STABILITY RESULTS FOR SOME FULLY NONLINEAR EIGENVALUE ESTIMATES

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ABSTRACT. In this paper, we give some stability estimates for the Faber-Krahn inequality relative to the eigenvalue $\lambda_k(\Omega)$ of the Hessian operator $S_k$, $1 \leq k \leq n$, in a reasonable bounded domain $\Omega$. Roughly speaking, we prove that if $\lambda_k(\Omega)$ is near to $\lambda_k(B)$, where $B$ is a ball which preserves an appropriate measure of $\Omega$, then, in a suitable sense, $\Omega$ is close to $B$.

1. INTRODUCTION

In this paper we prove some stability estimates for the eigenvalue $\lambda_k(\Omega)$ of the $k$-Hessian operator, that has the variational characterization

$$\lambda_k(\Omega) = \min \left\{ \int_\Omega (-u) S_k(D^2 u) \, dx, \ u \in \Phi_k^2(\Omega) \text{ and } \int_\Omega (-u)^{k+1} \, dx = 1 \right\}.$$ 

Here $\Omega$ is a bounded, strictly convex, open set of $\mathbb{R}^n$, $n \geq 2$, with $C^2$ boundary, $S_k(D^2 u)$, with $1 \leq k \leq n$, is the $k$-th elementary symmetric function of the eigenvalues of $D^2 u$ with $u \in C^2(\Omega)$, and $\Phi_k^2(\Omega)$ denotes the class of the admissible functions for $S_k$, the so-called $k$-convex functions (see Section 2 for the precise definitions). Notice that $S_1(D^2 u) = \Delta u$, the Laplacian operator, while $S_n(D^2 u) = \det D^2 u$, the Monge-Ampère operator.

It is known that, under suitable assumptions on $\Omega$, for this kind of operators a Faber-Krahn inequality holds, that is the eigenvalue $\lambda_k(\Omega)$ attains its minimum value on the ball $\Omega_{k-1}$, which preserves an appropriate curvature measure of $\Omega$, the $(k-1)$-th quermassintegral:

$$\lambda_k(\Omega) \geq \lambda_k(\Omega_{k-1}), \quad 1 \leq k \leq n$$

(see [7, 14]). Moreover, for $k = n$, $\lambda_n(\Omega)$ is also bounded from above by $\lambda_n(\Omega_0^*)$, with $|\Omega_0^*| = |\Omega|$ (see [5]). For sake of completeness, we recall that in the case of Neumann boundary condition, for $k = 1$, the reverse inequality in (1.1) holds (see [29, 36], [2] and [4, 11] for related results).

In [12] we give some stability estimates of (1.1), proving that

$$\frac{\lambda_k(\Omega) - \lambda_k(\Omega_{k-1})}{\lambda_k(\Omega)} \leq C_{n,k} \frac{|\Omega_{k-1}| - |\Omega|}{|\Omega_k^*|}, \quad 1 \leq k \leq n - 1,$$

for some constant $C_{n,k}$ depending only on $n$ and $k$, while for $k = n$,

$$\frac{\lambda_n(\Omega) - \lambda_n(\Omega_{n-1}^*)}{\lambda_n(\Omega)} \leq C_n \frac{|\Omega_{n-1}^*| - |\Omega|}{|\Omega_{n-1}|},$$

where $C_n$ which depends only on $n$. Roughly speaking, such inequalities state that if $\Omega$ is close to a ball with respect the $L^1$ norm, then their corresponding eigenvalues are near. Such result is in the spirit of a well-known result due to Payne and Weinberger for the Laplace operator (see [23]), and given in [6] for the $p$-Laplace operator (see also [23] for the best constant in the case $p = 2$, and [10] for the anisotropic case).

Date: March 27, 2013.
2000 Mathematics Subject Classification. 35P15, 35P30.
Key words and phrases. Eigenvalue problems, Hessian operators, stability estimates.
Viceversa, the aim of this paper is to prove some stability results which ensure that if $\lambda_k(\Omega)$ is near to $\lambda_k(\Omega^{*}_{k-1})$, then, in an appropriate sense, $\Omega$ is close to a suitable ball of $\mathbb{R}^n$ (see Section 2 for the precise statements).

There are several contributions in this direction, for the first eigenvalue of the Laplacian operator (see [22], [17]) or, more generally, for the $p$-Laplacian (see [3], [13]). In such papers, depending on the assumptions on $\Omega$, suitable notions of the distance between the set $\Omega$ and a ball are considered. In particular, under the convexity assumption on the domain, it seems natural to take into account the Hausdorff distance (see [22]), while, in a more general setting, such notion is replaced by the so-called Fraenkel asymmetry (see [3], [13]). Both arguments are considered in [17].

Dealing with convex sets, our aim is to prove some stability result for Hessian operators in the spirit of the results given in [17, 22]. In particular, we prove that for a strictly convex, smooth domain $\Omega$, such that

$$\lambda_k(\Omega) \leq \lambda_k(\Omega^{*}_{k-1})(1+\epsilon),$$

for some $\epsilon > 0$ sufficiently small, then there exist two balls $B_{t_1}$ and $B_{t_2}$ such that $B_{t_1} \subseteq \Omega \subseteq B_{t_2}$ and two suitable asymmetry coefficients of $\Omega$ with respect to $\Omega^{*}_{k-1}$ vanish when $\epsilon$ goes to zero. This will imply that the Hausdorff asymmetry of $\Omega$ is close to zero (see Section 2.4 for the precise definitions).

The paper is organized as follows. In Section 2, we recall some basic definitions of convex geometry, and the properties of symmetrization for quermassintegrals. Moreover, we summarize some useful results on the eigenvalue problem for Hessian operators. In sections 3 and 4 we state and prove the main results. We distinguish the case of the Monge-Ampère operator (see Section 3) from the case of $S_k$, $1 \leq k \leq n-1$. Our approach makes use of a quantitative version of a suitable isoperimetric inequality and a symmetrization for quermassintegral technique.

2. Notation and Preliminaries

Throughout the paper, we will denote with $\Omega$ a set of $\mathbb{R}^n$, $n \geq 2$ such that

$$\Omega$$

is a bounded, strictly convex, open set with $C^2$ boundary.

By strict convexity of $\Omega$ we mean that the Gauss curvature is strictly positive at every point of $\partial \Omega$.

Given a function $u \in C^2(\Omega)$, we denote by $\lambda(D^2 u) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ the vector of the eigenvalues of $D^2 u$. The $k$-Hessian operator $S_k(D^2 u)$, with $k = 1, 2, \ldots, n$, is

$$S_k(D^2 u) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \cdot \lambda_{i_2} \cdot \cdots \lambda_{i_k}.$$ 

Hence $S_k(D^2 u)$ is the sum of all $k \times k$ principal minors of the matrix $D^2 u$.

The $k$-Hessian operator can be written also in divergence form, that is

$$S_k(D^2 u) = \frac{1}{k} \sum_{i,j=1}^n (S_{ij}^k u_i)_j,$$

where $S_{ij}^k = \frac{\partial^2 S_k(D^2 u)}{\partial u_{ij}}$ (see for instance [32], [33], [35]).

Well known examples ok $k$-Hessian operators are $S_1(D^2 u) = \Delta u$, the Laplace operator, and $S_n(D^2 u) = \det(D^2 u)$, the Monge-Ampère operator.

It is well-known that $S_1(D^2 u)$ is elliptic. This property is not true in general for $k > 1$. As matter of fact, the $k$-Hessian operator is elliptic when it acts on the class of the so-called $k$-convex function, defined below.
**Definition 2.1.** Let $\Omega$ be as in (2.1). A function $u \in C^2(\Omega)$ is called a $k$-convex function (strictly $k$-convex) in $\Omega$ if

$$S_j(D^2u) \geq 0 \; (>0) \quad \text{for } j = 1, \ldots, k.$$  

We denote the class of $k$-convex functions in $\Omega$ such that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $u = 0$ on $\partial \Omega$ by $\Phi^k_+(\Omega)$.

Clearly, the set $\Phi^k_+(\Omega)$ coincides with the set of the convex and $C^2(\Omega)$ functions vanishing on $\partial \Omega$.

If we define with $\Gamma_k$ the following convex open cone

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n : S_1(\lambda) > 0, S_2(\lambda) > 0, \ldots, S_k(\lambda) > 0 \},$$

in [18] it is proven that $\Gamma_k$ is the cone of ellipticity of $S_k$. Hence the $k$-Hessian operator is elliptic with respect to the $k$-convex functions.

By definition, it follows that the $k$-convex functions are subharmonic in $\Omega$ and then negative in $\Omega$ if zero on $\partial \Omega$.

We go on by recalling some definitions of convex geometry which will be largely used in next sections. Standard references for this topic are [8], [28].

### 2.1. Quermassintegrals and the Aleksandrov-Fenchel inequalities

Let $K$ be a convex body, and let be $\rho > 0$. We denote by $|K|$ the Lebesgue measure of $K$, by $P(K)$ the perimeter of $K$ and by $\omega_n$ the measure of the unit ball in $\mathbb{R}^n$.

The well-known Steiner formula for the Minkowski sum is

$$|K + \rho B_1| = \sum_{i=0}^{n} \binom{n}{i} W_i(K) \rho^i.$$  

The coefficient $W_i(K)$, $i = 0, \ldots, n$, is known as the $i$-th quermassintegral of $K$. Some special cases are $W_0(K) = |K|$, $n W_1(K) = P(K)$, $W_n(K) = \omega_n$. If $K$ has $C^2$ boundary, with nonvanishing Gaussian curvature, the quermassintegrals can be related to the principal curvatures of $\partial K$. Indeed, in such a case

$$W_i(K) = \frac{1}{n} \int_{\partial K} H_{i-1} dH^{n-1}, \quad i = 1, \ldots, n.$$  

Here $H_j$ denotes the $j$-th normalized elementary symmetric function of the principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$ of $\partial K$, that is $H_0 = 1$ and

$$H_j = \binom{n-1}{j-1}^{-1} \sum_{1 \leq i_1 \leq \cdots \leq i_j \leq n-1} \kappa_{i_1} \cdots \kappa_{i_j}, \quad j = 1, \ldots, n-1.$$  

An immediate computation shows that if $B_R$ is a ball of radius $R$, then

$$W_i(B_R) = \omega_n R^{n-i}, \quad i = 0, \ldots, n.$$  

Moreover, the $i$-th quermassintegral, $0 \leq i \leq n$, rescales as

$$W_i(tK) = t^{n-i} W_i(K), \quad t > 0.$$  

The Aleksandrov-Fenchel inequalities state that

$$\left( \frac{W_i(K)}{\omega_n} \right)^{\frac{1}{n-i}} \geq \left( \frac{W_j(K)}{\omega_n} \right)^{\frac{1}{n-j}}, \quad 0 \leq i < j \leq n-1,$$

where the inequality is replaced by an equality if and only if $K$ is a ball.

In what follows, we use the Aleksandrov-Fenchel inequalities for particular values of $i$ and $j$. If $i = 1$, and $j = k - 1$, we have that

$$W_{k-1}(K) \geq \omega_n^{\frac{k-1}{n}} n^{-\frac{n-k+1}{n-1}} P(K)^{\frac{n-k+1}{n-1}}, \quad 3 \leq k \leq n.$$
When $i = 0$ and $j = 1$, we have the classical isoperimetric inequality:

$$P(K) \geq n\omega_n^{1/k} |K|^{1-1/k}.$$ 

Moreover, if $i = k - 1$ and $j = k$, we have

$$W_k(K) \geq \omega_n^{-(n+k)/n+k} W_{k-1}(K)^{n+k-1}.$$ 

It can be also shown a derivation formula for quermassintegral of level sets of a function $u \in \Phi_k^2(\Omega)$ (see [26]):

$$-\frac{d}{dt} W_k(\Omega_t) = \frac{n-k}{n} \int_{\Omega_t} H_k(\Sigma_t)|Du|^n d\mathcal{H}^{n-1},$$

where $\Sigma_t$ is the boundary of $\Omega_t = \{ -u > t \}$. Moreover, we recall the following equality (see [26] again):

$$\int_{\Omega_t} S_k(D^2u) dx = \frac{1}{k} \int_{\Sigma_t} H_{k-1}|Du|^k d\mathcal{H}^{n-1}.$$ 

2.2. Rearrangements for quermassintegrals. Now we recall some basic facts on rearrangements for quermassintegrals. For an exhaustive treatment of the properties of such rearrangements we refer the reader, for example, to [30], [34], [31].

Let $1 \leq k \leq n$, and denote by $\Omega_{k-1}^\ast$ the ball centered at the origin and with the same $W_{k-1}$ measure than $\Omega$, that is $W_{k-1}(\Omega_{k-1}^\ast) = W_{k-1}(\Omega)$.

The $(k-1)$-symmetrand of a function $u \in \Phi_k(\Omega)$, $k = 1, \ldots, n$, is the radially symmetric increasing function $u_{k-1}^\ast$, defined in the ball $\Omega_{k-1}^\ast$, which preserves the $W_{k-1}$ measure of the level sets of $u$. More precisely, we have that, for $x \in \Omega_{k-1}^\ast$,

$$u_{k-1}^\ast(x) = \inf \{ t \geq 0 : W_{k-1}(\Omega_t) \leq \omega_n |x|^{n-k+1}, Du \neq 0 \text{ on } \Sigma_t \},$$

where $\Sigma_t = \partial \Omega_t = \{ x \in \Omega : -u(x) > t \}$, with $t \geq 0$.

We stress that, for $k = 1$, $u_0^\ast(x)$ coincides with the classical Schwarz symmetrand of $u$, while for $k = 2$, $u_1^\ast(x)$ is the rearrangement of $u$ which preserves the perimeter of the level sets of $u$.

Denoting with $R$ the radius of $\Omega_{k-1}^\ast$, the following statements hold true (see [31], [34]):

1. $u_{k-1}^\ast(x) = \rho(r)$ for $r = |x|$, we have $\rho(0) = \min_{\Omega} u$ and $\rho(R) = 0$,
2. $\rho(r)$ is a negative and increasing function on $[0, R]$,
3. $\rho(r) \in C^{0,1}([0, R])$ and moreover $0 \leq \rho'(r) \leq \sup_{\Omega} |Du|$ almost everywhere.

If the function $u$ has convex level sets, the Aleksandrov–Fenchel inequalities (2.3) imply that $|\{-u > t\}| \leq |\{-u_{k-1}^\ast > t\}|$ and then, for any $p \geq 1$,

$$\|u\|_{L^p(\Omega)} \leq \|u_{k-1}^\ast\|_{L^p(\Omega_{k-1}^\ast)},$$

while, by property (1),

$$\|u\|_{L^\infty(\Omega)} = \|u_{k-1}^\ast\|_{L^\infty(\Omega_{k-1}^\ast)}.$$ 

Now, it is possible to define the following functional associated to the $k$-Hessian operator, known as $k$-Hessian integral:

$$I_k[u, \Omega] = \int_{\Omega} (-u) S_k(D^2u) dx.$$ 

In the radial case the Hessian integrals can be defined as follows:

$$I_k[u_{k-1}^\ast, \Omega_{k-1}^\ast] = \binom{n}{k} \omega_n \int_0^R f^{k+1}(\omega_n r^{n-k+1}) r^{n-k} dr$$

where $f(\omega_n |x|^{n-k+1}) = |\nabla u_{k-1}^\ast(x)|$. 
Finally we recall that for the Hessian integrals the following extension of Pólya-Szegő principle holds (see \[31\] [34]):

\begin{equation}
I_k[u, \Omega] \geq I_k[u^*_{k-1}, \Omega^*_{k-1}], \quad p \geq 1.
\end{equation}

2.3. Eigenvalue problems for $S_k$. In this subsection we give a quick review on the main properties of eigenvalues and eigenfunctions of the $k$-Hessian operators, namely the couples $(\lambda, u)$ which solve

\begin{equation}
\begin{cases}
S_k(D^2u) = \lambda(-u)^k & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}

The following existence result holds (see \[20\] for $k = n$, and \[35\], \[15\] in the general case):

**Theorem 2.1.** Let $\Omega$ as in (2.1). Then, there exists a positive constant $\lambda_k(\Omega)$ depending only on $n, k$, and $\Omega$, such that problem (2.8) admits a solution $u \in C^2(\Omega) \cap C^{1,1}(\Omega)$, negative in $\Omega$, for $\lambda = \lambda_k(\Omega)$ and $u$ is unique up to positive scalar multiplication. Moreover, $\lambda_k(\Omega)$ has the following variational characterization:

\[\lambda_k(\Omega) = \min_{\substack{u \in \Theta_k^1(\Omega), u \neq 0}} \frac{\int_\Omega (-u)S_k(D^2u) \, dx}{\int_\Omega (-u)^{k+1} \, dx}.\]

As matter of fact, if $k < n$ the above theorem holds under a more general assumption on $\Omega$, namely requiring that $\Omega$ is strictly $k$-convex (see \[35\], \[15\]).

As matter of fact, we observe that if $k = 1$, or $k = n$, $\lambda_k(\Omega)$ coincides respectively with the first eigenvalue of the Laplacian operator, or with the eigenvalue of the Monge-Ampère operator.

A simple but useful property of the eigenvalue $\lambda_k(\Omega)$ is that it rescales as

\begin{equation}
\lambda_k(t\Omega) = t^{-2k}\lambda_k(\Omega), \quad t > 0.
\end{equation}

If $k = 1$, the well-known Faber-Krahn inequality states that

\[\lambda_1(\Omega) \geq \lambda_1(\Omega^\#),\]

where $\Omega^\#$ is the ball centered at the origin with the same Lebesgue measure of $\Omega$. Moreover, the equality holds if $\Omega = \Omega^\#$.

In \[7\], \[14\] it is proved that if $k = n$ and $\Omega$ is a bounded strictly convex open set, then

\[\lambda_n(\Omega) \geq \lambda_n(\Omega^*_{n-1}).\]

In general, in \[14\] it is proved that if $\Omega$ is a strictly convex set such that the eigenfunctions have convex level sets, then, for $2 \leq k \leq n$,

\begin{equation}
\lambda_k(\Omega) \geq \lambda_k(\Omega^*_{k-1}).
\end{equation}

2.4. Asymmetry measures and isoperimetric deficit. A purpose of this paper is to prove that the difference between the two sides in (2.10) controls the exterior and interior deficiencies, defined as follows (see also \[17\] for $k = 1$). Given $\Omega$ bounded nonempty domain of $\mathbb{R}^n$, denoted by $R$ the radius of the ball $\Omega^*_{k-1}$, then the exterior and interior $k$-deficiency of $\Omega$ are, respectively, the nonnegative numbers

\begin{equation}
D_k(\Omega) = \frac{R_\Omega}{R} - 1, \quad d_k(\Omega) = 1 - \frac{r_\Omega}{R},
\end{equation}

where $r_\Omega$ and $R_\Omega$ are the inradius and the circumradius of $\Omega$. Such numbers give a measure of the asymmetry of $\Omega$ with respect to the ball with the same $(k-1)$-quermassintegral than $\Omega$. Furthermore, the deficiency of $\Omega$ is

\[\Delta(\Omega) = \frac{R_\Omega}{r_\Omega} - 1.\]
In order to have a measure of the asymmetry of $\Omega$ in terms of the Hausdorff distance $d$, we define the following coefficient:

\begin{equation}
\delta_{H}(\Omega) = \inf\{d(\Omega, \Omega_{n-1}^* + x_0), x_0 \in \mathbb{R}^n\}.
\end{equation}

We refer to $\delta_{H}$ as the Hausdorff asymmetry of $\Omega$.

In the class of convex sets, it is possible to obtain some stability estimates for the Aleksandrov-Fenchel inequalities \textit{2.3}. More precisely, if $s$ denotes the Steiner point of $\Omega$, then in \cite{16} it has been proved that

\begin{equation}
\begin{aligned}
d(\Omega, \Omega_{n-1}^* + s)^{(n+3)/2} \leq \tilde{C}_1 \frac{P(\Omega)^{(n^2-3)/2}}{|\Omega|^{(n+3)(n-2)/2}} \left[ \left( \frac{P(\Omega)}{\omega_n} \right)^n - \left( \frac{|\Omega|}{\omega_n} \right)^{n-1} \right], \\
d(\Omega, \Omega_{n-1}^* + s)^{(n+3)/2} \leq \tilde{C}_1 \frac{W_{n-2}(\Omega)W_{n-1}(\Omega)^{n-1}}{W_{k-1}(\Omega)^{n-k}} \left( \frac{W_{k}(\Omega)^{n-k+1}}{\omega_n} - W_{k-1}(\Omega)^{n-k} \right),
\end{aligned}
\end{equation}

where $\tilde{C}_1$ are two constants depending only on $n$, which can be explicitly determined. These estimates justify the definition of $\delta_{H}$.

As matter of fact, in \cite{16} it is observed that, the inequalities (2.13) can be rewritten as a Bonnesen-type inequality in terms of the inradius $r_{\Omega}$ and the circumradius $R_{\Omega}$ of $\Omega$:

\begin{equation}
\begin{aligned}
(R_{\Omega} - r_{\Omega})^{(n+3)/2} \leq \tilde{C}_1 \frac{P(\Omega)^{(n^2-3)/2}}{|\Omega|^{(n+3)(n-2)/2}} \left[ \left( \frac{P(\Omega)}{\omega_n} \right)^n - \left( \frac{|\Omega|}{\omega_n} \right)^{n-1} \right], \\
(R_{\Omega} - r_{\Omega})^{(n+3)/2} \leq \tilde{C}_1 \frac{W_{n-2}(\Omega)W_{n-1}(\Omega)^{n-1}}{W_{k-1}(\Omega)^{n-k}} \left( \frac{W_{k}(\Omega)^{n-k+1}}{\omega_n} - W_{k-1}(\Omega)^{n-k} \right).
\end{aligned}
\end{equation}

3. The case of the Monge-Ampère operator

In this section we consider the eigenvalue problem for the Monge-Ampère operator,

\begin{equation}
\begin{aligned}
-\det D^2u &= \lambda(-u)^n \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}

and we prove the first stability result, stated below.

\begin{Theorem}
Let $\Omega \subset \mathbb{R}^n$ be as in (2.1) such that

\begin{equation}
\lambda_n(\Omega) \leq (1 + \varepsilon) \lambda_n(\Omega_{n-1}^*),
\end{equation}

where $\varepsilon > 0$ is sufficiently small and $\Omega_{n-1}^*$ is the ball such that $W_{n-1}(\Omega) = W_{n-1}(\Omega_{n-1}^*)$. Then, if $\delta_{H}(\Omega)$ is the Hausdorff asymmetry (2.12), it holds that

\begin{equation}
\delta_{H}(\Omega) \leq C_n \varepsilon^{(n+1)(n+3)},
\end{equation}

where $C_n$ is a constant which depend only on $n$. Moreover, denoting by $d_n(\Omega)$ and $D_n(\Omega)$, respectively, the interior and exterior $n$-deficiency of $\Omega$ as in (2.11), we have the following:

(1) If $n = 2$, then

\begin{equation}
d_2(\Omega) \leq C_2 \sqrt{\varepsilon}, \quad D_2(\Omega) \leq C_2 \sqrt{\varepsilon},
\end{equation}

where $C_2$ denotes a positive constant which depends only on the dimension $n = 2$.

(2) If $n \geq 3$, then

\begin{equation}
d_n(\Omega) \leq C_n \varepsilon^{n+2}, \quad D_n(\Omega) \leq C_n \varepsilon^{(n+1)(n+3)},
\end{equation}

where $C_n$ depends only on $n$.

\begin{Remark}
The estimates (3.3) and (3.4) can be read as

\begin{equation}
\frac{P(\Omega) - P(B_{r_{\Omega}})}{P(\Omega)} \leq C_2 \sqrt{\varepsilon}, \quad \frac{P(B_{R_{\Omega}}) - P(\Omega)}{P(\Omega)} \leq C_2 \sqrt{\varepsilon},
\end{equation}

where $B_{r_{\Omega}}$ and $B_{R_{\Omega}}$ are the balls of radii $r_{\Omega}$ and $R_{\Omega}$, respectively.
\end{Remark}
Lemma 3.1. Under the hypotheses of Theorem 3.1, if \( u \) is the eigenfunction of \( H \), then for any \( \delta > 0 \) such that

\[
\| u \|_{L^{n+1}(\Omega)} = 1,
\]

it follows that

\[
W_{n-1}(\Omega_\delta) \geq W_{n-1}(\Omega)(1 - \max\{\varepsilon, 2\delta|\Omega|^{\frac{1}{n+1}}\}).
\]

Proof. For \( \delta > 0 \), we compute the Rayleigh quotient of the function \( \phi = u + \delta \) in \( \Omega_\delta \). Then,

\[
\lambda_n(\Omega_\delta) \leq \frac{\int_{\Omega_\delta}(-\phi)\det D^2\phi \, dx}{\int_{\Omega_\delta}(-\phi)^{n+1} \, dx} = \frac{\int_{\Omega_\delta}(-u - \delta)\det D^2u \, dx}{\int_{\Omega_\delta}(-u - \delta)^{n+1} \, dx}.
\]

Moreover, being \( u \) a solution of (3.1) with \( \lambda = \lambda_n(\Omega) \), we get, by Hölder inequality, and recalling that \( \int_{\Omega_\delta}(-u)^{n+1} \, dx = 1 \), that

\[
\int_{\Omega_\delta}(-u - \delta)\det D^2u \, dx = \lambda_n(\Omega) \int_{\Omega_\delta}(-u - \delta)(-u)^n \, dx
\leq \lambda_n(\Omega) \left( \int_{\Omega_\delta}(-u - \delta)^{n+1} \, dx \right)^{\frac{1}{n+1}} \left( \int_{\Omega_\delta}(-u)^n \, dx \right)^{\frac{n}{n+1}}
\leq \lambda_n(\Omega) \left( \int_{\Omega_\delta}(-u - \delta)^{n+1} \, dx \right)^{\frac{1}{n+1}}.
\]

Hence, combining the above estimate with (3.5) it follows that

\[
\lambda_n(\Omega_\delta) \leq \lambda_n(\Omega) \left( \int_{\Omega_\delta}(-u - \delta)^{n+1} \, dx \right)^{-\frac{n}{n+1}}
\]

On the other hand, by Minkowski inequality and choosing \( \delta < \frac{1}{2}|\Omega|^{\frac{1}{n+1}} \), we obtain that

\[
\left( \int_{\Omega_\delta}(-u - \delta)^{n+1} \, dx \right)^{\frac{1}{n+1}} \geq \left( \int_{\Omega_\delta}(-u)^{n+1} \, dx \right)^{\frac{1}{n+1}} - \left( \int_{\Omega_\delta}\delta^{n+1} \, dx \right)^{\frac{1}{n+1}}
\geq \left( 1 - \int_{\Omega_\delta}\delta^{n+1} \, dx \right)^{\frac{1}{n+1}} - \delta|\Omega_\delta|^{\frac{1}{n+1}}
\geq \left( 1 - \delta^{n+1}(|\Omega| - |\Omega_\delta|) \right)^{\frac{1}{n+1}} - \delta|\Omega_\delta|^{\frac{1}{n+1}}
\geq 1 - \delta(|\Omega| - |\Omega_\delta|)^{\frac{1}{n+1}} - \delta|\Omega_\delta|^{\frac{1}{n+1}} \geq 1 - 2\delta|\Omega|^{\frac{1}{n+1}}.
\]

Hence, from (3.6), (3.2) and Faber-Krahn inequality it follows that

\[
\lambda_n((\Omega_\delta)^*_n) \leq \lambda_n(\Omega_\delta) \leq (1 + \varepsilon)\lambda_n(\Omega^*_{n-1}(1 - 2\delta|\Omega|^{\frac{1}{n+1}})^{-n}
\]

which implies, by (2.9) that

\[
\left( \frac{W_{n-1}(\Omega_\delta)}{W_{n-1}(\Omega)} \right)^{2n} = \frac{\lambda_n(\Omega^*_n)}{\lambda_n((\Omega_\delta)^*_n)} \geq \frac{(1 - 2\delta|\Omega|^{\frac{1}{n+1}})^n}{1 + \varepsilon}.
\]
where we used that the balls $\Omega_{n-1}^*\setminus (\Omega_1)^*_{n-1}$ preserve, respectively, the $(n-1)$-th quermassintegral of $\Omega$ and $\Omega_1$. Hence, by (3.7) we get that
\[ \frac{W_{n-1}(\Omega_2)}{W_{n-1}(\Omega)} \geq \left( 1 - \frac{\epsilon + 2\delta |\Omega|^{\frac{1}{n+1}}}{1 + \epsilon} \right)^{1/2} \geq 1 - \max\{\epsilon, 2\delta |\Omega|^{\frac{1}{n+1}}\}, \]
obtaining the thesis.

The second lemma we need is the following.

**Lemma 3.2.** Under the hypotheses of Theorem 3.1 if $\Omega_t = \{-u > t\}$, and $u$ is the eigenfunction of (3.1) in $\Omega$ such that $\|u\|_{L^{n+1}(\Omega)} = 1$, then
\[ \lambda_n(\Omega) - \lambda_n(\Omega_{n-1}^*) \geq \int_{\Omega} (-u) \det D^2 u \, dx - \int_{\Omega_{n-1}^*} (-u_{n-1}^*) \det D^2 u_{n-1}^* \, dx \]
\[ \geq \frac{\int_{\Omega_{n-1}^*} (-u_{n-1}^*) \det D^2 u_{n-1}^* \, dx}{\int_{\Omega_{n-1}^*} (-u_{n-1}^*)^{n+1} \, dx} \left( \int_{\Omega_{n-1}^*} (-u_{n-1}^*)^{n+1} \, dx - 1 \right) \]

On the other hand, recalling that $u$ has normalized $L^{n+1}$ norm, the coarea formula and an integration by parts give that
\[ \int_{\Omega_{n-1}^*} (-u_{n-1}^*)^{n+1} \, dx - 1 = (n+1) \int_0^{+\infty} t^n \left( |\{-u_{n-1}^* > t\}| - |\{-u > t\}| \right) \, dt \]
\[ = (n+1) \omega_n^{1-n} \int_0^{+\infty} t^n (W_{n-1}(\Omega_t)^n - \omega_n^{n-1}|\Omega_t|) \, dt \]
Hence, joining (3.9) with the above equality, and using (3.2) we obtain that
\[ \int_0^{+\infty} t^n (W_{n-1}(\Omega_t)^n - \omega_n^{n-1}|\Omega_t|) \, dt \leq \frac{\omega_n^{n-1}}{n+1} \epsilon, \]
that is the thesis.

Last lemma plays a key role in order to obtain that the constant $C_n$ involved in (1) and (2) in Theorem 3.1 is independent on $\Omega$.

**Lemma 3.3.** Under the hypotheses of Theorem 3.1 it holds that
\[ |\Omega| \geq \tilde{C}_n [W_{n-1}(\Omega)]^n, \]
where $\tilde{C}_n$ denotes a positive constant depending only on $n$.

**Proof.** Let $u$ be an eigenfunction of (3.1) corresponding to the eigenvalue $\lambda = \lambda_n(\Omega)$. Then
\[ \det D^2 u = \lambda (-u)^n \quad \text{in} \ \Omega. \]
Integrating both sides in (3.10) on the level set \( \Omega_t = \{-u > t\} \), and denoting by \( \Sigma_t = \partial \Omega_t = \{-u = t\} \), we have

\[
(3.11) \quad \int_{\Omega_t} \det D^2u \, dx = \frac{1}{n} \int_{\Sigma_t} H_{n-1} |Du|^n \, d\mathcal{H}^{n-1} \geq
\]

\[
\geq \frac{1}{n} \left( \int_{\Sigma_t} H_{n-1} |Du|^{-1} \, d\mathcal{H}^{n-1} \right)^{n-1} = \frac{1}{n} \left( \frac{\omega_n}{\lambda_n} \right)^{n-1}.
\]

Last inequality follows by the Hölder inequality and the properties of quermassintegrals. Moreover, being \(|\Omega_t| \leq |\Omega|\), we have

\[
(3.12) \quad \left( \int_{\Omega_t} (-u)^n \, dx \right)^{\frac{1}{n}} \leq |\Omega|^{\frac{1}{n}} \|u\|_{L^\infty(\Omega)}.
\]

Putting together (3.11) and (3.12), by (3.10) we get

\[-\frac{d}{dt} W_{n-1}(\Omega_t) \geq n \omega_n^{1+\frac{1}{n}} \lambda^{-\frac{1}{n}} |\Omega|^{-\frac{1}{n}} \|u\|_{L^\infty(\Omega)}^{-1},\]

and, integrating between 0 and \( \|u\|_{L^\infty(\Omega)}^{-1} \),

\[W_{n-1}(\Omega) \geq n \omega_n^{1+\frac{1}{n}} \lambda^{-\frac{1}{n}} |\Omega|^{-\frac{1}{n}},\]

that is, being \( \lambda = \lambda_n(\Omega) \leq (1 + \epsilon) \lambda_n(\Omega_{n-1}^*) \),

\[
(3.13) \quad |\Omega|^{\frac{1}{n}} \geq n \omega_n^{1+\frac{1}{n}} W_{n-1}(\Omega)^{-1} \lambda_n(\Omega_{n-1}^*)^{-\frac{1}{n}} (1 + \epsilon)^{-\frac{1}{n}}.
\]

As matter of fact, being \( W_{n-1}(\Omega) = W_{n-1}(\Omega_{n-1}^*) \), properties (2.9) and (2.2) give that

\[
\lambda_n(\Omega_{n-1}^*) = \left( \frac{W_{n-1}(\Omega)}{\omega_n} \right)^{-2n} \lambda_n(B),
\]

where \( B = \{|x| < 1\} \). Then (3.13) becomes

\[|\Omega|^{\frac{1}{n}} \geq n \omega_n^{1+\frac{1}{n}} W_{n-1}(\Omega)^{-1} \lambda_n(B)^{-\frac{1}{n}} (1 + \epsilon)^{-\frac{1}{n}},\]

and this concludes the proof. \( \square \)

Now we are in position to prove the main theorem of this section.

**Proof of the Theorem 3.7** First of all, we observe that the quotient

\[\frac{W_{n-1}(K) - W_{n-1}(L)}{W_{n-1}(K)}\]

is rescaling invariant, hence we suppose that \( W_{n-1}(\Omega) = 1 \). Consequently, by Lemma 3.3 and the Aleksandrov-Fenchel inequality, we have that there exists two positive constants \( c_1(n) \) and \( c_2(n) \), which depend only on the dimension, such that

\[
(3.14) \quad c_1(n) \leq |\Omega| \leq c_2(n).
\]

For \( \delta \) as in Lemma 3.1 by (3.8) we get that

\[
\inf_{0 \leq t \leq \delta} \left\{ W_{n-1}(\Omega_t)^n - \omega_n^{n-1} |\Omega_t| \right\} \leq \frac{n+1}{\delta^{n+1}} \int_0^\delta t^n (W_{n-1}(\Omega_t)^n - \omega_n^{n-1} |\Omega_t|) \, dt
\]

\[
\leq \omega_n^{n-1} \frac{\epsilon}{\delta^{n+1}} = \omega_n^{n-1} \sqrt{\epsilon},
\]

where we finally choose \( \delta^{n+1} = \sqrt{\epsilon} \). Hence, this gives that there exists \( 0 \leq \tau \leq \delta \) such that

\[
(3.15) \quad W_{n-1}(\Omega_\tau)^n \leq \omega_n^{n-1} |\Omega_\tau| + \omega_n^{n-1} \sqrt{\epsilon}.
\]
Case $n = 2$. In such a case, (3.15) becomes

$$\frac{P(\Omega_\tau)^2}{4\pi} - |\Omega_\tau| \leq \sqrt\epsilon.$$ 

Then, denoting by $r_\tau$ and $R_\tau$ the inradius and the circumradius of $\Omega_\tau$ respectively, and by $\rho_\tau$ the radius of $\Omega_\tau$, using the Bonnesen inequality (see for example [24] and [112] for some related questions) we have

$$(\rho_\tau - r_\tau)^2 \leq (R_\tau - r_\tau)^2 \leq \sqrt\epsilon.$$ 

Being $2\pi\rho_\tau = P(\Omega)$, we have by Lemma 3.1 for $\epsilon$ sufficiently small, that

$$r_\tau \geq \frac{P(\Omega_\tau)}{2\pi} - \sqrt\epsilon \geq \frac{P(\Omega)}{2\pi} \left(1 - 2\sqrt\epsilon|\Omega|^{\frac{1}{n}}\right) - \sqrt\epsilon \geq R(1 - C_2|\Omega|^{\frac{1}{n}}\sqrt\epsilon),$$

where $R = \frac{P(\Omega)}{2\pi}$ is the radius of $\Omega_\tau^*$ and $C_2$ denotes a constant which depends only on the dimension $n = 2$. Being $r_\tau < r_\Omega$, by (3.14) we have that

$$(3.16) \quad d_2(\Omega) \leq 1 - \frac{r_\tau}{R} \leq C_2|\Omega|^{\frac{1}{n}}\sqrt\epsilon \leq C_2\sqrt\epsilon,$$

where $B_{r_\tau}$ is a ball of radius $r_\tau$ contained in $\Omega$. Then, by (3.16) and being $P(\Omega) = 2$, we have that

$$(3.17) \quad \left(\frac{P^2(\Omega)}{4\pi} - |\Omega|\right)^{\frac{1}{2}} \leq \left(\frac{P(B_{r_\Omega}) + C_2\sqrt\epsilon}{4\pi} - |B_{r_\Omega}|\right)^{\frac{1}{2}} \leq C_2\sqrt\epsilon,$$

where last inequality follows being $P(B_{r_\Omega}) \leq P(\Omega) = 2$. Then, (3.17), (3.14) and (2.13) give

$$\delta_H(\Omega) \leq C_2\sqrt\epsilon.$$ 

On the other hand, applying to (3.17) the well-known Bonnesen inequality, we get that

$$D_2(\Omega) \leq 2\pi(R_\Omega - r_\Omega) \leq C_2\sqrt\epsilon.$$ 

Case $n > 2$. From (3.15) and the Aleksandrov-Fenchel inequalities (2.4) with $k = n$, we have that

$$\frac{P(\Omega_\tau)^n}{n^n\omega_n} \leq |\Omega_\tau|^{1/n} + \sqrt\epsilon.$$ 

Hence, by (3.14) and for $\epsilon$ sufficiently small, an elementary inequality gives that

$$\frac{P(\Omega_\tau)^n}{n^n\omega_n} \leq |\Omega_\tau|^{1/n - 1} + C_n\sqrt\epsilon.$$ 

Then, applying the stability result (2.14), and using again (3.14), it follows that

$$(3.18) \quad (R_\tau - r_\tau)^{(n+3)/2} \leq C_n\frac{P(\Omega)^{(n^2 - 3)/2}}{|\Omega|^{(n+3)(n-2)/2}}\sqrt\epsilon \leq C_n\sqrt\epsilon,$$

where, as before, $R_\tau$ and $r_\tau$ are, respectively, the circumradius and the inradius of $\Omega_\tau$.

Now, let $\rho_\tau$ be the radius of the ball $(\Omega_\tau)^*_{\rho_\tau}$ having the same $W_{n-1}$ measure of $\Omega_\tau$. Similarly as before, being $\rho_\tau < R_\tau$, by (3.18), Lemma 3.1 and (3.14), for $\epsilon$ sufficiently small we have

$$r_\tau \geq \rho_\tau - C_n\epsilon^{1/n+3} = \frac{W_{n-1}(\Omega_\tau)}{\omega_n} - C_n\epsilon^{1/n+3} \geq \omega_n^{-1}W_{n-1}(\Omega_\tau)\left(1 - 2\epsilon^{1/(2n+2)}|\Omega|^{\frac{1}{n+3}}\right) - C_n\epsilon^{1/n+3} \geq$$

$$\geq R\left(1 - C_n\epsilon^{1/(2n+2)}\right),$$
where \( R = \omega_n^{-1}W_{n-1}(\Omega) \) is the radius of the ball \( \Omega_{n-1}^* \). Denoting again with \( r_\Omega \) the inradius of \( \Omega \), we have that \( r_\tau \leq r_\Omega \) and
\[
d_n(\Omega) \leq 1 - \frac{r_\tau}{R} \leq C_n \varepsilon^{1/(2n+2)}.
\]
As matter of fact, by the Aleksandrov-Fenchel inequalities, (3.19) and being \( W_{n-1}(\Omega) = 1 \), it follows that (3.20)
\[
\left[ \left( \frac{P(\Omega)}{n\omega_n} \right)^n - \left( \frac{|\Omega|}{\omega_n} \right)^{n-1} \right]^{\frac{2}{n+3}} \leq \left[ \left( \frac{W_{n-1}(\Omega)^n}{\omega_n^n} \right)^{n-1} - \left( \frac{|\Omega|}{\omega_n} \right)^{n-1} \right]^{\frac{2}{n+3}} \leq \left[ \left( \frac{W_{n-1}(B_{\Omega}) + C_n \varepsilon^{\frac{1}{n+3}}}{\omega_n} \right)^{n(n-1)} - \left( \frac{|B_{\Omega}|}{\omega_n} \right)^{n-1} \right]^{\frac{2}{n+3}} \leq \left[ \left( \frac{W_{n-1}(B_{\Omega})}{\omega_n} \right)^{n(n-1)} + C_n \varepsilon^{\frac{1}{n+3}} - \left( \frac{|B_{\Omega}|}{\omega_n} \right)^{n-1} \right]^{\frac{2}{n+3}} = C_n \varepsilon^{\frac{1}{n+1(n+3)}}.
\]
Hence, applying (2.13), from (3.20), (3.14) and being \( W_{n-1}(\Omega) = \omega_n R = 1 \), we get that
\[
\delta_H(\Omega) \leq C_n \varepsilon^{\frac{1}{(n+1)(n+3)}},
\]
while applying (2.14), we get
\[
D_n(\Omega) \leq \omega_n (R_\Omega - r_\Omega) \leq C_n \varepsilon^{\frac{1}{(n+1)(n+3)}},
\]
and this concludes the proof.

**Remark 3.2.** Under the assumption of Theorem 3.1 from (3.20) and (3.19) an estimate for the deficiency of \( \Omega \) holds, that is
\[
\Delta(\Omega) \leq C_n \varepsilon^{\frac{1}{(n+1)(n+3)}}.
\]

### 4. The case of the \( k \)-Hessian operator, \( 1 \leq k \leq n - 1 \)

In this section we consider the eigenvalue problem related to the \( k \)-Hessian operators, \( 1 \leq k \leq n - 1 \), namely
\[
\begin{align*}
S_k(D^2u) = \lambda(-u)^k & \quad \text{in } \Omega, \\
u = 0 & \quad \text{on } \partial\Omega,
\end{align*}
\]

obtaining the stability result as follows.

**Theorem 4.1.** Let \( 1 \leq k \leq n - 1 \), and \( \Omega \subset \mathbb{R}^n \) be as in (2.1) such that
\[
\lambda_k(\Omega) \leq (1 + \varepsilon)\lambda_k(\Omega_{k-1}^*),
\]
where \( \varepsilon > 0 \) is sufficiently small and \( \Omega_{k-1}^* \) is the ball such that \( W_{k-1}(\Omega) = W_{k-1}(\Omega_{k-1}^*) \). Moreover we suppose that the eigenfunctions related to \( \lambda_k(\Omega) \) have convex level sets. Then,
\[
\delta_H(\Omega) \leq C_{n,k} \varepsilon^{\frac{2}{n+3}},
\]
and
\[
d_k(\Omega) \leq C_{n,k} \varepsilon^\alpha, \quad D_k(\Omega) \leq C_{n,k} \varepsilon^{\frac{2}{n+3}},
\]
where \( \alpha = \max \left\{ \frac{1}{k+1}, \frac{2k}{(k+1)(n+3)} \right\} \), \( C_{n,k} \) is a positive constant which depends only on \( n \) and \( k \), and \( d_k(\Omega) \) and \( D_k(\Omega) \) are, respectively, the interior and exterior \( k \)-deficiency of \( \Omega \) as in (2.11).
Remark 4.1. As observed in Section 2.3, the additional hypothesis on the convexity of the level sets of the eigenfunctions corresponding to $\lambda_k(\Omega)$ is necessary to have that a Faber-Krahn inequality holds. On the other hand this assumption seems to be natural. Indeed, for $k = 1$ this is due to the Korevaar concavity maximum principle (see [19]), while it is trivial for $k = n$. For the $k$-Hessian operators, at least in the case $n = 3$ and $k = 2$, it is in [21] and [27] is proved that if $\Omega$ is sufficiently smooth, the eigenfunctions of $S_2$ have convex level sets. Up to our knowledge, the general case is an open problem.

Remark 4.2. Similarly as observed in Remark 3.1 by the estimates (4.2) we can obtain that
\[
\frac{W_{k-1}(\Omega) - W_{k-1}(B_{\delta n})}{W_{k-1}(\Omega)} \leq C_{n,k} e^{\alpha}, \quad \frac{W_{k-1}(B_{\delta n}) - W_{k-1}(\Omega)}{W_{k-1}(\Omega)} \leq C_{n,k} e^{\frac{2\alpha}{\delta n}}.
\]

Similarly to the case of the Monge-Ampère operator, to give the proof of Theorem 4.1 we first consider some preliminary results.

Lemma 4.1. Under the hypotheses of Theorem 4.1 if $u$ is the eigenfunction of $S_k$ in $\Omega$ such that $\|u\|_{L^{k+1}(\Omega)} = 1$, then for any $\delta$ such that $0 < \delta < \frac{1}{2} |\Omega|^{-\frac{1}{k+1}}$, we have
\[
W_{k-1}(\Omega_\delta) \geq W_{k-1}(\Omega)^{1 - (n - k + 1) \max \{\epsilon, 2\delta|\Omega|^{-\frac{1}{k+1}}\}}.
\]

Proof. For $\delta > 0$, we compute the Rayleigh quotient of the function $\phi = u + \delta$ in $\Omega_\delta$. Then,
\[
\lambda_k(\Omega_\delta) \leq \frac{\int_{\Omega_\delta} (-\delta) S_k(D^2u) \, dx}{\int_{\Omega_\delta} (-\delta)^{k+1} \, dx} = \frac{\int_{\Omega_\delta} (-u - \delta) S_k(D^2u) \, dx}{\int_{\Omega_\delta} (-u - \delta)^{k+1} \, dx}.
\]
\[
\int_{\Omega_\delta} (-u - \delta) S_k(D^2u) \, dx = \lambda_k(\Omega) \int_{\Omega_\delta} (-u - \delta)(-u)^k \, dx
\]
\[
\leq \lambda_k(\Omega) \left( \int_{\Omega_\delta} (-u - \delta)^{k+1} \, dx \right)^{\frac{1}{k+1}} \left( \int_{\Omega_\delta} (-u)^{k+1} \, dx \right)^{\frac{k}{k+1}}
\]
\[
\leq \lambda_k(\Omega) \left( \int_{\Omega_\delta} (-u - \delta)^{k+1} \, dx \right)^{\frac{k}{k+1}}.
\]

Hence, combining the above estimate with (4.4) it follows that
\[
\lambda_k(\Omega_\delta) \leq \lambda_k(\Omega) \left( \int_{\Omega_\delta} (-u - \delta)^{k+1} \, dx \right)^{-\frac{k}{k+1}}
\]

On the other hand, by Minkowski inequality and choosing $\delta < \frac{1}{2} |\Omega|^{-\frac{1}{k+1}}$, we obtain that
\[
\left( \int_{\Omega_\delta} (-u - \delta)^{k+1} \, dx \right)^{\frac{1}{k+1}} \geq \left( \int_{\Omega_\delta} (-u)^{k+1} \, dx \right)^{\frac{1}{k+1}} - \left( \int_{\Omega_\delta} \delta^{k+1} \, dx \right)^{\frac{1}{k+1}}
\]
\[
\geq \left( 1 - \int_{\Omega_\delta} \delta^{k+1} \, dx \right)^{\frac{1}{k+1}} - \delta |\Omega_\delta|^{\frac{1}{k+1}}
\]
\[
= \left( 1 - \delta^{k+1} (|\Omega| - |\Omega_\delta|) \right)^{\frac{1}{k+1}} - \delta |\Omega_\delta|^{\frac{1}{k+1}}
\]
\[
\geq 1 - \delta (|\Omega| - |\Omega_\delta|)^{\frac{1}{k+1}} - \delta |\Omega_\delta|^{\frac{1}{k+1}} \geq 1 - 2\delta |\Omega|^{\frac{1}{k+1}}.
\]

Hence, from (4.5), (4.1) and Faber-Krahn inequality it follows that
\[
\lambda_k((\Omega_\delta)_k) \leq \lambda_k(\Omega_k) \leq (1 + \epsilon) \lambda_k((\Omega_\delta)^{k+1}) (1 - 2\delta |\Omega|^{\frac{1}{k+1}})^{-k}.
\]
which implies, by (2.9), that

\[
(4.6) \quad \left( \frac{W_{k-1}(\Omega)}{W_{k-1}(\Omega)} \right)^{\frac{n}{k}} = \frac{\lambda_k(\Omega_{k-1}^*)}{\lambda_k((\Omega_\varepsilon_{k-1})^*)} \geq \frac{(1 - 2\delta|\Omega|^{1/n})^k}{1 + \varepsilon},
\]

where we used that the balls $\Omega_{k-1}^*$ and $(\Omega_\varepsilon)_{k-1}^*$ preserve, respectively, the $(k - 1)$-th quermassintegral of $\Omega$ and $\Omega_\varepsilon$. Hence, by (4.6) we get that

\[
\frac{W_{k-1}(\Omega)}{W_{k-1}(\Omega)} \geq \left( 1 - \varepsilon + 2\delta|\Omega|^{1/n} \right)^{\frac{n-k+1}{k}} \geq 1 - (n - k + 1) \max \{\varepsilon, 2\delta|\Omega|^{1/n}\},
\]

obtaining the thesis. \(\square\)

**Lemma 4.2.** Under the hypotheses of Theorem 4.1 if $u$ is the eigenfunction of $S_k$ in $\Omega$ such that $\|u\|_{\mathcal{H}^{k+1}(\Omega)} = 1$, we have that

\[
(4.7) \quad \frac{n(n - k + 1)^k}{k} \int_0^{\max(-u)} \frac{W_k(\Omega_t^{*})^{k+1} - W_k((\Omega_t^*)_{k-1}^{*})^{k+1}}{-\frac{d}{dt} W_{k-1}(\Omega_t)} dt \leq \varepsilon \lambda_k(\Omega_{k-1}^*).
\]

**Proof.** The divergence form of $S_k$ and the coarea formula give that (see also [31])

\[
\lambda_k(\Omega) = \frac{1}{k} \int_0^{\max(-u)} \frac{\int_{\Omega_t} H_{k-1}(\Sigma_t) \|Du\|^2 d\mathcal{H}^{n-1}}{d \mathcal{H}^n} dt = \int_0^{\max(-u)} \frac{W_k(\Omega_t)^{k+1}}{\left[-\frac{d}{dt} W_{k-1}(\Omega_t)\right]^k} dt.
\]

Moreover, being $\|u_{k-1}^*\|_{k+1} = \|u\|_{k+1} = 1$, we have

\[
(4.8) \quad \lambda_k(\Omega_{k-1}^*) \leq \frac{\int_{\Omega_{k-1}^*} (-u_{k-1}^*) S_k(D^2 u_{k-1}^*) dx}{\int_{\Omega_{k-1}^*} (-u_{k-1}^*)^{k+1} dx} \leq \int_{\Omega_{k-1}^*} (-u_{k-1}^*) S_k(D^2 u_{k-1}^*) dx = \frac{n(n - k + 1)^k}{k} \int_0^{\max(-u)} \frac{W_k((\Omega_t)^*_{k-1})^{k+1}}{\left[-\frac{d}{dt} W_{k-1}(\Omega_t)\right]^k} dt.
\]

Last equality follows from the symmetry of $u_{k-1}^*$ and being $W_{k-1}(\Omega_t) = W_{k-1}((\Omega_t)^*_{k-1})$. Hence, taking (4.8) and (4.9) and subtracting, from we have that

\[
\varepsilon \lambda_k(\Omega_{k-1}^*) \geq \lambda_k(\Omega) - \lambda_k(\Omega_{k-1}^*) \geq \frac{n(n - k + 1)^k}{k} \int_0^{\max(-u)} \frac{W_k(\Omega_t)^{k+1} - W_k((\Omega_t)^*_{k-1})^{k+1}}{\left[-\frac{d}{dt} W_{k-1}(\Omega_t)\right]^k} dt,
\]

that gives the thesis. \(\square\)

In the next result we prove a lower bound for $|\Omega|$ in term of $W_{k-1}(\Omega)$.

**Lemma 4.3.** Under the hypotheses of Theorem 4.1, it holds that

\[
|\Omega| \geq C_{n,k} W_{k-1}^{\frac{1}{k+1}}(\Omega),
\]

where $C_{n,k}$ denotes a positive constant depending only on $n$ and $k$. 
Proof. Let be \( u \) an eigenfunction corresponding to the eigenvalue \( \lambda = \lambda_k(\Omega) \) and such that \( \| u \|_{L^{k+1}} = 1 \). Then,

\[
(4.10) \quad S_k(D^2 u) = \lambda (-u)^k \quad \text{in} \ \Omega.
\]

Arguing as in Lemma 3.3 by (2.6) (2.5) and the Hölder inequality we have

\[
(4.11) \quad \int_{\Omega} S_k(D^2 u) dx = \frac{1}{k} \int_{\Gamma} \int_{L_t} H_{k-1} |Du|^k dH^{n-1} \geq C_{n,k} \frac{(W_k(\Omega_1))^{k+1}}{(- \frac{d}{dt} W_{k-1}(\Omega_t))^{1/k}}.
\]

We divide the proof in three cases.

Case \( k > \frac{n}{2} \). By Hölder inequality we have:

\[
(4.12) \quad \left( \int_{\Omega} (-u)^k dx \right)^{\frac{1}{k}} \leq |\Omega|^{\frac{1}{k}} \| u \|_{L^\infty(\Omega)}.
\]

Putting together (4.11) and (4.12), by (4.10) we get that

\[
W_k(\Omega_1)^{\frac{n+1}{n-k+1}} \geq C_{n,k} |\Omega|^{-\frac{1}{k}} \| u \|^{-1} \lambda^{-\frac{1}{k}}.
\]

Using the Aleksandrov-Fenchel inequalities (2.3) with \( j = k \) and \( i = k - 1 \), and integrating between 0 and \( \| u \|_{L^\infty(\Omega)} \), being \( k > \frac{n}{2} \) we get

\[
W_{k-1}(\Omega)^{\frac{n+1}{n-k+1}} \geq C_{n,k} \lambda^{-\frac{1}{k}} |\Omega|^{-\frac{1}{k}}.
\]

Being \( \lambda_k(\Omega) \leq (1 + \epsilon) \lambda_k(\Omega_1) \), and recalling the properties (2.9) and (2.2), we have:

\[
|\Omega|^\frac{1}{k} \geq C_{n,k} W_{k-1}(\Omega)^{-\frac{2}{n-k+1}} \lambda_k(\Omega_1)^{\frac{n}{n-k+1}} (1 + \epsilon)^{-\frac{n-k+1}{k}} = C_{n,k} W_{k-1}(\Omega)^{-\frac{2}{n-k+1}} \lambda_k(\Omega_1)^{\frac{n}{n-k+1}} (1 + \epsilon)^{-\frac{n-k+1}{k}} = C_{n,k} (1 + \epsilon)^{-\frac{n-k+1}{k}} W_{k-1}(\Omega)^{-\frac{1}{n-k+1}},
\]

and the first case is completed.

Case \( k < \frac{n}{2} \). By Hölder inequality, and being \( \| u \|_{k+1} = 1 \) we have:

\[
(4.13) \quad \left( \int_{\Omega} (-u)^k dx \right)^{\frac{1}{k}} \leq |\Omega|^{\frac{1}{k}} \left( \int_{\Omega} u^{k+1} dx \right)^{\frac{1}{k+1}} \leq |\Omega|^{\frac{1}{k}}.
\]

Then, joining (4.11) and (4.13), and using the Aleksandrov-Fenchel inequalities we get

\[
W_{k-1}(\Omega)^{\frac{n+1}{n-k+1}} \left( - \frac{d}{dt} W_{k-1}(\Omega_1) \right)^{\frac{1}{k}} \geq C_{n,k} |\Omega|^{-\frac{1}{k}} \lambda^{-\frac{1}{k}}.
\]

Integrating between 0 and \( \delta \) sufficiently small, we get that

\[
W_{k-1}(\Omega)^{\frac{n+1}{n-k+1}} - W_{k-1}(\Omega)^{\frac{n+1}{n-k+1}} \geq C_{n,k} |\Omega|^{-\frac{1}{k}} \lambda^{-\frac{1}{k}} \delta.
\]

Now we apply Lemma 4.1. Let \( \epsilon \) and \( \delta \) such that \( \epsilon < 2\delta |\Omega|^{\frac{1}{k+1}} < (n-k+1)^{-1} \). Hence, writing \( a = \frac{\epsilon}{\epsilon(n-k+1)} < 0 \), we get

\[
(4.14) \quad W_{k-1}(\Omega)^{\alpha} \left[ 1 - 2 \delta(n-k+1)|\Omega|^{\frac{1}{k+1}} \right] \geq C_{n,k} |\Omega|^{-\frac{1}{k+1}} \lambda^{-\frac{1}{k}} \delta.
\]

Moreover, if \( \delta \) is such that \( 2 \delta |\Omega|^{\frac{1}{k+1}} (n-k+1) \leq 1 - 2^{-\frac{1}{k+1}} \), from (4.14) we get

\[
W_{k-1}(\Omega)^{\alpha} \left[ -4 \alpha(n-k+1)|\Omega|^{\frac{1}{k+1}} \right] \geq C_{n,k} |\Omega|^{-\frac{1}{k+1}} \lambda^{-\frac{1}{k}} \delta,
\]

that is

\[
|\Omega|^\frac{1}{k} \geq C_{n,k} W_{k-1}(\Omega)^{\frac{n+1}{n-k+1}} \lambda^{-\frac{1}{k}} = C_{n,k}(1 + \epsilon)^{-\frac{1}{k}} \lambda_k(B_1)^{-\frac{1}{k}} W_{k-1}(\Omega)^{\frac{n+1}{n-k+1}},
\]
that is the thesis. 

Case $k = \frac{d}{2}$. Arguing as before, we get
\[
\log \left( \frac{W_{\underline{k}-1}(\Omega)}{W_{\underline{k}-1}(\Omega_\delta)} \right) \geq C_n |\Omega|^{-\frac{4}{n(n+2)}} \lambda^{-\frac{2}{n}} \delta.
\]

By Lemma 4.1, it follows that if $\epsilon < 2\delta|\Omega|^{\frac{2}{n+2}} < \frac{2}{n+2}$,
\[
- \log \left( 1 - \delta (n+2)|\Omega|^{\frac{2}{n+2}} \right) \geq C_n |\Omega|^{-\frac{4}{n(n+2)}} \lambda^{-\frac{2}{n}} \delta.
\]

Then, for $\delta$ such that $\delta(n+2)|\Omega|^{\frac{2}{n+2}} < \frac{1}{2(n+2)}$,
\[
2(n+2)|\Omega|^{\frac{2}{n+2}} \delta \geq C_n |\Omega|^{-\frac{4}{n(n+2)}} \lambda^{-\frac{2}{n}} \delta.
\]

Then, similarly as before,
\[
|\Omega|^{\frac{2}{n}} \geq C_n W_{\underline{k}-1}(\Omega)^{\frac{4}{n(n+2)}},
\]
and the proof of the Lemma is completed. \hfill \Box

Now we can prove the main theorem of this section.

**Proof of the Theorem 4.1** Without loss of generality, we may suppose that $W_{k-1}(\Omega) = 1$. Indeed, the quotient
\[
\frac{W_{k-1}(K) - W_{k-1}(L)}{W_{k-1}(K)}
\]
is rescaling invariant. Consequently, by Lemma 4.3 and the Aleksandrov-Fenchel inequality, we have that there exist two positive constants $c_1(n,k)$ and $c_2(n,k)$, which depend only on $n$ and $k$, such that

(4.15) $c_1(n,k) \leq |\Omega| \leq c_2(n,k).

The Hölder inequality gives that

(4.16) $\epsilon = \delta^{k+1} = \left( \int_0^\delta dt \right)^{k+1} \leq \left( \int_0^\delta \left( - \frac{d}{dt} W_{k-1}(\Omega_t) \right) dt \right)^k \int_0^\delta \left( - \frac{d}{dt} W_{k-1}(\Omega_t) \right)^k dt = |W_{k-1}(\Omega) - W_{k-1}(\Omega_\delta)|^k \int_0^\delta \left( - \frac{d}{dt} W_{k-1}(\Omega_t) \right)^k dt.$

Hence, for $\epsilon > 0$ sufficiently small, $\delta$ verifies the hypothesis in Lemma 4.1 and the inequalities (4.16), (4.3), (4.7) and (4.15) imply that

\[
\inf_{\tau \in [0,\delta]} \left( W_{k}(\Omega_t)^{k+1} - W_{k}((\Omega_t)^{*})^{k+1} \right) \leq \frac{1}{\epsilon} |W_{k-1}(\Omega) - W_{k-1}(\Omega_\delta)|^k \int_0^{\max(-u)} \left( - \frac{d}{dt} W_{k-1}(\Omega_t) \right)^k dt \leq C_{n,k} \epsilon^{-\frac{1}{n+2}} \lambda_k(\Omega_{k-1}).
\]

Hence, for some $\tau \in [0,\delta]$, we have that
\[
W_{k}(\Omega_t)^{k+1} \leq \frac{1}{\epsilon} W_{k}((\Omega_t)^{*})^{k+1} + C_{n,k} \epsilon^{-\frac{1}{n+2}},
\]
being $W_{k-1}(\Omega) = 1$. Moreover, an algebraic inequality and (4.15) give that
\[
\omega_n^{n-1}W_{k}(\Omega_t)^{n-k+1} \leq W_{k-1}(\Omega_t)^{n-k} + C_{n,k} \epsilon^{-\frac{1}{n+2}},
\]
Moreover, by (4.3) it follows that

\[
R_{\tau} - r_{\tau} \geq C_{n,k} \varepsilon^{\frac{1}{n+3}}.
\]

Hence, recalling (2.2), and being 

\[
\varepsilon \leq \frac{1}{n+1} \left( \frac{1}{|\Omega|^1} \right) \left( \frac{1}{n+1} \right) \left( \frac{1}{n+1} \right) \geq \frac{1}{n+1} \left( \frac{1}{|\Omega|^1} \right) \left( \frac{1}{n+1} \right) \left( \frac{1}{n+1} \right) = C_{n,k} \varepsilon^{\frac{1}{n+3}},
\]

where \( R_{\tau} = \left[ \omega_n^{-1} W_{k-1}(\Omega) \right]^{\frac{1}{n+1}} \) is the radius of \( \Omega_{\tau}^{k-1} \), and \( \varepsilon = \max \left\{ \frac{1}{n+1} \left( \frac{1}{|\Omega|^1} \right) \left( \frac{1}{n+1} \right) \left( \frac{1}{n+1} \right) \right\} \).

Hence, recalling (2.2) and being \( r_{\tau} \leq r_{\Omega} \) and \( W_{k-1}(\Omega) = 1 \), then

\[
d_{k}(\Omega) \leq \omega_n^{\frac{1}{n+1}} (R - r_{\tau}) \leq C_{n,k} \varepsilon^{\frac{1}{n+3}},
\]

that is the first estimate in (4.2). In order to obtain the remaining estimates of the theorem, using the Aleksandrov-Fenchel inequalities, (4.17) and being \( W_{k-1}(\Omega) = 1 \), we have that

\[
\left[ \left( \frac{P(\Omega)}{n \omega_n} \right)^{n} - \left( \frac{1}{\omega_n} \right)^{n-1} \right] \leq \left[ \left( \frac{W_{k-1}(\Omega)}{\omega_n} \right)^{n} - \left( \frac{1}{\omega_n} \right)^{n-1} \right] \leq \left[ \left( \frac{W_{k-1}(B_{r_{\Omega}}) + C_{n,k} \varepsilon^{\Delta}}{\omega_n} \right)^{n} - \left( \frac{1}{\omega_n} \right)^{n-1} \right] \leq \left[ \left( \frac{W_{k-1}(B_{r_{\Omega}})}{\omega_n} \right)^{n} + C_{n,k} \varepsilon^{\Delta} - \left( \frac{1}{\omega_n} \right)^{n-1} \right] = C_{n,k} \varepsilon^{\Delta}.
\]

Hence, (4.18) and (2.13) imply

\[
\delta_{H}(\Omega) \leq C_{n,k} \varepsilon^{\Delta},
\]

while, from (2.14) we get

\[
D_{k}(\Omega) \leq \omega_n^{\frac{1}{n+1}} (R_{\Omega} - r_{\Omega}) \leq C_{n,k} \varepsilon^{\Delta},
\]

and this concludes the proof.

\[\Box\]

**Remark 4.3.** Similarly as observed in Remark 3.2 from the proof of Theorem 4.1 it is possible to obtain that

\[
\Delta(\Omega) \leq C_{n,k} \varepsilon^{\Delta}.
\]

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