C*-ENVELOPES OF TENSOR ALGEBRAS OF PRODUCT SYSTEMS

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Abstract. Let \( P \) be a submonoid of a group \( G \) and let \( \mathcal{E} = (\mathcal{E}_p)_{p \in P} \) be a product system over \( P \) with coefficient C*-algebra \( A \). We show that the following C*-algebras are canonically isomorphic: the C*-envelope of the tensor algebra \( T_\mathcal{E}(\mathcal{E})^+ \) of \( \mathcal{E} \); the reduced cross-sectional C*-algebra of the Fell bundle associated to the canonical coaction of \( G \) on the covariance algebra \( A \times_T P \) of \( \mathcal{E} \); and the C*-envelope of the cosystem obtained by restricting the canonical gauge coaction on \( T_{\mathcal{E}}(\mathcal{E}) \) to the tensor algebra. As a consequence, for every submonoid \( P \) of a group \( G \) and every product system \( \mathcal{E} = (\mathcal{E}_p)_{p \in P} \) over \( P \), the C*-envelope \( C_{\text{env}}(T_{\mathcal{E}}(\mathcal{E})^+) \) automatically carries a coaction of \( G \) that is compatible with the canonical gauge coaction on \( T_{\mathcal{E}}(\mathcal{E}) \). This answers a question left open by Dor-On, Kakariadis, Katsoulis, Laca and Li. We also analyse co-universal properties of \( C_{\text{env}}(T_{\mathcal{E}}(\mathcal{E})^+) \) with respect to injective gauge-compatible representations of \( \mathcal{E} \). When \( \mathcal{E} = C^\infty \) is the canonical product system over \( P \) with one-dimensional fibres, our main result implies that the boundary quotient \( \partial T_{\mathcal{E}}(P) \) is canonically isomorphic to the C*-envelope of the closed non-selfadjoint subalgebra spanned by the canonical generating isometries of \( T_{\mathcal{E}}(P) \). Our results on co-universality imply that \( \partial T_{\mathcal{E}}(P) \) is a quotient of every nonzero C*-algebra generated by a gauge-compatible isometric representation of \( P \) that in an appropriate sense respects the zero element of the semilattice of constructible right ideals of \( P \).

1. Introduction

Let \( A \) be a C*-algebra and let \( \mathcal{E} : A \rightarrow A \) be a correspondence. The Toeplitz algebra \( T_\mathcal{E} \) of \( \mathcal{E} \) is the C*-algebra generated by the range of the canonical representation of \( \mathcal{E} \) on \( \mathcal{E}^+ = \bigoplus_{n \geq 0} \mathcal{E}^{\otimes n} \). Pimsner associated a C*-algebra \( O_\mathcal{E} \) to a faithful correspondence that unifies other constructions such as Cuntz algebras and crossed products by single automorphisms [26]. Pimsner’s C*-algebra was generalised later by Katsura to not necessarily faithful correspondences [16], and this construction of a C*-algebra out of a correspondence is now known as a Cuntz–Pimsner algebra. In a certain sense, the Cuntz–Pimsner algebra may be regarded as a crossed product of \( A \) by the given correspondence.

The Cuntz–Pimsner algebra \( O_\mathcal{E} \) is a quotient of \( T_\mathcal{E} \), and the corresponding quotient map is faithful on the copy of \( A \). In fact, \( O_\mathcal{E} \) has the following special feature: the canonical gauge action of the unit circle on \( T_\mathcal{E} \) induces a gauge action on \( O_\mathcal{E} \) for which a representation of \( O_\mathcal{E} \) is faithful on the fixed-point algebra if and only if it is faithful on \( A \) [16] Proposition 6.3. Although the definition of \( O_\mathcal{E} \) involves a notion of covariant representations, \( O_\mathcal{E} \) can also be characterised as the quotient of \( T_\mathcal{E} \) by its largest gauge-invariant ideal with trivial intersection with \( A \) [17] Proposition 7.14.

The copies of \( \mathcal{E} \) and \( A \) in \( T_\mathcal{E} \) generate a non-selfadjoint operator algebra \( T(\mathcal{E})^+ \), called the tensor algebra of \( \mathcal{E} \). The Shilov boundary for \( T(\mathcal{E})^+ \) is the largest ideal of \( T_\mathcal{E} \) such that the corresponding quotient map is completely isometric on \( T(\mathcal{E})^+ \). The quotient of \( T_\mathcal{E} \) by this ideal is (isomorphic to) the C*-envelope of \( T(\mathcal{E})^+ \), denoted by \( C_{\text{env}}(T(\mathcal{E})^+) \). The C*-envelope
of an operator system always exists by a result of Hamana [11], following the seminal work of Arveson in [1]. The existence of the C*-envelope for a non-unital operator algebra follows from the unital case and the work of Meyer on the unitization of an operator algebra [22].

While $C^*_\text{env}(\mathcal{T}(\mathcal{E})^+)$ is the smallest C*-algebra generated by a completely isometric copy of $\mathcal{T}(\mathcal{E})^+$, the Cuntz–Pimsner algebra $\mathcal{O}_\mathcal{E}$ is the smallest C*-algebra generated by an injective representation of $\mathcal{E}$ admitting a gauge action of $\mathbb{T}$ that is compatible with the canonical gauge action on $\mathcal{T}_\mathcal{E}$. Muhly and Solel initiated in [23] the investigation on the relationship between these C*-algebras. Under some assumptions on $\mathcal{E}$, they proved that the canonical representation of $\mathcal{E}$ in the $C^*$-envelope of $\mathcal{T}(\mathcal{E})^+$ induces an isomorphism $\mathcal{O}_\mathcal{E} \cong C^*_\text{env}(\mathcal{T}(\mathcal{E})^+)$ [23, Theorem 6.4]. Katsoulis and Kribs established such an isomorphism in the case of an arbitrary correspondence later in [15], following previous partial findings by Fowler, Muhly and Raeburn [10, Theorem 5.3]. Their result implies that $C^*_\text{env}(\mathcal{T}(\mathcal{E})^+)$ automatically carries a gauge action of the unit circle $\mathbb{T}$ for which the quotient map from $\mathcal{T}_\mathcal{E}$ is gauge-equivariant, and also connects notions of boundary quotients from the theories of selfadjoint and non-selfadjoint operator algebras.

Roughly speaking, a product system $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ over a monoid $P$ as introduced by Fowler [9] is a semigroup of correspondences over the same C*-algebra $A := \mathcal{E}_e$. A single correspondence naturally gives rise to a product system over $\mathbb{N}$, and up to isomorphism every product system over $\mathbb{N}$ is given by tensor powers of a single correspondence. If $P$ is left cancellative, there is a canonical representation of a product system $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ on the Hilbert $A$-module $\mathcal{E}^+ = \bigoplus_{p \in P} \mathcal{E}_p$, called the Fock representation. This generates a C*-algebra $\mathcal{T}_\mathcal{E}(\mathcal{E})$, called the Toeplitz algebra of $\mathcal{E}$. The non-selfadjoint closed subalgebra of $\mathcal{T}_\mathcal{E}(\mathcal{E})$ generated by the image of $\mathcal{E}$ under the Fock representation is called the tensor algebra of $\mathcal{E}$, and denoted by $\mathcal{T}_\mathcal{E}(\mathcal{E})^+$. As for a single correspondence, there are different notions of boundary quotients for a product system. In other words, there are different ways to associated a C*-algebra to a product system that is the smallest C*-algebra among those with a given property.

For $P$ a submonoid of a group $G$, a notion of strongly covariant representations of $\mathcal{E}$ was introduced in [29], following previous work on compactly aligned product systems over positive cones of quasi-lattice orders by Sims and Yeend [30]. The covariance algebra of $\mathcal{E}$, denoted by $A \times_\mathcal{E} P$, was then defined to be the universal C*-algebra for strongly covariant representations of $\mathcal{E}$. The C*-algebra $A \times_\mathcal{E} P$ does not depend on the group $G$, and the canonical representation of $\mathcal{E}$ in $A \times_\mathcal{E} P$ is always injective. In addition, $A \times_\mathcal{E} P$ satisfies a special property: it carries a canonical coaction of $G$ for which a representation of $A \times_\mathcal{E} P$ is faithful on the fixed-point algebra if and only if it is faithful on the copy of $A$ [29, Theorem 3.10]. Thus the reduced analogue of $A \times_\mathcal{E} P$, namely the reduced cross sectional C*-algebra of the Fell bundle associated to the canonical coaction on $A \times_\mathcal{E} P$, is the smallest $C^*$-algebra generated by an injective strongly covariant representation of $\mathcal{E}$ that admits a coaction of $G$ for which the quotient map from $\mathcal{T}_\mathcal{E}(\mathcal{E})$ is gauge-equivariant. Hence $A \times_\mathcal{E} P$ may be viewed as a generalisation of a Cuntz–Pimsner algebra of a single correspondence.

Dor-On, Kakariadis, Katsoulis, Laca and Li introduced in [6] the notion of a C*-envelope for a cosystem. A cosystem consists of an operator algebra equipped with an appropriately defined coaction of a discrete group $G$, and the C*-envelope of a cosystem takes the coaction on the operator algebra into account. Hence the resulting C*-algebra is the smallest C*-algebra carrying a coaction of $G$ that is generated by an equivariantly complete isometric copy of the underlying operator algebra. For $P$ a submonoid of a group $G$, the canonical gauge coaction of $G$ on $\mathcal{T}_\mathcal{E}(\mathcal{E})$ restricts to a coaction on the tensor algebra $\mathcal{T}_\mathcal{E}(\mathcal{E})^+$ (see [6, Proposition 4.1]). The C*-envelope of the corresponding cosystem has the C*-envelope $C^*_\text{env}(\mathcal{T}_\mathcal{E}(\mathcal{E})^+)$ as a canonical quotient C*-algebra. However, the question of whether this quotient map is an isomorphism or, equivalently, the question of whether the C*-envelope of $\mathcal{T}_\mathcal{E}(\mathcal{E})^+$ automatically
carries a coaction of $G$ for which the quotient map from $T\lambda(\mathcal{E})$ is gauge-equivariant was left open in [6]. By earlier results of Dor-On and Katsoulis, the quotient map was known to be an isomorphism in the case of compactly aligned product systems over positive cones of abelian lattice orders [5]. When $\mathcal{E}$ is a compactly aligned product system over an arbitrary right LCM submonoid of a group, the C*-envelope of the canonical cosystem associated to the tensor algebra $T\lambda(\mathcal{E})^+$ was shown to be canonically isomorphic to the reduced analogue of the covariance algebra of $\mathcal{E}$ in [6, Theorem 5.3]. But the relationship between these C*-algebras remained unclear for more general product systems.

In this paper our main goal is to show that all of the above notions of boundary quotients for a product system $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ coincide. We were motivated by the recent analysis of the connection between these C*-algebras in particular cases, see [10,12,13]. Precisely, we show that the C*-envelope of the tensor algebra $T\lambda(\mathcal{E})^+$ is canonically isomorphic to the reduced cross sectional C*-algebra of the Fell bundle associated to the canonical coaction of $G$ on the covariance algebra of $\mathcal{E}$, for $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ a product system over a submonoid $P$ of $G$. In particular, the C*-envelope of $T\lambda(\mathcal{E})^+$ necessarily carries a coaction of $G$, for which the quotient map from $T\lambda(\mathcal{E})$ is gauge-equivariant. Hence we also obtain that the C*-envelopes of $T\lambda(\mathcal{E})^+$ and of the cosystem arising from the canonical coaction on $T\lambda(\mathcal{E})^+$ are canonically isomorphic.

We point out that the original proof by Hamana of the existence of the C*-envelope for an operator system is rather abstract [11]. In contrast, an important consequence of our main theorem is the identification of the C*-envelope $C^*_{\text{env}}(T\lambda(\mathcal{E})^+)$ as the reduced analogue of the covariance algebra of $\mathcal{E}$, which is concretely defined. This yields an explicit description of the Shilov boundary ideal for $T\lambda(\mathcal{E})^+$ in several situations, such as when the underlying group is known to be exact. See Corollary 5.3.

We establish the existence of the isomorphisms mentioned above in Theorem 5.1. The proof builds on three main auxiliary results, which are proved in Sections 3 and 4. First, we show in Corollary 3.5 that the C*-envelope of $T\lambda(\mathcal{E})^+$ is a quotient of every other quotient of $T\lambda(\mathcal{E})$ by a gauge-invariant ideal with trivial intersection with the coefficient algebra $A$. In other words, any gauge-equivariant *-homomorphism of $T\lambda(\mathcal{E})$ that is faithful on the coefficient algebra restricts to a complete isometry on $T\lambda(\mathcal{E})^+$. This implies that $C^*_{\text{env}}(T\lambda(\mathcal{E})^+)$ is a quotient of both the covariance algebra of $\mathcal{E}$ and its reduced counterpart. The other two auxiliary results, namely Lemma 3.7 and Proposition 4.7, allow us to conclude that the image under the quotient map of the spectral subspaces associated to the coaction on $T\lambda(\mathcal{E})$ yields a topological $G$-grading for $C^*_{\text{env}}(T\lambda(\mathcal{E})^+)$.

After proving our main theorem in Section 5 we proceed to investigate co-universal properties for the C*-envelope of $T\lambda(\mathcal{E})^+$ in Section 6. We show that $C^*_{\text{env}}(T\lambda(\mathcal{E})^+)$ has the co-universal property for gauge-equivariant surjective *-homomorphisms of the cross sectional C*-algebra of the Fell bundle associated to the canonical coaction on $T\lambda(\mathcal{E})$ that are injective on $A$. See Corollary 6.2 for a precise statement. In particular this shows that the notion of C*-envelopes of cosystems and the assumption of compact alignment in [6, Theorem 4.9] are not needed to establish the existence of a C*-algebra with this property. If $\mathcal{E}$ is faithful, meaning that the left action of $A$ on each correspondence $\mathcal{E}_p$: $A \rightsquigarrow A$ is injective, then we show that $C^*_{\text{env}}(T\lambda(\mathcal{E})^+)$ is a quotient of every C*-algebra generated by an injective gauge-compatible representation of $\mathcal{E}$ that satisfies a certain condition, involving generating elements that correspond to the zero element of the semilattice of constructible right ideals of $P$ in a suitable sense (see Theorem 6.3). This result may be particularly helpful when we wish to decide whether $C^*_{\text{env}}(T\lambda(\mathcal{E})^+)$ is a *-homomorphic image of a given C*-algebra.

Although it is always helpful to have several descriptions for the same C*-algebra at hand, we believe that this becomes very explicit in our analysis of co-universal properties in
Section 6 in the case of the different descriptions for the C*-envelope $C^*_\text{env}(\mathcal{T}_\lambda(\mathcal{E})^+)$ obtained from Theorem 5.1. The co-universal property of $C^*_\text{env}(\mathcal{T}_\lambda(\mathcal{E})^+)$ given in Corollary 6.2 follows as a consequence of its defining property as the smallest C*-algebra generated by a completely isometric homomorphism of $\mathcal{T}_\lambda(\mathcal{E})^+$. On the other hand, the defining relations of the covariance algebra are our main tool in the proof of Theorem 6.3 which gives a stronger version of co-universal property of $C^*_\text{env}(\mathcal{T}_\lambda(\mathcal{E})^+)$ when $\mathcal{E}$ is faithful.

2. Background

In this section we review basic notions and results in the setting of non-selfadjoint operator algebras and of product systems that will be needed in the sequel, taking also the opportunity to establish our notation. We refer the reader to [21,22] for the theory of operator algebras and to [20,22] for the theory of Hilbert C*-modules. In this paper we will often use constructions of C*-algebras associated to Fell bundles, and their relationship to topologically graded C*-algebras in general. Our main reference for this part is [8].

2.1. Operator algebras and their C*-envelopes. An operator algebra on a Hilbert space $\mathcal{H}$ is a closed subalgebra of $\mathbb{B}(\mathcal{H})$. If $A$ is an operator algebra on $\mathcal{H}$, then for every $n \geq 1$ the algebra $M_n(A)$ of $n \times n$ matrices with entries in $A$ is an operator algebra on the direct sum $\mathcal{H}^n \cong \mathcal{C}^n \otimes \mathcal{H}$. We say that a homomorphism $\rho: A \to B$ between operator algebras is completely contractive (resp. completely isometric) if the induced homomorphism $\rho_n: M_n(A) \to M_n(B)$ is contractive (resp. isometric) for all $n \geq 1$.

A C*-cover of an operator algebra $A$ is a pair $(B, \rho)$, where $B$ is a C*-algebra and $\rho: A \to B$ is a completely isometric homomorphism such that $C^*(\rho(A)) = B$. The C*-envelope of an operator algebra $A$ is a C*-cover $(C^*_\text{env}(A), \iota)$ satisfying the following property: if $(B, \rho)$ is a C*-cover of $A$, then there exists a (necessarily unique and surjective) *-homomorphism $\pi: B \to C^*_\text{env}(A)$ such that $\pi \circ \rho = \iota$. The C*-envelope of $A$ exists, and is unique up to an isomorphism that identifies $A$. Hamana established the existence of the C*-envelope in the unital case in [11, Theorem 4.1], following the work of Arveson in [1]. The existence of the C*-envelope in the non-unital case follows from the work of Meyer on the unitization of an operator algebra [22]. See [2, Proposition 4.3.5] for further details.

Let $A$ be an operator algebra contained in a C*-algebra $B$ and suppose that $C^*(A) = B$. An ideal $J$ in $B$ is called a boundary ideal for $A$ if the quotient map $B \to B/J$ is completely isometric when restricted to $A$. We say that a boundary ideal $J$ of $B$ is the Shilov boundary for $A$ if it contains every other boundary ideal for $A$. An ideal $J$ in $B$ is the Shilov boundary for $A$ if and only if the quotient $B/J$ is canonically isomorphic to the C*-envelope of $A$.

2.2. Product systems. Let $A$ be a C*-algebra. A correspondence $\mathcal{E}: A \rightsquigarrow A$ consists of a right Hilbert $A$-module with a nondegenerate left action of $A$ implemented by a *-homomorphism $\varphi: A \to \mathbb{B}(\mathcal{E})$. We say that $\mathcal{E}$ is faithful if the left action of $A$ is injective. Notice that we have included nondegeneracy in our definition of a correspondence as we will be applying the main results of [29]. By [18, Remark 1.3] this condition automatically holds for the underlying correspondences of a product system over a monoid with nontrivial group of units (see also [6, Remark 2.2]).

Let $P$ be a submonoid of a group $G$. We denote the unit element of $P$ by $e$. A product system over $P$ of $A$-correspondences consists of:

(i) a correspondence $\mathcal{E}_p: A \rightsquigarrow A$ for each $p \in P \setminus \{e\}$;

(ii) correspondence isomorphisms $\mu_{p,q}: \mathcal{E}_p \otimes_A \mathcal{E}_q \xrightarrow{\sim} \mathcal{E}_{pq}$, also called multiplication maps, for all $p, q \in P \setminus \{e\}$.

In addition, we let $\mathcal{E}_e = A$ with the obvious structure of correspondence over $A$. The multiplication maps $\mu_{e,p}$ and $\mu_{p,e}$ implement the left and right actions of $A$ on $\mathcal{E}_p$, respectively. Thus $\mu_{e,p}(a \otimes \xi_p) = \varphi_p(a)\xi_p$ and $\mu_{p,e}(\xi_p \otimes a) = \xi_p a$ for all $a \in A$ and $\xi_p \in \mathcal{E}_p$, where
\(\varphi_p: A \to \mathcal{B}(E_p)\) denotes the left action of \(A\) on \(E_p\). The multiplication maps must be associative, meaning that the following diagram commutes for all \(p, q, r \in P\):

\[
\begin{array}{c}
(E_p \otimes_A E_q) \otimes_A E_r \\
\downarrow_{\mu_{p,q} \otimes 1} \\
E_p \otimes_A (E_q \otimes_A E_r) \\
\downarrow_{1 \otimes \mu_{q,r}} \\
E_p \otimes_A E_{qr}
\end{array}
\]

A product system \(E = (E_p)_{p \in P}\) will be called faithful if \(\varphi_p\) is injective for all \(p \in P\). We refer to [\textit{2}] for a more detailed account on product systems.

**Definition 2.1.** A representation of a product system \(E = (E_p)_{p \in P}\) in a C*-algebra \(B\) consists of linear maps \(\pi_p: E_p \to B\), for all \(p \in P \setminus \{e\}\), and a *-homomorphism \(\pi_e: A \to B\), satisfying the following two axioms:

(i) \(\pi_p(\xi)\pi_q(\eta) = \pi_{pq}(\xi \eta)\) for all \(p, q \in P, \xi \in E_p\), and \(\eta \in E_q\);  
(ii) \(\pi_p(\xi)^*\pi_p(\eta) = \pi_e(\langle \xi | \eta \rangle)\), for all \(p \in P\) and \(\xi, \eta \in E_p\).

We will say that a representation \(\pi = \{\pi_p\}_{p \in P}\) of a product system is injective if the *-homomorphism \(\pi_e\) is faithful. When \(\pi = \{\pi_p\}_{p \in P}\) is injective, \(\pi_p\) is automatically completely isometric for every \(p \in P\).

2.3. **The Fock representation.** There is a canonical injective representation associated to a product system \(E = (E_p)_{p \in P}\), which we now describe. Let \(E^+\) be the right Hilbert \(A\)-module given by the direct sum of all \(E_p\)'s. That is,

\[
E^+ = \bigoplus_{p \in P} E_p.
\]

We call \(E^+\) the Fock space of \(E\). For \(\xi \in E_p\), we define an operator \(\psi_p^+ (\xi)\) in \(\mathcal{B}(E^+)\) by setting for each \(\eta = \bigoplus_{s \in P} \eta_s \in E^+

\[
\psi_p^+ (\xi)(\eta)_s = \begin{cases} 
\mu_{p,p^{-1}s}(\xi \otimes \eta_{p^{-1}s}) & \text{if } s \in pP, \\
0 & \text{otherwise}.
\end{cases}
\]

We view \(E_{ps}\) as the correspondence \(E_p \otimes_A E_s\), using the correspondence isomorphism \(\mu_{p,s}^{-1}\). In this way, \(\psi_p^+ (\xi)^*(\eta)_s\) is the image of \(\eta_s\) in \(E_s\) under the operator defined on elements of the form \(\mu_{p,s}(\zeta_p \otimes \zeta_s)\) by the formula

\[
\psi_p^+ (\xi)^*(\mu_{p,s}(\zeta_p \otimes \zeta_s)) = \varphi_s(\langle \xi | \zeta_p \rangle)\zeta_s.
\]

So \(\psi_p^+ (\xi)^*\) is the adjoint of \(\psi_p^+ (\xi)\). This gives a representation \(\tilde{\psi}^+ = \{\psi_p^+\}_{p \in P}\) of \(E\) in \(\mathcal{B}(E^+)\) called the **Fock representation** of \(E\). This representation is injective because the action of \(A\) on the direct summand \(A = E_e \subset E^+\) is simply left multiplication in \(A\). See [\textit{9}] for further details.

**Definition 2.2.** We will refer to the C*-algebra generated by the range of \(\tilde{\psi}^+\) as the **Toeplitz algebra** of \(E\), and will denote it by \(T_\lambda(E)\). The **tensor algebra** of \(E\), denoted by \(T_\lambda(E)^+\), is the closed subalgebra of \(T_\lambda(E)\) generated by the range of \(\tilde{\psi}^+\). That is,

\[
T_\lambda(E)^+ = \text{span}\{\varphi_p(\xi) | p \in P, \xi \in E_p\}.
\]

The universal C*-algebra for representations of \(E\) as introduced by Fowler in [\textit{9}] Proposition 2.8] will also be useful in this paper. We will denote this C*-algebra by \(C^*_{\text{rep}}(E)\), and we let \(\tilde{t} = \{t_p\}_{p \in P}\) denote the universal representation of \(E\) in \(C^*_{\text{rep}}(E)\). Recall that the pair \((C^*_{\text{rep}}(E), \tilde{t})\) satisfies the following universal property: if \(\pi = \{\pi_p\}_{p \in P}\) is a representation of \(E\) in a C*-algebra \(B\), then there exists a unique *-homomorphism \(\tilde{\pi}: C^*_{\text{rep}}(E) \to B\) such that \(\tilde{\pi} \circ \tilde{t}_p = \pi_p\) for all \(p \in P\).
Remark 2.3. We have a few comments regarding the notation and terminology adopted in this paper:

1. We use the notation $C^*_\text{rep}(\mathcal{E})$ for Fowler’s universal $C^*$-algebra for representations of $\mathcal{E}$, and we simply refer to it as Fowler’s $C^*$-algebra. This differs from the notation and terminology used in Fowler’s original paper [9], where this $C^*$-algebra was denoted by $\mathcal{T}_\mathcal{F}$ and called the Toeplitz algebra of $\mathcal{E}$; this notation and terminology were later adopted in other references, such as [29, 22]. The reason for our choice of notation is that we believe that the symbols $\mathcal{T}_\mathcal{F}$ and $\mathcal{T}(\mathcal{E})$ are reminiscent of a universal analogue of $\mathcal{T}_\mathcal{F}(\mathcal{E})$, in the sense that we should expect a canonical isomorphism between $\mathcal{T}(\mathcal{E})$ and $\mathcal{T}_\mathcal{F}(\mathcal{E})$ in nice cases such as, for example, when the enveloping group of $P$ is amenable. But such an isomorphism does not exist in general, even in simple cases such as when $P = \mathbb{N}^2$ (see [24]).

2. Here we reserve the name “Toeplitz algebra” for $\mathcal{T}_\mathcal{F}(\mathcal{E})$; this aligns with the terminology arising from semigroup $C^*$-algebras and also with Pimsner’s original definition of the Toeplitz algebra of a single correspondence $\mathcal{E}_1$ as the $C^*$-algebra generated by what is now known as the Fock representation of $\mathcal{E}_1$ (see [19, 26]). We point out that in [6], $\mathcal{T}_\mathcal{F}(\mathcal{E})$ is called the Fock algebra of $\mathcal{E}$ (see [6, Definition 2.5]).

2.4. The coaction on the Toeplitz algebra. Let $G$ be a discrete group and let $\{u_g \mid g \in G\}$ be the canonical unitary generators of the full group $C^*$-algebra $C^*(G)$. Let $\delta_G$ be the $^\ast$-homomorphism $C^*(G) \to C^*(G) \otimes C^*(G)$ given by $\delta_G(u_g) = u_g \otimes u_g$. A (full) coaction of $G$ on a $C^*$-algebra $B$ is a nondegenerate and injective $^\ast$-homomorphism $\delta : B \to B \otimes C^*(G)$ satisfying the coaction identity

$$\delta((\delta \otimes \text{id}_{C^*(G)})\delta) = (\text{id}_B \otimes \delta_G)\delta.$$ 

We refer to the triple $(B, G, \delta)$ as a coaction. The coaction $(B, G, \delta)$ is called nondegenerate if $\delta(B)(1 \otimes C^*(G)) = B \otimes C^*(G)$. It is said to be normal if the $^\ast$-homomorphism $(\text{id}_B \otimes \lambda) \circ \delta : B \to B \otimes C^1(G)$ is injective, where $C^1(G)$ is the reduced group $C^*$-algebra of $G$ and $\lambda : C^*(G) \to C^1(G)$, $u_g \mapsto \lambda_g$ is the left regular representation. See [27] and also [7, Definition A.21]. Notice that it is not known whether all full coactions of discrete groups are automatically nondegenerate, see [14].

Let $(B, G, \delta)$ be a nondegenerate coaction. The spectral subspace at $g \in G$ is $B_g := \{b \in B \mid \delta(b) = b \otimes u_g\}$. We call $B_e = B^\delta$ the fixed-point algebra for the coaction $\delta$ of $G$ on $B$. The collection of subspaces $\{B_g \mid g \in G\}$ forms a topological $G$-grading for $B$. That is, $\{B_g \mid g \in G\}$ is a collection of linearly independent closed subspaces of $B$ with $B_{gh} \subset B_{gh}$ and $B_{g^{-1}} = B_g^\ast = B_g$ is dense in $B$, and there exists a conditional expectation $E^\delta : B \to B^\delta$ that vanishes on $B_g$ for $g \neq e$. The coaction $(B, G, \delta)$ is normal if and only if $E^\delta$ is faithful (see [27, Lemma 1.4]).

As in [6], we will need in the subsequent sections the normal coaction of $G$ on the Toeplitz algebra.

Proposition 2.4 ([6, Proposition 4.1]). Let $P$ be a submonoid of a group $G$ and let $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ be a product system over $P$ with coefficient $C^*$-algebra $A$. Let $\psi^+ = \{\psi^+_p\}_{p \in P}$ be the Fock representation of $\mathcal{E}$. Then there is a nondegenerate normal coaction

$$\bar{\delta} : \mathcal{T}_\mathcal{E}(\mathcal{E}) \to \mathcal{T}_\mathcal{E}(\mathcal{E}) \otimes C^*(G)$$

that sends a generator $\psi^+_p(\xi)$ to $\psi^+_p(\xi) \otimes u_p$, for $p \in P$ and $\xi \in \mathcal{E}_p$. Moreover, the spectral subspace $\mathcal{T}_\mathcal{E}(\mathcal{E})_g$ at $g \in G$ is the closed linear span of elements of the form

$$\psi^+_{p_1}(\xi_{p_1})\psi^+_{p_2}(\xi_{p_2})^* \cdots \psi^+_{p_{2k-1}}(\xi_{p_{2k-1}})\psi^+_{p_{2k}}(\xi_{p_{2k}})^*,$$

where $k \in \mathbb{N}$, $p_1p_2^{-1} \cdots p_{2k-1}p_{2k}^{-1} = g$ and $\xi_{p_i} \in \mathcal{E}_{p_i}$ for all $i \in \{1, 2, \ldots, 2k\}$.
We will write $E_\lambda$ for the (faithful) conditional expectation $E^\lambda : \mathcal{T}_\lambda(\mathcal{E}) \to \mathcal{T}_\lambda(\mathcal{E})_c$ associated to $\delta$.

Fowler’s $C^*$-algebra $C_{\text{rep}}^*(\mathcal{E})$ also carries a canonical nondegenerate coaction of $G$. Indeed, there is a canonical representation of $\mathcal{E}$ in $C_{\text{rep}}^*(\mathcal{E}) \otimes C^*(G)$ that sends $\xi_p \in \mathcal{E}_p$ to $\tilde{i}(\xi_p) \otimes u_p$. This induces a $*$-homomorphism $\tilde{\delta} : C_{\text{rep}}^*(\mathcal{E}) \to C_{\text{rep}}^*(\mathcal{E}) \otimes C^*(G)$ by the universal property of $C_{\text{rep}}^*(\mathcal{E})$. The triple $(C_{\text{rep}}^*(\mathcal{E}), G, \tilde{\delta})$ is a coaction, and as for $T_\lambda(\mathcal{E})$ the spectral subspace $C_{\text{rep}}^*(\mathcal{E})_g$ at $g \in G$ is the closed linear span of elements of the form

$$i(\xi_p_1)i(\xi_p_2)\ldots i(\xi_p_{2k-1})i(\xi_p_{2k})^*,$$

where $k \in \mathbb{N}$, $p_1p_2^{-1}\ldots p_{2k-1}p_{2k}^{-1} = g$ and $\xi_{p_i} \in \mathcal{E}_{p_i}$ for all $i \in \{1, 2, \ldots, 2k\}$. See, for example, \cite{29} Lemma 2.2.

3. Completely isometric homomorphisms of $T_\lambda(\mathcal{E})^+$

Our main goal in this section is to show that a gauge-equivariant quotient of $T_\lambda(\mathcal{E})$ contains a completely isometric copy of $T_\lambda(\mathcal{E})^+$ as long as the quotient map is faithful on $A$.

Throughout this section we let $P$ be a submonoid of a group $G$. Let $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ be a product system over $P$ and let $A := \mathcal{E}_e$ be its coefficient $C^*$-algebra.

**Proposition 3.1.** Let $\pi = \{\pi_p\}_{p \in P}$ be an injective representation of $\mathcal{E}$ in a $C^*$-algebra $B$ and suppose that the map that sends $\psi^+_p(\xi)$ to $\pi_p(\xi)$ for $p \in P$ and $\xi \in \mathcal{E}_p$ induces a completely contractive homomorphism $\tilde{\pi} : T_\lambda(\mathcal{E})^+ \to B$. Suppose, in addition, that there exists a conditional expectation $E_\pi : \tilde{\pi}(C_{\text{rep}}^*(\mathcal{E})) \to \tilde{\pi}(C_{\text{rep}}^*(\mathcal{E})_e)$ such that the diagram

$$
\begin{array}{ccc}
C_{\text{rep}}^*(\mathcal{E}) & \xrightarrow{\tilde{\pi}} & \tilde{\pi}(C_{\text{rep}}^*(\mathcal{E})) \\
E^\delta \downarrow & & \downarrow E_\pi \\
C_{\text{rep}}^*(\mathcal{E})_e & \xrightarrow{\tilde{\pi}} & \tilde{\pi}(C_{\text{rep}}^*(\mathcal{E})_e)
\end{array}
$$

commutes, where $\tilde{\pi} : C_{\text{rep}}^*(\mathcal{E}) \to B$ is the $*$-homomorphism obtained by universal property. Then $\tilde{\pi}$ is completely isometric.

**Proof.** We begin by proving that $\tilde{\pi}$ is isometric. To do so, it suffices to show that

$$\|\sum_{s \in F_1} \psi^+_s(\xi_s)\| = \|\sum_{s \in F_1} \pi_s(\xi_s)\|,$$

where $F_1 \subset P$ is a finite set and $\xi_s \in \mathcal{E}_s$ for all $s \in F_1$. Let $F_2 \subset P$ be another finite set and take an element $\eta = \sum_{r \in F_2} \eta_r \in \mathcal{E}^+$, where $\eta_r \in \mathcal{E}_r$ for all $r \in F_2$. Consider the set $F \subset P$ given by $F := \{sr \mid s \in F_1, r \in F_2\}$ and notice that

$$\|\sum_{s \in F_1} \psi^+_s(\xi_s)\eta\|^2 = \|\sum_{s \in F_1} \psi^+_s(\xi_s)(\sum_{s \in F_1} \psi^+_s(\xi_s)\eta)\| \geq \|\sum_{p \in F} (\eta'_p \mid \eta'_p)\|,$$

where $\eta'_p := \sum_{sr=p} \psi^+_s(\xi_s)\eta_r$ is simply the coordinate of $\sum_{s \in F_1} \psi^+_s(\xi_s)\eta$ at $p \in F$. Since $\pi$ is injective, we have that $\pi_\pi$ is isometric and so

$$\|\sum_{p \in F} \pi_\pi(\eta'_p)\| = \|\sum_{p \in F} (\eta'_p \mid \eta'_p)\| = \|\sum_{s \in F_1} \psi^+_s(\xi_s)\eta\|^2. \quad (3.2)$$

Also, observe that

$$\sum_{p,q \in F} \pi_\pi(\eta'_p)^* \pi_\pi(\eta'_q) = \left(\sum_{r \in F_2} \pi_r(\eta_r)\right)^* \left(\sum_{s,l \in F_1} \pi_s(\xi_s)^* \pi_l(\xi_l)\right) \left(\sum_{r \in F_2} \pi_r(\eta_r)\right) \leq \|\sum_{s \in F_1} \pi_s(\xi_s)\|^2 \sum_{r,r' \in F_2} \pi_r(\eta_r)^* \pi_{r'}(\eta_{r'}).$$
Applying the conditional expectation $E_\pi: \hat{\pi}(C^*_{\text{rep}}(\mathcal{E})) \to \hat{\pi}(C^*_{\text{rep}}(\mathcal{E}))_{\text{c}}$ to the resulting inequality and using that $E_\pi \circ \hat{\pi} = \hat{\pi} \circ E$, we deduce that
\[
\sum_{p \in F} \pi_{s}(\langle \eta_p' | \eta_p' \rangle) \leq \| \sum_{s \in F_1} \pi_{s}(\xi_s) \|_2^2 \sum_{r \in F_2} \pi_{r}(\langle \eta_r | \eta_r \rangle). \tag{3.3}
\]
Notice that
\[
\| \sum_{r \in F_2} \pi_{s}(\langle \eta_r | \eta_r \rangle) \| = \sum_{r \in F_2} \| \langle \eta_r | \eta_r \rangle \| = \| \eta \|^2,
\]
and hence (3.3) together with (3.2) yield the inequality
\[
\| \sum_{s \in F_1} \psi_s^+(\xi_s) \eta \|^2 \leq \sum_{s \in F_1} \| \pi_s(\xi_s) \|^2 \| \eta \|^2.
\]
This implies that $\| \sum_{s \in F_1} \psi_s^+(\xi_s) \| \leq \sum_{s \in F_1} \| \pi_s(\xi_s) \|$ because the above inequality holds for every element $\eta \in \mathcal{E}^+$ of finite support. Since $\hat{\pi}$ is contractive, we then conclude that $\hat{\pi}$ is isometric.

Next we show that $\hat{\pi}_n$ is isometric on $\mathcal{M}_n(T_\mathcal{E}(\mathcal{E})^+)$ also when $n \geq 2$. Let $b = (b_{i,j}) \in \mathcal{M}_n(T_\mathcal{E}(\mathcal{E})^+)$, where each entry $b_{i,j} \in T_\mathcal{E}(\mathcal{E})^+$ is a finite sum $\sum_{s=1}^n \psi_{s}^+(\xi_{s})$ with $\xi_{s} \in \mathcal{E}$. Similarly, let $\eta = (\eta_1, \ldots, \eta_n) \in (\mathcal{E}^+)^n$, where each $\eta_i$ is a finite sum of the form $\sum_{\pi_i = 1}^n \eta_{i,r} \in \mathcal{E}^+$ with $\eta_{i,r} \in \mathcal{E}_r$. We fix $i \in \{1, \ldots, n\}$ and for each $p \in P$ we let $\eta'_p$ be the coordinate of $\sum_{j=1}^n b_{i,j} \eta_j \in \mathcal{E}^+$ at $p \in P$. Then setting $F_i := \{p \in P | \eta'_p \neq 0\}$, we see that $F_i$ is a finite set with
\[
\langle \sum_{j=1}^n b_{i,j} \eta_j, \sum_{j=1}^n b_{i,j} \eta_j \rangle = \sum_{p \in F_i} \langle \eta'_p | \eta'_p \rangle.
\]
Thus as in the case $n = 1$ we have
\[
\| b \eta \|^2 = \| \sum_{i=1}^n \langle \sum_{j=1}^n b_{i,j} \eta_j, \sum_{j=1}^n b_{i,j} \eta_j \rangle \| = \| \sum_{i=1}^n \sum_{p \in F_i} \langle \eta'_p | \eta'_p \rangle \| = \| \sum_{i=1}^n \sum_{p \in F_i} \pi_{s}(\langle \eta'_p | \eta'_p \rangle) \|. \tag{3.4}
\]
If we set $\pi(\eta) := \sum_{\pi_i} \pi_{r}(\eta_{i,r})$ for $i = 1, \ldots, n$ and let $\pi_{n,1}(\eta) := (\pi(\eta_1), \ldots, \pi(\eta_n))$ regarded as an element of the direct sum $B^n$ with its $F_\pi$-structure of Hilbert $B$-module, we see that
\[
\sum_{p,q \in F_i} \pi_{p}(\eta'_p)^* \pi_{q}(\eta'_q) = \left( \sum_{j=1}^n \hat{\pi}(b_{i,j}) \pi(\eta_j) \right)^* \left( \sum_{j=1}^n \hat{\pi}(b_{i,j}) \pi(\eta_j) \right),
\]
and
\[
\langle \pi_{n,1}(\eta) | \pi_{n,1}(\eta) \rangle = \sum_{i=1}^n \pi(\eta_i)^* \pi(\eta_i) = \sum_{i=1}^n \sum_{r',r' \in F_i} \pi_{r}(\eta_{i,r})^* \pi_{r'}(\eta_{i,r'}).\]
Hence viewing $\hat{\pi}_n(b)$ as an adjointable operator on $B^n$ we conclude that
\[
\sum_{i=1}^n \pi_{p}(\eta'_p)^* \pi_{q}(\eta'_q) = \langle \hat{\pi}_n(b) \pi_{n,1}(\eta) | \hat{\pi}_n(b) \pi_{n,1}(\eta) \rangle \\
\leq \| \hat{\pi}_n(b) \|^2 \langle \pi_{n,1}(\eta) | \pi_{n,1}(\eta) \rangle \\
= \| \hat{\pi}_n(b) \|^2 \| \sum_{i=1}^n \sum_{r',r' \in F_i} \pi_{r}(\eta_{i,r})^* \pi_{r'}(\eta_{i,r'}) \|.
\]
Again applying $E_\pi$ to the resulting inequality and using that $E_\pi \circ \hat{\pi} = \hat{\pi} \circ E^2$, we obtain
\[
\sum_{i=1}^n \sum_{p \in F_i} \pi_{s}(\langle \eta'_p | \eta'_p \rangle) \leq \| \hat{\pi}_n(b) \|^2 \sum_{i=1}^n \pi_{s}(\langle \eta_i | \eta_i \rangle).
\]
Combining this with (3.4) and observing that $\| \sum_{i=1}^n \pi_{s}(\langle \eta_i | \eta_i \rangle) \| = \| \eta \|^2$, we conclude that
\[
\| b \eta \|^2 \leq \| \pi_n(b) \|^2 \| \eta \|^2.
\]
This shows that $\|b\| \leq \|\hat{\pi}_n(b)\|$, and so the equality holds because $\hat{\pi}_n$ is contractive. Hence $\hat{\pi}_n$ is isometric for all $n \geq 1$, proving that $\hat{\pi}$ is completely isometric as asserted.

The following is a concrete situation in which Proposition 3.1 applies.

**Corollary 3.5.** Let $\pi = \{\pi_p\}_{p \in P}$ be an injective representation of $E$ in a $C^*$-algebra $B$ and suppose that the map that sends $\psi_p^+(\xi)$ to $\pi_p(\xi)$ for $p \in P$ and $\xi \in E_p$ induces a $^*$-homomorphism $\hat{\pi}: \mathcal{T}_\lambda(E) \to B$. Suppose, in addition, that there exists a conditional expectation $\mathcal{E}_\pi: \hat{\pi}(\mathcal{T}_\lambda(E)) \to \hat{\pi}(\mathcal{T}_\lambda(E)_c)$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{T}_\lambda(E) & \xrightarrow{\hat{\pi}} & \hat{\pi}(\mathcal{T}_\lambda(E)) \\
\mathcal{E}_\lambda & \downarrow & \mathcal{E}_\pi \\
\mathcal{T}_\lambda(E)_c & \xrightarrow{\hat{\pi}} & \hat{\pi}(\mathcal{T}_\lambda(E)_c)
\end{array}
$$

commutes. Then the restriction of $\hat{\pi}$ to the tensor algebra $\mathcal{T}_\lambda(E)^+$ is completely isometric.

**Remark 3.6.** Notice that Corollary 3.5 immediately implies that if $\hat{\pi}: \mathcal{T}_\lambda(E) \to B$ is a $^*$-homomorphism and $(B, G, \gamma)$ is a coaction for which $\hat{\pi}$ is $\delta - \gamma$-equivariant, in the sense that $\gamma \circ \hat{\pi} = (\hat{\pi} \otimes \text{id}_{C^*(G)}) \circ \delta$, then the restriction of $\hat{\pi}$ to the tensor algebra is completely isometric provided it is injective on $A$.

The next result will be an important tool in the proof that the $C^*$-envelope of $\mathcal{T}_\lambda(E)^+$ also carries a conditional expectation as in Corollary 3.5, where the representation of $E$ in $C^*_\text{env}(\mathcal{T}_\lambda(E)^+)$ in this case is simply the one induced by the inclusion $\mathcal{T}_\lambda(E)^+ \hookrightarrow C^*_\text{env}(\mathcal{T}_\lambda(E)^+)$.

**Lemma 3.7.** Let $\pi = \{\pi_p\}_{p \in P}$ be a representation of $E$ in a $C^*$-algebra $B$ and suppose that the map that sends $\psi_p^+(\xi)$ to $\pi_p(\xi)$ for $p \in P$ and $\xi \in E_p$ induces a completely isometric homomorphism $\hat{\pi}: \mathcal{T}_\lambda(E)^+ \to B$. Then for every $n \geq 1$, finite sets $F_1, F_2, \ldots, F_n \subset P$ and choice of elements $\xi_p \in E_p$ for $p \in F_i$, $i = 1, \ldots, n$, we have that

$$
\| \sum_{i=1}^{n} \sum_{p \in F_i} \pi_p(\xi_p) \| \leq \| \sum_{i=1}^{n} \sum_{p,q \in F_i} \pi_p(\xi_p)^* \pi_q(\xi_q) \|.
$$

**Proof.** Suppose that $\hat{\pi}$ is completely isometric. We consider first the case $n = 1$. Let $F = F_1 \subset P$ be a finite set and for each $p \in F$, take $\xi_p \in E_p$. Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $A$ and for each $\lambda \in \Lambda$, consider the element $\eta_\lambda \in \mathcal{E}_c \subset \mathcal{E}^+$ determined by $u_\lambda$. That is, the coordinate of $\eta_\lambda$ at $p \in P$ is $u_\lambda$ when $p = e$, and zero for $p \neq e$. Then $\|\eta_\lambda\| = \|u_\lambda\| = 1$, and hence applying the operator $\sum_{p \in F} \psi_p^+(\xi_p) \in \mathcal{B}(\mathcal{E}^+)$ to $\eta_\lambda$ we deduce that

$$
\| \sum_{p \in F} (\xi_p u_\lambda | \xi_p u_\lambda) \| = \| (\sum_{p \in F} \psi_p^+(\xi_p)) \eta_\lambda \|^2 \leq \| \sum_{p \in F} \psi_p^+(\xi_p) \|^2.
$$

Since $\hat{\pi}$ is isometric, the corresponding norm inequality holds in $B$ with $\pi$ in place of $\psi^+$, and so

$$
\| \sum_{p \in F} \pi_p(\xi_p u_\lambda | \xi_p u_\lambda) \| \leq \| \sum_{p \in F} \pi_p(\xi_p) \|^2 = \| \sum_{p,q \in F} \pi_p(\xi_p)^* \pi_q(\xi_q) \|.
$$

Taking the limit over $\lambda$ we obtain the desired inequality.

Now let $n \geq 2$, and for each $i = 1, \ldots, n$, let $F_i \subset P$ be a finite set. For each $i \in \{1, \ldots, n\}$ and $p \in F_i$, choose $\xi_p \in E_p$. For each $i = 1, \ldots, n$, set $b_i := \sum_{p \in F_i} \psi_p^+(\xi_p)$ and consider the matrix

$$
b := \begin{bmatrix}
b_1 & 0 & \cdots & 0 \\
b_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
b_n & 0 & \cdots & 0
\end{bmatrix} \in M_n(\mathcal{T}_\lambda(E)^+).$$
That is, $b \in M_n(T_\lambda(\mathcal{E})^+)$ is the matrix whose $i$th entry of the first column is $b_i$ and the remaining columns vanish. Again take an approximate identity $(u_\lambda)_{\lambda \in \Lambda}$ for $A$ and let $\eta_\lambda = u_\lambda \in \mathcal{E}_e \subset \mathcal{E}^+$ be as above. Let $\eta_{\lambda,n} \in (\mathcal{E}^+)^n$ be given by

$$\eta_{\lambda,n} := (\eta_\lambda, 0, \ldots, 0).$$

Then $\|\eta_{\lambda,n}\| = \|\eta_\lambda\| = 1$, giving

$$\|b \eta_{\lambda,n}\|^2 \leq \|b\|^2. \quad (3.8)$$

Next we compute

$$\|b \eta_{\lambda,n}\|^2 = \|\sum_{i=1}^n (b_i \eta_\lambda | b_i \eta_\lambda)\| = \|\sum_{i=1}^n \sum_{p \in F_i} (\xi_p u_\lambda | \xi_p u_\lambda)\|, \quad (3.9)$$

and observe that $\hat{\pi}_n(b)^* \hat{\pi}_n(b)$ is the matrix whose first entry of the first row is

$$\sum_{i=1}^n \hat{\pi}(b_i)^* \hat{\pi}(b_i) = \sum_{i=1}^n \sum_{p,q \in F_i} \pi_p(\xi_p)^* \pi_q(\xi_q)$$

and the remaining entries are all zero. Thus using that $\hat{\pi}$ is completely isometric, we can replace $\|b\|^2$ in (3.8) by $\|\hat{\pi}_n(b)\|^2$, and combining the resulting inequality with (3.9) we obtain

$$\|\sum_{i=1}^n \sum_{p \in F_i} \pi_p(\xi_p u_\lambda | \xi_p u_\lambda)\| \leq \|\hat{\pi}_n(b)^* \hat{\pi}_n(b)\| = \|\sum_{i=1}^n \sum_{p,q \in F_i} \pi_p(\xi_p)^* \pi_q(\xi_q)\|.$$

Then the assertion follows by taking the limit over $\lambda$ in the inequality above. \qed

4. ON COVARIANCE ALGEBRAS AND THEIR GAUGE-EQUIVARIANT QUOTIENTS

In this section we first present a few technical results concerning the norm of an element in the fixed-point algebra of the covariance algebra of a product system $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$. Then we prove a result that will be used to show that the $C^*$-envelope of the tensor algebra of $\mathcal{E}$ carries a coaction of $G$ for which the quotient map $T_\lambda(\mathcal{E}) \to C^*_{\text{env}}(T_\lambda(\mathcal{E})^+)$ is gauge-equivariant.

As usual we let $P$ be a submonoid of a group $G$ and let $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ be a product system over $P$ with coefficient $C^*$-algebra $A$. We first recall the construction of the covariance algebra of $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ and its defining relations. We refer to [29] for further details. Let $F \subset G$ be a finite set. We set

$$K_F := \bigcap_{g \in F} gP.$$

For each $p \in P$ and each $F \subset G$ finite, we define an ideal $I_{p^{-1}(p \vee F)} \triangleleft A$ as follows. For $g \in F$, we let

$$I_{p^{-1}K_{(p,g)}} := \begin{cases} \ker \varphi_{p^{-1}r} & \text{if } K_{(p,g)} \neq \emptyset \text{ and } p \notin K_{(p,g)}, \\ A & \text{otherwise.} \end{cases}$$

We then set

$$I_{p^{-1}(p \vee F)} := \bigcap_{g \in F} I_{p^{-1}K_{(p,g)}}.$$

This yields a correspondence $\mathcal{E}_F : A \rightsquigarrow A$ by setting

$$\mathcal{E}_F := \bigoplus_{p \in P} \mathcal{E}_p I_{p^{-1}(p \vee F)}.$$

Let $\mathcal{E}_F^+$ be the right Hilbert $A$-module $\bigoplus_{g \in G} \mathcal{E}_g F$. For each $\xi \in \mathcal{E}_p$, we define an operator $\hat{t}_p^\mathcal{E}(\xi) \in \mathcal{B}(\mathcal{E}_F^+)$ so that it maps the direct summand $\mathcal{E}_g F$ into $\mathcal{E}_{pg} F$ for all $g \in G$. Explicitly,

$$\hat{t}_p^\mathcal{E}(\xi)(\eta_r) := \mu_{p,r}(\xi \otimes_A \eta_r), \quad \eta_r \in \mathcal{E}_r I_{r^{-1}(r \vee gF)}.$$
This is well defined because $I_{r-1}(\mathcal{F}) = I_{\{p\}}^{r-1}(\mathcal{F})$ for each $F \subset G$ finite and each $p \in P$. Its adjoint $t_p^* \circ (\xi^*)$ maps $\mu_{p,r}(\xi \otimes \eta_r)$ to $\phi_r((\xi_0 \otimes \eta_r) \eta_r).$ Again this is well defined since $I_{r-1}(\mathcal{F}) = I_{s-1}(\mathcal{F})$ for all $s \in pP$. Thus $t_F = \{t_p\}_{p \in P}$ is a representation of $\mathcal{F}$, and so it induces a $^*$-homomorphism $t_F: \mathcal{C}_{\text{rep}}(\mathcal{E}) \to \mathcal{B}(\mathcal{F}_F)$, still denoted by $t_F$ by abuse of language.

Let $Q^F_g$ be the projection of $\mathcal{E}_{\mathcal{F}}^c$ onto the direct summand $\mathcal{E}_{\mathcal{F}}^c$. Then

$$t_p^* \circ (\xi^*) = Q^F_g t_p \circ (\xi) \circ Q^F_g$$

for all $p \in P$ and $\xi \in \mathcal{E}_p$, which implies that $\mathcal{E}_p$ is invariant under the image of $\mathcal{C}_{\text{rep}}(\mathcal{E})_c$ under $t_F$. If $F_1 \subset F_2$ are finite subsets of $G$, then

$$I_{p-1}(\mathcal{F}) = I_{p-1}(\mathcal{F})$$

for all $p \in P$ and hence $\mathcal{E}_{F_2}$ may be regarded as a closed submodule of $\mathcal{E}_{F_1}$. In particular, we have

$$\|Q^F_g t_F (b) Q^F_g\| \leq \|Q^F_g t_F (b) Q^F_g\|$$

for all $b \in \mathcal{C}_{\text{rep}}(\mathcal{E})_c$. So let $F$ range in the directed set consisting of all finite subsets of $G$ ordered by inclusion and define an ideal $J_\mathcal{F} \subset \mathcal{C}_{\text{rep}}(\mathcal{E})_c$ by

$$J_\mathcal{F} := \{ b \in \mathcal{C}_{\text{rep}}(\mathcal{E})_c \mid \lim_F \|b\|_F = 0 \},$$

where $\|b\|_F := \|Q^F_g t_F (b) Q^F_g\|$.

**Definition 4.1 (29, Definition 3.2)**. A representation $\pi = \{\pi_p\}_{p \in P}$ of $\mathcal{E}$ in a $C^*$-algebra $B$ is strongly covariant if the induced $^*$-homomorphism $\tilde{\pi}: \mathcal{C}_{\text{rep}}(\mathcal{E}) \to B$ vanishes on $J_\mathcal{F}$.

Let $\mathcal{J}_{\infty} \subset \mathcal{C}_{\text{rep}}(\mathcal{E})$ be the ideal generated by $J_\mathcal{F}$. The covariance algebra of $\mathcal{E}$, denoted by $A \times_\mathcal{E} P$, is the quotient $C^*$-algebra $\mathcal{C}_{\text{rep}}(\mathcal{E})/\mathcal{J}_{\infty}$.

We let $q: \mathcal{C}_{\text{rep}}(\mathcal{E}) \to A \times_\mathcal{E} P$ be the quotient map and let $j = \{j_p\}_{p \in P}$ be induced representation of $\mathcal{E}$ in $A \times_\mathcal{E} P$, that is, $j_p = q \circ \tilde{i}_p$ for all $p \in P$. Then the pair $(A \times_\mathcal{E} P, j)$ has the following universal property: if $\pi = \{\pi_p\}_{p \in P}$ is a strongly covariant representation of $\mathcal{E}$ in a $C^*$-algebra $B$, then there is a unique $^*$-homomorphism $\tilde{\pi}: A \times_\mathcal{E} P \to B$ such that $\tilde{\pi} \circ j_p = \pi_p$ for all $p \in P$. It also follows that the group $G$ in question may be taken to be any group containing $P$ as a submonoid (see 29, Theorem 3.10).

By 29, Lemma 3.3, the ideal $J_{\infty}$ satisfies

$$J_{\infty} = \bigoplus_{g \in G} (J_{\infty} \cap \mathcal{C}_{\text{rep}}(\mathcal{E})_g) = \bigoplus_{g \in G} \mathcal{C}_{\text{rep}}(\mathcal{E})_g J_\mathcal{F},$$

and so the gauge coaction $(\mathcal{C}_{\text{rep}}(\mathcal{E}), G, \delta)$ gives rise to a canonical coaction

$$\delta: A \times_\mathcal{E} P \to (A \times_\mathcal{E} P) \otimes C^*(G)$$

with $\delta(j_p(\xi)) = j_p(\xi) \otimes u_p$ for all $p \in P$ and $\xi \in \mathcal{E}_p$. The spectral subspace $[A \times_\mathcal{E} P]_g$ at $g \in G$ is canonically isomorphic to the quotient $\mathcal{C}_{\text{rep}}(\mathcal{E})_g/\mathcal{C}_{\text{rep}}(\mathcal{E})_g J_\mathcal{F}$. See 29, Lemma 3.4 for further details. An important property of $A \times_\mathcal{E} P$ is that a $^*$-homomorphism $\tilde{\pi}: A \times_\mathcal{E} P \to B$ is faithful on the fixed-point algebra $(A \times_\mathcal{E} P)^\delta = [A \times_\mathcal{E} P]_e$ for $\delta$ if and only if it is faithful on the coefficient algebra $A$ (see 29, Theorem 3.10((C3))).

In the next lemma we will give a more explicit description of the ideal $J_\mathcal{F}$ and of the norm of an element in $(A \times_\mathcal{E} P)^\delta$.

**Lemma 4.2.** Let $q: \mathcal{C}_{\text{rep}}(\mathcal{E}) \to A \times_\mathcal{E} P$ be the quotient map and for each finite set $F \subset G$, let $J_{\mathcal{F}}$ be the kernel of the $^*$-homomorphism from $\mathcal{C}_{\text{rep}}(\mathcal{E})_c$ into $\mathcal{B}(\mathcal{F}_F)$ given by $b \mapsto Q^F_g t_F (b) Q^F_g$. Then

1. $J_\mathcal{F} = \bigcup_F J_{\mathcal{F},F}$.
(2) \( ||q(b)|| = \lim_{F \to \infty} ||b||_F \) for all \( b \in \mathcal{C}_{\text{rep}}^*(\mathcal{E}_e) \), where \( F \) ranges over the finite subsets of \( G \) directed by inclusion.

**Proof.** We begin by proving (1). It follows from the definition of \( J_e \) that \( J_e, F \subset J_e \) for all \( F \subset G \) finite. This gives the inclusion \( \bigcup_F J_e, F \subset J_e \). In order to establish the reverse inclusion, let \( b \in J_e \). Then \( \lim_F ||b||_F = 0 \), and thus given \( \varepsilon > 0 \), there exists a finite set \( F' \subset G \) such that \( ||b||_F < \varepsilon \) for all \( F \subset G \) finite such that \( F \supset F' \). Observe that \( ||b||_F \) is precisely the norm of the image of \( b \) in \( \mathcal{C}_{\text{rep}}^*(\mathcal{E}_e) / J_e, F \) under the quotient map. Thus we can find \( c \in J_e, F' \) such that

\[
||b||_{F'} = \inf \{ ||b + d|| \mid d \in J_e, F' \} \leq ||b + c|| < \varepsilon.
\]

This proves that \( b \) lies in the closure of \( \bigcup_F J_e, F \), giving the inclusion \( J_e \subset \bigcup_F J_e, F \).

Next we prove (2). Since \( J_e, F \subset J_e \), the quotient map \( \mathcal{C}_{\text{rep}}^*(\mathcal{E}_e) \to \mathcal{C}_{\text{rep}}^*(\mathcal{E}_e) / J_e = (A \times \mathcal{E})^\delta \) factors through \( \mathcal{C}_{\text{rep}}^*(\mathcal{E}_e) / J_e, F \). Denoting by \( q_F \) the quotient map \( \mathcal{C}_{\text{rep}}^*(\mathcal{E}_e) \to \mathcal{C}_{\text{rep}}^*(\mathcal{E}_e) / J_e, F \), we deduce that for all \( b \in \mathcal{C}_{\text{rep}}^*(\mathcal{E}_e) \)

\[
||q(b)|| \leq ||q_F(b)|| = ||b||_F.
\]

Because \( F \subset G \) is an arbitrary finite set, it follows that \( ||q(b)|| \leq \lim_{F \to \infty} ||b||_F \).

It remains to prove the inequality \( \lim_{F \to \infty} ||b||_F \leq ||q(b)|| \). Given \( \varepsilon > 0 \), let \( c \in J_e \) be such that \( ||q(b)|| \leq ||b + c|| < ||q(b)|| + \varepsilon \). Using (1), we may assume that \( c \in J_e, F' \) for some finite set \( F' \subset G \). It follows that

\[
\lim_{F \to \infty} ||b||_F \leq ||b||_{F'} \leq ||b + c|| < ||q(b)|| + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, we conclude that \( \lim_{F \to \infty} ||b||_F \leq ||q(b)|| \). This proves (2) and completes the proof of the lemma. \( \square \)

**Notation 4.3** (see [19 Section 2]). In what follows we will use the following notation:

- \( \mathcal{W}(P) \) will denote the set of words in \( P \) of even length.
- For a word \( \alpha = (p_1, p_2, \ldots, p_{2k-1}, p_{2k}) \in \mathcal{W}(P) \), we set
  \[
\hat{\alpha} := p_1 p_2^{-1} \cdots p_{2k-1} p_{2k}^{-1} \in G \quad \text{and} \quad \check{\alpha} = (p_{2k}, p_{2k-1}, \ldots, p_2, p_1) \in \mathcal{W}(P).
\]
- Given \( \alpha \in \mathcal{W}(P) \), we set
  \[
Q(\alpha) := \{ p_{2k}, p_{2k}^{-1}, p_{2k-1}, \ldots, \hat{\check{\alpha}} p_1 \}, \quad \text{and} \quad K(\alpha) := K(Q(\alpha)) = p_{2k} P \cap p_{2k}^{-1}, p_{2k-1} P, \ldots, \hat{\check{\alpha}} p_1 P.
\]

We refer to \( Q(\alpha) \) as the **iterated quotient set** of the word \( \alpha \). We will say that the word \( \alpha \) is **neutral** if \( \hat{\alpha} = e \). The set \( K(\alpha) \) is a **constructible right ideal** of \( P \) in the sense of Li [21].

**Remark 4.4.** For the purpose of this paper it is more convenient to work with the generalised iterated **right** quotient map \( \mathcal{W}(P) \to G \) that associates the product of right quotients \( \hat{\alpha} = p_1 p_2^{-1} \cdots p_{2k-1} p_{2k}^{-1} \in G \) to a word \( \alpha = (p_1, \ldots, p_{2k}) \in \mathcal{W}(P) \), rather than the iterated left quotient map used in [19 Section 2]. This is because we consider elements of the form

\[
\psi_{p_1}^+(\xi_{p_1}) \psi_{p_2}^+(\xi_{p_2})^* \cdots \psi_{p_{2k-1}}^+(\xi_{p_{2k-1}}) \psi_{p_{2k}}^+(\xi_{p_{2k}})^* \]

as a canonical spanning set of \( \mathcal{T}_\mathcal{E} \). By doing so we are following [29] and [6] and also avoid any further use of the nondegeneracy assumption on \( \mathcal{E} = (\mathcal{E}_P)_{P \subset P} \). As a result the meaning of \( Q(\alpha) \) and \( K(\alpha) \) here is slightly different from that in [19 Section 2].

**Lemma 4.5.** Let \( j = \{ j_P \}_{P \subset P} \) be the canonical representation of \( \mathcal{E} \) in \( A \times \mathcal{E} \). Consider a word \( \alpha = (p_1, p_2, \ldots, p_{2k-1}, p_{2k}) \in \mathcal{W}(P) \) and for each \( i = 1, \ldots, 2k \) let \( \xi_{p_i} \in \mathcal{E}_{p_i} \). Let \( F \subset G \) be a finite set containing the iterated quotient set \( Q(\alpha) \) of \( \alpha \).
(1) If $r \in P$ and $r \notin K(\alpha)$, then for all $\xi \in {\mathcal E}_r I_{r^{-1}(\Gamma \vee F)}$ we have

$$j_1(\xi_1)j_2(\xi_2)\cdots j_{p_{2k-1}}(\xi_{p_{2k-1}})j_{p_{2k}}(\xi_{p_{2k}})^*j_r(\xi) = 0.$$ 

(2) If $r \in K(\alpha)$, then $\hat{\alpha}r \in P$ and we have

$$j_1(\xi_1)j_2(\xi_2)^*\cdots j_{p_{2k-1}}(\xi_{p_{2k-1}})j_{p_{2k}}(\xi_{p_{2k}})^*j_r(\xi_\hat{\alpha}r) \subseteq j_1\hat{\alpha}(\xi_\hat{\alpha}r).$$

Proof. Part (1) was shown in the proof of [29, Lemma 3.6], and so we briefly indicate how the proof goes. Suppose that $r \notin K(\alpha)$ and let $\xi \in {\mathcal E}_r I_{r^{-1}(\Gamma \vee F)}$. If $rP \cap K(\alpha) = \emptyset$, then in $T_\chi(\mathcal{E})$ we have

$$\psi_1^+(\xi_1)\psi_2^+(\xi_2)^*\cdots \psi_{p_{2k-1}}^+(\xi_{p_{2k-1}})\psi_{p_{2k}}^+(\xi_{p_{2k}})^*\psi_r^+(\xi) = 0$$

because the restriction of $\psi_1^+(\xi_1)\psi_2^+(\xi_2)^*\cdots \psi_{p_{2k-1}}^+(\xi_{p_{2k-1}})\psi_{p_{2k}}^+(\xi_{p_{2k}})^*\psi_r^+(\xi)$ to a direct summand $E_s \subset \mathcal{E}$ does not vanish only if $s \in K(\alpha)$. This implies that

$$j_1(\xi_1)j_2(\xi_2)^*\cdots j_{p_{2k-1}}(\xi_{p_{2k-1}})j_{p_{2k}}(\xi_{p_{2k}})^*j_r(\xi) = 0$$

in $A \times \times P$. Assume now that $rP \cap K(\alpha) \neq \emptyset$ but $r \notin K(\alpha)$. Then there is $g \in Q(\alpha)$ such that $r \notin gP$. Since $F \supset Q(\alpha)$, we also have that $g \in F$. It follows that

$$\psi_1^+(\xi_1)\psi_2^+(\xi_2)^*\cdots \psi_{p_{2k-1}}^+(\xi_{p_{2k-1}})\psi_{p_{2k}}^+(\xi_{p_{2k}})^*\psi_r^+(\xi) = 0$$

in $T_\chi(\mathcal{E})$ because if $s \in P$ is such that $rs \in K(\alpha)$, we have $rs \in rP \cap gP$ and so $I_{r^{-1}(\Gamma \vee F)} \subset \ker \varphi_s$. Thus the corresponding element in $A \times P$ also vanishes. This gives part (1).

For part (2), notice that $K(\alpha) \subset \tilde{\alpha}p_1 P \cap P$. Thus if $r \in K(\alpha)$, we see that $\hat{\alpha}r \in \tilde{\alpha}\hat{\alpha}p_1 P = p_1 P \subset P$. So for $\xi \in {\mathcal E}_r$, we deduce that

$$j_1(\xi_1)j_2(\xi_2)^*\cdots j_{p_{2k-1}}(\xi_{p_{2k-1}})j_{p_{2k}}(\xi_{p_{2k}})^*j_r(\xi) \in j_1\hat{\alpha}(\xi_\hat{\alpha}r)$$

because $j$ is a representation of $\mathcal{E}$.

Corollary 4.6. Let $q: C^*_{top}(\mathcal{E}) \to A \times \times P$ be the quotient map and let $b \in C^*_{top}(\mathcal{E})_e$ be an element of the form $b = \sum_{i=1}^n b_i$, where $n \geq 1$ and each $b_i$ in turn has the form

$$b_i = \tilde{\iota}(\xi_1)^*\tilde{\iota}(\xi_2)^*\cdots \tilde{\iota}(\xi_{p_{2k}})^*$$

with $\alpha_i := (p_1, \ldots, p_{2k}) \in W(P)$ a neutral word. Let $F \subset G$ be a finite set containing the iterated quotient set $Q(\alpha_i)$ for every $i = 1, \ldots, n$. Then

$$\|q(b)\| = \|b\|_F.$$ 

Proof. It follows from part (2) of Lemma 4.2 that $\|q(b)\| \leq \|b\|_F$. In order to show that $\|b\|_F \leq \|q(b)\|$, let $F' \subset P$ be a finite set and for each $r \in F'$ take $\xi_r \in {\mathcal E}_r I_{r^{-1}(\Gamma \vee F)}$. Observe that because $F$ contains $Q(\alpha_i)$ for every $i = 1, \ldots, n$, it follows from Lemma 4.5 that for all $r \in F'$ and $i = 1, \ldots, n$ we have that $q(b_i)j_r(\xi_r) \in j_r(\mathcal{E}_r)$ in case $r \in K(\alpha_i)$, and $q(b_i)j_r(\xi_r) = 0$ in case $r \notin K(\alpha_i)$. Notice that if $r \in K(\alpha_i)$, then $q(b_i)j_r(\xi_r) = j_r(\eta_r)$ with $\eta_r := t_F(b_i)\xi_r \in {\mathcal E}_r I_{r^{-1}(\Gamma \vee F)}$ since $j = \{j_p\}_{p \in P}$ is a representation of $\mathcal{E}$. Now we have in $A \times \times P$ that

$$\|\sum_{r \in F'} j_r(\xi_r)^*q(b)^*q(b)j_r(\xi_r)\| \leq \|q(b)\|^2\|\sum_{r \in F'} j_r(\xi_r)\|.$$ 

Hence if we let $\xi := \sum_{r \in F'} \xi_r \in {\mathcal E}_F$, we see from the previous observation that the left-hand side above is precisely $\|t_F(b)\xi\|^2$ (see also the proof of [29, Lemma 3.6]). Also, $\|\sum_{r \in F'} j_r(\xi_r)\| = \|\xi\|^2$. So we have

$$\|t_F(b)\xi\|^2 \leq \|q(b)\|^2\|\xi\|^2,$$

which gives $\|b\|_F \leq \|q(b)\|$ because the set of elements of the form $\xi = \sum_{r \in F'} \xi_r \in {\mathcal E}_F$ with $F' \subset P$ finite is dense in $\mathcal{E}_F$. This shows that $\|q(b)\| = \|b\|_F$ as asserted.

The next is the main result of this section.
Proposition 4.7. Let \( \pi = \{ \pi_p \}_{p \in P} \) be an injective strongly covariant representation of \( \mathcal{E} \) in a \( C^* \)-algebra \( B \) and let \( \hat{\pi} : A \times \mathcal{E} \to B \) be the induced \(*\)-homomorphism. Then the following are equivalent:

1. There exists a conditional expectation \( E_\pi : \hat{\pi}(A \times \mathcal{E}) \to \hat{\pi}((A \times \mathcal{E})^\delta) \) such that the diagram

\[
\begin{array}{ccc}
A \times \mathcal{E} P & \xrightarrow{\hat{\pi}} & \hat{\pi}(A \times \mathcal{E} P) \\
E^\delta & \downarrow & \downarrow E_\pi \\
(A \times \mathcal{E} P)^\delta & \xrightarrow{\hat{\pi}} & \hat{\pi}((A \times \mathcal{E})^\delta)
\end{array}
\]

commutes.

2. For every \( n \geq 1 \), finite sets \( F_1, F_2, \ldots, F_n \subset P \) and elements \( \xi_p \in \mathcal{E}_p \) for \( p \in F_i \), \( i = 1, \ldots, n \), we have that

\[
\| \sum_{i=1}^n \pi_p(\langle \xi_p | \xi_p \rangle) \| \leq \| \sum_{i=1}^n \pi_p(\langle \xi_p \rangle^* \pi_q(\xi_q) \|.
\]

Proof. The implication (1) \( \Rightarrow \) (2) follows from evaluating \( E_\pi \) at \( \sum_{i=1}^n \sum_{q, \xi \in F_1} \pi_p(\langle \xi_p \rangle^* \pi_q(\xi_q) \) and using that \( E_\pi \) is contractive. Now assume that (2) holds. In order to establish the existence of \( E_\pi \), it suffices to show that for every finite set \( D \subset G \) and choice of elements \( c_g \in [A \times \mathcal{E}]_g \) for \( g \in D \), we have

\[
\| \hat{\pi}(c_g) \| \leq \| \sum_{g \in D} \hat{\pi}(c_g) \|.
\]

This is so because the map \( \sum \hat{\pi}(c_g) \to \hat{\pi}(c_e) \) then extends by continuity to give a conditional expectation \( E_\pi : \hat{\pi}(A \times \mathcal{E}) \to \hat{\pi}((A \times \mathcal{E})^\delta) \) as in the statement. We may as well assume that each \( c_g \) has the form \( c_g = q(b_g) \), where \( b_g = \sum_{i=1}^n b_{g,i} \in \mathcal{C}_\pi^\vee(\mathcal{E})_g \), \( n_g \geq 1 \) and \( b_{g,i} \) in turn has the form

\[
b_{g,i} = i (\xi_{p_1})^* i (\xi_{p_2})^* \cdots i (\xi_{p_{2k}})^* \]

with \( \alpha_i := (p_1, \ldots, p_{2k}) \in \mathcal{W}(P) \) satisfying \( \hat{\alpha}_i = \hat{g} \).

Let \( c = \sum_{g \in D} c_g \in A \times \mathcal{E} \) and \( b_g \in \mathcal{C}_\pi(\mathcal{E})_g \) as above with \( c_g = q(b_g) \). For each \( g \in D \), consider the union of iterated quotient sets \( F_0 := \bigcup_{i=1}^n Q(\alpha_i) \). Put \( F := \bigcup_{g \in D} F_0 \). Fix \( r \in P \) and let \( \xi_r \in \mathcal{E}_r L^{-1}(\mathcal{F}/P) \). It follows from Lemma 4.5 that \( c_{g,j,r}(\xi_r) = 0 \) if \( gr \notin P \) because then, in particular, \( r \notin K(\alpha_i) \) for all \( i = 1, \ldots, n_g \). In case \( gr \in P \) we have \( c_{g,j,r}(\xi_r) \in j_{gr}(\mathcal{E}_r) \) again by Lemma 4.5 because \( F \) contains \( Q(\alpha_i) \) for all \( i = 1, \ldots, n_g \). Consider the finite subset of \( P \) given by \( F_r := \{ gr \mid g \in D \text{ and } gr \in P \} \). For each \( p \in F_r \) let \( g \in D \) be such that \( p = gr \) and let \( \eta_p \in \mathcal{E}_p \) be the vector satisfying \( j_p(\eta_p) = c_{g,j,r}(\xi_r) \). Then

\[
\| \hat{\pi}(c) \|^2 \pi_p(\langle \xi_r | \xi_r \rangle) \geq \pi_r(\xi_r)^* \hat{\pi}(c)^* \hat{\pi}(c) \pi_r(\xi_r)
\]

\[
= \sum_{g,h \in D} \pi_r(\xi_r)^* \hat{\pi}(c_g)^* \hat{\pi}(c_h) \pi_r(\xi_r)
\]

\[
= \sum_{p \in F_r} \pi_p(\eta_p)^* \pi_q(\eta_q).
\]

Thus applying (2) with \( n = 1 \), \( F_1 = F_r \) and \( \eta_p \) playing the role of \( \xi_p \) for \( p \in F_r \), we obtain the inequality

\[
\| \sum_{p \in F_r} \pi_p(\langle \eta_p | \eta_p \rangle) \| \leq \| \hat{\pi}(c) \|^2 \| \pi_p(\langle \xi_r | \xi_r \rangle) \|.
\]

We compute

\[
\sum_{p \in F_r} \pi_p(\langle \eta_p | \eta_p \rangle) = \pi_r(\xi_r)^* \left( \sum_{g \in D} \hat{\pi}(c_g)^* \hat{\pi}(c_g) \right) \pi_r(\xi_r) \geq \pi_r(\xi_r)^* \hat{\pi}(c_e)^* \hat{\pi}(c_e) \pi_r(\xi_r).
\]

(4.9)
Now let $F' \subset P$ be a finite set and for each $r \in F'$, take $\xi_r \in \mathcal{E}_r I_{r, r'} \cap (r \vee F')$. For each $r \in F'$, set $F'_r = \{gr \mid g \in D \text{ and } gr \in P\}$. Then $F'_r$ is a finite subset of $P$, and for each $p \in F'_r$, take $g \in D$ such that $gr = p$. Let $\eta_p \in \mathcal{E}_p$ be the vector such that $j(\eta_p) = c_g j(\xi_r)$. Then as in (4.8) we obtain
\[
\|\hat{\pi}(c)\|^2 \sum_{r \in F'} \pi_r(\langle \xi_r \mid \xi_r \rangle) \geq \sum_{r \in F'} \sum_{g, h \in D} \pi_r(\langle \xi_r \rangle)^* \hat{\pi}(c)^* \hat{\pi}(c) \pi_r(\langle \xi_r \rangle) \\
= \sum_{r \in F'} \sum_{g, h \in D} \pi_r(\langle \xi_r \rangle)^* \hat{\pi}(c_g)^* \hat{\pi}(c_h) \pi_r(\langle \xi_r \rangle) \\
= \sum_{r \in F'} \sum_{p, q \in F'_r} \pi_p(\eta_p)^* \pi_q(\eta_q).
\]

Thus applying (2) to the sum in the last line above with $n = |F'|$, the finite sets $F'_r$ for $r \in F'$, and the elements $\eta_p \in \mathcal{E}_p$ for $p \in F_r$ and using (4.9), we deduce that
\[
\| \sum_{r \in F'} \pi_r(\langle \xi_r \rangle)^* \hat{\pi}(c)^* \hat{\pi}(c) \pi_r(\langle \xi_r \rangle) \| \leq \| \hat{\pi}(c) \|^2 \| \sum_{r \in F'} \pi_r(\langle \xi_r \rangle) \|.
\]

As in the proof of Corollary 4.6 let $\xi = \sum_{r \in F'} \xi_r \in \mathcal{E}_F$. Since $\pi = \{\pi_p\}_{p \in P}$ is injective, the left-hand side above is precisely $\|t_F(b_e)\xi\|^2$, and also $\| \sum_{r \in F'} \pi_r(\langle \xi_r \mid \xi_r \rangle) \| = \|\xi\|^2$. So we have
\[
\|t_F(b_e)\xi\|^2 \leq \|\hat{\pi}(c)\|^2 \|\xi\|^2,
\]
which implies that $\|b_e\|_F \leq \|\hat{\pi}(c)\|$. Combining this with Corollary 4.6 we obtain
\[
\|\hat{\pi}(c_e)\| = \|c_e\| = \|q(b_e)\| = \|b_e\|_F \leq \|\hat{\pi}(c)\|.
\]

This implies the existence of $E_{\pi^*}$, completing the proof of the proposition. \qed

5. The main theorem

We now combine the results from previous sections to show that the following three notions of boundary quotients associated to a product system $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ coincide: the $C^*$-envelope of the tensor algebra $\mathcal{T}_\lambda(\mathcal{E})^+$, in the sense of Arveson and Hamana; the $C^*$-envelope of the canonical cosystem associated to $\mathcal{T}_\lambda(\mathcal{E})^+$ by Dor-On, Kakariadis, Katsoulis, Laca and Li [6]; and the reduced cross sectional $C^*$-algebra of the Fell bundle associated to the canonical coaction on the covariance algebra of $\mathcal{E}$ as constructed in [29].

Notice that our approach only requires that $P$ be a submonoid of a group and our standard assumption that the left actions of $A$ on the underlying correspondences are nondegenerate, since these are precisely the assumptions adopted in [29]. Thus as opposed to [5,6,13] we do not require that $P$ be a right LCM monoid, and even for $P$ a right LCM monoid, we do not assume $\mathcal{E}$ to be compactly aligned. In case of compactly aligned product systems over right LCM monoids, we do not require the existence of a controlled map with abelian codomain as in [13 Theorem 6.1] (see [13 Definition 5.1]). Also, as a consequence of our main theorem, we deduce that the boundary quotient $\partial \mathcal{T}_\lambda(P)$ of the Toeplitz algebra $\mathcal{T}_\lambda(P)$ of a submonoid of a group is canonically isomorphic to the $C^*$-envelope of the closed non-selfadjoint subalgebra spanned by the canonical generating isometries of $\mathcal{T}_\lambda(P)$. This shows that the assumptions in Theorem 4.5 and Theorem 4.6 of [12] are not necessary.

Before proving our main theorem, we briefly recall the definition of the $C^*$-envelope of a cosystem from [6]. Let $A$ be an operator algebra. A coaction of a discrete group $G$ on $A$ is a completely isometric homomorphism $\gamma: A \to A \otimes C^*(G)$ such that $\sum A_g$ is norm-dense in $A$, where
\[
A_g = \{a \in A \mid \gamma(a) = a \otimes u_g\}.
\]
The triple $(A, G, \gamma)$ is called a cosystem. See [6, Definition 3.1] for further details. Notice that if $A$ is a $C^*$-algebra, $\gamma$ is automatically a nondegenerate coaction on $A$. 

Let \((A, G, \gamma)\) and \((B, G, \gamma')\) be two cosystems. A completely contractive homomorphism 
\(\phi: A \rightarrow B\) is said to be \(\gamma - \gamma'\)-equivariant or simply equivariant if 
\(\gamma' \circ \phi = (\phi \otimes \text{id}_{C^*(G)}) \circ \gamma\). 
A triple \((B, \rho, \gamma')\) is a \(C^*\)-cover for the cosystem \((A, G, \gamma)\) if \((B, G, \gamma')\) is a cosystem and the pair \((B, \rho)\) is a \(C^*\)-cover for \(A\) with \(\rho: A \rightarrow B\) being \(\gamma - \gamma'\)-equivariant. See [6, Definition 3.6]. 
The \(C^*\)-envelope of \((A, G, \gamma)\) is a \(C^*\)-cover for \((A, G, \gamma)\), denoted by 
\((C^*_{\text{env}}(A, G, \gamma), \iota_{\text{env}}, \gamma_{\text{env}})\)
 or simply \(C^*_{\text{env}}(A, G, \gamma)\), satisfying the following property: if \((B, \rho, \gamma')\) is a \(C^*\)-cover for 
\((A, G, \gamma)\), then there exists a \(\gamma' - \gamma_{\text{env}}\)-equivariant surjective \(\ast\)-homomorphism 
\(\phi: B \rightarrow C^*_{\text{env}}(A, G, \gamma)\) such that \(\phi \circ \rho = \iota_{\text{env}}\) [6 Definition 3.7]. The \(C^*\)-envelope of a cosystem 
\((A, G, \gamma)\) always exists by [6, Theorem 3.8], and is unique up to a canonical isomorphism.

Let \(\mathcal{E} = (\mathcal{E}_p)_{p \in P}\) be a product system and let \(\delta: \mathcal{T}_\lambda(\mathcal{E}) \rightarrow \mathcal{T}_\lambda(\mathcal{E}) \otimes C^*(G)\) be the canonical (normal) coaction of \(G\) on \(\mathcal{T}_\lambda(\mathcal{E})\). Let \(\delta^+\) denote the restriction of \(\delta\) to the tensor algebra \(\mathcal{T}_\lambda(\mathcal{E})^+\). This gives rise to the cosystem \((\mathcal{T}_\lambda(\mathcal{E})^+, G, \delta^+)\) as considered in [6,12,13].

**Theorem 5.1.** Let \(P\) be a submonoid of a group \(G\) and let \(\mathcal{E} = (\mathcal{E}_p)_{p \in P}\) be a product system over \(P\) with coefficient \(C^*\)-algebra \(A\). The following \(C^*\)-algebras associated to \(\mathcal{E}\) are canonically isomorphic:

1. the \(C^*\)-envelope \(C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)\);
2. the reduced cross sectional \(C^*\)-algebra of the Fell bundle \([A \times \mathcal{E} P]^g_{g \in G}\);
3. the \(C^*\)-envelope of the cosystem \((\mathcal{T}_\lambda(\mathcal{E})^+, G, \delta^+)\).

**Proof.** We begin by proving that the inclusion \(\iota: \mathcal{T}_\lambda(\mathcal{E})^+ \rightarrow C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)\) induces an isomorphism between the \(C^*\)-envelope \(C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)\) and the reduced cross sectional \(C^*\)-algebra \(C^*_r([A \times \mathcal{E} P]^g_{g \in G})\). Consider the \(\ast\)-homomorphism \(\phi_\lambda: \mathcal{T}_\lambda(\mathcal{E}) \rightarrow C^*_r([A \times \mathcal{E} P]^g_{g \in G})\) induced by the canonical representation of \(\mathcal{E}\) in \(C^*_r([A \times \mathcal{E} P]^g_{g \in G})\) (for the existence of \(\phi_\lambda\) see, for example, the discussion after [6, Proposition 5.4]). Then \(\phi_\lambda\) is injective on \(A\), and the canonical conditional expectation \(E_\lambda\) of \(C^*_r([A \times \mathcal{E} P]^g_{g \in G})\) onto the copy of \([A \times \mathcal{E} P]_e = ([A \times \mathcal{E} P]^\delta\) satisfies \(E_\lambda \circ \phi_\lambda = \phi_\lambda \circ E_\lambda\). Thus it follows from Corollary 3.5 that the restriction of \(\phi_\lambda\) to the tensor algebra \(\mathcal{T}_\lambda(\mathcal{E})^+\) is completely isometric.

We deduce that there exists a \(\ast\)-homomorphism 
\(\hat{\pi}: C^*_r([A \times \mathcal{E} P]^g_{g \in G}) \rightarrow C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)\)
such that \(\hat{\pi} \circ \phi_\lambda = \iota\) on \(\mathcal{T}_\lambda(\mathcal{E})^+\). In particular, \(\hat{\pi}\) is surjective. Since \(\hat{\pi}\) is injective on \(A\), it is also injective on the fixed-point algebra \([A \times \mathcal{E} P]_e = ([A \times \mathcal{E} P]^\delta\) by [20, Theorem 3.10].

The inclusion \(\iota: \mathcal{T}_\lambda(\mathcal{E})^+ \rightarrow C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)\) yields a canonical representation \(\iota = \{\iota_p\}_{p \in P}\) of \(\mathcal{E}\) in \(\mathcal{T}_\lambda(\mathcal{E})^+\), still denoted by \(\iota\) by abuse of language. Since \(\hat{\pi} \circ \phi_\lambda = \iota\) on \(\mathcal{T}_\lambda(\mathcal{E})^+\), it follows that \(\iota = \{\iota_p\}_{p \in P}\) is an injective strongly covariant representation of \(\mathcal{E}\). In addition, Lemma 3.7 implies that \(\iota = \{\iota_p\}_{p \in P}\) satisfies condition (2) of Proposition 4.1 since \(\iota: \mathcal{T}_\lambda(\mathcal{E})^+ \rightarrow C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)\) is completely isometric. Hence there exists a conditional expectation \(E_\lambda: C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+) \rightarrow \mathcal{T}_\lambda(\mathcal{E})^+\) satisfying \(E_\lambda \circ \iota = \hat{\pi} \circ E_\lambda\). Because \(E_\lambda\) is faithful and \(\hat{\pi}\) is faithful on \([A \times \mathcal{E} P]^\delta\), we deduce that \(\hat{\pi}\) is an isomorphism by [8, Theorem 19.5].

Finally, we establish the isomorphism between \(C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)\) and the \(C^*\)-envelope of the cosystem \((\mathcal{T}_\lambda(\mathcal{E})^+, G, \delta^+)\). It follows from the definition of \(C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+, G, \delta^+)\) that there exists a surjective \(\ast\)-homomorphism 
\(\phi: C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+, G, \delta^+) \rightarrow C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)\)
satisfying \(\phi \circ \iota_{\text{env}} = \iota\). To see that \(\phi\) is an isomorphism, notice that Fell’s absorption principle for Fell bundles, when combined with the canonical isomorphism \(C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+) \cong C^*_r([A \times \mathcal{E} P]^g_{g \in G})\) from the first part, gives that \(C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)\) carries a (normal) coaction 
\(\delta_\lambda: C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+) \rightarrow C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+) \otimes C^*(G)\)
satisfying \(\delta_\lambda(\iota_p(\xi)) = \iota_p(\xi) \otimes u_p\) for all \(p \in P\) and \(\xi \in \mathcal{E}_p\) (see [8, Proposition 18.24] and [6, Proposition 3.4]). Thus the inclusion \(\iota: \mathcal{T}_\lambda(\mathcal{E})^+ \rightarrow C^*_{\text{env}}(\mathcal{T}_\lambda(\mathcal{E})^+)\) is \(\delta^+ - \delta_\lambda\)-equivariant,
which implies that $(C^*_\text{env}(T_\lambda(\mathcal{E})), \iota, \delta_\lambda)$ is a $C^*$-cover for the cosystem $(T_\lambda(\mathcal{E})^+, G, \delta^+)$, Hence the defining property of $C^*_\text{env}(T_\lambda(\mathcal{E})^+)$ yields the desired inverse for $\phi$, proving that $\phi$ is an isomorphism.

**Remark 5.2.** We observe that the defining relations of the covariance algebra form an essential part of the proof of Proposition 4.7, and hence also of our proof that the $C^*$-envelope $C^*_\text{env}(T_\lambda(\mathcal{E})^+)$ necessarily carries a coaction for which the inclusion of $T_\lambda(\mathcal{E})^+$ is equivariant. In case of a more general operator algebra with a coaction, the question of whether the $C^*$-envelope of the corresponding cosystem coincides with the $C^*$-envelope of the underlying operator algebra remains open. More precisely, given a cosystem $(A, G, \gamma)$ that does not arise as the cosystem associated to the tensor algebra of a product system as above, it would be interesting to know whether the $C^*$-envelope of $A$ automatically carries a coaction of $G$ for which the inclusion $A \hookrightarrow C^*_\text{env}(A)$ is equivariant.

We highlight in the next corollary an important consequence of Theorem 5.1. See also the discussion after Proposition 5.4 in [0].

**Corollary 5.3.** Let $P$ be a submonoid of a group $G$ and let $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ be a product system over $P$. Let $q_\lambda : C^*_\text{reg}(\mathcal{E}) \to T_\lambda(\mathcal{E})$ be the *-homomorphism induced by the Fock representation of $\mathcal{E}$. If the sequence

$$0 \to J_\infty \xrightarrow{\gamma_\lambda} T_\lambda(\mathcal{E}) \xrightarrow{q_\lambda} C^*_\gamma([A \times_\mathcal{E} P]_g) \to 0$$

is exact (e.g. if $G$ is exact), then the Shilov boundary ideal for $T_\lambda(\mathcal{E})^+$ is the ideal of $T_\lambda(\mathcal{E})$ generated by

$$q_\lambda(J_e) = \{ c \in T_\lambda(\mathcal{E})_e | \lim_F \| c \|_F = 0 \},$$

where $F$ ranges over the finite subsets of $G$ ordered by inclusion and $\| c \|_F$ denotes the norm of the restriction of $c$ to $\mathcal{E}_F$.

We consider now the particular case in which $\mathcal{E} = C^P$ is the canonical product system over $P$ with one-dimensional fibres. The Fock representation of $C^P$ is simply the left regular representation $L : P \to B(\ell^2(P))$ on $\ell^2(P)$. Thus $T_\lambda(C^P) = T_\lambda(P)$ is the Toeplitz algebra of $P$, and the corresponding tensor algebra of $C^P$, which we denote by $T_\lambda(P)^+$, is then the closed non-selfadjoint subalgebra of $T_\lambda(P)$ generated by the isometries $\{ K_p | p \in P \}$. The fixed-point algebra $T_\lambda(P)_\gamma$, associated to the canonical coaction $\gamma$ of $G$ on $T_\lambda(P)$ coincides with the diagonal $C^*$-subalgebra $D_\gamma \subset T_\lambda(P)$, and so $T_\lambda(P)_\gamma$ is a commutative $C^*$-algebra. See, for example, [1, 19, 21] for further details on Toeplitz algebras of semigroups and other associated $C^*$-algebras.

There is a partial action $\gamma = (\{ A_g \}_{g \in G}, \{ \sigma_g \}_{g \in G})$ of $G$ on $D_\gamma$ such that $T_\lambda(P) \cong D_\gamma \rtimes_{\gamma,r} G$ canonically [4, Theorem 5.6.41] (see also [10, Section 4]). The Gelfand transform induces a partial action on $C(\Omega_P)$ and an isomorphism $T_\lambda(P) \cong C(\Omega_P) \rtimes_{\gamma} G$ in the natural way, where $\Omega_P$ denotes the spectrum of $D_\gamma$. The boundary quotient of $T_\lambda(P)$, denoted by $\partial T_\lambda(P)$, is the reduced partial crossed product $C(\partial \Omega_P) \rtimes_{\gamma} G$, where $\partial \Omega_P$ is the smallest closed nonempty $\gamma$-invariant subset of $\Omega_P$ (see [4, Definition 5.7.9]).

It follows from [19, Theorem 6.13] (see also [12, Theorem 3.14]) that $\partial T_\lambda(P)$ is isomorphic to the reduced crossed sectional $C^*$-algebra of the Fell bundle associated to the canonical coaction of $G$ on the covariance algebra of $C^P$, via an isomorphism that identifies the canonical generating isometries. That $\partial T_\lambda(P)$ is also canonically isomorphic to the $C^*$-envelope of the cosystem $(T_\lambda(P)^+, G, \delta^+)$ was established in [12, Theorem 4.4]. Although the authors proved in [12] that $\partial T_\lambda(P)$ is canonically isomorphic to the $C^*$-envelope of $T_\lambda(P)^+$ when either $\partial T_\lambda(P)$ is simple or $P$ is an Ore monoid, the question of whether $\partial T_\lambda(P)$ is always canonically isomorphic to $C^*_\text{env}(T_\lambda(P)^+)$ remained open even for group-embeddable right
LCM monoids. We can now answer this question in the affirmative for arbitrary submonoids of groups as an immediate application of Theorem 5.1.

**Corollary 5.4.** Let $P$ be a submonoid of a group. Then the boundary quotient $\partial T_\lambda(P)$ is isomorphic to the $C^*$-envelope of $T_\lambda(P)^+ = \overline{\text{span}}\{L_p \mid p \in P\}$, with an isomorphism that identifies the canonical generating isometries.

6. On co-universal properties for the $C^*$-envelope $C^*_\text{env}(T_\lambda(\mathcal{E})^+)$

For a compactly aligned product system $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ over a positive cone of a quasi-lattice order $(G, P)$, Carlsen, Larsen, Sims and Vittadello asked in [30] for the existence of a $C^*$-algebra satisfying a certain co-universal property with respect to injective gauge-compatible Nica covariant representations. In [3] they proved that under extra assumptions on $\mathcal{E}$, the reduced analogue of the Cuntz–Nica–Pimsner algebra as defined by Sims and Yeend in [30], denoted by $\mathcal{NO}_\mathcal{E}$, satisfies the desired properties. That is, $\mathcal{NO}_\mathcal{E}$ carries a coaction $\gamma$ of $G$ for which the quotient map from the Nica–Toeplitz algebra $\mathcal{NT}(\mathcal{E})$ onto $\mathcal{NO}_\mathcal{E}$ is gauge-equivariant, and if $(B, G, \gamma)$ is a coaction and $B$ is generated as a $C^*$-algebra by an injective Nica covariant representation of $\mathcal{E}$ that is gauge-compatible with $\gamma$ (see Definition 6.1 below), then there exists a surjective $^\ast$-homomorphism $\rho: B \to \mathcal{NO}_\mathcal{E}$ that identifies the corresponding copies of $\mathcal{E}$. More recently, in [6] Theorem 4.9 (see also [5]) the authors proved that for $\mathcal{E}$ a compactly aligned product system over a right LCM submonoid of $G$, the $C^*$-envelope of the cosystem $(T_\lambda(\mathcal{E})^+, G, \tilde{\delta}^+)$ always satisfies the co-universal property for injective gauge-compatible Nica covariant representations.

Motivated by the results mentioned above, we examine next co-universal properties for the $C^*$-envelope $C^*_\text{env}(T_\lambda(\mathcal{E})^+)$ for product systems over arbitrary submonoids of groups. Notice that if $(T_\lambda(\mathcal{E})_g)_{g \in G}$ is the Fell bundle associated to the coaction $\delta$ of $G$ on $T_\lambda(\mathcal{E})$, then the cross sectional $C^*$-algebra $C^*((T_\lambda(\mathcal{E})_g)_{g \in G})$ is canonically isomorphic to the Nica–Toeplitz algebra of $\mathcal{E}$ when $\mathcal{E}$ is a compactly aligned product system over a right LCM submonoid of a group (see [6] Proposition 4.3).

**Definition 6.1.** (see [3] Section 4 and [6] Definition 4.6). Let $\pi = \{\pi_p\}$ be a representation of $\mathcal{E} = (\mathcal{E}_p)_{p \in P}$ in a $C^*$-algebra $B$ and suppose that $C^*(\pi)$ admits a coaction $\gamma: C^*(\pi) \to C^*(\pi) \otimes C^*(G)$. We will say that $\pi$ is gauge-compatible with $\gamma$ if for all $p \in P$ and $\xi \in \mathcal{E}_p$ we have

$$\gamma(\pi_p(\xi)) = \pi_p(\xi) \otimes u_p.$$ 

Notice that if $(B, G, \gamma)$ is a coaction and $\pi = \{\pi_p\}_{p \in P}$ is a representation of $\mathcal{E}$ in $B$, then $\pi$ is gauge-compatible with $\gamma$ if and only if the induced $^\ast$-homomorphism $\tilde{\pi}: C^*_\text{rep}(\mathcal{E}) \to B$ is $\tilde{\delta} - \gamma$-equivariant, that is,

$$\gamma \circ \tilde{\pi} = (\tilde{\pi} \otimes \text{id}_{C^*(G)}) \circ \tilde{\delta}.$$ 

In case $\pi$ is gauge-compatible with $\gamma$ and $B = C^*(\pi)$, it follows that $(B, G, \gamma)$ is necessarily nondegenerate [14, Lemma 2.2]. When the coaction $\gamma$ is understood, we will simply say that $\pi$ is gauge-compatible.

**Corollary 6.2.** Let $(T_\lambda(\mathcal{E})_g)_{g \in G}$ be the Fell bundle associated to the gauge coaction $\tilde{\delta}$ on $T_\lambda(\mathcal{E})$. Then the $C^*$-envelope $C^*_\text{env}(T_\lambda(\mathcal{E})^+)$ satisfies the following properties:

1. There is a coaction $\Lambda: C^*_\text{env}(T_\lambda(\mathcal{E})^+) \to C^*_\text{env}(T_\lambda(\mathcal{E})^+) \otimes C^*(G)$ for which the representation of $\mathcal{E}$ induced by the inclusion $\iota: T_\lambda(\mathcal{E})^+ \hookrightarrow C^*_\text{env}(T_\lambda(\mathcal{E})^+)$ is gauge-compatible;
2. If $(B, G, \gamma)$ is a coaction and $\pi = \{\pi_p\}_{p \in P}$ is an injective representation of $\mathcal{E}$ in $B$ that is gauge-compatible with $\gamma$ and induces a surjective $^\ast$-homomorphism
\[\hat{\pi} : C^*((T_\Lambda(E))_g) \to B,\] then there exists a \(\gamma - \delta_\Lambda\)-equivariant surjective \(*\)-homomorphism \(\rho : B \to C^*_\text{env}(T_\Lambda(E)^+)\) such that the diagram

\[
\begin{array}{ccc}
C^*((T_\Lambda(E))_g) & \xrightarrow{\hat{\pi}} & B \\
\phi_\Lambda \circ \Lambda & \searrow & \\
& C^*_\text{env}(T_\Lambda(E)^+) & \\
\end{array}
\]

commutes, where \(\Lambda : C^*((T_\Lambda(E))_g) \to C^*_\text{env}(T_\Lambda(E)^+)\) is the left regular representation and \(\phi_\Lambda : T_\Lambda(E) \to C^*_\text{env}(T_\Lambda(E)^+)\) is the quotient map.

**Proof.** The normal coaction \((C^*_\text{env}(T_\Lambda(E)^+), G, \delta_\Lambda)\) as in the statement was already used in the proof of Theorem 5.1 and follows from the description of \(C^*_\text{env}(T_\Lambda(E)^+)\) as the reduced cross sectional \(C^*\)-algebra of the Fell bundle coming from the coaction \(\delta\) on the covariance algebra of \(\mathcal{E}\). In order to show that \(C^*_\text{env}(T_\Lambda(E)^+)\) also has property (2), let \(\hat{\pi} : C^*((T_\Lambda(E))_g) \to B\) be a surjective \(*\)-homomorphism induced by an injective representation \(\pi = \{\pi_g\}_{g \in G}\) of \(\mathcal{E}\) in \(B\) that is gauge-compatible with \(\gamma\). Let \((B_g)_{g \in G}\) be the Fell bundle associated to \((B, G, \gamma)\) and suppose first that \(\gamma\) is normal. Then \(B \cong C^*_r((B_g)_{g \in G})\) with an isomorphism that identifies the corresponding spectral subspaces. It follows that \(\hat{\pi}\) factors through \(T_\Lambda(E)\), giving a surjective \(*\)-homomorphism \(\hat{\pi}_s : T_\Lambda(E) \to B\) such that \(\hat{\pi} = \hat{\pi}_s \circ \Lambda\). Also, \(\hat{\pi}_s\) is injective on the copy of \(A\) and is \(\delta - \gamma\)-equivariant, and so the (faithful) conditional expectation of \(B\) onto \(B_e\) is compatible with \(E\). Hence Corollary 3.5 implies that the restriction of \(\hat{\pi}_s\) to the tensor algebra \(T_\Lambda(E)^+\) is completely isometric. It follows that there exists a surjective \(*\)-homomorphism \(\rho : B \to C^*_\text{env}(T_\Lambda(E)^+)\) such that \(\rho \circ \hat{\pi}_s |_{T_\Lambda(E)^+} = \iota\). Thus \(\phi_\Lambda = \rho \circ \hat{\pi}_s\), which yields \(\phi_\Lambda \circ \Lambda = \rho \circ \hat{\pi}\). This gives (2) when \(\gamma\) is a normal coaction. The general case follows from this one because there is a surjective \(*\)-homomorphism \(B \to C^*_r((B_g)_{g \in G})\) that identifies the fibres of \((B_g)_{g \in G}\) by [8, Theorem 19.5] and \(C^*_r((B_g)_{g \in G})\) carries a normal coaction of \(G\) for which the spectral subspace at \(g \in G\) is precisely the canonical copy of \(B^e_g\) (see [8, Proposition 18.24] and [6, Proposition 3.4]).

If \(\mathcal{E} = C^P\) is the canonical product system over \(P\) with one-dimensional fibres, so that \(T_\Lambda(E) = T_\Lambda(P)\), then \(C^*((T_\Lambda(P))_g)\) can be canonically identified with the universal Toeplitz algebra \(T_\Lambda(P)\) as introduced in [19]. Theorem 4.2 of [12] shows that \(\partial T_\Lambda(P)\) is also co-universal for gauge-equivariant nonzero representations of \(H\)'s semigroup \(C^*_s(P)\) [21, Definition 3.2]. If \(P\) does not satisfy the independence condition, \(C^*((T_\Lambda(P))_g)\) is then a proper quotient of \(C^*_s(P)\) (see [19, Corollary 3.23]). This implies in particular that \(C^*((T_\Lambda(E))_g)\) is in general not the largest \(C^*\)-algebra for which \(C^*_\text{env}(T_\Lambda(E)^+)\) has the co-universal property with respect to gauge-equivariant representations that are injective on the corresponding copy of \(A\). Under the assumption that \(\mathcal{E}\) is faithful, we give next a class of representations for which \(C^*_\text{env}(T_\Lambda(E)^+)\) has the co-universal property that is in general much larger than the class of injective gauge-compatible representations that induce a \(*\)-homomorphism of \(C^*((T_\Lambda(E))_g)\).

**Theorem 6.3.** Let \(\mathcal{E} = (E_p)_{p \in P}\) be a faithful product system over \(P\) with coefficient \(C^*\)-algebra \(A\). Let \((B, G, \gamma)\) be a coaction and let \(\pi = \{\pi_p\}_{p \in P}\) be an injective representation of \(\mathcal{E}\) in \(B\) such that \(B = C^*(\pi)\). Suppose, in addition, that \(\pi = \{\pi_p\}_{p \in P}\) is gauge-compatible with \(\gamma\) and satisfies the following property: for every neutral word \(\alpha = (p_1, p_2, \ldots, p_{2k-1}, p_{2k}) \in W(P)\) such that \(K(\alpha) = \emptyset\) and every choice of elements \(\xi_{p_i} \in E_{p_i}\) for \(i = 1, \ldots, 2k\), we have

\[
\pi_{p_1}(\xi_{p_1})\pi_{p_2}(\xi_{p_2})^* \cdots \pi_{p_{2k-1}}(\xi_{p_{2k-1}})\pi_{p_{2k}}(\xi_{p_{2k}})^* = 0.
\]

Then there exists a \(\gamma - \delta_\Lambda\)-equivariant surjective \(*\)-homomorphism \(\rho : B \to C^*_\text{env}(T_\Lambda(E)^+)\) such that \(\rho \circ \pi_p = \iota_p\) for all \(p \in P\).
**Proof.** Consider the \(*\)-homomorphism \(\hat{\pi}: C^*_\text{rep}(\mathcal{E}) \to B\) obtained by the universal property of \(C^*_\text{rep}(\mathcal{E})\). We will show that \(\ker \hat{\pi} \cap C^*_\text{rep}(\mathcal{E})_c\) is contained in \(J_e\). To do so, we begin by observing that a simple application of the \(C^*\)-axiom shows that

\[\pi_{p_1}(\xi_{p_1})\pi_{p_2}(\xi_{p_2})^* \cdots \pi_{p_{2k-1}}(\xi_{p_{2k-1}})\pi_{p_{2k}}(\xi_{p_{2k}})^* = 0\]

if \(K(\alpha) = \emptyset\) also when \(\alpha\) is not neutral since \(K(\alpha) = K(\tilde{\alpha}\alpha)\) and \(\tilde{\alpha}\alpha\) is then a neutral word (see, for example, [19, Proposition 2.6]).

We claim that if \(\alpha = (p_1, p_2, \ldots, p_{2k-1}, p_{2k})\) is a neutral word and \(F \subset G\) is a finite set containing the iterated quotient set \(Q(\alpha)\) of \(\alpha\), then for every \(r \in P\) such that \(r \not\in K(\alpha)\) and for every \(\xi_r \in \mathcal{E}_r I_{r-1}(r \vee F)\), we have that

\[\pi_{p_1}(\xi_{p_1})\pi_{p_2}(\xi_{p_2})^* \cdots \pi_{p_{2k-1}}(\xi_{p_{2k-1}})\pi_{p_{2k}}(\xi_{p_{2k}})^* \pi_r(\xi_r) = 0\]

Indeed, suppose that \(r \not\in K(\alpha)\). Then there is \(g \in Q(\alpha)\) such that \(r \not\in gP\). If \(rP \cap gP = \emptyset\), then \(K(\alpha) \cap rP \subset gP \cap rP = \emptyset\). Setting \(\beta := (p_1, p_2, \ldots, p_{2k-1}, p_{2k}, r, e)\), we see that \(K(\beta) = \emptyset\) and this yields

\[\pi_{p_1}(\xi_{p_1})\pi_{p_2}(\xi_{p_2})^* \cdots \pi_{p_{2k-1}}(\xi_{p_{2k-1}})\pi_{p_{2k}}(\xi_{p_{2k}})^* \pi_r(\xi_r) = 0\]

by the assumption and by nondegeneracy of the right action of \(A\) on \(\mathcal{E}_r\). Now if \(rP \cap gP \neq \emptyset\), we can find \(s \in rP \cap gP\) and because \(\mathcal{E}\) is faithful and \(r \not\in gP\), we obtain from the definition of \(I_{r-1}(r \vee F)\) that

\[I_{r-1}(r \vee F) \subset \ker \varphi_{r-1}s = \{0\}\]

Hence \(\xi_r = 0\), giving

\[\pi_{p_1}(\xi_{p_1})\pi_{p_2}(\xi_{p_2})^* \cdots \pi_{p_{2k-1}}(\xi_{p_{2k-1}})\pi_{p_{2k}}(\xi_{p_{2k}})^* \pi_r(\xi_r) = 0\]

This proves the claim.

Next take \(b \in \ker \hat{\pi} \cap C^*_\text{rep}(\mathcal{E})_c\). In order to show that \(b \in J_e\), let \(\varepsilon > 0\) and let \(b' \in C^*_\text{rep}(\mathcal{E})_c\) with \(\|b - b'\| < \frac{\varepsilon}{2}\) and such that \(b' = \sum_{i=1}^n b_i\), where each \(b_i\) is of the form

\[b_i = \hat{t}_{p_1}(\xi_{p_1})\hat{t}_{p_2}(\xi_{p_2})^* \cdots \hat{t}_{p_{2k-1}}(\xi_{p_{2k-1}})\hat{t}_{p_{2k}}(\xi_{p_{2k}})^*\]

and \(\alpha_i := (p_1, p_2, \ldots, p_{2k-1}, p_{2k}) \in W(P)\) is a neutral word. Let \(F := \bigcup_{i=1}^n Q(\alpha_i)\) and let \(\xi = \sum_{r \in F'} \xi_r \in \mathcal{E}_r I_{r-1}(r \vee F)\) with \(F' \subset P\) finite. Then since \(\hat{\pi}(b) = 0\), we have

\[\| \sum_{r \in F'} \pi_r(\xi_r)^* \pi_r(\xi_r) \| = \| \sum_{r \in F'} \pi_r(\xi_r)^* \pi_r(\xi_r) \| < \frac{\varepsilon^2}{4} \sum_{r \in F'} \pi_r(\|\xi_r\|)\]

Now we apply the usual argument: the left-hand side of the inequality above is precisely \(\|r(\xi)\|^2\) because \(\pi(\xi_r) = 0\) unless \(r \in K(\alpha_i)\) and \(\pi\) is injective on \(A\), while \(\sum_{r \in F'} \pi_r(\|\xi_r\|) = \|\xi\|^2\). Hence \(\|q(b')\| = \|b'\|_F < \frac{\varepsilon}{2}\), and thus \(\|q(b')\| < \varepsilon\), where \(q: C^*_\text{rep}(\mathcal{E}) \to A \times \mathcal{P}\) is the quotient map. Since \(\varepsilon > 0\) is arbitrary, we conclude that \(q(b) = 0\), that is, \(b \in J_e\) as wanted.

Finally, let \((B_g)_{g \in G}\) be the Fell bundle associated to \((B, G, \gamma)\). Because \(\ker \pi \cap C^*_\text{rep}(\mathcal{E})_c \subset J_e\), it follows that the restriction of the quotient map \(q: C^*_\text{rep}(\mathcal{E}) \to A \times \mathcal{P}\) to the spectral subspace \(C^*_\text{rep}(\mathcal{E})_g\) at \(g \in G\) factors through \(\hat{\pi}\). From this we obtain a morphism of Fell bundles \(\phi: (B_g)_{g \in G} \to ([A \times \mathcal{P}]_g)_{g \in G}\) such that \(\phi \circ \pi_p = j_p\) for all \(p \in P\) because \(\hat{\pi}\) is \(\delta - \gamma\)-equivariant. Such a morphism induces a \(*\)-homomorphism

\[\hat{\phi}: C^r_\text{env}(\mathcal{E})_g \to C^*_\text{rep}(\mathcal{E})_g \simeq C^*_\text{env}(T^*_\gamma(\mathcal{E})^+)\]

by [8] Proposition 21.3. Since the collection \((B_g)_{g \in G}\) is a topological grading for \(B\), it follows from [8] Theorem 19.5 that there exists a \(*\)-homomorphism \(\psi: B \to C^r_\text{env}(\mathcal{E})_g\) that identifies the fibres of \((B_g)_{g \in G}\). So setting \(\rho := \hat{\phi} \circ \psi\) we conclude that \(\rho: B \to C^*_\text{env}(T^*_\gamma(\mathcal{E})^+)\)
Remark 6.4. Whereas in the proof Corollary 6.2 it was more convenient to apply the defining property of $\mathcal{C}_{\text{env}}^*(T_\lambda(\mathcal{E})^+)$ as the smallest $\mathcal{C}^*$-algebra generated by a completely isometric copy of $T_\lambda(\mathcal{E})^+$, the defining relations of the covariance algebra of $\mathcal{E}$ were a crucial tool in the proof Theorem 6.3 above. This illustrates the importance of the distinct descriptions of $\mathcal{C}_{\text{env}}^*(T_\lambda(\mathcal{E})^+)$ provided by Theorem 5.1.

If $\mathcal{E} = \mathbb{C}^P$, the relation required in the statement of Theorem 6.3 corresponds to relation (T2) of [19] Definition 3.6. So Theorem 6.3 together with Corollary 5.4 imply that $\partial T_\lambda(P)$ has the co-universal property for a class of nonzero gauge-compatible isometric representations of $P$ that is in general strictly larger than the class of representations satisfying the defining relations of Li’s semigroup $\mathcal{C}^*$-algebra $\mathcal{C}_\lambda^*(P)$ (see [19] Proposition 3.22). Thus the next corollary provides a strengthening of [12] Theorem 4.2.

Corollary 6.5. Let $P$ be a submonoid of a group $G$. Let $(B,G,\gamma)$ be a coaction and suppose that $B$ is generated as a $\mathcal{C}^*$-algebra by a nonzero isometric representation $w: P \to B$ that is gauge-compatible with $\gamma$ and satisfies the following property: for every neutral word $\alpha = (p_1,p_2,\ldots,p_{2k-1},p_{2k}) \in W(P)$ such that $K(\alpha) = \emptyset$, we have

$$w_{p_1}w_{p_2}^* \cdots w_{p_{2k-1}} w_{p_{2k}}^* = 0.$$  

Then there exists a surjective $^*$-homomorphism $\rho: B \to \partial T_\lambda(P)$ such that $\rho \circ w_p = j_p$ for all $p \in P$, where $p \mapsto j_p$ is the canonical isometric representation of $P$ in $\partial T_\lambda(P)$.

Remark 6.6. We observe that in general $\mathcal{C}_{\text{env}}^*(T_\lambda(\mathcal{E})^+)$ does not have the co-universal property with respect to gauge-equivariant surjective $^*$-homomorphisms of Fowler’s $\mathcal{C}^*$-algebra $\mathcal{C}_{\text{rep}}^*(\mathcal{E})$ that are injective on $A$. Indeed, if $\mathcal{E} = \mathbb{C}^P$, then $\mathcal{C}_{\text{rep}}^*(\mathcal{E})$ is the universal $\mathcal{C}^*$-algebra for isometric representations of $P$ and hence has the full and reduced group $\mathcal{C}^*$-algebras $\mathcal{C}^*(G)$ and $\mathcal{C}^*_\lambda(G)$ as canonical gauge-equivariant quotients if $P \to G$ generates $G$ as a group. If $P$ is not left reversible, then the $^*$-homomorphism from $\mathcal{C}_{\text{rep}}^*(\mathcal{E})$ onto $\partial T_\lambda(P)$ induced by $j = \{j_p\}_{p \in P}$ does not factor through the group $\mathcal{C}^*$-algebra because $j_p j_p^* j_q j_q^* = 0$ in $\partial T_\lambda(P)$ if $pP \cap qP = \emptyset$, while $u_p^* u_q^* u_q u_p = 1$ in $\mathcal{C}^*(G)$. See [3] Example 4.7.

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