ANYONIC REALIZATION OF THE QUANTUM AFFINE LIE ALGEBRA $U_q(\hat{A}_{N-1})$

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Abstract

We give a realization of quantum affine Lie algebra $U_q(\hat{A}_{N-1})$ in terms of anyons defined on a two-dimensional lattice, the deformation parameter $q$ being related to the statistical parameter $\nu$ of the anyons by $q = e^{i\pi\nu}$. In the limit of the deformation parameter going to one we recover the Feingold-Frenkel fermionic construction of undeformed affine Lie algebra.

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1 Introduction

The connection between anyons and quantum groups was originally discovered in [2] for the case of $Sl_q(2)$ and later extended in [3] to $Sl_q(n)$ and in [4] to the other classical deformed algebras. For a review see [5].

Anyons are particles with any statistics [6, 7] that interpolate between bosons and fermions of which they can be considered, in some sense, as a deformation (for reviews see for instance [8, 9]). Anyons exist only in two dimensions because in this case the configuration space of collections of identical particles has some special topological properties allowing arbitrary statistics, which do not exist in three or more dimensions. Anyons are deeply connected to the braid group of which they are abelian representations, just like bosons and fermions are abelian representations of the permutation group. In fact, when one exchanges two identical anyons, it is not enough to compare their final configuration with the initial one, but instead it is necessary to specify also the way in which the two particles are exchanged, i.e. the way in which they braid around each other.

In the present paper we will use creation and destruction anyonic operators, equipped with a prescription about their braiding; such a prescription also induces an ordering among the anyons themselves [2].

However we remark that anyons can consistently be defined also on one dimensional chains; for simplicity, in this paper mainly we will use anyons defined on an infinite one-dimensional chain; at the end we will extend our results to anyons defined on a two-dimensional lattice.

Here we extend the realization of classical deformed algebras to the case of the deformed affine algebra $U_q(\hat{A}_{N-1})$. As the fermionic construction of undeformed affine algebras, the anyonic construction on a linear chain gives a value of the central charge equal to one. However at the end, see Sect. 3 we briefly discuss how to get central charges equal to an integer positive number, considering anyons in a two-dimensional lattice. In the case of deformed affine algebras two kinds of explicit realization are known and only for the case of affine $Sl_q(n)$: a construction in terms of an (infinite) combination of the Cartan-Chevalley generators of $Sl_q(\infty)$ build up with bilinears of deformed bosonic or fermionic oscillators ([10]) or a vertex operator construction ([11]).

The paper is organized as follows. In Sect. 2 after recalling the fermionic realization of $\hat{A}_{N-1}$, we present an anyonic realization by the use of anyons defined on a one-dimensional lattice $\mathbb{Z}$. In Sect. 3 we explain how more general representations of the quantum affine algebras can be built by means of anyons defined on a two-dimensional lattice; then we make some final remarks. The Appendix is devoted to a discussion of the ordering of anyons on a chain and on a two-dimensional lattice.
2 Anyonic construction of $U_q(\hat{A}_{N-1})$

Consider the affine Lie algebra $\hat{A}_{N-1}$ with generators denoted in the Cartan-Weyl basis by $h^m_i$ (Cartan generators) and $e^m_a$ (root generators) where $i = 1, \ldots, N - 1$ and $m \in \mathbb{Z}$. The root system of $A_{N-1}$ is given by $\Delta = \{\varepsilon_\rho - \varepsilon_\sigma, 1 \leq \rho \neq \sigma \leq N\}$, the $\varepsilon_\rho$'s spanning the dual of the Cartan subalgebra of $gl_N$. The generators satisfy the following commutation relations:

\[
\left[h^m_i, h^n_j\right] = \gamma m \delta_{m,-n} \delta_{ij} \quad (2.1a)
\]
\[
\left[h^m_i, e^n_a\right] = a^i e^{m+n}_a \quad (2.1b)
\]
\[
\left[e^m_a, e^n_b\right] = \begin{cases} 
\varepsilon(a, b) e^{m+n}_{a+b} & \text{if } a + b \text{ is a root} \\
\alpha^i h^m_i + \gamma m \delta_{m,-n} & \text{if } b = -a \\
0 & \text{otherwise} 
\end{cases} \quad (2.1c)
\]
\[
\left[h^m_i, \gamma\right] = \left[e^m_a, \gamma\right] = 0 \quad (2.1d)
\]

where $\varepsilon(a, b) = \pm 1$ is the usual 2-cocycle and $\gamma$ is the central charge of the algebra.

One can also use the Serre-Chevalley presentation, in which the algebra is described in terms of simple generators and relations (the Serre relations), the only data being the entries of the Cartan matrix $(a_{\alpha\beta})$ of the algebra. Let us denote the generators in the Serre-Chevalley basis by $h_\alpha$ and $e^\pm_\alpha$ where $\alpha = 0, 1, \ldots, N - 1$. The commutation relations are:

\[
\left[h_\alpha, h_\beta\right] = 0 \quad (2.2a)
\]
\[
\left[h_\alpha, e^\pm_\beta\right] = \pm a_{\alpha\beta} e^\pm_\beta \quad (2.2b)
\]
\[
\left[e^+_\alpha, e^-_\beta\right] = \delta_{\alpha\beta} h_\alpha \quad (2.2c)
\]

together with the Serre relations

\[
\sum_{\ell=0}^{1-a_{\alpha\beta}} (-1)^\ell \left( \frac{1}{\ell} - a_{\alpha\beta}\right) \left(e^\pm_\alpha\right)^{1-a_{\alpha\beta}-\ell} e^\pm_\beta \left(e^\pm_\alpha\right)^\ell = 0 \quad (2.3)
\]

The correspondence between the Serre-Chevalley and the Cartan-Weyl bases is the following ($i = 1, \ldots, N - 1$):

\[
h_i = h^0_i \\
h_0 = \gamma - \sum_{j=1}^{N-1} h^0_j \\
e^+_i = e^0_{\varepsilon_i-\varepsilon_{i+1}} \\
e^-_i = e^0_{\varepsilon_{i+1}-\varepsilon_i} \\
e^+_0 = e^1_{\varepsilon_{N-\varepsilon_1}} \\
e^-_0 = e^{-1}_{\varepsilon_1-\varepsilon_N} \quad (2.4)
\]
Let us recall now the fermionic realization of $\hat{A}_{N-1}$ in terms of creation and annihilation operators. Consider an infinite number of fermionic oscillators $c_\rho(r), c_\rho^\dagger(r)$ with $\rho = 1, \ldots, N$ and $r \in \mathbb{Z} + 1/2 = \mathbb{Z}'$, which satisfy the anticommutation relations

$$\{c_\rho(r), c_\sigma(s)\} = \{c_\rho^\dagger(r), c_\sigma^\dagger(s)\} = 0 \quad \text{and} \quad \{c_\rho^\dagger(r), c_\sigma(s)\} = \delta_{\rho \sigma} \delta_{rs} \quad (2.5)$$

the number operator being defined as usual by $n_\rho(r) = c_\rho^\dagger(r)c_\rho(r)$.

These oscillators are equipped with a normal ordered product such that

$$:c_\rho^\dagger(r)c_\sigma(s): = \begin{cases} c_\rho^\dagger(r)c_\sigma(s) & \text{if } s > 0 \\ -c_\sigma(s)c_\rho^\dagger(r) & \text{if } s < 0 \end{cases} \quad (2.6)$$

and therefore:

$$:n_\rho(r): = \begin{cases} n_\rho(r) & \text{if } r > 0 \\ n_\rho(r) - 1 & \text{if } r < 0 \end{cases} \quad (2.7)$$

Then a fermionic oscillator realization of the generators of $\hat{A}_{N-1}$ is given by

$$h_i^m = \sum_{r \in \mathbb{Z}'} (:c_\rho^\dagger(r)c_i(r + m) : - : c_{i+1}^\dagger(r)c_{i+1}(r + m) : ) \quad (i = 1, \ldots, N - 1), \quad (2.8a)$$

$$e_{\varepsilon_\rho - \varepsilon_\sigma}^m = \sum_{r \in \mathbb{Z}'} c_\rho^\dagger(r)c_\sigma(r + m), \quad \rho \neq \sigma, \quad 1 \leq \rho, \sigma \leq N \quad (2.8b)$$

Now, taking $e_{\varepsilon_\rho - \varepsilon_\sigma}^m$ and $e_{\varepsilon_\sigma - \varepsilon_\rho}^{-m}$ with let us say, $\rho > \sigma$ and $m \geq 0$, one can compute

$$\left[ e_{\varepsilon_\rho - \varepsilon_\sigma}^m, e_{\varepsilon_\sigma - \varepsilon_\rho}^{-m} \right] = \left[ \sum_{r \in \mathbb{Z}'} c_\rho^\dagger(r)c_\sigma(r + m), \sum_{s \in \mathbb{Z}'} c_\rho^\dagger(s)c_\rho(s - m) \right]$$

$$= \sum_{s \in \mathbb{Z}'} \left( c_\rho^\dagger(s - m)c_\rho(s - m) - c_\rho^\dagger(s)c_\rho(s) \right)$$

$$= \sum_{s \not\in [0,m]} (:n_\rho(s - m) : - : n_\sigma(s) :) + \sum_{s \in [0,m]} (:n_\rho(s - m) : - : n_\sigma(s) : +1)$$

$$= - \left( \sum_{k=\sigma}^{\rho-1} h_k^0 \right) + m \quad (2.9)$$

which proves that the central charge has value $\gamma = 1$ in the fermionic representation. Note that the value of the central charge is intimately related to the definition of the normal ordered product Eq. (2.7); different definitions, like:

$$:n_\rho(r) := n_\rho(r), \quad \forall r \in \mathbb{Z}' \quad (2.10a)$$

or $$:n_\rho(r) := n_\rho(r) - 1, \quad \forall r \in \mathbb{Z}', \quad (2.10b)$$
would lead to \( \gamma = 0 \).

It follows that a fermionic oscillator realization of the simple generators of \( \hat{A}_{N-1} \) in the Serre-Chevalley basis is given by \((\alpha = 0, 1, \ldots, N - 1)\)

\[
\begin{align*}
\hat{h}_\alpha &= \sum_{r \in \mathbb{Z}'} h_\alpha(r) \\
\hat{e}_\alpha^\pm &= \sum_{r \in \mathbb{Z}'} \hat{e}_\alpha^\pm(r)
\end{align*}
\]  

(2.11a)

where \((i = 1, \ldots, N - 1)\)

\[
\begin{align*}
\hat{h}_i(r) &= n_i(r) - n_{i+1}(r) =: n_i(r) - : n_{i+1}(r) : \\
\hat{h}_0(r) &= n_N(r) - n_1(r + 1) =: n_N(r) - : n_1(r + 1) + \delta_{r,-1/2} \\
\hat{e}_i^+(r) &= c_i^+(r)c_{i+1}(r), \\
\hat{e}_i^-(r) &= c_{i+1}(r)c_i(r),
\end{align*}
\]  

(2.12a)

\[
\begin{align*}
\hat{e}_0^+(r) &= c_N^+(r)c_1(r + 1), \\
\hat{e}_0^-(r) &= c_1^+(r + 1)c_N(r)
\end{align*}
\]  

(2.12b)

Inserting Eq. (2.12b) into Eq. (2.11a) and taking into account that the sum over \( r \) can be split into a sum of two convergent series only after normal ordering, one again checks that

\[
\hat{h}_0 = 1 + \sum_{r \in \mathbb{Z}'} : n_N(r) : - \sum_{r \in \mathbb{Z}'} : n_1(r) : = 1 - \sum_{j=1}^{N-1} h_j,
\]  

(2.13)

that is \( \gamma = 1 \).

In order to obtain an anyonic realization of \( U_q(\hat{A}_{N-1}) \), we will replace the fermionic oscillators by suitable anyons in the expressions of the simple generators of \( U_q(\hat{A}_{N-1}) \) in the Serre-Chevalley basis. We introduce therefore the following anyons defined on a one-dimensional lattice \((r \in \mathbb{Z}')\):

\[
a_\rho(r) = K_\rho(r)c_\rho(r) \quad 1 \leq \rho \leq N,
\]  

(2.14)

and similar expressions for the conjugated operator \( a_\rho^+(r) \), where the disorder factor \( K_\rho(r) \) is expressed as

\[
K_\rho(r) = \prod_{t \neq r} e^{-i\nu\Theta(r,t); n_\rho(t)}:
\]  

(2.15)

\( \nu \in [0, 2) \) being the statistical parameter and \( \Theta(r,t) \) a suitably defined angle (see Refs. [2, 3]).

The anyonic ordering being the natural order of the integers, in Eq. (2.15) the anyonic angle (see Appendix and Ref. [3]) can be chosen as

\[
\begin{align*}
\Theta(r,t) &= +\pi/2 \quad \text{if } r > t \\
\Theta(r,t) &= -\pi/2 \quad \text{if } r < t
\end{align*}
\]  

(2.16)
By means of Eq. (2.16) the disorder factor $K_{\rho}(r)$ can be rewritten, using the sign function $\varepsilon(t) = |t|/t$ if $t \neq 0$ and $\varepsilon(0) = 0$,

$$K_{\rho}(r) = q^{-\frac{1}{2}}\sum_{t \in \mathbb{Z}^{'}}\varepsilon(t-r):n_{\rho}(t):$$

(2.17)

with $q = \exp(i\pi\nu)$. By a direct calculation, one can prove that the $a$-type operators satisfy the following braiding relations for $r > s$:

$$a_{\rho}(r)a_{\rho}(s) + q^{-1}a_{\rho}(s)a_{\rho}(r) = 0$$
$$a_{\rho}^\dagger(r)a_{\rho}^\dagger(s) + q^{-1}a_{\rho}^\dagger(s)a_{\rho}^\dagger(r) = 0$$
$$a_{\rho}(r)a_{\rho}(s) + q\ a_{\rho}(s)a_{\rho}(r) = 0$$

(2.18)

and

$$a_{\rho}(r)a_{\rho}^\dagger(r) + a_{\rho}^\dagger(r)a_{\rho}(r) = 1$$
$$a_{\rho}(r)^2 = a_{\rho}^\dagger(r)^2 = 0$$

(2.19)

which shows that the operators $a_{\rho}(r), a_{\rho}^\dagger(s)$ are indeed anyonic oscillators with statistical parameter $\nu$. Notice that the last equations (2.19) do not contain any $q$-factor, the operators $a_{\rho}(r), a_{\rho}^\dagger(s)$ at the same point thus satisfying standard anticommutation relations.

Following Ref. [2], we also introduce twiddled anyonic oscillators:

$$\tilde{a}_{\rho}(r) = \tilde{K}_{\rho}(r)c_{\rho}(r) \quad 1 \leq \rho \leq N,$$

(2.20)

where:

$$\tilde{K}_{\rho}(r) = \prod_{t \neq r} e^{-i\nu\tilde{\Theta}(r,t):n_{\rho}(t):},$$

(2.21)

with the same statistical parameter $\nu$ and opposite braiding (and ordering) prescription, corresponding to the choice

$$\tilde{\Theta}(r,t) = -\pi/2 \quad \text{if } r > t$$
$$\tilde{\Theta}(r,t) = +\pi/2 \quad \text{if } r < t$$

(2.22)

and therefore to replace $q$ with $q^{-1}$ in Eqs. (2.17), (2.18).

Let us add that anyons with different indices $\rho, \sigma$ obey the standard anticommutation relations, for all values of $r, s \in \mathbb{Z}'$ and for all values of the anyonic parameter $\nu$.

Finally, the braiding relations between $a$-type and $\tilde{a}$-type anyons are given by

$$\left\{ \tilde{a}_{\rho}(r), a_{\rho}(s) \right\} = \left\{ \tilde{a}_{\rho}^\dagger(r), a_{\rho}^\dagger(s) \right\} = 0 \quad \text{for all} \quad r, s \in \mathbb{Z}'$$

(2.23)

$$\left\{ \tilde{a}_{\rho}^\dagger(r), a_{\rho}(s) \right\} = \left\{ \tilde{a}_{\rho}(r), a_{\rho}^\dagger(s) \right\} = 0 \quad \text{for all} \quad r \neq s \in \mathbb{Z}'$$

(2.24)
and

\[
\{ \tilde{a}_\rho (r), a^\dagger_\rho (r) \} = q \sum_{t \in \mathbb{Z}' } \varepsilon (t-r) : n_\rho (t) :
\]

\[
\{ \tilde{a}^\dagger_\rho (r), a_\rho (r) \} = q^{- \sum_{t \in \mathbb{Z}' } \varepsilon (t-r) : n_\rho (t) :}
\]

(2.25)

It is useful to remark that the following identities hold:

\[
a^\dagger_\rho (r) a_\rho (r) = \tilde{a}^\dagger_\rho (r) \tilde{a}_\rho (r) = n_\rho (r)
\]

(2.26)

and that normal ordering among anyons is defined exactly as in Eq. (2.7).

 Everywhere in this paper the generators \( E^+ \) will be expressed in terms of the oscillators \( a \) and the generators \( E^- \) in terms of the \( \tilde{a} \) as, in the deformed case, the \( E^- \)'s are not simply equal to \( (E^+)\dagger \), but also the exchange \( q \leftrightarrow q^{-1} \) must be performed.

We can now build an anyonic realization of the quantum affine Lie algebra \( U_q(\hat{A}_{N-1}) \) "anyonizing" Eqs. (2.11a-2.11b), i.e. replacing the fermionic oscillators \( c_\rho \) with the \( a_\rho \)'s in Eq. (2.12c), with the \( \tilde{a}_\rho \)'s in Eq. (2.12d), and, indifferently (see Eq. (2.26)), with the \( a_\rho \)'s or the \( \tilde{a}_\rho \)'s in the Cartan generators \( h_i \) (see Eqs. (2.12a) and (2.12b)).

**Theorem 1** An anyonic realization of the simple generators of the quantum affine Lie algebra \( U_q(\hat{A}_{N-1}) \) with central charge \( \gamma = 1 \) is given by \( (\alpha = 0, 1, \ldots, N-1) \)

\[
H_\alpha = \sum_{r \in \mathbb{Z}' } H_\alpha (r) \quad E^\pm_\alpha = \sum_{r \in \mathbb{Z}' } E^\pm_\alpha (r)
\]

(2.27)

where \( 1 \leq i \leq N-1 \)

\[
H_i (r) = h_i (r) = : n_i (r) : - : n_{i+1} (r) :
\]

\[
E^+_i (r) = a^+_i (r) a_{i+1} (r)
\]

\[
E^-_i (r) = \tilde{a}^+_i (r) \tilde{a}_i (r)
\]

(2.28a)

\[
H_0 (r) = h_0 (r) = : n_N (r) : - : n_1 (r+1) : + \delta_{r,-1/2}
\]

\[
E^+_0 (r) = q^{- \frac{1}{2} \varepsilon (r+1/2) } a^+_N (r) a_1 (r+1)
\]

\[
E^-_0 (r) = q^{- \frac{1}{2} \varepsilon (r+1/2) } \tilde{a}_1^+ (r+1) \tilde{a}_N (r)
\]

(2.28b)

**Proof**

We must check that the realization Eqs. (2.28a-2.28b) indeed satisfy the quantum affine Lie algebra \( U_q(\hat{A}_{N-1}) \) in the Serre-Chevalley basis, i.e. for \( \alpha, \beta = 0, 1, \ldots, N-1 \),

\[
[H_\alpha, H_\beta] = 0
\]

(2.29a)

\[
[H_\alpha, E^\pm_\beta] = \pm a_{\alpha\beta} E^\pm_\beta
\]

(2.29b)

\[
[E^+_\alpha, E^-_\beta] = \delta_{\alpha\beta} [H_\alpha]_{q^\gamma}
\]

(2.29c)
with the quantum Serre relations

\[ \sum_{\ell=0}^{1-a_{\alpha \beta}} (-1)^{\ell} \left[ \begin{array}{c} 1 - a_{\alpha \beta} \\ \ell \end{array} \right]_{q_{\alpha}} (E_{\alpha}^{\pm})^{1-a_{\alpha \beta}-\ell} E_{\beta}^{\pm} (E_{\alpha}^{\pm})^{\ell} = 0 \]  

where the notations are the standard ones, i.e.

\[ [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad \left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}, \quad [m]_q! = [1]_q \cdots [m]_q \]  

\( a_{\alpha \beta} \) being the Cartan matrix of \( \hat{A}_{N-1} \) given by (\( N \geq 3 \))

\[ a_{\alpha \beta} = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & 2 & -1 & \ddots & & & 0 \\ 0 & -1 & 2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & -1 & 2 & -1 & 0 \\ -1 & 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \]  

and, for \( N = 2 \), by

\[ a_{\alpha \beta} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \]  

Moreover in Eq. (2.29c) and Eq. (2.30) one has:

\[ q_{\alpha} = q^{(a_{\alpha}, a_{\alpha})/2} \]  

(\( a_{\alpha}, a_{\alpha} \)) being the length of the \( \alpha^{th} \) simple root \( a_{\alpha} \), and therefore \( q_{\alpha} = q \) for all \( \alpha = 0, 1, \ldots, N - 1 \).

The proof of this theorem will be based on the observation that the Eqs. (2.29a-2.29c) and (2.30), which define a generic deformed affine algebra \( U_q(\hat{A}) \), reduce to \( U_q(A) \) if the dot number 0 is removed, and to another finite dimensional algebra \( U_q(A') \) if the dot number 0 is kept and one or more other suitable dots are removed. In general, the relations defining \( U_q(\hat{A}) \) coincide with the overlap of those defining \( U_q(A) \) and \( U_q(A') \); therefore it will be enough to check that the equations defining \( U_q(\hat{A}) \) and \( U_q(A') \) are satisfied.

Thus our strategy will be the following:

• step 1) – prove that the generators written in the anyonic representation (see Eq. (2.27)) satisfy the relations (2.29a-2.29c) and (2.30) for \( \alpha, \beta \neq 0 \). This proof is performed using the technique outlined in Sect. 4 of [3], that is going through the following steps:
- step 1a) – individuate a set of generators \( \{ h_i, e_i^\pm; \ i \neq 0 \} \) of a low dimensional representation of \( U_q(\mathcal{A}) \);
- step 1b) – show that, once that the anyonic oscillators \( a, \tilde{a} \) are expressed in terms of the fermionic ones, the set \( \{ H_i, E_i^\pm; \ i \neq 0 \} \) is related to the set \( \{ h_i, e_i^\pm; \ i \neq 0 \} \) by iterated coproduct in \( U_q(\mathcal{A}) \) and thus is a representation of \( U_q(\mathcal{A}) \).

- \bullet \) step 2) – Repeat the same procedure for \( U_q(\mathcal{A}') \), through the following steps:

- step 2a) – individuate a set of generators \( \{ h'_\alpha, e'_\alpha^\pm; \ \alpha \in I \} \) of a low dimensional representation of \( U_q(\mathcal{A}') \); \( I \) is a suitable subset of the extended Dynkin graph of \( U_q(\hat{\mathcal{A}}) \), with \( 0 \in I \);
- step 2b) – show that, once that the anyonic oscillators \( a, \tilde{a} \) are expressed in terms of the fermionic ones, the set \( \{ H_\alpha, E_\alpha^\pm; \ \alpha \in I \} \) is related to the set \( \{ h'_\alpha, e'_\alpha^\pm; \ \alpha \in I \} \) by iterated coproduct in \( U_q(\mathcal{A}') \) and thus is a representation of \( U_q(\mathcal{A}') \).

In order to prove the present theorem, the step 1) is easily done once one realizes that, inserting Eqs. (2.14) and (2.20), the expressions (2.28a-2.28b) simplify and become

\[
E_\alpha^\pm(r) = e_\alpha^\pm(r) q^{\frac{1}{2} \sum_{i \in \mathbb{Z}} e^{(t-r)h_\alpha(t)}} \tag{2.35}
\]

where the generators \( h_\alpha(r) = H_\alpha(r)|_{q=1} = H_\alpha(r) \) and \( e_\alpha^\pm(r) = E_\alpha^\pm(r)|_{q=1} \), coincide with those defined in Eqs. (2.12a-2.12d), corresponding to the undeformed affine algebra \( \hat{\mathcal{A}}_{N-1} \).

In fact, for any fixed \( r \in \mathbb{Z} + 1/2 \), the set \( \{ h_i(r), e_i^\pm(r); \ i = 1, \ldots, N-1 \} \) is a representation of \( A_{N-1} \) of spin 0 and 1/2, and thus also of \( U_q(A_{N-1}) \) \( \{ 4 \} \); thanks to Eqs. (2.27), (2.36), \( H_\alpha, E_\alpha^\pm \) are the correct coproduct in \( U_q(A_{N-1}) \). Therefore Eqs. (2.29a-2.29k), (2.30) surely holds for \( \alpha, \beta \neq 0 \).

For the step 2), we cut the extended Dynkin diagram by deleting a dot \( \mu \neq 0 \); for any fixed \( r \in \mathbb{Z}', \) the set \( \{ h_0(r), e_0^\pm(r), h_i(r), e_i^\pm(r), h_j(r+1), e_j^\pm(r+1) \} \), where \( \mu + 1 \leq i \leq N-1 \) and \( 1 \leq j \leq \mu - 1 \), is a spin 0, 1/2 representation of \( A_{N-1} \), and thus a representation of \( U_q(A_{N-1}) \) (step 2a); again, thanks to Eqs. (2.27), (2.35), \( \{ H_\alpha, E_\alpha^\pm; \alpha \neq \mu \} \) is the correct coproduct in \( U_q(A_{N-1}) \). Therefore, Eqs. (2.29a-2.29k) and the quantum Serre relations (2.30) hold for any value of the indices \( \alpha, \beta \), only excluding the couples \( \alpha = 0, \beta = \mu \) or \( \alpha = \mu, \beta = 0 \). However, one notes that these equations obviously hold for \( |\alpha - \beta| \geq 2 \) where \( \alpha - \beta \) is taken modulo \( N \). Therefore, for \( N \geq 4 \), we take, for instance, \( \mu = 2 \) and for \( N = 3 \) it is enough to repeat the argument twice, once for \( \mu = 1 \) and once for \( \mu = 2 \).

To complete the proof, one has still to consider the case of \( U_q(\hat{\mathcal{A}}_1) \), whose Cartan matrix is reported in Eq. (2.33). In this case the previous argument fails to prove Eq. (2.29d) and the quantum Serre relations (2.30), which take now the following form

\[
\begin{align*}
(E_0^\pm)^3 E_1^\pm - \lambda \left( E_0^\pm \right)^2 E_0^\pm E_1^\pm + \lambda E_0^\pm E_1^\pm \left( E_0^\pm \right)^2 - E_1^\pm \left( E_0^\pm \right)^3 &= 0 \tag{2.36a} \\
(E_1^\pm)^3 E_0^\pm - \lambda \left( E_1^\pm \right)^2 E_0^\pm E_1^\pm + \lambda E_1^\pm E_0^\pm \left( E_1^\pm \right)^2 - E_0^\pm \left( E_1^\pm \right)^3 &= 0 \tag{2.36b}
\end{align*}
\]
where $\lambda = q^2 + q^{-2} + 1$. Such equations can however be explicitly checked by using the braiding properties of the anyonic oscillators $a$ and $\tilde{a}$.

The central charge is equal to 1 exactly for the same reasons of the undeformed case. As the deformation of any affine Lie algebra does not change its central charge, we will not repeat this argument in the following.

It is worthwhile to remark that the presence of a power of $q$ in front of the anyonic oscillators in the expressions (2.28b) of $E_0^\pm (r)$ (necessary in order to reproduce Eq. (2.35)) reflects the non vanishing of the central charge $\gamma$ and is related to the definition Eq. (2.7) of normal ordering; no power of $q$ in front of the anyonic oscillators in Eq. (2.28b) would be needed if normal ordering were defined according to Eq. (2.10a) or to Eq. (2.10b), corresponding to vanishing central charge.

Let us remark that we could prove Theorem 1 by using the embedding $U_q(\hat{A}_{N-1}) \subset U_q(A_\infty)$ according to a theorem due to Hayashi [10], that we report here in our notations:

**Theorem 2** Let $k_i, f_i^\pm (i \in \mathbb{Z})$ be the simple generators of $U_q(A_\infty)$, satisfying the following equations:

\[
\begin{align*}
\left[ k_i, k_j \right] &= 0 \quad (2.37a) \\
\left[ k_i, f_j^\pm \right] &= \pm a_{ij} f_j^\pm \quad (2.37b) \\
\left[ f_i^+, f_j^- \right] &= \delta_{ij} [k_i]_q \quad (2.37c) \\
\sum_{\ell=0}^{1-a_{ij}} (-1)^{\ell} \left[ \begin{array}{c}
1 - a_{ij} \\
\ell
\end{array} \right]_q \left( f_i^\pm \right)^{1-a_{ij}-\ell} f_j^\pm \left( f_i^\pm \right)^{\ell} &= 0 \quad (2.37d)
\end{align*}
\]

with the infinite dimensional Cartan matrix

\[
a_{ij} = \\
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & -1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

Then

\[
H_\alpha = \sum_{\ell \in \mathbb{Z}} k_{\alpha+N\ell} \quad \text{and} \quad E_\alpha^\pm = \sum_{\ell \in \mathbb{Z}} f_{\alpha+N\ell}^\pm q^{\frac{1}{2}} \sum_{m \in \mathbb{Z}} \varepsilon(m-\ell) k_{\alpha+Nm}
\]

where $\alpha = 0, 1, \ldots, N-1$, give a representation of the simple generators of $U_q(\hat{A}_{N-1})$.

By means of Eq. (2.35) one easily checks that the generators $H_\alpha$ and $E_\alpha^\pm$ defined in Eq. (2.27) coincide with those defined in Eq. (2.39), once that the identification

\[
\begin{align*}
h_\alpha(\ell - 1/2) &= k_{\alpha+N\ell} \\
c_\alpha^\pm(\ell - 1/2) &= f_{\alpha+N\ell}^\pm
\end{align*}
\]

(2.40)
where $\ell \in \mathbb{Z}$ and $\alpha = 0, 1, \ldots, N - 1$, is made.

### 3 More general representations and two dimensional anyons

In the previous section, we have built a representation of the deformed affine Lie algebras $U_q(\hat{A}_{N-1})$ by means of anyons defined on an infinite linear chain; as the corresponding fermionic representation, it has central charge $\gamma = 1$.

Representations with vanishing central charge could be built in the same way by using alternative normal ordering prescriptions Eq. (2.10a-2.10b).

Representations with $\gamma = 1$ and $\gamma = 0$ can be combined together by associating a representation to any horizontal line of a two-dimensional square lattice, infinite in one direction, let us say the horizontal one. So by $M$ copies of one-dimensional representations with central charge equal to 1 one can get representations with the value of the central charge equal to $M$. Note that by combining one representation with central charge equal to $M$ with a finite number of (one-dimensional) representations with vanishing value of the central charge one obtains an inequivalent representation with the same value of the central charge. We do not discuss here the problem of the irreducibility of these representations. The extensions to two dimensional lattice infinite in both directions can also be done, but it requires some care in the definition in order to avoid convergence problems.

We will show now that the use of anyons defined on a two-dimensional lattice naturally gives the coproduct of representations with the correct powers of the deformation parameter $q$. Each site of the two-dimensional square lattice is labelled by a vector $\mathbf{x} = (x_1, x_2)$; the first component $x_1 \in \mathbb{Z}'$ is the coordinate of a site on the line $x_2 \in \mathbb{Z}$. As it is shown in the Appendix in Eq. (4.8), in a suitable gauge, the angle $\Theta(\mathbf{x}, \mathbf{y})$ which enters into the definition of two-dimensional anyons can be chosen in such a way that

$$
\Theta(\mathbf{x}, \mathbf{y}) = \begin{cases} 
+\pi/2 & \text{if } x_2 > y_2 \\
-\pi/2 & \text{if } x_2 < y_2 
\end{cases}
$$

(3.1)

while when $\mathbf{x}$ and $\mathbf{y}$ lie on the same horizontal line, that is $x_2 = y_2$, the definitions of Sect. 2 hold.

Two-dimensional anyons still satisfy the braiding and anticommutations relations expressed in the general form in Eqs. (2.19-2.25).

Replacing in Sect. 2 one-dimensional anyons with two-dimensional ones, and making the corresponding replacements in the sums over the sites, we easily get that

$$
H_\alpha = \sum_{\mathbf{x}} H_\alpha(\mathbf{x}) \quad E_\alpha^\pm = \sum_{\mathbf{x}} E_\alpha^\pm(\mathbf{x})
$$

(3.2)

where the sum over the vector of the lattice has to be read as the sum on the (infinite) line
and the sum over the (finite) line $x_2$, that is

$$H_\alpha = \sum_{x_2} H_\alpha(\text{line } x_2)$$

$$E_\alpha^\pm = \sum_{x_2} \frac{1}{q_\alpha} \sum_{y_2} \varepsilon(y_2-x_2) H_\alpha(\text{line } y_2) E_\alpha^\pm(\text{line } x_2)$$

where $q_\alpha = q^{(a_\alpha, a_\alpha)/2}$ and $H_\alpha(\text{line } x_2)$, $E_\alpha^\pm(\text{line } x_2)$ are the simple generators of the $U_q(\hat{A})$ defined in the previous section. These equations show that the coproduct rules are fulfilled and the set $\{H_\alpha, E_\alpha^\pm\}$ gives a representation of the algebra $U_q(\hat{A})$ with central charge equal to the sum of the central charges associated to each line of the two-dimensional lattice.

It is important to remark that, for sake of simplicity, we have proved all the theorems of Sect. 2 by exploiting the representation of anyons in terms of fermions, the disorder operators giving the correct coproducts; however, this is not necessary and purely "anyonic" proofs of our Theorems can be given: it is easy to get convinced that the braiding and anticommutations relations of Eqs. (2.19-2.25) are the only relations necessary to prove that the simple generators built by means of anyons satisfy the commutations relations and the Serre equations defining the deformed affine algebras. We emphasize that anyons, defined by the braiding and anticommutations relations of Eqs. (2.19-2.25), have nothing to do with $q-\text{o}scillators$, that were even used to build representations of deformed algebras.

In the previous sections we have discussed the case of $|q| = 1$. The case of $q$ real can also be discussed and we refer to [3] for the definition of anyons for generic $q$.

In conclusions we have presented a method to get representations (in general reducible) of the affine untwisted quantum algebra $U_q(\hat{A}_{N-1})$ with positive integer value of the central charge. The role of the definition of the ordering of the anyons and of the normal ordering is essential in the above construction. One can naturally ask if this construction can also be generalized to the case of the other affine untwisted quantum algebras ($B, C, D$ series) and to the twisted affine algebras, or if a construction of a deformed Virasoro algebra can be obtained by means of anyons, or if anyons can also be used to realize affine deformed Lie superalgebras. On the same line, one could think that anyons are natural objects to use in statistical mechanical models in which quantum groups arise. We hope to address these issues in some future work.

4 Appendix

A crucial role in the construction of anyons in terms of fermions coupled to a Chern-Simons field, which endow them with a magnetic flux, is given by the angle $\Theta(x, y)$ under which the point $x$ is seen from the point $y$. As the angle $\Theta(x, y)$ appears in the exponent of the disorder operator $K(x)$ (2.13) multiplied by $i\nu$, $\nu$ being not integer, it must be thoroughly defined without any ambiguity (see [2] and references quoted therein). On a two-dimensional lattice, we fix a base point $B$ and associate to any point $x$ a path from $B$ to $x$ (see figure 1).
The angle \( \Theta(x, y) \) is defined as

\[
\Theta(x, y) = \hat{B}y^*x + \theta_0
\]  

(4.1)

where \( \theta_0 \) is a constant and \( \hat{B}y^*x \) is the angle under which the oriented path \( Bx \) is seen from a point \( y^* \) that will be eventually sent to \( y \) inside one of the four cells which have \( y \) as vertex. The point \( y^* \) is shifted from the point \( y \) by an arbitrarily small vector in order to define unambiguously the angle \( \hat{B}y^*x \) even if the path \( Bx \) passes through the site \( y \). The simplest choice is to fix \( B \) as the infinity point of the positive \( x \)-axis and to associate to each point \( x \) the horizontal straight line coming from \( B \) (see figure 2).

Choosing \( y^* = y + \varepsilon(u_x + u_y) \) with \( \varepsilon \to 0^+ \) and \( u_x, u_y \) unit vectors of the axis \( x \) and \( y \), one easily gets for two points on the same line \( (x_2 = y_2) \)

\[
\begin{align*}
\hat{B}y^*x &= 0 & \text{if } x_1 > y_1 \\
\hat{B}y^*x &= -\pi & \text{if } x_1 < y_1
\end{align*}
\]  

(4.2)

and therefore, fixing \( \theta_0 = \pi/2 \),

\[
\Theta(x, y) = \begin{cases} 
+\pi/2 & \text{if } x_1 > y_1 \\
-\pi/2 & \text{if } x_1 < y_1
\end{cases}
\]  

(4.3)

reproducing Eq. (2.16).

For two generic points \( x, y \) on the two-dimensional lattice, the angle \( \hat{B}y^*x \) can get any value in the interval \([ -\pi, \pi ) \). However, in the braiding relations, and therefore in all calculations of this paper, the only quantities coming into play are the differences \( \Theta(x, y) - \Theta(y, x) \). It is easy to check that, for any couple \( x, y \), the definition of figure 2 gives

\[
\Theta(x, y) - \Theta(y, x) = \begin{cases} 
+\pi & x_2 > y_2 \\
\phantom{+\pi} x_1 > y_1 \text{ if } x_2 = y_2 \\
-\pi & x_2 < y_2 \\
\phantom{-\pi} x_1 < y_1 \text{ if } x_2 = y_2
\end{cases}
\]  

(4.4)
The definition of the angle $\Theta(x, y)$ induces an ordering among the points of the lattice $\mathbb{Z}$. We say

\[
\begin{align*}
x \succ y & \quad \text{if} \quad \Theta(x, y) - \Theta(y, x) = +\pi \\
x \prec y & \quad \text{if} \quad \Theta(x, y) - \Theta(y, x) = -\pi
\end{align*}
\] (4.5)

This ordering coincides with the natural one for points on a horizontal line:

\[
x \succ y \iff x_1 > y_1 \text{ for } x_2 = y_2
\] (4.6)

As it is possible [3] to deform continuously the functions $\Theta(x, y)$ with the only restriction of keeping fixed the quantities $\Theta(x, y) - \Theta(y, x)$, we will choose

\[
\Theta(x, y) = \pm \frac{\pi}{2} \iff x \succ y
\] (4.7)

and therefore

\[
\Theta(x, y) = \begin{cases} 
  +\pi/2 & \text{if } x_2 > y_2 \\
  -\pi/2 & \text{if } x_2 < y_2
\end{cases}
\] (4.8)


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