Four spins correlation function of the $q$ states Potts model, for general values of $q$. Its percolation model limit $q \to 1$.

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Abstract.

Under the assumption that the product of two spin operators decomposes uniquely into the degenerate conformal fields $\{\Phi_{n',n}\}$, the general expression for the correlation function of four spins is defined for the $q$ states Potts model with $q$ taking general values in the interval $1 \leq q \leq 4$.

The limit of $q \to 1$ is considered in detail and the four spin function is obtained for the percolation model.
1 Introduction.

The renewed interest in the critical Potts model is related to the studies of three and four points cluster connectivities, which are connected to the corresponding spin correlation functions [1] - [5].

In the context of 2D conformal field theory, of the $q$ states Potts model [6, 7], the four spins function:

$$< \sigma(\infty) \sigma(1) \sigma(z, \bar{z}) \sigma(0)>$$ (1.1)

is a complicated object. This is because the spin operator is not a degenerate field, for general values of $q$, unlike for instance the energy operator which is the $\Phi_{1,2}$ degenerate field, in the conformal field theory classification, for general values of $q$ in the interval $1 \leq q \leq 4$.

For the degenerate operators, the conformal field theory provides well defined methods for calculation, in particular, of four-point functions [8, 7, 9]. For non-degenerate operators such techniques are absent.

In the next Section we shall suggest the method which allows to turn around this difficulty, in order to define the four spins function (1.1) for general values of $q$.

To be more precise, this is done under the assumption, yet to be justified, that the product of two spin operators decomposes uniquely into the degenerate conformal fields $\{\Phi_{n',n}\}$. This assumption is presented in some more details further below, in the next Section, in the set of remarks which follow the equation (2.36).

In the third Section we shall consider the limit $q \rightarrow 1$ for the function (1.1), which is the percolation model four spins function. This limit turns out to be very delicate.

The Section 4 is devoted to the discussion.
2 General expression for the four spins correlation function of the $q$ states Potts model.

The full space of degenerate operators of the conformal field theory with the central charge $c$ in the interval

$$0 \leq c \leq 1$$

(2.1)

is covered by the fields [5]:

$$\{\Phi_{n',n}\}$$

(2.2)

$n' = 1, 2, 3, \ldots, \infty, n = 1, 2, 3, \ldots, \infty$, having conformal dimensions

$$\{\Delta_{n',n}\}$$

(2.3)

which are given by the Kac formula

$$\Delta_{n',n} = \frac{(\alpha_- n' + \alpha_+ n)^2 - (\alpha_- + \alpha_+)^2}{4},$$

(2.4)

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1},$$

(2.5)

$$c = 1 - 24\alpha_0^2$$

(2.6)

Above is used the Coulomb Gas (CG) based parametrisation. $\alpha_-, \alpha_+$ are the charges of the screening operators, $\alpha_0$ is the background charge of the vacuum [7]. The CG vertex operators representing the primary fields (2.2) have the form:

$$V_{n',n}(z, \bar{z}) \equiv V_{\alpha_{n',n}}(z, \bar{z}) = \exp\{i\alpha_{n',n}\varphi(z, \bar{z})\}$$

(2.7)

$\varphi(z, \bar{z})$ is a free scalar field and the corresponding CG charges $\{\alpha_{n',n}\}$ have the form:

$$\alpha_{n',n} = \frac{1 - n'}{2}\alpha_- + \frac{1 - n}{2}\alpha_+$$

(2.8)

If, in turn, $\alpha_+^2$ is parametrized as

$$\alpha_+^2 = \frac{p + 1}{p}$$

(2.9)

then, for the parameter $p$ taking odd integer values

$$p = 3, 5, 7, \ldots, \infty,$$

(2.10)
the corresponding conformal field theories, with the central charge

$$c_p = 1 - \frac{6}{p(p+1)}, \quad (2.11)$$

correspond to minimal models representing the unitary set of Potts models, with \(q\) taking a discrete (infinite) set of values ranging from \(q = 2, p = 3, c = 1/2\) to \(q = 4, p \to \infty, c = 1\) [10]. More precisely, \(q\) is related to the parameter \(p\) by the formula:

$$\sqrt{q} = 2 \cos \frac{\pi}{p+1} \quad (2.12)$$

More details could be found in [6, 7, 10].

To come to \(c_p\) in (2.11), from the central charge expression in (2.6), the standard relations are to be used:

$$\alpha_+ = \sqrt{\frac{p+1}{p}}, \quad \alpha_+ \alpha_- = -1, \quad \alpha_- = -\sqrt{\frac{p}{p+1}}$$

$$\alpha_0 = \frac{\alpha_+ + \alpha_-}{2} = \frac{1}{2\sqrt{p(p+1)}} \quad (2.13)$$

For a particular unitary minimal model, with

$$p = 2s + 1 \quad (2.14)$$

\(s\) being positive integer, a finite set of primary fields (2.2) decouple, from the rest, by the operator algebra. They form a finite table (Kac table):

$$\{\Phi_{n',n}\}, \quad 1 \leq n' \leq 2s + 1, \quad 1 \leq n \leq 2s \quad (2.15)$$

In fact, the number of primary fields, of the corresponding model, is twice less, because of the symmetry: the fields

$$\Phi_{n,n} \quad \text{and} \quad \Phi_{2s+2-n',2s+1-n} \quad (2.16)$$

are identical. They have the same conformal dimensions and identical conformal theory properties.

The spin operator \(\sigma\), of the \(M_p\) minimal model (2.15), finds itself in the middle of the Kac table. Its two copies are the operators:

$$\Phi_{\frac{p+1}{p}, \frac{p-1}{p}} \quad \text{and} \quad \Phi_{\frac{p+1}{p}, \frac{p+1}{p}} \quad (2.17)$$
or, for \( p = 2s + 1 \):

\[
\Phi_{s+1,s} \quad \text{and} \quad \Phi_{s+1,s+1}
\]  

(2.18)

Now, for general (real) values of the parameter \( p \), in the interval \( 2 \leq p < \infty \), and hence the general values of the central charge (2.11), in the interval \( 0 \leq c \leq 1 \), the Kac table of the degenerate fields (2.2) is infinite, while the spin field, equations (2.17), (2.18), no longer belongs to the Kac table, its indices being real numbers, in general, instead of positive integers. Still the spin operator is represented by the two fields in (2.18). Their conformal dimensions are equal if \( s = (p - 1)/2 \).

The dimensions of the fields in (2.17) could still be calculated by the formula (2.4).

More explicitly, with the parametrization by the parameter \( p \), this formula is of the form:

\[
\Delta_{n',n} = \frac{(pn' - (p + 1)n)^2 - 1}{4p(p + 1)}
\]  

(2.19)

For \( n' = (p + 1)/2 \), \( n = (p - 1)/2 \), of the first field in (2.17), and also for \( n' = (p + 1)/2 \), \( n = (p + 1)/2 \), for the second field in (2.17), one gets the same value:

\[
\Delta_{\sigma} = \frac{(p - 1)(p + 3)}{16p(p + 1)}
\]  

(2.20)

– the conformal dimension of the spin field of the Potts model, for general values of the parameter \( p \).

The fact that the spin operator has two representations, as in (2.17), for general values of \( p \) (or \( c \), or \( q \)), in the same way as it was the case for the minimal models when \( p \) was odd integer, this fact will be important in the following.

In the context of a particular minimal model \( M_p \), with \( p \) being odd integer, the parameter

\[
s = \frac{p - 1}{2}
\]  

(2.21)

in (2.18) being positive integer, the correlation function of four spins (1.1) could readily be calculated with the Coulomb Gas technique [7, 9]. But for general values of \( p \) this way of calculating the four spin function is blocked: one would need to use a fractional number of screening operators, a fractional number of integrations, in the technique of [7, 9].
On the other hand, the four-point function
\[
< \sigma(\infty)\Phi_{n',n}(1)\Phi_{n',n}(z,\bar{z})\sigma(0) > \tag{2.22}
\]
could readily be calculated with the CG technique. It could be represented, in the CG technique, by the four-point function
\[
< V_{\alpha}^{+}(\infty)V_{\alpha_{n'},n}(1)V_{\alpha_{n'},n}(z,\bar{z})V_{\alpha}(0) >_{conf} \tag{2.23}
\]
The CG charge of the operators \(V_{\alpha_{n'},n}(1), V_{\alpha_{n'},n}(0)\) is given by the formula (2.8). The charge of the spin operator \(V_{\alpha}\) is also defined by the formula (2.8), but with the fractional indices \[n' = \frac{p+1}{2}, \quad n = \frac{p-1}{2}\] in the case of the first operator in (2.17):
\[
\alpha_{\sigma} = \frac{1 - \frac{p+1}{2}}{2} \alpha_{-} + \frac{1 - \frac{p-1}{2}}{2} \alpha_{+} = \frac{1}{4} \alpha_{-} + \frac{3 - p}{4} \alpha_{+} \tag{2.24}
\]
The charge of the operator \(V_{\alpha}^{+}(\infty), \) representing the spin operator at infinity, is CG conjugate:
\[
\alpha_{\sigma}^{+} = 2\alpha_{0} - \alpha_{\sigma} \tag{2.25}
\]
The total CG charge of the four vertex operators in (2.23) is
\[
2\alpha_{0} - \alpha_{\sigma} + 2\alpha_{n',n} + \alpha_{\sigma} = 2\alpha_{0} + 2\alpha_{n',n} = 2\alpha_{0} + (1 - n')\alpha_{-} + (1 - n)\alpha_{+} \tag{2.26}
\]
According to the CG technique \[7, 9\], this function requires \(n' - 1\) screening operators
\[
\int d^{2}u V_{\alpha_{-}}(u,\bar{u}), \quad V_{\alpha_{-}}(u,\bar{u}) = e^{i\alpha_{-}\phi(u,\bar{u})} \tag{2.27}
\]
and \(n - 1\) screening operators
\[
\int d^{2}v V_{\alpha_{+}}(v,\bar{v}), \quad V_{\alpha_{+}}(v,\bar{v}) = e^{i\alpha_{+}\phi(v,\bar{v})} \tag{2.28}
\]
This means that the four-point function (2.23), and hence the function (2.22), will be expressed by the multiple integral of \[9\] and it could be calculated.

Now, in more detail, the function (2.22) could be represented, in the \(s\) - channel decomposition (decomposition in powers of \(z, \bar{z}\)), as follows:
\[
< \sigma(\infty)\Phi_{n',n}(1)\Phi_{n',n}(z',\bar{z})\sigma(0) > \nonumber = \sum_{t',t} \frac{1}{|z|^{2\Delta_{\sigma} + 2\Delta_{n',n} - 2\Delta_{t',t}}} \times (D_{\sigma_{n',n}}(t',t))^{2} |F_{t',t}(z)|^{2} \tag{2.29}
\]
\[ F_{t',t}(z) \] is the conformal block function of the intermediate channel \((t', t)\), which is defined, in the standard way, by the Virassoro algebra \([8]\). This function depends also on the four external operators which dependence have been suppressed.

The operator algebra constants in \((2.29)\)

\[
\{ D_{\sigma,(n',n)}^{(t',t)} \} \equiv D_{\sigma,\Phi_{n',n}}^{(t',t)}
\]

– they are given by the multiple integrals of \([9]\). They are expressed finally by products of \(\Gamma\) functions.

When \(n'\) is given odd values, \(n' = 2l + 1, l = 0, 1, 2, 3, \ldots\),

\[
< \sigma(\infty) \Phi_{2l+1,n}(1) \Phi_{2l+1,n} (\bar{z}) \sigma(0) >
= \sum_{(t',t)} \frac{1}{|z|^{2\Delta_{\sigma}+2\Delta_{2l+1,n} - 2\Delta_{t',t}} (D_{\sigma,(2l+1,n)}^{(t',t)})^2 |F_{t',t}(z)|^2 ,
\]

in this case among the operators of the intermediate channels in \((2.31)\), \(\{ \Phi_{t',t} \} \), there is the spin operator \(\sigma\): the channel of the first operator in \((2.17)\) if \(n\) is odd, and the channel of the second operator in \((2.17)\) if \(n\) is even, Fig.1. The operator algebra constant for this particular channel,

\[
D_{\sigma,(2l+1,n)}^\sigma ,
\]

in the sum of \((2.31)\), corresponds to fusion:

\[
\sigma \times \Phi_{(2l+1,n)} \to \sigma
\]

But this process could be turned around as in Fig.2,

\[
D_{\sigma,(2l+1,n)}^\sigma = D_{\sigma,\sigma}^{(2l+1,n)}
\]

We remind that for the normalised operators, the operator algebra constants are symmetric in all three indices, i.e. in all three operators. In this way, from one particular intermediate channel of the four point function in \((2.31)\), we get the constants for the product of two spin operators producing the fields

\[
\{ \Phi_{2l+1,n} \}
\]
Figure 1: One intermediate channel, out of many, is that of \( \{ \Phi_{t',t} \} = \sigma \).

Figure 2: The symmetry of the operator algebra constants \( D^\sigma_{\sigma,(2l+1,n)} \), eq. (2.34).
In other words, we get the operator algebra constants for the intermediate channels of the function

\[ < \sigma(\infty)\sigma(1)\sigma(z, \bar{z})\sigma(0) > = \sum_{(l,n)} \frac{1}{|z|^{4\Delta_{\sigma}-2\Delta_{2l+1,n}}}(D_{\sigma,\sigma}^{(2l+1,n)})^{2}|F_{(2l+1,n)}(z)|^{2} \]  

(2.36)

Stating it again, the constants in the decomposition (2.36) above, by using the symmetry (2.34), Fig.2, could readily be calculated by the well defined integrals which define the four-point function (2.31). This is done by extracting an appropriate channel from it, from the sum in (2.31), Fig1.

Several remarks could be added at this point.

The three-point functions could be defined for any three operators \([11, 3, 12]\]. In general, they correspond to projecting a couple of operators on the third one. They do not, in general, represent the operator algebra constants of a particular theory, the constants which would appear, naturally, in the decomposition of four-point functions.

The constants \(D_{\sigma,\sigma}^{(2l+1,n)}\), eq.(2.32), which could also be calculated as three-point functions, they do represent the operator algebra constants, of the Potts model, as they originate in the four-point function (2.31). They could be called physical in the above sense. In this case the constants \(D_{\sigma,\sigma}^{(2l+1,n)}\), in (2.34), are also physical, they correspond to real processes in the model. They define, in particular, the four point function \(< \sigma \sigma \sigma \sigma >\), as in eq.(2.36).

As has been stated above with respect to the four-point function (2.29), the spin operator appears as one of the intermediate channels of this function in the case when the index \(n'\) is an odd integer. For \(n'\) even, the spin operator is not among the intermediate channel operators. Which means, in turn, that the primary operators \(\{\Phi_{n',n}\}\), with \(n'\) even, do not appear in the fusion of two spin operators. The corresponding operator algebra constants are zero. In this case the sum in (2.36) could be considered as being taken over all the primary fields, of the Potts model, in the intermediate channels.

We consider, but this is clearly an assumption, that the set of primary fields \(\{\Phi_{n',n}\}\)
is complete in the "neutral", or "even" sector: the sector which is generated by fusing an even number of spins, for the model with \( q \) general, a real number.

The spin type operators, or "odd" sector fields, are generated by fusing the spin operator with the fields \( \{ \Phi_{n',n} \} \), with the energy operator \( \Phi_{1,2} \) to begin with. The fusion rule is clear from the Coulomb Gas integral representation of the function \( \langle \sigma(\infty)\Phi_{n',n}(1)\Phi_{n',n}(z,\bar{z})\sigma(0) \rangle \), eq.(2.29). The sum over \((t',t)\), eq.(2.29), is being taken over fractional values: the fractional indices of the operator \( \sigma \), eq.(2.17), being shifted by integer values.

In other words, the fusion rule \( \sigma \times \Phi_{n',n} \rightarrow \sum_{t',t} \Phi_{t',t} \) is the same as that of minimal models. The indices of the operators \( \{ \Phi_{t',t} \} \), which we call spin type operators, have fractional values which are the same as those of the spin operator but shifted by integer values. For instance, \( \sigma \times \Phi_{1,2} = \Phi_{p+1,2} \times \Phi_{1,2} \rightarrow \Phi_{p+1,2} + \Phi_{p+3,2} \). \( \Phi_{p+1,2} \) is the spin operator, \( \sigma \), and \( \Phi_{p+3,2} \) is a particular spin type operator. The conformal dimensions of non-degenerate spin type operators \( \{ \Phi_{t',t} \} \) are still given by the formula (2.19), as follows from the Coulomb Gas. The operator algebra coefficients for fusions \( \sigma \times \Phi_{n',n} \rightarrow \sum_{t',t} \Phi_{t',t} \) could all be calculated by the corresponding Coulomb Gas integrals, the way this is done for a particular fusion channel, \( \sigma \times \Phi_{n',n} \rightarrow \sigma \), in the Section 3.

Under the assumption made above, on the completeness of the set of degenerate fields \( \{ \Phi_{n',n} \} \) for the even sector, the decomposition (2.36), for the function \( \langle \sigma \sigma \sigma \sigma \rangle \), will be exact. It defines the four-spin function of the Potts model.

We shall finish this Section by giving formulas for the operator algebra coefficients \( (D^{(2l+1,n)})^2 \) in (2.36).

As has been stated above, they are defined by a particular channel in the \( s \)-channel decomposition (2.31), for the four-point function \( \langle \sigma(\infty)\Phi_{2l+1,n}(1)\Phi_{2l+1,n}(z,\bar{z})\sigma(0) \rangle \). This function is well defined by the Coulomb Gas integrals:

\[
\langle \sigma(\infty)\Phi_{2l+1,n}(1)\Phi_{2l+1,n}(z,\bar{z})\sigma(0) \rangle > \\
\propto V_{\alpha_1}(\infty)V_{\alpha_2l+1,n}(1)V_{\alpha_{2l+1,n}}(z,\bar{z})V_{\alpha_1}(0) >_{\text{conf}} \quad (2.37)
\]

The cases of values of the index \( n \) being odd and even have to be considered separately.
For \( n \) taking odd values, \( n = 2k + 1 \), the total Coulomb Gas charge in the function (2.37) is given by:

\[
\alpha_\sigma^+ + 2\alpha_{2l+1,2k+1} + \alpha_\sigma
\]

\[
= 2\alpha_0 - \alpha_\sigma + (-2l\alpha_- - 2k\alpha_+) + \alpha_\sigma
\]

\[
= 2\alpha_0 - 2l\alpha_- - 2k\alpha_+
\]  

(2.38)

Accordingly, the Coulomb Gas function (2.37) \( (n = 2k + 1) \) requires \( 2l \) screenings \( \alpha_- \) and \( 2k \) screening \( \alpha_+ \). One obtains:

\[
<V_{\alpha_\sigma^+}(\infty)V_{\alpha_{2l+1,2k+1}}(1)V_{\alpha_{2l+1,2k+1}}(z, \bar{z})V_{\alpha_\sigma}(0)><_{\text{conf}}
\]

\[
= \prod_{i=1}^{2l} \int d^2u_i \prod_{j=1}^{2k} \int d^2v_j <V_{\alpha_\sigma^+}(\infty)V_{\alpha_{2l+1,2k+1}}(1)V_{\alpha_{2l+1,2k+1}}(z, \bar{z})V_{\alpha_\sigma}(0)
\]

\[
\prod_{i=1}^{2l} V_{\alpha_-}(u_i, \bar{u}_i) \prod_{j=1}^{2k} V_{\alpha_+}(v_j, \bar{v}_j) >_{(2\alpha_0)}
\]  

(2.39)

The indexing \(< ... >_{\text{conf}}, < ... >_{(2\alpha_0)}\) should be clear from the equations above, for those familiar with the CG technique.

Different channels in (2.31) correspond to different separations of screenings: in one subset, the screenings are integrated around 0 and \((z, \bar{z})\) while the rest of screenings are integrated around 1 and \(\infty\). In the limit of \(z \to 0\) this separation is exact and, accordingly, the coefficients \((D_{n}^{(t',t)})_{\sigma,(2l+1,2k+1)}^2\) are being equal to the product of two 3-point functions.

The particular channel, when \(\Phi(t',t) = \sigma\), is obtained when \(l\) screenings \(\alpha_-\) and \(k\) screenings \(\alpha_+\) are being integrated around 0 and \((z, \bar{z})\), while the remaining \(l\) and \(k\) screenings are integrated around 1 and \(\infty\): over the whole plane in fact, neglecting the presence of operators at 0 and \((z, \bar{z})\) and the other screenings running around them.

Identification of operators in the intermediate channels of (2.31), of the spin operator in particular, is obtained by calculating the total Coulomb Gas charge of two vertex operators in the product of (2.39), at 0 and \((z, \bar{z})\) for instance, and the cloud of screenings running around them. In particular, when the operators at 0 and \((z, \bar{z})\), \(V_{\alpha_\sigma}(0)\) and
$V_{2l+1,2k+1}(z, \bar{z})$, are surrounded by $l$ and $k$ screenings, the total charge of this subset of operators is given by:

$$\alpha_\sigma + \alpha_{2l+1,2k+1} + l\alpha_- + k\alpha_+ = \alpha_\sigma + (-l\alpha_- - k\alpha_+) + l\alpha_- + k\alpha_+ = \alpha_\sigma$$

(2.40)

So that the corresponding intermediate channel operator is in fact the spin operator.

In correspondence to the above described factorisation of the total set of operators (in the limit of $z \to 0$), into a subset of operators located around 0 and $(z, \bar{z})$ and another subset located around 1 and $\infty$, the coefficients, squared, $(D_{\sigma,(2l+1,2k+1)})^2$ are being expressed as a product of two 3-point functions:

$$(D^{(t',t)}_{\sigma,(2l+1,2k+1)})^2 = <\sigma(\infty)\Phi_{2l+1,2k+1}(1)\Phi_{t',t}(0)> <\Phi_{t',t}(\infty)\Phi_{2l+1,2k+1}(1)\sigma(0)>$$

$$\propto V_{\sigma,t}(\infty)V_{2l+1,2k+1}(1)V_{\sigma,t}(0) >_{conf} V_{\sigma,t}(\infty)V_{2l+1,2k+1}(1)V_{\sigma}(0) >_{conf}$$

(2.41)

$V_{2l+1,2k+1} \equiv V_{\sigma_{2l+1,2k+1}}$. The variables $z, \bar{z}$ of $V_{2l+1,2k+1}(z, \bar{z})$ in the second 3-point function are assumed to have been factored out by scaling.

In particular, for the spin operator in the intermediate channel:

$$(D_{\sigma,(2l+1,2k+1)})^2 = <\sigma(\infty)\Phi_{2l+1,2k+1}\sigma(0)> <\sigma(\infty)\Phi_{2l+1,2k+1}(1)\sigma(0)>$$

$$\propto V_{\sigma,t}(\infty)V_{2l+1,2k+1}(1)V_{\sigma}(0) >_{conf} V_{\sigma,t}(\infty)V_{2l+1,2k+1}(1)V_{\sigma}(0) >_{conf}$$

$$= C_{\sigma,t,\sigma_{2l+1,2k+1},\sigma} \times C_{\sigma,t,\sigma_{2l+1,2k+1},\sigma}$$

(2.42)

The $C$ constants are the operator algebra coefficients of the Coulomb Gas vertex operators.

The proportionality in (2.42) becomes the equality, when normalisation of the Coulomb
Gas operators is taken into account [12]. One gets:

\[
<V_{\alpha_+}^{+}(\infty)V_{2l+1,2k+1}(1)V_{\alpha_+}(0)> = <V_{\alpha_+}^{+}(\infty)V_{2l+1,2k+1}(1)V_{\alpha_+}(0)>^2 \\
= N(V_{\alpha_+})^2 \times N(V_{2l+1,2k+1})^2 \times (N(V_{\alpha_+}))^2 \times <\sigma(\infty)\Phi_{2l+1,2k+1}(1)\sigma(0)>^2 \\
= \frac{1}{N(V_{\alpha_+})^2} \times N_{2l+1,2k+1}^2 \times N(V_{\alpha_+})^2 \times (D_{\sigma,2l+1,2k+1}^\sigma)^2 \\
= \frac{1}{Z^2} N_{2l+1,2k+1}^2 (D_{\sigma,2l+1,2k+1}^\sigma)^2 \quad (2.43)
\]

Finally

\[(D_{\sigma,2l+1,2k+1}^\sigma)^2 = \frac{Z^2}{N_{2l+1,2k+1}^2} (C_{\alpha_+,\alpha_{2l+1,2k+1},\alpha_+})^2 \quad (2.44)
\]

Here \(Z\) is the Coulomb Gas partition function [12]:

\[
Z = \frac{-\rho}{(1-\rho)^2} \gamma(\rho')\gamma(\rho), \quad \rho' = \alpha_+^2, \quad \rho = \alpha_-^2, \quad \gamma(x) = \Gamma(x)/\Gamma(1-x) \quad (2.45)
\]

\(N_{2l+1,2k+1}\) is the norm of the operator \(V_{2l+1,2k+1} = V_{2l+1,2k+1}^\alpha\). Is used also the relation between the norms of the operators \(V_{\alpha_+}^+\) and \(V_{\alpha_+}^+\) [12]:

\[
N(V_{\alpha_+}^+) \equiv N(V_{\alpha_+}^+) = \frac{1}{ZN(V_{\alpha_+})} \quad (2.46)
\]
The Coulomb Gas structure constants are given by [9, 12]:

\[ C_{c,b,a} = \langle V_c(\infty)V_b(1)V_a(0) \rangle_{conf} \]

\[ = \frac{\rho^{-4k}}{Z} \times \prod_{i=1}^{l} \gamma(i\rho' - k) \times \prod_{j=1}^{k} \gamma(j\rho) \]

\[ \times \prod_{i=0}^{l-1} \gamma(1 - k + \alpha' + i\rho')\gamma(1 - k + \beta' + i\rho')\gamma(1 - k + \gamma' + i\rho') \]

\[ \times \prod_{j=0}^{k-1} \gamma(1 + \alpha + j\rho)\gamma(1 + \beta + j\rho)\gamma(1 + \gamma + j\rho) \]

\[ = \frac{\rho^{-4l}}{Z} \prod_{i=1}^{l} \prod_{j=1}^{k} \frac{1}{(i\rho' - j)^2} \times \prod_{i=1}^{l} \gamma(i\rho') \prod_{j=1}^{k} \gamma(j\rho) \]

\[ \times \prod_{i=0}^{l-1} \prod_{j=0}^{k-1} \gamma(1 + \alpha' + i\rho')\gamma(1 + \beta' + i\rho')\gamma(1 + \gamma' + i\rho') \]

\[ \times \prod_{j=0}^{k-1} \gamma(1 + \alpha + j\rho)\gamma(1 + \beta + j\rho)\gamma(1 + \gamma + j\rho) \]

- equations (4.4),(4.8) of [12], with the normalization factor $1/Z$ added. In the equations above $l$ and $k$ are the numbers of screening operators needed by the Coulomb Gas correlator $\langle V_c(\infty)V_b(1)V_a(0) \rangle_{conf}$. $a, b, c$ are the Coulomb Gas charges of the operators $V_a, V_b, V_c$. The parameters $\alpha', \beta', \gamma', \alpha, \beta, \gamma$ are given by:

\[ \alpha' = 2\alpha_a, \quad \beta' = 2\alpha_b, \quad \gamma' = 2\alpha_c, \]

\[ \alpha = 2\alpha_a, \quad \beta = 2\alpha_b, \quad \gamma = 2\alpha_c \]

Our excuses for using $\alpha$ (with indexes) for the Coulomb Gas charges is the previous equations, while also as a parameter (without indexes) in the equations (2.47), (2.48).

Similarly, our excuses for using $\gamma$ for the ratio of $\Gamma$ functions ($\gamma(x)$, eq. (2.45)) in the equation (2.47), while also for the parameters $\gamma, \gamma'$ in (2.47), (2.48).
The norm squared \( N_{2l+1,2k+1}^2 \) in the equation (2.44), of the operator \( V_{2l+1,2k+1} \equiv V_{\alpha,2l+1,2k+1} \), is given by the general equation [12]:

\[
N(V_{\alpha',n}) = \langle I^+(\infty) V_{\alpha',n}(1) V_{\alpha',n}(0) \rangle = \prod_{i=1}^{n'-1} \frac{\gamma(-1 + (i + 1)\rho)}{\gamma(i\rho)} \times \prod_{j=1}^{n-1} \frac{\gamma(-1 + (1 + j)\rho)}{\gamma(j\rho)}
\times \prod_{i=1}^{n'-1} \prod_{j=1}^{n-1} \frac{(-i + j\rho)^2}{(-i + j\rho)^2}
\]

(2.49)

\( I^+ \equiv V_{2\alpha_0} \) is the CG conjugate identity operator.

The equation (2.44) defines the structure constants \( (D_{\sigma,(2l+1,n)})^2 \), and hence the constants \( (D_{\sigma,2l+1,n})^2 \) in (2.36), in the case when the index \( n \) is an odd integer, \( n = 2k + 1 \). In the case of \( n \) being an even integer, \( n = 2k \), the expressions are differences. In this case the Coulomb Gas function (2.37),

\[
< V_{\alpha'}^+(\infty) V_{2l+1,2k}(1) V_{2l+1,2k}(z, \bar{z}) V_{\alpha}(0) >_{conf}
\]

(2.50)

will require \( 2l \) screenings \( \alpha_- \) and \( 2k - 1 \) screenings \( \alpha_+ \).

The spin operator appears in the intermediate \( s \) channel of the function (2.37), decomposed as in the equation (2.31), when \( l \) and \( k - 1 \) screenings are integrated around 0 and \((z, \bar{z})\), while \( l \) and \( k \) screenings are integrated around 1 and \( \infty \). In fact, in this case the total charge of the operators in the region around 0 and \((z, \bar{z})\) is given by:

\[
\alpha_{\sigma} + \alpha_{2l+1,2k} + l\alpha_- + (k - 1)\alpha_+ = \alpha_{\sigma} - l\alpha_- - (k - \frac{1}{2})\alpha_+ + l\alpha_- + (k - 1)\alpha_+ = \alpha_{\sigma} - \frac{1}{2}\alpha_+
\]

(2.51)

We remind that the spin operator, the first operator in (2.17),

\[
\sigma = \Phi_{\frac{p+1}{2}, \frac{p-1}{2}} \propto V_{\alpha_{\sigma}}
\]

(2.52)

has the CG charge:

\[
\alpha_{\sigma} = \alpha_{\frac{p+1}{2}, \frac{p-1}{2}} = \frac{1 - p}{4} \alpha_- + \frac{3 - p}{4} \alpha_+
\]

(2.53)
Then, by (2.51), the total charge in the 0, \((z, \bar{z})\) region is equal to:

\[
\alpha - \frac{1}{2} \alpha_+ = \frac{1-p}{4} \alpha_ - + \frac{1-p}{4} \alpha_+ = \alpha_{1+}^{1+} \tag{2.54}
\]

which is the charge of the second operator in (2.17), the second copy of the spin operator.

More precisely

\[
\alpha_{\frac{p+1}{2}, \frac{p+1}{2}} = \frac{1-p}{4} \alpha_ - + \frac{1-p}{4} \alpha_+ = 2\alpha_0 - \alpha = \alpha_+ \tag{2.55}
\]

This equality is easily verified. On one hand,

\[
\alpha_{\frac{p+1}{2}, \frac{p+1}{2}} = \frac{1-p}{4} (\alpha_ - + \alpha_+),
\]

\[
\alpha_ - = -\sqrt{\frac{p}{p+1}}, \quad \alpha_+ = \sqrt{\frac{p+1}{p}},
\]

\[
\alpha_ - + \alpha_+ = \frac{1}{\sqrt{p(p+1)}},
\]

\[
\alpha_{\frac{p+1}{2}, \frac{p+1}{2}} = \frac{1}{4} \left[ 1 - \frac{3-p}{4} - \frac{1}{\sqrt{p(p+1)}} \right] \tag{2.56}
\]

On the other hand

\[
\alpha_+ = 2\alpha_0 - \alpha = \alpha_ - + \alpha_+ - \alpha
\]

\[
= \frac{1}{\sqrt{p(p+1)}} - \frac{1}{4} \alpha_ - - \frac{3-p}{4} \alpha_+
\]

\[
= \frac{1}{\sqrt{p(p+1)}} + \frac{1-p}{4} \sqrt{\frac{p}{p+1}} - \frac{3-p}{4} \sqrt{\frac{p+1}{p}}
\]

\[
= \frac{1}{4\sqrt{p(p+1)}} \left[ 4 + (1-p)p - (3-p)(p+1) \right]
\]

\[
= \frac{1}{4\sqrt{p(p+1)}} (1-p) \tag{2.57}
\]

By (2.56) and (2.57), \(\alpha_{\frac{p+1}{2}, \frac{p+1}{2}} = \alpha_+\), as stated in (2.55).

Returning back to (2.54), one finds, by the total CG charge of the operators in the region 0, \((z, \bar{z})\), that the corresponding intermediate channel CG operator is

\[
V_{\alpha_+} \equiv V_{\alpha_+} \tag{2.58}
\]
When projected onto the operator $V_{\alpha_\sigma}(\infty)$, to form the corresponding 3-point function, one obtains the contribution of the $0, (z, \bar{z})$ region, which factors out from the rest of the operators, to be given by:

$$< V_{\alpha_\sigma}(\infty)V_{2l+1,2k}(1)V_{\alpha_\sigma}(0) > = C_{\alpha_\sigma,\alpha_{2l+1,2k},\alpha_\sigma} \quad (2.59)$$

The variables $z$ and $\bar{z}$ of the operator $V_{2l+1,2k}(z, \bar{z})$, initially, are assumed to have been factored out by scaling.

In a similar way as is describe above, one checks that the rest of the operators in (2.50), the operators $V_{\alpha_\sigma}(\infty)$ and $V_{2l+1,2k}(1)$ and $l$ and $k$ screenings integrated around, that this subset of operators correspond to the intermediate channel CG operator $V_{\alpha_\sigma}$. When projected onto the operator $V_{\alpha_\sigma}(0)$, to form the corresponding 3-point function, one finds that the contribution of the region $\infty, 1$ which also factors out (in the limit $z \to 0$), is given by:

$$< V_{\alpha_\sigma}(\infty)V_{2l+1,2k}(1)V_{\alpha_\sigma}(0) > = C_{\alpha_\sigma,\alpha_{2l+1,2k},\alpha_\sigma} \quad (2.60)$$

Now, in the present case of the index $n$, of $(D_{\sigma, (2l+1,n)})^2$, taking even values, the expression (2.44) is replaced by the following one:

$$(D_{\sigma, (2l+1,n)})^2 = \frac{Z}{N_{2l+1,2k}} \times C_{\alpha_\sigma,\alpha_{2l+1,2k},\alpha_\sigma} \times C_{\alpha_\sigma,\alpha_{2l+1,2k},\alpha_\sigma} \quad (2.61)$$

$Z$ is given by the equation (2.45), the normalisation constant squared $N_{2l+1,2k}$ is given by the general expression (2.49) and the Coulomb Gas constants in (2.61) are defined by the general formula (2.47).

In this way, by the equations (2.44), (2.61), the structure constants $(D_{\sigma, (2l+1,n)})^2$ are defined for all values of $n$, and hence the structure constants $D_{\sigma, (2l+1,n)}$ in (2.36), the $s$ channel decomposition of the four spin function $< \sigma(\infty)\sigma(1)\sigma(z, \bar{z})\sigma(0) >$.

It could be remarked that in the present case the expressions for the structure constants $(D_{\sigma, (2l+1,n)})^2$, given in terms of CG constants expressed by products of the usual $\Gamma$ functions, are by far much simpler to analyse and calculate as compared to more general expressions for the 3-point functions in [11, 3, 12], expressed in terms of $\Upsilon$ function.
It could also be remarked that the general formula (2.36), for the four spins function, might not be so straightforward to use in certain particular cases, for particular values of the parameter $p$. In particular, the reductions to a finite number of terms, in (2.36), in cases of minimal models, have to be analysed carefully, as certain decouplings are not explicit, not simply given by the corresponding $D$ constants being zero, see the remarks in [11] and [12].

One particular application of the general formula (2.36) will be considered in the next Section, the percolation model limit of the four spins function (2.36). The limit of $p \to 2$, $c \to 0$, or of

$$\alpha^2_\pm = \frac{3}{2} - \epsilon, \quad \epsilon \to 0,$$

this limit turns out to be fairly complicated.

3 The percolation model limit $q \to 1$.

The four spins correlation function in the percolation model has already been considered, in particular, in [13]. The limiting procedure for the spin operators, used in the paper [13], was somewhat forced, with the result that just one intermediate channel was selected, out of many, and not the most singular one in the limit $q \to 1$. In a sense, in this paper we shall complete the study of the four spins function which was started in [13].

Our problem will be to take, properly, the limit $q \to 0$ for the expression of the four spins function in (2.36). To take the limit we shall use the parameter $\alpha^2_\pm \equiv \rho$ taken in the form

$$\alpha^2_\pm \equiv \rho = \frac{3}{2} - \epsilon$$

The value $\alpha^2_\pm = 3/2$ corresponds to the $q = 1$, $c = 0$ point, so that the limit $q \to 1$, $c \to 1$ corresponds to taking the limit $\epsilon \to 0$ in (3.1). The operator algebra structure constants $(D_{\sigma,\sigma}^{(2l+1,n)})$ in (2.36) have different limits, when $\epsilon \to 0$, depending on values of $l$ and $n$. For some of them the limit is regular, they take finite or zero values when $\epsilon \to 0$. For others the limit is singular, they diverge as $1/\epsilon$ or $1/\epsilon^2$. 

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To summarize the possibilities, when $\epsilon \to 0$,

$$(D_{\sigma,\sigma}^{(2l+1,n)})^2 \to \begin{cases} \text{finite value or zero} \\ \sim 1/\epsilon \\ \sim 1/\epsilon^2 \end{cases}$$

(3.2)

depending on values of $l$ and $n$. The expressions for the coefficients $(D_{\sigma,\sigma}^{(2l+1,n)})^2$ are given by (2.44), for $n$ odd, $n = 2k + 1$, and by (2.61) for $n$ even, $n = 2k$. The tables in Fig.3 and Fig.4 summarize different limits for these coefficients, for different values of $l$ and $n$, odd and even. Most of the calculations have been done with the use of Mathematica.

Since the principal divergence is that of $1/\epsilon^2$, we shall define as the four spins function of the percolation model the limit:

$$\lim_{\epsilon \to 0} \epsilon^2 < \sigma(\infty)\sigma(1)\sigma(z,\bar{z})\sigma(0) >$$

(3.3)

In this limit only the $1/\epsilon^2$ intermediate channels will contribute, the rest will be suppressed.

Although, for $\epsilon$ small but different from zero, the other channels will perfectly be present. In particular, the channel of the energy operator $\Phi_{1,2}$, which is the cell of $l = 0$, $k = 1$ in the Fig.4. This channel has been selected in [13], it diverges as $1/\epsilon$.

Still in this paper we shall study the four spins function as defined by the limit (3.3). In this limit an infinity of intermediate channels will contribute, those which correspond to the diagonals $1/\epsilon^2$ in Fig.3.

The first diagonal is the one which starts with the cell $(l = 0, k = 1)$, the field $\Phi_{1,3}$, having the dimension

$$\Delta_{1,3} = \Delta(\Phi_{1,3}) = 2,$$

(3.4)

in the limit $\epsilon \to 0$.

It is easily checked that all the fields on this diagonal (the cells (0,1), (3,3), (6,5) and so on) have the same dimension, in the limit $\epsilon \to 0$, the same as the field $\Phi_{1,3}$ which is the base of this diagonal.
Figure 3: Different limits of the coefficients \( (D_{\sigma,\sigma}^{(2l+1,2k+1)})^2 \) for different values of \( l \) and \( k \). Empty cells correspond to finite or zero limiting values.
Figure 4: Different limits of the coefficients \( (D_{\sigma,\sigma}^{(2l+1,2k)})^2 \) for different values of \( l \) and \( k \).
The second $1/\epsilon^2$ diagonal is the one which starts with the field $\Phi_{1,7}$, the cell $l = 0$, $k = 3$, having the dimension

$$\Delta_{1,7} = 15,$$

in the limit. All the fields on this diagonal have the same dimension, that of the field $\Phi_{1,7}$ at the base.

And so on. The $1/\epsilon^2$ fields which are placed on the left of the first diagonal are the exact copies, by reflections with respect to the solid line diagonal in Fig.3, of the $1/\epsilon^2$ fields on the right: $(0, 1) \to (2, 0)$ and so on. They will not be considered as being different. This is similar to the doubling of primary fields in finite Kac tables, in the case of minimal models.

The contributions of all the $1/\epsilon^2$ fields along the first diagonal in Fig.3, to the expansion over the intermediate channels in the equation (2.36), are all the same. In particular, the conformal block functions $F_{2l+1,2k+1}(z)$ are the same for all the fields along the diagonal: they all have equal coefficients $\{k_i\}$,

$$F_{2l+1,2k+1}(z) = 1 + k_1 z + k_2 z^2 + k_3 z^3 + \ldots ,$$

in the limit $\epsilon \to 0$.

This is but with one exception, the operator algebra constants $(D_{\sigma,\sigma}^{(2l+1,2k+1)})^2$ are different for the $1/\epsilon^2$ fields along the diagonal. With some use of Mathematica one finds that

$$(D_{\sigma,\sigma}^{(1)})^2 = (D_{\sigma,\sigma}^{(0)})^2 \times \frac{1}{(2s + 1)^2}$$

(3.7)

Here $\Phi^{(1)}_{(s)}$ is the $1/\epsilon^2$ field, along the first diagonal, number $s$:

$$\Phi^{(1)}_{(0)} = \Phi_{1,3}, \text{the cell } (0,1), \text{the base}$$

$$\Phi^{(1)}_{(1)} = \Phi_{7,7}, \text{the cell } (3,3)$$

$$\Phi^{(1)}_{(2)} = \Phi_{13,11}, \text{the cell } (6,5)$$

and so on.
Then, in view of the above information, resuming the contributions of the $1/\ell^2$ fields along the first diagonal, to the expansion in (2.36), amounts to resuming the coefficients (3.7) over $s$:

$$
\sum_{s=0}^{\infty} (D_{\sigma,\sigma}^{(1)})^2 \times \sum_{s=0}^{\infty} \frac{1}{(2s+1)^2} = (D_{\sigma,\sigma}^{(1)})^2 \times \frac{\pi^2}{8},
$$

$$
(D_{\sigma,\sigma}^{(1)})_{eff} = (D_{\sigma,\sigma}^{(1)})^2 \times \frac{\pi^2}{8} \quad (3.9)
$$

Here $(D_{\sigma,\sigma}^{(1)})_{eff}$ is the effective constant for the first diagonal.

For the second diagonal, with the $\Phi_{(0)}^{(2)} = \Phi_{1,7}$ (cell $l = 0$, $k = 3$) at the base, one finds:

$$
(D_{\sigma,\sigma}^{(2)})^2 = (D_{\sigma,\sigma}^{(0)})^2 \times \left( \frac{3(s+1)}{(2s+1)(2n+3)} \right)^2
$$

$$
(D_{\sigma,\sigma}^{(2)})_{eff} = (D_{\sigma,\sigma}^{(0)})^2 \times \sum_{s=0}^{\infty} \left( \frac{3(s+1)}{(2s+1)(2n+3)} \right)^2 = (D_{\sigma,\sigma}^{(0)})^2 \times \frac{9\pi^2}{64} \quad (3.10)
$$

For the third diagonal, the field $\phi_{(0)}^{(3)} = \Phi_{1,11}$ (cell $l = 0$, $k = 5$) at the base:

$$
(D_{\sigma,\sigma}^{(3)})^2 = (D_{\sigma,\sigma}^{(0)})^2 \times \frac{15}{2} \times \frac{(s+1)(s+2)}{(2n+1)(2n+3)(2n+5)}^2
$$

$$
(D_{\sigma,\sigma}^{(3)})_{eff} = (D_{\sigma,\sigma}^{(0)})^2 \times \frac{2475\pi^2}{16384} \quad (3.11)
$$

Still one more diagonal, the fourth one, $\Phi_{(0)}^{(4)} = \Phi_{1,15}$ (cell $l = 0$, $k = 7$) at the base:

$$
(D_{\sigma,\sigma}^{(4)})^2 = (D_{\sigma,\sigma}^{(0)})^2 \times \frac{35}{2} \times \frac{(s+1)(s+2)(s+3)}{(2s+1)(2s+3)(2s+5)(2s+7)}^2
$$

$$
(D_{\sigma,\sigma}^{(4)})_{eff} = (D_{\sigma,\sigma}^{(0)})^2 \times \frac{20825\pi^2}{131072} \quad (3.12)
$$

The generalisation is evident. The numerical coefficient of the $s$-dependent factor, like 3, $\frac{15}{2}$, $\frac{35}{2}$, in the above formulas, is such that this factor, on the whole, becomes equal to 1 when $s = 0$.

One could provide the whole series, for the expansion (2.36), where the summation which remains is that over the diagonals, with coefficients $(D)^2$ effective and the conformal block functions $|F(z)|^2$ for the fields at the base, for each diagonal.
But here the numerical observation comes. One finds that the ratio of the first two
\((D)^2_{eff}\) coefficients:
\[
\frac{(D^{(2)}_{\sigma,\sigma})_{eff}^2}{(D^{(0)}_{\sigma,\sigma})_{eff}^2} \sim \frac{(D^{\Phi(2)}_{\sigma,\sigma})^2}{(D^{\Phi(0)}_{\sigma,\sigma})^2} \equiv \frac{(D^{\Phi_{1,7}}_{\sigma,\sigma})^2}{(D^{\Phi_{1,3}}_{\sigma,\sigma})^2}
\]
– this ratio is extremely small. With the use of Mathematica, one finds:
\[
(D^{\Phi_{1,3}}_{\sigma,\sigma})^2 = -\frac{25}{9437184} \times \frac{1}{\varepsilon^2} \simeq -2.6491 \times 10^{-6}
\]
\[
(D^{\Phi_{1,7}}_{\sigma,\sigma})^2 \simeq -8.0604 \times 10^{-38}
\]
Their ratio is equal to:
\[
\frac{(D^{\Phi_{1,7}}_{\sigma,\sigma})^2}{(D^{\Phi_{1,3}}_{\sigma,\sigma})^2} \simeq 3.0427 \times 10^{-32}
\]
which makes it reasonable to neglect the contribution of other diagonals, by keeping in (2.36) just the contribution of the first one, with the effective constant
\[
(D^{\Phi_{1,3}}_{\sigma,\sigma})_{eff}^2 = (D^{\Phi_{1,3}}_{\sigma,\sigma})^2 \times \frac{\pi^2}{8}
\]
and the corresponding conformal block function of the base field \(F_{1,3}(z)\).

We have calculated, with the standard technique based on the Virassoro algebra, the
first six coefficients of this function:
\[
F_{1,3}(z) = 1 + k_1 z + k_2 z^2 + k_3 z^3 + k_4 z^4 + k_5 z^5 + k_6 z^6 + \ldots
\]
\[
k_1 = 1
\]
\[
k_2 = \frac{16855}{18432} \approx 0.91444
\]
\[
k_3 = \frac{7639}{9216} \approx 0.82888
\]
\[
k_4 = \frac{769359145}{1019215872} \approx 0.75485
\]
\[
k_5 = \frac{235218217}{339738624} \approx 0.69235
\]
\[
k_6 = \frac{72083915765407}{112717121716224} \approx 0.63951
\]
Within the above described approximation (neglecting the higher diagonals) one finds the following expression for the four spins function:

$$\lim_{\epsilon \to 0} \epsilon^2 < \sigma(\infty)\sigma(1)\sigma(z, \bar{z})\sigma(0) > \approx \lim_{\epsilon \to 0} \epsilon^2 \frac{1}{|z|^{4\Delta_\epsilon-2\Delta_{1,3}}} (D_{\sigma,\sigma}^{(1,3)})^2 |F_{1,3}(z)|^2$$

$$= -\frac{25}{9437184} \frac{\pi^2}{8} |z|^{91/24} \times |F_{1,3}(z)|^2 \quad (3.19)$$

We remind that this function has been defined for the normalized spin operators, $< \sigma(z, \bar{z})\sigma(0) >= 1/|z|^{4\Delta_\epsilon}$.

$(D_{\sigma,\sigma}^{(1,3)})^2$ is negative because the norm squared $(N_{1,3})^2$ in eq.(2.44) is negative, the model is not unitary. We remind that the norm squared of CG vertex operators is defined by the equation (2.49). One can check that $(N_{1,3})^2 = -\frac{225}{64}$.

4 Discussion.

As has already been argued in [13], to realize on the lattice the four spins correlation function, of the conformal theory, might be difficult. The reason is that the lattice spins are, in general, linear combinations of the conformal theory spin-like operators. So that the lattice four spins correlation function will, in general, be a linear combination of different conformal theory four points correlation functions, of spin-like operators.

When doing different linear combinations of lattice defined four points connectivities, the number of possibilities is limited, on the lattice.

One may think of doing the analysis in the opposite direction: taking, in the conformal theory context, a linear combination of different spin-like operators, with undefined coefficients, calculate the corresponding four points functions and then try to fine tune the undefined coefficients so as to match a particular lattice defined four points connectivity.

To realise this approach might be easier in the context of a particular minimal model, where the number of spin-like operators, all known, is finite. Still it requires some work to
be done, in the conformal theory context, to calculate a number of four points functions which arise.

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