Stabilizing a solution to the inverse problem of signal separation based on parameters determining solution stability

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Abstract. This paper proposes a regularization algorithm that makes it possible to stabilize a solution to the inverse problem of separating signal sources—extraction of individual signals from an additive mixture of several signals. The signals come to measurement points from various signal sources inaccessible for direct measurement. A feature of the algorithm is the use of results of stability analysis. The feature simplifies the algorithm’s computational complexity. Stability analysis is accomplished by determining singular intervals for absolute, relative, and critical variation types of a signal formation model’s parameters. This paper also presents the results of computer modeling for the proposed algorithm.

1. Introduction
Signal separation is a solution to the problem of extracting individual signals from an additive mixture of several signals that come to measurement points from various signal sources inaccessible for direct measurement. That solution is needed in many practical fields such as monitoring and diagnosis of technical facilities (e.g., vibroacoustic diagnosis), communications, medical diagnosis, and speech processing.

The problem of signal separation relates to the class of inverse problems, which may be ill-posed, generally. From that, it follows that the solution may be unsteady because slight changes in the parameters of the mixing matrix $H$ of the signal formation model or in characteristics of source signals lead to impossibly large changes in the solution that is, an unstable computation of source signals [1]. For a stable solution, the parameters of the object described by the signal formation model must satisfy several prior restrictions [2]. For instance, the mixing matrix must be invertible; the polynomials describing the transfer functions of channels must not have common roots; the number of receivers must equal that of sources.

In practice, prior restrictions may be violated since object parameters may change because of the object’s evolution in time, measurement error, fabrication inaccuracies, and other causes that often are unpredictable. Thus, parameters of the mixing matrix $H$ and characteristics of source signals may cause a solution to the problem of signal separation to become unsteady and thus unsuitable for application.

At present, obtaining a stable solution to ill-posed problems involves using various methods. The most used among those methods are solution regularization methods such as deterministic
regularization [3,4] and statistical regularization [5,6]. These methods are appropriate to use if one knows a priori that the problem is ill-posed and the obtained solution unstable. But if object parameters change so that the problem becomes well-posed, the approximated regularized solution tends to an accurate solution because of the reduced regularization parameter in the stabilizing functional.

Regularization methods are used to obtain stable solutions to the problem of separating signal sources. The methods are well developed and versatile, but they have high computational complexity, hindering and limiting their use in stream data processing (e.g., in digital signal processing).

It is known [3,4] that the computational complexity of solution-regularization methods can be reduced significantly by using additional prior information in problem parameters. In the case at hand, such additional information reflects nuances of the signal separation problem—for example, in diagnostic problems using dynamic parameters (vibroacoustic diagnosis), communication problems (separation of voice signal sources and interference compensation), and the like. Data-processing systems for such problems are designed so that the signal separation problem is initially correct and its solution stable. Only anomalous changes in parameters of the mixing matrix $H$ and in characteristics of source signals may cause a steady solution to migrate toward an unstable one [7]. A converse process—an ill-posed problem transforming into a well-posed one—is also possible.

Thus, solution regularization is necessary only if stability analysis shows that the system moves into the domain of unstable solutions. Because the primary operation mode of signal separation systems in applied problems corresponds to a stable solution, computational expenses for solution stabilization are dramatically reduced.

Further, as problems move into the domain of unstable solutions from the domain of stable ones, it is possible to use the history of changes of parameters of the mixing matrix $H$ and source-signal characteristics as additional information for computing the regularization parameters of the stabilizing functional, thereby simplifying the calculations.

Thus, we can conclude that developing and investigating stabilization algorithms for solutions to the problem of signal separation using for calculation the parameters determining solution stability is of current importance.

2. Purpose of Investigation

To state the problem formally, we will consider an object signal formation model presented as a linear multivariable system that has $N$ inputs and $M$ outputs. The model’s input signals are $s_n(k)$, $n = 1, 2, ..., N$; output signals, $x_m(k)$, $m = 1, 2, ..., M$. The input signals are generated by various signal sources, and the output signals may be signals of various receivers such as sensors, measurement transducers, and antennas. Let us assume that each of the $M$ outputs of the multivariable system is connected with all the $N$ inputs through linear signal-transmission channels.

At any discrete instant of time $k$, the $M$-dimensional vector of sensor-measured discrete signals $\mathbf{x}(k) = [x_1(k), x_2(k), ..., x_M(k)]^T$ results from the $N$-dimensional vector of source signals $\mathbf{s}(k) = [s_1(k), s_2(k), ..., s_N(k)]^T$. The mathematical model of signal formation is described by an equation system of discrete convolution type (1), where the $m$th observed signal is an additive mixture of channel-distorted source signals and noise [1]; that is,

$$x_m(k) = \sum_{g=0}^{N-1} \sum_{l=1}^{M} h_{mn}(g, l) s_n(k - g) + y_m(k), \quad (1)$$

where $h_{mn}(g)$ is the element $M \times N$ of the $h(g)$ matrix for impulse characteristics of channels, and $\mathbf{y}(k) = [y_1(k), y_2(k), ..., y_M(k)]^T$ is the noise vector. For purposes of further discussion, we will assume that the $h_{mn}(g)$ impulse characteristics are finite and are represented by the counting number
The dynamic characteristics of channels $h_{mg}(g, \mathbf{l})$ are quasistationary in that they change depending on parameter vector $\mathbf{l}$ (time, temperature, location, etc.).

In the frequency domain, mathematical model (1) is described as

$$X(\omega) = H(\omega) \cdot S(\omega) + Y(\omega),$$

where $X(\omega) = [X_1(\omega), \ldots, X_M(\omega)]^T$ is the vector of observed signals, and it comprises Fourier transforms of receiver signals; $S(\omega) = [S_1(\omega), \ldots, S_N(\omega)]^T$ is the vector of source signals, and it comprises Fourier transforms of source signals; $Y(\omega) = [Y_1(\omega), \ldots, Y_M(\omega)]^T$ is the noise vector, comprising Fourier transforms of noise signals; $H(\omega) = \begin{pmatrix} H_{11}(\omega) & \cdots & H_{1N}(\omega) \\ \vdots & \ddots & \vdots \\ H_{M1}(\omega) & \cdots & H_{MN}(\omega) \end{pmatrix}$ is the mixing matrix ($M \times N$), comprising Fourier transforms of the channels. Signals of sources $S(\omega)$ and of noise $Y(\omega)$ are considered independent, and the channels can be modeled by spectral converters such as filters.

Generally, solving the problem of separating signal sources reduces to calculating the separating matrix $W(\omega)$, which is, in terms of specific criteria, equal or close to the matrix inverse to matrix $H(\omega)$. Thus, generally, the solution to the problem of separating source signals is the solution to system (1), and it can be expressed as

$$s_k(\omega) = \sum_{m=1}^{M} \sum_{g=0}^{G-1} w_{mg}(\omega) x_m(\omega - g),$$

where $W(\omega)$ is the matrix of impulse characteristics of tunable filters with $w_{mg}(\omega)$ elements. In the frequency domain, equation (3) can be written as

$$S(\omega) = W(\omega) X(\omega),$$

where $W(\omega) = H^{-1}(\omega)$.

Thus, solving the problem of separating of signal sources reduces to using a deterministic or statistical method to calculate the separating matrix $W(\omega)$ equal or close, in terms of specific criteria, to the matrix inverse to matrix $H(\omega)$.

We propose using singular intervals of parameters of the mixing matrix $H$ (the signal formation model), whose calculation algorithms are given in [2], as parameters determining the stability of the solution.

Stability verification consists in comparing calculated singular intervals with set intervals of stable separation.

The purpose of this paper is to develop and investigate a stabilization algorithm for a solution to the problem of signal separation using for calculation the parameters determining solution stability.

3. Regularization Algorithm for a Solution to the Problem of Signal Separation Using the Parameters Determining Solution Stability

If certain calculated singular intervals of the mixing matrix turn out to be less than the set intervals of stable separation (i.e., the solution is unstable), the parameters of the signal formation model should be adjusted so that the problem’s solution described by the separating matrix $W$ becomes stable.

The proposed algorithm for stabilizing a solution to the problem of signal separation is based on the premises of the Tikhonov regularization method [3,4].

The operator equation describing the signal formation model with disturbances is written as
where instead of the exact $H$ and $x$ their approximate values $\hat{H}$ and $\hat{x}$ are known whereby $\|H - \hat{H}\|_2 \leq \bar{\xi}_H$ and $\|x - \hat{x}\|_2 \leq \delta$, where $\delta$ and $\bar{\xi}_H$ are the upper estimates of absolute measurement error of the signals $x$ and the mixing matrix $H$. By using the singular value decomposition (SVD) of the mixing matrix, the regularized solution $s_\alpha$ based on the Tikhonov method [2] can be written as

$$s_\alpha = \sum_{n=1}^{N} f_n \frac{u_n^* \hat{x}}{\sigma_n^2} v_n,$$

where $\sigma_n$ are singular numbers, and $f_n$ are filtering multipliers of the form

$$f_n^{\text{Tikhonov}} = \frac{\sigma_n^2}{\sigma_n^2 + \alpha}, \quad (2)$$

where $\alpha > 0$ is the regularization parameter.

This paper proposes the regularizing operator $\tilde{W}_\alpha(\omega) = (\hat{H}(\omega) - \alpha \tilde{Z}_{\text{crit}}^\text{SVD}(\omega))^{-1}$ determined from the direction matrix $\tilde{Z}_{\text{crit}}^\text{SVD}$ obtained in [2]. Indeed, the model’s parameters need to be adjusted in the direction opposite to that which has caused instability—that is, the direction opposite to that determined by the direction matrix $\tilde{Z}_{\text{crit}}^\text{SVD}$ [8].

Then, by using the SVD of the direction matrix $-u_N(\omega)v_n^*(\omega)$, the regularized solution of the signal separation problem can be expressed as

$$S_\alpha(\omega) = \tilde{W}_\alpha(\omega) \tilde{X}(\omega) = \sum_{n=1}^{N} \tilde{f}_n(\omega) \frac{u_n^*(\omega) \hat{x}(\omega)}{\sigma_n(\omega)} v_n(\omega), \quad (3)$$

where the filtering multipliers are equal to

$$\tilde{f}_n(\omega) = \begin{cases} \sigma_n(\omega) \geq \alpha \\ \frac{\sigma_n(\omega)}{\sigma_n(\omega) + \alpha}, & \sigma_n(\omega) < \alpha \end{cases}. \quad (4)$$

If the model is substantially unstable, then, given the satisfied inequality $\sigma_N << \alpha$, the filtering multipliers (4) can be expressed as

$$\tilde{f}_n^{\text{mod}}(\omega) = \begin{cases} \frac{1, \sigma_n(\omega) \geq \alpha}{\sigma_n(\omega)}, & \sigma_n(\omega) < \alpha \end{cases}. \quad (5)$$

Expressions (3–5) represent a regularized solution to the multivariable problem of signal separation, and it is different in that it uses the proposed filtering multipliers (4,5) and in that $S_\alpha(\omega)$ and $\tilde{X}(\omega)$ are vectors and $\hat{H}(\omega)$ and $\tilde{W}_\alpha(\omega)$ are matrices, while in conventional univariate cases these components are scalars. It is important that for calculation of filtering multiplier (4), fewer computational operations are required as compared with multiplier (2).

We propose that the regularization parameter $\alpha$, given the known error $\bar{\xi}$ and $\delta$, be determined with the modified generalized residual method [3,9]. As it applies to the signal separation problem, the regularization parameter $\alpha > 0$ can be determined as the root of the equation

$$\left\| W_{\alpha} - \bar{x} \right\|_2^2 = \left\{ \bar{\xi} + \delta \|s_\alpha\|_2 \right\}^2 / \beta(r,M,N),$$

where $\alpha$ is the parameter $s_\alpha$, and the introduced multiplier $\beta(r,M,N) \geq 1$ serves as a scaling multiplier determined by the dimension of the problem $(M \times N)$ and by the measurement error of signal and channel parameters (determined, in particular, by the digit capacity $r$ of the analog-to-
digital converter [ADC]).

A feature of the proposed regularization algorithm, which is given in the table below, is the ability to use additional information obtained from stability analysis.

This simplifies the computational complexity of regularization.

Table 1. Generalized regularization algorithm for the solution to the problem of signal separation

| No. | Steps | Note |
|-----|-------|------|
| 1   | Set the initial iteration value $j = 1$, determine | Initialization at $\omega_0 = \omega_0, \ldots, \omega_{j-1}$ |
|     | $\alpha_{\min} = \min_\omega \| \Delta \mathbf{H}(\omega) \|_2$, $\alpha_{\max} = \max_\omega \sigma_{\max}(\mathbf{H}(\omega))$, $\alpha_j = \alpha_{\min}$, and determine SVD: $\mathbf{H}(\omega_j) = \mathbf{U}(\omega_j) \Sigma(\omega_j) \mathbf{V}^*(\omega_j)$ | |
| 2   | For each frequency $\omega_k$, verify the condition: | Adjustment of spectral matrices |
|     | $[ \Delta \mathbf{H}(\omega_k) ]_2 < \alpha_{\min}$, then | |
|     | calculate $\tilde{f}_n(\omega_k)$, $\mathbf{S}_n(\omega_k) = \sum_{n-1}^k \tilde{f}_n(\omega_k) \cdot u_n^*(\omega_k) \mathbf{X}(\omega_k) v_n(\omega_k)$ | Auxiliary function |
| 3   | $\rho(\alpha) = \| \mathbf{H} \mathbf{x}_0 - \mathbf{x} \|_2^2 / \| \mathbf{H} \|_2^2 \beta(r,M,N)$ | Determination of regularization parameter |
| 4   | If $\rho(\alpha_{\min}) \cdot \rho(\alpha_{\max}) > 0$, then $\alpha_j = \delta + \xi_j$, else find the root of the equation $\rho(\alpha) = 0$ for example, with the bisection method or the Newton method | |

Indeed, when control and stability algorithms are used as a complex, solutions are regularized only in the case of instability. Using the proposed filtering multipliers (4,5) also simplifies the computation complexity of regularization thanks to:

a) the decreased range $[\alpha_{\min}, \alpha_{\max}]$ for finding the regularization parameter (see step 1 in the table) or the iterative improvement of the range $[\alpha_{\min}, \alpha_{\max}]$, whose value can be determined as $\alpha_j = \alpha_{\min} - \varphi \cdot \alpha_{\max}$, $\alpha_j = \alpha_{\max} + \varphi \cdot \alpha_{\min}$, where the value $0 < \varphi < 1$ depends on the dynamics of changes in the model’s parameters;

b) calculation of some of the filtering multipliers, $n \leq N$, rather than all of them, $N$, for singular numbers satisfying the condition $\sigma_n < \alpha$ (this makes it possible to use the calculation results in the previous regularization steps); and

c) the decreased frequency range $[\omega_{\min}, \omega_{\max}]$, in which the parameters of the mixing matrix $\mathbf{H}(\omega)$ need to be regularized.

The presentation of the proposed algorithm in the frequency domain, with SVD in mind, enables parallel-in-time regularization in every frequency range, thereby saving regularization time.

The computational complexity of the regularization process can also be simplified by one-step regularization whereby the regularization parameter $\alpha(\omega)$ in the filtering multiplier (5) is calculated as

$$\alpha(\omega) = \sigma_{\max}(\mathbf{H}(\omega)) / \text{Cond}_{\mathbf{H}}.$$  

4. Modeling Results and Discussion

We have verified the conventional and the proposed regularized solutions using computer modeling techniques. The modeling involved the use of test problems with known signal sources and parameters
of the mixing matrix $H(\omega)$.

As part of modeling, each problem was preliminarily solved using regularization with known sources of signals $s_n(k)$ and with signals of receivers $\tilde{x}_n(k)$ measured with set error. Using prior information about the parameters of the mixing matrix and sources of signals (triangular test signals) allows us to consider such a test solution as a reference solution intended for comparative evaluation of the results of modeling the regularization algorithms proposed in this paper.

The quality of signal separation is measured by the vector $\psi = [\psi_1, \ldots, \psi_{Np}]$, each element of which is determined as the value of the separation error signal power $\sum_{k=0}^{K-1} (s_n(k) - s_{n,\alpha}(k))^2$ reduced to the power of the source signal:

$$\psi_{np} = \frac{\sum_{k=0}^{K-1} (s_n(k) - s_{n,\alpha}(k))^2}{\sum_{k=0}^{K-1} s_n^2(k)} \cdot 100\%,$$

where

$$s_n(k)$$ and $$s_{n,\alpha}(k) = \sum_{\omega=0}^{K-1} S_{n,\alpha}(\omega) \cdot e^{j2\pi\omega k}$$

are the exact and the regularized solutions, respectively.

Let us consider the signal formation model described by the mixing matrix $M = N = 2$ with frequency-dependent channels and known error $\xi_H$ and $\delta$. These types of error depend on the digit capacity of the ADC. To calculate the regularization parameter, we used the generalized residual method [3, 9] and stabilized the solution using the conventional and the proposed regularizing operators. Figure 1 shows the modeling results, where (a) is the reference (test) solution; (b) is the conventional (Tikhonov) solution; and (c) the proposed solution.

The modeling results show that with the selected regularizing operators, the regularization process in the multivariable problem at hand is convergent; that is, the regularization algorithm proposed by this paper can be used in practical applications. The proposed regularization algorithm and the most effective of the known ones (the Tikhonov method) are compatible with the separation error, but the proposed algorithm has less computational complexity.

![Figure 1. Relationship between separation error and ADC digit capacity.](image)

Figure 2 shows the dependence of the methodical $\psi_m(\alpha)$, random $\psi_r(\alpha)$ and general $\psi_g(\alpha)$ error sof signal separation of the proposed algorithm on the regularization parameter $\alpha$ for the signal
formation model described by the mixing matrix $M = N = 3$ with and the reduced error in measuring the channels and observed signals equal to 0.4%.

Let us evaluate the computational complexity of the proposed algorithm. As an example, we will consider a model having a mixing matrix with $M = N = 2$, where signals and channels are assigned with error corresponding to eight binary digits of the ADC. The plots obtained from the modeling are shown in figure 3. They represent the relationships between the condition number of the initial and the regularized solutions and the frequency.

From the plot in figure 3 it follows that the solution was regularized in the frequency range between 100 Hz and 375 Hz and that the parameters of the model’s mixing spectral matrices in the 0–100 Hz range remained unchanged—that is, regularization did not take place, thereby simplifying calculations.

This conclusion is confirmed by the relationship between the maximal $\sigma_1(\omega_f)$ and minimal $\sigma_2(\omega_f)$ singular numbers of spectral matrices $H(\omega_f)$, on the one hand, and the frequency at the set regularization parameter $\alpha$, on the other (figure 4). The intersection point of $\sigma_2(\omega_f)$ and the line $\alpha$ lies in the 100 Hz region; that is, only the minimal singular numbers of spectral matrices with a frequency of over 100 Hz underwent regularization.

**Figure 2.** Dependence of the separation error (solid line) on the regularization parameter $\alpha$ methodical component of the error (dotted line), random component, due to unstability solution (dot line).

**Figure 3.** Relationship between the condition number and frequency: a) unregularized solution; b) regularized solution.
This example allows us to evaluate the computational complexity of the proposed regularization algorithm as follows. The computational complexity of the calculated $n^{th}$ summand in the regularized solution $\tilde{f}_n(\omega) \cdot \frac{u_n(\omega)X(\omega)}{\sigma_n(\omega)}$ is designated $SL$. Then the computational complexity of the calculated regularized solution for the Tikhonov method equals $SL \cdot 375 \cdot 2$ because two singular numbers are used; and the computational complexity of the regularized solution calculated with the proposed algorithm is $SL \cdot (375 - 100) = SL \cdot 275$ because only one singular number is used, and the frequency range used for regularization is narrower almost by a factor of four. This example shows an almost three-fold decrease in computational complexity. If the condition $\sigma_n(\omega) << \sigma_{n-1}(\omega)$ is satisfied, the conclusion is valid that with an increase in $M$ and $N$, the improvement of computational complexity will be more pronounced.

Figure 5 shows an example of modeling the proposed regularization algorithm in the problem of separating triangular test signals.

These results of computer modeling and the comparison of the proposed algorithm with known algorithms have shown the validity of our theoretic conclusions, and the results confirm the benefits of the proposed regularization algorithm.
5. Main Conclusions
To stabilize a solution to the inverse problem of signal separation, we developed and investigated a regularization algorithm that provides stability for signal separation and stands out as having lower computational complexity.

A feature of the proposed regularization algorithms is the ability to use additional information obtained from stability analysis, i.e., complex use of regularization and stability analysis algorithms.

We propose using singular intervals of parameters of the mixing matrix for signal formation model as parameters determining the stability of the solution. Stability verification consists in comparing calculated singular intervals with set intervals of stable separation.

This simplifies the computational complexity of regularization thanks to:
- the decreased range for finding the regularization parameter or the iterative improvement of the range, depends on the dynamics of changes in the model's parameters;
- calculation of some of the filtering multipliers, rather than all of them, for singular numbers exceeding a certain threshold value (this makes it possible to use the calculation results in the previous regularization steps);
- the decreased frequency range, in which the parameters of the mixing matrix need to be regularized;
- the presentation of the proposed algorithm in the frequency domain, enables parallel-in-time regularization in every frequency range, thereby saving regularization time.

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