose \( \tau'_n \) in \( \{17\} \) of the form

\[
\tau'_n = \tau''_n + d_U' g_n
\]

for some null sequence \( \tau''_n \) in \( Z^1(U'_i, P^{(\ell-2)}) \) and some \( g_n \in P^{(\ell-2)} \). Since we may replace \( f_n \) with \( f_n - \partial_{\ell-1}(g_n) \), we may therefore assume that the sequence \( d_U' f_n \) is itself null.

However, since our assumed cases of Theorem 9.5 also give that \( H^1_m(U'_i, P^{(\ell-1)}) \) is Hausdorff, this now implies that we may modify \( f_n \) by another null sequence so that \( d_U' f_n = 0 \). So now \( (f_n)_n \) is a sequence in \( (P^{(\ell-1)})^U_i \), which is the corresponding module in the top structure complex of \( H^0_m(U'_i, P) = P^{U_i} \). An assumed case of Theorem 9.5 implies that this \( \Delta \)-module still has all nontrivial restrictions modest, and it is directed by \( (Z', Y' + U'_i, U') \preceq (Z', Y', U') \), so another appeal to our \( \preceq \)-inductive hypothesis completes the proof.

**Proposition 9.7.** With the same hypotheses as the preceding proposition, the image \( \partial_{\ell}(P^{(\ell-1)}) \) is relatively open-and-closed in

\[
\ker \left( P^{(\ell)} \xrightarrow{\partial_{\ell}} P^{(\ell+1)} \right).
\]

**Remark.** Note that this does not cover the case \( \ell = k \).

**Proof.** This will be another proof by \( \preceq \)-induction.

**Pure case** Once again, \( P^{(\ell)} \) itself is discrete by Proposition 7.2 so the result is trivial.

**Non-pure case** Now suppose that \( i \in [k] \) is such that \( Y' \preceq U'_i \), and that \( (f_n)_n \) is a null sequence in \( \ker \partial_{\ell+1} \).

Then \( \tilde{\nabla}^{U'_i} f_n \) is a null sequence in

\[
Z^1(U'_i, \ker \partial_{\ell+1}).
\]

Composing \( \tilde{\nabla}^{U'_i} f_n \) with the splitting given by Lemma 6.6 one obtains

\[
\tilde{\nabla}^{U'_i} f_n = \partial_{\ell} \sigma_n
\]

for some null sequence \( (\sigma_n)_n \) in \( Z^1(U'_i, P^{(\ell-1)}) \).

Now applying \( \varphi_{\ell} \), this becomes

\[
d_{U_i} f_n = \partial_{\ell} \sigma'_n,
\]

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so \((\sigma'_n)_n\) is a null sequence

\[
\ker(H^1_m(U'_i, P^{(\ell-1)}) \longrightarrow H^1_m(U'_i, P^{(\ell)})).
\]

Applying the hypothesis of our \(\preceq\)-induction to \(H^1_m(U'_i, \mathcal{P})\), this means that for \(n\) sufficiently large, \(\sigma'_n\) defines a class in \(\partial_{\ell-1}(H^1_m(U'_i, P^{(\ell-2)}))\), so its \(\partial_{\ell}\)-image is a coboundary. By one of the assumed cases of Theorem 9.5 we know that \(H^1_m(U'_i, P^{(\ell-1)})\) is Hausdorff, and so we may write

\[
d^{U'_i} f_n = d^{U'_i} \partial_{\ell}(g_n)
\]

for some null sequence \((g_n)_n\) in \(P^{(\ell-1)}\).

We may now replace \(f_n\) with \(f_n - \partial_{\ell}(g_n)\), and so reduce to the case in which \(f_n \in \ker(\partial_{\ell+1}((P^{(\ell)})^{U'_i}))\). Since \(\mathcal{P}^{U'_i}\) is directed by \((Z', Y' + U'_i, U') \preceq (Z', Y', U')\), another appeal to the inductive hypothesis completes the proof. \(\square\)

**Corollary 9.8.** Now suppose that Theorem 9.5 is known for any \(\ell_0\)-almost modest \((Z_1, Y_1, U_1)\)-\(\Delta\)-module for which \((Z_1, Y_1, U_1) \preceq (Z, Y, U)\), and let \(\mathcal{M}\) be as in the preceding subsection. Assume that \(W + Y + U_{[k]} = Z\). If \(p \geq 1\) or \(\mathcal{M}\) is strictly modest, then the middle homology of the sequence

\[
H^p(M^{(\ell-1)}) \longrightarrow H^p(M^{(\ell)}) \longrightarrow H^p(M^{(\ell+1)})
\]

is discrete.

**Proof.** Let \((Z', Y', U') := (Z, Y + W, U)\) and \(\mathcal{P} := H^p(\mathcal{M})\). By the assumed cases of Theorem 9.5 we already know that every nontrivial restriction of \(\mathcal{P}\) is modest, so the corollary follows by combining Propositions 9.6 and 9.7. \(\square\)

**Corollary 9.9.** In the same setting as above, the submodule

\[
\partial_{\ell} (Z^p(W, M^{(\ell-1)})) + B^p(W, M^{(\ell)}) \leq Z^p(W, M^{(\ell)})
\]

is closed for every \(p \geq 1\) and every \(\ell \in \{0, 1, \ldots, k - 1\}\).

**Proof.** The assumption of \(\preceq\)-preceding cases of Theorem 9.5 give that \(H^p(M^{(\ell)})\) is Hausdorff. Therefore it suffices to show that \(\partial_{\ell}^*(H^p(M^{(\ell-1)}))\) is relatively closed in \(H^p(M^{(\ell)})\), as follows from the preceding corollary. \(\square\)

**Corollary 9.10.** In the same setting as above, if \(p \geq 1\) or \(\mathcal{M}\) is strictly modest, then for each \(0 \leq \ell \leq k - 1\) the kernel

\[
\ker \left( H^p(M^{(\ell)}) \longrightarrow H^p(K_{\ell+1}) \right)
\]

is co-discrete over

\[
\text{img}(H^p(M^{(\ell-1)}) \longrightarrow H^p(M^{(\ell)})).
\]
Remark. Once again, we can prove this without yet knowing anything very precise about the structure of $K_{\ell+1}$ itself.

Proof. Let $A$ be the kernel in question, and observe the factorization

$$H^p(M^{(\ell)}) \rightarrow H^p(K_{\ell+1}) \rightarrow H^p(M^{(\ell+1)}).$$

This implies that $A \leq \ker \partial_{\ell+1}$, and on the other hand $A$ clearly contains $\partial_\ell(H^p(M^{(\ell-1)}))$, because $\partial_{\ell+1}\partial_\ell = 0$. So

$$\partial_\ell(H^p(M^{(\ell-1)})) \leq A \leq \ker \partial_{\ell+1},$$

and Corollary 9.8 has shown that the last module here is co-discrete over the smallest, so the same must be true of $A$. 

9.3 Proof of modesty

If $\ell_0 = k$, then $M_e = 0$ for all $e \subseteq [k]$ and $M_{[k]}$ is co-induced-of-locally-compact over $Y + U_{[k]}$, so one obtains the following from Lemma 3.5.

Lemma 9.11. If $\mathcal{M}$ is $k$-almost modest, then the top structure complex of $H^p(\mathcal{M})$ is

$$0 \rightarrow 0 \rightarrow \ldots \rightarrow 0 \rightarrow H^p(M_{[k]}) \rightarrow 0.$$

Thus $H^p(\mathcal{M})$ is $\ell_0$-almost modest, and modest in case $p \geq 1$ or $\mathcal{M}$ is modest.

We now focus on the case $\ell_0 < k$. This will use an inner induction on $\ell$.

The inductive hypothesis for that inner induction must carry rather more information than just the two assertions above: we will also need to deduce something about the structure of the cohomology groups $H^p(I_\ell)$ and $H^p(K_{\ell+1})$ for $0 \leq \ell \leq k-1$.

The basic building block of this section is the fact that each inclusion $I_\ell \hookrightarrow K_{\ell+1}$ is co-discrete once $\ell \geq \ell_0$, to which we will apply Lemma 3.7.

Each cycle of the induction will prove several properties about the intermediate modules of that complex.

Proposition 9.12. Fix $(Z, Y, U)$ and an $\ell_0$-almost modest $(Z, Y, U)$-$\Delta$-module $\mathcal{M}$, and suppose that Theorem 9.3 is already known for all almost or strictly modest $(Z_1, Y_1, U_1)$-$\Delta$-modules for which $(Z_1, Y_1, U_1) \not\geq (Z, Y, U)$.

The following hold for all $\ell_0 - 1 \leq \ell \leq k - 1$.

(i) The module $\text{coker}(H^p(M^{(\ell)}) \rightarrow H^p(I_\ell))$ is discrete for all $p \geq 0$. (Note that this co-discreteness holds for any $\mathcal{M}$ and even for $p = 0$ — this fact will be used later.)
(ii) If \( p \geq 0 \) and \( \sigma_n \in \mathcal{Z}^p(W, M^{(\ell)}) \) is a sequence of cocycles such that \( \partial_{\ell+1} \sigma_n \rightarrow 0 \) in \( \mathcal{Z}^p(W, I_\ell) \), then there are cocycles \( \sigma'_n \in \mathcal{Z}^p(W, M^{(\ell)}) \) such that \( \partial_{\ell+1} \sigma'_n = \partial_{\ell+1} \sigma_n \rightarrow 0 \).

(iii) The module \( \text{coker}(H^p(M^{(\ell)}) \rightarrow H^p(K_{\ell+1})) \) is locally compact, and discrete in case \( \ell \geq \ell_0 \) or \( p \geq 1 \).

(iv) The group \( H^p(K_{\ell+1}) \) is Hausdorff in its quotient topology for all \( p \geq 0 \).

(v) The kernel \( \text{ker}(H^p(K_{\ell+1}) \rightarrow H^p(M^{(\ell+1)})) \) is zero if \( p = 0 \) and discrete if \( p \geq 1 \).

We start the induction from \( \ell = \ell_0 - 1 \) in order to simplify the base case. On the one hand, properties (i)\( _{\ell_0-1} \) and (ii)\( _{\ell_0-1} \) both hold vacuously, and on the other one has the following.

Proof of (iii)\( _{\ell_0-1} \), (iv)\( _{\ell_0-1} \) and (v)\( _{\ell_0-1} \). By the \( \ell_0 \)-almost modesty of \( \mathcal{M} \), \( K_{\ell_0} \) is a submodule of \( M^{(\ell_0)} \) that is co-induced-of-locally-compact over \( Y + U_{[\ell]} \). The cohomology \( H^p(K_{\ell_0}) \) is therefore locally compact, and discrete if \( p \geq 1 \), by Proposition 3.3 and Lemma 3.5. This entails (iii)\( _{\ell_0-1} \) and (iv)\( _{\ell_0-1} \).

Property (v)\( _{\ell_0-1} \) is obvious if \( p \geq 1 \) owing to the discreteness of \( H^p(K_{\ell_0}) \) itself, and if \( p = 0 \) then the kernel in question is

\[ \text{ker}(K^W_0 \rightarrow M^W_0) = 0, \]

by the left-exactness of \( (-)^W \).

For the remainder of the induction on \( \ell \), it seems easiest to give a separate proof for each step in one inductive cycle. The reader will see that most steps do not need strictly all of the facts that have been proved before them, and so there is considerable arbitrariness about the order in which the above proposition is proved. Suppose now that \( \ell \geq \ell_0 \).

Proof that \( [(iv)\ell-1] \lor (v)\ell-1 \implies (i)\ell \). Applying the cohomology functor \( H^p(-) \) to the short exact sequence \( K_\ell \hookrightarrow M^{(\ell)} \rightarrow I_\ell \) gives a cohomology long exact sequence, in which the switchback homomorphisms define a sequence of isomorphisms

\[ \text{coker}(H^p(M^{(\ell)}) \rightarrow H^p(I_\ell)) \cong \ker(H^{p+1}(K_\ell) \rightarrow H^{p+1}(M^{(\ell)})), \]

and by Lemma 3.2 these maps are also continuous for the quotient topologies (even though \( H^p(I_\ell) \) is not yet known to be Hausdorff). Since the kernel on the right is discrete by property (v)\( _{\ell-1} \), so is the cokernel on the left.
Proof that \[(iv)_{\ell-1} \lor (v)_{\ell-1} \implies (ii)_{\ell}\]. Since \(\partial_{\ell+1} \sigma_n \to 0\) and the homomorphism \(\partial_{\ell+1} : M^{(\ell)} \to I_{\ell}\) is closed, we may certainly choose functions \(\kappa_n \in \mathcal{C}^p(W, M^{(\ell)})\) such that \(\partial_{\ell+1} \kappa_n = \partial_{\ell+1} \sigma_n\) and \(\kappa_n \to 0\). It remains to modify these \(\kappa_n\) so that they are still cocycles.

To do this, observe that \(d^W \kappa_n\) defines a cohomology class

\[
\ker \left( H^{p+1}(K_{\ell}) \to H^{p+1}(M^{(\ell)}) \right)
\]

which is discrete by \((v)_{\ell-1}\), and so for \(n\) sufficiently large we must have that \(d^W \kappa_n\) is a coboundary among \(K_\ell\)-valued functions, say \(d^W \kappa_n = d^W \alpha_n\) with \(\alpha_n \in \mathcal{C}^p(W, K_\ell)\). By \((v)_{\ell-1}\) and Theorem \ref{thm:discrete}\(\square\) we may also choose \(\alpha_n \to 0\). Now \(\sigma_n := \kappa_n - \alpha_n\) is a cocycle tending to zero and satisfying

\[
\partial_{\ell+1} \sigma_n = \partial_{\ell+1} \kappa_n = \partial_{\ell+1} \sigma_n,
\]

because \(\alpha_n\) takes values in \(K_\ell = \ker \partial_{\ell+1}\).

\[\square\]

Proof that \[(i)_{\ell} \lor (ii)_{\ell} \implies (iii)_{\ell}\]. Let \(A\) be the cokernel above.

**Step 1.** By Lemma \ref{lem:cohomology}\(.\) applied to the semi-functional inclusion \(I_\ell \hookrightarrow K_{\ell+1}\) (noting that ‘\(Y\)’ in that lemma is here the sum \(Y + U_{[k]}\)), the cokernel

\[
B := \text{coker} \left( H^p(I_\ell) \to H^p(K_{\ell+1}) \right)
\]

is discrete, using that \(\ell \geq \ell_0\) and so \(I_\ell \hookrightarrow K_{\ell+1}\) is co-discrete. On the other hand, \((i)_{\ell}\) gives that

\[
C := \text{coker} \left( H^p(M^{(\ell)}) \to H^p(I_\ell) \right)
\]

is discrete.

**Step 2.** We next show that the image of \(H^p(M^{(\ell)})\) in \(H^p(K_{\ell+1})\) is closed (equivalently, that the image of \(C\) in \(A\) is closed). Thus, suppose that \(\sigma_n \in \mathcal{Z}^p(W, M^{(\ell)})\) and \(\beta_n \in \mathcal{C}^{p-1}(W, K_{\ell+1})\) are such that

\[
\partial_{\ell+1} \sigma_n + d^W \beta_n \to 0.
\]

Letting \(\overline{\beta}_n\) be the quotient of \(\beta_n\) by \(I_\ell\) gives that \(d^W \overline{\beta}_n \to 0\) in \(B^p(W, K_{\ell+1}/I_\ell)\), so by Lemma \ref{lem:cohomology}\(.\) we may find \(\beta'_n \in \mathcal{C}^{p-1}(W, K_{\ell+1})\) tending to 0 such that \(d^W \beta'_n = d^W \overline{\beta}_n\). Replacing \(\beta_n\) by \(\beta_n - \beta'_n\), we may simply assume that \(d^W \beta_n = 0\).

However, having made that assumption, each function \(\partial_{\ell+1} \sigma_n + d^W \beta_n\) is now an element of \(\mathcal{Z}^p(W, I_\ell)\). Since this is co-discrete over

\[
\partial_{\ell}(\mathcal{Z}^p(W, M^{(\ell)})) + B^p(W, I_\ell) = \partial_{\ell}(\mathcal{Z}^p(W, M^{(\ell)}))
\]

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by property (i), once \( n \) is sufficiently large it follows that

\[
\partial_{\ell+1} \sigma_n + d^W \beta_n = \partial_{\ell+1} \sigma'_n
\]

for some \( \sigma'_n \in \mathbb{Z}^p(W, M^{(\ell)}) \). By (ii), we may also assume \( \sigma'_n \rightarrow 0 \), and now by the equivalence of (1) and (2) in Theorem 2.2 this implies that the homomorphism

\[
\text{H}^p(M^{(\ell)}) \rightarrow \text{H}^p(K_{\ell+1})
\]

has closed image.

**Step 3.** Steps 1 and 2 together give a presentation of Polish groups and continuous homomorphisms

\[
0 \rightarrow \text{img}(C \rightarrow A) \rightarrow A \rightarrow B \rightarrow 0
\]

in which \( \text{img}(C \rightarrow A) \) is discrete as a closed image of the countable discrete module \( C \). Since \( B \) is also discrete, so is \( A \).

**Remark.** As with the proof of Proposition 8.2, the argument above made essential use of the co-discreteness of \( \text{H}^p(M^{(\ell)}) \rightarrow \text{H}^p(I_{\ell}) \) in Step 2 above, not just the closure of its image.

**Proof that \((i) \lor (ii) \implies (iv)\).** This is trivial if \( p = 0 \) (since then there are no coboundaries), so assume \( p \geq 1 \). Suppose that \( f_n \in \mathcal{C}^{p-1}(W, K_{\ell+1}) \) is a sequence such that

\[
d^W f_n \rightarrow 0 \quad \text{in } \mathcal{C}^p(W, K_{\ell+1}).
\]

**Step 1.** Let

\[
\Phi : K_{\ell+1} \rightarrow K_{\ell+1}/I_{\ell} =: A
\]

be the quotient homomorphism, and let \( \overline{f}_n := \Phi f_n \), so we have

\[
d^W \overline{f}_n \rightarrow 0 \quad \text{in } \mathcal{C}^p(W, A).
\]

Proposition 3.3 implies, in particular, that \( \text{H}^p(K_{\ell+1}) \) is Hausdorff, and hence that the boundary homomorphism

\[
d : \mathcal{C}^{p-1}(W, K_{\ell+1}) \rightarrow \mathcal{B}^p(W, K_{\ell+1})
\]

has closed image. Theorem 2.2 therefore gives functions \( f'_n \in \mathcal{C}^{p-1}(W, K_{\ell+1}) \) tending to 0 such that

\[
d^W \overline{f}_n = d^W \overline{f}'_n,
\]
where \( \tilde{F}_n := \Phi f'_n \). Since \( f'_n \to 0 \), it suffices to prove the desired conclusion with \( f_n \) replaced by \( f_n - f'_n \), and this means we have reduced to the case \( d^W f_n = 0 \).

**Step 2.** So now \( d^W f_n \to 0 \) in \( Z^P(W, I_\ell) \). Since \( coker(H^P(M(\ell)) \to H^P(I_\ell)) \) is discrete by (i), it follows that in fact

\[
d^W f_n = \partial_{\ell+1} g_n \in B^p(W, I_\ell) = \partial_{\ell+1}(Z^P(W, M(\ell)))
\]

for all sufficiently large \( n \), say \( d^W f_n = \partial_{\ell+1} g_n \) with \( g_n \in Z^P(W, M(\ell)) \). By (ii), we may also assume \( g_n \to 0 \). On the other hand, each \( g_n \) defines an element of \( \ker(B \to H^p(M(\ell+1))) \).

Corollary 9.10 has shown that this kernel is co-discrete over \( \text{img}(H^p(M(\ell)) \to H^P(K_{\ell+1})) \) (recalling our assumption that \( p \geq 1 \)), and by Corollary 9.9 this image is closed. Therefore Theorem 2.2 implies that for \( n \) large enough we may write

\[
g_n = \partial_{\ell+1} h_n + d^W \alpha_n
\]

with \( (h_n, \alpha_n) \to 0 \). This now implies that

\[
d^W f_n = \partial_{\ell} g_n = d^W(\partial_{\ell} \alpha_n)
\]

with \( \partial_{\ell} \alpha_n \to 0 \), as required. \( \square \)

**Proof that \((iii)_\ell \vee (iv)_\ell \implies (v)_\ell\)**. If \( p = 0 \) this follows from the left-exactness of \( H^0 = (-)^W \), so assume \( p \geq 1 \).

If \( \ell + 1 = k \) then the homomorphism in question is the identity, so its kernel and cokernel are trivial. On the other hand, if \( \ell + 1 < k \), then let

\[
A := \ker(H^p(K_{\ell+1}) \to H^p(M^{(\ell+1)})),
\]

\[
B := \text{img}(H^p(M^{(\ell)} \to H^p(K_{\ell+1})),
\]

and

\[
C := \ker(B \to H^p(M^{(\ell+1)})) = A \cap B.
\]

By (iii) and (iv) together, \( B \) is a closed submodule of the Polish module \( H^p(K_{\ell+1}) \). Also, since \( \ell + 1 < k \), \( H^p(M^{(\ell+1)}) \) is already known to be Polish by our \( \leq \)-induction hypotheses. Since a continuous homomorphism of modules induces a continuous homomorphism between the cohomology groups, this implies that the kernel \( C \) is closed.

These modules fit into the commutative diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & A \longrightarrow \text{HP}(K_{\ell+1}) \longrightarrow \text{HP}(M^{(\ell+1)}) \\
& \uparrow & \uparrow \\
0 & \longrightarrow & C \longrightarrow B \longrightarrow \text{HP}(M^{(\ell+1)})
\end{array}
\]

in which the rows are both exact.

By (iii)\(_{\ell}\),
\[
\text{coker}(\text{HP}(M^{(\ell)}) \longrightarrow \text{HP}(K_{\ell+1})) = \text{HP}(K_{\ell+1})/B
\]
is locally compact. The continuous monomorphism \(A \longrightarrow \text{HP}(K_{\ell+1})\) quotients to a continuous and injective monomorphism
\[
A/C \longrightarrow \text{HP}(K_{\ell+1})/B,
\]
so \(A/C\) is also locally compact.

On the other hand, \(C\) may also be written as
\[
\text{img}\left(\ker(\text{HP}(M^{(\ell)}) \longrightarrow \text{HP}(M^{(\ell+1)})) \longrightarrow \text{HP}(K_{\ell+1})\right) = \text{img}\left(\frac{\ker(\text{HP}(M^{(\ell)}) \longrightarrow \text{HP}(M^{(\ell+1)}))}{\text{img}(\text{HP}(M^{(\ell-1)}) \longrightarrow \text{HP}(M^{(\ell)}))} \longrightarrow \text{HP}(K_{\ell+1})\right),
\]
since the composition \(\text{HP}(M^{(\ell-1)}) \longrightarrow \text{HP}(M^{(\ell)}) \longrightarrow \text{HP}(K_{\ell+1})\) is zero. By Corollary 9.8 this implies that \(C\) is an image of a discrete module. Since we have already argued that \(C\) must be Polish, it must itself be discrete, by Lemma 2.4. This now implies the same qualities for \(A\).

\(\square\)

**Completed proof of Proposition 9.12.** Properties (iv)\(_{\ell_0-1}\) and (v)\(_{\ell_0-1}\) start the induction, and then the implications proved above give
\[
[(\text{iv})_{\ell-1} \lor (\text{v})_{\ell-1}] \quad \implies \quad [(i)_{\ell} \lor (ii)_{\ell} \lor (iii)_{\ell} \lor (iv)_{\ell} \lor (v)_{\ell}] \quad \forall \ell \in \{\ell_0, \ell_0 + 1, \ldots, k - 1\},
\]
so the induction continues. \(\square\)

**Corollary 9.13.** With the same hypotheses as Proposition 9.12 the group \(\text{HP}(I_{\ell})\) is Hausdorff in its quotient topology for all \(p \geq 0\) and \(\ell_0 \leq \ell \leq k - 1\). (It is of course 0 if \(\ell < \ell_0\).)

**Proof.** This is trivial if \(p = 0\) since there are no 0-coboundaries, so suppose \(p \geq 1\). This case will follow from properties (i)\(_{\ell}\) and (ii)\(_{\ell}\) in Proposition 9.12.
Suppose $f_n \in \mathcal{C}^{p-1}(W, M^{(\ell-1)})$ is a sequence such that
\[ d^W \partial_\ell f_n \to 0 \quad \text{in } \mathcal{Z}^p(W, I_\ell). \]

Applying \((\text{ii})_{\ell-1}\) to the cocycles $\sigma_n = d^W f_n$ gives a sequence $g_n \in \mathcal{Z}^p(W, M^{(\ell-1)})$ tending to 0 such that $\partial_\ell d^W f_n = \partial_\ell g_n$.

So now each $g_n$ defines a cohomology class
\[ \ker \left( \mathcal{H}^p(M^{(\ell-1)}) \to \mathcal{H}^p(M^{(\ell)}) \right). \]

By Proposition 9.7 this kernel is co-discrete over
\[ \text{img} \left( \mathcal{H}^p(M^{(\ell-2)}) \to \mathcal{H}^p(M^{(\ell-1)}) \right), \]
so for all sufficiently large $n$ we may write $g_n = \partial_{\ell-1}(\kappa_n) + d^W h_n$ for some $\kappa_n$ and $h_n$. By Corollary 9.9 we can take $\kappa_n \to 0$ and $h_n \to 0$ as well. Since $\partial_{\ell+1} \partial_\ell = 0$, this gives
\[ d^W f_n = \partial_\ell g_n = d^W (\partial_\ell h_n), \]
where $\partial_\ell h_n \to 0$. By Theorem 2.2 this completes the proof.

\[ \square \]

**Corollary 9.14.** In the same setting as above, the submodule
\[ \partial_k \left( \mathcal{Z}^p(W, M^{(\ell)}) \right) + \mathcal{B}^p(W, K_{\ell+1}) \leq \mathcal{Z}^p(W, K_{\ell+1}) \]
is closed for every $p \geq 0$.

**Proof.** First suppose $p \geq 1$. By (iv)$_\ell$, $\mathcal{H}^p(K_{\ell+1})$ is Hausdorff, and hence Polish. On the other hand, by (iii)$_\ell$, it is co-discrete over its subgroup given by the image of $\mathcal{H}^p(M^{(\ell)})$. That image must therefore be closed, which at the level of cocycles gives the present assertion.

On the other hand, if $p = 0$, then
\[ \mathcal{Z}^0(W, I_\ell) = I_\ell \cap \mathcal{Z}^0(W, K_{\ell+1}) \]
is manifestly closed in $\mathcal{Z}^0(W, K_{\ell+1})$, and property (i)$_\ell$ gives that
\[ \partial_\ell (\mathcal{Z}^0(W, M^{(\ell-1)})) \]
is co-discrete inside $\mathcal{Z}^0(W, I_\ell)$, so this image must also be closed in $\mathcal{Z}^0(W, K_{\ell+1})$.

\[ \square \]
Completed proof of Theorem 9.5. As promised, this is proved by $\preceq$-induction. Fix $(Z, Y, U)$, and suppose the theorem is already known for all almost or strictly $(Z', Y', U')$-$\Delta$-modules for which $(Z', Y', U') \npreceq (Z, Y, U)$.

If $\ell_0 = k$ the result follows directly from Lemma 9.11 and no inductive hypotheses are necessary.

On the other hand, if $\ell_0 < k$, then the inductive hypothesis allows us to apply the results above. Property (iv)$_{k-1}$ of Proposition 9.12 gives that $H^p(M[k])$ is Hausdorff, and hence that $B^p(W, M[k])$ is closed in $Z^p(W, M[k])$. Next, Corollary 9.9 and the special case $\ell + 1 = k$ of Corollary 9.14 give the S-closure of $H^p_{in}(W, \mathcal{N})$ in its top structure complex. Lastly, Corollary 9.8 and property (iii)$_{k-1}$ of Proposition 9.12 give the $\ell_0$-almost discrete homology of the structure complex of $H^p_{in}(W, \mathcal{N})$.

10 Modesty of $\Delta$-submodules

Theorem 10.1. Every $\Delta$-submodule of a modest $\Delta$-module is modest.

The proof of Theorem 10.1 will make another use of the ‘differentiation’ trick outlined in Subsection 1.2, as a way of reducing this task to the study of a $\Delta$-morphism between $\Delta$-modules which come earlier in the order $\preceq$.

As for the proof of Theorem 9.5, the hypotheses of the $\preceq$-induction already give modesty of every nontrivial restriction $\mathcal{N} | e$, so we may focus on $e = [k]$; and for this, our main argument here will rely on an inner induction along the top structure complex of $\mathcal{N}$. However, this time we must treat separately the cases of pure and non-pure domain $\Delta$-modules, similarly to the auxiliary results of Subsection 9.2.

Remark. Unlike Theorem 9.5, the analog of this theorem fails for almost modest $\Delta$-modules. Once again, this is most easily seen with examples that are pure.

In the first place, one may choose any injective morphism $\varphi : A \rightarrow B$ of Abelian Lie groups whose image is not closed (such as $\mathbb{Z} \xrightarrow{\times \alpha} \mathbb{T}$ for some irrational $\alpha \in \mathbb{T}$), and interpret its image $\varphi(A)$ as a $\Delta$-submodule of the pure $(Z, Z, *)$-$\Delta$-module $B$. In this case the submodule does not even have Polish constituents.

However, problems can occur even if one assumes that the $\Delta$-submodule is Polish and is $S$-closed. For example, one may define a $(Z, Z, (Z, Z))$-$\Delta_{[2]}$-module by $M_0 = M_1 = 0, M_2 = M_{12} = \mathbb{R}^2$, with the obvious trivial or identity morphisms in all directions. This is a pure, inner, 1-almost modest $(Z, Z, (Z, Z))$-$\Delta$-module. However, it contains the $\Delta$-submodule with constituents $H_0 = H_1 = 0, H_2 = V$, and $H_{12} = \mathbb{R}^2$, where $V$ is any 1-dimensional subspace of $\mathbb{R}^2$. This submodule is not almost modest, since its top structure complex has homology isomorphic to $\mathbb{R}$ in both the first and second places.
In light of these very simple examples, I have not sought conditions on a \(\Delta\)-submodule of an almost modest \(\Delta\)-module to guarantee that the submodule is still almost modest. One could probably show that if such a \(\Delta\)-submodule is assumed to be Polish, then it must at least have structural homology that is co-induced from locally compact modules for the relevant subgroups; but this conclusion is too weak for us to then work safely with short exact sequences, in light of the examples at the end of Section 8. So we simply restrict our attention to Theorem 10.1 above. ☰

10.1 Proof in the pure case

Proof of Theorem 10.1 for pure \(\Delta\)-modules. This follows immediately from Proposition 7.2, which shows that all the possible modules in question are co-induced over \(Y\) from discrete \(Y\)-modules.

10.2 Strategy and preparation for the non-pure case

Now assume that there is \(i \in [k]\) such that \(U_i \not\leq Y\) (so, in particular, we cannot be in the base case of the \(\leq\)-induction). Fix such an \(i\) for the remainder of this section.

Consider taking differences along the subgroup \(U_i\); or rather, to be precise, applying the derivation-lifts \(\tilde{\nabla}_{\epsilon,e}^{\neq i}\) for \(u \in U_i\). Studying the images of this differencing will largely reduce our problem to combining the structures of various \(\Delta\)-modules that lie earlier in the partial order \(\leq\), and so are already understood by our inductive hypotheses.

We next list various preliminary conclusions that will be needed for the inductive argument.

Lemma 10.2. The \((Z, Y + U_i, U)\)-\(\Delta\)-module \(H_{s(U_i)}\) is modest.

Proof. Theorem 9.5 gives that \(M_{s(U_i)}\) is a modest \((Z, Y + U_i, U)\)-\(\Delta\)-module, and \(H_{s(U_i)}\) is a \(\Delta\)-submodule of it, so since \((Z, Y + U_i, U) \not\leq (Z, Y, U)\) this is an appeal to a strictly \(\leq\)-preceding case of Theorem 10.1.

Lemma 10.3. The \((Z, Y, U)\)-\(\Delta\)-module \(H_{s(U)} := H_{s(U_i)}\) is modest.

Proof. The restriction \(M_{s([k]\\setminus \{i\}))} \) is modest by Corollary 6.7. The restriction \(H_{s([k]\\setminus \{i\})} \) is a \(\Delta\)-submodule of \(M_{s([k]\\setminus \{i\})} \), so its modest follows by an appeal to Theorem 10.1 for the case of the \((k-1)\)-tuple of subgroups \((U_j)_{j \in [k]\\setminus \{i\}}\). Finally, the modesty of \(H_{s(U)}\) follows by another appeal to Corollary 6.7.

Lemma 10.4. The \((Z, Y + U_i, U)\)-\(\Delta\)-module \(H_{s(U_i), [k]}(U_i, H_{s(U)})\) is modest.

Proof. This now follows by another appeal to Theorem 9.5.

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Lemma 10.5. The top structure complexes of the \(\Delta\)-modules \(H^p_m(U_i, \mathcal{H}^e)\) are both split.

Proof. These both follow from Corollary \([6.3]\). For \(H^p_m(U_i, \mathcal{H}^e)\), this is justified by Lemma \([9.3]\) which gives that this cohomology \(\Delta\)-module is aggrandized from \([k] \setminus i\).

Now let us abbreviate \(\tilde{\nabla}^{e,e|\iota} = \nabla_{U_i}^e\) for any \(e\). If \(h \in H^e\), then, since \(\mathcal{H}\) is compatible with \(\tilde{\nabla}^{e,e|\iota}\), one has \(\tilde{\nabla}_{U_i}^e h \in H^e\) for any \(u \in U_i\), and the defining relations \((\Pi)\) of \(\nabla_{U_i}^e\) imply that the map

\[ u \mapsto \tilde{\nabla}_{U_i}^e h \]

is a 1-cocycle \(U_i \rightarrow H^e\). This therefore defines a homomorphism

\[ H^e \rightarrow Z^1(U_i, H^e) \quad \text{for each } e \subseteq [k], \]

and composing with the quotient by \(B^1(W; H^e)\) gives a homomorphism

\[ \Phi_e : H^e \rightarrow H^1_m(U_i, H^e). \]

By Lemma \([5.6]\) we may regard \(\mathcal{M}\) as a \((Z, Y + U_i, U)\)-\(\Delta\)-module (although it may not be modest as such). It is also easy to check that the homomorphisms \(\Phi_e\) intertwine the structure morphisms and derivation-lifts of \(\mathcal{H}\) (which are restricted from \(\mathcal{M}\)) with those of \(H^1_m(U_i, \mathcal{H}^e)\), and therefore \(\Phi = (\Phi_e)_e\) is a \(\Delta\)-morphism of \((Z, Y + U_i, U)\)-\(\Delta\)-modules. Let \(\mathcal{L} := \text{img } \Phi\).

Now, on the one hand, \(\mathcal{L}\) is a \((Z, Y + U_i, U)\)-\(\Delta\)-submodule of \(H^1_m(U_i, \mathcal{H}^e)\), which is modest by Lemma \([10.3]\) and Theorem \([9.5]\). Since \((Z, Y + U_i, U) \notin (Z, Y, U)\), the hypothesis of our \(\preceq\)-induction give that \(\mathcal{L}\) is itself a modest \((Z, Y + U_i, U)\)-\(\Delta\)-module.

On the other hand, if \(i \notin e\) then \(M_e = M_e \setminus i = M_e\), and therefore

\[ \Phi^o_e(M_e) = B^1(U_i, M_e) = B^1(U_i, M_e) \implies L_e = 0. \]

Therefore, by Lemma \([5.15]\) \(\mathcal{L}\) is also a modest \((Z, Y, U)\)-\(\Delta\)-module, and the above construction defines a surjective \(\Delta\)-morphism

\[ \Phi : \mathcal{H}^{Z,Y,U} \rightarrow \mathcal{L}^{Z,Y,U}. \]

Lastly, since Lemma \([10.3]\) already gives that every nontrivial restriction of \(\mathcal{H}\) is modest, a repeat outing for Proposition \([9.7]\) gives the following.
Lemma 10.6. For any $0 \leq \ell \leq k$, the kernel
\[ \ker \left( \text{H}_m^1(U_i, \text{H}^{(\ell-1)}) \rightarrow \text{H}_m^1(U_i, \text{H}^{(\ell)}) \right) \]
is co-discrete over
\[ \text{img}(\text{H}_m^1(U_i, \text{H}^{(\ell-2)}) \rightarrow \text{H}_m^1(U_i, \text{H}^{(\ell-1)})). \]

Remark. A little more work shows that the above construction of $\mathcal{L}$ actually yields a short exact sequence of $(Z,Y,U)$-$\Delta$-modules
\[ \mathcal{K} \hookrightarrow \mathcal{H} \rightarrow \mathcal{L}, \]
where $\mathcal{K} = (K_e)_e$ and $K_e$ consists of those $h \in H_e$ for which there is some $h' \in H_{e,i}$ such that $\nabla^{U_i} h = d^{U_i} h'$. However, I do not see that our earlier results about short exact sequences are of any direct help in proving the modesty of $\mathcal{H}$ (or of $\mathcal{K}$).

10.3 Proof in the non-pure case

To complete the proof, we must show that

- every $H_e$ is a closed submodule of $M_e$;
- the structure complexes of $\mathcal{H}$ are structurally closed;
- and the homology of those structure complexes is all co-induced from discrete modules over the relevant subgroups.

As remarked previously, our inductive hypotheses already give these results for every nontrivial restriction $\mathcal{H} |_{e'}$. Moreover, this now provides the input hypotheses for Propositions 9.6 and 9.7. Those results together give most cases of the above conclusions; it only remains to prove that $H_{[k]}$ is closed, that its submodule $\partial_k(H^{(k-1)})$ is closed, and that their quotient is co-induced over $Y + U_{[k]}$ from a discrete $(Y + U_{[k]})$-module. In fact, much as for Propositions 9.6 and 9.7 we will study the image $\partial_k(H^{(k-1)})$ first and then the larger module $H_{[k]}$.

Since $U_i \leq Y + U_{[k]}$, it will be clear that all of the $Z$-modules and homomorphisms that appear below are co-induced from $(Y + U_{[k]})$-modules and homomorphisms. As in the previous section, we may therefore simply assume that $Y + U_{[k]} = Z$.  

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Lemma 10.7. The submodule \( \partial_k(H^{(k-1)}) \) is closed in \( M_{[k]} \), and hence the restriction \( \partial_k|H^{(k-1)} \) of the boundary morphism is closed.

Proof. Suppose \( h_n \in H^{(k-1)} \) is a sequence such that \( \partial_k(h_n) \to 0 \) in \( M_{[k]} \). We need to show that we may produce the same image sequence \( \partial_k(h_n) \) with \( h_n \) itself tending to zero.

Step 1. Applying \( \bar{\nabla}^U_i : M_{[k]} \to C^1(U_i, M_{[k]}) \), which is continuous and commutes with \( \delta_i \), the above convergence gives

\[
\partial_i(\bar{\nabla}^U_i h_n) \to 0 \quad \text{in } Z^1(U_i, H_{[k]}).
\]

Let \([\partial_i(\bar{\nabla}^U_i h_n)]\) be the cohomology classes of these cocycles in \( H^1_m(U_i, H_{[k]}) \), so these lie in

\[
\partial_i(L^{(k-1)}) \leq \partial_i(H^1_m(U_i, H^{(k-1)}))
\]

(recalling the definition of \( L = \text{img } \Phi \) above).

Since \( L \) is modest, this left-hand image is a closed subgroup of \( L_{[k]} \). Therefore there is a sequence \( \sigma_n \in Z^1(U_i, H^{(k-1)}) \) tending to zero such that

- \([\sigma_n] \in L^{(k-1)} = \Phi^{(k-1)}(H^{(k-1)})\), and
- \([\partial_i(\bar{\nabla}^U_i h_n)] = \partial_i[\sigma_n] = [\partial_i(\sigma_n)]\).

Hence

\[
\partial_i(\bar{\nabla}^U_i h_n - \sigma_n) = d^U_i \beta_n \tag{20}
\]

for some \( \beta_n \in H_{[k]} \). This equation requires that \( d^U_i \beta_n \to 0 \), and because \( H^1_m(U_i, H_{[k]}) \) is Hausdorff (another feature of the modesty given by Lemma 10.4), we may actually assume \( \beta_n \to 0 \).

Step 2. Equation (20) implies that \( d^U_i \beta_n \) takes values in \( \partial_k(H^{(k-1)}) \), and so composing with the splitting morphism \( \partial_k(H^{(k-1)}) \to H^{(k-1)} \) promised by the first part of Lemma 10.5 gives that \( d^U_i \beta_n = d^U_i(\partial_i \gamma_n) \) for some null sequence \( \gamma_n \in H^{(k-1)} \). Substituting back into (20), we obtain

\[
\partial_i(\bar{\nabla}^U_i h_n - \sigma_n) = \partial_i d^U_i \gamma_n
\]

for all sufficiently large \( n \).

Since \([\sigma_n] = [\sigma_n + d^U_i \gamma_n]\) in \( H^1_m(U_i, H^{(k-1)}) \) (hence also in \( L^{(k-1)} \)), we may replace \( \sigma_n \) with \( \sigma_n + d^U_i \gamma_n \) without changing any of our previous conclusions about \( \sigma_n \), so now

\[
\partial_i(\bar{\nabla}^U_i h_n) = \partial_i(\sigma_n)
\]

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with $\sigma_n \to 0$ and $[\sigma_n] \in L^{(k-1)}$.

**Step 3.** Now let

$$\varphi^+: = \bigoplus_{|a|=k-1} \varphi^+_{a\setminus a}: H^{(k-1)} \to H^{(k-1)}.$$  

The condition that $[\sigma_n] \in L^{(k-1)}$ implies that

$$\varphi^+ \sigma_n \in B^1(U_i, H^{(k-1)}) = \bigoplus_{|a|=k-1} B^1(U_i, H_a).$$

The inductive hypotheses include all the conclusions of Theorem 9.5 and $H^1_m(U_i, H^{(k-1)})$ whenever $|a| = k-1$, including that the homomorphism

$$d^U_i: H^{(k-1)} \to B^1(W, H^{(k-1)})$$

is closed. We may therefore choose a null sequence $h'_n \in H^{(k-1)}$ such that $\varphi^+ \sigma_n = d^U_i h'_n$, and hence

$$\partial_k(d^U_i h_n) = \partial_k(\varphi^+ \sigma_n) = \partial_k(d^U_i h'_n).$$

Since we may replace $h_n$ with $h_n - h'_n$ without changing the desired conclusion, we may therefore assume that $d^U_i h_n$ takes values in $\ker(\partial_{k-1} H^{(k-1)})$.

**Step 4.** This implies that $\partial_k(h_n) \in H^1_{[k]}$. We know by Lemma 10.2 that $\mathcal{A}^{U_i}$ is modest, and this gives that $H^1_{[k]}$ is co-discrete over $\partial_k((H^{(k-1)})^{U_i})$. So $\partial_k(h_n) = \partial_k(s_n)$ for some $s_n \in (H^{(k-1)})^{U_i}$ for all sufficiently large $n$. By the structural closure implied by Lemma 10.2 we know we can choose $(s_n)_n$ null, so that completes the proof.

**Lemma 10.8.** The subgroup $\partial_k(H^{(k-1)})$ is relatively open in the subgroup $H^1_{[k]}$ (both as subgroups of $M_{[k]}$).

**Proof.** Suppose $h_n \in H^1_{[k]}$ is a null sequence: we must show that $h_n \in \partial_k(H^{(k-1)})$ once $n$ is sufficiently large.

**Step 1.** Applying the derivation-lift gives $\tilde{\nabla}^{U_i} h_n \to 0$. On the other hand, $\tilde{\nabla}^{U_i} h_n$ takes values in $H^1_{[k]}$. Let $[\tilde{\nabla}^{U_i} h_n]$ be the cohomology class in $H^1_m(U_i, H^1_{[k]})$.

Compose $\tilde{\nabla}^{U_i} h_n$ with the splitting $H^{(k-1)} \to H^{(k-1)}$ given by Lemma 10.5 and so obtain that $\tilde{\nabla}^{U_i} h_n = \partial_k \sigma_n$ for some $\sigma_n \in Z^1(U_i, H^{(k-1)})$ which also tends to 0.
Step 2. Applying $\varphi$ to the above equation gives $dU_i h_n = \partial_k(\sigma'_n)$, where $\sigma'_n = \varphi^* \sigma_n$.

This implies that the cohomology class $[\sigma'_n] \in H^1_m(U_i, H^{(k-1)})$ actually lies in

$$\ker \left( H^1_m(U_i, H^{(k-1)}) \rightarrow H^1_m(U_i, H_{[k]}) \right).$$

Therefore, for $n$ sufficiently large, Lemma 10.6 gives

$$\sigma'_n = \partial_{k-1} \tau_n + dU_i s_n$$

for some sequences $\tau_n \in Z^1(U_i, H^{(k-2)})$ and $s_n \in H^{(k-1)}$. By the closure of

$$\partial_{k-1}(Z^1(U_i, H^{(k-2)})) + B^1(U_i, H^{(k-1)})$$

promised by Lemma 10.6 these sequences may also be taken to be null.

Step 3. Since $\partial_k \partial_{k-1} = 0$, this expression for $\sigma'_n$ now implies

$$dU_i (h_n - \partial_k s_n) = 0.$$

Since $\partial_k s_n \in \partial_k(H^{(k-1)})$ and we still have $h_n - \partial_k s_n \rightarrow 0$, we may replace $h_n$ with $h_n - \partial_k s_n$ without changing the desired conclusion, and hence assume simply that each $h_n$ is $U_i$-invariant. Since Lemma 10.2 gives that the top structural homology of $\mathcal{M}^{U_i}$ is discrete, it follows that $H^1_{[k]}$ is co-discrete over $\partial_k((H^{(k-1)})^{U_i})$. This completes the proof.

Remark. Note that, similarly to the implication \((i) \lor (ii) \implies (iii)\) for Proposition 9.12 both of the lemmas above made essential use of the structural discreteness of the $\Delta$-modules appearing in Lemmas 10.3 in proving the closure of the groups $H_{[k]}$, via the fact that a convergent sequence in a discrete group is eventually constant.

\begin{flushright}
$\square$
\end{flushright}

Completed proof of Theorem 10.1. Lemma 10.7 gives the closure of $\partial_k(H^{(k-1)})$, and now Lemma 10.8 gives both the closure of $H_{[k]}$ (because any Cauchy sequence in $H_{[k]}$ must eventually stay within a single coset of $\partial_k(H^{(k-1)})$), and its co-discreteness over $\partial_k(H^{(k-1)})$. \hfill $\square$

11 Partial difference equations

11.1 Proof of Theorem A

Recall the construction of the solution $\Delta$-module of a PD$^{ce}$E in Example 5.20. It is clear that all constituents of these $\Delta$-modules are closed, and hence Polish. In our
abstract terminology, Theorem A asserts that this solution $\Delta$-module is 1-almost modest. We will prove that by giving an alternative construction of the solution $\Delta$-modules in terms of other more general operations on $\Delta$-modules.

For each $j \in \{0, 1, \ldots, k\}$, let $U^{(j)} = (U^{(j)}_\ell)_{\ell=1}^k$ be the subgroup tuple

$$U^{(j)}_\ell := \begin{cases} U_\ell & \text{if } \ell \leq j \\ \{0\} & \text{if } \ell > j. \end{cases}$$

Also, for each $j$, let $M_j = (M_j^e)_{e \subseteq [k]}$ be the solution $\Delta$-module of the PD $\mathcal{E}$ directed by $U^{(j)}$.

Note first of all that $M_0^\emptyset = 0 \forall j$ and $M_e^j = \mathcal{F}(Z, A)$ whenever $e \not\subseteq [j]$, since if $\ell \in e \setminus [j]$ then $U^{(j)}_\ell = \{0\}$, and so every $f \in \mathcal{F}(Z, A)$ trivially satisfies

$$d^{U^{(j)}_\ell} f = 0.$$  

In particular, $\mathcal{M}^0 = (M^0_e)_{e \subseteq [k]}$ is the solution $\Delta$-module of the PD $\mathcal{E}$ directed by $U^{(j)}$.

Next, for each $j$ let $\mathcal{M}^j := \mathcal{M}^j_{(\ell) \setminus \{j+1\}}$. It is easy to check that

$$M^{j+1}_{e \setminus \{j+1\}} \leq M^j_e \leq M^j_{e \setminus \{j+1\}} \forall j, e.$$  

(21)

We will prove by induction on $j$ that each $\mathcal{M}^j$ is a 1-almost modest $(Z, 0, U^{(j)})\Delta$-module, by showing how each $\mathcal{M}^{j+1}$ may be constructed in terms of $\mathcal{M}^j$. This will occupy the next few lemmas.

**Lemma 11.1.** Suppose for some $j \leq k - 1$ that $\mathcal{M}^j$ is already known to be a 1-almost modest $(Z, 0, U^{(j)})\Delta$-module.

Then the $(Z, 0, U^{(j)})\Delta$-module $\mathcal{M}^{j+1}_c$ and the $(Z, U^{(j)}_{j+1}, U^{(j)})\Delta$-module $(\mathcal{M}^{j} / \mathcal{M}^{j}_{c})U^{(j)}_{j+1}$ are both also $(Z, 0, U^{(j+1)})\Delta$-modules, and are 1-almost modest as such.

**Proof.** On the one hand, $\mathcal{M}^{j+1}$ is aggrandized from a 1-almost modest $(Z, 0, U^{(j)}_{[k] \setminus \{j+1\}})$-\Delta-module, so it may equally well be directed by $(Z, 0, U')$ for any subgroup-tuple $U' = (U'_i)_{i=1}^k$ such that $U'_i = U^{(j)}_i$ when $i \neq j + 1$, and it will still be 1-almost modest.
On the other hand, $\mathcal{M}^j / \mathcal{M}^j$ is now a 1-almost modest $(\mathbb{Z}, 0, U^{(j)})$-$\Delta$-module by part (1) of Proposition 8.2 about short exact sequences. In addition, this quotient $\Delta$-module has constituent equal to 0 in all positions $e \subseteq [k] \setminus \{j + 1\}$. Now Theorem 9.5 implies that $(\mathcal{M}^j / \mathcal{M}^j)^{U_{j+1}}$ is a 1-almost modest $(\mathbb{Z}, U_{j+1}, U^{(j)})$-$\Delta$-module which still has constituents zero at positions $e \subseteq [k] \setminus \{j + 1\}$.

However, since all the elements of $(\mathcal{M}^j e / \mathcal{M}^j e \{j+1\})^{U_{j+1}}$ are $U_{j+1}$-invariant, we may now interpret $(\mathcal{M}^j e / \mathcal{M}^j e \{j+1\})^{U_{j+1}}$ as a $(\mathbb{Z}, U_{j+1}, U^{(j)})$-$\Delta$-module, just by letting the derivation-lifts for the subgroup $U_{j+1}$ all be zero. Once again, this change of directing groups does not affect the 1-almost modesty of $\mathfrak{H}$.

Finally, an appeal to Lemma 5.15 now shows that (with these new, trivial derivation-lifts of $U_{j+1}$), $(\mathcal{M}^j / \mathcal{M}^j)^{U_{j+1}}$ is also a 1-almost modest $(\mathbb{Z}, 0, U^{(j+1)})$-$\Delta$-module, as required.

**Lemma 11.2.** For each $j \in \{0, 1, \ldots, k - 1\}$, if we regard the two $\Delta$-modules of the previous lemma as $(\mathbb{Z}, 0, U^{(j+1)})$-$\Delta$-modules, then there is a short exact sequence

$$\mathcal{M}^j \hookrightarrow \mathcal{M}^{j+1} \rightarrow (\mathcal{M}^j / \mathcal{M}^j)^{U_{j+1}}.$$

**Proof.** Given the preceding lemma, the inclusions (21) show that $\mathcal{M}^j \leq \mathcal{M}^{j+1}$ as $(\mathbb{Z}, 0, U^{(j+1)})$-$\Delta$-modules, so there is a short exact sequence

$$\mathcal{M}^j \hookrightarrow \mathcal{M}^{j+1} \hookrightarrow \mathcal{L}$$

with $\mathcal{L}$ equal to the $\Delta$-module of quotients $(M^j_{e \{j+1\}} e / M^j_{e \{j+1\}} e)$. It remains to show that $\mathcal{L} = (\mathcal{M}^j / \mathcal{M}^j)^{U_{j+1}}$:

- If $j + 1 \in e$, then $M^j_{e \{j+1\}}$ consists of those $f \in M^j_{e}$ such that

  $$d_u f \in M^j_{e \{j+1\}} \quad \forall u \in U_{j+1},$$

  hence

  $$M^j_{e \{j+1\}} / M^j_{e \{j+1\}} = \mathbb{Z}^0(U_{j+1}, M^j_{e} / M^j_{e \{j+1\}}).$$

- On the other hand, if $j + 1 \notin e$, then

  $$M^j_{e \{j+1\}} = M^j_{e} = M^j_{e \{j+1\}},$$

  and again one has the (now trivial) equality

  $$M^j_{e \{j+1\}} / M^j_{e \{j+1\}} = \mathbb{Z}^0(U_{j+1}, M^j_{e} / M^j_{e \{j+1\}}).$$

$\Box$
**Corollary 11.3.** The solution \(\Delta\)-module \(\mathcal{M}^3\) is \(1\)-almost modest for every \(j\).

**Proof.** This follows by induction on \(j\). We have already seen that \(\mathcal{M}^0\) is \(1\)-almost modest. If we assume that \(\mathcal{M}^3\) is \(1\)-almost modest, then Corollary 6.7 gives the same for \(\mathcal{M}^2\); part (1) of Proposition 8.2 gives the same for \(\mathcal{M}^3/\mathcal{M}^2\); Theorem 9.5 gives the same for \((\mathcal{M}^3/\mathcal{M}^2)^{U_3+1}\); and finally the previous lemma and part (2) of Proposition 8.2 give the same for \(\mathcal{M}^{j+1}\). \(\square\)

**Proof of Theorem A.** This is precisely the assertion that \(\mathcal{M}^k\) is \(1\)-almost modest in its top structure complex. \(\square\)

### 11.2 Multiple systems of equations

As a further illustration of our general theory, we are also able to study functions that simultaneously satisfy several PD\(\alpha\)Es. We now sketch this in the case of two systems.

Suppose that \(U = (U_i)_{i=1}^k\) and \(V = (V_j)_{j=1}^\ell\) are two tuples of closed subgroups of \(Z\). Let \(\mathcal{M}_0^0\) be the solution \((Z, 0, U)\)-\(\Delta\)-module associated to \(V\), so we have just shown that this is \(1\)-almost modest, and let

\[
\mathcal{M}_0^0 := \text{Ag}_{V}^{(U,V)} \mathcal{M}_0^0,
\]

where \((U, V)\) denotes the concatenated subgroup-tuple \((U_1, \ldots, U_k, V_1, \ldots, V_\ell)\).

Let \(U^{i'}\) for \(i' = 0, 1, \ldots, k\) be as before, and now define \(\mathcal{M}^{i'}\) to be the functional \((Z, 0, (U, V))\)-\(\Delta\)-module consisting of solutions to the PD\(\alpha\)E associated to \(V\) and also to \(U^{i'}\) (so this is compatible with our previous definition for \(i' = 0\)).

Now the exact same argument as before applies, giving by induction on \(i'\) that every \(\mathcal{M}^{i'}\) is \(1\)-almost modest. For \(i' = k\), this reaches the \((Z, 0, (U, V))\)-\(\Delta\)-module of simultaneous solutions to both systems of PD\(\alpha\)E. As a first consequence, we see that if \(U_k + V_\ell = Z\), then the module of simultaneous solutions to both PD\(\alpha\)E is co-discrete over the submodule of ‘degenerate solutions’: that is, those that may be decomposed as

\[
f = f_0 + \sum_{i=1}^k f_i + \sum_{j=1}^\ell f'_j,
\]

where the function \(f_i\) for \(1 \leq i \leq k\) solves the simpler pair of PD\(\alpha\)Es associated to \(U \upharpoonright [k] \setminus j\) and \(V\), and the function \(f'_j\) for \(1 \leq j \leq \ell\) solves the PD\(\alpha\)Es associated to \(U\) and \(V \upharpoonright [\ell] \setminus j\).
12 Zero-sum tuples

Now recall the zero-sum $\Delta$-module associated to data $U$ constructed in Example 5.22. To prove Theorem B we will show that this is always 2-almost modest.

The argument will be somewhat similar to the analysis of general $\Delta$-submodules of modest $\Delta$-modules in Section 10, but made simpler by some special features of the present situation.

For each $j$, let $U^{(j)}$ be the tuple of acting groups constructed from $U$ as in Subsection 11.1. Let

$$\Psi^j : \mathcal{D}^j \rightarrow \mathcal{L}^j$$

be the $\Delta$-morphism considered in Example 5.22 for the data $U^{(j)}$, so that $N^j := \ker \Psi^j$ is the associated zero-sum $\Delta$-module. Our ultimate interest is in $N^k$.

We start with the base clause of the induction.

**Lemma 12.1.** The $(Z, 0, U^{(0)})$-$\Delta$-module $N^0$ is 2-almost modest.

**Proof.** For $j = 0$, one has

- $P^0_e := \mathcal{F}(Z, A)^{\oplus e}$, as in Example 5.21,
- $L^0_e := \mathcal{F}(Z, A)$ for all nonempty $e$ and $L^0_\emptyset = 0$;
- and $\Psi^0_e((f_i)_{i \in e}) := \sum_{i \in e} f_i$.

The special feature of the case $j = 0$ is that each $\Psi^0_e$ is surjective. We have therefore constructed a short exact sequence

$$0 \rightarrow N^0 \rightarrow P^0 \Psi \rightarrow L^0 \rightarrow 0.$$

Next, $\mathcal{D}^0$ may be alternative written as

$$\bigoplus_{i=1}^k \text{Ag}_{\{i\}} \mathcal{D}^0_i,$$

where $\mathcal{D}^0_i$ is the $(Z, 0, (0))$-$\Delta$-module

$$0 \rightarrow \mathcal{F}(Z, A)$$

with trivial structure morphism and derivation-lifts.

Therefore, by Corollary 6.3 the homology of the structure complex of $\mathcal{D}^0$ at $e$ is all trivial whenever $|e| > 1$. When $e = \{i\}$, its structural homology at $(e, 1)$ is just $\mathcal{F}(Z, A) = \text{Coind}^Z_0 A$. Therefore $\mathcal{D}^0$ is 1-almost modest.
On the other hand, \( L^0 \) agrees with the constant \( \Delta \)-module \( (F(Z, A))_e \) at all \( e \) except \( e = \emptyset \). Since the constant \( \Delta \)-module has trivial structural homology everywhere except \( (\emptyset, 0) \), and we form \( L^0 \) by removing the module indexed by \( \emptyset \), it follows that the structure complex of \( L^0 \) at \( e \) has homology equal to \( F(Z, A) \) in position 1 and zero at all later positions. Therefore \( L^0 \) is also 1-almost modest.

Lastly, for each \( i \) one sees that \( \Psi_i : P^0_i \to L^0_i \) is just the identity morphism of \( F(Z, A) \), and so \( N^0_i = \ker \Psi^0_i = 0 \).

Putting these facts together, we may now apply part (3) of Proposition \( \ref{prop:props} \) to deduce that \( M^0 \) is 2-almost modest.

Now, for each \( j \leq k - 1 \), let \( M^i_j := M^i_{\{j\} \setminus \{j+1\}} \). By Corollary \( \ref{cor:cor} \) if \( M^i_j \) is 2-almost modest, then so is \( M^i_{j+1} \). Also, as for the modules of \( \text{PD}^e \mathcal{E} \) solutions, an easy check gives

\[
N^i_{e \setminus \{j+1\}} = N^i_{e \setminus \{j\}} \leq N^i_{e \setminus \{j+1\}} \leq N^i_e \quad \forall j, e,
\]

and so \( M^i_j \) may also be naturally identified with \( M^i_{j+1} \setminus \{j\} \setminus \{j+1\} \), and so interpreted as a \((Z, 0, U^{(j+1)})\)-\( \Delta \)-module.

Now suppose \( f \in N^i_{j+1} \). Then

\[
\nabla U_j^{j+1} f \in N^i_{e \setminus \{j+1\}} = N^i_{e \setminus \{j\}},
\]

so this defines a class in \( H^1_m(U_i, N^j_{e \setminus \{j+1\}}) \).

Now, by Lemma \( \ref{lem:lem5} \) we may interpret \( M^i_{j+1} \) as a \((Z, U_{j+1}, U^{(j+1)})\)-\( \Delta \)-module (although not expecting it to be 1-almost modest as such), and therefore this application of \( \nabla U_j^{j+1} \) defines a \( \Delta \)-morphism

\[
\Phi^j : M^i_{j+1} \to H^1_m(U_i, M^i_{j+1})
\]

of \((Z, U_{j+1}, U^{(j+1)})\)-\( \Delta \)-modules.

**Lemma 12.2.** If \( M^i_j \) is a 2-almost modest \((Z, 0, U^{(j)})\)-module, then \( \text{img} \Phi^j \) may be interpreted as a 2-modest \((Z, 0, U^{(j+1)})\)-\( \Delta \)-module.

**Proof.** Given this assumption, Corollary \( \ref{cor:cor} \) and Theorem \( \ref{thm:thm5} \) imply that \( H^1_m(U_{j+1}, M^i_{j+1}) \) is a 2-modest \((Z, U_{j+1}, U^{(j)})\)-\( \Delta \)-module, and as above it may be interpreted also as a \((Z, U_{j+1}, U^{(j+1)})\)-\( \Delta \)-module. Therefore Theorem \( \ref{thm:thm1} \) gives that \( \text{img} \Phi^j \) is a 2-modest \((Z, U_{j+1}, U^{(j+1)})\)-\( \Delta \)-submodule. On the other hand, if \( e \not\in j + 1 \), then \( N^i_{e \setminus \{j+1\}} = N^i_{e \setminus \{j\}} \), and so \( \Phi^j_e(f) \) is an \( N^i_{e \setminus \{j+1\}} \)-valued coboundary for every \( f \in N^i_{e \setminus \{j+1\}} \). Therefore \( \text{img} \Phi^j_e = 0 \) for all such \( e \), and the proof is completed by Lemma \( \ref{lem:lem5} \). \( \square \)
The next step is to determine \( \ker \Phi_j \).

If \( e \not\ni j + 1 \) then \( \ker \Phi^j_e = N^{j+1}_{e\setminus (j+1)} \), because \( N^{j+1}_{e\setminus (j+1)} = N^j_{e\setminus (j+1)} \) so every \( N^{j+1}_{e\setminus (j+1)} \)-valued coboundary is an \( N^j_{e\setminus (j+1)} \)-valued coboundary.

If \( e \ni j + 1 \), then \( f \in \ker \Phi^j_e \) if and only if \( \nabla U_{j+1} f = d_{U_{j+1}} g \) for some \( g \in N^j_{e\setminus (j+1)} \) . Now an easy check shows that this is equivalent to \( f \) being an element of \( N^j_e \) for which \( \nabla U_{j+1} f \) takes values in \( N^j_{e\setminus (j+1)} \).

As for Lemma 11.2, we may now form the \((Z, 0, U^{(j)})\)-\(\Delta\)-module \( \mathcal{N}^j_\cdot \) and interpret it as a \((Z, 0, U^{(j+1)})\)-\(\Delta\)-module because all elements are \( U_{j+1} \)-invariant. Having done so, we obtain the following direct analog of Lemma 11.2.

**Lemma 12.3.** There is a short exact sequence

\[
\mathcal{N}^j_\cdot \hookrightarrow \ker \Phi^j \rightarrow (\mathcal{N}^j_\cdot / \mathcal{N}^j_\cdot U_{j+1} )
\]

of \((Z, 0, U^{(j+1)})\)-\(\Delta\)-modules.

**Proof of Theorem B.** An induction on \( j \) shows that every \( \mathcal{N}^j_\cdot \) is a 2-almost modest \((Z, 0, U^{(j)})\)-\(\Delta\)-module. For \( j = 0 \) this is Lemma 12.1. Then, given the result for some \( j \leq k - 1 \), Lemma 12.3 and part (2) of Proposition 8.2 prove that \( \ker \Phi^j \) is a 2-almost modest \((Z, 0, U^{(j+1)})\)-module.

Finally, this now fits into the presentation

\[
\ker \Phi^j \hookrightarrow \mathcal{N}^{j+1}_\cdot \twoheadrightarrow \im \Phi^j
\]

in which the last \((Z, 0, U^{(j+1)})\)-\(\Delta\)-module is also known to be 2-almost modest by Lemma 12.2. Therefore the induction continues by another appeal to part (2) of Proposition 8.2.

13 Rudimentary quantitative results

13.1 Repairing approximate solutions

We next prove Theorem C. We first prove a version in which the error tolerances may depend on the underlying groups \( Z \) and \( U_i \), and will then remove that extra dependence by a compactness argument.

**Lemma 13.1 (Weak form of Theorem C).** Fix \( Z \) and a subgroup-tuple \( U \). For all \( k \geq 1 \) and \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that if \( f \in \mathcal{F}(Z) \) satisfies

\[
d_0(0, d^{U_1} \cdots d^{U_k} f) < \delta,
\]

then there is some \( g \in M \) for which \( d_0(f, g) < \varepsilon \).
Proof. This is proved by induction on \( k \). When \( k = 1 \) it is an easy exercise, so suppose \( k \geq 2 \) and that the result is known for any PD\(^{co}E\) of order \( k-1 \). Let \( M \) be the modules of solution to the PD\(^{co}E\) of \( U \), and let \( M' \) be the module of solutions to the PD\(^{co}E\) associated to \((U_1, \ldots, U_{k-1})\).

Now suppose that \((f_n)_n\) is a sequence in \( \mathcal{F}(Z) \) such that
\[
d^{U_1} \cdots d^{U_k} f_n \rightarrow 0.
\]
Then for each \( u \in U_k \), one has
\[
d^{U_1} \cdots d^{U_{k-1}} (d_u f_n) \rightarrow 0,
\]
so by the induction on \( k \) there are a null sequence \( f_{n,u} \) in \( \mathcal{F}(Z) \) and a sequence \( g_{n,u} \in M' \) such that \( d_u f_n = f_{n,u} + g_{n,u} \). By a simple measurable selection we may assume that each \( f_{n,u}(z) \) and \( g_{n,u}(z) \) is jointly measurable as a function of \((u,z)\).

Regarding each \( f_n \) and \( g_n \) as an element of \( \mathcal{C}^1(U_k, \mathcal{F}(Z)) \), it follows that
\[
\sigma_n := d^{U_k} f_n, = -d^{U_k} g_n.
\]
is both a null sequence, by the first expression, and an \( M' \)-valued 2-coboundary, by the second expression.

The proof of Theorem A showed that \( M' \) is the top module of a modest \( (Z, \{0\}, U) \)-\( \Delta \)-module, and so Theorem \( 9.5 \) implies, in particular, that \( H^2_m(U_k, M') \) is Hausdorff, and hence that \( B^2(U_k, M') \) is closed. Therefore Theorem \( 2.2 \) gives a null sequence \( g_{n,*} \in \mathcal{C}^1(U_k, M') \) such that \( \sigma_n = -d^{U_k} g_{n,*} \).

We now make two uses of this equation:

- Let \( g_{n,*} := g_{n,*} - g'_{n,*} \). This gives that \( d^{U_k} g_{n,*} = 0 \). Since \( H^1_m(U_k, \mathcal{F}(Z)) = 0 \), the latter implies that \( g_{n,*} = d^{U_k} g'_{n,*} \) for some \( g'_{n,*} \in \mathcal{F}(Z) \), and now since \( g'_{n,*} \) takes values in \( M' \), these \( g'_{n,*} \) all lie in \( M \).

- On the other hand, we have that \( f_n \) and \( g'_{n,*} \) are both null, and
\[
d^{U_k} (f_n + g'_{n,*}) = \sigma_n - \sigma_n = 0,
\]
so, using again that \( H^1_m(U_k, \mathcal{F}(Z)) = 0 \) (hence certainly Hausdorff), there is a null sequence \( f'_n \in \mathcal{F}(Z) \) such that \( f_n + g'_{n,*} = d^{U_k} f'_n \).

Finally, re-tracing our steps now yields
\[
d_u f_n = (f_{n,u} + g'_{n,u}) + (g_{n,u} - g'_{n,u}) = d_u (f'_n + g'_{n}).
\]
Therefore \( f_n = f'_n + g'_{n} + h_n \) for some \( h_n \in \mathcal{F}(Z)^{U_k} \). Since \( f'_n \) is null and \( (g'_{n} + h_n) \) lies in \( M + \mathcal{F}(Z)^{U_k} = M \), this completes the next step of the induction. \( \square \)
Proof of Theorem C. This proof is by compactness and contradiction. Suppose the result is false, and let \( \varepsilon > 0 \) be such that one can find a sequence

\[
(Z_n, U_{1,n}, \ldots, U_{k,n}, f_n)
\]

of data of the relevant kind such that

\[
d_0(0, d^{U_{1,n}} \cdots d^{U_{k,n}} f_n) \leq 2^{-n}
\]

but

\[
\min \{d_0(f_n, g) \mid g \in M_n \} \geq \varepsilon \quad \forall n,
\]

where \( M_n \) is the solution module of the PDCE defined by \( U_n \).

Let

\[
\overline{Z} := \prod_{n \geq 1} Z_n,
\]

\[
\overline{U}_i := \prod_{n \geq 1} U_{i,n}, \quad i = 1, 2, \ldots, k,
\]

and \( \overline{U} := (\overline{U}_i)_{i=1}^k \). Also let \( \overline{M} \) be the module of solutions to the PDCE associated to \( \overline{U} \), and define \( \overline{f}_n \in \mathcal{F}(\overline{Z}) \) by

\[
\overline{f}_n(\overline{z}) := f_n(z_n), \quad \text{where } \overline{z} = (z_1, z_2, \ldots).
\]

These now satisfy

\[
d_{\overline{m}_1} \cdots d_{\overline{m}_k} \overline{f}_n(\overline{z}) = d_{u_{1,n}} \cdots d_{u_{k,n}} f_n(z_n) \quad \forall \overline{m}_1, \ldots, \overline{m}_k, \overline{z},
\]

and hence

\[
d_0(0, d^{\overline{f}_1} \cdots d^{\overline{f}_k} \overline{f}_n) \longrightarrow 0 \quad \text{in } \mathcal{F}(\overline{Z}).
\]

Therefore Lemma 13.1 gives a sequence \( \overline{g}_n \in \overline{M} \) such that \( \overline{f}_n - \overline{g}_n \) is null. However, the fact that \( \overline{g}_n \in \overline{M} \) implies that for Haar-almost every choice of

\[
(z_1, z_2, \ldots, z_{n-1}, z_{n+1}, \ldots) \in \prod_{n' \geq 1, n' \neq n} Z_{n'},
\]

the restriction

\[
g_n(z_n) := \overline{g}_n(z_1, \ldots, z_n, \ldots)
\]

is a member of \( M_n \). On the other hand, the fact that \( \overline{f}_n - \overline{g}_n \) is null implies that, on average over such choices of \( (z_1, z_2, \ldots, z_{n-1}, z_{n+1}, \ldots) \), the quantity \( d_0(f_n, g_n) \) tends to zero. Therefore a suitable sequence of restrictions \( g_n \) gives \( g_n \in M \) and \( d_0(f_n, g_n) \longrightarrow 0 \), contradicting our assumptions. \( \square \)
Proof of Corollary C'. If \(f : Z \to \mathbb{D}\) and \(\|f\|_{U(U)}\) is close enough to 1, then this implies that \(|f(z)|\) is close to 1 for all \(z \in Z\) outside a set of small measure. We may therefore find a function \(f_1 : Z \to S^1\) which is very close to \(f\) in probability. If it is close enough, and if \(\|f\|_{U(U)}\) is close enough to 1, then \(\|f_1\|_{U(U)}\) will also be very close to 1. This now implies that \(f_1\) satisfies the conditions of Theorem C, written multiplicatively. One can therefore find an exact solution \(g : Z \to S^1\) close to \(f_1\), and hence close to \(f\).

Remark. Corollary C' is still far from even suggesting a conjecture for the inverse problem for the directional Gowers norms \(\| \cdot \|_{U(U)}\). As described in Subsection 1.3, this problem supposes that \(f : Z \to \mathbb{D}\) has \(\|f\|_{U(U)} > \delta\) for some fixed \(\delta > 0\), and asks for some structural conclusion about \(f\). However, the importance of cohomology does suggest that the following related inverse problem may be important:

Question 13.2. Suppose that \(f \in C^p(Z, \mathbb{T})\) is a \(p\)-cochain such that
\[
\int_{Z^{p+1}} \exp \left(2\pi i \cdot df(z_1, \ldots, z_{p+1})\right) m_Z(dz_1) \cdots m_Z(dz_{p+1}) > \delta
\]
for some \(\delta > 0\). What does this imply about the structure of \(f\)?

13.2 Independence from the underlying groups

We will now prove 'Theorem A', asserting that \(\varepsilon\) may be taken to depend only on \(k\) in both Theorem A. Theorem B', asserting the analogous independence in Theorem B, has an exactly similar proof to Theorem A', so we omit it. Theorem C is a crucial tool for this purpose. Let us reformulate this independence in terms of solution \(\Delta\)-modules.

Proof of Theorem A'. For a fixed choice of \(Z\) and \(U\), the existence of such an \(\varepsilon\) is precisely the discreteness of the structure homology of \(\mathcal{M}\), which holds because we have shown that \(\mathcal{M}\) is 1-almost modest.

It remains to show that \(\varepsilon\) may be chosen independently of \(Z\) and \(U\). Suppose that \(Z_n\) and \(U_n := (U_{1,n}, \ldots, U_{k,n})\) are sequences of ambient groups and subgroup tuples, let \(\mathcal{M}_n\) be the solution \(\Delta\)-modules of their PD\(^a\)Es, and suppose that for some \(\ell \leq k\) one can find a sequence of elements
\[
f_n \in \ker \partial_{\ell+1} \, \mathcal{M}_n \setminus \text{img} \, \partial_\ell \mathcal{M}_n
\]
with \(d_0(0, f_n) \to 0\).

Now construct \(Z, \overline{U}\) and \(\overline{f}_n\) just as in the proof of Theorem C. Let \(\overline{\mathcal{M}}\) be the solution \(\Delta\)-module for the PD\(^a\)E associated to \(\overline{U}\). Then the existence of some
\(\varepsilon > 0\) in Theorem A for this limiting PD\(^{\text{coE}}\)E implies that \(f_n \in \partial_k(M^{(\ell-1)})\) for all sufficiently large \(n\), allowing us to write

\[ f_n(z_n) = \overline{f_n(z)} = \partial_k\overline{g(z)} \quad \text{for some } \overline{g} \in M^{(\ell-1)}. \]

Similarly to the previous proof, this now implies that for a.e. choice of \((z_1, z_2, \ldots, z_{n-1}, z_{n+1}, \ldots)\), the restricted function defined by

\[ g(z_n) := \overline{g(z_1, \ldots, z_{n+1}, \ldots)} \]

are elements of the smaller modules \(M_n^{(\ell-1)}\). Hence \(f_n \in \partial_k(M_n^{(\ell-1)})\) for all sufficiently large \(n\), contradicting our assumptions. \(\square\)

### 13.3 Basic solutions are finite-dimensional

The following is an easy consequence of Theorem C, and may be of interest in its own right. It is a relative of \([2, \text{Theorem B}]\), which asserted that cohomology of general compact groups into discrete modules is all inflated from finite-dimensional quotient groups. We formulate the result for PD\(^{\text{coE}}\)Es, and leave the obvious analog for zero-sum tuples to the reader.

**Theorem 13.3.** Let \(Z\) be an ambient group, \(U = (U_i)_{i=1}^k\) a tuple of subgroups such that \(U_{[k]} = Z\), and let \(\mathcal{M} = (M_e)_e\) be the solution \(\Delta\)-module of the associated PD\(^{\text{coE}}\).

For any \(f \in M_{[k]}\), there are

- a finite-dimensional quotient \(q : Z \rightarrow Z' \leq \mathbb{T}^d\),
- a subgroup tuple \(U' = (U'_i)_{i=1}^k\) in \(Z'\),
- and a solution \(f'\) to the PD\(^{\text{coE}}\) on \(Z'\) associated to \(U'\)

such that

\[ f \in f' \circ q + \partial_k(M^{(k-1)}). \quad (22) \]

**Proof:** As is well-known, for every \(\varepsilon > 0\) there are some finite-dimensional quotient \(q : Z \rightarrow Z' \leq \mathbb{T}^d\) and some \(g \in \mathcal{F}(Z', \mathbb{T})\) such that

\[ d_0(f, g \circ q) < \varepsilon. \]

Letting \(U'_i := q(U_i)\) for each \(i\), this now implies that

\[ d_0(0, d^{U'_1} \cdots d^{U'_k} g) = d_0(0, (d^{U'_1} \cdots d^{U'_k} g) \circ q^{(k+1)}) = d_0((d^{U'_1} \cdots d^{U'_k} g) \circ q^{(k+1)}, d^{U_1} \cdots d^{U_k} f) < 2^k \varepsilon.\]
Using this and Theorem C, for any $\eta > 0$ there is some $\varepsilon > 0$ such that
\[ d_0(f, g \circ q) < \varepsilon \implies d_0(g, M'_{[k]}) < \eta, \]
where $\mathcal{M}'$ is the solution $\Delta$-module for $Z'$ and $U'$. Choosing $f' \in M'_{[k]}$ close enough to $g$, this now translates into
\[ d_0(0, f - f' \circ q) = d_0(f, f' \circ q) < \varepsilon + \eta. \]
Since $f' \circ q \in M_{[k]}$, so is $f - f' \circ q$.

Finally, if $\eta$ and then $\varepsilon$ were chosen small enough, Theorem $A'$ now implies that
\[ f - f' \circ q \in \partial_k(M^{(k-1)}). \]

\[ \square \]

**Remark.** The above argument actually gives the following more quantitative result: for each $k \geq 1$ there is some $\varepsilon > 0$ such that if $f \in M_{[k]}$ may be approximated in $d_0$ to within distance $\varepsilon$ by a function lifted from a $d$-dimensional quotient group, then it decomposes as in (22) using that same quotient group.

Theorem 13.3 begs the following question, which seems to lie beyond our current methods.

**Question 13.4.** Is it true that for each $k \geq 1$ there is a $d_0 \geq 1$, depending only on $k$, such that for any $f \in M_{[k]}$ one has a decomposition
\[ f \in f_1 \circ q_1 + \cdots + f_m \circ q_m + \partial_k(M^{(l-1)}), \]
where each $f_i \circ q_i$ is a solution lifted from some quotient $q_i : Z \twoheadrightarrow Z_i'$ for which $Z_i'$ is a subgroup of $\mathbb{T}^{d_0}$?

**Remark.** As a complement to Theorem 13.3 it should be possible to prove that if $Z$ is finite-dimensional a priori, then the discrete homology of all the higher structure complexes of $\mathcal{M}$ is finitely generated. More generally, within the class of almost modest $\Delta$-modules, one can isolate the subclass in which all the structural homology is not only discrete, but also finitely generated. If $Z$ (and so also all its subgroups) is finite-dimensional, then one should then be able to show, by the same arguments as above, that the classes of almost or strictly modest $(Z, Y, U)$-$\Delta$-modules whose discrete structural homology is finitely generated are closed under extensions, quotients, cohomology and submodules. Then the asserted finite-generation will follow as in the proof of Theorem A. The details are omitted here.
14 Analysis of some concrete examples

This section turns to a different aspect of the techniques developed above. Much of the work of analyzing modules of PD\textsuperscript{ce}E solutions or zero-sum tuples amounts to decomposing those modules into simpler pieces, for example in the sense of the short exact sequence of Lemma 11.2. Repeating this kind of decomposition eventually leads to modules that can be understood quite explicitly, usually cohomology groups with coefficients in some fixed Lie group.

Most of our arguments above concern one way or another in which the original, ‘large’ modules can be reconstructed from these ‘simple’ pieces. However, one may also use the classes in the resulting cohomology groups as obstructions to a certain kind of structure. Most obviously, this gives a way to prove that some PD\textsuperscript{ce}E solutions are non-degenerate.

This section revisits Examples 1.3 and 1.5, and offers a few more examples. This will include a (sketch) proof that the $\mathbb{Z}$-valued function in Example 1.5 is non-degenerate.

In these examples, we will not describe carefully all of the different cohomological invariants that can be obtained from them. Also, in many cases one finds that to understand the full module of PD\textsuperscript{ce}E-solutions or zero-sum tuples, one does not need to compute the exact structure complex of the associated $\Delta$-module at every position. Given a particular solution, one can often find an obstruction showing that it is non-degenerate with much less work. This is because in many cases one can foresee by inspection some vanishing or collapsing among the modules of the given $\Delta$-module, and this then justifies using a simplified presentation of the modules of degenerate solutions.

14.1 PD\textsuperscript{ce}E for linearly independent subgroups

As in Example 1.3, the subgroup tuple $U = (U_i)_{i=1}^k$ is linearly independent if

$$\forall (u_i)_i \in \prod_i U_i : \sum_i u_i = 0 \implies u_1 = u_2 = \ldots = u_k = 0.$$  

This is equivalent to the sum homomorphism

$$\prod_{i=1}^k U_i \rightarrow U_{[k]}$$

being injective, hence an isomorphism. Let us now assume that $Z = U_{[k]}$ for simplicity, so we may as well simply let $Z = \prod_{i=1}^k U_i$.  

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In this case, the original PD$ce$E may be solved using a generalization of the trick in Example [1.3]. Written out in full, a function $f \in \mathcal{F}(Z, A)$ satisfies that PD$ce$E if

$$\sum_{\eta \in \{0,1\}^k} (-1)^{|\eta|} f(z_1 + \eta_1 u_1, z_2 + \eta_2 u_2, \ldots, z_k + \eta_k u_k) \equiv 0,$$

where $|\eta| := |\{i \leq k \mid \eta_i = 1\}|$, and we write elements of $\mathcal{F}(Z, A)$ as functions of the separate coordinates in $\prod_{i=1}^k U_i$.

By Fubini’s Theorem, we may fix a tuple $(z_i)_i \in \prod_{i=1}^k U_i$ such that the above holds for a.e. $(u_i)_i \in \prod_{i=1}^k U_i$. Let $z^0_i := z_i$ and $z^1_i := z^0_i + u_i$ for each $i$. Using these new variables, the above can be re-arranged to give

$$f(z^1_1, \ldots, z^1_k) = \sum_{\eta \in \{0,1\}^k, |\eta| \leq k-1} (-1)^{k-|\eta|} f(z^\eta_1, \ldots, z^\eta_k).$$

Regarding this as a function of only $(z^1_1, \ldots, z^1_k)$, the right-hand side is manifestly an element of $\sum_{i=1}^k \mathcal{F}(Z, A)^U_i$, since very term on the right depends on $z^0_i$ for at least one $i$. Therefore all PD$ce$E-solutions are degenerate in this case.

14.2 Almost linearly independent subgroups

We now return to Examples [1.5] and [1.6].

First consider Example [1.5]. It had ambient group $\mathbb{T}^2$, but in Subsection [1.3] we discussed its close relative on $(\mathbb{Z}/N\mathbb{Z})^2$. In order to treat these together, fix a compact Abelian group $Z_0$, and let $Z := Z_0^2$ and

$$U_1 := (1,0) \cdot Z_0, \quad U_2 := (0,1) \cdot Z_0 \quad \text{and} \quad U_3 := (1,1) \cdot Z_0.$$ 

Let $\mathcal{M}$ be the solution $\Delta$-module of the associated PD$ce$E for $A$-valued functions.

Since any two of $U_1, U_2$ and $U_3$ are linearly independent, all nontrivial restrictions of $\mathcal{M}$ may be easily described as in the linearly independent case.

Now suppose that $f \in M_{[3]}$. Then $d^{U_3} f$ is a 1-cocycle $U_3 \to M_{[2]}$. If $f \in M_{[2]}$, then $d^{U_3} f$ is an $M_{[2]}$-valued coboundary by definition. Also, if $f \in M_{[1,3]}$, then by the linearly independent case we have $f = f_1 + f_3$ with $f_i$ being $U_i$-invariant for $i = 1, 3$, in which case we still have $d^{U_3} f = d^{U_3} f_1 \in B^1(U_3, M_{[2]})$, and similarly if $f \in M_{[2,3]}$. On the other hand, if $d^{U_3} f \in B^1(U_3, M_{[2]})$, then $f$ must lie in $\mathcal{F}(Z, A)^{U_3} + M_{[2]} = \partial_3(M_{[2]}^2)$. This proves in

$$\partial_3(M_{[2]}^2) = \ker (M_{[3]} \xrightarrow{d^{U_3}} B^1(U_3, M_{[2]}) \to H^1_\text{m}(U_3, M_{[2]})).$$

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So our next step is to compute this degree-1 cohomology group. To do, the result from the linearly independent case and a quick inspection show that the following sequence is exact:

$$0 \longrightarrow A \xrightarrow{a \mapsto (a, -a)} \mathcal{F}(Z, A)^{U_1} \oplus \mathcal{F}(Z, A)^{U_2} \longrightarrow M_{[2]} \longrightarrow 0.$$ 

From this, a piece of the resulting long exact sequence for $H^\ast_m(U_3, -)$ gives

$$\cdots \longrightarrow H^1_m(U_3, \mathcal{F}(Z, A)^{U_1}) \oplus H^1_m(U_3, \mathcal{F}(Z, A)^{U_2}) \longrightarrow H^1_m(U_3, M_{[2]}) \longrightarrow H^2_m(U_3, A) \longrightarrow H^2_m(U_3, \mathcal{F}(Z, A)^{U_1}) \oplus H^2_m(U_3, \mathcal{F}(Z, A)^{U_2}) \longrightarrow \cdots.$$

Now we observe that, since $U_1$ and $U_3$ are linearly independent and span $Z$, we have that $\mathcal{F}(Z, A)^{U_1}$ is isomorphic as a $U_3$-module to $\mathcal{F}(U_3, A)$, and similarly for $\mathcal{F}(Z, A)^{U_2}$. Therefore Lemma 3.1 gives

$$H^1_m(U_3, \mathcal{F}(Z, A)^{U_1}) \oplus H^1_m(U_3, \mathcal{F}(Z, A)^{U_2}) = H^2_m(U_3, M_{[2]}) \oplus H^2_m(U_3, \mathcal{F}(Z, A)^{U_1}) \oplus H^2_m(U_3, \mathcal{F}(Z, A)^{U_2}) = 0,$$

and so the above long exact sequence collapses to give an isomorphism

$$H^1_m(U_2, M_{[2]}) \cong H^2_m(U_3, A) \cong H^2_m(Z_0, A).$$

Therefore, if $H^2_m(Z_0, A) = 0$, then there are only degenerate solutions to our PD$^a$E. This is the case if $Z_0$ is either $\mathbb{T}$ or $\mathbb{Z}/N\mathbb{Z}$ and $A = \mathbb{T}$. In the second case this follows from Lemma [A.1], and in the first from [A.2].

On the other hand, Lemma [A.2] also gives $H^2_m(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}$, and digging into the proof of that lemma (details omitted), one obtains the generating cocycle

$$\sigma(z_1, z_2) := [\{z_1\} + \{z_2\}].$$

One may try to reconstruct a function in $M_{[3]}$ that gives rise to this $\sigma$, and (unsurprisingly) one finds that the function $f$ discussed in Example 1.5 has that property. This proves that

$$M_{[3]}/\partial_3(M^{(2)}) \cong \mathbb{Z},$$

and that the function $f$ from Example 1.5 generates this quotient. A similar analysis can be made for Example 1.6. Generalizing as above, we now take $Z = Z_0^3$ and the acting subgroups

$$U_1 := (1, 0, 0) \cdot Z_0, \quad U_2 := (1, -1, 0) \cdot Z_0,$$
$$U_3 := (0, 1, -1) \cdot Z_0 \quad \text{and} \quad U_4 := (0, 0, 1) \cdot Z_0. \quad (23)$$
Beware that these are not the direct analogs of the subgroups that were labelled ‘$U_i$’ in Example 1.6. However, if one applies differencing operators to equation (6), then the function $\sigma$ constructed there is a function $T^3 \to T$ which is annihilated by the partial differencing operator associated to the subgroups as in (23).

Thus, Example 1.6 leads to the PD ce

$$d^{U_i}d^{U_2}d^{U_3}d^{U_4}\sigma = 0$$

for $\sigma \in \mathcal{F}(Z, A)$.

Any three of the subgroups in (23) are linearly independent. Therefore, if $\mathcal{M}$ is the solution $\Delta_{[4]}$-module for these acting subgroups, then Subsection 14.1 gives that

$$M_e = \sum_{i \in e} \mathcal{F}(Z, A)^{U_i}$$

whenever $|e| \leq 3$. Another easy exercise now extends this to the fact that for each $e \in \binom{[4]}{3}$, the complex

$$0 \to A \xrightarrow{\partial_1} \bigoplus_{a \in \binom{[4]}{3}} \mathcal{F}(Z, A)^{U_a} \xrightarrow{\partial_2} \bigoplus_{i \in e} \mathcal{F}(Z, A)^{U_i} \xrightarrow{\partial_3} M_e$$

(24)

is exact. Note this is not literally the structure complex of $\mathcal{M}$ at $e$. It is an alternative resolution of $M_e$, obtained by inspection, which is simpler and will serve the same purpose.

Now, since any three of the subgroups (23) are linearly independent, Lemma 3.1 implies that

$$H^p_m(U_4, \mathcal{F}(Z, A)^{U_i}) = H^p_m(U_4, \mathcal{F}(Z, A)^{U_i+U_j}) = 0$$

whenever $i, j \in [3]$ and $p \geq 1$. Thus, given any short exact sequence that features either of the middle modules in (24) for $e = [3]$, the resulting long exact sequence for $H^*_m(U_4, -)$ collapses to a sequence of isomorphisms. Using this and the exactness of the whole sequence, one now obtains that

$$H^1_m(U_4, M_{[3]}) \cong H^2_m(U_4, \ker \partial_3) = H^2_m(U_4, \text{img} \partial_2) \cong H^3_m(U_4, \ker \partial_2) = H^3_m(U_4, \text{img} \partial_1),$$

and this is isomorphic to $H^3_m(Z_0, A)$.

If $\sigma \in M_{\{i,j,4\}}$ for any $i, j \in [3]$, then a simple check using (24) for $e = \{i,j,4\}$ shows that $d^{U_4}\sigma \in \mathcal{B}^1(U_4, M_{[3]})$. Therefore, if $\sigma \in M_{[4]}$ is such that $d^{U_4}\sigma$ leads to a nonzero class in $H^3_m(Z_0, A)$ under the above isomorphisms, then we know it is a non-degenerate solution.
This is exactly the situation for the function $\sigma$ given in Example 1.6, in that case we have $H^3_{\text{m}}(Z_0, A) = H^3_{\text{m}}(\mathbb{T}, \mathbb{T}) \cong \mathbb{Z}$ (Lemma A.4), and the $\sigma$ that we chose becomes a generator of that cohomology group (we omit the proof of this).

In light of Examples 1.5 and 1.6, it is natural to expect the following. I do not think a proof requires any deep ideas, but the calculation quickly becomes very tedious, so I have not pursued it to the end.

**Conjecture 14.1.** Fix $Z_0$ and $A$ as above, and also an integer $d \geq 2$, and let $Z := Z_0^d$ and

$$U_1 := (1, 0, \ldots, 0) \cdot Z_0, \quad \ldots, \quad U_d := (0, 0, \ldots, 1) \cdot Z_0, \quad \text{and} \quad U_{d+1} := (1, 1, \ldots, 1) \cdot Z_0.$$

These are clearly almost linearly independent: indeed, any $d$ out of these $d + 1$ subgroups provide a new isomorphism $Z \cong Z_0^d$.

Now let $\mathcal{M}$ be the solution $\Delta$-module for the PD$^{cE}$ associated to $U$. Then

$$M_{[d+1]} / \partial_{d+1}(M^{(d)}) \cong H^d_{\text{m}}(Z_0, A).$$

This suggests that the problem of finding cohomology groups can be embedded as a special case of the problem of solving PD$^{cE}$s. Certainly I do not know any way to solve Example 1.5 without knowing how to calculate the cohomology groups $H^2_{\text{m}}(Z_0, A)$ first.

### 14.3 Some miscellaneous further examples

The examples analyzed previously have all been of the either the kind which are truly polynomial on all cosets of the relevant subgroup, or obtained from the solutions to some cocycle equation.

It seems worth giving some more complicated examples to fill out this picture.

**Example 14.2.** Let $Z = \mathbb{T}^3$ and

$$U_1 = \{z_1 = 0\}, \quad U_2 = \{z_2 = 0\}, \quad U_3 = \{z_3 = 0\} \quad \& \quad U_4 = \{z_1 + z_2 + z_3 = 0\}.$$

These data are interesting for the following reason. Consider any three of these subgroups, say $U_1$, $U_2$ and $U_3$. Then the three intersections $U_1 \cap U_2$, $U_1 \cap U_3$ and $U_2 \cap U_3$ together generate the whole of $U_1 + U_2 + U_3 = Z$. If $q : Z \to Z'$ is any surjective homomorphism, then one must also have that the intersections $q(U_1) \cap q(U_2)$, $q(U_2) \cap q(U_3)$ and $q(U_1) \cap q(U_3)$ generate $q(U_1) + q(U_2) + q(U_3) = Z$. The same goes for any other three of the $U_i$s.
This now implies that for any other four-tuple of subgroups \( (U_i')_{i=1}^4 \) in \( Z' \), if \( q(U_i) \leq U_i' \) for each \( i \) then the \( U_i' \)'s also have the feature that the pairwise intersections of any three of them span \( Z' \). This implies that one can have non non-trivial homomorphism mapping \((Z, U)\) to a linearly independent or almost linear independent subgroup-tuple, and therefore non-degenerate solutions to the PD\( ^{\infty}\)E associated to \( U \) cannot simply have been ‘pulled back’ from cocycle-examples under such a homomorphism.

One finds such a nontrivial solution with target \( Z \): define \( f : \mathbb{T}^3 \rightarrow \mathbb{Z} \) by

\[
f(\theta_1, \theta_2, \theta_3) = \lfloor \{\theta_1\} + \{\theta_2\} + \{\theta_3\} \rfloor.
\]

Then this function satisfies the equation

\[
d^{U_1}d^{U_2}d^{U_3}d^{U_4}f = 0,
\]

because among real-valued functions it equals

\[
\{\theta_1\} + \{\theta_2\} + \{\theta_3\} - \{\theta_1 + \theta_2 + \theta_3\}.
\]

Let us use cohomological data to sketch a proof of the following.

**Lemma 14.3.** The function \( f \) above is a non-degenerate solution to this PD\( ^{\infty}\)E.

**Proof.** As remarked above, for any \( e \in \binom{[4]}{3} \), the corresponding tuple \( U \mid_{e} \) has the property that the pairwise intersections span the whole of \( Z \), but the triple intersection is trivial. More concretely, one may easily construct an isomorphism \((Z, U \mid_{e}) \cong (\mathbb{T}^3, V)\), where \( V \) is the collection of two-dimensional coordinate-subgroups of \( \mathbb{T}^3 \).

An simple relative of the argument in Subsection 14.1 now shows that any PD\( ^{\infty}\)E-solution associated to \( U \mid_{e} \) is an element of \( \sum_{i \in e} \mathcal{F}(Z, \mathbb{Z})^{U_i} \). In case \( e = [3] \), this sum of modules fits into the exact sequence

\[
\Gamma \hookrightarrow \bigoplus_{i \in e} \mathcal{F}(Z, \mathbb{Z})^{U_i} \twoheadrightarrow \sum_{i \in e} \mathcal{F}(Z, \mathbb{Z})^{U_i},
\]

where

\[
\Gamma := \{(m, n, p) \in \mathbb{Z}^3 \mid m + n + p = 0\} \cong \mathbb{Z}^2,
\]

and similarly for the other \( e \in \binom{[4]}{3} \).

**Step 1.** We first use this presentation for \( e = [3] \). If \( f \in M_{[3]} \), then \( d^{U_4}f \) is an element of \( \mathcal{Z}^1(U_4, M_{[3]}) \). If it lies in \( \mathcal{B}^1(U_4, M_{[3]}) \), then one has \( f = g + h \) for some \( g \in M_{[3]} \) and \( h \in \mathcal{F}(Z, \mathbb{Z})^{U_4} \), so this would be a degenerate solution.
Similarly, if \( f \in M_{\{i,j,4\}} \) for some distinct \( i, j \in [3] \), then \( f = f_i + f_j + f_4 \) for some \( U_i \)-invariant functions \( f_i \), and so \( d^{U_4} f = d^{U_4} f_i + d^{U_4} f_j \), which is still an element of \( B^1(U_4, M_{[3]}) \). The non-degenerate solutions \( f \) are therefore precisely those for which \( d^{U_4} f \) lies in a nontrivial class in \( H^1_m(U_4, M_{[3]}) \).

**Step 2.** To compute this cohomology group, we may use the above presentation to obtain the following piece of the resulting long exact sequence:

\[
\cdots \to H^1_m(U_4, \Gamma) \to H^1_m(U_4, \bigoplus_{i \in e} \mathcal{F}(\mathbb{Z}, \mathbb{Z})^{U_i}) \to H^1_m(U_4, M_{[3]}) \to \cdots
\]

Now, for each \( i = 1, 2, 3 \) one has

\[
H^p_m(U_4, \mathcal{F}(\mathbb{Z}, \mathbb{Z})^{U_i}) = H^p_m(U_4, \text{Coind}_{U_i}^\mathbb{Z}) \cong H^p_m(U_4, \text{Coind}_{U_4 \cap U_i}^\mathbb{Z}).
\]

Since \( U_4 \cap U_i \cong \mathbb{T} \), Lemma A.2 gives that this last cohomology group is 0 when \( p \) is odd, and is naturally isomorphic to \( U_4 \cap U_i \) when \( p \) is even. Lemma A.2 also gives a natural isomorphism \( H^2_m(U_4, \Gamma) \cong \widehat{U_4} \otimes \Gamma \) (which is \( \cong \mathbb{Z}^4 \), though not naturally).

Putting these calculations together, the above long exact sequence collapses to give

\[
H^1_m(U_4, M_{[3]}) \cong \ker \left( \widehat{U_4} \otimes \Gamma \to \bigoplus_{i=1}^3 U_4 \cap U_i \right).
\]

Here, \( \widehat{U_4} \otimes \Gamma \) is the group of zero-sum triples \((\chi_1, \chi_2, \chi_3)\) in \( \widehat{U_4} \). Such a triple lies in the kernel of the above homomorphism if and only if \( \chi_i(U_4 \cap U_i) = 0 \) for \( i = 1, 2, 3 \).

Now a simple exercise in linear algebra shows that the subgroup of zero-sum triples satisfying this condition is a copy of \( \mathbb{Z} \) generated by \((\chi_1, \chi_2, \chi_3)\), where

\[
\chi_i(\theta_1, \theta_2, \theta_3) = \theta_i \quad \forall (\theta_1, \theta_2, \theta_3) \in U_4.
\]

**Step 3.** Now one may simply seek a function on \( \mathbb{Z} \) which gives rise to this generator under the above sequence of reductions, and one finds that that function is precisely \( f \).

(This example \( f \) is still close to cohomological solutions, in that one has the identity

\[
f(\theta_1, \theta_2, \theta_3) = [\{\theta_1\} + \{\theta_2\}] + [\{\theta_1 + \theta_2\} + \{\theta_3\}],
\]

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as well as several similar identities for this same \( f \). However, neither of the two terms on the right-hand side here is annihilated by \( dV_1 dV_2 dU_3 dU_4 \), so this is not really a decomposition into ‘simpler’ solutions.)

Finally, we offer an example having some richer geometric meaning. Its description will take a little longer, and will depend on some understanding of nilrotations on nilmanifolds.

**Example 14.4.** Let
\[
G = \begin{pmatrix} 1 & R & R \\ 1 & R \\ 1 \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} 1 & Z & Z \\ 1 & Z \\ 1 \end{pmatrix}
\]
be the continuous Heisenberg group and its obvious lattice, respectively. The quotient \( G/\Gamma \) is a compact nilmanifold which is a circle bundle over \( G^{ab}/\Gamma^{ab} \cong \mathbb{T}^2 \), where \( G^{ab} \) and \( \Gamma^{ab} \) are the Abelianizations of \( G \) and \( \Gamma \).

The group \( G \) acts on \( G/\Gamma \) by left-multiplication on cosets. Considered as a measurable dynamical \( G \)-system preserving the Haar measure \( m_{G/\Gamma} \), it is a skew-product circle-extension of the action of \( G^{ab}/\Gamma^{ab} \) obtained by lifting the action of \( G^{ab} \) through the quotient homomorphism. For \( g \in G \), let \( T_g \otimes G/\Gamma \) be this measure-preserving transformation, and let \( R_g \otimes G^{ab}/\Gamma^{ab} \cong \mathbb{T}^2 \) be the torus-rotation that it extends. We shall see that a concrete description of this action in coordinates involves some functions that form a zero-sum tuple.

To coordinatize \( G/\Gamma \), let us use the fractional parts \( \{ \cdot \} \) to identify \( \mathbb{T}^2 \times \mathbb{T} \) with \([0, 1)^3\), and hence with the following fundamental domain for \( \Gamma \) in \( G \):
\[
\left\{ \begin{pmatrix} 1 & x & z \\ 1 & y \\ 1 \end{pmatrix} \right| (x, y, z) \in [0, 1)^3 \right\}.
\]

Then any element of \( G \) decomposes as
\[
\begin{pmatrix} 1 & a & c \\ 1 & b \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \{a\} \\ 1 & \{c - \lfloor b \rfloor \{a\}\} \\ 1 \end{pmatrix} \begin{pmatrix} 1 & \lfloor b \rfloor \{a\} \\ 1 & \{c - \lfloor b \rfloor \{a\}\} \\ 1 \end{pmatrix} \in \begin{pmatrix} 1 & \{a\} \\ 1 & \{c - \lfloor b \rfloor \{a\}\} \\ 1 \end{pmatrix} \Gamma.
\]

For each \( s = (s_1, s_2) \in \mathbb{T}^2 \), let
\[
g_s := \begin{pmatrix} 1 & s_1 & 0 \\ 1 & s_2 \\ 1 \end{pmatrix}.
\]

This has the property that \( R_g \) is the rotation of \( \mathbb{T}^2 \) by \( s \). In terms of the coordinates \((x, z) = (x_1, x_2, z) \in \mathbb{T}^2 \times \mathbb{T} \) introduced above, one may now compute that \( T_{g_s} \) acts by
\[
T_{g_s}(x, z) = (x + s, z + \sigma(s, x))
\]
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with the skew-rotating function

\[ \sigma(s, x) = \{s_1\}{x_2} - \lfloor \{x_2\} + \{s_2\}\rfloor\{x_1 + s_1\} \mod 1. \]

In terms of this function, a standard calculation gives

\[ T_{g_t} \circ T_{g_s}(x, z) = (x + s + t, z + \sigma(s, x) + \sigma(t, x + s)), \]

and similarly for longer compositions.

Since \( G \) is 1-step nilpotent, the commutator \([g_s, g_t] := g_s^{-1}g_t^{-1}g_sg_t\) must lie in the centre

\[ G_1 := \begin{pmatrix} 1 & 0 & \mathbb{R} \\ 1 & 0 & 1 \end{pmatrix}. \]

This means that \([T_{g_s}, T_{g_t}] = T_{[g_s, g_t]}\) is the transformation of \( T^2 \times T \) that corresponds to this central element, which must simply be a constant rotation of the last coordinate. An explicit calculation now shows that this rotation is by

\[ c(s, t) = \{s_1\}{t_2} - \{t_1\}{s_2} \mod 1. \]

Finally, if one writes out this commutation relation in terms of the skew-rotating function above, it reads

\[ \sigma(t, x) + \sigma(s, x + t) = \sigma(s, x) + \sigma(t, x + s) + c(s, t), \quad (25) \]

so by moving all terms to the right one obtains a zero-sum quintuple on the group \( Z = T^2 \times T^2 \times T^2 \) written in terms of the homomorphisms

\[ M_1(s, t, x) = (t, x), \quad M_1(s, t, x) = (s, x + t), \quad M_3(s, t, x) = (s, x), \]
\[ M_4(s, t, x) = (t, x + s), \quad M_5(s, t, x) = (s, t). \]

In a later section of the paper, we will show that this zero-sum quintuple cannot be decomposed into nontrivial zero-sum quadruples corresponding to the proper subcollections of these homomorphisms.

When written multiplicatively (that is, for \( S^1 \)-valued functions), equation (25) is the ‘Conze-Lesigne equation’. It first arose in the work of Conze and Lesigne on describing exact ‘characteristic factors’ for various multiple recurrence phenomena: see [6, 7, 8], and also the more recent works [27, 12, 19, 20, 39]. In those works, the key point was that (25) emerged from some more abstract considerations, and could then be used to reconstruct an action of a two-step nilpotent Lie group on a nilmanifold.

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Letting \( U_i := \ker M_i \) for \( i = 1, \ldots, 5 \), let us first difference the equation (25) by \( U_5 \) and re-arrange to obtain that
\[
d_u \sigma(t, x) - d_u \sigma(t, x + s) + d_u \sigma(s, x + t) - d_u \sigma(s, x) = 0.
\]

Now differencing further by \( U_2, U_3 \) and \( U_4 \), we find that \( d_u \sigma(t, x) \) takes values in the module \( M \) of solutions in \( \mathcal{F}(T^2, T) \) to the PD\(^\oplus E \) associated to \( (q(U_2), q(U_3), q(U_4)) \), where \( q \) is the quotient homomorphism \( \mathbb{Z} \to \mathbb{Z}/U_1 \approx T^2 \). However, this PD\(^\oplus E \) on \( T^2 \) is now just a copy of the problem studied in Subsection 14.2, so by the result of that subsection we have
\[
M = \mathcal{F}(T^2, T)^q(U_2) + \mathcal{F}(T, T)^q(U_3) + \mathcal{F}(T, T)^q(U_4).
\]

Therefore \( d^{U_5} \sigma \) defines a 1-cocycle into this module. Now a few more steps, along very similar lines to our previous analyses and omitted here, show that for our choice of \( \sigma \) this 1-cocycle defines a non-coboundary, and from this one may again deduce that we have a non-degenerate solution to the zero-sum equation (25).

\[\triangleright\]

15 Further questions

15.1 Continuous functions

All of our results have been about measurable functions \( Z \to A \). If one insists on continuous functions, then I do not know how to complete a similar analysis (unless \( A \) is a Euclidean space, for which easy arguments can then be made using Fourier analysis).

The problem is that our approach rests on reducing various calculations to cohomology, usually using the long exact sequence axiom. There is a cohomology theory for compact groups built using continuous cochains, \( H^*_\text{cts} \), but it does not satisfy this axiom if the ambient group \( Z \) is not pro-finite, such as a torus. I do not know how to get around this difficulty. See [2] for more discussion of the defects of \( H^*_\text{cts} \). Similar questions apply if one asks for smoothness or other forms of regularity.

15.2 More general groups

One can formulate the zero-sum problems for any tuple of closed subgroups in a compact group, say \( H_1, \ldots, H_k \leq G \). One can also formulate PD\(^\oplus E \), but for these it now matters in what order one applies the difference operators \( d^{H_i} \), unless the subgroups \( H_i \) all normalize each other. However, even for normal subgroups this generalization runs into difficulties if one tries to follow the approach of this paper.
The problem now is that if $M \in \mathsf{PMod}(G)$ and $H \leq G$, then $H^p_m(H, M)$ still makes sense as a topological group, but it is generally not still a $G$-module. One may need to set up a more general class of objects to account for this before any of the theory of $\Delta$-modules can be recovered.

A Some explicit calculations in group cohomology

In addition to the general overview of Subsection 3.1, this appendix collects some explicit calculations in group cohomology that were used during our analysis of the examples.

First, for any discrete group $G$, the restriction to measurable cochains in the definition of $H^p_m(G, -)$ is irrelevant, and so this theory simply agrees with classical group cohomology. That classical theory comes with a large arsenal of techniques for actually computing cohomology groups. In that theory, these techniques mostly stem from the ability to switch to any choice of injective resolution for a module of interest. (This does not generalize to the measurable theory for non-discrete $G$, because there are not enough injectives in $\mathsf{PMod}(G)$.)

One of the most classical calculations is that for cyclic groups.

**Lemma A.1.** For any $N \geq 1$ and any $(\mathbb{Z}/NZ)$-module $M$, say with action $R : \mathbb{Z}/NZ \curvearrowright M$, one has

$$H^p(\mathbb{Z}/NZ, M) = \begin{cases} M^{\mathbb{Z}/NZ} & \text{if } p = 0 \\ M^{\mathbb{Z}/NZ}/TM & \text{if } p \text{ even and } \geq 2 \\ \{m \in M \mid Tm = 0\}/(d_1m \mid m \in M) & \text{if } p \text{ odd}, \end{cases}$$

Here, $d_1 = R_1 - \text{id}$ as usual, and $T \in \text{End}_{\mathbb{Z}/NZ}(M)$ is the element $T = R_0 + R_1 + \ldots + R_{N-1}$.

This is usually proved by switching to some very simple injective resolutions that are available for cyclic groups: see, for instance, Section II.3 in Brown [4].

For groups that are not finite, fewer calculational methods are available. However, one theorem from classical group cohomology does pass through: the isomorphism

$$H^*_m(\mathbb{Z}, \mathbb{Z}) \cong H^*_\text{Cech}(B\mathbb{Z}, \mathbb{Z}),$$

where $B\mathbb{Z}$ is a choice of classifying space for $\mathbb{Z}$.

For Lie groups $Z$ this is proved in [38], and it is extended to general locally compact, second-countable groups in [2, Theorem E]. This isomorphism to classifying space cohomology, in its turn, is proved by showing that both sides are
isomorphic to a third cohomology theory, which may (in most cases) be taken to be that introduced by
Segal in [28]. Finally, the Čech cohomology of $B\mathbb{Z}$ can be accessed via a range of tools from more classical algebraic topology: this is explored in detail in Hofmann and Mostert [18].

This relation to classifying-spaces is the real workhorse for making explicit calculations in $H^*_m(\mathbb{Z},-)$). In many quite simple cases I do not know how to compute $H^*_m(\mathbb{Z},\mathbb{Z})$ without passing through this isomorphism, and thereby invoking some quite sophisticated homological algebra.

For $\mathbb{Z} = \mathbb{T}$, a suitable choice of classifying space is given by the infinite-dimensional complex projective space $\mathbb{C}P^\infty$; see, for instance, the sections on classifying spaces in [9]. For higher-dimensional tori one obtains a similar picture in terms of infinite-dimensional Stiefel manifolds. Using this, standard tools from algebraic topology give the following:

**Lemma A.2.** If $\mathbb{Z}$ is a compact connected Abelian group (such as a torus) and $\Gamma$ is a discrete $\mathbb{Z}$-module with trivial action, then as graded Abelian groups one has

$$H^*_m(\mathbb{Z},\Gamma) \cong H^*_\text{Čech}(B\mathbb{Z},\mathbb{Z}) \otimes \Gamma \cong \left(\mathbb{Z} \oplus \{0\} \oplus \hat{\mathbb{Z}} \oplus \{0\} \oplus (\hat{\mathbb{Z}} \times \hat{\mathbb{Z}}) \oplus \cdots\right) \otimes \Gamma,$$

and this isomorphism is natural in $\mathbb{Z}$ and $\Gamma$ (that is, both sides transform correctly under morphisms of either). Here ‘$\otimes$’ denotes the symmetric product.

More explicitly, this gives

$$H^p_m(\mathbb{Z},\Gamma) \cong \begin{cases} \hat{\mathbb{Z}}^{\otimes p/2} \otimes \Gamma & \text{if } p \text{ even} \\ 0 & \text{if } p \text{ odd} \end{cases},$$

(where $\hat{\mathbb{Z}}^{\otimes 0} := \mathbb{Z}$).

**Remark.** One can also recover $H^*_m(\mathbb{T}^d, -)$ using the presentation $\mathbb{Z}^d \hookrightarrow \mathbb{R}^d \twoheadrightarrow \mathbb{T}^d$ and the Lyndon-Hochschild-Serre spectral sequence. However, one still needs to use the fact that $H^p_m(\mathbb{R}^d, \mathbb{Z}) = 0$ for all $p \geq 1$, and this is effectively proved by using the classifying-space argument for $\mathbb{R}^d$. Since this argument works only for discrete modules, it begs the following elementary question for the measurable-cochains theory, which I believe is still open:

**Question A.3.** Is it true that $H^p_m(\mathbb{R}, -) = 0$ on the whole of $\mathbb{PMod}(\mathbb{R})$ for all $p \geq 1$?

This is known to hold for the intermediate cohomology theory $H^*_\text{Seg}$ of Segal mentioned above. That theory does not apply to all Polish modules, but it has a generalization, denoted $H^*_\text{ss}$, which does. However, it is not known whether $H^*_m(\mathbb{R}, -) \cong H^*\text{ss}(\mathbb{R}, -)$ in all cases. Once again, more details can be found in [2].

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Finally, [2, Theorem A] shows that $H^*_m(Z, M) = 0$ whenever $Z$ is compact and $M$ is a Fréchet space. Therefore, if we compute the long exact sequence in $H^*_m(Z, -)$ arising from the presentation $Z \hookrightarrow \mathbb{R} \rightarrow \mathbb{T}$, all regarded as $Z$-modules, it collapses to a sequence of isomorphisms

$$H^p_m(Z, T) \cong H^{p+1}_m(Z, Z) \quad \forall p \geq 1.$$ 

Combining this will Lemma A.2 gives the following.

**Lemma A.4.** If $Z$ is a compact connected Abelian group and $T$ is given the trivial $Z$-action, then

$$H^*_m(Z, T) = \begin{cases} 
\mathbb{T} & \text{if } p = 0 \\
\hat{\mathbb{Z}}^{\oplus (p+1)/2} \otimes \Gamma & \text{if } p \text{ odd} \\
0 & \text{if } p > 0 \text{ even.}
\end{cases}$$

□

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