Dark matter and Wigner’s third positive energy representation class

Bert Schroer
permanent address: Institut für Theoretische Physik
FU-Berlin, Arnimallee 14, 14195 Berlin, Germany
present address: CBPF, Rua Dr. Xavier Sigaud 150,
22290-180 Rio de Janeiro, Brazil

June 30, 2013

Abstract

The almost 7 decades lasting futile attempts to understand the possible physical content of the third Wigner representation class (the infinite spin class) came to a partial solution with the 2006 discovery of existence of string-localized spacetime covariantizations. This has led to a still ongoing vast generalization of renormalizability to fields with arbitrary high spin and a better understanding of the origin of partial invisibility as observed in the confinement of gluons and quarks.

The present note explains the total (non-gravitational) invisibility of fields associated to the third Wigner representation class. The last section presents a critical look at the possibility that third class Wigner matter may play a role in dark matter formation.

1 Wigner’s third positive energy representation class and dark matter

In a trailblazing 1939 publication on a representation theoretical approach to relativistic quantum theory, Eugene Wigner [1] classified all unitary irreducible ray representations of the Poincare group. In particular he found three classes of positive energy representations, of which the first two classes, consisting of massive and finite helicity massless particles, cover all the known particles of high energy physics. His third class, which also has zero mass but different transformation properties from the finite helicity representations, he referred to as infinite spin representation [1].

It is certainly a historical accident that the first observational indication of the existence of galactic “stuff” by the astrophysicist Zwicky was published

---

1In order to avoid incorrect physical associations we will refer to this rather large (continuous) set of representations as the “third Wigner class”.

---
in the same decade as Wigner’s particle classification, but it is the main aim of the present paper to argue that in hindsight both discoveries are connected by more than historical coincidences. More specifically, our conclusions will be that the characteristic property of the ”stuff” described by the third Wigner representation class is its inherent noncompact localization in spacetime. Being positive energy matter, it shares with ordinary matter the stability properties from the presence of an energy bottom as well as the coupling to gravity, but in all other aspect it is very different from matter as we know it.

Since causal localization is the defining principle of quantum field theory (QFT), it is not surprising that this difference to ordinary matter has its origin in significantly different localization properties. Indeed, whereas the first two Wigner classes are compact localizable in spacetime with the pointlike fields playing the rôle of singular (distributional) generators of local algebras of operators, the third class matter is ”irreducibly noncompact”; more precisely it is ”stuff” which can neither be subdivided into compact localized pieces nor be approximated by sequences of compact localized operators (spreading ordinary matter over unbounded spacetime regions). Its arena of action is not that of high energy collisions as realized under laboratory condition, rather it is ”galactic quantum matter par excellence” whose presence is only noticed through gravitational manifestations.

Whereas the representation theoretical content of the massive and massless finite helicity Wigner classes could be converted into covariant free fields associated with classical Lagrangians, all attempts to relate the third Wigner class to classical field structures failed. Since the only known way to study physical properties of quantum matter in the past was through Lagrangian quantization and perturbative expansions, the physical properties of third class Wigner matter remained unknown for a very long time. Wigner knew that the important missing property was its localization in spacetime; but his own as well as other attempts to solve this problem by looking for relativistic field equations associated to these representations remained without success. Wigner also looked at more direct formulations of quantum localization properties within those representations. His attempt (with one of his collaborators [2]) to adapt Born’s quantum mechanical localization to the invariant form of inner product in his representation theory did not lead to the desired covariance properties of causal localization which is inherent in quantum fields. Wigner’s failure of finding a pre-form of the field theoretic causal localization within his representation theory led him to believe that the conceptual understanding of QFT was incomplete since its causal localization principle should already be reflected in his particle classification which he considered to be the more profound way to understand QFT rather than quantizing classical fields. This may explain why, after having made important early contributions in joint work with Pascual Jordan, he felt that the conceptual understanding of QFT, even after the great advances in renormalized QED, was still incomplete.

In the 50s Arthur Wightman and Rudolf Haag began to explore Wigner’s representation theory and later Weinberg [3] gave an account of the ”covariantization” of Wigner’s unitary representations which led to a rather systematic
construction of free fields of all spins and helicities. But in all these contributions the employed method did not work for the third Wigner representation class. Weinberg’s consolation of these futile attempts was to say that "nature does not make use of it”.

Although in the 70s there was a theorem stating that the third Wigner representation was not compatible with pointlike fields [12], this did not stop further futile attempts to find descriptions in terms of pointlike fields. Only with the arrival of the new concept of modular localization, as the intrinsic (field-coordinatization independent) formulation of quantum causal localization, the age-old problem was finally solved in 2006. The solution came in two stages. The first one was the proof of a theorem [11], stating that whereas all positive energy representations can always be localized in arbitrarily narrow spacelike cones $C$

\[
C : x + \cup_{\lambda \geq 0} \lambda D, \text{ with core } x + \mathbb{R}_+ e, \ e^2 = -1
\]

where $D$ is a diamond-shaped double cone (the two-sided causal closure of a unit ball $|\vec{x}| \leq 1$ in Minkowski spacetime), only $m > 0$ and $m = 0$ finite helicity representations also allow compact modular localization in arbitrarily small double cones $D$. In other words whereas the localization for the massive and finite helicity zero mass can be sharpened to double cones $D$, this is not possible for the third class; in this case any attempt to construct a Wigner subspace localized in a compact spacetime region will lead to a null-space. This modular localization is precisely the intrinsic relativistic localization concept which Wigner was looking for. In this case it was not the lack of luck; in retrospect it is obvious that there was no chance to get to these concepts, inasmuch as there also was no possibility to unravel the Einstein-Jordan conundrum. Even though the latter led to the birth of QFT, it only found its complete resolution in terms of modular localization theory [5].

As the core of a double cone is a point, the core of spacelike cone a semi-infinite spacelike string; so the obvious idea, suggested by the above theorem, is that, as the generating covariant wave functions of first and second class Wigner representation can be chosen pointlike, those of the spacelike cones should be semi-infinite stringlike. With this conceptual guide, it was possible to construct the generating quantum fields of the third Wigner class in the form of stringlike fields $\psi(x,e)$ localized on semiinfinite spacelike lines $x + \mathbb{R}_+ e$ [9][10]. Possibly as a result of the use of the still little known concept of modular localization, this work was overlooked; in [11] where, in spite of a citation referring to a 1970 paper [12] in which the inconsistency with pointlike localization was proven, the old futile attempts to force this representation class into a pointlike (gauge) setting continued.

As emphasized by Wigner, the difference between the three representation classes has its origin in the Lorentz-subgroup (Wigner’s "little group") which leaves a typical vector in the energy-momentum spectrum invariant; the choice

---

2This concept was first used in order to understand the spacetime properties of the Zamolodchikov-Faddeev algebra [5] which led to a new constructive approach to integrable models [5][7].
\[ \vec{p} = (1, 0, 0, 0) \text{ for } m > 0 \text{ and } \vec{p} = (1, 0, 0, 1) \text{ for } m = 0 \] is the standard choice which permits the immediate identification with the rotation group in 3 dimensions respectively the noncompact Euclidean group in 2 dimensions \( E(2) \). Any representation for a general value of \( p \) requires the introduction of a family of Lorentz transformations \( B_p \) which transform the \( \vec{p} \) into \( p \). The representation theory of the \( E(2) \) invariance group which leaves \( \vec{p} \) fixed (Wigner’s little group) can be subdivided into degenerate representations (finite helicity representations), in which only the abelian rotation subgroup is nontrivial represented, and faithful representations of \( E(2) \). Since the differences with respect to localization properties and the related physical manifestations between the these two subclasses are actually much bigger than what is is revealed by group theory, the division into three representation classes is well justified.

The construction of stringlocal fields parallels in many aspects that of “covariantization” in terms of intertwiners which connect unitary Wigner representation with covariant pointlike tensorial/spinorial fields \( \Psi^{(A,B)}(x) \). In analogy to the well-known representation for pointlike fields, one expects a stringlike field to be of the form (for simplicity we specialize to self-conjugate bosonic fields)

\[
\Psi(x,e) = \frac{1}{(2\pi)^{3/2}} \int (e^{ipx} u(p, e) \cdot a^*(p) + h.c.) \frac{d^3p}{2p_0} 
\]

\[
D_\kappa(R(\Lambda, p)) u(\Lambda^{-1}p, e) = u(p, \Lambda e), \text{ with } (D_\kappa(z, \theta) \varphi)(k) = e^{iz\cdot k} \varphi(R_\theta^{-1}k) 
\]

\[
U(a, \Lambda) \Psi(x, e) U(a, \Lambda)^* = \Psi(x, \Lambda e) 
\]

Here the second line defines the intertwining property of \( u \) in terms of a representation \( D_\kappa \) of \( E(2) \) on \( L^2 \)-integrable wave functions \( \varphi(k) \) restricted to the circle of radius \( \kappa \), \( k^2 = \kappa^2 \); the notation \( R(\Lambda, p) = B_p^{-1} \Lambda B_{\Lambda^{-1}p} \) for the Wigner “rotation” is maintained except that the family of \( p \)-dependent Lorentz transformations \( B_p \) now transform the lightlike reference vector \( \vec{p} \) into on the generic position \( p \) so that the net Wigner “rotation” is a \( \Lambda, p \)-dependent \( E(2) \) transformation. The third class contains the resulting desired covariant transformation property of the stringlocal field. This has to be complemented by the (graded) commutation relations between two stringlocal fields in case of spacelike separation of their strings \( x + R, e, \) and \( x' + R, e' \). Fermionic commutation relations result from a representations of the two-fold covering of the abelian rotation. The stringlocal fields with different values of \( \alpha \) of their intertwiners \( u^\alpha(p, e) \) but the same value of the Pauli-Lubanski invariant \( \kappa \) are linear member of a localization class, i.e. their (graded) commutators vanish for relative spacelike separation of their strings \[9][10][13].

It is in the construction of these intertwiner functions that modular localization theory enters. The idea is to start from the tensor product of scalar wave functions on the mass hyperboloid with a wave function space on the \( d=1+2 \) de Sitter space \( e^2 = -1 \) on which an appropriate unitary representation of the homogeneous Lorentz group acts labeled by a complex number \( \alpha \) [13]. Both spaces admit an intrinsic compact localization with pointlike generators. The looked for representation of the little group corresponding to the value \( \kappa \) can
be obtained by a suitable projection [9], since it is a subgroup of the Lorentz group and the choice of the unitary representation of the Lorentz group guarantees that the looked for component occurs in the decomposition. The result is an irreducible representation which fits the sought intertwiner properties; in particular the trivial tensor product localization in Minkowski spacetime tensored with the d=1+2 de Sitter space representation has been converted into one which is compatible with that of a point \( x \) with an attached spacelike string \( x + \mathbb{R}_+ \epsilon, \epsilon^2 = -1 \). Instead of tensor product localization, one now encounters the nontrivial string-locality in Minkowski space.

The same idea has been applied to the first and second Wigner class [9][10][15] with the following results

- The massive class admits scalar representation of the form (2) for any spin since the relation between physical spin \( s \) and tensorial/spinorial field indices is limited to pointlike fields.

- In the finite helicity massless class, the stringlocal fields maintain the tensor character of the pointlike localization. The use of stringlocal fields resolves the clash between localization and the Hilbert space in case of potentials (\( A_\mu \)-vectorpotentials for \( s = 1 \), tensor potentials for \( s > 1 \)) in favor of stringlocal potentials. In the pointlike setting one has to formulate perturbation theory in Krein space (the well-known gauge theoretic BRST formalism). A pointlike Hilbert space description can only be obtained for gauge invariant observables (which excludes charge-carrying fields).

- In all cases (massive or massless) there exist stringlocal potentials of short distance dimension \( d_{sd} = 1 \) for all \( s \), whereas the \( d_{sd} \) for massive pointlike potentials increase as \( s + 1 \). This permits to write down interactions within the power-counting limitation for any \((m > 0, s)\) representation. In particular it is now possible to replace the gauge setting in Krein space by a stringlocal setting which is only subjected to the rules of causal localization. In many cases the results correspond to each other, but often the interpretation is totally different (couplings of neutral matter fields to massive vectormesons as compared to the Higgs mechanism) and in certain cases the gained conceptual inside goes much beyond to what can be attained in the gauge setting.

It is understandable that with so many new results about standard matter and its possible interactions, coming from a project whose original purpose was to unravel the localization of the third Wigner class, the question of physical aspects of the latter moved for a number of years into the background. Now, that these projects have develop their self-supporting momentum [16][17][18], it is time to return to the third Wigner class in order to learn more about its possible physical properties.

---

3 The growing number of results originating in modular theory well justify the at that time somewhat futuristic title of a paper by the late H.-J. Borchers entitled “On revolutionizing quantum field theory with Tomita’s modular theory” [19]
It is helpful to recall that the modern view of local quantum physics (LQP) about fields is that they correspond to "coordinatizations" (in analogy to geometry) of conceptually more intrinsic modular localization concepts: the net of local algebras $\mathcal{A}(\mathcal{O})$ representing operators localized in the spacetime region $\mathcal{O}$. This is a concept which uses the intrinsic modular data of each individual algebra and, at least in its most advanced form, tries to extract all physical properties (spacetime symmetries, inner symmetries) from the abstract relative modular positioning of a finite number copies of a "monad" (mathematically a hyper-finite type III$_1$ von Neumann factor) [20][21]. Contrary to an (unfortunately) widespread opinion, modular localized quantum matter cannot be subjected to dimensional reductions or embedding without violating its causal localization properties. Unlike classical field theory or quantum mechanics, causal locality in spacetime (including its dimensionality) is an inseparable attribute of quantum matter in local quantum physics.

The "coordinatization role" of fields with respect to the coordinatization-independent modular localization structure of local nets of algebras is best taken care of by collecting all (point- or string-like) fields which are relatively local to each other into one localization class (Borchers class) [22][23]. These fields are known to describe the same physics; in particular their $S$-matrices are identical. This does not only include the fields [2] which are linearly related to the Wigner creation/annihilation operators [9], but includes also nonlinear composites (Wick-ordered polynomials) in these fields. According to our best knowledge there are no pointlike composites in infinite spin field class. It is important to stress that any attempt to picture third class Wigner matter as just spread out standard matter is a misunderstanding; this noncompact matter is of a completely new kind which has no analog in QM.

According to standard interpretation of measurements, the result of a measurement in a counter (suitably idealized in order to avoid registering events in the vacuum [23]) is the localization of the measured object in the spacetime region (extension, duration) of the counter. Irreducibly noncompact matter (no subdivision, no approximation by sequences of standard localized matter) has different properties (or better lacks them) from global operators as e.g. conserved charges in ordinary matter and cannot be mimicked in terms of infinitely extended wave functions in QM. There is no way to register such matter in an earthly particle counter, its arena of action is galactic; by its invisible penetration of galaxies it changes the gravitational balance.

Such matter is expected to be also inert with respect to selfinteractions and in interactions with ordinary matter. It creation in a collision process between ordinary particles which collide within a compact spacetime region would violate causality in view of the noncompact stringlike extension. One would like to see this formally by coupling such matter. Such computations are complicated by the fact that its intertwining function $u(p, e)$ does not have a rational dependence

---

4Whereas it is fairly easy to maintain spacelike commutativity, a problem arises with causal completeness $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}'')$ [7].

5There have been attempts to construct point-local composites [9] but even recent refinements of this search did not reveal any (private communication by Köhler).
on these variables

\[ u^{(\alpha)}(p,e)(k) = e^{-i\alpha/2} \int d^2 z e^{i z \cdot k} (B_p \xi(z) \cdot e)^\alpha, \quad \alpha = -1 + i \rho \]  

(3)

\[ \xi(z) = \left( \frac{1}{2} (|z|^2 + 1, z_1, z_2, \frac{1}{2} (|z|^2 - 1)) \right) \]

where \( B_p \) is the previously introduced family of boosts. The calculation of the two-point function of string fields requires to carry out the k-integrals between two \( u's \) in closed form; although this can be done for class 1 and 2 stringlocal matter, this is impossible for the case at hand. On would expect that the loop integrations involving time-ordered products, which would arise in self couplings or interactions with normal matter, are incorrigibly divergent, indicating that noncompact cannot couple. Hopefully one can find ways to show this more explicit even in the absence of closed form expressions for propagators.

It is quite amusing to compare the darkness with the confinement mechanism within the new stringlocal setting; for simplicity we restrict the discussion to a pure Y-M interaction. Free vectorpotentials are well-defined stringlocal fields and their interacting counterparts are irreducible stringlocal fields since they cannot be represented as a seminfinite integration over a pointlike observable as their abelian counterparts. Their situation is opposite to that of third class Wigner matter; whereas the preservation of causality suggests that as the latter cannot react in a way which would lead to the disappearance of its noncompact localization, the appearance of gluons coming out of a compact collision region of normal matter would equally violate causality. But in the gluon case we have a much better chance to see in more details how the expected gluon confinement comes about, here the propagators are analytically accessible. In the new setting correlations involving interacting gluon potentials can be viewed as zero mass limits of interacting finite mass gluons whose stringlocal correlation functions are well-defined within the new stringlocal Hilbert space setting.[16][17][18].

In BRST perturbation theory such limits are known to lead to logarithmic infrared divergencies in the unphysical pointlike gluon field correlations; and since the momentum structure of the loop integrals for small \( p \) is not better for stringlocal potentials, this perturbative infrared divergence is likely to appear also for the physical stringlike gluon fields in Hilbert space. But what do such logarithmic divergencies in \( m_{gl} \to 0 \) mean in physical terms? In analogy to the on-shell perturbative \( \ln m \) leading logarithmic divergent terms whose resummation led to a vanishing scattering amplitude for charged particles[21], one expect that the resummation of the logarithmic \textit{off-shell} terms will lead to a vanishing of those off-shell correlations containing gluon fields so that only correlations of pointlike composites survive. This is the most perfect scenario for gluon confinement; in fact there is no alternative since the presence of interacting spacelike noncompact gluons creates havoc with causality. The difference between non-trivial off-shell and vanishing on-shell amplitudes in abelian gauge theories and vanishing off-shell gluon amplitudes (confinement) would then find its explanation in the the reducible/irreducible nature of their string fields. Whereas vanishing on-shell amplitudes require to look for the presence of charged parti-
cles in photon-inclusive cross sections, there is nothing to look for in the case of vanishing off-shell expectation values. Without stringlocal gluons we would not be able to define Y-M interactions but they get lost in the limit of $m \to 0$ when only pointlike generated composites survive.

Confined objects associated to interacting irreducible stringlike zero mass potentials are in a complementarity relations with free noncompact matter of the third Wigner class; whereas massless gluons cannot come out of a compact scattering region of ordinary matter, third class Wigner strings, once in the universe, are inert noncompact objects which only change the gravitation balance on large scales.

The proposal in this note is quite different from other attempted explanations coming from particle theory. There exists however phenomenological proposals which has a vague resemblance to noncompact localized matter. They uses quantum mechanical wave functions of galactic extension [25]. The authors solve a nonrelativistic Schrödinger equation for a particle with the absurdly small mass of $10^{-18} \text{eV}$ coupled to a Newtonian potential. Although quantum mechanics may have some use in the form of phenomenological description of galactic dark matter, only QFT with its modular localization property which permits intrinsically noncompact matter is capable to provide a conceptual understanding.

It should be clear that the interest in dark matter in the this work is less motivated by observational astrophysical facts; it is rather nourished by the philosophical (Einsteinian) belief that nature will not miss the chance to illustrate the natural theoretical appearance of the third Wigner positive energy representation class in terms of its hardware, especially because it shares the stability property and its coupling to gravity with the two particle physics classes.

Noncompact modular localized "stuff" is a new form of massless matter which does not share the fleeting properties of ordinary massless matter (photons) on galactic scales; it is therefore a viable candidate for explaining gravitational imbalances in galaxies. Since it depends on the Poincaré group in Minkowski spacetime, it is not clear what remains of these ideas in the context of local covariance in curved spacetime. A necessary prerequisite for the existence of irreducibly noncompact localized matter is certainly an unlimited spacelike extension.

If one slightly modifies Weinberg’s dictum [3] about third class Wigner matter "nature does not use it" by replacing nature by "nature as we thought to know it", the statement remains consistent with the present results. Its characteristic property is primarily the absence of all non-gravitational properties which we associate with matter as we know it. Any registering in particle counters of high energy laboratories or any observed reactions of astrophysical dark matter outside the omnipresent (shared by all positive energy matter) gravitational coupling falsifies the present idea; unlike all other proposals there is simply no room for accommodation. As long as the astrophysicists do not discover other manifestations than gravitational imbalances and as long as a particle counter is not activated by something which for as yet unknown good reasons should be identified with astrophysical dark matter, the stuff associated with the third Wigner class is a serious contender. It fulfills Poppers falsifi-
cation postulate for a scientific theory in a perfect way (without any range of accommodation).

If the present work leads to further investigations, and if in this way the powerful, but still widely unknown concept of modular localization attracts more attention which leads to more concrete computational results concerning interactions involving noncompact third class matter, the purpose of this note will have been accomplished.

Acknowledgements: The present results could not have been obtained without the prior joint work on the use of modular localization for the construction of string-local fields for any spin \([9]\). I thank Jens Mund and Jakob Yngvason for their continued critical interest. I am indebted to Edward Witten for prior informations concerning astrophysical facts about dark matter.

References

[1] E.P. Wigner, On unitary representations of the inhomogeneous Lorentz group, Ann. Math. 40, (1939) 149

[2] T. D. Newton and E. P. Wigner, Localized states for elementary systems, Review of Modern Physics 21, (1949) 400-428

[3] S. Weinberg, The Quantum Theory of Fields I, Cambridge University Press 1991

[4] R. Brunetti, D. Guido and R. Longo, Modular localization and Wigner particles, Rev. Math. Phys. 14, (2002) 759

[5] B. Schroer, Modular Wedge Localization and the d=1+1 Formfactor Program, Ann. Phys. 295 (1999), 190, and references therein

[6] G. Lechner, On the Construction of Quantum Field Theories with Factorizing S-Matrices, PhD thesis, arXiv:math-ph/0611050

[7] B. Schroer, The foundational origin of integrability in quantum field theory, Found. Phys. 43, (2013) 329; arXiv:1109.1212

[8] B. Schroer, The Einstein-Jordan conundrum and its relation to ongoing foundational research in local quantum physics, Eur. Phys. J. H 38, (2013) 137

[9] J. Mund, B. Schroer and J. Yngvason, Commun. Math. Phys. 268, (2006) 621

[10] J. Mund, B. Schroer and J. Yngvason, String-localized quantum fields from Wigner representations, Phys.Lett. B596 (2004) 156-162, arXiv:math-ph/0402043

[11] P. Schuster and N. Toro, A Gauge Field Theory of Continuous-Spin Particles, arXiv:3023225
[12] J. Yngvason, Zero-mass infinite spin representations of the Poincaré group and quantum field theory, Commun. Math. Phys. 18 (1970), 195

[13] J. Mund, String–localized quantum fields, modular localization, and gauge theories, New Trends in Mathematical Physics (V. Sidoravicius, ed.), Selected contributions of the XVth Int. Congress on Math. Physics, Springer, Dordrecht, 2009, pp. 495–508

[14] J. Bros and U. Moschella, Rev. Math. Phys. 8 (1996) 327

[15] M. Plaschke and J. Yngvason, Massless, String Localized Quantum Fields for Any Helicity, [arXiv:1111.5164]

[16] J. Mund, String-localized massive vector Bosons without ghosts and indefinite metric: The example of massive QED, to appear

[17] B. Schroer, Interactions with quadratic dependence on string-localized massive vectormesons: massive scalar quantum electrodynamics, [arXiv:1307.3469]

[18] J. Mund and B. Schroer, Adiabatic equivalence between string- and point-local fields and renormalizable couplings of massive vectormesons, to appear

[19] H-J. Borchers, On revolutionizing quantum field theory with Tomita’s modular theory, J. Math. Phys. 41, (2000) 860

[20] R. Kaehler and H.-P. Wiesbrock, Modular theory and the reconstruction of four-dimensional quantum field theories, Journal of Mathematical Physics 42, (2001) 74-86

[21] B. Schroer, Localization and the interface between quantum mechanics, quantum field theory and quantum gravity I, Stud. Hist. Philos. Mod. Phys.41, (2010) 104

[22] R. F. Streater and A. S. Wightman, PCT Spin&Statistics and all that, New York, Benjamin 1964

[23] R. Haag, Local Quantum Physics, Springer 1996

[24] D. Yenni, S. Frautschi and H. Suura, The infrared divergence phenomenon and high-energy processes, Ann. of Phys. 13, (1961) 370

[25] S. C. Spivey, Z. E. Musielak and J. L. Fry, Astronomical constraints on quantum theories of cold dark matter—I. Einasto density profile for galactic haloes, MNRAS 428 (2013) 713, and previous publications quoted therein.