A note on estimation of $\alpha$-stable CARMA processes sampled at low frequencies

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In this paper, we investigate estimators for symmetric $\alpha$-stable CARMA processes sampled equidistantly. Simulation studies suggest that the Whittle estimator and the estimator presented in García et al. (2011) are consistent estimators for the parameters of stable CARMA processes. For CARMA processes with finite second moments it is well-known that the Whittle estimator is consistent and asymptotically normally distributed. Therefore, in the light-tailed setting the properties of the Whittle estimator for CARMA processes are similar to those of the Whittle estimator for ARMA processes. However, in the present paper we prove that, in general, the Whittle estimator for symmetric $\alpha$-stable CARMA processes sampled at low frequencies is not consistent and highlight why simulation studies suggest something else. Thus, in contrast to the light-tailed setting the properties of the Whittle estimator for heavy-tailed ARMA processes can not be transferred to heavy-tailed CARMA processes. We elaborate as well that the estimator presented in García et al. (2011) faces the same problems. However, the Whittle estimator for stable CAR(1) processes is consistent.

AMS Subject Classification 2010: Primary: 62M86, 62F12, 62M10
Secondary: 60G10, 62F10

Keywords: autocovariance function, CARMA process, consistency, periodogram, Ornstein-Uhlenbeck process, stable Lévy process, state space model, Whittle estimator

1. Introduction

Continuous-time ARMA (CARMA) processes are the continuous-time versions of the well-known ARMA processes in discrete time. Since CARMA processes with finite second moments sampled equidistantly are weak ARMA processes, several methods to estimate the parameters of ARMA processes as, e.g., quasi-maximum likelihood estimators or Whittle estimators, can be utilized to estimate the parameters of CARMA processes. A challenge is that the distribution of the white noise of the
weak ARMA representation is highly dependent on the model parameters of the CARMA process and only a weak white noise instead of a strong white noise. Moreover, some attention has to be paid to guarantee the model identifiability. These generalisations are technical and take some effort. [Schlemm and Stelzer (2012)] derived a rigorous theory for quasi maximum-likelihood estimation for equidistantly sampled multivariate CARMA processes with finite second moments; see as well [Brockwell and Lindner (2019)] and [Brockwell et al. (2011)] for CARMA processes. The method itself is much older and applied in several disciplines, in particular, in economics, even though the statistical inference of that estimator was as far as we know not investigated earlier. Since this procedure essentially depends on the variance of the sampled white noise, it is unsuitable for estimating the parameters of heavy-tailed CARMA processes. The consistency and the asymptotic normality of the Whittle estimator for multivariate CARMA processes with finite second moments were recently derived in [Fasen-Hartmann and Mayer (2020)]. The limit covariance matrices of these estimators are slightly different to those of the corresponding ARMA estimators. Apart from that, the structure of the estimator is the same taking the identifiability assumptions into account.

The topic of this paper is the estimation of symmetric $\alpha$-stable CARMA processes with infinite second moments sampled at low frequencies. To the best of our knowledge there exist no estimators for heavy-tailed CARMA processes in the literature yet. Only stable Ornstein-Uhlenbeck processes, which correspond to the class of CARMA(1,0) processes, are investigated, e.g., in [Hu and Long (2007, 2009), Fasen (2013a) and Ljungdahl and Podolskij (2020)]. [Garcia et al. (2011)] proposed an indirect quasi-maximum likelihood method for stable CARMA processes. Their simulation study suggests that the estimator is converging when the number of observations tends to infinity, however they do not present a mathematical analysis of their estimator. In this paper we mainly investigate the Whittle estimator in more detail. The Whittle estimator was first introduced by [Whittle (1953)] to estimate the parameters of Gaussian ARMA processes and further explored in [Hannan (1973)]. The more general case of Whittle estimation of multivariate ARMA processes with finite second moments was topic of [Dunsmuir and Hannan (1976)]. [Mikosch et al. (1995)] observed that the same method also works for symmetric $\alpha$-stable ARMA processes with infinite second moments. This is not apparent since the spectral density is not defined for processes with infinite second moments and the Whittle estimator is based on its empirical version, the periodogram. [Mikosch et al. (1995)] derived the consistency and the convergence of the properly normalized and standardized Whittle estimator to a functional of stable random variables. Therefore, Whittle estimation is an estimation method for both heavy-tailed and light-tailed ARMA processes. The estimator is consistent in both settings which is a very important property. Indeed, the convergence rate of the Whittle estimator is $n^{1/\alpha}$ in the stable case which is even faster than in the case with finite second moments, where it is $n^{1/2}$. It would thus be desirable to have the same for heavy and light-tailed CARMA processes.

Similarly, for heavy-tailed fractional ARIMA processes, which exhibit long range dependence, the Whittle estimator is consistent and the asymptotic behaviour is known (see [Kokoszka and Taqqu (1996)]). The Whittle estimator was also used for parameter estimation of GARCH processes in [Giraitis and Robinson (2001) and Mikosch and Straumann (2002)]. For GARCH(1,1) processes [Mikosch and Straumann (2002)] pointed out that the Whittle estimator is consistent as long as the 4th moment is finite and inconsistent when the 4th moment is infinite. This is a very interesting statement to which we come back later.

Therefore, the behaviour of the Whittle estimator for heavy-tailed ARMA models and the fact that the Whittle estimator also works for light-tailed CARMA processes (see [Fasen-Hartmann and Mayer (2020)]) might suggest that the Whittle estimator is suitable for parameter estimation of heavy-tailed CARMA processes. In particular, there exist simulation studies which confirm this idea. However,
the main statement of the present paper is that the Whittle estimator for symmetric \( \alpha \)-stable CARMA processes is in general not a consistent estimator, even though there are simulation experiments which suggest something else. There is also evidence that the estimator of [García et al. (2011)](#) is not consistent as well. We elaborate that in more detail and present several arguments for our conjecture.

The paper is structured in the following way. We start with an introduction on symmetric \( \alpha \)-stable CARMA processes in Section 2 and present some basic facts on Whittle estimation for CARMA processes with finite second moments in Section 3 which motivates our approach. Then, in Section 4, we show the convergence of the Whittle function for symmetric \( \alpha \)-stable CARMA processes and deduce that the Whittle estimator is not consistent for general symmetric \( \alpha \)-stable CARMA processes. However, we show that the Whittle estimator is a consistent estimator for symmetric \( \alpha \)-stable CARA(1) processes. Finally, in Section 5, we demonstrate the theoretical results through a simulation study where we compare the performance of our estimator with the estimator proposed in [García et al. (2011)](#). The behaviour of the Whittle estimator and the behaviour of the estimator of [García et al. (2011)](#) is very similar in our simulation study. In particular, in the simulation setup of [García et al. (2011)](#) the Whittle estimator performs excellent and leads to the presumption that the Whittle estimator is converging. Conclusions are given in Section 6. Some auxiliary results on the asymptotic behaviour of the sample autocovariance function of symmetric \( \alpha \)-stable CARMA processes are postponed to the Appendix.

**Notation**

We write \( \mathcal{B}(A) \) for the Borel-\( \sigma \)-algebra of a set \( A \). The space \( L^p(A) \) for \( A \in \mathcal{B}(A) \) and \( 0 < p < \infty \) denotes the set of measurable functions \( f : A \rightarrow \mathbb{R} \) which satisfy \( \int_A |f(\cdot)|^p \, dt < \infty \). Furthermore, for some function \( f : A \rightarrow \mathbb{R} \) we write \( f^+ \) for the positive part \( f^+(\cdot) = \max\{0, f(\cdot)\} \) and \( f^- \) for the negative part \( f^-(\cdot) = \max\{0, -f(\cdot)\} \). The function \( \Gamma \) describes the Gamma function. The indicator function of a set \( A \) is denoted by \( 1_A \) and the signum function by \( \text{sign}(z) = 1_{\{z > 0\}} - 1_{\{z < 0\}} \). For the real and imaginary part of a complex valued \( z \) we use the notation \( \Re(z) \) and \( \Im(z) \), respectively. The \( N \)-dimensional identity matrix is denoted as \( I_N \). For some matrix \( A \), we write \( A^\top \) for its transpose. For the gradient operator we use the notation \( \nabla \) and we write consequently \( \nabla g(\vartheta_0) \) as shorthand for \( [\frac{\partial}{\partial \vartheta_1} g(\vartheta), \frac{\partial}{\partial \vartheta_2} g(\vartheta), \ldots]^{\top} |_{\vartheta = \vartheta_0} \) for any vector-valued function \( g \). The symbols \( \overset{\text{a.s.}}{\rightarrow}, \overset{p}{\rightarrow} \) and \( \overset{D}{\rightarrow} \) denote almost sure convergence, convergence in probability and convergence in distribution, respectively.

## 2. Symmetric \( \alpha \)-stable CARMA processes

In this paper, we consider a CARMA process driven by a symmetric \( \alpha \)-stable Lévy process.

**Definition 2.1.** A random variable \( Z \) is called \( \alpha \)-stable distributed, \( \alpha \in (0,2] \), if \( Z \) has the characteristic function \( \mathbb{E}(e^{itZ}) = \exp(\varphi_Z(t)) \) with

\[
\varphi_Z(z) = \begin{cases} 
-\sigma^\alpha |z|^\alpha (1 - i\beta(\text{sign}(z)) \tan \left( \pi \frac{\alpha}{2} \right)) + i\mu z, & \text{for } \alpha \neq 1, \\
-\sigma |z|^\alpha (1 + i\beta(\text{sign}(z))) \log |z| (\frac{\alpha}{\beta}) + i\mu z, & \text{for } \alpha = 1,
\end{cases}
\]

and \( \beta \in [-1, 1], \sigma > 0 \) and \( \mu \in \mathbb{R} \). The parameter \( \alpha \) is the index of stability. We write \( Z \sim \mathcal{S}_\alpha(\sigma, \beta, \mu) \).

In the case \( \beta = 0 \) and \( \mu = 0 \), the random variable \( Z \) is symmetric. For \( \alpha = 2 \) we get a normally distributed random variable. Furthermore, for \( \alpha \in (0,2) \) not all moments of \( Z \) exist. To be more
precise
\[ \mathbb{E}|Z|^p < \infty \quad \text{for} \ 0 < p < \alpha \quad \text{and} \quad \mathbb{E}|Z|^p = \infty \quad \text{for} \ p \geq \alpha. \]  

(2.1)

Important for us is likewise that in the symmetric case
\[ \lim_{n \to \infty} n^{1/\alpha} \mathbb{P}(|Z| > n^{1/\alpha}) = C_\alpha \sigma^\alpha \]

with
\[ C_\alpha = \begin{cases} \frac{1}{\Gamma(1 - \alpha) \cos(\pi \alpha/2)}, & \text{if} \ \alpha \neq 1 \ , \\ \frac{1}{\alpha}, & \text{if} \ \alpha = 1 \ . \end{cases} \]

(2.2)

see Property 1.2.15 of [Samorodnitsky and Taqqu (1994)] where more details on stable distributions can be found as well.

A one-sided Lévy process \( (L_t)_{t \geq 0} \) is a stochastic process with stationary and independent increments satisfying \( L_0 = 0 \) almost surely and having continuous in probability sample paths; see the book of [Sato (1999)] on Lévy processes. A two-sided Lévy process \( L = (L_t)_{t \in \mathbb{R}} \) can be constructed from two independent one-sided Lévy processes \( (L^{[1]}_t)_{t \geq 0} \) and \( (L^{[2]}_t)_{t \geq 0} \) through

\[ L_t = L^{[1]}_t 1_{\{t \geq 0\}} - \lim_{s \to t^-} L^{[2]}_s 1_{\{t < 0\}}, \quad t \in \mathbb{R}. \]

Then, an \( \alpha \)-stable Lévy process \( L^{(\alpha)} = (L^{(\alpha)}_t)_{t \in \mathbb{R}} \) is a Lévy process with increments

\[ L^{(\alpha)}_t - L^{(\alpha)}_s \sim \sigma(\alpha(t - s)^{1/\alpha}, \beta, \mu), \quad s < t, \]

and it is symmetric, if additionally \( \beta = \mu = 0 \).

In the following, we consider a parametric family of stationary CARMA processes. Let \( \Theta \subseteq \mathbb{R}^{N(\theta)}, N(\Theta) \in \mathbb{N}, \) be a parameter space, \( p \in \mathbb{N} \) be fixed and for any \( \theta \in \Theta \) let \( a_1(\theta), \ldots, a_p(\theta), c_0(\theta), \ldots, c_{p-1}(\theta) \in \mathbb{R}, a_p(\theta) \neq 0 \) and \( c_j(\theta) \neq 0 \) for some \( j \in \{0, \ldots, p-1\} \). Furthermore, define

\[
A(\theta) := \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \ldots & 0 & 1 \\
a_p(\theta) & a_{p-1}(\theta) & \ldots & -a_1(\theta)
\end{pmatrix} \in \mathbb{R}^{p \times p},
\]

\[
g(\theta) = \inf\{ j \in \{0, \ldots, p-1\} : c_l(\theta) = 0 \ \forall \ i > j \} \quad \text{with} \quad \inf\emptyset := p - 1,
\]

\[
c(\theta) := (c_{q_1(\theta)}(\theta), c_{q_2(\theta)-1}(\theta), \ldots, c_0(\theta), 0, \ldots, 0) \top \in \mathbb{R}^p.
\]

Throughout the paper we assume that the eigenvalues of \( A(\theta) \) have strictly negative real parts for \( \theta \in \Theta \). The family of CARMA processes \( Y(\theta) = (Y_t(\theta))_{t \geq 0} \) is then defined via the controller canonical state space representation: Let \( (L_t)_{t \geq 0} \) be a Lévy process and let \( (X_t(\theta))_{t \geq 0} \) be a strictly stationary solution to the stochastic differential equation

\[ dX_t(\theta) = A(\theta)X_t(\theta) \ dt + e_\theta \ dL_t, \quad t \geq 0, \]

(2.3a)
where $e_p$ denotes the $p$-th unit vector in $\mathbb{R}^p$. Then the process

$$Y_t(\vartheta) := c(\vartheta)^\top X_t(\vartheta), \quad t \geq 0,$$

(2.3b)
is called CARMA process of order $(p, q(\vartheta))$. If the Lévy process is $\alpha$-stable then $Y(\vartheta)$ is an $\alpha$-stable CARMA process. Necessary and sufficient conditions for the existence of strictly stationary CARMA processes are given in [Brockwell and Lindner (2009)]. In particular, the assumptions are satisfied for $\alpha$-stable Lévy processes and Lévy processes with finite second moments. A CARMA process can be interpreted as the stationary solution to the formal $p$-th order stochastic differential equation

$$a_\vartheta(D)Y_t(\vartheta) = c_\vartheta(D)D_t, \quad t \geq 0,$$

where $D$ denotes the differential operator with respect to $t$ and

$$a_\vartheta(z) := z^p + a_1(\vartheta)z^{p-1} + \ldots + a_p(\vartheta)$$

and $c_\vartheta(z) := c_0(\vartheta)z^q + c_1(\vartheta)z^{q-1} + \ldots + c_q(\vartheta)(\vartheta)$

are the autoregressive and the moving average polynomials, respectively. Hence, CARMA processes can be seen as the continuous-time analogue of discrete-time ARMA processes. Due to (2.3) the CARMA process $Y(\vartheta)$ also has the moving average (MA) representation

$$Y_t(\vartheta) = \int_{-\infty}^{\infty} g_\vartheta(t-s) dL_t^{(\alpha)}, \quad t \geq 0, \quad \text{with} \quad g_\vartheta(t) = c(\vartheta)^\top e^{A(\vartheta)t} e_p \mathbb{1}_{[0, \infty)}(t),$$

(2.4)

which plays an important role in this paper.

3. Whittle estimation for CARMA processes with finite second moments

Before we present the results for Whittle estimation of discrete-time sampled symmetric $\alpha$-stable CARMA processes $Y = (Y_t)_{t \geq 0} := (Y_t(\vartheta_0))_{t \geq 0} := Y(\vartheta_0)$, $\vartheta_0 \in \Theta$, with observations $Y_{\Delta t}, \ldots, Y_{n\Delta}$ ($\Delta > 0$ fixed), we repeat the basic results on Whittle estimation of CARMA processes with finite second moments which motivates our approach. This includes, in particular, Brownian motion driven CARMA processes which are 2-stable. For a CARMA process $Y(\vartheta)$ driven by a Lévy process $L$ with $\mathbb{E}(L_1) = 0$ and $0 < \mathbb{E}(L_1^2) = \sigma_L^2 < \infty$ the spectral density is

$$f_Y(\omega, \vartheta) = \frac{\sigma_L^2}{2\pi} \frac{|c_\vartheta(i\omega)|^2}{|a_\vartheta(i\omega)|^2}, \quad \omega \in \mathbb{R},$$

and for the discrete-time sampled process $Y^{(\Delta)}(\vartheta) := (Y_{k\Delta}(\vartheta))_{k \in \mathbb{N}}$ it is

$$f_Y^{(\Delta)}(\omega, \vartheta) = \frac{\sigma_L^2}{2\pi} \int_0^{\Delta} \left| \sum_{j=-\infty}^{\infty} g_\vartheta(u+j\Delta) e^{i\omega j} \right|^2 du$$

$$= \frac{\sigma_L^2}{2\pi} \int_0^{\Delta} \left| c(\vartheta)^\top e^{A(\vartheta)u} (I_p - e^{A(\vartheta)\Delta+i\omega \Delta})^{-1} e_p \right|^2 du, \quad \omega \in [-\pi, \pi],$$

see [Fasen (2013b), Example 2.4]. The empirical version of the spectral density is the periodogram

$$I_n(\omega) = \frac{1}{2\pi n} \left| \sum_{k=1}^{n} Y_k^{(\Delta)} e^{ik\omega} \right|^2 = \frac{1}{2\pi} \sum_{h=-n+1}^{n-1} \gamma_n(h) e^{-ih\omega}, \quad \omega \in [-\pi, \pi],$$

(3.1)
where
\[
\mathcal{Y}_n(h) := \frac{1}{n} \sum_{k=1}^{n-h} Y_k^{(\Delta)} Y_{k+h}^{(\Delta)} := \mathcal{Y}_n(-h) \quad \text{for } h = 0, \ldots, n-1,
\] (3.2)
is the empirical autocovariance function. The Whittle estimator is based on the distance between the spectral density and the periodogram. To be more precise, the Whittle estimator \( \hat{\theta}_n \) is the minimizer of the Whittle function
\[
W_n(\theta) := \frac{1}{2n} \sum_{j=-n+1}^{n} \left( f^{(\Delta)}_W(\omega_j, \theta)^{-1} I_n(\omega_j) + \log \left( f^{(\Delta)}_W(\omega_j, \theta) \right) \right), \quad \theta \in \Theta,
\]
with \( \omega_j = \frac{\pi j}{n} \) for \( j = -n+1, \ldots, n \). Fasen-Hartmann and Mayer (2020), Theorem 1, show that under very general assumptions the Whittle estimator is consistent and asymptotically normally distributed.

Since the spectral density depends on the variance \( \sigma^2 \) of the driving Lévy process, Fasen-Hartmann and Mayer (2020) present an adjusted version of the Whittle estimator which is independent of the variance parameter \( \sigma^2 \). However, to derive this estimator some background on the Kalman filter is required.

The Kalman filter is going back to Kalman (1960) and is well investigated in control theory (see Chui and Chen (2009)) and time series analysis (see Brockwell and Davis (1991)); in the context of CARMA processes we refer to Schlemm and Stelzer (2012). Therefore, note that the Riccati equation
\[
\Omega^{(\Delta)}(\theta) = e^{A(\theta)\Delta} \Omega^{(\Delta)}(\theta) e^{A(\theta)\Delta}^\top + \int_0^\Delta e^{A(\theta)u} u e_p^\top e_p e^{A(\theta)^\top} u du
\]
\[- \left( e^{A(\theta)\Delta} \Omega^{(\Delta)}(\theta) c(\theta) \right) \left( c(\theta)^\top \Omega^{(\Delta)}(\theta) c(\theta) \right)^{-1} \left( e^{A(\theta)\Delta} \Omega^{(\Delta)}(\theta) c(\theta) \right)^\top \]
has an unique positive semidefinite solution \( \Omega^{(\Delta)}(\theta) \) such that the Kalman gain matrix
\[
K^{(\Delta)}(\theta) = \left( e^{A(\theta)\Delta} \Omega^{(\Delta)}(\theta) c(\theta) \right) \left( c(\theta)^\top \Omega^{(\Delta)}(\theta) c(\theta) \right)^{-1}
\]
is well-defined. Define the polynomial \( \Pi^{(\Delta)} \) as
\[
\Pi^{(\Delta)}(z, \theta) := \left( 1 - c(\theta)^\top \left( I_N - (e^{A(\theta)\Delta} - K^{(\Delta)}(\theta)c(\theta)^\top) z \right)^{-1} K^{(\Delta)}(\theta) z \right), \quad z \in \mathbb{C},
\]
and its inverse
\[
\Pi^{(\Delta)}(z, \theta)^{-1} := 1 + c(\theta)^\top \sum_{j=1}^\infty \left( e^{A(\theta)\Delta} \right)^{j-1} K^{(\Delta)}(\theta) z^j, \quad z \in \mathbb{C},
\]
which gives \( \Pi^{(\Delta)}(z, \theta)^{-1} \Pi^{(\Delta)}(z, \theta) = z \). If the driving Lévy process \( L \) has finite second moments and mean zero, then
\[
\hat{e}_k^{(\Delta)}(\theta) = \Pi^{(\Delta)}(B, \theta) Y_k^{(\Delta)}(\theta), \quad k \in \mathbb{N},
\] (3.3)
is a sequence of uncorrelated identically distributed random variables. They are the noise of the ARMA representation of the discretely sampled CARMA process \( Y^{(\Delta)}(\theta) \) (cf. Schlemm and Stelzer (2012)). Define
\[
\sigma^2_{e^{(\Delta)}}(\theta) := \sigma^2 e(\theta)^\top \Omega^{(\Delta)}(\theta) c(\theta).
\] (3.4)
Then $\mathbb{E}(e^{\Delta}(\vartheta))^2 = \sigma^2_{\Delta}(\vartheta)$ is the variance of the white noise. By an application of Brockwell and Davis (1991), Theorem 11.8.3, the spectral density of $Y^{(\Delta)}(\vartheta)$ has the representation

$$f_Y^{(\Delta)}(\omega, \vartheta) = \frac{[\Pi^{(\Delta)}(e^{i\omega}, \vartheta)]^{-1} \sigma^2_{\Delta}(\vartheta)}{2\pi}, \quad \omega \in [-\pi, \pi].$$

Note that $\Pi^{(\Delta)}$ and hence, $\Pi^{(\Delta)}^{-1}$ do not depend on the variance $\sigma^2_L$ of the driving Lévy process. This motivates the definition of the (adjusted) Whittle function

$$W_n^{(\Delta)}(\vartheta) := \frac{\pi}{n} \sum_{j=-n+1}^n [\Pi^{(\Delta)}(e^{i\omega_j}, \vartheta)]^2 I_n(\omega_j)$$

and the (adjusted) Whittle estimator

$$\hat{\vartheta}_n^{(\Delta A)} := \arg \min_{\vartheta \in \Theta} W_n^{(\Delta)}(\vartheta).$$

The adjusted Whittle estimator has desirable properties in the light-tailed setting as derived in Fasen-Hartmann and Mayer (2020). To present these results we need some further assumptions.

**Assumption A.**

(A1) The parameter space $\Theta$ is a compact subset of $\mathbb{R}^N(\Theta)$.

(A2) The true parameter $\vartheta_0$ is an element of the interior of $\Theta$.

(A3) The eigenvalues of $A(\vartheta)$ have strictly negative real parts for $\vartheta \in \Theta$.

(A4) For all $\vartheta \in \Theta$ the zeros of $c_\vartheta(z) = c_0(\vartheta)z^{\vartheta} + c_1(\vartheta)z^{\vartheta-1} + \ldots + c_q(\vartheta)$ are different from the eigenvalues of $A(\vartheta)$.

(A5) For any $\vartheta, \vartheta' \in \Theta$ we have $(c(\vartheta), A(\vartheta)) \neq (c(\vartheta'), A(\vartheta'))$.

(A6) For all $\vartheta \in \Theta$ the spectrum of $A(\vartheta)$ is a subset of $\{z \in \mathbb{C} : -\frac{T}{2} < \Im(z) < \frac{T}{2}\}$.

(A7) For any $\vartheta_1, \vartheta_2 \in \Theta$, $\vartheta_1 \neq \vartheta_2$, there exists some $z \in \mathbb{C}$ with $|z| = 1$ and $\Pi^{(\Delta)}(z, \vartheta_1) \neq \Pi^{(\Delta)}(z, \vartheta_2)$.

(A8) The maps $\vartheta \mapsto A(\vartheta)$ and $\vartheta \mapsto c(\vartheta)$ are three times continuously differentiable.

Throughout the paper we suppose that Assumption A holds.

**Theorem 3.1 (Fasen-Hartmann and Mayer (2020), Theorem 1 and Theorem 4).** Let $L$ be a Lévy process with $\mathbb{E}(L_1) = 0$ and $\mathbb{E}(L_1^2) < \infty$.

(i) Then, $\hat{\vartheta}_n^{(\Delta A)} \overset{a.s.}{\rightarrow} \vartheta_0$ as $n \rightarrow \infty$.

(ii) Suppose further $\mathbb{E}(L_1^4) < \infty$ and for any $c \in \mathbb{C}^N(\Theta)$ there exists an $\omega^* \in [-\pi, \pi]$ such that $V_\vartheta |\Pi^{(\Delta)}(e^{i\omega^*}, \vartheta_0)|^{-2} \neq 0_N(\Theta)$. Then, there exists a positive definite matrix $\Sigma_{W(\Delta)} \in \mathbb{R}^{N(\Theta) \times N(\Theta)}$ such that as $n \rightarrow \infty$,

$$\sqrt{n} \left( \hat{\vartheta}_n^{(\Delta A)} - \vartheta_0 \right) \overset{d}{\rightarrow} \mathcal{N}(0, \Sigma_{W(\Delta)}).$$

If $L$ is a Brownian motion, then

$$\Sigma_{W(\Delta)} = 4\pi \left[ \int_{-\pi}^{\pi} V_\vartheta \log \left( |\Pi^{(\Delta)}(e^{i\omega}, \vartheta_0)|^{-2} \right) \top V_\vartheta \log \left( |\Pi^{(\Delta)}(e^{i\omega}, \vartheta_0)|^{-2} \right) d\omega \right]^{-1}.$$
4. Whittle estimation for symmetric $\alpha$-stable CARMA processes

The topic of this paper is Whittle estimation of the parameters of symmetric $\alpha$-stable CARMA processes. There are several arguments for the conjecture that the Whittle estimator might converge:

- For light-tailed CARMA processes the Whittle estimator is consistent and asymptotically normally distributed (see Theorem 3.1).

- In several simulations for symmetric $\alpha$-stable CARMA processes as, e.g., in the setup of García et al. (2011) (see Section 5), it seems that the Whittle estimator converges to the true parameter.

- In the context of ARMA processes the ideas of Whittle estimation for ARMA processes with finite second moments could be transferred to ARMA processes with infinite second moments (see Mikosch et al. (1995)). In particular, equidistant sampled CARMA processes with finite second moments have a weak ARMA representation.

The aim of this paper is to mathematically justify why the Whittle estimator for symmetric $\alpha$-stable CARMA processes is in general not converging. Therefore, we assume for the rest of the paper that the driving process $L$ of the CARMA process is a symmetric $\alpha$-stable Lévy process $L^{(\alpha)}$ with $L^{(\alpha)}_1 \sim S_{\alpha}(\sigma, 0, 0)$ for some $\sigma > 0$, $\alpha \in (0, 2)$ and that $Y$ is a symmetric $\alpha$-stable CARMA process with kernel function $g(t) = c^{-1}e^{i\omega t}e_\rho \mathbb{I}_{[0, \infty)}(t)$ as given in (2.4). In analogy to (3.6) for light-tailed CARMA processes, for symmetric $\alpha$-stable CARMA processes the (adjusted) Whittle function is

$$W_n^{(\alpha)}(\vartheta) := \frac{\pi}{n^{2/\alpha}} \sum_{j=-n+1}^{n} |\Pi^{(\Delta)}(e^{i\omega}, \vartheta)|^2 |I_n(\omega_j), \vartheta \in \Theta,$$

and the (adjusted) Whittle estimator is

$$\hat{\vartheta}^{(\Delta, \alpha)}_n := \arg \min_{\vartheta \in \Theta} W_n^{(\alpha)}(\vartheta).$$

**Theorem 4.1.** Suppose Assumption A holds. Define

$$W^{(\alpha)}(\vartheta) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \Pi^{(\Delta)}(e^{i\omega}, \vartheta) \right|^2 \sum_{j=-\infty}^{\infty} g(\Delta j-s) e^{-ij\omega} d\omega \ dL_{s^{(\alpha/2)}},$$

where $L_{s^{(\alpha/2)}} = (L_{s^{(\alpha/2)}}^{(\alpha/2)})_{t \geq 0}$ is an $\alpha/2$-stable Lévy process with $L^{(\alpha/2)}_1 \sim S_{\alpha/2}(\sigma^2 (C_{\alpha}/C_{\alpha/2})^{2/\alpha}, 1, 0)$ and the constants $C_{\alpha}$ and $C_{\alpha/2}$ are defined as in (2.2). Then, as $n \to \infty$,

$$\left(W_n^{(\alpha)}(\vartheta)\right)_{\vartheta \in \Theta} \overset{\theta}{\longrightarrow} (W^{(\alpha)}(\vartheta))_{\vartheta \in \Theta} \quad \text{in} \quad (C(\Theta), \| \cdot \|_\infty),$$

where $C(\Theta)$ is the space of continuous functions on $\Theta$ with the supremum norm $\| \cdot \|_\infty$.

**Proof.** We approximate $|\Pi^{(\Delta)}(e^{i\omega}, \vartheta)|^2$ by the Cesàro sum of its Fourier series of size $M$ for $M$ sufficiently large. Define

$$q_M(\omega, \vartheta) := \frac{1}{M} \sum_{j=0}^{M-1} \left( \sum_{|k| \leq j} b_k(\vartheta) e^{-ik\omega} \right) = \sum_{|k| < M} \left( 1 - \frac{|k|}{M} \right) b_k(\vartheta) e^{-ik\omega} \quad \text{with}$$

$$b_k(\vartheta) := \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Pi^{(\Delta)}(e^{i\omega}, \vartheta)|^2 e^{ik\omega} d\omega,$$
and
\[
W_{M,n}^{(\alpha)}(\vartheta) := \frac{\pi}{n^{2/\alpha}} \sum_{j=-n+1}^{n} q_{M}(\omega_j, \vartheta) \lambda^{(\alpha)}(\omega_j), \quad \vartheta \in \Theta.
\]

Let $\varepsilon_1 > 0$. A conclusion of Lemma 6 of [Fasen-Hartmann and Mayer (2020)] is that there exists an $M_0(\varepsilon_1) \in \mathbb{N}$ such that for $M \geq M_0(\varepsilon_1)$
\[
\sup_{\omega \in [-\pi, \pi]} \sup_{\vartheta \in \Theta} |q_{M}(\omega, \vartheta) - |\Pi^{(\Delta)}(e^{i\omega}, \vartheta)|^2| < \varepsilon_1.
\]

(4.1)

Similar arguments as in the proof of Proposition 2 in [Fasen-Hartmann and Mayer (2020)] yield
\[
\sup_{\vartheta \in \Theta} \left| W_n^{(\alpha)}(\vartheta) - W_{M,n}^{(\alpha)}(\vartheta) \right| \leq \frac{\varepsilon_1}{n^{2/\alpha - 1}} \gamma_n(0) \quad \text{for } M \geq M_0(\varepsilon_1).
\]

Due to Theorem A.3
\[
\frac{1}{n^{2/\alpha - 1}} \gamma_n(0) \overset{P}{\rightarrow} \int_{0}^{\Delta} \sum_{j=-\infty}^{\infty} g(\Delta j - s)^2 dL^{(\alpha/2)}_s \quad \text{as } n \to \infty.
\]

Therefore, we have for any $\varepsilon_2 > 0$
\[
\lim_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( \sup_{\vartheta \in \Theta} \left| W_n^{(\alpha)}(\vartheta) - W_{M,n}^{(\alpha)}(\vartheta) \right| > \varepsilon_2 \right) = 0.
\]

(4.2)

Furthermore, representation (5.1) gives
\[
W_{M,n}^{(\alpha)}(\vartheta) = \sum_{|k| < M} \left( 1 - \frac{|k|}{M} \right) b_k(\vartheta) \left( \frac{1}{2n} \sum_{|k| < n} \gamma_n(h) - \sum_{j=-n+1}^{n} e^{-i(k+h)\omega_j} \right)
\]
\[
= \sum_{|k| < M} \left( 1 - \frac{|k|}{M} \right) b_k(\vartheta) n^{-2/\alpha + 1} \gamma_n(-k).
\]

(4.3)

We define
\[
W_{M}^{(\alpha)}(\vartheta) := \sum_{|k| < M} \left( 1 - \frac{|k|}{M} \right) b_k(\vartheta) \int_{-\infty}^{\Delta} \sum_{j=-\infty}^{\infty} g(\Delta j + k) g(\Delta j - k) dL^{(\alpha/2)}_s.
\]

(4.4)

Due to Assumption (A8) and the definition of $\Pi^{(\Delta)}$, there exists a constant $C > 0$ such that for any $\delta > 0$
\[
\sup_{|\vartheta_1 - \vartheta_2| < \delta} \left| b_k(\vartheta_1) - b_k(\vartheta_2) \right| = \sup_{|\vartheta_1 - \vartheta_2| < \delta} \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( |\Pi^{(\Delta)}(e^{i\omega}, \vartheta_1)|^2 - |\Pi^{(\Delta)}(e^{i\omega}, \vartheta_2)|^2 \right) e^{ik\omega} d\omega \right\|
\]
\[
\leq \max_{|\vartheta_1 - \vartheta_2| < \delta} \max_{\omega \in [-\pi, \pi]} \left| \Pi^{(\Delta)}(e^{i\omega}, \vartheta_1) \right|^2 - \left| \Pi^{(\Delta)}(e^{i\omega}, \vartheta_2) \right|^2 \leq C \delta.
\]

This means that $(b_k(\vartheta))_{\vartheta \in \Theta}$ is uniformly continuous. By Theorem A.3 we have the joint convergence of the random vector $(\gamma_n(-M + 1), \ldots, \gamma_n(M - 1))$ implying with the representations (4.3), (4.4) and the continuous mapping theorem that
\[
(W_{M,n}^{(\alpha)}(\vartheta))_{\vartheta \in \Theta} \overset{\mathcal{D}}{\rightarrow} (W_{M}^{(\alpha)}(\vartheta))_{\vartheta \in \Theta} \quad \text{in } \left( \mathcal{C}(\Theta), \| \cdot \|_{\infty} \right).
\]

(4.5)
Furthermore,
\[
W_M^{(\alpha)}(\vartheta) = \int_0^\Delta \sum_{|k|<M} \left(1 - \frac{|k|}{M}\right) b_k(\vartheta) \sum_{j=-\infty}^\infty g(\Delta(j+k) - s)g(\Delta j - s) dL_s^{(\alpha/2)} \\
= \int_0^\Delta \sum_{|k|<M} \left(1 - \frac{|k|}{M}\right) b_k(\vartheta) \sum_{j=-\infty}^\infty g(\Delta j - s)g(\Delta\ell - s) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(j+k-\ell)\omega} d\omega\right] dL_s^{(\alpha/2)} \\
= \frac{1}{2\pi} \int_0^\Delta \left[\int_{-\pi}^{\pi} q_M(\omega, \vartheta) \left|\sum_{j=-\infty}^\infty g(\Delta j - s)e^{-ij\omega}\right|^2 d\omega\right] dL_s^{(\alpha/2)}.
\]

By this,
\[
W_M^{(\alpha)}(\vartheta) - W^{(\alpha)}(\vartheta) = \frac{1}{2\pi} \int_0^\Delta \left[\int_{-\pi}^{\pi} q_M(\omega, \vartheta) - |\Pi^{(\alpha)}(e^{i\omega}, \vartheta)|^2 \left|\sum_{j=-\infty}^\infty g(\Delta j - s)e^{-ij\omega}\right|^2 d\omega\right] dL_s^{(\alpha/2)}
\]
holds. Since the process \(L^{(\alpha/2)}\) is positive and increasing we obtain
\[
\sup_{\vartheta \in \Theta} |W_M^{(\alpha)}(\vartheta) - W^{(\alpha)}(\vartheta)| \\
\leq \frac{1}{2\pi} \int_0^\Delta \left[\int_{-\pi}^{\pi} \sup_{\vartheta \in \Theta} q_M(\omega, \vartheta) - |\Pi^{(\alpha)}(e^{i\omega}, \vartheta)|^2 \left|\sum_{j=-\infty}^\infty g(\Delta j - s)e^{-ij\omega}\right|^2 d\omega\right] dL_s^{(\alpha/2)} =: \tilde{W}_M^{(\alpha/2)}.
\]

Note that by Property 3.2.2 of Samorodnitsky and Taqqu (1994), \(\tilde{W}_M^{(\alpha/2)} \sim S_{\alpha/2}(\sigma_M, \beta_M, \mu_M)\) where \(\beta_M = 1, \mu_M = 0\) and \(\sigma_M^{\alpha/2} = \frac{\sigma^\alpha C_\alpha}{C_{\alpha/2}}\int_0^\Delta \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} q_M(\omega, \vartheta) - |\Pi^{(\alpha)}(e^{i\omega}, \vartheta)|^2 \left|\sum_{j=-\infty}^\infty g(\Delta j - s)e^{-ij\omega}\right|^2 d\omega\right] d\vartheta\].

Due to Assumption (A3) there exists a constant \(C > 0\) such that \((\sum_{j=0}^\infty ||e^{A(\Delta_j-s)}||)^2 < C\) for any \(s \in [0, \Delta]\). Thus,
\[
\int_0^\Delta \left[\int_{-\pi}^{\pi} \left|\sum_{j=-\infty}^\infty g(\Delta j - s)e^{-ij\omega}\right|^2 d\omega\right] ds < \frac{1}{2\pi} \int_0^\Delta \left[\int_{-\pi}^{\pi} ||e^{A(\Delta_j-s)}||^2 d\omega\right] d\vartheta < \infty.
\]

A conclusion of this and (4.1) is that \(\sigma_M^{\alpha/2} \xrightarrow{M \to \infty} 0\) and hence, the characteristic function \(\varphi_{\tilde{W}_M^{(\alpha/2)}}\) converges pointwise to \(\varphi_{\tilde{W}^{(\alpha/2)}} \equiv 1\). An application of Lévy's continuity theorem results then in \(\tilde{W}_M^{(\alpha/2)} \xrightarrow{p} 0 \) as \(M \to \infty\). Finally,
\[
\sup_{\vartheta \in \Theta} |W_M^{(\alpha)}(\vartheta) - W^{(\alpha)}(\vartheta)| \xrightarrow{p} 0 \quad \text{as } M \to \infty \quad (4.6)
\]
as well. In view of (4.2)-(4.6), Theorem 3.2 of Billingsley (1999) completes the proof. \(\square\)
Corollary 4.2. Let the assumptions of Theorem 4.1 hold. Define $G_{\theta, \theta_0} : [0, \Delta] \to \mathbb{R}$ as

$$G_{\theta, \theta_0}(u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \Pi^{(\Delta)}(e^{i\omega}, \cdot) \right|^2 \left| \Pi^{(\Delta)}(e^{i\omega}, \theta_0) \right|^2 \sum_{j=-\infty}^{\infty} g(j \Delta - u) e^{-ij\omega} \, d\omega.$$

Then,

$$W^{(\alpha)}(\theta) - W^{(\alpha)}(\theta_0) \sim S_{\alpha/2}(\sigma_{\theta, \theta_0}, \beta_{\theta, \theta_0}, 0)$$

is an $\alpha/2$-stable random variable with parameters

$$\beta_{\theta, \theta_0} = \frac{\int_{0}^{\Delta} (G_{\theta, \theta_0}(s))^{\alpha/2} - (G_{\theta, \theta_0}(s))^{\alpha/2} ds}{\int_{0}^{\Delta} |G_{\theta, \theta_0}(s)|^{\alpha/2} ds},$$

$$\sigma_{\theta, \theta_0}^{\alpha/2} = \frac{\alpha \sigma}{C_{\alpha/2}} \int_{0}^{\Delta} |G_{\theta_0}(s)|^{\alpha/2} ds.$$

4.1. Whittle estimation for symmetric $\alpha$-stable Ornstein-Uhlenbeck processes

An Ornstein-Uhlenbeck process $Y_t(\theta) = \int_{-\infty}^{t} e^{\theta(t-s)} dL_s^{(\alpha)}$, $t \geq 0$, sampled equidistantly has the AR(1) representation

$$Y_k^{(\Delta)}(\theta) = e^{\theta \Delta} Y_{k-1}(\theta) + \xi_k^{(\Delta)}(\theta),$$

where $\xi_k^{(\Delta)}(\theta) = \int_{(k-1)\Delta}^{k\Delta} e^{\theta(t-s)} dL_s^{(\alpha)}$, $k \in \mathbb{N}$, is an iid symmetric $\alpha$-stable sequence. Since the distribution of the white noise $\xi_k^{(\Delta)}(\theta)$ depends on $\theta$, the theory of Mikosch et al. (1995) cannot be applied directly to estimate $\theta$ in this setting even though we have an AR(1) representation. Thus, in this subsection we derive the consistency of the Whittle estimator for symmetric $\alpha$-stable Ornstein-Uhlenbeck processes.

Proposition 4.3. Let $Y_t(\theta) = \int_{-\infty}^{t} e^{\theta(t-s)} dL_s^{(\alpha)}$, $t \geq 0$, for $\theta \in \Theta \subseteq (-\infty, 0)$ and $\Theta$ compact be a family of symmetric $\alpha$-stable Ornstein-Uhlenbeck processes. Then, as $n \to \infty$,

$$W_n^{(\alpha)}(\theta) \xrightarrow{p} W_{OU}(\theta) S_{\alpha/2} \quad \text{in} \quad (\mathcal{C}(\Theta), \| \cdot \|_{\infty}),$$

where $S_{\alpha/2}$ is a positive $\alpha/2$-stable random variable and

$$W_{OU}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - e^{\theta \Delta + i\omega} \right|^2 \left| 1 - e^{\theta_0 \Delta + i\omega} \right|^2 \, d\omega, \quad \theta \in \Theta.$$

Proof. The Ornstein-Uhlenbeck process $Y(\theta)$ has the kernel function $g_{\theta}(t) = e^{\theta t} 1_{[0,\infty)}(t)$ and the transfer function $\Pi^{(\Delta)}(\theta) = 1 - e^{\theta \Delta}$. Therefore, an application of Theorem 4.1 yields as $n \to \infty$,

$$W_n^{(\alpha)}(\theta) \xrightarrow{p} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \Pi^{(\Delta)}(e^{i\omega}, \theta) \right|^2 \left| \sum_{j=-\infty}^{\infty} g_{\theta_0}(\Delta j - s) e^{-ij\omega} \right|^2 \, d\omega \, dL_s^{(\alpha)/2}$$

$$\xrightarrow{\Delta} \frac{1}{2\pi} \int_{0}^{\Delta} \left| \Pi^{(\Delta)}(e^{i\omega}, \theta) \right|^2 \left| \sum_{j=1}^{\infty} e^{\theta_0(\Delta j - s)} e^{-ij\omega} \right|^2 \, d\omega \, dL_s^{(\alpha)/2}.$$
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - e^{i\omega} \right|^2 \left| 1 - e^{i\omega} \right|^{-2} d\omega \int_0^\Delta e^{2i\omega \Delta - i\omega^2} dL_s^{(\alpha/2)}
\]
in \((\mathcal{C}(\Theta), \| \cdot \|_\infty)\). Define \(S_{\alpha/2}^* := \int_0^\Delta e^{2i\omega \Delta - i\omega^2} dL_s^{(\alpha/2)}\). Due to Property 3.2.2 of Samorodnitsky and Taqqu (1994),
\[
S_{\alpha/2}^* \sim S_{\alpha/2} \left( \frac{\alpha C_\alpha}{C_{\alpha/2}} \int_0^\Delta e^{\alpha \omega_s} d\omega_s \right)^{2/\alpha}, 1, 0
\]
which implies that \(S_{\alpha/2}^*\) is positive (see Proposition 1.2.11 of Samorodnitsky and Taqqu (1994)).

**Proposition 4.4.** Let the assumptions of Proposition 4.3 hold. Then, \(W_{OU}\) has a unique minimum in \(\vartheta_0\).

*Proof.* Proposition 8 of the Supplementary Material of Fasen-Hartmann and Mayer (2020) implies that under Assumptions (A1), (A4) and (A6)
\[
W_{OU}(\vartheta_0) = 1 < \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\|\Pi(\vartheta)\|^2}{\|\Pi(\vartheta)\|^2} d\omega = W_{OU}(\vartheta) \quad \text{for } \vartheta \neq \vartheta_0.
\]
Hence, \(\vartheta_0\) is indeed the unique minimum. \(\square\)

**Theorem 4.5.** Let the assumptions of Proposition 4.3 hold. Then, as \(n \to \infty\),
\[
\hat{\vartheta}_n^{(\alpha, \omega)} \xrightarrow{P} \vartheta_0.
\]

*Proof.* Proposition 4.3 and the Skorokhod's representation theorem give that there exists a probability space with processes \((W_n^*(\vartheta))_{\vartheta \in \Theta}\) and \((W^*(\vartheta))_{\vartheta \in \Theta}\) having the same distributions as \((W_n^{(\alpha)}(\vartheta))_{\vartheta \in \Theta}\) and \((W^{(\alpha)}(\vartheta))_{\vartheta \in \Theta}\), respectively, with
\[
\sup_{\vartheta \in \Theta} |W_n^*(\vartheta) - W^*(\vartheta)| \stackrel{a.s.}{\to} 0, \quad n \to \infty.
\]
With the same arguments as in the proof of Theorem 1 of Fasen-Hartmann and Mayer (2020), we can show that the minimizing arguments \(\hat{\vartheta}_n^*\) and \(\hat{\vartheta}_0^*\) of \((W_n^*(\vartheta))_{\vartheta \in \Theta}\) and \((W^*(\vartheta))_{\vartheta \in \Theta}\), respectively, satisfy, as \(n \to \infty\),
\[
\hat{\vartheta}_n^* \xrightarrow{a.s.} \vartheta_0^*, \quad \hat{\vartheta}_0^* \xrightarrow{P} \vartheta_0.
\]
which then implies \(\hat{\vartheta}_n^{(\alpha, \omega)} \xrightarrow{P} \vartheta_0^*\). Since \(\vartheta_0^*\) is a constant, convergence in distribution implies convergence in probability. \(\square\)

### 4.2. Whittle estimation for general symmetric \(\alpha\)-stable CARMA processes

**Theorem 4.6.** Consider the setting of Theorem 4.1 for a symmetric \(\alpha\)-stable CARMA\((p,q)\) process with \(p \geq 2\). Then, in general, the limit function \(W^{(\alpha)}\) of the Whittle function does not have a unique minimum in \(\vartheta_0\) and hence, the Whittle estimator is not consistent.

*Proof.* A necessary condition for the Whittle function \(W^{(\alpha)}\) to have a unique minimum in \(\vartheta_0\) is that \(W^{(\alpha)}(\vartheta) - W^{(\alpha)}(\vartheta_0)\) is a positive random variable for \(\vartheta \neq \vartheta_0\) and, hence, \(\hat{\beta}_{\vartheta, \vartheta_0}\) as defined in (4.2) is equal to 1.

However, this is not the case in general as can be seen in Example 4.7 and Example 4.8 which implies that the Whittle estimator is in general not consistent. \(\square\)
Example 4.7. We tackle the question, whether it is possible to find a model where $\beta_{\vartheta, \vartheta_0}$ is not equal to 1 for some $\vartheta \neq \vartheta_0$. Therefore, we consider symmetric 3/2-stable CARMA(2,0) processes with autoregressive and moving average polynomial

$$a_\vartheta(z) = z^\vartheta - (\vartheta - 2)z - 2\vartheta \quad \text{and} \quad c_\vartheta(z) = \vartheta - 2,$$

respectively. These CARMA processes have the state space representation

$$dX_t(\vartheta) = A(\vartheta)X_t(\vartheta)\,dt + e_2\,dL_t^{3/2}, \quad \text{and} \quad Y_t(\vartheta) := c(\vartheta)^\top X_t(\vartheta), \quad t \geq 0,$$

where

$$A(\vartheta) = \begin{pmatrix} 0 & 1 \\ 2\vartheta & \vartheta - 2 \end{pmatrix} \quad \text{and} \quad c(\vartheta)^\top = (\vartheta - 2, 0).$$

The true parameter is $\vartheta_0 = -3$. The behaviour of $\beta_{\vartheta, \vartheta_0}$ as defined in Corollary 4.2, the behaviour of the non-normalized positive part

$$\beta_{\vartheta, \vartheta_0}^+ := \int_0^\Delta (G_{\vartheta, \vartheta_0}^+(s))^{\alpha/2} ds$$

and the negative part

$$\beta_{\vartheta, \vartheta_0}^- := \int_0^\Delta (G_{\vartheta, \vartheta_0}^-(s))^{\alpha/2} ds,$$

respectively, are plotted as functions of $\vartheta$ for $\alpha = 1.5$ in Figure 1. As one can see, $\beta_{\vartheta, \vartheta_0}^+ > 0$ for all $\vartheta \in (-\infty, -3) \cup (-3, -2)$, and hence, $\beta_{\vartheta, \vartheta_0}^- < 1$. Of course, this holds independent of the choice of $\alpha$. Thus, $W^{(\alpha)}(\vartheta) - W^{(\alpha)}(\vartheta_0)$ is not a strictly positive random variable for $\vartheta \in (-\infty, -3) \cup (-3, -2)$ and hence, has not almost surely a unique minimum in $\vartheta_0$. Especially, $\beta_{\vartheta, \vartheta_0}^+ \to 0.8$ for $\vartheta \to -\infty$ in the case $\alpha = 1.5$.

![Figure 1: Behaviour of $\beta_{\vartheta, \vartheta_0}^+, \beta_{\vartheta, \vartheta_0}^-$ and $\beta_{\vartheta, \vartheta_0}^+$ in the CARMA(2,0) model of Example 4.7](image)

We set $\beta_{\vartheta_0, \vartheta_0} = 0$ to guarantee that $\beta_{\vartheta, \vartheta_0}$ is continuous.

Example 4.8. In view of Example 3.3 of García et al. (2011), we consider CARMA(2,1) processes with the parametrization (2.3) and $a_1(\vartheta) = \vartheta_1, \quad a_2(\vartheta) = \vartheta_2$ and $c_0(\vartheta) = \vartheta_3, \quad c_1(\vartheta) = 1, \quad \vartheta = (\vartheta_1, \vartheta_2, \vartheta_3) \in \Theta$. The kernel function $g_\vartheta$ in (2.4) has the representation

$$g_\vartheta(t) = \frac{(\vartheta_3 - \lambda^+(\vartheta))^t e^{-\lambda^+(\vartheta)t}}{\sqrt{\vartheta_1^2 - 4\vartheta_2}} - \frac{(\vartheta_3 - \lambda^-(\vartheta))^t e^{-\lambda^-(\vartheta)t}}{\sqrt{\vartheta_1^2 - 4\vartheta_2}}, \quad t \geq 0,$$
Therefore, the kernel function is

\[ \prod \] the AR simulation study and compare it with the behaviour of the estimator introduced in García et al. (2011).

In this section, we investigate the performance of the Whittle estimator for finite samples through a simulation study.

\section{5. Simulation study}

In this section, we investigate the performance of the Whittle estimator for finite samples through a simulation study and compare it with the behaviour of the estimator introduced in García et al. (2011). The estimator of García et al. (2011) is based on an indirect approach. Denoting the zeros of the AR(p) polynomial \( a(z) \) as \( \lambda_1, \ldots, \lambda_p \) which are assumed to be distinct and defining \( a_D(\lambda) = \prod_{j=1}^{p} (1 - e^{i \lambda_j}) \), it is well known (García et al., 2011, Proposition 3.1) that the sampled process \( Y^{(\Delta)} \) satisfies the equation

\[ a_D^{(\Delta)}(B)Y^{(\Delta)}_k = U_k^{(\Delta)}, \quad k \in \mathbb{N}, \quad (5.1) \]

where \( (U_k^{(\Delta)})_{k \in \mathbb{N}} \) is a \((p-1)\)-dependent sequence and B is the backshift operator with \( B Y^{(\Delta)}_k = Y^{(\Delta)}_{k-1} \). For CARMA processes with finite second moments, \( (U_k^{(\Delta)})_{k \in \mathbb{N}} \) is an MA\((p, p-1)\) process such that \( Y^{(\Delta)} \) is an ARMA\((p, p-1)\) process with an uncorrelated but not independent white noise. García et al. (2011) proposed to fit an ARMA\((p, p-1)\) model to the observations \( Y_1^{(\Delta)}, \ldots, Y_n^{(\Delta)} \) by standard maximum likelihood estimation for Gaussian ARMA models. The estimated autoregressive part of that ARMA model in discrete time is denoted by \( \hat{a}_D^{(\Delta)}(z) \) and the estimated moving average part we denote by \( \hat{c}_D^{(\Delta)}(z) \). The logarithmic zeros of \( \hat{a}_D^{(\Delta)}(z) \) divided by \( -\Delta \) are then estimators \( \hat{\lambda}_1, \ldots, \hat{\lambda}_p \) for the zeros \( \lambda_1, \ldots, \lambda_p \) of \( a(z) \). Hence, we obtain an estimator \( \hat{a}(z) \) for the autoregressive polynomial \( a(z) \). In a final step, the MA polynomial \( c(z) \) of the CARMA process is determined. Therefore the parameter \( \theta = (\hat{\theta}_1, \hat{\theta}_2) \) is divided in two parts where \( \hat{\theta}_1 \) models the AR coefficients and \( \hat{\theta}_2 \) the MA coefficients of the CARMA process. Now the autocorrelation function \( \rho^{(MA)}_{\hat{a}(\Delta), \hat{c}_D^{(\Delta)}}(k) = \rho^{(MA)}_{\hat{a}_D^{(\Delta)}, \hat{c}_D^{(\Delta)}}(k) \) for \( k = 1, \ldots, q \).

In the following, we use an Euler-Maruyama scheme for differential equations with initial value \( Y_0 = 0 \) and step size 0.01 to simulate \( \alpha \)-stable CARMA processes. We set \( \Delta = 1 \) as the distance between the discrete observations and \( \alpha = 1.5 \) for the stable index of the driving symmetric \( \alpha \)-stable Lévy process. We investigate the behaviour of the Whittle estimator and the estimator of García et al. (2011) for \( n = 500, 2000, 5000 \) based on 500 replications.

\[ \vartheta_0 = (1.9647, 0.0893, 0.1761). \]

which is non-negative and we take \( \alpha = 1.5 \). In this setting, we calculate \( \hat{\beta}_{\hat{\theta}, \vartheta_0} \) as a function of the components \( \hat{\theta}_1, \hat{\theta}_2 \) and \( \vartheta_0 \), respectively, where we fix the other two variables. Then the functions \( \beta_{\hat{\theta}, \vartheta_0}, \hat{\beta}_{\hat{\theta}, \vartheta_0} \) and \( \hat{\beta}_{\hat{\theta}, \vartheta_0} \) are plotted in Figure 2. In all three cases, the plots show that \( \hat{\beta}_{\hat{\theta}, \vartheta_0} > 0 \) for some \( \hat{\theta} \neq \vartheta_0 \) implying \( \beta_{\hat{\theta}, \vartheta_0} < 1 \). Therefore, if we only allow a single parameter to vary, the Whittle estimator converges to a function which has not an unique minimum in the true parameter. Hence, the Whittle estimator is not consistent. Again this statement is independent of the choice of \( \alpha \).
Figure 2: behaviour of $\beta_{\theta, \vartheta_0}, \beta_{\theta, \vartheta_0}^-$ and $\beta_{\theta, \vartheta_0}^+$ in the CARMA(2,1) model of Example 4.8 where $\vartheta$ originates from $\vartheta_0$ when we fix two components and vary the third one. We set $\beta_{\vartheta_0, \vartheta_0} = 0$ to guarantee that $\beta_{\theta, \vartheta_0}$ is continuous.

As first example, we simulate an Ornstein-Uhlenbeck process with $\vartheta_0 = -1$. The resulting sample mean, bias and sample standard deviation are given in Table 1. It seems that both the Whittle estimator and the estimator of García et al. (2011) converge to the true value. For the Whittle estimator this is consistent with Theorem 4.5. To compare the behaviour in the heavy-tailed setting with the behaviour in the light-tailed setting, we present a second simulation study where we use for the driving Lévy
Table 1: Estimation results for a symmetric 1.5-stable Ornstein-Uhlenbeck process with parameter $\vartheta_0 = -1$.

Table 2: Estimation results for a Brownian motion driven Ornstein-Uhlenbeck process with parameter $\vartheta_0 = -1$.

Table 3: Estimation results for a CARMA(2,0) process with parameter $\vartheta_0 = -3$.

Table 4: Estimation results for a CARMA(2,1) process with parameter $\vartheta_0 = -3$. 

Next, we simulate the CARMA(2,0) process of Example 4.7. Accordingly, the true value is $\vartheta_0 = -3$. The results are given in Table 3. As already argued in Example 4.7, the Whittle estimator is not a consistent estimator in this situation. This is confirmed by the simulation study. For $n = 5000$ the bias and standard deviation are even higher than for $n = 2000$. The estimator of García et al. (2011) behaves even worse. On the one hand, the bias and standard deviation of García et al. (2011) are quite high and not decreasing with increasing sample size. On the other hand, the estimation procedure of García et al. (2011) stops for every sample size for more than 1/5th of the replications. This can be traced back to an inadequate estimate of the zero of the AR polynomial, namely the real part of the estimated zero of the AR polynomial is less than 0 which means that the logarithm of this zero is not defined.

Finally, we investigate the CARMA(2,1) process of Example 4.8, see Table 4. Our simulation
results show the same findings as García et al. (2011); both estimators perform very well in this parameter setting. However, most of the time there is one parameter which has a slightly higher bias or standard deviation such that it is not apparent if the estimator is converging. Indeed, for the Whittle estimator we already showed in Example 4.8 that this is not the case and we guess that the same holds true for the estimator of García et al. (2011), although at the first view this seems to contradict the simulation study. But from the behaviour of \( \beta_{\theta, \theta_0} \) in Figure 2 we know that only in a small neighbourhood of \( \theta_0 \), the random variables \( W^{(\alpha)}(\theta) - W^{(\alpha)}(\theta_0) \) are not positive and outside this neighbourhood they are positive with probability one because \( \beta_{\theta, \theta_0} = 1 \). Although \( W^{(\alpha)}(\theta) \) has not a unique minimum in \( \theta_0 \), \( \theta_0 \) is close to the minimum of \( W^{(\alpha)}(\theta) \). Thus, the Whittle estimator is close to the true value \( \theta_0 \) as well.

| parameter | mean | bias | std. | mean | bias | std. | mean | bias | std. |
|-----------|------|------|------|------|------|------|------|------|------|
| \( \theta_0 = -3 \) | -3.4762 | 0.4762 | 1.2741 | -3.2902 | 0.2902 | 0.9367 | -3.3002 | 0.3002 | 0.9568 |

Table 3: Estimation results for the symmetric 1.5-stable CARMA(0,0) process of Example 4.7.

| parameter | mean | bias | std. | mean | bias | std. | mean | bias | std. |
|-----------|------|------|------|------|------|------|------|------|------|
| \( \theta_0 = -3 \) | -3.2473 | 0.2473 | 1.2220 | -3.8184 | 0.8164 | 1.1089 | -4.0770 | 1.0770 | 0.9238 |

Table 4: Estimation results for the symmetric 1.5-stable CARMA(2,1) process of Example 4.8.

| parameter | mean | bias | std. | mean | bias | std. | mean | bias | std. |
|-----------|------|------|------|------|------|------|------|------|------|
| \( \theta_1 = 1.9647 \) | 1.9520 | 0.0127 | 0.0516 | 1.9592 | 0.0055 | 0.0321 | 2.0069 | 0.0422 | 1.1890 |
| \( \theta_2 = 0.0893 \) | 0.1031 | 0.0138 | 0.0377 | 0.0940 | 0.0047 | 0.0224 | 0.0987 | 0.0094 | 0.0288 |
| \( \theta_3 = 0.1761 \) | -0.0144 | 0.1905 | 0.1836 | -0.0389 | 0.215 | 0.1681 | 0.1735 | 0.0026 | 0.0224 |

| parameter | mean | bias | std. | mean | bias | std. | mean | bias | std. |
|-----------|------|------|------|------|------|------|------|------|------|
| \( \theta_1 = 1.9647 \) | 2.0947 | 0.1300 | 0.4480 | 2.0138 | 0.0491 | 0.2405 | 2.0036 | 0.0389 | 0.1543 |
| \( \theta_2 = 0.0893 \) | 0.1462 | 0.0569 | 0.2160 | 0.0939 | 0.0046 | 0.0323 | 0.0930 | 0.0037 | 0.0300 |
| \( \theta_3 = 0.1761 \) | 0.2196 | 0.0435 | 0.1333 | 0.1877 | 0.0116 | 0.0487 | 0.1920 | 0.0159 | 0.0484 |
6. Conclusion

The simulation study confirms the theory that for symmetric $\alpha$-stable CARMA($p,q$) processes with $p \geq 2$ the Whittle estimator is in general not consistent. Similarly, it suggests that the estimator of Garcia et al. (2011) is not a consistent estimator as well, although for the special parameter setting of Example 4.8 both, the Whittle estimator and the estimator of Garcia et al. (2011), give quite reasonable results. In case of the Whittle estimator this effect is not surprising. Figure 2 suggests that $\Pr(W^{(\alpha)}(\vartheta) > W^{(\alpha)}(\vartheta_0))$ is quite high for $\vartheta \neq \vartheta_0$ such that the probability of a local minimum in $\vartheta_0$ is still high, even though $W^{(\alpha)}$ has not a unique minimum in $\vartheta_0$.

Essentially, the Whittle estimator is not a consistent for symmetric $\alpha$-stable CARMA($p,q$) processes since $(\varepsilon^{(\Delta)}_k(\vartheta))_{k \in \mathbb{N}}$, the noise in the MA representation (3.3), is a dependent sequence except for Ornstein-Uhlenbeck processes. Then, it is an iid sequence and therefore, it is not astonishing that the Whittle estimator is consistent. If the driving Lévy process has finite second moments $(\varepsilon^{(\Delta)}_k(\vartheta))_{k \in \mathbb{N}}$ is at least an uncorrelated sequence which is sufficient for the consistency of the Whittle estimator for CARMA processes (see Theorem 3.1).

For similar reasons, we also assume that the estimator of Garcia et al. (2011) is not a consistent estimator in general. In fact, if the driving Lévy process has finite second moments, the process $(U^{(\Delta)}_k)_{k \in \mathbb{N}}$ given in (5.1) is a MA($p-1$) process with a weak white noise. In contrast, for $\alpha$-stable CARMA($p,q$) models with $p \geq 2$, the sequence $(U^{(\Delta)}_k)_{k \in \mathbb{N}}$ has not a representation as a MA($p-1$) process with an iid noise and due to the lack of the second moments the sequence can not be uncorrelated as well. This dependence influences the estimator and prevents consistency. However, if $Y$ is an Ornstein-Uhlenbeck process, $(U^{(\Delta)}_k)_{k \in \mathbb{N}}$ is an iid sequence which implies that the estimator of Garcia et al. (2011) is consistent as well. The simulation results in Table 1 matches this insight.

In sum, a Lévy driven CARMA process with finite second moments sampled discretely has a weak ARMA representation, which can be used to estimate the CARMA parameters as, e.g., for quasi-maximum likelihood estimation (see Schlemm and Stelzer (2012)) or for Whittle estimation (see Fasen-Hartmann and Mayer (2020)). Beside the identifiability issue, a difficulty in this attempt is that the noise of the weak ARMA representation depends on the model parameters as we already saw for the Ornstein-Uhlenbeck process in Section 4.1. However, for heavy-tailed CARMA processes with finite second moments the estimators for heavy-tailed ARMA models as, e.g., the Whittle estimator do not work anymore because the noise in the ARMA representation of the discretely sampled CARMA process is neither independent nor uncorrelated. The same phenomena was also investigated for the parameter estimation of GARCH(1,1) processes with infinite 4th moment in Mikosch and Straumann (2002) where the Whittle estimator is inconsistent although it is consistent in the finite 4th moment case (see Giraitis and Robinson (2001) and Mikosch and Straumann (2002)). There as well the noise of the ARMA representation of the squared GARCH(1,1) process with finite 4th moments is a weak white noise which is in general not independent.

In conclusion, the analogy between parameter estimation for heavy-tailed CARMA and heavy-tailed ARMA processes with infinite variance is dangerous. For the estimation of heavy-tailed CARMA models different estimation approaches as for ARMA processes have to be developed, although they work for CARMA processes with finite second moments. This is topic of some future research.
A. Asymptotic behaviour of the sample autocovariance function of symmetric \( \alpha \)-stable CARMA processes

**Proposition A.1.** Let \( g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R} \) be bounded functions with \( g_1, g_2 \in L^\delta (\mathbb{R}) \) for some \( \delta < \min \{ \alpha, 1 \} \) and \( 0 < \int_{-\infty}^{\infty} \left| g_1(s)g_2(s) \right| ds < \infty \). Suppose \( \mathcal{L}(\alpha) \) is a symmetric \( \alpha \)-stable Lévy process with \( \alpha \in (0, 2) \) and \( L^\alpha_1 \sim S_\alpha(\sigma, 0, 0) \). Define the continuous-time MA processes

\[
Y^1_t = \int_{-\infty}^{\infty} g_1(t-s) dL^\alpha_s \quad \text{and} \quad Y^2_t = \int_{-\infty}^{\infty} g_2(t-s) dL^\alpha_s, \quad t \geq 0.
\]

Furthermore, \( G_{g_1, g_2} : [0, \Delta] \rightarrow \mathbb{R} \) is given as \( s \rightarrow \sum_{j=\infty}^{\infty} g_1(\Delta j-s) g_2(\Delta j-s) \) and suppose \( G_{g_1, g_2} \in L^{(\alpha/2)}[0, \Delta] \). Then, as \( n \rightarrow \infty \),

\[
\frac{1}{n^{2/\alpha}} \sum_{k=1}^{n} Y^1_{k\Delta} Y^2_{k\Delta} \overset{D}{\rightarrow} \int_0^\Delta G_{g_1, g_2}(s) dL^{(\alpha/2)}_s,
\]

where \( L^{(\alpha/2)} \) is the \( \alpha/2 \)-stable Lévy process of Theorem 4.1.

**Proof.** The proof is mostly the same as the proof of the asymptotic behaviour of the sample autocovariance function of a continuous-time moving average process in Theorem 3 of [Drapatz 2017] and is therefore omitted. \( \square \)

**Remark A.2.** Note that

\[
\int_0^\Delta G_{g_1, g_2}(s) dL^{(\alpha/2)}_s \sim S_{\alpha/2}(\sigma_{g_1, g_2}, \beta_{g_1, g_2}, 0),
\]

is an \( \alpha/2 \)-stable distribution with parameters

\[
\beta_{g_1, g_2} = \frac{\int_0^\Delta (G_{g_1, g_2}^+(s))^{\alpha/2} - (G_{g_1, g_2}^-(s))^{\alpha/2} ds}{\int_0^\Delta \left| G_{g_1, g_2}(s) \right|^\alpha ds},
\]

\[
\sigma_{g_1, g_2}^{\alpha/2} = \frac{\sigma^{\alpha} C_{\alpha}}{C_{\alpha/2}} \int_0^\Delta \left| G_{g_1, g_2}(s) \right|^\alpha ds,
\]

see Property 1.2.3 and 3.2.2 of [Samorodnitsky and Taqqu 1994].

**Theorem A.3.** Let \( Y \) be a symmetric \( \alpha \)-stable CARMA process with kernel function \( g(t) = e^{\gamma^T t} e^\mu \mathbb{1}_{[0,\infty)}(t) \) as given in (2.4) and \( \mathcal{P}_n(h) = -n+1, \ldots, n-1 \) be the sample autocovariance function as defined in (3.2). Then, for fixed \( m \in \mathbb{N} \) and as \( n \rightarrow \infty \),

\[
\frac{1}{n^{2/\alpha-1}} (\mathcal{P}_n(0), \ldots, \mathcal{P}_n(m)) \overset{D}{\rightarrow} \left( \int_0^\Delta \sum_{j=-\infty}^\infty g((\Delta j-s)^2) dL^{(\alpha/2)}_s, \ldots, \int_0^\Delta \sum_{j=-\infty}^\infty g((\Delta j-s)g((\Delta (j+m)-s)) dL^{(\alpha/2)}_s \right),
\]

where \( L^{(\alpha/2)} \) is the \( \alpha/2 \)-stable Lévy process of Theorem 4.1.
We obtain for \( \delta < \alpha/2 \) that
\[
\mathbb{E} \left| J_n^{[2]} \right|^\delta \leq n^{-2\delta/\alpha} \sum_{j=m+1}^{m} \sum_{k=0}^{m} c_k Y_{j+k}^{(\Delta)} Y_{k+j}^{(\Delta)} \leq n^{-2\delta/\alpha} m^{2\delta} \max_{k=0,\ldots,m} |c_k| \delta \mathbb{E} \left| Y_{1+k}^{(\Delta)} \right|^\delta \xrightarrow{n \to \infty} 0.
\]
Therefore, the second term \( J_n^{[2]} \) in (A.1) is negligible. For the first term \( J_n^{[1]} \) in (A.1) we define
\[
Y_i^{[1]} := \int_{-\infty}^{\infty} \sum_{k=0}^{m} c_k g(t+k\Delta-s) dL_s^{(\alpha)} \quad \text{and} \quad Y_i^{[2]} := \int_{-\infty}^{\infty} g(t-s) dL_s^{(\alpha)} , \quad t \geq 0.
\]
Thereby, we have
\[
\sum_{k=0}^{m} c_k Y_{j+k}^{(\Delta)} = \int_{-\infty}^{\infty} \sum_{k=0}^{m} c_k g((k+j)\Delta-s) dL_s^{(\alpha)} = Y_{j+k}^{[1]},
\]
\[
Y_{j}^{(\Delta)} = \int_{-\infty}^{\infty} g(j\Delta-s) dL_s^{(\alpha)} = Y_{j}^{[2]}.
\]
An application of Proposition [A.1] leads for \( n \to \infty \) to
\[
\frac{n}{n^{2/\alpha}} \sum_{k=0}^{m} c_k \tilde{Y}_n(k) = \frac{1}{n^{2/\alpha}} \sum_{k=1}^{n} Y_{k\Delta}^{[1]} Y_{k\Delta}^{[2]} + J_n^{[2]}
\]
\[
\xrightarrow{D} \int_{0}^{\Delta} \sum_{j=-\infty}^{\infty} \left( \sum_{k=0}^{m} c_k g(\Delta(k+j)-s) g(\Delta j-s) \right) dL_s^{(\alpha/2)}
\]
\[
= \sum_{k=0}^{m} \int_{0}^{\Delta} \sum_{j=-\infty}^{\infty} c_k g(\Delta(k+j)-s) g(\Delta j-s) dL_s^{(\alpha/2)}.
\]
Cramér-Wold completes the proof. \( \square \)

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