Casimir energy for acoustic phonons in graphene

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Abstract – We find the Casimir energy, at finite temperature, for acoustic phonons in a graphene sheet suspended over a rectangular trench, and the corresponding Casimir forces are interpreted as correction terms to the built-in tensions of the graphene. We show that these corrections generally break the tensional isotropy of the membrane, and can increase or decrease the membrane tension. We demonstrate that for a narrow rectangular trench with side-lengths in the order of few nanometers and few micrometers, these temperature corrections are expected to be noticeable (\(\sim 10^{-4} \text{N/m}\)) at room temperature. These corrections would be even more considerable by increasing the temperature, and can be applied for adjusting the built-in tension of the graphene. Consequently we introduce a corrected version for the fundamental resonance frequency of the graphene resonator.

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Introduction. – The Casimir effect, as one of the most important manifestations of the zero-point oscillations of quantum fields [1], nowadays has been investigated theoretically and experimentally for various systems with different boundary conditions and different geometries, see, e.g., [2–4] as review. As we know, the Casimir forces are observable mostly in microscale and smaller dimensions, so the Casimir effect can imply significant results for ultra-small structures, see [5–7] as review. Graphene, a single-atom-thick layer of carbon atoms covalently bonded in a honeycomb lattice, as the thinnest known material with remarkable mechanical and electrical properties, is expected to have many promising applications in ultra-small-dimension systems, see, e.g., [8–10].

In this paper, we investigate the physical effects of the zero-point oscillations for acoustic phonons in a monolayer graphene sheet suspended over a rectangular trench. Here the acoustic phonons are considered as the quanta of a massless bosonic field with two components, longitudinal acoustic (LA) and transverse acoustic (TA) fields, constrained by a Dirichlet boundary condition which results from the strong van der Waals (vdW) forces which clamp the graphene sheet to the sidewalls of the trench, see refs. [11–18]. So one can introduce a “phononic” Casimir energy for the graphene sheet by calculating the zero-point energy of these constrained acoustic fields in the suspended graphene. In fact here the graphene sheet can be considered as a phonon gas being confined in a 2-dimensional box with side-lengths equivalent to the trench side-lengths. As a result, the corresponding “phononic” Casimir forces can be defined by differentiating the above Casimir energy with respect to the trench side-lengths. As a result, the corresponding “phononic” Casimir forces can be defined by differentiating the above Casimir energy with respect to the side-lengths of the trench, see the second section. We would apply a new useful technique to find an exact expression for the Casimir energy being specifically useful for high temperatures, which has considerable importance for our work, since as we will see, the effective temperature of the graphene membrane is considerably smaller than the ordinary temperatures. Then we would interpret these Casimir forces as temperature-dependent corrections to the built-in tension of the membrane, and using the experimental values, we obtain numerical results for these temperature corrections, see the third section. Subsequently, utilizing these temperature corrections, we introduce a temperature-dependent corrected version for the fundamental resonance frequency of the graphene membrane.

Casimir energy for acoustic phonons in a membrane. – As we know, the dynamics of the acoustic modes in a membrane can be effectively described by the known
dispersion relations $\omega_f = v_f k$ in which $\omega$ and $k$ are the mode frequency and the mode wave number, respectively, of the acoustic modes, $v$ is the sound velocity in the membrane, and $f$ counts the longitudinal (LA) and transverse (TA) components. As a result for a fully clamped membrane, these acoustic modes can be regarded as oscillation modes of a massless 2-component bosonic field living in a rectangle with Dirichlet boundary conditions, with the known mode wave numbers

$$k_{n,m} = \pi \sqrt{n^2 a^2 + m^2 b^2}; \quad a \geq b, \quad n, m = 1, 2, \ldots,$$

(1)

in which “$a$” and “$b$” are the side-lengths of the rectangular trench, over which, the membrane is suspended. Now one can quantize the system simply by taking the oscillation modes of the mentioned acoustic fields, as quantum oscillators with mode frequencies $\omega_{n,m} = v_f k_{n,m}$ given through eq. (1). Then the zero-point energy of such a system at finite temperature, can be written as (see, e.g., [19])

$$E_0(a, b, T) = \frac{k_B T}{2} \times \sum_{l=1}^{\infty} \sum_{n,m=1}^{\infty} \ln \left[ \frac{2\pi k_B T}{\hbar} \right]^{2} \left( l^2 + \omega_{l,n,m}^2 \right),$$

(2)

in which “$T_m$” is the system temperature, and $k_B$ and $\hbar$ are the Boltzmann and the Planck constants, respectively. Note that, as we discussed before, the above zero-point energy can be considered actually as the vacuum energy of the membrane acoustic phonons considered as a phonon gas being confined in a 2-dimensional box with side-lengths “$a$” and “$b$”. So the regularized form of the above zero-point energy can be considered as the “phononic” Casimir energy of the membrane.

Now to find an explicit expression for the phononic zero-point energy (2), utilizing the Gamma function, we rewrite the above equation as a parametric integral;

$$E_0(a, b, T) = -\frac{k_B T}{2} \sum_{l} \sum_{n,m=1}^{\infty} \frac{\partial}{\partial s} \left( \frac{\hbar \nu}{2\pi k_B T} \right)^{2s} \times \left( l^2 + \lambda_{l,a,n^2}^2 + \lambda_{l,b,m^2}^2 \right)^{s}$$

$$= -\frac{k_B T}{2} \sum_{l} \sum_{n,m=1}^{\infty} \frac{\partial}{\partial s} \left( \frac{\hbar \nu}{2\pi k_B T} \right)^{2s} \times \int_{0}^{\infty} \frac{dt}{\Gamma(s)} \times \exp \left[ -t \left( l^2 + \lambda_{l,a,n^2}^2 + \lambda_{l,b,m^2}^2 \right) \right],$$

(3)

in which $\nu$ is an arbitrary parameter with the dimension of the inverse time, and we have introduced dimensionless variables $\lambda = \theta / T$ with effective temperatures

$$\theta_{l,a} = \frac{\hbar v_l}{2a k_B}, \quad \theta_{l,b} = \frac{\hbar v_l}{2b k_B}.$$

(4)

Note that it is specifically important, for our problem, to find an explicit expression for the zero-point energy for sufficiently high temperatures, since, as we will see in the third section the (larger) effective temperature $\theta_{l,a,b}$ of the suspended graphene membrane, is sufficiently lower than the ordinary temperatures. As we know, a limiting expression for the Casimir energy at high temperature, can be obtained generally by applying the known heat kernel expansion, see, e.g., [19,20]. But here we use a new useful technique to find an exact expression for the zero-point energy, being specifically useful for high temperatures. First separating $l = 0$ from the $l$-sum, we rewrite eq. (3) as

$$E_0(a, b, T) = -\frac{k_B T}{2} \sum_{l} \sum_{n,m=1}^{\infty} \frac{\partial}{\partial s} \left( \frac{\hbar \nu}{2\pi k_B T} \right)^{2s} \left[ I + II \right];$$

$$I \equiv \int_{0}^{\infty} \frac{dt}{\Gamma(s)} \sum_{n=1}^{\infty} \exp \left[ -t \lambda_{l,a,n^2}^2 \right] \times \sum_{m=1}^{\infty} \exp \left[ -t \lambda_{l,b,m^2}^2 \right],$$

$$II \equiv 2 \int_{0}^{\infty} \frac{dt}{\Gamma(s)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \exp \left[ -t l^2 \right] \times \exp \left[ -t \lambda_{l,a,n^2}^2 \right] \times \exp \left[ -t \lambda_{l,b,m^2}^2 \right].$$

(5)

Then, by applying the Poisson summation formula (see, e.g., [21])

$$\sum_{n=1}^{\infty} f(n) = -f(0) + \int_{0}^{\infty} f(x) dx + 2 \sum_{n=1}^{\infty} \int_{0}^{\infty} f(x) \cos(2\pi n x) dx$$

(6)

to the $m$-sum;

$$\sum_{m=1}^{\infty} \exp \left[ -t \lambda_{l,b,m^2}^2 \right] = -\frac{1}{2} + \frac{1}{2 \lambda_{l,b}} \sqrt{\frac{\pi}{t}} + \frac{1}{2 \lambda_{l,b}} \sqrt{\frac{\pi}{t}} \times \sum_{n=1}^{\infty} \exp \left[ -\frac{\pi^2 m^2}{\lambda_{l,b}^2} \right]$$

(7)

and substituting it in the expression $I$ in eq. (5), and after some calculations we find

$$I = -\frac{\zeta(2s)}{2 \lambda_{l,a}^2} + \frac{\sqrt{\pi}}{2 \lambda_{l,b}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \frac{\zeta(2s - 1)}{\lambda_{l,a}^{2s - 1}} + 2 \sqrt{\pi} \lambda_{l,b} \Gamma(s) \sum_{n,m=1}^{\infty} \left( \frac{n \lambda_{l,a}}{m \pi / \lambda_{l,b}} \right)^{s-1/2}$$

$$\times K_{-s+1/2} \left( 2 \pi n m \frac{\lambda_{l,a}}{\lambda_{l,b}} \right),$$

(8)

in which $K$ is a Bessel function of the second kind, and we have used the known Riemann zeta function
\begin{equation}
\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \text{ and the integral relation}
\int_{0}^{\infty} t^s \exp \left[-x^2 t - y^2 / t \right] \, dt = 2(x/y)^{-r-1} K_{r-1}(2xy).
\end{equation}

Then, for the expression \( \Pi \), applying the Poisson summation formula (6) to both \( m \)- and \( n \)-sum, and after some similar calculations, one can find

\begin{equation}
\Pi = \frac{\zeta(2s)}{2} - \frac{\sqrt{\pi}}{2} \left( \frac{1}{\lambda_{I,a}} + \frac{1}{\lambda_{I,b}} \right) \frac{\Gamma(s - 1/2)}{\Gamma(s)} \zeta(2s - 2) + \frac{\pi}{2\lambda_{I,a} \lambda_{I,b}} \frac{\Gamma(s)}{\Gamma(s)} \zeta(2s - 2) + \frac{2\sqrt{\pi}}{\lambda_{I,a} \lambda_{I,b}} \sum_{n=1}^{\infty} \left( \frac{l}{n\pi/\lambda_{I,a}} \right)^{s+1/2} K_{s+1/2}(\frac{2nl\pi}{\lambda_{I,a}}) - \frac{2\sqrt{\pi}}{\lambda_{I,a} \lambda_{I,b}} \sum_{n=1}^{\infty} \left( \frac{l}{n\pi/\lambda_{I,b}} \right)^{s+1/2} K_{s+1/2}(\frac{2nl\pi}{\lambda_{I,b}}) + \frac{2\pi}{\lambda_{I,a} \lambda_{I,b}} \sum_{n=1}^{\infty} \left( \frac{l}{n\pi/\lambda_{I,b}} \right)^{s+1} K_{s+1}(\frac{2nl\pi}{\lambda_{I,a}}) + \frac{2\pi}{\lambda_{I,a} \lambda_{I,b}} \sum_{n=1}^{\infty} \left( \frac{l}{n\pi/\lambda_{I,b}} \right)^{s+1} K_{s+1}(\frac{2nl\pi}{\lambda_{I,b}}) + \frac{4\pi}{\lambda_{I,a} \lambda_{I,b}} \sum_{n,m=1}^{\infty} \left( \frac{l}{n\pi/\lambda_{I,b}} \right)^{s+1} K_{s+1}(\frac{2nl\pi}{\lambda_{I,a}}) \times K_{s+1}(\frac{2l\sqrt{(n\pi/\lambda_{I,a})^2 + (m\pi/\lambda_{I,b})^2}}{\lambda_{I,a}}).
\end{equation}

Finally, performing the limit \( s \to 0 \) by noting that

\begin{equation}
limit_{s \to 0} \frac{g(s)}{\Gamma(s)} = g(0),
\end{equation}

the zero-point energy (5) turns to

\begin{equation}
E_0(a,b,T) = -\frac{k_B T}{2} \sum_{l} \left( \frac{\zeta(3)}{4\pi \lambda_{I,a} \lambda_{I,b}} - \frac{\pi}{12 \lambda_{I,a}} - \frac{\pi}{12 \lambda_{I,b}} \right) + \frac{\pi \lambda_{I,a}}{12 \lambda_{I,b}} - \ln \lambda_{I,a} + \sum_{n,m=1}^{\infty} \frac{\exp(-2mn\pi/\lambda_{I,a})}{m} + \frac{\exp(-2mn\pi/\lambda_{I,b})}{n} + 2 \sum_{n,m=1}^{\infty} \frac{m}{n} \left[ \frac{K_1(2mn\pi/\lambda_{I,a})}{\lambda_{I,a}} + \frac{K_1(2mn\pi/\lambda_{I,b})}{\lambda_{I,b}} \right] + 4 \sum_{n,m,l=1}^{\infty} \frac{l}{\sqrt{n^2 \lambda_{I,a}^2 + m^2 \lambda_{I,a}^2}}. \tag{11}
\end{equation}

Note that the first three terms of the above equation are the contributions of unbounded system (i.e., a rectangle with infinite side-lengths) which result in nonzero terms for the Casimir force at the limits \( a, b \to \infty \), note eq. (4). Hence these terms should be subtracted, to find the phononic Casimir energy of the membrane

\begin{equation}
E_C(a,b,T) = \frac{k_B T}{2} \sum_{l} \left( \frac{\ln \lambda_{I,a}}{2 \lambda_{I,b}} - \frac{\pi \lambda_{I,a}}{12 \lambda_{I,b}} \right) - \sum_{n,m=1}^{\infty} \frac{\exp(-2mn\pi/\lambda_{I,a})}{m} + \frac{\exp(-2mn\pi/\lambda_{I,b})}{n} - 2 \sum_{n,m=1}^{\infty} \frac{m}{n} \left[ \frac{K_1(2mn\pi/\lambda_{I,a})}{\lambda_{I,a}} + \frac{K_1(2mn\pi/\lambda_{I,b})}{\lambda_{I,b}} \right] - 4 \sum_{n,m,l=1}^{\infty} \frac{l}{\sqrt{n^2 \lambda_{I,a}^2 + m^2 \lambda_{I,a}^2}}. \tag{12}
\end{equation}

Note that in contrast to the heat kernel approach, our technique has resulted in an exact equation for the Casimir energy. In fact an exact term such as the third term of the above equation, which would turn out to be an important term, cannot be simply obtained through the heat kernel approach. For sufficiently large temperatures (with respect to the effective temperature \( \theta_{LA,b} \)), the last three terms can be neglected to find

\begin{equation}
E_C(a,b,T) \approx k_B T \left( \frac{4}{4} \sum_{l} \ln \left[ \frac{h v_1}{2a k_B T} - \frac{\pi b}{12 a} \right] - \sum_{n,m=1}^{\infty} \frac{\exp(-2mn\pi b/\lambda_{I,a})}{m} \right). \tag{13}
\end{equation}

Then the phononic Casimir forces can be defined as

\begin{equation}
F_{C,a}(a,b,T) \equiv -\frac{\partial E_C}{\partial a} \approx \frac{k_B T}{a} \left( \frac{1}{12 a} + \frac{b}{a} \right), \quad F_{C,b}(a,b,T) \equiv -\frac{\partial E_C}{\partial b} \approx \frac{k_B T}{b} \left( \frac{\pi b}{12 a} - \frac{b}{a} \right), \tag{14}
\end{equation}
in which

\begin{equation}
S(x) \equiv \sum_{n,m=1}^{\infty} 2n\pi \exp(-2mn\pi x); \quad x \leq 1.
\end{equation}

Through numerical computations, we have \( S(x) \geq S(1) \approx 0.01 \), hence the absolute value of \( F_{C,a} \) as well as \( F_{C,b} \) increases by decreasing the ratio \( b/a \). However \( F_{C,a} \) is always positive-valued (i.e., repulsive), while \( F_{C,b} \) is positive-valued for \( b/a \gtrsim 0.5 \), and negative-valued (i.e., attractive) for \( b/a \lesssim 0.5 \). Note that for a square trench \((a = b)\) we have

\begin{equation}
F_{C,a} = F_{C,b} \approx 0.25 \frac{k_B T}{a}, \tag{15}
\end{equation}

so for a square membrane the phononic Casimir force is always repulsive.
Note that an asymptotically appropriate expression for sufficiently small temperatures (in comparison to \( \theta \)), could be obtained by applying the Poisson sum (6) to the \( t \)-sum in eq. (5). Then after some similar calculations one would find

\[
E_C(a,b,T) = \left( \frac{\pi}{48} - \frac{\zeta(3)}{16\pi} \right) b \sum_{l} v_I - \frac{k_BT}{2} \left[ \frac{\pi}{12\lambda_{l,a}} + \frac{\pi}{12\lambda_{l,b}} - \frac{\zeta(3)}{4\pi\lambda_{l,a}\lambda_{l,b}} \right]
\]

\[
+ \sum_{n,m,l=1}^{\infty} \exp \left( -2l\pi \sqrt{n^2\lambda_{l,a}^2 + m^2\lambda_{l,b}^2} \right),
\]

(16)

which, for sufficiently small temperatures, can be approximated by its first line.

**Application to graphene resonator.** – As we know, the dynamics of a sufficiently thin membrane being perfectly flexible, with a sufficiently large pre-tension, can be described effectively by surface harmonic oscillation modes [22]. This is a valid consideration, e.g., for a suspended monolayer graphene sheet with a rather large initial tension arisen from the strong vdW adhesion which clamp the graphene sheet to the sidewalls of the trench [11–18]. Note however that this may not be a good approximation for thick multilayer graphene membranes, which have larger values of the bending rigidity being not negligible compared to the built-in tension of the graphene [11].

Now to find an appropriate physical interpretation for the phononic Casimir forces for such a fully clamped membrane with a sufficiently large pre-tension, we introduce the total ground-state energy (\( E_\theta \)) of the membrane as the sum of the (classical) tensional energy and the phononic Casimir energy. Then the change in \( E_\theta \), of a fully clamped membrane with the pre-tension \( \tau_0 \), due to infinitesimal changes \( \delta a \) and \( \delta b \), can be written as

\[
\frac{\partial E_C}{\partial \alpha} \delta a + (a \tau_0)\delta b + \frac{\partial E_C}{\partial b} \delta b = b \left( \tau_0 - \frac{1}{b} F_{C,a} \right) \delta a + a \left( \tau_0 - \frac{1}{a} F_{C,b} \right) \delta b.
\]

(17)

Hence one can take the quantum corrected tensions of the membrane as

\[
\tau_a = \tau_0 - \frac{1}{b} F_{C,a},
\]

\[
\tau_b = \tau_0 - \frac{1}{a} F_{C,b},
\]

(18)

i.e., the phononic Casimir forces contribute actually as quantum corrections to the pre-tension of the membrane. For sufficiently large temperatures the Casimir forces for the fully clamped membrane are given by eq. (14), so the corrected pre-tensions of the membrane over a rectangular trench can be written as

\[
\tau_{a,b}(a,b,T) \approx \tau_0 + \Delta_{a,b}(a,b,T),
\]

(19)

in which

\[
\Delta_a(a,b,T) \equiv -\frac{k_BT}{b^2} \left[ \frac{1}{2} \frac{b}{a} \left( \frac{b}{a} \right)^2 + \frac{1}{12} \left( \frac{b}{a} \right)^4 - \frac{\pi}{12} \right],
\]

\[
\Delta_b(a,b,T) \equiv \frac{k_BT}{b^2} \left[ -\frac{\pi}{12} \left( \frac{b}{a} \right)^2 + \frac{1}{12} \left( \frac{b}{a} \right)^4 - \frac{\pi}{12} \right],
\]

(20)

with \( S(x) \) as before. Note that for a square trench \( a = b \),

\[
\tau_a = \tau_b \approx \tau_0 - 0.25 \frac{k_BT}{a^2},
\]

(21)

while for a narrow trench \( b/a \ll 1 \) we have

\[
\tau_a(b,a,T) \approx \tau_0 \mp \frac{k_BT}{a^2} S \left( \frac{b}{a} \right),
\]

(22)

in which the signs “−” and “+” are referred to the indexes “\( a \)” and “\( b \)”, respectively. According to previous numerical computations (see below eq. (14)), for a rectangular trench, the phononic Casimir corrections generally break the tensional isotropy of the fully clamped membrane (i.e., \( \tau_a \neq \tau_b \)). However for a square trench, the phononic Casimir force always decreases the membrane pre-tension.

Now we can use experimental values to obtain numerical results for the above corrections. As we previously mentioned, a rather large tension is induced by the strong van der Waals adhesion between the graphene and the sidewalls of the trench. The (surface) pre-tension for suspended graphene sheets, at room temperature has been estimated as being of the order of \( 10^{-4} \)–\( 10^{-2} \) N/m, see, e.g., [16–18]. The one order of magnitude difference in these experimental values, has been attributed to, e.g., the external forces applied during the graphene fabrication, and/or the influences of the graphene adsorbates [17,23]. So the value of the vdW-induced tension of the graphene, can be taken as \( \sim 10^{-2} \) N/m. Having the acoustic wave velocities of the graphene as \( v_{TA} \approx 13.6 \) km/s and \( v_{LA} \approx 21.3 \) km/s (see, e.g., [24]), the (largest) effective temperature of a graphene sheet suspended over a rectangular trench with the side-lengths both in the order of few micrometers, would be obtained as \( \theta_{LA,b} \sim 0.1 \) K, see eq. (4) (having \( h \approx 1.05 \times 10^{-34} \) m\(^2\) kg/s and \( \kappa_B \approx 1.38 \times 10^{-23} \) m\(^2\) kg/s/K). This effective temperature is obviously far smaller than ordinary temperatures, hence the asymptotic expressions for the Casimir energy (13), Casimir forces (14), and the corrected tensions (19), are valid with a high degree of accuracy, for a wide range of accessible temperatures.

As one can see from eq. (19), at room temperature (\( \approx 300 \) K), the temperature corrections \( \Delta_{a,b} \) for a trench with side-lengths both in the order of few micrometers, would be obtained as \( \sim 10^{-10} \) N/m, which, in comparison to the built-in tension \( \tau_0 \sim 10^{-3} \) N/m, is completely neglectable.
negligible. However, the absolute value of the temperature correction, $|\Delta a, b|$, increases by decreasing the side-lengths $a, b$ as well as by decreasing the ratio $b/a$, since as one can see in the numerical plot of fig. 1, $S(x)$ increases extremely by reducing $x$. As a result, for $b$ and $a$, e.g., in the order of few nanometers and few micrometers, respectively, using eq. (22) with $S(0.001) \approx 1.3 \times 10^5$, one can find a value of the order of $10^{-4}$ N/m for $|\Delta a, b|$ at room temperature, which is noticeable in comparison to the built-in tension $\tau_0$. Note that for $b \approx 10^{-9}$ m, the effective temperature $\theta_{LA,b}$ would be in the order of 100 K, which is still sufficiently smaller than the room temperatures, so that the asymptotic expressions (13), (14) and (19) are still valid with a good degree of accuracy. These temperature corrections can have even more importance for graphene membranes at larger temperature, since actually the built-in tension of the graphene sheet decreases by increasing the temperature, see refs. [23,25,26], while $|\Delta a, b|$ increases by increasing the temperature. Then the change in the graphene tensions ($\Delta \tau a, b$), in terms of the temperature change ($\Delta T$), can be given as

$$\Delta \tau a, b \approx \frac{k_B \Delta T}{a^2} S \left( \frac{b}{a} \right).$$

These temperature corrections can be applied for adjusting the built-in tension of the graphene resonator, specifically at higher temperatures. But as we know, any change in the graphene tension would be observable as a change in its resonance frequencies, specifically the fundamental resonance frequency, see, e.g., [16]. However, as a result of anisotropic tension of the graphene due to the acoustic Casimir corrections, the fundamental resonance frequency of the graphene membrane [11], should be corrected as

$$f_{11} = \sqrt{\frac{1}{2\mu} \left( \frac{\tau_a}{a^2} + \frac{\tau_b}{b^2} \right)}$$

in which, $\mu \approx 10^{-6}$ kg/m$^2$ is the surface mass-density of the graphene (see, e.g., [11,16,17]), and $\tau a, b$ are given by eq. (19). Then for $b/a \ll 1$, using eq. (23), the change (in the squared value) of the fundamental resonance frequency (24), in terms of the temperature change, would be given as

$$\Delta f_{11}^2 = \frac{\Delta \tau_a}{a^2} + \frac{\Delta \tau_b}{b^2} \approx \frac{k_B \Delta T}{2\mu a^2 b^2} S \left( \frac{b}{a} \right).$$

Conclusion and remarks. – We have applied a useful technique (see eqs. (5)–(11)) to obtain the Casimir energy at finite temperature for acoustic phonons in a fully clamped sufficiently tensioned membrane suspended over a rectangular trench, see eq. (12), and the corresponding Casimir forces (14) have been interpreted as quantum temperature-dependent corrections to the built-in (surface) tensions of the membrane, see eqs. (18)–(20). We have shown that these temperature corrections generally break the tensional isotropy of a membrane over a rectangular trench, and can decrease or increase the pre-tensions of the membrane, while for a square trench, the corrections always decrease the membrane pre-tension, see the numerical discussions below eqs. (14) and (22).

We have obtained numerical results for a monolayer graphene sheet, using the experimental values given in refs. [16–18], and have demonstrated that for a narrow rectangular trench with side-lengths in the order of few nanometers and few micrometers, the temperature corrections to the pre-tensions, at room temperature, would be of the order of $10^{-4}$ N/m, which is expected to be noticeable in comparison to the vdW-induced built-in tension of the graphene sheet, see the numerical discussion above eq. (23). Consequently we have introduced a corrected version for the fundamental resonance frequency of the suspended graphene membrane, and have obtained its change in terms of the temperature change, see eq. (25). These temperature corrections would find even more importance for graphene membranes at larger temperature, since the built-in tension of the graphene decreases by increasing the temperature [23,25,26], while (the absolute value of) the temperature correction increases by increasing the temperature, see eq. (23). Hence, these temperature corrections can be applied for adjusting the built-in tension of the graphene resonator, specifically at higher temperatures.

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