GRIDLESS TWO-DIMENSIONAL DOA ESTIMATION WITH L-SHAPED ARRAY BASED ON
THE CROSS-COVARIANCE MATRIX

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ABSTRACT
The atomic norm minimization (ANM) has been successfully incorporated into the two-dimensional (2-D) direction-of-arrival (DOA) estimation problem for super-resolution. However, its computational workload might be unaffordable when the number of snapshots is large. In this paper, we propose two gridless methods for 2-D DOA estimation with L-shaped array based on the atomic norm to improve the computational efficiency. Firstly, by exploiting the cross-covariance matrix an ANM-based model has been proposed. We then prove that this model can be efficiently solved as a semi-definite programming (SDP). Secondly, a modified model has been presented to improve the estimation accuracy. It is shown that our proposed methods can be applied to both uniform and sparse L-shaped arrays and do not require any knowledge of the number of sources. Furthermore, since our methods greatly reduce the model size as compared to the conventional ANM method, and thus are much more efficient. Simulations results are provided to demonstrate the advantage of our methods.

Index Terms— 2-D DOA estimation, L-shaped array, atomic norm, cross-covariance matrix

1. INTRODUCTION
The problem of 2-dimensional (2-D) direction-of-arrival (DOA) estimation plays a fundamental role in array signal processing and is encountered in a variety of applications. These applications include, for instance, using an airborne or a spaceborne array to observe ground-based sources, or, estimating the channel of the massive multiple-input and multiple-output (MIMO) systems in wireless communications. In 2-D DOA estimation, the azimuth and elevation angles of the incident sources are jointly estimated by using planar arrays, which can be roughly classified into three categories: the rectangular arrays [1], the L/T-shaped arrays [2] and the cross arrays. The selection of the array geometry largely affects the estimation accuracy as well as the computational efficiency and has been extensively investigated in literature [2, 3]. In particular, the rectangular array can be regarded as the 2-D extension of the uniform linear array (ULA) and hence several computationally efficient methods have been proposed for 2-D DOA estimation in URAs [1, 4], among which the 2-D ESPRIT [4] is an easy-to-implement algorithm due to the shift invariance property of the array output. However, it is usually unapplicable to the L/T-shaped or cross arrays. The L-shaped array has the best estimation performance due to its larger array aperture as defined by the largest distance among the sensors [5]. Hence the L-shaped array has been often employed in dealing with 2-D DOA estimation problems and many methods have been proposed by exploring the structural information of the array geometry. In particular, the L-shaped array has an advantage that the cross-correlation matrix (CCM) between the received data of the two orthogonal ULAs can eliminate additive noises, based on which several methods have been proposed [2, 6, 7]. However, most of these methods have utilized the L-shaped array consisting of two ULAs, and hence they may suffer from the difficulty when some sensors are "missing". In addition, most of these methods rely on prior knowledge on the number of sources, which is actually unavailable in practice.

Recently, with the development of atomic norm theory, the atomic norm minimization (ANM) approach has been incorporated into the 2-D DOA estimation (a.k.a. the line spectral estimation) for super-resolution [8–10]. In particular, the ANM solves a semi-definite programming (SDP) problem by exploiting the structure of the two-level Toeplitz matrix. Compared to the conventional methods, the ANM method is immune to the correlation between the impinged source signals as well as the number of sources. Furthermore, it is shown that the angle ambiguity problem can be also solved [9], [11]. However, since ANM requires to solve an SDP problem, it incurs a high computational complexity, especially in the multi-snapshot case, which becomes near intractable for large-scale antennas systems. To deal with this problem, some latest studies decouple the two-level Toeplitz matrix into two Toeplitz matrices in one dimension [12, 13]. In this way, the new SDP formulation has a much reduced problem size and hence the computational efficiency can be improved. Nevertheless, this method can only handle single snapshot case and the extension to the multi-snapshot case is not straightforward.
To the best of our knowledge, most of the existing ANM-based 2-D DOA estimation methods are proposed based on the uniform or sparse rectangular array. Although the ANM-based methods are applicable to the L-shaped array by regarding it as a special case of the sparse rectangular array, the characteristics of the L-shaped array are not fully exploited in these methods to improve the estimation performance. For instance, the coprime linear array can be a great alternative for each ULA in the L-shaped array [14–18]. In this paper, we propose an ANM-based model for L-shaped array by employing the CCM and prove that the model can be efficiently solved as an SDP. We then modify the constraint of the problem to improve its estimation performance. The proposed methods are computationally much more efficient than the conventional ANM method and can be applied to sparse L-shaped array, which is usually difficult for many other methods [2, 6, 7]. Simulations are carried out to validate the effectiveness of our methods.

2. SIGNAL MODEL

Suppose $K$ far-field narrowband sources impinge onto an L-shaped array consisting of two ULAs with half-wavelength inter-element spacing as illustrated in Fig. 1. The ULAs along the $x$- and $y$-directions consist of $N_x$ and $N_y$ sensors, respectively. Note that the sensor at the origin is shared by the two ULAs, hence the total number of sensors in the L-shaped array is $N_x + N_y - 1$. In Fig. 1, the $\theta_k$ and $\phi_k$ denote the elevation and azimuth angles, respectively, and $\alpha_k$ and $\beta_k$ denote the electrical angles in $x$- and $y$-directions of the $k$-th signal, respectively. From basic geometric knowledge, the electrical angles have the following relations with respect to elevation and azimuth angles,

$$\phi_k = \tan^{-1} \left( \frac{\cos(\beta_k)}{\cos(\alpha_k)} \right)$$

$$\theta_k = \sin^{-1} \left( \frac{1}{\sqrt{\cos^2(\alpha_k) + \cos^2(\beta_k)}} \right).$$

Hence, when the electrical angles are retrieved, the elevation and azimuth angles can be uniquely determined by (1).

When $L$ snapshots are collected, the array output can be formulated as,

$$X = A_x S + V_x,$$

$$Y = A_y S + V_y,$$

where $X$ and $Y$ are the array outputs along $x$- and $y$-directions, respectively, $A_x = [a_x(\alpha_1), \ldots, a_x(\alpha_K)]$ and $A_y = [a_y(\beta_1), \ldots, a_y(\beta_K)]$ are the manifold of the array along $x$- and $y$-directions, respectively, $S$ denotes the waveform of the sources, and $V_x$ and $V_y$ are the additive noises received by the arrays along $x$- and $y$-directions, respectively.

The goal of 2-D DOA estimation is to estimate $\alpha_k$ and $\beta_k$ given the array output $X$ and $Y$. In the following, we will exploit the structural information of the L-shaped array and provide a super-resolution approach based on the atomic norm theory.

3. THE PROPOSED METHOD

From (2) and (3), we can easily obtain the CCM of the array output as,

$$R = E[YY^H] = A_y PA_x^H,$$

where $P \triangleq E[SS^H] = \text{diag}(p)$ denotes the covariance matrix of the sources with $p = [p_1, \ldots, p_K]^T$. Then, by vectorizing $R$ column by column, we can have,

$$r = \text{vec}(R) = \sum_{k=1}^{K} p_k b_k,$$

where $b_k = a_x^*(\alpha_k) \otimes a_y(\beta_k)$. Inspired by the atomic norm theory, we formally construct the following atom set,

$$\mathcal{A} = \left\{ b_k = a_x^*(\alpha_k) \otimes a_y(\beta_k) : \alpha_k, \beta_k \in [-90^\circ, 90^\circ] \right\}. $$

Then the 2-D DOA estimation can be accomplished by minimizing the following atomic norm,

$$\|r\|_A = \inf_{p_k, \alpha_k, \beta_k} \left\{ \sum_{k=1}^{K} p_k : r = \sum_{k=1}^{K} p_k b_k, p_k \in \mathbb{R}^+, b_k \in \mathcal{A} \right\}. $$

Although the atomic norm (7) is convex, it is a semi-infinite program with an infinite number of variables. To practically solve (7), inspired by Theorem 3 in [19], an SDP formulation of $\|r\|_A$ is provided in the following theorem.\(^1\)

**Theorem 1** \(\|r\|_A\) defined in (7) equals the optimal value of the following SDP:

$$\min_{t, \beta} \frac{1}{2N_x N_y} \left( t + tr[\mathbb{T}] \right)$$

s.t. \( t r^H \geq 0, \)

where $\mathbb{T}$ is a two-level Toeplitz matrix.

**Proof:** We first introduce the following lemma.

**Lemma 1** \([20]\)** Given $R = BB^H \succeq 0$, it holds that $r^H R^{-1} r = \min \|p\|_2^2$, subject to $Bp = r$.

\(^1\)Although a similar result is provided in [10], our proof is carried out from a different perspective and is simpler.
It follows from the constraint in (8) that $T \geq 0$ and $t \geq r^{HT^{-1}}r$. So, it suffices to show that

$$\|r\|_A = \min_T \frac{1}{2N_xN_y} (\text{tr}[T] + r^{HT^{-1}}r) \quad \text{s.t. } T \geq 0. \quad (9)$$

Let $T = BC^H = [BC^H][BC^H]^H$ be any feasible Vandermonde decomposition, where $B = [\ldots, b_k, \ldots]$, and $C = \text{diag}(\ldots, c_k, \ldots)$ with $c_k > 0$. Hence we have $\text{tr}[T] = N_xN_y \sum c_k$. According to Lemma 1, we have that

$$r^{HT^{-1}}r = \min_v \|v\|^2_2 \quad \text{s.t. } r = BC^Hv$$

(notation terms which are non-zero due to finite snapshot effects, we propose the following ANM approach,

$$BCB^H = \min_{p} \|C^{-\frac{1}{2}}p\|^2_2 \quad \text{s.t. } r = B_p$$

(10)

Based on (10), we have,

$$\min_{r,B^H} \frac{1}{2N_xN_y} (\text{tr}[T] + r^{HT^{-1}}r)$$

$$= \min_{r,B^H} \frac{\sqrt{N_xN_y}}{2} \sum_n c_n + \frac{1}{2N_xN_y} \operatorname{vec}^H C^{-1} \operatorname{vec} r$$

$$= \min_{r,B^H} \frac{\sqrt{N_xN_y}}{2} \sum_n c_n + \frac{1}{2N_xN_y} \sum_n p_n^2 c_n$$

$$= \min_{r,B^H} \sum_n p_n \quad \text{s.t. } B_p = r$$

$$= \|r\|_A.$$  

Hence, Theorem 1 can be concluded.

Note that the CCM is usually obtained with limited snapshots, as

$$\hat{R} = \frac{1}{L} YX^H,$$  

(12)

where $\hat{R}$ is error-contaminated due to finite snapshots. We denote the error component as

$$E = \hat{R} - R$$  

(13)

where $E$ consists of signal-signal and signal-noise cross correlation terms which are non-zero due to finite snapshot effect. By taking this error component into consideration, we propose the following ANM approach,

$$\min_{\hat{r},T} \frac{1}{2N_xN_y} (\text{tr}[\hat{T}] + \hat{r}^{HT^{-1}}\hat{r})$$

s.t. $\hat{r}^{HT^{-1}}\hat{r} \geq 0, \|\hat{r}\|_2 \leq \eta,$  

(14)

where $\hat{r} = \operatorname{vec}(\hat{R})$ and $\eta \geq \|E\|_F$ denotes the upper bound of the error energy.

Note that the DOAs of interest are actually encoded in the two-level Toeplitz matrix $T$. As long as $T$ is determined, the DOAs can be retrieved and automatically paired by using the generalized Vandermonde decomposition theorem given in [9].

**Remark 1** Compared to the ANM method in [9] and [10], which is time-consuming in the multiple snapshot case, our proposed method transforms the multiple snapshot model into the single snapshot model and the computational burden is greatly reduced as will be seen in simulations. Since we employ the CCM to eliminate the additive noise, the proposed method is named as cross-covariance ANM (CC-ANM).

We now consider the sparse L-shaped array where some sensors of the two ULAs fail to function or are missing. Let us further define the sensor index sets of the two linear arrays as $\Omega_x$ and $\Omega_y$, respectively. $^2$ Let $\Gamma_x$ be a selection matrix with respect to $\Omega_x$ such that the $m$-th row of $\Gamma_x$ contains all zeros but a single 1 at the $\Omega_{x_m}$-th position where $\Omega_{x_m}$ is the $m$-th element in $\Omega_x$. Similarly, we define $\Gamma_y$. By definition, the sample CCM can be denoted as $\hat{R}_{\Omega} = \Gamma_x \hat{R}_{\Omega y} \Gamma_y^H$ and its vectorized version is $\hat{r}_{\Omega} = \text{vec}(\hat{R}_{\Omega}) = (\Gamma_x^T \otimes \Gamma_y^T)\hat{r}$. Following the same manner as formulating problem (14), we propose the following SDP for the sparse L-shaped array,

$$\min_{\hat{r},\hat{T}} \frac{1}{2N_xN_y} (t + \text{tr}[T])$$

s.t. $\left[ \begin{array}{c} t \\ r^{HT} \end{array} \right] \geq 0, \|\hat{r} - \hat{r}_{\Omega}\|_2 \leq \eta,$

where $\hat{r}_{\Omega} = (\Gamma_x^T \otimes \Gamma_y^T)\hat{r}$.

**Remark 2** From problem (15) it can be seen that, our proposed method is still applicable even if some of the sensors in ULAs fail and hence is more reliable for practical applications. Furthermore, based on the atom norm theory, CC-ANM method does not require any knowledge of the number of sources.

### 4. THE MODIFIED CC-ANM

Although problem (14) or (15) can be efficiently solved by using CVX in a polynomial time, the appropriate value of user-defined parameter $\eta$ is usually hard to obtain. In this section, we propose a modified CC-ANM where the parameter can be easily determined. Note that $r_{\Omega} = r$ when no sensor fails, hence model (14) can be regarded as a special case of model (15). Without loss of generality, we use model (15) in this section and give the following theorem.

**Theorem 2** Suppose that the error component is given as $\varepsilon_{\Omega} = \hat{r}_{\Omega} - r_{\Omega}, R_{\Omega x}$, and $R_{\Omega y}$ are the covariance matrix of the linear arrays along x- and y-directions, respectively. Then, $\varepsilon_{\Omega}$ obeys the following asymptotic Gaussian distribution,

$$\varepsilon_{\Omega} \sim A_{S\lambda N}(0, Q),$$

(16)

where $Q = \frac{1}{L} R_{\Omega x} \otimes R_{\Omega y}$.

**Proof:** Inspired by [23], we first denote the $i$-th subvector (with length of $M_y$) of $\hat{r}_{\Omega}$ as $^3$

$$\hat{r}_{\Omega i} = \frac{1}{L} \sum_{t=1}^L y(t)x_i^*(t).$$

(17)

$^2$Detailed description of the sensor index set can be found in [21, 22].

$^3$$M_x, M_y$ denote the number of sensors along x- and y-directions, respectively.
Then we can establish the following relation,
\[
E[\hat{\Omega}(\hat{\Omega})^H] = \frac{1}{L^2} \sum_{i=1}^{L} \sum_{s=1}^{L} E[y(t)x_i^*(t)y^*(s)x_j(s)] = [\Omega][\Omega]^H + \frac{1}{L}[\Omega\Omega_{\ast}]_{ij}[\Omega\Omega_{\ast}],
\]
(18)
from which it can be concluded that,
\[
Q = E[(\hat{\Omega} - \Omega)(\hat{\Omega} - \Omega)^H] = \frac{1}{L}[\Omega\Omega_{\ast}]_{ij}[\Omega\Omega_{\ast}],
\]
(19)
According to Theorem 2, we can show that \(Q^{-\frac{1}{2}} \varepsilon_{\Omega}\) satisfies the standard Gaussian distribution, i.e., \(N(0, I)\), and its \(\ell_2\)-norm satisfies the chi-square distribution as,
\[
\|Q^{-\frac{1}{2}} \varepsilon_{\Omega}\|^2 \sim N_{\chi^2}(M_x, M_y).
\]
(20)
Based on the property of the chi-square distribution, the following inequality holds with probability \(1 - \kappa\),
\[
\|Q^{-\frac{1}{2}} \varepsilon_{\Omega}\|^2 \leq \beta,
\]
(21)
where \(\kappa\) is usually chosen to be a small value (e.g., \(10^{-4}\)) and \(\beta\) can be determined by using the Matlab routine chi2inv. Equation (21) can be regarded as the weighted least squares criterion and is a large-snapshot approximation to the maximum likelihood criterion [23]. As a result, we have the following modified SDP,
\[
\min_{T} \frac{1}{2\sqrt{N_xN_y}}(t + \text{tr}[T])
\]

s.t. \([t \quad r^H \quad T] \geq 0, \|Q^{-\frac{1}{2}} \varepsilon_{\Omega}\|^2 \leq \beta.
\]
(22)

After obtaining \(T\), the DOAs can be retrieved accordingly.

We name the proposed method as modified CC-ANM (MCC-ANM).

Before closing this section, we give a complexity comparison between our method and the traditional ANM method. In particular, the computational complexity of the traditional ANM method is \(O(n_1^2n_2^5)\), where \(n_1 = L^2 + P\) and \(n_2 = L + M_xM_y\) with \(P = 2M_xM_y - M_x - M_y + 1\) being the number of variables in \(T\). While in our methods, the computational complexity can be greatly reduced by noting that \(n_1 = 1 + P\) and \(n_2 = 1 + M_xM_y\). The superiority of our method will be further shown in the next section.

5. NUMERICAL RESULTS

In this section, two L-shaped arrays are considered for simulations: one consisting of two 5-element ULAs (Array 1) and the other consisting of two sparse linear arrays with \(\Omega_x = \Omega_y = \{1, 2, 3, 5\}\) (Array 2). We compare our proposed methods with ANM [9], which is the state-of-the-art gridless method for 2-D DOA estimation. All the compared methods are implemented by SDPT3.

![Fig. 2. RMSE and CPU time comparisons with Array 1.](image)

![Fig. 3. RMSE and CPU time comparisons with Array 2.](image)

Suppose two source signals impinge onto Array 1 from \(\alpha = [-25^\circ, 30^\circ]\) and \(\beta = [-35^\circ, 0^\circ]\). We examine the performance of our methods with comparison to ANM in terms of RMSE and CPU time, respectively. The SNR is set to 10dB and the number of snapshots varies from 50 to 200. We carry out 400 independent trials and show the statistical results in Fig. 2. From Fig. 2 (a), it can be seen that, the performance of these three methods is in general improved as the number of snapshots grows. Due to the modified constraint, MCC-ANM is superior to CC-ANM. Also, the performance gap between ANM and MCC-ANM becomes smaller when \(L\) gets large. Although ANM shows the best estimation performance, its computational workload can be unaffordable especially when \(L\) is large as shown in Fig. 2 (b). On the other hand, the computational complexity of our proposed methods is immune to the number of snapshots.

We then use Array 2 to replace Array 1 and carry out the previous experiment with the same settings. The simulation results are provided in Fig. 3. Clearly, MCC-ANM gives a better estimation performance than ANM does when \(L\) is large enough. More importantly, our proposed methods are much more computationally efficient than ANM in terms of the running time comparison.

6. CONCLUSION

In this paper, we have addressed the 2-D DOA estimation problem in the scenario of L-shaped arrays. By exploiting the characteristics of the L-shaped array, two gridless methods which can be applied to both uniform and sparse L-shaped arrays have been proposed. Simulation results show that our methods provide similar estimation performance to ANM method but with a much smaller computational workload.
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