INDESTRUCTIBILITY OF GENERICALLY STRONG CARDINALS

BRENT CODY AND SEAN COX

Abstract. Foreman [For13] proved the Duality Theorem, which gives an algebraic characterization of certain ideal quotients in generic extensions. As an application he proved that generic supercompactness of $\omega_1$ is preserved by any proper forcing. We generalize portions of Foreman’s Duality Theorem to the context of generic extender embeddings and ideal extenders (as introduced by Claverie [Cla10]). As an application we prove that if $\omega_1$ is generically strong, then it remains so after adding any number of Cohen subsets of $\omega_1$; however many other $\omega_1$-closed posets—such as $\text{Col}(\omega_1, \omega_2)$—can destroy the generic strength of $\omega_1$. This generalizes some results of Gitik-Shelah [GS89a] about indestructibility of strong cardinals to the generically strong context. We also prove similar theorems for successor cardinals larger than $\omega_1$.

1. Introduction

Many conventional large cardinal properties are witnessed by the existence of ultrafilters or elementary embeddings. Some of these properties can be viewed as special cases of more general concepts, namely generic large cardinal properties [For10, Remark 3.14], which are usually witnessed by generic ultrafilters and generic elementary embeddings, and which can be arranged to hold at small cardinals. For example, by collapsing below a measurable cardinal, one can arrange that there is a precipitous ideal $I$ on $\omega_1$ witnessing the generic measurability of $\omega_1$ in the sense that forcing with $P(\kappa)/I - \{0\}$ naturally yields a $V$-normal, $V$-ultrafilter on $P(\kappa)^V$ with a well-founded ultrapower in the extension (see [Cum10, Section 17.1]).

Preserving large cardinals through forcing by lifting elementary embeddings has been a major theme in set theory [Cum10]. In some cases, large cardinals can be made indestructible by wide classes of forcing. For example, given a supercompact cardinal $\kappa$, Laver [Lav78] used a preparatory forcing iteration to produce a model in which $\kappa$ remains supercompact, and the supercompactness of $\kappa$ is indestructible by any further $\kappa$-directed closed forcing. Gitik and Shelah [GS89b] proved that the strongness of a cardinal $\kappa$ can be made indestructible by $\kappa^+$-weakly closed forcing satisfying the Prikry condition. As shown by Hamkins [Ham00], one can obtain indestructibility results for many large cardinals by using a preparatory forcing he calls the lottery preparation.

It is natural to wonder whether similar results hold for some generic large cardinal properties. Let us first clear up some possibly confusing terminology in the literature, which will also help the reader more easily understand the division of Sections 3 and 4 in this paper and Claverie’s [Cla10] distinction between ideally strong and generically strong. Conventional large cardinals—such as measurable
and supercompact cardinals—admit equivalent definitions in terms of either embeddings or ultrafilters. This is not true of generic large cardinals. Let us say that a cardinal \( \kappa \) is:

- **generically measurable** iff there is a poset which forces that there is an elementary embedding \( j : V \rightarrow M \) with \( M \) wellfounded and \( \text{crit}(j) = \kappa \);
- **ideally measurable** iff there is a precipitous ideal on \( \kappa \); equivalently, \( \kappa \) is generically measurable and the poset which witnesses the generic measurability is of the form \( \varphi(\kappa)/I \) for some ideal \( I \) on \( \kappa \).
- **generically supercompact** iff for every \( \lambda > \kappa \) there is a poset which forces that there is an elementary embedding \( j : V \rightarrow M \) with \( M \) wellfounded, \( \text{crit}(j) = \kappa \), and \( j^\kappa \lambda \in M \).
- **ideally supercompact** iff \( \kappa \) is generically supercompact and for each \( \lambda \) the witnessing poset is of the form \( P(P_\kappa(\lambda))/I \) for some some normal precipitous ideal \( I \) on \( P_\kappa(\lambda) \). This is what Foreman [For13] calls generic supercompactness.

Kakuda [Kak81] and Magidor [Mag80] independently proved that if \( \kappa \) is a regular uncountable cardinal and \( P \) is a poset satisfying the \( \kappa \)-chain condition, then \( I \lesssim P(\kappa) \) is a precipitous ideal if and only if \( \models P \) “the ideal on \( \kappa \) generated by \( I \) is precipitous.” In particular, if \( \kappa \) is ideally measurable in the sense given above, then, then it remains so in any \( \kappa \)-c.c. forcing extension. Their proof also shows that generic measurability of \( \kappa \) is preserved by any \( \kappa \)-cc forcing; this is an easier result.

Foreman [For13] proved a Duality Theorem which gives an algebraic characterization of ideal quotients arising in many settings where generic large cardinal embeddings are lifted to generic extensions. As an application, he proved the following theorem:

**Theorem 1** (Foreman [For13]). If \( \omega_1 \) is generically supercompact or ideally supercompact, then it remains so after any proper forcing.

It should be noted that Foreman’s Theorem 1, and also our Theorem 2 below, require no “preparation forcing”. We caution the reader that what Foreman calls generic supercompactness is what we’re calling ideal supercompactness. We note that Foreman’s Duality Theorem is used in his proof of preservation of ideal supercompactness; but the Duality Theorem is not used in the proof of preservation of generic supercompactness.

Claverie [Cla10] introduced the notions of generically strong and ideally strong cardinals. These are defined in Section 2 below, but roughly: \( \kappa \) is generically strong iff there are arbitrarily strong wellfounded generic embeddings of \( V \); and \( \kappa \) is ideally strong iff its generic strongness is witnessed by what Claverie calls ideal extenders. In summary, just as precipitous ideals can be used to witness the generic measurability or supercompactness of a cardinal, precipitous ideal extenders can be used to witness generic strongness.

In this article we prove the following indestructibility result concerning ideally strong cardinals and generically strong cardinals:

**Theorem 2.** If \( \omega_1 \) is ideally strong or generically strong, then it remains so after forcing to add any number of Cohen subsets of \( \omega_1 \).
Let us remark that Theorem 2 is generalized below (see Theorem 3 on page 7) to include the case in which $\kappa > \omega_1$, assuming other technical requirements related to internal approachability.

Theorem 2 will follow from Theorem 4 and Theorem 6 below. Theorem 4—which applies to any generically strong setting—includes lifting a generic strongness embedding through Cohen forcing using a ‘surgery’ argument (see [Cum10, Theorem 25.1] and [CM14]) rather than a master condition argument which is possible in the supercompactness context. Theorem 6—which is specific to the ideally strong setting—extends portions of Foreman’s duality theory (see [For13] and [For10]) into the context of ideal extenders, and allows us to conclude that a certain $(\kappa, \lambda)$-ideal extender derived from a lifted embedding is precipitous. Theorems 4 and 6 are then used to prove Theorem 2 in Section 5.

It is natural to ask whether Theorem 2 can be strengthened; for example, whether the generic or ideal strongness of $\omega_1$ is necessarily preserved by all $\omega_1$-closed forcings (not just forcings to add Cohen subsets to $\omega_1$). The answer is no, for the following reason, as alluded to in Gitik-Shelah [GS89a] in the conventional large cardinal setting. Suppose $G$ is Col$(\kappa, \kappa^+)$-generic over $V$, and that in $V[G]$, the cardinal $\kappa$ is generically measurable. Then $V$ must have an inner model with a Woodin cardinal. Otherwise, the core model $K$ exists in $V$ and is absolute to $V[G]$; and moreover (see [CS12]) $V[G]$ believes that $K$ computes $\kappa^+$ correctly; but this is a contradiction, because $\kappa^+K \leq \kappa^+V < \kappa^+V[G]$.

## 2. Preliminaries

### 2.1. Ideal Extenders

We refer the reader to Kanamori [Kan03] for the definition of an extender.

Claverie [Cla10] gives a detailed account of ideal extenders and when ideal extenders are precipitous. We recount his main definitions in this section. Claverie defines a set $F$ to be a $(\kappa, \lambda)$-system of filters if $F \subseteq \{(a, x) \in [\lambda]^{<\omega} \times P([\kappa]^{<\omega}) : x \subseteq [\kappa]^{<\omega}\}$ and for each $a \in [\lambda]^{<\omega}$ the set $F_a := \{x : (a, x) \in F\}$ is a filter on $[\kappa]^{<\omega}$. The support of $F$ is supp$(F) = \{a \in [\lambda]^{<\omega} : F_a \neq \{[\kappa]^{<\omega}\}\}$. In what follows we will identify each finite subset of $\lambda$ with the corresponding unique increasing enumeration of its elements; in other words, $a \in [\lambda]^{<\omega}$ will be identified with the function $s_a : |a| \rightarrow a$ listing the elements of $a$ in increasing order. Given $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$, let $s : |a| \rightarrow |b|$ be the unique function such that $a(n) = b(s(n))$. For $x \in P([\kappa]^{<\omega})$, define $x_{a,b} = \{\langle u_i : i < |b|\rangle \in [\lambda]^{<\omega} : \langle u_{a(j)} : j < |a|\rangle \in x\}$. A $(\kappa, \lambda)$-system of filters $F$ is called compatible if for all $a \subseteq b$ with $a, b \in \text{supp}(F)$ one has $x \in F_a$ if and only if $x_{a,b} \in F_b$. Given $a, b \in [\lambda]^{<\omega}$ and $x \in F_a$, we define $F' = \text{span}\{F, (a, x)\}$, the span of $F$ and $(a, x)$, to be the smallest $(\kappa, \lambda)$-system of filters such that $F \subseteq F'$ and $(a, x) \in F'$.

If $F$ is a single filter on $\kappa$, the usual Boolean algebra $B_F := \wp(\kappa)/F$ is forcing equivalent to the poset $E_F$ whose conditions are filters of the form $F \cup \{S\}$, where $S \in F^+$; the poset $E_F$ is ordered by reverse inclusion. The following definition generalizes the latter poset to systems of filters:

**Definition 1.** [Cla10, Definition 4.3(v)] Given a $(\kappa, \lambda)$-system of filters $F$, the forcing associated to $F$ is denoted by $E_F$ and consists of all conditions $p = F^p$ where $F^p$ is a compatible $(\kappa, \lambda)$-system of filters, supp$(p) = \text{supp}(F^p) \subseteq \text{supp}(F)$.

---

1Here $F \cup \{S\}$ denotes the filter generated by $F \cup \{S\}$. 

is finite and $F^p$ is generated by one point $x \in F^+_a$ for some $a \in \text{supp}(p)$, i.e. $F^p = \text{span}\{F, (a, x)\}$. The ordering on $\mathbb{E}_F$ is defined by $p \leq q$ if and only if $\text{supp}(q) \subseteq \text{supp}(p)$ and for all $a \in \text{supp}(q)$, $F^q_a \subseteq F^p_a$, in other words $F^q \subseteq F^p$.

If $F$ is a compatible $(\kappa, \lambda)$-system of filters and $\hat{G}$ is the $\mathbb{E}_F$-name for the generic object, let $\hat{E}_F$ be the name for $\bigcup \hat{G}$. Then clearly $\hat{E}_F$ is forced to be a $(\kappa, \lambda)$-system of filters, and indeed, as shown in [Cla10], each $(\hat{E}_{F}^G)_a = (\bigcup G)_a$ is a $V$-ultrafilter. Claverie defines two additional combinatorial properties of $(\kappa, \lambda)$-systems of filters, potential normality and precipitousness (see [Cla10, Definition 4.4]) such that the following two facts hold. First, if $F$ is a compatible potentially normal $(\kappa, \lambda)$-system of filters, then $\hat{E}_F^G$ is a $(\kappa, \lambda)$-extender over $V$; secondly, if $F$ is in addition precipitous, then the generic ultrapower $j_{\hat{E}_F^G}: V \rightarrow \text{Ult}(V, \hat{E}_F^G)$ is well-founded.

**Definition 2.** [Cla10, Definition 4.5] Let $\kappa < \lambda$ be ordinals. $F$ is a $(\kappa, \lambda)$-ideal extender if it is a compatible and potentially normal $(\kappa, \lambda)$-system of filters such that for each $a \in \text{supp}(F)$, the filter $F_a$ is $<\kappa$-closed.

As noted in the introduction, precipitous $(\kappa, \lambda)$-ideal extenders can be used to witness generic strongness, in a way similar to that in which precipitous ideals can be used to witness generic measurability or generic supercompactness:

**Definition 3.** [Cla10, Definition 4.9] A regular cardinal $\kappa$ is called ideally strong if and only if for all $A \subseteq \text{ORD}$, $A \in V$, there is some precipitous $(\kappa, \lambda)$-ideal extender $F$ such that whenever $G$ is $\mathbb{E}_F$-generic, and $E_F^G$ is the corresponding $V$-(\kappa, \lambda)-extender, one has $A \in \text{Ult}(V, E_F^G)$.

Ideal strongness is a specific form of generic strongness, which is defined as follows:

**Definition 4.** [Cla10, Definition 4.20] A cardinal $\kappa$ is generically strong if for all $A$ there is a poset $\mathbb{P}$ such that in $V^\mathbb{P}$ there is a definable embedding $j: V \rightarrow M$ with $\text{crit}(j) = \kappa$, $M$ wellfounded, and $A \in M$.

Let us briefly mention a characterization of ideally strong cardinals which we will use in our proof of Theorem 2.

**Lemma 1.** A regular cardinal $\kappa$ is ideally strong (as in Definition 3) if and only if for every $\lambda > \kappa$ there is a precipitous $(\kappa, \lambda)$-ideal extender $F$ such that whenever $G$ is $(V, \mathbb{E}_F)$-generic the generic ultrapower $j_G: V \rightarrow M = \text{Ult}(V, E_F^G)$ satisfies the following properties.

1. $\text{crit}(j_G) = \kappa$
2. $j_G(\kappa) > \lambda$
3. $\mathcal{H}_\lambda^V \subseteq M$

One can also use the $V_\alpha$-hierarchy to characterize ideally strong cardinals. However, to obtain $V_\alpha \subseteq M = \text{Ult}(V, E_F^G)$, the length of the generic extender $E_F^G$ must be at least $|V_\alpha|^+$ (see [Kan03, Exercise 26.7]). For notational convenience, in what follows, we will use the characterization of ideally strong cardinals given in Lemma 1.

---

2**Definition 4** is equivalent to a first-order definition; in particular we could just require the domain of $j$ to be $H_{\theta_{\mathbb{P}, A}}^V$, where $\theta_{\mathbb{P}, A} \gg |\mathbb{P}|^+ V$ and $A \in H_{\theta_{\mathbb{P}, A}}$. Then $\theta_{\mathbb{P}, A}$ will always be uncountable in $V^\mathbb{P}$ and by well-known arguments such embeddings lift to domain $V$. 

Just as one can produce precipitous ideals by collapsing below large cardinals, Claverie proves [Cla10, Lemma 4.8] that if \( \kappa \) is \( \lambda \)-strong in \( V \) witnessed by a \( (\kappa, \lambda) \)-extender \( E \), and \( \mu < \kappa \) is a cardinal, then forcing with the Levy collapse \( \text{Col}(\mu, <\kappa) \) produces a model in which \( \mu^+ = \kappa \) and

\[
F = \{ (a, x) : x \subseteq [\kappa]^{|\kappa|} \text{ and } \exists y \text{ such that } (a, y) \in E \text{ and } y \subseteq x \}
\]
is a precipitous \( (\kappa, \lambda) \)-ideal extender.

Notice that if \( F \) is a filter, \( F_0 \) and \( F_1 \) are filter extensions of \( F \), and \( F_1 \not\subseteq F_0 \) as witnessed by some \( S \in F_1 - F_0 \), then the filter \( F^\theta_0 \) generated by \( F_0 \cup \{ S^c \} \) extends \( F_0 \) and is incompatible with \( F_1 \), in the sense that there is no proper filter extension of \( F^\theta_0 \cup F_1 \). The following lemma is a generalization of this fact, and will be used in our proof of Theorem 2.

**Lemma 2.** Given a \( (\kappa, \lambda) \)-system of filters \( F \), the associated forcing \( E_F \) is a separative poset.

**Proof.** Suppose \( p, q \in E_F \) are conditions and that \( p \not\leq q \). We have \( p = \text{span}(F, (a, x)) \) and \( q = \text{span}(F, (b, y)) \) for some \( a, b \in [\lambda]^{<\omega} \) and for some \( x \in (F_a)^+ \) and \( y \in (F_b)^+ \). Since \( p \not\leq q \) we have that \( F^s_a \not\leq F^s_b \) for some \( c \in \text{supp}(q) \). Let \( z \in F^s_a \setminus F^s_b \), then it follows that \( [\kappa]^{<\omega} \setminus z \in (F^s_b)^+ \subseteq F^+ \). Since \( F \) is a compatible \( (\kappa, \lambda) \)-system of filters and \( x \in F^s_a \), we have \( x_{a, a \cup c} \in F^s_{a, a \cup c} \) and \( ([\kappa]^{<\omega} \setminus z)_{c, a \cup c} \in (F^s_{a, a \cup c})^+ \). Thus, \( x_{a, a \cup c} \cap ([\kappa]^{<\omega} \setminus z)_{c, a \cup c} \subseteq F^s_{a, a \cup c} \), which means that \( r = \text{span}(F, (a \cup c, x_{a, a \cup c} \cap ([\kappa]^{<\omega} \setminus z)_{c, a \cup c})) \) is a condition in \( E_F \). Since \( x_{a, a \cup c} \in F^s_{a, a \cup c} \) and thus \( x \in F^s_{a, a \cup c} \), it follows that \( r \leq p \). Furthermore, since \( ([\kappa]^{<\omega} \setminus z)_{c, a \cup c} \in F^s_{a, a \cup c} \) we have \( [\kappa]^{<\omega} \setminus z \in F^s_{a, a \cup c} \), and thus \( r \neq \bot \). Therefore \( E_F \) is separative. \( \square \)

### 2.2. Internal Approachability

In this section we discuss the key technical issue which comes up in generalizing Theorem 2 to the case in which \( \kappa > \omega_1 \).

The following definition is standard; see Foreman-Magidor [FM95] for details:

**Definition 5.** Let \( \mu \) be a regular cardinal. A set \( X \) is called \( \mu \)-internally approachable if there is a \( \subseteq \)-increasing sequence \( \vec{N} = \langle N_i \mid i < \mu \rangle \) such that:

- \( N_i \cap \mu \in \mu \) for all \( i < \mu \);
- \( |N_i| < \mu \) for each \( i < \mu \);
- \( X = \bigcup_{i<\mu} N_i \);
- \( \langle N_i \mid i<j \rangle \in X \) for all \( j < \mu \)

We let \( \text{IA}_\mu \) denote the class of \( \mu \)-internally approachable sets.

Note that almost every countable subset of \( H_\theta \) (for \( \theta \) regular uncountable) is \( \omega \)-internally approachable; this is just because models of set theory are closed under finite sequences. A simple closing-off argument shows that for any regular \( \mu \leq \theta \), the set \( \text{IA}_\mu \cap \varphi_\mu^+(H_\theta) \) is always stationary. Note also that for any \( X \in \varphi_\mu^+(H_\theta) \): \( X \in \text{IA}_\mu \) iff \( H_X \in \text{IA}_\mu \), where \( H_X \) is the transitive collapse of \( X \). We refer the reader to Foreman-Magidor [FM95] for proofs of the following lemma.

**Lemma 3.** Suppose GCH holds. Let \( \theta \geq \mu \) be regular cardinals. Then the following sets are equal modulo the nonstationary ideal on \( \varphi_\mu^+(H_\theta) \):

- \( \text{IA}_\mu \cap \varphi_\mu^+(H_\theta) \);
- The set of \( N \in \varphi_\mu^+(H_\theta) \) such that \( <^\mu N \subset N \).
If $H$ is a countable transitive model and $P \in H$ is a poset, then one can use a diagonalization argument to build $(H, P)$-generic filters. If $H$ is uncountable then the diagonalization argument still works, provided that $H$ is internally approachable and believes $P$ is sufficiently closed and well-orderable:

**Lemma 4.** Suppose $H$ is a transitive ZF$^-$ model, $H \in IA_\mu$, $Q \in H$ is a poset, $H \models "Q is <\mu$-closed", and for every $N \in H$ if $|N|^V < \mu$ then $|N|^H < \mu$. Suppose $H$ has some wellorder of $Q$. Then for every $p \in Q$ there exists a $g \subset Q$ such that $p \in g$ and $g$ is $(H, Q)$-generic.

**Proof.** Fix some $p \in Q$. Let $\Delta \in H$ be a wellorder of $Q$, and let $\langle N_i \mid i < \mu \rangle$ witness that $H \in IA_\mu$. Note that without loss of generality we can assume that:

1. $p \in N_0$ and $\Delta \in N_0$;
2. for all $\ell < \mu$, $\langle N_i \mid i \leq \ell \rangle \in N_{\ell+1}$
3. $N_i < H$ for all $i < \mu$

For a set $N$ and a condition $r$, let us call $r$ a weakly $N$-generic condition if for every $D \in N$ which is dense, there is some $s \supseteq r$ such that $s \in D$; note we do not require that $s \in D \cap N$, so the upward closure of $r$ might fail to be an $(N, Q)$-generic filter in the usual sense.

Recursively construct a descending sequence $\langle p_i \mid i < \mu \rangle$ as follows:

- $p_0 := p$;
- Assuming $p_i$ has been defined and is an element of $N_{i+1}$, let $p_{i+1}$ be the $\Delta$-least weakly $N_{i+1}$-generic condition below $p_i$. Such a condition exists because $\{|[D \in N_{i+1} : D \subseteq Q] \text{ is dense below } p_i]|^H \leq |N_{i+1}|^H < \mu$ and $H \models "Q is <\mu$-closed.");
- Assuming $j < \mu$ is a limit ordinal and $\langle p_i \mid i < j \rangle$ has been defined, let $p_j$ be the $\Delta$-least lower bound of $\langle p_i \mid i < j \rangle$ in $Q$. Such a condition exists because $\langle p_i \mid i < j \rangle \in H$ follows from $\langle N_i \mid i < j \rangle \in H$ (and $H \models \exists Q$ is $<\mu$-closed).

A straightforward inductive proof shows that $p_j \in N_{j+1}$ for all $j$, that $\langle p_i \mid i < \ell \rangle \in N_{\ell+1}$ for all limit ordinals $\ell$, and that whenever $j$ is a successor ordinal then $p_j$ is a weakly generic condition for $N_j$. It follows (since $H = \bigcup_{i < \mu} N_i$) that $\langle p_i \mid i < \mu \rangle$ generates an $H$-generic filter. \hfill \Box

The role of $IA_\mu$ in the theory of generic embeddings (with critical point $\geq \omega_2$) has been explored in detail in Foreman-Magidor [FM95]. It also plays a role in the current paper.

**Definition 6.** Let $\mu$ be a regular cardinal and suppose $\kappa = \mu^+$. We say that $\kappa$ is generically strong on $IA_\mu$ if and only if for every $\lambda > \kappa$ and $A \in V$ there is an embedding $j : V \to M$ as in Definition 4, which in addition has the following properties in $V^\mathcal{P}$:

1. $M^{<\mu} \subseteq M$; and
2. $M \models H^V_\kappa \in IA_\mu$.

We say that $\kappa$ is ideally strong on $IA_\mu$ if and only if for every cardinal $\lambda \geq \kappa$ there is a precipitous $(\kappa, \lambda^+)$-ideal extender $F$, whose associated generic ultrapower $j : V \to M$ satisfies requirements (1) and (2).

**Remark 1.** Of course, if $\kappa$ is ideally strong on $IA_\mu$ then $\kappa$ is generically strong on $IA_\mu$. 
Remark 2. Notice that if $\kappa = \omega_1$ and $\mu = \omega$, then “$\kappa$ is generically strong on $\text{IA}_\mu$” is equivalent to “$\kappa$ is generically strong”, and “$\kappa$ is ideally strong on $\text{IA}_\mu$” is equivalent to “$\kappa$ is ideally strong”. This is simply because all the relevant models are closed under finite (i.e. $< \mu$) sequences.

Now we are ready to state a theorem generalizing Theorem 2 to include the case $\kappa > \omega_1$.

Theorem 3. Suppose $\mu$ is a regular cardinal. If $\kappa = \mu^+$ is generically strong on $\text{IA}_\mu$ then this is preserved by adding any number of Cohen subsets of $\kappa$. Moreover, if $\kappa$ is ideally strong on $\text{IA}_\mu$, then this is preserved by adding any number of Cohen subsets of $\kappa$.

The remainder of the paper is a proof of Theorem 3. Note that by Remark 2, Theorem 3 implies Theorem 2.

3. Lifting generic embeddings using surgery

Theorem 4. Suppose that $\mu \geq \omega$ is a regular cardinal and $\kappa = \mu^+$. Let $\theta$ and $\eta$ be cardinals with $\eta > \theta^+$, $\eta \geq \theta^c$, $\eta^+ = \eta$ and $\text{cf}(\eta) > \kappa$. Further suppose $E$ is a poset such that

\[ \models_{\mathcal{V}} \text{there is a } (\kappa, \eta^+)-\text{extender } \hat{E} \text{ over the ground model } \mathcal{V} \text{ with well-founded ultrapower } j_{\hat{E}} : \mathcal{V} \rightarrow \hat{\mathcal{M}} \text{ such that } \text{crit}(j_{\hat{E}}) = \kappa, j_{\hat{E}}(\kappa) > \eta^+, H_{\eta^+}^{\hat{\mathcal{M}}} \subseteq M, M^{<\mu} \subseteq M \text{ and } M \models H_{\eta^+}^{\mathcal{V}} \in \text{IA}_\mu. \]

Then given any condition $p \in \text{Add}(\kappa, \eta)^{\mathcal{V}}$, the poset $E$ also forces that there is an ultrafilter $g$ which is $(\mathcal{V}, \text{Add}(\kappa, \eta))$-generic with $p \in g$, and that the ultrapower map $j_{\hat{E}}$ can be lifted to $j_{\hat{E}}^{\mathcal{V}} : \mathcal{V}[g] \rightarrow M[j_{\hat{E}}^{\mathcal{V}}(g)]$ where $H_{\eta^+}^{\mathcal{V}[g]} \subseteq M[j_{\hat{E}}^{\mathcal{V}}(g)]$.

Proof. Let $\kappa \geq \omega_1^\mathcal{V}$ and suppose $\theta$ and $\eta$ are cardinals with $\eta > \theta^+$, $\eta \geq \theta^c$, $\eta^+ = \eta$ and $\text{cf}(\eta)^{\mathcal{V}} > \kappa$. Suppose there is a partial order $E$ such that if $G$ is $(\mathcal{V}, E)$-generic then there is a $V-(\kappa, \eta^+)$-extender $E \in \mathcal{V}' := \mathcal{V}[G]$ such that the ultrapower $j : \mathcal{V} \rightarrow M = \text{Ult}(V, E) \subseteq \mathcal{V}'$ is well-founded and has the following properties.

1. $\eta^+ < j(\kappa)$ and the critical point of $j$ is $\kappa$.
2. $H_{\eta^+}^{\mathcal{V}} \subseteq M$
3. $M^{<\mu} \subseteq M$
4. $M = \{ j(f)(a) \mid a \in [\eta^+]^{<\omega} \land f : [\kappa]^|a| \rightarrow \mathcal{V} \land f \in \mathcal{V} \}$
5. $M \models H_{\eta^+}^M \in \text{IA}_\mu$

Let us define $\mathbb{P} = \text{Add}(\kappa, \eta)^{\mathcal{V}}$ and fix a condition $p \in \mathbb{P}$. Our goal is to show that in $\mathcal{V}'$, there is a $(\mathcal{V}, \mathbb{P})$-generic filter $g$, with $p \in g$, such that the embedding $j$ can be lifted to domain $\mathcal{V}[g]$ in such a way that the lifted embedding $j^* : \mathcal{V}[g] \rightarrow M[j^*(g)]$ is a class of $\mathcal{V}'$ and $H_{\eta^+}^{\mathcal{V}[g]} \subseteq M[j^*(g)]$. To accomplish this we will build a $(\mathcal{V}, \mathbb{P})$-generic filter $g$ in $M$, an $(M, j(\mathbb{P}))$-generic filter $H$ in $\mathcal{V}[G]$ and then we will modify $H$ to obtain another $(M, j(\mathbb{P}))$-generic filter $H^*$ with $j^* g \subseteq H^*$.

Lemma 5. If $W \in ([H_{\eta^+}^M]^{<\kappa})^{\mathcal{V}}$ then $j \upharpoonright W \in M$.

Proof. Suppose $W \in ([H_{\eta^+}^M]^{<\kappa})^{\mathcal{V}}$. First notice that $W \in M$ since $([H_{\eta^+}^M]^{<\kappa})^{\mathcal{V}} \subseteq H_{\eta^+}^{\mathcal{V}} \subseteq M$. Furthermore, $j[W] \in M$ because the elements of $W$ can be enumerated in a $\kappa$ sequence $\hat{W}$, and then $j[W]$ is obtained as the range of the sequence $j(\hat{W}) \upharpoonright$.
κ ∈ M. By elementarity (W, ε) ≃ (j[W], ε), and thus the Mostowski collapse of (W, ε) equals that of (j[W], ε). Thus, in M, there are isomorphisms

\[(W, ε) \xrightarrow{π_0} \bar{W}, ε \xrightarrow{π_1} (j[W], ε)\]

where π₀ is the Mostowski collapse of W and π₁ is the inverse of the Mostowski collapse of j[W]. It follows that j \restriction W = π₁ ◦ π₀ ∈ M. □

![Diagram]

Claim 1. There is a (V, Add(κ, θ))-generic g ∈ M such that p ∈ g.

Proof of Claim 1. Working in M, we would like to apply Lemma 4 to the structure H = H_m^V and the poset Q = Add(κ, θ) ∈ H. We will show that our assumption M ⊨ H_m^V ∈ IA_µ implies that all of the other hypotheses of Lemma 4 hold. First let us show that, working in M, for N ∈ H, if |N|^M < µ then |N|^H < µ. Assume not, then for some N ∈ H we have |N|^M < µ and |N|^V = |N|^H ≥ µ. Let z ∈ P(N)^V with |z|^V = µ, then by elementarily, |j(z)|^M = µ. From Lemma 5, it follows that j \restriction z ∈ M and since \(|z|^V < \text{crit}(j)| = κ we have j(z) = j^n(z) ∈ M. Thus M ⊨ µ = |j(z)| = |j^n(z)| = |z| ≤ |N|, a contradiction. It is easy to verify that all of the other hypotheses of Lemma 4 are met, and hence we may apply Lemma 4 within M to obtain a (H, Add(κ, θ))-generic filter g with p ∈ g. Since H = H_m^V, it follows that g is also (V, Add(κ, θ))-generic. □

Claim 2. There is an (M, j(Add(κ, θ)))-generic filter h ∈ V' = V[G].

Proof. All of M’s dense subsets of j(Add(κ, θ)) are elements of j(H^V_η), so it suffices to prove that, in V[G], the model j(H^V_η) and the poset j(Add(κ, θ)) ⊆ j(H^V_η) satisfy all the requirements of Lemma 4, which will yield the desired generic object H. The proof uses the somewhat ad-hoc requirement (1) of Definition 6 (though this requirement is redundant when κ = ω₁, by Remark 2). We do not know if this requirement can be removed, but suspect it is possible.

Now clearly j(H^V_η) believes that j(Add(κ, θ)) is <j(κ)-closed, and so in particular <µ = j(µ)-closed. To show that j(H^V_η) and j(Add(κ, θ)) ⊆ H satisfy the remaining requirements of Lemma 4 (from the point of view of V[G]), it suffices to prove:

1. V[G] ⊨ |j(H^V_η)| ≤ µ; and
2. V[G] ⊨ j(H^V_η) is closed under < µ sequences.

To see item (1): since

\[M = \{ j(f)(a) \mid a \in [η^+]^{<ω} \land f : [κ]^{|a|} \to V \land f \in V \}\]

it follows that every element of j(H^V_η) is of the form j(f)(a) where a ∈ [η^+]^{<ω} and f : [κ]^{|a|} → H^V_η is some function in V. Thus, j(H^V_η)^{V[G]} ≤ (η^+)^V \cdot j(κ^{|a|})^{|a|} = η^+V; the last equality follows from our assumptions on cardinal arithmetic. Since
\( \eta^+ < j(\kappa) \) by assumption, then \( \eta^+ < j(\kappa) = j(\mu^+) = \mu^+ M \leq \mu^+[G] \). This completes the proof of item (1).

To see item (2): \( V \) believes that \( H^V_\eta \) is closed under \( < \mu \) sequences, so by elementarity \( M \) believes that \( j(H^V_\eta) \) is closed under \( < \mu \) sequences. The ad-hoc assumption (1) of Definition 6 says that \( M \) is closed under \( < \mu \) sequences in \( V[G] \). Thus item (2) trivially follows. \( \square \)

We will modify the \((M, j(\mathbb{P}))\)-generic \( h \) given by Claim 2, to obtain \( h^* \) such that \( h^* \) is still \((M, j(\mathbb{P}))\)-generic and \( j^* g \subseteq h^* \).

Working in \( V[G] \), we may define a function \( Q := \bigcup j^* g \) where \( \text{dom}(Q) = j[\theta] \times \kappa \). Given \( p \in h \subseteq \text{Add}(j(\kappa), j(\theta))^M \) we define \( p^* \) to be the function with \( \text{dom}(p^*) = \text{dom}(p) \) such that \( p^*(\alpha, \beta) = p(\alpha, \beta) \) if \( (\alpha, \beta) \in \text{dom}(p) - \text{dom}(Q) \) and \( p^*(\alpha, \beta) = Q(\alpha, \beta) \) if \( (\alpha, \beta) \in \text{dom}(Q) \cap \text{dom}(p) \). Observe that the subset of \( \text{dom}(p) \) on which modifications are made is contained in the range of \( j \):

\[ \Delta_p := \{ (\alpha, \beta) \in \text{dom}(p) \mid p^*(\alpha, \beta) \neq p(\alpha, \beta) \} \subseteq j[\theta] \times \kappa \leq \text{ran}(j) \].

Now we let \( h^* := \{ p^* \mid p \in h \} \in V[G] \) and argue that \( h^* \) is an \((M, j(\mathbb{P}))\)-generic filter.

**Lemma 6.** If \( p \in h \subseteq j(\mathbb{P}) \) then \( h^* \in j(\mathbb{P}) \).

**Proof.** It follows that \( p = j(f)(a) \) where \( a \in [\eta^+]^{< \omega} \) and \( f : [\kappa]^{|a|} \to \mathbb{P} \) is a function in \( V \). Let \( W := \bigcup \{ \text{dom}(q) \mid q \in \text{ran}(f) \} \subseteq \theta \times \kappa \). Clearly \( W \in (\mathbb{H}_\kappa)^{\mathbb{H}_\kappa} \) and thus \( j \upharpoonright W \in M \) by Lemma 5. Furthermore, we have \( \text{dom}(p) \subseteq j(W) \), which implies \( \text{dom}(p) \cap \text{ran}(j) \subseteq j[W] \). Observe that any modifications made in obtaining \( p^* \) from \( p \) must have been made over points in \( j[W] \); in other words, \( \Delta_p \subseteq j[W] \). Working in \( M \), we can use \( j \upharpoonright W \) and \( g \) to define a function \( Q_p \) with \( \text{dom}(Q_p) = j[W] \). Since \( p^* \) can be obtained by comparing \( p \) and \( Q_p \), it follows that \( p^* \) is in \( M \), and thus is a condition in \( j(\mathbb{P}) \). \( \square \)

**Lemma 7.** \( h^* \) is an \((M, j(\mathbb{P}))\)-generic filter.

**Proof.** Suppose \( A \) is a maximal antichain of \( j(\mathbb{P}) \) in \( M \) and let \( d := \bigcup \{ \text{dom}(r) \mid r \in A \} \). Since \( d \in M \) there is an \( a \in [\eta^+]^{< \omega} \) and a function \( f : \kappa \to [\theta \times \kappa]^{< \kappa} \) in \( V \) such that \( d = j(f)(a) \). It follows that \( W := \bigcup \text{ran}(f) \in (\mathbb{H}_\kappa)^{\mathbb{H}_\kappa} \) and hence \( j \upharpoonright W \in M \) by Lemma 5. Since \( d \in \text{ran}(j(f)) \) it follows that \( d \subseteq j(W) \) and hence \( d \cap \text{ran}(j) \subseteq j[W] \). We also have \( j(W) \subseteq M \). Working in \( M \), we can define a function \( Q_A \) with \( \text{dom}(Q_A) = j[W] \) by letting \( Q_A(\alpha, \beta) = i \) if and only if \( g(j^{-1}(\alpha, \beta)) = i \). Notice that \( Q_A = Q \upharpoonright j[W] \) and \( j[W] \subseteq M \). By the \(<j(\kappa)\)-distribution of \( j(\mathbb{P}) \) in \( M \) we have that \( h \upharpoonright j[W] \in M \), and since \( Q_A \in M \), there is an automorphism \( \pi_A \in M \) with \( j(W) \) which flips precisely those bits at which \( Q_A \) and \( h \upharpoonright j[W] \) disagree. Since \( \pi_A^{-1}[A] \in M \) is a maximal antichain in \( j(\mathbb{P}) \) and \( h \) is \((M, j(\mathbb{P}))\)-generic, it follows that there is a condition \( q \in \pi_A^{-1}[A] \cap h \) and by applying \( \pi_A \) we see that \( q^* = \pi_A(q) \in A \cap h^* \). \( \square \)

Since \( j^* g \subseteq h^* \), it follows that the embedding \( j : V \to M \) lifts to \( j^* : V[g] \to M[j^*(g)] \) where \( j^*(g) = h^* \). Since \( g \in M \) and \( H^V_{\eta^+} \subseteq M \) we have \( H^V_{\eta^+} = H^V_{\eta^+}[g] \subseteq M \leq M[H^*] \). This concludes the proof of Theorem 4. \( \square \)

Theorem 4 already implies the portion of Theorem 3 which deals with preservation of generically strong cardinals:
Corollary 5. If $\kappa = \mu^+$ is generically strong on $\text{IA}_\mu$ then it remains generically strong after adding any number of Cohen subsets of $\kappa$.

The next section combines Theorem 4 with a generalization of Foreman’s Duality Theorem to get preservation of ideally strong cardinals.

4. IDEAL EXTENDERS AND DUALITY

This section generalizes portions of the “duality theory” of Foreman [For13] to the context of extender embeddings.

4.1. Derived ideal extenders. The following definition and lemma are simple generalizations of the discussion in Section 3.2 of [For10].

Definition 7. Suppose $Q$ is a poset, $\kappa < \lambda$ are ordinals, and $\dot{E}$ is a $Q$-name such that

$$\forces_{Q} \dot{E} \text{ is a normal } (\kappa, \lambda)\text{-extender over } V$$

Then $F(\dot{E})$, called the ideal extender derived from $\dot{E}$, is defined by: $(a, S) \in F(\dot{E})$ iff:

- $a \in [\lambda]^{<\omega}$;
- $S \subseteq [\kappa]^{|a|}$; and
- $\langle \dot{a}, \dot{S} \rangle \in \dot{E}_{\text{ro}(Q)} = 1$.

The name “ideal extender” will be justified in Lemma 8 below.

Note we do not require that $Q$ forces $\text{Ult}(V, \dot{E})$ to be well-founded in Definition 7.

Lemma 8. Suppose $F(\dot{E})$ is the ideal extender derived from $\dot{E}$ as in Definition 7. Then:

1. $F(\dot{E})$ is indeed a $(\kappa, \lambda)$-ideal extender as in Definition 2; let $E_{F(\dot{E})}$ be the associated forcing as in Definition 1.
2. The map $\iota : E_{F(\dot{E})} \to \text{ro}(Q)$

   defined by

   $$p \mapsto [\check{F}^p \subseteq \check{E}]_{\text{ro}(Q)} = [\forall a \in \text{supp}(\check{p}) \forall S \in \check{F}^p \text{ } S \subseteq \check{E}_a]_{\text{ro}(Q)}$$

   is $\leq$ and $\perp$ preserving (but not necessarily regular).
3. If $H$ is $(V, Q)$-generic then $\bigcup \iota^{-1}[H] = E_H$.

Proof. (1) The fact that $F(\dot{E})$ is a $(\kappa, \lambda)$-ideal extender, as in Definition 2, follows from the fact that the relevant properties hold for $\dot{E}$ with boolean-value one in $V_{\text{ro}(Q)}$. It is easy to see that for each $a \in [\lambda]^{<\omega}$ the set $F(\dot{E})_a \subseteq [\kappa]^{|a|}$ is a filter since $[\dot{E}_a]$ is an ultrafilter over $V_{\text{ro}(Q)} = 1$. To see that $F(\dot{E})$ is compatible we observe that the coherence property of $(\kappa, \lambda)$-extenders [Kan03, (ii) on page 154] holds for $\dot{E}$ with boolean value one in $V_{\text{ro}(Q)}$, and this immediately implies that if $a \subseteq b \in \text{supp}(F)$ then $x \in F(\dot{E})_a \iff x_{a,b} \in F(\dot{E})_b$. The potential normality of $F(\dot{E})$ follows from the assumption that $1_Q$ forces $\dot{E}$ to be a normal extender; we refer the reader to the proof at the bottom of page 66 of Claverie [Cla10] for the details. Thus $F(\dot{E})$ is a $(\kappa, \lambda)$-ideal extender, as in Definition 2.
(2) Suppose \( p \leq E_F(q) \). This means that \( \text{supp}(q) \subseteq \text{supp}(p) \) and for every \( a \in \text{supp}(q) \) we have \( F_a^q \subseteq F_a^p \). Thus for \( a \in \text{supp}(q) \) we have \( [F_a^p \subseteq \hat{E}_a] \leq \rho_\mathbb{Q}(q) \) \( [F_a^q \subseteq \hat{E}_a] \). Thus \( \iota(p) = \bigland_{a \in \text{supp}(p)} [F_a^p \subseteq \hat{E}_a] \leq \rho_\mathbb{Q} \bigland_{a \in \text{supp}(q)} [F_a^q \subseteq \hat{E}_a] = \iota(q) \).

Suppose \( p, q \in E_F(\hat{\mathcal{E}}) \) where \( p = \text{span}(F(\hat{E}), (a, x)) \) and \( q = \text{span}(F(\hat{E}), (b, y)) \) and that \( \iota(p) \) and \( \iota(q) \) are compatible in \( \rho_\mathbb{Q}(q) \), say \( r \in \mathbb{Q} \) with \( r \leq \iota(p), \iota(q) \). Then forcing with \( \mathbb{Q} \) below \( r \) yields an extension in which there is a \((\kappa, \lambda)\)-extender \( F \) such that \( F_p \cup F_q \subseteq F \). Since \( x \in F_a \) and \( y \in F_b \), it follows that \( x_{a,a \cup b} \cap y_{b,a \cup b} \in F_{a \cup b} \) and hence \( x_{a,a \cup b} \cap y_{b,a \cup b} \in F(\hat{E})_{a \cup b} \). This implies that \( s := \text{span}(F(\hat{E}), (a \cup b, x_{a,a \cup b} \cap y_{b,a \cup b})) = F_{a \cup b} \) is a condition in \( E_F(\hat{\mathcal{E}}) \). Since \( x_{a,a \cup b} \in F_{a \cup b} \) is compatible \((\kappa, \lambda)\)-system of filters, it follows that \( x \in F_a^s \). Similarly, \( y \in F_b^s \). This implies that \( s \leq p, q \).

(3) To see that \( \bigcup \iota^{-1}(H) = \hat{\mathcal{E}}_H \), just note that \( \bigcup \iota^{-1}[H] = \bigcup \{ p \in E_F(\hat{\mathcal{E}}) : \iota(p) \in H \} = \{ p \in E_F(\hat{\mathcal{E}}) : F_p \subseteq \hat{\mathcal{E}}_H \} = \hat{\mathcal{E}}_H \). □

4.2. Duality for derived ideal extenders. In certain situations, the map \( \iota \) from Lemma 8 is a regular or even a dense embedding. In such situations we can nicely characterize the poset \( E_F(\hat{\mathcal{E}}) \) as a subalgebra of \( \mathbb{Q} \); and moreover if \( \text{Ult}(V, \hat{E}) \) was forced by \( \mathbb{Q} \) to be well-founded, then \( F(\hat{E}) \) will be a precipitous ideal extender.

In the following theorem, the poset which plays the role of the \( \mathbb{Q} \) from Definition 7 and Lemma 8 above is itself a quotient of a forcing associated with an ideal extender.

**Theorem 6.** Suppose \( F \) is a \((\kappa, \lambda)\)-ideal extender as in Definition 2 and that \( E_F \) is its associated forcing as in Definition 1. Suppose \( \mathbb{P} \) is a poset and \( \tau \) is a \( E_F \)-name such that \( E_F \) forces the following statement (here \( \hat{G}_F \) is the \( E_F \)-name for the \( E_F \)-generic object; so \( \bigcup \hat{G}_F \) is forced to be a \((\kappa, \lambda)\)-extender over \( V \)):

- \( \tau \) is \((V, \mathbb{P})\)-generic, and the ultrapower map \( j_{\hat{G}_F} \) can be lifted to domain \( V[\tau] \).

Then:

1. The map \( e : \mathbb{P} \to \text{ro}(E_F) \) defined by \( e(p) = \text{span}(F(\hat{E}), (a, x)) \) is a complete embedding (this is an abstract forcing fact);

2. Suppose \( g \) is \((V, \mathbb{P})\)-generic. In \( V[g] \) let:

   - \( \check{j} \) be the \( E_F \)-name for the lifting of \( j_{\hat{G}_F} \) to domain \( V[g] \) (notice that \( \check{\tau}_{\hat{G}_F} = g \) so the lifting is forced to exist by assumption);

   - \( \check{E} \) be the \( E_F \)-name for the \((\kappa, \lambda)\)-extender over \( V[g] \) derived from \( \check{j} \);

   - \( F(\check{E}) \) be the ideal extender derived from the \( E_F \)-name \( \check{E} \), as in Definition 7.

   - \( E_F(\check{E}) \) be the forcing in \( V[g] \) associated with \( F(\check{E}) \) as in Definition 1.

   Then in \( V[g] \) the map \( \iota : E_F(\check{E}) \to \text{ro}(E_F) \)

   \[ \iota \left( \frac{E_F}{E_F(\check{E})} \right) \]

   defined as in Lemma 8 is a dense embedding.

3. If the \((\kappa, \lambda)\)-ideal extender \( F \) was precipitous in \( V \) then \( F(\check{E}) \) is precipitous in \( V[g] \).

---

3We're assuming this lifting exists already in \( V^K \), not in some further forcing extension.
Proof: (1) is a well-known fact.

(2) A minor technical issue here is that although $E_F$ is separative by Lemma 2—and thus $E_F$ can directly be viewed as a dense subset of $\text{ro}(E_F)$—the quotient $Q := \frac{E_F}{\text{ro}(E_F)}$ is not separative; so $Q$ is technically not a subset of $\text{ro}(Q)$. Let $Q/\sim$ denote the separative quotient of $Q$. Since $Q/\sim$ is a dense subset of $\text{ro}(Q)$, it suffices to show that given any $[q]_\sim \in Q/\sim$ there is an $r \in E_{F(E)}$ with $\iota(r) \leq [q]_\sim$. Suppose $[q]_\sim \in Q/\sim$. Then $q = F^q = \text{span}\{F, (a, x)\}$ for some fixed $a \in [\lambda]^{<\omega}$ and some $x \in (F_a)^+$, and $q$ is compatible with every element of $e[q]$. Notice that $x \in (F(E)_0)^+$ since (a) $q$ is compatible with every element of $e[g]$ (b) $\emptyset \neq q \forces x \in \hat{E}$ and (c) the ideal dual to the filter $F(E)_0 = \{S \subseteq \kappa : [(\hat{a}, \hat{S}) \in \hat{E}]_{\text{ro}(Q)} = 1\}$ is $\{S \subseteq \kappa : [(\hat{a}, \hat{S}) \in \hat{E}] = 0\}$. Thus $r := \text{span}\{F(E)_0, (a, x)\}$ is a condition in $E_{F(E)}$ and by definition of $\iota$ we have $\iota(r) = [\forall b \in \text{supp}(r) \ F_b^r \subseteq \hat{F}], \text{ro}(Q)$. We will argue that $\iota(r) \leq [q]_\sim$. Notice that since $F \subseteq F(E)$, it follows that $q = \text{span}\{F, (a, x)\} \subseteq \text{span}\{F(E)_0, (a, x))\} = r$. Note that $q$ and $r$ are conditions in different posets, so this does not imply that $r$ extends $q$. However, since for all $b \in \text{supp}(q)$ we have $F_b^q \subseteq F_b^r$, it follows that if $H$ is $(V[g], \text{ro}(Q))$-generic and $\iota(r) \in H$ then $[q]_\sim \in H$. Since $\text{ro}(Q)$ is separative, it follows that $\iota(r) \leq [q]_\sim$ in $\text{ro}(Q)$.

(3) Assume the $(\kappa, \lambda)$-ideal extender $F$ is precipitous in $V$. Let $g$ be $(V, \mathbb{P})$-generic. By part 3 of Lemma 8, together with part (2) of the current theorem, generic ultrapowers of $V[g]$ by $E_{F(E)}$ correspond exactly to liftings of generic ultrapowers of $V$ by $F$; since these are wellfounded (by precipitousness of $F$) then the generic ultrapowers of $V[g]$ by $E_{F(E)}$ are also wellfounded. □

5. Proof of main theorem

In this section we combine the results of Section 3 and Section 4 to finish proving Theorem 3. Note that we only have to deal with preservation of ideally strong cardinals; preservation of generically strong cardinals was taken care of by Corollary 5.

Proof of Theorem 3. Suppose $\kappa = \mu^+$ is ideally strong and $\mu$ is a regular cardinal. Let $\theta > \kappa$ be a cardinal and let $g'$ be $(V, \text{Add}(\kappa, \theta))$-generic. Suppose $\kappa$ is not ideally strong in $V[g']$. Thus, in $V[g']$, for some cardinal $\eta > \kappa$ meeting all the hypotheses of Theorem 4, there is no $(\kappa, \eta^+)$-ideal extender witnessing that $\kappa$ is ideally $H_{\eta^+}$-strong. Let $p_0 \in g'$ force this. Let $F \subseteq V$ be a precipitous $(\kappa, \eta^+)$ ideal extender witnessing that $\kappa$ is generically $H_{\eta^+}$-strong in $V$. Let $G_F$ be a $(V, E_F)$-generic $V$-($\kappa, \eta^+$) extender so that the well-founded embedding $j_{G_F} : V \rightarrow M = \text{Ult}(V, G_F)$ satisfies $H_{\eta^+}^V \subseteq M$. By Theorem 4, it follows that we can find a $(V, \text{Add}(\kappa, \theta))$-generic filter $g$ with $p_0 \in g$ and $g \in M$, such that the embedding $j_{G_F}$ lifts to $j_{G_F}^* : V[g] \rightarrow M[j_{G_F}^*(g)]$, where $j_{G_F}^*(g) \in V[G_F]$ is some $(M, j(\text{Add}(\kappa, \theta)))$-generic filter. Thus there is an $E_F$-name $\tau$ with $\tau_{G_F} = g$ satisfying the hypothesis of Theorem 6 with $\mathbb{P} = \text{Add}(\kappa, \theta)^V$. Not only does the hypothesis to Theorem 6 hold, 4Recall that by Lemma 2, we know $E_F$ is separative and thus can be viewed as a dense subset of $\text{ro}(E_F)$, the target of the map $e$.
but Theorem 4 also shows that
\[ \mathcal{P}^V \] if the original ultrapower map \( j_{\mathcal{G}_F} : V \rightarrow M = \text{Ult}(V, \mathcal{G}_F) \)
satisfies \( H^V_q \subseteq M \), then the embedding lifted to \( j^*_{\mathcal{G}_F} : V[\tau] \rightarrow M[j^*_{\mathcal{G}_F}(\tau)] \) satisfies \( H^V_{q[\tau]} \subseteq M[j^*_{\mathcal{G}_F}(\tau)] \).

Applying Theorem 6, the embedding \( e : \mathbb{P} \rightarrow \text{ro}(\mathcal{E}_F) \) defined by \( e(p) = [\bar{p} \in \tau] \) is complete. As in the conclusion of Theorem 6, in \( V[g] \) we have the following objects: \( \tilde{j} \) is the \( E_F/e[g] \)-name \( j^*_{\mathcal{G}_F}/e[g] \) for the lift of \( j_{\mathcal{G}_F} \), \( E \) is the \( E_F/e[g] \)-name for the \( (\kappa, \eta^+)-\)extender over \( V[g] \) derived from \( \tilde{j} \), \( F(E) \) is the \( (\kappa, \eta^+)-\)ideal extender in \( V[g] \) derived from \( E, \mathcal{E}_F(E) \) is the associated forcing and \( \iota : \mathcal{E}_F(E) \rightarrow \text{ro}(\mathcal{E}_F(E)) \) is the natural dense embedding. Now if \( H_{F(E)} \) is \( (V[g], \mathcal{E}_F(E))-\)generic and \( H_F = \iota[H_{F(E)}] \) is the corresponding \( (V[g], \mathcal{E}_F(E))-\)generic filter, then by Lemma 8, \( \bigcup \iota^{-1}[H_F] = \hat{E}_{H_F} \)
is the \( V[g]-(\kappa, \eta)-\)extender derived from \( \tilde{j}_{H_F} \). Since \( \iota^{-1}[H_F] = H_{F(E)} \), we have \( \bigcup H_{F(E)} = \hat{E}_{H_F} \) and hence the ultrapower \( j_{H_{F(E)}} : V[g] \rightarrow \text{Ult}(V[g], \bigcup H_{F(E)}) \) by the \( V[g]-(\kappa, \eta)-\)extender \( \bigcup H_{F(E)} \) is well-founded. Furthermore, by (5.1), and the fact that \( j_{H_{F(E)}} = \tilde{j}_{H_F} = (j^*_{\mathcal{G}_F})_{H_F} \), it follows that \( H^V_{q[\tau]} \subseteq \text{Ult}(V[g], \bigcup H_{F(E)}) \). Thus, the \( (\kappa, \eta^+)-\)ideal extender \( H_{F(E)} \in V[g] \) witnesses the generic \( H_{\eta^+}\)-strength of \( \kappa \). This is a contradiction because \( p_0 \in g \) forces that such an ideal extender does not exist.

\[ \square \]

6. Questions

**Question 1.** Aside from adding Cohen subsets of \( \omega_1 \), what other posets will not destroy the ideal or generic strength of \( \omega_1 \)? Note that by the discussion in the introduction, only forcings which don’t collapse \( \omega_2 \) have any hope of (provably) preserving generic strength of \( \omega_1 \).

**Question 2.** Is the ideal strength of \( \omega_1 \) indestructible by Sacks(\( \omega_1, \theta \))?

**Question 3.** Regarding the main theorem in the case where \( \kappa \geq \omega_2 \): does the theorem still hold if we remove requirement (1) from Definition 6? That is, remove the requirement that \( ^{<\mu}M \subseteq M \)? The only place where this assumption was used was the proof of Claim 2 on page 8.

---

**References**

[Clai10] Benjamin Claverie. Ideals, ideal extenders and forcing axioms. PhD thesis, Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany, 2010.

[CM14] Brent Cody and Menachem Magidor. On supercompactness and the continuum function. *Ann. Pure Appl. Logic*, 165(2):620–630, 2014.

[CS12] Benjamin Claverie and Ralf Schindler. Woodin’s axiom (+), bounded forcing axioms, and precipitous ideals on \( \omega_1 \). *J. Symbolic Logic*, 77(2):475–498, 2012.

[Cum10] James Cummings. Iterated Forcing and Elementary Embeddings. In Akihiro Kanamori and Matthew Foreman, editors, *Handbook of Set Theory*, volume 2, chapter 14, pages 775–883. Springer, 2010.

[FM95] Matthew Foreman and Menachem Magidor. Large cardinals and definable counterexamples to the continuum hypothesis. *Ann. Pure Appl. Logic*, 76(1):47–97, 1995.

[For10] Matthew Foreman. Ideals and Generic Elementary Embeddings. In Akihiro Kanamori and Matthew Foreman, editors, *Handbook of Set Theory*, volume 2, chapter 13, pages 885–1147. Springer, 2010.
[For13] Matthew Foreman. Calculating quotient algebras of generic embeddings. *Israel Journal of Mathematics*, 193(1):309–341, 2013.

[GS89a] Moti Gitik and Saharon Shelah. On certain indestructibility of strong cardinals and a question of Hajnal. *Arch. Math. Logic*, 28(1):35–42, 1989.

[GS89b] Moti Gitik and Saharon Shelah. On certain indestructibility of strong cardinals and a question of Hajnal. *Archive of Mathematical Logic*, 28(1):35–42, 1989.

[Han00] Joel David Hamkins. The lottery preparation. *Annals of Pure and Applied Logic*, 101(2–3):103–146, 2000.

[Kak81] Yuzuru Kakuda. On a condition for Cohen extensions which preserve precipitous ideals. *J. Symbolic Logic*, 46(2):296–300, 1981.

[Kan03] Akihiro Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*. Springer, second edition, 2003.

[Lav78] Richard Laver. Making the supercompactness of $\kappa$ indestructible under $\kappa$-directed closed forcing. *Israel Journal of Mathematics*, 29(4):385–388, 1978.

[Mag80] Menachem Magidor. Precipitous ideals and $\Sigma^1_2$ sets. *Israel J. Math.*, 35(1-2):109–134, 1980.

(Brent Cody) Virginia Commonwealth University, Department of Mathematics and Applied Mathematics, 1015 Floyd Avenue, PO Box 842014, Richmond, Virginia 23284

E-mail address, B. Cody: bmcody@vcu.edu

URL: http://www.people.vcu.edu/~bmcody/

(Sean Cox) Virginia Commonwealth University, Department of Mathematics and Applied Mathematics, 1015 Floyd Avenue, PO Box 842014, Richmond, Virginia 23284

E-mail address, S. Cox: scox@vcu.edu

URL: http://skolemhull.wordpress.com/