A Free Group Approach to Error Correction

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Abstract

In this brief exploratory paper, we consider a free group based method of error correction. We present our method, which makes use of the universal property of free groups (UPFG). Through example we show that, in certain scenarios, our method of error correction is more efficient and more deterministic than minimum distance error correction. We argue that our method works well under specific conditions that have been shown to be met by certain neural systems. We hope that this might interest others in the role free groups might play in understanding how error correction of neural codes works.

Introduction

In addition to playing an important technological role, error correction is also being more greatly appreciated for the role it plays in neural systems, allowing for robust neural responses (see, for instance, [1], [2], [3]). In this paper, while we explicitly avoid thinking about error correction in any way other than mathematically, we will make assumptions that make sense for these neural systems.

In the rest of the paper, we think of error correction in the following way. In word space (i.e. the space of all words of a set length that we can send) we partition the space into disjoint sets, where we consider all the elements in a given disjoint set to be equivalent. In this way, we are defining an equivalence relation on the word space. The representative of each disjoint set (i.e. the way in which we refer to all the elements in a given equivalence class) we call a code word. Error correction comprises of determining, for a given word, what it’s code word is (or equivalently, what equivalence class the word belongs to). The process of determining the corresponding code word we call decoding a word. This is all equivalent to imagining that there are a set number of words that would be reasonable to be sent, and any deviation from those are words that have been corrupted, so decoding takes the form of finding the most likely sent word.

The partitioning of a given space into reasonable disjoint sets is by no means a trivial operation. Prentice et al., who have a similar view of error correction, use a hidden markov process to determine the code words and the partitioning of their space. In the rest of the paper, it will be assumed that the code words are already known, so we will not actually concern ourselves with constructing the partition.
Model

Let the set of generators, \( G \), be defined as \( G = S + B \), where \( S = \{1, 2, ..., n\} \), \( B \subset P(S) \) s.t. \( \forall b \in B, |b| > 1 \), and + is the set concatenation operator. We refer to \( B \) as the basis set. A simple example of this is, for \( n = 5 \), \( G = \{1, 2, 3, 4, 5, 12, 145\} \), where the last two elements are the elements of the basis set (and here 12 stands for \{1, 2\}).

Let the group of codewords, \((C, \ast_C)\), be defined as \( C = \{c_1, c_2, ..., c_m\} \), where \( \forall c_i \in C, c_i \in P(G) \) and \( \ast_C \) is some operation on the elements of \( C \) that meets the standard group criterion ([4]).

Finally, let the restricted free group of \( G \), \( \tilde{F}(G) \), be defined as the abeleanized group of all elements in the free group of \( G \), \( F(G) \), that are made up of, at most, each element of \( G \) once. For instance, while \( 121 \in F(G) \), \( 121 \notin \tilde{F}(G) \) because it has 1 twice.

Using the universal property of free groups (UPFG) ([4]), we get the following diagram,

\[
\begin{array}{ccc}
G & \xrightarrow{i} & \tilde{F}(G) \\
& \searrow{g} & \downarrow{\phi} \\
& & C
\end{array}
\]

(1)

where \( i \) is the inclusion map, \( g \) is a group function determining \( C \) from \( G \), and \( \phi \) is a unique homomorphism (given a specific \( g \)) from \( \tilde{F}(G) \) to \( C \). The UPFG tells us that we can relate \( g \) to \( \phi \) by \( \phi(A) = g(a_1) \ast_C ... \ast_C g(a_n) \), where \( A \in \tilde{F}(G) \) and \( A = a_1...a_n, a_i \in G \).

A final point on our model must be made. We define, \( \forall s \in S, g(s) = \text{id}_C \). With this definition, it is more clear what the role of the basis set, \( B \), is (as before, it’s role was unnecessary for ”building” \( \tilde{F}(G) \)); it is the generator of \((C, \ast_C)\) under \( g \).

Decoding

We now consider an element, \( \omega \), of \( \tilde{F}(G) \). \( \forall \omega \in \tilde{F}(G), \exists \{b_1, ..., b_k\}, b_i \in B \) and \( \{s_1, ..., s_l\}, s_j \in S \) s.t. \( \omega = b_1...b_ks_1...s_l \). Of course, this decomposition of \( \omega \) is by no means unique. We therefore define the decomposition of \( \omega \) as the pair \((\{b_i\}, \{s_j\})\), s.t. the size of \( \{b_i\} \) is the minimum of all decompositions of \( \omega \).

The decoding of \( \omega \) is given by \( \phi(\omega) \) (since \( \phi : \tilde{F}(G) \rightarrow C \)).

\[
\phi(\omega) = \phi(b_1...b_ks_1...s_l) = g(b_1) \ast_C ... \ast_C g(b_k) \ast_C g(s_1) \ast_C ... \ast_C g(s_l)
\]

(2)

By the defined decomposition of \( \omega \), this decoding is unique.
Example

To more clearly illustrate this error correction process, we give an example and show how this compares to minimum distance (MD) correction.

Let $G = \{1, 2, 3, 4, 12, 24\}$, $C = (\{0, 12, 24, 14\}, +)$. In a binary string representation, this corresponds to $G = \{1000, 0100, 0010, 0001, 1100, 0101\}$ and $C = (\{0000, 1100, 0101, 1001\}, + \text{mod}(2))$, if we take each number $1, \ldots, 4$ to be a position in a four bit string that has a one.

The results of applying our decoding method and applying MD decoding are given in Table 1. From this, we see that, first and foremost, the number of three way ties (as denoted by ?) is significantly less using the UPFG decoding as opposed to MD decoding (one vs. eight). Additionally, in every determined decoding, the two methods agree.

Error in decoding

There are two ways in which our error correction model can assign the wrong code word and thus, lead to an incorrect decoding. Because incorrect decoding signifies a break in our method, we discuss both error sources below.

An obvious source of error in our decoding method comes from the fact that the decoding process is entirely dependent on the specific basis elements being present. More specifically, in the example given in the example, we see that 1001 was assigned to itself, whereas one change to that, 0001, gets assigned a completely different codeword: the null word. Thus, if the error affects one of the elements in $B$, there is an error in our decoding.

However, if the probability that an error affects a basis element is very low - as it might be for certain neural systems where the basis elements are neurons with the same receptive field, or neurons that all don’t fire together - then this source of error wouldn’t be too detrimental to the decoding.

The other source of error arises from the fact that, if a number of additional elements of $S$ are added to a word then, for a sufficient number and position of these elements, a different decomposition will occur, leading to a different decoding. This is simply another way of saying that, like any error correcting method, if the sent word has too many corruptions, then our decoding won’t return the correct word.

Of course, if the generators are sufficiently disjoint (in terms of their Hamming distance) and the probability of adding any additional element from $S$ is small, then this also won’t be a large concern either. A more rigorous analysis of what the threshold for disjointness and probability needs to be in order for this error to be small will be worked out in a future paper.
| Word  | UPFG decoding | MD decoding |
|-------|---------------|-------------|
| 1111  | ?             | ?           |
| 1110  | 1100          | 1100        |
| 1101  | 1100          | ?           |
| 1011  | 1001          | 1001        |
| 0111  | 0101          | 0101        |
| 1100  | 1100          | 1100        |
| 1001  | 1001          | 1001        |
| 0011  | 0000          | ?           |
| 1010  | 0000          | ?           |
| 0101  | 0101          | 0101        |
| 0110  | 0000          | ?           |
| 1000  | 0000          | ?           |
| 0100  | 0000          | ?           |
| 0010  | 0000          | 0000        |
| 0001  | 0000          | ?           |
| 0000  | 0000          | 0000        |

Table 1

Discussion

We have presented a novel error correction model that uses the universal property of free groups (UPFG) to assign possible sent words (the codewords) to a neighborhood of possible received words. The subsequent collapsing of that neighborhood onto the codeword allows for error correction.

As illustrated in the example, our model can, in certain scenarios, be more deterministic and more efficient than the minimum distance (MD) decoding, as there are fewer undetermined decodings and fewer words to remember in order to correctly decode.

The full analysis of showing under what conditions the UPFG error correction does better than MD error correction given the strict two types of error in the decoding as discussed in the previous section has not been given and will have to be addressed in a future paper.

However, as discussed in the previous section, we believe that, for a system with disjoint enough codewords and basis elements that are not so susceptible to error, our method of decoding should work well. The fact that this type of system was found in salamander retinal ganglion cells ([1], [3]) bolsters our belief that UFPG decoding may play a role in error correcting in neural systems.

We hope that this brief exploratory paper convinces others that the role free groups, and the UFPG, plays in error correction might be an interesting and fruitful one, especially with regards to neural coding.

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