Symbolic dynamics for delayed maps and networks

Fatihcan M. Atay‡, Sarika Jalan§ and Jürgen Jost¶
Max Planck Institute for Mathematics in the Sciences
Inselstr. 22, 04103 Leipzig, Germany
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We introduce the concept of avoiding sets for investigating symbolic dynamics of discrete dynamical systems with time delays. Such sets arise from specific non-generating partitions of the phase space that restrict the occurrence of particular symbol sequences related to the characteristics of the dynamics. We use the theory in two applications to coupled map lattices, namely, determining unknown values of the transmission delays and detecting synchrony using measurements from a single node.

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Symbolic dynamics is a fundamental tool for describing the complicated time evolution of chaotic dynamical systems, the Smale horseshoe being a most famous prototype [1]. In symbolic dynamics, instead of representing a trajectory by a continuum of numbers one watches the alternation of finite symbols. In the process some information is “lost” but certain important invariants and robust properties of the dynamics may be kept [2]. Most studies of symbolic dynamics in the literature are based on the so-called generating partition [3] of the phase space, for which topological entropy achieves its maximum [4]. Symbolic dynamics based on generating partitions plays a crucial role in understanding many different properties of dynamical systems. However, finding generating partitions for general systems is a difficult problem and is an issue of ongoing research [5]. Some consequences of using misplaced partitions have been investigated in [6]. Nevertheless, certain non-generating partitions have recently been shown to have particular uses. Specifically, appropriately chosen partitions which restrict the appearance of some symbolic subsequences have been used for distinguishing random from deterministic time series [7], and for investigating the collective behavior of coupled systems [8]. In this paper, we use symbolic dynamics for the study of systems with time delays. The presence of time delays is a natural and important consequence of the finite speed of information transmission within a system or between coupled systems. However, the effects of delays on the symbolic dynamics have not received much attention so far. We develop an idea of restricted symbol subsequences for systems with delays, and present two applications of the method, namely detecting synchronization in a network using measurements from a single node and determining the unknown value of the delay in the system.

We introduce the concepts starting with the simplest system and moving on to more complicated ones. Let \( f : S \rightarrow S \) be a map on a subset \( S \) of \( \mathbb{R}^n \), and consider the dynamical system defined by the iteration rule

\[
x(t + 1) = f(x(t)),
\]

where the iteration step \( t \in \mathbb{Z} \) plays the role of discrete time. Let \( S_i : i = 1, \ldots, m \) be a partition of \( S \), i.e., a collection of mutually disjoint subsets satisfying \( \bigcup_{i=1}^{m} S_i = S \). We assume that the \( S_i \) are nonempty and \( m \geq 2 \) to prevent trivial cases. The symbolic dynamics corresponding to \( S \) is the sequence of symbols \( \{s_0, s_1, s_2, \ldots\} \) where \( s_i = i \) if \( x(t) \in S_i \) (shift of finite type [2]). We say the set \( S_i \) avoids \( S_j \) under \( f \) if

\[
f(S_i) \cap S_j = \emptyset.
\]

Clearly, if \( S_i \) avoids \( S_j \), so does any of its subsets. We also talk about a self-avoiding set if \( S \) holds with \( i = j \). The significance of avoiding sets is that if \( S_i \) avoids \( S_j \), then the symbolic dynamics for \( i \) cannot contain the symbol sequence \( ij \). The notion is extended in a straightforward way to the \( k \)th iterate of \( f \). Thus, if \( f^{k}(S_i) \cap S_j = \emptyset \), then the symbol sequence for the dynamics cannot contain the symbols \( i \) and \( j \) at two positions which are \( k - 1 \) symbols apart. This constrains the symbol sequences that can be generated by a given map, and provides a robust method to distinguish between different systems by inspecting their symbolic dynamics. As examples of avoiding sets, we mention that for the common unimodal maps of the interval \([0,1]\), such as the tent or logistic maps, the set \((x^*,1]\) and its subsets are self-avoiding, where \( x^* \) denotes the positive fixed point of \( f \).

More generally, partitions that contain avoiding sets can always be found. We give a constructive proof. Suppose one starts with some partition of \( m \) sets for which \( S \) does not hold for any \( i, j \). Now fix some pair \((i,j)\), \( i \neq j \). If \( f(S_i) \subset S_j \), then \( S_i \) can be arbitrarily partitioned into two parts which avoid each other. If, on the other hand, \( f(S_i) \not\subset S_j \), then let \( S_{m+1} = f^{-1}(S_j) \cap S_i \), and re-define \( S_i \) as \( S'_i = S_i \setminus S_{m+1} \) to obtain a new partition with \( m+1 \) sets, where \( S'_i \) avoids \( S_j \). The same argument can be used to construct self-avoiding sets: Assume \( f(S_i) \not\subset S_i \) (otherwise further partition \( S_i \) to obtain a set which is not invariant under \( f \) [13]), and define \( S_{m+1} = f^{-1}(S_i) \cap S_i \) and \( S'_i = S_i \setminus S_{m+1} \), so that \( S'_i \) is self-avoiding. Hence, it is possible to modify a given partition so that the sequence \( ij \) never occurs in the symbolic dynamics.
An extension of the system (1) is given by the delayed dynamics

\[ x(t+1) = (1-\varepsilon)f(x(t)) + \varepsilon f(x(t-\tau)), \]

where \( \tau \in \mathbb{Z}^+ \) is the time delay and \( \varepsilon \in [0, 1] \) is a parameter measuring the relative weight of the past in determining the next state. The significance of the delayed dynamics is that it governs the behavior of the synchronous solutions of coupled map networks with transmission delays, which are studied below. The domain \( S \) of the map \( f \) is required to be a convex set in order for the iterations (3) to be meaningful. The symbolic dynamics is defined as before, and avoiding sets give rise to forbidden sequences as follows: Let \( \bar{S} \) denote the complement of \( S \) and the observation that if \( \bar{S} \) is not possible for the delayed system (3), then the symbolic sequence

\[ i \ast \ldots \ast i j \]

is not possible for the delayed system (3). (Here \( \ast \ldots \ast \) denotes \( \tau - 1 \) arbitrary symbols.) This is a consequence of (3) and the observation that if \( f(x(t)) \) and \( f(x(t-\tau)) \) are both in the convex set \( \bar{S} \), then is their convex combination. Similarly, if \( S \) is self-avoiding under \( f \) and has convex complement, then any sequence of \( \tau + 1 \) symbols that begin and end with \( i \) cannot be followed by another \( i \). The important observation is that, although the dynamics of (3) varies greatly with \( \varepsilon \), the forbidden sequences are independent of the value of \( \varepsilon \).

The additional condition of convexity of the sets in case of the delayed dynamics does not present an extra restriction in many practical situations. In fact, one often measures a single component, say the first one, of the \( n \)-dimensional vector \( x = (x_1, \ldots, x_n) \). In this case, a simple partition of \( S \) given by the disjoint union \( S = S_1 \cup S_2 \), where

\[ S_1 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 < x^* \} \]

and \( x^* \) is a scalar threshold value, which can be chosen to make both \( S_1 \) and \( S_2 \) nonempty. It is easy to see that both \( S_1 \) and \( S_2 \) defined in this way are convex whenever \( S \) is convex. Such partitions are almost surely non-generating, so the corresponding symbol sequences do not capture all features of the dynamics. However, it will be seen that they still contain important information that can be utilized to study some important aspects about the delayed dynamics.

A more complicated system is the coupled map lattice model. We consider a more general form which allows arbitrary coupling topologies, including directed and/or weighted connections, as well as the information transmission delays along the connections:

\[ x_i(t+1) = f(x_i(t)) + \frac{1}{k_i} \sum_{j=1}^{N} a_{ij} [f(x_j(t-\tau)) - f(x_i(t))]. \]

Here \( x_i(t) \) is the state of the \( i \)-th unit at time \( t \), \( i = 1, \ldots, N \), \( a_{ij} \) is the nonnegative weight on the link from \( j \) to \( i \) (zero if there is no link), \( \varepsilon \in [0, 1] \) is the coupling strength, and \( k_i = \sum_j a_{ij} \) is the weighted sum of the incoming links to unit \( i \). (It is understood that the summation term is set to zero in (5) for any unit for which \( k_i \) is zero.) The delay \( \tau \) is the time it takes for the information from a unit to reach its neighbors and be processed. The system is said to synchronize if \( |x_i(t) - x_j(t)| \rightarrow 0 \) as \( t \rightarrow \infty \) for all \( i, j \) and all initial conditions from an open set. In this case, the state of every node asymptotically approaches the same synchronous solution \( x(t) \), whose dynamics is governed by (1) and (3), respectively, depending on whether the delay \( \tau \) is zero or nonzero.

It has been shown that the network (5) can synchronize even with positive delays, where the states are unaware of the present states of their neighbors but still can act in unison (10). The important distinction from the undelayed case, however, is that the synchronous dynamics \( x(t) \) is no longer identical to the isolated dynamics of the units, but is governed by the delayed equation (3). Consequently, the overall system (5) can exhibit a much wider range of behavior than its constituent units through the coordination of their actions (11).

Normally the symbol sequences observed from a node of a network can vary widely between the nodes. However, at the synchronized state \( x_i(t) = x(t) \) for all \( i \), so that the symbolic sequences observed from any node will be subject to the same constraints as that generated by (3) (16). This gives a method of detecting synchronization of the network by choosing an arbitrary node and calculating the transition probabilities of symbol subsequences: From the relative frequencies of occurrence of subsequences of the form (4) in the measured time series, one estimates the transition probabilities \( P(j|i* \ast \ldots \ast i)_{\tau+1} \), that is, the conditional probability that a sequence of length \( \tau + 1 \) starting and ending with \( i \) is followed by \( j \). Letting \( \xi^2 \) denote the average squared difference between the observed transition probabilities of the network and those of (3), synchronization is signaled when \( \xi^2 = 0 \).

Fig. 1 illustrates the synchronization detection method for several different networks and delay values. Here \( f \) is the chaotic tent map given by \( f(x) = 1 - 2|x - \frac{1}{2}| \) and the partition used is

\[ S_1 = [0, x^*], \quad S_2 = (x^*, 1], \]

where \( x^* = 2/3 \) is the fixed point of the map. Then \( S_2 \) is a self-avoiding set under \( f \). We evolve (5) starting from random initial conditions and estimate the transition probabilities using time series of length 1000 from
The network and determination of the value of the delay $\tau$ of coupling strengths using only measurements from an network can be accurately detected over the whole range both synchronized and unsynchronized behavior of the system (3). Hence, regardless of network topology and size, the presence of symbolic subsequences of the form (4) of observed symbolic dynamics. For this purpose, we check by standard time-series methods which use embedding time series is much shorter time than would be required to reconstruct the phase space [12, 13] for large networks. In our case, however, the length of the series is independent of the network size. Synchronization occurs when the variance of variables over the network given by $\sigma^2 = \left\langle \frac{1}{N} \sum_i (x_i(t) - \bar{x}(t))^2 \right\rangle_t$ drops to zero, where $\bar{x}(t) = \frac{1}{N} \sum_i x_i(t)$ denotes an average over the nodes of the network and $\langle \ldots \rangle_t$ denotes an average over time. As seen from Fig. 1 the region for synchronization exactly coincides with the region where the transition probabilities for the network are identical to those of the equation (3). Hence, regardless of network topology and size, both synchronized and unsynchronized behavior of the network can be accurately detected over the whole range of coupling strengths using only measurements from an arbitrarily selected node.

As a second application, we consider the reverse problem of determining the value of the delay $\tau$ in (3) from observed symbolic dynamics. For this purpose, we check the presence of symbolic subsequences of the form (4) of various lengths, knowing that such a sequence of length $\tau + 2$ would be forbidden. We plot the occurrence frequencies of (4) against $\tau$, and the point where the frequency is zero corresponds to the true value of $\tau$. (If the observations are contaminated by some small noise, then the true value of $\tau$ is found at the minimum of the curve.)

The same method works to find the value of the transmission delay in the network (5) from a knowledge of its synchrony. One can think of a situation where the value of $\tau$ is unknown, but the network is known to be synchronized or can be made to synchronize by the adjustment of available parameters. Then, using the measurements from a node and checking the presence of the forbidden sequences (4) as before, the value of $\tau$ can be obtained. An illustration is given in Fig. 2 for the tent map and the two-set partition (6), which plots the transition probability $P(2|2 \ast \cdots \ast 2 \tau+1)$ versus $\tau$, that is, the conditional probability that a sequence of length $\tau + 1$ starting and ending with 2 is followed by another 2. By the arguments following (3), such a sequence cannot occur for the dynamics (3) since $S_2$ is self-avoiding under $f$. Hence, the true value of $\tau$ is found at the point where the transition probability of the forbidden sequence drops to zero.

The foregoing ideas can be extended to systems with multiple delays, e.g., to the coupled map network

$$x_i(t+1) = f(x_i(t)) + \frac{\varepsilon}{k_i} \sum_{j=1}^{N} a_{ij} [f(x_j(t-\tau_{ij})) - f(x_i(t))],$$

where $\tau_{ij}$ denotes the transmission delay from $j$ to $i$. While the network (5) with a constant delay always admits a synchronous solution, one needs additional conditions in the case (7) of multiple delays. It can be shown that (7) admits non-constant synchronized solutions provided that each unit has the same fraction of weighted
symbolic sequence that if \( \varepsilon \) belongs to any convex set containing the latter. It follows

\[ \varepsilon F \]  

Further restrictions are obtained if some \( \lambda i \) 

time series measured from an arbitrarily selected node. The

FIG. 2: Detection of delay in a synchronized network using

coupled networks of 20 nodes with \( \varepsilon = 0.75 \), where the true

probability drops to zero. The figure is plotted for globally

\( \sum \tau \) 

This result is independent of the values of the

\( \varepsilon \) 

the coefficients \( \varepsilon \) 

the value of the delay is (a) 5 and (b) 6. The dotted lines show

results in the presence of 5% additive noise.

incoming connections having the same delay. In this case, it is possible to have a synchronous solution of the form

\[ x_i(t) = x(t) \] 

and end with \( \varepsilon \) 

where \( \tau_{\text{max}} \) is the maximum delay in the system, and

the coefficients \( \varepsilon \) are nonnegative numbers satisfying

\[ \sum_{m=0}^{\tau_{\text{max}}} \varepsilon \]  

It belongs to any convex set containing the latter. It follows

that if \( \hat{S}_j \) is convex and \( S_i \) avoids \( S_j \) under \( f \), then the symbolic sequence

\[ i i \ldots i j \]  

is not possible for \( \varepsilon \); that is, a sequence of consecutive

\( \varepsilon \)'s of length \( \tau_{\text{max}} + 1 \) cannot be followed by a \( j \). Note

that this result is independent of the values of the \( \varepsilon \)\( \varepsilon \)\( \varepsilon \). Further restrictions are obtained if some \( \varepsilon \) is zero, in which case the sequence \( \varepsilon \) will be forbidden even when the symbol at position \( \tau_{\text{max}} + 1 \) is replaced by an arbitrary symbol in the alphabet. As before, synchronization of the network can be detected by comparing the transition probabilities of symbolic sequences from an arbitrary node to those of \( \varepsilon \). Similarly, the unknown values of the delays in \( \varepsilon \) can be determined by systematically checking for the presence of sequences of various lengths that start with the symbol \( i \) and end with \( j \). By the same procedure, the unknown delays in the network

\[ (a) \] 

can be calculated from a knowledge of its synchrony.

In conclusion, we have used symbolic dynamics to study discrete-time systems and networks involving time delays. We have derived forbidden symbol sequences for the delayed system from the properties of the undelayed map. Although the partitions used are non-generating, the forbidden sequences are related to certain characteristics of the dynamics, and in particular to delays. Consequently, the value of the delay in the system can be determined by the presence and absence of such sequences. Conversely, a knowledge of the delays enables one to detect network synchronization using measurements from a single node. Computationally, the method has the advantage of being based on a phase-space partition that is much easier to obtain than a generating partition. Furthermore, based on measurements from as few as one single node of the network, rather short sequences of measurements can be utilized. The corresponding computations are fast and independent of the network size, and do not require knowledge of the connection structure. As such, the method can be a viable alternative to existing techniques for detecting synchronization based on phase-space reconstruction [14].

Discrete-time systems are a natural way to model dynamics from temporal measurements because of the nature of collected data, and the inclusion of time delays additionally enhances the models by enlarging the range of dynamics using only a few scalar parameters [10]. For models of the form \( \varepsilon \) or \( \varepsilon \), the knowledge of forbidden sequences provides an easy method to determine the admissible values of delays from measurement data. Symbolic dynamics therefore holds the promise of being a useful tool in the study of time series as well as time-delay systems and networks.

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[15] This is possible except for the trivial case when f is the identity map.
[16] The choice of the node is arbitrary so long as network is capable of chaotic synchronization, which is assumed to be the case here.