I. INTRODUCTION

In this work we compute the critical exponents $\nu$, $\eta$ and $\omega$ for the $Z_2$ and $O(N)$ symmetric scalar models in $d = 3$ Euclidean dimensions. We use the exact renormalization group (ERG) equation for effective average action $[1]$. The exact renormalization group (ERG) is a highly versatile method for tackling problems in statistical physics and quantum field theory. Its modern formulation has sprouted from Wilson’s approach to renormalization $[2]$. There are a number of other ways in modern physics to obtain critical exponents. Perhaps the first one to come to mind is lattice simulation. The Monte-Carlo (MC) simulations provide one of the most precise determination of the exponents for the Ising $[3]$ and XY $[4]$ universality classes. Generally, a larger lattice yields more precise predictions, but also increases the computational effort. The most commonly applied method in quantum field theory is the loop-expansion, which requires a smallness of the couplings in the Lagrangian. In fixed $d = 3$ dimensions, the Ising exponents have been computed up to six-loop order $[5]$ and the beta functions are determined at seven loops $[6]$. Wilson’s $d = 4 - \epsilon$ expansion has also been applied up to $\epsilon^9$ $[7]$. Presently, the most precise computation for the Ising exponents comes from the conformal bootstrap method (CB) $[8]$ using conformal field theory. This method also has a high computational cost, see Tab. II. of $[1]$ for instance. The last highlight on this list is the large-$N$ expansion, which is also a technique based on perturbation theory. It is applicable on theories, where the symmetry group corresponding to the symmetry of the Lagrangian is $O(N)$, $SO(N)$, $SU(N)$ and so on, with the small parameter being $1/N$.

The ERG has three important features: (i) it contains no small parameter, (ii) the predictions are systematically improved by the derivative expansion and (iii) one has to choose an arbitrary regulator function with some loose requirements $[1]$. In the absence of a small parameter the convergence of the derivative expansion (DE) was questioned, but recently arguments have been put forward that the DE is indeed convergent $[10]$ at least for the $Z_2$ and $O(N)$ symmetric models. The corrections were shown to be dampened by a factor of $1/4 \sim 1/9$, depending on the regulator function. The physical predictions depend on the regulator function at fixed order in the DE and the dependence decreases with increasing the order in the DE. This is similar to the renormalization scale dependence in perturbative quantum field theory.

Here we test our method of computing the critical exponents at the next-to-next-to-leading order (NNLO) of the DE on the $Z_2$ symmetric scalar model and generalize it to the $O(N)$ symmetric models. Our results complement those of Ref. $[11]$, where the authors employ the DE at NNLO as well, but there are key differences: (i) we do not use truncation of momenta in the derivation of our beta-functions, (ii) we employ Taylor expansion of the beta-functions in the field. These beta-functions describe the scale dependence of different functions depending on the field. The Taylor expansion reduces these to the beta-functions for coupling strengths corresponding to different vertices of the field. We compute the exponents with the exponential regulator, which is applicable at any order of the derivative expansion and (iii) also with a $\Theta$-type regulator $[12]$, which is the simplest applicable regulator at NNLO.

Increasing the number of terms in the Taylor expansion of the scale dependent functions, the values of the critical exponents fluctuate and eventually stabilize around their limiting values. Interestingly, we find that the exponents $\nu$, $\eta$ and $\omega$ of the $O(N)$ symmetric model are estimated remarkably well even at the zeroth order of the Taylor expansion in the field variable of the scale dependent functions corresponding to the NNLO of the DE. Furthermore, this fluctuation of the exponents at the NNLO is much less pronounced in the $O(N)$ symmetric case than in the $Z_2$ symmetric one.

We introduce the ERG briefly in Sec. II. The procedure we use to acquire the results is outlined in Sec. III. Our findings for the $Z_2$ symmetric model are detailed in Sec. V, while those of the $O(N)$ symmetric one can be found in Sec. VI.
II. EXACT RENORMALIZATION GROUP

The ERG uses functional integro-differential equations to describe the dependence of a theory on the variation of the characteristic energy scale. These equations can be used to describe non-perturbative phenomena. A widely used form of the ERG is the Wetterich equation \( \dot{\Gamma}_k \), which describes the scale dependence of the effective average action:

\[
\dot{\Gamma}_k = \frac{1}{2} \text{Str} \left[ \hat{R}_k (\Gamma_k^{(2)} + R_k)^{-1} \right]
\]

(1)

where the dot is an abbreviation for the operation \( k \partial_k \).

The functional \( \Gamma_k \) is the Legendre transform of the generating functional of the connected Green functions plus a scale-dependent mass term, called the regulator function \( R_k \), and \( \Gamma_k^{(2)} \) is the inverse propagator containing the physical mass. All the different formulations of the ERG equations require some sort of regularization. The regulator vanishes in the low energy limit of the theory. The super-trace contracts all momenta and group indices, therefore this equation can be viewed as a one-loop expression with an operator insertion \( (\hat{R}_k) \) and no external legs. The functional \( \Gamma_k \) possesses the symmetries of the original Lagrangian. In order to solve Eq. (1), one has to make an ansatz for \( \Gamma_k \) comprised of a finite number of functions, consistent with the symmetries of the original theory, and specify the regulator function.

A widely used approach in terms of the ansatz is the derivative expansion. In this method, the leading-order (or local potential approximation, LPA) only has a scale-dependent potential and a canonical kinetic term. An important feature of the exact renormalization group is that even the irrelevant couplings acquire non-trivial scale dependence during the RG-flow. This observation leads one to believe that the LPA prediction can be improved by including couplings, corresponding to scale dependent functions, which multiply all operators but the unit operator. Consequently, the next-to-leading order (NLO) introduces scale dependent functions multiplying every independent operator with two derivatives. Similarly, at the NNLO operators with four derivatives appear. This expansion decreases the truncation on the functional space of \( \Gamma_k \) order by order. One expects, that including higher orders in the derivative expansion improves the quality of the physical predictions. In fact, the convergence of this method has been demonstrated in Ref. [10] up to N3LO.

The dependence on the regulator is expected to vanish in the low-energy limit, \( k \to 0 \). As we study the critical theory, which is scale independent, we expect our physical predictions to be independent of the specific form of the regulator \( R_k \). This is strictly true only if we do not truncate the functional space. The dependence of the physical predictions and the magnitude of this spurious dependence on the regulator is somewhat similar to the renormalization scale dependence in the perturbative quantum field theory.

III. DERIVING THE BETA-FUNCTIONS

The system is critical in the Wilson-Fisher fixed point, which is the non-trivial solution of the fixed-point equation of the \( \beta \)-functions. We need to obtain the \( \beta \)-functions and the Wilson-Fisher fixed point to compute the critical exponents. The derivation of these \( \beta \)-functions is comprised of four steps for a given ansatz: (i) splitting the field to homogeneous and fluctuating pieces, (ii) functional Taylor expansion of Eq. (1) in powers of the fluctuating field, (iii) expansion in the momenta corresponding to the fluctuating field, and finally (iv) classification and sorting of the different types of loop integrals, called threshold integrals. We automated these steps in a Mathematica code.

A. Functional and momentum expansions

As an example, let us consider the ansatz for the \( Z_2 \) symmetric scalar model at \( \partial^2 \)-order:

\[
\Gamma_k[\phi] = \frac{1}{2} \int_x Z_k(\rho_x)(\partial \phi_x)^2 + \int_x U_k(\rho_x),
\]

(2)

where \( \rho_x = \phi_x^2/2 = \phi(x)^2/2 \). \( \int_x \equiv \int d^d x \) (and similarly \( \int_p = (2\pi)^{-d} \int d^d p \), to be used later), and \( (\partial f)^2 \equiv (\partial_p f)(\partial^\mu f) \) for any \( f \). The flow for \( U_k \) is obtained by setting the field \( \phi \) to be homogeneous \( \phi_x = \Phi \) (meaning \( \partial \Phi = 0 \)) and solving Eq. (1). In order to find \( \dot{Z}_k(\rho = \Phi^2/2) \) however, we expand Eq. (1) in terms a fluctuating field \( \eta_x \) around a constant background \( \phi_x = \Phi + \eta_x \) and collect the terms proportional to \( O(\eta^2) \). In momentum space, this is given by

\[
\int_Q \left( \dot{Z}_k(p) Q^2 + \dot{U}_k(p) + 2p\dot{U}_k^\prime(p) \right) \eta Q \eta_{-Q} =
\]

\[
= \int_{p,r} \dot{R}_k(p^2) G(p^2)(\eta \Gamma^{(3)})_{p,-r} G(r^2)(\eta \Gamma^{(3)})_{r,-p} G(p^2)
\]

\[
- \frac{1}{2} \int_{p} \dot{R}_k(p^2) G(p^2)(\eta \Gamma^{(4)})_{p,-p} G(p^2),
\]

(3)

with \( G(p^2) \) being the regularized propagator \( ((\Gamma_k^{(2)} + R_k)^{-1}) \), \( r = p \pm Q \), and

\[
(\eta \Gamma^{(3)})_{p,q} = \eta_{-p-q} \frac{\delta^{(3)} \Gamma}{\delta \phi_p \delta \phi_q \delta \phi_{-p-q}} \bigg|_{\phi_x = \Phi},
\]

\[
(\eta \Gamma^{(4)})_{p,q} = \int_{Q} \eta_Q \frac{\delta^{(4)} \Gamma}{\delta \phi_p \delta \phi_q \delta \phi_Q \delta \phi_{-Q}} \bigg|_{\phi_x = \Phi}.\]

(4)

Generally, in order to find \( \dot{F} \), where \( F \) multiplies an operator with \( n \) derivatives one has to collect terms propor-
tional to $O(p^n)$. We denote the momentum of the fluctuating field $\eta$ with $Q$ for transparency. In case, there are multiple $\eta$ fields in the same expression their momenta are denoted with $Q_1$, $Q_2$ and so on.

The left hand side of Eq. (3) shows, that in order to obtain $\dot{Z}_k(\rho)$, we have to expand the right hand side in $Q_\mu$ up to $Q^2$ and finally, identify the terms proportional to $Q^2$ as the beta function of $Z_k(\rho)$. The computations become naturally more complicated at NNLO, since there are multiple momenta $Q_i$. For the sake of concreteness, the complete ansatz for the $Z_2$ symmetric scalar model at the fourth order of the derivative expansion reads as

$$
\Gamma_k[\phi] = \frac{1}{2} \int_x Z_k(\rho_x)(\partial \phi_x)^2 + \int U_k(\rho)
$$

$$+ \frac{1}{2} \int_x W_k(\rho_x)(\partial_\mu \partial_\nu \phi_x)^2 + \frac{1}{2} \int_x H_k(\rho_x \phi_x)(\partial^2 \phi_x)^2 + \frac{1}{2} \int J_k(\rho_x)(\partial \phi_x)^4
$$

This form has been studied in great detail without and also with expansion in the fields. The scale dependent functions $W_k$, $H_k$ and $J_k$ are obtained from $\Gamma_k$ via

$$W_k(\rho) = \lim_{Q_1 \to +0} \left( \frac{\partial}{\partial Q_1^2} \right)^2 \Gamma^{(2)}_{Q_1, Q_2},
$$

$$H_k(\rho) = -\frac{1}{2} \lim_{Q_1, Q_2 \to +0} \frac{\partial}{\partial Q_1^2} \frac{\partial}{\partial Q_2^2} \Gamma^{(3)}_{Q_1, Q_2, Q_3},
$$

$$J_k(\rho) = \frac{1}{4} \lim_{Q_1, Q_2, Q_3 \to +0} \frac{\partial}{\partial Q_1^2} \frac{\partial}{\partial Q_2^2} \frac{\partial}{\partial Q_3^2} \Gamma^{(4)}_{Q_1, Q_2, Q_3, Q_4}
$$

as the coefficients of the integrands in the integrals

$$\int_{Q_1, \ldots, Q_n} \Pi_{i=1}^n \eta_0 \delta(\sum_{n=1}^n Q_i) = 0$$

for $n = 2$, $3$ and $4$. Note, that the scale dependent functions can be acquired by any permutation of the momentum indices $Q_i$ in the differentiation.

The $O(N)$ symmetric models introduce an additional index on the field corresponding to the symmetry group and can be generalized from the $Z_2$ symmetric models in a straightforward way.

### B. Threshold integrals

After sorting the different types of $\int_p$ integrals that appear in the formula of a general $F_3$ in the $Z_2$ symmetric model at NNLO, one finds three such types:

$$F^{d+a}_{m,n} = \int_p p^0 \frac{\hat{R}_k(p^2)}{G(p^2)^m},
$$

$$M^{d+a,b}_{m,b} = \int_p p^a (\partial^b \phi x^2)^b \frac{\hat{R}_k(p^2)}{G(p^2)^m},
$$

$$N^{d+a,b,c,\gamma}_{m,b,c} = \int_p p^a (\partial^b \phi x^2)^b (\partial^c \phi x^2)^c \frac{\hat{R}_k(p^2)}{G(p^2)^m}
$$

where $a$, $b$, $c$, and $\gamma$ are non-negative integers and $m$ is a positive one.

In the $O(N)$ symmetric models two types of propagators appear: one massive and one corresponding to the $N - 1$ Goldstone modes. This proliferates the types of threshold integrals.

### C. Regulator functions

The regulator itself is a function of the loop momentum squared $p^2$ and the running scale $k$. It is usually expressed as the function of the dimensionless ratio $y = p^2/k^2$:

$$R_k(p^2) = Z_k k^2 y r(y),
$$

where the explicit form of the regulator is defined by the function $r(y)$, $Z_k = 1$ at NPA and $Z_k = Z_k(\rho = \rho^*)$ at higher orders of the DE with $\rho^*$ being a reference value, detailed in Sect. 11.2.1. In general the form of the regulator is very flexible, yet it has to obey some requirements 1.

In order to obtain numerical results, one has to specify the regulator function. In this work we use two different types. The $\Theta_2$-regulator introduced in Ref. [12] reads as

$$r_\Theta(y) = \frac{(1 - y)^2}{y} \Theta(1 - y)
$$

where $\Theta(x)$ is the Heaviside step function. The regulator [13] is the simplest possible regulator which can be used in $\partial^4$-order calculations. The caveat is that it is not applicable beyond $\partial^2$-order due to the appearance of undefined Dirac-delta functionals $(\delta(0))$ in the final equations. Generally, at $\partial^n$-order the integral containing the highest $G$-derivative is:

$$M^{d+a, n}_{m,1} = \frac{\Omega_d}{(2\pi)^d} k^{d+a} \int dy y^{-1+d/2} (\partial_y^{-n} G(y)) \frac{\hat{R}_k(y)}{G(y)^{m}},
$$

where we have changed to the variable $y = p^2/k^2$. For the regulator [13] and $n = 4$ this integral takes the form

$$M^{d+a, 4}_{m,1} = -4\alpha^2 \frac{\Omega_d}{(2\pi)^d} (Z_k^2 k^{d+2}) \times
$$

$$\times \int dy y^{1-d/2}(y^2 - 1) \Theta(1 - y) \delta'(1 - y) G(y)^m
$$

$$= 8\alpha^2 (Z_k^2 k^{d+2}) \frac{\Omega_d}{(2\pi)^d} \frac{\Theta(0)}{G(1)^m}.
$$

We use the natural half-maximum convention $\Theta(0) \equiv \frac{1}{2}$ for the Heaviside step function. The integrals, which contain $\partial_y^2 G_k(y) = -2\alpha (Z_k k^2) \delta(1 - y)$ vanish, because the distributional product $x \delta(x)$ is zero and every integral contains $(1 - y)$ through $\hat{R}_k(y)$.

The second regulator we use here is called the exponential regulator

$$r_{exp}(y) = \alpha \frac{e^{-y}}{y},
$$

where $\alpha$ is a constant.
which is a $C^\infty$ function and has the advantage over the regulator containing the $\Theta$-function that it can be used at any orders of the derivative expansion. Both $\eta_\ast$ and $\eta_{\exp}$ remain unchanged in the $Z_2$ and $O(N)$ symmetric scalar models.

We vary the value of $\alpha$ and compute its effect on the critical exponents. We consider the extrema of these functions as the optimal values in our final predictions. This is the implementation of the the principle of minimal sensitivity (PMS) $\ddagger$ $\ddagger$. In practice, we locate the Wilson-Fisher fixed point for a fixed regulator for several values of $\alpha$, which simultaneously yields $\eta(\alpha)$ as the anomalous dimension is just a function of the couplings in the model. In each case we applied the PMS $\eta(\alpha)$ is either an upside or downside facing parabolid. The optimal value of $\eta_{\opt}$ is the minimum/maximum of this parabolid at $\alpha_{\opt}$ and we accept $\nu(\alpha_{\opt})$ and $\omega(\alpha_{\opt})$ as $\nu_{\opt}$ and $\omega_{\opt}$. In this sense, we only apply the PMS on the anomalous dimension.

Even the PMS optimized prediction $X_{\opt}$ for the critical exponent $X$ with the different regulators $X_{\Theta}^{\opt}$ and $X_{\exp}^{\opt}$ are different $X_{\Theta}^{\opt} \neq X_{\exp}^{\opt}$. We consider this a systematic uncertainty and consider our final prediction $X_{\fin}$ for the given exponent $X$ to be the average $X_{\fin} = (X_{\Theta}^{\opt} + X_{\exp}^{\opt})/2$ with the uncertainty $\delta X = |X_{\Theta}^{\opt} - X_{\fin}|$.

**D. Polynomial expansion and exponents**

In order to compute the critical exponents one has to use dimensionless quantities. The mass dimension of some are given as

$$[\phi] = (d - 2 + \eta_k)/2, \quad [U] = d,$$

where $\eta_k$ is the running anomalous dimension, which is defined by

$$\eta_k = -k \partial_k \ln Z_k(\rho^*).$$

(18)

The running anomalous dimension becomes the critical exponent $\eta$ in the fixed point. The Euclidean dimension $d$ is a continuous parameter in the beta functions of the dimensionless couplings, we set its value to $d = 3$ throughout this work. The beta functions for the dimensionless scale dependent functions are partial differential equations with the scale $k$ and the dimensionless field $\tilde{\rho}$ (we denote the dimensionless quantities with tilde) as independent variables. One strategy to solve these equations is to Taylor expand the dimensionless scale dependent functions in power of the dimensionless field around a reference point $\rho^*$

$$\tilde{F}_k(\tilde{\rho}) = \sum_{n=0}^{M_k} \frac{f_n(k)}{n!} (\tilde{\rho} - \rho^*)^n.$$  

(19)

This reduces the coupled set of partial differential equations to a coupled set of ordinary differential equations.

There are two well known choices for $\rho^*$. It can either be zero ($\rho^* = 0$) or the running minimum $\rho^* = \kappa_k$ of the most basic scale dependent function, the local potential $U_k$. Throughout this work we use $\rho^* = \kappa_k$, because it provides a faster convergence of the physical results with increasing $M_F$ than expanding around the vanishing field $\ddagger \ddagger$. We denote the highest power in the Taylor series of a general scale dependent function $F_k$ with $M_F$, if the subscript contains multiple capital Latin letters such as $M_{WH,J}$, it means that the scale dependent functions $W_k, H_k$ and $J_k$ are truncated at identical powers $M_{WH} = M_H = M_J = M_{WH,J}$.

The Wilson Fisher fixed point is the nontrivial fixed point solution of the beta functions. Once it is located, the critical value of the anomalous dimension $\eta$ is determined. The critical exponent of the correlation length $\nu$ and its subleading scale corrections $\omega, \omega_1$ are obtained by linearizing the RG-flow in the vicinity of the fixed point. The eigenvalues of the Jacobian matrix $J_{ij} = \partial \tilde{\rho}_i / \partial \tilde{\rho}_j$, with $\tilde{\rho}_i$ being a general dimensional coupling from the model, at the fixed point are $-\nu^{-1} < \omega < \omega_1 < \ldots$ in increasing order.

**IV. WYNN’S EPSILON ALGORITHM**

In many instances, the prediction of an exponent $X$ at successive orders of the DE, $X_{LPA}, X_{NLO}, X_{NNLO}$ and so on, form a convergent series alternating around the exact value $X$. This has been discussed in great detail in Ref. $\ddagger \ddagger$. In Ref. $\ddagger \ddagger$ the authors use the small parameter $1/4 - 1/9$ of the DE to improve their predictions on the critical exponents of the $O(N)$ symmetric scalar models at NNLO of the DE.

One may also turn to a similar, yet different approach to improve exponent predictions in the derivative expansion. Several series acceleration methods exist and are used successfully to accurately compute the limit of a slowly converging sequence. One of the most robust of these algorithms is Wynn’s epsilon algorithm $\ddagger \ddagger \ddagger$. It is already applicable if one only has the first three elements $a_1, a_2, a_3$ of a sequence $(a_n)$. In that case, the third element is improved as

$$\tilde{a}_3 = a_2 + \frac{1}{-a_1 + a_2 + -a_2 + a_3} = -\frac{a_2}{a_1} + \frac{1}{a_1 - 2a_2 + a_3}.$$  

(20)

Given the critical exponent $X$, this means, that the improved prediction of the DE is

$$\tilde{X} = \frac{-X_{nLO}^2 + X_{LPA}X_{NLO}}{X_{LPA} - 2X_{NLO} + X_{NNLO}}.$$  

(21)

The formula is even simpler for the anomalous dimension as the LPA prediction for it is zero. The only exception we cannot apply it accurately is the $\omega$ exponent in the $O(3)$ and $O(4)$ case (for this works prediction on those exponents see Tab. $\ddagger \ddagger$). In those instances $\omega$ increase monotonically in successive orders of the DE rather than
alternating around the exact prediction, there we simply cite our NNLO prediction as final results in Tab. [II]

In principle we use this method to accurately extrapolate to higher orders of the DE. This should reduce the uncertainty originating from the choice of the regulator, because with decreasing the truncation of the functional space of \( \Gamma_k \) has to decrease the dependence of physical predictions on the regulator. An other systematic source of error is that of the DE itself. If one insists on using Wynn’s epsilon algorithm, then it is necessary to compute the \( N^3\) LO prediction of the DE in order to give a conservative estimate on this error. Here, we keep the regulator uncertainty of the NNLO predictions fixed in an attempt to overestimate the total uncertainty of the improved prediction including that of the DE itself.

V. PREDICTIONS FOR THE \( Z_2 \) SYMMETRIC SCALAR MODEL

We derived the beta functions for the dimensionless scale dependent functions \( (U_k, Z_k, W_k, H_k, J_k) \) in the ansatz \( 5 \) using a Mathematica code. We verified the correctness of \( \tilde{U}_k \) and \( \tilde{Z}_k \) (at \( \partial^2 \)-order) to be the same as in the literature \([15,19]\). We expanded these functions in the powers of the field yielding the beta functions for the dimensionless couplings \( \tilde{f}_\alpha(k) \) in Eq. \( 19 \). We have calculated the effect of increasing \( M_F \) on the exponents. We start with the LPA, where the only scale dependent function is \( U_k \) and locate the Wilson-Fisher fixed point with truncation threshold \( M_U = 4 \). In the next step, we locate the fixed point for \( M_U = 5 \) using the previous fixed point solution with \( \tilde{u}_0 = 1 \) as initial value. After this, we move on to \( M_U = 6 \) using the previous fixed point solution with \( \tilde{u}_0 = 1 \) as initial value. In this iterative manner, we find the Wilson-Fisher fixed point for up to \( M_U = 8 \). At the NLO, we have an additional scale dependent function \( Z_k \) and nonzero anomalous dimension. We start with locating the fixed point at \( M_U = 8 \) and \( M_Z = 0 \), but including the effect of anomalous dimension and simply use the LPA values for \( M_U = 8 \) as initial value. Next, we apply to \( M_Z \) the iterative procedure used to find the fixed point for \( M_U = 8 \) at the LPA. We find the Wilson-Fisher fixed point for up to \( M_U = 8 \) and \( M_Z = 8 \). At NNLO, we have three scale dependent functions \( W_k, H_k \) and \( J_k \). We start looking for the Wilson-Fisher fixed point at \( M_U = M_Z = 8 \) with \( M_W = M_H = M_J = 0 \), and setting the initial values to be \( \tilde{w}_0 = h_0 = j_0 = 1 \) for the new couplings. Finally, we also apply here the previously described iterative algorithm but we increase simultaneously \( M_W, M_H \) and \( M_J \) and denote this value with \( M_{W,H,J} \). The upper limit where we have located the Wilson-Fisher fixed point is \( M_{W,H,J} = 7 \).

We have computed the fixed points with the two regulators discussed in Sect. [II] Using \( 13 \) with \( \alpha = 1/2 \) reduces the integrals \( 8 \) to linear combinations of the \( 2F_1 \) hypergeometric function, which greatly increases the speed of computations compared to \( 16 \) with any value of \( \alpha \).

The effect of the gradual inclusion of the new couplings can be seen on the left column of Fig. [I], which agrees with \( 13 \). The most important conclusion is that while at \( \partial^2 \)-order the contributions of the Taylor expansion in field variable become small for \( M_Z > 4 \) this threshold power value at \( \partial^4 \)-order is somewhat larger, \( M_{W,H,J} = 6 \). The magnitude of these contributions start to decrease monotonically for \( M_Z > 3 \) at NLO and \( M_{W,H,J} > 4 \) at NNLO. Next, we apply the principle of minimal sensitivity to \( M_{W,H,J} \geq 4 \), which corresponds to the last four data points in each row of Fig. [I]. We have found that starting from \( M_{W,H,J} = 5 \), the optimal values \( \alpha^{opt} \) for the regulators \( 13 \) and \( 16 \) stabilize at \( \alpha^{opt} = 0.3 \) and \( 0.8 \). Once we acquire the optimized results in this asymptotic regime, where each successive contribution from the Taylor expansion is smaller than the previous one, we fit a decaying function to these data points in an attempt to resum the corrections from the Taylor expansion. The model function in every instance is

\[ a + b \, e^{-c \, x} \sin(d \, x + e), \]  

(22)

with the independent variable being \( x \) and the fitted pa-
parameters \(a, b, d, e\) and \(c > 0\). This step is shown in Fig. 2. We consider our findings to be the \(M_{WHJ} \rightarrow \infty\) limit of these fitted functions, that is we identify the exponent as the fitted parameter \(a\) from the model function \([22]\). We do not apply Wynn’s epsilon algorithm here, because the corrections from increasing \(M_{WHJ}\) is not a simple alternating series. In the asymptotic regime, shown with the PMS optimized exponent on Fig. 2, these corrections alternate around their limiting value with periodicity of at least two. For instance, we expect that the correction from \(M_{WHJ} = 8\) increase the value of \(\nu^{opt}\) compared to \(M_{WHJ} = 7\) and the higher corrections to have smaller effect than this. The model function \([22]\) takes this into account correctly.

Every beta function contains terms proportional to \(\eta\) through \(R_k\). Considering only the exponents \(\nu\) and \(\eta\), the inclusion of these in \(\tilde{U}_k\) gives a 1% and 5% correction, while in \(\tilde{Z}_k\) they give 0.1% and 0.5% correction compared to not including those. We have also inspected the inclusion of these terms into \(\tilde{W}_k, \tilde{H}_k\) and \(\tilde{J}_k\) for the truncation \(M_U = 8\) and \(M_Z = 8\) with \(M_{WHJ} \leq 4\) and found that this characteristically gives a 0.02% and 0.008% correction to the exponents. We have neglected this correction in \(\tilde{W}_k, \tilde{H}_k\) and \(\tilde{J}_k\) for \(M_{WHJ} \geq 5\) and considered it as one source of uncertainty. The other source comes from the truncation of \(\tilde{U}_k\) and \(\tilde{Z}_k\). At NLO, we have computed the fixed point for truncation \(M_U = 9\), \(M_Z = 8\) and \(M_U = 9\), \(M_Z = 9\). We have found that the inclusion of the coupling \(\tilde{u}_0\) has negligible effect compared to the inclusion of \(\tilde{z}_9\). Our final predictions for the critical exponents of the \(Z_2\) symmetric model are shown in Tab. 1. The uncertainties for our results include the effect of the regulator dependence and the polynomial truncation. We obtained our improved result using Wynn’s epsilon algorithm, discussed in Sec. IV.

![Fig. 2. A decaying function fit on the PMS optimized values of the exponents of the \(Z_2\) symmetric scalar model at truncation \(M_{WHJ} = 4\) and above. The disks correspond to the values obtained with \(r_{\text{exp}}\), the squares to the values obtained with \(r_{\text{opt}}\). The dashed horizontal line shows the CB values.](image)

| Method          | \(\nu\)     | \(\eta\)   | \(\omega\)   |
|-----------------|--------------|-------------|--------------|
| LPA             | 0.6504(7)    | 0           | 0.654(1)     |
| NLO             | 0.6295(6)    | 0.0423(6)   | 0.841(3)     |
| NNLO            | 0.6302(4)    | 0.0347(8)   | 0.814(5)     |
| improved        | 0.6301(4)    | 0.0358(8)   | 0.819(5)     |
| \(\delta^3\), field exp. | 0.632   | 0.033       |
| \(\delta^3\), no field exp. | 0.63012(16) | 0.0362(12) | 0.832(14)   |
| MC              | 0.63062(10)  | 0.03627(10) | 0.832(6)     |
| six-loop PT     | 0.6304(13)   | 0.0335(25)  | 0.799(11)    |
| \(\epsilon^6\), epsilon exp. | 0.6292(5) | 0.0362(6) | 0.820(7)     |
| CB              | 0.629971(4)  | 0.0362978(20) | 0.82968(23) |

TABLE I. Our findings for the exponents of the \(Z_2\) symmetric scalar model in \(d = 3\) Euclidean dimensions (top four rows) for different orders of the DE and the improved, final prediction. The uncertainties are the sum of the uncertainties from the polynomial expansion and the regulator dependence. We compared these to some other methods: DE at NNLO (\(\delta^3\)) with field expansion [13], at N^3LO (\(\delta^3\)) without field expansion [11], MC [3], six-loop perturbation theory at fixed \(d = 3\) [6], \(d = 4 - \epsilon\) expansion at \(\epsilon^6\) [7] and the CB method [8].

VI. NNLO FOR THE O(N) SYMMETRIC SCALAR MODELS

A. Modifications compared to the \(Z_2\) symmetric case

There are more scale dependent functions in the \(O(N)\) symmetric scalar model beyond the LPA than in the \(Z_2\) symmetric one, due to an additional group index. At NLO, there are two instead of the one \(Z_k\), but at NNLO the number of independent scale dependent functions increase to ten, compared to the three \(W_k, H_k\) and \(J_k\). The
complete $\partial^4$-order ansatz is

$$
\Gamma_k(\vec{\phi}) = \int_x \left\{ U_k + \frac{1}{2} Z_k(\partial_\phi)^2 + \frac{1}{4} Y_k(\partial^2_\phi)^2 
+ \frac{1}{8} W_{1,k}(\partial_{\mu_1}\phi^a_{x})^2 + \frac{1}{8} W_{2,k}(\partial_{\mu_2}\phi^a_{x})^2 
+ \frac{1}{2} H_{1,k}(\partial_\phi^a)^2(\partial^2_{\phi^a}) + H_{2,k}(\partial_\phi^a)(\partial_\phi^b)(\partial^2_{\phi^b}) 
+ \frac{1}{4} H_{3,k}(\partial_\phi^a)^4 + \frac{1}{8} J_{5,k}(\partial_\rho^a)^4 
+ \frac{1}{2} J_{1,k}(\partial_\phi^a)^2(\partial^2_{\phi^a}) + \frac{1}{2} J_{2,k}(\partial_\phi^a)(\partial_\phi^a)(\partial_\phi^b)(\partial^2_{\phi^b}) 
+ \frac{1}{4} J_{3,k}(\partial_\phi^a)^2(\partial^2_{\phi^a}) + \frac{1}{4} J_{4,k}(\partial_\phi^a)(\partial_\phi^a)(\partial_\phi^b)(\partial^2_{\phi^b}) \right\}. 
$$

(23)

where $\vec{\phi}$ is the $N$ component scalar field and $\rho^a = \phi^a/2$ is the invariant under the $O(N)$ symmetry transformation. We have suppressed the field dependence of the scale dependent functions in (23) to be more transparent. Due to the appearance of the Goldstone modes in addition to the one massive mode in the $\tilde{Z}_2$ symmetric model, we have two anomalous dimensions corresponding to these modes:

$$\eta = -k \partial_\rho \ln Z_k(\rho^*)$$

(24)

$$\hat{\eta} = -k \partial_\rho \ln \left( Z_k(\rho^*) + \rho^a Y_k(\rho^*) \right) \equiv \eta - \partial \ln \tilde{Z}_k(\rho^*)$$

(25)

These anomalous dimensions are equal in the critical point. In our numerical check, we use this fact to ensure the correctness of our equations. Besides the field, the regulator function also receives $O(N)$ indices. We choose

$$R^{ab}(y) = \delta^{ab} Z_k(\rho^*) k^2 y r(y)$$

(26)

where $\delta^{ab}$ is the Kronecker-delta matrix, such that the regulator mass matrix is already diagonalized in the $O(N)$ space. In order to facilitate the bookkeeping of the $O(N)$ indices, we introduce projectors $P^{ab}_A$ with $(A = \parallel, \perp)$ to the radial $(P^{ab}_\parallel = e^a e^b)$ and perpendicular (Goldstone) $(P^{ab}_\perp = \delta^{ab} - e^a e^b)$ directions in the $O(N)$ space, with $e^a$ being the unit vector. The scale dependent functions $Y_k$, $W_{i,k}$, $H_{i,k}$ and $J_{i,k}$ are obtained by the same momentum derivatives (Eq. [6]) as $Z_k$, $W_k$, $H_k$ and $J_k$ in the $\tilde{Z}_2$ symmetric model as coefficients of the integrands in $\int Q_{1,\ldots,Q_n} \prod_{i=1}^n \eta_Q^{\parallel} \delta(\sum_{i=1}^n Q_i)$. The capital Latin letters correspond to either $\parallel$ or $(\perp, a)$. Using the projectors defined above one has

$$P^{ab}_\parallel \eta_Q = \eta_Q^{\parallel}$$

and

$$P^{ab}_\perp \eta_Q = \eta_Q^{\perp}.$$  

(27)

In this method, every $O(N)$ index is contracted in the final result, so that $\eta_Q^{\parallel}$ may occur only in pairs, such as $\eta_Q^{\parallel} \eta_Q^{\perp}$. For instance, the left-hand side the Wetterich equation for $O(\eta^2)$ Eq. [5] modifies to

$$\int Q \eta_Q^{\perp} \eta_Q^{\parallel} \left[ (\tilde{Z}_k + \tilde{Y}_k) Q^2 + (\tilde{W}_1 + \tilde{W}_2) Q^4 + \tilde{U}' \right]$$

$$+ \int Q \eta_Q^{\parallel} \eta_Q^{\parallel} \left[ (\tilde{Z}_k + \tilde{Y}_k) Q^2 + (\tilde{W}_1 + \tilde{W}_2) Q^4 + \tilde{U}' \right]$$

(28)

with the ansatz in Eq. [23].

We have followed the same steps of numerical analysis as we did for the $\tilde{Z}_2$ symmetric model. The system of $\beta$-functions are generated by a Mathematica code, which are then verified to reproduce the $\partial^2$-order results [20]. We applied the same iterative algorithm to find the Wilson Fisher fixed point for high values of truncation $M$ as for the $\tilde{Z}_2$ symmetric model. At the LPA, we have computed the exponents for up to $M_{\parallel} = 8$. In the NLO we have increased simultaneously the truncation $M_Z$ of $Z_k$ and $M_Y$ of $Y_k$ for up to $M_Z = M_Y = 4$ and denote this with $M_{\parallel Y}$. At NNLO, we have ten scale dependent functions. In order to make it easier to find the Wilson Fisher

---

**FIG. 3.** The dependence of the critical exponents $\nu$, $\eta$ and $\omega$ on the order of polynomial truncation for the $O(0)$ symmetric model at $M_{\parallel} = 8$. The vertical line separates our NLO results (left) from the NNLO ones (right). The dotted horizontal line shows the corresponding MC result. The continuous curve with disk markers belongs to the $\Theta$-type regulator with $\alpha = 1/2$, while the dashed curve with rectangle markers belong to the exponential-type regulator with $\alpha = 1$. At the points, where $\omega$ is not shown, it is a complex number.
fixed point, we further divide the iterative algorithm to three parts. First, we locate the fixed point for the truncation $M_U = 8$, $M_{ZY} = 4$, $M_{W^1} = M_{W^2} = 0$ with the initial values $\bar{w}_{1,0} = \bar{w}_{2,0} = 1$. In the next step, we use this fixed point as initial value with $\tilde{h}_{1,0} = \tilde{h}_{2,0} = \tilde{h}_{3,0} = 1$ for the truncation $M_D = 8$, $M_{ZY} = 4$, $M_{W^1} = M_{W^2} = 0$ and $M_{H^1} = M_{H^2} = M_{H^3} = 0$. In the last step we locate the fixed point with $M_{H} = 0$ ($i = 1, \ldots, 5$) also included. We denote this truncation with $M_{WHJ} = 0$ when all the NNLO level scale dependent functions are included with zeroth order truncation in their Taylor expansion. We have computed the exponents for up to $M_U = 8$, $M_{ZY} = 4$ and $M_{WHJ} = 3$.

The effect of the gradual inclusion of the new couplings for the $O(N)$ symmetric scalar model is shown in Figs. 3-7 for $N = 0 - 4$. We have also computed the exponents for the $N = 10$ and $N = 100$ cases but omitted to show their field dependence, as it is very small. The leading order of the DE, the local potential approximation (LPA) is exact for $O(N \to \infty)$. The anomalous dimension decreases monotonically at large $N$ values with increasing $N$ and vanishes completely in the limit $N \to \infty$. This means that the derivative expansion has to yield very precise predictions for the exponents for large $N$ values. This is reflected in the fact, that the field dependence is very small at $N = 10$ and at $N = 100$. We have chosen $N = 10$ and $100$ as benchmark points to compare our predictions with those of the large-$N$ expansion. We do show however the field dependence of the $O(1)$ symmetric model, which is different than the $Z_2$ symmetric one due to the different content of scale dependent functions. The two models should predict the same values for the critical exponents because of continuity of the equations in $N$. This feature is nicely shown in Fig. 4. Going back to the Figs. 6 - 7 we can clearly see, that the field expansion is very stable at NNLO even when one considers the correction of $M_{WHJ} = 1$ compared to $M_{WHJ} = 0$. Due to this smoothness of predictions from the field expansion at NNLO, we apply the principle of minimal sensitivity for $M_{WHJ} \geq 0$. In order to reduce the amount of computation, we have only looked for a PMS solution for the anomalous dimension and accepted the corresponding parameter value as the optimal $\alpha^{opt}$. We have found that for the regulator (13) $\alpha^{opt}$ fluctuates between 0.35 - 0.40 for different $N$ and $M_{WHJ}$ values while for (16) $\alpha^{opt}$ fluctuates between 0.8 - 0.9. We attempt to find the limiting value of the optimized exponents in the range $N = 0 - 4$ for $M_{WHJ} \to \infty$ in the same fashion as we did for the $Z_2$ symmetric model (see Fig. 2). As for $N = 10$ and $N = 100$ the fluctuation of the exponents is very small with varying $M_{WHJ}$; in these instances we consider our final predictions corresponding to $M_{WHJ} = 3$ with PMS optimization applied. We have computed the correction of including $M_{ZY} = 5$ in every instance of $N$ at NNLO and included those as uncertainties in our final results. An other source of uncertainty comes from the terms in the beta functions proportional to $\eta$ through $R_k$, just as we discussed at the end of Sec. 7. We have neglected these terms in the scale dependent functions corresponding to the NNLO. This constitutes an uncertainty orders of magnitude smaller than that of the truncation. The predictions computed with the different regulators (13) and (16) are very close to each other with $\alpha = 1/2$ and $\alpha = 1$, however the predictions from the $\Theta$-type regulator change more strongly with $\alpha$ than those from the exponential-type one. As we discussed in Sec. 11C, we consider our final predictions as the average of the optimized predictions from the two regulators (13) and (16). The deviation from the average is actually the largest source of uncertainty in our final results, shown in Tab. II.

**FIG. 4.** The dependence of the critical exponents $\nu$, $\eta$ and $\omega$ on the order of polynomial truncation for the $O(1)$ symmetric model at $M_U = 8$. The vertical line separates our NLO results (left) from the NNLO ones (right). The dotted horizontal line shows the corresponding CB result. The continuous curve with disk markers belongs to the $\Theta$-type regulator (13) with $\alpha = 1/2$, while the dashed curve with rectangle markers belong to the exponential-type regulator (16) with $\alpha = 1$.

**B. Numerical findings**

We have computed the critical exponents for the regulators (13) and (16). The former one with $\alpha = 1/2$ reduces a large number of the threshold integrals to $\text{2F1}$-type hypergeometric functions. This yields a significant speed boost in the computations compared to (16) with any value of $\alpha$. 

[13] [16]
VII. BRIEF SUMMARY OF THE $O(N)$ CRITICAL EXPOSNENTS FROM VARIOUS METHODS

The $O(N)$ symmetric scalar model was first introduced as the $n$-vector model as a generalization of some physically relevant models \[14\] in $d$ Euclidean dimensions. The $N = 0$ case describes the self avoiding walk \[33, 50\]. It is also noteworthy, that the $O(0)$ model probably does not have a Minkowskian counterpart, because in case the Euclidean dimension $d$ and $N$ are not positive integers the unitarity of the corresponding Minkowskian model is lost or at least highly nontrivial. At the level of the $n$-vector model, $O(1)$ model describes the Ising universality class. In the ERG however the $Z_2$ and $O(N)$ symmetric models at $N = 1$ are different because of the different content of scale dependent functions and the appearance of an additional, massless excitation in the $O(N)$ model. The $O(N)$ model however should reproduce the Ising exponents in the limit of $N \to 1$, due to the model being completely continuous in $N$. The $O(2)$ model is more commonly known as the $XY$-model, which is used to describe the phase transition in the superfluid helium-4.

The $O(3)$ model is also known as the Heisenberg model for ferromagnetism. Lastly but not the least, the $O(4)$ model can be considered as a Toy model for the standard model’s Higgs sector, but also applicable to chiral phase transitions.

Some of the most precise computations of the $O(N)$ critical exponents in $d = 3$ Euclidean dimensions are summarized in Tab. [III]. Comparing these with our findings, ’this work’ entry in the same table, one can see that the central values are in excellent agreement. The improved results of Ref. [11] take advantage of the convergence of the DE as well as the alternating behavior of the corrections from the successive orders of the DE. In contrast our improvement, the Wynn epsilon algorithmd detailed in Sec. [IV], is a brute force series acceleration method. Our uncertainties come solely from the choice of regulator and the polynomial truncation of the scale dependent functions. At NNLO, we are not able to accurately give the error of the DE. We also need the improved $N^3$LO predictions to give a conservative error bar, as the absolute difference of the Wynn improved NNLO and $N^3$LO predictions.

| $N$  | Order of DE | $\nu$  | $\eta$  | $\omega$ |
|------|-------------|--------|---------|---------|
| 0    | LPA         | 0.5927(5) | 0.0379(5) | 0.946(17) |
|      | NLO         | 0.5878(2) | 0.0296(2) | 0.896(4)  |
|      | NNLO        | 0.5875(2) | 0.0296(2) | 0.896(4)  |
| 1    | LPA         | 0.6505(8) | 0.0456(6) | 0.845(6)  |
|      | NLO         | 0.6278(4) | 0.0438(8) | 0.827(6)  |
|      | NNLO        | 0.6293(4) | 0.0438(8) | 0.827(6)  |
| 2    | LPA         | 0.7092(4) | 0.0467(6) | 0.785(6)  |
|      | NLO         | 0.6681(7) | 0.0467(6) | 0.785(6)  |
|      | NNLO        | 0.6721(13) | 0.0362(9) | 0.780(6)  |
| 3    | LPA         | 0.7621(12) | 0.0465(8) | 0.756(6)  |
|      | NLO         | 0.7066(7) | 0.0465(8) | 0.756(6)  |
|      | NNLO        | 0.7128(12) | 0.0356(11) | 0.766(10) |
| 4    | LPA         | 0.8059(12) | 0.0465(8) | 0.735(2)  |
|      | NLO         | 0.7418(7) | 0.0465(8) | 0.739(7)  |
|      | NNLO        | 0.7510(11) | 0.0340(12) | 0.766(10) |
| 10   | LPA         | 0.9212(23) | 0.0465(8) | 0.880(7)  |
|      | NLO         | 0.8786(4) | 0.0275(3) | 0.784(6)  |
|      | NNLO        | 0.8772(10) | 0.0223(2) | 0.808(2)  |
| 100  | LPA         | 0.9926(2)  | 0.0465(8) | 0.989(1)  |
|      | NLO         | 0.9898(1)  | 0.00298(1) | 0.978(1)  |
|      | NNLO        | 0.9889(2)  | 0.00264(2) | 0.977(1)  |

TABLE II. The main findings of this work. Our predictions for the critical exponents $\nu$, $\eta$ and $\omega$ at the LPA, NLO and NNLO of the DE for the $O(N)$ symmetric models in $d = 3$ Euclidean dimensions. These values are the average of the predictions computed from the $\Theta$-regulator [13] and the exponential regulator [16] and the deviation from the average is one source of the uncertainties. The other source of uncertainty correspond to the polynomial truncation of the scale dependent functions.

FIG. 5. The dependence of the critical exponents $\nu$, $\eta$ and $\omega$ on the order of polynomial truncation for the $O(2)$ symmetric model at $M_0 = 8$. The vertical line separates our NLO results (left) from the NNLO ones (right). The dotted horizontal line shows the corresponding CB result. The continuous curve with disk markers belongs to the $\Theta$-type regulator [13] with $\alpha = 1/2$, while the dashed curve with rectangle markers belong to the exponential-type regulator [16] with $\alpha = 1$. 

![Graph](image-url)
| $N$ | Method          | $\nu$   | $\eta$   | $\omega$  |
|-----|----------------|---------|----------|-----------|
| 0   | this work      | 0.5875(2) | 0.0311(2) | 0.903(4)  |
|     | $\partial^3$, raw | 0.5875 | 0.0292 | 0.901   |
|     | $\partial^4$, improved | 0.5876(2) | 0.0312(9) | 0.901(24) |
|     | MC [21, 22]    | 0.58759709(40) | 0.0310434(30) | 0.899(14) |
|     | six-loop PT    | 0.58821(11) | 0.028425(24) | 0.812(16) |
|     | $e^6$, c-exp.  | 0.5874(3) | 0.0310(77) | 0.841(13) |
|     | CB [23]        | 0.5876(12) | 0.0282(4) |          |
| 2   | this work      | 0.6717(13) | 0.03829(9) | 0.780(6)  |
|     | $\partial^3$, raw | 0.6732 | 0.0350 | 0.793   |
|     | $\partial^4$, improved | 0.6716(6) | 0.0380(13) | 0.791(8) |
|     | MC [4]         | 0.67169(7) | 0.038108(8) | 0.789(4) |
|     | six-loop PT    | 0.670315(15) | 0.035425(24) | 0.789(11) |
|     | $e^6$, c-exp.  | 0.669010(10) | 0.038060(6) | 0.804(3) |
|     | CB [24]        | 0.67181(1) | 0.0381184(4) | 0.794(8) |
| 3   | this work      | 0.7122(12) | 0.037711(7) | 0.766(9)  |
|     | $\partial^3$, raw | 0.7136 | 0.0347 | 0.773   |
|     | $\partial^4$, improved | 0.71149(9) | 0.037613(13) | 0.769(11) |
|     | MC [25, 29]    | 0.711610(10) | 0.037823(3) | 0.773   |
|     | six-loop PT    | 0.707235(35) | 0.035525(23) | 0.782(13) |
|     | $e^6$, c-exp.  | 0.705920(15) | 0.03785(3) | 0.795(7) |
|     | CB [27, 29]    | 0.712023(13) | 0.038513(6) | 0.791(22) |
| 4   | this work      | 0.7498(11) | 0.036012(12) | 0.766(10) |
|     | $\partial^3$, raw | 0.7500 | 0.0332 | 0.765   |
|     | $\partial^4$, improved | 0.74789(8) | 0.036012(7) | 0.761(12) |
|     | MC [26, 29]    | 0.74778(8) | 0.03604(1) | 0.765   |
|     | six-loop PT    | 0.7416(1) | 0.0350(45) | 0.774(20) |
|     | $e^6$, c-exp.  | 0.739735(35) | 0.036006(4) | 0.794(9) |
|     | CB [28, 30]    | 0.747287(67) | 0.037832(3) | 0.817(30) |
| 10  | this work      | 0.87712(10) | 0.023112(12) | 0.803(2)  |
|     | $\partial^3$, raw | 0.8771 | 0.0218 | 0.808   |
|     | $\partial^4$, improved | 0.87710(10) | 0.02316(6) | 0.807(7) |
|     | large-$N$      | 0.87(2) | 0.023(2) | 0.771(1) |
| 100 | this work      | 0.9886(2) | 0.0267(2) | 0.977(1)  |
|     | $\partial^3$, raw | 0.9887 | 0.0260 | 0.977   |
|     | $\partial^4$, improved | 0.98882(2) | 0.00268(4) | 0.9770(8) |
|     | large-$N$      | 0.9890(2) | 0.002681(1) | 0.9782(2) |

TABLE III. Critical exponents of the $O(N)$ symmetric scalar model in $d = 3$ Euclidean dimensions for several $N$ values with different methods: our improved predictions using Wynn’s epsilon algorithm, the DE at NNLO ($\partial^4$) without field expansion with raw (computed with the exponential regulator) and improved values [11]. Monte-Carlo simulations, six-loop perturbation theory at fixed $d = 3$ [5], $d = 4 - \epsilon$ expansion at $e^6$ [7], the conformal bootstrap method and the large-$N$ expansion [31, 33].

VIII. CONCLUSION

We have computed the critical exponents for the $Z_2$ and $O(N)$ symmetric scalar models in $d = 3$ Euclidean dimensions. We have employed the exact renormalization group equation for the effective average action. We have used the derivative expansion at NNLO (or $\partial^4$-order) and calculated the $\beta$-functions for the scale dependent functions, shown in [4] for the $Z_2$ and in [23] for the $O(N)$ symmetric models. In order to locate the Wilson-Fisher fixed point which is the nontrivial fixed point solution of the beta-functions, we have expanded the scale dependent functions in powers of the field. We interpret the scale dependent coefficients $f_n(k)$ from the Taylor expansion as effective coupling strengths for the interaction vertices of the field they multiply. We have located the fixed point in the theory space spanned by the (canonical mass) dimensionless couplings, with truncated Taylor series of the scale dependent functions. Our main findings for the $Z_2$ symmetric model shown in Tab. [4] are in agreement with predictions obtained using other methods. We have used the $Z_2$ symmetric model as a testing ground for the correctness of our Mathematica code. We then generalized this code for the $O(N)$ symmetric model and computed the critical exponents for some relevant $N$ values. We have tested the $O(N)$ Mathematica code for the $N = 1, 10$ and 100 cases. The $Z_2$ symmetric model belongs to the Ising universality class with one massive excitation. As opposed to this the $O(1)$ model is not the same in the sense, that it contains extra scale dependent functions and two anomalous dimensions [24] for a massive and a massless excitation. However, the exponents of the $O(1)$ symmetric model should coincide with those of the $Z_2$ symmetric one due to continuity of the beta-
functions in \( N \). We chose \( N = 10, 100 \) to be second and third benchmark points, because the effect of the derivative expansion is diminished with \( N \to \infty \), hence it can give very accurate results for large \( N \) values. Our main findings are summarized in Tab. II. A great advantage of our method is that it requires significantly less computer time than most of the other methods. For our highest employed polynomial truncation both for the \( Z_2 \) and \( O(N) \) symmetric models, the location of the Wilson Fisher fixed point roughly takes \( 1 - 2 \) hours, while computing the Jacobian matrix at the fixed point takes an additional hour on a single desktop PC.

In a recent paper [11] the authors have performed similar computations with the ERG. The differences are that (i) we have not truncated our formulas in the momenta (denoted here with \( Q_i \)); (ii) we have employed Taylor expansion for the scale dependent functions in powers of the field instead of shooting for a solution for the complete scale dependent functions; (iii) we have computed the exponents with the regulator [13], which is the simplest regulator at NNLO. Although this \( \Theta \)-regulator is argued to perform poorly in [10], we have found that it yields excellent predictions for the exponents in the models studied here. We also provide improved predictions using Wynn’s epsilon algorithm on our predictions of the DE, yielding central values which are in excellent agreement with other precise methods used to compute critical exponents.

Our method also produces the subleading scaling corrections \( \omega_i \) (from the eigenvalue spectrum \( -1/\nu < \omega < \omega_1 < \omega_2 < \ldots \) of the Jacobian of the beta-functions) as a byproduct of computing the exponents \( \nu \) and \( \omega \). The expansion of the scale dependent functions in powers of the field is also applicable to explore the phase structure of a model and the RG running of its couplings. The derivative expansion can also be improved to \( N^3\)LO (or \( Q^r \)-order) with some effort for the \( O(N) \) symmetric models, which would provide more precise exponent values for many cases of \( N \).

ACKNOWLEDGMENTS

The author would like to thank Z. Trócsányi for the careful reading of the manuscript.

[1] C. Wetterich, Exact evolution equation for the effective potential, Physics Letters B 301, 90 (1993)
[2] K. G. Wilson and J. Kogut, The renormalization group and the \( \epsilon \) expansion, Physics Reports 12, 75 (1974)
[3] M. Hasenbusch, Finite size scaling study of lattice models in the three-dimensional ising universality class, Phys. Rev. B 82, 17435 (2010)
[4] M. Hasenbusch, Monte carlo study of an improved clock model in three dimensions, Phys. Rev. B 100, 224517 (2019)
[5] R. Guida and J. Zinn-Justin, Critical exponents of the N-vector model, Journal of Physics A: Mathematical and General 31, 8103 (1998)
[6] O. Schnetz, Numbers and functions in quantum field theory, Phys. Rev. D 97, 085018 (2018)
[7] M. V. Kompaniets and E. Panzer, Minimally subtracted six-loop renormalization of \( \phi^4 \)-symmetric \( \phi^4 \) theory and critical exponents, Phys. Rev. D 96, 036016 (2017)
[8] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, Solving the 3d ising model with the conformal bootstrap, Phys. Rev. D 86, 025022 (2012)
[9] D. Simmons-Duffin, A semidefinite program solver for the conformal bootstrap, Journal of High Energy Physics 2015, 174 (2015)
[10] I. Balog, H. Chaté, B. Delamotte, M. Marohnić, and N. Wschebor, Convergence of nonperturbative approximations to the renormalization group, Phys. Rev. Lett. 123, 240604 (2019)
Appendix A: Technical details of numerical computations

We locate the Wilson-Fisher fixed point corresponding to the complete set of beta-functions \( \{ \beta_0 = 0 \} \) for the dimensionless couplings \( \tilde{g}_i \). In order to find the nontrivial root of this system of equations we have used the Affine Covariant Newton method with the iterative algorithm detailed in Sec. [A] and [B]. In the rare case it did not converge in 100 iterations we further applied the secant method. This requires two initial values, to obtain those we simply multiply the output from the Affine Covariant Newton method with the iterative algorithm.

The numerical integration of the threshold integrals \( L, M \) and \( N \) (from Sec. [III]) are computed with the optimized \texttt{NIntegrate} command of Mathematica, which selects the Gauss-Konrod quadrature formula as the most efficient numerical integration method.

In every instance we have worked with 12 or more digits of precision in our numerical computations.

Appendix B: Subleading scaling corrections

Our method also provides the scaling corrections \( \omega < \omega_1 < \omega_2 < \ldots \) to the correlation length as discussed in Sect. [IIID] These smallest one \( \omega \) is shown in Tab. [II] for the \( O(N) \) model at various \( N \) values. The larger scaling corrections \( \omega_1, \omega_2 \) are summarized in Tab. [IV].

Generally \( \omega_n \) becomes more susceptible to the polynomial truncation with increasing \( n \), \( \omega_1, \omega_2 \) are only stable in the first two or three significant digits with our employed truncation, detailed in Sec. [VIIB].
TABLE IV. The first two subleading scaling corrections $\omega_1$ and $\omega_2$ at the LPA, NLO and NNLO of the DE for the $O(N)$ symmetric models in $d = 3$ Euclidean dimensions. We have only kept the first few significant digits, which coincide for the predictions computed from the $\Theta$-regulator (13) and the exponential regulator (16).