Cosmic-ray hydrodynamics: Alfvén-wave regulated transport of cosmic rays

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ABSTRACT
Star formation in galaxies appears to be self-regulated by energetic feedback processes. Among the most promising agents of feedback are cosmic rays (CRs), the relativistic ion population of interstellar and intergalactic plasmas. In these environments, energetic CRs are virtually collisionless and interact via collective phenomena mediated by kinetic-scale plasma waves and large-scale magnetic fields. The enormous separation of kinetic and global astrophysical scales requires a hydrodynamic description. Here, we develop a new macroscopic theory for CR transport in the self-confinement picture, which includes CR diffusion and streaming. The interaction between CRs and electromagnetic fields of Alfvénic turbulence provides the main source of CR scattering, and causes CRs to stream along the magnetic field with the Alfvén velocity if resonant waves are sufficiently energetic. However, numerical simulations struggle to capture this effect with current transport formalisms and adopt regularization schemes to ensure numerical stability. We extend the theory by deriving an equation for the CR energy flux along the mean magnetic field and include a transport equation for the Alfvén-wave energy. We account for energy exchange of CRs and Alfvén waves via the gyroresonant instability and include other wave damping mechanisms. The resulting system of equations correctly recovers previous macroscopic CR transport formalisms in the steady-state limit. Using numerical simulations we demonstrate that our new theory self-regulates CR transport, and highlight differences to currently used algorithms. Our theory is free of tunable parameters and holds the promise to provide predictable simulations of CR feedback in galaxy formation.

Key words: cosmic rays – hydrodynamics – radiative transfer – methods: analytical – methods: numerical

1 INTRODUCTION
CRs are pervasive in galaxies and galaxy clusters and likely play an active role during the formation and evolution of these systems. CRs, magnetic fields, and turbulence are observed to be in pressure equilibrium in the midplane of the Milky Way (Boulares & Cox 1990), suggesting that CRs have an important dynamical role in maintaining the energy balance of the interstellar medium (ISM).

This equipartition could be the result of a self-regulated feedback process: provided that CR and magnetic midplane pressures are supercritical, their buoyancy force overcomes the magnetic tension of the dominant toroidal magnetic field, causing it to bend and open up (Parker 1966; Rodríguez et al. 2016). CRs stream and diffuse ahead of the gas into the halo along these open field lines and build up a pressure gradient. Once this gradient overcomes the gravitational attraction of the disc, it accelerates the gas, thereby driving a strong galactic outflow as shown in one-dimensional magnetic flux-tube models (Breitschwerdt et al. 1991; Zirakashvili et al. 1996; Ptuskin et al. 1997; Everett et al. 2008; Samui et al. 2018) and three-dimensional simulations (Uhlig et al. 2012; Booth et al. 2013; Salem & Bryan 2014; Pakmor et al. 2016; Simpson et al. 2016; Girichidis et al. 2016; Pfrommer et al. 2017b; Ruszkowski et al. 2017; Jacob et al. 2018). If the CR pressure is subcritical, the thermal gas can quickly radiate away the excess energy, thus approaching equipartition as a dynamical attractor solution.

Seemingly unrelated, at the centres of dense galaxy clusters the observed gas cooling and star formation rates are reduced to levels substantially below those expected from unimpeded cooling flows (Peterson & Fabian 2006). Most likely, a heating process associated with radio lobes that are inflated by jets from active galactic nuclei offsets radiative
cooking. Apparently, the cooking gas and nuclear activity are tightly coupled to a self-regulated feedback loop (McNamara & Nulsen 2007). A promising heating mechanism can be provided by fast-streaming CRs, which resonantly excite Alfvén waves through the “streaming instability” (Kulsrud & Pearce 1969). Scattering off of this wave field (partially) isotropizes these CRs in the reference frame of Alfvén waves, which causes CRs to stream down their gradient (Zweibel 2013). Damping of these waves transfers CR energy and momentum to the thermal gas at a rate that scales with the CR pressure gradient and provides an efficient means of suppressing the cooling catastrophe in cooling core clusters (Loewenstein et al. 1991; Guo & Oh 2008; Enßlin et al. 2011; Fujita & Ohira 2012; Pfrommer 2013; Jacob & Pfrommer 2017a,b).

Hence, in sharing energy and momentum with the thermal gas, CRs may play a critical role in galaxy formation and the evolution of galaxy clusters.

CRs interact with the thermal gas through particle collisions as well as through collisionless processes. Low-energy (MeV to GeV) CRs are important for collisional ionization and heating of the interstellar medium. In particular, the ability of CRs to deeply penetrate into molecular clouds (where ultra-violet and X-ray photons are absorbed) makes them prime drivers of cloud chemistry (Dalgarno 2006; Iley et al. 2018; Phan et al. 2018) and responsible for the evolution of these star-forming regions. Hadronic particle interactions generate secondary decay products that emit characteristic signatures from radio to gamma-ray energies, thereby enabling studies of the spatial and spectral CR distribution.

Energetic protons with energies of a few GeV, which dominate the total CR energy density, are mostly collisionless and interact via collective phenomena mediated by the ambient magnetic field. Being charged particles, CRs are bound to follow individual magnetic fields lines, which become modified as a result of the dynamical evolution of the CR distribution. Hence, in combination with the toroidal stretching of magnetic fields due to differential rotation of galactic discs, CR-generated gas motions can twist and fold magnetic structures, thereby amplifying and shaping galactic magnetic fields via a CR-driven dynamo (Hanasz et al. 2004).

Generally, these collective, collisionless interactions can be subdivided into CR transport processes at the microscale, the mesoscale and the macroscale. While CR interactions at the microscale are modelled with kinetic theory, CR transport at the macroscale is treated in the hydrodynamic picture in which the full phase-space information of CRs is condensed to a few variables that describe the system such as energy density, pressure, and number density. Interactions at the mesoscale combines elements of both descriptions and enables studies of, e.g., the structure of collisionless shocks (Caprioli & Spitkovsky 2013). Different scientific questions select the approach that is best suited for a problem at hand. While we always seek for clarity and apply Occam’s razor as a basic principle of model building, the richness of physics may force us to move elements from kinetic theory into the hydrodynamic picture to more faithfully capture the physics of CR transport on larger scales.

The kinetic picture of the underlying plasma assumes a sufficiently dense plasma that is well described by a distribution function. This is equivalent to requiring that many particles within a characteristic energy range be present on the plasma scale. Typically, problems such as the growth of kinetic instabilities and damping processes are addressed within kinetic theory. In particular, the non-resonant hybrid instability that excites right-handed circularly polarized Alfvén waves by the current of energetic protons, can potentially explain magnetic amplification and CR acceleration to (almost) PeV energies at supernova remnants (Bell 2004). Kinetic instabilities at shocks are important for energy exchange between electrons and protons and in building up the momentum spectrum of energetic particles (Spitkovsky 2008; Caprioli & Spitkovsky 2014). Thus, this approach provides a crucial input to modelling multi-frequency observations across the entire electromagnetic spectrum of supernova remnants (e.g., Morlino & Caprioli 2012; Blasi & Amato 2012), galaxies (Breitschwerdt et al. 2002; Recchia et al. 2016), and galaxy clusters (Brunetti & Lazarian 2011; Pinzke et al. 2017). However, to obtain a complete (non-linear) picture of a system, the dynamics on the CR gyroscale or at least the growth time-scale of a particular instability needs to be resolved. This requirement prohibits us from directly treating kinetic effects in global simulations of astrophysical objects such as galaxies or jets of active galactic nuclei.

Hence, to model CR transport in the ISM, the circum-galactic medium (CGM) or the intra-cluster medium (ICM), we have to resort to a hydrodynamic prescription. Traditionally, this was done by taking the kinetic-energy-weighted moment of the Fokker-Planck equation for CR transport, yielding the CR energy equation (Drury & Völk 1981; McKenzie & Völk 1982; Völk et al. 1984). This equation shows that CRs are transported through a combination of advection with the thermal gas as well as streaming and diffusion. In the ideal magneto-hydrodynamic (MHD) approximation, magnetic fields are flux-frozen into the thermal gas and thus advected with the flow. The collisionless CRs are bound to gyrate along magnetic field lines and are also advected alongside the moving gas. As CRs propagate along the mean field, they scatter at self-generated Alfvén waves, which causes them to stream down their gradient with a macroscopic velocity that is substantially reduced from their intrinsic relativistic speed. MHD turbulence that was driven at larger scales by energetic events and successively cascaded down in scale can also scatter CRs, redistributing their pitch angles, but conserving their energy (Zweibel 2017). This can be described as anisotropic diffusion where the main transport is along the local direction of the magnetic field (Shalchi 2009).

As a closure of these approaches, CR diffusion is modelled with a prescribed coefficient that is usually taken to be constant and not coupled to the physics of turbulence, and CR streaming is always assumed to be in steady state. However, neither of these two approaches is providing the correct prescription of CR transport (Wienet et al. 2017). Moreover, due to the non-linear property of the streaming equation, an ad-hoc regularization is applied that adds numerical diffusion to the solution (Sharma et al. 2010), questioning the results in regime of shallow gradients. Hence, these considerations reinforce the need for a novel description of CR transport that cures these weaknesses.

Recently, Jiang & Oh (2018) used an ad-hoc ansatz to reinterpret CR transport as a modification of radiation hydrodynamics. They showed numerically, that their result-
ing set of equations qualitatively captures the streaming-limit of CR transport. However, their equations are not conserving energy and, as we will show here, adopt an incomplete treatment of CR scattering. In this work, we provide a first-principle derivation of such an improved CR transport scheme while emphasizing the deep connection between radiation and CR hydrodynamics throughout this work.

This paper is organized as follows. In Section 2, we show the complete set of MHD and CR transport equations as a reference and derive the latter in the remainder of this work. In Section 3, we use the Eddington approximation for the two-moment approximation of CR transport. In Section 4, we derive equations accounting for the energy and pitch-angle scattering of CRs by Alfvénic turbulence. In Section 5, we introduce transport equations for Alfvén waves, which are coupled (i) to the gas via damping mechanisms and (ii) to the CR population by the streaming instability. We discuss the results and their consequences in Section 6. We show a numerical demonstration of our coupled transport equations for the energy densities contained in CRs and Alfvén waves in Section 7 and compare our theory to other approaches in the literature. We conclude in Section 8. In Appendix A, we show how pure CR diffusion emerges mathematically by neglecting the electric fields of Alfvén waves, thereby emphasizing the need for CR streaming for a full description of CR transport. In Appendix B, we discuss problems arising in the approximation of relativistic transport equations. We use the Heaviside system of units throughout this paper.

2 EQUATIONS OF CR HYDRODYNAMICS

The equations for ideal MHD coupled to non-thermal CR and Alfvén wave populations are given by:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]  
(1)

\[ \frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u} + P \mathbf{1} - \mathbf{B} \mathbf{B}) = 0 \]  
(2)

\[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{B} \mathbf{u} - \mathbf{u} \mathbf{B}) = 0 \]  
(3)

\[ \frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon \mathbf{u} + P \mathbf{1}) - (\mathbf{u} \cdot \nabla)P = -\frac{\rho}{\kappa} \left[ (\partial P)_{\text{cr}} + P_{\text{a,+}} + P_{\text{a,-}} \right] \] 
\[ + S_{\text{a,+}} + S_{\text{a,-}} \]  
(4)

where \( \mathbf{a} \) is the dyadic product of vectors \( \mathbf{a} \) and \( \mathbf{b} \). Gas density, mean velocity, and the local mean magnetic field are denoted as \( \rho \), \( \mathbf{u} \) and \( \mathbf{B} \). The total and MHD pressures are

\[ P_{\text{tot}} = P_{\text{th}} + \frac{B^2}{2} + P_{\text{cr}} + P_{\text{a,+}} + P_{\text{a,-}}, \]  
(5)

\[ P = P_{\text{th}} + \frac{B^2}{2} \]  
(6)

where \( P_{\text{th}} \) is the thermal pressure, \( P_{\text{cr}} \) is the CR pressure and \( P_{\text{a,+}} \) are the ponderomotive pressures due to presence of Alfvén waves. The total MHD energy density is

\[ \varepsilon = \frac{\rho u^2}{2} + \varepsilon_{\text{th}} + \varepsilon_{\text{B}} \]  
(7)

where \( \varepsilon_{\text{th}} \) and \( \varepsilon_{\text{B}} = B^2/2 \) are the thermal and magnetic energy densities. \( S_{\text{a,+}} \) and \( S_{\text{a,-}} \) are the source terms of thermal energy due to Alfvén wave energy losses as detailed in Section 5.

All pressures and the respective energy densities are related by equations of states:

\[ P_{\text{th}} = (\gamma_{\text{th}} - 1)\varepsilon_{\text{th}}, \quad \gamma_{\text{th}} = \frac{5}{3}, \]  
(8)

\[ P_{\text{cr}} = (\gamma_{\text{cr}} - 1)\varepsilon_{\text{cr}}, \quad \gamma_{\text{cr}} = \frac{4}{3}; \]  
(9)

\[ P_{\text{a,+}} = (\gamma - 1)e_{\text{a,+}}, \quad \gamma = \frac{3}{2}. \]  
(10)

We augment these evolution equations of MHD quantities by a CR-Alfvénic subsystem, which encompasses the hydrodynamics of CR transport that is mediated by Alfvén waves. As we will show in this work, this subsystem describes the anisotropic transport of CR energy density (\( \varepsilon_{\text{cr}} \)), CR energy flux density (\( f_{\text{cr}} \)), and Alfvén-wave energy density (\( e_{\text{a,+}} \)), where the ± signs denote co- and counter-propagating waves with respect to the large-scale magnetic field. Note that all four quantities are measured with respect to the comoving frame:

\[ \frac{\partial \varepsilon_{\text{cr}}}{\partial t} + \nabla \cdot \left[ \varepsilon_{\text{cr}} \mathbf{u} + P_{\text{cr}} + b f_{\text{cr}} \right] = \mathbf{u} \cdot \nabla P_{\text{cr}} \]
\[ - \frac{\rho_{\text{th}}}{\kappa} \left[ f_{\text{cr}} - \rho_{\text{th}} f_{\text{cr}} + P_{\text{cr}} \right] + \frac{\rho_{\text{th}}}{\kappa_{\text{cr}}} \left[ f_{\text{cr}} + \rho_{\text{th}} f_{\text{cr}} + P_{\text{cr}} \right]. \]  
(11)

\[ \frac{\partial f_{\text{cr}}}{\partial t} + \nabla \cdot (f_{\text{cr}} \mathbf{u}) + \frac{c^2}{3} \mathbf{B} \cdot \nabla = - (u \cdot \nabla) f_{\text{cr}} \]
\[ - \frac{c^2}{3\kappa_{\text{cr}}} \left[ f_{\text{cr}} - \rho_{\text{th}} f_{\text{cr}} + P_{\text{cr}} \right] - \frac{c^2}{3\kappa_{\text{cr}}} \left[ f_{\text{cr}} + \rho_{\text{th}} f_{\text{cr}} + P_{\text{cr}} \right]. \]  
(12)

\[ \frac{\partial e_{\text{a,+}}}{\partial t} + \nabla \cdot \left[ e_{\text{a,+}} + b_{\text{a,+}} \right] = \mathbf{u} \cdot \nabla e_{\text{a,+}} \]
\[ \pm \frac{\rho_{\text{th}}}{\kappa_{\text{a,+}}} \left[ f_{\text{cr}} \mp \rho_{\text{th}} f_{\text{cr}} + P_{\text{cr}} \right] - S_{\text{a,+}}. \]  
(13)

Here, \( c \) is the light speed (corresponding to intrinsic CR velocity in the ultra-relativistic approximation), \( \varepsilon_{\text{th}} = B / \sqrt{\rho} \) is the Alfvén velocity, \( B \) is the magnetic field strength and \( b = B / B \). The CR energy equation (11) contains source terms on the right-hand side that arise as a result of adiabatic changes and resonant scattering off of Alfvén waves. We refrain from including additional CR source and sink terms, as we focus solely on transport processes of CRs. Equations (11) and (12) fully describe CR diffusion and CR streaming in the self-confined picture. The right-hand side of the Alfvén-wave equation (13) shows loss terms \( S_{\text{a,+}} \) due to damping processes.

The CR-Alfvénic subsystem is closed by the grey approximation for the CR diffusion coefficient:

\[ \frac{1}{\kappa_{\text{a}}} = \frac{9\pi}{8} \frac{\Omega_{\text{a,+}}/2}{c^2} \left( 1 + \frac{2\epsilon_{\text{cr}}^2}{c^2} \right). \]  
(14)

Here, \( \Omega = Z e B / (\gamma m c) \) is the relativistic gyro frequency of a CR population with charge \( Z \) and characteristic Lorentz factor \( \gamma \), \( e \) is the elementary charge, and \( m \) is the particle rest mass. This equation links the transported CR energy density directly to the Alfvénic turbulence, described by its energy density \( e_{\text{a,+}} \). The total energy density of thermal gas, magnetic fields, CRs, and Alfvén waves is given by

\[ \varepsilon_{\text{tot}} = \frac{\rho u^2}{2} + \varepsilon_{\text{th}} + \varepsilon_{\text{B}} + \varepsilon_{\text{cr}} + \varepsilon_{\text{a,+}} + \varepsilon_{\text{a,-}}. \]  
(15)

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In the absence of explicit gain and loss terms, the total energy $E_{\text{tot}} = \int d^3 x \, \varepsilon_{\text{eq}}$ (where $x$ denotes the spatial coordinate) is a conserved quantity so that

$$\frac{\partial \varepsilon_{\text{tot}}}{\partial t} + \nabla \cdot \mathbf{F}_{\text{tot}} = 0,$$

where

$$\mathbf{F}_{\text{tot}} = \varepsilon_{\text{tot}} \mathbf{v} + \mathbf{F}_{\text{micro}} + \mathbf{F}_{\text{cr}} = [f_{\text{cr}} + \varepsilon_{\text{eq}} (\varepsilon_{\text{eq}} + \varepsilon_{\text{eq}} - (u \cdot B) B].$$

### 3 CR PHASE SPACE DYNAMICS

After summarizing the full set of equations for CR hydrodynamics, we will now derive them. Starting with the Vlasov equation, we discuss the Eddington approximation to the transport of the CR distribution function. In the next step, we will derive the CR fluid equations.

#### 3.1 Focused CR transport equation

The CR distribution lives in phase space that is spanned by the momentum and spatial coordinates $\mathbf{p}$ and $\mathbf{x}$, respectively, and is defined as

$$f = f(\mathbf{p}, \mathbf{x}, t) = \frac{d^6 N}{dx^3 \, dp^3}.$$  

It evolves according to the Vlasov equation

$$\frac{\partial f}{\partial t} + (\mathbf{u} + \mathbf{v}) \cdot \nabla_x f + \mathbf{F} \cdot \nabla_p f = 0,$$

where $\mathbf{v} = \mathbf{p} / (\gamma m)$ is the CR velocity field relative to the gas rest frame. CRs as charged particles are subject to the Lorentz force, which we split into contributions by large-scale and small-scale electromagnetic fields, respectively:

$$\mathbf{F} = \mathbf{F}_{\text{macro}} + \mathbf{F}_{\text{micro}}$$

$$= Z e \cdot \mathbf{B} + Z \left( \frac{\delta E + (u + \mathbf{v} \cdot \mathbf{B}) \cdot \mathbf{E}_\parallel}{c} \right).$$

The small-scale field fluctuations are provided by MHD waves, in particularly by Alfvén waves, which are generated by the CR-driven gyroresonant instability. Since these waves are the source of CR scattering, we denote their contribution to the Vlasov equation as:

$$\frac{\partial f}{\partial t} \bigg|_{\text{scatt}} = \mathbf{F}_{\text{micro}} \cdot \nabla_p f.$$  

We leave this term unspecified for now and return to it in Section 4.

CRs gyrate around large-scale magnetic fields on spatial and temporal scales that are small in comparison to any MHD scale. We can thus project out the full phase dynamics of CRs by taking the gyro average. Calculating this average of equation (19) results in the so called focused transport equation, which describes the gyrotrpic evolution of CRs. While Skillings (1971) performs this calculation in the Alfvén-wave frame, $u + v_b$, the identical result is obtained in the frame comoving with the mean gas velocity $u$ (Zank 2014). Using the latter result of the focused transport equation, we arrive at:

$$\frac{\partial f}{\partial t} + (u + \mu v_b) \cdot \nabla f$$

$$= \left[ 1 - \frac{3 \mu^2}{2} (b \cdot \nabla u \cdot b) - \frac{1}{2} \frac{\mu^2}{2} \nabla \cdot u \right] \frac{\partial f}{\partial p}$$

$$+ \left( u \delta b \cdot \mu \delta v \cdot u - 3 \mu (b \cdot \nabla u \cdot b) \right) \frac{1 - \mu^2}{2} \frac{\partial^2 f}{\partial \mu^2} = \frac{\partial f}{\partial t} \bigg|_{\text{scatt}}.$$  

Here, we use the conventional mixed coordinate system for phase space. While the ambient gas velocity $u$ and the direction of the large scale magnetic field $b = B/B$ are measured in the lab frame, the particle velocity $v$, momentum $p$ and the cosine of the pitch angle $\mu = \mathbf{v} / \mathbf{v}$ are given with respect to the comoving gas frame. A general discussion of the adiabatic terms and other pseudoforces of this equation is given in le Roux & Webb (2012).

The complexity of transport terms in equation (23) alone permits a general solution and we have to resort to approximations. In the following, we use a procedure which preserves the large-scale dynamics of the entire distribution in terms of thermodynamical quantities. To this end, we take moments of the momentum space variables $\mu$ and $p$ and describe the energy content in CRs and their transport properties in terms of an energy flux that is coupled to the Alfvén-wave dynamics.

#### 3.2 Eddington approximation

A similarly complex problem is the radiative transfer (RT) equation with its two phase space coordinates solid angle $\mathbf{n}$ and photon frequency. Powerful methods describing the transport of comoving radiation energy were pioneered by Mihalas & Weibel Mihalas (1984) and Castor (2007).

In the case of an optically thick medium, the Eddington approximation is a valuable tool to model the transport of radiation energy. In this approximation, the RT equation is expanded up to first order in $\mathbf{n}$ while assuming that the contribution from higher-order moments of the radiation distribution can be neglected. This assumption is justified in the optically thick medium because rapid scattering quickly damps any anisotropy.

A more accurate approximation of RT problems with a preferred direction is the assumption of plane-parallel or slab geometry. In this case, all quantities of the medium are taken to be constant on planes perpendicular to this particular direction $\mathbf{n}$. The RT equation can then be expressed in terms of the coordinate along $\mathbf{n}$ and the direction cosine $\mu$ between the orientation of a ray and $\mathbf{n}$. In this setting, the Eddington approximation for the radiation intensity $I$ simplifies to

$$I(\mu) = I_0 + I_1 \mu,$$

where we suppress the spatial dependence of the first- and second-order moments $I_0$ and $I_1$ in our notation. However, this simplified slab geometry is of limited use because it often does not apply to astrophysical problems at hand.

This is different for CR transport where the mean magnetic field is a priori known as a preferred direction of (gyro-phase averaged) motion. Thus, CR transport is locally akin to plane-parallel RT. To model CR transport with such an RT methodology, we have to account for the spatially and
temporarily varying plane and translate the corresponding terminologies.

The direction cosine \( \mu \) in RT is equivalent to the pitch-angle cosine \( \nu \) in CR transport. Thus, we expand equation (23) in moments of the pitch angle. This expansion has a long history in CR transport and is frequently revisited (see e.g. Klimas & Sandri 1971; Earl 1973; Zank et al. 2000; Litvinenko & Noble 2013). For completeness, we recall the derivation to introduce our notation.

In general, any complete basis of functions could be used to expand \( f \) in pitch-angle. Particularly useful are the Legendre polynomials, because of their geometric relationship to the pitch angle.\(^1\) Carrying out the complete expansion using these basis functions results in an infinite set of coupled differential equations. Even though this system captures the full dynamics of equation (23), it is not practicable because of the high degree of coupling between the transport terms. Thus, we expand equation (23) in moments of the pitch angle. This expansion has a quasi-linear approximation is valid in cases of self-confinement (RT) is equivalent to the pitch-angle average results in a compact form. The third term on the left-hand side corresponds to expanding the equation over momentum space, which yields an equation over momentum space, which yields:

\[
\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f_0 + \nabla \cdot (\mathbf{b} f_1) - \frac{1}{3} \nabla \cdot \mathbf{u} \frac{\partial f_0}{\partial \nu} + \frac{\partial f_0}{\partial \text{scatt}}. \quad (26)
\]

Equivalently, taking the \( \mu \)-moment of equation (23) yields:

\[
\frac{\partial f_1}{\partial t} + \mathbf{u} \cdot \nabla f_0 + \mathbf{u} \cdot \nabla f_1 + \left[ \frac{1}{3} \nabla \cdot \mathbf{u} - \frac{2}{5} (\mathbf{b} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} - \frac{1}{5} \nabla \cdot \mathbf{u} \right] \frac{\partial f_1}{\partial \nu} + \frac{\partial f_1}{\partial \text{scatt}}. \quad (27)
\]

The scattering terms on the right-hand side of equations (26) and (27) are calculated in Section 4.2.

A more complex expansion would use eigenfunctions of the scattering operator with a pitch-angle dependent scattering coefficient unnecessary, this more rigorous treatment would obfuscate the derivation and make our results inherently dependent on the actual form of \( f(\nu) \). Since we truncate the expansion after the first order and assume small anisotropies, we do not expect any change of the presented theory. Hence, our choice of a pitch-angle-averaged scattering rate represents a compromise between physical clarity and mathematical rigour.

### 3.3 Fluid equations

The CR energy density is given by

\[
\varepsilon_{\text{cr}} = \int d^3r \, T(p) f(p, \mu) = \int_0^\infty dp \, 4\pi p^2 T(p) f_0(p). \quad (28)
\]

where \( T(p) = \sqrt{\mathbf{p}^2 c^2 + m_e c^4} - mc^2 \) is the kinetic energy of CR particles. Such a CR population exerts a pressure

\[
P_{\text{cr}} = \int d^3r \, P_0 f(p, \mu) = \int_0^\infty dp \, 4\pi p^2 P_0 \left( \frac{5}{3} f_0(p) \right). \quad (29)
\]

on the surrounding plasma. Only the isotropic component of the CR distribution contributes to both quantities because any anisotropy vanishes as a result of pitch-angle integration. Pressure and energy are coupled via the equation of state

\[
P_{\text{cr}} = (\gamma_{\text{cr}} - 1)\varepsilon_{\text{cr}}, \quad (30)
\]

where the adiabatic index \( \gamma_{\text{cr}} = 4/3 \) holds in the ultra-relativistic limit that we are focusing on.

Similarly, we define the CR energy flux density \( f_{\text{cr}} \) and the CR pressure anisotropy \( K_{\text{cr}} \):

\[
f_{\text{cr}} = \int d^3r \, 4\pi p^2 T(p) \mu f_1(p), \quad (31)
\]

\[
K_{\text{cr}} = \int d^3r \, 4\pi p^2 P_0 \mu f_1(p). \quad (32)
\]

Algebraically, the same equation of state holds as for the CR energy density and pressure:

\[
K_{\text{cr}} = (\gamma_{\text{cr}} - 1)f_{\text{cr}}. \quad (33)
\]

The interpretation of \( f_{\text{cr}} \) becomes apparent after multiplying equation (26) by \( T(p) \) and successively integrating the equation over momentum space, which yields

\[
\frac{\partial f_{\text{scatt}}}{\partial t} + \mathbf{u} \cdot (\mathbf{v} f_{\text{cr}} + P_{\text{cr}}) + \mathbf{b} \cdot \mathbf{u} \cdot f_{\text{cr}} = \mathbf{u} \cdot \nabla P_{\text{cr}} + \frac{\partial f_{\text{scatt}}}{\partial \text{scatt}}. \quad (34)
\]

Hence, \( f_{\text{cr}} \) is the flux density of CR energy along the magnetic field. By analogy, \( K_{\text{cr}} \) is the corresponding (anisotropic) flux of CR pressure. The interpretation of the remaining terms in equation (34) is straightforward: the CR energy density is advected with the gas at velocity \( \mathbf{u} \) and subject to adiabatic changes.

We derive the transport equation for the flux density of CR energy, \( f_{\text{cr}} \), in the ultra-relativistic limit (\( c \rightarrow c \)) and show in Section 6.3 how to generalize this simplification to account for the transport of CR energy across the full momentum spectrum. Multiplying equation (27) by \( \varepsilon T(p) \) and integrating over momentum space yields

\[
\frac{\partial f_{\text{scatt}}}{\partial t} + \mathbf{u} \cdot (\mathbf{v} f_{\text{cr}}) + \frac{2}{3} \mathbf{b} \cdot \nabla f_{\text{cr}} = - (\mathbf{b} \cdot \nabla \mathbf{u}) \cdot \mathbf{b} f_{\text{cr}} + \frac{\partial f_{\text{scatt}}}{\partial \text{scatt}}. \quad (35)
\]

Here, we use equation (33) to cast the result in this compact form. The third term on the left-hand side corresponds to the Eddington term in RT. However, it differs from its
4 CR SCATTERING BY MAGNETIC TURBULENCE

In this section, we compute the scattering terms for the CR energy density and flux density while accounting for the Fokker-Planck coefficients of pitch-angle and momentum diffusion.

4.1 Pitch-angle scattering

In our derivation so far, we adopted the essential assumption of rapid CR scattering with Alfvén waves. In general this interaction is described by a non-linear stochastic process. If the magnetic perturbations $\delta B$ in the magnetic turbulence are small, $\delta B/B \approx 10^{-3}$ or less, this stochastic scattering process can be simplified and treated analytically. This is conventionally adopted within quasi-linear theory (QLT), where Boltzmann’s and Maxwell’s equation are evaluated up to linear order (Kulsrud 2004).

The wave-particle scattering can be provided by self-generated Alfvén waves through the gyro-resonant instability (Kulsrud & Pearce 1969): any residual anisotropy of the CR distribution can excite resonant Alfvén waves through the gyro-resonant instability (Kulsrud & Pearce 1969): any residual anisotropy of the generated Alfvén waves through the gyro-resonant instability (Kulsrud 2004).

Figure 1. The resonance condition for co-propagating Alfvén waves. For any given $\mu$ there is only one resonant wave polarization state of left- (L) or right- (R) handedness. At $\mu = \varepsilon_0/\varepsilon$ the resonant wave number $k_{\text{res}, \pm}$ becomes infinite and switches sign. This corresponds to a pitch angle of $90^\circ$ in the wave frame. By moving the pitch angle across this point, the type of wave polarization state that a CR can resonate with also changes. Thus, the point $k = \infty$ connects both wave spectra in terms of their resonant property of CR scattering. This connection enables us to compactify $k$-space to a circle onto which the $\mu$-axis can be mapped (via an Alexandroff compactification).

original appearance since it is projected onto the magnetic field that guides the anisotropic CR transport. This term can be interpreted as a source term: any spatial anisotropy as manifested by a gradient in $\varepsilon_{\text{cr}}$ gives rise to a change of the local anisotropy and hence to a flux of CR energy. The first term on the right-hand side accounts for the change of the local direction of reference and is thus the consequence of a pseudo-force.

Both equations fully describe the evolution of $\varepsilon_{\text{cr}}$ and $f_{\text{cr}}$ in our chosen geometry, i.e. along the local direction of the magnetic field. However, these equations are incomplete without specifying the scattering terms on the right-hand side.

$$\omega = \pm k_\parallel \varepsilon_\parallel \quad \text{for co-propagating waves, and}$$

$$\omega = \mp k_\parallel \varepsilon_\parallel \quad \text{for counter-propagating waves.}$$

A CR particle can always interact with two types of Alfvén waves: if the CR co-propagates with the wave, the mode needs to be right-handedly polarized, if it counter-propagates, the wave mode needs to be left-handedly polarized. From now on, we identify $k \equiv k_\parallel$, i.e., we drop the subscript on the wave number but retain its meaning. Combining the dispersion relation (37) and the resonance condition (36), we can derive a wave number for CRs that resonantly interact with Alfvén waves:

$$k_{\text{res}, \pm} = \frac{\Omega}{\mu \varepsilon_\parallel}$$

where we suppress the polarization sign that we encapsulate for co- and counter-propagating waves. (37)

$$R_{\pm}(k_{\text{res}, \pm}) = I^L_\pm(k_{\text{res}, \pm}) + I^R_\pm(k_{\text{res}, \pm}).$$

This enables us to define the total Alfvén wave energy density:

$$\varepsilon_{\pm} = \int_0^\infty d k \ E_\pm(k).$$

Because Alfvén waves are purely magnetic perturbations, there are no electric fields in their own frames. Hence, the interaction between Alfvén waves and CRs preserves their kinetic energies but changes their pitch angles. Mathematically, this scattering can be described as a diffusion
process in phase space (for the general case, see Schlickeiser (1989); and Teufel & Schlickeiser (2002) for our specific case). Thus, we have for pure pitch-angle scattering (Skilling 1971):
\[
\frac{\partial f}{\partial t}\bigg|_{\text{scatt}} = \frac{\partial }{\partial \mu} \left( \frac{1 - \mu^2}{2} \nu(p, \mu) \frac{\partial f}{\partial \mu} \right) \bigg|_{\text{wave}}
\] (42)
where the pitch angle has to be evaluated in the wave frame. The scattering frequencies for forward and backward propagating Alfvén waves are given by Schlickeiser (1989):
\[
\nu_{\pm}(p, \mu) = \frac{\pi \nu_{\text{res}, \pm} |R_\pm| (k_{\text{res}, \pm})}{\epsilon_B}
\] (43)
Pitch-angle scattering thus damps the CR anisotropy in the wave frame.

In the gas frame, propagating waves excite magnetic and electric fields. Accordingly, a scattering event implies an energy transfer between CRs and waves. Schlickeiser (1989) accounted for both pitch-angle and momentum diffusion in slab Alfvénic turbulence and found:
\[
\frac{\partial f}{\partial t}\bigg|_{\text{scatt}} = \frac{\partial }{\partial \mu} \left( D_{pp} \frac{\partial f}{\partial \mu} + D_{pp} \frac{\partial f}{\partial p} \right) + \frac{1}{p^2} \frac{\partial }{\partial p} p^2 \left( D_{pp} \frac{\partial f}{\partial \mu} + D_{pp} \frac{\partial f}{\partial p} \right).
\] (44)
The diffusion coefficients up to order \(O(\nu_{\text{res}}^2/\nu^2)\) are given by (Schlickeiser 1989; Dung & Schlickeiser 1990):
\[
D_{pp} = \frac{1 - \mu^2}{2} \left[ (1 - \mu \frac{\nu_{\text{L}}}{\nu})^2 \nu_v + (1 + \mu \frac{\nu_{\text{L}}}{\nu})^2 \nu_v \right],
\] (45)
\[
D_{pp} = \frac{1 - \mu^2}{2} \frac{\nu_{\text{res}}}{\nu} \left[ (1 - \mu \frac{\nu_{\text{L}}}{\nu}) \nu_v - (1 + \mu \frac{\nu_{\text{L}}}{\nu}) \nu_v \right],
\] (46)
\[
D_{pp} = \frac{1 - \mu^2}{2} \frac{\nu_{\text{res}}}{\nu} \frac{\nu_v}{\nu} (\nu_v + \nu_v),
\] (47)
where \(D_{pp}\) is the pitch-angle diffusion coefficient provided by magnetic fluctuations and \(D_{pp}\) is the momentum diffusion coefficient as a result of particle acceleration by fluctuating electric fields. The mixed coefficient \(D_{pp}\) contains elements of both scattering processes and formally derives as a result of cross-correlations between electric and magnetic turbulence.

### 4.2 CR Streaming

Evaluating equation (44) in terms of its moments is difficult, even in the ultra-relativistic limit. The unknown scattering frequency permits a direct calculation of the corresponding scattering terms.

This situation is reminiscent of RT. The analogue to the scattering by waves is the absorption and scattering of radiation by the gas. Our wave-scattering frequency is related to the absorption coefficient in RT. This coefficient has an intrinsic dependence on the photon frequency, as different absorption processes (i) operate in different frequency regimes and (ii) have a frequency dependence due to the underlying physical processes. In the context of RT, the absorption coefficient is often assumed to be constant. This strong assumption can be practically justified in cases where the dynamically interesting frequencies are confined to narrow bands. The resulting theory is called grey RT.

Cosmic-ray hydrodynamics

Here, we use a related approximation for CRs and define a reference kinetic energy \(E^*\) of typical CRs. These CRs resonate with Alfvén waves of wave numbers larger than \(k_{\text{res}, \pm} = \Omega'(\nu' \pm \nu_v)\), where \(\Omega'\) and \(\nu_v\) are the reference gyro-frequency and velocity at energy \(E^*\). In the following argument, we identify all occurring gyro frequencies with \(\Omega'\).

We further confine our analysis to isospectral Alfvén-wave intensities:
\[
I_\pm^E(k) = H(k - k_{\text{res}, \pm}) C_\pm \frac{1}{k^4},
\] (48)
\[
I_\pm^B(k) = H(k - k_{\text{res}, \pm}) C_\pm \frac{1}{k^4},
\] (49)
where \(C_\pm\) are normalisation constants, \(q\) is the spectral index and \(H\) is the Heaviside function. Using equation (41), we determine these constants to
\[
C_\pm = (q - 1) \frac{\epsilon_{\text{res}}}{\Omega' |q|} \frac{1}{(\nu' + \nu_v)^{q-1} + (\nu' - \nu_v)^{q-1}}.
\] (50)
Inserting this into equation (43) yields
\[
\nu_v = \pi \Omega' \frac{\epsilon_{\text{res}}}{\epsilon_B} (q - 1) \frac{|\nu' + \nu_v|^{q-1}}{(\nu' + \nu_v)|q|^{-1} + (\nu' - \nu_v)|q|^{-1}}.
\] (51)

This equation shows that it is impossible to fully embrace the idea of a grey transport theory that becomes trivially independent of pitch angle cosine \(\mu\). This would correspond to the case \(q = 1\), for which the wave spectra \(I_k^R \propto k^{-1}\) become degenerate as equation (41) diverges. For \(q > 1\), the isospectral scattering rate \(\nu_v\) is physically well defined and converges. However, in general different moments of the scattering rate of equation (44) cannot be solved in closed form except for the algebraically convenient choice of \(q = 2\), which we adopt here. It coincides with the upper limit of theoretically inferred spectral indices of 0.8 to 2.0 for the bulk of resonant wave numbers (Lazarian & Beresnyak 2006; Yan & Lazarian 2011). Assuming \(q = 2\) in equation (51), the pitch-angle averaged scattering frequencies are given by:
\[
\bar{\nu}_v = \frac{3}{2} \int_{-1}^{1} d\mu \frac{1 - \mu^2}{2} \nu_v = \frac{3\pi}{8} \frac{\epsilon_{\text{res}, \pm} |q|/2}{\epsilon_B} \left( 1 + \frac{2\nu_v^3}{\nu_v^2} \right).
\] (52)

Here, \((1 - \mu^2)/2\) is metrical factor connected to the pitch-angle gradient of equation (44). We checked that any different choice for \(1 < q < 2\) yields the exact same result for the different moments up to order \(O(\nu_{\text{res}}^2/\nu^2)\).

With every choice \(q \neq 1\) we encounter a well-known problem of QLT: for CRs with \(\mu = \pm \nu_v/|\nu_v|\) the scattering coefficient vanishes identically. Formally, these CRs cannot resonate with any wave. As this \(\mu\) corresponds to gyration nearly perpendicular to the large-scale magnetic field, this absence of scattering is commonly referred to as the 90°-problem. This problem can be resolved by two different arguments: (i) in the presence of dielectric effects the sharp resonance is broadened and CRs with wave vectors \(k_{\text{res}} = \infty\) in our definition are able to resonate with waves of finite wave number and (ii) a second-order treatment of the particle trajectories in small-scale turbulence, which includes a description of perturbed trajectories, introduces further resonance broadening.

As shown by theory and checked by simulations, diffusion coefficients in QLT underestimate their correct values even for \(\mu \approx \pm \nu_v/|\nu_v|\) (Shalchi 2005). Nevertheless the bulk of CRs are scattered with diffusion coefficients in accordance
with expectation of QLT. Hence, we expect the impact of second-order QLT to only marginally change the presented result (if at all).

Equipped with this approximation, we now evaluate moments of equation (44). Multiplying this equation by $T(p)$ and $\mu T(p)$, respectively, and integrating over momentum space results in

$$
\frac{\partial f_{\text{cr}}}{\partial t_{\text{scatt}}} = -3 \frac{v_a}{c^2} (\bar{v}_+ - \bar{v}_-) K_{\text{cr}} + 4 \frac{v_a^2}{c^2} (\bar{v}_+ - \bar{v}_-) P_{\text{cr}},
$$

(53)

$$
\frac{\partial f_{\text{cr}}}{\partial t_{\text{scatt}}} = -3 \frac{v_a}{c^2} (\bar{v}_+ - \bar{v}_-) f_{\text{cr}} + 4 \frac{v_a^2}{c^2} (\bar{v}_+ - \bar{v}_-) (e_{\text{cr}} + P_{\text{cr}}),
$$

(54)

where we used the ultra-relativistic approximation $v \rightarrow c$ again. The symmetry in these terms can be restored by using the equations of state linking energy density and pressure as well as their corresponding anisotropic fluxes. Thus, eliminating the CR pressure via equation (30) and the corresponding flux via equation (33), we arrive at

$$
\frac{\partial \kappa}{\partial t_{\text{scatt}}} = -\frac{v_a}{3 \kappa} \left[ f_{\text{cr}} - \kappa (e_{\text{cr}} + P_{\text{cr}}) \right]
$$

and

$$
\frac{\partial f_{\text{cr}}}{\partial t_{\text{scatt}}} = -\frac{c^2}{3 \kappa} \left[ f_{\text{cr}} - \kappa (e_{\text{cr}} + P_{\text{cr}}) \right] + \frac{v_a}{3 \kappa} \left[ f_{\text{cr}} + \kappa (e_{\text{cr}} + P_{\text{cr}}) \right],
$$

(55)

$$
\frac{\partial f_{\text{cr}}}{\partial t_{\text{scatt}}} = -\frac{c^2}{3 \kappa} \left[ f_{\text{cr}} - \kappa (e_{\text{cr}} + P_{\text{cr}}) \right] - \frac{c^2}{3 \kappa} \left[ f_{\text{cr}} + \kappa (e_{\text{cr}} + P_{\text{cr}}) \right],
$$

(56)

where the diffusion coefficients associated with either wave and are given by (see also Appendix A)

$$
\kappa = \frac{c^2}{3 \kappa_a}.
$$

(57)

The derivation of these equations concludes the proof of equations (11) and (12).

In deriving equations (55) and (56) we neglected every boundary term resulting from partial integrations in $p$. Formally, this imposes mathematical properties on the functional form of the CR proton distribution function that we locally approximate with a power law in momentum, $f \propto p^\beta$. To justify the neglect of boundary terms at low momenta, we require a low-momentum spectral index $\alpha_p > -1$, as the phase space volume element scales as $p^2 dp d\mu d\omega c$. In practice, a realistic CR distribution fulfills this constraint since at low particle energies, CRs suffer fast Coulomb interactions with the thermal plasma. Hence, the CR population quickly establishes a nearly constant low-momentum spectral index $\alpha_p \rightarrow 0$ (Enßlin et al. 2007). On the opposite side, our regularization constraint translates to a requirement for the high-momentum spectral index of $\alpha_p < -4$. Diffusive shock acceleration at strong shocks generates CRs with a spectral slope of $\alpha_p \approx -4.1$ and weaker shocks inject progressively softer spectra, thus meeting our requirement also holds in the high-energy regime (Amato & Blasi 2006). Moreover, the CR distribution exhibits an exponential cutoff at the maximum proton energy ($\sim 10^{15}$ eV for supernova remnants and $\sim 10^{20}$ eV for ultra high-energy CRs), which implies that there is no restricting mathematical precondition of our theory due to the spectral form of the CR distribution.

### 4.3 Galilean-invariant CR streaming

This form of equations (55) and (56) highlights the limit of purely Alfvénic transport: if one of both waves dominates, CRs constantly lose energy and get scattered until their flux approaches the Alfvénic limit:

$$
f_{\text{cr}} \rightarrow \pm v_a (e_{\text{cr}} + P_{\text{cr}}),
$$

(58)

We can understand this process in the wave frame: if the dominant wave scatters CRs, it isotropizes the CRs in its own frame. After the distribution reaches isotropy in the wave frame, the flux density of CR energy vanishes there by definition. A Galilean transformation into the gas frame demonstrates that the CR flux density is given by the limit (58). Hence CRs and their energy are transported with $\pm v_a$ with respect to the gas velocity. This transport mode is called streaming of CRs and is enforced in modern transport theories through a steady-state assumption (Zweibel 2013; Pfrommer et al. 2017a).

The above calculation had to be carried out to order $O(\bar{v}_a^2/c^2)$ in order to obtain a consistent result, namely a Galilean invariant expression for scattering. As can be inferred from equations (55) and (56), efficient scattering in the wave frame is necessary for a vanishing CR energy transfer and flux, which is the case of an isotropic CR distribution in one of the wave frames.

Calculations to lower order in the scattering terms fail to correctly account for the frame change and are thus incompatible with any Galilean invariant theory of CR transport. In Appendix A we explicitly demonstrate why a lower-order calculation up to $O(0)$ is inconsistent.

### 4.4 Flux-limited transport

In moment-based RT, there exists a simple physical constraint for the energy flux. Since photons travel with the speed of light $c$, the speed of the entire photon population is also limited to $c$:

$$
\frac{F_{\text{rad}}}{c E_{\text{rad}}} \leq 1,
$$

(59)

where $E_{\text{rad}}$ is the radiation energy and $F_{\text{rad}}$ is the associated flux. Both quantities are defined by analogy with their corresponding CR quantities.

A similar constraint must also hold for CRs. Consider an isolated population of CRs that carries a super-Alfvénic flux, $|f_{\text{cr}}| > v_a e_{\text{cr}}$. By means of equation (55) this flux density induces a strong energy transfer from CRs to Alfvén waves via the gyro-resonant instability. This possible mode of CR transport is unstable and rapidly decays to the Alfvén streaming limit on the growth timescale of the gyro-resonant instability (see Section 5.2). Thus, super-Alfvénic CR streaming can only exist as a transient phenomenon in a non-equilibrium situation! More formally, the presented argument states that CRs drift according to

$$
\left| \frac{f_{\text{cr}}}{v_a (e_{\text{cr}} + P_{\text{cr}})} \right| \leq 1.
$$

(60)

This is a posteriori justification of our initial assumption that $f_{\text{cr}} / f_0 \ll 1$ as equation (60) implies

$$
\left| \frac{f_1}{f_0} \right| \leq \frac{v_a}{v} \ll 1.
$$

(61)

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Note that both constraints are not enforced by physical limitations, as in the case of radiation, but due to the assumed self-confinement of CRs.

From the microscopic point of view, this argument holds for CRs at low to intermediate energies, which are indeed self-confined. For externally-confined CRs at energies \( E \gtrsim 200 \text{ GeV} \), this Alfvénic constraint needs to be replaced by equation (59). Since low- and intermediate-energy CRs dominate the CR energy density for normal momentum spectral indexes \( \alpha_p \lesssim -4.2 \) (assuming that the distribution function scales as \( f \propto p^\alpha \)), we conclude that equation (60) is valid for momentum-integrated quantities.

5 ALFVÉN WAVE DYNAMICS

In this section, we embrace the connection between CR and Alfvén-wave transport by deriving the energy equation for Alfvén waves in our framework. So far, there is only a limited literature on coupled transport of CRs and Alfvén waves available (e.g., Ko 1992; Jones 1993; Recchia et al. 2016; Zweibel 2017). Hence, we discuss different damping mechanisms and calculate the corresponding energy moments, to cast our treatment of the waves into a hydrodynamical picture. We furthermore show that the gyro-resonant instability acts as a source or sink of wave energy.

5.1 Macroscopic transport

The transport equations for MHD waves can be derived with the action principle and Whitham’s (1961) theory for waves (Dewar 1970). The result for Alfvén waves is (Jacques 1977)

\[
\frac{\partial E_a(k)}{\partial t} + \nabla \cdot (\mu v_a b) E_a(k) + \frac{1}{2} \nabla \cdot \mathbf{u} E_a(k) = \Gamma_a(k) E_a(k),
\]

where we included sources and sinks of wave energy on the right-hand side, which are denoted by their corresponding growth and decay times \( \Gamma_a(k) \). The interpretation of the left-hand side is straightforward: wave energy is transported with the Alfvén speed relative to the gas and experiences adiabatic changes due to the spectral wave pressure \( E_a(k)/2 \). This process is due to the ponderomotive force that Alfvén waves exert on the thermal gas (Achterberg 1981). The total energy contained in Alfvén waves is

\[
\varepsilon_{a,+} = \int_0^\infty dk E_a(k).
\]

The total wave pressure obeys the equation of state

\[
P_{a,+} = (\gamma_a - 1) \varepsilon_{a,+},
\]

with an adiabatic index of \( \gamma_a = 3/2 \). We can readly integrate the left-hand side of equation (62) over wave number space to obtain

\[
\frac{\partial \varepsilon_{a,+}}{\partial t} + \nabla \cdot [(u + \nu b) \varepsilon_{a,+}] + \frac{1}{2} \nabla \cdot \mathbf{u} \varepsilon_{a,+} = S_{\text{gri}+} - S_{a,+},
\]

where the Fourier integrated source termsock energy gains (via the gyro-resonant instability, gri for short) and losses are given by

\[
S_{\text{gri}+} = \int_0^\infty dk \Gamma_{\text{gri}+}(k) R_a(k), \quad \text{and}
\]

\[
S_{a,+} = \int_0^\infty dk \Gamma_{\text{loss}+}(k) E_a(k).
\]

In the following, we discuss different wave creation and annihilation processes, which are known to operate in ISM or ICM conditions and provide expressions for \( S_{a,+} \).

5.2 Gyro-resonant instability

As CRs drift with an anisotropy in the Alfven frame and gyrate around the mean magnetic field, collectively they excite Alfvén waves in resonance with their gyromotion. This effect is intimately related to CR scattering: any CR distribution with a residual anisotropy of pitch angles transfers energy to or extracts energy from the waves via scattering. For instance, if CRs are moving in the same direction as an Alfvén wave packet, but exceed the wave energy density, then these waves gain energy while the CR distribution loses energy. The growth rate of this process is (Kulsrud & Pearce 1969):

\[
\Gamma_{\text{gri}+} = \pm \int d^3p e^{-\frac{\pi\Omega^2 n_b}{s_B} (1 - \mu^2)} \frac{2}{2} \nabla \cdot \mathbf{u} \frac{\partial f}{\partial \mu} + \mathbf{u} \frac{\partial f}{\partial \mathbf{p}} \delta_k ((\mu v + v_a) - \Omega),
\]

where \( \mathbf{u} = \pm v_a b \) are wave speeds with respect to the rest frame of the gas. Dirac’s \( \delta \) distribution is the formal consequence of the gyro-resonance condition of equation (38).

Again, we account for the polarization dependence of the resonance by the definition of the resonant energy in equation (39). If we directly evaluate the \( d^3p \)-integral of equation (68), this definition and Dirac’s \( \delta \) distribution select the correct CR momenta and pitch angles, which are scattered by waves with a given \( k \).

To obtain the source function of Alfvén wave energy in equation (66), we integrate over \( k \)-space and evaluate \( R_a(k) \) at the zero of the argument of the \( \delta \) distribution. Accounting for the approximation of isospectral wave intensities as discussed in Sec. 4.2, we find in the ultra-relativistic limit:

\[
S_{\text{gri}+} = \pm \frac{v_a}{5 \kappa_a} \left[ f_{\text{cr}} \mp v_a (\kappa_{\text{cr}} + P_{\text{cr}}) \right].
\]

Comparing this result to the CR energy loss term on the right-hand side of equation (55) we find that the sum of CR wave energy is exactly conserved during gyro-resonant scattering. The derivation of equations (65) and (69) concludes the proof of equation (13).

5.3 Ion-neutral damping

One of the first damping mechanisms considered was the indirect damping of waves by the friction between ions and neutrals in a partially ionized medium (see Appendix C of Kulsrud & Pearce 1969). The process can be understood as follows: collisions between ions and neutrals maintain near equilibrium so that they share a similar temperature and mean velocity (modified by the square root of the mass ratio). The ions are additionally accelerated by the Lorenz force generated by the Alfvén waves. As before, the waves
lose energy due to this acceleration, while the ions gain this as kinetic energy. However, this force can be cancelled by friction between both particle species. In the end, the energy lost by waves is thermalized and heats both ions and neutrals.

We here account for the friction between ions (i), neutral hydrogen (H) and neutral helium (He). The damping rate for this three-component fluid was derived by Soler et al. (2016), whom we closely follow here. First, we consider the definition of the friction coefficient for collisions between ions and neutrals with small relative drift velocities:

\[
\frac{a_{\beta\beta'}}{n} = n \frac{m_i}{m_{\beta'}} \sigma_{\beta\beta'} \frac{4}{3} \sqrt{\frac{8k_B T}{\pi m_i}}
\]

where \(\beta, \beta' \in \{i, H, He\}\), \(m_{\beta} \) and \(m_{\beta'}\) are the reduced mass of either two species, \(n_i\) and \(n_{\beta'}\) are the number density and mass of species \(\beta\), \(T\) and \(k_B\) are temperature and Boltzmann’s constant, respectively. We implicitly assume that all plasma components share the same temperature. The momentum-transfer cross sections of interest are \(\sigma_{\beta i} = 10^{-18}\) m\(^2\) and \(\sigma_{\beta H} = 3 \times 10^{-19}\) m\(^2\). The resulting damping rate is given by

\[
\Gamma_{\text{in}} = \frac{1}{2} \left( \frac{\sigma_{\beta H}}{\rho_i} + \frac{\sigma_{\beta He}}{\rho_i} \right)
\]

where we neglect terms, which are second order in the collision frequencies \(a_{\beta\beta'}/\rho_i\) and \(\rho_i\) is the mass density of ions.

Since \(\Gamma_{\text{in}}\) is independent of wave number, we conclude that the total loss term of Alfvén waves by ion-neutral damping is given by

\[
S_{\text{all},k} = \Gamma_{\text{in}} \epsilon_{a,\pm}
\]

### 5.4 Non-linear Landau damping

The thermal gas can be directly heated via another mechanism. Consider two waves 1 and 2 with wave numbers \(k_i\) and wave frequencies \(\omega_i\) \((i \in \{1, 2\})\) that interact to form a beat wave, which propagates at the group velocity

\[
\nu_{\text{beat}} = \frac{\omega_1 - \omega_2}{k_1 - k_2}
\]

Associated with this beat wave is a second-order electric field, which accelerates thermal particles travelling at similar velocities. More formally, the two waves 1 and 2 interact through their beat wave at the Landau resonance with particles around the thermal speed \(\nu_{th}\):

\[
\nu_{\text{beat}} - \nu_{th} = 0
\]

In a linear perturbation analysis, Lee & Volk (1973) calculated the resulting damping of waves in a general setting. In a high-\(\beta\) plasma \((\beta_{\text{plasma}} = \nu_{th}^2/\nu_{\text{beat}}^2\)\), where thermal electrons and protons share the same temperature, the non-linear Landau (nll) damping rate can be approximated by (Volk & McKenzie 1981; Miller 1991)

\[
\Gamma_{\text{nll},k}(k) = \sqrt{\pi} \frac{\nu_{th}}{8} \frac{k}{\epsilon_B} \int_0^k dk' E_k(k')
\]

While this damping rate strictly only applies to waves of the same propagation direction, there can also be non-linear Landau damping between counter-propagating waves. However, this effect is smaller by an order of magnitude for high-\(\beta\) plasmas compared to the case of non-linear Landau of co-propagating waves (Miller 1991), hence we neglected this case here.

We can introduce a suitably averaged wave number \(\langle k \rangle\) (as in McKenzie & Bond 1983) so that the hydrodynamic version of equation (75) can be written as:

\[
S_{\text{all},k} = a \epsilon_{a,\pm}
\]

with an averaged wave number (Volk & McKenzie 1981):

\[
\langle k \rangle = \frac{1}{\epsilon_{a,\pm}} \int_0^k dk k E_k(k) \int_0^k dk' E_k(k')
\]

which, to order of magnitude, corresponds to the resonant wave number of CRs. Please note that our particular choice of the algebraic form of \(E_k(k) \sim k^{-2}\) formally gives rise to an ultra-violet divergence \((k \to \infty)\) of wave energy loss by virtue of equations (75) and (67). We remind the reader that this profile was an appropriate choice for intermediate wave numbers \((k \sim c/\Omega)\), where the turbulence is driven by the bulk of CRs. At larger wave numbers, i.e., in the inertial range and in the dissipation regime of the CR-driven turbulence, this spectrum is not applicable and would have to be modified to account for turbulent cascading and dissipation. This modification also cures the apparent ultra-violet divergence of the integral.

### 5.5 Turbulent and linear Landau damping

Magnetic turbulence becomes anisotropic through the elongation of wave packets along the mean magnetic field on scales much smaller than the injection scale (Goldreich & Sridhar 1995). Two interacting wave packets shear each other and cause field line wandering. As the two counter-propagating wave packets follow the perturbed field lines of their corresponding collision partner, they are distorted transverse to the mean magnetic field (Lithwick & Goldreich 2001). This process operates at the eddy turnover time and results in a cascade of energy to higher wave numbers \(k_{\parallel}\) (Farmer & Goldreich 2004).

It also acts as a damping process because it removes energy from scales where it was injected. The damping rate is minimized at the largest scale where waves are driven that obey the gyroresonance condition \(k_{\parallel,\text{max}} \sim k_{\parallel,\text{min}} \sim n_L\) and can be estimated as (Farmer & Goldreich 2004; Zweibel 2013):

\[
\Gamma_{\text{turb}} \approx \nu_{th} k_{\parallel,\min} \sqrt{\frac{k_{\text{mhd},\text{turb}}}{k_{\parallel,\min}}}
\]

where \(k_{\text{mhd},\text{turb}}\) is the wave number at which the large scale MHD turbulence is driven.

A related process is linear Landau damping of oblique waves (Zweibel 2017). Here the electric field of a single wave can interact with the gas through the Landau resonance. Since Alfvén waves constantly change their propagation angle relative to the mean magnetic field, this effect is directly linked to large-scale magnetic turbulence and the anisotropic
cascade. The corresponding damping rate can be estimated as
\[ \Gamma_{\|} = \frac{\sqrt{\pi}}{4} \sigma_k b_{\|,\text{min}} \sqrt{\beta_{\text{plasma}}} \frac{k_{\text{MHD,urb}}}{k_{\|,\text{min}}} . \] (80)

Combining both damping rates, the loss of total energy density by processes related to turbulence is
\[ S_{\text{urb+ll}} = (\Gamma_{\text{urb}} + \Gamma_{\text{ll}}) \rho_{\text{a},z} . \] (81)

6 DISCUSSION

After the derivation of the Alfvén wave-mediated CR transport equation, here we show how it relates to the classical streaming-diffusion equation that was previously used to model CR transport. We also show, how exactly pressure forces due to CRs emerge in our novel picture of CR transport and provide dynamic CR feedback by means of the Euler equation. We close by showing how to generalize our simplified picture of grey CR transport to include spectral information of CR momentum space and Alfvén wave-number space.

6.1 Relation to the streaming-diffusion equation

The streaming-diffusion equation can be derived by a Chapman–Enskog expansion of equations (23) and (41) (Skillings 1975; Schlickeiser 1989). In this expansion slow and fast time-scales are separated. While the macroscopic evolution of the CR–gas fluid proceeds on the slow time-scale, scattering and ballistic transport processes happen on the fast time-scale. To first order, the Chapman–Enskog expansion can be simplified as follows: since we are mainly interested in the dynamics on the slow time-scale, terms associated with fast time-scale are evaluated in their steady state. We apply this expansion to equation (12), which yields
\[ \mathbf{b} \cdot \nabla_{\text{Erf}} = -\frac{1}{\kappa} [f_{\text{cr}} - u_{\text{cr}}(\epsilon_{\text{cr}} + P_{\text{cr}}) ] , \] (82)
where the streaming velocity is given by
\[ u_{\text{cr}} = \frac{\bar{v}_c}{\lambda_c} \frac{v_{\perp}}{\bar{v}_+ + \bar{v}_-} , \] (83)
and the overall diffusion coefficient is
\[ \kappa = \frac{c^2}{3(\bar{v}_+ + \bar{v}_-)} . \] (84)
Comparing equation (82) to its original and complete evolution equation (12), it becomes clear that the Chapman–Enskog expansion approximates the flux in steady state. Inserting equation (82) into equation (11) results in:
\[ \frac{\partial \epsilon_{\text{cr}}}{\partial t} + \nabla \cdot [(u + u_{\text{ad}}) \epsilon_{\text{cr}} + P_{\text{cr}}] - \kappa \mathbf{b} \cdot \nabla_{\text{Erf}} \epsilon_{\text{cr}} = \] \[ + (u + u_{\text{ad}}) \cdot \nabla P_{\text{cr}} + \frac{\bar{v}_c}{\bar{v}_+ + \bar{v}_-} \frac{v_{\perp}^2}{c^2} (\epsilon_{\text{cr}} + P_{\text{cr}}) , \] (85)
which coincides with the streaming-diffusion equation, modified by the inclusion of last term (Ko 1992; Zweibel 2017; Pfrommer et al. 2017a). This term represents the second-order Fermi process, which accelerates CRs via electromagnetic interactions with Alfvén waves. Since both \( \bar{v}_c \) and \( \bar{v}_- \) are positive, this process always transfers energy from Alfvén waves to CRs.

To correctly account for the second-order Fermi process to order \( O(v_{\perp}^2/c^2) \), two parts of our derivation are necessary. First, the Galilean invariance of the scattering terms. Second, the inclusion of Alfvén waves to accurately estimate the scattering coefficient. It further guarantees that the total energy, \( E_{\text{tot}} \), in the system is conserved by this process.

6.2 Coupling to the thermal gas

So far, the thermal gas had a passive role in our description, as it solely provided the reference frame for the transport of CRs and Alfvén waves. However, both non-thermal components are able to interact with the thermal gas through the Lorentz force. The Euler equation for the thermal gas that includes Lorentz force and ponderomotive force densities \( F_{\text{ponder}} \) reads:
\[ \frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho uu + P_{\text{th}}) = \frac{j_{\text{gas}} \times \mathbf{B}}{c} + F_{\text{ponder}} . \] (86)
CRs contribute to the total force balance in two ways: (i) directly trough their dielectric current and (ii) indirectly by momentum exchange with Alfvén waves, which eventually decay and transfer their momentum to the gas. We characterize the dielectric contribution of CRs in the rapid-scattering limit, which significantly reduces their (population) drift velocity from their intrinsic particle velocity \( O(c) \). Hence, we are justified to describe the gas, CR current and large-scale electromagnetic fields in the non-relativistic limit. The component of the CR Euler equation perpendicular to the magnetic field reads
\[ \rho_{\text{cr}} \frac{\partial u_{\text{cr},\perp}}{\partial t} + \rho_{\text{cr}} (u_{\text{cr}} \cdot \nabla u_{\text{cr}})_{\perp} + \nabla_{\perp} P_{\text{cr}} = \frac{j_{\text{cr}} \times \mathbf{B}}{c} . \] (87)
Due to their low density, CRs exhibit low inertia and quickly adapt to any macroscopic force. Thus, we are justified to neglect both inertia terms in equation (87) and obtain
\[ \nabla_{\perp} P_{\text{cr}} = \frac{j_{\text{cr}} \times \mathbf{B}}{c} . \] (88)
Combining this equation with Ampere’s Law,
\[ \nabla \times \mathbf{B} = \frac{j_{\text{gas}} + j_{\text{cr}}}{c} , \] (89)
and inserting the result in equation (86), yields a expression for the Lorentz force exerted on the gas,
\[ F_{\text{Lorentz}} = \nabla \times \mathbf{B} \times \mathbf{B} - \nabla_{\perp} P_{\text{cr}} . \] (90)
We turn our attention towards the momentum balance parallel to the mean magnetic field. The definition of the CR energy flux density in equation (31) implies an expression for the CR momentum density parallel to magnetic field that reads in the ultra-relativistic limit
\[ \rho_{\text{cr}} u_{\text{cr}, \parallel} = \frac{1}{c^2} j_{\text{cr}} . \] (91)
In order to calculate CR dynamics, we only considered time-scales short in comparison to the gas dynamics. Here, we have to take the same limit when deriving the momentum transfer from CRs to Alfvén waves to the gas. We thus apply the Chapman–Enskog expansion to equation (35), which
here is equivalent to neglecting all inertia terms, yielding
\[ \frac{1}{3} \nabla \cdot \epsilon_{\text{cr}} = \frac{\partial (\rho u_{\text{cr}} c \epsilon_{\text{cr}})}{\partial t} \bigg|_{\text{scatt}}, \] (92)

or equivalently, using the equation of state (30):
\[ \nabla \cdot P_{\text{cr}} = \frac{\partial (\rho u_{\text{cr}} c \epsilon_{\text{cr}})}{\partial t} \bigg|_{\text{scatt}}. \] (93)

This momentum is transferred to Alfvén waves, induces a dielectric current and accelerates the thermal gas (Achterberg 1981). The associated ponderomotive force exerted on the gas is given by
\[ F_{\text{ponder}} = -\nabla \cdot \left( \frac{P_{\text{au}}}{P_{\text{au}} + P_{\text{al}}} \right) - \nabla \cdot P_{\text{cr}}. \] (94)

According to the first law of thermodynamics each conservative force field does work on the surroundings. The equation for mechanical energy is obtained by multiplying equation (86) by \( u \) and using the continuity equation:
\[ \frac{1}{2} \frac{\partial (\rho u^2)}{\partial t} + \nabla \cdot \left( \rho u \left( \frac{u^2}{2} \right) \right) = u_{\text{th}} + u_{\text{ponder}} + u_{\text{lorentz}}, \] (95)

where the volume work done by the thermal gas pressure, the ponderomotive force, and by the Lorentz force are, respectively,
\[ u_{\text{th}} = -u \cdot \nabla P_{\text{th}}, \] (96)
\[ u_{\text{ponder}} = -u \cdot \nabla (P_{\text{au}} + P_{\text{al}}) - u \cdot \nabla \cdot P_{\text{cr}}, \] (97)
\[ u_{\text{lorentz}} = +u \cdot \left( \nabla \times B \times B \right) - u \cdot \nabla \times (P_{\text{cr}}). \] (98)

Note that additionally accounting for CR inertia would force us to include \( O(u/c) \) terms to the underlying transport equations (19) and (23) in order to still conserve momentum and energy. Including those terms would only become important to next order, unnecessarily expand our already extensive derivation and obfuscate our transparent results. We further discuss this issue in Appendix B.

### 6.3 Spectral CR hydrodynamics

Here, we outline how to extend the presented theory to include the spectral dimension of the CR distribution, which would be equivalent to dropping our grey approximation of CR transport. This extension would enable us to drop some of our adopted assumptions and hence would provide a more accurate description of CR transport at the expense of being more complicated algebraically.

The fundamental assumption when evaluating moments of the focused transport equation (23) is the validity of the ultra-relativistic limit for the intrinsic CR speed. While this is certainly true for high-energy CRs, it fails for CR protons with a kinetic energy around their rest mass energy. This issue could be addressed by describing CR transport in a multi-spectral approach: instead of using the total energy of the entire CR population as the fundamental quantity, we could define spectral CR energy densities, i.e., integrated over a finite momentum range from \( p_i \) to \( p_{i+1} \) with \( i \in \{1, \ldots, N\} \). This would enable us to define a typical CR velocity of that spectral momentum range (thereafter called bin), which could be used instead of \( c \) in the transport equation. However, this approach would come with a price: because of the discreteness of the momentum bin, we would have to explicitly account for momentum boundary terms at every partial integration during the derivation of the transport equations (34) and (35).

The spectral bins could also be used to better account for the inherent momentum dependence of CR scattering. We adopted the approximation of replacing the momentum-dependent gyro-frequency by a typical value in order to obtain very compact expressions. Instead of choosing a reference gyro-frequency, we would be able to more accurately capture the typical momenta and associated gyro-frequencies in the multi-spectral approach.

So far, we adopted an isospectral ansatz for the Alfvén wave intensities to account for their inherent scale dependence. As discussed, this assumption has some drawbacks. In conjunction to or separate from the multi-spectral description for the CRs, it would be possible to also drop this approximation. The overall procedure would be the same: first, we would define wave number bins for the Alfvén wave intensities from \( k_i \) to \( k_{i+1} \) and describe the emerging dynamics in those bins. This directly enables a more accurate description of the gyro-resonant scattering process and nonlinear Landau damping, as both strongly depend on wave number.

All of our derivations rely on partial integrations. The described multi-spectral approach introduces new boundary terms for every partial integration. This inevitably would expand our evolution equations. Aiming for transparency in this work, we decided in favour of describing the CR distribution by only two independent thermodynamical quantities, namely an energy density and its corresponding flux. For this choice every boundary term vanishes identically and we obtain our compact results.

### 7 NUMERICAL DEMONSTRATION

In this section, we demonstrate the feasibility of our presented approach in one dimension that is oriented along a magnetic flux tube. We showcase the interplay of CR transport mediated by Alfvén wave dynamics in a few selected idealized cases and discuss the strengths and weaknesses of our approach in comparison to other approaches used in the literature.

#### 7.1 Methods

Here, we solely focus our attention to the dynamics of the new CR-Alfvén wave subsystem of the full set of hydrodynamical equations and leave a three-dimensional implementation and study of the dynamical impact of CRs to future work. Hence, we assume that the background gas is at rest and all MHD quantities are constant (\( \rho, \mathbf{B} = \text{const.}, \mathbf{B} = B_0 \mathbf{e}_x, \mathbf{u} = 0 \)). With this reduction, the CR transport and Alfvén wave equations (11), (12), and (13) simplify to the numerical standard form:
\[ \frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{Q})}{\partial x} = S(\mathbf{Q}), \] (99)
where the state and flux vectors are
\[
Q = \begin{bmatrix}
\epsilon_{a+} \\
f_{a+} \\
e_{a+} \\
\epsilon_{a-} \\
f_{a-} \\
e_{a-}
\end{bmatrix}, \quad F(Q) = \begin{bmatrix}
f_{\epsilon} \\
\epsilon_{\epsilon} / 3 + \epsilon_{a+} \\
\epsilon_{a+} \\
\epsilon_{\epsilon} / 3 - \epsilon_{a-} \\
\epsilon_{a-} \\
\epsilon_{a-}
\end{bmatrix},
\] (100)
while the sources are given by
\[
S(Q) = \begin{bmatrix}
-\epsilon_{a+} (f_{\epsilon} - \epsilon_{a+} \epsilon_{\epsilon} / 3) + \epsilon_{a+} (f_{\epsilon} + \epsilon_{a+} \epsilon_{\epsilon} / 3) \\
-\epsilon_{a-} (f_{\epsilon} - \epsilon_{a-} \epsilon_{\epsilon} / 3) - \epsilon_{a-} (f_{\epsilon} + \epsilon_{a-} \epsilon_{\epsilon} / 3) \\
\epsilon_{a+} (f_{\epsilon} - \epsilon_{a+} \epsilon_{\epsilon} / 3) - \epsilon_{a+} (f_{\epsilon} + \epsilon_{a+} \epsilon_{\epsilon} / 3) \\
\epsilon_{a-} (f_{\epsilon} - \epsilon_{a-} \epsilon_{\epsilon} / 3) - \epsilon_{a-} (f_{\epsilon} + \epsilon_{a-} \epsilon_{\epsilon} / 3)
\end{bmatrix}
\] (101)
where \(S_{\text{inj}}\) accounts for unresolved sources of Alfvén-wave energy. We only account for non-linear Landau damping (Section 5.4) and neglect other damping processes. Throughout this section we use internal code units (\(\nu_{a} = 1\)) and write for the diffusion coefficients:
\[
\frac{1}{3} \epsilon_{a \pm} = \chi \epsilon_{a \pm},
\] (102)
using equation (52).

We solve this equation with a finite volume scheme, which is second order by design. The hyperbolic eigenvalues of equation (99) have characteristic velocities \(c_{\epsilon} / \sqrt{3}\) and \(\pm \nu_{a}\). To avoid excessive numerical diffusion we separately calculate numerical fluxes for the CR subsystem (\(\epsilon_{a+}, f_{a+}\)) and for the Alfvén-wave system (\(\epsilon_{a-}, f_{a-}\)) respectively. The fluxes at the cell boundaries are determined via the localized Lax-Friedrichs approximate Riemann solver, which calculates fluxes for the left and right states (LeVeque 1992). For these states we use a space-time predictor, which approximates the left and right boundary values at a half-step akin to the MUSCL-Hancock method, which implicitly includes source terms (van Leer 1979). This predictor uses reconstructed state gradients, which are limited in characteristic variables by a minmod limiter (Toro 2009). The full time-step results from the divergence of the flux and an calculation of the source term. To resolve the small time-scales of scattering, we subcycle the source terms and implicitly update the state-vector in each cycle. A necessary condition for numerical convergence of a hyperbolic partial differential equation is a Courant-Friedrichs-Lewy (CFL) number less than unity; we adopt 0.3. Whether our scheme achieves its convergence order in practice remains to be seen. No analytic solution of the full set of equations is known to the authors, which precludes a formal convergence study.

In astrophysical environments under our consideration (ISM, CGM, ICM) and thus in simulations of those systems the light speed is \(10^{2}\) to \(10^{4}\) times larger than any MHD velocity. Thus, in order to follow CR dynamics that propagates information with the light speed, the resulting timestep is \(3 \times 10^{2}\) to \(3 \times 10^{4}\) times smaller by virtue of the CFL condition. Furthermore, this high signal velocity entails larger numerical diffusion for any Riemann solver of the same order. To reduce the diffusivity of the solution, we are either forced to increase the numerical order of our scheme or increase the spatial resolution, which would render most simulations unfeasible due to the increase in computational time.

Both problems can be addressed simultaneously by the reduced-speed-of-light approximation, which replaces the physical speed of light by an ad-hoc choice of a reduced value:
\[
c \to c_{\text{red}} < c.
\] (103)

However, to ensure physical validity of this approximation, the characteristic signal speed \(c_{\text{red}}\sqrt{\chi}\) of the CR subsystem has to be larger than any MHD velocity, which guarantees the correct propagation of information when coupled to MHD. This can be motivated by looking at the opposite case: if the CR signal speed is equal or smaller than the largest MHD velocity, then CRs are unable to outrun advection by the gas and hence, information contained in the CR distribution is transported differently in the numerical scheme in comparison to Nature.

### 7.2 Set-up and simulations

To demonstrate the emerging CR dynamics, we simulate three different cases, each of which probes a specific characteristics of CR transport. In Table 1 we summarize the adopted numerical parameters for our method.

| name | ICs | \(c_{\text{red}}\) | \(\chi\) | \(\alpha\) | \(S_{\text{inj}}\) |
|------|-----|----------------|-------|-------|-----------|
| \(t_{p,A,c100}\) | A | 100 | 5 \times 10^{7} | 5 \times 10^{10} | 1 \times 10^{-8} |
| \(t_{p,A,c100,ld}\) | A | 100 | 1 \times 10^{6} | 1 \times 10^{11} | 5 \times 10^{-6} |
| \(t_{p,A,c100,hd}\) | A | 100 | 1 \times 10^{6} | 1 \times 10^{12} | 5 \times 10^{-6} |
| \(t_{p,A,c10}\) | A | 10 | 5 \times 10^{7} | 5 \times 10^{10} | 1 \times 10^{-8} |
| \(t_{p,B,c100}\) | B | 100 | 5 \times 10^{7} | 5 \times 10^{10} | 1 \times 10^{-8} |
| \(t_{p,C,c10}\) | C | 10 | 5 \times 10^{7} | 5 \times 10^{10} | 1 \times 10^{-8} |
| \(t_{p,C,c100}\) | C | 100 | 5 \times 10^{7} | 5 \times 10^{10} | 1 \times 10^{-8} |

(i) We employ three different initial conditions (ICs).
(ii) Here, \(c_{\text{red}}\) the reduce speed of light, \(\chi\) is a numerical factor that describes the diffusion coefficient (see equations (102) and (52)), \(\alpha\) is the wave damping coefficient due to non-linear Landau damping for which we distinguish three cases: low, intermediate and high damping rates (labelled with ld, id and hd, respectively); \(S_{\text{inj}}\) accounts for unresolved sources of Alfvén-wave energy.

All these models are chosen to highlight different aspects of CR dynamics and to emphasis differences between the methods. They are all pathological, as we assume some functional forms of the physical quantities \(\epsilon_{a+}, f_{a+}\) and \(\epsilon_{a\pm}\)

Table 1. Adopted numerical parameters for simulations with our method.
which may not have a realization in reality. Now, we successively introduce the initial conditions for the simulations shown in this work.

### 7.2.1 Initial conditions A: isolated Gaussian

Our canonical example is a Gaussian distribution of CR energy density. We set up the energy flux density so that CRs stream initially with at the Alfvén velocity down their gradient. We also assume that there is a constant pool of Alfvén waves of both propagation directions initially present while the Gaussian contains additional wave energy. The specific initial conditions are given by

\[ g(x) = \exp(-40x^2), \]

\[ \varepsilon_{\text{cr}}(x) = g(x), \]

\[ f_{\text{cr}}(x) = \gamma_{\text{cr}} \varepsilon_{\text{cr}}(x) g(x), \]

\[ \varepsilon_{a,t}(x) = (1 + g(x)) \times 10^{-3}. \]

We use \([-1, 1]\) as our simulation domain.

### 7.2.2 Initial conditions B: Gaussian with background

The second set of initial conditions are given by

\[ g(x) = \exp(-40x^2), \]

\[ \varepsilon_{\text{cr}}(x) = 10 + g(x), \]

\[ f_{\text{cr}}(x) = \gamma_{\text{cr}} \varepsilon_{\text{cr}}(x) g(x), \]

\[ \varepsilon_{a,t}(x) = (1 + g(x)) \times 10^{-3}. \]

and follow the same reasoning as for initial conditions A, except that we place the Gaussian CRs distribution on top of a constant background of CR energy density, which has a 10 times larger amplitude in comparison to the Gaussian distribution. Setting up the flux this way ensures that only a 10 times larger amplitude in comparison to the Gaussian is streaming in the beginning while the background is kept at rest. Using this example, we can assess how the different numerical methods react to CR energy sources in the presence of an existing CR background. Here we simulate the domain \([-4, 4]\).

### 7.2.3 Initial conditions C: isolated box

The last set of initial conditions considered here is an isolated compact box of CR energy density, which is defined by

\[ g(x) = 1_{[-1/4, 1/4]}(x), \]

\[ \varepsilon_{\text{cr}}(x) = g(x), \]

\[ f_{\text{cr}}(x) = \gamma_{\text{cr}} g(x), \]

\[ \varepsilon_{a,t}(x) = (1 + g(x)) \times 10^{-3}, \]

where \(1_A\) is the characteristic function of the set \(A\) again, we add an initial background of Alfvén waves which is enhanced in the region containing CRs. Here, the computational domain is given by \([-1, 1]\).

These initial conditions serve as a formal example to investigate the characteristics of the hyperbolic part of our differential equations. It is unlikely that this extremely sharp transition between the CR plateau and the region outside is realized in nature as the flat plateau would have to be communicated instantaneously and initial CR confinement would have to be perfect.

### 7.3 CR streaming and diffusion

In Fig. 2 we show the temporal evolution of the isolated Gaussian initial conditions for \(\varepsilon_{\text{cr}}\) (model \(\text{tp\_A\_c100}\)). We adopt a reduced speed of light of \(c_{\text{red}} = 100\) and thus begin with one of the more natural set-ups.

The most prominent feature of the solution is the expanding plateau in the CR distribution, which propagates with the adiabatic Alfvén velocity \(\pm \gamma_{\text{cr}} c_A = \pm 4/3\). The effective CR streaming velocity \(u_{\text{st}} = f_{\text{cr}} / (\varepsilon_{\text{cr}} + P_{\text{cr}})\) is sub-Alfvénic in the plateau region, as can be inferred from the bottom-left panel of Fig. 2. The plateau is flat since the CR energy density flux approximately scales as \(f_{\text{cr}} \sim x\) in this region, which yields \(\partial f_{\text{cr}} / \partial x \sim \text{const}\). Hence, there is a coherent local CR energy loss which results in a decreasing energy level of the entire plateau. Co- and counter-propagating wave energy densities, \(\varepsilon_{a,t}\), are strongly damped as sub-Alfvénic streaming corresponds to a transfer of both wave energies to CRs because both wave types attempt to scatter CRs into their propagation direction. The injection of wave energy balances wave losses due to non-linear Landau damping and second-order Fermi processes at a low level of \(\varepsilon_{a,t} \sim 10^{-16}\).

The outer wings of the initial Gaussian CR population are spread out by CR diffusion because there is less wave energy available to efficiently scatter CRs. In these regions, the gyro-resonant instability decelerates CRs and transfers their kinetic energy to Alfvén waves. This results in an increase of wave energy of the outwards propagating mode. Exactly at the transition between plateau and wings, there are spikes in \(\varepsilon_{a,t}\). These correspond to fronts at which CRs are scattered most efficiently and hence, stream almost perfectly with \(f_{\text{cr}} \approx \pm \gamma_{\text{cr}} (\varepsilon_{\text{cr}} + P_{\text{cr}})\) so that residual growth of wave energy prevails over non-linear Landau damping.

As described, in the plateau region a large fraction of wave energy is damped. Hypothetical CR perturbations introduced there would not be efficiently scattered, because in order to do so, the waves would have to grow for approximately ten e-folding times to a level where the wave energy density would be large enough to affect the CR evolution. Hence these CR perturbations would propagate ballistically, an effect which we investigate now.

Using Fig. 3 we investigate how the reduced-speed-of-light approximation affects the overall solution. We accomplish this by simulating the time evolution of idealized initial box and Gaussian CR distributions, with each two values of \(c_{\text{red}}\) (models \(\text{tp\_A\_c100}\) and \(\text{tp\_A\_c10}\) for the Gaussian as well as \(\text{tp\_C\_c100}\) and \(\text{tp\_C\_c10}\) for the box simulations, see Table 1). Because the numerical scattering time scales as \(3x/c_{\text{red}}^2\), lowering \(c_{\text{red}}\) from 100 to 10 enables us to gain information about processes that usually happen at very small time-scales.

In the left column of Fig. 3 we display our solution for the box initial conditions in an \(x - t\) diagram. In this plot every straight line corresponds to a characteristic velocity \(u_{\text{char}}\), as \(x = u_{\text{char}} t\). The most prominent characteristics is the adiabatic Alfvén velocity \(\pm \gamma_{\text{cr}} c_A\), which encloses the extent of the evolved box. Visually, the true velocity appears to be
somewhat smaller, which results from the onset of diffusion at the box edges, causing them to spread apart.

In the pathological case of $c_{\text{red}} = 10$ (box initial conditions) we observe strong light-like characteristics propagating with velocity $\pm c_{\text{red}}/\sqrt{3}$. The initial sharp transition between the CR plateau and the region outside rapidly introduces an anisotropy via the geometric contribution of the Eddington term. As most of the initial wave energy has been used up to accelerate CRs via the second-order Fermi process, there is only a small amount of waves available to scatter CRs. Thus the anisotropically moving CRs cannot be efficiently scattered into one of the wave frames.

Furthermore, even though the CR gradient introduces anisotropy and should promote wave growth via the Eddington term, the growth rate is too small in order to efficiently reproduce waves. There are waves generated, but since the characteristics is a feature of small spatial extent travelling at large velocity its transition time is smaller than the wave growth time. The combination of both effects leads to incomplete scattering of these light-like characteristics, so that they propagate ballistically until they encounter one of the Alfvén characteristics. However, wave energy deposited by the light-like characteristics smoothes its wake. As the light-like characteristics interacts with the Alfvén characteristics, there is an evanescent wave transmitted and a reflected wave generated. While the evanescent wave damps instantaneously (the Alfvén mode prevails in presence of sufficiently energetic scattering waves), the reflected light-like characteristics propagates with a smaller amplitude in opposite direction.

The corresponding time-evolution of an isolated Gaussian for $c_{\text{red}} = 10$ and $c_{\text{red}} = 100$ is displayed in the right column of Fig. 3. Again, the entire CR population is enclosed by adiabatic Alfvénic characteristics that propagate at speed $\pm \gamma_{cr} v_a = \pm 4/3$. Here, the light-like characteristics are only present in the case $c_{\text{red}} = 10$ for an initial transient after which they quickly diffuse and vanish almost entirely. In the case of $c_{\text{red}} = 100$ there are no residual light-like characteristics visible and the evolution is completely smooth.

### 7.4 Impact of damping

In Fig. 4 we compare the influence of the damping coefficient $\alpha$ on the solution of isolated Gaussian simulations (initial conditions $\Lambda$). We show the results for simulations with $\alpha = 1 \times 10^{11}, 5 \times 10^{11}$ and $1 \times 10^{12}$ at $t = 0.04$ and $c_{\text{red}} = 100$ (models $\text{tp}_A c_{100} ld$, $\text{tp}_A c_{100} id$ and $\text{tp}_A c_{100} hd$). Here, we use a smaller CR-Alfvén wave coupling constant $\chi = 10^6$ to increase the relative impact of damping.

Corresponding to the notion of stronger damping, the
maximum wave energy decreases for increasing damping coefficients. The overall shape of $\varepsilon_{\text{cr}}$ remains similar while increasing damping coefficients yield broadened solutions of $\varepsilon_{\text{cr}}$. This behaviour is expected: as less wave energy is available to scatter the CRs into their frame, the mode of ballistic transport starts to influence the solution. Hence, CRs get less efficiently scattered in the direction opposing their current propagation direction. As a result, an increasing damping rate yields an increasing CR flux density and consequently a broader, more diffusive solution of $\varepsilon_{\text{cr}}$.

The particular numerical solutions presented here are clearly influenced by our choice of non-linear Landau damping. However, the overall trend remains the same for all damping processes. Consider two situations that start with the same CR distribution but exhibit varying damping strengths. The case of stronger damping implies a more evolved CR distribution with a larger spatial support in comparison to the situation with the weaker damping process.

Lowering the imposed dynamical Alfvén wave energy threshold by altering the injection rate does not change the presented qualitative results in terms of $\varepsilon_{\text{cr}}$ and $f_{\text{cr}}$. Doing so results in lower overall levels of wave energy density.

**7.5 Comparison to previous approaches**

Here, we compare our approach to two other approaches for CR transport in the literature: Sharma et al. (2010) model equilibrium CR streaming that is augmented with numerical diffusion to ensure numerical stability whereas we follow the non-equilibrium, wave-mediated CR transport with physically motivated pitch-angle scattering of CRs. On the other hand, Jiang & Oh (2018) employ an ad-hoc ansatz inspired by RT while we model CR transport in the Eddington approximation, self-consistently treating CR scattering up to $O(\alpha^2/\nu^2)$. In Table 2 we summarize the adopted numerical parameters for each of their methods.

We compare simulations of two set-ups: the evolution of an isolated Gaussian of $\varepsilon_{\text{cr}}$ (initial conditions A, see Fig. 5) and of a Gaussian CR distribution on a homogeneous background (initial conditions B, see Fig. 6). While we have discussed the evolution of an isolated Gaussian with our theory in Section 7.3, here we briefly comment on the additional features that the solution assumes when we consider the Gaussian on a homogeneous background.

As CRs are streaming away from the extremum, the wings of the Gaussian expand and the central extremum decreases. As a result, the background $\varepsilon_{\text{cr}}$ needs to respond to this change because the available volume for background CRs decreases since CRs cannot stream upwards their gra-
Figure 4. Effects of increasing the wave damping coefficient due to non-linear Landau damping, $\alpha$. As $\alpha$ increases, the initial Gaussian becomes broader due to lack of efficient scattering, which is accompanied by an increasing CR energy flux density $f_{cr}$. We use the same initial conditions for all three displayed values of $\alpha$ (shown in grey), adopt a reduced speed of light of $c_{\text{red}} = 100$, and choose a snapshot at $t = 0.04$.

Table 2. Adopted numerical parameters for simulations with the methods of Sharma et al. (2010) and Jiang & Oh (2018).

| Method of Sharma et al. (2010): | name | ICs | $\delta$ | $c_{\text{red}}$ | $\kappa_0$ |
|-------------------------------|------|-----|---------|-----------------|-----------|
| sc_A                          | A    | 100 | 100     | $10^{-6}$       |           |
| sc_B                          | B    | 100 | 100     | $10^{-6}$       |           |

| Method of Jiang & Oh (2018): | name | ICs | $\delta$ | $c_{\text{red}}$ | $\kappa_0$ |
|-------------------------------|------|-----|---------|-----------------|-----------|
| jo_A                          | A    | 100 | $10^{-6}$ | 100  |           |
| jo_B                          | B    | 100 | $10^{-6}$ | 100  |           |

(i) We employ two different initial conditions (ICs).
(ii) Here, $\delta$ is the regularization parameter of the streaming velocity in equation (118), $c_{\text{red}}$ the reduce speed of light that enters equation (123), and $\kappa_0$ is the floor value for the CR diffusion coefficient defined in equation (125).

7.5.1 Method of Sharma et al. (2010)

Neglecting CR diffusion and contributions from second-order Fermi processes, the steady-state version of the streaming-diffusion equation (85) reads in our simplified setting:

$$\frac{\partial \varepsilon_{cr}}{\partial t} + \frac{\partial}{\partial x} \left[ u_{st} (\varepsilon_{cr} + P_{cr}) \right] = u_{st} \frac{\partial P_{cr}}{\partial x},$$

where the streaming velocity is given by

$$u_{st} = -\varepsilon_{cr} \frac{\partial \varepsilon_{cr}}{\partial x}.$$

This equation cannot be integrated using conventional finite-volume numerical methods because the functional form of this streaming velocity implies that the equation is a highly non-linear diffusion equation. This problem was first analysed by Sharma et al. (2010), who suggested to regularize...
the streaming velocity of equation (117) via

\[
\tilde{u}_a = -v_a \tanh \left( \frac{1}{\delta} \frac{\partial \epsilon_{cr}}{\partial x} \right),
\]

where \( \delta = \text{const.} \) is a (small) regularization parameter. By analogy with our theory applied to steady state, we define the regularized CR energy flux density:

\[
f_{\text{reg}} = \tilde{u}_a (\epsilon_{cr} + P_{\text{cr}}).
\]

For \( \delta \to 0 \) the regularized streaming velocity matches its analytic counterpart. Inserting expression (118) for \( \tilde{u}_a \) into equation (116) yields

\[
\frac{\partial \epsilon_{cr}}{\partial t} + \tilde{u}_a \frac{\partial}{\partial x} (\epsilon_{cr} + P_{\text{cr}}) - \kappa_{\text{reg}} \frac{\partial^2 \epsilon_{cr}}{\partial x^2} = \tilde{u}_a \frac{\partial P_{\text{cr}}}{\partial x}, \quad \text{with (120)}
\]

\[
\kappa_{\text{reg}} = v_a \gamma_{\text{cr}} \epsilon_{cr} \frac{1}{\delta} \text{sech} \left( \frac{1}{\delta} \frac{1}{\partial^2 x} \right).
\]

Here, \( \kappa_{\text{reg}} \) is a numerical diffusion coefficient that depends on the regularization parameter \( \delta \), the CR energy density, and its gradient. For weak CR energy gradients \( (\partial \epsilon_{cr}/\partial x \ll \delta) \) numerical diffusion dominates the solution while steady-state CR streaming emerges for steep CR energy density gradients \( (\partial \epsilon_{cr}/\partial x \gg \delta) \).

With this choice, the streaming-diffusion equation is classified as a non-linear diffusion equation, even in the limit of negligible physical diffusion \( (\kappa = 0) \), and can be numerically integrated. In practical terms, this regularization attempts to emulate the steady state by reconstructing local CR streaming based on the energy gradient. The extension of equation (117) to values in between \( \epsilon_{cr} \) is justified, as this corresponds to a smooth transition between the two limiting stationary cases. The fact that we observe this behaviour in simulations of our new non-equilibrium CR transport model provides additional physical justification for this regularization.

We implement this regularized scheme by evaluating the gradient in equation (118) on cell faces while all gradients in equation (116) are discretized using central differences. We integrate the time derivative using an explicit super-timestepping Runge-Kutta method (Meyer et al. 2012), which permits us to treat the non-linear parabolic terms robustly and with large, second-order accurate time-steps, thereby circumventing the restrictive parabolic von-Neumann criterion. In particular, we adopt a regularization parameter of \( \delta = 0.005 \) and use 120 super-timesteps.

In the centre-column of Fig. 5 we show the time evolution of the isolated Gaussian of \( \epsilon_{cr} \) for the model by Sharma et al. (2010) (using model \( \text{sc}_A \)) and compare it to our numerical method. Both wings of the Gaussian are correctly transported with \( \pm \gamma_{\text{cr}} v_a = \pm 4/3 \). In this regime, the CRs transfer a sufficient amount of energy to resonant Alfvén...
waves in order to balance the damping of this wave type. In the assumed steady-state limit this corresponds to the regime in which one (i.e., the resonant) wave type dominates. Furthermore, the steady-state assumption implies that the CR distribution is isotropic in the frame of the dominant wave. As CRs stream away from the maximum, the spatial support of the Gaussian broadens. Energy conversion smooths out the initial maximum in $\varepsilon_{cr}$ so that it converges onto a plateau distribution. The residual gradient on the plateau continuously connects both wings in terms of energy flux density. This arises as a result of the diffusive nature of the regularization scheme, which would smooth any strong gradient of the energy flux.

In Fig. 6 we add a constant background of CR energy density (corresponding to the model $sc_B$). Now, the initial Gaussian is a small addition to the background, which is quickly erased by the diffusive nature of the regularization scheme. As a result, the CR energy converges to a constant value, losing any information about the initial conditions. Thus, in the picture of Sharma et al. (2010), the solution of this problem depends entirely on numerical diffusion and the specific choice of the regularization parameter.

In summary, the solution of the regularized steady-state equation (Sharma et al. 2010) is comparable to the solution obtained with our method in regions where the solution has a significant gradient. There, the gyro-resonant instability is capable of providing the necessary wave power to enforce the steady-state limit. However, this method does not allow to reconstruct the preferred streaming direction in regions of small CR gradients, which causes the diffusive regularization to dominate and thus results in unphysical solutions.

### 7.5.2 Method of Jiang & Oh (2018):

The fundamental idea of Jiang & Oh (2018) was to describe CR transport with the equations of RT while modifying the scattering terms in order to restore the strongly-coupled limit of CR transport. For a medium at rest their one-dimensional equations read:

\begin{align}
\frac{\partial \varepsilon_{cr}}{\partial t} + \frac{\partial f_{cr}}{\partial x} &= u_{st} \frac{\partial P_{cr}}{\partial x}, \\
\frac{1}{c_{\text{red}}^2} \frac{\partial^2 f_{cr}}{\partial t^2} + \frac{\partial P_{cr}}{\partial x} &= -\frac{1}{k} f_{cr},
\end{align}

where all quantities retain the same meaning as in the preceding sections. The resemblance to the transport part of our equations (34) and (35) is not incidental since both descriptions root in the same ideas that originate from RT. In multiple dimension, however, both transport theories differ fundamentally as Jiang & Oh (2018) model the transport of CR energy in terms of a three-dimensional flux while our theory is based on its projection onto the direction of the local magnetic field. The system of equations (122) and (123)
is closed by
\[ u_{st} = -v_2 \text{sgn} \left( \frac{\partial P_{cr}}{\partial x} \right) , \quad \text{and} \]
\[ \kappa = \kappa_0 + \left( \frac{\partial P_{cr}}{\partial x} \right)^{-1} v_2 (\varepsilon_{cr} + P_{cr}) , \]

(124)
(125)

where the first equation encodes the steady-state limit of the streaming velocity. The second equation contains a floor value for the diffusion coefficient \( \kappa_0 \) and a second term that is proportional to the Alfvén speed times the CR gradient length. This equation is an ad-hoc ansatz of mathematical nature and not derivable. Taking the steady state limit with \( \kappa_0 \to 0 \) of equation (123) results in
\[ f_{cr} = u_{st}(\varepsilon_{cr} + P_{cr}) , \]

(126)

which corresponds to the correct energy flux in the limit of streaming CRs. Thus, the equations of Jiang & Oh (2018) can be regarded as a compromise between our theory and that of Sharma et al. (2010).

This closure of the diffusion coefficient in equation (125) is the distinguishing feature of the formalism by Jiang & Oh (2018), as it aims at reconstructing an effective diffusion coefficient from local information of \( \varepsilon_{cr} \) only. Its particular choice roots in the idea that any CR pressure gradient excites Alfvén waves, which efficiently scatter CRs so that they are primarily transported via streaming. Because the scattering coefficient scales as \( \gamma \propto \varepsilon_{cr} \propto \kappa^{-1} \), a large gradient of \( \varepsilon_{cr} \) and consequently of steady-state wave energy implies a small diffusion coefficient. Conversely, CRs experience a large diffusion coefficient if the wave energy and hence the CR pressure gradient is negligible.

The numerical problems arising when evaluating the derivative of the discontinuity of the streaming velocity (i.e., \( \text{sgn}(\delta \varepsilon_{cr} / \delta x) \)) are naturally circumvented here, as the problematic term in equation (116) is replaced by the flux density \( f_{cr} \). Nevertheless, equation (123) depends non-linearly on \( \partial P_{cr} / \partial x \) through equation (125) and is thus classified as a Hamilton-Jacobi equation.\(^2\) As a consequence of this model, the closure alters the signal propagation speed of the entire system.

We implement a numerical solver similar that described by Jiang & Oh (2018). To this end we use the modified two-stage Runge-Kutta scheme to advance all quantities. The hyperbolic part of the equations is calculated using slope-limited piece-wise linear extrapolation to cell faces and the Lax-Friedrichs Riemann solver while the sources terms are treated implicitly. We calculate the diffusion coefficient of equation (125) before each stage as the mean of the left and right gradients of a cell. For this method, we use a constant floor value for the diffusion coefficient of \( \kappa_0 = 10^{-6} \).

We show the results of this scheme for the isolated Gaussian (defined by model jo_B) on the right hand side of Fig. 5. All displayed schemes show a flattening of the initial maximum of the CR pressure to a plateau where the gradient of \( P_{cr} \) approaches zero. This implies a large CR diffusion coefficient according to equation (125). The wings of the Gaussian are characterized by a sizeable CR gradient, which limits the diffusivity through the constant floor value of the diffusion coefficient \( \kappa_0 \). Here, the advective aspect of the scheme dominates, which results in a streaming CR distribution. The transition zone between plateau and wings is smeared out as this scheme overestimates the contribution of diffusion resulting from the intermediate gradient that develops there.

If we add a constant background to the Gaussian (model jo_A) and evolve the equations of Jiang & Oh (2018), the results are qualitatively comparable to that obtained by our new formulation but there are notable quantitative differences. The Gaussian is broadened, while maintaining a clear spatial separation to the background, which responds to the expanding enhancement of CR energy density. This scheme captures the undershoot at the position of the initial maximum of \( \varepsilon_{cr} \), however not to its full extent in comparison to the results of our new theory. The travelling plateaus seen in the evolution with our method are erased in the scheme of Jiang & Oh (2018) so that only a spike remains, which marks the transition between the moving wings and undershoot.

In conclusion, the transport equations by Jiang & Oh (2018) qualitatively agree better with our results and also show clear differences to the regularization approach of Sharma et al. (2010). This is the result of the adopted closure of Jiang & Oh (2018) for the diffusion coefficient, which attempts to emulate the actual CR dynamics more accurately. However, this comes with the drawback that in some cases the diffusion coefficient is overestimated which leads to excessive broadening of otherwise sharp transitions.

In Appendix B we discuss a serious shortcoming of coupling CR dynamics to MHD as proposed by Jiang & Oh (2018). This demonstrates that their theory is manifestly energy non-conserving, which questions the applicability for studies of CR feedback in galaxies.

8 CONCLUSIONS

We succeeded in developing a new macroscopic transport theory for CR transport, which includes both CR diffusion and streaming along magnetic field lines in the self-confinement picture: as CRs stream super-Alfvénically along the magnetic field, they resonantly excite Alfvén waves through the gyro-resonant instability. Scattering off of this wave field modulates the macroscopic mode of CR transport in interesting and non-trivial ways.

For the first time, we provide a mathematically rigorous derivation of the equations of CR hydrodynamics that are coupled to the evolution of Alfvén waves in the Eddington approximation of RT. We accomplish this by evaluating the zeroth- and first-order CR pitch-angle moment of the gyro-averaged CR transport equation and successive integration over CR-momentum space. As a result, we obtain two coupled evolution equations for the CR energy density \( \varepsilon_{cr} \) and its flux density \( f_{cr} \), which resemble the equations of classical radiation hydrodynamics.

However, both equations depend on CR scattering terms, which need to be specified to close this set of equations. Our key insight for evaluating CR scattering at mag-
ngetic turbulence consists in considering a reference kinetic energy of typical CRs, similar to the grey approximation of RT. This yields a pitch-angle-averaged scattering frequency that depends on the energy level of co- and counter-propagating waves, $\nu_{\perp} \propto \varepsilon_{\perp}$, and is not a constant value as often assumed in the literature. We explicitly demonstrate that CR-wave scattering terms to order $O(\nu_{\perp}^2/c^2)$ need to be considered in order to provide a Galilean invariant and flux-limited CR transport. A Chapman–Enskog expansion of this new set of equations (i.e., filtering out fast time-scales associated with non-equilibrium transients) enables us to restore CR diffusion and streaming in the stationary limit.

The dependence of the scattering rate on $\varepsilon_{\perp}$ immediately exemplifies the need to dynamically also evolve the Alfvén wave equations for self-consistency. To this end, we review the transport of wave energy and cast it into our new picture. We provide a complete review of all available wave damping processes such as sub-Alfvénically propagating CRs, non-linear Landau damping, turbulent and linear Landau damping, and show how their contributions change $\varepsilon_{\perp}$. Most importantly, we explicitly demonstrate that the energy lost by CRs owing to the gyro-resonant instability exactly matches the energy gained by Alfvén waves only if the calculation is done at least to order $O(\nu_{\perp}^2/c^2)$.

We finally show numerical solutions of our new CR-Alfvén wave subsystem in one dimension that is oriented along a magnetic flux tube. Our first-principle approach significantly advances over previous steady-state approaches because it enables us for the first time to include non-equilibrium kinetic effects such as non-linear Landau damping, second-order Fermi acceleration or energy transfer via the gyro-resonant instability in hydrodynamical settings. In particular, our numerical implementation enables to quantify the relative impact of these kinetic effects on CR transport and on how CR and wave-pressure gradients impact the dynamics of thermal plasma.

We provide a first parameter study of our CR transport theory and assess how it reacts to variations in

- the reduced speed of light, $c_{\text{red}}$: smaller values give access to processes that act on faster timescales but also promote (unphysical) ballistic CR transport;
- the wave damping coefficient $\alpha$ (due to the non-linear Landau process): larger values damp the peak wave energy, increase the CR flux density, and make CR transport more diffusive;
- the unresolved sources of Alfvén-wave energy, $S_{\text{inj}}$: as long as there is some initial wave energy, the solution for the peak wave energy (that determines the mode of CR transport) is independent on the exact wave amplitude.

We emphasize that our theory has no tunable free parameters; $c_{\text{red}}$ is chosen so that the solution does not depend on its specific value; the wave damping coefficient $\alpha$ and the inverse CR diffusion coefficient $\chi$ are given by MHD quantities and the characteristic gyro-frequency of our CR population in the grey approximation; and $S_{\text{inj}}$ does not impact on the solution as long as it does not become dynamically important. This should enable us to accurately capture momentum and energy deposition of propagating CRs in future simulations of galaxy formation.

Our numerical simulations recover CR streaming and diffusion at self-generated waves. Comparing our solutions to two previously suggested approaches, we highlight similarities and differences. Our approach recovers the equation of Sharma et al. (2010) in the physical steady-state limit (i.e., in the presence of sufficient Alfvén wave energy). However, their approach has the weakness of excessive numerical diffusion for small CR gradients. This problem is reinforced in the presence of highly stratified CR energy densities, which are inevitably encountered in simulations of galaxies and galaxy clusters. This renders the solution problematic if CRs are injected into an already pre-existing CR background as this represents the weak-gradient regime.

Jiang & Oh (2018) describe CR transport with the equations of RT while modifying the scattering terms. In particular, their ansatz for the diffusion coefficient scatters CRs into the comoving gas frame instead of the Alfvén frame. This inconsistent treatment of scattering results in non-conservation of energy when coupling CR transport to MHD. While their method shows some qualitative non-equilibrium aspects that result from the two-moment treatment of CR transport, their effective CR streaming velocity $u_{\parallel} \equiv f_{\text{cr}}/(\varepsilon_{\text{cr}} + P_{\text{cr}})$ does not converge to the correct solution.

Summarizing, our novel derivations of CR hydrodynamics holds the promise to provide a sustainable framework to assess the importance of CR momentum and energy feedback for galaxy formation and the cosmological evolution of cool core galaxy clusters.

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We start by expanding equation (44) to order $\mathcal{O}(v_{\perp})$ in $v_\parallel/v$, which yields

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \nu} \left[ 1 - \frac{\nu^2}{2} (v_\perp + v_\parallel) \frac{\partial f}{\partial \nu} + \frac{\partial f}{\partial \nu} \right]. \tag{A1}$$

In physical terms this expansion neglects any contribution from electric fields. Consequently, the interaction between waves and CRs is fully described by pitch-angle scattering against common belief a purely diffusing CR population is physically inconsistent in case of self-confined CRs.

We have

$$\frac{\partial}{\partial t} \mathcal{E}_\text{scatt} = 0, \tag{A2}$$

and

$$\frac{\partial}{\partial t} \mathcal{F}_\text{cr} = - (\breve{v}_+ + \breve{v}_-) f_\text{cr}. \tag{A3}$$
The physical interpretation is straight forward: as CRs scatter solely off of magnetic fields, there is no energy transfer between waves and CRs. Furthermore, according to this equation, the CR population will evolve towards an isotropic distribution in the gas frame. This is in direct contrast to our finding that the appropriate frame of isotropization is the frame moving with Alfvén waves. In order to account for the Doppler-shift between gas and propagating waves, terms of order $\nu_\perp/c$ are necessary, which precludes a derivation of this result to lowest order, $O(\nu_\perp^2)$, in the expansion.

Inserting these scattering terms in our fluid equations (34) and (35), we obtain a compact set of transport equations:

$$
\frac{\partial \epsilon_{\text{cr}}}{\partial t} + \mathbf{v} \cdot [\mathbf{u}(\epsilon_{\text{cr}} + P_{\text{cr}}) + \mathbf{b} f_{\text{cr}}] = \mathbf{u} \cdot \nabla P_{\text{cr}}
$$

(A4)

$$
\frac{\partial f_{\text{cr}}}{\partial t} + \nabla \cdot (\mathbf{u} f_{\text{cr}}) + \frac{\nu_\parallel}{3} \mathbf{b} \cdot \nabla \epsilon_{\text{cr}} = -\mathbf{b} \cdot \nabla \mathbf{u} \cdot f_{\text{cr}} - (\mathbf{v} \cdot \nabla + \mathbf{v} - \mathbf{u}) f_{\text{cr}}
$$

(A5)

In the following, we restrict ourselves to the special case of a background gas at rest and a homogeneous magnetic field (i.e., $\mathbf{u} = 0$ and $\mathbf{b} = \text{const.}$). Differentiating equation (A4) with respect to $t$, taking the gradient along the magnetic field direction of equation (A5), and combining the resulting equations, we obtain the telegraph equation:

$$
\frac{1}{c_{\text{red}}^2} \frac{\partial^2 \epsilon_{\text{cr}}}{\partial t^2} + \frac{\partial \epsilon_{\text{cr}}}{\partial t} = \nabla \cdot (\chi \mathbf{b} \mathbf{b} \cdot \nabla \epsilon_{\text{cr}}),
$$

(A6)

where again $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel$. This equation is a hyperbolic expansion of the usual (parabolic) diffusion equation, which is obtained by dropping the first term in the Chapman–Enskog expansion.

There has been a longstanding discussion concerning the validity of the telegraph equation to describe a diffusion process. The earliest work known to the authors which addresses this problem discusses the related process of heat transfer (Vernotte 1958). The same discussion has recently resurfaced in the context of CRs.

The fundamental solution of equation (A6) contains two singular wave fronts travelling at signal speed $\pi c/\sqrt{3}$ which decay at a typical rate of $\nu/t$ (Malkov & Sagdeev 2013). The existence of these characteristics casts doubt on the validity of the telegraph equation because direct numerical solutions of the underlying Boltzmann equation do not show these wave solutions (Litvinenko & Noble 2013, 2016) not even at early times. At times greater at $2/\nu/t$ the solutions to both, the telegraph and diffusion equation qualitatively match those of the Boltzmann equation. Thus, at times $t \lesssim 2/\nu/t$ in the ballistic regime of transport, the telegraph and the diffusion equation fail to correctly describe CR transport, while at later times, both equations reproduce the physical diffusion of CRs.

The telegraph equation has the mathematical appeal that it is a hyperbolic equation that contains a finite signal speed, whereas the diffusion equation apparently has an infinitely fast signal speed. Moreover, if a physical system has an intrinsic anisotropy, the telegraph equation preserves these anisotropic properties. On the contrary, those features are smeared out in the diffusive solution (Litvinenko & Noble 2016; Tautz & Lerche 2016).

In conclusion, physical diffusion can be well described by the mathematical diffusion equation as well as the telegraph equation after a few scattering times, while the latter preserves more physical properties. However, either equation is not suited to model CR transport in the self-confinement picture.

APPENDIX B: PROBLEMS ARISING IN APPROXIMATIONS OF RELATIVISTIC TRANSPORT EQUATIONS

First, we start the discussion with the set of MHD-CR equations as proposed by Jiang & Oh (2018). Omitting non-transport related CR source terms, their CR energy equation reads as

$$
\frac{\partial \epsilon_{\text{cr}}}{\partial t} + \nabla \cdot \mathbf{F}_{\text{cr}} = (\mathbf{u} + \mathbf{u}_\text{a}) \cdot \nabla P_{\text{cr}}
$$

(B1)

and is formulated in the lab frame. The right-hand side of this equation exhibits source terms due to adiabatic and Alfvén wave-excitation processes. The new idea by Jiang & Oh (2018) was to expand the transport description of CRs with a new, time-dependent equation borrowed form RT, which describes the evolution of the CR energy flux density $f_{\text{cr}}$. The RT equation for this quantity reads as:

$$
\frac{1}{c_{\text{red}}^2} \frac{\partial \mathbf{F}_{\text{cr}}}{\partial t} + \nabla P_{\text{cr}} = -\frac{1}{\kappa} (\mathbf{F}_{\text{cr}} - \mathbf{u}(\epsilon_{\text{cr}} + P_{\text{cr}})),
$$

(B2)

where $\kappa$ is chosen so that it mimics the special needs of CR transport. The particular forms of the closures for $u_a$ and $\kappa$ are given in Section 7. These CR equations are coupled to the thermal gas through modifications of the momentum equation

$$
\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{u} - \mathbf{B} \mathbf{B} + P) = \frac{1}{\kappa} (\mathbf{F}_{\text{cr}} - \mathbf{u}(\epsilon_{\text{cr}} + P_{\text{cr}}))
$$

(B3)

and the energy equation

$$
\frac{\partial \epsilon}{\partial t} + \nabla \cdot (\mathbf{u} \epsilon P - (\mathbf{B} \mathbf{u} + \mathbf{u} \mathbf{B} + P)) = -(\mathbf{u} + \mathbf{u}_\text{a}) \cdot \nabla P_{\text{cr}}
$$

(B4)

so that the sum of the corresponding right-hand sides of the gas and CR equations vanish identically. While the CR energy equation of Jiang & Oh (2018) is correct to order $O(1)$ in $\nu_\perp/c$, we showed in the main text that expansion to order $O(\nu_\perp/c)$ is necessary to obtain the functional form of the energy flux guessed in equation (B2). Thus, momentum and energy transport of the CR subsystem are provided to different orders in $\nu_\perp/c$. This inevitably leads to errors when scattering between CRs and Alfvén waves becomes weak (i.e., when $\mathbf{F}_{\text{cr}} \neq (\mathbf{u} + \mathbf{u}_\text{a})(\epsilon_{\text{cr}} + P_{\text{cr}})$). In this case, the net mechanical energy exchanged by CRs, Alfvén waves and the thermal gas is not necessarily zero. This can be seen by dotting the right-hand side of equation (B3) with $\mathbf{u}$ and comparing the result with the right hand side of equation (B4).

This conceptional problem is also plaguing approximate formulations of RT energy transport (see Mihalas & Weibel Mihalas 1984; Castor 2007, and references therein). The problem arises if the transport of thermal gas is modelled non-relativistically while simultaneously accounting for the inherent relativistic natures of radiation and CRs. In a complete relativistic description of the coupled energy-stress tensors most of these problems vanish at the expense of introducing a theory that is analytically and numerically difficult to solve. The usual Newtonian forms of the Euler equations
are restored in the limit $O(1)$ in $u/c$ of this relativistic theory. But this limit is too restrictive to correctly account for the scattering processes of radiation and CRs and is thus unable to correctly describe the transport of radiation or CRs.

Our decision to describe CR transport to higher order in $v_A/c$ in comparison to MHD was done in the light of this issue: it allows us to correctly capture scattering of CRs by Alfvén waves in an energy-conserving and comoving manner. Because transport terms in the comoving Boltzmann equation are tightly linked to the gas equations through the occurring pseudo-forces, we have to describe these and the momentum terms to same order. This is demonstrated in Section 5, where we apply the Chapman–Enskog expansion to enforce equal expansion orders of the ponderomotive forces. This connects the momentum exchange of CRs through the excitation of Alfvén waves to the thermal gas. Unfortunately, this expansion formally results in momentum non-conserving equations. Momentum conversation is asymptotically restored in the Newtonian limit together with strong wave-damping. In reality, these are weaker requirements than the assumption of strong coupling between Alfvén waves and CRs, as the former requirements are independent of a specific CR dynamics. We postpone a dedicated study to address this issue to future work.

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