Hamiltonian theory for the axial perturbations of a dynamical spherical background

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Abstract
We develop the Hamiltonian theory of axial perturbations around a general time-dependent spherical background spacetime. Using the fact that the linearized constraints are gauge generators, we isolate the physical and unconstrained axial gravitational wave in a Hamiltonian pair of variables. Then, switching to a more geometrical description of the system, we construct the only scalar combination of them. We obtain the well-known Gerlach and Sengupta scalar for axial perturbations, with no known equivalent for polar perturbations. The strategy, suggested and tested here, will be applied to the polar case in a separate article.

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1. Introduction and overview

Perturbation theory (both linear and higher order) is one of the most successful tools in general relativity (GR). It has been used to find the stability properties of a large variety of background solutions such as black holes, critical or cosmological solutions. It is also useful to model the evolution of dynamical processes in astrophysical scenarios that slightly deviate from an exact symmetry, such as the oscillations of a static spherical neutron star or a nearly spherical supernova explosion. In particular, it can be used to investigate the emission of gravitational waves in those processes.

A central problem in GR perturbation theory, inherited from the diffeomorphism invariance of the full theory, is that of isolating the physical degrees of freedom from the gauge-dependent information [1]. This can be done by imposing convenient gauge fixing conditions on the perturbations, as Regge and Wheeler [2] originally did in their study of perturbations of a Schwarzschild black hole. They and later Zerilli [3] succeeded in isolating the two physical degrees of freedom of the gravitational field around spherical vacuum, by taking suitable linear combinations of the remaining perturbations and their radial derivatives.

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These two variables further decouple due to their different properties under parity inversion: the Regge–Wheeler variable is axial, and the Zerilli variable is polar.

A more systematic treatment of the gauge freedom in GR perturbation theory was pioneered by Moncrief [4] in his Hamiltonian study of the nonspherical perturbations of Schwarzschild. In a Hamiltonian context, the four constraints obeyed by the 12 dynamical gravitational variables are the generators of the gauge transformations. Moncrief was able to use this information to perform several canonical transformations which reorganized the original six canonical pairs of variables into two physical pairs (equivalent to the Regge–Wheeler and Zerilli variables and their canonical momenta) and four gauge pairs in which the momenta were constrained to be zero, without any gauge fixing. The same technique was later applied to other spherical backgrounds with additional symmetries, such as Reissner–Nordström [5, 6], Oppenheimer–Snyder [7] or Friedmann–Robertson–Walker [8], but has never been applied to general spherically symmetric backgrounds, possibly highly time dependent. Hamiltonian perturbation theory has also been recently revisited in quantum gravity with a cosmological background [9]. A drawback of the Hamiltonian approach is that it is tied to a particular foliation of the background spacetime, and hence the geometric properties of the gauge-invariant variables under coordinate transformations involving time are far from obvious.

A Lagrangian formalism was introduced by Gerlach and Sengupta [10] (GS) to study perturbations around generic spherical spacetimes, in which the metric perturbation is geometrically split along the decomposition of the 4D manifold $M^4$ into the product of a general 2D Lorentzian manifold $M^2$ with boundary and the unit 2-sphere $S^2$. This is a highly geometrical framework, in which the meaning of the perturbations is transparent, and which also allows the construction of gauge-invariant variables. In the axial case, it has been possible to isolate the gravitational degree of freedom in a single scalar master variable which obeys a wave equation and can be coupled to any kind of matter, both in the background and the perturbations. This master scalar and its equation generalize the Regge–Wheeler variable and its equation to the axial perturbative problem around spherical symmetry for any reasonable matter model, and hence can be considered as the optimal framework for a perturbative study. Unfortunately, in the polar case there is not a master scalar valid for a generic spherical background and any matter model, though there are results for some particular cases. For instance, a master Zerilli scalar has been introduced by Sarbach and Tiglio [11] for a Schwarzschild background, which was later generalized to nonlinear electrodynamics [12], around any background solution of the theory. In [13, 14], the gauge-invariant combinations of the stress–energy tensor were also included but still on a vacuum background.

Both approaches to metric perturbation theory are complementary: the Hamiltonian approach offers a better framework to handle gauge invariance, while the Lagrangian approach gives a clearer picture of the geometrical structures being perturbed. This paper proposes a combination of both formalisms to construct gauge-invariant scalar perturbative variables containing all physically relevant information concerning the gravitational waves. We restrict ourselves to spherical backgrounds, but which can be highly dynamical. For definiteness, the dynamics will be introduced using a real massless scalar field, but could be done through any other matter model admitting a Hamiltonian description. This paper focuses on the axial subset of perturbations, for which the sought solution is the GS master scalar [10], previously found using the Lagrangian method only. We show how the Hamiltonian way allows a more systematic derivation of this object, and how both approaches mutually relate. Most important, this paper prepares the path toward a systematic analysis of the polar problem, for which a general gauge-invariant master scalar has never been found. Such analysis will be discussed in a second publication.
2. Hamiltonian perturbations in general relativity

2.1. ADM Hamiltonian formalism

Given the four-dimensional spacetime \( (M^4, (4)g_{\mu \nu}) \), we introduce a foliation of 3D spacelike slices as level surfaces of the time field \( t(x) \). The orthogonal vector \( u^\mu \) defines the projected metric \( (3)g_{\mu \nu} = (4)g_{\mu \nu} + u_\mu u_\nu \) on the slices. We introduce coordinates \( (t, x^i) \) adapted to the foliation, and work with three-dimensional objects. Only in the last part of this paper we shall use four-dimensional metric variables in order to compare our results with those from the GS formalism. Greek and Latin indices denote 4D and 3D tensors, respectively. A left superindex indicates dimensionality when confusion may arise.

The 4-metric is decomposed as customary in the lapse function, the shift vector and the 3-metric on the slices

\[
\alpha^{-2} \equiv (4)g^{tt}, \quad \beta_i \equiv (4)g_{ti}, \quad g_{ij} \equiv (4)g_{ij},
\]

with inverse

\[
g^{ij} = (4)g^{ij} + \alpha^{-2} \beta_i \beta_j ,
\]

with Latin indices always raised and lowered with \( g^{ij} \) and \( g_{ij} \).

The gravitational dynamical variables in the ADM Hamiltonian formalism are \( g_{ij} \) and their conjugated momenta:

\[
\Pi^i_{ij} \equiv (4)g^{ij} \left( K_{ij} - K^j_k \right), \quad \mu_g \equiv \sqrt{\det g_{ij}},
\]

where \( K_{ij} \) is the extrinsic curvature of the foliation hypersurfaces.

The spacetime will be assumed to contain a dynamical Klein–Gordon field \( \Phi \), whose evolution is controlled by the action

\[
S_{KG} = -\frac{1}{2} \int d^4x \sqrt{- (4)g} (4)g_{\mu \nu} \Phi,_{\mu} \Phi,_{\nu}
\]

\[
= \int d^4x \left[ \Pi \Phi,_{i} - \frac{\alpha}{2} \left( \frac{\Pi^2}{\mu_g} + \mu_g \Pi^j_{j} \Phi,_{i} \Phi,_{j} \right) - \beta^i (\Pi \Phi,_{i}) \right].
\]

Its canonical momentum has been defined as

\[
\Pi \equiv -\sqrt{- (4)g} (4)g^{i\mu} \Phi,_{\mu},
\]

and \((4)g \) denotes the determinant of the 4-metric. The complete action of the system, with coupling constant \( 16\pi G_N = 1 \) following [4], is given by

\[
S = S_G + S_{KG} = \int dt \int d^3x (\Pi^i_{ij} + \Pi \Phi,_{i} - \alpha \mathcal{H} - \beta^i \mathcal{H}_i).
\]

The Lagrange multipliers \( \alpha \) and \( \beta^i \) are associated with the constraints

\[
\mathcal{H} = \frac{1}{\mu_g} \left[ \Pi^i_{ij} \Pi,_{ij} - \frac{1}{2} (\Pi^j_{j})^2 \right] - \mu_g (3)R + \frac{1}{2} \left( \frac{\Pi^2}{\mu_g} + \mu_g \Pi^j_{j} \Phi,_{j} \Phi,_{j} \right),
\]

\[
\mathcal{H}_i = -2D_j \Pi^j_{ij} + \Pi \Phi,_{j},
\]
where $D_j$ is the covariant derivative associated with $g_{ij}$. Variation of action (7) with respect to $g_{ij}, \Pi_{ij}, \Phi$ and $\Pi$ gives the evolution equations for the corresponding conjugated variables.

2.2. Hamiltonian metric perturbations

Now suppose that the whole system is perturbed at first order. We define the following special notations:

\begin{align*}
C & \equiv \delta \alpha, \\
B^i & \equiv \delta (\beta^i), \\
h_{ij} & \equiv \delta (g_{ij}), \\
p^{ij} & \equiv \delta (\Pi_{ij}), \\
\varphi & \equiv \delta \Phi, \\
p & \equiv \delta \Pi.
\end{align*}

Note that we perturb the contravariant components of the shift vector because this will give rise to simpler equations, even though the comparison with GS variables will be slightly more involved because $\beta^i$ is better related to the 4-metric (see equation (1)).

Following Taub [15] and Moncrief [4] we shall obtain the equations for the linear perturbations using the Jacobi method of second variations. The idea is that the second variation of action (7), keeping only terms that are quadratic on first-order perturbations, gives an action functional for the perturbations,

\[
\frac{1}{2} \delta^2 S = \int d^4x \left[ p^{ij} h_{ij,t} + p_{\varphi,t} - C \delta (H) - B^i \delta (H_i) - \frac{\alpha}{2} \delta^2 (H) - \frac{\beta^i}{2} \delta^2 (H_i) \right].
\]

There are three kinds of terms. First we have kinetic terms, containing time derivatives of $h_{ij}$ and $\varphi$. Then we have the first variations of the constraints that, under a variation of the effective action (13) with respect to $B^i$ and $C$, give the constraints that must be obeyed by the perturbations

\[
\delta (H) = 0, \quad \delta (H_i) = 0.
\]

And finally we have the second variations of the constraints, which are quadratic in the perturbations $(h_{ij}, p^{ij}, \varphi, p)$, and will give the evolution equations for those perturbations. Even though we started with an exact Hamiltonian which was a linear combination of constraints, we end up having a quadratic Hamiltonian which does not vanish on shell.

The constraints $\mathcal{H}$ and $\mathcal{H}_i$ in general relativity are first-class constraints, and hence generators of gauge transformations on the constraint surface in phase space. This identifies the gauge orbits, but in general, it is not possible to separate explicitly the two physical degrees of freedom (four functions) from the four gauge variables and the four constrained variables in $g_{ij}$ and $\Pi^{ij}$. The situation in the linearized theory is simpler, but still only highly symmetric background scenarios allow the construction of gauge-invariant algebraic combinations of perturbations and their derivatives containing the physical information in the linearized approximation. (See [1] for a discussion of the importance of symmetry on the algebraic character of the combinations.) One of such background scenarios is a spherically symmetric spacetime, as we shall exploit for the rest of this paper.

For completeness, we provide the expressions for the first variations of the constraints:

\[
\delta (\mathcal{H}) = \frac{1}{\mu_g} \left( \Pi_{ij} - \frac{1}{2} g_{ij} \Pi^{ij} \right) \left( 2 p^{ij} + 2 h^i_k \Pi^{ij} - \frac{1}{2} h^k_l \Pi^{ij} \right) \\
+ \frac{1}{\mu_g} \left( \frac{1}{3} G^{ij} h_{ij} - D^i D^j h_{ij} + D^i D^j h_{ij} \right) \\
+ \frac{1}{4} h^i_k \left( - \frac{\mu^2}{\mu_g} + \mu_g \Phi, \Phi \right) + \frac{\Pi}{\mu_g} + \mu_g \Phi, \Phi \right) - \frac{1}{2} \mu_g h^{ij} \Phi, \Phi \right).
\]

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\( \delta \mathcal{H}_i = -2 D_k (h_{ij} \Pi^{ik} + g_{ij} p^{jk}) + \Pi^{ij} D_i h_{jj} + p \Phi_{,i} + \Pi \psi_{,i}. \)  
See [16] for intermediate expressions and techniques to compute these expressions. The corresponding expressions for their second variations are

\[
\delta^2 \mathcal{H} = \frac{1}{8 \mu_g} \left[ 8 p^2 + 8 \mu_g \{ G^{ij} (h_{ij} h_{kk} - 2 h_{ik} h_{jk}) + 16 P_{ij} P^{ij} - 8 P_{ij} P_{ij}^* \right] + 2 h^{ij} \left[ h^{kl} (8 \Pi_{ik} \Pi_{jl} - 4 \Pi_{ij} \Pi_{kl}) + h_{ij} \left( \Pi^2 - 2 \mu_g (5 R + 2 \Pi_{ik} \Pi_{jl} - \Pi^2 \Pi_{ij}^* \right) \right] - 8 \mu_g^2 \left( D_i h_{jk} - D_j h_{ik} - D_k D_i h_{j}^k + D_k D_j h_{i}^k \right) - 8 \Pi_{ik} P_{ij} + 32 \Pi_{ik} P_{ij}^* \right] + h_{ij}^2 \left( \Pi^2 + 2 \Pi_{ik} \Pi_{jl} - \Pi^2 \Pi_{ij}^* + 2 \mu_g (5 R) \right] + 8 h^{ij} \left( \Pi_{jk} \Pi_{ij} - 2 \Pi_{ij}^{*} \Pi_{kl} \right) - 8 \mu_g^2 \left( 4 D_j h_{j}^{ik} - D_k D^i h_{k}^j \right) - 8 \Pi p = -16 \Pi_{ij} P_{jk} + 8 \Pi_{ij} P_{jk}^* + \mu_g^2 \left[ 8 D_i \Phi D^i \varphi + h_{ij} \left( \Pi^2 - 2 \mu_g (5 R + 2 \Pi_{ik} \Pi_{jl} - \Pi^2 \Pi_{ij}^* \right) \right] + 2 h_{ij} D^i \Phi (4 h_{jk} D^j D^j \varphi - h_{ik} D^j \varphi) + 4 h_{ij} D^j h_{j}^k D^i \varphi + 4 h_{ij} (4 D^i \varphi + h_{ik} D^j \varphi) D^j \Phi \right] + 16 \left( D_i h_{ij} - D^i h_{ij} \right) D_k D_{ik} + 4 (2 D_j h_{jk} - 3 D_{jk} h_{ij}) D^i h_{ij} \right], \]

\[
\delta^2 \mathcal{H}_i = 2 p \psi_{,i} + 2 p^{ij} (D_i h_{jk} - D_j h_{ik} - 2 D_k h_{ij}) - 4 h_{ij} D_i p^{jk}. \]

### 3. Spherical background

Let us now restrict to a spherically symmetric background \( M^4 = M^2 \times S^2 \), where \( S^2 \) is the unit 2-sphere, and \( M^2 \) is a two-dimensional Lorentzian manifold with boundary. We shall use arbitrary coordinates \( x^A = (t, \rho) \) on \( M^2 \) and the usual spherical coordinates \( x^\alpha = (\theta, \phi) \) on \( S^2 \). Uppercase Latin indices \( A, B, C, \ldots \) denote objects on \( M^2 \), and lowercase Latin indices \( a, b, c, \ldots \) denote objects on \( S^2 \). The fact that we use arbitrary coordinates on \( M^2 \) will later allow us to keep track of the tensorial character of the different variables. Therefore, we do not impose any condition on the lapse or shift, apart from being consistent with spherical symmetry.

The 4-metric can be 2+2 decomposed as

\[
(\text{ds}^2)_4 = g_{AB}(x^D) \, dx^A \, dx^B + r^2 (x^D) \, d\Omega^2,
\]

with \( d\Omega^2 \) the round metric of the 2-sphere, and \( g_{AB} \) and \( r \) being a metric field and a scalar field on \( M^2 \), respectively. We define the vector field \( v_A = r^{-1} t_A \).

Using the radial coordinate \( \rho \) explicitly we can write the background spatial 3-metric as

\[
(\text{ds}^2)_3 = a^2 (t, \rho) \, dt^2 + r^2 (t, \rho) \, d\Omega^2.
\]

With a spherically symmetric lapse \( \alpha = \alpha(t, \rho) \) and shift vector \( \beta^i = (\beta(t, \rho), 0, 0) \) we have

\[
(\text{ds}^2)_4 = \left( -\alpha^2 + a^2 \beta^2 \right) \, dt^2 + 2 a^2 \beta \, dt \, d\rho + (\text{ds}^2)_3 = -\alpha^2 \, dt^2 + a^2 (d\rho + \beta \, dr)^2 + r^2 \, d\Omega^2,
\]

which takes the following matricial form:

\[
g_{AB} = \begin{pmatrix}
-a^2 + a^2 \beta^2 & a^2 \beta \\
a^2 \beta & a^2
\end{pmatrix}, \quad g^{AB} = \begin{pmatrix}
a^2 & -a^2 \beta \\
a^2 \beta & a^2 - a^2 \beta^2
\end{pmatrix}.
\]

The normal vector to the surfaces of constant \( t \) is \( u_\mu = (-\alpha, 0, 0, 0) \) or \( u^\mu = a^{-1} (-1, -\beta, 0, 0) \). Its orthogonal, radial vector is \( n^\mu = (0, a^{-1}, 0, 0) \) or \( n_\mu = a(\beta, 1, 0, 0) \). In order to work with more geometrical objects we define the following frame derivatives that act on any scalar...
We now derive the background equations, that will be later used to simplify the coefficients of the equations for the perturbations. It is convenient to define the following momentum-like variables, which have a definite tensorial character with respect to changes of the $\rho$ coordinate:

$$\Pi_1 \equiv \frac{a^2 \Pi^{00}}{\mu_g}, \quad \Pi_2 \equiv \frac{2r^2 \Pi^{00}}{\mu_g}, \quad \Pi_3 \equiv \frac{\Pi}{\mu_g}. \quad (25)$$

We can write the constraints in terms of these spherical variables:

$$H_{\mu g} = \frac{1}{\Pi_1} \left( \frac{\Pi_1}{2} - \Pi_2 \right) - \frac{1}{2} R + \frac{1}{2} (\Pi_3^2 + \Phi^2) = 0, \quad (26)$$

$$\frac{1}{\alpha} (r_{\rho} - \beta r_{,\rho}) = -\frac{2}{r^2} (r^2 \Pi_1)' + \frac{2r'}{r} \Pi_2 + \Pi_3 \Phi' = 0, \quad (27)$$

so that the action is

$$\frac{1}{4\pi} S = \int dt \int d\rho a r^2 \left[ 2\Pi_1 \frac{a_{,t}}{a} + 2\Pi_2 \frac{r_{,r}}{r} + \Pi_3 \Phi_{,t} - \frac{\mathcal{H}}{\mu_g} - \beta \frac{\mathcal{H}_{,\rho}}{\mu_g} \right]. \quad (28)$$

The evolution equations can be obtained by simple variation with respect to different variables:

$$\frac{1}{\alpha} (a_{,t} - (\beta a)_{,\rho}) = \frac{a}{2} (\Pi_1 - \Pi_2), \quad (29)$$

$$\frac{1}{\alpha} (r_{,t} - \beta r_{,\rho}) = -\frac{r}{2} \Pi_1, \quad (30)$$

$$\frac{1}{\alpha} (\Phi_{,t} - \beta \Phi_{,\rho}) = \Pi_3, \quad (31)$$

$$\frac{1}{\alpha} (\Pi_{1,t} - \beta \Pi_{1,\rho}) = \frac{3\Pi_1^2}{4} + \frac{1}{r^2} - \frac{r'}{r} \frac{(ar^2)'}{a^2 r} + \frac{1}{4} (\Pi_3^2 + \Phi^2), \quad (32)$$

$$\frac{1}{\alpha} (\Pi_{2,t} - \beta \Pi_{2,\rho}) = \frac{1}{2} (\Pi_1^2 + \Pi_2^2 - \Pi_1 \Pi_2) + \frac{2a' r'}{ar} - \frac{2(2r')^2}{ar} + \frac{1}{2} (\Pi_3^2 - \Phi^2), \quad (33)$$

$$\frac{1}{\alpha} (\Pi_{3,t} - \beta \Pi_{3,\rho}) = \frac{\Pi_3 (\Pi_1 + \Pi_2)}{2} + \frac{(ar^2 \Phi')'}{ar^2}. \quad (34)$$

Note that in spherical symmetry the restriction to vacuum, choosing Schwarzschild coordinates \((t, r = \rho)\), is given by $\Pi^{\mu\nu} = 0$, \((3)^3 R = 0\) and $\Phi = 0$, that is, $\Pi_1 = \Pi_2 = \Pi_3 = \Phi = 0$, which simplifies the previous expressions. In particular, constraints \((26)\) and \((27)\) are then trivially obeyed. This is the case studied by Moncrief [4], but here we want to analyze the general case.

4. Axial perturbations

4.1. Harmonic expansions

The tensor spherical harmonics form a complete basis on the 2-sphere to expand a tensor field of any rank. The appendix A gives the definitions for the harmonics that we shall need in this paper, as well as some of their basic properties and the relations with the harmonics used by Moncrief [4]. See [16] for full definitions in the arbitrary rank case and further properties. Tensor harmonics can be separated in two groups according to their polarity:
there are polar (or even, electric or poloidal) harmonics and axial (or odd, magnetic or toroidal) harmonics. In first-order perturbation theory around a spherical spacetime polar and axial harmonics decouple, and this paper will only deal with the axial part of the problem. Following Regge–Wheeler’s notation [2] for the metric perturbations and Moncrief’s notations [4] for the momentum and shift vector, we expand the perturbative variables in tensor spherical harmonics:

\[
\begin{align*}
    h_{ij} \, dx^i \, dx^j & = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left\{ -2(h_1)_m^l \, d\rho X_{l\,a}^m \, dx^a + (h_2)_m^l X_{l\,ab}^m \, dx^a \, dx^b \right\}, \\
    \frac{1}{\mu_g} p_{ij} \, dx^i \, dx^j & = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left\{ -2(\hat{p}_1)_l^m \, d\rho X_{l\,a}^m \, dx^a + (\hat{p}_2)_l^m X_{l\,ab}^m \, dx^a \, dx^b \right\}, \\
    B_i \, dx^i & = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} -(h_0)_l^m X_{l\,a}^m \, dx^a ,
\end{align*}
\]

Note that there are no axial perturbations of the 3D scalars \( \alpha, \Phi, \Pi \). That means that the scalar field plays no role from the perturbative point of view, though the background scalar field is still instrumental to allow for a general dynamical spacetime. As we will see, this does not imply any loss of generality. Different \((l, m)\) harmonic components also decouple around spherical symmetry, and so from now on we shall drop them from the perturbative variables, assuming that we work with a fixed pair of labels at any time. It is important to note that \( h_2, \hat{p}_2 \) and \( h_0 \) are scalars under changes of the \( \rho \) coordinate, but \( h_1 \) and \( \hat{p}_1 \) behave as components of a vector. In the language of 1D spacetimes, the latter are densities of weight +1, to be compensated with metric factors \( a \) to convert them into scalars. This will become clearer when comparing with the more geometrical GS approach.

The variables \((h_1, p_1)\) and \((h_2, p_2)\) form two pairs of canonically conjugated variables, whose evolution is partially determined by the arbitrary function \( h_0 \). For example, the evolution equations for the variables \( h_1 \) and \( h_2 \) can be easily obtained by perturbation of formula (3) after introducing the expansions (35)–(40)

\[
\begin{align*}
    \frac{1}{\alpha} \alpha(h_{1,t} - (\beta h_{1}),\rho) & = 2\hat{p}_1 + \Pi_1 h_1 + \frac{r^2}{\alpha} \left( \frac{h_0}{r^2} \right), \\
    \frac{1}{\alpha} \alpha(h_{2,t} - (\beta h_{2}),\rho) & = 2\hat{p}_2 + (\Pi_2 - \Pi_1) h_2 - \frac{2h_0}{\alpha}.
\end{align*}
\]

We shall later obtain the evolution equations for more convenient momentum variables.

4.2. Gauge-invariant perturbations

The action functional for the axial perturbations is

\[
\begin{align*}
    \frac{1}{2} \delta^2 S^{\text{axial}} & = \int dt \int d^3 x \left[ p^{ij} h_{ij,t} - B' \delta \delta (H_i) + \cdots \right], \\
    & = \int dt \left\{ \int d\rho \left( p_{1h_{1,t}} + p_{2h_{2,t}} \right) + H[h_0] + \cdots \right\}.
\end{align*}
\]
where the dots denote those terms coming from the second variation of the constraints, which we do not need to consider in this subsection. The functional $H$ will be defined below in terms of the first variation of the constraint. We have also defined

$$p_1 = \frac{2l(l+1)}{a} \hat{p}_1^*, \quad p_2 = \frac{a \lambda}{r^2} \hat{p}_2^*.$$  

(45)

where the star stands for complex conjugation, and we have defined

$$\lambda \equiv \frac{1}{2} (l - 1) l (l + 1) (l + 2).$$  

(46)

In terms of these variables, the perturbed constraint is given by

$$\delta(H^a)_{\text{axial}} = - \frac{X_{\mu \nu}}{l(l+1)} \left\{ \left( r^2 p_1 \right)_{\rho \mu} + 2 \frac{p_2}{a} + \lambda \frac{a \Pi_2 h_2}{r^2} + \frac{2l(l+1)}{a r^2} \left( \frac{r^2 \Pi h_1}{a} \right)_{\rho \mu} \right\}.$$  

(47)

which, in turn, defines the functional

$$H[h_0] = - \int d^3B \delta(H^i)_{\text{axial}}$$  

(48)

$$= \int d\rho \left\{ - r^2 \left( \frac{h_0}{r^2} \right)_{\rho} p_1 + 2h_0 p_2 + \lambda a \Pi_2 h_2 - \frac{2l(l+1)}{a r^2} \left( \frac{r^2 \Pi h_1}{a} \right)_{\rho} \right\}. $$  

(49)

This functional is the generator of gauge transformations, and of course commutes with itself on shell,

$$[H[f], H[g]] = \frac{1}{l(l+1)} \int d\rho \left\{ r^3 (f_{\rho \mu} g - g_{\rho \mu} f) \frac{1}{a \mu g} \right\}, $$  

(50)

for arbitrary scalar fields $f$ and $g$.

Following Moncrief [4] we perform two canonical transformations to separate the gauge-invariant information from the pure-gauge content in the canonical pairs $(h_1, p_1)$ and $(h_2, p_2)$. The first canonical transformation constructs the gauge-invariant combination $k_1$, also a vector component,

$$k_1 = h_1 + \frac{r^2}{2} \left( \frac{h_2}{r^2} \right)_{\rho}, \quad k_2 = h_2.$$  

(51)

It induces the following transformation on the momenta:

$$\pi_1 = p_1, \quad \pi_2 = p_2 + \frac{r^2 p_1_{\rho}}{2 r^2},$$  

(52)

and can be obtained from the generating function

$$F(p_1, p_2, k_1, k_2) = p_1 k_1 + p_2 k_2 - p_1 \frac{r^2}{2} \left( \frac{k_2}{r^2} \right)_{\rho}.$$  

(53)

In terms of the new variables we can write the first variation of the axial constraint as

$$\delta(H^i)_{\text{axial}} = \frac{X_{\mu \nu}}{l(l+1)} \left\{ \frac{2 \pi_2}{a} + \lambda \frac{a \Pi_2 k_2}{r^2} + \frac{2l(l+1)}{a r^2} \left[ \frac{r^2 \Pi_1}{a} \left( r^2 k_2 \right)_{\rho} \right]_{\rho} \right\}, $$  

(54)

which does not contain $\pi_1$ and therefore commutes with $k_1$. That is, $k_1$ is gauge invariant, as we had anticipated. This suggests the second canonical transformation:

$$Q_1 = k_1, \quad Q_2 = k_2.$$  

(55)
with conjugated momenta

\[ P_1 \equiv \pi_1 - l(l+1) \frac{r^2 \Pi_1}{a} \left( \frac{k_2}{r^2} \right)_{\rho}, \]

\[ P_2 \equiv \pi_2 + \frac{\lambda}{2r^2} a \Pi_2 k_2 + \frac{l(l+1)}{r^2} \left[ \frac{r^2 \Pi_1}{a} \left( k_1 - \frac{r^2}{2} \left( \frac{k_2}{r^2} \right)_{\rho} \right) \right]_{\rho} \].

We can obtain this canonical transformation from the generating function:

\[ F(P_1, P_2, k_1, k_2) = P_1 k_1 + P_2 k_2 + a l(l+1) \times \left\{ \frac{r^2 \Pi_1}{a} \left( \frac{k_2}{r^2} \right)_{\rho} \right\}_{\rho} - (l-1)(l+2) \frac{r^2}{8r^2} \Pi_2 k_2^2 \].

The first canonical transformation is independent of the dynamical content of the background spacetime, in the sense that it does not contain the background momenta \( /Pi_1, /Pi_2, /Pi_3 \). It is actually identical to that of Moncrief \[4\]. For the sake of clarity, we have separated the influence of the dynamical background into the second canonical transformation, which trivializes for any static background.

At this point we have isolated the physical information of the axial metric perturbation in the pair \((Q_1, P_1)\) while the \((Q_2, P_2)\) contains the gauge subsystem. \( P_2 \) is the generator of gauge transformations:

\[ \delta(H_a)_{\text{axial}} = \frac{X_a \mu_2}{l(l+1)} \frac{2 P_2}{a}, \]

and hence it is gauge invariant but constrained to vanish. Its conjugated variable \( Q_2 \) is gauge dependent, and its time evolution is determined by the arbitrary function \( h_0 \), which can be used to set any desired value for \( Q_2, t \).

### 4.3. Evolution equations

After replacing the new variables and integrating by parts a number of times, we get the following Jacobi action:

\[ \frac{1}{2} (\delta^2 S)_{\text{axial}} = \int dt \int d\rho \left[ P_1 (Q_1, t) - (\beta Q_1, \rho) \right] + P_2 (Q_2, t) - (\beta Q_2, \rho) + 2 P_2 h_0 - \alpha H^{(1)} \],

where we have defined the first-order quadratic Hamiltonian

\[ H^{(1)} \equiv \Pi_1 (P_1 Q_1 - P_2 Q_2) + \frac{1}{\alpha \lambda r^2} \left[ \left( \frac{r^2 P_1}{2} + l(l+1) \frac{r^2 \Pi_1}{a} Q_1 \right)_{\rho} - r^2 P_2 \right]^2 \]

\[ + \frac{a P_1^2}{2l(l+1)} + \frac{l(l+1)}{a} \left[ \frac{(l-1)(l+2)}{2r^2} + \frac{\Pi_1 (\Pi_2 - \Pi_1)}{2} + \dot{\Pi}_1 \right] Q_2^2. \]

The variation of the action with respect to \( h_0 \) gives the constraint that must be obeyed by the perturbations. This constraint now takes the simple form

\[ P_2 = 0. \]

This constraint is conserved in the evolution since variation with respect to \( Q_2 \) gives

\[ (r^2 P_2)_t = (\beta r^2 P_2, \rho). \]
As $P_2$ is the generator of the gauge transformations, its conjugated variable $Q_2$ is pure gauge. Its evolution equation comes from taking the variation of the action with respect to $P_2$:

$$\frac{1}{\alpha} Q_{2,t} - \beta Q_{2,\rho} = -2 \frac{h_0}{\alpha} - \Pi_1 Q_2 - \frac{1}{a} \left( \frac{r^2 P_1 + l(l+1)}{a} \frac{2r^2 \Pi_1}{a} Q_1 \right)_{,\rho} .$$  \tag{64}

The initial data for $Q_2$ are gauge, and its evolution is fully determined by the free function $h_0$. In particular, it is possible to choose $Q_2 = 0$ initially and take $h_0$ so that $Q_2 = 0$ at all times.

We can obtain the physically relevant equations by the variation of the action with respect to the variables $(Q_1, P_1)$. This gives rise to a system of two coupled second-order equations in $\rho$-derivatives, whose principal part is, in a matricial form,

$$\frac{(l-1)(l+2)}{2r^2} \frac{1}{\alpha} \left( \frac{2l(l+1)}{a} Q_1 \right)_{,t} = \left( -\Pi_1 \quad -1 \right) \frac{1}{a^2} \left( \frac{2l(l+1)}{a} Q_1 \right)_{,\rho\rho} + \cdots ,$$  \tag{65}

the dots denoting lower order terms in $\rho$-derivatives of $Q_1$ and $P_1$. We have divided $Q_1$ by $a$ to make it a scalar under changes of $\rho$ coordinate. This is a second order in the time evolution system, as corresponds to a single wave-like degree of freedom, but it apparently has fourth order in $\rho$-derivatives for generic values of the background variable $\Pi_1$. This is false because the $2 \times 2$ matrix has always vanishing square, and hence the system has third order at most. Actually it has second order, as can be checked by taking the matrix to its Jordan canonical form. Defining the combination

$$L \equiv P_1 + l(l+1) \frac{2\Pi_1}{a} Q_1,$$  \tag{66}

the system (65) is equivalent to the pair (we now use the dot and prime frame derivatives to simplify the expressions):

$$(r^2 L) = -2 \frac{Q_1}{a} ,$$  \tag{67}

$$\left( \frac{-2l Q_1}{r^2} \right)_{,t} = \frac{1}{\alpha} \left( \frac{r^2 (r^2 L)'}{2r^2} + \frac{\Pi_2 - \Pi_1}{2r^2} (r^2 L) - \frac{(l-1)(l+2)}{r^2} L ,$$  \tag{68}

which can be clearly combined into a single second-order equation for $L$, the sought generalization of the Regge–Wheeler equation for dynamical backgrounds.

When restricting to vacuum the variable $r L/\lambda$ is the Cunningham–Price–Moncrief master function [7] that obeys the Regge–Wheeler equation, though it is not immediately related to the Regge–Wheeler variable. Using the gauge $r = \rho$, $\beta = 0$ in vacuum we have $\Pi_1 = 0$, and hence $r L = r P_1$, while the Regge–Wheeler variable is $Q_1/(a^2 r)$. We have seen that the former is easily generalizable to a dynamical situation as given in (66), but not the latter, because it would require dividing by $\Pi_1$, which may vanish.

4.4. The master scalar perturbation

The physical variables $Q_1$ and $P_1$ are not scalars in $M^2$, and therefore their values depend upon the foliation we have chosen. It is better to describe the gravitational wave using not only gauge-invariant information, but also foliation-invariant information, that is, scalars in $M^2$. Studying the geometric properties of those variables under changes of foliation is not simple in the 3+1 notation. Following Gerlach and Sengupta [10], we change to the $M^2$-adapted framework introduced in section 3, in which the foliation is described by the orthonormal frame $(u^4, n^8)$. This allows us to look for scalars on $M^2$ as expressions which are frame
independent. The background momenta can be rewritten as

\[
\Pi_1 = -2u^A v_A, \quad (69)
\]
\[
\Pi_2 = -2u^A v_A - 2u^A \phi_A, \quad (70)
\]
\[
\Pi_3 = u^A \Phi_A. \quad (71)
\]

We see that \(\Pi_1\) and \(\Pi_3\) are essentially time components of vectors in \(M^2\). However, \(\Pi_2\) is a more complicated object.

In the GS formalism, the axial part of the metric perturbation is decomposed in tensor spherical harmonics:

\[
\delta (g_{\mu\nu}) dx^\mu dx^\nu \equiv h_{\mu\nu} dx^\mu dx^\nu = \sum_{l,m} \left\{ (h_A)^m_l X^m_{\mu \nu} dx^\mu dx^\nu + h^m_{l} X^m_{\mu \nu} dx^\mu dx^\nu \right\}. \quad (72)
\]

For a given pair \((l, m)\) the vector \(h_A\) and scalar \(h\) are related in the following way to the original Hamiltonian variables (35):

\[
h_1 = -(h_\rho)_{\text{GS}}, \quad (73)
\]
\[
h_2 = 2(h)_{\text{GS}}, \quad (74)
\]
\[
h_0 = \alpha^2 (h')_{\text{GS}}. \quad (75)
\]

The perturbations \(h_A\) and \(h\) are gauge dependent, but the following combination is gauge invariant:

\[
\kappa_A \equiv h_A - r^2 \left( \frac{h}{r^2} \right)_{,A}, \quad (76)
\]

and fully contains the axial information. Therefore, it must be given in terms of the gauge-invariant variables \(Q_1, P_1\) and \(P_2\):

\[
k_\rho = -Q_1, \quad (77)
\]
\[
k' = \alpha \left[ P_2 + \frac{\Pi_2}{2} h_2 \right] = \frac{1}{\lambda \alpha} \left\{ r^2 P_2 - \frac{1}{2} \left[ r^2 P_1 + 2(l + 1) \frac{r^2 \Pi_1}{\alpha} Q_1 \right]_{,\rho} \right\}. \quad (78)
\]

Those relations can be inverted, giving

\[
Q_1 = -k_\rho, \quad (79)
\]
\[
Q_2 = 2(h)_{\text{GS}}, \quad (80)
\]
\[
\frac{P_1}{l(l+1)} = -\epsilon^{AB} \kappa_{A,B} - 2(n^A u^B + n^B u^A) v_{A,B}, \quad (81)
\]
\[
\frac{2}{\alpha \alpha} \frac{r^2 P_2}{l(l+1)} = (l - 1)(l + 2)k' + \epsilon^{ABC} [r^A \epsilon^{AB} [r^{-2} \kappa_A]_{,B}]_{,C}. \quad (82)
\]

We see that the gauge invariant \(Q_1\) is the \(\rho\) component of the gauge-invariant vector \(-\kappa_A\). Then \(Q_2\) is a scalar in \(M^2\), but it is gauge dependent. The momentum \(P_1\) is the sum of two parts: the first one being a scalar (the curl of the vector \(\kappa_A\)) and the second one being essentially the off-diagonal component of the symmetric tensor \(v_{A,B} \kappa_{A,B}\). Therefore, \(P_1\) is not a component of a tensor itself. Finally, \(P_2\) is, apart from the factor \(\alpha \alpha\), the time component
of a contravariant vector. Again, it is important to stress that these properties are very easy to obtain in the GS formalism, but not in the original Hamiltonian formalism, where the variables are well adapted to a 3D point of view.

We want to construct a scalar from a linear combination of $Q_1$ and $P_1$, and perhaps their radial derivatives. We already know that $P_1$ is the sum of a scalar and a non-scalar, so that we have to find whether it is possible to cancel that non-scalar part using $Q_1$. It is possible because

$$-2(n^Au_B + n^Bu_A)v_Ak_B = -2\epsilon^{AB}v_Ak_B - 4(u^Av_A)(n^Bk_B).$$

(83)

The first term on the rhs is a scalar, and the second term is just $-2P_1Q_1$.

Therefore, the linear combination $L$ (66) that we defined in the previous section is the scalar we were looking for. It is related to the GS scalar variable as

$$-\frac{1}{r^2}L \equiv \frac{L}{l(l+1)} = \Pi_{GS} \equiv \epsilon^{AB}[r^{-2}\kappa_A],B.$$  

(84)

This is the most important result of this paper: we have constructed a gauge-invariant variable fully describing the physical content of the axial gravitational wave, and then we have shown that it is a scalar, so that it is also independent of the coordinate system used on the background spacetime. Note that no other independent scalar can be formed as a linear combination of $P_1$ and $Q_1$, and their derivatives. For the cases when $\Pi_1$ vanishes, which is equivalent to the background gauge condition $\rho = r$ and $\beta = 0$ (see equation (30)), the variable $P_1$ is already a scalar. Hence, it is again clear that $P_1$ is a more convenient variable than $Q_1$ for these cases.

Gerlach and Sengupta showed [10] that the master variable (84) obeys the following wave equation:

$$-\frac{1}{2r^2}(r^4\Pi_{GS})^A_{,A} + \frac{(l-1)(l+2)}{2} \Pi_{GS} = 8\pi\epsilon^{AB}\psi_A+B,$$

(85)

where the bar denotes the covariant derivative on the manifold $M^2$, and $\psi_A$ is a gauge-invariant axial perturbation of the energy–momentum tensor. This equation, equivalent to the pair (67)–(68) for scalar field matter, is valid for all background spherical spacetimes, and describes the evolution of the axial gravitational wave coupled to any matter model. Of course, different matter models will have additional variables and equations coupled to (85) but we stress the fact that both the form of $\Pi_{GS}$ and the form of (85) will remain unchanged.

The only issue we still need to analyze is whether the vector field $\kappa_A$, which could appear in the $\psi_A$ expression or in those other equations for the matter variables, can always be reconstructed from the $\Pi_{GS}$ scalar. An important equation in the GS formalism is

$$(l-1)(l+2)\kappa_A = 16\pi r^2\psi_A - \epsilon_{AB}(r^4\Pi_{GS})^B,$$

(86)

whose time component gives (82), after imposing the constraint $P_2 = 0$, and its radial component gives (67) for scalar field matter. This equation can be solved algebraically for $\kappa_A$ in terms of $\Pi_{GS}$ as long as $\psi_A$ does not contain the symmetrized derivative $\kappa(A|B)$ or higher derivatives of $\kappa_A$. Second and higher derivatives of $\kappa_A$ can be ruled out by requiring that the matter energy–momentum tensor must not contain second derivatives of the metric, because that would change the principal part of the Einstein equations. Symmetrized first-order derivatives of $\kappa_A$ cannot be ruled out on physical grounds because perturbation of covariant derivatives of tensor fields may introduce the term

$$\delta_{axial} = \Gamma_a^{\kappa |B|C} = X^a\kappa_{(B|C)}.$$

(87)

This is the only possible source of symmetrized derivatives of $\kappa_A$; all other perturbations of Christoffel give either $\Pi_{GS}$ or undifferentiated $\kappa_A$ terms. However, we have not found any standard matter model in which (87) appears under perturbation of its stress–energy tensor.
The combination of Hamiltonian gauge methods with the imposition of having a scalar field on $M^2$ has determined uniquely the Gerlach and Sengupta scalar, $\Pi_{GS}$, a variable containing all information on the axial gravitational wave and obeying a master wave equation. There are a number of ways of explaining the meaning of this variable (see, for instance, [17]), but probably the simplest one is given by the expression

$$\delta (R_{ABcd}) = -\frac{l(l+1)}{2}Y_\epsilon^{AB}\epsilon_{cd}r^2\Pi_{GS},$$

(88)

where $Y$ is the scalar harmonic.

5. Conclusions

The use of linear perturbation theory avoids the intrinsic nonlinear character of general relativity, but introduces new problems that must always be addressed in some form. A Hamiltonian approach to GR perturbations allows a natural discrimination of the gauge from the gauge-invariant information in the problem, but obscures the geometric character of the perturbative variables. A complementary Lagrangian approach offers a better geometric understanding, but gives no clues on how to separate the physical, unconstrained content of the model at hand. This paper proposes a combination of those two approaches to arrive at a scalar, gauge-invariant and unconstrained description of the linear perturbations in general relativity.

Around a spherically symmetric background the axial and polar subsets of perturbations decouple from each other; this paper has focused on the axial subset, for which a metric master scalar was already introduced by Gerlach and Sengupta [10] using a purely Lagrangian approach. Generalizing the Moncrief’s approach for a Schwarzschild background [4], we have reobtained this scalar for a general time-dependent background. First, we have isolated the gauge-invariant and unconstrained information in a Hamiltonian pair of variables ($Q_1, P_1$), each obeying a first order in time evolution equation. Then we have analyzed the geometric character of these two variables, showing that only a particular combination of them forms a scalar under transformations on the reduced (under spherical symmetry) spacetime. This scalar is, as was expected, the Gerlach and Sengupta master scalar.

The corresponding master scalar for the polar sector has never been found for a general time-dependent background, and there is no known obstruction for its existence. Such a variable would be relevant to study, for example, the matching problem through a timelike surface separating two different physical models (such as fluid and vacuum at a star surface [18]). Therefore, the open question is whether the same combination of techniques that we have used in this paper can be successfully applied to the polar gravitational wave. This analysis will be presented in a separate publication.

Many computations in this paper have been performed with the xAct [19–21] framework for tensor computer algebra. More precisely, we have used the package xPert for metric perturbation theory around general spacetimes to obtain the formulae of section 2. Then, we have expanded these formulae in tensor spherical harmonics with the package Harmonics.

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Appendix. Tensor spherical harmonics

It is possible to construct harmonic bases for tensor fields of arbitrary rank on the 2-sphere $S^2$. This appendix briefly summarizes the construction of those bases for scalars, vectors and symmetric tensors. For the general case see [1]. Coordinates on $S^2$ will be denoted with lowercase Latin letters $a, b, c, \ldots$ and the round metric will be $\gamma_{a b}$, with unit Gaussian curvature. Its associated Levi-Civita connection will be denoted with a colon, such that $\gamma_{a b}^{; c} = 0$. Finally, the volume form will be called $\epsilon_{a b}$. 

The spherical harmonics $Y_{m}^{l}$ form a basis for scalar fields on $S^2$. A vector basis can be constructed from them by differentiation, following Regge and Wheeler, as $\vec{Z}_{a} \equiv Y_{m}^{l}^{; a}$ and $X_{a} \equiv \epsilon_{a b} \gamma_{b c} Y_{m}^{l}^{; c}$, the former being polar and the latter being axial. Note that the labels $(l, m)$ are always implicitly assumed in the equations. A basis for polar symmetric tensors is given by $\gamma_{a b} Y_{m}^{l}$ (pure trace) and $Z_{a b} \equiv Y_{m}^{l} + \frac{l(l+1)}{2} \gamma_{a b} Y$ (traceless). Axial symmetric tensors can be expanded using the basis $X_{a b} \equiv (X_{a}^{; b} + X_{b}^{; a})/2$.

These tensor harmonics are related to those of Moncrief [4] as follows:

\begin{align}
(\hat{e}_1)_{ij} \, dx^i \, dx^j &= -2 \partial \rho X_{a} \, dx^a, \\
(\hat{e}_2)_{ij} \, dx^i \, dx^j &= X_{a b} \, dx^a \, dx^b, \\
(\hat{f}_1)_{ij} \, dx^i \, dx^j &= 2 \partial \rho Z_{a} \, dx^a, \\
(\hat{f}_2)_{ij} \, dx^i \, dx^j &= \partial \rho^2 Y, \\
(\hat{f}_3)_{ij} \, dx^i \, dx^j &= Y \gamma_{a b} \, dx^a \, dx^b, \\
(\hat{f}_4)_{ij} \, dx^i \, dx^j &= Y_{a}^{l}^{; b} \, dx^a \, dx^b.
\end{align}

Under integration over the 2-sphere the tensor spherical harmonics form an orthogonal basis, with the following normalizations:

\begin{align}
\int_{S^2} d\Omega \, Y^* \, Y' &= \delta_{l' m'} \delta_{m m'}, \\
\int_{S^2} d\Omega \, \gamma^{a b} \, Z_{a}^{*} \, Z_{b}' &= l(l+1) \delta_{l l'} \delta_{m m'}, \\
\int_{S^2} d\Omega \, \gamma^{a b} \, X_{a}^{*} \, X_{b}' &= l(l+1) \delta_{l l'} \delta_{m m'}, \\
\int_{S^2} d\Omega \, \gamma^{a b} \gamma^{c d} \, Z_{a c}^{*} \, Z_{b d}' &= \frac{1}{2} \frac{(l+2)!}{(l-2)!} \delta_{l l'} \delta_{m m'}, \\
\int_{S^2} d\Omega \, \gamma^{a b} \gamma^{c d} \, X_{a c}^{*} \, X_{b d}' &= \frac{1}{2} \frac{(l+2)!}{(l-2)!} \delta_{l l'} \delta_{m m'}.
\end{align}

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