ALGEBRAIC CYCLES OF A FIXED DEGREE

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Abstract. In this paper, the homotopy groups of Chow variety $C_{p,d}(\mathbb{P}^n)$ of effective $p$-cycles of degree $d$ are proved to be stable in the sense that $p$ or $n$ increases. We also obtain a negative answer to a question by Lawson and Michelsohn on homotopy groups for the space of degree two cycles.

1. Introduction

Let $\mathbb{P}^n$ be the complex projective space of dimension $n$ and let $C_{p,d}(\mathbb{P}^n)$ be the space of effective algebraic $p$-cycles of degree $d$ on $\mathbb{P}^n$. A fact proved by Chow and Van der Waerden is that $C_{p,d}(\mathbb{P}^n)$ carries the structure of a closed complex algebraic variety $[CW]$. Hence it carries the structure of a compact Hausdorff space.

Consider the spaces $D(d) := \lim_{p,q \to \infty} C_{p,d}(\mathbb{P}^{p+q})$ of cycles of a fixed degree (with arbitrary dimension and codimension), as introduced in $[LM]$, where the limit for $p$ is given by suspension $\Sigma : C_{p,d}(\mathbb{P}^n) \to C_{p+1,d}(\mathbb{P}^{n+1})$ and the limit for $q$ is induced by the inclusion $\mathbb{P}^{p+q} \subset \mathbb{P}^{p+q+1}$; i.e., a $p$-cycle in $C_{p,d}(\mathbb{P}^{p+q})$ is viewed as a $p$-cycle in $C_{p,d}(\mathbb{P}^{p+q+1})$.

Then there is a filtration (cf. $[LM]$ §7, $[L1]$)

$$BU = D(1) \subset D(2) \subset \cdots \subset D(\infty) = K(\text{even}, \mathbb{Z}),$$

where $BU = \lim_{q \to \infty} BU_q$ and $K(\text{even}, \mathbb{Z}) = \prod_{i=1}^{\infty} K(2i, \mathbb{Z})$ (the weak product of Eilenberg-MacLane spaces).

The inclusion map $D(d) \subset D(\infty)$ induces maps on homology and homotopy groups. It was proved in $[LM]$ that $D(1) \subset D(\infty)$ induces an injective map on homotopy groups. Moreover, as abstract groups $\pi_*(D(1)) \cong \pi_*(D(\infty))$.

The following natural question was proposed in $[LM]$:

Question 1.1. Is $\pi_*(D(d)) \to \pi_*(D(\infty))$ injective for $d \geq 1$?

An affirmative answer to Question $[LM]$ implies that $\pi_*(D(d)) \cong \pi_*(D(\infty))$ as abstract groups.

The first main result in this paper is the following negative answer to Question $[LM]$ for $d = 2$.

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Theorem 1.2. There is an integer $k > 0$ such that the induced map $\pi_k(D(2)) \to \pi_k(D(\infty))$ from the inclusion $D(2) \subset D(\infty)$ is not injective. Moreover, there is an integer $k > 0$ such that, as abstract groups, $\pi_k(D(2)) \not\cong \pi_k(D(\infty))$.

The proof of this theorem is based on Theorem 1.3 below and direct calculations under the assumption of a positive answer to Question 1.1.

The second main result is the following:

Theorem 1.3. $\pi_k(D(d)) \cong \pi_k(C_{p,d}(\mathbb{P}^n))$ for $k \leq \min\{2p + 1, 2(n - p)\}$.

The proof of Theorem 1.3 is given in section 3.

2. Homology groups of the space of algebraic cycles with degree two

Note that $C_{p,1}(\mathbb{P}^n)$ is the Grassmannian $G(p + 1, n + 1)$ of linear $p$-spaces in $\mathbb{P}^n$. Furthermore,

\begin{equation}
C_{p,2}(\mathbb{P}^n) = SP^2(G(p + 1, n + 1)) \cup Q_{p,n},
\end{equation}

where $SP^i(X)$ denotes the $i$-th symmetric product $X$ and $Q_{p,n}$ consists of effective irreducible $p$-cycles of degree 2 in $\mathbb{P}^n$. Every degree 2 variety of dimension $p$ in $\mathbb{P}^n$ is contained as a hypersurface in a linear space of dimension $p + 1$ (cf. [GH, pp. 173-4]). Hence $Q_{p,n}$ is a fiber bundle over the Grassmannian $G(p + 2, n + 1)$ with fiber $S$, the space of all smooth quadrics in $\mathbb{P}^{p+1}$. Note that $S$ is isomorphic to $\mathbb{P}^{(n+1)-1} - SP^2(\mathbb{P}^{p+1})$, i.e., the complement of non-irreducible quadrics (which is a pair of $p$-planes) in the space of all quadric hypersurfaces in $\mathbb{P}^{p+1}$.

To prove Theorem 1.2 we assume that the answer to Question 1.1 is affirmative for $d = 2$. Then $\pi_{2k}(D(2))$ is a subgroup of $\mathbb{Z}$ and so $\pi_{2k}(D(2)) \cong 0$ or $\mathbb{Z}$. Note that the map $\pi_{2k}(D(1)) \to \pi_{2k}(D(\infty)) \cong \mathbb{Z}$ is multiplication by $(k - 1)!$ (cf. [LM], Theorem 4.4) and it factors through $\pi_{2k}(D(2))$. So $\pi_{2k}(D(2))$ is non-trivial and $\pi_{2k}(D(2)) \cong \mathbb{Z}$ for all $k$ if Question 1.1 has a positive answer. Similarly, $\pi_{2k-1}(D(2)) = 0$ by assuming a positive answer to Question 1.1.

By Theorem 1.3 we have

$$\pi_k(C_{p,2}(\mathbb{P}^n)) \cong \pi_k(D(2)) = \left\{ \begin{array}{ll} \mathbb{Z}, & k \leq 2p + 1 \text{ and even}, \\ 0, & k \leq 2p + 1 \text{ and odd}. \end{array} \right.$$ 

In the following computation, we take $p = 4, d = 2$ as our example.

Lemma 2.1. Let $X \to B$ be a fibration between CW complexes with fiber $F$. Suppose that $B$ is simply connected, $H_{2q}(B, Q)$ is finite dimensional, and $H_{2i-1}(B, Q)$ and $H_{2i-1}(F, Q)$ vanish. Then $H_k(X, Q) \cong \bigoplus_{i+j=k} H_i(B, Q) \otimes H_j(F, Q)$; that is, the Leray spectral sequence degenerates at $E^2$.

Proof. By Leray’s Theorem for singular homology, we get the $E^2$ term

$$E^2_{p,q} = H_p(B, H_q(F, Q)) \cong H_p(B, Q) \otimes H_q(F, Q), \quad d^2 : E^2_{p,q} \to E^2_{p-2,q+1}$$

since $B$ is simply connected.

From the assumption, all odd dimensional homology groups of $B$ and $F$ vanish, so at least one of $E^2_{p,q}$ and $E^2_{p-3,q+2}$ vanishes. This implies that $d^2$ is a zero map. Hence we get $E^3_{p,q} = E^2_{p,q}$ and $d^3 : E^2_{p,q} \to E^2_{p-3,q+2}$. By the same reason, $d^3 = d^4 = \cdots = 0$. Therefore, the Leray spectral sequence degenerates at $E^2$, i.e.,

$$\bigoplus_{p+q=k} H_p(B, Q) \otimes H_q(F, Q) \cong \bigoplus_{p+q=k} E^2_{p,q} = \bigoplus_{p+q=k} E^\infty = H_k(X, Q).$$

\[\square\]
Proposition 2.2. Let $X$ be a connected CW complex such that

$$\pi_k(X) = \begin{cases} 
\mathbb{Z}, & 0 < k \leq 9 \text{ and even;} \\
0, & k \leq 9 \text{ and odd.}
\end{cases}$$

Then the first 10 Betti numbers $\beta_i(X)$ of $X$ are

$$\beta_i(X) = \begin{cases} 
1, 1, 2, 3, 5, & \text{for } i = 0, 2, 4, 6, 8, \\
0, & \text{for } i = 1, 3, 5, 7, 9.
\end{cases}$$

Proof. Let $\cdots \to Y_n \to Y_{n-1} \to \cdots \to Y_1 = K(\pi_1(X), 1)$ be the Postnikov approximation of $X$, where $Y_n \to Y_{n-1}$ is a fibration with $K(\pi_n(X), n)$ as fibers (cf. [W], Chapter IX). For a fixed $n$, we have isomorphisms of homotopy groups $\pi_q(X) \cong \pi_q(Y_n)$ and homology groups $H_q(X, \mathbb{Q}) \cong H_q(Y_n, \mathbb{Q})$ for $1 \leq q \leq n$. Therefore, the first 10 Betti numbers of $X$ coincide with those of $Y_n$.

Note that $Y_2$ is homotopy equivalent (denoted by $\simeq$) to $K(\mathbb{Z}, 2)$ since $Y_1$ is contractible. Since $Y_3 \to Y_2$ is a fibration with $K(\pi_3(X), 3) \simeq \ast$ as fibers, we get $Y_3 \simeq Y_2$. Note that $Y_4 \to Y_3$ is a fibration with $K(\pi_4(X), 4) = K(\mathbb{Z}, 4)$ as fibers, we obtain $H_k(X, \mathbb{Q}) \cong \bigoplus_{i+j=k} H_i(Y_3, \mathbb{Q}) \otimes H_j(K(\mathbb{Z}, 4), \mathbb{Q})$ by Lemma 2.1. Using Lemma 2.1 for several times, we get (modulo $H_*(\ast, \mathbb{Q})$ for $* > 9$)

$$H_*(X, \mathbb{Q}) \cong H_*(Y_9, \mathbb{Q}) \\
\cong H_*(Y_8, \mathbb{Q}) \quad (\text{since } Y_5 \simeq Y_6) \\
\cong H_*(Y_7, \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 8), \mathbb{Q}) \quad (\text{since } K(\mathbb{Z}, 8) \to Y_6 \to Y_7 \text{ is a fibration}) \\
\cong H_*(Y_6, \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 8), \mathbb{Q}) \\
\cong H_*(Y_5, \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 6), \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 8), \mathbb{Q}) \\
\cong H_*(K(\mathbb{Z}, 2), \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 4), \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 6), \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 8), \mathbb{Q}).$$

Therefore, the first 10 Betti numbers $\beta_i(X)$ of $X$ are given as follows:

$$\beta_i(X) = \begin{cases} 
1, 1, 2, 3, 5, & \text{for } i = 0, 2, 4, 6, 8; \\
0, & \text{for } i = 1, 3, 5, 7, 9. 
\end{cases} \quad \square$$

The combination of Theorem 1.3 and Proposition 2.2 implies the following result:

Corollary 2.3. If the answer to Question 1.1 is affirmative for $d = 2$, then the first 10 Betti numbers of $C_{4,2}(\mathbb{P}^n)$ $(n \geq 9)$ are given by

$$\beta_i(C_{4,2}(\mathbb{P}^n)) = \begin{cases} 
1, 1, 2, 3, 5, & \text{for } i = 0, 2, 4, 6, 8; \\
0, & \text{for } i = 1, 3, 5, 7, 9. 
\end{cases}$$

The proof of Proposition 2.2 actually shows the following result:

Remark 2.4. Let $M$ be a connected CW complex such that $\pi_k(X) = 0$ for $k$ odd and $\pi_k(M) \cong \mathbb{Z}$ for $k$ positive even integers. Then

$$\text{rank}(H_k(M)) = \begin{cases} 
p(k), & \text{if } k \text{ is even}, \\
0, & \text{if } k \text{ is odd},
\end{cases}$$

where $p(k)$ represents the number of all possible partitions of $k$.

Examples of such a CW complex $M$ include the infinite product $\prod_{i=1}^{\infty} K(\mathbb{Z}, 2i)$ (with the weak topology) of Eilenberg-MacLane spaces and $BU = \lim_{q \to \infty} BU_q$. Although the homotopy types of these topological spaces are different, their corresponding Betti numbers coincide.
Now we will compute Betti numbers of \( C_{4,2}(\mathbb{P}^n) \) \((n \geq 9)\) in a different way. Since \( C_{p,2}(\mathbb{P}^n) - \text{SP}^2(G(p+1,n+1)) = Q_{p,n} \), we have \( H_i(C_{p,2}(\mathbb{P}^n),\text{SP}^2(G(p+1,n+1)) \cong H_i^{BM}(Q_{p,n}) \) for all \( i \), where \( H_i^{BM} \) denotes the Borel-Moore homology. Let \( A_{p,n} \) be the fiber bundle over \( G(p+2,n+1) \) whose fiber is the space of all quadric hypersurfaces in \( \mathbb{P}^{p+1} \) and let \( B_{p,n} \) be the fiber bundle over \( G(p+2,n+1) \) whose fiber is the space of pairs of hyperplanes in \( \mathbb{P}^{p+1} \). From the definition of \( Q_{p,n} \), we have \( H_i(A_{p,n},B_{p,n}) \cong H_i^{BM}(Q_{p,n}) \) for all \( i \). In particular,

\[
H_i(C_{4,2}(\mathbb{P}^n),\text{SP}^2(G(5,n+1)) \cong H_i(A_{4,n},B_{4,n})
\]

for \( i \geq 0 \) and \( n \geq 9 \).

**Lemma 2.5.** Let \( A_{4,n}, B_{4,n} \) be defined as above,

\[
\beta_i(\text{SP}^2(G(5,n+1))) = \begin{cases} 
1, 1, 3, 5, 11, & \text{for } i = 0, 2, 4, 6, 8; \\
0, & \text{for } i \text{ odd;}
\end{cases}
\]

\[
\beta_i(A_{4,n}) = \begin{cases} 
1, 2, 4, 7, 12, & \text{for } i = 0, 2, 4, 6, 8; \\
0, & \text{for } i \text{ odd;}
\end{cases}
\]

\[
\beta_i(B_{4,n}) = \begin{cases} 
1, 2, 5, 9, 17, & \text{for } i = 0, 2, 4, 6, 8; \\
0, & \text{for } i \text{ odd.}
\end{cases}
\]

**Proof.** To show the first formula, we note that all the odd Betti numbers of \( G(5,n+1) \) are zero and the first five even Betti numbers of \( G(5,n+1) \) are given by

\[
\beta_i(G(5,n+1)) = 1, 1, 2, 3, 5 \text{ for } i = 0, 2, 4, 6, 8.
\]

Therefore all the odd Betti numbers of \( \text{SP}^2(G(5,n+1)) \) vanish and the first five even Betti numbers of \( \text{SP}^2(G(5,n+1)) \) are given by (a special case of MacDonald’s formula \[M\])

\[
\beta_i(\text{SP}^2(G(5,n+1))) = 1, 1, 3, 5, 11 \text{ for } i = 0, 2, 4, 6, 8.
\]

To show the second formula, we note that \( A_{4,n} \) is a fiber bundle over \( G(6,n+1) \) with fibers the space of all quadric hypersurfaces in \( \mathbb{P}^5 \); i.e., fibers are isomorphic to \( \mathbb{P}^{20} \). By Lemma 2.1 all the odd Betti numbers of \( A_{4,n} \) vanish since both \( G(6,n+1) \) and \( \mathbb{P}^{20} \) only carry non-vanishing even Betti numbers. Again, by Lemma 2.1

\[
\beta_k(A_{4,n}) = \bigoplus_{i+j=k} \beta_{2i}(G(6,n+1)) \cdot \beta_{2j}(\mathbb{P}^{20}).
\]

The first five even Betti numbers of \( G(6,n+1) \) are given by

\[
\beta_k(G(6,n+1)) = 1, 1, 2, 3, 5 \text{ for } i = 0, 2, 4, 6, 8.
\]

Hence from equation (2.4), we get the first five even Betti numbers of \( \tilde{Y} \):

\[
\beta_k(A_{4,n}) = 1, 2, 4, 7, 12 \text{ for } i = 0, 2, 4, 6, 8.
\]

To show the third formula, we note that \( B_{4,n} \) is a fiber bundle over \( G(6,n+1) \) with fibers the space of pairs of hyperplanes in \( \mathbb{P}^{p+1} \); i.e., fibers are isomorphic to \( \text{SP}^2(\mathbb{P}^5) \). By Lemma 2.1 all the odd Betti numbers of \( B_{4,n} \) vanish and the even Betti numbers of \( B_{4,n} \) are given by the formula

\[
\beta_k(B_{4,n}) = \bigoplus_{i+j=k} \beta_{2i}(G(6,n+1)) \cdot \beta_{2j}(\text{SP}^2(\mathbb{P}^5)).
\]
The first five Betti numbers of $\text{SP}^2(\mathbb{P}^5)$ are given as follows (cf. [M]):

$$\beta_i(\text{SP}^2(\mathbb{P}^5)) = 1, 1, 2, 2, 3 \text{ for } i = 0, 2, 4, 6, 8.$$ 

Therefore the five Betti numbers of $Z$ are given by the formula

(2.7) \hspace{1cm} \beta_i(B_{4,n}) = 1, 2, 5, 9, 17 \text{ for } i = 0, 2, 4, 6, 8. \hspace{1cm} \square

**Proposition 2.6.** The relations among the first 10 Betti numbers of $C_{4,2}(\mathbb{P}^n)$ ($n \geq 9$) are given as follows:

$$\beta_{2i}(C_{4,2}(\mathbb{P}^n)) - \beta_{2i+1}(C_{p,2}(\mathbb{P}^n)) = 1, 1, 2, 3, 6 \text{ for } i = 0, 1, 2, 3, 4.$$ 

In particular, $\beta_5(C_{4,2}(\mathbb{P}^n)) \geq 6$.

**Proof.** Set $M = C_{4,2}(\mathbb{P}^n)$ and $X = \text{SP}^2G(5, n + 1)$. From the long exact sequence of homology groups for the pair $(M, X)$, we have

(2.8) \hspace{1cm} \cdots \rightarrow H_i(X) \rightarrow H_i(M) \rightarrow H_i(M, X) \rightarrow H_{i-1}(X) \rightarrow \cdots.

Since $H_{2i-1}(X) = 0$ for all $i$, equation (2.8) breaks into exact sequences

(2.9) \hspace{1cm} 0 \rightarrow H_{2i+1}(M) \rightarrow H_{2i+1}(M, X) \rightarrow H_{2i}(X) \rightarrow H_{2i}(M) \rightarrow H_{2i}(M, X) \rightarrow 0.

Set $Y = A_{4,n}$ and $Z = B_{4,n}$. From the long exact sequence of homology groups for the pair $(Y, Z)$, we have

(2.10) \hspace{1cm} \cdots \rightarrow H_i(Z) \rightarrow H_i(Y) \rightarrow H_i(Y, Z) \rightarrow H_{i-1}(Z) \rightarrow \cdots.

Since $H_{2i-1}(Y) = 0$ and $H_{2i-1}(Z) = 0$ for all $i$, equation (2.10) breaks into exact sequences

(2.11) \hspace{1cm} 0 \rightarrow H_{2i+1}(Y, Z) \rightarrow H_{2i}(Z) \rightarrow H_{2i}(Y) \rightarrow H_{2i}(Y, Z) \rightarrow 0.

From equations (2.3), (2.4) and (2.11), we have

$$\beta_{2i+1}(M) - \beta_{2i}(Z) + \beta_{2i}(Y) + \beta_{2i}(X) - \beta_{2i}(M) = 0$$

i.e.,

(2.12) \hspace{1cm} \beta_{2i+1}(C_{4,2}(\mathbb{P}^n)) - \beta_{2i}(B_{4,n}) + \beta_{2i}(A_{4,n}) + \beta_{2i}(\text{SP}^2G(5, n + 1)) - \beta_{2i}(C_{4,2}(\mathbb{P}^n)) = 0.

Now the proposition follows from equation (2.12) and Lemma 2.5. \hspace{1cm} \square

The contradiction between Corollary 2.3 and Proposition 2.6 comes from the assumption that the answer to Question 1.1 for $d = 2$ is affirmative. Therefore the answer to Question 1.1 for $d = 2$ is negative; i.e., the induced map $\pi_*(\mathcal{D}(d)) \rightarrow \pi_*(\mathcal{D}(\infty))$ by inclusion is not always injective for $d = 2$. This completes the proof of Theorem 1.2.

**Remark 2.7.** We actually used the assumption that $\pi_*(\mathcal{D}(d)) \cong \pi_*(\mathcal{D}(\infty))$ are isomorphisms as abstract groups for $k \leq 9$ in the proof of Theorem 1.2. Hence $\pi_*(\mathcal{D}(2))$ is not isomorphic to $\pi_*(\mathcal{D}(\infty))$ for all $*$ as abstract abelian groups.
3. Proof of Theorem [L3]

In this section we will prove Theorem [L3]. The method comes from Lawson’s proof of the Complex Suspension Theorem [L1]; i.e., the complex suspension to the space of \( p \)-cycles yields a homotopy equivalence to the space of \( (p+1) \)-cycles. Here we briefly review the general construction of such a homotopy equivalence. For details, the reader is referred to [L1], [L2] and [F].

Fix a hyperplane \( \mathbb{P}^n \subset \mathbb{P}^{n+1} \) and a point \( \mathbb{P}^0 \in \mathbb{P}^{n+1} - \mathbb{P}^n \). For any non-negative integer \( p \) and \( d \), set

\[
T_{p+1,d}(\mathbb{P}^{n+1}) := \{ c = \sum n_i V_i \in C_{p+1,d}(\mathbb{P}^{n+1}) | \dim(V_i \cap \mathbb{P}^n) = p, \forall i \}
\]

(where \( d = 0 \), \( C_{p,0}(\mathbb{P}^n) \) is defined to be the empty cycle).

**Proposition 3.1 ([L1]).** The set \( T_{p+1,d}(\mathbb{P}^{n+1}) \) is Zariski open in \( C_{p+1,d}(\mathbb{P}^{n+1}) \). Moreover, \( T_{p+1,d}(\mathbb{P}^{n+1}) \) is homotopy equivalent to \( C_{p,d}(\mathbb{P}^n) \). In particular, their corresponding homotopy groups are isomorphic, i.e.,

\[
\pi_*(T_{p+1,d}(\mathbb{P}^{n+1})) \cong \pi_*(C_{p,d}(\mathbb{P}^n)).
\]

Fix a linear embedding \( \mathbb{P}^{n+1} \subset \mathbb{P}^{n+2} \) and two points \( x_0, x_1 \in \mathbb{P}^{n+2} - \mathbb{P}^n \). Each projection \( p_i : \mathbb{P}^{n+2} - \{x_0\} \to \mathbb{P}^{n+1} \) \( (i = 0,1) \) gives us a holomorphic line bundle over \( \mathbb{P}^{n+1} \).

Let \( D \in C_{p+1,e}(\mathbb{P}^{n+2}) \) be an effective divisor of degree \( e \) in \( \mathbb{P}^{n+2} \) such that \( x_0, x_1 \) are not in \( D \). Any effective cycle \( c \in C_{p+1,d}(\mathbb{P}^{n+1}) \) can be lifted to a cycle with support in \( D \), defined as follows:

\[
\Psi_D(c) = (\Sigma x_0 c) \cdot D.
\]

The map \( \Psi(c,D) := \Psi_D \) is a continuous map in the variables \( c \) and \( D \). Hence we have a continuous map \( \Phi_D : C_{p+1,d}(\mathbb{P}^{n+1}) \to C_{p+1,de}(\mathbb{P}^{n+2} - \{x_0, x_1\}) \). The composition of \( \Phi_D \) with the projection \( (p_0)_* \) is \( (p_0)_* \circ \Phi_D = e \) (multiplication by the integer \( e \) in the monoid, \( e \cdot c = c + \cdots + c \) for \( e \) times). The composition of \( \Phi_D \) with the projection \( (p_1)_* \) gives us a transformation of cycles in \( \mathbb{P}^{n+1} \) which makes most of them intersect properly to \( \mathbb{P}^n \). To see this, consider the family of divisors \( tD, \ 0 \leq t \leq 1 \), given by scalar multiplication by \( t \) in the line bundle \( p_0 : \mathbb{P}^{n+2} - \{x_0\} \to \mathbb{P}^{n+1} \).

Assume \( x_1 \) is not in \( tD \) for all \( t \). Then the above construction gives us a family transformation

\[
F_{tD} := (p_1)_* \circ \Psi_{tD} : C_{p+1,d}(\mathbb{P}^{n+1}) \to C_{p+1,de}(\mathbb{P}^{n+2})
\]

for \( 0 \leq t \leq 1 \). Note that \( F_{0D} \equiv d(\text{multiplication by } d) \).

The question is for a fixed \( c \), which divisors \( D \in C_{n+1,e}(\mathbb{P}^{n+2}) \) \( (x_0 \notin D) \) and \( x_1 \) is not in \( \bigcup_{0 \leq t \leq 1} tD \) have the property that

\[
F_{tD}(c) \in T_{p+1,de}(\mathbb{P}^{n+1})
\]

for all \( 0 < t \leq 1 \).

Set \( B_c := \{ D \in C_{n+1,e}(\mathbb{P}^{n+2}) | F_{tD}(c) \text{ is not in } T_{p+1,de}(\mathbb{P}^{n+1}) \text{ for some } 0 < t \leq 1 \} \), i.e., all degree \( e \) divisors on \( \mathbb{P}^{n+2} \) such that some component of

\[
(p_1)_* \circ \Psi_{tD}(c) \subset \mathbb{P}^n
\]

for some \( t > 0 \).

**Proposition 3.2 ([L1]).** For \( c \in C_{p+1,d}(\mathbb{P}^{n+1}) \), \( \text{codim}_C B_c \geq \binom{p+e+1}{e} \).
In this construction, if we take $e = 1$, then $F_{tD}$ maps $C_{p+1,d}(\mathbb{P}^{n+1})$ to itself, i.e.,

$$F_{tD} := (p_1)_* \circ \Psi_{tD} : C_{p+1,d}(\mathbb{P}^{n+1}) \to C_{p+1,d}(\mathbb{P}^{n+1}).$$

Moreover, the image of $F_{tD}$ is in the Zariski open subset $T_{p+1,d}(\mathbb{P}^{n+1})$ if $D$ is not $B_c$. We can find such a $D$ if codim$\mathbb{C}B_c \geq \binom{n+1+1}{p+1+1} = p + 2$ is positive.

Suppose now that $f : S^k \to C_{p+1,d}(\mathbb{P}^{n+1})$ is a continuous map for $0 < k \leq 2p+2$. We may assume that $f$ is piecewise linear up to homotopy. Then the map $f$ is homotopic to a map $S^k \to T_{p+1,d}(\mathbb{P}^{n+1})$. To see this, we consider the family

$$F_{tD} \circ f : S^k \to C_{p+1,d}(\mathbb{P}^{n+1}), \quad 0 \leq t \leq 1,$$

where $D$ lies outside the union $\bigcup_{x \in S^n} B_{f(x)}$. This is a set of real codimension larger than or equal to $2(p + 2) - (k + 1)$. Therefore, $2(p + 2) - (k + 1) \geq 1$, i.e., $k \leq 2p + 2$, so such a $D$ exists. This proves that the map $i_* : \pi_k(T_{p+1,d}(\mathbb{P}^{n+1})) \to \pi_k(C_{p+1,d}(\mathbb{P}^{n+1}))$ induced by inclusion $i : T_{p+1,d}(\mathbb{P}^{n+1}) \hookrightarrow C_{p+1,d}(\mathbb{P}^{n+1})$ is surjective if $k \leq 2p + 2$.

Similarly, suppose that $g : (D^{k+1}, S^k) \to (C_{p+1,d}(\mathbb{P}^{n+1}), T_{p+1,d}(\mathbb{P}^{n+1}))$ is a pair of continuous maps for $0 < k \leq 2p+1$. Then the map $g$ can be deformed through a map of pairs to $\tilde{g} : (D^{k+1}, S^k) \to (T_{p+1,d}(\mathbb{P}^{n+1}), T_{p+1,d}(\mathbb{P}^{n+1}))$ if $2(p + 2) - (k + 2) \geq 1$, i.e., $k \leq 2p + 1$. This proves that the map $i_* : \pi_k(T_{p+1,d}(\mathbb{P}^{n+1})) \to \pi_k(C_{p+1,d}(\mathbb{P}^{n+1}))$ induced by inclusion $i : T_{p+1,d}(\mathbb{P}^{n+1}) \hookrightarrow C_{p+1,d}(\mathbb{P}^{n+1})$ is injective if $k \leq 2p + 1$.

Therefore,

$$(3.2) \quad \pi_k(T_{p+1,d}(\mathbb{P}^{n+1})) \cong \pi_k(C_{p+1,d}(\mathbb{P}^{n+1}))$$

for $0 \leq k \leq 2p + 1$.

The combination of equations (3.1) and (3.2) gives us the following result:

**Proposition 3.3.** The complex suspension $\Sigma : C_{p,d}(\mathbb{P}^{n}) \to C_{p+1,d}(\mathbb{P}^{n+1})$ induces an isomorphism

$$(3.3) \quad \Sigma_* : \pi_k(C_{p,d}(\mathbb{P}^{n})) \cong \pi_k(C_{p+1,d}(\mathbb{P}^{n+1}))$$

for $0 \leq k \leq 2p + 1$.

As a corollary, we get the simply connectedness of $C_{p,d}(\mathbb{P}^{n})$, which has been obtained using general position arguments by Lawson (LI), the proof of Lemma 2.6):

**Corollary 3.4 (LI).** The Chow variety $C_{p,d}(\mathbb{P}^{n})$ is simply connected for integers $p, d, n \geq 0$.

**Proof.** Since $C_{0,d}(\mathbb{P}^{n})$ can be identified with the $d$-th symmetric product $SP^d(\mathbb{P}^{n})$ of $\mathbb{P}^{n}$ and $SP^d(\mathbb{P}^{n})$ is path connected, we have $\pi_0(C_{0,d}(\mathbb{P}^{n})) = 0$ for all $d, n \geq 0$. Repeating using equation (3.3), we know $\pi_0(C_{p,d}(\mathbb{P}^{n})) = 0$ for all $p, d, n \geq 0$. Moreover, since $SP^d(\mathbb{P}^{n})$ is simply connected for all $d, n \geq 0$, we have $\pi_1(C_{0,d}(\mathbb{P}^{n})) = 0$ for all $d, n \geq 0$. Repeating using equation (3.3), we get

$$\pi_1(C_{p,d}(\mathbb{P}^{n})) \cong \pi_1(C_{p-1,d}(\mathbb{P}^{n-1})) \cong \cdots \cong \pi_1(C_{0,d}(\mathbb{P}^{n-p})) = 0$$

for all $p, d, n \geq 0$. \qed

Now we study the connectedness of maps induced by the inclusion $i : \mathbb{P}^{n} \hookrightarrow \mathbb{P}^{n+1}$.
Proposition 3.5. For any integer \( d \geq 1 \), the inclusion \( i : C_{p,d}(\mathbb{P}^n) \hookrightarrow C_{p,d}(\mathbb{P}^{n+1}) \) induces an isomorphism

\[
\pi_k(C_{p,d}(\mathbb{P}^n)) \cong \pi_k(C_{p,d}(\mathbb{P}^{n+1}))
\]

for \( 0 \leq k \leq 2(n-p) \).

Remark 3.6. By using Proposition 3.5 we give another possibly more elementary proof of Corollary 3.4. If \( n = p \), then \( C_{p,d}(\mathbb{P}^n) \) is a point and so it is simply connected. If \( n = p + 1 \), then \( C_{p,d}(\mathbb{P}^n) \cong \mathbb{P}^{(n+1) - 1} \) so it is simply connected. If \( n - p \geq 2 \), then \( \pi_k(C_{p,d}(\mathbb{P}^n)) \cong \pi_k(C_{p,d}(\mathbb{P}^{n-1})) \cong \cdots \cong \pi_k(C_{p,d}(\mathbb{P}^{p+1})) = 0 \) for \( k \leq 1 \) by using Proposition 3.5 and so \( C_{p,d}(\mathbb{P}^n) \) is simply connected.

Proposition 3.5 can be used to compute the second homotopy group of Chow varieties.

Corollary 3.7. For \( d \geq 1 \) and \( n > p \geq 0 \), we have \( \pi_2(C_{p,d}(\mathbb{P}^n)) \cong \mathbb{Z} \) and hence \( H_2(C_{p,d}(\mathbb{P}^n)) \cong \mathbb{Z} \).

Proof. Replacing \( \pi_k \) by \( \pi_2 \) in Remark 3.6 yields the proof of the first statement. The second statement is a result of the first statement, Corollary 3.4 and the Hurewicz isomorphism theorem.

Lawson’s idea in the proof of the Complex Suspension Theorem in [L1] can be used to prove Proposition 3.5.

For any non-negative integer \( p \) and \( d \), set

\[
U_{p,d}(\mathbb{P}^{n+1}) := \{ c = \sum n_i V_i \in C_{p,d}(\mathbb{P}^{n+1})| V^0 \text{ is not in } \bigcup V_i \}.
\]

Proposition 3.5 follows directly from the combination of Lemmas 3.8 and 3.9 below:

Lemma 3.8. \( U_{p,d}(\mathbb{P}^{n+1}) \) is homotopy equivalent to \( C_{p,d}(\mathbb{P}^n) \). In particular, their corresponding homotopy groups are isomorphic, i.e.,

\[
\pi_*(U_{p,d}(\mathbb{P}^{n+1})) \cong \pi_*(C_{p,d}(\mathbb{P}^n)).
\]

Proof. Let \( p_0 : \mathbb{P}^{n+1} - \mathbb{P}^0 \to \mathbb{P}^n \) be the canonical projection away from \( \mathbb{P}^0 \in \mathbb{P}^{n+1} - \mathbb{P}^n \). Then \( p_0 \) induces a deformation retract from \( U_{p,d}(\mathbb{P}^{n+1}) \) to \( C_{p,d}(\mathbb{P}^n) \).

To see this, note that \( p_0 \) is a holomorphic line bundle and let \( F_t : (\mathbb{P}^{n+1} - \mathbb{P}^0) \times \mathbb{C} \to \mathbb{P}^{n+1} - \mathbb{P}^0 \) denote the scalar multiplication by \( t \in \mathbb{C} \) in this bundle. This map \( F_t \) is holomorphic (in fact, algebraic) and satisfies \( F_1 = id_{\mathbb{P}^{n+1} - \mathbb{P}^0} \) and \( F_0 = p_0 \). Hence \( F_t \) induces a family of continuous maps \( (F_t)_* : U_{p,d}(\mathbb{P}^{n+1}) \to C_{p,d}(\mathbb{P}^n) \).

Therefore, \( (p_0)_* \) is a deformation retraction.

Lemma 3.9. The inclusion \( i : U_{p,d}(\mathbb{P}^{n+1}) \hookrightarrow C_{p,d}(\mathbb{P}^{n+1}) \) is \( 2(n-p) \)-connected.

Proof. By definition, it is enough to show that the induced maps on homotopy groups

\[
i_* : \pi_k(U_{p,d}(\mathbb{P}^{n+1})) \to \pi_k(C_{p,d}(\mathbb{P}^{n+1}))
\]

are isomorphisms for \( k \leq 2(n-p) \). Let \( f : S^k \to C_{p,d}(\mathbb{P}^{n+1}) \) be a continuous map for \( k \leq 2(n-p) \). We may assume \( f \) to be piecewise linear up to homotopy. Then \( f \) is homotopic to a map \( S^k \to U_{p,d}(\mathbb{P}^{n+1}) \). To see this, we first note that the union

\[
\bigcup_{x \in S^k} f(x)
\]
is a set of real codimension $\geq 2(n + 1) - 2p - k \geq 2 > 0$. So we can find a point $Q \in \mathbb{P}^{n+1} - \mathbb{P}^n$ such that $Q$ is not in $\bigcup_{x \in S^k} f(x)$. Let $G_t$ be a family of automorphism of $\mathbb{P}^{n+1}$ mapping $\mathbb{P}^0$ to $Q$ but preserving $\mathbb{P}^n$. Composing with the automorphism $G_t$, we obtain the family $G_t \circ f : S^k \to C_{p,d}(\mathbb{P}^{n+1})$ such that $G_0 \circ f = f$ and $G_1 \circ f : S^k \to U_{p,d}(\mathbb{P}^{n+1})$. Hence $i_* f$ is surjective for $k \leq 2(n - p)$. 

Similarly, suppose $g$ is a map of pairs $g : (D_{k+1}, S^k) \to (C_{p,d}(\mathbb{P}^{n+1}), U_{p,d}(\mathbb{P}^{n+1}))$. Then the map can be deformed through a map of pairs to one with image in $U_{p,d}(\mathbb{P}^{n+1})$ if $k \leq 2(n - p)$. Therefore, $i_* f$ is injective for $k \leq 2(n - p)$. □

The proof of Theorem 1.3 By Proposition 3.3, $\pi_k(C_{p,d}(\mathbb{P}^n))$ is stable when $n \to \infty$. By the combination of equations (3.3) and (3.4), we have the isomorphism $\pi_k(C_{p,d}(\mathbb{P}^n)) \cong \lim_{m,q \to \infty} \pi_k(C_{p,q,d}(\mathbb{P}^{n+m+q}))$ for $0 \leq k \leq 2p + 1$ and $k \leq 2(n - p)$. This completes the proof of Theorem 1.3 □

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