A Lorentz covariant representation of bound state wave functions

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We present a method enabling us to write in relativistic manner the wave functions of some particular two particle bound state models in quantum mechanics. The idea is to expand the bound state wave function in terms of free states and to introduce the potential energy of the bound system by means of the 4-momentum of an additional constituent, supposed to represent in a global way some hidden degrees of freedom. The procedure is applied to the solutions of the Dirac equation with confining potentials which are used to describe the quark antiquark bound states representing a given meson state.

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I. INTRODUCTION

About ten years ago a Lorentz covariant stationary expression for the internal wave function of a meson has been introduced [1]. The meson was supposed to be made of a free $q\bar{q}$ pair and of a collective excitation of a background field representing the time averaged result of the continuous series of quantum fluctuations taking place in the bound system. In momentum space the generic form of the single meson state was written as:

$$|\mathcal{M}(P)\rangle = \int d^3 k_1 \frac{m_1}{e_1} d^3 k_2 \frac{m_2}{e_2} d^4 Q$$

$$\times \delta^{(4)}(k_1 + k_2 + Q - P)\varphi(k_1, k_2; Q)$$

$$\times \bar{u}_{s_2}(k_2) \Gamma \mathcal{M} v_{s_1}(k_1) \Phi^\dagger(Q) b_{s_1}^\dagger(k_1) a_{s_2}^\dagger(k_2)|0\rangle$$

(1)

where $a^\dagger$, $b^\dagger$ are free creation operators of the valence $q\bar{q}$ pair, $u$ and $v$ are free Dirac spinors, $\Gamma \mathcal{M}$ is a Dirac matrix ensuring the relativistic coupling of the quark spins. The collective excitation is represented by $\Phi^\dagger(Q)$ where $Q^\mu$ is the difference between the bound state 4-momentum and the sum of the free quarks 4-momenta.

The ket $|\mathcal{M}(P)\rangle$ is not the output of a dynamical scheme. In fact, it is only a phenomenological form with suitable features, like normalizability, Lorentz covariance, fulfillment of the mass shell constraints by the meson and the quark momenta.

Recently we found out that it is possible to derive an expression like (1) from the solutions of some bound state problems in relativistic quantum mechanics which allow the independent treatment of the quarks while preserving the translational invariance of the wave function. Observing that these requirements cannot be simultaneously fulfilled if the bound system is made of a $q\bar{q}$ pair only, we supposed the existence of an additional constituent which represents some hidden degrees of freedom and show that the wave function can be written in a form like (1).

In the following we present a method enabling us to relate the relativistic invariant function $\varphi(k_1, k_2; Q)$ in Eq. (1) with the bound state wave function in quantum mechanics and the 4th component of the momentum carried by the additional constituent, $Q^4$, with the potential energy of the bound system.

The second section is devoted to the particular models in relativistic quantum mechanics for bound states which are adequate to our purpose. The bound state wave function factorizes into solutions of the Dirac equation with confining potentials and a function describing the effect of the hidden degrees of freedom on the quark system.

By comparing these models with the bound state models in field theory and establishing a correspondence between their elements having similar rôles we make the conjecture that the additional constituent represents in global way the quantum fluctuations of the background field which generate the binding.

In the third section we present the general method leading to a relativistic representation of the bound state wave function in the specific cases presented above. The method is applied to the analytical solutions of the Dirac equation with confining potentials which are used to write the meson wave function.

In the last section we give some brief comments on the relation existing between various approaches to the bound state problem. Also, commenting upon the particular way of including the quantum fluctuations in a relativistic stationary representation of a bound system we conclude that our method is an alternative to the light cone formalism for low and intermediate energy.

II. BOUND STATE MODELS

As mentioned in the introduction, our way to ensure the quark independence without violating the invariance at translations of the bound state wave function is to assume the existence of an additional constituent of the bound system. This is reflected by the appearance in
rather like an environment because it is not perceived through its specific properties, but through its effect on the embedded quarks. This remark shall be enforced in the next section when making a comparison between the representations of the bound systems in quantum mechanics and field theory.

III. LORENTZ COVARIANT FORM OF THE WAVE FUNCTION $\Psi_M$

The first step of the procedure leading to a relativistic expression for the wave function it to expand $\Psi$ in terms of free states whose transformation properties at boosts are known.

Turning to the first case and noticing that $\Psi(\vec{r}_1, \vec{r}_2, \vec{p})$ in Eq. (2) is a $4 \times 4$ matrix, we write:

$$\Psi(\vec{r}_1, \vec{r}_2, \vec{p}) = \sum_{\epsilon_1, \epsilon_2, s_1, s_2} \int d^3k_1 d^3k_2 e^{i(\vec{k}_1+\vec{k}_2)(\vec{p}-\vec{r})} \tilde{\psi}_{s_1, s_2}^{(\epsilon_1, \epsilon_2)}(\vec{k}_1, \vec{k}_2) w_{\epsilon_1, s_1}(\vec{k}_1) \bar{w}_{\epsilon_2, s_2}(\vec{k}_2)$$

where $w_{\epsilon, s}$ is the general notation for free Dirac spinors, $\epsilon$ denotes the sign of the energy, $s$ is the spin projection on an arbitrary axis. By $\vec{k} = (1 - \eta) \epsilon_1 \vec{k}_1 - \eta \epsilon_2 \vec{k}_2, \vec{p} = \vec{r}_1 - \vec{r}_2$ and $\vec{K} = \epsilon_1 \vec{k}_1 + \epsilon_2 \vec{k}_2$ we denoted the relative momentum, relative position vector and total momentum of the quark pair respectively. According to the general principles of the quantum mechanics, the coefficient $\tilde{\psi}_{s_1, s_2}^{(\epsilon_1, \epsilon_2)}(\vec{k}_1, \vec{k}_2)$ in (8) writes as

$$\tilde{\psi}_{s_1, s_2}^{(\epsilon_1, \epsilon_2)}(\vec{k}_1, \vec{k}_2) = \int d^3r_1 d^3r_2 d^3\rho \Psi(\vec{r}_1, \vec{r}_2, \vec{p}) \times e^{-i(\vec{k}_1+\vec{k}_2)(\vec{r}_1-\vec{r}_2) \vec{K}} w_{\epsilon_1, s_1}(\vec{k}_1) \bar{w}_{\epsilon_2, s_2}(\vec{k}_2)$$

and is the probability amplitude to find in the bound system a free $q\bar{q}$ pair with the quantum numbers $\{\epsilon_1, \vec{k}_1, s_1\}$ and $\{\epsilon_2, \vec{k}_2, s_2\}$ and a stationary wave with momentum $-\epsilon_1 \vec{k}_1 - \epsilon_2 \vec{k}_2$.

In the second case, using the same notations as above, the single particle wave functions $\psi$ in (4), reads:

$$\psi(\vec{r} - \vec{p}) = \sum_{\epsilon, s} \int d^3k \frac{m}{e_k} \tilde{\psi}_{s}^{(\epsilon)}(k) w_{\epsilon, s}(k) e^{i\vec{k}(\vec{r} - \vec{p})}$$

where

$$\tilde{\psi}_{s}^{(\epsilon)}(k) = \int d^3r \bar{w}_{\epsilon, s}(k) \psi(\vec{r}) e^{-i\vec{k}\vec{r}}$$

is the probability amplitude to find a free quark with positive (negative) energy, spin $s$ and momentum $\vec{k}$ in a bound state characterized by $\psi$. Taking into account the similar relations existing for $\bar{\psi}$ and for the charge conjugated solution $\bar{\psi}$ we write the expansion coefficient of the wave function (9) as follows:
\[
\tilde{\Psi}^{(s_1,s_2)}(\vec{k}_1,\vec{k}_2) = \int d^3\rho \ e^{i\vec{\rho}(\vec{k}_1 + \vec{k}_2)} \tilde{\psi}^{(s_1)}(\vec{k}_1) \tilde{\psi}^{(s_2)}(\vec{k}_2) \phi(\rho - \vec{k}_1 - \vec{k}_2). \tag{12}
\]

From the above examples it results that the additional constituent contributes to the total momentum with \( \vec{Q} = -\epsilon_1 \vec{k}_1 - \epsilon_2 \vec{k}_2 \), which may be seen as the reaction of the environment with respect to the independent motion of the quarks.

A similar relation is supposed to hold for the 4th components of the momenta where the "energy" of the additional constituent, \( Q^0 \), is the binding energy of the system, defined as the remaining part from the meson mass after extracting the energies of the free quarks. Then we have:

\[
Q^0 = M - \epsilon_1 \sqrt{k_1^2 + m_1^2} - \epsilon_2 \sqrt{k_2^2 + m_2^2}. \tag{13}
\]

In an ideal model the relation (13), like the potential similar with (14) when introducing the effective potential as the remaining part in the effective QCD Lagrangian after removing the kinetic terms \( \mathcal{L} \).

We remark that, according to the above definitions, the "square mass" of the effective constituent, \( Q^2 = Q_0^2 - \vec{Q}^2 \), is not a constant and therefore, as we already supposed, \( Q^0 \) is not the 4-momentum of an elementary particle.

Now, introducing the occupation numbers of the free states and recalling that in field theory the negative energy creation operators are in fact annihilation operators of positive energy particles which give zero when acting on the vacuum, we get the following expression for the ket representing a single meson at rest:

\[
\left| \mathcal{M}(M,0) \right> = \int d^3k_1 \frac{m_1}{e_1} d^3k_2 \frac{m_2}{e_2} d^4Q \frac{\delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{Q}) \delta(\epsilon_1 + \epsilon_2 + Q_0 - M)}{e_1 e_2} \times \sum_{s_1,s_2} \tilde{\Psi}^{(s_1,s_2)}(\vec{k}_1,\vec{k}_2) \tilde{\phi}(\vec{k}_1) \tilde{\phi}(\vec{k}_2) \alpha^1(Q) a_{s_1}^\dagger(\vec{k}_1) a_{s_2}^\dagger(\vec{k}_2) b_{s_1}(\vec{k}_1) b_{s_2}(\vec{k}_2) \left| 0 \right>. \tag{14}
\]

We mention that the quark (antiquark) creation operators, \( a^\dagger \) (\( b^\dagger \)) are simple tools reflecting the quark statistics, not the result of a canonical quantification formalism. Extending this notation to the additional constituent we denote by \( a^1(Q) \) the creation of an effective excitation with momentum \( Q^0 \). The quark operators and \( \alpha \) represent independent elements in a stationary representation and we find reasonable to assume they commute with each other. However, we notice that it doesn’t make any sense to write commutations relations among \( \alpha \)-s because they do not represent elementary excitations and hence they are not quantified. The best we can do is to deal with them in such a way as to ensure the overall conservation of the energy and momentum \( M \) in the physical amplitudes which make use of expressions like (14).

The last step towards a relativistic representation of quark antiquark bound state is to write the expansion coefficients \( \tilde{\Psi}^{(s_1,s_2)} \) in relativistic invariant form. This can be easily done by replacing the 4th component of a vector \( V^\mu \) in the rest frame by \( (P^\mu V^\nu) M^{-1} \), where \( V^\mu \) is the vector \( V^\mu \) in the reference frame where the meson momentum is \( P^\mu = (E, \vec{P}) \); also, the scalar products like \( \vec{k}_i \cdot \vec{k}_j \) must be written as \( (P \cdot k_1)(P \cdot k_2) M^{-2} - (k_1 \cdot k_2) \).

Then turning to Eq.(11) we observe that it is equivalent with (14) if \( \varphi(k_1,k_2;Q) \) is identified with \( \tilde{\Psi}^{(s_1,s_2)}(k_1,k_2) \) in the first case and with \( \tilde{\psi}^{(s_1)}(k_1) \tilde{\psi}^{(s_2)}(k_2) \phi(Q) \) in the second case, written with the Lorentz invariant notations defined previously.

IV. EXAMPLES

The single particle Dirac equation

The method presented in the previous section is now applied to the cases where the single particle wave functions \( \psi \) are simple, analytical solutions of the Dirac equation with confining potentials of the type \( \mathcal{V} = \begin{pmatrix} V_{1,2} & 0 \\ 0 & -V_{2,1} \end{pmatrix} \).

i. Linear rising potentials

First we consider the case of linear rising potentials where

\[
V_{1r} = \zeta r, \tag{15}
\]
\[ \mathcal{V}_2(\vec{r}) = -2m + \sqrt{\frac{\zeta}{\xi}} (1 + 2\vec{\sigma} \cdot \vec{L}) + \frac{\xi}{r}, \]  
\[ (16) \]

where the parameters \( \zeta \) and \( \xi \) characterize the potential.

The simplest analytical solutions having the angular momentum \( J \), magnetic number \( M_J \), and energy \( E_J = m + 2\sqrt{\frac{\xi}{\zeta}}J \) are written as follows:

\[ \psi^R_{JM_J}(\vec{r}) = \begin{pmatrix} \gamma^R e^{-\sqrt{\frac{\xi}{\zeta}}r} \mathcal{Y}^{(J+\frac{1}{2}, \frac{1}{2})}_{JM_J} \\ -i\sqrt{\frac{\xi}{\zeta}} \gamma^R e^{-\sqrt{\frac{\xi}{\zeta}}r} \mathcal{Y}^{(J-\frac{1}{2}, \frac{1}{2})}_{JM_J} \end{pmatrix}, \]
\[ (17) \]

where \( \mathcal{Y}^{JM_J} \) are eigenfunctions of the total angular momentum \( J \), and \( \rho = J + \frac{1}{2} \).

Projecting \( \psi \) on the free Dirac solutions, the expression takes the form \( (10) \). Using the relativistic notation mentioned in the preceding section we get in the case \( J = 1/2, M_J = r \):

\[ \tilde{\psi}^{(+)}_{sr}(k) = \Omega^{(+)}(k) \bar{u}_s(k) u_r(k), \]
\[ \tilde{\psi}^{(-)}_{sr}(k) = \Omega^{(-)}(k) \bar{u}_s(k) \left( \frac{\gamma \cdot P}{M} \right) v_r(k), \]
\[ (18, 19) \]

where \( \gamma^\mu, \mu = 0, 1, 2, 3 \) are Dirac matrices, \( M \) is the meson mass and \( \Omega^{(\pm)}(k) \) is given by the invariant form:

\[ \Omega^{(\pm)}(k) = \sqrt{\frac{2m}{\xi}} \frac{\xi}{\sqrt{e(k) \pm m}} \left[ \sqrt{\xi} \frac{\sqrt{e(k) \pm m}}{e(k) \pm m} \Sigma(k) \right] \]
\[ \pm \left( 1 + \sqrt{\xi} \frac{1}{e(k) \pm m} \right) \Delta(k) \]
\[ (20) \]

where

\[ \Sigma(k) = \frac{\xi \xi - 3k^2}{(\xi + k^2)^3} \]
\[ (21) \]

and

\[ \Delta(k) = \frac{k}{(\xi + k^2)^2} \]
\[ (22) \]

with \( k \) representing the Lorentz invariant expression \( k = \sqrt{(P^\mu k_\mu)^2 - m^2} \) and \( e(k) = \sqrt{k^2 + m^2} \).

ii. Oscillator potential

In the case of the oscillator potential where

\[ \mathcal{V}_1 = \lambda \omega^2 r^2, \]
\[ \mathcal{V}_2 = -2m + \lambda - 2\omega(1 + \vec{\sigma} \cdot \vec{L}), \]
\[ (23, 24) \]

the simplest analytical solutions of the Dirac equation are:

\[ \psi^J_{JM_J}(\vec{r}) = \begin{pmatrix} r^{J-\frac{1}{2}} e^{-\frac{1}{2} \lambda \omega r^2} \mathcal{Y}^{(J-\frac{1}{2})}_{JM_J} \\ -i\omega r^{J+\frac{1}{2}} e^{-\frac{1}{2} \lambda \omega r^2} \mathcal{Y}^{(J+\frac{1}{2})}_{JM_J} \end{pmatrix}, \]
\[ (25) \]

Proceeding as above, and using the same notations, in the case \( J = 1/2, M = r \) we get:

\[ \tilde{\psi}^{(+)}_{sr}(k) = \sqrt{\frac{e(k) + m}{2m}} \left[ 1 - \frac{e(k) - m}{\omega} \right] e^{\frac{k^2}{2 \omega}} \bar{u}_s(k) u_r(k), \]
\[ \tilde{\psi}^{(-)}_{sr}(k) = \sqrt{\frac{e(k) + m}{2m}} \left[ \frac{e(k) + m}{\omega} - 1 \right] e^{\frac{k^2}{2 \omega}} \bar{u}_s(k) \left( \frac{\gamma \cdot P}{M} \right) u_r(k). \]
\[ (26, 27) \]

In all the above cases it is possible to define the charge conjugated solutions \( \psi^c = i\gamma_2 \gamma_0 \tilde{\psi}^T \). As a result \( u \rightarrow v \) and \( \psi^{(\pm)} \rightarrow \psi^{c(\pm)} \) in \( (18, 27) \).

Closing this section, we mention that the single particle solutions of the Dirac equation with confining potentials also can be used in the case where the confining potential depends on the relative quark coordinates if \( \mathcal{V}^{(1,2)}(\vec{r}_1 - \vec{r}_2) \) can be written as the limit of \( \mathcal{V}^{(1)}(\vec{r}_1 - \vec{p}_1) + \mathcal{V}^{(2)}(\vec{r}_2 - \vec{p}_2) \) when \( \vec{p}_1 \rightarrow \vec{r}_2 \) and \( \vec{p}_2 \rightarrow \vec{r}_1 \). Then the two particle problem separates into two independent single particle problems and the coefficient \( \tilde{\psi}^{(+, +)}_{s_1, s_2}(k_1, k_2) \) can be calculated like in the previous cases.

V. COMMENTS AND CONCLUSIONS

Now we comment briefly upon the features of the different relativistic approaches to the bound state problem in order to get a deeper understanding of the subject.

First we consider the approaches in relativistic quantum mechanics and observe that their common features are the existence of an attractive potential well and of stationary wave functions with finite norms in the space of the relative coordinates. Such approaches are not really relativistic because the interaction potential cannot be written in a boosted reference frame.

In the approaches to the bound state problem derived from the field theory, the interaction is the result of the continuous exchange of quanta \( \mathcal{E} \) between the constituents. In this case, the iterative solution of the dynamical equation is expressed in terms of a relativistic interaction kernel and of free propagators. As a result the solution is Lorentz covariant but has a fluctuating character because the bound state appears to be made of an indefinite number of free constituents.

The two representations look quite different. However, from consistency arguments one may suppose the existence of a well defined correspondence between their elements having similar roles in the bound system. In this sense we think reasonable to suppose that the stationary wave function in the quantum mechanical approach is the result of a time average over a finite time \( T \) of the fluctuating solution in the field theory. This means that if the observation time is longer than \( T \), the fluctuating set of free particles in the field approach appears as a small,
stationary set made of free particles, and of an effective excitation of the background field which gives rise to the binding.

In this paper we have shown that a similar representation emerges from a particular set of models with confining potentials, if one assumes that the bound system contains beside the pair of quarks an additional effective constituent. Owing to this one we succeeded to write the bound state wave function as a superposition of free states and to escape the problems raised by the presence of the interaction potentials by giving a Lorentz covariant meaning to the potential energy of the bound system. Furthermore, by relating the effective constituent to the quantum fluctuations of the background field generating the binding we provided a justification for the existence of some spacial degrees of freedom accompanying the interaction potential. These ones, which are quite unusual in quantum mechanics, in our models are the natural consequence of the imperfect cancellation of the vector momenta during the quantum fluctuations.

Also, as an unquantized element of the bound system, the additional constituent creates the possibility to bypass the Dirac no-go theorem which states that a bound system with a fixed number of particles can be quantized in relativistic manner only if the generators of the symmetry group depend explicitly on the interaction potential.

Concluding these comments, one can say that the additional constituent makes our method an alternative to the light cone formalism for low and intermediate energy, where only the average effect of the quantum fluctuations can be observed.

Now, comparing the two particular models presented in the second section, we remark that the first one is much like the usual potential models, where the quarks are the sources of the forces which bind them together. However, the influence of the environment which manifests through the random motion of the center of mass of the $qq$ pair around some fixed point is not to be neglected, because it is essential for the relativistic treatment of the bound system and, moreover, it may offer a solution to the old center of mass problem in relativistic quantum mechanics.

In our second case the confining forces are mainly due to the environment which looks like a glue in which both quarks are embedded. This model is similar with the bound state models where the wave function is factorized in terms of constituent wave functions and the confinement is the result of the independent interaction of the quarks with an effective constituent like, for instance, the bag in the bag models.

We further remark that in both these models, as an effect of the quark independence, the mass center of the quark pair is not at rest in the rest frame of the meson. The center of mass is at rest in the limit $\kappa \to \infty$ and $\delta \to \infty$ when the functions $\phi$ simulate the $\delta^{(3)}(\vec{k}_1 + \vec{k}_2)$ and the quarks lose their independence. In this case the first model transforms into a classical two body one and has a well defined nonrelativistic limit, while the wave function of the second model writes as

$$\Psi(\vec{r}_1, \vec{r}_2, \vec{p}_0) \to \int d^3k_1 \frac{m_1}{e_1} d^3k_2 \frac{m_2}{e_2} dQ^0 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) e^{i(\vec{k}_1 - \vec{k}_2)(\vec{r}_1 - \vec{r}_2)} \tilde{\psi}_s^{(+)}(\vec{k}_1) \tilde{\psi}_s^{(+)}(\vec{k}_2) \ldots,$$

and it may lack a suitable classical correspondent.

A last comment concerns the place of the time in the present formalism. Our models are stationary quantum mechanical models where time is a simple parameter, not a real coordinate like in the field approach. In the correspondence we established between the two representations of a bound system we conjectured that the stationary wave function in quantum mechanics is the result of a time average over a finite time $T$ of the fluctuating solution in the field approach, where $T$ is a physical parameter of the model, not a coordinate.

This supposition makes the coordinate representation inadequate for the relativistic treatment of a bound system and explains why it is impossible to relate the iterative field solution in the coordinate representation with the stationary solution in the quantum mechanical approach, or to give a relativistic meaning to the last one.

In this paper we have shown that the adequate representation is the momentum one, under the condition to find a suitable way to take into account the stationary effect of the continuous series of quantum fluctuations generating the binding.

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