Quantum MERA Channels

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(Dated: February 10, 2022)

Tensor networks representations of many-body quantum systems can be described in terms of quantum channels. We focus on channels associated with the Multi-scale Entanglement Renormalization Ansatz (MERA) tensor network that has been recently introduced to efficiently describe critical systems. Our approach allows us to compute the MERA correspondent to the thermodynamic limit of a critical system introducing a transfer matrix formalism, and to relate the system critical exponents to the convergence rates of the associated channels.

PACS numbers: 03.67.-a,05.30.-d,89.70.-a

Understanding the properties of strongly interacting many-body quantum system is central in many areas of physics. Whenever it is hard to devise reliable analytical approaches, as in many situations of experimental relevance, ingenious numerical methods are necessary to grasp the essential properties of these systems. Our ability of simulating them is based on the possibility to find an efficient description of their ground state. This is the case, for example, of White’s Density Matrix Renormalization Group [1] which can be recast in terms of Matrix Product States (MPS) [2, 3, 4, 5, 6]. Such representations are characterized by a simple tensor decomposition of the many-body wave-function which allows one to efficiently compute all the relevant observables of the system (e.g. energy, local observables, and correlation functions), and ii) to reduce the effective number of parameters over which the numerical optimization needs to be performed. MPS fulfill these requirements and can be used to describe faithfully the ground states of not critical, short range one-dimensional many-body Hamiltonians at zero-temperature. However MPS typically fail to provide an accurate description in other relevant situations, i.e. when the system is critical, in higher physical dimensions or if the model possesses long-range couplings. Several proposals have been put forward to overcome this problem. Projected Entangled Pair States (PEPS) [7] generalize MPS in dimensions higher than one. Weighted graph states [8] can deal with long-range correlations. In this Letter we focus on a solution recently proposed by Vidal [9] who introduced a tensor structure based on the so called Multiscale Entanglement Renormalization Ansatz (MERA). The MERA tensor network satisfies both the constraints i) and ii) and accommodates the scale invariance typical of critical systems [10, 11]. The relevance of this approach might represent a major breakthrough in our simulation capabilities [12] and motivates an intensive study of the MERA [13, 14].

Here we point out a previously unnoticed connection between the MERA and the theory of completely positive quantum maps [15] establishing a link between two important areas of quantum information science. This allows us to introduce a transfer matrix formalism in the same spirit as it has been done for MPS [3, 5], providing new tools to compute physical observables using MERA. The main outcomes of our work are i) a method for determining the properties of critical many-body systems in the thermodynamic limit and ii) a connection between the critical exponents governing the decay of correlation functions and the eigenvalues of the MERA transfer matrix. As a consequence this yields a full characterization of the asymptotic properties of one-dimensional critical systems. The paper is organized as follows: after a brief review of the MERA, we show how the local expectation values and correlations functions can be casted in terms of the vectors \( \{ \xi_{\ell} \}_{\ell} \) represented by the blue inverted Ys; and the type-\( \{ \chi_{\ell} \}_{\ell} \) tensors \( \lambda_{\ell_{1},\ell_{2}}^{\chi} \) represented by the red Xs; the type-\( \{ \lambda_{\ell_{1},\ell_{2}}^{\chi} \}_{\ell_{1},\ell_{2}} \) tensors \( \lambda_{\ell_{1},\ell_{2}}^{\chi} \) represented by the blue inverted Ys; and the type-\( \{ \gamma_{\ell} \}_{\ell} \) tensor \( C_{\ell_{1},\ell_{2},\ell_{3}}^{\gamma} \) represented by the green semi-circle. As shown in Fig. the \( \chi \)'s, the \( \lambda \)'s are coupled together to form a triangular structure with the vectors \( \xi_{\ell} \) as the closing element of the top: any two joined legs from any two distinct nodes indicate saturation of the associated indices. Consequently, the tensor \( T \) associated with the \( N \) qudit state \( | \psi \rangle \) is written as a network of \( O(N) \) tensors organized in \( O(\log_2 N) \) different levels composed by one layer of \( \chi \) ten-
FIG. 1: Representation of a MERA decomposition of the tensors connected with one layer of composed by contraction of tensors $\mathcal{M}_5$ (shown in the inset). It is composed by contraction of tensors $\mathcal{M}_5$ associated with the local operator $\Theta_j$ (black rectangle). What makes the MERA decomposition a convenient one is the assumption that the $\chi$'s and $\lambda$'s satisfy special contraction rules. Specifically, for each $\chi$ and $\lambda$ let us define its adjoint $\bar{\chi}$ and $\bar{\lambda}$ as the tensors of elements $\lambda_{u_1 u_2} \equiv (\lambda_{u_1 u_2})^*$ and $\chi_{\bar{u}_1 \bar{u}_2} \equiv (\chi_{\bar{u}_1 \bar{u}_2})^*$. With this definition the MERA contraction rules are $\bar{\chi}_{\bar{u}_1 \bar{u}_2} \lambda_{u_1 u_2} = \delta_{u_1,\bar{u}_1} \delta_{u_2,\bar{u}_2}$, $\delta_{\bar{v}_1,\bar{u}_1} \delta_{\bar{v}_2,\bar{u}_2}$ and $\delta_{\bar{v}_1,\bar{u}_1} \delta_{\bar{v}_2,\bar{u}_2}$ respectively. The light blue region represents the causal cone associated with the dimensions of their indices are upper-bounded by a fixed constant (for easy of notation we omit the labels expressing the position of the tensors in the network).

Local observables and quantum channels:- We first show how the average of local observables can be related to the study of concatenated quantum channels. Given an operator $\Theta_j$ which acts not trivially on no more than three consecutive qudits (say the $(j-1)$th, $j$th and $(j+1)$th), the quantity $\langle \Theta_j \rangle = \langle \psi | \Theta_j | \psi \rangle$ requires to perform contractions only over the $\chi$’s and $\lambda$’s belonging to the causal cone of the triple $j-1$, $j$, and $j+1$. A compact expression is obtained by grouping these tensors in compounds composed by 2 $\chi$’s and by 3 $\lambda$’s (see inset of Fig. 1). This forms $m = \log_2 (N/4)$ non-necessarily identical type-$(\chi_1 \lambda_2)$ tensors $\mathcal{M}_5 \equiv \chi_{\lambda}$ where the products $\chi$ and $\lambda$ are defined by $|\lambda| = \lambda_{u_1 u_2} \equiv \chi_{\bar{u}_1 \bar{u}_2}$, $\lambda_{u_1 u_2} \equiv \chi_{\bar{u}_1 \bar{u}_2}$, and $\lambda_{u_1 u_2} \equiv \chi_{\bar{u}_1 \bar{u}_2}$. For each one of the $m$ tensors $\mathcal{M}_5$ we can then introduce two families of operators $\hat{L}_r \rho$ and $\hat{R}_r \rho$ acting on the Hilbert space $H^{\otimes 3}_d$ and labeled through the composed index $r \equiv (r_1, r_2, r_3)$ with $r_{1,2,3}$ being $d$-dimensional. In the computational basis they are defined by the matrices $|\xi_{u_1 u_2, \xi u_3}|L_r|\xi_{\ell_1, \xi_{\ell_2, \xi_{\ell_3}}}$ and $|\xi_{u_1 u_2, \xi u_3}|R_r|\xi_{\ell_1, \xi_{\ell_2, \xi_{\ell_3}}}$ of elements $[M^r_{u_1 u_2 u_3}]_{r_1, \ell_2, \ell_3, r_2, r_3}$ and $[M^r_{u_1 u_2 u_3}]_{r_1, \ell_1, \ell_2, r_2, r_3}$ respectively. They are related through a reshuffling of the input and output qudits, i.e. $\hat{R}_r = \Pi | L_r \rho \rangle = \hat{P}_r \hat{L}_r$, where $\hat{P}_r$ is the unitary transformation which exchanges the first and the third qudit. Most importantly, according to the contraction rules defined previously, they satisfy the normalization conditions $\sum_r \hat{L}_r \hat{L}^\dagger_r = \hat{I}^{\otimes 3} = \sum_r \hat{R}_r \hat{R}^\dagger_r$, with $\hat{I}$ being the identity operator of $H_d$. This implies that $\{\hat{L}_r\}_r$ can be used to define a completely positive, unital, not necessarily trace preserving super-operators $\Phi^{(L)}_r$, which transforms the linear operator $\hat{\Theta}$ of $H^{\otimes 3}_d$ into $\Phi^{(L)}_r(\hat{\Theta}) = \sum_r \hat{L}_r \hat{\Theta} L^\dagger_r$. Analogously $\{\hat{R}_r\}_r$ defines the map $\Phi^{(R)}_r$ which is related with $\Phi^{(L)}_r$ through the identity $\Phi^{(R)}_r = \Pi \circ \Phi^{(L)}_r \circ \Pi$, where "" indicates the composition of super-operators. We also introduce the vector of $H^{\otimes 4}_d$, $|\hat{\Theta}_j\rangle \equiv \sum_{\ell_1, \ell_2, \ell_3, \ell_4} |\ell_1, \ell_2, \ell_3, \ell_4 \rangle$, which without loss of generality is assumed to be normalized, and define $\hat{\rho}_C$ the three sites reduced density matrix obtained by tracing $|\hat{\Theta}_j\rangle |\hat{\Theta}_j\rangle$ over one of the 4 qudits. With these definitions one can finally write the expectation value of $\Theta_j$ as $\langle \Theta_j \rangle = \langle \psi | \hat{\Theta}_j | \psi \rangle$, which is related with $\Phi^{(L)}_r |\hat{\Theta}_j\rangle = \Phi^{(L)}_r |\hat{\Theta}_j\rangle$, and where (enumerating from the lower MERA level of Fig. 1) $\Phi^{(L)}_r$ is either the map $\Phi^{(L)}_r$ or $\Phi^{(R)}_r$ associated with the $k$-th tensor $\mathcal{M}_5$ of the causal cone (which one depends upon $N$ and $j$). The operator $\hat{B}_j \rho = \hat{B}_j \rho$ is thus obtained by applying to the observable $\hat{\Theta}_j$ a sequence of $m$ super-operators associated to the MERA causal cone. We can then write

$$\langle \hat{\Theta}_j \rangle = \text{Tr} \left[ \Phi^{(1-m)}_r (\hat{\rho}_C) \hat{\Theta}_j \right],$$

where $\Phi^{(1-m)}_r \equiv \Phi^{(1)}_r \circ \ldots \circ \Phi^{(m)}_r$, with $\Phi^{(k)}_r$ being the super-operator $\Phi^{(k)}_r$ in Schrödinger picture. By construction the $\Phi^{(k)}_r$ (and hence $\Phi^{(1-m)}_r$) are Completely Positive, Trace Preserving (CPT) maps, i.e. quantum channels with Kraus operators defined by either the set $\{\hat{L}_r\}_r$ or $\{\hat{R}_r\}_r$. Equation (1) establishes a formal equivalence between the MERA tensor network and the successive application of a family of CPT maps (the QuMERA family). Since it holds for all the local observable $\Theta_j$ this implies that $\Phi^{(1-m)}_r (\hat{\rho}_C)$ coincides with the reduced density matrix $\hat{\rho}_j$ of the input state $|\psi\rangle$ associated with the qudits $j-1$, $j$ and $j+1$, i.e.

$$\Phi^{(1-m)}_r (\hat{\rho}_C) = \hat{\rho}_j.$$
According to the above derivation, in the limit of large \( m \) Eq. (2) converges toward the reduced density matrix \( \hat{\rho}_T \) of three consecutive qudits of the systems \( \lim_{m \to \infty} \Phi^{(1-m)}(\hat{\rho}_C) = \hat{\rho}_T \). Of course the above limit should not depend upon the particular causal cone "trajectory" one chooses to follow (the system is translational invariant). Without loss of generality we can thus pick the one associated with the central sites of the MERA, i.e. the one associated with the causal cone of \( N/2 \)-th qudit. This allows us to identify all the \( \Phi^{(k)} \) of \( \Phi^{(1-m)} \) with maps of the form \( \Phi^{(k)} \). A further simplification arises by enforcing the scale invariance property of the system. This can be done for instance by assuming that all the tensors \( \chi \)'s and \( \lambda \)'s of the MERA to be identical \( \chi \) and by requiring \( \Phi^{(L)} = \Phi^{(R)} = \Phi \). With this assumption all the sequence \( \Phi^{(1-m)} \) can now be written as a composition of \( m \) identical quantum channels, i.e.

\[
\Phi^{(1-m)} = \Phi \circ \cdots \circ \Phi = [\Phi]^m. \tag{3}
\]

By general results on quantum channels the vast majority of CPT maps are known to be mixing (or relaxing) \[16, 17, 18, 19\]. This means that for a generic choice of \( \Phi \), in the limit \( m \to \infty \) the transformation \[3\] will send all input states into a unique fix point identified as the unique eigen-operator of \( \Phi \) associated with its largest eigenvalue. As in the case of MPS \[2, 22\], we can now provide a simplified expression for the thermodynamic limit \( \hat{\rho}_T \) of the reduced density matrix \( \hat{\rho}_T \). Of course the above limit can be computed along the same lines presented for local observables. Here we specialize in the two point correlation functions as the generalization is straightforward. Consider then the expectation value \( \langle \hat{\Theta}_i \hat{\Theta}_j \rangle \) of \( \hat{\Theta}_i \) and \( \hat{\Theta}_j \) being two local operators acting on (say) the \( i \)-th and \( j \)-th qudit respectively. In this case the causal cone is formed by two single sites causal cones which intercept at the MERA level \( \bar{m} = \text{int}[\log_2(i-j)] - 1 \), see Fig. 2. The resulting expectation values can then be written as

\[
\langle \hat{\Theta}_i \hat{\Theta}_j \rangle = \text{Tr}[(\Phi^{(1-m)}_i \otimes \Phi^{(1-m)}_j)(\hat{\rho}_{ij}^{(m)}) (\hat{\Theta}_i \otimes \hat{\Theta}_j)], \tag{6}
\]

where \( \Phi^{(1-m)}_{i,j} \) are the two CPT maps of the two single-site causal cones associated with the sites \( i \) and \( j \) respectively (light blue regions of Fig. 2). The operator \( \hat{\rho}_{ij}^{(m)} \) instead is a \( 6 \) qudits state associated with the last \( m - \bar{m} \) levels of the MERA. It is obtained from the 4 qubit state \( |\text{hat} \rangle \) through the application of a quantum channel \( \Psi^{(m-\bar{m})} \) which, similarly to \( \Phi^{(1-m)}_{i,j} \), originates from a proper concatenation of CPT maps associated with \( M_5 \) or with the type-\( ^5 \chi \) and type-\( ^4 \chi \) tensors \( M_9 = \lambda \chi \) and \( M_7 = \lambda \chi \). Since this applies to all the two sites observable, we can then conclude that \( \Phi^{(1-m)}_{i,j}(\hat{\rho}_{ij}^{(m)}) \) must coincide with the reduced density matrix \( \hat{\rho}_{ij} \) of \( |\text{hat} \rangle \) associated with the sites \( i \) and \( j \).

Let us focus then on the thermodynamic limit of the correlation function \( \Delta_{ij} = \langle \hat{\Theta}_i \hat{\Theta}_j \rangle - \langle \hat{\Theta}_i \rangle \langle \hat{\Theta}_j \rangle \) which for Hamiltonian systems at criticality decays as \( |i - j|^{-\nu} \). Under the same assumptions used to derive Eqs. (3) and (4) we can assume \( j \) to be the central site of the MERA (i.e. \( j = N/2 \)). Suppose then that the associated map \( \Phi \) is mixing with fix point \( \hat{\rho}_T \). For any input state \( \hat{\rho}_{ij} \) of the
sites $i$ and $j$ we then have \( \lim_{\tilde{m} \to \infty} (I_i \otimes \Phi_j^{(1-\tilde{m})})(\tilde{\rho}_{ij}) = \tilde{\rho}_i \otimes \tilde{\rho}_j \), with $\tilde{\rho}_i \equiv \text{Tr}_j[\tilde{\rho}_{ij}]$ and $I_i$ being the identity super-operator of the site $i$. The speed of convergence, evaluated through the trace distance, is exponentially fast \( m \gg \kappa \) in $\tilde{m}$ and, a part from some constant prefactor, can be upper-bounded by the quantity $m^\nu \kappa^{\tilde{m}}$ with $\kappa < 1$ being the modulus of the largest eigenvalue of $\Phi$ whose associated eigenvector contribute in the expansion of Eq. \( (6) \). This is sufficient for claiming that the distance between $(\Phi_i^{(1-\tilde{m})} \otimes \Phi_j^{(1-\tilde{m})})(\tilde{\rho}_{ij})$ and $\Phi_i^{(1-\tilde{m})}(\tilde{\rho}_i) \otimes \Phi_j^{(1-\tilde{m})}(\tilde{\rho}_j)$ is bounded by $\propto \kappa^{2\tilde{m}}$. Thus we can write $\log_2(\langle \Delta_{ij} \rangle) \leq 2\tilde{m} \log_2 \kappa + O(\log_2 \tilde{m})$, which through the definition of $\tilde{m}$ provides a bound for the critical exponent $\nu$ associated to the observable $\tilde{\Theta}$ in terms of the properties of the map $\Phi$, i.e. $\nu \geq -2 \log_2 \kappa$. In effect one can show that such bound is tight, i.e.

\[
\nu = -2 \log_2 \kappa .
\]

This can be seen for instance by expressing Eq. \( (6) \) in the Liouville space formalism as in Eq. \( (1) \), and expanding the transfer matrices $E_{\Phi_i} \otimes E_{\Phi_j}$ in Jordan blocks (the calculation is similar to the MPS analysis of Ref. \[22\]). A numerical test of Eq. \( (7) \) on a MERA state approximating the ground energy of an Ising chain up to $10^{-4}$ accuracy yielded $\kappa \approx 0.915, 0.49, 0.52$ to be compared with the exact values $\kappa_{Ising} \approx 0.917, 0.46, 0.50$ associated with the $x, y$ and $z$ two-point correlation functions. Similar results have been obtained for the XXZ model.

**Concluding remarks:** Equations \( (6) \) and \( (7) \) constitute the main results of our analysis. As already discussed by Vidal \[3\], MERA networks are able to describe algebraic decaying correlations. In this work we put on firm grounds this observation giving an explicit expression of the critical exponents in terms of properties of the associated QuMERA channels. The combination of this approach with conformal field theory methods may provide a powerful tool to achieve a complete description of one-dimensional critical quantum systems. Similarly our findings yield a natural connection between the tensor network description of the thermodynamic limit of critical systems and the master equation formalism. Combining these results with the algorithms presented in \[10\,13\] one can exploit the introduction of the transfer operator \[14\] studying directly the infinite size system improving simulation efficiency.

The results presented here can be easily extended in several ways. For instance since a binary tree can be seen as a MERA with the disentanglers $\chi$ set to the identity, all the arguments presented previously can be easily adapted to this case. Similarly also the thermodynamical limit for MPS \[5\] can be described in terms of repeated application of CPT maps (here the operator \[15\] reduces to the MPS transfer matrix). More generally our approach can be adapted to any tensor network by associating it with a family of CPT transformations which, properly concatenated, allows one to compute the local observables of the system. In this perspective the quantum circuit \[3\,14\] associated with the tensor network can be seen as a unitary dilatation or, Stinespring representation \[13\], of the corresponding CPT family. Finally a generalization to higher spatial dimensions seems straightforward.

We thank M. Rizzi for useful discussions. This work was in part founded by the Quantum Information research program of Centro “Ennio De Giorgi” of SNS.

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