Elliptic solutions to the KP hierarchy and elliptic Calogero–Moser model

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Received 16 February 2021, revised 1 June 2021
Accepted for publication 10 June 2021
Published 30 June 2021

Abstract
We consider solutions of the Kadomtsev–Petviashvili hierarchy which are elliptic functions of \(x = t_1\). It is known that their poles as functions of \(t_2\) move as particles of the elliptic Calogero–Moser model. We extend this correspondence to the level of hierarchies and find the Hamiltonian \(H_k\) of the elliptic Calogero–Moser model which governs the dynamics of poles with respect to the \(k\)th hierarchical time. The Hamiltonians \(H_k\) are obtained as coefficients of the expansion of the spectral curve near the marked point in which the Baker–Akhiezer function has essential singularity.

Keywords: integrable systems, elliptic Calogero–Moser model, Kadomtsev–Petviashvili hierarchy

1. Introduction
The investigation of dynamics of poles of singular solutions to nonlinear integrable equations was initiated in the seminal paper [1], where it was shown that poles of elliptic and rational solutions to the Korteweg–de Vries and Boussinesq equations move as particles of the integrable many-body Calogero–Moser system [2–5] with some restrictions in the phase space. As it was proved in [6, 7], this connection becomes most natural for the more general Kadomtsev–Petviashvili (KP) equation, in which case there are no restrictions in the phase space for the Calogero–Moser dynamics of poles.

The KP equation is the first member of an infinite hierarchy of consistent integrable equations with infinitely many independent variables (times) \(t = \{t_1, t_2, t_3, \ldots\}\) (the KP hierarchy). In [8], Shiota has shown that the correspondence between rational solutions to the
KP equation and the Calogero–Moser system with rational potential can be extended to the level of hierarchies: the evolution of poles with respect to the higher times $t_k$ of the KP hierarchy was shown to be governed by the higher Hamiltonians $H_k = \text{Tr} L_k$ of the integrable Calogero–Moser system, where $L$ is the Lax matrix. Later this correspondence was generalized to trigonometric solutions of the KP hierarchy (see [9, 10]).

A natural generalization of rational and trigonometric solutions are elliptic (double periodic in the complex plane) solutions. Elliptic solutions to the KP equation

$$3u_{t_2 t_2} = (4u_{t_1} - 12uu_x - u_{xxx})_x$$  

(where $x = t_1$) were studied by Krichever in [11], where it was shown that poles $x_i$ of the elliptic solutions

$$u = -\sum_{i=1}^N \wp(x - x_i) + 2c$$  

as functions of $t_2$ move according to the equations of motion

$$\dot{x}_i = 4\sum_{k \neq i} \wp' (x_i - x_k)$$

of the Calogero–Moser system of particles with the elliptic interaction potential $\wp(x_i - x_j)$ ($\wp$ is the Weierstrass $\wp$-function). Here dot means derivative with respect to the time $t_2$. See also the review [12]. The Calogero–Moser system is Hamiltonian with the Hamiltonian

$$H = \sum_i p_i^2 - 2\sum_{i<j} \wp(x_i - x_j)$$

and the Poisson brackets $\{x_i, p_k\} = \delta_{ik}$. Note that $\dot{x}_i = \partial H / \partial p_i = 2p_i$. It is known [13] that the elliptic Calogero–Moser system is integrable, i.e., there are $N$ independent integrals of motion $H_k$ in involution.

The aim of this paper is to establish the precise correspondence between the flows of the KP hierarchy parametrized by the times $t_m$ and the Hamiltonian flows of the hierarchy of the elliptic Calogero–Moser systems. In short, the result is as follows. Let the function $\lambda(z)$ be determined from the equation of the Calogero–Moser spectral curve in the form

$$\det_{N \times N} ((z + \zeta(\lambda))I - L(\lambda)) = 0,$$

where $I$ is the unity matrix, $L(\lambda)$ is the Lax matrix of the Calogero–Moser system depending on the spectral parameter $\lambda$ and $\zeta(\lambda)$ is the Weierstrass $\zeta$-function. We show that the function $\lambda(z)$ expanded as $z \to \infty$ as

$$\lambda(z) = -Nz^{-1} + \sum_{m \geq 1} H_m z^{-m-1}$$

is the generating function for the Calogero–Moser Hamiltonians $H_m$ corresponding to the flows $t_m$ of the KP hierarchy. We find first few Hamiltonians explicitly. In the rational and trigonometric limit it is possible to find them for any $m$ in terms of traces of the Lax matrix and the result coincides with what was previously known (see [8, 10]).
2. The KP hierarchy

We begin with a short review of the KP hierarchy (see [14] for more details). The KP hierarchy is an infinite set of evolution equations in the times \( t = \{ t_1, t_2, t_3, \ldots \} \) for functions of a variable \( x \). In the Lax formulation of the hierarchy, the main object is the pseudo-differential operator

\[
\mathcal{L} = \partial_x + \sum_{k \geq 1} u_k \partial_x^{-k},
\]

where the coefficient functions \( u_k \) are functions of \( x \) and \( t \). The equations of the KP hierarchy are encoded in the Lax equations

\[
\partial_t u_m \mathcal{L} = [A_m, \mathcal{L}], \quad A_m = (\mathcal{L}^m)_{+},
\]

where \( (\ldots)_{+} \) means taking the purely differential part of a pseudo-differential operator. In particular, we have \( \partial_{t_1} \mathcal{L} = \partial_x \mathcal{L} \), i.e., \( \partial_{t_1} u_k = \partial_x u_k \) for all \( k \geq 1 \). This means that the evolution in \( t_1 \) is simply a shift of \( x \) : \( u_k(x,t) = u_k(x + t_1,t_2,t_3,\ldots) \).

An equivalent formulation of the KP hierarchy is through the zero curvature (Zakharov–Shabat) equations

\[
\partial_t A_m - \partial_{t_{m-1}} A_n + [A_m, A_n] = 0.
\]

The simplest nontrivial equation, i.e., (1) is obtained for \( u = u_1 \) with \( m = 2, n = 3 \).

A common solution to the KP hierarchy is provided by the tau-function \( \tau = \tau(x,t) \). The coefficient functions \( u_k \) of the Lax operator can be expressed through the tau-function. For example,

\[
u_1(x,t) = u(x,t) = \partial_x^2 \log \tau(x,t).
\]

The whole hierarchy is encoded in the bilinear relation [14, 15]

\[
\oint_{\infty} \exp(-x'z + (t_2 - t_1)z')e^{-D(z)\tau(x,t)}(e^{D(z)\tau(x',t')})dz = 0
\]

valid for all \( x, x', t, t' \), where

\[
\xi(t,z) = \sum_{k \geq 1} t_k z^k
\]

and \( D(z) \) is the differential operator

\[
D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_z^k.
\]

The integration contour is a big circle around infinity separating the singularities coming from the exponential factor from those coming from the tau-functions.

Let us point out an important corollary of the bilinear relation. Applying the operator \( D'(\mu) = -\sum_{k \geq 1} \mu^{-k} \partial_z^k \) to (11) and putting \( x = x', t = t' \) after that, we obtain

\[
-\sum_{k \geq 1} \oint_{\infty} \mu^{-k} e^D(e^{-D(z)\tau}(e^{D(z)\tau}))dz + \oint_{\infty} D'(\mu)(e^{-D(z)\tau})(e^{D(z)\tau})dz = 0
\]
or
\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{z}{(e^{-D(z)\tau})(e^{D(z)\tau})} \, dz = D'(\mu)\tau - D'(\mu)\tau \partial_{x}\tau.
\]

Taking the residues in the left-hand side, we get the equation
\[
\frac{(e^{D(\mu)\tau})(e^{-D(\mu)\tau})}{\tau^2} = 1 - D'(\mu)\partial_{x} \log \tau.
\] (13)

The zero curvature equation (9) are compatibility conditions of the auxiliary linear problems
\[
\partial_{t}m_{\psi} = A_{\mu} \psi
\] (14)

for the wave function \(\psi = \psi(x, t, z)\) depending on the spectral parameter \(z\). In particular, at \(k = 2\) we have the equation
\[
\partial_{t}^{2} \psi = \partial_{x}^{2} \psi + 2u \psi.
\] (15)

One can also introduce the adjoint wave function \(\psi^{\ast}\) satisfying the adjoint equation (14):
\[
-\partial_{t} \psi^{\ast} = A_{\mu}^{\dagger} \psi^{\ast},
\] (16)

where the \(^{\dagger}\)-operation is defined as \((f(x) \circ \partial_{x})^{\dagger} = (-\partial_{x})^{n} \circ f(x)\). In \([14, 15]\) it is shown that the wave functions can be expressed through the tau-function in the following way:
\[
\psi(x, t, z) = e^{\psi(x, t, z)} e^{D(z)\tau(x, t)} \tau(x, t),
\] (17)

\[
\psi^{\ast}(x, t, z) = e^{-\psi(x, t, z)} e^{D(z)\tau(x, t)} \tau(x, t).
\] (18)

Note that in terms of the wave functions the equation (13) can be written in the form
\[
\partial_{t} \psi = \partial_{1} \log \tau(x, t) = \text{res}_{\infty}(z^{n} \psi(x, t, z)\psi^{\ast}(x, t, z)),
\] (19)

where \(\text{res}\) is defined as \(\text{res}_{\infty}(z^{-n}) = \delta_{n1}\).

3. Elliptic solutions

The ansatz for the tau-function of elliptic (double-periodic in the complex plane) solutions to the KP hierarchy is
\[
\tau = e^{Q(x, t)} \prod_{i=1}^{N} \sigma(x - x_{j}(t)),
\] (20)

where
\[
Q(x, t) = c(x + t_{1})^{2} + (x + t_{1})A(t_{2}, t_{3}, \ldots) + B(t_{2}, t_{3}, \ldots)
\]

with a constant \(c\), a linear function
\[
A(t_{2}, t_{3}, \ldots) = A_{0} + \sum_{j>2} a_{j} f_{j}
\] (21)
and some function $B(t_2, t_1, \ldots)$. The coefficients $c$, $a_j$ here are not arbitrary but are determined by algebro-geometric data of the spectral curve, and, in particular, by the choice of the local parameter around the marked point of the curve. The function $\sigma$ in equation (20) is the Weierstrass $\varphi$-function

$$\sigma(x) = \sigma(x|\omega, \omega') = x \prod_{s \neq 0} \left(1 - \frac{x}{s}\right) e^{\frac{x}{s} + \frac{x^2}{2s^2}},$$

with quasi-periods $2\omega$, $2\omega'$ such that $\text{Im}(\omega'/\omega) > 0$. It is connected with the Weierstrass $\zeta$- and $\varphi$-functions by the formulas $\zeta(x) = \sigma'(x)/\sigma(x)$, $\varphi(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x)$. The monodromy properties of the function $\sigma(x)$ are

$$\sigma(x + 2\omega) = -e^{2i(x + \omega)}\sigma(x), \quad \sigma(x + 2\omega') = -e^{2i(x + \omega')}\sigma(x),$$

where the constants $\eta = \zeta(\omega)$, $\eta' = \zeta(\omega')$ are related by $\eta \omega' - \eta' \omega = \pi i/2$. The roots $x_i$ are assumed to be all distinct. Correspondingly, the function $u = \partial_x^2 \log \tau$ is an elliptic function with double poles at the points $x_i$:

$$u = -\sum_{i=1}^{N} \varphi(x - x_i) + 2c.$$  

(23)

The poles depend on the times $t_1, t_2, t_3, \ldots$. The dependence on $t_1$ is especially simple: since the solution must depend on $x + t_1$, we have $\partial_{t_1} x_i = -1$.

Let $\Delta(\mu)$ be the difference operator

$$\Delta(\mu) = e^{D(\mu)} + e^{-D(\mu)} - 2.$$  

(24)

Substituting the ansatz (20) into equation (13), we get:

$$e^{G + (x + t_1)\Delta(\mu)A} \prod_i \frac{\sigma(x - e^{D(\mu)} x_i)\sigma(x - e^{-D(\mu)} x_i)}{\sigma^2(x - x_i)}$$

$$= 1 + 2c\mu^{-2} - D'(\mu)A - \sum_k D'(\mu) x_k \varphi(x - x_k),$$

where

$$G = 2c\mu^{-2} + \mu^{-1}(e^{D(\mu)} - e^{-D(\mu)})A + \Delta(\mu)B.$$  

The right-hand side is an elliptic function of $x$ with periods $2\omega$, $2\omega'$. Therefore, for the left-hand side be also an elliptic function of $x$ with the same periods the following relations have to be satisfied:

$$\left\{ \begin{array}{l}
\exp\left(-2\eta \Delta(\mu) \sum_k x_k + 2\omega \Delta(\mu) A\right) = 1 \\
\exp\left(-2\eta' \Delta(\mu) \sum_k x_k + 2\omega' \Delta(\mu) A\right) = 1
\end{array} \right.$$
from which it follows that
\[ \Delta(\mu)A = 2n' \eta - 2n \eta', \quad \Delta(\mu) \sum_k x_k = 2n' \omega - 2n \omega' \] with integer \( n, n' \).

The right hand sides do not depend on \( \mu \). Expanding the equalities in powers of \( \mu \), one sees that the left hand sides are \( O(\mu^{-2}) \) as \( \mu \to \infty \), therefore, \( n = n' = 0 \) and we have
\[
\Delta(\mu)A = 0, \quad \Delta(\mu) \sum_k x_k = 0. \tag{25}
\]

The first equation is satisfied if \( A \) is a linear function of times as in (21). The second equation means that
\[
(1 - e^{-D(\mu)}) \sum_i x_i = -(1 - e^{D(\mu)}) \sum_i x_i. \tag{26}
\]

Note that the functions (17) and (18) with \( \tau \) as in (20) are double-Bloch functions, i.e., they satisfy the monodromy properties \( \psi(x + 2 \omega) = B \psi(x), \psi(x + 2 \omega') = B' \psi(x) \) with some Bloch multipliers \( B, B' \). Any non-trivial double-Bloch function (i.e. not an exponential function) must have poles in \( x \) in the fundamental domain. The Bloch multipliers of the function (17) are
\[
B = e^{2 \omega' (\zeta(\omega') - 2 \omega'(e^{-\zeta(\omega')}) \sum_s \zeta_s)},
B' = e^{2 \omega' (\zeta(\omega') - 2 \omega'(e^{-\zeta(\omega')}) \sum_s \zeta_s)}, \tag{27}
\]
where
\[
\alpha(z) = 2cz^{-1} + \sum_{j \geq 2} \frac{a_j}{j} z^{-j} \tag{28}
\]
and the coefficients \( a_j \) are those entering (21). Equation (26) means that the Bloch multipliers of the adjoint wave function \( \psi^* \) are \( B^{-1} \) and \( B'^{-1} \).

Let us introduce the elementary double-Bloch function \( \Phi(x, \lambda) \) defined as
\[
\Phi(x, \lambda) = \frac{\sigma(x + \lambda)}{\sigma(\lambda) \sigma(x)} e^{-\zeta(\lambda x)} \tag{29}
\]
(\( \zeta(\lambda) \) is the Weierstrass \( \zeta \)-function). The monodromy properties of the function \( \Phi \) are
\[
\Phi(x + 2 \omega, \lambda) = e^{2i(\zeta(\lambda) \omega - \zeta(\omega) \lambda)} \Phi(x, \lambda),
\]
\[
\Phi(x + 2 \omega', \lambda) = e^{2i(\zeta(\lambda) \omega' - \zeta(\omega') \lambda)} \Phi(x, \lambda),
\]
so it is indeed a double-Bloch function. The function \( \Phi \) has a simple pole at \( x = 0 \) with residue 1:
\[
\Phi(x, \lambda) = \frac{1}{x} + \alpha_1 x + \alpha_2 x^2 + \cdots, \quad x \to 0,
\]
where \( \alpha_1 = -\frac{1}{2} \zeta'(\lambda), \alpha_2 = -\frac{1}{2} \zeta'(\lambda) \). We will often suppress the second argument of \( \Phi \) writing simply \( \Phi(x) = \Phi(x, \lambda) \). We will also need the \( x \)-derivative \( \Phi'(x, \lambda) = \partial_x \Phi(x, \lambda) \).
Equations (17), (18) and (20) imply that the wave functions $\psi, \psi^*$ have simple poles at the points $x_i$. One can expand the wave functions using the elementary double-Bloch functions as follows:

$$\psi = e^{ik + \alpha(k - z) + \xi(t, z)} \sum_i c_i \Phi(x - x_i, \lambda)$$

(30)

$$\psi^* = e^{-ik - n(k - z) - \xi(t, z)} \sum_i c_i^* \Phi(x - x_i, -\lambda)$$

(31)

(this is similar to expansion of a rational function in a linear combination of simple fractions). Here $c_i, c_i^*$ are expansion coefficients which do not depend on $x$ and $k$ is an additional spectral parameter. Note that the normalization of the functions (17) and (18) implies that $c_i$ and $c_i^*$ are $O(\lambda)$ as $\lambda \to 0$. One can see that (30) is a double-Bloch function with Bloch multipliers

$$B = e^{2\omega k - \xi(\lambda) - 2\zeta(\omega) \lambda}, \quad B' = e^{2\omega' k - \xi(\lambda) - 2\zeta(\omega') \lambda}$$

(32)

and (31) has Bloch multipliers $B^{-1}$ and $B'^{-1}$. These Bloch multipliers should coincide with (27).

Therefore, comparing (27) with (32), we get

$$2\omega(k - \zeta(\lambda) - z + \alpha(z)) + 2\zeta(\omega) \left( \lambda + (e^{-Dc(z) - 1} \sum_i x_i) \right) = 2\pi in,$$

$$2\omega'(k - \zeta(\lambda) - z + \alpha(z)) + 2\zeta(\omega') \left( \lambda + (e^{-Dc(z) - 1} \sum_i x_i) \right) = 2\pi in'$$

with some integer $n, n'$. Regarding these equations as a linear system, we obtain the solution

$$k - z + \alpha(z) - \zeta(\lambda) = 2n' \zeta(\omega) - 2n \zeta(\omega'),$$

$$\lambda + (e^{-Dc(z) - 1} \sum_i x_i) = 2n \omega' - 2n' \omega.$$ 

Shifting $\lambda$ by a suitable vector of the lattice spanned by $2\omega, 2\omega'$, one gets zeros in the right hand sides of these equalities, so we can write

$$k = z - \alpha(z) + \zeta(\lambda),$$

$$\lambda = (1 - e^{-Dc(z)}) \sum_i x_i.$$ 

These two equations for three variables $k, z, \lambda$ determine the spectral curve. Below we will obtain another description of the spectral curve as the spectral curve of the Calogero–Moser system (given by the characteristic polynomial of the Lax matrix $L(\lambda)$ for the Calogero–Moser system). It appears in the form $R(k, \lambda) = 0$, where $R(k, \lambda)$ is a polynomial in $k$ whose coefficients are elliptic functions of $\lambda$ (see below in section 5). These coefficients are integrals of motion in involution. The spectral curve in the form $R(k, \lambda) = 0$ appears if one excludes $z$ from the equation (33). Equivalently, one can represent the spectral curve as a relation connecting two variables $z$ and $\lambda$:

$$R(z - \alpha(z) + \zeta(\lambda), \lambda) = 0.$$ 

(34)
Let us write the second equation in (33) as the expansion in powers of $z$:

$$
\lambda = -\sum_{m \geq 1} z^{-m} \hat{h}_m X, \quad X := \sum_i x_i,
$$

(35)

where $\hat{h}_m$ are differential operators of the form

$$
\hat{h}_m = -\frac{1}{m} \partial_m + \text{higher order operators in } \partial_1, \partial_2, \ldots, \partial_{m-1}.
$$

(36)

For example, the first few are

$$
\hat{h}_1 = -\partial_1, \quad \hat{h}_2 = \frac{1}{2}(\partial_1^2 - \partial_2), \quad \hat{h}_3 = \frac{1}{6}(-\partial_1^3 + 3\partial_1\partial_2 - 2\partial_3).
$$

As is explained above, the coefficients in the expansion (35) are integrals of motion, i.e.,

$$
\partial_t \hat{h}_m X = 0 \text{ for all } j, m.
$$

It then follows from the equation $\partial_t x_i = -1$ and from the explicit form of the operators $\hat{h}_m$ that $\partial_t \hat{h}_m X = 0$ and $\partial_t \partial_{m} X = 0$. A simple inductive argument then shows that $\partial_t \partial_m X = 0$ for all $j, m$. This means that $-\hat{h}_m X = \frac{1}{m} \partial_m X$ and $X$ is a linear function of the times:

$$
X = \sum_i x_i = X_0 - Nt_1 + \sum_{m \geq 2} V_m t_m
$$

(37)

with some constants $V_m$ (velocities of the ‘center of masses’ of the points $x_i$ multiplied by $N$). Therefore, the second equation in (33) can be written as

$$
\lambda = D(z) \sum_i x_i = -Nz^{-1} + \sum_{j \geq 2} z^{-j} V_j.
$$

(38)

In what follows we will show that $H_m = -\frac{1}{m+1} V_{m+1}$ are Hamiltonians for the dynamics of the poles in $t_m$, with $H_2$ being the standard Calogero–Moser Hamiltonian.

4. Dynamics of poles with respect to $t_2$

The coefficient $u$ in the linear problem (15)

$$
\partial_t^2 \psi - \partial_x^2 \psi - 2u \psi = 0
$$

(39)

is an elliptic function of $x$ of the form (23). Therefore, one can find solutions which are double-Bloch functions of the form (30).

The next procedure is standard after the work [11]. We substitute $u$ in the form (23) and $\psi$ in the form (30) into the left-hand side of (39) and cancel the poles at the points $x = x_i$. The highest poles are of third order but it is easy to see that they cancel identically. It is a matter of direct calculation to see that the conditions of cancellation of second and first order poles have the form

$$
c_i \dot{x}_i = -2k c_i - 2 \sum_{j \neq i} c_j \Phi(x_i - x_j),
$$

(40)

$$
\dot{c}_i = (k^2 - \alpha^2 + 4c - 2\alpha_1) c_i - 2 \sum_{j \neq i} c_j \Phi'(x_i - x_j) - 2\alpha_1 \sum_{j \neq i} \psi(x_i - x_j),
$$

(41)
where dot means the \( t_2 \)-derivative. Introducing \( N \times N \) matrices

\[
L_{ij} = -\frac{1}{2} \delta_{ij} k_i - (1 - \delta_{ij}) \Phi(x_i - x_j),
\]

\[
M_{ij} = \delta_{ij}(k^2 - z^2) + \varphi(\lambda) + 4c - 2\delta_{ij} \sum_{k \neq i} \varphi(x_i - x_k) - 2(1 - \delta_{ij}) \Phi'(x_i - x_j),
\]

we can write the above conditions as a system of linear equations for the vector \( \mathbf{c} = (c_1, \ldots, c_N)^T \):

\[
\begin{cases}
L(\lambda)\mathbf{c} = k\mathbf{c} \\
\dot{\mathbf{c}} = M(\lambda)\mathbf{c}.
\end{cases}
\]

(44)

Differentiating the first equation in (44) with respect to \( t_2 \), we arrive at the compatibility condition of the linear problems (44):

\[
(\dot{L} + [L, M]) \mathbf{c} = 0.
\]

(45)

The Lax equation \( \dot{L} + [L, M] = 0 \) is equivalent to the equations of motion of the elliptic Calogero–Moser system (see [12] for the detailed calculation). Our matrix \( M \) differs from the standard one by the term \( \delta_{ij}(k^2 - z^2) \) but it does not affect the compatibility condition. It follows from the Lax representation that the time evolution is an isospectral transformation of the Lax matrix \( L \), so all traces \( \text{Tr} \, L^m \) and the characteristic polynomial \( \det(L - kI) \), where \( I \) is the unity matrix, are integrals of motion. Note that the Lax matrix is written in terms of the momenta \( p_i \) as follows:

\[
L_{ij} = -\delta_{ij} p_i - (1 - \delta_{ij}) \Phi(x_i - x_j).
\]

(46)

A similar calculation shows that the adjoint linear problem for the function (31) leads to the equations

\[
\begin{cases}
\mathbf{c}^T L(\lambda)\mathbf{c} = k\mathbf{c}^T \\
\dot{\mathbf{c}}^T = -\mathbf{c}^T M(\lambda)
\end{cases}
\]

with the compatibility condition \( \mathbf{c}^T (\dot{L} + [L, M]) = 0 \).

5. The spectral curve

The first of the equation (44) determines a connection between the spectral parameters \( k, \lambda \) which is the equation of the spectral curve:

\[
R(k, \lambda) = \det(kI - L(\lambda)) = 0.
\]

(48)

As it was already mentioned, the spectral curve is an integral of motion. The matrix \( L = L(\lambda) \), which has an essential singularity at \( \lambda = 0 \), can be represented in the form \( L = V \tilde{L} V^{-1} \), where matrix elements of \( \tilde{L} \) do not have essential singularities and \( V \) is the diagonal matrix \( V_{ij} = \delta_{ij} e^{-\zeta(\lambda)k_i} \). Therefore,

\[
R(k, \lambda) = \sum_{m=0}^{N} R_m(\lambda) k^m,
\]
where the coefficients $R_m(\lambda)$ are elliptic functions of $\lambda$ with poles at $\lambda = 0$. The functions $R_m(\lambda)$ can be represented as linear combinations of the $\wp$-function and its derivatives. Coefficients of this expansion are integrals of motion. Fixing values of these integrals, we obtain via the equation $R(k, \lambda) = 0$ an algebraic curve $\Gamma$ which is an $N$-sheet covering of the initial elliptic curve $E$ realized as a factor of the complex plane with respect to the lattice generated by $2\omega, 2\omega^\prime$.

Example ($N = 2$):

$$\det_{2 \times 2}(kI - L(\lambda)) = k^2 + k(p_1 + p_2) + p_1p_2 + \wp(x_1 - x_2) - \wp(\lambda).$$

Example ($N = 3$):

$$\det_{3 \times 3}(kI - L(\lambda)) = k^3 + k^2(p_1 + p_2 + p_3)$$

$$+ k(p_1p_2 + p_1p_3 + p_2p_3 + \wp(x_{12}) + \wp(x_{13}) + \wp(x_{23}) - 3\wp(\lambda))$$

$$+ p_1p_2p_3 + p_1\wp(x_{23}) + p_2\wp(x_{13}) + p_3\wp(x_{12}) - \wp(\lambda)(p_1 + p_2 + p_3) - \wp'(\lambda),$$

where $x_{ik} = x_i - x_k$.

In a neighborhood of $\lambda = 0$ the matrix $\tilde{L}$ can be written as

$$\tilde{L} = -\lambda^{-1}(E - I) + O(1),$$

where $E$ is the rank 1 matrix with matrix elements $E_{ij} = 1$ for all $i, j = 1, \ldots, N$. The matrix $E$ has eigenvalue 0 with multiplicity $N - 1$ and another eigenvalue equal to $N$. Therefore, we can write $R(k, \lambda)$ in the form

$$R(k, \lambda) = \det (kI + E^{-1} - \lambda^{-1} - f_1(\lambda))$$

$$= (k + (N-1)\lambda^{-1} - f_N(\lambda)) \prod_{i=1}^{N-1} (k - \lambda^{-1} - f_i(\lambda)), \quad (49)$$

where $f_i$ are regular functions of $\lambda$ at $\lambda = 0$: $f_i(\lambda) = O(1)$ as $\lambda \to 0$. This means that the function $k$ has simple poles on all sheets at the points $P_j (j = 1, \ldots, N)$ of the curve $\Gamma$ located above $\lambda = 0$. Its expansion in the local parameter $\lambda$ on the sheets near these points is given by the multipliers in the right-hand side of (49):

$$k = \lambda^{-1} + f_j(\lambda) \quad \text{near } P_j, \quad j = 1, \ldots, N - 1,$$

$$k = -(N-1)\lambda^{-1} + f_N(\lambda) \quad \text{near } P_N. \quad (50)$$

The $N$th sheet is distinguished, as it can be seen from (50). As in [11], we call it the upper sheet. Note that equations (33) and (38) imply

$$k(\lambda) = -\frac{N-1}{\lambda} + O(1) \quad \text{as } \lambda \to 0,$$

so the expansion (38) is the expansion of $\lambda(z)$ on the upper sheet of the spectral curve in a neighborhood of the point $P_N$. 

10
6. Dynamics in higher times

Our basic tool is equation (19). Substituting \( \tau(x, t) \) in the form (20) and \( \psi, \psi^* \) in the form (30) and (31) in it, we have:

\[
\sum_i \partial_{m_i} x_i \Phi(x-x_i) + C(t_2, t_3, \ldots) = \text{res}_\infty \left( z^m \sum_{i,j} c_i c_j^* \Phi(x-x_i, \lambda) \Phi(x-x_j, -\lambda) \right).
\] (51)

Equating the coefficients in front of the second order poles at \( x = x_i \), we obtain

\[
\partial_{m_i} x_i = \text{res}_\infty \left( z^m c_i^* c_i \right) = \text{res}_\infty \left( z^m e^T E_i c \right),
\] (52)

where \( E_i \) is the diagonal matrix with 1 at the place \( ii \) and zeros otherwise. At \( m = 1 \) this reads

\[
\text{res}_\infty (z e^T c) = -N.
\]

Summing the equation (52) over \( i \), we get

\[
\text{res}_\infty (z^m e^T c) = \partial_{m} \sum_i x_i.
\]

It then follows from these equations that

\[
(e^T c) = -N/z^2 + \sum_m z^{-m-1} \partial_m \sum_i x_i = -\lambda'(z).
\] (53)

The absence of terms with non-negative powers of \( z \) in the right-hand side (which would not change the residue) follows from the above mentioned fact that \( c \) and \( c^* \) are \( O(\lambda) = O(z^{-1}) \) as \( z \to \infty \). The last equality in (53) follows from (38). Equation (53) is an important non-trivial relation which will allow us to identify the Hamiltonians for the higher flows \( t_m \).

Now let us note that according to (46) \( E_i = -\partial_{p_i} L \). Therefore, we can continue the chain of equalities (52) as follows:

\[
\partial_{m_i} x_i = \text{res}_\infty \left( z^m e^T E_i c \right) = -\text{res}_\infty \left( z^m e^T \partial_{p_i} L c \right)
\]

\[
= -\partial_{p_i} \text{res}_\infty \left( z^m e^T L c \right) + \text{res}_\infty \left( z^m \partial_{p_i} e^T L c \right) + \text{res}_\infty \left( z^m e^T L \partial_{p_i} c \right)
\]

\[
= -\partial_{p_i} \text{res}_\infty \left( z^m e^T L c \right) + \text{res}_\infty \left( z^m \partial_{p_i} e^T k c \right) + \text{res}_\infty \left( z^m e^T k \partial_{p_i} c \right)
\]

\[
= -\partial_{p_i} \text{res}_\infty \left( z^m e^T L c \right) + \partial_{p_i} \text{res}_\infty \left( z^m e^T k c \right) - \text{res}_\infty \left( z^m \partial_{p_i} k e^T c \right)
\]

\[
= -\partial_{p_i} \text{res}_\infty \left( z^m e^T (L - k I) c \right) - \text{res}_\infty \left( z^m \partial_{p_i} k e^T c \right)
\]

\[
= \text{res}_\infty \left( z^m \lambda'(z) \partial_{p_i} k \right)
\]
(the last equality follows from (44) and (53)). Here \( \partial_{\lambda} k = \partial_{\lambda} k(\lambda, I)|_{\lambda=\text{const.}} \), where \( I \) is the full set of integrals of motion. From (33) we see that
\[
\partial_{\pi^k} = \partial_{\pi^k}(\lambda, I)\bigg|_{\lambda=\text{const.}},
\]
where \( I \) is the full set of integrals of motion. From (33) we see that
\[
\partial_{\pi^k} = (1 - \alpha'(z))\partial_{\pi^z}(\lambda, I)\bigg|_{\lambda=\text{const.}}.
\]
We consider \( z \) as an independent variable, so we can write
\[
0 = \frac{dz}{d\pi} = \partial_{\pi^z}(\lambda, I)\bigg|_{\lambda=\text{const.}} + \partial_{\lambda} z \bigg|_{I=\text{const.}}\partial_{\pi^\lambda}.
\]
Therefore, we have the first set of the Hamiltonian equations
\[
\partial_{x^i} = -\text{res}_{\infty}(z^m(1 - \alpha'(z))\partial_{\pi^\lambda}) = \partial_{\pi^k} H_m, \quad (55)
\]
where the Hamiltonian \( H_m \) is
\[
H_m = H^{(\alpha)}_m + 2c H^{(\alpha)}_m + \sum_{j=2}^{m-1} a_j H^{(\alpha)}_{m-j-1}, \quad (56)
\]
(see (28)). It is the linear combination of the Hamiltonians
\[
H^{(\alpha)}_m = -\text{res}_{\infty}(z^m \lambda(z)) \quad (57)
\]
with constant coefficients. The latter implicitly depend on \( \alpha(z) \) through the parametrization of the spectral curve (34).

In their turn, the Hamiltonians \( H^{(\alpha)}_m \) are linear combinations of the basic Hamiltonians \( H_m \) defined at \( \alpha(z) = 0 \) by
\[
H_m = -\text{res}_{\infty}(z^m \lambda_0(z)), \quad (58)
\]
where \( \lambda_0(z) \) is defined through the equation of the spectral curve
\[
R(z + \zeta(\lambda_0), \lambda_0) = \det((z + \zeta(\lambda_0))I - L(\lambda_0)) = 0. \quad (59)
\]
Then
\[
\lambda(z) = \lambda_0(z - \alpha(z)) = \lambda_0(z) - \alpha(z)\lambda_0'(z) + \frac{1}{2} \alpha^2(z)\lambda_0''(z) + \cdots
\]
and so we see that the Hamiltonians (57) are indeed linear combinations of the \( H_m \)'s with constant coefficients.

The remaining set of Hamiltonian equations can be obtained by differentiating (52) with respect to \( t_2 \) and using (44) and (47):
\[
2\partial_{p^i} = \partial_{x^i} = \text{res}_{\infty}(z^m e^T E_i e) + \text{res}_{\infty}(z^m e^T E_i \dot{e})
\]
\[
= \text{res}_{\infty}(z^m e^T [E_i, M] e)
\]
Now, it is a matter of direct verification to see that
\[ [E_i, M] = 2\partial_{x_i}L. \] (60)

Therefore, we can write
\[ \partial_{\alpha_n}p_i = \text{res}_\infty (z^m e^{zT} \partial_{x_n} L). \]

Repeating the transformations presented above in detail, we have:
\[ \partial_{\alpha_n}p_i = \text{res}_\infty (z^m e^{zT} \partial_{x_n} k) = -\text{res}_\infty (z^m \alpha'(z) \partial_{x_n} k) \]

The same argument as above shows that
\[ \partial_{x_n}k = (1 - \alpha'(z))\partial_{x_n}z(\lambda, \mathbf{I})|_{\lambda = \text{const.}} \] (61)

and
\[ \partial_{x_n}z = -\frac{\partial_{\alpha_n} \lambda}{\lambda'(z)}. \]

Therefore, we obtain the second set of Hamiltonian equations for the dynamics of poles:
\[ \partial_{\alpha_n}p_i = \text{res}_\infty (z^m(1 - \alpha'(z))\partial_{x_n} \lambda) = -\partial_{x_n}H_m. \] (62)

Let us find \( H_m \) explicitly in terms of \( H_m \) for the case when \( a_j = 0, \ c \neq 0 \). In this case \( \alpha(z) = 2cz^{-1} \) and we have
\[ \lambda(z) = \lambda_0(z) - \sum_{j,n \geq 1} (2cz)^{n+j-1} H_{j-1} z^{-j-2n}, \] (63)

\[ H_m = H_m^{(\alpha)} + 2c H_m^{(\alpha)} \] and from (63) we see that
\[ H_m^{(\alpha)} = H_m + \sum_{j=1}^{[m/2]} (2c)^j \begin{pmatrix} m-j \\ j \end{pmatrix} H_{m-2j}. \]

Therefore,
\[ H_m = H_m + \sum_{j=1}^{[m/2]} (2c)^j \left[ \begin{pmatrix} m-j \\ j \end{pmatrix} + \begin{pmatrix} m-j+1 \\ j-1 \end{pmatrix} \right] H_{m-2j}. \] (64)

In particular, \( H_3 = H_3 + 6cH_1 \) which agrees with the result of the paper [12].
7. Calculation of the Hamiltonians

In order to find the Hamiltonians explicitly, we use the description of the spectral curve given in the paper [16]:

$$\sum_{j=0}^{N} I_j T_{N-j}(k|\lambda) = 0, \quad (65)$$

where $T_j(k|\lambda)$ are polynomials in $k$ of degree $N$ such that

$$\partial_k T_n(k|\lambda) = n T_n(k|\lambda) - 1, \quad (66)$$

and $I_j$ are integrals of motion. The first few are

$$I_0 = 1,$$
$$I_1 = \sum_j p_j,$$
$$I_2 = \sum' \left( \frac{1}{2} p_j p_j + \frac{1}{2} \psi(x_j) \right),$$
$$I_3 = \sum' \left( \frac{1}{2} p_j p_j p_j + \frac{1}{2} \psi(x_j) \psi(x_k) \right),$$
$$I_4 = \sum' \left( \frac{1}{2} p_j p_j p_j p_j + \frac{1}{2 (2!)^2} \psi(x_j) \psi(x_k) \psi(x_m) \right),$$
$$I_5 = \sum' \left( \frac{1}{2} p_j p_j p_j p_j p_j + \frac{1}{2 (2!)^3} \psi(x_j) \psi(x_k) \psi(x_m) \psi(x_n) \right),$$

where $\sum'$ means summation over distinct indices. Recalling the equation of the spectral curve in terms of $z$ and $\lambda$, let us also introduce $S_n(z|\lambda) = \zeta(z + \zeta(\lambda)|\lambda)$, then

$$\partial_z S_n(z|\lambda) = n S_{n-1}(z|\lambda). \quad (68)$$

For example,

$$T_5(k|\lambda) = k^5 - 10 \psi(\lambda) k^3 - 10 \psi'(\lambda) k^2 - 5 \psi''(\lambda) k - 2 \psi'(\lambda) \psi(\lambda).$$

We have

$$\zeta(\lambda) = \frac{1}{\lambda} - \frac{g_2 \lambda^3}{2 \cdot 3 \cdot 5} + O(\lambda^{-5}),$$

where

$$g_2 = 60 \sum_{i \neq 0} \frac{1}{i^3}, \quad s = 2m \omega + 2m' \omega', \quad m, m' \in \mathbb{Z}.$$

Expanding $S_5(z|\lambda)$ in $z$ using the above formula for $T_5$, we get:

$$S_5(z|\lambda) = z^5 + \frac{5 z^4}{\lambda} - \frac{g_2}{2} \left( \frac{1}{\lambda} + 5 z + \frac{10}{3} z^2 \lambda + \frac{10}{3} z^3 \lambda^2 + \frac{1}{6} z^4 \lambda^3 \right) + O(z^{-1}).$$
Note that if we introduce the gradation such that deg $z = 1$, deg $\lambda = -1$, then deg $g_2 = 4$, deg $S_n = n$. Note also that in the rational limit $g_2 = 0$ and the equation of the spectral curve becomes linear in $\lambda^{-1}$ (see below in the next section). This can be only in the case if $S_n(z|\lambda) = z^n - nz^{n-1}\lambda^{-1}$ in the rational limit (the coefficient is found from the condition (68)).

In the non-degenerate case we have

$$S_n(z|\lambda) = z^n - \frac{nz^{n-1}}{\lambda} + g_2 O(z^{n-4})$$  \hspace{1cm} (69)

or

$$S_n(z|\lambda) = z^n - \frac{nz^{n-1}}{\lambda} + g_2 A_n z^{n-4} + g_2 B_n I_1 z^{n-5} + O(z^{n-6}),$$ \hspace{1cm} (70)

where $A_n$ and $B_n$ are some constant coefficients. ($I_1$ comes from the expansion $\lambda = -Nz^{-1} - \frac{1}{2}z^{-2} + O(z^{-3})$.) Therefore, we can write

$$S_N(z|\lambda) = z^N - \frac{Nz^{N-1}}{\lambda} + g_2 (A_N z^{N-4} + B_N I_1 z^{N-5}) + O(z^{N-6}),$$

$$S_{N-1}(z|\lambda) = z^{N-1} - \frac{(N-1)z^{N-2}}{\lambda} + g_2 A_{N-1} z^{N-5} + O(z^{N-6}),$$ \hspace{1cm} (71)

$$S_{N-j}(z|\lambda) = z^{N-j} - \frac{(N-j)z^{N-j-1}}{\lambda} + O(z^{N-j}), \hspace{0.5cm} j = 2, 3, 4, 5.$$

and the equation of the spectral curve (65) acquires the form

$$z^N + \sum_{i=1}^{5} I_i z^{N-i} + g_2 A_N z^{N-4} + g_2 (A_{N-1} + B_N) I_1 z^{N-5} + O(z^{N-6})$$

$$= -\frac{1}{\lambda} \left( Nz^{N-1} + \sum_{i=1}^{5} (N-i) I_i z^{N-i} \right).$$

Expressing $\lambda$ as a function of $z$ from here, we have:

$$H_1 = -I_1,$$

$$H_2 = I_1^2 - 2I_2,$$

$$H_3 = -I_1 + 3I_1 I_2 - 3I_3,$$ \hspace{1cm} (72)

$$H_4 = I_1^4 - 4I_1 I_2 + 2I_2^2 + 4I_1 I_3 - 4I_4 + \text{const.},$$

$$H_5 = -I_1^5 + 5I_1^2 I_2 - 5I_1 I_3^2 + 5I_2 I_3 - 5I_1 I_4 - 5I_5 + g_2 K I_1,$$

where $K = (N + 1)A_N - N(A_{N-1} + B_N)$, or, explicitly,
\[ H_1 = - \sum_i p_i, \]
\[ H_2 = \sum_i p_i^2 - \sum_{i \neq j} \psi(x_{ij}), \]
\[ H_3 = - \sum_i p_i^3 + 3 \sum_{i \neq j} p_i \psi(x_{ij}), \]
\[ H_4 = \sum_i p_i^4 - 2 \sum_{i \neq j} p_i p_j \psi(x_{ij}) - 4 \sum_{i \neq j} p_i^2 \psi(x_{ij}) \]
\[ + \sum_{i \neq j} \psi^2(x_{ij}) + 2 \sum_{i \neq j} \psi(x_{ij}) \psi(x_{jk}) + \text{const.}, \]
\[ H_5 = - \sum_i p_i^5 + 5 \sum_{i \neq j} (p_i^3 + p_i^2 p_j) \psi(x_{ij}) - 5 \sum_{i \neq j} p_i \psi^2(x_{ij}) \]
\[ - 5 \sum_{i \neq j} p_i \psi(x_{ij}) \psi(x_{jk}) - 5 \sum_{i \neq j} \psi(x_{ij}) \psi(x_{jk}) + \text{const.} \cdot \sum_i p_i. \]

(73)

These are indeed the Hamiltonians of the elliptic Calogero–Moser model. It is easy to see that they satisfy the property
\[ H_{m-1} = - \frac{1}{m} \sum_i \partial_{H_i} H_m. \]

(74)

Indeed, we have
\[ \lambda(z) = -Nz^{-1} + \sum_{m \geq 2} \frac{z^{-m}}{m} V_m = -Nz^{-1} + \sum_{m \geq 1} z^{-m-1} H_m, \]

so
\[ V_m = \partial_{\lambda_n} \sum_i x_i = \sum_i \partial_{\lambda_i} H_m = -mH_{m-1}. \]

One can see that the higher Hamiltonians will consist from the principal part and other terms as follows:
\[ H_n = (-1)^n \sum_{|\mu|=n} C^n_\mu I_\mu + g_2 \sum_{|\nu|=n-4} B^n_\nu I_\nu + \cdots, \]

(75)

where the first sum is taken over Young diagrams \( \mu \) of \( n = |\mu| \) boxes, \( I_\mu = I_{\mu_1} I_{\mu_2} \cdots I_{\ell(\mu)} \), where \( \ell(\mu) \) is the number of non-empty rows of the diagram \( \mu \) and \( C^n_\mu \) is the matrix of the transition from the basis of elementary symmetric polynomials to the basis of power sums.

8. Rational and trigonometric limits

In the rational limit \( \omega_1, \omega_2 \to \infty, \sigma(\lambda) = \lambda, \Phi(x, \lambda) = (x^{-1} + \lambda^{-1})e^{-x/\lambda} \) and the equation of the spectral curve becomes
\[ \det \left( L_{\text{rat}} - (E - I) \lambda^{-1} - (z + \lambda^{-1}) I \right) = 0, \]

(76)
where

$$(L_{\text{rat}})_{ij} = -\delta_{ij} p_i - \frac{1 - \delta_{ij}}{x_i - x_j}$$

(77)

is the Lax matrix of the rational Calogero–Moser model. Rewriting the equation of the spectral curve in the form

$$\det \left( I - E \frac{L^{-1}}{L_{\text{rat}}} - zI \right) = 0$$

and using the property $\det(I + Y) = 1 + \text{Tr} Y$ for any matrix $Y$ of rank 1, we get

$$\lambda = -\text{tr} \left( E \frac{1}{zL_{\text{rat}} - I} \right) = -\sum_{n \geq 0} z^{-n-1} \text{tr} L^n_{\text{rat}},$$

(78)

where we use the well-known property $\text{tr}(EL_{\text{rat}}) = \text{tr}(L_{\text{rat}})$. So the Hamiltonians are $H_m = \text{tr} L^m_{\text{rat}}$ which agrees with Shiota’s result \[8\].

The trigonometric limit is more tricky. Let $\pi i/\gamma$ be period of the trigonometric (or hyperbolic) functions (the second period tends to infinity). The Weierstrass functions in this limit become

$$\sigma(x) = \gamma^{-1} e^{\frac{x^2}{4\gamma^2}} \sinh(\gamma x), \quad \zeta(x) = \gamma \coth(\gamma x) - \frac{1}{3} \gamma^2 x.$$

The tau-function for trigonometric solutions is

$$\tau(x, t) = \prod_{i=1}^{N} (e^{2\gamma x} - e^{2\gamma x(t)}),$$

(79)

so we should consider

$$\tau(x, t) = \prod_{i=1}^{N} \sigma(x - x_i) e^{\frac{x^2}{2} \gamma^2 (x-x_i)^2 + \gamma(x+x_i)},$$

(80)

With this choice, equation (33) acquires the form $k = z + \zeta(\lambda) + \frac{1}{3} \gamma^2 \lambda$ or

$$k = z + \gamma \coth(\gamma \lambda).$$

(81)

The trigonometric limit of the function $\Phi(x, \lambda)$ is

$$\Phi(x, \lambda) = \gamma (\coth(\gamma x) + \coth(\gamma \lambda)) e^{-\gamma x \coth(\gamma \lambda)}.$$

Therefore, the equation of the spectral curve can be written in the form

$$\det \left( W^{1/2}LW^{-1/2} + \gamma (1 - \coth(\gamma \lambda))(E - I) - (z + \gamma \coth(\gamma \lambda))I \right) = 0,$$

(82)

where $W = \text{diag}(w_1, w_2, \ldots, w_N)$ and

$$L_{ij} = -\delta_{ij} p_i - \frac{(1 - \delta_{ij})\gamma}{\sinh(\gamma(x_i - x_j))} = -\delta_{ij} p_i - 2\gamma(1 - \delta_{ij}) \frac{w_i^{1/2} w_j^{1/2}}{w_i - w_j}$$

(83)

is the Lax matrix of the trigonometric Calogero–Moser model. Here and below we use the notation $w_i = e^{2\gamma x_i}$. 
After the transformations similar to the rational case equation (82) can be brought to the form
\[
\gamma(1 - \coth(\gamma \lambda)) \text{tr} \left[ W^{-1/2} E W^{1/2} \frac{1}{z I - (L - \gamma I)} \right] = 1
\]
or
\[
\lambda = \frac{1}{2\gamma} \log \left[ 1 - 2\gamma \text{tr} \left( W^{-1/2} E W^{1/2} \frac{1}{z I - (L - \gamma I)} \right) \right]. \tag{84}
\]
Applying the formula \( \det (I + Y) = 1 + \text{tr} Y \) for any matrix \( Y \) of rank 1 in the opposite direction, we have
\[
\lambda = \frac{1}{2\gamma} \log \det \left[ I - 2\gamma W^{-1/2} E W^{1/2} \frac{1}{z I - (L - \gamma I)} \right]. \tag{85}
\]
Now we are going to use the identity
\[
[L, W] = 2\gamma(W^{1/2} E W^{1/2} - W) \tag{86}
\]
which can be easily checked. With the help of this identity, we can transform (85) as follows:
\[
2\gamma \lambda = \log \det \left( I - W^{-1} L W \frac{1}{z I - (L - \gamma I)} + \frac{L}{z I - (L - \gamma I)} 
- \frac{2\gamma}{z I - (L - \gamma I)} \right) 
= \log \det \left[ \left( I - \frac{2\gamma}{z I - (L - \gamma I)} \right) \frac{z I - (L - \gamma I)}{z I - (L + \gamma I)} \right. 
\times \left. \left( I - W^{-1} L W \frac{1}{z I - (L - \gamma I)} + \frac{L}{z I - (L - \gamma I)} - \frac{2\gamma}{z I - (L - \gamma I)} \right) \right] 
= \log \det \left[ \left( I - \frac{2\gamma}{z I - (L - \gamma I)} \right) \left( I - W^{-1} L W \frac{1}{z I - (L + \gamma I)} \right. 
+ \frac{L}{z I - (L + \gamma I)} \right. 
\left. + \frac{2\gamma}{z I - (L + \gamma I)} \right] 
\times \left( I - W^{-1} L W \frac{1}{z I - (L - \gamma I)} \right) \left( z I - (L + \gamma I) \right) 
- W^{-1} L W + L \right] 
= \log \det \frac{z I - (L + \gamma I)}{z I - (L - \gamma I)}.
\]
Therefore, we get
\[ \lambda = \frac{1}{2\gamma} \text{tr} \left( \log(I - z^{-1}(L + \gamma I)) - \log(I - z^{-1}(L - \gamma I)) \right) \]

\[ = -\frac{1}{2\gamma} \text{tr} \sum_{m \geq 1} \frac{z^{-m}}{m} ((L + \gamma I)^m - (L - \gamma I)^m) \tag{87} \]

and

\[ H_m = \frac{1}{2\gamma(m + 1)} \text{tr} \left( (L + \gamma I)^{m+1} - (L - \gamma I)^{m-1} \right) \tag{88} \]

which agrees with the result of paper [10].

**Acknowledgments**

We thank I Krichever for illuminating discussions. The research of AZ has been funded within the framework of the HSE University Basic Research Program and the Russian Academic Excellence Project ‘5-100’.

**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

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