Gauss maps of harmonic and minimal great circle fibrations

Ioannis Fourtzis 1 · Michael Markellos 1 · Andreas Savas-Halilaj 1

Received: 17 August 2022 / Accepted: 27 January 2023 / Published online: 13 February 2023
© The Author(s) 2023

Abstract
We investigate Gauss maps associated to great circle fibrations of the euclidean unit 3-sphere $S^3$. We show that the associated Gauss map to such a fibration is harmonic, respectively minimal, if and only if the unit vector field generating the great circle foliation is harmonic, respectively minimal. These results can be viewed as analogues of the classical theorem of Ruh and Vilms about the harmonicity of the Gauss map of a minimal submanifold in the euclidean space. Moreover, we prove that a harmonic or minimal unit vector field on $S^3$, whose integral curves are great circles, is a Hopf vector field.

Keywords
Hopf vector fields · Great circle fibration · Gauss maps · Maximum principle

Mathematics Subject Classification
Primary 53C43 · 58E20 · 53C24 · 53C40 · 53C42 · 57K35

1 Introduction

Suppose that $(M, g)$ is an $m$-dimensional manifold equipped with a Riemannian metric $g$. A unit vector field $\xi \in \mathfrak{X}(M)$ can be regarded as a “graphical” map from $M$ to its unit tangent bundle $UM$ equipped with the Sasaki metric $g_S$. There are two natural functionals that we may consider in the space of unit vector fields. The first one is the energy functional

\[ E(\xi) = \frac{m}{2} \text{vol}(M, g) + \frac{1}{2} \int_M |\nabla \xi|^2 d\text{vol}_g. \]

M. Markellos & A. Savas-Halilaj would like to acknowledge support by (HFRI) Grant No:133.

[1] Section of Algebra and Geometry, University of Ioannina, 45110 Ioannina, Greece
Critical points of the energy functional with respect to variations through nearby unit vector fields are called harmonic unit vector fields. The second functional is the volume functional
\[ V(\xi) = \int_M \sqrt{\det(I + (\nabla \xi)^T \circ (\nabla \xi))} \, d\text{vol}_g, \]
whose critical points are called minimal unit vector fields.

One can exploit the complex structure \( \mathcal{J} \) of \( \mathbb{C}^{n+1} = \mathbb{R}^{2n+1} \times \mathbb{R}^{n+1} \) to generate interesting unit vector fields on the euclidean unit sphere \( S^{2n+1} \). More precisely, if \( v \) is the unit normal of \( S^{2n+1} \) then \( \xi = \mathcal{J} v \) gives rise to a unit vector field whose integral curves are great circles. A unit vector field of this type is called Hopf vector field and the corresponding quotient map \( f : S^{2n+1} \to \mathbb{C}P^n \) is called Hopf fibration, where \( \mathbb{C}P^n \) stands for the complex projective space equipped with its standard Fubini-Study metric. It turns out that the Hopf fibration is a harmonic Riemannian submersion. Additionally, the Hopf fibration is a space equipped with its standard Fubini-Study metric. It turns out that the Hopf fibration is a harmonic Riemannian submersion. Additionally, the Hopf fibration is a minimal map, i.e. its graph \( \Gamma_f : S^{2n+1} \to S^{2n+1} \times \mathbb{C}P^n \) is a minimal submanifold; see [27]. Let us point out here that \( \mathbb{C}P^1 \) is isometric with the sphere \( S^2(1/2) \) of radius 1/2. Composing the Hopf fibration with an isometry of \( S^3 \) and the homothety from \( S^2(1/2) \) into \( S^2 \), we obtain again a submersion with totally geodesic fibres. For simplicity, we call all these maps Hopf fibrations.

Hopf vector fields are simultaneously harmonic and minimal unit vector fields. Moreover, they are the unique global minima of the energy functional restricted to unit vector fields on the 3-sphere; see [7]. Surprisingly, for spheres of dimension greater than 3, Hopf vector fields are unstable critical points of the energy functional; see [37]. Analogously, Hopf vector fields are absolute volume minimisers in their homology classes in \( US^3 \). According to a result due to Gluck and Ziller [18], the converse is also true. Again, for spheres of dimensions greater than 3, the Hopf vector fields are not minimisers of the volume functional; see [24, 28]. Let us mention that for spheres \( S^{2n+1}(r) \) of arbitrary radius \( r > 0 \) the situation is different. For \( n > 1 \), the stability of the Hopf fields depends on the radius of the sphere; see [6, 12–14].

The classification of harmonic or minimal unit vector fields on the sphere \( S^{2n+1} \) as well as the classification of harmonic or minimal maps \( f : S^{2n+1} \to \mathbb{C}P^n \) is an open problem, even when \( n = 1 \). For example, the Hopf vector fields are the only known examples of harmonic or minimal unit vector fields on \( S^3 \). Moreover, according to a conjecture of Eells, any harmonic map \( f : S^3 \to S^2 \) must be weakly conformal; see [2, Note 10.4.1, page 422] or [35, page 730]. Therefore, an interesting question is whether a harmonic map \( f : S^3 \to S^2 \) can be written as the composition of a Hopf fibration with a conformal map from \( S^2 \) to \( S^2 \).

We investigate harmonic and minimal unit vector fields on \( S^3 \) with totally geodesic integral curves. According to a beautiful result of Gluck and Warner [19], any great circle fibration of \( S^3 \) generates a graphical surface in \( S^2 \times S^2 \). Generically, at least locally, the converse is also true; see Theorem 3.6. In particular, the fibration is smooth and globally defined on \( S^3 \) if and only if the graphical surface is generated by a smooth strictly length decreasing map. The idea to establish this duality is to consider the map which assigns each great circle of the foliation to the 2-plane of \( \mathbb{R}^4 \) containing the circle. This mapping may be considered as a Gauss map associated to the great circle fibration. Therefore, there is a huge class of great circle fibrations of \( S^3 \). Moreover, the corresponding quotient maps are homotopic to the Hopf fibration and so they are homotopically non-trivial. Let us emphasise here that great sphere fibrations of the sphere are of particular interest because of their relation with the Blaschke conjecture; for more details on the subject see for example [22, 25, 26, 39].

We show that a unit vector field generating a great circle fibration is minimal if and only if the Gauss map is a minimal surface. More precisely, we prove the following result.
Theorem A  A unit vector field with closed totally geodesic integral curves defined in an open connected neighbourhood $V$ of $\mathbb{S}^3$ is minimal if and only if its corresponding Gauss map generates a minimal surface in $\mathbb{S}^2 \times \mathbb{S}^2$.

Following the terminology introduced by Baird in [3], the Gauss map of a submersion $f : M \to N$ between oriented Riemannian manifolds of dimensions $m$ and $n$, respectively, is the map $G : M \to G_{m-n}(M)$ which associates to each point $x \in M$ the tangent plane to the fibre of $f$ passing through $x$. Here, $G_{m-n}(M)$ denotes the Grassmann bundle over $M$, whose fibre at each point $x \in M$ is the Grassmannian of oriented $(m-n)$-planes in $T_x M$. Observe that if $g : N \to N$ is a diffeomorphism, then the submersions $g \circ f$ and $f$ have the same Gauss map. If $n = m - 1$, then the kernel of $f$ is generated by a unit vector field. Hence, the Gauss map $G$ may be regarded as a section of the unit tangent bundle $UM$ of $M$. Our next result is an analogue of the classical theorem of Ruh and Vilms [30] in the case of great circle fibrations of $\mathbb{S}^3$ and their corresponding quotient maps.

Theorem B  Let $\zeta : V \subset \mathbb{S}^3 \to U\mathbb{S}^3$ be a unit vector field whose integral curves are closed great circles defined in an open connected neighbourhood $V$ of $\mathbb{S}^3$ and $f : V \subset \mathbb{S}^3 \to \mathbb{S}^2$ be the corresponding quotient map. Then $f$ is a harmonic map if and only if $\zeta : V \to U\mathbb{S}^3$ is a harmonic unit vector field.

In the sequel we prove the following global result which generalises a previous one by Han and Yim in [21].

Theorem C  A harmonic unit vector field $\zeta$ on $\mathbb{S}^3$, whose integral curves are great circles, is a Hopf vector field and the corresponding quotient map $f : \mathbb{S}^3 \to \mathbb{S}^2$ is a Hopf fibration.

The main ingredient in the proof of Theorem C is the (formal) second fundamental form $\varphi$ of the horizontal distribution, i.e. the perpendicular distribution to the line bundle spanned by the vector field $\zeta$. It turns out that $\varphi$ satisfies a Ricatti-type ODE along the integral curves of $\zeta$, and on the horizontal distribution it satisfies a Codazzi type system of PDEs; see Sect. 3. Using the harmonicity of the vector field, we arrive at the conclusion that the squared mean curvature $(\text{trace } \varphi)^2$ of the horizontal distribution is a subharmonic function. Then, from the maximum principle and the Bernstein-type theorem for strictly length decreasing minimal maps in [32, Theorem A], we obtain that $\varphi$ is an orthogonal complex structure. This leads us to the conclusion that $\zeta$ is a Hopf vector field.

Next we turn our attention to minimal unit vector fields on $\mathbb{S}^3$ and prove the following result.

Theorem D  A minimal unit vector field $\zeta$ on $\mathbb{S}^3$, whose integral curves are great circles, is a Hopf vector field and the corresponding quotient map $f : \mathbb{S}^3 \to \mathbb{S}^2$ is a Hopf fibration.

In the proof of Theorem D, we use the geometric setup of Gluck and Warner [19] to show that if the mean curvature of the “graphical” map $\zeta : (M, \zeta^* g_S) \to (UM, g_S)$ vanishes, then the corresponding graphical surface $G$ in $\mathbb{S}^2 \times \mathbb{S}^2$ is minimal. Again from the Bernstein-type theorem in [32, Theorem A], we obtain that $G$ is totally geodesic, from where we deduce that $\zeta$ is a Hopf vector field.
2 Geometry of the tangent bundle

2.1 The unit tangent bundle

Let us collect here some basic facts about the geometry of the tangent bundle $TM$ of a manifold $(M, g)$. There is a natural Riemannian metric on $TM$ whose construction goes back to the seminal paper of Sasaki [29]. He uses the metric $g$ to construct a metric $g_S$ on $TM$, which nowadays is called the Sasaki metric. The construction of $g_S$ is based on a natural splitting of the tangent bundle $TTM$ of $TM$ into a vertical and horizontal sub-bundle by means of the Levi-Civita connection $\nabla$ of $g$. Let us briefly recall the construction of the Sasaki metric, following closely the material in [4, 5, 9].

Let $(M, g)$ be an $m$-dimensional Riemannian manifold and denote by $\pi_M : TM \rightarrow M$ the canonical projection from the tangent bundle $TM$ on $M$. The kernel of $d\pi_M$ gives rise to the vertical distribution $\mathcal{V}$ of $TM$, which is smooth and $m$-dimensional, i.e.

$$\mathcal{V}_{(x, v)} = \ker d\pi_M |_{(x, v)}.$$  

We will use the Levi-Civita connection $\nabla$ of $g$ to introduce the horizontal distribution on the tangent bundle $TM$. This can be achieved through the connection map $K : TTM \rightarrow TM$ which is defined in the following way: For fixed vector $X \in T_{(x, v)}TM$ consider a smooth curve $\gamma(s) = (\alpha(s), v(s)), s \in (-\varepsilon, \varepsilon)$, in the tangent bundle $TM$ such that

$$\gamma(0) = (\alpha(0), v(0)) = (x, v) \quad \text{and} \quad \gamma'(0) = X$$

and define

$$K(X) = K(\gamma'(0)) = \nabla_{\gamma'(0)}v.$$  

Then we define the horizontal distribution as the kernel of the connection map $K$, i.e.

$$\mathcal{H}_{(x, v)} = \ker K_{(x, v)}.$$  

From (2.1), we can readily see that $\mathcal{H}_{(x, v)}$ is $m$-dimensional and if $X \in \mathcal{V}_{(x, v)} \cap \mathcal{H}_{(x, v)}$, then $X = 0$. Hence, at any point $(x, v) \in TM$, we have the following decomposition

$$T_{(x, v)}TM = \mathcal{V}_{(x, v)} \oplus \mathcal{H}_{(x, v)}.$$  

The Sasaki metric $g_S$ is a Riemannian metric that makes the above sum orthogonal, i.e.

$$g_S(X, Y) = g(d\pi_M(X), d\pi_M(Y)) + g(K(X), K(Y)),  \quad (2.2)$$

for any $X, Y \in TTM$. Observe now that the projection map $\pi_M : (TM, g_S) \rightarrow (M, g)$ becomes a Riemannian submersion. Any vector $X \in T_{(x, v)}TM$ can be written in the form

$$X = X^{\text{ver}} + X^{\text{hor}},$$

where $X^{\text{ver}}$ stands for the vertical and $X^{\text{hor}}$ for the horizontal component of $X$. Moreover, for any $w \in T_xM$, there exists a unique $w^{\text{hor}}_{(x, v)} \in \mathcal{H}_{(x, v)}$ and a unique $w^{\text{ver}}_{(x, v)} \in \mathcal{V}_{(x, v)}$ such that

$$d\pi_M|_{(x, v)}(w^{\text{hor}}_{(x, v)}) = w \quad \text{and} \quad K_{(x, v)}(w^{\text{ver}}_{(x, v)}) = w.$$  

In this case, the vector $w^{\text{ver}}$ is called the vertical lift and $w^{\text{hor}}$ the horizontal lift of $w$. The horizontal lift of a vector field $w \in \mathfrak{X}(M)$ is the vector field $w^{\text{hor}} \in \mathfrak{X}(TM)$ whose value
at a point \((x, v)\) is the horizontal lift of \(w_x\) to \((x, v)\). The vertical lift \(w^{\text{ver}}\) of \(w\) is defined similarly. There is also a natural complex structure \(J_{TM}\) on the manifold \(TM\), i.e. we define

\[
J_{TM}w^{\text{hor}} = w^{\text{ver}} \quad \text{and} \quad J_{TM}w^{\text{ver}} = -w^{\text{hor}},
\]

for any \(w \in \mathfrak{X}(M)\).

Consider now the unit tangent bundle \(UM\) of \(M\), i.e. \(UM = \{(x, v) \in TM : g(v, v) = 1\}\). The unit tangent bundle \(UM\) is an embedded hypersurface of \(TM\), and we can equip it with the induced from \(g_S\) Riemannian metric. One can readily check that the vector field \(\eta(x, v) = v^{\text{ver}}(x, v)\) is a unit normal of \(UM\). Note that if \(w \in \mathfrak{X}(M)\), then

\[
g_S\left(w_{(x,v)}^{\text{hor}}, \eta(x,v)\right) = g_S\left(w_{(x,v)}^{\text{ver}}, v^{\text{ver}}(x,v)\right) = 0.
\]

Consequently, the horizontal lift \(w^{\text{hor}}\) of a vector field \(w \in \mathfrak{X}(M)\) is always tangent to \(UM\). On the other hand, the vertical lift \(w^{\text{ver}}\) is not necessarily tangent to \(UM\). Following the terminology introduced by Boeckx and Vanhecke [5], we call tangential lift \(w^{\text{tan}}\) of \(w\) the tangential to \(UM\) component of \(w^{\text{ver}}\), i.e.

\[
w_{(x,v)}^{\text{tan}} = w_{(x,v)}^{\text{ver}} - g_S\left(w_{(x,v)}^{\text{ver}}, v^{\text{ver}}(x,v)\right) \eta(x,v) = w_{(x,v)}^{\text{ver}} - g(w, v)\eta(x,v).
\]

At the point \((x, v) \in UM\), the tangent space of \(UM\) can be written as

\[
T_{(x,v)}UM = \left\{ w_1^{\text{hor}} + w_2^{\text{tan}} : w_1, w_2 \in T_xM \right\}
\]

\[
= \left\{ w_1^{\text{hor}} + w_2^{\text{ver}} : w_1, w_2 \in T_xM, \ g(w_2, v) = 0 \right\}.
\]

On the unit tangent bundle \(UM\), there is an analogue of the Hopf vector field defined from the complex structure of \(TM\). More precisely, the vector field \(\xi_U\) given by

\[
(\xi_U)_{(x,v)} = -J(\eta_{(x,v)}) = v^{\text{hor}}_{(x,v)},
\]

where \((x, v) \in UM\), is unit, tangent along the unit tangent bundle \(UM\) and is called the geodesic flow vector field.

The formula in the next lemma will be used later and its proof can be found in [4, page 176] or in [5, page 82].

**Lemma 2.1** For any \(X, Y \in \mathfrak{X}(M)\), the following formula holds

\[
\nabla_{X^{\text{hor}}}^{g_S} Y^{\text{hor}} |_{(x,v)} = (\nabla_X Y)^{\text{hor}} |_{(x,v)} - \frac{1}{2} \left(R_M(X_x, Y_x)v\right)^{\text{tan}}, \quad (x, v) \in TM,
\]

where \(R_M\) stands for the Riemann curvature operator of the metric \(g\).

### 2.2 Critical points of the energy and volume functional

A unit vector field \(\zeta\) on \((M, g)\) can be regarded as the embedding \(\zeta : M \to UM\) given by \(\zeta(x) = (x, \zeta_x)\). Let us collect in the proposition below the Euler-Lagrange equations for the critical points of the energy and volume functional as well as some useful identities. For the proofs, we refer to [15, 36–38].

**Proposition 2.2** Let \(\zeta : M \to UM\) be a unit vector field, which we view as a mapping of the form \(\zeta(x) = (x, \zeta_x)\), for any \(x \in M\).
(a) The differential of $\zeta$ is given by
\[ d\zeta(X) = X^{\text{hor}} + (\nabla_X\zeta)^{\text{tan}}, \]
for all vector fields $X \in \mathfrak{X}(M)$. Moreover, we have that
\[ (\nabla_X\zeta)^{\text{tan}} = (\nabla_X\zeta)^{\text{ver}}, \]
for all vector fields $X \in \mathfrak{X}(M)$.

(b) The induced via $\zeta$ Riemannian metric $\zeta^*g_S$ on $M$ satisfies
\[ (\zeta^*g_S)(X, Y) = g_S(d\zeta(X), d\zeta(Y)) = g(X, Y) + g(\nabla_X\zeta, \nabla_Y\zeta), \]
for all $X, Y \in \mathfrak{X}(M)$.

(c) It holds that $g = \zeta^*g_S$ if and only if the vector field $\zeta$ is parallel.

(d) The vector field $\zeta$ is a critical point of the energy functional with respect to variations through nearby unit vector fields if and only if
\[ \Delta\zeta + |\nabla\zeta|^2\zeta = 0, \]
where $\Delta$ is the rough Laplacian of $\zeta$, i.e. $\Delta = \sum_{i=1}^{m} (\nabla_{\alpha_i}\nabla_{\alpha_i}\zeta - \nabla_{\nabla_{\alpha_i}\alpha_i}\zeta)$, where $\{\alpha_1, \ldots, \alpha_m\}$ is a local orthonormal frame of $M$ with respect to $g$.

(e) Consider the $(1, 1)$-tensor $\phi$ given by the formula $\phi(X) = \nabla_X\zeta$ for any $X \in \mathfrak{X}(M)$. Then:

\[ (e_1) \] At any $x \in M$, there exist orthonormal frames $\{v_0, v_1, \ldots, v_{m-1}\}$ and $\{\beta_1, \ldots, \beta_{m-1}\}$ with respect to $g$ such that
\[ \phi(v_0) = 0, \quad \phi(v_1) = \lambda_1\beta_1, \ldots, \phi(v_{m-1}) = \lambda_{m-1}\beta_{m-1}, \]
where $\lambda_1 \geq \cdots \geq \lambda_{m-1} \geq 0$. The numbers $\lambda_i, i \in \{1, \ldots, m-1\}$, are the singular values of $\phi$ at the point $x \in M$.

\[ (e_2) \] With the above ordering, the singular values give rise to continuous functions on $M$. Moreover, they are smooth on open subsets where the corresponding multiplicities are constant, and the corresponding eigenspaces are smooth distributions.

(f) The vectors
\[ e_0 = v_0, \quad e_1 = \frac{v_1}{\sqrt{1 + \lambda_1^2}}, \ldots, \quad e_{m-1} = \frac{v_{m-1}}{\sqrt{1 + \lambda_{m-1}^2}}, \]
form an orthonormal basis at the point $x \in M$ with respect to the induced Riemannian metric $\zeta^*g_S$. Moreover, the vectors
\[ \xi_1 = \frac{-\lambda_1 v_1^{\text{hor}} + \beta_1^{\text{ver}}}{\sqrt{1 + \lambda_1^2}}, \ldots, \quad \xi_{m-1} = \frac{-\lambda_{m-1} v_{m-1}^{\text{hor}} + \beta_{m-1}^{\text{ver}}}{\sqrt{1 + \lambda_{m-1}^2}}, \]
form an orthonormal basis of the normal bundle of the embedding $\zeta$ at $\zeta(x)$.

(g) The mean curvature vector $H_{\zeta}$ of the embedding $\zeta$ is given by the formula
\[ H_{\zeta} = \sum_{\alpha=1}^{m-1} \frac{1}{\sqrt{1 + \lambda_\alpha^2}} \left\{ \langle \nabla^2_{v_0,v_0} \zeta, \beta_\alpha \rangle + \sum_{i=1}^{m-1} \frac{\langle \nabla_{v_i,v_i} \zeta, \beta_\alpha \rangle - \lambda_\alpha\lambda_i R_M(v_\alpha, v_i, \zeta, \beta_i)}{1 + \lambda_i^2} \right\} \xi_\alpha. \]
where \( R_M \) stands for the Riemannian curvature operator of \( g \) and
\[
\nabla^2_{v,w} \zeta = \nabla_v \nabla_w \zeta - \nabla_{\nabla_v w} \zeta, \quad v, w \in \mathfrak{X}(M),
\]
is the Hessian of the vector field \( \zeta \); compare with Corollary 1(b) in [27].

(h) The vector field \( \zeta \) is a critical point of the volume functional if and only if the isometric embedding \( \xi : (M, \xi^* g_S) \to (UM, g_S) \) is minimal.

3 Fibrations with totally geodesic fibres

Let \( \zeta \) be a unit vector field with closed totally geodesic integral curves, defined in an open connected neighbourhood \( V \) of \( S^3 \). We denote by \( V = \text{span} \{ \zeta \} \) the vertical line bundle and by \( \mathcal{H} = V^\perp \) the horizontal plane bundle generated by \( \zeta \).

3.1 Great circle foliations and the Ricatti equation

Consider the tensor \( \varphi : \mathcal{H} \to \mathcal{H} \), given by
\[
\varphi(v) = -\nabla_v \zeta,
\]
for all \( v \in \mathcal{H} \), where \( \nabla \) stands for the standard Levi-Civita connection of \( S^3 \). The tensor \( \varphi \) is the (formal) second fundamental form of \( \mathcal{H} \). It is well known that \( \varphi \) satisfies the equations
\[
\nabla^2_{\zeta} \varphi = \varphi^2 + I \quad \text{and} \quad (\nabla^2_{v} \varphi)w - (\nabla^2_{w} \varphi)v = 0,
\]
for any pair of vector fields \( v, w \) on \( \mathcal{H} \); see for example [2, page 313]. Denote by \( J \) the complex structure of \( \mathcal{H} \) and by \( \{ \alpha_2, \alpha_3 \} = J \alpha_2 \) a local orthonormal frame on \( V \). Moreover, denote by \( \varphi_{ij} = \langle \varphi(\alpha_i), \alpha_j \rangle \) the components of \( \varphi \) with respect to the orthonormal frame \( \{ \alpha_2, \alpha_3 \} \). Then,
\[
\text{trace } \varphi = \varphi_{22} + \varphi_{33} = -\text{div}(\zeta).
\]
From the Ricatti equation in (3.1), we obtain the following:

**Lemma 3.1** Let \( \gamma \) be an integral curve of \( \zeta \) and \( \{ \alpha_2, \alpha_3 \} \) be a parallel orthonormal frame along \( \gamma^* \mathcal{H} \). Then:

(a) The components \( \varphi_{ij} \), \( i, j \in \{ 2, 3 \} \), of \( \varphi \) with respect to this frame, satisfy the ODEs
\[
\begin{align*}
\zeta(\varphi_{22}) &= 1 + \varphi_{22}^2 + \varphi_{23} \varphi_{32}, \\
\zeta(\varphi_{33}) &= 1 + \varphi_{33}^2 + \varphi_{23} \varphi_{32}, \\
\zeta(\varphi_{23}) &= \varphi_{23}(\varphi_{22} + \varphi_{33}), \\
\zeta(\varphi_{32}) &= \varphi_{32}(\varphi_{22} + \varphi_{33}).
\end{align*}
\]
(b) The functions \( \text{trace } \varphi \), \( \text{trace}(\varphi \circ J) \) and \( \text{det } \varphi \) satisfy the ODEs
\[
\begin{align*}
\zeta(\text{trace}(\varphi \circ J)) &= (\text{trace } \varphi)(\text{trace}(\varphi \circ J)), \\
\zeta(1 + \text{det } \varphi) &= (\text{trace } \varphi)(1 + \text{det } \varphi), \\
\zeta(\text{trace } \varphi) &= (\text{trace } \varphi)^2 - 2(1 + \text{det } \varphi) + 4,
\end{align*}
\]
and
\[
\zeta \{ (\text{trace } \varphi)^2 - 4 \text{ det } \varphi \} = 2(\text{trace } \varphi) \{ (\text{trace } \varphi)^2 - 4 \text{ det } \varphi \}.
\]
3.2 Great circle foliations and length decreasing maps

Due to an impressive result of Gluck and Warner [19], there is a relation between great circle foliations of the 3-sphere and length decreasing maps between two-dimensional euclidean spheres. This construction is obtained as follows: Associate at each point \( x \in S^3 \) the 2-dimensional subspace of \( \mathbb{R}^4 \) spanned by the great circle of the fibration passing through \( x \). In this way, we obtain a map with values in the Grassmann space \( G_2(\mathbb{R}^4) \simeq S^2 \times S^2 \) of oriented 2-planes in \( \mathbb{R}^4 \). It turns out that the image of this particular map is a two-dimensional surface in \( S^2 \times S^2 \) which is the graph of a strictly length decreasing map. The converse is also true, i.e. any strictly length decreasing map between 2-dimensional euclidean unit spheres, gives rise to a great circle foliation of the 3-sphere. To make the paper self-contained, let us describe here this duality following our exposition.

Denote by \( \Lambda^2(\mathbb{R}^4) \) the dual space of all alternative multilinear forms of degree 2. Elements of \( \Lambda^2(\mathbb{R}^4) \) are called 2-vectors. Given vectors \( v_1 \) and \( v_2 \) of \( \mathbb{R}^4 \), the exterior product \( v_1 \wedge v_2 \) is the linear map which on an alternating form \( \Omega \) of degree 2 takes the value

\[
(v_1 \wedge v_2)(\Omega) = \Omega(v_1, v_2).
\]

The exterior product is linear in each variable separately. Interchanging two elements, the sign of the product changes and if two variables are the same the exterior product vanishes. A 2-vector \( \omega \) is called simple or decomposable if it can be written as a single wedge product of vectors, that is \( \omega = v_1 \wedge v_2 \). Note that there are 2-vectors that are not simple. One can verify that the exterior product \( v_1 \wedge v_2 \) is zero if and only if the vectors are linearly dependent. Moreover, if \( \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \) consists of a basis for \( \mathbb{R}^4 \), then the collection \( \{\varepsilon_i \wedge \varepsilon_j : 1 \leq i < j \leq 4\} \) consists of a basis of \( \Lambda^2(\mathbb{R}^4) \). Therefore, the dimension of the vector space of 2-vectors is 6.

We can equip \( \Lambda^2(\mathbb{R}^4) \) with a natural inner product \((\cdot, \cdot)\). Indeed, define

\[
(v_1 \wedge v_2, w_1 \wedge w_2) = (v_1, w_1)(v_2, w_2) - (v_1, w_2)(v_2, w_1),
\]
on simple 2-vectors and then extend linearly. Note that, if \( \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \) is an orthonormal basis of \( \mathbb{R}^4 \), then the 2-vectors \( \{\varepsilon_i \wedge \varepsilon_j : 1 \leq i < j \leq 4\} \) consist of an orthonormal basis for the exterior power \( \Lambda^2(\mathbb{R}^4) \).

Each simple 2-vector represents a unique 2-dimensional subspace of \( \mathbb{R}^4 \). Moreover, if \( \omega_1 \) and \( \omega_2 \) are simple 2-vectors representing the same subspace, then there exists a nonzero real number \( \lambda \) such that \( \omega_1 = \lambda \omega_2 \). Therefore, there is an obvious equivalence relation in the space of simple 2-vectors such that the space of equivalence classes is in a one-to-one correspondence with the space of 2-dimensional subspaces of \( \mathbb{R}^4 \). Additionally, we can consider another relation on the set of nonzero simple 2-vectors: \( \omega_1 \) and \( \omega_2 \) are called equivalent if and only if \( \omega_1 = \lambda \omega_2 \) for some positive number \( \lambda \). Denote by \([\omega]\) the class containing all simple 2-vectors that are equivalent to \( \omega \). The equivalence classes now obtained are called oriented 2-dimensional subspaces of \( \mathbb{R}^4 \), and the space \( G_2(\mathbb{R}^4) \) of all equivalence classes is called Grassmann space of oriented 2-planes of \( \mathbb{R}^4 \). Consequently, a plane \( \Pi \) in \( \mathbb{R}^4 \) can be associated with the equivalence class \([\omega]\) of the 2-vector \( \omega = v_1 \wedge v_2 \), where \( \{v_1, v_2\} \) is an orthonormal basis of \( \Pi \).

There exists a natural linear endomorphism \(*\) of \( \Lambda^2(\mathbb{R}^4) \) which maps a 2-plane \( \Pi \) in \( \mathbb{R}^4 \) into its orthogonal complement \( \Pi^\perp \). Specifically, if \( \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \) is the standard orthonormal basis of \( \mathbb{R}^4 \) and

\[
\omega = \alpha_{12}\varepsilon_1 \wedge \varepsilon_2 + \alpha_{13}\varepsilon_1 \wedge \varepsilon_3 + \alpha_{14}\varepsilon_1 \wedge \varepsilon_4 + \alpha_{23}\varepsilon_2 \wedge \varepsilon_3 + \alpha_{24}\varepsilon_2 \wedge \varepsilon_4 + \alpha_{34}\varepsilon_3 \wedge \varepsilon_4,
\]

we define
\[ \ast \omega = \alpha_{34} \varepsilon_1 \wedge \varepsilon_2 - \alpha_{24} \varepsilon_1 \wedge \varepsilon_3 + \alpha_{23} \varepsilon_1 \wedge \varepsilon_4 + \alpha_{14} \varepsilon_2 \wedge \varepsilon_3 - \alpha_{13} \varepsilon_2 \wedge \varepsilon_4 + \alpha_{12} \varepsilon_3 \wedge \varepsilon_4. \]

The operator \( \ast \) is called the \textit{Hodge star operator}. Let us mention here that \( \ast \) is an isometry and it satisfies

\[ \ast^2 = \ast \circ \ast = I. \]

Using elementary arguments, one can show that a nonzero 2-vector \( \omega \) is simple if and only if \( \omega \wedge \ast \omega = 0 \) or, equivalently, if and only if \( (\omega, \ast \omega) = 0 \). Hence, we may represent the space of oriented two planes in \( \mathbb{R}^4 \) in the form

\[ \mathbb{G}_2(\mathbb{R}^4) = \{ [\omega] : \omega \in \Lambda^2(\mathbb{R}^4), \| \omega \|_{\Lambda^2(\mathbb{R}^4)} = 1 \text{ and } (\omega, \ast \omega) = 0 \}. \]

The Hodge star operator has eigenvalues +1 and −1, both of multiplicity 3. In particular, the corresponding eigenspaces of the \( \ast \) are 3-dimensional and they are given by

\[ E_- = \{ \omega \in \Lambda^2(\mathbb{R}^4) : \ast \omega = -\omega \} \quad \text{and} \quad E_+ = \{ \omega \in \Lambda^2(\mathbb{R}^4) : \ast \omega = \omega \}. \]

The eigenspaces \( E_- \) and \( E_+ \) are mutually perpendicular and

\[ \Lambda^2(\mathbb{R}^4) = E_- \oplus E_+. \]

As a matter of fact, any element \( \omega \in \Lambda^2(\mathbb{R}^4) \) can be uniquely written in the form

\[ \omega = \frac{\omega - \ast \omega}{2} \oplus \frac{\omega + \ast \omega}{2}, \quad \text{(3.5)} \]

where the first term in the right-hand side of (3.5) belongs to \( E_- \) and the second to \( E_+ \). Moreover, if \( \{ \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \} \) is the standard basis of \( \mathbb{R}^4 \), then the collection

\[ \left\{ \frac{\varepsilon_1 \wedge \varepsilon_2 - \varepsilon_3 \wedge \varepsilon_4}{2}, \frac{\varepsilon_1 \wedge \varepsilon_3 + \varepsilon_2 \wedge \varepsilon_4}{2}, \frac{\varepsilon_1 \wedge \varepsilon_4 - \varepsilon_2 \wedge \varepsilon_3}{2} \right\}, \]

forms an orthogonal basis of \( E_- \) and

\[ \left\{ \frac{\varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4}{2}, \frac{\varepsilon_1 \wedge \varepsilon_3 - \varepsilon_2 \wedge \varepsilon_4}{2}, \frac{\varepsilon_1 \wedge \varepsilon_4 + \varepsilon_2 \wedge \varepsilon_3}{2} \right\}, \]

forms an orthogonal basis of \( E_+ \). Consider now the euclidean spheres

\[ S^2_- = \{ \omega \in E_- : \| \omega \|_{\Lambda^2(\mathbb{R}^4)} = 1/\sqrt{2} \} \quad \text{and} \quad S^2_+ = \{ \omega \in E_+ : \| \omega \|_{\Lambda^2(\mathbb{R}^4)} = 1/\sqrt{2} \}. \]

\textbf{Lemma 3.2} The Grassmann space \( \mathbb{G}_2(\mathbb{R}^4) \) can be identified with the direct product \( S^2_- \times S^2_+ \).

\textbf{Proof} Let \( \Pi = [\omega] \in \mathbb{G}_2(\mathbb{R}^4) \) and assume that the representative \( \omega \) is chosen to have the form \( \omega = v_1 \wedge v_2 \), where \( \{v_1, v_2\} \) is an orthonormal basis of the plane \( \Pi \). Observe that if \( \{w_1, w_2\} \) is another orthonormal frame of \( \Pi \) with the same orientation as \( \{v_1, v_2\} \), then \( v_1 \wedge v_2 = w_1 \wedge w_2 \). Now, the map

\[ [\omega] \mapsto \frac{\omega - \ast \omega}{2} \oplus \frac{\omega + \ast \omega}{2} \quad \text{(3.6)} \]

is well defined and gives rise to a bijection between \( \mathbb{G}_2(\mathbb{R}^4) \) and \( S^2_- \times S^2_+ \). \( \square \)
Let us denote by \( \pi \) the quotient map of the foliation, i.e. the map which identifies points of the same integral curve of \( \zeta \). Since, at the moment, our considerations have local nature, without loss of generality, we may assume that \( \pi(V) \subset S^3 \) is simply connected. Define the maps \( h_{\pm} : \pi(V) \subset S^2 \to S^2 \) given by
\[
h_{\pm}(\pi(x)) = \frac{x \wedge \zeta_x \pm *(x \wedge \zeta_x)}{2}, \tag{3.7}
\]
where \( x \in V \). We now give an alternative proof of a beautiful result first proved by Gluck and Warner [19].

**Lemma 3.3** Let \( \zeta \) be a unit vector field with closed totally geodesic integral curves defined in an open connected neighbourhood \( V \) of \( S^3 \) such that \( \pi(V) \) is simply connected. Then:

(a) The maps \( h_{\pm} \) are well-defined and smooth.
(b) At least one of the maps \( h_- \) or \( h_+ \) is a diffeomorphism. Moreover, either \( h_+ \circ h_-^{-1} \) or \( h_- \circ h_+^{-1} \) is strictly length decreasing, i.e. its differential has norm less than one. In particular, \( G = (h_-, h_+) \) gives rise to a strictly length decreasing graphical submanifold.
(c) If \( h_- \) is a diffeomorphism, then the singular values \( \mu_1 \) and \( \mu_2 \) of \( h_+ \circ h_-^{-1} \) are related with the second fundamental form \( \varphi \) by
\[
\mu_1 \circ \pi = \frac{\sqrt{(1 - \det \varphi)^2 + (\text{trace } \varphi)^2} - \sqrt{|\varphi|^2 - 2 \det \varphi}}{1 + \det \varphi + \text{trace } (\varphi \circ J)},
\]
and
\[
\mu_2 \circ \pi = \frac{\sqrt{(1 - \det \varphi)^2 + (\text{trace } \varphi)^2} + \sqrt{|\varphi|^2 - 2 \det \varphi}}{1 + \det \varphi + \text{trace } (\varphi \circ J)}.
\]

**Proof** (a) Consider points \( x, y \in V \) such that \( \pi(x) = \pi(y) \). Then, \( x \) and \( y \) belong in the same circle of the foliation from where we deduce that \( x \wedge \zeta_x = y \wedge \zeta_y \). Therefore, \( h_{\pm}(\pi(x)) = h_{\pm}(\pi(y)) \) and \( h_{\pm} \) is well defined. Smoothness of the maps \( h_{\pm} \) is clear.

(b) Let us compute the differentials of \( h_{\pm} \circ \pi \). Fix a point \( x \in V \) and consider a local oriented orthonormal basis \( \{\alpha_1 = \zeta, \alpha_2, \alpha_3\} \) of \( T_x V \). Then,
\[
*(x \wedge \alpha_1) = \alpha_2 \wedge \alpha_3, \quad *(x \wedge \alpha_2) = \alpha_1 \wedge \alpha_3, \quad *(x \wedge \alpha_3) = -x \wedge \alpha_2,
\]
\[
*(x \wedge \alpha_2) = -\alpha_1 \wedge \alpha_3, \quad *(x \wedge \alpha_3) = x \wedge \alpha_1, \quad *(\alpha_2 \wedge \alpha_3) = x \wedge \alpha_1.
\]

Differentiating (3.7) with respect to \( \alpha_2 \) and \( \alpha_3 \), we get that
\[
d(h_- \circ \pi)(\alpha_2) = (1 - \varphi_{23})\frac{x \wedge \alpha_3 - \alpha_1 \wedge \alpha_2}{2} - \varphi_{23}\frac{x \wedge \alpha_2 + \alpha_1 \wedge \alpha_3}{2}, \tag{3.8}
\]
\[
d(h_- \circ \pi)(\alpha_3) = -\varphi_{33}\frac{x \wedge \alpha_3 - \alpha_1 \wedge \alpha_2}{2} - (1 + \varphi_{32})\frac{x \wedge \alpha_2 + \alpha_1 \wedge \alpha_3}{2}, \tag{3.9}
\]
\[
d(h_+ \circ \pi)(\alpha_2) = -(1 + \varphi_{23})\frac{x \wedge \alpha_3 + \alpha_1 \wedge \alpha_2}{2} - \varphi_{23}\frac{x \wedge \alpha_2 - \alpha_1 \wedge \alpha_3}{2}, \tag{3.10}
\]
\[
d(h_+ \circ \pi)(\alpha_3) = -\varphi_{33}\frac{x \wedge \alpha_3 + \alpha_1 \wedge \alpha_2}{2} + (1 - \varphi_{32})\frac{x \wedge \alpha_2 - \alpha_1 \wedge \alpha_3}{2}. \tag{3.11}
\]

Observe that the vectors
\[
\left\{ \frac{x \wedge \alpha_3 - \alpha_1 \wedge \alpha_2}{\sqrt{2}}, \frac{x \wedge \alpha_2 + \alpha_1 \wedge \alpha_3}{\sqrt{2}} \right\}
\]
form an orthonormal basis of the tangent space at $h_-(\pi(x))$ of $\mathbb{S}_-^2 \subset E_-$ and
\[
\left\{ \frac{x \wedge \alpha_3 + \alpha_1 \wedge \alpha_2}{\sqrt{2}}, \frac{x \wedge \alpha_2 - \alpha_1 \wedge \alpha_3}{\sqrt{2}} \right\}
\]
form an orthonormal basis of the tangent space at $h_+(\pi(x))$ of $\mathbb{S}_+^2 \subset E_+$. We claim now that either
\[
D_- = \det \begin{bmatrix} 1 - \varphi_{23} & -\varphi_{33} \\ -\varphi_{22} & 1 - \varphi_{32} \end{bmatrix} = -1 - \det \varphi + \varphi_{23} - \varphi_{32} = -1 - \det \varphi - \text{trace}(\varphi \circ J) \neq 0,
\]
everywhere on $V$, or
\[
D_+ = \det \begin{bmatrix} -1 - \varphi_{23} & -\varphi_{33} \\ -\varphi_{22} & 1 - \varphi_{32} \end{bmatrix} = -1 - \det \varphi + \varphi_{32} - \varphi_{23} = -1 - \det \varphi + \text{trace}(\varphi \circ J) \neq 0,
\]
everywhere in $V$. To this end, we will show at first that $1 + \det \varphi$ is strictly positive on $V$. Indeed! Suppose to the contrary that there exists a point $x_0$ where $(1 + \det \varphi)(x_0) \leq 0$. From the second equation of (3.3), it follows that $1 + \det \varphi \leq 0$ along the integral curve $\gamma$ of $\zeta$ passing through $x_0$. Hence, the third equation of (3.3) becomes
\[
\zeta(\text{trace } \varphi) \geq (\text{trace } \varphi)^2 + 4,
\]
along $\gamma$. Since by assumption $\gamma$ is closed, trace $\varphi$ attains its maximum at a point $x_{\text{max}}$ on $\gamma$. Then
\[
0 \geq (\text{trace } \varphi)^2(x_{\text{max}}) + 4,
\]
which is a contradiction. Hence, $1 + \det \varphi$ is strictly positive on $V$. Now, from the first equation of (3.3) and the first equation of (3.2), it follows that $\varphi_{23} - \varphi_{32}$ is nowhere zero. Consequently, either $\varphi_{23} - \varphi_{32}$ is strictly negative and $D_-$ is strictly negative in $V$, or $\varphi_{23} - \varphi_{32}$ is strictly positive and $D_+$ is strictly negative in $V$. Since $\pi(V)$ is simply connected, we deduce that at least one of the maps $h_\pm$ is a diffeomorphism and $G$ is graphical.

Without loss of generality, assume that $h_-$ is a diffeomorphism. Then, $\varphi_{23} - \varphi_{32} < 0$ everywhere in $V$. In this case, we will show that $h_+ \circ h_-^{-1}$ is strictly length decreasing, i.e.
\[
|d(h_+ \circ h_-^{-1})(d\pi(a))| < |d\pi(a)|,
\]
or, equivalently,
\[
|d(h_+ \circ \pi)(\alpha)|^2 < |d(h_- \circ \pi)(\alpha)|^2,
\]
for any vector $\alpha \in \mathcal{H}$. Indeed! If $\alpha = \kappa_1 \alpha_2 + \kappa_2 \alpha_3$, then from (3.8), (3.9), (3.10) and (3.11), we get that (3.12) holds if and only if
\[
\varphi_{23} \kappa_1^2 + (\varphi_{33} - \varphi_{22})\kappa_1 \kappa_2 - \varphi_{32} \kappa_2^2 < 0,
\]
for any $\kappa_1, \kappa_2 \in \mathbb{R}$. On the other hand, (3.13) holds for any $\kappa_1, \kappa_2$, if and only if the matrix
\[
A = \begin{bmatrix} \frac{1}{2} \varphi_{23} & \frac{1}{2} (\varphi_{33} - \varphi_{22}) \\ \frac{1}{2} (\varphi_{33} - \varphi_{22}) & -\varphi_{32} \end{bmatrix}
\]
has negative eigenvalues or, equivalently, if and only if
\[
\text{trace } A = \varphi_{23} - \varphi_{32} < 0 \quad \text{and} \quad 4 \det A = -(\text{trace } \varphi)^2 + 4 \det \varphi > 0.
\]
The validity of the first condition is clear. Suppose now to the contrary, that there is a point \( x_0 \in V \), where

\[
(\text{trace } \varphi)^2(x_0) - 4 \det \varphi(x_0) \geq 0.
\]

From (3.4), it follows that the same inequality holds along the integral curve \( \gamma \) of \( a_1 \) passing through \( x_0 \). From the third identity of (3.3), we obtain that along \( \gamma \) it holds

\[
\alpha_1(\text{trace } \varphi) = (\text{trace } \varphi)^2 - 2 \det \varphi + 2 = (\text{trace } \varphi)^2 - 4 \det \varphi + 2(\det \varphi + 1)
\geq 2(\det \varphi + 1)
> 0,
\]

which leads to a contradiction. Therefore, \( h_+ \circ h_-^1 \) is a strictly length decreasing map.

(c) To compute the singular values of the map \( h_+ \circ h_-^1 \) we proceed as follows. Fix a point \( x_0 \) in \( V \) and suppose that \( \{a_1 = \xi, a_2, a_3\} \in T_x V \) and \( \{\beta_2, \beta_3\} \in T_{\pi(x)} S^2 \) are orthonormal bases of the singular decomposition of \( \pi \). Then, from (3.8), (3.9), (3.10), and (3.11), we get the expressions for the singular values of \( h_+ \circ h_-^1 \). This completes the proof. □

Let us discuss now the converse. Suppose that \( G : V \subset S^2_- \to S^2_- \times S^2_+ \cong \mathbb{G}_2(\mathbb{R}^4) \) is the graph of a smooth map \( g : V \subset S^2_- \to S^2_+ \), where \( V \) is an open simply connected domain of \( S^2 \). Then, for any \( x \in V \), the element

\[
G(x) = x \oplus g(x) \tag{3.14}
\]

describes a plane in \( \mathbb{R}^4 \) and \( G(x) \cap S^3 \) gives rise to a great circle of \( S^3 \). To explicitly describe the plane generated by \( G(x) \), we will use the quaternionic structure of \( \mathbb{R}^4 \). As a vector space, the *quaternions* are

\[
\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in \mathbb{R}\}.
\]

They become an associative algebra with 1 as the multiplicative unit via

\[
i^2 = j^2 = k^2 = -1, \quad i \cdot j = -j \cdot i = k, \quad j \cdot k = -k \cdot j = i, \quad k \cdot i = -i \cdot k = j.
\]

The standard euclidean inner product and the norm on \( \mathbb{R}^4 \cong \mathbb{H} \) can be written in the form

\[
\langle x, y \rangle = \text{Re}(x \cdot \bar{y}) = \text{Re}(\bar{x} \cdot y) \quad \text{and} \quad |x|^2 = x \cdot \bar{x} = x \cdot x, \quad \text{for any } x, y \in \mathbb{H}.
\]

The standard outer product in \( \mathbb{R}^3 \) can be regarded as the map \( \times : \text{Im} \mathbb{H} \times \text{Im} \mathbb{H} \to \text{Im} \mathbb{H} \) given by

\[
x \times y = \text{Im}(x \cdot y), \quad \text{for any } x, y \in \text{Im} \mathbb{H}.
\]

We denote by \( \text{Re} \mathbb{H} \) the one-dimensional linear subspace spanned by the element 1 and \( \text{Im} \mathbb{H} \) the orthogonal complement of \( \text{Re} \mathbb{H} \). Hence, an element \( x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H} \) can be described by its *real part* \( \text{Re}(x) = x_0 \) and its *imaginary part* \( \text{Im}(x) = x_1 i + x_2 j + x_3 k \). Moreover, the *conjugate* \( \bar{x} \) of \( x \) is defined to be the quaternionic number

\[
\bar{x} = x_0 - x_1 i - x_2 j - x_3 k.
\]

Let us collect in the following lemma the most important properties of the quaternionic multiplications; for more details see [20, page 186].

**Lemma 3.4** The following identities hold:
(a) For any \( x, y, z \in \mathbb{H} \), we have \( \langle z \cdot x, z \cdot y \rangle = \langle x, y \rangle |z|^2 = \langle x \cdot z, y \cdot z \rangle \).

(b) For any \( x \in \text{Im} \mathbb{H} \), we have \( x^2 = -|x|^2 \).

(c) For any \( x, y \in \mathbb{H} \), we have \( \vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x} \).

(d) For any \( x, y \in \text{Im} \mathbb{H} \), we have \( x \cdot y + y \cdot x = -2 \langle x, y \rangle \); hence orthogonal imaginaries anti-commute.

Consider now the unit sphere \( S^3 \subset \mathbb{H} \) as the subset of quaternions of length 1. Moreover, we consider \( S^2_{\pm} \subset \text{Im} \mathbb{H} \subset \mathbb{H} \) as the subset of pure imaginary quaternions with length \( 1/\sqrt{2} \).

Under these considerations, the space \( E_- \) is spanned by the vectors
\[
\left\{ \frac{1 + i - j \wedge k}{2}, \frac{1 + j + i \wedge k}{2}, \frac{1 + k - i \wedge j}{2} \right\}
\]
and \( E_+ \) by the vectors
\[
\left\{ \frac{1 + i + j \wedge k}{2}, \frac{1 + j - i \wedge k}{2}, \frac{1 + k + i \wedge j}{2} \right\}.
\]

By straightforward elementary computations, we obtain the following lemma.

**Lemma 3.5** Let \( x = (x_1/\sqrt{2}, x_2/\sqrt{2}, x_3/\sqrt{2}) \in V \subset S^2_- \) and \( g \) be the map given in (3.14). Then the following facts hold:

(a) If \( g(x) \neq -x \), then the 2-plane \( G(x) = x \oplus g(x) \) is generated by the orthonormal vectors
\[
\xi(x) = \frac{x + g(x)}{|x + g(x)|} \in \text{Im} \mathbb{H} \quad \text{and} \quad \eta(x) = \sqrt{2} g(x) \cdot \xi(x) = \frac{1 - 2g(x) \cdot x}{\sqrt{2} |x + g(x)|} \in \mathbb{H}.
\]

(b) If \( g(x) = -x \), then the 2-plane \( G(x) = x \oplus (-x) \) is generated by orthonormal vectors which have one of the following forms:

\[
\xi_1(x) = \frac{x \times i}{|x \times i|} \in \text{Im} \mathbb{H} \quad \text{and} \quad \eta_1(x) = -\sqrt{2} x \cdot \xi_1(x) = \frac{-\sqrt{2} x \cdot (x \times i)}{|x \times i|} \in \text{Im} \mathbb{H}, \quad \text{if } x_1 \neq \pm 1,
\]
\[
\xi_2(x) = \frac{x \times j}{|x \times j|} \in \text{Im} \mathbb{H} \quad \text{and} \quad \eta_2(x) = -\sqrt{2} x \cdot \xi_2(x) = \frac{-\sqrt{2} x \cdot (x \times j)}{|x \times j|} \in \text{Im} \mathbb{H}, \quad \text{if } x_2 \neq \pm 1,
\]
\[
\xi_3(x) = \frac{x \times k}{|x \times k|} \in \text{Im} \mathbb{H} \quad \text{and} \quad \eta_3(x) = -\sqrt{2} x \cdot \xi_3(x) = \frac{-\sqrt{2} x \cdot (x \times k)}{|x \times k|} \in \text{Im} \mathbb{H}, \quad \text{if } x_3 \neq \pm 1.
\]

Now we give the classification of Gluck and Warner [19] of the great circle fibrations of the 3-sphere following our exposition.

**Theorem 3.6** Suppose that \( G : V \subset S^2_- \rightarrow S^2_- \times S^2_+ \simeq G_2(\mathbb{R}^4) \) is the graph of a smooth map \( g : V \subset S^2_- \rightarrow S^2_+ \), where \( V \) is a simply connected open domain. The following statements hold:

(a) Let \( x_0 \) be a point in \( V \). Then, the circle \( G(x_0) \cap S^3 \) can be represented by
\[
S(x_0, t) = \cos t \xi(x_0) + \sin t \eta(x_0),
\]
where \( \xi \) and \( \eta \) are the vectors obtained in Lemma 3.5. In particular:

(a1) If \( g(x_0) \neq \pm x_0 \), there exists an open neighbourhood \( U_{x_0} \subset V \) and a positive number \( \varepsilon_{x_0} > 0 \), such that the map \( S : U_{x_0} \times (-\varepsilon_{x_0}, \varepsilon_{x_0}) \rightarrow S^3 \) given by
\[
S(x, t) = \cos t \xi(x) + \sin t \eta(x) = \left( \cos t + \sqrt{2} \sin t g(x) \right) \cdot \xi(x),
\]
is a diffeomorphism.
(a2) If \( g(x) = x \) in an open set \( U_{x_0} \) around \( x_0 \), then \( S : U_{x_0} \times (-\pi/2, \pi/2) \to S^3 \) given by

\[
S(x, t) = \cos t \, \xi(x) + \sin t \, \eta(x) = -\sqrt{2} \, \cos tx + \sin t,
\]

is a diffeomorphism.

(a3) If \( g(x) = -x \) in an open set \( U \) around \( x_0 \) which does not contain the points \( \pm k/\sqrt{2} \),

then there exists a sufficiently small open neighbourhood \( U_{x_0} \subset U \) and a positive number \( \varepsilon_{x_0} > 0 \) such that the map \( S : U_{x_0} \times (-\varepsilon_{x_0}, \varepsilon_{x_0}) \to S^3 \) given by

\[
S(x, t) = \cos t \, \xi(x) + \sin t \, \eta(x) = \frac{\cos tx \, k - \sqrt{2} \, \sin tx \cdot (x \times k)}{|x \times k|},
\]

is a diffeomorphism.

In each of the cases (a1), (a2) and (a3), the map \( \pi : S(U_{x_0} \times (-\varepsilon_{x_0}, \varepsilon_{x_0})) \subset S^3 \to U_{x_0} \)
given by

\[
\pi(\cos t \, \xi(x) + \sin t \, \eta(x)) = (\xi(x) \cdot g(x) \cdot \xi(x)) = x,
\]

is a submersion with totally geodesic fibres, which are generated by the unit vector field \( \xi = dS_{x_0}(\partial_t) \). Additionally, if \( h : S^2_- \to S^2_+ \) is a diffeomorphism, then \( h \circ \pi \) gives rise to a submersion whose fibres are generated again by \( \xi \).

(b) The great circle foliation

\[
\mathcal{F} = \bigcup_{x \in U_{x_0}} G(x) \cap S^3
\]
is smooth if and only if \( g : U_{x_0} \subset S^2_- \to S^2_+ \) is strictly length decreasing.

(c) If \( g : S^2_- \to S^2_+ \) is strictly length decreasing, then the maps \( \pi \) given in (a) generate a globally defined submersion from \( S^3 \) onto \( S^2_- \) with totally geodesic fibres. Moreover, the quotient map \( \pi \) is a Hopf fibration, and \( \xi \) is a Hopf vector field, if and only if the map \( g \) is constant.

The proof of Theorem 3.6 follows by direct computations and using Lemmas 3.4 and 3.5.

3.3 Applications

Let us collect some immediate applications of the classification theorem of the great circle fibrations of the 3-sphere.

Corollary 3.7 (Gluck [16]) Let \( \xi \) be a unit vector field on \( S^3 \) with totally geodesic integral curves. The dual 1-form associated with \( \xi \) gives rise to a contact structure of \( S^3 \).

Proof Let us denote with \( \omega \) the associated 1-form to \( \xi \), i.e. the form given by \( \omega(\alpha) = \langle \xi, \alpha \rangle \), for all tangent vectors \( \alpha \). Recall that \( \omega \) is a contact form in \( S^3 \) if and only if \( \omega \wedge d\omega \neq 0 \). Let now \( \{\alpha_1 = \xi, \alpha_2, \alpha_3\} \) be a local orthonormal frame and \( \varphi \) the tensor introduced in the Sect. 3. We compute

\[
(\omega \wedge d\omega)(\alpha_1, \alpha_2, \alpha_3) = d\omega(\alpha_2, \alpha_3) = \alpha_2(\omega(\alpha_3)) - \alpha_3(\omega(\alpha_2)) - \omega([\alpha_2, \alpha_3])
\]
\[
= \langle \nabla_{\alpha_2} \alpha_1, \alpha_3 \rangle - \langle \nabla_{\alpha_3} \alpha_1, \alpha_2 \rangle = -\varphi_{23} + \varphi_{32}
\]

where \( J \) stands for the complex structure of \( \mathcal{H} \). Recall that in the proof of Lemma 3.3(b) we proved that \( \text{trace}(\varphi \circ J) \) is nowhere zero. Hence, \( \omega \) is a contact form. This completes the proof. \( \square \)
Corollary 3.8 (Gluck & Gu [17]) Let ζ be a divergence-free unit vector field on $\mathbb{S}^3$ with totally geodesic integral curves. Then ζ is a Hopf vector field.

**Proof** Since ζ is divergence-free, then trace $\varphi = 0$. Then, from Lemma 3.1(b) it follows that det $\varphi = 1$. Hence, from Lemma 3.3(c) it follows that the singular values of $g$ are equal. Consequently, $g$ is a conformal map. This implies that the graph of $g$ is a minimal surface in $\mathbb{S}^2 \times \mathbb{S}^2$, see for example [2, Proposition 4.5.3], [10, Example 3] or [27]. On the other hand, $g$ is strictly length decreasing. From [32, Theorem A], it follows that $g$ must be constant. Consequently, from Theorem 3.6(c) we deduce that $\pi$ is a Hopf fibration and so ζ is a Hopf vector field. This completes the proof. $\square$

Corollary 3.9 (Heller [23]) Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be a weakly conformal submersion with totally geodesic fibres. Then the map $f$ is the composition of a Hopf fibration with a conformal diffeomorphism of $\mathbb{S}^2$.

**Proof** Choose a local orthonormal frame $\{\alpha_1, \alpha_2, \alpha_3\}$, such that $\alpha_1 \in \ker df$. Denote also by $\lambda^2$ the conformal factor of $f$, i.e.

$$\lambda^2 = \langle df(\alpha_2), df(\alpha_2) \rangle = \langle df(\alpha_3), df(\alpha_3) \rangle > 0. \quad (3.15)$$

In this case it holds $|df|^2 = 2\lambda^2 > 0$ and so $\lambda$ is a smooth function. Differentiating the identity of (3.15) with respect to $\alpha_1$, we deduce that

$$\lambda \alpha_1(\lambda) = \langle \nabla_{\alpha_1}^f df(\alpha_2), df(\alpha_2) \rangle = \langle B_f(\alpha_1, \alpha_2) + df(\nabla_{\alpha_1} \alpha_2), df(\alpha_2) \rangle$$

$$= \langle B_f(\alpha_1, \alpha_2), df(\alpha_2) \rangle + \langle \nabla_{\alpha_2}^f df(\alpha_1) - df(\nabla_{\alpha_2} \alpha_1), df(\alpha_2) \rangle$$

$$= \lambda^2 \varphi_{22},$$

where $\nabla^f$ is the connection of the induced by $f$ vector bundle $f^*T\mathbb{S}^2$ and

$$B_f(\alpha_i, \alpha_j) = \nabla_{\alpha_i}^f df(\alpha_j) - df(\nabla_{\alpha_i} \alpha_j)$$

is the Hessian of $f$. Similarly, we get that

$$\lambda \alpha_1(\lambda) = \langle \nabla_{\alpha_1}^f df(\alpha_3), df(\alpha_3) \rangle = \lambda^2 \varphi_{33}.$$

Moreover,

$$0 = \alpha_1 \langle df(\alpha_2), df(\alpha_3) \rangle$$

$$= \langle B_f(\alpha_1, \alpha_2) + df(\nabla_{\alpha_1} \alpha_2), df(\alpha_3) \rangle + \langle B_f(\alpha_1, \alpha_3) + df(\nabla_{\alpha_1} \alpha_3), df(\alpha_2) \rangle$$

$$= \langle B_f(\alpha_1, \alpha_2), df(\alpha_3) \rangle + \langle B_f(\alpha_1, \alpha_3), df(\alpha_2) \rangle + \lambda^2 \langle \nabla_{\alpha_1} \alpha_2, \alpha_3 \rangle + \lambda^2 \langle \nabla_{\alpha_1} \alpha_3, \alpha_2 \rangle$$

$$= -\langle df(\nabla_{\alpha_2} \alpha_1), df(\alpha_3) \rangle - \langle df(\nabla_{\alpha_3} \alpha_1), df(\alpha_2) \rangle$$

$$= \lambda^2 (\varphi_{23} + \varphi_{32}).$$

From the above relations, we see that

$$\varphi_{23} = -\varphi_{32} \quad \text{and} \quad \varphi_{22} = \varphi_{33} = \alpha_1(\log \lambda). \quad (3.16)$$

As a consequence, $|\varphi|^2 = 2 \det \varphi$. Hence, from Lemma 3.3(c) we see that the singular values of $g$ are everywhere equal. As in the proof of Corollary 3.8, we show that $g$ is constant. This implies that $\alpha_1$ is a Hopf vector field and the corresponding projection $\pi$ a Hopf fibration. Observe that there exists a diffeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $f = h \circ \pi$. Since $f$ is a weakly conformal submersion and $\pi$ a Riemannian submersion, we deduce that $h$ is a conformal diffeomorphism. This completes the proof. $\square$
Remark 3.10 In [23] Heller proves a more general result. More precisely, he shows that up to conformal transformations of \( S^2 \) and \( S^3 \), every conformal fibration of \( S^3 \) by circles (not necessarily great circles) is the Hopf fibration.

Corollary 3.11 (Escobales [11]) Suppose that \( f : V \subset S^3 \to S^2 \) is a submersion with totally geodesic fibres and equal constant singular values defined in an open neighbourhood \( V \) of \( S^3 \). Then \( f \) is a Hopf fibration.

Proof By assumption, the singular values of \( f \) are \( 0, \lambda, \lambda \), where \( \lambda \) is a nonzero constant. From the formulas (3.16), we deduce that \( \varphi_{22} = \varphi_{33} = 0 \) and \( \varphi_{23} + \varphi_{32} = 0 \). From Lemma 3.1(b), it follows that \( \det \varphi = 1 \), which implies that \( \varphi_{23} = -1 \) and \( \varphi_{32} = 1 \). Now from Lemma 3.3(c) the singular values of \( g \) are zero, which implies that \( g \) is constant. Following the same lines as in Corollary 3.9, we deduce that \( f \) is a Hopf fibration. This completes the proof. \( \square \)

4 Harmonic and minimal unit vector fields

Let us assume now that \( g : V \subset S^2_+ \to S^2_+ \) is strictly length decreasing, where \( V \) is an open and simply connected domain. Fix a point \( x_0 \in S^2_+ \). From Theorem 3.6, it follows that there exists a sufficiently small neighbourhood \( U_{x_0} \) around the point \( x_0 \) where the map \( \xi \) is a diffeomorphism. Hence, by setting \( y = \xi(x) \) and \( h = \sqrt{2} g \circ \xi^{-1} \), we may reparametrise the foliation by the map \( \vartheta : \xi(U_{x_0}) \times [-\pi, \pi] \to S^3 \) given by

\[
\vartheta(y, t) = \cos t y + \sin t h(y) \cdot y.
\]

Observe that \( h \) is a smooth map from an open domain of the unit 2-sphere \( S^3 \cap \text{Im} \mathbb{H} \) with values in the same sphere. According to Theorem 3.6, we have that

\[
\xi(\vartheta(y, t)) = h(y) \cdot \vartheta(y, t) \quad \text{and} \quad \pi(\vartheta(y, t)) = \frac{\overline{y} \cdot h(y) \cdot y}{\sqrt{2}}.
\]

On the other hand, one can readily check that for any \( (y, t) \) it holds

\[
\pi(\vartheta(y, t)) = \frac{\vartheta(y, t) \cdot h(y) \cdot \vartheta(y, t)}{\sqrt{2}}, \tag{4.1}
\]

Hence, for \( p = \vartheta(y, t) \) we deduce that the unit vector field \( \xi \) generating the leaves of the foliation is related to \( \pi \) via the formulas

\[
\xi(p) = p \cdot f(p) \quad \text{and} \quad f(p) = \overline{p} \cdot \xi(p), \tag{4.2}
\]

for any \( p \in V \subset S^3 \), where \( f = \sqrt{2} \pi \).

4.1 Harmonic unit vector fields

We would like to relate the Hessian of the vector field \( \xi \) with the Hessian of the quotient map \( f \) given in (4.2). We need the following lemma.

Lemma 4.1 Let \( \xi \) be a unit vector field with closed totally geodesic integral curves defined in an open connected neighbourhood \( V \) of \( S^3 \) and \( f : V \subset S^3 \to S^2 \) be the corresponding quotient map given in (4.2). Assume that \( f(V) \) is simply connected. Then,

\[
\langle df(\alpha_i), df(\alpha_j) \rangle = \langle \alpha_i, \alpha_j \rangle + \langle \varphi(\alpha_i), \varphi(\alpha_j) \rangle - \langle \varphi(\alpha_i), J\alpha_j \rangle - \langle \varphi(\alpha_j), J\alpha_i \rangle, \tag{4.3}
\]

⃜ Springer
where \( i, j \in \{2, 3\}, \alpha_2, \alpha_3 \in \mathcal{H} \) and \( J \) is the complex structure of \( \mathcal{H} \).

**Proof** \( \) Observe at first that any tangent vector \( \alpha \in T_pS^3 \) satisfies the equations

\[
\alpha = -p \cdot \bar{p} \quad \text{and} \quad \alpha = -p \cdot \bar{p} \cdot p.
\]  

(4.4)

Indeed! Consider a curve \( \sigma : (-\varepsilon, \varepsilon) \to S^3 \) such that \( \sigma(0) = p \) and \( \sigma'(0) = \alpha \). Differentiating the expression \( \sigma \cdot \bar{\sigma} = 1 \), and estimating at \( t = 0 \), we get the first identity of (4.4). The second follows immediately from the first. Differentiating the second equation of (4.2) with respect to \( \alpha_i, i, j \in \{2, 3\} \), we get

\[
df(\alpha_i) = \bar{\alpha}_i \cdot \xi + \bar{p} \cdot D_{\alpha_i} \xi = \bar{\alpha}_i \cdot \xi + \bar{p} \cdot \nabla_{\alpha_i} \xi = \bar{\alpha}_i \cdot \xi - \bar{p} \cdot \varphi(\alpha_i),
\]

(4.5)

where \( D \) is the standard connection of \( \mathbb{H} = \mathbb{R}^4 \). Using Lemma 3.4 and (4.4), we obtain

\[
\langle df(\alpha_i), df(\alpha_j) \rangle = \langle \bar{\alpha}_i \cdot \xi - \bar{p} \cdot \varphi(\alpha_i), \bar{\alpha}_j \cdot \xi - \bar{p} \cdot \varphi(\alpha_j) \rangle
\]

\[
= \langle \alpha_i, \alpha_j \rangle + \langle \varphi(\alpha_i), \varphi(\alpha_j) \rangle + \langle \alpha_i, \bar{p} \cdot \xi, \varphi(\alpha_j) \rangle + \langle \alpha_j, \bar{p} \cdot \xi, \varphi(\alpha_i) \rangle.
\]

(4.6)

For any \( p \in S^3 \), the vectors \( \{\alpha_1 = \xi, \alpha_2, \alpha_3, p\} \) form a basis of \( \mathbb{H} \). We decompose the vector \( \alpha_2 \cdot \bar{p} \cdot \alpha_3 \) in terms of \( \alpha_1, \alpha_2, \alpha_3 \) and \( p \). We observe that

\[
\langle \alpha_2 \cdot \bar{p} \cdot \alpha_3, \alpha_2 \rangle = \langle \alpha_2 \cdot \bar{p} \cdot \alpha_3, \alpha_3 \rangle = 0.
\]

Furthermore,

\[
\langle \alpha_2 \cdot \bar{p} \cdot \alpha_3, p \rangle = \langle \alpha_2 \cdot \bar{p} \cdot \alpha_3, \bar{p} \rangle = -(\alpha_2 \cdot \bar{\alpha}_3, 1) = 0,
\]

(4.7)

since \( \langle \alpha_2, \alpha_3 \rangle = 0 \). Thus, \( \alpha_2 \cdot \bar{p} \cdot \alpha_3 = \pm \alpha_1 \). Choosing positive orientation, we get

\[
\begin{align*}
\alpha_1 \cdot \bar{p} \cdot \alpha_2 &= -\alpha_2 \cdot \bar{p} \cdot \alpha_1 = \alpha_3, \\
\alpha_3 \cdot \bar{p} \cdot \alpha_1 &= -\alpha_1 \cdot \bar{p} \cdot \alpha_3 = \alpha_2, \\
\alpha_2 \cdot \bar{p} \cdot \alpha_3 &= -\alpha_3 \cdot \bar{p} \cdot \alpha_2 = \alpha_1.
\end{align*}
\]

(4.8)

Using (4.4), we obtain

\[
\begin{align*}
\alpha_1 \cdot \bar{p} \cdot \alpha_1 &= -\alpha_1 \cdot \bar{\alpha}_1 \cdot p = -p, \\
\alpha_2 \cdot \bar{p} \cdot \alpha_2 &= \alpha_3 \cdot \bar{p} \cdot \alpha_3 = -p.
\end{align*}
\]

(4.9)

From (4.6) and the equations (4.8), we get (4.3). This completes the proof. \( \square \)

**Definition 4.2** \( \) Let \( F : (M, g_M, \nabla^M) \to (N, g_N, \nabla^N) \) be a smooth map between Riemannian manifolds, \( \nabla^F \) be the induced by \( F \) connection of the pullback bundle \( F^*TN \) and

\[
B_F(X, Y) = \nabla^F_X dF(Y) - dF(\nabla^M_X Y), \quad X, Y \in \mathfrak{X}(M),
\]

be the Hessian of \( F \). The trace

\[
\tau_F = \text{trace}_{g_M} B_F
\]

is called the tension field of \( F \). The map \( F \) is called harmonic if \( \tau_F = 0 \).

**Proposition 4.3** \( \) Let \( \xi \) be a unit vector field with closed totally geodesic integral curves defined in an open connected neighbourhood \( V \) of \( S^3 \) and \( f : V \subset S^3 \to S^2 \) be the corresponding
where \( i \), \( j \in \{2, 3\} \), \( \{\alpha_2, \alpha_3\} \) is a local orthonormal frame of \( \mathcal{H} \), and \( B_f \) is the Hessian of the map \( f : (S^3, g_{S^3}) \to (S^2, g_{S^2}) \). In particular,

\[
\Delta \zeta + |\nabla \zeta|^2 \zeta = p \cdot \tau_f,
\]

where \( \Delta \) stands for the rough Laplacian operator and \( \tau_f \) for the tension field of \( f \).

**Proof** Consider a local orthonormal frame \( \{\alpha_1 = \zeta, \alpha_2, \alpha_3\} \) such that \( \alpha_1 \in \mathcal{V} \) and \( \alpha_2, \alpha_3 \in \mathcal{H} \). Recall that

\[
\Delta \alpha_1 = \sum_{i=2}^{3} (\nabla_{\alpha_i} \nabla_{\alpha_1} - \nabla_{\nabla_{\alpha_i} \alpha_1}).
\]

Differentiating the identity \( \alpha_1(p) = p \cdot f(p) \) with respect to \( \alpha_j, j \in \{2, 3\} \), we get

\[
D_{\alpha_j} \alpha_1 = \alpha_j \cdot f + p \cdot df(\alpha_j).
\]  (4.11)

Differentiating (4.11) with respect to \( \alpha_i, i \in \{2, 3\} \), we have

\[
D_{\alpha_i} D_{\alpha_j} \alpha_1 = \nabla_{\alpha_i} \alpha_j \cdot f - \langle \alpha_i, \alpha_j \rangle \alpha_1 - \langle df(\alpha_i), df(\alpha_j) \rangle \alpha_1
+ \alpha_j \cdot df(\alpha_i) + \alpha_i \cdot df(\alpha_j) + p \cdot \nabla_{\alpha_i} df(\alpha_j).
\]  (4.12)

Using (4.11) and (4.12), we deduce

\[
\nabla_{\alpha_i, \alpha_j}^2 \alpha_1 = p \cdot B_f(\alpha_i, \alpha_j) - (\langle \alpha_i, \alpha_j \rangle + \langle df(\alpha_i), df(\alpha_j) \rangle) \alpha_1
+ \alpha_j \cdot df(\alpha_i) + \alpha_i \cdot df(\alpha_j) - \langle \alpha_i, \varphi(\alpha_j) \rangle \alpha_1 + \langle \varphi(\alpha_i), \alpha_j \rangle \alpha_1.
\]  (4.13)

From the formulas (4.8)-(4.9), we get

\[
\alpha_i \cdot \overline{p} \cdot \varphi(\alpha_j) = -\langle \varphi(\alpha_j), \alpha_i \rangle p + \langle \varphi(\alpha_j), J\alpha_i \rangle \alpha_1,
\]

\[
\alpha_j \cdot \overline{p} \cdot \varphi(\alpha_i) = -\langle \varphi(\alpha_i), \alpha_j \rangle p + \langle \varphi(\alpha_i), J\alpha_j \rangle \alpha_1.
\]  (4.14)

Combining Lemma 3.4, (4.4), (4.5) and (4.14), we obtain

\[
\alpha_i \cdot df(\alpha_j) + \alpha_j \cdot df(\alpha_i) = -\langle \alpha_i \cdot \overline{p} \cdot \alpha_j \cdot \overline{p} + \alpha_j \cdot \overline{p} \cdot \alpha_i \cdot \overline{p} \rangle \alpha_1
- \alpha_i \cdot \overline{p} \cdot \varphi(\alpha_j) - \alpha_j \cdot \overline{p} \cdot \varphi(\alpha_i)
= 2\langle \alpha_i, \alpha_j \rangle \alpha_1 + \langle \varphi(\alpha_j), \alpha_i \rangle + \langle \varphi(\alpha_i), \alpha_j \rangle \rangle p
- \langle \varphi(\alpha_i), J\alpha_j \rangle \alpha_1 + \langle \varphi(\alpha_i), J\alpha_i \rangle \alpha_1.
\]  (4.15)

Substituting (4.15) in (4.13) and using (4.3), we obtain the desired formula (4.10). This completes the proof of the proposition. \( \square \)

As a direct consequence of Proposition 4.3, we derive the following theorem.

**Theorem B** Let \( \zeta \) be a unit vector field with closed totally geodesic integral curves defined in an open connected neighbourhood \( V \) of \( S^3 \) and \( f : V \subset S^3 \to S^2 \) be the corresponding quotient map. Then \( f \) is a harmonic map if and only if \( \zeta : V \to US^3 \) is a harmonic unit vector field.
Let us prove now our next main theorem.

**Theorem C** A harmonic unit vector field $\zeta$ on $S^3$, whose integral curves are great circles, is a Hopf vector field and the corresponding quotient map $f : S^3 \to S^2$ is a Hopf fibration.

**Proof** Consider a local orthonormal frame $\{\alpha_1 = \zeta, \alpha_2, \alpha_3\}$ such that $\alpha_2, \alpha_3 \in \mathcal{H}$. Denote by $f : S^3 \to S^2$ the quotient map associated to $\zeta$. Let us introduce the functions

$$v = \text{trace } \varphi = \varphi_{22} + \varphi_{33} \quad \text{and} \quad u = \text{trace}(\varphi \circ J) = \varphi_{32} - \varphi_{23}. \quad (4.16)$$

The functions $v$ and $u$ are smooth and globally defined on $S^3$. We have that

$$\Delta \alpha_1 + |\nabla \alpha_1|^2 \alpha_1 = 0.$$

From the equations

$$\langle \Delta \alpha_1, \alpha_2 \rangle = 0 = \langle \Delta \alpha_1, \alpha_3 \rangle$$

we obtain

$$\alpha_2(\varphi_{22}) + \alpha_3(\varphi_{32}) = (\varphi_{22} - \varphi_{33})(\nabla_{\alpha_3} \alpha_3, \alpha_2) + (\varphi_{23} + \varphi_{32})(\nabla_{\alpha_2} \alpha_2, \alpha_3) \quad (4.17)$$

and

$$\alpha_2(\varphi_{23}) + \alpha_3(\varphi_{33}) = (\varphi_{33} - \varphi_{22})(\nabla_{\alpha_2} \alpha_2, \alpha_3) + (\varphi_{23} + \varphi_{32})(\nabla_{\alpha_3} \alpha_3, \alpha_2). \quad (4.18)$$

Since

$$(\nabla^\mathcal{H}_{\alpha_2} \varphi) \alpha_3 = (\nabla^\mathcal{H}_{\alpha_3} \varphi) \alpha_2,$$

we obtain the following system of PDEs

$$\alpha_3(\varphi_{22}) - \alpha_2(\varphi_{32}) = (\varphi_{22} - \varphi_{33})(\nabla_{\alpha_2} \alpha_2, \alpha_3) - (\varphi_{23} + \varphi_{32})(\nabla_{\alpha_3} \alpha_3, \alpha_2) \quad (4.19)$$

and

$$\alpha_3(\varphi_{23}) - \alpha_2(\varphi_{33}) = (\varphi_{23} + \varphi_{32})(\nabla_{\alpha_2} \alpha_2, \alpha_3) + (\varphi_{22} - \varphi_{33})(\nabla_{\alpha_3} \alpha_3, \alpha_2). \quad (4.20)$$

Subtracting (4.20) from (4.17), adding (4.18) and (4.19) and keeping in mind (3.3) and (4.16), we see that the functions $u$ and $v$ satisfy the system of differential equations

$$\alpha_1(v) = v^2 - 2(1 + \det \varphi) + 4, \quad \alpha_1(u) = uv, \quad \alpha_2(u) = \alpha_3(v) \text{ and } \alpha_3(u) = -\alpha_2(v). \quad (4.21)$$

From (4.21) and (3.3), we deduce that

$$\Delta v = \alpha_1 \alpha_1(v) + \alpha_2 \alpha_2(v) + \alpha_3 \alpha_3(v) - \nabla_{\alpha_2} \alpha_2(v) - \nabla_{\alpha_3} \alpha_3(v)$$

$$= 2uv \alpha_1(v) - 2v(1 + \det \varphi) - [\alpha_2, \alpha_3](u) - v \alpha_1(v)$$

$$+ [\alpha_2, \nabla_{\alpha_2} \alpha_3] \alpha_2(u) - [\alpha_3, \nabla_{\alpha_3} \alpha_2] \alpha_3(u)$$

$$= u^2 v + v \alpha_1(v) - 2v(1 + \det \varphi) = v(u^2 + v^2 - 4 \det \varphi)$$

$$= v(|\varphi|^2 - 2 \det \varphi).$$

Since

$$|\varphi|^2 - 2 \det \varphi \geq 0,$$

we obtain that

$$\Delta v^2 = 2v \Delta v + 2|\nabla v|^2 = 2v^2(|\varphi|^2 - 2 \det \varphi) + 2|\nabla v|^2 \geq 0.$$

\( \square \)
From the maximum principle, it follows that the function $v$ is constant. On the other hand

$$v = \varphi_{22} + \varphi_{33} = -\text{div}(\alpha_1)$$

and by the Stokes’ Theorem, we conclude that $v = \text{trace} \varphi = 0$. Moreover, from the first equation of (4.21) we deduce that $\det \varphi = 1$. From Lemma 3.3(c), we see that the associated graph in $S^2_+ \times S^2_-$ is generated by a conformal and strictly length decreasing mapping $g$. Note that conformality implies minimality; see [2, Proposition 4.5.3], [10, Example 3] or [27]. From [32, Theorem A], it follows that $g$ must be constant. Then, from Theorem 3.6(c) we deduce that $\alpha_1$ is a Hopf vector field.

\[\square\]

### 4.2 Minimal unit vector fields

The unit tangent bundle of $S^3$ can be identified with the *Stiefel manifold* $V_2(\mathbb{R}^4)$, i.e. the set of orthonormal two-frames in $\mathbb{R}^4$. $V_2(\mathbb{R}^4)$ is diffeomorphic with the coset space $SO(4)/SO(2)$. Using the quaternionic structure of $\mathbb{R}^4$, we can identify $US^3$ with the product of unit spheres $S^3 \times S^2$. Indeed, one can easily verify that the map $\Phi : US^3 \to S^3 \times S^2$ given by

$$\Phi(x, v) = (x, \overline{x} \cdot v)$$

is a diffeomorphism. The map $T : SO(4)/SO(2) \cong US^3 \to G_2(\mathbb{R}^4) \cong S^2_- \times S^2_+$ given by

$$T(x, v) = x \wedge v = \left(\frac{x \wedge v - \ast(x \wedge v)}{2}, \frac{x \wedge v + \ast(x \wedge v)}{2}\right)$$

is called the **Stiefel bundle map**. $(US^3, g_S)$ is isometric to the Stiefel manifold of orthonormal 2-frames of $\mathbb{R}^4$ equipped with the homogeneous metric resulting from its diffeomorphism with $SO(4)/SO(2)$ ([18, page 180]). The Grassmann manifold $G_2(\mathbb{R}^4)$ is isometric to the product $S^2_- \times S^2_+$. It turns out that $T : (US^3, g_S) \to (S^2_- \times S^2_+, g_{S^2_- \times S^2_+})$ is a Riemannian submersion; see for example ([19, pages 121-123 and Remark 7.4]) and ([18, page 180]). For reader’s convenience we include a short proof of this fact.

**Proposition 4.4** The Stiefel bundle map $T : (US^3, g_S) \to (S^2_- \times S^2_+, g_{S^2_- \times S^2_+})$ is a Riemannian submersion whose fibres are generated by the geodesic flow vector field $\xi_U$ of $US^3$.

**Proof** Let $(x, v) \in US^3$ and $Z \in T_{(x,v)}US^3$ and let $\gamma : (-\varepsilon, \varepsilon) \to US^3$ be a smooth curve such that $\gamma(0) = (x, v)$ and $\gamma'(0) = Z$. Then, $\gamma$ can be written in the form $\gamma = (\alpha, V)$, where $\alpha = \pi_{S^3} \circ \gamma$ is the projection of the curve $\gamma$ to $S^3$ and $V$ is a unit vector field along $\alpha$. Note that $\alpha(0) = x$ and $V(0) = v$. Furthermore, we have that

$$Z = w_1^{\text{hor}} + w_2^{\text{tan}},$$

where $w_1 = \alpha'(0) \in T_x S^3$ and $w_2 = \frac{\nabla V}{\partial t}(0) = V'(0) + \langle w_1, v \rangle x = V'(0) - \langle V'(0), x \rangle x$ (since $\langle \alpha(t), V(t) \rangle = 0$). Differentiating $T \circ \gamma = \alpha \wedge V$ with respect to $t$ and estimating at $t = 0$, we get

$$dT_{(x,v)}(Z) = dT_{(x,v)}(w_1^{\text{hor}} + w_2^{\text{tan}}) = w_1 \wedge v + x \wedge w_2.$$ (4.22)

Fix a point $x \in S^3$ and consider a local orthonormal frame $\{v, \alpha_2, \alpha_3\}$ of $T_x S^3$. Observe that $Z \in \ker dT_{(x,v)}$ if and only if

$$w_1 \wedge v + x \wedge w_2 = 0.$$
Decomposing the vectors \( w_1 \) and \( w_2 \) in terms of \( v, \alpha_2 \) and \( \alpha_3 \), we see that \( w_1 \in \text{span} \, v \) and \( w_2 = 0 \). Hence, the Stiefel bundle map is a submersion and the geodesic flow vector field \( \xi_U \) generates the fibres of \( T \). Furthermore, \( Z \in \{ \ker dT(x, v) \}^\perp \) if and only if

\[
\langle w_1, v \rangle = 0.
\]  

Differentiating the expression \( \langle V, V \rangle = 1 \) with respect to \( t \) and evaluating at \( t = 0 \), we obtain

\[
\langle w_2, v \rangle = 0.
\]  

It remains to show that the map \( T \) is a Riemannian submersion. Indeed, choose vector fields

\[
Z_1 = w_1^{\text{hor}} + w_2^{\text{tan}} \in \{ \ker dT(x, v) \}^\perp \quad \text{and} \quad Z_2 = w_3^{\text{hor}} + w_4^{\text{tan}} \in \{ \ker dT(x, v) \}^\perp.
\]

We set

\[
A_1 = \frac{w_1 \wedge v + x \wedge w_2 - \ast(w_1 \wedge v + x \wedge w_2)}{2},
A_2 = \frac{w_3 \wedge v + x \wedge w_4 - \ast(w_3 \wedge v + x \wedge w_4)}{2}.
\]

Similarly,

\[
B_1 = \frac{w_1 \wedge v + x \wedge w_2 + \ast(w_1 \wedge v + x \wedge w_2)}{2},
B_2 = \frac{w_3 \wedge v + x \wedge w_4 + \ast(w_3 \wedge v + x \wedge w_4)}{2}.
\]

Using (2.2), (4.23) and (4.24), we have

\[
g_{S_2^- \times S_2^+}(dT(x, v)(Z_1), dT(x, v)(Z_2)) = (A_1, A_2) + (B_1, B_2)
= (w_1 \wedge v + x \wedge w_2, w_3 \wedge v + x \wedge w_4)
= (w_1, w_3) + (w_2, w_4) - \langle w_2, x \rangle \langle w_4, x \rangle
= g_S(Z_1, Z_2).
\]

This completes the proof of the proposition. \( \square \)

We would like to relate the mean curvature of the vector field \( \xi \) with the mean curvature of the corresponding graph. We need the following lemma.

**Lemma 4.5** Let \( \xi : V \subset S^3 \to US^3 \) be a unit vector field with closed totally geodesic integral curves defined in an open connected neighbourhood \( V \) of \( S^3 \), \( \pi : V \subset S^3 \to S^- \) be the quotient map given in (4.1) and \( G = (h_-, h_+) : \pi(V) \subset S^- \to S^- \times S^- \) be its corresponding graph. Assume that \( \pi(V) \) is simply connected.

(a) The quotient map \( \pi : (V \subset S^3, \xi^*g_S) \to (S^-_2, G^*g_{S^-_2 \times S^-_2}) \) is a Riemannian submersion. Moreover, we have that \( T \circ \xi = G \circ \pi \); compare also with [19, equation (7.12) page 126].

(b) The following formula holds

\[
dT(A_\xi(X, Y)) = A_G(d\pi(X), d\pi(Y)) + dG(B_\pi(X, Y)),
\]

for all vector fields \( X, Y \) on \( (V, \xi^*g_S) \) perpendicular to \( \xi \), where \( T \) is the Stiefel map and \( B_\pi \) is the Hessian of \( \pi : (V, \xi^*g_S) \to (S^-_2, G^*g_{S^-_2 \times S^-_2}) \). In particular,

\[
dT(H_\xi) = H_G + dG(\tau_\pi), \quad \text{for all vector fields } X, Y \text{ on } (V, \xi^*g_S),
\]

where \( \tau_\pi \) is the tension field of \( \pi : (V, \xi^*g_S) \to (S^-_2, G^*g_{S^-_2 \times S^-_2}) \).
Proof (a) Without loss of generality, let us assume that the map \( h_- \) is diffeomorphism. From the definition of the Stiefel map and (3.7), we conclude that
\[
T(\xi(p)) = T(p, \xi_p) = p \wedge \xi_p = h_-(\pi(p)) \oplus h_+(\pi(p)) = G(\pi(p)),
\]
for all \( p \in V \); compare also with the computations in [19, page 122]. Hence, \( T \circ \xi = G \circ \pi \).

Since the unit vector field \( \xi \) has geodesic integral curves, making use of (2.4), we get
\[
\ker dT = \{ W = X^\text{hor} + Y^\text{tan} \in T_\xi U^3 : \langle X, \xi \rangle = 0 \}
\]
for details see also [2, Lemma 4.5.1, page 119].

Similarly, we get
\[
\ker dT^* = \{ W = X^\text{hor} + Y^\text{tan} \in T_\xi U^3 : \langle \xi^*g_S(X, \xi) \rangle = 0 \}. \tag{4.26}
\]
We claim now that \( \pi : (V, \xi^*g_S) \to (\mathbb{S}^2_+, \xi^*g_{\mathbb{S}^2_+ \times \mathbb{S}^2_+}) \) is a Riemannian submersion. Indeed, from the equations (2.3), (2.4), (4.26) and the fact that \( T \) is a Riemannian submersion, we obtain
\[
G^*g_S(\mathbb{S}^2_+ \times \mathbb{S}^2_+) (d\pi(X), d\pi(Y)) = g_{\mathbb{S}^2_+ \times \mathbb{S}^2_+}(dG(d\pi(X)), dG(d\pi(Y)))
\]
\[
\quad = g_{\mathbb{S}^2_+ \times \mathbb{S}^2_+}(dT(d\xi(X)), dT(d\xi(Y)))
\]
\[
\quad = g_S(X^\text{hor} + (\nabla_X \xi)^\text{tan}, Y^\text{hor} + (\nabla_Y \xi)^\text{tan})
\]
\[
\quad = \xi^*g_S(X, Y),
\]
for all vector fields \( X, Y \) on \( V \subset \mathbb{S}^3 \) perpendicular to \( \xi \) with respect to \( \xi^*g_S \).

(b) Using the Koszul formula and the fact that \( T \) is a Riemannian submersion we deduce that for any pair of \( v_1, v_2 \in \ker dT \), we have
\[
B_T(v_1, v_2) = \nabla_T^\pi dT(v_2) - dT(\nabla_{v_1}^\pi v_2) = 0, \tag{4.27}
\]
for details see also [2, Lemma 4.5.1, page 119]. Since \( T \circ \xi = G \circ \pi \), we have
\[
B_T \circ \xi = B_{G \circ \pi}.
\]

From the composition formula, we obtain that for all vector fields \( X, Y \in \mathfrak{X}(V) \), it holds
\[
dT(A_\xi(X, Y)) + B_T(d\xi(X), d\xi(Y)) = dG(B_\pi(X, Y)) + A_G(d\pi(X), d\pi(Y)). \tag{4.28}
\]
Using (2.3) and (4.26), we see that \( \xi^*g_S(X, \xi) = 0 \), then \( d\xi(X) \in \ker dT \). Therefore, from (4.27) we get
\[
dT(A_\xi(X, Y)) = dG(B_\pi(X, Y)) + A_G(d\pi(X), d\pi(Y)), \tag{4.29}
\]
for all vector fields \( X, Y \) on \( V \subset \mathbb{S}^3 \) perpendicular to \( \xi \) with respect to \( \xi^*g_S \). Using the Koszul formula, we get
\[
2\xi^*g_S(\nabla_\xi \xi^*g_S, X) = -2\xi^*g_S([[\xi, X], \xi]) = -2(\xi, X, \xi) = 2\langle \nabla_\xi \xi, X \rangle = 0.
\]
for all vector fields \( X \) on \( V \subset \mathbb{S}^3 \) such that \( \xi^*g_S(X, \xi) = \langle X, \xi \rangle = 0 \). Moreover, \( \nabla_\xi \xi^*g_S = 0 \). Using the identity
\[
\nabla_{X^\text{hor}} Y^\text{hor} = (\nabla_X Y)^\text{hor} - \frac{1}{2}(R_{\mathbb{S}^3}(X, Y)v)^\text{tan},
\]
of Lemma 2.1, we deduce that
\[
A_\xi(\xi, \xi) = \nabla_{\xi^\text{hor}}^\xi \xi^\text{hor} - d\xi(\nabla_\xi \xi^*g_S, \xi) = 0. \tag{4.30}
\]
Similarly, we get
\[
B_T(d\xi(\xi), d\xi(\xi)) = 0, B_\pi(\xi, \xi) = 0. \tag{4.31}
\]
Taking trace, with respect to the metric $\zeta^* g_S$, at the two sides of (4.28), using the fact that the quotient map

$$\pi : (V \subset S^3, \zeta^* g_S) \to (S^2_-, G^* g_{S^2_\times S^2_+})$$

is a Riemannian submersion, (4.29), (4.30) and (4.31), we get (4.25). \hfill \Box

Now we are ready to prove Theorem A.

**Theorem A** A unit vector field with closed totally geodesic integral curves defined in an open connected neighbourhood $V$ of $S^3$ is minimal if and only if its corresponding Gauss map generates a minimal surface in $S^2 \times S^2$.

**Proof** We follow the notation of the above lemma. Without loss of generality we assume that $\pi(V)$ is simply connected. Suppose at first that $\zeta$ is a minimal unit vector field. Then, $H_\zeta = 0$. Using (4.25), we obtain that $HG = 0$, i.e. the strictly length decreasing map $g$ is minimal. Conversely, let us assume that $g$ is a minimal map. Then,

$$dT(H_\zeta) = dG(\tau_\pi).$$

Since $(\xi_U)_\zeta = \zeta^{\text{hor}} = d\zeta(\zeta)$, we observe that the mean curvature $H_\zeta$ is perpendicular to the geodesic flow vector field, i.e. $H_\zeta \in \ker dT \perp$. Furthermore, $dT(H_\zeta)$ is normal to the graph of $g$.

For all vector fields $X$ on $V \subset S^3$ such that $\zeta^* g_S(X, \zeta) = 0$. Using (4.32), we have that $dT(H_\zeta) = 0$. Since $T$ is an isometry restricted to $\ker dT \perp$, we deduce that $H_\zeta = 0$, i.e. the vector field $\zeta$ is a minimal unit vector field. This completes the proof. \hfill \Box

As an immediate consequence of Theorem A and the Bernstein-type Theorem A in [32] we obtain the following uniqueness-type theorem.

**Theorem D** A minimal unit vector field $\zeta$ on $S^3$, whose integral curves are great circles, is a Hopf vector field and the corresponding quotient map $f : S^3 \to S^2$ is a Hopf fibration.

**Remark 4.6** Let us conclude the paper with some comments and final remarks.

(a) As we already have seen, great circle fibrations of $S^3$ can be identified with graphical surfaces of $S^2 \times S^2$. Torralbo and Urbano [34] established a local correspondence between (non-complex) minimal surfaces of $S^2 \times S^2$ and a certain pair of minimal surfaces of $S^3$.

(b) Let $f : S^3 \to S^2$ be a smooth map and $\lambda_1 = 0 \leq \lambda_2 \leq \lambda_3$ be its singular values. The 2-dilation $\text{Dil}_2(f) = \sup \lambda_2 \lambda_3$ of $f$ measures how much $f$ contracts the areas of compact surfaces in $S^3$. In [1], it is shown that if $\text{Dil}_2(f) < 2$ then $f$ is null-homotopic.

Since any submersion with totally geodesic fibres is homotopically non-trivial, it follows that the 2-dilation of any such map is greater or equal than 2.

(c) An interesting problem is to classify minimal maps $f : S^3 \to S^2$ with totally geodesic fibres. The minimal map $f$ becomes harmonic when we equip $S^3$ with the graphical metric $g = g_{S^3} + f^* g_{S^2}$, where $g_{S^3}$ is the metric of $S^3$ and $g_{S^2}$ is the metric of $S^2$. It seems that the methods developed in the present paper cannot be easily used, since in the structure equations (3.1) the curvature operator of the graphical metric shows up; for more information regarding this problem we refer to [27].
(d) In [31] it is shown that the graphical mean curvature flow (GMCF for short) will smoothly deform any strictly length decreasing map \( g : S^2 \to S^2 \) into a constant map. Moreover, this process will preserve the strict length decreasing property. Hence, GMCF will generate a smooth deformation of a unit vector field with totally geodesic fibres into a Hopf vector field.

(e) A similar problem in submanifold geometry is the classification of minimal \( m \)-dimensional submanifolds in \( S^n \) with index of relative nullity at least \( m - 2 \). Such submanifolds are foliated by \((m - 2)\)-dimensional totally geodesic spheres. Under the assumption of completeness, it turns out that any such submanifold is either totally geodesic or has dimension 3. In the latter case, there are plenty of examples, even compact ones. For more details, we refer to [8, 33].

Acknowledgements The authors would like to express their gratitude to the referee for the careful reading of the manuscript and for the valuable comments.

Funding Open access funding provided by HEAL-Link Greece.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Assimos, R., Savas-Halilaj, A., Smoczyk, K.: Graphical mean curvature flow with bounded bi-Ricci curvature. Calc. Var. Partial Differ. Equ. 62, 1 (2023)
2. Baird, P., Wood, J.: Harmonic morphisms between Riemannian manifolds, London Mathematical Society Monographs. New Series, p. 29. Oxford University Press, Oxford (2003)
3. Baird, P.: The Gauss map of a submersion, pp. 8–24. The Australian National University, Centre for Mathematical Analysis, Canberra AUS, Miniconference on Geometry and Partial Differential Equations (1986)
4. Blair, D.E.: Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Mathematics, p. 203. Birkhäuser Boston Ltd, Boston (2010)
5. Boeckx, E., Vanhecke, L.: Harmonic and minimal vector fields on tangent and unit tangent bundles. Differ. Geom. Appl. 13, 77–93 (2000)
6. Borrelli, V., Gil-Medrano, O.: A critical radius for unit Hopf vector fields on spheres. Math. Ann. 334, 731–751 (2006)
7. Brito, F.: Total bending of flows with mean curvature correction. Diff. Geom. Appl. 334, 157–163 (2000)
8. Dajczer, M., Kasioumis, Th., Savas-Halilaj, A., Vlachos, Th.: Complete minimal submanifolds with nullity in Euclidean spheres. Comment. Math. Helv. 93, 645–660 (2018)
9. Dombrowski, P.: On the geometry of the tangent bundle. J. Reine Angew. Math. 210, 73–88 (1962)
10. Eells, J.: Minimal graphs. Manuscr. Math. 28, 101–108 (1979)
11. Escoberas, R.: Riemannian submersions with totally geodesic fibers. J. Differ. Geom. 10, 253–276 (1975)
12. Gil-Medrano, O.: Volume minimising unit vector fields on three dimensional space forms of positive curvature. Calc. Var. Partial Differ. Equ. 61, 66 (2022)
13. Gil-Medrano, O., Hurtado, A.: Volume, energy and generalized energy of unit vector fields on Berger spheres: stability of Hopf vector fields. Proc. R. Soc. Edinb. Sect. A Math. 135, 789–813 (2005)
14. Gil-Medrano, O., Llinares-Fuster, E.: Second variation of volume and energy of vector fields. Stability of Hopf vector fields. Math. Ann. 320, 531–545 (2001)
15. Gil-Medrano, O., Llinares-Fuster, E.: Minimal unit vector fields. Tohoku Math. J. 54, 71–84 (2002)
16. Gluck, H.: Great circle fibrations and contact structures on the 3-sphere. Geom. Dedic. 72, 1–19 (2022)
17. Gluck, H., Gu, W.: Volume-preserving great circle flows on the 3-sphere. Geom. Dedic. 88, 259–282 (2001)
18. Gluck, H., Ziller, W.: On the volume of a unit vector field on the three-sphere. Comment. Math. Helv. 61, 177–192 (1986)
19. Gluck, H., Warner, F.: Great circle fibrations of the three-sphere. Duke Math. J. 50, 107–132 (1983)
20. Gluck, H., Warner, F., Ziller, W.: The geometry of the Hopf fibrations. L’Enseign. Math. 32, 173–198 (1986)
21. Han, D.-S., Yim, J.-W.: Unit vector fields on spheres, which are harmonic maps. Math. Z. 227, 83–92 (1998)
22. Harrison, M.: Skew and sphere fibrations arXiv:2203.16412, 1–33 (2022)
23. Heller, S.: Conformal fibrations of $S^3$ by circles, Harmonic maps and differential geometry, Contemp. Math., 542, Amer. Math. Soc., Providence, RI, pp 195–202 (2011)
24. Johnson, D.L.: Volumes of flows. Proc. Am. Math. Soc. 104, 923–932 (1988)
25. McKay, B.: The Blaschke conjecture and great circle fibrations of spheres. Am. J. Math. 126, 1155–1191 (2004)
26. McKay, B.: Summary of progress on the Blaschke conjecture, arXiv:1309.1326v5, pp. 1–23 (2016)
27. Markellos, M., Savas-Halilaj, A.: Rigidity of the Hopf fibration. Calc. Var. Partial Differ. Equ. 60, 171 (2021)
28. Pedersen, S.L.: Volumes of vector fields on spheres. Trans. Am. Math. Soc. 336, 69–78 (1993)
29. Sasaki, S.: On the differential geometry of tangent bundles of Riemannian manifolds. Tohoku Math. J. 10, 338–354 (1958)
30. Ruh, E.A., Vilms, J.: The tension field of the Gauss map. Trans. Am. Math. Soc. 149, 569–573 (1970)
31. Savas-Halilaj, A., Smoczyk, K.: Evolution of contractions by mean curvature flow. Math. Ann. 361, 725–740 (2015)
32. Savas-Halilaj, A., Smoczyk, K.: Bernstein theorems for length and area decreasing minimal maps. Calc. Var. Partial Differ. Equ. 50, 549–577 (2014)
33. Savas-Halilaj, A.: On deformable minimal hypersurfaces in space forms. J. Geom. Anal. 23, 1032–1057 (2013)
34. Torralbo, F., Urbano, F.: Minimal surfaces in $S^2 \times S^2$. J. Geom. Anal. 25, 1132–1156 (2015)
35. Wang, G.: $S^1$-invariant harmonic maps from $S^3$ to $S^2$. Bull. Lond. Math. Soc. 32, 729–735 (2000)
36. Wiegmink, G.: Total bending of vector fields on Riemannian manifolds. Math. Ann. 303, 325–344 (1995)
37. Wood, C.M.: On the energy of a unit vector field. Geom. Dedic. 64, 319–330 (1997)
38. Yampolsky, A.: On the mean curvature of a unit vector field. Publ. Math. 60, 131–155 (2002)
39. Yang, C.T.: Smooth great circle fibrations and an application to the topological Blaschke conjecture. Trans. Am. Math. Soc. 320, 504–524 (1990)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.