Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra, and $Y_\hbar(\mathfrak{g})$, $U_q(L\mathfrak{g})$ the corresponding Yangian and quantum loop algebra, with deformation parameters related by $q = e^{\pi i \hbar}$. When $\hbar$ is not a rational number, we constructed in [10] an exact, faithful functor $\Gamma$ from the category of representations of $Y_\hbar(\mathfrak{g})$ to those of $U_q(L\mathfrak{g})$, whose restrictions to $\mathfrak{g}$ and $U_q\mathfrak{g}$ respectively are integrable and in category $\mathcal{O}$. The functor $\Gamma$ is governed by the additive difference equations defined by the commuting fields of the Yangian, and restricts to an equivalence on an explicitly defined subcategory of representations of $Y_\hbar(\mathfrak{g})$. Assuming further that $\text{Im} \hbar \neq 0$, and that $\mathfrak{g}$ is finite–dimensional so that the categories in question are the finite–dimensional representations of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$, we construct in this paper a tensor structure on $\Gamma$ when both $U_q(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$ are endowed with the Drinfeld coproduct. This tensor structure arises from the abelian $q$KZ equations defined by the commutative part $R_0$ of the $R$–matrix of $Y_\hbar(\mathfrak{g})$. Along the way, we show that the deformed Drinfeld coproduct is a rational function of the deformation parameter, that it endows the finite–dimensional representations of $U_q(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$ with the structure of meromorphic tensor categories, and that $R_0^\circ$ gives rise to a meromorphic braiding on $\text{Rep}_{\mathbb{C}}(Y_\hbar(\mathfrak{g}))$.

1. Introduction

1.1. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, and $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$ the Yangian and quantum loop algebra of $\mathfrak{g}$. When $\hbar \in \mathbb{C} \setminus \mathbb{Q}$, so that $q = e^{\pi i \hbar}$ is not a root of unity, we constructed in [10] an exact, faithful functor $\Gamma$ from the category of non–congruent finite–dimensional representations of $Y_\hbar(\mathfrak{g})$ to those of $U_q(L\mathfrak{g})$. The functor $\Gamma$ is governed by the additive difference equations defined by the commuting fields of the Yangian, and restricts to an equivalence on an explicitly defined subcategory of representations of $Y_\hbar(\mathfrak{g})$. Assuming further that $\text{Im} \hbar \neq 0$, and that $\mathfrak{g}$ is finite–dimensional so that the categories in question are the finite–dimensional representations of $Y_\hbar(\mathfrak{g})$ and $U_q(L\mathfrak{g})$, we construct in this paper a tensor structure on $\Gamma$ when both $U_q(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$ are endowed with the Drinfeld coproduct. This tensor structure arises from the abelian $q$KZ equations defined by the commutative part $R_0$ of the $R$–matrix of $Y_\hbar(\mathfrak{g})$. Along the way, we show that the deformed Drinfeld coproduct is a rational function of the deformation parameter, that it endows the finite–dimensional representations of $U_q(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$ with the structure of meromorphic tensor categories, and that $R_0^\circ$ gives rise to a meromorphic braiding on $\text{Rep}_{\mathbb{C}}(Y_\hbar(\mathfrak{g}))$.
1.2. The Drinfeld coproduct on $U_q(Lg)$ was defined in [5] and involves formal infinite sums of elements in $U_q(Lg) \otimes^2$. Composing with the $\mathbb{C}^\times$–action on the first factor yields a deformed coproduct, which is an algebra homomorphism

$$\Delta_\zeta : U_q(Lg) \rightarrow U_q(Lg)((\zeta^{-1})) \otimes U_q(Lg)$$

where $\zeta$ is a formal variable [12, §6]. This coproduct is coassociative in the sense that $\Delta_\zeta \otimes 1 \circ \Delta_\zeta = 1 \otimes \Delta_\zeta \circ \Delta_\zeta \otimes 1$ [13, Lemma 3.2].

We show in Section 3 that $\Delta_\zeta$ is analytically well–behaved in the following sense. If $V, W$ are finite–dimensional representations of $U_q(Lg)$, the action of $U_q(Lg)$ on $V((\zeta^{-1})) \otimes W$ obtained via $\Delta_\zeta$ is the Laurent expansion at $\infty$ of a family of actions of $U_q(Lg)$ on $V \otimes W$, whose matrix coefficients are rational functions of $\zeta \in \mathbb{C}^\times$. We denote $V \otimes W$ endowed with this action by $V \otimes_\zeta W$.

The tensor product $\otimes_\zeta$ gives $\text{Rep}_{fd}(U_q(Lg))$ the structure of a meromorphic tensor category in the sense of [18]. This category is strict in that for any $V_1, V_2, V_3 \in \text{Rep}_{fd}(U_q(Lg))$, the identification of vector spaces

$$(V_1 \otimes_\zeta V_2) \otimes_\zeta V_3 = V_1 \otimes_\zeta_2 (V_2 \otimes_\zeta V_3)$$

intertwines the action of $U_q(Lg)$. Meromorphic (braided) tensor categories were introduced by Soibelman in [18] to formalise the structure of the category of finite–dimensional representations of $U_q(Lg)$ endowed with the standard (Kac–Moody) tensor product and the $R$–matrix $R(\zeta)$. The observation that such a structure also arises from the Drinfeld coproduct and the commutative part of the $R$–matrix (see §1.7–1.9 below) seems to be new.

1.3. In a related vein, a Drinfeld coproduct was defined for the double Yangian $DY_h(g)$ by Khoroshkin–Tolstoy [14]. We prove similarly in §3 that this coproduct can be deformed by using the translation action of $\mathbb{C}$ on $Y_h(g)$, and understood as giving $\text{Rep}_{fd}(Y_h(g))$ the structure of a meromorphic tensor category whose tensor product satisfies

$$(V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3 = V_1 \otimes_{s_1+s_2} (V_2 \otimes_{s_2} V_3)$$

for any $V_1, V_2, V_3$ and $s_1, s_2 \in \mathbb{C}$.
1.4. Before stating our main result, let us recall the notion of non–congruent representation of $Y_h(\mathfrak{g})$ [10, §5.1]. Let $I$ be the set of vertices of the Dynkin diagram of $\mathfrak{g}$, and let $\{\xi_{i,r}, x_{i,r}^\pm \}_{i \in I, r \in \mathbb{N}}$ be the loop generators of $Y_h(\mathfrak{g})$ (see [4], or §2 for definitions). Consider the generating series

$$\xi_i(u) = 1 + h \sum_{r \geq 0} \xi_{i,r} u^{-r} \quad \text{and} \quad x_i^\pm(u) = h \sum_{r \geq 0} x_{i,r}^\pm u^{-r-1}$$

On a finite–dimensional representation $V$, these series are expansions at $u = \infty$ of $\text{End}(V)$–valued rational functions [10, Prop. 3.6]. $V$ is called non–congruent if, for any $i \in I$ the poles of $x_i^+(u)$ (resp. $x_i^-(u)$) do not differ by non–zero integers.

1.5. If $V_1, V_2$ are non–congruent finite–dimensional representations of $Y_h(\mathfrak{g})$, the Drinfeld tensor product $V_1 \otimes V_2$ is generically non–congruent in $s$. The following is the main result of this paper.

**Theorem.** Assume that $\text{Im} \ h \neq 0$. Then,

(i) There exists a meromorphic $\text{End}(V_1 \otimes V_2)$–valued function $\mathcal{J}_{V_1, V_2}(s)$ which is natural in $V_1, V_2$ and such that

$$\mathcal{J}_{V_1, V_2}(s) : \Gamma(V_1) \otimes \Gamma(V_2) \longrightarrow \Gamma(V_1 \otimes_s V_2)$$

is an isomorphism of $U_q(L\mathfrak{g})$–modules, where $\zeta = e^{2\pi i s}$.

(ii) $\mathcal{J}$ defines a meromorphic tensor structure on $\Gamma$. That is, the following diagram is commutative for any non–congruent $V_1, V_2, V_3 \in \text{Rep}_{id}(Y_h(\mathfrak{g}))$

\[
\begin{array}{ccc}
\Gamma(V_1) \otimes_{\zeta_1} \Gamma(V_2) & \otimes_{\zeta_2} \Gamma(V_3) & \longrightarrow \Gamma(V_1) \otimes_{\zeta_1 \zeta_2} (\Gamma(V_2) \otimes_{\zeta_2} \Gamma(V_3)) \\
\mathcal{J}_{V_1, V_2}(s_1) \otimes 1 & 1 \otimes \mathcal{J}_{V_2, V_3}(s_2) & \\
\Gamma(V_1 \otimes_{s_1} V_2) \otimes_{\zeta_2} \Gamma(V_3) & \longrightarrow & \Gamma(V_1) \otimes_{\zeta_1 \zeta_2} (\Gamma(V_2 \otimes_{s_2} V_3) \\
\mathcal{J}_{V_1 \otimes_{s_1} V_2, V_3}(s_2) & \longrightarrow & \mathcal{J}_{V_1, V_2 \otimes_{s_2} V_3}(s_1 + s_2) \\
\Gamma((V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3) & \longrightarrow & \Gamma(V_1 \otimes_{s_1 + s_2} (V_2 \otimes_{s_2} V_3))
\end{array}
\]

where $\zeta_i = \exp(2\pi i s_i)$.

1.6. Just as the functor $\Gamma$ is governed by the abelian, additive difference equations defined by the commuting fields $\xi_i(u)$ of the Yangian, the tensor structure $\mathcal{J}_{V_1, V_2}(s)$ arises from another such difference equation, namely an abelianisation of the $q$ KZ equations on $V_1 \otimes V_2$ introduced in [9, 17]. Specifically, let

$$\mathcal{R}^0(s) \sim 1 + h \frac{\Omega_h}{s} + \cdots$$
be the diagonal part of the $R$–matrix of $Y_h(g)$ acting on $V_1 \otimes V_2$, where $Ω_h \in h \otimes h$ is the Cartan part of the Casimir tensor of $g$. The abelianised $q$KZ equation is the equation $Φ(s + 1) = R^0(s)Φ(s)$, for a meromorphic $V_1 \otimes V_2$–valued function $Φ$. It admits a canonical fundamental solution $Φ^+(s)$ which is holomorphic and invertible on a right half–plane, and has an asymptotic expansion of the form $(1 + ϕ_1 s^{-1} + \cdots) \cdot s^{-Ω_h}$ as $s \to \infty$ with $\text{Re } s >> 0$ (see \cite{2, 3, 16} or \cite{10, §4}). The twist $J_{V_1,V_2}(s)$ is then equal to $Φ^+(s + 1)^{-1}$, which is a regularisation of the infinite product

$$\cdots R^0(s + 3)R^0(s + 2)R^0(s + 1)$$

1.7. The construction of $R^0(s)$ as a meromorphic function of $s$ does in fact require some work. $R^0(s)$ was constructed as a formal infinite product with values in the double Yangian $DY_h(g)$ by Khoroshkin–Tolstoy \cite[Thm. 5.2]{14}. Simple calculations show however that this product does not converge on tensor products of finite–dimensional representations. To regularise it, we notice in Section 4 that $R^0(s)$ formally satisfies an (abelian) additive difference equation whose step is a multiple of $h$. We then show that the coefficient matrix $A(s)$ of this equation can be interpreted as a rational function of $s$, and define $R^0(s)$ as one of the canonical fundamental solutions of the difference equation. Let us outline this approach in some more detail.

1.8. Let $b_{ij} = d_i a_{ij}$ the entries of the symmetrized Cartan matrix of $g$. Let $T$ be an indeterminate, and $B(T) = ([b_{ij}]_T)$ the corresponding matrix of $T$–numbers. Then, there exists an integer $l \geq 1$ such that $B(T)^{-1} = [l]^{-1}_T C(T)$, where the entries of $C(T)$ are Laurent polynomials in $T$ with coefficients in $\mathbb{N}$ \cite{14}.

Consider the following $GL(V_1 \otimes V_2)$–valued function of $s \in \mathbb{C}$

$$A(s) = \exp \left( - \sum_{i,j \in I} c_{ij}^{(r)} \int_{C} t_i'(v) \otimes t_j \left( v + s + \frac{(l + r)h}{2} \right) dv \right)$$

where

- $c_{ij}(T) = \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} T^r$ are the entries of $C(T)$.
- the contour $C$ encloses all poles of $ξ_i(u)^{±1}$ on $V_1$.
- $t_i(u) = \log(ξ_i(u))$ is defined by choosing a branch of the logarithm.
- $s \in \mathbb{C}$ is such that $v \to t_j(v + s + (l + r)h/2)$ is analytic on $V_2$ within $C$, for every $j \in I$ and $r \in \mathbb{Z}$ such that $c_{ij}^{(r)} \neq 0$.

We prove in Section 4.3 that $A$ extends to a rational function of $s$ which has the following expansion near $s = \infty$

$$A(s) = 1 - lh^2 \frac{Ω_h}{s^2} + O(s^{-3})$$
1.9. The infinite product $R_0$ considered in [14] formally satisfies

$$R_0(s + l\hbar) = A(s)R_0(s)$$

This difference equation is regular (that is, the coefficient of $s^{-1}$ in the expansion of $A(s)$ at $s = \infty$ is zero), and therefore admits two canonical meromorphic fundamental solutions $R_{0, \pm}(s)$. The latter are uniquely determined by the requirement that they be holomorphic and invertible for $\pm \text{Re}(s/\hbar) >> 0$, and such that $R_{0, \pm}(s)$ possesses an asymptotic expansion as $s \to \infty$ with $\pm \text{Re}(s/\hbar) >> 0$ (see e.g., [2, 3, 16] or [10, §4]). Explicitly,

$$R_{0, +}(s) = \prod_{n \geq 0} A(s + nl\hbar)^{-1}$$

$$R_{0, -}(s) = \prod_{n \geq 1} A(s - nl\hbar)$$

The functions $R_{0, \pm}(s)$ are regularisations of $R_0$, and we show in Theorem 4.8 that they define meromorphic commutativity constraints on $\text{Rep}_{fd}(Y_\hbar(g))$ endowed with the Drinfeld tensor product $\otimes$.  

1.10. We now define the twist $J(s)$ as

$$J(s) = e^{h\gamma \Omega} \prod_{m \geq 1} R_{0, +}(s + m)e^{-\hbar \Omega m}$$

where $\gamma$ is the Euler-Mascheroni constant, and show that it converges provided $\text{Im} \hbar \neq 0$. As mentioned above, $J(s)$ is equal to $\Phi^+(s + 1)^{-1}$, where $\Phi^+$ is a canonical fundamental solution of the abelian $q$KZ equations $\Phi(s + 1) = R_{0, +}(s)\Phi(s)$.  

1.11. We conjecture that the meromorphic twist $J(s)$ also yields a (non–meromorphic) tensor structure on the functor $\Gamma$, when the categories $\text{Rep}_{fd}(Y_\hbar(g))$ and $\text{Rep}_{fd}(U_q(Lg))$ are endowed with the standard monoidal structure arising from the Kac–Moody coproducts.

More precisely, the Drinfeld and Kac–Moody coproducts on $U_q(Lg)$ are related by a twist, given by the lower triangular part $\mathcal{R}_{U_q(Lg)}(\zeta)$ of the universal $R$–matrix [6]. A similar statement holds for $Y_\hbar(g)$. Composing, we obtain a meromorphic tensor structure $J(s)$ on $\Gamma$ relative to the standard monoidal structures

$$\Gamma(V_1(s)) \otimes \Gamma(V_2) \xrightarrow{\mathcal{R}_{U_q(Lg)}(\zeta)} \Gamma(V_1) \otimes \zeta \Gamma(V_2)$$

$$J_{V_1, V_2}(s) \Gamma(V_1 \otimes V_2) \xrightarrow{\mathcal{R}_{Y_\hbar(g)}(s)} \Gamma(V_1 \otimes_s V_2)$$
We conjecture that \( J_{V_1, V_2} (s) \) is analytic in \( s \), and can therefore be evaluated at \( s = 0 \), thus yielding a tensor structure on \( \Gamma \) with respect to the standard coproducts. We will return to this in [11].

1.12. We note that the results of [10] hold for an arbitrary symmetrisable Kac–Moody algebra \( g \). Although we restricted ourselves to the case of a finite–dimensional semisimple \( g \) in this paper, it seems likely that our results hold in the general case as well. The main obstacle in working in this generality is the construction of \( R^0 \) for arbitrary \( g \). Once this is achieved, the proof of Theorem 6.1 should carry over verbatim.

1.13. **Outline of the paper.** In Section 2, we review the definitions of \( Y_\hbar (g) \) and \( U_q (Lg) \). Section 3 is devoted to defining the Drinfeld coproduct on \( U_q (Lg) \) and \( Y_\hbar (g) \). We give a construction of the diagonal part of the \( R \)–matrix of \( Y_\hbar (g) \) in §4. Section 5 reviews the definition of the functor \( \Gamma \) defined in [10]. The main result of this paper is given in Theorem 6.1 of §6.

2. **Yangians and quantum loop algebras**

2.1. Let \( g \) be a complex, semisimple Lie algebra and \((\cdot, \cdot)\) the non–degenerate, invariant bilinear form on \( g \) normalised so that the squared length of short roots is 2. Let \( h \subset g \) be a Cartan subalgebra of \( g \), \( \{ \alpha_i \}_{i \in I} \subset h^* \) a basis of simple roots of \( g \) relative to \( h \) and \( a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \) the entries of the corresponding Cartan matrix \( A \). Set \( d_i = (\alpha_i, \alpha_i)/2 \in \mathbb{N} \), so that \( d_i a_{ij} = d_j a_{ji} \) for any \( i, j \in I \).

2.2. **The Yangian \( Y_\hbar (g) \).** Let \( \hbar \in \mathbb{C} \). The Yangian \( Y_\hbar (g) \) is the \( \mathbb{C} \)–algebra generated by elements \( \{ x^\pm_{i,r}, \xi_{i,r} \}_{i \in I, r \in \mathbb{N}} \), subject to the following relations

Y1 \( \) For any \( i, j \in I \), \( r, s \in \mathbb{N} \)

\[ [\xi_{i,r}, \xi_{j,s}] = 0 \]

Y2 \( \) For \( i, j \in I \) and \( r, s \in \mathbb{N} \)

\[ [\xi_{i,0}, x^\pm_{j,s}] = \pm d_i a_{ij} x^\pm_{j,s} \]

Y3 \( \) For \( i, j \in I \) and \( r, s \in \mathbb{N} \)

\[ [\xi_{i,r+1}, x^\pm_{j,s}] - [\xi_{i,r}, x^\pm_{j,s+1}] = \pm \hbar d_i a_{ij} (\xi_{i,r} x^\pm_{j,s} + x^\pm_{j,s} \xi_{i,r}) \]

Y4 \( \) For \( i, j \in I \) and \( r, s \in \mathbb{N} \)

\[ [x^\pm_{i,r+1}, x^\pm_{j,s}] - [x^\pm_{i,r}, x^\pm_{j,s+1}] = \pm \hbar d_i a_{ij} (x^\pm_{i,r} x^\pm_{j,s} + x^\pm_{j,s} x^\pm_{i,r}) \]

Y5 \( \) For \( i, j \in I \) and \( r, s \in \mathbb{N} \)

\[ [x^+_i, x^-_j] = \delta_{ij} \xi_{i,r+s} \]
acts on $Y$. 2.4. Shift automorphism. where $a \in \mathbb{C}$, $y$ is one of $\xi_i, x_i^\pm$. In terms of the generating series introduced in 2.3, $\tau_a(y(u)) = y(u - a)$

Given a representation $V$ of $Y_h$ and $a \in \mathbb{C}$, set $V(a) = \tau_a^*(V)$. 

Proposition. [10, Proposition 2.3] The relations (Y1), (Y2)–(Y3), (Y4), (Y5), (Y6) are equivalent to the following identities in $Y_h$ by

$$\begin{align*}
\xi_i(u) &= 1 + h \sum_{r \geq 0} \xi_{i,r} u^{-r-1} \quad \text{and} \quad x_i^\pm(u) = h \sum_{r \geq 0} x_i^{\pm,r} u^{-r-1} \\
\xi_i(u), \xi_j(v) &= 0 \\
[\xi_i, x_j^\pm(u)] &= \pm d_i a_{ij} x_j^\pm(u) \\
(u - v \mp a) \xi_i(u) x_j^\pm(v) &= (u - v \mp a) x_j^\pm(v) \xi_i(u) - 2 a x_j^\pm(u) \xi_i(u) \\
[\xi_i, x_j^\pm(u), x_j^\pm(v)] &= (u - v \mp a) x_j^\pm(u) x_j^\pm(v) + h \left( [x_i^{\pm,0}, x_j^\pm(v)] - [x_i^\pm(u), x_j^{\pm,0}] \right) \\
(u - v)[x_i^\pm(u), x_j^\pm(v)] &= -\delta_{ij} h (\xi_i(u) - \xi_i(v)) \\
\sum_{\pi \in \mathfrak{S}_m} \left[ x_i^{\pm}(u_{\pi(1)}), x_i^{\pm}(u_{\pi(2)}), \cdots, x_i^{\pm}(u_{\pi(m)}), x_j^\pm(v) \cdots \right] &= 0 \\
\end{align*}$$

2.3. Assume henceforth that $h \neq 0$, and define $\xi_i(u), x_i^\pm(u) \in Y_h[[u^{-1}]]$ 

(Y6) Let $i \neq j \in I$ and set $m = 1 - a_{ij}$. For any $r_1, \cdots, r_m \in \mathbb{N}$ and $s \in \mathbb{N}$

$$\sum_{\pi \in \mathfrak{S}_m} \left[ x_i^{\pm,r_1}, x_i^{\pm,r_2}, \cdots, x_i^{\pm,r_m}, x_j^\pm(s) \cdots \right] = 0$$

For any $y \in Y$ are equivalent to the following identities in $Y_h[[u^{-1}]]$.
2.5. Quantum loop algebra $U_q(Lg)$. Let $q \in \mathbb{C}^\times$ be of infinite order. For any $i \in I$, set $q_i = q^{d_i}$. We use the standard notation for Gaussian integers
\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}
\]
\[
[n]_q! = [n]_q[n-1]_q \cdots [1]_q
\]
\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}
\]

The quantum loop algebra $U_q(Lg)$ is the $\mathbb{C}$–algebra generated by elements \{\(\Psi_{i,r}^\pm\)\}_{i \in I, r \in \mathbb{N}}, \{X_{i,k}^\pm\}_{i \in I, k \in \mathbb{Z}}$, subject to the following relations

(QL1) For any $i, j \in I$, $r, s \in \mathbb{N}$,
\[
[\Psi_{i+r}^\pm, \Psi_{j+s}^\pm] = 0 \quad [\Psi_{i+r}^\pm, \Psi_{j+s}^\mp] = 0 \quad \Psi_{i+0}^+ \Psi_{i+0}^- = 1
\]

(QL2) For any $i, j \in I$, $k \in \mathbb{Z}$,
\[
\Psi_{i,0}^+ X_{j,k}^\pm = q_i^{\pm d_{ij}} X_{j,k}^\pm
\]

(QL3) For any $i, j \in I$, $\varepsilon \in \{\pm\}$ and $l \in \mathbb{Z}$
\[
\Psi_{i+l}^\varepsilon X_{j,l}^\pm - q_i^{\pm a_{ij}} X_{j,l}^\pm \Psi_{i+l}^\varepsilon = 0
\]
for any $l \in \mathbb{Z}_{\geq 0}$ if $\varepsilon = +$ and $k \in \mathbb{Z}_{< 0}$ if $\varepsilon = -$

(QL4) For any $i, j \in I$ and $k, l \in \mathbb{Z}$
\[
\Psi_{i,k+l}^\pm X_{j,l}^\pm - q_i^{\pm a_{ij}} X_{j,l}^\pm \Psi_{i,k+l}^\pm = 0
\]

(QL5) For any $i, j \in I$ and $k, l \in \mathbb{Z}$
\[
[X_{i,k}^+, X_{j,l}^-] = \frac{\delta_{ij} \Psi_{i,k+l}^- - \Psi_{i,k+l}^+}{q_i - q_i^{-1}}
\]

where $\Psi_{i, \mp k}^\pm = 0$ for any $k \geq 1$.

(QL6) For any $i \neq j \in I$, $m = 1 - a_{ij}, k_1, \ldots, k_m \in \mathbb{Z}$ and $l \in \mathbb{Z}$
\[
\sum_{\pi \in \mathfrak{S}_m} \sum_{s=0}^{m} (-1)^s \left[ \sum_{\pi \in \mathfrak{S}_m} \chi_{i,k_{\pi(1)}}^\pm \cdots \chi_{i,k_{\pi(s)}}^\pm X_{j,l}^\pm \chi_{i,k_{\pi(s+1)}}^\pm \cdots \chi_{i,k_{\pi(m)}}^\pm \right] = 0
\]

2.6. Define $\Psi_i(z)^+, X_i^+(z)^+ \in U_q(Lg)[[z^{-1}]]$ and $\Psi_i(z)^-, X_i^-(z)^- \in U_q(Lg)[[z]]$ by
\[
\Psi_i(z)^+ = \sum_{r \geq 0} \Psi_{i,r}^+ z^{-r} \quad \Psi_i(z)^- = \sum_{r \leq 0} \Psi_{i,r}^- z^{-r}
\]
\[
X_i^+(z)^+ = \sum_{r \geq 0} X_{i,r}^+ z^{-r} \quad X_i^-(z)^- = - \sum_{r < 0} X_{i,r}^- z^{-r}
\]

**Proposition.** [10, Proposition 2.7] The relations (QL1), (QL2)–(QL3), (QL4), (QL5), (QL6) imply the following relations in $U_q(Lg)[z, w; z^{-1}, w^{-1}]$

(QL1) For any $i, j \in I$, and $h, h' \in h$,
\[
[\Psi_i(z)^+, \Psi_j(w)^+] = 0
\]
Given a representation $V$, we have

\[ \Psi_{1,0}^+ \alpha^\pm(z)^+ \left( \Psi_{1,0}^+ \right)^{-1} = q_i^{\pm a_{ij}} \alpha^\pm_j(z)^+ \]

(2.7. Shift automorphism. The group $\mathbb{C}^*$ of dilations of the complex plane acts on $U_q(Lg)$ by

\[ \tau_\alpha(Y_k) = \alpha^k Y_k \]

where $\alpha \in \mathbb{C}^*$, $Y$ is one of $\Psi_i^\pm$, $\alpha_i^\pm$. In terms of the generating series of 2.6, we have

\[ \tau_\alpha(Y(z)^\pm) = Y(\alpha^{-1}z)^\pm \]

Given a representation $V$ of $U_q(Lg)$ and $\alpha \in \mathbb{C}^*$, we denote $\tau_{\alpha}^*(V)$ by $V(\alpha)$.}

(2.8. Rationality. The following rationality property is due to Beck–Kac [1] and Hernandez [13] for $U_q(Lg)$ and to authors for $Y_h(g)$. In the form below, the result appears in [10].

\[ \xi_i(u) \in \text{End}(V_{\mu})[[u^{-1}]] \quad \text{and} \quad x_i^\pm(u) \in \text{Hom}(V_{\mu}, V_{\mu \pm \alpha_i})[[u^{-1}]] \]
defined in 2.3 are the expansions at \( \infty \) of rational functions of \( u \).

Specifically, let \( t_{i,1} = \xi_{i,1} - \frac{\hbar}{2} \xi_{i,0}^2 \in Y_\hbar(\mathfrak{g})^b \). Then,

\[
x_i^\pm(u) = 2d_i \hbar u^{-1} \left( 2d_i \mp \frac{\text{ad}(t_{i,1})}{u} \right)^{-1} x_i^\pm \,
\]

and

\[
\xi_i(u) = 1 + [x_i^+(u), x_i^-] \quad (\text{ii})
\]

Let \( \mathcal{V} \) be a \( U_q(L\mathfrak{g}) \)-module on which the operators \( \{\Psi_{i,0}^\pm\}_{i \in \mathbb{I}} \) act semisimply with finite–dimensional weight spaces. Then, for every weight \( \mu \) of \( \mathcal{V} \) and \( \varepsilon \in \{\pm\} \), the generating series

\[
\Psi_i(z)\varepsilon \in \text{End}(\mathcal{V}_\mu)[[z^{\mp 1}]] \quad \text{and} \quad \mathcal{X}_i(z)\varepsilon \in \text{Hom}(\mathcal{V}_\mu, \mathcal{V}_{\mu \pm \alpha_i})[[z^{\mp 1}]]
\]
defined in 2.6 are the expansions of rational functions \( \Psi_i(z), \mathcal{X}_i(z) \)

at \( z = \infty \) and \( z = 0 \). Specifically, let \( H_{i,\pm 1} = \pm \Psi_{i,0}^\pm \Psi_{i,\pm 1}^\pm (q_i - q_i^{-1}) \).

Then,

\[
\mathcal{X}_i(z)\varepsilon = \left( 1 - \varepsilon \frac{\text{ad}(H_{i,1}^+)}{[2]_i z} \right)^{-1} \mathcal{X}_{i,0}^\varepsilon
\]

and

\[
\Psi_i(z) = \Psi_{i,0}^- + (q_i - q_i^{-1})[\mathcal{X}_i^+(z), \mathcal{X}_i^-]
\]

2.9. Poles of finite–dimensional representations. By Proposition 2.8, we can define, for a given \( V \in \text{Rep}_0(Y_\hbar(\mathfrak{g})) \), a subset \( \sigma(V) \subset \mathbb{C} \) consisting of the poles of the rational functions \( \xi_i(u)^\pm, x_i^\pm(u) \).

Similarly, for any \( \mathcal{V} \in \text{Rep}_0(U_q(L\mathfrak{g})) \), we define a subset \( \sigma(\mathcal{V}) \subset \mathbb{C}^\times \) consisting of the poles of the functions \( \Psi_i(z)^\pm, \mathcal{X}_i^\pm(z) \).

2.10. The following result will be needed later.

Lemma. Let \( V \) be a finite–dimensional representation of \( Y_\hbar(\mathfrak{g}) \) and \( i, k \in \mathbb{I} \).

If \( u_0 \) is a pole of \( x_k^+(u) \), then \( u_0 \pm \frac{\hbar d_ia_{ik}}{2} \) are poles of \( \xi_i(u)^\pm \).

Proof. Consider the relation (\( \mathcal{V}'3 \)) of Proposition 2.3 and its inverse, as follows (here \( b = \hbar d_ia_{ik}/2 \)).

\[
\text{Ad}(\xi_i(u)) x_k^+(v) = \frac{u - v + b}{u - v - b} x_k^+(v) - \frac{2b}{u - v - b} x_k^+(u - b)
\]

\[
\text{Ad}(\xi_i(u))^{-1} x_k^+(v) = \frac{u - v - b}{u - v + b} x_k^+(v) + \frac{2b}{u - v + b} x_k^+(u + b)
\]
Differentiating the first identity and using the fact that
\[
\frac{d}{du} \text{Ad}(\xi_i(u))x_k^+(v) = \text{Ad}(\xi_i(u)) [\xi_i(u)^{-1}\xi'_i(u), x_k^+(v)]
\]
shows that
\[
[\xi_i(u)^{-1}\xi'_i(u), x_k^+(v)] = \left(\frac{1}{u-v+b} - \frac{1}{u-v-b}\right) x_k^+(v) + \frac{1}{u-v-b} x_k^+(u-b) - \frac{1}{u-v+b} x_k^+(u+b) \quad (2.1)
\]
Thus, if \(x_k^+(v)\) has a pole at \(u_0\) of order \(N\), then multiplying both sides by \((v-u_0)^N\) and letting \(v \to u_0\) we get:
\[
[\xi_i(u)^{-1}\xi'_i(u), X] = \left(\frac{1}{u-u_0+b} - \frac{1}{u-u_0-b}\right) X
\]
where \(X = (v-u_0)^N x_k^+(v)\big|_{v=u_0}\). Hence the logarithmic derivative of \(\xi_i(u)\) has poles at \(u_0 \pm b\), which implies that \(u_0 \pm b\) must be poles of \(\xi_i(u)^\pm\). The argument for \(x_k^-(v)\) is same as above, upon replacing \(b\) by \(-b\).

3. The Drinfeld coproduct

In this section, we show that the deformed Drinfeld coproduct (see [12, 13] for \(U_q(Lg)\) and [5, 14] for the double Yangian \(DY_h(g)\)) is a rational function of the deformation parameter, and that it defines a meromorphic tensor product on the category of finite–dimensional representations of \(U_q(Lg)\) and \(Y_h(g)\).

3.1. Drinfeld coproduct on \(U_q(Lg)\). Let \(\mathcal{V}, \mathcal{W} \in \text{Rep}_{td}(U_q(Lg))\). Define an action of the generators of \(U_q(Lg)\) on \(\mathcal{V} \otimes \mathcal{W}\) depending on \(\zeta \in \mathbb{C}^\times\) as follows
\[
\Delta_\zeta(\Psi^\pm_{i,\pm m}) = \sum_{p+q=m} \zeta^{p+q} \Psi^\pm_{i,\pm q} \otimes \Psi^\pm_{i,\pm q} \quad (3.1)
\]
\[
\Delta_\zeta(\mathcal{X}^+_{i,k}) = \zeta^k \mathcal{X}^+_{i,k} \otimes 1 + \oint_{C_2} \Psi_i(\zeta^{-1} w) \otimes \mathcal{X}^+_{i}(w) w^{k-1} dw \quad (3.2)
\]
\[
\Delta_\zeta(\mathcal{X}^-_{i,k}) = \oint_{C_1} \mathcal{X}^-_{i}(w) \otimes \Psi_i(\zeta w) \zeta^k w^{k-1} dw + 1 \otimes \mathcal{X}^-_{i,k} \quad (3.3)
\]
where
- \(C_1, C_2 \subset \mathbb{C}^\times\) are Jordan curves which do not enclose 0.\(^1\)
- \(C_1\) (resp. \(C_2\)) encloses \(\sigma(\mathcal{V})\) (resp. \(\sigma(\mathcal{W})\)).
- \(\oint_C f(w) dw = (2\pi i)^{-1} \int_C f(w) dw\)
- \(\zeta\) is taken large enough that \(\Psi_i(\zeta^{-1} w)\) (resp. \(\Psi_i(\zeta w)\)) is regular on \(\mathcal{V}\) (resp. \(\mathcal{W}\)) when \(w\) is inside \(C_2\) (resp. \(C_1\)).

\(^1\)By a Jordan curve, we shall mean a disjoint union of simple, closed curves the inner domains of which are pairwise disjoint.
The corresponding generating series \( \Delta_\zeta(\Psi_i(z)\pm) \), \( \Delta_\zeta(\mathcal{X}_i^\pm(z)) \) are the expansions at \( z = \infty, 0 \) of the rational functions

\[
\Delta_\zeta(\Psi_i(z)) = \Psi_i(\zeta^{-1}z) \otimes \Psi_i(z)
\]
\[
\Delta_\zeta(\mathcal{X}_i^+(z)) = \mathcal{X}_i^+(\zeta^{-1}z) \otimes 1 + \int_{C_2} \frac{z^{-1}w}{z-w} \Psi_i(\zeta^{-1}w) \otimes \mathcal{X}_i^+(w) \, dw
\]
\[
\Delta_\zeta(\mathcal{X}_i^-(z)) = \int_{C_1} \frac{z^{-1}w}{z-w} \mathcal{X}_i^-(w) \otimes \Psi_i(\zeta w) \, dw + 1 \otimes \mathcal{X}_i^-(z)
\]

where the integrals are understood to mean the (rational) function of \( z \) defined for \( z \) outside of \( \zeta C_1, C_2 \).

3.2.

Theorem.

(i) The formulae (3.1)–(3.3) define an action of \( U_q(L\mathfrak{g}) \) on \( \mathcal{V} \otimes \mathcal{W} \). The resulting representation is denoted by \( \mathcal{V} \otimes_\zeta \mathcal{W} \).

(ii) The action of \( U_q(L\mathfrak{g}) \) on \( \mathcal{V} \otimes_\zeta \mathcal{W} \) is a rational function of \( \zeta \), with poles contained in \( \sigma(\mathcal{W})\sigma(\mathcal{V})^{-1} \).

(iii) The identification of vector spaces

\[
(\mathcal{V}_1 \otimes_{\zeta_1} \mathcal{V}_2) \otimes_{\zeta_2} \mathcal{V}_3 = \mathcal{V}_1 \otimes_{\zeta_1 \zeta_2} (\mathcal{V}_2 \otimes_{\zeta_2} \mathcal{V}_3)
\]

intertwines the action of \( U_q(L\mathfrak{g}) \).

(iv) The following holds for any \( \zeta, \zeta' \in \mathbb{C}^\times \),

\[
\mathcal{V} \otimes_{\zeta \zeta'} \mathcal{W} = \mathcal{V}(\zeta) \otimes_{\zeta'} \mathcal{W}
\]

and \( \mathcal{V}(\zeta') \otimes_\zeta \mathcal{W}(\zeta') = (\mathcal{V} \otimes_\zeta \mathcal{W})(\zeta') \), where \( \mathcal{V}(\zeta) = \tau_\zeta^* \mathcal{V} \).

(v) The poles of \( \Psi_i(z), \mathcal{X}_i^\pm(z) \) on \( \mathcal{V} \otimes_\zeta \mathcal{W} \) are contained in \( (\zeta \sigma(\mathcal{V})) \cup \sigma(\mathcal{W}) \).

Proof. (iv) and (v) are clear.

To prove (ii), let \( \{ w_j \}_{j \in J} \subset \mathbb{C}^\times \) be the poles of \( \mathcal{X}_i^+(w) \) on \( \mathcal{W} \), and

\[
\mathcal{X}_i^+(w) = \mathcal{X}_i^+_{i,0} + \sum_{j \in J, n \geq 1} \mathcal{X}_i^+_{i,j,n}(w - w_j)^{-n}
\]

its corresponding partial fraction decomposition. Since \( C_2 \) encloses all \( w_j \), and \( \Psi_i(\zeta^{-1}w)w^{k-1} \) is regular inside \( C_2 \), (3.2) yields

\[
\Delta_\zeta(\mathcal{X}_i^+_{i,k}) = \zeta^k \mathcal{X}_i^+_{i,k} \otimes 1 + \sum_{j,n} \partial_w^{(n-1)} \left( \Psi_i(\zeta^{-1}w)w^{k-1} \right)|_{w=w_j}
\]

where \( \partial_w^{(p)} = \partial^p / p! \). This is clearly a rational function of \( \zeta \), whose poles are a subset of the points \( \zeta = w_j w_k' \), where \( w_k' \) is a pole of \( \Psi_i(w) \) on \( \mathcal{V} \). A similar argument shows that \( \Delta_\zeta(\mathcal{X}_i^-_{i,k}) \) is also a rational function whose poles are contained in \( \sigma(\mathcal{W})\sigma(\mathcal{V})^{-1} \).

A direct proof of (i) and (iii) can be given along the lines of that of Theorem 3.4 below. Alternatively, (i) and (iii) can be deduced from [12, 13].
as follows. Expanding \( \Delta_\zeta(\Psi_{i,m}^\pm) \) and \( \Delta_\zeta(\mathcal{X}_{i,k}^\pm) \) as Laurent series in \( \zeta^{-1} \) yields the following for any \( m \in \mathbb{N} \) and \( k \in \mathbb{Z} \):

\[
\Delta_\zeta(\Psi_{i,m}^\pm) = \sum_{n=0}^{m} \zeta^{\pm n} \Psi_{i,\pm n}^\pm \otimes \Psi_{i,\pm(m-n)}^\pm
\]

\[
\Delta_\zeta(\mathcal{X}_{i,k}^+) = \zeta^k \mathcal{X}_{i,k}^+ \otimes 1 + \sum_{l \geq 0} \zeta^{-l} \int_{C_2} \Psi_{i,-l}^- \otimes \mathcal{X}_i^+(w) w^{k+l-1} dw
\]

\[
= \zeta^k \mathcal{X}_{i,k}^+ \otimes 1 + \sum_{l \geq 0} \zeta^{-l} \Psi_{i,-l}^- \otimes \mathcal{X}_{i,k+l}^+
\]

\[
\Delta_\zeta(\mathcal{X}_{i,k}^-) = \sum_{l \geq 0} \zeta^{-l} \int_{C_1} \mathcal{X}_i^-(w) \otimes \Psi_{i,l}^- w^{k+l-1} dw + 1 \otimes \mathcal{X}_{i,k}^-
\]

where the third and fifth equalities follow by a deformation of contour, and the fact that \( C_1 \) (resp. \( C_2 \)) encloses all the poles of \( \mathcal{X}_i^- (w) \) (resp. \( \mathcal{X}_i^+ (w) \)) and does not enclose 0.

Comparing with the formulae in [12, 13], we see that for every \( X \in U_q (Lg) \):

\[
\Delta_\zeta(X) = \Delta^{(H)}_{\zeta^{-1}} (\tau_\zeta(X))
\]

where \( \Delta^{(H)} \) is the deformed Drinfeld coproduct defined by Hernandez. (i) and (iii) now follow from [12, §6], [13, Lemma 3.2] and the fact that it is sufficient to establish them when \( \zeta \) is a formal variable.

3.3. Drinfeld coproduct on \( Y_h(g) \). Let now \( V, W \in \text{Rep}_{\text{id}}(Y_h(g)) \). Define an action of the generators of \( Y_h(g) \) on \( V \otimes W \) depending on \( s \in \mathbb{C} \) by

\[
\Delta_s(\xi_i(u)) = \xi_i(u - s) \otimes \xi_i(u)
\]

\[
\Delta_s(x_i^+(u)) = x_i^+(u - s) \otimes 1 + \int_{C_2} \frac{1}{u - v} \xi_i(v - s) \otimes x_i^+(v) dv
\]

\[
\Delta_s(x_i^-(u)) = \int_{C_1} \frac{1}{u - v - s} x_i^-(v) \otimes \xi_i(v + s) dv + 1 \otimes x_i^-(u)
\]

where \( C_1, C_2 \) are contours enclosing \( \sigma(V), \sigma(W) \) respectively, and \( s \) is such that \( \xi_i(v - s) \) (resp. \( \xi_i(v + s) \)) is regular on \( V \) (resp. \( W \)) when \( v \) lies inside \( C_2 \) (resp. \( C_1 \)). The integral in (3.5) (resp. (3.6)) is understood to mean the rational function of \( u \) it defines in the domain where \( u \) is not enclosed by \( C_2 \) (resp. \( C_1 \)).
In terms of the generators \( \{ \xi_{i,r}, x_{i,r}^\pm \} \) the above formulae read
\[
\Delta_s(\xi_{i,r}) = \tau_s(\xi_{i,r}) \otimes 1 + \hbar \sum_{p+q=r-1} \tau_s(\xi_{i,p}) \otimes \xi_{i,q} + 1 \otimes \xi_{i,r}
\]
\[
\Delta_s(x_{i,r}^+) = \tau_s(x_{i,r}^+) \otimes 1 + \hbar^{-1} \int_{C_2} \xi_i(v-s) \otimes x_{i,r}^+(v)v' dv
\]
\[
\Delta_s(x_{i,r}^-) = \hbar^{-1} \int_{C_1} x_{i,r}^-(v) \otimes \xi_i(v+s)(v+s)' dv + 1 \otimes x_{i,r}^-
\]

3.4. Theorem.

(i) The formulae (3.4)–(3.6) define an action of \( Y_h(g) \) on \( V \otimes W \). The resulting representation is denoted by \( V \otimes_s W \).

(ii) The action of \( Y_h(g) \) on \( V \otimes_s W \) is a rational function of \( s \), with poles contained in \( \sigma(W) - \sigma(V) \).

(iii) The identification of vector spaces
\[
(V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3 = V_1 \otimes_{s_1+s_2} (V_2 \otimes_{s_2} V_3)
\]
intertwines the action of \( Y_h(g) \).

(iv) The following holds for any \( s, s' \in \mathbb{C} \),
\[
V \otimes_{s+s'} W = V(s) \otimes_{s'} W
\]
and \( V(s') \otimes_s W(s') = (V \otimes_s W)(s') \), where \( V(s) = \tau_s^* V \).

(v) The poles of \( \xi_i(u), x_{i,r}^+(u) \) on \( V \otimes_s W \) are contained in \( (s + \sigma(V)) \cup \sigma(W) \).

Proof. (iv) and (v) are clear, and (ii) is proved as in Theorem 3.2.

To prove (i), we may assume by (iv) that \( \sigma(V) \cap \sigma(W) = \emptyset \), and that \( s = 0 \). We need to verify that the relations of the Yangian hold. We choose the contours \( C_1 \) and \( C_2 \) enclosing \( \sigma(V) \) and \( \sigma(W) \) respectively, such that they do not intersect. The relation (Y1) holds trivially. The Serre relation (Y6) easily reduces to its special case for zero modes, which is then a consequence of the other relations. Thus it remains to verify (Y2)–(Y5) only. The relations (Y2) and (Y3) are checked in Section 3.5, (Y4) in Section 3.6 and (Y5) in Section 3.7.

The proof of (iii) is given in Section 3.8. \( \square \)

3.5. Proof of (Y2) and (Y3). We prove these relations for the + case only and consequently drop the superscript. Recall that these two relations can be uniformly written as (see (Y3) of Proposition 2.3):
\[
\xi_i(u_1)x_j(u_2)\xi_i(u_1)^{-1} = \frac{u_1 - u_2 + a}{u_1 - u_2 - a} x_j(u_2) - \frac{2a}{u_1 - u_2 - a} x_j(u_1 - a)
\]
where \( a = \hbar d a_{ij}/2 \). In the computations below, \( u_1 \) and \( u_2 \) are assumed to be large enough, so that the assumptions imposed following equations (3.4)–(3.6) hold. To be more precise, we assume that \( u_2 \) and \( u_1 - a \) are not
enclosed by the contour $C_2$.

Applying the coproduct to the left–hand side gives

$$\Delta(\text{L.H.S.}) = \xi_i(u_1)x_j(u_2)\xi_i(u_1)^{-1} \otimes 1 + \oint_{C_2} \frac{1}{u_2 - v} \xi_i(v) \otimes \xi_i(u_1)x_j(v)\xi_i(u_1)^{-1} \, dv$$

$$= \xi_i(u_1)x_j(u_2)\xi_i(u_1)^{-1} \otimes 1 + \oint_{C_2} \frac{u_1 - v + a}{(u_2 - v)(u_1 - v - a)} \xi_i(v) \otimes x_j(v) \, dv$$

Similarly, we can compute the coproduct of the right–hand side

$$\Delta(\text{R.H.S.}) = \xi_i(u_1)x_j(u_2)\xi_i(u_1)^{-1} \otimes 1 + \oint_{C_2} \frac{u_1 - u_2 + a}{u_1 - u_2 - a} \frac{1}{u_2 - v} \frac{1}{u_1 - u_2 - a} \frac{1}{u_1 - v - a} \xi_i(v) \otimes x_j(v) \, dv$$

The equality of the two expressions now follows from the identity

$$\frac{u_1 - u_2 + a}{u_1 - u_2 - a} \frac{1}{u_2 - v} \frac{1}{u_1 - u_2 - a} \frac{1}{u_1 - v - a} = \frac{u_1 - v + a}{(u_2 - v)(u_1 - v - a)}$$

3.6. Proof of (Y4). Again we check this relation for the $+ \text{ case only}$. An easy computation shows that

$$\Delta(x_{i,m}x_{j,n}) = x_{i,m}x_{j,n} \otimes 1 + \frac{1}{\hbar} \oint_{C_2} v^n x_{i,m} \xi_j(v) \otimes x_j(v) \, dv + \frac{1}{\hbar} \oint_{C_2} \frac{v^n \xi_i(v) x_{j,n} \otimes x_i(v)}{x_j(v)} \, dv + \frac{1}{\hbar^2} \oint_{C_2} v_1^n \xi_i(v_1) \xi_j(v_2) \otimes x_i(v_1) x_j(v_2) \, dv_1 \, dv_2$$

We need to verify that the following equation holds after applying $\Delta$:

$$x_{i,r+1}x_{j,s} - x_{i,r}x_{j,s+1} - ax_{i,r}x_{j,s} = x_{j,s}x_{i,r+1} - x_{j,s+1}x_{i,r} + ax_{j,s}x_{i,r}$$

Now we apply $\Delta$ to both sides of this equation. In order to make the computation readable we will write four terms of $\Delta(x_{i,m}x_{j,n})$ separately.

The first terms of $\Delta(\text{L.H.S.})$ and $\Delta(\text{R.H.S.})$ are respectively:

$$(x_{i,r+1}x_{j,s} - x_{i,r}x_{j,s+1} - ax_{i,r}x_{j,s}) \otimes 1$$

$$(x_{j,s}x_{i,r+1} - x_{j,s+1}x_{i,r} + ax_{j,s}x_{i,r}) \otimes 1$$

which cancel because of relation (Y4).

The Second term of $\Delta(\text{L.H.S.})$ and the third term of $\Delta(\text{R.H.S.})$ are respectively

$$\oint_{C_2} (v^n x_{i,r+1} - v^{s+1} x_{i,r} - av^n x_{i,r}) \xi_j(v) \otimes x_j(v) \, dv$$

$$\oint_{C_2} \xi_j(v) (v^n x_{i,r+1} - v^{s+1} x_{i,r} + av^n x_{i,r}) \otimes x_j(v) \, dv$$
which cancel because of the following version of (Y2) and (Y3)

\((x_{i,r+1} - vx_{i,r} - ax_{i,r})\xi_j(v) = \xi_j(v)(x_{i,r+1} - vx_{i,r} + ax_{i,r})\)

Similarly the third term of \(\Delta(\text{L.H.S.})\) cancels with the second term of \(\Delta(\text{R.H.S.})\).

The fourth terms of \(\Delta(\text{L.H.S.})\) and \(\Delta(\text{R.H.S.})\) are respectively

\[
\mathcal{B} = \oint_{C_1} \oint_{C_2} \frac{1}{(u_1 - u_2)(u_2 - v_1)} \left( \xi_i(v_2)x_j^-(v_1) \otimes x_i^+(v_2)\xi_j(v_1) \\
- x_j^-(v_1)\xi_i(v_2) \otimes \xi_j(v_1)x_i^+(v_2) \right) dv_2 dv_1
\]

These terms cancel because of the following relation (see (Y4) of Proposition 2.3).

\((v_1 - v_2 - a)x_i(v_1)x_j(v_2) = (v_1 - v_2 + a)x_j(v_2)x_i(v_1) + \hbar \left( [x_i, x_j(v_2)] - [x_i(v_1), x_j, 0] \right)\)

3.7. Proof of (Y5). We need to check that

\([x_i^+(u_1), x_j^-(u_2)] = \delta_{ij} \frac{\hbar}{u_1 - u_2} (\xi_i(u_2) - \xi_i(u_2))\)

Applying \(\Delta\) to the left–hand side yields

\[
\Delta(\text{L.H.S.}) = \oint_{C_1} \frac{1}{u_2 - v}[x_i^+(u_1), x_j^-(v)] \otimes \xi_j(v) dv + \\
\oint_{C_2} \frac{1}{u_1 - v}\xi_i(v) \otimes [x_i^+(v), x_j^-(u_2)] dv + \mathcal{B}
\]

where the term \(\mathcal{B}\) is given by

\[
\mathcal{B} = \oint_{C_1} \oint_{C_2} \frac{1}{(u_1 - u_2)(u_2 - v_1)} \left( \xi_i(v_2)x_j^-(v_1) \otimes x_i^+(v_2)\xi_j(v_1) \\
- x_j^-(v_1)\xi_i(v_2) \otimes \xi_j(v_1)x_i^+(v_2) \right) dv_2 dv_1
\]

We claim that \(\mathcal{B} = 0\). Thus \(\Delta(\text{L.H.S.}) = 0\) if \(i \neq j\). Now assume \(i = j\).
\[ \Delta(\text{L.H.S.}) = \oint_{C_1} \frac{h}{(u_2 - v)(u_1 - v)} (\xi_i(v) - \xi_i(u_1)) \otimes \xi_i(v) \, dv \\
+ \oint_{C_2} \frac{h}{(u_1 - v)(v - u_2)} \xi_i(v) \otimes (\xi_i(u_2) - \xi_i(v)) \, dv \\
= \oint_{C_1 + C_2} \frac{h}{(u_1 - v)(u_2 - v)} \xi_i(v) \otimes \xi_i(v) \, dv \\
= \frac{h}{u_1 - u_2} \oint_{C_1 + C_2} \left( \frac{1}{u_2 - v} - \frac{1}{u_1 - v} \right) \xi_i(v) \otimes \xi_i(v) \, dv \\
= \frac{h}{u_1 - u_2} (\xi_i(u_2) \otimes \xi_i(u_2) - \xi_i(u_1) \otimes \xi_i(u_1)) \\
= \Delta(\text{R.H.S.}) \]

**Proof that** $B = 0$. For this we need the following variant of relation (3.7) of Proposition 2.3.

\[ (u - v)[\xi_i(u), x_j^\pm(v)] = \pm a \{ \xi_i(u), x_j^\pm(v) - x_j^\pm(u) \} \quad (3.7) \]

where $a = \hbar d_i a_{ij} / 2$ and $\{x, y\} = xy + yx$.

The integrand of $B$ can be simplified in two different ways. First we write

\[ \xi_i(v_2)x_j^-(v_1) \otimes x_i^+(v_2)\xi_j(v_1) - x_j^-(v_1)\xi_i(v_2) \otimes x_j^+(v_1)x_i^+(v_2) = \\
[\xi_i(v_2), x_j^-(v_1)] \otimes x_i^+(v_2)\xi_j(v_1) - x_j^-(v_1)\xi_i(v_2) \otimes [\xi_j(v_1), x_i^+(v_2)] \]

Now using the relation (3.7) and the fact that $C_1$ and $C_2$ are non-intersecting, we obtain the following:

\[ B = \oint_{C_1} \oint_{C_2} \frac{h}{(v_1 - v_2)(u_1 - v_2)(u_2 - v_1)} \left( \xi_i(v_2)x_j^-(v_1) \otimes x_i^+(v_2)\xi_j(v_1) - x_j^-(v_1)\xi_i(v_2) \otimes x_i^+(v_1)x_i^+(v_2) \right) \, dv_2 \, dv_1 \]

In obtaining the expression above, we use the fact that $\xi_i(v_2) \otimes 1$ and $1 \otimes \xi_j(v_1)$ are analytic within $C_2$ and $C_1$ respectively.

Now if we write

\[ \xi_i(v_2)x_j^-(v_1) \otimes x_i^+(v_2)\xi_j(v_1) - x_j^-(v_1)\xi_i(v_2) \otimes x_j^+(v_1)x_i^+(v_2) = \\
- \xi_i(v_2)x_j^-(v_1) \otimes [\xi_j(v_1), x_i^+(v_2)] + [\xi_i(v_2), x_j^-(v_1)] \otimes \xi_j(v_1)x_i^+(v_2) \]

and use the relation (3.7) as before, we obtain:

\[ B = \oint_{C_1} \oint_{C_2} \frac{h}{(v_1 - v_2)(u_1 - v_2)(u_2 - v_1)} \left( \xi_i(v_2)x_j^-(v_1) \otimes x_i^+(v_2)\xi_j(v_1) - x_j^-(v_1)\xi_i(v_2) \otimes x_i^+(v_1)x_i^+(v_2) \right) \, dv_2 \, dv_1 \]
Thus $\mathcal{B} = -\mathcal{B}$ and hence $\mathcal{B} = 0$.

3.8. We have to show that, for $y = \xi_i, x_i^\pm$,

$$(\Delta_{s_1} \otimes 1) (\Delta_{s_2} (y(u))) = (1 \otimes \Delta_{s_2}) (\Delta_{s_1+s_2} (y(u)))$$

For $\xi_i(u)$ both sides of this equation yield $\xi_i(u-s_1-s_2) \otimes \xi_i(u-s_2) \otimes \xi_i(u)$. Now we verify this equation for $x_i^+(u)$. Let $C_j$ be a contour enclosing $\sigma(V_j)$ $(j = 1, 2, 3)$. Then we obtain

\[
L.H.S. = x_i^+(u-s_1-s_2) \otimes 1 \otimes 1 + \oint_{C_2} \frac{1}{u-v-s_2} \xi_i(v-s_1) \otimes x_i^+(v) \otimes 1 \, dv \\
+ \oint_{C_3} \frac{1}{u-v} \xi_i(v-s_1-s_2) \otimes \xi_i(v-s_2) \otimes x_i^+(v) \, dv
\]

\[
R.H.S. = x_i^+(u-s_1-s_2) \otimes 1 \otimes 1 + \oint_{C_2} \frac{1}{u-v-s_2} \xi_i(v-s_1) \otimes x_i^+(v) \otimes 1 \, dv + \\
\oint_{C_3 \cup (s_2+C_2)} \frac{1}{u-v} \xi_i(v-s_1-s_2) \otimes \left( \oint_{C_3} \frac{1}{v-v'} \xi_i(v'-s_2) \otimes x_i^+(v') \, dv' \right) \, dv
\]

where $C_3$ is again a contour enclosing $\sigma(V_3)$ lying entirely within $C_3$. The equality of these two expressions follows by integrating the last term of the latter with respect to $v$ (note that the integrand has only simple poles at $v = v'$ in the variable $v$).

The proof for $x_i^-(u)$ is similar.

4. THE $R^0$-MATRIX OF THE YANGIAN

In this section, we construct the commutative part $R^0(u)$ of the $R$–matrix of the Yangian and show that it defines a meromorphic commutativity constraint on $\text{Rep}_\text{id}(Y_h(\mathfrak{g}))$, when the latter is equipped with the Drinfeld tensor product defined in $\S3$.

The matrix $R^0(u)$ was constructed by Khoroshkin–Tolstoy as a formal infinite product whose terms lie in the double Yangian $DY_h(\mathfrak{g})$ [14, Thm. 5.2]. Our starting point is the observation that $R^0(u)$ formally satisfies an additive difference equation whose coefficient matrix $A(s)$ we show to be a rational function on finite–dimensional representations of $Y_h(\mathfrak{g})$. By taking the left and right canonical fundamental solutions of this equation, we construct two regularisations $R^{0,\pm}(u)$ of $R^0(u)$ which are meromorphic functions of the spectral parameter $u$. We begin by constructing $A(u)$ in Section 4.3 below.

4.1. The $T$–Cartan matrix of $\mathfrak{g}$. Let $A = (a_{ij})$ be the Cartan matrix of $\mathfrak{g}$ and $B = (b_{ij})$ its symmetrisation, where $b_{ij} = d_i a_{ij}$. Let $T$ be an indeterminate, and $B(T) = (b_{ij}|T) \in GL_1(\mathbb{C}[T^{\pm 1}])$ the corresponding matrix of
$T$–numbers. Then, there exists a positive number $l = l(g)$ such that

$$B(T)^{-1} = \frac{1}{|l|_T} C(T) \tag{4.1}$$

where the entries of $C(T)$ are Laurent polynomials in $T$ with positive integer coefficients \cite[Table 7.6]{14}. We denote the entries of the matrix $C(T)$ by $c_{ij}(T) = \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} T^r$, and note that $c_{ji}(T) = c_{ij}(T^{-1})$.

4.2. matrix logarithms. We shall need the following result

**Proposition.** Let $V$ be a complex, finite–dimensional vector space, and $\xi : \mathbb{C} \to \text{End}(V)$ a function such that

- $\xi(u)$ is a rational, and $\xi(\infty) = 1$.
- $[\xi(u), \xi(v)] = 0$ for any $u, v \in \mathbb{C}$.

Let $\sigma(\xi) \subset \mathbb{C}$ be the set of poles of $\xi(u)^{\pm 1}$, and define the cut–set $X(\xi)$ by

$$X(\xi) = \bigcup_{a \in \sigma(\xi)} [0, a] \tag{4.2}$$

where $[0, a]$ is the line segment joining 0 and $a$. Then, there is a unique single–valued, holomorphic function $t(u) = \log(\xi(u)) : \mathbb{C} \setminus X(\xi) \to \text{End}(V)$ such that

$$\exp(t(u)) = \xi(u) \quad \text{and} \quad t(\infty) = 0 \tag{4.3}$$

Moreover, $[t(u), t(v)] = 0$ for any $u, v \in \mathbb{C}$, and $t(u)' = \xi(u)^{-1} \xi'(u)$.

**Proof.** The equation (4.3) uniquely defines $t(u)$ as a holomorphic function near $u = \infty$. To continue $t(u)$ meromorphically, note first that the semisimple and unipotent factors $\xi_S(u), \xi_U(u)$ of the multiplicative Jordan decomposition of $\xi(u)$ are rational functions of $u$ since $[\xi(u), \xi(v)] = 0$ for any $u, v$ (see e.g., \cite[Lemma 4.12]{10}). Thus,

$$t_N(u) = \log(\xi_U(u)) = \sum_{k \geq 1} (-1)^{k-1} \frac{(\xi_U(u) - 1)^k}{k}$$

is a well–defined rational function of $u \in \mathbb{C}$ whose poles are contained in those of $\xi(u)$.

To define $\log(\xi_S(u))$ consistently, note that the eigenvalues of $\xi(u)$ are rational functions of the form $\prod_{j}(u - a_j)(u - b_j)^{-1}$. Since, for $a \in \mathbb{C}^\times$, the function $\log(1 - au^{-1})$ is single–valued on the complement of the interval $[0, a]$, where log is the standard determination of the logarithm, we may define a single–valued, holomorphic function $\log(\xi_S(u))$ on the complement of the intervals $[0, a]$, where $a$ ranges over the (non–zero) zeros and poles of the eigenvalues of $\xi(u)$.

Finally, we set

$$t(u) = t_N(u) + t_S(u)$$

The fact that $[t(u), t(v)] = 0$ is clear from the construction, or from the fact that it clearly holds for $u, v$ near $\infty$. Finally, the derivative of $t(u)$ can be
computed by differentiating the identity \( \exp(t(u)) = \xi(u) \), and using the formula for the left–logarithmic derivative of the exponential function (see, e.g., [7]).

**Definition.** If \( V \) is a finite–dimensional representation of \( Y_h(\mathfrak{g}) \), and \( \xi_i(u) \) is the rational function \( \xi_i(u) = 1 + h \sum_{r \geq 0} \xi_i,u^{-r-1} \) given by Proposition 2.8, the corresponding logarithm will be denoted by \( t_i(u) \).

4.3. **The operator** \( \mathcal{A}_{V_1,V_2}(s) \). Let \( V_1, V_2 \) be two finite–dimensional representations of \( Y_h(\mathfrak{g}) \). Let \( \mathcal{C}_1 \) be a contour enclosing the poles of the operators \( \xi_i(u)^{\pm 1} \) on \( V_1 \), and consider the following operator on \( V_1 \otimes V_2 \)

\[
\mathcal{A}_{V_1,V_2}(s) = \exp \left( - \sum_{i,j \in I} \sum_{r \in \mathbb{Z}} c_{ij}^{(r)} \oint_{\mathcal{C}_1} t_i'(v) \otimes t_j \left( v + s + \frac{(l+r)h}{2} \right) dv \right)
\]

where \( s \in \mathbb{C} \) is such that \( t_j(v + s + h(l+r)/2) \) is an analytic function of \( v \) within \( \mathcal{C} \) for every \( j \in I \) and \( r \in \mathbb{Z} \) such that \( c_{ij}^{(r)} \neq 0 \) for some \( i \in I \).

**Theorem.**

(i) \( \mathcal{A}_{V_1,V_2}(s) \) extends to a rational function of \( s \) which is regular at \( \infty \), and such that

\[
\mathcal{A}_{V_1,V_2}(s) = 1 - lh^2 \frac{\Omega_h}{s^2} + O(s^{-3})
\]

where \( \Omega_h = \sum_i d_i h_i \otimes \omega_i^\vee \in \mathfrak{h} \otimes \mathfrak{h} \).

(ii) For any \( s, s' \) we have \( [\mathcal{A}_{V_1,V_2}(s), \mathcal{A}_{V_1,V_2}(s')] = 0 \).

(iii) For any \( V_1, V_2, V_3 \in \text{Rep}_{\text{fd}}(Y_h(\mathfrak{g})) \), we have

\[
\mathcal{A}_{V_1 \otimes s_1, V_2 \otimes s_2}(s_2) = \mathcal{A}_{V_1,V_2}(s_1 + s_2) \mathcal{A}_{V_2,V_3}(s_2) \\
\mathcal{A}_{V_1 \otimes s_2, V_2 \otimes s_1}(s_1 + s_2) = \mathcal{A}_{V_1,V_3}(s_1 + s_2) \mathcal{A}_{V_1,V_2}(s_1)
\]

(iv) The following (shifted) unitary condition holds

\[
\mathcal{A}_{V_1,V_2}(s - lh) = \sigma \circ \mathcal{A}_{V_2,V_1}(-s) \circ \sigma
\]

where \( \sigma \) is the flip of the tensor factors.

(v) For every \( a, b \in \mathbb{C} \) we have

\[
\mathcal{A}_{V_1(a),V_2(b)}(s) = \mathcal{A}_{V_1,V_2}(s + a - b)
\]

**Proof.** Properties (ii),(iii) and (v) follow from the definition of \( \mathcal{A} \), and the fact that \( t_i(u) \) are primitive with respect to the Drinfeld coproduct. For the proof of the remaining properties, we work in the following more general situation.

Let \( V,W \) be complex, finite–dimensional vector spaces, \( A,B : \mathbb{C} \to \text{End}(V) \) be rational functions satisfying the assumptions of Proposition 4.2,
and let \( \log A(v) \), \( \log B(v) \) be the corresponding logarithms. Let \( \sigma(A) \), \( \sigma(B) \) denote the set of poles of \( A(v)^{\pm 1} \) and \( B(v)^{\pm 1} \) respectively. Set

\[
X(s) = \exp \left( \oint_{C_1} A(v)^{-1} A'(v) \otimes \log(B(v + s)) \, dv \right)
\]

where \( C_1 \) encloses \( \sigma(A) \), and \( s \) is such that \( \log(B(v + s)) \) is analytic within \( C_1 \).

**Claim 1.** The operator \( X(s) \in \text{End}(V \otimes W) \) is a rational function of \( s \), regular at \( \infty \), and has the following Taylor series expansion near \( \infty \)

\[
X(s) = 1 + \left( A_0 \otimes B_0 \right) s - 2 + O(s^{-3})
\]

where \( A(v) = 1 + A_0 v^{-1} + O(v^{-2}) \) and \( B(v) = 1 + B_0 v^{-1} + O(v^{-2}) \).

Note that this claim implies the first part of Theorem 4.3, since

\[
\mathcal{A}_{V_1,V_2}(s) = \prod_{i,j \in I} \exp \left( \oint_C t_i'(v) \otimes t_j \left( v + s + \frac{(l+r) \hbar}{2} \right) \, dv \right)^{-c_{ij}^{(r)}}
\]

\[
= 1 - \hbar^2 s^{-2} \sum_{i,j \in I} c_{ij} (\xi_{i,0} \otimes \xi_{j,0}) + O(s^{-3})
\]

\[
= 1 - l \hbar^2 \Omega_\hbar s^{-2} + O(s^{-3})
\]

since \( c_{ij}(T)|_{T=1} \) is the \((i,j)\) entry of \( l \cdot B^{-1} \).

Part (iv) of Theorem 4.3 is a consequence of the following claim, together with the fact that \( c_{ij}^{(r)} = c_{ij}^{(-r)} \).

**Claim 2.** \( X(s) = \exp \left( \oint_{C_2} \log(A(v - s)) \otimes B(v)^{-1} B'(v) \, dv \right) \), where \( C_2 \) encloses \( \sigma(B) \) and \( s \in \mathbb{C} \) is such that \( \log(A(v - s)) \) is analytic within \( C_2 \).

We prove these claims in §4.4 and 4.5 respectively. \( \square \)

### 4.4. Proof of Claim 1

Since \( A(v) \) commutes with itself for different values of \( v \), the semisimple and unipotent parts \( A(v) = A_S(v)A_U(v) \) of the Jordan decomposition of \( A(v) \) are rational functions of \( v \) [10, Lemma 4.12]. Since the logarithmic derivative of \( A(v) \) separates the two additively, we can treat the semisimple and unipotent cases separately.

The semisimple case reduces to the scalar case, i.e., when \( V \) is one-dimensional and

\[
A(v) = \prod_j \frac{v - a_j}{v - b_j} = 1 + \left( \sum_j b_j - a_j \right) v^{-2} + O(v^{-3})
\]
for some $a_j, b_j \in \mathbb{C}$. In this case,

$$X(s) = \exp \left( \sum_j \oint_{C_1} \left( \frac{1}{v - a_j} - \frac{1}{v - b_j} \right) \otimes \log(B(v + s))dv \right)$$

$$= \exp \left( \sum_j 1 \otimes (\log(B(s + a_j)) - \log(B(s + b_j))) \right)$$

$$= \prod_j 1 \otimes B(s + a_j)B(s + b_j)^{-1}$$

which is clearly a rational function of $s$ such that

$$X(s) = 1 + s^{-2} \left( \sum_j b_j - a_j \right) \otimes B_0 + O(s^{-3})$$

Assume now that $A(v)$ is unipotent. In this case,

$$\log(A(v)) = \sum_{k \geq 1} (-1)^{k-1} \frac{(A(v) - 1)^k}{k} = A_0v^{-1} + O(v^{-2})$$

is given by a finite sum, and is therefore a rational function of $v$. Decomposing it into partial fractions yields

$$\log(A(v)) = \sum_{j \in J, n \in \mathbb{N}} \frac{N_{j,n}}{(v - a_j)^{n+1}}$$

where $J$ is a finite indexing set, $a_j \in \mathbb{C}$ and $\sum_j N_{j,0} = A_0$. In this case we obtain

$$X(s) = \exp \left( \sum_{j \in J, n \in \mathbb{N}} -(n + 1)N_{j,n} \otimes \frac{\partial^{n+1}}{(n + 1)!} \log(B(v)) \bigg|_{v=s+a_j} \right)$$

This is again a rational function of $s$ since the $N_{j,n}$ are nilpotent and pairwise commute. Moreover,

$$X(s) = 1 + s^{-2} \sum_j N_{j,0} \otimes B_0 + O(s^{-3})$$

4.5. **Proof of Claim 2.** Let $X(A), X(B) \subset \mathbb{C}$ be defined by (4.2), and $C_1, C_2$ be two contours enclosing $X(A)$ and $X(B)$ respectively. For each $s \in \mathbb{C}$ such
that $C_1 + s$ is outside of $C_2$, we have

\[
\int_{C_1} A(v)^{-1} A'(v) \otimes \log(B(v + s)) \, dv \\
= - \int_{C_1} \log(A(v)) \otimes B(v + s)^{-1} B'(v + s) \, dv \\
= \int_{C_2 - s} \log(A(v)) \otimes B(v + s)^{-1} B'(v + s) \, dv \\
= \int_{C_2} \log(A(w - s)) \otimes B(w)^{-1} B'(w) \, dv
\]

where the first equality follows by integration by parts, the second by a deformation of contour since the integrand is regular at $v = \infty$ and has zero residue there, and the third by the change of variables $w = v + s$.

4.6. Commutation relations. Let $C_{1,k}^\pm, C_{2,k}^\pm$ be two contours enclosing the poles of $x_k^\pm(u)$ on $V_1, V_2$ respectively, and $a_1, a_2 : \mathbb{C} \to \text{End}(V_i)$ two meromorphic functions which are analytic within these contours, and take values in the commutative subalgebras generated by the operators $\{\xi_{i,r}\}_{i \in \mathbf{I}, r \in \mathbb{N}}$. Define

\[
X_k^{\pm,1} = \oint_{C_{1,k}^\pm} a_1(v)x_k^\pm(v) \otimes a_2(v) \, dv \quad \text{and} \quad X_k^{\pm,2} = \oint_{C_{2,k}^\pm} a_1(v) \otimes a_2(v)x_k^\pm(v) \, dv
\]

**Proposition.** The following commutation relations hold

\[
\text{Ad}(\mathcal{A}_{V_1,V_2}(s))X_k^{\pm,1} = \oint_{C_{1,k}^+} a_1(v)x_k^\pm(v) \otimes a_2(v)\xi_k(v + s + l\hbar)^{\pm 1}\xi_k(\nu + s)^{-1} \, dv \\
\text{Ad}(\mathcal{A}_{V_1,V_2}(s))X_k^{\pm,2} = \oint_{C_{2,k}^+} a_1(v)\xi_k(v - s)^{\pm 1}\xi_k(\nu - s - l\hbar)^{\mp 1} \otimes a_2(v)x_k^\pm(v) \, dv
\]

**Proof.** We only prove the first relation. The second one follows from the first and the unitarity property of Theorem 4.3. We begin by computing the commutation between $X_k^{\pm,1}$ and a typical summand in $\log \mathcal{A}_{V_1,V_2}(s)$. Set
b = ±hd, a_{ik}/2. By (2.1),

\[
\int_C \left(\int_{C_{i,k}^\pm} a_1(u)v^t_j(u) \otimes t_j(u + s) \, du, X_k^{\pm,1}\right)
\]

\[
= \int_C \int_{C_{i,k}^\pm} a_1(v)[v^t_j(u), x_k^\pm(v)] \otimes t_j(u + s) a_2(v) \, dvdu
\]

\[
= \int_C \int_{C_{i,k}^\pm} \frac{1}{u - v + b} a_1(v)x_k^\pm(v) \otimes t_j(u + s) a_2(v) \, dvdu
\]

\[
- \int_C \int_{C_{i,k}^\pm} \frac{1}{u - v - b} a_1(v)x_k^\pm(v) \otimes t_j(u + s) a_2(v) \, dvdu
\]

\[
+ \int_C \int_{C_{i,k}^\pm} \frac{1}{u - v - b} a_1(v)x_k^\pm(u - b) \otimes t_j(u + s) a_2(v) \, dvdu
\]

\[
- \int_C \int_{C_{i,k}^\pm} \frac{1}{u - v + b} a_1(v)x_k^\pm(u + b) \otimes t_j(u + s) a_2(v) \, dvdu
\]

\[
= \int_{C_{i,k}^\pm} a_1(v)x_k^\pm(v) \otimes (t_j(v - b + s) - t_j(v + b + s)) a_2(v) \, dv
\]

where the third identity follows by choosing the contour \( C \) so that it encloses \( C_{1,k}^\pm \) and its translates by ±b, and by using the fact that \( s \) is such that \( t_j(u + s) \) is holomorphic inside \( C \).

Let the indeterminate \( T \) of Section 4.1 act as the difference operator \( Tt_j(v) = t_j(v + h/2) \). Then,

\[
\sum_{i,j \in I} \left(\int_C \left(\int_{C_{i,k}^\pm} a_1(v)x_k^\pm(v) \otimes a_2(v)c_{ij}(T)(T^{\pm b_{ik}} - T^{\pm b_{jk}}) t_j(v + s) \, dv\right) \right)
\]

\[
= \sum_{i,j \in I} \int_{C_{i,k}^\pm} a_1(v)x_k^\pm(v) \otimes a_2(v)(T^{\pm l} - T^{\pm l}) t_j(v + s) \, dv
\]

where the second equality follows from (4.1). The claimed identity easily follows from this. \( \square \)

4.7. The abelian \( R \)-matrix of \( Y_h(\mathfrak{g}) \). Let \( V_1, V_2 \in \text{Rep}_{\text{ad}}(Y_h(\mathfrak{g})) \), and let \( \mathcal{A}_{V_1,V_2}(s) \in GL(V_1 \otimes V_2) \) be the operator defined in 4.3. Consider the additive difference equation

\[
\mathcal{R}_{V_1,V_2}(s + lh) = \mathcal{A}_{V_1,V_2}(s)\mathcal{R}_{V_1,V_2}(s)
\]

where \( l \in \mathbb{N}^\times \) is given by (4.1). This equation admits two canonical meromorphic fundamental solutions \( \mathcal{R}_{V_1,V_2}^{0,\pm}(s) \) which are uniquely determined by the following requirements (see \textit{e.g.}, [2, 3, 16] or [10, §4])

- \( \mathcal{R}_{V_1,V_2}^{0,\pm}(s) \) is holomorphic and invertible for \( \pm \text{Re}(s/h) >> 0 \).
\bullet \mathcal{R}_{V_1,V_2}^{0,\pm}(s)\text{ possesses an asymptotic expansion of the form}

\mathcal{R}_{V_1,V_2}^{0,\pm}(s) \sim 1 + \mathcal{R}_{0}^{\pm}s^{-1} + \mathcal{R}_{1}^{\pm}s^{-2} + \cdots

as \ s \to \infty \text{ with } \pm \text{Re}(s/\hbar) \gg 0.

Explicitly, since \ A_{V_1,V_2}(s) = 1 + O(s^{-2}),
\mathcal{R}_{V_1,V_2}^{0,+}(s) = \prod_{n \geq 0} A_{V_1,V_2}(s + nl\hbar)^{-1} \\
\mathcal{R}_{V_1,V_2}^{0,-}(s) = \prod_{n \geq 1} A_{V_1,V_2}(s - nl\hbar)

4.8. The following is the main result of this section.

**Theorem.** The \(GL(V_1 \otimes V_2)\)-valued functions \(\mathcal{R}_{V_1,V_2}^{0,\pm}(s)\) have the following properties

(i) \([\mathcal{R}_{V_1,V_2}^{0,\pm}(s), \mathcal{R}_{V_1,V_2}^{0,\pm}(s')] = 0\) for any \(s, s'\).

(ii) The map \(\sigma \circ \mathcal{R}_{V_1,V_2}^{0,\pm}(s) : V_1 \otimes_s V_2 \to (V_2 \otimes_s V_1)(s)\) with \(\sigma\) the flip of tensor factors, is a morphism of \(Y_{\hbar}(g)\)-modules, which is natural in \(V_1\) and \(V_2\).

(iii) For any \(V_1, V_2, V_3 \in \text{Rep}_{\text{id}}(Y_{\hbar}(g))\) we have
\begin{align*}
\mathcal{R}_{V_1 \otimes_s V_2, V_3}^{0,\pm}(s_2) &= \mathcal{R}_{V_1,V_2}^{0,\pm}(s_1 + s_2) \mathcal{R}_{V_2,V_3}^{0,\pm}(s_2) \\
\mathcal{R}_{V_1, V_2 \otimes_s V_3}^{0,\pm}(s_1 + s_2) &= \mathcal{R}_{V_1,V_3}^{0,\pm}(s_1 + s_2) \mathcal{R}_{V_1,V_2}^{0,\pm}(s_1)
\end{align*}

(iv) The following unitary condition holds

\(\mathcal{R}_{V_1,V_2}^{0,+}(s)^{-1} = \sigma \circ \mathcal{R}_{V_2,V_1}^{0,-}(-s) \circ \sigma\)

(v) For \(a, b \in \mathbb{C}\) we have
\(\mathcal{R}_{V_1,V_2}^{0,\pm}(a \otimes_s V_2)(b) = \mathcal{R}_{V_1,V_2}^{0,\pm}(s + a - b)\)

(vi) \(\mathcal{R}_{V_1,V_2}^{0,+}(s)\) and \(\mathcal{R}_{V_1,V_2}^{0,-}(s)\) have the same asymptotic expansion as \(s \to \infty\) with 1-jet
\(\mathcal{R}_{V_1,V_2}^{0,\pm}(s) \sim 1 + \hbar \Omega s^{-1} + O(s^{-2}) \quad (4.4)\)

**Proof.** Properties (iv), (v), (vi) and (vii) follow from those of Theorem 4.3 respectively. Part (iii) is a consequence of Proposition 4.9 below, which in turn follows from Proposition 4.6. \(\square\)
4.9. Let $X^\pm_k, X^\pm_{k\pm 1}$ be the operators defined in 4.6.

**Proposition.** We have the following commutation relations

\[
\text{Ad}(R^0_{V_1, V_2}(s)) X^\pm_k = \oint \frac{a_1(v) x^\pm_k(v) \otimes a_2(v) \xi_k(v + s)^\pm \, dv}{C_{k,k}}
\]

\[
\text{Ad}(R^0_{V_1, V_2}(s)) X^\pm_{k\pm 1} = \oint \frac{a_1(v) (v + s)^\pm \otimes a_2(v) x^\pm_k(v) \, dv}{C_{k,k}}
\]

**Proof.** This follows from Proposition 4.6 and the definition of $R^0_{V_1, V_2}(s)$. □

5. The functor $\Gamma$

We review below the main construction of [10]. Assume henceforth that $\mathbb{h} \in \mathbb{C} \setminus \mathbb{Q}$, and that $q = e^{\pi i \mathbb{h}}$.

5.1. **Non–congruent representations.** We shall say that $V \in \text{Rep}_{fd}(Y_\mathbb{h}(g))$ is *non–congruent* if, for any $i \in \mathcal{I}$, the poles of $x^+_i(u)$ (resp. $x^-_i(u)$) are not congruent modulo $\mathbb{Z}$. Let $\text{Rep}_{NC}(Y_\mathbb{h}(g))$ be the full subcategory of $\text{Rep}_{fd}(Y_\mathbb{h}(g))$ consisting of non–congruent representations.

5.2. **Additive difference equations.** The functor $\Gamma$ is governed by the abelian, additive difference equations

\[
\phi_i(u + 1) = \xi_i(u) \phi_i(u)
\]

defined by the commuting fields $\xi_i(u) = 1 + h \xi_{i,0} u^{-1} + \cdots$ on a finite–dimensional representation $V$ of $Y_\mathbb{h}(g)$.

Let $\phi^\pm_i(u)$ be the canonical fundamental solutions of (5.1). $\phi^\pm_i(u)$ are uniquely determined by the requirement that they be holomorphic and invertible for $\pm \text{Re}(u) >> 0$, and admit an asymptotic expansion of the form

\[
\phi^\pm_i(u) \sim (1 + \phi^0_i u^{-1} + \phi^1_i u^{-2} \cdots) (\pm u)^{h \xi_{i,0}}
\]

in any right (resp. left) halfplane (see e.g., [2, 3, 16] or [10, §4]). $\phi^+_i(u), \phi^-_i(u)$ are regularisations of the formal infinite products

\[
\xi_i(u)^{-1} \xi_i(u + 1)^{-1} \xi_i(u + 2)^{-1} \cdots \text{ and } \xi_i(u - 1) \xi_i(u - 2) \xi_i(u - 3) \cdots
\]

respectively. They are explicitly given by

\[
\phi^+_i(u) = e^{-\gamma h \xi_{i,0}} \xi_i(u)^{-1} \prod_{n \geq 1} \xi_i(u + n)^{-1} e^{h \xi_{i,0}/n}
\]

\[
\phi^-_i(u) = e^{-\gamma h \xi_{i,0}} \prod_{n \geq 1} \xi_i(u - n) e^{h \xi_{i,0}/n}
\]

where $\gamma$ is the Euler–Mascheroni constant.
Let \( S_i(u) = (\phi_i^+(u))^{-1} \phi_i^-(u) \) be the connection matrix of (5.1). Thus, \( S_i(u) \) is 1–periodic in \( u \) and therefore a function of \( z = \exp(2\pi i u) \). It is moreover regular at \( z = 0, \infty \) and

\[
S_i(0) = e^{\pi i b_i,0} = S_i(\infty)^{-1}
\]

Explicitly,

\[
S_i(u) = \lim_{n \to \infty} \xi_i(u+n) \cdots \xi_i(u+1) \xi_i(u-1) \cdots \xi_i(u-n)
\]

5.3. The functor \( \Gamma \). Given \( V \in \text{Rep}_{fd}^\text{NC}(Y_h(\mathfrak{g})) \), define the action of the generators of \( U_q(L\mathfrak{g}) \) on \( \Gamma(V) = V \) as follows.

(i) For any \( i \in I \), the generating series \( \Psi_i(z)^+ \) (resp. \( \Psi_i(z)^- \)) of the commuting generators of \( U_q(L\mathfrak{g}) \) acts as the Taylor expansions at \( z = \infty \) (resp. \( z = 0 \)) of the rational function

\[
\Psi_i(z) = S_i(u)|_{e^{2\pi i u} = z}
\]

To define the action of the remaining generators of \( U_q(L\mathfrak{g}) \), let \( c_i^\pm \in \mathbb{C}^\times \) be scalars such that

\[
c_i^- c_i^+ = d_i \Gamma(hd_i)^2
\]

and define \( g_i^\pm(u) : \mathbb{C} \to GL(V) \) by

\[
g_i^-(u) = c_i^- \phi_i^-(u) \quad \text{and} \quad g_i^+(u) = c_i^+ \phi_i^+(u+1)^{-1}
\]

Explicitly,

\[
g_i^-(u) = c_i^- e^{-\gamma h \xi_i,0} \prod_{n \geq 1} \xi_i(u-n)_\mu e^{h \xi_i,0/n}
\]

(5.2)

\[
g_i^+(u) = c_i^+ \prod_{n \geq 1} e^{-h \xi_0/n} \xi_i(u+n)_\mu e^{h \xi_i,0}
\]

(5.3)

(ii) For any \( i \in I \) and \( k \in \mathbb{Z} \), \( X_{i,k}^\pm \) acts as the operator

\[
X_{i,k}^\pm = \oint_{C_i^\pm} e^{2\pi i ku} g_i^\pm(u) x_i^\pm(u) \, du
\]

where the Jordan curve \( C_i^\pm \) encloses the poles of \( x_i^\pm(u) \) and none of their \( \mathbb{Z}^\times \)-translates. 2

5.4. Let \( \Pi \subset \mathbb{C} \) be a subset such that \( \Pi \pm \frac{\Pi}{2} \subset \Pi \). Let

\[
\text{Rep}_{fd}^\Pi(Y_h(\mathfrak{g})) \subset \text{Rep}_{fd}(Y_h(\mathfrak{g}))
\]

be the full subcategory of consisting of the representations \( V \) such that \( \sigma(V) \subset \Pi \).

Similarly, let \( \Omega \subset \mathbb{C}^\times \) be a subset stable under multiplication by \( q^{\pm 1} \). We define \( \text{Rep}_{fd}^\Omega(U_q(L\mathfrak{g})) \) to be the full subcategory of \( \text{Rep}_{fd}(U_q(L\mathfrak{g})) \) consisting of those \( V \) such that \( \sigma(V) \subset \Omega \).

\[2\text{Note that such a curve exists for any } i \in I \text{ since } V \text{ is non–congruent.} \]
5.5. \textbf{Theorem.}

(i) The above operators give rise to an action of $U_q(L\mathfrak{g})$ on $V$. They therefore define an exact, faithful functor

\[ \Gamma : \text{Rep}^{NC}_{\text{fd}}(Y_\hbar(\mathfrak{g})) \rightarrow \text{Rep}_{\text{fd}}(U_q(L\mathfrak{g})) \]

(ii) The functor $\Gamma$ is compatible with shift automorphisms. That is, for any $V \in \text{Rep}^{NC}_{\text{fd}}(Y_\hbar(\mathfrak{g}))$ and $a \in \mathbb{C}$,

\[ \Gamma(V(a)) = \Gamma(V)(e^{2\pi i a}) \]

(iii) Let $\Pi \subset \mathbb{C}$ be a non–congruent subset such that $\Pi \pm \frac{1}{2} \hbar \subset \Pi$. Then, $\text{Rep}^\Pi_{\text{fd}}(Y_\hbar(\mathfrak{g}))$ is a subcategory of $\text{Rep}^{NC}_{\text{fd}}(Y_\hbar(\mathfrak{g}))$, and $\Gamma$ restricts to an isomorphism of abelian categories.

\[ \Gamma_{\Pi} : \text{Rep}^\Pi_{\text{fd}}(Y_\hbar(\mathfrak{g})) \sim \rightarrow \text{Rep}^\Omega_{\text{fd}}(U_q(L\mathfrak{g})) \]

where $\Omega = \exp(2\pi i \Pi)$.

(iv) $\Gamma_{\Pi}$ preserves the $q$–characters of Knight and Frenkel–Reshetikhin \cite{8, 15}.

6. Tensor structure on $\Gamma$

6.1. Let $V_1, V_2$ be finite–dimensional representations of $Y_\hbar(\mathfrak{g})$. Define an $GL(V_1 \otimes V_2)$–valued function $\mathcal{J}_{V_1,V_2}(s)$ by

\[ \mathcal{J}_{V_1,V_2}(s) = e^{b_\hbar \Omega} \prod_{m \geq 1} \mathcal{R}_{V_1,V_2}^{0,+}(s + m) e^{-\frac{b_\hbar \Omega}{m}} \]

\textbf{Theorem.}

(i) $\mathcal{J}_{V_1,V_2}(s)$ is a meromorphic function of $s$ which is is natural in $V_1, V_2$.

(ii) If $V_1$ and $V_2$ are non–congruent,

\[ \mathcal{J}_{V_1,V_2}(s) : \Gamma(V_1) \otimes \zeta \Gamma(V_2) \rightarrow \Gamma(V_1 \otimes_s V_2) \]

is an isomorphism of $U_q(L\mathfrak{g})$–modules for any $s \notin \sigma(V_2) - \sigma(V_1) + \mathbb{Z}$, where $\zeta = e^{2\pi is}$. 
(iii) For any non-congruent finite-dimensional representations $V_1, V_2, V_3$, the following is a commutative diagram

\[
\begin{array}{ccc}
\Gamma(V_1) \otimes_{\zeta_1} \Gamma(V_2) & \xrightarrow{\mathcal{J}_{V_1,V_2}(s_1) \otimes 1} & \Gamma(V_1) \otimes_{\zeta_1 \zeta_2} \Gamma(V_2) \\
\Gamma(V_1 \otimes_{s_1} V_2) \otimes_{\zeta_2} \Gamma(V_3) & \xrightarrow{\mathcal{J}_{V_1,s_1} V_2 \otimes_{s_1} V_3(s_2)} & \Gamma(V_1 \otimes_{s_1 + s_2} V_2 \otimes_{s_2} V_3) \\
\Gamma((V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3) & \xrightarrow{\mathcal{J}_{V_1,s_1} V_2 \otimes_{s_1} V_2 \otimes_{s_2} V_3(s_1 + s_2)} & \Gamma((V_1 \otimes_{s_1} V_2) \otimes_{s_2} V_3)
\end{array}
\]

where $\zeta = \exp(2\pi is_1)$.

Note that the condition $s \not\in \sigma(V_2) - \sigma(V_1) + \mathbb{Z}$ is equivalent to $V_1 \otimes_s V_2$ being non-congruent, which is required in order to define $\Gamma(V_1 \otimes_s V_2)$.

**Proof.** The convergence of this infinite product follows from properties (i) and (iv) of $\mathcal{R}_{V_1,V_2}^{0,+}(s)$ given in §4.7. Also, naturality follows from that of $\mathcal{R}_{V_1,V_2}^{0,+}(s)$, property (iii) of §4.7. Thus it remains to show that $\mathcal{J}_{V_1,V_2}(s)$ is an intertwiner between $\Gamma(V_1) \otimes_{\zeta} \Gamma(V_2)$ and $\Gamma(V_1 \otimes_s V_2)$.

Given an element $x \in U_q(Lg)$, let us denote its action on $\Gamma(V_1) \otimes_{\zeta} \Gamma(V_2)$ by $x'$ and its action on $\Gamma(V_1 \otimes_s V_2)$ by $x''$. We need to prove that

\[\mathcal{J}_{V_1,V_2}(s) x' \mathcal{J}_{V_1,V_2}(s)^{-1} = x'' \text{ for every } x \in U_q(Lg)\]

Since $\xi_i(u)$ are group-like in the Drinfeld coproduct, so are the fundamental solutions and the connection matrix of the difference equation $\phi_i(u + 1) = \xi_i(u) \phi_i(u)$. This observation implies that $\Psi_i(z)' = \Psi_i(z)''$. Moreover, $\mathcal{R}_{V_1,V_2}^{0,+}(s)$ and hence $\mathcal{J}(s)$ commute with these elements. This proves the required relation for $\{\Psi_i(z)\}_{i \in I}$.

Note that $g_i^{\pm}(u)$ are not group-like because of the appearance of the scalars $c_i^{\pm}$. But we have $\Delta(g_i^{\pm}(u)) = \frac{1}{c_i^{\pm}} g_i^{\pm}(u) \otimes g_i^{\pm}(u)$. Now we prove the relation for $\mathcal{X}_{t,0}^{+}$. The proof for $\mathcal{X}_{t,0}^{-}$ is similar. This will prove the theorem, since $\{\Psi_i(z)\}$ together with $\{\mathcal{X}_{t,0}^{\pm}\}$ generate $U_q(Lg)$ as an algebra. An easy computation with the definitions of the functor $\Gamma$ and the Drinfeld
coproduct shows that
\[
\left( X_{i,0}^+ \right)' = \oint_{C_1} g_i^+(v) x_i^+(v) \otimes 1 \, dv \\
+ \frac{1}{c_i^+ c_i^-} \oint_{C_2} g_i^+(v-s) g_i^-(v-s) \xi_i(v-s) \otimes g_i^+(v) x_i^+(v) \, dv \tag{6.1}
\]
\[
\left( X_{i,0}^+ \right)'' = \frac{1}{c_i^+} \oint_{C_1} g_i^+(v) x_i^+(v) \otimes g_i^+(v+s) \, dv \\
+ \frac{1}{c_i^+} \oint_{C_2} g_i^+(v-s) \xi_i(v-s) \otimes g_i^+(v) x_i^+(v) \, dv \tag{6.2}
\]

Let us compute the action of \( \text{Ad}(J_{V_1, V_2}(s)) \) on the first term on the right–hand side of (6.1) using Proposition 4.9. We will also need the following relation
\[
\text{Ad}(e^{\Omega_h})(x_i^+(v) \otimes 1) = x_i^+(v) \otimes e^{\xi_i(v)}
\]
\[
\text{Ad}(J_{V_1, V_2}(s)) \left( \oint_{C_1} g_i^+(v) x_i^+(v) \otimes 1 \, dv \right) \\
= \oint_{C_1} g_i^+(v) x_i^+(v) \otimes e^{\gamma h \xi_i(v)} \prod_{n \geq 1} \xi_i(v + s + n) e^{-h \xi_i(v)/n} \, dv \\
= \oint_{C_1} g_i^+(v) x_i^+(v) \otimes \left( c_i^+ \right)^{-1} g_i^+(v+s) \, dv
\]
by definition of \( g_i^+(v) \) given in (5.3). Thus we obtain the first term on the right–hand side of (6.2). A similar computation can be carried out for the second term of the right–hand side of (6.1) which proves that
\[
J_{V_1, V_2}(s)(X_{i,0}^+)' J_{V_1, V_2}(s)^{-1} = (X_{i,0}^+)''
\]
This completes the proof of the first part of the theorem. Note that the second part is an easy consequence of Theorem 4.8 (vii). \( \square \)

References
1. J. Beck and V.G. Kac, Finite–dimensional representations of quantum affine algebras at roots of unity, Journal of the AMS 9 (1996), no. 2, 391–423.
2. G.D. Birkhoff, General theory of linear difference equations, Trans. Amer. Math. Soc. 12 (1911), 243–284.
3. A. Borodin, Isomonodromy transformations of linear systems of difference equations, Annals of Mathematics 160 (2004), 1141–1182.
4. V. G. Drinfeld, A new realization of Yangians and quantum affine algebras, Soviet Math. Dokl. 36 (1988), no. 2, 212–216.
5. V.G. Drinfeld, A new realization of Yangians and quantized affine algebras, Preprint FTINT (1986), 30–86.
6. B. Enriquez, S. Khoroshkin, and S. Pakuliak, Weight functions and Drinfeld currents, Comm. Math. Phys. 276 (2007), 691–725.
7. J. Faraut, Analysis on Lie groups, an introduction, Cambridge University Press, 2008.
8. E. Frenkel and N. Yu. Reshetikhin, *The q-characters of representations of quantum affine algebras and deformed W-algebras*, Recent Developments in Quantum Affine Algebras and Related Topics (Raleigh, NC), Contemp. Math., vol. 248, AMS, Providence, RI, 1999, pp. 163–205.

9. I. B. Frenkel and N. Yu. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. 146 (1992), 1–60.

10. S. Gautam and V. Toledano Laredo, *Yangians and quantum loop algebras II: Equivalence of categories via abelian difference equations*, (2013), arXiv:1310.7318.

11. ———, *Analytic properties of the R–matrix of the Yangian*, in preparation (2014).

12. D. Hernandez, *Representations of quantum affinizations and fusion product*, Transform. Groups 10 (2005), no. 2, 163–200.

13. ———, *Drinfeld coproduct, quantum fusion tensor category and applications*, Proc. Lond. Math. Soc. (3) 95 (2007), no. 3, 567–608.

14. S. Khoroshkin and V. Tolstoy, *Yangian double*, Lett. Math. Phys. 36 (1996), 373–402.

15. H. Knight, *Spectra of tensor products of finite dimensional representations of Yangians*, J. Algebra 174 (1995), 187–196.

16. I.M. Krichever, *Analytic theory of difference equations with rational and elliptic coefficients and the Riemann–Hilbert problem*, Russian Math. Surveys 59 (2004), no. 6, 1117–1154.

17. F. A. Smirnov, *Form factors in completely integrable models of quantum field theory*, Advanced series in mathematical physics, vol. 14, World Scientific, 1992.

18. Y. S. Soibelman, *The meromorphic braided category arising in quantum affine algebras*, Int. Math. Res. Not. 19 (1999), 1067–1079.

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027

E-mail address: sachin@math.columbia.edu

Department of Mathematics, Northeastern University, 360 Huntington Avenue, Boston, MA 02115

E-mail address: V.ToledanoLaredo@neu.edu