The rigorous derivation of the Linear Landau equation from a particle system in a weak-coupling limit

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Abstract

We consider a system of $N$ particles interacting via a short-range smooth potential, in a weak-coupling regime. This means that the number of particles $N$ goes to infinity and the range of the potential $\epsilon$ goes to zero in such a way that $N\epsilon^2 = \alpha$, with $\alpha$ diverging in a suitable way. We provide a rigorous derivation of the Linear Landau equation from this particle system. The strategy of the proof consists in showing the asymptotic equivalence between the one-particle marginal and the solution of the linear Boltzmann equation with vanishing mean free path. This point follows [3] and makes use of technicalities developed in [16]. Then, following the ideas of Landau, we prove the asymptotic equivalence between the solutions of the Boltzmann and Landau linear equation in the grazing collision limit.

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1 Introduction

1.1 The Boltzmann-Grad limit

In kinetic theory a gas is described by a system of small indistinguishable interacting particles. The evolution of this system is quite complicated since the order of particles involved is quite large. For this reason it is interesting to consider the system from a statistical point of view. The starting point is a system of $N$ particles having unitary mass and moving in a domain $D \subseteq \mathbb{R}^3$. These particles can interact by means of a short-range radial potential $\Phi$. The microscopic state of the system is given by the position and velocity variables denoted by $q_N = (q_1, q_2, ..., q_N)$ and $v_N = (v_1, v_2, ..., v_N)$, where $q_i, v_i$ are respectively position and velocity of the $i$-th particle. The time is denoted by $\tau$. Throughout the paper we will use bold letters for vectors of variables.

Let $\epsilon > 0$ be a parameter denoting the ratio between typical macroscopic and microscopic scales, say the inverse of the number of atomic diameters necessary to fill a centimeter. If we want a macroscopic description of the system it is natural to introduce macroscopic variables defined by

$$x_N = q_N \epsilon \quad t = \tau \epsilon \quad (1.1)$$

where $x_N = (x_1, x_2, ..., x_N)$ are the macroscopic position and $t$ is the macroscopic time variable. Notice that the velocities are unscaled. From the Liouville equation for the particle dynamic it is possible to derive a hierarchy of equations for the $j$-particles marginal probability density function, with $j \leq N$.

In the case of hard spheres we found the following BBGKY hierarchy

$$\left( \partial_t + v_j \cdot \nabla x_j \right) f^N_j = (N - j)\epsilon^2 \sum_{k=1}^{j} \int_{\mathbb{R}^3} dv_{j+1} \int_{\nu(v_k - v_{j+1}) \geq 0} |\nu \cdot (v_k - v_{j+1})| \left[ f^N_{j+1}(x_1, v_1, ..., x_k, v'_k, ..., x_j, v_j, x_k - \eta \epsilon, v'_j + 1) - f^N_{j+1}(x_1, v_1, ..., x_j, v_j, x_k + \eta \epsilon, v'_j + 1) \right]$$

$$\quad (1.2)$$

where $\nu = \frac{x_{j+1} - x_k}{|x_{j+1} - x_k|}$ and $v'_k = v_k - \nu [\nu \cdot (v_k - v_{j+1})]$, $v'_j + 1 = v_{j+1} + \nu [\nu \cdot (v_k - v_{j+1})]$. Equations 1.2 were first formally derived by [4], then a rigorous analysis has been done by [20, 19, 18, 17].

Scaling according to $N \to \infty$ and $\epsilon \to 0$, in such a way that $N\epsilon^2 \cong 1$, we are in a low-density regime suitable for the description of a rarified gas. This kind of scaling is usually called the Boltzmann-Grad limit. The formal Boltzmann-Grad limit in the BBGKY gives a new hierarchy of equations called
the Boltzmann hierarchy. The central idea in kinetic theory is the concept of propagation of chaos, namely, if the initial datum factorizes, i.e. \( f_{0,j}(x_j, v_j) = \prod_{i=1}^{j} f_{0,1}(x_i, v_i) \), then also the solution at time \( t \) factorizes:

\[
f_j(x_j, v_j) = \prod_{i=1}^{j} f_1(x_i, v_i).
\]

(1.3)

Actually the Boltzmann hierarchy admits factorized solutions so that it is compatible with the propagation of chaos and under this hypothesis, which however must be proved from a rigorous viewpoint, the first equation of this hierarchy is the Boltzmann equation

\[
\partial_t f + v \cdot \nabla_x f = \int dv_1 \int B(v, v' - v_1) \left[ f(x, v') f(x, v_1') - f(x, v) f(x, v_1) \right].
\]

(1.4)

However, as soon as \( \epsilon > 0 \) propagation of chaos does not hold because the evolution creates correlation between particles so that we cannot describe the system in terms of a single equation for the one-particle marginal and this is the reason why the Boltzmann equation can describe in a more manageable way the statistical evolution of a gas.

The validity of the Boltzmann equation is a fundamental problem in kinetic theory. It consists in proving that the solution of the BBGKY hierarchy for hard spheres converge in the Boltzmann-Grad limit to the solution of the Boltzmann hierarchy. This means that the propagation of chaos is recovered in the limit.

The rigorous derivation of the Boltzmann equation was first proved by Lanford in 1975 [14] in the case of a hard spheres system for a small time. The main idea of the Lanford work is to write the solution of the BBGKY hierarchy for hard spheres and of the Boltzmann hierarchy as a perturbative series of the free evolution and then prove that the series solution of the BBGKY converge to the series solution of the Boltzmann hierarchy.

More recently Gallagher, Saint-Raymond and Texier [8] and Pulvirenti, Saffirio and Simonella [16] proved the rigorous derivation of the Boltzmann equation, for a small time, starting from a system of particles interacting by means of a short-range potential providing an explicit rate of convergence. In the case of a short-range potential the starting hierarchy is no more the BBGKY hierarchy but the Grad hierarchy, that was developed by Grad in [10].
1.2 The linear case

The linear Boltzmann equation describes the evolution of a tagged particle in a random stationary background at equilibrium and reads as follows

$$
\partial_t g^\alpha + v \cdot \nabla_x g^\alpha = \alpha \int dv_1 M_\beta(v_1) \int dv B(v, v - v_1) \left[ g^\alpha(x, v') - g^\alpha(x, v) \right]
$$

(1.5)

where $M_\beta(v_1) = \frac{1}{C_\beta} e^{-\beta |v_1|^2}$ and $C_\beta$ is chosen in such a way that $\int dv_1 M_\beta(v_1) = 1$. The linear Boltzmann equation can be obtained from the equation (1.4) setting $f(x, v) = g^\alpha(x, v) M_\beta(v)$ and $f(x, v_1) = M_\beta(v_1)$, $g^\alpha$ is the evolution of the perturbation in the stationary background given by $M_\beta(v)$.

The derivation of the linear Boltzmann equation from an hard spheres system has been proved for an arbitrary time by Spohn, Lebowitz [15] and more recently quantitative estimates on the rate of convergence have been obtained by Bodineau, Gallagher and Saint-Raymond [3]. A different type of linear Boltzmann equation has been derived in the case of a Lorentz gas in Refs [9, 1].

1.3 A different scaling

A different scaling can be used to study a different regime from the low density. In case of particles interacting by means of a short-range radial potential $\Phi$, we rescale position and time as in (1.1) but we set $N \epsilon^2 \cong \epsilon^{-1}$ and $\Phi(q) = \epsilon^{-\frac{1}{2}} \Phi(\frac{q}{\epsilon})$. This scaling is called the weak-coupling limit since the density of the particle is diverging in the limit but this is balanced by the interaction that becomes weaker. This weak interaction between particles is called also a “grazing collision” since it changes only slightly the velocity of a particle. The kinetic equation derived from this scaling is the Landau equation

$$
\partial_t f + v \cdot \nabla_x f = \int dv_1 \nabla_v \cdot \left[ \frac{A}{|v - v_1|} P^\perp_{(v-v_1)} (\nabla_v - \nabla_{v_1}) f(v)f(v_1) \right]
$$

(1.6)

where $A$ is a suitable constant and $P^\perp_{(v-v_1)}$ is the projector on the orthogonal subspace to that generated by $v - v_1$.

The Landau equation was derived in a formal way by Landau in [13] starting from the Boltzmann equation in the so-called grazing collision limit. It rules the dynamics of a dense gas with weak interaction between particles. Recently Bobylev, Pulvirenti and Saffirio proved in [2] a result of consistency, but the problem of the rigorous derivation of Landau equation is still open even for short
times.

Also in the case of the Landau equation it is possible to consider the evolution of a perturbation of the stationary solution. This evolution is given by the following linear Landau equation

\[
\partial_t g + v \cdot \nabla_x g = \frac{1}{|V|^3} A \int dv_1 M_\beta(v_1) \left[ |V|^2 \Delta g(v) - (V, D^2(g)V) - 4V \cdot \nabla_v g(v) \right]
\]  

(1.7)

where \(D^2(g)\) is the hessian matrix of \(g\) with respect to the velocity variables and \(A\) is a suitable constant.

Recently Desvillettes and Ricci [6] and Kirkpatrick [12] proved a rigorous derivation for a type of linear Landau equation in two dimensions starting from a Lorentz gas. In this case the velocity of the test particles does not change and the equation obtained is a diffusion of the velocity on the unitary sphere.

1.4 Main theorem

In this paper we prove the rigorous derivation of the linear Landau equation starting from a system of particles. These particles interact by means of a two-body short-range smooth potential and we consider an initial datum which is a perturbation of the equilibrium. We rescale the variables describing the particles system according to (1.1). Simultaneously we set \(N \epsilon^2 \simeq \alpha\) and \(\Phi(q) = \frac{1}{\sqrt{\alpha}} \Phi(\frac{q}{\epsilon})\). This gives us an intermediate scaling between the low density and the weak-coupling and allows us to use the properties of both. Thanks to the low density properties of the scaling as first step we prove that the dynamics of the particles system is near to the solution of the linear Boltzmann equation. In a second step using the weak-coupling properties of the scaling we show that the solution of the linear Landau equation is near to the solution of the linear Boltzmann equation. More precisely let \(f_1^N\) be the one particle marginal distribution and let \(g^\alpha\) be the solution of the linear Boltzmann equation, then we are able to prove that

\[
\|f_1^N(x,v) - g^\alpha(x,v)M_\beta(v)\|_\infty \to 0.
\]  

(1.8)
Then, denoting with $g$ the solution of the linear Landau equation, it results that

$$
\| g(x, v) - g^\alpha(x, v) \|_H \to 0
$$

(1.9)

where $H = L^2(\Gamma \times \mathbb{R}^3, dxd\mu)$, with $d\mu = M_\beta(v)dv$. 
2 Dynamics and statistical description of the motion

2.1 Hamiltonian system

We consider a system of \(N\) indistinguishable particles with unitary mass moving in a torus \(\Gamma_\epsilon = [0, \frac{1}{\epsilon})^3 \subset \mathbb{R}^3\) with \(\epsilon > 0\). The particles interact by means of a two body positive, radial and not increasing potential \(\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}\). We assume also that \(\Phi\) is short-range, namely \(\Phi(q) = 0\) if \(|q| > 1\), moreover \(\Phi \in C^2(\mathbb{R}^3)\). The Hamiltonian of the system is given by

\[
H = \frac{1}{2} \sum_{i=1}^{N} |v_i|^2 + \frac{1}{2} \sum_{i,j=1, i\neq j}^{N} \Phi(q_i - q_j) \tag{2.1}
\]

where \(q_i, v_i\) are respectively position and velocity of the \(i\)-th particle.

The Newton equations are the following

\[
\frac{d^2 q_i}{d\tau^2}(\tau) = \sum_{i\neq j} F(q_i(\tau) - q_j(\tau)) \tag{2.2}
\]

for \(i = 1, ..., N\), where \(F(q_i - q_j) = -\nabla \Phi(q_i - q_j)\) and \(\tau\) is the time variable. The hypothesis that we made on the potential ensure the existence and uniqueness of the solution of the (2.2).

2.2 Scaling

We rescale the system from microscopic coordinates \((q, \tau)\) to macroscopic ones in the following way.

We set

\[
x = \epsilon q \quad t = \epsilon \tau \tag{2.3}
\]

where \(x, t\) are respectively the macroscopic position variable and the macroscopic time variable. We set \(Nc^2 \cong \alpha\), with \(\alpha \cong (\log \log N)^{\frac{1}{4}}\), and we also assume that \(|Nc^2 - \alpha| \rightarrow 0\). With this scaling the density of the gas and the inverse of the mean free path are diverging in the limit. This means that a given particle experiences an high number of interaction per unit time. To balance this divergence we rescale also the potential in the following way

\[
\Phi \rightarrow \alpha^{-\frac{1}{2}} \Phi \tag{2.4}
\]
Figure 2.1: Here $\omega = \omega(\nu, V)$ is the unit vector bisecting the angle between $-V$ and $V'$, $\nu$ is the unit vector pointing from the particle with velocity $v_1$ to the particle with velocity $v_2$ when they are about to collide. We denote with $\beta$ the angle between $-V$ and $\omega$, with $\varphi$ the angle between $-V$ and $\nu$, with $\rho = \sin \varphi$ the impact parameter and with $\theta$ the deflection angle. It results that $\theta = \pi - 2\beta$.

In the microscopic variables the equations of motion read as

$$
\frac{d^2 x_i}{dt^2}(\tau) = \frac{1}{\epsilon \sqrt{\alpha}} \sum_{i \neq j} -\nabla \Phi \left( \frac{x_i(t) - x_j(t)}{\epsilon} \right).
$$

(2.5)

From now we shall work in macroscopic variables unless explicitly indicated.

2.3 The scattering of two particles

In this section we want to give a picture of the scattering between two particles. We turn back to microscopic variables where the potential is assumed to have range one. Let $q_1, v_1, q_2, v_2$ be positions and velocities of two particles which are performing a collision. This two-body problem can be reduced to a central-force problem if we set the origin of the coordinates $c$ in the center of mass

$$
c = \frac{q_1 + q_2}{2}
$$

(2.6)

Thanks to the conservation of the angular momentum we have that the scattering takes place on a plane. We define $V = v_1 - v_2$ as the incoming relative velocity and $V' = v'_1 - v'_2$ as the outgoing relative velocity with
Figure 2.2: We denote with $\sigma \in S^2\left(\frac{v_1 + v_2}{2}\right)$ the direction of $V'$ and with $\theta$ the angle between $V$ and $V'$.

Another useful way to represent the collision between two particles is the so called $\sigma$-representation (Figure 2.2). With this notation the post collisional velocities can be written as follow

$$
\begin{align*}
  v'_1 &= v_1 - \omega [\omega \cdot V] \\
  v'_2 &= v_2 + \omega [\omega \cdot V]
\end{align*}
$$

(2.7)

We can now define the scattering operator $I$, a map defined over

$$
\{(\nu, V) \in S^2 \times \mathbb{R}^3 \setminus \{0\} \text{ s.t. } V \cdot \nu \leq 0\}
$$

(2.9)

by

$$
I(\nu, V) = \left(\nu', V'\right)
$$

(2.10)
\[ \begin{cases} V' = V - 2\omega (\omega \cdot V) \\ \nu' = -\nu + 2\omega (\omega \cdot \nu) \end{cases} \] (2.11)

From the definition of \( \nu' \) and \( V' \) we have that \( \nu \cdot V = -\nu' \cdot V' \). It follows that \( I \) sends incoming configuration in outgoing configuration. The main property of \( I \) is given by the following lemma, proved in [16].

**Lemma 2.1.** \( I \) is an invertible transformation that preserves the Lebesgue measure.

We conclude this section with an estimate for the angle \( \theta \), for which a complete proof can be found in [6].

**Lemma 2.2.** Let \( \Phi \) be a potential satisfying our assumption and let \( \theta(\rho, \alpha) \) be the scattering angle in function of the impact parameter \( \rho \). Then the following estimate holds true:

\[ \theta(\rho, \alpha) \leq \frac{-2}{|V|^2 \sqrt{\alpha}} \gamma(\rho) + \frac{1}{|V|^4 \alpha} M(\rho, \alpha) \] (2.12)

where

\[ \gamma(\rho) = \frac{1}{|\rho|} \int \frac{\rho \Phi' (|\rho|)}{|u|} \frac{du}{\sqrt{1 - u^2}} \] (2.13)

and \( M(\rho, \alpha) \) is positive bounded functions.

**Remark 2.3.** Formula (2.12) points out that when \( \alpha \to \infty \) the collision becomes grazing.

### 2.4 Statistical description

Now we want to describe our system from a statistical point of view. We will denote the phase space as

\[ \Lambda_N = \left\{ z_N \in (\Gamma \times \mathbb{R}^3)^N \right\} \] (2.14)

where \( z_N = (z_1, z_2, \ldots, z_N) \), \( z_i = (x_i, v_i) \) and \( \Gamma \) is the torus of unitary side.

We consider a probability density function \( W_{0,N} \) defined on \( \Lambda_N \). The time evolution of \( W_{0,N} \) is given by the solution \( W_N \) of the following Liouville equation

\[ \begin{cases} \partial_t W_N + \mathcal{L}_N W_N = 0 \\ W_N(0) = W_{0,N} \end{cases} \] (2.15)
where $\mathcal{L}_N = \mathcal{L}_N^0 + \mathcal{L}_N^I$ with

$$\mathcal{L}_N^0 = \sum_{i=1}^{N} v_i \cdot \nabla z_i \quad (2.16)$$

$$\mathcal{L}_N^I = \frac{1}{\epsilon} \sum_{i,j=1}^{N} F_{i,j} \cdot \nabla v_i \quad (2.17)$$

and $F_{i,j} = -\frac{1}{\sqrt{\alpha}} \nabla \Phi \left( \frac{x_i(t) - x_j(t)}{\epsilon} \right)$. We suppose that $W_{0,N}$ is symmetric in the exchange of particles, and hence $W_N(t)$ is still symmetric for any positive times.

The marginals distribution of the measure $W_N(t)$ are defined as

$$f_N^j(z_j, t) = \int dz_{j+1}...dz_N W_N(z_N, t) \quad (2.18)$$

Nevertheless, it is more convenient to work with the reduced marginals $\tilde{f}_N^j(z_j, t)$ that read as follow

$$\tilde{f}_N^j(z_j, t) = \int_{S(x_j)^{N-j}} dz_{j+1}...dz_N W_N(z_N, t) \quad (2.19)$$

where

$$S(x_j)^{N-j} = \{ z = (x, v) \in \Gamma \times \mathbb{R}^3 \mid |x - x_k| > \epsilon \ \forall \ 1 \leq k \leq j \} \quad (2.20)$$

As can be easily seen the reduced marginals are asymptotically equivalent (for $\epsilon \to 0$) to the standard marginals.

For the reduced marginals it is possible to derive from the Liouville equation the following hierarchy of equations, called the Grad hierarchy (GH),

$$(\partial_t + \mathcal{L}_j) f_N^j = \sum_{m=0}^{N-j-1} A_{j+1+m}^N f_{j+1+m}^N \quad 0 \leq j \leq N \quad (2.21)$$

where

$$A_{j+1+m}^N (z_j, t) = \binom{N-j-1}{m} (N-j) \epsilon^2 \sum_{i=1}^{j} \int_{S^2} d\nu \chi_{\{|x_i - x_i| < \epsilon\}}(\nu)$$

$$\int_{\mathbb{R}^3} dv_{j+1} (v_{j+1} - v_j) \cdot \nu \int_{\Delta_m(x_{j+1})} d\Delta_{j+1+m} f_{j+1+m}^N (z_j, x_i + \nu v_{j+1}, z_{j+1}, m, t) \quad (2.22)$$
and \( z_{j+1,m} = (z_{j+1}, \ldots, z_{j+1+m}) \). The set \( \Delta_m(x_{j+1}) \) is defined as follows

\[
\Delta_m(x_{j+1}) = \{ z_{j+1,m} \subset S(x_j)^m \text{ such that } \forall l = j + 2, \ldots, j + 1 + m, \text{ there exists a choice of index } h_1, \ldots, h_r \in \{ j + 2, \ldots, j + 1 + m \} \text{ such that } |x_l - x_{h_1}| \leq \epsilon, |x_{h_k-1} - x_{h_k}| \leq \epsilon \text{ for } k = 2, \ldots, r \text{ and } \min_{i \in \{ l, h_1, \ldots, h_r \}} |x_i - x_{j+1}| \leq \epsilon \}
\]

(2.23)

This hierarchy was first introduced by Grad [10]. Actually in views of the Boltzmann-Grad limit only the first equation of this hierarchy was considered. The full hierarchy was introduced and derived by King in [11]. A complete derivation of this hierarchy can also be found in [8] and [16].

It is possible to represent the solution of the Grad hierarchy as a series obtained by iterating the Duhamel formula. It results that

\[
\tilde{f}_j^N(t) = \sum_{n=0}^{\infty} G_{j,n}^r(t)f_0^N
\]

(2.24)

where

\[
f_0^N = \int_{S(x_j)^N-j} dz_{j+1} \ldots dz_N W_{0,N}(z_N)
\]

(2.25)

and \( G_{j,n}^r(t) \) is defined for \( n \leq N - j \) as

\[
G_{j,n}^r(t) = \sum_{m_1, \ldots, m_n \geq 0} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n
\]

(2.26)

and it is identically equal to zero for \( n > N - j \). The operator \( S_j^r(t) \) is the interacting flow operator:

\[
S_j^r(t)g(z_j) = g(T_j^r(-t)z_j)
\]

(2.27)

where \( T_j^r(t) \) is the solution of the Newton equation (2.5). We call this series the Grad series solution (GSS).
Next we introduce the following hierarchy of equations, called the intermediate hierarchy (IH)

\[(\partial_t + L_j) f_j^N = (N - j) \epsilon^2 C_{j+1}^\epsilon(f_{j+1}^N)\]  

\[C_{j+1}^\epsilon(f_{j+1}^N) = \sum_{k=1}^{N-j} \int dv_{j+1} \int_{\nu \cdot (v_k - v_{j+1}) \geq 0} d\nu \cdot (v_k - v_{j+1})\]

\[\left[f_j^N(x_1, v_1, ..., x_k, v_k', ..., x_j, v_j, x_k - \eta \epsilon, v_j') - f_j^N(x_1, v_1, ..., x_j, v_j, x_k + \eta \epsilon, v_j')\right]\]  

This hierarchy is formally similar to the BBGKY hierarchy for hard spheres but the collision operator appearing in IH is different. Indeed, in the IH we have that the transferred momentum is

\[p = (V \cdot \omega) \omega\]  

while in hard spheres it is

\[p = (V \cdot \nu) \nu\]  

Note that it may be convenient to express \(\nu\) in terms of \(\omega\), which is the parameter appearing in the expression of the outgoing velocities. However, as described in [16], this is a delicate point and we prefer to avoid it, working as much as possible with formula (2.29). We want to notice also that \(A_{j+1}^\epsilon f_{j+1}^N = C_{j+1}^\epsilon(f_{j+1}^N)\), i.e. the first term in the sum on the right hand side of equation (2.24) is the collision term that arise in the IH case. As we will see this will be the only \(O(1)\) term as \(\epsilon \to 0\).

Also for IH we can write the following formal series for the solution, that we will call intermediate series solution (ISS)

\[f_j^N(t) = \sum_{n=0}^{\infty} Q_{j,n}^\epsilon(t)f_{0,j}^N\]  

where the operator \(Q_{j,n}^\epsilon(t)\) is defined for \(n \leq N - j\) as

\[Q_{j,n}^\epsilon(t) = (N - j) ... (N - j - n + 1) \epsilon^{2n} \int_0^t dt_1 ... \int_0^{t_{n-1}} dt_n S_j^\epsilon(t - t_1)C_{j+1}^\epsilon S_{j+1+m_1}^\epsilon(t_1 - t_2)...C_{j+n}^\epsilon S_{j+n}^\epsilon(t_n) f_{0,j+n}^N\]  

and it is identically equal to zero for \(n > N - j\).

Finally we observe that by sending \(\epsilon \to 0\), \(N \to \infty\), \(N \epsilon^2 \to \alpha\) in the IH we obtain, formally, the
following hierarchy, called the Boltzmann hierarchy (BH)

\[(\partial_t + v \cdot \nabla_x) f_j = \alpha C_{j+1}(f_j) \quad 0 \leq j \quad (2.34)\]

\[C_{j+1}(f_j) = \sum_{k=1}^{j} \int_{\mathbb{R}^3} dv_{j+1} \int_{\nu \cdot (v_k - v_{j+1}) \geq 0} d\nu |\nu \cdot (v_k - v_{j+1})| \]

\[\left[ f_{j+1}(x_1, v_1, ..., x_k, v_k, ..., x_j, v_j, v_{j+1}) - f_{j+1}(x_1, v_1, ..., x_j, v_j, v_{j+1}) \right] \quad (2.35)\]

If we assume the propagation of chaos, i.e. that \(f_j = f_1^{\otimes j}\), the first equation of this infinite hierarchy becomes the Boltzmann equation.

The series solution for the Boltzmann hierarchy (BSS) is the following

\[f_j^n(t) = \sum_{n=0}^{\infty} Q_{j,n}^n(t) f_{0,j+n} \quad (2.36)\]

where \(f_{0,j+n}\) is the \(j+n\) particles initial datum and \(Q_{j,n}^n(t)\) is defined as follows

\[Q_{j,n}^n(t) = \alpha^n \int_0^t dt_1 ... \int_0^{t_{n-1}} dt_n S_j(t - t_1) C_{j+1} S_{j+1+m_1}(t_1 - t_2) ... C_{j+n} S_{j+n}(t_n) f_{0,j+n} \quad (2.37)\]

where \(S_j(t)\) is the free flow operator, i.e.

\[S_j(t) g^j(z_j) = g(x_j - v_j t). \quad (2.38)\]
3 Linear regime

In this section we formally derive the linear Boltzmann and Landau equations. First we define the Gibbs measure defined by

\[ M_{N,\beta}(z_n) = C_{N,\beta} e^{-\beta H_N(z_n)} \]  \hspace{1cm} (3.1)

where \( \beta > 0 \) and \( C_{N,\beta} \) is chosen so that

\[ \int_{\mathcal{X}_N} M_{N,\beta}(z_n) dz_n = 1 \]  \hspace{1cm} (3.2)

The Gibbs measure is an invariant measure for the gas dynamics and (3.1) is a stationary solution of the Liouville equation.

In case of the Boltzmann and Landau equations, a stationary solution is given by the Maxwellian distribution (free gas)

\[ M_\beta(v) = C_\beta e^{-\frac{\beta}{2} |v|^2} \]  \hspace{1cm} (3.3)

where \( \beta > 0 \) and \( C_\beta \) is such that

\[ \int_{\Gamma \times \mathbb{R}^3} M_\beta(v) dx dv = 1. \]  \hspace{1cm} (3.4)

Moreover a stationary solution of the Boltzmann hierarchy is

\[ M^{\otimes j}_\beta(v_j) = \prod_{i=1}^{j} M_\beta(v_i). \]  \hspace{1cm} (3.5)

Now we consider the Liouville equation (2.15) with initial datum given by

\[ W_{0,N}(z_N) = M_{N,\beta}(z_N)g_0(x_1, v_1) \]  \hspace{1cm} (3.6)

where \( g_0 \in L^\infty (\Gamma \times \mathbb{R}^3) \) is a perturbation on the first particle such that \( \int dz_1 M_{N,\beta}(z_1)g_0(x_1, v_1) = 1. \)

**Theorem 3.1.** Let \( W^N \) be the solution of the Liouville equation (2.15) with initial datum (3.6) and let \( f^N_j \) be the \( j \)-particles reduced marginal. Then for any \( 1 \leq j \leq N \) the following bound holds

\[ \sup_t f^N_j(z_j, t) \leq M_{N,\beta}(z_j)\|g_0\|_\infty \leq M^{\otimes j}_\beta(z_j)\|g_0\|_\infty \]  \hspace{1cm} (3.7)
Proof. From the choice of the initial datum we have that

$$f_0^N(z_N) \leq M_{N,\beta}(z_N)\|g_0\|_{\infty}$$

(3.8)

Since the maximum principle holds for the Liouville equation and \(M_{N,\beta}(z_N)\) is a stationary solution we have that

$$W^N(z_N,t) \leq M_{N,\beta}(z_N)\|g_0\|_{\infty}$$

(3.9)

This implies the (3.7) since \(M_{N,\beta}(z_j) \leq M_{\beta}^{\otimes j}(z_j)\) by the positivity of the interaction.

3.1 Linear Boltzmann equation and asymptotics

In this section we derive the linear Boltzmann equation from the non linear one and study its asymptotic behavior for \(\alpha \to \infty\). Suppose that the initial datum of the Boltzmann hierarchy (2.35) is

$$f_{0,j}(x_1, v_1, \ldots, x_j, v_j) = M_{\beta}(v_1) \ldots M_{\beta}(v_j) g_0(x_1, v_1)$$

(3.10)

with \(g_0(x_1, v_1) \in L^\infty(\Gamma)\). Since the Maxwellian distribution is a stationary solution of the equations we look for a solution at time \(t\) given by

$$f_{\alpha,j}(z_j, t) = M_{\beta}(v_1) \ldots M_{\beta}(v_j) g_{\alpha}(x_1, v_1, t).$$

(3.11)

From (3.10) and (2.35) we have that (3.11) is a solution of the Boltzmann hierarchy if \(g_{\alpha}\) satisfies the following equation

$$M_{\beta}(v) \left( \partial_t g_{\alpha} + v \cdot \nabla_x g_{\alpha} \right) = \alpha \int dv_1 \int_{\nu \cdot V > 0} dv' \cdot V \left[ M_{\beta}(v') M_{\beta}(v_1) g_{\alpha}(x, v') - M_{\beta}(v) M_{\beta}(v_1) g_{\alpha}(x, v) \right]$$

(3.12)

Since \(M_{\beta}(v') M_{\beta}(v_1) = M_{\beta}(v) M_{\beta}(v_1)\) the equation (3.12) becomes the Linear Boltzmann equation

$$\partial_t g_{\alpha} + v \cdot \nabla_x g_{\alpha} = Q_B(g_{\alpha}),$$

(3.13)

where

$$Q_B(g_{\alpha}) = \alpha \int dv_1 M_{\beta}(v_1) \int_{\nu \cdot V > 0} dv' \cdot V \left[ g_{\alpha}(x, v') - g_{\alpha}(x, v) \right].$$

(3.14)
We are interested to investigate the behavior of $Q_B$ when $\alpha \to \infty$. We denote with $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ an orthonormal base of $\mathbb{R}^3$ such that $\hat{e}_1 = \frac{V}{|V|}$. Now we consider the semisphere $S^2_+ = \{ \nu \in s^2 | \nu \cdot V > 0 \}$.

For a fixed $\nu$ in this semisphere the scattering takes place in the plane generated by $\hat{e}_1$ and $\nu$. An orthonormal base of the scattering plane is given by the vectors $\hat{e}_1$ and $\hat{e} (\psi) = \hat{e}_2 \cos \psi + \hat{e}_3 \sin \psi$, calling with $\psi$ the angle between $\hat{e}_2$ and $\hat{e}$. We also denote with $\varphi$ the angle between $\hat{e}_1$ and $\nu$.

![Diagram](image)

Figure 3.1: A representation of a three dimensional scattering.

From the $\sigma - representation$ (2.8) we have that

$$v' = c + r\sigma$$  \hspace{1cm} (3.15)

where $r = \frac{|V|}{2}$ and $c = \frac{v_r}{2}$. Notice that in our coordinates it results that

$$\sigma = \cos \theta \hat{e}_1 - \sin \theta \hat{e}(\psi)$$  \hspace{1cm} (3.16)
We denote with $v'(\theta)$ the post collisional velocity in function of the scattering angle $\theta$

$$v'(\theta) = c + r \cos \theta \hat{e}_1 - r \sin \theta \hat{e}(\psi)$$

(3.17)

This implies that

$$g^\alpha(v'(\theta)) = g^\alpha(c + r \cos \theta \hat{e}_1 - r \sin \theta \hat{e}(\psi))$$

(3.18)

For sake of brevity we will not take care of the dependence of $g$ from the spatial variable. Let us consider the Taylor expansion of $g$ with respect to $\theta$ up to the second order. We have

$$g^\alpha(v') - g^\alpha(v) = g^\alpha(v'(\theta)) - g^\alpha(v'(0))$$

$$= \theta \nabla_v g^\alpha(v) \cdot \frac{dv'}{d\theta}(0) + \frac{\theta^2}{2} \left[ \nabla_v g^\alpha \cdot \frac{d^2v'}{d\theta^2}(0) + \left( \left. \frac{dv'}{d\theta}(0) \right| D_v^2 g^\alpha \frac{dv'}{d\theta}(0) \right) \right] + o(\theta^2)$$

(3.19)

where $D_v^2 g^\alpha$ is the hessian matrix of $g^\alpha$ with respect to the velocity. A simple calculation gives us that

$$\frac{dv'}{d\theta}(0) = -r \hat{e}(\psi)$$

(3.20)
\[ \frac{d^2 \psi'}{d\theta^2} (0) = -r \hat{e}_1 \]  

(3.21)

It can be easily seen that the integration of the first term is zero by symmetry. Moreover from Lemma 2.2 we have that

\[ \theta^2 (\rho, \alpha) \leq \frac{4}{|V|^4 \alpha} \gamma^2 (\rho) + o(\alpha^{-1}) \]  

(3.22)

From this remark and by equations (3.19) and (3.14) we have that

\[ Q_B = \int dv_1 M_\beta (v_1) \int_{\nu \cdot V > 0} d\nu |\nu \cdot V| \frac{2}{|V|^4} \gamma (\rho)^2 \left[ -\frac{1}{2} V \cdot \nabla g^a (v) + \frac{|V|^2}{4} \left( \hat{e} (\psi), D^2 \hat{e} (\psi) \right) \right] + o(\alpha^{-1}) \]  

(3.23)

From the change of variables \( \nu \rightarrow \psi, \varphi \), since \( d\nu = \sin \varphi d\varphi d\psi \), we have that

\[ Q_B = \int dv_1 M_\beta (v_1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi |\nu \cdot V| \sin \varphi \frac{2}{|V|^4} \gamma (\rho)^2 \left[ -\frac{1}{2} V \cdot \nabla g^a (v) + \frac{|V|^2}{4} \left( \hat{e} (\psi), D^2 \hat{e} (\psi) \right) \right] + o(\alpha^{-1}) \]  

(3.24)

Since \( |\nu \cdot V| = |V| \cos \varphi \) and \( \rho = \sin \varphi \), it results that \( \cos \varphi d\varphi = d\rho \) and so

\[ Q_B = \int dv_1 M_\beta (v_1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \left[ -\frac{1}{2} V \cdot \nabla g^a (v) + \frac{|V|^2}{4} \left( \hat{e} (\psi), D^2 \hat{e} (\psi) \right) \right] + o(\alpha^{-1}) = \]

\[ \int dv_1 M_\beta (v_1) \frac{1}{|V|^3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \left[ \frac{|V|^2}{2} \left( \hat{e} (\psi), D^2 \hat{e} (\psi) \right) - V \cdot \nabla g^a (v) \right] \frac{1}{1} \int dp \rho \gamma (\rho)^2 + o(\alpha^{-1}) \]  

(3.25)

From the definition of \( \hat{e} (\psi) \), and since \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \psi d\psi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \psi d\psi = \frac{\pi}{2} \) and \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \psi \cos \psi d\psi = 0 \), we have that

\[ Q_B = \int dv_1 M_\beta (v_1) \frac{1}{|V|^3} \left[ |V|^2 \left( \hat{e}_2, D^2 \hat{e}_2 \right) + |V|^2 \left( \hat{e}_3, D^2 \hat{e}_3 \right) - 4 V \cdot \nabla g^a (v) \right] \frac{\pi}{4} \int dp \rho \gamma (\rho)^2 + o(\alpha^{-1}) \]  

(3.26)

Now since the laplacian is the trace of the Hessian matrix and it is invariant under changes of coordi-
nates we have that
\[ \triangle g(v) = (\hat{e}_1, D^2 \hat{e}_1) + (\hat{e}_2, D^2 \hat{e}_2) + (\hat{e}_3, D^2 \hat{e}_3) \quad (3.27) \]
and so
\[ |V|^2 (\hat{e}_2, D^2 \hat{e}_2) + |V|^2 (\hat{e}_3, D^2 \hat{e}_3) = |V|^2 \triangle g^\alpha(v) - (V, D^2 V) \quad (3.28) \]
Thanks to (3.28) we finally arrive to
\[ Q_B(g) = B \int d\nu_1 M_\beta(v_1) \frac{1}{|V|^3} \left[ |V|^2 \triangle g^\alpha(v) - (V, D^2 V) - 4V \cdot \nabla_v g^\alpha(v) \right] + o(\alpha) \]
\[ = Q_L(g) + o(\alpha) \quad (3.29) \]
where
\[ B = \frac{\pi}{4} \int_{-1}^{1} d\rho \rho \gamma(\rho)^2 \quad (3.30) \]

3.2 Linear Landau equation

In this subsection we will show that the linear operator \( Q_L(g) \) is indeed the linear Landau operator obtained by the full nonlinear equations. Consider
\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f &= C_L(f) \\
f(x,v,0) &= f_0(x,v)
\end{align*}
\]}

whit
\[ C_L(f) = A \int d\nu_1 \nabla_v \cdot \left[ \frac{1}{|v - v_1|} P_{(v-v_1)} \nabla_v f(v_1) f(v) \right] \quad (3.31) \]
where \( A > 0 \) is a suitable constant and \( P_{(v-v_1)} \) is the projector on the orthogonal subspace to \( v - v_1 \).

Also in this case we consider a perturbation of the stationary state. We set \( f(v) = M_\beta(v) g(v) \) and \( f(v_1) = M_\beta(v_1) \) in (3.31). This represents a single particle perturbed in a stationary background.

With this choice equation (3.31) becomes
\[ M_\beta(v) (\partial_t g + v \cdot \nabla_x g) = K(g) \]
\[ K(g) = A \int d\nu_1 \nabla_v \cdot \left[ \frac{1}{|v - v_1|} P_{(v-v_1)} \nabla_v M_\beta(v_1) g(v) \right] \]
We suppose to have all the necessary regularity to give sense to the following calculations. We start
from the gradient term which leads to

\[ K(g) = A \int dv_1 \nabla_v \left[ \frac{1}{|V|} P^\perp_V (M_\beta(v) M_\beta(v_1) \nabla_v g(v) - 2v_1 \beta M_\beta(v_1) h(v) + 2v_1 \beta M_\beta(v_1) g(v)) \right] \]

\[ = A \int dv_1 M_\beta(v_1) \nabla_v \cdot \left[ \frac{1}{|V|} P^\perp_V (M_\beta(v) \nabla_v g(v) - 2\beta M_\beta(v_1) g(v)(V)) \right] \quad (3.32) \]

Notice that \( P^\perp_V (2\beta M_\beta(v_1) g(v)(V)) = 0 \), this yields

\[ K(g) = A \int dv_1 M_\beta(v_1) \nabla_v \cdot \left[ \frac{1}{|V|} P^\perp_V (M_\beta(v) \nabla_v g(v)) \right] \quad (3.33) \]

We also notice that \( \nabla_v \frac{1}{|V|} \) is parallel to \( V \), we calculate the divergence and obtain

\[ \nabla_v \cdot \left[ \frac{1}{|V|} P^\perp_V (M_\beta(v) \nabla_v g(v)) \right] = \nabla_v \left[ \frac{1}{|V|} \right] \cdot P^\perp_V (M_\beta(v) \nabla_v g(v)) + \frac{1}{|V|} \nabla_v \cdot P^\perp_V (M_\beta(v) \nabla_v g(v)) = \]

\[ \frac{1}{|V|} \nabla_v \cdot P^\perp_V (M_\beta(v) \nabla_v g(v)) \]

Therefore by (3.33) we have

\[ K(g) = A \int dv_1 M_\beta(v_1) \frac{1}{|V|} \nabla_v \cdot [M_\beta(v) P^\perp_V (\nabla_v g(v))] \quad (3.34) \]

We calculate again the divergence

\[ \nabla_v \cdot [M_\beta(v) P^\perp_V (\nabla_v g(v))] = -2\beta v_1 M_\beta(v) \cdot P^\perp_V (\nabla_v g(v)) + M_\beta(v) \nabla_v \cdot [P^\perp_V (\nabla_v g(v))] = \]

\[ -2\beta (v - v_1) M_\beta(v) \cdot P^\perp_V (\nabla_v g(v)) - 2\beta v_1 M_\beta(v) \cdot P^\perp_V (\nabla_v g(v)) + M_\beta(v) \nabla_v \cdot [P^\perp_V (\nabla_v g(v))] = \]

\[ -2\beta v_1 M_\beta(v) \cdot P^\perp_V (\nabla_v g(v)) + M_\beta(v) \nabla_v \cdot [P^\perp_V (\nabla_v g(v))] \quad (3.35) \]

From (3.35) and (3.34) we arrive to

\[ K(g) = A \int dv_1 M_\beta(v_1) \frac{1}{|V|} \left\{ -2\beta v_1 M_\beta(v) \cdot P^\perp_V (\nabla_v g(v)) + M_\beta(v) \nabla_v \cdot [P^\perp_V (\nabla_v g(v))] \right\} = \]

\[ A \int dv_1 M_\beta(v_1) \frac{1}{|V|} \left[ -2\beta v_1 M_\beta(v) \cdot P^\perp_V (\nabla_v g(v)) \right] + \]

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We now use (3.40) and (3.41) together with (3.38) to get

\[ A \int dv_1 M_\beta(v_1) \frac{1}{|V|} M_\beta(v) \nabla_v \cdot [P_\nabla (\nabla_v g(v))] \]  

(3.36)

Now we work on the first term of the right hand side of (3.36). Since \(-2\beta v_1 M_\beta(v_1) = \nabla_v M_\beta(v_1)\), by means of the divergence Theorem we have that

\[ A \int dv_1 M_\beta(v_1) \frac{1}{|V|} [-2\beta v_1 M_\beta(v) \cdot P_\nabla (\nabla_v g(v))] = M_\beta(v) A \int dv_1 (-2\beta v_1) M_\beta(v_1) \cdot \frac{P_\nabla (\nabla_v g(v))}{|V|} = \]

\[ -M_\beta(v) A \int dv_1 M_\beta(v_1) \nabla_{v_1} \cdot \left[ \frac{P_\nabla (\nabla_v h(v))}{|V|} \right] = -M_\beta(v) A \int dv_1 M_\beta(v_1) \frac{1}{|V|} \nabla_{v_1} \cdot [P_\nabla (\nabla_v g(v))] \]  

(3.37)

From (3.37) and (3.36) we arrive to

\[ K(g) = M_\beta(v) A \int dv_1 M_\beta(v_1) \frac{1}{|V|} (\nabla_v - \nabla_{v_1}) \cdot [P_\nabla (\nabla_v g(v))] \]  

(3.38)

Now we want to calculate \( \nabla_v \cdot [P_\nabla (\nabla_v g(v))] \) and \( \nabla_{v_1} \cdot [P_\nabla (\nabla_v g(v))] \). First we observe that

\[ P_\nabla (\nabla_v g(v)) = \nabla_v g(v) - \frac{(V, \nabla_v g(v)) V}{|V|^2} \]  

(3.39)

and so

\[ \nabla_{v_1} \cdot [P_\nabla (\nabla_v g(v))] = \nabla_{v_1} \cdot \left[ \nabla_v g(v) - \frac{(V, \nabla_v g(v)) V}{|V|^2} \right] = -\nabla_{v_1} \cdot \left[ \frac{(V, \nabla_v g(v)) V}{|V|^2} \right] = \]

\[ 2 \frac{(V, \nabla_v g(v))}{|V|^2} \]  

(3.40)

For the other term we have that

\[ \nabla_v \cdot [P_\nabla (\nabla_v g(v))] = \nabla_v \cdot \left[ \nabla_v g(v) - \frac{(V, \nabla_v g(v)) V}{|V|^2} \right] = \triangle g(v) - \nabla_v \cdot \left[ \frac{(V, \nabla_v g(v)) V}{|V|^2} \right] = \]

\[ \triangle g(v) - 2 \frac{(V, \nabla_v g(v))}{|V|^2} + \frac{(V, D^2 V)}{|V|^2} \]  

(3.41)

We now use (3.40) and (3.41) together with (3.38) to get

\[ K(g) = M_\beta(v) A \int dv_1 M_\beta(v_1) \frac{1}{|V|^2} \left[ |V|^2 \triangle g(v) - (V, D^2 V) - 4V \cdot \nabla_v g(v) \right] \]  

(3.42)
Finally we can define the linear Landau equation

$$\partial_t g + v \cdot \nabla_x g = \tilde{Q}_L(g) \tag{3.43}$$

where $\tilde{Q}_L$ is the linear Landau operator defined as

$$\tilde{Q}_L(g) = A \int dv_1 M_\beta(v_1) \frac{1}{|V|^3} \left[ |V|^2 \Delta g(v) - (V, D^2 V) - 4V \cdot \nabla_v g(v) \right] \tag{3.44}$$

Notice that $\tilde{Q}_L$ and $Q_L$ are the same operator if $A = B$. The constant $A$ is precisely characterized by the formal derivation of the Landau equation from a system of particles and it has the following value

$$A = \frac{1}{8\pi} \int_0^{+\infty} dr r^3 \hat{\Phi}(r)^2 \tag{3.45}$$

where $\hat{\Phi}(|k|) = \int dx \Phi(|x|) e^{-ik \cdot x}$. It can be easily proved that $A = B$ by following the calculations made in [12] and, therefore, that $\tilde{Q}_L = Q_L$. 

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4 Continuity estimates

In this section we will prove some useful estimates for the operators arising in the series solution of the hierarchies. Observe that in the case of $\alpha = 1$ these estimates are enough to prove the convergence of the series solution for a small time. In our case since $\alpha \to \infty$ the time of the convergence of the series is going to zero. As we will see in the next section we can still use these estimates in the linear case thanks to the a priori estimate.

We define the following norm

$$\|f_j(z_j)\|_\beta = \sup_{z_j \in \Lambda_j} \left( e^{\beta H(z_j)} f_j(z_j) \right)$$

(4.1)

where the Hamiltonian $H(z_j)$ in macroscopic variables reads as

$$H(z_j) = \frac{1}{2} \sum_{i=1}^j |v_i|^2 + \frac{1}{2\sqrt{\alpha}} \sum_{i,k=1, i \neq k}^j \Phi(\frac{x_i - x_k}{\epsilon})$$

(4.2)

For sake of simplicity we don’t indicate the dependence from $j$ in the definition of $\| \cdot \|_\beta$. Notice also that the norm depends on $\alpha$ but not in a harmful way.

Since we are interested in the linear regime we will take as initial datum a perturbation of the stationary state, as we have seen in section 3.1 and 3.2. We assume that the initial datum of GH and IH has the form

$$f_{N,0}^j(z_j) = M_{N,\beta}(z_j)g_0(x_1, v_1)$$

(4.3)

We assume also that the initial data for the Boltzmann hierarchy is

$$f_0^\alpha(z_j) = M^\alpha_{\beta}(v_j)g_0(x_1, v_1)$$

(4.4)

Notice that the estimates that we will prove work also in case of a general $f_0$ with $\|f_0\|_\beta < \infty$ for a $\beta > 0$.

4.1 Estimates of the operators

We start by estimating the operator appearing in GSS
Lemma 4.1. Let \( g_j^N \) be a sequence of continuous functions with \( g_j^N = 0 \) for \( j > N \) and suppose that
\[
\|g_j^N\|_\beta \leq C^j \quad (4.5)
\]
Then for \( \beta' < \beta \) there exist a constant \( C_1 = C_1(\beta, \beta', g_j^N) \) such that for \( \epsilon \) small enough and \( \forall j \geq 0 \)
\[
\|G^*_{j, \epsilon}(t)g_j^N(z_j)\|_{\beta'} \leq (C_1\alpha t)^n \quad (4.6)
\]
Proof. From the definition of the operator \( A_{j+1+m}^\epsilon g_j^N \) we have that
\[
e^{\beta' H(z_j)}|A_{j+1+m}^\epsilon g_j^N(z_j)| \leq C^{j+1}C^m e^{3m} \epsilon^2 N^{m+1} \sum_{i=1}^{j} \int dv_{j+1} (|v_i| + |v_{j+1}|) e^{-\beta - \beta'} v_i v_{j+1}^{\frac{1}{2}} \quad (4.7)
\]
since
\[
\int \Delta_{m(x_{j+1})} dz_{j+1, m} f_{j+1+m}(z_j, x_{j+1, m}, t) \leq C^m \epsilon^{3m} \quad (4.8)
\]
and
\[
\|g_j^{N+1+m}\|_\beta \leq C^{j+1}C^m \quad (4.9)
\]
Now since \( \epsilon^2 N \equiv \alpha \) we have that
\[
e^{3m} \epsilon^2 N^{m+1} \leq \alpha (Ce\alpha)^m \quad (4.10)
\]
and so
\[
e^{\beta' H(z_j)}|A_{j+1+m}^\epsilon g_j^N(z_j)| \leq n\alpha (Ce\alpha)^m \quad (4.11)
\]
We can choose \( \epsilon \) small enough, since \( \alpha \approx \sqrt{\log\log N} \), to have that \( C\alpha < 1 \). We perform the sum over \( m \) to obtain
\[
\sum_{m \geq 0} (C\alpha)^m \leq C \quad (4.12)
\]
that leads us to
\[
\|A_{j+1+m}^\epsilon g_j^N(z_j)\|_{\beta'} \leq n\alpha C \quad (4.13)
\]
Now since for any \( \beta > 0 \) it results that
\[
\|S_j^\epsilon(t)g_j^N\|_\beta = \|g_j^N\|_\beta \quad (4.14)
\]
we can alternate estimate (4.13) and (4.14) and performe the time integrals in (2.26). This gives us that
\[
\| G_{j,n}^\epsilon(t) g_j^N(z_j) \|_{\beta'} \leq (C_1 \alpha t)^n
\] (4.15)

In the same way we can estimate the operators \( Q_{j,n}^\epsilon(t) \) and \( Q_{j,n}^\alpha(t) \) and prove the following lemma

**Lemma 4.2.** Let \( Q_{j,n}^\epsilon(t) \) and \( Q_{j,n}^\alpha(t) \) be defined respectively as in (2.33) and in (2.37). Let also \( g_j^N, g_j \) be sequence of continuous functions with \( g_j^N = 0 \) for \( j > N \) suppose that
\[
\| g_j^N \|_{\beta} \leq C^j
\] (4.16)
\[
\| g_j \|_{\beta} \leq C^j
\] (4.17)
then there exist constants \( C_2 \) and \( C_3 \) such that for \( \epsilon \) small enough and \( \beta' < \beta \)
\[
\| Q_{j,n}^\epsilon(t) g_j^N \|_{\beta'} \leq (C_2 \alpha t)^n
\] (4.18)
\[
\| Q_{j,n}^\alpha(t) g_j \|_{\beta'} \leq (C_3 \alpha t)^n
\] (4.19)

### 4.2 Estimates for an arbitrary time

Now we want to use the a priori estimate to prove the convergence of the series solution for an arbitrary time. The main idea is to separate the interval \([0, t]\) in \( s \in \mathbb{N} \) parts of length \( h \) such that
\[
t = sh
\] (4.20)
and write \( f_{1,s}^N(t), f_{1,s}^\alpha(t) \) and \( f_{1,s}^N(t) \) in terms of a finite sum plus a remainder. We use the technique used by Bodineau, Gallagher and Saint-Raymond [3]. It consists in bounding the number of interactions in an interval \([ih, (i+1)h]\) \( 0 \leq i < s \) by \( 2^i - 1 \) and send the time \( h \) to zero in a suitable way.

In literature there is another method, which is employed by Colangeli, Pezzoti and Pulvirenti in [5], that consists in taking \( h \) smaller than the Lanford time of the convergence of the series solutions and then bounding in a suitable way the number of creations in each interval. We cannot use this method since in our case the time of the convergence of the series is going to zero.
We can write the solution at time $t$ of the GH as the evolution of a time $h$ of the solution at time $t-h$

$$\tilde{f}_1^N(t) = \sum_{j_1=0}^{\infty} G_{1,j_1}^\epsilon(h) \tilde{f}_{j_1+1}^N(t-h) \quad (4.21)$$

We introduce the Grad truncated series solution (GTS) by truncating the series (4.21) at $j_1 = 2^1 - 1 = 1$. We obtain

$$\tilde{f}_1^N(t) = \sum_{j_1=0}^{1} G_{1,j_1}^\epsilon(h) \tilde{f}_{j_1+1}^N(t-h) + \tilde{R}_{1,1}(t-h,t) \quad (4.22)$$

$$\tilde{R}_{1,1}(t-h,t) = \sum_{j_1=2}^{\infty} G_{1,j_1}^\epsilon(h) \tilde{f}_{j_1+1}^N(t-h) \quad (4.23)$$

Now we can iterate this procedure on $\tilde{f}_{j_1+1}^N(t-h)$. We have that

$$\tilde{f}_{j_1+1}^N(t-h) = \sum_{j_2=0}^{\infty} G_{j_1+1,j_2}^\epsilon(h) \tilde{f}_{j_2+1}^N(t-h) \quad (4.24)$$

We truncate again the series at $j_2 = 2^2 - 1$ and we arrive to

$$\tilde{f}_{j_1+1}^N(t-h) = \sum_{j_2=0}^{2^2-1} G_{j_1+1,j_2}^\epsilon(h) \tilde{f}_{j_2+1}^N(t-2h) + \tilde{R}_{j_1+1,2}(t-2h,t-h) \quad (4.25)$$

where

$$\tilde{R}_{j_1+1,2}(t-2h,t-h) = \sum_{p=4}^{N-j_1-1} G_{j_1+1,2}^\epsilon(h) \tilde{f}_{j_1+1+p}^N(t-2h) \quad (4.26)$$

From (4.25) and (4.22) we have

$$\tilde{f}_1^N(t) = \sum_{j_1=0}^{1} \sum_{j_2=0}^{2^2-1} G_{1,j_1}^\epsilon(h) G_{j_1+1,j_2}^\epsilon(h) \tilde{f}_{j_2+1}^N(t-2h) + \tilde{R}_N^2(t) \quad (4.27)$$

where $\tilde{R}_N^2(t)$ takes into account the evolution of the remainders of each truncation and reads as follows

$$\tilde{R}_N^2(t) = \tilde{R}_{1,1}(t-h,t) + \sum_{j_1=0}^{1} G_{1,j_1}^\epsilon(h) \tilde{R}_{j_1+1,2}(t-2h,t-h) \quad (4.28)$$

We iterate this procedure with a sequence of cutoffs $2^i - 1$, this leads to

$$\tilde{f}_1^N(t) = \tilde{f}_{1,s}^N(t) + \tilde{R}_N^S(t) \quad (4.29)$$
where, denoting with $P_i = 1 + \sum_{k=1}^{i} j_k$ the number of particles after $i$ iterations,

$$\tilde{f}_{1,s}^N(t) = \sum_{j_1=0}^{1} \sum_{j_s=0}^{2^{i-1}-1} G_{1,j_1}^s(h) G_{P_{1},j_2}^s(h) \ldots G_{P_{i-1,j_s}}^s(h) f_{0}^N$$

(4.30)

$$\tilde{R}_{1}^N(t) = \sum_{i=1}^{s} \sum_{j_1=0}^{1} \sum_{j_{i-1}=0}^{2^{i-1}-1} G_{1,j_1}^i(h) G_{P_{i-1},j_2}^i(h) \ldots G_{P_{i-2,j_{i-1}}}^i(h) \tilde{R}_{P_{i-1,i}}(t - ih, t - (i - 1)h)$$

(4.31)

$$\tilde{R}_{P_{i-1,i}}(t - ih, t - (i - 1)h) = \sum_{p=2^i}^{N-P_{i-1}} G_{P_{i-1,p}}^i(h) \tilde{f}_{P_{i-1+p}}^N$$

(4.32)

We use the same procedure for the series solution of the Boltzmann hierarchy and we obtain the truncated Boltzmann solution (BTS)

$$f_{1,s}^N(t) = \sum_{j_1=0}^{1} \sum_{j_s=0}^{2^{i-1}-1} Q_{1,j_1}^s(h) Q_{P_{1},j_2}^s(h) \ldots Q_{P_{i-1,j_s}}^s(h) f_{0}^N$$

(4.33)

$$R^s(t) = \sum_{i=1}^{s} \sum_{j_1=0}^{1} \sum_{j_{i-1}=0}^{2^{i-1}-1} Q_{1,j_1}^i(h) Q_{P_{i},j_2}^i(h) \ldots Q_{P_{i-2,j_{i-1}}}^i(h) R_{P_{i-1,i}}(t - ih, t - (i - 1)h)$$

(4.34)

$$R_{P_{i-1,i}}(t - ih, t - (i - 1)h) = \sum_{p=2^i}^{N-P_{i-1}} Q_{P_{i-1,p}}^i(h) f_{P_{i-1+p}}^N$$

(4.35)

We also define the intermediate truncated solution (ITS)

$$f_{1,s}^N(t) = \sum_{j_1=0}^{1} \sum_{j_s=0}^{2^{i-1}-1} Q_{1,j_1}^i(h) Q_{P_{1},j_2}^i(h) \ldots Q_{P_{i-1,j_s}}^i(h) f_{0}^N$$

(4.36)

Now we want to prove an estimate for the remainder term.

**Theorem 4.3.** Let $\tilde{R}_{N}^N(t), R^s(t)$ be defined respectively as in (4.31) and (4.34). Then the following estimate holds

$$\|\tilde{R}_{N}^N(t)\|_{\infty} + \|R^s(t)\|_{\infty} \leq \|g_0\|_{\infty} \left( C (\alpha t)^2 \right)^2$$

(4.37)
Proof. Thanks to the semigroup property we have that

\[
\tilde{R}_s^N(t) = \sum_{i=1}^{s} \sum_{j_1=0}^{1} \ldots \sum_{j_{i-1}=0}^{2^{i-1}-1} G_{1,P_{i-1}-1}^\gamma((i-1)h) \tilde{R}_{P_{i-1},i}(t - ih, t - (i-1)h)
\]  

(4.38)

Now from the steps of Lemma 4.1 it follows that

\[
\|G_{1,P_{i-1}-1}^\gamma((i-1)h) \tilde{R}_{P_{i-1},i}(t - ih, t - (i-1)h)\|_\infty \leq (C\alpha(i-1)h)^{P_{i-1} - 1} \|\tilde{R}_{P_{i-1},i}(t - ih, t - (i-1)h)\|_2
\]  

(4.39)

Furthermore we have that

\[
\|\tilde{R}_{P_{i-1},i}(t - ih, t - (i-1)h)\|_2 \leq \sum_{p=2^i}^{N-P_{i-1}} (C\alpha h)^p \|f_{P_{i-1}+p}^{N}\|_2 \leq \|g_0\|_\infty \sum_{p=2^i}^{N-P_{i-1}} (C\alpha h)^p
\]  

(4.40)

We use together the last two estimates and that

\[
C\alpha h < \frac{1}{2}
\]

and we arrive to

\[
\|\tilde{R}_s^N(t)\|_\infty \leq \|g_0\|_\infty \sum_{i=1}^{s} \sum_{j_1=0}^{1} \ldots \sum_{j_{i-1}=0}^{2^{i-1}-1} (C\alpha h)^{P_{i-1} - 1} (C\alpha h)^{2^i} \leq \|g_0\|_\infty \sum_{i=1}^{s} 2^i (C\alpha h)^{2^i}
\]

\[
\leq \|g_0\|_\infty \sum_{i=1}^{s} \left(\frac{C(\alpha t)^2}{s}\right)^{2^i}
\]  

(4.41)

In the last step we used that \(h = \frac{4}{s}\) and that \(i(i-1) \leq 2^i\). Now we assume also that

\[
\frac{C(\alpha t)^2}{s} < \frac{1}{2}
\]  

(4.42)

and we finally arrive to

\[
\|\tilde{R}_s^N(t)\|_\infty \leq \|g_0\|_\infty \left(\frac{C(\alpha t)^2}{s}\right)^{2^i}
\]  

(4.43)
The estimate for $R^e(t)$ can be obtained in the same way.

Thanks to Theorem 4.3 we can work directly on the truncated series since we have an estimate on the remainders. We want to prove that the GTS is close to the ITS as $\epsilon \rightarrow 0$. We have

**Theorem 4.4.** Let $\tilde{f}_{j,s}^N(t), f_{j,s}^N(t)$ be respectively the solution of the first equation of GH and IH. Then the following estimate holds for all $t \geq 0$

$$\|\tilde{f}_{j,s}^N(t) - f_{j,s}^N(t)\|_\infty \leq \|g_0\|_\infty 2^{s(s+1)} (C\alpha t)^{2s+1} \epsilon$$ (4.44)

**Proof.** The definition of the truncated solution series leads to

$$\tilde{f}_{j,s}^N(t) - f_{j,s}^N(t) = \sum_{j_1=0}^{2} \sum_{j_2=0}^{2s-1} \left[ G_{1,j_1}^\epsilon(h) G_{1,j_2}^\epsilon(h) \ldots G_{P_{l-1},j_s}^\epsilon(h) - Q_{1,j_1}^\epsilon(h) Q_{1,j_2}^\epsilon(h) \ldots Q_{P_{l-1},j_s}^\epsilon(h) \right] f_{0}^N$$ (4.45)

Now from the semigroup property and the identity

$$a^n - b^n = \sum_{i=1}^{n} a^{i-1}(a - b)b^{n-i}$$ (4.46)

we have

$$G_{1,j_1}^\epsilon(h) G_{1,j_2}^\epsilon(h) \ldots G_{P_{l-1},j_s}^\epsilon(h) - Q_{1,j_1}^\epsilon(h) Q_{1,j_2}^\epsilon(h) \ldots Q_{P_{l-1},j_s}^\epsilon(h) =$$

$$\sum_{l=1}^{s} G_{1,P_{l-1}-1}^\epsilon((l-1)h) \left[ G_{P_{l-1},j_l}^\epsilon(h) - Q_{P_{l-1},j_l}^\epsilon(h) \right] Q_{P_{l-1},P_{l-1}}^\epsilon((s-l)h)$$ (4.47)

Since the operator $Q_{P_{l-1},j_l}^\epsilon(h)$ is the first term not equal to zero in the asymptotic of the operator $G_{P_{l-1},j_l}^\epsilon(h)$ we obtain that

$$G_{P_{l-1},j_l}^\epsilon(h) - Q_{P_{l-1},j_l}^\epsilon(h) f_{0}^N(0) = \sum_{m_1, \ldots, m_{j_l} \geq 0, \sum_{i=1}^{j_l} m_i \neq 0}^{h} \int_{0}^{t_{j_l-1}} dt_{j_l-2} \ldots \int_{0}^{t_{j_l-1}} dt_{j_l}$$

$$S_{P_{l-1},j_l}^\epsilon(h-t_{j_l}) A_{P_{l-1},j_l}^\epsilon \sum_{i=1}^{j_l} m_i$$

$$A_{P_{l-1},j_l}^\epsilon \sum_{i=1}^{j_l} m_i$$

$$S_{P_{l-1},j_l}^\epsilon(h-t_{j_l}) A_{P_{l-1},j_l}^\epsilon \sum_{i=1}^{j_l} m_i$$

$$S_{P_{l-1},j_l}^\epsilon(h-t_{j_l}) A_{P_{l-1},j_l}^\epsilon \sum_{i=1}^{j_l} m_i$$

The same steps of Lemma 4.1 lead to

$$\|G_{P_{l-1},j_l}^\epsilon(h) - Q_{P_{l-1},j_l}^\epsilon(h) f_{0}^N(0)\|_{\beta'} \leq (C\alpha h)^{j_l} \|g_0\|_\infty \epsilon$$ (4.49)
From (4.49) and (4.47) we arrive to

$$\sum_{l=1}^{s} \|G^e_{1, P_{l-1}}((l-1)h) \left[ G^e_{P_{l-1}, j_l} (h) - Q^e_{P_{l-1}, j_l} (h) \right] Q^e_{P_{l-1}, P_s-P_l} ((s-l)h) f_{N,e} \|_{\infty} \leq \|g_0\|_{\infty} \epsilon \left( C\alpha t \right)^{P_s-1}$$

(4.50)

We perform the sum over $j_1, j_2, ..., j_s$ and we finally have that

$$\|f_{N,1, s}^e(t) - f_{N,1, s}^N(t)\|_{\infty} \leq \|g_0\|_{\infty} 2^{s(s+1)} \epsilon \left( C\alpha t \right)^{2s+1}$$

(4.51)

Thanks to this theorem we can reduce us to study only the convergence of the ITS to the BTS.
5 Convergence to Linear Boltzmann equation

5.1 The Boltzmann backward flow and the Interacting backward flow

In this section we will represent in a convenient way the series (2.32) and (2.36) for the first-particle marginal. These series solutions can be represented graphically as a trees expansion. We define an n-collision tree graph as the following collection of integer

\[ \Gamma(n) = \{ (i_1, ..., i_n) \in \mathbb{N}^n \mid i_k \leq k \} \]  

(5.1)

Roughly speaking, this integer represent the label of the particle that creates a new particle in a creation term. In Figure (5.1) we give a picture of the tree (1,1,2).

![Tree Graph Representation](image)

Figure 5.1: A representation of the tree graph (1,1,2). At the time \( t_1 \) we create the particle 2 on the particle 1. Then at time \( t_2 \) we create the particle 3 on the particle 1. Finally at time \( t_3 \) the particle 4 is created on the particle 2.

We define the following collections of variables for the ITS...
\[ \sigma_n = (\sigma_1, \ldots, \sigma_n) \quad \sigma_i = \pm 1 \quad \sigma_n = \prod_{i=1}^{n} \sigma_i \]  
(5.2)

\[ t_n = (t_1, \ldots, t_n) \]  
(5.3)

\[ w_n = (w_1, \ldots, w_n) \]  
(5.4)

\[ \nu_n = (\nu_1, \ldots, \nu_n) \]  
(5.5)

Here \( t_1, \ldots, t_n \) are the time variables appearing in the time integrals, while \( w_i \) and \( \nu_i \) are the velocity and the impact parameter that appears in the creation of the \((i+1)\)-th particle. Fixed these variables we can construct the interacting backwards flow (IBF). We define the IBF at time \( s \in (t_k, t_{k+1}) \) as

\[ \zeta^\epsilon(s) = (r^\epsilon_1(s), \xi^\epsilon_1(s), \ldots, r^\epsilon_{i+k}(s), \xi^\epsilon_{i+k}(s)) \]  
(5.6)

where \( r_i(s), \xi_i(s) \) are respectively position and velocity of the \( i \)-Th particle at time \( s \). At time \( t \) we have that \( \zeta^\epsilon(t) = (x_1, v_1) \), then we go back in time with the interacting flow defined as the solution of equation (2.5). Between time \( t \) and time \( t - t_1 \) we set \( \zeta^\epsilon(s) = T^\epsilon_1(-s) (r^\epsilon_1(t), \xi^\epsilon_1(t)) \). Then at time \( t - t_1 \) we create a new particle in position \( r_2(t - t_1) = r^\epsilon_1(t - t_1) + \epsilon \nu_1 \) with velocity \( \xi_2(t - t_1) = w_1 \) in a pre-collisional state if \( \sigma_1 = +1 \) or in post collisional one if \( \sigma_1 = -1 \). Between time \( t - t_1 \) and \( t - t_1 - t_2 \) we set the IBF as \( \zeta^\epsilon(s) = T^\epsilon_2(-t + t_1 + s) (r^\epsilon_1(t - t_1), \xi^\epsilon_1(t - t_1), r^\epsilon_2(t - t_1 - t_2), \xi^\epsilon_2(t - t_1)) \). In this way we create a new particle at time \( t - t_1 - t_2 \) in position \( r^\epsilon_2(t - t_1 - t_2) + \epsilon \nu_2 \) with velocity \( w_2 \) in pre-collisional or post-collisional configuration that depends on \( \sigma_2 \). We iterate this procedure and we define the IBF up to time 0 by alternating the creation of new particles with the interacting flow \( T^\epsilon_j \).

For sake of simplicity we define the following time variables

\[ \tau_k = t - \sum_{i=1}^{k} t_i \]  
(5.7)

With this definition \( \tau_k \) are the backward times of a creation.

We can write the one particle marginal in a more manageable way thanks to the IBF
\[ f_{1,s}^N(t) = \sum_{j_1=0}^{N-1} \cdots \sum_{j_s=0}^{N-P_s-1} (N-1)...(N-P_s-1) (\epsilon^2)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \sigma_{P_s-1} I_{\sigma_{P_s-1}}(z_j, t) \] (5.8)

with

\[ I_{\sigma_{P_s-1}}(z_j, t) = \int dt_{P_s-1} dw_{P_s-1} d\nu_{P_s-1} \prod_{k=1}^{P_s-1} B^\epsilon (\nu_k, w_{1+k} - \xi_{ik}(\tau_k)) f_{0,P_s}^N(\zeta^\epsilon(0)) \] (5.9)

and

\[ B^\epsilon (\nu_k, w_{1+k} - \xi_{ik}(\tau_k)) = |\nu_k \cdot (w_{1+k} - \xi_{ik}(\tau_k))| |\chi \{ |r_{k+1}(\tau_k) - r_{ik}(\tau_k)| > \epsilon \} \cdot \chi \{ \sigma_k \nu_k \cdot (w_{1+k} - \xi_{ik}(\tau_k)) \geq 0 \} \] (5.10)

With a similar procedure we can build the Boltzmann backward flow (BBF) but we have to take into account the following difference:

- The flow between two creation is the free flow and not the interacting flow;
- The new particle in each creation is created in the position of his progenitor, i.e. \( r_{ik}(\tau_k) = r_{k+1}(\tau_k) \);
- There is no constraint on \( \nu_k \) other than the one implied by the value of \( \sigma_k \);
- if \( \sigma_k = +1 \) before going back in time we have to change the velocities from post collisional in pre-collisional according to the scattering rules.

Taking into account these differences, we define the BBF at time \( s \in (t_k, t_{k+1}) \) as

\[ \zeta(s) = (r_1(s), \xi_1(s), ..., r_{1+k}(s), \xi_{1+k}(s)) \] (5.11)

We use the BBF to write the one particle marginal of the Boltzmann equation as

\[ f_{1,s}(t) = \sum_{j_1=0}^{N-1} \cdots \sum_{j_s=0}^{N-P_s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \sigma_{P_s-1} I_{\sigma_{P_s-1}}(z_j, t) \] (5.12)

\[ I_{\sigma_{P_s-1}}(z_j, t) = \int dt_{P_s-1} dw_{P_s-1} d\nu_{P_s-1} \prod_{k=1}^{P_s-1} B f_{0,P_s}(\zeta(0)) \] (5.13)
We also define the vectors of the only velocities at time $s \in (t_k, t_{k+1})$ as

$$\xi^s(s) = (\xi^1(s), \ldots, \xi^{1+k}(s))$$ (5.14)

$$\xi(s) = (\xi_1(s), \ldots, \xi_{1+k}(s))$$ (5.15)

Figure 5.2: We used a dashed line to evidence the virtual trajectory of the first particle.

Now we define the virtual trajectory of the particle $i$ in the BBF in an inductive way. We set $\zeta^1(s) = (r_1(s), \xi_1(s))$, then we define the inductive step

$$\zeta^i(s) = (r^i(s), \xi^i(s)) = \begin{cases} 
(r_i(s), \xi_i(s)) & s \in [0, t_{i-1}] \\
(r^j(s), \xi^j(s)) & s \in (t_{i-1}, t] \end{cases}$$ (5.16)

where $j \in \{1, \ldots, i-1\}$ is the progenitor of the particle $i$, i.e. the particle where we create the particle $i$. With this definition the virtual trajectory of a particle $i$ is its backward trajectory until his creation,
before of his creation it is the virtual trajectory of its progenitors.

5.2 Estimate of the recollision

We want to take advantage of the tree expansion to estimate the difference between the intermediate truncated solution and the Boltzmann truncated solution by estimating the difference between the IBF and the BBF. The main difference between the IBF and the BBF are the recollision, i.e. an interaction between particles which is not a creation. This can happen only in the IBF and creates correlations.

First we consider some cutoffs on the integration variables and estimate the complementary term, denoting the various cutoff with an apex Err$_i$. We establish some obvious estimates useful in the following.

\[
\sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} 1 \leq 2^{s(s+1)} \tag{5.17}
\]

\[
P_s - 1 \leq \sum_{i=1}^{s} 2^i \leq 2^{s+1} \tag{5.18}
\]

\[
|f_{0,P_s}(\zeta(0))| \leq C e^{-\frac{\beta}{2}|\xi(0)|^2} \tag{5.19}
\]

\[
\|f_{0,P_s}(\zeta(0))\| \leq C e^{-\frac{\beta}{2}|\xi(0)|^2} \tag{5.20}
\]

We also denote with \(d\Lambda_{P_s-1} = dt_{P_s-1}d\omega_{P_s-1}d\nu_{P_s-1}\).

First we consider the error coming from the difference \(|\alpha - N\epsilon^2|\).

**Lemma 5.1.** Suppose that \(|\alpha - N\epsilon^2| \leq \epsilon\) and let

\[
f_{1,s}^{N,\text{Err}_1} = \sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \sum_{\tau_{P_s-1}} \int d\Lambda_{P_s-1} P_{s-1} \prod_{k=1}^{P_{s-1}} B^k f_{0,P_s}(\zeta(0)) \tag{5.21}
\]

Then

\[
\|f_{1,s} - f_{1,s}^{N,\text{Err}_1}\|_{\infty} \leq \|g_0\|_{\infty} (Ct)^{2^s+1} 2^{s(s+1)} \epsilon \tag{5.22}
\]

**Proof.** We recall that

\[
f_{1,s}^{N} = \sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (N-1) \ldots (N-P_s-1) (\epsilon^2)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \sum_{\tau_{P_s-1}} \int d\Lambda_{P_s-1} P_{s-1} \prod_{k=1}^{P_{s-1}} B^k f_{0,P_s}(\zeta(0))
\]

and since

\[
|(N-1) \ldots (N-P_s-1) (\epsilon^2)^{P_s-1} - (\alpha)^{P_s-1}| \leq 2^{s+1} \alpha^{2^{s+1}} |\alpha - N\epsilon^2|
\]

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it results that
\[ \| f_{1,s}^N - f_{1,s}^{N,\text{Err}} \|_\infty \leq \| g_0 \|_\infty 2^{s(s+1)^2} (C\alpha t)^{2^{s+1}} \epsilon \] (5.23)
\[ \square \]

Next we control the terms \( \prod_{k=1}^{P_s-1} B^r \) and \( \prod_{k=1}^{P_s-1} B \). For \( \lambda \in (0, 1) \) we define the indicator function
\[ \chi_\lambda = \chi \left\{ \prod_{k=1}^{P_s-1} B^r \leq \epsilon^{-\lambda} \right\} \] (5.24)
and
\[ \chi = \chi \left\{ \prod_{k=1}^{P_s-1} B \leq \epsilon^{-\lambda} \right\} . \] (5.25)

The following lemma gives us an estimate for the complementary terms.

**Lemma 5.2.** Let
\[ f_{1,s}^{N,\text{Err}} = \sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \int d\Lambda_{P_s-1} \chi \prod_{k=1}^{P_s-1} B^r f_{0,P_s}^{N}(\zeta(0)) \] (5.26)

and
\[ f_{1,s}^{\alpha,\text{Err}} = \sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \int d\Lambda_{P_s-1} \chi \prod_{k=1}^{P_s-1} B f_{0,P_s}(\zeta(0)) \] (5.27)

Then
\[ \| f_{1,s}^{N,\text{Err}} - f_{1,s}^{N,\text{Err}} \|_\infty + \| f_{1,s}^{\alpha,\text{Err}} - f_{1,s}^{\alpha,\text{Err}} \|_\infty \leq \epsilon^\lambda \| g_0 \|_\infty (C\alpha t)^{2^{s+1}} 2^{s(s+1)} \] (5.28)

**Proof.** We prove the estimate only for \( B^r \), the one for \( B \) can be obtained along the same lines. We have that

\[ | f_{1,s}^{N} - f_{1,s}^{N,\lambda} | \leq \sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (C\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \int d\Lambda_{P_s-1} \chi \left\{ \prod_{k=1}^{P_s-1} B^r > \epsilon^{-\lambda} \right\} \prod_{k=1}^{P_s-1} B^r f_{0,P_s}^{N}(\zeta(0)) | \leq \epsilon^\lambda \| g_0 \|_\infty \sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (C\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \int d\Lambda_{P_s-1} \prod_{k=1}^{P_s-1} B^r |^2 e^{-\frac{2}{3}\| \zeta(0) \|^2} \] (5.29)

where we used that \( 1 = \epsilon^{-\lambda} \epsilon^\lambda \leq \epsilon^\lambda \prod_{k=1}^{P_s-1} B^r \). Now we observe that
\[
\sum_{\Gamma(P_s-1)} P_s-1 \prod_{k=1}^{P_s-1} |B'|^2 \leq 2^{P_s-1} \prod_{k=1}^{P_s-1} (P_s) v_k^2 + \sum_{i=1}^{P_s} v_i^2 \tag{5.30}
\]

Therefore:

\[
|f_{1,s}^N - f_{1,s}^{N,\lambda}| \leq \epsilon^\lambda \|g_0\|_\infty (C\alpha)^{2(s+1)} \sum_{j_1=0}^{1} \sum_{j_2=0}^{2^{s}-1} \int d\Lambda_{P_s-1} e^{-\frac{2}{\epsilon} |\xi^\epsilon(0)|^2} \prod_{k=1}^{P_s-1} \bigg[(P_s) v_{k+1}^2 e^{-\frac{\beta}{\epsilon} |v_k+1|^2} + \sum_{i=1}^{P_s} v_i^2 e^{-\frac{\beta}{\epsilon} |v_i|^2} \bigg] \leq \epsilon^\lambda \|g_0\|_\infty (C\alpha^t)^{2(s+1)} 2^{s(s+1)} \tag{5.31}
\]

The next step is to consider an energy cutoff. We define

\[
\chi_{\lambda,E}^\epsilon = \chi \left\{ \prod_{k=1}^{P_s-1} B \leq \epsilon^{-\lambda} \right\} \chi \left\{ |\xi^\epsilon(0)| \leq 2E \right\} \tag{5.32}
\]

and

\[
\chi_{\lambda,E} = \chi \left\{ \prod_{k=1}^{P_s-1} B \leq \epsilon^{-\lambda} \right\} \chi \left\{ |\xi(0)| \leq 2E \right\} \tag{5.33}
\]

The following estimate holds true

**Lemma 5.3.** Let

\[
f_{1,s}^{N,\text{Err3}} = \sum_{j_1=0}^{1} \sum_{j_2=0}^{2^{s}-1} (\alpha)^{P_s-1} \prod_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \int d\Lambda_{P_s-1} \chi_{\lambda,E}^\epsilon \prod_{k=1}^{P_s-1} B^\epsilon f_{0,P_s}^N (\zeta^\epsilon(0)) \tag{5.34}
\]

and let

\[
f_{1,s}^{\alpha,\text{Err3}} = \sum_{j_1=0}^{1} \sum_{j_2=0}^{2^{s}-1} (\alpha)^{P_s-1} \prod_{\Gamma(P_s-1)} \sum_{\sigma_{P_s-1}} \int d\Lambda_{P_s-1} \chi_{\lambda,E} \prod_{k=1}^{P_s-1} B f_{0,P_s} (\zeta(0)) \tag{5.35}
\]

Then it results:

\[
\|f_{1,s}^{N,\text{Err2}} - f_{1,s}^{N,\text{Err3}}\|_\infty + \|f_{1,s}^{\alpha,\text{Err2}} - f_{1,s}^{\alpha,\text{Err3}}\|_\infty \leq \|g_0\|_\infty e^{-\frac{\beta}{\epsilon} B^2} (C\alpha^t)^{2^{s+1}} 2^{s(s+1)} \tag{5.36}
\]

**Proof.** We give a proof only for \(f_{1,s}^{N,\text{Err2}} - f_{1,s}^{N,\text{Err3}}\|_\infty\), the other one can be proved in the same way.
We have that

\[
|f_{1,s}^{N,\text{Err2}} - f_{1,s}^{N,\text{Err3}}| \leq \|g_0\|_{\infty} \sum_{j_1=0}^{1} \sum_{j_2=0}^{2^s-1} (C\alpha)^{P_{s}-1} \sum_{\Gamma(P_{s}-1)} d\Lambda_{P_{s}-1} \prod_{k=1}^{P_{s}-1} B^k \chi \{|\zeta^*(0)| > 2E\} e^{-\frac{2}{\beta}|\zeta^*(0)|^2}
\]

\[
\leq \|g_0\|_{\infty} e^{-\beta E^2} (C\alpha t)^{2^s+1} 2^{s(s+1)} \tag{5.37}
\]

where we used that \(e^{-\frac{2}{\beta}|\zeta^*(0)|^2} \leq e^{-\frac{\beta}{4}|\zeta^*(0)|^2} e^{-\beta E^2} \).

The next cutoff regards the time variables. We want to separate enough the time between two creation, i.e. we want that \(t_i - t_{i-1} > \delta \forall 0 < i \leq P_s - 1 \). We define

\[
\chi_{\lambda,E,\delta} = \chi \left\{ \prod_{k=1}^{P_{s}-1} B^k \leq e^{-\lambda} \right\} \chi \{|\zeta^*(0)| \leq 2E\} \chi \{t_i - t_{i-1} > \delta, 0 < i \leq P_s - 1 \} \tag{5.38}
\]

and

\[
\lambda_{\lambda,E,\delta} = \chi \left\{ \prod_{k=1}^{P_{s}-1} B \leq e^{-\lambda} \right\} \chi \{|\zeta(0)| \leq 2E\} \chi \{t_i - t_{i-1} > \delta, 0 < i \leq P_s - 1 \} \tag{5.39}
\]

For the complementary set we have the following lemma

\textbf{Lemma 5.4.} Let

\[
f_{1,s}^{N,\text{Err4}} = \sum_{j_1=0}^{1} \sum_{j_2=0}^{2^s-1} (\alpha)^{P_{s}-1} \sum_{\Gamma(P_{s}-1)} \sigma_{P_{s}-1} \int d\Lambda_{P_{s}-1} \chi_{\lambda,E,\delta} \prod_{k=1}^{P_{s}-1} B^k f_{0,P_s}(\zeta^*(0)) \tag{5.40}
\]

and let

\[
f_{1,s}^{\alpha,\text{Err4}} = \sum_{j_1=0}^{1} \sum_{j_2=0}^{2^s-1} (\alpha)^{P_{s}-1} \sum_{\Gamma(P_{s}-1)} \sigma_{P_{s}-1} \int d\Lambda_{P_{s}-1} \chi_{\lambda,E,\delta} \prod_{k=1}^{P_{s}-1} B f_{0,P_s}(\zeta(0)) \tag{5.41}
\]

Then the following estimate holds

\[
\|f_{1,s}^{N,\text{Err3}} - f_{1,s}^{N,\text{Err4}}\|_{\infty} + \|f_{1,s}^{\alpha,\text{Err3}} - f_{1,s}^{\alpha,\text{Err4}}\|_{\infty} \leq e^{-\lambda}\|g_0\|_{\infty} (C\alpha t)^{2^s+1} 2^{(s+2)(s+1)} \delta \tag{5.42}
\]

\textbf{Proof.} As the other lemma we give a proof only for the term \(\|f_{1,s}^{N,\text{Err3}} - f_{1,s}^{N,\text{Err4}}\|_{\infty}\) since for the other
one the proof is similar. We have that

$$|f_{1,s}^{N,Err_3} - f_{1,s}^{N,Err_4}| \leq \epsilon^{-\lambda}\|g_0\|_\infty \sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_{s-1})} f_{\lambda,E} \sum_{\sigma_{P_s-1}} \int d\Lambda_{P_{s-1}} \left( \chi_{\lambda,E}^c - \chi_{\lambda,E,\delta}^c \right) e^{-\frac{q}{2}t(0)^2}$$

(5.43)

There are $P_s - 1$ choices of time variables such that $t_i - t_{i-1} \leq \delta$, this gives us that

$$|f_{1,s}^{N,Err_3} - f_{1,s}^{N,Err_4}| \leq \epsilon^{-\lambda}\|g_0\|_\infty (C\alpha t)^{2s+1} 2^{(s+2)(s+1)} \frac{\delta}{t}$$

(5.44)

Finally we introduce the last cutoff in the integrals. We define the indicator function

$$\chi_{\lambda,E,\delta,q}^c = \chi_{\lambda,E,\delta}^c \chi \{|\omega_k \cdot (w_{k+1} - \xi_{ik}(\tau_k))| \geq \epsilon^q, |\rho_k| \geq \epsilon^q, 1 \leq k \leq P_s - 1\}$$

(5.45)

and

$$\chi_{\lambda,E,\delta,q} = \chi_{\lambda,E,\delta} \chi \{|\omega_k \cdot (w_{k+1} - \xi_{ik}(\tau_k))| \geq \epsilon^q, |\rho_k| \geq \epsilon^q, 1 \leq k \leq P_s - 1\}$$

(5.46)

With this cutoff we are neglecting the grazing and the central velocities in the creation of new particles.

We have the following estimate:

**Lemma 5.5.** Let

$$f_{1,s}^{N,Err_5} = \sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_{s-1})} \sigma_{P_s-1} \int d\Lambda_{P_{s-1}} \chi_{\lambda,E,\delta,q}^c \prod_{k=1}^{P_s-1} B f_{0,P_s}(\zeta^c(0))$$

(5.47)

and let

$$f_{1,s}^{\alpha,Err_5} = \sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_{s-1})} \sigma_{P_s-1} \int d\Lambda_{P_{s-1}} \chi_{\lambda,E,\delta,q} \prod_{k=1}^{P_s-1} B f_{0,P_s}(\zeta^c(0))$$

(5.48)

Then:

$$\|f_{1,s}^{N,Err_3} - f_{1,s}^{N,Err_4}\|_\infty + \|f_{1,s}^{\alpha,Err_4} - f_{1,s}^{\alpha,Err_5}\|_\infty \leq \|g_0\|_\infty \epsilon^{\frac{q}{2}} (C\alpha t)^{2s+1} 2^{(s+2)(s+1)}$$

(5.49)

with $0 < q < 1$. 41
Proof. We have that
\[
| f_{1,s}^{N,Err4} - f_{1,s}^{N,Err5} | \leq e^{-\lambda} \| g_0 \|_\infty \sum_{j_1=0}^{2^s-1} \sum_{j_s=0}^{2^{s-1}} (C\alpha)^{P_s-1} \sum_{\Gamma(P_{s-1})} d\Lambda_{P_{s-1}} (\chi_{\lambda,E,\delta} - \chi_{\lambda,E,\delta,q}) e^{-\frac{q}{2} |\xi(0)|^2} 
\]  
(5.50)
This means that there exist a \( k \) such that \( |\omega_k \cdot (v_{k+1} - \xi_{\lambda_k}^\epsilon (\tau_k)) | \leq \epsilon^q \). If \( |(v_{k+1} - \xi_{\lambda_k}^\epsilon (\tau_k)) | \leq \epsilon^\frac{q}{2} \) then we simply have that
\[
| f_{1,s}^{N,Err4} - f_{1,s}^{N,Err5} | \leq \| g_0 \|_\infty \epsilon^{\frac{q}{2} - \lambda} (C\alpha)^{2^{s+1}} 2^{(s+2)(s+1)} 
\]  
(5.51)
Otherwise if \( |(v_{k+1} - \xi_{\lambda_k}^\epsilon (\tau_k)) | > \epsilon^\frac{q}{2} \) it results that \( |\cos \gamma| \leq \epsilon^\frac{q}{2} \), where \( \gamma \) is the angle between \( v_{k+1} - \xi_{\lambda_k}^\epsilon (\tau_k) \) and \( \omega_k \). Therefore \( |\frac{\pi}{2} - \gamma| \leq C\epsilon^\frac{q}{2} \) and, fixed \( v_{k+1} - \xi_{\lambda_k}^\epsilon (\tau_k) , \omega_k \) must be in a set of measure bounded by \( C\epsilon^q \). The case \( \rho_k \leq \epsilon^q \) can be easily estimated, since \( d\nu_k = \rho_k d\rho_k d\psi \). We have that
\[
| f_{1,s}^{N,Err4} - f_{1,s}^{N,Err5} | \leq \| g_0 \|_\infty \epsilon^{q - \lambda} (C\alpha)^{2^{s+1}} 2^{(s+2)(s+1)} 
\]  
(5.52)
From (5.51) and (5.52) we arrive to
\[
\| f_{1,s}^{N,Err4} - f_{1,s}^{N,Err5} \|_\infty \leq \| g_0 \|_\infty \epsilon^{\frac{q}{2} - \lambda} (C\alpha)^{2^{s+1}} 2^{(s+2)(s+1)} 
\]  
(5.53)

We are now in position to estimate the difference between the BBF and the IBF.

We define the following set
\[
N^{P_s}(\epsilon_0) = \left \{ (t_{P_{s-1}}, \nu_{P_{s-1}}, w_{P_{s-1}}) \in \mathbb{R}^{P_{s-1}} \times S^2(P_{s-1}) \times \mathbb{R}^3(P_{s-1}) \mid \min_{i<k} \min_{\tau \in [0,t_{i-1}]} d(r_i(\tau), r_k(\tau)) < \epsilon_0 \right \} 
\]  
(5.54)
where \( d(\cdot, \cdot) \) denotes the distance over the torus \( \Gamma \). This set is completely defined via the BBF and it is the set of variables for which a recollision can appear. At this point we need to prove that the measure of the set \( N^{P_s}(\epsilon_0) \) is small, taking into account also the constraints given by \( \chi_{\lambda,E,\delta,q} \) and \( \chi_{\lambda,E,\delta,q} \). This smallness is proved in [16] in the case of particles moving in the whole \( \mathbb{R}^3 \) instead that in a torus. In the following lemma we adapt this result to the present context by using also some geometrical estimate proved in [3].

Lemma 5.6. Let \( \chi_{\lambda,E,\delta,q} \) be defined as in (5.46) and let \( \chi \left \{ N^{P_s}(\epsilon_0) \right \} \) be the characteristic function of
the set (5.54). Then it results that

\[
\sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^{s-1}} (\alpha)^{P_{s-1}} \sum_{\Gamma(P_{s-1})} \sum_{\sigma_{P_{s-1}}} \sigma_{P_{s-1}} \int d\Lambda P_{s-1} \chi_{\lambda,E,\delta,q} \chi \{N^{P_s}(\epsilon_0)\} P_{s-1} \prod_{k=1}^{P_s} B f_0^{N,P_s}(\zeta(0)) \leq \|g_0\|_{\infty} \cdot (C\alpha t)^{2^{s+1} - E^2} 2^{(s+4)(s+1)} \left( \frac{\epsilon_0^2 - \lambda}{\epsilon_0^2} + \frac{\epsilon_0^2 - \lambda}{\delta^2} + \frac{\epsilon_0^2 - \lambda}{\epsilon_0^2} \right) \tag{5.55}
\]

We leave the proof of this lemma in the appendix II.

Thanks to these estimates we can now give a proof of the convergence of the IBF to the BBF and then of the one particle marginal of the GH to the solution of the Boltzmann equation. First we choose the magnitude of the parameters in the following way

\[
\alpha \cong C (\log \log N)^{\frac{1}{2}} \quad s \cong \frac{\log \log N}{2 \log 2} \tag{5.56}
\]

Furthermore we have that

\[
2^{s+1} \leq 2 (\log N)^{\frac{1}{2}} \tag{5.57}
\]

\[
2^{(s+2)(s+1)} \leq 2 (\log N)^{\frac{\log \log N}{2 \log 2}} \tag{5.58}
\]

\[
(C\alpha t)^{2^{s+1}} \leq (C \log \log N)^{\sqrt{\log N}} \tag{5.59}
\]

\[
N \epsilon^2 \leq C (\log \log N)^{\frac{1}{2}} \tag{5.60}
\]

\[
\epsilon \leq C \left( \frac{\log \log N}{N^2} \right)^{\frac{1}{4}} \tag{5.61}
\]

We also set \( \epsilon_0 = \epsilon^\frac{1}{2}, \delta = \epsilon^\frac{1}{2}, E = \frac{\sqrt{\log N}}{2}, \) and we fix \( q = \frac{1}{8} \) and \( \lambda = \frac{1}{32} \). We have the following theorem

**Theorem 5.7.** Let \( \tilde{f}_1^N(t) \) be the one particle marginal of the Grad hierarchy with initial datum as (3.6) and let \( f_1^\alpha(t) \) be the solution of the Boltzmann equation with initial datum as (3.10). Then \( \forall t \in [0, T] \) it results that

\[
\| \tilde{f}_1^N(t) - f_1^\alpha(t) \|_{\infty} \to 0 \tag{5.62}
\]

for \( N \to \infty, \epsilon \to 0, |Ne^2 - \alpha| \leq \epsilon \).

**Proof.** We have

\[
\| \tilde{f}_1^N(t) - f_1^\alpha(t) \|_{\infty} \leq \| \tilde{f}_1^{N,\alpha}(t) - f_1^{N,\alpha}(t) \|_{\infty} + \| f_1^{N,\alpha}(t) - f_1^\alpha(t) \|_{\infty} + \| R_N^e(t) \|_{\infty} + \| R^e(t) \|_{\infty} \tag{5.63}
\]
From Theorems 4.3 and 4.4 it results that

\[
\|f_{1,s}^N(t) - f_{1,s}^\alpha(t)\|_\infty \leq \|g_0\|_\infty \epsilon(C\alpha t)^{2s+1} \leq \|g_0\|_\infty (C \log \log N)^{\log \log N \sqrt{\log N}} \leq \frac{(\log N)^4}{N^2}
\]

(5.64)

\[
\|R_N(t)\|_\infty + \|R^s(t)\|_\infty \leq \|g_0\|_\infty \left(\frac{C (\alpha t)^2}{s}\right)^2 \leq \frac{C\|g_0\|_\infty}{\log \log N}
\]

(5.65)

We have to work on the term \(\|f_{1,s}^N(t) - f_{1,s}^\alpha(t)\|_\infty\). First it results that

\[
\|f_{1,s}^N(t) - f_{1,s}^\alpha(t)\|_\infty \leq \sum_{l=1}^{5} \|f_{1,s}^{N,E_{\ell}} - f_{1,s}^{\alpha,E_{\ell}}\|_\infty + \sum_{l=0}^{5} \|f_{1,s}^{\alpha,E_{\ell}}(t) - f_{1,s}^{\alpha,E_{\ell}}(t)\|_\infty
\]

\[
\|f_{1,s}^{N,E_{\ell}}(t) - f_{1,s}^{\alpha,E_{\ell}}(t)\|_\infty
\]

(5.66)

where \(f_{1,s}^{N,E_{\ell}}(t) = f_{1,s}^N(t)\) and \(f_{1,s}^{\alpha,E_{\ell}}(t) = f_{1,s}^\alpha(t)\). We focus on the last term, it results that

\[
|f_{1,s}^{N,E_{\ell}}(t) - f_{1,s}^{\alpha,E_{\ell}}(t)| \leq \sum_{j_1=0}^{1} \sum_{j_s=0}^{2^s-1} (C\alpha)^{P_s-1} \sum_{\sigma P_s-1} \sum_{\Gamma(P_s-1)} \int d\Lambda_{P_s-1} |\chi_{\lambda,E,\delta,q} f_{0,P_s}^N(\zeta(0)) - \chi_{\lambda,E,\delta,q} f_{0,P_s}(\zeta(0))|
\]

(5.67)

Now we split the integrals by using the indicator functions \(1 - \chi \{N^P(\epsilon_0)\}\) and \(\chi \{N^P(\epsilon_0)\}\). In the first case since we are outside the set \(N^P(\epsilon_0)\) the particles must be at a distance greater than \(\epsilon_0\), this implies that \(M_{N,\beta}(z_N) = C_{N,\beta}e^{-\frac{2\epsilon_0}{\alpha\epsilon_0^2}}\) and that \(\chi_{\lambda,E,\delta,q} = \chi_{\lambda,E,\delta,q}\). Then we have that

\[
\int d\Lambda_{P_s-1} (1 - \chi \{N^P(\epsilon_0)\}) \chi_{\lambda,E,\delta,q} |f_{0,P_s}^N(\zeta(0)) - f_{0,P_s}(\zeta(0))| \leq
\]

\[
\int d\Lambda_{P_s-1} (1 - \chi \{N^P(\epsilon_0)\}) \chi_{\lambda,E,\delta,q} \left[|f_{0,P_s}^N(\zeta(0)) - f_{0,P_s}(\zeta(0))| + |f_{0,P_s}(\zeta(0)) - f_{0,P_s}(\zeta(0))|\right]
\]

(5.68)

From the definition of the initial datum it turns out that

\[
|f_{0,P_s}^N(\zeta(0)) - f_{0,P_s}(\zeta(0))| \leq \|g_0\|_\infty |C_{P_s,\beta} - C_{P_s,\beta}^N|
\]

(5.69)
A straightforward calculation from the definition (3.2) and (3.3) gives us that

$$|C_{P_\alpha,\beta} - C_{P_\beta}^\epsilon| \leq 2^{2(s+1)}\epsilon^3$$  \hspace{1cm} (5.70)

Moreover outside the set $N^{P_\epsilon}(\epsilon_0)$ the velocities of the BBF and of the IBF are the same and also $p_1^\epsilon(s) = p_1(s)$ $0 \leq s \leq t$, it follows that

$$|f_{0,P_\epsilon}(\zeta^\epsilon(0)) - f_{0,P_\epsilon}(\zeta(0))| = |C_{N,\beta}e^{-\frac{\beta}{2}(|\zeta^\epsilon(0)|^2)}g_0(p_1^\epsilon(0), \xi^\epsilon_1(0)) - C_{N,\beta}e^{-\frac{\beta}{2}(|\zeta(0)|^2)}g_0(r_1^\epsilon(0), \xi^\epsilon_1(0))| = 0$$  \hspace{1cm} (5.71)

Finally we have that

$$\sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (C_\alpha)^{P_\epsilon-1} \sum_{\sigma_{P_\epsilon-1}} \sum_{\Gamma(P_\epsilon-1)} \int d\Lambda_{P_\epsilon-1} \left( 1 - \chi \{ N^{P_\epsilon}(\epsilon_0) \} \right) \chi_{\lambda,E,\delta,q} f_{0,P_\epsilon}^{N}(\zeta^\epsilon(0)) - f_{0,P_\epsilon}(\zeta(0)) \leq (C_\alpha)^{s+1} 2^{2s+1}\epsilon^{3-\lambda}$$  \hspace{1cm} (5.72)

In the second case we use the estimates of Lemma 5.6 to obtain that

$$\sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} (C_\alpha)^{P_\epsilon-1} \sum_{\sigma_{P_\epsilon-1}} \sum_{\Gamma(P_\epsilon-1)} \int d\Lambda_{P_\epsilon-1} \chi \{ N^{P_\epsilon}(\epsilon_0) \} | \chi_{\lambda,E,\delta,q} f_{0,P_\epsilon}^{N}(\zeta^\epsilon(0)) - \chi_{\lambda,E,\delta,q} f_{0,P_\epsilon}^{N}(\zeta(0))| \leq \epsilon^\lambda \|g_0\|_\infty \left( C_\alpha \right)^{2^{s+1}} E^{82(s+4)(s+1)} \left( \frac{\epsilon^\lambda}{\epsilon_0^0} + \frac{\epsilon^\lambda \delta}{\epsilon_0^0} + \frac{\epsilon^\lambda \delta^2}{\epsilon_0^0} \right) \leq \|g_0\|_\infty \epsilon^\frac{1}{2^s} (C \log \log N)^{\sqrt{\log N}} (\log N)^{4 \log \log N}$$  \hspace{1cm} (5.73)

We have proved that

$$\|f_{1,s,E_{\text{err}}}(t) - f_{1,s,E_{\text{err}}}(t)\|_\infty \leq \|g_0\|_\infty \epsilon^\frac{1}{2^s} (C \log \log N)^{\sqrt{\log N}} (\log N)^{4 \log \log N}$$  \hspace{1cm} (5.74)

The remainders can be easily handled with the estimates proved in Lemmas 5.1-5.5. It follows that

$$\sum_{l=1}^{5} \|f_{1,s,E_{\text{err}}-l}(t) - f_{1,s,E_{\text{err}}-l}(t)\|_\infty + \sum_{l=0}^{5} \|f_{1,s,E_{\text{err}}-l}(t) - f_{1,s,E_{\text{err}}}(t)\|_\infty \leq \|g_0\|_\infty (C_\alpha)^{2^{s+1}} 2^{(s+2)(s+1)} \left( \epsilon^\frac{\lambda}{2^s} + \epsilon^\frac{\lambda \delta}{2^s} + \epsilon^\frac{\lambda \delta^2}{2^s} \right) \leq \|g_0\|_\infty (C \log \log N)^{\sqrt{\log N}} (\log N)^{\frac{\log \log N}{\log 2}} \left( \epsilon^\frac{1}{2^s} + \frac{1}{N} \right)$$  \hspace{1cm} (5.75)
Summarizing, we have that

\[ \| \tilde{f}_1^N(t) - f_1^N(t) \|_\infty \leq \| \tilde{f}_1^N(t) - f_1^N(t) \|_\infty + \| f_1^N(t) - f_1^{\alpha}(t) \|_\infty + \| \tilde{R}_N(t) \|_\infty + \| R(t) \|_\infty \leq \]

\[ \frac{C \| g_0 \|_\infty}{\log \log N} + \| g_0 \|_\infty (C \log \log N)^{\sqrt{\log N}} (\log N)^{\left(\frac{\log \log N}{\log N}\right)^3} \left( \epsilon^{\frac{1}{20}} + \frac{1}{N} \right) \]

If we send \( N \to \infty, \epsilon \to 0 \) with \( N \epsilon^2 \cong C (\log \log N) \frac{1}{2} \) we obtain the proof of the theorem.
6 From Linear Boltzmann to Linear Landau

6.1 Existence of semigroups

In this section we want to prove that the solution of the Linear Boltzmann equation converges as $\alpha \to \infty$ to the solution of the Linear Landau equation. For this purpose we rewrite in the following way the linear Boltzmann and Landau equations

\[
\begin{align*}
\partial_t f &= G_\alpha(f) \\
f(x,v,0) &= f_0(x,v)
\end{align*}
\] (6.1)

\[
\begin{align*}
\partial_t f &= G(f) \\
f(x,v,0) &= f_0(x,v)
\end{align*}
\] (6.2)

where

\[
G_\alpha(f) = Q_B(f) - v \cdot \nabla_x f
\] (6.3)

and

\[
G(f) = Q_L(f) - v \cdot \nabla_x f.
\] (6.4)

Now we want to set the problem in the Hilbert space $H = L^2(\Gamma \times \mathbb{R}^3, dxd\mu)$ where $d\mu = M_\beta(v)dv$. This space arises naturally from the definition of the operators $G$ and $G_\alpha$. Indeed, we have that $G_\alpha$ and $G$ are unbounded linear operators densely defined respectively on $D(G_\alpha) = H^1(\Gamma, dx) \times L^2(\mathbb{R}^3, d\mu)$ and $D(G) = H^1(\Gamma, dx) \times H^2(\mathbb{R}^3, d\mu)$, where $H^1$ and $H^2$ denote the usual Sobolev spaces.

The main motivation to introduce $H$ is the following lemma:

**Lemma 6.1.** The operators $Q_B(f)$ and $Q_L(f)$ are well defined as self-adjoint operators on $L^2(\mathbb{R}^3, d\mu)$ and $H^2(\mathbb{R}^3, d\mu)$ respectively. Moreover for the operators $G$ and $G_\alpha$, defined in (6.3) and (6.4), we have that $\forall f \in H$ and $\forall g \in D(G)$

\[
(G_\alpha^* f, f) = (f, G_\alpha f) \leq 0
\] (6.5)

and

\[
(G^* g, g) = (g, Gg) \leq 0
\] (6.6)

i.e. $G_\alpha$ and $G$ are dissipative operators. Furthermore $G_\alpha$ and $G$ are closed operators.
We give the proof of this lemma in the appendix I.

Thanks to these properties of the operators we can use the following theorem.

**Theorem 6.2.** ([7]) Let $A$ be a linear operator densely defined on a linear subspace $D(A)$ of the Hilbert space $H$. If both $A$ and $A^*$ are dissipative operators then $A$ generate a contraction semigroup on $H$.

This theorem ensures the existence of $T_\alpha(t)$ and $T(t)$, the semigroups with infinitesimal generator given by $G_\alpha$ and $G$ respectively. Indeed, from Lemma 6.1 we have that $G_\alpha$ and $G$ are closed operators and that $G_\alpha^*$ and $G^*$ are dissipative operators, then we have the existence of $T_\alpha(t)$ and $T(t)$.

### 6.2 Convergence of the semigroups

The last step of our proof is to show that the semigroup generated by $G_\alpha(f)$ strongly converges to the semigroup generated by $G(f)$ in the limit $\alpha \to 0$. We use the following theorem, that gives necessary and sufficient conditions for the convergence.

**Theorem 6.3.** (Trotter-Kato). Let $A$ and $A_n$ be the generators of the contraction semigroups $T(t)$ and $T_n(t)$ respectively. Let $D$ be a core for $A$. Suppose that $D \subseteq D(A_n) \forall n$ and that $\forall f \in D A_n f \to A f$. Then

$$\|T_n f - T f\|_H \to 0 \text{ as } n \to +\infty$$

\(\forall f \in H\) and uniformly for $t \in [0,T]$ for any $T > 0$.

A proof of this theorem can be found in [7].

We want to apply this theorem to prove that $T_\alpha f \to Tf$. We note that $D = C_{p,0}^\infty(\Gamma) \times C_{0}^\infty(\mathbb{R}^3)$ is a core for $G_\alpha$ and $G$ as follows by a direct inspection. Then we use the steps of section 3.2 to prove the strong convergence of the operators on this set.

**Theorem 6.4.** Let $G_\alpha$ and $G$ be defined as in (6.3) and (6.4). Then $\forall f \in D$ it results that

$$\|(G_\alpha - G) f\|_H \to 0 \text{ as } \alpha \to \infty$$

**Proof.** First we define the following operator

$$Q_\beta^\alpha(f) = \alpha \int dv_1 M_\beta(v_1) \int_{\nu \cdot V > 0} d\nu |\nu \cdot V| x \left\{ |V| \geq \alpha^{-\frac{1}{2}} \right\} \left[ f(v') - f(v) \right]$$

\(6.8\)
This is the Linear Boltzmann operator with a $\alpha$-depending cutoff on the small relative velocities. Observe that $Q^c_B$ and $Q_B$ are asymptotically equivalent as $\alpha \to \infty$. Indeed, we have that $\forall f \in D_0$

\[
\| (Q^c_B - Q_B) f \|_{H}^2 = \int dx \int d\mu(v) \left| \alpha \int d\mu(v_1) \int_{\nu \cdot V > 0} \chi \{ |V| < \alpha^{-\frac{4}{15}} \} |\nu \cdot V| [f(x, v') - f(x, v)] \, d\nu \right|^2 \\
\leq C \| f \|_\infty^2 \int dx d\mu(v) \left| \int d\nu_1 A \frac{\mu(v_1)}{|V|^3} \chi \{ |V| < \alpha^{-\frac{4}{15}} \} \right|^2 \\
\leq C \| f \|_\infty^2 \int dx d\mu(v) \left( A \frac{\mu(v_1)}{|V|^3} \right) ^2 (6.9)
\]

Since

\[
\int \chi \{ |V| < \alpha^{-\frac{4}{15}} \} \, d\nu_1 \leq C \alpha^{-\frac{4}{15}}, (6.10)
\]

we arrive to

\[
\| (Q^c_B - Q_B) f \|_{H}^2 \leq C \alpha^{-\frac{4}{15}}. (6.11)
\]

We put the same cutoff on the operator $Q_L$ and we define

\[
Q^c_L = A \int d\nu_1 M_\beta(v_1) \frac{1}{|V|^3} \left[ |V|^2 \Delta f(v) - (V, D^2 V) - 4V \cdot \nabla_v f(v) \right] \chi \{ |V| \geq \alpha^{-\frac{4}{15}} \} (6.12)
\]

Then $\forall f \in D$ we have

\[
\| (Q^c_L - Q_L) f \|_{H}^2 = \int dx d\mu(v) \left| \int d\nu_1 A \frac{\mu(v_1)}{|V|^3} \chi \{ |V| \geq \alpha^{-\frac{4}{15}} \} \right|^2 \\
\leq C(A, f) \alpha^{-\frac{11}{15}} (6.13)
\]

Now we want to prove that $Q^c_B$ converges strongly to $Q^c_L$ when $\alpha \to +\infty$. We have that for all $f \in D$

\[
\| (Q^c_B - Q^c_L) f \|_{H}^2 = \int dx d\mu(v) \\
\left| \int d\mu(v_1) \left\{ \alpha \int_{\nu \cdot V > 0} \chi \{ |V| \geq \alpha^{-\frac{4}{15}} \} |\nu \cdot V| [f(x, v') - f(x, v)] \, d\nu \right. \right|^2 \\
A \frac{1}{|V|^3} \left[ |V|^2 \Delta_v f(x, v) - (V, D_v^2 f(x, v))V - 4V \cdot \nabla_v f(x, v) \right] \chi \{ |V| \geq \alpha^{-\frac{4}{15}} \} (6.14)
\]
We perform the same steps of section 3 to obtain:

\[
\| (Q^c_B - Q^c_L) f \|_{H}^2 \leq C \int dx \int d\mu(v) \int d\mu(v_1) \int_{\nu \cdot V > 0} \alpha \chi \left\{ |V| \geq \alpha^{-\frac{3}{2}} \right\} |\nu \cdot V| o(\alpha^{-1})^2 \tag{6.15}
\]

For the second term we have to go further in the Taylor expansion and use the Lagrange form for the remainder term. From Lemma (2.2) it results that

\[
o(\alpha^{-1}) = \frac{M^2(\rho, \alpha)}{|V|^8 \alpha^2} + \frac{\theta^3}{3!} f'''(\xi) \tag{6.16}
\]

for a certain \(\xi \in [0, \theta]\). Therefore

\[
\int dx \int d\mu(v) \int d\mu(v_1) \int_{\nu \cdot V > 0} \alpha \chi \left\{ |V| \geq \alpha^{-\frac{3}{2}} \right\} |\nu \cdot V| o(\alpha^{-1})^2 \leq \int dx \int d\mu(v) \int d\mu(v_1) \int_{\nu \cdot V > 0} \alpha \chi \left\{ |V| \geq \alpha^{-\frac{3}{2}} \right\} |\nu \cdot V| \left[ \frac{M^2(\rho, \alpha)}{|V|^8 \alpha^2} + \frac{\theta^3}{3!} f'''(\xi) \right]^2 \tag{6.17}
\]

Thanks to formula (2.12) we have that

\[
|\theta^3(\rho, \alpha)| \leq C \left( \alpha^{-\frac{3}{2}} \frac{\gamma^3(\rho)}{|V|^6} + \alpha^{-3} \frac{M^3(\rho, \alpha)}{|V|^{12}} \right) \tag{6.18}
\]

Furthermore from (3.18) it follows that

\[
|f'''(\xi)| \leq C(f)|V| \tag{6.19}
\]

and then we can write

\[
\| (Q^c_B - Q^c_L) f \|_{H}^2 \leq C(f, \gamma, M) \int dx \int d\mu(v) \left[ \int d\mu(v_1) \left( \frac{\alpha^{-1}}{|V|^6} + \alpha^{-\frac{3}{2}} \frac{\gamma^3(\rho)}{|V|^6} + \alpha^{-2} \frac{M^3(\rho, \alpha)}{|V|^{12}} \chi \left\{ |V| \geq \alpha^{-\frac{3}{2}} \right\} \right)^2 \right] \tag{6.20}
\]
A change of variables on the right hand side of (6.20) gives us
\[
\int d\mu(v_1) \left( \frac{\alpha^{-1}}{|V|^6} + \frac{\alpha^{-\frac{1}{2}}}{|V|^4} + \frac{\alpha^{-2}}{|V|^2} \right) \chi \left\{ |V| \geq \alpha^{-\frac{1}{2}} \right\} \leq C \int_{\alpha^{-\frac{1}{6}}}^{\infty} dr \left( \frac{\alpha^{-1}}{r^4} + \frac{\alpha^{-\frac{1}{2}}}{r^2} + \frac{\alpha^{-2}}{r^8} \right) \leq C \alpha^{\frac{1}{2}}
\]
(6.21)

From formula (6.20) and (6.21) we have
\[
\| (Q_B - Q_L) f \|^2_{H^2} \leq C_2 (\gamma M) \alpha^{-\frac{1}{16}}
\]
(6.22)

Then we have
\[
\| (Q_B - Q_L) f \|^2_{H^2} \leq C \alpha^{-\frac{1}{16}}
\]
(6.23)

and this proves our theorem.

Finally we use Theorem 6.3 and Theorem 6.4 to prove that the solution of the linear Boltzmann equation converge to the solution of the linear Landau equation.

**Theorem 6.5.** Let \( g^\alpha(x,v,t) \) be the solution of the linear Boltzmann equation and let \( g(x,v,t) \) be the solution of the linear Landau equation. Suppose that the initial datum of both equations is given by \( g_0(x,v) \). Then it results that
\[
\| g^\alpha(x,v) - g(x,v) \|_H \to 0
\]
(6.24)

when \( \alpha \to 0 \).

**Proof.** Since \( g^\alpha(t) = T_\alpha(t) g_0(x,v) \) and \( g(t) = T(t) g_0(x,v) \) we have that
\[
\| g^\alpha(x,v) - g(x,v) \|_H = \| T_\alpha(t) g_0(x,v) - T(t) g_0(x,v) \|_H
\]
(6.25)

From Theorem 6.3 and Theorem 6.4 we have that the right hand side of (6.25) goes to zero when \( \alpha \to 0 \) and the theorem is proved.
7 Proof of the main theorem

In this section we summarize all the estimates obtained and we finally give a proof that the solution of the first equation of the Grad hierarchy converge to the solution of the linear Landau equation in the scaling $N \epsilon^2 \to \alpha$ with $\alpha \cong C (\log \log N)^{\frac{1}{2}}$.

**Theorem 7.1.** Let $\tilde{f}_1^N(t)$ be the first-particle marginal of the solution of the Liouville equation with initial datum given by $W_{0,N}(z_N) = M_{N,\beta}(z_N) g_0(x_1, v_1)$, and let $g(t)$ be the solution of the linear Landau equation with initial datum given by $g(x, v, 0) = g_0(x, v)$. Then $\forall t > 0$

$$\|\tilde{f}_1^N(x, v, t) - M_{\beta}(v) g(x, v, t)\|_H \to 0$$

when $N \to \infty$, with $N \epsilon^2 \cong (\log \log N)^{\frac{1}{2}}$.

**Proof.** First we want to estimate the following difference

$$\|\tilde{f}_1^N(t) - f_1^\alpha(t)\|_H \quad (7.1)$$

Since $\int dx \int d\mu(v) |\tilde{f}_1^N(t) - f_1^\alpha(t)|^2 \leq C \|\tilde{f}_1^N(t) - f_1^\alpha(t)\|_\infty^2$ it results that

$$\|\tilde{f}_1^N(t) - f_1^\alpha(t)\|_H \leq C \|\tilde{f}_1^N(t) - f_1^\alpha(t)\|_\infty \quad (7.2)$$

From Theorem 5.7 we have that

$$\|\tilde{f}_1^N(t) - f_1^\alpha(t)\|_\infty \to 0 \quad (7.3)$$

and then

$$\|\tilde{f}_1^N(t) - f_1^\alpha(t)\|_H \to 0 \quad (7.4)$$

As we have seen the solution of the first equation of the BH with initial data given by (3.10) has the form

$$f_1^\alpha(x, v, t) = M_{\beta}(v) g^\alpha(x, v, t) \quad (7.5)$$

where $g^\alpha(x, v)$ is the solution of the Linear Boltzmann equation. Then we have proved that

$$\|\tilde{f}_1^N(t) - M_{\beta}(v) g^\alpha(x, v, t)\|_H \to 0 \quad (7.6)$$
Since
\[ ||M_\beta(v)g^\alpha(x,v,t) - M_\beta(v)g(x,v,t)||_H \leq C||g^\alpha(x,v) - g(x,v,t)||_H \] (7.7)

Thanks to Theorem (6.4) we have that
\[ ||g^\alpha(x,v,t) - g(x,v,t)||_H \to 0 \] (7.8)

From (7.7) and (7.4) we arrive to
\[ ||\tilde{f}_N^N(x,v,t) - M_\beta(v)g(x,v,t)||_H \leq ||\tilde{f}_N^N(t) - f_1^N(t)||_H + C||g^\alpha(x,v,t) - g(x,v,t)||_H \] (7.9)

Finally we estimate the difference between the reduced marginal and the standard marginal. We have
\[
|\tilde{f}_N^N(x,v,t) - \tilde{f}_1^N(x,v,t)| = |\int dz_{j+1}...dz_N W_N(z_N,t) \left( 1 - \chi \{ S(x_1)^{N-1} \} \right) |
\leq N | \int dx_1 dv_1 \tilde{f}_2^N(x,v,x_1,v_1,t) |
\leq CN\epsilon^3
\] (7.10)

Then it results
\[ ||\tilde{f}_N^N(x,v,t) - \tilde{f}_1^N(x,v,t)||_\infty \] (7.11)

We send \( \alpha \to \infty, N \to \infty \) with \( N\epsilon^2 \equiv \alpha \equiv (\log \log N)^{1/2} \) and we obtain the proof of the theorem.

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8 Appendix I, Proof of Lemma 6.1

Here we gives a proof of the Lemma 6.1.

Proof. First we want to prove that the operator $Q_L$ is self-adjoint. It results that

$$(f, Q_L(g))_{L^2(d\mu)} = \int dv \int dw M_\beta(v) M_\beta(w) \frac{1}{|V|^3} \left[ |V|^2 \Delta g - (V, D_v^2(g)V) - 4V \cdot \nabla_v g \right] f(v) \quad (8.1)$$

We integrate by parts the first term. We have

$$\int dv \int dw M_\beta(v) M_\beta(w) \frac{1}{|V|^3} f \Delta g =$$

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[ -\nabla f \cdot \nabla g \frac{1}{|V|} + 2\beta v \cdot \nabla g \frac{1}{|V|} f + \frac{V \cdot \nabla g}{|V|^3} f \right] \quad (8.2)$$

For the second term it results that

$$-\int dv \int dw M_\beta(v) M_\beta(w) \frac{1}{|V|^3} (V, H_v(g)V) =$$

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[ \frac{V \cdot \nabla g}{|V|^3} f + \frac{(V \cdot \nabla g) (V \cdot \nabla f)}{|V|^3} - 2\beta (v \cdot V) \frac{V \cdot \nabla g}{|V|^3} f \right] \quad (8.3)$$

We put together these two terms with the last one, this gives us

$$(f, Q_L(g))_{L^2(d\mu)} = \int dv \int dw M_\beta(v) M_\beta(w) \left[ -\nabla f \cdot \nabla g \frac{1}{|V|} + \frac{(V \cdot \nabla g) (V \cdot \nabla f)}{|V|^3} - 2\beta (v \cdot V) \frac{V \cdot \nabla g}{|V|^3} f \right] +$$

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[ 2\beta \frac{v \cdot \nabla g}{|V|} f - 2\beta (v \cdot V) \frac{V \cdot \nabla g}{|V|^3} f \right] \quad (8.4)$$

Now we observe that $v = w + V$ and that $2\beta w M_\beta(w) = -\nabla w M_\beta(w)$, we also integrate by parts with respect to the variable $w$ in the second terms of (7.4) and we arrive to

$$\int dv \int dw M_\beta(v) M_\beta(w) \left[ 2\beta \frac{v \cdot \nabla g}{|V|} f - 2\beta (v \cdot V) \frac{V \cdot \nabla g}{|V|^3} f \right] =$$

$$\int dv \int dw M_\beta(v) \nabla w M_\beta(w) \left[ -\frac{\nabla g}{|V|} f + \frac{V (V \cdot \nabla g)}{|V|^3} f \right] =$$

$$\int dv \int dw M_\beta(v) M_\beta(w) 2 \frac{V \cdot \nabla g}{|V|^3} f \quad (8.5)$$

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This yields

\[(f, Q_L (g))_{L^2(d\mu)} = \int dv \int dw M_\beta (v) M_\beta (w) \left[ -\frac{\nabla f \cdot \nabla g}{|V|} + \frac{(V \cdot \nabla g) (V \cdot \nabla f)}{|V|^3} \right] \quad (8.6)\]

Another integration by parts leads to

\[(f, Q_L (g))_{L^2(d\mu)} = \int dv \int dw M_\beta (v) M_\beta (w) \left[ g \frac{\Delta f}{|V|} - 2\beta \frac{(v \cdot \nabla f) g}{|V|} - \frac{(V \cdot \nabla f) g}{|V|^3} - \frac{(V, D^2(f)V) g}{|V|^3} + 2\beta \frac{(v \cdot V) (V \cdot \nabla f) g}{|V|^3} \right] = \]

\[\int dv \int dw M_\beta (v) M_\beta (w) \left[ g \frac{\Delta f}{|V|} - \frac{(V, D^2(f)V) g}{|V|^3} - 2\frac{(v \cdot \nabla f) g}{|V|^3} \right] + \]

\[\int dv \int dw M_\beta (v) M_\beta (w) \left[ -2\beta \frac{(v \cdot \nabla f) g}{|V|} + 2\beta \frac{(v \cdot V) (V \cdot \nabla f) g}{|V|^3} \right] \quad (8.7)\]

We integrate by parts the last term with respect to \(w\), it gives us

\[\int dv \int dw M_\beta (v) M_\beta (w) \left[ -2\beta \frac{(v \cdot \nabla f) g}{|V|} + 2\beta \frac{(v \cdot V) (V \cdot \nabla f) g}{|V|^3} \right] = \]

\[\int dv \int dw M_\beta (v) M_\beta (w) - 2\frac{V \cdot \nabla f}{|V|^3} g \quad (8.8)\]

We use together (8.8) and (8.7) and we finally arrive to

\[(f, Q_L (g))_{L^2(d\mu)} = \int dv \int dw M_\beta (v) M_\beta (w) \left[ g \frac{\Delta f}{|V|} - \frac{(V, D^2(f)V) g}{|V|^3} - 4\frac{(V \cdot \nabla f) g}{|V|^3} \right] = \]

\[(Q_L (f), g)_{L^2(d\mu)} \quad (8.9)\]

Obviously \(D(Q_L) = D(Q_L^*)\) and so \(Q_L\) is self-adjoint.

We now prove that the linear Boltzmann operator \(Q_B\) is self-adjoint. We have

\[(f, Q_B (g))_{L^2(d\mu)} = \int dv \int dw \int dv' M_\beta (v) M_\beta (w) |\nu \cdot V| f(v) \left[ g(v') - g(v) \right] = \]

\[\int dv \int dw \int dv' M_\beta (v) M_\beta (w) |\nu \cdot V| f(v) g(v') - \int dv \int dw \int dv' M_\beta (v) M_\beta (w) |\nu \cdot V| f(v) g(v) \quad (8.10)\]

In the first term of the sum we change variables in the integration by using the map defined in formula.
This gives us
\[
\int dv \int dw \int d\nu M_\beta(v)M_\beta(w)|\nu \cdot V|f(v)g(v') = \int dv' \int dw' \int d\nu M_\beta(v)M_\beta(w)|\nu \cdot V|f(v')g(v)
\]
(8.11)

Now we use Lemma 2.1 that gives us that
\[
dv' dw' d\nu' = dvdwd\nu,
\]
this with (8.10) leads to
\[
(f, Q_B(g))_{L^2(d\mu)} = \int dv \int dw \int d\nu M_\beta(v)M_\beta(w)|\nu \cdot V|g(v) \left[ f(v') - f(v) \right] = (Q_B(f), g)_{L^2(d\mu)}
\]
(8.12)

Formula (6.5) and (6.6) can be proved simply with some integration by parts in the definition of the operators $Q_B$ and $Q_L$. This leads to
\[
(f, Q_B f) \leq 0
\]
(8.13)
\[
(f, Q_L f) \leq 0
\]
(8.14)
\[
(f, Q_B(f))_{L^2(d\mu)} = \int dv \int dw \int d\nu M_\beta(v)M_\beta(w)|\nu \cdot V| \left[ f(v)f(v') - f^2(v) \right]
\]
(8.15)

Another change of variables in the integration gives us
\[
(f, Q_B(f))_{L^2(d\mu)} = \int dv \int dw \int d\nu M_\beta(v)M_\beta(w)|\nu \cdot V| \left[ f(v)f(v') - f^2(v') \right]
\]
(8.16)

We sum together these two equality, this leads to
\[
2(f, Q_B(f))_{L^2(d\mu)} \leq -\int dv \int dw \int d\nu M_\beta(v)M_\beta(w)|\nu \cdot V| \left[ f(v') - f(v) \right]^2 \leq 0
\]
(8.17)

From formula (8.6) we have
\[
(f, Q_L(f))_{L^2(d\mu)} = \int dv \int dw M_\beta(v)M_\beta(w) \left[ \left( \nabla f \cdot \nabla f \right)^2 - |\nabla f|^2 \right]
\]
(8.18)
and, since $\left( \nabla \cdot \nabla f \right)^2 - |\nabla f|^2 \leq 0$, it results that
\[
(f, Q_L(f))_{L^2(d\mu)} \leq 0
\]
(8.19)

Now we observe that
\[
(f, -v \cdot \nabla_x f) = (v \cdot \nabla_x f, f) = (f, v \cdot \nabla_x f) = 0
\]
(8.20)
and we arrive to

\[(f, G_\alpha f) = (G_\alpha^* f, f) = (L f, f) + ((v \cdot \nabla_x f, f)) = (L f, f) \leq 0\]  \hspace{1cm} (8.21)

With similar steps it is possible to prove the (6.5).

Since \(D(G)\) and \(D(G_\alpha)\) are dense in \(H\) by the Von Neumann Theorem we have that \(G_\alpha^{**} = G_\alpha\) (8.22)

but \(G_\alpha^{**} = G_\alpha\) and so \(G_\alpha\) is closed. This can be proved in the same way for \(G\).
9 Appendix II, estimate of the recollision set

Lemma 9.1. Let $\chi_{\lambda,E,\delta,q}$ be defined as in (5.46) and let $\chi \{ N_{P_s}(\epsilon_0) \}$ be the characteristic function of the set (5.54). Then it results that

$$
\sum_{j=0}^{2s-1} \sum_{j=0}^{2s-1} (\alpha)^{P_s-1} \sum_{\Gamma(P_s-1)} \sigma_{P_s-1} \int d\Lambda P_s-1 \chi_{\lambda,E,\delta,q} \left\{ N_{P_s}(\epsilon_0) \right\} \prod_{k=1}^{P_s-1} B f_0, P_s(\zeta(0)) \leq \left\| g_0 \right\|_\infty (C_{\alpha})^{2s+1} 2^{(s+4)(s+1)} \left( \epsilon_0^{2s-\lambda} + \frac{\epsilon_0^{2s-\lambda}}{\gamma} + \epsilon_0^{\frac{s}{2} - \lambda} \right)
$$

(9.1)

Proof. First we observe that

$$
N_{P_s}(\epsilon_0) = \bigcup_{i=1}^{k} \bigcup_{k=2}^{P_s} N_{i,k}(\epsilon_0)
$$

(9.2)

where

$$
N_{i,k}(\epsilon_0) = \left\{ (t_{P_s-1}, \nu_{P_s-1}, w_{P_s-1}) \in \mathbb{R}^{P_s-1} \times \mathbb{R}^{3(P_s-1)} \left| \min_{i<k} \min_{\tau \in [0,t_{i-1}]} d(r_i(\tau), r_k(\tau)) < \epsilon_0 \right\} \right.
$$

(9.3)

We also define a subsequence $t^q$ of the times $t_1, \ldots, t_n$ associated to the virtual trajectory of particles $i$ and $k$. We put $t^0$ as the time in which the two virtual trajectory merge, then we consider the ordered union of the times of creations in the virtual trajectory of particles $i$ and $k$ (Figure 9.1).

For a point in $N_{i,k}(\epsilon_0)$ there exist

$$
\tau^* = \max \left\{ \tau \in [0,t_{i-1}] \left| d(p_i(\tau), p_k(\tau)) < \epsilon_0 \right\} \right.
$$

(9.4)

It must be $\tau^* \in [t^l, t^{l+1})$ for some $l \geq 0$. With this definition $l$ represents the total number of creation after the time $t^0$ in the virtual trajectory of the particles $i$ and $k$. For $q \in [0,l]$ we define

$$
Y^q = r^k(t^q) - r^i(t^q)
$$

(9.5)

$$
\xi^q_i = \xi_i(\tau) \quad \xi^q_k = \xi_k(\tau) \quad \text{for} \ \tau \in (t^{q+1}, t^q)
$$

(9.6)

$$
W^q = \xi^q_k - \xi^q_i
$$

(9.7)

Observe that, since we are considering only one tree, it will be always $Y^0 = 0$.

First suppose that $l = 0$, this means that the particles $i$ and $k$ have a recollision after the creation.
of the particle $k$. This can happen in two cases. In the first case the particles $i$ and $h$ do not separate enough after the creation. In the second case the particles, after being separated enough, perform a recollision since the trajectory on the torus have no dispersive properties.

In the first case it must be

$$|W^0|(t^1 - t^0) \leq \epsilon_0$$

(9.8)

We recall that the cutoff (5.46) implies that $(t^1 - t^0) \geq \delta$ and that $|W^0| \geq \epsilon^q$. Then the particles must be separated at least by a distance of $\delta \epsilon^q$. We choose the parameters in such a way that

$$\epsilon_0 \leq \delta \epsilon^q$$

(9.9)

and this gives us that the (9.8) cannot happen.

In the second case we prove that $W^0$ must be in a set of small measure. There exist a $\tau > \delta$ such that
\[ d(r^i(t^0 - \tau), r^k(t^0 - \tau)) \leq \epsilon_0 \]  
(9.10)

We use the correspondence between the torus and the whole space with periodic structure. We have that
\[ (r^i(t^0) - \tau \xi^i(t^0)) - (r^k(t^0) - \tau \xi^k(t^0)) \in \bigcup_{p \in \mathbb{Z}^3} B_{\epsilon_0}(p) \]  
(9.11)

Thanks to the energy cutoff we have that
\[ |W^0| \leq 4E \]  
and so
\[ \tau W^0 \in \left( \bigcup_{p \in \mathbb{Z}^3} B_{\epsilon_0}(r^i(t^0) - r^k(t^0) + p) \right) \cap B_{4Et}(0) \]  
(9.12)

Suppose that \( |r^i(t^0) - r^k(t^0) + p| < \frac{1}{4} \). This can happen only for a value of \( p \) since the distance between the centers of the spheres is 1. Taking \( \hat{v} \) a unit vector normal to \( r^i(t^0) - r^k(t^0) + p \) it results that
\[ \tau |W^0 \cdot \hat{v}| \leq \epsilon_0 \]  
(9.13)

and then
\[ |W^0 \cdot \hat{v}| \leq \frac{\epsilon_0}{\delta} \]  
(9.14)

This implies that \( W^0 \) is in the intersection of \( B_{4E}(0) \) and a cylinder of radius \( \frac{\epsilon_0}{\delta} \) and so in a set of measure bounded by \( C E \frac{\epsilon_0^2}{\delta^2} \). Suppose now \( |r^i(t^0) - r^k(t^0) + p| \geq \frac{1}{4} \) and that \( \epsilon \) is small enough, then \( W^0 \) is in the intersection of \( B_{4E}(0) \) and some cone of vertex 0 and solid angle \( C \epsilon_0^2 \) and these cones are at most \( (8Et)^3 \). Finally putting together these two estimates gives us that \( W^0 \) must be in a suitable set \( B_{k,0} \) such that
\[ |B_{k,0}| \leq C \left( E \frac{\epsilon_0^2}{\delta^2} + (Et)^3 \epsilon_0^2 \right) \]  
(9.15)

We can now suppose that \( l \geq 1 \). The \( \epsilon_0 \)-overlap is verified only if
\[ Y^l - \tau W^l \in \bigcup_{p \in \mathbb{Z}^3} B_{\epsilon_0}(p) \]  
(9.16)

for some \( \tau \in [0, t^l) \). Moreover it results that
\[ Y^l = \tilde{p} - \sum_{q=0}^{l-1} W^q \left( t^q - t^{q+1} \right) = \tilde{p} - W^0 t^0 + \sum_{q=1}^{l} (W^{q-1} - W^q) t^q \]  
(9.17)
where \( \hat{p} \in \mathbb{Z}^3 \) is chosen in such a way that the right hand side of equation (9.17) is a point in the torus.

Now we prove that it must be
\[
\sum_{q=1}^{l} |W^q - W^{q-1}| > \epsilon_0^2
\]  
(9.18)

Otherwise it would be \(|W^q - W^0| \leq \epsilon_0^2\) for all \(q\), then using (9.17) and (9.16) it results that
\[
W^0(\tau + t^0 - t^l) \in \bigcup_{p \in \mathbb{Z}^3} B_{(\epsilon_0 + \epsilon_3^0 t^l)}(p)
\]  
(9.19)

Since \(\tau + t^0 - t^l \geq \delta\) we can perform the same steps of estimate (9.15) to prove that in this case \(W^0\) must be in a set \(B_{k,1}\) of measure bounded by
\[
C \left( E^4 \epsilon_0^4 + (Et)^3 \epsilon_0^4 \right)
\]  
(9.20)

Condition (9.16) implies that
\[
| (Y^l + \hat{p}) \wedge \hat{W}| \leq \epsilon_0 
\]  
(9.21)

with \(\hat{W} = \frac{W^l}{|W^l|}\). Then from (9.17) we have that
\[
| (\hat{p} - W^0 t^0) \wedge \hat{W} - \sum_{q=1}^{l} [(W^q - W^{q-1}) \wedge \hat{W}] t^q | \leq \epsilon_0
\]  
(9.22)

Now suppose that
\[
\sum_{q=1}^{l} |(W^q - W^{q-1}) \wedge \hat{W}| \leq \epsilon_0^2
\]  
(9.23)

from (9.18) it must exist a \(\bar{q} \in \{1, \ldots, l\}\) such that
\[
U = U^{\bar{q}} = W^{\bar{q}} - W^{\bar{q}-1}
\]  
(9.24)

has modulus
\[
|U| > \frac{\epsilon_0^2 \bar{q}}{l}
\]  
(9.25)

Moreover from (9.21) it results that
\[
|U \wedge \hat{W}| \leq \epsilon_0^3
\]  
(9.26)
We set $\hat{U} = \frac{U}{|U|}$, this gives us

$$|\hat{U} \wedge \hat{W}| \leq (P_s - 1)\epsilon_0^{\frac{1}{5}}$$  \hspace{1cm} (9.27)

Thanks to cutoff (5.46) it results that $|W^0| > \epsilon^q$, that with (9.23) gives us that

$$|\hat{W}^0 \wedge \hat{W}| \leq \epsilon_0^{\frac{2}{5} - q}$$  \hspace{1cm} (9.28)

This with (9.27), assuming $q = \frac{1}{8}$, finally gives

$$|\hat{W}^0 \wedge \hat{U}| \leq C\epsilon_0^{\frac{1}{5}}(P_s - 1)$$  \hspace{1cm} (9.29)

We have two cases, if $\sum_{q=1}^{l} |(W^q - W^{q-1}) \wedge \hat{W}| \leq \epsilon_0^{\frac{2}{5}}$ then it results that $|\hat{W}^0 \wedge \hat{U}| \leq C\epsilon_0^{\frac{4}{5}}(P_s - 1)$.

Otherwise we have that $\sum_{q=1}^{l} |(W^q - W^{q-1}) \wedge \hat{W}| > \epsilon_0^{\frac{2}{5}}$. This implies that for some $q^*$ we have

$$|(W^{q^*} - W^{q^*-1}) \wedge \hat{W}| > \frac{\epsilon_0^{\frac{2}{5}}}{l}$$  \hspace{1cm} (9.30)

From (9.17) it follows that

$$|\left(\hat{p} - W^0\right) \wedge \hat{W} - \sum_{q=1}^{l} \left[(W^q - W^{q-1}) \wedge \hat{W}\right] t^q| \leq \epsilon_0$$  \hspace{1cm} (9.31)

and then

$$|\left(\hat{p} - W^0\right) \wedge \hat{W} - |(W^{q^*} - W^{q^*-1}) \wedge \hat{W}| t^{q^*} - \sum_{q=1,q\neq q^*}^{l} \left[(W^q - W^{q-1}) \wedge \hat{W}\right] t^q| \leq \epsilon_0$$  \hspace{1cm} (9.32)

This last formula implies that, for a fixed $\hat{p}$, $t^{q^*}$ must be in a interval of length smaller than

$$\epsilon_0 |(W^{q^*} - W^{q^*-1}) \wedge W^l|^{-1}$$  \hspace{1cm} (9.33)

that from (9.29) is bounded by $\epsilon_0^2 (P_s - 1)$. Since the possible choices of $\hat{p}$ are at most $(CEt)^3$ it results that $t^{q^*}$ is in a set of measure bounded by

$$\epsilon_0^2 (P_s - 1) (CEt)^3$$  \hspace{1cm} (9.34)
We summarize as follows. We denote with $V_{r_1}$ and $V'_{r_1}$ respectively the outgoing and incoming relative velocities of the collision at time $\tau_{r_1}$ in the BBF. Let $t^q = t_{r_2}$ and $U^q = U_{r_2}$ that is a function of $V_{r_2}, \nu_{r_2}$ only. We have that

$$\chi_{\lambda,E,\delta,q} \{ N^{P_s}(\epsilon_0) \} \leq \chi_{\lambda,E,\delta,q} \sum_{r=1}^{P_s-1} \chi \{ V_r \in B_{r,0} \cup B_{r,1} \} +$$

$$\sum_{i,k} \sum_{l=1}^{n_{ik}} \chi_{\lambda,E,\delta,q} \{ N^q_{i,k}(\epsilon_0) \} + \sum_{r_1=r_{r_1}+1}^{P_s-1} \chi_{\lambda,E,\delta,q} \{ |V'_{r_1} \wedge U_{r_2}| \leq C_0^4 (P_s - 1) \} \quad (9.35)$$

where $n_{ik}$ are the total number of creation in the virtual trajectory of the particles $i$ and $k$ between the time $t^0$ and the time $t$.

$$N^q_{i,k}(\epsilon_0) = \{ t_{P_s-1}, \nu_{P_s-1}, w_{P_s-1} \}$$

the virtual trajectories of $i$ and $k$ satisfies (9.16), (9.36)

$$\text{with } |W^q - W^{q-1} \wedge \hat{W}| \geq \frac{\epsilon_0^3}{l}$$

We now estimate the three terms in the right hand side of (9.35). For the first term by a simple change of variables we have that

$$\sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} \chi_{\lambda,E,\delta,q} \sum_{r=1}^{P_s-1} \chi \{ V_r \in B_{r,0} \cup B_{r,1} \} \prod_{k=1}^{P_s-1} B f^N_{0,P_s}(\zeta(0)) \sum_{i,k} \sum_{l=1}^{n_{ik}} \sum_{q^*=1}^{P_s-1} \chi_{\lambda,E,\delta,q} \chi \{ N^q_{i,k}(\epsilon_0) \} \prod_{k=1}^{P_s-1} B f^N_{0,P_s}(\zeta(0)) \leq$$

$$e^{-\lambda \|g_0\|_\infty} (C \alpha t)^{2^{s+1}} C \left( \frac{E_0^3}{\delta^2} + (Et)^3 \epsilon_0^3 \right) \quad (9.37)$$

For the second term from (9.34) it follows that

$$\sum_{j_1=0}^{1} \ldots \sum_{j_s=0}^{2^s-1} \chi_{\lambda,E,\delta,q} \sum_{r=1}^{P_s-1} \chi \{ V_r \in B_{r,0} \cup B_{r,1} \} \prod_{k=1}^{P_s-1} B f^N_{0,P_s}(\zeta(0)) \leq$$

$$e^{-\lambda \|g_0\|_\infty} (C \alpha t)^{2^{s+1}} \frac{E^3 g(s+4)(s+1) \epsilon_0^4}{2} \quad (9.38)$$
The last term to be estimated is

\[
\sum_{j_1=0}^{1} \sum_{j_2=0}^{2^r-1} \sum_{p_r} \sum_{\Gamma(p_r-1)} \sum_{\sigma_{p_r-1}} \int d\Lambda_{p_{r-1}} \chi_{\alpha,\eta,\delta,q} \sum_{k_1=1}^{p_r-1} \sum_{k_2=k_1+1}^{p_r-1} \chi \left\{ \left| V_{k_1} \hat{\Lambda} \hat{N} \right| \leq C \epsilon_0^4 (P_s - 1) \right\} \prod_{k=1}^{p_r-1} B f_{0,p_r}(\zeta(0))
\]

(9.39)

We first consider the set

\[
\int d\Lambda_{p_{r-1}} \chi_{\alpha,\eta,\delta,q} \sum_{k_1=1}^{p_r-1} \sum_{k_2=k_1+1}^{p_r-1} \chi \left\{ \left| V_{k_1} \hat{\Lambda} \hat{N} \right| \leq C \epsilon_0^4 (P_s - 1) \right\} e^{-\frac{2}{2} |\zeta'(0)|^2}
\]

(9.40)

We change the integration variables in the following way

\[
(v_{k_1}, w_{k_1}, v_{k_2}, w_{k_2}) \rightarrow \left( v_{k_1}', V_{k_1}', v_{k_2}, V_{k_2}' \right)
\]

(9.41)

where \( V_{k_1}' = v_{k_1}' - \zeta_i (\tau_{r_k}) \) and \( V_{k_2} = w_{k_1} - \zeta_i (\tau_{r_k}) \). From Lemma 2.1 it follows that (9.41) is a change of variables that preserve the measure. Thanks to this change of variables a simple calculation leads to

\[
\int d\Lambda_{p_{r-1}} \chi_{\alpha,\eta,\delta,q} \sum_{k_1=1}^{p_r-1} \sum_{k_2=k_1+1}^{p_r-1} \chi \left\{ \left| V_{k_1} \hat{\Lambda} \hat{N} \right| \leq C \epsilon_0^4 (P_s - 1) \right\} e^{-\frac{2}{2} |\zeta'(0)|^2} \leq E^5 \epsilon_0^2 2^{s+1}
\]

(9.42)

This implies that

\[
\sum_{j_1=0}^{1} \sum_{j_2=0}^{2^r-1} \sum_{p_r} \sum_{\Gamma(p_r-1)} \sum_{\sigma_{p_r-1}} \int d\Lambda_{p_{r-1}} \chi_{\alpha,\eta,\delta,q} \sum_{k_1=1}^{p_r-1} \sum_{r_2=k_1+1}^{p_r-1} \chi \left\{ \left| V_{k_1} \hat{\Lambda} \hat{N} \right| \leq C \epsilon_0^4 (P_s - 1) \right\} \prod_{k=1}^{p_r-1} B f_{0,p_r}(\zeta(0)) \leq e^{-\lambda} \| g_0 \| \infty \cdot (C \alpha t)^{2^{s+1}} 2^{(s+2)(s+1)} E^5 \epsilon_0^2
\]

(9.43)

Finally from these estimates we arrive to

\[
\sum_{j_1=0}^{1} \sum_{j_2=0}^{2^r-1} \sum_{p_r} \sum_{\Gamma(p_r-1)} \sum_{\sigma_{p_r-1}} \int d\Lambda_{p_{r-1}} \chi_{\alpha,\eta,\delta,q} \chi \left\{ N^{p_r} (\epsilon_0) \right\} \prod_{k=1}^{p_r-1} B f_{0,p_r}(\zeta(0)) \leq e^{-\lambda} \| g_0 \| \infty \cdot (C \alpha t)^{2^{s+1}} E^5 2^{(s+4)(s+1)} \left( \frac{2}{\epsilon_0^4} + \frac{4}{\delta^2} + \frac{2}{\delta} \right)
\]

(9.44)
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