Permutationally Invariant Codes
for Quantum Error Correction

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Abstract

A permutationally invariant n-bit code for quantum error correction can be realized as a subspace stabilized by the non-Abelian group $S_n$. The code is spanned by bases for the trivial representation, and all other irreducible representations, both those of higher dimension and orthogonal bases for the trivial representation, are available for error correction.

A number of new (non-additive) binary codes are obtained, including two new 7-bit codes and a large family of new 9-bit codes. It is shown that the degeneracy arising from permutational symmetry facilitates the correction of certain types of two-bit errors. The correction of two-bit errors of the same type is considered in detail, but is shown not to be compatible with single-bit error correction using 9-bit codes.

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1 Introduction

Quantum error correction is now well-developed in the case of those stabilizer codes [4, 6], which arise as subspaces stabilized by Abelian subgroups of the Pauli group. These codes, also known as “additive codes,” can be regarded as an extension of classical binary codes over $\mathbb{Z}_2$ to codes over $\text{GF}(4)$ which satisfy an additional orthogonality condition. They generalize the classical notion of distance and thus seem best suited to situations in which all one-bit errors are equally likely and the noise is uncorrelated.

There are other approaches to fault tolerant computation which use structures which are resistant to decoherence, e.g., topological quantum computation and decoherence free (DF) subspaces or subsystems. (See [11] and [17] respectively for further discussion and references.) Some physical implementations may also be designed to protect against certain types of errors. Much of the current analysis is based on simple models using independent errors. In more realistic models some types of correlated errors may be more probable than arbitrary two-bit errors (and possibly even than certain one-bit errors). Hybrid approaches to fault tolerance which combine resiliency (either through encoding or hardware design) with error correction may require codes with properties different from those stabilized by Abelian subgroups of the Pauli group.

It is now known [20] that other types of quantum codes, often called “non-additive,” exist. Although some attempts [21] have been made to develop classes of non-additive codes, much of this work, e.g., [13] [12], has been for non-binary codes. In this paper we consider a natural generalization of stabilizer codes to binary codes associated with the action of non-Abelian groups. We concentrate our attention on the symmetric group as a case study, and call a code on which the symmetric group acts trivially permutationally invariant. We will be particularly interested in the use of higher dimensional representations for the correction of two-bit errors, and the ways in which the degeneracy associated with permutational invariance of code words allows the correction of more two-bit errors than would be expected by simple dimensional arguments.

We find a number of new codes. In particular, we give two new 7-bit codes which are impervious to exchange and can correct all one-bit errors together with some rather special two-bit errors. We show that the classical 5-bit repetition code can correct more two-bit quantum errors than those associated with a single type of one-bit error. We show that there is a large family of permutationally invariant 9-bit codes in addition to the simple one found in [22]. Unfortunately, none of these 9-bit codes is as powerful for two-bit error correction as one might expect.

Although the discovery of new codes is always of interest, we emphasize that our primary goal is to study permutationally invariant codes as examples of codes
obtained from the action of a non-Abelian group. These non-Abelian groups will, typically, be more general than subgroups of the Pauli group.

It is worth pointing out some significant differences between our approach and the “Clifford codes” associated with “nice error bases” as proposed by Knill [14] and developed by Klappenecker and Rötteler [12, 13]. Their approach, which considers generalizations of the Pauli group for \( d > 2 \), yields non-stabilizer codes only for \( d \geq 4 \); we obtain new non-stabilizer codes for \( d = 2 \). (Although our approach could, in principle, be applied for any \( d \), we study only \( d = 2 \).) In the KKR approach, the code is associated with a normal subgroup \( N \) of an error group, but need not come from bases for the trivial representation of \( N \). We retain the requirement that a code subspace is spanned by bases for the trivial representation of a group, but the non-Abelian group defining our code need not be associated with an error group in the sense of Knill [14]. From a formal point of view, our group and error sets reside in an operator algebra associated with the usual Pauli group, but we do not use this structure.

It was recognized earlier [1, 10], in the context of DF subspaces, that quantum error correcting codes can be obtained as stabilizers for non-Abelian groups. However, the use of higher dimensional irreducible representations for error correction was not explored. Moreover, the original philosophy underlying the DF approach to fault tolerant quantum computation, namely, to avoid anything which might perturb the system out of the stable subspace, is antithetical to active error correction. In [2, 9] the use of encoding to facilitate universal computation, rather than error correction, was introduced. Another important development was the generalization of DF subspaces to DF subsystems [16], in which the code itself can transform as a higher dimensional representation. The notion of stabilizer was then modified in [10, 24] to encompass DF subsystems as well. There is now an extensive literature on various aspects of both DF subspaces and systems, including proposals for hybridization of DF methods with active error correction, and scenarios in which DF encoding can replace active error correction. We refer the reader to [17] for references and further discussion.

Although motivated by the expected utility of codes capable of correcting specific set of correlated errors, we do not present a physical model leading to such sets. We deal only with construction of codes, leaving their application within a full-fledged scheme for fault tolerance for further investigation.

The rest of this paper is organized as follows. In the next section, we outline the basic set-up and notation we will use. We describe different classes of conditions associated with one-bit errors in Section 3 and analyze them in Section 4. In Section 5, we consider two-bit error correction. In Section 6 we first consider some explicit examples of codes for \( n = 5, 7 \) or 9; we then show that none of the 9-bit codes can correct all double errors of one type.


2 Preliminaries

2.1 Stabilizers and error sets

In the general situation, we have a set of errors $\mathcal{E} = \{e_1, e_2 \ldots e_M\}$ which we want to correct. We will also have a unitary group $\mathcal{G}$ which acts on the vector space $\mathbb{C}^{2^n}$. The elements of both $\mathcal{E}$ and $\mathcal{G}$ will be linear operators which act on $\mathbb{C}^{2^n}$. Typically, these will be non-trivial linear combinations of elements of the Pauli group, rather than simply tensor products of Pauli matrices. In particular, we can consider $S_n$ as the group generated by the exchange operators $E_{rs}$ which can be written as

$$E_{rs} = \frac{1}{2} [I \otimes I + X_r \otimes X_s + Y_r \otimes Y_s + Z_r \otimes Z_s]$$  \quad (1)$$

where $X_k, Y_k, Z_k$ denote the action of the $\sigma_x, \sigma_y$ and $\sigma_z$ operators on bit $k$. Note that the set $\{E_{1s} : s = 2 \ldots n\}$ suffices to generate the group $S_n$.

Since $\mathbb{C}^{2^n}$ is invariant under the action of $\mathcal{G}$, it can be decomposed into invariant subspaces corresponding to irreducible representations of $\mathcal{G}$. As is well-known $[8, 23]$, those subspaces corresponding to inequivalent representations are orthogonal, and those for equivalent representations can be chosen orthogonal. We want to exploit the freedom in the latter to construct codes with particular properties, and use the additional orthogonality from inequivalent representations for error correction.

Now suppose there is a subspace $T$ which is stabilized by $\mathcal{G}$, in the sense $g|w\rangle = |w\rangle$ for all $g \in \mathcal{G}$ and all $|w\rangle \in T$. Consider a subset of errors $\mathcal{E}'$ which is invariant under $\mathcal{G}$ in the sense $ge_pg^{-1} \in \mathcal{E}'$ for $e_p \in \mathcal{E}'$. Then the space $\mathcal{E}'(T)$ spanned by $\{e_p|w\rangle : e_p \in \mathcal{E}', |w\rangle \in T\}$ is also invariant under $\mathcal{G}$ since

$$ge_p|w\rangle = (ge_pg^{-1})g|w\rangle = e_q|w\rangle.$$  \quad (2)

Hence the space $\mathcal{E}'(T)$ can be decomposed into an orthogonal sum corresponding to irreducible representations of $\mathcal{G}$. Since the span of $\mathcal{E}'$ itself is invariant under $\mathcal{G}$, it too can be decomposed into a sum of irreducible subspaces. In fact, one can regard the two spaces $\mathcal{E}'(T)$ and span$\{\mathcal{E}'\}$ as being decomposed in parallel into orthogonal sums corresponding to irreducible representations of $\mathcal{G}$.

For example, the set of single bit flips $\mathcal{E}_X = \{X_1, X_2, \ldots X_n\}$ is invariant under $S_n$. In fact, its span is isomorphic to the standard $n$-dimensional representation of $S_n$, which decomposes into the sum of the trivial representation, spanned by $\sum_r X_r$, and the $(n-1)$-dimensional irreducible representation, spanned by $X_1 - X_2, \ldots, X_1 - X_n$. Similar considerations hold for the errors $\{Y_1, \ldots Y_n\}$, and $\{Z_1, \ldots Z_n\}$.
The resulting linear combinations of errors \( X_p - X_q \) may not be invertible. However, this poses no problems for error correction because we will only need to “invert” when a measurement shows we are in \((X_p - X_q)T\) which is orthogonal to the null space of \((X_p - X_q)\).

In view of their role as bases for the trivial representation, it is useful to define the average errors \( \overline{X}, \overline{Y}, \overline{Z} \) as

\[
\overline{X} = \frac{1}{n} \sum_{k=1}^{n} X_k \quad \overline{Y} = \frac{1}{n} \sum_{k=1}^{n} Y_k \quad \overline{Z} = \frac{1}{n} \sum_{k=1}^{n} Z_k
\]

Note that \( X_r - X_s = (X_1 - X_s) - (X_1 - X_r) \), and recall that a code that can correct errors in a set \( E \) can also correct any complex linear combination of these errors. Thus, the error sets \( E = \{ I, X_k, Y_k, Z_k, k = 1 \ldots n \} \) and \( E = \{ I, \overline{X}, \overline{Y}, \overline{Z}, X_1 - X_k, Y_1 - Y_k, Z_1 - Z_k, k = 2 \ldots n \} \) are equivalent.

### 2.2 Notation

The \( 2^n \)-dimensional complex vector space \( \mathbb{C}^{2^n} \) has an orthonormal basis \( \{|v\rangle = |v_1, v_2 \ldots v_n||\rangle \} \) indexed by binary \( n \)-tuples \( v = (v_1, v_2 \ldots v_n) \in (\mathbb{Z}_2)^n \). If an orthonormal basis \( |0\rangle, |1\rangle \) for \( \mathbb{C}^2 \) is fixed, this is simply the basis of tensor products of the form \( |v_1 \rangle \otimes |v_2 \rangle \otimes \ldots \otimes |v_n \rangle \) with each \( v_i \in \mathbb{Z}_2 \). The symmetric group \( S_n \) acts on \( \mathbb{C}^{2^n} \) via a natural action on these basis vectors; if \( P \) takes \( (1 \ldots n) \mapsto (i_1 \ldots i_n) \), then \( P|v_1, v_2 \ldots v_n\rangle = |v_{i_1}, v_{i_2} \ldots v_{i_n}\rangle \). Define

\[
W_k = \text{span}\{ |v\rangle : \text{wt}(v) = k \}
\]

where \( \text{wt}(v) \) is the number of \( k \) for which \( v_k = 1 \). (This is the classical Hamming weight of \( v \).) Then \( \mathbb{C}^{2^n} = \bigoplus_{k=0}^{n} W_k \) is the orthogonal direct sum of the \( W_k \). Moreover, each \( W_k \) is invariant under \( S_n \) and can be further decomposed into an orthogonal sum of spaces affording inequivalent irreducible representations of \( S_n \). This yields an orthogonal decomposition of \( \mathbb{C}^{2^n} \) into irreducible subspaces. However, unlike the regular representation, some irreducible representations occur more than once in \( \mathbb{C}^{2^n} \), and others not at all. Appendix describes the decomposition of \( W_k \) into irreducible subspaces for \( n = 5, 7, 9 \).

Each \( W_k \) contains the trivial representation, for which we introduce the basis vector

\[
W_k = \sum_{\text{wt}(v)=k} |v\rangle = \sum_{P} P|1\ldots10\ldots0\rangle
\]
where the second sum ranges over those permutations \( \mathcal{P} \) which yield distinct vectors \( |v\rangle \). Thus \( \langle W_k, W_k \rangle = \binom{n}{k} \). Occasionally we will use the normalized vectors \( \tilde{W}_k = \binom{n}{k}^{-1/2} W_k \). Although normalized vectors are useful for many purposes, those denoted \( W_k \) are more convenient in combinatoric computations.

Finally, we will make repeated use of the combinatoric identity

\[
\binom{N}{K} - \binom{N}{K-J} = \frac{N-2K+J}{N+J} \binom{N+J}{K}
\]

(7)

which holds for \( J = 1,2 \) and is easy to verify. We will occasionally use the convention that \( \binom{N}{K} = 0 \) when \( N < K \).

### 2.3 Codes

Given a (possibly non-Abelian) group \( \mathcal{G} \), we define a code \( \mathcal{C} \) as a subspace of \( \mathbb{C}^{2^n} \) which is stabilized by \( \mathcal{G} \) in the sense

\[
g |v\rangle = |v\rangle \quad \text{for all } g \in \mathcal{G} \text{ and all } |v\rangle \in \mathcal{C}.
\]

(8)

If \( \mathcal{C} \) has dimension \( 2^m \), then one can effectively encode \( m \) logical binary units in \( n \) physical qubits. We will restrict ourselves here to the simple case of \( 1 \) to \( n \) encoding, for which \( m = 1 \) and \( \mathcal{C} \) is two-dimensional. A code is often specified by an orthonormal basis for \( \mathcal{C} \), in which case each basis vector, or “code word” can be regarded as a basis for the trivial representation of \( \mathcal{G} \). In the case of two-dimensional codes, we can interpret these basis vectors as a logical 0 and 1, and will label them \( |c_0\rangle \) and \( |c_1\rangle \) accordingly.

We now consider two-dimensional codes for the group \( \mathcal{G} = S_n \). If \( |v\rangle = |v_1, v_2, \ldots, v_n\rangle \) is a basis vector of \( \mathbb{C}^{2^n} \) of weight \( k \), (or, equivalently, a binary \( n \)-tuple of weight \( k \)) then \( \{g|v\rangle : g \in S_n\} \) is the set of all basis vectors of weight \( k \). Therefore, any vector satisfying \( g|v\rangle = |v\rangle \) for all \( g \in S_n \) must have the form \( \sum_k a_k W_k \), so that we can write a **permutationally invariant code** as a pair of basis vectors of the form

\[
|c_0\rangle = \sum_k a_k W_k \quad \text{and} \quad |c_1\rangle = \sum_k b_k W_k
\]

(9)

for some complex numbers \( a_k \) and \( b_k \) with \( \sum_k a_k b_k = 0 \). A vector \( \sum_k d_k W_k \) is called **even** (resp. **odd**) if \( d_k \) is nonzero only when \( k \) is even (resp. odd).

Note that we have defined a code so that the individual basis vectors are permutationally invariant. This is a stronger requirement than that the subspace defined by the code is invariant under \( S_n \). However, the distinction is unlikely to matter in practice. In the case of two-dimensional codes, the two types of
invariance are equivalent whenever \( n > 3 \). In general one can have an invariant subspace of dimension \( 2^n \) only if it can be written as a direct sum of irreducible subspaces whose dimensions sum to \( 2^n \); this will usually consist of \( 2^n \) copies of the trivial representation, in which case the code words are also invariant.

We will be primarily interested in codes of the form (9) which also satisfy the following two conditions (which together imply that \( n \) is odd).

I) \( b_k = a_{n-k} \) or, equivalently, \( |c_1\rangle = (\otimes_j X_j)|c_0\rangle \).

II) \( c_0 \) is even and \( c_1 \) is odd or, equivalently, \( (\otimes_j Z_j)|c_\ell\rangle = (-1)^\ell|c_\ell\rangle \).

When (I) and (II) both hold, we can write

\[
|c_0\rangle = \sum_{j=0}^{(n-1)/2} a_{2j} W_{2j}, \quad |c_1\rangle = \sum_{j=0}^{(n-1)/2} a_{n-2j-1} W_{2j+1}.
\] (10)

In addition to simplifying the analysis and ensuring that certain inner products are zero, these assumptions serve another purpose. They ensure that the logical \( X \) and \( Z \) operations can be implemented on the code words by \( \otimes_j X_j \) and \( \otimes_j Z_j \) respectively. Since the actual use of codes in fault tolerant computation requires a mechanism for implementing gates on the code words [7], this is an important consideration. Moreover, there is little loss of generality in this assumption. The operators \( \otimes_j X_j \) and \( \otimes_j Z_j \) lie in the commutant of \( S_n \). Therefore, they necessarily map invariant subspaces of \( S_n \) to invariant subspaces of \( S_n \). In the case of the code space, we require the stronger condition that

\[
[\otimes_j X_j]|C\rangle = |C\rangle \quad \text{and} \quad [\otimes_j Z_j]|C\rangle = |C\rangle.
\] (11)

When (11) holds, there is no loss of generality in assuming (I) and (II). These simply restrict the choice of basis in a way that is convenient and can always be satisfied.

Our goal is to construct a permutationally invariant 2-dimensional code that can correct all single qubit errors, and to examine the types of two-bit errors that can be corrected.

As noted at the end of Section 1, non-Abelian stabilizers were considered previously in the context of DF subspace codes. Conversely, one can consider a permutationally invariant code as a DF subspace which arises from the highly idealized situation in which a quantum computer is completely insulated from its environment, but the qubits are the spin components of identical particles which interact.\(^1\) Then, as discussed in [22], even in the absence of spin-spin

\(^1\)For a complete analogy to DF subspaces, only those particles within each logical encoded unit would be permitted to interact with each other. Interactions between particles in different units, would lead to exchange errors between units. However, since these would appear as single bit errors in each logical unit, they would also be correctable.
interactions, the Pauli principle induces an effective interaction between qubits whose DF subspace group is precisely that generated by exchanges. In essence, the Pauli principle requires correlations between the spatial and spin components so that spatial interactions (such as the Coulomb interaction) affect the spin components. The result of tracing over the spatial component yields a completely positive map on the spin components, as in the standard noise model. Only a fully symmetric spin function allows the full wave function to be a product (with an anti-symmetric spatial function) consistent with the Pauli principle for fermions. Thus, a DF subspace is precisely one which transforms as the trivial (or fully symmetric) representation of $S_n$. Although this is not a very realistic DF scenario, it is useful to see how codes constructed for different purposes can be interpreted within the DF subspace, as well as the stabilizer, formalism.

3 Error correction conditions

The now well-known necessary and sufficient condition \cite{3,15} for the code $C$ to correct errors in a set $\mathcal{E} = \{e_1 \ldots e_M\}$ can be stated as

$$\langle e_p c_i, e_q c_j \rangle = \delta_{ij} d_{pq} \quad \forall \quad e_p, e_q \in \mathcal{E}. \quad (12)$$

where the matrix $d_{pq}$ does not depend on $i, j$. One often chooses codes for which $d_{pq} = \delta_{pq} \mu_p$, but that is not necessary. Indeed, the requirement

$$d_{pq} \equiv \langle e_p c_0, e_q c_0 \rangle = \langle e_p c_1, e_q c_1 \rangle, \quad (13)$$

which is implicit in (12), implies that one can always transform the error set into a modified one $\tilde{\mathcal{E}}$ for which the stronger condition $\tilde{d}_{pq} = \delta_{pq} \mu_p$ holds.

Strictly speaking one can only determine whether or not a particular set of errors is correctable; not whether a particular error or type of error is “correctable”. However, it is often natural to look for codes for which the set of correctable errors includes all errors of a particular type, e.g., the one-bit errors. One can then ask what additional errors could be added to this subset to yield a set $\mathcal{E}$ satisfying (12). In our discussion of such situations, the subset involved may be implied by the context.

We will find it useful to think of (12) as defining a pair of matrices $D^{ij}$ with elements $d^{ii}_{pq} = \langle e_p c_i, e_q c_i \rangle$ for $i = 0,1$ and a matrix $B = D^{01}$ with elements $b_{pq} \equiv d^{01}_{pq} = \langle e_p c_0, e_q c_1 \rangle$. Then (12) is equivalent to the requirements $B = 0$ and $D^{00} = D^{11}$. (Because $d^{10}_{pq}$ and $d^{01}_{pq}$ are complex conjugates, $D^{01} = 0 \Leftrightarrow D^{10} = 0$. Hence we need not consider $D^{10}$ explicitly and will use only $B \equiv D^{01}$.) For simplicity, we omit the superscript in $d^{i}_{pq}$.
For example, the 9-bit permutationally invariant code in [22] corrects single qubit errors as well as the Pauli exchange errors (transpositions) $E_{rs}$, for the 36 unordered pairs $r, s$. We can consider the above matrices with respect to the errors $\mathcal{E} = \{I, E_{rs}, X_1, \ldots, X_9, Y_1, \ldots, Y_9, Z_1, \ldots, Z_9\}$. It was shown that $D^{00} = D^{11}$ and has the block diagonal form

$$
\begin{pmatrix}
D_0 & 0 & 0 & 0 \\
0 & D_{XX} & 0 & 0 \\
0 & 0 & D_{YY} & 0 \\
0 & 0 & 0 & D_{ZZ}
\end{pmatrix}
$$

(14)

where $D_0$ is a $37 \times 37$ rank one matrix and the $9 \times 9$ matrices $D_{XX}, D_{YY}, D_{ZZ}$ correspond to the one-bit errors indicated by the subscripts. These all have the cyclic form

$$
\begin{pmatrix}
a & b & \ldots & b \\
b & a & \ldots & b \\
\vdots & \vdots & & \vdots \\
b & b & \ldots & a
\end{pmatrix}
$$

(15)

For any permutationally invariant code the blocks $D_{XX}, D_{YY}, D_{ZZ}$ necessarily have the form (15). Such matrices can always be diagonalized by a change of basis to $(1, 1 \ldots 1)$ and its orthogonal complement. This corresponds to replacing the errors $\{f_1, f_2 \ldots f_n\}$ by the corresponding average $\{\bar{f}\}$ and a suitable orthogonalization of $\{f_j - f_k, k = 2 \ldots n\}$ where $f$ denotes any of $X, Y, Z$.

Now the orthogonality of subspaces associated with different irreducible representations ensures that

$$
\langle \bar{f}W_j, (g_r - g_s)W_k \rangle = 0
$$

(16)

for all $j, k$ and any choice of $\bar{f} = I, X, Y, Z$ and $g = X, Y, Z$. Alternatively, we can show this directly by observing that the exchange operator $E_{rs}$ is unitary so that

$$
\langle \bar{f}W_j, (g_r - g_s)W_k \rangle = \langle E_{rs} \bar{f}E_{rs}(E_{rs}W_j), E_{rs}(g_r - g_s)E_{rs}(E_{rs}W_k) \rangle = \langle \bar{f}W_j, (g_s - g_r)W_k \rangle = -\langle \bar{f}W_j, (g_r - g_s)W_k \rangle
$$

which implies (16). For such codes, each of the matrices $D^{ii}, (i = 0, 1)$ and $B$
have the form below (which we write only for $D$) with respect to the order in (11).

$$
\begin{pmatrix}
  d_{II} & d_{IX} & d_{IY} & d_{IZ} \\
  d_{XI} & d_{XX} & d_{XY} & d_{XZ} \\
  d_{YI} & d_{YX} & d_{YY} & d_{YZ} \\
  d_{ZI} & d_{ZX} & d_{ZY} & d_{ZZ}
\end{pmatrix}
\begin{pmatrix}
  0 \\
  D_{XX} & D_{XY} & D_{XZ} \\
  D_{YX} & D_{YY} & D_{YZ} \\
  D_{ZX} & D_{ZY} & D_{ZZ}
\end{pmatrix}
$$

Conditions (I) and (II) immediately give many additional zero entries. One nice way to see which entries are zero is to observe that $\otimes Z_k$ commutes with $Z_r$ and anti-commutes with $X_r$ and $Y_r$ for all $r$. Thus, for every one-bit error $e_p$, $e_p(\otimes Z_k) = e_p^Z (\otimes Z_k) e_p$, where

$$
\epsilon_p^Z = \begin{cases} 
  +1 & \text{for } e_p \in \{I, \overline{Z}, (Z_r - Z_s)\} \\
  -1 & \text{for } e_p \in \{X, \overline{Y}, (X_r - X_s), (Y_r - Y_s)\}
\end{cases}
$$

Also, $\otimes Z_k$ is unitary so that

$$
\langle e_p c_i, e_q c_j \rangle = \langle (\otimes Z_k) e_p c_i, (\otimes Z_k) e_q c_j \rangle = \epsilon_p^Z \epsilon_q^Z \langle e_p (\otimes Z_k) c_i, e_q (\otimes Z_k) c_j \rangle = \epsilon_p^Z \epsilon_q^Z (-1)^{i+j} \langle e_p c_i, e_q c_j \rangle.
$$

From this we can conclude the following.

A) When $i = j$, $\langle e_p c_i, e_q c_i \rangle = 0$ whenever $\epsilon_p^Z \neq \epsilon_q^Z$. Thus, $d_{IX} = d_{IY} = d_{XZ} = d_{YZ} = 0$ and $D_{XX} = D_{YX} = 0$.

B) When $i \neq j$, $\langle e_p c_i, e_p c_j \rangle = 0$ whenever $\epsilon_p^Z = \epsilon_q^Z$, from which we can conclude $b_{IZ} = b_{ZX} = b_{XY} = b_{YX} = 0$ and $B_{XX} = B_{YX} = B_{YY} = B_{XZ} = B_{XY} = B_{YX} = 0$.

Combining this with $d_{fg} = 0 \iff d_{gf} = 0$ and $D_{fg} = 0 \iff D_{gf} = 0$, we find that the $D^{d_i}$ have the form

$$
\begin{pmatrix}
  d_{II} & 0 & 0 & d_{IZ} \\
  0 & d_{XX} & d_{XY} & 0 \\
  0 & d_{YX} & d_{YY} & 0 \\
  d_{ZI} & 0 & 0 & d_{ZZ}
\end{pmatrix}
\begin{pmatrix}
  0 & D_{XX} & D_{XY} & 0 \\
  D_{YX} & D_{YY} & 0 \\
  0 & 0 & D_{ZZ}
\end{pmatrix}
$$
and $B$ has the form

$$
\begin{pmatrix}
0 & b_{IX} & b_{IY} & 0 \\
b_{XI} & 0 & 0 & b_{XZ} \\
b_{YI} & 0 & 0 & b_{YZ} \\
0 & b_{ZX} & b_{ZY} & 0 \\
0 & 0 & 0 & B_{XZ} \\
0 & 0 & 0 & B_{YZ} \\
B_{ZX} & B_{ZY} & 0 & 0
\end{pmatrix}
$$

(20)

Now, observe that $\otimes_k X_k$ commutes with $X_r$ and anti-commutes with $Y_r$ and $Z_r$. Proceeding as above, we find

$$
\epsilon_p^X = \begin{cases} 
+1 & \text{for } e_p \in \{I, \overline{X}, (X_r - X_s)\} \\
-1 & \text{for } e_p \in \{Y, Z, (Y_r - Y_s), (Z_r - Z_s)\}
\end{cases}
$$

(21)

and

$$
\langle e_p c_i, e_q c_j \rangle = \langle (\otimes_k X_k) e_p c_i, (\otimes_k X_k) e_q c_j \rangle \\
= \epsilon_p^X \epsilon_q^X \langle e_p (\otimes_k X_k) c_i, e_q (\otimes_k X_k) c_j \rangle \\
= \epsilon_p^X \epsilon_q^X \langle e_p c_{i+1}, e_q c_{j+1} \rangle,
$$

where we interpret $i + 1$ and $j + 1$ mod 2. Thus we can conclude

C) When $i = j$, condition (12) holds whenever $\epsilon_p^X = \epsilon_q^X$. (This means, in particular, that the diagonal entries and blocks of $D_{00}^0$ and $D_{11}^1$ agree.)

D) When $i = j$ and $\epsilon_p^X \neq \epsilon_q^X$, condition (12) can only be satisfied if $\langle e_p c_i, e_q c_i \rangle = 0$ for $i = 0, 1$. Thus we must have $d_{IZ} = d_{XY} = 0$ and $D_{XY} = 0$.

E) When $i \neq j$, $\langle e_p c_0, e_q c_1 \rangle = \pm \langle e_p c_1, e_q c_0 \rangle = \pm \langle e_q c_0, e_p c_1 \rangle$. Thus, we can conclude, e.g., that matrix entries $b_{XZ} = 0 \Leftrightarrow b_{ZX} = 0$ and blocks $B_{XZ} = 0 \Leftrightarrow B_{ZX} = 0$, so it suffices to check entries of $B$ above the main diagonal.

Thus, when conditions (I) and (II) are satisfied, we find that sufficient (and necessary) conditions for (12) to hold are that

- All off-diagonal entries and blocks in (19) are zero,
- All remaining entries in (20) are zero.

Moreover, it suffices to check matrix elements above the main diagonal in (19) and (20).

We can break these conditions into several groups, which will turn out to be related or equivalent.
a) Conditions \( b_{IX} = b_{IY} = b_{ZX} = b_{ZY} = 0 \), which are equivalent to 
\[ \langle c_0, (X \pm iY)c_1 \rangle = \langle Zc_0, (X \pm Y)c_1 \rangle = 0. \] 
These will yield just two conditions when the \( a_k \) are real.

b) Conditions \( d_{IZ} = d_{XY} = 0 \), i.e., 
\[ \langle c_0, Zc_0 \rangle = \langle Xc_0, Yc_0 \rangle = 0. \] 
These will reduce to one condition when the \( a_k \) are real.

c) The block conditions \( B_{XZ} = B_{YZ} = 0 \), which are equivalent to 
\[ \langle [(X_1 - X_r) \pm i(Y_1 - Y_r)]c_0, (Z_1 - Z_s)c_1 \rangle = 0 \text{ for } 2 \leq r, s \leq n. \]

d) The block condition \( D_{XY} = 0 \), which is equivalent to 
\[ \langle (X_1 - X_r)c_0, (Y_1 - Y_s)c_0 \rangle = 0 \text{ for } 2 \leq r, s \leq n. \]

We will see that for codes which satisfy conditions (I) and (II) and have all coefficients real, conditions (c) on blocks will be satisfied whenever (a) holds; and conditions (d) on blocks will be satisfied whenever (b) holds. Thus, we will only need to satisfy three non-linear equations for such codes. We can summarize this as follows.

**Theorem 1** Assume that the coefficients \( a_k \) associated with a permutationally invariant code which has the form (10) and length \( n \) are all real. Such a code can correct all one-bit errors if and only if the following equations hold.

\[
0 = \frac{n+1}{2} \left( \frac{n}{n+1} \right)^{a_{n+1}^2} + 2 \sum_{m=1}^{\lfloor (n-1)/4 \rfloor} a_{2m} a_{n-2m+1} 2m \binom{n}{2m} \tag{22}
\]

\[
0 = \frac{n+1}{2} \left( \frac{n}{n-1} \right)^{a_{n+1}^2} + 2 \sum_{m=0}^{\lfloor (n-3)/4 \rfloor} a_{2m} a_{n-2m-1} (n-2m) \binom{n}{2m} \tag{23}
\]

\[
0 = \sum_{m=0}^{(n-1)/2} a_{2m}^2 (n-4m) \binom{n}{2m} \tag{24}
\]

The theorem will follow from the analysis in the next section. The result can be extended to complex \( a_k \) as discussed in Appendix C. As noted before, it is implicit in (10) that \( n \) is odd.
4 Error condition analysis

4.1 Conditions of type (a) — $c_0, c_1$ orthogonality

First we give expressions for the action of the average errors $X, Y$ and $Z$. It is easy to check that for $0 \leq k \leq n$

$$nZ W_k = (n - 2k)W_k$$

$$nX W_k = (k + 1)W_{k+1} + (n - k + 1)W_{k-1}$$

$$i nY W_k = (k + 1)W_{k+1} - (n - k + 1)W_{k-1},$$

where it is understood that if $m < 0$ or $m > n$, $W_m$ should be replaced by zero; for example, $nX W_0 = W_1$.

To analyze the requirement $b_{IX} = b_{IY} = b_{ZX} = b_{ZY} = 0$, it is equivalent (and somewhat easier) to use the conditions

$$0 = \langle c_0, (X \pm iY) c_1 \rangle$$

$$0 = \langle Zc_0, (X \pm iY) c_1 \rangle$$

When condition (I) holds, these conditions become

$$n\langle c_0, (X + iY)c_1 \rangle = 2 \sum_{k=1}^{n} k \binom{n}{k} a_k a_{n-k+1} = 0$$

$$n^2\langle Zc_0, (X + iY)c_1 \rangle = 2 \sum_{k=1}^{n} (n - 2k)k \binom{n}{k} a_k a_{n-k+1} = 0$$

$$n\langle c_0, (X - iY)c_1 \rangle = 2 \sum_{k=0}^{n-1} (n - k) \binom{n}{k} a_k a_{n-k-1} = 0$$

$$n^2\langle Zc_0, (X - iY)c_1 \rangle = 2 \sum_{k=0}^{n-1} (n - 2k)(n - k) \binom{n}{k} a_k a_{n-k-1} = 0$$

Thus far, condition (I) has played a minor role and one can easily obtain more general conditions by replacing $a_{n-k}$ by $b_k$ above. Now, however, we make explicit use of the fact that all products have the form $a_k a_{n-k+1}$ to conclude that the real parts of the expressions in (30) and (31) and in (32) and (33) agree up to sign, which leads to the following lemma.

**Lemma 2** When the coefficients $a_k$ are all real, the equations (30) to (33) are equivalent in pairs, (30) $\leftrightarrow$ (31) and (32) $\leftrightarrow$ (33). When $n$ is odd these reduce
to
\[
0 = \sum_{k=1}^{(n-1)/2} a_k a_{n-k+1} 2^k \binom{n}{k} + a_{n+1}^2 \frac{n+1}{2} \binom{n}{n+1} \tag{34}
\]
\[
0 = \sum_{k=0}^{(n-3)/2} a_k a_{n-k-1} 2^k (n-k) \binom{n}{k} + a_{n+1}^2 \frac{n+1}{2} \binom{n}{n+1} \tag{35}
\]
When \( n \) is even, similar expressions hold with upper limits of \( n/2 \) and \( n/2 - 1 \) respectively but without any extra square terms analogous to \( a_{n/2+1}^2 \).

When condition (II) holds, equations (34) and (35) reduce to (22) and (23), which proves the first part of Theorem 1.

**Proof:** First, observe that for \( m \neq \frac{n+1}{2} \), the term \( a_m a_{n-m+1} \) occurs twice in (30), once for \( m = k \) and once for \( m = n - k + 1 \). Thus, the coefficient of \( a_m a_{n-m+1} \) is
\[
m \binom{n}{m} + (n-m+1) \binom{n}{n-m+1} = 2m \binom{n}{m}
\]
The coefficient of the same term in (31) is
\[
(n-2m) m \binom{n}{m} + [n-2(n-m+1)] (n-m+1) \binom{n}{n-m+1} = -2m \binom{n}{m}.
\]
To obtain a general proof and reduction to (22), it suffices to make the change of variable \( k \to n - k + 1 \) when \( k \geq n+3 \) in (30) and (31) and, as above, use the elementary identity
\[
m \binom{n}{m} = (n-m+1) \binom{n}{n-m+1} \tag{36}
\]
Similarly, the change of variable \( k \to n - k - 1 \) for \( k \geq \frac{n+1}{2} \) in (32) and (33) yields (23). QED

### 4.2 Conditions of type (b) — off-diagonal conditions

Using (25) to (27) one finds that
\[
\langle nXW_k, i n\overline{Y}W_k \rangle = \langle W_k, n\overline{Z}W_k \rangle = (n-2k) \binom{n}{k}
\]
\[
\langle nXW_k, i n\overline{Y}W_{k+2} \rangle = -\langle nXW_{k+2}, i n\overline{Y}W_k \rangle
\]
\[
\langle nXW_j, i n\overline{Y}W_\ell \rangle = 0 \quad \text{if } j - \ell \neq \pm 2, 0.
\]
From these relations, it follows that when all $a_k$ and $b_k$ are real

$$n^2\langle Xc_j, iYc_j \rangle = n\langle c_j, Zc_j \rangle = \sum_k \alpha_k^2(n-2k)\binom{n}{k}$$

(37)

where $\alpha_k$ equals $a_k$ or $b_k$ according as $j$ equals 0 or 1. Thus, we can conclude that when $a_k, b_k$ are real, the off-diagonal conditions $d_{IZ} = d_{XY} = 0$ hold if and only if

$$\sum_k a_k^2(n-2k)\binom{n}{k} = \sum_k b_k^2(n-2k)\binom{n}{k} = 0.$$  

(38)

This reduces to (24) when conditions I and (II) are satisfied. If some $a_k, b_k$ are not real, then (37) holds with $\alpha_k^2$ replaced by $|a_k|^2$ or $|b_k|^2$, but the additional condition (92) is needed to ensure that the imaginary part of $\langle Xc_i, iYc_i \rangle$ is zero, as discussed in Appendix C.

4.3 Conditions of type (c) — block $c_0, c_1$ orthogonality.

It will again be useful to replace the separate $X, Y$ equations by their sums and differences. The requirement that the blocks $B_{XZ} = B_{YZ} = 0$ is equivalent to

$$\langle [(X_1 - X_r) \pm i(Y_1 - Y_s)] c_0, (Z_1 - Z_s) c_1 \rangle = 0.$$  

(39)

for $2 \leq r, s \leq n$. We now need results from Appendix A. When conditions (I) and (II) hold, equations (80a), (80b) and (80c) imply that (39) is equivalent to the following pair of equations

$$\sum_{m=0}^{(n-3)/2} \overline{a}_{2m}a_{n-2m-1}\binom{n-2}{2m} = 0, \text{ and}$$

(40)

$$\sum_{m=1}^{(n-1)/2} \overline{a}_{2m}a_{n-2m+1}\binom{n-2}{2m-2} = 0.$$  

(41)

To see that these are equivalent to (22) and (23), again make a change of variable of the form $k \rightarrow n-k \mp 1$ in the second half of each sum and use the identities

$$\binom{n-2}{k} + (1-\delta_{k0})\binom{n-2}{n-k-1} = \binom{n-1}{k} = \frac{n-k}{n}\binom{n}{k},$$

$$\binom{n-2}{k-2} + (1-\delta_{k1})\binom{n-2}{n-k-1} = \binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}.$$
4.4 Conditions of type (d)— block off-diagonal conditions

We now consider the condition \( d_{XY} = 0 \) which means that

\[
\langle (X_1 - X_r)c_0, i(Y_1 - Y_s)c_0 \rangle = 0
\]

for all choices of \( 2 \leq r, s \leq n \). The crucial fact is that the inner products of this type with \( r = s \) and \( r \neq s \) differ only by a factor of 2 as shown by (78) in Appendix A.

**Theorem 3** When conditions (I) and (II) are satisfied and the \( a_k \) are real, (42) holds if and only if (24) does.

**Proof:** It follows from (79b) and (79c) that when \( 1, r, s \) are distinct

\[
\langle (X_1 - X_r)c_0, i(Y_1 - Y_r)c_0 \rangle = 2 \langle (X_1 - X_r)c_0, i(Y_1 - Y_s)c_0 \rangle
\]

\[
= 2A_0 + 2 \sum_{m=1}^{(n-3)/2} \left( \frac{n-2}{2m} \right) A_m
\]

(43)

where \( A_m = |a_{2m}|^2 - |a_{2m+2}|^2 + a_{2m} \bar{a}_{2m+2} - \bar{a}_{2m} a_{2m+2} \). Thus when the \( a_k \) are real, (42) will be satisfied for all choices of \( r, s \) if

\[
a_0^2 + \sum_{m=1}^{(n-3)/2} a_{2m}^2 \left( \frac{n-2}{2m} \right) - \left( \frac{n-2}{2(m-1)} \right) - (n-2)a_{n-1}^2 = 0.
\]

(44)

One can then conclude that (44) is equivalent to (24) if \( n \left[ \frac{(n-2)}{2m} - \frac{(n-2)}{2(m-1)} \right] = (n-4m)\left( \frac{n}{2m} \right) \) which follows from (7) with \( N = n-2, K = 2m, J = 2 \).

5 Two-bit errors

5.1 Some special types of two-bit errors

The standard 7-bit CSS code [18] can correct two-bit errors of the form \( X_rZ_s \) but not those of the form \( X_rX_s \) or \( Z_rZ_s \). The last two are far more likely to occur, especially for nearest neighbors. We now consider the effect of two-bit errors of the same type, which we call “double” errors, on permutationally invariant codes.

Recall that exchange errors have the form (1). Permutationally invariant codes are designed so that exchange errors are degenerate with the identity, i.e.,
$E_{rs}|c_j⟩ = |c_j⟩$ for $j = 0, 1$. Now consider the following three errors

$$F_{rs} = \frac{1}{2} \left[ I \otimes I - X_r \otimes X_s - Y_r \otimes Y_s + Z_r \otimes Z_s \right]$$ (45a)

$$G_{rs} = \frac{1}{2} \left[ I \otimes I + X_r \otimes X_s - Y_r \otimes Y_s - Z_r \otimes Z_s \right]$$ (45b)

$$H_{rs} = \frac{1}{2} \left[ I \otimes I - X_r \otimes X_s + Y_r \otimes Y_s - Z_r \otimes Z_s \right]$$ (45c)

and observe that

- $F_{rs}$ exchanges two bits and multiplies by $-1$ if and only if the values of the bits are different.

- $G_{rs}$ flips the two bits $r$ and $s$ if and only if they are the same.

- $H_{rs}$ flips the two bits $r$ and $s$ and then multiplies by $-1$ if and only if they are the same.

In a product basis of the form $|00⟩, |01⟩, |10⟩, |11⟩$ these operators are represented by the matrices

$$E_{rs} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad F_{rs} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G_{rs} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad H_{rs} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Any code which can correct all errors of the type $E_{rs}, F_{rs}, G_{rs}, H_{rs}$ can also correct any error of the form $Z_rZ_s, X_rX_s, Y_rY_s,$ since an error of one type can be written as a linear combination of those of the other. For permutationally invariant codes, these two types of errors are actually equivalent.

**Theorem 4** If $|ψ⟩$ is permutationally invariant (i.e., $E_{rs}|ψ⟩ = |ψ⟩$ for all $r,s$), then the operators $F_{rs}, G_{rs}$ and $H_{rs}$ have the same effect on $|ψ⟩$ as $Z_rZ_s, X_rX_s,$ and $Y_rY_s$ respectively, i.e., $F_{rs}|ψ⟩ = Z_rZ_s|ψ⟩$, $G_{rs}|ψ⟩ = X_rX_s|ψ⟩$ and $H_{rs}|ψ⟩ = Y_rY_s|ψ⟩$.

**Proof:** First note that $E_{rs} + F_{rs} = I + Z_rZ_s$. Then

$$F_{rs}|ψ⟩ = (I + Z_rZ_s - E_{rs})|ψ⟩ = |ψ⟩ + Z_rZ_s|ψ⟩ - |ψ⟩ = Z_rZ_s|ψ⟩$$

The other two cases are done similarly using $E_{rs} + G_{rs}$ and $E_{rs} + H_{rs}$ respectively.
5.2 Two-bit error correction conditions

We begin with some simple, but fundamental, results. The first follows from the fact that all double errors preserve parity.

**Theorem 5** Whenever condition (II) is satisfied,

\[ \langle e_p c_0, e_q c_1 \rangle = 0 \] \hspace{1cm} (46)

for any pair of errors in the set \{I, Z_r Z_s, X_r X_s, Y_r Y_s\} or, equivalently, in the set \{I, E_r s, F_r s, G_r s, H_r s\}.

The next theorem says that all inner products of the form \( \langle Z_r Z_s c_j, Z_q Z_t c_j \rangle \), \( \langle Z_r Z_s c_j, X_q X_t c_j \rangle \) etc. are independent of \( j = 0, 1 \). It follows easily from the equivalence of condition (I) to \( \langle \otimes_k X_k | c_0 \rangle = | c_1 \rangle \), and the fact that \( \otimes_k X_k \) is a unitary operator which commutes with any error of the form \( X_r X_s, Y_r Y_s \) or \( Z_r Z_s \).

**Theorem 6** Whenever condition (I) is satisfied,

\[ \langle f_r f_s c_0, g_q g_t c_0 \rangle = \langle f_r f_s c_1, g_q g_t, c_1 \rangle \] \hspace{1cm} (47)

where \( f, g \) denote any of \{X, Y, Z\} (the same as well as different) and \( r, s, q, t \) are arbitrary.

One is often interested in knowing which two-bit errors can be corrected in addition to one-bit errors. Conditions involving the average error \( ZZ \) can be readily calculated by noting that

\[ \sum_{r \neq s} Z_r Z_s = \sum_{r,s} Z_r Z_s - \sum_r Z_r^2 = \left( \sum_r Z_r \right)^2 - nI = (nZ)^2 - nI. \] \hspace{1cm} (48)

Combining this with (25), one finds

\[ n(n-1) ZZ W_k = \left( \sum_{r \neq s} Z_r Z_s \right) W_k = [(n-2k)^2 - n] W_k. \] \hspace{1cm} (49)

The additional conditions needed to correct all errors of the form \( Z_r Z_s \) include \( \langle Z c_0, ZZ c_0 \rangle = 0 \) and \( \langle ZZ c_0, (X \pm iY) c_1 \rangle = 0 \). The latter gives the following pair of conditions

\[ 0 = 2 \sum_{k=1}^{n} [(n-2k)^2 - n] k \binom{n}{k} a_k a_{n-k+1} \] \hspace{1cm} (50)

\[ 0 = 2 \sum_{k=0}^{n-1} [(n-2k)^2 - n(n-k) \binom{n}{k} a_k a_{n-k-1} \] \hspace{1cm} (51)
Using (25) and (49), one finds that \( \langle Zc_0, ZZc_0 \rangle = d_{Z,ZZ} = 0 \) is equivalent to

\[
0 = \sum_k |a_k|^2(n - 2k)[(n - 2k)^2 - n] \binom{n}{k},
\]

(52)

Although one can write down a similar set of conditions for the correction of errors of the form \( X_rX_s \), it is probably easier to use the following observation.

**Theorem 7** A permutationally invariant code \( |c_0\rangle, |c_1\rangle \) which satisfies conditions (I) and (II) [and corrects a specified set of one-bit errors] can correct all errors of the form \( Z_r Z_s \) if and only if the code

\[
|C_j\rangle = H^\otimes n|c_0\rangle + (-1)^j H^\otimes n|c_1\rangle
\]

(53)

can correct all errors of the form \( X_r X_s \).

The map \( |c_j\rangle \mapsto |C_j\rangle \) in (53) consists of a Hadamard gate acting on all qubits, followed by an effective Hadamard operation on the resulting code words themselves. This is extremely useful and is its own inverse. We will refer to it as the “Hadamard code map”.

### 5.3 Degeneracy enhancement of classical codes

It follows from Theorems 5 and 6 that every permutationally invariant code for which both conditions (I) and (II) are satisfied can correct all double errors of the form \( Z_r Z_s, X_r X_s, Y_r Y_s \) provided that we do not also require single bit errors to be correctable. For example, the 3-bit repetition code \( |c_0\rangle = |000\rangle, |c_1\rangle = |111\rangle \) is generally regarded as able to correct all single bit flips, but no other errors. However, one could instead use it to correct all double bit flips, at the expense of the ability to correct any single bit errors. The theorem above says that it can do even more — it can correct all two-bit errors of the same type. Although this might seem surprising at first, it is easy to understand why it is true. For this code \( Z_r Z_s |c_j\rangle = |c_j\rangle \) so that \( Z_r Z_s \) is degenerate with the identity. Similarly, \( Y_r Y_s \) is degenerate with \( X_r X_s \). Note that this degeneracy extends to any n-bit repetition code

\[
|c_0\rangle = |00 \ldots 0\rangle = W_0, \quad |c_1\rangle = |11 \ldots 1\rangle = W_n.
\]

(54)

When \( n \geq 5 \), the simple repetition code (54) can correct all single and all double bit flips. Indeed, for \( n = 5 \), this is just a classical code for two-bit error correction. Applying the Hadamard code map (53) to (54) yields a code which
can correct all single and double phase errors. In fact, omitting the normalizing coefficients, this code is
\[ |C_0\rangle = H^\otimes n |c_0\rangle + H^\otimes n |c_1\rangle = \sum_m W_{2m} \]  
\[ |C_1\rangle = H^\otimes n |c_0\rangle - H^\otimes n |c_1\rangle = \sum_m W_{2m+1} \] (55a) (55b)

Because the phase errors preserve parity, the necessary and sufficient conditions for a code satisfying conditions (I) and (II) to correct both single and double phase errors are

\[ 0 = \langle c_0, Zc_0 \rangle \] (56a)  
\[ 0 = \langle Zc_0, ZZc_0 \rangle \] (56b)  
\[ 0 = \langle (Z_1 - Z_t)c_0, Z_1 Z_s c_0 \rangle = 0 \quad (t \neq s). \] (56c)

Note that since \(\langle (Z_1 - Z_t)c_0, ZZc_0 \rangle = 0\), the single and double-Z errors which transform as the \((n-1)\)-dimensional representation are orthogonal if and only if (56c) holds. In fact, as shown after (86) in Appendix A, (56c) is redundant, i.e., it is satisfied whenever (56a) and (56b) hold. Thus, one finds that the necessary and sufficient conditions for a code satisfying conditions (I) and (II) to correct single and double phase errors are (38) [which becomes (89) when \(a_k\) is complex] and (52), which we rewrite below.

\[ 0 = \sum_k |a_k|^2 (n - 2k) \binom{n}{k} = 0 \] (57a)  
\[ 0 = \sum_k |a_k|^2 (n - 2k)[(n - 2k)^2 - n] \binom{n}{k} = 0. \] (57b)

When \(n = 5\), the pair of equations in (57) has exactly one solution (up to normalization), namely \(|a_0|^2 = |a_2|^2 = |a_4|^2\). Choosing identical phases yields the code in (55). In addition to correcting all one and two-bit phase errors, it can also correct all errors of the form \(X_rX_s\) and \(Y_rY_s\). Choosing other phases yields other codes and taking the Hadamard code map yields classical codes for two-bit error correction that are distinct from (54). These also satisfy conditions (I) and (II) and, hence, can correct all double \(Z_rZ_s\) and \(Y_rY_s\) errors as well as single and double bit flips when used as quantum codes.

When \(n \geq 7\) and odd, the pair of equations (57) has infinitely many solutions in addition to \(|a_0|^2 = |a_2|^2 \ldots = |a_{n-1}|^2\). Taking the Hadamard transform then yields infinitely many classical codes for two bit error correction.
5.4 Higher dimensional representations

In this section we take some preliminary steps toward exploiting higher dimensional irreducible representations for correction of errors in addition to one-bit errors. First, we review the mutually orthogonal subspaces required for the correction of single errors. The operators \( I, X, Y, Z \) acting on the code words \(|c_0\rangle, |c_1\rangle\) require four pairs of one-dimensional subspaces which transform as the trivial representation. The three sets of differences \( X_1 - X_r, Y_1 - Y_r, Z_1 - Z_r \) acting on the code words require three pairs of subspaces of dimension \( n-1 \) which transform as the even \((n-1)\)-dimensional representation. But (as described in Appendix B) the decomposition of \( C_{2n}^2 \) into an orthogonal sum of irreducible subspaces includes other irreducible representations of \( S_n \).

The next irreducible representation has dimension \( \frac{n(n-3)}{2} \) and arises in the decomposition of double errors of one type, e.g., \( f_{rs} = X_r X_s \). Consider the subspace generated by \( f_{rs} W_k \) for \( k = 2 \) (or \( k = n-2 \)) as \( r, s \) run through all \( \frac{n(n-1)}{2} \) combinations of \( r < s \). This subspace splits into an orthogonal direct sum consisting of

- a 1-dimensional subspace spanned by \( \overline{f} f W_k \) where \( \overline{f} f = \binom{n}{2}^{-1} \sum_{r \neq s} f_{rs} \) is the average error of this type,
- an \( (n-1) \)-dimensional subspace spanned by the vectors \( f_r W_k \) for \( r = 2, 3 \ldots n \) where \( \overline{f}_r = \sum_{s=2}^{n} f_{1s} - \sum_{s \neq r} f_{rs} \quad r = 2, 3 \ldots n \), and
- an \( \frac{n(n-3)}{2} \)-dimensional subspace which is obtained by taking the orthogonal complement of the vectors \( \overline{f} f W_k \) and \( f_r W_k \) in \( \text{span}\{f_{rs} W_k\} \).

As described in Section 2.1, the error set \( \{f_{rs}\} \) can be correspondingly decomposed into bases for representations of \( S_n \) with dimensions 1, \( n-1 \), and \( \frac{n(n-3)}{2} \).

For an explicit example of the last type of error, consider \( n = 4 \). Then \( W_2 \) splits into three subspaces, corresponding to irreducible representations of dimensions 1, 3 and 2. The last is spanned by the vectors:

\[
2f_{12} - f_{13} - f_{14} - f_{23} - f_{24} + 2f_{34} \\
f_{12} + f_{13} - 2f_{14} - 2f_{23} + f_{24} + f_{34}
\]

There is a sense in which these errors are rather delocalized, since they act on all six pairs of qubits. Although one could eliminate some pairs by a different choice of basis vectors, one can not, e.g., eliminate all terms of the form \( f_{j4} \) involving the 4th qubit. This delocalization is, unfortunately, the antithesis of what one might want in certain situations, such as errors between nearest neighbors.

We now focus on \( n = 7 \) as an example and note that \( C_{2^7}^2 \) can be decomposed into an orthogonal sum of irreducible subspaces spanned by
- 8 orthogonal bases for the trivial 1-dimensional representation,
- 6 orthogonal bases for the even 6-dimensional representation,
- 4 orthogonal bases for a 14-dimensional representation,
- 2 orthogonal bases for a second, inequivalent, 14-dimensional representation.

Thus, correcting the one-bit errors requires all of the available 1 and 6-dimensional representations. However, the two types of 14-dimensional representations are available to correct two-bit errors. One of these 14-dimensional representations has spin $\frac{3}{2}$ and is associated with the partition [5, 2]. This is the $\frac{n(n-3)}{2}$-dimensional representation which arises in the decomposition of $f_{rs}$ errors described above, and can be used to correct the corresponding subclass of double errors. There are three kinds of double errors, those from $Z_rZ_s$, from $X_rX_s$, and from $Y_rY_s$. This would seem to require six orthogonal subspaces which transform as this 14-dimensional representation; however, we have only four — one each from $W_2, W_3, W_4, W_5$. Nevertheless, Theorems 5 and 6 imply that all three types of double errors can be corrected.

This is indeed the case and is the result of degeneracy. For permutationally invariant codes,

\[(X_rX_s + Z_rZ_s + Y_rY_s)\ket{\psi} = \ket{\psi}.\]  

Thus, it suffices to correct any two of $X_rX_s$, $Z_rZ_s$, $Y_rY_s$ to ensure that all three types of errors can be corrected, and this requires only four 14-dimensional subspaces, exactly what one has available when $n = 7$. Thus, a 7-bit permutationally invariant code which can correct all one-bit errors can also correct all errors in the 14-dimensional irreducible components of the decompositions of $X_rX_s$, $Z_rZ_s$, and $Y_rY_s$. This implies that arbitrary double errors would be corrected about 2/3 of the time. Similarly, a 9-bit code could correct them about 3/4 of the time. Unfortunately, the other 1/3 (or 1/4) of the time, the procedure does not simply fail to detect the error — it incorrectly interprets a two-bit error as a one-bit error and the attempted correction actually introduces additional errors.

We next consider the case $n = 9$. Correcting the one-bit errors uses 8 of the 10 available 1-dimensional representations and 6 of the 8 available 8-dimensional representations. In addition to the six 27-dimensional representations, two 1-dimensional representations and two 8-dimensional representations are also potentially available to correct some two-bit errors. Correcting one type of $f_{rs}$ double errors would require a pair of 1-dimensional, 8-dimensional and 27-dimensional subspaces. Based on dimensional considerations, one might expect to correct one type of double error completely using a 9-bit permutationally invariant code. Unfortunately, as will be shown in Section 6.5, this is not possible.
One can still ask what additional errors can be corrected with permutationally invariant 9-bit codes. The operators $I, X, Y, Z$ acting on the code words $|c_0\rangle, |c_1\rangle$ generate an 8-dimensional space. Taking the orthogonal complement in the 10-dimensional subspace spanned by $\{W_0, W_1 \ldots W_9\}$ yields a two-dimensional subspace. There is a family of linear operators which map $|c_0\rangle, |c_1\rangle$ to a pair of orthogonal vectors in this two-dimensional subspace. Any member of this family can be chosen as an additional correctable error. Similarly, there will be a set of correctable errors which transform as the 8-dimensional representation and whose action on the code words spans the orthogonal complement of the one-bit errors in $\text{span}(\bigoplus_{k=1}^{8} U_k^8)$. Although a procedure for obtaining these operators can be written down, we have been unable to characterize them in a useful way.

6 Special cases

6.1 $n = 5$

When $n = 5$, conditions (I) and (II) hold, and all $a_k$ are real, the three necessary and sufficient conditions in Theorem 1 become

\[
\begin{align*}
a_2a_4 &= 0 \\
a_0a_4 + 3a_2^2 &= 0 \\
a_0^2 + 2a_2^2 - 3a_4^2 &= 0.
\end{align*}
\]

It is easy to verify that these have no non-trivial solution. This is not surprising. It is well-known that the 5-bit code for correcting all one-bit quantum errors is essentially unique and is not permutationally symmetric.

Nevertheless, there is still something to be learned by looking at 5-bit codes. As discussed in section 5.3 the simple repetition code

\[
|c_0\rangle = |00000\rangle, \quad |c_1\rangle = |11111\rangle
\]

(59)

corrects both all single and all double bit flips, and

\[
\begin{align*}
|C_0\rangle &= H^\otimes 5 |c_0\rangle + H^\otimes 5 |c_1\rangle = W_0 + W_2 + W_4 \\
|C_1\rangle &= H^\otimes 5 |c_0\rangle - H^\otimes 5 |c_1\rangle = W_5 + W_3 + W_1
\end{align*}
\]

(60a)

(60b)
corrects all single and double phase errors. In fact, when $n = 5$, equations (57a) and (57b) imply that the only codes satisfying conditions (I) and (II) are those with $|a_0|^2 = |a_2|^2 = |a_4|^2$.

Moreover, both codes can correct all double errors of the form $X_jX_k$, $Y_jY_k$, $Z_jZ_k$. To see this, note that $Z_jZ_k|\psi\rangle = |\psi\rangle$ on the span of (59) so that (58) implies
\(X_r X_s |\psi\rangle = -Y_r Y_s |\psi\rangle\), i.e., the pair \(\{Z_r Z_s, I\}\) is degenerate and this induces a degeneracy on the pair \(\{X_r X_s, Y_r Y_s\}\).

Thus, the 5-bit codes \((59)\) and \((60)\) can each correct more types of quantum errors than one might expect from their classical distance properties. They are optimal for the correction of all one-bit and two-bit errors of a particular type (phase or bit flip) and can not correct additional one-bit errors. Nevertheless they can correct additional types of two-bit errors.

6.2 \(n = 7\)

When \(n = 7\), conditions (I) and (II) hold, and all \(a_k\) are real, the three conditions in Theorem 4 become

\[
\begin{align*}
3a_2a_6 + 5a_4^2 &= 0 \quad (61a) \\
a_0a_6 + 15a_2a_4 &= 0 \quad (61b) \\
a_0^2 + 9a_2^2 - 5a_4^2 - 5a_6^2 &= 0. \quad (61c)
\end{align*}
\]

It is not hard to see that \(a_6 = 0\) implies all \(a_k = 0\). Therefore we can divide through by \(a_6\) or, equivalently, assume without loss of generality that \(a_6 = 1\). Then \((61a)\) and \((61b)\) imply \(a_2 = -\frac{5}{3}a_4^2\) and \(a_0 = 25a_4^3\). Letting \(x = a_4^2\) and substituting in \((61c)\) yields

\[
125x^3 + 5x^2 - x - 1 = 0,
\]

to which \(x = \frac{1}{5}\) is the only real solution, giving

\[
a_0 = \pm \sqrt{5}, \quad a_2 = -\frac{1}{3}, \quad a_4 = \pm \frac{1}{\sqrt{5}}, \quad a_6 = 1.
\]

It is then straightforward to verify that both signs in the normalized vector

\[
|c_0\rangle = \frac{1}{8} \left[ \pm \sqrt{15} \hat{W}_0 - \sqrt{7} \hat{W}_2 \pm \sqrt{21} \hat{W}_4 + \sqrt{21} \hat{W}_6 \right] \quad (62)
\]

yield acceptable codes. This gives two distinct new codes when \(n = 7\).

It is interesting — and a good check — to write the vectors \(\overline{X}|c_1\rangle, \overline{Y}|c_1\rangle, \overline{Z}|c_0\rangle\) and see that together with \(|c_0\rangle\) they form an orthogonal set. Up to normalizing scalars, we have

\[
\begin{align*}
\overline{Z}|c_0\rangle &= \sqrt{\frac{1}{7}} \left[ \pm \sqrt{35} \hat{W}_0 - \sqrt{3} \hat{W}_2 \mp \hat{W}_4 - 5 \hat{W}_6 \right] \\
\overline{X}|c_1\rangle &= \sqrt{\frac{1}{7}} \left[ \sqrt{7} \hat{W}_0 + \sqrt{3}(2 \pm \sqrt{5}) \hat{W}_2 \pm (\pm 4 - \sqrt{5}) \hat{W}_4 + (\pm \sqrt{5} - 2) \hat{W}_6 \right] \\
\overline{Y}|c_1\rangle &= \sqrt{\frac{1}{7}} \left[ - \sqrt{7} \hat{W}_0 + \sqrt{3}(2 \mp \sqrt{5}) \hat{W}_2 \mp (\pm 4 + \sqrt{5}) \hat{W}_4 + (\mp \sqrt{5} - 2) \hat{W}_6 \right]
\end{align*}
\]
Remark: One might ask if one can obtain additional permutationally invariant 7-bit codes by allowing complex coefficients. In that case the following equations are necessary and sufficient.

\[0 = 10|a_4|^2 + 3(\bar{a}_2 a_6 + a_2 \bar{a}_6) \quad (63a)\]
\[0 = \bar{a}_2 a_6 - a_2 \bar{a}_6 \quad (63b)\]
\[0 = (\bar{a}_0 a_6 + a_0 \bar{a}_6) + 15(\bar{a}_2 a_4 + a_2 \bar{a}_4) \quad (63c)\]
\[0 = (a_0 \bar{a}_6 - \bar{a}_0 a_6) + 5(a_2 \bar{a}_4 - \bar{a}_2 a_4) \quad (63d)\]
\[0 = |a_0|^2 + 9|a_2|^2 - 5|a_4|^2 - 5|a_6|^2 \quad (63e)\]
\[0 = (\bar{a}_0 a_2 - a_0 \bar{a}_2) + 10(\bar{a}_2 a_4 - a_2 \bar{a}_4) + 5(\bar{a}_4 a_6 - a_4 \bar{a}_6). \quad (63f)\]

As in the real case, \(a_6 = 0\) forces all coefficients to be zero, so we can assume \(a_6 = 1\). Then (63b) implies that \(a_2 = -\frac{5}{3}|a_4|^2\). However, (63a) and (63b) yield a pair of linear equations for \(\text{Im} \ a_0\) and \(\text{Im} \ a_4\) which have a non-zero solution if and only if \(a_2 = +1\) which is not consistent with \(a_2 = -\frac{5}{3}|a_4|^2\). Thus, there are no permutationally invariant 7-bit codes other than those in (62).

6.3 n = 9

When \(n = 9\), conditions (I) and (II) hold, and all \(a_k\) are real, the three conditions in Theorem 1 become

\[a_2 a_8 + 7a_4 a_6 = 0 \quad (64a)\]
\[35a_4^2 + a_0 a_8 + 28a_2 a_6 = 0 \quad (64b)\]
\[a_0^2 + 20a_2^2 + 14a_4^2 - 28a_6^2 - 7a_8^2 = 0. \quad (64c)\]

We now show that these have have infinitely many solutions. First, suppose \(a_8 = 0\), so that the first equation becomes \(a_4 a_6 = 0\). If \(a_6 = 0\), then all the coefficients are zero. If \(a_4 = 0\) then we find the only possibility is \(a_2 = 0\) and \(a_0^2 = 28a_6^2\), giving the two solutions found in (22). To find the remaining solutions we may assume \(a_8 = 1\).

If \(a_6 = 0\), then also \(a_2 = 0\) and \(a_0 = -35a_4^2\) where \(a_4^2 = x\) is the positive root of the quadratic equation \(175x^2 + 2x - 1 = 0\). This gives two more solutions. If \(a_0 = 0\), then we find all of the remaining coefficients depend on \(a_6^2 = t\), where \(t\) is a positive root of the cubic \(f(t) = (28^3/5)t^3 + (2\cdot 28^2/5)t^2 - 4t - 1:\)

\[a_4 = (28/5)t, \quad a_6 = \pm \sqrt{t}, \quad a_2 = (-7 \cdot 28)t.\]

Since \(f(0) < 0\) and \(f(1) > 0\), \(f\) does have a positive root (approximately \(t = 0.478\)), and this gives two more solutions.
There are no further solutions with any of the $a_k$ equal to zero, so now assume they are all nonzero. Writing $x = a_6^2$ and $t = a_4$, it follows that

$$a_2 = \pm 7\sqrt{xt} \quad a_0 = 7(28tx - 5t^2)$$

where $x$ and $t$ satisfy the equation

$$x^2(5488t^2) + 4x(-490t^3 + 35t^2 - 1) + (175t^4 + 2t^2 - 1) = 0.$$ Using Maple, one can verify that there are infinitely many values of $t$ (e.g., all for which $-0.25 < t < 0.4$) for which this quadratic in $x$ has at least one positive solution. Thus there are infinitely many solutions in which all the coefficients $a_{2m}$ are non-zero real numbers.

### 6.4 Conditions for double error correction with 9-bit codes

As discussed in Section 5.2, a 9-bit permutationally invariant codes which can correct all errors of type $Z_1 Z_2$ as well as all one-bit errors, must satisfy at least 9 conditions. In the notation of Section 3, there are six of the form $b_{1X} = b_{1Y} = b_{2X} = b_{2Y} = b_{Z1X} = b_{Z2Y} = 0$ and three of the form $d_{1X} = d_{1Z} = d_{Z1Z} = 0$.

First, consider the conditions (30), (31) and (50) which correspond to the requirements $\langle f_{c_0}, (X + iY) c_1 \rangle = 0$ with $f = I, Z$ or $ZZ$. For $n = 9$ these are equivalent to

$$\begin{align*}
\bar{a}_2a_8 + 7\bar{a}_4a_6 + 7\bar{a}_6a_4 + \bar{a}_8a_2 &= 0 \quad (65a) \\
5\bar{a}_2a_8 + 7\bar{a}_4a_6 - 21\bar{a}_6a_4 - 7\bar{a}_8a_2 &= 0 \quad (65b) \\
2\bar{a}_2a_8 - 7\bar{a}_4a_6 + 5\bar{a}_6a_4 &= 0. \quad (65c)
\end{align*}$$

These can be treated as a set of 3 linear equations in the 4 unknowns, $\bar{a}_2a_8, \bar{a}_4a_6, \bar{a}_6a_4, \bar{a}_8a_2$ from which one finds that the group (65) is equivalent to

$$\begin{align*}
\bar{a}_2a_8 &= -\bar{a}_8a_2 = -i\nu \quad (66a) \\
\bar{a}_4a_6 &= -\bar{a}_6a_4 = i\frac{3}{7}\nu \quad (66b)
\end{align*}$$

for some real parameter $\nu$. Note also that $\text{Re}\bar{a}_4a_6 = \text{Re}\bar{a}_2a_8 = 0$ implies that any real solutions must have $a_4a_6 = a_2a_8 = 0$. However, all such solutions have been found above and none satisfy the additional requirements below. Hence, correcting all double-$Z$ errors does require complex coefficients.

Next we consider the conditions (32), (33) and (51) which correspond to the requirements $\langle f_{c_0}, (X - iY) c_1 \rangle = 0$ with $f = I, Z$ or $ZZ$. These become

$$\begin{align*}
\bar{a}_0a_8 + 28\bar{a}_2a_6 + 70|a_4|^2 + 28\bar{a}_6a_2 + \bar{a}_8a_0 &= 0 \quad (67a) \\
9\bar{a}_0a_8 + 140\bar{a}_2a_6 + 70|a_4|^2 - 84\bar{a}_6a_2 - 7\bar{a}_8a_0 &= 0 \quad (67b) \\
9\bar{a}_0a_8 + 56\bar{a}_2a_6 - 70|a_4|^2 + 5\bar{a}_8a_0 &= 0. \quad (67c)
\end{align*}$$
which can be rewritten as

\[
\begin{align*}
\Re \overline{a}_0 a_8 + 28 \Re \overline{a}_2 a_6 + 35 |a_4|^2 &= 0 \\
7 \Re \overline{a}_0 a_8 + 28 \Re \overline{a}_2 a_6 - 35 |a_4|^2 &= 0 \\
\Im \overline{a}_0 a_8 + 14 \Im \overline{a}_2 a_6 &= 0
\end{align*}
\]

(68a) (68b) (68c)

since (67a) and (67b) have the same real part while (67b) and (67c) have the same imaginary part. Equations (68a) and (68b) are equivalent to

\[
\begin{align*}
\Re \overline{a}_0 a_8 &= \frac{35}{3} |a_4|^2 \\
\Re \overline{a}_2 a_6 &= -\frac{5}{3} |a_4|^2.
\end{align*}
\]

(69a) (69b)

To these conditions we need to add the requirements

\[
\langle \overline{X} c_j, i \overline{Y} c_j \rangle = \langle c_j, \overline{Z} c_j \rangle = \langle \overline{Z} c_j, ZZ c_j \rangle = 0
\]

which become.

\[
\begin{align*}
|a_0|^2 + 20|a_4|^2 + 14|a_4|^2 - 28|a_6|^2 - 7|a_8|^2 &= 0 \\
9|a_0|^2 + 40|a_2|^2 - 14|a_4|^2 - 35|a_8|^2 &= 0 \\
\Im \overline{a}_0 a_2 + 21 \Im \overline{a}_2 a_4 + 35 \Im \overline{a}_4 a_6 + 7 \Im \overline{a}_6 a_8 &= 0.
\end{align*}
\]

(70a) (70b) (70c)

The last equation (70c) is precisely the condition \( \Im \langle \overline{X} c_j, i \overline{Y} c_j \rangle = 0 \) and is obtained as the reduction of (92) when \( n = 9 \) and conditions I and II hold.

To recap, we have three groups of equations; namely, (65) from the conditions \( b_f X + ib_f Y = 0 \), (67) from the conditions \( b_f X - ib_f Y = 0 \), and (70) from the conditions \( d_{XY} = d_{IZ} = d_{Z,ZZ} = 0 \). In what follows, we will use the equivalent conditions (66) in place of (65), and (69) or (68c) in place of (67).

### 6.5 Limits on correction of \( ZZ \) and one-bit errors

To analyze the conditions obtained above, write \( a_k = x_k + iy_k \). We can assume without loss of generality that \( a_8 = 1 \); then (66a) implies that \( a_2 = iv \) and (69) implies \( y_6 = -\frac{5}{3v} |a_4|^2 \). We also have that (66b) implies \( a_6 = \frac{3v}{7 |a_4|^2} a_4 i \) from which we can conclude \( y_6 = \frac{3v}{7 |a_4|^2} \). Equating the two expressions above for \( y_6 \) yields

\[
\nu^2 = \frac{35 |a_4|^4}{9 x_4}
\]

(71)

which implies that \( x_4 < 0 \). Under the assumption \( a_8 = 1 \), (70c) becomes

\[
\begin{align*}
0 &= \nu \left( x_0 - 21x_4 + 15 - \frac{7}{\nu} y_6 \right) \\
&= \nu \left( \frac{35}{3} |a_4|^2 + 21 |x_4| + 15 + \frac{35}{3\nu^2} |a_4|^2 \right)
\end{align*}
\]

(72)
The last equation implies that either $\nu = 0$ or $a_4 = 0$, either of which generates only a trivial solution. Thus there is no non-trivial solution to the seven equations (65), (67) and (70c).

By Theorem 7 this implies that there is no 9-bit permutationally invariant code which can correct all one-bit errors as well as one type of double error.

One might wonder if there is a 9-bit code which satisfies all the conditions above, except (70c). Such a code would still be of some interest. It would be able to correct all single and double $Z$ errors, and detect all single $X$ and $Y$ errors. However, it would not be able to correct $X_k$ and $Y_k$ errors because it could not reliably distinguish between them. Unfortunately, even this is not possible.

We return to the equations (65) and (67) and observe that there are infinitely many solutions that can be expressed using one complex variable $a_4$, or two real variables $x_4, y_4$, in either case with the constraint $\text{Im} a_4 = x_4 < 0$. Let $x = -x_4 > 0$ and $y = y_4$. Then we have

\[
\begin{align*}
a_0 &= \frac{35}{3} \left(1 - 2i \frac{y_4}{x_4}\right) |a_4|^2 \\
a_2 &= i\nu = \pm i \frac{\sqrt{35}}{3} \frac{|a_4|^2}{\sqrt{|x_4|}} \\
a_4 &= x_4 + iy_4 \\
a_6 &= \pm i \frac{\sqrt{35}}{7} \frac{a_4}{\sqrt{|x_4|}} \\
a_8 &= 1
\end{align*}
\]

Substituting into (70a) and (70b) yields two equations in two unknowns which have no solution. Thus, there is no 9-bit code which satisfies all the desired equations except (70c).

We also considered the possibility of dropping all $Y_k$ conditions to find a code which could correct all errors of the form single $X_k$, single $Z_k$ and double $Z_jZ_k$. However, this is as restrictive as dropping only (70c).

7 Concluding Remarks

Permutationally invariant codes which can correct all one-bit errors require a minimum of seven qubits. We have shown that there are two distinct 7-bit codes of this type. Although one might expect that 9-bit codes could also correct one class of double errors, a detailed analysis shows that this is not possible. Even a 9-bit code which could correct all one-bit errors of the form $X_k$ and $Z_k$ and all two bit errors of the form $Z_jZ_k$ does not exist. If one modifies this to the
requirement that the code be able to correct all one bit errors of the form $X_k$ and double errors of the form $X_jX_k$ and $Z_jZ_k$, this can be done. However, it does not require 9-bits; it can be achieved using the simple 5-bit repetition code \(^{(59)}\) which can correct all double errors of the form $X_jX_k$ and $Y_jY_k$ as well.

Permutationally invariant codes are highly degenerate, since all \(\binom{n}{2}\) exchange errors are equivalent to the identity. As discussed in Section \(5.3\) and illustrated by the 5-bit repetition code, this degeneracy can sometimes lead to enhanced ability to correct two-bit errors. However, there are also limitations on their ability to correct all two-bit errors of a given type as well as all one-bit errors, as shown by our analysis of 9-bit codes. Although the reasons for this remain unclear, it may be that the “degeneracy enhancement” also gives hidden constraints, i.e., that one is implicitly trying to correct more two-bit errors than those from which the conditions were obtained.

We have concentrated here on the construction of permutationally invariant codes. Actual implementation would require a number of additional considerations. For example, one would need a mechanism for initializing the computer in states corresponding to $|c_0\rangle \otimes |c_0\rangle \ldots |c_0\rangle$. One could then obtain any state of the form $|c_{k_1}\rangle \otimes |c_{k_2}\rangle \ldots |c_{k_m}\rangle$ with $k_i \in \{0, 1\}$ by application of $\otimes_{j=1}^{n} X_{kn+j}$ for suitable choices of $\kappa$. One also needs a mechanism for decoding, including a set of measurements which can distinguish between the different error subspaces, as well as a circuit for implementing the error correction process.

Finally, one needs a set of gates for universal computation. As noted in Section \(2.3\) the logical $X$ and $Z$ operations are easily implemented as products of their single-bit counterparts. The logical $Y$ is given by the product $i[\otimes_j X_j][\otimes_j Z_j] = (-1)^{(n-1)/2}[\otimes_j Y_j]$ when $n$ is odd. Moreover, the $X,Y,Z$ gates all lie in the commutant of $S_n$. This implies that all gates needed for universal computation (including a non-trivial two-bit gate) also lie in the commutant since they can be written as linear combinations of these operations, the identity, and their products. (Alternatively, one could observe that all logical actions on code words must be permutationally symmetric and, hence, lie in the commutant of $S_n$.) However, this does not necessarily mean that all the desired gates can be implemented as products of a small set of one and two-bit gates. We leave the question of a practical implementation of a universal set of gates on code words for further investigation.

We have only begun to explore the potential of non-Abelian stabilizer codes for quantum error correction; other examples should be studied. In addition to the issues identified above, there may be others which arise if one wants to combine non-Abelian stabilizers with other approaches to fault tolerant computation.
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A Differences of one-bit errors

In this section we will need some additional notation. Let $\varepsilon_r$ denote the binary n-tuple with components $\varepsilon_j = \delta_{jr}$ so that $v + \varepsilon_r$ has components $v_j + \delta_{jr}$ with addition mod 2. Let $\mathbf{1}$ be the binary n-tuple with all elements equal to 1. We will use $s(v) = \{ j : v_j \neq 0 \}$ to denote the support of of $v = (v_1, \ldots, v_n)$.

It will be convenient to also introduce the vector

$$V_k(r, s) = \sum_{\text{wt}(v) = k, \quad r, s \notin s(v)} (|v + \varepsilon_r| - |v + \varepsilon_s|)$$

which is well-defined for $k = 0, 1, \ldots (n - 2)$, and has the following properties when $r \neq s$.

$$\langle V_k(r, s), W_\ell \rangle = 0 \quad \text{for all } r, s, k, \ell$$

$$\langle V_k(r, s), V_\ell(q, t) \rangle = 0 \quad \text{for } m \neq \ell \quad \forall \ r, s, q, t$$

$$\langle V_k(r, s), V_k(r, s) \rangle = 2 {n-2 \choose k}$$

$$\langle V_k(r, s), V_k(r, t) \rangle = {n-2 \choose k} \quad \text{for } r, s, t \text{ all distinct.}$$

These are all straightforward, except (77) which follows from

$$\langle V_k(r, s), V_k(r, t) \rangle = {n-3 \choose k} + {n-3 \choose k-1}$$

and the easily verified combinatoric identity ${n-2 \choose k} = {n-3 \choose k} + {n-3 \choose k-1}$.

An important consequence of (76) and (77) is that they imply that, for $s \neq t$,

$$\langle V_k(r, s), V_k(r, s) \rangle = 2 \langle V_k(r, s), V_k(r, t) \rangle.$$  

(78)

This result plays an essential role in section 4.4.
Our main results are that, for any code of the general form (9),

\[(Z_r - Z_s)|c_0\rangle = -2 \sum_{k=1}^{n-1} a_k V_{k-1}(r, s) \quad (79a)\]

\[(X_r - X_s)|c_0\rangle = \sum_{k=0}^{n-2} (a_k - a_{k+2}) V_k(r, s) \quad (79b)\]

\[i(Y_r - Y_s)|c_0\rangle = \sum_{k=0}^{n-2} (a_k + a_{k+2}) V_k(r, s) \quad (79c)\]

with similar equations for \(|c_1\rangle\) and \(b_k\). Under the assumption that conditions (I) and (II) hold, we find the following variants useful

\[(Z_r - Z_s)|c_1\rangle = -2 \sum_{m=0}^{(n-1)/2} a_{n-2m-1} V_{2m}(r, s) \quad (80a)\]

\[\left[(X_r - X_s) + i(Y_r - Y_s)\right]|c_0\rangle = 2 \sum_{m=0}^{(n-3)/2} a_{2m} V_{2m}(r, s) \quad (80b)\]

\[\left[(X_r - X_s) - i(Y_r - Y_s)\right]|c_0\rangle = -2 \sum_{m=1}^{(n-1)/2} a_{2m} V_{2m-2}(r, s). \quad (80c)\]

To prove (79a) and (80) in the case of \(Z_r\), it suffices to observe that

\[(Z_r - Z_s)W_k = \begin{cases} 
0 & \text{for } k = 0, n \\
-2V_{k-1}(r, s) & \text{for } 1 \leq k \leq n - 1
\end{cases} \quad (81)\]

which is easily verified.

Equations (79b) and (79c) can be verified by some rather straightforward, but tedious, computations and combinatorics. One approach is to write out the effect of the errors \(X_r\) and \(Y_r\). Since these results are identical except for the signs of some terms, we introduce

\[\omega_{XY} \equiv \begin{cases} 
+1 & \text{for } X \\
-1 & \text{for } iY
\end{cases} \quad (82)\]

and write the equations only for \(X_r\) with the understanding that these results hold for \(Y_r\) with the sign changes indicated by \(\omega_{XY}\)

\[X_r W_0 = |\varepsilon_r\rangle \]

\[X_r W_k = \sum_{\text{wt}(v) = k} |v + \varepsilon_r\rangle + \omega_{XY} \sum_{\text{wt}(u) = k-1} |u\rangle \quad \text{for } 1 \leq k \leq n - 1 \]

\[X_r W_n = \omega_{XY} |T + \varepsilon_r\rangle. \]

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For distinct $r$, $s$, we want to determine the effect of the differences $X_r - X_s$, $Y_r - Y_s$ on the $W_k$, and for this purpose, the following expression, which we write only for $2 \leq k \leq n - 2$, is useful.

\[
X_r W_k = \sum_{\text{wt}(v) = k} |v + \varepsilon_r\rangle + \sum_{\text{wt}(v) = k - 1, r, s \notin s(v)} |v + \varepsilon_s + \varepsilon_r\rangle
\]
\[
+ \omega_{XY} \sum_{\text{wt}(u) = k - 1, r, s \notin s(u)} |u\rangle + \omega_{XY} \sum_{\text{wt}(u) = k - 2, r, s \notin s(u)} |u + \varepsilon_s\rangle
\]  

(83)

Then (83) implies

\[
(X_r - X_s)W_0 = |\varepsilon_r\rangle - |\varepsilon_s\rangle = V_0(r, s)
\]
\[
(X_r - X_s)W_1 = \sum_{j \neq r, s} (|\varepsilon_j + \varepsilon_r\rangle - |\varepsilon_j + \varepsilon_s\rangle) = V_1(r, s)
\]
\[
(X_r - X_s)W_k = V_k(r, s) - \omega_{XY} V_{k - 2}(r, s)
\]
\[
(X_r - X_s)W_{n - 1} = \omega_{XY} \sum_{j \neq r, s} (|\bar{I} + \varepsilon_j + \varepsilon_r\rangle - |\bar{I} + \varepsilon_j + \varepsilon_s\rangle) = -\omega_{XY} V_{n - 3}(r, s)
\]
\[
(X_r - X_s)W_n = \omega_{XY}(|\bar{I} + \varepsilon_r\rangle - |\bar{I} + \varepsilon_s\rangle) = -\omega_{XY} V_{n - 2}(r, s).
\]

To analyze double phase errors, first observe that when $\langle f c_i, \sum_{j} Z c_j \rangle = 0$ (with $f = X, Y, Z$), the analogous inner products involving $(n - 1)$-dimensional representations will be zero if and only if $\langle (f_1 - f_t) c_i, Z_r Z_s c_j \rangle = 0$. By considering the action of the transposition $(1t)$, one can show that this holds whenever $\{1, t\} = \{r, s\}$ or $\{1, t\} \cap \{r, s\} = \emptyset$. Hence, it suffices to consider $r = 1$ and $t \neq s$, in which case one can use (79a) and $Z_1 Z_s = I - Z_1 (Z_1 - Z_s)$ to conclude that

\[
Z_1 Z_s c_0 = |c_0\rangle + 2 \sum_{k=1}^{n-1} a_k Z_1 V_{k - 1}(1, s).
\]  

(84)

We will also need the formula

\[
\langle Z_1 V_k(1, s), V_k(1, t) \rangle = \binom{n - 3}{k - 1} - \binom{n - 3}{k} = \frac{2k + 2 - n}{n - 2} \binom{n - 2}{k}
\]  

(85)

which follows from (7) with $N = n - 3$.

We now let $f = Z$. Using (79a) again and (83) with $k = 2m - 1$, one finds $\langle (Z_1 - Z_t) c_0, Z_1 Z_s c_0 \rangle = 0$ if and only if

\[
\sum_{m=1}^{(n-1)/2} |a_{2m}|^2 \frac{4m - n}{n - 2} \binom{n - 2}{2m} = 0.
\]  

(86)
Then it follows from \( \frac{2k-n}{n-2} \binom{n-2}{k-1} = \frac{(2k-n)(n-k)}{n(n-1)(n-2)} \binom{n}{k} \) that equation (86) is equivalent to equation (56b) minus \( n^2 - n \) times equation (36a). Thus, the “block” conditions for double phase errors do not add additional constraints when conditions (I) and (II) hold.

The cases \( f = X \pm iY \), and \( i = 0, j = 1 \), can be dealt with similarly, but are not needed here. We note only that, unlike the case \( f = Z \), they do generate additional constraints.

### B Decomposition into irreducibles

In view of the repeated use of decompositions into irreducible subspaces, we explicitly write out some of them. Recall that \( C^n = \bigoplus_{k=0}^{n} \mathcal{W}_k \); each \( \mathcal{W}_k \) is the eigenspace of the operator \( S_z \equiv \frac{1}{2} \sum_k Z_k = \frac{n}{2} \mathbb{Z} \) with eigenvalue \( \frac{1}{2}(n-2k) \). Each \( \mathcal{W}_k \) can be further decomposed into a direct sum of subspaces which transform as irreducible representations of \( S_n \). In fact, the action of \( S_n \) on \( \mathcal{W}_k \) or \( \mathcal{W}_{n-k} \) is its action on sets of size \( k \) in \( \{1, \ldots, n\} \). For \( 0 \leq k \leq \lfloor \frac{n}{2} \rfloor \), \( \mathcal{W}_k \) is known to decompose into a sum of irreducible subspaces indexed by the partitions \( [n-j, j] \) for \( j = 0, \ldots, k \), each appearing once. Physicists may recognize that this is equivalent to a decomposition into simultaneous eigenspaces of \( S_z \) and the total spin \( 2 \) operator \( \mathbf{S} = S^2 = S_x^2 + S_y^2 + S_z^2 \) with eigenvalue \( s(s+1) \) for \( s = \frac{n}{2}, \frac{n}{2} - 1, \ldots, \frac{1}{2}[n-2k] \).

To facilitate use in counting arguments, as in Section 5.4, we adopt the convention of labeling an irreducible subspace (in part) by its dimension. Thus, \( \mathcal{U}^d_k \) denotes an irreducible subspace of \( \mathcal{W}_k \) with dimension \( d \).

For \( n = 5 \)

\[
\begin{align*}
\mathcal{W}_0 & = \mathcal{U}_0^1 \\
\mathcal{W}_1 & = \mathcal{U}_1^1 \oplus \mathcal{U}_1^4 \\
\mathcal{W}_2 & = \mathcal{U}_2^1 \oplus \mathcal{U}_2^4 \oplus \mathcal{U}_3^5 \\
\mathcal{W}_3 & = \mathcal{U}_3^1 \oplus \mathcal{U}_3^4 \oplus \mathcal{U}_3^5 \\
\mathcal{W}_4 & = \mathcal{U}_4^1 \oplus \mathcal{U}_4^4 \\
\mathcal{W}_5 & = \mathcal{U}_5^1
\end{align*}
\]

\[\text{In this one paragraph, we use the familiar } S_x, S_y, S_z, \text{ rather than the equivalent } \frac{n}{2} \mathbf{X}, \frac{n}{2} \mathbf{Y}, \frac{\mathbb{Z}}{2}, \text{ to denote the components of spin, and trust that context suffices to distinguish them from the symmetric group denoted } S_n, \text{ which is a very different entity.}\]
For $n = 7$

\[
\begin{align*}
\mathcal{W}_0 &= U^1_0 \\
\mathcal{W}_1 &= U^1_1 \oplus U^8_1 \\
\mathcal{W}_2 &= U^1_2 \oplus U^6_2 \oplus U^1_{14} \\
\mathcal{W}_3 &= U^1_3 \oplus U^6_3 \oplus U^8_3 \oplus U^1_{14} \oplus U^1_{14} \\
\mathcal{W}_4 &= U^1_4 \oplus U^6_4 \oplus U^8_4 \oplus U^1_{14} \oplus U^1_{14} \\
\mathcal{W}_5 &= U^1_5 \oplus U^6_5 \oplus U^8_5 \\
\mathcal{W}_6 &= U^1_6 \oplus U^6_6 \oplus U^8_6 \oplus U^8_{14} \\
\mathcal{W}_7 &= U^1_7 \\
\end{align*}
\]

where $U^1_{14}$ denotes the irreducible representation associated with the partition $[5, 2]$ and $U^1_{14}$ a second, distinct, 14-dimensional irreducible representation associated with $[4, 3]$. In terms of spin, $U^1_{14}$ has $s = \frac{3}{2}$ and $U^1_{14}$ has $s = \frac{1}{2}$.

For $n = 9$

\[
\begin{align*}
\mathcal{W}_0 &= U^1_0 \\
\mathcal{W}_1 &= U^1_1 \oplus U^8_1 \\
\mathcal{W}_2 &= U^1_2 \oplus U^8_2 \oplus U^8_2 \\
\mathcal{W}_3 &= U^1_3 \oplus U^8_3 \oplus U^8_3 \oplus U^8_4 \oplus U^1_{12} \\
\mathcal{W}_4 &= U^1_4 \oplus U^8_4 \oplus U^8_4 \oplus U^8_{12} \oplus U^1_{12} \\
\mathcal{W}_5 &= U^1_5 \oplus U^8_5 \oplus U^8_5 \oplus U^8_5 \oplus U^8_{12} \\
\mathcal{W}_6 &= U^1_6 \oplus U^8_6 \oplus U^8_6 \oplus U^8_6 \oplus U^8_{12} \\
\mathcal{W}_7 &= U^1_7 \oplus U^8_7 \oplus U^8_7 \\
\mathcal{W}_8 &= U^1_8 \oplus U^8_8 \\
\mathcal{W}_9 &= U^1_9
\end{align*}
\]

C Complex coefficients

If the $a_k$ are not real, then one must modify the analysis in Section 3 accordingly, and require both real and imaginary parts of the resulting equations to be zero. We again use the classification of error conditions described at the end of Section 3. We omit the details and summarize the results.
a) Setting the real parts of (30)-(33) to zero yields the pair of equations

\[
0 = \frac{n+1}{2} \left( \frac{n}{n+1} \right) a_{n+1}^2 + 2 \sum_{k=1}^{(n-1)/2} k \binom{n}{k} \text{Re}(\overline{a}_k a_{n-k+1}) \tag{87}
\]

\[
0 = \frac{n+1}{2} \left( \frac{n}{n-1} \right) a_n^2 + 2 \sum_{k=0}^{(n-3)/2} (n-k) \binom{n}{k} \text{Re}(\overline{a}_k a_{n-k-1}). \tag{88}
\]

The imaginary parts of both (30) and (32) are always zero and do not place any additional restrictions on \( a_k \). Setting the imaginary parts of (31) and (33) to zero yields the conditions

\[
0 = \frac{1}{2} \sum_{k=1}^{(n-1)/2} \text{Im}\{\overline{a}_k a_{n-k+1}\} k(n-2k+1) \binom{n}{k}. \tag{89}
\]

\[
0 = \sum_{k=0}^{(n-1)/2} \text{Im}\{\overline{a}_k a_{n-k-1}\} (n-k)(n-2k-1) \binom{n}{k}. \tag{90}
\]

b) The condition (38) from the off diagonal terms in \( D \) becomes

\[
0 = \sum_{k=0}^{n} |a_k|^2 (n-2k) \binom{n}{k}. \tag{91}
\]

(This is sufficient to ensure that \( d_{IZ} = 0 \) and \( D_{IZ} = 0 \), as well as that the real part of (43) is zero.) To ensure that the imaginary part of \( \langle X c_i, i Y c_i \rangle \) is zero we must also require

\[
0 = \sum_{k=1}^{n-1} \text{Im}\{a_{k+1} a_{k-1}\} (n-2) \binom{n}{k-1}. \tag{92}
\]

(This also ensures that the imaginary part of (43) is zero so that \( d_{XY} = 0 \) and \( D_{XY} = 0 \).)

As before, we analyze the “block” conditions only under the assumption that conditions (I) and (II) hold. As for real coefficients, these conditions do not yield new requirements.

c) Setting the imaginary parts of (40) and (41) to zero yields conditions equivalent to (39) and (40).

d) The expression in (43) gives two conditions. The first is equivalent to (91) and the second to (92) with \( k = 2m+1 \).
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