Curve graphs for Artin–Tits groups of type $B$, $\tilde{A}$ and $\tilde{C}$ are hyperbolic

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Abstract

The graph of irreducible parabolic subgroups is a combinatorial object associated to an Artin–Tits group $A$ defined so as to coincide with the curve graph of the $(n+1)$-times punctured disk when $A$ is Artin’s braid group on $(n+1)$ strands. In this case, it is a hyperbolic graph, by the celebrated Masur–Minsky’s theorem. Hyperbolicity of the graph of irreducible parabolic subgroups for more general Artin–Tits groups is an important open question. In this paper, we give a partial affirmative answer.

For $n \geq 3$, we show that the graph of irreducible parabolic subgroups associated to the Artin–Tits group of spherical type $B_n$ is also isomorphic to the curve graph of the $(n+1)$-times punctured disk; hence, it is hyperbolic.

For $n \geq 2$, we show that the graphs of irreducible parabolic subgroups associated to the Artin–Tits groups of euclidean type $\tilde{A}_n$ and $\tilde{C}_n$ are isomorphic to some subgraphs of the curve graph of the $(n+2)$-times punctured disk which are not quasi-isometrically embedded. We prove nonetheless that these graphs are hyperbolic.

1. Introduction and Background

An Artin–Tits group is a group defined by a presentation involving a finite set of generators $S$ (the standard generators) and where all the relations are as follows: every pair $(a,b)$ of standard generators satisfies at most one balanced relation of the form

$$\Pi(a,b;m_{a,b}) = \Pi(b,a;m_{a,b}),$$

with $m_{a,b} \geq 2$ and where for $k \geq 2$,

$$\Pi(a,b;k) = \begin{cases} (ab)^{\frac{k}{2}} & \text{if } k \text{ is even}, \\ (ab)^{k-1} & \text{if } k \text{ is odd}. \end{cases}$$

This presentation can be encoded by a Coxeter graph $\Gamma$. The vertices of $\Gamma$ are in bijection with the set $S$. Two distinct vertices $a, b$ of $\Gamma$ are connected by a labeled edge if and only if either they satisfy no relation, in which case the label is $\infty$, or $m_{a,b} > 2$, in which case the label is $m_{a,b}$.

The Artin–Tits group defined by the Coxeter graph $\Gamma$ will be denoted by $A_{\Gamma}$. Because all the relations in the given presentation of $A_{\Gamma}$ are balanced, there is a homomorphism $\epsilon_{\Gamma} : A_{\Gamma} \rightarrow \mathbb{Z}$ assigning to each element of $A_{\Gamma}$ the exponent sum of any word on $S$ representing it.
The group $A_\Gamma$ is said to be irreducible if $\Gamma$ is connected and dihedral if $\Gamma$ has two vertices. The quotient by the normal subgroup generated by the squares of the elements in $S$ is a Coxeter group denoted by $W_\Gamma$. The Artin–Tits group $A_\Gamma$ is said to be of spherical type if $W_\Gamma$ is finite.

Given a proper subset $\emptyset \neq X \subset S$, the proper subgroup of $A_\Gamma$ generated by $X$ is called a standard parabolic subgroup of $A_\Gamma$; it is naturally isomorphic to the Artin–Tits group $A_\Xi$ defined by the subgraph $\Xi$ of $\Gamma$ induced by the vertices in $X$ [29]. A subgroup $P$ of $A_\Gamma$ is called parabolic if it is conjugate to a standard parabolic subgroup.

The flagship example of an Artin–Tits group (of spherical type) is the braid group on $(n+1)$ strands — that is, the Artin–Tits group defined by the graph $A_n$ shown in Figure 1(a). This group is isomorphic to the Mapping Class Group of an $(n+1)$-times punctured closed disk $D_{n+1}$, which is the group of isotopy classes of orientation-preserving homeomorphisms of $D_{n+1}$ which induce the identity on the boundary of $D_{n+1}$.

The curve graph $CG(D_{n+1})$ of the $(n+1)$-times punctured disk $D_{n+1}$ is the graph whose vertices are the isotopy classes of essential simple closed curves in $D_{n+1}$ (closed curves without auto-intersection and enclosing at least 2 and at most $n$ punctures) and where two vertices are joined by an edge if the corresponding isotopy classes of curves are distinct and admit disjoint representatives. This is a connected graph whenever $n \geq 3$ (see [17, section 4.1.1]), which we will suppose henceforth. The graph $CG(D_{n+1})$ is equipped with the combinatorial metric $d_{D_{n+1}}$ defined by declaring each edge to have length one. There is a natural action of the Artin–Tits group $A_{A_n}$ on the set of isotopy classes of essential simple closed curves in $D_{n+1}$; this action preserves adjacency in $CG(D_{n+1})$; hence, $A_{A_n}$ acts by isometries on the curve graph of $D_{n+1}$.

Following Masur–Minsky’s celebrated theorem [21, Theorem 1.1], $CG(D_{n+1})$ is a hyperbolic metric space. Furthermore, the Artin–Tits group $A_{A_n}$ (viewed as the mapping class group of $D_{n+1}$) is a hierarchically hyperbolic space [3–5, 28], with respect to the projections to the curve graphs of subsurfaces of the disk. A natural and challenging question is whether any irreducible Artin–Tits group $A_\Gamma$ (not necessarily of spherical type) admits such a hierarchical structure. A first step forward is to define a hyperbolic space on which $A_\Gamma$ acts in the same way as the braid group acts on the curve graph of the punctured disk.

In [13], Cumplido, Gebhardt, González-Meneses and Wiest explain a one-to-one correspondence between isotopy classes of simple closed curves in $D_{n+1}$ and proper irreducible parabolic subgroups of $A_{A_n}$ which allows them to translate the definition of the curve graph $CG(D_{n+1})$ in purely algebraic terms. This definition can then be generalized to any Artin–Tits group as follows:

**Definition 1.1** [13] [24, Definition 4.1]. Let $A_\Gamma$ be an Artin–Tits group. Two distinct proper irreducible parabolic subgroups $P$ and $Q$ are called adjacent if one of the following conditions holds:
\[ P \subset Q \text{ or } Q \subset P, \]
\[ P \cap Q = \{1\} \text{ and } pq = qp \text{ for all } p \in P \text{ and } q \in Q. \]

The graph of irreducible parabolic subgroups of \( A_\Gamma \) is the graph \( C_{parab}(\Gamma) \) whose vertices are the proper irreducible parabolic subgroups of \( A_\Gamma \) and where two vertices are connected by an edge if and only if they correspond to adjacent parabolic subgroups.

The graph \( C_{parab}(\Gamma) \) is equipped with a metric, declaring each edge to have length one. We denote by \( d_1 \) the distance on \( C_{parab}(\Gamma) \). There is a natural simplicial action of \( A_\Gamma \) on \( C_{parab}(\Gamma) \), by conjugation on parabolic subgroups. This action will be denoted on the right: given a proper irreducible parabolic subgroup \( P \) of \( A_\Gamma \) and \( g \in A_\Gamma \), the parabolic subgroup \( g^{-1}Pg \) will be denoted by \( P^g \). Accordingly, we will always use the exponent notation for conjugacy in a group \( G \): given \( g, h \in G \), \( h^g = g^{-1}hg \).

Note that \( C_{parab}(\Gamma) \) is empty if \( \Gamma \) consists of a single vertex (\( A_\Gamma \) is cyclic). If \( A_\Gamma \) is dihedral, then by [24, Lemma 5.2, Theorem 5.3], \( C_{parab}(\Gamma) \) is not connected and has infinite diameter, unless \( \Gamma \) consists of two vertices with no edge, in which case \( A_\Gamma \cong \mathbb{Z}^2 \) and \( C_{parab}(\Gamma) \) consists of two vertices and a single edge between them. Also, if \( \Gamma \) is not connected, \( C_{parab}(\Gamma) \) is easily shown to have diameter 2. In this paper, we will always assume that \( \Gamma \) is connected and has at least three vertices.

Most of the known properties of the graph \( C_{parab}(\Gamma) \) are gathered in [11] and [24]. For example, if \( A_\Gamma \) is irreducible and of spherical type (\( \Gamma \) with at least three vertices), then \( C_{parab}(\Gamma) \) is connected and has infinite diameter ([24, Lemma 5.2] and [11, Corollary 4.13]). Finally, when \( \Gamma = A_n \), \( C_{parab}(A_n) \) is isomorphic to the curve graph \( CG(\mathbb{D}_{n+1}) \) and it is hyperbolic in virtue of Masur–Minsky’s theorem.

In this paper, we study the graphs of irreducible parabolic subgroups of three infinite families of Artin–Tits groups closely related to Artin’s braid groups, whose defining Coxeter graphs are depicted in Figure 1(b)–(d). The group \( A_{B_n} \) is of spherical type, while \( A_{\tilde{A}_n} \) and \( A_{\tilde{C}_n} \) are of euclidean type. While Artin–Tits groups of spherical type were long well known ([7], for instance), the structure of euclidean Artin–Tits groups was elucidated only recently [23], see also [25].

Here is a brief summary of the results in the paper. Firstly, we recall that the Artin–Tits group \( A_{B_n} \) can be realized as a subgroup of index \( (n + 1)! \) of Artin’s braid group on \( (n + 1) \) strands \( A_{A_n} \) [19]. We shall prove that this inclusion induces a graph isomorphism between the respective graphs of irreducible parabolic subgroups:

**Theorem 1.2.** For \( n \geq 3 \), the graph \( C_{parab}(B_n) \) is isomorphic to \( CG(\mathbb{D}_{n+1}) \) — and to \( C_{parab}(A_n) \). Therefore, \( C_{parab}(B_n) \) is hyperbolic.

We then focus on the Artin–Tits groups \( A_{\tilde{A}_n} \) and \( A_{\tilde{C}_n} \). Our arguments build on two classical embeddings between the Artin–Tits groups involved. Firstly, there is an embedding of \( A_{\tilde{A}_n} \) in \( A_{B_{n+1}} \) which yields a semi-direct product decomposition \( A_{B_{n+1}} \cong A_{\tilde{A}_n} \rtimes \mathbb{Z} \) [19]. Second, \( A_{\tilde{C}_n} \) embeds as a subgroup of index \( (n + 1)(n + 2) \) in Artin’s braid group on \( (n + 2) \) strands \( A_{A_{n+1}} \) — in a very similar way as \( A_{B_n} \) embeds in \( A_{A_n} \) [1].

We prove the following theorem.

**Theorem 1.3.** Let \( n \geq 2 \). Let \( Z = A \) or \( C \).

(i) The graph \( C_{parab}(Z_n) \) is connected.

(ii) The graph \( C_{parab}(Z_n) \) is isomorphic to a subgraph \( K_Z \) of \( CG(\mathbb{D}_{n+2}) \).

(iii) The graph \( C_{parab}(Z_n) \) has infinite diameter.

(iv) The graph \( C_{parab}(Z_n) \) is hyperbolic.
We already point out that part (i) can be proven using the same argument as given in [24, Lemma 5.2]; actually, we have the following lemma.

**Lemma 1.4.** Suppose that $\Gamma$ is a connected graph with at least three vertices. Then $C_{\text{parab}}(\Gamma)$ is connected.

The embedding promised by part (ii) is described in Corollary 4.6 for $\tilde{A}_n$, and Corollary 5.6 for $\tilde{C}_n$. Once (ii) is proven, (iii) follows easily after showing that the isomorphic image $K_Z$ of $C_{\text{parab}}(\tilde{Z}_n)$ is dense in $CG(\mathbb{D}_{n+2})$ (Propositions 4.7 and 5.7). Finally, we observe that $K_Z$ is not quasi-isometrically embedded in $CG(\mathbb{D}_{n+2})$ — see Proposition 4.9 and Remark 5.9. Therefore, hyperbolicity of $C_{\text{parab}}(\tilde{Z}_n)$ is not immediate and to establish it, we rely on a theorem of Kate Vokes [30] which allows to prove the hyperbolicity of some subgraphs of the curve graph. Whatsoever, it is important to note that our proofs of the hyperbolicity of $C_{\text{parab}}(B_n)$, $C_{\text{parab}}(\tilde{A}_n)$ and $C_{\text{parab}}(\tilde{C}_n)$ strongly depend on the hyperbolicity of curve graphs; it would be highly desirable to obtain independent algebraic proofs.

As explained in [11], Theorem 1.2 can be rephrased by saying that the union $X_{NP}(B_n)$ of the normalizers of the proper irreducible standard parabolic subgroups of $A_{B_n}$ is a hyperbolic structure on $A_{B_n}$, answering partially [11, Conjecture 4.7]. Finally, we exhibit another hyperbolic structure on $A_{B_n}$ and partially answer [11, Conjectures 4.2 and 4.18]:

**Theorem 1.5.** Let $n \geq 3$. Let $X_P(B_n)$ be the union of the proper irreducible standard parabolic subgroups of $A_{B_n}$ and the center of $A_{B_n}$. Then $X_P(B_n)$ is a hyperbolic structure on $A_{B_n}$ which is not equivalent to $X_{NP}(B_n)$.

The paper is arranged as follows. Section 2 contains all the prerequisites for our results and some notation that will be used throughout the paper. In Section 2.1, we review some results on parabolic subgroups of Artin–Tits groups; in Section 2.2, we recall the correspondence between essential simple closed curves in $\mathbb{D}_{n+1}$ and proper irreducible parabolic subgroups of $A_{A_n}$ and we introduce useful notation; in Section 2.3, we present the above-mentioned theorem of Vokes [30], Theorem 1.2 is proved in Section 3. Sections 4 and 5 are devoted to the study of $C_{\text{parab}}(\tilde{A}_n)$ and $C_{\text{parab}}(\tilde{C}_n)$, respectively. Finally, in Section 6 we prove Theorem 1.5.

## 2. Prerequisites

### 2.1. Artin–Tits groups and Coxeter groups

Let $A_\Gamma$ be any Artin–Tits group with standard generators $S$. Let $W_\Gamma = A_\Gamma/\langle\langle s^2 \mid s \in S\rangle\rangle$ be the associated Coxeter group. The canonical projection $\pi : A_\Gamma \to W_\Gamma$ admits a set section $\nu$ defined as follows — see, for instance, [6, Theorem 3.3.1(ii)]. For $s \in S$, denote by $\bar{s}$ its image in $W_\Gamma$ and $\bar{S} = \{\bar{s} \mid s \in S\}$. Let $w \in W_\Gamma$ and let $\bar{s_1} \cdots \bar{s_r}$ be a reduced expression for $w$, meaning a shortest word representative for $w$ on $\bar{S}$; then $\nu(w) = s_1 \cdots s_r$. The kernel of the projection $\pi$ is called the pure Artin–Tits group (or colored Artin–Tits group) and is denoted as $PA_\Gamma$.

Given a subset $X$ of $S$, the standard parabolic subgroup of $A_\Gamma$ generated by $X$ is denoted by $A_X$.

**Lemma 2.1** [27, Theorem 4.1]. Let $X, Y \subset S$. The following are equivalent.

(i) The subgroups $A_X$ and $A_Y$ are conjugate in $A_\Gamma$.

(ii) The sets $X, Y$ are conjugate in $A_\Gamma$.
LEMMA 2.2 [27, Corollary 4.2]. Let $s, t \in S$; then $s$ and $t$ are conjugate in $A_\Gamma$ if and only if there is a path in $\Gamma$ which connects $s$ and $t$ and follows only edges with odd labels.

In the rest of this section, we assume that $A_\Gamma$ is of spherical type. In this case, $W_\Gamma$ contains a unique longest element $w_0$ (see, for instance, [14, Lemma 4.6.1]). Denote its lift $\nu(w_0)$ in $A_\Gamma$ by $\Delta_\Gamma$. Whenever $\Gamma$ is connected, it is known that the center of $A_\Gamma$ is cyclic generated by $\Delta_\Gamma$ or $\Delta_\Gamma^2$ [7, Theorem 7.2]. Any proper irreducible parabolic subgroup $P$ of $A_\Gamma$ is itself an irreducible Artin–Tits group of spherical type. The center of $P$ is a cyclic group generated by an element $z_P$ (actually we have the generators $z_P$ and $z_P^{-1}$ and we choose $z_P$ so that its exponent sum $\epsilon_\Gamma(z_P)$ is positive). We will always refer to this particular element as the central element of $P$.

The following two results will be used throughout the paper. The first one says in particular (with $g = 1$) that the element $z_P$ determines completely the subgroup $P$.

PROPOSITION 2.3 [27, Theorem 5.2]. Let $A_\Gamma$ be an Artin–Tits group of spherical type. Let $P, Q$ be two irreducible parabolic subgroups of $A_\Gamma$ and let $g \in A_\Gamma$. Then $Q = P^g$ if and only if $z_Q = z_P^g$.

The second result reduces the definition of adjacency in the graph of irreducible parabolic subgroups to a very simple commutation condition between the respective central elements.

PROPOSITION 2.4 [13, Theorem 2.2]. Let $A_\Gamma$ be an Artin–Tits group of spherical type. Let $P, Q$ be two distinct irreducible parabolic subgroups of $A_\Gamma$. Then $P, Q$ are adjacent (Definition 1.1) if and only if $z_P$ and $z_Q$ commute.

2.2. Braids, curves and parabolic subgroups

Recall that the braid group on $(n+1)$ strands — or Artin–Tits group $A_{A_n}$ — can be identified with the Mapping Class Group of a closed disk with $(n+1)$ punctures $D_{n+1}$. Assume that $D_{n+1}$ is the closed disk in the complex plane of radius $\frac{n+2}{2}$ centered at $\frac{n+2}{2}$ and the punctures are at the integer numbers $1 \leq i \leq n + 1$. For $1 \leq i \leq n$, the standard generator $\sigma_i$ of $A_{A_n}$ corresponds to a clockwise half-Dehn twist along the horizontal segment $[i, i+1]$. The group $A_{A_n}$ naturally acts — on the right — on the set of isotopy classes of essential simple closed curves in $D_{n+1}$. In the sequel we will simply write “essential curve” or even “curve” instead of “isotopy class of essential simple closed curve”; accordingly, we will say that two distinct curves are disjoint if the corresponding isotopy classes admit disjoint representatives. The result of the action of a braid $y$ on a curve $C$ will be denoted by $C^y$. Finally, note that a curve $C$ in $D_{n+1}$ divides the disk in two connected components naturally referred to as the interior and the exterior of $C$.

Let $I$ be a proper subinterval of $[n] = \{1, \ldots, n\}$, that is,

$$\emptyset \neq I \subsetneq [n], \quad [(i < j < k) \land (i, k \in I)] \implies j \in I.$$

This defines a proper irreducible standard parabolic subgroup of $A_{A_n}$, generated by $\{\sigma_i \mid i \in I\}$; denote this subgroup by $A_I$. Let $m = \min(I)$ and $k = \#I$; the standard or round curve associated to $I$ is the isotopy class of a geometric circle surrounding the $(k + 1)$ punctures $m, \ldots, m + k$. Let us denote this curve by $C_I$.

As explained in [13, Section 2], there is a one-to-one correspondence

$$\{\text{curves in } D_{n+1}\} \xrightarrow{f_n} \{\text{proper irreducible parabolic subgroups of } A_{A_n}\}$$

which induces a graph isomorphism $\mathcal{G}(D_{n+1}) \to \mathcal{C}_{parab}(A_n)$. To a curve $C$ in $D_{n+1}$, we associate the subgroup $f_n(C)$ of $A_{A_n}$ consisting of all isotopy classes of homeomorphisms of $D_{n+1}$.
whose support is enclosed by $C_i$; this is a proper irreducible parabolic subgroup. In particular, given a proper subinterval $I$ of $[n]$, we have $f_n(C_I) = A_I$. The inverse correspondence is given by the — well defined — formula $A_I^y \mapsto C_{I^y}$, for any proper subinterval $I$ of $[n]$ and any $y \in A_{A_n}$.

Let us see that the adjacency condition given in Proposition 2.4 turns $A_0$ into a graph isomorphism. Let $C$ be a curve in $\mathbb{D}_{n+1}$, let $P = f_n(C)$ and let $z_P$ be the central element of $P$. If $C$ surrounds at least three punctures, then $z_P$ is the Dehn twist around the curve $C$. Otherwise, $z_P$ is the half-Dehn twist along an arc connecting the two punctures in the interior of $C$ and which does not intersect $C$. Now, given two parabolic subgroups $P_1 = f_n(C_1)$ and $P_2 = f_n(C_2)$, $z_{P_1}$ and $z_{P_2}$ commute if and only if $C_1$ and $C_2$ are disjoint.

Before going on, we introduce a set of special braids which will play an important role in the paper. Let $n \geq 3$; let $p, q, r$ be positive integers with $1 \leq p \leq q$ and $q + 1 \leq r \leq n + 1$. We define

$$\xi_{p,q,r} = \Pi_{i=q}^{r-1} \sigma_i \cdots \sigma_{i-(q-p)}.$$ 

In this positive braid, the strands numbered $p, \ldots, q$ end at positions $p + r - q, \ldots, r$ without crossings between them and the strands numbered $q + 1, \ldots, r$ end at positions $p, \ldots, p + r - q - 1$ without crossings between them. As an example, Figure 2(i) shows $\xi_{3,4,8} \in A_{A_9}$.

For $1 \leq i \leq n$, define also $a_i = \sigma_i \cdots \sigma_1$, $b_i = \sigma_i \cdots \sigma_n$, and $a_0 = b_{n+1}$ is the trivial braid. Equivalently, $a_i = \xi_{1,i,i+1}$ and $b_i = \xi_{i,i,n+1}$. For $y \in A_{A_n}$, we denote by $\pi_y$ the permutation in $S_{n+1}$ associated to $y$. As usual when composing braids, we use the otherwise unusual convention that permutations are composed from left to right. Notice that $\pi_{a_i}(i + 1) = 1$, for all $0 \leq i \leq n$ and $\pi_{b_i}(i) = n + 1$, for all $1 \leq i \leq n + 1$.

**Lemma 2.5.** Let $I$ be a proper subinterval of $[n]$, $m_I = \min(I)$ and $k_I = \#I$, so that the circle $C_I$ in $\mathbb{D}_{n+1}$ surrounds the punctures $m_I$ to $m_I + k_I$. Let $0 \leq i_0 \leq n$.

(i) If $i_0 + 1 < m_I$, that is, if the puncture $i_0 + 1$ is to the left of $C_I$, then $C_I^{a_{i_0}} = C_I$.

(ii) If $i_0 + 1 > m_I + k_I$, that is, if the puncture $i_0 + 1$ is to the right of $C_I$, then $C_I^{a_{i_0}} = C_{I'}$, where $I' = \{i + 1 \mid i \in I\}$.

![Figure 2](attachment:image.png)

**Figure 2.** (i) The braid $\xi_{3,4,8} \in A_{A_9}$. (ii) Lemma 2.5(i)-(ii): if the puncture $i_0 + 1$ is not surrounded by the curve $C_I$, then $C_I^{a_{i_0}}$ is standard again. (iii) Lemma 2.5(iii): the puncture $i_0 + 1$ is surrounded by $C_I$ (and $m_I > 1$). The curve $C_I$ is preserved by the action of the first crossings of $a_{i_0}$; the action of the braid $\xi_1 = \xi_{2,m_I,m_I+k_I}$ standardizes $C_I^{a_{i_0}}$. (iv) Lemma 2.6(iii): the puncture $i_0$ is surrounded by $C_I$ (and $m_I + k_I < n + 1$). The action of the braid $\xi_I = \xi_{m_I,m_I+k_I-1,n}$ standardizes the curve $C_I^{a_{i_0}}$. 
(iii) If \( m_I \leq i_0 + 1 \leq m_I + k_I \), that is, if the puncture \( i_0 + 1 \) is in the interior of \( \mathcal{C}_I \), then \( \mathcal{C}_I^{i_0} = \mathcal{C}_I^{m_I + 1} \) is not standard (except if \( m_I = 1 \)) and \( \mathcal{C}_I^{m_I} = \mathcal{C}_I^{m_I + k_I} = \mathcal{C}_{[1,k_I]} \). We will write \( \xi_I = \xi_{2,m_I,m_I + k_I} \) (it does not depend on \( i_0 \)).

Proof. The contents of Lemma 2.5 are depicted in Figure 2(ii)–(iii). Only the third case might need a short proof: it suffices to observe that the crossings \( \sigma_{i_0}, \ldots, \sigma_{m_I} \) fix the curve \( \mathcal{C}_I \) as they are inner to it, so it only remains the action of \( \sigma_{m_I - 1} \cdots \sigma_1 = a_{m_I - 1} \).

Lemma 2.6. Let \( I \) be a proper subinterval of \([n]\), \( m_I = \min(I) \) and \( k_I = \#I \), so that the circle \( \mathcal{C}_I \) in \( \mathbb{D}_{n+1} \) surrounds the punctures \( m_I \) to \( m_I + k_I \). Let \( 1 \leq j_0 \leq n + 1 \).

(i) If \( j_0 < m_I \), that is, if the puncture \( j_0 \) is to the left of \( \mathcal{C}_I \), then \( \mathcal{C}_I^{j_0} = \mathcal{C}_{I'} \), where \( I' = \{ i - 1 \mid i \in I \} \).

(ii) If \( j_0 > m_I + k_I \), that is, if the puncture \( j_0 \) is to the right of \( \mathcal{C}_I \), then \( \mathcal{C}_I^{j_0} = \mathcal{C}_I \).

(iii) If \( m_I \leq j_0 \leq m_I + k_I \), that is, if the puncture \( j_0 \) is in the interior of \( \mathcal{C}_I \), then \( \mathcal{C}_I^{j_0} = \mathcal{C}_I^{m_I + k_I} \) is not standard (except if \( m_I + k_I = n + 1 \)) and \( \mathcal{C}_I^{j_0} = \mathcal{C}_{[n-k_I+1,n]} \). We will write \( \xi'_I = \xi_{m_I,m_I + k_I,n} \) (it does not depend on \( j_0 \)).

Proof. Similar to Lemma 2.5. An example of (iii) is depicted in Figure 2(iv).

Remark 2.7. We note that \( \xi_I \) has its first strand straight and \( \xi'_I \) has its last strand straight.

2.3. Hyperbolicity for some graphs of curves

In this section, we present a specialization of a theorem of Kate Vokes, which we will use as a criterion for proving the hyperbolicity of some subgraphs of the curve graph of the punctured disk. Consider the \( n \)-times punctured disk \( \mathbb{D}_n \). A subsurface of \( \mathbb{D}_n \) is (the isotopy class of) a connected subsurface \( X \) of \( \mathbb{D}_n \) so that every boundary component of \( X \) is either \( \partial \mathbb{D}_n \) or an essential curve in \( \mathbb{D}_n \). A simple closed curve in \( X \) is essential (in \( X \)) if it cannot be isotoped in \( X \) to a point, a puncture or a boundary component of \( X \). By an annulus in \( \mathbb{D}_n \) we mean the subsurface consisting of a tubular neighborhood of some essential curve in \( \mathbb{D}_n \). Given a curve \( \mathcal{C} \) in \( \mathbb{D}_n \) and a subsurface \( X \) of \( \mathbb{D}_n \), we say that \( \mathcal{C} \) and \( X \) are disjoint if they admit disjoint representatives; \( X \) is said to be a witness for \( \mathcal{C} \) if \( \mathcal{C} \) and \( X \) are not disjoint. In particular, \( X \) is not a witness for any of its boundary components. Two subsurfaces are disjoint if they admit disjoint representatives.

Theorem 2.8 [30, Corollary 1.5]. Let \( \hat{\mathcal{C}} \) be a family of curves in \( \mathbb{D}_n \); let \( \mathcal{K} \) be the subgraph of \( \mathcal{CG}(\mathbb{D}_n) \) induced by \( \hat{\mathcal{C}} \), equipped with the combinatorial metric \( d_\mathcal{C} \) (each edge has length one). Let \( \mathcal{X} \) be the set of witnesses for \( \mathcal{K} \), which is the set of all subsurfaces of \( \mathbb{D}_n \) which are a witness for every element of \( \hat{\mathcal{C}} \). Suppose that

(i) \( \mathcal{K} \) is connected,

(ii) the action of \( PA_{n-1} \) on \( \mathbb{D}_n \) induces an isometric action of \( PA_{A_{n-1}} \) on \( \mathcal{K} \),

(iii) \( \mathcal{X} \) contains no annulus,

(iv) no two elements of \( \mathcal{X} \) are disjoint.

Then \( \mathcal{K} \) is hyperbolic.

Remark 2.9. Let us check that the hypothesis of Theorem 2.8 matches the hypothesis of [30, Corollary 1.5], namely that \( \mathcal{K} \) is a twist-free multicurve graph having no pair of disjoint witnesses. The second half is exactly our clause (iv). The definition of a twist-free multicurve
that each vertex of $K$ graph ([30, Definition 2.1]) consists of clauses (1)–(5). To see clauses (2) and (4), observe that each vertex of $K$ being a curve in $\mathbb{D}_n$ is in particular a multicurve and that two adjacent vertices in $K$ are disjoint curves. Clauses (1) and (5) correspond to (i) and (iii) of Theorem 2.8, respectively. Clause (3) is adapted into clause (ii) of Theorem 2.8. A priori, the results in [30] work for compact surfaces (possibly with boundary). However, according to [22, Section 2.3], punctures can be treated as boundary components, so the results of [30] apply to punctured surfaces as well. In the case of the punctured disk, we have to replace the whole braid group by the pure braid group since mapping classes in $S$ are required to fix the boundary pointwise. In the sequel we will find it more convenient to maintain the difference between punctures of the disk and “real” boundaries, thinking of punctures as “distinguished boundary components.”

3. The Graph $\mathcal{C}_{\text{parab}}(B_n)$

A proper irreducible standard parabolic subgroup of $A_{B_n}$ is determined by a proper subinterval of $[n]$: for any proper subinterval $I$ of $[n]$, we denote by $B_I$ the proper irreducible standard parabolic subgroup of $A_{B_n}$, generated by $\{\tau_i \mid i \in I\}$.

There is a monomorphism

$$\eta_n : A_{B_n} \to A_{A_n},$$

$$\tau_i \mapsto \begin{cases} \sigma_i^2 & \text{if } i = 1, \\ \sigma_i & \text{if } 2 \leq i \leq n. \end{cases}$$

The image of $\eta_n$ is the subgroup $\mathfrak{P}_1$ of $(n + 1)$-strands $1$-pure braids, that is, the subgroup of all $(n + 1)$ strands braids in which the first strand ends in the first position. In other words, a braid $y$ on $(n + 1)$ strands belongs to $\mathfrak{P}_1$ if and only if $\pi_y(1) = 1$, where $\pi_y = \pi(y)$ is the permutation in $S_{n+1} = W_{A_n}$ associated to $y$. A presentation for $\mathfrak{P}_1$ was given by Wei-Liang Chow [12] in 1948; for a proof that $\eta_n$ defines an isomorphism between $A_{B_n}$ and $\mathfrak{P}_1$, the reader may consult [19].

For the rest of this section, given a proper subinterval $I$ of $[n]$, we shall denote $m_I = \min(I)$ and $k_I = \#I$. The central elements $z_{A_I}$ of $A_I$ and $z_{B_I}$ of $B_I$ are given by the following formulae (see [26, Lemmas 3.1 and 4.1]):

$$z_{A_I} = \begin{cases} \sigma_{m_I}^2 & \text{if } k_I = 1, \\ \prod_{i=1}^{m_I-k_I} (\sigma_{m_I} \cdot \cdots \cdot \sigma_{m_I+k_I-1}) & \text{if } k_I \geq 2, \end{cases}$$

$$z_{B_I} = \begin{cases} \tau_{m_I} & \text{if } k_I = 1, \\ \prod_{i=1}^{m_I-k_I} (\tau_{m_I} \cdot \cdots \cdot \tau_{m_I+k_I-1}) & \text{if } k_I \geq 2 \text{ and } 1 \not\in I, \\ \tau_{m_I} & \text{if } k_I \geq 2 \text{ and } 1 \in I. \end{cases}$$

The proof of the next lemma follows from an easy computation left to the reader.

Lemma 3.1. Let $I$ be a proper subinterval of $[n]$. We have

(i) $\eta_n(B_I) = A_I \cap \mathfrak{P}_1$,

(ii) $\eta_n(z_{B_I}) = z_{A_I}$, except if $I = \{1\}$, in which case $\eta_n(z_{B_I}) = z_{A_I}^2$.

Proposition 3.2. Let $I, J$ be proper subintervals of $[n]$ and let $g \in A_{B_n}$. The following are equivalent:

(i) $B_I^g = B_J$,

(ii) $C_{\eta_n(g)} = C_J$.

Proof. Note that using the isomorphism $f_n$ from Section 2.2, (ii) is equivalent to $A_{B_I}^{\eta_n(g)} = A_J$. 

Assume (i). According to Proposition 2.3, we have \( z_B^g = z_{B_J} \). Assume first that \( I = \{ 1 \} \) (hence \( J = \{ 1 \} \), by Lemmas 2.1 and 2.2) so that \( \tau_I^g = \pi_1 \), which yields \( (\sigma_I^g)^{\eta}(g) = \sigma_I^g \) after applying the monomorphism \( \eta \). Then [18, Theorem 2.2] ensures that also \( (\sigma_I^g)^{\eta}(g) = \sigma_I^g \) whenever \( A_I^J(g) = A_J \). If on the contrary \( I, J \neq \{ 1 \} \), Lemma 3.1(ii) yields that \( z_{A_I^J}(g) = z_{A_J} \) from which \( A_I^J(g) = A_J \) follows using Proposition 2.3.

Conversely, assume (ii). By Proposition 2.3, we get \( z_{A_I^J}(g) = z_{A_J} \). By Lemma 2.1, again \( I = \{ 1 \} \) if and only if \( J = \{ 1 \} \), as \( \eta \) is 1-pure. In this case, it follows that \( (\sigma_I^g)^{\eta}(g) = \sigma_I^g \); as \( \eta \) is injective, we get \( \tau_I^g = \pi_1 \), which is to say \( B_I^g = B_J \). If \( I \neq \{ 1 \} \), from the relation \( z_{A_I}(g) = \), and using Lemma 3.1(ii) we get \( z_{B_I}^g = z_{B_J} \), which, by Proposition 2.3, implies (i).

**Proposition 3.3.** Let \( I, J \) be proper subintervals of \([n] \) and let \( g \in A_{B_n} \). The following are equivalent:

1. \( B_I^g \) and \( B_J \) are adjacent in \( C_{parab}(B_n) \).
2. \( C_{I}^{\eta}(g) \) and \( C_J \) are adjacent in \( \mathcal{CG}(\mathbb{D}_{n+1}) \).

**Proof.** By Proposition 2.4, (i) is equivalent to saying that \( z_{B_I}^g \) and \( z_{B_J} \) commute; by Lemma 3.1(ii) (and injectivity of \( \eta \)), this is equivalent to \( z_{A_I}^g \) and \( z_A \) commuting (notice that \( \sigma^g_I \) and \( \sigma_A \) have the same centralizer in \( A_n \), by [18, Theorem 2.2]). Using Proposition 2.4 again, this is equivalent in turn to \( A_I^J(g) \) and \( A_J \) being adjacent in \( C_{parab}(A_n) \). Using the isomorphism \( f_n^{-1} \) from Section 2.2, this is also equivalent to (ii).

Our next goal is to show that each curve \( C \) in \( \mathbb{D}_{n+1} \) can be written \( C = C_{\eta}^I(g) \) for some proper subinterval \( I \) of \([n] \) and some \( g \in A_{B_n} \). Observe that \( \mathfrak{P}_1 \) has index \((n + 1)\) in \( A_{A_n} \). The braids \( a_i \) introduced in Section 2.2 enumerate the cosets of \( \mathfrak{P}_1 \); given \( y \in A_{A_n} \), there is a unique \( i \in \{ 0, \ldots, n \} \) so that \( ya_i \in \mathfrak{P}_1 \).

**Proposition 3.4.** Let \( C \) be a curve in \( \mathbb{D}_{n+1} \). There exists \( \alpha \in \mathfrak{P}_1 \) such that \( C^\alpha \) is standard.

**Proof.** Let \( \zeta \) be any braid such that \( C^\zeta \) is a round curve, say \( C_J \), and write \( m = \min(J) \) and \( k = \# J \). Let \( i_0 = \pi_{\zeta}(1) - 1 \), in such a way that \( \zeta a_i \) is 1-pure. We use Lemma 2.5. If \( i_0 + 1 < m \) or \( i_0 + 1 > m + k \), then \( C_{\zeta a_i} = C_{\zeta}^J \) is standard and we can take \( \alpha = \zeta a_i \). Otherwise, suppose that \( m \leq i_0 + 1 \leq m + k \). If \( C_{\zeta a_i} = C_{\zeta}^J \) is not standard, we have \( m > 1 \) and \( C_{\zeta a_i} \xi_J = (C_J^{m-1}) \xi_J = C_{[1,k]} \), so we can take \( \alpha = \zeta a_i \) and \( \xi_J \) is 1-pure as \( \zeta a_i \) and \( \xi_J \) are 1-pure (Remark 2.7).

Now, the following achieves the proof of Theorem 1.2.

**Corollary 3.5.** The assignment \( B_I^g \mapsto C_{I}^{\eta}(g) \), where \( I \) is a proper subinterval of \([n] \) and \( g \in A_{B_n} \), defines a graph isomorphism \( f_n \) from \( C_{parab}(B_n) \) to \( \mathcal{CG}(\mathbb{D}_{n+1}) \).

**Proof.** By Proposition 3.2, \( f_n \) is a well-defined injective map. By Proposition 3.4, \( f_n \) is surjective. Moreover, by Proposition 3.3, both \( f_n \) and its inverse are graph homomorphisms.

4. The Graph \( C_{parab}(\tilde{A}_n) \)

Let us start with a description of the proper irreducible standard parabolic subgroups of \( A_{A_n} \).

We say that a proper subset \( I \) of \( \{ 0, \ldots, n \} \) is a proper cyclic subinterval if \( I \) is either a proper
subinterval of \(\{0,\ldots,n\}\) or the union of two proper subintervals of \(\{0,\ldots,n\}\) of the form \([l,n]\) and \([0,k]\), for some \(k, l\) with \(1 \leq k + 1 < l \leq n\). A proper irreducible standard parabolic subgroup of \(\tilde{A}_n\) is determined by a proper cyclic subinterval of \(\{0,\ldots,n\}\): for any proper cyclic subinterval \(I\) of \(\{0,\ldots,n\}\), we denote by \(\tilde{A}_I\) the proper irreducible standard parabolic subgroup of \(\tilde{A}_n\) generated by \(\{\tilde{\sigma}_i \mid i \in I\}\).

According to [19], there is a monomorphism

\[
\theta_n : A_{\tilde{\Delta}_n} \rightarrow A_{B_{n+1}}
\]

\[
\tilde{\sigma}_i \mapsto \begin{cases} 
\tau_{i+1} & \text{if } i \geq 1, \\
\tau_{n+1}^{-1} \cdots \tau_3^{-1} \tau_1^{-1} \tau_2^{-1} \tau_3 \cdots \tau_{n+1} & \text{if } i = 0.
\end{cases}
\]

**Proposition 4.1** [19]. Let \(\rho = (\tau_1 \tau_2 \cdots \tau_{n+1})^{-1} \in A_{B_{n+1}}\).

(i) \(\rho^{-(n+1)}\) is the central element of \(A_{B_{n+1}}\).

(ii) \(\theta_n(\tau_i) = \theta_n(\tilde{\sigma}_i)^{\rho}\) for \(0 \leq i \leq n - 1\) and \(\theta_n(\tilde{\sigma}_n)^{\rho} = \theta_n(\tilde{\sigma}_0)\).

(iii) The group \(A_{B_{n+1}}\) can be decomposed as the semi-direct product \(A_{B_{n+1}} = \theta_n(\tilde{A}_n) \rtimes \langle \rho \rangle\), where the action of \(\rho\) is given by conjugation, as in (ii).

**Proposition 4.2.** If \(P\) is a proper irreducible parabolic subgroup of \(\tilde{A}_n\), then \(\theta_n(P)\) is a proper irreducible parabolic subgroup of \(A_{B_{n+1}}\).

**Proof.** It is enough to check the result for standard parabolic subgroups. So, suppose that \(P = \tilde{A}_I\) for some proper cyclic subinterval of \(\{0,\ldots,n\}\). If \(I\) is a subinterval of \(\{0,\ldots,n\}\) which does not contain 0, then \(\theta_n(P)\) is the subgroup of \(A_{B_{n+1}}\) generated by \(\{\tau_{i+1} \mid i \in I\}\), which is a proper irreducible standard parabolic subgroup. If \(I\) is a subinterval of \(\{0,\ldots,n\}\) which contains 0, then in view of Proposition 4.1(iii), \(\theta_n(\tilde{A}_I)^{\rho} = (\tau_{i+2} : i \in I)\), whence \(\rho\) conjugates \(\theta_n(\tilde{A}_I)\) to a proper irreducible standard parabolic subgroup of \(A_{B_{n+1}}\). Similarly, if \(I\) is of the form \([l,n] \cup [0,k]\), with \(1 \leq k + 1 < l \leq n\), then \(\theta_n(\tilde{A}_I)^{\rho^{n-l+2}} = (\tau_2, \ldots, \tau_{k+n-l+3})\), whence \(\theta_n(\tilde{A}_I)\) is again conjugate to a proper irreducible standard parabolic subgroup of \(A_{B_{n+1}}\).

**Proposition 4.3.** Let \(P, Q\) be proper irreducible parabolic subgroups of \(\tilde{A}_n\), the following are equivalent:

(i) \(P\) and \(Q\) are adjacent in \(C_{\text{parab}}(\tilde{A}_n)\),

(ii) \(\theta_n(P)\) and \(\theta_n(Q)\) are adjacent in \(C_{\text{parab}}(B_{n+1})\).

**Proof.** Recall that the adjacency is defined in Definition 1.1. The equivalence follows easily in view of the injectivity of \(\theta_n\).

Following the scheme of Section 3, we now want to characterize those parabolic subgroups of \(A_{B_{n+1}}\) which are of the form \(\theta_n(P)\) for some proper irreducible parabolic subgroup \(P\) of \(A_{\tilde{\Delta}_n}\). Recall the graph isomorphism \(\theta_{n+1} : C_{\text{parab}}(B_{n+1}) \rightarrow CG(\mathbb{D}_{n+2})\) — see Corollary 3.5. We will say that a proper irreducible parabolic subgroup \(P\) of \(A_{B_{n+1}}\) is a braid subgroup if there is some \(I \subset \{2, \ldots, n+1\}\) so that \(P\) is conjugate to \(B_I\). Notice that the non-cyclic braid subgroups are exactly the non-cyclic proper irreducible parabolic subgroups which are isomorphic to an Artin–Tits group \(A_k, k \geq 2\), hence the name.

**Proposition 4.4.** Let \(Q\) be a proper irreducible parabolic subgroup of \(A_{B_{n+1}}\), the following are equivalent.
(i) There exists a proper irreducible parabolic subgroup $P$ of $A_{\tilde{A}_n}$ such that $Q = \theta_n(P)$.
(ii) $Q$ is a braid subgroup of $A_{B_{n+1}}$.
(iii) The curve $\mathcal{C} = \mathcal{S}_{n+1}(Q)$ does not surround the first puncture of $\mathbb{D}_{n+2}$.

Proof. (i) $\implies$ (ii). Let $P$ be a proper irreducible parabolic subgroup of $A_{\tilde{A}_n}$ with the claimed property. Suppose first that $P$ is cyclic, conjugate to some $\langle \sigma_1 \rangle$; then by the formulae defining $\theta_n$, $\theta_n(P)$ is conjugate to $\langle \sigma_2 \rangle$ hence is a cyclic braid subgroup. Suppose then that $P$ is not cyclic, that is, $P$ is isomorphic to a braid group of type $A_k, k \geq 2$. Note that $\theta_n(P)$ is isomorphic to $P$; by Proposition 4.2, it is an irreducible parabolic subgroup; hence it is a braid subgroup.

(ii) $\iff$ (iii). We have the following chain of equivalences. $Q$ is a braid subgroup of $A_{B_{n+1}}$ $\iff Q = B_I^r$ for some $x \in A_{B_{n+1}}$ and some $I \subseteq \{2, \ldots, n + 1\} \iff \mathcal{S}_{n+1}(Q) = C_I^{y_{n+1}(x)}$ for some $x \in A_{B_{n+1}}$ and some $I \subseteq \{2, \ldots, n + 1\} \iff \mathcal{C} = \mathcal{S}_{n+1}(Q) = C_I^r$, for some $y$ 1-pure and some $I \subseteq \{2, \ldots, n + 1\} \iff \mathcal{C} = \mathcal{S}_{n+1}(Q)$ does not surround the first puncture. The right-to-left direction of the latter equivalence uses Proposition 3.4 which allows to standardize any curve through a 1-pure braid.

(iii) $\implies$ (i). We need to show that every braid subgroup $Q$ satisfies that $Q = \theta_n(P)$ for some proper irreducible parabolic subgroup $P$ of $A_{\tilde{A}_n}$. Assume firstly that $Q$ is standard; that is, $Q = B_I$, for $I \subseteq \{2, \ldots, n + 1\}$. Setting $I' = \{i - 1 \mid i \in I\}$, we see that $Q = \theta_n(\tilde{A}_I')$, as desired. If $Q$ is not standard, let $\zeta \in A_{B_{n+1}}$ be such that $Q^\zeta = B_I$ is standard, for $I \subseteq \{2, \ldots, n + 1\}$ and let $I' = \{i - 1 \mid i \in I\}$ so that $Q^\zeta = B_I = \theta_n(\tilde{A}_I')$. If $\zeta = \theta_n(x)$ for some $x \in A_{\tilde{A}_n}$ we are done as $Q = \theta_n((\tilde{A}_I')^{\theta_n(x)^{-1}} = \theta_n(\tilde{A}_I^{-1})$ is the image of a proper irreducible parabolic subgroup of $A_{\tilde{A}_n}$, because the contrary $\zeta$ is not in the image of $\theta_n$, there is some $r \in \mathbb{Z}$ and $x \in A_{\tilde{A}_n}$ such that $\zeta^\rho = \theta_n(x)$ — see Proposition 4.1(iii). Using Proposition 4.1(ii), we have $Q^{\zeta^\rho} = (Q^\zeta)^{\rho} = (\theta_n(\tilde{A}_I'))^{\rho} = \theta_n(\tilde{A}_{\{i + r \mid i \in I'\}})$, where in the last term, all indices are taken modulo $(n + 1)$. But this is equivalent to saying that $Q = \theta_n(\tilde{A}_{\{i + r \mid i \in I'\}})$, which achieves the proof. 

Notation 4.5. Let $K_A$ be the subgraph of $CG(\mathbb{D}_{n+2})$ induced by the curves which do not surround the first puncture of $\mathbb{D}_{n+2}$.

Corollary 4.6. The assignment $P \mapsto \mathcal{S}_{n+1}(\theta_n(P))$, where $P$ is a proper irreducible parabolic subgroup of $A_{\tilde{A}_n}$, defines a graph isomorphism $\Theta_n$ from $C_{parab}(\tilde{A}_n)$ to $K_A$. In particular, we have $d_{\mathbb{D}_{n+2}}(\Theta_n(P), \Theta_n(P')) \leq d_{\tilde{A}_n}(P, P')$ for all proper irreducible parabolic subgroups $P$ and $P'$ of $A_{\tilde{A}_n}$.

Proof. Recall that $\mathcal{S}_{n+1}$ is the isomorphism from Corollary 3.5. Because $\theta_n$ is injective and according to Proposition 4.2, the formula $P \mapsto \mathcal{S}_{n+1}(\theta_n(P))$ defines an injective map $\Theta_n$ from the set of vertices of $C_{parab}(\tilde{A}_n)$ to the set of curves in $\mathbb{D}_{n+2}$. By Proposition 4.4, the image of this map is the set of vertices of $K_A$. By Proposition 4.3, both $\Theta_n$ and its inverse are graph homomorphisms.

Proposition 4.7. The subgraph $K_A$ is 1-dense in $CG(\mathbb{D}_{n+2})$; as a consequence, the graph $C_{parab}(\tilde{A}_n)$ has infinite diameter.

Proof. Let $\mathcal{C}$ be any curve in $\mathbb{D}_{n+2}$; we may suppose that $\mathcal{C}$ surrounds the first puncture. We need to find a curve $\mathcal{C}'$ in $K_A$ such that $\mathcal{C}'$ and $\mathcal{C}$ are disjoint. Suppose first that $\mathcal{C}$ is standard; if it surrounds only the first two punctures — that is, $\mathcal{C} = \mathcal{C}_{\{1\}}$ — we can take $\mathcal{C}' = \mathcal{C}_{\{3\}}$; otherwise, we can take $\mathcal{C}' = \mathcal{C}_{\{2\}}$. If $\mathcal{C}$ is not standard, by Proposition 3.4, there is a 1-pure
braid $\alpha$ so that $C^\alpha$ is standard — and still surrounding the first puncture. By the above line of argument, there is $C''$ in $K_A$ disjoint from $C^\alpha$ and it suffices to take $C' = C''\alpha^{-1}$.

Let us show the second part of the statement. Let $P$ be any vertex of $C_{parab}(\tilde{A}_n)$ and let $M > 0$. We shall find a vertex $P'$ of $C_{parab}(\tilde{A}_n)$ so that $d_{\tilde{A}_n}(P, P') > M$. We know that $CG(D_{n+2})$ has infinite diameter [21, Proposition 4.6]; in particular, there exists a curve $C$ in $D_{n+2}$ so that $d_{D_{n+2}}(\Theta_n(P), C) > M + 1$. As we have just seen, there is a curve $C'$ from $K_A$ such that $d_{D_{n+2}}(C, C') \leq 1$. Let $P'$ be the proper irreducible parabolic subgroup of $A_{\tilde{A}_n}$ such that $\Theta_n(P') = C'$. We deduce by Corollary 4.6 that $d_{\tilde{A}_n}(P, P') > M$.

It would be conceivable that the subgraph $K_A$ is quasi-isometrically embedded in $CG(D_{n+2})$, which would imply the hyperbolicity of $C_{parab}(\tilde{A}_n)$. However, this is not the case, as we will now see. We firstly show that $K_A$ matches hypotheses (i)–(iii) of Theorem 2.8.

**Lemma 4.8.** (i) $K_A$ is connected. (ii) The action of the pure braid group $PA_{A_{n+1}}$ on $D_{n+2}$ induces an isometric action on $K_A$. (iii) No annulus in $D_{n+2}$ can be a witness for all vertices of $K_A$.

**Proof.** (i) This follows from Lemma 1.4 and Corollary 4.6. (ii) The restriction to the pure braids of the natural action of $A_{A_{n+1}}$ on $D_{n+2}$ provides a simplicial action of $PA_{A_{n+1}}$ on $K_A$. (iii) Given an essential curve $C$ in $D_{n+2}$, we will see that there always exists some curve $c$ in $K_A$ which is disjoint from the annulus determined by $C$. Assume first that $C$ does not surround the first puncture, so that $C$ is a curve in $K_A$; then $C$ itself can be isotoped so that it does not intersect the annulus it determines. Suppose on the contrary that $C$ surrounds the first puncture; if the exterior of $C$ contains at least 2 punctures, we can take $c$ to be any curve in the exterior of $C$. Otherwise the interior of $C$ contains $n + 1 \geq 3$ punctures and we can choose $c$ to be any curve surrounded by $C$ and not enclosing the first puncture.

As the next proposition is not needed in the sequel, its proof is only sketched and we refer the reader to [30] for a precise statement of the results used throughout. We denote by $d_{K_A}$ the distance in the graph $K_A$.

**Proposition 4.9.** The subgraph $K_A$ is not quasi-isometrically embedded in $CG(D_{n+2})$. More precisely, given any $M > 0$, there exists a pair of curves $a, b$ in $D_{n+2}$ not surrounding the first puncture with the following properties:

- $a, b$ are disjoint from the circle $C_{(1)}$, so that $d_{D_{n+2}}(a, b) \leq 2$,
- $d_{K_A}(a, b) > M$.

**Proof.** In view of Lemma 4.8 (see also Remark 2.9) and according to [30, Corollary 1.2], we have a distance formula in $K_A$ from which we can deduce the claim. Fix $M > 0$. Fix any curve $a$ not surrounding the first puncture and disjoint from the circle $C_{(1)}$. Let $D$ be the subdisk in $D_{n+2}$ enclosed by $C_{(1)}$ and let $X = D_{n+2} \setminus D$. Note that the subsurface $X$ is homeomorphic to a disk with $(n + 1)$ punctures and is a witness for every vertex of $K_A$. Note also that $a$ is a curve in $X$. Let $C_0$ be the constant associated to $K_A$ by [30, Corollary 1.2], let $C > C_0$ and let $K_1 = K_1(C)$, $K_2 = K_2(C)$ as given by [30, Corollary 1.2]. Define an element $f$ of $A_{A_{n+1}}$ by choosing a pseudo-Anosov mapping class of $X$ which fixes each puncture of $X$ (a pseudo-Anosov pure braid on $(n + 1)$ strands) and doubling its first strand. Then $f$ acts loxodromically on the curve graph of $X$ [21, Proposition 4.6] and, choosing $b$ as the image of $a$ under a sufficiently high power of $f$, we can arrange that the distance $d_X(a, b)$ (in the curve graph of $X$) is bigger than $\max\{C, MK_1 + K_2\}$. Notice that $b$ is disjoint from $C_{(1)}$ and that $b$ does not surround the first puncture in $D_{n+2}$. The distance formula [30, Corollary 1.2] then says in particular
that $K_1.d_{K_A}(a, b) + K_2$ is bounded from below by a sum of positive terms to which $d_X(a, b)$ contributes. It follows in particular that $d_{K_A}(a, b) \geq \frac{d_X(a, b) - K_2}{K_1} > M$, as desired. This finishes the proof of Proposition 4.9.

We are now ready for proving the hyperbolicity of $C_{parab}(\tilde{A}_n)$. As $C_{parab}(\tilde{A}_n)$ is isomorphic to $K_A$, it suffices to prove that $K_A$ is hyperbolic. This will follow from Theorem 2.8 after we check the remaining hypothesis (iv). The next lemma describes all possible witnesses for $K_A$; its proof will achieve the demonstration. Throughout, $p_1$ denotes the first puncture of $D_{n+2}$.

Lemma 4.10. Let $X$ be a subsurface of $D_{n+2}$. Then $X$ is a witness for $K_A$ if and only if one of the following holds.

(i) $X = D_{n+2}$ or $X = D_{n+2} \setminus D$, where $D$ is the interior of an essential curve surrounding $p_1$ and exactly one other puncture.

(ii) $X$ is the interior of an essential curve surrounding $p_1$ and $n$ other punctures.

(iii) $X = X' \setminus D$, where $X'$ is the interior of an essential curve surrounding $p_1$ and $n$ other punctures and $D$ is the interior of an essential curve surrounding $p_1$ and exactly one other puncture.

We will say that $X$ is a witness of type (i), (ii) or (iii). Two witnesses for $K_A$ are never disjoint.

Proof. The three types of subsurfaces in Lemma 4.10 are depicted in Figure 3. Firstly, we check that all subsurfaces (i)–(iii) are witnesses for $K_A$: we see that the only curves which can fail to be witnessed by $X$ must surround the first puncture. Conversely, let $X$ be a witness for $K_A$. We shall distinguish two cases.

First case. Suppose that $\partial D_{n+2}$ is a boundary component of $X$. Assume that $X \neq D_{n+2}$. Therefore, there is at least some essential curve $C$ of $D_{n+2}$ which is a boundary component of $X$. Assume that $C'$ is another essential curve of $D_{n+2}$ which is a boundary component of $X$. Notice that $C$ and $C'$ cannot be nested as $X$ has to be connected. Then at least one of $C$ or $C'$ does not surround $p_1$ and this provides a particular curve of $K_A$ for which $X$ is not a witness, a contradiction. Therefore, $X$ has exactly one essential curve $C$ of $D_{n+2}$ as a boundary component and $C$ must surround $p_1$. Moreover, $C$ must surround exactly two punctures; otherwise, there would exist a curve $c$ in the interior of $C$, not surrounding $p_1$, and $X$ would fail to be a witness for this curve $c$. Letting $D$ be the interior of $C$, we have shown that whenever $X$ has $\partial D_{n+2}$ as a boundary component, $X = D_{n+2} \setminus D$ has to be of type (i).

Second case. Suppose that the boundary $\partial D_{n+2}$ is not a boundary component of $X$. As $X$ is connected, $X$ has exactly one outermost boundary component which is an essential curve $C$ of $D_{n+2}$. Again, $C$ must surround $p_1$; otherwise, $C$ is a curve from $K_A$ which is disjoint from $X$ and $X$ fails to be a witness for $K_A$. Moreover, $C$ must surround $(n + 1)$ punctures, otherwise

\[ Figure 3. Shaded, the different types of witnesses for $K'$. The big dot represents $p_1$. \]
the exterior of \( \mathcal{C} \) would contain at least two punctures and there would exist a curve \( c \) from \( \mathcal{K}_A \) entirely contained in \( \mathbb{D}_{n+2} \setminus X \), contradicting that \( X \) is a witness for \( \mathcal{K}_A \). If \( X \) has no other boundary component, we have shown that \( X \) is of type (ii).

Finally, suppose that \( X \) has another boundary component. This must be an essential curve \( \mathcal{C}' \) of \( \mathbb{D}_{n+2} \) which is nested in \( \mathcal{C} \). Let \( \mathcal{C}'' \) be another putative boundary component of \( X \) nested in \( \mathcal{C} \). Then \( \mathcal{C}' \) and \( \mathcal{C}'' \) cannot be nested, as \( X \) is connected. Therefore, only one of \( \mathcal{C}', \mathcal{C}'' \) can surround \( p_1 \): one of \( \mathcal{C}', \mathcal{C}'' \) is a curve from \( \mathcal{K}_A \) for which \( X \) is not a witness, contradiction. Therefore, there is exactly one boundary component \( \mathcal{C}' \) of \( X \) nested in \( \mathcal{C} \) and \( \mathcal{C}' \) must surround exactly two punctures, for the same reasons as in the first case. Taking \( X' \) to be the interior of \( \mathcal{C} \) and \( D \) to be the interior of \( \mathcal{C}' \), we have shown that \( X = X' \setminus D \) is of type (iii).

Finally, the last claim follows from a direct case-by-case inspection. \( \square \)

The proof of the statements of Theorem 1.3 concerning \( \mathcal{C}_{\text{parab}}(\tilde{A}_n) \) is now complete: (i) is Lemma 1.4, (ii) is the statement of Corollary 4.6, (iii) is the statement of Proposition 4.7 and (iv) results from Lemmas 4.8, 4.10 and Theorem 2.8.

We conclude with a generalization of Propositions 2.3 and 2.4 to the case of the Artin–Tits group \( A_{\tilde{A}_n} \). To the best of our knowledge, this was not previously written in the literature. Before proceeding, notice that if \( P \) is a proper irreducible parabolic subgroup of \( A_{\tilde{A}_p} \), \( P \) is an irreducible Artin–Tits group of spherical type; therefore, we can define the central element \( z_P \) of \( P \), as in Section 2.1.

**Proposition 4.11.** Let \( P \) and \( Q \) be two proper irreducible parabolic subgroups of \( A_{\tilde{A}_n} \). Let \( g \in A_{\tilde{A}_n} \). Then \( P^g = Q \) if and only if \( z_P^g = z_Q \).

**Proof.** The direct implication is obvious, considering the centers. Let us prove the converse. Given any proper irreducible parabolic subgroup \( P \) of \( A_{\tilde{A}_n} \), Proposition 4.2 says that \( \theta_n(P) \) is a proper irreducible subgroup of \( A_{\tilde{B}_{n+1}} \). As \( \theta_n \) induces an isomorphism between \( P \) and \( \theta_n(P) \), \( \theta_n(z_P) \) generates the center of \( \theta_n(P) \). By the formulae defining \( \theta_n \), we see that the exponent sums \( \epsilon_{\tilde{B}_{n+1}}(\theta_n(x)) \) and \( \epsilon_{\tilde{A}_n}(x) \) coincide, for every \( x \in A_{\tilde{A}_n} \). We deduce that \( \theta_n(z_P) = z_{\theta_n(P)} \), for every proper irreducible subgroup of \( A_{\tilde{A}_n} \). Now, assume \( z_P^g = z_Q \). We have, on the one hand, \( \theta_n(z_Q) = z_{\theta_n(Q)} \), while, on the other hand,

\[
\theta_n(z_P^g) = \theta_n(z_P)^{\theta_n(g)} = \theta_n(z_{\theta_n(P)^{\theta_n(g)}}) = z_{\theta_n(P)^{\theta_n(g)}} = z_{\theta_n(P)}.
\]

We deduce \( z_{\theta_n(P)} = z_{\theta_n(Q)} \) whence by Proposition 2.3, \( \theta_n(P^g) = \theta_n(Q) \) and by injectivity of \( \theta_n \), \( P^g = Q \) as desired. \( \square \)

**Proposition 4.12.** Let \( P \) and \( Q \) be two distinct proper irreducible parabolic subgroups of \( A_{\tilde{A}_n} \). Then \( P \) and \( Q \) are adjacent if and only if \( z_P \) and \( z_Q \) commute.

**Proof.** Suppose that \( P \) and \( Q \) are adjacent — see Definition 1.1. Then it is clear that \( z_P \) and \( z_Q \) commute with each other.

Conversely, suppose that \( z_P \) and \( z_Q \) commute. Then \( \theta_n(z_P) = z_{\theta_n(P)} \) and \( \theta_n(z_Q) = z_{\theta_n(Q)} \) commute, whence by Proposition 2.4, \( \theta_n(P) \) and \( \theta_n(Q) \) are adjacent in \( \mathcal{C}_{\text{parab}}(B_{n+1}) \). By Proposition 4.3, this implies that \( P \) and \( Q \) are adjacent. \( \square \)

5. The Graph \( \mathcal{C}_{\text{parab}}(\tilde{C}_n) \)

Let us start with a description of the proper irreducible standard parabolic subgroups of \( A_{\tilde{C}_n} \). A proper irreducible standard parabolic subgroup of \( A_{\tilde{C}_n} \) is determined by a proper subinterval of
[\[n + 1\]: for any proper subinterval \(I\) of \([n + 1]\), we denote by \(\tilde{C}_I\) the proper irreducible standard parabolic subgroup of \(A_{\tilde{C}_n}\) generated by \(\{\tilde{\tau}_i \mid i \in I\}\).

The following facts can be found in [1, Section 4]. There is a monomorphism

\[
\lambda_n : A_{\tilde{C}_n} \rightarrow A_{A_{n+1}}
\]

\[
\tilde{\tau}_i \mapsto \begin{cases} 
\sigma_i^2 & \text{if } i = 1 \text{ or } i = n + 1, \\
\sigma_i & \text{if } 2 \leq i \leq n.
\end{cases}
\]

The image of \(\lambda_n\) is the subgroup \(\mathcal{P}\) of \((n + 2)\)-strands braids in which the first strand ends in the first position and the \((n + 2)\)nd strand ends in the \((n + 2)\)nd position. In other words, a braid \(y\) on \((n + 2)\) strands belongs to \(\mathcal{P}\) if and only if \(\pi_y(1) = 1\) and \(\pi_y(n + 2) = n + 2\), where \(\pi_y = \pi(y)\) is the permutation in \(S_{n+2} = W_{A_{n+1}}\) associated to \(y\). We shall call these braids \((1, (n + 2))\)-pure.

Although \(A_{\tilde{C}_n}\) is not of spherical type, we observe that each proper irreducible parabolic subgroup of \(A_{\tilde{C}_n}\) is an irreducible Artin–Tits group of spherical type (as is the case for every proper irreducible parabolic subgroup of a euclidean Artin–Tits group). This allows to associate to each proper irreducible parabolic subgroup \(P\) of \(A_{\tilde{C}_n}\) its central element \(z_P\), as in Section 2.1. The following is the analogue of Lemma 3.1.

**Lemma 5.1.** Let \(I\) be a proper subinterval of \([n + 1]\). Then

(i) \(\lambda_n(\tilde{C}_I) = A_I \cap \mathcal{P}\),

(ii) \(\lambda_n(z_{\tilde{C}_I}) = \begin{cases} 
z_{A_I} & \text{if } I \neq \{1\}, \{n + 1\} \\
z_{A_I}^n & \text{if } I = \{1\} \text{ or } I = \{n + 1\}.
\end{cases}\)

**Proof.** As \(I\) is a proper subinterval, by symmetry, we can assume that \(n + 1\) does not lie in \(I\). Then \(\tilde{C}_I\) is a standard parabolic subgroup of the Artin–Tits subgroup of \(A_{\tilde{C}_n}\) generated by \(\tilde{\tau}_i, i = 1, 2, \ldots, n\), which is an Artin–Tits group of type \(B_n\) and the restriction of \(\lambda_n\) to this subgroup coincides with the map \(\eta_n\) (followed by the embedding \(A_{A_n} \subseteq A_{A_{n+1}}\)). Then the lemma is a consequence of Lemma 3.1. \(\square\)

As \(A_{\tilde{C}_n}\) is not of spherical type, we do not know a priori the analogues of Propositions 2.3 and 2.4. However, these analogues hold, as shown in the next Propositions 5.2 and 5.3 — compare Propositions 4.11 and 4.12.

**Proposition 5.2.** Let \(I, J\) be proper subintervals of \([n + 1]\) and let \(g \in A_{\tilde{C}_n}\). The following three statements are equivalent:

(i) \(\tilde{C}_I^g = \tilde{C}_J\),

(ii) \(z_{\tilde{C}_I}^g = z_{\tilde{C}_J}\),

(iii) \(C_{\lambda_n(g)} = C_J\).

**Proof.** Note that using the isomorphism \(I_{n+1}\) from Section 2.2, (iii) is equivalent to \(A_J^{\lambda_n(g)} = A_J\). Assume (i). Considering the center, we get immediately (ii).

Let us show that (ii) implies (iii). Assume first that \(I = \{1\}\). Then \(z_{\tilde{C}_J} = z_{\tilde{C}_I} = \tilde{\tau}_1^g\). By Lemma 2.2, this forces \(J = \{1\}\), so that \(\tilde{\tau}_1^g = \tilde{\tau}_1\). It follows that \(\sigma_{1}^g = \sigma_{1}^g\) and from [18, Theorem 2.2], we deduce that \(\sigma_{1}^{\lambda_n(g)} = \sigma_{1}\). Finally, \(A_{\{1\}}^{\lambda_n(g)} = A_{\{1\}}\) as desired. The proof is similar if we assume \(I = \{n + 1\}\). Suppose then that \(I, J \neq \{1\}, \{n + 1\}\). By Lemma 5.1(ii), we have \(z_{A_{1}}^{\lambda_n(g)} = z_{A_{J}}\) and by Proposition 2.3, we obtain \(A_{I}^{\lambda_n(g)} = A_{J}\) as desired.
Finally, assume (iii) and let us show (i). We have, as \( \lambda_n(g) \in \mathcal{P} \) and using Lemma 5.1(i),
\[
\lambda_n(\widetilde{C}_I^g) = (\lambda_n(\widetilde{C}_I))^{\lambda_n(g)} = (A_I \cap \mathcal{P})^{\lambda_n(g)} = A_I^{\lambda_n(g)} \cap \mathcal{P}^{\lambda_n(g)} = A_I \cap \mathcal{P} = \lambda_n(\widetilde{C}_J)
\]
and the injectivity of \( \lambda_n \) ensures that \( \widetilde{C}_I^g = \widetilde{C}_J \), as required. \( \square \)

**Proposition 5.3.** Let \( I, J \) be proper subintervals of \([n + 1]\) and let \( g \in A_{\widetilde{C}_n} \). The following are equivalent.

- (i) \( \widetilde{C}_I^g \) and \( \widetilde{C}_J \) are adjacent in \( \mathcal{C}_{\text{parab}}(\widetilde{C}_n) \).
- (ii) \( z_{\widetilde{C}_I^g}^n \) and \( z_{\widetilde{C}_J} \) commute.
- (iii) \( A_I^{\lambda_n(g)} \) and \( \mathcal{C}_J \) are adjacent in \( \mathcal{C}_G(\mathbb{D}_{n+2}) \).

**Proof.** (i) \( \Rightarrow \) (ii) is proven in the same way as the direct implication of Proposition 4.12.

Suppose (ii). By Lemma 5.1(ii) (and [18, Theorem 2.2] for the case where \( I \) or \( J = \{1\} \) or \( \{n + 1\} \)), we obtain that \( z_{A_I}^{\lambda_n(g)} \) and \( z_{A_J} \) commute, which is to say, according to Proposition 2.4, that \( A_I^{\lambda_n(g)} \) and \( A_J \) are adjacent in \( \mathcal{C}_{\text{parab}}(A_{n+1}) \). Applying the isomorphism \( f_{n+1}^{-1} \), we obtain (iii).

Suppose (iii) and let us show (i). Using the isomorphism \( f_{n+1} \), \( A_I^{\lambda_n(g)} \) and \( A_J \) are adjacent. Suppose first that \( A_I^{\lambda_n(g)} \subset A_J \). Then we have, as \( \lambda_n(g) \in \mathcal{P} \) and using Lemma 5.1(i),
\[
\lambda_n(\widetilde{C}_I^g) = \lambda_n(\widetilde{C}_I)^{\lambda_n(g)} = (A_I \cap \mathcal{P})^{\lambda_n(g)} = A_I^{\lambda_n(g)} \cap \mathcal{P} \subset A_J \cap \mathcal{P} = \lambda_n(\widetilde{C}_J).
\]
It follows that \( \lambda_n(\widetilde{C}_I^g) \subset \lambda_n(\widetilde{C}_J) \) and injectivity of \( \lambda_n \) shows that \( \widetilde{C}_I^g \subset \widetilde{C}_J \). The proof when \( A_J \subset A_I^{\lambda_n(g)} \) is similar. Assume finally that \( A_I^{\lambda_n(g)} \cap A_J = \{1\} \) and that both subgroups commute. Then we have, as \( \lambda_n(g) \in \mathcal{P} \) and using Lemma 5.1(i) and the injectivity of \( \lambda_n \),
\[
\lambda_n(\widetilde{C}_I^g \cap \widetilde{C}_J) = \lambda_n(\widetilde{C}_I^g) \cap \lambda_n(\widetilde{C}_J) = (A_I^{\lambda_n(g)} \cap \mathcal{P}) \cap (A_J \cap \mathcal{P}) = A_I^{\lambda_n(g)} \cap A_J \cap \mathcal{P} = \{1\}
\]
and \( \widetilde{C}_I^g \cap \widetilde{C}_J = \{1\} \). As \( \lambda_n(\widetilde{C}_I^g) \) is a subgroup of \( A_I^{\lambda_n(g)} \), \( \lambda_n(\widetilde{C}_J) \) is a subgroup of \( A_J \) and \( A_I^{\lambda_n(g)} \) and \( A_J \) commute mutually and again by injectivity of \( \lambda_n \), \( \widetilde{C}_I^g \) and \( \widetilde{C}_J \) commute. Therefore, we have shown that \( \widetilde{C}_I^g \) and \( \widetilde{C}_J \) are adjacent. \( \square \)

By contrast with the embedding \( \eta_n \) of \( A_{B_n} \) in \( A_{\mathcal{A}_n} \) (Section 3), not all curves in \( \mathbb{D}_{n+2} \) can be obtained as \( \widetilde{C}_I^{\lambda_n(g)} \), for a proper subinterval \( I \) of \([n + 1]\) and \( g \in A_{\widetilde{C}_n} \).

**Proposition 5.4.** Let \( \mathcal{C} \) be a curve in \( \mathbb{D}_{n+2} \); the following are equivalent.

- (i) There exist a proper subinterval \( I \) of \([n + 1]\) and \( g \in A_{\widetilde{C}_n} \) such that \( \mathcal{C} = \mathcal{C}_I^{\lambda_n(g)} \).
- (ii) \( \mathcal{C} \) does not surround both the first and the last punctures.

**Proof.** (i) \( \Rightarrow \) (ii) Suppose that \( \mathcal{C} = \mathcal{C}_I^{\lambda_n(g)} \) for some proper subinterval \( I \) of \([n + 1]\) and some \( g \in A_{\widetilde{C}_n} \). The curve \( \mathcal{C}_I \) cannot surround both the first and the last punctures as it is standard; assume, for instance, that it does not surround the first puncture (the other case is similar). Then as \( \lambda_n(g) \in \mathcal{P} \), we see that \( \mathcal{C}_I^{\lambda_n(g)} \) does not surround the first puncture either.

(ii) \( \Rightarrow \) (i). Suppose that \( \mathcal{C} \) does not surround both the first and the last punctures. We must show that \( \mathcal{C} \) can be transformed into a standard curve by the action of some braid \( \beta \) in \( \mathcal{P} \). Suppose, for instance, that \( \mathcal{C} \) does not surround the first puncture (the other case is similar). By Proposition 3.4, we know that there exist a 1-pure braid \( \alpha \) and a proper subinterval \( I \) of \([n + 1]\) such that \( \mathcal{C}_I^\alpha = \mathcal{C}_I \) is a standard curve surrounding punctures \( m, \ldots, m+k \), for some \( m \geq 2, k \geq 1 \).
Let $\pi_n \in \mathfrak{S}_{n+2}$ be the permutation associated to $\alpha$. Let $j_0 = \pi_n(n + 2) \in \{2, \ldots, n + 2\}$. Recall the braid $b_{j_0}$ from Section 2.2; note that $\alpha b_{j_0} \in \mathcal{P}$. We use Lemma 2.6. If $j_0 < m$ or $j_0 > m + k$, then $C^b_{j_0}$ is standard so we can take $\beta = \alpha b_{j_0}$. Otherwise $m \leq j_0 \leq m + k$. If $C^b_{j_0} = C^b_{m+k}$ is not standard, we have $m + k < n + 2$ and $C^b_{j_0} = C^b_{[n+2-k,n+1]}$. By Remark 2.7, $\xi_j'$ is $(n + 2)$-pure; it is also 1-pure because $m \geq 2$. Therefore, we can take $\beta = \alpha b_{j_0} \xi_j'$.

**Notation 5.5.** Let $\mathcal{K}_C$ be the subgraph of $\mathcal{CG}(\mathcal{D}_{n+2})$ induced by the curves which do not surround both the first and the last punctures of $\mathcal{D}_{n+2}$.

**Corollary 5.6.** The assignment $\tilde{\mathcal{C}}_I^g \mapsto \mathcal{C}_I^{\Lambda_n(g)}$, where $I$ is a proper subinterval of $[n+1]$ and $g \in A_{\tilde{\mathcal{C}}_n}$, defines a graph isomorphism $\Lambda_n$ from $\mathcal{C}_{parab}(\tilde{\mathcal{C}}_n)$ to $\mathcal{K}_C$. In particular, we have $d_{\mathcal{D}_{n+2}}(\Lambda_n(P), \Lambda_n(P')) \leq d_{\tilde{\mathcal{C}}_n}(P, P')$ for all proper irreducible parabolic subgroups $P$ and $P'$ of $A_{\tilde{\mathcal{C}}_n}$.

**Proof.** By Proposition 5.2, the assignment $\Lambda_n$ is a well-defined injective map. By Proposition 5.4, the image of this map is $\mathcal{K}_C$. By Proposition 5.3, both $\Lambda_n$ and its inverse are graph homomorphisms.

**Proposition 5.7.** The subgraph $\mathcal{K}_C$ is 1-dense in $\mathcal{CG}(\mathcal{D}_{n+2})$; as a consequence, the graph $\mathcal{C}_{parab}(\tilde{\mathcal{C}}_n)$ has infinite diameter.

**Proof.** This amounts to show that given a curve $\mathcal{C}$ surrounding both the first and the last punctures, it is possible to find another curve $c$ disjoint from $\mathcal{C}$ and such that $c$ does not surround both the first and the last punctures. If the exterior of $\mathcal{C}$ contains at least two punctures, we take any curve $c$ in the exterior of $\mathcal{C}$. Otherwise, the interior of $\mathcal{C}$ contains $n + 1 \geq 3$ punctures and we can choose in the interior of $\mathcal{C}$ any curve which does not surround both the first and the last punctures. For the second part, we can argue following the same lines as in the proof of the second part of Proposition 4.7.

To conclude our study we will now show that $\mathcal{K}_C$ is hyperbolic. In the next two lemmas, we show that $\mathcal{K}_C$ satisfies the hypothesis of Theorem 2.8, from which we conclude that $\mathcal{K}_C$, and hence, $\mathcal{C}_{parab}(\tilde{\mathcal{C}}_n)$ is hyperbolic.

**Lemma 5.8.** (i) $\mathcal{K}_C$ is connected. (ii) The natural action of the pure braid group $PA_{\mathcal{A}_{n+1}}$ on $\mathcal{D}_{n+2}$ induces an action by isometries on $\mathcal{K}_C$. (iii) No annulus in $\mathcal{D}_{n+2}$ can be a witness for all vertices of $\mathcal{K}_C$.

**Proof.** The proof is identical to the proof of Lemma 4.8.

**Remark 5.9.** The same argument as in the proof of Proposition 4.9 shows that the embedding of $\mathcal{K}_C$ in $\mathcal{CG}(\mathcal{D}_{n+2})$ is not quasi-isometric.

For the following lemma, we denote by $p_1$ the first puncture and by $p_{n+2}$ the last puncture.

**Lemma 5.10.** Let $X$ be a subsurface of $\mathcal{D}_{n+2}$. Then $X$ is a witness for $\mathcal{K}_C$ if and only if one of the following holds.

(i) $X = \mathcal{D}_{n+2}$ or $X = \mathcal{D}_{n+2} \setminus D$, where $D$ is the interior of an essential curve surrounding $p_1$ and $p_{n+2}$ and no other puncture.
(ii) $X$ is the interior of an essential curve surrounding $p_1, p_{n+2}$ and exactly $(n - 1)$ other punctures.

(iii) $X = X' \setminus D$, where $X'$ is the interior of an essential curve surrounding $p_1, p_{n+2}$ and exactly $(n - 1)$ other punctures and $D$ is the interior of an essential curve surrounding $p_1$ and $p_{n+2}$ and no other puncture.

We will say that $X$ is a witness of type (i), (ii) or (iii). Two witnesses for $K_C$ are never disjoint.

Proof. Mutatis mutandis, the proof is the same as the proof of Lemma 4.10. □

The proof of the statements of Theorem 1.3 concerning $C_{parab}(\tilde{C}_n)$ is now complete: (i) is Lemma 1.4, (ii) is the statement of Corollary 5.6, (iii) is the statement of Proposition 5.7 and (iv) results from Lemmas 5.8, 5.10 and Theorem 2.8.

6. Hyperbolic Structures on Artin–Tits Groups

In this section, we briefly review some of the — known or conjectural — hyperbolic structures on Artin–Tits groups presented in [11] and we show some connections with our results. A generating set $X$ of a group $G$ is a hyperbolic structure if the Cayley graph $\Gamma(G, X)$ of $G$ with respect to $X$ is a hyperbolic metric space.

6.1. Hyperbolic structures on $A_{B_n}$

Let $A_\Gamma$ be an Artin–Tits group of spherical type and consider the following generating sets for $A_\Gamma$.

- $X_{NP}(\Gamma)$ is the union of the normalizers of the proper irreducible standard parabolic subgroups of $A_\Gamma$.
- $X_P(\Gamma)$ is the union of the proper irreducible standard parabolic subgroups of $A_\Gamma$ and the cyclic subgroup generated by the square of the element $\Delta$.
- $X_{abs}(\Gamma)$ is the lift of the longest element of the corresponding Coxeter group.

By [9, Theorem 1], we know that $X_{abs}(\Gamma)$ is a hyperbolic structure on $A_\Gamma$; moreover, this is the only one of the three sets which is known to be a hyperbolic structure for all $\Gamma$. Both $X_{NP}(A_n)$ and $X_P(A_n)$ are hyperbolic structures; indeed, $\text{Cay}(A_{A_n}, X_{NP}(A_n))$ is quasi-isometric to $CG(D_n)$ [11, Proposition 3.2], while $\text{Cay}(A_{A_n}, X_P(A_n))$ is quasi-isometric to $A_G(D_{n+1})$, the graph of arcs in $D_{n+1}$ both of whose endpoints lie in the boundary $\partial D_{n+1}$ [11, Proposition 3.4].

A classical argument (see [11, Lemma 2.5, Proposition 4.4]) shows that (except for dihedral Artin–Tits groups), $\text{Cay}(A_\Gamma, X_{NP}(\Gamma))$ is quasi-isometric to the graph of irreducible parabolic subgroups of $A_\Gamma$. Therefore, Theorem 1.2 can be rephrased by saying that $X_{NP}(B_n)$ is a hyperbolic structure on $A_{B_n}$. We shall prove an analogous statement for the generating set $X_P(B_n)$. Again, this will be obtained by comparing $\text{Cay}(A_{B_n}, X_P(B_n))$ and $\text{Cay}(A_{A_n}, X_P(A_n))$.

Recall from Section 3 the monomorphism $\eta_n : A_{B_n} \rightarrow A_{A_n}$ whose image is the subgroup $\Phi_1$ of 1-pure braids on $(n + 1)$ strands. For $1 \leq i \leq n$, let $a_i = \sigma_1 \cdots \sigma_1$ and $a_0 = Id$; for each $y \in A_{A_n}$, there is a unique $i \in \{0, \ldots, n\}$ such that $ya_i \in \Phi_1$. For any braid $y \in A_{A_n}$, we denote by $\pi_y$ the associated permutation in $\mathfrak{S}_{n+1}$. The next lemma resembles Lemma 2.5.

**Lemma 6.1.** Let $I$ be a proper subinterval of $[n]$, $m = \min(I)$ and $k = \#I$, so that the circle $C_I$ in $D_{n+1}$ surrounds the punctures $m$ to $m + k$. Let $0 \leq i_0, j_0 \leq n$. Let $g \in A_I$ and suppose that $z = a_{i_0}^{-1} ga_{j_0}$ is 1-pure.
(i) If $i_0 + 1 < m$, then $z = g \in A_I$.
(ii) If $i_0 + 1 > m + k$, then $z = sh(g) \in A_{I'}$, where $I' = \{i + 1 \mid i \in I\}$.
(iii) If $m \leq i_0 + 1 \leq m + k$, then $z^\xi_I \in A_{\{I, \ldots, k\}}$.

Here, $\xi_I \in \mathfrak{P}_1$ is the braid defined in Section 2.2 which satisfies $A_I^m = A_{[1,k]}$ and $sh$ denotes the shift homomorphism $\sigma_i \mapsto \sigma_{i+1}$ from $A_{[1,n-1]}$ to $A_{[2,n]}$.

Proof. Firstly observe that as $z$ is 1-pure, we must have $\pi_g(i_0 + 1) = j_0 + 1$. Moreover, as $g \in A_I$, $\pi_g(i) = i$ for all $i \in \{1, \ldots, m-1\} \cup \{m + k + 1, \ldots, n + 1\}$. In particular, if $i_0 + 1 < m$ or $i_0 + 1 > m + k$, we must have $i_0 + 1 = j_0 + 1$, whence $i_0 = j_0$.

(i) Suppose that $i_0 + 1 < m$; as we have just seen, $z = a^{-1}_{i_0} g a_{i_0}$. But $a_{i_0}$ commutes with all letters $\sigma_i$, $i \in I$ whence $z = g$.

(ii) Suppose that $i_0 + 1 > m + k$; again $z = a^{-1}_{i_0} g a_{i_0}$. We have, for all $i \in I$,
\[
a_{i_0}^{-1} a_i a_{i_0} = (\sigma_1^{-1} \cdots \sigma_{i_0}^{-1}) \sigma_i (\sigma_{i_0} \cdots \sigma_1) = \sigma_1^{-1} \cdots \sigma_{i_0}^{-1} (\sigma_{i+1}^{-1} \sigma_i \sigma_{i+1}) \sigma_1 \cdots \sigma_i = \sigma_{i+1},
\]
and the claim follows.

(iii) Suppose that $m \leq i_0 + 1 \leq m + k$; then also $m \leq j_0 + 1 \leq m + k$. We have
\[
z = a_{i_0}^{-1} g a_{j_0} = (\sigma_1^{-1} \cdots \sigma_{m-1}^{-1}) (\sigma_{m-1}^{-1} g_{j_0} \cdots \sigma_m) (\sigma_{m-1} \cdots \sigma_1) = (\sigma_1^{-1} \cdots \sigma_{m-1}^{-1}) g'(\sigma_{m-1} \cdots \sigma_1),
\]
where $g' = (\sigma_{m-1}^{-1} \cdots \sigma_{i_0}^{-1}) g(\sigma_{j_0} \cdots \sigma_m)$ (note that the first and third factors may be trivial if $i_0 + 1 = m$ or $j_0 + 1 = m$). Since all crossings involved are inner to $\mathcal{C}_I$, $g' \in A_I$. We deduce that $z = g' a_{i_0}^{-1} \in A_I^{m-1}$ and then by definition of $\xi_I$, $z^\xi_I \in A_{[1,k]}$ as claimed.

\[\square\]

Proposition 6.2. Let $n \geq 3$. The monomorphism $\eta_n : A_{B_n} \rightarrow A_{A_n}$ induces a quasi-isometry between $\text{Cay}(A_{B_n}, X_P(B_n))$ and $\text{Cay}(A_{A_n}, X_P(A_n))$.

Proof. Throughout the proof, the notation $\|x\|_{X_P(B_n)}$ means the word length of $x \in A_{B_n}$ with respect to the generating set $X_P(B_n)$.

We know by Lemma 3.1(i) that given a proper subinterval $I$ of $[n]$, $\eta_n(B_I) = A_I \cap \mathfrak{P}_1$; therefore, for $g \in A_{B_n}$, $g \in B_I$ is equivalent to $\eta_n(g) \in A_I$. Similarly, $g = \Delta_{B_{B_n}}^{2k}$ for some $k \in \mathbb{Z}$ is equivalent to $\eta_n(g) = \Delta_{B_{A_n}}^{4k}$ for some $k \in \mathbb{Z}$. Therefore, $\eta_n$ induces a 1-Lipschitz map from $\text{Cay}(A_{B_n}, X_P(B_n))$ to $\text{Cay}(A_{A_n}, X_P(A_n))$.

We define a map $\psi_n : A_{A_n} \rightarrow A_{B_n}$ in the following way. Given $y \in A_{A_n}$, let $i \in \{0, \ldots, n\}$ (it is unique!) be such that $g_{ai} \in \mathfrak{P}_1 = Im(\eta_n)$, and define $\psi(y) = \eta_n^{-1}(g_{ai})$. Let us see that $\psi_n$ is a quasi-inverse for $\eta_n$. Indeed, we have by construction, for $x \in A_{B_n}$, $\psi_n \circ \eta_n(x) = x$. Conversely, for $y \in A_{A_n}$, $y$ and $\eta_n \circ \psi_n(y)$ differ by $a_i$, for some $0 \leq i \leq n$; however, $a_i$ can be written as a product of at most two elements in $X_P(A_n)$, whence the distance between $y$ and $\eta_n \circ \psi_n(y)$ in $\text{Cay}(A_{A_n}, X_P(A_n))$ is at most 2.

We show finally that the map $\psi_n$ is Lipschitz. Let $y, y' \in A_{A_n}$ be adjacent in $\text{Cay}(A_{A_n}, X_P(A_n))$; write $g = y^{-1} y'$. This means that $g \in X_P(A_n)$, that is, $g = \Delta_{A_n}^{2k}$ for $k \in \mathbb{Z}$,
or \( g \in A_I \) for some proper subinterval \( I \) of \([n]\). We have unique \( i_0, j_0 \) so that \( ya_{i_0} \in \Psi_1 \) and \( y'a_{j_0} \in \Psi_1 \). Then also \( a_{i_0}^{-1}ga_{j_0} \) is 1-pure. Let \( x = \eta_n^{-1}(ya_{i_0}) \) and \( x' = \eta_n^{-1}(y'a_{j_0}) \); we must estimate \( \|x^{-1}x'\|_{X_P(B_n)} \).

If \( g = \Delta_{B_n}^\pm \), \( g \) is pure. As \( a_{i_0}^{-1}ga_{j_0} \) is 1-pure, we must have \( i_0 + 1 = \pi_g(i_0 + 1) = j_0 + 1 \), whence \( i_0 = j_0 \). We deduce, as \( g \) is central, \( \eta_n(x^{-1}x') = a_{i_0}^{-1}ga_{j_0} = g = \Delta_{A_n}^\pm \) and \( x^{-1}x' = \Delta_{B_n}^\pm \). In summary, \( \|x^{-1}x'\|_{X_P(B_n)} \leq \|\Delta_{B_n}\|_{X_P(B_n)} + 1 \) (and this bound is 1 if \( k \) is even).

If \( g \in A_I \) for some proper subinterval \( I \) of \([n]\), then Lemma 6.1 says that there is some proper subinterval \( J \) of \([n]\) so that either \( a_{i_0}^{-1}ga_{j_0} \in A_J \) or \( (a_{i_0}^{-1}ga_{j_0})_{\xi_1} \in A_J \). Pulling back to \( A_{B_n} \) this assertion, we see using Lemma 3.1(i) that either \( x^{-1}x' \in B_J \) or \( (x^{-1}x')_{\eta_n^{-1}(\xi)} \in B_J \). It follows that \( \|x^{-1}x'\|_{X_P(B_n)} \leq 1 + \|\eta_n^{-1}(\xi)\|_{X_P(B_n)} \). This is uniformly bounded as the set of proper connected subintervals of \([n]\) is finite.

We can now complete the proof of Theorem 1.5. By [11, Proposition 3.4], \( X_P(A_n) \) is a hyperbolic structure on \( A_{A_n} \), that is, \( Cay(A_{A_n}, X_P(A_n)) \) is hyperbolic. By Proposition 6.2, \( Cay(A_{B_n}, X_P(B_n)) \) is hyperbolic as well, that is, \( X_P(B_n) \) is a hyperbolic structure on \( A_{B_n} \). From Proposition 6.2 and Theorem 1.2, we have quasi-isometries between \( Cay(A_{B_n}, X_P(B_n)) \) and \( Cay(A_{A_n}, X_P(A_n)) \) on the one hand and between \( Cay(A_{B_n}, X_{N_P}(B_n)) \) and \( Cay(A_{A_n}, X_{N_P}(A_n)) \) on the other hand. But we also know from Proposition [11, Proposition 4.19] that the hyperbolic structures \( X_P(A_n) \) and \( X_{N_P}(A_n) \) on \( A_{A_n} \) are not equivalent. This shows the second part of the theorem.

PROBLEM 6.3. As the graph of irreducible parabolic subgroups of a dihedral Artin–Tits group is not connected, the only infinite family of Artin–Tits groups of spherical type for which the hyperbolicity of the graph of irreducible parabolic subgroups is still open is the type \( D_n \). The Artin–Tits group of type \( D_n \) can be seen as an index 2 subgroup of the quotient of an Artin–Tits group of type \( B_n \) by the normal subgroup generated by the standard generator \( \tau_1 \) [2]. It is interesting to ask whether this embedding can be used to establish the hyperbolicity of \( C_{parab}(D_n) \). A positive answer to this question would likely give some hints to attack also the case of euclidean Artin–Tits groups of types \( \tilde{B} \) and \( \tilde{D} \).

6.2. Non-spherical type

Even if \( A_{\Gamma} \) is not of spherical type, we can extend the definition of \( X_{N_P}(\Gamma) \) and \( X_P(\Gamma) \), just dropping the powers of \( \Delta_{\Gamma} \) in the definition of \( X_P(\Gamma) \). We will see that \( X_{N_P}(\tilde{A}_n) \) is a hyperbolic structure on \( A_{\tilde{A}_n} \), for \( Z = A \) or \( C \). We leave open whether \( X_P(\tilde{Z}_n) \) is a hyperbolic structure on \( A_{\tilde{Z}_n} \).

The proof rests on a technical result — simultaneous standardization of adjacent proper irreducible parabolic subgroups — which generalizes [20, Proposition 4.4] and [13, Section 11] and could be interesting on its own. The result is split into the next two propositions.

PROPOSITION 6.4. Let \( P, Q \) be adjacent proper irreducible parabolic subgroups of \( A_{\tilde{A}_n} \). Then there exists \( s \in A_{\tilde{A}_n} \) so that \( P^s \) and \( Q^s \) are standard.

Proof. By Proposition 4.3, \( \theta_n(P) \) and \( \theta_n(Q) \) are adjacent in \( C_{parab}(B_{n+1}) \). As \( A_{B_{n+1}} \) is of spherical type, by [13, Section 11], there exists \( \zeta \in A_{B_{n+1}} \) so that \( \theta_n(P)^\zeta \) and \( \theta_n(Q)^\zeta \) are standard parabolic subgroups of \( A_{B_{n+1}} \). By Proposition 4.4, \( \theta_n(P) \), \( \theta_n(Q) \) and their respective standard conjugates \( \theta_n(P)^\zeta \) and \( \theta_n(Q)^\zeta \) are braid subgroups of \( A_{B_{n+1}} \), that is, \( \theta_n(P)^\zeta = B_I \) and \( \theta_n(Q)^\zeta = B_J \) for some proper subintervals \( I \) and \( J \) of \([n+1]\) not containing 1. Let \( I' = \{ i - 1 \mid i \in I \} \) and \( J' = \{ j - 1 \mid j \in J \} \), so that \( \theta_n(P)^\zeta = \theta_n(A_{I'}) \) and \( \theta_n(Q)^\zeta = \theta_n(A_{J'}) \).
Suppose first that \( \zeta = \theta_n(s) \) for some \( s \in A_{\tilde{A}_n} \). Then we have \( \theta_n(P^s) = \theta_n(P)\zeta = \theta_n(\tilde{A}_I') \) and \( \theta_n(Q^s) = \theta_n(Q)\zeta = \theta_n(\tilde{A}_J') \). We deduce from the injectivity of \( \theta_n \) that \( P^s = \tilde{A}_I' \) and \( Q^s = \tilde{A}_J' \) are both standard, showing our claim.

If on the contrary \( \zeta \) is not in the image of \( \theta_n \), in view of Proposition 4.1(iii), we can write \( \zeta = \zeta_0 \rho^r \), where \( \zeta_0 = \theta_n(s) \) for some \( s \in A_{\tilde{A}_n} \) and \( r \in \mathbb{Z} \). Using Proposition 4.1(ii), we then have

\[
\theta_n(P^s) = \theta_n(P)\zeta_0 = \theta_n(P)\zeta_0^{\rho^r} = \theta_n(\tilde{A}_I')^{\rho^r} = \theta_n(\tilde{A}_{I''})
\]

and

\[
\theta_n(Q^s) = \theta_n(Q)\zeta_0 = \theta_n(Q)\zeta_0^{\rho^r} = \theta_n(\tilde{A}_J')^{\rho^r} = \theta_n(\tilde{A}_{J''}),
\]

where \( I'' = \{ i' - r \mid i' \in I' \} \), \( J'' = \{ j' - r \mid j' \in J' \} \) and the indices are taken modulo \((n + 1)\).

We obtain that \( P^s = \tilde{A}_{I''} \) and \( Q^s = \tilde{A}_{J''} \) are both standard, as needed. \( \square \)

**Proposition 6.5.** Let \( P, Q \) be adjacent proper irreducible parabolic subgroups of \( A_{\tilde{C}_n} \). Then there exists \( s \in A_{\tilde{C}_n} \) so that \( P^s \) and \( Q^s \) are standard.

**Proof.** Recall the graph isomorphism \( \Lambda_n \) from Corollary 5.6: \( C_1 = \Lambda_n(P) \) and \( C_2 = \Lambda_n(Q) \) are disjoint curves in \( \mathbb{D}_{n+2} \) which do not surround both the first and the last punctures. Under this isomorphism, standard parabolic subgroups of \( A_{\tilde{C}_n} \) are in correspondence with standard curves of \( \mathbb{D}_{n+2} \). So our claim will follow from proving that there exists \( \zeta_0 \in \mathbb{Q} = \lambda_n(A_{\tilde{C}_n}) \) so that both \( C_1^{\zeta_0} \) and \( C_2^{\zeta_0} \) are standard; in this way, setting \( s = \lambda_n^{-1}(\zeta_0) \) will prove the claim.

Recall the graph isomorphism \( \mathcal{S}_{n+1} \) from Corollary 3.5 which identifies \( \mathcal{G}(\mathbb{D}_{n+2}) \) and \( \mathcal{C}_{\text{parab}}(B_{n+1}) \). Because \( A_{B_{n+1}} \) is of spherical type, simultaneous standardization in \( A_{B_{n+1}} \) (see [13, Section 11]) yields \( g \in A_{B_{n+1}} \) such that \( \mathcal{S}_{n+1}^{-1}(C_1)^g \) and \( \mathcal{S}_{n+1}^{-1}(C_2)^g \) are standard parabolic subgroups of \( A_{B_{n+1}} \). Setting \( \alpha = \eta_{n+1}(g) \), we obtain that \( C_1^{\alpha} \) and \( C_2^{\alpha} \) are disjoint standard curves, say \( C_I \) and \( C_J \) for some proper subintervals \( I, J \) of \([n + 1] \). Note that \( \alpha \) is already \( 1 \)-pure.

If \( \alpha \in \mathbb{Q} \), we are done by setting \( \zeta_0 = \alpha \). Otherwise, as in the proof of Proposition 5.3, let \( \pi_0 \) be the permutation in \( \mathcal{S}_{n+2} \) associated to \( \alpha, j_0 = \pi_0(n + 2) = \{2, \ldots, n - 1\} \) and \( b_{j_0} = \sigma_{j_0} \cdots \sigma_{n-1} \). Then \( ab_{j_0} \in \mathbb{Q} \).

If both \( C_I^{b_{j_0}} \) and \( C_J^{b_{j_0}} \) are standard again, then we can choose \( \zeta_0 = \alpha b_{j_0} \). Suppose then that \( C_I^{b_{j_0}} \) is not standard; then (Lemma 2.6(iii)) \( C_I \) must surround the puncture \( j_0 \). Recall that \( C_I \) did not surround both the first and the last puncture and that \( \pi_0(1) = 1 \) and \( \pi_0(n + 2) = j_0 \); it follows that \( C_I = C_I^{\alpha} \) cannot surround both the punctures 1 and \( j_0 \); hence, it cannot surround the first puncture.

If \( C_J^{b_{j_0}} \) is standard, then the braid \( \xi_j' \) from Lemma 2.6 satisfies that \( \mathcal{C}_e^{ab_{j_0} \xi_j'} \) is standard for \( e = 1, 2 \); moreover, \( \xi_j' \in \mathbb{Q} \) (Remark 2.7 and because \( C_I \) does not surround the first puncture). Therefore, we can choose \( \zeta_0 = \alpha b_{j_0} \xi_j' \in \mathbb{Q} \). If \( C_J^{b_{j_0}} \) is not standard, then again by Lemma 2.6(iii), \( C_J \) must surround the puncture \( j_0 \) and both \( C_I, C_J \) have to be nested. This situation is depicted in Figure 4. Assume that \( C_J \) is in the interior of \( C_I \). Let \( m_I = \min I, m_J = \min J, k_I = \#I \) and \( k_J = \#J \). We define

\[
\xi = \xi_j' \xi_{m_I, m_I + k_I - k_J, n + n - k_J, k_I}.
\]

We note that \( \xi \) does not use any letter \( \sigma_{j+1} \), so that \( \xi \in \mathbb{Q} \). Moreover, \( C_I^{b_{j_0} \xi} \) and \( C_J^{b_{j_0} \xi} \) are standard. This concludes the proof choosing \( \zeta_0 = \alpha b_{j_0} \xi \). \( \square \)

**Corollary 6.6.** Assume that \( \Gamma \) is either \( \tilde{A}_n \) or \( \tilde{C}_n \) \((n \geq 2)\). Then \( \mathcal{C}_{\text{parab}}(\Gamma) \) is quasi-isometric to \( \text{Cay}(A_\Gamma, X_{NP}(\Gamma)) \) and \( X_{NP}(\Gamma) \) is a hyperbolic structure on \( A_\Gamma \).
Figure 4 (colour online). Example in the 10-strands braid group \((n + 2 = 10)\). The standard curves \(C_I\) and \(C_J\) surround the puncture \(j_0\) and are nested with \(C_J\) in the interior of \(C_I\). The puncture \(j_0\) is depicted in red. Here we have \(m_I = 2, m_J = 4, k_I = 5\) and \(k_J = 2\). (a) shows the action of the braid \(b_{j_0}\) (which satisfies \(ab_{j_0} \in \Psi\)). (b) shows the action of \(\xi_J\); we have \(ab_{j_0} \xi_J \in \Psi\). Finally, (c) shows the action of \(\xi_{2,4,7} \in \Psi\).

Proof. By Theorem 1.3(i), \(C_{parab}(\Gamma)\) is connected; the proper irreducible standard parabolic subgroups form a finite set representing all the orbits of vertices under the natural action of \(A_\Gamma\) and by Propositions 6.4 and 6.5, the finite set consisting of all edges bounded by two standard parabolic subgroups is a set of representatives of the orbits of edges under the action of \(A_\Gamma\). By [11, Lemma 2.5], this implies that \(C_{parab}(\Gamma)\) and \(Cay(A_\Gamma, X_{NP}(\Gamma))\) are quasi-isometric. By Theorem 1.3(iv), \(X_{NP}(\Gamma)\) is a hyperbolic structure on \(A_\Gamma\).

Now, we comment on the last generating set considered above: \(X_{abs}(\Gamma)\) is the set of absorbable elements. The concept of an absorbable element was introduced in the context of a finite-type Garside group [9]. The only irreducible Artin–Tits groups which are finite-type Garside groups are those of spherical type [7]. However, the euclidean Artin–Tits groups under study in this paper possess a Garside structure of infinite type — see [15, 16]. Moreover, this situation can be generalized: every euclidean Artin–Tits group embeds in a group with an infinite-type Garside structure, the so-called crystallographic Garside group [23]. It has been shown recently that the construction of absorbable elements and the additional length graph from [9] can be generalized to such an infinite-type Garside structure, yielding again a hyperbolic graph. This allows to establish the acylindrical hyperbolicity of every irreducible euclidean Artin–Tits group [8].

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