NOTES ON RIDGE FUNCTIONS AND NEURAL NETWORKS

Vugar E. Ismailov

May 29, 2020
To the Memory of My Parents
Preface

These notes are about ridge functions. Recent years have witnessed a flurry of interest in these functions. Ridge functions appear in various fields and under various guises. They appear in fields as diverse as partial differential equations (where they are called plane waves), computerized tomography and statistics. These functions are also the underpinnings of many central models in neural networks.

We are interested in ridge functions from the point of view of approximation theory. The basic goal in approximation theory is to approximate complicated objects by simpler objects. Among many classes of multivariate functions, linear combinations of ridge functions are a class of simpler functions. These notes study some problems of approximation of multivariate functions by linear combinations of ridge functions. We present here various properties of these functions. The questions we ask are as follows. When can a multivariate function be expressed as a linear combination of ridge functions from a certain class? When do such linear combinations represent each multivariate function? If a precise representation is not possible, can one approximate arbitrarily well? If well approximation fails, how can one compute/estimate the error of approximation, know that a best approximation exists? How can one characterize and construct best approximations?

We also study properties of generalized ridge functions, which are very much related to linear superpositions and Kolmogorov’s famous superposition theorem. These notes end with a few applications of ridge functions to the problem of approximation by single and two hidden layer neural networks with a restricted set of weights.

We hope that these notes will be useful and interesting to both researchers and students.
Contents

Introduction 5

1 Properties of linear combinations of ridge functions 13
   1.1 Review of some results on ridge functions . 14
      1.1.1 $\mathcal{R}(a^1,\ldots,a^r)$ – ridge functions with fixed directions 14
      1.1.2 $\mathcal{R}_r$ – ridge functions with variable directions . 18
   1.2 Representation of multivariate functions by linear combinations of ridge functions . 24
      1.2.1 Two representation problems . 24
      1.2.2 Cycles . 25
      1.2.3 Minimal cycles and the main results . 30
      1.2.4 Corollaries . 33
   1.3 Characterization of an extremal sum of ridge functions . 36
      1.3.1 Exposition of the problem . 37
      1.3.2 The characterization theorem . 38
      1.3.3 Construction of an extremal element . 49
      1.3.4 Density of ridge functions and some problems . 53
   1.4 Sums of continuous ridge functions . 57
      1.4.1 Exposition of the problem . 57
      1.4.2 The representation theorem . 58
   1.5 On the proximinality of ridge functions . 63
      1.5.1 Problem statement . 63
      1.5.2 Proximinality of $R_b(X)$ in $B(X)$ . 64
      1.5.3 Proximinality of $R_c(X)$ in $C(X)$ . 66
   1.6 On the approximation by weighted ridge functions . 71
      1.6.1 Problem statement . 71
      1.6.2 Characterization of the best approximation . 72
      1.6.3 Calculation of the approximation error . 76
2 Approximation of multivariate functions by sums of univariate functions

2.1 Characterization of some bivariate function classes by formulas for the error of approximation

2.1.1 Exposition of the problem

2.1.2 Definition of the main classes

2.1.3 Construction of an extremal element

2.1.4 Characterization of $V_c(R)$

2.1.5 Classes $V_c^-(R), U(R)$ and $U_c^-(R)$

2.2 Approximation by sums of univariate functions on certain domains

2.2.1 Problem statement

2.2.2 The maximization process

2.2.3 $E$-bolts

2.2.4 Error estimates

2.3 On the theorem of M. Golomb

2.3.1 History of Golomb’s formula

2.3.2 Measures supported on projection cycles

3 General ridge functions and linear superpositions

3.1 Some results on the representation by linear superpositions

3.2 Kolmogorov’s superposition theorem and its extension

3.3 Some other superposition theorems

4 Applications to neural networks

4.1 Single hidden layer neural networks

4.1.1 Problem statement

4.1.2 Density results

4.1.3 A necessary condition for the representation by neural networks

4.1.4 Approximation error and extremal networks

4.2 Two hidden layer neural networks

4.2.1 Relation of the Kolmogorov superposition theorem to two hidden layer neural networks

4.2.2 The main result

References
Introduction

Recent years have seen a growing interest in the study of special multivariate functions called ridge functions. A ridge function, in its simplest format, is a multivariate function of the form \( g(a \cdot x) \), where \( g : \mathbb{R} \to \mathbb{R} \), \( a = (a_1, \ldots, a_d) \) is a fixed vector (direction) in \( \mathbb{R}^d \setminus \{0\} \), \( x = (x_1, \ldots, x_d) \) is the variable and \( a \cdot x \) is the standard inner product. In other words, a ridge function is a multivariate function constant on the parallel hyperplanes \( a \cdot x = c \), \( c \in \mathbb{R} \). These functions arise naturally in various fields. They arise in computerized tomography (see, e.g., \([72-74,97,106,111]\)), in statistics (see, e.g., \([13,14,27,33,42]\)) and neural networks (see, e.g., \([22,58,90,94,100,119,123,125]\)). These functions are also used in modern approximation theory as an effective and convenient tool for approximating complicated multivariate functions (see, e.g., \([38,57,66,101,114,118,137]\)).

It should be remarked that long before the appearance of the name “ridge”, these functions have been used in the theory of partial differential equations under the name of plane waves (see, e.g., \([69]\)). For example, assume that \((\alpha_i, \beta_i)\), \(i = 1, \ldots, r\), are pairwise linearly independent vectors in \( \mathbb{R}^2 \). Then the general solution to the homogeneous partial differential equation

\[
\prod_{i=1}^{r} \left( \alpha_i \frac{\partial}{\partial x} + \beta_i \frac{\partial}{\partial y} \right) u(x, y) = 0
\]

are all functions of the form

\[
u(x, y) = \sum_{i=1}^{r} g_i (\beta_i x - \alpha_i y)
\]

for arbitrary continuous univariate functions \( g_i, i = 1, \ldots, r \). Here the derivatives are understood in the sense of distributions.

The term “ridge function” was coined by Logan and Shepp in their seminal paper \([97]\) devoted to the basic mathematical problem of computerized
tomography. This problem consists of reconstructing a given multivariate function from values of its integrals along certain straight lines in the plane. The integrals along parallel lines can be considered as a ridge function. Thus, the problem is to reconstruct \( f \) from some set of ridge functions generated by the function \( f \) itself. In practice, one can consider only a finite number of directions along which the above integrals are taken. Obviously, reconstruction from such data needs some additional conditions to be unique, since there are many functions \( g \) having the same integrals. For uniqueness, Logan and Shepp [97] used the criterion of minimizing the \( L_2 \) norm of \( g \). That is, they found a function \( g(x, y) \) with the minimum \( L_2 \) norm among all functions, which has the same integrals as \( f \). More precisely, let \( D \) be the unit disk in the plane and an unknown function \( f(x, y) \) be square integrable and supported on \( D \). We are given projections \( P_f(t, \theta) \) (integrals of \( f \) along the lines \( x \cos \theta + y \sin \theta = t \)) and looking for a function \( g = g(x, y) \) of minimum \( L_2 \) norm, which has the same projections as \( f : P_g(t, \theta_j) = P_f(t, \theta_j) \), \( j = 0, 1, ..., n - 1 \), where the angles \( \theta_j \) generate equally spaced directions, i.e. \( \theta_j = \frac{j \pi}{n}, j = 0, 1, ..., n - 1 \). The authors of [97] showed that this problem of tomography is equivalent to the problem of \( L_2 \)-approximation of the function \( f \) by sums of ridge functions with the equally spaced directions \( (\cos \theta_j, \sin \theta_j) \), \( j = 0, 1, ..., n - 1 \). They gave a closed-form expression for the unique function \( g(x, y) \) and showed that the unique polynomial \( P(x, y) \) of degree \( n - 1 \) which best approximates \( f \) in \( L_2(D) \) is determined from the above \( n \) projections of \( f \) and can be represented as a sum of \( n \) ridge functions.

Kazantsev [72] solved the above problem of tomography without requiring that the considered directions are equally spaced. Marr [106] considered the problem of finding a polynomial of degree \( n - 2 \), whose projections along lines joining each pair of \( n \) equally spaced points on the circumference of \( D \) best matches the given projections of \( f \) in the sense of minimizing the sum of squares of the differences. Thus we see that the problems of tomography give rise to an independent study of approximation theoretic properties of the following set of linear combinations of ridge functions:

\[
\mathcal{R}(a^1, ..., a^r) = \left\{ \sum_{i=1}^{r} g_i (a^i \cdot x) : g_i : \mathbb{R} \to \mathbb{R}, i = 1, ..., r \right\},
\]

where directions \( a^1, ..., a^r \) are fixed and belong to the \( d \)-dimensional Euclidean space. Note that the set \( \mathcal{R}(a^1, ..., a^r) \) is a linear space.

Ridge function approximation also appears in statistics in Projection Pur-
suit. This term was introduced by Friedman and Tukey [32] to name a technique for the explanatory analysis of large and multivariate data sets. This technique seeks out “interesting” linear projections of the multivariate data onto a line or a plane. Projection Pursuit algorithms approximate a multivariate function $f$ by sums of ridge functions with variable directions, that is, by functions from the set

$$\mathcal{R}_r = \left\{ \sum_{i=1}^{r} g_i (a^i \cdot x) : a^i \in \mathbb{R}^n \setminus \{0\}, \; g_i : \mathbb{R} \to \mathbb{R}, \; i = 1, \ldots, r \right\}.$$  

Here $r$ is the only fixed parameter, directions $a^1, \ldots, a^r$ and functions $g_1, \ldots, g_r$ are free to choose. The first method of such approximation was developed by Friedman and Stuetzle [33]. Their approximation process called Projection Pursuit Regression (PPR) operates in a stepwise and greedy fashion. The process does not find a best approximation from $\mathcal{R}_r$, it algorithmically constructs functions $g_r \in \mathcal{R}_r$, such that $\|g_r - f\|_{L^2} \to 0$, as $r \to \infty$. At stage $m$, PPR looks for a univariate function $g_m$ and direction $a^m$ such that the ridge function $g_m (a^m \cdot x)$ best approximates the residual $f(x) - \sum_{j=1}^{m-1} g_j (a^j \cdot x)$.

Projection pursuit regression has been proposed as an approach to bypass the curse of dimensionality and now is applied to prediction in applied sciences. In [13,14], Candes developed a new approach based not on stepwise construction of approximation but on a new transform called the ridgelet transform. The ridgelet transform represents general functions as integrals of ridgelets – specifically chosen ridge functions.

The significance of approximation by ridge functions is well understood from its role in the theory of neural networks. Ridge functions appear in the definitions of many central neural network models. It is a broad knowledge that neural networks are being successfully applied across an extraordinary range of problem domains, in fields as diverse as finance, medicine, engineering, geology and physics. Generally speaking, neural networks are being introduced anywhere that there are problems of prediction, classification or control. Thus not surprisingly, there is a great interest to this powerful and very popular area of research (see, e.g., [119] and a great deal of references therein). An artificial neural network is a way to perform computations using networks of interconnected computational units vaguely analogous to neurons simulating how our brain solves them. An artificial neuron, which forms the basis for designing neural networks, is a device with $d$ real inputs.
and an output. This output is generally a ridge function of the given inputs. In mathematical terms, a neuron may be described as

\[ y = \sigma(w \cdot x - \theta), \]

where \( x = (x_1, ..., x_d) \in \mathbb{R}^d \) are the input signals, \( w = (w_1, ..., w_d) \in \mathbb{R}^d \) are the synaptic weights, \( \theta \in \mathbb{R} \) is the bias, \( \sigma \) is the activation function and \( y \) is the output signal of the neuron. In a layered neural network the neurons are organized in the form of layers. We have at least two layers: an input and an output layer. The layers between the input and the output layers (if any) are called hidden layers, whose computation nodes are correspondingly called hidden neurons or hidden units. The output signals of the first layer are used as inputs to the second layer, the output signals of the second layer are used as inputs to the third layer, and so on for the rest of the network. Neural networks with this kind of architecture is called a Multilayer Feedforward Perceptron (MLP). This is the most popular model among other neural network models. In this model, a neural network with a single hidden layer and one output represents a function of the form

\[ \sum_{i=1}^{r} c_i \sigma(w_i \cdot x - \theta_i). \]

Here the weights \( w_i \) are vectors in \( \mathbb{R}^d \), the thresholds \( \theta_i \) and the coefficients \( c_i \) are real numbers and the activation function \( \sigma \) is a univariate function. We fix only \( \sigma \) and \( r \). Note that the functions \( \sigma(w_i \cdot x - \theta_i) \) are ridge functions. Thus it is not surprising that some approximation theoretic problems related to neural networks have strong association with the corresponding problems of approximation by ridge functions.

It is clear that in the special case, linear combinations of ridge functions turn into sums of univariate functions. This is also the simplest case. The simplicity of the approximation apparatus itself guarantees its utility in applications where multivariate functions are constant obstacles. In mathematics, this type of approximation has arisen, for example, in connection with the classical functional equations [11], the numerical solution of certain PDE boundary value problems [9], dimension theory [132,133], etc. In computer science, it arises in connection with the efficient storage of data in computer databases (see, e.g., [140]). There is an interesting interconnection between the theory of approximation by univariate functions and problems of equilibrium construction in economics (see [136]).
Linear combinations of ridge functions with fixed directions allow a natural generalization to functions of the form 
\( g(\alpha_1(x_1) + \cdots + \alpha_n(x_n)) \), where \( \alpha_i(x_i), i = 1, \ldots, n \), are real univariate functions. Such a generalization has a strong association with linear superpositions. A linear superposition is a function expressed as the sum

\[
\sum_{i=1}^{r} g_i(h_i(x)), \ x \in X,
\]

where \( X \) is any set (in particular, a subset of \( \mathbb{R}^d \)), \( h_i : X \to \mathbb{R}, i = 1, \ldots, r \), are arbitrarily fixed functions, and \( g_i : \mathbb{R} \to \mathbb{R}, i = 1, \ldots, r \). Note that here we deal with more complicated composition than the composition of a univariate function with the inner product. A starting point in the study of linear superpositions was the well-known superposition theorem of Kolmogorov [83] (see also the paper on Kolmogorov’s works by Tikhomirov [139]). This theorem states that for the unit cube \( I^d, I = [0,1], d \geq 2 \), there exist \( 2d + 1 \) functions \( \{s_q\}_{q=1}^{2d+1} \subset C(I^d) \) of the form

\[
s_q(x_1, \ldots, x_d) = \sum_{p=1}^{d} \varphi_{pq}(x_p), \ \varphi_{pq} \in C(I), \ p = 1, \ldots, d, \ q = 1, \ldots, 2d + 1
\]

such that each function \( f \in C(I^d) \) admits the representation

\[
f(x) = \sum_{q=1}^{2d+1} g_q(s_q(x)), \ x = (x_1, \ldots, x_d) \in I^d, \ g_q \in C(\mathbb{R}).
\]

Thus, any continuous function on the unit cube can be represented as a linear superposition with the fixed inner functions \( s_1, \ldots, s_{2d+1} \). In literature, these functions are called universal functions or the Kolmogorov functions. Note that all the functions \( g_q(s_q(x)) \) in the Kolmogorov superposition formula are general ridge functions, since each \( s_q \) is a sum of univariate functions.

In these notes, we consider some problems of approximation of multivariate functions by linear combinations of ridge functions with fixed directions, general ridge functions and feedforward neural networks. The notes consist of four chapters.

Chapter 1 is devoted to the approximation from some sets of ridge functions with arbitrarily fixed directions in \( C \) and \( L_2 \) metrics. First, we study problems of representation of multivariate functions by linear combinations
of ridge functions. Then, in case of two fixed directions and under suitable conditions, we give complete solutions to three basic problems of uniform approximation, namely, problems on existence, characterization, and construction of a best approximation. We also study problems of well approximation (approximation with arbitrarily small degree of accuracy) and representation of continuous multivariate functions by sums of two continuous ridge functions. The reader will see the main difficulties and remained open problems in the uniform approximation by sums of more than two ridge functions. For $L_2$ approximation, a number of summands does not play such an essential role as it plays in the uniform approximation. In this case, it is known that a best approximation always exists and unique. For some special domains in $\mathbb{R}^d$, we characterize and then construct the best approximation. We also give an explicit formula for the approximation error.

Chapter 2 is devoted to the simplest type of ridge functions – univariate functions. Note that a ridge function depends only on one variable if its direction coincides with the coordinate direction. Thus, in case of coincidence of all given directions with the coordinate directions, the problem of ridge function approximation turns into the problem of approximation of multivariate functions by sums of univariate functions. In this chapter, we first consider the approximation of a bivariate function $f(x, y)$ by sums $\varphi(x) + \psi(y)$ on a rectangular domain $R$. We construct special classes of continuous functions depending on a numerical parameter and characterize each class in terms of the approximation error calculation formulas. This parameter will show which points of $R$ the calculation formula involves. We will also construct a best approximating sum $\varphi_0(x) + \psi_0(y)$ to a function from constructed classes. Then we develop a method for obtaining explicit formulas for the error of approximation of bivariate functions, defined on a union of rectangles, by sums of univariate functions. It should be remarked that formulas of such type were known only for functions defined on a rectangle with sides parallel to the coordinate axes. Our method, based on a maximization process over certain objects, called “closed bolts”, allows the consideration of functions defined on hexagons, octagons and stairlike polygons with sides parallel to the coordinate axes. At the end of this chapter we discuss one important result from Golomb’s paper [37]. This paper, published in 1959, made a start of a systematic study of approximation of multivariate functions by various compositions, including sums of univariate functions. In [37], along with many other results, Golomb obtained a duality formula for the error of approximation to a multivariate function from the set of sums of univariate functions.
Unfortunately, his proof had a gap, which was 24 years later pointed out by Marshall and O’Farrell [107]. But the question if Golomb’s formula was correct, remained unsolved. In Chapter 2, we show that Golomb’s formula is correct, and moreover it holds in a stronger form.

Chapter 3 tells us about some problems concerning general ridge functions $g(\alpha_1(x_1) + \cdots + \alpha_n(x_n))$ and linear superpositions. We consider the problem of representation of general functions by linear superpositions. We show that if some representation by linear superpositions, in particular by linear combinations of general ridge functions, holds for continuous functions, then it holds for all functions. This leads us to extensions of many superpositions theorems (such as the well-known Kolmogorov superposition theorem, Ostrand’s superposition theorem, etc.) from continuous to arbitrarily behaved multivariate functions. Concerning general ridge functions, we see that every multivariate function can be written as a general ridge function or as a sum of finitely many such functions.

Chapter 4 is about neural network approximation. We consider a single and two hidden layer feedforward neural network models with a restricted set of weights. Such network models are important from the point of view of practical applications. We study approximation properties of single hidden layer neural networks with weights varying on a finite set of directions and straight lines. We give several necessary and sufficient conditions for well approximation by such networks. For a set of weights consisting of two directions (and two straight lines), we show that there is a geometrically explicit solution to the problem. We also obtain a sufficient condition for a such network to be a best approximation (or extremal network). Regarding two hidden layer feedforward neural networks, we prove that two hidden layer neural networks with $d$ inputs, $d$ neurons in the first hidden layer, $2d + 2$ neurons in the second hidden layer and with a specifically constructed sigmoidal and infinitely differentiable activation function can approximate any continuous multivariate function with arbitrary precision. We see that for such approximation a finite number of fixed weights (precisely, $d$ fixed weights) are sufficient.

There are topics related to ridge functions that are not presented here. The glaring omission is that of smoothness in ridge function representation, i.e., smoothness of each ridge function component if a finite linear combination of them is smooth. See [12,117] for a precise exposition of the problem and [1,2,86-88,120] for the corresponding results. We also do not address, for example, problems of interpolation on straight lines by ridge functions, linear
independence and spanning by linear combinations of ridge monomials in the spaces of homogeneous and algebraic polynomials of a fixed degree, integral representations of functions where the kernel is a ridge function. These and similar topics may be found in the monograph by Pinkus [117].
Chapter 1

Properties of linear combinations of ridge functions

In this chapter, we consider approximation-theoretic problems arising in ridge function approximation. First we review some results on approximation by sums of ridge functions with both fixed and variable directions. Then we analyze the problem of representability of an arbitrary multivariate function by linear combinations of ridge functions with fixed directions. In the special case of two fixed directions, we characterize a best uniform approximation from the set of sums of ridge functions with these directions. For a class of bivariate functions we use this result to construct explicitly a best approximation. Questions on existence of a best approximation are also studied. We also study problems of well approximation (approximation with any degree of accuracy) and representation of continuous multivariate functions by sums of two continuous ridge functions. The reader will see the main difficulties and remained open problems in the uniform approximation by sums of more than two ridge functions. For $L_2$ approximation, a number of summands does not play such an essential role as it plays in the uniform approximation. In this case, it is known that a best approximation is always exists and unique. For some special domains in $\mathbb{R}^n$, we characterize and then construct the best approximation. We also give an explicit formula for the approximation error.
1.1 Review of some results on ridge functions

In this section we review some known results on approximation from the sets \( \mathcal{R}(a^1, \ldots, a^r) \) and \( \mathcal{R}_r \).

1.1.1 \( \mathcal{R}(a^1, \ldots, a^r) \) – ridge functions with fixed directions

It is clear that well approximation (approximation with any degree of accuracy) of a given multivariate function \( f : X \to \mathbb{R} \) from some normed space by using elements of the set \( \mathcal{R}(a^1, \ldots, a^r) \) is not always possible. The value of the approximation error depends not only on the approximated function \( f \) but also on geometrical structure of the given set \( X \). For example, if \( X \) has an interior point, then the error of approximation cannot equal to zero for many functions \( f \notin \mathcal{R}(a^1, \ldots, a^r) \) (see [95]). This fact gives rise to some problems on approximate or exact computations of the approximation error and algorithms for constructing best approximating ridge sums.

Serious difficulties arise when such problems are considered in continuous function spaces endowed with the uniform norm. For example, consider the algorithm for constructing a best approximation, called the Diliberto-Straus algorithm. The essence of this algorithm is as follows. Let \( X \) be a compact subset of \( \mathbb{R}^d \) and \( A_i \) be a best approximation operator from the space of continuous functions \( C(X) \) to the subspace of ridge functions

\[ G_i = \{ g_i(a^i \cdot x) : g_i \in C(\mathbb{R}), x \in X \}, \quad i = 1, \ldots, r. \]

That is, for each function \( f \in C(X) \), the function \( A_i f \) is a best approximation to \( f \) from \( G_i \). Set

\[ Tf = (I - A_r)(I - A_{r-1}) \cdots (I - A_1)f, \]

where \( I \) is the identity operator. It is clear that

\[ Tf = f - g_1 - g_2 - \cdots - g_r, \]

where \( g_k \) is a best approximation from \( G_k \) to the function \( f - g_1 - g_2 - \cdots - g_{k-1}, \ k = 1, \ldots, r \). Consider powers of the operator \( T \): \( T^2, T^3 \) and so on. Is the sequence \( \{T^n f\}_{n=1}^\infty \) convergent? In case of an affirmative answer, which function is the limit of \( T^n f \), as \( n \to \infty \)? One may expect that the sequence \( \{T^n f\}_{n=1}^\infty \) converges to \( f - g^* \), where \( g^* \) is a best approximation from \( \mathcal{R}(a^1, \ldots, a^r) \) to \( f \). This conjecture was first formulated by Diliberto and Straus [26] in 1951 for the uniform approximation of a multivariate function,
defined on the unit cube, by sums of univariate functions (that is, sums of ridge functions with the coordinate directions). But later it was shown by Aumann [4] that the algorithm may not converge for the cases \( r > 2 \). For \( r = 2 \), the sequence \( \{T^n f\}_{n=1}^\infty \) converges to \( f - g_0 \) over certain convex compact sets \( X \), where \( g_0 \) is a best approximation from \( R(\mathbf{a}^1, \mathbf{a}^2) \) (see [61,117]). In general case, when \( r > 2 \) no efficient algorithm is known for a best uniform approximation from the space \( R(\mathbf{a}^1, ..., \mathbf{a}^r) \). Note that in \( L_2 \) metric, the Diliberto-Straus algorithm converges to the desired function for an arbitrary number of distinct directions (see [118]).

One of the basic problems concerning the approximation by sums of ridge functions with fixed directions is the problem of verifying if a given function \( f \) belongs to the space \( R(\mathbf{a}^1, ..., \mathbf{a}^r) \). This problem has a simple solution if the space dimension \( d = 2 \) and a given function \( f(x, y) \) has partial derivatives up to \( r \)-th order. For the representation of \( f(x, y) \) in the following form

\[
f(x, y) = \sum_{i=1}^{r} g_i(a_i x + b_i y),
\]

it is necessary and sufficient that

\[
\prod_{i=1}^{r} \left( b_i \frac{\partial}{\partial x} - a_i \frac{\partial}{\partial y} \right) f = 0. \tag{1.1}
\]

This recipe is valid for all continuous bivariate functions provided that the derivatives are understood in the generalized sense.

Unfortunately, such simple verification does not carry over to the representation \( f(\mathbf{x}) = \sum_{i=1}^{r} g_i (\mathbf{a}^i \cdot \mathbf{x}) \), \( \mathbf{x} = (x_1, ..., x_d) \), if the dimension \( d > 2 \). Below we formulate two results on the representation of a multivariate function by a sum of ridge functions with fixed directions.

**Proposition 1.1** (Diaconis, Shahshahani [25]). Let \( \mathbf{a}^1, ..., \mathbf{a}^r \) be pairwise linearly independent vectors in \( \mathbb{R}^d \). Let for \( i = 1, 2, ..., r \), \( H^i \) denote the hyperplane \( \{ \mathbf{c} \in \mathbb{R}^d : \mathbf{c} \cdot \mathbf{a}^i = 0 \} \). Then a function \( f \in C^r(\mathbb{R}^d) \) can be represented as

\[
f(\mathbf{x}) = \sum_{i=1}^{r} g_i (\mathbf{a}^i \cdot \mathbf{x}) + P(\mathbf{x}),
\]
where $P(x)$ is a polynomial of degree not more than $r$, if and only if

$$\prod_{i=1}^{r} \sum_{s=1}^{d} c^i_s \frac{\partial f}{\partial x_s} = 0,$$

for all vectors $c^i = (c^i_1, c^i_2, \ldots, c^i_d) \in H^i, i = 1, 2, \ldots, r$.

There are examples showing that one cannot simply dispense with the polynomial $P(x)$ in the above proposition (see [25]). In fact, the polynomial term is necessary in the proof of the sufficient part of the proposition.

Lin and Pinkus [95] obtained more general result on the representation by ridge functions. We need some notation to present their theorem. Each polynomial $p(x_1, \ldots, x_d)$ generates the differential operator $p\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)$. Let $P(a^1, \ldots, a^r)$ denote the set of polynomials which vanish on all the lines \{λ$a^i$, $\lambda \in \mathbb{R}$, $i = 1, \ldots, r$. Obviously, this is an ideal in the ring of all polynomials. Let $Q$ be the set of polynomials $q = q(x_1, \ldots, x_d)$ such that $p\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right) q = 0$, for all $p(x_1, \ldots, x_d) \in P(a^1, \ldots, a^r)$.

**Proposition 1.2** (Lin, Pinkus [95]). Let $a^1, \ldots, a^r$ be pairwise linearly independent vectors in $\mathbb{R}^d$. A function $f \in C(\mathbb{R}^d)$ can be expressed in the form

$$f(x) = \sum_{i=1}^{r} g_i(a^i \cdot x),$$

if and only if $f$ belongs to the closure of the linear span of $Q$.

In [120], A.Pinkus considers the problems of smoothness and uniqueness in ridge function representation. For a given function $f \in \mathcal{R}(a^1, \ldots, a^r)$, he poses and answers the following questions. If $f$ belongs to some smoothness class, what can we say about the smoothness of the functions $g_i$? How many different ways can we write $f$ as a linear combination of ridge functions?

The above problem of representation of a fixed multivariate function by ridge functions gives rise to the problem of representation of some classes of functions by such sums. For example, one may consider the following problem. Let $X$ be a subset of the $d-$dimensional Euclidean space. Let $C(X)$, $B(X)$, $T(X)$ denote the set of continuous, bounded and all real functions defined on $X$ correspondingly. In the first case, we additionally suppose that $X$ is a compact set. Let $\mathcal{R}_c(a^1, \ldots, a^r)$ and $\mathcal{R}_b(a^1, \ldots, a^r)$ denote the subspaces of $\mathcal{R}(a^1, \ldots, a^r)$ comprising only sums with continuous and bounded
terms \( g_i(a^i \cdot x) \), \( i = 1, \ldots, r \), correspondingly. The following questions naturally arise: For which sets \( X \), one can claim that \( C(X) = R_c(a^1, \ldots, a^r) \), \( B(X) = R_b(a^1, \ldots, a^r) \), and \( T(X) = R(a^1, \ldots, a^r) \)? The first two problems in a more general setting were solved in Sternfeld [131,132] (see also Chapter 3). The third problem will be solved in the next section. Let us briefly discuss some results of Sternfeld in compliance with the case of representation by sums of ridge functions. Let we are given directions \( a^1, \ldots, a^r \in \mathbb{R}^d \setminus \{0\} \) and a set \( X \subseteq \mathbb{R}^d \). A family \( F = \{a^1, \ldots, a^r\} \) uniformly separates points of \( X \) if there exists a number \( 0 < \lambda \leq 1 \) such that for each pair \( \{x_j\}_{j=1}^m \), \( \{z_j\}_{j=1}^m \) of disjoint finite sequences in \( X \), there exists some direction \( a^k \in F \) so that if from the two sequences \( \{a^k \cdot x_j\}_{j=1}^m \) and \( \{a^k \cdot z_j\}_{j=1}^m \) we remove a maximal number of pairs of points \( a^k \cdot x_j \) and \( a^k \cdot z_j \) with \( a^k \cdot x_j = a^k \cdot z_j \), then there remains at least \( \lambda m \) points in each sequence (or, equivalently, at most \( (1 - \lambda)m \) pairs can be removed). Sternfeld [132], in particular, proved that a family of directions \( F = \{a^1, \ldots, a^r\} \) uniformly separates points of \( X \) if and only if \( R_b(a^1, \ldots, a^r) = B(X) \). In [132], he also obtained a practically convenient sufficient condition for the equality \( R_b(a^1, \ldots, a^r) = B(X) \). To describe this condition, define the set functions
\[
\tau_i(Z) = \{x \in Z : |p_i^{-1}(p_i(x)) \cap Z| \geq 2\},
\]
where \( Z \subset X \), \( p_i(x) = a^i \cdot x \), \( i = 1, \ldots, r \), and \( |Y| \) denotes the cardinality of a considered set \( Y \). Define \( \tau(Z) \) to be \( \bigcap_{i=1}^r \tau_i(Z) \) and define \( \tau^2(Z) = \tau(\tau(Z)) \), \( \tau^3(Z) = \tau(\tau^2(Z)) \) and so on inductively.

**Proposition 1.3** (Sternfeld [132]). If \( \tau^n(X) = \emptyset \) for some \( n \), then \( R_b(a^1, \ldots, a^r) = B(X) \). If \( X \) is a compact subset of \( \mathbb{R}^d \), and \( \tau^n(X) = \emptyset \) for some \( n \), then \( R_c(a^1, \ldots, a^r) = C(X) \).

The sufficient condition “\( \tau^n(X) = \emptyset \) for some \( n \)” turns out to be also necessary for the case \( r = 2 \). In this case the equality \( R_b(a^1, a^2) = B(X) \) is equivalent to the equality \( R_c(a^1, a^2) = C(X) \). In another work [131], Sternfeld obtained a measure-theoretic necessary and sufficient condition for the equality \( R_c(a^1, \ldots, a^r) = C(X) \). Let \( p_i(x) = a^i \cdot x \), \( i = 1, \ldots, r \), \( X \) be a compact set in \( \mathbb{R}^d \) and \( M(X) \) be a class of measures defined on some field of subsets of \( X \). The family \( F = \{a^1, \ldots, a^r\} \) uniformly separates measures of the class \( M(X) \) if there exists a number \( 0 < \lambda \leq 1 \) such that for each measure \( \mu \) in \( M(X) \) the equality \( \|\mu \circ p_k^{-1}\| \geq \lambda \|\mu\| \) holds for some direction \( a^k \in F \). Sternfeld [131,134], in particular, proved that the equality
\( \mathcal{R}_c(\mathbf{a}^1, \ldots, \mathbf{a}^r) = C(X) \) holds if and only if the family of directions \( \{\mathbf{a}^1, \ldots, \mathbf{a}^r\} \) uniformly separates measures of the class \( C(X)^* \) (that is, the class of regular Borel measures). Besides, he proved that \( \mathcal{R}_b(\mathbf{a}^1, \ldots, \mathbf{a}^r) = B(X) \) if and only if the family of directions \( \{\mathbf{a}^1, \ldots, \mathbf{a}^r\} \) uniformly separates measures of the class \( l_1(X) \) (that is, the class of finite measures defined on countable subsets of \( X \)). Since \( l_1(X) \subset C(X)^* \), the first equality \( \mathcal{R}_c(\mathbf{a}^1, \ldots, \mathbf{a}^r) = C(X) \) implies the second equality \( \mathcal{R}_b(\mathbf{a}^1, \ldots, \mathbf{a}^r) = B(X) \). The inverse is not true (see [131]).

It should be remarked that the above results of Sternfeld were obtained for more general functions, than linear combinations of ridge functions, namely for functions of the form \( \sum_{i=1}^r g_i(h_i(x)) \), where \( h_i \) arbitrarily fixed functions (bounded or continuous) defined on \( X \).

### 1.1.2 \( \mathcal{R}_r \) – ridge functions with variable directions

Obviously, the set \( \mathcal{R}_c(\mathbf{a}^1, \ldots, \mathbf{a}^r) \) is not dense in \( C(\mathbb{R}^d) \) in the topology of uniform convergence on compact subsets of \( \mathbb{R}^d \). Density here does not hold because the number of considered directions is finite. If consider all the possible directions, then the set \( \mathcal{R} = \text{span}\{g(\mathbf{a} \cdot \mathbf{x}) : g \in C(\mathbb{R}), \mathbf{a} \in \mathbb{R}^d \setminus \{0\}\} \) will be certainly dense in the space \( C(\mathbb{R}^d) \) in the above mentioned topology.

In order to be sure, it is enough to consider only the functions \( e^{\mathbf{a} \cdot \mathbf{x}} \in \mathcal{R} \), the linear span of which is dense in \( C(\mathbb{R}^d) \) by the Stone-Weierstrass theorem. In fact, for density it is not necessary to comprise all directions. The following theorem shows how many directions in totality satisfy the density requirements.

**Proposition 1.4** (Vostrecov and Kreines [142], Lin and Pinkus [95]).

For density of the set

\[
\mathcal{R}(\mathcal{A}) = \text{span}\{g(\mathbf{a} \cdot \mathbf{x}) : g \in C(\mathbb{R}), \mathbf{a} \in \mathcal{A} \subset \mathbb{R}^d\}
\]

in \( C(\mathbb{R}^d) \) (in the topology of uniform convergence on all compacta) it is necessary and sufficient that the only homogeneous polynomial which vanishes identically on \( \mathcal{A} \) is the zero polynomial.

Since in the definition of \( \mathcal{R}(\mathcal{A}) \) we vary over all univariate functions \( g \), allowing one direction \( \mathbf{a} \) is equivalent to allowing all directions \( k\mathbf{a} \) for every real \( k \). Thus it is sufficient to consider only the set \( \mathcal{A} \) of directions normalized to the unit sphere \( S^{n-1} \). For example, if \( \mathcal{A} \) is a subset of the sphere \( S^{n-1} \),
which contains an interior point (interior point with respect to the induced topology on $S^{n-1}$), then $\mathcal{R}(A)$ is dense in the space $C(\mathbb{R}^d)$. The proof of Proposition 1.4 highlights an important fact that the set $\mathcal{R}(A)$ is dense in $C(\mathbb{R}^d)$ in the topology of uniform convergence on compact subsets if and only if $\mathcal{R}(A)$ contains all the polynomials (see [95]).

Representability of polynomials by sums of ridge functions is a building block for many results. In many works (see, e.g., [119]), the following fact is fundamental: Every multivariate polynomial $h(x) = h(x_1, ..., x_d)$ of degree $k$ can be represented in the form

$$h(x) = \sum_{i=1}^{l} p_i(a^i \cdot x),$$

where $p_i$ is a univariate polynomial, $a^i \in \mathbb{R}^d$, and $l = \binom{d-1+k}{k}$. For example, for the representation of a bivariate polynomial of degree $k$, it is needed $k + 1$ univariate polynomials and $k + 1$ directions (see [97]). The proof of this fact is organized so that the directions $a^i$, $i = 1, ..., k + 1$, are chosen once for all multivariate polynomials of $k$-th degree. At one of the seminars in the Technion – Israel Institute of Technology in 2007, A. Pinkus posed two problems:

1) Can every multivariate polynomial of degree $k$ be represented by less than $l$ ridge functions?

2) How large is the set of polynomials represented by $l - 1$, $l - 2$, ... ridge functions?

Note that for bivariate polynomials the 1-st problem is solved positively, that is, the number $l = k + 1$ can be reduced. Indeed, for a bivariate polynomial $P(x, y)$ of $k$-th degree, there exists a large set of real numbers $c_0, ..., c_k$ such that

$$\sum_{i=0}^{k} c_i \frac{\partial^k}{\partial x^i \partial y^{k-i}} P(x, y) = 0.$$

Further the numbers $c_i$, $i = 0, ..., k$, can be selected to enjoy the property that the polynomial $\sum_{i=0}^{k} c_i t^i$ has distinct real zeros. Then it is not difficult to verify that the differential operator $\sum_{i=0}^{k} c_i \frac{\partial^k}{\partial x^i \partial y^{k-i}}$ can be written in the form

$$\prod_{i=1}^{k} \left( b_i \frac{\partial}{\partial x} - a_i \frac{\partial}{\partial y} \right),$$

Electronic copy available at: https://ssrn.com/abstract=3618165
for some pairwise linearly independent vectors \((a_i, b_i)\), \(i = 1, ..., k\). Now from the above criterion (1.1) we obtain that the polynomial \(P(x, y)\) can be represented as a sum of \(k\) ridge functions. Note that the problem of representation of a multivariate algebraic polynomial \(P(x)\) in the form \(\sum_{i=0}^{r} g_i(a^i \cdot x)\) with minimal \(r\) was extensively studied in the monograph of Pinkus [117].

In connection with the 2-nd problem of Pinkus, V. Maiorov [103] studied certain geometrical properties of the manifold \(R_r\). Namely, he estimated the \(\varepsilon\)-entropy numbers in terms of smaller \(\varepsilon\)-covering numbers of the compact class formed by the intersection of the class \(R_r\) with the unit ball in the space of polynomials of degree at most \(s\) on \(\mathbb{R}^d\). Let \(E\) be a Banach space and let for \(x \in E\) and \(\delta > 0\), \(S(x, \delta)\) denote the ball of radius \(\delta\) centered at the point \(x\). For any positive number \(\varepsilon\), the \(\varepsilon\)-covering number of a set \(F\) in the space \(E\) represents the quantity

\[
L_{\varepsilon}(F, E) = \min \left\{ N : \exists x_1, ..., x_N \in F \text{ such that } F \subset \bigcup_{i=1}^{N} S(x_i, \varepsilon) \right\}
\]

The \(\varepsilon\)-entropy of \(F\) is defined as the number \(H_{\varepsilon}(F, E) \overset{d.e.f.}{=} \log_2 L_{\varepsilon}(F, E)\). The notion of \(\varepsilon\)-entropy has been devised by A.N.Kolmogorov (see [84,85]) to classify compact metric sets according to their massivity.

In order to formulate Maiorov's result, let \(P^d_s\) be the space of all polynomials of degree at most \(s\) on \(\mathbb{R}^d\), \(L_q = L_q(I)\), \(1 \leq q \leq \infty\), be the space of \(q\)-integrable functions on the unit cube \(I = [0, 1]^d\) with the norm \(\|f\|_q = \left(\int_I |f(x)|^q dx\right)^{1/q}\), \(BL_q\) be the unit ball in the space \(L_q\), and \(B_q P^d_s = BL_q \cap P^d_s\) be the unit ball in the space \(P^d_s\) equipped with the \(L_q\) metric.

**Proposition 1.5** (Maiorov [103]). Let \(r, s \in \mathbb{N}\), \(1 \leq q \leq \infty\), \(0 < \varepsilon < 1\). The \(\varepsilon\)-entropy of the class \(B_q P^d_s \cap R_r\), in the space \(L_q\) satisfies the inequalities

1) \[
c_1 rs \leq \frac{H_{\varepsilon}(B_q P^d_s \cap R_r, L_q)}{\log_2 \frac{1}{\varepsilon}} \leq c_2 rs \log_2 \frac{2 \varepsilon s^{d-1}}{r},
\]

for \(r \leq s^{d-1}\).

2) \[
c'_1 s^d \leq \frac{H_{\varepsilon}(B_q P^d_s \cap R_r, L_q)}{\log_2 \frac{1}{\varepsilon}} \leq c'_2 s^d,
\]

for \(r > s^{d-1}\). In these inequalities \(c_1, c_2, c'_1, c'_2\) are constants depending only on \(d\).
Let us consider $\mathcal{R}_r$ as a subspace of some normed linear space $X$ endowed with the norm $\|\cdot\|_X$. The error of approximation of a given function $f \in X$ by functions $g \in \mathcal{R}_r$ is defined as follows

$$E(f, \mathcal{R}_r, X) \overset{\text{def}}{=} \inf_{g \in \mathcal{R}_r} \|f - g\|_X.$$  

Let $B^d$ denote the unit ball in the space $\mathbb{R}^d$. Besides, let $\mathbb{Z}^d_+$ denote the lattice of nonnegative multi-integers in $\mathbb{R}^d$. For $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d_+$, set $|k| = k_1 + \cdots + k_d$, $x^k = x_1^{k_1} \cdots x_d^{k_d}$ and

$$D^k = \frac{\partial^{|k|}}{\partial^{k_1}x_1 \cdots \partial^{k_d}x_d}.$$  

The Sobolev space $W^m_p(B^d)$ is the space of functions defined on $B^d$ with the norm

$$\|f\|_{m,p} = \left\{ \begin{array}{ll}
\left( \sum_{0 \leq |k| \leq m} \|D^k f\|_p^p \right)^{1/p}, & \text{if } 1 \leq p < \infty \\
\max_{0 \leq |k| \leq m} \|D^k f\|_\infty, & \text{if } p = \infty.
\end{array} \right.$$  

Here

$$\|h(x)\|_p = \left\{ \begin{array}{ll}
\left( \int_{B^n} |h(x)|^p \, dx \right)^{1/p}, & \text{if } 1 \leq p < \infty \\
\text{ess sup}_{x \in B^d} |h(x)|, & \text{if } p = \infty.
\end{array} \right.$$  

Let $S^m_p(B^d)$ be the unit ball in $W^m_p(B^d)$:

$$S^m_p(B^d) = \{ f \in W^m_p(B^d) : \|f\|_{m,p} \leq 1 \}.$$  

In 1999, Maiorov [102] proved the following result

**Proposition 1.6** (Maiorov [102]). Assume $m \geq 1$ and $d \geq 2$. Then for each $r \in \mathbb{N}$ there exists a function $f \in S^m_2(B^d)$ such that

$$E(f, \mathcal{R}_r, L^2) \geq Cr^{-m/(d-1)}, \quad (1.2)$$

where $C$ is a constant independent of $f$ and $r$.

For $d = 2$, this inequality was proved by Oskolkov [114]. In [102], Maiorov also proved that for each function $f \in S^m_2(B^d)$

$$E(f, \mathcal{R}_r, L^2) \leq Cr^{-m/(d-1)}. \quad (1.3)$$
Thus he established the following order for the error of approximation to functions in $S^m_2(B^d)$ from the class $R_r$:

$$E(S^m_2(B^d), R_r, L_2) \overset{def}{=} \sup_{f \in S^m_2(B^d)} E(f, R_r, L_2) \asymp r^{-m/(d-1)}.$$ 

Pinkus [119] revealed that the upper bound (1.3) is also valid in $L_p$ metric ($1 \leq p \leq \infty$). In other words, for every function $f \in S^m_2(B^d)$

$$E(f, R_r, L_p) \leq C r^{-m/(d-1)}.$$

These inequalities were successfully applied to some problems of approximation of multivariate functions by neural networks with a single hidden layer. Recall that such networks are given by the formula $\sum_{i=1}^r c_i \sigma(w^i \cdot x - \theta_i)$. By $M_r(\sigma)$ let us denote the set of all single hidden layer networks with the activation function $\sigma$. That is,

$$M_r(\sigma) = \left\{ \sum_{i=1}^r c_i \sigma(w^i \cdot x - \theta_i) : \ c_i, \theta_i \in \mathbb{R}, \ w^i \in \mathbb{R}^d \right\}.$$

The above results on ridge approximation from $R_r$ enable us to estimate the rate with which the approximation error $E(f, M_r(\sigma), L_2)$ tends to zero. First note that $M_r(\sigma) \subset R_r$, since each function of the form $\sigma(w \cdot x - \theta)$ is a ridge function with the direction $w$. Thus the lower bound (1.2) holds also for the set $M_r(\sigma)$: there exists a function $f \in S^m_2(B^d)$, for which

$$E(f, M_r(\sigma), L_2) \geq C r^{-m/(d-1)}.$$

It remains to see whether the upper bound (1.3) is valid for $M_r(\sigma)$. Clearly, it cannot be valid if $\sigma$ is an arbitrary continuous function. Here we are dealing with the question if there exists a function $\sigma^* \in C(\mathbb{R})$, for which

$$E(f, M_r(\sigma^*), L_2) \leq C r^{-m/(d-1)}.$$

This question is answered affirmatively by the following result.

**Proposition 1.7** (Maiorov, Pinkus [99]). There exists a function $\sigma^* \in C(\mathbb{R})$ with the following properties

1) $\sigma^*$ is infinitely differentiable and strictly increasing;
2) $\lim_{t \to \infty} \sigma(t) = 1$ and $\lim_{t \to -\infty} \sigma(t) = 0$;
3) for every $g \in \mathcal{R}_r$ and $\varepsilon > 0$ there exist $c_i, \theta_i \in \mathbb{R}$ and $w^i \in \mathbb{R}^d$ satisfying

$$\sup_{x \in B^d} \left| g(x) - \sum_{i=1}^{r+d+1} c_i \sigma(w^i \cdot x - \theta_i) \right| < \varepsilon.$$ 

Temlyakov [138] considered the approximation from some certain subclass of $\mathcal{R}_r$ in $L_2$ metric. More precisely, he considered the approximation of a function $f \in L_2(D)$, where $D$ is the unit disk in $\mathbb{R}^2$, by functions $\sum_{i=1}^r g_i(a^i \cdot x) \in \mathcal{R}_r \cap L_2(D)$, which satisfy the additional condition $\|g_i(a^i \cdot x)\|_2 \leq B \|f\|_2, i = 1, ..., r$ ($B$ is a given positive number). Let $\sigma^B_r(f)$ be the error of this approximation. For this approximation error, the author of [138] obtained upper and lower bounds. Let for $\alpha > 0$, $H^\alpha(D)$ denote the set of all functions $f \in L^2(D)$, which can be represented in the form

$$f = \sum_{n=1}^\infty P_n,$$

where $P_n$ are bivariate algebraic polynomials of total degree $2^n - 1$ satisfying the inequalities

$$\|P_n\|_2 \leq 2^{-\alpha n}, n = 1, 2, ...$$

**Proposition 1.8** (Temlyakov [138]). 1) For every $f \in H^\alpha(D)$, we have

$$\sigma^1_r(f) \leq C(\alpha)r^{-\alpha}.$$

2) For any given $\alpha > 0$, $B > 0$, $r > 1$, there exists a function $f \in H^\alpha(D)$ such that

$$\sigma^B_r(f) \geq C(\alpha, B)(r \ln r)^{-\alpha}.$$

Petrushev [116] proved the following interesting result: Let $X_k$ be the $k$ dimensional linear space of univariate functions in $L_2[-1, 1]$, $k = 1, 2, ...$. Besides, let $B^d$ and $S^{d-1}$ denote correspondingly the unit ball and unit sphere in the space $\mathbb{R}^d$. If $X_k$ provides order of approximation $O(k^{-m})$ for univariate functions with $m$ derivatives in $L_2[-1, 1]$ and $\Omega_k$ are appropriately chosen finite sets of directions distributed on $S^{d-1}$, then the space $Y_k = \text{span}\{p_k(a \cdot x) : p_k \in X_k, a \in \Omega_k\}$ will provide approximation of order $O(k^{-m-d/2+1/2})$. 

Electronic copy available at: https://ssrn.com/abstract=3618165
for every function $f \in L_2(B^d)$ with smoothness of order $m + d/2 - 1/2$. Thus, Petrushev showed that the above form of ridge approximation has the same efficiency of approximation as the traditional multivariate polynomial approximation.

Many other results concerning the approximation of multivariate functions by functions from the set $\mathcal{R}_r$ and their applications in neural network theory may be found in [43,90,94,99,119,123,125].

1.2 Representation of multivariate functions by linear combinations of ridge functions

1.2.1 Two representation problems

Let $X$ be a subset of $\mathbb{R}^d$ and $\{a^i\}_{i=1}^r$ be arbitrarily fixed nonzero directions (vectors) in $\mathbb{R}^d$. Consider the following set of linear combinations of ridge functions.

$$\mathcal{R}(a^1, \ldots, a^r; X) = \left\{ \sum_{i=1}^r g_i(a^i \cdot x), \ x \in X, \ g_i : \mathbb{R} \to \mathbb{R}, \ i = 1, \ldots, r \right\}$$

In this section, we are going to deal with the following two problems:

**Problem 1.** What conditions imposed on $f : X \to \mathbb{R}$ are necessary and sufficient for the inclusion $f \in \mathcal{R}(a^1, \ldots, a^r; X)$?

**Problem 2.** What conditions imposed on $X$ are necessary and sufficient that every function defined on $X$ belongs to the space $\mathcal{R}(a^1, \ldots, a^r; X)$?

As noticed in Section 1.1, Problem 1 was considered for continuous functions in [95] and a theoretical result was obtained. It was also noticed there that the similar problem of representation of $f$ in the form $\sum_{i=1}^r g_i(a^i \cdot x) + P(x)$ with polynomial $P(x)$ was solved for continuously differentiable functions in [25]. Problem 2 was solved in [10] for finite subsets $X$ of $\mathbb{R}^d$ and in [81] for the case when $r = d$ and $a^i$ are the coordinate directions.

Here we consider both Problem 1 and Problem 2 without imposing on $X$, $f$ and $r$ any conditions. In fact, we solve these problems for more general,
than \( \mathcal{R}(a^1, ..., a^r; X) \), set of functions. Namely, we solve them for the set

\[
\mathcal{B}(X) = \mathcal{B}(h_1, ..., h_r; X) = \left\{ \sum_{i=1}^{r} g_i(h_i(x)), \ x \in X, \ g_i : \mathbb{R} \to \mathbb{R}, \ i = 1, ..., r \right\},
\]

where \( h_i : X \to \mathbb{R}, \ i = 1, ..., r \), be arbitrarily fixed functions. In particular, the functions \( h_i, \ i = 1, ..., r \), may be equal to scalar products of the variable \( x \) with some vectors \( a^i, \ i = 1, ..., r \). Only in this special case, we have \( \mathcal{B}(h_1, ..., h_r; X) = \mathcal{R}(a^1, ..., a^r; X) \).

### 1.2.2 Cycles

The main idea leading to solutions of the above problems is in using new objects called **cycles** with respect to \( r \) functions \( h_i : X \to \mathbb{R}, \ i = 1, ..., r \) (and in particular, with respect to \( r \) directions \( a^1, ..., a^r \)). In the sequel, by \( \delta_A \) we will denote the characteristic function of a set \( A \subset \mathbb{R} \). That is,

\[
\delta_A(y) = \begin{cases} 
1, & \text{if } y \in A \\
0, & \text{if } y \notin A.
\end{cases}
\]

**Definition 1.1.** Given a subset \( X \subset \mathbb{R}^d \) and functions \( h_i : X \to \mathbb{R}, \ i = 1, ..., r \). A set of points \( \{x_1, ..., x_n\} \subset X \) is called a cycle with respect to the functions \( h_1, ..., h_r \) (or, concisely, a cycle if there is no confusion), if there exists a vector \( \lambda = (\lambda_1, ..., \lambda_n) \) with the nonzero real coordinates \( \lambda_i, \ i = 1, ..., n \), such that

\[
\sum_{j=1}^{n} \lambda_j \delta_{h_i(x_j)} = 0, \ i = 1, ..., r. \tag{1.4}
\]

If \( h_i = a^i \cdot x, \ i = 1, ..., r \), where \( a^1, ..., a^r \) are some directions in \( \mathbb{R}^d \), a cycle, with respect to the functions \( h_1, ..., h_r \), is called a cycle with respect to the directions \( a^1, ..., a^r \).

Let for \( i = 1, ..., r \), the set \( \{h_i(x_j), \ j = 1, ..., n\} \) have \( k_i \) different values. Then it is not difficult to see that Eq. (1.4) stands for a system of \( \sum_{i=1}^{r} k_i \) homogeneous linear equations in unknowns \( \lambda_1, ..., \lambda_n \). If this system has any solution with the nonzero components, then the given set \( \{x_1, ..., x_n\} \) is a cycle. In the last case, the system has also a solution \( m = (m_1, ..., m_n) \) with
the nonzero integer components \( m_i, \ i = 1, \ldots, n \). Thus, in Definition 1.1, the vector \( \lambda = (\lambda_1, \ldots, \lambda_n) \) can be replaced with a vector \( m = (m_1, \ldots, m_n) \) with \( m_i \in \mathbb{Z}\setminus\{0\} \).

For example, the set \( l = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)\} \) is a cycle in \( \mathbb{R}^3 \) with respect to the functions \( h_i(z_1, z_2, z_3) = z_i, \ i = 1, 2, 3 \). The vector \( \lambda \) in Definition 1.1 can be taken as \((2, 1, 1, 1, -1)\).

In case \( r = 2 \), the picture of cycles becomes more clear. Let, for example, \( h_1 \) and \( h_2 \) be the coordinate functions on \( \mathbb{R}^2 \). In this case, a cycle is the union of some sets \( A_k \) with the property: each \( A_k \) consists of vertices of a closed broken line with the sides parallel to the coordinate axis. These objects (sets \( A_k \)) have been exploited in practically all works devoted to the approximation of bivariate functions by univariate functions, although under various different names (see, for example, [76, chapter 2]). If the functions \( h_1 \) and \( h_2 \) are arbitrary, the sets \( A_k \) can be described as a trace of some point traveling alternatively in the level sets of \( h_1 \) and \( h_2 \), and then returning to its primary position. It should be remarked that in the case \( r > 2 \), cycles do not admit such a simple geometric description. We refer the reader to Braess and Pinkus [10] for the description of cycles when \( r = 3 \) and \( h_i(x) = a_i \cdot x, x \in \mathbb{R}^2, a_i \in \mathbb{R}^2\setminus\{0\}, i = 1, 2, 3 \).

Let \( T(X) \) denote the set of all functions on \( X \). With each pair \( \langle p, \lambda \rangle \), where \( p = \{x_1, \ldots, x_n\} \) is a cycle in \( X \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a vector known from Definition 1.1, we associate the functional

\[
G_{p,\lambda}: T(X) \to \mathbb{R}, \quad G_{p,\lambda}(f) = \sum_{j=1}^{n} \lambda_j f(x_j).
\]

In the following, such pairs \( \langle p, \lambda \rangle \) will be called cycle-vector pairs of \( X \). It is clear that the functional \( G_{p,\lambda} \) is linear and \( G_{p,\lambda}(g) = 0 \) for all functions \( g \in \mathcal{B}(h_1, \ldots, h_r; X) \).

**Lemma 1.1.** Let \( X \) have cycles and \( h_i(X) \cap h_j(X) = \emptyset \), for all \( i, j \in \{1, \ldots, r\}, i \neq j \). Then a function \( f : X \to \mathbb{R} \) belongs to the set \( \mathcal{B}(h_1, \ldots, h_r; X) \) if and only if \( G_{p,\lambda}(f) = 0 \) for any cycle-vector pair \( \langle p, \lambda \rangle \) of \( X \).

**Proof.** The necessity is obvious, since the functional \( G_{p,\lambda} \) annihilates all members of \( \mathcal{B}(h_1, \ldots, h_r; X) \). Let us prove the sufficiency. Introduce the notation

\[
Y_i = h_i(X), \ i = 1, \ldots, r;
\]

\[
\Omega = Y_1 \cup \ldots \cup Y_r.
\]
Consider the following set.

\[ \mathcal{L} = \{ Y = \{ y_1, \ldots, y_r \} : \text{if there exists } x \in X \text{ such that } h_i(x) = y_i, \ i = 1, \ldots, r \} \]

(1.5)

Note that \( \mathcal{L} \) is not a subset of \( \Omega \). It is a set of some certain subsets of \( \Omega \).

Each element of \( \mathcal{L} \) is a set \( Y = \{ y_1, \ldots, y_r \} \subset \Omega \) with the property that there exists \( x \in X \) such that \( h_i(x) = y_i, \ i = 1, \ldots, r \).

In what follows, all the points \( x \) associated with \( Y \) by (1.5) will be called \((\ast)\)-points of \( Y \).

It is clear that the number of such points depends on \( Y \) as well as on the functions \( h_1, \ldots, h_r \), and may be greater than 1. But note that if any two points \( x_1 \) and \( x_2 \) are \((\ast)\)-points of \( Y \), then \( h_i(x_1) = h_i(x_2), \ i = 1, \ldots, r \), whence

\[ 1 \cdot \delta_{h_i(x_1)} + (-1) \cdot \delta_{h_i(x_2)} \equiv 0, \ i = 1, \ldots, r. \]

The last identity means that the set \( p_0 = \{ x_1, x_2 \} \) forms a cycle and \( \lambda_0 = (1; -1) \) is an associated vector. Then by the sufficiency condition, \( G_{p_0, \lambda_0}(f) = 0 \), which yields that \( f(x_1) = f(x_2) \).

Let now \( Y^* \) be the set of all \((\ast)\)-points of \( Y \). Since we have already known that \( f(Y^*) \) is a single number, we can define the function

\[ t : \mathcal{L} \rightarrow \mathbb{R}, \ t(Y) = f(Y^*). \]

Or, equivalently, \( t(Y) = f(x) \), where \( x \) is an arbitrary \((\ast)\)-point of \( Y \).

Consider now a class \( \mathcal{S} \) of functions of the form \( \sum_{j=1}^{k} r_j \delta_{D_j} \), where \( k \) is a positive integer, \( r_j \) are real numbers and \( D_j \) are elements of \( \mathcal{L} \), \( j = 1, \ldots, k \).

We fix neither the numbers \( k, r_j \), nor the sets \( D_j \). Clearly, \( \mathcal{S} \) is a linear space. Over \( \mathcal{S} \), we define the functional

\[ F : \mathcal{S} \rightarrow \mathbb{R}, \ F \left( \sum_{j=1}^{k} r_j \delta_{D_j} \right) = \sum_{j=1}^{k} r_j t(D_j). \]

First of all, we must show that this functional is well defined. That is, the equality

\[ \sum_{j=1}^{k_1} r'_j \delta_{D'_j} = \sum_{j=1}^{k_2} r''_j \delta_{D''_j} \]

always implies the equality

\[ \sum_{j=1}^{k_1} \delta_{h_i(x_1)} + \sum_{j=1}^{k_2} \delta_{h_i(x_2)} \equiv 0, \ i = 1, \ldots, r. \]
\[ \sum_{j=1}^{k_1} r'_j t(D'_j) = \sum_{j=1}^{k_2} r''_j t(D''_j). \]

In fact, this is equivalent to the implication

\[ \sum_{j=1}^{k} r_j \delta_{D_j} = 0 \implies \sum_{j=1}^{k} r_j t(D_j) = 0, \quad \text{for all } k \in \mathbb{N}, r_j \in \mathbb{R}, D_j \subset \mathcal{L}. \quad (1.6) \]

Suppose that the left-hand side of the implication (1.6) be satisfied. Each set \( D_j \) consists of \( r \) real numbers \( y^j_1, \ldots, y^j_r \), \( j = 1, \ldots, k \). By the hypothesis of the lemma, all these numbers are different. Therefore,

\[ \delta_{D_j} = \sum_{i=1}^{r} \delta_{y^j_i}, \quad j = 1, \ldots, k. \quad (1.7) \]

Eq. (1.7) together with the left-hand side of (1.6) gives

\[ \sum_{i=1}^{r} \sum_{j=1}^{k} r_j \delta_{y^j_i} = 0. \quad (1.8) \]

Since the sets \( \{ y^1_i, y^2_i, \ldots, y^k_i \} \), \( i = 1, \ldots, r \), are pairwise disjoint, we obtain from (1.8) that

\[ \sum_{j=1}^{k} r_j \delta_{y^j_i} = 0, \quad i = 1, \ldots, r. \quad (1.9) \]

Let now \( x_1, \ldots, x_k \) be some \((*)\)-points of the sets \( D_1, \ldots, D_k \) respectively. Since by (1.5), \( y^j_i = h_i(x_j) \), for \( i = 1, \ldots, r \) and \( j = 1, \ldots, k \), it follows from (1.9) that the set \( \{ x_1, \ldots, x_k \} \) is a cycle. Then by the condition of the sufficiency, \( \sum_{j=1}^{k} r_j \bar{f}(x_j) = 0 \). Hence \( \sum_{j=1}^{k} r_j t(D_j) = 0 \). We have proved the implication (1.6) and hence the functional \( F \) is well defined. Note that the functional \( F \) is linear (this can be easily seen from its definition).

Consider now the following space:

\[ \mathcal{S}' = \left\{ \sum_{j=1}^{k} r_j \delta_{\omega_j} \right\}, \]

where \( k \in \mathbb{N}, r_j \in \mathbb{R}, \omega_j \subset \Omega \). As above, we do not fix the parameters \( k, r_j \) and \( \omega_j \). Clearly, the space \( \mathcal{S}' \) is larger than \( \mathcal{S} \). Let us prove that the
functional $F$ can be linearly extended to the space $S'$. So, we must prove that there exists a linear functional $F' : S' \to \mathbb{R}$ such that $F'(x) = F(x)$, for all $x \in S$. Let $H$ denote the set of all linear extensions of $F$ to subspaces of $S'$ containing $S$. The set $H$ is not empty, since it contains a functional $F$. For each functional $v \in H$, let $\text{dom}(v)$ denote the domain of $v$. Consider the following partial order in $H$: $v_1 \preceq v_2$, if $v_2$ is a linear extension of $v_1$ from the space $\text{dom}(v_1)$ to the space $\text{dom}(v_2)$. Let now $P$ be any chain (linearly ordered subset) in $H$. Consider the following functional $u$ defined on the union of domains of all functionals $p \in P$:

$$u : \bigcup_{p \in P} \text{dom}(p) \to \mathbb{R}, \quad u(x) = p(x), \text{ if } x \in \text{dom}(p)$$

Obviously, this functional is well defined and linear. Besides, the functional $u$ provides an upper bound for $P$. We see that the arbitrarily chosen chain $P$ has an upper bound. Then by Zorn’s lemma, there is a maximal element $F' \in H$. We claim that the functional $F'$ must be defined on the whole space $S'$. Indeed, if $F'$ is defined on a proper subspace $D \subset S'$, then it can be linearly extended to a space larger than $D$ by the following way: take any point $x \in S' \setminus D$ and consider the linear space $D' = \{D + \alpha x\}$, where $\alpha$ runs through all real numbers. For an arbitrary point $y + \alpha x \in D'$, set $F''(y + \alpha x) = F'(y) + \alpha b$, where $b$ is any real number considered as the value of $F''$ at $x$. Thus, we constructed a linear functional $F'' \in H$ satisfying $F' \leq F''$. The last contradicts the maximality of $F'$. This means that the functional $F'$ is defined on the whole $S'$ and $F \leq F'$ ($F'$ is a linear extension of $F$).

Define the following functions by means of the functional $F'$:

$$g_i : Y_i \to \mathbb{R}, \quad g_i(y_i) \overset{\text{def}}{=} F'(\delta_{y_i}), \quad i = 1, \ldots, r.$$ Let $x$ be an arbitrary point in $X$. Obviously, $x$ is a $(\ast)$-point of some set $Y = \{y_1, \ldots, y_r\} \subset \mathcal{L}$. Thus,

$$f(x) = t(Y) = F(\delta_Y) = F\left(\sum_{i=1}^{r} \delta_{y_i}\right) = F'\left(\sum_{i=1}^{r} \delta_{y_i}\right) = \sum_{i=1}^{r} F'(\delta_{y_i}) = \sum_{i=1}^{r} g_i(y_i) = \sum_{i=1}^{r} g_i(h_i(x)).$$

\[\boxdot\]
1.2.3 Minimal cycles and the main results

**Definition 1.2.** A cycle \( p = \{x_1, ..., x_n\} \) is said to be minimal if \( p \) does not contain any cycle as its proper subset.

For example, the set \( l = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (1,1,1)\} \) considered above is a minimal cycle with respect to the functions \( h_i(z_1, z_2, z_3) = z_i, \ i = 1, 2, 3 \). Adding the point \((0,1,1)\) to \( l \), we will have a cycle, but not minimal. The vector \( \lambda \) associated with \( l \cup \{(0,1,1)\} \) can be taken as \((3, -1, -1, -2, 2, -1)\).

A minimal cycle \( p = \{x_1, ..., x_n\} \) has the following obvious properties:

(a) The vector \( \lambda \) associated with \( p \) through Eq. (1.4) is unique up to multiplication by a constant;

(b) If in (1.4), \( \sum_{j=1}^{n} |\lambda_j| = 1 \), then all the numbers \( \lambda_j, j = 1, ..., n \), are rational.

Thus, a minimal cycle \( p \) uniquely (up to a sign) defines the functional

\[
G_p(f) = \sum_{j=1}^{n} \lambda_j f(x_j), \quad \sum_{j=1}^{n} |\lambda_j| = 1.
\]

**Lemma 1.2.** The functional \( G_{p, \lambda} \) is a linear combination of functionals \( G_{p_1}, ..., G_{p_k} \), where \( p_1, ..., p_k \) are minimal cycles in \( p \).

**Proof.** Let \( (p, \lambda) \) be a cycle-vector pair of \( X \), where \( p = \{x_1, ..., x_n\} \) and \( \lambda = (\lambda_1, ..., \lambda_n) \). Let \( p_1 = \{y_1^1, ..., y_{s_1}^1\}, s_1 < n \), be a minimal cycle in \( p \) and

\[
G_{p_1}(f) = \sum_{j=1}^{s_1} \nu_j f(y_j^1), \quad \sum_{j=1}^{s_1} |\nu_j| = 1.
\]

Without loss of generality, we may assume that \( y_1^1 = x_1 \). Put

\[
t_1 = \frac{\lambda_1}{\nu_1}.
\]

Then the functional \( G_{p_\lambda} - t_1 G_{p_1} \) has the form

\[
G_{p_\lambda} - t_1 G_{p_1} = \sum_{j=1}^{n_1} \lambda_j f(x_j^1),
\]
where \( x_j^1 \in p, \lambda_j^1 \neq 0, j = 1, \ldots, n_1 \). Clearly, the set \( l_1 = \{ x_1^1, \ldots, x_{n_1}^1 \} \) is a cycle in \( p \) with the associated vector \( \lambda^1 = (\lambda_1^1, \ldots, \lambda_{n_1}^1) \). Besides, \( x_1 \notin l_1 \).

Thus, \( n_1 < n \) and \( G_{l_1, \lambda^1} = G_{p, \lambda} - t_1 G_{p_1} \). If \( l_1 \) is minimal, then the proof is completed. Assume \( l_1 \) is not minimal. Let \( p_1 = \{ y_1^2, \ldots, y_{s_2}^2 \} \), \( s_2 < n_1 \), be a minimal cycle in \( l_1 \) and

\[
G_{p_2}(f) = \sum_{j=1}^{s_2} \nu_j^2 f(y_j^2), \quad \sum_{j=1}^{s_2} |\nu_j^2| = 1.
\]

Without loss of generality, we may assume that \( y_1^2 = x_1^1 \). Put

\[
t_2 = \frac{\lambda_1^1}{\nu_1^2}.
\]

Then the functional \( G_{l_1, \lambda^1} - t_2 G_{p_2} \) has the form

\[
G_{l_1, \lambda^1} - t_2 G_{p_2} = \sum_{j=1}^{n_2} \lambda_j^2 f(x_j^2),
\]

where \( x_j^2 \in l_1, \lambda_j^2 \neq 0, j = 1, \ldots, n_2 \). Clearly, the set \( l_2 = \{ x_1^2, \ldots, x_{n_2}^2 \} \) is a cycle in \( l_1 \) with the associated vector \( \lambda^2 = (\lambda_1^2, \ldots, \lambda_{n_2}^2) \). Besides, \( x_1^2 \notin l_2 \).

Thus, \( n_2 < n_1 \) and \( G_{l_2, \lambda^2} = G_{l_1, \lambda^1} - t_2 G_{p_2} \). If \( l_2 \) is minimal, then the proof is completed. Let \( l_2 \) be not minimal. Repeating the above process for \( l_2 \), then for \( l_3 \), etc., after some \( k - 1 \) steps we will come to a minimal cycle \( l_{k-1} \) and the functional

\[
G_{l_{k-1}, \lambda^{k-1}} = G_{l_{k-2}, \lambda^{k-2}} - t_{k-1} G_{p_{k-1}} = \sum_{j=1}^{n_{k-1}} \lambda_j^{k-1} f(x_j^{k-1}).
\]

Since the cycle \( l_{k-1} \) is minimal,

\[
G_{l_{k-1}, \lambda^{k-1}} = t_k G_{l_{k-1}}, \quad \text{where} \quad t_k = \sum_{j=1}^{n_{k-1}} |\lambda_j^{k-1}|.
\]

Now putting \( p_k = l_{k-1} \) and considering the above chain relations between the functionals \( G_{l_i, \lambda^i}, i = 1, \ldots, k - 1 \), we obtain that

\[
G_{p, \lambda} = \sum_{i=1}^{k} t_i G_{p_i}.
\]
Theorem 1.1. Let \( X \subset \mathbb{R}^d \) and \( h_1, \ldots, h_r \) be arbitrarily fixed real functions on \( X \). The following assertions are valid.

1) Let \( X \) have cycles with respect to the functions \( h_1, \ldots, h_r \). A function \( f : X \to \mathbb{R} \) belongs to the space \( B(h_1, \ldots, h_r ; X) \) if and only if \( G_p(f) = 0 \) for any minimal cycle \( p \subset X \).

2) Let \( X \) have no cycles. Then \( B(h_1, \ldots, h_r ; X) = T(X) \).

Proof. 1) The necessity is clear. Let us prove the sufficiency. On the strength of Lemma 1.2, it is enough to prove that if \( G_{p,\lambda}(f) = 0 \) for any cycle-vector pair \( \langle p,\lambda \rangle \) of \( X \), then \( f \in B(X) \).

Consider a system of intervals \( \{(a_i, b_i) \subset \mathbb{R}\}_{i=1}^r \) such that \( (a_i, b_i) \cap (a_j, b_j) = \emptyset \) for all the indices \( i, j \in \{1, \ldots, r\} \), \( i \neq j \). For \( i = 1, \ldots, r \), let \( \tau_i \) be one-to-one mappings of \( \mathbb{R} \) onto \( (a_i, b_i) \). Introduce the following functions on \( X \):

\[
h'_i(x) = \tau_i(h_i(x)), \quad i = 1, \ldots, r.
\]

It is clear that any cycle with respect to the functions \( h_1, \ldots, h_r \) is also a cycle with respect to the functions \( h'_1, \ldots, h'_r \), and vice versa. Besides, \( h'_i(X) \cap h'_j(X) = \emptyset \), for all \( i, j \in \{1, \ldots, r\} \), \( i \neq j \). Then by Lemma 1.1,

\[
f(x) = g'_1(h'_1(x)) + \cdots + g'_r(h'_r(x)),
\]

where \( g'_1, \ldots, g'_r \) are univariate functions depending on \( f \). From the last equality we obtain that

\[
f(x) = g'_1(\tau_1(h_1(x))) + \cdots + g'_r(\tau_r(h_r(x))) = g_1(h_1(x)) + \cdots + g_r(h_r(x)).
\]

That is, \( f \in B(X) \).

2) Let \( f : X \to \mathbb{R} \) be an arbitrary function. First suppose that \( h_i(X) \cap h_j(X) = \emptyset \), for all \( i, j \in \{1, \ldots, r\} \), \( i \neq j \). In this case, the proof is similar to and even simpler than that of Lemma 1.1. Indeed, the set of all \((*)\)-points of \( Y \) consists of a single point, since otherwise we would have a cycle with two points, which contradicts the hypothesis of the 2-nd part of the theorem. Further, well definition of the functional \( F \) becomes obvious, since the left-hand side of (1.6) also contradicts the nonexistence of cycles. Thus, as in the proof of Lemma 1.1, we can extend \( F \) to the space \( \mathcal{S}' \) and then obtain the desired representation for the function \( f \). Since \( f \) is arbitrary, \( T(X) = B(X) \).

Using the techniques from the proof of the 1-st part of the theorem, one can easily generalize the above argument to the case when the functions \( h_1, \ldots, h_r \) have arbitrary ranges.
Theorem 1.2. \( \mathcal{B}(h_1, \ldots, h_r; X) = T(X) \) if and only if \( X \) has no cycles with respect to the functions \( h_1, \ldots, h_r \).

Proof. The sufficiency immediately follows from Theorem 1.1. To prove the necessity, assume that \( X \) has a cycle \( p = \{x_1, \ldots, x_n\} \). Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a vector associated with \( p \) by Eq. (1.4). Consider a function \( f_0 \) on \( X \) with the property: \( f_0(x_i) = 1 \), for indices \( i \) such that \( \lambda_i > 0 \) and \( f_0(x_i) = -1 \), for indices \( i \) such that \( \lambda_i < 0 \). For this function, \( G_{p,\lambda}(f_0) \neq 0 \). Then by Theorem 1.1, \( f_0 \notin \mathcal{B}(X) \). Hence \( \mathcal{B}(X) \neq T(X) \). The contradiction shows that \( X \) does not admit cycles.

1.2.4 Corollaries

From Theorems 1.1 and 1.2 we obtain the following corollaries for the ridge function representation.

**Corollary 1.1.** Let \( X \subset \mathbb{R}^d \) and \( a^1, \ldots, a^r \in \mathbb{R}^d \setminus \{0\} \).

1) Let \( X \) have cycles with respect to the directions \( a^1, \ldots, a^r \). A function \( f : X \to \mathbb{R} \) belongs to the space \( \mathcal{R}(a^1, \ldots, a^r; X) \) if and only if \( G_p(f) = 0 \) for any minimal cycle \( p \subset X \).

2) Let \( X \) have no cycles. Then every function \( f : X \to \mathbb{R} \) belongs to the space \( \mathcal{R}(a^1, \ldots, a^r; X) \).

**Corollary 1.2.** \( \mathcal{R}(a^1, \ldots, a^r; X) = T(X) \) if and only if \( X \) has no cycles with respect to the directions \( a^1, \ldots, a^r \).

Note that solutions to Problems 1 and 2 are given by Corollaries 1.1 and 1.2, correspondingly. Although it is not always easy to find all cycles of a given set \( X \) and even to know if \( X \) possesses a single cycle, Corollaries 1.1 and 1.2 are of more practical than theoretical character. Particular cases of Problems 1 and 2 evidence in favor of our opinion. For example, for the problem of representation by sums of two ridge functions, the picture of cycles is completely describable (see the beginning of this section). The interpretation of cycles with respect to three directions in the plane can be found in Braess and Pinkus [10]. A geometric description of cycles with respect to 4 and more directions is quite complicated and requires deep techniques from geometry and graph theory. This is not within the aim of our study.
From the last corollary, it follows that if representation by sums of ridge functions with fixed directions $a^1, ..., a^r$ is valid in the class of continuous functions (or in the class of bounded functions), then such representation is valid in the class of all functions. For a rigid mathematical formulation of this result, let us introduce the notation:

$$R_c(a^1, ..., a^r; X) = \left\{ \sum_{i=1}^{r} g_i(a^i \cdot x), \ x \in X, \ g_i(a^i \cdot x) \in C(X), \ i = 1, ..., r \right\}$$

and

$$R_b(a^1, ..., a^r; X) = \left\{ \sum_{i=1}^{r} g_i(a^i \cdot x), \ x \in X, \ g_i(a^i \cdot x) \in B(X), \ i = 1, ..., r \right\}$$

Here $C(X)$ and $B(X)$ denote the spaces of continuous and bounded functions defined on $X \subset \mathbb{R}^d$ correspondingly (for the first space, the set $X$ is supposed to be compact). As we know (see Section 1.1) from the results of Sternfeld it follows that the equality $R_c(a^1, ..., a^r; X) = C(X)$ implies the equality $R_b(a^1, ..., a^r; X) = B(X)$. In other words, if every continuous function is represented by sums of ridge functions (with fixed directions!), then every bounded function also obeys such representation (with bounded summands). Corollaries 1.1 and 1.2 allow us to obtain the following result.

**Corollary 1.3.** Let $X$ be a compact subset of $\mathbb{R}^d$ and $a^1, ..., a^r$ be given directions in $\mathbb{R}^d \setminus \{0\}$. If $R_c(a^1, ..., a^r; X) = C(X)$, then $R(a^1, ..., a^r; X) = T(X)$.

**Proof.** If every continuous function defined on $X \subset \mathbb{R}^d$ is represented by sums of ridge functions with the directions $a^1, ..., a^r$, then it can be shown by applying the same idea (as in the proof of Theorem 1.2) that the set $X$ has no cycles with respect to the given directions. Only, because of continuity, Urysohn’s great lemma should be taken into account. That is, it should be taken into account that, by assuming the existence of a cycle $p_0 = \{x_1, ..., x_n\}$ with an associated vector $\lambda_0 = (\lambda_1, ..., \lambda_n)$, we can deduce from Urysohn’s great lemma the existence of a continuous function $u : X \to \mathbb{R}$ satisfying

1) $u(x_i) = 1$, for indices $i$ such that $\lambda_i > 0$
2) $u(x_j) = -1$, for indices $j$ such that $\lambda_j < 0$,
3) $-1 < u(x) < 1$, for all $x \in X \setminus p_0$. 

34
These properties would mean that \( G_{p_0, \lambda_0}(u) \neq 0 \implies u \notin \mathcal{R}_{c}(a^1, ..., a^r; X) \implies \mathcal{R}_{c}(a^1, ..., a^r; X) \neq C(X) \).

But if \( X \) has no cycles with respect to the directions \( a^1, ..., a^r \), then by Corollary 1.2, \( \mathcal{R}(a^1, ..., a^r; X) = T(X) \).

Let us now give some examples of sets over which the representation by linear combinations of ridge functions is possible.

(1) Let \( r = 2 \) and \( X \) be the union of two parallel lines not perpendicular to the given directions \( a^1 \) and \( a^2 \). Then \( X \) has no cycles with respect to \( \{a^1, a^2\} \). Therefore, by Corollary 1.2, \( \mathcal{R}(a^1, a^2; X) = T(X) \).

(2) Let \( r = 2, a^1 = (1, 1), a^2 = (1, -1) \) and \( X \) be the graph of the function \( y = \arcsin(\sin x) \). Then \( X \) has no cycles and hence \( \mathcal{R}(a^1, a^2; X) = T(X) \).

(3) Let now given \( r \) directions \( \{a^j\}_{j=1}^r \) and \( r + 1 \) points \( \{x^i\}_{i=1}^{r+1} \subset \mathbb{R}^d \) such that

\[
\begin{align*}
a^1 \cdot x^i &= a^1 \cdot x^j \neq a^1 \cdot x^2, \quad \text{for } 1 \leq i, j \leq r + 1, i, j \neq 2 \\
a^2 \cdot x^i &= a^2 \cdot x^j \neq a^2 \cdot x^3, \quad \text{for } 1 \leq i, j \leq r + 1, i, j \neq 3 \\
&\vdots \\
a^r \cdot x^i &= a^r \cdot x^j \neq a^r \cdot x^{r+1}, \quad \text{for } 1 \leq i, j \leq r.
\end{align*}
\]

The simplest data realizing these equations are the basis directions in \( \mathbb{R}^d \) and the points \( (0, 0, ..., 0), (1, 0, ..., 0), (0, 1, ..., 0), ..., (0, 0, ..., 1) \). From the first equation we obtain that \( x^2 \) cannot be a point of any cycle in \( X = \{x^1, ..., x^{r+1}\} \). Sequentially, from the second, third, ..., \( r \)-th equations it follows that the points \( x^3, x^4, ..., x^{r+1} \) also cannot be points of cycles in \( X \) respectively. Thus the set \( X \) does not contain cycles at all. By Corollary 1.2, \( \mathcal{R}(a^1, ..., a^r; X) = T(X) \).

(4) Let given directions \( \{a^j\}_{j=1}^r \) and a curve \( \gamma \) in \( \mathbb{R}^d \) such that for any \( c \in \mathbb{R} \), \( \gamma \) has at most one common point with at least one of the hyperplanes \( a^j \cdot x = c, j = 1, ..., r \). Clearly, the curve \( \gamma \) has no cycles and hence \( \mathcal{R}(a^1, ..., a^r; \gamma) = T(\gamma) \).

Braess and Pinkus [10] considered the partial case of Problem 2: characterize a set of points \( \{x^1, ..., x^k\} \subset \mathbb{R}^d \) such that for any data \( \{\alpha_1, ..., \alpha_k\} \subset \mathbb{R} \)
there exists a function $g \in \mathcal{R} (\mathbf{a}^1, ..., \mathbf{a}^r; \mathbb{R}^d)$ satisfying $g(x^i) = \alpha_i, i = 1, ..., k$.

In connection with this problem, they introduced the notion of the $NI$-property (non interpolation property) and $MNI$-property (minimal non interpolation property) of a finite set of points as follows:

Given directions $\{\mathbf{a}^j\}_{j=1}^r \subset \mathbb{R}^d \setminus \{0\}$, we say that a set of points $\{x^i\}_{i=1}^k \subset \mathbb{R}^d$ has the $NI$-property with respect to $\{\mathbf{a}^j\}_{j=1}^r$, if there exists $\{\alpha_i\}_{i=1}^k \subset \mathbb{R}$ such that we cannot find a function $g \in \mathcal{R} (\mathbf{a}^1, ..., \mathbf{a}^r; \mathbb{R}^d)$ satisfying $g(x^i) = \alpha_i, i = 1, ..., k$. We say that the set $\{x^i\}_{i=1}^k \subset \mathbb{R}^d$ has the $MNI$-property with respect to $\{\mathbf{a}^j\}_{j=1}^r$, if $\{x^i\}_{i=1}^k$ but no proper subset thereof has the $NI$-property.

It follows from Corollary 1.2 that a set $\{x^i\}_{i=1}^k$ has the $NI$-property if and only if $\{x^i\}_{i=1}^k$ contains a cycle with respect to the functions $h_i = \mathbf{a}^i \cdot x$, $i = 1, ..., r$ (or, simply, to the directions $\mathbf{a}^i$, $i = 1, ..., r$) and the $MNI$-property if and only if the set $\{x^i\}_{i=1}^k$ itself is a minimal cycle with respect to the given directions. Taking into account this argument and Definitions 1.1 and 1.2, we obtain that the set $\{x^i\}_{i=1}^k$ has the $NI$-property if and only if there is a vector $\mathbf{m} = (m_1, ..., m_k) \in \mathbb{Z}^k \setminus \{0\}$ such that

$$\sum_{j=1}^k m_j g(\mathbf{a}^j \cdot x^i) = 0,$$

for $i = 1, ..., r$ and all functions $g : \mathbb{R} \to \mathbb{R}$. This set has the $MNI$-property if and only if the vector $\mathbf{m}$ has the additional properties: it is unique up to multiplication by a constant and all its components are different from zero.

This special consequence of Corollary 1.2 was proved in [10].

### 1.3 Characterization of an extremal sum of ridge functions

The approximation problem considered in this section is to approximate a continuous multivariate function $f(x) = f(x_1, ..., x_d)$ by sums of two ridge functions in the uniform norm. We give a necessary and sufficient condition for a sum of two ridge functions to be a best approximation to $f(x)$. This main result is next used in a special case to obtain an explicit formula for the approximation error and to construct a best approximation. The problem of well approximation by such sums is also considered.
1.3.1 Exposition of the problem

Consider the following set of sums of ridge functions

\[ \mathcal{R} = \mathcal{R}(a,b) = \{ g_1(a \cdot \mathbf{x}) + g_2(b \cdot \mathbf{x}) : g_i \in C(\mathbb{R}), i = 1, 2 \} \]

That is, we fix directions \( a \) and \( b \) and consider linear combinations of ridge functions with these directions.

Let \( f(\mathbf{x}) \) be a given continuous function on some compact subset \( Q \) of \( \mathbb{R}^d \). We want to find conditions that are necessary and sufficient for a function \( g_0 \in \mathcal{R}(a,b) \) to be an extremal element (or a best approximation) to \( f \). In other words, we want to characterize such sums \( g_0(\mathbf{x}) = g_1(a \cdot \mathbf{x}) + g_2(b \cdot \mathbf{x}) \) of ridge functions that

\[ \| f - g_0 \| = \max_{\mathbf{x} \in Q} |f(\mathbf{x}) - g_0(\mathbf{x})| = E(f), \]

where

\[ E(f) = E(f, \mathcal{R}) \overset{\text{def}}{=} \inf_{g \in \mathcal{R}(a,b)} \| f - g \| \]

is the error in approximating from \( \mathcal{R}(a,b) \). The other related problem is how to construct these sums of ridge functions. We also want to know if we can approximate well, i.e. for which compact sets \( Q, \mathcal{R}(a,b) \) is dense in \( C(Q) \) in the topology of uniform convergence. It should be remarked that solutions to these problems may be useful in connection with the study of partial differential equations. For example, assume that \((a_1,b_1)\) and \((a_2,b_2)\) are linearly independent vectors in \( \mathbb{R}^2 \). Then the general solution of the homogeneous partial differential equation

\[ \left( a_1 \frac{\partial}{\partial x} + b_1 \frac{\partial}{\partial y} \right) \left( a_2 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} \right) u(x,y) = 0 \] (1.10)

are all functions of the form

\[ u(x,y) = g_1(b_1 x - a_1 y) + g_2(b_2 x - a_2 y) \] (1.11)

for arbitrary \( g_1 \) and \( g_2 \). In [36], Golitschek and Light described an algorithm that computes the error of approximation of a continuous real-valued function \( f(x,y) \) by solutions of equation (1.10), provided that \( a_1 = b_2 = 1, a_2 = b_1 = 0 \). Using our main result (Theorem 1.3), one can characterize those solutions (1.11) that are extremal to a given function \( f(x,y) \). For certain class of
functions \( f(x,y) \), one can also easily calculate the approximation error and construct one extremal solution (see Theorems 1.4 and 1.5).

The problem of approximating by functions from the set \( \mathcal{R}(a,b) \) arises in other contexts too. Buck [11] studied the classical functional equation: given \( \beta(t) \in C[0,1], \ 0 \leq \beta(t) \leq 1 \), for which \( u \in C[0,1] \) does there exist \( \varphi \in C[0,1] \) such that

\[
\varphi(t) = \varphi(\beta(t)) + u(t).
\]

He proved that the set of all \( u \) satisfying this condition is dense in the set

\[
\{ v \in C[0,1] : v(t) = 0 \ \text{whenever} \ \beta(t) = t \}
\]

if and only if \( \mathcal{R}(a,b) \) with the unit directions \( a = (1;0) \) and \( b = (0,1) \) is dense in \( C(K) \), where \( K = \{(x,y) : y = x \text{ or } y = \beta(x), \ 0 \leq x \leq 1 \} \).

Although there are enough reasons to consider approximation problems associated with the set \( \mathcal{R}(a,b) \) in an independent way, one may ask why sums of only two ridge functions are considered instead of sums with an arbitrary number of terms. We will try to answer this fair question in Section 1.3.4.

### 1.3.2 The characterization theorem

Let \( Q \) be a compact subset of \( \mathbb{R}^d \) and \( a, b \in \mathbb{R}^d \setminus \{0\} \).

**Definition 1.3.** A finite or infinite ordered set \( p = (p_1, p_2, \ldots) \subset Q \) with \( p_i \neq p_{i+1} \), and either \( a \cdot p_1 = a \cdot p_2, b \cdot p_2 = b \cdot p_3, a \cdot p_3 = a \cdot p_4, \ldots \) or \( b \cdot p_1 = b \cdot p_2, a \cdot p_2 = a \cdot p_3, b \cdot p_3 = b \cdot p_4, \ldots \) is called a path with respect to the directions \( a \) and \( b \).

This notion (in two-dimensional case) was introduced by Braess and Pinkus [10]. They showed that paths give geometric means of deciding if a set of points \( \{x^i\}_{i=1}^m \subset \mathbb{R}^2 \) has the NI property (see Section 1.2.4). Ismailov and Pinkus [63] used these objects to study the problem of interpolation on straight lines by linear combinations of a finite number of ridge functions with fixed directions. In [44,51,53] paths were generalized to those with respect to two functions. The last objects turned out to be useful in problems of approximation and representation by sums of compositions of fixed multivariate functions with univariate functions.

If \( a \) and \( b \) are the coordinate vectors in \( \mathbb{R}^2 \), then Definition 1.3 defines a **bolt of lightning**. The idea of bolts was first introduced in Diliberto and
Straus [26], where these objects are called “permissible lines”. They appeared further in a number of papers, although under several different names (see, e.g., [18,29,34,36,55,56,60,62,76,79,82,93,107,108,113]). Note that the term “bolt of lightning” is due to Arnold [3].

For the sake of brevity, we use the term “path” instead of the expression “path with respect to the directions a and b”.

The length of a path is the number of its points. A single point is a path of the unit length. A finite path \((p_1, p_2, \ldots, p_{2n})\) is said to be closed if \((p_1, p_2, \ldots, p_{2n}, p_1)\) is a path.

We associate each closed path \(p = (p_1, p_2, \ldots, p_{2n})\) with the functional

\[
G_p(f) = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^{k+1} f(p_k).
\]

This functional has the following obvious properties:

(a) If \(g \in \mathcal{R}(a,b)\), then \(G_p(g) = 0\).
(b) \(\|G_p\| \leq 1\) and if \(p_i \neq p_j\) for all \(i \neq j, 1 \leq i, j \leq 2n\), then \(\|G_p\| = 1\).

**Lemma 1.3.** Let a compact set \(Q\) have closed paths. Then

\[
\sup_{p \subset Q} |G_p(f)| \leq E(f),
\]

where the sup is taken over all closed paths. Moreover, inequality (1.12) is sharp, i.e. there exist functions for which (1.12) turns into equality.

**Proof.** Let \(p\) be a closed path of \(Q\) and \(g\) be any function from \(\mathcal{R}(a,b)\). Then by the linearity of \(G_p\) and properties (a) and (b),

\[
|G_p(f)| = |G_p(f-g)| \leq \|f-g\|.
\]

Since the left-hand and the right-hand sides of (1.13) do not depend on \(g\) and \(p\) respectively, it follows from (1.13) that

\[
\sup_{p \subset Q} |G_p(f)| \leq \inf_{\mathcal{R}(a,b)} \|f-g\|.
\]

Now we prove the sharpness of (1.12). By assumption \(Q\) has closed paths. Then \(Q\) has closed paths \(p' = (p'_1, \ldots, p'_{2m})\) such that all points \(p_1, \ldots, p_{2m}\) are distinct. In fact, such special paths can be obtained from any closed
path \( p = (p_1, ..., p_{2n}) \) by the following simple algorithm: if the points of the path \( p \) are not all distinct, let \( i \) and \( k > 0 \) be the minimal indices such that \( p_i = p_{i+2k} \); delete from \( p \) the subsequence \( p_{i+1}, ..., p_{i+2k} \) and call \( p \) the obtained path; repeat the above step until all points of \( p \) are all distinct; set \( p' := p \).

On the other hand there exist continuous functions \( h = h(x) \) on \( Q \) such that \( h(p'_i) = 1, i = 1, 3, ..., 2m - 1, h(p'_i) = -1, i = 2, 4, ..., 2m \) and \(-1 < h(x) < 1 \) elsewhere. For such functions we have

\[
G_{p'}(h) = \|h\| = 1 \quad (1.15)
\]

and

\[
E(h) \leq \|h\|, \quad (1.16)
\]

where the last inequality follows from the fact that \( 0 \in R(a, b) \). From (1.14)-(1.16) it follows that

\[
\sup_{p \subset Q} |G_p(h)| = E(h).
\]

Lemma 1.4. Let \( Q \) be a convex compact subset of \( \mathbb{R}^d \), \( f(x) \in C(Q) \). For a vector \( e \in \mathbb{R}^d \setminus \{0\} \) and a real number \( t \) set

\[
Q_t = \{x \in Q : e \cdot x = t\}, \quad T_h = \{t \in \mathbb{R} : Q_t \neq \emptyset\}.
\]

The functions

\[
g_1(t) = \max_{x \in Q_t} f(x), \quad t \in T_h \quad \text{and} \quad g_2(t) = \min_{x \in Q_t} f(x), \quad t \in T_h
\]

are defined and continuous on \( T_h \).

The proof of this lemma is not difficult and can be obtained by the well-known elementary methods of mathematical analysis.

Definition 1.4. A finite or infinite path \( (p_1, p_2, ...) \) is said to be extremal for a function \( u(x) \in C(Q) \) if \( u(p_i) = (-1)^i \|u\|, i = 1, 2, ... \) or \( u(p_i) = (-1)^{i+1} \|u\|, i = 1, 2, ... \)

Theorem 1.3. Let \( Q \subset \mathbb{R}^d \) be a convex compact set satisfying the following condition.
Condition (A): For any path \( q = (q_1, q_2, \ldots, q_n) \subset Q \) there exist points \( q_{n+1}, q_{n+2}, \ldots, q_{n+s} \in Q \) such that \( (q_1, q_2, \ldots, q_{n+s}) \) is a closed path and \( s \) is not more than some positive integer \( N_0 \) independent of \( q \).

A necessary and sufficient condition for a function \( g_0 \in \mathcal{R}(a, b) \) to be an extremal element to the given function \( f \in C(Q) \) is the existence of a closed or infinite path \( l = (p_1, p_2, \ldots) \) extremal for the function \( f_1(x) = f(x) - g_0(x) \).

It should be remarked that satisfaction of the above condition (A) strongly depends on the fixed directions \( a \) and \( b \). For example, in the familiar case of a square \( S \subset \mathbb{R}^2 \) there are many directions which are not allowed. If it is possible to reach a corner of the square with not more than one of the two directions orthogonal to \( a \) and \( b \) respectively (we don’t differentiate between directions \( c \) and \( -c \)), the triple \( (S, a, b) \) does not satisfy condition (A) of the theorem. Here are simple examples: Let \( S = [0; 1]^2, a = (1; 0), b = (1; 1) \). Then the ordered set \{((0; 1), (1; 0), (1; 1))\} is a path in \( S \) which can not be made closed. In this case, \((1; 1)\) is not reached with the direction orthogonal to \( b \). Let now \( a = \left(1; \frac{1}{2}\right), b = (1; 1) \). Then the corner \((1; 1)\) is reached with none of the directions orthogonal to \( a \) and \( b \) respectively. In this case, for any positive integer \( N_0 \) and any point \( q_0 \) in \( S \) one can chose a point \( q_1 \in S \) from a sufficiently small neighborhood of the corner \((1; 1)\) so that any path containing \( q_0 \) and \( q_1 \) has the length more than \( N_0 \). These examples and a little geometry show that if a compact convex set \( Q \subset \mathbb{R}^2 \) satisfies condition (A) of Theorem 1.3, then any point in the boundary of \( Q \) must be reached with each of the two directions orthogonal to \( a \) and \( b \) respectively. If \( Q \subset \mathbb{R}^d, a, b \in \mathbb{R}^d\setminus\{0\}, d > 2, \) there are many directions orthogonal to \( a \) and \( b \). In this case, condition (A) requires that any point in the boundary of \( Q \) should be reached with at least two directions orthogonal to \( a \) and \( b \) respectively.

Proof. Necessity. Let \( g_0 = g_{1,0}(a \cdot x) + g_{2,0}(b \cdot x) \) be an extremal element from \( \mathcal{R}(a, b) \) to \( f \). We must show that if there is not a closed path extremal for \( f_1 \), then there exists a path extremal for \( f_1 \) with the infinite length (number of points). Suppose the contrary. Suppose that there exists a positive integer \( N \) such that the length of each path extremal for \( f_1 \) is not more than \( N \). Set the following functions:

\[
 f_n = f_{n-1} - g_{1,n-1} - g_{2,n-1}, \quad n = 2, 3, \ldots
\]
where
\[
g_{1,n-1}(a \cdot x) = \frac{1}{2} \left( \max_{y \in Q} f_{n-1}(y) + \min_{y \in Q} f_{n-1}(y) \right)
\]
\[
g_{2,n-1}(b \cdot x) = \frac{1}{2} \left( \max_{y \in Q} (f_{n-1}(y) - g_{1,n-1}(a \cdot y)) \right.
\]
\[
+ \min_{y \in Q} (f_{n-1}(y) - g_{1,n-1}(a \cdot y)) \right).
\]

By Lemma 1.4, all the functions \( f_n(x) \), \( n = 2, 3, \ldots \), are continuous on \( Q \). By assumption \( g_0 \) is a best approximation to \( f \). Hence \( \|f_1\| = E(f) \). Now we show that \( \|f_2\| = E(f) \). Indeed, for any \( x \in Q \)
\[
f_1(x) - g_{1,1}(a \cdot x) \leq \frac{1}{2} \left( \max_{y \in Q} f_1(y) - \min_{y \in Q} f_1(y) \right) \leq E(f) \quad (1.17)
\]
and
\[
f_1(x) - g_{1,1}(a \cdot x) \geq \frac{1}{2} \left( \min_{y \in Q} f_1(y) - \max_{y \in Q} f_1(y) \right) \geq -E(f). \quad (1.18)
\]
Using the definition of \( g_{2,1}(b \cdot x) \), for any \( x \in Q \) we have
\[
f_1(x) - g_{1,1}(a \cdot x) - g_{2,1}(b \cdot x)
\]
\[
\leq \frac{1}{2} \left( \max_{y \in Q} (f_1(y) - g_{1,1}(a \cdot y)) - \min_{y \in Q} (f_1(y) - g_{1,1}(a \cdot y)) \right)
\]
and
\[
f_1(x) - g_{1,1}(a \cdot x) - g_{2,1}(b \cdot x)
\]
\[
\leq \frac{1}{2} \left( \min_{y \in Q} (f_1(y) - g_{1,1}(a \cdot y)) - \max_{y \in Q} (f_1(y) - g_{1,1}(a \cdot y)) \right).
\]
Using (1.17) and (1.18) in the last two inequalities, we obtain that for any \( x \in Q \)
\[-E(f) \leq f_2(x) = f_1(x) - g_{1,1}(a \cdot x) - g_{2,1}(b \cdot x) \leq E(f).\]

Therefore,
\[\|f_2\| \leq E(f).\] (1.19)

Since \( f_2(x) - f(x) \) belongs to \( R(a, b) \), we deduce from (1.19) that
\[\|f_2\| = E(f).\]

By the same way, one can show that \( \|f_3\| = E(f) \), \( \|f_4\| = E(f) \), and so on. Thus we can write
\[\|f_n\| = E(f), \text{ for any } n.\]

Let us now prove the implications
\[f_1(p_0) < E(f) \Rightarrow f_2(p_0) < E(f)\] (1.20)

and
\[f_1(p_0) > -E(f) \Rightarrow f_2(p_0) > -E(f),\] (1.21)

where \( p_0 \in Q \). First, we are going to prove the implication
\[f_1(p_0) < E(f) \Rightarrow f_1(p_0) - g_{1,1}(a \cdot p_0) < E(f).\] (1.22)

There are two possible cases.

1) \( \max_{y \in Q} f_1(y) = E(f) \) and \( \min_{y \in Q} f_1(y) = -E(f) \). In this case, \( g_{1,1}(a \cdot p_0) = 0 \). Hence
\[f_1(p_0) - g_{1,1}(a \cdot p_0) < E(f).\]

2) \( \max_{y \in Q} f_1(y) = E(f) - \varepsilon_1 \) and \( \min_{y \in Q} f_1(y) = -E(f) + \varepsilon_2 \),
where \( \varepsilon_1, \varepsilon_2 \) are nonnegative real numbers with the sum \( \varepsilon_1 + \varepsilon_2 \neq 0 \). In this case,
\[f_1(p_0) - g_{1,1}(a \cdot p_0) \leq \max_{y \in Q} f_1(y) - g_{1,1}(a \cdot p_0) = \]
\[= \frac{1}{2} \left( \max_{y \in Q} f_1(y) - \min_{y \in Q} f_1(y) \right) = \]

Electronic copy available at: https://ssrn.com/abstract=3618165
\[ E(f) - \frac{\varepsilon_1 + \varepsilon_2}{2} < E(f) \]

Thus we have proved (1.22). Using this method, we can also prove that

\[ f_1(p_0) - g_{1,1}(a \cdot p_0) < E(f) \Rightarrow f_1(p_0) - g_{1,1}(a \cdot p_0) - g_{2,1}(b \cdot p_0) < E(f). \]  

(1.23)

Now (1.20) follows from (1.22) and (1.23). By the same way we can prove (1.21). It follows from implications (1.20) and (1.21) that if \( f_2(p_0) = E(f) \), then \( f_1(p_0) = E(f) \) and if \( f_2(p_0) = -E(f) \), then \( f_1(p_0) = -E(f) \). This simply means that each path extremal for \( f_2 \) will be extremal for \( f_1 \).

Now we show that if any path extremal for \( f_1 \) has the length not more than \( N \), then any path extremal for \( f_2 \) has the length not more than \( N - 1 \). Suppose the contrary. Suppose that there is a path extremal for \( f_2 \) with the length equal to \( N \). Denote it by \( q = (q_1, q_2, ..., q_N) \). Without loss of generality we may assume that \( b \cdot q_{N-1} = b \cdot q_N \). As it has been shown above, the path \( q \) is also extremal for \( f_1 \). Assume that \( f_1(q_N) = E(f) \). Then there is not any \( q_0 \in Q \) such that \( q_0 \neq q_N \), \( a \cdot q_0 = a \cdot q_N \) and \( f_1(q_0) = -E(f) \). Indeed, if there was such \( q_0 \) and \( q_0 \notin q \), the path \( (q_1, q_2, ..., q_N, q_0) \) would be extremal for \( f_1 \). But this would contradict our assumption that any path extremal for \( f_1 \) has the length not more than \( N \). Besides, if there was such \( q_0 \) and \( q_0 \in q \), we could form some closed path extremal for \( f_1 \). This also would contradict our assumption that there does not exist a closed path extremal for \( f_1 \).

Hence

\[ \max_{y \in Q, a \cdot y = a \cdot q_N} f_1(y) = E(f), \quad \min_{y \in Q, a \cdot y = a \cdot q_N} f_1(y) > -E(f). \]

Therefore,

\[ |f_1(q_N) - g_{1,1}(a \cdot q_N)| < E(f). \]

From the last inequality it is easy to obtain that (see the proof of implications (1.20) and (1.21))

\[ |f_2(q_N)| < E(f). \]

This means, on the contrary to our assumption, that the path \( (q_1, q_2, ..., q_N) \) can not be extremal for \( f_2 \). Hence any path extremal for \( f_2 \) has the length not more than \( N - 1 \).

By the same way, it can be shown that any path extremal for \( f_3 \) has the length not more than \( N - 2 \), any path extremal for \( f_4 \) has the length
not more than \(N - 3\) and so on. Finally, we will obtain that there is not a path extremal for \(f_{N+1}\). Hence there is not a point \(p_0 \in Q\) such that \(\|f_{N+1}(p_0)\| = \|f_{N+1}\|\). But by lemma 1.4, all the functions \(f_2, f_3, \ldots, f_{N+1}\) are continuous on the compact set \(Q\); hence the norm \(\|f_{N+1}\|\) must be attained. This contradiction means that there exists a path extremal for \(f_1\) with the infinite length.

**Sufficiency.** Let a path \(l = (p_1, p_2, \ldots, p_{2n})\) be closed and extremal for \(f_1\). Then

\[
|G_l(f)| = \|f - g_0\|. \tag{1.24}
\]

By Lemma 1.3,

\[
|G_l(f)| \leq E(f). \tag{1.25}
\]

It follows from (1.24) and (1.25) that \(g_0\) is a best approximation.

Let now a path \(l = (p_1, p_2, \ldots, p_n, \ldots)\) be infinite and extremal for \(f_1\). Consider the sequence \(l_n = (p_1, p_2, \ldots, p_n), \quad n = 1, 2, \ldots,\) of finite paths. By condition (A) of the theorem, for each \(l_n\) there exists a closed path \(l_{m_n} = (p_1, p_2, \ldots, p_n, q_{n+1}, \ldots, q_{n+m_n})\), where \(m_n \leq N_0\). Then for any positive integer \(n\),

\[
|G_{l_{m_n}}^n(f)| = |G_{l_{m_n}}^n(f - g_0)| \leq \frac{n\|f - g_0\| + m_n\|f - g_0\|}{n + m_n} = \|f - g_0\|
\]

and

\[
|G_{l_{m_n}}^n(f)| \geq \frac{n\|f - g_0\| - m_n\|f - g_0\|}{n + m_n} = \frac{n - m_n}{n + m_n} \|f - g_0\|.
\]

It follows from the above two inequalities for \(|G_{l_{m_n}}^n(f)|\) that

\[
\sup_{l_{m_n}^n} |G_{l_{m_n}}^n(f)| = \|f - g_0\|.
\]

This together with Lemma 1.3 give that

\[
\|f - g_0\| \leq E(f).
\]

Hence \(g_0\) is a best approximation.

Theorem 1.3 has been proved by using only methods of classical analysis. By implementing more deep techniques from functional analysis we will see
below that condition (A) and the convexity assumption on a compact set $Q$
can be dropped.

**Theorem 1.4.** Assume $Q$ is a compact subset of $\mathbb{R}^d$. A function $G_0 \in \mathcal{R}$
is a best approximation to a function $f \in C(Q)$ if and only if there exists a closed or infinite path $p = (p_1, p_2, ...)$ extremal for the function $f - G_0$.

**Proof.** *Sufficiency.* There are two possible cases. The first case happens when there exists a closed path $(p_1, ..., p_{2n})$ extremal for the function $f - G_0$. Let us check that in this case, $f - G_0$ is a best approximation. Indeed, on the one hand, the following equalities are valid.

\[
\left| \sum_{i=1}^{2n} (-1)^i f(p_i) \right| = \left| \sum_{i=1}^{2n} (-1)^i [f - G_0](p_i) \right| = 2n \|f - G_0\|.
\]

On the other hand, for any function $G \in \mathcal{R}$, we have

\[
\left| \sum_{i=1}^{2n} (-1)^i f(p_i) \right| = \left| \sum_{i=1}^{2n} (-1)^i [f - G](p_i) \right| \leq 2n \|f - G\|.
\]

Therefore, $\|f - G_0\| \leq \|f - G\|$ for any $G \in \mathcal{R}$. That is, $G_0$ is a best approximation.

The second case happens when we do not have closed paths extremal for $f - G_0$, but there exists an infinite path $(p_1, p_2, ...)$ extremal for $f - G_0$. To analyze this case, consider the following linear functional

\[
l_q : C(Q) \to \mathbb{R}, \quad l_q(F) = \frac{1}{n} \sum_{i=1}^{n} (-1)^i F(q_i),
\]

where $q = \{q_1, ..., q_n\}$ is a finite path in $Q$. It is easy to see that the norm $\|l_q\| \leq 1$ and $\|l_q\| = 1$ if and only if the set of points of $q$ with odd indices $O = \{q_i \in q : i is an odd number\}$ do not intersect with the set of points of $q$ with even indices $E = \{q_i \in q : i is an even number\}$. Indeed, from the definition of $l_q$ it follows that $|l_q(F)| \leq \|F\|$ for all functions $F \in C(Q)$, whence $\|l_q\| \leq 1$. If $O \cap E = \emptyset$, then for a function $F_0$ with the property $F_0(q_i) = -1$ if $i$ is odd, $F_0(q_i) = 1$ if $i$ is even and $-1 < F_0(x) < 1$ elsewhere on $Q$, we have $|l_q(F_0)| = \|F_0\|$. Hence, $\|l_q\| = 1$. Recall that such a function $F_0$ exists on the basis of Urysohn’s great lemma.
Note that if \( q \) is a closed path, then \( l_q \) annihilates all members of the class \( \mathcal{R} \). But in general, when \( q \) is not closed, we do not have the equality \( l_q(G) = 0 \), for all members \( G \in \mathcal{R} \). Nonetheless, this functional has the important property that

\[
|l_q(g_1 + g_2)| \leq \frac{2}{n}(\|g_1\| + \|g_2\|), \tag{1.26}
\]

where \( g_1 \) and \( g_2 \) are ridge functions with the directions \( a_1 \) and \( a_2 \), respectively, that is, \( g_1 = g_1(a_1 \cdot x) \) and \( g_2 = g_2(a_2 \cdot x) \). This property is important in the sense that if \( n \) is sufficiently large, then the functional \( l_q \) is close to an annihilating functional. To prove (1.26), note that \( |l_q(g_1)| \leq \frac{2}{n}\|g_1\| \) and \( |l_q(g_2)| \leq \frac{2}{n}\|g_2\| \). These estimates become obvious if consider the chain of equalities \( g_1(a_1 \cdot q_1) = g_1(a_1 \cdot q_2), g_1(a_1 \cdot q_3) = g_1(a_1 \cdot q_1), \ldots \) (or \( g_1(a_1 \cdot q_2) = g_1(a_1 \cdot q_3), g_1(a_1 \cdot q_1) = g_1(a_1 \cdot q_5), \ldots \)) for \( g_1(a_1 \cdot x) \) and the corresponding chain of equalities for \( g_2(a_2 \cdot x) \).

Now consider the infinite path \( p = (p_1, p_2, \ldots) \) and form the finite paths \( p_k = (p_1, \ldots, p_k), k = 1, 2, \ldots \). For ease of notation, let us set \( l_k = l_{p_k} \). The sequence \( \{l_k\}_{k=1}^{\infty} \) is a subset of the unit ball of the conjugate space \( C^*(Q) \). By the Banach-Alaoglu theorem, the unit ball is weak* compact in the weak* topology of \( C^*(Q) \) (see [122, p.68]). It follows from this theorem that the sequence \( \{l_k\}_{k=1}^{\infty} \) must have weak* cluster points. Suppose \( l^* \) denotes one of them. Without loss of generality we may assume that \( l_k \xrightarrow{\text{weak*}} l^* \), as \( k \to \infty \).

From (1.26) it follows that \( l^*(g_1 + g_2) = 0 \). That is, \( l^* \in \mathcal{R}^\perp \), where the symbol \( \mathcal{R}^\perp \) stands for the annihilator of \( \mathcal{R} \). Since in addition \( \|l^*\| \leq 1 \), we can write that

\[
|l^*(f)| = |l^*(f - G)| \leq \|f - G\|, \tag{1.27}
\]

for all functions \( G \in \mathcal{R} \). On the other hand, since the infinite bolt \( p \) is extremal for \( f - G_0 \)

\[
|l_k(f - G_0)| = \|f - G_0\|, \quad k = 1, 2, \ldots
\]

Therefore,

\[
|l^*(f)| = |l^*(f - G_0)| = \|f - G_0\|. \tag{1.28}
\]

From (1.27) and (1.28) we conclude that

\[
\|f - G_0\| \leq \|f - G\|,
\]

for all \( G \in \mathcal{R} \). In other words, \( G_0 \) is a best approximation to \( f \). We proved the sufficiency of the theorem.
Necessity. The proof of this part is mainly based on the following result of Singer: Let $X$ be a compact space, $U$ be a linear subspace of $C(X)$, $f \in C(X) \setminus U$ and $u_0 \in U$. Then $u_0$ is a best approximation to $f$ if and only if there exists a regular Borel measure $\mu$ on $X$ such that

1. The total variation $\|\mu\| = 1$;
2. $\mu$ is orthogonal to the subspace $U$, that is, $\int_X u d\mu = 0$ for all $u \in U$;
3. For the Jordan decomposition $\mu = \mu^+ - \mu^-$,

$$f(x) - u_0(x) = \begin{cases} \|f - u_0\| & \text{for } x \in S^+, \\ -\|f - u_0\| & \text{for } x \in S^-, \end{cases}$$

where $S^+$ and $S^-$ are closed supports of the positive measures $\mu^+$ and $\mu^-$, respectively.

Let us show how we use this theorem in the proof of necessity part of our theorem. Assume $G_0 \in \mathcal{R}$ is a best approximation. For the subspace $\mathcal{R}$, the existence of a measure $\mu$ satisfying the conditions (1)-(3) is a direct consequence of Singer’s result. Let $x_0$ be any point in $S^+$. Consider the point $y_0 = a_1 \cdot x_0$ and a $\delta$-neighborhood of $y_0$. That is, choose an arbitrary $\delta > 0$ and consider the set $I_\delta = (y_0 - \delta, y_0 + \delta) \cap a_1 \cdot Q$. Here, $a_1 \cdot Q = \{a_1 \cdot x : x \in Q\}$. For any subset $E \subset \mathbb{R}$, put

$$E^i = \{x \in Q : a_i \cdot x \in E\}, \ i = 1, 2.$$ 

Clearly, for some sets $E$, one or both the sets $E^i$ may be empty. Since $I_\delta \cap S^+$ is not empty (note that $x_0 \in I_\delta^1$), it follows that $\mu^+(I_\delta^1) > 0$. At the same time $\mu(I_\delta) = 0$, since $\mu$ is orthogonal to all functions $g_1(a_1 \cdot x)$. Therefore, $\mu^-(I_\delta) > 0$. We conclude that $I_\delta^1 \cap S^-$ is not empty. Denote this intersection by $A_\delta$. Tending $\delta$ to 0, we obtain a set $A$ which is a subset of $S^-$ and has the property that for each $x \in A$, we have $a_1 \cdot x = a_1 \cdot x_0$. Fix any point $x_1 \in A$. Changing $a_1$, $\mu^+$, $S^+$ to $a_2$, $\mu^-$ and $S^-$ correspondingly, repeat the above process with the point $y_1 = a_2 \cdot x_1$ and a $\delta$-neighborhood of $y_1$. Then we obtain a point $x_2 \in S^+$ such that $a_2 \cdot x_2 = a_2 \cdot x_1$. Continuing this process, one can construct points $x_3, x_4$, and so on. Note that the set of all constructed points $x_i, i = 0, 1, ...$, forms a path. By Singer’s above result, this path is extremal for the function $f - G_0$. We have proved the necessity and hence Theorem 1.4.

Theorem 1.4, in a more general setting, was proven in Pinkus [117, p.99] under additional assumption that $Q$ is convex. Convexity assumption was
made to guarantee continuity of the following functions

\[ g_{i}(t) = \max_{x \in Q} a_{i} \cdot x = t \quad \text{and} \quad g_{2,i}(t) = \min_{x \in Q} a_{i} \cdot x = t \]

where \( F \) is an arbitrary continuous function on \( Q \). Note that in the proof of Theorem 1.4 we did not need continuity of these functions.

It is well known that characterization theorems of this type are very essential in approximation theory. Chebyshev was the first to prove a similar result for polynomial approximation. Khavinson [79] characterized extremal elements in the special case of the problem considered here. His case allows the approximation of a continuous bivariate function \( f(x, y) \) by functions of the form \( \varphi(x) + \psi(y) \).

1.3.3 Construction of an extremal element

In 1951, Diliberto and Straus [26] established a formula for the error in approximating bivariate functions by sums of univariate functions. Their formula contains the supremum over all closed bolts. Although the formula is valid for all continuous functions, it is not easily calculable. Therefore, it does not give the desired effect if one is interested in the precise value of the approximation error. After this general result some authors started to seek easily calculable formulas for the approximation error by considering not the whole space, but some subsets of continuous functions (see, for example, [4, 7, 55, 56, 79, 121]). These subsets were chosen so that they could provide precise and easy computation of the approximation error. Since the set of ridge functions contain univariate functions as its proper subset, one may ask for explicit formulas for the error in approximating by sums of ridge functions.

In this section, we see how with the use of Theorem 1.3 (or 1.4) it is possible to find the error and an extremal element in approximating a continuous function by sums of ridge functions. We restrict ourselves to \( \mathbb{R}^2 \). To make the problem more precise, let \( \Omega \) be a compact set in \( \mathbb{R}^2 \), \( f(x_1, x_2) \in C(\Omega) \), \( \mathbf{a} = (a_1, a_2) \), \( \mathbf{b} = (b_1, b_2) \) be linearly independent vectors. We want, in some conditions on \( f \) and \( \Omega \), to establish a formula for an easy and direct computation of the error in approximating from \( R(\mathbf{a}, \mathbf{b}) \).

Theorem 1.5. Let

\[ \Omega = \left\{ \mathbf{x} \in \mathbb{R}^2 : c_1 \leq \mathbf{a} \cdot \mathbf{x} \leq d_1, \quad c_2 \leq \mathbf{b} \cdot \mathbf{x} \leq d_2 \right\}, \]
where \( c_1 < d_1 \) and \( c_2 < d_2 \). Let a function \( f(x) \in C(\Omega) \) have the continuous partial derivatives \( \frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2^2} \) and for any \( x \in \Omega \)

\[
\frac{\partial^2 f}{\partial x_1 \partial x_2} (a_1 b_2 + a_2 b_1) - \frac{\partial^2 f}{\partial x_1^2} a_2 b_2 - \frac{\partial^2 f}{\partial x_2^2} a_1 b_1 \geq 0.
\]

Then

\[
E(f) = \frac{1}{4} \left( f_1(c_1, c_2) + f_1(d_1, d_2) - f_1(c_1, d_2) - f_1(d_1, c_2) \right),
\]

where

\[
f_1(y_1, y_2) = f \left( \frac{y_1 b_2 - y_2 a_2}{a_1 b_2 - a_2 b_1}, \frac{y_2 a_1 - y_1 b_1}{a_1 b_2 - a_2 b_1} \right).
\]

Proof. Introduce the new variables

\[
y_1 = a_1 x_1 + a_2 x_2, \quad y_2 = b_1 x_1 + b_2 x_2.
\]

Since the vectors \((a_1, a_2)\) and \((b_1, b_2)\) are linearly independent, for any \((y_1, y_2) \in Y\), where \( Y = [c_1, d_1] \times [c_2, d_2] \), there exists only one solution \((x_1, x_2) \in \Omega\) of the system (1.30). The coordinates of this solution are

\[
x_1 = \frac{y_1 b_2 - y_2 a_2}{a_1 b_2 - a_2 b_1}, \quad x_2 = \frac{y_2 a_1 - y_1 b_1}{a_1 b_2 - a_2 b_1}.
\]

The linear transformation (1.31) transforms the function \( f(x_1, x_2) \) to the function \( f_1(y_1, y_2) \). Consider the approximation of \( f_1(y_1, y_2) \) from the set

\[
Z = \{ z_1(y_1) + z_2(y_2) : z_i \in C(\mathbb{R}), \; i = 1, 2 \}.
\]

It is easy to see that

\[
E(f, R) = E(f_1, Z).
\]

With each rectangle \( S = [u_1, v_1] \times [u_2, v_2] \subset Y \) we associate the functional

\[
L(h, S) = \frac{1}{4} (h(u_1, u_2) + h(v_1, v_2) - h(u_1, v_2) - h(v_1, u_2)), \quad h \in C(Y).
\]

This functional has the following obvious properties:
(i) \( L(z, S) = 0 \) for any \( z \in \mathcal{Z} \) and \( S \subset Y \).

(ii) For any point \( (y_1, y_2) \in Y \), \( L(f_1, Y) = \sum_{i=1}^{4} L(f_1, S_i) \), where \( S_1 = [c_1, y_1] \times [c_2, y_2] \), \( S_2 = [y_1, d_1] \times [y_2, d_2] \), \( S_3 = [c_1, y_1] \times [y_2, d_2] \), \( S_4 = [y_1, d_1] \times [c_2, y_2] \).

By the conditions of the theorem, it is not difficult to verify that 
\[
\frac{\partial^2 f_1}{\partial y_1 \partial y_2} \geq 0 \quad \text{for any} \quad (y_1, y_2) \in Y.
\]

Integrating both sides of the last inequality over arbitrary rectangle \( S = [u_1, v_1] \times [u_2, v_2] \subset Y \), we obtain that 
\[
L(f_1, S) \geq 0. \quad (1.33)
\]

Set the function 
\[
f_2(y_1, y_2) = L(f_1, S_1) + L(f_1, S_2) - L(f_1, S_3) - L(f_1, S_4). \quad (1.34)
\]

It is not difficult to verify that the function \( f_1 - f_2 \) belongs to \( \mathcal{Z} \). Hence 
\[
E(f_1, \mathcal{Z}) = E(f_2, \mathcal{Z}). \quad (1.35)
\]

Calculate the norm \( \|f_2\| \). From the property (ii), it follows that 
\[
f_2(y_1, y_2) = L(f_1, Y) - 2(L(f_1, S_3) + L(f_1, S_4))
\]
and 
\[
f_2(y_1, y_2) = 2(L(f_1, S_1) + L(f_1, S_2)) - L(f_1, Y).
\]

From the last equalities and (1.33), we obtain that 
\[
|f_2(y_1, y_2)| \leq L(f_1, Y), \quad \text{for any} \quad (y_1, y_2) \in Y.
\]

On the other hand, one can check that 
\[
f_2(c_1, c_2) = f_2(d_1, d_2) = L(f_1, Y) \quad (1.36)
\]
and 
\[
f_2(c_1, d_2) = f_2(d_1, c_2) = -L(f_1, Y). \quad (1.37)
\]

Therefore, 
\[
\|f_2\| = L(f_1, Y). \quad (1.38)
\]

Electronic copy available at: https://ssrn.com/abstract=3618165
Note that the points \((c_1, c_2), (c_1, d_2), (d_1, d_2), (d_1, c_2)\) in the given order form a closed path with respect to the directions \((0, 1)\) and \((1, 0)\). We conclude from (1.36)-(1.38) that this path is extremal for \(f_2\). By Theorem 1.3, \(z_0 = 0\) is a best approximation to \(f_2\). Hence
\[
E(f_2, Z) = L(f_1, Y).
\] (1.39)

Now from (1.32),(1.35) and (1.39) we finally conclude that
\[
E(f, R) = L(f_1, Y) = \frac{1}{4} \left( f_1(c_1, c_2) + f_1(d_1, d_2) - f_1(c_1, d_2) - f_1(d_1, c_2) \right),
\]
which is the desired result.

**Corollary 1.4.** Let all the conditions of Theorem 1.5 hold and \(f_1(y_1, y_2)\) is the function defined in (1.29). Then the function \(g_0(y_1, y_2) = g_{1,0}(y_1) + g_{2,0}(y_2)\), where
\[
g_{1,0}(y_1) = \frac{1}{2} f_1(y_1, c_2) + \frac{1}{2} f_1(y_1, d_2) - \frac{1}{4} f_1(c_1, c_2) - \frac{1}{4} f_1(d_1, d_2),
\]
\[
g_{2,0}(y_2) = \frac{1}{2} f_1(c_1, y_2) + \frac{1}{2} f_1(d_1, y_2) - \frac{1}{4} f_1(c_1, d_2) - \frac{1}{4} f_1(d_1, c_2)
\]
and \(y_1 = a_1 x_1 + a_2 x_2, y_2 = b_1 x_1 + b_2 x_2,\) is a best approximation from the set \(R(a, b)\) to the function \(f\).

**Proof.** It is not difficult to verify that the function \(f_2(y_1, y_2)\) defined in (1.34) has the form
\[
f_2(y_1, y_2) = f_1(y_1, y_2) - g_{1,0}(y_1) - g_{2,0}(y_2).
\]
On the other hand, we know from the proof of Theorem 1.5 that
\[
E(f_1, Z) = \|f_2\|.
\]
Therefore, the function \(g_{1,0}(y_1) + g_{2,0}(y_2)\) is a best approximation to \(f_1\). Then the function \(g_{1,0}(a \cdot x) + g_{2,0}(b \cdot x)\) is an extremal element from \(R(a, b)\) to \(f(x)\). □

**Remark 1.1.** Rivlin and Sibner [121], and Babaev [7] proved Theorem 1.5 for the case in which \(a\) and \(b\) are the unit vectors. Our proof of Theorem 1.5 is different, short and elementary. Moreover, it has turned out to be useful in constructing of an extremal element (see the proof of Corollary 1.4).
1.3.4 Density of ridge functions and some problems

Let $a^1$ and $a^2$ be nonzero directions in $\mathbb{R}^d$. One may ask the following question: are there cases in which the set $\mathcal{R}(a^1,a^2)$ is dense in the space of all continuous functions? Undoubtedly, a positive answer depends on the geometrical structure of compact sets over which all the considered functions are defined. This problem may be interesting in the theory of partial differential equations. Take, for example, equation (1.10). A positive answer to the problem means that for any continuous function $f$ there exist solutions of the given equation uniformly converging to $f$.

It should be remarked that our problem is a special case of the problem considered by Marshall and O'Farrell. In [107], they obtained a necessary and sufficient condition for a sum $A_1 + A_2$ of two subalgebras to be dense in $C(U)$, where $C(U)$ denotes the space of real-valued continuous functions on a compact Hausdorff space $U$. Below we describe Marshall and O'Farrell's result for sums of ridge functions.

Let $X$ be a compact subset of $\mathbb{R}^d$. The relation on $X$, defined by setting $x \approx y$ if $x$ and $y$ belong to some path in $X$, is an equivalence relation. The equivalence classes we call orbits.

**Theorem 1.6.** Let $X$ be a compact subset of $\mathbb{R}^d$ with all its orbits closed. The set $\mathcal{R}(a^1,a^2)$ is dense in $C(X)$ if and only if $X$ contains no closed path with respect to the directions $a^1$ and $a^2$.

The proof immediately follows from proposition 2 in [108] established for the sum of two algebras. Since that proposition was given without proof, for completeness of the exposition we give the proof of Theorem 1.6.

**Proof. Necessity.** If $X$ has closed paths, then $X$ has closed paths $p' = (p'_1, ..., p'_{2m})$ such that all points $p'_1, ..., p'_{2m}$ are distinct. In fact, such special paths can be obtained from any closed path $p = (p_1, ..., p_{2n})$ by the following simple algorithm: if the points of the path $p$ are not all distinct, let $i$ and $k > 0$ be the minimal indices such that $p_i = p_{i+2k}$; delete from $p$ the subsequence $p_{i+1}, ..., p_{i+2k}$ and call $p$ the obtained path; repeat the above step until all points of $p$ are all distinct; set $p' := p$. By Urysohn's great lemma, there exist continuous functions $h = h(x)$ on $X$ such that $h(p'_i) = 1$, $i = 1, 3, ..., 2m-1$, $h(p'_i) = -1$, $i = 2, 4, ..., 2m$ and $-1 < h(x) < 1$ elsewhere. Consider the measure
where $\delta_{p'_i}$ is a point mass at $p'_i$. For this measure, $\int_X h \, d\mu_{p'_i} = 1$ and $\int_X g \, d\mu_{p'_i} = 0$ for all functions $g \in \mathcal{R}(a^1, a^2)$. Thus the set $\mathcal{R}(a^1, a^2)$ cannot be dense in $C(X)$.

**Sufficiency.** We are going to prove that the only annihilating regular Borel measure for $\mathcal{R}(a^1, a^2)$ is the zero measure. Suppose, contrary to this assumption, there exists a nonzero annihilating measure on $X$ for $\mathcal{R}(a^1, a^2)$. The class of such measures with total variation not more than 1 we denote by $S$. Clearly, $S$ is weak-* compact and convex. By the Krein-Milman theorem, there exists an extreme measure $\mu$ in $S$. Since the orbits are closed, $\mu$ must be supported on a single orbit. Denote this orbit by $T$.

For $i = 1, 2$, let $X_i$ be the quotient space of $X$ obtained by identifying the points $y$ and $z$ whenever $a^i \cdot y = a^i \cdot z$. Let $\pi_i$ be the natural projection of $X$ onto $X_i$. For a fixed point $t \in X$ set $T_1 = \{t\}$, $T_2 = \pi_1^{-1}(\pi_1 T_1)$, $T_3 = \pi_2^{-1}(\pi_2 T_2)$, $T_4 = \pi_1^{-1}(\pi_1 T_3)$, ... Obviously, $T_1 \subset T_2 \subset T_3 \subset \cdots$. Therefore, for some $k \in \mathbb{N}$, $|\mu|(T_{2k}) > 0$, where $|\mu|$ is a total variation measure of $\mu$. Since $\mu$ is orthogonal to every continuous function of the form $g(a^1 \cdot x)$, $\mu(T_{2k}) = 0$. From the Haar decomposition $\mu(T_{2k}) = \mu^+(T_{2k}) - \mu^-(T_{2k})$ it follows that $\mu^+(T_{2k}) = \mu^-(T_{2k}) > 0$. Fix a Borel subset $S_0 \subset T_{2k}$ such that $\mu^+(S_0) > 0$ and $\mu^-(S_0) = 0$. Since $\mu$ is orthogonal to every continuous function of the form $g(a^2 \cdot x)$, $\mu(\pi_2^{-1}(\pi_2 S_0)) = 0$. Therefore, one can chose a Borel set $S_1$ such that $S_1 \subset \pi_2^{-1}(\pi_2 S_0) \subset T_{2k+1}$, $S_1 \cap S_0 = \emptyset$, $\mu^+(S_1) = 0$, $\mu^-(S_1) \geq \mu^+(S_0)$. By the same way one can chose a Borel set $S_2$ such that $S_2 \subset \pi_1^{-1}(\pi_1 S_1) \subset T_{2k+2}$, $S_2 \cap S_1 = \emptyset$, $\mu^-(S_2) = 0$, $\mu^+(S_2) \geq \mu^-(S_1)$, and so on.

The sets $S_0, S_1, S_2, \ldots$are pairwise disjoint. For otherwise, there would exist positive integers $n$ and $m$, with $n < m$ and a path $(y_n, y_{n+1}, \ldots, y_m)$ such that $y_i \in S_i$ for $i = n, \ldots, m$ and $y_m \in S_m \cap S_n$. But then there would exist paths $(z_1, z_2, \ldots, z_{n-1}, y_n)$ and $(z_1, z'_2, \ldots, z'_{n-1}, y_m)$ with $z_i$ and $z'_i$ in $T_i$ for $i = 2, \ldots, n - 1$. Hence, the set

$$\{z_1, z_2, \ldots, z_{n-1}, y_n, y_{n+1}, \ldots, y_m, z'_1, \ldots, z'_{n-1}, z'_2, z_1\}$$

would contain a closed path. This would contradict our assumption on $X$. 

54
Now, since the sets $S_0, S_1, S_2, \ldots$ are pairwise disjoint, and $|\mu|(S_i) \geq \mu^+(S_0) > 0$ for each $i = 1, 2, \ldots$, it follows that the total variation of $\mu$ is infinite. This contradiction completes the proof. \qed

The following corollary concerns the problem considered by Colitschek and Light in [36].

**Corollary 1.5.** Let $D$ be a compact subset of $\mathbb{R}^2$ with all its orbits closed. Let $W$ denote the set of all solutions of the wave equation

$$\frac{\partial^2 w}{\partial s \partial t}(s, t) = 0, \quad (s, t) \in D.$$  

Then

$$\inf_{w \in W} \|f - w\| = 0$$

for any continuous function $f(s, t)$ on $D$ if and only if $D$ contains no closed bolt of lightning.

**Proof.** Let $\pi_1$ and $\pi_2$ denote the usual coordinate projections, viz: $\pi_1(s, t) = s$ and $\pi_2(s, t) = t, (s, t) \in \mathbb{R}^2$. Set $S = \pi_1(D)$ and $T = \pi_2(D)$. It is easy to see that

$$W = \left\{ w \in C(D) : w(s, t) = x(s) + y(t), \quad x \in C^2(S), \quad y \in C^2(T) \right\}.$$  

Set

$$\widetilde{W} = \left\{ w \in C(D) : w(s, t) = x(s) + y(t), \quad x \in C(S), \quad y \in C(T) \right\}.$$  

Since the set $W$ is dense in $\widetilde{W}$,

$$\inf_{w \in W} \|f - w\| = \inf_{w \in \widetilde{W}} \|f - w\|.$$  

But by Theorem 1.6, the equality

$$\inf_{w \in \widetilde{W}} \|f - w\| = 0$$

holds for any $f \in C(D)$ if and only if $D$ contains no closed bolt of lightning. \qed

55
Let us indicate difficulties with the sum of more than two ridge functions. Consider the set
\[ R(a^1,\ldots,a^r) = \left\{ \sum_{i=1}^{r} g_i(a^i \cdot x) : g_i \in C(\mathbb{R}), i = 1,\ldots,r \right\}, \]
where \( a^1,\ldots,a^r \) are pairwise linearly independent vectors in \( \mathbb{R}^d \setminus \{0\} \). Let \( r \geq 3 \). How can we define a path? Recall that in the case when \( r = 2 \), a path is an ordered set of points \((p_1, p_2, \ldots, p_n)\) in \( \mathbb{R}^d \) with edges \( p_i p_{i+1} \) in alternating hyperplanes. The first, the third, the fifth,... hyperplanes (also the second, the fourth, the sixth,... hyperplanes) are parallel. If not differentiate between parallel hyperplanes, the path \((p_1, p_2, \ldots, p_n)\) can be considered as a trace of some point traveling in two alternating hyperplanes. In this case, if the point starts and stops at the same location (i.e., if \( p_n = p_1 \)) and \( n \) is an odd number, then the path functional
\[ G(f) = \frac{1}{n-1} \sum_{i=1}^{n-1} (-1)^{i+1} f(p_i), \]
annihilates each sum of ridge functions with the two fixed directions. The picture becomes more complicated when the number of directions more than two. The simple generalization of the above-mentioned arguments demands a point traveling in three or more alternating hyperplanes. But in this case the appropriate generalization of the functional \( G \) does not annihilate functions from \( R(a^1,\ldots,a^r) \).

There were several attempts to fill this gap in the special case when \( r = d \) and \( a^1,\ldots,a^r \) are the unit vectors. Unfortunately, all these attempts failed (see, for example, the attempts in [26,37] and the refutations in [4,28,109]).

At the end we want to draw the readers attention to the following problems. All these problems are general and not solved by the methods introduced in this section.

Let \( Q \) be a compact subset of \( \mathbb{R}^d \). Consider the approximation of a continuous function defined on \( Q \) by functions from \( R(a^1,\ldots,a^r) \). Let \( r \geq 3 \).

**Problem 3.** Characterize those functions from \( R(a^1,\ldots,a^r) \) that are extremal to a given continuous function.

**Problem 4.** Establish explicit formulas for the error in approximating from \( R(a^1,\ldots,a^r) \) and construct a best approximation.

**Problem 5.** Find necessary and sufficient geometrical conditions for the set \( R(a^1,\ldots,a^r) \) to be dense in \( C(Q) \).
It should be remarked that in [108], Problem 5 was set up for the sum of \( r \) subalgebras of \( C(Q) \). Lin and Pinkus [95] proved that the set \( \mathcal{R}(a^1, ..., a^r) \) (\( r \) may be very large) is not dense in \( C(\mathbb{R}^d) \) in the topology of uniform convergence on compact subsets of \( \mathbb{R}^d \). That is, there are compact sets \( Q \subset \mathbb{R}^d \) such that \( \mathcal{R}(a^1, ..., a^r) \) is not dense in \( C(Q) \). In the case \( r = 2 \), Theorem 1.6 complements this result, by describing compact sets \( Q \subset \mathbb{R}^2 \), for which \( \mathcal{R}(a^1,a^2) \) is dense in \( C(Q) \).

1.4 Sums of continuous ridge functions

In this section, we find geometric means of deciding if any continuous multivariate function can be represented by a sum of two continuous ridge functions.

1.4.1 Exposition of the problem

In this section, we will consider the following representation problem associated with the set \( \mathcal{R}(a^1, ..., a^r) \).

**Problem 6.** Let \( X \) be a compact subset of \( \mathbb{R}^d \). Give geometrical conditions that are necessary and sufficient for

\[
\mathcal{R}(a^1, ..., a^r) = C(X),
\]

where \( C(X) \) is the space of continuous functions on \( X \) furnished with the uniform norm.

We solve this problem for \( r = 2 \) and indicate some difficulties related to the case \( r \geq 3 \). In the sequel, we will use the notation

\[
H_1 = H_1(X) = \{ g_1(a^1 \cdot x) : g_1 \in C(\mathbb{R}) \},
\]

\[
H_2 = H_2(X) = \{ g_2(a^2 \cdot x) : g_2 \in C(\mathbb{R}) \}.
\]

Note that by this notation, \( \mathcal{R}(a^1,a^2) = H_1 + H_2 \).

At the end of this section, we generalize the obtained result from \( H_1 + H_2 \) to the set of sums \( g_1(h_1(x)) + g_2(h_2(x)) \), where \( h_1, h_2 \) are fixed continuous functions on \( X \).
1.4.2 The representation theorem

Theorem 1.7. Let \( X \) be a compact subset of \( \mathbb{R}^d \). The equality

\[
H_1 (X) + H_2 (X) = C (X)
\]

holds if and only if \( X \) contains no closed path and there exists a positive integer \( n_0 \) such that the lengths of paths in \( X \) are bounded by \( n_0 \).

Proof. Necessity. Let \( H_1 + H_2 = C (X) \). Consider the linear operator

\[
A : H_1 \times H_2 \rightarrow C (X), \quad A [(g_1, g_2)] = g_1 + g_2,
\]

where \( g_1 \in H_1, g_2 \in H_2 \). The norm on \( H_1 \times H_2 \) we define as

\[
\|(g_1, g_2)\| = \|g_1\| + \|g_2\|.
\]

It is obvious that the operator \( A \) is continuous with respect to this norm. Besides, since \( C (X) = H_1 + H_2 \), \( A \) is a surjection. Consider the conjugate operator

\[
A^* : C (X)^* \rightarrow [H_1 \times H_2]^*, \quad A^* [G] = (G_1, G_2),
\]

where the functionals \( G_1 \) and \( G_2 \) are defined as follows

\[
G_1 (g_1) = G (g_1), g_1 \in H_1; \quad G_2 (g_2) = G (g_2), g_2 \in H_2.
\]

An element \((G_1, G_2)\) from \([H_1 \times H_2]^*\) has the norm

\[
\|(G_1, G_2)\| = \max \{\|G_1\|, \|G_2\|\}. \tag{1.40}
\]

Let now \( p = (p_1, ..., p_m) \) be any path with different points: \( p_i \neq p_j \) for any \( i \neq j, 1 \leq i, j \leq m \). We associate with \( p \) the following functional over \( C (X) \)

\[
L [f] = \frac{1}{m} \sum_{i=1}^{m} (-1)^{i-1} f (p_i).
\]

Since \( |L(f)| \leq \|f\| \) and \( |L(g)| = \|g\| \) for a continuous function \( g(x) \) such that \( g(p_i) = 1 \), for odd indices \( i \), \( g(p_j) = -1 \), for even indices \( j \) and \(-1 < g(x) < 1\) elsewhere, we obtain that \( \|L\| = 1 \). Let \( A^* [L] = (L_1, L_2) \). One can easily verify that

\[
\|L_i\| \leq \frac{2}{m}, i = 1, 2.
\]
Therefore, from (1.40) we obtain that
\[ \|A^* [L]\| \leq \frac{2}{m}. \]  
(1.41)

Since \( A \) is a surjection, there exists \( \delta > 0 \) such that
\[ \|A^* [G]\| \geq \delta \|G\| \quad \text{for any functional } \ G \in C (X)^* \]
Hence
\[ \|A^* [L]\| \geq \delta. \]  
(1.42)

Now from (1.41) and (1.42) we conclude that
\[ m \leq \frac{2}{\delta}. \]

This means that for a path with different points, \( n_0 \) can be chosen as \( \left\lceil \frac{2}{\delta} \right\rceil + 1 \).

Let now \( p = (p_1, \ldots, p_m) \) be a path with at least two coinciding points. Then we can form a closed path with different points. This may be done by the following way: let \( i \) and \( j \) be indices such that \( p_i = p_j \) and \( j - i \) takes its minimal value. Note that in this case all the points \( p_i, p_{i+1}, \ldots, p_{j-1} \) are distinct. Now if \( j - i \) is an even number, then the path \( (p_i, p_{i+1}, \ldots, p_{j-1}) \), and if \( j - i \) is an odd number, then the path \( (p_{i+1}, \ldots, p_{j-1}) \) is a closed path with different points. It remains to show that \( X \) cannot possess closed paths with different points. Indeed, if \( q = (q_1, \ldots, q_{2k}) \) is a path of this type, then the functional \( L \), associated with \( q \), annihilates all functions from \( H_1 + H_2 \).

On the other hand, \( L [f] = 1 \) for a continuous function \( f \) on \( X \) satisfying the conditions \( f (t) = 1 \) if \( t \in \{q_1, q_3, \ldots, q_{2k-1}\} \); \( f (t) = -1 \) if \( t \in \{q_2, q_4, \ldots, q_{2k}\} \); \( f (t) \in (-1; 1) \) if \( t \in X \setminus q \). This implies on the contrary to our assumption that \( H_1 + H_2 \neq C (X) \). The necessity has been proved.

Sufficiency. Let \( X \) contains no closed path and the lengths of all paths are bounded by some positive integer \( n_0 \). We may suppose that any path has different points. Indeed, in other case we can form a closed path, which contradicts our assumption.

For \( i = 1, 2 \), let \( X_i \) be the quotient space of \( X \) obtained by identifying the points \( a \) and \( b \) whenever \( g (a) = g (b) \) for each \( g \) in \( H_i \). Let \( \pi_i \) be the natural projection of \( X \) onto \( X_i \). For a point \( t \in X \) set \( T_1 = \pi_1^{-1} (\pi_1 t), T_2 = \pi_2^{-1} (\pi_2 T_1), \ldots \). By \( O (t) \) denote the orbit of \( X \) containing \( t \). Since the length of any path in \( X \) is not more than \( n_0 \), we conclude that \( O (t) = T_{n_0} \). Since
X is compact, the sets $T_1, T_2, ..., T_{n_0}$, hence $O(t)$, are compact. By Theorem 1.6 (see Section 1.3), $H_1 + H_2 = C(X)$.

Now show that $H_1 + H_2$ is closed in $C(X)$. Set

$$H_3 = H_1 \cap H_2.$$ 

Let $X_3$ and $\pi_3$ be the associated quotient space and projection. Fix some $a \in X_3$. Show, within conditions of our theorem, that if $t \in \pi_3^{-1}(a)$, then $O(t) = \pi_3^{-1}(a)$. The inclusion $O(t) \subset \pi_3^{-1}(a)$ is obvious. Suppose that there exists a point $t_1 \in \pi_3^{-1}(a)$ such that $t_1 \notin O(t)$. Then $O(t) \cap O(t_1) = \emptyset$. By $X|O$ denote the factor space generated by orbits of $X$. $X|O$ is a normal topological space with its natural factor topology. Hence we can construct a continuous function $u \in C(X|O)$ such that $u(O(t)) = 0$, $u(O(t_1)) = 1$. The function $u(x) = u(O(x))$, $x \in X$, is continuous on $X$ and belongs to $H_3$ as a function being constant on each orbit. But, since $O(t) \subset \pi_3^{-1}(a)$ and $O(t_1) \subset \pi_3^{-1}(a)$, the function $v(x)$ can not take different values on $O(t)$ and $O(t_1)$. This contradiction means that there is not a point $t_1 \in \pi_3^{-1}(a)$ such that $t_1 \notin O(t)$. Thus,

$$O(t) = \pi_3^{-1}(a) \quad (1.43)$$

for any $a \in X_3$ and $t \in \pi_3^{-1}(a)$.

Now prove that there exists a positive real number $c$ such that

$$\sup_{z \in X_3} \var f \leq c \sup_{y \in X_2} \var f$$ \quad (1.44)

for all $f$ in $H_1$. Note that for $Y \subset X$, $\var f$ is the variation of $f$ on the set $Y$. That is,

$$\var f = \sup_{x,y \in Y} |f(x) - f(y)|.$$ 

Due to (1.43), inequality (1.44) can be written in the following form

$$\sup_{t \in X} \var f \leq c \sup_{t \in X} \var f$$ \quad (1.45)

for all $f \in H_1$.

Let $t \in X$ and $t_1, t_2$ be arbitrary points of $O(t)$. Then there is a path $(b_1, b_2, ..., b_m)$ with $b_1 = t_1$ and $b_m = t_2$. Besides, by the condition, $m \leq n_0$.
Let first $a^2_1 b_1 = a^2_2 b_2, a^1_1 b_2 = a^1_3 b_3, \ldots, a^2_m b_{m-1} = a^2_m b_m$. Then for any function $f \in H_1$

$$|f(t_1) - f(t_2)| = |f(b_1) - f(b_2) + \ldots - f(b_m)| \leq$$

$$\leq |f(b_1) - f(b_2)| + \ldots + |f(b_{m-1}) - f(b_m)| \leq \frac{n_0}{2} \sup_{t \in X} \text{var} f. \quad (1.46)$$

It is not difficult to verify that inequality (1.46) holds in all other possible cases of the path $(b_1, \ldots, b_m)$. Now from (1.46) we obtain (1.45), hence (1.44), where $c = \frac{n_0}{2}$. In [108], Marshall and O’Farrell proved the following result (see Proposition 4 in [108]): Let $A_1$ and $A_2$ be closed subalgebras of $C(X)$ that contain the constants. Let $(X_1, \pi_1)$, $(X_2, \pi_2)$ and $(X_3, \pi_3)$ be the quotient spaces and projections associated with the algebras $A_1$, $A_2$ and $A_3 = A_1 \cap A_2$ respectively. Then $A_1 + A_2$ is closed in $C(X)$ if and only if there exists a positive real number $c$ such that

$$\sup_{z \in X_3 \pi_3^{-1}(z)} \text{var} f \leq c \sup_{y \in X_2 \pi_2^{-1}(y)} \text{var} f$$

for all $f \in A_1$.

By this proposition, (1.44) implies that $H_1 + H_2$ is closed in $C(X)$. Thus we finally obtain that $H_1 + H_2 = C(X)$. \qed

Paths with respect to two directions are explicit objects and give geometric means of deciding if $H_1 + H_2 = C(X)$. Let us show this in the example of the bivariate ridge functions $g_1 = x_1 + x_2$ and $g_2 = x_1 - x_2$. If $X$ is the union of two parallel line segments in $\mathbb{R}^2$, not parallel to any of the lines $x_1 + x_2 = 0$ and $x_1 - x_2 = 0$, then Theorem 1.7 holds. If $X$ is any bounded part of the graph of the function $x_2 = \text{arcsin} \left(\sin x_1\right)$, then Theorem 1.7 also holds. Let now $X$ be the set

$$\{(0, 0), (1, -1), (0, -2), (-1\frac{1}{2}, -\frac{1}{2}), (0, 1), (\frac{3}{2}, \frac{1}{2}), (0, -\frac{1}{2}), (-\frac{3}{8}, -\frac{1}{8}), (0, \frac{1}{4}), (\frac{5}{16}, \frac{1}{16}), \ldots\}.$$

In this case, there is no positive integer bounding lengths of all paths. Thus Theorem 1.7 fails. Note that since orbits of all paths are closed, Theorem 1.6 from the previous section shows $H_1 + H_2$ is dense in $C(X)$.

If $X$ is any set with interior points, then both Theorem 1.6 and Theorem 1.7 fail, since any such set contains the vertices of some parallelogram with sides parallel to the directions $a^1$ and $a^2$, that is a closed path.
To solve Problem 6 for the general case in which \( r \geq 3 \) is more difficult than to solve it for \( r = 2 \). In this case, we even don’t know what objects will be an appropriate generalization of paths (see Section 1.3.4). Representation by sums of continuous ridge functions requires more complicated relations between points of \( X \) than relations induced by paths with respect to only two directions. If disregard continuity, we have seen in Section 1.2 that cycles with respect to \( n \) directions are able to solve the representation problem. But when some topology is involved, the picture is quite different. No one knows a geometrically explicit solution to the problem of representation of continuous multivariate functions by sums of continuous ridge functions. Nevertheless, it should be noted that this problem in quite abstract form was solved by Sternfeld. His solution involves a family of functions that separates regular Borel measures on a given compact set \( X \). A family \( F = \{ h \} \subseteq C(X) \) is said to be a measure separating family (m.s.f.) if there exists a number \( 0 < \lambda \leq 1 \) such that for any measure \( \mu \) in \( C(X)^* \), the inequality \( \| \mu \circ h^{-1} \| \geq \lambda \| \mu \| \) holds for some \( h \in F \). Sternfeld [131], in particular, proved that \( R(a^1,...,a^r) = C(X) \) if and only if the family \( \{ a^i \cdot x, i = 1, ..., r \} \) is a m.s.f.

Definition 1.5. Let \( X \) be a compact set in \( \mathbb{R}^d \) and \( h_i \in C(X), i = 1, 2 \). A finite ordered subset \( (p_1, p_2, ..., p_m) \) of \( X \) with \( p_i \neq p_{i+1} (i = 1, ..., m - 1) \), and either \( h_1 (p_1) = h_2 (p_2) \), \( h_2 (p_2) = h_1 (p_3) \), \( h_1 (p_3) = h_2 (p_4) \), ..., or \( h_2 (p_1) = h_2 (p_2) \), \( h_1 (p_2) = h_1 (p_3) \), \( h_2 (p_3) = h_2 (p_4) \), ... is called a path with respect to the functions \( h_1 \) and \( h_2 \) or shortly an \( h_1 \)-\( h_2 \) path.

Theorem 1.8. Let \( X \) be a compact subset of \( \mathbb{R}^d \). Every function \( f \in C(X) \) admits a representation

\[
    f(x) = g_1(h_1(x)) + g_2(h_2(x)), \quad g_1, g_2 \in C(\mathbb{R})
\]

if and only if the set \( X \) contains no closed \( h_1 \)-\( h_2 \) path and there exists a positive integer \( n_0 \) such that the lengths of \( h_1 \)-\( h_2 \) paths in \( X \) are bounded by \( n_0 \).

The proof can be carried out by the same arguments as above.
It should be noted that Theorem 1.8 was first proved by Khavinson in his monograph [76]. Khavinson’s proof (see [76, p.87]) used theorems of Sternfeld [132] and Medvedev [76, Theorem 2.2], whereas our proof, which generalizes the ideas of Khavinson, was based on the above proposition of Marshall and O’Farrell.

1.5 On the proximinality of ridge functions

In this section, using two results of Garkavi, Medvedev and Khavinson [35], we give sufficient conditions for proximinality of sums of two ridge functions with bounded and continuous summands in the spaces of bounded and continuous multivariate functions, respectively. In the first case, we give an example which shows that the corresponding sufficient condition cannot be made weaker for certain subsets of \( \mathbb{R}^n \). In the second case, we obtain also a necessary condition for proximinality. All the results are furnished with plenty of examples. The results, examples and following discussions naturally lead us to a conjecture on the proximinality of the considered class of ridge functions.

1.5.1 Problem statement

Let \( E \) be a normed linear space and \( F \) be its subspace. We say that \( F \) is proximinal in \( E \) if for any element \( e \in E \) there exists at least one element \( f_0 \in F \) such that

\[
\| e - f_0 \| = \inf_{f \in F} \| e - f \|.
\]

In this case, the element \( f_0 \) is said to be extremal to \( e \).

We are interested in the problem of proximinality of the set of linear combinations of ridge functions in the spaces of bounded and continuous functions respectively. This problem will be considered in the simplest case when the class of approximating functions is the set

\[
\mathcal{R} = \mathcal{R} (\mathbf{a}^1, \mathbf{a}^2) = \{ g_1 (\mathbf{a}^1 \cdot \mathbf{x}) + g_2 (\mathbf{a}^2 \cdot \mathbf{x}) : g_i : \mathbb{R} \to \mathbb{R}, i = 1, 2 \}.
\]

Here \( \mathbf{a}^1 \) and \( \mathbf{a}^2 \) are fixed directions and we vary over \( g_i \). It is clear that this is a linear space. Consider the following three subspaces of \( \mathcal{R} \). The first is obtained by taking only bounded sums \( g_1 (\mathbf{a}^1 \cdot \mathbf{x}) + g_2 (\mathbf{a}^2 \cdot \mathbf{x}) \) over some set \( X \).
in $\mathbb{R}^n$. We denote this subspace by $\mathcal{R}_a(X)$. The second and the third are subspaces of $\mathcal{R}$ with bounded and continuous summands $g_i(a^i \cdot x)$, $i = 1, 2$, on $X$ respectively. These subspaces will be denoted by $\mathcal{R}_b(X)$ and $\mathcal{R}_c(X)$. In the case of $\mathcal{R}_c(X)$, the set $X$ is considered to be compact.

Let $B(X)$ and $C(X)$ be the spaces of bounded and continuous multivariate functions over $X$ respectively. What conditions must one impose on $X$ in order that the sets $\mathcal{R}_a(X)$ and $\mathcal{R}_b(X)$ be proximinal in $B(X)$ and the set $\mathcal{R}_c(X)$ be proximinal in $C(X)$? We are also interested in necessary conditions for proximinality. It follows from one result of Garkavi, Medvedev and Khavinson (see theorem 1 [35]) that $\mathcal{R}_a(X)$ is proximinal in $B(X)$ for all subsets $X$ of $\mathbb{R}^n$. There is also an answer (see theorem 2 [35]) for proximinality of $\mathcal{R}_b(X)$ in $B(X)$. This will be discussed in Section 1.5.2. Is the set $\mathcal{R}_b(X)$ always proximinal in $B(X)$? There is an an example of a set $X \subset \mathbb{R}^n$ and a bounded function $f$ on $X$ for which there does not exist an extremal element in $\mathcal{R}_b(X)$.

In Section 1.5.3, we will obtain sufficient conditions for the existence of extremal elements from $\mathcal{R}_c(X)$ to an arbitrary function $f \in C(X)$. Based on one result of Marshall and O’Farrell [108], we will also give a necessary condition for proximinality of $\mathcal{R}_c(X)$ in $C(X)$. All the theorems, following discussions and examples of the paper will lead us naturally to a conjecture on the proximinality of the subspaces $\mathcal{R}_b(X)$ and $\mathcal{R}_c(X)$ in the spaces $B(X)$ and $C(X)$ respectively.

The reader may also be interested in the more general case with the set $\mathcal{R} = \mathcal{R}(a^1, ..., a^r)$. In this case, the corresponding sets $\mathcal{R}_a(X)$, $\mathcal{R}_b(X)$ and $\mathcal{R}_c(X)$ are defined similarly. Using the results of [35], one can obtain sufficient (but not necessary) conditions for proximinality of these sets. This needs, besides paths, the consideration of some additional and more complicated relations between points of $X$. Here we will not consider the case $r \geq 3$, since our main purpose is to draw the reader’s attention to the arisen problems of proximinality in the simplest case of approximation. For the existing open problems connected with the set $\mathcal{R}(a^1, ..., a^r)$, where $r \geq 3$, see [53] and [118].

### 1.5.2 Proximinality of $R_b(X)$ in $B(X)$

Let $a^1$ and $a^2$ be two different directions in $\mathbb{R}^n$. In the sequel, we will use paths with respect to the directions $a^1$ and $a^2$. Recall that a length of a
path is the number of its points and can be equal to $\infty$ if the path is infinite. A singleton is a path of the unit length. We say that a path $(x^1, ..., x^n)$ belonging to some subset $X$ of $\mathbb{R}^n$ is irreducible if there is not another path $(y^1, ..., y^l) \subset X$ with $y^1 = x^1$, $y^l = x^m$ and $l < m$.

The following theorem follows from theorem 2 of [35].

**Theorem 1.9.** Let $X \subset \mathbb{R}^n$ and the lengths of all irreducible paths in $X$ be uniformly bounded by some positive integer. Then each function in $B(X)$ has an extremal element in $R_b(X)$.

There are a large number of sets in $\mathbb{R}^n$ satisfying the hypothesis of this theorem. For example, if a set $X$ has a cross section according to one of the directions $a^1$ or $a^2$, then the set $X$ satisfies the hypothesis of Theorem 1.9. By a cross section according to the direction $a^1$ we mean any set $X_{a^1} = \{x \in X : a^1 \cdot x = c\}, c \in \mathbb{R}$, with the property: for any $y \in X$ there exists a point $y^1 \in X_{a^1}$ such that $a^2 \cdot y = a^2 \cdot y^1$. By the similar way, one can define a cross section according to the direction $a^2$. For more on cross sections in problems of proximinality of sums of univariate functions see [34,77].

Regarding Theorem 1.9 one may ask if the condition of the theorem is necessary for proximinality of $R_b(X)$ in $B(X)$. While we do not know a complete answer to this question, we are going to give an example of a set $X$ for which Theorem 1.9 fails. Let $a^1 = (1; -1)$, $a^2 = (1; 1)$. Consider the set

$$X = \{(2; \frac{2}{3}), (\frac{2}{3}; -\frac{2}{3}), (0; 0), (1; 1), (1 + \frac{1}{2}; 1 - \frac{1}{2}), (1 + \frac{1}{2} + \frac{1}{4}; 1 - \frac{1}{2} + \frac{1}{4}), (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}; 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}), ...\}.$$

In what follows, the elements of $X$ in the given order will be denoted by $x^0, x^1, x^2, ...$. It is clear that $X$ is a path of the infinite length and $x^n \to x^0$, as $n \to \infty$. Let $\sum_{n=1}^{\infty} c_n$ be any divergent series with the terms $c_n > 0$ and $c_n \to 0$, as $n \to \infty$. Besides let $f_0$ be a function vanishing at the points $x^0, x^2, x^4, ...$, and taking values $c_1, c_2, c_3, ...$ at the points $x^1, x^3, x^5, ...$ respectively. It is obvious that $f_0$ is continuous on $X$. The set $X$ is compact and satisfies all the conditions of Theorem 1.6. By that theorem, $\overline{R_c(X)} = C(X)$. Therefore, for any continuous function on $X$, thus for $f_0$,

$$\inf_{g \in R_c(X)} \|f_0 - g\|_{C(X)} = 0. \quad (1.47)$$

65
Since \( \mathcal{R}_c(X) \subset \mathcal{R}_b(X) \), we obtain from (1.47) that
\[
\inf_{g \in \mathcal{R}_b(X)} \| f_0 - g \|_{B(X)} = 0. \tag{1.48}
\]

Suppose that \( f_0 \) has an extremal element \( g_1^0 (a^1 \cdot x) + g_2^0 (a^2 \cdot x) \) in \( \mathcal{R}_b(X) \). By the definition of \( \mathcal{R}_b(X) \), the ridge functions \( g_i^0, i = 1, 2 \), are bounded on \( X \). From (1.48) it follows that \( f_0 = g_1^0 (a^1 \cdot x) + g_2^0 (a^2 \cdot x) \). Since \( a^1 \cdot x^{2n} = a^1 \cdot x^{2n+1} \) and \( a^2 \cdot x^{2n+1} = a^2 \cdot x^{2n+2} \), for \( n = 0, 1, \ldots \), we can write
\[
\sum_{n=0}^{k} c_{n+1} = \sum_{n=0}^{k} \left[ f(x^{2n+1}) - f(x^{2n}) \right]
\]
\[
= \sum_{n=0}^{k} \left[ g_2^0(x^{2n+1}) - g_2^0(x^{2n}) \right] = g_2^0(a^2 \cdot x^{2k+1}) - g_2^0(a^2 \cdot x^0). \tag{1.49}
\]
Since \( \sum_{n=1}^{\infty} c_n = \infty \), we deduce from (1.49) that the function \( g_2^0 (a^2 \cdot x) \) is not bounded on \( X \). This contradiction means that the function \( f_0 \) does not have an extremal element in \( \mathcal{R}_b(X) \). Therefore, the space \( \mathcal{R}_b(X) \) is not proximinal in \( B(X) \).

1.5.3 Proximinality of \( \mathcal{R}_c(X) \) in \( C(X) \)

In this section, give sufficient conditions and also a necessary condition for proximinality of \( \mathcal{R}_c(X) \) in \( C(X) \).

**Theorem 1.10.** Let the system of independent vectors \( a^1 \) and \( a^2 \) has a complement to a basis \( \{a^1, \ldots, a^n\} \) in \( \mathbb{R}^n \) with the property: for any point \( x^0 \in X \) and any positive real number \( \delta \) there exist a number \( \delta_0 \in (0, \delta] \) and a point \( x^\sigma \) in the set
\[
\sigma = \{ x \in X : a^2 \cdot x^0 - \delta_0 \leq a^2 \cdot x \leq a^2 \cdot x^0 + \delta_0 \},
\]
such that the system
\[
\begin{cases}
    a^2 \cdot x' = a^2 \cdot x^\sigma \\
    a^1 \cdot x' = a^1 \cdot x \\
    \sum_{i=3}^{n} |a^i \cdot x' - a^i \cdot x| < \delta
\end{cases}
\]
has a solution \( x' \in \sigma \) for all points \( x \in \sigma \). Then the space \( \mathcal{R}_c(X) \) is proximinal in \( C(X) \).
Proof. Introduce the following mappings and sets:

\[ \pi_i : X \to \mathbb{R}, \pi_i(x) = a^i \cdot x, \quad Y_i = \pi_i(X), \quad i = 1, ..., n. \]

Since the system of vectors \{a^1, ..., a^n\} is linearly independent, the mapping \( \pi = (\pi_1, ..., \pi_n) \) is an injection from \( X \) into the Cartesian product \( Y_1 \times ... \times Y_n \). Besides, \( \pi \) is linear and continuous. By the open mapping theorem, the inverse mapping \( \pi^{-1} \) is continuous from \( Y = \pi(X) \) onto \( X \). Let \( f \) be a continuous function on \( X \). Then the composition \( f \circ \pi^{-1}(y_1, ..., y_n) \) will be continuous on \( Y \), where \( y_i = \pi_i(x), \quad i = 1, ..., n \), are the coordinate functions.

Consider the approximation of the function \( f \circ \pi^{-1} \) by elements from

\[ G_0 = \{ g_1(y_1) + g_2(y_2) : \ g_i \in C(Y_i), \ i = 1, 2 \} \]

over the compact set \( Y \). Then one may observe that the function \( f \) has an extremal element in \( R_c(X) \) if and only if the function \( f \circ \pi^{-1} \) has an extremal element in \( G_0 \). Thus the problem of proximinality of \( R_c(X) \) in \( C(X) \) is reduced to the problem of proximinality of \( G_0 \) in \( C(Y) \).

Let \( T, T_1, ..., T_{m+1} \) be metric compact spaces and \( T \subseteq T_1 \times ... \times T_{m+1} \). For \( i = 1, ..., m \), let \( \varphi_i \) be the continuous mappings from \( T \) onto \( T_i \). In [35], the authors obtained sufficient conditions for proximinality of the set

\[ C_0 = \{ \sum_{i=1}^{n} g_i \circ \varphi_i : \ g_i \in C(T_i), \ i = 1, ..., m \} \]

in the space \( C(T) \) of continuous functions on \( T \). Since \( Y \subseteq Y_1 \times Y_2 \times Z_3 \), where \( Z_3 = Y_3 \times ... \times Y_n \), we can use this result in our case for the approximation of the function \( f \circ \pi^{-1} \) by elements from \( G_0 \). By this theorem, the set \( G_0 \) is proximinal in \( C(Y) \) if for any \( y_2^0 \in Y_2 \) and \( \delta > 0 \) there exists a number \( \delta_0 \in (0, \delta) \) such that the set \( \sigma(y_2^0, \delta_0) = [y_2^0 - \delta_0, y_2^0 + \delta_0] \cap Y_2 \) has \( (2, \delta) \) maximal cross section. The last means that there exists a point \( y_2^p \in \sigma(y_2^0, \delta_0) \) with the property: for any point \( (y_1, y_2, z_3) \in Y \), with the second coordinate \( y_2 \) from the set \( \sigma(y_2^0, \delta_0) \), there exists a point \( (y_1', y_2^p, z_3') \in Y \) such that \( y_1 = y_1' \) and \( \rho(z_3, z_3') < \delta \), where \( \rho \) is a metrics in \( Z_3 \). Since these conditions are equivalent to the conditions of Theorem 1.10, the space \( G_0 \) is proximinal in the space \( C(Y) \). Then by the above conclusion, the space \( R_c(X) \) is proximinal in \( C(X) \). \( \square \)

Let us give some simple examples of compact sets satisfying the hypothesis of Theorem 1.10. For the sake of brevity, we restrict ourselves to the case \( n = 3 \).

67
(a) Let $X$ be a closed ball in $\mathbb{R}^3$, $a^1$ and $a^2$ be two arbitrary orthogonal directions. Then Theorem 1.10 holds. Note that in this case, we can take $\delta_0 = \delta$ and $a^3$ as an orthogonal vector to both the vectors $a^1$ and $a^2$.

(b) Let $X$ be the unite cube, $a^1 = (1; 1; 0)$, $a^2 = (1; -1; 0)$. Then Theorem 1.10 also holds. In this case, we can take $\delta_0 = \delta$ and $a^3 = (0; 0; 1)$. Note that the unit cube does not satisfy the hypothesis of the theorem for many directions (take, for example, $a^1 = (1; 2; 0)$ and $a^2 = (2; -1; 0)$).

In the following example, one can not always chose $\delta_0$ as equal to $\delta$.

(c) Let $X = \{(x_1, x_2, x_3) : (x_1, x_2) \in Q, 0 \leq x_3 \leq 1\}$, where $Q$ is the union of two triangles $A_1B_1C_1$ and $A_2B_2C_2$ with the vertices $A_1 = (0; 0)$, $B_1 = (1; 2)$, $C_1 = (2; 0)$, $A_2 = (1\frac{1}{2}; 1)$, $B_2 = (2\frac{1}{2}; -1)$, $C_2 = (3\frac{1}{2}; 1)$. Let $a^1 = (0; 1; 0)$ and $a^2 = (1; 0; 0)$. Then it is easy to see that Theorem 1.10 holds (the vector $a^3$ can be chosen as $(0; 0; 1)$). In this case, $\delta_0$ can not be always chosen as equal to $\delta$. Take, for example, $x^0 = (1\frac{3}{4}; 0; 0)$ and $\delta = 1\frac{3}{4}$. If $\delta_0 = \delta$, then the second equation of the system (1.50) has not a solution for a point $(1; 2; 0)$ or a point $(2\frac{1}{2}; -1; 0)$. But if we take $\delta_0$ not more than $1\frac{1}{4}$, then for $x' = x^0$ the system has a solution. Note that the last inequality $|a^3 \cdot x' - a^3 \cdot x| < \delta$ of the system can be satisfied with the equality $a^3 \cdot x' = a^3 \cdot x$ if $a^3 = (0; 0; 1)$.

It should be remarked that the results of [35] tell nothing about necessary conditions for proximinality of the spaces considered there. To fill this gap in our case, we want to give a necessary condition for proximinality of $\mathcal{R}_c(X)$ in $C(X)$. Our result will be based on the result of Marshall and O’Farrell given below. First, let us introduce some notation. By $\mathcal{R}^i_c$, $i = 1, 2$, we will denote the set of continuous ridge functions $g(a^i \cdot x)$ on the given compact set $X \subset \mathbb{R}^n$. Note that $\mathcal{R}_c = \mathcal{R}^1_c + \mathcal{R}^2_c$. Besides, let $\mathcal{R}^3_c = \mathcal{R}^1_c \cap \mathcal{R}^2_c$. For $i = 1, 2, 3$, let $X_i$ be the quotient space obtained by identifying points $y_1$ and $y_2$ in $X$ whenever $f(y_1) = f(y_2)$ for each $f$ in $\mathcal{R}^i_c$. By $\pi_i$ denote the natural projection of $X$ onto $X_i$, $i = 1, 2, 3$. Note that we have already dealt with the quotient spaces $X_1$, $X_2$ and the projections $\pi_1$, $\pi_2$ in the previous section. Recall that the relation on $X$, defined by setting $y_1 \approx y_2$ if $y_1$ and $y_2$ belong to some path, is an equivalence relation and the equivalence classes are called
orbits. By \(O(t)\) denote the orbit of \(X\) containing \(t\). For \(Y \subset X\), let \(\text{var}_Y f\) be the variation of a function \(f\) on the set \(Y\). That is,

\[
\text{var}_Y f = \sup_{x,y \in Y} |f(x) - f(y)|.
\]

**Theorem 1.11.** Suppose that the space \(\mathcal{R}_c(X)\) is proximinal in \(C(X)\). Then there exists a positive real number \(c\) such that

\[
\sup_{t \in X} \text{var}_{O(t)} f \leq c \sup_{t \in X} \text{var}_{\pi^{-1}(\pi(t))} f
\]

for all \(f \in \mathcal{R}_c^1\).

The proof is simple. In [108], Marshall and O’Farrell proved the following result (see proposition 4 in [108]): Let \(A_1\) and \(A_2\) be closed subalgebras of \(C(X)\) that contain the constants. Let \((X_1, \pi_1)\), \((X_2, \pi_2)\) and \((X_3, \pi_3)\) be the quotient spaces and projections associated with the algebras \(A_1\), \(A_2\) and \(A_3 = A_1 \cap A_2\) respectively. Then \(A_1 + A_2\) is closed in \(C(X)\) if and only if there exists a positive real number \(c\) such that

\[
\sup_{z \in X_3} \text{var}_{\pi_3^{-1}(z)} f \leq c \sup_{y \in X_2} \text{var}_{\pi_2^{-1}(y)} f
\]

for all \(f \in A_1\).

If \(\mathcal{R}_c(X)\) is proximinal in \(C(X)\), then it is necessarily closed and therefore, by the above proposition, (1.52) holds for the algebras \(A_i^1 = \mathcal{R}_c^1\), \(i = 1, 2, 3\). The right-hand side of (1.52) is equal to the right-hand side of (1.51). Let \(t\) be some point in \(X\) and \(z = \pi_3(t)\). Since each function \(f \in \mathcal{R}_c^2\) is constant on the orbit of \(t\) (note that \(f\) is both of the form \(g_1(a^1 \cdot x)\) and of the form \(g_2(a^2 \cdot x)\)), \(O(t) \subset \pi_3^{-1}(z)\). Hence,

\[
\sup_{t \in X} \text{var}_{O(t)} f \leq c \sup_{z \in X_3} \text{var}_{\pi_3^{-1}(z)} f
\]

From (1.52) and (1.53) we obtain (1.51).

Note that the inequality (1.52) provides not worse but less practicable necessary condition for proximinality than the inequality (1.51) does. On the other hand, there are many cases in which both the inequalities are equivalent. For example, let the lengths of irreducible paths of \(X\) are bounded

Electronic copy available at: https://ssrn.com/abstract=3618165
by some positive integer $n_0$. In this case, it can be shown that the inequality (1.52), hence (1.51), holds with the constant $c = \frac{n_0}{2}$ and moreover $O(t) = \pi_3^{-1}(z)$ for all $t \in X$, where $z = \pi_3(t)$ (see the proof of theorem 5 in [53]). Therefore, the inequalities (1.51) and (1.52) are equivalent for the considered class of sets $X$. The last argument shows that all the compact sets $X \subset \mathbb{R}^n$ over which $R_c(X)$ is not proximinal in $C(X)$ should be sought in the class of sets having irreducible paths consisting sufficiently large number of points. For example, let $I = [0, 1]^2$ be the unit square, $a^1 = (1; 1)$, $a^2 = (1; \frac{1}{2})$. Consider the path

$$l_k = \{(1; 0), (0; 1), (\frac{1}{2}; 0), (0; \frac{1}{2}), (\frac{1}{4}; 0), \ldots, (0; \frac{1}{2^k})\}.$$  

It is clear that $l_k$ is an irreducible path with the length $2k + 2$, where $k$ may be very large. Let $g_k$ be a continuous univariate function on $\mathbb{R}$ satisfying the conditions: $g_k(\frac{1}{2^i}) = i$, $i = 0, \ldots, k$, $g_k(t) = 0$ if $t < \frac{1}{2^i}$, $i - 1 \leq g_k(t) \leq i$ if $t \in (\frac{1}{2^i-1}, \frac{1}{2^i})$, $i = 1, \ldots, k$, and $g_k(t) = k$ if $t > 1$. Then it can be easily verified that

$$\sup_{t \in X} \text{var}_{\pi_2^{-1}(\pi_3(t))} g_k(a^1 \cdot x) \leq 1. \quad (1.54)$$

Since $\max_{x \in I} g_k(a^1 \cdot x) = k$, $\min_{x \in I} g_k(a^1 \cdot x) = 0$ and $\text{var}_{x \in O(t_1)} g_k(a^1 \cdot x) = k$ for $t_1 = (1; 0)$, we obtain that

$$\sup_{t \in X} \text{var}_{O(t)} g_k(a^1 \cdot x) = k. \quad (1.55)$$

Since $k$ may be very large, from (1.54) and (1.55) it follows that the inequality (1.51) cannot hold for the function $g_k(a^1 \cdot x) \in R_c^1$. Thus the space $R_c(I)$ with the directions $a^1 = (1; 1)$ and $a^2 = (1; \frac{1}{2})$ is not proximinal in $C(I)$.

It should be remarked that if a compact set $X \subset \mathbb{R}^n$ satisfies the hypothesis of Theorem 1.10, then the length of all irreducible paths are uniformly bounded (see the proof of Theorem 1.10 and lemma in [35]). We have already seen that if the last condition does not hold, then the proximinality of both $R_c(X)$ in $C(X)$ and $R_b(X)$ in $B(X)$ fail for some sets $X$. Besides the examples given above and in Section 1.5.2, one can easily construct many other examples of such sets. All these examples, Theorems 1.9, 1.10, 1.11 and the subsequent remarks justify the statement of the following conjecture:

**Conjecture.** Let $X$ be some subset of $\mathbb{R}^n$. The space $R_b(X)$ is proximinal in $B(X)$ and the space $R_c(X)$ is proximinal in $C(X)$ (in this case, $X$ is
considered to be compact) if and only if the lengths of all irreducible paths of $X$ are uniformly bounded.

**Remark 1.2.** Medvedev’s result (see [76, p.58]), which later came to our attention, in particular, says that the set $R_c(X)$ is closed in $C(X)$ if and only if the lengths of all irreducible paths of $X$ are uniformly bounded. Thus, in the case of $C(X)$, the necessity of the above conjecture was proved by Medvedev.

**Remark 1.3.** Note that there are situations in which a continuous function (a specially chosen function on a specially constructed set) has an extremal element in $R_b(X)$, but not in $R_c(X)$ (see [76, p.73]). One subsection of [76] (see p.68) was devoted to the proximinality of sums of two univariate functions with continuous and bounded summands in the spaces of continuous and bounded bivariate functions, respectively. If $X \subset \mathbb{R}^2$ and $a^1, a^2$ be linearly independent directions in $\mathbb{R}^2$, then the linear transformation $y_1 = a^1 \cdot x$, $y_2 = a^2 \cdot x$ reduces the problems of proximinality of $R_b(X)$ in $B(X)$ and $R_c(X)$ in $C(X)$ to the problems considered in that subsection. But in general, when $X \subset \mathbb{R}^n, n > 2$, our case cannot be obtained from that of [76].

### 1.6 On the approximation by weighted ridge functions

In this section, we characterize the best $L_2$ approximation to a multivariate function by linear combinations of ridge functions multiplied by some fixed weight functions. In the special case when the weight functions are constants, we propose explicit formulas for both the best approximation and approximation error.

#### 1.6.1 Problem statement

Ridge approximation in $L_2$ was actively studied in the late 90’s by K.I. Oskolkov [114], V.E. Maiorov [102], A. Pinkus [118], V.N. Temlyakov [138], P. Petrushev [116] and other researchers.

Let $D$ be the unit disk in $\mathbb{R}^2$. In [97], Logan and Shepp along with other results gave a closed-form expression for the best $L_2$ approximation to a function $f(x_1, x_2) \in L_2(D)$ from the set
Their solution requires that the directions $\mathbf{a}_1, \ldots, \mathbf{a}_r$ be equally-spaced and involves finite sums of convolutions with explicit kernels. In the $n$ dimensional case, we obtained an expression of simpler form for the best $L_2$ approximation to square-integrable multivariate functions over a certain domain, provided that $r = n$ and the directions $\mathbf{a}_1, \ldots, \mathbf{a}_r$ are linearly independent (see [52]).

In this section, we consider the approximation from the more general set

$$\mathcal{R} (\mathbf{a}_1, \ldots, \mathbf{a}_r; w_1, \ldots, w_r) = \left\{ \sum_{i=1}^r w_i(x) g_i (\mathbf{a}_i \cdot x) : g_i : \mathbb{R} \to \mathbb{R}, \ i = 1, \ldots, r \right\},$$

where $w_1, \ldots, w_r$ are fixed multivariate functions. We are going to characterize the best $L_2$ approximation from this set for the case $r \leq n$. Then, in the special case when the weight functions $w_1, \ldots, w_r$ are constants, we will prove two theorems on explicit formulas for the best approximation and the error of approximation, respectively. Unfortunately, we do not yet know any reasonable answer to these problems in other possible cases of $r$.

### 1.6.2 Characterization of the best approximation

Let $X$ be a subset of $\mathbb{R}^n$ with a finite Lebesgue measure. Consider the approximation of a function $f(x) = f(x_1, \ldots, x_n)$ in $L_2(X)$ from the manifold $\mathcal{R} (\mathbf{a}_1, \ldots, \mathbf{a}_r; w_1, \ldots, w_r)$, where $r \leq n$. We suppose that the functions $w_i(x)$ and the products $w_i(x) \cdot g_i (\mathbf{a}_i \cdot x), \ i = 1, \ldots, r$, belong to the space $L_2(X)$. Besides, we assume that the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_r$ are linearly independent. We say that a function $g^0_w = \sum_{i=1}^r w_i(x) g_i^0 (\mathbf{a}_i \cdot x)$ in $\mathcal{R} (\mathbf{a}_1, \ldots, \mathbf{a}_r; w_1, \ldots, w_r)$ is the best approximation (or extremal) to $f$ if

$$\|f - g^0_w\|_{L_2(X)} = \inf_{g \in \mathcal{R}(\mathbf{a}_1, \ldots, \mathbf{a}_r; w_1, \ldots, w_r)} \|f - g\|_{L_2(X)}.$$

Let the system of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_r, \mathbf{a}_{r+1}, \ldots, \mathbf{a}_n\}$ be a completion of the system $\{\mathbf{a}_1, \ldots, \mathbf{a}_r\}$ to a basis in $\mathbb{R}^n$. Let $J : X \to \mathbb{R}^n$ be the linear transformation given by the formulas

$$y_i = \mathbf{a}_i \cdot x, \quad i = 1, \ldots, n. \quad (1.56)$$
Since the vectors $a^i$, $i = 1,\ldots,n$, are linearly independent, it is an injection. The Jacobian $\det J$ of this transformation is a constant different from zero.

Let the formulas

$$x_i = b^i \cdot y, \quad i = 1,\ldots,n,$$

stand for the solution of linear equations (1.56) with respect to $x_i$, $i = 1,\ldots,n$.

Introduce the notation

$$Y = J(X)$$

and

$$Y_i = \{y_i \in \mathbb{R} : y_i = a^i \cdot x, \quad x \in X\}, \quad i = 1,\ldots,n.$$  

For any function $u \in L^2(X)$, put

$$u^* = u^*(y) \overset{\text{def}}{=} u(b^1 \cdot y,\ldots,b^n \cdot y).$$

It is obvious that $u^* \in L^2(Y)$. Besides,

$$\int_Y u^*(y) \, dy = |\det J| \cdot \int_X u(x) \, dx \quad (1.57)$$

and

$$\|u^*\|_{L^2(Y)} = |\det J|^{1/2} \cdot \|u\|_{L^2(X)}. \quad (1.58)$$

Set

$$L^i_2 = \{w^*_i(y)g_i(y_i) \in L^2(Y)\}, \quad i = 1,\ldots,r.$$  

We need the following auxiliary lemmas.

**Lemma 1.5.** Let $f(x) \in L^2(X)$. A function $\sum_{i=1}^r w_i(x)g^0_i(a^i \cdot x)$ is extremal to the function $f(x)$ if and only if $\sum_{i=1}^r w^*_i(y)g^0_i(y_i)$ is extremal from the space $L^1_2 \oplus \ldots \oplus L^r_2$ to the function $f^*(y)$.

Due to (1.58) the proof of this lemma is obvious.
Lemma 1.6. Let \( f(x) \in L_2(X) \). A function \( \sum_{i=1}^{r} w_i(x) g_i^0(a^i \cdot x) \) is extremal to the function \( f(x) \) if and only if
\[
\int_X \left( f(x) - \sum_{i=1}^{r} w_i(x) g_i^0(a^i \cdot x) \right) w_j(x) h(a^j \cdot x) \, dx = 0
\]
for any ridge function \( h(a^j \cdot x) \) such that \( w_j(x) h(a^j \cdot x) \in L_2(X) \) \( j = 1, ..., r \).

Lemma 1.7. The following formula is valid for the error of approximation to a function \( f(x) \) in \( L_2(X) \) from \( \mathcal{R}(a^1, ..., a^r; w_1, ..., w_r) \):
\[
E(f) = \left( \| f(x) \|_{L_2(X)}^2 - \left\| \sum_{i=1}^{r} w_i(x) g_i^0(a^i \cdot x) \right\|_{L_2(X)}^2 \right)^{1/2},
\]
where \( \sum_{i=1}^{r} w_i(x) g_i^0(a^i \cdot x) \) is the best approximation to \( f(x) \).

Lemmas 1.6 and 1.7 follow from the well-known facts of functional analysis that the best approximation of an element \( x \) in a Hilbert space \( H \) from a linear subspace \( Z \) of \( H \) must be the image of \( x \) via the orthogonal projection onto \( Z \) and the sum of squares of norms of orthogonal vectors is equal to the square of the norm of their sum.

We say that \( Y \) is an \( r \)-set if it can be represented as \( Y_1 \times ... \times Y_r \times Y_0 \), where \( Y_0 \) is some set from the space \( \mathbb{R}^{n-r} \). In special case, \( Y_0 \) may be equal to \( Y_{r+1} \times ... \times Y_n \), but it is not necessary. By \( Y^{(i)} \), we denote the Cartesian product of the sets \( Y_1, ..., Y_r, Y_0 \) except for \( Y_i, i = 1, ..., r \). That is, \( Y^{(i)} = Y_1 \times ... \times Y_{i-1} \times Y_{i+1} \times ... \times Y_r \times Y_0, i = 1, ..., r \).

Theorem 1.12. Let \( Y \) be an \( r \)-set. A function \( \sum_{i=1}^{r} w_i(x) g_i^0(a^i \cdot x) \) is the best approximation to \( f(x) \) if and only if
\[
g_j^0(y_j) = \frac{1}{\int_{Y^{(j)}} w_j^2(y) dy^{(j)}} \int_{Y^{(j)}} \left( f^*(y) - \sum_{i=1, i\neq j}^{r} w_i^*(y) g_i^0(y) \right) w_j^*(y) dy^{(j)}, j = 1, ..., r.
\]
Proof. Necessity. Let a function \( \sum_{i=1}^{r} w_i(x) g_i^0 (a^i \cdot x) \) be extremal to \( f \). Then by Lemma 1.5, the function \( \sum_{i=1}^{r} w_i^*(y) g_i^0 (y_i) \) in \( L_2^1 \oplus \ldots \oplus L_2^r \) is extremal to \( f^* \). By Lemma 1.6 and equality (1.57),

\[
\int_{Y} f^*(y) w_j^*(y) h(y_j) \, dy = \int_{Y} w_j^*(y) h(y_j) \sum_{i=1}^{r} w_i^*(y) g_i^0 (y_i) \, dy \quad (1.60)
\]

for any product \( w_j^*(y) h(y_j) \) in \( L_2^j \), \( j = 1, \ldots, r \). Applying Fubini's theorem to the integrals in (1.60), we obtain that

\[
\int_{Y(j)} h(y_j) \left( \int_{Y^*(j)} f^*(y) w_j^*(y) \, dy \right) \, dy_j = \int_{Y(j)} w_j^*(y) \left( \int_{Y^*(j)} \sum_{i=1}^{r} w_i^*(y) g_i^0 (y_i) \, dy \right) \, dy_j.
\]

Since \( h(y_j) \) is an arbitrary function such that \( w_j^*(y) h(y_j) \in L_2^j \),

\[
\int_{Y^*(j)} f^*(y) w_j^*(y) \, dy = \int_{Y^*(j)} w_j^*(y) \sum_{i=1}^{r} w_i^*(y) g_i^0 (y_i) \, dy, \quad j = 1, \ldots, r.
\]

Therefore,

\[
\int_{Y^*(j)} w_j^{*2}(y) g_j^0 (y_j) \, dy = \int_{Y^*(j)} \left( f^*(y) - \sum_{i=1, i \neq j}^{r} w_i^*(y) g_i^0 (y_i) \right) \, dy, \quad j = 1, \ldots, r.
\]

Now, since \( y_j \notin Y^*(j) \), we obtain (1.59).

Sufficiency. Note that all the equalities in the proof of the necessity can be obtained in the reverse order. Thus, (1.60) can be obtained from (1.59). Then by (1.57) and Lemma 1.6, we finally conclude that the function \( \sum_{i=1}^{r} w_i(x) g_i^0 (a^i \cdot x) \) is extremal to \( f(x) \). \( \square \)
In the following, $|Q|$ will denote the Lebesgue measure of a measurable set $Q$. The following corollary is obvious.

**Corollary 1.6.** Let $Y$ be an $r$-set. A function $\sum_{i=1}^{r} g_i^0 (a^i \cdot x)$ in $\mathcal{R} (a^1, ..., a^r)$ is the best approximation to $f(x)$ if and only if

$$g_j^0 (y_j) = \frac{1}{|Y(j)|} \int_{Y(j)} \left( f^*(y) - \sum_{i=1}^{r} g_i^0 (y_i) \right) dy^{(j)}, \quad j = 1, ..., r.$$ 

In [52], this corollary was proven for the case $r = n$.

### 1.6.3 Calculation of the approximation error

In this section, we are going to establish explicit formulas for both the best approximation and approximation error, provided that the weight functions are constants. In this case, since we vary over $g_i$, the set $\mathcal{R} (a^1, ..., a^r; w_1, ..., w_r)$ coincides with $\mathcal{R} (a^1, ..., a^r)$. Thus, without loss of generality, we may assume that $w_i(x) = 1$ for $i = 1, ..., r$.

For brevity of the further exposition, introduce the notation

$$A = \int_Y f^* (y) dy \quad \text{and} \quad f_i^* = f_i^* (y_i) = \int_{Y^{(i)}} f^* (y) dy^{(i)}, \quad i = 1, ..., r.$$ 

The following theorem is a generalization of the main result of [52] from the case $r = n$ to the cases $r < n$.

**Theorem 1.13.** Let $Y$ be an $r$-set. Set the functions

$$g_1^0 (y_1) = \frac{1}{|Y(1)|} f_1^* - (r - 1) \frac{A}{|Y|}$$

and

$$g_j^0 (y_j) = \frac{1}{|Y(j)|} f_j^*, \quad j = 2, ..., r.$$
Then the function \( \sum_{i=1}^r g_i^0 (a_i \cdot x) \) is the best approximation from \( \mathcal{R}(a^1, ..., a^r) \) to \( f(x) \).

The proof is the same as in [52]. It is sufficient to verify that the functions \( g_j^0 (y_j) \), \( j = 1, ..., r \), satisfy the conditions of Corollary 1.6. This becomes obvious if note that

\[
\sum_{i=1}^r \frac{1}{|Y^{(j)}| |Y^{(i)}|} \int_{Y^{(j)}} \left[ \int_{Y^{(i)}} f^* (y) dy^{(i)} \right] dy^{(j)} = (r - 1) \frac{1}{|Y|} \int_Y f^* (y) dy
\]

for \( j = 1, ..., r \).

**Theorem 1.14.** Let \( Y \) be an \( r \)-set. Then the error of approximation to a function \( f(x) \) from the set \( \mathcal{R}(a^1, ..., a^r) \) can be calculated by the formula

\[
E(f) = |\text{det} J|^{-1/2} \left( \| f^* \|_{L_2(Y)}^2 - \sum_{i=1}^r \frac{1}{|Y^{(i)}|^2} \| f_i^* \|_{L_2(Y)}^2 + (r - 1) \frac{A^2}{|Y|} \right)^{1/2}.
\]

**Proof.** From Eq. (1.58), Lemma 1.7 and Theorem 1.13, it follows that

\[
E(f) = |\text{det} J|^{-1/2} \left( \| f^* \|_{L_2(Y)}^2 - I \right)^{1/2},
\]

where

\[
I = \left\| \sum_{i=1}^r \frac{1}{|Y^{(i)}|} f_i^* - (r - 1) \frac{A}{|Y|} \right\|_{L_2(Y)}^2.
\]

The integral \( I \) can be written as a sum of the following four integrals:

\[
I_1 = \sum_{i=1}^r \frac{1}{|Y^{(i)}|^2} \| f_i^* \|_{L_2(Y)}^2, \quad I_2 = \sum_{i=1}^r \sum_{j=1}^r \frac{1}{|Y^{(i)}| |Y^{(j)}|} \int_Y f_i^* f_j^* dy,
\]

\[
I_3 = -(r - 1) \frac{1}{|Y|} A \sum_{i=1}^r \frac{1}{|Y^{(i)}|} \int_Y f_i^* dy, \quad I_4 = (r - 1)^2 \frac{A^2}{|Y|}.
\]
It is not difficult to verify that

\[
\int_{Y} f_i^* f_j^* \, dy = \left| Y_0 \times \prod_{\substack{k=1 \\text{to} \ r \ \text{but not} \ i,j}} Y_k \right| A^2, \text{ for } i, j = 1, \ldots, r, \ i \neq j, \quad (1.62)
\]

and

\[
\int_{Y} f_i^* \, dy = \left| Y_0 \times \prod_{\substack{k=1 \\text{to} \ r \ \text{but not} \ i}} Y_k \right| A, \text{ for } i = 1, \ldots, r. \quad (1.63)
\]

Considering (1.62) and (1.63) in the expressions of \( I_2 \) and \( I_3 \) respectively, we obtain that

\[
I_2 = r(r - 1) \frac{A^2}{|Y|} \quad \text{and} \quad I_3 = -2r(r - 1) \frac{A^2}{|Y|}.
\]

Therefore,

\[
I = I_1 + I_2 + I_3 + I_4 = \sum_{i=1}^{r} \frac{1}{|Y(0)|^2} \| f_i^* \|^2_{L^2(Y)} - (r - 1) \frac{A^2}{|Y|}. \quad (1.64)
\]

Now the last equality with (1.61) complete the proof.

**Example.** Consider the following set

\[
X = \{ x \in \mathbb{R}^4 : y_i = y_i(x) \in [0; 1], \ i = 1, \ldots, 4 \},
\]

where

\[
\begin{aligned}
    y_1 &= x_1 + x_2 + x_3 - x_4 \\
    y_2 &= x_1 + x_2 - x_3 + x_4 \\
    y_3 &= x_1 - x_2 + x_3 + x_4 \\
    y_4 &= -x_1 + x_2 + x_3 + x_4
\end{aligned}
\]

Let the function

\[
f = 8x_1 x_2 x_3 x_4 - \sum_{i=1}^{4} x_i^4 + 2 \sum_{i=1}^{3} \sum_{j=i+1}^{4} x_i^2 x_j^2
\]
be given on \( X \). Consider the approximation of this function from the set 
\( \mathcal{R}(a^1, a^2, a^3) \), where 
\( a^1 = (1; 1; 1; -1) \), \( a^2 = (1; 1; -1; 1) \), \( a^3 = (1; -1; 1; 1) \).

Putting \( a^4 = (-1; 1; 1; 1) \), we complete the system of vectors \( a^1, a^2, a^3, a^4 \) to

the basis \( \{a^1, a^2, a^3, a^4\} \) in \( \mathbb{R}^4 \). The linear transformation \( J \) defined by (1.64) maps the set \( X \) onto the set \( Y = [0; 1]^4 \). The inverse transformation is given by the formulas

\[
\begin{align*}
    x_1 &= \frac{1}{4} y_1 + \frac{1}{4} y_2 + \frac{1}{4} y_3 - \frac{1}{4} y_4 \\
    x_2 &= \frac{1}{4} y_1 + \frac{1}{4} y_2 - \frac{1}{4} y_3 + \frac{1}{4} y_4 \\
    x_3 &= \frac{1}{4} y_1 - \frac{1}{4} y_2 + \frac{1}{4} y_3 + \frac{1}{4} y_4 \\
    x_4 &= -\frac{1}{4} y_1 + \frac{1}{4} y_2 + \frac{1}{4} y_3 + \frac{1}{4} y_4
\end{align*}
\]

It can be easily verified that \( f^* = y_1 y_2 y_3 y_4 \) and \( Y \) is a 3-set with \( Y_i = [0; 1], i = 1, 2, 3 \). Besides, \( Y_0 = [0; 1] \). After easy calculations we obtain that

\( A = \frac{1}{16}; \ f_i^* = \frac{1}{8} y_i \) for \( i = 1, 2, 3; \) \( \det J = -16; \ \|f^*\|_{L^2(Y)}^2 = \frac{1}{8}; \ \|f_i^*\|_{L^2(Y)}^2 = \frac{1}{192}, i = 1, 2, 3 \). Now from Theorems 1.13 and 1.14 it follows that the function

\( \frac{1}{8} \sum_{i=1}^3 (a^i \cdot x) - \frac{1}{8} \) is a best approximation from \( \mathcal{R}(a^1, a^2, a^3) \) to \( f \) and \( E(f) = \frac{1}{576} \sqrt{2\sqrt{47}} \).

**Remark 1.4.** Most of the material in this chapter is to be found in

[47,49,50,52-54,64,66].
Chapter 2

Approximation of multivariate functions by sums of univariate functions

It is clear that in the special case, when directions of ridge functions coincide with the coordinate directions, the problem of approximation by linear combinations of these functions turn into the problem of approximation by sums of univariate functions. This is also the simplest case in ridge function approximation. The simplicity of the approximation guarantees its practicability in application areas, where complicated multivariate functions are main obstacles. In mathematics, this type of approximation has arisen, for example, in connection with the classical functional equations [11], the numerical solution of certain PDE boundary value problems [9], dimension theory [132,133], etc. In this chapter, we obtain some results concerning the problem of best approximation by sums of univariate functions.

Most of the material of this chapter is taken from [48,55,56,59].

2.1 Characterization of some bivariate function classes by formulas for the error of approximation

This section is devoted to calculation formulas for the error of approximation of bivariate functions by sums of univariate functions. Certain classes of
bivariate functions depending on some numerical parameter are constructed
and characterized in terms of the approximation error calculation formulas.

2.1.1 Exposition of the problem

The approximation problem considered here is to approximate a continuous
and real-valued function of two variables by sums of two continuous functions
of one variable. To make the problem precise, let $Q$ be a compact set in the
$xOy$ plane. Consider the approximation of a continuous function $f \in C(Q)$
by functions from the manifold $D = \{ \varphi(x) + \psi(y) \}$, where $\varphi(x), \psi(y)$
are defined and continuous on the projections of $Q$ into the coordinate axes $x$
and $y$, respectively. The approximation error is defined as the distance from
$f$ to $D$:

$$ E(f) = \text{dist}(f, D) = \sup_{D} \| f - \varphi - \psi \|_{C(Q)} = $$

$$ = \sup_{D} \max_{(x,y) \in Q} |f(x,y) - \varphi(x) - \psi(y)|. $$

A function $\varphi_0(x) + \psi_0(y)$ from $D$, if it exists, is called an extremal element
or a best approximating sum if

$$ E(f) = \| f - \varphi_0 - \psi_0 \|_{C(Q)}. $$

To show that $E(f)$ depends also on $Q$, in some cases to avoid confusion, we
will write $E(f, Q)$ instead of $E(f)$.

In this section we deal with calculation formulas for $E(f)$. In 1951 Diliberto
and Straus published a paper [26], in which along with other results
they established a formula for $E(f, R)$, where $R$ here and throughout this
section is a rectangle with sides parallel to the coordinate axes, containing
supremum over all closed lightning bolts. Later the same formula was es-
tablished by other authors differently, in cases of both rectangle (see [113])
and more general sets (see [79], [107]). Although the formula was valid for
all continuous functions, it was not easily calculable. Some authors started
to seek easily calculable formulas for the approximation error for some sub-
sets of continuous functions. Rivlin and Sibner [121] proved a result, which
allow one to find the exact value of $E(f, R)$ for a function $f(x,y)$ having the
continuous and nonnegative derivative $\frac{\partial^2 f}{\partial x \partial y}$. This result in a more general
case (for functions of $n$ variables) was proved by Flatto [30]. Babaev [6] gen-
eralized Rivlin and Sibner’s result (as well as Flatto’s result, see [7]). More
precisely, he considered the class $M(R)$ of continuous functions $f(x, y)$ with the property
\[
\Delta_{h_1,h_2} f = f(x, y) + f(x + h_1, y + h_2) - f(x, y + h_2) - f(x + h_1, y) \geq 0
\]
for each rectangle $[x, x + h_1] \times [y, y + h_2] \subset R$, and proved that if $f(x, y)$ belongs to $M(R)$, where $R = [a_1, b_1] \times [a_2, b_2]$, then
\[
E(f, R) = \frac{1}{4} [f(a_1, a_2) + f(b_1, b_2) - f(a_1, b_2) - f(b_1, a_2)].
\]
As seen from this formula, to calculate $E(f)$ it is sufficient to find only values of $f(x, y)$ at the vertices of $R$. One can see that the formula also gives a sufficient condition for membership in the class $M(R)$, i.e. if
\[
E(f, S) = \frac{1}{4} [f(x_1, y_1) + f(x_2, y_2) - f(x_1, y_2) - f(x_2, y_1)],
\]
for a given $f$ and for each $S = [x_1, x_2] \times [y_1, y_2] \subset R$, then the function $f(x, y)$ is from $M(R)$.

Our purpose is to construct new classes of continuous functions, which will depend on a numerical parameter, and characterize each class in terms of the approximation error calculation formulas. The mentioned parameter will show which points of $R$ the calculation formula involves. We will also construct a best approximating sum $\varphi_0 + \psi_0$ to a function from constructed classes.

### 2.1.2 Definition of the main classes

Let throughout this section $R = [a_1, b_1] \times [a_2, b_2]$ be a rectangle and $c \in (a_1, b_1]$. Denote $R_1 = [a_1, c] \times [a_2, b_2]$ and $R_2 = [c, b_1] \times [a_2, b_2]$. It is clear that $R = R_1 \cup R_2$ and if $c = b_1$, then $R = R_1$.

We associate each rectangle $S = [x_1, x_2] \times [y_1, y_2]$ lying in $R$ with the following functional:
\[
L(f, S) = \frac{1}{4} [f(x_1, y_1) + f(x_2, y_2) - f(x_1, y_2) - f(x_2, y_1)].
\]

**Definition 2.1.** *We say that a continuous function $f(x, y)$ belongs to the class $V_c(R)$ if*

82
1) $L(f, S) \geq 0$, for each $S \subset R_1$;
2) $L(f, S) \leq 0$, for each $S \subset R_2$;
3) $L(f, S) \geq 0$, for each $S = [a_1, b_1] \times [y_1, y_2]$, $S \subset R$.

It can be shown that for any $c \in (a_1, b_1]$ the class $V_c(R)$ is not empty. Indeed, one can easily verify that the function

$$v_c(x, y) = \begin{cases} w(x, y) - w(c, y), & (x, y) \in R_1 \\ w(c, y) - w(x, y), & (x, y) \in R_2 \end{cases}$$

where $w(x, y) = \left(\frac{x-a_1}{b_1-a_1}\right)^n \cdot y$ and $n \geq \log_2 \frac{b_1-a_1}{c-a_1}$, satisfies conditions 1)-3) and therefore belongs to $V_c(R)$. The class $V_c(R)$ has the following obvious properties:

a) For given functions $f_1, f_2 \in V_c(R)$ and numbers $\alpha_1, \alpha_2 \geq 0, \alpha_1 f_1 + \alpha_2 f_2 \in V_c(R)$. $V_c(R)$ is a closed subset of the space of continuous functions.

b) $V_{b_1}(R) = M(R)$.

c) If $f$ is a common element of $V_{c_1}(R)$ and $V_{c_2}(R)$, $a_1 < c_1 < c_2 \leq b_1$ then $f(x, y) = \varphi(x) + \psi(y)$ on the rectangle $[c_1, c_2] \times [a_2, b_2]$.

The properties a) and b) are clear. The property c) also becomes clear if note that according to the definition of the classes $V_{c_1}(R)$ and $V_{c_2}(R)$, for each rectangle

$$S \subset [c_1, c_2] \times [a_2, b_2]$$

we have

$$L(f, S) \leq 0 \text{ and } L(f, S) \geq 0,$$

respectively. Hence

$$L(f, S) = 0 \text{ for each } S \subset [c_1, c_2] \times [a_2, b_2].$$

Thus it is not difficult to understand that $f$ is of the form $\varphi(x) + \psi(y)$ on the rectangle $[c_1, c_2] \times [a_2, b_2]$.

**Lemma 2.1.** Assume a function $f(x, y)$ has the continuous derivative $\frac{\partial^2 f}{\partial x \partial y}$ on the rectangle $R$ and satisfies the following conditions

1) $\frac{\partial^2 f}{\partial x \partial y} \geq 0$, for all $(x, y) \in R_1$;
2) $\frac{\partial^2 f}{\partial x \partial y} \leq 0$, for all $(x, y) \in R_2$;
3) $\frac{df(a_1, y)}{dy} \leq \frac{df(b_1, y)}{dy}$, for all $y \in [a_2, b_2]$. 

83
Then \( f(x, y) \) belongs to \( V_c(R) \).

The proof of this lemma is very simple and can be obtained by integrating both sides of inequalities in conditions 1)-3) through sets \([x_1, x_2] \times [y_1, y_2] \subset R_1\), \([x_1, x_2] \times [y_1, y_2] \subset R_2\) and \([y_1, y_2] \subset [a_2, b_2]\), respectively.

Example 2.1. Consider the function \( f(x, y) = y \sin \pi x \) on the unit square \( K = [0, 1] \times [0, 1] \) and rectangles \( K_1 = [0, \frac{1}{2}] \times [0, 1], K_2 = [\frac{1}{2}, 1] \times [0, 1] \). It is not difficult to verify that this function satisfies all conditions of the lemma and therefore belongs to \( V_2(K) \).

2.1.3 Construction of an extremal element

The following theorem is valid.

**Theorem 2.1.** The approximation error of a function \( f(x, y) \) from the class \( V_c(R) \) can be calculated by the formula

\[
E(f, R) = L(f, R_1) = \frac{1}{4} \left[ f(a_1, a_2) + f(c, b_2) - f(a_1, b_2) - f(c, a_2) \right].
\]

Let \( y_0 \) be any solution from \([a_2, b_2]\) of the equation

\[
L(f, Y) = \frac{1}{2} L(f, R_1), \quad Y = [a_1, c] \times [a_2, y].
\]

Then the function \( \varphi_0(x) + \psi_0(y) \), where

\[
\varphi_0(x) = f(x, y_0),
\]

\[
\psi_0(y) = \frac{1}{2} \left[ f(a_1, y) + f(c, y) - f(a_1, y_0) - f(c, y_0) \right]
\]

is a best approximating sum from the manifold \( D \) to \( f \).

To prove this theorem we need the following lemma.

**Lemma 2.2.** Let \( f(x, y) \) be a function from \( V_c(R) \) and \( X = [a_1, x] \times [y_1, y_2] \) be a rectangle with fixed \( y_1, y_2 \in [a_2, b_2] \). Then the function \( h(x) = L(f, X) \) has the properties:

1) \( h(x) \geq 0 \), for any \( x \in [a_1, b_1] \);
2) \( \max_{[a_1,b_1]} h(x) = h(c) \) and \( \min_{[a_1,b_1]} h(x) = h(a_1) = 0. \)

**Proof.** If \( X \subset R_1 \), then the validity of \( h(x) \geq 0 \) follows from the definition of \( V_c(R) \). If \( X \) is from \( R \) but not lying in \( R_1 \), then by denoting \( X' = [x,b_1] \times [y_1,y_2], S = X \cup X' \) and using the obvious equality

\[
L(f, S) = L(f, X) + L(f, X')
\]

we deduce from the definition of \( V_c(R) \) that \( h(x) \geq 0 \).

To prove the second part of the lemma, it is enough to show that \( h(x) \) increases on the interval \( [a_1,c] \) and decreases on the interval \( [c,b_1] \). Indeed, if \( a_1 \leq x_1 \leq x_2 \leq c \), then

\[
h(x_2) = L(f, X_2) = L(f, X_1) + L(f, X_{12}), \tag{2.1}
\]

where \( X_1 = [a_1,x_1] \times [y_1,y_2], X_2 = [a_1,x_2] \times [y_1,y_2], X_{12} = [x_1,x_2] \times [y_1,y_2] \). Taking into consideration that \( L(f, X_1) = h(x_1) \) and \( X_{12} \) lies in \( R_1 \) we obtain from (2.1) that \( h(x_2) \geq h(x_1) \). If \( c \leq x_1 \leq x_2 \leq b_1 \), then \( X_{12} \) lies in \( R_2 \) and we obtain from (2.1) that \( h(x_2) \leq h(x_1) \).

**Proof of Theorem 2.1.** It is obvious that \( L(f, R_1) = L(f - \varphi - \psi, R_1) \) for each sum \( \varphi(x) + \psi(y) \). Hence

\[
L(f, R_1) \leq \| f - \varphi - \psi \|_{C(R_1)} \leq \| f - \varphi - \psi \|_{C(R)}.
\]

Since a sum \( \varphi(x) + \psi(y) \) is arbitrary, \( L(f, R_1) \leq E(f, R) \). To complete the proof it is sufficient to construct a sum \( \varphi_0(x) + \psi_0(y) \) for which the equality

\[
\| f - \varphi_0 - \psi_0 \|_{C(R)} = L(f, R_1) \tag{2.2}
\]

holds.

Consider the function

\[
g(x,y) = f(x,y) - f(x,a_2) - f(a_1,y) + f(a_1,a_2).
\]

This function has the following obvious properties

1) \( g(x,a_2) = g(a_1,y) = 0; \)
2) \( L(f, R_1) = L(g, R_1) = \frac{1}{4} g(c,b_2); \)
3) \( E(f, R) = E(g, R); \)
4) The function of one variable \( g(c,y) \) increases on the interval \([a_2,b_2].\)
The last property of $g$ allows us to write that

$$0 = g(c, a_2) \leq \frac{1}{2} g(c, b_2) \leq g(c, b_2).$$

Since $g(x, y)$ is continuous, there exists at least one solution $y = y_0$ of the equation

$$g(c, y) = \frac{1}{2} g(c, b_2)$$

or, in other notation, of the equation

$$L(f, Y) = \frac{1}{2} L(f, R_1), \text{ where } Y = [a_1, c] \times [a_2, y_0],$$

Introduce the functions

$$\varphi_1(x) = g(x, y_0),$$

$$\psi_1(y) = \frac{1}{2} (g(c, y) - g(c, y_0)),$$

$$G(x, y) = g(x, y) - \varphi_1(x) - \psi_1(y).$$

Calculate the norm of $G(x, y)$ on $R$. Consider the rectangles $R' = [a_1, b_1] \times [y_0, b_2]$ and $R'' = [a_1, b_1] \times [a_2, y_0]$. It is clear that

$$\|G\|_{C(R)} = \max \left\{ \|G\|_{C(R')}, \|G\|_{C(R'')} \right\}.$$ 

First calculate the norm $\|G\|_{C(R')}$:

$$\|G\|_{C(R')} = \max_{(x, y) \in R'} |G(x, y)| = \max_{y \in [y_0, b_2]} \max_{x \in [a_1, b_1]} |G(x, y)|. \quad (2.3)$$

For a fixed point $y$ (we keep it fixed until (2.6)) from the interval $[y_0, b_2]$ we can write that

$$\max_{x \in [a_1, b_1]} G(x, y) = \max_{x \in [a_1, b_1]} (g(x, y) - g(x, y_0)) - \psi_1(y) \quad (2.4)$$

and

$$\min_{x \in [a_1, b_1]} G(x, y) = \min_{x \in [a_1, b_1]} (g(x, y) - g(x, y_0)) - \psi_1(y). \quad (2.5)$$

By Lemma 2.2, the function

$$h_1(x) = 4L(f, X) = g(x, y) - g(x, y_0), \text{ where } X = [a_1, x] \times [y_0, y],$$

86
reaches its maximum on $x = c$ and minimum on $x = a_1$:

$$\max_{x \in [a_1, b_1]} h_1(x) = g(c, y) - g(c, y_0)$$

$$\min_{x \in [a_1, b_1]} h_1(x) = g(a_1, y) - g(a_1, y_0) = 0.$$  

Considering these facts in (2.4) and (2.5) we obtain that

$$\max_{x \in [a_1, b_1]} G(x, y) = g(c, y) - g(c, y_0) - \psi_1(y) = \frac{1}{2} (g(c, y) - g(c, y_0)),$$

$$\min_{x \in [a_1, b_1]} G(x, y) = -\psi_1(y) = -\frac{1}{2} (g(c, y) - g(c, y_0)).$$

Consequently,

$$\max_{x \in [a_1, b_1]} |G(x, y)| = \frac{1}{2} (g(c, y) - g(c, y_0)) \quad \text{(2.6)}.$$  

Taking (2.6) and the 4-th property of $g$ into account in (2.3) yields

$$\|G\|_{C(R')} = \frac{1}{2} (g(c, b_2) - g(c, y_0)) = \frac{1}{4} g(c, b_2).$$

Similarly it can be shown that

$$\|G\|_{C(R')} = \frac{1}{4} g(c, b_2).$$  

Hence

$$\|G\|_{C(R)} = \frac{1}{4} g(c, b_2) = L(f, R_1).$$

But by the definition of $G$,

$$G(x, y) = g(x, y) - \varphi_1(x) - \psi_1(y) = f(x, y) - \varphi_0(x) - \psi_0(y),$$

where

$$\varphi_0(x) = \varphi_1(x) + f(x, a_2) - f(a_1, a_2) + f(a_1, y_0) = f(x, y_0),$$

$$\psi_0(y) = \psi_1(y) + f(a_1, y) - f(a_1, y_0) =$$

$$= \frac{1}{2} (f(a_1, y) + f(c, y) - f(a_1, y_0) - f(c, y_0)).$$

87
Therefore, 
\[ \|f - \varphi_0 - \psi_0\|_{C(R)} = L(f, R_1). \]

We proved (2.2) and hence Theorem 2.1. Note that the function \( \varphi_0(x) + \psi_0(y) \) is a best approximating sum from the manifold \( D \) to \( f \).

**Remark 2.1.** In the special case \( c = b_1 \), Theorem 2.1 turns into Babaev’s result from [6].

**Corollary 2.1.** Let a function \( f(x, y) \) have the continuous derivative \( \frac{\partial^2 f}{\partial x \partial y} \) on the rectangle \( R \) and satisfy the following conditions
\begin{enumerate}
  \item \( \frac{\partial^2 f}{\partial x \partial y} \geq 0 \), for all \((x, y) \in R_1; \)
  \item \( \frac{\partial^2 f}{\partial x \partial y} \leq 0 \), for all \((x, y) \in R_2; \)
  \item \( \frac{df(a_1, y)}{dy} \leq \frac{df(b_1, y)}{dy} \), for all \( y \in [a_2, b_2]. \)
\end{enumerate}

Then
\[ E(f, R) = L(f, R_1) = \frac{1}{4} [f(a_1, a_2) + f(c, b_2) - f(a_1, b_2) - f(c, a_2)]. \]

The proof of this corollary can be obtained directly from Lemma 2.1 and Theorem 2.1.

**Remark 2.2.** Rivlin and Sibner [121] proved Corollary 2.1 in the special case \( c = b_1 \).

**Example 2.2.** As we know (see Example 2.1) the function \( f = y \sin \pi x \) belongs to \( V_{1/2}^c(K) \), where \( K = [0, 1] \times [0, 1] \). By Theorem 2.1, \( E(f, K) = \frac{1}{4} \) and the function \( \frac{1}{2} \sin \pi x + \frac{1}{2}y - \frac{1}{4} \) is a best approximating sum.

The following theorem shows that in some cases the approximation error formula in Theorem 2.1 is valid for more general sets than rectangles with sides parallel to the coordinate axes.

**Theorem 2.2.** Let \( f(x, y) \) be a function from \( V_c(R) \) and \( Q \subset R \) be a compact set which contains all vertices of \( R_1 \) (points \((a_1, a_2), (a_1, b_2), (c, a_2), (c, b_2)\)). Then
\[ E(f, Q) = L(f, R_1) = \frac{1}{4} [f(a_1, a_2) + f(c, b_2) - f(a_1, b_2) - f(c, a_2)]. \]
Proof. Since $Q \subset R$, $E(f, Q) \leq E(f, R)$. On the other hand by Theorem 2.1, $E(f, R) = L(f, R_1)$. Hence $E(f, Q) \leq L(f, R_1)$. It can be shown, as it has been shown in the proof of Theorem 2.1, that $L(f, R_1) \leq E(f, Q)$. But then automatically $E(f, Q) = L(f, R_1)$.

Example 2.3. Calculate the approximation error of the function $f(x, y) = -(x - 2)^{2n}y^m$ ($n$ and $m$ are positive integers) on the domain $Q = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq (x - 1)^2 + 1\}$.

It can be easily verified that $f \in V_2(R)$, where $R = [0, 4] \times [0, 2]$. Besides, $Q$ contains all vertices of $R_1 = [0, 2] \times [0, 2]$. Consequently, by Theorem 2.2, $E(f, Q) = L(f, R_1) = 2^{2(n-1)+m}$.

2.1.4 Characterization of $V_c(R)$

The following theorem characterizes the class $V_c(R)$ in terms of the approximation error calculation formulas.

Theorem 2.3. The following conditions are necessary and sufficient for a continuous function $f(x, y)$ belong to $V_c(R)$:

1) $E(f, S) = L(f, S)$, for each rectangle $S = [x_1, x_2] \times [y_1, y_2], S \subset R_1$;
2) $E(f, S) = -L(f, S)$, for each rectangle $S = [x_1, x_2] \times [y_1, y_2], S \subset R_2$;
3) $E(f, S) = L(f, S_1)$, for each rectangle $S = [a_1, b_1] \times [y_1, y_2], S \subset R$ and $S_1 = [a_1, c] \times [y_1, y_2]$.

Proof. The necessity easily follows from the definition of $V_c(R)$, Babaev’s above-mentioned result (see Section 2.1.1) and Theorem 2.1. The sufficiency is clear if pay attention to the fact that $E(f, S) \geq 0$.

2.1.5 Classes $V_c^-(R), U(R)$ and $U_c^-(R)$

By $V_c^-(R)$ we denote the class of functions $f(x, y)$ such that $-f \in V_c(R)$. It is clear that $E(f, R) = -L(f, R_1)$ for each $f \in V_c^-(R)$.

We define $U_c(R), a_1 \leq c < b_1$, as a class of continuous functions $f(x, y)$ with the properties

1) $L(f, S) \leq 0$, for each rectangle $S = [x_1, x_2] \times [y_1, y_2], S \subset R_1$;
2) \( L(f, S) \geq 0 \), for each rectangle \( S = [x_1, x_2] \times [y_1, y_2] \), \( S \subset R_2 \);
3) \( L(f, S) \geq 0 \), for each rectangle \( S = [a_1, b_1] \times [y_1, y_2] \), \( S \subset R \).

Using the same techniques in the proof of Theorem 2.1 it can be shown that the following theorem is valid:

**Theorem 2.4.** The approximation error of a function \( f(x, y) \) from the class \( U_c(R) \) can be calculated by the formula

\[
E(f, R) = L(f, R_2) = \frac{1}{4} [f(c, a_2) + f(b_1, b_2) - f(c, b_2) - f(b_1, a_2)].
\]

Let \( y_0 \) be any solution from \([a_2, b_2]\) of the equation

\[
L(f, Y) = \frac{1}{2} L(f, R_2), \quad Y = [c, b_1] \times [a_2, y].
\]

Then the function \( \varphi_0(x) + \psi_0(y) \), where

\[
\varphi_0(x) = f(x, y_0), \quad \psi_0(y) = \frac{1}{2} [f(c, y) + f(b_1, y) - f(c, y_0) - f(b_1, y_0)],
\]

is a best approximating sum from the manifold \( D \) to \( f \).

By \( U_c^{-}(R) \) denote the class of functions \( f(x, y) \) such that \(-f \in U_c(R)\). It is clear that \( E(f, R) = -L(f, R_2) \) for each \( f \in U_c^{-}(R) \).

**Remark 2.3.** The correspondingly modified versions of Theorems 2.2, 2.3 and Corollary 2.1 are valid for the classes \( V_c^{-}(R), U_c(R) \) and \( U_c^{-}(R) \).

**Example 2.4.** Consider the function \( f(x, y) = (x - \frac{1}{2})^2 y \) on the unit square \( K = [0, 1] \times [0, 1] \). It can be easily verified that \( f \in U_c^{-}(K) \). Hence, by Theorem 2.4, \( E(f, K) = \frac{1}{16} \) and the function \( \frac{1}{2} (x - \frac{1}{2})^2 + \frac{1}{8} y - \frac{1}{16} \) is a best approximating function.

### 2.2 Approximation by sums of univariate functions on certain domains

The purpose of this section is to develop a method for obtaining explicit formulas for the error of approximation of bivariate functions by sums of
univariate functions. It should be remarked that formulas of this type were known only for functions defined on a rectangle with sides parallel to the coordinate axes. Our method, based on a maximization process over closed bolts, allows the consideration of functions defined on hexagons, octagons and stairlike polygons with sides parallel to the coordinate axes.

2.2.1 Problem statement

Let $Q$ be a compact set in $\mathbb{R}^2$. Consider the approximation of a continuous function $f \in C(Q)$ by functions from the set $D = \{ \varphi(x) + \psi(y) \}$, where $\varphi(x), \psi(y)$ are defined and continuous on the projections of $Q$ into the coordinate axes $x$ and $y$, respectively. The approximation error is defined as follows

$$E(f, Q) = \inf_{\varphi + \psi \in D} \| f - \varphi - \psi \|_{C(Q)}.$$

Our purpose is to develop a method for obtaining explicit formulas providing precise and easy computation of $E(f, Q)$ for polygons $Q$ with sides parallel to the coordinate axes. This method will be based on the herein developed closed bolts maximization process and can be used in alternative proofs of the known results from [6], [59] and [121]. First, we show efficiency of the method in the example of a hexagon with sides parallel to the coordinate axes. Then we formulate an analogous theorem for staircase polygons and two theorems for octagons, which can be proved in a similar way, and touch some aspects of the question about the case of an arbitrary polygon with sides parallel to the coordinate axes. The condition posed on sides of polygons (being parallel to the coordinate axes) is essential for our method. This has several reasons, which get clear through the proof of Theorem 2.5. Here we are able to explain one of these reasons: by theorem 3 from [34], a continuous function $f(x, y)$ defined on a polygon with sides parallel to the coordinate axes has an extremal element, the existence of which is required in our method. Now let $K$ be a rectangle (not speaking about polygons) with sides not parallel to the coordinate axes. Does any function $f \in C(K)$ have an extremal element? No one knows (see[34]).

In the sequel, all the considered polygons are supposed to have sides parallel to the coordinate axes.
2.2.2 The maximization process

Let $H$ be a closed hexagon. It is clear that $H$ can be uniquely represented in the form

$$H = R_1 \cup R_2,$$  \hspace{1cm} (2.7)

where $R_1, R_2$ are rectangles and there does not exist any rectangle $R$ such that $R_1 \subset R \subset H$ or $R_2 \subset R \subset H$.

We associate each closed bolt $p = \{p_1, p_2, \cdots p_{2n}\}$ with the following functional

$$l(f, p) = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^{k-1} f(p_k).$$

Denote by $M(H)$ the class of bivariate continuous functions $f$ on $H$ satisfying the condition

$$f(x_1, y_1) + f(x_2, y_2) - f(x_1, y_2) - f(x_2, y_1) \geq 0$$

for any rectangle $[x_1, x_2] \times [y_1, y_2] \subset H$.

**Theorem 2.5.** Let $H$ be a hexagon and (2.7) be its representation. Let $f \in M(H)$. Then

$$E(f, H) = \max \{|l(f, h)|, |l(f, r_1)|, |l(f, r_2)|\},$$  \hspace{1cm} (2.8)

where $h, r_1, r_2$ are closed bolts formed by vertices of the polygons $H, R_1, R_2$ respectively.

**Proof.** Without loss of generality, we may assume that the rectangles $R_1$ and $R_2$ are of the following form

$$R_1 = [a_1, a_2] \times [b_1, b_3], \quad R_2 = [a_1, a_3] \times [b_1, b_2], \quad a_1 < a_2 < a_3, \quad b_1 < b_2 < b_3.$$ 

Introduce the notation

$$f_{11} = f(a_1, b_1), \quad f_{12} = -f(a_1, b_2), \quad f_{13} = -f(a_1, b_3);$$
$$f_{21} = -f(a_2, b_1), \quad f_{22} = -f(a_2, b_2), \quad f_{23} = f(a_2, b_3);$$
$$f_{31} = -f(a_3, b_1), \quad f_{32} = f(a_3, b_2).$$  \hspace{1cm} (2.9)
It is clear that

\[
|l(f, r_1)| = \frac{1}{4} (f_{11} + f_{13} + f_{23} + f_{21}),
\]

\[
|l(f, r_2)| = \frac{1}{4} (f_{11} + f_{12} + f_{32} + f_{31}),
\]

\[
|l(f, h)| = \frac{1}{6} (f_{11} + f_{13} + f_{23} + f_{22} + f_{32} + f_{31}).
\]  

(2.10)

Let \( p = \{p_1, p_2, \cdots p_{2n}\} \) be any closed bolt. We group the points \( p_1, p_2, \cdots p_{2n} \) by putting

\[ p_+ = \{p_1, p_3, \cdots p_{2n-1}\}, \quad p_- = \{p_2, p_4, \cdots p_{2n}\}. \]

First, assume that \( l(f, p) \geq 0 \). We apply the following algorithm, which we call the maximization process over closed bolts, to \( p \).

**Step 1.** Consider sequentially the units \( p_ip_{i+1} \) \( (i = 1, 2n, p_{2n+1} = p_1) \) with the vertices \( p_i(x_i, y_i) \), \( p_{i+1}(x_{i+1}, y_{i+1}) \) having equal abscissae: \( x_i = x_{i+1} \). Four cases are possible.

1) \( p_i \in p_+ \) and \( y_{i+1} > y_i \). In this case, replace the unit \( p_ip_{i+1} \) by a new unit \( q_iq_{i+1} \) with the vertices \( q_i = (a_1, y_i) \), \( q_{i+1} = (a_1, y_{i+1}) \).

2) \( p_i \in p_+ \) and \( y_{i+1} < y_i \). In this case, replace the unit \( p_ip_{i+1} \) by a new unit \( q_iq_{i+1} \) with the vertices \( q_i = (a_2, y_i) \), \( q_{i+1} = (a_2, y_{i+1}) \) if \( b_2 < y_i \leq b_3 \) or with the vertices \( q_i = (a_3, y_i) \), \( q_{i+1} = (a_3, y_{i+1}) \) if \( b_1 \leq y_i \leq b_2 \).

3) \( p_i \in p_- \) and \( y_{i+1} < y_i \). In this case, replace \( p_ip_{i+1} \) by a new unit \( q_iq_{i+1} \) with the vertices \( q_i = (a_1, y_i) \), \( q_{i+1} = (a_1, y_{i+1}) \).

4) \( p_i \in p_- \) and \( y_{i+1} > y_i \). In this case, replace \( p_ip_{i+1} \) by a new unit \( q_iq_{i+1} \) with the vertices \( q_i = (a_2, y_i) \), \( q_{i+1} = (a_2, y_{i+1}) \) if \( b_2 < y_{i+1} \leq b_3 \) or with the vertices \( q_i = (a_3, y_i) \), \( q_{i+1} = (a_3, y_{i+1}) \) if \( b_1 \leq y_{i+1} \leq b_2 \).

Since \( f \in M(H) \), it is not difficult to verify that

\[
f(p_i) - f(p_{i+1}) \leq f(q_i) - f(q_{i+1}) \quad \text{for cases } 1 \text{ and } 2,
\]

\[
-f(p_i) + f(p_{i+1}) \leq -f(q_i) + f(q_{i+1}) \quad \text{for cases } 3 \text{ and } 4.
\]  

(2.11)

It is clear that after Step 1 the bolt \( p \) will be replaced by the ordered set \( q = \{q_1, q_2, \cdots, q_{2n}\} \). We do not say a bolt but an ordered set because of a possibility of coincidence of some successive points \( q_i, q_{i+1} \) (this, for example, may happen if the 1-st case takes place for the units \( p_{i-1}p_i \) and \( p_{i+1}p_{i+2} \)). Let us exclude simultaneously successive and coincident points from \( q \). Then we obtain some closed bolt, which we denote by \( q' = \{q'_1, q'_2, \cdots, q'_{2n}\} \). It is not
difficult to understand that all points of the bolt \( q' \) are located on straight lines \( x = a_1, \ x = a_2, \ x = a_3 \).

From inequalities (2.11) and the fact that \( 2m \leq 2n \), we deduce that
\[
l(f, p) \leq l(f, q'). \tag{2.12}
\]

**Step 2.** Consider sequentially units \( q'_i q'_{i+1} \ (i = 1, 2m, q'_{2m+1} = q'_1) \) with the vertices \( q'_i = (x'_i, y'_i), \ q'_{i+1} = (x'_{i+1}, y'_{i+1}) \) having equal ordinates: \( y'_i = y'_{i+1} \). The following four cases are possible.

1) \( q'_i \in q'_+ \) and \( x'_{i+1} > x'_i \). In this case, replace the unit \( q'_i q'_{i+1} \) by a new unit \( p'_i p'_{i+1} \) with the vertices \( p'_i = (x'_i, b_1), \ p'_{i+1} = (x'_{i+1}, b_1) \).

2) \( q'_i \in q'_+ \) and \( x'_{i+1} < x'_i \). In this case, replace the unit \( q'_i q'_{i+1} \) by a new unit \( p'_i p'_{i+1} \) with the vertices \( p'_i = (x'_i, b_2), \ p'_{i+1} = (x'_{i+1}, b_2) \) if \( x'_i = a_3 \) and with the vertices \( p'_i = (x'_i, b_3), \ p'_{i+1} = (x'_{i+1}, b_3) \) if \( x'_i = a_2 \).

3) \( q'_i \in q'_- \) and \( x'_{i+1} < x'_i \). In this case, replace \( q'_i q'_{i+1} \) by a new unit \( p'_i p'_{i+1} \) with the vertices \( p'_i = (x'_i, b_1), \ p'_{i+1} = (x'_{i+1}, b_1) \).

4) \( q'_i \in q'_- \) and \( x'_{i+1} > x'_i \). In this case, replace \( q'_i q'_{i+1} \) by a new unit \( p'_i p'_{i+1} \) with the vertices \( p'_i = (x'_i, b_2), \ p'_{i+1} = (x'_{i+1}, b_2) \) if \( x'_i = a_3 \) and with the vertices \( p'_i = (x'_i, b_3), \ p'_{i+1} = (x'_{i+1}, b_3) \) if \( x'_i = a_2 \).

It is easy to see that after Step 2 the bolt \( q' \) will be replaced by the bolt \( p' = \{p'_1, p'_2, \ldots p'_{2m}\} \) and
\[
l(f, q') \leq l(f, p'). \tag{2.13}
\]

From (2.12) and (2.13) we obtain that
\[
l(f, p) \leq l(f, p'). \tag{2.14}
\]

It is clear that each point of the set \( p'_+ \) coincides with one of the points \( (a_1, b_1), (a_2, b_3), (a_3, b_2) \) and each point of the set \( p'_- \) coincides with one of the points \( (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_3, b_1) \). Denote by \( m_{ij} \) the number of points of the bolt \( p' \) coinciding with the point \( (a_i, b_j), \ i, j = 1, 3, \ i+j \neq 6 \). By (2.9), we can write that
\[
l(f, p') = \frac{1}{2m} \sum_{i,j=1}^{3} m_{ij} f_{ij}. \tag{2.15}
\]

On the straight line \( x = a_i \) or \( y = b_i, \ i = 1, 3, \) the number of points of the set \( p'_+ \) is equal to the number of points of the set \( p'_- \). Hence
\[
m_{11} = m_{12} + m_{13} = m_{21} + m_{31}; \ m_{23} = m_{22} + m_{21} = m_{13}; \ m_{32} = m_{31} = m_{12} + m_{22}.
\]
From these equalities we deduce that
\[ m_{11} = m_{12} + m_{21} + m_{22}; \quad m_{13} = m_{21} + m_{22}; \quad m_{23} = m_{21} + m_{22}; \quad m_{31} = m_{12} + m_{21}. \]  

(2.16)

Consequently,
\[ 2m = \sum_{i,j=1,3 \atop i+j \leq 5} m_{ij} = 4m_{12} + 4m_{21} + 6m_{22}. \]

(2.17)

Considering (2.16) and (2.17) in (2.15) and taking (2.10) into account, we obtain that
\[ l(f, p') = \frac{4m_{12} |l(f, r_2)| + 4m_{21} |l(f, r_1)| + 6m_{22} |l(f, h)|}{4m_{12} + 4m_{21} + 6m_{22}} \]
\[ \leq \max \{ |l(f, r_1)|, |l(f, r_2)|, |l(f, h)| \}. \]

Therefore, due to (2.14),
\[ l(f, p) \leq \max \{ |l(f, r_1)|, |l(f, r_2)|, |l(f, h)| \}. \]

(2.18)

Note that in the beginning of the proof the bolt \( p \) has been chosen so that \( l(f, p) \geq 0 \). Let now \( p = \{ p_1, p_2, \ldots, p_{2n} \} \) be any closed bolt such that \( l(f, p) \leq 0 \). Since \( l(f, p'') = -l(f, p) \geq 0 \) for the bolt \( p'' = \{ p_2, p_3, \ldots, p_{2n}, p_1 \} \), we obtain from (2.18) that
\[ -l(f, p) \leq \max \{ |l(f, r_1)|, |l(f, r_2)|, |l(f, h)| \}. \]

(2.19)

From (2.18) and (2.19) we deduce on the strength of arbitrariness of \( p \) that
\[ \sup_{p \in H} \{ |l(f, p)| \} = \max \{ |l(f, r_1)|, |l(f, r_2)|, |l(f, h)| \}, \]

(2.20)

where the \( \sup \) is taken over all closed bolts of the hexagon \( H \).

The hexagon \( H \) satisfies the conditions of Theorem 1.10 on the existence of a best approximation. By theorem 2 [79] (see Section 2.3), we obtain that
\[ E(f, H) = \sup_{p \in H} \{ |l(f, p)| \}. \]

(2.21)

From (2.20) and (2.21) we finally conclude that
\[ E(f, H) = \max \{ |l(f, r_1)|, |l(f, r_2)|, |l(f, h)| \}. \]

\( \square \)
Corollary 2.2. Let a function $f(x,y)$ have the continuous nonnegative derivative $\frac{\partial^2 f}{\partial x \partial y}$ on $H$. Then the formula (2.8) is valid.

The proof is very simple and can be obtained by integrating the inequality $\frac{\partial^2 f}{\partial x \partial y} \geq 0$ over an arbitrary rectangle $[x_1, x_2] \times [y_1, y_2] \subset H$ and applying Theorem 2.5.

The method used in the proof of Theorem 2.5 can be generalized to obtain similar results for stairlike polygons. For example, let $S$ be a closed polygon of the following form

$$S = \bigcup_{i=1}^{N-1} P_i,$$

where $N \geq 2$, $P_i = [a_i, a_{i+1}] \times [b_1, b_{N+1-i}]$, $i = 1, N - 1$, $a_1 < a_2 < \cdots < a_N$, $b_1 < b_2 < \cdots < b_N$. Such polygons will be called stairlike polygons (see [56]).

A closed $2m$-gon $F$ with sides parallel to the coordinate axes is called a maximal $2m$-gon of the polygon $S$ if $F \subset S$ and there is no another $2m$-gon $F'$ such that $F \subset F' \subset S$. Clearly, if $F$ is a maximal $2m$-gon of the polygon $S$, then $m \leq N$. A closed bolt formed by the vertices of a maximal polygon $F$ is called a maximal bolt of $S$. By $S^B$ denote the set of all maximal bolts of the stairlike polygon $S$.

Theorem 2.6. Let $S$ be a stairlike polygon. The approximation error of a function $f \in M(S)$ can be computed by the formula

$$E(f, S) = \max \{ |r(f, h)|, \ h \in S^B \}.$$

For the proof of this theorem see [56].

2.2.3 $E$-bolts

The main idea in the proof of Theorem 2.5 can be successfully used in obtaining formulas of type (2.8) for functions $f(x,y)$ defined on another simple polygons. The following two theorems include cases of some octagons and can be proved in a similar way.
Theorem 2.7. Let \( a_1 < a_2 < a_3 < a_4, b_1 < b_2 < b_3 \) and \( Q \) be an octagon of the following form

\[
Q = \bigcup_{i=1}^{4} R_i,
\]

where

\[
R_1 = [a_1, a_2] \times [b_1, b_2], R_2 = [a_2, a_3] \times [b_1, b_2], R_3 = [a_3, a_4] \times [b_1, b_2], R_4 = [a_2, a_3] \times [b_2, b_3].
\]

Let \( f \in M(Q) \). Then the following formula holds

\[
E(f, Q) = \max \{ |l(f,q)|, |l(f,r_{123})|, |l(f,r_{124})|, |l(f,r_{234})|, |l(f,r_{24})| \},
\]

where \( q, r_{123}, r_{124}, r_{234}, r_{24} \) are closed bolts formed by the vertices of the polygons \( Q, R_1 \cup R_2 \cup R_3, R_1 \cup R_2 \cup R_4, R_2 \cup R_3 \cup R_4 \) and \( R_2 \cup R_4 \), respectively.

Theorem 2.8. Let \( a_1 < a_2 < a_3 < a_4, b_1 < b_2 < b_3 \) and \( Q \) be an octagon of the following form

\[
Q = \bigcup_{i=1}^{3} R_i,
\]

where \( R_1 = [a_1, a_4] \times [b_1, b_2], R_2 = [a_1, a_2] \times [b_2, b_3], R_3 = [a_3, a_4] \times [b_2, b_3] \).

Let \( f \in M(Q) \). Then

\[
E(f, Q) = \max \{ |l(f,r)|, |l(f,r_{12})|, |l(f,r_{13})| \},
\]

where \( r, r_{12}, r_{13} \) are closed bolts formed by the vertices of the polygons \( R = [a_1, a_4] \times [b_1, b_3], R_1 \cup R_2, R_1 \cup R_3 \), respectively.

Although the closed bolts maximization process can be applied to bolts of an arbitrary polygon, some combinatorial difficulties arise when grouping values at points of maximized bolts (bolts obtained after the maximization process, see (2.15)-(2.18)). While we do not know a complete answer to this problem, we can describe points of a polygon \( F \) with which points of maximized bolts coincide and state a conjecture concerning the approximation error.

Let \( F = A_1A_2...A_{2n} \) be any polygon with sides parallel to the coordinate axes. The vertices \( A_1, A_2, ..., A_{2n} \) in the given order form a closed bolt, which we denote by \( r_F \). By \([r_F]\) denote the length of \( r_F \). In our case, \([r_F] = 2n\).

Definition 2.2. Let \( F \) and \( S \) be polygons with sides parallel to the coordinate axes. We say that the closed bolt \( r_F \) is an \( e \)-bolt (extended bolt) of \( S \)
if \( r_F \subset S \) and there does not exist any polygon \( F' \) such that \( F \subset F' \), \( r_{F'} \subset S \), \( [r_{F'}] \leq [r_F] \).

For example, in Theorem 2.8 the octagon \( Q \) has 3 e-bolts. They are \( r, r_{12} \) and \( r_{13} \). In Theorem 2.7, the octagon \( Q \) has 5 e-bolts, which are \( q, r_{123}, r_{124}, r_{234} \) and \( r_{24} \). The polygon \( S_{2n} = \bigcup_{i=1}^{n-1} R_i \), where \( R_i = [a_i, a_{i+1}] \times [b_i, b_{n+1-i}], i = 1, n-1, a_1 < a_2 < ... < a_n, b_1 < b_2 < ... < b_n \) has exactly \( 2^{n-1} - 1 \) e-bolts. It is not difficult to observe that the set of points of a closed bolt obtained after the maximization process is a subset of the set of points of all e-bolts. This condition and Theorems 2.5-2.8 justify the statement of the following conjecture:

Let \( S \) be any polygon with sides parallel to the coordinate axes and \( f \in M(S) \). Then

\[ E(f, S) = \max_{h \in S^E} \{ |l(f, h)| \}, \]

where \( S^E \) is a set of all e-bolts of the polygon \( S \).

### 2.2.4 Error estimates

Theorem 2.5 allows us to consider classes wider than \( M(H) \) and establish sharp estimates for the approximation error.

**Theorem 2.9.** Let \( H \) be a hexagon and (2.7) be its representation. The following sharp estimates are valid for a function \( f(x, y) \) having the continuous derivative \( \frac{\partial^2 f}{\partial x \partial y} \) on \( H \):

\[ A \leq E(f, H) \leq BC + \frac{3}{2} (B |l(g, h)| - |l(f, h)|), \quad (2.22) \]

where

\[ B = \max_{(x, y) \in H} \left| \frac{\partial^2 f(x, y)}{\partial x \partial y} \right|, \quad g = g(x, y) = x \cdot y, \]

\[ A = \max \{|l(f, h)|, |l(f, r_1)|, |l(f, r_2)|\}, \quad C = \max \{|l(g, h)|, |l(g, r_1)|, |l(g, r_2)|\}, \]

where \( h, r_1, r_2 \) are closed bolts formed by vertices of the polygons \( H, R_1 \) and \( R_2 \), respectively.
Remark 2.4. Inequalities similar to (2.22) were established in Babaev [8] for the approximation of a function \( f(x) = f(x_1, ..., x_n) \), defined on a parallelepiped with sides parallel to the coordinate axes, by sums \( \sum_{i=1}^{n} \varphi_i(x \setminus x_i) \). For the approximation of bivariate functions, Babaev’s result contains only rectangular case.

Remark 2.5. Estimates (2.22) are easily calculable in contrast to those established in [5] for continuous functions defined on certain domains, which are different from polygons.

To prove Theorem 2.9 we need the following lemmas.

Lemma 2.3. Let \( X \) be a normed space, \( F \) be a subspace of \( X \). The following inequality is valid for an element \( x = x_1 + x_2 \) from \( X \):

\[
|E(x_1) - E(x_2)| \leq E(x) \leq E(x_1) + E(x_2),
\]

where

\[
E(x) = E(x, F) = \inf_{y \in F} \|x - y\|.
\]

Lemma 2.4. If \( f \in M(H) \), then

\[
|l(f, r_i)| \leq \frac{3}{2} |l(f, h)|, i = 1, 2.
\]

Lemma 2.3 is obvious. To prove Lemma 2.4, note that for any \( f \in M(H) \)

\[
6 |l(f, h)| = 4 |l(f, r_i)| + 4 |l(f, r_3)|, \quad i = 1, 2,
\]

where \( r_3 \) is a closed bolt formed by the vertices of the rectangle \( R_3 = H \setminus R_i \).

Proof. It is not difficult to verify that if \( \frac{\partial^2 u}{\partial x \partial y} \geq 0 \) on \( H \) for some \( u(x, y) \), \( \frac{\partial^2 u(x, y)}{\partial x \partial y} \in C(H) \), then \( u \in M(H) \) (see the proof of Corollary 2.2). Set \( f_1 = f + Bg \). Since \( \frac{\partial^2 f_1}{\partial x \partial y} \geq 0 \) on \( H \), \( f_1 \in M(H) \). By Lemma 2.4,

\[
|l(f_1, r_i)| \leq \frac{3}{2} |l(f_1, h)|, i = 1, 2.
\]

(2.23)
Theorem 2.5 implies that

\[ E(f_1, H) = \max \{ |l(f_1, h)|, |l(f_1, r_1)|, |l(f_1, r_2)| \} \]. \hspace{1cm} (2.24)

We deduce from (2.23) and (2.24) that

\[ E(f_1, H) \leq \frac{3}{2} |l(f_1, h)|. \]

First, let the closed bolt \( h \) start at the point \((a_1, b_1)\). Then it is clear that

\[ E(f_1, H) \leq \frac{3}{2} l(f_1, h). \] \hspace{1cm} (2.25)

By Lemma 2.3,

\[ E(f, H) - E(Bg, H) \leq E(f_1, H). \] \hspace{1cm} (2.26)

Inequalities (2.25) and (2.26) yield

\[ E(f, H) \leq BE(g, H) + \frac{3}{2} l(f_1, h). \] \hspace{1cm} (2.27)

Since the functional \( l(f, h) \) is linear,

\[ l(f_1, h) = l(f, h) + Bl(g, h). \]

Considering this expression of \( l(f_1, h) \) in (2.27), we obtain that

\[ E(f, H) \leq BE(g, H) + \frac{3}{2} Bl(g, h) + \frac{3}{2} l(f, h). \] \hspace{1cm} (2.28)

Now consider the function \( f_2 = Bg - f \). Obviously, \( \frac{\partial^2 f_2}{\partial x \partial y} \geq 0 \) on \( H \). It can be shown, in the same way as (2.28) has been obtained, that

\[ E(f, H) \leq BE(g, H) + \frac{3}{2} Bl(g, h) - \frac{3}{2} l(f, h). \] \hspace{1cm} (2.29)

From (2.28) and (2.29) it follows that

\[ E(f, H) \leq BE(g, H) + \frac{3}{2} Bl(g, h) - \frac{3}{2} |l(f, h)|. \] \hspace{1cm} (2.30)

Since \( g \in M(H) \) and \( h \) starts at the point \((a_1, b_1)\), we have \( l(g, h) \geq 0 \).
Let now \( h \) start at a point such that \( l(u, h) \leq 0 \) for any \( u \in M(H) \). Then in a similar way as above we can prove that

\[
E(f, H) \leq BE(g, H) - \frac{3}{2} Bl(g, h) - \frac{3}{2} |l(f, h)|, \tag{2.31}
\]

where \( l(g, h) \leq 0 \). From (2.30), (2.31) and the fact that \( E(g, H) = C \) (in view of Theorem 2.5), it follows that

\[
E(f, H) \leq BC + \frac{3}{2} (B |l(g, h)| - |l(f, h)|).
\]

The upper bound in (2.22) has been established. Note that it is attained by \( f = g = xy \).

The proof of the lower bound in (2.22) is simple. One of the obvious properties of the functional \( l(f, p) \) is that \( |l(f, p)| \leq E(f, H) \) for any continuous function \( f \) on \( H \) and a closed bolt \( p \). Hence,

\[
A = \max \{|l(f, h)|, |l(f, r_1)|, |l(f, r_2)|\} \leq E(f, H).
\]

Note that by Theorem 2.5 the lower bound in (2.22) is attained by an arbitrary function from \( M(H) \).

Remark 2.6. Using Theorems 2.7 and 2.8 one can obtain sharp estimates of type (2.22) for bivariate functions defined on the corresponding simple polygons with sides parallel to the coordinate axes.

2.3 On the theorem of M. Golomb

Let \( X_1, ..., X_n \) be compact spaces and \( X = X_1 \times \cdots \times X_n \). Consider the approximation of a function \( f \in C(X) \) by sums \( g_1(x_1) + \cdots + g_n(x_n) \), where \( g_i \in C(X_i), i = 1, ..., n \). In [37], M.Golomb obtained a formula for the error of this approximation in terms of measures constructed on special points of \( X \), called “projection cycles”. However, his proof had a gap, which was pointed out later by Marshall and O’Farrell [107]. But the question if the formula was correct, remained open. The purpose of this section is to prove that Golomb’s formula is valid, and moreover it holds in a stronger form.

101

Electronic copy available at: https://ssrn.com/abstract=3618165
2.3.1 History of Golomb’s formula

Let \( X_i, i = 1, \ldots, n, \) be compact Hausdorff spaces. Consider the approximation to a continuous function \( f, \) defined on \( X = X_1 \times \cdots \times X_n, \) from the manifold

\[
M = \left\{ \sum_{i=1}^{n} g_i(x_i) : g_i \in C(X_i), \; i = 1, \ldots, n \right\}.
\]

The approximation error is defined as the distance from \( f \) to \( M: \)

\[
E(f) \overset{\text{def}}{=} \text{dist}(f, M) = \inf_{g \in M} \| f - g \|_{C(X)}.
\]

The well-known duality relation says that

\[
E(f) = \sup_{\mu \in M^\perp \| \mu \| \leq 1} \left| \int_X f \, d\mu \right|,
\]

where \( M^\perp \) is the space of regular Borel measures annihilating all functions in \( M \) and \( \| \mu \| \) stands for the total variation of a measure \( \mu. \) It should be noted that the sup in (2.32) is attained by some measure \( \mu^* \) with total variation \( \| \mu^* \| = 1. \) We are interested in the problem: is it possible to replace in (2.32) the class \( M^\perp \) by some subclass of it consisting of measures of simple structure? For the case \( n = 2, \) this problem was first considered by Diliberto and Straus [26]. They showed that the measures generated by closed bolts are sufficient for the equality (2.32).

In case of general topological spaces, a lightning bolt is defined similarly to the case \( \mathbb{R}^2. \) Let \( X = X_1 \times X_2 \) and \( \pi_i \) be the projections of \( X \) onto \( X_i, \; i = 1, 2. \) A lightning bolt (or, simply, a bolt) is a finite ordered set \( \{a_1, \ldots, a_k\} \) contained in \( X, \) such that \( a_i \neq a_{i+1}, \) for \( i = 1, 2, \ldots, k - 1, \) and either \( \pi_1(a_1) = \pi_1(a_2), \; \pi_2(a_2) = \pi_2(a_3), \; \pi_1(a_3) = \pi_1(a_4), \ldots, \) or \( \pi_2(a_1) = \pi_2(a_2), \; \pi_1(a_2) = \pi_1(a_3), \; \pi_2(a_3) = \pi_2(a_4), \ldots. \) A bolt \( \{a_1, \ldots, a_k\} \) is said to be closed if \( k \) is an even number and the set \( \{a_2, \ldots, a_k, a_1\} \) is also a bolt.

Let \( l = \{a_1, \ldots, a_{2k}\} \) be a closed bolt. Consider a measure \( \mu_l \) having atoms \( \pm \frac{1}{2k} \) with alternating signs at the vertices of \( l. \) That is,

\[
\mu_l = \frac{1}{2k} \sum_{i=1}^{2k} (-1)^{i-1} \delta_{a_i}, \quad \mu_l = \frac{1}{2k} \sum_{i=1}^{2k} (-1)^i \delta_{a_i},
\]

102
where $\delta_{a_i}$ is a point mass at $a_i$. It is clear that $\mu_l \in M^\perp$ and $\|\mu_l\| \leq 1$. $\|\mu_l\| = 1$ if and only if the set of vertices of the bolt $l$ having even indices does not intersect with that having odd indices. The following duality relation was first established by Diliberto and Straus [26]

$$E(f) = \sup_{l \subset X} \left| \int_X f d\mu_l \right|,$$  \hspace{1cm} (2.33)

where $X = X_1 \times X_2$ and the sup is taken over all closed bolts of $X$. In fact, Diliberto and Straus obtained the formula (2.33) for the case when $X$ is a rectangle in $\mathbb{R}^2$ with sides parallel to the coordinate axis. The same result was independently proved by Smolyak (see [113]). Yet another proof of (2.33), in the case when $X$ is a Cartesian product of two compact Hausdorff spaces, was given by Light and Cheney [93]. For $X$'s other than a rectangle in $\mathbb{R}^2$, the theorem under some additional assumptions appeared in the works [62,79,107]. But we shall not discuss these works here.

Golomb’s paper [37] made a start to a systematic study of approximation of multivariate functions by various compositions, including sums of univariate functions. Golomb generalized the notion of a closed bolt to the $n$-dimensional case and obtained the analogue of formula (2.33) for the error of approximation from the manifold $M$. The objects introduced in [37] were called projection cycles and they are defined as sets of the form

$$p = \{b_1, \ldots, b_k; c_1, \ldots, c_k\} \subset X,$$  \hspace{1cm} (2.34)

with the property that $b_i \neq c_j$, $i, j = 1, \ldots, k$ and for all $\nu = 1, \ldots, n$, the group of the $\nu$-th coordinates of $c_1, \ldots, c_k$ is a permutation of that of the $\nu$-th coordinates of $b_1, \ldots, b_k$. Some points in the $b$-part ($b_1, \ldots, b_k$) or $c$-part ($c_1, \ldots, c_k$) of $p$ may coincide. The measure associated with $p$ is

$$\mu_p = \frac{1}{2k} \left( \sum_{i=1}^k \delta_{b_i} - \sum_{i=1}^k \delta_{c_i} \right).$$

It is clear that $\mu_p \in M^\perp$ and $\|\mu_p\| = 1$. Besides, if $n = 2$, then a projection cycle is the union of closed bolts after some suitable permutation of its points. Golomb’s result states that

$$E(f) = \sup_{p \subset X} \left| \int_X f d\mu_p \right|,$$  \hspace{1cm} (2.35)

Electronic copy available at: https://ssrn.com/abstract=3618165
where $X = X_1 \times \cdots \times X_n$ and the sup is taken over all projection cycles of $X$.

It can be proved that in the case $n = 2$, the formulas (2.33) and (2.35) are equivalent. Unfortunately, the proof of (2.35) had a gap, which was pointed out many years later by Marshall and O’Farrell [107]. But the question if the formula (2.35) was correct, remained unsolved (see also the more recent monograph by Khavinson [76]). Note that Golomb’s result was used and cited in the literature, for example, in works [75,126].

In the following subsection, we will construct families of normalized measures (that is, measures with the total variation equal to 1) on projection cycles. Each measure $\mu_p$ defined above will be a member of some family. We will also consider minimal projection cycles and measures constructed on them. By properties of these measures, we show that Golomb’s formula (2.35) is valid in a stronger form.

### 2.3.2 Measures supported on projection cycles

Let us give an equivalent definition of a projection cycle. This will be useful in constructing of certain measures having simple structure and capability of approximating arbitrary measures in $M^\perp$.

In the sequel, $\chi_a$ will denote the characteristic function of a single point set $\{a\} \subset \mathbb{R}$.

**Definition 2.3.** Let $X = X_1 \times \cdots \times X_n$ and $\pi_i$ be the projections of $X$ onto the sets $X_i$, $i = 1, \ldots, n$. We say that a set $p = \{x_1, \ldots, x_m\} \subset X$ is a projection cycle if there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ with nonzero real coordinates such that

\[
\sum_{j=1}^{m} \lambda_j \chi_{\pi_i(x_j)} = 0, \quad i = 1, \ldots, n. \tag{2.36}
\]

Let us give some explanatory remarks concerning Definition 2.3. Fix the subscript $i$. Let the set $\{\pi_i(x_j), j = 1, \ldots, m\}$ have $s_i$ different values, which we denote by $\gamma_{i1}^i, \gamma_{i2}^i, \ldots, \gamma_{is_i}^i$. Then (2.36) implies that

\[
\sum_{j} \lambda_j = 0,
\]

104
where the sum is taken over all \( j \) such that \( \pi_i(x_j) = \gamma_i^k, k = 1, \ldots, s_i \). Thus for fixed \( i \), we have \( s_i \) homogeneous linear equations in \( \lambda_1, \ldots, \lambda_m \). The coefficients of these equations are the integers 0 and 1. By varying \( i \), we obtain \( s = \sum_{i=1}^n s_i \) such equations. Hence (2.36), in its expanded form, stands for the system of these equations. One can observe that if this system has a solution \( (\lambda_1, \ldots, \lambda_m) \) with nonzero real components \( \lambda_i \), then it also has a solution \( (n_1, \ldots, n_m) \) with nonzero integer components \( n_i, i = 1, \ldots, m \). This means that in Definition 2.3, we can replace the vector \( \lambda \) by the vector \( n = (n_1, \ldots, n_m) \), where \( n_i \in \mathbb{Z}\{0\}, i = 1, \ldots, m \). Thus, Definition 2.3 is equivalent to the following definition.

**Definition 2.4.** A set \( p = \{x_1, \ldots, x_m\} \subset X \) is called a projection cycle if there exist nonzero integers \( n_1, \ldots, n_m \) such that

\[
\sum_{j=1}^m n_j \chi_{\pi_i(x_j)} = 0, \quad i = 1, \ldots, n. \tag{2.37}
\]

**Lemma 2.5.** Definition 2.4 is equivalent to Golomb’s definition of a projection cycle.

**Proof.** Let \( p = \{x_1, \ldots, x_m\} \) be a projection cycle with respect to Definition 2.4. By \( b \) and \( c \) denote the set of all points \( x_i \) such that the integers \( n_i \) associated with them in (2.37) are positive and negative correspondingly. Write out each point \( x_i \) \( n_i \) times if \( n_i > 0 \) and \( -n_i \) times if \( n_i < 0 \). Then the set \( \{b; c\} \) is a projection cycle with respect to Golomb’s definition (see Introduction). The inverse is also true. Let a set \( p_1 = \{b_1, \ldots, b_k; c_1, \ldots, c_k\} \) be a projection cycle with respect to Golomb’s definition. Here, some points \( b_i \) or \( c_i \) may be repeated. Let \( p = \{x_1, \ldots, x_m\} \) stand for the set \( p_1 \), but with no repetition of its points. Let \( n_i \) show how many times \( x_i \) appear in \( p_1 \). We take \( n_i \) positive if \( x_i \) appears in the \( b \)-part of \( p_1 \) and negative if it appears in the \( c \)-part of \( p_1 \). Clearly, the set \( \{x_1, \ldots, x_m\} \) is a projection cycle with respect to Definition 2.4, since the integers \( n_i, i = 1, \ldots, m \), satisfy (2.37). \( \square \)

In the sequel, we will use Definition 2.3. A pair \( \langle p, \lambda \rangle \), where \( p \) is a projection cycle in \( X \) and \( \lambda \) is a vector associated with \( p \) by (2.36), will be called a “projection cycle-vector pair” of \( X \). To each such pair \( \langle p, \lambda \rangle \) with \( p = \{x_1, \ldots, x_m\} \) and \( \lambda = (\lambda_1, \ldots, \lambda_m) \), we correspond the measure

105
\[ \mu_{p,\lambda} = \frac{1}{\sum_{j=1}^{m} |\lambda_j|} \sum_{j=1}^{m} \lambda_j \delta_{x_j}. \]  

(2.38)

Clearly, \( \mu_{p,\lambda} \in M^\perp \) and \( \|\mu_{p,\lambda}\| = 1 \). We will also deal with measures supported on some certain subsets of projection cycles called \textit{minimal projection cycles}. A projection cycle is said to be minimal if it does not contain any projection cycle as its proper subset. For example, the set \( p = \{(0,0,0), (0,0,1), (1,0,0), (1,0,1), (1,1,1)\} \) is a minimal projection cycle in \( \mathbb{R}^3 \), since the vector \( \lambda = (2,-1,-1,-1,1) \) satisfies Eq. (2.36) and there is no such vector for any other subset of \( p \). Adding one point \((0,1,1)\) from the right to \( p \), we will also have a projection cycle, but not minimal. Note that in this case, \( \lambda \) can be taken as \( (3,-1,-1,-2,2,-1) \).

\textbf{Remark 2.7.} A minimal projection cycle under the name of a \textit{loop} was introduced and used in the works of Klopotowski, Nadkarni, Rao [80,81].

To prove our main result we need some auxiliary facts.

\textbf{Lemma 2.6.} (1) The vector \( \lambda = (\lambda_1, \ldots, \lambda_m) \) associated with a minimal projection cycle \( p = (x_1, \ldots, x_m) \) is unique up to multiplication by a constant.

(2) If in (1), \( \sum_{j=1}^{m} |\lambda_j| = 1 \), then all the numbers \( \lambda_j \), \( j = 1, \ldots, m \), are rational.

\textit{Proof.} Let \( \lambda^1 = (\lambda_1^1, \ldots, \lambda_m^1) \) and \( \lambda^2 = (\lambda_1^2, \ldots, \lambda_m^2) \) be any two vectors associated with \( p \). That is,

\[ \sum_{j=1}^{m} \lambda_j^1 \chi_{\pi_i(x_j)} = 0 \quad \text{and} \quad \sum_{j=1}^{m} \lambda_j^2 \chi_{\pi_i(x_j)} = 0, \quad i = 1, \ldots, n. \]

After multiplying the second equality by \( c = \frac{\lambda_j^1}{\lambda_j^2} \) and subtracting from the first, we obtain that

\[ \sum_{j=2}^{m} (\lambda_j^1 - c\lambda_j^2) \chi_{\pi_i(x_j)} = 0, \quad i = 1, \ldots, n. \]

Now since the cycle \( p \) is minimal, \( \lambda_j^1 = c\lambda_j^2 \), for all \( j = 1, \ldots, m \).

The second part of the lemma is a consequence of the first part. Indeed, let \( n = (n_1, \ldots, n_m) \) be a vector with the nonzero integer coordinates associated
with \( p \). Then the vector \( \lambda' = (\lambda'_1, ..., \lambda'_m) \), where \( \lambda'_j = \frac{n_j}{\sum_{j=1}^{m} n_j} \), \( j = 1, ..., m \), is also associated with \( p \). All coordinates of \( \lambda' \) are rational and therefore by the first part of the lemma, it is the unique vector satisfying \( \sum_{j=1}^{m} |\lambda'_j| = 1 \). \( \square \)

By this lemma, a minimal projection cycle \( p \) uniquely (up to a sign) defines the measure

\[
\mu_p = \sum_{j=1}^{m} \lambda_j \delta_{x_j}, \quad \sum_{j=1}^{m} |\lambda_j| = 1.
\]

**Lemma 2.7.** Let \( \mu \) be a normalized orthogonal measure on a projection cycle \( l \subset X \). Then it is a convex combination of normalized orthogonal measures on minimal projection cycles of \( l \). That is,

\[
\mu = \sum_{i=1}^{s} t_i \mu_i, \quad \sum_{i=1}^{s} t_i = 1, \quad t_i > 0,
\]

where \( l_i, i = 1, ..., s, \) are minimal projection cycles in \( l \).

This lemma follows from the result of Navada (see theorem 2 of [112]): Let \( S \subset X_1 \times \cdots \times X_n \) be a finite set. Then any extreme point of the convex set of measures \( \mu \) on \( S, \mu \in M^\perp, \|\mu\| \leq 1 \), has its support on a minimal projection cycle contained in \( S \).

**Remark 2.8.** In the case \( n = 2 \), Lemma 2.7 was proved by Medvedev (see [76, p.77]).

**Lemma 2.8** (see [76, p.73]). Let \( X = X_1 \times \cdots \times X_n \) and \( \pi_i \) be the projections of \( X \) onto the sets \( X_i, i = 1, ..., n \). In order that a measure \( \mu \in C(X)^* \) be orthogonal to the subspace \( M \), it is necessary and sufficient that

\[
\mu \circ \pi_i^{-1} = 0, \quad i = 1, ..., n.
\]

**Lemma 2.9** (see [76, p.75]). Let \( \mu \in M^\perp \) and \( \|\mu\| = 1 \). Then there exist a net of measures \( \{\mu_\alpha\} \subset M^\perp \) weak* converging in \( C(X)^* \) to \( \mu \) and satisfying the following properties:

1) \( \|\mu_\alpha\| = 1 \);
2) The closed support of each \( \mu_\alpha \) is a finite set.
Our main result is the following theorem.

**Theorem 2.10.** The error of approximation from the manifold \( M \) obeys the equality

\[
E(f) = \sup_{\mu \in \mathcal{X}} \left| \int_X f \, d\mu \right|,
\]

where the sup is taken over all minimal projection cycles of \( X \).

**Proof.** Let \( \tilde{\mu} \) be a measure with finite support \( \{x_1, \ldots, x_m\} \) and orthogonal to the space \( M \). Put \( \lambda_j = \tilde{\mu}(x_j), j = 1, \ldots, m \). By Lemma 2.8, \( \tilde{\mu}(\pi^{-1}_i(\pi_i(x_j))) = 0 \), for all \( i = 1, \ldots, n, j = 1, \ldots, m \). Fix the indices \( i \) and \( j \). Then we have the equation \( \sum_k \lambda_k = 0 \), where the sum is taken over all indices \( k \) such that \( \pi_i(x_k) = \pi_i(x_j) \). Varying \( i \) and \( j \), we obtain a system of such equations, which concisely can be written as

\[
\sum_{k=1}^m \lambda_k \chi_{\pi_i(x_k)} = 0, \quad i = 1, \ldots, n.
\]

This means that the finite support of \( \tilde{\mu} \) forms a projection cycle. Therefore, a net of measures approximating the given measure \( \mu \) in Lemma 2.9 are all of the form (2.38).

Let now \( \mu_{p,\lambda} \) be any measure of the form (2.38). Since \( \mu_{p,\lambda} \in M^\perp \) and \( \|\mu_{p,\lambda}\| = 1 \), we can write

\[
\left| \int_X f \, d\mu_{p,\lambda} \right| = \left| \int_X (f - g) \, d\mu_{p,\lambda} \right| \leq \|f - g\|,
\]

where \( g \) is an arbitrary function in \( M \). It follows from (2.39) that

\[
\sup_{(p,\lambda)} \left| \int_X f \, d\mu_{p,\lambda} \right| \leq E(f),
\]

where the sup is taken over all projection cycle-vector pairs of \( X \).

Consider the general duality relation (2.32). Let \( \mu_0 \) be a measure attaining the supremum in (2.32) and \( \{\mu_{p,\lambda}\} \) be a net of measures of the form (2.38) approximating \( \mu_0 \) in the weak* topology of \( C(X)^* \). We already know that
this is possible. For any \( \varepsilon > 0 \), there exists a measure \( \mu_{p_0, \lambda_0} \) in \( \{\mu_{p, \lambda}\} \) such that
\[
\left| \int_X f \, d\mu_0 - \int_X f \, d\mu_{p_0, \lambda_0} \right| < \varepsilon.
\]
From the last inequality we obtain that
\[
\left| \int_X f \, d\mu_{p_0, \lambda_0} \right| > \left| \int_X f \, d\mu_0 \right| - \varepsilon = E(f) - \varepsilon.
\]
Hence,
\[
\sup_{(p, \lambda)} \left| \int_X f \, d\mu_{p, \lambda} \right| \geq E(f). \tag{2.41}
\]
From (2.40) and (2.41) it follows that
\[
\sup_{(p, \lambda)} \left| \int_X f \, d\mu_{p, \lambda} \right| = E(f). \tag{2.42}
\]
By Lemma 2.7,
\[
\mu_{p, \lambda} = \sum_{i=1}^{s} t_i \mu_{l_i},
\]
where \( l_i, i = 1, \ldots, s, \) are minimal projection cycles in \( p \) and \( \sum_{i=1}^{s} t_i = 1, \ t_i > 0. \) Let \( k \) be an index in the set \( \{1, \ldots, s\} \) such that
\[
\left| \int_X f \, d\mu_{l_k} \right| = \max \left\{ \left| \int_X f \, d\mu_{l_i} \right|, \ i = 1, \ldots, s \right\}.
\]
Then
\[
\left| \int_X f \, d\mu_{p, \lambda} \right| \leq \left| \int_X f \, d\mu_{l_k} \right|. \tag{2.43}
\]
Now since
\[ \left| \int_X f \, d\mu_l \right| \leq E(f), \]

for any minimal cycle \( l \), from (2.42) and (2.43) we obtain the assertion of the theorem. \( \square \)

**Remark 2.9.** Theorem 2.10 not only proves Golomb’s formula, but also improves it. Indeed, based on Lemma 2.5, one can easily observe that the formula (2.35) is equivalent to the formula

\[ E(f) = \sup_{\langle p, \lambda \rangle} \left| \int_X f \, d\mu_{p, \lambda} \right|, \]

where the sup is taken over all projection cycle-vector pairs \( \langle p, \lambda \rangle \) of \( X \) provided that all the numbers \( \lambda_i / \sum_{j=1}^{m} |\lambda_j|, \ i = 1, \ldots, m, \) are rational. But by Lemma 2.6, minimal projection cycles enjoy this property.
Chapter 3

General ridge functions and linear superpositions

A ridge function \(g(a \cdot x)\) with a direction \(a \in \mathbb{R}^n\{0\}\) admits a natural generalization to a multivariate function of the form \(g(\alpha_1(x_1) + \cdots + \alpha_n(x_n))\), where \(\alpha_i(x_i), \ i = 1, n,\) are real, presumably well behaved, fixed univariate functions.

We know from Section 1 that finitely many directions \(a^j\) are not enough for sums \(\sum g_j(a^j \cdot x)\) to approximate multivariate functions. However, we will see in this section that sums of the form \(\sum g_j(\alpha^j_1(x_1) + \cdots + \alpha^j_n(x_n))\) with finitely many \(\alpha^j_i(x_i)\) is capable not only approximating multivariate functions but also precisely representing them. First we study the problem of representation of a function \(f : X \to \mathbb{R}\), where \(X\) is any set, as a linear superposition \(\sum_j g_j(h_j(x))\) with arbitrary but fixed functions \(h_j : X \to \mathbb{R}\). Then we apply the obtained result and the famous Kolmogorov superposition theorem to prove representability of an arbitrarily behaved multivariate function in the form of a general ridge function \(\sum g_j(\alpha^j_1(x_1) + \cdots + \alpha^j_n(x_n))\).

The material of this chapter is taken from [49].

Electronic copy available at: https://ssrn.com/abstract=3618165
3.1 Some results on the representation by linear superpositions

Let $X$ be any set and $h_i : X \to \mathbb{R}, \ i = 1, ..., r,$ be arbitrarily fixed functions. Consider the set

$$B(X) = B(h_1, ..., h_r; X) = \left\{ \sum_{i=1}^{r} g_i(h_i(x)), \ x \in X, \ g_i : \mathbb{R} \to \mathbb{R}, \ i = 1, ..., r \right\}$$

(3.1)

Members of this set will be called linear superpositions with respect to the functions $h_1, ..., h_r$ (see [141]). Note that sums of general ridge functions is a special case of linear superpositions. In Section 1.2, we considered linear superpositions defined on a subset of the $d$-dimensional Euclidean space, while here $X$ is a set of arbitrary nature. As in Section 1.2, we are interested in the question: what conditions on $X$ guarantee that each function on $X$ will be in the set $B(X)$? The simplest case $X \subset \mathbb{R}^d, \ r = d$ and $h_i$ are the coordinate functions was solved in [81]. See also [76, p.57] for the case $r = 2$.

By $B_c(X)$ and $B_b(X)$ denote the right hand side of (3.1) with continuous and bounded $g_i : \mathbb{R} \to \mathbb{R}, \ i = 1, ..., r,$ respectively. Our starting point is the well-known superposition theorem of Kolmogorov [83]. It states that for the unit cube $I^d, \ I = [0,1], \ d \geq 2,$ there exists $2d + 1$ functions $\{s_q\}_{q=1}^{2d+1} \subset C(I^d)$ of the form

$$s_q(x_1, ..., x_d) = \sum_{p=1}^{d} \varphi_{pq}(x_p), \ \varphi_{pq} \in C(I), \ p = 1, ..., d, \ q = 1, ..., 2d + 1 \quad (3.2)$$

such that each function $f \in C(I^d)$ admits the representation

$$f(x) = \sum_{q=1}^{2d+1} g_q(s_q(x)), \ x = (x_1, ..., x_d) \in I^d, \ g_q \in C(\mathbb{R}). \quad (3.3)$$

Note that functions $g_q(s_q(x))$, involved in the right hand side of (3.3), are general ridge functions. In our notation, (3.3) means that $B_c(s_1, ..., s_{2d+1}; I^d) = C(I^d)$. This surprising and deep result, which solved (negatively) Hilbert’s 13-th problem, was improved and generalized in several directions. It was first observed by Lorentz [98] that the functions $g_q$ can be replaced by a single continuous function $g$. Sprecher [128] showed that the theorem can be proven
with constant multiples of a single function \( \varphi \) and translations. Specifically, \( \varphi_{pq} \) in (3.2) can be chosen as \( \lambda^p \varphi(x_p + \varepsilon q) \), where \( \varepsilon \) and \( \lambda \) are some positive constants. Fridman [31] succeeded in showing that the functions \( \varphi_{pq} \) can be constructed to belong to the class \( \text{Lip}(1) \). Vitushkin and Henkin [141] showed that \( \varphi_{pq} \) cannot be taken to be continuously differentiable.

Ostrand [115] extended the Kolmogorov theorem to general compact metric spaces. In particular, he proved that for each compact \( B \) ric spaces. In particular, he proved that for each compact metric space \( X \) there exist continuous real functions \( \{ \alpha_i \}_{i=1}^{2d+1} \subset C(X) \) such that \( B_c(\alpha_1, \ldots, \alpha_{2d+1}; X) = C(X) \). Sternfeld [130] showed that the number \( 2d + 1 \) cannot be reduced for any \( d \)-dimensional space \( X \). Thus, the number of terms in the Kolmogorov superposition theorem is the best possible.

Some papers of Sternfeld were devoted to the representation of continuous and bounded functions by linear superpositions. Let \( C(X) \) and \( B(X) \) denote the space of continuous and bounded functions on some set \( X \) respectively (in the first case, \( X \) is supposed to be a compact metric space). Let \( F = \{ h \} \) be a family of functions on \( X \). \( F \) is called a uniformly separating family \( (u.s.f.) \) if there exists a number \( 0 < \lambda \leq 1 \) such that for each pair \( \{ x_j \}_{j=1}^m \), \( \{ z_j \}_{j=1}^m \) of disjoint finite sequences in \( X \), there exists some \( h \in F \) so that if from the two sequences \( \{ h(x_j) \}_{j=1}^m \) and \( \{ h(z_j) \}_{j=1}^m \) in \( h(X) \) we remove a maximal number of pairs of points \( h(x_{j_1}) \) and \( h(z_{j_2}) \) with \( h(x_{j_1}) = h(z_{j_2}) \), there remains at least \( \lambda m \) points in each sequence (or, equivalently, at most \( (1 - \lambda)m \) pairs can be removed). Sternfeld [132] proved that for a finite family \( F = \{ h_1, \ldots, h_r \} \) of functions on \( X \), being a \( u.s.f. \) is equivalent to the equality \( B_u(h_1, \ldots, h_r; X) = B(X) \), and that in the case where \( X \) is a compact metric space and the elements of \( F \) are continuous functions on \( X \), the equality \( B_c(h_1, \ldots, h_r; X) = C(X) \) implies that \( F \) is a \( u.s.f. \). Thus, in particular, Sternfeld obtained that the formula (3.3) is valid for all bounded functions, where \( g_q \) are bounded functions depending on \( f \) (see also [76, p.21]).

Let \( X \) be a compact metric space. The family \( F = \{ h \} \subset C(X) \) is said to be a measure separating family \( (m.s.f.) \) if there exists a number \( 0 < \lambda \leq 1 \) such that for any measure \( \mu \in C(X)^* \), the inequality \( \| \mu \circ h^{-1} \| \geq \lambda \| \mu \| \) holds for some \( h \in F \). Sternfeld [131] proved that \( B_u(h_1, \ldots, h_r; X) = C(X) \) if and only if the family \( \{ h_1, \ldots, h_r \} \) is a \( m.s.f. \). In [132], it was also shown that if \( r = 2 \), then the properties \( u.s.f. \) and \( m.s.f. \) are equivalent. Therefore, the equality \( B_u(h_1, h_2; X) = B(X) \) is equivalent to \( B_c(h_1, h_2; X) = C(X) \). But for \( r > 2 \), these two properties are no longer equivalent. That is, \( B_u(h_1, \ldots, h_r; X) = B(X) \) does not always imply \( B_c(h_1, \ldots, h_r; X) = C(X) \).
Our purpose is to consider the above mentioned problem of representation by linear superpositions without involving any topology (that of continuity or boundedness). We start with characterization of those sets $X$ for which $\mathcal{B}(h_1, \ldots, h_r; X) = T(X)$, where $T(X)$ is the space of all functions on $X$. As in Section 1.2, this will be done in terms of cycles. We claim that nonexistence of cycles in $X$ is equivalent to the equality $\mathcal{B}(X) = T(X)$ for an arbitrary set $X$. In particular, we show that $\mathcal{B}_c(X) = C(X)$ always implies $\mathcal{B}(X) = T(X)$. This implication will enable us to obtain some new results, namely extensions of the previously known theorems from continuous to discontinuous multivariate functions. For example, we will prove that the formula (3.3) is valid for all discontinuous multivariate functions $f$ defined on the unite cube $I^d$, where $g_q$ are univariate functions depending on $f$.

### 3.2 Kolmogorov’s superposition theorem and its extension

In this subsection, we show that if some representation by linear superpositions holds for continuous functions, then it holds for all functions. This will lead us to natural extensions of some known superposition theorems (such as Kolmogorov’s superposition theorem, Ostrand’s superposition theorem, etc) from continuous to discontinuous functions.

In the sequel, by $\chi_A$ we will denote the characteristic function of a set $A \subset \mathbb{R}$. That is,

$$
\chi_A(y) = \begin{cases} 
1, & \text{if } y \in A \\
0, & \text{if } y \notin A.
\end{cases}
$$

The following definition is a generalized version of Definition 1.1 from Section 1.2, where in connection with ridge functions only subsets of $\mathbb{R}^d$ were considered.

**Definition 3.1.** Given an arbitrary set $X$ and functions $h_i : X \to \mathbb{R}$, $i = 1, \ldots, r$. A set of points $\{x_1, \ldots, x_n\} \subset X$ is called to be a cycle with respect to the functions $h_1, \ldots, h_r$ (or, concisely, a cycle if there is no confusion), if there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ with the nonzero real coordinates

114
\[ \sum_{j=1}^{n} \lambda_j \chi_{h_i(x_j)} = 0, \ i = 1, \ldots, r. \] (3.4)

A cycle \( p = \{x_1, \ldots, x_n\} \) is said to be minimal if \( p \) does not contain any cycle as its proper subset.

Note that in this definition the vector \( \lambda = (\lambda_1, \ldots, \lambda_n) \) can be chosen so that it has only integer components. Indeed, let for \( i = 1, \ldots, r \), the set \( \{h_i(x_j), \ j = 1, \ldots, n\} \) have \( k_i \) different values. Then it is not difficult to see that Eq. (3.4) stands for a system of \( \sum_{i=1}^{r} k_i \) homogeneous linear equations in unknowns \( \lambda_1, \ldots, \lambda_n \). This system can be written in the matrix form \((\lambda_1, \ldots, \lambda_n) \times C = 0\), where \( C \) is an \( n \) by \( \sum_{i=1}^{r} k_i \) matrix. The basic property of this matrix is that all of its entries are 0’s and 1’s and no row or column of \( C \) is identically zero. Since Eq. (3.4) has a nontrivial solution \((\lambda'_1, \ldots, \lambda'_n) \in \mathbb{R}^n\) and all entries of \( C \) are integers, by applying the Gauss elimination method we can see that there always exists a nontrivial solution \((\lambda_1, \ldots, \lambda_n) \) with the integer components \( \lambda_i, \ i = 1, \ldots, n \).

For a number of simple examples, see Section 1.2.

Let \( T(X) \) denote the set of all functions on \( X \). With each pair \( \langle p, \lambda \rangle \), where \( p = \{x_1, \ldots, x_n\} \) is a cycle in \( X \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a vector known from Definition 3.1, we associate the functional

\[ G_{p,\lambda} : T(X) \to \mathbb{R}, \quad G_{p,\lambda}(f) = \sum_{j=1}^{n} \lambda_j f(x_j). \]

In the following, such pairs \( \langle p, \lambda \rangle \) will be called cycle-vector pairs of \( X \). It is clear that the functional \( G_{p,\lambda} \) is linear. Besides, \( G_{p,\lambda}(g) = 0 \) for all functions \( g \in \mathcal{B}(h_1, \ldots, h_r; X) \). Indeed, assume that (3.4) holds. Given \( i \leq r \), let \( z = h_i(x_j) \) for some \( j \). Hence, \( \sum_{j} (h_i(x_j) = z) \lambda_j = 0 \) and \( \sum_{j} (h_i(x_j) = z) \lambda_j g_i(h_i(x_j)) = 0 \). A summation yields \( G_{p,\lambda}(g_i \circ h_i) = 0 \). Since \( G_{p,\lambda} \) is linear, we obtain that

\[ G_{p,\lambda}(\sum_{i=1}^{r} g_i \circ h_i) = 0. \]

A minimal cycle \( p = \{x_1, \ldots, x_n\} \) has the following obvious properties:

(a) The vector \( \lambda \) associated with \( p \) by Eq. (3.4) is unique up to multiplication by a constant;

(b) If in (3.4), \( \sum_{j=1}^{n} |\lambda_j| = 1 \), then all the numbers \( \lambda_j, \ j = 1, \ldots, n \), are rational.
Thus, a minimal cycle \( p \) uniquely (up to a sign) defines the functional

\[
G_p(f) = \sum_{j=1}^{n} \lambda_j f(x_j), \quad \sum_{j=1}^{n} |\lambda_j| = 1.
\]

**Proposition 3.1.**
1) Let \( X \) have cycles. A function \( f : X \to \mathbb{R} \) belongs to the space \( B(h_1, \ldots, h_r; X) \) if and only if \( G_p(f) = 0 \) for any minimal cycle \( p \subset X \) with respect to the functions \( h_1, \ldots, h_r \).
2) Let \( X \) has no cycles. Then \( B(h_1, \ldots, h_r; X) = T(X) \).

**Proposition 3.2.** \( B(h_1, \ldots, h_r; X) = T(X) \) if and only if \( X \) has no cycles.

These propositions are proved by the same way as Theorems 1.1 and 1.2. We use these propositions to obtain our main result (see Theorem 3.1 below).

The condition whether \( X \) have cycles or not, depends both on \( X \) and the functions \( h_1, \ldots, h_r \). In the following, we see that if \( h_1, \ldots, h_r \) are “nice” functions (smooth functions with the simple structure. For example, ridge functions) and \( X \subset \mathbb{R}^d \) is a “rich” set (for example, the set with interior points), then \( X \) has always cycles. Thus the representability by linear combinations of univariate functions with the fixed “nice” multivariate functions requires at least that \( X \) should not possess interior points. The picture is quite different when the functions \( h_1, \ldots, h_r \) are not “nice”. Even in the case when they are continuous, we will see that many sets of \( \mathbb{R}^d \) (the unite cube, any compact subset of that, or even the whole space \( \mathbb{R}^d \) itself) may have no cycles. If disregard the continuity, there exists even one function \( h \) such that every multivariate function is representable as \( g \circ h \) over any subset of \( \mathbb{R}^d \). First, let us introduce the following definition.

**Definition 3.2.** Let \( X \) be a set and \( h_i : X \to \mathbb{R}, i = 1, \ldots, r \), be arbitrarily fixed functions. A class \( A(X) \) of functions on \( X \) will be called a “permissible function class” if for any minimal cycle \( p \subset X \) with respect to the functions \( h_1, \ldots, h_r \) (if it exists), there is a function \( f_0 \) in \( A(X) \) such that \( G_p(f_0) \neq 0 \).

Clearly, \( C(X) \) and \( B(X) \) are both permissible function classes (in case of \( C'(X), X \) is considered to be a normal topological space).

**Theorem 3.1.** Let \( A(X) \) be a permissible function class. If \( A(X) \subset B(h_1, \ldots, h_r; X) \), then \( B(h_1, \ldots, h_r; X) = T(X) \).

116
The proof is simple and based on Propositions 3.1 and 3.2. Assume for a moment that $X$ admits a cycle $p$. By Proposition 3.1, the functional $G_p$ annihilates all members of the set $B(h_1, ..., h_r; X)$. By Definition 3.2 of permissible function classes, $A(X)$ contains a function $f_0$ such that $G_p(f_0) \neq 0$. Therefore, $f_0 \notin B(h_1, ..., h_r; X)$. We see that the embedding $A(X) \subset B(h_1, ..., h_r; X)$ is impossible if $X$ has a cycle. Thus $X$ has no cycles. Then by Proposition 3.2, $\mathcal{B}(h_1, ..., h_r; X) = T(X)$.

In the “if part” of Theorem 3.1, instead of $\mathcal{B}(h_1, ..., h_r; X)$ and $A(X)$ one can take $\mathcal{B}_c(h_1, ..., h_r; X)$ and $C(X)$ (or $\mathcal{B}_b(h_1, ..., h_r; X)$ and $B(X)$) respectively. That is, the following corollaries are valid.

**Corollary 3.1.** Let $X$ be a set and $h_i : X \to \mathbb{R}$, $i = 1, ..., r$, be arbitrarily fixed bounded functions. If $\mathcal{B}_b(h_1, ..., h_r; X) = B(X)$, then $\mathcal{B}(h_1, ..., h_r; X) = T(X)$.

**Corollary 3.2.** Let $X$ be a normal topological space and $h_i : X \to \mathbb{R}$, $i = 1, ..., r$, be arbitrarily fixed continuous functions. If $\mathcal{B}_c(h_1, ..., h_r; X) = C(X)$, then $\mathcal{B}(h_1, ..., h_r; X) = T(X)$.

The main advantage of Theorem 3.1 is that we need not check directly if the set $X$ has no cycles, which in many cases may turn out to be very tedious task. Using this theorem, we can extend free-of-charge the existing superposition theorems from the classes $B(X)$ or $C(X)$ (or some other permissible function classes) to all functions defined on $X$. For example, this theorem allows us to extend the Kolmogorov superposition theorem from continuous to all multivariate functions.

**Theorem 3.2.** Let $d \geq 2$, $\mathbb{I} = [-1; 1]$, and $\varphi_{pq}$, $p = 1, ..., d$, $q = 1, ..., 2d + 1$, be the universal continuous functions in (3.2). Then each multivariate function $f : \mathbb{I}^d \to \mathbb{R}$ can be represented in the form

$$f(x) = \sum_{q=1}^{2d+1} g_q(\sum_{p=1}^{d} \varphi_{pq}(x_p)), \; x = (x_1, ..., x_d) \in \mathbb{I}^d.$$ 

where $g_q$ are univariate functions depending on $f$.

It should be remarked that Sternfeld [132], in particular, obtained that the formula (3.3) is valid for functions $f \in B(\mathbb{I}^d)$ provided that $g_q$ are bounded.
functions depending on $f$ (see [76, chapter 1] for more detailed information and interesting discussions).

Let $X$ be a compact metric space and $h_i \in C(X), i = 1, ..., r$. The result of Sternfeld (see Section 3.1) and Corollary 3.1 give us the implications

$$B_c(h_1, ..., h_r; X) = C(X) \Rightarrow B_b(h_1, ..., h_r; X) = B(X) \Rightarrow B(h_1, ..., h_r; X) = T(X).$$

The first implication is invertible when $r = 2$ (see [132]). We want to show that the second is not invertible even in the case $r = 2$. The following interesting example is due to Khavinson [76, p.67].

Let $X \subset \mathbb{R}^2$ consist of a broken line whose sides are parallel to the coordinate axis and whose vertices are

$$(0; 0), (1; 0), (1; 1), (1 + \frac{1}{2^2}; 1), (1 + \frac{1}{2^2}; 1 + \frac{1}{2^2}), (1 + \frac{1}{2^2} + \frac{1}{3^2}; 1 + \frac{1}{2^2}), ...$$

We add to this line the limit point of the vertices $(\frac{\pi^2}{6}, \frac{\pi^2}{6})$. Let $r = 2$ and $h_1, h_2$ be the coordinate functions. Then the set $X$ has no cycles with respect to $h_1$ and $h_2$. By Proposition 3.1, every function $f$ on $X$ is of the form $g_1(x_1) + g_2(x_2), (x_1, x_2) \in X$. Now construct a function $f_0$ on $X$ as follows. On the link joining $(0; 0)$ to $(1; 0)$ $f_0(x_1, x_2)$ continuously increases from 0 to 1; on the link from $(1; 0)$ to $(1; 1)$ it continuously decreases from 1 to 0; on the link from $(1; 1)$ to $(1 + \frac{1}{2^2}; 1)$ it increases from 0 to $\frac{1}{2}$; on the link from $(1 + \frac{1}{2^2}; 1)$ to $(1 + \frac{1}{2^2}; 1 + \frac{1}{2^2})$ it decreases from $\frac{1}{2}$ to 0; on the next link it increases from 0 to $\frac{1}{3}$, etc. At the point $(\frac{\pi^2}{6}, \frac{\pi^2}{6})$ set the value of $f_0$ equal to 0. Obviously, $f_0$ is a continuous functions and by the above argument, $f_0(x_1, x_2) = g_1(x_1) + g_2(x_2)$. But $g_1$ and $g_2$ cannot be chosen as continuous functions, since they get unbounded as $x_1$ and $x_2$ tends to $\frac{\pi^2}{6}$. Thus, $B(h_1, h_2; X) = T(X)$, but at the same time $B_c(h_1, h_2; X) \neq C(X)$ (or, equivalently, $B_b(h_1, h_2; X) \neq B(X)$).

### 3.3 Some other superposition theorems

We have seen in the previous subsection that the unit cube in $\mathbb{R}^d$ has no cycles with respect to some $2d + 1$ continuous functions (namely, the Kolmogorov functions $s_q$ (3.2)). From the result of Ostrand [115] (see Section 3.1) and Corollary 3.2 it follows that compact sets $X$ of finite dimension also lack
cycles with respect to a certain family of finitely many continuous functions on $X$. Namely, the following generalization of Ostrand's theorem is valid.

**Theorem 3.3.** For $p = 1, 2, ..., m$ let $X_p$ be a compact metric space of finite dimension $d_p$ and let $n = \sum_{p=1}^{m} d_p$. There exist continuous functions $\alpha_{pq} : X_p \to [0, 1]$, $p = 1, ..., m$, $q = 1, ..., 2n + 1$, such that every real function $f$ defined on $\prod_{p=1}^{m} X_p$ is representable in the form

$$f(x_1, ..., x_m) = \sum_{q=1}^{2n+1} g_q(\sum_{p=1}^{m} \alpha_{pq}(x_p)).$$

where $g_q$ are real functions depending on $f$. If $f$ is continuous, then the functions $g_q$ can be chosen continuous.

Note that Ostrand proved "if $f$ is continuous..." part of Theorem 3.3, while we prove the validity of (3.5) for discontinuous $f$.

One may ask if there exists a finite family of functions $\{h_i : \mathbb{R}^d \to \mathbb{R}\}_{i=1}^{n}$ such that any subset of $\mathbb{R}^d$ does not admit cycles with respect to this family? The answer is positive. This follows from the result of Demko [23]: there exist $2d + 1$ continuous functions $\varphi_1, ..., \varphi_{2d+1}$ defined on $\mathbb{R}^d$ such that every bounded continuous function on $\mathbb{R}^d$ is expressible in the form $\sum_{i=1}^{2d+1} g \circ \varphi_i$ for some $g \in C(\mathbb{R})$. This theorem together with Corollary 3.1 yield that every function on $\mathbb{R}^d$ is expressible in the form $\sum_{i=1}^{2d+1} g_i \circ \varphi_i$ for some $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, ..., 2d + 1$. We do not yet know if $g_i$ here can be replaced by a single univariate function. We also don't know if the number $2d + 1$ can be reduced so that the whole space of $\mathbb{R}^d$ (or any $d$-dimensional compact subset of that, or at least the unit cube $I^d$) has no cycles with respect to some continuous functions $\varphi_1, ..., \varphi_k : \mathbb{R}^d \to \mathbb{R}$, where $k < 2d + 1$. One of the basic results of Sternfeld [130] says that the dimension of a compact metric space $X$ equals $d$ if and only if there exist functions $\varphi_1, ..., \varphi_{2d+1} \in C(X)$ such that $B_c(\varphi_1, ..., \varphi_{2d+1}; X) = C(X)$ and for any family $\{\psi_i\}_{i=1}^{k} \subset C(X)$, $k < 2d + 1$, we have $B_c(\psi_1, ..., \psi_k; X) \neq C(X)$. In particular, from this result it follows that the number of terms in the Kolmogorov superposition theorem cannot be reduced. But since the equalities $B_c(X) = C(X)$ and $B(X) = T(X)$ are not equivalent, the above question on the nonexistence of cycles in $\mathbb{R}^d$ with respect to less than $2d + 1$ continuous functions is far from trivial.

If disregard the continuity, one can construct even one function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that the whole space $\mathbb{R}^d$ will not possess cycles with respect to $\varphi$ and
therefore, every function $f : \mathbb{R}^d \to \mathbb{R}$ will admit the representation $f = g \circ \varphi$ with some univariate $g$ depending on $f$. Our argument easily follows from Corollary 3.2 and the result of Sprecher [127]: for any natural number $d$, $d \geq 2$, there exist functions $h_p : I \to \mathbb{R}$, $p = 1, \ldots, d$, such that every function $f \in C(I^d)$ can be represented in the form

$$f(x_1, \ldots, x_d) = g \left( \sum_{p=1}^{d} h_p(x_p) \right),$$

(3.6)

where $g$ is a univariate (generally discontinuous) function depending on $f$.

Note that the function involved in the right hand side of (3.6) is a general ridge function. Thus, the result of Sprecher together with our result means that every multivariate function $f$ is representable as a general ridge function $g(\cdot)$ and if $f$ is continuous, then $g$ can be chosen continuous as well.

**Remark 3.1.** Concerning genuine ridge functions $g(a \cdot x)$, representation of every multivariate function by linear combinations of such functions may not be possible over many sets in $\mathbb{R}^d$. For example, this is not possible for sets having interior points. More precisely, assume we are given finitely many nonzero directions $a^1, \ldots, a^r$ in $\mathbb{R}^d$. Then $\mathcal{R}(a^1, \ldots, a^r; X) \neq T(X)$ for any set $X \subset \mathbb{R}^d$ with a nonempty interior. Indeed, let $y$ be a point in the interior of $X$. Consider vectors $b^i$, $i = 1, \ldots, r$, with sufficiently small coordinates such that $a^i \cdot b^i = 0$, $i = 1, \ldots, r$. Note that the vectors $b^i$, $i = 1, \ldots, r$, can be chosen pairwise linearly independent. With each vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$, $\varepsilon_i \in \{0, 1\}$, $i = 1, \ldots, r$, we associate the point

$$x_\varepsilon = y + \sum_{i=1}^{r} \varepsilon_i b^i.$$

Since the coordinates of $b^i$ are sufficiently small, we may assume that all the points $x_\varepsilon$ are in the interior of $X$. We correspond each point $x_\varepsilon$ to the number $(-1)^{|\varepsilon|}$, where $|\varepsilon| = \varepsilon_1 + \cdots + \varepsilon_r$. One may easily verify that the pair $\langle \{x_\varepsilon\}, \{(-1)^{|\varepsilon|}\} \rangle$ is a cycle-vector pair of $X$. Therefore, by Proposition 3.2, $\mathcal{R}(a^1, \ldots, a^r; X) \neq T(X)$.

Note that the above method of construction of the set $\{x_\varepsilon\}$ is due to Lin and Pinkus [95].

**Remark 3.2.** A different generalization of ridge functions was considered in Lin and Pinkus [95]. This generalization involves multivariate functions
of the form $g(Ax)$, where $x \in \mathbb{R}^d$ is the variable, $A$ is a fixed $d \times n$ matrix, $1 \leq n < d$, and $g$ is a real-valued function defined on $\mathbb{R}^n$. For $n = 1$, this reduces to a ridge function.
Chapter 4

Applications to neural networks

Neural networks have increasingly been used in many areas of applied sciences. Most of the applications employ neural networks to approximate complex nonlinear functional dependencies on a high dimensional data set. The theoretical justification for such applications is that any continuous function can be approximated within an arbitrary precision by carefully selecting parameters in the network. The most commonly used model of neural networks is the multilayer feedforward perceptron (MLP) model. This model consists of a finite number of successive layers. The first and the last layers are called the input and the output layers, respectively. The intermediate layers are called hidden layers. MLP models are usually classified not by their number of layers, but by their number of hidden layers. In this chapter, we study approximation properties of the single and two hidden layer feedforward perceptron models. Our analysis is based on ridge functions and the Kolmogorov superposition theorem.

The material of this chapter may be found in [43,45,46,65].

4.1 Single hidden layer neural networks

In this section, we consider single hidden layer neural networks with a set of weights consisting of a finite number of directions or straight lines. We characterize compact sets $X$ in the $d$-dimensional space such that the network can approximate any continuous function over $X$. In the special case, when weights vary only on two straight lines, we give a lower bound for the
approximation error and find a sufficient condition for the network to be a best approximation.

4.1.1 Problem statement

It is well known that neural networks are very powerful tools for approximating complicated multivariate functions, which are the major obstacle in applied mathematics. Approximation capabilities of networks have been investigated in a great deal of works over the last 30 years (see, e.g., [15-17,19-21,24,39-41,67,68,70,71,89-92,96,104,105,110,119,123,125,135]). In this section, we are interested in questions of density of a single hidden layer perceptron model. A typical density result shows that this model can approximate an arbitrary function in a given class with any degree of accuracy.

A single hidden layer perceptron model with $r$ units in the hidden layer and input $x = (x_1, ..., x_d)$ evaluates a function of the form

$$
\sum_{i=1}^{r} c_i \sigma(w^i \cdot x - \theta_i),
$$

where the weights $w^i$ are vectors in $\mathbb{R}^d$, the thresholds $\theta_i$ and the coefficients $c_i$ are real numbers and the activation function $\sigma$ is a univariate function, which is considered to be continuous here. Note that in Eq (4.1) the functions $\sigma(w^i \cdot x - \theta_i)$ are ridge functions. For various activation functions $\sigma$, it has been proved in a number of papers that one can approximate arbitrarily well a given continuous function by functions of the form (4.1) ($r$ is not fixed!) over any compact subset of $\mathbb{R}^d$. In other words, the set

$$
\mathcal{M}(\sigma) = \text{span} \{ \sigma(w \cdot x - \theta) : \theta \in \mathbb{R}, w \in \mathbb{R}^d \}
$$

is dense in the space $C(\mathbb{R}^d)$ in the topology of uniform convergence on all compacta (see, e.g., [17,21,41,67,68]). The most general result of this type belongs to Leshno, Lin, Pinkus and Schocken [89]. They proved that the necessary and sufficient condition for a continuous activation function to have the density property is that it not be a polynomial. This result shows the efficacy of the single hidden layer perceptron model within all possible choices of the activation function $\sigma$, provided that $\sigma$ is continuous. In fact, density of the set $\mathcal{M}(\sigma)$ also holds for some reasonable sets of weights and thresholds. (see[119]).
Some authors showed that a single hidden layer perceptron with a suitably restricted set of weights can also have the u.a.p. (universal approximation property). For example, White and Stinchcombe [135] proved that a single layer network with a polygonal, polynomial spline or analytic activation function and a bounded set of weights has the u.a.p. Ito [68] investigated this property of networks using monotone sigmoidal functions (tending to 0 at minus infinity and 1 at infinity), with only weights located on the unit sphere. We see that weights required for the u.a.p. are not necessary to be of an arbitrarily large magnitude. But what if they are too restricted. How can one learn approximation properties of networks with an arbitrarily restricted set of weights? This problem is too general to be solved completely in this form. But there are some cases which deserve a special attention. The most interesting case is, of course, neural networks with weights varying on a finite set of directions or lines. To the best of our knowledge, approximation capabilities of such networks have not been studied yet. More precisely, let $W$ be a set of weights consisting of a finite number of vectors (or straight lines) in $\mathbb{R}^d$. It is clear that if $w$ varies only in $W$, the set $\mathcal{M}(\sigma)$ cannot be dense in the topology of uniform convergence on all compacta. In this case, one may want to determine boundaries of efficacy of the model. Over which compact sets $X \subset \mathbb{R}^d$ does the model preserve its general propensity to approximate arbitrarily well every continuous multivariate function? In Section 4.1.2, we will consider this problem and give both sufficient and necessary conditions for well approximation by networks with weights from a finite set of directions or lines. For a set $W$ of weights consisting of two vectors, we show that there is a geometrically explicit solution to the problem. In this special case, we also touch some aspects of the exact representation by neural networks.

Clearly, well approximation by neural networks with weights varying only on two directions or straight lines is not always possible. If such networks cannot approximate a prescribed multivariate function with arbitrarily small degree of accuracy, one may be interested in the error of approximation. In Section 4.1.3, we will give an explicit lower bound for the approximation error and find a sufficient condition for a neural network to be a best approximation.
4.1.2 Density results

In this subsection we give a sufficient and also a necessary conditions for approximation by neural networks with finitely many weights and with weights varying on a finite set of straight lines (through the origin).

Let $X$ be a compact subset of $\mathbb{R}^d$. Consider the following set functions

$$\tau_i(Z) = \{x \in Z : |p_i^{-1}(p_i(x)) \cap Z| \geq 2\}, \quad Z \subset X, \ i = 1, \ldots, k,$$

where $p_i(x) = a_i \cdot x$, $|Y|$ denotes the cardinality of a considered set $Y$. Define $\tau(Z)$ to be $\bigcap_{i=1}^{k} \tau_i(Z)$ and define $\tau^2(Z) = \tau(\tau(Z))$, $\tau^3(Z) = \tau(\tau^2(Z))$ and so on inductively. These functions first appeared in the work [132] by Sternfeld, where he investigated problems of representation by linear superpositions.

Clearly, $\tau(Z) \supseteq \tau^2(Z) \supseteq \tau^3(Z) \supseteq \ldots$ It is possible that for some $n$, $\tau^n(Z) = \emptyset$. In this case, one can see that $Z$ does not contain a cycle. In general, if some set $Z \subset X$ forms a cycle, then $\tau^n(Z) = Z$. But the reverse is not true. Indeed, let $Z = X = \{(0, 0, \frac{1}{2}), (0, 0, 1), (0, 1, 0), (1, 1, 0), (\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)\}$, $a^i, i = 1, 2, 3$, are the coordinate directions in $\mathbb{R}^3$. It is not difficult to verify that $X$ does not possess cycles with respect to these directions and at the same time $\tau(X) = X$ (and so $\tau^n(X) = X$ for every $n$).

Consider the linear combinations of ridge functions with fixed directions $a^1, \ldots, a^k$

$$\mathcal{R}(a^1, \ldots, a^k) = \left\{ \sum_{i=1}^{k} g_i (a^i \cdot x) : g_i \in C(\mathbb{R}), \ i = 1, \ldots, k \right\}. \quad (4.2)$$

Let $K$ be a family of functions defined on $\mathbb{R}^d$ and $X$ be a subset of $\mathbb{R}^d$. By $K_X$ we will denote the restriction of this family to $X$. Thus $\mathcal{R}_X(a^1, \ldots, a^k)$ stands for the set of sums of ridge functions in (4.2) defined on $X$.

The following theorem is a special case of the known general result of Sproston and Strauss [129] established for the sum of subalgebras of $C(X)$.

**Theorem 4.1.** Let $X$ be a compact subset of $\mathbb{R}^d$. If $\bigcap_{n=1,2,\ldots} \tau^n(X) = \emptyset$, then the set $\mathcal{R}_X(a^1, \ldots, a^k)$ is dense in $C(X)$.

**Lemma 4.1.** If $\mathcal{R}_X(a^1, \ldots, a^k)$ is dense in $C(X)$, then the set $X$ does not contain a cycle with respect to the directions $a^1, \ldots, a^k$.
Proof. Suppose the contrary. Suppose that the set $X$ contains cycles. Each cycle $l = (x_1, \ldots, x_n)$ and the associated vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ generate the functional

$$G_{l, \lambda}(f) = \sum_{j=1}^{n} \lambda_j f(x_j), \quad f \in C(X).$$

Clearly, $G_{l, \lambda}$ is linear and continuous with the norm $\sum_{j=1}^{n} |\lambda_j|$. It is not difficult to verify that $G_{l, \lambda}(g) = 0$ for all functions $g \in \mathcal{R}(a^1, \ldots, a^k)$. Let $f_0$ be a continuous function such that $f_0(x_j) = 1$ if $\lambda_j > 0$ and $f_0(x_j) = -1$ if $\lambda_j < 0$, $j = 1, \ldots, n$. For this function, $G_{l, \lambda}(f_0) \neq 0$. Thus, we have constructed a nonzero linear functional which belongs to the annihilator of the manifold $\mathcal{R}_X (a^1, \ldots, a^k)$. This means that $\mathcal{R}_X (a^1, \ldots, a^k)$ is not dense in $C(X)$. The obtained contradiction proves the lemma. \qed

Now we are able to step forward from ridge function approximation to neural networks. Let $\sigma \in C(\mathbb{R})$ be a continuous activation function. For a subset $W \subset \mathbb{R}^d$, let $\mathcal{M}(\sigma; W, \mathbb{R})$ stand for the set of neural networks with weights from $W$. That is,

$$\mathcal{M}(\sigma; W, \mathbb{R}) = \text{span}\{\sigma(w \cdot x - \theta) : w \in W, \theta \in \mathbb{R}\}$$

**Theorem 4.2.** Let $\sigma \in C(\mathbb{R}) \cap L_p(\mathbb{R})$, where $1 \leq p < \infty$, or $\sigma$ be a continuous, bounded, nonconstant function, which has a limit at infinity (or minus infinity). Let $W = \{a^1, a^2, \ldots, a^k\} \subset \mathbb{R}^d$ be the given set of weights and $X$ be a compact subset of $\mathbb{R}^d$. Then the following assertions are valid:

1. if $\cap_{n=1,2,\ldots} \tau^n(X) = \emptyset$, then the set $\mathcal{M}_X(\sigma; W, \mathbb{R})$ is dense in the space of all continuous functions over $X$.

2. if $\mathcal{M}_X(\sigma; W, \mathbb{R})$ is dense in $C(X)$, then the set $X$ does not contain cycles.

**Proof.** Part (1). Let $X$ be a compact subset of $\mathbb{R}^d$ for which $\cap_{n=1,2,\ldots} \tau^n(X) = \emptyset$. By Theorem 4.1, the set $\mathcal{R}_X (a^1, \ldots, a^k)$ is dense in $C(X)$. This means that for any positive real number $\varepsilon$ there exist continuous univariate functions $g_i$, $i = 1, \ldots, k$ such that

$$\left| f(x) - \sum_{i=1}^{k} g_i (a^i \cdot x) \right| < \frac{\varepsilon}{k + 1} \quad (4.3)$$
for all $x \in X$. Since $X$ is compact, the sets $Y_i = \{a^i \cdot x : x \in X\}$, $i = 1, 2, ..., k$ are also compacts. In 1947, Schwartz [124] proved that continuous and $p$-th degree Lebesgue integrable univariate functions or continuous, bounded, nonconstant functions having a limit at infinity (or minus infinity) are not mean-periodic. Note that a function $f \in C(\mathbb{R}^d)$ is called mean periodic if the set $span \{f(x - b) : b \in \mathbb{R}^d\}$ is not dense in $C(\mathbb{R}^d)$ in the topology of uniform convergence on compacta (see [124]). Thus, Schwartz proved that the set $span \{\sigma(y - \theta) : \theta \in \mathbb{R}\}$ is dense in $C(\mathbb{R})$ in the topology of uniform convergence. We learned about this result from Pinkus [119, page 162]. This density result means that for the given $\varepsilon$ there exist numbers $c_{ij}, \theta_{ij} \in \mathbb{R}$, $i = 1, 2, ..., k$, $j = 1, ..., m_i$ such that

$$
\left| g_i(y) - \sum_{j=1}^{m_i} c_{ij} \sigma(y - \theta_{ij}) \right| < \frac{\varepsilon}{k + 1}
$$

(4.4)

for all $y \in Y_i$, $i = 1, 2, ..., k$. From (4.3) and (4.4) we obtain that

$$
\left\| f(x) - \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} \sigma(a^i \cdot x - \theta_{ij}) \right\|_{C(X)} < \varepsilon.
$$

(4.5)

Hence $\mathcal{M}_X(\sigma;W,\mathbb{R}) = C(X)$.

Part (2). Let $X$ be a compact subset of $\mathbb{R}^d$ and the set $\mathcal{M}_X(\sigma;W,\mathbb{R})$ be dense in $C(X)$. Then for an arbitrary positive real number $\varepsilon$, inequality (4.5) holds with some coefficients $c_{ij}, \theta_{ij}$, $i = 1, 2, ..., k$, $j = 1, ..., m_i$. Since for each $i = 1, 2, ..., k$, the function $\sum_{j=1}^{m_i} c_{ij} \sigma(a^i \cdot x - \theta_{ij})$ is a function of the form $g_i(a^i \cdot x)$, the subspace $\mathcal{R}_X(a^1, ..., a^k)$ is dense in $C(X)$. Then by Lemma 4.1, the set $X$ contains no cycle.

The above theorem still holds if the set of weights $W = \{a^1, a^2, ..., a^k\}$ is replaced by the set $W_1 = \{t_1a^1, t_2a^2, ..., t_ka^k : t_1, t_2, ..., t_k \in \mathbb{R}\}$. In fact, for the set $W_1$, the above restrictions on the activation function $\sigma$ may be weakened.

**Theorem 4.3.** Let $\sigma \in C(\mathbb{R})$ and assume $\sigma$ is not a polynomial. Let $W_1 = \{t_1a^1, t_2a^2, ..., t_ka^k : t_1, t_2, ..., t_k \in \mathbb{R}\} \subset \mathbb{R}^d$ be the given set of weights and $X$ be a compact subset of $\mathbb{R}^d$. Then the following assertions are valid:
(1) if $\cap_{n=1,2,...}^n (X) = \emptyset$, then the set $\mathcal{M}_X(\sigma; W_1, \mathbb{R})$ is dense in the space of all continuous functions over $X$.

(2) if $\mathcal{M}_X(\sigma; W_1, \mathbb{R})$ is dense in $C(X)$, then the set $X$ does not contain cycles.

The proof of this theorem is similar to that of Theorem 4.2 and based on the following result of Leshno, Lin, Pinkus and Schocken [89]: if $\sigma$ is not a polynomial, then the set $\text{span} \{\sigma(ty - \theta) : t, \theta \in \mathbb{R}\}$ is dense in $C(\mathbb{R})$ in the topology of uniform convergence.

The above example with the set $\{(0, 0, \frac{1}{2}), (0, 0, 1), (0, 1, 0), (1, 0, 1), (1, 1, 0), (\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$ shows that the sufficient condition in part (1) of Theorem 4.2 is not necessary. The necessary condition in part (2), in general, is not sufficient. But it is not easily seen. Here, is the nontrivial example showing that nonexistence of cycles is not sufficient for the density $\mathcal{M}_X(\sigma; W_1, \mathbb{R}) = C(X)$.

By $S_i$, $i = 1,4$, denote the closed discs with the unit radius and centered at the points $(-2; 0)$, $0; 2)$, $(2; 0)$ and $(0; -2)$ respectively. Consider a continuous function $f_0$ such that $f_0(x) = 1$ for $x \in (S_1 \cup S_3) \cap X$, $f_0(x) = -1$ for $x \in (S_2 \cup S_4) \cap X$, and $-1 < f_0(x) < 1$ elsewhere on $\mathbb{R}^2$. Let $p = (y^1, y^2, ...)$ be any infinite path in $X$. Note that the points $y^i$, $i = 1, 2, ..., $ are alternatively in the sets $(S_1 \cup S_3) \cap X$ and $(S_2 \cup S_4) \cap X$. Obviously,

$$E(f_0, X) \overset{def}{=} \inf_{g \in \mathcal{R}_X(a^1, a^2)} \|f_0 - g\|_{C(X)} \leq \|f_0\|_{C(X)} = 1.$$ (4.6)
For each positive integer \( k = 1, 2, \ldots \), set \( p_k = (y^1, \ldots, y^k) \) and consider
the path functionals

\[
G_{p_k}(f) = \frac{1}{k} \sum_{i=1}^{k} (-1)^{i-1} f(y^i).
\]

\( G_{p_k} \) is a continuous linear functional obeying the following obvious properties:

1. \( \|G_{p_k}\| = G_{p_k}(f_0) = 1; \)
2. \( G_{p_k}(g_1 + g_2) \leq \frac{2}{k}(\|g_1\| + \|g_2\|) \) for ridge functions \( g_1 = g_1(a^1 \cdot x) \) and \( g_2 = g_2(a^2 \cdot x) \).

By property (1), the sequence \( \{G_{p_k}\}_{k=1}^{\infty} \) has a weak* cluster point. This point will be denoted by \( G \). By property (2), \( G \in \mathcal{R}_X(a^1, a^2)^\perp \). Therefore,

\[
1 = G(f_0) = G(f_0 - g) \leq \|f_0 - g\|_{C(X)} \quad \text{for any } g \in \mathcal{R}_X(a^1, a^2).
\]

Taking inf over \( g \) in the right-hand side of the last inequality, we obtain that \( 1 \leq E(f_0, X) \). Now it follows from (4.6) that \( E(f_0, X) = 1 \). Recall that \( \mathcal{M}_X(\sigma; W, \mathbb{R}) \subset \mathcal{R}_X(a^1, a^2) \). Thus

\[
\inf_{h \in \mathcal{M}_X(\sigma; W, \mathbb{R})} \|f - h\|_{C(X)} \geq 1.
\]

The last inequality finally shows that \( \mathcal{M}_X(\sigma; W, \mathbb{R}) \neq C(X) \).

For neural networks with weights consisting of only two vectors (or directions) the problem of density becomes more clear. In this case, under some minor restrictions on \( X \), the necessary condition in part (2) of Theorem 4.2 (nonexistence of cycles) is also sufficient for the density of \( \mathcal{M}_X(\sigma; W, \mathbb{R}) \) in \( C(X) \). These restrictions are imposed on the following equivalent classes of \( X \) induced by paths. The relation \( x \sim y \) when \( x \) and \( y \) belong to some path in a given compact set \( X \subset \mathbb{R}^d \) defines an equivalence relation. Recall that the equivalence classes are called orbits (see Section 1.3.4).

**Theorem 4.4.** Let \( \sigma \in C(\mathbb{R}) \cap L_p(\mathbb{R}), \) where \( 1 \leq p < \infty \), or \( \sigma \) be a continuous, bounded, nonconstant function, which has a limit at infinity (or minus infinity). Let \( W = \{a^1, a^2\} \subset \mathbb{R}^d \) be the given set of weights and \( X \) be
a compact subset of $\mathbb{R}^d$ with all its orbits closed. Then $\mathcal{M}_X(\sigma; W, \mathbb{R})$ is dense in the space of all continuous functions over $X$ if and only if $X$ contains no closed paths with respect to the directions $a^1$ and $a^2$.

**Proof. Sufficiency.** Let $X$ be a compact subset of $\mathbb{R}^d$ with all its orbits closed. Besides, let $X$ contain no closed paths. By Theorem 1.6 (see Section 1.3.4), the set $\mathcal{R}_X(a^1, a^2)$ is dense in $C(X)$. This means that for any positive real number $\varepsilon$ there exist continuous univariate functions $g_1$ and $g_2$ such that

$$|f(x) - g_1(a^1 \cdot x) - g_2(a^2 \cdot x)| < \frac{\varepsilon}{3} \quad (4.7)$$

for all $x \in X$. Since $X$ is compact, the sets $Y_i = \{a^i \cdot x : x \in X\}, \ i = 1, 2,$ are also compacts. As mentioned above, Schwartz [124] proved that continuous and $p$-th degree Lebesgue integrable univariate functions or continuous, bounded, nonconstant functions having a limit at infinity (or minus infinity) are not mean-periodic. Thus, the set

$$\text{span} \{\sigma(y - \theta) : \theta \in \mathbb{R}\}$$

is dense in $C(\mathbb{R})$ in the topology of uniform convergence. This density result means that for the given $\varepsilon$ there exist numbers $c_{ij}, \theta_{ij} \in \mathbb{R}, \ i = 1, 2, \ j = 1, \ldots, m_i$ such that

$$\left|g_i(y) - \sum_{j=1}^{m_i} c_{ij} \sigma(y - \theta_{ij})\right| < \frac{\varepsilon}{3} \quad (4.8)$$

for all $y \in Y_i, \ i = 1, 2$. From (4.7) and (4.8) we obtain that

$$\left\|f(x) - \sum_{i=1}^{2} \sum_{j=1}^{m_i} c_{ij} \sigma(a^i \cdot x - \theta_{ij})\right\|_{C(X)} < \varepsilon. \quad (4.9)$$

Hence $\overline{\mathcal{M}_X(\sigma; W, \mathbb{R})} = C(X)$.

**Necessity.** Let $X$ be a compact subset of $\mathbb{R}^n$ with all its orbits closed and the set $\mathcal{M}_X(\sigma; W, \mathbb{R})$ be dense in $C(X)$. Then for an arbitrary positive real number $\varepsilon$, inequality (4.9) holds with some coefficients $c_{ij}, \theta_{ij}, \ i = 1, 2, \ j = 1, \ldots, m_i$. Since for $i = 1, 2, \sum_{j=1}^{m_i} c_{ij} \sigma(a^i \cdot x - \theta_{ij})$ is a function of the form $g_i(a^i \cdot x)$, the subspace $\mathcal{R}_X(a^1, a^2)$ is dense in $C(X)$. Then by Theorem 1.6, the set $X$ contains no closed path. \qed
Remark 4.1. It can be shown that the necessity of the theorem is valid without any restriction on orbits of \(X\). Indeed if \(X\) contains a closed path, then it contains a closed path \(p = (x^1, \ldots, x^{2m})\) with different points. The functional \(G_p = \sum_{i=1}^{2m} (-1)^{i-1} f(x^i)\) belongs to the annihilator of the subspace \(R_X(a^1, a^2)\). There exist nontrivial continuous functions \(f_0\) on \(X\) such that \(G_p(f_0) \neq 0\) (take, for example, any continuous function \(f_0\) taking values +1 at \(\{x^1, x^3, \ldots, x^{2m-1}\}\), −1 at \(\{x^2, x^4, \ldots, x^{2m}\}\) and \(-1 < f_0(x) < 1\) elsewhere). This shows that the subspace \(R_X(a^1, a^2)\) is not dense in \(C(X)\). But in this case, the set \(M_X(\sigma; W, \mathbb{R})\) cannot be dense in \(C(X)\). The obtained contradiction means that our assumption is not true and \(X\) contains no closed path.

Theorem 4.4 remains valid if the set of weights \(W = \{a^1, a^2\}\) is replaced by the set \(W_1 = \{t_1a^1, t_2a^2 : t_1, t_2 \in \mathbb{R}\}\). In fact, for the set \(W_1\), the required conditions on \(\sigma\) may be weakened. As in Theorem 4.3, the activation function \(\sigma\) can be taken only non-polynomial.

Theorem 4.5. Let \(\sigma \in C(\mathbb{R})\) and assume \(\sigma\) is not a polynomial. Let \(a^1\) and \(a^2\) be fixed vectors and \(W_1 = \{t_1a^1, t_2a^2 \in \mathbb{R}, i = 1, 2\}\) be the set of weights. Let \(X\) be a compact subset of \(\mathbb{R}^d\) with all its orbits closed. Then \(M_X(\sigma; W_1, \mathbb{R})\) is dense in the space of all continuous functions over \(X\) if and only if \(X\) contains no closed paths with respect to the directions \(a^1\) and \(a^2\).

The proof is analogous to that of Theorem 4.4 and based on the above mentioned result of Leshno, Lin, Pinkus and Schocken [89].

Examples:

(a) Let \(a^1\) and \(a^2\) be two noncollinear vectors in \(\mathbb{R}^2\). Let \(B = B_1 \ldots B_k\) be a broken line with the sides \(B_iB_{i+1}, i = 1, \ldots, k - 1\), alternatively perpendicular to \(a^1\) and \(a^2\). Besides, let \(B\) does not contain vertices of any parallelogram with sides perpendicular to these vectors. Then the set \(M_B(\sigma; \{a^1, a^2\}, \mathbb{R})\) is dense in \(C(B)\).

(b) Let \(a^1\) and \(a^2\) be two noncollinear vectors in \(\mathbb{R}^2\). If \(X\) is the union of two parallel line segments, not perpendicular to any of the vectors \(a^1\) and \(a^2\), then the set \(M_X(\sigma; \{a^1, a^2\}, \mathbb{R})\) is dense in \(C(X)\).

(c) Let now \(a^1\) and \(a^2\) be two collinear vectors in \(\mathbb{R}^2\). Note that in this case any path consisting of two points is automatically closed. Thus the
set $\mathcal{M}_X(\sigma; \{a^1, a^2\}, \mathbb{R})$ is dense in $C(X)$ if and only if $X$ contains no path different from a singleton. A simple example is a line segment not perpendicular to the given direction.

(d) Let $X$ be a compact set with an interior point. Then Theorem 4.4 fails, since any such set contains vertices of some parallelogram with sides perpendicular to the given directions $a^1$ and $a^2$, that is a closed path.

### 4.1.3 A necessary condition for the representation by neural networks

In this subsection we give a necessary condition for the representation of functions by neural networks with weights from a finitely many straight lines. Before formulating our result, we introduce new objects - semicycles with respect to given $k$ directions $a^1, a^2, \ldots, a^k \in \mathbb{R}^d \setminus \{0\}$.

**Definition 4.1.** A set of points $l = (x^1, \ldots, x^n) \subset \mathbb{R}^d$ is called a semicycle with respect to the directions $a^1, a^2, \ldots, a^k$ if there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \setminus \{0\}$ such that for any $i = 1, \ldots, k$

$$\sum_{j=1}^{n} \lambda_j \delta_{a^i \cdot x^j} = \sum_{s=1}^{r_i} \lambda_i \delta_{a^i \cdot x^s}, \quad \text{where } r_i \leq k. \quad (4.10)$$

Here $\delta_a$ is the characteristic function of the single point set $\{a\}$. Note that for $i = 1, \ldots, k$, the set $\{\lambda_{i_s}, s = 1, \ldots, r_i\}$ is a subset of the set $\{\lambda_j, j = 1, \ldots, n\}$. Thus, Eq. (4.10) means that for each $i$, we actually have at most $k$ terms in the sum $\sum_{j=1}^{n} \lambda_j \delta_{a^i \cdot x^j}$.

Recall that if in (4.10) for any $i = 1, \ldots, k$ we have

$$\sum_{j=1}^{n} \lambda_j \delta_{a^i \cdot x^j} = 0,$$

then the set $l = (x^1, \ldots, x^n)$ is a cycle with respect to the directions $a^1, a^2, \ldots, a^k$ (see Section 1.2). Thus a cycle is a special case of a semicycle.

Let, for example, $k = 2$, $a^1 \cdot x^1 = a^1 \cdot x^2$, $a^2 \cdot x^2 = a^2 \cdot x^3$, $a^1 \cdot x^3 = a^1 \cdot x^4, \ldots, a^2 \cdot x^{n-1} = a^2 \cdot x^n$. Then it is not difficult to see that for a vector
\[ \lambda = (\lambda_1, \ldots, \lambda_n) \] with the components \[ \lambda_j = (-1)^j, \]
\[ \sum_{j=1}^{n} \lambda_j \delta_{\mathbf{a}^1, \mathbf{x}^j} = \lambda_n \delta_{\mathbf{a}^1, \mathbf{x}^n}, \]
\[ \sum_{j=1}^{n} \lambda_j \delta_{\mathbf{a}^2, \mathbf{x}^j} = \lambda_1 \delta_{\mathbf{a}^2, \mathbf{x}^1}. \]

Thus, by Definition 4.1, the set \( l = \{x^1, \ldots, x^n\} \) forms a semicycle with respect to the directions \( \mathbf{a}^1 \) and \( \mathbf{a}^2 \).

Note that one can construct many semicycles by adding not more than \( k \) arbitrary points to a cycle with respect to the directions \( \mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^k \). Note also that in the case \( k = 2 \), a path with respect to two directions \( \mathbf{a}^1 \) and \( \mathbf{a}^2 \) (see Section 1.3) satisfies Eq. (4.10). That is, every path with respect to directions \( \mathbf{a}^1 \) and \( \mathbf{a}^2 \) is a semicycle with respect to these directions. But Eq. (4.10) may allow also some union of paths.

A cycle (or semicycle) \( l \) is called a \( q \)-cycle (\( q \)-semicycle) if the vector \( \lambda \) associated with \( l \) can be chosen so that \[ |\lambda_i| \leq q, \quad i = 1, \ldots, n, \] and \( q \) is the minimal number with this property.

The semicycle considered above is a 1-semicycle. If in that example, \( \mathbf{a}^2 \cdot \mathbf{x}^{n-1} = \mathbf{a}^2 \cdot \mathbf{x}^1 \), then the set \( \{x_1, x_2, \ldots, x_{n-1}\} \) is a 1-cycle. Let us give a simple example of a 2-cycle with respect to the directions \( \mathbf{a}^1 = (1, 0) \) and \( \mathbf{a}^2 = (0, 1) \). Consider the union

\[ \{0, 1\}^2 \cup \{0, 2\}^2 = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0), (0, 2), (2, 0)\}. \]

It is easy to see that this set is a 2-cycle with the associated vector \((2, 1, 1, -1, -1, -1, -1)\). Similarly, one can construct a \( q \)-cycle or \( q \)-semicycle for any positive integer \( q \).

**Theorem 4.6.** Let \( \mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^k \in \mathbb{R}^d \), \( W = \{t_1 \mathbf{a}^1, t_2 \mathbf{a}^2, \ldots, t_k \mathbf{a}^k : t_1, t_2, \ldots, t_k \in \mathbb{R}\} \) and \( \mathcal{M}_X(\sigma; W, \mathbb{R}) = C(X) \). Then \( X \) contains no cycle and the lengths (number of points) of all \( q \)-semicycles in \( X \) are bounded by some positive integer.

**Proof.** Let \( \mathcal{M}_X(\sigma; W, \mathbb{R}) = C(X) \). Then \( \mathcal{R}_1 + \mathcal{R}_2 + \ldots + \mathcal{R}_k = C(X) \), where

\[ \mathcal{R}_i = \{g_i(\mathbf{a}^i \cdot \mathbf{x}) : g_i \in C(\mathbb{R})\}, \quad i = 1, 2, \ldots, k. \]
Consider the linear space

\[ \mathcal{U} = \prod_{i=1}^{k} \mathcal{R}_i = \{(g_1, \ldots, g_k) : \ g_i \in \mathcal{R}_i, \ i = 1, \ldots, k\} \]

e Kurdish with the norm

\[ \|(g_1, \ldots, g_k)\| = \|g_1\| + \cdots + \|g_k\|. \]

By \( \mathcal{U}^* \) denote the dual space of \( \mathcal{U} \). Each functional \( F \in \mathcal{U}^* \) can be written as

\[ F = F_1 + \cdots + F_k, \]

where the functionals \( F_i \in \mathcal{R}^*_i \) and

\[ F_i(g_i) = F[(0, \ldots, g_i, \ldots, 0)], \ i = 1, \ldots, k. \]

We see that the functional \( F \) determines the collection \( (F_1, \ldots, F_k) \). Conversely, every collection \( (F_1, \ldots, F_k) \) of continuous linear functionals \( F_i \in \mathcal{R}^*_i, \ i = 1, \ldots, k \), determines the functional \( F_1 + \cdots + F_k \), on \( \mathcal{U} \). Considering this, in what follows, elements of \( \mathcal{U}^* \) will be denoted by \( (F_1, \ldots, F_k) \).

It is not difficult to verify that

\[ \|(F_1, \ldots, F_k)\| = \max\{\|F_1\|, \ldots, \|F_k\|\}. \quad (4.11) \]

Let \( l = (x^1, \ldots, x^n) \) be any \( q \)-semicycle (with respect to the directions \( a^1, a^2, \ldots, a^k \)) in \( X \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a vector associated with it. Consider the following functional

\[ G_{l, \lambda}(f) = \sum_{j=1}^{n} \lambda_j f(x^j), \ f \in C(X). \]

Since \( l \) satisfies (4.10), for each function \( g_i \in \mathcal{R}_i, \ i = 1, \ldots, k \), we have

\[ G_{l, \lambda}(g_i) = \sum_{j=1}^{n} \lambda_j g_i(a^j \cdot x^j) = \sum_{s=1}^{r_i} \lambda_{i_s} g_i(a^i \cdot x^{i_s}), \quad (4.12) \]

where \( r_i \leq k \). That is, for each set \( \mathcal{R}_i \), \( G_{l, \lambda} \) can be reduced to a functional defined with the help of not more than \( k \) points of the semicycle \( l \).
Consider the operator

\[ A : \mathcal{U} \to C(X), \quad A[(g_1, \ldots, g_k)] = g_1 + \cdots + g_k. \]

Clearly, \( A \) is a linear continuous operator with the norm \( \| A \| = 1 \). Besides, since \( \mathcal{R}_1 + \mathcal{R}_2 + \ldots + \mathcal{R}_k = C(X) \), \( A \) is a surjection. Consider also the conjugate operator

\[ A^* : C(X)^* \to \mathcal{U}^*, \quad A^*[H] = (F_1, \ldots, F_k), \]

where \( F_i(g_i) = H(g_i) \), for any \( g_i \in \mathcal{R}_i \), \( i = 1, \ldots, k \). Set \( A^*[G_{l,\lambda}] = (G_1, \ldots, G_k) \). From (4.12) it follows that

\[ |G_i(g_i)| = |G_{l,\lambda}(g_i)| \leq \|g_i\| \sum_{s=1}^{r_i} |\lambda_{i_s}| \leq kq \|g_i\|, \quad i = 1, \ldots, k, \]

Therefore,

\[ \|G_i\| \leq kq, \quad i = 1, \ldots, k. \]

From (4.11) we obtain that

\[ \|A^*[G_{l,\lambda}]\| = \|(G_1, \ldots, G_k)\| \leq kq. \quad (4.13) \]

Since \( A \) is a surjection, there exists a positive real number \( \delta \) such that

\[ \|A^*[H]\| > \delta \|H\| \]

for any functional \( H \in C(X)^* \) (see [122, p.100]). Taking into account that \( \|G_{l,\lambda}\| = \sum_{j=1}^n |\lambda_j| \), for the functional \( G_{l,\lambda} \) we have

\[ \|A^*[G_{l,\lambda}]\| > \delta \sum_{j=1}^n |\lambda_j|. \quad (4.14) \]

It follows from (4.13) and (4.14) that

\[ \delta < \frac{kq}{\sum_{j=1}^n |\lambda_j|}. \]

The last inequality shows that \( n \) (the length of the arbitrarily chosen \( q \)-semicycle \( l \)) cannot be as great as possible, otherwise \( \delta = 0 \). This simply
means that there must be some positive integer bounding the lengths of all \(q\)-semicycles in \(X\).

It remains to show that there are no cycles in \(X\). Indeed, if \(l = (x_1, \ldots, x^n)\) is a cycle in \(X\) and \(\lambda = (\lambda_1, \ldots, \lambda_n)\) is a vector associated with it, then the above functional \(G_{l,\lambda}\) annihilates all functions from \(R_1 + R_2 + \ldots + R_k\). On the other hand, \(G_{l,\lambda}(f) = \sum_{j=1}^n |\lambda_j| \neq 0\) for a continuous function \(f\) on \(X\) satisfying the conditions \(f(x^j) = 1\) if \(\lambda_j > 0\) and \(f(x^j) = -1\) if \(\lambda_j < 0\), \(j = 1, \ldots, n\). This implies that \(R_1 + R_2 + \ldots + R_k \neq C(X)\). Since \(M_X(\sigma; W, \mathbb{R}) \subseteq R_1 + R_2 + \ldots + R_k\), we obtain that \(M_X(\sigma; W, \mathbb{R}) \neq C(X)\) on the contrary to our assumption. \(\square\)

**Remark 4.2.** Assume \(M_X(\sigma; W, \mathbb{R})\) is dense in \(C(X)\). Is it necessarily closed? Theorem 4.6 may describe cases when it is not. For example, let \(a^1 = (1; -1), a^2 = (1; 1), W = \{a^1, a^2\}\) and \(\sigma\) be any continuous, bounded and nonconstant function, which has a limit at infinity. Consider the set \(X = \{(2; \frac{2}{3}), (\frac{2}{3}; \frac{2}{3}), (0; 0), (1; 1), (1 + \frac{1}{2}; 1 - \frac{1}{2}), (1 + \frac{1}{2} + \frac{1}{4}; 1 - \frac{1}{2} + \frac{1}{4}), (1 + \frac{1}{2} + \frac{1}{8}; 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}), \ldots\}\).

It is clear that \(X\) is a compact set with all its orbits closed. (In fact, there is only one orbit, which coincides with \(X\)). Hence, by Theorem 4.4, \(M_X(\sigma; W, \mathbb{R}) = C(X)\). But by Theorem 4.6, \(M_X(\sigma; W, \mathbb{R}) \neq C(X)\). Therefore, the set \(M_X(\sigma; W, \mathbb{R})\) is not closed in \(C(X)\).

### 4.1.4 Approximation error and extremal networks

If well approximation by neural networks is not possible, one may be interested in the error of this approximation. Below for one special class of bivariate functions, we give an easily calculable lower bound for the error of approximation by neural networks with any continuous activation function and weights varying on two lines (through the origin).

Let \(\sigma\) be a continuous activation function and \(W = \{ka, tb : k, t \in \mathbb{R}\}\) be the set of weights, where \(a, b\) are linearly independent vectors in \(\mathbb{R}^2\). For a compact set \(\Omega\) in \(\mathbb{R}^2\), the error of approximation of a given function \(f \in C(\Omega)\) with networks from \(M_\Omega(\sigma; W, \mathbb{R})\) is denoted by \(E(f, M)\). That is,

\[
E(f, M) \overset{\text{def}}{=} \inf_{g \in M_\Omega(\sigma; W, \mathbb{R})} \|f - g\|.
\]
The following theorem shows that in certain cases it is possible to compute
the precise value of $E(f, \mathcal{M})$ quite easily.

**Theorem 4.7.** Let

\[ \Omega = \{ \mathbf{x} \in \mathbb{R}^2 : c_1 \leq \mathbf{a} \cdot \mathbf{x} \leq d_1, \ c_2 \leq \mathbf{b} \cdot \mathbf{x} \leq d_2 \}, \]

where $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ are linearly independent vectors, $c_1 < d_1$
and $c_2 < d_2$. Let a function $f \in C(\Omega)$ have the continuous partial derivatives
\[
\frac{\partial^2 f}{\partial x_1^2}, \ \frac{\partial^2 f}{\partial x_1 \partial x_2}, \ \frac{\partial^2 f}{\partial x_2^2} \text{ and for any } \mathbf{x} = (x_1, x_2) \in \Omega
\]
\[
\left[ a_1 \frac{\partial}{\partial x_1} - a_2 \frac{\partial}{\partial x_2} \right] \left[ b_1 \frac{\partial}{\partial x_1} - b_2 \frac{\partial}{\partial x_2} \right] f \leq 0.
\]

Then
\[ E(f, \mathcal{M}) \geq \frac{1}{4} \left( f_1(c_1, c_2) + f_1(d_1, d_2) - f_1(c_1, d_2) - f_1(d_1, c_2) \right), \]
where
\[ f_1(y_1, y_2) = f \left( \frac{y_1 b_2 - y_2 a_2}{a_1 b_2 - a_2 b_1}, \frac{y_2 a_1 - y_1 b_1}{a_1 b_2 - a_2 b_1} \right). \]

**Proof.** Introduce the new variables
\[ y_1 = a_1 x_1 + a_2 x_2, \quad y_2 = b_1 x_1 + b_2 x_2. \quad (4.15) \]

Since the vectors $(a_1, a_2)$ and $(b_1, b_2)$ are linearly independent, for any
$(y_1, y_2) \in Y$, where $Y = [c_1, d_1] \times [c_2, d_2]$, there exists only one solution
$(x_1, x_2) \in \Omega$ of the system (4.15). The coordinates of this solution are
\[ x_1 = \frac{y_1 b_2 - y_2 a_2}{a_1 b_2 - a_2 b_1}, \quad x_2 = \frac{y_2 a_1 - y_1 b_1}{a_1 b_2 - a_2 b_1}. \quad (4.16) \]

The linear transformation (4.16) transforms the function $f(x_1, x_2)$ to the
function $f_1(y_1, y_2)$. Consider the approximation of $f_1(y_1, y_2)$ from the set
\[ \mathcal{Z} = \{ z_1(y_1) + z_2(y_2) : z_i \in C(\mathbb{R}), \ i = 1, 2 \}. \]

Note that
\[ E(f, \mathcal{M}) \geq E(f, \mathcal{R}) = E(f_1, \mathcal{Z}), \quad (4.17) \]
where $\mathcal{R}$ is the set of linear combinations of ridge functions with the directions $\mathbf{a}$ and $\mathbf{b}$.

With each rectangle $S = [u_1, v_1] \times [u_2, v_2] \subset Y$ we associate the functional

$$L(h, S) = \frac{1}{4} (h(u_1, u_2) + h(v_1, v_2) - h(u_1, v_2) - h(v_1, u_2)), \quad h \in C(Y).$$

This functional has the following obvious properties:

(i) $L(z, S) = 0$ for any $z \in \mathcal{Z}$ and $S \subset Y$.

(ii) For any point $(y_1, y_2) \in Y$, $L(f_1, Y) = \sum_{i=1}^{4} L(f_1, S_i)$, where $S_1 = [c_1, y_1] \times [c_2, y_2], S_2 = [y_1, d_1] \times [y_2, d_2], S_3 = [c_1, y_1] \times [y_2, d_2], S_4 = [y_1, d_1] \times [c_2, y_2]$.

It is not difficult to verify that

$$\frac{\partial^2 f_1}{\partial y_1 \partial y_2} \geq 0 \quad \text{for any} \quad (y_1, y_2) \in Y.$$

Integrating both sides of the last inequality over arbitrary rectangle $S = [u_1, v_1] \times [u_2, v_2] \subset Y$, we obtain that

$$L(f_1, S) \geq 0. \quad (4.18)$$

Set the function

$$f_2(y_1, y_2) = L(f_1, S_1) + L(f_1, S_2) - L(f_1, S_3) - L(f_1, S_4).$$

It is not difficult to verify that the function $f_1 - f_2$ belongs to $\mathcal{Z}$. Hence

$$E(f_1, \mathcal{Z}) = E(f_2, \mathcal{Z}). \quad (4.19)$$

Calculate the norm $\|f_2\|$. From the property (ii), it follows that

$$f_2(y_1, y_2) = L(f_1, Y) - 2(L(f_1, S_3) + L(f_1, S_4))$$

and

$$f_2(y_1, y_2) = 2(L(f_1, S_1) + L(f_1, S_2)) - L(f_1, Y).$$

From the last equalities and (4.18), we obtain that

$$|f_2(y_1, y_2)| \leq L(f_1, Y), \quad \text{for any} \quad (y_1, y_2) \in Y.$$
On the other hand, one can check that

$$f_2(c_1, c_2) = f_2(d_1, d_2) = L(f_1, Y)$$  \hspace{1cm} (4.20)

and

$$f_2(c_1, d_2) = f_2(d_1, c_2) = -L(f_1, Y).$$  \hspace{1cm} (4.21)

Therefore,

$$\|f_2\| = L(f_1, Y).$$  \hspace{1cm} (4.22)

Note that the points $(c_1, c_2), (c_1, d_2), (d_1, d_2), (d_1, c_2)$ in the given order form a closed bolt. We conclude from (4.20)-(4.22) that $z_0 = 0$ is a best approximation to $f_2$. Hence

$$E(f_2, Z) = L(f_1, Y).$$  \hspace{1cm} (4.23)

Now from (4.17),(4.19) and (4.23) we finally obtain that

$$E(f, M) \geq L(f_1, Y) = \frac{1}{4}(f_1(c_1, c_2) + f_1(d_1, d_2) - f_1(c_1, d_2) - f_1(d_1, c_2)).$$

The last inequality completes the proof.

Let, for example, $(a_1, a_2)$ and $(b_1, b_2)$ be the coordinate vectors $(1,0)$ and $(0,1)$, correspondingly. As a set $\Omega$ take the unit square $[0,1]^2$. Let $f_0(x_1, x_2) = (x_1 - \frac{1}{2})(x_2 - \frac{1}{2})$. This function satisfies all the conditions of Theorem 4.7. The approximating set of networks $M$ has members of the form

$$\sum_{i=1}^{n_1} c_i \sigma(k_i x_1 - \theta_i) + \sum_{j=1}^{n_2} d_j \sigma(t_j x_2 - \lambda_j),$$  \hspace{1cm} (4.24)

where $c_i, d_j, \theta_i, \lambda_j$ are arbitrary real numbers, $k_i$ and $t_j$ are real numbers different from zero and $n_1, n_2$ are positive integers. Applying Theorem 4.7, we obtain that the error of approximation $E(f_0, M)$ of the function $f_0$ by networks of the form (4.24) is not less than $\frac{1}{4}$. On the other hand note that $E(f_0, M) \leq \|f_0\| = \frac{1}{4}$. Thus, $E(f_0, M) = \frac{1}{4}$.

Now we give a sufficient condition for a network with weights from the set of two lines (through the origin) to be an extremal element. To make the problem more precise, fix a function $\sigma \in C(\mathbb{R})$ and vectors $a_1, a_2 \in \mathbb{R}^d \setminus \{0\}$. Consider neural networks from the set $\mathcal{M}(\sigma; W, \mathbb{R})$, where $W = \{k_1 a_1, k_2 a_2: k_1, k_2 \in \mathbb{R}\}$. Let $f(x)$ be a given continuous function on some
compact subset $Q$ of $\mathbb{R}^d$. We want to find sufficient conditions for a network $\Xi \in \mathcal{M}_Q(\sigma; W, \mathbb{R})$ to be an extremal element (or a best approximation) to $f$.

The following theorem is valid.

**Theorem 4.8.** Let $Q$ be a compact subset of $\mathbb{R}^d$. A network $\Xi(x) \in \mathcal{M}_Q(\sigma; W, \mathbb{R})$ is extremal to the given function $f(x) \in C(Q)$ if there exists a closed or infinite path $l = (p_1, p_2, ...)$ such that $f(p_i) - \Xi(p_i) = (-1)^i \| f - \Xi \|, i = 1, 2, ...$ or $f(p_i) - \Xi(p_i) = (-1)^{i+1} \| f - \Xi \|, i = 1, 2, ...$

**Proof.** Let $f \in C(Q)$, $\Xi \in \mathcal{M}_Q(\sigma; W, \mathbb{R})$, $l = (p_1, p_2, ..., p_{2n})$ be a closed path in $Q$ and $f(p_i) - \Xi(p_i) = (-1)^i \| f - \Xi \|, i = 1, 2, ...$ or $f(p_i) - \Xi(p_i) = (-1)^{i+1} \| f - \Xi \|, i = 1, 2, ...$

Consider the functional

$$G_l(f) = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^{k+1} f(p_k).$$

Note that for any network $g \in \mathcal{M}_Q(\sigma; W, \mathbb{R})$, $G_l(g) = 0$. That is, the functional $G_l$ belongs to the annihilator of the set $\mathcal{M}_Q(\sigma; W, \mathbb{R})$.

It can be easily verified that

$$|G_l(f)| = \| f - \Xi \|. \quad (4.25)$$

and

$$|G_l(f)| \leq E(f). \quad (4.26)$$

It follows from (4.25),(4.26) and the definition of $E(f)$ that $\Xi$ is an extremal element.

Let now a path $l = (p_1, p_2, ..., p_n, ...)$ be infinite and $f(p_i) - \Xi(p_i) = (-1)^i \| f - \Xi \|, i = 1, 2, ...$ or $f(p_i) - \Xi(p_i) = (-1)^{i+1} \| f - \Xi \|, i = 1, 2, ...$

Without loss of generality we may assume that all the points $p_i$ are distinct (in other case, we could form a closed path and prove in a few lines as above that $\Xi$ is an extremal element). Consider the sequence $l_n = (p_1, p_2, ..., p_n)$, $n = 1, 2, ...$, of finite paths and the path functionals

$$F_{l_n}(f) = \frac{1}{n} \sum_{i=1}^{n} (-1)^{i-1} f(p_i).$$

Unlike $G_l$, these functionals do not belong to the annihilator of the set $\mathcal{M}_Q(\sigma; W, \mathbb{R})$. But it can be easily verified that $\| F_{l_n} \| = 1$ for all $n \in N$. Indeed, $\| F_{l_n}(w) \| \leq \| w \|$ for all continuous functions $w$ over $Q$ and $\| F_{l_n}(w_0) \| = 140$
\[ \|w_0\| \] for a continuous function taking values +1 at the points \( p_i \in l_n \) with odd indices \( i \), −1 at the points \( p_i \in l_n \) with even indices \( i \) and from the interval \((-1;1)\) at all other points of \( Q \). By the well-known result of functional analysis (any bounded set in \( E^* \), dual for a separable Banach space \( E \), is precompact in the weak* topology), the sequence \( \{F_{l_n}\}_{n=1}^{\infty} \) has a weak* cluster point. Denote it by \( F \). Note that for any \( n \in N \)

\[ |F_{l_n}(g_1 + g_2)| \leq \frac{2}{n}(\|g_1\| + \|g_2\|), \]

where \( g_1 = g_1(a^1 \cdot x) \) and \( g_2 = g_2(a^2 \cdot x) \) are arbitrary ridge functions with directions \( a^1 \) and \( a^2 \). Besides, it is clear that \( \|F\| \leq 1 \). Since every network \( g \in \mathcal{M}_Q(\sigma; W, \mathbb{R}) \) can be represented as a sum \( g_1(a^1 \cdot x) + g_2(a^2 \cdot x) \), it follows from the last two properties of the functional \( F \) that

\[ |F(f)| = |F(f - g)| \leq \|f - g\|, \tag{4.27} \]

for all \( g \in \mathcal{M}_Q(\sigma; W, \mathbb{R}) \). Taking \( \inf \) over \( g \) in the left-hand side of (4.27), we obtain that

\[ |F(f)| \leq E(f). \tag{4.28} \]

Since at the points \( p_1, p_2, ..., p_n, ... \) the function \( f(x) - \Xi(x) \) reaches alternatively its minimal and maximal values, for each finite path \( l_n = (p_1, p_2, ..., p_n), n \in \mathbb{N} \),

\[ |F_{l_n}(f - \Xi)| = \|f - \Xi\|. \]

Hence

\[ |F(f)| = |F(f - \Xi)| = \|f - \Xi\|. \tag{4.29} \]

Now by (4.28) and (4.29), we finally conclude that \( \Xi \) is an extremal element.

\[ \square \]

### 4.2 Two hidden layer neural networks

A single hidden layer perceptron is able to approximate a given data with any degree of accuracy. But in applications it is necessary to define how many neurons one should take in a hidden layer. The more the number of
neurons, the more the probability of the network to give precise results. Un-
fortunately, practicality decreases with the increase of the number of neurons
in the hidden layer. In other words, single hidden layer perceptrons are not
always effective if the number of neurons in the hidden layer is prescribed. In
this section, we show that this phenomenon is no longer true for perceptrons
with two hidden layers. We prove that a two hidden layer neural network
with \( d \) inputs, \( d \) neurons in the first hidden layer, \( 2d+2 \) neurons in the second
hidden layer and with a specifically constructed sigmoidal and infinitely dif-
ferentiable activation function can approximate any continuous multivariate
function with arbitrary accuracy.

4.2.1 Relation of the Kolmogorov superposition theo-
rem to two hidden layer neural networks

Note that if \( r \) is fixed in (4.1), then the set

\[
\mathcal{M}_r(\sigma) = \left\{ \sum_{i=1}^r c_i \sigma(w^i \cdot x - \theta_i) : c_i, \theta_i \in \mathbb{R}, w \in \mathbb{R}^d \right\}
\]

is no longer dense in in the space \( C(\mathbb{R}^d) \) (in the topology of uniform conver-
gence on compact sets) for any activation function \( \sigma \). The set \( \mathcal{M}_r(\sigma) \) will
not be dense even if we variate also over all univariate continuous functions
\( \sigma \) (see [95, theorem 5.1]). In the following, we will see that this property
of single hidden layer neural networks does not carry over to networks with
more than one hidden layer.

A two hidden layer network is defined by iteration of the one hidden layer
model. The output of two hidden layer perceptron with \( r \) units in the first
layer, \( s \) units in the second layer and the input \( x = (x_1, \ldots, x_d) \) is

\[
\sum_{i=1}^s d_i \sigma \left( \sum_{j=1}^r c_{ij} \sigma(w^{ij} \cdot x - \theta_{ij}) - \gamma_i \right).
\]

Here \( d_i, c_{ij}, \theta_{ij}, \gamma_i \) are real numbers, \( w^{ij} \) are vectors of \( \mathbb{R}^d \) and \( \sigma \) is a fixed
univariate function.

In many applications, it is very convenient to take \( \sigma \) as a function tending
to 0 at minus infinity and 1 at infinity. In neural network literature, such
functions are called sigmoidal functions.
In this section, we prove that there exist neural networks with infinitely differentiable sigmoidal activation function \( \sigma \), \( d \) units in the first layer, \( 2d + 2 \) units in the second layer and having the ability to approximate every continuous multivariate function with arbitrary accuracy. The idea behind the proof of this result is very much connected to the Kolmogorov superposition theorem (see Section 3.1). This theorem has been much discussed in neural network literature (see, e.g., [119]). In our opinion, the most remarkable application of the Kolmogorov superposition theorem to neural networks was given by Maiorov and Pinkus [99]. They showed that there exists a sigmoidal, strictly increasing, analytic activation function for which a fixed number of units in both hidden layers are sufficient to approximate arbitrarily well any continuous multivariate function. Namely, the authors of [99] proved the following theorem.

**Theorem 4.9 (Maiorov and Pinkus [99]).** There exists an activation function \( \sigma \) which is analytic, strictly increasing and sigmoidal and has the following property: For any \( f \in C[0,1]^d \) and \( \varepsilon > 0 \), there exist constants \( d_i \), \( c_{ij} \), \( \theta_{ij} \), \( \gamma_i \), and vectors \( w^{ij} \in \mathbb{R}^d \) for which

\[
\left| f(x) - \sum_{i=1}^{6d+3} d_i \sigma \left( \sum_{j=1}^{3d} c_{ij} \sigma(w^{ij} \cdot x - \theta_{ij}) - \gamma_i \right) \right| < \varepsilon \tag{4.30}
\]

for all \( x = (x_1, ..., x_d) \in [0,1]^d \).

This theorem is based on the following version of the Kolmogorov superposition theorem given by Lorentz [98] and Sprecher [128].

**Theorem 4.10 (Kolmogorov’s superposition theorem).** For the unit cube \( I^d \), \( I = [0,1] \), \( d \geq 2 \), there exists constants \( \lambda_q > 0 \), \( q = 1, ..., d \), \( \sum_{q=1}^{d} \lambda_q = 1 \), and nondecreasing continuous functions \( \phi_p : [0,1] \to [0,1] \), \( p = 1, ..., 2d + 1 \), such that every continuous function \( f : I^d \to \mathbb{R} \) admits the representation

\[
f(x_1, ..., x_d) = \sum_{p=1}^{2d+1} g \left( \sum_{q=1}^{d} \lambda_q \phi_p(x_q) \right) \tag{4.31}
\]

for some \( g \in C[0,1] \) depending on \( f \).
In the next subsection, using the general ideas developed in [99], we show that the bounds of units in hidden layers in (4.30) may be chosen even equal to the bounds in the Kolmogorov superposition theorem. More precisely, these bounds can be taken as $2d + 2$ and $d$ instead of $6d + 3$ and $3d$. To attain this purpose, we change the "analyticity" of $\sigma$ to "infinite differentiability". In addition, near infinity we assume that $\sigma$ is "$\lambda$-strictly increasing" instead of being "strictly increasing".

4.2.2 The main result

We begin this subsection with a definition of a $\lambda$-monotone function. Let $\lambda$ be any nonnegative number. A real function $f$ defined on $(a; b)$ is called $\lambda$-increasing ($\lambda$-decreasing) if there exists an increasing (decreasing) function $u : (a, b) \rightarrow \mathbb{R}$ such that $|f(x) - u(x)| \leq \lambda$, for all $x \in (a, b)$. If $u$ is strictly increasing (or strictly decreasing), then the above function $f$ is called a $\lambda$-strictly increasing (or $\lambda$-strictly decreasing) function. Clearly, $0$-monotonicity coincides with the usual concept of monotonicity and a $\lambda_1$-monotone function is $\lambda_2$-monotone if $\lambda_1 \leq \lambda_2$. It is also clear from the definition that a $\lambda$-monotone function behaves like a usual monotone function as $\lambda$ gets very small.

Our purpose is to prove the following theorem.

**Theorem 4.11.** For any positive numbers $\alpha$ and $\lambda$, there exists a $C^\infty(\mathbb{R})$, sigmoidal activation function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ which is strictly increasing on $(-\infty, \alpha)$, $\lambda$-strictly increasing on $[\alpha, +\infty)$, and satisfies the following property: For any $f \in C[0, 1]^d$ and $\varepsilon > 0$, there exist constants $d_p, c_{pq}, \theta_{pq}, \gamma_p$, and vectors $w^{pq} \in \mathbb{R}^d$ for which

$$
|f(x) - \sum_{p=1}^{2d+2} d_p \sigma \left( \sum_{q=1}^d c_{pq} \sigma (w^{pq} \cdot x - \theta_{pq}) - \gamma_p \right)| < \varepsilon \quad (4.32)
$$

for all $x = (x_1, ..., x_d) \in [0, 1]^d$.

**Proof.** Let $\alpha$ be any positive number. Divide the interval $[\alpha, +\infty)$ into the segments $[\alpha, 2\alpha], [2\alpha, 3\alpha], ...$. Let $h(t)$ be any strictly increasing, infinitely differentiable function on $[\alpha, +\infty)$ with the properties
1) \(0 < h(t) < 1\) for all \(t \in [\alpha, +\infty)\);
2) \(1 - h(\alpha) \leq \lambda\);
3) \(h(t) \to 1\), as \(t \to +\infty\).

The existence of a strictly increasing smooth function satisfying these properties is easy to verify. Note that from conditions (1)-(3) it follows that any function \(f(t)\) satisfying the inequality \(h(t) < f(t) < 1\) for all \(t \in [\alpha, +\infty)\), is \(\lambda\)-strictly increasing and \(f(t) \to 1\), as \(t \to +\infty\).

We are going to construct \(\sigma\) obeying the required properties in stages. Let \(\{u_n(t)\}_{n=1}^{\infty}\) be the sequence of all polynomials with rational coefficients defined on \([0, 1]\). First, we define \(\sigma\) on the closed intervals \([(2m-1)\alpha, 2m\alpha]\), \(m = 1, 2, ...,\) as the function

\[
\sigma(t) = a_m + b_m u_m(t) \left(\frac{t}{\alpha} - 2m + 1\right), \quad t \in [(2m-1)\alpha, 2m\alpha],
\]

or equivalently,

\[
\sigma(\alpha t + (2m-1)\alpha) = a_m + b_m u_m(t), \quad t \in [0, 1],
\]

where \(a_m\) and \(b_m \neq 0\) are appropriately chosen constants. These constants are determined from the condition

\[
h(t) < \sigma(t) < 1,
\]

for all \(t \in [(2m-1)\alpha, 2m\alpha]\). There is a simple procedure for determining a suitable pair of \(a_m\) and \(b_m\). Indeed, let

\[
M = \max h(t), \quad A_1 = \min u_m(t) \left(\frac{t}{\alpha} - 2m + 1\right), \quad A_2 = \max u_m(t) \left(\frac{t}{\alpha} - 2m + 1\right),
\]

where in all the above max and min, the variable \(t\) runs over the closed interval \([(2m-1)\alpha, 2m\alpha]\). Note that \(M < 1\). If \(A_1 = A_2\) (that is, if the function \(u_m\) is constant on \([0, 1]\)), then we can set \(\sigma(t) = (1 + M)/2\) and easily find a suitable pair of \(a_m\) and \(b_m\) from (4.33). Let now \(A_1 \neq A_2\) and \(y = a + bx, \ b \neq 0\), be a linear function mapping the segment \([A_1, A_2]\) into \((M, 1)\). Then it is enough to take \(a_m = a\) and \(b_m = b\).

At the second stage we define \(\sigma\) on the intervals \([2m\alpha, (2m+1)\alpha]\), \(m = 1, 2, ...,\) so that it is in \(C^{\infty}(\mathbb{R})\) and satisfies the inequality (4.35). Finally, in all of \((-\infty, \alpha)\) we define \(\sigma\) while maintaining the \(C^{\infty}\) strict monotonicity.
property, and also in such a way that \( \lim_{t \to -\infty} \sigma(t) = 0 \). We obtain from the properties of \( h \) and the condition (4.35) that \( \sigma(t) \) is a \( \lambda \)-strictly increasing function on the interval \([\alpha, +\infty)\) and \( \sigma(t) \to 1 \), as \( t \to +\infty \).

From the above construction of \( \sigma \), that is, from (4.34) it follows that for each \( m = 1, 2, ..., \) there exists numbers \( A_m, B_m \) and \( r_m \) such that

\[
u_m(t) = A_m \sigma(\alpha t - r_m) - B_m, \tag{4.36}\]

where \( A_m \neq 0 \).

Let \( f \) be any continuous function on the unit cube \([0, 1]^d\). By the Kolmogorov superposition theorem the expansion (4.31) is valid for \( f \). For the exterior continuous univariate function \( g(t) \) in (4.31) and for any \( \varepsilon > 0 \) there exists a polynomial \( u_m(t) \) of the above form such that

\[
|g(t) - u_m(t)| < \frac{\varepsilon}{2(2d + 1)},
\]

for all \( t \in [0, 1] \). This together with (4.36) means that

\[
|g(t) - [a \sigma(\alpha t - r) - b]| < \frac{\varepsilon}{2(2d + 1)}, \tag{4.37}
\]

for some \( a, b, r \in \mathbb{R} \) and all \( t \in [0, 1] \).

Substituting (4.37) in (4.31) we obtain that

\[
\left| f(x_1, ..., x_d) - \sum_{p=1}^{2d+1} \left( a \sigma \left( \alpha \sum_{q=1}^{d} \lambda_q \phi_p(x_q) - r \right) - b \right) \right| < \frac{\varepsilon}{2}, \tag{4.38}
\]

for all \((x_1, ..., x_d) \in [0, 1]^d\).

For each \( p \in \{1, 2, ..., 2d + 1\} \) and \( \delta > 0 \) there exist constants \( a_p, b_p \) and \( r_p \) such that

\[
|\phi_p(x_q) - [a_p \sigma(\alpha x_q - r_p) - b_p]| < \delta, \tag{4.39}
\]

for all \( x_q \in [0, 1] \). Since \( \lambda_q > 0, q = 1, ..., d, \sum_{q=1}^{d} \lambda_q = 1 \), it follows from (4.39) that

\[
\left| \sum_{q=1}^{d} \lambda_q \phi_p(x_q) - \sum_{q=1}^{d} \lambda_q a_p \sigma(\alpha x_q - r_p) - b_p \right| < \delta, \tag{4.40}
\]

for all \((x_1, ..., x_d) \in [0, 1]^d\).
Now since the function \( a\sigma(\alpha t - r) \) is uniformly continuous on every closed interval, we can choose \( \delta \) sufficiently small and obtain from (4.40) that

\[
\left| \sum_{p=1}^{2d+1} a\sigma \left( \alpha \cdot \sum_{q=1}^{d} \lambda_q \phi_p(x_q) - r \right) - \sum_{p=1}^{2d+1} a\sigma \left( \alpha \cdot \left[ \sum_{q=1}^{d} \lambda_q a_p \sigma(\alpha x_q - r_p) - b_p \right] - r \right) \right| < \frac{\varepsilon}{2}.
\]

This inequality may be rewritten as

\[
\left| \sum_{p=1}^{2d+1} a\sigma \left( \alpha \cdot \sum_{q=1}^{d} \lambda_q \phi_p(x_q) - r \right) - \sum_{p=1}^{2d+1} d_p \sigma \left( \sum_{q=1}^{d} c_{pq} \sigma(\mathbf{w}_{pq} \cdot \mathbf{x} - \theta_{pq}) - \gamma_p \right) \right| < \frac{\varepsilon}{2}.
\]

From (4.38) and (4.41) it follows that

\[
\left| f(\mathbf{x}) - \left[ \sum_{p=1}^{2d+1} d_p \sigma \left( \sum_{q=1}^{d} c_{pq} \sigma(\mathbf{w}_{pq} \cdot \mathbf{x} - \theta_{pq}) - \gamma_p \right) - s \right] \right| < \varepsilon,
\]

where \( s = (2d + 1)b \). Since the constant \( s \) can be written in the form

\[
s = d\sigma \left( \sum_{q=1}^{d} c_q \sigma(\mathbf{w}_q \cdot \mathbf{x} - \theta_q) - \gamma \right),
\]

from (4.42) we finally obtain the validity of (4.32).

The next theorem follows from Theorem 4.11 easily, since the Kolmogorov superposition theorem is valid for all compact sets of \( \mathbb{R}^d \).

**Theorem 4.12.** Let \( Q \) be a compact set in \( \mathbb{R}^d \). For any numbers \( \alpha \in \mathbb{R} \) and \( \lambda > 0 \), there exists a \( C^\infty(\mathbb{R}) \), sigmoidal activation function \( \sigma : \mathbb{R} \to \mathbb{R} \) which is strictly increasing on \((-\infty, \alpha)\), \( \lambda \)-strictly increasing on \([\alpha, +\infty)\), and satisfies the following property: For any \( f \in C(Q) \) and \( \varepsilon > 0 \) there exist real numbers \( d_i, c_{ij}, \theta_{ij}, \gamma_i \), and vectors \( \mathbf{w}_{ij} \in \mathbb{R}^d \) for which

\[
\left| f(\mathbf{x}) - \sum_{i=1}^{2d+2} d_i \sigma \left( \sum_{j=1}^{d} c_{ij} \sigma(\mathbf{w}_{ij} \cdot \mathbf{x} - \theta_{ij}) - \gamma_i \right) \right| < \varepsilon
\]

for all \( \mathbf{x} = (x_1, \ldots, x_d) \in Q \).
Remark 4.3. It is easily seen in the proof of Theorem 4.11 that all the weights \( w_{ij} \) are fixed (see (4.41)). Namely, \( w_{ij} = \alpha e^j \), for all \( i = 1,...,2d+2 \), \( j = 1,...,d \), where \( e^j \) is the \( j \)-th coordinate vector of the space \( \mathbb{R}^d \).

Remark 4.4. In some literature, a single hidden layer perceptron is defined as the function

\[
\sum_{i=1}^{r} c_i \sigma (w^i \cdot x - \theta_i) - c_0.
\]

A two hidden layer network then takes the form

\[
\sum_{i=1}^{s} d_i \sigma \left( \sum_{j=1}^{r} c_{ij} \sigma (w^{ij} \cdot x - \theta_{ij}) - \gamma_i \right) - d_0. \tag{4.43}
\]

The proof of Theorem 4.11 shows that for networks of type (4.43) the theorem is valid if we take \( 2d+1 \) neurons in the second hidden layer (instead of \( 2d+2 \) neurons as above). That is, there exist networks of type (4.43) having the universal approximation property and for which the number of units in the hidden layers is equal to the number of summands in the Kolmogorov superposition theorem.

Remark 4.5. It is known that the \( 2d+1 \) in the Kolmogorov superposition theorem is minimal (see Sternfeld [130]). Thus it is doubtful if the number of neurons in Theorems 4.11 and 4.12 can be reduced.

Remark 4.6. Inequality (4.37) shows that single hidden layer neural networks of the form (4.43) with the activation function \( \sigma \) and with only one neuron in the hidden layer can approximate any continuous function on the interval \([0,1]\) with arbitrary precision. Since the number \( b \) in (4.37) can always be written as \( b = a_1 \sigma (0 \cdot t - r_1) \) for some \( a_1 \) and \( r_1 \), we see that two neurons in the hidden layer are sufficient for traditional single hidden layer neural networks with the activation function \( \sigma \) to approximate continuous functions on \([0,1]\). Applying the linear transformation \( x = a + (b-a)t \) it can be proven that the same argument holds for any interval \([a,b]\).

Remark 4.7. Inequality (4.37) and Theorem 4.12 show only existence of an infinitely differentiable, almost monotone, sigmoidal activation function such that the corresponding single hidden layer neural networks with
two hidden neurons and two hidden layer neural networks with \(3d + 2\) hidden neurons can approximate arbitrarily well any continuous univariate and \(d\)-variable function, respectively. In [39,40], such a sigmoidal function was constructed algorithmically for both single and two hidden layer neural networks, and the applicability of the obtained results were illustrated with various numerical examples.
References

[1] Aliev R.A., Ismailov V.E., On a smoothness problem in ridge function representation, Adv. Appl. Math. 73 (2016), 154-169.
[2] Aliev R.A., Asgarova A.A., Ismailov V.E., A note on continuous sums of ridge functions, J. Approx. Theory 237 (2019), 210-221.
[3] Arnold V.I., On functions of three variables, (Russian) Dokl. Akad. Nauk SSSR 114 (1957), 679-681; English transl. in Amer. Math. Soc. Transl. 28 (1963), 51-54.
[4] Aumann G., Approximative nomographie, II, Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. (1959), 103-109.
[5] Babaev M-B.A., Ismailov V.E., Two-sided estimates for the best approximation in domains different from the parallelepiped, Funct. Approx. Comment. Math. 25(1997), 121-128.
[6] Babaev M-B.A., Approximation of polynomials in two variables by functions of the form $\varphi(x) + \psi(y)$, (Russian) Dokl. Akad. Nauk. SSSR 193 (1970), 967-969; English transl. in Soviet. Math. Dokl. 11 (1970), 1034-1036.
[7] Babaev M-B.A., Extremal elements and the value of the best approximation of a monotone function on $\mathbb{R}^n$ by sums of functions of fewer variables, (Russian) Dokl. Akad. Nauk. SSSR 265 (1982), 11-13; English transl. in Soviet. Math. Dokl. 26 (1982), 1-4.
[8] Babaev M-B.A., On obtaining close estimates in the approximation of functions of many variables by sums of functions of a fewer number of variables, (Russian) Mat. Zametki 12 (1972), 105-114; English transl. in Math. Notes of the Acad. of Sciences of the USSR 12 (1972), 495-500.
[9] Bank R.E., An automatic scaling procedure for a D'Yakanov-Cunn iteration scheme, Linear Algebra Appl., 28 (1979), 17-33.
[10] Braess D., Pinkus A., Interpolation by ridge functions, J. Approx. Theory 73 (1993), 218-236.
[11] Buck R.C., On approximation theory and functional equations, J. Approx. Theory, 5 (1972), 228-237.
[12] Buhmann M.D., Pinkus A., Identifying linear combinations of ridge functions, Adv. in Appl. Math. 22 (1999), 103-118.
[13] Candes E.J., Ridgelets: estimating with ridge functions, Ann. Statist. 31 (2003), 1561-1599.
[14] Candes E.J., Ridgelets: theory and applications. Ph.D. Thesis, Technical Report, Department of Statistics, Stanford University.
[15] Cao F., Lin S., Xu Z., Approximation capability of interpolation neural networks, Neurocomputing 74 (2010), 457-460.
[16] Cao F., Lin S., Xu Z., Constructive approximate interpolation by neural networks in the metric space, Math. Comput. Modelling 52 (2010), 1674-1681.
[17] Chen T., Chen H., Approximation of continuous functionals by neural networks with application to dynamic systems, IEEE Trans. Neural Networks 4 (1993), 910-918.
[18] Cheney E.W., Light W.A., Approximation Theory in Tensor Product Spaces. Lecture Notes in Math., 1169, Springer-Verlag, Berlin, 1985.
[19] Costarelli D., Spigler R, Constructive approximation by superposition of sigmoidal functions, Anal. Theory Appl. 29 (2013), no. 2, 169-196.
[20] Costarelli D., Spigler R, Approximation results for neural network operators activated by sigmoidal functions, Neural Networks 44 (2013), 101-106.
[21] Cybenko G., Approximation by superpositions of a sigmoidal function, Math. Control, Signals, and Systems 2 (1989), 303-314.
[22] Dahmen W., Micchelli C.A., Some remarks on ridge functions, Approx. Theory Appl. 3 (1987), 139-143.
[23] Demko S., A superposition theorem for bounded continuous functions, Proc. Amer. Math. Soc. 66 (1977), 75-78.
[24] DeVore R.A., Oskolkov K.I., Petrushev P.P., Approximation by feed-forward neural networks, Ann. Numer. Math. 4 (1997), 261-287.
[25] Diaconis P., Shahshahani M., On nonlinear functions of linear combinations, SIAM J. Sci. Stat. Comput. 5 (1984), 175-191.
[26] Diliberto S.P., Straus E.G., On the approximation of a function of several variables by the sum of functions of fewer variables, Pacific J. Math. 1 (1951), 195-210.
[27] Donoho D.L., Johnstone I.M., Projection-based approximation and a duality method with kernel methods, Ann. Statist. 17 (1989), 58-106.
[28] Dyn N., A straightforward generalization of Diliberto and Straus algorithm does not work, J. Approx. Theory, 30 (1980), 247-250.
[29] Dyn N., Light W.A., Cheney E.W., Interpolation by piecewise-linear radial basis functions, J. Approx. Theory. 59 (1989), 202-223.
[30] Flatto L., The approximation of certain functions of several variables by sums of functions of fewer variables, Amer. Math. Monthly 73 (1966), 131-132.
[31] Fridman B.L., An improvement in the smoothness of the functions in A. N. Kolmogorov’s theorem on superpositions, (Russian) Dokl. Akad. Nauk SSSR 177 (1967), 1019-1022.
[32] Friedman J.H., Tukey J.W., A Projection Pursuit Algorithm for Exploratory Data Analysis, IEEE Transactions on Computers C-23 (1974), 881-890.
[33] Friedman J.H., Stuetzle W., Projection pursuit regression, J. Amer. Statist. Assoc. 76 (1981), 817-823.
[34] Garkavi A.L., Medvedev V.A., Khavinson S.Ya., On the existence of a best uniform approximation of functions of two variables by sums of the type \( \varphi(x) + \psi(y) \), (Russian) Sibirskii Mat. Zh., 36 (1995), 819-827; English transl. in Siberian Math. J., 36 (1995), 707-713.
[35] Garkavi A.L., Medvedev V.A., Khavinson S.Ya., On the existence of a best uniform approximation of a function of several variables by the sum of functions of fewer variables, (Russian) Mat. Sbornik 187 (1996), 3-14; English transl. in Sbornik Math. 187 (1996), 623-634.
[36] Golitschek M.v., Light W.A., Approximation by solutions of the planar wave equation, SIAM J. Numer. Anal. 29 (1992), 816-830.
[37] Golomb M., Approximation by functions of fewer variables, On numerical approximation. Proceedings of a Symposium. Madison 1959. Edited by R.E.Langer. The University of Wisconsin Press, 275-327.
[38] Gordon Y., Maiorov V., Meyer M., Reisner S., On the best approximation by ridge functions in the uniform norm, Constr. Approx. 18 (2002), 61-85.
[39] Guliyev N.J., Ismailov V.E., On the approximation by single hidden layer feedforward neural networks with fixed weights, Neural Networks 98 (2018), 296-304.
[40] Guliyev N.J., Ismailov V.E., Approximation capability of two hidden layer feedforward neural networks with fixed weights, Neurocomputing 316 (2018), 262-269.
[41] Hornik K., Approximation capabilities of multilayer feedforward networks, Neural Networks 4 (1991), 251-257.
[42] Huber P. J., Projection pursuit, Ann. Statist. 13 (1985), 435-475.
[43] Ismailov V.E., Approximation by neural networks with weights varying on a finite set of directions, J. Math. Anal. Appl. 389 (2012), 72-83.
[44] Ismailov V.E., A note on the representation of continuous functions by linear superpositions, Expo. Math. 30 (2012), 96-101.
[45] Ismailov V.E., Approximation by Neural Networks with a Restricted Set of Weights, book chapter (chapter 6), in: Advances in Mathematics Research (NOVA Science Publishers, USA), Volume 16, 2011, pp. 193-206.
[46] Ismailov V.E., Approximation capabilities of neural networks with weights from two directions. Azerb. J. Math. 1 (2011), no. 1, 122–128.
[47] Ismailov V.E., On the proximinality of ridge functions, Sarajevo J. Math. 5(17) (2009), no. 1, 109-118.
[48] Ismailov V.E., On the theorem of M. Golomb, Proc. Indian Acad. Sci. Math. Sci. 119 (2009), no. 1, 45-52.
[49] Ismailov V.E., On the representation by linear superpositions, J. Approx. Theory 151 (2008), 113-125.
[50] Ismailov V.E., On the approximation by weighted ridge functions. Analele Universitatii de Vest din Timisoara, Ser. Mat.-Inform. 46 (2008), 75–83.
[51] Ismailov V.E., On the approximation by compositions of fixed multivariate functions with univariate functions, Studia Mathematica 183 (2007), 117-126.
[52] Ismailov V.E., On the best $L_2$ approximation by ridge functions, Appl. Math. E-Notes, 7 (2007), 71-76.
[53] Ismailov V.E., Representation of multivariate functions by sums of ridge functions, J. Math. Anal. Appl. 331 (2007), 184-190.
[54] Ismailov V.E., Characterization of an extremal sum of ridge functions, J. Comput. Appl. Math. 205 (2007), 105-115.
[55] Ismailov V.E., On error formulas for approximation by sums of univariate functions, Int. J. Math. Math. Sci. 2006 (2006), Article ID 65620, 11 pp.
[56] Ismailov V.E., On methods for computing the exact value of the best approximation by sums of functions of one variable. (Russian), Sibirskii Mat. Zh. 47 (2006), no. 5, 1076-1082; English transl. Siberian Math. J. 47 (2006), no. 5, 883-888.
[57] Ismailov V.E., Approximation by sums of ridge functions with fixed directions, (Russian) Algebra i Analiz 28 (2016), no. 6, 20-69; English transl. in St. Petersburg Math. J. 28 (2017), no. 6, 741-772.

[58] Ismailov V.E., Savas E., Measure theoretic results for approximation by neural networks with limited weights, Numer. Funct. Anal. Optim. 38 (2017), no. 7, 819-830.

[59] Ismailov V.E., On some classes of bivariate functions characterized by formulas for the best approximation. Radovi Matematicki 13 (2004), 53-62.

[60] Ismailov V.E., On discontinuity of the best approximation of a continuous function. Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. 23 (2003), no. 4, 57-60.

[61] Ismailov V.E., Alternating algorithm for the approximation by sums of two compositions and ridge functions, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. 41 (2015), no. 1, 146-152.

[62] Ismailov V.E., Theorem on lightning bolts for elementary domains. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. 17 (2002), 78-85.

[63] Ismailov V.E., Pinkus A., Interpolation on lines by ridge functions, J. Approx. Theory 175 (2013), 91-113.

[64] Ismailov V.E., A note on the equioscillation theorem for best ridge function approximation, Expo. Math. 35 (2017), no. 3, 343-349.

[65] Ismailov V.E., On the approximation by neural networks with bounded number of neurons in hidden layers, J. Math. Anal. Appl. 417 (2014), no. 2, 963-969.

[66] Ismailov V.E., A review of some results on ridge function approximation, Azerb. J. Math. 3 (2013), no.1, 3-51.

[67] Ito Y., Approximation of functions on a compact set by finite sums of a sigmoid function without scaling, Neural Networks 4 (1991), no. 6, 817-826.

[68] Ito Y., Approximation of continuous functions on $\mathbb{R}^d$ by linear combinations of shifted rotations of a sigmoid function with and without scaling, Neural Networks 5 (1992), 105-115.

[69] John F., Plane Waves and Spherical Means Applied to Partial Differential Equations, Interscience, New York, 1955.

[70] Kainen P.C., Kürkova V., Vogt A., Best approximation by Heaviside perceptron networks, Neural Networks 13 (2007), no. 7, 695-697.

[71] Kainen P.C., Kürkova V., An Integral Upper Bound for Neural Network Approximation, Neural Computation 21 (2009), 2970-2989.

[72] Kazantsev I., Tomographic reconstruction from arbitrary directions using ridge functions, Inverse Problems 14 (1998), 635-645.
[73] Kazantsev I., Tomographic reconstruction using ridge functions, Proceedings of 1st World Congress on Industrial Process Tomography, Buxton, Derbishyre, UK, April 14-17, 1999, pp. 433-437.

[74] Kazantsev I., Lemahieu I., Reconstruction of elongated structures using ridge functions and natural pixels, Inverse Problems 16 (2000), 505-517.

[75] Kelley C.T., A note on the approximation of functions of several variables by sums of functions of one variable, J. Approx. Theory 13 (1981), no. 3, 179-189.

[76] Khavinson S.Ya., Best approximation by linear superpositions (approximate nomography), Translated from the Russian manuscript by D. Khavinson. Translations of Mathematical Monographs, 159. American Mathematical Society, Providence, RI, 1997, 175 pp.

[77] Khavinson, S.Ya., Some approximation properties of linear superpositions, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 39 (1995), 63–73; English transl. in Russian Math. (Iz. VUZ) 39 (1995), 60-70.

[78] Khavinson, S.Ya., Representation of functions of two variables by the sums $\varphi(x) + \psi(y)$, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1985, no. 2, 66–73; English transl. in Soviet Math. (Iz. VUZ) 29 (1985) no. 2, 81-90.

[79] Khavinson S.Ya., A Chebyshev theorem for the approximation of a function of two variables by sums of the type $\varphi(x) + \psi(y)$, (Russian) Izv. Akad. Nauk. SSSR Ser. Mat. 33 (1969), 650-666; English transl. in Math. USSR Izv. 3 (1969), 617-632.

[80] Klopotowski A., Nadkarni M.G., Bhaskara Rao K.P.S., Geometry of good sets in n-fold Cartesian product, Proc. Indian Acad. Sci. Math. Sci. 114 (2004), 181-197.

[81] Klopotowski A., Nadkarni M.G., Bhaskara Rao K.P.S., When is $f(x_1, x_2, ..., x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n)$?, Proc. Indian Acad. Sci. Math. Sci. 113 (2003), 77-86.

[82] Klopotowski A., Nadkarni M.G., Shift invariant measures and simple spectrum, Colloq. Math. 84/85 (2000), 385-394.

[83] Kolmogorov A.N., On the representation of continuous functions of many variables by superposition of continuous functions of one variable and addition. (Russian), Dokl. Akad. Nauk SSSR 114 (1957), 953-956; English transl. in Amer. Math. Soc. Transl. (2) 28 (1963), 55-59.

[84] Kolmogorov A.N., Asymptotic characteristics of some completely bounded metric spaces, (Russian) Dokl. Akad. Nauk SSSR 108 (1956), 585-589.
[85] Kolmogorov A.N., Tikhomirov V.M., $\varepsilon$-entropy and $\varepsilon$-capacity of sets in function spaces, (Russian) Uspehi Mat. Nauk no. 2 (86), 14 (1959), 3-86; English transl. in Amer. Math. Soc. Transl. (2) 17 (1961), 277-364.

[86] Konyagin S.V., Kuleshov A.A., On the continuity of finite sums of ridge functions, (Russian) Mat. Zametki 98 (2015), 308-309; English transl. in Math. Notes 98 (2015), 336-338.

[87] Konyagin S.V., Kuleshov A.A., Maiorov V.E., Some problems in the theory of ridge functions, (Russian) Tr. Mat. Inst. Steklova 301 (2018), Kompleksnyi Analiz, Matematicheskaya Fizika i Prilozheniya, 155-181; English transl. in Proc. Steklov Inst. Math. 301 (2018), no. 1, 144-169.

[88] Kuleshov A.A., On some properties of smooth sums of ridge functions, (Russian) Tr. Mat. Inst. Steklova 294 (2016), Sovremennye Problemy Matematiki, Mekhaniki i Matematicheskoi Fiziki. II, 99-104.

[89] Leshno M., Lin V.Ya., Pinkus A., Schocken S., Multilayer feedforward networks with a non-polynomial activation function can approximate any function, Neural Networks 6 (1993), 861-867.

[90] Li X., Interpolation by ridge polynomials and its application in neural networks, J. Comput. Appl. Math. 144 (2002) 197-209.

[91] Li H.X., Lee E.S., Interpolation functions of feedward neural networks, Comput. Math. Appl. 46 (2003), 1861-1874.

[92] Li H.X., Li L.X., Wang J.Y., Interpolation representation of feedforward neural networks, Math. Comput. Modelling 37 (2003) 829-847.

[93] Light W.A., Cheney E.W., On the approximation of a bivariate function by the sum of univariate functions, J. Approx. Theory 29 (1980), 305-323.

[94] Light W.A., Ridge functions, sigmoidal functions and neural networks. Approximation theory VII (Austin, TX, 1992), 163-206.

[95] Lin V.Ya, Pinkus A., Fundamentality of ridge functions, J. Approx. Theory 75 (1993), 295-311.

[96] Llanas B., Sainz F.J., Constructive approximate interpolation by neural networks, J. Comput. Appl. Math. 188 (2006) 283-308.

[97] Logan B.F., Shepp L.A., Optimal reconstruction of a function from its projections, Duke Math. J. 42 (1975), 645-659.

[98] Lorentz G.G., Metric entropy, widths, and superpositions of functions, Amer. Math. Monthly 69 (1962), 469–485.

[99] Maiorov V., Pinkus A., Lower bounds for approximation by MLP neural networks, Neurocomputing 25 (1999), 81-91.
[100] Maiorov V.E., Meir R., On the near optimality of the stochastic approximation of smooth functions by neural networks, Adv. Comput. Math. 13 (2000), no. 1, 79-103.

[101] Maiorov V., Meir R., Ratsaby J., On the approximation of functional classes equipped with a uniform measure using ridge functions, J. Approx. Theory 99 (1999), 95-111.

[102] Maiorov V.E., On best approximation by ridge functions, J. Approx. Theory 99 (1999), 68-94.

[103] Maiorov V.E., Geometric properties of the ridge function manifold, Adv. Comput. Math. 32 (2010), 239–253.

[104] Makovoz Y., Uniform approximation by neural networks. J. Approx. Theory 95 (1998), 215-228.

[105] Makovoz Y., Random approximants and neural networks. J. Approx. Theory 85 (1996), no. 1, 98-109.

[106] Marr R.B., On the reconstruction of a function on a circular domain from a sampling of its line integrals. J. Math. Anal. Appl. 45 (1974), 357-374.

[107] Marshall D.E. and O’Farrell A.G., Approximation by a sum of two algebras. The lightning bolt principle, J. Funct. Anal. 52 (1983), 353-368.

[108] Marshall D.E. and O’Farrell A.G., Uniform approximation by real functions, Fund. Math. 104 (1979), 203-211.

[109] Medvedev V.A., Refutation of a theorem of Diliberto and Straus, Mat. zametki 51 (1992), 78-80; English transl. in Math. Notes 51 (1992), 380-381.

[110] Mhaskar H.N., On the tractability of multivariate integration and approximation by neural networks, J. Complexity 20 (2004), 561-590.

[111] Natterer F., The Mathematics of Computerized Tomography, Wiley, New York, 1986.

[112] Navada K.G., Some remarks on good sets, Proc. Indian Acad. Sci. Math. Sci. 114 (2003), No.4, 389-397.

[113] Ofman Ju.P., Best approximation of functions of two variables by functions of the form \( \varphi(x) + \psi(y) \), (Russian) Izv. Akad. Nauk. SSSR Ser. Mat. 25 (1961), 239-252; English transl. in Amer. Math. Soc. Transl. 44 (1965), 12-28.

[114] Oskolkov K.I., Ridge approximation, Fourier-Chebyshev analysis, and optimal quadrature formulas, (Russian) Tr. Mat. Inst. Steklova 219 (1997), 269–285; English transl. in Proc. Steklov Inst. Math. 219 (1997), 265-280.
[115] Ostrand P.A., Dimension of metric spaces and Hilbert’s problem 13, Bull. Amer. Math. Soc. 71 (1965), 619-622.
[116] Petrushev P.P., Approximation by ridge functions and neural networks, SIAM J. Math. Anal. 30 (1998), 155-189.
[117] Pinkus A., Ridge functions, Cambridge Tracts in Mathematics, 205. Cambridge University Press, 2015, 207 pp.
[118] Pinkus A., Approximating by ridge functions, in: Surface Fitting and Multiresolution Methods, (A.Le Méhauté, C.Rabut and L.L.Schumaker, eds), Vanderbilt Univ.Press (Nashville), 1997, 279-292.
[119] Pinkus A., Approximation theory of the MLP model in neural networks, Acta Numerica 8 (1999), 143-195.
[120] Pinkus A., Smoothness and uniqueness in ridge function representation, Indag. Math. (N.S.) 24 (2013), no. 4, 725-738.
[121] Rivlin T.J., Sibner R.J., The degree of approximation of certain functions of two variables by a sum of functions of one variable, Amer. Math. Monthly 72 (1965), 1101-1103.
[122] Rudin W., Functional analysis. Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, 424 pp.
[123] Sanguineti M., Universal Approximation by Ridge Computational Models and Neural Networks: A Survey, The Open Applied Mathematics Journal 2 (2008), no. 1, 31-58.
[124] Schwartz L., Theorie generale des fonctions moyenne-periodiques, Ann. Math. 48 (1947), 857-928.
[125] Shin Y., Ghosh J., Ridge polynomial networks, IEEE Transactions on Neural Networks 6 (1995), 610-22.
[126] Sprecher D.A., On the existence of best approximations and representations in several variables, J. Reine Angew. Math. 234 (1969), 152-162.
[127] Sprecher D.A., A representation theorem for continuous functions of several variables, Proc. Amer. Math. Soc. 16 (1965), 200-203.
[128] Sprecher D.A., An improvement in the superposition theorem of Kolmogorov, J. Math. Anal. Appl. 38 (1972), 208–213.
[129] Sproston J.P. and Strauss D., Sums of subalgebras of C(X), J. London Math. Soc. 45 (1992), 265–278.
[130] Sternfeld Y., Dimension, superposition of functions and separation of points, in compact metric spaces, Israel J. Math. 50 (1985), 13-53.
[131] Sternfeld Y., Uniform separation of points and measures and representation by sums of algebras, Israel J. Math. 55 (1986), 350-362.
[132] Sternfeld Y., Uniformly separating families of functions, Israel J. Math. 29 (1978), 61-91.

[133] Sternfeld Y., Dimension theory and superpositions of continuous functions, Israel J. Math. 20 (1975), no. 3-4, 300-320.

[134] Sternfeld Y., Superpositions of continuous functions, J. Approx. Theory 25 (1979), no. 4, 360-368.

[135] Stinchcombe M. and White H., Approximating and learning unknown mappings using multilayer feedforward networks with bounded weights, in Proceedings of the IEEE 1990 International Joint Conference on Neural Networks, 1990, Vol. 3, IEEE, New York, 7-16.

[136] Strulovici B.H., Weber T.A., Additive Envelopes of Continuous Functions, Operations Research Letters 38 (2010), 165-168.

[137] Sun X., Cheney E.W., The fundamentality of sets of ridge functions, Aequationes Math. 44 (1992), 226-235.

[138] Temlyakov V.N., On approximation by ridge functions, Preprint. Department of Mathematics, University of South Carolina, 1996.

[139] Tikhomirov V.M., The works of A. N. Kolmogorov on ε-entropy of function classes and superpositions of functions, (Russian) Uspehi Mat. Nauk 18 (1963) no. 5 (113), 55-92; English transl. in Russian Math. Surveys 18 (1963), no. 5, 51-87.

[140] Trofimov V.N., Hariton L.R., On the error of uniform approximation of functions of two variables by a sum of functions of one variable, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1979, no. 8, 70-73; English transl. in Soviet Mathematics (Izvestiya VUZ. Matematika), 1979, no. 8, 71-74.

[141] Vitushkin A.G., Henkin G.M., Linear superpositions of functions. (Russian), Uspehi Mat. Nauk 22 (1967), no. 1 (133), 77-124; English transl. in Russian Math. Surveys 22 (1967), no. 1, 77-125.

[142] Vostrecov B.A., Kreines M.A., Approximation of continuous functions by superpositions of plane waves, (Russian) Dokl. Akad. Nauk SSSR 140 (1961), 1237-1240; English transl. in Soviet Math. Dokl. 2 (1961), 1326-1329.