Concise Derivation of Complex Bayesian Approximate Message Passing via Expectation Propagation

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Abstract—In this paper, we address the problem of recovering complex-valued signals from a set of complex-valued linear measurements. Approximate message passing (AMP) is one state-of-the-art algorithm to recover real-valued sparse signals. However, the extension of AMP to complex-valued case is nontrivial and no detailed and rigorous derivation has been explicitly presented. To fill this gap, we extend AMP to complex Bayesian approximate message passing (CB-AMP) using expectation propagation (EP). This novel perspective leads to a concise derivation of CB-AMP without sophisticated transformations between the complex domain and the real domain. In addition, we have derived state evolution equations to predict the reconstruction performance of CB-AMP. Simulation results are presented to demonstrate the efficiency of CB-AMP and state evolution.

Index Terms—Compressed sensing, complex-valued approximate message passing, expectation propagation, state evolution.

I. INTRODUCTION

Compressed sensing (CS) aims to undersample high-dimensional signals yet accurately reconstruct them by exploiting their structure [1, 2]. To this end, a plethora of methods have been proposed in the past years [3]. Among others, approximate message passing (AMP) [4] proposed by Donoho et al. is one state-of-the-art algorithm to recover sparse signals. As an efficient application of belief propagation [5, 6], AMP has found various applications in solving linear inverse problems. Moreover, AMP has been extended to Bayesian AMP (B-AMP) [7, 8] and general linear mixing problems [9–11]. However, most of the existing works focus on the case of real-valued signals and measurements, while in many applications, e.g., communication [10], magnetic resonance imaging [12], and radar imaging [13], etc., it is more convenient to represent signals in the complex-domain [14]. Though it can be transformed and processed in the real domain, it is beneficial to deal with complex-valued signals in a straightforward way since their real and imaginary components are often either both zero or both non-zero simultaneously [14, 15].

The extension of AMP to deal with complex-valued signals with complex-valued measurements has already been considered in [10, 15–18]. In [15], the authors proposed one kind of complex approximate message passing (CAMP) algorithm. However, the extension of AMP to CAMP is sophisticated. A more compact form of CAMP is proposed in [10, 16–18]. To the best of our knowledge, although such extensions have been considered, no detailed and rigorous derivation has been explicitly presented. In [19], we derived the original AMP algorithm from the expectation propagation (EP) [20, 21] perspective, which unveils the intrinsic connection between AMP and EP. Nevertheless, it only deals with real-valued sparse signals with Laplace prior, which limits its use in more general problems. In this paper we further extend it to complex Bayesian AMP (CB-AMP), i.e., complex-valued signal reconstruction with general known prior distribution. This novel perspective leads to a concise and natural extension from AMP to CB-AMP, without sophisticated transformations between the complex domain and the real domain. In addition, we have also derived state evolution equations to predict the reconstruction performance of CB-AMP. The superiority of CB-AMP is demonstrated via simulation results, which are consistent with the prediction results of state evolution equations.

II. DERIVATION OF CB-AMP VIA EP

A. System Model

Consider a complex-valued linear system of the form

\[ y = Ax + w, \tag{1} \]

where \( x \in \mathbb{C}^N \) is the unknown complex signal, \( A \in \mathbb{C}^{M \times N} \) is the measurement matrix, \( w \in \mathbb{C}^M \) is the additive complex Gaussian noise with zero mean and covariance matrix \( \sigma^2 I_M \), where \( I_M \) is the identity matrix of size \( M \). The complex Gaussian distribution of \( w \) is denoted by \( \mathcal{CN}(w; 0, \sigma^2 I_M) \). The prior distribution of signal \( x \) is supposed to be known and has a separable form

\[ p_0(x) = \prod_{i=1}^{N} p_0(x_i). \tag{2} \]

The goal is to estimate \( x \) from the noisy observations \( y \) given \( A \) and the statistical information of \( x \) and \( w \) using the minimum mean square error (MMSE) criterion. It is well
known that the MMSE estimate of $x_i$ is the posterior mean, i.e., \( \hat{x}_i = \int x p(x|y) \, dx \), where \( p(x|y) \) is the marginal distribution of the joint posterior distribution

\[
p(x|y) = \frac{p(y|x) p_0(x)}{p(y)} \propto \prod_{a=1}^{\mathcal{M}} p(y_a|x) \prod_{i=1}^{\mathcal{N}} p_0(x_i),
\]

where \( \propto \) denotes identity between two distributions up to a normalization constant. Under the statistical assumption of measurement noise \( w \), the conditional distribution of the \( a \)-th element of \( y, y_a \), given \( x \) can be explicitly represented as

\[
p(y_a|x) = \frac{1}{\sqrt{\pi \sigma^2}} \exp \left( -\frac{1}{\sigma^2} y_a - \sum_i A_{ai} x_i^2 \right).
\]

Message passing algorithms \([5], [6], [22]\) provide a family of efficient methods to (approximately) compute the marginals. The basic paradigm is well illustrated via factor graph \([6]\) which represents the statistical dependencies between random variables. The factorization in (3) can be encoded in a factor graph \( \mathcal{G} = (\mathcal{V}, \mathcal{F}, \mathcal{E}) \), where \( \mathcal{V} = \{i\} \) is the set of variable nodes, \( \mathcal{F} = \{a\} \) is the set of factor nodes and \( \mathcal{E} \) denotes the set of edges. In the sequel, we assume that the measurement matrix \( A \) is a homogenous matrix whose elements admit i.i.d. distribution with mean zero and variance \( \gamma \).

**B. Approximate inference using EP**

Given the factor graph representation, the marginals can be computed distributively via local message passing \([6], [20], [21]\). The projection of a particular distribution \( p \) into a distribution set \( \Phi \) is defined as \([21]\)

\[
\text{Proj}_\Phi[p] = \arg\min_{q \in \Phi} D(p||q),
\]

where \( D(p||q) \) denotes the Kullback-Leibler divergence.

Denote by \( m_{i \rightarrow a}^{t} (x_i) \) and \( m_{a \rightarrow i}^{t} (x_i) \) the message from variable node \( i \) to factor node \( a \) in the \( t \)-th iteration and the message in the opposite direction, respectively. Then, the message passing update rules of EP read \([20], [21]\)

\[
m_{i \rightarrow a}^{t+1} (x_i) \propto \frac{\text{Proj}_\Phi[p_0(x_i) \prod_b m_{b \rightarrow i}^{t} (x_i)]}{m_{i \rightarrow a}^{t} (x_i)},
\]

\[
m_{a \rightarrow i}^{t} (x_i) \propto \frac{1}{m_{a \rightarrow i}^{t} (x_i)} \text{Proj}_\Phi \left[ m_{i \rightarrow a}^{t} (x_i) \right] \times \int \prod_{j \neq i} m_{j \rightarrow a}^{t} (x_j) p(y_a|x). \quad (7)
\]

After projection, each message \( m_{j \rightarrow a}^{t} (x_j) \) from variable node \( j \) to factor node \( a \) is approximated as complex Gaussian density function \( \mathcal{CN}(x_j; \hat{x}_{j \rightarrow a}^{t}, \nu_{j \rightarrow a}^{t}) \), thus, under the product measure \( \prod_{j \neq i} m_{j \rightarrow a}^{t} (x_j) \), the random variables \( x_j, j \neq i \) are independent complex Gaussian random variables. Define \( Z_{ai} = \sum_{j \neq i} A_{aj} x_j \), so that \( Z_{ai} \) is a complex Gaussian random variable with mean and variance, respectively,

\[
Z_{ai}^{t+1} = \sum_{j \neq i} A_{aj} \hat{x}_{j \rightarrow a}^{t+1},
\]

\[
V_{a \rightarrow i}^{t+1} = \sum_{j \neq i} |A_{aj}|^2 \nu_{j \rightarrow a}^{t+1}.
\]

Then, we obtain

\[
\int \prod_{j \neq i} m_{j \rightarrow a}^{t} (x_j) p(y_a|x) \propto \mathcal{CN}(x_i; y_a - Z_{ai}^{t+1}, \sigma^2 + V_{a \rightarrow i}^{t+1}),
\]

which implies that \( x_i \) also admits complex Gaussian distribution. In this case, the projection operation in (7) reduces to identity operation, so that

\[
m_{a \rightarrow i}^{t+1} (x_i) \propto \mathcal{CN}(x_i; \hat{x}_{i \rightarrow a}^{t+1}, \nu_{i \rightarrow a}^{t+1}).
\]

Next we evaluate the message \( m_{i \rightarrow a}^{t+1} (x_i) \). The marginal posterior density estimate of \( x_i \), i.e., the marginal belief estimate \( m_{i \rightarrow a}^{t+1} (x_i) \), is defined as

\[
m_{i \rightarrow a}^{t+1} (x_i) = \text{Proj}_\Phi \left[ p_0(x_i) \prod_a m_{a \rightarrow i}^{t} (x_i) \right].
\]

According to the product rule of Gaussian functions \([23]\), we have

\[
\prod_a m_{a \rightarrow i}^{t} (x_i) \propto \mathcal{CN}(x_i; R_i^t, \Sigma_i^t),
\]

where

\[
\Sigma_i^t = \sum_a \frac{|A_{ai}|^2}{\sigma^2 + V_{a \rightarrow i}^{t+1}},
\]

\[
R_i^t = \sum_a A_{ai}^\ast (y_a - Z_{ai}^{t+1})^2 \sigma^2 + V_{a \rightarrow i}^{t+1}.
\]

and \((\cdot)^\ast\) denotes conjugate operation.

For notational brevity, we introduce a family of density functions

\[
p(x; R, \Sigma) \equiv \frac{p_0(x) \exp \left[ -\frac{1}{\Sigma} |x - R|^2 \right]}{\Sigma},
\]

where \( z(R, \Sigma) = \int p_0(x) \exp \left[ -\frac{1}{\Sigma} |x - R|^2 \right] dx \) is the normalization constant. As in \([8], [18]\), the corresponding mean and variance are denoted as

\[
f_a(R, \Sigma) = \int x p(x; R, \Sigma) \, dx,
\]

\[
f_c(R, \Sigma) = \int |x - f_a(R, \Sigma)|^2 p(x; R, \Sigma) \, dx.
\]

Combining \([13], [16], [17] and [18]\), we obtain the tentative approximation of the posterior mean and variance of \( x_i \) in the \( (t+1) \)-th iteration, which are denoted by \( f_a(R_i^t, \Sigma_i^t) \) and \( f_c(R_i^t, \Sigma_i^t) \), respectively. Then, using projection operation \([12]\) and moment matching, we project the posterior belief to the complex Gaussian distribution set, yielding

\[
m_{i \rightarrow a}^{t+1} (x_i) \propto \mathcal{CN}(x_i; \hat{x}_{i \rightarrow a}^{t+1}, \nu_{i \rightarrow a}^{t+1}),
\]

where

\[
\hat{x}_{i \rightarrow a}^{t+1} = f_a(R_i^t, \Sigma_i^t),
\]

\[
\nu_{i \rightarrow a}^{t+1} = f_c(R_i^t, \Sigma_i^t).
\]
\[ \nu_{i+1}^{t+1} = f_c \left( R_i^t, \Sigma_i^t \right). \]

According to (10) and (12), the message from variable node \( i \) to factor node \( a \) is evaluated by
\[ m_{i \rightarrow a}^{t+1}(x_i) \propto CN \left( x_i; \hat{x}_{i \rightarrow a}^{t+1}, \nu_{i \rightarrow a}^{t+1} \right), \quad (20) \]
where
\[ \frac{1}{\nu_{i \rightarrow a}^{t+1}} = 1 - \frac{|A_{ai}|^2}{\sigma^2 + V_{a \rightarrow i}^t}, \quad (21) \]
\[ \hat{x}_{i \rightarrow a}^{t+1} = \nu_{i \rightarrow a}^{t+1} \left( \hat{x}_{i \rightarrow a}^{t+1} - \frac{A_{ai}^* (y_a - Z_{a \rightarrow i}^t)}{\sigma^2 + V_{a \rightarrow i}^t} \right). \quad (22) \]

Now we have closed the message computation. However, about \( O(MN) \) messages need to be computed. In the sequel, we further reduce the number of messages per iteration to \( O(M+N) \) by neglecting the high order terms in large system limit.

C. Reducing the number of messages

Define
\[ Z_a^t = \sum_i A_{ai} \hat{x}_{i \rightarrow a}^t, \quad (23) \]
\[ V_a^t = \sum_i |A_{ai}|^2 \nu_{i \rightarrow a}^t, \quad (24) \]

Then, it can be easily seen that (18) and (19) can be rewritten as
\[ Z_{a \rightarrow i}^t = Z_a^t - A_{ai} \hat{x}_{i \rightarrow a}^t, \quad (25) \]
\[ V_{a \rightarrow i}^t = V_a^t - |A_{ai}|^2 \nu_{i \rightarrow a}^t. \quad (26) \]

Neglecting the high order term \( |A_{ai}|^2 \nu_{i \rightarrow a}^t \) in (26), we have
\[ V_{a \rightarrow i}^t \approx V_a^t, \quad (27) \]
which is independent of \( i \).

The simplification of \( Z_{a \rightarrow i}^t \) is not that trivial since we should be careful to keep the Onsager reaction term in approximating \( Z_{a \rightarrow i}^t \). From (21), neglecting the high order term \( |A_{ai}|^2 / (\sigma^2 + V_{a \rightarrow i}^t) \), we obtain
\[ \nu_{i \rightarrow a}^{t+1} \approx \nu_{i \rightarrow a}^t, \quad (28) \]
so that
\[ V_a^t \approx \sum_i |A_{ai}|^2 \nu_{i}^t. \quad (29) \]

Substituting (28) and (27) into (22), we have
\[ \hat{x}_{i \rightarrow a}^{t+1} \approx \hat{x}_{i \rightarrow a}^t - \nu_{i+1} A_{ai}^* (y_a - Z_{a \rightarrow i}^t) / (\sigma^2 + V_a^t). \quad (30) \]

Combining (25), (27), and (30), we have
\[ Z_{a \rightarrow i}^t = Z_a^t - A_{ai} \hat{x}_{i \rightarrow a}^t + \nu_{i}^t |A_{ai}|^2 (y_a - Z_{a \rightarrow i}^t) / (\sigma^2 + V_a^t). \quad (31) \]
which leads to a further approximation
\[ \hat{x}_{i \rightarrow a}^{t+1} \approx \hat{x}_{i \rightarrow a}^t - \nu_{i+1} A_{ai}^* (y_a - Z_{a \rightarrow i}^t) / (\sigma^2 + V_a^t), \]
\[ Z_a^t + A_{ai} \hat{x}_{i \rightarrow a}^t \approx |A_{ai}|^2 (y_a - Z_{a \rightarrow i}^t) / (\sigma^2 + V_a^t). \]

where the last step is approximated by neglecting the high order terms. Then, \( Z_a^t \) defined in (23) can be approximated as
\[ Z_a^t \approx \sum_i A_{ai} \hat{x}_{i}^t - \frac{(y_a - Z_{a \rightarrow i}^t)}{\sigma^2 + V_a^t}. \quad (33) \]

Substituting (27) into (14) leads to
\[ \Sigma_i^t \approx \left( \sum_a |A_{ai}|^2 \right)^{-1}. \quad (34) \]

Substituting (27) and (31) into (15), we have
\[ R_i^t \approx \sum_a A_{ai}^*\left( y_a - Z_{a \rightarrow i}^t \right) / \sigma^2 + V_a^t + \Sigma_i \sum_a |A_{ai}|^2 \hat{x}_{i}^t - \sum_a A_{ai}^*\left( y_a - Z_{a \rightarrow i}^t \right) / \sigma^2 + V_a^t \approx \hat{x}_{i}^t + \Sigma_i \sum_a A_{ai}^*\left( y_a - Z_{a \rightarrow i}^t \right) / \sigma^2 + V_a^t. \quad (35) \]

where in the last step we have neglected high order term and used the relationship (34).

At this step, we finally obtain the complex Bayesian approximate message passing (CB-AMP) as shown in algorithm (1) which is the same as that in (13) and (10) (note that some notational modification is needed to match (10)).

Algorithm 1 CB-AMP

1) Initialization: \( t = 1, \hat{x}_{i}^1 = \int x_i p_0(x_i) dx_i, \nu_{i}^1 = \int |x_i - \hat{x}_{i}^1| p_0(x_i) dx_i, i = 1, \ldots, N, V_a^0 = 1, Z_{a \rightarrow i}^0 = y_a, a = 1, \ldots, M \)
2) Factor node update: For \( a = 1, \ldots, M \)
\[ V_a^t = \sum_i |A_{ai}|^2 \nu_{i}^t, \]
\[ Z_a^t = \sum_i A_{ai} \hat{x}_{i}^t - \frac{V_a^t}{\sigma^2 + V_a^t} (y_a - Z_{a \rightarrow i}^{t-1}). \]
3) Variable node update: For \( i = 1, \ldots, N \)
\[ \Sigma_i^t = \sum_a |A_{ai}|^2 / (\sigma^2 + V_a^t)^{-1}, \]
\[ R_i^t = \hat{x}_{i}^t + \Sigma_i \sum_a A_{ai}^* (y_a - Z_{a \rightarrow i}^t) / (\sigma^2 + V_a^t), \]
\[ \nu_{i}^{t+1} = f \left( R_i^t, \Sigma_i^t \right), \]
\[ \hat{x}_{i}^{t+1} = f_c \left( R_i^t, \Sigma_i^t \right). \]
4) Set \( t \leftarrow t + 1 \) and proceed to step 2) until a predefined number of iterations or other termination conditions are satisfied.
As is shown in [8], the state evolution (or cavity method) uses a statistical analysis of the messages at iteration \( t \), in the large system limit, to derive their distributions at iteration \( t+1 \). Define
\[
V^t = \frac{1}{N} \sum_{i=1}^{N} v^t_i, \quad E^t = \frac{1}{N} \sum_{i=1}^{N} |\hat{x}_i^t - x_i|^2. \tag{37}
\]

where step \( \approx \) in (38) is due to the assumption that \( \sum |A_{ai}|^2 \nu^t_{j-a} \) is independent of \( \mu \) such that it is canceled out in the denominator and numerator; step \( \approx \) in (38) is attributed to the assumption that \( A_{ai} \) admits i.i.d. distribution with mean zero and variance \( \gamma \) so that \( \sum |A_{ai}|^2 \approx \sum \gamma = M\gamma \).

Denote by \( r^t_i = \sum_a A^*_a w_a + \sum_{j \neq i} A^*_a \sum_{j \neq i} A_{aj} (x_j - \hat{x}_{j-a}^t) \), then \( r^t_i \) is a complex random variable with respect to the distribution of the measurement matrix elements and the complex Gaussian noise \( w_a \sim CN(w_a; 0, \sigma^2) \). By central limit theorem, it can be verified that \( r^t_i \) is a complex Gaussian random variable with zero mean and variance \( M\gamma (\sigma^2 + \gamma NE^t) \). Thus, \( R^t_i \) can be reformulated as
\[
R^t_i = x_i + \sqrt{\frac{\sigma^2 + \gamma NE^t}{M\gamma}} z, \tag{39}
\]

where \( z \sim CN(z; 0, 1) \) is a complex Gaussian random variable with zero mean and unit variance.

Then, from (34), we obtain that
\[
\Sigma_i^t \approx \frac{\sigma^2 + \gamma N V^t}{M\gamma}. \tag{40}
\]

Thus, the MMSE estimate of \( x_i \) at the \((t+1)\)-th iteration is given by \( f_\alpha(\Sigma_i^t, R^t_i) \), and the corresponding MSE reads
\[
E^{t+1} = \int dx_i P_0(x_i) \int Dz f_\alpha(\Sigma_i^t, R^t_i) - x_i|^2, \tag{41}
\]

where \( P_0(x_i) \) is the prior distribution defined in (3) and \( Dz \) is the unit complex Gaussian measure \( Dz = e^{-|z|^2} dz/\pi \).

According to (37), the average variance estimate at the \((t+1)\)-th iteration is given by
\[
V^{t+1} = \int dx_i P_0(x_i) \int Dz f_\alpha(\Sigma_i^t, R^t_i). \tag{42}
\]

So that (41) and (42) constitute the state evolution equations for CB-AMP.

IV. SIMULATION RESULTS

We evaluate the performance of CB-AMP for reconstruction of complex-valued sparse signals. The elements of measurement matrix \( A \) are generated using i.i.d. complex Gaussian distribution with mean zero and variance \( \gamma = 1/N \).

The complex-valued sparse signals are assumed to follow \( \rho \)-sparse Bernoulli-Gaussian distribution, i.e., \( p_0(x) = \prod_{i=1}^{N} (\frac{1 - \rho}{2} + \rho CN(x_i; \mu, \tau) \), where \( \rho \approx 1, \mu, \tau \) are known. In this case, after some algebra, the posterior mean and variance defined in (17) and (19) can be calculated as
\[
f_\alpha(R, \Sigma) = \frac{\exp(\frac{|m|^2}{\Sigma} - \frac{|m|^2}{\tau})}{\exp(\frac{|m|^2}{\Sigma} - \frac{|m|^2}{\tau}) + 1}, \tag{43}
\]
\[
f_c(R, \Sigma) = \frac{1}{2} \left| \frac{V}{\Sigma} \right| - \frac{|m|^2 + V}{Z(R, \Sigma)} \tag{44}
\]
\[
V = \frac{\tau \Sigma}{\tau + \Sigma}, \tag{45}
\]
\[
m = \frac{\tau R + \Sigma \mu}{\tau + \Sigma}, \tag{46}
\]
\[
Z(R, \Sigma) = \frac{1}{\tau} \exp\left(\frac{-|m|^2}{\Sigma}\right) + \frac{\rho}{\tau} \exp\left(\frac{|m|^2}{\Sigma} - \frac{|m|^2}{\tau} - \frac{|R|^2}{\Sigma}\right), \tag{47}
\]

In the noisy case, the MSE performances of different methods are depicted in Fig.1 when \( N = 10^4, \alpha = M/N = 0.5, \rho = 0.1, \mu = 0, \tau = 1 \). Other simulation scenarios are omitted due to lack of space. Compared with the real AMP method, which converts the complex signal to the real domain before processing, CB-AMP improves the MSE evidently and converges more quickly. In addition, the theoretical state evolution prediction matches closely with the experimental result, implying that the performance of CB-AMP can be accurately predicted by state evolution.

In the noiseless case, i.e., \( \sigma^2 = 0 \), the phase transition curves are shown in Fig.2. For both real AMP and CB-AMP, the signal length is \( N = 1000 \), and number of iterations is set to be \( T = 500 \). The phase transition curves display the relationship between the measurement rate \( \alpha \) and the sparsity rate \( \rho \) at a success rate of 50\%, where the success of recovering the original signal is stated if the mean square error \( MSE < 10^{-4} \). The line \( \alpha = \rho \) indicates the maximum-a-posterior (MAP) threshold. As shown in Fig.2, CB-AMP
improves the phase transition curve of real AMP significantly, which is attributed to the structured sparsity of the complex signal.

![Graph 1](image1.png)

**Figure 1.** MSE versus the number of iterations. \( N = 10^3, \alpha = 0.5, \rho = 0.1, \mu = 0, \tau = 1 \).

![Graph 2](image2.png)

**Figure 2.** Phase transition curve. \( N = 10^3, \mu = 0, \tau = 1 \). Number of iterations \( T = 500 \). Success is stated if \( MSE < 10^{-4} \).

**V. CONCLUSION**

In this paper, we considered the problem of recovering complex-valued signals from a set of complex-valued linear measurements. Using EP, we have extended the AMP algorithm to complex-valued Bayesian AMP (CB-AMP). This novel perspective leads to a more concise and natural derivation of CB-AMP, without resorting to sophisticated transformations between the complex domain and the real domain. State evolution equations for CB-AMP are also derived. Simulation results demonstrate that CB-AMP outperforms real AMP in the complex-valued case and that state evolution predicts the reconstruction performance of CB-AMP accurately.

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