Existence and regularity results for Fully Non Linear Operators on the model of the pseudo Pucci’s operators

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1 Introduction

This paper is devoted to the existence and regularity of viscosity solutions for a class of degenerate operators, on the model of the pseudo $p$-Laplacian.

Recall that the pseudo-$p$-Laplacian, for $p > 1$ is defined by:

$$\tilde{\Delta}_p u := \sum_1^N \partial_i (|\partial_i u|^{p-2} \partial_i u).$$

When $p > 2$, it is degenerate elliptic at any point where even only one derivative $\partial_i u$ is zero.

Using classical methods in the calculus of variations, equation

$$\tilde{\Delta}_p u = (p-1)f$$

has solutions in $W^{1,p}$, when for example $f \in L^{p'}$. The regularity results are obtained through specific variational technics, see [13], [11]. When $p < 2$, Lipschitz regularity is a consequence of [13].

When $p > 2$ things are more delicate. Note that in [8], for some fixed non negative numbers $\delta_i$, the following widely degenerate equation was considered

$$\sum_i \partial_i (|\partial_i u| - \delta_i)^{p-1} \frac{\partial_i u}{|\partial_i u|} = (p-1)f.$$  \hfill (1.2)

The authors proved that the solutions of (1.2) are in $W^{1,q}_{loc}$ when $f \in L^{\infty}_{loc}$. As a consequence, by the Sobolev Morrey’s imbedding, the solutions are Hölder’s continuous for any exponent $\gamma < 1$. 

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The Lipschitz interior regularity for (1.1) has been very recently proved by
the second author in [12]. The regularity obtained concerns Lipschitz con-
tinuity for viscosity solutions. Since weak solutions are viscosity solutions, (see also
[3]), she obtains Lipschitz continuity for weak solutions when the forcing term
is in \( L^{\infty}_{\text{loc}} \).

At the same time, in [7], the local Lipschitz regularity of the solutions of
(1.2) has been proved when either \( N = 2, p \geq 2 \) and \( f \in W^{1,p}_{\text{loc}} \) or \( N \geq 3, p \geq 4, \)
and \( f \in W^{1,\infty}_{\text{loc}} \). Remark that (1.2) can also be written formally as
\[
\sum_i(|\partial_i u| - \delta_i)^{p-2} \partial_i u = f.
\]

Hence viscosity solutions have an obvious definition, and with the methods
employed in [12], one can prove, in particular, that the solutions are Hölder's
continuous for any exponent \( \gamma < 1 \). Unfortunately the Lipschitz continuity for
viscosity solution of (1.2) cannot be obtained in the same way.

Let us state the precise assumptions that hold in this paper and present our
main result. Fix \( \alpha > 0 \), and for any \( q \in \mathbb{R}^N \) let \( \Theta_\alpha(q) \) be the diagonal matrix
with entries \( |q_i|^\alpha \) on the diagonal, and let \( X \) be a symmetric matrix.

Let \( F \) be defined on \( \mathbb{R}^N \times \mathbb{R}^N \times S \), continuous in all its arguments, which
satisfies \( F(x, 0, M) = F(x, p, 0) = 0 \) and

(H1) For any \( M \in S \) and \( N \in S, N \geq 0, \) for any \( x \in \Omega \)
\[
\lambda \text{tr}(\Theta_\alpha(q) N \Theta_\alpha(q)) \leq F(x, q, M + N) - F(x, q, M) \leq \Lambda \text{tr}(\Theta_\alpha(q) N \Theta_\alpha(q))
\]

(H2) There exist \( \gamma_F \in [0, 1] \) and \( c_{\gamma_F} > 0 \) such that for any \( (q, X) \in \mathbb{R}^N \times S \)
\[
|F(x, q, X) - F(y, q, X)| \leq c_{\gamma_F} |x - y|^{\gamma_F} |q|^\alpha |X|
\]

(H3) There exists \( \omega_F \) a continuous function on \( \mathbb{R}^+ \) such that \( \omega_F(0) = 0 \), and as
soon as \( (X, Y) \) satisfy for some \( m > 0 \)
\[
-m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq m \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]
then
\[
F(x, m(x-y), X) - F(x, m(x-y), Y) \leq \omega_F(m|x-y|^\alpha) + o(m|x-y|^\alpha) (1.6)
\]
when \( m \) goes to infinity.

(H4) There exists \( c_F \) such that for any \( p, q \in \mathbb{R}^N \), for all \( x \in \mathbb{R}^N \), \( X \in S \)

\[
|F(x, p, X) - F(x, q, X)| \leq c_F||p||^\alpha - ||q||^\alpha||X||
\] (1.7)

Note that the pseudo-Pucci’s operators, for \( 0 < \lambda < \Lambda \)

\[
\mathcal{M}^+_\alpha(q, X) = \Lambda tr((\Theta_\alpha(q)X\Theta_\alpha(q))^+) - \lambda tr((\Theta_\alpha(q)X\Theta_\alpha(q))^-)
\]

\[
= \sup_{M \leq A \leq M} tr(A\Theta_\alpha(q)X\Theta_\alpha(q))
\]

and

\[
\mathcal{M}^-_\alpha(q, X) = -\mathcal{M}^+_\alpha(q, -X)
\]
satisfy all the assumptions above.

We will also consider equations with lower order terms. Precisely, let \( h \) defined on \( \mathbb{R}^N \times \mathbb{R}^N \), continuous with respect to its arguments, which satisfies on any bounded domain \( \Omega \)

\[
|h(x, q)| \leq c_{h,\Omega}(||q||^{1+\alpha} + 1)
\] (1.8)

Our main result is the following.

**Theorem 1.1.** Let \( \Omega \) be a bounded domain and \( f \) be continuous and bounded in \( \Omega \). Under the conditions (1.6), (1.3), (1.7), and (1.8), let \( u \) be a solution of

\[
F(x, \nabla u, D^2u) + h(x, \nabla u) = f \quad \text{in} \quad \Omega.
\] (1.9)

Then, for any \( \Omega' \subset \subset \Omega \), there exists \( C_{\Omega'} \), such that for any \( (x, y) \in \Omega' \)

\[
|u(x) - u(y)| \leq C_{\Omega'}|x - y|.
\]

This will be a consequence of the more general result Theorem 3.1 in section three.

We shall construct in Section 4 a super-solution of (1.9) which is zero on the boundary. Ishii’s Perron method, since the comparison principle holds, leads to the following existence’s result:

**Theorem 1.2.** Suppose that \( \Omega \) is a bounded \( C^2 \) domain and let \( F \) and \( h \) satisfy (1.6), (1.3), (1.7), (H3), and (1.8). Then for any \( f \in C(\bar{\Omega}) \) there exists \( u \) a viscosity solution of

\[
\left\{ \begin{array}{ll}
F(x, \nabla u, D^2u) + h(x, \nabla u) = f(x) & \text{in} \ \Omega \\
u = 0 & \text{on} \ \partial\Omega
\end{array} \right.
\]

Furthermore \( u \) is Lipschitz continuous in \( \Omega \).
Finally in the last section we prove that the strong maximum principle holds. Let us end this introduction by saying a few words about the principal eigenvalues and eigenfunctions, on the model of [4]. Indeed the regularity and existence results obtained above allow to prove the existence of a principal eigenvalue as long as the operator, $F$ is in addition homogeneous, precisely:

For any $(x, p, X)$ and any $s \in \mathbb{R}$ and $t \geq 0$:

$$F(x, sp, tX) = |s|^\alpha t F(x, p, X).$$

We also suppose that $h$ is continuous with values in $\mathbb{R}^N$, and that $h(x, p) = h(x) \cdot p|p|^\alpha$. Then we can define the two values

$$
\mu^+ = \{ \mu \in \mathbb{R}, \exists \phi > 0 \text{ in } \Omega, F(x, \nabla \phi, D^2 \phi) + h(x) \cdot \nabla \phi |\nabla \phi|^\alpha + \mu \phi^{\alpha+1} \leq 0 \}.
$$

and

$$
\bar{\mu}^- := \sup \{ \mu \in \mathbb{R}, \exists \psi < 0 \text{ in } \Omega, F(x, \nabla \psi, D^2 \psi) + h(x) \cdot \nabla \psi |\nabla \psi|^\alpha + \mu |\psi|^\alpha \psi \geq 0 \}.
$$

As in previous works e.g. [4], [2] and [5] it is easy to prove that below $\mu^+$ the classical property of maximum principle holds, i.e. if $\tau < \bar{\mu}^+$ and $u$ is a solution of

$$F(x, \nabla u, D^2 u) + h(x) |\nabla u|^\alpha \nabla u + \tau |u|^\alpha u \geq 0$$

such $u \leq 0$ on $\partial \Omega$ then $u \leq 0$ in $\Omega$. Similarly, for any $\tau < \mu^-$, the minimum principle holds.

Furthermore, one can prove the existence of $\psi^+ > 0$ and $\psi^- < 0$ solution, respectively of

$$F(x, \nabla \psi^+, D^2 \psi^+) + \mu^+(\psi^+)^{\alpha+1} = 0 \text{ in } \Omega, \psi^+ = 0 \text{ on } \partial \Omega,$$

$$F(x, \nabla \psi^-, D^2 \psi^-) + \mu^-|\psi^-|^\alpha \psi^- = 0 \text{ in } \Omega, \psi^- = 0 \text{ on } \partial \Omega.$$

Thus the values $\mu^+$ and $\mu^-$ are called ”principal eigenvalues”.

We will not give proofs for these last results which can be obtained arguing as in [5], and using the comparison principle in Theorem 4.1 and the Lipschitz a priori bounds.

Many questions concerning these very degenerate operators are still open. To name a few let us mention:

Does Alexandroff Bakelman Pucci ’s inequality hold true, similarly to the cases treated in [14]?
Is some Harnack inequality true (still as in [14])? Even for the pseudo-\( p \)-Laplacian this is not known. Finally the further step in regularity is naturally the \( C^1 \) regularity. Even in the case \( f = 0 \) and \( N = 2 \) it does not seem easy to obtain.

2 Examples

Example 1: Let

\[ F(x, p, X) := \text{tr}(L(x)\Theta_\alpha(p)X\Theta_\alpha(p)L(x)). \]

if \( L(x) \) is a Lipschitz and bounded matrix such that \( \sqrt{\lambda I} \leq L \leq \sqrt{\Lambda I} \) then conditions (1.3) and (1.4) are obviously satisfied. In order to check the condition (H3), one uses the right inequality in (1.5) multiplied by \( \begin{pmatrix} L(x)\Theta_\alpha(p) \\ L(y)\Theta_\alpha(p) \end{pmatrix} \) on the right and by its transpose on the left. Hence

\[
F(x, m(x - y), X) - F(x, m(x - y), -Y) \\
\leq m^r(L(x) - L(y))\Theta_\alpha^2(m(x - y))(L(x) - L(y)) \\
\leq m^{\alpha+1}|x - y|^{2+\alpha}
\]

Let us check now condition (1.7)

\[
F(x, p, X) - F(x, q, X) = \text{tr}(L(x)(\Theta_\alpha(p) - \Theta_\alpha(q))X(\Theta_\alpha(p) + \Theta_\alpha(q))L(x)) \\
\leq \Lambda|X|(|\Theta_\alpha(p)^2 - \Theta_\alpha(q)^2|)
\]

which yields the result.

Example 2: We define

\[ F(x, p, X) := a(x)\mathcal{M}_\alpha^\pm(p, X). \]

If \( a \) is a Lipschitz function such that \( a(x) \geq a_o > 0 \) then conditions (1.3) and (1.4) are satisfied. Let us check the condition (H3).

Recall the following standard extremality property of the Pucci’s operators

\[ \mathcal{M}^+(X) \leq \mathcal{M}^+(-Y) + \mathcal{M}^+(X + Y) \quad \text{and} \quad \mathcal{M}^-(X) \leq \mathcal{M}^-(Y) + \mathcal{M}^+(X + Y). \]
Using the identity

\[ a(x)M^\pm_a(X) = \mathcal{M}^\pm(a(x)\Theta_a(p)X\Theta_a(p)) = \mathcal{M}^\pm(\sqrt{a(x)\Theta_a(p)X\Theta_a(p)}\sqrt{a(x)}) \]

we have

\[
\mathcal{M}^\pm(\sqrt{a(x)\Theta_a(p)X\Theta_a(p)}\sqrt{a(x)}) \leq \mathcal{M}^\pm(\sqrt{a(y)\Theta_a(p)(-Y)\Theta_a(p)}\sqrt{a(y)}) \\
+ \mathcal{M}^\pm\left[\sqrt{a(x)\Theta_a(p)X\Theta_a(p)}\sqrt{a(x)} + \sqrt{a(y)\Theta_a(p)Y\Theta_a(p)}\sqrt{a(y)}\right].
\]

Multiplying (1.5), by the matrix

\[
\begin{pmatrix}
\sqrt{a(x)\Theta_a(p)} & 0 \\
0 & \sqrt{a(y)\Theta_a(p)}
\end{pmatrix}
\]

on the left and on the right, one obtains that for \( p = m(x - y) \),

\[
\sqrt{a(x)\Theta_a(p)X\Theta_a(p)}\sqrt{a(x)} + \sqrt{a(y)\Theta_a(p)Y\Theta_a(p)}\sqrt{a(y)} \\
\leq m(\sqrt{a(x)} - \sqrt{a(y)})^2\Theta_a(p)^2 \\
\leq m^{a+1}|x - y|^a \frac{(a(x) - a(y))^2}{(\sqrt{a(x)} + \sqrt{a(y)})^2}I \\
\leq (\text{Lip } a)^2 m^{a+1}\frac{|x - y|^{a+2}}{4a_o}I.
\]

In particular

\[
\mathcal{M}^+(\sqrt{a(x)\Theta_a(p)X\Theta_a(p)}\sqrt{a(x)}) + \sqrt{a(y)\Theta_a(p)Y\Theta_a(p)}\sqrt{a(y)} \\
\leq \Lambda(\text{Lip } a)^2 m^{a+1}\frac{|x - y|^{a+2}}{4a_o}I.
\]

Let us check finally (1.7), for that, it is clear that one can suppose \( a(x) = 1 \), we write

\[
|M^\pm_a(p, X) - M^\pm_a(q, X)| \leq \mathcal{M}^+(\Theta_a(p)X\Theta_a(p) - \Theta_a(q)X\Theta_a(q)) \\
= \frac{1}{2}\mathcal{M}^+[(\Theta_a(p)) - \Theta_a(q))X(\Theta_a(p) + \Theta_a(q)) + (\Theta_a(p) + \Theta_a(q))X(\Theta_a(p) - \Theta_a(q))] \\
\leq \Lambda|\Theta_a^2(p) - \Theta_a^2(q)||X|
\]

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3 Proof of Lipschitz regularity.

In this section we prove our main result:

**Theorem 3.1.** Let $f$ and $g$ be continuous and bounded in $\Omega$, while $F$, $\Omega$ and $h$ satisfy the hypothesis in Theorem 1.1. Suppose that $u$ is a bounded USC sub-solution of

$$F(x, \nabla u, D^2u) + h(x, \nabla u) \geq f \text{ in } \Omega$$

and $v$ is a bounded LSC super-solution of

$$F(x, \nabla v, D^2v) + h(x, \nabla v) \leq g \text{ in } \Omega.$$ 

Then for any $\Omega' \subset \subset \Omega$ there exists $C_{\Omega'}$, such that for any $(x, y) \in \Omega' 

$$u(x) \leq v(y) + \sup_{\Omega}(u - v) + C_{\Omega'}|x - y|.$$ 

We start by recalling some general facts.

If $\psi : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$, let $D_1\psi$ denotes the gradient in the first $N$ variables and $D_2\psi$ the gradient in the last $N$ variables.

In the proof of Theorem 3.1 we shall need the following technical lemma.

**Lemma 3.2.** Suppose that $u$ and $v$ are respectively USC and LSC functions such that, for some constant $M > 1$ and for some function $\Phi$

$$u(x) - v(y) - M|x - x_o|^2 - M|y - x_o|^2 - \Phi(x, y)$$

has a local maximum in $(\bar{x}, \bar{y})$ where $\phi$ is $C^2$.

Then for any $\iota$, there exist $X_\iota, Y_\iota$ such that

$$(D_1\Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), X_\iota) \in J^{2,+}u(\bar{x}),$$

$$(-D_2\Phi(\bar{x}, \bar{y}) - 2M(\bar{y} - x_o), -Y_\iota) \in J^{2,-}v(\bar{y})$$

with

$$-(\frac{1}{\iota} + |A| + 1) \left( \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right) \leq \left( \begin{array}{cc} X_\iota - 2MI & 0 \\ 0 & Y_\iota - 2MI \end{array} \right) \leq (A + \iota A^2) + \left( \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right)$$

and $A = D^2\Phi(\bar{x}, \bar{y})$. 

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This is a direct consequence of a famous Lemma by Ishii [16]. For the convenience of the reader the proof of Lemma 3.2 is given in the appendix. In the sequel, for some \( M \), we will use Lemma 3.2 with \( \Phi(x, y) := Mg(x - y) \), where \( g \) is some functions which is \( C^2 \) except at 0, to be defined later. Denoting by \( H_1(x) := D^2 g(x) \), then

\[
D^2 \Phi = M \begin{pmatrix}
H_1(\bar{x} - \bar{y}) & -H_1(\bar{x} - \bar{y}) \\
-H_1(\bar{x} - \bar{y}) & H_1(\bar{x} - \bar{y})
\end{pmatrix}
\]

Choosing \( \iota = \frac{1}{4M|H_1(x)|} \) and defining \( \bar{H}(x) := H_1(x) + \frac{2}{4|H_1(x)|} H_2^2(x) \), one has

\[
D^2 \Phi + \iota (D^2 \Phi)^2 = M \begin{pmatrix}
\bar{H}(\bar{x} - \bar{y}) & -\bar{H}(\bar{x} - \bar{y}) \\
-\bar{H}(\bar{x} - \bar{y}) & \bar{H}(\bar{x} - \bar{y})
\end{pmatrix}.
\]

Remark that \( |A| = 2M|H_1(\bar{x} - \bar{y})| \). We give some precisions on the choice of \( g \): We will assume that \( g \) is radial, say there exists some continuous function \( \omega \) on \( \mathbb{R}^+ \), such that \( g(x) = \omega(|x|) \) and \( \omega \) is supposed to satisfy:

\[
\omega(0) = 0, \quad \omega \text{ is } C^2 \text{ on } \mathbb{R}^+, \quad \omega(s) > 0, \quad \omega'(s) > 0 \text{ and } \omega''(s) < 0 \text{ on } ]0, 1[. \tag{3.1}
\]

For \( x \neq 0 \), it is well known that \( Dg(x) = \omega'(|x|) \frac{x}{|x|} \) and

\[
D^2 g(x) = \left( \omega''(|x|) - \frac{\omega'(|x|)^2}{|x|^2} \right) x \otimes x + \frac{\omega'(|x|)}{|x|} I.
\]

For \( \iota \leq \frac{1}{4|D^2 g(x)|} \), there exist constants \( \gamma_H \in \left[ \frac{1}{2}, \frac{3}{2} \right] \), \( \beta_H \geq \frac{1}{2} \) such that

\[
D^2 g + 2\iota (D^2 g)^2(x) = \left( \beta_H \omega''(|x|) - \gamma_H \frac{\omega'(|x|)}{|x|} \right) x \otimes x + \gamma_H \frac{\omega'(|x|)}{|x|} I. \tag{3.2}
\]

For \( |x| < 1 \) and \( \epsilon > 0 \), we shall use the following set:

\[
I(x, \epsilon) := \{ i \in [1, N], |x_i| \geq |x|^{1+\epsilon} \}.
\]

We also define the diagonal matrix \( \Theta(x) \) with entries \( \Theta_{ii}(x) = \left| \frac{\omega'(|x|) x_i}{|x|} \right|^2 \).

A consequence of (3.2) is the following Proposition proved in [12].
Proposition 3.3 ([12]). 1) If $\alpha \leq 2$, for all $x \neq 0$, $|x| < 1$, $\Theta(x)\tilde{H}(x)\Theta(x)$ has at least one eigenvalue smaller than

$$\frac{N^{-\alpha}}{\beta_H \omega''(|x|)(\omega'(|x|))^\alpha}. \quad (3.3)$$

2) If $\alpha > 2$, for all $x \neq 0$, $|x| < 1$, for any $\epsilon > 0$ such that $I(x, \epsilon) \neq \emptyset$, and such that

$$\beta_H \omega''(|x|)(1 - N|x|^{2\epsilon}) + \gamma_H N|x|^{2\epsilon} \omega'(|x|) \leq \frac{\omega''(|x|)}{4} < 0, \quad (3.4)$$

then $\Theta(x)\tilde{H}(x)\Theta(x)$ possesses at least one eigenvalue smaller than

$$\frac{1 - N|x|^{2\epsilon}}{\#I(x, \epsilon)} (\omega'(|x|))^\alpha \omega''(|x|) |x|^{(\alpha - 2)\epsilon}. \quad (3.5)$$

[Proof of Theorem 3.1] Borrowing ideas from [15], [1], [6], for some $x_o \in B_r$ we define the function

$$\psi(x, y) = u(x) - v(y) - \sup (u - v) - M\omega(|x - y|) - M|x - x_o|^2 - M|y - x_o|^2;$$

$M$ is a large constant and $\omega$ is a function satisfying (3.1), both to be defined more precisely later.

If there exists $M$ independent of $x_o \in B_r$ such that $\psi(x, y) \leq 0$ in $B_r^2$, by taking $x = x_o$ and using $|x_o - y| \leq 2$ one gets

$$u(x_o) - v(y) \leq \sup (u - v) + 3M\omega(|x_o - y|).$$

So making $x_o$ vary we obtain that for any $(x, y) \in B_r^2$

$$u(x) - v(y) \leq \sup (u - v) + M\omega(|x - y|).$$

This proves the theorem when $\omega$ behaves like $s$ near zero. This can be done once the case where $\omega(s) = s^\gamma$ is treated for $\gamma \in [0, 1]$, i.e the Hölder’s analogous result.

In order to prove that $\psi(x, y) \leq 0$ in $B_r^2$, suppose by contradiction that the supremum of $\psi$, achieved on $(\bar{x}, \bar{y}) \in B_r^2$, is positive. If we have chosen $M$ such that

$$M(1 - r)^2 > 4(|u|_\infty + |v|_\infty), \quad \text{and} \quad M > \frac{2|u|_\infty + 2|v|_\infty}{\omega(\delta)}, \quad (3.6)$$

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we would get that $|\bar{x} - x_o|, |\bar{y} - x_o| < \frac{1-\epsilon}{2}$. Hence, by (3.6), $\bar{x}$ and $\bar{y}$ are in $B_{\frac{1-\epsilon}{2}}$ i.e. in $B_1$. Furthermore, always using (3.6), the positivity of the supremum of $\psi$ leads to $|\bar{x} - \bar{y}| < \delta$.

As it is shown later the contradiction will be found by choosing $\delta$ small enough and $M$ large enough depending on $(r, \alpha, \lambda, \Lambda, N)$.

We proceed using Lemma 3.2 and so, for all $\iota > 0$ there exist $X_\iota$ and $Y_\iota$ such that

$$(q + 2M(x - x_o), X_\iota) \in \mathcal{T}^{2+} u(\bar{x}) \text{ and } (q - 2M(y - x_o), -Y_\iota) \in \mathcal{T}^{2-} v(\bar{y})$$

with $q = M\omega'(|\bar{x} - \bar{y}|) \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|}$. Furthermore, still using the above notations i.e. $g(x) = \omega(|x|)$, and choosing $\iota = \frac{1}{4M|D^2g(x)|}$, for $\bar{H} = (D^2g(x) + \frac{1}{2|D^2g(x)|}D^2g(x)^2)$, we have that

$$- \left( I + M|\bar{H}| \right) \left( \begin{array} {cc} I & 0 \\ 0 & I \end{array} \right) \leq \left( \begin{array} {cc} X_\iota - (2M + 1)I & 0 \\ 0 & Y_\iota - (2M + 1)I \end{array} \right) \leq M \left( \begin{array} {cc} \bar{H} & -\bar{H} \\ -\bar{H} & \bar{H} \end{array} \right).$$

(3.7)

From now on we will drop the $\iota$ for $X$ and $Y$. Recall that $\Theta(q)$ is the diagonal matrix such that $(\Theta)_{ii}(q) = (|q_i|)^{\frac{1}{\alpha}}$.

In order to end the proof we will prove the following claims.

**Claims.** There exists $\hat{\tau} > 0$, such that, if $\delta$ is small enough and $|x - y| < \delta$ the matrix $\Theta(X + Y)\Theta$ has one eigenvalue $\mu_1$ such that

$$\mu_1(\Theta(X + Y)\Theta) \leq -cM^{\alpha+1}|\bar{x} - \bar{y}|^{-\hat{\tau}}$$

(3.8)

There exist $\tau_i < \hat{\tau}$ and $c_i$ for $i = 1, \ldots, 4$ such that the four following assertions hold:

$$\text{for all } j \geq 2 \mu_j(\Theta(X + Y)\Theta) \leq c_1M^{\alpha+1}|\bar{x} - \bar{y}|^{-\tau_1},$$

(3.9)

$$|F(\bar{x}, q^x, X) - F(\bar{x}, q, X)| + |F(\bar{y}, q^y, -Y) - F(\bar{y}, q, -Y)| \leq c_2M^{\alpha+1}|\bar{x} - \bar{y}|^{-\tau_2}$$

(3.10)

$$|F(\bar{x}, q, X) - F(\bar{y}, q, X)| + |F(\bar{x}, q, -Y) - F(\bar{y}, q, -Y)| \leq c_3M^{1+\alpha}|\bar{x} - \bar{y}|^{-\tau_3},$$

(3.11)
\[ |h(\bar{x}, q^x)| + |h(\bar{y}, q^y)| \leq c_4 M^{1+\alpha} |\bar{x} - \bar{y}|^{-\tau_4}. \]  

(3.12)

From all these claims, by taking \( \delta \) small enough depending only on \( c_i \) and \( \tau_i, \lambda, \Lambda, \alpha, N, r \) one gets

\[ F(\bar{x}, q^x, X) - F(\bar{y}, q^y, -Y) + h(\bar{x}, q^x) - h(\bar{y}, q^y) \leq -\frac{\lambda c}{2} M^{\alpha+1} |\bar{x} - \bar{y}|^{-\tau}. \]

Precisely one needs to take \( \delta \) such that

\[ c_2 \delta^{-\tau_2+\hat{\tau}} + c_3 \delta^{\tau-\tau_3} + c_4 \delta^{\tau-\tau_4} + \Lambda c_1 \delta^{\tau-\tau_1} < \frac{\lambda c}{2}. \]

Finally, one can conclude as follows

\[ f(\bar{x}) \leq F(\bar{x}, q^x, X) + h(\bar{x}, q^x) \]
\[ \leq F(\bar{y}, q^y, -Y) + h(\bar{y}, q^y) - \frac{\lambda c}{2} M^{\alpha+1} |\bar{x} - \bar{y}|^{-\tau} \]
\[ \leq -\frac{\lambda c}{2} M^{\alpha+1} |\bar{x} - \bar{y}|^{-\tau} + g(\bar{y}). \]

This contradicts the fact that \( f \) and \( g \) are bounded, as soon as \( \delta \) is small or \( M \) is large enough. And then in order to get the desired result it is sufficient to prove in all cases \((3.8), (3.9), (3.10), (3.11), (3.12)\).

So to prove the claims, we will use inequality \((3.7)\) which has three important consequences for \( \Theta(X + Y - 2(2M + 1)I)\Theta \):

1. As is well known the second inequality in \((3.7)\) gives
   \((X + Y - 2(2M + 1)I) \leq 0\), then also \( \Theta(X + Y - 2(2M + 1)I)\Theta \leq 0 \). In particular
   all the eigenvalues of \( \Theta(X + Y)\Theta \) are less than \( 6M|\Theta|^2 \).  
   \[(3.13)\]
2. By Proposition \(3.3\), \( \Theta(\bar{H})\Theta \) has a large negative eigenvalue, let \( e \) be the corresponding eigenvector. Multiplying by \( \begin{pmatrix} e \\ -e \end{pmatrix} \) on the right and by its transpose on the left of \((3.7)\), one gets, using \((3.3)\) that, for some positive constant \( c \), when \( \alpha \leq 2 \),
   \[ \mu_1(\Theta(X + Y - 2(2M + 1)I)\Theta) \leq c M^{1+\alpha} \omega''(|\bar{x} - \bar{y}|)(\omega'(|\bar{x} - \bar{y}|))^\alpha; \]
   \[(3.14)\]
   this in particular implies that
   \[ \mu_1(\Theta(X + Y)\Theta) \leq c M^{1+\alpha} \omega''(|\bar{x} - \bar{y}|)(\omega'(|\bar{x} - \bar{y}|))^\alpha + 6M|\Theta|^2. \]
When \( \alpha \geq 2 \), if (3.4) holds, using (3.3),
\[
\mu_1(\Theta(X+Y)\Theta) \leq cM^{1+\alpha}\omega''(|\bar{x}-\bar{y}|)(\omega'(|\bar{x}-\bar{y}|))^\alpha|\bar{x}-\bar{y}|^{2\alpha} + 6M|\Theta|^2. \tag{3.15}
\]

3. Finally, using (3.7), we obtain an upper bound for \(|X| + |Y|\) i.e.
\[
|X| + |Y| \leq CM(|D^2g| + 1). \tag{3.16}
\]

remarking that \( |\bar{H}| \leq \frac{3}{2}|D^2g(x)| \).

We will need to detail the cases \( \omega(s) \simeq s \) or \( \omega(s) = s^\gamma \) both when \( \alpha \leq 2 \) or \( \alpha \geq 2 \).

**Proofs of the claims when \( \omega(r) = r^\gamma \) and \( \alpha \leq 2 \).**

In this case, \( \omega(s) = s^\gamma \), \( \omega'(s) = \gamma s^{\gamma - 1} \) and \( \omega''(s) = -\gamma(1 - \gamma)s^{\gamma - 2} \), \( q = M\gamma|x - \bar{y}|^{\gamma - 1} - \bar{y}|x - \bar{y}|^{\gamma - 1} \), \( q = q + 2M(\bar{x} - x_o) \), \( q = q - 2M(\bar{y} - x_o) \). By (3.14), since \( \gamma \in (0, 1) \), \( \Theta(X + Y - 2(2M + 1)I)\Theta \) has one eigenvalue less than
\[
-\frac{\gamma(1 - \gamma)}{4}M^{\alpha + 1}|\bar{x} - \bar{y}|^{\gamma - 2 + (\gamma - 1)\alpha}.
\]

While \( 6|\Theta|^2 \leq 6M^{\alpha}\gamma\alpha|\bar{x} - \bar{y}|^{(\gamma - 1)\alpha} \). Consequently, as soon as \( \delta \) is small enough, \( \Theta(X + Y)\Theta \) has at least one eigenvalue less than \( -\frac{\gamma(1 - \gamma)}{4}M^{\alpha + 1}|\bar{x} - \bar{y}|^{\gamma - 2 + (\gamma - 1)\alpha} + 6M|\Theta|^2 \leq -\frac{\gamma(1 - \gamma)}{4}M^{\alpha + 1}|\bar{x} - \bar{y}|^{\gamma - 2 + (\gamma - 1)\alpha} + 6M^{\alpha + 1}d^{\alpha}|\bar{x} - \bar{y}|^{(\gamma - 1)\alpha} \leq -\frac{\gamma(1 - \gamma)}{4}M^{\alpha + 1}|\bar{x} - \bar{y}|^{\gamma - 2 + (\gamma - 1)\alpha} + 6M^{\alpha + 1}d^{\alpha}|\bar{x} - \bar{y}|^{(\gamma - 1)\alpha} \)
\( \cdot \) \( \leq \frac{\gamma(1 - \gamma)}{4}M^{\alpha + 1}|\bar{x} - \bar{y}|^{\gamma - 2 + (\gamma - 1)\alpha} \). This proves (3.8) with \( \hat{\tau} = 2 - \gamma + (1 - \gamma)\alpha \), \( \gamma < 1 \), and \( c = \frac{\gamma(1 - \gamma)}{4}M^\alpha \).

Now using (3.14) and the above estimate on \( M|\Theta|^2 \), (3.9) holds with \( \tau_1 = (1 - \gamma)\alpha \).

Note that
\[
|D^2g(\bar{x} - \bar{y})| \leq \gamma(N - \gamma)|\bar{x} - \bar{y}|^{\gamma - 2},
\]
and recall that \( |\bar{H}| \leq \frac{3}{2}|D^2g| \), and then, by (3.16),
\[
|X| + |Y| \leq 6\gamma(N - \gamma + 3)M|\bar{x} - \bar{y}|^{\gamma - 2}. \tag{3.17}
\]

Consequently (3.11) holds with \( \tau_2 = (2 - \gamma) + (1 - \gamma)\alpha - \gamma_F \) and \( c_2 = 12c_\gamma\gamma^{1 + \alpha}(N + 3 - \gamma) \) using hypothesis (1.4).

To prove (3.10) we will use the following universal inequality : For any \( Z \) and \( T \) in \( \mathbb{R}^N \)
\[
||Z|^\alpha - |T|^\alpha| \leq \sup(1, \alpha)|Z - T|^\gamma(1, \alpha)(|Z| + |T|)^{(\alpha - 1)^+} \tag{3.18}
\]
in the form

\[ |q|^\alpha - |q|^\alpha | \leq 2^a \sup(1, \alpha) M^\alpha |\bar{x} - \bar{y}|^{(\gamma - 1)(\alpha - 1)} \]

Hence using (3.17), (3.10) holds with \( \tau_3 = (2 - \gamma) + (1 - \gamma)(\alpha - 1)^+ \), and \( c_3 = c_F 2^{1 + \alpha} (\gamma + 1)^{(\alpha - 1)^+} \). Finally (3.12) holds with \( \tau_4 = (1 - \gamma)(1 + \alpha) \) and \( c_4 = 2c_{h, F} ((\gamma + 3)^{1 + \alpha} + 1) \).

**Proofs of the claims when** \( \omega(r) = r^\gamma \) **and** \( \alpha \geq 2 \). The function \( \omega \) is the same than in the previous case. In order to use the result in Proposition 3.3 we need (3.4) to be satisfied. For that aim we take \( \epsilon > 0 \) such that \( \epsilon < \inf(\frac{\gamma}{2}, \frac{1 - \gamma}{2}) \).

Let

\[ \delta_N := \exp\left(\frac{-\log(2N(4 - \gamma)) + \log(1 - \gamma)}{2\epsilon}\right) \tag{3.19} \]

and assume \( \delta < \delta_N \). In particular, since there exists \( i \in [1, N] \) such that

\[ |\bar{x}_i - \bar{y}_i|^2 \geq \frac{|\bar{x} - \bar{y}|^2}{N} \geq |\bar{x} - \bar{y}|^{2 + 2\epsilon}, \]

for \( \alpha \geq 2 \), using the definition of \( \delta_N \) in (3.19), \( I(\bar{x} - \bar{y}, \epsilon) \neq \emptyset \). Furthermore for \( |\bar{x} - \bar{y}| < \delta \leq \delta_N \)

\[
\frac{1}{2} \omega''(|\bar{x} - \bar{y}|)(1 - N)|\bar{x} - \bar{y}|^{2\epsilon} + \frac{3N}{2} |\bar{x} - \bar{y}|^{2\epsilon} \omega'(|\bar{x} - \bar{y}|) \leq \frac{1}{2} \omega''(|\bar{x} - \bar{y}|) \\
+ \frac{N}{2} |\bar{x} - \bar{y}|^{2\epsilon} (\gamma(1 - \gamma) + 3\gamma)|\bar{x} - \bar{y}|^{\gamma - 2} \\
\leq \frac{1}{4} \gamma(\gamma - 1)|\bar{x} - \bar{y}|^{\gamma - 2} = \frac{\omega''(|\bar{x} - \bar{y}|)}{4},
\]

and then (3.4) is satisfied. We are in a position to apply (3.15), and \( \Theta(X + Y)\Theta \) has at least one eigenvalue \( \mu_1 \) less than \( -(\gamma(1 - \gamma)\frac{1}{4})M^{\alpha + 1}|\bar{x} - \bar{y}|^{\gamma - 2 + (\gamma - 1)\alpha + \epsilon} + 6M|\Theta|^2 \), hence

\[
\mu_1 \leq -(\gamma(1 - \gamma)\frac{1}{4})M^{\alpha + 1}|\bar{x} - \bar{y}|^{\gamma - 2 + (\gamma - 1)\alpha + \epsilon} \\
+ 6M^{1 + \alpha}\gamma^\alpha |\bar{x} - \bar{y}|^{(\gamma - 1)\alpha} \\
\leq -(\gamma(1 - \gamma)\frac{1}{8})M^{\alpha + 1}|\bar{x} - \bar{y}|^{\gamma - 2 + (\gamma - 1)\alpha + \epsilon}
\]
for $|\bar{x} - \bar{y}| \leq \delta$ small enough, hence (3.8) holds with $\bar{\tau} = 2 - \gamma + (1 - \gamma) - \epsilon$.

Note that (3.9), (3.11) (3.10) and (3.12) have already been proved in the previous case, since the sign of $\alpha - 2$ does not play a role. Recall then that $\gamma_1 = (-\gamma + 1)\alpha$, and $c_1 = 6\gamma^{1+\alpha}(N - \gamma + 3)$, while $\gamma_2 = (2 - \gamma) + (1 - \gamma)\alpha - \gamma F < \bar{\tau}$ by the choice of $\epsilon$, and $c_2 = 12\gamma F \gamma^{1+\alpha}(N + 3 - \gamma)$.

Finally $\gamma_3 = (2 - \gamma) + (\alpha - 1)(\gamma - 1)$ and $c_3 = \gamma F 2^{1+\alpha}(\gamma + 1)(\alpha-1)^+$, and (3.12) still holds with $\gamma_4 = (1 - \gamma)(1 + \alpha)$.

Let us observe that in the hypothesis of Theorem 3.1 we have proved that $u$ and $v$ satisfy, for any $\gamma \in (0, 1)$,

$$u(x) \leq v(y) + \sup_{\Omega}(u - v) + c_{\gamma, r}|x - y|^\gamma \quad (3.20)$$

This will be used in the next cases.

**Proofs of the claims when $\omega(r) \approx r$ and $\alpha \leq 2$.** We choose $\tau \in (0, \inf(\gamma F, 1, \frac{1}{2}))$ and $\gamma \in (1, \inf(\bar{\tau} + \frac{\alpha}{\gamma}))$. We define $\omega(s) = s - \omega o s^{1+\bar{\tau}}$, where $s < s_o = \left(\frac{1}{(1+\bar{\tau})\omega_o}\right)^\frac{1}{\bar{\tau}}$ and $\omega_o$ is chosen so that $s_o > 1$. We suppose that $\delta^\bar{\tau}\omega_o(1 + \tau) < \frac{1}{2}$, which ensures that

$$s < \delta \frac{1}{2} \leq \omega'(s) < 1, \quad \omega(s) \geq \frac{s}{2} \quad (3.21)$$

We suppose that

$$M \frac{\delta \tau}{(1 + \tau)} > 2\sup u, \quad M > 1 \quad \text{and} \quad M \frac{1 - r}{2} > 2\sup u \quad (3.22)$$

which implies in particular (3.6).

Here $|D^2g(\bar{x} - \bar{y})| \leq \frac{N-1}{|\bar{x} - \bar{y}|} + \omega_o \tau (1 + \tau)|\bar{x} - \bar{y}|^{-1+\bar{\tau}} \leq (N - 1 + \omega_o \tau (1 + \tau))|\bar{x} - \bar{y}|^{-1}$, $|H| \leq \frac{3}{2} |D^2g(\bar{x} - \bar{y})|$ and then (3.16) is nothing else but

$$|X| + |Y| \leq 6M(|D^2g(\bar{x} - \bar{y})| + 1) \leq 6M(N - 1 + \omega_o \tau (1 + \tau))|\bar{x} - \bar{y}|^{-1}. \quad (3.23)$$

Furthermore $q = M\omega'(1 - \bar{y})|\bar{x} - \bar{y}|\frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|} \in q^\tau = q + 2M(\bar{x} - x_o), \quad q^\gamma = q - 2M(\bar{y} - x_o)$. Using (3.20) in $B_{1/2}$, for all $\gamma < 1$,

$$M|\bar{x} - x_o|^2 + M|\bar{y} - x_o|^2 + \sup(u - v) \leq u(\bar{x}) - v(\bar{y}) \leq \sup(u - v) + c_{\gamma, r}|\bar{x} - \bar{y}|^\gamma$$

and then
\[ |\bar{y} - x_0| + |\bar{x} - x_0| \leq \left( \frac{c_{\gamma,r}|\bar{x} - \bar{y}|}{M} \right)^{\frac{1}{2}}. \] (3.24)

Then taking \( \delta \) small enough, more precisely if \((c_{\gamma,r}\delta)^{\frac{1}{2}} < \frac{1}{4}\) by (3.21),

\[ \frac{M}{2} \leq |q| \leq M, \quad \frac{M}{4} \leq |q^x|, |q^y| \leq \frac{5M}{4} \] (3.25)

Then we derive from (3.14) that \( \Theta(X + Y - 2(2M + 1)I)\Theta \) has at least one eigenvalue less than

\[ -\frac{\omega_o\tau(1 + \tau)}{4}M^{\alpha+1}|\bar{x} - \bar{y}|^{\tau-1} \] (3.26)

Since \( M|\Theta|^2 \leq M^{1+\alpha} \), (3.9) holds with \( \tau_1 = 0 < 1 - \tau \), and \( c_1 = 6 \), while (3.11) is satisfied with \( \tau_2 = -\gamma_F + 1 < 1 - \tau \), and \( c_2 = c_{\gamma_F}(6 + 2\omega_o\tau(1 + \tau)) \).

To check (3.10), by (3.18), (3.25), (3.24), and (3.23)

\[ |q_i^x|^\alpha - |q_i^y|^\alpha |X_{ii}| \leq (6 + 2\omega_o\tau(1 + \tau))M^{1+\frac{\inf(1,\alpha)}{2}}c_{\gamma,r}^\alpha \frac{M^\alpha}{\inf(1,\alpha)^{\frac{\gamma}{2}}} |\bar{x} - \bar{y}|^{\inf(1,\alpha)^{\frac{\gamma}{2}}} \]

Hence, by using \( \inf(1,\alpha)\gamma > 2\tau \), (3.10) holds with \( \tau_3 = 1 - \frac{\inf(1,\alpha)}{2}\gamma \) and \( c_3 = 2c_F(6 + 2\omega_o\tau(1 + \tau))(c_{\gamma,r})^{\frac{\alpha}{2}} \) if \( \alpha \leq 1 \) and \( c_3 = 2c_F(6 + 2\omega_o\tau(1 + \tau)c_{\gamma,r}^{\frac{1}{2}}\alpha^{3^{\alpha-1}}) \) if \( \alpha \geq 1 \).

Finally \( \tau_4 = 0 \) and \( c_4 = c_{h,\Omega}(2^{1+\alpha} + 1) \) are convenient for (3.12).

**Proofs of the claims when \( \omega(r) \simeq r \) and \( \alpha \geq 2 \)**

In order to use the result in Proposition 3.3 we need (3.4) to be satisfied. For that aim we take \( \tau \) and \( \epsilon > 0 \) such that

\[ 0 < \tau < \inf\left(\frac{1}{\alpha}, \gamma_F\right), \quad 1 > \gamma > \tau \alpha, \quad \text{and} \quad \frac{\tau}{2} < \epsilon < \inf\left(\frac{\gamma_F - \tau}{\alpha - 2}, \frac{\gamma_F - \tau}{\alpha - 2}\right). \] (3.27)

Let us define \( \omega, s_o \), as in the case \( \alpha \leq 2 \). We suppose \( \delta < \delta_N \) where

\[ \delta_N : = \inf\left(\exp\frac{\log(\omega_o(1 + \tau)\tau) - \log(2N(1 + \omega_o\tau(1 + \tau)))}{2\epsilon - \tau}, \frac{\exp - \log(2\omega_o(1 + \tau))}{\tau}\right) \] (3.28)
In particular since there exists $i$ such that $|\bar{x}_i - \bar{y}_i|^2 \geq \frac{1}{2} |\bar{x} - \bar{y}|^2 \geq |\bar{x} - \bar{y}|^{2+2\epsilon}$, by \[3.28\], $I(\bar{x} - \bar{y}, \epsilon) \neq \emptyset$. Furthermore, recall that by \[3.28\], $1 \geq \omega'(|\bar{x} - \bar{y}|) \geq \frac{1}{2}$ and

$$
\frac{1}{2} \omega''(|\bar{x} - \bar{y}|) + \frac{N}{2} \omega_0 \tau (1 + \tau) |\bar{x} - \bar{y}|^{\tau - 1 + 2\epsilon} + \frac{3}{2} N |\bar{x} - \bar{y}|^{2\epsilon - 1} \omega'(|\bar{x} - \bar{y}|)
$$

$$\leq \frac{1}{2} \omega''(|\bar{x} - \bar{y}|) + \frac{N}{2} (\omega_0 \tau (1 + \tau) + 3) |\bar{x} - \bar{y}|^{2\epsilon - 1}
$$

$$\leq -\frac{1}{4} \omega_0 (1 + \tau) |\bar{x} - \bar{y}|^{1 + \tau} = \frac{\omega''(|\bar{x} - \bar{y}|)}{4},
$$

and then \[3.4\] holds. We still assume that \[3.22\] holds.

As in the case $\alpha \leq 2$, using \[3.20\], one has, for $\delta$ small enough, \[3.25\] still holds.

The hypothesis \[3.28\] ensures that $\Theta(X + Y - 2(2M + 1)I) \Theta$ has at least one eigenvalue less than

$$\omega_o \tau (1 + \tau) M^{1+\alpha} |\bar{x} - \bar{y}|^{-1 + \tau + (\alpha - 2)\epsilon}
$$

and then using the fact that $\Theta(X + Y) \Theta \leq 6M|\Theta|^2 \leq 6M^{1+\alpha}$, by \[3.27\] and for $\delta$ small enough, \[3.8\] holds with $\hat{\tau} = (2 - \alpha)\epsilon + 1 - \tau$ and $c = \frac{\omega_0 \tau (1 + \tau)}{8}$.

Furthermore \[3.9\] holds with $\tau_1 = 0$, and $c_1 = 6$.

As in the previous case, \[3.23\] holds, and then \[3.11\] holds with $\tau_2 = 1 - \gamma_F < 1 - \tau + (2 - \alpha)\epsilon$ and $c_2 = c_{\gamma_F} (6 + 2\omega_0 \tau (1 + \tau))$.

Now using \[3.18\], \[3.23\], \[3.25\], \[3.24\], one has

$$||q^x|^{\alpha} - |q|^\alpha||X| \leq \alpha (M|\bar{x} - x_0|)(\frac{5M}{4})^{\alpha - 1} M |\bar{x} - \bar{y}|^{-1} \leq c_3 |\bar{x} - \bar{y}|^{\frac{3}{2} - 1} M^{1+\alpha}
$$

and then \[3.10\] holds with $\tau_3 = 1 - \frac{3}{2} < 1 - \tau + (2 - \alpha)\epsilon$ and $c_3 = 2(c_{\gamma,r})(2)^{\alpha - 1}$.

Note finally that

$$|h(\bar{x}, q^x)| + |h(\bar{y}, q^y)| \leq 2c_h \left(\frac{5M}{4}\right)^{\frac{1}{2}}
$$

and then \[3.12\] holds with $\tau_4 = 0$ and $c_4 = 2^{2+\alpha} c_h$. 

4 Existence of solutions.

As it is classical, see e. g. [9], the existence’s Theorem 1.2 will be proved once the following Propositions are known:

**Proposition 4.1.** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^N$ and that $F$ satisfies (1.3), (H3), (1.4), (1.7). Suppose that $h$ is continuous and it satisfies (1.8). Let $u$ be a USC sub-solution of
\[
F(x, \nabla u, D^2 u) + h(x, \nabla u) - \beta(u) \geq f \text{ in } \Omega
\]
and $v$ be a LSC super-solution of
\[
F(x, \nabla v, D^2 v) + h(x, \nabla v) - \beta(v) \leq g \text{ in } \Omega
\]
where $\beta$, $f$ and $g$ are continuous.

Suppose that either $\beta$ is increasing and $f \geq g$, or $\beta$ is nondecreasing and $f > g$. If $u \leq v$ on $\partial \Omega$, then $u \leq v$ in $\Omega$.

**Proposition 4.2.** Suppose that the assumptions in Proposition 4.1 hold, and that $f$ is continuous and bounded and $\beta$ is increasing. If $\underline{u}$ is a USC sub-solution, and $\overline{u}$ is a LSC super-solution of the equation
\[
F(x, \nabla u, D^2 u) + h(x, \nabla u) - \beta(u) = f, \text{ in } \Omega,
\]
such that $\underline{u} = \overline{u} = \varphi$ on $\partial \Omega$. Then there exists $u$ a viscosity solution of the equation with $\underline{u} \leq u \leq \overline{u}$ in $\Omega$, and $u = \varphi$ on $\partial \Omega$.

The proofs of these two Propositions can be done by using the classical tools, see [9].

**Remark 4.3.** One can get the same existence’s result when $\beta = 0$, by using a standard approximation procedure and the stability of viscosity solutions.

Nevertheless the proof of Theorem 1.2 requires the existence of a supersolution which is zero on the boundary when $\beta = 0$:

**Proposition 4.4.** Suppose that $\Omega$ is a bounded $C^2$ domain, and that $F$ and $h$ satisfy the hypothesis in Proposition 4.2. Then for any $f$ continuous and bounded, there exist a super-solution and a sub-solution of of
\[
F(x, \nabla u, D^2 u) + h(x, \nabla u) = f \text{ in } \Omega
\]
which are zero on the boundary.
Proof of Proposition 4.4:

Let us recall that the distance to the boundary \( d \) satisfies everywhere: \( d \) is semi concave or equivalently there exists \( C_1 \) such that

\[
D^2d \leq C_1 I.
\]

In the following lines we will make the computations as if \( d \) is \( C^2 \), it is not difficult to see that the required inequalities hold also in the viscosity sense.

We now choose \( k \) large such that

\[
(k + 1)(\sum_1^N |\partial_i d|^{2+\alpha})^{\frac{1+\alpha}{2+\alpha}} \geq 2C_1 N^{\frac{1+\alpha}{2+\alpha}} (1 + \text{diam } \Omega).
\]

This can be done since \( \sum_1^N (\partial_i d)^2 = 1 \leq (\sum_1^N |\partial_i d|^{2+\alpha})^{\frac{2}{2+\alpha}} N^{\frac{1+\alpha}{2+\alpha}} . \) We will choose later \( M \) large and define

\[
\psi(x) = M(1 - \frac{1}{(1 + d)^k}).
\]

Clearly

\[
\nabla \psi = M \frac{k \nabla d}{(1 + d)^{k+1}}, \quad D^2 \psi = \frac{Mk}{(1 + d)^{k+2}} ((1 + d) D^2 d - (k + 1) \nabla d \otimes \nabla d)
\]

and then choosing \( k \) such that \( \lambda (k + 1) N^{-\frac{1}{2+\alpha}} \geq 3N \lambda C_1 + 2C_h (1 + \text{diam } \Omega) \).

\[
F(x, \nabla \psi, D^2 \psi) + h(x, \nabla \psi) \leq \frac{(Mk)^{\alpha+1}}{(1 + d)^{k+2+(k+1)\alpha}} ((1 + d) M^+_{\alpha} (\nabla d, D^2 d) - (k + 1) M^-_{\alpha} (\nabla d, D^2 d)) + C_h |\nabla \psi|^{1+\alpha}
\]

\[
\leq \frac{(Mk)^{\alpha+1}}{(1 + d)^{k+2+(k+1)\alpha}} [(1 + d) \lambda C_1 \sum |\partial_i d|^{\alpha} - (k + 1) \lambda \sum |\partial_i d|^{\alpha+2}] + C_h \frac{(Mk)^{\alpha+1}}{(1 + d)^{(k+1)(1+\alpha)}}
\]

\[
\leq \frac{(Mk)^{\alpha+1}}{(1 + d)^{k+2+(k+1)\alpha}} (2N \lambda C_1 - \lambda (k + 1) N^{-\frac{1}{2+\alpha}})
\]

\[
+ C_h \frac{(Mk)^{\alpha+1}}{(1 + d)^{(k+1)(1+\alpha)}}
\]

\[
\leq \frac{(k + 1) \lambda N^{-\frac{1}{2+\alpha}} (Mk)^{\alpha+1}}{4(1 + d)^{k+2+(k+1)\alpha}}.
\]

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It is clear that one can choose $M$ large enough as soon as $k$ is fixed as above in order that

$$F(x, \nabla \psi, D^2 \psi) + h(x, \nabla \psi) \leq -\|f\|_\infty.$$ 

A similar computation leads to:

$$F(x, \nabla (-\psi), D^2 (-\psi)) + h(x, \nabla - \psi) \geq \|f\|_\infty.$$

5 The strong Maximum Principle

Theorem 5.1. Suppose that $u$ is a supersolution of the equation $F(x, \nabla u, D^2 u) \leq 0$ in a domain $\Omega$ and that $u \geq 0$. Then either $u > 0$ in $\Omega$ or $u \equiv 0$.

Proof. One can suppose that $u > 0$, on $B(x_1, |x_1 - x_o|)$, $u(x_o) = 0$, $R = |x_1 - x_o|$ and we can assume that the annulus $\frac{R}{2} \leq |x - x_1| \leq \frac{3R}{2}$ is included in $\Omega$. Let \( w \) be defined as

$$w(x) = m(e^{-c|x-x_1|} - e^{-cR})$$

for some $c$ and $m$ to be chosen. Without loss of generality we will suppose that $x_1 = 0$ and denote $r := |x - x_1| = |x|$. We choose $m$ so that on $r = \frac{R}{2}$, $w \leq u$. In the sequel for simplicity we replace $m$ by 1.

One has

$$\nabla w = \frac{-cx}{r}e^{-cr}, \quad D^2 w = e^{-cr}(c^2 \frac{2}{r^2} + \frac{1}{r} x \otimes x) - \frac{c}{r} I$$

and then, using the usual notation $\Theta(\nabla w)$, $H := \Theta(\nabla w)D^2 w\Theta(\nabla w)$, i.e.

$$He^{c(\alpha + 1)r} = \left(\frac{c}{r}\right)^{\alpha} \left(\frac{c^2}{r^2} + \frac{c}{r^3} \vec{i} \otimes \vec{i} - \frac{c}{r^3} \vec{j} \otimes \vec{j}\right)$$

where $\vec{i} = \sum |x_i|^{\frac{2}{\alpha}} x_i e_i$ and $\vec{j} = \sum |x_i|^{\frac{2}{1+\alpha}} e_i$.

We need to evaluate the eigenvalues of $H$ and in particular prove that

$$\mathcal{M}^-(H) > 0.$$ 

For that aim let us note that $(\vec{i}, \vec{j})^\perp$ is in the kernel of $H$. We introduce $a = \frac{1}{r^2} + \frac{c}{r}$ and $b = -\frac{c}{r}$. Then the non zero eigenvalues of $He^{-c\alpha e^{c\alpha}}$ are given by

$$\mu^\pm = \frac{a|\vec{i}|^2 + b|\vec{j}|^2}{2} \pm \sqrt{\left(\frac{a|\vec{i}|^2 + b|\vec{j}|^2}{2}\right)^2 - ab(|\vec{i}|^2|\vec{j}|^2 - (\vec{i} \cdot \vec{j})^2)}.$$
Note that there exist constants $c_i(N, \alpha)$ for $i = 1, \cdots, 4$, such that
\[
c_1(N, \alpha) \left( \frac{R}{2} \right)^{\alpha + 2} \leq c_1(N, \alpha) r^{\alpha + 2} \leq |\vec{i}|^2 \leq c_2(N, \alpha) r^{\alpha + 2} \leq c_2(N, \alpha) \left( \frac{3R}{2} \right)^{\alpha + 2}
\]
and
\[
c_3(N, \alpha) \left( \frac{R}{2} \right)^{\alpha} \leq c_3(N, \alpha) r^{\alpha} \leq |\vec{j}|^2 \leq c_4(N, \alpha) r^{\alpha} \leq c_4(N, \alpha) \left( \frac{3R}{2} \right)^{\alpha}.
\]
Note that one can choose $c$ large enough in order that for some constant $c_5(N, \alpha)$
\[
a|\vec{i}|^2 + b|\vec{j}|^2 \geq c_1(N, \alpha) \left( \frac{R}{2} \right)^{\alpha + 2} \frac{c^2}{r^2} - c_4(N, \alpha) \left( \frac{3R}{2} \right)^{\alpha} \frac{c}{r}
\]
\[
\geq c_5(N, \alpha) c^2.
\]
On the other hand one can assume $c$ large enough in order that
\[
4|ab|(|\vec{i}|^2|\vec{j}|^2 - (\vec{i} \cdot \vec{j})^2) \leq \frac{4c^3}{r^2} c_2(N, \alpha) c_4(N, \alpha) \left( \frac{3R}{2} \right)^{2\alpha + 2}
\]
\[
\leq c_6(N, \alpha) c^3
\]
\[
< \left[ \left( \frac{\lambda + \Lambda}{\Lambda - \lambda} \right)^2 - 1 \right] (c_5(N, \alpha) c^2)^2
\]
\[
\leq \left[ \left( \frac{\lambda + \Lambda}{\Lambda - \lambda} \right)^2 - 1 \right] \left( a|\vec{i}|^2 + b|\vec{j}|^2 \right)^2.
\]
In particular this implies
\[
\lambda \mu^+ + \Lambda \mu^- = \left( \frac{a|\vec{i}|^2 + b|\vec{j}|^2}{2} \right) \left( \lambda + \Lambda \right) + \left( \lambda - \Lambda \right) \sqrt{1 + 4 \frac{|ab|(|\vec{i}|^2|\vec{j}|^2 - (\vec{i} \cdot \vec{j})^2)}{(a|\vec{i}|^2 + b|\vec{j}|^2)^2}} > 0
\]
i.e. $\mathcal{M}^-(H) > 0$. Using the comparison principle in the annulus $\{ \frac{R}{2} \leq |x - x_1| \leq \frac{3R}{2} \}$ one obtains that $u \geq w$.

Observe that $w$ touches $u$ by below on $x_o$, and then, since $w$ is $C^2$ around $x_o$, by the definition of viscosity solution
\[
F(x_o, \nabla w(x_o), D^2 w(x_o)) \leq 0.
\]
This contradicts the above computation. \hfill \Box

**Remark 5.2.** As it is well known, the above proof can be used to see that on a point of the boundary where the interior sphere condition is satisfied, the Hopf principle holds.
6 Appendix: Proof of Lemma 3.2

The proof of Lemma 3.2 is based on the following Lemma by Ishii

**Lemma 6.1** (Ishii). Let $A$ be a symmetric matrix on $\mathbb{R}^{2N}$. Suppose that $U \in USC(\mathbb{R}^{N})$ and $V \in USC(\mathbb{R}^{N})$ satisfy $U(0) = V(0)$ and, for all $(x, y) \in (\mathbb{R}^{N})^2$,

$$U(x) + V(y) \leq \frac{1}{2}(t^t x + t^t y)A \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

Then, for all $\iota > 0$, there exist $X_U^\iota \in S$, $X_V^\iota \in S$ such that $(0, X_U^\iota) \in \bar{J}^{2,+}U(0)$, $(0, X_V^\iota) \in \bar{J}^{2,+}V(0)$

and

$$-(\frac{1}{\iota} + |A|) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_U^\iota & 0 \\ 0 & X_V^\iota \end{pmatrix} \leq (A + \iota A^2).$$

We can now start the proof of Lemma 3.2. The second order Taylor’s expansion for $\Phi$, gives that for all $\epsilon > 0$ there exists $r > 0$ such that, for $|x - \bar{x}|^2 + |\bar{y} - y|^2 \leq r^2$,

$$u(x) - u(\bar{x}) - \langle D_1 \Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), x - \bar{x} \rangle +$$
$$+ v(\bar{y}) - v(y) - \langle D_2 \Phi(\bar{x}, \bar{y}) + 2M(\bar{y} - y_o), y - \bar{y} \rangle \leq \frac{1}{2} (t^t (x - \bar{x}), t^t (y - \bar{y})) (D^2 \Phi(\bar{x}, \bar{y}) + \epsilon I) \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} + M(|x - \bar{x}|^2 + |y - \bar{y}|^2).$$

We now introduce the functions $U$ and $V$ defined, in the closed ball $|x - \bar{x}|^2 + |\bar{y} - y|^2 \leq r^2$, by

$$U(x) = u(x + \bar{x}) - \langle D_1 \Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), x \rangle - u(\bar{x}) - M|x|^2$$

and

$$V(y) = -v(y + \bar{y}) - \langle D_2 \Phi(\bar{x}, \bar{y}) + 2M(\bar{y} - y_o), y \rangle + v(\bar{y}) - M|y|^2$$

which we extend by some convenient negative constants in the complementary of that ball (see [16] for details). Observe first that

$$(0, X_U^\iota) \in \bar{J}^{2,+}U(0), \ (0, X_V^\iota) \in \bar{J}^{2,-}V(0)$$
is equivalent to
\[
(D_1 \Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), X^U + 2MI) \in \mathcal{J}^2+ u(\bar{x})
\]
and
\[
(-D_2 \Phi(\bar{x}, \bar{y}) - 2M(\bar{y} - x_o), -X^V - 2MI) \in \mathcal{J}^2_2 v(\bar{y}).
\]
We can apply Lemma 6.1 which gives that, for any $\iota > 0$, there exists $(X_\iota, Y_\iota)$ such that
\[
(D_1 \Phi(\bar{x}, \bar{y}) + 2M(\bar{x} - x_o), X_\iota) \in \mathcal{J}^2_2 u(\bar{x})
\]
and
\[
(-D_2 \Phi(\bar{x}, \bar{y}) - 2M(\bar{y} - x_o), -Y_\iota) \in \mathcal{J}^2_2 v(\bar{y}).
\]
Choosing $\epsilon$ such that $2\epsilon|D^2\Phi(\bar{x}, \bar{y})| + \epsilon + \iota(\epsilon)^2 < 1$, one gets
\[
-(\frac{1}{\iota} + |D^2\Phi| + 1) \left( \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right) \leq \left( \begin{array}{cc} X_\iota - 2MI & 0 \\ 0 & Y_\iota - 2MI \end{array} \right)
\]
\[
\leq (D^2\Phi + \iota(D^2\Phi)^2) + \left( \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right).
\]

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