A Two-fluid Model for Plasma with Prandtl Number Correction

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Abstract

A two-fluid model is derived from the plasma kinetic equations using the moment model reduction method. The moment method we adopt was recently developed with a globally hyperbolic regularization where the moment model attained is locally well-posed in time. Based on the hyperbolic moment model with well-posedness, the Maxwellian iteration method is utilized to get the closure relations for the resulted two-fluid model. By taking the Shakhov collision operator in the Maxwellian iteration, the two-fluid model inherits the correct Prandtl number from the plasma kinetic equations. The new model is formally the same as the five-moment two-fluid model except for the closure relations, where the pressure tensor is anisotropic and the heat flux is presented. This provides the model the capacity to depict problems with anisotropic pressure tensor and large heat flux.

Keywords: Plasma kinetic equations; Maxwellian iteration; two-fluid model; moment model reduction

1 Introduction

The multi-species plasma model was proposed to capture dynamics in systems with multiple species of charges [17]. Each species is treated separately and coupled together through electromagnetic and collisional processes. In the collisionless problems, the evolution may be most accurately [24, 31] modeled using the kinetic theory such as the Vlasov-Maxwell equations, where the plasma is described by the distribution functions of all species, and the electromagnetic fields are governed by Maxwell equations. However, it is extremely expensive to numerically solve the plasma kinetic equations due to the high dimensionality of the plasma kinetic equations [26].

Several reduced models were introduced [10, 21] to approximate the plasma kinetic equations. When plasma behaves like an electrically conducting fluid, where the motion of the electrons and ions is locked together by electrostatic forces, the MHD model is utilized [8]. In the MHD model, the plasma is treated as a single electrically conducting fluid. Several algorithms have been designed based on the MHD model, which are already used to simulate several plasma phenomena successfully [16, 23]. It was pointed out that the MHD model ignores the electron mass and the finite Larmor radius effects [3]. This may lead to the limitation to treat the Hall effect and diamagnetic terms [14]. Hall effect can be added to the MHD equations and we get the Hall-MHD model [12]. Though a distinction is made between the bulk plasma...
velocity and electron velocity in the Hall-MHD model, electron inertia and displacement current are still ignored, and the plasma is assumed to be quasi-neutral [11].

In order to capture the separate motion of the electrons and ions, the two-fluid plasma model was proposed without adding the complexity of the kinetic model [26]. The electromagnetic fields are also modeled using Maxwell equations of electromagnetism. This model was derived [23, 19] by taking moments of the plasma kinetic equations for each species and is also a generalization of the MHD model. The two-fluid plasma model retains electron inertia effects and displacement current. However, directly taking moments of the plasma kinetic equations will lead to an unclosed equation system and different closure methods will deduce various kinds of two-fluid plasma models. For example, assuming there is no heat flow, one obtains the five-moment ideal two-fluid equations with scalar fluid pressure [11, 26]. The strongly collisional plasma can be accurately described by the five-moment model. In the weaker collisional regime, the anisotropies increase that the five-moment model is not valid any more [11]. Several higher-order moment descriptions of the plasma have been developed respectively for the collisionless and collisional transport in plasma. For example, the ten- and sixteen-moment hydrodynamic models are derived for the collisionless regime [32, 27, 22]. The ten- and thirteen-moment models for capturing collisional transport in mixed gases and magnetic plasma are also derived [21, 10, 29, 13, 9].

For reduced models involving only a few macroscopic variables, it is of great importance to predict the physical Prandtl number. However, only a few works have been done to preserve the correct Prandtl number, though various versions of the two-fluid model have been derived. In this paper, we are aiming at deriving a new two-fluid model based on the plasma kinetic model with the correct Prandtl number. The method we adopt is based on the moment model reduction method developed in [11, 13] and the classical Maxwellian iteration [13, 30]. First, the distribution function is expanded into a Hermite series [3], and the hyperbolic moment equations (HME) are derived for the plasma kinetic model [6] under the framework [2]. The method in [2] was first developed for the Boltzmann equation, and the distribution function is approximated by the Hermite expansion around local Maxwellian. The globally hyperbolic regularization method [1] is adopted to get the hyperbolic moment equations, which is locally well-posed in time. Here we follow the same approach therein to get the HME system for the plasma kinetic model with Shakhov collision operator. Then, the Maxwellian iteration is applied to get the closure relations for the shear stress and heat flux based on the HME system. These closure relations have the same form as that in [4], but the Prandtl number in the Shakhov collision model is inherited to the resulted two-fluid model. Consequently, the new model is equipped with the capacity to predict the correct Prandtl number, which is also formally validated by the demonstrative examples. In comparison, this new reduced model is formally the same as the five-moment model in [11] except for the closure relations. The five-moment model has isotropic pressure and neglects the heat flux. In this new model, the pressure tensor is anisotropic and the heat flux is presented, where both terms are expressed by the density, macroscopic velocity and temperature. We hope that the new reduced model may provide improved performance for the problem with not negligible anisotropic pressure tensor and heat flux for different Prandtl numbers.

The rest of this paper is arranged as follows. In Section 2 the two-fluid model for the plasma and the plasma kinetic model are introduced briefly. In Section 3 we make the Hermite expansion to the kinetic plasma model and then derive the hyperbolic moment equations (HME). The new reduced two-fluid model is deduced in Section 4, where the closure relations for the shear stress and the heat flux are derived by Maxwellian iteration. We present demonstrative examples in Section 5 to validate our new model. A short conclusion in Section 6 closes the
2 Preliminary

In the collisionless plasma where the collective interactions dominate the plasma dynamics, the Vlasov-Maxwell (VM) equations are mostly used to describe the evolution of the plasma. When the distribution function of the particles is not far from Maxwellian, the movement of the plasma can be described by the two-fluid model [11, 18]. In this section, we will introduce the two-fluid plasma model and the VM model briefly.

2.1 Two-fluid plasma model

The two-fluid model, which captures the separate motion of the electrons and ions, is derived by taking moments of the plasma kinetic model for each species. This process eliminates the microscopic velocity space, and at the same time, the macroscopic variables, including the density, momentum and energy, are utilized to describe the evolution of the plasma. The detailed form of the two-fluid model is as below

\[
\frac{dn_\beta}{dt} + n_\beta (\nabla_x \cdot u_\beta) = 0, \\
m_\beta n_\beta \frac{du_\beta}{dt} = n_\beta q_\beta [E + u_\beta \times B] - \nabla_x p_\beta - \nabla_x \cdot \sigma_\beta, \\
\frac{3}{2} n_\beta \frac{dT_\beta}{dt} = -n_\beta T_\beta (\nabla_x \cdot u_\beta) - \nabla_x \cdot q_\beta - \sigma_\beta \cdot \nabla_x u_\beta,
\]

where \( \beta = i, e \), represents electrons and ions respectively. The expression \( \sigma \cdot \nabla_x u \) is defined as

\[
\sigma \cdot \nabla_x u = \sum_{i,j=1}^{3} \sigma_{ij} \frac{\partial u_j}{\partial x_i}.
\]

\( m_\beta \) and \( q_\beta \) is the mass and charge of the particles. \( n_\beta, u_\beta \) and \( T_\beta \) represent the density, macroscopic velocity and temperature. Moreover, \( p_\beta \) is the isotropic pressure, \( \sigma_\beta \) is the shear stress tensor, and \( q_\beta \) is the heat flux. \( E \) and \( B \) are the electric and magnetic field respectively, which is decided by the Maxwell equations, the exact form of which is as below

\[
\partial B \partial t + \nabla_x \times E = 0, \\
\frac{1}{c^2} \frac{\partial E}{\partial t} - \nabla_x \times B = -\nu_0 j, \\
\nabla_x \cdot E = \frac{\rho_e}{\epsilon_0}, \quad \nabla_x \cdot B = 0.
\]

Here \( \nu_0 \) and \( \epsilon_0 \) are the permeability and permittivity of free space [10], and \( c = (\nu_0 \epsilon_0)^{-1/2} \) is the speed of light. \( \rho_e \) and \( j \) are the charged density and current density defined by

\[
\rho_e = \sum_{\beta=i,e} q_\beta n_\beta, \quad j(t, x) = \sum_{\beta=i,e} q_\beta n_\beta u_\beta.
\]
the anisotropic stress $\sigma_\beta$ and the heat flux $q_\beta$ are set as zero. Combined with the equation of state for the ideal gas, we can get the closed five-moment ideal two-fluid model \[11\]. Similarly, we can also get the ten-moment two-fluid equations \[10\]. In \[4\], the closure of $\sigma$ and $q$ are derived in the kinetic theory of gas with the BGK collision term

$$q = -\kappa \nabla_x T, \quad \sigma = -2\mu \left( \frac{1}{2} \left( \nabla_x u + (\nabla_x u)^T \right) - \frac{1}{3} I (\nabla_x \cdot u) \right),$$

(2.5)

where $\kappa$ is the coefficient of heat conductivity and $\mu$ is the coefficient of viscosity. Since only one species is considered in the kinetic theory of gas, the footnote index $\beta$ in (2.5) is eliminated. However, in (2.5) the BGK collision model will lead to wrong Prandtl number, which equals $1$ for the ideal gas, while the correct one equals $2/3$. Therefore, other collision models should be utilized to get the correct Prandtl number.

Based on the Shakhov collision model, we will try to deduce the two-fluid model with the correct Prandtl number based on the kinetic plasma equations. In the next section, the plasma kinetic equations will be introduced briefly.

### 2.2 Plasma kinetic equations

In the plasma kinetic equations, the plasma is described by the distribution functions $f(t, x, v)$, which depends on time, physical space and microscopic velocity space. The plasma kinetic equations then have the form below

$$\frac{\partial f_\beta}{\partial t} + v \cdot \nabla_x f_\beta + \frac{q_\beta}{m_\beta} (E + v \times B) \cdot \nabla_v f_\beta = \frac{\partial f_\beta}{\partial t} \bigg|_{\text{collision}}, \quad (x, v) \in \Omega \times \mathbb{R}^3,$$

(2.6)

where $\beta$ also represents the ions, or electrons respectively, and $\Omega \in \mathbb{R}^3$. The electron-magnetic fields $(E, B)$ are given by the classical Maxwell system (2.3). In the VM model, where for the low-collisional plasma, the collisions are neglected and collective interactions are assumed to dominate the plasma dynamics, which means that

$$\frac{\partial f_\beta}{\partial t} \bigg|_{\text{collision}} = 0.$$  

(2.7)

In what follows, we are focusing on the derivation of the closed two-fluid model with correct Prandtl number. Therefore, the Shakhov collision model is applied \[25\], whose exact form is

$$Q_{\text{Shakhov}}(f_\beta) = \nu_\beta (f_\beta^s - f_\beta), \quad f_\beta^s = P_\beta(t, x, v) M_\beta(t, x, v),$$

(2.8)

with

$$P_\beta = 1 + \frac{(1 - \Pr)(v - u_\beta) \cdot q_\beta}{(D + 2) \rho_\beta(t, x) T_\beta(t, x)} \left( \frac{|v - u_\beta|^2}{T_\beta(t, x)} - (D + 2) \right), \quad D = 3,$$

(2.9)

where $\nu_\beta$ is the collision frequencies, $D$ is the dimension number of the microscopic velocity space and $M_\beta(t, x, v)$ is Maxwellian, which has the form

$$M_\beta(t, x, v) = \frac{n_\beta}{(2\pi T_\beta)^{3/2}} \exp \left( -\frac{(v - u_\beta)^2}{2T_\beta} \right).$$

(2.10)

Moreover, $\Pr$ is the Prandtl number which is decided by the type of the particles. $u_\beta$, $T_\beta$ and $q_\beta$ are the macroscopic variables defined in the last section, whose relationships with the
distribution function $f_{\beta}(t, x, v)$ are as below

\[
\begin{align*}
    n_{\beta} &= \int_{\mathbb{R}^3} v f_{\beta} \, dv, \\
    n_{\beta} u_{\beta} &= \int_{\mathbb{R}^3} v f_{\beta} \, dv, \\
    \frac{3}{2} n_{\beta} T_{\beta} &= \frac{1}{2} \int_{\mathbb{R}^3} |v - u_{\beta}|^2 f_{\beta} \, dv, \\
    q_{\beta} &= \frac{1}{2} \int_{\mathbb{R}^3} |v - u|^2 (v - u_{\beta}) f_{\beta} \, dv, \\
    p_{\beta,ij} &= \int_{\mathbb{R}^3} (v_i - u_{\beta,i})(v_j - u_{\beta,j}) f_{\beta} \, dv, \\
    p_{\beta} &= \frac{1}{3} \sum_{i=1}^{3} p_{\beta,ii}.
\end{align*}
\]

(2.11)

The pressure $p_{\beta}$ is defined as

\[
p_{\beta} = \frac{1}{3} \sum_{i=1}^{3} p_{\beta,ii}.
\]

(2.12)

Since the plasma kinetic equation is seven-dimensional, it is only used to simulate the plasma to capture the essential physics [26]. Some simplified models are introduced to simulate the evolution of the plasma, just by taking moments of the plasma kinetic equations.

Different collision models and closure methods will lead to different two-fluid models. In the following sections, we will consider the Shakhov collision model purposely for a correct Prandtl number in the reduced model.

3 Moment Model Reduction

In this section, we will first derive the hyperbolic moment model for the plasma kinetic equations, which is the base for us to deduce the new two-fluid model. For simplicity, we consider at first the single-species case of the non-relativistic electrons under the self-consistent electromagnetic fields while the ions are treated as a uniform fixed background. The whole procedure can be trivially extended to the general plasma.

Without loss of generality, we consider the dimensionless form of the governing equations for one species. Eliminating the footnote index for the distribution function $f_{\beta}(t, x, v)$, the dimensionless equation for the electrons is as below

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = \nu(f^s - f),
\]

\[(x, v) \in \Omega \times \mathbb{R}^3, \quad (3.1)\]

and the dimensionless form for the classical Maxwell system is

\[
\begin{cases}
    \frac{\partial E}{\partial t} - \nabla_x \times B &= -j, \\
    \frac{\partial B}{\partial t} + \nabla_x \times E &= 0, \\
    \nabla_x \cdot E &= \rho - h, \\
    \nabla_x \cdot B &= 0,
\end{cases}
\]

(3.2)

where the relations of the current density $j$ and charge density $\rho$ with the distribution function $f$ are reduced into

\[
\begin{align*}
    j(t, x) &= \int_{\mathbb{R}^3} f(t, x, v) v \, dv, \\
    \rho(t, x) &= \int_{\mathbb{R}^3} f(t, x, v) \, dv,
\end{align*}
\]

(3.3)

with $h$ the initial background density satisfying

\[
\int_{\Omega} (\rho - h) \, dx = 0.
\]

(3.4)
The relationship between the macroscopic variables and the distribution function is listed in (2.11). Besides, the Maxwellian is reduced into
\[ M(t, x, v) = \frac{\rho}{(2\pi T)^{3/2}} \exp\left(\frac{-(v - u)^2}{2T}\right). \]  
(3.5)

Following the method in [3], we approximate the distribution function using Hermite basis functions as
\[ f(v) \approx \sum_{|\alpha| \leq M} f_{\alpha} \mathcal{H}_{\alpha}(\xi), \quad \xi = \frac{v - u}{\sqrt{T}}, \]  
(3.6)

where \( M \) is the expansion order, and \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) is a three-dimensional multi-index. The basis functions \( \mathcal{H}_{\alpha} \) are defined as
\[ \mathcal{H}_{\alpha}(\xi) = \frac{3}{\sqrt{2\pi}} \prod_{d=1}^{3} \frac{1}{\sqrt{T^{\frac{\alpha_d+1}{2}}}} \text{He}_{\alpha_d}(\xi_d) \exp\left(\frac{-\xi_d^2}{2}\right), \]  
(3.7)

where \( \text{He}_n(x) \) is the Hermite polynomial
\[ \text{He}_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right). \]  
(3.8)

For convenience, \( \text{He}_n(x) \) is taken as zero if \( n < 0 \), thus \( \mathcal{H}_{\alpha}(\xi) \) is zero when any component of \( \alpha \) is negative. Based on the expansion (3.6), we can find that the Maxwellian \( M \) in (3.5) equals
\[ M(t, x, v) = f_0(t, x) \mathcal{H}_{\alpha}(\xi), \quad \xi = \frac{v - u}{\sqrt{T}}, \]  
(3.9)

and it holds for the coefficients \( f_{\alpha} \) that
\[ f_0 = \rho(t, x), \quad f_{e_i} = 0, \quad \sum_{d=1}^{3} f_{2e_d} = 0, \quad i = 1, 2, 3. \]  
(3.10)

Moreover, the relationship between heat flux \( q_i \), shear stress \( \sigma_{ij} \) and the expansion coefficients \( f_{\alpha} \) are
\[ \sigma_{ij} = (1 + \delta_{ij})f_{e_i+e_j}, \quad q_i = 2f_{3e_i} + \sum_{d=1}^{3} f_{2e_d+e_i}, \quad i, j = 1, 2, 3, \]  
(3.11)

where the relationship between the shear stress \( \sigma_{ij} \) and the distribution function is
\[ \sigma_{ij} = \int_{\mathbb{R}} \left((v_i - u_i)(v_j - u_j) - \frac{1}{3} \delta_{ij} |v - u|^2\right) f dv, \quad i, j = 1, 2, 3. \]  
(3.12)

With the equation of state of the ideal gas, we can derive the relationship between the pressure tensor and the shear stress as
\[ p = \rho T, \quad \sigma_{ij} = p_{ij} - \delta_{ij} p. \]  
(3.13)

Below we briefly introduce the moment systems for (3.1), and refer [6] for the detailed procedure. With the relationship
\[ \frac{\partial \mathcal{H}_{\alpha}(\xi)}{\partial v_j} \mathcal{H}_{\alpha}(\xi) = -\mathcal{H}_{\alpha+e_j}(\xi), \]  
(3.14)
\[ v_j \mathcal{H}_{\alpha}(\xi) = T \mathcal{H}_{\alpha+e_j}(\xi) + u_j \mathcal{H}_{\alpha}(\xi) + \alpha_j \mathcal{H}_{\alpha-e_j}(\xi), \]  
(3.15)
we can get that
\[ \nabla_v \cdot \left[(\mathbf{E} + \mathbf{v} \times \mathbf{B})f\right] = \]
\[ - \sum_{d=1}^{3} \left[ E_d - \sum_{k,m} \varepsilon_{dkm} u_k B_m \right] f_{\alpha - \epsilon_d} - \sum_{d,k,m=1}^{3} \varepsilon_{dkm} (\alpha_k + 1) B_m f_{\alpha - \epsilon_d + \epsilon_k}, \]

where the Levi-Civita symbols \( \varepsilon_{dkm} \) are defined as
\[ \varepsilon_{dkm} = \begin{cases} 
1, & d \neq k \neq m \text{ cyclic permutation of } 1,2,3, \\
-1, & d \neq k \neq m \text{ anti-cyclic permutation of } 1,2,3, \\
0, & (d-k)(k-m)(m-d) = 0.
\end{cases} \]

The Shakhov collision term can be expanded as
\[ \nu (f^s - f) = \nu \sum_{\alpha \in \mathbb{N}^3} Q_\alpha \mathcal{H}_{T,\alpha}(\xi), \quad \xi = \frac{\mathbf{v} - \mathbf{u}}{\sqrt{T}}, \]

with
\[ Q_\alpha = \begin{cases} 
0, & |\alpha| < 2, \\
\frac{1 - \Pr}{5} q_j - f_\alpha, & \alpha = 2e_i + e_j, \quad i,j = 1,2,3, \\
f_\alpha, & \text{otherwise}.
\end{cases} \]

By plugging the expansion (3.6) into (3.1), the general moment equations can be obtained as
\[ \begin{align*}
\frac{\partial f_\alpha}{\partial t} + \sum_{d=1}^{3} \left[ \frac{\partial u_d}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_d}{\partial x_j} - E_d - \sum_{k,m=1}^{3} \varepsilon_{dkm} u_k B_m \right] f_{\alpha - \epsilon_d} \\
- \sum_{d,k,m=1}^{3} \varepsilon_{dkm} (\alpha_k + 1) B_m f_{\alpha - \epsilon_d + \epsilon_k} + \frac{1}{2} \left( \frac{\partial T}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial T}{\partial x_j} \right) \sum_{d=1}^{3} f_{\alpha - 2\epsilon_d} \\
+ \sum_{j,d=1}^{3} \left[ \frac{\partial u_d}{\partial x_j} (T f_{\alpha - \epsilon_d - e_j} + (\alpha_j + 1) f_{\alpha - \epsilon_d + e_j} (1 - H(|\alpha| - M))) \\
+ \frac{1}{2} \frac{\partial T}{\partial x_j} (T f_{\alpha - 2\epsilon_d - e_j} + (\alpha_j + 1) f_{\alpha - 2\epsilon_d + e_j} (1 - H(|\alpha| - M))) \right] \\
+ \sum_{j=1}^{3} \left( T \frac{\partial f_{\alpha - e_j}}{\partial x_j} + u_j \frac{\partial f_{\alpha}}{\partial x_j} + (\alpha_j + 1) \frac{\partial f_{\alpha + e_j}}{\partial x_j} (1 - H(|\alpha| - M)) \right) = \nu Q_\alpha, \quad 0 \leq |\alpha| \leq M,
\end{align*} \]

where \( M \) is the expansion order and \( H[x] \) is defined as
\[ H[x] = \begin{cases} 
1, & x = 0, \\
0, & \text{otherwise}.
\end{cases} \]

Similar to that in [3], we can deduce the conservation of mass, momentum and the equation for
the temperature as
\[
\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \left( u_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial u_j}{\partial x_j} \right) = 0,
\]
\[
\frac{\partial u_d}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_d}{\partial x_j} + \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial p_d}{\partial x_j} = E_d + \sum_{k,m}^{3} \epsilon_{km} u_k B_m, \tag{3.22}
\]
\[
\rho \left( \frac{\partial T}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial T}{\partial x_j} \right) + \frac{2}{3} \sum_{j=1}^{3} \left( \frac{\partial p_d}{\partial x_j} + \sum_{d=1}^{3} \frac{\partial u_d}{\partial x_j} \right) = 0.
\]
Substituting \( p_{ij} \) in (3.22) with (3.12), we can derive that
\[
\frac{\partial u_d}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_d}{\partial x_j} + \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial \rho}{\partial x_j} + \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial p_d}{\partial x_j} = E_d + \sum_{k,m}^{3} \epsilon_{km} u_k B_m, \tag{3.23}
\]
\[
\rho \left( \frac{\partial T}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial T}{\partial x_j} \right) + \frac{2}{3} \sum_{j=1}^{3} \left( \frac{\partial q_j}{\partial x_j} + \sum_{d=1}^{3} \frac{\partial u_d}{\partial x_j} \right) = 0.
\]
Finally, substituting \( \frac{\partial u_d}{\partial t} \) and \( \frac{\partial T}{\partial t} \) in (3.20) with (3.23), we can get the quasi-linear system for the higher order moments of the Shakhov collision term as
\[
F_\alpha = -\nu f_\alpha, \quad \forall |\alpha| = 2,
\]
\[
F_\alpha = \nu \left( 1 - \frac{\Pr}{5} q_j - f_\alpha \right), \quad \alpha = 2e_i + e_j, \quad i, j = 1, 2, 3, \tag{3.24}
\]
\[
F_\alpha = -\nu f_\alpha, \quad \forall 3 \leq |\alpha| \leq M \text{ and } \alpha \neq 2e_i + e_j, \quad i, j = 1, 2, 3,
\]
where
\[
F_\alpha = \frac{\partial f_\alpha}{\partial t} + \sum_{j=1}^{3} \left( T \frac{\partial f_{\alpha - e_j}}{\partial x_j} + u_j \frac{\partial f_\alpha}{\partial x_j} + (\alpha_j + 1) \frac{\partial f_{\alpha + e_j}}{\partial x_j} (1 - H[|\alpha| - M]) \right)
\]
\[
+ \sum_{j=1}^{3} \sum_{d=1}^{3} \left( T^d f_{\alpha - e_d - e_j} + (\alpha_j + 1) f_{\alpha - e_d + e_j} (1 - H[|\alpha| - M]) - \frac{p_d}{3\rho} \sum_{k=1}^{3} f_{\alpha - 2e_k} \right) \frac{\partial u_d}{\partial x_j}
\]
\[
+ \sum_{k=1}^{3} \sum_{d=1}^{3} \left( T^d f_{\alpha - 2e_k - e_j} + (\alpha_j + 1) f_{\alpha - 2e_k + e_j} (1 - H[|\alpha| - M]) \right) \left( -T \frac{p_d}{2\rho \partial x_j} + \frac{1}{6\rho} \sum_{d=1}^{3} \frac{\partial p_j}{\partial x_j} \right)
\]
\[
- \sum_{j=1}^{3} \sum_{d=1}^{3} \frac{f_{\alpha - e_d}}{\rho} \frac{\partial p_d}{\partial x_j} - \frac{1}{3\rho} \left( \sum_{k=1}^{3} f_{\alpha - 2e_k} \right) \sum_{j=1}^{3} \frac{\partial q_j}{\partial x_j} - \sum_{d,k,m=1}^{3} \varepsilon_{km} (\alpha_k + 1) B_m f_{\alpha - e_d + e_k}. \tag{3.25}
\]
Together with (3.23) and (3.24), we derive the hyperbolic moment system for the one species plasma kinetic equations (3.11).
It is known that one can deduce the Euler and Navier-Stokes equations from the Boltzmann equation based on the zeroth- and first-order expansions of the distribution function. For the similarity of the Boltzmann and the plasma kinetic equation, we will try to deduce the two-fluid plasma model based on the moment equations.

4 New Two-Fluid Model

To derive our new two-fluid model based on HME for the plasma kinetic equations, we carry out a Maxwellian iteration to give the closure relations. The Maxwellian iteration was introduced by Ikenberry and Truesdell [13, 30] as a technique to derive the NSF and Burnett equations from the moment equations. It is then used to analyze the order for the magnitude of each moment, which is known as the COET (Consistently Ordered Extended Thermodynamics) method.

Here, Maxwellian iteration is utilized to derive models with the first order of accuracy. To begin with, we introduce the scaling \( t = \tau / \epsilon \) and \( x_i = x_i' / \epsilon \), and rewrite the moment equations \((3.24)\) with time and spatial variables \( t' \) and \( x_i' \). Thus, a factor \( \epsilon^{-1} \) is introduced to the right-side of \((3.24)\). In this section, we will work on the scaled equations, and the prime symbol \( t' \) and \( x_i' \) will be omitted. With some rearrangement, \((3.24)\) is rewritten as

\[
F_\alpha = -\frac{\nu}{\epsilon} f_\alpha, \quad \forall |\alpha| = 2,
\]

\[
F_\alpha = \frac{\nu}{\epsilon} \left( \frac{1 - \operatorname{Pr}_5}{5} q_j - f_\alpha \right), \quad \alpha = 2e_i + e_j, \quad i, j = 1, 2, 3,
\]

\[
F_\alpha = -\frac{\nu}{\epsilon} f_\alpha, \quad \forall 3 \leq |\alpha| \leq M, \text{ and } \alpha \neq 2e_i + e_j, \quad i, j = 1, 2, 3.
\]

Assuming that the asymptotic expansions for all the moments have the form

\[
f_\alpha = f_\alpha^{(0)} + \epsilon f_\alpha^{(1)} + \epsilon^2 f_\alpha^{(2)} + \cdots,
\]

we will begin the iteration based on \((4.1)\). In the original Maxwellian iteration method, we require that \( f_\alpha^{(0)} \) is the local Maxwellian \( M(t, x, v) \). The corresponding assumption for the coefficients is

\[
f_\alpha^{(0)} = \begin{cases} 
\rho, & \text{if } |\alpha| = 0, \\
0, & \text{otherwise.}
\end{cases}
\]

Then noting that \( f_\alpha = 0 \) if \( |\alpha| = 1 \) based on \((3.10)\), the iteration begins from \( |\alpha| \geq 2 \), which can be written as the iterative scheme below

\[
f_\alpha^{(n+1)} = -g_\alpha \left( f_\beta^{(n)} \right)_{|\beta| \in \mathbb{N}^3}, \quad \forall \alpha \in \mathbb{N}^3 \text{ and } |\alpha| \geq 2, \quad n = 0, 1, 2, \cdots
\]

and the macroscopic variables \( \rho, \ u \) and \( \mathcal{T} \) are treated as

\[
\rho = \mathcal{O}(1), \quad u = \mathcal{O}(1), \quad \mathcal{T} = \mathcal{O}(1).
\]

When \( |\alpha| = 2 \), \((4.1)\) is changed into

\[
f_\alpha = -\frac{\epsilon}{\nu} F_\alpha, \quad \forall |\alpha| = 2.
\]

In the iteration, \( \{f_\alpha, |\alpha| > 2\} \) are treated as of high order, and the terms \( \{f_\alpha, |\alpha| = 2\} \) in \( F_\alpha \) are also omitted, due to the \( \epsilon \) on the right hand side. Thus, we can easily deduce the approximation
to $f_{\alpha}$ when $|\alpha| = 2$ as

$$f_{2e_i}^{(1)} = -\frac{1}{\nu} \left[ -\frac{1}{3} \sum_{j=1}^{3} \left( \sum_{d=1}^{3} p_{jd} \frac{\partial u_d}{\partial x_j} + \frac{\partial q_{ij}}{\partial x_j} \right) + \rho T \frac{\partial u_i}{\partial x_i} \right], \quad i = 1, 2, 3, \quad (4.7)$$

$$f_{e_i+e_k}^{(1)} = -\frac{1}{\nu} T \rho \left( \frac{\partial u_d}{\partial x_j} + \frac{\partial u_j}{\partial x_d} \right), \quad i, k = 1, 2, 3, \quad i \neq k.$$

Similarly, when $|\alpha| = 3$, (4.1) is changed into

$$f_{\alpha} = -\frac{\epsilon}{\nu} F_{\alpha} + \frac{1}{5} \Pr q_j, \quad \alpha = 2e_i + e_j, \quad i, j = 1, 2, 3, \quad (4.8)$$

$$f_{\alpha} = -\frac{\epsilon}{\nu} F_{\alpha}, \quad \alpha = e_i + e_j + e_k, \quad i, j, k = 1, 2, 3, \quad \alpha \neq j \neq k.$$

From (3.11), we can find that $q$ is at the same order as $\{f_{\alpha}, |\alpha| = 3\}$. Assuming that the asymptotic expansion for heat flux $q_i$ is

$$q_i = q_i^{(0)} + \epsilon q_i^{(1)} + \epsilon^2 q_i^{(2)} + \cdots, \quad i = 1, 2, 3, \quad (4.9)$$

then $q_i^{(0)} = 0$ due to (4.3). Thus it holds for $\{f_{\alpha}^{(1)}, |\alpha| = 3\}$ that

$$f_{2e_i+e_k}^{(1)} = -\frac{1}{\nu} T \left( \frac{-T}{2} \frac{\partial \rho}{\partial x_k} + \frac{1}{6} \sum_{d=1}^{3} \frac{\partial \rho_{dd}}{\partial x_k} + \frac{1}{5} \Pr q_k^{(1)} \right), \quad i, k = 1, 2, 3, \quad (4.10)$$

$$f_{e_i+e_j+e_k}^{(1)} = 0, \quad i, j, k = 1, 2, 3, \quad \alpha \neq j \neq k.$$

With the relationship between the pressure tensor and the temperature (2.12) and (3.13), (4.10) is reduced into

$$f_{2e_i+e_k}^{(1)} = -\frac{1}{2\nu} \rho T \frac{\partial T}{\partial x_k} + \frac{1}{5} \Pr q_k^{(1)}, \quad i, k = 1, 2, 3, \quad (4.11)$$

$$f_{e_i+e_j+e_k}^{(1)} = 0, \quad i, j, k = 1, 2, 3, \quad \alpha \neq j \neq k.$$

Moreover, with the same method, we can derive that for the higher order of coefficients,

$$f_{\alpha}^{(1)} = 0, \quad |\alpha| \geq 4. \quad (4.12)$$

For now, the first order of iteration is finished. From (3.11) and the expression of $f_{\alpha}^{(1)}$, we can get the approximation to the shear stress and heat flux at order $O(\epsilon)$

$$\sigma_{ii} = 2f_{2e_i} \approx -\frac{2\epsilon}{\nu} \left[ -\frac{1}{3} \sum_{j=1}^{3} \left( \sum_{d=1}^{3} p_{jd} \frac{\partial u_d}{\partial x_j} + \frac{\partial q_{ij}}{\partial x_j} \right) + \rho T \frac{\partial u_i}{\partial x_i} \right], \quad i = 1, 2, 3, \quad (4.13)$$

$$\sigma_{ij} = f_{e_i+e_j} \approx -\frac{\epsilon}{\nu} \rho T \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i \neq j, \quad i, j = 1, 2, 3, \quad (4.14)$$

$$q_i = 2f_{3e_i} + \sum_{d=1}^{3} f_{2e_d+e_i} \approx -\frac{1}{\Pr} \frac{5 \epsilon}{2\nu} \rho T \frac{\partial T}{\partial x_i}, \quad i = 1, 2, 3. \quad (4.15)$$

From (4.14) and (4.15), we can find that $\sigma_{ij}$ and $q_i$ are all at order $O(\epsilon)$. Substituting (4.14) and (4.15) into (4.13) and omitting the terms at order $O(\epsilon^2)$, we can get the simplified form of (4.13) with (3.12)

$$\sigma_{ii} \approx -\frac{2\epsilon}{\nu} \rho T \left( \frac{\partial u_i}{\partial x_i} - \frac{1}{3} \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} \right), \quad i = 1, 2, 3. \quad (4.16)$$
Introducing the parameter viscosity conductivity $\mu$ and thermal conductivity $\kappa$, we can get the final closure term by letting $\epsilon \rightarrow 1$ as

$$\sigma_{ij} = -\mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k} \right), \quad q_i = -\kappa \frac{\partial T}{\partial x_i}, \quad i = 1, 2, 3,$$

(4.17)

where

$$\mu = \frac{\rho T}{\nu}, \quad \kappa = \frac{1}{\Pr} \frac{5 \rho T}{2 \nu}.$$  

(4.18)

Remark 1. From the definition of Prandtl number

$$\Pr = \frac{5 \mu}{2 \kappa},$$

(4.19)

we can find that (4.18) could give the correct Prandtl number for any type of particles. However, if the BGK collision model is utilized instead of the Shakhov collision model in the deduction, the wrong Prandtl number $\Pr = 1$ will be derived.

For the moment, we have got the approximation of the macroscopic variables such as the shear stress $\sigma_{ij}$ and the heat flux $q_i$ with the density $\rho$, macroscopic velocity $u$ and temperature $T$. Next, we will deduce the new reduced model with these approximations and the moment equations (3.22) and (3.23). Rewriting (3.23) into the vector form and adopting (4.17), we can get the equation system as

\[
\begin{cases}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \\
\frac{\rho}{2} \frac{\partial u}{\partial t} = -\nabla_p - \nabla \cdot \sigma + \rho (E + u \times B), \\
\frac{3}{2} \rho \frac{\partial T}{\partial t} = [(-p I - \sigma) \cdot \nabla] \cdot u - \nabla \cdot q,
\end{cases}
\]

(4.20)

where $\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + u \cdot \nabla$, and $I$ is the $3 \times 3$ unit matrix with

$$\sigma = (\sigma_{ij})_{3 \times 3} = -\mu \left[ \nabla u + (\nabla u)^T - \frac{2}{3} (\nabla \cdot u) I \right],$$

$$p = \rho T, \quad q = -\kappa \nabla x T, \quad \mu = \frac{\rho T}{\nu}, \quad \kappa = \frac{1}{\Pr} \frac{5 \rho T}{2 \nu}.$$  

(4.21)

Until now, we have deduced the reduced model for (3.1). Compared with the two-fluid model in Section 2.1, we can find (4.21) has the same form of (2.5), but with different coefficients of viscosity and thermal conductivity. This is due to the different collision models and closure methods, where the Shakhov collision model is adopted here and will inherit the correct Prandtl number from the plasma kinetic equations.

Since the shear stress and the heat flux are all expressed by density, macroscopic velocity and the temperature, this new reduced model has the same number of degree of freedom as the five-moment two-fluid model, which greatly decreases the computational complexity compared with the plasma kinetic equations. More importantly, the shear stress and heat flux are not simply set as zero, which is done in the five-moment model [29], but expressed by the macroscopic variables such as the density, macroscopic velocity and temperature. Thus, it is expected that this new reduced model could be more capable to describe the problems with anisotropic pressure tensor and large heat flux compared to the five-moment model.
Though this new model is only deduced for the single-species of plasma, it can be extended naturally to the plasma kinetic equation \((2.0)\). For the collisionless plasma where the collective interactions dominate the plasma dynamics, we can neglect the momentum and heat transfer between different species. Thus, we can derive the new two-fluid model for multi-species of plasma with the same Maxwellian iteration. In this case, the new reduced model \((4.20)\) is changed into

\[
\begin{align*}
\text{density:} & \quad \frac{dn_\beta}{dt} + n_\beta (\nabla_x \cdot u_\beta) = 0, \\
\text{momentum:} & \quad m_\beta n_\beta \frac{du_\beta}{dt} = n_\beta q_\beta [E + u_\beta \times B] - \nabla_x p_\beta - \nabla_x \cdot \sigma_\beta, \\
\text{energy:} & \quad \frac{3}{2} n_\beta \frac{dT_\beta}{dt} = -n_\beta T_\beta (\nabla_x \cdot u_\beta) - \nabla_x \cdot q_\beta - \sigma_\beta \cdot \nabla_x u_\beta,
\end{align*}
\]

with the closure relations

\[
\sigma_\beta = -\mu_\beta \left[ \nabla_x u_\beta + (\nabla_x u_\beta)^T - \frac{2}{3}(\nabla_x \cdot u_\beta) I \right],
\]

\[
p = n_\beta T_\beta, \quad q_\beta = -\kappa_\beta \nabla_x T_\beta, \quad \mu_\beta = \frac{n_\beta T_\beta}{\nu_\beta}, \quad \kappa_\beta = \frac{1}{Pr} \frac{5 n_\beta T_\beta}{2 \nu_\beta}.
\]

**Remark 2.** We only derive the new reduced model for the plasma which does not contain the interactions between different species. For the general case, the corresponding new reduced model can also be derived but with some differences in the closure relations.

For the different collision models, the same Maxwellian iteration method can also be utilized to get the reduced models for the plasma kinetic equations, but the final form of the two-fluid model may be different.

## 5 Model Validation

In this section, the validation of this new model is studied, especially for the problem where Prandtl number will bring negligible effect. To show the capability of the new model, the steady-state and time-dependent problems are tested.

**First example** In this example, we focus on the steady-state problem. The Hartmann flow problem is often used to test the resistive, viscous MHD models near the high collision regime \([16, 21]\). Below, a similar problem is examined to show that the new two-fluid model can capture correct Prandtl number.

The problem setting shown in Figure 1 is a plasma set between two infinitely large plates. Two infinite conducting plates are driving shear flow along \(x\)-direction with different velocities and temperatures. A constant magnetic field is imposed along \(z\)-direction, which drives a current in \(y\)-direction. The flow generated along \(y\) is interacting with \(B_z\), which will suppress the viscous boundary layer. Since we are focusing on the effect of Prandtl number, the dimensionless problem of only one species with a fixed background is tested. To further reduce this problem, we just set the viscosity and thermal conductivity \(\mu\) and \(\kappa\) in \([12, 22]\) as constant. Moreover, the walls are no-slip, but no adiabatic.
For the steady state problem, the governing equations are reduced into

\[
\sum_{j=1}^{3} \frac{\partial}{\partial x_j} (\rho u_j) = 0,
\]

\[
\sum_{j=1}^{3} u_j \frac{\partial u_d}{\partial x_j} + \rho \sum_{j=1}^{3} \left( \frac{\partial \sigma_{jd}}{\partial x_j} + \delta_{jd} \frac{\partial p}{\partial x_j} \right) = E_d + \sum_{k,m} \epsilon_{dkm} u_k B_m,
\]

(5.1)

\[
\rho \sum_{j=1}^{3} u_j \frac{\partial T}{\partial x_j} + \frac{2}{3} \sum_{j=1}^{3} \left( \frac{\partial q_j}{\partial x_j} + \sum_{d=1}^{3} (\sigma_{jd} + \delta_{jd} p) \frac{\partial u_d}{\partial x_j} \right) = 0,
\]

with the closure

\[
p = \rho T, \quad \sigma = -\mu \left[ \nabla_x u + (\nabla_x u)^T - \frac{2}{3} (\nabla_x \cdot u) I \right], \quad q = -\kappa \nabla_x T,
\]

(5.2)

where \(x_1, x_2, x_3\) are corresponding to \(x, y, z\). Different boundary conditions will lead to different steady state solutions. In this test, the plasma is assumed to be incompressible and the initial density is set as \(\rho = 1\). The velocity and temperature of the upper wall are set as \(u_{up} = 1\), and \(T_{up} = 1\), while those of the bottom wall are set as \(u_{bottom} = 0\), and \(T_{bottom} = 0\). The height between the two plates is \(L = 1\), with the imposed magnetic field \(B_z = 1\). For this problem is designed for a slab geometry, there is no gradient in the \(x\) or \(y\) direction, which means that

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0.
\]

(5.3)

Then from the continuity equation, we can derive

\[
u_z = 0.
\]

(5.4)

Steady state Faraday’s Law in a slab geometry [21] could give

\[
E_x = E_y = 0.
\]

(5.5)

Then the governing system (5.1) is reduced into

\[
\mu \frac{\partial^2 u_x}{\partial z^2} = -\rho u_y B_z,
\]

\[
\mu \frac{\partial^2 u_y}{\partial z^2} = \rho u_x B_z,
\]

(5.6)

\[
\kappa \frac{\partial^2 T}{\partial z^2} = -\mu \left( \left( \frac{\partial u_x}{\partial z} \right)^2 + \left( \frac{\partial u_y}{\partial z} \right)^2 \right).
\]
Then we can get the solution to the steady state problem as

$$u_x(z) = a_1 \exp(bz) + a_2 \exp(-bz) + a_3 \exp(\tilde{b}z) + a_4 \exp(-\tilde{b}z),$$

$$u_y(z) = -i[a_1 \exp(bz) + a_2 \exp(-bz) - a_3 \exp(\tilde{b}z) - a_4 \exp(-\tilde{b}z)],$$

$$T(z) = -\frac{4\Pr}{5} a_1 a_3 \left[ \exp((b + \tilde{b})z) - \exp((b - \tilde{b})z) - \exp((-b + \tilde{b})z) + \exp((-b - \tilde{b})z) \right] + Cz,$$

where

$$a_1 = \frac{1}{2(\exp(b) - \exp(-b))}, \quad a_2 = -\frac{1}{2(\exp(b) - \exp(-b))}, \quad a_3 = \frac{1}{2(\exp(\tilde{b}) - \exp(-\tilde{b}))}, \quad a_4 = -\frac{1}{2(\exp(\tilde{b}) - \exp(-\tilde{b}))}, \quad b = \frac{1}{\sqrt{2\mu}}(1 + i),$$

$$C = 1 + \frac{4\Pr}{5} a_1 a_3 \left[ \exp\left(-\frac{\sqrt{2}}{\sqrt{\mu}}\right) - \exp\left(-\frac{\sqrt{2i}}{\sqrt{\mu}}\right) - \exp\left(\frac{\sqrt{2i}}{\sqrt{\mu}}\right) + \exp\left(\frac{\sqrt{2}}{\sqrt{\mu}}\right) \right].$$

In Figure 2, the solutions of the temperature for different Prandtl numbers are plotted where the viscous conductivity is set as $\mu = 0.01$. We can see that the appearance of the temperature with different Prandtl numbers varies greatly, which means Prandtl number is quite important to get the correct model. The new reduced two-fluid model could inherit the correct Prandtl number from the plasma kinetic equations, and could be more capable to describe the behavior of the plasma.

![Figure 2: Temperature for different Prandtl numbers.](image)

![Figure 3: Time-dependent problem.](image)
Second example  In this test, we are focusing on the time-dependent problem, where a plasma set between two infinitely large plates is also studied, but one plate is at \( z = 0 \), and the other is at \( z = +\infty \). The periodic boundary is utilized in the \( x \)-direction. The setting of the problem is shown in Figure 3. A constant magnetic field of strength \( B_y = 1 \) acts in the direction of the \( y \) axis. The density and velocity of the left wall are set as \( \rho_{\text{left}} = 1 \) and \( u_{\text{left}} = 0 \). The initial temperature is changing with \( x \) as

\[
T_{\text{left}} = 1 - 0.5 \cos(2\pi x).
\]

The initial condition of the particles is set as

\[
\rho = 1, \quad u = 0, \quad T = 1.
\]

The similar problem is studied in [7, 28] to show the ghost effect brought by the kinetic theory. In this problem setting, there is also no gradient in \( y \)-direction, and we can derive

\[
\frac{\partial}{\partial y} = 0, \quad u_2 = 0.
\]

The external electric field is added to balance the self-consistent electric field. Thus the system (5.1) is reduced into

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_1}{\partial x_1} + \frac{\partial \rho u_3}{\partial x_3} = 0,
\]

\[
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_3 \frac{\partial u_1}{\partial x_3} - \frac{4\mu}{3\rho} \frac{\partial^2 u_1}{\partial x_1^2} - \frac{\mu}{3\rho} \frac{\partial^2 u_1}{\partial x_3^2} = -\frac{\partial T}{\partial x_1} - \frac{T}{\rho} \frac{\partial \rho}{\partial x_1} = -Bu_3,
\]

\[
\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x_1} + u_3 \frac{\partial u_3}{\partial x_3} - \frac{4\mu}{3\rho} \frac{\partial^2 u_3}{\partial x_1^2} - \frac{\mu}{3\rho} \frac{\partial^2 u_3}{\partial x_3^2} = \frac{\partial T}{\partial x_3} + \frac{T}{\rho} \frac{\partial \rho}{\partial x_3} = Bu_1,
\]

\[
\frac{\partial T}{\partial t} + u_1 \frac{\partial T}{\partial x_1} + u_3 \frac{\partial T}{\partial x_3} + \sum_{i=1,3} \frac{8\mu}{9\rho} \left( \frac{\partial u_i}{\partial x_i} \right)^2 - \frac{2\kappa}{3\rho} \frac{\partial^2 T}{\partial x_i^2} + \frac{2\kappa}{\beta} \frac{\partial \rho}{\partial x_i} = 0.
\]

The numerical solutions at time \( t = 0.01 \) with different Prandtl numbers are studied. The upwind scheme is adopted to approximate the first-order derivatives with the central difference scheme to approximate the second-order derivatives in (5.12). In the numerical test, the grid size is \( N = 400 \). Due to the similar behavior of the solutions with different Prandtl numbers, Figure 4 shows the behavior of \( \rho, u_1, u_3 \) and \( T \) for Prandtl number 2/3 with \( \mu = 0.01 \). We can find \( u_3 \) and \( T \) are all changing periodically in the \( x \)-direction, which is consistent with the problem setting. To show the affection of Prandtl number, the local numerical solutions with different Prandtl numbers at different \( x \) positions are plotted. Figure 5 illustrates the differences of the local solution \( \rho, u_1, u_3 \) and \( T \) at position \( x = 0 \) and 0.5 with different Prandtl numbers.

We can find that for the time-dependent problem, even with a quite small time length, the numerical solutions vary greatly with different Prandtl numbers. This shows that Prandtl number will greatly affect the numerical solution and is an important factor to correctly describe the behavior of the plasma. The new reduced two-fluid model maintains the correct Prandtl number from the kinetic plasma model for any type of particles, and may be more capable to capture the behavior of the particles, which we will verify in future work.
Figure 4: Numerical solution with Prandtl number \( Pr = 2/3 \) at \( t = 0.01 \).

Figure 5: Numerical solutions with different Prandtl numbers at \( t = 0.01 \). The top column is at \( x = 0 \), and the bottom column is at \( x = 0.5 \).

6 Conclusion Remarks

We derived a new two-fluid model for the plasma using Maxwell iteration based on the HME system. With the Shakhov collision model, the new two-fluid model takes the correct Prandtl number in its closure relations. Though it has the same degree of freedom as the five-moment two-fluid model, this new reduced model is expected to have improved performance for problems with anisotropic pressure tensor and large heat flux, since anisotropic pressure tensor and heat flux are expressed by the density, macroscopic velocity and temperature instead of simply setting as zero. It is our future work to investigate by numerical simulations the performance of the new model in practical applications.

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