THE MORSE-SARD-BROWN THEOREM FOR FUNCTIONALS ON BOUNDED-FRÉCHET-FINSLER MANIFOLDS

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ABSTRACT. In this paper we study Lipschitz-Fredholm vector fields on Bounded-Fréchet-Finsler manifolds. In this context we generalize the Morse-Sard-Brown theorem, asserting that if $M$ is a connected smooth bounded-Fréchet-Finsler manifold endowed with a strengthened connection $\mathcal{K}$ and if $\xi$ is a smooth Lipschitz-Fredholm vector field on $M$ with respect to $\mathcal{K}$ which satisfies condition (CV). Then, for any smooth functional $l$ on $M$ which is associated to $\xi$, the set of the critical values of $l$ is of first category in $\mathbb{R}$. Therefore, the set of the regular values of $l$ is a residual Baire subset of $\mathbb{R}$.

1. Introduction

The notion of a Fredholm vector field on a Banach manifold $B$ with respect to a connection on $B$ was introduced by Tromba [12]. Such vector fields arise naturally in non-linear analysis from variational problems. There are geometrical objects such as harmonic maps, geodesics and minimal surfaces which arise as the zeros of a Fredholm vector field. Therefore, it would be valuable to study the critical points of functionals which are associated to Fredholm vector fields. In [13], Tromba proved the Morse-Sard-Brown theorem for this type of functionals in the case of Banach manifolds. Such a theorem would have applications to problems in the calculus of variations in the large such as Morse theory [11] and index theory [12].

The purpose of this paper is to extend the theorem of Tromba ([13 Theorem 1 ¨(MSB)]) to a new class of generalized Fréchet manifolds, the so-called bounded Fréchet manifolds, which was introduced in [8]. Such spaces arise in geometry and physical field theory and have many desirable properties. For instance, the space of all smooth sections of a fiber bundle (over closed or non-compact manifolds), which is the foremost example of infinite dimensional manifolds, has the structure of a bounded Fréchet manifold, see [8 Theorem 3.34]. The idea to introduce this category of manifolds was to overcome some permanent difficulties (i.e. problems of intrinsic nature) in the theory of Fréchet spaces. For example, the lack of a non-trivial topological group structure on the general linear group of a Fréchet space. As for the importance of bounded Fréchet manifolds, we refer to [4], [3] and [8].

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Essentially, to define the index of Fredholm vector fields we need the stability of Fredholm operators under small perturbation but this is unobtainable in the case of proper Fréchet spaces (non-normable spaces) in general, see [1]. Also, we need a subtle notion of a connection via a connection map but (because of the aforementioned problem) such a connection can not be constructed for Fréchet manifolds in general (cf. [2]). However, in the case of bounded Fréchet manifolds under the global Lipschitz assumption on Fredholm operators, the stability of Lipschitz-Fredholm operators was established in [4]. In addition, the notion of a connection via a connection map was defined in [3]. By using these results, we introduce the notion of a Lipschitz-Fredholm vector field in Section 3. With regard to a kind of compactness assumption (condition (WCV)), which one needs to impose on vector fields, we will be interested in manifolds which admit a Finsler structure. We then define Finsler structures for bounded Fréchet manifolds in Section 4. Finally, after we explained all subsequent portions for proving the Morse-Sard-Brown theorem, we formulate the theorem in the setting of Finsler manifolds in Section 5. A key point in the proof of the theorem is Proposition 5.2 which in its simplest form says that a Lipschitz-Fredholm vector field \( \xi \) near origin locally has a representation of the form \( \xi(u, v) = (u, \eta(u, v)) \), where \( \eta \) is some smooth map. Indeed, this is a consequence of the inverse function theorem (Theorem 5.1). One of the most important advantage of the category of bounded Fréchet manifold is the availability of the inverse function theorem in the sense of Nash and Moser (see [8]).

Morse theory and index theories for Fréchet manifolds have not been developed. Nevertheless, our approach provides some essential tools (such as connection maps, covariant derivatives, Finsler structures) which would create a proper framework for these theories.

2. Preliminaries

In this section we summarize all the necessary preliminary material that we need for a self contained presentation of the paper. We shall work in the category of smooth manifolds and bundles. We refer to [3] for the basic geometry of bounded Fréchet manifolds.

A Fréchet space \((F, d)\) is a complete metrizable locally convex space whose topology is defined by a complete translational-invariant metric \(d\). A metric with absolutely convex balls will be called a standard metric. Note that every Fréchet space admits a standard metric which defines its topology: If \(\alpha_n\) is an arbitrary sequence of positive real numbers converging to zero and if \(\rho_n\) is any sequence of continuous seminorms defining the topology of \(F\). Then

\[
d_{\alpha, \rho}(e, f) := \sup_{n \in \mathbb{N}} \alpha_n \frac{\rho_n(e - f)}{1 + \rho_n(e - f)}
\]  

(2.1)
is a metric on \( F \) with the desired properties. We shall always define the topology of Fréchet spaces with this type of metrics. Let \( (E, g) \) be another Fréchet space and let \( \mathcal{L}_{g,d}(E, F) \) be the set of all linear maps \( L : E \to F \) such that

\[
\text{Lip}(L)_{g,d} := \sup_{x \in E \setminus \{0\}} \frac{d(L(x),0)}{g(x,0)} < \infty.
\]

We abbreviate \( \mathcal{L}_g(E) := \mathcal{L}_{g,g}(E, E) \) and write \( \text{Lip}(L)_g = \text{Lip}(L)_{g,g} \) for \( L \in \mathcal{L}_g(E) \). The metric \( D_{g,d} \) defined by

\[
D_{g,d} : \mathcal{L}_{g,d}(E, F) \times \mathcal{L}_{g,d}(E, F) \to [0, \infty), \quad (L, H) \mapsto \text{Lip}(L - H)_{g,d},
\]

is a translational-invariant metric on \( \mathcal{L}_{d,g}(E, F) \) turning it into an Abelian topological group (see [6, Remark 1.9]). The latter is not a topological vector space in general, but a locally convex vector group with absolutely convex balls. The topology on \( \mathcal{L}_{d,g}(E, F) \) will always be defined by the metric \( D_{g,d} \). We shall always equip the product of any finite number \( k \) of Fréchet spaces \( (F_i, d_i), 1 \leq i \leq k \), with the maximum metric

\[
d_{\max}((x_1, \ldots, x_k), (y_1, \ldots, y_k)) := \max_{1 \leq i \leq k} d_i(x_i, y_i).
\]

Let \( E, F \) be Fréchet spaces, \( U \) an open subset of \( E \), and \( P : U \to F \) a continuous map. Let \( CL(E, F) \) be the space of all continuous linear maps from \( E \) to \( F \) topologized by the compact-open topology. We say \( P \) is differentiable at the point \( p \in U \) if there exists a linear map \( dP(p) : E \to F \) with

\[
d P(p)h = \lim_{t \to 0} \frac{P(p + th) - P(p)}{t},
\]

for all \( h \in E \). If \( P \) is differentiable at all points \( p \in U \), if \( dP(p) : U \to CL(E, F) \) is continuous for all \( p \in U \) and if the induced map \( P' : U \times E \to F, (u, h) \mapsto dP(u)h \) is continuous in the product topology, then we say that \( P \) is Keller-differentiable. We define \( P^{(k+1)} : U \times E^{k+1} \to F \) inductively by

\[
P^{(k+1)}(u, f_1, ..., f_k+1) = \lim_{t \to 0} \frac{P^{(k)}(u + tf_{k+1})(f_1, ..., f_k) - P^{(k)}(u)(f_1, ..., f_k)}{t}.
\]

If \( P \) is Keller-differentiable, \( dP(p) \in \mathcal{L}_{d,g}(E, F) \) for all \( p \in U \), and the induced map \( dP(p) : U \to \mathcal{L}_{d,g}(E, F) \) is continuous, then \( P \) is called \( b \)-differentiable. We say \( P \) is \( MC^0 \) and write \( P^0 = P \) if it is continuous. We say \( P \) is an \( MC^1 \) and write \( P^{(1)} = P' \) if it is \( b \)-differentiable. Let \( \mathcal{L}_{d,g}(E,F)_0 \) be the connected component of \( \mathcal{L}_{d,g}(E,F) \) containing the zero map. If \( P \) is \( b \)-differentiable and if \( V \subseteq U \) is a connected open neighborhood of \( x_0 \in U \), then \( P'(V) \) is connected and hence contained in the connected component \( P'(x_0) + \mathcal{L}_{d,g}(E,F)_0 \) of \( P'(x_0) \) in \( \mathcal{L}_{d,g}(E,F) \). Thus, \( P'|V - P'(x_0) : V \to \mathcal{L}_{d,g}(E,F)_0 \) is again a map between subsets of Fréchet spaces. This enables a recursive definition: If \( P \) is \( MC^1 \) and \( V \) can be chosen
for each \(x_0 \in U\) such that \(P' |_V - P'(x_0) : V \to \mathcal{L}_{d,g}(E, F)_0\) is \(MC^{k-1}\), then \(P\) is called an \(MC^k\)-map. We make a piecewise definition of \(P^{(k)}\) by \(P^{(k)} |_V := (P' |_V - P'(x_0))^{(k-1)}\) for \(x_0\) and \(V\) as before. The map \(P\) is \(MC^\infty\) if it is \(MC^k\) for all \(k \in \mathbb{N}_0\). We shall denote the derivative of \(P\) at \(p\) by \(D P(p)\).

A bounded Fréchet manifold is a Hausdorff second countable topological space with an atlas of coordinate charts taking their values in Fréchet spaces such that the coordinate transition functions are all \(MC^\infty\)-maps.

3. Lipschitz-Fredholm vector fields

Throughout the paper we assume that \((F, d)\) is a Fréchet space and \(M\) is a bounded Fréchet manifold modelled on \(F\). Let \(\{\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}\) be a compatible atlas for \(M\). The latter gives rise to a trivializing atlas \(\{(\pi^{-1}_M(U_\alpha), \psi_\alpha)\}_{\alpha \in \mathcal{A}}\) on the tangent bundle \(\pi_M : TM \to M\), with

\[
\psi_\alpha : \pi^{-1}_M(U_\alpha) \to \varphi_\alpha(U_\alpha) \times F, \quad j^1_p(f) \mapsto (\varphi_\alpha(p), (\varphi_\alpha \circ f)'(0)),
\]

where \(j^1_p(f)\) stands for the 1-jet of an \(MC^\infty\)-mapping \(f : \mathbb{R} \to M\) that sends zero to \(p \in M\). Let \(N\) be another bounded Fréchet manifold and \(h : M \to N\) an \(MC^k\)-map. The tangent map \(Th : TM \to TN\) is defined by \(Th(j^1_p(f)) = j^1_{h(p)}(h \circ f)\). Let \(\Pi_{TM} : T(TM) \to TM\) be an ordinary tangent bundle over \(TM\) with the corresponding trivializing atlas \(\{((\Pi^{-1}_{TM})(\pi^{-1}_M(U_\alpha)), \widetilde{\psi}_\alpha)\}_{\alpha \in \mathcal{A}}\). A strengthened connection map on the tangent bundle \(TM\) (possible also for general vector bundles) was defined in [3]. A strengthened connection map for \(TM\) is a map \(\mathcal{K} : T(TM) \to TM\), which is fully determined by its local form:

\[
\mathcal{K}_\alpha := \varphi_\alpha \circ \mathcal{K} \circ (\widetilde{\Phi}_\alpha)^{-1},
\]

\[
\varphi_\alpha(U_\alpha) \times F \times F \times F \to \varphi_\alpha(U_\alpha) \times F, \quad \mathbb{K}_\alpha = (f, g, h, k) = (f, k + \tau_\alpha(f, g)h),
\]

for a family of mappings

\[
\tau_\alpha : \varphi_\alpha(U_\alpha) \times F \to \mathcal{L}_d(F)^\times.
\]

Here \(\mathcal{L}_d(F)^\times\) is a subset of \(\mathcal{L}_d(F)\) consists of invertible mappings. The mapping \(\tau_\alpha\) is \(MC^{k-1}\) in the sense that the map

\[
\widehat{\tau}_\alpha : (\varphi_\alpha(U_\alpha) \times F) \times F \to F \times F, \quad (x, y, h) \mapsto (\tau_\alpha(x, y)h, \tau^{-1}_\alpha(x, y)(h))
\]

is \(MC^{k-1}\). It follows of course that \(\mathbb{K}\) is of class \(MC^{k-1}\). A strengthened connection on \(M\) is a strengthened connection map on the tangent bundle \(\pi_M : TM \to M\). A strengthened connection \(\mathcal{K}\) is linear if and only if it is linear on the fibers of the tangent map. Locally \(T\pi\) is the map \(U_\alpha \times F \times F \times F \to U_\alpha \times F\) defined by \(T\pi(f, \xi, h, \gamma) = (f, h)\), hence locally its fibers are the spaces \(\{f\} \times F \times \{h\} \times F\). Therefore, \(\mathcal{K}\) is linear on these fibers if and only if
the maps \((g, k) \mapsto k + \tau_\alpha(f, g)h\) are linear, and this means that the mappings \(\tau_\alpha\) need to be linear with respect to the second variable.

**Remark 3.1.** If \(\varphi : U \subset M \to F\) is a local coordinate chart for \(M\), then a vector field \(\xi\) on \(M\) induces a vector field \(\xi\) on \(F\) called the local representative of \(\xi\) by the formula \(\xi(x) = T\varphi \cdot \xi(\varphi^{-1}(x))\). We shall frequently use \(\xi\) itself to denote this local representation.

In the following we adapt the Elliason’s definition of covariant derivative [5].

**Definition 3.2.** Let \(\pi_M : TM \to M\) be the tangent bundle over \(M\). Let \(N\) be a bounded Fréchet manifold modelled on \(F\), \(\lambda : N \to M\) a Fréchet vector bundle with fiber \(F\), and \(K_\lambda\) a strengthened connection map on \(TN\). If \(\xi : M \to N\) is a smooth section of \(\lambda\), we define the covariant derivative of \(\xi\) at \(p \in M\) to be the bundle map \(\nabla\xi : TM \to N\) given by

\[
\nabla\xi(p) = K_\lambda \circ T_p \xi, \quad T_p \xi = T\xi|_{T_p M}.
\]

In a local coordinate chart \((U, \Phi)\) it becomes

\[
\nabla\xi(x) \cdot y = D\xi(x) \cdot y + \tau_\Phi(x, \xi(x))y.
\]

Where \(\tau_\Phi\) is the component for \(K_\lambda\) with respect to the chart \((U, \Phi)\).

The covariant derivative \(\nabla\xi(p)\) is a linear map from the tangent space \(T_p M\) to \(F_p := \lambda^{-1}(p)\). This is because it is the combination of the tangent map \(T_p \xi\) that maps \(T_p M\) linearly into \(T\xi(p)N\) with \(K_\lambda\) which is a linear map from \(T\xi(p)N\) to \(F_p\).

**Definition 3.3** ([4], Definition 3.2). Let \((F, d)\) and \((E, g)\) be Fréchet spaces. A map \(\varphi \in \mathcal{L}_{g,d}(E, F)\) is called Lipschitz-Fredholm operator if it satisfies the following conditions:

1. The image of \(\varphi\) is closed.
2. The dimension of the kernel of \(\varphi\) is finite.
3. The co-dimension of the image of \(\varphi\) is finite.

We denote by \(\mathcal{LF}(E, F)\) the set of all Lipschitz-Fredholm operators from \(E\) into \(F\). For \(\varphi \in \mathcal{LF}(E, F)\) we define the index of \(\varphi\) as follows:

\[
\text{Ind } \varphi = \text{dim ker } \varphi - \text{codim Img } \varphi.
\]

**Theorem 3.4** ([4], Theorem 3.2). \(\mathcal{LF}(E, F)\) is open in \(\mathcal{L}_{g,d}(E, F)\) with respect to the topology defined by the Metric \((2.2)\). Furthermore, the function \(T \to \text{Ind } T\) is continuous on \(\mathcal{LF}(E, F)\), hence constant on connected components of \(\mathcal{LF}(E, F)\).

Now we define a Lipschitz-Fredholm vector field on \(M\) with respect to a connection on \(M\).
Definition 3.5. A smooth vector field $\xi : M \to TM$ is called Lipschitz-Fredholm with respect to a strengthened connection $\mathcal{K} : T(TM) \to TM$ if for each $p \in M$, $\nabla \xi(p) : T_pM \to T_pM$ is a linear Lipschitz-Fredholm operator. The index of $\xi$ at $p$ is defined to be the index of $\nabla \xi(p)$, that is

$$\text{Ind} \nabla \xi(p) = \dim \ker \nabla \xi(p) - \dim \text{Im} \nabla \xi(p).$$

By Theorem 3.4, if $M$ is connected the index is independent of the choice of $p$ and the common integer is called the index of $\xi$, while if $M$ is not connected the index is constant on components and we shall require it to be the same on all components of $M$.

Remark 3.6. Note that the notion of Lipschitz-Fredholm vector field depends on the choice of $\mathcal{K}$. If $p$ is a zero of $\xi$, $\xi(p) = 0$, then by Definition 3.2 we have $\nabla \xi(p) = \nabla _p \xi$ and hence the covariant derivative at $p$ does not depend on $\mathcal{K}$. In this case, the derivative of $\xi$ at $p$, $D \xi(p)$, can be viewed as a linear endomorphism from $T_pM$ into itself.

4. Finsler structures

A Finsler structure on the bounded Fréchet manifold $M$ is defined in the same way as in the case of Fréchet manifolds (see [1] for the definition of Fréchet-Finsler manifolds). However, we need a countable family of seminorms on its Fréchet model space $F$ which defines the topology of $F$. As mentioned in Preliminaries, we always define the topology of a Fréchet space by a metric with absolutely convex balls. One reason for this consideration is that a metric with this property can give us back original seminorms. More precisely:

Remark 4.1 ([8], Theorem 3.4). Assume that $(E, g)$ is a Fréchet space and $g$ is a metric with absolutely convex balls. Let $B^g_{\frac{1}{2}} (0) := \{ y \in E \mid g(y, 0) < \frac{1}{2} \}$, and suppose $U_i$’s, $i \in \mathbb{N}$, are convex subsets of $B^g_{\frac{1}{2}} (0)$. Define the Minkowski functionals

$$\| v \|_i := \inf \{ \epsilon > 0 \mid \epsilon \in \mathbb{R}, \frac{1}{\epsilon} \cdot v \in U_i \}.$$

These Minkowski functionals are continuous seminorms on $E$. A collection $\{ \| v \|_i \}_{i \in \mathbb{N}}$ of these seminorms gives the topology of $E$.

Definition 4.2. Let $F$ be as before. Let $X$ be a topological space and $V = X \times F$ the trivial bundle with fiber $F$ over $X$. A Finsler structure for $V$ is a collection of functions

$$\| \cdot \|_n : V \to \mathbb{R}^+, \quad n \in \mathbb{N},$$

such that

1. For $b \in X$ fixed, $\| (b, x) \|_n = \| x \|_b^n$ is a collection of seminorms on $F$ which gives the topology of $F$. 

(2) Given $K > 1$ and $x_0 \in X$, there exits a neighborhood $\mathcal{U}$ of $x_0$ such that
\[
\frac{1}{K} \| f \|_x^n \leq \| f \|_x^n \leq K \| f \|_x^n
\]
for all $x \in \mathcal{U}$, $n \in \mathbb{N}$, $f \in F$.

Let $\pi_M : TM \to M$ be the tangent bundle and let $\| \cdot \|_n : TM \to \mathbb{R}^+$ be a collection of functions, $n \in \mathbb{N}$. We say \{\| \cdot \|_n\}_{n \in \mathbb{N}} is a Finsler structure for $TM$ if for a given $m_0 \in M$ and any open neighborhood $U$ of $m_0$ which trivializes the tangent bundle $TM$, i.e.
\[
\psi : \pi_M^{-1}(U) \approx U \times (F_{m_0} := \pi_M^{-1}(m_0)),
\]
\{\| \cdot \|_n \circ \psi^{-1}\}_{n \in \mathbb{N}} is a Finsler structure for $U \times F_{m_0}$.

**Definition 4.3.** A bounded-Frèchet-Finsler manifold is a bounded Frèchet manifold together with a Finsler structure on its tangent bundle.

**Proposition 4.4.** Let $N$ be a paracompact bounded Frèchet manifold modelled on a Frèchet space $(E, g)$. If all seminorms $\| \cdot \|_i$, $i \in \mathbb{N}$, (which are defined as in Remark 4.1) are smooth maps on $E \setminus \{0\}$, then $N$ admits a partition of unity. Moreover, $N$ admits a Finsler structure.

**Proof.** See [1], Propositions 3 and 4. \hfill \Box

If \{\| \cdot \|_n\}_{n \in \mathbb{N}} is a Finsler structure for $M$ then eventually we can obtain a graded Finsler structure, denoted by $(\| \cdot \|_n)_{n \in \mathbb{N}}$, for $M$. Let $(\| \cdot \|_n)_{n \in \mathbb{N}}$ be a graded Finsler structure for $M$. We define the length of piecewise $MC^1$-curve $\gamma : [a, b] \to M$ by
\[
L^n(\gamma) = \int_a^b \| \gamma'(t) \|_{\gamma(t)}^n \, dt.
\]
On each connected component of $M$, the distance is defined by
\[
\rho^n(x, y) = \inf_{\gamma} L^n(\gamma),
\]
where infimum is taken over all continuous piecewise $MC^1$-curve connecting $x$ to $y$. Thus, we obtain an increasing sequence of pseudometrics $\rho^n(x, y)$ and define the distance $\rho$ by
\[
\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho^n(x, y)}{1 + \rho^n(x, y)}.
\]

**Lemma 4.5 (1, Lemma 2).** A collection $\{\sigma^i\}_{i \in \mathbb{N}}$ of pseudometrics on $F$ defines a unique topology $\mathcal{T}$ such that for every sequence $(x_n)_{n \in \mathbb{N}} \subset F$, we have $x_n \to x$ in topology $\mathcal{T}$ if and only if $\sigma^i(x_n, x) \to 0$, for all $i \in \mathbb{N}$. The topology is Hausdorff if and only if $x = y$ when all $\sigma^i(x, y) = 0$. In addition,
\[
\sigma(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\sigma^n(x, y)}{1 + \sigma^n(x, y)}
\]
is a pseudometric on $F$ defines the same topology.
With the aid of this lemma, the proof of the following theorem is close to the usual proof given for Banach manifolds (cf. [10]).

**Theorem 4.6.** Suppose $M$ is connected and endowed with a Finsler structure $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$. Then the distance $\rho$ defined by (4.2) is a metric for $M$. Furthermore, the topology induced by this metric coincides with the original topology of $M$.

**Proof.** The distance $\rho$ is pseudometric by Lemma 4.5. We prove that $\rho(x_0, y_0) > 0$ if $x_0 \neq y_0$. Let $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$ be a collection of all seminorms on $F$ (which are defined as in Remark 4.1). Given $x_0 \in M$, let $\varphi : U \to F$ be a chart for $M$ with $x_0 \in U$ and $\varphi(x_0) = u_0$. Let $y_0 \in M$ and $\gamma(t)$ an $MC^1$-curve $\gamma : [a, b] \to M$ connecting $x_0$ to $y_0$. Let $B_r(u_0)$ be a ball with center $u_0$ and radius $r > 0$. Choose $r$ small enough so that $U := \varphi^{-1}(B_r(u_0)) \subset U$ and for a given $K > 1$

$$
\frac{1}{K} \| f \|_{L^\infty} \leq \| f \|_x \leq K \| f \|_{L^0}
$$

for all $x \in U$, $n \in \mathbb{N}$, $f \in F$. Let $I = [a, b]$ and $\mu(t) = \varphi \circ \gamma(t)$. If $\gamma(I) \subset U$, then let $\beta = b$. Otherwise, let $\beta$ be the first $t > 0$ such that $\| \mu(t) - u_0 \|_n = r$ for all $n \in \mathbb{N}$. Then, since for $x \in U$ the map $\phi(x) : T_x M \to F$ given by $j_x \mapsto \varphi(x)$ is a homeomorphism it follows that for all $n \in \mathbb{N}$

$$
\int_a^b \| \gamma'(t) \|_{\gamma(t)}^n \, dt \geq \frac{1}{K} \int_a^\beta \| \phi^{-1}(x) \circ \mu'(t) \|_{x_0}^n \, dt \geq k_1 \int_a^\beta \| \mu'(t) \|_n \, dt
$$

$$
\geq k_1 \int_a^\beta \mu'(t) \, dt ||_n = k_1 \| \mu(\beta) - \mu(a) \|_n \quad \text{for some } k_1 > 0.
$$

(The last inequality follows from [7, Theorem 2.1.1]). Thereby, if $x_0 \neq y_0$ then $\rho^n(x_0, y_0) > 0$ and hence $\rho(x_0, y_0) > 0$. Now we prove that the topology induced by $\rho$ coincides with the topology of $M$. By virtue of Lemma 4.5, we only need to show that $\{\rho^n\}_{n \in \mathbb{N}}$ induces the topology which is consistent with the topology of $M$. If $x_i \to x_0$ in $M$ then eventually $x_i \in U$.

Define $\lambda_i : [0, 1] \to U$, an $MC^1$-curve connecting $x_0$ to $x_i$, by $t \varphi(x_i)$. Then, for all $n \in \mathbb{N}$

$$
\rho^n(x_i, x_0) \leq L^n(\lambda_i) = \int_0^1 \| \lambda'_i \|_{\lambda_i(t)}^n \, dt = \int_0^1 \| \varphi(x_i) \|_{\varphi(x_i)}^n \, dt
$$

$$
\leq K \int_0^1 \| \varphi(x_i) \|_{x_0}^n \, dt = K \| \varphi(x_i) \|_n
$$

But $\varphi(x_i) \to 0$ as $x_i \to x_0$, thereby $\rho^n(x_i, x_0) \to 0$ for all $n \in \mathbb{N}$. Conversely, if for all $n \in \mathbb{N}$, $\rho^n(x_i, x_0) \to 0$ then eventually we can choose $r$ small enough so that $x_i \in U$. Then, for all $n \in \mathbb{N}$ we have $\| \varphi(x_i) \|_{x_0} \leq K \rho^n(x_i, x_0)$ so $\| \varphi(x_i) \|_{x_0} \to 0$ in $T_{x_0} M$, whence $\varphi(x_i) \to 0$. Therefore, $x_i \to x_0$ in $U$ and hence in $M$. \qed

The metric $\rho$ is called the Finsler metric for $M$ and it is bounded by 1.
5. Morse-Sard-Brown Theorem

In this section we prove the Morse-Sard-Brown theorem for functionals on bounded-Frèchet-Finsler manifolds. The proof relies on the following inverse function theorem.

**Theorem 5.1** ([6], Proposition 7.1. Inverse Function Theorem for $MC^k$-maps). Let $(E, g)$ be a Frèchet space with standard metric $g$. Let $U \subset E$ be open, $x_0 \in U$ and $f : U \subset E \to E$ an $MC^k$-map, $k \geq 1$. If $f'(x_0) \in \text{Aut}(E)$, then there exists an open neighborhood $V \subseteq U$ of $x_0$ such that $f(V)$ is open in $E$ and $f|_V : V \to f(V)$ is an $MC^k$-diffeomorphism.

The following consequence of this theorem is an important technical tool.

**Proposition 5.2** (Local representation). Let $F_1, F_2$ be Frèchet spaces and $U$ an open subset of $F_1 \times F_2$ with $(0,0) \in U$. Let $E_2$ be another Frèchet space and $\phi : U \to F_1 \times E_2$ an $MC^\infty$-map with $\phi(0,0) = (0,0)$. Assume that the partial derivative $D_1 \phi(0,0) : F_1 \to F_1$ with respect to the first variable is linear isomorphism. Then there exists a local $MC^\infty$-diffeomorphism $\psi$ from an open neighborhood $V_1 \times V_2 \subseteq F_1 \times F_2$ of $(0,0)$ onto an open neighborhood of $(0,0)$ contained in $U$ such that $\phi \circ \psi(u,v) = (u, \mu(u,v))$, where $\mu : V_1 \times V_2 \to E_2$ is an $MC^\infty$-mapping.

**Proof.** Let $\phi = \phi_1 \times \phi_2$, where $\phi_1 : U \to F_1$ and $\phi_2 : U \to E_2$. By assumption we have $D_1 \phi_1(0,0) = D_1 \phi(0,0)|_{F_1} \in \text{Iso}(F_1, F_1)$. Define the map
\[
g : U \subset F_1 \times F_2 \to F_1 \times E_2, \\
g(u_1, u_2) = (\phi_1(u_1, u_2), u_2)
\]
locally at $(0,0)$. Therefore, for all $u = (u_1, u_2) \in U$, $f_1 \in F_1$, $f_2 \in F_2$ we have
\[
Dg(u) \cdot (f_1, f_2) = \begin{pmatrix} D_1 \phi_1(u) & D_2 \phi_1(u) \\ 0 & \text{Id}_{E_2} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}
\]
and hence $Dg(u)$ is a linear isomorphism at $(0,0)$. By the inverse function theorem, there are open sets $U'$ and $V = V_1 \times V_2$ and an $MC^\infty$-diffeomorphism $\Psi : V \to U'$ such that $(0,0) \in U' \subset U$, $g(0,0) \in V \subset F_1 \times E_2$, and $\Psi^{-1} = g|_{U'}$. Hence if $(u, v) \in V$, then
\[
(u, v) = (g \circ \Psi)(u, v) = g(\Psi_1(u, v), \Psi_2(u, v)) = (\phi_1 \circ \Psi_1(u, v), \Psi_2(u, v)),
\]
where $\Psi = \Psi_1 \times \Psi_2$. This shows that $\Psi_2(v, v) = v$ and $(\phi_1 \circ \Psi)(u, v) = u$. Define $\eta = \phi_2 \circ \Psi$, then
\[
(\phi \circ \Psi)(u, v) = (\phi_1 \circ \Psi(u, v), \phi_2 \circ \Psi(u, v)) = (u, \eta(u, v)).
\]
This completes the proof. □

In the sequel, we assume that $M$ is connected and endowed with a Finsler structure $\{ \| \cdot \|^n \}_{n \in \mathbb{N}}$ and a Finsler metric $\rho$. 

Definition 5.3. Let \( l : M \to \mathbb{R} \) be an \( MC^\infty \)-functional and \( \xi : TM \to M \) a smooth vector field. By saying that \( l \) and \( \xi \) are associated we mean \( Dl(p) = 0 \) if and only if \( \xi(p) = 0 \). A point \( p \in M \) is called a critical point for \( l \) if \( Dl(p) = 0 \). The corresponding value \( l(p) \) is called a critical value. Values other than critical are called regular values. The set of all critical points of \( l \) is denoted by \( \text{Crit}_l \).

The following is our version of the compactness condition due to Tromba [11].

Condition (CV). Let \((m_i)_{i \in \mathbb{N}} \) be a sequence in \( M \). We say that a vector field \( \xi : M \to TM \) satisfies condition (CV) if \( \| \xi(m_i) \|^n \to 0 \) for all \( n \in \mathbb{N} \), then \((m_i)_{i \in \mathbb{N}} \) has a convergent subsequence.

It follows immediately from the definition that the set of zeros of a vector field \( V \) that satisfies the condition (CV) is compact.

A subset \( G \) of a Fréchet space \( E \) is called topologically complemented or split in \( E \) if there is another subspace \( H \) of \( E \) such that \( E \) is homeomorphic to the topological direct sum \( G \oplus H \). In this case we call \( H \) a topological complement of \( G \) in \( E \).

We will need the following facts:

Theorem 5.4 ([8], Theorem 3.14). Let \( E \) be a Fréchet space. Then

1. Every finite-dimensional subspace of \( E \) is closed.
2. Every closed subspace \( G \subset E \) with \( \text{codim}(G) = \dim(E/G) < \infty \) is topologically complemented in \( E \).
3. Every finite-dimensional subspace of \( E \) is topologically complemented.
4. Every linear isomorphism between the direct sum of two closed subspaces and \( E \), \( G \oplus H \to E \), is a homeomorphism.

The proof of the Morse-Srad-Brown theorem requires Proposition 5.2 and Theorem 5.4. Except the arguments which involve these results and the Finslerian nature of manifolds, the rest of arguments are similar to that of Banach manifolds case, see [13, Theorem 1].

Theorem 5.5 (Morse-Sard-Brown Theorem). Assume that \((M, \rho)\) is endowed with a strengthened connection \( K \). Let \( \xi \) be a smooth Lipschitz-Fredholm vector field on \( M \) with respect to \( K \) which satisfies condition (CV). Then, for any \( MC^\infty \)-functional \( l \) on \( M \) which is associated to \( \xi \), the set of its critical values \( l(\text{Crit}_l) \) is of first category in \( \mathbb{R} \). Therefore, the set of the regular values of \( l \) is a residual Baire subset of \( \mathbb{R} \).

Proof. We can assume \( M = \bigcup_{i \in \mathbb{N}} M_i \), where all the \( M_i \)'s are closed balls of radius \( i \) about some fixed point \( m_0 \in M \) with respect to the Finsler metric \( \rho \). Thus to conclude the proof
it suffices to show that the image \( l(C_B) \) of the set \( C_B \) of the zeros of \( \xi \) in some closed set \( B \) is compact without interior.

Let \( B \) be a closed set and \( C_B \) as before. If \( p \in C_B \) then eventually \( \xi(p) = 0 \). Since \( C_B \) is compact we only need to show that for a neighborhood \( U \) of \( p \), \( l(C_B \cap \overline{U}) \) is compact without interior. In other words, we can work locally. Therefore, we may assume without loss of generality that \( p = 0 \in F \) and \( \xi, l \) are defined locally on an open neighborhood of \( p \). An endomorphism \( D\xi(p) : F \to F \) is a Lipschitz-Fredholm operator because \( \xi \) is a Lipschitz-Fredholm vector field (see Remark 3.6). Thereby, in the light of Theorem 5.4 it has the split image \( F_1 \) with the topological complement \( F_2 \) and the split kernel \( E_2 \) with the topological complement \( E_1 \). Moreover, \( D\xi(p) \) maps \( E_1 \) isomorphically onto \( F_1 \) so we can identify \( F_1 \) with \( E_1 \). Then by Proposition 5.2 there is an open neighborhood \( U \subset E_1 \times E_2 \) of \( p \) such that \( \xi(u, v) = (u, \eta(u, v)) \) for all \( (u, v) \in U \), where \( \eta : U \to F_2 \) is an \( MC^\infty \)-map. Thus, if \( \xi(u, v) = 0 = (u, \eta(u, v)) \) then \( u = 0 \). Therefore, in this local representation, the zeros of \( \xi \) (and hence critical points of \( l \)) in \( \overline{U} \) are in \( \overline{U}_1 := \overline{U} \cap \{0\} \times E_2 \). The restriction of \( l, \overline{l}_{U_1} : \overline{U}_1 \to \mathbb{R} \), is again \( MC^\infty \) and \( C_B \cap \overline{U} = C_B \cap \overline{U}_1 \) so \( l(C_B \cap \overline{U}) = l(C_B \cap \overline{U}_1) \).

We have for some constant \( k \in \mathbb{N} \), \( \dim \overline{U}_1 = \dim E_2 = k \) because \( \xi(p) \) is a Lipschitz-Fredholm operator and \( E_2 \) is its kernel. Thus, by the classical Sard’s theorem \( l(C_B \cap \overline{U}_1) \) has measure zero (note that \( MC^k \)-differentiability implies the usual \( C^k \)-differentiability for maps of finite dimensional manifolds). Therefore, since \( C_B \cap \overline{U}_1 \) is compact it follows that \( l(C_B \cap \overline{U}_1) \) is compact without interior and hence \( l(C_B \cap \overline{U}) \) is compact without interior. \( \square \)

**Remark 5.6.** From the proceeding proof we have that \( \dim F_2 = m \), where \( m \in \mathbb{N} \) is constant. Thus, the index of \( \xi \) is the following:

\[ \text{Ind} \xi = \dim E_2 - \dim F_2 = k - m. \]

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