New Bounds on the Biplanar and $k$-Planar Crossing Numbers

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Abstract

The biplanar crossing number of a graph $G$ is the minimum number of crossings over all possible drawings of the edges of $G$ in two disjoint planes. We present new bounds on the biplanar crossing number of complete graphs and complete bipartite graphs. In particular, we prove that the biplanar crossing number of complete bipartite graphs can be approximated to within a factor of 3, improving over the best previously known approximation factor of 4.03. For complete graphs, we provide a new approximation factor of 3.17, improving over the best previous factor of 4.34. We provide similar improved approximation factors for the $k$-planar crossing number of complete graphs and complete bipartite graphs, for any positive integer $k$. We also investigate the relation between (ordinary) crossing number and biplanar crossing number of general graphs in more depth, and prove that any graph with a crossing number of at most 10 is biplanar.

1 Introduction

An embedding (or drawing) of a graph $G$ in the Euclidean plane is a mapping of the vertices of $G$ to distinct points in the plane and a mapping of edges to smooth curves between their corresponding vertices. A planar embedding is a drawing of the graph such that no two edges cross each other, except for possibly in their endpoints. A graph that admits such a drawing is called planar. A biplanar embedding of a graph $G = (V,E)$ is a decomposition of the graph into two graphs $G_1 = (V,E_1)$ and $G_2 = (V,E_2)$ such that $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, together with planar embeddings of $G_1$ and $G_2$. In this case, we call $G$ biplanar. Biplanar embeddings are central to the computation of thickness of graphs [12], with applications to VLSI design [13]. It is well-known that planarity can be recognized in linear time, while biplanarity testing is NP-complete [11].

Let $cr(G)$ be the minimum number of edge crossings over all drawings of $G$ in the plane, and let $cr_k(G)$ be the minimum of $cr(G_1) + cr(G_2) + \cdots + cr(G_k)$ over all possible decompositions of $G$ into $k$ subgraphs $G_1, G_2, \ldots, G_k$. We call $cr(G)$ the crossing number of $G$, and $cr_k(G)$ the $k$-planar crossing number of $G$. Throughout this paper, we only consider simple drawings for each subgraph $G_i$, in which no two edges cross more than once, and no three edges cross at a point (such drawings are sometimes called nice drawings). Moreover, we denote by $n$ the number of vertices, and by $m$ the number of edges of a graph.

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Determining the crossing number of complete graphs and complete bipartite graphs has been the subject of extensive research over the past decades. In 1955, Zarankiewicz [19] conjectured that the crossing number \( cr(K_{p,q}) \) of the complete bipartite graph \( K_{p,q} \) is equal to

\[
Z(p, q) := \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p - 1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q - 1}{2} \right\rfloor.
\]

He also established a drawing with that many crossings. In 1960, Guy [7] conjectured that the crossing number \( cr(K_n) \) of the complete graph \( K_n \) is equal to

\[
Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor \left\lfloor \frac{n - 2}{2} \right\rfloor \left\lfloor \frac{n - 3}{2} \right\rfloor.
\]

Both conjectures have remained open after more than six decades. For the biplanar case, even formulating such conjectures seems to be hard. As noted in [4], techniques like embedding method and the bisection width method which are useful for bounding ordinary crossing numbers do not seem applicable to the biplanar case.

In 1971, Owens [13] described a biplanar embedding of \( K_n \) with almost \( \frac{n^2}{2} Z(n) \) crossings. In 2006, Czabarka et al. [4] presented a biplanar embedding for \( K_{p,q} \) with about \( \frac{2}{9} Z(p, q) \) crossings. They also proved that \( cr_2(K_n) \geq n^4/952 \) and \( cr_2(K_{p,q}) \geq p(p-1)q(q-1)/290 \). Shahrokhi et al. [16] generalized these lower bounds to the \( k \)-planar case. Recently, Pach et al. [14] proved that for every graph \( G \) and any positive integer \( k \), \( cr_k(G) \leq \left( \frac{1}{2} \right)^k cr(G) \). This includes as a special case the inequality \( cr_2(G) \leq \frac{3}{8} cr(G) \), originally proved by Czabarka et al. [5].

**Our results.** In this paper, we present several new bounds for approximating the biplanar and \( k \)-planar crossing number of complete graphs and complete bipartite graphs. Given a positive integer \( k \) and a real constant \( \alpha \geq 1 \), we say that \( cr_k(K_n) \) is approximated to within a factor of \( \alpha \), if there is an upper bound \( f(n) \) and a lower bound \( g(n) \) on the value of \( cr_k(K_n) \) such that \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \leq \alpha \). Here, \( \alpha \) is called an asymptotic approximation factor for \( cr_k(K_n) \).

Similarly, we say that \( cr_k(K_{p,q}) \) is approximated to within a factor of \( \alpha \), if there is an upper bound \( f(p, q) \) and a lower bound \( g(p, q) \) on the value of \( cr_k(K_{p,q}) \) such that \( \lim_{p,q \to \infty} \frac{f(p, q)}{g(p, q)} \) exists and is no more than \( \alpha \). The results presented in this paper are summarized below:

- We prove that for all \( p, q \geq 21 \), \( cr_2(K_{p,q}) \geq p(p-1)q(q-1)/216 \). This significantly improves the best current lower bound of \( cr_2(K_{p,q}) \geq p(p-1)q(q-1)/290 \), due to Czabarka et al. [4]. Combined with the upper bound of \( cr_2(K_{p,q}) \leq \frac{2}{9} Z(p, q) + o(p^2q^2) \) [4], our result implies an asymptotic approximation factor of 3 for \( cr_2(K_{p,q}) \), improving over the best previously known approximation factor of 4.03.

- For complete graphs, we show that \( cr_2(K_n) \geq \frac{n^4}{952} \), improving the best current lower bound of \( cr_2(K_n) \geq \frac{n^4}{952} \) [4]. Combined with the upper bound of \( cr_2(K_n) \leq \frac{7}{24} Z(n) + o(n^4) \) due to Owens [13], we achieve an asymptotic approximation factor of 3.17 for \( cr_2(K_n) \), improving the best previously known approximation factor of 4.34.

\(^1\)By definition, \( f(x, y) = o(g(x, y)) \) if \( \lim_{x,y \to \infty} \frac{f(x,y)}{g(x,y)} = 0 \).
Table 1: Summary of the asymptotic approximation factors for the biplanar and k-planar crossing number of complete graphs and complete bipartite graphs.

| Crossing Number | Asymptotic Approx Factor | Reference |
|-----------------|--------------------------|-----------|
| $cr_2(K_{p,q})$ | 4.03                     | [4]       |
|                 | 3                        | [This work] |
| $cr_2(K_n)$    | 4.34                     | [4, 13]   |
|                 | 3.17                     | [This work] |
| $cr_k(K_{p,q})$| 13.5                     | [14, 16]  |
|                 | 9.15                     | [This work] |
| $cr_k(K_n)$    | 13.5                     | [14, 16]  |
|                 | 7.25                     | [This work] |

- We investigate the relation between $cr(G)$ and $cr_2(G)$ in general graphs, and pose a new problem of finding the maximum integer $\xi(r)$, for a given integer $r \geq 0$, such that $cr(G) \leq \xi(r)$ implies $cr_2(G) \leq r$, for all graphs $G$. For the special case of $r = 0$, we show that $\xi(r) \geq 10$. It implies that any graph $G$ that can be drawn in the plane with at most 10 crossings is biplanar.

- We extend our lower bounds for the biplanar crossing number to the $k$-planar case, for any positive integer $k$. In particular, we show that for sufficiently large $n$, $cr_k(K_n) \geq n^4/(232k^2)$, improving the best current lower bound of $cr_k(K_n) \geq n^4/(432k^2)$, due to Shahrokhi et al. [16]. Considering the recent upper bound of $cr_k(K_n) \leq \frac{2}{k^2} Z(n)$ proved by Pach et al. [14], we obtain an asymptotic approximation factor of 7.25 for $cr_k(K_n)$, improving the best current approximation factor of 13.5 available for $cr_k(K_n)$.

- Finally, we prove that for any positive integer $k$, $cr_k(K_{p,q}) \geq p(p-1)q(q-1)/(73.2k^2)$, improving the current lower bound of $cr_k(K_{p,q}) \geq p(p-1)q(q-1)/(108k^2)$ due to Shahrokhi et al. [16]. Combined with the upper bound of $cr_k(K_n) \leq \frac{2}{k^2} Z(p,q)$ [14], we obtain an asymptotic approximation factor of 9.15 for $cr_k(K_{p,q})$, improving the best current factor of 13.5.

A summary of the asymptotic approximation factors for the biplanar and $k$-planar crossing number of $K_n$ and $K_{p,q}$ is presented in Table 1.

2 Preliminaries

One of the main combinatorial tools typically used for deriving lower bounds on the crossing number of graphs is the counting method (see, e.g., [8, 15]). We use the following generalization of the counting method in this paper.

Lemma 1 (Counting method). Let $G$ be a simple graph that contains $\alpha$ copies of a subgraph $H$. If in every $k$-planar drawing of $G$, each crossing of the edges belongs to at most $\beta$ copies of $H$, then

$$cr_k(G) \geq \left\lceil \frac{\alpha}{\beta} cr_k(H) \right\rceil.$$
Proof. Let $D$ be a $k$-planar drawing of $G$, realizing $cr_k(G)$. For each of the $\alpha$ copies of $H$, $D$ contains a $k$-planar drawing with at least $cr_2(H)$ crossings. Since each crossing is counted at most $\beta$ times by our assumption, the lemma statement follows. Note that a ceiling is put in the right-hand side of the inequality, because $cr_k(G)$ is always an integer. \hfill \qed \\

The following lemma provides another main ingredient used throughout this paper.

**Lemma 2.** Let $\mathcal{G}$ be a hereditary class of graphs which is closed under removing edges. Let $f$ be a linear function $f(x) = cx$, for some constant $c$, and let $g$ be an arbitrary function. If for every graph $G$ in $\mathcal{G}$, $cr(G) \geq f(m) - g(n)$, then $cr_k(G) \geq f(m) - k \cdot g(n)$ for all $G \in \mathcal{G}$ and all positive integers $k$.

**Proof.** Fix a graph $G \in \mathcal{G}$. Let $G = \bigcup_{i=1}^{k} G_i$ be a decomposition of $G$ into $k$ subgraphs $G_i = (V, E_i)$ such that $\sum_{i=1}^{k} cr(G_i)$ is minimum. By the hereditary property of $\mathcal{G}$, each $G_i$ is in $\mathcal{G}$, and hence $cr(G_i) \geq f(m_i) - g(n)$, where $m_i = |E_i|$. Therefore, $cr_k(G) = \sum_{i=1}^{k} cr(G_i) \geq \sum_{i=1}^{k} (f(m_i) - g(n)) = c \sum_{i=1}^{k} m_i - \sum_{i=1}^{k} g(n) = f(m) - k \cdot g(n)$. \hfill \qed

### 3 Lower Bounds for Complete Bipartite Graphs

In this section, we provide new lower bounds on the biplanar crossing number of complete bipartite graphs. In particular, we improve the following bound due to Czabarka et al. [4] which states that for all $p, q \geq 10$,

$$cr_2(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{290}.$$  

From Euler’s formula, we have $cr(G) \geq m - 3(n - 2)$ for simple graphs, and $cr(G) \geq m - 2(n - 2)$ for bipartite graphs. Using Lemma 2, we immediately get a lower bound of $cr_2(G) \geq m - 6(n - 2)$ for simple graphs, and a lower bound of $cr_2(G) \geq m - 4(n - 2)$ for bipartite graphs.

To establish stronger lower bounds, we need to incorporate more powerful ingredients. A graph is called $k$-planar, if it can be drawn in the plane in such a way that each edge has at most $k$ crossings. It is known that every 1-planar drawing of any 1-planar graph has at most $n - 2$ crossings [6]. (Note the difference between $k$-planar graphs, and $k$-planar crossing numbers.) Removing one edge per crossing yields a planar graph. Therefore, every 1-planar bipartite graph has at most $3n - 6$ edges. Karpov [9] proved that for every 1-planar bipartite graph with at least 4 vertices, the inequality $m \leq 3n - 8$ holds. In a recent work, Angelini et al. [2] proved that for every 2-planar bipartite graph we have $m \leq 3.5n - 7$. We use these results to obtain the following stronger lower bound.

**Lemma 3.** For every bipartite graph $G$ with $n \geq 4$,

$$cr_2(G) \geq 3m - 17n + 38.$$  

**Proof.** Let $G$ be a bipartite graph with $n$ vertices and $m$ edges. Fix a drawing of $G$ with a minimum number of crossings. If $m > 3.5n - 7$, then by [2], there must be an edge in the drawing with at least three crossings. We repeatedly remove such an edge until we reach a drawing with $[3.5n - 7]$ edges. Then by Karpov’s result there must be an edge in the drawing
with at least two crossings. Similarly we repeatedly remove such an edge until we reach a drawing with $3n - 8$ edges. Let $G'$ be the bipartite graph corresponding to the remaining drawing. Now,

$$
\begin{align*}
\text{cr}(G) &\geq 3(m - \lfloor 3.5n - 7 \rfloor) + 2(\lfloor 3.5n - 7 \rfloor - (3n - 8)) + \text{cr}(G') \\
&\geq 3(m - \lfloor 3.5n - 7 \rfloor) + 2(\lfloor 3.5n - 7 \rfloor - (3n - 8)) + (3n - 8) - 2(n - 2) \\
&\geq 3m - \lfloor 3.5n - 7 \rfloor - (3n - 8) - 2(n - 2) \\
&\geq 3m - 8.5n + 19.
\end{align*}
$$

Applying Lemma 2 yields $\text{cr}_2(G) \geq 3m - 17n + 38$.

For complete bipartite graphs, Lemma 3 implies that $\text{cr}_2(K_{p,q}) \geq 3pq - 17(p + q) + 38$, for all $p, q \geq 2$. We use Lemma 3 along with a counting argument to obtain the following improved bound on $\text{cr}_2(K_{p,q})$.

**Theorem 4.** For all $p, q \geq 21$,

$$
\text{cr}_2(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{216}.
$$

**Proof.** Using the counting method (Lemma 1) for $K_{n,n}$ and $K_{n+1,n}$ we have

$$
\text{cr}_2(K_{n+1,n}) \geq \left\lceil \frac{n + 1}{n - 1} \text{cr}_2(K_{n,n}) \right\rceil.
$$

This is because $K_{n+1,n}$ contains $n + 1$ copies of $K_{n,n}$, and each crossing realized by two edges, belongs to at most $\binom{n-1}{n-2} = n - 1$ of these copies. Using a similar argument for $K_{n+1,n}$ and $K_{n+1,n+1}$, we get

$$
\text{cr}_2(K_{n+1,n+1}) \geq \left\lceil \frac{n + 1}{n - 1} \left\lceil \frac{n + 1}{n - 1} \text{cr}_2(K_{n,n}) \right\rceil \right\rceil. \quad (1)
$$

By Lemma 3, $\text{cr}_2(K_{15,15}) \geq 203$. Plugging into (1), yields $\text{cr}_2(K_{16,16}) \geq 266$. Now, we use the recurrence relation (1) iteratively from $n = 15$ to 21 to get

$$
\text{cr}_2(K_{21,21}) \geq 817. \quad (2)
$$

We can now apply the counting method on $K_{21,21}$ and $K_{p,q}$ to obtain

$$
\text{cr}_2(K_{p,q}) \geq \frac{\binom{p}{2} \binom{q}{2}}{(p-2)!(q-2)!} \text{cr}_2(K_{21,21}) = \frac{p(p-1)q(q-1)}{21 \times 20 \times 21 \times 20} \text{cr}_2(K_{21,21}).
$$

Plugging (2) in the above inequality yields the theorem statement. \qed

### 4 Biplanar Crossing Number of Complete Graphs

We now consider the biplanar crossing number of complete graphs. Czabarka *et al.* [4] used a probabilistic method to prove that for large values of $n$,

$$
\text{cr}_2(K_n) \geq \frac{n^4}{952}.
$$

We improve this lower bound using the counting method.
Theorem 5. For all \( n \geq 24 \),
\[
cr_2(K_n) \geq \frac{n(n-1)(n-2)(n-3)}{698}.
\]

Proof. We know from [1] that for every \( G \) with \( n \geq 3 \), \( cr(G) \geq 5m - \frac{139}{6}(n-2) \). Applying Lemma 2, we get
\[
cr_2(G) \geq 5m - \frac{139}{3}(n-2).
\]
This in particular implies that \( cr_2(K_{25}) \geq 435 \). Now, we use the counting method (Lemma 1) on \( K_{25} \) and \( K_n \) to get
\[
cr_2(K_n) \geq \frac{n}{25} \frac{cr_2(K_{25})}{n-4} \geq \frac{n(n-1)(n-2)(n-3)}{25 \times 24 \times 23 \times 22 \times 435},
\]
which implies the theorem statement. \( \square \)

We can slightly improve this result, using an iterative counting method similar to what we used in the previous section.

Theorem 6. For large values of \( n \),
\[
cr_2(K_n) \geq \frac{n^4}{694}.
\]

Proof. Using the counting method (Lemma 1) for \( K_n \) and \( K_{n+1} \) we have,
\[
cr_2(K_{n+1}) \geq \left\lceil \frac{(n+1)cr_2(K_n)}{n-3} \right\rceil.
\]
Starting from \( cr_2(K_{25}) \geq 435 \), we use the recurrence relation (3) iteratively from \( n = 25 \) to \( 57 \) to obtain \( cr_2(K_{57}) \geq 13667 \). Now, we use the counting method on \( K_{57} \) and \( K_n \) to get
\[
cr_2(K_n) \geq \frac{n}{57} \frac{cr_2(K_{57})}{n-4} \geq \frac{n(n-1)(n-2)(n-3)}{57 \times 56 \times 55 \times 54} \geq \frac{n(n-1)(n-2)(n-3)}{693.9},
\]
which implies \( cr_2(K_n) \geq \frac{n^4}{694} \) for \( n \) sufficiently large. \( \square \)

5 The Maximum Crossing Number that Implies Biplanarity

Czabarka et al. [5] defined \( c^* \) as the smallest constant such that for every graph \( G \), \( cr_2(G) \leq c^* \cdot cr(G) \). They proved that \( 0.067 \leq c^* \leq \frac{3}{8} = 0.375 \). It is known that \( cr(K_n) \leq \frac{n^4}{694} \) [18].

By Theorem 6, for \( n \) sufficiently large, \( cr_2(K_n) \geq \frac{n^4}{694} \). Therefore, our results from Section 4 imply an improved bound of \( c^* \geq \frac{64}{694} \approx 0.092 \). In a more general sense, we are interested in the following problem.

Problem. Given a positive integer \( r \), find the largest integer \( \xi(r) \) such that for every graph \( G \), \( cr(G) \leq \xi(r) \) implies \( cr_2(G) \leq r \).
For the special case of \( r = 0 \), the problem is to find the largest integer \( \xi \) such that drawing a graph with at most \( \xi \) crossings in the plane guarantees that the graph is biplanar. As proved by Battle et al. [3] and Tutte [17], \( K_5 \) is not biplanar. Moreover, we know that \( cr(K_9) = 36 \) [10]. Therefore, \( \xi(0) < 36 \).

The inequality \( cr_2(G) \leq \frac{3}{2}cr(G) \), due to Czabarka et al. [5], implies that if \( cr(G) \leq 2 \), then \( G \) is biplanar. Therefore, \( \xi(0) \geq 2 \). We can strengthen this bound as follows. Recall that by Kuratowski’s theorem, every nonplanar graph contains a subdivision of \( K_{3,3} \) or \( K_5 \). Therefore, there is no nonplanar graph with less than 9 edges. This leads to the following observation.

**Observation 1.** Every graph with at most 8 edges is planar. The only nonplanar graph with 9 edges is \( K_{3,3} \), and the only nonplanar graphs with 10 edges are \( K_5 \), \( K_{3,3} \) with an extra edge, and \( K_{3,3} \) with a subdivided edge.

From this simple observation, we can infer that \( \xi(0) \geq 4 \) as follows. Suppose a graph \( G \) is drawn in the plane with at most 4 crossings. The number of edges involved in these four crossings is at most 8. If we remove these 8 edges from the drawing, the remaining drawing has no crossing. Moreover, the subgraph of \( G \) that contains only these 8 (or fewer) edges is planar by Observation 1. Therefore, \( G \) is the union of two planar graphs, and hence is biplanar. We will significantly improve this lower bound in the following theorem.

**Theorem 7.** Every graph \( G \) with \( cr(G) \leq 10 \) is biplanar. In other words, \( \xi(0) \geq 10 \).

**Proof.** Let \( G \) be a graph with \( cr(G) \leq 10 \). Fix a drawing of \( G \) with a minimum number of crossings. We repeatedly remove an edge from the drawing that involves in a maximum number of crossings until there remains no more crossings. Let \( G_1 \) be the graph corresponding to the remaining drawing, and \( G_2 \) be the graph formed by the removed edges. Clearly, \( G_2 \) has at most 10 edges. Moreover, \( G_1 \) is planar by construction. If \( G_2 \) has 8 or less edges, then it is planar by Observation 1, and we are done. Otherwise, \( G_2 \) has 9 or 10 edges. Note that removing any of these edges from \( G \) has removed at least one crossing. Therefore, removing any of these edges, except possibly the first one, has removed exactly one crossing from \( G \). By Observation 1, if \( G_2 \) is not planar, then it is either \( K_5 \), \( K_{3,3} \), \( K_{3,3} \) with a subdivided edge, or \( K_{3,3} \) with an extra edge. In the former two cases, let \( e \) be the last edge removed from \( G \). Clearly, \( e \) was crossing exactly one edge \( f \) in \( G_1 \) just before removal. Therefore, switching \( e \) and \( f \) between \( G_1 \) and \( G_2 \) keeps \( G_1 \) planar. Moreover, the new \( G_2 \) is planar, because it contains no subdivision of \( K_5 \) and \( K_{3,3} \). Hence, \( G \) is biplanar in the first two cases. In the latter two cases, i.e., when \( G_2 \) is a \( K_{3,3} \) with a subdivided edge or a \( K_{3,3} \) with an extra edge, \( G_2 \) has exactly 10 edges. Therefore, removing any of these edges from \( G \) has removed exactly one crossing, which means that any edge in \( G \) is crossing at most one edge. If \( G_2 \) is a \( K_{3,3} \) with a subdivided edge, let \( e \) be any edge of \( G_2 \) except the two edges forming the subdivided edge, and if \( G_2 \) is a \( K_{3,3} \) with an extra edge, let \( e \) be any edge of \( G_2 \) except this extra edge.

We know that \( e \) was crossing exactly one edge \( f \) in \( G \). Moreover, \( f \) was only crossing \( e \) in \( G \), and hence, it remains in \( G_1 \) after removing \( e \). Similar to the previous case, switching \( e \) and \( f \) between \( G_1 \) and \( G_2 \) completes the proof. \( \square \)
6  
\textit{k-Planar Crossing Number of } K_n \text{ and } K_{p,q}

In this section, we provide improved lower bounds on the k-planar crossing number of complete bipartite and complete graphs. Shahrokhi \textit{et al.} [16] proved that for any positive integer \(k\), and sufficiently large integers \(p\), \(q\), and \(n\):

\[
\text{cr}_k(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{108k^2},
\]

and

\[
\text{cr}_k(K_n) \geq \frac{n(n-1)(n-2)(n-3)}{432k^2}.
\]

We improve these results using the ideas developed in Sections 2 and 3.

\textbf{Theorem 8.} For all \(p, q \geq 8k + 2\),

\[
\text{cr}_k(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{73.2k^2}.
\]

\textit{Proof.} We apply the counting method (Lemma 1) on \(K_{8k+2,8k+2}\) and \(K_{p,q}\). As noted in the proof of Lemma 3, for every bipartite graph \(G\), \(\text{cr}(G) \geq 3m - 8.5n + 19\). Therefore, by Lemma 2, \(\text{cr}_k(G) \geq 3m - (8.5n - 19)k\). This yields

\[
\text{cr}_k(K_{8k+2,8k+2}) \geq 56k^2 + 43k + 12.
\]

Hence,

\[
\text{cr}_k(K_{p,q}) \geq \frac{(p-8k+2)(q-8k+2)\text{cr}_k(K_{8k+2,8k+2})}{(p-2)(q-2)(8k+2)(8k+1)(8k+2)(8k+1)} = \frac{p(p-1)q(q-1)\text{cr}_k(K_{8k+2,8k+2})}{(8k+2)(8k+1)(8k+2)(8k+1)} \geq \frac{p(p-1)q(q-1)}{512k^2},
\]

which completes the proof. \(\Box\)

\textbf{Theorem 9.} For all \(n \geq 14k - 3\),

\[
\text{cr}_k(K_n) \geq \frac{n(n-1)(n-2)(n-3)}{232k^2}.
\]

\textit{Proof.} We use the counting method (Lemma 1) for \(K_{14k-3}\) and \(K_n\). Recall that for every \(G\) with \(n \geq 3\), \(\text{cr}(G) \geq 5m - \frac{139}{6} (n-2)\) [1]. Therefore, \(\text{cr}_k(G) \geq 5m - \frac{139}{6} (n-2)k\) by Lemma 2. Thus,

\[
\text{cr}_k(K_{14k-3}) \geq \frac{497}{3}k^2 - \frac{775}{6}k + 30.
\]

Therefore,

\[
\text{cr}_k(K_n) \geq \frac{\binom{n}{14k-3} \text{cr}_k(K_{14k-3})}{\binom{n}{14k-7}} = \frac{n(n-1)(n-2)(n-3)\text{cr}_k(K_{14k-3})}{(14k-3)(14k-4)(14k-5)(14k-6)},
\]

which implies the theorem. \(\Box\)
7 Conclusion

In this paper, we presented several improved bounds on the biplanar and \( k \)-planar crossing number of complete graphs and complete bipartite graphs. An obvious open problem is whether the asymptotic approximation factors presented in this paper can be further improved. We also posed an open problem of finding the largest positive integer \( \xi(r) \) such that \( cr(G) \leq \xi(r) \) implies \( cr_2(G) \leq r \). In particular, we proved that \( 10 \leq \xi(0) \leq 35 \). This definition can be easily generalized to the \( k \)-planar case: given positive integers \( k \) and \( r \), find the largest integer \( \xi_k(r) \) such that \( cr(G) \leq \xi_k(r) \) implies \( cr_k(G) \leq r \). Determining the value of \( \xi_k(r) \) is an intriguing problem, even for the special case of \( r = 0 \).

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