THE MINIMAL MEASUREMENT NUMBER FOR LOW-RANK MATRICES 
RECOVERY

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ABSTRACT. The paper presents several results that address a fundamental question in low-rank matrices recovery: how many measurements are needed to recover low rank matrices? We begin by investigating the complex matrices case and show that 4nr − 4r^2 generic measurements are both necessary and sufficient for the recovery of rank-r matrices in \( \mathbb{C}^{n \times n} \) by algebraic tools developed in [10]. Thus, we confirm a conjecture which is raised by Eldar, Needell and Plan for the complex case. We next consider the real case and prove that the bound 4nr − 4r^2 is tight provided \( n = 2^k + r, k \in \mathbb{Z}^+ \). Motivated by Vinzant’s work, we construct 11 matrices in \( \mathbb{R}^{4 \times 4} \) by computer random search and prove they define injective measurements on rank-1 matrices in \( \mathbb{R}^{4 \times 4} \). This disproves the conjecture raised by Eldar, Needell and Plan for the real case. Finally, we use the results in this paper to investigate the phase retrieval by projection and show fewer than 2n−1 orthogonal projections are possible for the recovery of \( x \in \mathbb{R}^n \) from the norm of them, which gives a negative answer for a question raised in [1].

1. INTRODUCTION

1.1. Problem setup. The problem of low-rank matrix recovery attracted many attention recently since it is widely used in image processing, system identification and control, Euclidean embedding, and recommender systems. Suppose that the matrix \( Q \in \mathbb{H}^{n \times n} \) with rank(\( Q \)) \( \leq r \), where \( \mathbb{H} \) is either \( \mathbb{R} \) or \( \mathbb{C} \). The information we gather about \( Q \) is

\[
\mathbf{b}_j := \langle A_j, Q \rangle := \text{trace}(A_j^* Q), \quad j = 1, \ldots, m
\]

where \( A_j \in \mathbb{H}^{n \times n}, j = 1, \ldots, m \). The aim of the low-rank matrix recovery is to recover \( Q \) from \( \mathbf{b} = [\mathbf{b}_1, \ldots, \mathbf{b}_m] \in \mathbb{H}^m \).

For a given \( \mathcal{A} := \{A_1, \ldots, A_m\} \subset \mathbb{H}^{n \times n} \), we define the map \( \mathbf{M}_{\mathcal{A}} : \mathbb{H}^{n \times n} \rightarrow \mathbb{H}^m \) by

\[
\mathbf{M}_{\mathcal{A}}(Q) = [\mathbf{b}_1, \ldots, \mathbf{b}_m].
\]

Set

\[
\mathcal{L}_r^\mathbb{H} := \{X \in \mathbb{H}^{n \times n} : \text{rank}(X) \leq r \}.
\]

We say the matrices set \( \mathcal{A} := \{A_1, \ldots, A_m\} \) has the low-rank matrix recovery property for \( \mathcal{L}_r^\mathbb{H} \) if the map \( \mathbf{M}_{\mathcal{A}} \) is injective on \( \mathcal{L}_r^\mathbb{H} \). Naturally, we are interested in the minimal \( m \) for which the map \( \mathbf{M}_{\mathcal{A}} \) is injective on \( \mathcal{L}_r^\mathbb{H} \).

There are many convex programs for the recovery of the low-rank matrix \( Q \) from \( \mathbf{M}_{\mathcal{A}}(Q) \). A well-known one is nuclear-norm minimization which requires \( m = Cnr \) random linear measurements for the recovery of rank-\( r \) matrices in \( \mathbb{H}^{n \times n} \) [6][8][7]. Despite many literatures on low-rank matrices recovery, there remains a fundamental lack of understanding about the theoretical limit of the number of the cardinality of \( \mathcal{A} \) which has the low-rank matrix recovery property for \( \mathcal{L}_r^\mathbb{H} \). This paper focusses on the problem of the minimal measurements number for the recovery of low-rank matrix. We state the problem as follows:

**Problem 1** What is the minimal measurement number \( m \) for which there exists \( \mathcal{A} := \{A_1, \ldots, A_m\} \subset \mathbb{H}^{n \times n} \) so that \( \mathbf{M}_{\mathcal{A}} \) is injective on \( \mathcal{L}_r^\mathbb{H} \)?

The aim of this paper is to addresses Problem 1 under many different settings.

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1.2. Related work. A related problem to low-rank matrices recovery is phase retrieval, which is to recover the rank-one matrix \(xx^* \in \mathbb{H}^{n \times n}\) from the measurements \(|\langle \phi_j, x \rangle|^2 = \langle \phi_j \phi_j^*, xx^* \rangle, j = 1, \ldots, m\), where \(\phi_j \in \mathbb{H}^n\) and \(x \in \mathbb{H}^n\). In the context of phase retrieval, one is interested in the minimal measurement number \(m\) for which the map \(M_{\Phi}\) is injective on \(S_1^H\) where \(\Phi := \{\phi_1, \ldots, \phi_m \phi_m^*\}\) and \(S_1^H := \{X \in \mathbb{H}^{n \times n} : \text{rank}(X) \leq r, X^* = X\}, r \in \mathbb{Z}\). It is known that in the real case \(\mathbb{H} = \mathbb{R}\) one needs at least \(m \geq 2n - 1\) vectors so that \(M_{\Phi}\) is injective on \(S_1^R\). For the complex case \(\mathbb{H} = \mathbb{C}\), the same problem remain open. Balan, Casazza and Edidin in [2] show that \(M_A\) is injective on \(S_1^C\) if \(m \geq 4n - 2\) and \(\phi_1, \ldots, \phi_m\) are generic vectors in \(\mathbb{C}^n\). In [3], Bandeira, Cahill, Mixon and Nelson conjectured the following (a) if \(m < 4n - 4\) then \(M_{\Phi}\) is not injective on \(S_1^C\); (b) if \(m \geq 4n - 4\) then \(M_{\Phi}\) is injective on \(S_1^C\) for generic vectors \(\phi_j, j = 1, \ldots, m\). The part (b) of the conjecture is proved by Conca, Edidin, Hering and Vinzant in [10] by employing algebraic tools. They also confirm part (a) for the case where \(n\) is in the form of \(2^k + 1, k \in \mathbb{Z}\). Recently, in [19], a counterexample is presented disproving part (a) of this conjecture. In fact, [19] gives \(11 = 4n - 5 \leq 4n - 4\) vectors \(\phi_1, \ldots, \phi_{11} \in \mathbb{C}^4\) and prove that \(M_{\Phi}\) is injective on \(S_1^C\) by algebraic computation where \(\Phi = \{\phi_1 \phi_1^*, \ldots, \phi_{11} \phi_{11}^*\}\).

In context of low-rank matrix recovery, it is Eldar, Needell and Plan [11] that show that Gaussian matrices \(A_1, \ldots, A_m\) has low-rank matrix recovery property for \(L_C^C\) with probability 1 (see also [14, 15, 18]) provided \(r \leq n/2\). Naturally, one may be interested in whether the number \(4nr - 4r^2\) is tight. In [11], the authors made the following conjecture:

**Conjecture 1.1.** [11] If \(m < 4nr - 4r^2\) then \(M_A\) is not injective on \(L_r^C\).

1.3. Our contribution. The aim of this paper is to address Problem 1 by employing algebraic tools which are developed in [10][19]. In Section 2, we consider the case \(\mathbb{H} = \mathbb{C}\) and prove that \(M_A\) is injective on \(L_r^C\) if \(A\) contains \(m \geq 4nr - 4r^2\) generic matrices \([A_1, \ldots, A_m] \in \mathbb{C}^{mn^2}\). Compared to the results of [11], we do not require the \(m\) matrices are Gaussian matrices. Hence, our result does not suffer from probabilistic qualifiers on the injective (E.g., injective “with probability 1”). We also show that the bound \(4nr - 4r^2\) is tight which means \(M_A\) is not injective on \(L_r^C\) provided \(m < 4nr - 4r^2\) and hence confirm Conjecture 1.1 for the complex case, i.e., \(\mathbb{H} = \mathbb{C}\). We turn to the real case in Section 3 and prove the bound \(4nr - 4r^2\) is tight provided \(n\) is in the form of \(2^k + r\). Inspired by the work of [19], we use computer random search to construct a counterexample for the case \(n = 4, r = 1\). In fact, we present \(11 = 4n - 5\) matrices \(A_1, \ldots, A_{11} \in \mathbb{R}^{4 \times 4}\) and prove \(M_A\) is injective on \(L_r^R\) using Vinzant’s test with disproving Conjecture 1.1 for the real case. We next consider the recovery of the symmetric matrix and investigate the minimal measurement number \(m\) for which there exists \(A = \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n \times n}\) so that \(M_A\) is injective on \(S_1^R\). In Section 4, we apply the results to study the phase retrieval by projection. Set \(W_j := \text{span}\{u_{j, 1}, \ldots, u_{j, d_j}\} \subset \mathbb{R}^n\) and \(P_j : \mathbb{R}^n \to W_j\) is an orthogonal projection. Following [1][5], phase retrieval by projection is to recover \(x \in \mathbb{R}^n\) up to a unimodular constant from \(\|P_j x\|_2\). We say that \([W_j]_{j=1}^m\) yields phase retrieval if for all \(x, y \in \mathbb{R}^n\) satisfying \(\|P_j x\| = \|P_j y\|\) for all \(j = 1, \ldots, m\) then \(x = \pm y\). In [5], Cahill, Casazza, Peterson and Woodland proved that \(2n - 1\) projections are enough for phase retrieval. Particularly, they showed that phase retrieval can be done in \(\mathbb{R}^n\) with \(2n - 1\) subspaces each of any dimension less than \(n - 1\). A question is also raised in [1] which states can phase retrieval be done in \(\mathbb{R}^n\) with fewer than \(2n - 1\) projections? Using the results in this paper, we present a positive answer for the question provided \(n\) is in the form of \(2^k + 1\). We also give a negative answer for the case \(n = 4\) by constructing \(6 = 2n - 2\) subspaces \(W_1, \ldots, W_6 \subset \mathbb{R}^4\) and prove they are phase retrieval by computational algebra.

2. The recovery of complex low rank matrices

We first recall the following lemma

**Lemma 2.1.** [11] Suppose that \(r \leq n/2\). The map \(M_A\) is not injective on \(L_r^H\) if and only if there is a nonzero \(Q \in L_r^H\) for which

\[
M_A(Q) = 0.
\]

According to Lemma 2.1, the set \(L_r^C\) plays an important role in the investigation of the \(M_A\). Recall that \(\text{rank}(Q) \leq r\) is equivalent to the vanishing of all \((r + 1) \times (r + 1)\) minors of \(Q\). Hence, \(L_r^C\) is an affine variety in \(\mathbb{C}^{n^2}\) with the dimension \(2nr - r^2\) [12] Prop. 12.2 and the degree \(d_{n, r} := \prod_{i=0}^{n-r-1} \frac{(a+i)!d!}{(r+i)!r!(n-r+i)!}\) [12] Ex.
Note these \((r + 1) \times (r + 1)\) minors are homogeneous polynomials in the entries of \(Q\). Thus the projectivization of \(L^C_r\) is a projective variety in \(\mathbb{P}(\mathbb{C}^{n^2})\) and it is called as determinant variety. Throughout the paper, by \(m\) generic matrices in \(\mathbb{H}^{n \times n}\) we mean \([A_1, \ldots, A_m]\) corresponds to a point in a non-empty Zariski open subset of \(\mathbb{H}^{mn^2}\) which is also open and dense in the Euclidean topology (see \cite{10} Section 2.2).

We next state the main result of this section:

**Theorem 2.1.** Suppose that \(r \leq n/2\). Consider \(m\) matrices \(A = \{A_1, \ldots, A_m\} \subset \mathbb{C}^{n \times n}\) and the mapping \(M_A : \mathbb{C}^{n \times n} \to \mathbb{C}^m\). The following holds

(a) If \(m \geq 4nr - 4r^2\) then \(M_A\) is injective on \(L^C_r\) for generic matrices \(A_1, \ldots, A_m\).

(b) If \(m < 4nr - 4r^2\), then \(M_A\) is not injective on \(L^C_r\).

**Proof.** We use \(G_{m,n}\) to denote the matrices set \(([A_1, \ldots, A_m], [Q]) \in \mathbb{P}(\mathbb{C}^{n \times n} \times \cdots \times \mathbb{C}^{n \times n}) \times \mathbb{P}(\mathbb{C}^{n \times n})\) which satisfies the following property:

\[
\text{rank}(Q) \leq 2r \quad \text{and} \quad \langle A_j, Q \rangle = 0, \quad \text{for all } 1 \leq j \leq m.
\]

Note that \(G_{m,n}\) is defined by the vanish of homogeneous polynomials in the entries of \(A_j\) and \(Q\). Thus \(G_{m,n}\) is a projective variety of \(\mathbb{P}(\mathbb{C}^{n \times n} \times \cdots \times \mathbb{C}^{n \times n}) \times \mathbb{P}(\mathbb{C}^{n \times n})\). We next consider the dimension of the projective complex variety \(G_{m,n}\). We let \(\pi_1\) and \(\pi_2\) be projections onto the first and the second coordinates, respectively, i.e.,

\[
\pi_1([A_1, \ldots, A_m], [Q]) = [A_1, \ldots, A_m], \quad \pi_2([A_1, \ldots, A_m], [Q]) = Q.
\]

We claim that \(\pi_2(G_{m,n}) = \mathcal{P}L^C_{2r}\) where

\[
\mathcal{P}L^C_{2r} := \{Q \in \mathbb{P}(\mathbb{C}^{n \times n}) : \text{rank}(Q) \leq 2r\}.
\]

Indeed, for any fixed \(Q_0 \in \mathcal{P}L^C_{2r}\), there exists a matrix \(A_0 \in \mathbb{C}^{n \times n}\) satisfying \(\langle A_0, Q_0 \rangle = 0\) since \(\langle A_0, Q_0 \rangle\) is a linear equation about the entries of \(A_0\). This implies that \(([A_0, \ldots, A_0], [Q_0]) \in G_{m,n}\) and \(\pi_2([A_0, \ldots, A_0], [Q_0]) = Q_0\). Thus we have \(\pi_2(G_{m,n}) = \mathcal{P}L^C_{2r}\). Note that \(\mathcal{P}L^C_{2r} \subset \mathbb{P}(\mathbb{C}^{n \times n})\) is a projective variety with \(\dim(\pi_2(G_{m,n})) = 4nr - 4r^2 - 1\).

We next consider the dimension of the preimage \(\pi_2^{-1}(Q_0) \subset \mathbb{P}(\mathbb{C}^{n \times n} \times \cdots \times \mathbb{C}^{n \times n})\) for a fixed \(Q_0 \in \mathbb{P}(\mathbb{C}^{n \times n})\). A simple observation is that

\[
\langle A_j, Q_0 \rangle = 0
\]

defines a nonzero linear equation on the entries of \(A_j\). For each \(A_j\) the linear equation \(\langle A_j, Q_0 \rangle = 0\) defines a hyperplane of dimension \(n^2 - 1\) in \(\mathbb{C}^{n^2} \cong \mathbb{C}^{n \times n}\). Hence, after projectivization, the preimage \(\pi_2^{-1}(Q_0)\) has dimension \(m(n^2 - 1) - 1 = mn^2 - m - 1\). Then, according to \cite{12} Cor.11.13

\[
\dim(G_{m,n}) = \dim(\pi_2(G_{m,n})) + \dim(\pi_2^{-1}(Q_0)) = (4nr - 4r^2 - 1) + (mn^2 - m - 1) = mn^2 + 4nr - 4r^2 - m - 1.
\]

If \(m \geq 4nr - 4r^2\), then

\[
\dim(\pi_1(G_{m,n})) \leq \dim(G_{m,n}) = mn^2 + 4nr - 4r^2 - m < mn^2 - 1.
\]

Here, we use the result which states the dimension of the projection is less or equal to the dimension of the original variety \cite{12} Cor.11.13. Note the dimension of the \(\mathbb{P}(\mathbb{C}^{n \times n} \times \cdots \times \mathbb{C}^{n \times n})\), which is the target of the projection \(\pi_1\), is \(mn^2 - 1\). The \(\langle A_j, Q_0 \rangle = 0\) shows the dimension of \(\pi_1(G_{m,n})\) is strictly less than \(mn^2 - 1\) provided \(m \geq 4nr - 4r^2\). This means the image of the projection \(\pi_1\) lies in a hyper-surface which is defined by the vanish of some polynomials. We arrive at (a).

We next turn to (b). For \(A = \{A_1, \ldots, A_m\}\), we set

\[
Z_A := \{Q \in \mathbb{P}(\mathbb{C}^{n \times n}) : \langle A_j, Q \rangle = 0, \quad j = 1, \ldots, m\}.
\]

Note that \(Z_A\) is a linear subspace in \(\mathbb{P}(\mathbb{C}^{n \times n})\) with \(\dim(Z_A) \geq n^2 - 1 - m\). The projective variety \(\mathcal{P}L^C_{2r} \subset \mathbb{P}(\mathbb{C}^{n \times n})\) has dimension \(4nr - 4r^2 - 1\). If \(m \leq 4nr - 4r^2 - 1\), then

\[
\dim(Z_A) + \dim(\mathcal{P}L^C_{2r}) \geq n^2 - 1,
\]
which implies that (see [12 Prop.11.4])

\[ Z_A \cap \mathcal{PL}_{2r}^C \neq \emptyset. \]

Hence, if \( m \leq 4nr - 4r^2 - 1 \) there exits a non-zero matrix \( Q_0 \in Z_A \cap \mathcal{PL}_{2r}^C \) satisfying

\[ \langle A_j, Q_0 \rangle = 0, \quad j = 1, \ldots, m \]

which implies \( M_A \) is not injective on \( \mathcal{L}_{2r}^C \).

\[ \square \]

Remark 1. We also can use the technology in the proof of Theorem 2.1 to study the weak recovery, which means to recover a fixed \( Q_0 \in \mathcal{L}_{2r}^C \) from \( M_A(Q_0) \) (see also [11]). As shown in [11], to ensure \( M_A \) has the weak recovery property, we only need show that \( Q = Q_0 \) if \( M_A(Q - Q_0) = 0 \) and \( Q \in \mathcal{L}_{2r}^C \). Note that

\[ \{ Q - Q_0 : Q \in \mathcal{L}_{2r}^C \} \subset \mathbb{C}^{n \times n} \]

is an affine variety with dimension \( 2nr - r^2 \). Then using a similar method with the proof of Theorem 2.1 we can show that \( M_A \) is injective on \( \{ Q - Q_0 : Q \in \mathcal{L}_{2r}^C \} \subset \mathbb{C}^{n \times n} \) if \( m \geq 2nr - r^2 + 1 \) and \( A_1, \ldots, A_m \) are \( m \) generic matrices.

Remark 2. It will be very interesting to construct \( m = 4nr - 4r^2 \) deterministic matrices \( A_1, \ldots, A_m \) so that \( M_A \) is injective on \( \mathcal{L}_{2r}^C \). In the context of phase retrieval, such constructions are presented in [4] and [16]. In fact, [4] and [16] present \( 4n - 4 \) deterministic Hermite matrices and prove they define an injective measurement on \( S_{2r}^C \). It will be interesting to extend the results and methods of [4] and [16] to low-rank matrices recovery.

3. The recovery of real low rank matrices

In this section, we consider the case where \( \mathbb{H} = \mathbb{R} \). Then we have

**Theorem 3.1.** Suppose that \( r \leq n/2 \). Consider \( m \) matrices \( A = \{ A_1, \ldots, A_m \} \subset \mathbb{R}^{n \times n} \) and the mapping \( M_A : \mathbb{R}^{n \times n} \to \mathbb{R}^m \). The following holds

(a) If \( m \geq 4nr - 4r^2 \) then \( M_A \) is injective on \( \mathcal{L}_{2r}^C \) for generic matrices \( A_1, \ldots, A_m \).

(b) Suppose that \( n = 2^k + r, k \in \mathbb{Z}_+ \), or \( n = 2r + 1 \). If \( m < 4nr - 4r^2 \), then \( M_A \) is not injective on \( \mathcal{L}_{2r}^C \).

**Proof.** The proof of Part (a) is similar with the proof of (a) in Theorem 2.1 and hence we omit it here. We next turn to (b). Following the notation from the proof of Theorem 2.1 we set

\[ Z_A := \{ Q \in \mathbb{P}(\mathbb{C}^{n \times n}) : \langle A_j, Q \rangle = 0, \quad j = 1, \ldots, m \}. \]

Note that \( Z_A \) is a linear space and \( \dim(Z_A) \geq n^2 - 1 - m \). To state conveniently, set

\[ \mathcal{PL}_{2r}^C := \{ Q \in \mathbb{P}(\mathbb{C}^{n \times n}) : \text{rank}(Q) \leq 2r \}. \]

The \( \mathcal{PL}_{2r}^C \) is a projective variety in \( \mathbb{C}^{n^2} \) and \( \dim(\mathcal{PL}_{2r}^C) \geq 4nr - 4r^2 - 1 \). Note that when \( m \leq 4nr - 4r^2 - 1 \),

\[ \dim Z_A + \dim(\mathcal{PL}_{2r}^C) \geq n^2 - 1, \]

which implies that \( \mathcal{PL}_{2r}^C \cap Z_A \neq \emptyset \) [12 Prop.11.4]. According to Lemma 3.1 the variety \( \mathcal{PL}_{2r}^C \) has odd degree provided \( n = 2^k + r \). Note that \( Z_A \) is a linear space hence the intersection between \( \mathcal{PL}_{2r}^C \) and \( Z_A \) also has odd degree, which implies that the intersection \( \mathcal{PL}_{2r}^C \cap Z_A \) has a real point since any projective variety with odd degree defined over \( \mathbb{R} \) has real point. Thus there exists a nonzero real matrix \( Q_0 \in \mathcal{PL}_{2r}^C \cap Z_A \), which implies that \( M_A \) is not injective on \( \mathcal{L}_{2r}^C \).

\[ \square \]

According to [12 Ex. 19.10], the degree of the projective variety of \( \mathcal{PL}_{2r}^C \),

\[ d_{n,2r} := \prod_{i=0}^{n-2r-1} \frac{(n + i)! \cdot i!}{(2r + i)! \cdot (n - 2r + i)!}. \]

Then

**Lemma 3.1.** For \( n = 2^k + r, k \in \mathbb{Z}_+ \) or \( n = 2r + 1 \), \( d_{n,2r} \) is an odd integer.
Proof. We first consider the case where \( n = 2r + 1 \). A simple calculation shows that

\[
d_{2r+1,2r} = \frac{(2r + 1)!}{(2r)!} = 2r + 1,
\]

which implies that \( d_{n,2r} \) is odd provided \( n = 2r + 1 \).

We next assume that \( n = 2^k + r, k \in \mathbb{Z}_+ \). Note that

\[
d_{n,2r} = \prod_{i=0}^{n-2r-1} \frac{(n+i)! \cdot i!}{(2r+i)! \cdot (n-2r-i)!} = \prod_{i=0}^{n-2r-1} \frac{(n+i) \cdots (n+i-(2r-1))}{(i+1) \cdots (i+2r)}
\]

\(= \prod_{i=0}^{2^k-r-1} \frac{(2^k+i+r) \cdots (2^k+i-r+1)}{(i+1) \cdots (i+2r)}.
\]

Here, in the last equality, we use the assumption of \( n = 2^k + r \). To state conveniently, we use \( p_2(m) \) to denote the highest power of 2 dividing \( m \in \mathbb{Z} \). Then

\[
p_2(d_{n,2r}) = \sum_{i=0}^{2^k-r-1} \left( \sum_{j=-(r-1)}^{r} p_2(2^k+i+j) - \sum_{j=1}^{2r} p_2(i+j) \right)
\]

\[
= \sum_{i=0}^{2^k-r-1} \left( \sum_{j=1}^{r} p_2(2^k+i+1-j) - \sum_{j=1}^{2r} p_2(i+j+r) \right).
\]

Here, in the last equality, we use the fact of \( p_2(i+j) = p_2(2^k+i+j) \) provided \( i+j \leq 2^k-1 \). We first consider the first term in (3.3), i.e.,

\[
\sum_{i=0}^{2^k-r-1} \sum_{j=1}^{r} p_2(2^k+i+1-j) = \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+1-j) + \sum_{i=0}^{2^k-3r-1} \sum_{j=1}^{r} p_2(2^k+i+2r+1-j)
\]

\[
= \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+1-j) + \sum_{i=0}^{2^k-3r-1} \sum_{j=1}^{r} p_2(i+2r+1-j)
\]

\[
= \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+1-j) + \sum_{i=0}^{2^k-3r-1} \sum_{j=1}^{r} p_2(i+j+r).
\]

Then, combining (3.3) and (3.4), we obtain that

\[
p_2(d_{n,2r}) = \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+1-j) - \sum_{i=2^k-3r}^{2^k-r-1} \sum_{j=1}^{r} p_2(i+j+r)
\]

\[
= \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+1-j) - \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+j+2r)
\]

\[
= \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+j-r) - \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+j+2r)
\]

\[
= \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+j-r) - \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+j-2r)
\]

\[
= \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+j) - \sum_{i=0}^{2r-1} \sum_{j=1}^{r} p_2(2^k+i+j-2r)
\]

\[
= \sum_{s=1}^{2r-1} (b_s p_2(2^k+s) - b_{2r-s} p_2(2^k-s)) = 0.
\]
Here, in the last equality, \( b_s := \# \{(i, j) : i + j = s, 0 \leq i \leq r - 1, 1 \leq j \leq r \} \) and we use the fact of \( b_s = b_{2r-s} \) and \( p_2(2^k + s) = p_2(2^k - s) \).

Theorem 3.1 shows the bound \( 4nr - 4r^2 \) is tight provided \( n \) is in the form of \( 2^k + r \) or \( 2r + 1 \). One may be interested in whether the bound \( 4nr - 4r^2 \) is tight in general. The next counterexample shows that for the case where \( (n, r) = (4, 1) \) there exist 11 \( 4n - 5 \) matrices \( A = \{A_1, \ldots, A_{11}\} \subset \mathbb{R}^{4 \times 4} \) so that \( M_A \) is injective on \( \mathbb{L}_1^5 \subset \mathbb{R}^{4 \times 4} \). And hence the bound is not tight provided \( n = 4, r = 1 \). We list the 11 matrices as follows which are obtained by computer random search:

\[
\begin{align*}
A_1 &= \begin{pmatrix} -4 & 1 & 3 & 4 \\ -4 & 4 & 4 & 3 \\ 4 & -3 & 0 & -3 \\ 0 & -4 & 2 & 1 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} 0 & 3 & -1 & -1 \\ 0 & -2 & -1 & 2 \\ 0 & 3 & -2 & 3 \\ -1 & -1 & -3 & 2 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} -1 & -4 & -1 & -1 \\ 4 & 0 & -1 & 1 \\ 1 & -4 & -4 & -2 \\ 0 & -1 & 2 & 2 \end{pmatrix}, \\
A_4 &= \begin{pmatrix} -2 & -2 & 4 & 3 \\ -2 & 0 & 2 & 3 \\ 1 & -2 & -4 & 3 \\ -3 & 3 & 4 & -2 \end{pmatrix}, \\
A_5 &= \begin{pmatrix} 4 & 2 & -4 & -4 \\ -4 & -3 & 0 & -4 \\ 1 & -4 & 4 & -2 \\ 3 & 0 & 2 & 0 \end{pmatrix}, \\
A_6 &= \begin{pmatrix} 2 & 2 & 3 & 4 \\ 2 & -4 & 3 & 1 \\ 1 & 4 & 2 & -1 \\ -1 & -3 & 2 & 0 \end{pmatrix}, \\
A_7 &= \begin{pmatrix} 2 & 1 & 4 & 0 \\ -1 & -3 & 0 & -1 \\ 1 & 4 & 2 & -1 \\ 0 & 3 & 0 & 4 \end{pmatrix}, \\
A_8 &= \begin{pmatrix} 2 & -1 & 4 & -4 \\ -2 & 2 & 3 & -1 \\ -1 & 1 & 4 & -1 \\ -3 & -4 & 4 & 3 \end{pmatrix}, \\
A_9 &= \begin{pmatrix} -4 & 0 & 2 & -1 \\ 4 & 1 & 0 & 4 \\ -1 & -3 & 4 & 1 \\ -3 & 2 & 4 & -2 \end{pmatrix}, \\
A_{10} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 3 & 0 & -2 & -4 \\ -4 & -2 & -4 & 0 \\ 4 & 3 & 2 & -2 \end{pmatrix}, \\
A_{11} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ -4 & 0 & 2 & -1 \\ 4 & 1 & 0 & 4 \\ -1 & -3 & 4 & 1 \end{pmatrix}.
\end{align*}
\]

We show the map \( M_A \) associated with the 11 matrices is injective on \( \mathbb{L}_1^4 \):

**Theorem 3.2.** Set \( A = \{A_1, \ldots, A_{11}\} \) where \( A_j, j = 1, \ldots, 11 \), are defined in (3.5). Then the map \( M_A \) is injective on \( \mathbb{L}_1^5 \subset \mathbb{R}^{4 \times 4} \).

**Proof.** To this end, we only need prove the set

\[ \{Q \in \mathbb{R}^{4 \times 4} : \langle A_j, Q \rangle = 0, j = 1, \ldots, 11, \text{rank}(Q) \leq 2 \} \]

has only zero matrix.

We build the proof following the ideas of Vinzant [19, Theorem 1]. In fact, we use Vinzant’s test, which is stated in Algorithm 1 to verify the map \( M_A \) is injective on \( \mathbb{L}_1^5 \subset \mathbb{R}^{4 \times 4} \). We next explain the reason why Algorithm 1 works. Any \( 4 \times 4 \) real matrix can be written as

\[ Q = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}, \]

where \( x_{jk}, 1 \leq j \leq 4, 1 \leq k \leq 4, \) are 16 variables. Set

\[ \ell_j := \langle A_j, Q \rangle, j = 1, \ldots, 11 \]

and we use \( m_{jk} \) to denote the determinant of the sub-matrix formed by deleting the \( j \)th row and \( k \)th column from the matrix \( Q \). Note that both \( \ell_j \) and \( m_{jk} \) are polynomials about \( x_{11}, \ldots, x_{44} \). We recall the fact rank\((Q) \leq 2\) is equivalent to the vanish of \( m_{jk}, j = 1, \ldots, 4, k = 1, \ldots, 4 \). Hence, he map \( M_A \) is injective if and only if the polynomial system

\[ m_{11} = m_{12} = \cdots = m_{44} = \ell_1 = \cdots = \ell_{11} = 0 \]

has nonzero real solution \( (x_{11}, \ldots, x_{44}) \in \mathbb{R}^{16} \). A simple observation is that if \( x^0 := (x_{11}^0, x_{12}^0, \ldots, x_{44}^0) \in \mathbb{R}^{16} \) is a root of (3.7), then \( f(x^0) = 0 \) for any \( f \) in the ideal generated by the set of polynomials \{\( m_{11}, \ldots, m_{44}, \ell_1, \ldots, \ell_{11} \}\). To state conveniently, we use the notation \( \langle m_{11}, \ldots, m_{44}, \ell_1, \ldots, \ell_{11} \rangle \) to denote the ideal. We use the computer algebra software **maple** to compute a Gröbner basis of the ideal and elimination. The result is a polynomial \( f_0 \in Q[x_{43}, x_{44}] \) (see Appendix A), which is a homogeneous polynomial of degree 20. Then \( f_0(x_{43}^0, x_{44}^0) = 0 \) if \( x^0 = (x_{11}, x_{12}, \ldots, x_{44}) \in \mathbb{R}^{16} \) is a root of (3.7) since \( f_0 \in \langle m_{11}, \ldots, m_{44}, \ell_1, \ldots, \ell_{11} \rangle \). We claim that \( f_0 \) only has real root \((0, 0)\). Indeed, if \( f_0 \) has a nonzero real solution \( (x_{43}^0, x_{44}^0) \) then \( x_{44}^0 \neq 0 \) (otherwise, \( x_{43}^0 = x_{44}^0 = 0 \) since \( f_0 \) is a homogeneous polynomial). Note that if \( f_0(x_{43}^0, x_{44}^0) = 0 \) then \( f_0(\lambda x_{43}^0, \lambda x_{44}^0) = 0 \) for any \( \lambda \in \mathbb{C} \). Without loss of generality, we suppose that \( x_{44}^0 = 1 \). Then using Sturm sequences, we can verify the univariate polynomial \( f_0(x_{43}, 1) \) has no real solutions, which implies that the real root of \( f_0(x_{43}, x_{44}) \) is only \((0, 0)\).

We claim that there is nonzero root to (3.7) with \( x_{43} = 0, x_{44} = 0 \). We can verify the claim still by Gröbner basis. For any \( \lambda \in \mathbb{C}, (\lambda x_{11}^0, \lambda x_{12}^0, \ldots, \lambda x_{44}^0) \) is a solution to (3.7) if \( (x_{11}^0, x_{12}^0, \ldots, x_{44}^0) \) is a
proof. A simple observation is that the map \( M(b) \). We set the proof of Part (a) is similar with the proof of (a) in Theorem 2.1 and we omit it here. We next turn to 3.1.

We post the code for these computation in Maple at [http://lsec.cc.ac.cn/~xuzq/LowRank.htm](http://lsec.cc.ac.cn/~xuzq/LowRank.htm) □

**Algorithm 1** Vinzant’s test for injective of the map \( M_A \)

**Inputs:** \( m=11 \), the matrices \( A_1, \ldots, A_m \) which are given in (3.5),

\[
Q = \begin{pmatrix}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{21} & x_{22} & x_{23} & x_{24} \\
x_{31} & x_{32} & x_{33} & x_{34} \\
x_{41} & x_{42} & x_{43} & x_{44}
\end{pmatrix}.
\]

1. Set \( Q_{j,k}, 1 \leq j, k \leq 4 \) is the sub-matrix of \( Q \) formed by deleting \( j \)th column and \( k \)th row from \( Q \).
2. Set \( \ell_j = \langle A_j, Q \rangle, j = 1, \ldots, m \) and \( m_{j,k} = \text{det}(Q_{j,k}), 1 \leq j, k \leq 4 \).
3. Computer Gröbner basis of the ideal \( \langle \ell_1, \ldots, \ell_m, m_{11}, m_{12}, \ldots, m_{44} \rangle \) and obtain that \( f_0 \in \mathbb{Q}[x_{4,3}, x_{4,4}] \).
4. Use Sturm Sequence to compute the number of real roots of \( f_0(x_{4,3}, 1) \).
5. if the number of real roots of \( f_0(x_{4,3}, 1) \) is 0 then
6. for all \( j, k \in [1, 4] \times [1, 4] \) do
7. Check whether 1 \( \in \langle x_{j,k} - 1, x_{4,3}, x_{4,4}, \ell_1, \ldots, \ell_m, m_{11}, \ldots, m_{44} \rangle \) by computing Gröbner basis
8. if 1 \( \in \langle x_{j,k} - 1, x_{4,3}, x_{4,4}, \ell_1, \ldots, \ell_m, m_{11}, \ldots, m_{44} \rangle \) then
9. \( r_{j,k} = 1 \)
10. else
11. \( r_{j,k} = 0 \), “FAIL”
12. end if
13. if \( r_{j,k} = 1 \) for all \( j, k \in [1, 4] \times [1, 4] \) then
14. “INJECTIVE”
15. end if
16. end for
17. else
18. “FAIL”
19. end if

3.1. Symmetric matrix. We next consider the symmetric matrix which will be helpful for the investigation of phase retrieval by projection. Recall that

\[
S_r^\mathbb{R} := \{ X \in \mathbb{R}^{n \times n} : \text{rank}(X) \leq r, X^\top = X \}.
\]

**Theorem 3.3.** Suppose that \( r \leq n/2 \). Consider \( m \) matrices \( A_1, \ldots, A_m \in \mathbb{R}^{n \times n} \) and the mapping \( M_A : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^m \) where \( A = \{ A_1, \ldots, A_m \} \). The following holds:

(a) If \( m \geq 2nr + r - 2r^2 \) then \( M_A \) is injective on \( S_r^\mathbb{R} \) for generic matrices \( A_1, \ldots, A_m \).

(b) If \( n = 2^k + r \) and \( m < 2nr + r - 2r^2 \), then \( M_A \) is not injective on \( S_r^\mathbb{R} \).

**Proof.** A simple observation is that the map \( M_A \) is injective on \( S_r^\mathbb{R} \) if and only if there is a nonzero \( Q \in S_r^\mathbb{R} \) for which \( M_A(Q) = 0 \). Thus, we only need show that \( Q = 0 \) provided \( Q \in S_r^\mathbb{R} \) and \( M_A(Q) = 0 \). Recall that \( S_r^\mathbb{R} \) is an affine algebraic variety with dimension \( \binom{n+1}{2} - \binom{r+1}{2} \), which implies that \( \dim(S_r^\mathbb{R}) = 2nr + r - 2r^2 \). The proof of Part (a) is similar with the proof of (a) in Theorem 2.1 and we omit it here. We next turn to (b). We set

\[
Z_A := \{ Q \in \mathbb{P}(\mathbb{C}^{n \times n}) : \langle A_j, Q \rangle = 0, Q^\top = Q, \ j = 1, \ldots, m \},
\]

\[
\mathcal{PS}_{2r}^\mathbb{C} := \{ X \in \mathbb{P}(\mathbb{C}^{n \times n}) : \text{rank}(X) \leq 2r, X^\top = X \}.
\]
As introduced in Section 1.3, we say that
\[ \|A\|_2 \]
Based on Theorem 3.3, we show that the bound 2
\[ \subset \{ \]
Then
\[ \text{when } m \leq 2nr + r - 2r^2 - 1, \]
\[ \dim Z_A + \dim(\mathcal{PS}_{2r}^C) \geq \frac{n(n + 1)}{2} - 1, \]
which implies that \( \mathcal{PS}_{2r}^C \cap Z_A \neq \emptyset \). Note that \( \mathcal{PS}_{2r}^C = \mathcal{PS}_{2r}^C \cap \mathcal{PL}_{2r} \). Then for any \( \mathcal{H}_{jk} := \{ X \in \mathcal{P}(\mathbb{C}^{n \times n}) : x_{jk} = x_{kj} \} \).

According to Lemma 3.11, the variety \( \mathcal{PL}_{2r} \) has odd degree provided \( n = 2^k + r \), which implies that the degree of \( \mathcal{PS}_{2r}^C \) is odd if \( n = 2^k + r \) since \( \mathcal{PS}_{2r}^C \) is the intersection of \( \mathcal{PL}_{2r} \) and some linear spaces. Note that \( Z_A \) is a linear space hence the intersection between \( \mathcal{PS}_{2r}^C \) and \( Z_A \) also has odd degree, which implies that the intersection \( \mathcal{PS}_{2r}^C \cap Z_A \) has a real point since any projective variety with odd degree defined over \( \mathbb{R} \) has real point.

\[ \square \]

**Remark 3.** When \( r = 1 \), the bound \( 2nr + r - 2r^2 \) is reduced to \( 2n - 1 \). A natural question is whether the bound \( 2n - 1 \) is tight for the recovery of the symmetric rank-1 matrix. For the case \( n = 4 \), one can construct 6 = 2n - 2 matrices which are injective on \( \mathcal{PS}_1^R \) (see Theorem 4.2), which implies that the bound \( 2nr + r - 2r^2 \) is not tight for \( n = 4, r = 1 \).

**Remark 4.** If we require \( A_j \) is in the form of \( a_ja_j^\top \) with \( a_j \in \mathbb{R}^n \), then the bound \( 2n - 1 \) is tight. In fact, \( \langle A_j, Q \rangle = |\langle a_j, x \rangle|^2 \) provided \( Q = xx^\top \in \mathbb{R}^{n \times n} \). According to the result from phase retrieval [3], \( M_A \) is injective on \( S_R \) if and only if \( \{ a_1, \ldots, a_m \} \subset \mathbb{R}^n \) satisfies the finite complement property, i.e., for every subset \( I \subset \{ 1, \ldots, m \} \) either \( \{ a_j \}_{j \in I} \) or \( \{ a_j \}_{j \notin I} \) spans \( \mathbb{R}^n \), which implies the bound \( 2n - 1 \) is tight provided the measurement matrices \( A_j \) is in the form of \( a_ja_j^\top \).

### 4. Phase Retrieval by Projections

Recall that we use \( P_j : \mathbb{R}^n \rightarrow W_j \) to denote an orthogonal projection where \( W_j \subset \mathbb{R}^n \) is a subspace. As introduced in Section 1.3, we say that \( \{ W_j \}_{j=1}^m \) yields phase retrieval if for all \( x, y \in \mathbb{R}^n \) satisfying \( \|P_jx\| = \|P_jy\| \) for all \( j = 1, \ldots, m \) then \( x = \pm y \). The following theorem shows that \( 2n - 1 \) projections are enough for phase retrieval by projection.

**Theorem 4.1.** [2] Phase retrieval can be done in \( \mathbb{R}^n \) with \( 2n - 1 \) subspaces each of any dimension less than \( n - 1 \).

The problem is also raised in [1] which states can phase retrieval be done in \( \mathbb{R}^n \) with fewer than \( 2n - 1 \) projections? Based on Theorem 3.3.11, we show that the bound \( 2n - 1 \) is tight provided \( n = 2^k + 1, k \in \mathbb{Z}_{\geq 1} \). Particularly, we have

**Corollary 4.1.** Suppose that \( n \) is in the form of \( 2^k + 1 \). Given any subspaces \( \{ W_j \}_{j=1}^m \) in \( \mathbb{R}^n \) with \( m < 2n - 1 \), there exist \( x, y \in \mathbb{R}^n \) with \( x \neq \pm y \) so that \( \|P_jx\| = \|P_jy\|, j = 1, \ldots, m \).

**Proof.** Suppose that \( \{ u_{j,1}, \ldots, u_{j,d_j} \} \) is an orthonormal basis of \( W_j \). Then for any \( x \in \mathbb{R}^n \)

\[ \|P_jx\|^2 = \langle A_j, xx^\top \rangle, \quad j = 1, \ldots, m \]

where

\[ (4.8) \]

\[ A_j := u_{j,1}u_{j,1}^\top + \cdots + u_{j,d_j}u_{j,d_j}^\top. \]

Then \( \{ W_j \}_{j=1}^m \) allow phase retrieval if and only if the map \( M_A \) is injective on \( S_R^+ \) where \( A = \{ A_1, \ldots, A_m \} \) and \( A_j, j = 1, \ldots, m \) are defined in (4.8). The part (b) in Theorem 3.3 shows that \( M_A \) is not injective if \( m < 2n - 1 \) and \( n = 2^k + 1 \) which implies the corollary.

\[ \square \]

Naturally, one may be interested in whether the bound \( 2n - 1 \) is tight when \( n \neq 2^k + 1 \). We give a negative answer by presenting a counterexample for the case where \( n = 4 \). In fact, we present 7 subspaces in \( \mathbb{R}^4 \),
which is obtained by the computer search. Set

\begin{align}
U & := [u_1, u_2, \ldots, u_6] = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & -4 & 2 \\
0 & 0 & 0 & 1 & -3 & 3
\end{pmatrix}, \quad V := [v_1, v_2, \ldots, v_6] = \begin{pmatrix}
0 & 5 & -1 & -1 & -17 & -5 \\
-5 & 0 & 0 & 5 & 4 & 4 \\
2 & -2 & 0 & -3 & -2 & 2 \\
-2 & 1 & 0 & 0 & -3 & -1
\end{pmatrix}
\end{align}

and

\begin{align}
W_j & := \text{span}\{u_j, v_j\} \subset \mathbb{R}^4, \quad j = 1, \ldots, 6.
\end{align}

**Theorem 4.2.** Suppose that $W_1, \ldots, W_6$ are defined in (4.10). Then the phase retrieval by projection can be done in $\mathbb{R}^4$ with the 6 subspaces $W_1, \ldots, W_6$.

**Proof.** To this end, we only need show that the set

\begin{align}
\{Q \in \mathbb{R}^{4 \times 4} : \text{rank}(Q) \leq 2, Q^\top = Q, \quad (A_j, Q), j = 0, 1, \ldots, 6\},
\end{align}

has only zero matrix, where

\begin{align}
A_j = \frac{1}{\|u_j\|^2} u_j u_j^\top + \frac{1}{\|v_j\|^2} v_j v_j^\top, \quad j = 1, \ldots, 6.
\end{align}

Any symmetric $4 \times 4$ matrix can be written as

\begin{align}
Q = \begin{pmatrix}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{12} & x_{22} & x_{23} & x_{24} \\
x_{13} & x_{23} & x_{33} & x_{34} \\
x_{14} & x_{24} & x_{34} & x_{44}
\end{pmatrix}.
\end{align}

Then we can verify $\mathcal{M}_A$ is injective on $\mathbb{S}_1^4$ with $A = \{A_1, \ldots, A_6\}$ by using a similar method with Algorithm 1. In fact, we verify $\mathcal{M}_A$ is injective by Algorithm 1 with inputting $m = 6$, $A_1, \ldots, A_m$, and $Q$, which are given in (4.11) and (4.12), respectively. In Line 3 of Algorithm 1, we obtain $f_0 \in \mathcal{Q}[x_{34}, x_{44}]$ by computing the Gröbner basis, which is shown as follows:

\[ f_0(x_{34}, x_{44}) = \begin{align}
& 519966562263644355495348491970361616539511911263753956248783182163623419737152512 \cdot x_{34}^{10} \\
& + 32857924978094588040378180884308246920612283125532941525689039614516382331655616 \cdot x_{34}^8x_{44}^4 \\
& + 14889376593364459642443824031426913082957091595139086664937634093652895722168 \cdot x_{34}^6x_{44}^6 \\
& - 12339400486809177169040569636150505525212099725117855573783266945351610854716 \cdot x_{34}^8x_{44}^4 \\
& + 5608302528871717048293812205661442812900643057674097309748633281558249915632343 \cdot x_{34}^4x_{44}^8 \\
& + 862820283255455843496154023668088334391251839891193130388856747170901880736209512 \cdot x_{34}^8x_{44}^4 \\
& + 7757543201949880820388834227841551121001844781939267906144764361343765018047929 \cdot x_{34}^6x_{44}^6 \\
& + 47027735438311757878033351572290668783205695745022607215936313709974378602178127500 \cdot x_{34}^4x_{44}^6 \\
& + 200124464535786576051259835437973957238302830519817242734748938937577103039639000 \cdot x_{34}^8x_{44}^4 \\
& + 61075717770585934448520379034324236932067749445810309769599665348820297371910000 \cdot x_{34}^4x_{44}^4 \\
& + 8636626929016108140668062419995447162569962558969976455134579423688060006725000 \cdot x_{14}^8.
\end{align}\n
We post the code for these computation in Maple at [http://lsec.cc.ac.cn/~xuzq/LowRank.htm](http://lsec.cc.ac.cn/~xuzq/LowRank.htm)
Appendix A. The $f_0(x_{43}, x_{44})$ which is used in the proof of Theorem 3.3
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