An efficient spectral method for solving third-kind Volterra integral equations with non-smooth solutions

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Received: 29 June 2022 / Revised: 3 April 2023 / Accepted: 9 May 2023 / Published online: 24 May 2023
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Abstract
This paper is concerned with the numerical solution of Volterra integral equations of the third kind with non-smooth solutions based on the recursive approach of the spectral Tau method. To this end, a new set of the fractional version of canonical basis polynomials (called FC-polynomials) is introduced. The approximate polynomial solution (called Tau-solution) is expressed in terms of FC-polynomials. The fractional structure of Tau-solution allows recovering the standard degree of accuracy of spectral methods even in the case of non-smooth solutions. The convergence analysis of the method is carried out. The obtained numerical results show the accuracy and efficiency of the method compared to other existing methods.

Keywords Fractional recursive Tau method · Third-kind Volterra integral equation · Fractional canonical polynomials · Convergence analysis · Non-smooth solutions

Mathematics Subject Classification 65M70 · 35R09

1 Introduction

Integral equations appeared for the first time in a work by V. Volterra in 1884, where he studied the solution of an electrostatic problem (Volterra 1884), and later used by the same author in the modeling of population growth (Volterra 1928). Many mathematical models that arise in various problems of physics, biology, chemistry, engineering, etc. are based on integral equations Wazwaz (2011); Corduneanu (2010); Rahman (2007). Some of the most
well-known numerical techniques used to approximate solutions of integral equations are: multi-step methods (Cardone et al. 2018; Bellen et al. 1990; Conte and Paternoster 2009; Hoppensteadt et al. 2007), spectral methods (Cardone et al. 2018; Shen et al. 2011; Brunner 2017), product integration methods (Linz 1971; Diogo et al. 2004), Adomian decomposition method, homotopy perturbation method, Picard method (Wazwaz et al. 2013; Wazwaz 2011; Cherruault et al. 1992), etc. Linear Volterra integral equations (VIEs) of the general form

\[ a(t)y(t) = g(t) + \int_{0}^{t} k(t, s)y(s)ds, \quad t \in \Omega = [0, 1], \]  

(1)

where \( a(t) = 0 \) at a finite number of points in \( \Omega \) are called third-kind VIEs. In 1896, Volterra studied the solvability of the form (1) when \( k(0, 0) = 0 \) and \( k(0, 0) \neq 0 \) on \( (0, 1) \) ([41], Nota III). The corresponding integral operator in Eq. (1) is non-compact on \( C(\Omega) \) if \( k(0, 0) \neq 0 \) and its spectrum will be uncountable (Brunner 2017). This is a fundamental property for investigating the existence and uniqueness of solution for Eq. (1). In 1911, the third-kind VIEs with various types of kernel singularities were studied by Evans in Evans (1911). Third-kind VIEs with weakly singular kernels have been considered in Pereverzev and Prössdorf (1997); Allaei et al. (2017); Brunner (2017) and references therein. The aim of this paper is to present a numerical method for solving a class of third-kind VIEs with weakly singular kernels

\[ t^\beta y(t) = t^\beta g(t) + \int_{0}^{t} (t - s)^{\gamma - 1}s^{\beta - \gamma} H(t, s)y(s)ds, \quad t \in \Omega, \]  

(2)

with \( \beta \geq \gamma, \quad 0 < \gamma \leq 1, \quad \beta > 0 \) and assume that

\[ \gamma = \frac{p_1}{q_1}, \quad \beta = \frac{p_2}{q_2}, \]  

(3)

where \( p_i, q_i \in \mathbb{N} \) and \( lcm(p_i, q_i) = 1 \) for \( i = 1, 2 \). We denote the least common multiple of two positive integers \( a \) and \( b \) by \( lcm(a, b) \). The given functions \( g \) and \( H \) are continuous on the \( \Omega \) and \( D := \{(t, s) : 0 \leq s \leq t \leq 1\} \), respectively, and \( y(t) \) is the unknown function. Equation (2) arises frequently in the modeling of many applied science problems, including bioscience modeling, ecological competition systems, and population growth. The book (Ladopoulos 2000), contains various physical and engineering applications of singular integral equations. Equation (2) can be written in the form of the equivalent cordial integral equation

\[ y(t) = g(t) + Qy(t), \]  

(4)

where

\[ Qy(t) = \int_{0}^{t} t^{-\beta}(t - s)^{\gamma - 1}s^{\beta - \gamma} H(t, s)y(s)ds. \]  

(5)

is called cordial Volterra integral operator. This is a bounded linear operator on \( C(\Omega) \). As G. Vainikko remarked in his first work on cordial Volterra equations (Vainikko 2009) a typical example of a kernel that generates a cordial integral operator is

\[ K(t, s) = t^{-\beta}s^{\beta - 1}, \]  

with \( \beta > 0 \). The interest on this type of kernel (which is also studied, for example, in Diogo et al. (2004) and Allaei et al. (2017)) is motivated by the fact that it arises in connection with a heat conduction problem with mixed boundary conditions, analyzed for the first time by Bartoshevich (1975). Moreover, there is a well-known connection between Volterra equations of the form (2) and fractional order equations. Namely, if \( H(t, s) \equiv 1 \) and \( \beta = \gamma \)
this integral operator is the Riemann–Liouville operator of order γ. Integral and integrodifferential equations of fractional order appear in many mathematical models of physics and engineering (see for example (Diethelm 2010)) which gives an additional motivation for studying equations of the form (2). In 2015, Allaei et al. (2017) studied the existence and uniqueness of the solutions of Eq. (4):

**Theorem 1**  The cordial integral operator $K$ with $\beta \geq \gamma$ is compact if $H(0, 0) = 0$, otherwise, it is a non-compact operator with the uncountable spectrum

$$\triangle_K = \{0\} \cup \{H(0, 0)B(\gamma, 1 - \gamma + \beta + \lambda); \ Re(\lambda) \geq 0\}.$$  

Also, the Eq. (4) has a unique solution $y \in C(I)$ if $1 \notin \triangle_K$. Here, $B(., .)$ denotes the Beta function and

$$Kt^\lambda = \Theta(\lambda)t^\lambda; \ \Theta(\lambda) = \int_0^1 \Theta(t)t^\lambda dt,$$

with $\Theta(t) = t^{\beta-\gamma}(1-t)^{\nu-1}$.

In the last years, some numerical methods have been introduced to solve Eq. (2) including: collocation methods with modified graded meshes (Allaei et al. 2017), operational matrix methods with hat functions (Nemati and Lima 2018), multi-step collocation methods (Shayanfard et al. 2019), collocation methods (Song et al. 2019; Talaei and Micula 2022), Legendre-Galerkin methods (Cai 2020), Bernstein approximation techniques (Usta 2021). However, there is very little work on third-kind Volterra equations with non-smooth solutions via spectral methods.

The spectral methods are a class of applicable numerical techniques for obtaining approximate solutions of functional equations, based on polynomials basis functions such as Chebyshev, Legendre, Jacobi, etc. The three most commonly used techniques among spectral methods are: Galerkin, collocation, and Tau method (Shen et al. 2011). These methods have exponential rate of convergence in solving the problems with smooth solutions (Canuto et al. 2006; Brunner 2017; Hesthaven et al. 2009). Since spectral methods with usual polynomial basis have low convergence order in the case of non-smooth solutions, we need to use appropriate basis in this case. A useful technique to solve this problem is to use a fractional version of polynomial basis functions (see Conte et al. (2020); Azizipour and Shahmorad (2022); Kazem et al. (2013); Talaei et al. (2019); Talaei (2019); Talaei and Shahmorad (2022) and references therein). Based on this motivational background, the main focus of this paper is to develop a new version of recursive approach to the Tau method to solve Eq. (2) by introducing a fractional set of the canonical polynomial basis (FC-polynomials). The FC-polynomials are constructed by a simple recursive algorithm. The approximate solution of the problem is obtained as a linear combination of FC-polynomials that is called Tau-solution. The unknown coefficients ($\tau$ parameters) in the Tau-solution are calculated by solving a linear algebraic system. This paper contains the following sections:

- Sect. 2: Some definitions and Theorems about shifted fractional Legendre polynomials on $[0, 1]$ are presented.
- Sect. 3: A new fractional version of recursive Tau method to solve Eq. (2) is introduced.
- Sect. 4: The convergence of this method is analyzed.
- Sect. 5: Some examples are given to show the accuracy of the method in comparison with other existing methods.
- Sect. 6: Conclusions and further work.
2 Fractional Legendre polynomials

In this section, we first provide the definition and some preliminary results for shifted fractional Legendre polynomials, and then introduce some projection and interpolation operators with error estimates that play a key role in the convergence analysis of the proposed method. In this section $c$ denotes a generic positive constant independent of $n$. We recall the following preliminaries which are needed in the sequel:

- The shifted Jacobi $L^2$ space on $\Omega$:

$$L^2_{w^{\theta, \xi}}(\Omega) := \{ u : \Omega \to \mathbb{R} ; \| u \|_{w^{\theta, \xi}} < \infty \},$$

in which $w^{\theta, \xi}(t) = (1 - t)^{\theta} t^{\xi}$ and $\| . \|_{\theta, \xi}$ is the shifted Jacobi $L^2$-norm defined by

$$\| u \|^2_{w^{\theta, \xi}} = (u, u)_{w^{\theta, \xi}} := \int_\Omega u^2(t) w^{\theta, \xi}(t) dt.$$

(We will use the notations $(., .)$, $L^2(\Omega)$ and $\| . \|$ when $\theta = \xi = 0$.)

- $\delta_i u(t):= (\delta / \delta t) u(t)$,
- $\mathbb{N}_0:= \{0, 1, 2, \ldots \}$,
- $\mathbb{P}_n:= \text{span}\{1, t, \ldots, t^n\}$,
- $\mathbb{M}_n^\theta:= \text{span}\{1, t^\theta, \ldots, t^n^\theta\}$,
- $\mathbb{Q}_n^\theta:= \text{span}\{1, t^\theta, \ldots, t^n^\theta\} \otimes \text{span}\{s^\theta, \ldots, s^n^\theta\}$,
- $\mathbb{V}_n^\theta:= \text{span}\{P_{0, \theta}(t), \ldots, P_{n, \theta}(t)\}$,

The shifted Jacobi polynomials on $\Omega$ denoted by $J_n^{\theta, \xi}(t)$ are orthogonal with respect to weight function $w^{\theta, \xi}(t)$ and the parameters $\theta, \xi > -1$, i.e.,

$$\int_\Omega J_m^{\theta, \xi}(t) J_n^{\theta, \xi}(t) w^{\theta, \xi}(t) dt = h_n^{\theta, \xi} \delta_{mn}, \quad m, n \geq 0,$$

in which

$$h_n^{\theta, \xi} = \| J_n^{\theta, \xi} \|_{w^{\theta, \xi}}^2 = \frac{\Gamma(n + \theta + 1) \Gamma(n + \xi + 1)}{(2n + \theta + \xi + 1)n! \Gamma(n + \theta + \xi + 1)},$$

and $\delta_{mn}$ is the Kronecker delta. These polynomials have the following explicit formula

$$J_i^{\theta, \xi}(t) = \sum_{j=0}^i \frac{(-1)^i-j \Gamma(i + \xi + 1) \Gamma(i + \theta + \xi + j + 1)}{\Gamma(\xi + j + 1) j! \Gamma(i + \theta + \xi + 1)(i - j)!} t^j.$$

The shifted Jacobi orthogonal projection $\Pi_n^{\theta, \xi} : L^2_{w^{\theta, \xi}}(\Omega) \to \mathbb{P}_n$ is defined by

$$\Pi_n^{\theta, \xi} u(t):= \sum_{i=0}^n u_i J_i^{\theta, \xi}(t); \quad u_i = \frac{1}{h_i^{\theta, \xi}} (u, J_i^{\theta, \xi})_{w^{\theta, \xi}},$$

The Lagrange interpolating operator $I_n : C(\Omega) \to \mathbb{P}_n$ associated with the $(n + 1)$-degree Legendre-Gauss points $\{ t_i \}_{i=0}^n$, the roots of $J_n^{0,0}(t)$, is denoted:

$$I_n u(t):= \sum_{i=0}^n u(t_i) F_i(t); \quad F_i(t) = \prod_{j=0, j \neq i}^n \frac{t - t_j}{t_i - t_j},$$
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which satisfies

\[ I_n u(t_i) = u(t_i), \quad i = 0, ..., n, \]

where \( F_i(t) \) is the Lagrange interpolation basis function. Define

\[ \|u\|_\infty := \sup\{|u(t)|; \quad t \in \Omega\}. \]

**Lemma 1** *(Xie et al. (2012), Lemma 3.1)* Assume that \( \partial^m_t u(t) \in L^2(\Omega) \), then

\[ \|u - \Pi^0_n u\|_\infty \leq cn^{\frac{3}{2} - m} \|\partial^m_t u(t)\|, \]

for \( 1 \leq m \leq n + 1 \).

**Lemma 2** *(Xie et al. (2012), Lemma 3.2)* Assume that \( \partial^m_t u(t) \in L^2(\Omega) \), then

\[ \|u - I_n u\|_\infty \leq Cn^{\frac{3}{2} - m} \|\partial^m_t u(t)\|, \]

for \( 1 \leq m \leq n + 1 \).

The fractional Legendre polynomials (FLPs) \( \{P_{i,\theta}(t)\}_i^{\infty} \) on \( \Omega \) are a class of fractional order shifted Jacobi polynomials (Shen and Wang 2016) with index \( \theta = \xi = 0 \) which are defined by

\[ P_{i,\theta}(t) := i^{0,0}_i(t^\theta), \quad i \in \mathbb{N}_0, \quad \theta \in (0, 1]. \]

These polynomials satisfy the following orthogonal condition with respect to \( \omega_\theta(t) = t^\theta - 1 \)

\[ \int_0^1 P_{i,\theta}(t) P_{j,\theta}(t) \omega_\theta(t) dt = \frac{\delta_{ij}}{2i + 1}. \tag{9} \]

and can be generated by the three-term recurrence relation as follows

\[ P_{i+1,\theta}(t) = \frac{(2i + 1)(2\theta - 1)}{i + 1} P_{i,\theta}(t) - \frac{i}{i + 1} P_{i-1,\theta}(t), \quad i = 1, 2, ..., \]

\[ P_{0,\theta}(t) = 1, \quad P_{1,\theta}(t) = 2\theta - 1. \]

The explicit form of the FLPs is given by

\[ P_{i,\theta}(t) = \sum_{j=0}^{i} C_{i,j} t^j; \quad C_{i,j} = \frac{(-1)^{i-j}(i + j)!}{(i - j)!j!}, \quad i = 0, 1, ... \tag{10} \]

Consider the fractional Legendre orthogonal projection \( \Pi_{n,\theta} : L^2_{\omega_\theta}(\Omega) \rightarrow \mathbb{V}^\theta_n \) defined by

\[ \Pi_{n,\theta} u(t) = \sum_{i=0}^{n} a_i P_{i,\theta}(t). \tag{11} \]

From orthogonality condition (9), all the coefficients \( a_i \) can be obtained as

\[ a_i = (2i + 1) \int_0^1 u(t) P_{i,\theta}(t) \omega_\theta(t) dt, \quad i = 0, 1, ..., n. \]
The fractional Lagrange interpolation operator $I_{n,\theta} : C(\Omega) \to \mathbb{M}_n^{\theta}$ based on \( t_i^{1/\theta} \) for \( i = 0, \ldots, n \), the roots of $P_{n+1, \theta}(t)$ defined by

$$I_{n,\theta}u(t) = \sum_{i=0}^{n} u(t_i^{1/\theta}) F_i,_{\theta}(t), \quad F_i,_{\theta}(t) = \prod_{j=0, j \neq i}^{n} \frac{t_i^{1/\theta} - t_j^{1/\theta}}{t_i^{1/\theta} - t_j}.$$  

(12)

The following Lemmas provide the error estimates in the $L^\infty$-norm:

**Lemma 3** Let $\partial_t^m u(t^{1/\theta}) \in L^2(\Omega)$, then

$$\|u - \Pi_{n,\theta} u\|_\infty \leq cn^{3-m} \|\partial_t^m u(t^{1/\theta})\|,$$

for $1 \leq m \leq n + 1$.

**Proof** Using the variable transformation $t = s^{1/\theta}$ and setting $\widehat{u}(s) = u(s^{1/\theta})$, by Lemma 1, we have

$$\|u - \Pi_{n,\theta} u\|_\infty = \|\widehat{u} - \Pi_n^{0,0} \widehat{u}\|_\infty \leq C n^{3-m} \|\partial_t^m u(s^{1/\theta})\|.$$  

(13)

**Lemma 4** Let $\partial_t^m u(t^{1/\theta}) \in L^2(\Omega)$, then

$$\|u - I_{n,\theta} u\|_\infty \leq C n^{3-m} \|\partial_t^m u(t^{1/\theta})\|,$$

for $1 \leq m \leq n + 1$.

**Proof** We have $I_{n,\theta}u(t) = I_n u(t^{1/\theta})$. The result is obtained in the same way as Lemma 3, but using lemma 2 instead of Lemma 1.

Given a two-index $\mathbf{m} = (m_1, m_2)$ of nonnegative integers, we set $|\mathbf{m}| = m_1 + m_2$ and

$$\partial^\mathbf{m} u(t, s) = \frac{\partial^{m_1} u(t, s)}{\partial t^{m_1} \partial s^{m_2}},$$

and define the $L^2$ space on $\Omega^2 := \Omega \times \Omega$:

$$L^2(\Omega^2) := \{ u : \Omega^2 \to \mathbb{R} ; \|u\|_{\Omega^2} = \int_{\Omega^2} u^2(t, s) \, dt \, ds < \infty \}.$$  

For any two-variable continuous function $u(t, s)$ on $\Omega^2$, we define the two-variable fractional Lagrange interpolation operator $\mathbf{I}_{n,\theta} : C(\Omega^2) \to \mathbb{Q}_n^{\theta}$, satisfying

$$\mathbf{I}_{n,\theta} u(t_i^{1/\theta}, s_j^{1/\theta}) = u(t_i^{1/\theta}, s_j^{1/\theta}), \quad i, j = 0, \ldots, n.$$  

It can be written in the form

$$\mathbf{I}_{n,\theta} u(t, s) = \sum_{i=0}^{n} \sum_{j=0}^{n} u(t_i^{1/\theta}, s_j^{1/\theta}) F_i,_{\theta}(t_i) F_j,_{\theta}(s_j).$$  

(14)

**Lemma 5** Assume that $\partial^\mathbf{m} u(t^{1/\theta}, s^{1/\theta}) \in L^2(\Omega^2)$ for $1 \leq |\mathbf{m}| \leq n + 1$, and let $\mathbf{I}_{n,\theta} u(t, s)$ be its interpolation polynomial defined in (14), then the following estimate holds

$$\|u(t, s) - \mathbf{I}_{n,\theta} u(t, s)\|_\infty \leq cn^{3-\mathbf{m}} |u|_{\mathbf{m}}.$$  

(15)
in which \( \tilde{m} = \min\{m_1, m_2\} \) and
\[
|u|_m := \left( \| \partial_t^{m_1} u(t^{1/\theta}, s^{1/\theta}) \|_{\Omega^2}^2 + \| \partial_s^{m_2} u(t^{1/\theta}, s^{1/\theta}) \|_{\Omega^2}^2 \right)^{1/2}.
\]

**Proof** We have
\[
I_{n,\theta} u(t, s) = I_{n,\theta} \circ \left( I_{n,\theta} u(t, s) \right)
\]

Hence, using Lemma 4 leads to
\[
\| u(t, s) - I_{n,\theta} u(t, s) \|_{\infty} \leq \| u(t, s) - I_{n,\theta} u(t, s) \|_{\infty} + \| I_{n,\theta} \circ (u(t, s) - I_{n,\theta} u(t, s)) \|_{\infty}.
\]
The rest of the proof is similar as for Lemma 3.

See further details in (Canuto et al. (2006), page 3.14, Shen et al. (2011), sec. 8.4.3).

### 3 Fractional recursive Tau method

The Tau method was introduced by Lanczos (1956). The key idea of this method is to obtain a polynomial solution by adding a perturbation term to the right-hand side of the problem. The concept of the canonical polynomials was first introduced in this method as approximation solution basis. In 1969, Ortiz (1969) developed the recursive approach of the Tau method to a general class of ordinary differential equations. This approach to the Tau method was extended later to solve a certain class of functional problems (see Eldaou and Khajah (1997); El-Daou and Al-Hamad (2012) and references therein). In Conte et al. (2020); Talaei et al. (2019), the authors studied and investigated the recursive approach to the Tau method to solve a class of weakly singular Volterra integral equations. In this section, we intend to carry out a new formulation of the Tau method to solve Eq. (4) following the idea in Talaei et al. (2019).

Define the following notations:
\[
\delta := \text{lcm}(q_1, q_2), \quad \theta_1 := \frac{\delta}{q_1}, \quad \theta_2 := \frac{\delta}{q_2},
\]
\[
\sigma_1 := p_1 \theta_1, \quad \sigma_2 := p_2 \theta_2,
\]
\[
\alpha := \frac{1}{\delta},
\]
therefore, we can write
\[
\gamma = \sigma_1 \alpha, \quad \beta = \sigma_2 \alpha.
\]

Assume that \( \tilde{H}(t, s) := I_{n,\alpha} H(t, s) \) is the two-variable fractional interpolation approximation of \( H(t, s) \), i.e.,
\[
\tilde{H}(t, s) = \sum_{i=0}^{n} \sum_{j=0}^{n} \tilde{h}_{i, j} F_{i,\alpha}(t) F_{j,\alpha}(s) = \sum_{i=0}^{n} \sum_{j=0}^{n} \tilde{h}_{i, j} t^i s^j, \quad (17)
\]
in which \( \tilde{h}_{i, j} = H(t^{1/\alpha}, s^{1/\alpha}) \). Now, we define a linear operator \( \tilde{L} \) (called cordial operator) related to kernel function \( \tilde{H}(t, s) \) as follows
\[
\tilde{L} : \mathbb{M}_n^{\alpha} \rightarrow \mathbb{M}_n^{\alpha}, \quad (n \leq \tilde{n}),
\]
\[
(\tilde{L} y)(t) := y(t) - \int_0^t (t - s)^{-\sigma_2 \alpha} (t - s)^{\sigma_1 \alpha - 1} s^{(\sigma_2 - \sigma_1) \alpha} \tilde{H}(t, s) y(s) \, ds. \quad (18)
\]
By applying \( \tilde{L} \) on \( t^{\alpha} \), we have
\[
\tilde{L}(t^{\alpha}) = t^{\alpha} - \int_0^t (t-s)^{-\sigma_2 \alpha} (t-s)^{\sigma_1 \alpha-1} s^{(\sigma_2-\sigma_1)\alpha} \tilde{H}(t,s) s^\alpha ds
\]
\[
= t^{\alpha} - \sum_{i,j=0}^n h_{i,j} B (\sigma_1 \alpha, (r+j+\sigma_2-\sigma_1)\alpha+1) t^{(r+i+j)\alpha}
\]
\[
:= \sum_{\ell=r}^{r+\vartheta_n} S_{\ell,r} t^{\alpha}, \quad \vartheta_n \in \{0, 1, \ldots, 2n\}.
\] (19)

Assume that \( \tilde{g}(t) := \Pi_{n,\alpha} g(t) \) is the fractional projection of \( g \) into the space \( \mathbb{W}^0_n \) as follows
\[
\tilde{g}(t) = \sum_{i=0}^n \tilde{g}_i P_{1,\alpha}(t) = \sum_{i=0}^n g_i t^{\alpha}.
\] (20)

The main idea of the Tau method (Ortiz 1969) is to find a polynomial solution \( y_n(t) \in \mathbb{W}^\alpha_n \) which is the exact solution (called Tau-solution) of the perturbed problem
\[
\tilde{L} y_n(t) = \tilde{g}(t) + \mathcal{H}_n(t).
\] (21)

The polynomial \( \mathcal{H}_n(t) \) is called a perturbation term. It follows from (19) that
\[
\text{deg } [\tilde{L} y_n(t) - \tilde{g}(t)] \leq (n + \vartheta_n) \alpha,
\]
in which \( \vartheta_n \) is called the height of the operator \( \tilde{L} \). Thus, \( \mathcal{H}_n(t) \) can be defined in terms of FLPs as follows
\[
\mathcal{H}_n(t) = \sum_{r \in S} \tau_{n,r} P_{n+\vartheta_n-r,\alpha}(t).
\] (22)

The set \( S \) and the unknown parameters \( \tau_{n,r} \) are determined when finding the Tau-solution of Eq. (21). Define the cordial Volterra integral operator with respect to \( \tilde{H}(t,s) \) as
\[
\tilde{K} y(t) := \int_0^t (t-s)^{-\beta} (t-s)^{\gamma-1} s^{\beta-\gamma} \tilde{H}(t,s) y(s) ds.
\] (23)

According to Theorem 1, the sufficient condition for the existence of unique solution to the perturbed problem (21) is that \( 1 \notin \triangle_{\tilde{L}} \), i.e.,
\[
h_{0,0} B(\gamma, 1 - \gamma + \beta + r \alpha) \neq 1, \quad r = 0, 1, \ldots.
\] (24)

**Definition 1** For all \( r \geq 0 \), the \( \varphi_r(t) \) are called fractional canonical polynomials (FC-polynomials) associated with the linear operator \( \tilde{L} \) if
\[
(\tilde{L} \varphi_r)(t) = t^{\alpha}.
\]

**Theorem 2** Assume that \( \vartheta_n > 0 \) and all of the above notations and condition (24) hold. Then the FC-polynomials are generated by a recursive relation of the form
\[
\varphi_{r+\vartheta_n}(t) = \frac{1}{S_{r+\vartheta_n,r}} \left( t^{\alpha} - \sum_{\ell=r}^{r+\vartheta_n-1} S_{\ell,r} \varphi_\ell(t) \right), \quad r = 0, 1, \ldots.
\] (25)
In particular, for $\vartheta_n = 0$ we have
\[
\varphi_r(t) = \frac{1}{S_{r,r}} t^{r\alpha}, \quad r = 0, 1, \ldots, \tag{26}
\]
in which
\[
S_{r,r} = 1 - h_{0,0} B (\sigma_1 \alpha, (r + \sigma_2 - \sigma_1) \alpha + 1).
\]

**Proof.** By using of Definition 1 in (19) we get
\[
\tilde{L}(t^{r\alpha}) = \sum_{t=r}^{r+\vartheta_n} S_{t,t} t^{r\alpha} = S_{r+\vartheta_n,r} t^{(r+\vartheta_n)\alpha} + \sum_{\ell=r}^{r+\vartheta_n} S_{\ell,r} \tilde{L} \varphi_\ell(t).
\]

From the linearity of operator $\tilde{L}$ we obtain
\[
\tilde{L} \left( t^{r\alpha} - \sum_{\ell=r}^{r+\vartheta_n-1} S_{\ell,r} \varphi_\ell(t) \right) = S_{r+\vartheta_n,r} t^{(r+\vartheta_n)\alpha},
\]
so
\[
\varphi_{r+\vartheta_n}(t) = \frac{1}{S_{r+\vartheta_n,r}} \left( t^{r\alpha} - \sum_{\ell=r}^{r+\vartheta_n-1} S_{\ell,r} \varphi_\ell(t) \right), \quad r = 0, 1, \ldots,
\]
because of Definition 1. In view of the above process, we can derive (26) for $\vartheta_n = 0$. □

From relation (25), in the case of $\vartheta_n > 0$, the FC-polynomials $\{\varphi_r(t)\}_{r=0}^{\infty}$ generated by the finite set of polynomials $\{\psi_0(t), \ldots, \psi_{\vartheta_n-1}(t)\}$ that are called undefined FC-polynomials. Therefore, one can rewrite them as follows
\[
\varphi_r(t) = \psi_r(t) + \sum_{j=0}^{\vartheta_n-1} d_{r,j} \varphi_j(t), \tag{27}
\]
where $\psi_r(t)$ are called associated FC-polynomials. Now, set
\[
\psi_r(t) = 0, \quad d_{r,j} = \delta_{r,j}; \quad r, j = 0, \ldots, \vartheta_n - 1, \tag{28}
\]
in which $\delta_{r,j}$ is the Kronecker delta function. Substituting (27) in (25) yields
\[
\psi_{r+\vartheta_n}(t) + \sum_{j=0}^{\vartheta_n-1} d_{r+\vartheta_n,j} \varphi_j(t) = \frac{1}{S_{r+\vartheta_n,r}} \left( t^{r\alpha} - \sum_{\ell=r}^{r+\vartheta_n-1} S_{\ell,r} \left( \psi_\ell(t) + \sum_{j=0}^{\vartheta_n-1} d_{\ell,j} \varphi_j(t) \right) \right) = \frac{1}{S_{r+\vartheta_n,r}} \left( t^{r\alpha} - \sum_{\ell=r}^{r+\vartheta_n-1} S_{\ell,r} \psi_\ell(t) \right) + \sum_{j=0}^{\vartheta_n-1} \left( \frac{-1}{S_{r+\vartheta_n,r}} \left( \sum_{\ell=r}^{r+\vartheta_n-1} d_{\ell,j} S_{\ell,r} \right) \varphi_j(t) \right). \tag{29}
\]

Comparison of the both sides of (29) gives
\[
\begin{cases}
\psi_{r+\vartheta_n}(t) = \frac{1}{S_{r+\vartheta_n,r}} \left( t^{r\alpha} - \sum_{\ell=r}^{r+\vartheta_n-1} S_{\ell,r} \psi_\ell(t) \right), \quad r \geq 0, \\
d_{r+\vartheta_n,j} = \frac{-1}{S_{r+\vartheta_n,r}} \left( \sum_{\ell=r}^{r+\vartheta_n-1} d_{\ell,j} S_{\ell,r} \right), \quad r \geq 0, \quad j = 0, \ldots, \vartheta_n - 1.
\end{cases} \tag{30}
\]
The relations (28) and (30) allow to generate the FC-polynomials by a simple recursive procedure.

**Theorem 3** Assume that $H_n(t)$ is of the from (22) and all of the above notations and condition (24) hold. Then, the exact polynomial solution (Tau-solution) of Eq. (21) is given by

$$y_n(t) = \sum_{r=0}^{n} g_r \psi_r(t) + \sum_{r=0}^{n} \tau_{n,r} \left( \sum_{j=0}^{n+\theta_n-r} C_{n+\theta_n-r,j} \psi_j(t) \right),$$

(31)

where $\tau_{n,r}$ are determined by solving the $\theta_n$-dimensional linear system of algebraic equations (called Tau-system)

$$\mathcal{M}\overline{\tau} = \mathcal{B},$$

(32)

where

$$\mathcal{M} = \begin{pmatrix}
\sum_{j=0}^{n+\theta_n} C_{n+\theta_n,j} \psi_j & \sum_{j=0}^{n+\theta_n-1} C_{n+\theta_n-1,j} \psi_j & \cdots & \sum_{j=0}^{n+1} C_{n+1,j} \psi_j \\
\sum_{j=0}^{n+\theta_n} C_{n+\theta_n,j} \psi_j & \sum_{j=0}^{n+\theta_n-1} C_{n+\theta_n-1,j} \psi_j & \cdots & \sum_{j=0}^{n+1} C_{n+1,j} \psi_j \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{j=0}^{n+\theta_n} C_{n+\theta_n,j} \psi_j & \sum_{j=0}^{n+\theta_n-1} C_{n+\theta_n-1,j} \psi_j & \cdots & \sum_{j=0}^{n+1} C_{n+1,j} \psi_j
\end{pmatrix},$$

$$\overline{\tau} = (\tau_{n,0}, \tau_{n,1}, \cdots, \tau_{n,\theta_n-1})^T,$$

and

$$\mathcal{B} = (-\sum_{r=0}^{n} g_r \rho_{r,0}, -\sum_{r=0}^{n} g_r \rho_{r,1}, \cdots, -\sum_{r=0}^{n} g_r \rho_{r,\theta_n-1})^T.$$

**Proof** From relation (21) and (22), we have

$$\widehat{L} y_n(t) = \sum_{r=0}^{n} g_r t^\alpha + \sum_{r=0}^{n} \tau_{n,r} \left( \sum_{j=0}^{n+\theta_n-r} C_{n+\theta_n-r,j} t^j \psi_j(t) \right).$$

(33)

Thus by Definition of 1 and linearity of $\widehat{L}$

$$\widehat{L} \left( y_n(t) - \sum_{r=0}^{n} \tau_{n,r} \left( \sum_{j=0}^{n+\theta_n-r} C_{n+\theta_n-r,j} \psi_j(t) \right) - \sum_{r=0}^{n} g_r \psi_r(t) \right) = 0.$$  

(34)

Because of Theorem 1, i.e., $1 \notin \Delta_{\overline{L}}$, we find

$$y_n(t) = \sum_{r=0}^{n} g_r \psi_r(t) + \sum_{r=0}^{n} \tau_{n,r} \left( \sum_{j=0}^{n+\theta_n-r} C_{n+\theta_n-r,j} \psi_j(t) \right).$$

(35)

In view of relation (27), we can rewrite (35) in the form

$$y_n(t) = \sum_{r=0}^{n} g_r \psi_r(t) + \sum_{r=0}^{n} \tau_{n,r} \left( \sum_{j=0}^{n+\theta_n-r} C_{n+\theta_n-r,j} \psi_j(t) \right).$$
\[ + \sum_{\ell=0}^{n} \left( \sum_{r=0}^{n} g_r d_{r,\ell} + \sum_{r \in S} \tau_{n,r} \left( \sum_{j=0}^{n+\vartheta n-r} C_{n+\vartheta n-r,j} d_{j,\ell} \right) \right) \varphi_{\ell}(t). \] (36)

In order to leave out the undefined canonical polynomials in (36) the parameters \( \tau_{n,r} \) are determined in such a way that the coefficients of \( \varphi_{\ell}(t), \ell = 0, ..., \vartheta n - 1 \) are equal to zero

\[ \sum_{r \in S} \tau_{n,r} \left( \sum_{j=0}^{n+\vartheta n-r} C_{n+\vartheta n-r,j} d_{j,\ell} \right) = -\sum_{r=0}^{n} g_r d_{r,\ell}, \quad \ell = 0, ..., \vartheta n - 1, \] (37)

therefore \( S = \{0, 1, ..., \vartheta n - 1\} \) and Tau-system (32) is derived. Finally, the Tau-solution of the perturbed problem (21) becomes

\[ y_n(t) = \sum_{r=0}^{n} g_r \psi_r(t) + \sum_{r=0}^{\vartheta n-1} \tau_{n,r} \left( \sum_{j=0}^{n+\vartheta n-r} C_{n+\vartheta n-r,j} \psi_j(t) \right). \] (38)

**Corollary 1** From (28) and (30), we have that

\[ \text{deg} [\psi_{\vartheta n}(t)] = 0, \quad \text{deg} [\psi_{\vartheta n+1}(t)] = \alpha, \ldots, \text{deg} [\psi_{\vartheta n+\alpha}(t)] = r\alpha, \]

therefore,

\[ \text{deg} [y_n(t)] = \text{deg} \left[ \sum_{i=0}^{n} g_i \psi_i(t) + \sum_{i \in S} \tau_{n,i} \left( \sum_{j=0}^{n+\vartheta n-i} C_{n+\vartheta n-i,j} \psi_j(t) \right) \right] \leq n\alpha. \]

**Corollary 2** For \( \vartheta n = 0 \) we have \( \mathcal{H}_n(t) \equiv 0 \), and the Tau-solution is obtained as follows

\[ y_n(t) = \sum_{r=0}^{n} g_r \varphi_r(t). \]

**Corollary 3** For \( g \equiv 0 \), from relation (31) we have

\[ y_n \equiv 0 \iff \tau = 0. \]

**Corollary 4** The dimension of the Tau-system remains fixed and independent of the degree of the Tau-solution if \( H(t, s) \in \mathbb{Q}^\alpha_n \) for \( n \in \mathbb{N}_0 \).

### 4 Convergence analysis

In this section, we provide a convergence analysis of the method.

**Theorem 4** Let \( y_n \) and \( y \) be the solution of Eqs. (21) and (4), respectively. If the following conditions are fulfilled:

1. \( \partial_t^{m} g(t^{1/\alpha}) \in L^2(\Omega) \) for \( m \geq 1 \),
2. \( \partial_t^{m} H(t^{1/\alpha}, s^{1/\alpha}) \in L^2(\Omega^2) \) for \( |m| \geq 1 \),

then for sufficiently large \( n \) the following error estimate holds

\[ \| y - y_n \|_\infty \leq c n^{\frac{3}{2} - \tilde{m}} \left( B(\gamma, \beta - \gamma + 1) H |m| \| y \|_\infty + \| \partial_t^{\tilde{m}} g(t^{1/\alpha}) \| \right), \] (39)

where \( \tilde{m} = \min\{\tilde{m}, \tilde{\tilde{m}}\} \) and \( c \) is a positive constant independent of \( n \).
Proof By subtracting (21) from (4) yields

\[ y - y_n = g - \tilde{g} + \mathcal{K}y - \tilde{\mathcal{K}}y_n - \mathcal{H}_n. \]  

(40)

By setting \( e_n := y - y_n \) we obtain

\[ \mathcal{K}y - \tilde{\mathcal{K}}y_n = \mathcal{K}y - \tilde{\mathcal{K}}y + \mathcal{K}(y - y_n) = \mathcal{K}y - \tilde{\mathcal{K}}y + \mathcal{K}e_n - (\mathcal{K}e_n - \tilde{\mathcal{K}}e_n). \]  

(41)

From (40) and (41) it follows that

\[ e_n = \int_0^t t^{-\beta} (t - s)^{-1} s^{\beta - \gamma} H(t, s)e_n(s)ds + J_1 + J_2 - J_3 - \mathcal{H}_n. \]  

(42)

where

\[ J_1 = g - \tilde{g}, \quad J_2 = (\mathcal{K} - \tilde{\mathcal{K}})y, \quad J_3 = (\mathcal{K} - \tilde{\mathcal{K}})e_n. \]

Consequently,

\[ |e_n(t)| \leq \|H\|_{\infty} \int_0^t t^{-\beta} (t - s)^{-1} s^{\beta - \gamma} |e_n(s)|ds + |J_1| + |J_2| + |J_3| + |\mathcal{H}_n|. \]  

(43)

By the Gronwall’s inequality (Lemma 3.5; Ref. Ma and Huang (2021)), we obtain

\[ \|e_n\|_{\infty} \leq c \left( \|J_1\|_{\infty} + \|J_2\|_{\infty} + \|J_3\|_{\infty} + \|\mathcal{H}_n\|_{\infty} \right). \]  

(44)

By Lemma 3, we have

\[ \|J_1\|_{\infty} \leq cn^{\frac{3}{2}-m} \|\partial_t^m g(t^{1/\alpha})\|. \]  

(45)

By definition of operators \( \mathcal{K} \) and \( \tilde{\mathcal{K}} \),

\[ \mathcal{K}y - \tilde{\mathcal{K}}y = \int_0^t t^{-\beta} (t - s)^{-1} s^{\beta - \gamma} (H(t, s) - \tilde{H}(t, s))y(s)ds \]  

(46)

Thanks to Lemma 5, we obtain

\[ |(\mathcal{K}y)(t) - (\tilde{\mathcal{K}}y)(t)| \leq \int_0^t t^{-\beta} (t - s)^{-1} s^{\beta - \gamma} |H(t, s) - \tilde{H}(t, s)||y(s)|ds \leq B(\gamma, \beta - \gamma + 1)\|H - \tilde{H}\|_{\infty}\|y\|_{\infty} \leq cB(\gamma, \beta - \gamma + 1)n^{\frac{3}{2}-m}\|H\|_{\mathbf{m}}\|y\|_{\infty}, \]  

(47)

so

\[ \|J_2\|_{\infty} \leq cB(\gamma, \beta - \gamma + 1)n^{\frac{3}{2}-m}\|H\|_{\mathbf{m}}\|e_n\|_{\infty}. \]  

(48)

In the same way, for \( J_3 \) we have

\[ \|J_3\|_{\infty} \leq cB(\gamma, \beta - \gamma + 1)n^{\frac{3}{2}-m}\|H\|_{\mathbf{m}}\|e_n\|_{\infty}. \]  

(49)

According to Conte et al. (2020), the sequence of dual spaces \( \{\mathbb{W}_n^{\alpha, \perp}\} \) have the following property

\[ \cdots \mathbb{W}_{n+2}^{\alpha, \perp} \subset \mathbb{W}_{n+1}^{\alpha, \perp} \subset \mathbb{W}_n^{\alpha, \perp}; \quad \text{diam}(\mathbb{W}_n^{\alpha, \perp}) \to 0, \quad n \to \infty, \]
therefore,

$$\|H_n\|_\infty \rightarrow 0, \quad n \rightarrow \infty,$$

(50)
because of $H_n(t) \in W_n^\perp$. Finally, from relations (45), (46), (49) and (50) we obtain the estimate (39) provided $n$ is sufficiently large. \qed

5 Numerical results

This section contain some examples to illustrate the significance of the method. These examples were also studied in recent works (Cai 2020; Nemati et al. 2021; Allaei et al. 2017; Shayanfard et al. 2019; Wang et al. 2021; Nemati and Lima 2018). All of the numerical calculation are performed on computer using a program written in Maple 2018. The order of convergence is numerically computed by using the following formula

$$(O.C)_n := \log_2 \left( \frac{\|e_{n/2}\|_\infty}{\|e_n\|_\infty} \right).$$

Here, we state an algorithm to summarize the steps of the method:

**Algorithm:** Fractional recursive Tau method

**Input:** Function $f(t)$, $H(t, s)$ and the values of $\beta$ and $\gamma$.

**Step 1:** Compute the values of $\sigma_1$, $\sigma_2$ and $\alpha$.

**Step 2:** Compute $\tilde{L}(t^\alpha) = \sum_{\ell=r}^{r+\eta_n} S_{\ell, r} t^{r\alpha}$ from (19).

**Step 3:** Construct $\psi_r(t)$ and $d_{r, j}$ for $r \geq 0$ and $j \in S$ from (30).

**Step 4:** Solve the linear system (32).

**Output:** Construct the Tau-solution $y_n(t)$ from (31).

**Example 1** Usta (2021); Shayanfard et al. (2019) Consider the following third-kind Volterra integral equation

$$t^{1/2} y(t) = t^2 - B(\frac{1}{2}, \frac{9}{2}) t^4 + \int_0^t (t - s)^{-1/2} s^2 y(s) ds,$$

(51)

with the exact solution $y(t) = t^{3/2}$. Based on the Theorem 1, the associated cordial Volterra integral operator is

$$\mathcal{K}y(t) = \int_0^t t^{-1/2} (t - s)^{-1/2} s^2 y(s) ds,$$

which is compact, and therefore the Eq. (51) has unique solution. Now, we apply the algorithm to obtain the Tau-solution. Thus

$$\begin{cases}
\psi_{r+4}(t) = \frac{1}{S_{r+4, r}} \left( t^{r\alpha} - \sum_{\ell=r}^{r+3} S_{\ell, r} \psi_{\ell}(t) \right), & r \geq 0, \\
d_{r+4, j} = -\frac{1}{S_{r+4, r}} \left( \sum_{\ell=r}^{r+3} d_{\ell, j} S_{\ell, r} \right), & r \geq 0, \quad j = 0, \ldots, 3,
\end{cases}$$

(52)
where
\[
\psi_0(t) = \cdots = \psi_3(t) = 0, \quad d_{i,j} = \delta_{i,j}, \quad i, j = 0, ..., 3,
\]
and so for \( n = 8 \), we obtain \( \tau_{8,0} = \tau_{8,1} = \tau_{8,2} = \tau_{8,3} = 0 \). Then,
\[
y_8(t) = \sum_{r=0}^{8} g_r \psi_r(t) + \sum_{r=0}^{3} \tau_{8,r} \left( \sum_{j=0}^{12-r} C_{12-r,j} \psi_j(t) \right),
\]
\[
eq t^{\frac{3}{2}}.
\]

Since it provides the exact solution of Eq. (51), our method is more efficient than the methods described in Usta (2021); Shayanfard et al. (2019).

**Example 2** Consider the following third-kind Volterra integral equation
\[
t^2 \frac{3}{2} y(t) = t^{\frac{47}{12}} \left( 1 - \frac{\Gamma \left( \frac{1}{3} \right) \Gamma \left( \frac{55}{12} \right)}{\pi \sqrt{3} \Gamma \left( \frac{59}{12} \right)} \right) + \frac{\sqrt{3}}{3\pi} \int_0^t (t-s)^{-\frac{2}{3}} s^{\frac{1}{3}} y(s) ds, \tag{53}
\]
and the exact solution \( y(t) = t^{\frac{13}{3}} \). The integral equation (53) is a well-known Lighthill model that describes the temperature distribution on the surface of a projectile moving through a laminar layer (Lighthill 1950). Based on Theorem 1, the associated cordial Volterra integral operator
\[
\mathcal{K} y(t) = \int_0^t t^{-\frac{3}{2}} (t-s)^{-\frac{2}{3}} \sqrt{3} s^{\frac{1}{3}} y(s) ds
\]
is non-compact with the uncountable spectrum
\[
\Delta_{\mathcal{K}} = \{0\} \cup \{ \frac{\sqrt{3}}{3\pi} B \left( \frac{1}{3}, 1 + \frac{1}{3} + \lambda \right); \quad Re(\lambda) \geq 0 \},
\]
An efficient spectral method...

Fig. 1 Absolute error function $|e_n(t)|$ for various $n$ for Example 2

Fig. 2 Absolute error function $|e_n(t)|$ for various $n$ for Example 2

Table 1 Comparison the values of $\|e_n\|_\infty$ of our method and Ref. Cai (2020) versus $n$ for Example 2

| n   | 6     | 8     | 10    | 12    | 14    | 16    |
|-----|-------|-------|-------|-------|-------|-------|
| Our method | 5.17e−03 | 9.61e−05 | 5.37e−08 | 3.61e−10 | 9.79e−12 | 5.28e−13 |
| CPU-Time(s) | 1.529  | 2.917  | 2.184  | 2.464  | 3.089  | 3.370  |
| (O.C)$_n$       | -      | 9.39   | 18.49  | 23.77  | 26.48  | 27.44  |
| Ref. Cai (2020) | 1.09e−05 | 1.48e−06 | 3.15e−07 | 8.84e−08 | 2.99e−08 | 7.71e−08 |

therefore, the Eq. (53) has unique solution if $1 \notin \triangle$, i.e.,

$$1 - \frac{\sqrt{3}}{3\pi} B\left(\frac{1}{3}, 1 + \frac{1}{3} + \lambda\right) \neq 0.$$  

By implementation of the algorithm, we obtain

$$\begin{align*}
\varphi_r(t) &= \frac{1}{1 - \frac{\sqrt{3}}{3\pi} B\left(\frac{1}{3}, \frac{1+r}{3} + 1\right)} t^r, \quad r \geq 0, \\
y_n(t) &= \sum_{r=0}^{n} g_r \varphi_r(t).
\end{align*}$$

The linear variation of error versus the degree of the Tau-solution in semi-log representation is displayed in Fig. 1. The behavior of absolute error function for different values of $n$ on the interval $[0, 1]$ is shown in Fig. 2. Table 1 shows the comparison of maximum absolute error of the method with the Ref. Cai (2020). The estimates of the convergence order increase with the approximation degree $n$ which confirms the high-order of accuracy of the proposed method regardless of the singular behavior of the exact solution. Also, the results of Table 2 show that our method has higher accuracy than the method described in Cai (2020).

**Example 3** Allaei et al. (2017) Consider the following third-kind Volterra integral equation

$$ty(t) = \frac{6}{7} t^3 \sqrt{t} + \int_{0}^{t} \frac{1}{2} y(s) ds,$$

(54)
### Table 2  Comparison results of Example 2

| Our method          | Ref. Allaei et al. (2017) | Ref. Wang et al. (2021) | Ref. Nemati et al. (2021) | Ref. Nemati and Lima (2018) |
|---------------------|---------------------------|-------------------------|---------------------------|----------------------------|
| (n=14)              | (m=3, N=256)              | (m=3, N=256)            | (ν = γ = 0, M=5, k=6)     | (n=192)                    |
| 9.79e−12            | 5.13e−9                   | 3.66e−12                | 2.02−10                   | 5.16e−9                    |
| (CPU-Time(s)=3.089) |                           |                         |                           |                            |

**Fig. 3**  The plot of $\| e \|_n$ for various $n$ for Example 3

**Fig. 4**  Absolute error function $|e_n(t)|$ for various $n$ Example 3
Table 3 The values of $\|e_n\|_\infty$ versus $n$ for Example 3

| n   | 4       | 6       | 8       | 10      | 12      | 14      |
|-----|---------|---------|---------|---------|---------|---------|
| $\|e_n\|_\infty$ | 1.14e−03 | 1.73e−04 | 4.56e−05 | 1.61e−05 | 6.84e−06 | 3.30e−06 |
| CPU-Time(s) | 0.515   | 0.609   | 0.624   | 0.734   | 0.858   | 0.0.873 |
| $(O.C)_n$ | -       | 4.72    | 4.64    | 4.64    | 4.66    | 4.68    |

Table 4 Comparison results of Example 3

| Our method | Ref. Allaei et al. (2017) | Ref. Wang et al. (2021) | Ref. Nemati et al. (2021) | Ref. Nemati and Lima (2018) |
|------------|---------------------------|-------------------------|---------------------------|-------------------------------|
| (n=20)     | (m=2, N=256)              | (m=2, N=256)            | (ν = γ = 0, M=5, k=6)    | (n=192)                       |
| 6.02e−07   | 1.30e−5                   | 2.46e−9                 | 2.69–8                    | 3.46e−8                       |
| (CPU-Time(s)=1.591) |                         |                         |                           |                               |

with the exact solution $y(t) = t^{\frac{5}{2}}$. This integral equations arise in the modeling of heat conduction problems with mixed-type boundary conditions problem. Based on the Theorem 1, the associated cordial Volterra integral operator

$$K y(t) = \frac{1}{2} \int_0^t t^{-\frac{3}{2}} y(s) ds$$

is non-compact with the uncountable spectrum

$$\Delta_K = [0] \cup \left\{ \frac{1}{2(1 + \lambda)} : \Re(\lambda) \geq 0 \right\},$$

therefore, the Eq. (54) has unique solution if $1 \notin \Delta$, i.e.,

$$1 - \frac{1}{2(1 + \lambda)} \neq 0.$$

The errors of the Tau-solution for different values of $n$ are listed in Fig 3 showing the spectral accuracy of our method for non-smooth solutions. The behavior of absolute error function for different values of $n$ on the interval $[0, 1]$ is shown in Fig. 4. Table 3 and 4 reports the efficiency of our method.

**Example 4** Nemati et al. (2021) Consider the following third-kind Volterra integral equation

$$t^{\frac{3}{2}} y(t) = \frac{t^{\frac{3}{10}}}{10} \left( 1 - \frac{\Gamma(\frac{19}{10})}{\sqrt{2\pi} \Gamma(\frac{43}{10})} \right) + \int_0^t \frac{\sqrt{2}}{2\pi} (t-s)^{-\frac{1}{2}} y(s) ds;$$

(55)

with the exact solution $y(t) = t^{\frac{9}{2}}$. Based on the Theorem 1, the associated cordial Volterra integral operator

$$K y(t) = \frac{\sqrt{2}}{2\pi} \int_0^t t^{-\frac{3}{2}} (t-s)^{-\frac{1}{2}} y(s) ds$$
Table 5 The values of $\|e_n\|_\infty$ versus $n$ for Example 4

| n   | 6     | 8     | 10    | 12    | 14    | 16    |
|-----|-------|-------|-------|-------|-------|-------|
| $\|e_n\|_\infty$ | 2.34e−05 | 3.08e−06 | 6.55e−07 | 1.86e−07 | 6.40e−08 | 2.54e−08 |
| CPU-Time(s) | 1.154 | 1.373 | 1.482 | 1.763 | 1.919 | 2.262 |
| $(O.C)\_n$ | - | 7.36 | 7.08 | 6.98 | 6.94 | 6.92 |

Table 6 Comparison results of Example 4

| Our method (n=14) | Ref. Nemati et al. (2021) with (M=5, k=5) $(\nu = \gamma =0.5)$ | (ν = γ =0) | (ν = γ =−0.5) |
|-------------------|---------------------------------------------------------------|--------------|-----------------|
| 6.40e−8           | 4.99e−8                                                       | 1.48e−7      | 4.99e−7         |
| (CPU-Time(s)= 1.919) |                                                              |              |                 |

Fig. 5 The plot of $\|e\|_n$ for various $n$ for Example 4

is non-compact with the uncountable spectrum

$$\Delta_K = \{0\} \cup \left\{ \frac{\sqrt{2}}{2\pi} B(\frac{1}{2}, 2 + \lambda); \ Re(\lambda) \geq 0 \right\},$$

therefore, the Eq. (55) has unique solution if $\frac{1}{\lambda} \notin \Delta$, i.e.,

$$1 - \frac{\sqrt{2}}{2\pi} B(\frac{1}{2}, 2 + \lambda) \neq 0.$$

The numerical results of Fig 5 and Table 5 show the exponential rate of convergence of the method. The behavior of absolute error function for different values of $n$ on the interval $[0, 1]$ is shown in Fig. 6. In Table 6, the error norm of the method is compared with the error norm in the case of Jacobi wavelets method. The comparison results confirm the accuracy of our method with a small number of basis FC-polynomials.
Fig. 6 Absolute error function $|e_n(t)|$ for various $n$ for Example 4

Table 7 The values of $\|e_n\|_\infty$ versus $n$ for Example 5

| $n$ | 6    | 8    | 10   | 12   | 14   | 16   |
|-----|------|------|------|------|------|------|
| $\|e_n\|_\infty$ | 8.29e−04 | 1.61e−05 | 2.64e−06 | 6.48e−08 | 4.18e−09 | 1.81e−09 |
| CPU-Time(s) | 3.619 | 5.226 | 6.022 | 6.989 | 7.878 | 10.093 |
| $(O.C)_n$ | -    | 9.41  | 10.51 | 13.64 | 14.81 | 13.12 |

Example 5 Consider the following third-kind Volterra integral equation

$$ty(t) = g(t) + \int_0^t (t - s)^{-\frac{1}{3}} s^2 y(s) \, ds$$  \hspace{1cm} (56)

with the exact solution $y(t) = t^{\frac{1}{2}} \sin(t)$. Based on the Theorem 1, the associated cordial Volterra integral operator

$$K_y(t) = \int_0^t t^{-\frac{1}{3}} (t - s)^{-\frac{1}{3}} s^2 y(s) \, ds$$

is compact and the Eq. (56) has unique solution. Figure 7 and Table 7 show the error norm for different values of $n$. Looking at the graphics displayed in Fig. 7, it shows that the logarithm of the absolute error depends linearly on $n$. This means that the error has an exponential decay. The behavior of absolute error function for different values of $n$ on the interval $[0, 1]$ is shown in Fig. 8. This figure shows that this method is effective even for small values of $n$.

6 Conclusion and future work

The Tau recursive method is applied using a new class of fractional order polynomials. These fractional order polynomials are generated based on a simple recursive algorithm. The performance of this method, in comparison with existing techniques, is illustrated by a set of numerical examples. The success of this method results from the introduction of fractional polynomials, which allow the Tau-solution to have a similar behavior to the one of the
non-smooth solution. As a future work, this method can be extended to the linear/nonlinear integro-differential integral equations of the third kind.

Acknowledgements The second author acknowledges financial support from FCT, through grants UIDB/04621/2020, UIDP/04621/2020.

Declarations

Conflict of interest The authors declared no potential conflict of interests with respect to the research, authorship, and/or publication of this manuscript.
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