A Suggested Answer To Wallstrom’s Criticism: 
Zitterbewegung Stochastic Mechanics II

Maaneli Derakhshani*

August 1, 2016

Institute for History and Foundations of Science & Department of Mathematics, Utrecht University, Utrecht, The Netherlands

Abstract

The "zitterbewegung stochastic mechanics" (ZSM) answer to Wallstrom’s criticism, introduced in the companion paper [1], is extended to many particles. We first formulate the many-particle generalization of Nelson-Yasue stochastic mechanics (NYSM), incorporating external and classical interaction potentials. Then we formulate the many-particle generalization of the classical zitterbewegung $zbw$ model introduced in Part I, for the cases of free particles, particles interacting with external fields, and classically interacting particles. On the basis of these developments, ZSM is constructed for classically free particles, as well as for particles interacting both with external fields and through inter-particle scalar potentials. Throughout, the beables of ZSM (based on the many-particle formulation) are made explicit. Subsequently, we assess the plausibility and generalizability of the $zbw$ hypothesis. We close with an appraisal of other proposed answers, and compare them to ZSM.

*Email: maanelid@yahoo.com and m.derakhshani@uu.nl

1 Address when this work was initiated: Department of Physics & Astronomy, University of Nebraska-Lincoln, Lincoln, NE 68588, USA.
1 Introduction

This paper is a direct continuation of the preceding paper, Part I \[1\]. There we proposed an answer to the Wallstrom criticism of stochastic mechanical theories by modifying Nelson-Yasue stochastic mechanics (NYSM) for a single non-relativistic particle with the following hypothesis: Nelson’s hypothetical stochastic ether medium that drives the conservative diffusions of the particle, also induces mean harmonic oscillations of zitterbewegung (zbw) frequency in the particle’s instantaneous mean rest frame. We then showed that the resulting phase for these zbw oscillations implies the quantization condition that Wallstrom criticizes, and dynamically evolves by the Hamilton-Jacobi-Madelung (HJM) equations. This allowed us to recover the Schrödinger equation for single-valued wavefunctions with (potentially) multi-valued phases, for the cases of a free particle and a particle interacting with external fields (the latter of which we illustrated with the two-dimensional central potential problem). We termed this modification of NYSM "zitterbewegung stochastic mechanics" or ZSM.

The approach of this paper is similar to that of Part I. In section 2, we formulate the many-particle generalization of NYSM and point out where in the derivation of the many-particle Schrödinger equation the Wallstrom criticism applies. Section 3 formulates the classical model of constrained zitterbewegung motion for the cases of many free particles, many particles interacting with external fields, and classically interacting particles. Section 4 generalizes ZSM to the cases of many free particles, many particles interacting with external fields, and classically interacting particles; throughout, the beables\[2\] of ZSM are made explicit. Section 5 assesses the plausibility and generalizability of the zbw hypothesis through multiple considerations. Finally, Section 6 appraises other proposed answers to Wallstrom’s criticism, and compares them to ZSM.

\[2\]This term was coined by J.S. Bell\[2\] as a play on “observables” in standard quantum mechanics. It refers to “those elements which might correspond to elements of reality, to things which exist. Their existence does not depend on ‘observation.’ Indeed observation and observers must be made out of beables”\[3\].
2 Nelson-Yasue Stochastic Mechanics for Many Particles

The first non-relativistic $N$-particle extension of stochastic mechanics was given by Loffredo and Morato [4], who used the Guerra-Morato variational formulation. However, as noted in footnote 8 of Part I [1], the Guerra-Morato formulation is not applicable to ZSM because the Guerra-Morato variational principle requires that the $S$ function is always single-valued. Koide [8] has given a brief two-particle extension of the non-relativistic Nelson-Yasue formulation, for the case of a classical interaction potential, but otherwise no comprehensive $N$-particle extension has been given (to the best of our knowledge). Accordingly, we shall develop the $N$-particle extension of NYSM before extending ZSM to the many-particle case. This will also be useful for identifying the various points of demarcation between NYSM and ZSM in the many-particle formulation. For completeness, we will incorporate coupling of the particles to external (scalar and vector) potentials and to each other through scalar interaction potentials.

As in the single-particle formulation of NYSM [3][10][11], we hypothesize that the vacuum of 3-D space is pervaded by a homogeneous and isotropic ether fluid with classical stochastic fluctuations that impart a frictionless, conservative diffusion process to a point particle of mass $m$ and charge $e$ immersed within the ether. Accordingly, for $N$ point particles of masses $m_i$ and charges $e_i$ immersed in the ether, each particle will in general have its position 3-vector $q_i(t)$ constantly undergoing diffusive motion with drift, as modeled by the first-order forward stochastic differential equations

$$d\mathbf{q}_i(t) = \mathbf{b}_i(q(t), t)dt + d\mathbf{W}_i(t). \quad (1)$$

Here $q(t) = \{q_1(t), q_2(t), ..., q_N(t)\} \in \mathbb{R}^{3N}$, $\mathbf{b}_i(q(t), t)$ is the deterministic mean forward drift velocity of the $i$-th particle (which in general may be a function of the positions of all the other particles, such as in the case of particles interacting with each other gravitationally and/or electrostatically), and $\mathbf{W}_i(t)$ is the Wiener process modeling the $i$-th particle’s interaction with the ether fluctuations.

The Wiener increments $d\mathbf{W}_i(t)$ are assumed to be Gaussian with zero mean, independent of $d\mathbf{q}_i(s)$ for $s \leq t$, and with variance

$$E_t[d\mathbf{W}_{in}(t)d\mathbf{W}_{in}(t)] = 2\nu_i\delta_{nm}dt, \quad (2)$$

where $E_t$ denotes the conditional expectation at time $t$. We then hypothesize that the magnitudes of the diffusion coefficients $\nu_i$ are given by

$$\nu_i = \frac{\hbar}{2m_i}. \quad (3)$$

In addition to (1), we can also consider the backward stochastic differential equations

$$d\mathbf{q}_i(t) = \mathbf{b}_{i*}(q(t), t)dt + d\mathbf{W}_{i*}(t), \quad (4)$$

where $\mathbf{b}_{i*}(q(t), t)$ are the mean backward drift velocities, and $d\mathbf{W}_{i*}(t)$ are the backward Wiener processes. As in the single-particle case, the $d\mathbf{W}_{i*}(t)$ have all the properties of $d\mathbf{W}_i(t)$ except that they are independent of the $d\mathbf{q}_i(s)$ for $s \geq t$. With these conditions on $d\mathbf{W}_i(t)$ and $d\mathbf{W}_{i*}(t)$, Eqs. (1) and (4) respectively define forward and backward Markov processes for $N$ particles on $\mathbb{R}^3$ (or, equivalently, for a single particle on $\mathbb{R}^{3N}$).

Associated to the trajectories $\mathbf{q}_i(t)$ is the $N$-particle probability density $\rho(q, t) = n(q, t)/N$ where $n(q, t)$ is the (fixed) number of particles per unit volume. Corresponding to (1) and (4), then, are the $N$-particle forward and backward Fokker-Planck equations

$$\frac{\partial \rho(q, t)}{\partial t} = -\sum_{i=1}^{N} \nabla_i \cdot [\mathbf{b}_i(q, t)\rho(q, t)] + \sum_{i=1}^{N} \frac{\hbar}{2m_i} \nabla_i^2 \rho(q, t), \quad (5)$$

and

$$\frac{\partial \rho(q, t)}{\partial t} = -\sum_{i=1}^{N} \nabla_i \cdot [\mathbf{b}_{i*}(q, t)\rho(q, t)] - \sum_{i=1}^{N} \frac{\hbar}{2m_i} \nabla_i^2 \rho(q, t), \quad (6)$$

The stochastic action used by Guerra-Morato is defined by a conditional expectation, making it intrinsically single-valued over the diffusion process [3][10][11].
where we assume that the solutions $\rho(q,t)$ in each time direction satisfy the normalization condition
\begin{equation}
\int_{\mathbb{R}^N} \rho_0(q)d^{3N}q = 1.
\end{equation}

Up to this point, (5) and (6) describe independent diffusion processes in opposite time directions. To fix the diffusion process uniquely for both time directions, we must impose certain time-symmetric kinematic constraints on (5) and (6).

First, we impose the average of (5) and (6) to obtain the $N$-particle continuity equation
\begin{equation}
\frac{\partial \rho(q,t)}{\partial t} = -\sum_{i=1}^{N} \nabla_i \cdot [v_i(q,t)\rho(q,t)],
\end{equation}
where
\begin{equation}
v_i(q,t) = \frac{1}{2} (b_i(q,t) + b_i^*(q,t))
\end{equation}
is the current velocity field of the $i$-th particle.

Second, we require that $v_i(q,t)$ is equal to the gradient of a scalar potential $S(q,t)$ (if we allowed $v_i(q,t)$ a non-zero curl, then the time-reversal operation would change the orientation of the curl, thus distinguishing time directions [12, 13]); and in the case of particles classically interacting with an external vector potential $A^{ext}(q,t)$, the current velocities get modified by the usual expression
\begin{equation}
v_i(q,t) = \frac{\nabla_i S(q,t)}{m_i} - \frac{e_i}{m_i c} A^{ext}(q_i,t).
\end{equation}
So (8) becomes
\begin{equation}
\frac{\partial \rho(q,t)}{\partial t} = -\sum_{i=1}^{N} \nabla_i \cdot \left[ \left( \frac{\nabla_i S(q,t)}{m_i} - \frac{e_i}{m_i c} A^{ext}_i(q,t) \right) \rho(q,t) \right],
\end{equation}
which is now a time-reversal invariant evolution equation for $\rho$. The function $S$ is an $N$-particle velocity potential, defined here as a field over $N$ Gibbsian ensembles of fictitious, non-interacting, identical point charges, where each member of the $i$-th ensemble has a different initial position (hence the dependence of $S$ on the generalized coordinates $q_i$) and different initial irrotational mean flow velocity given by (10). We make no assumptions at this level as to whether or not $S$ can be written as a sum of single-particle velocity potentials. Rather, this will depend on the initial conditions and constraints specified for a system of $N$ Nelsonian particles, as well as the dynamics we obtain for $S$. For example, for $N$ particles constrained to interact with each other through a classical Newtonian gravitational and/or electrostatic potential, and $S$ evolving by the $N$-particle generalization of the quantum Hamilton-Jacobi equation (which will turn out to be the case), we will find that $S$ won’t be decomposable into a sum as long as the interactions are appreciable. On the other hand, for $N$ non-interacting particles, we will find that $S$ evolving by the quantum Hamilton-Jacobi equation can (in many cases) be written as $\sum_{i=1}^{N} S_i(q_i,t)$.

Third, we subtract (6) from (5) to get
\begin{equation}
u_i(q,t) = \frac{\hbar}{2m_i} \frac{\nabla_i \rho(q,t)}{\rho(q,t)} = \frac{1}{2} \left[ b_i(q,t) - b_i^*(q,t) \right],
\end{equation}
which formally defines the osmotic velocity field of the $i$-th particle. From (10) and (12), we then have $b_i = v_i + u_i$ and $b_i^* = v_i - u_i$, which when inserted back into (5) and (6) returns (11). Thus, $\rho$ is fixed as the unique, single-time, ‘equilibrium’ distribution for the solutions of (1) and (4).

As in the single-particle case, we can give physical meaning to the osmotic velocities by analogy with the Einstein-Smoluchowski theory: We postulate the presence of an external “osmotic” potential (which we will formally write as a field on the $N$-particle configuration space, in analogy with a classical $N$-particle
external potential), \( U(q,t) \), which couples to the \( i \)-th particle as \( R(q,t) = \mu U(q,t) \) (we assume that the coupling constant \( \mu \) is identical for particles of the same species), and imparts to the \( i \)-th particle a momentum, \( \nabla_i R(q,t)|_{q_i} = q_i(t) \). This momentum then gets counter-balanced by the ether fluid’s osmotic impulse pressure, \((\hbar/2m_i) \nabla_i \ln[n(q,t)]|_{q_i} = q_i(t)\). So the \( N \)-particle osmotic velocity is the equilibrium velocity acquired by the \( i \)-th particle when \( \nabla_i R/m_i = (\hbar/2m_i) \nabla_i \rho/\rho \) (using \( \rho = n/N \)), which implies that \( \rho \) depends on \( R \) as \( \rho = e^{2R/\hbar} \) for all times.

It might be thought that, as an external potential (in the sense of a potential not sourced by the particle), it should be reasonable to assume that \( R \) is a separable function of the \( N \) coordinates so that we can write \( R(q,t) = \sum_{i=1}^{N} R_i(q_i, t) \). However, we know from the single-particle case that the evolution of \( R \) depends on the evolution of \( S \) (through the continuity equation for \( \rho \)), and that the evolution of \( S \) depends on the classical potential \( V \). Since, for many particles, \( V \) can be an interaction potential (such as an \( N \)-particle Coulomb potential), and since we expect to find that the \( N \)-particle evolution equations for \( R \) and \( S \) are the \( N \)-particle generalizations of the HJM equations, we should expect \( R \) to possibly depend on the positions of all the other particle coordinates as a consequence of its nonlinear coupling to \( S \).

From a more physical point of view, it would be reasonable to expect that \( R \) functionally depends on the coordinates of all the other particles if either (i) the source of the potential \( U \) dynamically couples to all the particles in such a way that the functional dependence of \( U \) is determined by the magnitude of inter-particle physical interactions, or (ii) \( U \) is an independently existing field in space-time that directly exchanges energy-momentum with the particles. Since, by Nelson’s hypothesis, each particle undergoes a conservative diffusion process through the ether, on the average, the energy-momentum of each particle is a constant (assuming no time-dependent classical external potentials are present). This suggests that the source of \( U \) should be Nelson’s ether\(^5\) (otherwise the diffusions would not be conservative). So the functional dependence of \( U \) must be determined by the (hypothetical) dynamical coupling of the ether to the particles, and whether or not the particles classically interact with one another. In this way, it is conceivable how \( U \) could have a non-separable functional dependence on the coordinates associated with all the particles. Moreover, we should expect the ‘strength’ of the non-separability (i.e., the inter-particle correlations) of \( U \) to be proportional to the strength of the classical interactions between the particles. (As it turns out, a dust grain undergoing Brownian motion in a nonequilibrium plasma induces an electrostatic osmotic potential from the plasma through an analogous mechanism to what we’ve sketched here\(^6\); moreover, the corresponding Fokker-Planck equation for the stationary probability distribution in velocity space is formally equivalent to Eq. (5) here.)

Since we do not at present have a physical model for Nelson’s ether and its dynamical interactions with the particles, in practice, hypothesis (i) in the previous paragraph gets implemented via Eq. (11) (which, as we’ve noted, equivalently describes the time-evolution of \( R \) and thereby the time-evolution of the coupling of the particles to \( U \)) and Yasue’s stochastic variational principle for the particles. Thus, for \( N \) particles constrained to interact with each other through a classical Newtonian gravitational and/or electrostatic potential, and \( R \) coupled to \( S \) by the \( N \)-particle HJM equations, we will indeed see that \( R \) (and hence \( \rho \)) is not separable, from which we can deduce that \( U \) will also not be factorizable. On the other hand, in the case of non-interacting particles, we will find that it is possible to write \( R(q,t) = \sum_{i=1}^{N} R_i(q_i, t) \) (hence \( \rho(q,t) = \prod_{i=1}^{N} \rho_i(q_i, t) \)). So, for now, we will keep writing the general form \( R = R(q,t) \).

Now we need to construct the \( N \)-particle generalizations of Nelson’s mean forward and backward derivatives. This generalization is straightforwardly given by

\[
Dq_i(t) = \lim_{\Delta t \to 0^+} E_t \left[ \frac{q_i(t + \Delta t) - q_i(t)}{\Delta t} \right],
\]

and

\[
D_q q_i(t) = \lim_{\Delta t \to 0^+} E_t \left[ \frac{q_i(t) - q_i(t - \Delta t)}{\Delta t} \right].
\]

By the Gaussianity of \( dW_i(t) \) and \( dW_{i*}(t) \), we obtain \( Dq_i(t) = b_i(q(t), t) \) and \( D_q q_i(t) = b_{i*}(q(t), t) \). To compute \( Db_i(q(t), t) \) (or \( D_q b_i(q(t), t) \)), we expand \( b_i \) in a Taylor series up to terms of order two in \( dq_i(t) \):

\(^5\)So the idea would be that the ether fluid produces a potential field \( U \) that imparts a momentum of \( \nabla_i(\mu U) \) to each particle, causing the particles to scatter through the ether constituents and thereby experience a counter-balancing osmotic impulse pressure of magnitude \((\hbar/2m_i) \nabla_i \ln[n(q)]\).
\[
db_i(q(t), t) = \frac{\partial b_i(q(t), t)}{\partial t} dt + \sum_{i=1}^{N} dq_i(t) \cdot \nabla b_i(q(t), t)|_{q_i=q_i(t)} + \sum_{i=1}^{N} \frac{1}{2} \sum_{m,n} dq_m(t)dq_n(t) \frac{\partial^2 b_i(q(t), t)}{\partial q_m \partial q_n}|_{q_i=q_i(t)} + \ldots,
\]

(15)

From (1), we can replace \( dq_i(t) \) by \( dW_i(t) \) in the last term, and when taking the conditional expectation at time \( t \) in (13), we can replace \( dq_i(t) \cdot \nabla b_i|_{q_i=q_i(t)} \) by \( b_i(q(t), t) \cdot \nabla b_i|_{q_i=q_i(t)} \) since \( dW_i(t) \) is independent of \( q_i(t) \) and has mean 0. From (2), we then obtain

\[
Db_i(q(t), t) = \left[ \frac{\partial}{\partial t} + \sum_{i=1}^{N} b_i(q(t), t) \cdot \nabla_i + \sum_{i=1}^{N} \frac{\hbar}{2m_i} \nabla^2_i \right] b_i(q(t), t),
\]

(16)

and likewise

\[
D_{i,*} b_i(q(t), t) = \left[ \frac{\partial}{\partial t} + \sum_{i=1}^{N} b_{i,*}(q(t), t) \cdot \nabla_i - \sum_{i=1}^{N} \frac{\hbar}{2m_i} \nabla^2_i \right] b_{i,*}(q(t), t).
\]

(17)

Using (16-17), and assuming the particles also couple to an external electric potential, \( \Phi_{i}^{ext}(q_i(t), t) \), as well as to each other by the Coulomb interaction potential \( \Phi_{i}^{int}(q_i(t), q_j(t)) = \frac{1}{2} \sum_{i=1=1}^{N} \frac{e_i}{|q_i(t) - q_j(t)|} \) we can then construct the \( N \)-particle generalization of Yasue’s ensemble-averaged, time-symmetric mean action:

\[
J(q) = \int_{\mathbb{R}^{2N}} d^Nq \rho(q, t) \int_{t_i}^{t_f} \sum_{i=1}^{N} \left\{ \frac{1}{2} \left[ \frac{1}{2} m_i b_i^2 + \frac{1}{2} m_i b_{i,*}^2 \right] + \frac{e_i}{c} A_{i}^{ext} \cdot \frac{1}{2} (D + D_*) q_i(t) - e_i \left[ \Phi_{i}^{ext} + \Phi_{i}^{int} \right] \right\} dt
\]

\[
= \int_{\mathbb{R}^{2N}} d^Nq \rho \int_{t_i}^{t_f} \sum_{i=1}^{N} \left\{ \frac{1}{2} m_i v_i^2 + \frac{1}{2} m_i u_i^2 + \frac{e_i}{c} A_{i}^{ext} \cdot v_i - e_i \left[ \Phi_{i}^{ext} + \Phi_{i}^{int} \right] \right\} dt,
\]

(18)

where we note that \( v_i(q_i(t), t) = \frac{1}{2} (D + D_*) q_i(t) \).

Upon imposing the conservative diffusion constraint through the \( N \)-particle generalization of Yasue’s variational principle

\[
J(q) = \text{extremal},
\]

(19)

a straightforward computation (see Appendix A) shows that (19) implies

\[
\sum_{i=1}^{N} m_i \frac{1}{2} [D_i, D + DD_*] q_i(t) = \sum_{i=1}^{N} e_i \left[ -\frac{1}{c} \partial_t A_i^{ext} - \nabla_i \left( \Phi_{i}^{ext} + \Phi_{i}^{int} \right) + \frac{v_i}{c} \times \left( \nabla_i \times A_{i}^{ext} \right) \right]|_{q_i=q_i(t)}.
\]

(20)

Moreover, since the \( \delta q_i(t) \) are independent (as we show in Appendix A), it follows from (20) that we have the equations of motion

\[
m_i a_i(t, t) = \frac{m_i}{2} [D_i, D + DD_*] q_i(t) = \left[ -\frac{e_i}{c} \partial_t A_i^{ext} - e_i \nabla_i \left( \Phi_{i}^{ext} + \Phi_{i}^{int} \right) + \frac{e_i}{c} v_i \times \left( \nabla_i \times A_{i}^{ext} \right) \right]|_{q_i=q_i(t)},
\]

(21)

for \( i = 1, \ldots, N \). Applying the mean derivatives in (20), using that \( b_i = v_i + u_i \) and \( b_{i,*} = v_i - u_i \), and replacing \( q(t) \) with \( q \) in the functions on both sides, straightforward manipulations show that (20) turns into

\[
\sum_{i=1}^{N} m_i \left[ \partial_t v_i + v_i \cdot \nabla_i v_i - u_i \cdot \nabla_i u_i - \frac{\hbar}{2m_i} \nabla^2_i u_i \right]
\]

\[
= \sum_{i=1}^{N} \left[ -\frac{e_i}{c} \partial_t A_i^{ext} - e_i \nabla_i \left( \Phi_{i}^{ext} + \Phi_{i}^{int} \right) + \frac{e_i}{c} v_i \times \left( \nabla_i \times A_{i}^{ext} \right) \right].
\]

(22)

Using (10) and (12), integrating both sides of (22), and setting the arbitrary integration constants equal to zero, we then obtain the \( N \)-particle quantum Hamilton-Jacobi equation
\[-\partial_t S(q,t) = \sum_{i=1}^{N} \left[ \nabla_i S(q,t) - \frac{\hbar c}{2 m_i} \Phi_i^{{\text{ext}}}(q_i, t) \right]^2 + \sum_{i=1}^{N} e_i \left[ \Phi_i^{{\text{ext}}}(q_i, t) + \Phi_i^{{\text{int}}}(q_i, t) \right] - \sum_{i=1}^{N} \frac{\hbar^2}{2 m_i} \nabla_{i}^2 \sqrt{\rho(q,t)}, \tag{23}\]

which describes the sum total energy field over the \(N\) statistical ensembles, and, upon evaluation at \(q = q(t)\), the sum total energy of the actual particles along their actual mean trajectories. So (11) and (23) together define the \(N\)-particle HJM equations.

Note that, as a consequence of the non-separability of \(\Phi_i^{{\text{int}}}(q_i, t)\), we will not be able to write (23) as a sum of total energies for each particle (unless the particles are sufficiently spatially separated from each other that we can effectively neglect this interaction term), which means \(S(q,t) \neq \sum_{i=1}^{N} S_i(q_i, t)\). Indeed, as a consequence of this non-separability, we can now see from the coupling of (11) and (23) that \(R\) (and hence \(U\)) will also be non-factorizable since its evolution depends on \(\nabla_i S\) through (11). We can make this more explicit by writing the general solutions, \(S\) and \(R\), to (23) and the differentiated form of (11), respectively. For (23), the general solution takes the form

\[
S(q,t) = \sum_{i=1}^{N} \int p_i(q,t) \cdot dq_i - \sum_{i=1}^{N} \int \left[ \frac{|p_i(q,t) - \frac{\hbar c}{2 m_i} \Phi_i^{{\text{ext}}}(q_i, t)|^2}{2 m_i} + e_i \left[ \Phi_i^{{\text{ext}}}(q_i, t) + \Phi_i^{{\text{int}}}(q_i, t) \right] - \frac{\hbar^2}{2 m_i} \nabla_i^2 \sqrt{\rho(q,t)} \right] \, dt. \tag{24}\]

For the differentiated form of (11), the general solution \(R\) can be found most easily by first solving (11) directly in terms of \(\rho\) and then using the relation \(\rho = e^{2R/\hbar}\). Rewriting (11) as \(\left( \partial_t + \sum_i^N v_i \cdot \nabla_i \right) \rho = -\rho \sum_i^N \nabla_i \cdot v_i\), we have \((d/dt)ln[\rho] = -\sum_i^N \nabla_i \cdot v_i\). Solving this last expression yields

\[
\rho(q,t) = \rho_0(q_0) e^{\int_0^t \left( \sum_i^N \nabla_i \cdot v_i \right) dt'}. \tag{25}\]

The osmotic potential obtained from \(\rho\) then takes the form

\[
R(q,t) = R_0(q_0) - \left( \frac{\hbar}{2} \right) \int_0^t \left( \sum_i^N \nabla_i \cdot v_i \right) dt'. \tag{26}\]

Accordingly, we see clearly that \(R\) depends on \(S\) through \(v_i\), and that \(S\) depends on \(R\) through the quantum kinetic. So the non-separability of \(\Phi_i^{{\text{int}}}\) alone entails non-factorizability of \(S(q,t)\), which entails non-factorizability of \(R(q,t)\), which entails non-factorizability of the quantum kinetic. That is, the nonlinear coupling between (24) and (26) entails that \(S\) is actually non-factorizable by virtue of the non-separability of \(\Phi_i^{{\text{int}}}\) and (as a consequence thereof) that the quantum kinetic is non-factorizable. Thus we’ve explicitly shown, from the \(N\)-particle HJM equations, that the presence of classical interactions between Nelsonian particles means that the \(N\)-particle osmotic potential cannot be written as a factorizable sum of \(N\) osmotic potentials associated to each particle.

Let us now combine (11) and (23) into an \(N\)-particle Schrödinger equation and write down the most general form of the \(N\)-particle wavefunction. To do this, we first need to impose the \(N\)-particle generalization of the quantization condition

\[
\sum_{i=1}^{N} \oint_L \nabla_i S(q,t) \cdot dq_i = nh, \tag{27}\]

which, by (26), also entails quantization of the osmotic potential sourced by the ether. Then we can combine (11) and (23) into

\[\text{In Part I, we explained that we prefer to call the “quantum potential” the “quantum kinetic” in order to emphasize its physical origin in the kinetic energy term associated with the osmotic velocity of a Nelsonian particle.}\]
\[ i\hbar \frac{\partial \psi(q,t)}{\partial t} = \sum_{i=1}^{N} \left[ -\frac{i\hbar \nabla_i - \frac{2\pi}{\hbar} A_i^{ext}(q_i,t)}{2m_i} + e_i \left( \Phi_i^{ext}(q_i,t) + \Phi_i^{int}(q_i,q_j) \right) \right] \psi(q,t), \]  

(28)

where the single-valued \( N \)-particle wavefunction in polar form is

\[ \psi(q,t) = \sqrt{\rho(q,t)}e^{iS(q,t)/\hbar}. \]

Note that, as in the single-particle case, this wavefunction must be interpreted at least partially as an epistemic field in the sense that it is defined in terms of the ensemble variables \( \rho \) and \( S \).

Now, consider the case of 2 distinguishable particles, where particle 1 is associated with a wavepacket \( \psi_A \) and particle 2 is associated with a packet \( \psi_B \). If, initially, the particles are classically non-interacting and there are no correlations between them, then the joint wavefunction is the product state (suppressing the \( t \) variable for simplicity)

\[ \psi_f(q_1,q_2) = \psi_A(q_1)\psi_B(q_2). \]

(29)

We can also construct a non-factorizable solution of (28) by writing

\[ \psi_{nf}(q_1,q_2) = \text{Norm} [\psi_A(q_1)\psi_B(q_2) + \psi_C(q_1)\psi_D(q_2)]. \]

(30)

If the summands in (30) negligibly overlap by virtue of either \( \psi_A \cap \psi_C \approx O \) or \( \psi_B \cap \psi_D \approx O \) (Norm = normalization factor), then the system wavefunction is ‘effectively factorizable’; that is, the 2-particle wavefunction associated with the actual particles at time \( t \) is effectively either \( \psi_f = \psi_A(q_1)\psi_B(q_2) \) or \( \psi_f = \psi_C(q_1)\psi_D(q_2) \).

On the other hand, if we ‘turn on’ the classical interaction \( \Phi_c^{int} \), evolution by (28) will make the overlap of the summands non-negligible, and the system wavefunction will not be effectively factorizable \[16\]. Consequently, from (30), we will have a non-separable 2-particle velocity potential

\[ S_{nf}(q_1,q_2) = S_A(q_1) + S_C(q_1) + S_B(q_2) + S_D(q_2) + \text{consts.} \]

(31)

The probability density will also be non-factorizable since it becomes

\[ \rho_{nf}(q_1,q_2) = \text{Norm}^2 \left\{ e^{2(R_{A1}+R_{B1})/\hbar} + e^{2(R_{C1}+R_{D1})/\hbar} \right. \]

\[ + 2e^{(R_{A1}+R_{C1}+R_{B2}+R_{D2})/\hbar}\cos [(S_{A1}+S_{B2}-S_{C1}-S_{D2})/\hbar] \left. \right\} \]

(32)

And by taking \( \ln[\rho] \) and multiplying by \( \hbar/2 \), the corresponding non-separable 2-particle osmotic potential takes the form

\[ R_{nf}(q_1,q_2) = \frac{3}{2} \{ R_A(q_1) + R_C(q_1) + R_B(q_2) + R_D(q_2) \} \]

\[ + \frac{\hbar}{2} \{ \cos [(S_{A1}+S_{B2}-S_{C1}-S_{D2})/\hbar] \} + \text{consts.} \]

(33)

By the mathematical equivalence of (28) with the set (11)-(23)-(27), we can see that (33) and (31) will be coupled solutions of (11) and (23), respectively. On the other hand, when the summands of \( \psi_{nf} \) have effectively disjoint support in configuration space (e.g., in the case of particles sufficiently separated that their classical interaction can be neglected), the system wavefunction becomes effectively factorizable again. In this case, the system velocity potential is either \( S_f = S_{A1} + S_{B2} \) or \( S_f = S_{C1} + S_{D2} \), the probability density reduces to \( \rho_f \approx N^2 \left\{ e^{2(R_{A1}+R_{B2})/\hbar} + e^{2(R_{C1}+R_{D2})/\hbar} \right\} \), and the system osmotic potential is either \( R_f = R_{A1} + R_{B2} \) or \( R_f = R_{C1} + R_{D2} \).

Incidentally, this latter case most clearly illustrates how, from the stochastic mechanics viewpoint, the wavefunction plays the role of an epistemic variable while also reflecting some of the ontic properties of the physical system: The modulus-square of the factorizable two-particle wavefunction describes the position density for a statistical ensemble of two-particle systems, while the \( R \) and \( S \) functions encoded in the factorizable two-particle wavefunction represent the possible \( R \) and \( S \) functions that the actual particles actually ‘have’ at time \( t \); concurrently, the possible \( R \) and \( S \) functions for the two-particle system reflect objectively real

\[ ^7 \text{Following up on our comment in footnote 3 of Part I, we point out again that the epistemic nature of the N-particle Nelson-Yasue wavefunction is not in contradiction with the axioms of the Pusey-Barrett-Rudolph theorem [13]. The “ontic state space” of N-particle NYSM, which includes the ontic N-particle osmotic potential, } R(q,t) = \mu U(q,t), \text{ is in general not separable, as we are about to see.} \]
properties of Nelson’s ontic ether, insofar as $R_{A1}$ ($R_{B2}$) and $R_{C1}$ ($R_{D2}$) correspond to (effectively) disjoint regions of the ontic osmotic potential sourced by the ether $U_{A1}$ ($U_{B2}$) and $U_{C1}$ ($U_{D2}$), and insofar as $S_{A1}$ ($S_{B2}$) and $S_{C1}$ ($S_{D2}$) reflect the irrotationality of the ether in regions $A$ and $B$ and regions $C$ and $D$. Thus we’ve confirmed the properties of the osmotic potential and its relation to the velocity potential that we observed from the solutions of the $N$-particle HJM equations, for the cases of classically interacting and non-interacting distinguishable particles.

However, we should note that the linearity of (28) entails non-factorizable solutions for the case of classically non-interacting identical bosons. (To justify the symmetrization postulates, we can import Bacciagaluppi’s finding [17] that the symmetrization postulates are derivable from the assumption of symmetry of the Nelsonian particle trajectories in configuration space.) For identical bosons, we simply replace $\psi_C(q_1)\psi_D(q_2)$ in (30) with $\psi_A(q_2)\psi_B(q_1)$, and similarly for $S_{nf}$, $\rho_{nf}$, and $R_{nf}$. Then, if particle 1 and particle 2 start out without any classical interaction, we will initially have $\psi_A \cap \psi_B \approx 0$ (approximately, because the wavepackets never have completely disjoint support in configuration space, even in the non-interacting case); if the packets of these particles then move towards each other and overlap such that $(q_1 > q_2)^2 \leq \sigma_A^2 + \sigma_B^2$, where $\sigma_A$ and $\sigma_B$ are the widths of the packets, the resulting wavefunction of the 2-particle system will be given by (30) with $\psi_A \cap \psi_B \neq 0$ [16]. Physically, the appreciable overlap of the wavepackets implies that the initially independent osmotic potentials possibly associated with particle 1 ($R_{A1}$ or $R_{B1}$) and particle 2 ($R_{A2}$ or $R_{B2}$), respectively, become non-separable by virtue of their joint support in configuration space becoming non-negligible. So the resulting motion of particle 1 will have a non-separable physical dependence on part of the osmotic potentials possibly associated with particle 2 (and vice versa), a dependence which is instantaneous between the particles in 3-space (since the $N$-particle quantum kinetic in (23) acts instantaneously on the two particles at time $t$). Of course, for classically non-interacting identical particles, the 2-particle wavefunction will satisfy $\psi_A \cap \psi_B = 0$ again once the wavepackets pass each other and their overlap becomes negligible; but if the particles are classically interacting via $\Phi^{cert}$ the non-separability will persist until the particles are sufficiently spatially separated that $\Phi^{cert} \approx 0$.

Thus the linearization of the HJM equations into Schrödinger’s equation, through the use of condition (27), makes possible non-separable/non-local correlations between (distinguishable or identical) particles not admitted by the HJM equations alone (since the solutions of the HJM equations don’t generally satisfy the superposition principle without (27), as we know from Wallstrom [19]). In fact, such solutions tell us that the two-particle wavefunction for identical bosons (interacting or non-interacting) must always be given by (30), where the joint support of the summands never completely vanishes and can increase appreciably due to (classical or non-classical) interactions between the particles [16].

This last realization complicates the interpretation of the space in which Nelson’s ether lives versus the space in which the particles live: we started out by postulating that the ether lives in 3-D space, but have found that once the constraints (19) and (27) are imposed, the $R$ and $S$ functions (which, as we’ve seen, reflect objectively real properties of the ether) are in general not factorizable, and thus (mathematically) always live in $3N$-dimensional configuration space. If we take this mathematical non-factorizability of $R$ and $S$ as a literal indication about the ontic nature of the ether, then this would seem to force us to infer that the ether must actually live in $3N$-dimensional configuration space, and therefore regard configuration space as an ontic space in its own right. We could then say (to whatever extent one finds this plausible) that the ether and osmotic potential live in configuration space, but that there are still $N$ ontic particles living in an (also) ontic 3-D space, and postulate that the two sets of beables can somehow causally interact with each other via the set (1)-(4)-(21), despite living in independent ontic spaces. (This situation is analogous to a common interpretation of the de Broglie-Bohm theory, where the fundamental ontology consists of an ontic wavefunction living in an ontic 3N-dimensional configuration space, and $N$ ontic particles living in an ontic 3-D space; one then postulates a one-way causal relationship between the wavefunction and the $N$ particles via the "guiding equation" [16] [20] [21].)

Alternatively, if one finds it unintelligible to say that beables living in two independent ontic spaces can causally interact, we could suppose (in analogy with Albert’s “flat-footed” interpretation of the de Broglie-
Bohm theory [22, 23] that the representation of \( N \) particles in 3-D space is a mathematical fiction and that the ontic description is actually a single particle in 3N-dimensional configuration space. This has the virtue that it is straightforward to assert that this single particle causally interacts with Nelson’s ether (since they both live in the same ontic space). The cost is that one now has to employ a complicated (philosophical) functional analysis [22, 23, 24] of how the form of the interaction potential \( \Phi^\text{int}(\mathbf{q}_i, \mathbf{q}_j) \) in the Quantum-Hamilton-Jacobi equation (23) makes it possible to recover \( N \) particles in 3-D space as an emergent ontology; additionally, this view seems logically inconsistent with the fact that the non-factorizable \( R \) and \( S \) functions are a consequence of extremizing the mean action Eq. (18), defined in terms of \( N \) contributions, if there aren’t really \( N \) particles diffusing in 3-D space to which those \( N \) contributions correspond.

A third possibility is that the configuration-space representation of \( R \) and \( S \) is somehow just an abstract encoding of a complicated array of ontic fields in space-time that nonlocally connect the motions of the particles. In practice, we might implement this by analogy with Norsen’s “TELB” approach to the de Broglie-Bohm theory [25, 26]: Taylor-expand the \( R \) and \( S \) functions in configuration space into \( N \) one-particle \( R \) and \( S \) functions, each coupled to a countably infinite hierarchy of “entanglement fields” in space-time that implement the nonlocal connections between the motions of the particles. The upshot of this approach is that one can maintain that Nelson’s ether lives in plain-old 3-D space along with \( N \) particles. A drawback is the immense complexity of positing a countable infinity of ontic fields in space-time, in order to reproduce all the information encoded in the \( R \) and \( S \) functions in configuration space. To be sure, this last possibility is more speculative than the former two (since it would be non-trivial to actually construct such a variant of NYSM); but we think it is ultimately the most intelligible and fruitful one for stochastic mechanics (for reasons discussed in sections 4 and 5).

Of course, the validity of this last construction of non-factorizable solutions for NYSM depends on the plausibility of imposing (27). But such a condition is arbitrary from the point of view of (11) and (23), insofar as we have reconstructed those equations from the Nelson-Yasue assumptions. This, in essence, is Wallstrom’s criticism applied to the \( N \)-particle case. Our task then is to reformulate \( N \)-particle NYSM into \( N \)-particle ZSM.

### 3 Classical Model of Constrained Zitterbewegung Motion for Many Particles

In developing \( N \)-particle ZSM, it will be helpful to first develop the \( N \)-particle version of our classical \( zbw \) model, for free particles, particles interacting with external fields, and particles interacting with each other through Coulomb forces. As we will see, even at the classical level, the \( N \)-particle extension turns out to be non-trivial.

#### 3.1 Free \( zbw \) particles

Let us now suppose we have \( N \) identical, non-interacting \( zbw \) particles in space-time, and no external fields present. In other words, the \( i \)-th particle has rest mass \( m_i \) (taking \( i = 1, \ldots, N \)) and is rheonomically constrained to undergo an unspecified oscillatory process with constant angular frequency \( \omega_{ci} \) about some fixed point in 3-space \( \mathbf{q}_{0i} \) in a Lorentz frame where \( \mathbf{v}_i = d\mathbf{q}_{0i}/dt = 0 \). Then, in a fixed Lorentz frame where \( \mathbf{v}_i \neq 0 \), the \( zbw \) phase for the \( i \)-th free particle takes the form (using \( \theta_i = -\frac{\omega_{ci}}{m_i} S_i = -\frac{1}{\hbar} S_i \))

\[
\delta S_i(\mathbf{q}_i(t), t) = (\mathbf{p}_i \cdot \delta \mathbf{q}_i(t) - E_i \delta t),
\]

where \( E_i = \gamma_i m_i c^2 \). So for each particle, we will have

\[
\oint_{L} \delta S_i(\mathbf{q}_i(t), t) = \oint_{L} (\mathbf{p}_i \cdot \delta \mathbf{q}_i(t) - E_i \delta t) = \hbar,
\]

which implies

\[
\sum_{i=1}^{N} \oint_{L} \delta S_i(\mathbf{q}_i(t), t) = \sum_{i=1}^{N} \oint_{L} (\mathbf{p}_i \cdot \delta \mathbf{q}_i(t) - E_i \delta t) = \hbar.
\]

In the non-relativistic limit, the \( i \)-th \( zbw \) phase is
\[ S_i(q_i(t), t) \approx m_i v_i \cdot q_i(t) - \left( m_i c^2 + \frac{m_i v_i(q_i(t), t)^2}{2} \right) t + \hbar \phi_i, \]  

and satisfies the classical HJ equation

\[ E_i(q_i(t), t) = -\partial_t S_i(q_i, t)|_{q_i=q_i(t)} = \frac{(\nabla_i S_i(q_i, t))^2}{2m_i}|_{q_i=q_i(t)} + m_i c^2. \]  

We can also define the total system energy as the sum of the individual energies of each \( zbw \) particle:

\[ E(q(t), t) = -\partial_t S(q(t), t)|_{q=q(t)} = \sum_{i=1}^{N} \frac{(\nabla_i S(q(t), t))^2}{2m_i}|_{q_i=q_i(t)} + \sum_{i=1}^{N} m_i c^2, \]

where we have used \( E = -\partial_t S = \sum_{i=1}^{N} E_i = -\sum_{i=1}^{N} \partial_t S_i = -\partial_t \sum_{i=1}^{N} S_i \). Accordingly, we can define the 'joint phase' of the \( N \)-particle system as the sum

\[ S(q(t), t) = \sum_{i=1}^{N} S_i(q_i(t), t) \approx \sum_{i=1}^{N} m_i v_i(q_i(t), t) \cdot q_i(t) - \left( \sum_{i=1}^{N} m_i c^2 + \sum_{i=1}^{N} \frac{m_i v_i(q(t), t)^2}{2} \right) t + \hbar \sum_{i=1}^{N} \phi_i, \]

which satisfies (39). Correspondingly, we can rewrite (36) as

\[ \sum_{i=1}^{N} \oint_{L} \nabla_i S|_{q_i=q_i(t)} \cdot \delta q_i(t) = \hbar, \]

for displacements along closed loops with time held fixed. We are now ready to formulate the HJ statistical mechanics for \( N \) free particles.

### 3.2 Classical Hamilton-Jacobi statistical mechanics for free \( zbw \) particles

If the actual positions of the \( zbw \) particles are unknown, then \( q_i(t) \) gets replaced by \( q_i \), and we now consider \( N \) non-interacting Gibbsian statistical ensembles of \( zbw \) particles (where the \( i \)-th ensemble reflects the unknown position of the \( i \)-th actual particle). The non-relativistic joint \( zbw \) phase then becomes a field

\[ S(q(t), t) \approx \sum_{i=1}^{N} m_i v_i(q(t), t) \cdot q_i - \sum_{i=1}^{N} \left( m_i c^2 + \frac{m_i v_i(q(t), t)^2}{2} \right) t + \sum_{i=1}^{N} \hbar \phi_i, \]

where \( v_i(q(t), t) = \nabla_i S(q(t), t)/m_i \) and satisfies

\[ \sum_{i=1}^{N} \oint_{L} \nabla_i S \cdot dq_i = \hbar, \]

and

\[ E(q, t) = -\partial_t S = \sum_{i=1}^{N} \left[ \frac{(\nabla_i S)^2}{2m_i} + m_i c^2 \right]. \]

The physical independence of the particles further implies

\[ E_i = -\partial_t S_i = \frac{(\nabla_i S_i)^2}{2m_i} + m_i c^2, \]

where

\[ S(q, t) = \sum_{i=1}^{N} S_i(q_i, t), \]

and
\[ \int_L \nabla_i S_i \cdot dq_i = n \hbar. \]  

(47)

As (42) is defined from the sum of \( N \) independent ensemble phase fields, Eq. (46), the corresponding velocity fields, \( v_i(q,t) \), are also physically independent of one another. Consequently, for the trajectory fields obtained from integrating \( v_i(q,t) \), the associated \( N \)-particle probability density \( \rho(q,t) = n(q,t)/N \) can be taken in most cases to be factorizable into a product of \( N \) independent probability densities (for simplicity, we ignore the special case of classical correlations corresponding to when \( \rho \) is a mixture of factorizable densities; but see [13] for a discussion of classical correlations in a related context):

\[ \rho(q,t) = \prod_i \rho_i(q_i,t), \]  

(48)

where (48) satisfies \( \rho(q,t) \geq 0 \), the normalization condition \( \int_{\mathbb{R}^3N} \rho_0(q)d^3N q = 1 \), and evolves by the \( N \)-particle continuity equation

\[ \frac{\partial \rho}{\partial t} = -\sum_{i=1}^N \nabla_i \cdot \left[ \left( \frac{\nabla_i S_i}{m_i} \right) \rho \right], \]  

(49)

which by (48) implies

\[ \frac{\partial \rho_i}{\partial t} = -\nabla_i \cdot \left[ \left( \frac{\nabla_i S_i}{m_i} \right) \rho_i \right]. \]  

(50)

We can then combine (44) and (49) to obtain a single-valued \( N \)-particle classical wavefunction \( \psi(q,t) = \sqrt{\rho_0(q_1 - v_1 t, ..., q_N - v_N t)e^{iS(q,t)/\hbar}} \) satisfying the \( N \)-particle nonlinear Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = \sum_{i=1}^N \left[ \frac{\hbar^2}{2m_i} \nabla_i^2 + \frac{\hbar^2}{2m_i} \frac{\nabla_i^2 |\psi|}{|\psi|} + m_i c^2 \right] \psi, \]  

(51)

which implies

\[ i\hbar \frac{\partial \psi_i}{\partial t} = \left[ \frac{\hbar^2}{2m_i} \nabla_i^2 + \frac{\hbar^2}{2m_i} \frac{\nabla_i^2 |\psi_i|}{|\psi_i|} + m_i c^2 \right] \psi_i, \]  

(52)

since

\[ \psi(q,t) = \prod_i \psi_i(q_i,t). \]  

(53)

Having completed the description of \( N \) free particles, we now develop the slightly less trivial case of \( zbw \) particles interacting with external fields.

### 3.3 External fields interacting with \( zbw \) particles

To describe the interaction of our \( zbw \) particles with external fields, consider first the change in the \( zbw \) phase of the \( i \)-th particle in its rest frame:

\[ \delta \theta_i(t_0) = \omega_{ci} \delta t_0 = \frac{1}{\hbar} \left( m_i c^2 \right) \delta t_0. \]  

(54)

The coupling of the particle to (say) the Earth’s external gravitational field leads to a small correction (in the now instantaneous rest frames of the particles) as follows:

\[ \delta \theta_i(q_{0i}, t_0) = [\omega_{ci} + \kappa_i(q_{0i})] \delta t_0 = \frac{1}{\hbar} \left[ m_i c^2 + m_i \Phi_{gi}^{ext}(q_{0i}) \right] \delta t_0, \]  

(55)

where \( \kappa_i = \omega_{ci} \Phi_{gi}^{ext} / c^2 \). As in the single particle case, we have approximated the coupling as point-like since we assume \( |q_i| \gg \lambda_{ci} \). Supposing also that the \( zbw \) particles carry charge \( c_i \) (so that they now become classical charged oscillators of some identical type), their point-like couplings to a space-time varying external electric field lead to additional (small) phase shifts of the form
\[\delta \theta_i(q_{0i}, t_0) = [\omega_{ci} + k_i(q_{0i}) + \varepsilon_i(q_{0i}, t_0)] \delta t_0 = \frac{1}{h} \left[ m_i c^2 + m_i \Phi^\text{ext}_{gi}(q_{0i}) + e_i \Phi^\text{ext}_{ei}(q_{0i}, t_0) \right] \delta t_0, \quad (56)\]

where \( \varepsilon_i = \omega_{ci} (e_i/m_i c^2) \Phi^\text{ext}_{ei}. \)

Transforming to the lab frame where the \( i \)-th \( z \) bullet particle has nonzero but variable translational velocity, (56) becomes

\[\delta \theta_i(q, t) = \left[ \omega_{dBi} + k_i(q(t)) + \varepsilon_i(q(t), t) \right] \gamma_i \left( \delta t - \frac{v_{0i}(q_t, t) \cdot \delta q_i(t)}{c^2} \right) = \frac{1}{h} \left[ (\gamma_i m_i c^2 + \gamma_i m_i \Phi^\text{ext}_{gi} + e_i \Phi^\text{ext}_{ei}) \delta t - (\gamma_i m_i c^2 + \gamma_i m_i \Phi^\text{ext}_{gi} + e_i \Phi^\text{ext}_{ei}) \frac{v_{0i} \cdot \delta q_i(t)}{c^2} \right] \]

\[= \frac{1}{h} \left( E_i \delta t - p_i \cdot \delta q_i(t) \right), \quad (57)\]

where \( E_i = \gamma_i m_i c^2 + \gamma_i m_i \Phi^\text{ext}_{gi} + e_i \Phi^\text{ext}_{ei} \) and \( p_i = m_i v_i = (\gamma_i m_i c^2 + \gamma_i m_i \Phi^\text{ext}_{gi} + e_i \Phi^\text{ext}_{ei}) (v_{0i}/c^2). \) Incorporating coupling to an external vector potential, we have \( v_i \to v'_i = v_i + e_i A^\text{ext}_i / \gamma_i m_i c \) (where \( \gamma_i \) depends on the time-dependent \( v_i \)).

Now, even under the physical influence of the external fields, the phase of the \( i \)-th particle’s oscillation is a well-defined function of its space-time location. Thus, if we displace the \( i \)-th particle around a closed loop, the phase change is still given by

\[\oint_L \delta \theta_i = \frac{1}{h} \oint_L [E_i \delta t - p'_i \cdot \delta q_i(t)] = 2\pi n, \quad (58)\]

or

\[\oint_L \delta S_i = \oint_L [p'_i \cdot \delta q_i(t) - E_i \delta t] = nh. \quad (59)\]

Accordingly, we will also have

\[\sum_{i=1}^N \oint_L \delta S_i = \sum_{i=1}^N \oint_L [p'_i \cdot \delta q_i(t) - E_i \delta t] = nh. \quad (60)\]

Moreover, for the special case of a loop in which time is held fixed, we have

\[\oint_L \delta S_i = \oint_L \nabla_i S_i \big|_{q_i=q_i(t)} \cdot \delta q_i(t) = \oint_L p'_i \cdot \delta q_i(t) = nh, \quad (61)\]

or

\[\oint_L m_i v_i \cdot \delta q_i(t) = nh - \frac{\varepsilon_i}{c} \oint_L A^\text{ext}_i \cdot \delta q_i(t). \quad (62)\]

Likewise

\[\sum_{i=1}^N \oint_L \delta S_i = \sum_{i=1}^N \oint_L \nabla_i S_i \big|_{q_i=q_i(t)} \cdot \delta q_i(t) = \sum_{i=1}^N \oint_L p'_i \cdot \delta q_i(t) = nh, \quad (63)\]

which is equivalent to

\[\sum_{i=1}^N \oint_L m_i v_i \cdot \delta q_i(t) = nh - \sum_{i=1}^N \frac{\varepsilon_i}{c} \oint_L A^\text{ext}_i \cdot \delta q_i(t). \quad (64)\]

Integrating (57) and rewriting in terms of \( S_i \), we obtain

\[S_i = \int dS_i = \int [p'_i \cdot \delta q_i(t) - E_i dt] - h \phi_i, \quad (65)\]

and thus
\[ S = \sum_{i=1}^{N} S_i = \sum_{i=1}^{N} \left[ \mathbf{p}_i' \cdot d\mathbf{q}_i(t) - E_i dt \right] - \sum_{i=1}^{N} \hbar \phi_i, \]  

(66)

When \( v_i \ll c \)

\[ S \approx \sum_{i=1}^{N} \left[ m_i \mathbf{v}_i' \cdot d\mathbf{q}_i(t) - \frac{1}{2m_i} \left( \mathbf{p}_i - \frac{e_i}{c} \mathbf{A}_i^{\text{ext}} \right)^2 + m_i \Phi_{gi}^{\text{ext}} + e_i \Phi_{ei}^{\text{ext}} \right] dt - \sum_{i=1}^{N} \hbar \phi_i, \]

(67)

and satisfies

\[ - \partial_t S|_{\mathbf{q}_i=\mathbf{q}_i(t)} = \sum_{i=1}^{N} \frac{\left( \nabla_i S - \frac{\Phi_{qi}^{\text{ext}}}{2m_i} \right)^2}{2m_i}|_{\mathbf{q}_i=\mathbf{q}_i(t)} + \sum_{i=1}^{N} \left[ m_i c^2 + m_i \Phi_{gi}^{\text{ext}} + e_i \Phi_{ei}^{\text{ext}} \right], \]

(68)

where the kinetic velocity, \( \mathbf{v}_i = (1/m_i) \nabla_i S|_{\mathbf{q}_i=\mathbf{q}_i(t)} - e_i \mathbf{A}_i^{\text{ext}}/m_i c \), satisfies the classical Newtonian equation of motion

\[ m_i \mathbf{q}_i'(t) = \left( \frac{\partial}{\partial t} + \mathbf{v}_i \cdot \nabla_i \right) \left[ \nabla_i S - \frac{e_i}{c} \mathbf{A}_i^{\text{ext}} \right]_{\mathbf{q}_i=\mathbf{q}_i(t)} = -\nabla_i \left[ m_i \Phi_{gi}^{\text{ext}} + e_i \Phi_{ei}^{\text{ext}} \right]_{\mathbf{q}_i=\mathbf{q}_i(t)} - \frac{e_i}{c} \frac{\partial \mathbf{A}_i^{\text{ext}}}{\partial t} \left|_{\mathbf{q}_i=\mathbf{q}_i(t)} \right. + \frac{e_i}{c} \mathbf{v}_i \times \mathbf{B}_i^{\text{ext}}. \]

(69)

As in the previous section, we now want to extend our model to a classical HJ statistical mechanics for \( N \)-particles.

### 3.4 Classical Hamilton-Jacobi statistical mechanics for \( zbw \) particles interacting with external fields

If in the lab frame we do not know the actual positions of the \( zbw \) particles, then \( \mathbf{q}_i(t) \) gets replaced by \( \mathbf{q}_i \), and the phase (67) becomes a field over \( N \) non-interacting Gibbsian statistical ensembles of \( zbw \) particles (one ensemble reflecting each actual particle). In the \( v_i \ll c \) approximation

\[ S(q,t) = \sum_{i=1}^{N} \int m_i \mathbf{v}_i'(\mathbf{q}_i,t) \cdot d\mathbf{q}_i \]

(70)

\[ - \sum_{i=1}^{N} \int \left( m_i c^2 + \frac{1}{2m_i} \left[ \mathbf{p}_i(\mathbf{q}_i,t) - \frac{e_i}{c} \mathbf{A}_i^{\text{ext}}(\mathbf{q}_i,t) \right]^2 + m_i \Phi_{gi}^{\text{ext}}(\mathbf{q}_i) + e_i \Phi_{ei}^{\text{ext}}(\mathbf{q}_i,t) \right) dt - \sum_{i=1}^{N} \hbar \phi_i. \]

To obtain the equations of motion for \( S \) and \( \mathbf{v}_i \) we will now apply the classical analogue of Yasue’s \( N \)-particle variational principle, in anticipation of the method we will use for constructing \( N \)-particle ZSM (we did not do this in the free-particles case because there the dynamics of the particles is trivial).

First we introduce the \( N \)-particle ensemble-averaged phase (inputting limits between initial and final states),

\[ J(q) = \int_{\mathbb{R}^N} d^N \mathbf{q} \rho(q,t) \sum_{i=1}^{N} \left[ \int_{q_i}^{q_i'} m_i \mathbf{v}_i' \cdot d\mathbf{q}_i(t) - \int_{t_i}^{t_i'} \left( m_i c^2 + \frac{1}{2m_i} \left[ \mathbf{p}_i - \frac{e_i}{c} \mathbf{A}_i^{\text{ext}} \right]^2 + m_i \Phi_{gi}^{\text{ext}} + e_i \Phi_{ei}^{\text{ext}} \right) dt \right] - \hbar \phi_i \]

(71)

where the equated expressions are related by the usual Legendre transformation. Imposing the variational constraint
a straightforward computation exactly along the lines of the Appendix yields (69). And, upon replacing \( q_i(t) \) by \( q_i \), we obtain the equation of motion for the acceleration field \( a(q, t) \):

\[
\begin{align*}
m_i a_i &= \left( \frac{\partial}{\partial t} - v_i \cdot \nabla_i \right) \left( \nabla_i S - \frac{e_i}{c} A_i^{ext} \right) \\
&= -\nabla_i \left[ m_i \Phi_{gi}^{ext} + e_i \Phi_i^{ext} \right] - \frac{e_i}{c} \frac{\partial A_i^{ext}}{\partial t} + \frac{e_i}{c} \mathbf{v}_i \times \mathbf{B}_i^{ext},
\end{align*}
\]

(73)
where \( v_i = (1/m_i) \nabla_i S - e_i A_i^{ext}/m_i c \) corresponds to the kinetic velocity field over the \( i \)-th ensemble.

Integrating both sides of (73), summing over all \( N \) terms, and setting the integration constants equal to the rest masses, we then obtain the classical \( N \)-particle Hamilton-Jacobi equation for (70)

\[
-\partial_t S = \sum_{i=1}^{N} \left( \nabla_i S - \frac{e_i}{m_i} A_i^{ext} \right)^2 + \sum_{i=1}^{N} \left[ m_i c^2 + m_i \Phi_{gi}^{ext} + e_i \Phi_i^{ext} \right].
\]

(74)
Correspondingly, the probability density \( \rho(q, t) \) now evolves by the modified \( N \)-particle continuity equation

\[
\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{N} \nabla_i \cdot \left( \frac{\nabla_i S}{m_i c} - \frac{e_i}{m_i} A_i^{ext} \right) \rho,
\]

(75)
which preserves the normalization, \( \int \rho_0 \rho^{3N} q = 1 \). As in the free particle case, each element of the \( i \)-th ensemble is a \( zbw \) particle, making the phase \( S \) a single-valued function of \( q \) and \( t \) (up to an additive integer multiple of \( 2\pi \)) and implying

\[
\sum_{i=1}^{N} \int_L \nabla_i S \cdot dq_i = nh.
\]

(76)
Then we can combine (74-75) into the nonlinear Schrödinger equation

\[
\begin{align*}
\hbar \frac{\partial \psi}{\partial t} &= \sum_{i=1}^{N} \left( \frac{-i \hbar \nabla_i - \frac{e_i}{c} A_i^{ext}}{2m_i} \right)^2 + \frac{\hbar^2}{2m_i} \frac{\nabla_i^2 |\psi|}{|\psi|} + m_i \Phi_{gi}^{ext} + e_i \Phi_i^{ext} + m_i c^2 \psi,
\end{align*}
\]

(77)
with \( N \)-particle wavefunction \( \psi(q, t) = \sqrt{\rho(q, t)} e^{iS(q, t)/\hbar} \), which is single-valued because of (76). We can also obtain the single-particle versions of (74-77) in the case that \( S, \rho, \text{ and } \psi \) satisfy the factorization conditions (46), (48), and (53), respectively.

We are now ready to develop the non-trivial case of classically interacting \( zbw \) particles.

### 3.5 Classically interacting \( zbw \) particles

For simplicity we will consider just two \( zbw \) particles classically interacting through a scalar potential in the lab frame, under the assumptions that \( v_i \ll c \) and no external potentials are present. (Restricting the particles to the non-relativistic regime also avoids complications associated with potentials sourced by relativistic particles [27] [28].) In particular, we suppose that the particles interact through the Coulomb potential

\[
V_c^{int}(q_1(t), q_2(t)) = \sum_{i=1}^{2} e_i \Phi_c^{int}(q_1(t), q_2(t)) = \frac{e_1 e_2}{|q_1(t) - q_2(t)|},
\]

(78)
where we recall \( \Phi_c^{int}(q_i(t), q_j(t)) = \frac{1}{2} \sum_{j \neq i} \frac{e_j}{|q_i(t) - q_j(t)|} \). Note that we make the point-like interaction assumption \( |q_1(t) - q_2(t)| \gg \lambda_c \). So the motions of the particles are not physically independent in the lab frame, and this implies that the \( zbw \) oscillation of particle 1 (particle 2) in the lab frame is physically dependent on the position of particle 2 (particle 1), through the interaction potential (78). We can represent this physical dependence of the \( zbw \) oscillations by a non-separable joint phase change, which involves contributions from both particles in the form
\[
\delta \theta_{\text{joint}}^{\text{ab}}(q_1(t), q_2(t), t) = \left[ \sum_{i=1}^{2} \omega_{ci} + \sum_{i=1}^{2} \omega_{ci} \left( \frac{v_i^2}{2c^2} \right) + \sum_{i=1}^{2} \omega_{ci} \left( \frac{e_i \Phi_{\text{int}}}{m_i c^2} \right) \right] |_{q_i = q_i(t)} \left[ \delta t - \sum_{i=1}^{2} \frac{V_{0i}}{c^2} \cdot \delta q_i(t) \right] |_{q_i = q_i(t)}
\]

\[
= \delta \theta_{1}^{\text{rest}}(q_01(t), q_2(t), t) = \left[ \omega_{c1} + \omega_{c2} \left( \frac{v_1^2}{2c^2} \right) + \sum_{i=1}^{2} \omega_{ci} \left( \frac{e_i \Phi_{\text{int}}}{m_i c^2} \right) \right] |_{q_i = q_i(t)} \left[ \delta t - \sum_{i=1}^{2} \omega_{ci} \left( \frac{V_{0i}}{c^2} \right) \right] |_{q_i = q_i(t)} \cdot \delta q_2(t)
\]

\[
= \frac{1}{\hbar} \left[ \left( m_1 c^2 + \frac{m_2 v_1^2}{2} + V_{\text{int}}^c \right) |_{q_1 = q_1(t)} \delta t - \sum_{i=1}^{2} \frac{m_i v_i}{c} \cdot \delta q_i(t) \right].
\]

(79)

Not surprisingly, when \(|q_1(t) - q_2(t)|\) becomes sufficiently great that \(V_{\text{int}}^c\) is negligible, (79) reduces to a sum of the physically independent phase changes associated with particle 1 and particle 2, respectively.

Now, even though the particles don’t have physically independent phases because of \(V_{\text{int}}^c\), it is clear that the \(\text{zbw}\) oscillation of particle 1 (particle 2) still has a well-defined individual phase at all times. Moreover, we can deduce from (79) the individual (‘conditional’) phase of a particle, given its physical interaction with the other particle via (78), in much the same way that “conditional wavefunctions” for subsystems of particles can be deduced from the universal wavefunction in the de Broglie-Bohm theory [29, 24].

To motivate this, let us first ask: in the instantaneous rest frame (IRF) of (say) particle 1, how will the \(\text{zbw}\) oscillation change in time for a co-moving observer that’s continuously monitoring the oscillation? The phase change associated with particle 1 in its IRF can be obtained from (79) simply by subtracting \(\omega_{c2} \delta t\) and setting \(v_1 = 0\), giving

\[
\delta \theta_{1}^{\text{rest}}(q_01(t), q_2(t), t) = \left[ \omega_{c1} + \omega_{c2} \left( \frac{v_2^2}{2c^2} \right) + \sum_{i=1}^{2} \omega_{ci} \left( \frac{e_i \Phi_{\text{int}}}{m_i c^2} \right) \right] |_{q_i = q_i(t)} \left[ \delta t - \sum_{i=1}^{2} \omega_{ci} \left( \frac{V_{0i}}{c^2} \right) \right] |_{q_i = q_i(t)} \cdot \delta q_2(t)
\]

\[
= \frac{1}{\hbar} \left[ \left( m_1 c^2 + \frac{m_2 v_2^2}{2} + V_{\text{int}}^c \right) |_{q_1 = q_1(t)} \delta t - \sum_{i=1}^{2} \frac{m_i v_i}{c} \cdot \delta q_i(t) \right].
\]

(80)

where \(q_01(t)\) denotes the transverse coordinate of particle 1 in its IRF (which, of course, changes as a function of time due to the Coulomb interaction). In other words, (80) tells us how the Compton frequency of particle 1, \(\omega_{c1}\), gets modulated by the physical coupling of particle 1 to particle 2, in the IRF of particle 1. Thus (80) represents the conditional phase change of particle 1 in its IRF. We can also confirm that when \(\Phi_{\text{int}}^c \approx 0\) the velocity of particle 2 no longer depends on the position of particle 1 at time \(t\), leaving \(\delta \theta_{1}^{\text{rest}} = \omega_{c1} \delta t_0\). Likewise we can obtain the conditional \(\text{zbw}\) phase of particle 2 in its IRF.

The conditional \(\text{zbw}\) phase of particle 1 in the lab frame where \(v_1 \neq 0\) is just

\[
\delta \theta_{1}^{\text{lab}}(q_1(t), q_2(t), t) = \left[ \omega_{c1} + \omega_{c2} \left( \frac{v_1^2}{2c^2} \right) + \sum_{i=1}^{2} \omega_{ci} \left( \frac{e_i \Phi_{\text{int}}}{m_i c^2} \right) \right] |_{q_i = q_i(t)} \left[ \delta t - \sum_{i=1}^{2} \omega_{ci} \left( \frac{V_{0i}}{c^2} \right) \right] |_{q_i = q_i(t)} \cdot \delta q_i(t)
\]

\[
= \frac{1}{\hbar} \left[ \left( m_1 c^2 + \frac{m_2 v_1^2}{2} + V_{\text{int}}^c \right) |_{q_1 = q_1(t)} \delta t - \sum_{i=1}^{2} \frac{m_i v_i}{c} \cdot \delta q_i(t) \right].
\]

(81)

Equivalently, we can obtain (81) by just subtracting \(\omega_{c2} \delta t\) from (79). And likewise for the conditional \(\text{zbw}\) phase of particle 2 in the lab frame.

Recall that, by hypothesis, each \(\text{zbw}\) particle is essentially a harmonic oscillator. This means that when \(V_{\text{int}}^c \approx 0\) each particle has its own well-defined phase at each point along its space-time trajectory. Consistency with this hypothesis also means that when \(V_{\text{int}}^c > 0\) the joint phase must be a well-defined function of the space-time trajectories of both particles (since we posit that both particles remain harmonic oscillators despite having their oscillations physically coupled by \(V_{\text{int}}^c\)). Then for a closed loop \(L\), along which each particle can be physically or virtually displaced, the joint phase in the lab frame will satisfy

\[
\sum_{i=1}^{2} \int_L \delta \theta_{\text{joint}}^{\text{lab}} = 2\pi n,
\]

(82)
and for a loop in which time is held fixed,

$$\sum_{i=1}^{2} \oint_L \mathbf{p}_i \cdot \delta \mathbf{q}_i(t) = nh. \tag{83}$$

It also follows from (82) and (83) that

$$\oint_L \delta \theta_1^{\text{lab}} = 2\pi n, \tag{84}$$

and

$$\oint_L \mathbf{p}_1 \cdot \delta \mathbf{q}_1(t) = nh, \tag{85}$$

where this time the closed-loop integration involves keeping the coordinate of particle 2 fixed while particle 1 is displaced along \( L \). From (82-85), it will also be the case that

$$\sum_{i=1}^{2} \oint_L \delta \theta_1^{\text{lab}} = 2\pi n, \tag{86}$$

and

$$\oint_L \delta \theta_1^{\text{lab}} = 2\pi n. \tag{87}$$

Integrating (79) and multiplying through by \( h \) yields (setting \( S_1^{\text{lab}} = S \))

$$S = \sum_{i=1}^{2} \oint_L \mathbf{p}_i \cdot d\mathbf{q}_i(t) - \sum_{i=1}^{2} \oint_L \left( m_i c^2 + \frac{m_i v_i^2}{2} + \epsilon_i \Phi_i^{\text{int}} \right) dt - \sum_{i=1}^{2} \oint_L \mathbf{q}_i(t), \tag{88}$$

and evolves by

$$- \partial_t S |_{\mathbf{q}_i=\mathbf{q}_i(t)} = \sum_{i=1}^{2} \frac{m_i c^2}{2} + \sum_{i=1}^{2} \frac{(\nabla_i S)^2}{2m_i} |_{\mathbf{q}_i=\mathbf{q}_i(t)} + V_i^{\text{int}}. \tag{89}$$

The conditional phase \( S_1^{\text{lab}} = S_1 \) and its equation of motion only differ from (88-89) by subtracting \( m_2 c^2 t - \phi_2 \). Analogous considerations apply to particle 2. Finally, the acceleration of the \( i \)-th particle is obtained from the equation of motion

$$m_i \ddot{\mathbf{q}}_i(t) = \left[ \partial_t \mathbf{p}_i + \mathbf{v}_i \cdot \nabla_i \mathbf{p}_i \right] |_{\mathbf{q}_i=\mathbf{q}_i(t)} = -\nabla_i V_i^{\text{int}} |_{\mathbf{q}_i=\mathbf{q}_i(t)}. \tag{90}$$

Another, more convenient way of modeling the case of two classically interacting \( zb \) particles is by exploiting the well-known fact that a two-particle system with an interaction potential of the form (78) has an equivalent Hamiltonian of the form (ignoring the trivial CM motion)

$$E_{rel} = \frac{p_{rel}^2}{2\mu} + V_{rel}(|\mathbf{q}_{rel}(t)|) + \mu c^2, \tag{91}$$

where the reduced mass \( \mu = m_1 m_2/(m_1 + m_2) \) and \( V_{rel}(|\mathbf{q}_{rel}(t)|) = V_c^{\text{int}}(|\mathbf{q}_1(t) - \mathbf{q}_2(t)|) \). In other words, (91) describes a fictitious \( zb \) particle of mass \( \mu \) and relative coordinate \( \mathbf{q}_{rel}(t) \), moving in an “external” potential \( V_{rel}(|\mathbf{q}_{rel}(t)|) \). This fictitious particle then has a Compton frequency, \( \omega_{c}^{\text{red}} = \mu c^2 / h \), and an associated phase change in the lab frame of the form

$$\delta \theta_{rel}(\mathbf{q}_{rel}(t)) = \left( \omega_{c}^{\text{red}} + \omega_{e}^{\text{red}} \frac{p_{rel}^2}{2c^2} + \omega_{e}^{\text{red}} \frac{V_{rel}(|\mathbf{q}_{rel}(t)|)}{\mu c^2} \right) \frac{dt}{\mu c^2} - \frac{V_{rel}(\mathbf{q}_{rel}(t)) \cdot \delta \mathbf{q}_{rel}(t)}{c^2}. \tag{92}$$

Upon integration, this of course gives
\[ S_{\text{rel}} = -\hbar \partial_{r_{\text{rel}}} = \int [p_{\text{rel}} \cdot dq_{\text{rel}}(t) - E_{\text{rel}} dt] - \hbar \phi_{\text{rel}}, \]  

which evolves in time by the HJ equation

\[ - \partial_t S_{\text{rel}}|_{q_{\text{rel}}=q_{\text{rel}}(t)} = \mu c^2 + \frac{\left(\nabla_{r_{\text{rel}}} S_{\text{rel}}\right)^2}{2 \mu} |_{q_{\text{rel}}=q_{\text{rel}}(t)} + V_{\text{rel}}, \]

and gives the equation of motion

\[ \mu \ddot{q}_{\text{rel}}(t) = [\partial_t p_{\text{rel}} + v_{\text{rel}} \cdot \nabla_{r_{\text{rel}}} p_{\text{rel}}]|_{q_{\text{rel}}=q_{\text{rel}}(t)} = -\nabla_{r_{\text{rel}}} V_{\text{rel}}|_{q_{\text{rel}}=q_{\text{rel}}(t)}. \]  

Since this situation is formally equivalent to the case of a single \( zbw \) particle moving in an external field, we can immediately see that it follows

\[ \oint_L \delta S_{\text{rel}} = n\hbar, \]

and

\[ \oint_L p_{\text{rel}} \cdot \delta q_{\text{rel}}(t) = n\hbar. \]

Furthermore, the physical equivalence between this coordinatization and the original two-particle coordinatization establishes that if phase quantization holds in one coordinatization it must hold in the other.

While we considered here only two \( zbw \) particles classically interacting through an electric scalar potential, all our considerations straightforwardly generalize to the case of many \( zbw \) particles classically interacting through electric scalar potentials as well as magnetic vector potentials (and likewise for the gravitational analogues).

### 3.6 Classical Hamilton-Jacobi statistical mechanics for two interacting \( zbw \) particles

For a statistical ensemble of two classically interacting particles, we replace the trajectories \( \{q_1(t), q_2(t)\} \) with the coordinates \( \{q_1, q_2\} \). Then the non-relativistic joint phase field in the lab frame is obtained from (88) as

\[ S(q_1, q_2, t) = \sum_{i=1}^{2} \int p_i \cdot dq_i - \sum_{i=1}^{2} \int \left[ m_i c^2 + \frac{m_i v^2_i(q_1, q_2, t)}{2} + e_i \Phi_{\text{int}}(q_1, q_2) \right] dt - \sum_{i=1}^{2} \hbar \phi_i, \]

and evolves by

\[ - \partial_t S = \sum_{i=1}^{2} \frac{m_i c^2}{2} + \sum_{i=1}^{2} \frac{(\nabla_i S)^2}{2m_i} + V_{\text{int}}^i, \]

where \( v_i(q_1, q_2) = \nabla_i S(q_1, q_2, t)/m_i \). Since each member of the \( i \)-th ensemble is a \( zbw \) particle, it follows that

\[ \sum_{i=1}^{2} \oint_L \nabla_i S \cdot dq_i = n\hbar, \]

where \( L \) is now a mathematical loop in the 2-particle configuration space, along which a fictitious \( zbw \) particle in the \( i \)-th ensemble can be displaced.

The two-particle probability density for the joint ensemble \( \rho(q_1, q_2, t) \geq 0 \) evolves by the two-particle continuity equation

\[ \frac{\partial \rho}{\partial t} = -\sum_{i=1}^{2} \nabla_i \cdot \left[ \left( \frac{\nabla_i S}{m_i} \right) \rho \right], \]

and allows us to define the time-symmetric, ensemble-averaged, two-particle Lagrangian.
\[ J(q_1, q_2) = \int_{\mathbb{R}^6} d^3q_1 d^3q_2 \rho(q_1, q_2, t) \sum_{i=1}^2 \left[ \int_{q_i}^{q_f} m_i \dot{v}_i \cdot dq_i(t) - \int_{q_i}^{q_f} \left( m_i c^2 + \frac{p_i^2}{2m_i} + e_i \Phi_{c_i}^{\text{int}} \right) dt - h\Phi_i \right] \]

where the equated expressions are related by the usual Legendre transformation. Imposing

\[ J(q_1, q_2) = \text{extremal}, \]

straightforward manipulations along the lines of those in the Appendix yield (90). And, upon replacing \( q_i(t) \) with \( q_i \), the classical Newtonian equation for the acceleration field \( a_i(q_1, q_2, t) \):

\[ m_i a_i = \partial_t p_i + \dot{v}_i \cdot \nabla_i p_i = -\nabla_i V_c^{\text{int}}. \]

Now, we can obtain the conditional \( zbw \) phase field for particle 1 by evaluating the joint phase field at the actual position of particle 2 at time \( t \), i.e., \( S(q_1, q_2(t), t) = S_1(q_1, t) \). Taking the total time derivative we have

\[ \partial_t S_1(q_1, t) = \partial_t S(q_1, q_2(t))|_{q_2=q_2(t)} + \frac{dq_2(t)}{dt} \cdot \nabla_2 S(q_1, q_2(t))|_{q_2=q_2(t)}, \]

where the conditional velocities

\[ \frac{dq_1(t)}{dt} = v_1(q_1, t)|_{q_1=q_1(t)} = \frac{\nabla_1 S_1(q_1, t)}{m_1}|_{q_1=q_1(t)}, \]

and

\[ \frac{dq_2(t)}{dt} = v_2(q_2, t)|_{q_2=q_2(t)} = \frac{\nabla_2 S_2(q_2, t)}{m_2}|_{q_2=q_2(t)}, \]

the latter defined from the conditional phase field \( S_2(q_2, t) \) for particle 2. Inserting (105) into the left hand side of (99) and adding the corresponding term on the right hand side, we then find that the conditional phase field for particle 1 evolves by a ‘conditional HJ equation’, namely

\[ -\partial_t S_1 = m_1 c^2 + \frac{(\nabla_1 S_1)^2}{2m_1} + \frac{(\nabla_2 S_2)^2}{2m_2}|_{q_2=q_2(t)} - \frac{dq_2(t)}{dt} \cdot \nabla_2 S|_{q_2=q_2(t)} + V_c^{\text{int}}(q_1, q_2(t)), \]

where \( V_c^{\text{int}}(q_1, q_2(t)) \) is the ‘conditional potential’ for particle 1; that is, the potential field that particle 1, at location \( q_1 \), would ‘feel’ given the actual location of particle 2. The solution of (108) can be verified as

\[ S_1 = \int p_1 \cdot dq_1 - \int \left[ m_1 c^2 + m_1 v_1^2 \frac{1}{2} + m_1 v_2^2 \frac{1}{2} - p_2 \cdot \frac{dq_2(t)}{dt} + V_c^{\text{int}} \right] dt - h\Phi_1. \]

Notice here that the conditional phase field is a field on 3-D space. This makes perfect sense since, after all, the conditional phase refers to the phase associated to the \( zbw \) oscillation of particle 1, a real physical oscillation in 3-D space. It can also be verified that when (109) is evaluated at \( q_1 = q_1(t) \), it is equivalent to \( S_{\text{joint}}^{\text{lab}}(q_1(t), q_2(t), t) - m_2 c^2 t + h\Phi_2 \). Once again, since the conditional \( zbw \) phase field for particle 1 is a field over a statistical ensemble of fictitious, identical, non-interacting \( zbw \) particle 1’s (each fictitious member of the ensemble representing a possible position, velocity, and phase that the actual \( zbw \) particle 1 could occupy at time \( t \)), it will be the case that

\[ \oint_L \nabla_1 S_1 \cdot dq_1 = nh, \]

where \( L \) is a mathematical loop in 3-D space along which a fictitious \( zbw \) particle in the \( i = 1 \) conditional ensemble can be displaced.

Likewise, we can obtain the conditional probability density for particle 1 by writing \( \rho(q_1, q_2(t), t) = \rho_1(q_1, t) \). Taking the total time derivative gives

\[ \partial_t \rho_1(q_1, t) = \partial_t \rho(q_1, q_2(t))|_{q_2=q_2(t)} + \frac{dq_2(t)}{dt} \cdot \nabla_2 \rho(q_1, q_2(t))|_{q_2=q_2(t)}. \]
Inserting this on the left hand side of (101) and adding the corresponding term on the right hand side, we obtain the conditional continuity equation for particle 1:
\[ \partial_t \rho_1 = -\nabla_1 \cdot \left[ \left( \frac{\nabla_1 S_1}{m_1} \right) \rho_1 \right] - \nabla_2 \cdot \left[ \left( \frac{\nabla_2 S}{m_2} \right) \rho \right] \bigg|_{q_2=q_2(t)} + \frac{d q_2(t)}{dt} \cdot \nabla_2 \rho |_{q_2=q_2(t)}, \]  
which implies \( \rho_1(q_1, t) \geq 0 \) and preservation of the normalization \( \int_{\mathbb{R}^3} \rho_1(q_1, 0) = 1 \). With the conditional acceleration field of particle 1:

\[ J_1(q_1) = \int_{\mathbb{R}^3} d^3 q_1 \rho_1(q_1, t) \left[ \int_{t_i}^{t_f} m_1 v_1 \cdot d q_1(t) - \int_{t_i}^{t_f} \left( \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} - p_2 \cdot \frac{d q_2(t)}{dt} + V^\text{int} \right) \right] dt - \hbar \phi_1, \]  
where it can be readily confirmed that the equated lines are related by the Legendre transformation. Imposing

\[ J_1 = \text{extremal}, \]  
where the subscript 1 denotes that the variation is only with respect to \( q_1(t) \), straightforward manipulations analogous to those in the Appendix yield, upon replacing \( q_1(t) \) with \( q_1 \), the classical equation of motion for the conditional acceleration field of particle 1:

\[ m_1 a_1(q_1, t) = \left[ \partial_t p_1 + v_1 \cdot \nabla_1 p_1 \right] (q_1, t) = -\nabla_1 V^\text{int}(q_1, q_2(t)). \]

The conditional phase field, probability density, etc., for particle 2, are developed analogously.

We now turn to the formulation of our classical statistical mechanics in terms of the reduced mass \( zbw \) particle. Replacing \( q_{rel}(t) \) with \( q_{rel} \), the reduced mass \( zbw \) phase field

\[ S_{rel}(q_{rel}, t) = \int p_{rel} \cdot dq_{rel} - \int \left( \mu c^2 + \frac{p_{rel}^2}{2 \mu} + V_{rel} \right) dt - \hbar \phi_{rel}, \]

evolves by the reduced mass HJ equation

\[ -\partial_t S_{rel} = \mu c^2 + \frac{(\nabla_{rel} S_{rel})^2}{2 \mu} + V_{rel}, \]

and satisfies

\[ \oint_L \nabla_{rel} S_{rel} \cdot dq_{rel} = nh, \]

where \( L \) is a mathematical loop in 3-D space. Introducing the probability density for the reduced mass \( zbw \) particle, \( \rho_{rel}(q_{rel}, t) \geq 0 \), it is straightforward to show it evolves by the continuity equation

\[ \frac{\partial \rho_{rel}}{\partial t} = -\nabla_{rel} \cdot \left[ \left( \frac{\nabla_{rel} S_{rel}}{m_{rel}} \right) \rho_{rel} \right], \]

which preserves the normalization \( \int_{\mathbb{R}^3} d^3 q_{rel} \rho_{rel}(q_{rel}, 0) = 1 \). Using this density to introduce

\[ J_{rel}(q_{rel}) = \int_{\mathbb{R}^3} d^3 q_{rel} \rho_{rel}(q_{rel}, t) \left[ \int_{q_{rel}}^{q_{rel}} \mu v_{rel} \cdot dq_{rel}(t) - \int_{t_i}^{t_f} \left( \mu c^2 + \frac{p_{rel}^2}{2 \mu} + V_{rel} \right) dt - \hbar \phi_{rel} \right] \]

\[ = \int_{\mathbb{R}^3} d^3 q_{rel} \rho_{rel} \int_{t_i}^{t_f} \left\{ \frac{1}{2} \mu v_{rel}^2 - \mu c^2 - V_{rel} \right\} dt - \hbar \phi_{rel}, \]

and imposing the constraint

\[ J_{rel} = \text{extremal}, \]

we obtain after manipulations (and replacing \( q_{rel}(t) \) by \( q_{rel} \)) the equation of motion

\[ \mu a_{rel}(q_{rel}, t) = \partial_t p_{rel} + v_{rel} \cdot \nabla_{rel} p_{rel} = -\nabla_{rel} V_{rel}(|q_{rel}|). \]
Let us now recover the nonlinear Schrödinger equations for each of the three cases we’ve considered.

The combination of (99)-(101) gives

\[ i\hbar \frac{\partial \psi}{\partial t} = \sum_{i=1}^{2} \left[ -\frac{\hbar^2}{2m_i} \nabla_i^2 + \frac{\hbar^2}{2m_i} \frac{\nabla_i^2 |\psi|}{|\psi|} + mc^2 \right] \psi + V^{\text{int}} \psi, \]

(123)

where \( \psi(q_1, q_2, t) = \sqrt{\rho(q_1, q_2, t)} e^{iS(q_1, q_2, t)/\hbar} \) is single-valued by (100).

Combining (108) and (112) gives the conditional nonlinear Schrödinger equation for particle 1:

\[ i\hbar \frac{\partial \psi_1}{\partial t} = -\frac{\hbar^2}{2m_1} \nabla_1^2 \psi_1 - \frac{\hbar^2}{2m_1} \nabla_1^2 |\psi_1|_{q_2=q_2(t)} + V^{\text{int}}(q_1, q_2(t)) \psi_1 + m_1 c^2 \psi_1 \]

\[ + i\hbar \frac{\partial q_2(t)}{\partial t} \cdot \nabla_2 \psi |_{q_2=q_2(t)} + \left( \frac{\hbar^2}{2m_1} \nabla_1^2 |\psi_1| \right) \psi_1 + \left( \frac{\hbar^2}{2m_2} \nabla_2^2 |\psi| \right) |_{q_2=q_2(t)} \psi_1, \]

(124)

where \( \psi(q_1, q_2, t) = \psi_1(q_1, t) = \sqrt{\rho_1(q_1, t)} e^{iS_1(q_1, t)/\hbar} \) is the conditional classical wavefunction for particle 1, and satisfies single-valuedness as a consequence of (110). Here \( d\delta q_2(t)/dt = (\hbar/m_2) \text{Im}\{\nabla_2 \ln(\psi_2)\} |_{q_2=q_2(t)} \), where \( \psi_2 = \psi_2(q_2, t) \) is the conditional wavefunction for particle 2 and satisfies a conditional nonlinear Schrödinger equation analogous to (124). Note that also that (124) can be obtained by taking the total time derivative of the conditional wavefunction for particle 1

\[ \frac{\partial \psi_1(q_1, q_2, t)}{\partial t} = \frac{\partial \psi_1(q_1, q_2, t)}{\partial q_2} \frac{\partial q_2(t)}{\partial t} + \frac{\partial \psi_1(q_1, q_2, t)}{\partial q_2} \cdot \nabla_2 \psi |_{q_2=q_2(t)}, \]

(125)

inserting this on the left hand side of (123), adding the corresponding term on the right hand side, and subtracting \( m_2 c^2 \psi_2 \).

Finally, combining (117-119) gives the nonlinear Schrödinger equation for the fictitious reduced mass particle:

\[ i\hbar \frac{\partial \psi_{\text{rel}}}{\partial t} = \left[ -\frac{\hbar^2}{2\mu} \nabla_{\text{rel}}^2 + \frac{\hbar^2}{2\mu} \frac{\nabla_{\text{rel}}^2 |\psi_{\text{rel}}|}{|\psi_{\text{rel}}|} + \mu c^2 \right] \psi_{\text{rel}} + V(|q_{\text{rel}}|) \psi_{\text{rel}}, \]

(126)

where \( \psi_{\text{rel}}(q_{\text{rel}}, t) = \sqrt{\rho_{\text{rel}}(q_{\text{rel}}, t)} e^{iS_{\text{rel}}(q_{\text{rel}}, t)/\hbar} \) is a single-valued classical wavefunction. As with the linear Schrödinger equation of quantum mechanics, it is easily verified that (126) can be obtained from (123) by transforming the two-particle Hamiltonian operator to the center of mass and relative coordinates.

This completes the development of the classical HJ statistical mechanics for two classically interacting zbw particles. The generalization to \( N \) zbw particles interacting through their electric scalar and magnetic vector potentials (and the gravitational analogues thereof) is straightforward, but will not be given here due to unnecessary mathematical complexity.

### 3.7 Remarks on close-range interactions

Throughout we have assumed the point-like interaction case, \( q_{\text{rel}}(t) = |q_1(t) - q_2(t)| \gg \lambda_c \). But what changes when \( q_{\text{rel}}(t) \sim \lambda_c? \) Not much. To show this, we adopt the approach of Zelevinsky [20] in modeling the deviation from point-like interactions with a Darwin interaction term as follows.

Consider the (hypothesized) 3-D zbw oscillation/fluctuation around the relative coordinate, \( q_{\text{rel}}(t) + \delta q(t) \), where \( \delta q_{\text{max}} = \{\delta q_{\text{max}}(t)\} = \lambda_c \). Taylor expand the (Coulomb or Newtonian) interaction potential into \( V_{\text{int}}(|q_{\text{rel}}(t) + \delta q(t)|) \approx V_{\text{int}}(|q_{\text{rel}}(t)|) + \delta q(t) \cdot \nabla V_{\text{int}}(|q_{\text{rel}}(t)|) + \frac{1}{2} \sum_{i,j} \delta q_i(t) \delta q_j(t) \partial_i \partial_j V_{\text{int}}(|q_{\text{rel}}(t)|) \). Then, under the reasonable assumptions that the mean and variance of the fluctuations are given by \( \left< \delta q(t) \right> = 0 \) and \( \left< \delta q(t)^2 \right> = \lambda_c^2 / 2 \), the average potential \( \left< V_{\text{int}}(|q_{\text{rel}}(t) + \delta q(t)|) \right> = V_{\text{int}}(|q_{\text{rel}}(t)|) + \frac{1}{2} \lambda_c^2 \). Finally, approximating \( \left< \delta q(t)^2 \right> = \frac{1}{2} \lambda_c^2 \), we find that the perturbation of the potential due to the fluctuations is \( \delta V \approx \frac{1}{12} \lambda_c^2 \nabla^2 V_{\text{int}} = \frac{1}{12} \lambda_c^2 4\pi K \delta(q) \), if the interaction potential is of the general form, \( V_{\text{int}}(q) = K \delta(q) \), where \( K \) is a constant.

Note that because the zbw oscillation is a (rheonomic) constraint on each particle, the Coulomb interaction between them never causes their oscillations to deviate from simple harmonic motion (even though their oscillation frequencies can slightly shift by an amount of the order \( (\omega V_{\text{int}})/\hbar) \); so phase/momentum quantization for each particle is not altered, even when \( q_{\text{rel}}(t) \sim \lambda_c \). Alternatively, we could relax the zbw constraint by assuming that when \( q_{\text{rel}}(t) \sim \lambda_c \), a slight deviation from simple harmonic motion occurs because the Coulomb
4 Zitterbewegung Stochastic Mechanics

4.1 Free zbw particles

We take as our starting point the hypothesis that \( N \) particles of rest masses, \( m_i \), and 3-D space positions, \( \mathbf{q}_i(t) \), are immersed in Nelson’s hypothesized ether and undergo conservative diffusion processes according to the stochastic differential equations

\[
d\mathbf{q}_i(t) = \mathbf{b}_i(q(t), t)dt + d\mathbf{W}_i(t),
\]

and

\[
d\mathbf{q}_i(t) = \mathbf{b}_i(q(t), t)dt + d\mathbf{W}_i(t),
\]

where the forward Wiener processes \( d\mathbf{W}_i(t) \) satisfy \( E_t [d\mathbf{W}_i] = 0 \) and \( E_t [d\mathbf{W}_i^2] = (\hbar/m_i) dt \), and the backward Wiener processes satisfy the same conditions but are independent of \( d\mathbf{q}_i(s) \) for \( s \geq t \). Note that we take the \( \mathbf{b}_i \) \( (\mathbf{b}_{i*}) \) to be functions of all the particle positions, \( q(t) = \{ \mathbf{q}_1(t), \mathbf{q}_2(t), ..., \mathbf{q}_N(t) \} \in \mathbb{R}^{3N} \). The reasons for this are: (1) all the particles are continuously exchanging energy-momentum with a common background medium (Nelson’s ether) and thus are in general physically connected in their motions through the ether via \( \mathbf{b}_i \) \( (\mathbf{b}_{i*}) \), insofar as the latter are constrained by the physical properties of the ether; and (2) the dynamical equations and initial conditions for the \( \mathbf{b}_i \) \( (\mathbf{b}_{i*}) \) are what will determine the specific situations under which the latter will be effectively separable functions of the particle positions and when they cannot be effectively separated.

Hence, at this level, it is only sensible to write \( \mathbf{b}_i \) \( (\mathbf{b}_{i*}) \) as functions of all the particle positions at a single time.

As in the single particle case, in order to incorporate the zbw oscillation as a property of each particle, we must amend Nelson’s original phenomenological hypotheses about his ether and particles with the \( N \)-particle generalizations of the new phenomenological hypotheses we introduced in Part I:

1. Nelson’s ether is not only a stochastically fluctuating medium in space-time, but an oscillating medium with a spectrum of angular frequencies superposed at each point in 3-space. More precisely, we imagine the ether as a continuous (or effectively continuous) medium composed of a countably infinite number of fluctuating, stationary, spherical waves superposed at each point in space, with each wave having a different fixed angular frequency, \( \omega_i^0 \), where \( i \) denotes the \( i \)-th ether mode. The relative phases between the modes are taken to be random so that each mode is effectively uncorrelated with every other mode.

2. The particles of rest masses \( m_i \), located at positions \( \mathbf{q}_{0i} \) in their respective instantaneous mean translational rest frames (IMTRFs), i.e., the frames in which \( D\mathbf{q}_i(t) = D\mathbf{q}_i(t) = 0 \), are bounded to harmonic oscillator potentials with fixed natural frequencies \( \omega_{0i} = \omega_{0i} = (1/\hbar) m_i c^2 \). In keeping with the phenomenological approach of ZSM, and the approach taken by de Broglie and Bohm with their zbw models, we need not specify the precise physical nature of these harmonic oscillator potentials; this task is left for a future physical model of the ZSM particle.

3. Each particle’s center of mass, as a result of being immersed in the ether, undergoes an approximately frictionless translational Brownian motion (due to the homogeneous and isotropic ether fluctuations that couple to the particles by possibly electromagnetic, gravitational, or some other means), as modeled by Eqs. (127) and (128); and, in their respective IMTRFs, undergo driven oscillations about \( \mathbf{q}_{0i} \) by coupling to a narrow band of ether modes that resonantly peak around their natural frequencies. However, in order that the oscillation of each particle doesn’t become unbounded in kinetic energy, there must be some mechanism by which the particles dissipate energy back into the ether so that, on the average, a steady-state equilibrium regime is reached for their oscillations. So we posit that on short relaxation time-scales, \( \tau_i \), which are identical for particles of identical rest masses, the mean energy absorbed
from the driven oscillation by the resonant ether modes equals the mean energy dissipated back to the ether by a given particle. Thus, in the steady-state regime, each particle undergoes a constant mean \( zbw \) oscillation about its \( q_{0i} \) in its IMTRF, as characterized by the fluctuation-dissipation relation, \( \langle H_i \rangle_{\text{steady-state}} = \hbar \omega_{ci} = m_i c^2 \), where \( \langle H_i \rangle_{\text{steady-state}} \) is the conserved mean energy due to the steady-state oscillation of the \( i \)-th particle. Accordingly, if, relative to the ether, all the particles have zero mean translational motion, then we will have \( \sum_i \langle H_i \rangle_{\text{steady-state}} = \sum_i \hbar \omega_{ci} = \sum_i m_i c^2 = \text{const.} \)

It follows then that, in the IMTRF of the \( i \)-th particle, the mean \( zbw \) phase change is given by
\[
\delta \bar{\theta}_i = \omega_{ci} \delta t_0 = \frac{m_i c^2}{\hbar} \delta t_0,
\]
and the corresponding absolute mean phase is
\[
\bar{\theta}_i = \omega_{ci} t_0 + \phi_i = \frac{m_i c^2}{\hbar} t_0 + \phi_i.
\]
Then the joint (mean) phase for all the particles will just be
\[
\bar{\theta} = \sum_{i=1}^{N} \bar{\theta}_i = \sum_{i=1}^{N} (\omega_{ci} t_0 + \phi_i) = \sum_{i=1}^{N} \left( \frac{m_i c^2}{\hbar} t_0 + \phi_i \right).
\]

As in the single particle case, we cannot talk of the \( zbw \) phase other than in the IMTRFs of the particles, because we cannot transform to a frame in which \( \dot{d}q(t)/dt = 0 \), as this expression is undefined for the Wiener process.

Now, Lorentz transforming back to the lab frame
\[
Dq_i(t) = b_i(q_i(t), t) \neq 0,
\]
and
\[
Dq^*_i(t) = b^*_i(q_i(t), t) \neq 0,
\]
and approximating the transformation for non-relativistic velocities so that \( \gamma = 1/\sqrt{(1 - b_i^2/c^2)} \approx 1 + b_i^2/2c^2 \), the forward and backward joint phase changes become
\[
\delta \bar{\theta}_+(q(t), t) = \sum_{i=1}^{N} \frac{\omega_{ci}}{m_i c^2} [E_{i+}(q(t), t) \delta t - m_i b_i(q(t), t) \cdot \delta \dot{q}_i(t)] = \frac{1}{\hbar} \left[ \sum_{i=1}^{N} E_{i+}(q(t), t) \delta t - \sum_{i=1}^{N} m_i b_i(q(t), t) \cdot \delta \dot{q}_i(t) \right],
\]
and
\[
\delta \bar{\theta}_-(q(t), t) = \sum_{i=1}^{N} \frac{\omega_{ci}}{m_i c^2} [E_{i-}(q(t), t) \delta t - m_i b^*_i(q(t), t) \cdot \delta \dot{q}_i(t)] = \frac{1}{\hbar} \left[ \sum_{i=1}^{N} E_{i-}(q(t), t) \delta t - \sum_{i=1}^{N} m_i b^*_i(q(t), t) \cdot \delta \dot{q}_i(t) \right],
\]
where, without any time-symmetric dynamical constraints on the diffusion process,
\[
E_{i+}(q(t), t) = m_i c^2 + \frac{1}{2} m_i b_i^2,
\]
and
\[
E_{i-}(q(t), t) = m_i c^2 + \frac{1}{2} m_i b^*_i,
\]

Since each \( zbw \) particle is essentially a harmonic oscillator, each particle has its own well-defined phase at each point along its space-time trajectory (when \( b_i(q, t) \approx \sum_i b_i(q_i, t) \)). Consistency with this hypothesis also means that when \( b_i(q, t) \neq \sum_i b_i(q_i, t) \), the joint phase must be a well-defined function of the space-time trajectories of all particles (since we posit that all particles remain harmonic oscillators despite having their oscillations physically coupled through the common ether medium they interact with). Thus, even before
imposing time-symmetric constraints on the diffusion process, we can see that for a closed loop $L$ along which each particle can be physically or virtually displaced, it follows that the forward and backward joint phase changes, (134) and (135), each equal $2\pi n$. And this holds for a closed loop in which both time and position change, as well as a closed loop with time held constant.

In the lab frame, the forward and backward stochastic differential equations for the translational motion are again given by (127) and (128), and the corresponding forward and backward Fokker-Planck equations take the form

$$\frac{\partial \rho(q,t)}{\partial t} = - \sum_{i=1}^{N} \nabla_i \cdot [b_i(q,t)\rho(q,t)] + \sum_{i=1}^{N} \frac{\hbar}{2m_i} \nabla_i^2 \rho(q,t),$$

and

$$\frac{\partial \rho(q,t)}{\partial t} = - \sum_{i=1}^{N} \nabla_i \cdot [b_i^*(q,t)\rho(q,t)] - \sum_{i=1}^{N} \frac{\hbar}{2m_i} \nabla_i^2 \rho(q,t).$$

Moreover, if we now impose the time-symmetric kinematic constraints

$$v_i(q,t) = \frac{1}{2} [b_i(q,t) + b_i^*(q,t)] = \frac{\nabla_i S(q,t)}{m_i}|_{q_i = q_i^*(t)},$$

and

$$u_i(q,t) = \frac{1}{2} [b_i(q,t) - b_i^*(q,t)] = \frac{\hbar}{2m_i} \nabla_i \rho(q,t) \rho(q,t)|_{q_i = q_i^*(t)},$$

then (138) and (139) reduce to

$$\frac{\partial \rho(q,t)}{\partial t} = - \sum_{i=1}^{N} \nabla_i \cdot \left[ \frac{\nabla_i S(q,t)}{m_i} \rho(q,t) \right],$$

with $b_i = v_i + u_i$ and $b_i^* = v_i - u_i$.

As we did for $N$-particle NYSM, we now postulate here the presence of an external (to the particle) osmotic potential, $U(q,t)$, which couples to the $i$-th particle as $R(q(t),t) = \mu U(q(t),t)$ (assuming that the coupling constant $\mu$ is identical for particles of the same species), and imparts to the $i$-th particle a momentum, $\nabla_i R(q,t)|_{q_i = q_i^*(t)}$. This momentum then gets counter-balanced by the ether fluid’s osmotic impulse pressure, $(\hbar/2m_i) \nabla_i \ln n(q,t)|_{q_i = q_i^*(t)}$, so that the $N$-particle osmotic velocity is the equilibrium velocity acquired by the $i$-th particle when $\nabla_i R/m_i = (\hbar/2m_i) \nabla_i \rho/\rho$ (using $\rho = n/N$), which implies $\rho = e^{2R/\hbar}$ for all times. As discussed in section 2, it is expected that $R$ generally depends on the coordinates of all the other particles. The reasons, to remind the reader, are that: (i) we argued, for reasons of consistency, that $U$ should be sourced by the ether, and (ii) since the particles continuously exchange energy-momentum with the ether, the functional dependence of $U$ will be determined by the dynamical coupling of the ether to the particles as well as the magnitude of the inter-particle physical interactions (whether through a classical inter-particle potential or, in the free particle case, just through the ether). To make this last point more explicit, suppose two classically non-interacting $zbw$ particles of identical mass, each initially driven in their oscillations and translational motions by effectively independent regions of oscillating ether, each region sourcing the osmotic potentials $U_1(q_1,t)$ and $U_2(q_2,t)$, move along trajectories that cause the spatial support of their dynamically relevant regions of oscillating ether to significantly overlap; then the particles will be exchanging energy-momentum with a common region of oscillating ether modes, leading to an osmotic potential sourced by this common region of oscillating ether that depends on the motions (hence positions) of both particles, i.e., $U(q_1, q_2)$. Indeed, this common region of oscillating ether will drive the subsequent mean $zbw$ oscillations and translational Brownian motions of both particles, leading to a joint phase $\theta(q_1, q_2, t)$ whose gradient with respect to the $i$-th particle coordinate gives rise to the current velocity of the $i$-th particle (as we will see), and to an osmotic counter-balancing of $\nabla_i U(q_1, q_2, t)$, which gives rise to the osmotic velocity of the $i$-th particle (as we’ve already seen). Mathematically, the non-linear coupling between the osmotic potential and the evolution of the joint phase of the $zbw$ particles can be seen by writing the solution to (142), which from
section 2 is

$$\rho(q, t) = \rho_0(q_0) \exp[- \int_0^t \left( \sum_i \nabla_i \cdot \mathbf{v}_i \right) dt'] = \rho_0(q_0) \exp[- \int_0^t \left( \sum_i \frac{\nabla_i^2 S}{m_i} \right) dt'],$$

(143)

giving

$$R(q, t) = R_0(q_0) - \left( \frac{\hbar}{2} \right) \int_0^t \left( \sum_i \frac{\nabla_i^2 S}{m_i} \right) dt',$$

(144)

where \( S \) plays the role of the joint phase via \( \tilde{\theta} = -\frac{i}{\hbar} S \) (as we will see). Then we can infer from (144) that if a narrow bandwidth of common ether modes is driving the \( zbw \) oscillations of both particles (as described in hypothesis 3 above), the evolution of the osmotic potential (sourced by the common ether modes) will develop functional dependence on the positions of both particles. The precise form of this functional dependence and how it evolves in time will depend on the evolution equation for \( S \), which we of course need to specify (but already know will end up being the \( N \)-particle quantum HJ equation).

To obtain the time-symmetric dynamics for the mean translational motions of the \( N \) particles, i.e., the dynamics for \( S \), we integrate the time-asymmetric joint phases, (134) and (135), and then average the two to get

$$\tilde{\theta}(q(t), t) = \sum_{i=1}^N \frac{\omega_i}{m_i c^2} \left[ \int_{E_i(q(t), t)}^{E_i(q(t), t) + \Delta E_i} \mathbf{b}_i(q(t), t) \cdot d\mathbf{q}_i(t) \right] + \sum_{i=1}^N \phi_i,$$

(145)

where, from the kinematic constraints (140) and (141), we have

$$E_i(q(t), t) = m_i c^2 + \frac{1}{2} \left[ \frac{1}{2} m_i b_i^2 + \frac{1}{2} m_i b_i^2 \right] = m_i c^2 + \frac{1}{2} m_i v_i^2 + \frac{1}{2} m_i u_i^2.$$

(146)

Then the ensemble-averaged, time-symmetric particle phase is given by

$$J(q) = \int_{\mathbb{R}^N} d^N q \rho(q, t) \tilde{\theta}(q(t), t) = \int_{\mathbb{R}^N} d^N q \sum_{i=1}^N \frac{\omega_i}{m_i c^2} \left[ \int_{E_i}^{E_i + \Delta E_i} \mathbf{E}_i dt - \int_{q_{i,1}}^{q_{i,2}} \mathbf{E}_i \cdot d\mathbf{q}_i(t) \right] + \int_{\mathbb{R}^N} d^N q \rho \sum_{i=1}^N \phi_i,$$

(147)

which by the time-symmetric mean Legendre transformation\(^9\)

$$L_i = \frac{1}{2} \left[ (m \mathbf{b}_i) \cdot \mathbf{b}_i + (m \mathbf{b}_{i+}) \cdot \mathbf{b}_{i+} \right] - \frac{1}{2} \left( (E_{i+} + E_{i-}) = (m \mathbf{v}_i) \cdot \mathbf{v}_i + (m \mathbf{u}_i) \cdot \mathbf{u}_i - E_i \right),$$

(148)

and recalling that \( \tilde{\theta} = -\frac{i}{\hbar} S \), can be seen equivalent to Eq. (18) in section 2 (with the potentials set to zero and modulo the rest energy terms). Applying the conservative diffusion constraint through the variational principle implies

$$J(q) = extremal,$$

(149)

Straightforward computation shows that this yields

$$\sum_{i=1}^N \frac{m_i}{2} \left[ D_i S + DD_i \right] \mathbf{q}_i(t) = 0.$$

(150)

\(^9\)The time-symmetric mean Legendre transformation corresponds to taking the Legendre transforms of the forward and backward Lagrangians, respectively, and then taking the average of the Legendre transformed forward and backward Lagrangians, as given by (148).
Moreover, since the $\delta q_i(t)$ are independent (as shown in Appendix A), it follows from (150) that we have the individual equations of motion

$$m_i a_i(q(t), t) = \frac{m_i}{2} [D_i D + DD_i] q_i(t) = 0. \tag{151}$$

By applying the mean derivatives in (150), using that $b_i = v_i + u_i$ and $b_i \perp = v_i - u_i$, and replacing $q(t)$ with $q$ on both sides, straightforward manipulations give

$$\sum_{i=1}^N m_i \left[ \partial_t v_i + v_i \cdot \nabla_i v_i - u_i \cdot \nabla_i u_i - \frac{\hbar}{2m_i} \nabla_i^2 u_i \right] = 0, \tag{152}$$

Computing the derivatives in (152), we obtain

$$\sum_{i=1}^N m_i a_i(q(t), t) = \sum_{i=1}^N m_i \left[ \frac{\partial v_i(q, t)}{\partial t} + v_i(q, t) \cdot \nabla_i v_i(q, t) - u_i(q, t) \cdot \nabla_i u_i(q, t) - \frac{\hbar}{2m_i} \nabla_i^2 u_i(q, t) \right] \bigg|_{q_i=q_i(t)} = 0. \tag{153}$$

Integrating both sides of (153), setting the arbitrary integration constants equal to the rest energies, and replacing $q(t)$ with $q$, we then have the $N$-particle Quantum Hamilton-Jacobi equation

$$- \partial_t S(q, t) = \sum_{i=1}^N m_i c^2 + \sum_{i=1}^N \frac{(\nabla_i S(q, t))^2}{2m_i} - \sum_{i=1}^N \frac{\hbar^2}{2m_i} \frac{\nabla_i^2 \sqrt{\rho(q, t)}}{\sqrt{\rho(q, t)}} \tag{154}$$

for the sum total energy field over $N$ Gibbsian statistical ensembles of $zbw$ particles; and, upon evaluation at $q = q(t)$, the sum total energy of the actual particles along their actual mean trajectories. We can now see explicitly that the evolution equation for the time-symmetric phase, Eq. (145), under the conservative diffusion constraint (149), will just be (154). Additionally, from (140) and (145) we make the identification

$$p_i(q(t), t) = -\hbar \nabla_i \theta(q(t)) \big|_{q_i=q_i(t)} = \nabla_i S(q, t) \big|_{q_i=q_i(t)} \tag{155},$$

which establishes the $i$-th Nelsonian current velocity as the $i$-th translational mean velocity component of (145), and the velocity potential $S$ as the joint mean phase of the $zbw$ particles undergoing conservative diffusions.

The general solution of (154), i.e., the joint phase field of the $zbw$ particles in the lab frame (in which the current velocities of the $zbw$ particles are non-zero), is clearly of the form

$$S(q, t) = \sum_{i=1}^N \int p_i(q, t) \cdot dq_i - \sum_{i=1}^N \int E_i(q, t) dt - \sum_{i=1}^N \hbar \phi_i. \tag{156}$$

The dynamics for (156) clearly differs from the dynamics of the joint phase of the free classical $zbw$ particles by the presence of the quantum kinetic in (154). As in the single-particle case, the two phases are formally connected by the ‘classical limit’ $(\hbar/2m_i) \to 0$, but this is only formal since such a limit corresponds to deleting the presence of the ether, thereby also deleting the physical mechanism that causes the $zbw$ particles to oscillate at their Compton frequencies. The physically realistic ‘classical limit’ for the phase (156) corresponds to situations where the quantum kinetic and its gradient are negligible, which will occur (as in the dBB theory) whenever the center of mass of a system of interacting particles is sufficiently large and environmental decoherence is appreciable \cite{31, 32, 33}.

Since each $zbw$ particle is posited to essentially be a harmonic oscillator of (unspecified) identical type, each particle has its own well-defined phase at each point along its space-time trajectory (when $v_i(q, t) \approx \sum_{i=1}^N v_i(q_i, t)$). Consistency with this means that when $v_i(q, t) \neq \sum_{i=1}^N v_i(q_i, t)$, the joint mean phase must be a well-defined function of the time-symmetric mean trajectories of all particles (since we posit that all particles remain harmonic oscillators despite having their oscillations physically coupled through the common ether medium they interact with). Then, for a closed loop $L$ along which each particle can be physically or virtually displaced, it follows that
particles will be entangled in their joint phase (163) and their joint osmotic potential obtained from (162):

\[ \sum_{i=1}^{N} \oint_L \delta_i S(q(t), t) = \sum_{i=1}^{N} \oint_L \left[ p_i(q(t), t) \cdot \delta q_i(t) - E_i(q(t), t) \delta t \right] = nh. \]  

(157)

And for a closed loop \( L \) with \( \delta t = 0 \), we have

\[ \sum_{i=1}^{N} \oint_L d_i S(q(t), t) = \sum_{i=1}^{N} \oint_L p_i \cdot d q_i = \sum_{i=1}^{N} \oint_L \nabla_i S(q(t)) = nh. \]  

(158)

If we also consider the joint phase field, \( S(q, t) \) which is a function over \( N \) Gibbsian statistical ensembles of \( zw \) particles (an ensemble for each coordinate \( q_i \) in \( S \)), then, by the same physical reasoning applied to each member of the \( i \)-th ensemble, we will have

\[ \sum_{i=1}^{N} \oint_L d_i S(q(t), t) = \sum_{i=1}^{N} \oint_L p_i \cdot d q_i = \sum_{i=1}^{N} \oint_L \nabla_i S(q(t)) = nh. \]  

(159)

It also clear that (159) implies phase quantization for each individual particle ensemble, upon keeping all but the \( i \)-th coordinate fixed and performing the closed-loop integration.

Notice that (159) implies quantization of the osmotic potential as well, due to the coupling of \( S \) to \( R \) (hence \( U \)) via (144). This makes physical sense since, as we observed earlier, the oscillating ether drives the \( zw \) oscillations of the particles while also sourcing the osmotic potential that imparts the osmotic velocities to the particles.

Combining (142), (154), and (159), we can construct the \( N \)-particle Schrödinger equation

\[ i\hbar \frac{\partial \psi(q, t)}{\partial t} = \sum_{i=1}^{N} \left[ -\frac{\hbar^2}{2m_i} \nabla_i^2 + m_i c^2 \right] \psi(q, t), \]  

(160)

where the \( N \)-particle wavefunction \( \psi(q, t) = \sqrt{\rho(q, t)} e^{i S(q, t)/\hbar} \) is single-valued by (159). Since the solution space of the set (142), (154), and (159) is equivalent to the solution space of (160), any non-factorizable wavefunctions that can be constructed for (160) will also be solutions (in \( \rho \) and \( S \) variables) of (142), (154), and (159). As an example, let us reconsider two identical, classically non-interacting bosons with initial wavefunction \(^{10}\)

\[ \psi(q_1, q_2) = \text{Norm} \left[ \psi_A(q_1) \psi_B(q_2) + \psi_A(q_2) \psi_B(q_1) \right], \]  

(161)

where particle 1 is associated with wavepacket \( \psi_A \) and particle 2 is associated with wavepacket \( \psi_B \), and the wavepackets satisfy \( \psi_A \cap \psi_B = \emptyset \). Then, if the packets of these particles move towards each other and overlap such that \( \langle q_1 > - \langle q_2 > \rangle^2 \leq \sigma_A^2 + \sigma_B^2 \), the subsequent wavefunction of the 2-particle system will be (161) but with \( \psi_A \cap \psi_B \neq \emptyset \). Moreover, if we rewrite (161) in \( \rho \) and \( S \) variables

\[ \rho(q_1, q_2) = \text{Norm}^2 \left\{ e^{2(R_{A1} + R_{B2})/\hbar} + e^{2(R_{A2} + R_{B1})/\hbar} + 2e^{(R_{A1} + R_{B2} + R_{A2} + R_{B1})/\hbar} \cos [(S_{A1} + S_{B2} - S_{A2} - S_{B1})/\hbar] \right\}, \]  

(162)

and

\[ S(q_1, q_2) = S_A(q_1) + S_B(q_2) + S_A(q_2) + S_B(q_1) + \text{consts}, \]  

(163)

then (162) satisfies (142), and (163) is a solution of (154) and satisfies the constraint (159). That is, the two particles will be entangled in their joint phase (163) and their joint osmotic potential obtained from (162):

\[ R(q_1, q_2) = \frac{3}{2} \left\{ R_A(q_1) + R_B(q_2) + R_A(q_2) + R_B(q_1) \right\} \]  

(164)

\[ + \frac{\hbar}{2} \cos [(S_{A1} + S_{B2} - S_{A2} - S_{B1})/\hbar] + \text{consts}. \]

---

\(^{10}\)The Nelsonian derivation of the symmetry postulates given by Bacciagaluppi in \[17\], which allows us to write down a wavefunction like (161) (or its anti-symmetric counterpart), is consistent with the assumptions of ZSM and carries over without any change.
In fact, this scenario of entanglement formation between two identical bosons is equivalent to the scenario we considered earlier for our justification of why the osmotic potential should have functional dependence on the positions of both particles: Eq. (163) is the joint phase that develops between the two particles from having their \( \text{zbw} \) oscillations driven by a common region of oscillating ether that forms when \( \langle q_1 \rangle - \langle q_2 \rangle \leq \sigma_1^2 + \sigma_2^2 \). Likewise, (164) is the joint osmotic potential that arises from this common region of oscillating ether sourcing the osmotic potential.

Additionally, Eqs. (142), (154), and (159) answer how the non-local functional dependence of (164) on the positions of the two particles changes in time: for classically non-interacting particles, the non-local correlations become negligible when the 3-D spatial separation between the particles becomes sufficiently large, i.e., when the overlap of the wavepackets in the summands of (161) becomes negligible. Of course, the correlations never completely vanish because the overlap of the wavepackets in the summands of (161) never completely vanishes, implying that the common region of oscillating ether that physically connects the mean \( \text{zbw} \) oscillations and translational Brownian motions of the particles must, in some sense, extend over macroscopic distances in 3-D space. \( ^1 \) That is, if we view the ether as a medium in 3-D space and not in 3N-dimensional configuration space, even though (163-164) are non-separable fields on configuration space. This is last (TELB) view is indeed the one we take, since, as stated earlier, we think it’s the most plausible one among the present options.

To be sure, the interpretive issues we discussed in section 2 for NYSM apply just as well to ZSM. To review the options, one might view the mathematical non-factorizability of (163-164) as indicating that the oscillating ether medium lives in 3N-dimensional configuration space instead of 3D-space. Or, one might view the configuration space representation (163-164) as a mathematically convenient encoding of a much more complicated 3-D space representation of the joint phase field and joint osmotic potential of the particles, making it conceptually more problematic to imagine the oscillating ether as a medium in 3-D space. In the former case, we then have the options of: (1) viewing the \( \text{zbw} \) particles as living in 3-D space, and positing a law-like dynamical relationship between the particles in 3-D space and the oscillating ether in 3N-dimensional configuration space; and (2) viewing the particles in 3-D space as a fictitious representation of a single real \( \text{zbw} \) particle living at a definite point in 3N-dimensional configuration space, and taking the physical interactions between this particle and the ether as occurring in the configuration space. In the latter case, since both the particles and the oscillating ether would live in 3-D space (the TELB view), their physical interactions would occur there as well.

As with NYSM, the drawback of option 1 in the former case is that it seems mysterious and implausible that two sets of beables, living in completely independent physical spaces, should have a law-like dynamical relationship between them (i.e., why should oscillations of an ether medium in a 3N-dimensional configuration space ‘drive’ the mean \( \text{zbw} \) oscillations of particles at definite points in a 3-D space?). The drawback of option 2 is that while it’s conceptually more plausible how oscillations of the ether could drive the mean oscillations of the \( \text{zbw} \) particles (since they both live in the same physical space), it would then be necessary to employ a complicated philosophical functionalist analysis of the \( N \)-particle Quantum Hamilton-Jacobi equation, in order to derive the image of \( N \) \( \text{zbw} \) particles moving in 3-D space; and we would be in the seemingly paradoxical situation of having derived the \( N \)-particle QHJ equation from an ensemble-averaged Lagrangian defined from \( N \) contributions, under the starting hypothesis that there really are \( N \) particles diffusing in a 3-D space. Of course, the main shortcoming of the TELB view is that it remains speculative at the moment, since no such formulation of NYSM or ZSM exists at present; but it is not implausible that such a formulation can be constructed, and we have already sketched in section 2 one way it could be done. Thus we assume, provisionally, that a TELB formulation of ZSM exists and awaits discovery (unless shown otherwise), and base our interpretation of the beables of ZSM on this provisional assumption.

It is interesting to observe that the existence of entangled solutions such as (163-164) is a consequence of four physical constraints we’ve used in our construction of ZSM: (1) time-reversal invariance of the probability density via (142); (2) the conservative diffusion constraint on the ensemble-averaged, time-symmetric, \( N \)-particle action via (149); (3) single-valuedness of the joint phase field (up to an integer multiple of \( 2\pi \)) via (149); and (4) the requirement that the particles, under the evolution constraints (140-159), satisfy a natural notion of identically under exchange of their coordinates, thereby yielding the symmetrization postulates associated with bosons (and fermions) \( ^{17} \) (though let us be clear that for classically interacting non-identical

---

\(^{11}\) More precisely, we have in mind that the regions of oscillating ether immediately surrounding each particle will directly drive their respective \( \text{zbw} \) oscillations, while the ether in between the two particles will nonlocally encode physical correlations between the immediate regions of ether surrounding each particle, in a way consistent with the conservative diffusion constraint (149), even if the two particles are macroscopically separated in 3-D space. Of course, the exact details of how Nelson’s ether (under the amendments 1-3) would accomplish this await the construction of a physical model for it.

28
particles, entangled solutions can also arise by virtue of the previous three physical constraints). So ZSM offers a novel way to understand the emergence of continuous-variable entanglement nonlocality in terms of deeper ‘subquantum’ principles. One could then study how relaxing these physical constraints might lead to experimentally testable differences from the entangled solutions of the $N$-particle Schrödinger equation, in experimental tests of Bell inequalities for continuous-variable correlations \[19\].

Now, since we wish to view the particles as living at definite points in 3-D space, and their $zbw$ oscillations as occurring in 3-D space, we should find a way of constructing the phase field associated with the $i$-th particle’s $zbw$ oscillation in 3-D space. To do this, we can construct the conditional phase field and conditional osmotic approximation) is given by

\[
\delta \theta_{\text{joint}+}(\mathbf{q}_1, \mathbf{q}_2, t) = \frac{1}{\hbar} \left[ \sum_{i=1}^{2} m_i c^2 + \sum_{i=1}^{2} \frac{b_i^2}{2c^2} + \sum_{i=1}^{2} \frac{\omega_i b_i^2}{2c^2} + \sum_{i=1}^{2} \frac{\omega_i b_i^2}{2c^2} \right] \left( \delta t - \frac{2}{\hbar} \sum_{i=1}^{2} m_i \mathbf{b}_i \cdot \delta \mathbf{q}_i(t) \right),
\]

and $\delta \theta_{\text{joint}−}$ differs by only $\mathbf{b}_i \rightarrow \mathbf{b}_i′$. Incorporating coupling to an external vector potential, we then have $\mathbf{b}_i \rightarrow \mathbf{b}_i′ = \mathbf{b}_i + e_i \mathbf{A}_i^{\text{ext}}/\hbar c$ and likewise for $\mathbf{b}_i$. When $|\mathbf{b}_1(t) - \mathbf{b}_2(t)|$ becomes sufficiently great that $\mathbf{V}_i^{\text{int}}$ is negligible, (166) reduces to an effectively separable sum of the mean forward phase changes associated with particle 1 and particle 2, respectively. (Effectively, because the ether will of course still physically correlate the phase changes of the particles, even if negligibly.)

We can then write (dropping the “lab” superscript hereafter)

\[
\delta \bar{\theta}_{\text{joint}+}(\mathbf{q}_1, \mathbf{q}_2, t) = \frac{1}{\hbar} \left[ \mathbf{E}_{\text{joint}+} \delta t - \sum_{i=1}^{2} m_i \mathbf{b}_i′ \cdot \delta \mathbf{q}_i(t) \right],
\]

and

\[
\delta \bar{\theta}_{\text{joint}−}(\mathbf{q}_1, \mathbf{q}_2, t) = \frac{1}{\hbar} \left[ \mathbf{E}_{\text{joint}−} \delta t - \sum_{i=1}^{2} m_i \mathbf{b}_i′ \cdot \delta \mathbf{q}_i(t) \right].
\]

---

\[19\] Which we subject again to the hypothetical constraint of no electromagnetic radiation emitted when there is no translational motion; or the constraint that the oscillation of the charge is radially symmetric so that there is no net energy radiated; or, if the ether turns out to be electromagnetic in nature as Nelson suggested \[11\], then that the steady-state $zbw$ oscillations of the particles are due to a balancing between the time-averaged electromagnetic energy absorbed via the driven oscillations of the particle charges, and the time-averaged electromagnetic energy radiated back to the ether by the particles.

29
As in the classical case, we can readily construct from (167-168) the corresponding conditional phase change for particle 1 (particle 2) in the lab frame and/or IMTRF of particle 1 (particle 2).

Because each \( zbw \) particle is essentially a harmonic oscillator, when \( V_{int} \approx 0 \), each particle has its own well-defined phase at each point along its mean forward/backward space-time trajectory. Consistency with this entails that for \( V_{int} > 0 \) the joint phase must be a well-defined function of the mean forward/backward space-time trajectories of both particles (since we again posit that both particles remain harmonic oscillators even when physically coupled by \( V_{int} \)). Then for a closed loop \( L \), along which each particle can be physically or virtually displaced, the mean forward joint phase in the lab frame will satisfy

\[
\sum_{i=1}^{2} \oint_L \delta_1 \tilde{\theta}_{joint+} = 2 \pi n, \tag{169}
\]

and for a loop with time held fixed

\[
\sum_{i=1}^{2} \oint_L b_i^t \cdot \delta q_i(t) = nh, \tag{170}
\]

and likewise for the mean backward joint phase. It also follows from (169-170) that

\[
\oint_L \delta_1 \tilde{\theta}_{joint+} = 2 \pi n, \tag{171}
\]

and

\[
\oint_L b_i^t \cdot \delta q_1(t) = nh, \tag{172}
\]

where the closed-loop integral here keeps the coordinate of particle 2 fixed while particle 1 is displaced along \( L \).

In the lab frame, the forward and backward stochastic differential equations for the translational motion are then given by

\[
dq_i(t) = b_i^t(q(t),t)dt + dW_i(t), \tag{173}
\]

and

\[
dq_i(t) = b_i^{t*}(q(t),t)dt + dW_{i*}(t), \tag{174}
\]

with corresponding Fokker-Planck equations

\[
\frac{\partial \rho(q,t)}{\partial t} = -\sum_{i=1}^{2} \nabla_i \cdot \left[ \left( b_i^t(q,t) - \frac{e_i}{m_i c} A_i^{ext}(q,t) \right) \rho(q,t) \right] + \sum_{i=1}^{2} \frac{\hbar}{2m_i} \nabla_i^2 \rho(q,t), \tag{175}
\]

and

\[
\frac{\partial \rho(q,t)}{\partial t} = -\sum_{i=1}^{2} \nabla_i \cdot \left[ \left( b_i^{t*}(q,t) - \frac{e_i}{m_i c} A_i^{ext}(q,t) \right) \rho(q,t) \right] - \sum_{i=1}^{2} \frac{\hbar}{2m_i} \nabla_i^2 \rho(q,t). \tag{176}
\]

Imposing the time-symmetric kinematic constraints

\[
v_i = \frac{1}{2} [b_i + b_{i*}] = \frac{\nabla_i S}{m_i} - \frac{e_i}{m_i c} A_i^{ext}, \tag{177}
\]

and

\[
u_i = \frac{1}{2} [b_i - b_{i*}] = \frac{\hbar}{2m_i} \frac{\nabla_i \rho}{\rho}, \tag{178}
\]

then (175-176) reduce to
\[
\frac{\partial \rho}{\partial t} = - \sum_{i=1}^{2} \nabla_i \cdot \left[ \left( \frac{\nabla_i S}{m_i} - \frac{e_i}{m_i c} \mathbf{A}_i^{ext} \right) \rho \right].
\]  
(179)

So \( \mathbf{b}'_i = \mathbf{v}'_i + \mathbf{u}_i \), and \( \mathbf{b}'_s = \mathbf{v}'_s - \mathbf{u}_s \), recalling that \( \mathbf{v}'_i = \mathbf{v}_i + (e_i / m_i c) \mathbf{A}_{i}^{ext} \), \( \mathbf{b}_i = \mathbf{b}'_i - (e_i / m_i c) \mathbf{A}_i^{ext} \), and \( \mathbf{b}_{is} = \mathbf{b}'_s - (e_i / m_i c) \mathbf{A}_i^{ext} \). Moreover, the solution of (179) is just

\[
\rho(q, t) = \rho_0(q_0) \exp \left[ - \int_0^t \left[ \sum_{i=1}^{2} \left( \frac{\nabla_i^2 S}{m_i} - \frac{e_i}{m_i c} \nabla_i \cdot \mathbf{A}_i^{ext} \right) \right] dt' \right].
\]  
(180)

Here again we postulate an osmotic potential to which each particle couples via \( R(q(t), t) = \mu U(q(t), t) \), which imparts momentum \( \nabla_i R(q(t)) |_{q_i = q_i(t)} \) that is counter-balanced by \( \langle \mathbf{h} / 2m_i \rangle \nabla_i \ln[p(q(t))]|_{q_i = q_i(t)} \), giving the equilibrium velocity \( \nabla_i R/m_i = \langle \mathbf{h} / 2m_i \rangle \nabla_i \rho / \rho \). Thus \( \rho = e^{2R/\hbar} \) for all times and

\[
R(q, t) = R_0(q_0) - \left( \frac{\hbar}{2} \right) \int_0^t \left[ \sum_{i=1}^{2} \left( \frac{\nabla_i^2 S}{m_i} - \frac{e_i}{m_i c} \nabla_i \cdot \mathbf{A}_i^{ext} \right) \right] dt',
\]  
(181)

where \( S \) will end up playing the role of the joint phase via \( \theta_{joint} = -\frac{1}{\hbar} S \).

Now, integrating (167-168) and then averaging the two, we obtain

\[
\tilde{\theta}_{joint} = \frac{1}{\hbar} \left[ \int E_{joint} dt - \sum_{i=1}^{2} \int q_i \mathbf{b}_i + \mathbf{b}'_s \cdot d\mathbf{q}_i(t) \right] + \sum_{i=1}^{2} \phi_i,
\]  
(182)

where

\[
E_{joint} = \frac{1}{2} \left[ E_{joint} + E_{joint} - \sum_{i=1}^{2} \frac{m_i c^2}{2} + \sum_{i=1}^{2} \left( \frac{1}{2} m_i \mathbf{v}_i^2 + \frac{1}{2} m_i \mathbf{b}_i^2 \right) + \sum_{i=1}^{2} V_i^{ext} + V_c^{int} \right]
\]  
(183)

Then

\[
J(q) = \int_{\mathbb{R}^6} d^6 \mathbf{q} \left( \int_0^t dt \tilde{\theta}_{joint}(q(t), t) \right) = \frac{1}{\hbar} \left[ \int_{t_i}^{t_f} E_{joint} dt - \sum_{i=1}^{2} \int_{q_i}^{q_i(t)} m_i \mathbf{v}'_i \cdot d\mathbf{q}_i(t) \right] + \sum_{i=1}^{2} \int_{\mathbb{R}^6} d^6 \mathbf{q} \phi_i
\]  
(184)

which by the time-symmetric mean Legendre transformation

\[
L = \sum_{i=1}^{2} \frac{1}{2} \left[ (m_i \mathbf{b}'_i) \cdot \mathbf{b}_i + (m_i \mathbf{b}'_s) \cdot \mathbf{b}_s \right] - \frac{1}{2} (E_{joint} + E_{joint} - \sum_{i=1}^{2} (m_i \mathbf{v}'_i) \cdot \mathbf{v}_i + (m_i \mathbf{u}_i) \cdot \mathbf{u}_s - E_{joint}),
\]  
(185)

and using \( \tilde{\theta}_{joint} = -\frac{1}{\hbar} S \), is equivalent to Eq. (18) in section 2. Applying

\[
J(q) = \text{extremal},
\]  
(186)

we have

\[
\sum_{i=1}^{2} \frac{m_i}{2} [D_{i} D + DD_{i}] \mathbf{q}_i(t) = \sum_{i=1}^{2} \epsilon_i \left[ -\frac{1}{c} \partial_t \mathbf{A}^{ext} \cdot \nabla \left( \Phi^{ext} + \Phi^{int} \right) + \frac{\mathbf{v}_i}{c} \times \left( \nabla \times \mathbf{A}^{ext}_i \right) \right] |_{q_i=q_i(t)},
\]  
(187)

and from the independent \( \delta \mathbf{q}_i(t) \), the individual equations of motion
\[ m_i \mathbf{a}_i(q(t), t) = \frac{m_i}{2} [\mathcal{D}_c \mathcal{D} + \mathcal{D}_c \mathcal{D}_c] \mathbf{q}_i(t) = \left[ -\frac{e_i}{c} \partial_t \mathbf{A}^{\text{ext}}_i - e_i \nabla_i \left( \Phi^{\text{ext}}_i + \Phi^{\text{int}}_i \right) + \frac{e_i}{c} \mathbf{v}_i \times (\nabla_i \times \mathbf{A}^{\text{ext}}_i) \right] \bigg|_{\mathbf{q}_i = \mathbf{q}_i(t)}. \]  

(188)

Applying the mean derivatives, using that \( \mathbf{b}_i = \mathbf{v}_i + \mathbf{u}_i, \mathbf{b}_{\text{i},*} = \mathbf{v}_i - \mathbf{u}_i, \) and replacing \( q(t) \) with \( q \) on both sides, (187) becomes

\[ \sum_{i=1}^{2} m_i \left[ \partial_t \mathbf{v}_i + \mathbf{v}_i \cdot \nabla_i \mathbf{v}_i - \mathbf{u}_i \cdot \nabla_i \mathbf{u}_i - \frac{\hbar}{2m_i} \nabla^2 \mathbf{u}_i \right] \]

\[ = \sum_{i=1}^{2} \left[ -\frac{e_i}{c} \partial_t \mathbf{A}^{\text{ext}}_i - e_i \nabla_i \left( \Phi^{\text{ext}}_i + \Phi^{\text{int}}_i \right) + \frac{e_i}{c} \mathbf{v}_i \times (\nabla_i \times \mathbf{A}^{\text{ext}}_i) \right]. \]

(189)

Identifying

\[ \mathbf{p}_i = -\left( \hbar \nabla_i \partial_{\text{joint}} + \frac{e_i}{c} \mathbf{A}^{\text{ext}}_i \right) = \left( \nabla_i S - \frac{e_i}{c} \mathbf{A}^{\text{ext}}_i \right), \]

(190)

using (177-178) in (189), integrating both sides, and setting the arbitrary integration constants equal to the particle rest energies, we then get

\[ E(q, t) = -\partial_t S(q, t) = \sum_{i=1}^{2} m_i c^2 + \sum_{i=1}^{2} \left[ \nabla_i S(q, t) - \frac{\hbar}{2m_i} \mathbf{A}^{\text{ext}}_i(q, t) \right]^2 \]

\[ + \sum_{i=1}^{2} e_i \left[ \Phi^{\text{ext}}_i(q, t) + \Phi^{\text{int}}_i(q, t) \right] - \sum_{i=1}^{2} \frac{\hbar^2}{2m_i} \nabla^2 \sqrt{\rho(q, t)} \]

(191)

The general solution of (191) is clearly of the form

\[ S = \left( \sum_{i=1}^{2} \int \mathbf{p}_i' \cdot d\mathbf{q}_i - \int E dt \right) - \sum_{i=1}^{N} \hbar \phi_i. \]

(192)

As in the classical model, we make the natural assumption that the presence of classical external potentials doesn’t alter the harmonic nature of the mean \( \text{zbw} \) oscillations. Moreover, since each \( \text{zbw} \) particle is a harmonic oscillator, each particle has its own well-defined mean phase at each point along its time-symmetric mean trajectory. Accordingly, when \( \Phi^{\text{int}}_c \) is not negligible, the joint phase must be a well-defined function of the time-symmetric mean trajectories of both particles (since we posit that all particles remain harmonic oscillators despite having their oscillations physically coupled through \( \Phi^{\text{int}}_c \) and through the common ether medium they interact with). So for a closed loop \( L \) along which each particle can be physically or virtually displaced, it follows that

\[ \sum_{i=1}^{2} \oint_L \delta \mathbf{S} = \sum_{i=1}^{2} \oint_L \left[ \mathbf{p}_i' \cdot \delta \mathbf{q}_i(t) - E \delta t \right] = nh. \]

(193)

and

\[ \sum_{i=1}^{2} \oint_L \delta \mathbf{S} = \sum_{i=1}^{2} \oint_L \mathbf{p}_i' \cdot \delta \mathbf{q}_i(t) = \sum_{i=1}^{2} \oint_L \nabla_i S|_{\mathbf{q}_i = \mathbf{q}_i(t)} \cdot \delta \mathbf{q}_i(t) = nh, \]

(194)

for a closed loop \( L \) with \( \delta t = 0 \). For the joint phase field \( S(q, t) \), we can apply the same physical reasoning to each member of the \( i \)-th ensemble to obtain

\[ \sum_{i=1}^{2} \oint_L d \mathbf{S} = \sum_{i=1}^{2} \oint_L \mathbf{p}_i' \cdot d \mathbf{q}_i = \sum_{i=1}^{2} \oint_L \nabla_i S \cdot d \mathbf{q}_i = nh. \]

(195)

Clearly (195) implies phase quantization for each individual particle ensemble, upon keeping all but the \( i \)-th coordinate fixed and performing the closed-loop integration. Combining (195), (191), and (179), we can
construct the 2-particle Schrödinger equation for classically interacting \( zbw \) particles in the presence of external fields

\[
\frac{i\hbar}{\partial t} \psi(q_1, q_2, t) = \sum_{i=1}^{2} \left[ \frac{-i\hbar \nabla_i - \frac{\partial}{\partial t} A_{\text{ext}}^i(q_i, t)}{2m_i} \right]^2 + m_i c^2 + e_i \left( \Phi_{\text{ext}}^i(q_i, t) + \phi_{\text{int}}^i(q_i, q_j) \right) \psi(q_1, q_2, t),
\]

where \( \psi(q_1, q_2, t) = \sqrt{\rho(q_1, q_2, t)} e^{iS(q_1, q_2, t)/\hbar} \) is single-valued via (195).

We would now like to have a statistical description of the evolution of the conditional phase field and conditional probability density associated to each \( zbw \) particle. For simplicity, we first set \( A_{\text{ext}}^i = \Phi_{\text{ext}}^i = 0 \).

We then obtain the conditional \( zbw \) phase field for particle 1 by writing \( S(q_1, q_2(t), t) = S_1(q_1, t) \). Taking the total time derivative gives

\[
\frac{\partial t}{\partial t} S_1(q_1, t) = \frac{\partial t}{\partial t} S(q_1, q_2(t))|_{q_2=q_2(t)} + \frac{dq_2(t)}{dt} \cdot \nabla_2 S(q_1, q_2, t)|_{q_2=q_2(t)},
\]

where the conditional velocities

\[
v_1(q_1, t)|_{q_1=q_1(t)} = \frac{\nabla_1 S_1(q_1, t)}{m_1}|_{q_1=q_1(t)} = \frac{dq_1(t)}{dt},
\]

and

\[
v_2(q_2, t)|_{q_2=q_2(t)} = \frac{\nabla_2 S_2(q_2, t)}{m_2}|_{q_2=q_2(t)} = \frac{dq_2(t)}{dt},
\]

the latter defined from the conditional phase field, \( S_2(q_2, t) \), for particle 2. Likewise, for the conditional density for particle 1, \( \rho(q_1, q_2(t), t) = \rho_1(q_1, t) \) and

\[
\frac{\partial t}{\partial t} \rho_1(q_1, t) = \frac{\partial t}{\partial t} \rho(q_1, q_2(t))|_{q_2=q_2(t)} + \frac{dq_2(t)}{dt} \cdot \nabla_2 \rho(q_1, q_2(t))|_{q_2=q_2(t)}.
\]

Inserting (200) on the left hand side of (179) and adding the corresponding term on the right hand side, we obtain the conditional continuity equation for particle 1:

\[
\frac{\partial t}{\partial t} \rho_1 = -\nabla_1 \cdot \left( \frac{\nabla_1 S_1}{m_1} \rho_1 \right) - \nabla_2 \cdot \left( \frac{\nabla_2 S}{m_2} \rho \right)|_{q_2=q_2(t)} + \frac{dq_2(t)}{dt} \cdot \nabla_2 \rho|_{q_2=q_2(t)},
\]

which implies \( \rho_1(q_1, t) \geq 0 \) and preservation of the normalization \( \int_{\mathbb{R}^3} \rho_1(q_1, 0) = 1 \). Similarly, inserting (197) into the left hand side of (191) and adding the corresponding term on the right hand side, we find that the conditional phase field for particle 1 evolves by the conditional quantum Hamilton-Jacobi equation

\[
-\frac{\partial t}{\partial t} S_1 = m_1 c^2 + \frac{(\nabla_1 S_1)^2}{2m_1} + \frac{(\nabla_2 S)^2}{2m_2}|_{q_2=q_2(t)} - \frac{dq_2(t)}{dt} \cdot \nabla_2 S|_{q_2=q_2(t)} + V_{\text{int}}(q_1, t) - \frac{\hbar^2 \nabla_1^2 \rho_1}{\sqrt{\rho_1}} - \frac{\hbar^2 \nabla_2^2 \rho}{\sqrt{\rho}}|_{q_2=q_2(t)}
\]

where \( V_{\text{int}}(q_1, t) \) is the 'conditional interaction potential' for particle 1. The solution of (201) can be verified as

\[
\rho_1 = \rho_{01} \exp[- \int_0^t \left( \nabla_1 \cdot v_1(q_1, t) + \nabla_2 \cdot v_2(q_1, q_2, t) \right)|_{q_2=q_2(t)} dt'],
\]

from which we extract the conditional osmotic potential

\[
R_1 = R_{01} - \frac{(\hbar/2)}{\int_0^t \left( \nabla_1 \cdot v_1(q_1, t) + \nabla_2 \cdot v_2(q_1, q_2, t) \right)|_{q_2=q_2(t)} dt'},
\]

while the solution of (202) is

\[
S_1 = \int \mathbf{p}_1 \cdot dq_1 - \int \left[ m_1 c^2 + \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} - \mathbf{p}_2 \cdot \frac{dq_2(t)}{dt} + V_{\text{int}} + \frac{\hbar^2 \nabla_1^2 \rho_1}{\sqrt{\rho_1}} + \frac{\hbar^2 \nabla_2^2 \rho}{\sqrt{\rho}} \right]|_{q_2=q_2(t)} dt' - \hbar \phi_1.
\]

33
Hence (204) allows us to consistently ascribe a region of oscillating ether in 3-D space that sources a local (i.e., in 3-D space) osmotic potential that imparts the osmotic momentum to particle 1. Likewise, (205) lets us ascribe a region of oscillating ether in 3-D space that directly drives the \( zbw \) oscillation of particle 1 in 3-D space. Note that when (205) is evaluated at \( q_1 = q_1(t) \), it is equivalent to \( S(q_1(t), q_2(t), t) = m_2 c^2 t + \hbar \phi_2 \). As in the classical model, since the conditional \( zbw \) phase field for particle 1 is a field over a Gibbsian statistical ensemble of \( zbw \) particles (each fictitious member of the ensemble representing a possible position, velocity, and phase that the actual \( zbw \) particle could have at time \( t \)), it follows that

\[
\oint_L \nabla_1 S_1 \cdot dq_1 = nh,
\]

where \( L \) is a mathematical loop in 3-D space along which a fictitious \( zbw \) particle in the \( i = 1 \) conditional ensemble can be displaced.

With these results in hand, the conditional forward and backward stochastic differential equations for particle 1 can be straightforwardly obtained writing \( b_1 = v_1 + u_1, \ b_{1*} = v_1 - u_1 \), and inserting these expressions into (173) and (174), respectively.

Also like in the classical model, if we use the conditional probability density for particle 1 to define the time-symmetric, ensemble-averaged, conditional Lagrangian

\[
J_1(q_1) = \int_{\mathbb{R}^3} d^3 q_1 \rho_1 \left[ \int_{t_1}^{t_2} \left( \sum_{i=1}^{n} m_i v_i \cdot dq_i(t) \right) - \int_{t_1}^{t_2} \left( m_1 c^2 + \sum_{i=1}^{n} \left[ \frac{1}{2} m_i v_i^2 + \frac{1}{2} m_i u_i^2 \right] + V_c^{int} \right) dt \right] - \int_{\mathbb{R}^3} d^3 q_1 \rho_1 h \phi_1
\]

and then impose

\[
J_1 = \text{extremal},
\]

we get the conditional mean acceleration for particle 1:

\[
m_1 a_1(q_1(t), t) = \frac{m_1}{2} \left[ D_s D + DD_s \right] q_1(t) = -\nabla_1 V_c^{int}(q_1, q_2(t))|_{q_1=q_1(t)}
\]

\[
\frac{m_1}{D t} \frac{D v_1(q_1(t), t)}{dq_1} = \left[ \partial_t p_1 + v_1 \cdot \nabla_1 p_1 \right] (q_1(t), t)|_{q_1=q_1(t)} = -\nabla_1 \left[ V_c^{int}(q_1, q_2(t)) - \frac{\hbar^2}{2 m_1} \nabla_1^2 \sqrt{\rho_1(q_1, t)} \right] |_{q_1=q_1(t)},
\]

and likewise for particle 2. In fact, (209) is what we would obtain from computing the derivatives in (188) (modulo the external potentials) and subtracting out the \( u_i \) dependent terms. Of course, it should be said that we cannot obtain (202) simply by integrating (209) and the analogous expression for particle 2, and then summing up the terms. Because we obtained (209) directly from the full configuration space fields \( S \) and \( \rho \), themselves obtained from extremizing (184).

For particle 2, the conditional phase field, probability density, etc., are defined analogously.

Finally, combining (206), (209), and (201) gives us the conditional Schrödinger equation for particle 1:

\[
\frac{i \hbar}{m_1} \frac{\partial \psi_1}{\partial t} = -\frac{\hbar^2}{2 m_1} \nabla_1^2 \psi_1 - \frac{\hbar^2}{2 m_1} \nabla_1^2 \psi|_{q_2=q_2(t)} + V_c^{int}(q_1, q_2(t)) \psi_1
\]

\[
+ m_1 c^2 \psi_1 + \frac{i \hbar}{m_1} \frac{dq_2(t)}{dt} \cdot \nabla_2 \psi|_{q_2=q_2(t)},
\]

where \( \psi_1(q_1, t) = \sqrt{\rho_1(q_1, t)} e^{i S_1(q_1, t) / \hbar} \) is the single-valued conditional wavefunction for particle 1, and \( dq_2(t) / dt = (\hbar / m_2) \text{Im} \left[ \nabla_2 \text{Im} (\psi_2) \right] |_{q_2=q_2(t)} \), where \( \psi_2 = \psi_2(q_2, t) \) is the conditional wavefunction for particle 2, satisfying the analogous conditional Schrödinger equation. Like in the classical case, (210) can also be obtained from writing

\[
\partial_t \psi_1(q_1, t) = \partial_t \psi(q_1, q_2, t)|_{q_2=q_2(t)} + \frac{dq_2(t)}{dt} \cdot \nabla_2 \psi(q_1, q_2, t)|_{q_2=q_2(t)},
\]
inserting this on the left hand side of (196), adding the corresponding term on the right hand side, and subtracting $m_2c^2\psi_1$ (again, modulo the external potentials).

The development of ZSM in relative coordinates is formally identical to the case of a single $zbw$ particle in an external potential, and need not be explicitly given here.

This completes the formulation of ZSM for $N$-particles interacting with classical fields.

4.3 Remark on close-range interactions

Since the quantum kinetic doesn’t depend on the inter-particle separation, its presence in the equation of motion (209) doesn’t introduce any fundamentally new complications for the description of two-particle scattering in ZSM. So the account we gave of two-particle scattering in subsection 3.7 carries over to classically interacting particles in ZSM.

5 Plausibility of the Zitterbewegung Hypothesis

Ultimately, the plausibility of our suggested answer to Wallstrom hinges (in no particular order) on the plausibility of the $zbw$ hypothesis, its incorporation into NYSM, and the generalizability of ZSM. So we should ask if: 1) ZSM can be consistently generalized to relativistic flat and curved spacetimes; 2) the $zbw$ hypothesis can be generalized to incorporate electron spin; 3) ZSM has a conceivable field-theoretic extension; 4) a self-consistent physical model of the $zbw$ particle, Nelson’s ether (suitably amended for ZSM), and dynamical interaction between the two, can be constructed; and 5) ZSM suggests testable new predictions and/or offers novel solutions to open problems in the foundations of quantum mechanics that justify its mathematical and conceptual complexity (relative to other hidden variable approaches to solving the measurement problem, such as the dBB theory).

Can ZSM be consistently generalized to relativistic flat and curved spacetimes? We have implicitly assumed throughout our paper that this is possible, based on our repeated use of the next-to-leading order approximation of the Lorentz transformation. But there is also good reason to expect that relativistic generalizations of ZSM to flat and curved spacetimes do exist. Stochastic mechanics based on the Guerra-Morato variational principle has already been given a consistent generalization to the case of relativistic spacetimes (flat and curved) by Dohrn and Guerra \cite{34, 35, 36} as well as Serva \cite{37}. An attempt was made by Zastawniak to give a relativistic flat-spacetime generalization of Yasue’s variational principle \cite{38}, but it seems problematic since it doesn’t address the problem of not having a normalizable spacetime probability density when the metric is not positive-definite. Fortunately, this problem can be resolved in the approaches of Dohrn-Guerra and Serva, and there seems to be no obstacle in adapting Dohrn and Guerra’s methods or Serva’s method to extend Yasue’s variational principle to flat and curved spacetimes (currently in progress by us). Once done, we see no fundamental reason why a corresponding generalization of ZSM cannot be given.

Can the $zbw$ hypothesis be generalized to incorporate electron spin? It seems to us plausible that it can. As is well-known, in standard relativistic quantum mechanics for spin-1/2 particles, the Dirac equation implies $zbw$ of the corresponding velocity operator \cite{39}. What’s more, realist versions of relativistic quantum mechanics for spin-1/2 particles - the Bohm-Dirac theory \cite{10, 11}, the “zig-zag” model of de Broglie-Bohm theory by Colin & Wiseman \cite{11} and Struyve \cite{12}, and the stochastic mechanical models of the Dirac electron by de Angelis et al. \cite{13} and Garbaczewski \cite{14} - all predict $zbw$ as a real, continuous oscillation of the particle beable. In the de Broglie-Bohm theories, the $zbw$ arises from imposing Lorentz invariance and the Dirac spinor algebra on the dynamics of the wavefunction (described by Dirac spinors in the Bohm-Dirac theory, or Weyl spinors in the zig-zag model), and then using this wavefunction in the definition of the guiding equation for the de Broglie-Bohm particle. Likewise, in the stochastic mechanical theories, the $zbw$ beable arises from constructing Nelsonian diffusion processes from the Dirac wavefunction. The description of a physically real spin-based $zbw$ can also be implemented in classical physics, namely in the Barut-Zanghì model of a classical Dirac electron \cite{15, 16, 17, 18}, which turns into the usual flat-space and curved-space versions of the Dirac equation (in the proper-time formulation) upon first-quantization by the standard methods \cite{19, 20}. Here it is the imposition of relativistic covariance and the Dirac spinor algebra that leads to classical equations of motion for a massless (non-radiating) point charge circularly orbiting a center of mass, the former moving with speed $c$ and the latter moving translationally with sub-luminal relativistic speeds. So it is plausible to imagine a relativistic generalization of ZSM in which the Barut-Zanghì model of a $zbw$ particle is implemented into a relativistic version of the Nelson-Yasue diffusion process (e.g., along
the lines of Dohrn and Guerra), under the hypothesis that Nelson’s ether has vorticity that imparts to the massless point charge a mean rotational motion of speed $c$ and angular momentum $\hbar/2$, and derive from this spin-based $zbw$ a relativistic generalization of the quantization condition, along with the Dirac equation for a double-valued Dirac spinor wavefunction. (The approaches of de Angelis et al. and Garbaczewski don’t seem adequate for this task because they don’t actually derive the zitterbewegung and Dirac equation from Nelson-Yasue diffusions; rather, they start from the Dirac equation and Dirac spinor wavefunction, and show that Nelsonian diffusions can be associated to them.) The non-relativistic limit of this ZSM theory should presumably then recover non-relativistic ZSM for a spinning $zbw$ particle with angular momentum magnitude $\hbar/2$, along with a vorticity term added to the current velocity (as is known to arise from the non-relativistic limit of the relativistic guiding equation under Gordon decomposition in the Bohm-Dirac theory [51, 52]). Alternatively, we might try deducing a non-relativistic ZSM theory directly from Takabayasi’s non-relativistic generalization of the Madelung fluid to spin-1/2 motion [53]. These tasks remain for a future paper.

Does ZSM have a field-theoretic generalization that recovers the predictions of relativistic quantum field theory for fermions and bosons? A generalization of ZSM to massive scalar or spinor fields seems in-principle unproblematic, but a generalization to massless fields (such as to describe the photon or gluon, which have no measured rest mass) would seem, at first sight, difficult (though not necessarily impossible [54]. Another possibility is to note that one can reproduce nearly all the predictions of the Standard Model (SM) with a pilot-wave model for point-like fermions in which the Dirac sea is taken seriously (i.e., taken as ontological) [55]. In this model, no beables are introduced for the massless bosons, yet it recovers nearly all the predictions of the SM. So we might try constructing a version of relativistic ZSM for spin-1/2 particles in which the Dirac sea for fermions is taken seriously, and check if it can recover nearly all the predictions of the SM as well. If one insists on adding beables for the bosons, perhaps one could adapt the approach of Nielsen et al. [59, 60], who show how to introduce a Dirac sea for bosons in second-quantized field theory based on massive hypermultiplets. Finally, it seems plausible that one could make a ZSM generalization of bosonic string theory by constructing a Nelson-Yasue version of the model of Santos and Escobar [61], who use the Guerra-Morato variational principle to construct a stochastic mechanics of the open bosonic string (the idea being that the open bosonic string’s IRF oscillations would play the role of the $zbw$, and would be hypothesized to be dynamically driven by resonant coupling to the ZSM version of Nelson’s ether). All this remains for future work.

Can a self-consistent dynamical model of the $zbw$ particle, Nelson’s ether, and the physical interaction between the two, be constructed? We see no principled obstacle to this possibility. Furthermore, physical models of a real classical $zbw$ particle have been constructed in the context of stochastic electrodynamics (SED), by Rueda & Cavalleri [62], Rueda [63, 64], de la Peña & Cetó [55], and Haisch & Rueda [65]. These models involve treating the electron as a structured object composed of a point charge with negligible (or zero) mass, harmonically bound to some non-charged center of mass, and driven to oscillate at near or equal to the speed of light (i.e., Compton frequency) by resonant modes of a classically fluctuating electromagnetic zero-point field. Additionally, in Rueda’s model [63, 64], not only does the classical zero-point field drive the $zbw$ oscillations, but the frequency cut-off generated by the $zbw$ results in a non-dissipative, (effectively) Markovian diffusion process with diffusion coefficient $\hbar/2m$. Of course, these SED-based approaches should be cautioned; SED is know to have difficulties as a viable theory of quantum electrodynamical phenomena [67, 68], and it is not clear that these difficulties can be resolved (but see [69, 70, 71, 72, 73] for recent counter-arguments). Furthermore, we expect that any realistic physical model of the $zbw$ particle should consistently incorporate the Higgs mechanism (or some subquantum generalization thereof) [74] as the process by which the self-stable $zbw$ harmonic potential of rest-mass $m$ is formed in the first place. Nevertheless, these SED-based models can at least be viewed as proofs of principle that the $zbw$ hypothesis can be implemented in a concrete model; and, in a future paper, we will show how one of these SED-based models can in fact recover the quantization condition as an effective condition. But the task of constructing a physical model of the $zbw$ particle, the ZSM version of Nelson’s ether, and the physical/dynamical interaction between the two, which also incorporates

---

For example, we might consider introducing small rest masses for the photon and gluon consistent with experimental bounds, which for the photon is $< 10^{-14} \text{eV}/c^2$ [54] and for the gluon $< 0.0002 \text{eV}/c^2$ [55], if both masses are to be produced by the Higgs mechanism. This would, of course, change the gauge symmetries of QED and QCD, but not in a way that can be experimentally discerned at energy scales above these lower-bounds [56].

14The single different prediction appears to be that this Dirac sea pilot-wave model predicts fermion number conservation, whereas the Standard Model predicts a violation of fermion number for sufficiently high energies (so-called anomalies of the Standard Model). To the best of our knowledge, no evidence has been found for fermion number violation thus far [77]. But as Colin and Struyve point out [55], even if fermion number violation is eventually observed, it may still be possible to model it in a Dirac sea picture.
spin and can be used to recover the Dirac/Pauli/Schrödinger equation, remains for future work.

Lastly, does ZSM suggest testable new predictions and/or novel solutions to open foundational problems in quantum mechanics? We claim it does. Since the equilibrium density \( \rho = |\psi|^2 \), ZSM’s statistical predictions in equilibrium will agree with all the statistical predictions of non-relativistic quantum mechanics. But if \( \rho \neq |\psi|^2 \), then we should expect differences such as position and momentum measurements with more precision than allowed by Heisenberg’s uncertainty principle \([72]\). Accordingly, it would be possible, in principle, to experimentally detect the stochasticity of the particle trajectories, hence deviations from the mean trajectories satisfying the quantization condition. Under what physical conditions might we see nonequilibrium fluctuations? The most obvious possibility seems to be by measuring the position or momentum of a Nelsonian particle on time-scales comparable to or shorter than the correlation time of the ether fluctuations. For ZSM, insofar as it’s based on Nelson’s diffusion process, the correlation time-scale of the fluctuations is infinitesimal because the noise is assumed to be white. Nelson stressed, however, that his white-noise (Markovian) assumption was only a simplifying one \([11]\); so one could instead consider a colored-noise (non-Markovian) description of conservative diffusions to which Nelson’s white-noise description is a long-time approximation (as is the case with all other known statistical fluctuation phenomena in nature \([77]\)). Then the true fluctuation time-scale would be finite (as also suggested by relativistic considerations \([37]\)) and one could work out the expected experimental signatures of the nonequilibrium dynamics on timescales comparable to some hypothetical finite correlation time \( \tau_{\text{noise}} \) (work on this is currently underway by us). In this connection, Montina’s theorem \([78]\) says that any ontic theory compatible with the predictions of a system with Hilbert space dimensionality \( k \) must contain at least \( 2k - 2 \) continuous ontic variables (which corresponds to the number of arguments in an \( N \)-particle wavefunction after normalization and fixing of the overall phase), if the theory has deterministic or stochastic Markovian dynamics (i.e., a dynamics that is local in time); likewise, Montina’s theorem implies that an ontic theory with non-Markovian dynamics (i.e., dynamics which is nonlocal in time) could have fewer continuous ontic variables than \( 2k - 2 \), and Montina himself has demonstrated this in a toy model of a single ontic variable with stochastic evolution driven by time-correlated (colored) noise that exactly reproduces any unitary evolution of a qubit (\( \psi \) for a qubit has two degrees of freedom) \([79]\) \([80]\). So it seems plausible to expect that a colored-noise extension of stochastic mechanics (which entails non-Markovian evolution for the particle trajectories), formulated in the TELB way \([25]\) \([26]\), would make it possible to express all the physical degrees of freedom encoded in the \( N \)-particle wavefunction in terms of \( S \) and \( R \) fields on 3-space (a pair for each particle), and only a finite number of supplementary continuous ontic variables on 3-space (which would encode the non-local correlations between the particles arising from their coupling to the common oscillating ether). This would also make ZSM seem more natural, insofar as we want to view the joint \( zbw \) phase for an \( N \)-particle system as associated with real physical oscillations about the actual 3-space locations of \( N \) particles, and insofar as we conjecture the ether to be a medium that fundamentally lives in 3-space instead of configuration space.

6 Comparison to Other Answers

Several other answers to Wallstrom’s criticism have been offered in the context of stochastic mechanics \([81]\) \([7]\) \([82]\) \([83]\) \([84]\) \([85]\). Here we briefly review and assess each approach, and compare them to ZSM.

Smolin proposed \([82]\) that Wallstrom’s criticism could be answered by allowing discontinuities in the wavefunction - that is, for a given multi-valued wavefunction, one could introduce discontinuities at the multi-valued points to make it single-valued. The example he used is stochastic mechanics on \( S^1 \), where he argued that although the resultant wavefunction is not single-valued and smooth, it is well-known that almost every wavefunction in the Hilbert space \( L^2(S^1) \) is discontinuous at one or many points, and yet each wavefunction is normalizable and gives well-defined (i.e., single-valued) current velocities. Smolin’s proposal seems incomplete, however, in that a number of essential questions were left unaddressed. If the wavefunction is allowed to

---

\(^{15}\) Everything we have said here is of course also true of the dBB theory \([75]\). However, in our view, a proper understanding of the origin of randomness in the dBB theory (the ‘typicality’ approach of Durr-Goldstein-Zanghì \([23]\)) entails that the existence of quantum nonequilibrium subsystems in the observable universe is extremely improbable, even in the context of early universe cosmology (for a different view, see \([70]\)). By contrast, we will suggest here that this limitation of the dBB theory does not necessarily apply to ZSM.

\(^{16}\) Of course, this idea could also be explored in NYSM with the quantization condition imposed ad-hoc. The advantage of ZSM, though, is that it makes the idea worth taking seriously as a possibility since ZSM gives an independent justification for the (more basic) quantization condition, without which the stochastic mechanics approach would be neither empirically viable nor plausible.
be discontinuous, won’t expectation values of the momentum and kinetic energy operators lead to divergences at the discontinuous points, and doesn’t this contradict experimental facts? If the wavefunction is discontinuous, how then can one interpret $|\psi|^2$ as a probability density since, by definition, a probability density must be smooth in order to have a globally well-defined, conserved evolution? Even if Smolin’s proposal works for the multiply connected configuration space of the unit circle, how will it work in the more general cases of simply connected configuration spaces of dimensionality $3N$? Wallstrom emphasizes, after all, that the inequivalence he observes between the HJM equations and Schrödinger’s equation applies to simply connected configuration spaces of two dimensions or greater [7]. (See also [76] for a critique of Smolin’s approach.) To compare with ZSM, these concerns don’t arise - the derived wavefunctions are single-valued and smooth, and ZSM works for the general case of simply connected 3N-dimensional configuration space.

Carlen & Loffredo [81] considered stochastic mechanics on $S^1$ and suggested to introduce a stochastic analogue of the quantization condition, which they argue is related in a natural way to the topological properties of $S^1$. They then showed that this stochastic analogue of the quantization condition establishes mathematical equivalence between stochastic mechanics and quantum mechanics on $S^1$. However, the difficulty with taking their proposal as a general answer is that it seems to only work in the special case of $S^1$, whereas Wallstrom’s criticism applies to simply connected configuration spaces of two dimensions or greater, as mentioned earlier.

Fritsche & Haugk [83] attempted to answer Wallstrom by motivating the quantization condition from the physical requirement that the probability density, $|\psi|^2$, should always be normalizable. To accomplish this, they first required that the velocity potential, $S$, be single-valued on a closed loop (in analogy with the definition of a single-valued magnetic scalar potential) via jump discontinuities. Constructing the wavefunction from this $S$ function through an approach equivalent to Nelson’s Newtonian formulation of stochastic mechanics, they then argued that the only way $|\psi|^2$ can remain normalizable for a superposition of two eigenstates is if the phase difference between the eigenstates satisfies the quantization condition. The main problem with their approach lies in the their non-trivial assumption that $S$ can have jump discontinuities. As pointed out by Wallstrom [8] [7], allowing jump discontinuities in $S$ implies that $\nabla\psi = \frac{i}{\hbar} (\nabla R + i \nabla S)\psi$ develops a singularity, which is physically inadmissible. Accordingly, the same technical concerns we raised towards Smolin’s proposal apply here as well. We note, by contrast, that in ZSM, $\nabla S$ is always continuous even though $S$ is in general discontinuous (e.g., at nodal points of the probability density).

Wallstrom made the observation [7] that if one takes the quantization condition as an initial condition on the current velocity, then the time-evolution of the HJM equations will ensure that it is valid for all future times, in analogy with Kelvin’s circulation theorem from classical fluid mechanics. So one might think to use this as a justification for the quantization condition in the context of the HJM equations. As he pointed out, however, this seems to require an extreme form of fine-tuning (why should the initial condition on the current velocity correspond exactly to the quantization condition?), and it is not clear that this initial condition would be stable for interacting particles. By contrast, we saw in ZSM that the $z_{bw}$ hypothesis combined with the Lorentz transformation implies the quantization condition so that it is not the result of fine-tuning (other than the assumption that the mean oscillation frequency in the IMTRF is of fixed Compton magnitude). Moreover, we showed that in the case of classically interacting $z_{bw}$ particles, it can be plausibly argued that the quantization condition remains stable.

Bacciagaluppi [86] suggested that when the external potential $V$ has time-dependence, the complement of the nodal set of $\rho$ may become simply connected in a neighborhood of a given time $t$. In other words, the time-dependence of $V$ may make it possible to eliminate the nodes of $\rho$ around which a multi-valued $S$ accumulates values other than $\hbar n$ (because $S$ would have to be single-valued in that neighborhood of $t$). While Bacciagaluppi’s suggestion was intended as an abstract, mathematical argument, it is interesting to note that his proposal seems relevant to measurement situations when the interaction of a system with a pointer apparatus entails a time-dependent $V$; in other words, Bacciagaluppi’s suggestion might be used to argue that energy-momentum quantization arises as a dynamical effect of measurement interactions, as opposed to a measurement-independent property of particles in bound states (as in ZSM). We find this an intriguing possibility, but the technical details need to be developed for it to become a serious proposal.

Grössing et al. [88] constructed a model of a classical “walking bouncer” particle (essentially a harmonic oscillator of natural frequency $\omega_0$) coupled to a dissipative thermal environment which imparts a stochastic, periodic, driving force. They then showed that in the large friction limit the mean stochastic dynamics of the bouncer satisfies what amounts to the quantization condition. They claim “this condition resolves the problem discussed by Wallstrom [20] about the single-valuedness of the quantum mechanical wavefunctions and eliminates possible contradictions arising from Nelson-type approaches to model quantum mechanics.” It
is unclear to us that their model involves physically consistent assumptions,\textsuperscript{17} but setting aside this concern, the main difficulty we see with their claim is that they don’t show how to derive the HJM equations from their model (although they do show that their model yields the energy spectrum of the quantum harmonic oscillator), which is the context in which Wallstrom’s critique applies. In addition, it is unclear to us that their model is consistent with NYSM since Nelson’s diffusion process is a conservative one while their model assumes a dissipative diffusion process in a thermal environment. No such (apparent) inconsistency exists for ZSM, since we implemented the $zw$ hypothesis into NYSM in a manner consistent with Nelson’s (suitably generalized) ether hypothesis. Nevertheless, in our view, Grössing et al.’s model (if it can be shown physically consistent) has value as a proof-of-principle that one can construct a physical model of a classical, harmonically oscillating particle coupled to some fluctuating, oscillating, ether-like background medium, and dynamically obtain the quantization condition.

Schmelzer\textsuperscript{51} argued that in order to obtain empirical equivalence with quantum mechanics, it is sufficient for stochastic mechanics to only recover wavefunctions with simple zeros. He then showed that if one invokes the postulate, $0 < \Delta \rho(x) < \infty$ almost everywhere when $\rho(x) = 0$, one obtains the quantization condition for simple zeros, i.e., where $n = \pm 1$. He also showed that this postulate corresponds to an “energy balance” constraint, namely, that the total energy density of the Nelsonian particle remains finite. Schmelzer suggested that it remains for subsequent theories to somehow dynamically justify the energy balance constraint. In our view, Schmelzer does not adequately justify his claim that simple zeros are sufficient to recover empirical equivalence with quantum mechanics (e.g., how can this account for energy level shifts in the hydrogen atom described by the Rydberg formula?) but if this can be shown, then we would concur that his proposal seems to be a non-circular, non-ad-hoc, empirically adequate justification for a limited version of the quantization condition. In ZSM, by contrast, the full quantization condition is obtained from the phase of the hypothesized $zw$ particle(s), with the proviso that it should be understood as a phenomenological stepping-stone to a physical theory of Nelson’s (suitable modified) ether, the $zw$ particle, and the dynamical interaction between the two.

Caticha and his collaborators\textsuperscript{39, 90} have offered two routes to answering Wallstrom within the context of his “entropic dynamics” (ED) framework (essentially, a Bayesian inference version of stochastic mechanics). In the first route, Caticha appeals to Pauli\textsuperscript{91}, who suggested that the criterion for admissibility for wavefunctions is that they must form a basis for a representation of the transformation group for a given eigenvalue problem. He then suggests that this criterion is “extremely natural” from the perspective of a theory of inference since “in any physical situation symmetries constitute the most common and most obviously relevant pieces of information”\textsuperscript{39}. However, it should be noted that Pauli’s criterion, more precisely, is that “repeated actions of the operators corresponding to physical quantities should not lead outside the domain of square-integrable eigenfunctions”\textsuperscript{91}. In other words, Pauli’s criterion just requires that wavefunctions continue to satisfy the linearity of Schrödinger’s equation (i.e., the superposition principle), even after being acted upon by operators for physical quantities. But insofar as ED attempts to recover the Schrödinger equation from the HJM equations, such a criterion cannot be invoked in entropic dynamics without begging the question. In the second route, Bartolomeo and Caticha\textsuperscript{90} take inspiration from Takabayasi’s generalization of the HJM equations to a spinning fluid\textsuperscript{53}; they propose to interpret their postulated “drift potential”, $\phi(x, t)$, as an angle describing particle spin, and thereby argue that the change of $\phi$ along a closed loop in space must equal $2\pi n$. In fact, this argument is conceptually equivalent to the ones given by de Broglie\textsuperscript{92-94} and Bohm\textsuperscript{91, 95, 96}, and which we’ve used in ZSM. On the other hand, it should be noted that Bartolomeo and Caticha don’t actually model spin in ED, nor do they suggest to connect spin to the dynamical influence of an ether or background field (in contrast to ZSM). Indeed, Bartolomeo and Caticha admit that “ED is a purely epistemic theory. It does not attempt to describe the world.... In fact ED is silent on the issue of what causative power is responsible for the peculiar motion of the particles”\textsuperscript{90}. From our point of view, this makes their argument for the quantization condition less compelling than the one offered by ZSM, and ED less compelling as a satisfactory theory of quantum phenomena compared to the (programmatic) ontological approach offered by ZSM. Nevertheless, to whatever extent one views the Bayesian inference approach to physics as valuable and interesting, it appears that one can give a somewhat non-ad-hoc justification for the quantization condition via ED.

\textsuperscript{17}They assume that their dissipative thermal environment corresponds to a classical “zero-point field” of Ornstein-Uhlenbeck statistical type, unknown positive temperature, and that imparts to the bouncer a total energy of $\hbar \omega_0/2$. But the zero-point fields of QED and SED are, by construction, frequency-cubed-dependent in their spectral density, non-dissipative in that they produce no Einstein-Hopf drag force, and non-thermal in that the zero-point motion they induce on charged particles persists at zero temperature\textsuperscript{87, 88}. 

39
7 Conclusion

We have extended both our classical zb model and ZSM to the cases of free particles, particles in external fields, and classically interacting particles. Along the way, we have made explicit the beables of ZSM and suggested three possible approaches for parsing the beables into local vs. nonlocal types. In addition, we have given arguments for the plausibility of the zb hypothesis and suggested new lines of research that could be pursued from the foundation provided here. We have also reviewed and compared several other proposals for answering the Wallstrom criticism, arguing that ZSM is the most general and viable approach of all of them presently.

We wish to emphasize, once more, that ZSM should not be viewed as a proposal for a fundamental physical theory of non-relativistic quantum phenomena; rather, it should be viewed as a provisional, phenomenological theory that provides the conceptual and mathematical scaffolding for an eventual physical theory of Nelson’s ether (amended for ZSM), the zb particle, and the dynamical coupling between the two.

In his 1994 paper [2], Wallstrom wrote: “There seems to be nothing within the particle-oriented world of stochastic mechanics which can lead to what is effectively a condition on the ‘wave function’.” We have shown, with the example of ZSM, that this claim can no longer be sustained for all formulations of stochastic mechanics.

8 Acknowledgments

I wish to thank Guido Bacciagaluppi, Dieter Hartmann, and Herman Batelaan for helpful discussions and encouragement throughout this work. I especially thank Guido Bacciagaluppi for a careful reading of this paper and several useful suggestions for improvements.

A Proof of the Stochastic Variational Principle

Let \( q_i'(t) = q_i(t) + \delta q_i(t) \) be sample-wise variations of the sample paths \( q_i(t) \), with end-point constraints \( \delta q_i(t_f) = \delta q_i(t_i) = 0 \). Then, using \( b_i = Dq_i(t) \) and \( b_{i*} = D_s q_i(t) \), the condition

\[
J(q) = \int_{t_1}^{t_f} dt \sum_{i=1}^{N} \left\{ \frac{1}{2} \left[ \frac{1}{2} m_i (Dq_i(t))^2 + \frac{1}{2} m_i (D_s q_i(t))^2 \right] + \frac{c_i}{c} A^{ext}_i \cdot \frac{1}{2} (D + D_s) q_i(t) - e_i \left( \Phi^{ext}_i + \Phi^{int}_i \right) \right\}
\]

is equivalent to the variation,

\[
\delta J(q) = J(q') - J(q),
\]

up to first order in \( ||\delta q_i(t)|| \). So (213) gives

\[
\delta J = \int_{t_1}^{t_f} dt \sum_{i=1}^{N} \left\{ \left[ \frac{1}{2} m_i (Dq_i(t) \cdot D\delta q_i(t) + D_s q_i(t) \cdot D_s \delta q_i(t)) \right] + \frac{c_i}{c} A^{ext}_i \cdot \frac{1}{2} (D\delta q_i(t) + D_s \delta q_i(t)) + \frac{c_i}{c} (\delta q_i(t) \cdot \nabla_i A^{ext}_i) \right\} v_i - e_i \nabla_i \left( \Phi^{ext}_i + \Phi^{int}_i \right) \cdot \delta q_i(t) \bigg|_{q_i = q_i(t)} dt,
\]

Now, for an arbitrary function \( f_i(q(t), t) \), we have the relations
\[
\int_{\mathbb{R}^3} d^3N q_0 \int_{t_I}^{t_F} \sum_{i=1}^{N} [f_i(q(t), t) D \delta q_i(t)] dt = - \int_{\mathbb{R}^3} d^3N q_0 \int_{t_I}^{t_F} \sum_{i=1}^{N} [\delta q_i(t) D_s f_i(q(t), t)] dt,
\]
and
\[
\int_{\mathbb{R}^3} d^3N q_0 \int_{t_I}^{t_F} \sum_{i=1}^{N} [f_i(q(t), t) D_s \delta q_i(t)] dt = - \int_{\mathbb{R}^3} d^3N q_0 \int_{t_I}^{t_F} \sum_{i=1}^{N} [\delta q_i(t) D f_i(q(t), t)] dt,
\]
and
\[
\frac{1}{2} (D + D_s) f_i(q(t), t) = \left\{ \partial_t + \frac{1}{2} [D q_i(t) + D_s q_i(t)] \cdot \nabla_i \right\} f_i(q(t), t) \big|_{q_i = q_i(t)}.
\]
So, using Eq. (9) in section 2, the integrand of (214) becomes
\[
\delta J = \int_{\mathbb{R}^3} d^3N q_0 \int_{t_I}^{t_F} \sum_{i=1}^{N} \left\{ \frac{M_i}{2} [D_s D + DD_s] q_i(t) - \frac{e_i}{c} \mathbf{v}_i \times (\nabla_i \times \mathbf{A}_i^{ext}) + \frac{e_i}{c} \partial_i \mathbf{A}_i^{ext} + e_i \nabla_i \left[ \Phi_i^{ext} + \Phi_i^{int} \right] \right\} |_{q_i = q_i(t)} \delta q_i(t) dt + \vartheta(||\delta q_i||).
\]
From the variational constraint (212-213), and using the fact that the arbitrary sample-wise variations (i.e., the virtual displacements in the generalized coordinates) \( \delta q_i(t) \) are independent for all \( i \) by D’Alembert’s principle [97], it follows that the first-order variation of \( J \) must be zero for each \( \delta q_i(t) \). Moreover, since the ensemble-average is a positive linear functional, we will have the equations of motion
\[
\sum_{i=1}^{N} \frac{M_i}{2} [D_s D + DD_s] q_i(t) = \sum_{i=1}^{N} e_i \left[ -\frac{1}{c} \partial_t \mathbf{A}_i^{ext} - \nabla_i \left( \Phi_i^{ext} + \Phi_i^{int} \right) + \left( \frac{\mathbf{v}_i}{c} \right) \times \left( \nabla_i \times \mathbf{A}_i^{ext} \right) \right] |_{q_i = q_i(t)},
\]
and
\[
\frac{M_i}{2} [D_s D + DD_s] q_i(t) = \left[ -\frac{e_i}{c} \partial_t \mathbf{A}_i^{ext} - e_i \nabla_i \left( \Phi_i^{ext} + \Phi_i^{int} \right) + \frac{e_i}{c} \mathbf{v}_i \times \left( \nabla_i \times \mathbf{A}_i^{ext} \right) \right] |_{q_i = q_i(t)},
\]
for each time \( t \in [t_I, t_F] \) with probability one.

References

[1] M. Derakhshani. A suggested answer to wallstrom’s criticism: Zitterbewegung stochastic mechanics i. 2016, http://arxiv.org/abs/1510.06391.
[2] J. S. Bell. *Speakable and Unspeakable in Quantum Mechanics*, chapter The theory of local beables, pages 52–62. Cambridge University Press, Cambridge, 2004.
[3] J. S. Bell. *Speakable and Unspeakable in Quantum Mechanics*, chapter Beables for quantum field theory, pages 173–180. Cambridge University Press, Cambridge, 2004.
[4] M. I. Loffredo and L. M. Morato. Stochastic quantization for a system of n identical interacting boson particles. *J. Phys. A Math. Theor.* 40, 2007, http://www.matapp.unimib.it/bertacchi/workshop07/morato.pdf.
[5] F. Guerra and L. Morato. Quantization of dynamical systems and stochastic control theory. *Phys. Rev. D*, 27:1774–1786, 1983.
[6] T. C. Wallstrom. On the derivation of the schroedinger equation from stochastic mechanics. *Foundations of Physics Letters*, 2:113–126, 1989.

[7] T. C. Wallstrom. Inequivalence between the schroedinger equation and the madelung hydrodynamic equations. *Phys. Rev. A*, 49:1613–1617, 1994.

[8] T. Koide. Classicalization of quantum variables and quantum-classical hybrids. *Phys. Lett. A* 379, 36, pages 2007-2012, 2015, http://arxiv.org/abs/1412.6321v3.

[9] E. Nelson. Derivation of the schroedinger equation from newtonian mechanics. *Phys. Rev.*, 150:1079–1085, 1966.

[10] E. Nelson. *Dynamical Theories of Brownian Motion*. Princeton University Press, Princeton, 1967. "Osmotic velocity", Chapters 4 and 13. https://web.math.princeton.edu/nelson/books/bmotion.pdf.

[11] E. Nelson. *Quantum Fluctuations*. Princeton University Press, Princeton, 1985. "Nelson-Yasue", pp. 73-77; "Physical interpretation of the wave function", pp. 81; "identical particles", pp. 100; "non-equilibrium physics", pp. 117-119; "electromagnetic origin", pp. 65; "conservative diffusions", pp. 65; "electromagnetic potentials", pp. 72-76; "nodal point", pp. 77-82; "Markovianity", pp. 128-130. https://web.math.princeton.edu/nelson/books/qf.pdf.

[12] L. de la Pena and A. M. Cetto. Does quantum mechanics accept a stochastic support? *Found. Phys. 12* 1017-37, 1982.

[13] G. Bacciagaluppi. Non-equilibrium in stochastic mechanics. *Journal of Physics: Conference Series*, 361:012017, 2012, http://iopscience.iop.org/1742-6596/361/1/012017/.

[14] B. I. Lev, V. B. Tymchyshyn, and A. G. Zagorodny. Brownian particle in non-equilibrium plasma. *Condensed Matter Physics*, Vol. 12, No. 4, pp. 593-602, 2009.

[15] M. F. Pusey, J. Barrett, and T. Rudolph. On the reality of the quantum state. *Nature Physics*, 8:475–478, 2012, http://arxiv.org/abs/1111.3328.

[16] P. R. Holland. *The Quantum Theory of Motion: An Account of the de Broglie-Bohm Causal Interpretation of Quantum Mechanics*. Cambridge University Press, Cambridge, 1993. "Hamilton-Jacobi formulation", "Gibbsian statistical ensemble", "nonlinear Schrödinger equation", and "continuity equation", Chapter 2; "polar decomposition", pp. 68-72; "nodal point(s)", pp. 85-86.

[17] G. Bacciagaluppi. Derivation of the symmetry postulates for identical particles from pilot-wave theories. 2003, http://arxiv.org/abs/quant-ph/0302099.

[18] S. D. Bartlett, T. Rudolph, and R. W. Spekkens. Reconstruction of gaussian quantum mechanics from liouville mechanics with an epistemic restriction. *Phys. Rev. A* 86, 012103, 2012, http://arxiv.org/abs/1111.5057.

[19] E. G. Cavalcanti, C. J. Foster, M. D. Reid, and P. D. Drummond. Bell inequalities for continuous-variable correlations. *Phys. Rev. Lett.* 99, 210405, 2007, http://arxiv.org/abs/0705.1385.

[20] D. Bohm and B. J. Hiley. *The Undivided Universe: An Ontological Interpretation of Quantum Theory*. Routledge, 1995.

[21] J. S. Bell. *Speakable and Unspeakable in Quantum Mechanics*, chapter Quantum mechanics for cosmologists, pages 117–138. Cambridge University Press, Cambridge, 2004.

[22] D. Z. Albert. *Bohmian Mechanics and Quantum Theory: An Appraisal*, chapter Elementary Quantum Metaphysics, page 277. Springer, 1996.

[23] D. Z. Albert. *The Wave Function: Essays on the Metaphysics of Quantum Mechanics*, chapter Wave Function Realism, page 52. Oxford University Press, 2013.

[24] D. Z. Albert. *After Physics*, chapter Quantum Mechanics and Everyday Life, page 124. Harvard University Press, 2015.
[25] T. Norsen. The theory of (exclusively) local beables. *Found. Phys.*, 40:1858, 2010, http://arxiv.org/abs/0909.4553.

[26] T. Norsen, D. Marian, and X. Oriols. Can the wave function in configuration space be replaced by single-particle wave functions in physical space? *Synthese Special Issue (forthcoming): Space-time and the wave function*, pages 1–24, 2014, http://arxiv.org/abs/1410.3676.

[27] A. Komar. Interacting relativistic particles. *Phys. Rev. D*, 18:1887, 1978.

[28] F. Rohrlich. Relativistic hamiltonian dynamics i. classical mechanics. *Annals of Physics*, 117:292–322, 1979.

[29] D. Duerr, S. Goldstein, and N. Zanghi. Quantum equilibrium and the origin of absolute uncertainty. *Journal of Statistical Physics*, 67:843–907, 1992, http://arxiv.org/abs/quant-ph/0308039v1.

[30] V. Zelevinsky. *Quantum Physics Volume 1: From Basics to Symmetries and Perturbations*, chapter Hydrogen Fine Structure, page 551. WILEY-VCH, 2011.

[31] V. Allori. *Decoherence and the Classical Limit of Quantum Mechanics*. PhD thesis, Universita degli Studi di Genova, 2001, http://www.niu.edu/ vallori/tesi.pdf.

[32] G. E. Bowman. On the classical limit in bohm’s theory. *Foundations of Physics, Vol. 35, 4*, pp. 605-625, 2005.

[33] X. Oriols, D. Tena, and A. Benseny. Natural classical limit for the center of mass of many-particle quantum systems. 2016, https://arxiv.org/abs/1602.03988.

[34] D. Dohrn and F. Guerra. Nelson’s stochastic mechanics on riemannian manifolds. *Lettere al Nuovo Cimento*, 22:121–127, 1978.

[35] D. Dohrn, F. Guerra, and P. Ruggiero. *Feynman Path Integrals (Lecture Notes in Physics vol 106)*, chapter Spinning particles and relativistic particles in the framework of Nelson’s stochastic mechanics. Berlin: Springer, 1979.

[36] D. Dohrn and F. Guerra. Compatibility between the brownian metric and the kinetic metric in nelson stochastic quantization. *Phys. Rev. D*, 31:2521–2524, 1985.

[37] M. Serva. Relativistic stochastic processes associated to klein-gordon equation. *Ann. Inst. Henri Poincare*, 49:415–432, 1988.

[38] T. Zastawniak. A relativistic version of nelson’s stochastic mechanics. *Europhysics Letters*, 13:13–17, 1990.

[39] W. Greiner. *Relativistic Quantum Mechanics: Wave Equations*. Springer, 2000.

[40] P. R. Holland. The dirac equation in the de broglie-bohm theory of motion. *Found. Phys.*, 22:1287–1301, 1992.

[41] S. Colin and H. M. Wiseman. The zig-zag road to reality. *J. Phys. A.*, 44:345304, 2011, http://arxiv.org/abs/1107.4909.

[42] W. Struyve. On the zig-zag pilot-wave approach for fermions. *J. Phys. A.*, 45:195307, 2012, http://arxiv.org/abs/1201.4169.

[43] G. F. de Angelis, G. Jona-Lasinio, M. Serva, and N. Zanghi. Stochastic mechanics of a dirac particle in two spacetime dimensions. *J. Phys. A: Math. Gen.*, 19:865, 1986, http://iopscience.iop.org/0305-4470/19/6/017/pdf/ja196p865.pdf.

[44] P. Garbaczewski. Relativistic problem of random flights and nelson’s stochastic mechanics. *Phys. Lett. A*, 164:6–16, 1992, http://www.fiz.uni.opole.pl/pgar/documents/pla92b.pdf.

[45] A. O. Barut and N. Zanghi. Classical model of the dirac electron. *Phys. Rev. Lett.*, 52:2009, 1984.
[46] A. O. Barut and M. Pavsic. Classical model of the Dirac electron in curved space. *Classical and Quantum Gravity*, 4:L41, 1987.

[47] A. O. Barut and N. Unal. Generalization of the Lorentz-Dirac equation to include spin. *Phys. Rev. A*, 40:5404, 1989.

[48] A. O. Barut, C. Onem, and N. Unal. The classical relativistic two-body problem with spin and self-interactions. *J. Phys. A.*, 23:1113, 1990.

[49] A. O. Barut and M. Pavsic. Quantisation of the classical relativistic zitterbewegung in the Schrödinger picture. *Classical and Quantum Gravity*, 4:L131, 1987.

[50] A. O. Barut and I. H. Duru. Path integral formulation of quantum electrodynamics from classical particle trajectories. *Physics Reports*, 172:1–32, 1989.

[51] P. Holland and C. Philippidis. Implications of Lorentz covariance for the guidance equation in two-slit quantum interference. *Phys. Rev. A* 67, 062105, 2003, http://arxiv.org/abs/quant-ph/0302076.

[52] G. Bacciagaluppi. Nelsonian mechanics revisited. *Found. Phys. Lett.* 12, 1–16, 1999, http://arxiv.org/abs/quant-ph/9811040.

[53] T. Takabayasi. Vortex, spin and triad for quantum mechanics of spin-spin particle.i. *Prog. Theoretical Physics*, Vol. 70, No. 1, 1983, http://ptp.oxfordjournals.org/content/70/1/1.refs.

[54] E. Adelberger, G. Dvali, and A. Gruzinov. Photon mass bound destroyed by vortices. *Phys. Rev. Lett.*, 98:010402, 2007, http://arxiv.org/abs/hep-ph/0306245.

[55] F. J. Yndurain. Limits on the mass of the gluon. *Phys. Lett. B*, 345:524–526, 1995.

[56] A. S. Goldhaber and M. M. Nieto. Photon and graviton mass limits. *Rev. Mod. Phys.*, 82, pp. 939-979, 2010, http://arxiv.org/abs/0809.1003.

[57] G. Durieux, J. Gerard, F. Maltoni, and C. Smith. Three-generation baryon and lepton number violation at the LHC. *Phys. Lett. B*, 721:82–85, 2013, http://arxiv.org/abs/1210.6598.

[58] S. Colin and W. Struyve. A Dirac sea pilot-wave model for quantum field theory. *J. Phys. A.*, 40:7309–7342, 2007, http://arxiv.org/abs/quant-ph/0701085.

[59] H. B. Nielsen and M. Ninomiya (1998). Dirac sea for bosons. http://arxiv.org/abs/hep-th/9808108.

[60] Y. Habara, Y. Nagatani, H. B. Nielsen, and M. Ninomiya. Dirac sea and hole theory for bosons i - a new formulation of quantum field theories. *International Journal of Modern Physics A*, 23:2733–2769, 2008, http://arxiv.org/abs/hep-th/0603242.

[61] L. F. Santos and C. O. Escobar. Stochastic motion of an open bosonic string. *Phys. Lett. A*, 256, 89-94, 1999, http://arxiv.org/abs/quant-ph/9806044v2.

[62] A. Rueda and G. Cavalleri. Zitterbewegung in stochastic electrodynamics and implications on a zero-point field acceleration mechanism. *Il Nuovo Cimento C*, 6:239–260, 1983.

[63] A. Rueda. Stochastic electrodynamics with particle structure part i: Zero-point induced brownian behavior. *Found. Phys. Lett.*, 6:75–108, 1993.

[64] A. Rueda. Stochastic electrodynamics with particle structure part ii - towards a zero-point induced wave behavior. *Found. Phys. Lett.*, 6:139–166, 1993.

[65] L. de la Pena and A. M. Cetto. *The Quantum Dice: An Introduction to Stochastic Electrodynamics*, chapter 12. Kluwer Academic Publisher, 1996.

[66] B. Haisch and A. Rueda. On the relation between a zero-point-field-induced inertial effect and the Einstein-de Broglie formula. *Phys. Lett. A*, 268:224–227, 2000, http://xxx.lanl.gov/abs/gr-qc/9906084.
[67] D. T. Pope, P. D. Drummond, and W. J. Munro. Disagreement between correlations of quantum mechanics and stochastic electrodynamics in the damped parametric oscillator. Phys. Rev. A, 62:042108, 2000, http://arxiv.org/abs/quant-ph/0003131.

[68] M. Genovese, G. Brida, M. Gramegna, F. Piacentini, E. Predazzi, and I. Ruo-Berchera. Experimental tests of hidden variable theories from dbb to stochastic electrodynamics. Journal of Physics: Conference Series, 67:012047, 2007, http://iopscience.iop.org/1742-6596/67/1/012047.

[69] A. Valdes-Hernandez, L. de la Pena, and A. M. Cetto. Bipartite entanglement induced by a common background (zero-point) radiation field. Found. Phys., 41:843–862, 2011.

[70] A. M. Cetto, L. de la Pena, and A. Valdes-Hernandez. Quantization as an emergent phenomenon due to matter-zeropoint field interaction. Journal of Physics: Conference Series, 361:012013, 2012.

[71] L. de la Pena, A. M. Cetto, and A. Valdes-Hernandez. Quantum behavior derived as an essentially stochastic phenomenon. Physica Scripta, 2012:014008, 2012.

[72] A. M. Cetto and L. de la Pena. Radiative corrections for the matter–zeropoint field system: establishing contact with quantum electrodynamics. Physica Scripta, 2012:014009, 2012.

[73] A. M. Cetto, L. de la Pena, and A. Valdes-Hernandez. Emergence of quantization: the spin of the electron. Journal of Physics: Conference Series, 504:012007, 2014.

[74] R. Penrose. The Road to Reality, chapter 25.2 The zigzag picture of the electron. Alfred A. Knopf, 2005.

[75] P. Pearle and A. Valentini. Encyclopaedia of Mathematical Physics, chapter Generalizations of Quantum Mechanics. Elsevier, North-Holland, 2006, http://arxiv.org/abs/quant-ph/0506115.

[76] A. Valentini. Inflationary cosmology as a probe of primordial quantum mechanics. Phys. Rev. D, 063513, 2010.

[77] P. Hanggi and P. Jung. Advances in Chemical Physics, Volume LXXXIX, chapter Colored Noise In Dynamical Systems, pages 239–326. Wiley & Sons, Inc., 1995.

[78] A. Montina. Exponential complexity and ontological theories of quantum mechanics. Phys. Rev. A 77, 022104, 2008, http://arxiv.org/abs/0711.4770.

[79] A. Montina. State space dimensionality in short memory hidden variable theories. Phys. Rev. A 83, 032107, 2011, http://arxiv.org/abs/1008.4415.

[80] A. Montina. Dynamics of a qubit as a classical stochastic process with time-correlated noise: minimal measurement invasiveness. Phys. Rev. Lett. 108, 160501, 2012, http://arxiv.org/abs/1108.5138.

[81] E. A. Carlen and M. I. Loffredo. The correspondence between stochastic mechanics and quantum mechanics on multiply connected configuration spaces. Phys. Lett. A, 141:9–13, 1989.

[82] L. Smolin (2006). Could quantum mechanics be an approximation to another theory? http://arxiv.org/abs/quant-ph/0609109.

[83] L. Fritsche and M. Haugk (2009). Stochastic foundation of quantum mechanics and the origin of particle spin. http://arxiv.org/abs/0912.3442.

[84] I. Schmelzer (2011). An answer to the wallstrom objection against nelsonian stochastics. http://arxiv.org/abs/1101.5774.

[85] G. Groessing, J. M. Pascasio, and H. Schwabl. A classical explanation of quantization. Found. Phys., 41:1437–1453, 2011, http://arxiv.org/abs/0812.3561.

[86] G. Bacciagaluppi. Endophysics, Time, Quantum and the Subjective, chapter A Conceptual Introduction to Nelson’s Mechanics, pages 367–88, http://philsci–archive.pitt.edu/8853/. Singapore: World Scientific, 2005.
[87] T. H. Boyer. *Foundation of Radiation Theory and Quantum Electrodynamics*, chapter A brief survey of stochastic electrodynamics, pages 141–162. Plenum, New York, 1980.

[88] P. W. Milonni. *The Quantum Vacuum: An Introduction to Quantum Electrodynamics*. Academic Press, New York, 1994.

[89] A. Caticha. Entropic dynamics, time and quantum theory. *J. Phys. A.* 44:225303, 2011, http://arxiv.org/abs/1005.2357.

[90] D. Bartolomeo and A. Caticha. Entropic dynamics: The Schroedinger equation and its bohmian limit. In *MaxEnt 2015, the 35th International Workshop on Bayesian Inference and Maximum Entropy Methods in Science and Engineering*, 2015, http://arxiv.org/abs/1512.09084.

[91] W. Pauli. *General Principles of Quantum Mechanics*, chapter Stationary States as Eigenvalue Problem, pages 47–48. Springer-Verlag, 1980.

[92] Louis-Victor de Broglie. *On the Theory of Quanta*. PhD thesis, University of Paris, 1925.

[93] O. Darrigol. *Strangeness and soundness in Louis de Broglie’s early works*. L.S. Olschki, 1994.

[94] D. Bohm. *Observation and Interpretation: A Symposium of Philosophers and Physicists*, chapter A proposed explanation of quantum theory in terms of hidden variables at a sub-quantum-mechanical level, pages 33–40. Butterworths Scientific Publications, 1957.

[95] D. J. Bohm and B. J. Hiley. The de Broglie pilot wave theory and the further development of new insights arising out of it. *Foundations of Physics, Vol. 12, No. 10, pp. 1001-1016*, 1982.

[96] D. Bohm. *Wholeness and the Implicate Order*, chapter Explanation of the quantization of action, pages 122–133. Routledge, 2002.

[97] S. Ray and J. Shamana. On virtual displacement and virtual work in lagrangian dynamics. *European Journal of Physics*, 27:311–329, 2006, http://arxiv.org/abs/physics/0510204.