Numerical Solution for A Special Class of optimal Control Problem by using Hermite polynomial

Abstract- In this paper, a numerical solution for solving a special class of optimal control problems is considered. The main idea of the solution is to parameterize the state space by approximating the state function using a linear combination of Hermite polynomial with unknown coefficients an iterative method is proposed in order to facilitate the computation of unknown coefficients. Some illustrated examples are included to test the efficiency of algorithm.

Keywords- Hermite polynomial, Optimal control problems, State parameterization.

1. Introduction

Optimal control has many applications in every area of science and engineering. And has been studied by many researches [1-4]. Since the analytic solution is not always available for optimal control problems, therefore a numerical solution must be found. Numerical methods for solving optimal control problem are vary in their approach and complexity. In [5], the authors suggested a new algorithm for solving optimal control problems and controlled duffing oscillator using Chebyshev polynomial as a basis function. While numerical solution for solving optimal control problems based on state parameterization technique were consider in [6] and [7]. Furthermore the fundamental of control parameterization method and solving its various applications were introduced in [8]. In addition, control parameterization technique for discrete value control problems was considered in [9]. In recent year different approximate methods and many algorithms has been introduced to solve the optimal control problems [10-13].

The organization of this paper is presented into the following sections. In section 2 the Hermite polynomial which are used as a basis function are reviewed briefly. Section 3, is about mathematical formulation of optimal control problem in section 4, the proposed algorithm is derived. While section 5 includes numerical example and results. Finally, the paper is concluded in section 6.

2. Hermite polynomials

In mathematics, the Hermite polynomials are a classical orthogonal polynomial sequence that arises in probability, such as the Edgeworth series, in combinatorics, as an example of an Appell sequence, obeying the umbral calculus, and in physics, where they give rise to the Eigenstates of the quantum harmonic oscillator. They are named in honor of Charles Hermite. "in a sense to be described below, of the form $H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}$ for $n=1,2,3,...$

The first four Hermite polynomials are

$H_0(t) = 1$

$H_1(t) = 2t$

$H_2(t) = 4t^2 - 2$

$H_3(t) = 8t^3 - 12t$

$H_4(t) = 16t^4 - 48t^2 + 12$

1-1 Definition: "For $n \in N$, we define Hermite polynomials $H_n(t)$ by $\sum_{n=0}^{\infty} \frac{H_n(t)}{n!} r^n = e^{2tr-r^2}$ for $|r| < \infty$ (1)"

To find $H_n(t)$ expand the right hand side of (1) as a Maclaurin series in $r$ and equate coefficients. From Equation (1) we derive the closed expression

$H_n(t) = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} (2t)^{n-2k}$ (2)

Where $[t]$ denoted the largest integer less than or equal to $t$. checking with $n=0,1,2,...$ We find that (2) yields the expected Hermite polynomials."

3. Mathematical formulate

The process illustrated by the following system of nonlinear differential equation on the final time interval $[0,1]$ is consider

$u(t) = f(t, x(t), \dot{x}(t))$ (3)

With initial condition $x(0) = x_0, x(1) = x_1$ (4)

Where $x(.): [0,1] \rightarrow R$ is the state variable, $u(.): [0,1] \rightarrow R$ is the control variable, and $f$ is a real valued continuously differential function. Along with the controlled process (3-4) a cost functional of the form

\[ J = \int_{0}^{1} L(x(t), u(t), \dot{x}(t)) dt \]

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J = ∫₀¹ L(t, x(t), u(t))dt

(5)

is defined.

There are admissible control are always assume that pass through (0, x₀) and (1, x₁) and in the set of controls, the control variable is searched which minimizes J and call it optimal control.

4. The proposed algorithm

The following approximate for x(·) is first consider which is in terms the Hermite polynomials Hₖ(·), k = 0, 1, 2

x₁(t) = a₀H₀(t) + a₁H₁(t) + a₂H₂(t)

(6)

Using the boundary condition (4), yields:

a₀ = x₀ + 2a₂ and a₁ = \frac{x₁-x₀}{2} - 2a₂

(7)

By substitution of (7) into (6), we obtain

x₁(t) = a₂H₂(t) + \left(\frac{x₁-a₂}{2} - a₂\right)H₁(t) + \left(x₀ + 2a₂\right)H₀(t)

(8)

The control variable u(·) are then obtained using eq.(3). Then, substituting x₁(t) and u(t), we obtain J as a function of a₂. The solution of the optimal control problem (3-4) is J(a’) (a’ is the value which minimizes J(a₃)).

The state and control variables are also found from a’ approximately.

In the second step, the following approximated is use

x₂(t) = x₁(t) + a₁H₁(t) + a₂H₂(t) + a₃H₃(t)

(9)

Using the boundary conditions (4) one can obtain

x₂(0) = x₁(0) + a₁H₁(0) + a₂H₂(0) + a₃H₃(0)

(10)

x₂(1) = x₁(1) + a₁H₁(1) + a₂H₂(1) + a₃H₃(1)

(11)

From (10-11) we have

a₂ = 0 and a₁ = 2a₃

(12)

In this case the solution of optimal control problem (3-4) is J(a’) where a’ is the value which minimizes J(a₃).

In general, the approximate solution in the nᵗʰ step will be

xₙ(t) = xₙ₋₁(t) + aₙ₋₁Hₙ₋₁(t) + aₙHₙ(t) + aₙ₊₁Hₙ₊₁(t)

(13)

Using the first condition x(0)=x₀ to get

xₙ(0) = xₙ₋₁(0) + aₙ₋₁Hₙ₋₁(0) + aₙHₙ(0) + aₙ₊₁Hₙ₊₁(0) - x₀ + aₙ₋₁Hₙ₋₁(0) + aₙHₙ(0) + aₙ₊₁Hₙ₊₁(0) = 0

(14)

Form the second condition of (4) we obtained

aₙ₋₁Hₙ₋₁(1) + aₙHₙ(1) + aₙ₊₁Hₙ₊₁(1) = 0

(15)

We solve the equation (14) and (15) simultaneously to obtain aₙ₋₁ and aₙ as a function of aₙ₊₁ as follows:

Multiply eq.(14) and (15)by Hₙ(1)and Hₙ(0) respectively ,yields:

Hₙ(1)(aₙ₋₁Hₙ₋₁(0) + aₙHₙ(0) + aₙ₊₁Hₙ₊₁(0)) = 0

Hₙ(0)(aₙ₋₁Hₙ₋₁(1) + aₙHₙ(1) + aₙ₊₁Hₙ₊₁(1)) = 0

From the above equations ,one can get

aₙ₋₁ = \frac{Hₙ(0)Hₙ₊₁(1) - Hₙ(1)Hₙ₊₁(0)}{Hₙ₋₁(0)Hₙ₊₁(1) - Hₙ₋₁(1)Hₙ₊₁(0)}aₙ₊₁

aₙ = \frac{Hₙ₋₁(0)Hₙ(1) - Hₙ₋₁(1)Hₙ(0)}{Hₙ₋₁(0)Hₙ₊₁(1) - Hₙ₋₁(1)Hₙ₊₁(0)}aₙ₊₁

The denominator in Eq.(16) and (17) are not zero as illustrate in the following lemma.

Lemma (1):

The result of

Hₙ₋₁(0)Hₙ(1) - Hₙ₋₁(1)Hₙ(0)

is not zero.

Proof : If n is even ,then (18) becomes

H₂m₋₁(0)H₂m(1) - H₂m₋₁(1)H₂m(0) , m=0,1,2

Since we have H₂m(0) = (-1)^m \frac{(2m)!}{m!}

And H₂m₋₁(0) = 0 , Therefore

H₂m₋₁(1)H₂m(0) - H₂m₋₁(0)H₂m(1) ≠ 0

Now if n is odd ,then Hₙ(0) = 0

⇒ Hₙ₋₁(1)Hₙ(0) - Hₙ₋₁(0)Hₙ(1) = -Hₙ₋₁(0)Hₙ(1)

Hₙ₋₁(0) = H₂m(0) m = 0,1,2, ...

Hence

-H₂m(0)Hₙ(1) = (-1)^m \frac{(2m)!}{m!}Hₙ(1)

= (-1)^{m+1} \frac{(2m)!}{m!}Hₙ(1)

Therefore

Hₙ₋₁(1)Hₙ(0) - Hₙ₋₁(0)Hₙ(1) ≠ 0

The proposed algorithm can be summarized by the following steps:

Step 1: Choose an ε > 0.

Step 2: For n=1 , calculate :

x₁(t) = a₂H₂(t) + \left(\frac{x₁-a₂}{2} - a₂\right)H₁(t) + \left(x₀ + 2a₂\right)H₀(t)

And then calculate a₂.

Step 3: For n=2 , calculate

x₂(t) = x₁(t) + a₁H₁(t) + a₂H₂(t) + a₃H₃(t)

Set a₀=0 and a₁=2a₃ calculate a₃,

Step 4: For n ⇒ n + 1 , calculate

xₙ(t) = xₙ₋₁(t) + aₙ₋₁Hₙ₋₁(t) + aₙHₙ(t) + aₙ₊₁Hₙ₊₁(t)

Set

aₙ₋₁ = \frac{Hₙ(0)Hₙ₊₁(1) - Hₙ(1)Hₙ₊₁(0)}{Hₙ₋₁(0)Hₙ₊₁(1) - Hₙ₋₁(1)Hₙ₊₁(0)}aₙ₊₁

aₙ = \frac{Hₙ₋₁(0)Hₙ(1) - Hₙ₋₁(1)Hₙ(0)}{Hₙ₋₁(0)Hₙ₊₁(1) - Hₙ₋₁(1)Hₙ₊₁(0)}aₙ₊₁

5. Numerical Examples

The efficiency of the proposed algorithms is illustrated by some examples which have analytical solutions, so that the validation of the method can
be allowed by comparing with the results of the exact solution.

Example (1)
This example concerns with the minimization of
\[ J = \frac{1}{2} \left( x(t) - \frac{1}{2} u^2(t) \right) dt \]  
(19)

Subject to
\[ \dot{x}(t) = u(t) - x(t) \]  
(20)

With boundary conditions
\[ x(0) = 0, \quad x(1) = \frac{1}{2} \left( 1 - \frac{1}{e} \right)^2 \]  
(21)

Where the analytical solution is:
\[ x(t) = 1 - \frac{1}{2} e^{t-1} + \left( \frac{1}{3e} - 1 \right) e^{-t} \]  
(22)
\[ u(t) = 1 - e^{t-1} \]  
(23)

Consider on approximation of \( x_1(t) \) to be:
\[ x_1(t) = a_0 h_0(t) + a_1 h_1(t) + a_2 h_2(t) \]  
(24)

Using the boundary conditions (21) yields:
\[ a_0 = 2a_2 \]  
(25)
\[ a_1 = \frac{1}{4} \left( 1 - \frac{1}{e} \right)^2 - 2a_2 \]  
(26)

Relations (25-26) are substituted into (24) to get the
\[ x_1(t) = 2a_2 h_0 + \left( \frac{1}{4} \left( 1 - \frac{1}{e} \right)^2 - 2a_2 \right) h_1 + a_2 h_2 \]  
(27)

The control variable \( u(t) \) can be found from Eq.(20) with the use of Eq.(24) to be
\[ u(t) = 2a_2 h_0 + \left( \frac{1}{4} \left( 1 - \frac{1}{e} \right)^2 - 2a_2 \right) h_1 + a_2 h_2 + 2a_2 h_0 + \left( \frac{1}{4} \left( 1 - \frac{1}{e} \right)^2 - 2a_2 \right) h_1 + a_2 h_2 \]  
(28)

Then substituted the Eqs.(24)and (26) into Eq.(19)
,we obtain J as a function of \( a^2 \)
\[ J = \frac{1038}{1947} - \frac{44}{15} a_2^2 - \frac{1081}{1801} a_2 \]  

The value which minimize J is \( a^* = a_2 = -0.1023 \)
then \( J(a^*) = 0.08401526 \)
In addition \( a_0 = -0.2046 \) and \( a_1 = 0.3045 \).
The state and control variables can be calculated approximately as
\[ x_1 = \frac{5485}{9007} t - \frac{7370}{1801} t^2 \]
\[ u = \frac{5485}{9407} - \frac{9428}{4504} t - \frac{7370}{1801} t^2 \]

Now the approximated solution can be modified as below
\[ x_2(1) = x_1(1) + a_1 h_1(1) + a_2 h_2(1) + a_3 h_3(1) \]  
(29)

And the results of repeated the above procedure are summarized as follows:
\[ a_1 = - \frac{3599}{1801} + \frac{1}{4} \left( 1 - \frac{1}{e} \right)^2 + 2a_3 \]
\[ a_2 = \frac{1399}{2306} t - \frac{7370}{1801} t^2 + \frac{2191}{1153} t^3 \]
\[ u = \frac{1399}{2306} - \frac{4871}{1153} t - \frac{2191}{1153} t^3 \]

And the value of \( J^* : 0.08401684 \).

The approximate results are listed in table (1) and are plotted in Figure (1) and Figure (2).

| time | x    | u    | u    |
|------|------|------|------|
| 0    | 0    | 0.6089 |
| 0.1  | 0.0568 | 0.5839 |
| 0.2  | 0.1054 | 0.5007 |
| 0.3  | 0.1459 | 0.5093 |
| 0.4  | 0.1781 | 0.4597 |
| 0.5  | 0.2022 | 0.4020 |
| 0.6  | 0.2181 | 0.3360 |
| 0.7  | 0.2258 | 0.2619 |
| 0.8  | 0.2253 | 0.1796 |
| 0.9  | 0.2166 | 0.0891 |
| 1    | 0.1998 | -0.0096 |
| exact | 0.0840456 |

Figure (1) State vector
Figure (2) Optimal control vector

Example (2): The performance index to be minimized is
\[ J = \frac{1}{2} \int_{0}^{1} \left( 3x^2(t) + u^2(t) \right) dt \]  
(30)
\[ \dot{x}(t) = u(t) - x(t) \]  
(31)
\[ x(0) = 0 \quad x(1) = 2 \]  
(32)
we will solved by expanding \( x(t) \) into two order Hermite series.
\[ N = 2 \] the state variable can be written as
\[ x_1(t) = a_0 H_0(t) + a_1 H_1(t) + a_2 H_2(t) \]  
(33)
And the same steps above in example (1) we obtain
\[ x_1 = \frac{4}{7} t + \frac{10}{7} t^2 \]  
(34)
\[ u = \frac{4}{7} t + \frac{24}{7} t + \frac{10}{7} t^2 \]  
and the result value of \( J = 6.1905 \).
The modified equation of \( x_1 \) is
\[ x_2 = x_1 + a_0 H_0 + a_2 H_2 + a_3 H_3 \]  
(35)
And re-sequencing solution steps, such as the first example. The value of \( J \) is 6.0693.
The approximate results are listed in table (2).

| \( n \) | \( J \) |
|-------|-------|
| 1     | 6.6667 |
| 2     | 6.1905 |
| 2(modified method) | 6.0693 |

Example (3):
Consider the following quadratic optimal control problem
Minimize
\[ J = \int_{0}^{1} \left( x^2(t) + u^2(t) \right) dt \]  
(36)
\[ \dot{x}(t) = u(t) \]  
(37)
we approximate the state variable by 2nd order series of unknown parameters.
\[ x_1 = a_0 H_0 + a_1 H_1 + a_2 H_2 \]  
(38)
The first result of \( x_1 \) is
\[ x_1 = \frac{17}{44} t + \frac{5}{44} t^2 \]  
\[ u = \frac{17}{44} t + \frac{10}{44} t \]  
the value of \( J = 0.3286 \).And the value of \( J \) becomes 0.32857867 after use the modified equation \( x_2 = x_1 + a_1 H_1 + a_2 H_2 + a_3 H_3 \) 
And we can use other the approximated solution x as
\[ x_2 = x_1 + a_0 H_0 + a_2 H_2 + a_3 H_3 \]  
And the value of \( J = 0.328587046 \). See Table (3).

Table (3)

| \( n \) | \( J \) |
|-------|-------|
| 2     | 0.3286 |
| 2(modified method) | 0.32857867 |

6. Conclusion
The proposed algorithm for treating optimal control problem depending on Hermite polynomial and their propertied provided a simple way to obtain an optimal control with fast convergence.

References
[1] A. A. Abdurrahman, “Numerical Solution of Optimal Control Problems Using New Third Kind Chebyshev Wavelets Operational Matrix of Integration”, Eng. & Tech. Journal .Vol.32, Part (B), No.1 , 2014
[2] Ch. T. & Y.C.,2010, “An approximate method for numerically solving fractional order optimal control problems of general form”, Computers and Mathematics with Applications 59, PP 1644–1655.
[3] k. B.& D. A., 2013 “A Numerical Approach for Solving the optimal Control Problems Using the Boubakn Polynomial Expansion Scheme”. Journal of interpolation and Approximation in scientific computing, PP 1-18.
[4] M. F.& G. , E., 2016 “A Chebyshev technique for the Solution of optimal control problem with nonlinear programming methods”, Mathematics and computer in simulation V.121, pp.95-105.
[5] k. B.&D. A., 2012 "Application of Chebyshev polynomials to derive efficient algorithms for the solution of optimal control problems”. Scientia Leonica 19(3), 795-805,2012.
[6] k. B.&D. A.,2012“Numerical Solution of nonlinear optimal control problems based on state parameterization”, IJST, (Special issue-Mathematics) ;pp. 331-340.
[7] M. H.&B. A., “A Numerical Method for Solving optimal control problems using state parameterization”. Numan Algo.42:PP 165-169, 2006.
[8] L. Q.and L. R., 2014, “The control parameterization methods for nonlinear optimal control”, A survey. Journal of industrial and management optimization Vol.10, No.1, pp.(275-309).
[9] L. H. and T. K., “Control parameterization enhancing technique for optimal discrete –valued control problems”, Automatica Vol.35, pp.1401-1407, 1999.
[10] H.J.& A. M., 2014 “An Iterative Technique for Solving a Class of Nonlinear Quadratic Optimal control Problems Using Chebyshev Polynomials”, I.J. Intelligent Systems and Applications, Vol.06, PP.53-57.

[11] E. T., “Eigen function approximation methods for linearly-solvable optimal control problems”, in IEEE ADPRL, 2009.

[12] I. Chr., J. C. and B. K., 2007 “Classical and relaxed optimization methods for Optimal Control Problems”, International Mathematical Forum, Vol.2, No.30, pp.1477 – 1498,

[13] H. R., S. S., H. A., A. J. 2012 January “An Iterative procedure for optimal control of bilinear systems”, International Journal of Instrumentation and Control Systems (IJICS) Vol.2, No.1.

[14] H. A. Ali, 2012 ”Approximate Solution for Linear Time- Delayed Improper Integral Equation Using Orthogonal Polynomials”, Eng. & Tech. Journal, Vol.30, No.1.

Author(s) biography
Saba Sattar Hasen,: Msc., applied mathematics, applied science dep., UOT,.Baghdad-Iraq.
She researched in optimal control and numerical analysis field and this paper specializes in optimal control problems. Lecturer Hasen.