Mutually Equidistant Spheres that Intersect

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ABSTRACT. The setting for this brief paper is $\mathbb{R}^3$. Distance between two spheres is understood as distance $\delta$ between spherical centers. For instance, a Reuleaux tetrahedron $T$ is the intersection of four unit balls satisfying $\delta = 1$ pairwise. Volume and surface area of $T$ are already well-known; our humble contribution is to calculate the mean width of $T$.

Two earlier papers [1, 2] were devoted to convex hulls involving disks. We will here discuss intersections of balls. It is natural to compute volume and surface area of such intersections. A third quantity, mean width, is “a new measure on three-dimensional solids that enjoys equal rights with volume and surface area” [3]. While results for $VL$ and $AR$ have appeared many times in the past, some of our expressions for $MW$ may be new.

1. Dihedron (Symmetric Lens)

Consider, for simplicity, two unit spheres in $\mathbb{R}^3$ that pass through each other’s centers. The region enclosed by both spheres is called a spherical dihedron, using language of [4, 5, 6, 7]. It is more commonly called a symmetric lens. Starting with spheres

$$\left( x - \frac{1}{2} \right)^2 + y^2 + z^2 = 1, \quad \left( x + \frac{1}{2} \right)^2 + y^2 + z^2 = 1$$

and defining

$$f(x, y) = \sqrt{1 - \left( x + \frac{1}{2} \right)^2 - y^2}, \quad a(x) = \sqrt{1 - \left( x + \frac{1}{2} \right)^2}$$

we obtain

$$VL = 8 \int_0^{1/2} a(x) \int_0 f(x, y) dy dx = \frac{5\pi}{12},$$

$$AR = 8 \int_0^{1/2} a(x) \int_0 \sqrt{1 + f_x^2 + f_y^2} dy dx = 2\pi.$$

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Interiors of both faces have mean curvature 1. The (unique) edge is a circle with radius \( f(0,0) = \sqrt{3}/2 \), hence it has circumference \( \sqrt{3}\pi \). The dihedral angle at the edge is \( \pi/3 \). By the “indirect approach” in [1], we have

\[
MW = \frac{1}{2\pi} AR + \frac{1}{4\pi} \frac{\pi}{3} \sqrt{3}\pi = 1 + \frac{\pi}{4\sqrt{3}}.
\]

An obvious generalization allows \( 0 < \delta < 2 \), where distance \( \delta \) between spherical centers was taken to be 1 above. Substantial work [8, 9, 10, 11, 12, 13, 14, 15, 16, 17] has been performed on intersections of arbitrary collections of spheres in \( \mathbb{R}^3 \). We merely quote an extended volume result [18] for two unit spheres:

\[
VL = \frac{4\pi}{3} \left( 1 - \frac{3}{4} \delta + \frac{1}{16} \delta^3 \right).
\]

A symmetric lens is the union of two spherical caps. A different parametrization makes use of the angular radius \( \varphi \) of either cap (also called the angle of aperture or colatitude angle of the cap). From \( \delta = 2\cos(\varphi) \), it follows that [19]

\[
VL = \frac{2\pi}{3} \left( 2 - 3\cos(\varphi) + \cos(\varphi)^3 \right),
\]

\[
AR = 4\pi \left( 1 - \cos(\varphi) \right),
\]

\[
MW = 2 - 2\cos(\varphi) + \left( \frac{\pi}{2} - \varphi \right) \sin(\varphi).
\]

An unfortunate ambiguity appears in Figure 5f of [19]: the line segment is intended to be normal to the rightmost spherical cap (not tangent to the leftmost spherical cap). Thus the indicated angle \( \alpha \) is indeed equal to \( \varphi \) (and not equal to the contact angle \( \pi/2 - \varphi \)). Confusion occurs because \( \alpha \) is chosen close to \( \pi/4 \) in Figure 5f; the center picture in Figure 3 of [20] helps to clarify matters.

We mention another error in the literature. The mean width in [21], although defined to be \( 2 \cdot MW \) (often encountered), gives results inconsistent with [19]. See [22] for another treatment of overlapping spheres.

Higher \( n \)-dimensional results for both \( VL \) and \( AR \) can be inferred from [23, 24, 25, 26] that involve the Gauss hypergeometric function:

\[
VL = \frac{\pi^{n/2}}{\Gamma(1 + n/2)} \left[ 1 - \frac{2\Gamma(1 + n/2)}{\sqrt{\pi}\Gamma((n + 1)/2)} \, _2F_1 \left( \frac{1}{2}, \frac{1 - n}{2}, \frac{3}{2}, \cos(\varphi)^2 \right) \cos(\varphi) \right],
\]

\[
AR = \frac{2\pi^{n/2}}{\Gamma(n/2)} \left[ 1 - \frac{2\Gamma(n/2)}{\sqrt{\pi}\Gamma((n - 1)/2)} \, _2F_1 \left( \frac{1}{2}, \frac{3 - n}{2}, \frac{3}{2}, \cos(\varphi)^2 \right) \cos(\varphi) \right].
\]

We have not attempted to find an analogous formula for \( MW \). Older alternative expressions for \( VL \) and \( AR \) are found by unraveling the quermassintegrals \( W_0 \) and \( k W_1 \) in [20] (formula 69). New and old results agree. Also, \( (2\Gamma(1 + k/2)/\pi^{k/2})W_{k-1} \) coincides with \( MW \) when \( k = 3 \). This is as far as our analysis has gone.
2. Trihedron

Consider now three unit spheres in $\mathbb{R}^3$ that pass through each other’s centers. The region enclosed by all spheres is called a spherical trihedron, using language of [4, 5, 6, 7]. Starting with spheres

$$
\left( x - \frac{1}{\sqrt{3}} \right)^2 + y^2 + z^2 = 1,
$$

$$
\left( x + \frac{1}{2\sqrt{3}} \right)^2 + \left( y - \frac{1}{2} \right)^2 + z^2 = 1,
$$

$$
\left( x + \frac{1}{2\sqrt{3}} \right)^2 + \left( y + \frac{1}{2} \right)^2 + z^2 = 1
$$

and defining

$$
f(x, y) = \sqrt{1 - \left( x - \frac{1}{\sqrt{3}} \right)^2 - y^2}, \quad a(y) = \frac{1}{\sqrt{3}} - \sqrt{1 - y^2}, \quad c(y) = -\frac{1}{\sqrt{3}}y
$$

we obtain

$$
VL = 12 \int_0^{1/2} \int_{a(y)}^{c(y)} f(x, y) \, dx \, dy = \frac{1}{12} \left[ 2\sqrt{2} + 24\pi - 57 \text{arcsec}(3) \right],
$$

$$
AR = 12 \int_0^{1/2} \int_{a(y)}^{c(y)} \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy = 6 \left[ \pi - 2 \text{arcsec}(3) \right].
$$

Each of the three edges is a circular arc with radius $\sqrt{3}/2$. Focus on the arc that lies entirely in the $xz$-plane. From

$$
\left( x + \frac{1}{2\sqrt{3}} \right)^2 + z^2 = 1 - \frac{1}{4} = \left( \frac{\sqrt{3}}{2} \right)^2
$$

we deduce that the circle center is $(-1/(2\sqrt{3}), 0, 0)$ and hence the subtended angle is

$$
2 \arccos \left( \frac{1/(2\sqrt{3})}{\sqrt{3}/2} \right) = 2 \text{arcsec}(3)
$$

by symmetry. As before, we have

$$
MW = \frac{1}{2\pi} AR + \frac{3}{4\pi} \left( 2 \text{arcsec}(3) \right) \sqrt{3} = \frac{12\pi - (24 - \sqrt{3}) \pi \text{arcsec}(3)}{4\pi}
$$

$$
= 1.182061751038757...
$$

Incidently, the vertex-to-vertex distance $\lambda$ here is $2\sqrt{2}/3$ and $VL/\lambda^3 = 0.154...$, consistent with Figure 1 in [7]. See [18] for extended volume results.
3. Tetrahedron
Consider finally four unit spheres in $\mathbb{R}^3$ that pass through each other’s centers. The region enclosed by all spheres is called a spherical tetrahedron or Reuleaux tetrahedron. Our analysis of this solid is complicated due to a narrow sliver in the half-space $z < 0$. Starting with spheres

$$\left(x - \frac{1}{\sqrt{3}}\right)^2 + y^2 + z^2 = 1, \quad \left(x + \frac{1}{2\sqrt{3}}\right)^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = 1,$$

$$\left(x + \frac{1}{2\sqrt{3}}\right)^2 + \left(y + \frac{1}{2}\right)^2 + z^2 = 1, \quad x^2 + y^2 + \left(z - \frac{2}{\sqrt{3}}\right)^2 = 1$$

and defining

$$f(x, y) = \sqrt{1 - \left(x - \frac{1}{\sqrt{3}}\right)^2 - y^2}, \quad g(x, y) = \sqrt{\frac{2}{3} - \sqrt{1 - x^2 - y^2}},$$

$$a(y) = \frac{1}{\sqrt{3}} - \sqrt{1 - y^2}, \quad b(y) = \frac{1}{\sqrt{6}} \left(\frac{1}{\sqrt{2}} - \sqrt{3 - 4y^2}\right), \quad c(y) = -\frac{1}{\sqrt{3}}y$$

we obtain [27, 28]

$$VL = 12 \int_0^{1/2} b(y) \int_0^{1/2} c(y) f(x, y) dx dy + 6 \int_0^{1/2} c(y) f(x, y) dx dy \left[f(x, y) - g(x, y)\right] dx dy$$

$$= \frac{1}{12} \left[3\sqrt{2} + 32\pi - 81 \text{arcsec}(3)\right],$$

$$AR = 12 \int_0^{1/2} b(y) \int_0^{1/2} c(y) \sqrt{1 + f_x^2 + f_y^2} dx dy + 6 \int_0^{1/2} c(y) \left(\sqrt{1 + f_x^2 + f_y^2 + \sqrt{1 + g_x^2 + g_y^2}}\right) dx dy$$

$$= 2 \left[4\pi - 9 \text{arcsec}(3)\right].$$

Each of the six edges is a circular arc with radius $\sqrt{3}/2$. Focus again on the arc that lies entirely in the $xz$-plane. Previously it started at $(0, 0, \sqrt{2}/3)$ and ended at $(0, 0, -\sqrt{2}/3)$; now it ends at $(1/\sqrt{3}, 0, 0)$ where the arc meets the bottom face. Hence the subtended angle is half its previous value. We consequently have

$$MW = \frac{1}{2\pi} AR + \frac{6\pi}{4\pi \text{arcsec}(3)} \sqrt{3} = \frac{16\pi - (36 - \sqrt{3}\pi) \text{arcsec}(3)}{4\pi}$$

$$= 1.006582094946935...$$
which falls between the minimum width 1 and the maximum width $\sqrt{3} - 1/\sqrt{2} = 1.0249...$, as expected [29].

Incidently, the vertex-to-vertex distance $\lambda$ here is 1 and $VL/\lambda^3 = 0.422...$, consistent with Figure 1 in [7]. See [18] for extended volume results.

The **Meissner tetrahedron**, obtained by rounding off three edges of the Reuleaux tetrahedron to force a constant width, possesses measures [29, 30, 31, 32]

$$VL' = \frac{1}{12} \left[ 8 - 3\sqrt{3}\text{arcsec}(3) \right] \pi < VL,$$

$$AR' = \frac{1}{2} \left[ 4 - \sqrt{3}\text{arcsec}(3) \right] \pi < AR$$

and, of course, $MW' = 1 < MW$.

## 4. Miscellanea

The symmetric lens arises as a solution of certain geometric optimization problems [33, 34, 35]. We report on three other such “important” solids here. In the same papers, the **right circular cylinder with hemispherical ends** is featured. Assuming the radius is 1 and the cylinder length is $\ell$, we easily have [36]

$$VL = \left( \ell + \frac{4}{3} \right) \pi,$$

$$AR = 2 (\ell + 2) \pi,$$

$$MW = \frac{1}{2} (\ell + 4).$$

The measures of a hemisphere, as an aside, are given in standard tables [37, 38] (although not spherical caps in general).

The **symmetric segment** or **spherical slice** appears in [39, 40, 41, 42]. This solid is obtained by removing two diametrically-opposed spherical caps from the unit ball, each of angular radius $\varphi$. Clearly [19]

$$VL = \frac{2\pi}{3} \left( 2 + \sin(\varphi)^2 \right) \cos(\varphi),$$

$$AR = 2\pi \left( 2 \cos(\varphi) + \sin(\varphi)^2 \right),$$

$$MW = 2 \cos(\varphi) + \varphi \sin(\varphi).$$

Figure 5e of [19] is unambiguous (unlike Figure 5f, as discussed earlier). We are, however, unable to find agreement with the quermassintegrals in [20] (formula 68) when $k = 3$. 
The cap body of a ball appears in the same papers, as well as in [43, 44, 45, 46, 47, 48]. This solid is the convex hull of the unit ball with a line segment passing symmetrically through its center. Another name for this is 1-tangential body. With an interpretation of \( \varphi \) identical to above, we find that [19]

\[
VL = \frac{2\pi}{3} \frac{1 + \cos(\varphi)^2}{\cos(\varphi)},
\]

\[
AR = 2\pi \frac{1 + \cos(\varphi)^2}{\cos(\varphi)},
\]

\[
MW = \frac{1 + \cos(\varphi)^2}{\cos(\varphi)}.
\]

Figure 5g of [19] is unambiguous. Again, we are unable to find agreement with the quermassintegrals in [20] (formula 70) when \( k = 3 \).

A less important example (evidently) might be called the “ring body of a ball”. This solid is the convex hull of the unit ball with a circle suspended symmetrically above its equator. Its measures are given in [19] – see Figure 5h – but since it does not appear elsewhere in the literature, we omit further discussion.

In closing, here is an unanswered question. The region enclosed by the six spheres:

\[
\left( x - \frac{1}{\sqrt{2}} \right)^2 + y^2 + z^2 = 1, \quad \left( x + \frac{1}{\sqrt{2}} \right)^2 + y^2 + z^2 = 1,
\]

\[
x^2 + \left( y - \frac{1}{\sqrt{2}} \right)^2 + z^2 = 1, \quad x^2 + \left( y + \frac{1}{\sqrt{2}} \right)^2 + z^2 = 1,
\]

\[
x^2 + y^2 + \left( z - \frac{1}{\sqrt{2}} \right)^2 = 1, \quad x^2 + y^2 + \left( z + \frac{1}{\sqrt{2}} \right)^2 = 1
\]

is called a spherical hexahedron (cube). What are exact expressions for \( VL, AR \) and \( MW \)? The dihedral angle at any edge is \( \pi/3 \). While the spheres are not mutually equidistant, we can still define \( \lambda \) to be the adjacent vertex-to-vertex distance, which here is \( 2/\sqrt{3} \). Numerical work gives \( VL/\lambda^3 = 1.508.. \), consistent with Figure 1 in [7]. A similar question can be asked about the spherical dodecahedron, for which \( VL/\lambda^3 \) was given in [7] to be approximately 7.86.

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Mutually Equidistant Spheres that Intersect

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