A Hybrid Dynamical Extension of Averaging

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\textbf{Abstract}—We extend a smooth dynamical systems averaging technique to a class of hybrid systems with a limit cycle that is particularly relevant to the synthesis of stable legged gaits. After introducing a definition of hybrid averageability sufficient to recover the classical result, we provide a simple illustration of its applicability to legged locomotion and conclude with some rather more speculative remarks concerning the prospects for further generalization of these ideas.

I. \textbf{Introduction}

The emergence of physically motivated and mathematically tractable hybrid models [1], [2], [3] offers the prospect of extending classical ideas and techniques of dynamical systems theory for application to new control settings. In this paper we work at the intersection of a class of tractable hybrid legged locomotion models [1] with a class of well-behaved hybrid limit cycle models [3] to generalize an initial result [4] on the stability of “averageable” hybrid oscillators. Specifically, we extend a classical smooth dynamical averaging technique to a class of hybrid systems with a limit cycle that is particularly relevant to the synthesis of stable gaits.

While our present technical focus precludes more than a simple illustrative example, our aim in pursuing this generalization beyond the narrow sufficient conditions imposed in [4] is to enable much broader subsequent applicability of this result to stable composition of dynamical gaits in a new family of legged machines [5].

In the classical (i.e. smooth) setting, the effect of weak periodic perturbations can be approximated by “averaging” continuous dynamics over one period [6, Ch. 4.1]. This technique has two benefits for assessing stability: first, as entailed by the passage to a return map, it reduces the state dimension by 1 (as the “averaged” variable is removed); second, it formally justifies the neglect of certain complicated transient dynamical effects that, on average, do not affect the system’s asymptotic behavior. In Def. 1 we generalize such classically averageable systems to hybrid models whose continuous flow is punctuated by a single discrete reset. This reset introduces an abrupt change in the system’s state that precludes application of the classical result. However, by incorporating and appropriately analyzing the reset map’s contribution to the hybrid system’s Poincaré map, we provide conditions under which the cumulative effect of many iterations of hybrid flow-and-reset dynamics can be approximated using a single classical system’s flow. We envision that future work may yield generalizations to systems with multiple domains and time scales, and provide some indications for how such generalizations might be obtained (see Sec. IV).

A. \textbf{Relation to Classical Averaging}

Classical averaging [6, Ch. 4.1] yields a method of approximating (with error bounds) solutions of the $T$-periodic vector field (5) using the averaged vector field (6). As in the classical case, our results guarantee equivalence in stability type to a simpler approximant (named the \emph{averaged system}) of the system of interest. Specifically, we show that if the return map of the averaged system has a hyperbolic periodic orbit, then so does the original system, and

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{We define a class of averageable systems (Def. 1) with a single domain, fast ($x_1$) and slow ($x_2$) coordinates, with general conditions on the flow (orange), and requirements on fixed points of the flow and the reset, $x^\star$. The calculations of II-A show how to construct a new hybrid system with constant flow time (yellow, guard $\mathcal{G} := \{x_1^\star \times x_2\}$ for any hybrid system with variable flow time (purple, guard $\mathcal{G} = \gamma^{-1}(0)$) by augmenting with a flow-to-reset map.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{A Hybrid Dynamical Extension of Averaging}
\end{figure}

B. \textbf{Contributions and Organization}

Sec. II introduces an averaging result for hybrid systems in a single domain with non-overlapping guards and fixed time-of-flow (Lemma 1). Subsequently, in Thm. 1, we provide a condition under which the more general case (where the flow time between resetting is not constant) reduces to the former case (by appropriately redefining the reset map). Sec. III illustrates the utility of this result to the stability analysis of physically interesting models by application of Cor. 1 to the vertical hopper of Fig. 2, as well as accompanying simulation to indicate the physical relevance of this theory. Sec. IV provides some examples to demonstrate the present limits of this theory, and a conclusion.

II. \textbf{Averaging in \textit{Single-Mode} Hybrid Systems}

We introduce our class of “averageable” hybrid systems by first imposing the continuous dynamics properties arising in the classical setting that entail a flow with a “fast” coordinate whose influence over a “long” period turns out to be negligible (Def. 1(iii)). We then augment the classical setup by introducing a guard and associated reset map whose base and tangent properties guarantee the overall return map will be hyperbolic (Def. 1(iv)–(vi)) affording the carry-over of stability conclusions to the perturbed system. The essential result in Lemma 1 follows from two applications of the implicit function theorem (adapted for this purpose in Lemma 2) to track the fixed points of a (first partially averaged and then fully averaged) approximant to the original Poincaré map together with calculations (relegated to the Appendix) bearing on the order of magnitude perturbation in the variational flow and tangent maps. We find it convenient to precede this definition with some calculations in Sec. II-A bearing on the structure of the time-to-reset event map that enter into our sufficient conditions in the “averageable systems” Def. 1 of Sec. II-B.

A. \textbf{Transformation from Variable to Constant Flow Time}

We seek a hybrid dynamical extension of averaging. This involves introducing a guard surface into the state space of an ODE with a
cyclic variable that triggers a discrete reset of the system state at trajectory crossing events. The time elapsed between these events generally varies with initial condition, complicating our analysis. Thus before proceeding, we digress to provide a construction that enables us to transform a system with variable flow time to a system with constant flow time that has equivalent \(^1\) Poincaré (i.e., flow-and-reset) dynamics.

To that end, let: \(\mathcal{X}\) be an open subset of Euclidean space; \(F : \mathcal{X} \rightarrow T^2\) be a \(C^2\) vector field; \(R : \mathcal{X} \rightarrow \mathcal{X}\) be a \(C^1\) function; \(\gamma : \mathcal{X} \rightarrow \mathcal{X}\) be a \(C^1\) function such that with \(\mathcal{G} = \gamma^{-1}(0)\) we have \(D\gamma \neq 0\); \(\epsilon \in \mathcal{G}\) (hence \(\gamma(x) = 0\)). Let \(\Phi : \mathcal{X} \rightarrow \mathcal{X}\) denote the maximal flow associated with \(F\); recall that \(\mathcal{F} \subset \mathcal{X} \times \mathcal{X}\) is an open set containing \((0) \times \mathcal{X}\) [7, Ch. 8 §7]. Applying the implicit function theorem [7, App. IV] to the equation

\[
\gamma \circ \Phi(t, x) = 0
\]

with respect to \(t\) at \((0, x_0) \in \mathcal{F}\) yields a \(C^1\) time-to-event map \(\tau : \mathcal{U} \rightarrow \mathcal{X}\) where \(\mathcal{U} \subset \mathcal{X}\) is a neighborhood of \(x_0^+\) [7, Ch. 11 §2]. Note that the image of \(\tau\) includes negative times.

Now suppose \(\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2\) where \(\mathcal{X}_1 \subset \mathcal{X}\) and \(\mathcal{X}_2 \subset \mathcal{R}^n\) are open, and let \(x^* = (x_1^*, x_2^*)\). Let \(U_2 = \{x_2 \in \mathcal{X}_2 : (x_1^*, x_2) \in \mathcal{U}\}\) and define \(\mathcal{R} : U_2 \rightarrow \mathcal{X}_2\) by

\[
\forall x_2 \in U_2 : \mathcal{R}(x_2) := \pi_2 \circ R \circ \Phi(\tau(x_1^*, x_2), (x_1^*, x_2)).
\]

Note that the derivative (i.e. gradient) of \(\tau\) can be computed at \(x^+\) by differentiating (1) with respect to \(x^+\)

\[
D\gamma \cdot (D_1\Phi \cdot D\tau + D_2\Phi) = 0 \quad \implies \quad D\tau = -\frac{D\gamma}{D_1\Phi \cdot D\gamma}.
\]

Differentiating (2) with respect to \(x^+\) and substituting using (3) we conclude that

\[
D\mathcal{R} = D\pi_2\left(D_2R - \frac{D\gamma \cdot D_2\gamma}{D_1\Phi \cdot D\gamma}\right).
\]

### B. Averaging for Single-mode Hybrid Systems

**Definition 1** (average single-mode hybrid system). Given a separation of time-scales parameter \(\varepsilon \geq 0\), \(H = (\mathcal{X}, F, \mathcal{G}, R, x^+\)) is a single-mode hybrid dynamical system averageable at \(x^+\) if:

(i) \(\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2\) is a domain comprised of an open interval around the origin, \(\mathcal{X}_1 \subset \mathcal{R},\) containing the "phase" coordinate and a topological \(n\)-dimensional ball \(\mathcal{X}_2 \subset \mathcal{R}^n\) containing the remaining coordinate;

(ii) \(F : \mathcal{X} \rightarrow \mathcal{R}^{n+1}\) is a \(\varepsilon\)-parameterized vector field with the form

\[
\dot{x} = F(x) = e_1 + \varepsilon \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix},
\]

where: \(e_1\) is the canonical unit vector of \(\mathcal{X}_1 \subset \mathcal{X}\), and \(F_1 : \mathcal{X} \rightarrow \mathcal{R}, F_2 : \mathcal{X} \rightarrow \mathcal{R^n}\) are \(C^2\) \(\varepsilon\)-parameterized maps;

(iii) \(x^+ = (x_1^+, x_2^+)\) is such that the averaged vector field \(\overline{F} : \mathcal{X}_2 \rightarrow \mathcal{R}^n\) defined by

\[
\overline{F}(x_2) := \frac{1}{x_1^+} \int_0^{x_1^+} F_2([x_2^+]) d\sigma
\]

has an equilibrium at \(x_2^+\);

(iv) \(\mathcal{G} \subset \mathcal{X}\) is a codimension-1 embedded submanifold called a guard that (i) contains \(x^+\) and (ii) can be specified (not uniquely) as

\[\text{For 3, we get}
\]

\[
\frac{dx_2}{dx_1} = \frac{\varepsilon F_2(x)}{1 + \varepsilon F_1(x)} = \varepsilon f(x, \varepsilon),
\]

which [6, (4.1.1)] regards as a non-autonomous \(x_1\)-varying dynamical system. Note that the averaged vector field (6) is the \(x_1\)-average of (11) at \(\varepsilon = 0\), coinciding with the definition in [6, (4.1.2)].

In fact for every \(x_2 \in \mathcal{X}_2\) we have \(\overline{F} = \mathcal{I}\) then:

\[
\text{An alternate way to state the last condition is that} \quad (w, e_1) \neq 0 \quad \text{where} \quad w \quad \text{is a vector normal to} \quad \mathcal{G}.
\]

\[
\text{When} \quad x_1 \circ \mathcal{R} \quad \text{is constant but nonzero, a shift of the} \quad x_1 \quad \text{coordinate ensures} \quad \mathcal{R} \quad \text{satisfies the stated hypothesis. When} \quad x_1 \circ \mathcal{R} \quad \text{is not constant,} \quad R \quad \text{can be augmented with a flow-to-event map as in II-A to satisfy the stated hypothesis.}
\]

\[
\text{Note that we impose as a condition that} \quad \mathcal{S}_0 \quad \text{is a constant matrix that does not vary with} \quad x_2; \quad \text{this property is employed in the proof of Lemma 1 through Appendix 2.}
\]

\[\text{Used in the proof of Lemma 1 through Lemma 2.}\]
generalizing to cases with multiple disjoint guards in the same
paper all pertain to hybrid systems with a single guard; however,
Remark 3 (Multiple hybrid modes). The theoretical statements in this
case where the hybrid modes reside in different spaces lies
outside the scope of this paper (but see Sec. IV-1 for a discussion).
Lemma 1 provides a similar conclusion as [6, Thm. 4.1.1(ii)], except
this second point is proven in Appendix 1. By the implicit function
theorem [7, App. IV], there exists a C¹ function \( \rho : \mathcal{E} \rightarrow \mathcal{P} \) defined
over a sufficiently small open interval \( \mathcal{E} \subset \mathbb{R} \) containing the origin
such that \( \rho(\varepsilon) \equiv 0 \), i.e. \( \forall \varepsilon \in \mathcal{E} : \mathcal{P}(\rho(\varepsilon)) = \rho(\varepsilon) \).

Proof of Lemma 1 (based on [6]). The ODE (5) satisfies all conditions
required for the proof of [6, Thm. 4.1.1(i)] on the set \( t \in (0,x_1^2) \).
Construct the change of coordinates \( x_2 = y + \varepsilon w(y,t,\varepsilon) \), as in [6], so
that the \( \theta \)-augmented dynamics of (5) and (6) become (respectively)
the autonomous systems
\[
\begin{align*}
\dot{y} &= \varepsilon f(y) + \varepsilon^2 f_1(\theta,y,\varepsilon), & \dot{\theta} &= 1, \\
\dot{y} &= \varepsilon \tilde{f}(y), & \dot{\theta} &= 1,
\end{align*}
\]
where \( (\theta,y) \in \mathcal{X} \) and \( f_1 \) is a lumped remainder term. Define \( Q : \mathcal{X}_2 \rightarrow \mathcal{X}_2 \) and \( \widetilde{Q} : \mathcal{X}_2 \rightarrow \mathcal{X}_2 \) as the time-\( \varepsilon \)-flows for (14) and (15)
from \( x_1 = 0 \), i.e. for every \( x_2 \in \mathcal{X}_2 \) let
\[
(\varepsilon) \times (0,2) \cup \mathcal{Q}(x_2) = \pi_2 \circ \Phi(x_1^2,0,x_2), \quad \mathcal{Q}(x_2) = \pi_2 \circ \Phi(x_1^2,0,x_2),
\]
where \( \Phi \) is the flow for (14) and \( \widetilde{\Phi} \) is the flow for (15). Finally, define
\[
P := \mathcal{R} \circ \mathcal{Q}, \quad \mathcal{P} := \mathcal{R} \circ \widetilde{Q}.
\]
Since \( x_1^2 \) is an equilibrium of (15), \( \widetilde{Q}(x_2^2) = 0 \). Using [7, pg. 300],
the spatial derivative of the flow around an equilibrium is that of a
linear time-invariant system, whence
\[
\mathcal{D}\mathcal{Q}(x_2^2) = \exp(\varepsilon x_1^2 \mathcal{D}\tilde{f}(x_2)) = I + \varepsilon x_1^2 \mathcal{D}\tilde{f}(x_2^2) + O(\varepsilon^2).
\]
Note that \( x_2^2 \) is a fixed point of \( \mathcal{Q} \) since it is both an equilibrium of \( \mathcal{P} \) as well as a fixed point of \( \mathcal{R} \) (Def. 1(b)), and therefore
\[
\mathcal{D}\mathcal{P}(x_2^2) = \mathcal{D}\mathcal{R}(x_2^2) = \mathcal{D}\tilde{f}(x_2^2) + O(\varepsilon^2).
\]
From (19) and Def. 1(vi), we have the Taylor expansion
\[
\mathcal{D}\mathcal{P}(x_2^2) = S_0 + \varepsilon(S_1(x_2^2) + \varepsilon S_2(x_2^2) + \varepsilon^2 S_3(x_2^2)) + O(\varepsilon^3).
\]
First, note that since \( S_0 \) is invertible (Def. 1(vi)), the fixed point is
hyperbolic and hence isolated for \( \varepsilon > 0 \) sufficiently small.
Additionally, we know that
\[
\begin{align*}
\begin{array}{l}
\mathcal{P}(\varepsilon)(0) = x_2^2 \quad \text{(from (17), (18) and Def. 1(b))}, \\
\mathcal{D}\mathcal{P}(x_2^2)|_{\varepsilon=0} = S_0(x_2^2) \text{ is invertible (from Def. 1(vi)), and} \\
\varepsilon \text{ term in (19) is invertible (from Def. 1(vi)).}
\end{array}
\end{align*}
\]
Applying Lemma 2, we conclude that \( \mathcal{P} \) has a family of fixed points
specified by a map \( \rho : \mathcal{E} \rightarrow \mathcal{X}_2 \) satisfying \( \rho(0) = x_2^2 \).
In Appendix 2, we show that
\[
\mathcal{D}\mathcal{R}(\rho(\varepsilon)) = \mathcal{D}\mathcal{R}(x_2^2) + O(\varepsilon^2).
\]
Lemma 2 again to conclude that 
\[ P(\varepsilon) = D\phi(x_0) + O(\varepsilon^2). \]
Together with (20), we conclude
\[ D\phi(\varepsilon) = (D\phi(x_0) + O(\varepsilon^2))(D\phi(x_0) + O(\varepsilon^2)) = D(\phi(x_0) + O(\varepsilon^2)). \]
Since the \( \varepsilon \)-expansion of (21) is identical to (19), we can apply Lemma 2 again to conclude that \( P \) has a family of fixed points specified by \( \rho: \tilde{E} \to \tilde{X}_2 \) satisfying \( \rho(0) = \tilde{x}_2 \). Reusing Appendix 3 and the argument in (20) for the first equality below, we see that
\[ D\phi(\varepsilon) = D\phi(\varepsilon) + O(\varepsilon^2) \equiv D\phi(x_0) + O(\varepsilon^2). \]
Thus, we have shown that the eigenvalues of \( D\phi(\varepsilon) \) (the linearization of the return map at the fixed point of (5)) are \( \varepsilon \)-close to eigenvalues of \( D\phi(x_0) \), which has the simple form (19).

Remark 5 (Lower and upper bounds on \( \varepsilon \)). The conclusion of the preceding Lemma is formally valid only for \( \varepsilon > 0 \) sufficiently small. However, it may be possible in practice to obtain lower or upper bounds on the allowable range for \( \varepsilon \).

(a) Since we have invoked IFT in Lemma 2, it is straightforward (if tedious) to bound the size of the neighborhood in which (5) has a periodic orbit as in [8, Supplement 2.5A]. Alternatively, singular perturbation methods [9] may provide lower bounds on values of \( \varepsilon \) that ensure the conclusions of Lemma 2 hold.

(b) An obstruction to enlarging the upper bound on \( \varepsilon \) appears in our example in Sec. III: the quotient (25) is only valid when \( \varepsilon_1 > 0 \), which is violated when \( \varepsilon > \omega \).

Combining the constructions in II-A with the conclusions of Lemma 1, we obtain conditions under which a general averageable system (Def. 1) may be approximated by its average.

Theorem 1 (Averaging with variable flow time). Let \( (X, F, \mathcal{S}, R, x^*) \) be an averageable system (Def. 1). Then for all \( \varepsilon > 0 \) sufficiently small, eigenvalues of the linearization of the flow-and-reset map (10) are \( O(\varepsilon^2) \)-close to those of \( D\phi(x_0) := D\phi(x_0) \cdot (I + \varepsilon x_1D\tilde{f}(x_2)) \). Thus the asymptotic dynamics of the averaged system approximate those of the original system.

Proof. Let \( H = (X, F, \{x_1^*\} \times \mathcal{X}_2, (0, R)) \) be as in Remark 1. Note that \( H \) is an averageable system (Def. 1) with constant flow time (Def. 2), whence we can apply Lemma 1. We conclude that \( D\phi \circ D\phi(x_0) = D\phi \circ D\phi(x_0) + O(\varepsilon^2) \), where \( x_2 \) is the fixed point of \( R \circ Q \). As in II-A, the linearization \( D\phi \) is identical for \( H \) and \( \tilde{H} \); we conclude that the linearization of the return map of \( H \) satisfies \( D\phi(x_2) = D\phi(x_2) + O(\varepsilon^2) = D\phi \circ D\phi(x_0) + O(\varepsilon^2) \).

C. Stability of Orthogonally Reset Averageable Systems

We focus in this section on the case where \( S_0 \) is an orthogonal matrix. Our motivation for this comes from the form of the \( \varepsilon \)-parameterized return map (19). If \( S_0 \) has eigenvalues which are not strictly on the unit circle, the asymptotic behavior is dominated by \( S_0 \) for small \( \varepsilon \) (rendering the continuous dynamics irrelevant). In this subsection we examine contractiveness of \( DPDTDP \) as a sufficient condition for stability, in which case the relevant property of \( S_0 \) is orthogonality, a special case of having eigenvalues on the unit circle. Even though we only discuss a scalar example in this paper (Sec. III), orthogonal reset systems appear widely in hybrid system models for locomotion (e.g. [10], [11], [12]).

Using Thm. 1, we expose in a simple formula (Cor. 1) the relative contributions of the continuous (specified by \( F \)) and discrete (specified by \( R \)) dynamics to the overall (in)stability of the averageable system.

Definition 3 (Averageable system with orthogonal reset). An averageable system \( (X, F, \mathcal{S}, R, x^*) \) has orthogonal reset if:

(a) \( S_0 \) in (7) is an orthogonal matrix;
(b) \( W := S_0 \cdot S_1 + x_1D\tilde{f} \) is invertible at \( x_2 \).

Remark 6. The first condition in the preceding definition implies that \( S_0 \) is invertible. The second condition ensures Def. 1(iv)(c) is satisfied. Thus orthogonal resetting provides one route to ensure the hypotheses of 1(vi) are satisfied.

Corollary 1 (Stability of averageable system with orthogonal reset). An averageable system \( (X, F, \mathcal{S}, R, x^*) \) with orthogonal reset has an exponentially stable periodic orbit if \( W(x_2)^T + W(x_2)^T < 0 \) (negative definite), where \( W \) is as defined in Def. 3.

Proof. The linearization of the averaged return map at \( x_2 \) is \( D\phi = D\phi - D\phi = (S_0 + \varepsilon S_1)(I + \varepsilon x_1D\tilde{f}) = S_0 + \varepsilon (S_1 + x_1S_0D\tilde{f}) + O(\varepsilon^2) = S_0(I + \varepsilon W) + O(\varepsilon^2) \).

For arbitrary unit vector \( v \),
\[
\|S_0(I + \varepsilon W)v\|^2 - \|v\|^2 = v^T(I + \varepsilon(W^T + W))v - v^Tv + O(\varepsilon^2) = \varepsilon v^T(W^T + W)v + O(\varepsilon^2).
\]
For small \( \varepsilon > 0 \), we see that the right hand is negative since \( W^T + W < 0 \) by assumption. Thus \( DP \) is a contraction, whence the return map \( P \) is locally exponentially stable. Using Thm. 1, we conclude for \( \varepsilon > 0 \) sufficiently small that the return map of the original system \( P = R \circ Q \) has an exponentially stable fixed point nearby.

III. APPLICATION: 1DOF VERTICAL HOPPER

The application domain that motivated the preceding theoretical developments is legged locomotion on land. A well-known model for running is that of a mass suspended on a massless leg by a (physical or virtual) spring [13], [14]—the so-called Spring-Loaded Inverted Pendulum (SLIP) [15]. Variants of this model have been analyzed extensively in the literature, e.g. see [16] for analysis of a one-degree-of-freedom (1DOF) restriction of this planar point-mass model whose energizing input is inspired by the empirically successful strategies reported in [17]. Informed by the structure of the averaging results of Sec. II, we propose an alternative energization scheme (intuitively similar but physically distinct from [17]) and apply Thm. 1 to establish an analogous stability result.

The physical model is illustrated in Fig. 2: a unit mass restricted to travel along its vertical axis with an attached massless leg. The physical model is illustrated in Fig. 2: a unit mass restricted to travel along its vertical axis with an attached massless leg. The physical model is illustrated in Fig. 2: a unit mass restricted to travel along its vertical axis with an attached massless leg.
A. Averageable Vertical Hopper Model

In the absence of damping and actuation, the spring-mass hopper exhibits constant stance duration [17]. Introducing even small amounts of damping or actuation generally leads to variable stance durations, and this effect can influence the system’s stability properties. In what follows, we show that introducing viscous drag and periodic forcing results in stance duration that varies weakly in a sense made precise in (31), account for this effect in the linearized return map (32), and assess stability of hopping in the resulting nonconservative system.

1) Continuous dynamics: Introducing viscous drag \(-\varepsilon \ddot{z}\) and actuation \(u\), the hopper’s equations of motion are

\[
\ddot{z} = \begin{cases} u - g - \varepsilon \beta \dot{z} & \text{(stance);} \\ -g & \text{(flight).} \end{cases}
\]  

We consider the empirically-motivated weakly nonlinear periodic energization strategy from [4, eq. (8)] in stance, which involves defining phase-energy coordinates \(x := (\psi, a)\) (where \(x_k = \psi\) denotes the “phase” coordinate of Def. 1, and \(x_2 = a\) denotes the remaining coordinate) such that

\[ a \sin \psi = z_0 - z, \quad \alpha_\omega \cos \psi = -\dot{z}. \]  

With this notation in force, and having met the requirements of Def. 1(i) for \(\psi\) in some open interval \(X_1 := (-\Psi, \Psi) \subset \mathbb{S}^1\) and \(a \in X_2 := \mathbb{R}_+\), the feedback law from [4, eq. (8)] becomes

\[ u(x) := (g + \omega^2 a \sin \psi - \varepsilon \omega k \cos \psi), \]  

where the first two summands can be thought of as instantiating a virtual Hooke’s law spring, and the last summand’s negative damping\(^{14}\) term serves to supply the system with energy [12, 18]. In the phase-energy coordinates, the closed-loop dynamics are given by [19, Eqn. (10.36)]

\[
\dot{x} = \begin{bmatrix} \omega \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} \frac{1}{\alpha a} v(x) \sin \psi \\ -\frac{\omega a}{v(x)} \cos \psi \end{bmatrix},
\]  

where \(v(\psi, a) := a(\alpha \beta - k) \cos \psi\). With

\[ x_1^* = \pi, \quad x_2^* = k/\beta, \]  

a straightforward computation yields

\[
\int_0^\pi F_2 \left[ \begin{bmatrix} \sin \psi \\ \frac{1}{a} \cos \psi \end{bmatrix} \right] d\psi = -\int_0^\pi v(x) \cos \psi d\psi = \frac{\pi(k - x_2^* \beta)}{2} = 0,
\]  

so \(x^*\) is a fixed point of the averaged vector field

\[
\tilde{f}(a) = \frac{k - a \beta}{2 \omega},
\]  

satisfying Def. 1(iii)–(iii).

2) Guard set: The guard set is defined by the physical liftoff event when the normal force at the toe-ground interface goes to 0, i.e. \(u - g - \varepsilon \beta \dot{r} = 0\). From (24), this occurs at the zeros of

\[
\gamma(x) := \omega \tan \psi - \varepsilon \left( \frac{k}{a} - \beta \right).
\]  

Note that \(\gamma(x^*) = 0\) at the fixed point of the averaged vector field (28), satisfying Def. 1(iv).

3) Reset map: As in [4], the massless in-flight leg is reset to its nominal length, \(z_0\). It follows from (23) that the touchdown phase, \(\psi\) is identically zero since \(z = \rho\) at the touchdown event. Noting from (23) that \(\dot{z} = a\omega\) at touchdown, and recalling that the mechanical energy \(\dot{\epsilon}^2/2 + gz\) is conserved in flight, we can solve for \(a\) at touchdown, yielding

\[
R(x) = \left[ \sqrt{a^2 \cos^2 \psi - 2ga \sin \psi/\omega^2} \right],
\]  

Note that \(\pi_1 R \equiv 0\) and \(\pi_2 R(x^*) = x_2^*,\) satisfying Def. 1(v).

B. Averaging the Vertical Hopper Model

With \(X_1\) and \(X_2\) as defined above, \(X = X_1 \times X_2, \) \(F\) as in (25), \(\mathcal{G} = \gamma^{-1}(0)\) with \(\gamma\) as in (29), \(R\) as in (30), and \(x^* \in \mathcal{X}\) as in (26), the system \(H = (X, F, \mathcal{G}, R, x^*)\) satisfies conditions (i)–(v) from Def. 1. From (29),

\[
D\gamma(x^*) = \left[ \sec^2 \pi \frac{\epsilon \dot{\epsilon}}{\omega^2} \right] = \left[ 1 \frac{\epsilon \dot{\epsilon}^2}{\omega^2} \right],
\]  

whence by (25) we see that \(D\gamma \cdot F(x^*) = \omega + O(\epsilon^2)\). Noting further that \(D R \cdot F(x^*) = g/\omega\), we find that (4) simplifies to

\[
\mathbb{N} = 1 - \frac{\epsilon g \beta^2}{k \omega^3},
\]  

where the second summand was introduced by the variable flow time, and would have been neglected if a constant stance duration approximation were employed. Using (28) we can check that

\[
W := S_1 + \pi D \tilde{f} = -g \beta^2 \frac{\beta \pi}{k \omega^2} < 0,
\]  

satisfying Def. 1(vi). We have now checked all the conditions of Thm. 1, and from (32) it is clear that \(a S_0 = 1\) satisfies the “orthogonality” condition, and \(b) W < 0\) satisfies the rank condition of Def. 3. We conclude from Cor. 1 that the vertical hopper has a stable fixed point that is \(\varepsilon\)-close\(^{15}\) to (26).

Remark 7 (approximating continuous control with discrete steps). Note that the averaged vector field (28) has the form of a proportional controller on total energy. Thus Thm 1 enables us to conclude that the cumulative control effect on body height from multiple isolated steps through a second-order ODE is approximately equivalent to that of a first-order ODE acting on body height (as shown in Fig. 3(middle)).

\(^{15}\)Practitioners may wish to note that \(\varepsilon\)-closeness in state corresponds to \(\varepsilon\)-closeness in energy in mechanical systems like this hopper.
C. Simulation Results

Using parameters \( \omega = 50 \text{ rad/s}, k = 0.4 \text{ N-s/m}^2, \beta = 10 \text{ N/(m/s)} \) and \( \epsilon = 2 \), numerical simulations of the vertical hopper show that a) the fixed point of the averaged system is approximately 0.15 mm away from the original system's fixed point (Fig. 3 middle, difference between purple and dashed gray \( a^* \)), and b) the residual error between trajectories of the averaged and original systems are an order of magnitude smaller than \( a^* = k/\beta = 0.04 \text{ m} \) (Fig. 3, bottom).

IV. CONCLUSIONS AND FUTURE WORK

This paper presents, to the best of our knowledge, the first instance of a generalization to hybrid systems of a classical averaging result. Thus, Thm. 1 joins a growing body of cases wherein suitably constructed hybrid systems [2], [3], [20], [21], [22], [23] admit an appropriately restated version of classical dynamical systems results, with useful applications to new engineering settings. Application to a simple 1DOF model relevant to legged locomotion (Sec. III) indicates that stability analyses of limit cycles in higher dimensional systems [5], [12] could be greatly simplified, in analogy to the simple construction afforded by [4] relative to the initial controllers of [12]. This is an avenue of ongoing research being undertaken by the authors, and would add to the large body of emerging engineering-motivated research to develop approximations of the behavior of nonlinear dynamical systems near reference trajectories [24] (limit cycles in the case of this paper).

We conclude with some examples that demonstrate limitations of the present theory, and in doing so, motivate future theoretical work.

1) Extension to multiple domains: Intuitively, conditions 1(ii)–(v) together ensure that all non–phase coordinates vary slowly with respect to the phase. Robotic systems in steady–state operation—with asymptotically stable limit cycles—are one (important) class of systems our results apply to, but by no means the only.

Generalizing the results herein to hybrid systems with multiple modes or overlapping guards presents a number of challenges. With multiple domains, there is no privileged set of coordinates shared across disjoint portions of state space, so it is not obvious how to generalize conditions (ii) and (iv) in Def. 1. With overlapping guards, the return map is generally discontinuous [25, Table 3] or at least nonsmooth [2, Sec. 4.2], so it is not obvious how to generalize conditions (v) and (vi) in Def. 1.

2) Effects of \( \epsilon \)-perturbation: As discussed in the remarks after Def. 1, large–\( \epsilon \) limits of classical averaging conclusions can be found in the literature (e.g. [9, pg. 17]). Numerically, as well as from our empirical experience in [4], the vertical hopper example in Sec. III retains the asymptotic behavior of (28) for large \( \epsilon \) (Sec. III-C).

3) Rank condition in Def. 1(vii)(c): We emphasize that this condition is necessary for averaging along the lines of hyperbolicity in the classical theory [6, Thm. 4.1.1]. Indeed, consider \( H = (X, F, \mathcal{G}, R, x^*) \) with

\[
X = \mathbb{R}^2, \quad F = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathcal{G} = \{ x^*_1 \} \times \mathbb{R}, \quad R = \begin{bmatrix} 0 & x_2 + \epsilon x^*_1 x_2 \end{bmatrix}
\]

and \( x^* = (x^*_1, 0) \) for any \( x^*_1 > 0 \), and observe that \( \bar{f} = -x_2 \) and \( D\bar{f} = -1 \). Since \( S_1 + x^*_1 D\bar{f} = 0 \), \( H \) violates Def. 1(vii)(c). Note that the linearization of the return map,

\[
D\bar{f} \cdot D\bar{Q}(x) = (1 + \epsilon x^*_1)(1 - \epsilon x^*_1) + O(\epsilon^2) = 1 + O(\epsilon^2),
\]

is not hyperbolic to \( O(\epsilon) \), so we cannot assess stability using Thm. 1.

16With Mathematica 10, using NDSolve and WhenEvent.

4) Multiple fast coordinates: The form of (5) does not apply directly to systems where there are multiple fast coordinates. However, we plan (in future work) to exploit slow phase difference dynamics, as previously demonstrated in the classical averaging context [26].

APPENDIX

1) Calculation of \( D_1 \zeta \) for proof of Lemma 2: Using (12) and (13), we can calculate

\[
\begin{align*}
V^{-1} D_1 \zeta (p^*, \epsilon)V &= \begin{bmatrix} \frac{1}{2} I_m & 0 \\ I_{n-m} & 0 \end{bmatrix} \left( \begin{bmatrix} 0 \\ U - I_{n-m} \end{bmatrix} + \epsilon V A_1 V^{-1} \right).
\end{align*}
\]

In the limit \( \epsilon \to 0 \), the top m rows have rank \( m \) since \( A_1 \) is full rank, while the bottom \( n - m \) rows evaluate to \( U - I_{n-m} \); since \( U \) has no unity eigenvalues, \( U - I_{n-m} \) is also full rank.

For \( \epsilon \neq 0 \), the argument is unchanged for the first \( m \) rows. For the bottom rows, the entries of \( U - I_{n-m} \) dominate those of \( \epsilon V A_1 V^{-1} \), and so by continuity of eigenvalues with matrix entries, the right hand side is full rank for sufficiently small \( |\epsilon| \).

2) Calculation of \( D\bar{R}(\rho(\epsilon)) \) for (20): Using the \( \epsilon \)-expansion of \( D\bar{R} \) in Def. 1(vii),

\[
||D\bar{R}(\rho(\epsilon)) - D\bar{R}(\rho(0))|| \leq \epsilon L_1 ||\rho(\epsilon) - \rho(0)|| = O(\epsilon^2),
\]

where \( L_1 \) is the Lipschitz constant for \( S_1 \), and we used the fact that \( S_0(\rho(\epsilon)) = S_0(\rho(0)) \) as assumed in Def. 1(vi).

3) Calculation of \( DQ(\rho(\epsilon)) \) for (21): The time-varying system (14) can be interpreted as time-invariant in \( x := (\theta, b) \) coordinates,

\[
\dot{x} = \begin{bmatrix} 1 & \epsilon \\ \epsilon & 1 \end{bmatrix} f(x, \epsilon) + \epsilon^2 f_1(x, \epsilon) := \bar{F}(x).
\]

Let the time-\( t \) flow of the \( \bar{F} \) system be denoted by \( \bar{\Phi}(\tau) \). We follow [7, pg. 319] to compute the spatial Jacobian of the flow. Note that \( DQ(\rho(\epsilon)) = \pi_2 D_2 \bar{\Phi}(0, \rho(\epsilon)) \) by definition, where \( \pi_2 \) is the projection on to the second component. Let \( \bar{x} := (0, \rho(\epsilon)) \). As in [7],

\[
\frac{d}{dt} D\bar{\Phi}(\bar{x}) = A(t) D\bar{\Phi}(\bar{x}),
\]

where

\[
A(t) := D\bar{F}(\bar{\Phi}(\bar{x})) = \begin{bmatrix} 0 & 0 \\ 0 & \epsilon^2 D_1 f_1 + \epsilon^2 D_2 f_1 \end{bmatrix}.
\]

The solution of this time-varying linear system from initial condition \( D\bar{F}(\bar{\Phi}(\bar{x})) = I \) can be computed using the Peano-Baker series. Since we are only interested in the lower right block of (37),

\[
DQ(\rho(\epsilon)) = I + \int_0^\tau \int_0^\tau D\bar{F}(\bar{\Phi}(\tau) \bar{x}) dt + O(\epsilon^2)
\]

\[
= I + \epsilon \int_0^\tau \left[ (D\bar{F}(\bar{\Phi}(\tau) \bar{x}) - D\bar{F}(x^*_2)) + D\bar{F}(x^*_2) \right] dt + O(\epsilon^2)
\]

\[
= I + \epsilon \int_0^\tau D\bar{F}(\bar{\Phi}(\tau) \bar{x}) dt + O(\epsilon^2),
\]

where it is understood that \( D\bar{F} \) only takes the second component of \( \bar{\Phi}(\bar{x}) \) as argument (notation overloaded for brevity).

By continuity of the flow with respect to initial conditions [7, Thm 8.4], we know that the \( \epsilon \)-perturbation of the initial condition, \( (0, x^*_2) \) from the usage of Lemma 2 to find a fixed point of \( \bar{F} \), only affects the nominal solution \( x_{nom}(t) \equiv (0, x_2^*) \) in an \( O(\epsilon) \) way, \( \pi_2 \bar{\Phi}(\bar{x}) = x^*_2 + O(\epsilon) \). Additionally, since \( D\bar{F} \) is Lipschitz continuous,

\[
D\bar{F} \circ \pi_2 \bar{\Phi}(0, x^*_2 + O(\epsilon)) = D\bar{F}(x^*_2) + L \cdot O(\epsilon),
\]
where $L$ is the Lipschitz constant, and (38) simplifies to
\[ DQ(p(\epsilon)) = 1 + \epsilon x_1^* Df(x_2) + O(\epsilon^2). \]  
(39)

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