ψ*-closed sets in fuzzy topological spaces

M. A. Abd Allah* and A. S. Nawar

Abstract

In this paper, we introduce a new class of fuzzy sets, namely, fuzzy ψ*-closed sets for fuzzy topological spaces, and some of their properties have been proved. Further, we introduce fuzzy ψ*-continuous, fuzzy ψ*-irresolute functions, and fuzzy ψ*-closed (open) functions, as applications of these fuzzy sets, fuzzy T_{1/5}-spaces, fuzzy T_{ψ^*}
-spaces, and fuzzy ψ^*T_{1/5}-spaces.

Keywords: Fuzzy ψ*-closed sets, Fuzzy T_{1/5}-spaces, Fuzzy ψ*-continuous, Fuzzy ψ*-closed functions

Mathematical subject classification: 54 C 10, 54 A 40, 54 A 05

Introduction

Zadeh [1] introduced the fundamental concept of fuzzy sets and fuzzy set operations in 1965. Fuzzy topology was introduced by Chang [2] in 1965. Subsequently, many researchers have worked on various basic concepts from general topology using fuzzy sets and developed the theory of fuzzy topological spaces [3–7]. Muthukumaraswamy and Devi [8] introduced fuzzy generalized α–closed and fuzzy α–generalized closed (briefly fα-g-closed and fαg-closed) sets in fuzzy topological space in 2004. Abd Allah and Nawar [9] introduced and studied ψ*-closed sets in topological space in 2014. In this paper, we introduced another new notion of fuzzy generalized closed set called fuzzy ψ*-closed sets, which is properly placed in between the class of fuzzy α-closed sets and the class of fuzzy generalized α-closed sets. The structure of the rest of this paper is as follows. The “Preliminaries” section introduces the necessary definitions of fuzzy α-closed sets and fuzzy generalized α-closed sets. In the “Fuzzy ψ*-closed sets in fts” section, we introduce the definition of fuzzy ψ*-closed sets in fuzzy topological spaces and proved some of their properties. In the “Fuzzy ψ*-continuous and fuzzy ψ*-irresolute functions in fts” section, we identify the concept of fuzzy ψ*-continuous and fuzzy ψ*-irresolute functions and fuzzy ψ*-closed (open) functions and introducing some of their properties. Further, new classes of spaces, namely, fuzzy T_{1/5}-spaces, fuzzy T_{ψ^*}-spaces, and fuzzy ψ^*T_{1/5}-spaces, are introduced in the “Applications of Fψ*-closed sets” section.

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**Preliminaries**

Throughout this paper, \((G, \tau)\) and \((H, \sigma)\) (or simply, \(G\) and \(H\)) always mean fuzzy topological spaces. The members of \(\tau\) are called fuzzy open sets, and their complements are fuzzy closed sets. And \(\phi : (G, \tau) \to (H, \sigma)\) (or simply, \(\phi : G \to H\)) denotes a mapping \(\phi\) from \(G\) to \(H\).

For a fuzzy set \(D\) of \((G, \tau)\), fuzzy closure and fuzzy interior of \(D\) denoted by \(\text{cl}(D)\) and \(\text{int}(D)\), respectively and are defined by \(\text{cl}(D) = \bigwedge \{E : E\text{ is fuzzy closed set of }G, E \supseteq D, 1 - E \in \tau\}\) and \(\text{int}(D) = \bigvee \{S : S\text{ is fuzzy open set of }G, S \subseteq D, S \in \tau\}\) [10].

**Definition 2.1** A fuzzy set \(D\) of a fts \(G\) is called fuzzy \(\alpha\)-open (briefly, F-open) if \(D \subseteq \text{int}(\text{cl}(D))\) and a fuzzy \(\alpha\)-closed (briefly, F\(\alpha\)-closed) if \(D \supseteq \text{cl}(\text{int}(D))\) [4]; the intersection of all fuzzy \(\alpha\)-closed sets of \((G, \tau)\) containing \(D\) is called fuzzy \(\alpha\)-closure of a fuzzy subset \(D\) of \(G\) and is denoted by \(\text{acl}(D)\).

**Definition 2.2** A fuzzy set \(D\) of a fts \(G\) is called fuzzy generalized \(\alpha\)-closed (briefly, \(\text{Fg}\alpha\)-closed) [8] if \(\text{acl}(D) \subseteq U\) whenever \(D \subseteq U\) and \(U\) is fuzzy \(\alpha\)-open in \((G, \tau)\). The complement of \(\text{Fg}\alpha\)-closed set is called F\(\alpha\)-open set.

**Definition 2.3** Let \((G, \tau)\) and \((H, \sigma)\) be two fuzzy topological spaces. A function \(\phi : (G, \tau) \to (H, \sigma)\) is called as follows:

(i) \(\text{Fa-continuous}\) [10] if \(\phi^{-1}(V)\) is \(\text{ Fa}\)-closed in \(G\), for each \(V \in \text{FC}(H)\);

(ii) \(\text{Fg}\alpha\)-continuous [8] if \(\phi^{-1}(V)\) is \(\text{Fg}\alpha\)-closed in \(G\), for each \(V \in \text{FC}(H)\);

(iii) \(\text{F-irresolute}\) [11] if \(\phi^{-1}(V)\) is \(\text{F}\)-closed in \(G\), for each \(V \in \text{FC}(H)\).

**Definition 2.4** A function \(\phi : (G, \tau) \to (H, \sigma)\) is said to be fuzzy-open (fuzzy-closed) [2] if the image of every fuzzy open (fuzzy-closed) set in \(G\) is fuzzy-open (fuzzy-closed) set in \(H\).

**Fuzzy \(\psi\)-closed sets in fts**

In this section, we introduce fuzzy \(\psi\)-closed sets in fuzzy topological space and discuss some of its characterizations and relationships with other notions.

**Definition 3.1** A fuzzy set \(D\) in \((G, \tau)\) is called fuzzy \(\psi\)-closed (\(\text{F}\psi\)-closed) if \(\text{acl}(D) \subseteq U\) whenever \(D \subseteq U\) and \(U\) is F\(\gamma\)-open in \((G, \tau)\). The complement of \(\text{F}\psi\)-closed set is called \(\text{F}\psi\)-open set.

The class of fuzzy \(\psi\)-closed sets of fts \((G, \tau)\) is denoted by \(\text{F}\psi\mathcal{C}(G)\).

**Proposition 3.1** Every fuzzy \(\alpha\)-closed set is fuzzy \(\psi\)-closed.

**Proof** Let \(D\) be a \(\text{Fa}\)-closed set in \((G, \tau)\), and since every \(\text{Fa}\)-closed set is \(\text{Fg}\alpha\)-closed. Then, \(\text{acl}(D) \subseteq U\) whenever \(D \subseteq U\) and \(U\) is \(\text{Fa}\)-open in \((G, \tau)\), and since every \(\text{Fa}\)-open set is \(\text{Fg}\alpha\)-open. So, \(\text{acl}(D) \subseteq U\) whenever \(D \subseteq U\) and \(U\) is \(\text{Fg}\alpha\)-open in \((G, \tau)\). Thus, \(D\) is \(\text{F}\psi\)-closed.

The converse of Proposition 3.1 needs not be true as seen from the following example.

**Example 3.1** Let \(G = \{a, b, c\}\) with fuzzy topology \(\tau = \{0, 1, \{a_{0.5}, b_{0.2}, c_{0.7}\}, \{a_{0.7}, b_{0.8}, c_{0.3}\}, \{a_{0.5}, b_{0.2}, c_{0.3}\}, \{a_{0.7}, b_{0.8}, c_{0.7}\}\}\). The fuzzy subset \(D = \{a_{0.4}, b_{0.8}, c_{0.7}\}\) is \(\text{F}\psi\)-closed set in \((G, \tau)\) but not \(\text{Fa}\)-closed set since \(\text{cl}(\text{int}(\text{cl}(D))) = \{a_{0.5}, b_{0.8}, c_{0.7}\}\).

**Proposition 3.2** Every fuzzy \(\psi\)-closed set is fuzzy \(\text{g}\alpha\)-closed set.

**Proof** Follows from the fact that every \(\text{Fa}\)-open set is \(\text{Fg}\alpha\)-open.
The converse of Proposition 3.2 needs not be true as seen from the following example.

**Example 3.2** In Example 3.1, the fuzzy subset $D = \{a_{0.5}, b_{0.3}, c_{0.7}\}$ is Fgα-closed set in $(G, r)$ but not $F^\alpha$-closed set.

**Proposition 3.3** If $D$ and $E$ are $F^\alpha$-closed sets in $(G, r)$, then $D \cup E$ is also $F^\alpha$-closed set in $(G, r)$.

**Proof** If $D \cup E \subseteq U$ and $U$ are Fgα-open, then $D \subseteq U$ and $E \subseteq U$. Since $D$ and $E$ are $F^\alpha$-closed, $\text{acl}(D) \subseteq U$ and $\text{acl}(E) \subseteq U$, and hence $\text{acl}(D \cup E) = \text{acl}(D) \cup \text{acl}(E) \subseteq U$. Thus, $D \cup E$ is $F^\alpha$-closed set in $(G, r)$.

**Proposition 3.4** If $D$ is Fgα-open set and fuzzy $\psi^*$-closed set in $(G, r)$, then $D$ is fuzzy $\alpha$-closed set in $(G, r)$.

**Proof** Since $D \subseteq D$ and $D$ is Fgα-open set and $F^\alpha$-closed, then $\text{acl}(D) \subseteq D$. Since $D \subseteq \text{acl}(D)$, then $D = \text{acl}(D)$, and thus $D$ is Fgα-open set in $(G, r)$.

**Proposition 3.5** Every fuzzy $\psi^*$-open set is fuzzy ga-open.

**Proof** Let $D \in F^\alpha\sigma(G)$. Then, $1 - D \in F^\alpha\sigma(G)$ and hence Fgα-closed set in $(G, r)$ by Proposition 3.2. This implies that $D$ is Fgα-open set in $(G, r)$. Hence, every $F^\alpha$-open set in $G$ is Fgα-open set in $G$.

**Proposition 3.6** If $D$ is $F^\alpha$-closed set in $(G, r)$ and $D \subseteq E \subseteq \text{acl}(A)$, then $E$ is $F^\alpha$-closed set of $(G, r)$.

**Proof** Let $U$ be a Fgα-open subset of $(G, r)$ such that $E \subseteq U$. Then, $D \subseteq U$ and since $D \in F^\alpha\sigma(G)$, then $\text{acl}(D) \subseteq U$. Now, $\text{acl}(E) \subseteq \text{acl}(D) \subseteq U$. Then, $E \in F^\alpha\sigma(G)$.

**Corollary 3.1** If $D$ is $F^\alpha$-open set in $(G, r)$ and $\text{aint}(D) \subseteq E \subseteq D$, then $E$ is $F^\alpha$-open set.

**Proof** Let $D \in F^\alpha\sigma(G)$, and $\text{aint}(D) \subseteq E \subseteq D$. Then, $1 - D \in F^\alpha\sigma(G)$, and $1 - D \subseteq 1 - E \subseteq \text{acl}(1 - D)$. By Proposition 3.6, $1 - B \in F^\alpha\sigma(G)$. Hence, $E \in F^\alpha\sigma(G)$.

**Definition 3.2** For any fuzzy set $D$ in a fts $G$, we have the fuzzy $\psi^*$-interior of $D$ (briefly $\psi^*$-$\text{int}(D)$) is the union of all fuzzy $\psi^*$-open sets of $G$ contained in $D$. That is, $\psi^* - \text{int}(D) = \{E : E \subseteq D, E \text{ is } F^\alpha - \text{open in } G\}$.

**Definition 3.3** Let $(G, r)$ be a fuzzy topological space. Then, for a fuzzy subset $D$ of $G$, the fuzzy $\psi^*$-closure of $D$ (briefly $\psi^*$-$\text{cl}(D)$) is the intersection of all fuzzy $\psi^*$-closed sets of $G$ containing $D$. That is, $\psi^* - \text{cl}(D) = \{E : E \supseteq D, E \text{ is } F^\alpha - \text{closed in } G\}$.

**Proposition 3.7** For any fuzzy sets $D$ and $B$ in a fts $G$, we have as follows:

(i) $\psi^* - \text{int}(D) \subseteq D$.

(ii) $D$ is $F^\alpha - \text{open} \Leftrightarrow \psi^* - \text{int}(D) = D$.

(iii) $\psi^* - \text{int}(\psi^* - \text{int}(D)) = \psi^* - \text{int}(D)$.

(iv) If $D \subseteq B$, then $\psi^* - \text{int}(D) \subseteq \psi^* - \text{int}(B)$.

**Proof** (i) Follows from Definition 3.3.

(ii) Let $D \in F^\alpha\sigma(G)$. Then, $D \subseteq \psi^* - \text{int}(D)$. By using (i), we get $D = \psi^* - \text{int}(D)$. Conversely, assume that $D = \psi^* - \text{int}(D)$. By using Definition 3.3, $D \in F^\alpha\sigma(G)$.

(iii) By using (ii), we get $\psi^* - \text{int}(\psi^* - \text{int}(D)) = \psi^* - \text{int}(D)$.

(iv) Since $D \subseteq E$ by using (ii), $\psi^* - \text{int}(D) \subseteq D \subseteq E$. That is, $\psi^* - \text{int}(D) \subseteq E$. By (iii), $\psi^* - \text{int}(D) \subseteq \psi^* - \text{int}(E)$. Thus, $\psi^* - \text{int}(D) \subseteq \psi^* - \text{int}(E)$.

**Proposition 3.8** For any fuzzy sets $D$ and $E$ in a fts $G$, we have as follows:
(i) \( \psi^* - \text{int}(D \land E) = \psi^* - \text{int}(D) \land \psi^* - \text{int}(E) \).

(ii) \( \psi^* - \text{int}(D \lor E) \geq \psi^* - \text{int}(D) \lor \psi^* - \text{int}(E) \).

**Proof**

(i) Since \( D \land E \leq D \) and \( D \land E \leq E \), by using Proposition 3.7 (iv), we get \( \psi^* - \text{int}(D \land E) \leq \psi^* - \text{int}(D) \) and \( \psi^* - \text{int}(D \land E) \leq \psi^* - \text{int}(E) \). Thus,

\[ \psi^* - \text{int}(D \land E) \leq \psi^* - \text{int}(D) \land \psi^* - \text{int}(E). \]  

(1)

By using Proposition 3.7 (i), we have \( \psi^* - \text{int}(D) \leq D \) and \( \psi^* - \text{int}(E) \leq E \). This implies that \( \psi^* - \text{int}(D) \land \psi^* - \text{int}(E) \leq D \land E \). Now applying Proposition 3.7 (iv), we get \( \psi^* - \text{int}(\psi^* - \text{int}(D) \land \psi^* - \text{int}(E)) \leq \psi^* - \text{int}(D \land E) \). By (1), \( \psi^* - \text{int}(\psi^* - \text{int}(D) \land \psi^* - \text{int}(E)) \leq \psi^* - \text{int}(D \land E) \). By using Proposition 3.7 (iii),

\[ \psi^* - \text{int}(D \land E) \land \psi^* - \text{int}(E) \leq \psi^* - \text{int}(D \land E). \]  

(2)

Forms (1) and (2), \( \psi^* - \text{int}(D \land E) = \psi^* - \text{int}(D) \land \psi^* - \text{int}(E) \).

(ii) Since \( D \leq D \lor E \) and \( E \leq D \lor E \), by using Proposition 3.7 (iv), we have \( \psi^* - \text{int}(D) \leq \psi^* - \text{int}(D \lor E) \) and \( \psi^* - \text{int}(E) \leq \psi^* - \text{int}(D \lor E) \). Thus, \( \psi^* - \text{int}(D \lor E) \leq \psi^* - \text{int}(D \lor E) \).

The equality in Proposition 3.8 (ii) need not be hold as seen from the following example.

Example 3.3 In Example 3.1, consider \( D = [a_{0.4}, b_{0.8}, c_{0.7}] \) and \( E = [a_{0.6}, b_{0.8}, c_{0.3}] \). Then, \( \psi^* - \text{int}(D) = 0 \), and \( \psi^* - \text{int}(E) = [a_{0.6}, b_{0.2}, c_{0.3}] \). That implies \( \psi^* - \text{int}(D) \lor \psi^* - \text{int}(E) = [a_{0.6}, b_{0.2}, c_{0.3}] \). Now, \( D \lor E = [a_{0.6}, b_{0.8}, c_{0.7}] \); it follows that \( \psi^* - \text{int}(D \lor E) = [a_{0.6}, b_{0.8}, c_{0.7}] \). Then, \( \psi^* - \text{int}(D \lor E) \neq \psi^* - \text{int}(D) \lor \psi^* - \text{int}(E) \).

**Proposition 3.9** For any fuzzy set \( D \) in a fts \( G \), we have as follows:

(i) \( (\psi^* - \text{int}(D))^c = \psi^* - \text{cl}(D^c) \)

(ii) \( (\psi^* - \text{cl}(D))^c = \psi^* - \text{int}(D^c) \)

**Proof**

(i) By using Definition 3.3, \( \psi^* - \text{int}(D) = \lor \{ E : E \leq D, \ E \in F \psi^*\text{O}(G) \} \). Taking complement on both sides, we get as follows:

\[ (\psi^* - \text{int}(D))^c = (\lor \{ E : E \leq D, \ E \in F \psi^*\text{open in } G \})^c \]

\[ = \inf \{ E^c : E^c \geq D^c, \ E^c \in F \psi^*\text{closed in } G \}. \]

Replacing \( E \) by \( C \), we get

\[ (\psi^* \text{int}(D))^c = \land \{ C : C \geq D^c, \ C \in F \psi^*\text{ - closed in } G \}. \]

By Definition 3.4, \( (\psi^* - \text{int}(D))^c = \psi^* - \text{cl}(D^c) \).

(ii) By using (i), \( (\psi^* - \text{int}(D))^c = \psi^* - \text{cl}(D^c) = \psi^* - \text{cl}(D) \). Taking complement on both sides, we get \( \psi^* - \text{int}(D^c) = (\psi^* - \text{cl}(D))^c \).

**Proposition 3.10** Let \( D \) be a fuzzy set in a fts \( G \). Then, \( D \in F \psi^*\text{C}(G) \) if and only if \( D^c \) is \( F \psi^*\text{-open}. \)

**Proposition 3.11** For any fuzzy sets \( D \) and \( E \) in a fts \( G \), we have as follows:

(i) \( D \leq \psi^* - \text{cl}(D) \)

(ii) \( D \) is \( F \psi^*\text{-closed} \iff \psi^* - \text{cl}(D) = D \)

(iii) \( \psi^* - \text{cl}(\psi^* - \text{cl}(D)) = \psi^* - \text{cl}(D) \)
(iv) If \( D \leq E \), then \( \psi^* \cdot \text{cl}(D) \leq \psi^* \cdot \text{cl}(E) \)

**Proof**

(i) Follows from Definition 3.4.

(ii) Let \( D \in F \psi^* C(G) \). By using Proposition 3.10, \( D^* \in F \psi^* O(G) \). By using Proposition 3.9 (ii), \( \psi^* \cdot \text{int}(D^*) = D^* \Rightarrow (\psi^* \cdot \text{cl}(D))^\circ = \psi^* \cdot \text{cl}(D) = D \).

(iii) By using (ii), we get \( \psi^* - \text{cl}(\psi^* - \text{cl}(D)) = \psi^* - \text{cl}(D) \).

(iv) If \( D \land E \leq D \) and \( D \land E \leq E \) By using Proposition 3.7 (iv), \( \psi^* - \text{int}(E^*) \leq \psi^* - \text{int}(D^*) \). Taking complement on both sides, we get \( (\psi^* - \text{int}(E^*))^c \geq (\psi^* - \text{int}(D^*))^c \). By using Proposition 3.9 (ii), \( \psi^* - \text{cl}(E) \geq \psi^* - \text{cl}(D) \).

**Proposition 3.12**

Let \( D \) be a fuzzy set in a fts \( G \). Then, \( \text{int}(D) \leq \alpha - \text{int}(D) \leq \psi^* - \text{int}(D) \leq \psi^* - \text{cl}(D) \leq \alpha - \text{cl}(D) \leq \text{cl}(D) \).

**Proof**

It follows from the definition of corresponding operators.

**Proposition 3.13**

For any fuzzy sets \( D \) and \( E \) in a fts \( G \), we have as follows:

(i) \( \psi^* \cdot \text{cl}(D \lor E) = \psi^* \cdot \text{cl}(D) \lor \psi^* \cdot \text{cl}(E) \)

(ii) \( \psi^* \cdot \text{cl}(D \land E) \leq \psi^* \cdot \text{cl}(D) \land \psi^* \cdot \text{cl}(E) \)

**Proof**

(i) Since \( \psi^* \cdot \text{cl}(D \lor E) = \psi^* \cdot \text{cl}((D \lor E)^c)^c \), by using Proposition 3.9 (i), we have \( \psi^* \cdot \text{cl}(D \lor E) = (\psi^* \cdot \text{cl}(D \lor E))^c = (\psi^* \cdot \text{cl}(D))^c \lor (\psi^* \cdot \text{cl}(E))^c = (\psi^* \cdot \text{int}(D)^c \lor (\psi^* \cdot \text{int}(E))^c) \).

By using Proposition 3.9 (i), we have \( \psi^* \cdot \text{cl}(D \lor E) = \psi^* \cdot \text{cl}(D^c) \lor \psi^* \cdot \text{cl}(E^c) = \psi^* \cdot \text{cl}(D) \lor \psi^* \cdot \text{cl}(E) \).

(ii) Since \( D \land E \leq D \) and \( D \land E \leq E \), by using Proposition 3.11 (iv), we have \( \psi^* - \text{cl}(D \land E) \leq \psi^* - \text{cl}(D) \) and \( \psi^* - \text{cl}(D \land E) \leq \psi^* - \text{cl}(E) \). This implies that \( \psi^* - \text{cl}(D \land E) \leq \psi^* - \text{cl}(D) \land \psi^* - \text{cl}(E) \).

**Proposition 3.14**

For any fuzzy sets \( D \) and \( E \) in a fts \( G \), we have as follows:

(i) \( \psi^* - \text{cl}(D) \geq D \lor \psi^* - \text{cl}(\psi^* - \text{int}(D)) \)

(ii) \( \psi^* - \text{int}(\psi^* - \text{cl}(D)) \leq \text{int}(\psi^* - \text{cl}(D)) \)

(iii) \( \text{int}(\psi^* - \text{cl}(D)) \leq \text{int}(\psi^* - \text{cl}(D)) \)

(iv) \( \text{int}(\psi^* - \text{cl}(D)) \geq \text{int}(\psi^* - \text{cl}(D)) \)

**Proof**

(i) By Proposition 3.11 (i), \( D \leq \psi^* - \text{cl}(A) \). Again, using Proposition 3.7 (i), \( \psi^* - \text{int}(D) \leq D \). Then, \( \psi^* - \text{cl}(\psi^* - \text{int}(D)) \leq \psi^* - \text{cl}(D) \).

(ii) By Proposition 3.7 (i), \( \psi^* - \text{int}(D) \leq D \). Again, using Proposition 3.11(i), \( D \leq \psi^* - \text{cl}(D) \). Then, \( \psi^* - \text{int}(D) \leq \psi^* - \text{int}(\psi^* - \text{cl}(D)) \). Then, \( \psi^* - \text{int}(D) \leq \psi^* - \text{int}(\psi^* - \text{cl}(D)) \).

(iii) By Proposition 3.12, \( \psi^* - \text{cl}(D) \leq \text{cl}(D) \). We get \( \text{int}(\psi^* - \text{cl}(D)) \leq \text{int}(\text{cl}(D)) \).

(iv) By (i), \( \psi^* - \text{cl}(D) \leq D \lor \psi^* - \text{cl}(\psi^* - \text{int}(D)) \). Then, \( \text{int}(\psi^* - \text{cl}(D)) \geq \text{int}(D \lor \psi^* - \text{cl}(\psi^* - \text{int}(D))) \). Since \( \text{int}(D \lor E) \geq \text{int}(D) \lor \text{int}(E) \), \( \text{int}(\psi^* - \text{cl}(D)) \geq \text{int}(D) \lor \text{int}(\psi^* - \text{cl}(\psi^* - \text{int}(D))) \).

**Fuzzy \( \psi^* \)-continuous and fuzzy \( \psi^* \)- irresolute functions in FTS**

As application of fuzzy \( \psi^* \)-closed set, we identify some types of fuzzy functions and introducing some of their properties.
**Definition 4.1** A function $\phi : (G, r) \to (H, s)$ is said to be fuzzy $\psi^*$-continuous (F$\psi^*$-continuous) if $\phi^{-1}(V)$ is F$\psi^*$-closed in $H$, for each fuzzy closed set $V$ in $H$.

**Proposition 4.1** Every F$\alpha$-continuous function is F$\psi^*$-continuous.

**Proof** Let $V \in FC(H)$. Since $\phi$ is F$\alpha$-continuous, then $\phi^{-1}(V)$ is F$\alpha$-closed in $G$. Since every F$\alpha$-closed set is F$\psi^*$-closed set, then $\phi^{-1}(V) \in F\psi^*C(G)$. Thus, $\phi$ is F$\psi^*$-continuous.

The converse of Proposition 4.1 need not be true as seen from the following example.

**Example 4.1** Suppose that $G = \{a, b, c\}$ with fuzzy topology $\tau = \{0, 1, \{a_{0.5}, b_{0.2}, c_{0.7}\}, \{a_{0.7}, b_{0.8}, c_{0.3}\}, \{a_{0.5}, b_{0.2}, c_{0.3}\}, \{a_{0.7}, b_{0.8}, c_{0.7}\}\}$ and $H = \{x, y, z\}$ with fuzzy topology $\sigma = \{0, 1, \{x_{0.8}, y_{0.2}, z_{0.3}\}\}$. Let $\phi : (G, r) \to (H, s)$ be defined by $\phi(a) = x$, $\phi(b) = y$, and $\phi(c) = z$. $\phi$ is F$\psi^*$-continuous function, but it is not a F$\alpha$-continuous function, since $V = \{x_{0.2}, y_{0.8}, z_{0.7}\} \in FC(H)$ but $\phi^{-1}(V) \notin FA\psi^*C(G)$.

**Proposition 4.2** Every F$\psi^*$-continuous function is F$\gamma$-continuous.

**Proof** Let $V \in FC(H)$. Since $\phi$ is F$\psi^*$-continuous, then $\phi^{-1}(V) \in F\psi^*C(G)$. By Proposition 3.1, every F$\psi^*$-closed set is F$\gamma$-closed set; then, $\phi^{-1}(V) \in F\gamma C(G)$. Thus, $\phi$ is F$\gamma$-continuous.

The converse of Proposition 4.2 need not be true as seen from the following example.

**Example 4.2** Suppose that $G = \{a, b, c\}$ with fuzzy topology $\tau = \{0, 1, \{a_{0.5}, b_{0.2}, c_{0.7}\}, \{a_{0.7}, b_{0.8}, c_{0.3}\}, \{a_{0.5}, b_{0.2}, c_{0.3}\}, \{a_{0.7}, b_{0.8}, c_{0.7}\}\}$ and $H = \{x, y, z\}$ with fuzzy topology $\sigma = \{0, 1, \{x_{0.5}, y_{0.6}, z_{0.3}\}\}$. Let $\phi : (G, r) \to (H, s)$ be defined by $\phi(a) = x$, $\phi(b) = y$, and $\phi(c) = z$. $\phi$ is F$\gamma$-continuous function, but it is not a F$\psi^*$-continuous function, since $V = \{x_{0.5}, y_{0.4}, z_{0.7}\} \in FC(Y)$ but $\phi^{-1}(V) \notin F\psi^*C(X)$.

**Definition 4.2** A function $\phi : (G, r) \to (H, s)$ is said to be F$\psi^*$-irresolute (F$\psi^*$-irr) if $\phi^{-1}(V) \in F\psi^*C(G)$, for each F$\psi^*$-closed set $V$ in $H$.

**Proposition 4.3** Every F$\psi^*$-irresolute function is F$\psi^*$-continuous.

**Proof** It follows from the definitions.

The converse of Proposition 4.3 need not be true as seen from the following example.

**Example 4.3** In the Example 4.1, Let $\phi : (G, r) \to (H, s)$ be defined by $\phi(a) = x$, $\phi(b) = y$, and $\phi(c) = z$. $\phi$ is F$\psi^*$-continuous function, but it is not a F$\psi^*$-irresolute function, since $V = \{x_{0.2}, y_{0.7}, z_{0.4}\} \in F\psi^*C(H)$ but $\phi^{-1}(V) \notin F\psi^*C(G)$.

**Proposition 4.4** Let $\phi : G \to H$ and $\gamma : H \to W$ be any two functions. Then, as follows:

(i) $\gamma \circ \phi$ is F$\psi^*$-continuous if $g$ is fuzzy continuous, and $\phi$ is F$\psi^*$-continuous.

(ii) $\gamma \circ \phi$ is F$\psi^*$-irresolute if both $\phi$ and $g$ are F$\psi^*$-irresolute.

(iii) $\gamma \circ \phi$ is F$\psi^*$-continuous if $g$ is F$\psi^*$-continuous, and $\phi$ is F$\psi^*$-irresolute.

**Proof** Let $V \in FC(W)$. Since $\gamma$ is fuzzy continuous, then $\gamma^{-1}(V) \in FC(H)$. Since $\phi$ is F$\psi^*$-continuous, then we have $\phi^{-1}(\gamma^{-1}(V)) \in F\psi^*C(G)$. Consequently, $\gamma \circ \phi$ is F$\psi^*$-continuous.

(ii) - (iii) By similarity.

**Applications of F$\psi^*$-closed sets**

As applications of F$\psi^*$-closed sets, three fuzzy spaces, namely, fuzzy $T_{1/5}$-spaces, fuzzy $T_{1/5}^\psi$-spaces, and fuzzy $\psi^*T_{1/5}$-spaces are introduced.
We introduce the following definitions.

**Definition 5.1** A fuzzy topological space \((G, r)\) is called as follows:

(i) Fuzzy \(T_{1/5}\)-space if every \(F_{\alpha}\)-closed set in \(G\) is a \(F\)-closed set in \(G\).

(ii) Fuzzy \(T_{1/5}^{\psi}\)-space if every \(F_{\psi}\)-closed set in \(G\) is a \(F\)-closed set in \(G\).

(iii) Fuzzy \(T_{1/5}^{\psi}\)-space if every \(F_{\psi}\)-closed set in \(G\) is a \(F\)-closed set in \(G\).

**Proposition 5.1** If \(\phi : G \rightarrow H\) is \(F_{\psi}\)-continuous and \(G\) is fuzzy \(T_{1/5}^{\psi}\)-space; then \(\phi\) is \(F\)-continuous.

**Proof** Let \(V \in FC(H)\); since \(f\) is \(F_{\psi}\)-continuous, then \(\phi^{-1}(V) \in FC(G)\). Since \(G\) is \(F\) \(T_{1/5}^{\psi}\)-space, then \(\phi^{-1}(V)\) is \(F\)-closed set in \(G\). Thus, \(\phi\) is \(F\)-continuous.

**Proposition 5.2** If \(\phi : G \rightarrow H\) is \(F_{\psi}\)-irresolute and \(G\) is fuzzy \(T_{1/5}^{\psi}\)-space, then \(\phi\) is \(F\)-continuous.

**Proof** By Theorem 5.1.

**Proposition 5.3** If \(\phi : G \rightarrow H\) is \(F_{\alpha}\)-continuous and \(G\) is fuzzy \(T_{1/5}^{\psi}\)-space, then \(\phi\) is \(F_{\psi}\)-continuous.

**Proof** Let \(V \in FC(H)\); since \(\phi\) is \(F_{\alpha}\)-continuous, then \(\phi^{-1}(V)\) is \(F_{\alpha}\)-closed set in \(G\).

Since \(G\) is \(F_{\psi}\) \(T_{1/5}^{\psi}\)-space, then \(\phi^{-1}(V)\) is \(F_{\psi}\)-closed set in \(G\). Thus, \(\phi\) is \(F_{\psi}\)-continuous.

**Proposition 5.4** Let \(\phi : G \rightarrow H\) be onto \(F_{\psi}\)-irresolute and \(F\)-closed. If \(G\) is fuzzy \(T_{1/5}^{\psi}\)-space, then \(H\) is also a fuzzy \(T_{1/5}^{\psi}\)-space.

**Proof** Let \(V \in FC(H)\); since \(f\) is \(F_{\psi}\)-irresolute, then \(\phi^{-1}(V) \in FC(G)\). Since \(G\) is \(F\) \(T_{1/5}^{\psi}\)-space, then \(\phi^{-1}(V)\) is \(F\)-closed set in \(G\). Since \(\phi\) is \(F\)-closed and onto, then we have \(V\) is \(F\)-closed. Therefore, \(H\) is also a \(F\) \(T_{1/5}^{\psi}\)-space.

**Proposition 5.5** Let \(G, H, W\) be \(f\)-ss, and \(\phi : G \rightarrow H\), \(\gamma : H \rightarrow W\) and \(\gamma \circ \phi : G \rightarrow W\) be functions, then if \(\phi\) is \(F\)-irresolute function and \(\gamma\) is \(F_{\psi}\)-continuous function, such that \(H\) is fuzzy \(T_{1/5}^{\psi}\)-space. Then, \(\gamma \circ \phi\) is \(F\)-continuous function.

**Proof** Let \(U \in FC(W)\); since \(\gamma\) is \(F_{\psi}\)-continuous, then \(\gamma^{-1}(U)\) is \(F_{\psi}\) \(T_{1/5}^{\psi}\)-space, then \(\gamma^{-1}(U)\) is \(F\)-closed set in \(H\). But \(\phi\) is \(F\)-irresolute function, then \(\phi^{-1}(\gamma^{-1}(U))\) is \(F\)-closed set in \(G\). But \(\phi^{-1}(\gamma^{-1}(U)) = (\gamma \circ \phi)^{-1}(U)\). Therefore, \(\gamma \circ \phi\) is \(F\)-continuous function.

**Definition 5.2** A map \(\phi : (G, r) \rightarrow (H, \sigma)\) is said to be \(F_{\psi}\)-open (\(F_{\psi}\)-closed) if the image of every open (closed) fuzzy set in \(G\) is \(F_{\psi}\)-open (closed) set in \(H\).

**Proposition 5.6** Every fuzzy-open map is fuzzy \(\psi\)-open map.

**Proof** The proof follows from the Definition 5.2.

The converse of Proposition 5.6 need not be true as seen from the following example.

**Example 5.1** Suppose that \(G = \{a, b, c\}\) with fuzzy topology \(r = \{0, 1, \{a_{0.8}, b_{0.2}, c_{0.3}\}\}\), and \(H = \{x, y, z\}\) with fuzzy topology \(\sigma = \{0, 1, \{x_{0.35}, y_{0.2}, z_{0.7}\}, \{x_{0.7}, y_{0.8}, z_{0.3}\}, \{x_{0.5}, y_{0.8}, z_{0.3}\}\}\). Let \(\phi : (G, r) \rightarrow (H, \sigma)\) be defined by \(\phi(a) = x, \phi(b) = y, \phi(c) = z\). \(\phi\) is \(F_{\psi}\)-open map, but it is not a \(F\)-open map, since \(G = \{a_{0.8}, b_{0.2}, c_{0.3}\} \in FO(G)\) but \(\phi(G) \notin FO(H)\).

**Proposition 5.7** Every fuzzy-closed map is \(F_{\psi}\)-closed map.

**Proof** The proof follows from the Definition 5.2.

The converse of Proposition 5.7 need not be true as seen from the following example.
Example 5.2 In the Example 5.1, let $\phi : (G, r) \to (H, \sigma)$ be defined by $\phi (a) = x$, $\phi (b) = y$, and $\phi (c) = z$. $\phi$ is $F\psi^*$-closed map, but it is not an F-closed map, since $V = \{a_{0.2}, b_{0.8}, c_{0.7}\} \in FC(G)$ but $\phi(V) \notin FC(H)$.

Proposition 5.8 If $\phi : G \to H$ is F-closed map and $\gamma : H \to W$ is $F\psi^*$-closed map, then $\gamma \circ \phi : G \to W$ is $F\psi^*$-closed map.

Conclusion
In this paper, we have defined a new class of fuzzy sets, namely, fuzzy $\psi^*$-closed sets for fuzzy topological spaces, which is properly placed in between the class of fuzzy $\alpha$-closed sets and the class of fuzzy generalized $\alpha$-closed sets. We have also investigated some properties of these fuzzy sets. Fuzzy $\psi^*$-continuous, fuzzy $\psi^*$-irresolute functions, and fuzzy $\psi^*$-closed (open) functions have been introduced. We have proved that every $F\psi^*$-continuous function is $F\alpha_{\gamma}$-continuous, but the converse need not be true, and the composition of two $F\psi^*$-irresolute functions is $F\psi^*$-irresolute. Fuzzy $T_{1/5}$-spaces, fuzzy $T_{1/5}\psi^*$-spaces, and fuzzy $\psi^*T_{1/5}$-spaces have been established as applications of fuzzy $\psi^*$-closed set. In the future, we will generalize this class of fuzzy sets in fuzzy bitopological spaces, and some applied examples should be given.

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