Universal aspects in the behavior of the entanglement spectrum in one dimension: Scaling transition at the factorization point and ordered entangled structures

S. M. Giampaolo,1 S. Montangero,2 F. Dell’Anno,3 S. De Siena,4,5 and F. Illuminati4,5,*
1University of Vienna, Faculty of Physics, Boltzmannasse 5, 1090 Vienna, Austria
2Institut für Quanteninformationsverarbeitung, Universität Ulm, D-89069 Ulm, Germany
3Liceo Statale P. E. Imbriani, via Pescatori 155, I-83100 Avellino, Italy
4Dipartimento di Ingegneria Industriale, Università degli Studi di Salerno, Via Giovanni Paolo II, I-84084 Fisciano (SA), Italy
5CNISM - Consorzio Nazionale Interuniversitario per le Scienze Fisiche della Materia, Unità di Salerno, I-84084 Fisciano (SA), Italy

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We investigate the scaling of the entanglement spectrum and of the Rényi block entropies and determine its universal aspects in the ground state of critical and noncritical one-dimensional quantum spin models. In all cases, the scaling exhibits an oscillatory behavior that terminates at the factorization point and whose frequency is universal. Parity effects in the scaling of the Rényi entropies for gapless models at zero field are thus shown to be a particular case of such universal behavior. Likewise, the absence of oscillations for the Ising chain in transverse field is due to the vanishing value of the factorizing field for this particular model. In general, the transition occurring at the factorizing field between two different scaling regimes of the entanglement spectrum corresponds to a quantum transition to the formation of finite-range, ordered structures of quasidimers, quasitrimeros, and quasipolymers. This entanglement-driven transition is superimposed to and independent of the long-range magnetic order in the broken symmetry phase. Therefore, it conforms to recent generalizations that identify and classify the quantum phases of matter according to the structure of ground-state entanglement patterns. We characterize this form of quantum order by a global order parameter of entanglement defined as the integral, over blocks of all lengths, of the Rényi entropy of infinite order. Equivalently, it can be defined as the integral of the bipartite single-copy or geometric entanglement. The global entanglement order parameter remains always finite at fields below the factorization point and vanishes identically above it.

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I. INTRODUCTION

During the past decade, the use of entanglement in the study of complex quantum systems has developed at an increasingly fast pace.1–5 Extended analyses have established the monotonic scaling of the von Neumann entropy in the ground state of spin models,6–10 the so-called area law, and its profound relations both with conformal field theory11–13 (CFT) and with the so-called “majorization” of the entanglement along renormalization group flows in quantum spin chains.11,12

The von Neumann entanglement entropy is in general extremely hard to compute and its measurement requires complete state tomography.1 On the other hand, it has been argued that the Rényi entropies of higher order contain substantial information about the universal properties of a quantum many-body system.13–15 In particular, the Rényi entropy of order 2 is directly related to the purity of the ground-state block reduced density matrix, the so-called Schmidt gap, plays the role of an effective order parameter in one-dimensional quantum spin-1 models endowed with Haldane topological phases.29

Indeed, these quantities play a relevant role in determining the scaling properties of numerical algorithms based on matrix product states.21–24 The Rényi entropies are also a useful tool to determine the continuous or discontinuous nature of a phase transition25 and to estimate quasi-long-range order in low-dimensional systems.26 Furthermore, the concept of topological entanglement entropy27,28 can be extended to the Rényi entropies for which it has been shown to coincide with the total quantum dimension.13 Concerning the relation of the entanglement spectrum to quantum phase transitions, it has been recently shown that a particular subset of such spectrum, that is the difference between the two largest eigenvalues of the reduced density matrix, the so-called Schmidt gap, plays the role of an effective order parameter in one-dimensional quantum spin-1 models endowed with Haldane topological phases.29

Above all, there is a growing awareness that the entanglement properties of quantum ground states provide the most fundamental characterization of quantum phases of matter beyond the traditional approach based on symmetry breaking, especially when considering quantum phases of matter that are established in the absence of symmetry breaking and local order parameters, and are due, e.g., to the presence of hidden topological order associated to a nonvanishing topological component of the ground-state entanglement entropy.27,28,30

In this perspective, the entanglement spectrum and the topological components of the Rényi entropies are being actively investigated in various problems at the cutting edge of condensed matter physics, including Bose-Hubbard spin liquids;31 frustrated models on nontrivial lattice geometries,32 non-Abelian fractional Hall systems,18 and low-dimensional gapless models.33,34

When the Rényi entropies are computed for blocks A of ℓ spins in the ground state of critical models at zero field, it is observed that, for order α > 2, they are characterized by large subleading corrections that violate monotonicity in
the size of the block ($\ell$), while no effect appears in the Ising chain at any value of the transverse field.\textsuperscript{17,35–37} The presence of this oscillatory behavior has been explained thus far as a peculiar characteristic of critical models, due to parity effects, i.e., depending on whether the number of spins in a given partition is even or odd.

In this work, we show that the oscillatory scaling is a universal feature of the entanglement spectra and of the Rényi entropies in the ground state of critical and noncritical one-dimensional models and that this behavior is related to the existence of a factorizing field, located always below or coinciding with the critical field,\textsuperscript{38–42} at which the system Hamiltonian admits at least one fully separable ground state. Using exact analytical methods for integrable models and numerical methods for nonintegrable ones, we show that all one-dimensional models admitting ground-state factorization exhibit two different scaling regimes, separated by the factorizing field $h_f$. In the region $h<h_f$, the Schmidt gap and the Rényi block entropies exhibit a nonmonotonic scaling in the size $\ell$ of the block, while monotonicity is restored for $h>h_f$. These oscillations are associated to a series of crossovers between the two entangled eigenvectors of the reduced block density matrix associated to the two largest eigenvalues. By increasing the size of the block, the Schmidt gap closes exponentially in the symmetry-broken phase but with a ratio that increases as $h$ decreases for $h_f<h<h_c$. On the contrary, it becomes independent of $h$ as soon as $h<h_f$. We will relate this general scaling structure in one dimension to a finite-range, entanglement-driven order that is captured by introducing a global order parameter of entanglement.

The paper is organized as follows. In Sec. II, we introduce the class of $XY$ and $XX$ models in transverse field and discuss, using exact analytic methods, the scaling of the Rényi block entropies in the ground state of such models, either gapped or gapless, in the thermodynamics limit. In all cases, we show that the factorizing field $h_f$ separates two different scaling regimes in the behavior of the Rényi block entropies, either as functions of the transverse field $h$ or of the size of the block $\ell$. In Sec. III, we analyze the behavior of the entanglement spectrum as a function of $\ell$ and we show how the oscillatory behavior is associated to crossovers between the eigenvectors of the block reduced density matrix corresponding to the two largest eigenvalues. As a consequence, we show how the Schmidt gap, i.e., the difference between the two largest eigenvalues, captures most of the significant aspects of the behavior of the full entanglement spectrum. In Sec. IV, we analyze the detailed features of the oscillatory scaling, showing that it is characterized by a set of frequencies that depend only on the normalized transverse field $h/h_f$. As a consequence, a series of ordered entangled structures in terms of quasidisorders, quasitrimers, and quasipolymers is identified for all fields below the factorization point. This form of quantum order is characterized by a global parameter of entanglement defined as the integral, over all block lengths, of the Rényi entropy of infinite order. In Sec. V, we extend our investigation to nonintegrable models of the $XYZ$ class. At variance with the $XY$, $XX$, and Ising classes, nonintegrable models may not always admit ground-state factorization points. Using numerical methods, in particular the density matrix renormalization group (DMRG) algorithm on systems of large but finite size (up to 128 spins), we discuss analogies and differences between the scaling of the entanglement spectrum in models that admit and models that do not admit factorized ground states.

II. SCALING OF THE RÉNYI BLOCK ENTROPIES IN INTEGRABLE MODELS

To set the stage, let us first consider the class of translationally invariant, one-dimensional $XY$ spin-$\frac{1}{2}$ Hamiltonians

$$H_{xy} = \frac{1}{2} \sum_i (1 + \gamma) \sigma_i^x \sigma_{i+1}^x + (1 - \gamma) \sigma_i^y \sigma_{i+1}^y - h \sum_i \sigma_i^z,$$

(1)

where $\sigma_i^\alpha (\alpha = x, y, z)$ stands for the spin-$\frac{1}{2}$ Pauli operator on site $i$, $h$ is the external transverse field, and $\gamma$ is the anisotropy, taking values in the interval $[0, 1]$, whose extremes correspond, respectively, to the fully isotropic, gapless $XX$ model and to the maximally anisotropic, gapped Ising chain. For every $\gamma$, when $h$ takes the value $h_f = \sqrt{1 - \gamma^2}$ the system admits two degenerate, fully factorized ground states that are products of single-site states.\textsuperscript{38,39} For anisotropy $\gamma \neq 0$, such local states are not, in general, eigenstates of $\sigma_z$. Hence, their tensor product is not an eigenstate of the parity operator along $z$. On the contrary, their coherent linear symmetric superpositions (with positive or negative relative sign) define two degenerate ground states of definite parity (respectively even or odd) whose entanglement, in general, does not vanish. For $\gamma = 0$, the two factorized ground states collapse into a single state that preserve all the symmetries of the Hamiltonian. In the thermodynamic limit, regardless of the value of $\gamma$, at $h_c = 1$ the system undergoes a quantum phase transition in the $XY$ plane. For $\gamma > 0$ and $h < h_c$, the system is characterized by a twofold-degenerate ground space and a gapped energy spectrum with a magnetic order along the $x$ axis, while for $\gamma = 0$ and $h < h_c$, the ground state is unique and the energy spectrum is gapless.

The presence of possible degeneracies in the ground space and the fact that different ground states may be characterized by different entanglement properties require, before proceeding further, that we fix the class of ground states on which the investigation shall be focused. Indeed, in order to capture the possible universal aspects in the scaling behavior of the entanglement spectrum and of the Rényi block entropies, one needs to treat the gapped case $\gamma \neq 0$ consistently with the isotropic case $\gamma = 0$ and with the paramagnetic phase $h > h_c$, since in these two instances the system Hamiltonian admits only a single, nondegenerate ground state that preserves all the symmetries. Therefore, from now on we will consider, also in the magnetically ordered phases of gapped models ($\gamma \neq 0$ and $h < h_c$), only the class of ground states that preserve all the symmetries of the Hamiltonian, including parity. That is, we will consider only states of definite fixed parity, either even or odd. Of course, also the case of states with no definite parity is physically very relevant. On the other hand, it is technically much more complex since in such instance one needs to resort to challenging numerical investigations even for the simplest, exactly solvable Hamiltonians such as the one-dimensional $XY$ model. Therefore, the case of...
parity-nonconserving ground states will be attacked and investigated carefully in a separate work.

Models of the $XY$ and $XX$ class play a relevant role in the field of quantum statistical mechanics because they are exactly solved by resorting to the Jordan-Wigner transformations of the spin operators into pseudofermion operators. Indeed, given a bipartition of our one-dimensional lattice of total size $N$ into a block of $\ell$ spins and a remainder of size $N - \ell$, it is always possible to obtain, for ground states of fixed parity, an exact analytical expression of the reduced block density matrix $\rho_\ell$, i.e., of the block reduced state obtained by tracing the ground state over the degrees of freedom of the remainder. Knowledge of $\rho_\ell$ allows us to compute exactly the associated Rényi block entropies, defined as

$$S_\alpha(\ell) = \frac{1}{1 - \alpha} \ln \left[ \text{Tr}(\rho_\ell^\alpha) \right]. \quad (2)$$

In Fig. 1, we report the behavior of the Rényi entropies $S_\alpha(\ell, h)$, both in the gapless and gapped cases, as functions of the external field $h$ for blocks of spins of different size $\ell$. All calculations are exact and are performed for the one-dimensional $XY$, $XX$, and Ising models in the thermodynamic limit. We observe that both in the critical and noncritical cases the entropy $S_1(\ell, h)$, i.e., the block von Neumann entropy, exhibits a monotonic behavior both in $h$ and in $\ell$. However, as soon as $\alpha > 1$ increases, an oscillatory behavior is observed in the region $h < h_f$. Specifically, for $1 < \alpha \leq 2$ and $h < h_f$, the Rényi entropies violate monotonicity in $h$ while remaining monotonically nondecreasing in $\ell$: $S_\alpha(\ell + 1, h) > S_\alpha(\ell, h)$. The amplitude of the oscillations increases as $\ell$ and $\alpha$ increase. For $\alpha > 2$ the scaling becomes nonmonotonic also in the block size: $S_\alpha(\ell + 1, h) > S_\alpha(\ell, h)$. As already discussed previously, notice that at the factorizing field $h_f$, the entanglement does not vanish, in general, in the parity-preserving ground states. However, unambiguous signatures of factorization are readily identified also in the symmetry-protected sectors. In particular, for every value of $\ell$, at factorization the block entanglement in parity-symmetric ground states becomes independent on the relative distance between the different spins in the block. Vice versa, all entanglement-related quantities vanish, correctly, in the nonsymmetric fully separable ground states at the factorization point.

The central role of the factorizing field for the scaling of the Rényi block entropies is illustrated also in Fig. 2, where they are analyzed, for different values of the external field, as functions of the block size $\ell$ both in gapped and gapless models. For $h < h_c$, the entropies of order $\alpha > 2$ exhibit damped oscillations that violate the area-law behavior. As $h$ is increased, the frequency and the amplitude of the oscillations decrease; the area-law monotonic scaling is restored exactly at factorization. Oscillations are exponentially damped in noncritical (anisotropic) models and only weakly algebraically damped in critical (isotropic) systems. As the monotonic area-law behavior is restored at the factorization point, which, for gapped models, means well before the quantum critical point, the oscillatory scaling behavior must be directly related to the patterns of entanglement in the ground state, independently of the onset of the long-range magnetic order at the critical point.

III. SCALING OF THE ENTANGLEMENT SPECTRUM

In order to gain a full understanding of the anomalous scaling in relation to the patterns of ground-state entanglement, we turn now to the analysis of the entanglement spectrum, that is, the entire set $\{s_k\}$ of the eigenvalues of the reduced density matrix $\rho_\ell$. The oscillatory behavior of the Rényi entropies becomes even more pronounced as the order $\alpha$ grows. Correspondingly, it follows from the definition of the Rényi entropies [Eq. (2)] that the relative weight of the largest eigenvalues of the reduced density matrix $\rho_\ell$ increases. Since the characterizations either in terms of the full entanglement spectrum or in terms of the entire hierarchy of the Rényi...
entropies must be equivalent, we may expect that only the largest eigenvalues of $\rho_f$ will contribute significantly to the scaling of the Rényi entropies. Indeed, this is verified both in the gapped and gapless cases, as shown in Fig. 3.

From Fig. 3 we see that the two largest eigenvalues in the entanglement spectrum exhibit an oscillating scaling behavior analogous to that of the Rényi entropies, with growing frequency of the oscillations as the size of the partition increases. Moreover, for partitions of increasing size $\ell$, the ending point of the oscillations converges, asymptotically, to the factorization point $h_f$, where $h_f < h_c$ in gapped systems and coincides with it in the critical model. The approach to quantum criticality is signaled by the opening of the Schmidt gap, i.e., the difference between the two largest eigenvalues. Classical saturation to a paramagnetic ground state at high fields $h$ occurs asymptotically for $h > h_c$ in the gapped case, and instantaneously at $h_c \equiv h_f$ in the gapless case.

Given these results, aside from considering the entire entanglement spectrum, it is sensible to analyze in more detail the scaling of the Schmidt gap $\Delta$, whose importance has been recently discussed in the analysis of the quantum phase diagram of some one-dimensional spin-1 models. In Fig. 4, we report the behavior of the Schmidt gap $\Delta$ as function of the block size $\ell$ for different values of the external field $h$ both in critical and noncritical models. As $\ell$ increases, the Schmidt gap exhibits an overall envelope that is exponentially decreasing in the ordered phase $h < h_c$, in full agreement with the presence of a gap in the energy spectrum. On the other hand, aside from the overall exponential envelope, Fig. 4 shows two clearly different behaviors: one for $h_f < h < h_c$ and one for $h < h_f$. In the first region, the Schmidt gap undergoes a net exponential decay as $h$ decreases. When the system enters the region $h < h_f$, an oscillating behavior is superimposed on the exponential envelope, due to a series of crossovers between the eigenvectors corresponding to the two largest eigenvalues of the reduced block density matrix. Contrary to what happens in the first region, the argument of the exponential decay becomes independent of the external field. Being independent of $h$, the argument of the exponential can then be evaluated analytically at factorization. We obtain that $\Delta(\ell)$ for $h \leq h_f$ obeys the relation $\Delta(\ell) = \chi(y)^{-\ell} f(h,\gamma,\ell)$, where $\chi(y) = \frac{1}{\pi} \int_0^{2\pi} \frac{\text{sign}[\cos(\phi)] - (1 - y^2)^{1/2} + iy \sin(\phi)}{\text{d}\phi}$, and $f(h,\gamma,\ell)$ contains slowly, nonexponentially decaying oscillating terms and tends to unity as $h$ tends to $h_f$.

In the gapless case (see inset of Fig. 4), for all $h < h_c$ the Schmidt gap exhibits a power-law decay modulated by an oscillating dependence on $\ell$, in agreement with the fact that the system is in a critical region and that, being $h_f = h_c$, the external field lies always below its factorizing value. These observations indicate that there are universal aspects in the scaling of the entanglement spectrum in one dimension that are independent of the critical or noncritical nature of the systems considered. These universal effects can then be captured, as reported in Fig. 5, by analyzing the reduced Schmidt gap $\Delta_{\text{red}} \equiv \Delta(h,\gamma)/\Delta(h_f,\gamma)$, i.e., the ratio of the Schmidt gap parametrically dependent on the external field $h$ and on the

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**FIG. 3.** (Color online) Entanglement spectrum $\{\lambda_k\}$ as a function of the external field $h$ for different values of the block size $\ell$. Upper panel: anisotropic gapped models ($\gamma=0.4$). Lower panel: isotropic gapless models ($\gamma=0$). In both cases, the relevant contributions are mainly due to the two largest eigenvalues (red line and blue line) compared to the sum of the remaining eigenvalues (green line), the more so as factorization and criticality are approached. Oscillations increase with increasing size of the block, and terminate asymptotically at the factorization point. Criticality is signaled by the opening of the Schmidt gap between the two largest eigenvalues.

**FIG. 4.** (Color online) Schmidt gap $\Delta$ as function of the block size $\ell$ for different values of $h$ and $\gamma$. Main panel: gapped case ($\gamma=0.4$). Black squares: $h=0.5 h_f \geq 0.458$; red circles: $h=h_f \geq 0.916$; green up triangles: $h_f < h = 0.95 h_c$; blue down triangles: $h_c < h = 1.2$. Inset: gapless case ($\gamma=0$). Black squares: $h=0.5$; red circles: $h = h_f = h_c = 1$.
anisotropy $\gamma$ to the Schmidt gap evaluated at the factorizing field $h_f$, as a function of the block size $\ell$. Figure 5 shows that the oscillation frequency in the scaling of the reduced entanglement spectrum is a universal aspect common to critical and noncritical models, the former differing from the latter only in the overall exponentially decaying envelope whose presence depends on the existence of a gap in the energy spectrum. The nonperfect superposition of the different curves is partly due to the finite numerical precision and likely also due to the fact that this is an effective, emerging form of universality.

The scaling of the Schmidt gap with the external field $h$ is reported in Fig. 6. It reports the Schmidt gap as a function of $\delta h = |h - h_c|$ for different values of the block size $\ell$. On the contrary, when $h < h_f$, Fig. 6 shows that $\mathcal{E}_G^{(2)}$ and hence the distance of the ground state from the set of biseparable states oscillates as a function of $\ell$, with the amplitude of the oscillations increasing for decreasing $h$ and $\gamma$. In particular, recalling the definition of $\mathcal{E}_G^{(2)}$, we see from Fig. 3 that at sufficiently small fields the ground state is closest to the biseparable state $|\psi_2\rangle \otimes |\phi_{N-2}\rangle$. This implies that well below factorization ($h \ll h_f$), the ground state tends to order in dimerized domains, i.e., entangled structures that involve only two spins in a maximally entangled Bell state. A similar oscillating behavior in the eigenstates of the block reduced density matrix is observed in models that possess exactly dimerized ground states, either due to the presence of competing interactions of different spatial range or due to complex spatial patterns of the nearest-neighbor interaction couplings, as is the case, respectively, of the $J_1$-$J_2$ model at the Majumdar-Ghosh point $47$ and of models admitting exactly dimerized ground states, either due to the presence of a gap at the factorizing field, and $\Delta$ opens, independently of $\ell$, at the critical point $h_c$.

**IV. FORMATION OF ORDERED ENTANGLMENT STRUCTURES AND ORDER PARAMETER OF ENTANGLEMENT**

The above results for the scaling of the entanglement spectrum and of the Rényi entropies imply that the factorizing field $h_f$ is itself to be considered a quantum critical point, of a different nature with respect to the standard one $h_c$ and coinciding with it for gapless models. Exactly at $h = h_f$, one-dimensional systems undergo a further phase change beyond the quantum phase transition occurring at the critical point $h_c$. In the region $h < h_f \leq h_c$, a finite-range, entanglement-driven order is established that is superimposed to the global magnetic one driven by spontaneous symmetry breaking.

The implications of the anomalous scaling of the entanglement spectrum for a transition between different patterns of entanglement in the ground state can be understood by recalling that the Rényi entropy of infinite order $S_\infty = -\ln \lambda_{\text{max}}$ coincides with the single-copy bipartite entanglement $44,45$ and is monotonic in the bipartite geometric entanglement $\mathcal{E}_G^{(2)} = 1 - \lambda_{\text{max}}$, where $\lambda_{\text{max}}$ is the largest eigenvalue of the ground-state block reduced density matrix $\rho_\ell$. Given a system of total size $N$, $\mathcal{E}_G^{(2)}$ is the minimum distance between the ground state and the set of pure biseparable states $|\psi_\ell\rangle \otimes |\phi_{N-\ell}\rangle$, where $|\psi_\ell\rangle$ is a state of a block $\ell$, and $|\phi_{N-\ell}\rangle$ is a state of the remainder of the chain. From the last panel of Fig. 2 one sees that above factorization, $S_\infty$ and hence also $\mathcal{E}_G^{(2)}$ increase monotonically in $\ell$. On the contrary, when $h < h_f$, Fig. 2 shows that $\mathcal{E}_G^{(2)}$ and hence the distance of the ground state from the set of biseparable states oscillates as a function of $\ell$, with the amplitude of the oscillations increasing for decreasing $h$ and $\gamma$. In particular, recalling the definition of $\mathcal{E}_G^{(2)}$, we see from Fig. 3 that at sufficiently small fields the ground state is closest to the biseparable state $|\psi_2\rangle \otimes |\phi_{N-2}\rangle$. This implies that well below factorization ($h \ll h_f$), the ground state tends to order in dimerized domains, i.e., entangled structures that involve only two spins in a maximally entangled Bell state. A similar oscillating behavior in the eigenstates of the block reduced density matrices is observed in models that possess exactly dimerized ground states, either due to the presence of competing interactions of different spatial range or due to complex spatial patterns of the nearest-neighbor interaction couplings, as is the case, respectively, of the $J_1$-$J_2$ model at the Majumdar-Ghosh point $47$ and of models admitting exactly dimerized ground states, either due to the presence of a gap at the factorizing field, and $\Delta$ opens, independently of $\ell$, at the critical point $h_c$. However, at variance with such models, in the class of the $XY$ models with nearest-neighbor interactions and spatially constant coupling amplitudes, the oscillatory behavior holds...
only for the Rényi entropies of order \( \alpha \geq 2 \) and therefore dimerization is not exact. The fact that dimerization is only partial is due to the fact that the relative weight of all the remaining eigenstates of the reduced density matrix with respect to the one of largest amplitude never becomes so small to be put to zero, contrary to what happens, e.g., in the Majumdar-Ghosh model for which it vanishes at each fixed value of \( \ell \).

From Fig. 5 we see that by increasing the strength of the transverse field from values \( h \ll h_f \) to values progressively closer to the factorization point, the frequency of the oscillations reduces progressively. This change in the periodicity of the oscillations signals the growth in the spatial dimension of the ordered entangled structures. The latter gradually evolve from strongly entangled domains of quasidimers, leaving place to larger structures ranging from quasitrimers up to quasipolymers, until their size diverges exactly at the factoring field. In other words, as one moves away from vanishingly small values of the external field and approaches the factorization point, the associated entangled structures become spatially more extended. At the same time, these extended structures develop close to factorization and involve ever larger number of sites, the associated entanglement becomes smaller and smaller in each pair of sites, in full agreement with the behavior of the pairwise concurrence (entanglement of formation) close to a factorization point.\(^5\)

As the oscillatory behavior is maximized by the single-copy entanglement \( S_\infty(\ell,h) \), the finite-range, entanglement-driven order below factorization is naturally characterized by an order parameter of entanglement defined as the integral of \( S_\infty(\ell,h) \) over partitions of all sizes:

\[
\Gamma = -\sum_{\ell=1}^{\infty} \min[S_\infty(\ell+1,h) - S_\infty(\ell,h),0]. \tag{3}
\]

In Fig. 7, we report the contour plot of \( \Gamma \) for the class of one-dimensional XY models. It vanishes identically for \( h > h_f \). Approaching the Ising limit, it is strongly reduced, corresponding to strongly damped oscillations. Approaching the XX limit, as the oscillations become long ranged, \( \Gamma \) increases and diverges exactly at \( \gamma = 0 \), consistently with the fact that in gapless models below the critical point the oscillations persist, with weak algebraic decay, also for large \( \ell \).

Equation (3) and Fig. 7 show that the quasidimerized order below factorization is exclusively due to the entanglement properties of each individual ground state and, therefore, unlike magnetic order, it is not a consequence of ground-state degeneracy and symmetry breaking.

V. SCALING OF THE ENTANGLEMENT SPECTRUM IN NONINTEGRABLE MODELS

We will now show that the results of the above analysis extend as well to generic nonintegrable models of the XYZ and Heisenberg types:

\[
H_{\text{xyz}} = \frac{1}{2} \sum_{i,l} J_x \sigma_i^x \sigma_l^x + J_y \sigma_i^y \sigma_l^y + J_z \sigma_i^z \sigma_l^z - h \sum_i \sigma_i^z. \tag{4}
\]

Here, \( J_\mu \) are the spin-spin couplings along the \( \mu = x,y,z \) directions and, without loss of generality, we set

\[
J_z = 1 \geq |J_x|, |J_y|. \tag{5}
\]

These systems undergo a quantum phase transition at \( h = h_c \), while, contrary to the integrable cases discussed above, factorized ground states do not necessarily exist. At each finite value of the external field, the factorizing field is defined as \( h = h_f = \sqrt{(J_x + J_y)(J_x + J_z)} \), and therefore it exists if and only if \( J_z \geq -J_x \). The models in Eq. (4) are not exactly solvable. In order to determine the block reduced density matrices, we diagonalized the system by means of the density matrix renormalization group\(^{51–53} \) (DMRG) applied to open chains of up to 128 spins. We kept up to \( m = 16 \) states of the reduced density matrix, letting the truncation error stay well below \( 10^{-7} \) at each step.

For models admitting a factorization point, the results of the XY case carry over essentially unmodified to the general XYZ instance. In Fig. 8, we report the scaling of the Rényi entropies in XYZ models that possess a factorizing field \( h_f \) (\( J_z \geq -J_x \)) and the scaling of the Schmidt gap as a function of the block size \( \ell \) for different values of the external field. The qualitative behavior of these two quantities is analogous to that of the XY models and confirms the existence of an entanglement-induced order of quasidimerized domains in the region \( h < h_f \).

For models that do not admit a factorized ground state at any finite value of the magnetic field, the scaling behavior of the Rényi entropies and of the entanglement spectrum is rather different, as reported in Fig. 9. At variance with models that do admit ground-state factorization, the area-law scaling with the size of the block \( S_\infty(\ell+1,h) > S_\infty(\ell,h) \) appears to be always violated for all values of \( h \), as one can see from Fig. 9. However, all the Rényi entropies behave smoothly and do not acquire local maxima. On the other hand, when
considered as functions of the block size $\ell$ (at fixed external field $h$), the block entropies exhibit the same oscillating behavior as in models with factorized ground states. These features can be intuitively understood by considering that in these models factorization, so to speak, is shifted away towards infinitely large values of $h$, indeed corresponding to a classical saturation. The latter is the physical limiting situation in which an extremely strong transverse field orients all spins along the field direction, resulting in a classical paramagnetic state that is naturally a product of single-site paramagnets, with all quantum correlations washed away. Indeed, as we can see from the left inset of Fig. 9, the true quantum oscillations in the Schmidt gap appear to be confined up to a value of the external field $h = \sqrt{(J_x + J_z)(-J_x - J_z)}$, which, somehow, corresponds to an inverted, decomplexified factorization point. Therefore, a sort of would-be factorizing field seems to play a very important role also in models that do not admit a physical factorized ground state.

A complete, rigorous confirmation of these behaviors for all values of the transverse field $h$ in nonintegrable models is beyond our current numerical possibilities for two reasons: first, the differences between Rényi entropies of the same order $\alpha$ and different partition size $\ell$ fall off extremely rapidly with $h$; moreover, the limitations on current algorithms do not allow a reliable analysis for partitions of size $\ell > 4$. This is due to the fact that the algorithm is based on the direct computation of the two-site density matrix of the two central sites of the DMRG ansatz. The further density matrices, e.g., the four-site density matrix, are obtained by repeated merging of two physical sites in one computational site. This protocol is very effective in obtaining all quantities of interest but, as the local dimension of the computational sites increases exponentially with the number of physical sites grouped together, it prevents at the moment going beyond blocks of size $\ell > 4$. If further confirmed by future more powerful numerical analysis, the reported behavior would imply that the entanglement-induced order can exist also in disordered phases with vanishing local order parameters.

VI. DISCUSSION AND OUTLOOK

We have investigated the scaling behavior of the entanglement spectrum and of the Rényi entanglement entropies in the ground state of one-dimensional gapped and gapless quantum spin models. We have showed that a violation of the area-law scaling behavior occurs both in the critical and noncritical cases for all Rényi entropies of order $\alpha > 2$ and for external fields $h < h_f < h_c$ in models admitting ground-state factorization. An analogous behavior is observed in the Schmidt gap, which is the difference between the two largest eigenvalues of the block reduced density matrix.

The existence of a factorizing field and the associated anomalous scaling of the entanglement spectrum correspond to the existence of a quantum regime superimposed to and distinct from the ordered and disordered phases associated to the existence of ground states breaking some symmetry of the Hamiltonian. Unlike long-range magnetic order due to spontaneous symmetry breaking, this finite-range, entanglement-driven order does not rely on the onset of ground-state degeneracy, but is rather due to the entanglement properties of each ground state of fixed parity. The factorizing field plays a key role in the understanding of the anomalous scaling of the entanglement spectrum by clarifying the origin of this phenomenon even in the gapless case in which the factorization point and the critical point coincide. It also explains why the anomalous scaling is not observed in systems, such as the Ising model, for which $h_f = 0$. For nonintegrable models that do not admit factorization, the oscillatory scaling appears to extend to all values of the external field, while the nonmonotonic behavior of the Schmidt gap appears to be...
confined in the region \( h < h = \sqrt{(J_x + J_y)(-J_y - J_z)} \). However, further numerical analyses will be required in order to reach a definite assessment for such models.

As already mentioned in the Introduction,\(^2,28,30\) a new quantum revolution is taking place in condensed matter physics inasmuch as it is being recognized that there are kinds of quantum phases that are truly and uniquely characterized by their entanglement properties, the paramount example being the quantum spin-liquid phase associated to a nonvanishing topological entanglement entropy for some classes of two-dimensional, geometrically frustrated quantum spin systems (e.g., the Heisenberg model on the kagome lattice). In fact, this fundamental realization has motivated the characterization and classification of quantum phases according to equivalence classes of entanglement: for instance, ground states endowed with topological order are characterized by a constant, topological entanglement entropy of infinite spatial range. Therefore, they can not be adiabatically connected to ground states with short-range entanglement patterns without crossing a quantum phase transition. In particular, phases endowed with topological order are not compatible with factorized ground states, the latter being a particular element of the equivalence classes of states with short-range entanglement.\(^34,55\) The above holds true (although a rigorous proof is still lacking) for two- and higher-dimensional systems, where topological effects can occur independently of the presence or absence of symmetries. In one-dimensional systems, it has been conjectured that topological phases can occur only if the associated ground states share some given symmetry and can not be adiabatically deformed into each other without crossing a quantum phase transition, if the deformation preserves that symmetry,\(^56-58\) leading to a classification of gapped symmetric phases in one-dimensional spin systems.\(^59\) This is certainly not the case for the exchange quantum spin-$\frac{1}{2}$ chains that we have discussed in this work. On the other hand, the conjecture puts the role of the factorized ground state at the center stage of the investigation, and for models that are thought of possessing true symmetry-protected topological phases in one dimension, our methods for the study of ground-state factorization could be applied in order to verify the conjecture in explicit model Hamiltonians in one dimension.

Frustration, a crucial ingredient for the realization of topological phases in two dimensions, appears to be an important factor in one-dimensional systems as well. We have noticed that the single-copy entanglement \( S_\infty \) is monotonic in the bipartite geometric entanglement \( \mathcal{E}_G^{(2)} \), which in turn has been shown to be a universal lower bound to ground-state frustration, independently of the spatial dimension.\(^60,61\) This correspondence suggests the existence of an intimate relation between the scaling of the entanglement spectrum, the tendency to form ordered entangled structures below the factorization point, and the frustration of purely quantum origin. In such a perspective, the present investigation might be further fruitfully extended to geometrically frustrated models and systems with topological order and other classes of exotic quantum phases.

Finally, the fact that the factorization point has been identified to constitute the boundary point for different types of scaling of the ground-state entanglement suggests that the presence of factorization might impose fundamental limits to Trotterization and other classes of adiabatic methods\(^62,63\) for the simulatability of quantum many-body Hamiltonians. This crucial point of great practical importance will be the subject of forthcoming work.

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*Corresponding author: illuminati@sa.infn.it

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