Abstract. In this paper we tried to condense the determinant of $n$ square matrix to the determinant of $(n - 1)$ square matrix with the mathematical proof.

Key words. Matrix, Condensation, Determinants.

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1. Introduction. We can write the well-known algorithm of Dodgson, concerning the $n$ square matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ as follows:

$$
\det (a_{i,j})_{1 \leq i,j \leq n} \det (a_{i,j})_{i \neq k, l \neq k, l} = \det (a_{i,j})_{i \neq l, j \neq k} \det (a_{i,j})_{i \neq l, j \neq k},
$$

for all $k, l = 1, ..., n$ considering $k < l$ (see S.Kouachi, S.Abdelmalek and B.Rebai [2]).

This formula enables us condense the determinant of $n$ square matrix to the determinant of 2 square matrix. The elements of 2 square matrix are the determinants of $(n - 1)$ square matrix.

In the same way we try to create a formula that enables us condense the determinant of $n$ square matrix to the determinant of $(n - 1)$ square matrix. The elements of $(n - 1)$ square matrix are the determinants of 2 square matrix.

For example $n = 7$

$$
\begin{vmatrix}
2 & 5 & 4 & 7 & 6 & 1 & 2 \\
0 & 1 & 3 & 8 & 8 & 1 & 5 \\
9 & 4 & 7 & 8 & 9 & 8 & 6 \\
7 & 8 & 4 & \sqrt{3} & 2 & 0 & 8 \\
11 & 2 & 5 & 4 & 5 & \frac{1}{2} & 5 \\
5 & 7 & 8 & 6 & 1 & 0 & 5 \\
9 & 2 & 3 & 5 & 8 & 5 & 3
\end{vmatrix}
$$

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We denote by $\det (A)$. We need a lemma.

**Lemma 2.1.** We need a lemma.

**2. RESULTS.** We need some notations:

For this purpose, we need some notations:

The $(n - k) \times (n - l)$ matrix obtained from $A$ by removing the $i_1^{th}$, $i_2^{th}$, $i_3^{th}$ rows and the $j_1^{th}$, $j_2^{th}$, $j_3^{th}$ columns is denoted by $(a_{i,j})_{i \neq i_1, i_2, \ldots, i_n \neq j_1, j_2, \ldots, j_k}$.

We denote by $\det_{\alpha \leq i,j \leq \beta} (a_{i,j})_{i \neq i_1, i_2, \ldots, i_k \neq j_1, j_2, \ldots, j_k}$ the $k \times \beta - \alpha$ to the determinant of the $(\beta - \alpha - k + 1)$ square matrix obtained from $A$ by removing the $(\alpha - 1)$ first rows and columns, by removing the $(n - \beta)$ last rows and columns and by removing $i_1^{th}$, $i_2^{th}$, $i_3^{th}$ rows and the $j_1^{th}$, $j_2^{th}$, $j_3^{th}$ columns.

2. RESULTS. We need a lemma.

**Lemma 2.1.** If $a_{11} = 0$, then we get the following formula:

$$\det_{1 \leq i,j \leq n-1} \left[ \begin{array}{cc} a_{1,1} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{array} \right] = 0.$$  \hspace{1cm} (2.1)

**Proof.** To prove the formula (2.1), we let $a_{11} = 0$. Thus, the first term of this formula will be as follows:

$$\det_{1 \leq i,j \leq n-1} \left[ \begin{array}{cc} 0 & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{array} \right] = \det_{1 \leq i,j \leq n-1} \left[ (-a_{1,j+1}a_{i+1,1}) \right] =$$
One of the main results of the paper is the following:

**Theorem 2.2.** Let the \( n \) square matrix \( A = (a_{i,j})_{1 \leq i,j \leq n} \). For all \( n > 2 \), the following formula is realised

\[
(a_{1,1})^{n-2} \det_{1 \leq i,j \leq n} \begin{bmatrix} (a_{i,j})_{1 \leq i,j \leq n} \end{bmatrix} = \det_{1 \leq i,j \leq n-1} \begin{bmatrix} a_{1,1} & a_{1,j+1} & 0 & \cdots & 0 \\ a_{1+1,1} & a_{1,j+1+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-1,j+1} & a_{n,j} & \cdots & a_{n-1,j+1} & 0 \\ a_{n,j+1} & a_{n,j} & \cdots & a_{n-1,j+1} & 0 \end{bmatrix}. \tag{2.2}
\]

We notice that this formula enables us to condense the determinant of \( n \) square matrix to the determinant of \( (n-1) \) square matrix. The elements of \( (n-1) \) square matrix are the determinants of \( 2 \) square matrix.

**Proof.** To prove formula (2.2) there are two cases:

The first case when \( a_{1,1} = 0 \), the proof of formula (2.2) is the same as the one of lemma (2.1).

The second case when \( a_{1,1} \neq 0 \), we prove the formula (2.2) inductively.

For \( n = 3 \), we find that the proof of the formula (2.2) is evident.

For \( n = 4 \), we find that the proof of the formula (2.2) is evident with simple calculations.

When \( n > 4 \), we suppose the formula (2.2) is correct for \( (n-1) \) and we prove it for \( n \).

To prove the case of \( (n > 4) \) we use formula (1.1), and to choose \( \begin{bmatrix} (a_{i,j})_{i\neq k, l} \end{bmatrix} \neq 0 \) with \( k > 1 \) (this choice is possible).

We assume without loss of generality that \( \begin{bmatrix} (a_{i,j})_{i\neq n-1, n} \\ j\neq n-1, n \end{bmatrix} \neq 0 \), and this means that \( k = n-1, l = n \), thus formula (1.1) will be as follows:

\[
\det_{1 \leq i,j \leq n} \begin{bmatrix} (a_{i,j})_{1 \leq i,j \leq n} \end{bmatrix} \det_{i \neq n-1, n, j \neq n-1, n} \begin{bmatrix} (a_{i,j})_{i\neq n-1, n} \\ j\neq n-1, n \end{bmatrix} = \det_{i \neq n, j \neq n-1} \begin{bmatrix} (a_{i,j})_{i\neq n} \\ j\neq n-1 \end{bmatrix} \det_{i \neq n, j \neq n-1, n} \begin{bmatrix} (a_{i,j})_{i\neq n} \\ j\neq n \end{bmatrix} \begin{bmatrix} (a_{i,j})_{i\neq n-1} \\ j\neq n-1 \end{bmatrix} \tag{2.3}
\]
We apply the formula (2.2) for \((n - 1)\) on the second side of formula (2.3), so we get:

\[
(a_{11})^{n-3} \det \left( [a_{i,j}]_{i\neq n} \right) = \det_{1 \leq i,j \leq n-2} \left[ \det \left[ \begin{array}{ccc} a_{1,1} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{array} \right] \right] \quad (2.4)
\]

\[
(a_{11})^{n-3} \det \left( [a_{i,j}]_{i\neq n} \right) = \det_{1 \leq i,j \leq n-1} \left[ \det \left[ \begin{array}{ccc} a_{1,1} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{array} \right] \right]_{i\neq n-2}^{j\neq n-2} \quad (2.5)
\]

\[
(a_{11})^{n-3} \det \left( [a_{i,j}]_{i\neq n} \right) = \det_{1 \leq i,j \leq n-1} \left[ \det \left[ \begin{array}{ccc} a_{1,1} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{array} \right] \right]_{i\neq n-1}^{j\neq n-2} \quad (2.6)
\]

\[
(a_{11})^{n-3} \det \left( [a_{i,j}]_{i\neq n} \right) = \det_{1 \leq i,j \leq n-1} \left[ \det \left[ \begin{array}{ccc} a_{1,1} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{array} \right] \right]_{i\neq n-2}^{j\neq n-1} \quad (2.7)
\]

By using the formulas (2.4)-(2.7), the formula (2.3) will be as follows:

\[
\left( (a_{11})^{n-3} \right)^2 \det \left( [a_{i,j}]_{1 \leq i,j \leq n} \right) \det \left( [a_{i,j}]_{i\neq n-1,n} \right) = \\
\det_{1 \leq i,j \leq n-2} \left[ \det \left( [a_{i,j}]_{i\neq n-1,n} \right) \right] \det_{1 \leq i,j \leq n-1} \left[ \det \left( [a_{i,j}]_{i\neq n-1,n} \right) \right]_{i\neq n-2}^{j\neq n-2}. 
\]

To simplify the above formula, we put this notation \(d_{i,j} = \frac{a_{1,1}}{a_{i+1,1}} a_{1,j+1} a_{i+1,j+1} \), thus it will be as follows:

\[
\left( (a_{11})^{n-3} \right)^2 \det \left( [a_{i,j}]_{1 \leq i,j \leq n} \right) \det \left( [a_{i,j}]_{i\neq n-1,n} \right) = \\
\det \left[ \det_{1 \leq i,j \leq n-2} \left( [d_{i,j}] \right) \right] \det_{1 \leq i,j \leq n-1} \left[ \det \left( [d_{i,j}]_{i\neq n-1,n} \right) \right]_{i\neq n-2}^{j\neq n-2}. 
\]

We can write it as follows:

\[
\left( (a_{11})^{n-3} \right)^2 \det \left( [a_{i,j}]_{1 \leq i,j \leq n} \right) \det \left( [a_{i,j}]_{i\neq n-1,n} \right) = \\
= \det \left[ \det_{1 \leq i,j \leq n-1} \left( [d_{i,j}]_{i\neq n-1,n} \right) \right] \det_{1 \leq i,j \leq n-1} \left[ \det \left( [d_{i,j}]_{i\neq n-1,n} \right) \right]_{i\neq n-2}^{j\neq n-2}. 
\]
But by applying formula (2.3) for \((n - 1)\) on \((d_{i,j})_{1 \leq i, j \leq n-1}\) we get:

\[
\begin{align*}
\text{det} \left[ (d_{i,j})_{1 \leq i, j \leq n-3} \right] \text{det} \left[ (d_{i,j})_{1 \leq i, j \leq n-1} \right] &= \text{det} \left[ (d_{i,j})_{1 \leq i, j \leq n-1} \right] \\
\text{det} \left[ (d_{i,j})_{i \neq n-1 \atop j \neq n-1} \right] &= \text{det} \left[ (d_{i,j})_{i \neq n-1 \atop j \neq n-2} \right].
\end{align*}
\]

Thus formula (2.8) will be as follows:

\[
\left\{ (a_{11})^{n-3} \right\}^2 \text{det} \left[ (a_{i,j})_{1 \leq i, j \leq n} \right] \text{det} \left[ (a_{i,j})_{i \neq n-1, n \atop j \neq n-1, n} \right] = \text{det} \left[ (d_{i,j})_{1 \leq i, j \leq n-3} \right] \text{det} \left[ (d_{i,j})_{1 \leq i, j \leq n-1} \right].
\]

(2.9)

But

\[
\text{det} \left[ (a_{i,j})_{i \neq n-1, n \atop j \neq n-1, n} \right] = \text{det} \left[ (a_{i,j})_{1 \leq i, j \leq n-2} \right].
\]

By applying formula (2.2) for \((n - 2)\) on \((a_{i,j})_{1 \leq i, j \leq n-2}\) we get:

\[
(a_{11})^{n-4} \text{det} \left[ (a_{i,j})_{1 \leq i, j \leq n-2} \right] = \text{det} \left[ \begin{array}{cc} a_{11} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{array} \right].
\]

the same as:

\[
(a_{11})^{n-4} \text{det} \left[ (a_{i,j})_{1 \leq i, j \leq n-2} \right] = \text{det} \left[ (d_{i,j})_{1 \leq i, j \leq n-3} \right].
\]

(2.10)

Finally, formula (2.9) will be as follows:

\[
\left\{ (a_{11})^{n-3} \right\}^2 \text{det} \left[ (a_{i,j})_{1 \leq i, j \leq n} \right] \text{det} \left[ (a_{i,j})_{1 \leq i, j \leq n-2} \right] = \text{det} \left[ (d_{i,j})_{1 \leq i, j \leq n-3} \right] \text{det} \left[ (d_{i,j})_{1 \leq i, j \leq n-1} \right].
\]

And by using formula (2.10), it will be as follows:

\[
\left\{ (a_{11})^{n-3} \right\}^2 \text{det} \left[ (a_{i,j})_{1 \leq i, j \leq n} \right] \text{det} \left[ (a_{i,j})_{1 \leq i, j \leq n-2} \right] = (a_{11})^{n-4} \text{det} \left[ (a_{i,j})_{1 \leq i, j \leq n-2} \right] \text{det} \left[ (d_{i,j})_{1 \leq i, j \leq n-1} \right].
\]

At the end, we get:

\[
(a_{11})^{n-2} \text{det} \left[ (a_{i,j})_{1 \leq i, j \leq n} \right] = \text{det} \left[ (d_{i,j})_{1 \leq i, j \leq n-1} \right].
\]

And like this we have proved formula (2.2) for \(n\).
We can generalize theorem (2.2) by the following theorem:

**Theorem 2.3.** Let the $n$ square matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$.

For all $n > 2$, we can generalize the formula (2.2) as follows:

$$(a_{k,l})^{n-2} \det \left[ (a_{i,j})_{1 \leq i,j \leq n} \right] = \det \left[ 1 \leq i,j \leq n-1, [\det (A_{i,j})] \right], \quad 1 \leq k, l \leq n \quad (2.11)$$

when

$$A_{(i,j)} = \begin{cases} 
(a_{i,j} \ a_{i,l}) & \text{if } j < l, i < k \\
(a_{i,l} \ a_{i,j+1}) & \text{if } j \geq l, i < k \\
(a_{k,j} \ a_{k,l}) & \text{if } j < l, i \geq k \\
(a_{k,l} \ a_{k,j+1}) & \text{if } j \geq l, i \geq k \\

& \text{when } B_{i,j} = \begin{cases} 
(a_{k,l} \ a_{k,j}) & \text{if } j < l, i < k \\
(a_{k,l} \ a_{k,j+1}) & \text{if } j \geq l, i < k \\
(a_{k,j} \ a_{k,l}) & \text{if } j < l, i \geq k \\
(a_{k,l} \ a_{k,j+1}) & \text{if } j \geq l, i \geq k \\

\end{cases}.

\]

**Proof.** To prove formula (2.11), we move the element $a_{k,l}$ from its position to the position of the element $a_{1,1}$ in matrix $A$ by using determinants properties in order to apply formula (2.2).

So, we replace row $k$ and row $(k-1)$ by each other. Then, the new row $(k-1)$ and row $(k-2)$ by each other, and so on till row $k$ in matrix $A$ will be the first row. On the other side, we replace column $l$ and column $(l-1)$ by each other. Then, the new column $(l-1)$ and column $(l-2)$ by each other, and so on till column $l$ in matrix $A$ will be the first column. We get a new matrix $B$ that realises:

$$\det A = (-1)^{(k-1)+(l-1)} \det B. \quad (2.13)$$

We apply formula (2.2) on matrix $B$, we get:

$$(a_{k,l})^{n-2} \det B = \det \left[ 1 \leq i,j \leq n-1, [\det (B_{i,j})] \right] \quad (2.14)$$

when

$$B_{i,j} = \begin{cases} 
(a_{k,l} \ a_{k,j}) & \text{if } j < l, i < k \\
(a_{k,l} \ a_{k,j+1}) & \text{if } j \geq l, i < k \\
(a_{k,j} \ a_{k,l}) & \text{if } j < l, i \geq k \\
(a_{k,l} \ a_{k,j+1}) & \text{if } j \geq l, i \geq k \\

\end{cases}.

by using determinant properties, we get:
Condensation of Determinants

\[
\det (B_{i,j}) = \begin{cases} 
(-1)^2 \begin{vmatrix} a_{i,j} & a_{i,l} \\ a_{k,j} & a_{k,l} \end{vmatrix} & \text{if } j < l, i < k \\
(-1) \begin{vmatrix} a_{i,l} & a_{i,j+1} \\ a_{k,l} & a_{k,j+1} \end{vmatrix} & \text{if } j \geq l, i < k \\
(-1) \begin{vmatrix} a_{k,j} & a_{k,l} \\ a_{i+1,j} & a_{i+1,l} \end{vmatrix} & \text{if } j < l, i \geq k \\
a_{k,l} \begin{vmatrix} a_{k,j+1} \\ a_{i+1,l} \end{vmatrix} & \text{if } j \geq l, i \geq k
\end{cases}
\]

Thus, we write \((\det (B_{i,j}))_{1 \leq i,j \leq n-1}\) as a block matrix

\[
(\det (B_{i,j}))_{1 \leq i,j \leq n-1} = \\
(k-1) \begin{pmatrix} (-1)^2 \begin{vmatrix} a_{i,j} & a_{i,l} \\ a_{k,j} & a_{k,l} \end{vmatrix} & \begin{vmatrix} a_{i,l} & a_{i,j+1} \\ a_{k,l} & a_{k,j+1} \end{vmatrix} \end{pmatrix} \quad (n-l) \begin{pmatrix} a_{i,l} & a_{i,j+1} \\ a_{k,l} & a_{k,j+1} \\ a_{k,l} & a_{k,j+1} \\ a_{i+1,l} & a_{i+1,j+1} \end{pmatrix} \\
(n-k) \begin{pmatrix} (-1) \begin{vmatrix} a_{k,j} & a_{k,l} \\ a_{i+1,j} & a_{i+1,l} \end{vmatrix} \end{pmatrix} \end{pmatrix}
\]

and it can be written:

\[
(\det (B_{i,j}))_{1 \leq i,j \leq n-1} = (-1)^{(k-1)+(l-1)} \begin{pmatrix} a_{i,j} & a_{i,l} \\ a_{k,j} & a_{k,l} \\ a_{k,j} & a_{k,l} \\ a_{i+1,j} & a_{i+1,l} \end{pmatrix} \begin{pmatrix} a_{i,l} & a_{i,j+1} \\ a_{k,l} & a_{k,j+1} \\ a_{k,l} & a_{k,j+1} \\ a_{i+1,l} & a_{i+1,j+1} \end{pmatrix} \begin{pmatrix} a_{i,j} & a_{i,l} \\ a_{k,j} & a_{k,l} \\ a_{k,j} & a_{k,l} \\ a_{i+1,j} & a_{i+1,l} \end{pmatrix}
\]

According to formula (2.12), we can write the last formula as follows:

\[
(\det (B_{i,j}))_{1 \leq i,j \leq n-1} = (-1)^{(k-1)+(l-1)} \det \begin{pmatrix} a_{i,j} \\ a_{k,j} \\ a_{i+1,j} \end{pmatrix} \det \begin{pmatrix} a_{i,l} \\ a_{k,l} \\ a_{i+1,l} \end{pmatrix} \det \begin{pmatrix} a_{i,j} \\ a_{k,j} \\ a_{i+1,j} \end{pmatrix}
\]

Thus, formula (2.14) will be:

\[
(a_{k,l})^{n-2} \det B = (-1)^{(k-1)+(l-1)} \det \begin{pmatrix} a_{i,j} \\ a_{k,j} \\ a_{i+1,j} \end{pmatrix} \det \begin{pmatrix} a_{i,l} \\ a_{k,l} \\ a_{i+1,l} \end{pmatrix} \det \begin{pmatrix} a_{i,j} \\ a_{k,j} \\ a_{i+1,j} \end{pmatrix}
\]

So

\[
(a_{k,l})^{n-2} \left[ (-1)^{(k-1)+(l-1)} \det B \right] = \det \begin{pmatrix} a_{i,j} \\ a_{k,j} \\ a_{i+1,j} \end{pmatrix} \det \begin{pmatrix} a_{i,l} \\ a_{k,l} \\ a_{i+1,l} \end{pmatrix} \det \begin{pmatrix} a_{i,j} \\ a_{k,j} \\ a_{i+1,j} \end{pmatrix}
\]

By using formula (2.13), we get formula (2.11).
3. Application. In this section, we’ll show the main results we have found. These results are the construction of an easy and simplified algorithm which compute the determinant of any matrix (see [1]). For this, we give the following proposition:

**Proposition 3.1.** For \( l = 1, \ldots, n \):

\[
\left(a_{(1,l)}\right)^{n-2} \det \left[ (a_{(i,j)})_{1 \leq i,j \leq n} \right] = \det_{1 \leq i,j \leq n-1} (|A_{i,j}|) \tag{3.1}
\]

when

\[
A_{i,j} = \begin{cases} 
(a_{1,l}, a_{1,j+1}) & \text{if } l \leq j \\
(a_{i+1,l}, a_{i+1,j+1}) & \text{if } j < l
\end{cases}
\]

**Proof.** By putting \( k = 1 \) in formula (2.11) we get formula (3.1).

**Algorithm 3.2.** This algorithm can be described in the following steps:

1. Let \( n \times n \) square matrix \( A \) (we wish to compute its determinant).
2. If \( n = 2 \), we compute \( |A| \) by the known method, else,
3. if all the elements of the first row of matrix \( A \) are nil, then \( |A| = 0 \), else,
4. the first non nil elements in the first row is in \( l^{th} \) column. we form the \((n-1) \times (n-1) \) square matrix \( B = (b_{i,j})_{1 \leq i,j \leq n-1} \), Its elements are the determinants of \( 2 \times 2 \) square matrix

\[
b_{i,j} = \begin{cases} 
(a_{1,l}, a_{1,j+1}) & \text{if } l \leq j \\
(a_{i+1,l}, a_{i+1,j+1}) & \text{if } j < l
\end{cases}
\]

5. So,

\[
det A = \frac{det B}{(a_{(1,l)})^{n-2}}
\]

6. Let \( A = B \). We repeat the previous steps until we find the determinant.

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