Mutual entropy production in bipartite systems

Giovanni Diana and Massimiliano Esposito

Complex Systems and Statistical Mechanics, University of Luxembourg, L-1511 Luxembourg, Luxembourg
E-mail: g.diana.mail@gmail.com and massimiliano.esposito@uni.lu

Received 8 November 2013
Accepted for publication 20 February 2014
Published 17 April 2014

Online at stacks.iop.org/JSTAT/2014/P04010
doi:10.1088/1742-5468/2014/04/P04010

Abstract. It was recently shown by Barato et al (2013 Phys. Rev. E 87 042104) that the mutual information at the trajectory level of a bipartite Markovian system is not bounded by the entropy production. In the same way as Gaspard showed (2004 J. Stat. Phys. 117 599) that the entropy production is not directly related to the Shannon entropy at the trajectory level but is in fact equal to its difference from the so-called time-reversed Shannon entropy, we show in this paper that the difference between the mutual information and its time-reversed form is equal to the mutual entropy production (MEP), i.e. the difference between the full entropy production and that of the two marginal processes. Evaluation of the MEP is in general a difficult task due to non-Markovian effects. For bipartite systems, we provide closed expressions in various limiting regimes which we verify by numerical simulations.

Keywords: coarse-graining (theory), stochastic processes (theory), stationary states
1. Introduction

A major achievement of the last decade has been to establish a nonequilibrium thermodynamic description for small systems described by Markovian stochastic dynamics [1]–[6]. This theory, called stochastic thermodynamics, has close and remarkable connections with information theory. Various central quantities are expressed in terms of mathematical objects commonly used in information theory. The entropy of the system is given by its Shannon entropy [7, 8] and the entropy production is a relative entropy between the probability of forward and time-reversed trajectories [9]–[11]. In a nonequilibrium steady state, this latter can be expressed as a difference between the entropy rate [8, 12] associated with forward and time-reversed trajectories [13]–[15].

Beside these formal connections to information theory, the framework of stochastic thermodynamics has been used in the recent years to revisit the finite-time aspects of various problems involving information processing which were originally proposed for reversible transformations. Such problems include for example the thermodynamic description of systems subjected to feedback [16]–[32] (which systematizes the study of Maxwell demons and Szilard engines), Landauer’s principle [32]–[36], Bennet’s reversible computing and kinetic proofreading [6, 32], [37]–[44]. Experimental verifications of these results have also been proposed [45]–[49].

One of the important new contributions has been to establish a systematic thermodynamic description of measurement, feedback and erasure for systems controlled by external time-dependent forces [17, 18, 24, 25, 28, 34, 35]. The key quantity is the mutual information established as a result of the interaction between the system and the measurement apparatus. Unfortunately, in the case of autonomous systems performing some sensing tasks, such as electronic detectors [50]–[52] and biochemical cell receptors
[53, 54], these notions do not extend because the mutual information does not change in time. Such continuous measurement processes require a different treatment and no unified approach is currently available. Moreover, recent studies have considered the mutual information rate defined at the trajectory level and found that it is not bounded by, or related to, the entropy production [54, 55].

In this paper, we introduce the concept of mutual entropy production (MEP) and show that it can be expressed as the difference between the mutual information associated with the forward and time-reversed trajectories. This result can be put in parallel with the result obtained by Gaspard [13] showing that for Markov processes the entropy production is not immediately related to the information-theoretic entropy rate but can be expressed as the difference between the entropy rate associated with the forward and time-reversed trajectories. The MEP rate, as well as the mutual information rate, is in general difficult to calculate because coarse-graining introduces non-Markovian features [15, 55]. For bipartite systems, we find limiting cases where the MEP rate can be obtained analytically and expressed in terms of quantities previously studied in [4] and easy to calculate.

The plan of the paper is as follows. In section 2 we review the different ways to define entropy rates for general random processes. In section 3 we introduce the MEP and consider the limits where it can be obtained analytically. In section 4 we verify our predictions numerically.

2. Entropy and entropy production rates

We consider an arbitrary stationary random process of duration \( t \) observed in discrete time with a time step \( \tau \). Each realization of the process gives rise to a sequence \( Z_N = \{z_1, z_2, \ldots, z_N\} \) with a probability \( P(Z_N) \), where \( N = t/\tau \).

The Shannon entropy in the space of all possible trajectories and its corresponding time-reversed entropy are defined as

\[
H(N, \tau) \equiv -\langle \ln P(Z_N) \rangle, \\
H^R(N, \tau) \equiv -\langle \ln P(Z_N^R) \rangle,
\]

where \( Z_N^R \) denotes the time-reversed sequence of \( Z_N \) and the averages \( \langle \cdot \rangle \) are taken over the ensemble of sequences of length \( N \).

The entropy production is defined as the difference between the Shannon entropy and its corresponding time-reversed entropy,

\[
\Delta_iS(N, \tau) \equiv H^R(N, \tau) - H(N, \tau) \geq 0.
\]

It is always non-negative because, using (1) and (2), it can be expressed as a relative entropy (Kullback–Leibler divergence) between \( P(Z_N) \) and \( P(Z_N^R) \). It vanishes if and only if the probability of any trajectory coincides with the probability of its corresponding time-reversed trajectory, \( P(Z_N^R) = P(Z_N) \), i.e. when the dynamics satisfies detailed balance. We note that this definition of entropy production is mathematical and its relation to the thermodynamic notion of entropy production is difficult to assess in general. For Markovian processes, this connection can be made explicitly [13] (see also [10, 56, 57]).
Nevertheless, following [15, 58], we will use (3) as the definition of entropy production for an arbitrary random process. A closely related definition is also used for classical and quantum Hamiltonian systems [11, 59].

The Shannon entropy, as well as the entropy production, depends on both $N$ and $\tau$. We will now consider the rates associated with these quantities in the large $N$ limit and at fixed $\tau$. The entropy rates at fixed $\tau$ corresponding to (1) and (2) are defined as [8, 12]

$$\dot{H}(\tau) \equiv \lim_{N \to \infty} \frac{H(N, \tau)}{N\tau}, \quad (4)$$

$$\dot{H}^R(\tau) \equiv \lim_{N \to \infty} \frac{H^R(N, \tau)}{N\tau}. \quad (5)$$

The corresponding entropy production rate at fixed $\tau$ is thus defined as

$$\dot{S}_i(\tau) \equiv \lim_{N \to \infty} \frac{\Delta_i S(N, \tau)}{N\tau}, \quad (6)$$

and can be expressed in terms of the entropy rates (4) and (5),

$$\dot{S}_i(\tau) = \dot{H}^R(\tau) - \dot{H}(\tau). \quad (7)$$

For a Markovian process described by a transition rate matrix $W_{z'z}$ (characterizing the probability per unit time to jump from a state $z$ to a state $z'$), the entropy rates (4) and (5) can be expressed as [13]

$$\dot{H}(\tau) = - \sum_{z \neq z'} W_{zz'} p(z') \ln W_{zz'} + B(\tau), \quad (8)$$

$$\dot{H}^R(\tau) = - \sum_{z \neq z'} W_{zz'} p(z') \ln W_{z'z} + B(\tau), \quad (9)$$

where $B(\tau) \equiv - \sum_{z \neq z'} W_{zz'} p(z') \ln \tau/e + O(\tau)$. The symbol $\sum_{z \neq z'}$ denotes a summation over $z$ and $z'$ such that $z \neq z'$. The crucial observation made in [13] is that while (8) and (9) depend on $\tau$ via $B(\tau)$, the entropy rate $\dot{S}_i$ does not and coincides with the well known entropy production rate [57, 60, 61]

$$\sigma \equiv \sum_{z \neq z'} W_{zz'} p(z') \ln \frac{W_{zz'} p(z')}{W_{z'z} p(z)} \geq 0. \quad (10)$$

We now turn back to the general $N$- and $\tau$-dependent definition of Shannon entropy and entropy production (1)–(3). Instead of considering rates associated with the large $N$ limit at fixed $\tau$, we now take the large $N$ limit at fixed duration of the process $t = N\tau$. This limiting procedure provides the continuous-time limit of the entropy production,

$$\Delta_i S(t) \equiv \lim_{N \to \infty} \Delta_i S(N, t/N). \quad (11)$$

Taking the time derivative of $\Delta_i S(t)$ is an alternative way compared to (6) to define an entropy production rate,

$$\dot{S}_i(t) \equiv \frac{d\Delta_i S(t)}{dt}. \quad (12)$$

doi:10.1088/1742-5468/2014/04/P04010
In general this rate depends on \( t \). Since \( \Delta_i S_i(t) \bigg|_{t=0} = 0 \) and assuming that \( \dot{S}_i(t) \) eventually reaches a constant asymptotic value, the short and long time limits of \( \dot{S}_i(t) \) can be expressed as

\[
\dot{S}_i(0) = \lim_{t \to 0} \frac{\Delta_i S_i(t)}{t},
\]

\[
\dot{S}_i(\infty) = \lim_{t \to \infty} \frac{\Delta_i S_i(t)}{t}.
\]

A remarkable feature of Markovian processes is that \( \dot{S}_i(t) \) becomes independent of \( t \) and equal to (10). Therefore, for stationary Markov processes the rates (6) and (12) coincide with (10), namely

\[
\dot{S}_i = \dot{\tilde{S}}_i = \sigma_Z.
\]

However, for general non-Markovian processes these entropy production rates do not necessarily coincide.

3. Mutual information and mutual entropy production

We now consider a Markovian random process \( Z = (X, Y) \) producing sequences in a space spanned by the pair of variables \( z_i = (x_i, y_i) \). The random processes \( X \) and \( Y \) are a coarse-grained description of the joint Markovian process \( Z \) and are thus in general non-Markovian.

A measure of the correlation between \( X \) and \( Y \) is the non-negative mutual information defined as [8]

\[
I(N, \tau) \equiv H_X(N, \tau) + H_Y(N, \tau) - H_{XY}(N, \tau).
\]

In analogy to what was done in the preceding section for the Shannon entropy, we introduce the time-reversed mutual information as

\[
I^R(N, \tau) \equiv H^R_X(N, \tau) + H^R_Y(N, \tau) - H^R_{XY}(N, \tau).
\]

Pushing the analogy further, we introduce the concept of mutual entropy production (MEP), which measures the difference between the entropy production of the joint process \( \Delta_i S_{XY} \) and the entropy production of the marginal processes \( \Delta_i S_X \) and \( \Delta_i S_Y \),

\[
\Delta_i S^M(N, \tau) \equiv \Delta_i S_{XY}(N, \tau) - \Delta_i S_X(N, \tau) - \Delta_i S_Y(N, \tau).
\]

Using (16) and (17) with (3), we find that the MEP can be expressed as the difference between the mutual information and its time-reversal form,

\[
\Delta_i S^M(N, \tau) = I(N, \tau) - I^R(N, \tau).
\]

This result is reminiscent of (3) and provides a connection between mutual information and entropy production.
The MEP is obviously zero when the two variables are independent. Using the log-sum rule, we find that $\Delta_i S_{XY}(N, \tau) \geq \Delta_i S_X(N, \tau) \geq 0$ and $\Delta_i S_{XY}(N, \tau) \geq \Delta_i S_Y(N, \tau) \geq 0$.

The following general inequality ensues:

$$-\Delta_i S_{XY}(N, \tau) \leq \Delta_i S^M(N, \tau) \leq \Delta_i S_{XY}(N, \tau).$$

(20)

The lower bound is reached when the two variables are completely correlated, i.e. when all the transitions involve a simultaneous change in the two variables. Indeed, in such a case the entropy production values for both $X$ and for $Y$ become equal to each other and to the total entropy production.

3.1. Bipartite networks

To proceed with our analysis, we assume that the joint Markov process occurs on a bipartite network, where each transition between states $z$ can involve a jump either in $x$ or in $y$ but not in both. The transition matrix of the joint process is thus of the form

$$W \equiv \begin{cases} 1 - R_y \tau & \text{if } y = y' \text{ and } x = x' \\ w_{xx'} \tau & \text{if } y = y' \text{ and } x \neq x' \\ w_{yy'} \tau & \text{if } x = x' \text{ and } y \neq y' \\ 0 & \text{if } x \neq x' \text{ and } y \neq y' \end{cases},$$

(21)

where $R_y \equiv \sum_{x' \neq x} w_{xx'} + \sum_{y' \neq y} w_{yy'}$ is the decay rate from state $(x,y)$. This transition matrix satisfies the normalization condition $\sum_{x,y} W_{xx'} = 1$.

In the continuous-time limit, the probability to find a system described by the transition matrix (21) in a given state $(x,y)$ satisfies the Markovian master equation

$$\frac{d}{dt} p(x, y) = \sum_{x'} J_{xx'}(y) + \sum_{y'} J_{yy'}(x),$$

(22)

where

$$J_{xx'}(y) \equiv w_{xx'} p(x', y) - w_{x'y} p(x, y),$$

$$J_{yy'}(x) \equiv w_{yy'} p(x, y') - w_{y'y} p(x, y).$$

(23)

Since the joint process $Z$ is Markovian, the different definitions of the entropy production rate all coincide,

$$\dot{S}_{i,XY} = \dot{S}_{i,XY} = \sigma_{XY},$$

(24)

where

$$\sigma_{XY} = \sum_{y,x \neq x'} w_{yy'} p(x', y) \ln \frac{w_{x'y} p(x', y)}{w_{x'y} p(x, y)} + \sum_{x,y \neq y'} w_{xx'} p(x, y') \ln \frac{w_{y'y} p(x, y')}{w_{y'y} p(x, y)}.$$

(25)

If the stochastic network contains multiple edges between pairs of nodes, the summations over pairs of states in (25) must contain a summation over all these edges. In other words, if the net transition rate between two states is in fact the sum of rates associated
Mutual entropy production in bipartite systems

with different physical mechanisms ν such as reservoirs or chemical reactions (e.g. \( w'_y x = \sum_\nu w'_y x(\nu) \)), the summation in (25) has to also contain the sum over ν [62, 63].

Since the random processes X and Y constitute a coarse-grained description of the joint process Z, they are in general non-Markovian. As a result, the MEP rate defined using the limiting procedure (12),

\[
S^i_M(t) = \sigma_{XY} - \hat{S}_{i,X}(t) - \hat{S}_{i,Y}(t),
\]

(26)
does not necessarily coincide with the rate defined using (6),

\[
\dot{S}^i_M(\tau) = \sigma_{XY} - \dot{S}_{i,X}(\tau) - \dot{S}_{i,Y}(\tau).
\]

(27)

### 3.2. Decomposition of the entropy production

Even though the MEP rates are often difficult to evaluate, we will see in sections 3.3 and 3.4 that under special conditions they can be expressed in terms of much simpler quantities which appear in the general decomposition of the joint entropy production proposed in [4]. For bipartite networks this decomposition reads

\[
\sigma_{XY} = \sigma^{(1)}_X + \sigma^{(2)}_X + \sigma^{(3)}_X \geq 0,
\]

(28)

where

\[
\sigma^{(1)}_X = \sum_{x \neq x'} w_{xx'} p(x') \ln \frac{w_{xx'} p(x')}{w_{x'x} p(x)} \geq 0,
\]

(29)

\[
\sigma^{(2)}_X = \sum_{x,y \neq y'} w_{y'y} p(x, y') \ln \frac{w_{y'y} p(y'|x)}{w_{y'x} p(y|x)} \geq 0,
\]

(30)

\[
\sigma^{(3)}_X = \sum_{x \neq x'} w_{xx'} p(x') \sum_y f_{yx'} \ln \frac{f_{yx'}}{f_{yx}} \geq 0.
\]

(31)

In these definitions we introduced the coarse-grained rates between \( x' \) and \( x \),

\[
\overline{w}_{xx'} = \sum_y w_{xx'} p(y|x'),
\]

(32)

where \( p(y|x) \) is the conditional probability of finding \( y \) given \( x \), as well as

\[
f_{yx'} \equiv \frac{w_{yx'}}{w_{xx'}} p(y|x'),
\]

(33)

the fraction of jumps between \( x' \) and \( x \) occurring at a given value of \( y \), which is normalized by \( \sum_y f_{yx'} = 1 \).

The decomposition (28) is particularly useful when considering a description of the system in terms of the variable \( x \) whereas \( y \) has been coarse-grained. Indeed, the term \( \sigma^{(1)}_X \) can be seen as an effective entropy production rate at the coarse-grained level and \( \sigma^{(2)}_X \) as an average over \( x \) of the various entropy productions due to the dynamics in \( y \) at a given \( x \). The last term \( \sigma^{(3)}_X \) quantifies the asymmetry between the fraction of jumps occurring at a given \( y \) between \( x' \) and \( x \), and \( f_{yx'} = \frac{w_{yx'}}{w_{xx'}} p(y|x') \), the fraction of jumps occurring at the same \( y \) between \( x \) and \( x' \). This quantity is thus large when most of the transitions

doi:10.1088/1742-5468/2014/04/P04010
Mutual entropy production in bipartite systems

between \( x' \) and \( x \) occur at a given value of \( y \) while most of the transitions between \( x \) and \( x' \) occur at a different value of \( y \).

Analogously to (28), by exchanging the roles of \( X \) and \( Y \), we obtain the symmetric decomposition

\[
\sigma_{XY} = \sigma_Y^{(1)} + \sigma_Y^{(2)} + \sigma_Y^{(3)},
\]

which is more relevant when the coarse-grained variable is \( x \) instead of \( y \).

By comparing (25) with the definition (30), we find that the Markovian entropy production rate for bipartite networks can be expressed as the sum

\[
\sigma_{XY} = \sigma_X^{(2)} + \sigma_Y^{(2)}.
\]

This property implies the following useful identities:

\[
\sigma_X^{(2)} = \sigma_Y^{(1)} + \sigma_Y^{(3)}, \quad \sigma_Y^{(2)} = \sigma_X^{(1)} + \sigma_X^{(3)}.
\]

3.3. Short-time limit of the rates

To obtain an exact analytical expression for the MEP (26), we will consider in this section its short-time limit under stationary conditions,

\[
\dot{S}_i^M(0) = \lim_{t \to 0} \frac{\Delta_i S_i^M(t)}{t} = \sigma_{XY} - \dot{S}_{i,X}(0) - \dot{S}_{i,Y}(0).
\]

We start with the continuous-time limit of the MEP (11), which can be expressed as

\[
\Delta_i S_i^M(t) = \sum_Z \mathcal{P} \left( \ln \frac{\mathcal{P}}{\mathcal{P}^R} - \ln \frac{\sum_X \mathcal{P}}{\sum_X \mathcal{P}^R} - \ln \frac{\sum_Y \mathcal{P}}{\sum_Y \mathcal{P}^R} \right),
\]

where the probabilities \( \mathcal{P} \equiv \mathcal{P}(Z) \) and \( \mathcal{P}^R \equiv \mathcal{P}(Z^R) \) of the trajectories \( Z = (X, Y) \) are the continuous-time analogue of the discrete-time probabilities used in section 2.

If we denote by \( (X^l, Y^m) \) a trajectory with \( l \) transitions in \( X \) and \( m \) transitions in \( Y \), and if the initial state \((x, y)\) is drawn from the stationary probability \( p(x, y) \), we get

\[
\mathcal{P}(X^0, Y^0) = p(x, y)(1 - tR^y_y),
\]

\[
\mathcal{P}(X^1, Y^0) = p(x, y)t_w x y z,
\]

\[
\mathcal{P}(X^0, Y^1) = p(x, y)t_w x y y',
\]

\[
\mathcal{P}(X^l, Y^m) = \mathcal{O}(t^2), \quad \text{if } l + m > 1.
\]

Using these expressions in (38), we find that

\[
\Delta_i S_i^M(t) = \sum_{x,x',y} t w_{y}^{x} x p(x, y) \left( \ln \frac{w_{x}^{x} p(x, y)}{w_{x}^{x} p(x', y)} - \ln \frac{w_{x}^{x} p(x)}{w_{x}^{x} p(x')} \right)
\]

\[
+ \sum_{x,y,y'} t w_{y}^{x} y p(x, y) \left( \ln \frac{w_{y}^{y} p(x, y)}{w_{y}^{y} p(x, y')} - \ln \frac{w_{y}^{y} p(y)}{w_{y}^{y} p(y')} \right) + \mathcal{O}(t^2),
\]

which using (13) and (29) leads to

\[
\dot{S}_i^M(0) = \sigma_{XY} - \sigma_X^{(1)} - \sigma_Y^{(1)}.\]
The rate of MEP in the short-time limit is thus given by the entropy production of the joint system minus the sum of the effective entropy productions resulting from a coarse-graining over $x$ and $y$ respectively. Using the decomposition (28) and the relations (36), this result can also be rewritten as

$$\hat{S}^M_i(0) = \sigma^{(3)}_X + \sigma^{(3)}_Y \geq 0.$$  \hspace{1cm} (42)

This important result shows that the short-time limit of the MEP rate does not depend explicitly on the terms $\sigma^{(1)}$ and $\sigma^{(2)}$, which characterize the dissipation along a given coordinate of the bipartite network. Instead, it can be exclusively expressed in terms of the $\sigma^{(3)}$, which characterize an intrinsically mixed source of dissipation.

We now turn to the mutual information rate in the short-time limit and establish a connection with the work presented in [54, 55]. The mutual information (16) in continuous time can be expressed as

$$I(t) = \sum_z P \left( \ln P - \ln \sum_x P - \ln \sum_y P \right).$$  \hspace{1cm} (43)

Using the short-time probabilities (39), all terms proportional to $\ln t$ cancel out so that only the constant and linear terms in $t$ survive. We thus obtain

$$I(t) = M + t \sum_{x',xy} p(x,y) w_{y,x}^{x'} \ln \frac{w_{y,x}^{x'}}{w_{x}^{x'}} + t \sum_{y',yx} p(x,y) w_{y',y}^{x} \ln \frac{w_{y',y}^{x}}{w_{y}^{y}} + O(t^2),$$  \hspace{1cm} (44)

where $M$ is the mutual information associated with the steady-state probabilities,

$$M \equiv \sum_{x,y} p(x,y) \ln \frac{p(x,y)}{p(x)p(y)} \geq 0.$$  \hspace{1cm} (45)

The mutual information rate in the short-time limit is therefore given by

$$\dot{I}(0) = \left. \frac{dI(t)}{dt} \right|_{t=0} = \sum_{xy} p(x,y) \left( \sum_{x'(\neq x)} w_{y,x}^{x'} \ln \frac{w_{y,x}^{x'}}{w_{x}^{x'}} + \sum_{y'(\neq y)} w_{y',y}^{x} \ln \frac{w_{y',y}^{x}}{w_{y}^{y}} \right).$$  \hspace{1cm} (46)

We note that this quantity corresponds precisely to the upper bound of the mutual information rate $\dot{I}(\tau)$ in the limit $\tau \to 0$ found in [54, 55], namely

$$\dot{I}(0) \geq \dot{I}(0).$$  \hspace{1cm} (47)

Analogously, the time-reversed mutual information rate $\dot{I}^R(0)$ reads

$$\dot{I}^R(0) = \left. \frac{dI^R(t)}{dt} \right|_{t=0} = \sum_{xy} p(x,y) \left( \sum_{x'(\neq x)} w_{y,x}^{x'} \ln \frac{w_{y,x}^{x'}}{w_{x}^{x'}} + \sum_{y'(\neq y)} w_{y',y}^{x} \ln \frac{w_{y',y}^{x}}{w_{y}^{y}} \right).$$  \hspace{1cm} (48)

Therefore, we can also express the mutual information rate in the short-time limit as

$$\hat{S}^M_i(0) = \dot{I}(0) - \dot{I}^R(0) \geq 0.$$  \hspace{1cm} (49)

doi:10.1088/1742-5468/2014/04/P04010
3.4. Timescale separation

A regime of timescale separation occurs whenever the transitions in one of the two variables $X$ or $Y$ happen at a much higher rate than the other. In this section, we will assume that $Y$ is faster than $X$, thus the rates $w_{y'y}^x$ are much larger than $w_{yy}^x$. To discuss this regime, we multiply the rates $w_{y'y}^x$ by a scaling factor $\gamma$. As shown in [4], the marginal probability $p(x)$ always satisfies a master equation of the form

$$
\frac{d}{dt}p(x) = \sum_{x'}(w_{x'y}^xp(x') - w_{y'x}^xp(x)),
$$

in terms of the effective rates $w_{x'y}^x$ introduced in (32). This equation is not closed since the effective rates depend on the conditional probabilities $p(y|x')$ which require the solution of the full joint dynamics (22). However, in the regime of timescale separation, these probabilities can be obtained by finding the stationary state of the closed Markovian master equation (valid when $\gamma \to 0$)

$$
\frac{d}{dt}p(y|x) = \sum_{y'}(w_{y'y}^xp(y'|x) - w_{y'y}^xp(y|x)),
$$

and used to calculate the effective rates (32) perturbatively to order $\gamma$ [4]. As a result, (50) becomes a closed Markovian master equation and the entropy production rate for $X$ is thus given by

$$
\dot{S}_{i,X} = \sigma_X^{(1)} + O(\gamma^2).
$$

The MEP rate (26) therefore reduces to

$$
\dot{S}_i^M(t) = \sigma_{XY} - \sigma_X^{(1)} - \dot{S}_{i,Y}(t) + O(\gamma^2) = \sigma_X^{(3)} + \sigma_X^{(2)} - \dot{S}_{i,Y}(t) + O(\gamma^2).
$$

To proceed, we consider the special case where the fast transitions between $y$ states do not depend on the states $x$. Using (32) and (33), we find that $\overline{w}_{x'y} = w_{x'y}$ and $f_{y'y}^x = p(y|x')$. Also, the conditional probabilities in (51) become independent of $x$, i.e. $p(y|x) = p(y)$. The dynamics for $Y$ thus also becomes Markovian and

$$
\dot{S}_{i,Y} = \sigma_Y^{(1)}.
$$

By inserting (54) into (53), we find that $\dot{S}_i^M$ is independent of $t$ and, consistently with (41), coincides with the short-time limit $\dot{S}_i^M(0)$, also expressed as (42). Since the transition rates in $x$ do not depend on $y$, we also note from the definitions (29) and (30) that $\sigma_Y^{(1)}$ and $\sigma_X^{(2)}$ are equal. Thus the relations (36) imply that $\sigma_Y^{(3)} = 0$. As a result, in this particular case the MEP rate reduces to

$$
\dot{S}_i^M = \sigma_X^{(3)} + O(\gamma^2).
$$

We now turn to the situation where the fast process $Y$ at fixed $x$ is locally at equilibrium for all $x$ in the limit of $\gamma \to 0$, i.e. the conditional probabilities $p(y|x)$ satisfy the detailed balance relation

$$
w_{y'y}^xp(y|x) = w_{y'y}^x p(y|x).
$$

We now turn to the situation where the fast process $Y$ at fixed $x$ is locally at equilibrium for all $x$ in the limit of $\gamma \to 0$, i.e. the conditional probabilities $p(y|x)$ satisfy the detailed balance relation

$$
w_{y'y}^xp(y|x) = w_{y'y}^x p(y|x).
$$

doi:10.1088/1742-5468/2014/04/P04010
As $\gamma \to 0$, $\sigma_X^{(2)}$ is of order $\gamma^2$, therefore from the relations (36) $\sigma_Y^{(1)}$ and $\sigma_Y^{(3)}$ must also be of the same order,

$$\sigma_X^{(2)} = \sigma_Y^{(1)} = \sigma_Y^{(3)} = \mathcal{O}(\gamma^2).$$

If we consider times $t$ shorter than the typical time needed for transitions between $x$ states to occur, i.e. $t \ll 1/\bar{w}_{xx'} \sim 1/\gamma$, the states $x$ are frozen and

$$\Delta_i S_Y(t) = \left\langle \ln \frac{\sum_x p(x) P(Y|x)}{\sum_x p(x) P(Y|R|x)} \right\rangle_Y,$$

where $p(x)$ is the probability to sample a trajectory starting (and thus staying) in $x$. Using the log-sum rule in (58), we find that

$$\dot{S}_Y(t) \leq \sigma_X^{(2)} \sigma_Y^{(2)} = \mathcal{O}(\gamma^2) \quad \text{for} \quad t \ll 1/\bar{w}_{xx'} \sim 1/\gamma.\quad (59)$$

This result is consistent with (42) as can be verified using (57).

However, for generic regimes of timescale separation we have that

$$\dot{S}_Y(t) = \sigma_X^{(2)} + \mathcal{O}(\gamma).\quad (60)$$

Therefore, since

$$\sigma_X^{(1)}, \sigma_X^{(3)}, \sigma_Y^{(2)} = \mathcal{O}(\gamma), \quad \sigma_Y^{(1)}, \sigma_Y^{(3)}, \sigma_X^{(2)} = \mathcal{O}(1);$$

the evaluation of the MEP rate (53) crucially depends on the corrections to (60), which are in general difficult to compute.

4. Application

In order to verify the results of section 3.4, we consider the bipartite model depicted in figure 1 in the steady-state regime. We also impose the condition

$$a \equiv k_{on}^+/k_{off}^+ = k_{on}^-/k_{off}^-.$$ \quad (62)

The Markovian entropy production of the joint system reads [60]

$$\sigma = J_c \ln \frac{r_0^+ r_1^+}{r_0^- r_1^-} + J_{on} \ln \frac{k_{on}^+ w_{on}^+}{k_{on}^- w_{on}^-} + J_{off} \ln \frac{k_{on}^+ w_{off}^+}{k_{on}^- w_{off}^-},$$

where

$$J_c = r_0^+ p(\text{off},0) - r_0^- p(\text{on},0),$$

$$J_{on} = w_{on}^+ p(\text{on},1) - w_{on}^- p(\text{on},0),$$

$$J_{off} = w_{off}^+ p(\text{off},1) - w_{off}^- p(\text{off},0)$$

are respectively the counterclockwise probability currents associated with the large and the two small cycles in figure 1. Equilibrium requires the three affinities (i.e. the logarithms in (63)) to vanish.

In figure 2, we calculated numerically $\Delta_i S^M/N\tau$ and $\Delta_i S_Y/N\tau$ for this model by generating Markovian discrete-time trajectories in the joint space $(X,Y)$. To interpolate between the two regimes of timescale separation we introduced a scaling parameter $\lambda$ multiplying $r_0$ and $r_1$ and varying from $10^{-3}$ to $10^3$,

$$r_0^\pm \rightarrow \lambda r_0^\pm, \quad r_1^\pm \rightarrow \lambda r_1^\pm.$$ \quad (65)
Figure 1. Four-state model of a bipartite system made of two states $x = \text{off}, \text{on}$ and two states $y = 0, 1$. Each direct transition is associated with a rate with a superscript ‘+’. The reverse transition is associated with the corresponding rate with superscript ‘−’ (not displayed).

Figure 2. Upper panel: comparison between the numerical $\Delta_i S_Y / N\tau$ (black, solid) and its asymptotic values $\sigma_Y^{(2)}$ (red, dashed) and $\sigma_Y^{(1)}$ (green, dot-dashed) as a function of the scaling parameter $\lambda$ introduced in equation (65). Lower panel: comparison between $\Delta_i S_M / N\tau$ (black, solid), $\sigma_X^{(3)}$ (red, dashed) and $\sigma_Y^{(3)}$ (green, dot-dashed). Markovian discrete-time trajectories of length $N = 10^8$ and with time step $\tau = 10^{-4}$ have been considered. The set of parameters used is $a = 10$, $r_0^+ = 0.15$, $r_1^+ = 0.1$, $r_0^- = 0.1$, $r_1^- = 0.2$, $w_{\text{on}}^+ = w_{\text{off}}^+ = 0.1$, $w_{\text{on}}^- = w_{\text{off}}^- = 1.3$, $k_{\text{off}}^+ = 0.4$, $k_{\text{off}}^- = 0.3$. 

doi:10.1088/1742-5468/2014/04/P04010
With this choice $\lambda \ll 1$ corresponds to the regime discussed in section 3.4 where $Y$ is faster than $X$ and $\lambda$ plays the same role as the perturbative parameter $\gamma$. For $\lambda \gg 1$ the roles of $X$ and $Y$ are exchanged and the comparison with section 3.4 is obtained by rescaling all the timescales by a factor $\lambda$ and then identifying $\gamma$ as $\lambda^{-1}$.

Numerically, $\Delta_i S_X/N\tau$ is almost zero over the whole range in $\lambda$ and is therefore not shown. This is related to the fact that $\sigma_X^{(1)}$ is always zero for this model since states with different $x$ are connected by a single edge. When $\lambda$ is small, $Y$ is faster than $X$ and the fast conditional dynamics of $Y$ at fixed state $x$ will reach a nonequilibrium steady state obtained from (51). As predicted by (60), $\Delta_i S_Y/N\tau$ converges rapidly to $\sigma_X^{(2)}$. However, as explained below that equation, this convergence does not imply that the asymptotic value of the MEP will coincide with $\sigma_X^{(3)}$. The discrepancy between the two is related to the dependence of the fast rates on $X$, quantified in this model by the value of $a$ defined in (62). We checked numerically that when $a$ approaches unity, this difference vanishes, consistently with our result in (55). Finally, the slow dynamics in $X$ becomes Markovian and, using (52), $\Delta_i S_X/N\tau$ tends to $\sigma_X^{(1)}$, which is always zero in this model (not plotted).

We now turn to the opposite regime of timescale separation at large values of $\lambda$, where $X$ is faster than $Y$. Therefore, when referring to the results of section 3.4, the roles of $X$ and $Y$ have to be interchanged. The fast conditional dynamics of $X$ at fixed state $y$ reaches a steady state given by the solution of equation (51). In this case the steady state corresponds to an equilibrium steady state since transitions in $x$ are due to a single edge. Detailed balance is thus satisfied inside each state $y$. Under this condition we expect from (57) that $\sigma_X^{(3)} = \sigma_Y^{(2)} = \mathcal{O}(\lambda^{-1})$. We also note that $\Delta_i S_Y/N\tau$ approaches the Markovian rate $\sigma_Y^{(1)}$ as expected from (52) and that the MEP becomes exactly $\sigma_Y^{(3)}$ as predicted in equation (59). Finally, according to (58), the entropy production $\Delta_i S_X/N\tau$ remains very close to zero (not plotted).

As a physical application of this model, we consider a system made of two capacitively coupled single level quantum dots, as depicted in figure 3. It is defined from the general model by assuming that the second and third affinities in (63) are the same, namely under the condition

$$w_{\text{on}}^- w_{\text{off}}^+ = w_{\text{on}}^+ w_{\text{off}}^-.$$  
(66)

doi:10.1088/1742-5468/2014/04/P04010

Figure 3. Maxwell’s demon model. The upper single level quantum dot $X$ (in contact with a cold lead) is sensing via capacitive coupling the presence or absence of an electron in the lower dot $Y$ (in contact with two warmer leads). The diagram shows the possible transitions between the four states of the model and their corresponding directed rates.
Its entropy production is therefore of the form
\[ \sigma = J_e \ln \frac{r_0^+ r_1^-}{r_0^- r_1^+} + (J_{on} + J_{off}) \ln \frac{k_{on} w_{on}^+}{k_{on} w_{on}^-}. \]

(67)

In this model \( Y \) is a single level quantum dot in contact with two leads at the same temperature but different chemical potentials, while \( X \) is a second single level quantum dot capacitively coupled to the first dot and in contact with a lead at a lower temperature. Such models have been used to describe single electron detectors in electron counting statistics [50, 64, 65]. In [51] this model has been used to show that in a finely tuned regime, dot \( X \) can play the role of an ideal Maxwell demon acting on dot \( Y \). In this regime \( X \) perfectly monitors the state of \( Y \) and is able to generate a force on the electron transfers in \( Y \). This force is of a purely entropic nature since it does not affect the energy balance of \( Y \) but only its entropy balance.

In this paper we focus on a broader regime where entropic as well as energetic effects come into play to enable \( X \) to accurately sense the electron transfers in and out of \( Y \), by causally correlating its state with the state of \( Y \). Typically, if an electron enters (exits) \( Y \) from one of its two leads, the state of \( X \) has to immediately become empty (filled). This implies that the dynamics of \( X \) has to be much faster than \( Y \) and that the mutual information between \( X \) and \( Y \) has to be large. The counterclockwise probability flux along the large cycle also has to be large in order to generate a causal response of \( X \) to the electron transfers happening in \( Y \). In the perfectly accurate sensing regime, the jumps from \( y = 0 \) to \( y = 1 \) (from \( y = 1 \) to \( y = 0 \)) exclusively occur at \( x = \text{on} \) (\( x = \text{off} \)). Using (31), one verifies that in this case the ratios of \( f_{x,(\nu)}^{y} \) and \( f_{y,(\nu)}^{x} \) introduced in (33) become singular and lead to a divergence of \( \sigma_Y^{(3)} \). \( \nu \) denotes whether the left or right lead of dot \( Y \) triggered the transition. This suggests that \( \sigma_Y^{(3)} \) could be a measure of the sensing accuracy. Different regimes of operation of the Maxwell’s demon model are displayed in figure 4. We observe that high \( \sigma_Y^{(3)} \) indeed corresponds to the regime of accurate sensing. Since in this regime the conditional dynamics in \( Y \) at fixed \( y \) equilibrates, we have \( \sigma_Y^{(2)} = 0 \), thus, using (42), (57) and (59), we find that \( \dot{S}_1 = \sigma_Y^{(3)} \). In other words, \( \sigma_Y^{(3)} \), and thus the MEP, seems to provide a good measure of the sensing accuracy in this model.

Finally, we observe that in the ideal Maxwell demon regime, \( \sigma_Y^{(1)} \) can be interpreted as the entropy production generated by a Markov dynamics in \( Y \) with rates phenomenologically modified as proposed in [30] to account for a Maxwell demon feedback [51]. However, such a phenomenological approach neglects \( \sigma_Y^{(3)} \), which is by far the dominant contribution to the total entropy production of the process and which diverges in the regime of perfect detection.

5. Conclusions

We introduced in this paper the notion of mutual entropy production (MEP) and showed that it can be expressed as the difference between the mutual information rate and the time-reversed mutual information rate. This result is analogous to the expression of the entropy production as the difference between the time-reversed entropy rate and the entropy rate found by Gaspard in [13]. The MEP is in general hard to evaluate due to the
Mutual entropy production in bipartite systems

Figure 4. Maxwell’s demon model in a regime of (a) low flux circulation and low mutual information, (b) low flux and high mutual information, (c) high flux circulation and low mutual information, (d) high flux circulation and high mutual information. The latter corresponds to a regime of timescale separation (slow transitions are dashed). The MEP rate $\dot{S}_1^M(0)$ (42), $\sigma_X^{(3)}$ and $\sigma_Y^{(3)}$ (31), the mutual information $M$ (45) and the mutual information rate $I(0)$ (46) are displayed in the different regimes.

non-Markovian character induced by coarse-graining procedures. However, for a bipartite system we were able to provide explicit expressions in the short-time limit and in the presence of timescale separation between its components. We also verified the accuracy of these results numerically using a four-state model system. Finally, by considering a model of two capacitively coupled quantum dots where one dot monitors the other, we found that the MEP provides a measure of the sensing accuracy. The physical relevance of the MEP beyond this model remains, however, an open problem.

Acknowledgment

This work is supported by the National Research Fund, Luxembourg in the frame of project FNR/A11/02.

References

[1] Sekimoto K, 2010 Stochastic Energetics (Berlin: Springer)
[2] Seifert U, 2012 Rep. Prog. Phys. 75 126001
[3] Jarzynski C, 2011 Ann. Rev. Condens. Matter Phys. 2 329
[4] Esposito M, 2012 Phys. Rev. E 85 041125
[5] Zhang X J, Qian H and Qian M, 2012 Phys. Rep. 510 1
[6] Ge H, Qian M and Qian H, 2012 Phys. Rep. 510 87
[7] Shannon C E, 1948 Bell Syst. Techn. J. 27 379
[8] Cover T M and Thomas J A, 2006 Elements of Information Theory (New York: Wiley)

doi:10.1088/1742-5468/2014/04/P04010
Mutual entropy production in bipartite systems

[9] Qian M P and Qian M, 1985 Chin. Sci. Bull. 30 445
[10] Crooks G E, 1999 Phys. Rev. E 60 2721
[11] Kawai R, Parrondo J M R and Van den Broeck C, 2007 Phys. Rev. Lett. 98 080602
[12] Gaspard P and Wang X J, 1993 Phys. Rep. 235 291
[13] Gaspard P, 2004 J. Stat. Phys. 117 599
[14] Andrieux D, Gaspard P, Ciliberto S, Garnier N, Joubaud S and Petrosyan A, 2007 Phys. Rev. Lett. 98 150601
[15] Roldán E and Parrondo J M R, 2012 Phys. Rev. E 85 031129
[16] Sagawa T and Ueda M, 2008 Phys. Rev. Lett. 100 080403
[17] Sagawa T and Ueda M, 2009 Phys. Rev. Lett. 102 250602
[18] Sagawa T and Ueda M, 2010 Phys. Rev. Lett. 104 090602
[19] Horowitz J M and Vaikuntanathan S, 2010 Phys. Rev. E 82 061120
[20] Averin D V, Möttönen M and Pekola J P, 2011 Phys. Rev. B 84 245448
[21] Horowitz J and Parrondo J M P, 2011 Europhys. Lett. 95 10005
[22] Horowitz J and Parrondo J M P, 2011 New J. Phys. 13 123019
[23] Averin D and Seifert U, 2011 Europhys. Lett. 94 10001
[24] Abreu D and Seifert U, 2012 Phys. Rev. Lett. 108 030601
[25] Sagawa T and Ueda M, 2012 Phys. Rev. Lett. 109 180602
[26] Mandal D and Jarzynski C, 2012 Proc. Nat. Acad. Sci. USA 109 11641
[27] Barato A C and Seifert U, 2013 Europhys. Lett. 101 60001
[28] Sagawa T and Ueda M, 2010 Phys. Rev. Lett. 104 090602
[29] Munakata T and Rosinberg M L, 2012 J. Stat. Mech. P05010
[30] Esposito M and Schaller G, 2012 Europhys. Lett. 99 30003
[31] Kundu A, 2012 Phys. Rev. E 86 021107
[32] Leff H and Rex A F, 2002 Maxwell's Demon 2: Entropy, Classical and Quantum Information, Computing (Boca Raton, FL: CRC Press)
[33] Andrieux D and Gaspard P, 2008 Europhys. Lett. 81 28004
[34] Esposito M and Van den Broeck C, 2011 Europhys. Lett. 95 40004
[35] Lan G, Sartori P, Neumann S, Sourjik V and Tu Y, 2012 Nature Phys. 8 422
[36] Sartori P and Pigolotti S, 2013 Phys. Rev. Lett. 110 188101
[37] Bennett C H, 1973 IBM J. Res. Dev. 17 525
[38] Bennett C H, 1979 BioSystems 11 85
[39] Andrieux D and Gaspard P, 2008 Proc. Nat. Acad. Sci. USA 105 9451
[40] Lan G, Sartori P, Neumann S, Sourjik V and Tu Y, 2012 Nature Phys. 8 422
[41] Sartori P and Pigolotti S, 2013 Phys. Rev. Lett. 110 188101
[42] Murugan A, Huse D A and Leibler S, 2012 Proc. Nat. Acad. Sci. USA 109 12034
[43] Still S, Sikav D A, Bell A J and Crooks G E, 2012 Phys. Rev. Lett. 109 120604
[44] Serreli V, Lee C F, Kay E R and Leigh D A, 2007 Nature 445 523
[45] Toyabe S, Sagawa T, Ueda M, Muneyuki E and Sano M, 2010 Nature Phys. 6 988
[46] Bérut A, Arakelyan A, Petrosyan A, Ciliberto S, Dillenschneider R and Lutz E, 2012 Nature 483 187
[47] Orlov A O, Lent C S, Thorpe C C, Boechler G P and Snider G L, 2012 Japan. J. Appl. Phys. 51 06FE10
[48] Jun Y and Bechhoefer J, 2012 Phys. Rev. E 86 061106
[49] Bulnes Cuetara G, Esposito M and Gaspard P, 2011 Phys. Rev. B 84 165114
[50] Strasberg P, Schaller G, Brandes T and Esposito M, 2013 Phys. Rev. Lett. 110 040601
[51] Bulnes Cuetara G, Esposito M and Gaspard P, 2013 arXiv:1305.1830
[52] Mehta P and J S D, 2012 Proc. Nat. Acad. Sci. USA 109 17978
[53] Barato A C, Hartich D and Seifert U, 2013 Phys. Rev. E 87 042104
[54] Barato A C, Hartich D and Seifert U, 2013 arXiv:1306.1698
[55] Maes C, 2003 Séminaire Poissonacé 2 29
[56] Seifert U, 2005 Phys. Rev. Lett. 95 040602
[57] Gomez-Marín A, Parrondo J M R and Van den Broeck C, 2008 Phys. Rev. E 78 011107
[58] Crooks G E, 2009 Rev. Mod. Phys. 81 1665
[59] Schnakenberg J, 1976 Rev. Mod. Phys. 48 571
[60] Nicolas G and Prigogine I, 1977 Self-Organization in Non-Equilibrium Systems (New York: Wiley)
[61] Esposito M and Van den Broeck C, 2010 Phys. Rev. E 82 011143
[62] Van den Broeck C and Esposito M, 2010 Phys. Rev. E 82 011144
[63] Schaller G, Kießlich G and Brandes T, 2010 Phys. Rev. B 82 041303
[64] Sanchez R, Lopez R, Sanchez D and Buttiker M, 2010 Phys. Rev. Lett. 104 076801

doi:10.1088/1742-5468/2014/04/P04010