A Simple Approximation Algorithm for Vector Scheduling and Applications to Stochastic Min-Norm Load Balancing

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Abstract

We consider the Vector Scheduling problem on identical machines: we have $m$ machines, and a set $J$ of $n$ jobs, where each job $j$ has a processing-time vector $p_j \in \mathbb{R}_+^d$. The goal is to find an assignment $\sigma : J \rightarrow [m]$ of jobs to machines so as to minimize the makespan $\max_{i \in [m]} \max_{r \in [d]} \left( \sum_{j \in J} \sigma(j) = i \right) p_{j,r}$. A natural lower bound on the optimal makespan is $lb := \max \{ \max_{j \in J,r \in [d]} p_{j,r}, \max_{r \in [d]} \left( \sum_{j \in J} p_{j,r} / m \right) \}$. Our main result is a very simple $O(\log d)$-approximation algorithm for vector scheduling with respect to the lower bound $lb$: we devise an algorithm that returns an assignment whose makespan is at most $O(\log d) \cdot lb$.

As an application, we show that the above guarantee leads to an $O(\log \log m)$-approximation for stochastic minimum-norm load balancing (StochNormLB). In StochNormLB, we have $m$ identical machines, a set $J$ of $n$ independent stochastic jobs whose processing times are nonnegative random variables, and a monotone, symmetric norm $f : \mathbb{R}^m \rightarrow \mathbb{R}_+$. The goal is to find an assignment $\sigma : J \rightarrow [m]$ that minimizes the expected $f$-norm of the machine-load vector $\overrightarrow{load}^\sigma$, where the $i$th coordinate of $\overrightarrow{load}^\sigma$ is the (random) total processing time assigned to machine $i$. Our $O(\log \log m)$-approximation guarantee is in fact much stronger: we obtain an assignment that is simultaneously an $O(\log \log m)$-approximation for StochNormLB with all monotone, symmetric norms. Next, this approximation factor significantly improves upon the $O(\log m / \log \log m)$-approximation in (Ibrahimpur and Swamy, FOCS 2020) for StochNormLB, and is a consequence of a more-general black-box reduction that we present, showing that a $\gamma(d)$-approximation for $d$-dimensional vector scheduling with respect to the lower bound $lb$ yields a simultaneous $\gamma(\log m)$-approximation for StochNormLB with all monotone, symmetric norms. We emphasize that it is crucial for this reduction that the approximation guarantee for vector scheduling is with respect to the lower bound $lb$.

1 Introduction

We consider the well-studied Vector Scheduling problem on identical machines, a natural generalization of the (scalar) makespan minimization problem to a setting with $d \geq 1$ dimensions. We have a set $J$ of $n$ jobs that are to be processed on exactly one of $m$ identical machines. We use $[N]$ to denote $\{1, \ldots, N\}$. Each job $j \in J$ has a processing-time (or size) vector $p_j = (p_{j,1}, \ldots, p_{j,d}) \in \mathbb{R}_+^d$, where $p_{j,r}$ denotes the size of job $j$ in dimension $r \in [d]$; for example, dimensions could correspond to processor cycles, storage space, or network bandwidth needed to complete the job. An assignment $\sigma : J \rightarrow [m]$ of jobs to machines induces an $(m \times d)$-dimensional load vector $\overrightarrow{load}^\sigma$ (one entry per machine-dimension pair): for machine $i \in [m]$ and dimension $r \in [d]$, the load in the $r$th dimension of machine $i$ is $\overrightarrow{load}^\sigma_{i,r} := \sum_{j \in J, \sigma(j) = i} p_{j,r}$. The makespan objective of an assignment $\sigma$ is defined as $\max_{i \in [m], r \in [d]} \overrightarrow{load}^\sigma_{i,r}$ i.e., the maximum load across all machines and all dimensions. The goal in vector scheduling is to find an assignment $\sigma$ that minimizes the makespan.

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Vector scheduling (a.k.a. vector load balancing) was first considered by Chekuri and Khanna [4] who gave an $O(\log^2 d)$-approximation algorithm for the problem. Meyerson, Roytman and Tagiku in [16] improved the approximation guarantee to a factor $O(\log d)$; in fact, they gave an $O(\log d)$-competitive algorithm for the online version of the problem where jobs arrive one at a time and have to be assigned irrevocably to a machine on arrival. The current best approximation for the problem is an $O(\log d/\log \log d)$-competitive algorithm by Im, Kell, Kulkarni and Panigrahi [14]. In terms of hardness, very recently, Sai Sandeep [18] showed that, for any $\epsilon > 0$, (offline) vector scheduling is hard to approximate to a factor $O((\log d)^{1-\epsilon})$ under some complexity theoretic assumptions.

A natural lower bound on the optimal makespan in vector scheduling is given by:

$$
\text{lb} := \max \left\{ \max_{\text{dimension } r \in [d]} \frac{1}{m} \cdot \max_{\text{job } j \in J} p_{j,r}, \frac{1}{m} \cdot \max_{\text{dimension } r \in [d]} \sum_{\text{job } j \in J} p_{j,r} \right\}. 
$$

The first term above arises because each job has to assigned to some machine, and the second term arises because in each dimension $r$, a total load of $\sum_j p_{j,r}$ is distributed across $m$ machines.

Motivated by applications in stochastic load balancing, the main question that initiated this work is: how well can we approximate vector scheduling with respect to the natural lower bound $\text{lb}$? We devise a simple $\text{lb}$-relative randomized $O(\log d)$-approximation algorithm (see Theorem 1.1).

We make a few remarks comparing this result with some of of the (previously mentioned) prior work on vector scheduling. The $O(\log d)$ approximation guarantee of Meyerson et al. [16] is also with respect to $\text{lb}$, although it is not explicitly stated in this form. While their guarantee is deterministic and holds also in the online setting, our chief notable feature is the simplicity of our algorithm and analysis.\footnote{The algorithm of Meyerson et al. is also fairly simple—it assigns the recently-arrived job to the machine that leads to the least increase in an exponential potential function—but the analysis is slightly involved.}

From a pure norm without the knowledge of the job-size realizations. We sometimes use the randomness in the job-size distributions. We emphasize that all jobs are assigned to machines up front without the knowledge of the job-size realizations. We sometimes use Stoch-$f$-LB to explicitly indicate the norm $f$ that we are considering in an instance of StochNormLB.
Kleinberg, Rabani and Tardos [15] were the first to consider the stochastic makespan minimization problem (StochNormLB when \( f \) is the \( \ell_\infty \) norm) and gave an \( O(1) \)-approximation algorithm. Subsequent work by [6, 8, 17, 2, 3, 11, 5, 13] have focused on obtaining approximation algorithms for various StochNormLB settings: (i) unrelated-machines setting where job-size distributions can depend on the machine; (ii) the norm \( f \) is an \( \ell_p \) norm or a Top \( \ell \) norm (where Top \( \ell \)(\( x \)) is defined as the sum of \( \ell \) largest coordinates of \( x \)); and (iii) job-sizes distributions are Bernoulli, Exponential, Poisson random variables, or even deterministic \((X_j = p_j \) with probability 1\). We elaborate on related work in Section 1.2. The result most relevant to this work is the \( O(\log m / \log \log m) \)-approximation algorithm of [11] for StochNormLB when \( f \) is an arbitrary monotone symmetric norm and jobs are arbitrarily distributed. In this work, we give an exponential improvement in the approximation guarantee for StochNormLB when machines are identical, and to do this we crucially use the lb-relative approximation algorithm for vector scheduling.

1.1 Our Results and Techniques

We formally state our contributions in this work with a brief overview of the techniques used. Our first main result is a simple \( O(\log d) \)-approximation algorithm for vector scheduling where the approximation guarantee is with respect to the natural lower bound.

**Theorem 1.1.** There is a randomized algorithm for the \( d \)-dimensional vector scheduling problem that computes an assignment \( \sigma \) with makespan at most \( O(\log d) \cdot lb \). The algorithm runs in time polynomial in the input size, and succeeds in finding an approximate assignment with probability at least 2/3.

Our vector-scheduling algorithm in Theorem 1.1 is extremely simple to implement since it is based on randomly sampling jobs to be processed on a machine. Its analysis is equally simple because it only uses the well-known Chernoff tail bounds. By scaling job-size vectors, we may assume without loss of generality that \( lb = 1 \). In particular, this implies \( p_{j,r} \in [0, 1] \) for any job \( j \in J \) and dimension \( r \in [d] \), and \( \sum_j p_{j,r} \leq m \) for any \( r \). For simplicity, further assume that \( \sum_j p_{j,r} = m \) for all \( r \). Consider a random job-set \( S \subseteq J \) where job \( j \) is independently included in \( S \) with probability \( c \log d/m \) for a sufficiently large constant \( c \geq 1 \). By our choice of \( S \), for any dimension \( r \in [d] \), the expected total size of jobs in \( S \) in that dimension is \( c \log d \). By Chernoff tail bounds, in any dimension \( r \), we have \( \Pr[1 \leq \sum_{j \in S} p_{j,r} \leq O(\log d)] \leq 1/poly(d) \). By union bound, with some positive probability, \( \sum_{j \in S} p_{j,r} \) is in \([1, O(\log d)]\) simultaneously for all dimensions \( r \in [d] \). We assign the job-set \( S \) to one of the \( m \) machines, and recurse on the residual instance with job-set \( J \setminus S \) and \( m - 1 \) identical machines; note that \( lb \) for the residual instance is at most 1. We formalize the argument in Section 2.

Our second main result is an \( O(\log \log m) \)-approximation for StochNormLB.

**Theorem 1.2.** We can compute (in polynomial time) an assignment \( \sigma : J \to [m] \) such that, with probability at least 2/3, \( \sigma \) is an \( O(\log \log m) \)-approximation to the given instance of StochNormLB.

In fact, we prove a stronger result: we return an assignment that simultaneously achieves an \( O(\log \log m) \)-approximation for all monotone, symmetric norms.

**Theorem 1.3.** Consider \( n \) stochastic jobs \( \{X_j \}_{j \in J} \) and \( m \) identical machines. We can compute (in polynomial time) an assignment \( \sigma : J \to [m] \) such that, with probability at least 2/3, \( \sigma \) is simultaneously an \( O(\log \log m) \)-approximation to all Stoch-g-LB instances where \( g : \mathbb{R}^m \to \mathbb{R}_{\geq 0} \) is a monotone, symmetric norm. Moreover, given an lb-relative \( \gamma(d) \)-approximation algorithm \( A \) for \( d \)-dimensional vector scheduling, we can use one call of \( A \) to obtain an \( O(\gamma(\log m)) \) simultaneous-approximation for Stoch-g-LB for all monotone, symmetric norms \( g \).

The randomization above stems from our randomized guarantee for vector scheduling. We remark that the probability of success in Theorem 1.3 is of a detectable (polytime-verifiable) event (see Remark 1).
Therefore, we can lower the failure probability to \( \delta \), for any \( \delta > 0 \), by repeating the algorithm in Theorem 1.3 \( O(\log(1/\delta)) \) times.

At a high level, to prove Theorem 1.3 we utilize two key ideas from [11]. First, a technical result from [11] called as approximate stochastic majorization (Theorem 3.2) gives a reduction from the problem of approximating all monotone symmetric norms to that of approximating only a small collection of Top \( r \) norms. Formally, a simultaneous \( \alpha \)-approximation to all Stoch-Top \( r \)-LB instances with \( \ell \in \{1, 2, 4, \ldots, 2^{\log_2 m} \} \) implies a simultaneous \( O(\alpha) \)-approximation for all StochNormLB instances (see Theorem 3.3).

Second, we specialize the \( O(1) \)-approximation algorithm for Stoch-Top \( r \)-LB found in [11] to the identical-machines setting. We first note that, modulo some technical assumptions, Stoch-Top \( r \)-LB problem on identical machines is essentially an instance of 1-dimensional vector scheduling where the (scalar) size of a job \( j \) is a certain sophisticated function of its job-size distribution \( X_j \). More crucially, we show that an \( lb \)-relative \( \alpha \)-approximate assignment for this 1-dimensional vector scheduling instance yields an \( O(\alpha) \)-approximate assignment for the Stoch-Top \( r \)-LB instance. Since the expression of \( lb \) for \( d \)-dimensional vector scheduling has a certain independence among its \( d \) dimensions, we can approximate the simultaneous Stoch-Top \( r \)-LB problem (for all powers-of-2 \( \ell \)) via the \( O(\log m) \)-dimensional vector scheduling problem. As Theorem 2.1 gives us an \( O(\log d) \)-approximation for \( d \)-dimensional vector scheduling, we get an \( O(\log \log m) \)-approximation in Theorems 1.2 and 1.3. We remark that our algorithms for stochastic load balancing applications only require black-box access to \( lb \)-relative approximation algorithms for vector scheduling. If the vector-scheduling algorithm is deterministic (randomized), then our algorithm for Theorem 1.3 is also deterministic (randomized).

1.2 Related Work

The \( d \)-dimensional vector scheduling problem was first considered by Chekuri and Khanna in [4] where they gave an \( O(\log^2 d) \)-approximation algorithm and showed that the problem is NP-hard to approximate within any constant factor. They also gave a PTAS when \( d \) is a fixed constant. Meyerson et al. [16] gave an improved \( O(\log d) \)-approximation and the current best approximation factor of \( O(\log d/\log \log d) \) is due to Harris and Srinivasan [10] and Im et al. [14]. The results in [16, 14] also hold in the online setting, and the result of [10] works even in the unrelated-machines setting where the size vector of a job \( j \) can depend on the machine that it is processed on. Recently, Sai Sandeep [18] gave very strong inapproximability results for (offline) vector scheduling indicating that the current best results are almost optimal: assuming \( \text{NP} \not\subseteq \text{ZPTIME}(n^{O(\log n)}) \), they rule out an \( O((\log d)^{1-\epsilon}) \)-approximation for any \( \epsilon > 0 \), and under the weaker assumption of \( \text{P} \not\subseteq \text{NP} \), they rule out a \( \text{poly}(\log d) \)-approximation. We also mention that factor \( O(\log d/\log \log d) \) is the best possible competitive ratio for any (deterministic or randomized) online vector scheduling algorithm (see [14, 1]).

Growing interest in optimization under uncertainty has led to some recent work on stochastic load balancing under various norm objectives. As mentioned before, Kleinberg et al. [15] were the first to investigate stochastic load balancing on identical machines for the makespan objective (i.e., Stoch-\( \ell_{\infty} \)-LB) and gave an \( O(1) \)-approximation algorithm for the problem. This result was generalized to the unrelated-machines setting by Gupta, Kumar, Nagarajan and Shen [7, 8], and subsequently, Molinaro [17] generalized the result to all \( \ell_p \) norms. Ibrahimpur and Swamy [11, 12] were the first to consider StochNormLB in its most general form (unrelated machines, arbitrary monotone symmetric norm objectives, and arbitrary job-size distributions) and gave the following approximation guarantees: (i) \( O(1) \)-approximation when jobs are weighted Bernoulli random variables; (ii) \( O(1) \)-approximation when \( f \) is a Top \( r \) norm; and (iii) \( O(\log m/\log \log m) \)-approximation for the most general setting. With an eye towards obtaining small approximation factors, Goel and Indyk [6] considered stochastic makespan minimization (on identical machines) when job-sizes follow a structured distribution. Among other results, they obtained a simple 2-approximation when jobs are Poisson-distributed. Very recently, there have been two works on improving the approximation factor for
stochastic min-norm load balancing with Poisson jobs (PoisNormLB). First, De, Khanna, Li and Nikpey [5] gave a PTAS for the makespan objective when machines are identical. Second, Ibrahimpur and Swamy [13] gave a $(2 + \epsilon)$-approximation for the most general version of PoisNormLB (arbitrary monotone symmetric norm objectives and unrelated-machines setting), and a PTAS when machines are identical.

2 Vector Scheduling

Recall that in $d$-dimensional vector scheduling, we have a set $J$ of vector jobs and $m$ identical machines. The size-vector of job $j$ is $p_j \in \mathbb{R}^d_{\geq 0}$. To keep the notation simple, we reserve $i \in [m]$ to index the machine-set, $j \in J$ to index the job-set, and $r \in [d]$ to index the dimensions. We seek an assignment $\sigma : J \rightarrow [m]$ that minimizes the makespan $\max_{i,r}(\sum_{j: \sigma(j)=i} p_{j,r})$. The natural lower bound $lb$ on the optimal makespan is $\max\{\max_{j,r} p_{j,r}, \max_r \sum_j p_{j,r}/m\}$ (see (1)). By scaling, we may assume without loss of generality that $lb = 1$; this implies $p_{j,r} \in [0, 1]$ for all $j, r$, and $\sum_j p_{j,r} \leq m$ for all $r$.

We prove our main result on vector scheduling (Theorem 1.1) by proving a slightly stronger result.

**Theorem 2.1.** Consider an instance of vector scheduling with $p_{j,r} \in [0, 1]$ for all $j \in J, r \in [d]$, and $\sum_{j \in J} p_{j,r} \leq m \log d$ for all $r \in [d]$. There is a randomized algorithm that produces an assignment $\sigma$ whose makespan is $O(\log d)$. The algorithm succeeds with probability at least $2/3$.

The only tool that we use in the proof of Theorem 2.1 is Chernoff tail bounds, which we state below.

**Lemma 2.2 (Chernoff Bounds).** Let $\{Z_j\}$ be a finite collection of independent $[0, 1]$-bounded random variables, and let $S = \sum_j Z_j$ denote their sum. Then,

(a) for any $\mu \geq \mathbb{E}[S]$ and $\delta \geq 0$, we have $\Pr[S \geq (1 + \delta)\mu] \leq \exp(-\frac{\delta^2 \mu}{2 + \delta})$.

(b) for any $\mu \leq \mathbb{E}[S]$ and $\delta \in [0, 1]$, we have $\Pr[S \leq (1 - \delta)\mu] \leq \exp(-\frac{\delta^2 \mu}{2})$.

As mentioned in Section 1.1, our algorithm for vector scheduling is based on finding a subset of jobs $S \subseteq J$ that can all be processed on a single machine by incurring a load of at most $O(\log d)$ in each dimension $r \in [d]$, while ensuring that the residual problem is also a valid sub-instance of the problem with $m - 1$ machines. We formalize the argument below.

**Lemma 2.3.** Suppose $m \geq 7$ and $d \geq 2$. Consider a random job-set $S \subseteq J$ where job $j \in J$ is independently included in $S$ with probability $q := 7/m$. With probability at least $1/2$, for all $r \in [d]$ we have $\sum_{j \in S} p_{j,r} \leq 14 \log d$ and $\sum_{j \notin S} p_{j,r} \leq (m - 1) \log d$.

**Proof.** The proof is based on a straightforward application of Chernoff bounds. By our choice of $q$, for any $r$ we have $\mathbb{E}[\sum_{j \in S} p_{j,r}] \leq 7 \log d$. Using Lemma 2.2(a) with $\delta = 1$, for any $r$ we have:

$$\Pr[\sum_{j \in S} p_{j,r} \geq 14 \log d] \leq \exp\left(-\frac{7 \log d}{3}\right) \leq \frac{1}{d^2}.$$  

Next, we use Chernoff bounds for the lower tail to prove the remaining size-bound on jobs in $J \setminus S$. Let $B := \{r \in [d] : \sum_{j \in J} p_{j,r} > (m - 1) \log d\}$. Since $m \geq 7$, for any $r \in B$ we have $\mathbb{E}[\sum_{j \in S} p_{j,r}] \geq (1 - 1/m) \cdot 7 \log d \geq 6 \log d$. Using Lemma 2.2(b) with $\delta = 5/6$ we get:

$$\Pr[\sum_{j \in S} p_{j,r} \leq \log d] \leq \exp\left(-\frac{25}{36} \cdot \frac{6 \log d}{2}\right) \leq \frac{1}{d^2}.$$  

By union-bound, with probability at least $1/2$, we have $\sum_{j \in S} p_{j,r} \leq 14 \log d$ for all $r \in [d]$, and $\sum_{j \in J} p_{j,r} \geq \log d$ for all $r \in B$. The desired bounds follow. ■
Proof of Theorem 2.1. We call an instance of vector scheduling as a valid instance with \( m \) machines if \( p_{j,r} \in [0,1] \) for all \( j, r \), and \( \sum_j p_{j,r} \leq m \log d \) for all \( r \). Note that in Theorem 2.1 we are given a valid instance with \( m \) machines. We will assume that \( d \geq 2 \), since otherwise a greedy list-scheduling algorithm gives an easy 2-approximation.

Let \( N := \left\lceil \log_2 (3m) \right\rceil = O(\log m) \). We now describe a randomized algorithm for vector scheduling. The overall procedure has at most \( m \) iterations, and if the algorithm is successful, it produces an assignment with makespan \( O(\log d) \). Consider the start of an iteration \( t \) for some \( t \geq 1 \). We have a valid instance with \( m - t + 1 \) identical machines. If \( m - t + 1 = 6 \), then we simply assign all remaining jobs to a single machine; note that \( \sum_j p_{j,r} \leq 6 \log d \) for all dimensions \( r \), so we are not assigning too much load to this machine in any dimension. Otherwise, \( m - t + 1 \geq 7 \). Consider random subsets \( S_1, \ldots, S_N \) where each \( S_t \subseteq J, \ell \in [N] \) is obtained by independently including job \( j \) in \( S_t \) with probability \( q := 7/m \). By Lemma 2.3, with probability at most \( 2^{-N} \leq \frac{1}{3m} \), none of the \( S_t \) sets satisfy the conclusion of the lemma; in this case we terminate the algorithm with a failure. Otherwise, for some \( \ell \in [N] \), we have for all \( r \), \( \sum_{j \in S_t} p_{j,r} \leq 14 \log d \) and \( \sum_{j \in J \setminus S_t} p_{j,r} \leq (m - t) \log d \). In this case, we assign all jobs in \( S_t \) to one machine and end the iteration with a valid instance with \( m - t + 1 \) machines and residual job-set \( J \setminus S_t \). The probability that the algorithm terminates with a failure in any given iteration is at most \( 1/3m \), so the overall failure probability of the algorithm is at most \( 1/3 \). In other words, with probability at least \( 2/3 \), we obtain an assignment whose makespan is at most \( 14 \log d \).

We remark that if we assume \( \sum_j p_{j,r} \leq m \cdot U \) for all dimensions \( r \) in the statement of Theorem 2.1 (where \( U \) does not depend on \( m \)), then we can easily adapt the above arguments to obtain makespan \( O(\max(\log d, U)) \). Suppose \( U > \log d \). Sampling jobs as before with probability \( 7/m \) now ensures that the random subset \( S \) satisfies \( \Pr \left[ \sum_{j \in S} p_{j,r} \leq 14U, \sum_{j \in J \setminus S} p_{j,r} \leq (m - 1)U \right] \geq 1 - \frac{2}{\ell} \), for \( m \geq 7 \). Given this, we proceed exactly as in Theorem 2.1, where a valid instance with \( m \) machines now means that the total load in each dimension \( r \) is at most \( m \cdot U \).

3 Stochastic Minimum-Norm Load Balancing

We now utilize our results on vector scheduling to derive approximation algorithms for stochastic minimum norm load balancing (StochNormLB), crucially leveraging the fact that our approximation guarantee for vector scheduling is with respect to the natural lower bound \( lb \) (see (1)).

Recall that in StochNormLB we have a set \( J \) of \( n \) independent stochastic jobs and \( m \) identical machines. The processing time of a job \( j \) is a nonnegative random variable \( X_j \) whose distribution is given to us. (We will only however need a certain kind of access to these distributions.) We are also given a monotone symmetric norm \( f : \mathbb{R}^m \to \mathbb{R}_{\geq 0} \). Our goal in StochNormLB is to find an assignment \( \sigma \) that minimizes \( \mathbb{E}[f(\text{load}^\sigma)] \), where \( \text{load}^\sigma_i := \sum_{j, \sigma(j) = i} X_j \) for \( i \in [m] \). Also recall that we use Stoch-\( g \)-LB to explicitly indicate the norm \( g \) that we are considering in an instance of StochNormLB.

In this section we prove Theorem 1.3 by giving a simultaneous \( O(\log \log m) \)-approximation for all Stoch-\( g \)-LB instances where \( g \) is a monotone, symmetric norm. Theorem 1.2 immediately follows from Theorem 1.3. The proof of Theorem 1.3 is based on a reduction to vector scheduling with \( O(\log m) \)-dimensional vectors, by utilizing suitable tools from prior work [15, 11]. We first introduce some notation. For a nonnegative random variable \( Z \) and a scalar \( \theta \), we define the truncated random variable \( Z^{< \theta} := Z \cdot 1_{Z < \theta} \) as the random variable that takes size \( Z \) when the event \( \{ Z < \theta \} \) happens, and size 0 otherwise. We also define the exceptional random variable \( Z^{\geq \theta} := Z \cdot 1_{Z \geq \theta} \) as the random variable that takes size \( Z \) when the event \( \{ Z \geq \theta \} \) happens, and size 0 otherwise. Note that \( Z = Z^{< \theta} + Z^{\geq \theta} \). We reserve \( g : \mathbb{R}^m \to \mathbb{R}_{\geq 0} \) to denote an arbitrary monotone symmetric norm and \( Y = (Y_1, \ldots, Y_m) \) to denote an arbitrary \( m \)-dimensional random vector whose coordinates \( \{ Y_i \}_{i \in [m]} \) form an independent collection of nonnegative random variables. For
brevity, we will say that $Y$ follows a product distribution on $\mathbb{R}_{\geq 0}^m$. Observe that load vectors $\text{load}^\sigma$ that arise in StochNormLB instances with $m$ machines follow a product distribution on $\mathbb{R}_{\geq 0}^m$.

Let $\text{POS} = \text{POS}(m) := \{1, 2, 4, \ldots, 2^{\lceil \log_2 m \rceil} \}$. The first tool that we utilize shows importantly that we can control $\mathbb{E}[g(Y)]$ by controlling a collection of simpler norms called $\text{Top}_\ell$ norms (Theorem 3.2).

**Definition 3.1 (Top$_\ell$ norm).** For any $\ell \in [m]$, the Top$_\ell$ norm is defined as follows: for $x \in \mathbb{R}_{\geq 0}^m$, $\text{Top}_\ell(x)$ is the sum of the $\ell$ largest coordinates of $x$, i.e., $\text{Top}_\ell(x) = \sum_{i=1}^{\ell} x_i^\uparrow$, where $x^\uparrow$ denotes the vector $x$ with its coordinates sorted in non-increasing order.

Like $\ell_p$ norms, Top$_\ell$ norms give us a way to interpolate between the max-norm (Top$_1$) and the sum-norm (Top$_m$). A well-known mathematical result called the majorization inequality [9] (see also [2]) shows that controlling all Top$_\ell$ norms gives us a handle on all monotone symmetric norms: for $x, y \in \mathbb{R}_{\geq 0}^m$, if $\text{Top}_\ell(x) \leq \text{Top}_\ell(y)$ for all $\ell \in [m]$, then $g(x) \leq g(y)$ for every monotone, symmetric norm $g$. Recently, in [11], we proved a substantial stochastic generalization (qualitatively speaking) of this result.

**Theorem 3.2 (Approximate Stochastic Majorization, Theorem 5.2 in [12]).** Let $Y = (Y_1, \ldots, Y_m)$ and $W = (W_1, \ldots, W_m)$ follow product distributions on $\mathbb{R}_{\geq 0}^m$, and let $\alpha$ be a positive scalar.

(i) If $\mathbb{E}[\text{Top}_\ell(Y)] \leq \alpha \mathbb{E}[\text{Top}_\ell(W)]$ for all $\ell \in [m]$, then $\mathbb{E}[g(Y)] \leq 28 \cdot \alpha \mathbb{E}[g(W)]$.

(ii) If $\mathbb{E}[\text{Top}_\ell(Y)] \leq \alpha \mathbb{E}[\text{Top}_\ell(W)]$ for all $\ell \in \text{POS}(m)$, then $\mathbb{E}[g(Y)] \leq 56 \cdot \alpha \mathbb{E}[g(W)]$.

By Theorem 3.2(ii), the task of proving Theorem 1.3 reduces to proving the following.

**Theorem 3.3.** Consider $n$ stochastic jobs $\{X_j\}_{j \in J}$ and $m$ identical machines. With probability at least $2/3$, we can obtain an assignment $\sigma : J \rightarrow [m]$ that is simultaneously an $O(\log \log m)$-approximation to all Stoch-Top$_\ell$-LB instances with $\ell \in \text{POS}$.

We prove Theorem 3.3 using an approximation algorithm for $|\text{POS}|$-dimensional vector scheduling with respect to the natural lower bound $lb$. Since $|\text{POS}| = O(\log m)$, and Section 2 presents such an $O(\log d)$-approximation algorithm for $d$-dimensional vector scheduling, we obtain the desired result.

### 3.1 Handling a single Top$_\ell$ norm

We next discuss how to tackle stochastic load balancing with a single Top$_\ell$ norm. Combining the ingredients here for all $\ell \in \text{POS}$ will yield our reduction to $|\text{POS}|$-dimensional vector scheduling (see Section 3.2).

We fix $\ell \in \text{POS}$ throughout this subsection. In [11], the authors give an $O(1)$-approximation for Stoch-Top$_\ell$-LB, even in the more-general setting of unrelated machines via an LP-based algorithm, but for identical machines, the underlying arguments can be substantially simplified. In particular, [11] show that, $\mathbb{E}[\text{Top}_\ell(Y)]$ can be approximated via a separable expression (in the $Y_i$s) when $Y$ follows a product distribution (Theorem 3.4), and show how one can deal with this proxy expression by generalizing certain arguments from [15]. Mimicking the proof strategy in [15], which gives an $O(1)$-approximation for Stoch-Top$_1$-LB, then shows that Stoch-Top$_\ell$-LB can be reduced to (scalar) makespan minimization (i.e., 1-dimensional vector scheduling). The following technical theorem gives a handle on $\mathbb{E}[\text{Top}_\ell(Y)]$ when $Y$ follows a product distribution on $\mathbb{R}_{\geq 0}^m$.

**Theorem 3.4.** Let $\theta$ be a positive scalar.

(i) (Lemma 4.1 in [12]) If $\sum_{i \in [m]} \mathbb{E}[Y_i^\geq \theta] \leq \ell \theta$, then $\mathbb{E}[\text{Top}_\ell(Y)] \leq 2\ell \theta$.

(ii) (Lemma 4.2 in [12]) If $\sum_{i \in [m]} \mathbb{E}[Y_i^\geq \theta] > \ell \theta$, then $\mathbb{E}[\text{Top}_\ell(Y)] > \ell \theta / 2$. 

7
The upshot of Theorem 3.4 is that if \( \theta \) is such that \( \sum_{i \in [m]} E[Y_i^{\geq \theta}] \) is roughly \( \ell \theta \), then the separable expression \( \sum_{i \in [m]} E[Y_i^{\geq \theta}] \) is a constant-factor proxy for \( E[Top_\ell(Y)] \). In StochNormLB, we have \( Y_i = \frac{\text{load}_j^\sigma}{\ell} \) (where \( \sigma : J \rightarrow [m] \) is the job-assignment), which is a sum of independent random variables, and we next need to address how to obtain a handle on \( E[Y_i^{\geq \theta}] \). Similar to [15, 8, 11], given the “scale” \( \theta \), we can decompose any instance of Stoch-Top_\ell-LB into an “easy” sub-instance, for which it is trivial to bound \( E[Y_i^{\geq \theta}] \) and any assignment yields an \( O(1) \)-approximation, and a (more) “difficult” sub-instance, where we need some technical results to control \( E[Y_i^{\geq \theta}] \) and obtain an \( O(1) \)-approximation. Lemmas 3.5 and 3.7 indicate how one may define these two subinstances.

**Lemma 3.5.** Let \( \theta \) be a positive scalar. Consider an instance of Stoch-Top_\ell-LB where for each job \( j \), the distribution of \( X_j \) is supported on \( \{0\} \cup [\theta, \infty) \) i.e., \( \Pr[0 < X_j < \theta] = 0 \).

(i) If \( \sum_{j \in J} E[X_j] \leq \ell \theta \), then for any assignment \( \sigma \), we have \( E[\text{Top}_\ell(\text{load}^\sigma)] \leq 2\ell \theta \).

(ii) If \( \sum_{j \in J} E[X_j] > \ell \theta \), then for any assignment \( \sigma \), we have \( E[\text{Top}_\ell(\text{load}^\sigma)] > \ell \theta / 2 \).

**Proof.** Fix an assignment \( \sigma \) and let \( Y := \text{load}^\sigma \) denote the induced random load vector. By Theorem 3.4, it suffices to argue that \( \sum_{i \in [m]} E[Y_i^{\geq \theta}] = \sum_{j \in J} E[X_j] \). Since the distribution of each \( X_j \) is supported on \( \{0\} \cup [\theta, \infty) \), the distribution of the \( i \)-th machine’s load \( Y_i = \sum_{j : \sigma(j) = i} X_j \) is also supported on \( \{0\} \cup [\theta, \infty) \). Thus,

\[
\sum_{i \in [m]} E[Y_i^{\geq \theta}] = \sum_{i \in [m]} E[Y_i] = \sum_{i \in [m]} \sum_{j : \sigma(j) = i} E[X_j] = \sum_{j \in J} E[X_j],
\]

and so both parts follow.

Thus, the difficulty in bounding \( E[Y_i^{\geq \theta}] \) (where \( Y_i = \frac{\text{load}_j^\sigma}{\ell} \)) arises when the \( X_j \) random variables are supported on \( [0, \theta] \). We need the notion of effective size to deal with such difficult instances.

**Definition 3.6 (Effective Size).** For a nonnegative random variable \( Z \) and a scalar parameter \( \lambda > 1 \), the \( \lambda \)-effective size \( \beta_\lambda(Z) \) of \( Z \) is defined as \( \log_\lambda E[Z^\alpha] \). We define \( \beta_1(Z) \) to be \( E[Z] \).

The benefit of effective sizes follows because Lemmas A.1 and A.2 (from [11]) show that, for independent random variables \( \{Z_j\} \), we can control \( E[(\sum_j Z_j^{\geq \theta})^{\geq \theta}] \) by bounding the effective sizes of certain random variables related to the \( Z_j^{\geq \theta} \) random variables. We leverage this to obtain the following result, which is implicit in [11] and is a generalization of Lemma 3.4 from [15]; for completeness, we present the proof in Appendix A, since this statement does not explicitly appear in prior work.

**Lemma 3.7.** Let \( \theta \) be a positive scalar and let \( \lambda := \frac{2m}{\ell} \). Consider an instance of stochastic Top_\ell-norm load balancing where for each job \( j \in J \), the distribution of \( X_j \) is supported on \( [0, \theta] \) i.e., \( \Pr[X_j \geq \theta] = 0 \).

(i) If \( \sum_{j \in J} \beta_\lambda(X_j/4\theta) \leq 8m \), then for any assignment \( \sigma \) we have \( E[\text{Top}_\ell(\text{load}^\sigma)] \leq 32 \cdot \alpha \cdot \ell \theta \), where \( \alpha := \max\{1, \max_{i \in [m]} \sum_{j : \sigma(j) = i} \beta_\lambda(X_j/4\theta)\} \).

(ii) If \( \sum_{j \in J} \beta_\lambda(X_j/4\theta) > 8m \), then for any assignment \( \sigma \) we have \( E[\text{Top}_\ell(\text{load}^\sigma)] > \ell \theta / 2 \).

Lemmas 3.5 and 3.7 yield the following approach for obtaining a constant-factor approximation for Stoch-Top_\ell-LB. Let \( \text{OPT}_\ell \) denote the optimal value. For a given scalar \( \theta \), “split” each job \( j \) as \( X_j = X_j^{< \theta} + X_j^{\geq \theta} \). This yields the exceptional sub-instance with the exceptional job-variables \( \{X_j^{\geq \theta}\}_j \), and the
truncated sub-instance with the truncated job-variables \( \{X_j^{<\theta}\}_j \). Clearly, for any assignment \( \sigma \), if \( Y \) and \( Y' \) denote the respective load vectors in these two sub-instances, we have \( \text{load}_\sigma^Y = Y + Y' \).

Let \( \lambda = \left[ \frac{2m}{\ell t} \right] \). Now, we can use binary search to find the right threshold \( t_\ell \), such that for \( \theta = t_\ell \), the conditions in part (i) of Lemmas 3.5 and 3.7 hold, and for some \( \theta \geq t_\ell / (1 + \epsilon) \), the opposite is true. This is because Lemmas 3.5 and 3.7 imply that for \( \theta \) large enough, the conditions in part (i) of these lemmas hold; we flesh out the binary-search interval at the end of Section 3.2. So we have: (1) \( \sum_{j \in J} E[X_j^{<\ell t}] \leq \ell t \) and \( \sum_{j \in J} \beta_j(X_j^{<\ell t} / \ell t) \leq 8m \); and (2) for some \( \theta \geq t_\ell / (1 + \epsilon) \), we have \( \sum_{j \in J} E[X_j^{>\theta}] > 8m \). Due to (2), we obtain that the optimum value for the exceptional or truncated sub-instance is \( \Omega(\ell t) \), and hence \( \text{OPT}_\ell = \Omega(\ell t) \). Due to (1), by part (i) of Lemma 3.5, the expected \( \text{Top}_\ell \)-load of the assignment \( \sigma \) in the exceptional sub-instance is \( O(\ell t) \). Also, considering deterministic scheduling with job sizes \( p_j := \beta_j(X_j^{<\ell t} / \ell t) \) (which is at most 1), we can easily compute an assignment \( \sigma \) that assigns total \( p_j \)-load at most \( \sum_{j \in J} p_j + 1 \leq 9 \) on any machine; part (i) of Lemma 3.7 then shows that the expected \( \text{Top}_\ell \)-load of \( \sigma \) in the truncated sub-instance is also \( O(\ell t) \) (at most \( 32 \cdot 9 \cdot \ell t \)). Therefore, we have \( E[\text{Top}_\ell(\text{load}_\sigma)] = O(\ell t) \), showing that \( \sigma \) is an \( O(1) \)-approximate solution to the given instance of Stoch-Top\( _\ell \)-LB.

### 3.2 Proof of Theorem 3.3: simultaneous approximation for StochNormLB via vector scheduling

We now prove Theorem 3.3 and, as a consequence, Theorem 1.3 as well. Recall that we seek an assignment \( \sigma \) is simultaneously a \( O(\log \log m) \)-approximation to Stoch-Top\( _\ell \)-LB for all \( \ell \in \text{POS} = \{1, 2, 4, \ldots, 2^{|x_{2^m}|}\} \). Let \( \mathcal{A} \) be a \( \gamma(d) \)-approximation algorithm for the \( d \)-dimensional vector scheduling problem with respect to the natural lower bound \( lb \). As we show below, we use \( \mathcal{A} \) in a black-box fashion, so if \( \mathcal{A} \) is deterministic (respectively randomized), then our simultaneous-approximation for StochNormLB is also deterministic (respectively randomized).

The idea is simple: we essentially follow the approach used to obtain an \( O(1) \)-approximation for Stoch-Top\( _\ell \)-LB (outlined above) for each \( \ell \in \text{POS} \) separately. More precisely, we first find the right \( t_\ell \), for each \( \ell \in \text{POS} \) separately, via binary search. That is, for each \( \ell \in \text{POS} \), letting \( \lambda_\ell = \left[ \frac{2m}{\ell t} \right] \), we have that: (i) \( \sum_{j \in J} E[X_j^{<\ell t}] \leq \ell t \), (ii) \( \sum_{j \in J} \beta_j(X_j^{<\ell t} / \ell t) \leq 8m \), and (iii) \( \text{OPT}_\ell = \Omega(\ell t) \). We defer the details of this binary search to the end of the proof. If we consider the exceptional and truncated sub-instances for an index \( \ell \in \text{POS} \), then, as before, the expected \( \text{Top}_\ell \)-load for the exceptional sub-instance is \( O(\ell t) \), under any assignment. So we only need to consider the truncated sub-instances for \( \ell \in \text{POS} \) (with the truncated random variables \( \{X_j^{<\ell t}\}_j \)).

Define \( p_{j,\ell} := \beta_j(X_j^{<\ell t} / \ell t) \) for each job \( j \in J \) and index \( \ell \in \text{POS} \). By Lemma 3.7 (i), it order to obtain expected \( \text{Top}_\ell \)-load of \( O(\alpha) \cdot \text{OPT}_\ell \) for each \( \ell \in \text{POS} \), it suffices to find an assignment \( \sigma \) such that, for each \( \ell \in \text{POS} \), the total \( p_{j,\ell} \)-load on each machine \( i \) is at most \( \alpha \). In other words, considering the job vectors \( p_j = (p_{j,\ell})_{\ell \in \text{POS}} \in \mathbb{R}_{\geq 0}^{|\text{POS}|} \) for each job \( j \in J \), we seek a solution to this vector-scheduling instance with makespan \( \alpha \).

When we have only one index \( \ell \) (i.e., Stoch-Top\( _\ell \)-LB), it is easy to argue (e.g., using Graham’s list scheduling) that \( \sum_{j \in J} p_{j,\ell} = O(m) \) and \( p_{j,\ell} \leq 1 \) for all \( j, \ell \), implies that we can assign the jobs so that the total \( p_{j,\ell} \)-load on each machine is \( O(1) \). In our vector-scheduling instance, we have a bound on the total-load \( p_{j,\ell} \) load on each coordinate \( \ell \) (due to (ii)), and we know that \( p_{j,\ell} \leq 1 \) for all \( j, \ell \). Thus, the natural lower bound \( lb \) for this vector-scheduling instance is at most \( 8 \), and what we now need is an \( lb \)-relative approximation algorithm for vector scheduling. We can therefore utilize \( \mathcal{A} \) to obtain makespan \( \gamma(\log m) \) for our vector-scheduling instance, which thus yields an \( O(\gamma(\log m)) \) approximation to Stoch-Top\( _\ell \)-LB for all \( \ell \in \text{POS} \). Theorem 2.1 yields \( \gamma = O(\log d) \), which yields a simultaneous \( O(\log \log m) \)-approximation.

Finally, we furnish the details of the binary-search procedure to find the \( t_\ell s \). Fix an index \( \ell \in \text{POS} \).
We first establish suitable upper and lower bounds for \( t_\ell \). Define \( \kappa := \max_{j \in J} E[X_j] \) (which can be easily computed from input data). Let \( W \) denote the random load vector induced by an optimal solution and \( \text{OPT}_\ell := E[\text{Top}_\ell(W)] \) denote the optimal value. By the contrapositive of part (ii) of Lemmas 3.5 and 3.7, for any \( \theta \geq 2\text{OPT}_\ell/\ell \) the following inequalities hold: \( \sum_{j \in J} E[X_j^{2\theta}] \leq \ell \theta \) and \( \sum_{j \in J} E[X_j^{<\theta}] / \ell \theta \leq 8m \). In particular, since \( \text{OPT}_\ell \leq E[\sum_{i \in [m]} W_i] = \sum_{j \in J} E[X_j] \leq nk \), these two inequalities hold for \( \theta = hi := 2n\kappa \geq 2\text{OPT}_\ell/\ell \). Next, observe that for any job \( k \) and \( \theta = E[X_k] / (m + 2) \), we have
\[
\sum_j E[X_j^{\geq \theta}] \geq E[X_k^{\geq \theta}] \geq E[X_k] - \theta = (m + 1)\theta > \theta.
\]
Thus, for any \( \theta \leq \text{low} := \kappa / (m + 2) \), we have \( \sum_j E[X_j^{\geq \theta}] > \ell \theta \).

Let \( \epsilon > 0 \) be a small constant (say, \( 1/1000 \)). Then, by doing a binary search in the interval \([\text{low}, \text{hi}]\), we can find scalars \( t_\ell \) and \( t_\ell' \) satisfying:

(i) \( t_\ell' < t_\ell \leq t_\ell'(1 + \epsilon) \).

(ii) \( \sum_{j \in J} E[X_j^{\geq t_\ell}] \leq \ell t_\ell \) and \( \sum_{j \in J} \beta_\ell(X_j^{<t_\ell} / 4t_\ell) \leq 8m \).

(iii) \( \sum_{j \in J} E[X_j^{\geq t_\ell'}] > \ell t_\ell' \) or \( \sum_{j \in J} \beta_\ell(X_j^{<t_\ell'} / 4t_\ell') > 8m \).

Properties (i) and (iii) imply that \( \text{OPT}_\ell > \ell t_\ell'/2 \geq \ell t_\ell / 2(1 + \epsilon) \).

**Remark 1.** If the algorithm \( A \) is randomized (as in Theorem 2.1), then we detect if the assignment \( \sigma \) returned by solving vector scheduling satisfies \( \sum_{j: \sigma(j) = i} p_{j, \ell} \leq O(\log |\text{POS}|) \) for all \( \ell \in \text{POS} \) and \( i \in [m] \), i.e., is the approximation guaranteed by \( A \), and if not, return failure. Then, the success probability in the statement of Theorem 3.3 is the probability of success of \( A \) (which lower bounds the probability of obtaining a simultaneous \( O(\log \log m) \)-approximation).

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A Omitted proofs from Section 3

We give a proof of Lemma 3.7 in this section. The following two technical lemmas on effective sizes will be useful to us.

**Lemma A.1** (Lemma 3.8 in [12]). Let $Z$ be a nonnegative random variable and $\lambda \geq 1$. If $\beta_\lambda(Z) \leq b$, then

$$\Pr[Z \geq b + c] \leq \lambda^{-c}, \text{ for any } c \geq 0.$$ 

Furthermore, if $\lambda \geq 2$, then

$$E[Z^{\beta_\lambda(Z)+1}] \leq (\beta_\lambda(Z) + 3)/\lambda.$$

**Lemma A.2** below is a partial converse to Lemma A.1.
Lemma A.2 (Lemma 3.9 in [12]). Let $\lambda \geq 1$ be an integer and $\theta$ be a positive scalar. Let $S = \sum_j Z_j$ be a sum of independent $[0, \theta]$-bounded random variables. Then,

$$E[S^{\geq \theta}] \geq \frac{\theta}{4\lambda} \cdot \left( \sum_j \beta_\lambda \left( \frac{Z_j}{4\theta} \right) - 6 \right).$$

Proof of Lemma 3.7. Recall that $\ell \in [m]$ is a positive integer, $\theta$ is a positive scalar, and $\lambda := \lceil \frac{2m}{\ell} \rceil$. Fix an assignment $\sigma$ of jobs to machines, and let $Y := \frac{1}{4\theta} \cdot \text{load}^\sigma$ denote the induced load vector scaled down by a factor $4\theta$. For the first part, suppose that the total effective size of all jobs is at most $8m$: $\sum_{j \in J} \beta_\lambda \left( \frac{X_j}{4\theta} \right) \leq 8m$. We want to argue that the expected Top$_\ell$ norm of the induced load vector is bounded from above. Recall that $\alpha := \max\{1, \max_{i \in [m]} \sum_{j: \sigma(j) = i} \beta_\lambda (X_j/4\theta)\}$ is the maximum of $1$ and the makespan of $\sigma$ w.r.t. $\beta_\lambda (\cdot)$-sizes. Since $X_j$s are independent random variables, for any $i \in [m]$ we have $\beta_\lambda (Y_i) = \sum_{j: \sigma(j) = i} \beta_\lambda \left( \frac{X_j}{4\theta} \right) \leq \alpha$. By Lemma A.1:

$$E[Y_i^{\alpha+1}] \leq E[Y_i^{\beta_\lambda(Y_i)+1}] \leq \beta_\lambda(Y_i) + 3 \leq \frac{\alpha + 3}{\lambda}.$$ 

Summing over all $i \in [m]$, we get $\sum_{i \in [m]} E[Y_i^{\alpha+1}] \leq (\alpha + 3) \frac{m}{\lambda} \leq (\alpha + 3) \ell$, where we use that $\lambda = \lceil \frac{2m}{\ell} \rceil \geq m/\ell$. Using Theorem 3.4(i), with the scalar parameter $\alpha + 3$, we get $E[\text{Top}_\ell(Y)] \leq 2(\alpha + 3) \ell$. By homogeneity of norms (and noting that $\alpha \geq 1$), we get

$$E[\text{Top}_\ell(\text{load}^\sigma)] = 4\theta E[\text{Top}_\ell(Y)] \leq 8(\alpha + 3) \ell \theta \leq 32 \cdot \alpha \cdot \ell \theta.$$

For the second part, suppose that the total effective size of all jobs is larger than $8m$ i.e., $\sum_{j \in J} \beta_\lambda \left( \frac{X_j}{4\theta} \right) > 8m$. We want to argue that the expected Top$_\ell$ norm of the induced load vector is bounded from below. For any $i \in [m]$, $Y_i = \sum_{j: \sigma(j) = i} (X_j/4\theta)$ is a sum of independent $[0,1]$-bounded random variables. By Lemma A.2, we get:

$$\sum_{i \in [m]} E[Y_i^{\geq 1}] \geq \frac{1}{4\lambda} \cdot \sum_{i \in [m]} \left( \sum_{j: \sigma(j) = i} \beta_\lambda \left( \frac{X_j}{4\theta} \right) - 6 \right) = \frac{1}{4\lambda} \cdot \left( \sum_{j \in J} \beta_\lambda \left( \frac{X_j}{4\theta} \right) - 6m \right) > \frac{(8m - 6m)}{4 \cdot \frac{2m}{\ell}} = \ell/4,$$

where we use $\lambda \geq 2m/\ell$. Using Theorem 3.4(ii), with the scalar parameter $1/4$, we get $E[\text{Top}_\ell(Y)] > \ell/8$. Subsequently, $E[\text{Top}_\ell(\text{load}^\sigma)] = 4\theta E[\text{Top}_\ell(Y)] > \ell \theta/2$. \hfill \blacksquare