Models for the Displacement Calculus

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Abstract. The displacement calculus $\mathbf{D}$ is a conservative extension of the Lambek calculus $\mathbf{L}1$ (with empty antecedents allowed in sequents). $\mathbf{L}1$ can be said to be the logic of concatenation, while $\mathbf{D}$ can be said to be the logic of concatenation and intercalation. In many senses, it can be claimed that $\mathbf{D}$ mimics $\mathbf{L}1$ in that the proof theory, generative capacity and complexity of the former calculus are natural extensions of the latter calculus. In this paper, we strengthen this claim. We present the appropriate classes of models for $\mathbf{D}$ and prove some completeness results; strikingly, we see that these results and proofs are natural extensions of the corresponding ones for $\mathbf{L}1$.

1 Introduction

The displacement calculus $\mathbf{D}$ is a quite well-studied extension of the Lambek calculus $\mathbf{L}1$ (with empty antecedents allowed in sequents). In many papers (see [9], [12] and [11]), $\mathbf{D}$ has proved to provide elegant accounts of a variety of linguistic phenomena of English, and of Dutch, namely a processing interpretation of the so-called Dutch cross-serial dependencies.

The hypersequent format $\mathbf{hD}$ of displacement calculus is a pure sequent calculus free of structural rules which subsumes the sequent calculus for $\mathbf{L}1$. The Cut elimination algorithm for $\mathbf{hD}$ provided in [12] mimics the one of Lambek’s [5] syntactic calculus (with some minor differences concerning the possibility of empty antecedents). Like $\mathbf{L}1$, $\mathbf{D}$ enjoys some nice properties such as the subformula property, decidability, the finite reading property and the focalisation property ([7]).

Like $\mathbf{L}1$, $\mathbf{D}$ is known to be NP-complete [6]. Concerning (weak) generative capacity, $\mathbf{D}$ recognises the class of well-nested multiple context-free languages ([13]). In this respect, the result on generative capacity generalises the result that states that $\mathbf{L}1$ recognises the class of context-free languages. One point of divergence in terms of generative capacity is that $\mathbf{D}$ recognises the class of the permutation closures of context-free languages ([10]). Finally, it is important to note that a Pentus-like upper bound theorem for $\mathbf{D}$ is not known.

In this paper we present natural classes of models for $\mathbf{D}$. Several strong completeness results are proved, in particular strong completeness w.r.t. the class

* Research partially supported by SGR2014-890 (MACDA) of the Generalitat de Catalunya, and MINECO project APCOM (TIN2014-57226-P).

1 Not to be confused with the hypersequents of Avron ([1]).
of residuated displacement algebras (a natural extension of residuated monoids). Powerset frames for L1 are of interest from the linguistic point of view because of their relation to language models. Powerset residuated displacement algebras over displacement algebras are given, which generalise the powerset residuated monoids over monoids, as well as over free monoids. Strong completeness results for the so-called implicative fragment of D, which is very relevant linguistically, is proved in the spirit of Buszkowski (2), but the construction is more subtle.

The structure of the paper is as follows. In Section 2 we present the basic proof-theoretic tools (useful for the construction of canonical models) which we shall employ for the study of D from a semantic point of view. In Section 3 we provide the proof of two strong completeness of what we call the implicative fragment w.r.t. powerset DAs over standard DAs (with a countably infinite set of generators) and L-models respectively.

2 The Categorical calculus cD and the Hypersequent Calculus hD

D is model-theoretically motivated, and the key to its conception is the use of many-sorted universal algebra (3), namely ω-sorted universal algebra. Here, we assume a version of many-sorted algebra such that the sort domains of an ω-sorted algebra A are non-empty. With this condition we avoid some pathologies which arise in a naïve version of many-sorted universal algebra (cf. 3, and 4).

Some definitions are needed. Let M = (|M|, +, 0, 1) be a free monoid where 1 is a distinguished element of the set of generators X of M. We call such an algebra a separated monoid. Given an element a ∈ |M|, we can associate to it a number, called its sort as follows:

(1) s(1) = 1
s(a) = 0 if a ∈ X and a ≠ 1
s(w1 + w2) = s(w1) + s(w2)

This induction is well-defined since M is free and 1 is a (distinguished) generator; the sort function s(·) in a separated monoid simply counts the number of separators an element contains.

Definition 1. (Sort Domains)
Where M = (|M|, +, 0, 1) is a separated monoid, the sort domains |M|i of sort i are defined as follows:

|M|i = {a ∈ |M| : s(a) = i}, i ≥ 0

It is readily seen that for every i, j ≥ 0, |M|i ∩ |M|j = ∅ iff i ≠ j.

Definition 2. (Standard Displacement Algebra)
The standard displacement algebra (or standard DA) defined by a separated monoid (|M|, +, 0, 1) is the ω-sorted algebra with the ω-sorted signature ΣD = (+, {x_i}_{i≥0}, 0, 1) with sort functionality ((i, j → i + j)_{i,j≥0}, (i, j → i + j − 1)_{i,j≥0})_{i,j≥0}, +, {x_i}_{i≥0}, 0, 1):
where:

| operation | which is |
|-----------|----------|
| $+ : |M|_i \times |M|_j \rightarrow |M|_{i+j}$ | as in the separated monoid |
| $\times_k : |M|_i \times |M|_j \rightarrow |M|_{i+j-1}$ | $\times_k(s, t)$: the result of replacing the $k$-th separator in $s$ by $t$ |

The sorted types of $\mathcal{D}$, which we will interpret residuating w.r.t the sorted operations in Definition 2, are defined by mutual recursion in Figure 1. We let $T_p = \bigcup_{i \geq 0} T_{p_i}$. A subset $B$ of $|M|$ is called a same-sort subset iff there exists an $i \in \omega$ such that for every $a \in B$, $s(a) = i$. $\mathcal{D}$ types are to be interpreted as same-sort subsets of $|M|$. I.e. every inhabitant of $\llbracket A \rrbracket$ has the same sort. The

$$T_{p_i} := Pr_i,$$

where $Pr_i$ is the set of atomic types of sort $i$

$T_{p_0} := I$ \hspace{1cm} Continuous unit

$T_{p_1} := J$ \hspace{1cm} Discontinuous unit

$T_{p_{i+j}} := T_{p_i} \cdot T_{p_j}$ \hspace{1cm} continuous product

$T_{p_j} := T_{p_i} \setminus T_{p_{i+j}}$ \hspace{1cm} under

$T_{p_i} := T_{p_{i+j}} / T_{p_j}$ \hspace{1cm} over

$T_{p_{i+j}} := T_{p_{i+1}} \ominus_k T_{p_j}$ \hspace{1cm} discontinuous product

$T_{p_j} := T_{p_{i+1}} \lceil_k T_{p_{i+j}}$ \hspace{1cm} extract

$T_{p_{i+1}} := T_{p_{i+1}} \uparrow_k T_{p_j}$ \hspace{1cm} infix

**Fig. 1.** The sorted types of $\mathcal{D}$

intuitive semantic interpretation of the connectives is shown in Figure 2; this interpretation is called the standard interpretation. Observe that for any type $A \in T_p$, the interpretation of $A$, i.e. $\llbracket A \rrbracket$, is contained in $M_{s(A)}$, where the sort map $s(\cdot)$ for the set $T_p$, is such that

(2) \hspace{1cm} $s(p) = i$ \hspace{1cm} for $p \in Pr_i$

$s(I) = 0$

$s(J) = 1$

$s(A \cdot B) = s(A) + s(B)$

$s(A \setminus B) = s(B) - s(A)$

$s(B / A) = s(B) - s(A)$

$s(A \ominus_k B) = s(A) + s(B) - 1$

$s(A \lceil_k B) = s(B) - s(A) + 1$

$s(B \uparrow_k A) = s(B) - s(A) + 1$

2.1 $\mathcal{D}$ and its Categorical Presentation $c\mathcal{D}$

In [14] $\mathcal{D}$ is presented as a categorical calculus:
Let us define the formal definition of a model. A model $\mathcal{M} = (\mathcal{A}, v)$ comprises a (residuated) $\Sigma_D$-algebra and a $\omega$-sorted mapping $v : Pr \to Tp[\mathcal{C}]$ called a valuation. The mapping $\tilde{v}$ is the unique function which extends $v$ and which is such that $\tilde{v}(A \ast B) = \tilde{v}(A) \ast \tilde{v}(B)$ (if $\ast$ is a binary connective of $\mathcal{C}$) and $\tilde{v}(\ast A) = \ast \tilde{v}(A)$ (if $\ast$ is a unary connective of $\mathcal{C}$). Finally, a $0$-ary connective is mapped into the corresponding unit of $\mathcal{A}$. Needless to say, the mappings $v$ and $\tilde{v}$ preserve the sorting regime.

Let us see that $\mathcal{D}$ (with all the connectives) is strongly complete w.r.t. $\mathcal{RD}$. Soundness is trivial because we are considering the categorical calculus $c\mathcal{D}$. For

$$\begin{align*}
[J] &= \{0\} & \text{continuous unit} \\
[J] &= \{1\} & \text{discontinuous unit} \\
[A \ast B] &= \{s_1 + s_2| s_1 \in [A] \text{ and } s_2 \in [B]\} & \text{product} \\
[A \ast C] &= \{s_2| \forall s_1 \in [A], s_1 + s_2 \in [C]\} & \text{under} \\
[C \ast B] &= \{s_1| s_2 \in [B], s_1 + s_2 \in [C]\} & \text{over} \\
[A \circ_k B] &= \{x_k(s_1, s_2)| s_1 \in [A] \text{ and } s_2 \in [B]\} & k > 0 \text{ discontinuous product} \\
[A \circ_k C] &= \{s_2| \forall s_1 \in [A], x_k(s_1, s_2) \in [C]\} & k > 0 \text{ infix} \\
[C \uparrow_k B] &= \{s_1| s_2 \in [B], x_k(s_1, s_2) \in [C]\} & k > 0 \text{ extract}
\end{align*}$$

Fig 2. Standard semantic interpretation of $\mathcal{D}$ types

(3) $A \to A$ Axiom

$A b \to C$ iff $A \to C$ \iff $B \to A \\uparrow C$ \text{ Rescont}

$A \circ_1 B \to C$ \iff $A \to C \uparrow_1 B$ \iff $B \to A \\uparrow_1 C$ \text{ Resdiscont}

$A \ast I \leftrightarrow A \leftrightarrow I \ast A$ \text{ Continuous associativity}

$A \circ_i (B \circ_j C) \leftrightarrow (A \circ_i B) \circ_{i+j-1} C$ \text{ Discontinuous associativity}

$A \circ_i (B \circ_j C) \leftrightarrow (A \circ_i B) \circ_{i+j-1} C$, if $i \leq j \leq 1 + s(B) - 1$

$A \circ_i B \circ_j C \leftrightarrow (A \circ_{i-s(B)+1} C) \circ_{j} B$, if $j > i + s(B) - 1$ Mixed permutation

$A \circ_i C \circ_{j} B \leftrightarrow (A \circ_j B) \circ_{i+s(B)-1} C$, if $j < i$

$A \ast B \circ C \leftrightarrow (A \circ C) \ast B$, if $1 \leq i \leq s(A)$ Mixed associativity

$A \ast B \circ C \leftrightarrow A \ast (B \circ_{i-s(C)} C)$, if $s(A) + 1 \leq i \leq s(A) + s(B)$

From $A \to B$ and $B \to C$ we have $A \to C$ Transitivity

In Figure [2] we find the axiomatisation of the class of DAs $\mathcal{DA}$. Just as in the case of $\mathcal{L}_1$, the natural class of algebras is the class of residuated monoids $\mathcal{RM}$, in the case of $\mathcal{D}$, the natural class of algebras is the class of residuated displacement algebras (residuated DAs) $\mathcal{RD}$.

One can restrict the definition of the sorted types. Let $\mathcal{C}$ be a subset of the connectives considered in the definition of types in Figure [1]. We define $Tp[\mathcal{C}]$ as the least set of sorted types generated by $Pr$ and the set of connectives $\mathcal{C}$. If the context is clear, we will write $Tp$ instead of $Tp[\mathcal{C}]$. 

Let us define the formal definition of a model. A model $\mathcal{M} = (\mathcal{A}, v)$ comprises a (residuated) $\Sigma_D$-algebra and a $\omega$-sorted mapping $v : Pr \to Tp[\mathcal{C}]$ called a valuation. The mapping $\tilde{v}$ is the unique function which extends $v$ and which is such that $\tilde{v}(A \ast B) = \tilde{v}(A) \ast \tilde{v}(B)$ (if $\ast$ is a binary connective of $\mathcal{C}$) and $\tilde{v}(\ast A) = \ast \tilde{v}(A)$ (if $\ast$ is a unary connective of $\mathcal{C}$). Finally, a $0$-ary connective is mapped into the corresponding unit of $[\mathcal{A}]$. Needless to say, the mappings $v$ and $\tilde{v}$ preserve the sorting regime.

Let us see that $\mathcal{D}$ (with all the connectives) is strongly complete w.r.t. $\mathcal{RD}$. Soundness is trivial because we are considering the categorical calculus $c\mathcal{D}$. For
Continuous associativity
\[ x + (y + z) \approx (x + y) + z \]

Discontinuous associativity
\[ x \times_i (y \times_j z) \approx (x \times_i y) \times_{i+j-1} z \]
\[ (x \times_i y) \times_j z \approx x \times_i (y \times_{j-i+1} z) \] if \( i \leq j \leq 1 + s(y) - 1 \)

Mixed permutation
\[ (x \times_i y) \times_j z \approx (x \times_{j-s(y)+1} y) \times_i z \] if \( j > i + s(y) - 1 \)
\[ (x \times_i z) \times_j y \approx (x \times_j y) \times_{i+s(y)-1} z \] if \( j < i \)

Mixed associativity
\[ (x + y) \times_i z \approx (x \times_i z) + y \] if \( 1 \leq i \leq s(x) \)
\[ (x + y) \times_i z \approx x + (y \times_{s(x)} z) \] if \( x + 1 \leq i \leq s(x) + s(y) \)

Continuous unit and discontinuous unit
\[ 0 + x \approx x \approx x + 0 \] and \[ 1 \times_1 x \approx x \approx x \times_1 1 \]

Fig. 3. Axiomatization of \( \mathcal{DA} \)

completeness, we can define the well-known Lindenbaum-Tarski construction to see that \( \mathcal{CD} \) is strongly complete w.r.t. \( \mathcal{RD} \). The canonical model is \( (\mathcal{L}, v) \) where \( \mathcal{L} = (\mathcal{Tp}/\theta, \circ, (\circ_i)_{i>0}, \\setminus, \setminus, (\setminus_i)_{i>0}, (\setminus_i)_{i>0}, \cdot, \triangleright, \leq) \) where the interpretation of the new symbols is as expected. Let \( \theta_R \) be the equivalence relation on \( \mathcal{Tp} \) defined as follows: \( A \theta_R B \) if \( R \vdash_{\mathcal{CD}} A \rightarrow B \) and \( R \vdash_{\mathcal{CD}} B \rightarrow A \), where \( R \) is a set of non-logical axioms. Using the usual tonicity properties for the connectives of \( \mathcal{Tp} \), one proves that \( \theta_R \) is a congruence. Where \( A \) is a type, \( \overline{A} \) is an element of \( \mathcal{Tp}/\theta_R \), i.e. \( \mathcal{Tp} \) modulo \( \theta_R \). We define \( \overline{A} \leq \overline{B} \) if \( R \vdash_{\mathcal{CD}} A \rightarrow B \). We define the valuation \( v \) as \( v(p) = \overline{p} \) (\( p \) is a primitive type). We have that for every type \( A \), \( \overline{\circ(A)} = \overline{A} \). Finally, one has that \( (\mathcal{L}, v) \models A \rightarrow B \) if \( R \vdash_{\mathcal{CD}} A \rightarrow B \). From this, we infer the following theorem:

Theorem 1. The calculus \( \mathcal{CD} \) is strongly complete w.r.t. \( \mathcal{RD} \).

Since \( \mathcal{DA} \) is a variety\(^2\), (see Figure 3), it is closed by subalgebras, direct products and homomorphic images, which give additional DAs.

We have other interesting examples of DAs, for instance the powerset \( DA \) over \( A = (|A|, +, \times_{\circ_i}, \setminus_0, 0, 1) \), which we denote \( \mathcal{P}(A) \). We have:

\[ (4) \mathcal{P}(A) = (|\mathcal{P}(A)|, \cdot, \triangleright, \setminus, 1) \]

The notation of the carrier set of \( \mathcal{P}(A) \) presupposes that its members are same-sort subsets; notice that \( \emptyset \) vacuously satisfies the same-sort condition. Where \( A, B \) and \( C \) denote same-sort subsets of \( |A| \), the operations \( \setminus, \cdot \) and \( \triangleright \) are defined as follows:

\(^2\) The term *equational class* is sometimes used in the literature.
It is readily seen that for every \( A \), \( \mathcal{P}(A) \) is in fact a DA. Notice that every sort domain \( |\mathcal{P}(A)|_i \) is a collection of same-sort subsets, that the sort domains of \( \mathcal{P}(A) \) are non-empty, but no longer satisfy that \( |\mathcal{P}(A)|_i \cap |\mathcal{P}(A)|_j = \emptyset \) if \( i \neq j \), since the empty set \( \emptyset \in |\mathcal{P}(A)|_i \) for every \( i \geq 0 \). A residuated powerset displacement algebra over a displacement algebra \( \mathcal{P}(A) \) is the following:

\[
\mathcal{P}(A) = \langle |\mathcal{P}(A)|, \cdot, \setminus, /, \{, \emptyset, \}, \{, \emptyset, \}, \{, \emptyset, \}, (, \emptyset, ) \rangle
\]

where \( \setminus, /, \{, \emptyset, \}, \{, \emptyset, \}, \{, \emptyset, \} \) are defined as follows:

\[\begin{align*}
A \setminus B &= \{d \mid \text{for every } a \in A, a + d \in B \} \\
B / A &= \{d \mid \text{for every } a \in A, d + a \in B \} \\
B i A &= \{d \mid \text{for every } a \in A, d \times a \in B \} \\
A i B &= \{d \mid \text{for every } a \in A, a \times d \in B \}
\end{align*}\]

The class of powerset residuated DAs over a DA is denoted \( \mathcal{PRDD} \). The class of powerset residuated DAs over a standard DA is denoted \( \mathcal{PRSD} \). Finally, the subclass of \( \mathcal{PRSD} \) which is formed by powerset residuated algebras over finitely-generated standard DA are known simply as \( L\)-models.

Every standard DA \( \mathcal{A} \) has two remarkable properties, namely the property that sort domains \( |\mathcal{A}|_i \) (for \( i > 0 \)) can be defined in terms of \( |\mathcal{A}|_0 \), and the property that every element \( a \) of a sort domain \( |\mathcal{A}|_i \) is decomposed uniquely around the separator 1:

\[
|\mathcal{A}|_i = |\mathcal{A}|_0 \circ \{1\} \circ \cdots \circ \{1\} \circ |\mathcal{A}|_0
\]

\[
(i - 1) \text{'} s
\]

(8) For \( i > 0 \), if \( a_0 + 1 + \cdots + 1 + a_i = b_0 + 1 + \cdots + 1 + b_i \) then \( a_k = b_k \) for \( 0 \leq k \leq i \)

Standard DAs, as their name suggests, are particular cases of (general) DAs:

**Lemma 1.** The class of standard DAs is a subclass of the class of DAs.\(^{\text{3}}\)

**Proof.** We define a useful notation which will help us to prove the lemma. Where \( \mathcal{A} = (|\mathcal{A}|, +, (\times)_i)_{i \geq 0}, 0, 1 \) is a standard DA, let \( a \) be an arbitrary element of sort \( s(a) \). We associate to every \( a \in |\mathcal{A}| \) a sequence of elements \( a_0, \ldots, a_{s(A)} \). We have the following vectorial notation:

\[
\overrightarrow{a}_i = \begin{cases} a_i & \text{if } i = j \\ \overrightarrow{a}_{i+1} + a_j & \text{if } j - i > 0 \end{cases}
\]

Since \( \mathcal{A} \) is a standard DA, the \( a_i \) associated to a given \( \overrightarrow{a} \) are unique (by freeness of the underlying monoid). We have that \( a = \overrightarrow{a}_0^{s(A)} \), and we write \( \overrightarrow{a} \) in place of \( \overrightarrow{a}_0^{s(A)} \). Consider arbitrary elements \( \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} \) of \( |\mathcal{A}| \):

\(^{3}\) Later we see that the inclusion is proper.
• Continuous associativity is obvious.

• Discontinuous associativity. Let $i, j$ be such that $i \leq j \leq i + s(\overrightarrow{a}) - 1$:

$$\overrightarrow{b} \times_j \overrightarrow{c} = \overrightarrow{b}^{i-1} + \overrightarrow{c} + \overrightarrow{b}^{(b)}_j,$$

therefore:

$$\overrightarrow{a} \times_i (\overrightarrow{b} \times_j \overrightarrow{c}) = \overrightarrow{a}^{i-1} + \overrightarrow{b}^{(b)}_j + \overrightarrow{a}^{s(\overrightarrow{c})}_i \quad (\ast)$$

On the other hand, we have that:

$$\overrightarrow{a} \times_i \overrightarrow{b} = \overrightarrow{a}^{i-1} + \overrightarrow{b} + \overrightarrow{a}^{s(\overrightarrow{c})}_i = \overrightarrow{a}^{i-1} + \overrightarrow{b}^{j-1} + \overrightarrow{a}^{s(\overrightarrow{c})}_i + \overrightarrow{a}^{s(\overrightarrow{c})}_i \quad \text{\textup{(i+j-1)-th separator}}$$

It follows that:

$$(\overrightarrow{a} \times_i \overrightarrow{b}) \times_{i+j-1} \overrightarrow{c} = \overrightarrow{a}^{i-1} + \overrightarrow{b}^{j-1} + \overrightarrow{a}^{s(\overrightarrow{c})}_i + \overrightarrow{a}^{s(\overrightarrow{c})}_i \quad (\ast\ast)$$

By comparing the right hand side of ($\ast$) and ($\ast\ast$), we have therefore:

$$\overrightarrow{a} \times_i (\overrightarrow{b} \times_j \overrightarrow{c}) = (\overrightarrow{a} \times_i \overrightarrow{b}) \times_{i+j-1} \overrightarrow{c}$$

• Mixed Permutation. Consider $\overrightarrow{a} \times_i \overrightarrow{b}$ and suppose that $i + s(\overrightarrow{b}) - 1 < j$:

$$\overrightarrow{a} \times_i \overrightarrow{b} = \overrightarrow{a}^{i-1} + \overrightarrow{b} + \overrightarrow{a}^{j-s(\overrightarrow{b})}_i + \overrightarrow{a}^{s(\overrightarrow{c})}_j \quad j-s(\overrightarrow{b})+s(\overrightarrow{c})-1=\overrightarrow{b}-1 \text{ separators}$$

It follows that:

$$(\overrightarrow{a} \times_i \overrightarrow{b}) \times_j \overrightarrow{c} = \overrightarrow{a}^{i-1} + \overrightarrow{b} + \overrightarrow{a}^{j-s(\overrightarrow{b})}_i + \overrightarrow{a}^{s(\overrightarrow{c})}_i \quad \text{\textup{(i+j-1)-th separator}}} \quad \text{\textup{(\ast\ast\ast)}}$$

Since $i + s(\overrightarrow{b}) - 1 < j$, then $i < j - s(\overrightarrow{b}) + 1$. Then we have that:

$$\overrightarrow{a} \times_{j-s(\overrightarrow{b})+1} \overrightarrow{c} = \overrightarrow{a}^{i-1} + \overrightarrow{a}^{j-s(\overrightarrow{b})}_i + \overrightarrow{a}^{s(\overrightarrow{c})}_j \quad \text{\textup{(i+j-1)-th separator}}} \quad \text{\textup{(\ast\ast\ast\ast)}}$$

By comparing the right hand side of ($\ast\ast\ast$) and ($\ast\ast\ast\ast$), we have therefore:

$$(\overrightarrow{a} \times_i \overrightarrow{b}) \times_j \overrightarrow{c} = (\overrightarrow{a} \times_{j-s(\overrightarrow{b})+1} \overrightarrow{c}) \times_i \overrightarrow{b}$$

• Mixed associativity. There are two cases: $i \leq s(\overrightarrow{a})$ or $i > s(\overrightarrow{a})$. Considering the first one, this is true for:

$$(\overrightarrow{a} \times \overrightarrow{b}) \times \overrightarrow{c} = (\overrightarrow{a}^{i-1} + \overrightarrow{b}^{s(\overrightarrow{a})}) \times \overrightarrow{c} = \overrightarrow{a}^{i-1} + \overrightarrow{c} + \overrightarrow{a}^{s(\overrightarrow{a})} \times \overrightarrow{b} = (\overrightarrow{a} \times \overrightarrow{c}) \times \overrightarrow{b}$$

The other case corresponding to $i > s(\overrightarrow{a})$ is completely similar.

• The case corresponding to the units is completely trivial.

\[\square\]
2.2 The Hypersequent Calculus $h\mathcal{D}$

We will now consider the string-based hypersequent syntax from [8]. The reason for using the prefix hyper in the term sequent is that the data-structure used in hypersequent antecedents is quite nonstandard. A fundamental tool to build the data-structure of a sequent calculus for $\mathcal{D}$ is the notion of type-segment, $\text{seg}$. For any type of sort 0 $\text{seg}(A) = \{A\}$. If $s(A) > 0$ then $\text{seg}(A) = \{\sqrt{A}, \ldots, \sqrt[n]{A}\}$. We call $\text{seg}(A)$ the set of type-segments of $A$. If $\mathcal{C}$ is a set of connectives, we can now define the set of type-segments corresponding to the set $\mathcal{T}_p[\mathcal{C}]$ of types generated by the connectives $\mathcal{C}$ as $\text{seg}[\mathcal{C}] = \bigcup_{A \in \mathcal{T}_p[\mathcal{C}]} \text{seg}(A)$. Type segments of sort 0 are types. But, type segments of sort greater than 0 are no longer types. Strings of type segments can form meaningful logical material like the set of configurations, which we now define. Where $\mathcal{C}$ is a set of connectives the configurations $\mathcal{O}[\mathcal{C}]$ are defined in BNF unambiguously by mutual recursion as follows, where $\Lambda$ is the empty string and 1 is the metalinguistic separator:

\begin{equation}
\mathcal{O}[\mathcal{C}] ::= A
\mathcal{O}[\mathcal{C}] ::= 1, \mathcal{O}[\mathcal{C}]
\mathcal{O}[\mathcal{C}] ::= A, \mathcal{O}[\mathcal{C}] \quad \text{for } s(A) = 0
\mathcal{O}[\mathcal{C}] ::= \sqrt{A}, \mathcal{O}[\mathcal{C}], \sqrt[\alpha]{A}, \ldots, \sqrt[n]{A}, \mathcal{O}[\mathcal{C}], \sqrt[\alpha]{\sqrt[\beta]{\mathcal{O}[\mathcal{C}]}} \quad \text{for } s(A) > 0
\end{equation}

The intuitive semantic interpretation of the last clause from (10) consists of elements $\alpha_0 + \beta_1 + \alpha_1 + \ldots + \alpha_{n-1} + \beta_n + \alpha_n$ where $\alpha_0 + 1 + \alpha_1 + \ldots + \alpha_{n-1} + 1 + \alpha_n \in [A]$ and $\beta_1, \ldots, \beta_n$ are the interpretations of the intercalated configurations.

If the context is clear we will write $\mathcal{O}$ for $\mathcal{O}[\mathcal{C}]$, and likewise $\mathcal{T}_p$ and $\text{seg}$. The syntax in which $\mathcal{O}$ has been defined is called string-based hypersequent syntax. An equivalent syntax for $\mathcal{O}$ is called tree-based hypersequent syntax. As was defined in [9], [12]. For proof-search and human readability, the tree-based notation is more convenient than the string-based notation, but for semantic purposes, the string-based notation turns out to be very useful since the canonical model construction considered in Section 3 relies on the set of type-segments.

In string-based notation the figure $\overline{\Lambda}$ of a type $\Lambda$ is defined as follows:

\begin{equation}
\overline{\Lambda} = \begin{cases}
A & \text{if } s(A) = 0 \\
\sqrt[\alpha]{\overline{\Lambda}}, 1, \sqrt[\alpha]{\overline{\Lambda}}, \ldots, \sqrt[n]{\overline{\Lambda}}, 1, \sqrt[n]{\overline{\Lambda}} & \text{if } s(A) > 0
\end{cases}
\end{equation}

The sort of a configuration is the number of metalinguistic separators it contains. We have $\mathcal{O} = \bigcup_{i \geq 0} \mathcal{O}_i$, where $\mathcal{O}_i$ is the set of configurations of sort $i$. We define a more general notion of configuration, namely preconfiguration. If $V$ denotes $\text{seg}[\mathcal{C}] \cup \{1\}$, a preconfiguration $\Delta$ is simply a word of $V^*$. Obviously, we have that $\mathcal{O} \subseteq V^*$. A preconfiguration $\Delta$ is proper iff $\Delta \not\in \mathcal{O}$. As in the case of configurations, preconfigurations have a sort.

Where $\Gamma$ and $\Phi$ are configurations and the sort of $\Gamma$ is at least 1, $\Gamma|_k \Phi$ ($k > 0$) signifies the configuration which is the result of replacing the $k$-th separator in $\Gamma$ by $\Phi$. The notation $\Delta(\Gamma)$, which we call a configuration with a distinguished configuration $\Gamma$ abbreviates the following configuration: $\Delta_0, \Gamma_0, \Delta_1, \ldots, \Delta_{s(\Gamma)}, \Gamma_{s(\Gamma)}, \Delta_{s(\Gamma)+1}$, where $\Delta_i \in \mathcal{O}$ but $\Delta_0$ and $\Delta_{s(\Gamma)+1}$ are possibly proper preconfigurations. When a type-occurrence $\Lambda$ in a configuration is written without vectorial notation, that means the sort of $\Lambda$ is 0. However, when one writes the
models for the displacement calculus, this does not mean that the sort of \( A \) is necessarily greater than 0.

A hypersequent \( \Gamma \Rightarrow A \) comprises an antecedent configuration in string-based notation of sort \( i \) and a succedent type \( A \) of sort \( i \). The hypersequent calculus for \( D \) is as shown in Figure 4. The following lemma is useful for the strong completeness results of section 3:

**Lemma 2.** Recall that \( \mathcal{O} \) is a subset of \( V^* = (seg[C] \cup \{1\})^* \). We have that:

i) \( \mathcal{O} \) is closed by concatenation and intercalation.

ii) If \( \Delta \in V^* \), \( \Gamma \in \mathcal{O} \), and \( \Delta, \Gamma \in \mathcal{O} \), then \( \Delta \in \mathcal{O} \). Similarly, if we have \( \Gamma, \Delta \in \mathcal{O} \) instead of \( \Delta, \Gamma \in \mathcal{O} \). Finally, If \( \Delta \in V^* \), \( \Gamma \in \mathcal{O} \), and \( \Delta \uparrow, \Gamma \in \mathcal{O} \), then \( \Delta \in \mathcal{O} \).
Proof. Propositions (i) and ii) are both proved via the BNF derivations of (11). The details of the proof are rather tedious but not difficult.

What is the connection between the calculi \( \mathbf{cD} \) and \( \mathbf{hD} \)? In [14] a (faithful) embedding translation is proved. Let \( \Delta \) denote a configuration. We define its type-equivalent \( \Delta^* \), which is a type which has the same algebraic meaning as \( \Delta \).

Via the BNF formulation of \( O[\mathbf{C}] \) in (10) one defines recursively \( \Delta^* \) as follows:

\[
\begin{align*}
\Lambda^* &= I \\
(A, I)^* &= I \\
(A, A_1, \ldots, A_n, I)^* &= (\cdots (A \circ_1 A_1^* \cdots) \cdots) \circ_{s(A)+1} (A_1^* \cdots A_n^*)^* \\
(A_1^* \cdots A_n^*)^* &= A_1^* \cdots A_n^* \\
\end{align*}
\]

The semantic interpretation of a configuration \( \Delta \) (for a given valuation \( v \)) is 
\( \hat{v}(\Delta) = \hat{v}(\Delta^*) \). The embedding translation is as follows. For any \( \Delta \in O, cD \vdash \Delta^* \rightarrow A \) iff \( hD \vdash \Delta \Rightarrow A \).

2.3 Some special DAs

The standard DA \( \mathcal{S} \), induced by the separated monoid with generator set \( V = \text{seg} \cup \{1\} \), plays an important role. The interpretation of the signature \( \Sigma_D \) in \( |\mathcal{S}| \) is:

\[
\begin{align*}
\mathcal{S} = (V^*, +, \{[i]\}_{i > 0}, A, 1)
\end{align*}
\]

Here, + denotes concatenation, and \( \{[i]\}_{i > 0} \) \( i \)-th intercalation. We have seen in Section 2 that \( O \) is closed by concatenation + and intercalation \( \{[i]\}_{i > 0} \), i.e. \( \mathcal{C} = (O, +, \{[i]\}_{i > 0}, A, 1) \) is a \( \Sigma_D \)-subalgebra of the standard DA \( \mathcal{S} \). Since \( \mathcal{DA} \) is a variety \( \mathcal{C} \) is a (general) DA, concretely a nonstandard DA. To see that \( \mathcal{C} \) cannot be standard we notice that the sort domains of \( \mathcal{C} \) are not separated by \( \{1\} \). Recall that \( |\mathcal{C}| = \bigcup \mathcal{O}_i \ (|\mathcal{C}|_i = \mathcal{O}_i \), for every \( i \geq 0 \). We have that:

\[
\begin{align*}
\text{(13) For } i > 0, |\mathcal{C}|_i \neq \bigcup_{i \geq 0} \mathcal{O}_0 \cdots \circ \mathcal{O}_0 \\
\text{Because, for example, let us take } \overrightarrow{p_i \downarrow p} = \sqrt[p_i \downarrow p]{1}, \sqrt[p_i \downarrow p]{p}, \text{ where } p \in \text{Pr}_0. \\
\text{The type } \overrightarrow{p_i \downarrow p} \text{ has sort } 1, \text{ but clearly neither } \sqrt[p_i \downarrow p]{1} \text{ nor } \sqrt[p_i \downarrow p]{p} \text{ are members of } \mathcal{O}_0. \text{ In fact, we have the proper inclusion:} \\
\text{(14) For } i > 0, \bigcup_{i \geq 0} \mathcal{O}_0 \cdots \circ \mathcal{O}_0 \subset |\mathcal{C}|_i \\
\text{It follows that the class of standard DAs is a proper subclass of the class of general DAs.}
\end{align*}
\]

\( \footnote{We recall that varieties are closed by subalgebras, homomorphic images, and direct products.} \)
2.4 Synthetic Connectives and the Implicative fragment

From a logical point of view, synthetic connectives abbreviate formulas in sequent systems. They form new connectives with left and right sequent rules. Using a linear logic slogan, synthetic connectives help to eliminate some bureaucracy in Cut-free proofs and in the (syntactic) Cut-elimination algorithms (see [14]). We consider here a set of synthetic connectives which are of linguistic interest:

- The binary non-deterministic implications \( \uparrow \), and \( \downarrow \).
- The unary connectives \( \vartriangleright -1 \), \( \vartriangleright -1 \), and \( (\vartriangleright k)_{k>0} \), which are called respectively left projection, right projection, and split.

Together with the binary deterministic implications \( \setminus \), \( / \), \( (\uparrow i)_{i>0} \), \( (\downarrow i)_{i>0} \), these constitute what we call *implicative* connectives. These connectives are incorporated in the recursive definitions of \( Tp \), \( seg \), and \( O \). We denote this implicative fragment as \( D[\rightarrow] \). We write also \( Tp[\rightarrow] \), \( seg[\rightarrow] \), and \( O[\rightarrow] \), although, as usual, if the context is clear we will avoid writing \( [\rightarrow] \). The intuitive semantic interpretation of the implicative connectives can be found in Figure 5. Figure 6 and Figure 7 correspond to their hypersequent rules.

Besides the usual continuous and discontinuous implications, the nondeterministic discontinuous implications are used to account for particle shift nondeterminism where the object can be intercalated between the verb and the particle, or after the particle. For a particle verb like *call+1+up* we can give the lexical assignment \( \sigma^{-1}(\vartriangleright (N\setminus S)\uparrow N) \). Projections can be used to account for the cross-serial dependencies of Dutch. The split connective can be used for parentheticals like *fortunately* with the type assignment \( \vartriangleright S_{i+1}S \).

![Fig. 5. Semantic interpretation in standard DAs for the set of synthetic connectives](image)

3 Strong Completeness of the implicative fragment w.r.t. \( L \)-models

In this section we prove two strong completeness theorems in relation to the implicative fragment. In order to prove them, we demonstrate first strong completeness of \( hD[\rightarrow] \) w.r.t. powerset residuated DAs over standard DAs with a countable set of generators.
Let \( V = \text{seg}[-|] \cup \{1\} \). Clearly, \( V \) is countably infinite since \( \text{seg}[-|] \) is the countable union \( \bigcup_{i} \text{seg}[-|]_{i} \), where each \( \text{seg}[-|]_{i} \) is also countably infinite. Let us consider the standard DA \( S \) (from (12)), induced by the (countably) infinite set of generators \( V \):

\[
S = (V^{*}, +, \{1\}_{i>0}, A, 1)
\]

We define some notation:

**Definition 3.** For any type \( C \in \text{Tp}[\rightarrow] \) and set \( R \) of non-logical axioms:

\[
[C]_{R} = \{ \Delta : \Delta \in O \text{ and } R \vdash \Delta \Rightarrow C \}
\]

In practice, when the set of hypersequents \( R \) is clear from the context, we simply write \([C]\) instead of \([C]_{R}\).

**Lemma 3.** (Truth Lemma)

Let \( \mathcal{P}(S) \) be the powerset residuated DA over the standard DA \( S \) from (12). Let \( v_{R} \) be the following valuation on the powerset \( \mathcal{P}(S) \):

\[
\text{For every } p \in \text{Pr}, \quad v_{R}(p) = [p]_{R}
\]

Let \( \mathcal{M} = (\mathcal{P}(S), v_{R}) \) be called as usual the canonical model. The following equality holds:

\[
\text{For every } C \in \text{Tp}[\rightarrow], \quad v_{R}(C) = [C]_{R}
\]
Proof. We proceed by induction on the structure of type $C$; we will write $\hat{v}$ instead of $\hat{v}_R$, and $[\cdot]$ instead of $[\cdot]_R$; we will say that an element $\Delta \in \hat{v}(A)$ is correct if $\Delta \in \mathcal{O}(C)$.

- $C$ is primitive. True by definition.

- $C = B^\uparrow_i A$. Let us see:

  $$[B^\uparrow_i A] \subseteq \hat{v}(B^\uparrow_i A)$$

  Let $\Delta$ be such that $R \vdash \Delta \Rightarrow B^\uparrow_i A$. Let $\Gamma_A \in \hat{v}(A)$. By induction hypothesis (i.h.), $\hat{v}(A) = [A]$. Hence, $R \vdash \Gamma_A \Rightarrow A$ We have:

  $$\Delta \Rightarrow B^\uparrow_i A \quad \frac{\Gamma_A \Rightarrow B}{\Delta \mid \Gamma_A \Rightarrow B} \quad \text{Cut}$$

  By i.h., $\hat{v}(B) = [B]$. It follows that $\Delta \mid \Gamma_A \in \hat{v}(B)$, hence $\Delta \in \hat{v}(B^\uparrow_i A)$. Whence, $[B^\uparrow_i A] \subseteq \hat{v}(B^\uparrow_i A)$.

  Conversely, let us see:

  $$\hat{v}(B^\uparrow_i A) \subseteq [B^\uparrow_i A]$$

  Let $\Delta \hat{v}(B^\uparrow_i A)$. By i.h, $\hat{v}(A) = [A]$. For any type $A$, we have eta-expansion, i.e. $\hat{\overrightarrow{A}} \Rightarrow \hat{\overrightarrow{A}}$. Hence, $\overrightarrow{A} \in \hat{v}(A)$. We have that $\Delta \mid \overrightarrow{A} \in \hat{v}(B)$. By i.h., $\Delta \mid \overrightarrow{A} \Rightarrow B$. Since $\overrightarrow{A}$ is correct, and by i.h. $\Delta \mid \overrightarrow{A}$ is correct, by Lemma 2 $\Delta$ is correct. By applying the $\uparrow_i$ right rule to the provable hypersequent $\Delta \mid \overrightarrow{A} \Rightarrow B$ we get:

  $$\Delta \Rightarrow B^\uparrow_i A$$

  This ends the case of $B^\uparrow_i A$.

- $C = A\downarrow_i B$. Completely similar to case $B^\uparrow_i A$.

- $C = B/A$ or $A\setminus B$. Similar to the discontinuous case.

- Nondeterministic connectives. Consider the case $C = \hat{B}^\uparrow_i A$.

  $$[\hat{B}^\uparrow_i A] \subseteq \hat{v}(\hat{B}^\uparrow_i A)$$

  Let $\Gamma_A \in \hat{v}(A)$. By i.h, $\Gamma_A \Rightarrow A$. Let $\Delta \Rightarrow B^\uparrow_i A$. By $s(B) - s(A) + 1$ applications of $\hat{\uparrow}$ left rule, we have

  $$\frac{\Delta \Rightarrow B^\uparrow_i A}{\Gamma_A \Rightarrow \hat{B} \Rightarrow B, \text{ by eta-expansion}}$$

  By $s(B) - s(A) + 1$ Cut applications with $\Delta \Rightarrow B^\uparrow_i A$, we get:

  $$\Delta \mid \Gamma_A \Rightarrow B$$

  $^5$ Recall that a priori $\Delta \in \mathcal{S}$, which is equal to $(\mathsf{Seg}[-] \cup \{1\})^\ast$.

  $^6$ By simple induction on the structure of types.
Hence, for $i = 1, \ldots, s(B) - s(A) + 1$, by i.h. $\Delta|_{i}A \in \hat{\mathcal{v}}(B).$ Hence, $\Delta \in \hat{\mathcal{v}}(B \upharpoonright A)$.

Conversely, let us see: $$\hat{\mathcal{v}}(B \upharpoonright A) \subseteq [B \upharpoonright A]$$

By i.h, we see that $A \in \hat{\mathcal{v}}(A).$ Let $\Delta \in \hat{\mathcal{v}}(B \upharpoonright A).$ This means that for every $i = 1, \ldots, s(B) - s(A) + 1$ $\Delta|_{i}A \in \hat{\mathcal{v}}(B).$ By i.h., $\Delta|_{i}A \Rightarrow B.$ By a similar reasoning to the deterministic case $C = B \uparrow_{1}A$, we see that $\Delta$ is correct. We have that:

$$\Delta|_{1}A \Rightarrow B \quad \cdots \quad \Delta|_{s(B) - s(A) + 1}A \Rightarrow B \quad \uparrow R$$

- $C = A \downarrow B$ is completely similar to the previous one.

- $C = \alpha^{-1}A.$ Let us see: 

$$\lceil \alpha^{-1}A \rceil \subseteq \hat{\mathcal{v}}(\alpha^{-1}A)$$

Let $\Delta \in \lceil \alpha^{-1}A \rceil.$ Hence, $\Delta \Rightarrow \alpha^{-1}A.$ We have that:

$$\alpha^{-1}L$$

$$\Delta \Rightarrow \alpha^{-1}A \quad \alpha^{-1}A, 1 \Rightarrow A \quad \text{Cut}$$

By i.h., $\Delta, 1 \in \hat{\mathcal{v}}(A).$ Hence, $\Delta \in \hat{\mathcal{v}}(\alpha^{-1}A).$

Conversely, let us see: $$\hat{\mathcal{v}}(\alpha^{-1}A) \subseteq \lceil \alpha^{-1}A \rceil$$

Let $\Delta \in \hat{\mathcal{v}}(\alpha^{-1}A).$ By definition, $\Delta, 1 \in \hat{\mathcal{v}}(A).$ By i.h., $\Delta, 1 \Rightarrow A,$ and by lemma 2 $\Delta$ is correct. By application of $\alpha^{-1}$ right rule, we get: $$\Delta \Rightarrow \alpha^{-1}A$$

This proves the converse.

- $C = \beta^{-1}A$ is completely similar to the previous one.

- $C = \gamma^{*}A.$ Let us see: 

$$\lceil \gamma^{*}A \rceil \subseteq \hat{\mathcal{v}}(\gamma^{*}A)$$

Let $\Delta \Rightarrow \gamma^{*}A.$ We have that:

$$\gamma^{*}L$$

$$\Delta \Rightarrow \gamma^{*}A \quad \gamma^{*}A, k \Rightarrow A \quad \text{Cut}$$
By induction on the structure of $O$.

Conversely, let us see that:

$$\hat{\nu}(\vdash^* A) \subseteq \vdash^* A$$

Let $\Delta \in \hat{\nu}(\vdash^* A)$. By definition, $\Delta \vdash^* A \in \hat{\nu}(A)$. By i.h. and lemma 2 $\Delta$ is correct and $\Delta \vdash^* A \Rightarrow A$. By application of the $\vdash^*$ right rule:

$$\Delta \Rightarrow \vdash^* A$$

Hence, $\Delta \in [\vdash^* A]$. □

By induction on the structure of $O$, see (10), one proves the following lemma:

**Lemma 4. (Identity lemma)**

For any $\Delta \in O$, $\Delta \in \hat{\nu}(\Delta)$.

Let $\Delta (\Gamma_1 \cdots \Gamma_n \vdash A_1 \cdots A_n)$ be the result of replacing every type-occurrence $A_i$ with $\Gamma_i$.

Recall that we have fixed a set of hypersequents $R$. We have the lemma:

**Lemma 5.** $M = (\mathcal{P}(S), \nu) \vDash R$

Proof. Let $(\Delta \Rightarrow A) \in R$. For every type-occurrence $A_i$ in $\Delta$ (we suppose that the sequence of type-occurrences in $\Delta$ is $(A_i)_{i=1,\ldots,n}$), we have by the Truth Lemma that $\hat{\nu}(A_i) = [A_i]_R$. For any $\Gamma_i \in \hat{\nu}(A_i)$, we have by the Truth Lemma that $R \vdash \Gamma_i \Rightarrow A_i$. Since $(\Delta \Rightarrow A) \in R$, we have then that $R \vdash \Delta \Rightarrow A$. By $n$ applications of the Cut rule with the premises $\Gamma_i$ we get from $R \vdash \Delta \Rightarrow A$ that $R \vdash \Delta (\Gamma_1 \cdots \Gamma_n \vdash A_1 \cdots A_n) \Rightarrow A$. We have that $\hat{\nu}(\Delta) = \{\Delta (\Gamma_1 \cdots \Gamma_n \vdash A_1 \cdots A_n) : \Gamma_i \in \hat{\nu}(A_i)\}$. Since, we have $R \vdash \Delta (\Gamma_1 \cdots \Gamma_n \vdash A_1 \cdots A_n) \Rightarrow A$, again by the Truth Lemma, $\hat{\nu}(\Delta) \subseteq \hat{\nu}(A)$. We have then that $\hat{\nu}(\Delta) \subseteq [\Delta \Rightarrow A]$. We are done. □

**Theorem 2.** $D[\Rightarrow]$ is strongly complete w.r.t. the class $\mathcal{P}_{RSD}$.

Proof. Suppose $\mathcal{P}_{RSD}(R) \vDash \Delta \Rightarrow A$. Hence, in particular this is true of the canonical model $M$. Since $\Delta \in \hat{\nu}(\Delta)$, it follows that $\Delta \in \hat{\nu}(A)$. By the Truth Lemma, $\hat{\nu}(A) = [A]_R$. Hence, $R \vdash \Delta \Rightarrow A$. We are done. □

We shall also prove strong completeness w.r.t. L-models over the set of connectives $\Sigma[\Rightarrow]_\text{-split}$, where $\text{split} = \{\vdash^*: k > 0\}$. Since the canonical model $\mathcal{S}$ is countably infinite, $|\mathcal{S}|$ is in bijection with a set $V_1 = (a_i)_{i>0} \cup \{1\}$ via a mapping $\Phi$. Let $\mathcal{A}$ be the standard DA associated to $V_1$. $\Phi$ extends to an isomorphism of standard DAs between $\mathcal{S}$ and $\mathcal{A}$, and then induces an isomorphism $\Phi$ of residuated powerset DAs over standard DAs. Let $\mathcal{B}$ be a standard DA generated by the finite set of generators $V_2 = \{a, b, 1\}$. We have that $|\mathcal{A}| = V_1^\ast$, and $|\mathcal{B}| = V_2^\ast$. Let $\rho$ be the following injective mapping from $V_1$ into $V_2^\ast$: 

$$\rho(a_i) = a_i \vdash^* 1^\ast$$

$$\rho(a_i) = a_i \vdash^* b^\ast$$

$$\rho(1) = 1 \vdash^* 1^\ast$$

We have that $\rho(V_1) = \rho(V_1^\ast) = V_2^\ast$. Since $\mathcal{S}$ is strongly complete w.r.t. $\mathcal{P}_{RSD}$, $\mathcal{A}$ is strongly complete w.r.t. $\mathcal{S}$. Hence, $\mathcal{B}$ is strongly complete w.r.t. $\mathcal{A}$. We are done. □
The mapping $\rho$ extends recursively to the morphism of standard DAs. Clearly $\rho$ is injective by freeness of the underlying free monoids $|A|$ and $|B|$. The mapping $\rho$ is a monomorphism of DAs which induces a monomorphism $\bar{\rho}$ of residuated powerset DAs over DAs. Let $A$, $B$ and $C$ range over subsets of $|A|$ such that they are non-empty and different from $\{\epsilon\}$. Since $\rho$ is injective, so is $\bar{\rho}$. The following equalities hold:

\[
\begin{align*}
(15) & \quad \rho(1) = 1 \quad \rho(a_i) = a + b^i + a \\
(16) & \quad \bar{\rho}(A \cdot B) = \bar{\rho}(A) \cdot \bar{\rho}(B) \quad \bar{\rho}(A\triangle B) = \bar{\rho}(A)\triangle \bar{\rho}(B) \\
& \quad \bar{\rho}(A \downarrow B) = \bar{\rho}(A) \downarrow \bar{\rho}(B) \quad \bar{\rho}(\overline{A} \uparrow B) = \bar{\rho}(\overline{A}) \uparrow \bar{\rho}(B) \\
& \quad \bar{\rho}(A \uparrow B) = \bar{\rho}(A) \uparrow \bar{\rho}(B) \quad \bar{\rho}(\overline{A} \downarrow B) = \bar{\rho}(\overline{A}) \downarrow \bar{\rho}(B)
\end{align*}
\]

The equalities (16) are due to the fact that 1) $\bar{\rho}$ is a monomorphism of DAs, 2) we can apply cancellation, and 3) the subsets considered are non-empty and different from $\{\epsilon\}$. Since $\rho$ is injective, arbitrary families of (same-sort) subsets satisfy $\rho(\bigcap_{i \in I} X_i) = \bigcap_{i \in I} \rho(X_i)$. Moreover, using (16) one proves:

\[
\begin{align*}
(17) & \quad \bar{\rho}( \bigcap_{i=1}^{s(B) - s(A) + 1} A \uparrow_i B) = \bigcap_{i=1}^{s(B) - s(A) + 1} \bar{\rho}(A) \uparrow \bar{\rho}(B) \\
& \quad \bar{\rho}( \bigcup_{i=1}^{s(B) - s(A) + 1} A \downarrow_i B) = \bigcup_{i=1}^{s(B) - s(A) + 1} \bar{\rho}(A) \downarrow \bar{\rho}(B)
\end{align*}
\]

Recall that $v$ is the valuation of the canonical model $\mathcal{P}(S)$. Consider the following composition of mappings: $\Pr \xrightarrow{\nu} \mathcal{P}(S) \xrightarrow{\Phi} \mathcal{P}(A) \xrightarrow{\rho} \mathcal{P}(B)$. Put $w = \rho \circ \Phi \circ v$. We have that $\hat{w} = \bar{\rho} \circ \Phi \circ \bar{v}$. In order to prove the last equality we have to see that $\bar{\rho} \circ \Phi \circ \bar{v}$ is a monomorphism of DAs. For example, if $A$ and $B$ are types, one has:

\[
(\bar{\rho} \circ \Phi \circ \bar{v})(B \uparrow_k A) = \bar{\rho}((\Phi(\bar{v})(B) \uparrow_k \bar{v})(A)) = \bar{\rho}(\Phi(\bar{v})(B) \uparrow_k \bar{v})(A), \quad \Phi \text{ is an isomorphism of DAs} = \bar{\rho}(\Phi(\bar{v})(B) \uparrow_k \bar{v})(A)), \quad \rho \text{ satisfies (16)}
\]

Similar computations give the desired equalities for the remaining considered implicative connectives. Given a set of non-logical axioms $R$, $R \vdash_{\text{bfD}} \Delta \Rightarrow A$ iff $\bar{v}(\Delta) \subseteq \bar{v}(A)$ (we write $\bar{v}$ instead of $\bar{v}_R$) iff $(\Phi \circ \bar{v})(\Delta) \subseteq (\Phi \circ \bar{v})(A)$ iff $(\rho \circ \Phi \circ \bar{v})(\Delta) \subseteq (\rho \circ \Phi \circ \bar{v})(A)$. We have proved:

**Theorem 3.** $\mathcal{D}[\Sigma \rightarrow \text{split}]$ is strongly complete w.r.t. L-models.

**Corollary 1.** $\mathcal{D}[\Sigma \rightarrow \text{split}]$ is strongly complete w.r.t. powerset residuated DAs overs standard DAs with 3 generators.

---

7 Since the underlying structures are free monoids we can apply left/right cancellation.
8 There is a unique morphism of DAs extending $w$.
9 Including also projection connectives.
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