Optimal Harvesting Strategy of a Discretization Fractional-Order Biological Model

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Abstract
Optimal control methods are used to get an optimal policy for harvesting renewable resources. In particular, we investigate a discretization fractional-order biological model, as well as its behavior through its fixed points, is analyzed. We also employ the maximal Pontryagin principle to obtain the optimal solutions. Finally, numerical results confirm our theoretical outcomes.

Keywords: Discrete fractional-order, ratio-dependent prey-predator, optimal strategy

1-Introduction
Harvesting is an important theme in renewable resources management, so that the dilemma of harvest biological systems have been analyzed and investigated by many researchers to get optimal exploitation polices.

The books of C.W. Clark [1], and Mark Kot [2] are extremely applicable and relevant to the optimal harvesting problems. Rassi and Jerry[3] related to the maximization of the total net gains derived by the harvesting of the resources. They also developed and studied the exploitation policy to the optimal control problems.

The original work of Lotka and Volterra [4, 5] is inception of the prey-predator theory, then after it became the most important subject in mathematical ecology. So that many

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authors have been adapted the work of Lotka-Volterra system employing difference equations, ordinary differential equations or partial differential equations, as well as fractional-order derivative [6-13], and references therein. However, there are many types of functional response, namely Holling type I, II, III, and type IV, Beddington-Deaugelis, Leslie-Gower, Corwley-Marin, and others [8,14,15,16].

Fractional-order-derivative provides a precise description of the dynamics of biological or epidemiological models due to in consideration of information about a population memory compared to the other descriptions for that many researchers prefer to model their systems by fractional-order derivative. For more details about the fractional-order derivative we refer to these references [6,11,12,17-19].

A general system of two dimensional prey-predator without harvesting is described by ordinary differential equations as following:

\[
\begin{align*}
\frac{dx(t)}{dt} &= xf(x) - g(x,y)y \\
\frac{dy(t)}{dt} &= dg(x,y)y - c(y)y(t)
\end{align*}
\]  (1)

Here the variables \(x(t)\), and \(y(t)\) denote to the size of prey and predator population at time \(t\), respectively. Parameter \(d\) is the conversion rate. Parameter \(c\) denotes the death natural rate of predator species. The function \(f\) represents growth rate of prey, while the function \(g(x,y)\) is called the functional response of predator to prey density. This work is organized as follows: The fractional-order derivative model is described in section 2, as well as its discretization is analyzed and investigated through its fixed points. Then we extend the discretization system to an optimal control problem, this is done in section 3. In section 4 numerical results are presented to clarify the theoretical analysis. A discussion follows in section 5.

2-The fractional-order derivative system, and its discretization

**Definition 1 [20]** The \(\theta\) - order Caputo differential operator is defined as follows:

\[
D^\theta f(x) = 1^{1-\theta}f(x), \quad \theta > 0
\]

Such that \(l = [\theta]\), and \(l^\beta u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \mu)^{\beta-1}u(t)dt, \quad \beta > 0.

\(l^\beta\) represents the \(\beta\) order Riemann-Liouville integral operator, \(\Gamma(.)\) denotes the Gamma function.

In this work the ratio-dependent predator–prey or Michaelis–Menten type prey–predator model [21] is modified to the following fractional-order model.

\[
\begin{align*}
D^\theta x(t) &= x(t)(1 - x(t)) - \frac{x(t)y(t)}{ax(t) + by(t)} - h_1x(t) \\
D^\theta y(t) &= \frac{dx(t)y(t)}{ax(t) + by(t)} - cy(t) - h_2y(t)
\end{align*}
\]  (2)

Where \(x(t)\), and \(y(t)\) denote the densities of prey, and predator species at time \(t\), respectively. In this system the prey grows logistically. The parameter \(d\) represents the conversion rate part from the prey species to the predator species. The parameter \(c\) denotes the death rate of predator species. The functional response is the ratio-dependent predator-prey. \(a, and b\) are the half saturation constants. \(h_1, and h_2\) are the rate harvesting or the removal rate of prey and predator, respectively. Throughout this article we assume that \(h_2 = 0, and h_1 = h\).
Applying discretization method to the fractional-order system (2). For more details we refer to [6,10]. The system (2) is reduced to

\[
\begin{align*}
x_{n+1} &= x_n + \frac{s^\theta}{\Gamma(\theta+1)} \left[ x_n(1-x_n) - \frac{x_n y_n}{(a x_n + b y_n)} - h x_n \right] \\
y_{n+1} &= y_n + \frac{s^\theta}{\Gamma(\theta+1)} \left[ -\frac{d x_n y_n}{(a x_n + b y_n)} - c y_n \right]
\end{align*}
\]

(3)

**Definition 2** [22]: Let \( \bar{x}_{t+1} = f(x_t) \) \( t = 2, 3, \ldots \)
be a discrete time system the point \( e^* \) is called a fixed point of equation (4) if \( e^* = f(e^*) \). If \( |\lambda_i| < 1 \) for \( i = 1, 2, \ldots, n \) \( \lambda_i \) are the eigenvalues of the Jacobian matrix \( J \) at \( e^* \) then it is called local stable point. Otherwise \( e^* \) is called unstable point. While if \( |\lambda_i| = 1 \) for some \( 1 \leq i \leq n \) then \( e^* \) is called a non-hyperbolic point.

The system (3) has the following fixed points:

1- The \( e_0 = (0,0) \) is the trivial fixed point which always exists, while the fixed point \( e_1 = (1-h,0) \) exists only when \( 1 > h \).

2- The unique positive fixed point \( e_2 = (x_p, y_p) \) exits if \( bd(1-h) > (d-ae) \) and \( d > ae \) where \( x_p = \frac{bd(1-h)-(d-ae)}{bd} \) and \( y_p = \frac{(d-ae)}{bc} x_p \).

To discuss the dynamic behavior of the system (3) we have to compute the Jacobian matrix of (3). The Jacobian matrix at \( (x, y) \) is as follows:

\[
J(x,y) = \begin{bmatrix}
\frac{j_{11}}{j_{12}}
\frac{j_{12}}{j_{22}}
\end{bmatrix}
\]

Where \( j_{11} = 1 + m - 2mx - \frac{mby^2}{k} - mh \), \( j_{12} = -\frac{max^2}{k} \), \( j_{21} = \frac{mbdy^2}{k} \), \( j_{22} = 1 - mc + \frac{madx^2}{k} \).

For the local stability of the fixed points \( e_0 \) and \( e_1 \) of system (3) we have the following theorem.

**Theorem 1**

1- The \( e_0 \) is never to be locally stable point.

2- The \( e_1 \) is locally stable if \( h \in \left( \frac{m-2}{m}, 1 \right) \), and \( c \in \left( \frac{d}{a}, \frac{2a+md}{am} \right) \)

**Proof**: The Jacobian matrices at \( e_0 \) and \( e_1 \) are

\[
J_{e_0} = \begin{bmatrix}
1 + m - mh & 0 \\
0 & 1 - mc
\end{bmatrix}
\]

and

\[
J_{e_1} = \begin{bmatrix}
1 - m + mh & \frac{-m}{a} \\
0 & 1 - mc + \frac{md}{a}
\end{bmatrix}
\]

respectively.

Now the eigenvalues of \( J_{e_0} \) are \( \lambda_1 = 1 + m(1-h), \lambda_2 = 1 - mc \). Since \( h \) is always less than \( 1 \), therefore \( \lambda_1 > 1 \), and the point \( e_0 \) is never to be stable point.

The eigenvalues of \( J_{e_1} \) are \( \lambda_1 = 1 - m + mh), \lambda_2 = 1 - mc + \frac{md}{a} \) hence if \( h \in \left( \frac{m-2}{m}, 1 \right) \), then \( -2 + m < mh < m \), and \( \lambda_1 < 1 \). Now we assume that \( c \in \left( \frac{d}{a}, \frac{2a+md}{am} \right) \) this gives \( \frac{d}{a} < c < \frac{2a+md}{am} \), and \( \lambda_2 < 1 \). Therefore the \( e_1 \) is locally stable.

**Lemma 1** [22] Let \( P(x) = x^2 + px + q_1 \), if the following conditions hold:

1. \( P(1) > 0 \)
2. \( P(-1) > 0 \)

3. \( q_1 < 1 \).

Then the roots of \( P(x) \) are inside the unit disk.

**Theorem 2**

The point \( e_2 \) is locally stable if \( h \in (\text{Max}\{\frac{z_5}{z_6}, \frac{z_1}{z_2}\}, \frac{z_3}{z_4}\}) \), and \( c \in \left(\frac{\text{adx}^2}{k}, \text{Max}\{\frac{1}{m} + \frac{\text{adx}^2}{k}, \frac{2}{m} + \frac{\text{adx}^2}{k}\}\right) \) where

\[
\begin{align*}
    z_1 &= -c + 2x^* + \frac{k}{2} + \frac{\text{adx}^2}{k} - \frac{2\text{dx}^2}{k} - 2\text{cy}^2 - \frac{2bym^2}{k} + \frac{2bm^2y^2}{k} + \frac{2bcy^2}{k} - \frac{2abm^2}{k}, \\
    z_2 &= \frac{\text{adx}^2}{k} - c, \\
    z_3 &= 4 - 2mc + \frac{2\text{madx}^2}{k} + 2m - 2mx^* - \frac{2bym^2}{k} - 2m^2c + 2m^2x^*c + \frac{2bcy^2}{k} + \frac{2bmy^2}{k} - \frac{2admx^2}{k}, \\
    z_4 &= 2m - m^2c + \frac{2bmy^2}{k} - \frac{2admx^2}{k}, \\
    z_5 &= 1 - 2x^* - \frac{by^2}{k} - c - mc - 2mcx^* + \frac{bx^2}{k} + \frac{bmy^2}{k} - \frac{admx^2}{k} + \frac{2admx^2}{k}, \\
    z_6 &= 1 - mc + \frac{\text{admx}^2}{k}.
\end{align*}
\]

Proof: Jacobian matrix at \( e_2 \) is

\[
    J_{e_2} = \begin{bmatrix}
        1 + m - 2mx^* - \frac{\text{mby}^2}{k} - \frac{\text{max}^2}{k} & -\frac{\text{max}^2}{k} \\
        \frac{\text{mbdy}^2}{k} & 1 - mc + \frac{\text{madx}^2}{k}
    \end{bmatrix}
\]

Then the characteristic polynomial of \( J_{e_2} \) is given as follows:

\[
P(\lambda) = \lambda^2 + p\lambda + q, \quad \text{where} \quad p = -2 + mc - \frac{\text{madx}^2}{k} - m + 2mx^* + \frac{\text{bmy}^2}{k} + \text{mh}. \quad \text{And} \quad q = 1 + m - 2mx^* - \frac{\text{bmy}^2}{k} - \text{mh} - mc - m^2c + 2m^2x^*c + \frac{bcy^2}{k} + \frac{bmc^2}{k} + \frac{admx^2}{k} + \frac{2admx^2}{k} - \frac{2admx^2}{k} - \frac{\text{admx}^2}{k} - \frac{\text{admx}^2}{k}.
\]

If \( h < \text{Min}\left\{\frac{z_1}{z_2}, \frac{z_1}{z_4}\right\} \), with \( \frac{\text{adx}^2}{k} > \frac{z_5}{z_6} \) then the condition 1 and 2 in lemma 1 hold , while if \( h > \frac{z_5}{z_6} \) then the condition 3 in lemma 1 holds . Therefore the point \( e_2 \) is local stable point.

**3-Optimal harvesting approach.**

This part of the article deals with the optimal harvesting amounts so that the system (3) becomes as follows

\[
\begin{align*}
    x_{n+1} &= x_n + \frac{s^9}{\Gamma(\theta+1)} \left[ x_n \left(1 - x_n \right) - \frac{x_n y_n}{(a x_n + b y_n)} - h_n x_n \right] \\
    y_{n+1} &= y_n + \frac{s^9}{\Gamma(\theta+1)} \left[ \frac{d x_n y_n}{(a x_n + b y_n)} - c y_n \right]
\end{align*}
\]  

(5)

The all parameter are the same previous interpolation, while the parameter \( h_n \) represents the control variable. We form the objective functional as follows:

\[
    J(h_n) = \sum_{n=0}^{T-1} c_1 h_n x_n - c_2 h_n
\]  

(6)

Subject to the considered system (5) the parameters \( c_1 \) and \( c_2 \) are positive constants. Now we have to find out the optimal solution \( h_n^* \) that satisfies \( J(h_n^*) = \text{Max} \{J(h_n)\} \) for all \( 0 \leq h_n \leq h_{\text{Max}} \), \( h_{\text{Max}} \) represents the maximum harvesting. We apply the Pontryagin’s Maximum Principle [1,3, 23-25] to get the necessary conditions for the optimal variable control and corresponding states.
Theorem 3
If \( h_n^* \) represents an optimal solution with the optimal corresponding states \( x_n^* \) and \( y_n^* \), then for \( n = 1,2,\ldots,T-1 \) the adjoint functions \( \lambda_n \) and \( \mu_n \) exist that satisfy:

\[
\lambda_n = c_1 h_n + \lambda_{n+1} [1 + m - 2m x - \frac{mb y^2}{\dot{y}} - mh_n] + \mu_{n+1} [\frac{mb y^2}{\dot{y}}]
\]

\[
\mu_n = \lambda_{n+1} - \frac{\max^2}{k} + \mu_{n+1} [1 - mc + \frac{\max^2}{k}]
\]

(7)

\( \lambda_T = 0, \mu_T = 0, \) and \( m = \frac{s^0}{\Gamma(\theta + 1)} \). The optimal control is given by \( h_n^* = \frac{c_1 x_n - \lambda_{n+1} x_n}{2c_2} \) for \( 0 < \frac{c_1 x_n - \lambda_{n+1} x_n}{2c_2} < h_{\text{max}} \), while \( h_n^* = h_{\text{max}} \) if \( \frac{c_1 x_n - \lambda_{n+1} x_n}{2c_2} > h_{\text{max}} \).

Proof:
The Hamiltonian function is

\[
H_n = c_1 h_n x_n - c_2 h_n^2 + \lambda_{n+1} \left[ x_n + \frac{s^0}{\Gamma(\theta + 1)} [x_n (1 - x_n) - \frac{x_n y_n}{(a x_n + by_n)} - h_n x_n] + \mu_{n+1} [y_n + \frac{s^0}{\Gamma(\theta + 1)} \left( \frac{dx y_n}{a x_n + by_n} - cy_n \right) \right].
\]

By the necessary conditions of Pontryagin maximum principle, we have for \( n = 1,2,\ldots,T-1 \).

\[
\lambda_n = c_1 h_n + \lambda_{n+1} [1 + m - 2m x - \frac{mb y^2}{\dot{y}} - mh_n] + \mu_{n+1} [\frac{mb y^2}{\dot{y}}]
\]

And \( \mu_n = \lambda_{n+1} - \frac{\max^2}{k} + \mu_{n+1} [1 - mc + \frac{\max^2}{k}] \). Now the optimal variable will be \( h_n^* = \frac{c_1 x_n - \lambda_{n+1} x_n}{2c_2} \) for \( 0 < \frac{c_1 x_n - \lambda_{n+1} x_n}{2c_2} < h_{\text{max}} \), and \( h_n^* = h_{\text{max}} \) if \( \frac{c_1 x_n - \lambda_{n+1} x_n}{2c_2} > h_{\text{max}} \).

4-Numerical results
This section verifies the effectiveness of our theoretical results, so that some numerical simulations are given. To confirm the behavior of the system (3) through the local stability of its fixed points. Some numerical simulations have been given. To confirm of the point \( e_1 = (1 - h, 0) \) is local stable point we use the following values of parameters: \( a = 0.6; b = 0.8; d = 0.3; c = 0.6; h = 0.1; \alpha = 0.98 \), and the initial point is \( (1.8, 1.9) \). Hence the condition 2 in Theorem (1) is established. Figure 1 displays the local stability of \( e_1 \).
Figure 1: Local stability of \( e_1 \) is illustrated in this figure.

For the point \( e_2 \) the values of parameters are set as follows: \( a = 0.4; b = 0.8; d = 0.33; c = 0.4; h = 0.15; \alpha = 0.98 \) and the initial point is \((0.4,0.5)\). Hence the Theorem 2 is verified, and the point is stable. This is displayed in Figure 2. Trajectories of the prey species and the predator species as a function of time which Indicates that the point \( e_2 \) is local stability. This is done in Figure 3.

Figure 2: This figure shows the point \( e_2 \) is locally stable point.
Figure 3: This figure shows the time series of prey density and predator density that indicates the local stability of $e_2$.

We use and employ iterative method to find the optimal control solution. We use an iterative algorithm. For more details we refer to [5, 14, 25]. The values of parameters as follows: $a = 0.3; b = 0.8; d = 0.45; c = 0.5; \alpha = 0.98, c_1 = 0.2, . \text{ and } c_2 = 0.2$ with initial guess $x_0 = 0.4, \text{ and } y_0 = 0.5$ for prey, and predator, respectively. We obtain the total net optimal harvesting is $J(h_n) = 0.1090$. Figure 4 shows the optimal solution variable as function of time, while Figures 5-6 indicate the effect of optimal solution and the fixed harvesting amount on the prey, predator, respectively.

Figure 4: The optimal solution of the system 5 is plotted as function of time.
Figure 5: This figure shows prey species in system (6) with control, without control, and with fixed harvest amount.

Figure 6: The predator species in system (6) is plotted with control, without control, and with fixed harvest amount.

5- Discussions and Conclusions

In this paper, a discretization of fractional-order prey-predator system with ratio-dependent predator–prey functional response has been presented and analyzed. The local stability of its fixed point is studied. Our analysis shows the considered system has three fixed points as well as the trivial fixed point is never to be stable point, while the other points are...
locally stable under certain conditions. We also conclude that the equilibrium harvesting amount as well as any constant harvesting amount cannot be the optimal solution. We can see in Figures 4 and 5 that the level of prey species density, predator density with optimal control are lower than their equilibrium level. It is also seen that the heavily harvesting will lead to increase the possibility of extinction.

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