QUANTUM AND SPECTRAL PROPERTIES OF THE LABYRINTH MODEL

YUKI TAKAHASHI

Abstract. We consider the Labyrinth model, which is a two-dimensional quasicrystal model. We show that the spectrum of this model, which is known to be a product of two Cantor sets, is an interval for small values of the coupling constant. We also consider the density of states measure of the Labyrinth model, and show that it is absolutely continuous with respect to Lebesgue measure for almost all values of coupling constants in the small coupling regime.

1. Introduction

1.1. Quasicrystal and the Labyrinth model. The Fibonacci Hamiltonian is a central model in the study of electronic properties of one-dimensional quasicrystals. It is given by the following bounded self-adjoint operator in $l^2(\mathbb{Z})$:

$$ (H_{\lambda,\beta}\psi)(n) = \psi(n + 1) + \psi(n - 1) + \lambda \chi[1-\alpha,1](n\alpha + \beta \mod 1)\psi(n), $$

where $\alpha = \frac{\sqrt{5} - 1}{2}$ is the frequency, $\beta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the phase, and $\lambda > 0$ is the coupling constant. By the minimality of the circle rotation and strong operator convergence, the spectrum is easily seen to be independent of $\beta$. With this specific choice of $\alpha$, when $\beta = 0$ the potential of (1.1) coincides with the Fibonacci substitution sequence (for the precise definition, see section 2). Papers on this model include [7], [9], [10], [11], [14]. In [10], the authors showed that for sufficiently small coupling constant, the spectrum is a dynamically defined Cantor set, and the density of states measure is exact dimensional. Later, this result was extended for all values of the coupling constant [14].

In physics papers, it is more traditional to consider off-diagonal model, but in fact they are known to be very similar. The operator of the off-diagonal model has the following form:

$$ (H_\omega\psi)(n) = \omega(n + 1)\psi(n + 1) + \omega(n)\psi(n - 1), $$

where the sequence $\omega$ is in the hull of the Fibonacci substitution sequence. For the precise definition of hull, see (2.1). This sequence takes two positive real values.
say 1 and $a$. Let

$$\lambda = \frac{|a^2 - 1|}{a},$$

and call this the coupling constant. The spectral properties of $H_\omega$ do not depend on the particular choice of $\omega$, and depend only on the coupling constant $\lambda$. Recent mathematics papers discussing this operator include $[10, 27, 42]$. In [27] the authors considered tridiagonal substitution Hamiltonians, which include both (1.1) and (1.2) as special cases.

It is natural to consider higher dimensional models, but that is known to be extremely difficult. To get an idea of spectral properties of higher dimensional quasicrystals, simpler models have been considered. In two dimensional case, we have, for example, the square Fibonacci Hamiltonian $[13]$, the square tiling, and the Labyrinth model. The Labyrinth model is the main subject of this paper. These models are separable, so the existing results of one-dimensional models can be applied in the study of their spectral properties.

The square Fibonacci Hamiltonian is constructed by two copies of the Fibonacci Hamiltonian. Namely, this operator acts on $l^2(\mathbb{Z}^2)$, and is given by

$$[H_{\lambda_1, \lambda_2, \beta_1, \beta_2} \psi](m, n) = \psi(m + 1, n) + \psi(m - 1, n) + \psi(m, n + 1) + \psi(m, n - 1) + \lambda_1 \chi_{[1-\alpha, 1)}(m\alpha + \beta_1 \mod 1) + \lambda_2 \chi_{[1-\alpha, 1)}(n\alpha + \beta_2 \mod 1) \psi(m, n),$$

where $\alpha = \frac{\sqrt{5} - 1}{2}$, $\beta_1, \beta_2 \in \mathbb{T}$, and $\lambda_1, \lambda_2 > 0$. It is known that the spectrum of this operator is given by the sum of the spectra of the one-dimensional models, and the density of states measure of this operator is the convolution of the density of states measures of the one-dimensional models. See, for example, the appendix in [13]. Recently, it was shown that for small coupling constants the spectrum of the square Fibonacci Hamiltonian is an interval $[10]$. Furthermore, it was shown that for almost all pairs of the coupling constants, the density of states measure is absolutely continuous with respect to Lebesgue measure in weakly coupled regime $[13]$. 

![Figure 1. The square tiling (left) and the Labyrinth model (right).](image-url)
The square tiling is constructed by two copies of off-diagonal models. The operator acts on $l^2(\mathbb{Z}^2)$, and is given by

$$[H_{\omega_1,\omega_2}\psi](m, n) = \omega_1(m + 1) \psi(m + 1, n) + \omega_1(m) \psi(m - 1, n)$$

$$+ \omega_2(n + 1) \psi(m, n + 1) + \omega_2(n) \psi(m, n - 1),$$

where the sequences $\omega_1$ and $\omega_2$ are in the hull of the Fibonacci substitution sequence. All vertices are connected horizontally and vertically. See Figure 1. It has been mainly studied numerically by physicists (e.g., [17], [19], [25]). By repeating the argument from [13], one can show that the analogous results of the square Fibonacci Hamiltonian hold for the square tiling. Recently, [18] considered the square tridiagonal Fibonacci Hamiltonians, which include the square Fibonacci Hamiltonian and the square tiling as special cases.

The operator of the Labyrinth model is given by:

$$\hat{H}_{\omega_1,\omega_2}\psi(m, n) = \omega_1(m + 1)\omega_2(n + 1)\psi(m + 1, n + 1)$$

$$+ \omega_1(m + 1)\omega_2(n)\psi(m + 1, n - 1)$$

$$+ \omega_1(m)\omega_2(n + 1)\psi(m - 1, n + 1)$$

$$+ \omega_1(m)\omega_2(n)\psi(m - 1, n - 1),$$

where the sequences $\omega_1$ and $\omega_2$ are in the hull of the Fibonacci substitution sequence. It is constructed by two copies of off-diagonal models. All vertices are connected diagonally, and the strength of the bond is equal to the product of the sides of the rectangle. See Figure 1. Compare with Figure 2 from [33]. Without loss of generality, we can assume that $\omega_1$ and $\omega_2$ take values in \{1, $a_1$\} and \{1, $a_2$\}, respectively. We denote the corresponding coupling constants by $\lambda_1$ and $\lambda_2$. It can be shown that the spectral properties do not depend on the specific choice of $\omega_1$ and $\omega_2$, and only depend on the coupling constants $\lambda_1$ and $\lambda_2$. Unlike the square Fibonacci Hamiltonian or the square tiling, the spectrum is the product (not the sum) of the spectra of the two one-dimensional models, and the density of states measure is not the convolution of the density of states measures of the one-dimensional models. This model was suggested in the late 1980s in [33], and so far this has been studied mostly by physicists, and their work is mainly relied on numerics [4, 32, 33, 34, 36, 37, 38, 39, 40, 43]. Sire considered this model in [33] and the numerical experiments suggested that the density of states measure is absolutely continuous for small coupling constants and singular continuous for large coupling constants. By a heuristic argument, the author also estimated the critical value of which the transition from zero measure spectrum to positive measure spectrum occurs, and showed that it agrees with numerical experiment. In some papers, other substitution sequences, e.g., silver mean sequence or bronze mean sequence, are used to define the Labyrinth model. We consider more general cases in this paper, and give rigorous proofs to some of the physicists’ conjectures. In physicists’ work, the coupling constants of two substitution sequences $\omega_1$ and $\omega_2$ are set as equal, but we consider the case that they may be different. We denote the spectrum of (1.4) by $\hat{\Sigma}_{\lambda_1,\lambda_2}$, and the density of states measures of (1.2) and (1.4) by $\nu_\lambda$ and $\hat{\nu}_{\lambda_1,\lambda_2}$, respectively. The following theorems are the main results of this paper.

The square tiling is constructed by two copies of off-diagonal models. The operator acts on $l^2(\mathbb{Z}^2)$, and is given by

$$[H_{\omega_1,\omega_2}\psi](m, n) = \omega_1(m + 1) \psi(m + 1, n) + \omega_1(m) \psi(m - 1, n)$$

$$+ \omega_2(n + 1) \psi(m, n + 1) + \omega_2(n) \psi(m, n - 1),$$

where the sequences $\omega_1$ and $\omega_2$ are in the hull of the Fibonacci substitution sequence. All vertices are connected horizontally and vertically. See Figure 1. It has been mainly studied numerically by physicists (e.g., [17], [19], [25]). By repeating the argument from [13], one can show that the analogous results of the square Fibonacci Hamiltonian hold for the square tiling. Recently, [18] considered the square tridiagonal Fibonacci Hamiltonians, which include the square Fibonacci Hamiltonian and the square tiling as special cases.

The operator of the Labyrinth model is given by:

$$\hat{H}_{\omega_1,\omega_2}\psi(m, n) = \omega_1(m + 1)\omega_2(n + 1)\psi(m + 1, n + 1)$$

$$+ \omega_1(m + 1)\omega_2(n)\psi(m + 1, n - 1)$$

$$+ \omega_1(m)\omega_2(n + 1)\psi(m - 1, n + 1)$$

$$+ \omega_1(m)\omega_2(n)\psi(m - 1, n - 1),$$

where the sequences $\omega_1$ and $\omega_2$ are in the hull of the Fibonacci substitution sequence. It is constructed by two copies of off-diagonal models. All vertices are connected diagonally, and the strength of the bond is equal to the product of the sides of the rectangle. See Figure 1. Compare with Figure 2 from [33]. Without loss of generality, we can assume that $\omega_1$ and $\omega_2$ take values in \{1, $a_1$\} and \{1, $a_2$\}, respectively. We denote the corresponding coupling constants by $\lambda_1$ and $\lambda_2$. It can be shown that the spectral properties do not depend on the specific choice of $\omega_1$ and $\omega_2$, and only depend on the coupling constants $\lambda_1$ and $\lambda_2$. Unlike the square Fibonacci Hamiltonian or the square tiling, the spectrum is the product (not the sum) of the spectra of the two one-dimensional models, and the density of states measure is not the convolution of the density of states measures of the one-dimensional models. This model was suggested in the late 1980s in [33], and so far this has been studied mostly by physicists, and their work is mainly relied on numerics [4, 32, 33, 34, 36, 37, 38, 39, 40, 43]. Sire considered this model in [33] and the numerical experiments suggested that the density of states measure is absolutely continuous for small coupling constants and singular continuous for large coupling constants. By a heuristic argument, the author also estimated the critical value of which the transition from zero measure spectrum to positive measure spectrum occurs, and showed that it agrees with numerical experiment. In some papers, other substitution sequences, e.g., silver mean sequence or bronze mean sequence, are used to define the Labyrinth model. We consider more general cases in this paper, and give rigorous proofs to some of the physicists’ conjectures. In physicists’ work, the coupling constants of two substitution sequences $\omega_1$ and $\omega_2$ are set as equal, but we consider the case that they may be different. We denote the spectrum of (1.4) by $\hat{\Sigma}_{\lambda_1,\lambda_2}$, and the density of states measures of (1.2) and (1.4) by $\nu_\lambda$ and $\hat{\nu}_{\lambda_1,\lambda_2}$, respectively. The following theorems are the main results of this paper.
Theorem 1.1. The spectrum $\hat{\Sigma}_{\lambda_1, \lambda_2}$ is a Cantor set of zero Lebesgue measure for sufficiently large coupling constants and is an interval for sufficiently small coupling constants.

Theorem 1.2. For any $E \in \mathbb{R}$,

$$\hat{\nu}_{\lambda_1, \lambda_2} \left( (\infty,E] \right) = \int_{\mathbb{R}^2} \chi_{(\infty,E]}(xy) d\nu_{\lambda_1}(x) d\nu_{\lambda_2}(y).$$

The density of states measure $\hat{\nu}_{\lambda_1, \lambda_2}$ is singular continuous for sufficiently large coupling constants. Furthermore, there exists $\lambda^* > 0$ such that for almost every pair $(\lambda_1, \lambda_2) \in [0, \lambda^*) \times [0, \lambda^*)$, the density of states measure $\hat{\nu}_{\lambda_1, \lambda_2}$ is absolutely continuous with respect to Lebesgue measure.

1.2. Structure of the paper. In section 2, we introduce metallic mean sequences and prove some lemmas. We then define off-diagonal model and discuss necessary results. In section 3 we define the Labyrinth model, and using the results in section 2, we prove Theorem 1.1 and 1.2.

2. Preliminaries

2.1. Linearly recurrent sequences. We recall some basic facts about subshifts over two symbols.

An alphabet is a finite set of symbols called letters. A word on $A$ is a finite nonempty sequence of letters. Write $A^+$ for the set of words. For $u = u_1u_2 \cdots u_n \in A^+$, $|u| = n$ is the length of $u$. Define the shift $T$ on $A^\mathbb{Z}$ by

$$(Tx)_n = x_{n+1}.$$  

Assume that $A^\mathbb{Z}$ is equipped with the product topology. A subshift $(X,T)$ on an alphabet $A$ is a closed $T$-invariant subset $X$ of $A^\mathbb{Z}$, endowed with the restriction of $T$ to $X$, which we denote again by $T$. Given $u = u_1u_2 \cdots u_n \in A^+$ and an interval $J = \{i, \cdots, j\} \subset \{1, 2, \cdots, n\}$, we write $u_J$ to denote the word $u_iu_{i+1} \cdots u_j$. A factor of $u$ is a word $v$ such that $v = u_J$ for some interval $J \subset \{1, 2, \cdots, n\}$. We extend this definition in obvious way to $u \in A^\mathbb{Z}$. The language $L(X)$ of a subshift $(X,T)$ is the set of all words that are factors of at least one element of $X$.

Definition 2.1. Let $(X,T)$ be a subshift. We say that $x \in X$ is linearly recurrent if there exists a constant $K > 0$ such that for every factor $u,v$ of $x$, $K|u| < |v|$ implies that $u$ is a factor of $v$.

We say that a subshift is linearly recurrent if it is minimal and contains a linearly recurrent sequence. Note that if a subshift is linearly recurrent, then by minimality, all sequences belonging to $X$ are linearly recurrent.

2.2. Metallic mean sequence. Let $A = \{a,b\}$ be an alphabet, and consider the following substitution:

$$P_s : \begin{cases} a \rightarrow a^s b \\ b \rightarrow a \end{cases}$$

where $s$ is a positive integer. Consider the iteration of $P_s$ on $a$. For example, if $s = 1$,

$$a \rightarrow ab \rightarrow aba \rightarrow abaab \rightarrow abaaba \rightarrow \cdots$$
Let us write the \( n \)th iteration as \( C_s(n) \). It is easy to see that

\[
C_s(n + 1) = (C_s(n))^s C_s(n - 1).
\]

Therefore, for any \( s \in \mathbb{N} \) we can define a sequence \( \{u_s(k)\}_{k=1}^{\infty} \) by \( u_s = \lim_{n \to \infty} C_s(n) \). They are called metallic mean sequences. In particular, when \( s = 1 \), it is called the Fibonacci substitution sequence or golden mean sequence. When \( s = 2, 3 \), they are called the silver mean sequence and bronze mean sequence, respectively.

We define the hull \( \Omega_{a,b}^{(s)} \) of \( u_s \) by

\[
\Omega_{a,b}^{(s)} = \left\{ \omega \in \{a,b\}^\mathbb{Z} \mid \text{every factor of } \omega \text{ is a factor of } u_s \right\}.
\]

It is well known that \( \Omega_{a,b}^{(s)} \) is compact and \( T \)-invariant and \((\Omega_{a,b}^{(s)}, T)\) is linearly recurrent. See for example, [30] and references therein.

**Remark 2.1.** Let us define a rotation sequence \( v_{a,b,s,\beta} \) by

\[
v_{a,b,s,\beta}(n) = \begin{cases} a & \text{if } n \alpha + \beta \mod 1 \in [1 - \alpha, 1) \\ b & \text{o.w.,} \end{cases}
\]

where \( \alpha \) is given by

\[
\alpha = \frac{1}{1 + s + \frac{1}{s + \frac{1}{s + \cdots}}} = \frac{1}{s + \frac{2}{s + \frac{4}{s + \cdots}}} = \frac{1}{2} - \sqrt{s^2 + 4}.
\]

It is easy to see that the potential of the Fibonacci Hamiltonian \([1]\) is \( v_{\lambda,0,1,\beta} \). It is well known that \( v_{a,b,s,0} = u_s \), so there is no need to distinguish the rotation sequence and substitution sequence. However, it seems that it is more common to use the rotation sequence in the definition of the on-diagonal model and use the substitution sequence in the definition of the off-diagonal model. It is also known that

\[
\Omega_{a,b}^{(s)} = \bigcup_{\beta \in \mathbb{T}} v_{a,b,s,\beta}.
\]

See, for example [26].

**Remark 2.2.** There seems to be a minor confusion about substitution sequences and rotation sequences in some papers. Let

\[
\alpha^* = \frac{1}{s + \frac{1}{s + \frac{1}{s + \cdots}}}.
\]

Using \( \alpha^* \), define \( v_{a,b,s,\beta}^* \) analogously. In some papers it is stated that \( v_{a,b,s,0}^* \in \Omega_{a,b}^{(s)} \), but this is obviously not true. What is true is that \( v_{a,b,1,0}^* = v_{b,a,1,0} \), so when \( s = 1 \) there is no actual harm.
We simply write \( \Omega_{a,b} \) as \( \Omega \) below when there is no chance of confusion.

2.3. Necessary results. We will need the following definition and subsequent lemmas later.

**Definition 2.2.** Let \((X,T)\) be a linearly recurrent subshift, and let \(x, y \in \mathcal{L}(X)\). If there exist disjoint intervals \(J_1\) and \(J_2\) such that

1) \(J_i \subset \{1, 2, \cdots, |x|\}\) for \(i = 1, 2,\)
2) \(J_1 = J_2 + k\) for some odd number \(k\), and
3) \(x_{J_1} = x_{J_2} = y,\)

we say that \(y\) is odd-twin in \(x\). Define even-twin analogously. For example, in the case of the subshift \((\Omega^{(1)}, T)\), \(ab\) is odd-twin in \(abaab\), and even-twin in \(abaab\).

**Lemma 2.1.** For any \(k \geq 1\), there exists \(x \in \mathcal{L}(\Omega^{(s)})\) such that \(|x| \leq 3|C_s(k)|\) and \(C_s(k)\) is odd-twin in \(x\).

**Proof.** In the proof below, we simply write \(C_s(n)\) as \(C(n)\). Recall that \(C(n)\) satisfies the concatenation rule

\[
C(n + 1) = C(n)^a C(n - 1).
\]

Therefore, it is easy to see that \(C(k)C(k)\) and \(C(k)C(k - 1)C(k)\) are both factors of \(C(k + 3)\). If \(|C(k)|\) is odd, \(x = C(k)C(k)\) satisfies the desired properties. Suppose \(|C(k)|\) is even. Note that the sequence

\[
\{ |C(n)| \mod 2 \}
\]

repeats 1, 1, 1, 1, \cdots if \(s\) is even, and 1, 0, 1, 1, 0, 1 \cdots if \(s\) is odd. Since \(|C(k)|\) is even, \(s\) has to be odd. Therefore \(|C(k - 1)|\) is odd, so \(x = C(k)C(k - 1)C(k)\) satisfies the desired properties. \(\square\)

**Lemma 2.2.** For every \(s \in \mathbb{N}\), there exists a constant \(K_s > 0\) such that for any \(x, y \in \mathcal{L}(\Omega^{(s)})\), \(K_s |y| < |x|\) implies \(y\) is odd-twin in \(x\). Analogous results hold for even-twins.

**Proof.** Let us show the statement for odd-twin. The latter statement is immediate. Let \(K > 0\) be a number such that for any \(x, y \in \mathcal{L}(\Omega^{(s)})\), \(y\) is a factor of \(x\) whenever \(K|y| < |x|\). Let \(y \in \mathcal{L}(\Omega^{(s)})\). Take \(k > 0\) such that

\[
|C_s(k - 1)| \leq K|y| < |C_s(k)|.
\]

Then \(y\) is a factor of \(C_s(k)\), and since \(|C_s(k)| < (s + 1)|C_s(k - 1)|\), we have \(|C_s(k)| < (s + 1)K|y|\). Therefore, by Lemma 2.1 there exists \(v \in \mathcal{L}(\Omega^{(s)})\) such that \(|v| < 3(s + 1)K|y|\) and \(C_s(k)\) is odd-twin in \(v\). Since \(y\) is a factor of \(C_s(k)\), \(y\) is odd-twin in \(v\). Therefore,

\[
K_s := K \cdot 3(s + 1)K = 3(s + 1)K^2
\]

satisfies the desired properties. \(\square\)
2.4. **The off-diagonal model.** Let $a, b > 0$ be real numbers, and let $s$ be a positive integer. Let $\omega \in \Omega^{(s)}_{a,b}$. We define a Jacobi matrix $H_\omega$ acting on $l^2(\mathbb{Z})$ by

$$(H_\omega \psi)(n) = \omega(n+1)\psi(n+1) + \omega(n)\psi(n-1),$$

and set

$$\lambda = \left| \frac{a^2 - b^2}{ab} \right|.$$ 

We call this $\lambda$ the **coupling constant**. We only consider the case that $a > b$. The argument is completely analogous in the case $a < b$. By appropriate scaling, we can always assume $b = 1$. We assume this scaling all throughout this section. Note that this coincides with the definition (1.3). See also Remark 2.3 below. We call this family of self-adjoint operators $\{H_\omega\}$ the **off-diagonal model**. By a well known argument (minimality of the subshift and strong operator convergence), one can see that the spectrum of $H_\omega$ is independent of the specific choice of $\omega$ and depends only on $\lambda$ and $s$.

**Definition 2.3.** We define the trace map $T_s$ by

$$T_s = U^s \circ P,$$

where

$$U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2xz - y \\ x \\ z \end{pmatrix} \quad \text{and} \quad P \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \\ y \end{pmatrix}.$$

Let $\ell_\lambda$ be the line given by

$$\ell_\lambda = \left\{ \left( \frac{E^2 - a^2 - \frac{1}{2}E}{2a}, \frac{E}{2} \right) : E \in \mathbb{R} \right\},$$

and call this the **line of initial condition**. We define the map $J_\lambda(\cdot)$ by

$$(2.2) \quad J_\lambda : E \mapsto \ell_\lambda(E).$$

The function

$$G(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$$

is invariant under the action of $T_s$ and hence preserves the family of surfaces

$$S_V = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2xyz - 1 = \frac{V^2}{4} \right\}.$$

It is easy to see that $\ell_\lambda \subset S_\lambda$.

The following can be proven by repeating the argument of [27].

**Theorem 2.1.** We have

$$\sigma(H_\omega) = \{ E \in \mathbb{R} : \text{the forward semi-orbit of } J_\lambda(E) \text{ is bounded} \}.$$

Notice that it is clear from this theorem that the spectrum of $H_\omega$ depends only on $\lambda$ and $s$. We denote it by $\Sigma_\lambda$ below.

**Remark 2.3.** Let us define $\ell'_\lambda$ by

$$\ell'_\lambda = \left\{ \left( \frac{E^2 - \lambda E - \frac{1}{2}E}{2}, \frac{E - \lambda}{2} \right) : E \in \mathbb{R} \right\}.$$
and the map $J'_\lambda(\cdot)$ by

$$J'_\lambda : E \mapsto \ell'_\lambda(E).$$

It is easy to see that $\ell'_0 = \ell_0$ and $\ell'_\lambda \subset S_\lambda$. Then, the exact same statement in Theorem 2.1 holds with $H_{\lambda,\beta}$ and $J'$ instead of $H_\omega$ and $J$, respectively. See, for example, [10]. This is why we defined the coupling constant of off-diagonal model by (1.3).

By Theorem 2.1 we immediately get the following:

**Corollary 2.1.** The spectrum $\Sigma_\lambda$ contains 0.

**Proof.** Note that

$$(2.3) \quad J'(0) = \left( -\frac{a^2 + 1}{2a}, 0, 0 \right).$$

It is easy to see that this point is periodic under the action of $T_s$. \hfill \square

In what follows we are going to use some notations and results from the theory of hyperbolic dynamical systems, see [20] for some background on this subject.

Let us denote by $\Lambda_\lambda$ the set of points whose orbits are bounded under $T_s$. The following theorem was first proven in [35] for the Fibonacci Hamiltonian.

**Theorem 2.2** ([2], see also Theorem 4.1 from [26]). The set $\Lambda_\lambda$ is a compact locally maximal $T_s$-invariant transitive hyperbolic subset of $S_\lambda$, and the periodic points of $T_s$ form a dense subset of $\Lambda_\lambda$.

We also have the following:

**Theorem 2.3** (Corollary 2.5 of [16]). The forward semi-orbit of a point $p \in S_\lambda$ is bounded if and only if $p$ lies in the stable lamination of $\Lambda_\lambda$.

The following theorem was proven in [14] for the Fibonacci Hamiltonian case, and recently it was extended to tridiagonal Fibonacci Hamiltonians in [18]. It follows by repeating the argument of [14].

**Theorem 2.4.** For all $\lambda > 0$, the intersections of the curve of initial condition $\ell_\lambda$ with the stable lamination is transverse.

**Corollary 2.2.** The spectrum $\Sigma_\lambda$ is a dynamically defined Cantor set.

Now we define the density of states measure. The definition is analogous for higher dimensional models.

**Definition 2.4.** Denote by $H_\omega^{(N)}$ the restriction of $H_\omega$ to the interval $[0, N-1]$ with Dirichlet boundary conditions. The density of states measure $\nu_\lambda$ of $H_\omega$ is given by

$$\nu_\lambda \left( (-\infty, E] \right) = \lim_{N \to \infty} \frac{1}{N} \# \left\{ \text{eigenvalues of } H_\omega^{(N)} \text{ that are in } (-\infty, E] \right\},$$

where $E \in \mathbb{R}$.

The limit does not depend on the specific choice of $\omega$, and depends only on $\lambda$ and $s$. In fact, the convergence is uniform in $\omega$. This was shown in a more general setting [23].
It is well known that $\Sigma_0 = [-2,2]$, and

$$\nu_0 ((-\infty, E]) = \begin{cases} 
0 & E \leq -2 \\
\frac{1}{\pi} \arccos(-\frac{E}{2}) & -2 < E < 2 \\
1 & E \geq 2.
\end{cases}$$

Figure 2. The Markov partition for the map $A$.

Let us write $S = S_0 \cap \{(x, y, z) \in \mathbb{R}^3 \mid |x| \leq 1, |y| \leq 1, |z| \leq 1\}$. The trace map $T_s$ restricted to $S$ is a factor of the hyperbolic automorphism $A$ of $\mathcal{T} = \mathbb{R}^2/\mathbb{Z}^2$ given by

$$A : \left( \begin{array}{c} \theta \\ \varphi \end{array} \right) \mapsto \left( \begin{array}{c} s \\ 1 \end{array} \right) \left( \begin{array}{c} \theta \\ \varphi \end{array} \right).$$

The semi-conjugacy is given by the map

$$F : (\theta, \varphi) \mapsto (\cos 2\pi(\theta + \varphi), \cos 2\pi\theta, \cos 2\pi\varphi).$$

A Markov partition of $\mathcal{A} : \mathcal{T}^2 \to \mathcal{T}^2$ when $s = 1$ is shown in Figure 2. Compare with Figure 5 from [3]. For other values of $s \in \mathbb{N}$, the only difference is the slope of the stable and unstable manifolds. Its image under the map $F : \mathcal{T}^2 \to S$ is a Markov partition for the pseudo-Anosov map $T_s : S \to S$. Write

$$I = \{(t, t) \mid 0 \leq t \leq 1/2\}.$$ 

Note that $F(I) \subset \ell_0$. The following lemma is immediate:

**Lemma 2.3.** The push-forward of the normalized Lebesgue measure on $I$ under the semi-conjugacy $F$, which is a probability measure on $\ell_0 \cap S$, corresponds to the free density of states measure (2.4) under the identification (2.2).
Consider the union of elements of the Markov partition of $T^2$, 1, 2, 4, 5, and 6, as in Figure 3. Compare with Figure 3 from [11]. Let us denote the image of this union of elements under $F$ by $R_0$, and the continuation of $R_0$ in $\lambda > 0$ by $R_\lambda$. The following statement can be proven by repeating the proof of Claim 3.2 of [11].

**Proposition 2.1.** Consider the measure of maximal entropy of $T_s|_{\Lambda_\lambda}$ and restrict it to $R_\lambda$. Normalize this measure and project it to $\ell_\lambda$ along the stable manifolds. Then, the resulting probability measure on $\ell_\lambda$ corresponds to the density of states measure $\nu_\lambda$ under the identification (2.2).

This immediately implies the following:

**Theorem 2.5.** For every $\lambda > 0$, the density of states measure $\nu_\lambda$ is exact-dimensional. That is, for $\nu_\lambda$-almost every $E \in \mathbb{R}$, we have
\[
\lim_{\epsilon \downarrow 0} \frac{\log \nu_\lambda(E - \epsilon, E + \epsilon)}{\log \epsilon} = d_\lambda,
\]
where $d_\lambda$ satisfies
\[
\lim_{\lambda \downarrow 0} d_\lambda = 1.
\]

**Proof.** The first claim is an immediate consequence of Proposition 2.1. The second claim follows verbatim from the repetition of Theorem 1.1 of [11].

We also have the following:
Proposition 2.2 [29]. The stable and unstable Lyapunov exponents are analytic functions of \( \lambda > 0 \).

For any Cantor set \( K \), we denote the thickness of \( K \) by \( \tau(K) \). For the definition of thickness, see for example, chapter 4 of [28]. By Theorem 2 of [7] and by repeating the proof of Theorem 1.1 of [10], we obtain the following:

**Theorem 2.6.** We have
\[
\lim_{\lambda \to \infty} \dim_H \Sigma_\lambda = 0, \quad \text{and} \quad \lim_{\lambda \downarrow 0} \tau(\Sigma_\lambda) = \infty.
\]

Proposition 2.3. The density of states measure \( \nu_\lambda \) is symmetric with respect to the origin. In particular, the spectrum \( \Sigma_\lambda \) is symmetric with respect to the origin.

**Proof.** Denote by \( H_\omega^{(N)} \) the restriction of \( H_\omega \) to the interval \([0, N-1]\) with Dirichlet boundary conditions. Let \( \psi \) be an eigenvector of \( H_\omega^{(N)} \) and \( E \) be the corresponding eigenvalue. Let us define \( \phi \in l^2([0, N-1]) \) by
\[
\phi(n) = (-1)^n \psi(n) \quad (n = 0, 1, \ldots, N - 1).
\]

Then, since
\[
(H_\omega^{(N)} \phi)(n) = \omega(n+1)\phi(n+1) + \omega(n)\phi(n-1)
\]
\[
= (-1)^{n+1} \omega(n+1)\psi(n+1) + (-1)^n \omega(n)\psi(n-1)
\]
\[
= (-1)^{n+1} E \psi(n)
\]
\[
= -E \phi(n),
\]

\(-E\) is also an eigenvalue of \( H_\omega^{(N)} \). Therefore, the set of eigenvalues of \( H_\omega^{(N)} \) is symmetric with respect to the origin. Therefore, for any interval \( A \subset (0, \infty) \),
\[
\nu_\lambda (A) = \lim_{N \to \infty} \frac{1}{N} \# \left\{ \text{eigenvalues of } H_\omega^{(N)} \text{ that are in } A \right\}
\]
\[
= \lim_{N \to \infty} \frac{1}{N} \# \left\{ \text{eigenvalues of } H_\omega^{(N)} \text{ that are in } (-A) \right\}
\]
\[
= \nu_\lambda (-A).
\]

This concludes the first claim. Since the spectrum is the topological support of the density of states measure, the second claim also follows. \( \square \)

3. The Labyrinth Model

Let \( a_i, b_i > 0 \) \((i = 1, 2)\) be real numbers, and let \( s \) be a positive integer. Let \( \omega_i \in \Omega_{a_i, b_i}^{(s)} \) \((i = 1, 2)\) and \( \lambda_i \) be the corresponding coupling constants.

3.1. The Labyrinth model. We define the Labyrinth model. Write
\[ A^e = \{(m, n) \in \mathbb{Z}^2 \mid m + n \text{ is even}\}, \quad \text{and} \quad A^o = \{(m, n) \in \mathbb{Z}^2 \mid m + n \text{ is odd}\}. \]

Using \( \omega_1, \omega_2 \), we realign the lattices of \( A^e \) and \( A^o \). See Figure 4. We denote this again by \( A^e \) and \( A^o \) (we use this identification freely). We define the operator
\[ \hat{H}_{\omega_1, \omega_2}, \] which acts on \( l^2(A^e \cup A^o) \), by

\[
\begin{align*}
[\hat{H}_{\omega_1, \omega_2} \psi](m, n) &= \omega_1(m + 1)\omega_2(n + 1)\psi(m + 1, n + 1) \\
&\quad + \omega_1(m + 1)\omega_2(n)\psi(m + 1, n - 1) \\
&\quad + \omega_1(m)\omega_2(n + 1)\psi(m - 1, n + 1) \\
&\quad + \omega_1(m)\omega_2(n)\psi(m - 1, n - 1).
\end{align*}
\]

Every lattice is connected diagonally and the strength of the bond is equal to the product of the sides of the rectangle. With appropriate scaling, we can always assume that \( b_i = 1 \) (\( i = 1, 2 \)). We assume this scaling throughout this section. In a similar way, we define the operators \( \hat{H}_{e, \omega_1, \omega_2} \) and \( \hat{H}_{o, \omega_1, \omega_2} \), which act on \( l^2(A^e) \) and \( l^2(A^o) \), respectively. From here, we drop the subscripts \( \omega_1, \omega_2 \) if no confusion can arise. Note that

\[ \hat{H} = \hat{H}^e \oplus \hat{H}^o. \]

It is natural to expect that the spectral properties of \( \hat{H}^e \) and \( \hat{H}^o \) are the same, and in fact, the spectra and the density of states measures coincide for the three operators. For the proof, we need the notion of Delone dynamical systems, and linear repetitivity of Delone dynamical systems. See, for example, \[22\].

**Proposition 3.1.** Let us denote the density of states measures of \( \hat{H}^e, \hat{H}^o \) and \( \hat{H} \) by \( \hat{\nu}^e, \hat{\nu}^o \) and \( \hat{\nu} \), respectively. Then, \( \hat{\nu}^e, \hat{\nu}^o \) and \( \hat{\nu} \) define the same measure. In particular, the spectra of \( \hat{H}^e, \hat{H}^o \) and \( \hat{H} \) all coincide.

**Proof.** By Lemma \[22\], \( A^e \) and \( A^o \) are linearly repetitive. Therefore, by Theorem 6.1 of \[22\], Theorem 3 of \[23\] and Lemma \[22\], we have \( \hat{\nu}^e = \hat{\nu}^o \). Since \( \hat{H} = \hat{H}^e \oplus \hat{H}^o \), we get \( \hat{\nu}^e = \hat{\nu}^o = \hat{\nu} \).

By this proposition, we will restrict our attention to \( \hat{H} \) below.

**3.2. The spectrum of the Labyrinth model.** We start by proving that the spectrum of \( \hat{H}_{\omega_1, \omega_2} \) is given by the product of the spectra of off-diagonal models.
Proposition 3.2. We have
\[ \sigma(\hat{H}_{\omega_1, \omega_2}) = \Sigma_{\lambda_1} \cdot \Sigma_{\lambda_2}. \]
In particular, the spectrum \( \sigma(\hat{H}_{\omega_1, \omega_2}) \) does not depend on particular choice of \( \omega_1 \) and \( \omega_2 \) and only depends on the coupling constants \( \lambda_1, \lambda_2 \).

In the proof below, we simply write \( H_{\omega_1}, H_{\omega_2} \) and \( \hat{H}_{\omega_1, \omega_2} \) as \( H_1, H_2 \) and \( \hat{H} \), respectively.

Proof. Let \( U \) be the unique unitary map from \( l^2(\mathbb{Z}) \otimes l^2(\mathbb{Z}) \) to \( l^2(\mathbb{Z}^2) \) so that for \( \psi_1, \psi_2 \in l^2(\mathbb{Z}) \), the elementary tensor \( \psi_1 \otimes \psi_2 \) is mapped to the element \( \psi \) of \( l^2(\mathbb{Z}^2) \) given by \( \psi(m, n) = \psi_1(m)\psi_2(n) \). We have
\[
\begin{align*}
\hat{H}U(\psi_1 \otimes \psi_2) (m, n) &= \omega_1(m + 1)\omega_2(n + 1)\psi_1(m + 1)\psi_2(n + 1) \\
&\quad + \omega_1(m + 1)\omega_2(n)\psi_1(m + 1)\psi_2(n - 1) \\
&\quad + \omega_1(m)\omega_2(n + 1)\psi_1(m - 1)\psi_2(n + 1) \\
&\quad + \omega_1(m)\omega_2(n)\psi_1(m - 1)\psi_2(n - 1) \\
&= [H_1\psi_1](m) [H_2\psi_2](n) \\
&= [U(H_1 \otimes H_2)(\psi_1 \otimes \psi_2)](m, n),
\end{align*}
\]
for all \((m, n) \in \mathbb{Z}^2\). Therefore,
\[
(U^* \hat{H}U)(\psi_1 \otimes \psi_2) = (H_1 \otimes H_2)(\psi_1 \otimes \psi_2).
\]
Since the linear combinations of elementary tensors are dense in \( l^2(\mathbb{Z}) \otimes l^2(\mathbb{Z}) \), we get \( U^* \hat{H}U = H_1 \otimes H_2 \). Therefore, the result follows from Theorem VIII 33 of [31].

Let us denote the spectrum of \( \hat{H}_{\omega_1, \omega_2} \) by \( \hat{\Sigma}_{\lambda_1, \lambda_2} \).

Proof of Theorem 1.2. By Theorem 2.6 we have
\[
\tau(\Sigma_{\lambda_1}), \tau(\Sigma_{\lambda_2}) > 1 + \sqrt{2}
\]
for sufficiently small coupling constants. Therefore, by Theorem 1.4 of [41], \( \Sigma_{\lambda_1}, \Sigma_{\lambda_2} \) is an interval for sufficiently small \( \lambda_1, \lambda_2 \). Combining this with Proposition 3.2, \( \hat{\Sigma}_{\lambda_1, \lambda_2} \) is an interval for sufficiently small coupling constants.

By Theorem 2.6 we get
\[
\dim_H \Sigma_{\lambda_1} + \dim_H \Sigma_{\lambda_2} < 1
\]
for sufficiently large \( \lambda_1, \lambda_2 \). Notice that, by the symmetry of \( \Sigma_{\lambda_1} \) and \( \Sigma_{\lambda_2} \), we have
\[
\hat{\Sigma}_{\lambda_1, \lambda_2}^+ = \exp (\log \Sigma_{\lambda_1}^+ + \log \Sigma_{\lambda_2}^+),
\]
where \( A^+ \) denotes \( A \cap (0, \infty) \). Therefore, since
\[
\dim_H \log \Sigma_{\lambda_i}^+ = \dim_H \Sigma_{\lambda_i} (i = 1, 2),
\]
by Proposition 1 of chapter 4 in [28], \( \log \Sigma_{\lambda_1}^+ + \log \Sigma_{\lambda_2}^+ \) has zero Lebesgue measure. Hense, for sufficiently large \( \lambda_1, \lambda_2, \hat{\Sigma}_{\lambda_1, \lambda_2} \) is a Cantor set of zero Lebesgue measure. \( \square \)
3.3. Density of states measure of the Labyrinth model. In this section, we will prove Theorem 1.2. If there is no chance of confusion, we simply write the density of states measures of \( \hat{H}, H_1 \) and \( H_2 \) as \( \hat{\nu}, \nu_1 \) and \( \nu_2 \), respectively.

**Proof of (1.5).** The proof is essentially the repetition of the proof of Proposition A.3 of [13]. For the reader’s convenience, we will repeat the argument.

Denote by \( \hat{H}_j^{(N)} (j = 1, 2) \) the restriction of \( H_j \) to the interval \([0, N - 1]\) with Dirichlet boundary conditions. Denote the corresponding eigenvalues and eigenvectors by \( E_{j,k}^{(N)}, \phi_{j,k}^{(N)} \), where \( j = 1, 2 \) and \( 1 \leq k \leq N \). Recall that we have

\[
\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N \mid E_{j,k}^{(N)} \in (-\infty, E) \right\} = \nu_j ((-\infty, E))
\]

for \( E \in \mathbb{R} \).

Similarly, we denote by \( \hat{H}^{(N)} \) the restriction of \( \hat{H} \) to \([0, N - 1]^2\) with Dirichlet boundary conditions. Denote the corresponding eigenvalues and eigenvectors by \( E_k^{(N)}, \phi_k^{(N)} \ (1 \leq k \leq N^2) \). Then, we have

\[
\lim_{N \to \infty} \frac{1}{N^2} \# \left\{ 1 \leq k \leq N^2 \mid E_k^{(N)} \in (-\infty, E) \right\} = \hat{\nu} ((-\infty, E)).
\]

The eigenvectors \( \phi_{j,k}^{(N)} \) of \( \hat{H}_j^{(N)} \) form an orthonormal basis of \( l^2([0, N - 1]) \). Thus, the associated elementary tensors

\[
\phi_{1,k_1}^{(N)} \otimes \phi_{2,k_2}^{(N)} (1 \leq k_1, k_2 \leq N)
\]

form an orthonormal basis of \( l^2([0, N - 1]) \otimes l^2([0, N - 1]) \), which is canonically isomorphic to \( l^2([0, N - 1]^2) \). Moreover, the vector in (3.2) is an eigenvector of \( \hat{H}^{(N)} \), corresponding to the eigenvalue \( E_{1,k_1}^{(N)} \cdot E_{2,k_2}^{(N)} \). By counting dimensions, these eigenvalues exhaust the entire set \( \left\{ E_k^{(N)} \mid 1 \leq k \leq N^2 \right\} \). Therefore, for \( E \in \mathbb{R} \),

\[
\# \left\{ 1 \leq k \leq N^2 \mid E_k^{(N)} \in (-\infty, E) \right\} = \# \left\{ 1 \leq k_1, k_2 \leq N \mid E_{1,k_1}^{(N)} \cdot E_{2,k_2}^{(N)} \in (-\infty, E) \right\}.
\]

Let \( \nu_j^{(N)} (j = 1, 2) \) be the probability measures on \( \mathbb{R} \) with \( \nu_j^{(N)}(E_{j,k}^{(N)}) = 1/N \ (k = 1, 2, \cdots, N) \). Similarly, Let \( \hat{\nu}^{(N)} \) be the probability measure on \( \mathbb{R} \) with \( \hat{\nu}^{(N)}(E_k^{(N)}) = 1/N^2 \ (k = 1, 2, \cdots, N^2) \). Then, by the above argument, we get

\[
\hat{\nu}^{(N)} ((-\infty, E]) = \int_{\mathbb{R}^2} \chi_{(-\infty,E)}(xy) \, d\nu_1^{(N)}(x) \, d\nu_2^{(N)}(y).
\]

By (3.1), \( \nu_i^{(N)} \) converges weakly to \( \nu_i \) (see, for example, chapter 13 of [21]). Therefore, \( \nu_1^{(N)} \times \nu_2^{(N)} \) converges weakly to \( \nu_1 \times \nu_2 \). By Theorem 13.16 of [21], we have

\[
\lim_{N \to \infty} \int_{\mathbb{R}^2} \chi_{(-\infty,E]}(xy) \, d\nu_1^{(N)}(x) \, d\nu_2^{(N)}(y) = \int_{\mathbb{R}^2} \chi_{(-\infty,E]}(xy) \, d\nu_1(x) \, d\nu_2(y).
\]

The result follows from this.

Let us define Borel measures \( \tilde{\nu}_i (i = 1, 2) \) on \( \mathbb{R} \) by

\[
\tilde{\nu}_i(A) = \nu_i(e^A),
\]

where \( A \subset \mathbb{R} \) is a Borel set. Then, the following holds.
Lemma 3.1. The density of states measure of the Labyrinth model \( \hat{\nu} \) is given by
\[
\hat{\nu}(A) = 2 \left\{ (\hat{\nu}_1 \times \hat{\nu}_2)(\log A^+) + (\hat{\nu}_1 \ast \hat{\nu}_2)(\log A^-) \right\},
\]
where \( A \) is a Borel set, and \( A^+ = A \cap (0, \infty) \) and \( A^- = (-A) \cap (0, \infty) \).

Proof. Let \( A \subset (0, \infty) \) be a Borel set. Using Fubini’s Theorem and change of coordinates, we get
\[
\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \chi_A(xy) \, d\nu_1(x) \, d\nu_2(y) = \int_{\mathbb{R}^+} \left( \int_{\mathbb{R}^+} \chi_A(xy) \, d\nu_1(x) \right) \, d\nu_2(y)
= \int_{\mathbb{R}^+} \left( \int_{\mathbb{R}} \chi_A(e^xy) \, d\nu_1(x) \right) \, d\nu_2(y)
= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(e^xy) \, d\nu_1(x) \, d\nu_2(y)
= \int_{\mathbb{R}^2} \chi_{\log A}(x+y) \, d\nu_1(x) \, d\nu_2(y)
= (\hat{\nu}_1 \ast \hat{\nu}_2)(\log A).
\]
Combining this with Proposition 2.3, the result follows. \( \square \)

Therefore, the absolute continuity of \( \hat{\nu} \) is equivalent to the absolute continuity of \( \hat{\nu}_1 \ast \hat{\nu}_2 \).

Theorem 3.2 from [13] implies the following:

Theorem 3.1. Let \( \mathcal{J} \subset \mathbb{R} \) be an interval. Assume that for \( \lambda \in \mathcal{J} \), \( \nu_\lambda \) is the density of states measure of \( H_\Lambda \). Let \( \gamma : \mathbb{R} \to \mathbb{R} \) be a diffeomorphism, and define a Borel measure \( \mu_\lambda \) by
\[
\mu_\lambda(A) = \nu_\lambda(\gamma(A)),
\]
where \( A \subset \mathbb{R} \) is a Borel set. Then, for any compactly supported exact-dimensional measure \( \eta \) on \( \mathbb{R} \) with
\[
\dim_H \eta + \dim_H \mu_\lambda > 1
\]
for all \( \lambda \in \mathcal{J} \), the convolution \( \eta \ast \mu_\lambda \) is absolutely continuous with respect to Lebesgue measure for almost every \( \lambda \in \mathcal{J} \).

Proof of Theorem 1.2. By Theorem 1.1 \( \Sigma_{\lambda_1, \lambda_2} \) is a Cantor set of zero Lebesgue measure for sufficiently large coupling constants. Therefore, since the density of states measure \( \hat{\nu}_{\lambda_1, \lambda_2} \) is supported on \( \Sigma_{\lambda_1, \lambda_2} \), it has to be singular continuous for sufficiently large \( \lambda_1, \lambda_2 \).

By Theorem 2.5 there exists \( \lambda^* > 0 \) such that \( \dim_H \nu_\lambda > \frac{1}{2} \) for all \( \lambda \in [0, \lambda^*) \). Recall that \( 0 \in \Sigma_\lambda \). Recall also that \( \Sigma_\lambda \) is the set of intersections between the stable laminations and the line \( \ell_\lambda \). Therefore, since the stable laminations and \( \ell_\lambda \) both depend smoothly on \( \lambda \), we can write
\[
\Sigma_\lambda \cap (0, \infty) = \bigcup_{n=1}^{\infty} K_n^{(i)}(\lambda_i) \quad (i = 1, 2),
\]
where \( K_n^{(i)}(\lambda_i) \) are Cantor sets which depend naturally on \( \lambda_i \). Let us define Borel measures \( \hat{\nu}_{\lambda_i}^{(n)} \) \( (i = 1, 2, \ n \in \mathbb{N}) \) by
\[
\hat{\nu}_{\lambda_i}^{(n)}(A) = \nu_{\lambda_i} |_{K_n^{(i)}(\lambda_i)}(e^A),
\]
where $A \subset \mathbb{R}$ is a Borel set. Then, by Theorem 3.1, for each $(m, n) \in \mathbb{N} \times \mathbb{N}$, $\tilde{\nu}^{(m)}_{\lambda_1} * \tilde{\nu}^{(n)}_{\lambda_2}$ is absolutely continuous for almost all $(\lambda_1, \lambda_2)$. This implies that $\tilde{\nu}_{\lambda_1} * \tilde{\nu}_{\lambda_2}$ is absolutely continuous for almost all $(\lambda_1, \lambda_2)$. □

Acknowledgements

The author would like to acknowledge the invaluable contributions of Anton Gorodetski and David Damanik. The author would also like to thank May Mai and William Yessen for their many helpful conversations, and to the anonymous referee for many helpful suggestions and remarks.

References

[1] J. Bellissard, B. Iochum, E. Scoppola, D. Testard, Spectral properties of one dimensional quasi-crystals, Commun. Math. Phys. 125 (1989), 527–543.
[2] S. Cantat, Bers and Hénon, Painlevé and Schrödinger, Duke Math. Journal 149 (2009), 411–460.
[3] M. Casdagli, Symbolic dynamics for the renormalization map of a quasiperiodic Schrödinger equation, Commun. Math. Phys. 107 (1986), 295–318.
[4] V.Z. Cerovski, M. Schreiber, U. Grimm, Spectral and diffusive properties of silver-mean quasicrystals in one, two, and three dimensions, Physical Review B 72 (2005), 054203.
[5] D. Damanik, Substitution Hamiltonians with bounded Trace map orbits, Journal of Mathematical Analysis and Applications 249 (2000), 393–411.
[6] D. Damanik, M. Embree, A. Gorodetski, Spectral properties of Schrödinger operators arising in the study of Quasicrystals, Mathematics of aperiodic order, Birkhäuser, 2015.
[7] D. Damanik, M. Embree, A. Gorodetski, S. Tcheremchantsev, The fractal dimension of the spectrum of the Fibonacci Hamiltonian, Commun. Math. Phys. 280 (2008), 499–516.
[8] D. Damanik, A. Gorodetski, Almost sure frequency independence of the dimension of the spectrum of Sturmian Hamiltonians, Communi. Math. Phys. 337 (2015), 1241–1253.
[9] D. Damanik, A. Gorodetski, Hyperbolicity of the trace map for the weakly coupled Fibonacci Hamiltonian, Nonlinearity 22 (2009), 123–143.
[10] D. Damanik, A. Gorodetski, Spectral and quantum dynamical properties of the weakly coupled Fibonacci Hamiltonian, Commun. Math. Phys. 305 (2011), 221–277.
[11] D. Damanik, A. Gorodetski, The density of states measure of the weakly coupled Fibonacci Hamiltonian, Geom. Funct. Anal. 22 (2012), 976–989.
[12] D. Damanik, A. Gorodetski, Qing-Hui Liu, Yan-Hui Qu, Transport exponents of Sturmian Hamiltonians, Journal of Funct. Anal. 269 (2015), 1404–1440.
[13] D. Damanik, A. Gorodetski, B. Solomyak, Absolutely continuous convolutions of singular measures and an application to the square Fibonacci Hamiltonian, Duke Math. Journal 164 (2015), 1603–1640.
[14] D. Damanik, A. Gorodetski, W. Yessen, The Fibonacci Hamiltonian, preprint [arXiv:1403.7823].
[15] D. Damanik, D. Lenz, Linear repetitivity. I. Uniform subadditive ergodic theorems and applications, Discrete Comput. Geom. 26 (2001), 411–428.
[16] D. Damanik, P. Munger, W. Yessen, Orthogonal polynomials on the unit circle with Fibonacci Verblunsky coefficients, I. The essential support of the measure, J. Approx. Theory 173 (2013), 56–88.
[17] S. Even-Dar Mandel, R. Lifshitz, Electronic energy spectra of square and cubic Fibonacci quasicrystals, Philosophical Magazine 88 (2008), 2261–2273.
[18] J. Fillman, Y. Takahashi, W. Yessen, Mixed spectral regimes for square Fibonacci Hamiltonians, to appear in J. Fract. Geom.
[19] R. Ilan, E. Liberty, S. Even-Dar Mandel, R. Lifshitz, Electrons and phonons on the square Fibonacci tiling, Ferroelectrics 305 (2004), 15–19.
[20] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, Cambridge, 1996.
[21] A. Klenke, *Probability Theory: A Comprehensive Course*, Universitext, 2014.

[22] C. Lagarias, B. Pleasants, Repetitive Delone sets and quasicrystals, *Ergod. Th. and Dynam. Sys.* 23 (2003), 831–867.

[23] D. Lenz, P. Stollmann, An ergodic theorem for Delone dynamical systems and existence of the integrated density of states, *J. Anal. Math.* 97 (2005), 1–24.

[24] D. Lenz, P. Stollmann, Delone dynamical systems and associated random operators, *Operator Algebras and Mathematical Physics, Theta Bucharest* (2003), 267–285.

[25] R. Lifshitz, The square Fibonacci tiling, *Journal of Alloys and Compounds* 342 (2002), 186–190.

[26] M. Mei, Spectra of discrete Schrödinger operators with primitive invertible substitution potentials, *J. Math. Phys.* 55 (2014), 082701.

[27] M. Mei, W. Yessen, Tridiagonal substitution Hamiltonians, *Math. Model. Nat. Phenom.* 9 (2014), 204–238.

[28] J. Palis, F. Takens, *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*, Cambridge University Press, Cambridge, 1993.

[29] M. Pollicott, Analyticity of dimensions for hyperbolic surface diffeomorphisms, *Proc. Amer. Math. Soc.* 143 (2015), 3465–3474.

[30] M. Queffélec, *Substitution Dynamical Systems - Spectral Analysis*, Lecture Notes in Mathematics, 2010.

[31] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I*, Cambridge University Press, Cambridge, 1990.

[32] S. Rolof, S. Thiem, M. Schreiber, Electronic wave functions of quasiperiodic systems in momentum space, *The European Physical Journal B* 86 (2013), 372.

[33] C. Sire, Electronic spectrum of a 2D quasi-crystal related to the octagonal quasi-periodic tiling, *Europhys. Lett.* 10 (1989), 483–488.

[34] C. Sire, R. Mosseri, J.-F. Sadoc, Geometric study of a 2D tiling related to the octagonal quasiperiodic tiling, *J. Physique* 55 (1989), 3463–3476.

[35] A. Sütő, The spectrum of a quasiperiodic Schrödinger operator, *Commun. Math. Phys.* 111 (1987), 409–415.

[36] S. Thiem, M. Schreiber, Generalized inverse participation numbers in metallic-mean quasiperiodic systems, *Physics of Condensed Matter* 84 (2011).

[37] S. Thiem, M. Schreiber, Quantum diffusion in separable d-dimensional quasiperiodic tilings, preprint (arXiv: 1212.6337).

[38] S. Thiem, M. Schreiber, Renormalization group approach for the wave packet dynamics in golden-mean and silver-mean Labyrinth tilings, *Physical Review B* 85 (2012), 224205.

[39] S. Thiem, M. Schreiber, Similarity of eigenstates in generalized Labyrinth tilings, *Journal of Physics Conference Series* 226 (2010), 012029.

[40] S. Thiem, M. Schreiber, Wave functions, quantum diffusion and scaling exponents in golden-mean quasiperiodic tilings, *Journal of Physics Condensed Matter* 25 (2013), 075503.

[41] Y. Takahashi, Products of two Cantor sets, preprint (arXiv: 1601.01370).

[42] W. Yessen, Spectral analysis of tridiagonal Fibonacci Hamiltonians, *J. Spectr. Theory* 3 (2013), 101–128.

[43] H.Q. Yuan, U. Grimm, P. Repetowicz, M. Schreiber, Energy spectra, wave functions and quantum diffusion for quasi periodic systems, *Physical Review B* 62 (2000), 15569.

Department of Mathematics, University of California, Irvine, CA 92697, USA
E-mail address: takahasy@math.uci.edu