A Proof of the Isometric Embedding Theorem in Euclidean Space of Dimension Three.

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Abstract

A proof of the isometric embedding of a given two-metric in $E^3$ of low differentiability class. The method uses the theory of first order partial differential equations. The curvature of the metric plays no role in the proof.

Introduction.

Non-linear first order systems of partial differential equations giving the transformation of the components of a metric expressed in one parameter system to its expression in components in another parameter system play a vital role in the embedding problem. Although the systems are non-linear in the partial derivatives they are linear in the algebraic sense in the components of the metric. Thus we can make use of the consistency theorem of linear algebra, namely: a linear algebraic system is consistent (i.e. has a solution) if and only if the rank of the augmented matrix is equal to the rank of the coefficient matrix. In our case the elements of the coefficient matrix are the squares of the partial derivatives of the parameters of one parameter system with respect to parameters in a second parameter system. The consistency theorem thus gives rise to two linear partial differential equations of first order the existence of whose solutions is sufficient for the existence of solutions of the original non-linear PDE systems. The applications of this idea are given in two parts in this paper. In both parts the metric discussed is presented in orthogonal parameters.

In Part Two we give sufficient conditions on the components of any presented metric for it to be isometrically embedded as a surface in $E^3$. The
sufficient conditions are in the form of two equations involving the solutions of two linear PDEs of first order whose solutions are determined by respective initial conditions. The initial conditions may be any $C^1$ functions and are treated as unknown functions to be determined using Parts Two and Three.

In Part Three we present sufficient conditions on functions that they be components of a given metric presented $a\'priori$ in geodesic parameters. This part is purely intrinsic.

In Part Four we put the two parts together in two differential equations in which the initial conditions of Part Two play the role of unknown functions. The two differential equations in Part Four form a system of ordinary differential equations and so can be solved by appropriate initial conditions. Because the sufficiency condition of Part Two is satisfied by the system some isometric embedding exists. Because the sufficiency condition of Part Three is satisfied as well this embedding is an isometric embedding of the $a\'priori$ given metric.

Part Five presents a simple example illustrating the method.

1 Main theorem.

Any 2-metric can be locally isometrically embedded in $E^3$ in the form

$$X(u, v) = x(u, v)i + y(u, v)j + vk$$

where $u$ and $v$ are orthogonal parameters on the embedded surface and $i, j, k$ is an orthonormal basis of $E^3$.

We assume that the components of the metric are in differentiability class $C^1$. We refer to $u$ and $v$ as level parameters.

2 Preliminary. Isometric Embedding of a special metric.

Consider the system $S_0$ defined as

$$E = x_u^2 + y_u^2$$

$$0 = x_ux_v + y_uy_v$$

$$G - 1 = x_v^2 + y_v^2$$
with \( G > 1, E \) and \( G \) functions of \( u, v \).

Theorem 1. If \( S_0 \) is satisfied by \( C^1 \) functions \( x(u, v), y(u, v), E(u, v), G(u, v) \), then the surface defined above by \( X(u, v) \) is an isometric embedding of the metric
\[
\omega = E(u, v)du^2 + G(u, v)dv^2.
\]

Proof. The metric components induced by the ambient space \( E^3 \) on \( X \) are:
\[
X^2_u = x^2_u + y^2_u,
\]
\[
X_u \cdot X_v = x_u x_v + y_u y_v,
\]
\[
X^2_v = x^2_v + y^2_v + 1.
\]

Therefore \( E = X^2_u, F = X_u \cdot X_v = 0, G = X^2_v \). Thus \( X \) is, in fact, a regular surface since \( EG - F^2 > 0 \). Hence, \( X \) is an isometric embedding of the metric \( \omega \). QED theorem 1.

The system \( S_0 \) says that the Euclidean plane metric \( \omega_0 = dx^2 + dy^2 \) has components \( E(u, v), 0, G(u, v) - 1 \) in parameter system \( u, v \) and the functions \( x(u, v), y(u, v) \) give the transformation of parameters from the \( u, v \) parameters to the canonical \( x, y \) parameters of the plane.

Consider the inverse system \( S_{-1}^0 \) which says that the plane metric \( \omega_0 \) has components \( 1, 0, 1 \) in canonical parameters \( x, y \) and the functions \( u(x, y), v(x, y) \) give the transformation of parameters from the \( x, y \) parameters to the \( u, v \) parameters. \( S_{-1}^0 \) is the system
\[
1 = E(u, v)u^2_x + (G(u, v) - 1)v^2_x. \quad (1)
\]
\[
0 = E(u, v)u_x u_y + (G(u, v) - 1)v_x v_y. \quad (2)
\]
\[
1 = E(u, v)v^2_y + (G(u, v) - 1)v^2_y. \quad (3)
\]

\( S_0 \) has a solution \( x(u, v), y(u, v), E(u, v), G(u, v) \), if and only if \( S_{-1}^0 \) has a solution \( E(u, v), G(u, v), u(x, y), v(x, y) \). The functions \( x(u, v), y(u, v) \) give the transformation of parameters from the \( u, v \) parameters to the canonical \( x, y \) parameters of the plane. The functions \( u(x, y), v(x, y) \) give the inverse parameter transformation. We will work with \( S_{-1}^0 \) to show that it has, in fact, such solutions. We make use of the theory of first order partial differential equations. \[1\]

Lemma 1. The two linear first order PDEs
\[
u_x - u_y = 0, \quad (4)
\]
\[ v_x + v_y = 0. \] \hspace{1cm} (5)

have Cauchy problem solutions

\[ u = a(x + y), \] \hspace{1cm} (6)

\[ v = b(x - y) \] \hspace{1cm} (7)

with initial conditions respectively on \( y = 0 : u = a(x) \) and \( v = b(x) \). \( a \) and \( b \) are arbitrary \( C^1 \) functions chosen to have non-zero first derivatives. The proof is by inspection. QED Lemma 1.

The two solutions provide a one-to-one parameter transformation from the \( xy \) plane to the \( uv \) plane whose Jacobian evaluated on \( y = 0 \) has the value

\[ J(x, 0) = -2a'(x)b'(x), \]

where the prime indicates the first derivative. The parameter transformation is thus locally invertible in a neighborhood of the initial curve.

Noting that \( x \) and \( y \) are functions of \( u \) and \( v \), we define

\[ E(u, v) = \frac{1}{2}(a'(x + y))^{-2}. \] \hspace{1cm} (8)

\[ G(u, v) - 1 = \frac{1}{2}(b'(x - y))^{-2}. \] \hspace{1cm} (9)

The data space of functions \( a(x), b(x) \) is thus mapped to the space of functions \( E(u, v), G(u, v) \).

Claim. The system \( S_0^{-1} \) is satisfied by the functions \( E(u, v), G(u, v), u(x, y), v(x, y) \).

Proof of Claim. By substitution. See subsection 2.1 below on calculation.

QED Claim.

Therefore \( S_0 \) has a solution \( x(u, v), y(u, v), E(u, v), G(u, v) \).

By Theorem 1

\[ X(u, v) = x(u, v)i + y(u, v)j + vk \]

is an isometric embedding in \( E^3 \) of the metric defined by

\[ \omega = Edu^2 + Gdv^2. \]
2 Preliminary. Isometric Embedding of a special metric.

2.1 Calculation.

Substitute from the definitions of $E$, $G$ into the right sides of the equations of systems $S_{-1}$ using the solutions $(6, 7)$ of the PDEs to show that the equations are satisfied.

\[ \frac{1}{2}(a'(x + y))^{-2}u_x^2 + (G(u, v) - 1)v_x^2 \]

\[ = \frac{1}{2}(a'(x + y))^{-2}(a'(x + y))^2 + \frac{1}{2}(b'(x - y)^2)(b'(x - y)^2) = 1. \]

Therefore the first equation of $S_{-1}$ is satisfied.

\[ \frac{1}{2}(a'(x + y))^{-2}a'(x + y)a'(x + y) + \frac{1}{2}(b'(x - y)^2)b'(x - y)b'(x - y)(-1) = 0. \]

Therefore the second equation of $S_{-1}$ is satisfied.

\[ \frac{1}{2}(a'(x + y))^{-2}(a'(x + y))^2 + \frac{1}{2}(b'(x - y)^2)(b'(x - y)(-1))^2 = 1. \]

Therefore the third equation of $S_{-1}$ is satisfied. QED Claim.

The parameter transformation $(6), (7)$ has non-zero Jacobian in a neighborhood of the initial curve and thus can be inverted there. Denote the inverse parameter transformation by

\[ x = H(u, v), y = H^*(u, v). \]

Thus by Theorem 1, we have proved

Theorem 2.

\[ X(u, v) = H(u, v)i + H^*(u, v)j + vk \]

is an isometric embedding in $E^3$ of the metric

\[ \omega = Edu^2 + Gdv^2 \]

where $E$ and $G$ are defined above depending on the choice of initial conditions $a$ and $b$. (That is, $E$ and $G$ are not given functions a priori.)
3 Transformation of components of a given metric expressed in geodesic parameters.

We assume that the components of the given metric are in class $C^1$. We define the given metric by

$$\omega = d\hat{u}^2 + \hat{G}(\hat{u}, \hat{v})d\hat{v}^2.$$  

Transform the metric to a system of orthogonal parameters $u, v$ in which the components of $\omega$ are $R(u, v), 0, S(u, v)$:

$$\omega = Rd\hat{u}^2 + Sd\hat{v}^2.$$  

Then the components in the two systems are related by a system $\hat{S}$ defined as:

$$1 = Ru^2 + Sv^2.$$  

$$0 = Ru_\hat{u}u_\hat{v} + Sv_\hat{u}v_\hat{v}.$$  

$$\hat{G}(\hat{u}, \hat{v}) = Ru^2 + Sv^2.$$  

The system is algebraically linear in the unknowns $R, S$ with the squared derivatives considered as coefficients to be determined independently below (10, 11). Therefore a necessary and sufficient condition for there to be an algebraic solution for $R, S$ is that the rank of the augmented matrix equals the rank of the coefficient matrix. The augmented matrix is

$$\begin{bmatrix}
  u_\hat{u}^2 & v_\hat{u}^2 & 1 \\
  u_\hat{u}u_\hat{v} & v_\hat{u}v_\hat{v} & 0 \\
  u_\hat{v}^2 & v_\hat{v}^2 & \hat{G}(\hat{u}, \hat{v})
\end{bmatrix}.$$  

The determinant of the augmented matrix is

$$\begin{vmatrix}
  u_\hat{u}^2 & v_\hat{u}^2 & 1 \\
  u_\hat{u}u_\hat{v} & v_\hat{u}v_\hat{v} & 0 \\
  u_\hat{v}^2 & v_\hat{v}^2 & \hat{G}(\hat{u}, \hat{v})
\end{vmatrix} = u_\hat{v}v_\hat{v} \begin{vmatrix}
  u_\hat{u} & v_\hat{u} & \hat{G}(\hat{u}, \hat{v}) \\
  u_\hat{u} & v_\hat{u} & u_\hat{v} & v_\hat{v} \\
  u_\hat{u} & v_\hat{u} & u_\hat{v} & v_\hat{v}
\end{vmatrix} = \left(u_\hat{v}v_\hat{v} + \hat{G}(\hat{u}, \hat{v})u_\hat{u}v_\hat{u} \right) \begin{vmatrix}
  u_\hat{u} & v_\hat{u} & \hat{G}(\hat{u}, \hat{v}) \\
  u_\hat{u} & v_\hat{u} & u_\hat{v} & v_\hat{v} \\
  u_\hat{u} & v_\hat{u} & u_\hat{v} & v_\hat{v}
\end{vmatrix} = 0.$$
We will insure (see Claim below) that the determinant is not zero. Thus the consistency condition is
\[
\left(u_{\hat{v}}v_{\hat{v}} + \hat{G}(\hat{u}, \hat{v})u_{\hat{u}}v_{\hat{u}}\right) = 0.
\]
The partial derivatives are to be determined as solutions of Cauchy problems for the first order linear PDEs
\[
u_{\hat{u}} - u_{\hat{v}} = 0, \quad \hat{G}(\hat{u}, \hat{v})v_{\hat{u}} + v_{\hat{v}} = 0.
\]
Solutions of these two PDEs satisfy the consistency condition as can be seen by substitution. Note, however, that they are not necessary for consistency: the identity transformation satisfies the system \(S\) with \(R = S = 1\) but does not satisfy the PDEs.

Solutions of Cauchy problems for these PDEs provide a solution of the system \(S\) for \(R\) and \(S\) by algebraic methods: Substitute the solutions into \(S\) and solve by Cramer’s Rule for \(R\) and \(S\). Since the system is consistent we need only two of the equations. We choose the first two.

\[
1 = Ru_{\hat{u}}^2 + Sv_{\hat{v}}^2.
\]
\[
0 = Ru_{\hat{u}}u_{\hat{v}} + Sv_{\hat{u}}v_{\hat{v}}.
\]
Substitute from the first order PDEs:
\[
1 = Ru_{\hat{v}}^2 + S\left(\frac{v_{\hat{v}}^2}{G^2}\right).
\]
\[
0 = Ru_{\hat{v}}^2 - S\frac{v_{\hat{v}}^2}{G}.
\]
Solve by Cramer’s rule:
\[
R(u, v) = u_{\hat{v}}^{-2} \frac{\hat{G}(\hat{u}, \hat{v})}{\hat{G}(\hat{u}, \hat{v}) + 1}.
\]
\[
S(u, v) = v_{\hat{v}}^{-2} \frac{\hat{G}^2(\hat{u}, \hat{v})}{\hat{G}(\hat{u}, \hat{v}) + 1}.
\]
The theory of first order PDEs is explained clearly in [1]. The theory is particularly simple for the linear PDEs above. We solve initial value problems for these. A general solution of the initial value problem for the first PDE (10) above is worked out in the first problem in John’s book, p. 15. With appropriate change of notation the solution is

$$u = h(\hat{u} + \hat{v})$$

with initial conditions on $\hat{v} = 0 : u = h(\hat{u})$. We stipulate that the initial value satisfies $h_{\hat{u}}(\hat{u}) \equiv h'(\hat{u}) \neq 0$.

A general solution of the second (11) is:

$$v = \hat{h}(\hat{u}, \hat{v})$$

with chosen initial value $\hat{h}(\hat{u}) = \hat{h}(\hat{u}, 0)$. We also choose the initial value to satisfy that $v_{\hat{v}}(\hat{u}, 0)$ is not zero.

Thus, for all $\hat{u}$, $\hat{v}$

$$u_{\hat{v}} = h'(\hat{u} + \hat{v}), v_{\hat{v}} = \hat{h}_{\hat{v}}(\hat{u}, \hat{v}).$$

Claim. The Jacobian of the transformation of parameters given by the solutions of the two PDEs is not zero at the initial curve (and hence in a neighborhood of the initial curve).

Proof of Claim. : The IC implies $\hat{h}_{\hat{v}}$ is not zero (see 11).

$$J = \begin{vmatrix} u_{\hat{v}} & u_{\hat{u}} \\ v_{\hat{v}} & v_{\hat{u}} \end{vmatrix} = \begin{vmatrix} h'(\hat{u}) & h'(\hat{u}) \\ \hat{h}_{\hat{u}} & \hat{h}_{\hat{v}} \end{vmatrix} = h'(\hat{u})(\hat{h}_{\hat{u}} - \hat{h}_{\hat{v}}) = h'(\hat{u})(v_{\hat{v}} - v_{\hat{u}}) = h'(\hat{u})[-\hat{G}(\hat{u}, \hat{v})v_{\hat{u}} - v_{\hat{u}}].$$

Thus $J = h'(\hat{u})[-v_{\hat{u}}(\hat{G} + 1)] \neq 0$.

QED Claim.

Hence, using the solutions,

$$R(u, v) = [u_{\hat{v}}]^{-2} \frac{\hat{G}}{\hat{G} + 1}.$$

$$S(u, v) = [\hat{h}_{\hat{v}}(\hat{u}, \hat{v})]^{-2} \frac{\hat{G}^2}{\hat{G} + 1}.$$

Thus $R(u, v)$ is known because it is determined by the given metric and the initial conditions. The value of $S(u, v)$ depends on the choice of the IC on $\hat{v}=0$, that is, on the choice of $\hat{h}(\hat{u}, 0)$ such that $\hat{h}_{\hat{v}}(\hat{u}, 0)$ is not zero.
Note that $R$ and $S$ are given here as functions of $\hat{u}$ and $\hat{v}$. We want them as functions of $u$ and $v$. By what we have just shown, the transformation of parameters is locally one-to-one and has a $C^1$ inverse transformation giving $\hat{u}$, $\hat{v}$ in terms of $u$, $v$.

Represent the inverse by:

$$\hat{u} = f(u,v), \tag{14}$$
$$\hat{v} = g(u,v) \tag{15}$$

and substitute into the expressions for $R$ and $S$:

$$R(u,v) = [u_\theta]^2 \frac{\hat{G}(f(u,v), g(u,v))}{\hat{G}(f(u,v), g(u,v)) + 1}.$$  

$$S(u,v) = [\hat{h}(f(u,v), g(u,v))]^2 \frac{\hat{G}(f(u,v), g(u,v))}{\hat{G}(f(u,v), g(u,v)) + 1}.$$  

Thus, $R(u,v)$ and $S(u,v)$ are determined by the choice of the initial conditions $\hat{h}(\hat{u}) = \hat{h}(\hat{u}, 0)$ and $h(\hat{u}) = h(\hat{u} + 0)$.

Thus we have proved

Theorem 3. $R(u,v)$ and $S(u,v)$ are components of $\omega$.

Remark. We could have simplified the argument a bit by choosing the initial condition for (10) such that the solution is $u = \hat{u} + \hat{v}$ but that would have precluded the change of metric components to other geodesic parameters (e.g., it would produce a contradiction when $R = 1$). Since this has no consequences for our embedding proof but simplifies the notation, in what follows we will assume that the solution is $u = \hat{u} + \hat{v}$.

4 Proof of the embedding theorem.

We return to Lemma 1, section 2.

The parametric form of the solutions of the PDEs in Lemma 1 is as follows.

For the first PDE (4) we choose parameters $s = x + y$, $t = y$.

A parametric solution of the first PDE is (with John’s notation $z$ replaced by $u$ in p. 15, example 1, [1]):

$$x = s - t,$$
4 Proof of the embedding theorem.

\[ y = t, \]
\[ u = a(s). \]

The initial curve, on which \( t = 0 \), is
\[ x = s, \]
\[ y = 0, \]
\[ u = a(s). \]

For the second PDE (5) we choose parameters \( \sigma = x - y, \tau = y \). This is possible because the two PDEs are independent.
\[ x = \sigma + \tau, \]
\[ y = \tau, \]
\[ v = b(\sigma). \]

The initial curve for the second PDE, on which \( \tau = 0 \), is
\[ x = \sigma, \]
\[ y = 0, \]
\[ v = b(\sigma). \]

(With \( z \) replaced by \( v \) in [1].)

Note that the characteristic curves of the PDEs satisfy respectively \( \frac{du}{d\tau} = 0, \frac{dv}{d\tau} = 0 \). Compare [1], p.14. Also, note that the solution functions have the same form as the initial conditions. Therefore we have

Lemma 2. In results making use of solutions of the PDEs we may assume, without loss of generality, that \( y = 0, x = s = \sigma, \) and \( t = \tau = 0 \).

Consider the ODE system
\[ \frac{1}{2}(a'(s))^2 = \frac{\hat{G}(f(a(s), b(s)), g(a(s), b(s)))}{\hat{G}(f(a(s), b(s)), g(a(s), b(s))) + 1}. \]

\[ \frac{1}{2}(b'(s))^2 + 1 = \left[ \hat{h}_\nu(f(a(s), b(s)), g(a(s), b(s))) \right]^2 \frac{[\hat{G}(f(a(s), b(s)), g(a(s), b(s)))]^2}{\hat{G}(f(a(s), b(s)), g(a(s), b(s))) + 1}. \]
Put in standard form $a' = \ldots$, $b' = \ldots$ choosing the plus signs for the square roots, e.g. For given initial values of $a$ and $b$ at a point there exists a solution $a(s), b(s)$ of the system in a neighborhood of the initial point satisfying the initial conditions. [2]

We choose the initial point to be $s = 0$ and the initial values of the unknown functions to be $a(0) = 0$, $b(0) = 0$. Thus a solution of exists satisfying these initial conditions.

Claim. The standard forms are real. $a'$ is real by inspection. For $b'$ we must show that the right side of the second equation is greater than one at the initial point. This is equivalent to showing that

$$\hat{h} \hat{v}(\hat{u}, 0) = \hat{v} \hat{u}(\hat{u}, 0) = \frac{1}{2} \hat{v}^2 \hat{u}(\hat{u}, 0).$$

By choosing the initial value $\hat{h}(\hat{u}, 0)$ such that $\hat{h} \hat{u}(\hat{u}, 0)$ is small but not zero in absolute value, the inequality will hold. QED Claim.

To be definite we choose the plus signs for the square roots.

Theorem 2, Section 2 (2.1) gives a sufficient condition for the embedding of a metric $\omega = Edu^2 + Gdv^2$ where

$$E(u, v) = \frac{1}{2}(a'(x + y))^{-2},$$

$$G(u, v) - 1 = \frac{1}{2}(b'(x - y))^{-2}.$$

In Theorem 2, $a(x)$ and $b(x)$ are arbitrary $C^1$ initial data for two PDEs of first order. Therefore we may choose the initial data $a(x), b(x)$ to be the solution functions of the initial value problem for the ODE system. Therefore, by Theorem 2, $\omega$ thus defined is isometrically embeddable. It remains to show that $\omega$ is the given metric, i.e., we

Claim: $E = R$ and $G = S$. That is,

$$\frac{1}{2}(a'(x + y))^{-2} = u_0^{-2} \frac{\hat{G}(f(u, v), g(u, v))}{G(f(u, v), g(u, v)) + 1}. \hspace{1cm} (16)$$

$$\frac{1}{2}(b'(x - y))^{-2} = \hat{h}^{-2}(f(u, v), g(u, v)) \frac{[\hat{G}(f(u, v), g(u, v))]}{G(f(u, v), g(u, v)) + 1} - 1. \hspace{1cm} (17)$$
Recall that functions $f$ and $g$ were defined in Section 3 depending on solutions of initial value problems for PDEs (10, 11).

Now apply the parametric forms above of the solutions in Lemma 1, Section 2. From the solutions to the PDEs of Lemma 1 in parametric form we obtain for the left side of (16)

$$\frac{1}{2}(a'(s))^{-2}.$$ However, by the solution of the ODE system above using our choice of initial condition for the first PDE (10) (i.e., $u = \hat{u} + \hat{v}$, cf. Remark Section 3) the left side is equal to

$$\frac{\hat{G}(f(a(s), b(s)), g(a(s), b(s)))}{G(f(a(s), b(s)), g(a(s), b(s)) + 1}.$$ Again using the parametric solutions of the PDEs of Lemma 1, i.e., $u = a(s)$, $v = b(s)$, we obtain for the left side of equation (16):

$$\frac{\hat{G}(f(u, v), g(u, v))}{G(f(u, v), g(u, v) + 1}.$$ Which is the right side of (16). QED (16).

Now apply the parametric forms of the PDEs in Lemma 1 to (17): For the left side of (17) we obtain

$$\frac{1}{2}(b'(\sigma)^{-2}.$$ In the ODE system the parameter $s$ may be replaced by $\sigma$. The left side of (17) yields

$$\hat{h}^{-2}(f(a(\sigma), b(\sigma)), g(a(\sigma), b(\sigma))) \frac{[\hat{G}(f(a(\sigma), b(\sigma)), g(a(\sigma), b(\sigma)))]^2}{G(f(a(\sigma), b(\sigma)), g(a(\sigma), b(\sigma)) + 1} - 1.$$ Since the equations for the solutions of the PDEs are the same as the equations for the initial conditions we may use the conditions given by $y = 0$. This yields

$$x = s = \sigma.$$
That is, on the initial plane $x$ is the parameter. Evaluate the preceding expression on $y = 0$:

$$
\hat{h}_{\hat{v}}^{-2}(f(a(s), b(s)), g(a(s), b(s))) \frac{[\hat{G}(f(a(s), b(s)), g(a(s), b(s)))]^2}{\hat{G}(f(a(s), b(s)), g(a(s), b(s))) + 1} - 1.
$$

Again using the parametric solutions of the PDEs we obtain for the left side

$$
\hat{h}_{\hat{v}}^{-2}(f(u, v), g(u, v)) \frac{[\hat{G}(f(u, v), g(u, v))]^2}{\hat{G}(f(u, v), g(u, v)) + 1} - 1.
$$

which is the right side of (17). QED claim.

This concludes the proof of the isometric embedding theorem.

5 An example to illustrate the method.

Given the metric

$$
\omega = d\hat{u}^2 + d\hat{v}^2,
$$

i.e., the Euclidean/Riemannian metric, with $\hat{G} = 1$ (cf. Section 3). The first order PDEs become

\begin{align*}
  u_{\hat{u}} - u_{\hat{v}} &= 0, \\
  v_{\hat{u}} + v_{\hat{v}} &= 0.
\end{align*}

(18) (19)

A solution of (18) is

$$
u = \hat{u} + \hat{v} \quad \text{with initial value } u = \hat{u} \text{ at } \hat{v} = 0.
$$

A solution of (19) is

$$
v = \epsilon(\hat{u} - \hat{v}) \quad \text{with initial value } v = \epsilon\hat{u} \text{ at } \hat{v} = 0.
$$

These two solutions give a transformation of parameters from the given parameter system $\hat{u}, \hat{v}$ to a new parameter system $u, v$ for $\omega$. The Jacobian of the transformation has the value $J = -2\epsilon$. We choose $\epsilon \neq 0$. Then the inverse parameter transformation exists locally, is one-to-one and is in differentiability class $C^1$. The inverse transformation can be found explicitly; for most examples the explicit inverse cannot be found, we only know that it exists. The inverse is

$$
\hat{u} = \frac{u + v}{2\epsilon}, \quad \hat{v} = \frac{u - v}{2\epsilon}.
$$
By definitions (14,15)

\[ f(u, v) = \frac{u + v}{2\epsilon}, \quad g(u, v) = \frac{u - v}{2\epsilon}. \]

Now apply this to the ODE system using \( u = a(s), \ v = b(s), \ \hat{G} = 1 : \)

\[ \frac{1}{2}(a'(s))^2 = \frac{1}{2}. \]

\[ \frac{1}{2}(b'(s))^{-2} + 1 = \left[ \hat{h}_\theta(f(a(s), b(s)), g(a(s), b(s))) \right]^{-1}. \]

Or

\[ [a'(s)]^2 = 1. \]

\[ [b'(s)]^{-2} + 2 = [\hat{h}_\theta(\frac{a(s) + b(s)}{2\epsilon}, a(s) - b(s))^{-2}. \]

Put the system in the standard form \( a' = ..., \ b' = ... \) choosing the plus sign for the square roots for uniformity. Then

\[ a'(s) = 1. \]

\[ b'(s) = \left( [\hat{h}_\theta(\frac{a(s) + b(s)}{2\epsilon}, a(s) - b(s))]^{-2} - 2 \right)^{-\frac{1}{2}}. \]

A solution of the first equation is \( a(s) = s. \) Substitute into the second equation:

\[ b'(s) = \left( [\hat{h}_\theta(\frac{s + b(s)}{2\epsilon}, \frac{s - b(s)}{2\epsilon})]^{-2} - 2 \right)^{-\frac{1}{2}}. \]

Solve the initial value problem for the second equation with initial point \( s = 0, \) initial value \( b(0) = 0. \)

The function \( \hat{h}_\theta \) depends on the choice of the initial condition \( \hat{h}(\hat{u}, 0) \) for the PDE \( v_\hat{u} + v_\hat{v} = 0 \) for which a solution is

\[ \hat{h}(\hat{u}, \hat{v}) = \epsilon(\hat{u} - \hat{v}). \]

Therefore, for all \( \hat{u}, \hat{v}, \)

\[ \hat{h}_\theta(\hat{u}, \hat{v}) = -\epsilon. \]
Thus the second equation becomes

\[ b'(s) = \left( [-(\epsilon)]^{-2} - 2 \right)^{\frac{1}{2}} = (\epsilon^{-2} - 2)^{-\frac{1}{2}} \]

which is real if \(|\epsilon|\) is small. Thus a solution of the ODE system is

\[ a(s) = s, \]
\[ b(s) = (\epsilon^{-2} - 2)^{-\frac{1}{2}} s. \]

By the parametric solutions of the PDEs at the beginning of the preceding section 4,

\[ x + y = s, \]
\[ x - y = \sigma, \]
\[ u = s, \]
\[ v = (\epsilon^{-2} - 2)^{-\frac{1}{2}} \sigma. \]

Eliminating \(s\) and \(\sigma\) we find a relation between the \(x, y\) and \(u, v\) parameters:

\[ u = x + y, \]
\[ v = (\epsilon^{-2} - 2)^{-\frac{1}{2}} (x - y). \]

The inverse transformation is

\[ x = H(u, v) = \frac{u((\epsilon^{-2} - 2)^{-\frac{1}{2}} + v}{2(\epsilon^{-2} - 2)^{-\frac{1}{2}}}, \]
\[ y = H^*(u, v) = \frac{v - u(\epsilon^{-2} - 2)^{-\frac{1}{2}}}{-2(\epsilon^{-2} - 2)^{-\frac{1}{2}}}. \]

By Theorem 2, the isometric embedding of \(\omega\) in level parameters \(u, v\) is

\[ X(u, v) = \frac{u((\epsilon^{-2} - 2)^{-\frac{1}{2}} + v}{2(\epsilon^{-2} - 2)^{-\frac{1}{2}}} i + \frac{v - u(\epsilon^{-2} - 2)^{-\frac{1}{2}}}{-2(\epsilon^{-2} - 2)^{-\frac{1}{2}}} j + v k. \]

Referring now to section 2 we find

\[ E(u, v) = \frac{1}{2} (a'(s))^{-2} = \frac{1}{2}. \]
5 An example to illustrate the method.

\[ G(u, v) = 1 + \frac{1}{2}(b'(\sigma))^{-2} = 1 + \frac{1}{2}((\epsilon^{-2} - 2)^{-\frac{1}{2}})^{-2} = 1 + \frac{1}{2}(\epsilon^{-2} - 2). \]

This may be confirmed by direct calculation from our formula for \( X(u, v) \).

The metric in level parameters is

\[ \omega = \frac{1}{2} du^2 + \frac{1}{2} [\epsilon^{-2}] dv^2. \]

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