ON THE OPTIMAL VOLUME UPPER BOUND FOR KÄHLER
MANIFOLDS WITH POSITIVE RICCI CURVATURE

KEWEI ZHANG

Abstract. Using δ-invariants and Newton–Okounkov bodies, we derive the
optimal volume upper bound for Kähler manifolds with positive Ricci cur-
vature, from which we get a new characterization of the complex projective
space.

1. Introduction

Let \((X, g)\) be an \(m\)-dimensional Riemannian manifold such that
\[
\text{Ric}(g) \geq (m - 1)g.
\]
Then the well-known Bishop–Gromov volume comparison says that
\[
\text{Vol}(X, g) \leq \text{Vol}(S^m, g_{S^m}),
\]
and the equality holds if and only if \((X, g)\) is isometric to the standard \(m\)-sphere
\(S^m\). However, suppose in addition that \(X\) has a complex structure \(J\) such that
\((X, g, J)\) is Kähler, then Liu [19] shows that this volume upper bound is never
sharp (unless \(X = \mathbb{P}^1\)), in the sense that there exists a dimensional gap \(\epsilon(n) > 0\)
such that
\[
\text{Vol}(X, g) \leq \text{Vol}(S^m, g_{S^m}) - \epsilon(n).
\]
This distinguishes the Kählerian geometry from the Riemannian case (see also Li–
Wang [18] for related discussions using holomorphic bisectional curvatures). So it
is natural to ask what the optimal volume upper bound should be. The purpose
of this paper is to answer this question by using some new input from algebraic
geometry.

Our main result can be stated as follows, which substantially improves the bound
given by the Bishop–Gromov Theorem.

Theorem 1.1. Let \((X, \omega)\) be an \(n\)-dimensional Kähler manifold with
\[
\text{Ric}(\omega) \geq (n + 1)\omega.
\]
Then one has
\[
\int_X \omega^n \leq (2\pi)^n,
\]
and the equality holds if and only if \((X, \omega)\) is holomorphically isometric to \((\mathbb{P}^n, \omega_{FS})\),
where \(\omega_{FS}\) denotes the Fubini–Study metric.

This result generalizes the works of Berman–Berndtsson [1] and K. Fujita [11]
(see also Y. Liu [20]) to general Kähler classes and gives a new characterization of
the complex projective space in terms of the Ricci curvature and volume.

Kähler manifolds with positive Ricci curvature are automatically Fano. These are
simply connected projective manifolds with many additional algebraic properties.
So in what follows, unless otherwise specified, we will always assume that $X$ is an $n$-dimensional Fano manifold. Note that the Picard group $\text{Pic}(X) \cong H^2(X, \mathbb{Z})$ is a finitely generated torsion free Abelian group, hence a lattice. So the isomorphism classes of line bundles on $X$ are in one-to-one correspondence with the lattice points of $H^2(X, \mathbb{Z})$. Also note that the Kähler cone $\mathcal{K}(X)$ of $X$ coincides with its ample cone, as $H^2(X, \mathbb{R}) = H^{1,1}(X, \mathbb{R})$ by Kodaira vanishing and Hodge decomposition. So any Kähler class in $\mathcal{K}(X)$ can be approximated by a sequence of rational classes (corresponding to ample $\mathbb{Q}$-line bundles).

Based on this, we will prove Theorem 1.1 from an algebraic viewpoint, which requires several new tools that have been developed in the K-stability theory. First of all, we will reformulate the curvature condition in terms of the greatest Ricci lower bound, which in turn can be related to the algebraic $\delta$-invariant thanks to Berman–Boucksom–Jonsson [3, Theorem C] (see also Cheltsov–Rubinstein–Zhang [7, Theorem 5.7]). Then by modifying the argument of K. Fujita [11], we get the desired volume upper bound for Kähler classes. The essential difficulty of Theorem 1.1 lies in the characterization of the equality. For this, we will compute the anti-canonical Seshadri constant as in [11]. However, to deal with general Kähler classes, we need some new observations from convex geometry. In particular, Newton–Okounkov bodies and the positivity criterion of Kûronya–Lozovanu [14] will be crucially used in our argument.

The rest of this paper is organized as follows. In Section 2 we review some necessary notions and tools from the literature. In Section 3 we prove Theorem 1.1 by assuming that $[\omega]$ is (a multiple of) a rational class. In Section 4, we prove Theorem 1.1 in full generality. We end this paper by proposing a conjecture in Section 5.

**Remark 1.2.** After completing this paper, the author was kindly informed by Feng Wang that he had also independently obtained our volume upper bound by adapting the argument of [11] to a sequence of conic KE metrics.

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2. Preliminaries

In this section $X$ is assumed to be an $n$-dimensional Fano manifold.

2.1. The volume function on the Néron–Severi space.

Note that, $H^{1,1}(X, \mathbb{R})$ can be identified with the Néron–Severi space $N^1(X)_{\mathbb{R}}$, which consists of numerical equivalence classes of $\mathbb{R}$-divisors on $X$. One can define a continuous volume function $\text{Vol}(\cdot)$ on $N^1(X)_{\mathbb{R}}$. When restricted to the Kähler cone $\mathcal{K}(X)$ (i.e., the ample cone), $\text{Vol}(\cdot)$ is the usual volume for Kähler classes (which will be treated as ample $\mathbb{R}$-divisors in what follows).

Also recall that, a class $\xi \in N^1(X)_{\mathbb{R}}$ is called nef if for every curve $C$ on $X$

$$\xi \cdot C \geq 0.$$ 

For nef classes $\xi$, $\text{Vol}(\xi)$ is simply equal to the top self-intersection number $\xi^n$.

A class $\xi \in N^1(X)$ is called big if

$$\text{Vol}(\xi) > 0.$$
For more details on this subject, we refer the reader to the standard reference [15].

2.2. The greatest Ricci lower bound.

Let \( \mathcal{K}(X) \) denote the Kähler cone of \( X \). For any Kähler class \( \xi \in \mathcal{K}(X) \), one can naturally define its greatest Ricci lower bound \( \beta(X, \xi) \) to be

\[
(2.1) \quad \beta(X, \xi) := \sup\{ \mu > 0 \mid \exists \text{ Kähler form } \omega \in 2\pi \xi \text{ s.t. } \text{Ric}(\omega) \geq \mu \omega \}.
\]

Note that, by the Calabi–Yau theorem, given any Kähler form \( \alpha \in 2\pi c_1(X) \), one can always find \( \omega_0 \in 2\pi \xi \) such that \( \text{Ric}(\omega_0) = \alpha > 0 \). By compactness of \( X \) we see \( \text{Ric}(\omega_0) \geq \epsilon \omega_0 \) for some \( \epsilon > 0 \). So \( \beta(X, \xi) \) is always a positive number. On the other hand, \( \beta(X, \xi) \) is naturally bounded from above by the Seshadri constant

\[
(2.2) \quad \epsilon(X, \xi) := \sup\{ \mu > 0 \mid c_1(X) - \mu \xi \text{ is nef} \}.
\]

Thus we always have

\[
(2.3) \quad 0 < \beta(X, \xi) \leq \epsilon(X, \xi).
\]

When \( \xi = c_1(L) \) for some ample \( \mathbb{Q} \)-line bundle \( L \), we will write

\[
\beta(X, L) := \beta(X, c_1(L))
\]

for ease of notation.

Remark 2.1. When \( \xi = 2\pi c_1(X) \), the greatest Ricci lower bound was first studied by Tian [26], although it was not explicitly defined there. It was first explicitly defined by Rubinstein in [22, 23], and was later further studied by Székelyhidi [25], Li [17], Song–Wang [24], Cable [5], et al.

2.3. The \( \delta \)-invariant.

Let \( L \) be an ample \( \mathbb{Q} \)-line bundle on \( X \). Following [12, 4], the \( \delta \)-invariant of \( L \) is defined by

\[
(2.4) \quad \delta(X, L) := \inf_{v \in \text{Val}_X} \frac{A_X(v)}{S_L(v)}.
\]

Here \( \text{Val}_X \) denotes the space of valuations over \( X \), \( A_X(v) \) denotes the log discrepancy of \( v \), and

\[
S_L(v) := \frac{1}{\text{Vol}(L)} \int_0^\infty \text{Vol}(v(L) \geq x) dx
\]

denotes the expected vanishing order of \( L \) with respect to \( v \). Note that \( \delta \)-invariant is also called stability threshold in the literature, which plays important roles in the study of K-stability and has attracted intensive research attentions. When \( L = -K_X \), it was proved by the author in the appendix of his joint work with Cheltsov–Rubinstein [7] that

\[
\beta(X, -K_X) = \min\{1, \delta(X, -K_X)\}.
\]

For arbitrary ample \( \mathbb{Q} \)-line bundles, we have the following general result, which is essentially contained in the work of Berman–Boucksom–Jonsson [3], giving a geometric interpretation of the \( \delta \)-invariants on Fano manifolds.

Theorem 2.2 ([3]). Let \( L \) be an ample \( \mathbb{Q} \)-line bundle on a Fano manifold \( X \). Then one has

\[
\beta(X, L) = \min\{ \epsilon(X, L), \delta(X, L) \}.
\]

\footnote{We put a factor \( 2\pi \) in the definition for convenience.}
Proof. For any smooth semi-positive \((1,1)\)-form \(\theta\) on \(X\), \(\theta\)-twisted \(\beta\), \(\epsilon\) and \(\delta\)-invariants were introduced in [3] (in fact they allow \(\theta\) to be currents, but we do not need this here). More precisely, put

\[
\beta_{\theta}(X,L) := \sup\{ \mu > 0 \mid \exists \omega \in 2\pi c_1(L) \text{ s.t. } \text{Ric}(\omega) \geq \mu \omega + \theta \},
\]

\[
\epsilon_{\theta}(X,L) := \sup\{ \mu > 0 \mid c_1(X) - \mu c_1(L) - [\theta]/2\pi \text{ is nef} \}
\]

and

\[
\delta_{\theta}(X,L) := \inf_{v \in \text{Val}_X} \frac{A_\theta(v)}{S(v)}.
\]

It follows clearly from the definition that

\[
\beta(X,L) = \sup_{\theta} \beta_{\theta}(X,L).
\]

Moreover, as \(\theta\) is smooth, one has [3, Example 3.2]

\[
\delta(X,L) = \delta_{\theta}(X,L).
\]

Now using [3, Theorem C]

\[
\beta_{\theta}(X,L) = \min\{ \epsilon_{\theta}(X,L), \delta_{\theta}(X,L) \}
\]

for any smooth \(\theta\), we obtain

\[
\beta(X,L) = \sup_{\theta} \min\{ \epsilon_{\theta}(X,L), \delta(X,L) \}.
\]

Using the fact \(\epsilon_{\theta}(X,L) \leq \epsilon(X,L)\), we finish the proof. \(\square\)

Regarding the \(\delta\)-invariant, [4, Theorem D] implies the following volume upper bound (we will recall its proof in Section 3)

\[
\delta(X,L)^n \text{Vol}(L) \leq (n+1)^n.
\]

This inequality reveals the deep relationship between singularities and volumes of linear systems. A combination of Theorem 2.2 and (2.8) gives

**Proposition 2.3.** Let \(L\) be an ample \(\mathbb{Q}\)-line bundle on a Fano manifold \(X\), then one has

\[
\beta(X,L)^n \text{Vol}(L) \leq (n+1)^n.
\]

Note that, in the toric case when \(L = -K_X\), this was first obtained by Berman–Berndtsson [1] using analytic methods.

### 2.4. Newton–Okounkov bodies and positivity of \(\mathbb{R}\)-line bundles.

Let \(Y\) be an \(n\)-dimensional projective manifold. Choose an admissible flag of subvarieties

\[Y_\bullet : Y = Y_0 \supseteq Y_1 \supseteq ... \supseteq Y_{n-1} \supseteq Y_n = \{\text{pt.}\},\]

such that each \(Y_i\) is an irreducible subvariety of codimension \(i\) and smooth at the point \(Y_n\). Then any big class \(\xi \in N^1(Y)_{\mathbb{R}}\) can be associated with a convex body \(\Delta_{Y_\bullet}(\xi)\) in \((\mathbb{R}_{\geq 0})^n\), which is called the Newton–Okounkov body of \(\xi\) with respect to the flag \(Y_\bullet\). This generalizes the classical polytope construction for divisors on toric varieties. A crucial fact is that

\[
\text{Vol}(\xi) = n! \text{Vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(\xi)).
\]

In this way one can study the volume function \(\text{Vol}(\cdot)\) on \(N^1(Y)_{\mathbb{R}}\) using convex geometry. For more details of this construction we refer the reader to [16].
It turns out that Newton–Okounkov bodies can also help us visualize the positivity of \( \mathbb{R} \)-line bundles. More precisely, for any big \( \mathbb{R} \)-divisor \( D \), one can define its \textit{restricted base loci} by

\[
B_-(D) := \bigcup_A B(D + A),
\]

where the union is over all ample \( \mathbb{Q} \)-divisors \( A \) on \( Y \) and \( B(\cdot) \) denotes the stable base loci (cf. [10]). Then \( B_-(D) \) captures the non-nef locus of \( D \) (see [10, Example 1.18]). Indeed, suppose that there exists some curve \( C \) intersecting negatively with \( D \), then by adding a small amount of ample \( \mathbb{Q} \)-divisor \( A \), one still has \( (D + A) \cdot C < 0 \), which implies that \( C \subset B(D + A) \) and hence \( C \subset B_-(D) \).

The recent result of Küronya–Lozovanu [14, Theorem A] says that one can characterize the restricted base loci using Newton–Okounkov bodies.

**Theorem 2.4 ([14]).** Let \( D \) be a big \( \mathbb{R} \)-divisor. Then the following are equivalent.

1. \( q \notin B_-(D) \).
2. There exists an admissible flag \( Y_\bullet \) with \( Y_n = \{ q \} \) such that the origin \( 0 \in \Delta_{Y_\bullet}(D) \subset \mathbb{R}^n \).
3. For any admissible flag \( Y_\bullet \) with \( Y_n = \{ q \} \), one has \( 0 \in \Delta_{Y_\bullet}(D) \subset \mathbb{R}^n \).

Let us also record the following useful translation property of Newton–Okounkov bodies.

**Proposition 2.5.** [14, Proposition 1.6]. Let \( \xi \) be a big \( \mathbb{R} \)-divisor and \( Y_\bullet \) an admissible flag on \( Y \). Then for any \( t \in [0, \tau(\xi, Y_1)) \) we have

\[
\Delta_{Y_\bullet}(\xi)_{\nu_1 \geq t} = \Delta_{Y_\bullet}(\xi - tY_1) + te_1,
\]

where \( \tau(\xi, F) := \sup\{ \mu > 0 | \xi - \mu Y_1 \text{ is big} \} \) denotes the pseudo-effective threshold, \( \nu_1 \) denotes the first coordinate of \( \mathbb{R}^n \) and \( e_1 = (1, 0, ..., 0) \in \mathbb{R}^n \).

3. **Rational classes**

In this Section, we will verify Theorem 1.1 for rational classes. More precisely, we prove the following

**Theorem 3.1.** Let \( L \) be an ample \( \mathbb{Q} \)-line bundle on a Fano manifold \( X \). Then one has

\[
\beta(X, L)^n \text{Vol}(L) \leq (n + 1)^n,
\]

with equality if and only if \( X \) is biholomorphic to \( \mathbb{P}^n \).

**Proof.** In the view of Proposition 2.3, it remains to characterize the equality. We follow the argument of K. Fujita [11]. By rescaling, we assume \(^2\)

\[
\beta(X, L) = 1 \text{ and } \text{Vol}(L) = (n + 1)^n.
\]

Then Theorem 2.2 implies that \( \delta(X, L) \geq 1 \) and \( \epsilon(X, L) \geq 1 \).

\(^2\)It is possible that \( \beta(X, L) \) is irrational before rescaling, but this will not cause issues for our argument. In fact we believe that one always has \( \beta(X, L) \in \mathbb{Q} \).
Pick any point $p \in X$ and let $\hat{X} \xrightarrow{\sigma} X$ be the blow-up at $p$. Let $E$ be the exceptional divisor of $\sigma$. Then one has
\[ A_X(E) \geq S_L(E), \]
amely \(^3\)
\[ n = A_X(E) \geq S_L(E) = \frac{1}{\text{Vol}(L)} \int_0^\infty \text{Vol}(\sigma^*L - xE)dx \]
\[ \geq \frac{1}{\text{Vol}(L)} \int_0^{\sqrt[n]{\text{Vol}(L)}} (\text{Vol}(L) - x^n)dx \]
\[ = \frac{n}{n+1} \sqrt[n]{\text{Vol}(L)}. \]

Here we used [11, Theorem 2.3(1)]. Now the condition $\text{Vol}(L) = (n+1)^n$ implies that
\[ \text{Vol}(\sigma^*L - xE) = \text{Vol}(L) - x^n \]
for any $x \in [0, n+1]$. So [11, Theorem 2.3(2)] implies that
\[ \sigma^*L - (n+1)E \]
is nef as well. Thus $X \cong \mathbb{P}^n$ by [8, 13]. □

4. General Kähler classes

Let us attend to general Kähler classes. The main result of this section is the following.

**Theorem 4.1.** Let $\xi$ be a Kähler class of a Fano manifold $X$. Then one has
\[ \beta(X, \xi)^n \text{Vol}(\xi) \leq (n+1)^n, \]
with equality if and only if $X$ is biholomorphic to $\mathbb{P}^n$.

**Remark 4.2.** Using Bishop–Gromov, one can quickly derive
\[ \beta(X, \xi)^n \text{Vol}(\xi) \leq \frac{2^n+n!(2n-1)^n}{(2n)!}. \]
However this bound is much worse than $(n+1)^n$ (especially when $n$ is large).

The proof of Theorem 4.1 will be divided into several steps. We first show the inequality by approximation and then characterize the equality using Newton–Okounkov bodies. We begin with

**Lemma 4.3.** The greatest Ricci lower bound $\beta(X, \cdot)$ is a lower semi-continuous function on $\mathcal{K}(X)$.

**Proof.** Let $\{e_1, ..., e_\rho\}$ be a basis of $H^{1,1}(X, \mathbb{R})$, then any Kähler class $\xi \in \mathcal{K}(X)$ can be written as
\[ \xi = \sum_{i=1}^\rho a_i e_i \]
for some $a_i \in \mathbb{R}$. For each $i \in \{1, ..., \rho\}$ choose a smooth real $(1, 1)$-form $\eta_i \in e_i$.\(^3\)

---

\(^3\)This in fact gives the proof of (2.8).
Now assume that there exists $\omega \in 2\pi \xi$ such that $\text{Ric}(\omega) > \mu\omega$ for some $\mu > 0$. For any $\vec{\epsilon} = (\epsilon_1, ..., \epsilon_\rho) \in \mathbb{R}^\rho$ with $||\vec{\epsilon}|| \ll 1$, we put
$$\omega_{\vec{\epsilon}} := \omega + \sum_{i=1}^\rho \epsilon_i \eta_i.$$ Then for $||\vec{\epsilon}|| \ll 1$ one also has $\text{Ric}(\omega_{\vec{\epsilon}}) > \mu\omega_{\vec{\epsilon}}$. So the lower semi-continuity of $\beta(X, \cdot)$ follows.

As a consequence we get the following volume upper bound for general Kähler classes in terms of its greatest Ricci lower bound.

**Proposition 4.4.** Let $\xi$ be a Kähler class on an $n$-dimensional Fano manifold $X$. Then one has
$$\beta(X, \xi)^n \text{Vol}(\xi) \leq (n + 1)^n. \quad (4.1)$$

*Proof.* Choose a sequence of ample $\mathbb{Q}$-line bundles $L_i$ such that $L_i \to \xi$ in $\mathcal{N}_1^1(X)_{\mathbb{R}}$.

By Lemma 4.3 we have
$$\beta(X, \xi) \leq \liminf_i \beta(X, L_i).$$
So for any $\epsilon > 0$ and $i \gg 1$, one has
$$\beta(X, L_i) \geq \beta(X, \xi) - \epsilon.$$ Now Proposition 2.3 implies that
$$(\beta(X, \xi) - \epsilon)^n \text{Vol}(L_i) \leq (n + 1)^n.$$ Using the continuity $\text{Vol}(L_i) \to \text{Vol}(\xi)$ and sending $\epsilon \to 0$, we get
$$\beta(X, \xi)^n \text{Vol}(\xi) \leq (n + 1)^n.$$ 

Therefore, to finish the proof of Theorem 4.1, it remains to show that the equality of (4.1) is exactly obtained by $\mathbb{P}^n$. Let us prepare the following lemma.

**Lemma 4.5.** Let $X$ be a projective manifold. Pick any point $p \in X$ and let $\hat{X} \xrightarrow{\sigma} X$ be the blow-up at $p$. Let $E$ be the exceptional divisor of $\sigma$. Let $\xi \in \mathcal{N}_1^1(X)$ be an ample $\mathbb{R}$-line bundle.

1. For any $x \in \mathbb{R}_{\geq 0}$, one has
$$\text{Vol}(\sigma^* \xi - xE) \geq \text{Vol}(\xi) - x^n.$$  

2. Suppose in addition that $X$ is Fano and that $\xi$ satisfies
$$\beta(X, \xi)^n \text{Vol}(\xi) = (n + 1)^n,$$
then for any $x \in [0, \text{Vol}(\xi)^{1/n}]$,
$$\text{Vol}(\sigma^* \xi - xE) = \text{Vol}(\xi) - x^n.$$
Proof. This first part follows from [11, Theorem 2.3(1)] by approximation. Indeed, let \( L_i \) be a sequence of ample \( \mathbb{Q} \)-line bundles such that \( L_i \to \xi \) in \( N^1(X)_{\mathbb{R}} \). Then for any \( x \in \mathbb{R}_{\geq 0} \), [11, Theorem 2.3(1)] says that
\[
\mathrm{Vol}(\sigma^*(L_i) - xE) \geq \mathrm{Vol}(L_i) - x^n,
\]
so the assertion follows by the continuity of \( \mathrm{Vol}(\cdot) \).

For the second part, we rescale \( \xi \) such that \( \beta(X,\xi) = 1 \) and \( \mathrm{Vol}(\xi) = (n+1)^n \).

Let \( L_i \) be a sequence of ample \( \mathbb{Q} \)-line bundles such that \( L_i \to \xi \) in \( N^1(X)_{\mathbb{R}} \). For any \( \epsilon > 0 \) and \( i \gg 1 \), Theorem 2.2 and Lemma 4.3 implies that
\[
\delta(X,L_i) \geq \beta(X,L_i) \geq 1 - \epsilon.
\]
Thus we get
\[
n = A_X(E) \geq (1 - \epsilon)S_{L_i}(E) = \frac{1 - \epsilon}{\mathrm{Vol}(L_i)} \int_0^\infty \mathrm{Vol}(\sigma^*L_i - xE)dx.
\]
Letting \( i \to \infty \), by dominated convergence theorem and by sending \( \epsilon \to 0 \), we get
\[
n \geq \frac{1}{\mathrm{Vol}(\xi)} \int_0^\infty \mathrm{Vol}(\sigma^*\xi - xE)dx,
\]
so that (recall \( \mathrm{Vol}(\xi) = (n+1)^n \))
\[
n \geq \frac{1}{\mathrm{Vol}(\xi)} \int_0^{\mathrm{Vol}(\xi)^{1/n}} (\mathrm{Vol}(\xi) - x^n)dx = \frac{n}{n+1} \mathrm{Vol}(\xi)^{1/n} = n.
\]
This gives
\[
\mathrm{Vol}(\sigma^*\xi - xE) = \mathrm{Vol}(\xi) - x^n
\]
for \( x \in [0,\mathrm{Vol}(\xi)^{1/n}] \) as claimed. \( \square \)

As we have seen in the proof of Theorem 3.1, [11, Theorem 2.3(2)] says that, for ample \( \mathbb{Q} \)-line bundles, the condition
\[
\mathrm{Vol}(\sigma^*L - xE) = \mathrm{Vol}(L) - x^n, \text{ } x \in [0,a]
\]
implies that \( \sigma^*L - xE \) is nef for \( x \in [0,a] \), whose proof however heavily relies on the rationality of \( L \) and the ampleness criterion of [9]. To prove the same assertion for general ample \( \mathbb{R} \)-line bundles, there is some subtlety involved if we follow Fujita’s original argument. So to avoid potential issues, we take an alternative approach, using Newton–Okounkov bodies.

**Proposition 4.6.** Let \( X \) be a projective manifold. Pick any point \( p \in X \) and let \( X \xrightarrow{\sigma} X \) be the blow-up at \( p \). Let \( E \) be the exceptional divisor of \( \sigma \). Let \( \xi \in N^1(X)_{\mathbb{R}} \) be an ample \( \mathbb{R} \)-divisor. Suppose that there exists \( a > 0 \) such that
\[
\mathrm{Vol}(\sigma^*\xi - xF) = \mathrm{Vol}(\xi) - x^n \text{ for any } x \in [0,a].
\]
Then \( \sigma^*\xi - xF \) is nef for any \( x \in [0,a] \).
Proof. First of all, there exists some $\epsilon > 0$ such that $\sigma^*\xi - \epsilon E$ is nef for any $\epsilon \in [0, \epsilon]$. Indeed, as an ample $\mathbb{R}$-divisor, we may write $\xi$ as $\xi = \sum_{i=1}^{r} a_i A_i$, where, for each $i$, $a_i > 0$ and $A_i$ is an ample divisor on $X$. Let

$$
\epsilon_i := \inf \frac{A_i \cdot C}{\text{mult}_p(C)}
$$

denote the Seshari constant of each $A_i$ at $p$ (where the inf is over all the curves passing through $p$). Then $\epsilon_i > 0$ by the ampleness of $A_i$ and we can take

$$
\epsilon := \sum_{i=1}^{r} a_i \epsilon_i > 0
$$

to fulfill our purpose.

Now we argue by contradiction. Suppose that $\sigma^*\xi - \epsilon_0 E$ is not nef for some $\epsilon_0 \in (\epsilon, a)$. Then there exists some curve $C$ intersecting $\sigma^*\xi - \epsilon_0 E$ negatively. Note that such $C$ necessarily intersects $E$ by projection formula. Thus the restricted base loci $B_{\sigma^*\xi - \epsilon_0 E}$ intersect $E$ as well. So we can pick a point $q \in B_{\sigma^*\xi - \epsilon_0 E} \cap E$ and build an admissible flag $Y_\bullet$ on $\hat{X}$ with

$$
Y_1 = E \text{ and } Y_n = \{q\}.
$$

Then we get the Newton–Okounkov body $\Delta_{Y_\bullet}(\sigma^*\xi)$ and by Proposition 2.5, $\Delta_{Y_\bullet}(\sigma^*\xi - \epsilon_0 E)$ can be obtained from $\Delta_{Y_\bullet}(\sigma^*\xi)$ by truncating and translating in the first coordinate $\nu_1$ of $\mathbb{R}^n$. Now using the condition (recall (2.9))

$$
\text{Vol}(\sigma^*\xi - \epsilon_0 E) = \text{Vol}(\xi) - \epsilon^a \text{ for } \epsilon \in [0, a]
$$

and the Brunn–Minkowski inequality in convex geometry, we see that, in the region $\{0 \leq \nu_1 \leq a\}$, $\Delta_{Y_\bullet}(\sigma^*\xi)$ has to be a convex cone over the $(n - 1)$-dimensional convex set $\Sigma := \Delta_{Y_\bullet}(\sigma^*\xi) \cap \{\nu_1 = a\}$. Note that the cone $\Delta_{Y_\bullet}(\sigma^*\xi) \cap \{0 \leq \nu_1 \leq a\}$ necessarily contains the line segment $\{(t, 0, ..., 0) | t \in [0, \epsilon]\}$ by Theorem 2.4 and Proposition 2.5, so the cone property forces $\Delta_{Y_\bullet}(\sigma^*\xi)$ to contain the whole line segment $\{(t, 0, ..., 0) | t \in [0, a]\}$.

Intuitively, one has the following picture.

\begin{center}
\includegraphics[width=0.5\textwidth]{picture.png}
\end{center}

\textbf{Figure 1.} $\Delta_{Y_\bullet}(\sigma^*\xi) \cap \{0 \leq \nu_1 \leq a\}$
In particular, this implies that $0 \in \Delta_{Y^\bullet}(\sigma^*\xi - x_0E)$ and hence $q \notin B_-(\sigma^*\xi - x_0E)$ by Theorem 2.4, which contradicts our choice of $q$. Thus $\sigma^*\xi - xE$ is nef for any $x \in [0, a]$. 

Thus we have generalized all the ingredients in the proof of Theorem 3.1 to ample $\mathbb{R}$-line bundles. So Theorem 4.1 is proved.

Finally, we are able to prove the main result of this paper.

**Theorem 4.7** (=Theorem 1.1). Let $(X, \omega)$ be an $n$-dimensional Kähler manifold with $\text{Ric}(\omega) \geq (n + 1)\omega$. Then one has
\[
\int_X \omega^n \leq (2\pi)^n,
\]
and the equality holds if and only if $(X, \omega)$ is holomorphically isometric to $(\mathbb{P}^n, \omega_{FS})$.

**Proof.** The volume upper bound is a direct consequence of Proposition 4.4, and the equality holds if and only if $X \cong \mathbb{P}^n$, in which case, the equality $\int_{\mathbb{P}^n} \omega = (2\pi)^n$ implies that
\[
[\omega] = 2\pi c_1(O_{\mathbb{P}^n}(1)).
\]
So $\bar{\partial}\partial f$-lemma gives some $f \in C^\infty(\mathbb{P}^n, \mathbb{R})$ such that
\[
\sqrt{-1}\partial\bar{\partial}f = \text{Ric}(\omega) - (n + 1)\omega \geq 0,
\]
which forces $f$ to be a constant. Thus $\omega$ satisfies the Kähler–Einstein equation
\[
\text{Ric}(\omega) = (n + 1)\omega.
\]
Now by the uniqueness of KE metrics [2], we obtain $\omega = \omega_{FS}$ up to an automorphism. \hfill \Box

5. Final remark

Let us recall a classical result in Cheeger–Colding theory.

**Theorem 5.1.** [6, Theorem A.1.10]. There exists $\epsilon(m) > 0$, such that if $(M, g)$ is an $m$-dimensional Riemannian manifold with $\text{Ric}(g) \geq (m - 1)g$ and $\text{Vol}(M, g) \geq \text{Vol}(S^m) - \epsilon(m)$, then $M$ is diffeomorphic to $S^m$.

So it is reasonable to believe that the following holds in Kähler geometry.

**Conjecture 5.2.** There exists $\epsilon(n) > 0$, such that if $(X, \omega)$ is an $n$-dimensional Kähler manifold with $\text{Ric}(\omega) \geq (n + 1)\omega$ and $\int_X \omega^n \geq (2\pi)^n - \epsilon(n)$, then $X$ is biholomorphic to $\mathbb{P}^n$.

Some evidence comes from the work of Liu–Zhuang [21]. Indeed, assume that $X$ is not biholomorphic to $\mathbb{P}^n$, then [21] implies that the anti-canonical Seshadri constant cannot be bigger than $n$. This implies that, for any Kähler class $\xi$ with $\epsilon(X, \xi) = 1$, $\sigma^*\xi - xE$ is not nef when $x > n$. This gives an explicit gap between $X$ and $\mathbb{P}^n$ in terms of the Seshadri constant, from which one can possibly derive a volume gap. But this seems to be a highly non-trivial problem.
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Beijing International Center for Mathematical Research, Peking University.
E-mail address: kwzhang@pku.edu.cn