BIFURCATION ANALYSIS OF A MOSQUITO POPULATION MODEL FOR PROPORTIONAL RELEASING STERILE MOSQUITOES

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Abstract. To reduce or eradicate mosquito-borne diseases, one of effective methods is to control the wild mosquito populations by using the sterile insect technique. Dynamical models with different releasing strategies of sterile mosquitoes have been proposed and investigated in the recent work by Cai et al. [SIAM. J. Appl. Math. 75(2014)], where some basic analysis on the dynamics are given and some complicated dynamical behaviors are found by numerical simulations. While their findings seem exciting and promising, yet the models could exhibit much more complex dynamics than it has been observed. In this paper, to further study the impact of the sterile insect technique on controlling the wild mosquito populations, we systematically study bifurcations and dynamics of the model with a proportional release rate of sterile mosquitoes by bifurcation method. We show that the model undergoes saddle-node bifurcation, subcritical and supercritical Hopf bifurcations, and Bogdanov-Takens bifurcation as the values of parameters vary. Some numerical simulations, including the bifurcation diagram and phase portraits, are also presented to illustrate the theoretical conclusions. These rich and complicated bifurcation phenomena can be regarded as a complement to the work by Cai et al. [SIAM. J. Appl. Math. 75(2014)].

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1. **Introduction.** In recent years, people have paid great attention to the mosquito-borne diseases, such as malaria, dengue, etc., due to the significant mortality and morbidity burden and huge economic impact associated with mosquito-borne diseases [6, 28]. While a variety of control approaches have been explored, most of them are chemically-based, and the effectiveness of this measure has been hampered by the appearance of insecticide-resistant strains [1, 3]. Fortunately, transgenic technologies, including the endosymbiotic bacterium *Wolbachia* and sterile insect techniques (SIT), are being explored to control or eradicate the mosquito-borne diseases [16, 14, 27, 31]. One of the most important and challenging questions in releasing sterile insect is to determine suitable releasing strategies and understand the dynamics of the interactive wild and sterile mosquitoes.

For the last few years, some mathematical models have been formulated to explore the dynamics of the interactive wild and sterile mosquitoes [2, 9, 10, 19, 20, 5, 15]. In [5], Cai et al. introduced three mathematical models with different releasing strategies of sterile mosquitoes to investigate the interactive dynamics of mosquito populations, and they analyzed the existence and stability of possible equilibria and established the threshold dynamics. In particular, when it is assumed that the releasing rate is proportional to the wild mosquito population size, some interesting dynamical phenomena are presented for the following model

\[
\begin{align*}
\frac{dw}{dt} &= \left( \frac{aw}{1+w+g} - (\mu_1 + \xi_1(w+g)) \right) w, \\
\frac{dg}{dt} &= bw - (\mu_2 + \xi_2(w+g))g, 
\end{align*}
\]

(1)

where \(w(t)\) is the number of wild mosquitoes and \(g(t)\) is the number of sterile mosquitoes at time \(t\), respectively. \(a > 0\) is the number of wild offspring produced per mate, \(\mu_i > 0\) and \(\xi_i > 0\), \(i = 1, 2\), are the density independent and dependent death rates of the wild and sterile mosquitoes, respectively. \(bw\) is the release rate of the sterile mosquitoes, which is proportional to the wild mosquito population size, \(b > 0\) is the release rate coefficient. We refer the readers to [5] for more biological implications.

In [5], the authors have proved that there exists a critical value \(b_0\) implicitly for the release rate coefficient \(b\) in system (1). When \(b > b_0\), all solutions of system (1) approach the origin for any positive initial population. This means that the mosquitoes can be eliminated. When \(b < b_0\), there exist two positive equilibria, one of which is always a saddle point, and the other is always a stable focus or node under the following hypothesis:

\[\mu_1 \leq \mu_2 \quad \text{and} \quad \xi_1 \leq \xi_2.\]

Thus model (1) has no periodic orbits with the above hypothesis. However, by numerical simulations, several examples have been given to show the existence of a stable limit cycle when the above hypothesis is not satisfied, and the nonlinear dynamics for model (1) is still not well-understood.

In this paper, to further study the nonlinear dynamics in model (1), we remove the restriction: \(\mu_1 \leq \mu_2\) and \(\xi_1 \leq \xi_2\), assumed in [5], *i.e.*, we assume that

\[\mu_1 > \mu_2 \quad \text{or} \quad \xi_1 > \xi_2,\]

our qualitative and bifurcation analysis reveals that model (1) can exhibit complex dynamics and bifurcations, such as saddle-node bifurcation, subcritical and supercritical Hopf bifurcations, and Bogdanov-Takens bifurcation as the values of
parameters vary. Thus, there exist some parameter values such that model (1) exhibits a stable or unstable limit cycle, and there exist some other parameter values such that model (1) exhibits an unstable homoclinic cycle. We also presented numerical examples, including bifurcation diagram and corresponding phase portraits, to illustrate the theoretical results. These rich and complicated bifurcation phenomena, which have not been explored in [5], can be seen as a complement to the work by Cai et al. [5].

The organization of this paper is as follows: In Section 2, we discuss the type and stability of equilibria. In Section 3, we show that system (1) undergoes saddle-node bifurcation, supercritical and subcritical Hopf bifurcations, and Bogdanov-Takens bifurcation as the parameters vary. A brief discussion is given in Section 4.

2. Type and stability of equilibria. Define the set \( \Omega \) as follows

\[
\Omega := \left\{ (w, g) \mid 0 \leq w \leq \frac{a}{\xi_1}, \ 0 \leq g \leq \frac{b a}{\mu_2 \xi_1} \right\}.
\]

It is easy to verify that \( \Omega \) is a positive invariant set for system (1) (see [5]). In the following, we shall consider system (1) in \( \Omega \).

Obviously, system (1) always has the origin \((0,0)\) as a trivial equilibrium and no other boundary equilibria. The positive equilibrium \( E^* \) of system (1) satisfies the following equations:

\[
a w
\frac{1 + w + g}{1 + w + g} = \mu_1 + \xi_1 (w + g), \quad bw = (\mu_2 + \xi_2 (w + g)) g.
\]

Equations in (2) can be rewritten, in terms of \( N = w + g \), as

\[
a N (\mu_2 + \xi_2 N) = (1 + N)(\mu_1 + \xi_1 N)(b + \mu_2 + \xi_2 N),
\]

\[
w = \frac{1}{a}(1 + N)(\mu_1 + \xi_1 N), \quad g = \frac{b(1 + N)(\mu_1 + \xi_1 N)}{a(\mu_2 + \xi_2 N)}.
\]

Define

\[
F(N) := (1 + N)(\mu_1 + \xi_1 N)(b + \mu_2 + \xi_2 N) - a N (\mu_2 + \xi_2 N)
\]

\[
= \xi_1 \xi_2 N^3 + \left( \xi_1 (b + \mu_2) + \xi_2 (\mu_1 + \xi_1 - a) \right) N^2
\]

\[+ (\mu_1 \xi_2 + b (\mu_1 + \xi_1) + (\mu_1 + \xi_1 - a) \mu_2) N + \mu_1 (b + \mu_2).
\]

From the first equation of (3), it follows that the existence of a positive solution to \( F(N) = 0 \) is equivalent to the existence of positive solution to the following equation:

\[
\frac{(1 + N)(\mu_1 + \xi_1 N) - a N (\mu_2 + \xi_2 N)}{(1 + N)(\mu_1 + \xi_1 N)} + b = 0.
\]

Let

\[
G(N) := \frac{(a N - (1 + N)(\mu_1 + \xi_1 N))(\mu_2 + \xi_2 N)}{(1 + N)(\mu_1 + \xi_1 N)}.
\]

then there exists a positive equilibrium of (1) if and only if there exists a positive solution \( N \) to equation \( G(N) = b \).

The following quadratic equation, whose real roots are the same as those of \( G(N) = 0 \),

\[
a N - (1 + N)(\mu_1 + \xi_1 N) = - (\xi_1 N^2 - (a - \mu_1 - \xi_1) N + \mu_1) = 0,
\]
has two positive roots
\[ N_{1,2} = \frac{1}{2\xi_1} \left( (a - \mu_1 - \xi_1) \pm \sqrt{(a - \mu_1 - \xi_1)^2 - 4\mu_1\xi_1} \right), \] (5)

if
\[ a - \mu_1 - \xi_1 > 0 \quad \text{and} \quad (a - \mu_1 - \xi_1)^2 - 4\mu_1\xi_1 > 0, \]
or, equivalently,
\[ a > (\sqrt{\mu_1} + \sqrt{\xi_1})^2. \] (6)

Suppose condition (6) is satisfied, and let \( \bar{N} \) be the point in \((N_1, N_2)\) such that \( G'(\bar{N}) = 0 \). We then define the threshold release value of sterile mosquitoes as
\[ b_0 := G(\bar{N}). \] (7)

Thus, we have \( b_0 > 0 \), and \( G(N) = b \) has two positive roots \( N_1^* < N_2^* \), lying in \((N_1, N_2)\), if and only if \( b < b_0 \), and a unique positive root \( N_1^* = N_2^* = \bar{N} \) if \( b = b_0 \).

The Jacobian matrix at a positive equilibrium \( E \) is given as
\[ J(E(w, g)) = \begin{pmatrix} \frac{aw(1+g)}{1+N} - \xi_1w - \left( \frac{aw}{1+N} + \xi_1 \right)w & -\mu_2 - \xi_2(N + g) \\ b - \xi_2g & -\mu_1 - \xi_2(N + g) \end{pmatrix}, \]
and
\[ \text{Det}J = \frac{w}{1+N} F'(N), \quad \text{Tr}J = \frac{1}{a} A(N), \]
where
\[ A(N) := (a(\mu_1 - \mu_2) + \mu_1(\xi_2 - \xi_1 - \mu_1)) + (a(\xi_1 - 2\xi_2) + \xi_1(\xi_2 - \xi_1))N + \xi_1(\xi_2 - 2\xi_1)N^2. \]

Thus, from Theorem 3.1 in [5] and Theorems 7.1-7.3 in [30], we can easily get the following results.

**Theorem 2.1.** Assume condition (6) holds, system (1) always has a boundary equilibrium \((0,0)\), moreover

(i): if \( b > b_0 \), system (1) has no positive equilibrium;

(ii): if \( b = b_0 \), system (1) has a unique positive equilibrium \( E^*(w^*, g^*) \), which is a saddle-node if \( A(N^*) \neq 0 \) and a cusp if \( A(N^*) = 0 \), where
\[ w^* = \frac{1}{a} (1 + \bar{N})(\mu_1 + \xi_1\bar{N}), \quad g^* = \frac{b(1 + \bar{N})(\mu_1 + \xi_1\bar{N})}{a(\mu_2 + \xi_2\bar{N})}, \quad N^* = w^* + g^*; \] (8)

(iii): if \( b < b_0 \), system (1) has two positive equilibria \( E_1^*(w_1^*, g_1^*) \) and \( E_2^*(w_2^*, g_2^*) \), and \( E_1^* \) is a saddle, \( E_2^* \) is locally asymptotically stable if \( A(N_2^*) < 0 \) and unstable if \( A(N_2^*) > 0 \) and a center-type equilibrium if \( A(N_2^*) = 0 \), where
\[ w_i^* = \frac{1}{a} (1 + N_i^*)(\mu_1 + \xi_1N_i^*), \quad g_i^* = \frac{b(1 + N_i^*)(\mu_1 + \xi_1N_i^*)}{a(\mu_2 + \xi_2N_i^*)}, \quad i = 1, 2. \] (9)

where \( b_0 \) is defined in (7), \( \bar{N} \) satisfies \( G'(\bar{N}) = 0 \), \( N_i^*, i = 1, 2 \), satisfy \( G(N) = b \), and \( N_1 < N_1^* < \bar{N} < N_2^* < N_2 \), with \( N_i \) given in (5).

The phase portraits are shown in Figure 1.
3. **Bifurcation analysis.** From Theorem 2.1, we can know that system (1) may exhibit Hopf bifurcation around the equilibrium \( E^*_2(w^*_2, g^*_2) \), and Bogdanov-Takens bifurcation around the equilibrium \( E^*(w^*, g^*) \). To simplify our calculation in discussing the bifurcations of system (1), we focus on some special cases in system (1).

Let \( a = 1 \) and \((w, g) = (1/2, 1/2)\) be the positive equilibrium point \( E^*_2(w^*_2, g^*_2) \) or \( E^*(w^*, g^*) \) so that \( N = 1 \) in system (1) in the following analysis.

### 3.1. Saddle-node bifurcation.

From Theorem 2.1, we know that

\[
SN = \{(a, b, \mu_1, \mu_2, \xi_1, \xi_2) : a > (\sqrt{\mu_1} + \sqrt{\xi_1})^2, A(N^*) \neq 0, b = b_0\}
\]

is a saddle-node bifurcation surface. When the parameters pass from one side of the surface to the other side, the number of positive equilibria of system (1) changes from zero to two, the saddle-node bifurcation yields two positive equilibria. This implies that there exists a critical releasing rate \( b_0 \) such that both of the wild and sterile mosquitoes go extinct when the releasing rate of sterile mosquitoes \( b \) is greater than \( b_0 \), and coexistence for model (1) is certain in the form of a positive equilibrium for certain choices of initial values when \( b = b_0 \).

### 3.2. Hopf bifurcation.

We firstly consider the Hopf bifurcation around the equilibrium \( E^*_2 \).

Because \( E^*_2(w^*_2, g^*_2) = (1/2, 1/2) \), thus the parameters of Eq.(2) have to satisfy

\[
\mu_1 = \frac{1}{4} - \xi_1, \quad \mu_2 = b - \xi_2.
\]

Firstly, we look for some parameter values such that system (1) has an equilibrium \( E^*_2(1/2, 1/2) \) with \( \text{Tr}(J(E^*_2)) = 0 \) and \( \text{Det}(J(E^*_2)) > 0 \).

By direct calculation from the Jacobian matrix \( J(E^*_2(1/2, 1/2)) \), we have \( \text{Tr}J = \frac{1}{4} \left( \frac{3}{8} - 2\xi_1 - 4\mu_2 - 6\xi_2 \right) = 0 \), and by equation (10) we have

\[
\xi_1 = \frac{3}{8} - 2\mu_2 - 3\xi_2 = \frac{3}{8} - 2b - \xi_2.
\]
On the other hand, \( \text{Det}(J(E_2^2)) = \frac{w_2^4}{1 + N_2^2} F'(N_2^2) = \frac{1}{4} F'(1) \). Thus, we need to have
\[
F'(1) = 3\xi_1 \xi_2 + 2[(\mu_2 + \xi_2)\xi_1 + \xi_1 \xi_2 - \xi_2(3\mu_1 + 4\xi_1) + \xi_1 \mu_2] + [\mu_1 \xi_2 + \mu_2 \xi_1 + (\mu_2 + \xi_2)\mu_1 - (3\mu_1 + 4\xi_1)\mu_2]
\[
= 2\xi_1 \mu_2 - 4\mu_1 \xi_2 - 2\mu_1 \mu_2
\]
\[
= 2[b(\frac{1}{2} - 4b - 2\xi_2) - \frac{1}{4}\xi_2]
\]
\[
> 0.
\]

Thus, we can choose \( b \) and \( \xi_2 \) as dependent parameters and establish the following results.

**Theorem 3.1.** Let \( \xi_1 = \frac{3}{8} - 2b - \xi_2, \quad \mu_1 = 2b + \xi_2 - \frac{1}{8}, \quad \mu_2 = b - \xi_2 \) and
\[
b(\frac{1}{2} - 4b - 2\xi_2) - \frac{1}{4}\xi_2 > 0,
\]
then system (1) may occur Hopf bifurcation around the positive equilibrium \( E_2^2(\frac{1}{2}, \frac{1}{2}) \).

**Remark 1.** From Theorem 3.1, we know that the parameters regions satisfy
\[
\frac{1}{8} < 2b + \xi_2 < \frac{3}{8}, \quad b(\frac{1}{2} - 4b - 2\xi_2) > \frac{1}{4}\xi_2, \quad b > \xi_2.
\]

By further simplification, it is not difficult to get the parameters regions for necessary conditions to occur Hopf bifurcation are
\[
\Omega_1 := \{(b, \xi_2) | \max\{0, \frac{1}{8} - 2b\} < \xi_2 < \frac{2b(1 - 8b)}{1 + 8b}, \frac{1}{24} < b < \frac{1}{8}\}.
\]

From \( \text{Tr} J(E_2^2) = \frac{1}{4}\left(\frac{3}{4} - 2\xi_1 - 4\mu_2 - 6\xi_2\right) \), we can easily check the transversality condition \( \frac{\partial}{\partial \xi_1}(\text{Tr} J(E_2^2)) | \xi_1 = \frac{3}{8} - 2b_2 - 3\xi_2 = -\frac{1}{2} \neq 0 \). Next we shall investigate the nondegeneracy condition and the stability of the bifurcating periodic orbit at the positive equilibrium \( E_2^2(\frac{1}{2}, \frac{1}{2}) \) of system (1) by calculating the first Lyapunov number.

Substituting the conditions in Theorem 3.1 to system (1), we get the following system
\[
\begin{align*}
\frac{dw}{dt} &= w\left[\frac{w}{1 + w + g} - (2b + \xi_2 - \frac{1}{8}) - (\frac{3}{8} - 2b - \xi_2)(w + g)\right], \\
\frac{dg}{dt} &= bw - [b + \xi_2(w + g - 1)]g.
\end{align*}
\]
(12)

Translate \( E_2^2 \) to the origin by letting \( x = w - \frac{1}{2}, \ y = g - \frac{1}{2}, \) the Taylor expansion of system (12) around the origin takes the form
\[
\begin{align*}
\frac{dx}{dt} &= (b + \frac{\xi_2}{2})x + (-\frac{1}{4} + b + \frac{\xi_2}{2})y + (-\frac{3}{32} + 2b + \xi_2)x^2 + (-\frac{9}{16} + 2b + \xi_2)xy \\
&\quad + \frac{1}{32}y^2 - \frac{9}{64}x^3 - \frac{3}{64}x^2y + \frac{5}{64}xy^2 - \frac{1}{64}y^3 + O(|x, y|^4), \\
\frac{dy}{dt} &= (b - \frac{\xi_2}{2})x - (b + \frac{\xi_2}{2})y - \xi_2xy - \xi_2y^2 + O(|x, y|^4).
\end{align*}
\]
(13)

Make a change of variables as follow
\[
\begin{pmatrix}
x \\ y
\end{pmatrix} = \begin{pmatrix}
a(a - \frac{1}{4}) & -(a - \frac{1}{4})c \\ -(a^2 + c^2) & 0
\end{pmatrix} \begin{pmatrix}
X \\ Y
\end{pmatrix},
\]
then system (13) can be written as

\[
\frac{dX}{dt} = -cY + f(X, Y), \\
\frac{dY}{dt} = cX + g(X, Y),
\]

where

\[
f(X, Y) = \frac{1}{4} (a + 4c^2)\xi_2 X^2 + \frac{1}{4} (1 + 4a)\xi_2 XY + O(|X, Y|^4),
\]

\[
g(X, Y) = \frac{1}{128(-1 + 4a)c} (-256a^4 + 16a^3 (3 + 16b) - 8a(-9 + 32b)c^2 - 16c^4 + a^2(3 - 320c^2 + 64b(-1 + 16c^2)) + 64a(4a^2 - 4c^2 + a(-1 + 16c^2))\xi_2 X^2 + \frac{3a}{64} - ab + a^2(\frac{3}{8} + 2b) + \frac{(9 - 32b)c^2}{16} + (-\frac{3a}{4} + 2a^2 - c^2)\xi_2 XY - \frac{1}{128}((-1 + 4a)c(-3 + 64b + 32\xi_2)Y^2 - \frac{9((1 - 4a)^2c^2)}{1024}Y^3 + \frac{3(-1 + 4a)c(-9a + 32a^2 - 4c^2)}{1024}XY^2 + \frac{(192a^4 - 256a^4 - 24ac^2 + 80c^4 + a^2(-27 + 256c^2))}{1024}X^2Y - \frac{(a + 4c^2)(-3a + 16a^2 + 4c^2)^2}{1024c(-1 + 4a)}X^3 + O(|X, Y|^4),
\]

and \(a = b + \frac{\xi_2}{2}, \ c = \sqrt{\frac{1}{2}b(\frac{1}{2} - 4b - 2\xi_2)} - \frac{\xi_2}{8} \).

The Liapunov number (Perko [22]) can be expressed as

\[
C_1 = \frac{1}{16}[(f_{xxx} + f_{xyy} + g_{xx}Y + g_{yy}Y) + \frac{1}{c}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy})]\big|_{X=Y=0} = \frac{1}{32768|2b(-1 + 8b) + (1 + 8b)\xi_2|}(\xi_2^2 (2b^2(9 - 128b + 448b^2)) + b(-27 + 168b - 179b^2 - 8192b^3)\xi_2 + (9 + 60b + 544b^2 + 8192b^3)\xi_2^3 - 8(5 - 8b + 256b^3)\xi_2^4 - 2048b\xi_2^4)).
\]

(15)

**Theorem 3.2.** When (10) and (11) are satisfied, and \((b, \xi_2) \in \Omega_1\), then

(i): if \(C_1 < 0\), system (1) exhibits supercritical Hopf bifurcation, and a stable limit cycle occurs around \(E_2^\prime(\frac{1}{2}, \frac{1}{2})\);

(ii): if \(C_1 > 0\), system (1) exhibits subcritical Hopf bifurcation, and an unstable limit cycle occurs around \(E_2^\prime(\frac{1}{2}, \frac{1}{2})\);

(iii): if \(C_1 = 0\), system (1) may exhibit degenerate Hopf bifurcation, and multiple limit cycles may occur around \(E_2^\prime(\frac{1}{2}, \frac{1}{2})\).

Where \(C_1\) is given in (15).

The existence of one stable (or unstable) limit cycle is shown in Figure 2.
3.3. Bogdanov-Takens bifurcation. In this section, we shall consider Bogdanov-Takens bifurcation of system (1) around the equilibrium $E^*(w^*, g^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$. We follow the techniques and steps of Huang et al. [12] to prove the existence of Bogdanov-Takens bifurcation. From the element bifurcation theory [7, 17], in order for a Bogdanov-Takens bifurcation of system (1) in a small neighborhood of the equilibrium $E^*$ to occur, we need to have $\text{Det}(J(E^*)) = 0$ and $\text{Tr}(J(E^*)) = 0$.

Recall that when $E^*(w^*, g^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$, $a = 1$ and $\text{Tr}(J(E^*)) = 0$, the equations (10) and (11) are satisfied, that is we have

$$\xi_1 = \frac{3}{8} - 2\mu_2 - 3\xi_2, \quad \mu_1 = 2\mu_2 + 3\xi_2 - \frac{1}{8}, \quad \mu_2 = b - \xi_2.$$  

(16)

On the other hand, $\text{Det}(J(E^*)) = \frac{w^*}{1+N} F' (\tilde{N}) = \frac{1}{4} F'(1) = \frac{1}{2} b \left( \frac{1}{2} - 4b - 2\xi_2 \right) - \frac{1}{8} \xi_2$. Thus, when $\text{Det}(J(E^*)) = 0$, we have

$$\xi_2 = \frac{2b(1-8b)}{1+8b}.$$  

(17)

From equations (16) and (17), we see that $\xi_1$, $\xi_2$, $\mu_1$ and $\mu_2$ can be expressed by $b$ as follow

$$\xi_1 = \frac{3-8b}{8(1+8b)}, \quad \xi_2 = \frac{2b(1-8b)}{1+8b}, \quad \mu_1 = \frac{24b-1}{8(1+8b)}, \quad \mu_2 = \frac{b(24b-1)}{1+8b},$$  

(18)

where $\frac{1}{24} < b < \frac{1}{8}$.

**Theorem 3.3.** If (18) is satisfied, then the unique positive equilibrium $E^*\left(\frac{1}{2}, \frac{1}{2}\right)$ of system (1) is a cusp of codimension 2 (Bogdanov-Takens singularity).

**Proof.** Under the conditions (18), the system (1) can be rewritten as

$$\frac{dw}{dt} = w \left[ \frac{w}{1+w+g} - \frac{24b-1}{8(1+8b)} - \frac{3-8b}{8(1+8b)} (w+g) \right],$$

$$\frac{dg}{dt} = b \left[ w - g - \frac{2(1-8b)}{1+8b} (w+g-1) g \right],$$  

(19)

where $\frac{1}{24} < b < \frac{1}{8}$. 

**Figure 2.** (a) An unstable limit cycle created by the subcritical Hopf bifurcation; (b) A stable limit cycle created by the supercritical Hopf bifurcation.
Firstly, we let \( x_1 = g - \frac{1}{2} \) and \( x_2 = w - \frac{1}{2} \), system (19) can be transformed into

\[
\frac{dx_1}{dt} = a_{10}x_1 + a_{02}x_2 + a_{12}x_1x_2 + a_{11}x_1^2 + O(|x_1, x_2|^3),
\]

\[
\frac{dx_2}{dt} = b_{10}x_1 + b_{02}x_2 + b_{22}x_2^2 + b_{12}x_1x_2 + b_{11}x_1^2 + O(|x_1, x_2|^3),
\]

where

\[
a_{10} = -\frac{2b}{1 + 8b}, \quad a_{02} = \frac{16b^2}{1 + 8b}, \quad a_{12} = -\frac{2b(1 - 8b)}{1 + 8b}, \quad a_{11} = -\frac{2b(1 - 8b)}{1 + 8b},
\]

\[
b_{10} = -\frac{1}{4(1 + 8b)}, \quad b_{02} = \frac{2b}{1 + 8b}, \quad b_{11} = \frac{1}{32}, \quad b_{12} = -\frac{9 + 8b}{16(1 + 8b)}, \quad b_{22} = \frac{104b - 3}{32(1 + 8b)}.
\]

Secondly, we make the transformation \( y_1 = x_1, \ y_2 = \alpha_{10}x_1 + \alpha_{02}x_2 \), which brings system (20) into

\[
\frac{dy_1}{dt} = y_2 + \alpha_1 y_1^2 + \alpha_2 y_1y_2 + O(|y_1, y_2|^3),
\]

\[
\frac{dy_2}{dt} = \beta_1 y_1^2 + \beta_2 y_1y_2 + \beta_3 y_2^2 + O(|y_1, y_2|^3),
\]

where

\[
\alpha_1 = \frac{8b - 1}{4}, \quad \alpha_2 = \frac{8b - 1}{8b}, \quad \beta_1 = \frac{-448b^2 + 48b - 3}{128(1 + 8b)},
\]

\[
\beta_2 = \frac{64b - 320b^2 - 3}{128b(1 + 8b)}, \quad \beta_3 = \frac{104b - 3}{512b^2}.
\]

By the results in [22] (see also Lemma 3.1 in [11] or [12]), the system (21) is equivalent to the following system in some small neighborhood of \((0, 0)\)

\[
\frac{dz_1}{dt} = z_2,
\]

\[
\frac{dz_2}{dt} = c_1z_1^2 + c_2z_1z_2 + O(|z_1, z_2|^3),
\]

where

\[
c_1 = \beta_1 = \frac{-448b^2 + 48b - 3}{128(1 + 8b)}, \quad c_2 = \beta_2 + 2\alpha_1 = \frac{4096b^3 - 320b^2 - 3}{128b(1 + 8b)}.
\]

It is easy to show that \( c_1 < 0 \) and \( c_2 < 0 \) when \( b \in (1/24, 1/8) \). In fact, let

\[
f(b) = -448b^2 + 48b - 3, \quad g(b) = 4096b^3 - 320b^2 - 3.
\]

Obviously, \( f(b) < 0 \) for any \( b \), \( g(b) \) has two extreme points, \( g(b) = 0 \) has only one real root \( b = \frac{1}{8} \), and \( g(b) < 0 \) for \( b < \frac{1}{8} \).

Therefore, by [8], the equilibrium \( E^*(\frac{1}{2}, \frac{1}{2}) \) is a cusp of codimension 2.

Theorem 3.3 indicates that system (1) may exhibit Bogdanov-Takens bifurcation if the bifurcation parameters are chosen suitably. Actually, we have the following theorem.

**Theorem 3.4.** When \( \mu_2 = \frac{b(24b - 1)}{1 + 8b} \), \( \xi_2 = \frac{2b(1 - 8b)}{1 + 8b} \) and \( b \in (\frac{1}{24}, \frac{3}{32}) \cup (\frac{3}{32}, \frac{1}{8}) \), system (1) undergoes Bogdanov-Takens bifurcation in a small neighborhood of the unique positive equilibrium \( E^*(\frac{1}{2}, \frac{1}{2}) \) as \( (\xi_1, \mu_1) \) varies near \( (\frac{3 - 8b}{8(1 + 8b)}, \frac{24b - 1}{8(1 + 8b)}) \). Hence, there exist some parameter values such that system (1) has an unstable limit cycle, and
there exist some other parameter values such that system (1) has an unstable homoclinic loop. The bifurcation portrait and corresponding phase portraits are shown in Figure 3.

Proof. We choose \( \xi_1 \) and \( \mu_1 \) as bifurcation parameters, and consider the following unfolding system

\[
\frac{dw}{dt} = w \left( \frac{w}{1 + w + g} - \frac{24b - 1}{8(1 + 8b)} + \lambda_1 \right) - \left( \frac{3 - 8b}{8(1 + 8b)} + \lambda_2 \right)(w + g),
\]

\[
\frac{dg}{dt} = bw - \left( \frac{b(24b - 1)}{1 + 8b} + \frac{2b(1 - 8b)}{1 + 8b} \right)(w + g),
\]

where \((\lambda_1, \lambda_2)\) is a parameter vector in a small neighborhood of \((0, 0)\).

Let \( x_1 = g - \frac{1}{2} \), \( x_2 = w - \frac{1}{2} \). Then system (23) can be transformed into

\[
\frac{dx_1}{dt} = \alpha_{10}x_1 + \alpha_{02}x_2 + \alpha_{12}x_1x_2 + \alpha_{11}x_1^2,
\]

\[
\frac{dx_2}{dt} = \mu_{00} + \beta_{10}x_1 + \beta_{02}x_2 + \beta_{11}x_1^2 + \beta_{12}x_1x_2 + \beta_{22}x_2^2 + R_1(x_1, x_2),
\]

where

\[
\alpha_{10} = -\frac{2b}{1 + 8b}, \quad \alpha_{02} = \frac{16b^2}{1 + 8b}, \quad \alpha_{12} = -\frac{2b(1 - 8b)}{1 + 8b}, \quad \alpha_{11} = -\frac{2b(1 - 8b)}{1 + 8b},
\]

\[
\mu_{00} = -\frac{1}{2}(\lambda_1 + \lambda_2), \quad \beta_{10} = -\frac{1}{4}(\frac{1}{1 + 8b} + 2\lambda_2), \quad \beta_{02} = \frac{2b}{1 + 8b} - \lambda_1 - \frac{3}{2}\lambda_2,
\]

\[
\beta_{11} = \frac{1}{32}, \quad \beta_{12} = -\frac{9 + 8b}{16(1 + 8b)} - \lambda_2, \quad \beta_{22} = \frac{104b - 3}{32(1 + 8b)} - \lambda_2,
\]

and \( R_1 \) is a \( C^\infty \) function at least of the third order with respect to \((x_1, x_2)\). Let \( x = x_1, y = \alpha_{10}x_1 + \alpha_{02}x_2 + \alpha_{12}x_1x_2 + \alpha_{11}x_1^2, \)

then system (24) can be written as

\[
\dot{x} = y,
\]

\[
\dot{y} = \alpha_{02}\mu_{00} + \alpha_1x + \alpha_2y + \alpha_3x^2 + \alpha_4xy + \alpha_5y^2 + R_2(x, y),
\]

where

\[
\alpha_1 = \alpha_{02}\beta_{10} - \alpha_{10}\beta_{02} + \alpha_{12}\mu_{00}, \quad \alpha_2 = \alpha_{10} + \beta_{02},
\]

\[
\alpha_3 = \alpha_{02}\beta_{11} - \alpha_{11}\beta_{02} - \alpha_{10}\beta_{12} + \frac{\beta_{22}}{\alpha_{02}} + \alpha_{12}\beta_{10},
\]

\[
\alpha_4 = \beta_{12} - \frac{2\alpha_{10}\beta_{22}}{\alpha_{02}} - \frac{\alpha_{12}\alpha_{10}}{\alpha_{02}} + 2\alpha_{11}, \quad \alpha_5 = \frac{\beta_{22}}{\alpha_{02}} + \frac{\alpha_{12}}{\alpha_{02}},
\]

and \( R_2 \) is a \( C^\infty \) function at least of the third order with respect to \((x, y)\).

Next, we introduce a new time variable \( \tau \) by \( dt = \left( 1 - \frac{\beta_{22} + \alpha_{12}x}{\alpha_{02}} \right) d\tau \). Rewriting \( \tau \) as \( t \), we have from (26) that

\[
\dot{x} = (1 - \frac{\beta_{22} + \alpha_{12}x}{\alpha_{02}})y,
\]

\[
\dot{y} = (1 - \frac{\beta_{22} + \alpha_{12}}{\alpha_{02}})x(\alpha_{02}\mu_{00} + \alpha_1x + \alpha_2y + \alpha_3x^2 + \alpha_4xy + \alpha_5y^2 + R_2(x, y)).
\]

Let
\[ X = x, \ Y = y \left( 1 - \frac{\beta_{22} + \alpha_{12}}{\alpha_{02}} x \right), \]

then system (27) can be rewritten as

\[
\begin{align*}
\dot{X} &= Y, \\
\dot{Y} &= \psi_1 + \psi_2 X + \psi_3 Y + \psi_4 X^2 + \psi_5 XY + R_3(X, Y, \lambda_1, \lambda_2),
\end{align*}
\]

where \( R_3 \) is a \( C^\infty \) function at least of the third order with respect to \((X, Y)\), whose coefficients depend smoothly on \( \lambda_1 \) and \( \lambda_2 \), and

\[
\begin{align*}
\psi_1 &= \alpha_{02} \mu_{00}, \quad \psi_2 = \alpha_{02} \beta_{10} - \alpha_{10} \beta_{02} - \alpha_{12} \mu_{00} - 2 \beta_{22} \mu_{00}, \quad \psi_3 = \beta_{02} + \alpha_{10}, \\
\psi_4 &= \alpha_{02} \beta_{11} - \alpha_{11} \beta_{02} + \frac{\beta_{22} \alpha_{10}}{\alpha_{02}} + (2 \frac{\beta_{22} + \alpha_{12}}{\alpha_{02}} \beta_{02} - \beta_{12}) \alpha_{10} \\
&\quad - \beta_{10}(2 \beta_{22} + \alpha_{12}) + \frac{\beta_{22} - \alpha_{12}}{\alpha_{02}} \mu_{00}, \\
\psi_5 &= \beta_{12} + 2 \alpha_{11} - \frac{\beta_{02} \beta_{22} + \alpha_{12} \beta_{02} + 3 \alpha_{10} \beta_{22} + 2 \alpha_{10} \alpha_{12}}{\alpha_{02}}.
\end{align*}
\]

Notice that

\[
\psi_4 = \frac{-448 \alpha_{10}^2 + 48 \beta_{12} + 3}{128(1+8b)} + h_1(\lambda_1, \lambda_2),
\]

where \( h_1(\lambda_1, \lambda_2) \) is a function with respect to \((\lambda_1, \lambda_2)\), whose coefficients depend smoothly on \( b \), we can get that \( \psi_4 < 0 \) when \( \lambda_i \) are small.

Make the following change of variables

\[ x = X, \ y = \frac{Y}{\sqrt{-\psi_4}}, \ \tau = \sqrt{-\psi_4} t, \]

then system (28) becomes

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\frac{\psi_1}{\psi_4} - \frac{\psi_2 x}{\psi_4} + \frac{\psi_3 x}{\sqrt{-\psi_4}} y - x^2 + \frac{\psi_5}{\sqrt{-\psi_4}} xy + R_4(x, y, \lambda_1, \lambda_2),
\end{align*}
\]

where \( R_4 \) is a \( C^\infty \) function at least of the third order with respect to \((x, y)\), whose coefficients depend smoothly on \( \lambda_1 \) and \( \lambda_2 \). Let

\[ X = x + \frac{\psi_2}{2\psi_4}, Y = y, \]

then system (29) can be written as

\[
\begin{align*}
\dot{X} &= Y, \\
\dot{Y} &= \frac{\psi_1}{\psi_4} + \frac{\psi_2}{4\psi_4} + \left( \frac{\psi_3}{\sqrt{-\psi_4}} - \frac{\psi_2 \psi_5}{2\psi_4 \sqrt{-\psi_4}} \right) Y - X^2 + \frac{\psi_5}{\sqrt{-\psi_4}} XY + R_5(X, Y, \lambda_1, \lambda_2),
\end{align*}
\]

where \( R_5 \) is a \( C^\infty \) function at least of the third order with respect to \((X, Y)\), and the coefficients depend smoothly on \( \lambda_1 \) and \( \lambda_2 \).

Notice that

\[
\psi_5 = \frac{4096 \alpha_{10}^3 - 32 \beta_{12}^2 - 3}{128(1+8b)} + h_2(\lambda_1, \lambda_2),
\]

where \( h_2(\lambda_1, \lambda_2) \) is a function with respect to \((\lambda_1, \lambda_2)\), whose coefficients depend smoothly on \( b \), we can get that \( \psi_5 \neq 0 \) (in fact, \( \psi_5 < 0 \)) when \( \lambda_i \) are small.

Make the change of variables one more time by setting
\[
x = \frac{\psi_3^2}{\psi_4} X, \quad y = \frac{\psi_2^2}{\psi_4 \sqrt{4}} Y, \quad \tau = -\frac{\sqrt{-\psi_4}}{\psi_5} t,
\]
then we obtain the versal unfolding of system (23)
\[
\dot{x} = y, \quad \dot{y} = \tau_1 + \tau_2 y + x^2 + xy + R_6(x, y, \lambda_1, \lambda_2),
\]
where \(R_6\) is a \(C^\infty\) function at least of the third order with respect to \((x, y)\), whose coefficients depend smoothly on \(\lambda_1\) and \(\lambda_2\), and
\[
\tau_1 = \frac{\psi_1 \psi_4^2}{\psi_4^4} - \frac{\psi_2^2}{4 \psi_4^4}, \quad \tau_2 = -\frac{\psi_2 \psi_5^2}{2 \psi_4^2}.
\]
By some simple computation, we obtain that
\[
\tau_1 = \frac{Q_1^4(b)(\lambda_1 + \lambda_2)}{16b^2(1 + 8b)^2c_3^2(b)} - \frac{Q_3^4(b)}{1024b^4(1 + 8b)^2c_2^4(b)} ((27 - 288b + 6144b^2 + 21504b^3)
- 4386816b^6 + 23920640b^5 + 70254592b^6 + 2348810240b^7) \lambda_1^2 + 2(27 + 72b
+ 4224b^2 + 76800b^3 - 513288b^4 - 4685824b^5 - 94371840b^6
+ 3221225472b^7) \lambda_1 \lambda_2 + (27 + 432b + 3072b^2 + 144384b^3 - 574688b^4
- 3303044b^5 - 130023424b^6 + 4026531840b^7) \lambda_2^2) + O(\lambda_1, \lambda_2^3),
\]
\[
\tau_2 = \frac{1}{64b^2c_3^2(b)} ((27 - 64b + 14976b^2 - 282624b^3 + 2863104b^4 - 32899072b^5
+ 251658240b^6 - 536870912b^7) \lambda_1 + 3(9 - 288b + 6528b^2 - 98304b^3 + 987136b^4
- 13500416b^5 + 75497427b^6) \lambda_2) + \frac{1}{8192b^4c_3^2(b)} ((-243 + 6480b
- 136512b^2 + 1935360b^3 + 34615296b^4 - 110970240b^5 + 20226244608b^6
- 32187088896b^7 + 260650827760b^8 - 16355235463168b^9
+ 136339441844224b^{10} - 422212465065984b^{11} \lambda_1^2 - 2(243 - 1944b + 5184b^2
+ 622080b^3 - 87699456b^4 + 1458143232b^5 - 24847319040b^6 + 338297880576b^7
- 1844151582720b^8 + 19318762897408b^9 - 181414941853040b^{10}
+ 351843720888320b^{11} \lambda_1 \lambda_2 + (-243 - 2592b + 84672b^2 - 3843072b^3
+ 132820992b^4 - 1651507200b^5 + 33413136384b^6 - 28335459328b^7
+ 1014551805952b^8 - 32152125177856b^9 + 212205744160768b^{10}) \lambda_2^2)
+ O(\lambda_1, \lambda_2^3),
\]
where \(Q_1(b) = 3 + 320b^2 - 4096b^3, Q_2(b) = 3 - 48b + 448b^2\).

Since
\[
\left| \frac{\partial(\tau_1, \tau_2)}{\partial(\lambda_1, \lambda_2)} \right|_{\lambda=0} = \frac{(-3+32b)(-3-320b^2+4096b^3)^5}{2^{15}(1+8b)(3-48b+448b^2)^5} \neq 0
\]
when \(b \in \left(\frac{1}{32}, \frac{3}{32}\right) \cup \left(\frac{3}{32}, \frac{1}{16}\right)\), the above parameter transformation is a homeomorphism in a small neighborhood of the origin, and \(\tau_1\) and \(\tau_2\) are independent parameters.

By the results in Perko [22], we obtain the following local representations of the bifurcation curves up to second-order approximations for system (23):
(i) The saddle-node bifurcation curve is \( SN = \{ (\tau_1, \tau_2) \mid \tau_1 = 0, \tau_2 \neq 0 \} = \{(\lambda_1, \lambda_2) \mid \frac{3 + 320b^2 - 4096b^3}{16b^4}((\lambda_1 + \lambda_2) + \frac{(-3 - 320b^2 + 4096b^3)^3}{1024b^4(1 + 8b)^4(3 - 48b + 448b^2)^4}\) \}

\( \lambda_1^2 + 2(27 + 72b + 422b^2 + 7680b^3 - 5132288b^4 - 4685824b^5 - 94371840b^6 + 32212254720\lambda_1\lambda_2 + (27 + 432b + 3072b^2 + 144384b^3 - 5746688b^4 - 33030144b^5 - 4026531840b^6)\lambda_2^2 = 0\} \).

(ii) The Hopf bifurcation curve is \( H = \{ (\tau_1, \tau_2) \mid \tau_2 = \sqrt{-\tau_1}, \tau_1 < 0 \} = \{(\lambda_1, \lambda_2) \mid \frac{3 + 320b^2 - 4096b^3}{16b^4}((\lambda_1 + \lambda_2) + \frac{(3 + 320b^2 - 4096b^3)^3}{4096b^4(1 + 8b)^2(3 - 48b + 448b^2)^4}\) \}

\( |\lambda_1| + 2|\lambda_2| - 30528 - 33030144b^5 + 8509456384b^6 - 906805480b^6 - 7263748096b^6 - 101946753024b^7 + 155960999936b^8 - 6133213298688b^9 + 52776558133248b^{10}\lambda_1\lambda_2 + (-243 - 9072b + 33984b^2 - 3446784b^3 + 79441920b^4 + 382795776b^5 + 8509456384b^6 - 906805480b^6 - 17904649152b^8 - 7284264534016b^9 + 65970697666560b^{10})\lambda_2^2 = 0\} \).

(iii) The homoclinic bifurcation curve is \( HL = \{ (\tau_1, \tau_2) \mid \tau_2 = \frac{2}{3}\sqrt{-\tau_1}, \tau_1 < 0 \} = \{(\lambda_1, \lambda_2) \mid \frac{3 + 320b^2 - 4096b^3}{16b^4}((\lambda_1 + \lambda_2) + \frac{(3 + 320b^2 - 4096b^3)^3}{1024b^4(1 + 8b)^2(3 - 48b + 448b^2)^4}\) \}

\( \lambda_1^2 + 2(-243 - 4752b + 6336b^2 - 1916928b^3 + 62926848b^4 + 111869952b^5 + 7263748096b^6 - 101946753024b^7 + 155960999936b^8 - 6133213298688b^9 + 52776558133248b^{10}\lambda_1\lambda_2 + (-243 - 9072b + 33984b^2 - 3446784b^3 + 79441920b^4 + 382795776b^5 + 8509456384b^6 - 906805480b^6 - 17904649152b^8 - 7284264534016b^9 + 65970697666560b^{10})\lambda_2^2 = 0\} \).

The Bogdanov-Takens bifurcation diagram and corresponding phase portraits of system (23) with \( b = \frac{35}{800} \) are given in Figure 3. These bifurcation curves \( H, HL \) and \( SN \) divide the small neighborhood of the origin in the parameter \( (\lambda_1, \lambda_2) \)-plane into four regions (see Figure 3(a)).

(a) When \( (\lambda_1, \lambda_2) = (0, 0) \), the unique positive equilibrium is a cusp of codimension 2.

(b) There are no equilibria when the parameters lie in region I (see Figure 3(b)), all solutions will pass through the \( g \)-axis and go out of the first quadrant.

(c) When the parameters lie on the curve \( SN \), there is a positive equilibrium, which is a saddle-node.
(d) Two positive equilibria, one is an unstable focus and the other is a saddle, will occur through the saddle-node bifurcation when the parameters cross $SN$ into region II (see Figure 3(c)).

(e) An unstable limit cycle will appear through the subcritical Hopf bifurcation when the parameters cross $H$ into region III (see Figure 3(d)), where the focus is stable, whereas the focus is an unstable one with multiplicity one when the parameters lie on the curve $H$.

(f) An unstable homoclinic cycle will occur through the homoclinic bifurcation when the parameters pass region III and lie on the curve $HL$ (see Figure 3(e)).

(g) The relative location of one stable and one unstable manifold of the saddle $E^*_1$ will be reversed when the parameters cross III into region IV (compare Figure 3(c) and Figure 3(f)).

4. Discussion. In most dynamical models arising in applications, there are inevitably some parameters associated with these models. If the dynamics in these models will change due to the parameters vary, then we need to know that which
are the crucial parameters to effect the dynamics? Can we find all the dynamics when the parameters vary around the crucial parameters? Bifurcation theory is a useful tool to answer these problems. Complex dynamics and bifurcations, such as Hopf bifurcation and Bogdanov-Takens bifurcation etc., have been observed and studied in epidemic models [25, 4], information diffusion model [23] and predator-prey models [29, 26, 13, 18, 24, 21]. The existence of positive equilibria, homoclinic loop and limit cycles, arising from the various bifurcations in system (1), means that the interactive wild and sterile mosquitoes can persist in the form of periodic coexistent oscillations or positive steady states.

In paper [5], the authors obtained a threshold releasing rate and discussed only the local stability of equilibria by assuming $\mu_1 \leq \mu_2$ and $\xi_1 \leq 2\xi_2$. In the present research, we have provided additional bifurcation analysis for the model [5], and showed that the existence of saddle-node bifurcation, subcritical and supercritical Hopf bifurcations, and Bogdanov-Takens bifurcation when the values of parameters are changed by assuming $\mu_1 > \mu_2$ or $\xi_1 > 2\xi_2$. These rich and complex bifurcation phenomena support the numerical observations in [5].

The release rate coefficient $b$ in system (1) is biologically significant, and it is considered as a critical index to closely keep sampling or surveillance of the wild mosquitoes by letting the release rate be proportional to the wild mosquito population size. By maintaining surveillance or closely sampling, we can determine a threshold release value. By theoretical analysis, we have shown that if the release rate coefficient $b$ is below the threshold value given in (7), then (1) undergoes some complex bifurcation phenomena, such as subcritical and supercritical Hopf bifurcations, and a stable or unstable limit cycle will occur (see Figure 2). A stable limit cycle describes the sustained periodic oscillations, which helps to explain the real observations and also provides useful guidelines in mosquitoes control and diseases prevention. These results can be very useful for identifying threshold conditions to determine when the interacting mosquito population will tend to a sustained periodic outbreak. Hence, the strategies and the timing for releasing sterile mosquitoes is important in reducing disease outbreak level. The existence of periodic oscillation in system (1) also demonstrates that the use of the sterile insect techniques may have an adverse effect, and cause a periodic outbreak of mosquito population. This means that increasing the releasing rate of the sterile mosquitoes only reduce the number of wild mosquitoes, but can not eradicate them. Thus, finding a more efficient release strategies will be important in combating mosquito-borne diseases.

Finally, we would like to point out that a special case is investigated throughout the bifurcation analysis, i.e., let the positive equilibria be $\left(\frac{1}{2}, \frac{1}{2}\right)$. It will be very interesting and challenging to study the effect of the proportional releasing sterile mosquitoes on the wild mosquitoes by performing a complete bifurcation analysis for system (1). We leave these for future investigation.

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