ON PRODUCTS OF HARMONIC FORMS

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Abstract. We prove that manifolds admitting a Riemannian metric for which products of harmonic forms are harmonic satisfy strong topological restrictions, some of which are akin to properties of flat manifolds. Others are more subtle, and are related to symplectic geometry and Seiberg-Witten theory.

We also prove that a manifold admits a metric with harmonic forms whose product is not harmonic if and only if it is not a rational homology sphere.

1. Introduction

On a general Riemannian manifold, wedge products of harmonic forms are not usually harmonic. But there are some examples where this does happen, like compact globally symmetric spaces. For these the harmonic forms coincide with the invariant ones, and the latter are clearly closed under products. See [2], pp. 10-13.

Sullivan [8] observed that “There are topological obstructions for M to admit a metric in which the product of harmonic forms is harmonic.” The reason Sullivan gave is that if the product of harmonic forms is harmonic, then the rational homotopy type is a formal consequence of the cohomology ring. Therefore, manifolds which are not formal in this sense cannot admit a metric for which the products of harmonic forms are harmonic.

This motivates the following:

Definition 1. A Riemannian metric is called (metrically) formal if all wedge products of harmonic forms are harmonic.

A closed manifold is called geometrically formal if it admits a formal Riemannian metric.

Thus geometric formality implies formality in the sense of Sullivan. Compact globally symmetric spaces are metrically formal, as are arbitrary Riemannian metrics on rational homology spheres. Further examples can be generated by taking products, because the product of two formal metrics is again formal.

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In this paper we describe a number of elementary topological obstructions for geometric formality of closed oriented manifolds. These obstructions are independent of formality in the sense of rational homotopy theory, and are often nonzero on formal manifolds. The simplest obstruction is the product of the first Betti number and the Euler characteristic. In small dimensions these elementary obstructions are strong enough to imply:

**Theorem 2.** If $M$ is a closed oriented geometrically formal manifold of dimension $\leq 4$, then $M$ has the real cohomology algebra of a compact globally symmetric space.

It is however not true that $M$ is a globally symmetric space, even up to homotopy. We give many examples of this in dimensions 3 and 4.

We also give examples of 4-manifolds which do have the real cohomology algebra of a compact symmetric space, but are not geometrically formal. This is detected by some more subtle obstructions coming from symplectic geometry and Seiberg-Witten gauge theory.

The pattern of the arguments presented here is that metric formality is a weakening of a reduction of holonomy. For example, it implies that harmonic forms have constant length, though it does not imply that they are parallel. Nevertheless, the more harmonic forms there are, the stronger the constraints.

In Section 7 we prove that every manifold which is not a rational homology sphere admits a non-formal Riemannian metric.

This paper was motivated by an example pointed out by D. Toledo, which we describe in Section 2, and which is related to joint work in progress of the author with H. Endo. Further impetus came from a question posed by D. Huybrechts and U. Semmelmann concerning products of harmonic forms on Calabi-Yau manifolds, see [4].

### 2. Motivation: dimension two

Let us consider first the case of a closed oriented surface $\Sigma$. If its genus is 0 or 1, then there are globally symmetric Riemannian metrics. If the genus of $\Sigma$ is $\geq 2$, there are nontrivial harmonic 1-forms for all metrics, but every 1-form has zeros. In this case every wedge product of 1-forms also has zeros, but for cohomological reasons cannot vanish identically in all cases. The only harmonic 2-forms are the constant multiples of the Riemannian volume form, so there cannot be any formal Riemannian metric. This proves Theorem 2 in the 2-dimensional case.
The argument above, pointed out to the author by D. Toledo, shows that harmonicity of products of harmonic forms can fail on compact locally rather than globally symmetric spaces, contradicting a statement in [3], p. 158.

Note that $\Sigma$ is formal in the sense of Sullivan, but that the nonvanishing of $b_1(\Sigma) \cdot \chi(\Sigma)$ obstructs geometric formality.

On the sphere every metric is formal, because there are no interesting harmonic forms. However, when there are enough harmonic forms, the harmonicity of their products is a restriction on the metric enforcing rigidity.

**Theorem 3.** Every formal Riemannian metric on the two-torus is flat.

**Proof.** Let $g$ be a formal metric on $T^2$, and $\alpha$ a nontrivial harmonic 1-form. Then $\ast \alpha$ and

$$\alpha \wedge \ast \alpha = |\alpha|^2 dvol_g$$

are also harmonic and so $\alpha$ has constant length. In particular it has no zeros. It follows that $\alpha(p)$ and $\ast \alpha(p)$ span $T_p \Sigma$ for all $p \in \Sigma$. As $|a|$ is constant for every constant linear combination $a$ of $\alpha$ and $\ast \alpha$, the Bochner formula

$$\Delta(a) = \nabla^* \nabla a + Ric(a)$$

allows us to compute:

$$0 = \Delta \left( \frac{1}{2} |a|^2 \right) = g(\nabla^* \nabla a, a) - |\nabla a|^2 = -g(Ric(a), a) - |\nabla a|^2.$$

This shows that the (Ricci) curvature is everywhere nonpositive. But by the Gauß-Bonnet theorem this implies that $g$ is flat. \hfill \Box

### 3. Elementary obstructions

Let $M$ be a closed oriented manifold of dimension $n$, and $g$ a formal Riemannian metric on $M$. As usual, we extend $g$ to spaces of differential forms.

**Lemma 4.** The inner product of any two harmonic $k$-forms is a constant function. In particular, the length of any harmonic form is constant.

**Proof.** That the length of any harmonic form is constant follows from equation (1). The more general statement follows by polarisation. \hfill \Box
Lemma 5. Suppose $\alpha_1, \ldots, \alpha_m$ are orthogonal harmonic $k$-forms. Then

$$\alpha = \sum_{i=1}^{m} f_i \alpha_i$$

is harmonic if and only if the functions $f_i$ are all constant.

Proof. If $\alpha$ is harmonic, then $g(\alpha, \alpha_i) = f_i |\alpha_i|^2$ is constant by Lemma 4. Using that the length of $\alpha_i$ is also constant by Lemma 4, we conclude that $f_i$ is constant.

The converse is trivial. \qed

Lemma 4 implies that harmonic forms which are linearly independent at some point are linearly independent everywhere. Systems of linearly independent harmonic forms can be orthonormalised using constant coefficients.

We can now generalise the discussion in Section 2 to higher dimensions.

Theorem 6. Suppose the closed oriented manifold $M^n$ is geometrically formal. Then

1. the real Betti numbers of $M$ are bounded by $b_k(M) \leq b_k(T^n)$, and
2. if $n = 4m$, then $b_{2m}^+(M) \leq b_{2m}^+(T^n)$.
3. The first Betti number $b_1(M) \neq n - 1$.

Proof. Fix a formal Riemannian metric on $M$. It follows from the above Lemmas that the number of linearly independent harmonic $k$-forms is at most the rank of the vector bundle $\Lambda^k$. Similarly, when the dimension is $4m$, the number of self-dual or anti-self-dual harmonic forms in the middle dimension is bounded by the rank of $\Lambda_{\pm}^{2m}$.

Suppose now that $\alpha_1, \ldots, \alpha_{n-1}$ are linearly independent harmonic 1-forms. Then $*(\alpha_1 \wedge \ldots \wedge \alpha_{n-1})$ is also a harmonic 1-form, and is linearly independent of $\alpha_1, \ldots, \alpha_{n-1}$. Thus $b_1(M) \geq n - 1$ implies $b_1(M) = n$. \qed

There is an uncanny similarity here with the classification of flat Riemannian manifolds [10], which satisfy all the conclusions of Theorem 6. We can push this further:

Theorem 7. Suppose the closed oriented manifold $M^n$ is geometrically formal. If $b_1(M) = k$, then there is a smooth submersion $\pi : M \to T^k$, for which $\pi^*$ is an injection of cohomology algebras. In particular, if $b_1(M) = n$, then $M$ is diffeomorphic to $T^n$. In this case every formal Riemannian metric is flat.
Proof. Fix a formal Riemannian metric $g$ on $M$. We consider the Albanese or Jacobi map $\pi : M \to T^k$ given by integration of harmonic 1-forms. As the harmonic 1-forms have constant lengths and inner products, $\pi$ is a submersion. It induces an isomorphism on $H^1$, and products of linearly independent harmonic 1-forms are never zero, but are harmonic because the metric is formal.

In the case $b_1(M) = n$, we conclude that $M$ is a covering of $T^n$, and is therefore a torus itself. Every formal metric on $T^n$ must be flat because it admits an orthonormal framing by harmonic 1-forms.

In the case $b_1(M) = 1$ we have a partial converse to Theorem 7:

**Theorem 8.** Let $M$ be a closed oriented $n$-manifold which fibers smoothly over $S^1$. If $b_1(M) = 1$ and $b_k(M) = 0$ for $1 < k < n - 1$, then $M$ is geometrically formal.

Proof. Suppose $M$ fibers over $S^1$ with fiber $F$ and monodromy diffeomorphism $\phi : F \to F$. By Moser’s Lemma, we may assume that $\phi$ preserves a volume form $\epsilon$ on $F$, so that its pullback to $F \times \mathbb{R}$ descends to $M$ as a closed form which is a volume form along the fibers. We can find a Riemannian metric on $M$ for which $**\epsilon = \alpha$ is the closed 1-form defining the fibration over $S^1$, and has constant length. Then $\alpha$ and $\epsilon$ generate the harmonic forms in degree 1 and $n - 1$, and their product is harmonic.

Flat manifolds satisfy further topological restrictions, for example their Euler characteristics vanish. In our present context we have:

**Theorem 9.** Suppose the closed oriented manifold $M^n$ is geometrically formal.

1. If $b_k(M) \neq 0$, then $e(\Lambda^k) = 0$, and
2. if $n = 4m$ and $b^\pm_{2m}(M) \neq 0$, then $e(\Lambda^2m) = 0$.

In particular the Euler characteristic of $M$ vanishes if $b_1(M) \neq 0$.

Proof. This follows from the obstruction-theory definition of the Euler class, and the fact that every nontrivial harmonic form has no zeros because of (1).

4. Dimension three

If $M$ is a closed oriented geometrically formal 3-manifold, then by Theorem 8 we have $b_1(M) \in \{0, 1, 3\}$. If the first Betti number is maximal, then Theorem 7 says that $M$ is the 3-torus. At the other extreme, if the first Betti number is zero, then $M$ is a rational homology sphere. Clearly every metric on every such manifold is formal.
Thus, the only interesting case is that of first Betti number one. Then the real cohomology algebra is that of the globally symmetric space $S^2 \times S^1$, so that Theorem 2 is proved in the 3-dimensional case.

Theorems 7 and 8 imply:

**Corollary 10.** Let $M$ be a closed oriented 3-manifold with $b_1(M) = 1$. Then $M$ is geometrically formal if and only if it fibers over $S^1$.

This includes many non-symmetric manifolds. Thurston has proved that every 3-manifold which fibers over $S^1$ carries a unique locally homogeneous geometry. It is not clear whether the induced metric is formal, even when the first Betti number is one.

5. **Dimension four**

If $M$ is a closed oriented geometrically formal 4-manifold, then by Theorem 8 we have $b_1(M) \in \{0, 1, 2, 4\}$. If the first Betti number is maximal, then Theorem 7 says that $M$ is the 4-torus.

5.1. **First Betti number = 2.** In this case the Euler characteristic vanishes by Theorem 6, and $b_2(M) = 2$.

Fix a formal Riemannian metric $g$. If $\alpha$ and $\beta$ are harmonic 1-forms generating $H^1(M)$, then they are pointwise linearly independent. Therefore $\omega = \alpha \wedge \beta$ is a non-zero harmonic 2-form with square zero. Thus the intersection form of $M$ is indefinite, and we conclude $b_2^+ = b_2^- = 1$. This means that the real cohomology ring of $M$ is the same as that of the globally symmetric space $S^2 \times T^2$.

We know from Theorem 7 that $M$ fibers over $T^2$. The above discussion shows that the fiber is nontrivial in homology.

There are many examples of such manifolds, other than $S^2 \times T^2$. If $N$ is any 3-manifold with $b_1(N) = 1$ which fibers over the circle, then the product $M = N \times S^1$ is a 4-manifold with the real cohomology ring of $S^2 \times T^2$. By Corollary 10 it is geometrically formal, because we can take a product metric which on $N$ is the formal metric constructed in the proof of Theorem 8.

5.2. **First Betti number = 1.** If the first Betti number is one, the Euler characteristic vanishes by Theorem 6, and therefore $b_2(M) = 0$. In this case $M$ has the real cohomology algebra of the globally symmetric space $S^3 \times S^1$. Theorems 7 and 8 imply:

**Corollary 11.** Let $M$ be a closed oriented 4-manifold with $b_1(M) = 1$ and $b_2(M) = 0$. Then $M$ is geometrically formal if and only if it fibers over $S^1$. 
This includes many non-symmetric manifolds. The simplest example is a product of $S^1$ with a rational homology 3-sphere which is not symmetric.

5.3. **First Betti number** = 0. From Theorem 6 we know $b_2^+ \leq 3$. If $b_2^+ > 0$, then there are nontrivial self-dual harmonic forms. By (1) they have no zeros and so define almost complex structures compatible with the orientation of $M$. It follows that $b_2^+$ is odd. Similarly, if $b_2^- > 0$, then there are almost complex structures compatible with the orientation of $\overline{M}$ and $b_2^-$ must be odd.

Suppose now that $b_2^+(M) = 3$. Then the self-dual harmonic forms trivialise $\Lambda^2_+$, and each defines an almost complex structure with trivial first Chern class (because the pointwise orthogonal complement of each in $\Lambda^2$ is trivial). Thus $0 = c_1(M) = 2\chi(M) + 3\sigma(M) = 4 + 5b_2^+ - b_2^- = 19 - b_2^-$, which contradicts $b_2^- \leq 3$. Therefore $b_2^+ = 3$ is not possible, and similarly $b_2^- = 3$ is not possible either.

Thus the only possible values for $b_2^\pm$ are 0 and 1, and all combinations occur for the globally symmetric spaces $S^4$, $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$. This finally completes the proof of Theorem 2 in the 4-dimensional case.

Any other example must have the same real cohomology ring as one of the above. In fact, other examples exist for each cohomological type. In the case of $S^4$ any rational homology 4-sphere will do. In the case of $\mathbb{C}P^2$, there is the Mumford surface $\overline{\mathbb{C}}$, an algebraic surface of the form $\mathbb{C}H^2/\Gamma$ with the same rational cohomology as $\mathbb{C}P^2$. The Kähler form is of course harmonic, it generates the cohomology and its square is harmonic. Reversing the orientation of the Mumford surface we obtain an example with the cohomology ring of $\overline{\mathbb{C}P^2}$. Finally, in the case of $S^2 \times S^2$, there is also a locally Hermitian symmetric algebraic surface $M$ of general type with the same real cohomology ring, due to Kuga, cf. [1] p. 237. In this case $M$ is of the form $(\mathbb{H}^2 \times \mathbb{H}^2)/\Gamma$, and the harmonic forms are generated by a self-dual and an anti-self-dual harmonic 2-form (with respect to the locally symmetric metric). These are Kähler forms for $M$ and $\overline{M}$ respectively, and are therefore parallel and have harmonic products.

6. **Obstructions from symplectic geometry**

In this section we discuss relations between harmonicity of products of harmonic forms on 4-manifolds on the one hand, and existence of symplectic structures on the other. This leads to further obstructions to geometric formality.
Let $M$ denote a closed oriented 4-manifold with a Riemannian metric $g$. Suppose that $b_2^+(M) > 0$. Then there is a nontrivial $g$-self-dual harmonic 2-form $\omega$. If the product

$$\omega \wedge \omega = \omega \wedge *\omega = |\omega|^2 \text{dvol}_g$$

is harmonic, then $\omega$ has constant length, and in particular has no zeros. It is then a symplectic form on $M$ compatible with the orientation, and $g$ is an almost Kähler metric.

There are 4-manifolds for which the elementary obstructions of Section 3 vanish, but which are not geometrically formal because they do not admit any symplectic structure:

**Example 12.** Let $X$ be $\mathbb{C}P^2$, $S^2 \times S^2$, or the Kuga or Mumford surface. Let $N$ be a rational homology 4-sphere whose fundamental group has a nontrivial finite quotient. Then $M = X \# N$ has the real cohomology ring of the geometrically formal manifold $X$, but is not itself geometrically formal because it does not admit any symplectic structure by the result of [6].

There is another application of the relationship between harmonicity of products of harmonic forms and symplectic structures. Namely we can show that on certain manifolds all products of certain harmonic forms are non-harmonic. This is obviously much stronger than geometric non-formality.

For an example, consider the smooth manifold $M$ underlying a complex K3 surface. Then $M$ is simply connected with $b_2^+ = 3$ and $b_2^- = 19$. The elementary considerations in Section 5 already show that $M$ is not geometrically formal. We can sharpen this as follows:

**Proposition 13.** Let $g$ be an arbitrary Riemannian metric on the K3 surface $M$. If $\alpha$ is a $g$-anti-self-dual harmonic 2-form, then it must have a zero. In particular the wedge product $\alpha \wedge \beta$ is not harmonic for any $\beta$ unless it vanishes identically. For example, if $\alpha$ is nontrivial then $\alpha \wedge \alpha$ is not harmonic.

**Proof.** Suppose $\alpha$ is nontrivial and anti-self-dual. We have

$$\alpha \wedge \alpha = -\alpha \wedge *\alpha = -|\alpha|^2 \text{dvol}_g,$$

which is harmonic if and only if the norm of $\alpha$ is constant. If it is constant, it must be a non-zero constant, and then the above equation shows that $\alpha$ is a symplectic form inducing the opposite (non-complex) orientation on $M$. In particular, $M$ must have non-trivial Seiberg-Witten invariants, see [9]. But the K3 surface contains smoothly
embedded (-2)-spheres, which become (+2)-spheres when the orienta-
tion is reversed, showing that all the Seiberg-Witten invariants vanish, see [5].

**Remark 14.** The vanishing of the Seiberg-Witten invariants for \( \overline{M} \) can also be proved without appealing to the vanishing theorem for spheres of positive self-intersection. For a scalar-flat Calabi-Yau metric the Seiberg-Witten equations on \( \overline{M} \) have no solution, though they do have a (unique) solution on \( M \).

This can be generalised quite substantially. If \( M \) has an indefinite intersection form, there are both self-dual and anti-self-dual harmonic forms for all metrics. If the square of such a form is harmonic, it is a symplectic form on \( M \), respectively \( \overline{M} \). But by the results of [5], mani-
ifolds which are symplectic for both choices of orientation are quite rare. Thus, the above proof generalises to many cases to show that for all metrics on certain 4-manifolds, all self-dual and/or all anti-self-dual harmonic forms must have zeros and non-harmonic squares. This generalises the existence of zeros of harmonic 1-forms on surfaces, cf. Section 2.

In the case of complex surfaces, Theorem 1 of [5] implies the following:

**Theorem 15.** Let \( M \) be a compact complex surface of general type. Assume one of the following conditions holds:

1. \( K_M \) is not ample, or
2. \( c_1^2(M) \) is odd, or
3. the signature \( \sigma(M) \) is negative, or is zero and \( M \) is not uni-
formised by the polydisk.

Then for every Riemannian metric \( g \) on \( M \), all \( g \)-anti-self-dual har-
monic 2-forms have zeros and non-harmonic squares.

In the first case, the argument is the same as in the proof of Proposition 13, because ampleness of \( K_M \) only fails if there are rational curves of negative self-intersection. Note that we only need smoothly rather than holomorphically embedded spheres, so one can replace condition 1. by a weaker assumption.

**7. General existence of non-formal metrics**

Having seen that only very few manifolds are geometrically formal, we now want to show that even these tend to also have non-formal metrics. The two-dimensional case of the following result was already proved in Section 2.
Theorem 16. A closed oriented manifold admits a non-formal Riemannian metric if and only if it is not a rational homology sphere.

Proof. It is clear that every metric on every rational homology sphere is formal because there are no nontrivial harmonic forms.

Conversely, assume that $M$ is a manifold with a non-zero Betti number $b_k(M)$, for $0 < k < \text{dim}(M)$. Let $g$ be a Riemannian metric which has positive curvature operator on an open set, say a ball $B \subset M$, and assume it is formal.

If $\alpha$ is a nontrivial $g$-harmonic $k$-form, then

$$\alpha \wedge \ast \alpha = |\alpha|^2 dvol_g$$

shows that $\alpha$ has constant length. Therefore, the Bochner-Weitzenböck formula

$$\Delta(\alpha) = \nabla^* \nabla \alpha + R(\alpha)$$

for $k$-forms allows us to compute:

$$0 = \Delta\left(\frac{1}{2}|\alpha|^2\right) = g(\nabla^* \nabla \alpha, \alpha) - |\nabla \alpha|^2 = -g(R(\alpha), \alpha) - |\nabla \alpha|^2.$$

Here the term $R$ is positive on $B$, because there the curvature operator is positive. Thus $\alpha$ vanishes identically on $B$. As $\alpha$ is harmonic, the unique continuation principle implies that $\alpha$ vanishes on all of $M$, contradicting the assumption that $\alpha$ is nontrivial.

Remark 17. The above proof shows that there is an open set of non-formal metrics in the space of all Riemannian metrics (with the $C^\infty$ topology, say) on any manifold which is not a rational homology sphere.

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