Abstract

We present \( \sigma \)-strongly functionally discrete mappings which expand the class of \( \sigma \)-discrete mappings and generalize Banach’s theorem on analytically representable functions

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1. Introduction

Let \( X \) be a topological space and \( \mathcal{A} \) be a family of subsets of \( X \). We define classes \( \mathcal{F}_\alpha(\mathcal{A}) \) and \( \mathcal{G}_\alpha(\mathcal{A}) \) in the following way: \( \mathcal{F}_0(\mathcal{A}) = \mathcal{A}, \mathcal{G}_0(\mathcal{A}) = \{ X \setminus A : A \in \mathcal{A} \} \) and for all \( 1 \leq \alpha < \omega_1 \) let

\[
\mathcal{F}_\alpha(\mathcal{A}) = \left\{ \bigcap_{n=1}^{\infty} A_n : A_n \in \bigcup_{\beta < \alpha} \mathcal{G}_\beta(\mathcal{A}), \ n = 1, 2, \ldots \right\},
\]

\[
\mathcal{G}_\alpha(\mathcal{A}) = \left\{ \bigcup_{n=1}^{\infty} A_n : A_n \in \bigcup_{\beta < \alpha} \mathcal{F}_\beta(\mathcal{A}), \ n = 1, 2, \ldots \right\}.
\]

If \( \mathcal{A} \) is the collection of all /functionally/ closed subsets of \( X \), then elements of \( \mathcal{F}_\alpha(\mathcal{A}) \) or \( \mathcal{G}_\alpha(\mathcal{A}) \) are called sets of the \( \alpha \)'th /functionally/ multiplicative class or sets of the \( \alpha \)'th /functionally/ additive class, respectively; elements of the family \( \mathcal{F}_\alpha(\mathcal{A}) \cap \mathcal{G}_\alpha(\mathcal{A}) \) are called /functionally/ ambiguous sets of the class \( \alpha \).

A mapping \( f : X \to Y \) between topological spaces belongs to the \( \alpha \)'th /functionally/ Lebesgue class, if the preimage \( f^{-1}(V) \) of any open set \( V \subseteq Y \) is of the \( \alpha \)'th /functionally/ additive class \( \alpha \) in \( X \). The collection of all mappings of the \( \alpha \)'th /functionally/ Lebesgue class we denote by \( H_\alpha(X,Y) / K_\alpha(X,Y) \). Notice that \( H_\alpha(X,Y) = K_\alpha(X,Y) \) for any perfectly normal space \( X \) and any topological space \( Y \).

By \( C(X,Y) \) we denote the class of all continuous mappings between \( X \) and \( Y \).

Let \( \Phi_1(X,Y) = H_1(X,Y) \) and for all \( 1 < \alpha < \omega_1 \) the symbol \( \Phi_\alpha(X,Y) \) stands for the collection of all mappings between \( X \) and \( Y \) which are pointwise limits of sequences of mappings from \( \bigcup_{\beta < \alpha} \Phi_\beta(X,Y) \). The next result is the classical Banach’s theorem [1].

**Theorem A.** Let \( X \) be a metric space, \( Y \) be a metric separable space and \( 0 < \alpha < \omega_1 \). Then

1. \( \Phi_\alpha(X,Y) = H_\alpha(X,Y), \) if \( \alpha < \omega_0 \),

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In Section 5 we present classes Λα which contain three approximation lemmas which are crucial in the proof of the main theorem. Let Λα(X, Y) = \bigcup_{n=1}^{\infty} B_n of subsets of X such that every family B_n is discrete in X and the preimage f^{-1}(V) of any open set V in Y is a union of sets from B. Class of all σ-discrete mappings between X and Y is denoted by Σ(X, Y). In [5, Theorem 7] Hansell proved the following generalization of Banach’s theorem.

**Theorem B.** Let X be a perfect space, Y be a metric space and 0 < α < ω₁. Then

(i) Φα(X, Y) ∩ Σ(X, Y) = Hα(X, Y) ∩ Σ(X, Y), if α < ω₀,
(ii) Φα(X, Y) ∩ Σ(X, Y) = Hα+1(X, Y) ∩ Σ(X, Y), if α ≥ ω₀.

In this paper we develop technique from [5] and [3] and prove an analogue of Theorem B for an arbitrary topological space X. To do this we introduce classes of mappings Σα(Y, X) of functionally open subsets of the class α in X and a family U = \bigcup_{n=1}^{\infty} U_n of functionally open subsets of X, where U_n = (U_B : B ∈ B_n), such that every family U_n is discrete in X, \overline{B} ⊆ U_B for every B ∈ B_n and the preimage f^{-1}(V) of any open set V in Y is a union of sets from B. Properties of this class are studied in Section 2. Let us observe that the class Σα(Y, X) coincides with the class of all σ-discrete mappings of the α’th Lebesgue class in case X is a perfectly normal space and Y is a metric space. Auxiliary technical propositions are gathered in Section 3. The forth section contains three approximation lemmas which are crucial in the proof of the main theorem. In Section 5, we present classes Λα which are close to classes \Phiα: let Λ1(X, Y) = Σ1(Y, X), and for all 1 < α < ω₁ let Λα(X, Y) be the collection of all mappings between X and Y which are pointwise limits of sequences of mappings from \bigcup_{\beta<\alpha} Λβ(X, Y). The theorem below is the main result of the paper.

**Theorem C.** Let X be a topological space, Y be a metric space and 0 < α < ω₁. Then

(i) Λα(X, Y) = Σα(Y, X), if α < ω₀,
(ii) Λα(X, Y) = Σα+1(Y, X), if α ≥ ω₀.

An example at the end of the fifth section shows that the assertion on X in Theorem B is essential.

2. Properties of σ-strongly functionally discrete mappings

**Definition 1.** A family A = (A_i : i ∈ I) of subsets of a topological space X is called

1. discrete, if every point of X has an open neighborhood which intersects with at most one set from A;
2. strongly discrete, if there exists a discrete family (U_i : i ∈ I) of open subsets of X such that \overline{A_i} ⊆ U_i for every i ∈ I;
3. strongly functionally discrete or, briefly, sfd-family, if there exists a discrete family (U_i : i ∈ I) of functionally open subsets of X such that \overline{A_i} ⊆ U_i for every i ∈ I;
4. well strongly functionally discrete of well sfd-family, if there exist discrete families (A_i : i ∈ I) of functionally open sets and (F_i : i ∈ I) of functionally closed sets such that A_i ⊆ F_i ⊆ U_i for every i ∈ I.
Notice that \((4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)\) for any \(X\); a topological space \(X\) is collectionwise normal if and only if every discrete family in \(X\) is strongly discrete; if \(X\) is normal then \((2) \Leftrightarrow (3)\).

**Definition 2.** Let \(\mathcal{P}\) be a property of a family of sets. A family \(\mathcal{A}\) is called a \(\sigma\)-\(\mathcal{P}\)-family if \(\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n\), where every family \(\mathcal{A}_n\) has the property \(\mathcal{P}\).

**Definition 3.** A family \(\mathcal{B}\) of sets of a topological space \(X\) is called a base for a mapping \(f : X \to Y\) if the preimage \(f^{-1}(V)\) of an arbitrary open set \(V\) in \(Y\) is a union of sets from \(\mathcal{B}\).

**Definition 4.** If a mapping \(f : X \to Y\) has a base which is a \(\sigma\)-\(\mathcal{P}\)-family, then we say that \(f\) is a \(\sigma\)-\(\mathcal{P}\) mapping.

The collection of all \(\sigma\)-\(\mathcal{P}\) mappings between \(X\) and \(Y\) we will denote by

- \(\Sigma(X,Y)\), if \(\mathcal{P}\) is a property of discreteness;
- \(\Sigma^*(X,Y)\), if \(\mathcal{P}\) is a property of a strong discreteness;
- \(\Sigma^f(X,Y)\), if \(\mathcal{P}\) is a property of a strong functional discreteness.

By \(\Sigma^f_\alpha(X,Y) / \Sigma^*_{\alpha}(X,Y)\) we denote the collection of all mappings between \(X\) and \(Y\) which has a \(\sigma\)-sfd base of functionally ambiguous /multiplicative/ sets of the class \(\alpha\) in \(X\).

**Remark 1.** For any spaces \(X\) and \(Y\) the following relations holds:

1. \(\Sigma^f_\beta(X,Y) \subseteq \Sigma^f_\alpha(X,Y)\) if \(0 \leq \beta < \alpha < \omega_1\);
2. \(\Sigma^f_\beta(X,Y) = \Sigma^f_{\beta+1}(X,Y)\) if \(0 \leq \beta < \omega_1\);
3. \(\Sigma^f_\alpha(X,Y) \subseteq C(X,Y)\).

Let us observe that every continuous mapping \(f : X \to Y\) is \(\sigma\)-strongly functionally discrete if either \(X\), or \(Y\) is a metric space, since every metric space has \(\sigma\)-sfd base of open sets. Clearly, every mapping with values in a second countable space is \(\sigma\)-sfd. In [4] Hansell proved that any Borel measurable mapping \(f : X \to Y\) is \(\sigma\)-discrete if \(X\) is a complete metric space and \(Y\) is a metric space.

**Lemma 1.** Let \(0 \leq \alpha < \omega_1\), \(X\) be a topological space, \((U_i : i \in I)\) be a locally finite family of functionally open sets in \(X\), \((B_i : i \in I)\) be a family of sets of the \(\alpha\)'th functionally additive /multiplicative/ class in \(X\) such that \(B_i \subseteq U_i\) for every \(i \in I\). Then the set \(B = \bigcup_{i \in I} B_i\) is of the \(\alpha\)'th functionally additive /multiplicative/ class \(\alpha\) in \(X\).

**Proof.** For \(\alpha = 0\) we consider the case each \(B_i\) is functionally closed and take a continuous function \(f_i : X \to [0,1]\) such that \(B_i = f_i^{-1}(0)\) and \(X \setminus U_i = f_i^{-1}(1)\), \(i \in I\). Then the function \(f(x) = \min_{i \in I} f_i(x)\) is continuous and \(B = f^{-1}(0)\).

Assume that our proposition is true for all \(0 \leq \xi < \alpha\) and prove it for \(\xi = \alpha\). If \(\alpha\) is a limit ordinal then we take an increasing sequence of ordinals \((\alpha_n)_{n=1}^{\infty}\) which converges to \(\alpha\). If \(\alpha = \beta + 1\) then we put \(\alpha_n = \beta\) for every \(n \in \mathbb{N}\).

Let \(B_i\) be a set of the \(\alpha\)'th functionally additive class \(\alpha\) and \((B_{i,n})_{n=1}^{\infty}\) be a sequence of sets of the \(\alpha_n\)'th functionally multiplicative classes such that \(B_i = \bigcup_{n=1}^{\infty} B_{i,n}\). By the inductive assumption
the set $F_n = \bigcup_{i \in I} B_{i,n}$ belongs to the $\alpha_n$’th functionally multiplicative class for every $n$. Hence, the set $B = \bigcup_{i \in I} B_i = \bigcup_{n=1}^{\infty} F_n$ is of the $\alpha$’th functionally additive class.

Now assume that $B_i$ belongs to the $\alpha$’th functionally multiplicative class for every $i \in I$ and take a sequence $(B_{i,n})_{n=1}^{\infty}$ of sets of the $\alpha_n$’th functionally additive classes such that $B_i = \bigcap_{n=1}^{\infty} B_{i,n}$. Notice that each set $G_{i,n} = B_{i,n} \cap U_i$ is of the $\alpha_n$’th functionally additive class, $G_{i,n} \subseteq U_i$ and $B_i = \bigcap_{n=1}^{\infty} G_{i,n}$. Then $G_n = \bigcup_{i \in I} G_{i,n}$ belongs to the $\alpha_n$’th functionally additive class for every $n$.

Hence, the set $B = \bigcap_{n=1}^{\infty} G_n$ if of the $\alpha$’th functionally multiplicative class.

**Corollary 2.** For any $0 \leq \alpha < \omega_1$ a union of an sfd-family of sets of the $\alpha$’th functionally additive/multiplicative class in a topological space is a set of the same class.

**Lemma 3.** Let $X$ be a topological space and $f \in \Sigma^f(X,Y)$. Then $f$ has a $\sigma$-sfd base $B$ which is a union of a sequence of well sfd-families.

**Proof.** Let $B' = \bigcup_{n=1}^{\infty} B'_n$ be a base for $f$, where $B'_n$ is an sfd-family for every $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ and $B \in B'_n$ we take a functionally open set $U_{B,n}$ and a sequence of functionally closed sets $(F_{B,m})_{m=1}^{\infty}$ such that the family $(U_{B,n} : B \in B'_n)$ is discrete, $B \subseteq U_{B,n}$ and $U_{B,n} = \bigcup_{m=1}^{\infty} F_{B,m}$ for every $B \in B'$. For all $n, m \in \mathbb{N}$ we put

$$B_{n,m} = (B \cap F_{B,m} : B \in B'_n) \quad \text{and} \quad B = \bigcup_{n,m=1}^{\infty} B_{n,m}.$$  

It is easy to see that each $B_{n,m}$ is well sfd-family and $B$ is a base for $f$.

**Theorem 4.** Let $0 < \alpha < \omega_1$, $X$ be a topological space and $Y$ be a topological space with a $\sigma$-disjoint base. Then

$$K_\alpha(X,Y) \cap \Sigma^f(X,Y) \subseteq \Sigma^f_\alpha(X,Y).$$

**Proof.** Let $f \in K_\alpha(X,Y) \cap \Sigma^f(X,Y)$. According to Lemma 3 there exists a base $B = \bigcup_{m=1}^{\infty} B_m$ for $f$ such that each $B_m = (B_{i,m} : i \in I_m)$ is well sfd-family. For all $m$ and $i \in I_m$ we take a functionally open set $U_{i,m}$ and a functionally closed set $F_{i,m}$ in $X$ such that $B_{i,m} \subseteq F_{i,m} \subseteq U_{i,m}$ and the family $(U_{i,m} : i \in I_m)$ is discrete.

Consider a $\sigma$-disjoint base $V = \bigcup_{n=1}^{\infty} V_n$ of open sets in $Y$. Since $f \in K_\alpha(X,Y)$, for every $V \in V$ there exists a sequence $(A_{k,V})_{k=1}^{\infty}$ of sets of functionally multiplicative classes $< \alpha$ in $X$ such that $f^{-1}(V) = \bigcup_{k=1}^{\infty} A_{k,V}$. For $m, n, k \in \mathbb{N}$ we put

$$B_{m,n,k} = (F_{i,m} \cap A_{k,V} : i \in I_m, V \in V_n \quad \text{and} \quad B_{i,m} \subseteq f^{-1}(V)).$$

Notice that each family $B_{m,n,k}$ consists of functionally ambiguous sets of the class $\alpha$ and is strongly functionally discrete in $X$, since the family $B_m$ is strongly functionally discrete and for any nonempty set $B_{i,m} \in B_m$ there is at most one set $V \in V_n$ such that $B_{i,m} \subseteq f^{-1}(V)$. Let

$$B_0 = \bigcup_{m,n,k=1}^{\infty} B_{m,n,k}.$$
We show that $B_0$ is a base for $f$. Fix $V \in \mathcal{V}$ and verify that

$$f^{-1}(V) = \bigcup_{m,k=1}^{\infty} \bigcup_{i \in I_m, b_{i,m} \subseteq f^{-1}(V)} (F_{i,m} \cap A_{k,V}).$$

Since $A_{k,V} \subseteq f^{-1}(V)$ for every $k$, the set in the right side of the equality is contained in $f^{-1}(V)$. On the other hand, if $x \in f^{-1}(V)$ then $x \in A_{k,V}$ for some $k$. Moreover, $B$ is a base for $f$, consequently, there are $m$ and $i \in I_m$ such that $x \in B_{i,m} \subseteq f^{-1}(V)$. Then $x \in F_{i,m}$.

**Proposition 5.** Let $0 < \alpha < \omega_1$, $X$ and $Y$ be topological spaces. Then

$$\Sigma_\alpha^f(X,Y) \subseteq K_\alpha(X,Y) \cap \Sigma^f(X,Y).$$

**Proof.** Let $f \in \Sigma_\alpha^f(X,Y)$. Clearly, $f \in \Sigma^f(X,Y)$. We show that $f \in K_\alpha(X,Y)$. Let $V$ be an open set in $Y$ and $B = \bigcup_{m=1}^{\infty} B_m$ be a base for $f$ such that each family $B_m$ is strongly functionally discrete in $X$ and consists of functionally ambiguous sets of the class $\alpha$. Then there exists a subfamily $B_V \subseteq B$ such that $f^{-1}(V) = \bigcup B_V$. For every $m \in \mathbb{N}$ we denote $B_m = (B \in B_V : B \in B_m)$. Corollary 2 implies that every set $B_m = \bigcup B'_m$ belongs to the $\alpha$’th functionally additive class in $X$. Moreover, $f^{-1}(V) = \bigcup_{m=1}^{\infty} B_m$. Hence, $f \in K_\alpha(X,Y)$.

Theorem 4 and Proposition 5 imply

**Theorem 6.** Let $0 < \alpha < \omega_1$, $X$ be a topological space and $Y$ be a space with a $\sigma$-disjoint base. Then

$$K_\alpha(X,Y) \cap \Sigma^f(X,Y) = \Sigma_\alpha^f(X,Y).$$

**Proposition 7.** Let $0 \leq \alpha < \omega_1$, $X$, $Y$, and $Z$ be topological spaces, $f \in \Sigma_\alpha^f(X,Y)$, $g \in \Sigma^f(Y,Z)$ and let $h : X \rightarrow Y \times Z$ be defined by

$$h(x) = (f(x), g(x))$$

for every $x \in X$. Then $h \in \Sigma_\alpha^f(X,Y \times Z)$.

**Proof.** Let $B_f = \bigcup_{n=1}^{\infty} B_{n,f}$ and $B_g = \bigcup_{m=1}^{\infty} B_{m,g}$ be $\sigma$-sfd bases of functionally ambiguous sets of the class $\alpha$ for $f$ and $g$, respectively. For all $n, m \in \mathbb{N}$ we put

$$B_{n,m} = \{ B_f \cap B_g : B_f \in B_{n,f}, B_g \in B_{m,g} \}.$$ 

It is easy to see that $B = \bigcup_{n,m=1}^{\infty} B_{n,m}$ is a $\sigma$-sfd base for $h$ which consists of functionally ambiguous sets of the class $\alpha$ in $X$.

**Definition 5.** We say that a family $(A_i : i \in I)$ is a partition of a space $X$ if $X = \bigcup_{i \in I} A_i$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

**Proposition 8.** Let $0 \leq \alpha < \omega_1$, $(X_n : n \in \mathbb{N})$ be a partition of a topological space $X$ by functionally ambiguous sets of the class $\alpha$, $(f_n)_{n=1}^{\infty}$ be a sequence of mappings from $\Sigma_\alpha^f(X,Y)$ and $f(x) = f_n(x)$ if $x \in X_n$ for some $n$. Then $f \in \Sigma_\alpha^f(X,Y)$.

**Proof.** Let $B_n$ be a $\sigma$-sfd base for a mapping $f_n$ which consists of functionally ambiguous sets of the class $\alpha$ in $X$. Let

$$B = \bigcup_{n=1}^{\infty} (B \cap X_n : B \in B_n).$$

It is easy to see that $B$ is a $\sigma$-sfd base for $f$ which consists of functionally ambiguous sets of the $\alpha$’th class.
3. Auxiliary facts on functionally measurable sets

The proofs of the next two lemmas are completely similar to the proofs of Theorem 2 from \[6\, p.\ 350\] and Theorem 2 from \[6, p.\ 357\].

**Lemma 9.** Let \(0 < \alpha < \omega_1\) and \(X\) be a topological space. Then for any disjoint sets \(A, B \subseteq X\) of the \(\alpha\)'th functionally multiplicative class there exists a functionally ambiguous set \(C\) of the class \(\alpha\) such that \(A \subseteq C \subseteq X \setminus B\).

**Lemma 10.** If \(A\) is a functionally ambiguous set of the \((\alpha + 1)\)'th class in a topological space \(X\), where \(\alpha\) is a limit ordinal, then there exists a sequence \((A_n)_{n=1}^\infty\) of functionally ambiguous sets of classes \(< \alpha\) such that

\[
A = \bigcup_{n=1}^\infty \bigcap_{k=0}^\infty A_{n+k} = \bigcap_{n=1}^\infty \bigcup_{k=0}^\infty A_{n+k}.
\]

(1)

The definition of sfd-family easily implies the following fact.

**Lemma 11.** Let \(A_1, \ldots, A_n\) be sfd-families of subsets of a topological space \(X\), \(A_k = \bigcup A_k\) for \(k = 1, \ldots, n\) and the family \((A_k : k = 1, \ldots, n)\) is strongly functionally discrete. Then the family \(A = \bigcup_{k=1}^n A_k\) is strongly functionally discrete.

**Lemma 12.** Let \(0 < \alpha < \omega_1\), \(A\) be a disjoint \(\sigma\)-sfd family of functionally additive sets of the \(\alpha\)'th class in a topological space \(X\). Then for any \(A \in A\) there exists an increasing sequence \((D_n^A)_{n=1}^\infty\) of functionally ambiguous sets of the class \(\alpha\) such that \(A = \bigcup_{n=1}^\infty D_n^A\) and the family \((D_n^A : A \in A)\) is strongly functionally discrete for every \(n \in \mathbb{N}\).

If \(\alpha = \beta + 1\), then every set \(D_n^A\) can be chosen from the \(\beta\)'th functionally multiplicative class.

**Proof.** Let \(\alpha = 1\) and \(A = \bigcup_{k=1}^\infty A_k\), where \(A_k\) is an sfd-family of sets of the first functionally additive class and \((\cup A_k) \cap (\cup A_j) = \emptyset\) for \(k \neq j\). For every \(A \in A\) we take an increasing sequence \((B_n^A)_{n=1}^\infty\) of functionally closed sets such that \(A = \bigcup_{n=1}^\infty B_n^A\). Now for all \(A \in A\) and \(n \in \mathbb{N}\) we put

\[
F_n^A = \begin{cases} B_n^A, & \text{if } A \in A_k \text{ for } k \leq n, \\ \emptyset, & \text{if } A \in A_k \text{ for } k > n. \end{cases}
\]

Then \((F_n^A : A \in A) = \bigcup_{k \leq n} (B_n^A : A \in A_k)\) for every \(n\). Since every family \(B_k = (B_n^A : A \in A_k)\) is strongly functionally discrete, the set \(B_k = \bigcup B_k\) is functionally closed. Moreover, \(B_k \cap B_m = \emptyset\) for all \(k \neq m\). Since \((B_k : k = 1, \ldots, n)\) is an sfd-family, Lemma 11 implies that \((F_n^A : A \in A)\) is also sfd-family. Moreover, \(A = \bigcup_{n=1}^\infty F_n^A\) for every \(A \in A\).

Assume that the assertion of lemma is true for all \(1 \leq \xi < \alpha\) and verify it for \(\xi = \alpha\). Consider a disjoint sequence of sfd-families \(A_k\) which consist of sets of the \(\alpha\)'th functionally additive class.

For every \(A \in A\) we take an increasing sequence \((B_n^A)_{n=1}^\infty\) such that \(A = \bigcup_{n=1}^\infty B_n^A\). We may assume that every set \(B_n^A\) belongs to the \(\alpha_n\)'th functionally multiplicative class, where \(\alpha_n = \beta\) for every \(n \in \mathbb{N}\) if \(\alpha = \beta + 1\), and \((\alpha_n)_{n=1}^\infty\) is an increasing sequence of ordinals such that \(\alpha = \sup \alpha_n\) if \(\alpha\) is a limit ordinal.
Fix \( n \in \mathbb{N} \) and for every \( k = 1, \ldots, n \) we denote \( B_{k,n} = (B^A_n : A \in \mathcal{A}_k) \) and \( B_{k,n} = \bigcup B_{k,n} \). Since \( B_{1,n}, \ldots, B_{n,n} \) are mutually disjoint sets of the \( \alpha_n \)’th functionally multiplicative class, Lemma 9 implies that there exist mutually disjoint functionally ambiguous sets \( C_{1,n}, \ldots, C_{n,n} \) of the class \( \alpha_n \) such that \( B_{k,n} \subseteq C_{k,n} \) for every \( k = 1, \ldots, n \). By the inductive assumption \( C_{k,n} = \bigcup_{m=1}^{\infty} C_{k,n,m} \) and for every \( m \in \mathbb{N} \) the family \( (C_{k,n,m} : k = 1, \ldots, n) \) is strongly functionally discrete and consists of functionally ambiguous sets of the class \( \alpha_n \).

Now for all \( n, m \in \mathbb{N} \) and \( A \in \mathcal{A} \) we put

\[
D^A_{n,m} = \begin{cases} 
B^A_n \cap C_{k,m,n}, & \text{if } A \in \mathcal{A}_k \text{ for } k \leq n, \\
\emptyset, & \text{if } A \in \mathcal{A}_k \text{ for } k > n. 
\end{cases}
\]

Then \( (D^A_{n,m} : A \in \mathcal{A}) = \bigcup_{k \leq n} (B^A_n \cap C_{k,m,n} : A \in \mathcal{A}_k) \).

Fix \( n, m \in \mathbb{N} \) and for every \( k = 1, \ldots, n \) we put

\[
D_k = (B^A_n \cap C_{k,m,n} : A \in \mathcal{A}_k).
\]

Notice that every family \( D_k \) is strongly functionally discrete and consists of functionally ambiguous sets of the class \( \alpha \). Then the set \( D_k = \bigcup D_k \) is functionally ambiguous of the class \( \alpha \) for \( k = 1, \ldots, n \). Moreover, \( D_k \cap D_m = \emptyset \) for all \( k \neq m \). Since the family \( (D_k : k = 1, \ldots, n) \) is strongly functionally discrete, Lemma 11 implies that \( (D^A_{n,m} : A \in \mathcal{A}) \) is an sfd-family. Moreover, \( A = \bigcup_{n,m=1}^{\infty} D^A_{n,m} \) for every \( A \in \mathcal{A} \).

In case \( \alpha = \beta + 1 \) for all \( n, m \in \mathbb{N} \) and \( A \in \mathcal{A} \) we choose an increasing sequence \( (D^A_{n,m,k})_{k=1}^{\infty} \) of functionally multiplicative class \( \beta \) such that \( D^A_{n,m} = \bigcup_{k=1}^{\infty} D^A_{n,m,k} \). Clearly, \( (D^A_{n,m,k} : A \in \mathcal{A}) \) is an sfd-family for all \( n, m, k \) and \( A = \bigcup_{n,m,k=1}^{\infty} D^A_{n,m,k} \).

**Lemma 13.** Let \( 0 \leq \alpha < \omega_1 \), \( X \) be a topological space, \( \mathcal{A} \) be an \( \sigma \)-sfd family of sets of the \( \alpha \)’th functionally multiplicative class such that \( \bigcup \mathcal{A} = X \). Then there exists a sequence \( (\mathcal{A}_n)_{n=1}^{\infty} \) of families of sets of the \( \alpha \)’th functionally multiplicative class such that

1. \( \bigcup_{n=1}^{\infty} \mathcal{A}_n \prec \mathcal{A} \),
2. \( \mathcal{A}_n \prec \mathcal{A}_{n+1} \),
3. \( \mathcal{A}_n \) is an sfd-family,
4. \( \bigcup_{n=1}^{\infty} \mathcal{A}_n = X \)

for every \( n \in \mathbb{N} \).

**Proof.** Let \( \mathcal{A} = \bigcup_{k=1}^{\infty} \mathcal{B}_k \) and let \( \mathcal{B}_k \) be an sfd-family of sets of the \( \alpha \)’th functionally multiplicative class. For every \( k \in \mathbb{N} \) we put

\[
\mathcal{C}_k = (B \setminus \bigcup_{j<k} \mathcal{B}_j : B \in \mathcal{B}_k) \text{ and } \mathcal{C} = \bigcup_{k=1}^{\infty} \mathcal{C}_k.
\]

Then \( \mathcal{C} \) is a disjoint \( \sigma \)-sfd family of sets of the \( (\alpha + 1) \)’th functionally additive class, \( \mathcal{C} \prec \mathcal{A} \) and \( \bigcup \mathcal{C} = X \). By Lemma 12 for every \( C \in \mathcal{C} \) there exists an increasing sequence \( (D^C_n)_{n=1}^{\infty} \) of sets of the \( \alpha \)’th functionally multiplicative class such that \( C = \bigcup_{n=1}^{\infty} D^C_n \) and the family \( (D^C_n : C \in \mathcal{C}) \) is strongly functionally discrete for every \( n \in \mathbb{N} \).

It remains to put \( \mathcal{A}_n = \left( \bigcup_{k \leq n} D^C_k : C \in \mathcal{C} \right) \) for \( n \in \mathbb{N} \).
4. Approximation lemmas

**Definition 6.** We say that a sequence \((f_n)_{n=1}^\infty\) of mappings \(f_n : X \to Y\) is stably convergent to \(f : X \to Y\) and denote this fact by \(f_n \overset{\text{st}}{\to} f\), if for every \(x \in X\) there exists \(n_0 \in \mathbb{N}\) such that \(f_n(x) = f(x)\) for all \(n \geq n_0\).

If \(A \subseteq Y^X\), then the symbol \(\overline{A}^{\text{st}}\) stands for the set of all stable limits of sequences from \(A\).

**Lemma 14.** Let \(X, Y\) be topological spaces, \(((B_{i,m} : i \in I_m))_{m=1}^\infty\) be a sequence of sfd-families of sets of the \((\alpha + 1)\)th functionally ambiguous sets in \(X\), \(\alpha\) be a limit ordinal, let the family \((B_m = \bigcup_{i \in I_m} B_{i,m} : m \in \mathbb{N})\) be a partition of \(X\), \(((y_{i,m} : i \in I_m))_{m=1}^\infty\) be a sequence of points from \(Y\) and let \(f : X \to Y\) be defined by

\[
f(x) = y_{i,m},
\]

if \(x \in B_{i,m}\) for some \(m \in \mathbb{N}\) and \(i \in I_m\). Then \(f \in \Sigma_{<\alpha}(X,Y)\).

**Proof.** Fix \(m \in \mathbb{N}\). Since \(B_m\) is functionally ambiguous set of the class \((\alpha + 1)\) in \(X\) by Corollary \[\text{1}\]\ Lemma \[\text{10}\] implies that there exists a sequence \((C_{m,n})_{n=1}^\infty\) of functionally ambiguous sets of classes \(<\alpha\) such that

\[
B_m = \bigcup_{n=1}^\infty \bigcap_{k=0}^{n} C_{m,n+k} = \bigcap_{n=1}^\infty \bigcup_{k=0}^{n} C_{m,n+k}.
\]

Moreover, there exists a discrete family \((U_{i,m} : i \in I_m)\) of functionally open sets in \(X\) such that \(B_{i,m} \subseteq U_{i,m}\) for every \(i \in I_m\).

For all \(m, n \in \mathbb{N}\) we put

\[
D_{m,n} = C_{m,n} \setminus \bigcup_{k<m} C_{k,n}.
\]

Notice that every \(D_{m,n}\) is a functionally ambiguous set of a class \(<\alpha\). Moreover,

\[
(\forall x \in X) (\exists m_x \in \mathbb{N}) (\exists n_x \in \mathbb{N}) (\forall n \geq n_x) (x \in D_{m_x,n}).
\]

Indeed, if \(x \in X\), then there exists a unique number \(m_x\) such that \(x \in B_{m_x}\) and \(x \not\in B_k\) for all \(k \neq m_x\). Then the equality \([\text{3}]\) implies that there are numbers \(N_1, \ldots, N_{m_x}\) such that

\[
x \not\in \bigcup_{n \geq N_k} C_{k,n} \text{ if } k < m_x \text{ and } x \in \bigcap_{n \geq N_{m_x}} C_{m_x,n}.
\]

Hence, for all \(n \geq n_x = \max\{N_1, \ldots, N_{m_x}\}\) we have \(x \in D_{m_x,n}\).

Let \(y_0\) be any point from \(\{y_{i,1} : i \in I_1\}\). Fix \(n \in \mathbb{N}\) and for all \(x \in X\) let

\[
f_n(x) = \begin{cases} y_{i,m}, & \text{if } x \in D_{m,n} \cap U_{i,m} \text{ for some } m < n \text{ and } i \in I_m, \\ y_0, & \text{otherwise.} \end{cases}
\]

Observe that \(f_n : X \to Y\) is defined correctly, since the family \((U_{i,m} : i \in I_m)\) is discrete for every \(m\) and the family \((D_{m,n} : m < n)\) is disjoint.

We show that \(f_n \in \Sigma_{<\alpha}(X,Y)\). For \(m = 1, \ldots, n-1\) let \(B_m = (D_{m,n} \cap U_{i,m} : i \in I_m)\) and let \(B_n\) be a family which consists of the set \((X \setminus \bigcup_{m<n} \bigcup B_m)\). Clearly, \(B = \bigcup_{m=1}^n B_m\) is \(\sigma\)-sfd family of functionally ambiguous sets of classes \(<\alpha\). It follows from the definition of \(f_n\) that \(B\) is a base for \(f_n\).

It remains to prove that \(f_n \overset{\text{st}}{\to} f\) on \(X\). Indeed, if \(x \in X\), then there exist \(m \in \mathbb{N}\) and \(i \in I_m\) such that \(x \in B_{i,m} \subseteq U_{i,m}\). Then \(f(x) = y_{i,m}\). It follows from \([\text{3}]\) that there exists a number \(n_0 > m\) with \(x \in D_{m,n}\) for all \(n \geq n_0\). Then \(f_n(x) = y_{i,m}\) for all \(n \geq n_0\). Hence, \(f_n(x) \to f(x)\) for all \(n \geq n_0\).
Lemma 15. Let \( \alpha < \omega_1 \) be a limit ordinal, \( X \) be a topological space, \( Y \) be a metric space and \( f \in \Sigma^f_{\alpha+1}(X,Y) \). Then there exists a sequence of mappings \( f_n \in \Sigma^f_{\leq \alpha}(X,Y)^{st} \) which is uniformly convergent to \( f \) on \( X \).

Proof. Let \( \mathcal{B} \) be a \( \sigma \)-sfd base for \( f \) which consists of functionally ambiguous sets of the class \( (\alpha+1) \) in \( X \). For every \( n \in \mathbb{N} \) we consider a covering \( \mathcal{U}_n \) of \( Y \) by open balls of diameters \( < \frac{1}{n} \) and put

\[
\mathcal{B}_n = \{ B \in \mathcal{B} : \exists U \in \mathcal{U}_n \ | \ B \subseteq f^{-1}(U) \}.
\]

Then \( \mathcal{B}_n \) is a \( \sigma \)-sfd family of functionally ambiguous sets of the class \( (\alpha+1) \), diam \( f(B) < \frac{1}{n} \) for every \( B \in \mathcal{B}_n \) and \( X = \bigcup \mathcal{B}_n \) for every \( n \in \mathbb{N} \).

Fix \( n \in \mathbb{N} \) and let \( \mathcal{B}_n = \bigcup_{m=1}^{\infty} \mathcal{B}_{n,m} \), where \( \mathcal{B}_{n,m} \) is an sfd-family of functionally ambiguous sets of the class \( (\alpha+1) \). We put \( \mathcal{A}_{n,1} = \mathcal{B}_{n,1} \) and \( \mathcal{A}_{n,m} = (B \setminus (\bigcup_{k<m} \mathcal{B}_{n,k}) : B \in \mathcal{B}_{n,m}) \) for \( m > 1 \). Notice that for every \( m \in \mathbb{N} \) the set \( \mathcal{A}_{n,m} = \bigcup \mathcal{A}_{n,m} \) is functionally ambiguous of the class \( (\alpha+1) \) and the family \( \mathcal{A}_m = \mathcal{A}_{n,m} \) is a partition of \( X \). For every \( A \in \mathcal{A}_{n,m} \) we choose an arbitrary point \( y_{n,m}^A \in f(A) \). We define a mapping \( f_n : X \to Y \) by

\[
f_n(x) = y_{n,m}^A,
\]

if \( x \in A \) for some \( m \in \mathbb{N} \) and \( A \in \mathcal{A}_{n,m} \). Then \( f_n \in \Sigma^f_{\leq \alpha}(X,Y)^{st} \) by Lemma [14].

It remains to verify that \( (f_n)_{n=1}^{\infty} \) converges uniformly to \( f \). Indeed, if \( x \in X \) and \( n \in \mathbb{N} \), then \( f_n(x) = y_{n,m}^A \in f(A) \) for some \( m \in \mathbb{N} \) and \( A \in \mathcal{A}_{n,m} \). Since \( A \subseteq B \) for some \( B \in \mathcal{B}_{n,m} \),

\[
d(f(x), f_n(x)) \leq \text{diam} f(B) < \frac{1}{n},
\]

which completes the proof.

Lemma 16. Let \( 0 < \alpha < \omega_1 \), \( X \) be a topological space, \( (Y,d) \) be a metric space, \( f : X \to Y \) be a mapping, \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) be families of subsets of \( X \) such that

(i) \( \mathcal{A}_k \) is an sfd-family of sets of the \( \alpha \)th functionally multiplicative class;

(ii) \( \mathcal{A}_{k+1} \prec \mathcal{A}_k \) for \( k < n \);

(iii) \( \text{diam}(f(A)) < \frac{1}{2k+2} \) for all \( A \in \mathcal{A}_k \)

for every \( k = 1, \ldots, n \). Then there exists a mapping \( g \in \Sigma^f_{\alpha}(X,Y) \) such that the inclusion \( x \in \bigcup \mathcal{A}_k \) for \( k = 1, \ldots, n \) implies the inequality

\[
d(f(x), g(x)) < \frac{1}{2k}.
\]

Proof. Let \( \mathcal{A}_k = (A_{i,k} : i \in I_k) \) and \( (U_{i,k} : i \in I_k) \) be discrete families of functionally open sets in \( X \) such that \( A_{i,k} \subseteq U_{i,k} \) for every \( i \in I_k \) and \( k = 1, \ldots, n \).

By Lemma [9] there exists a family \( (B_{i,1} : i \in I_1) \) of functionally ambiguous sets of the class \( \alpha \) such that \( A_{i,1} \subseteq B_{i,1} \subseteq U_{i,1} \). Since \( \mathcal{A}_2 \prec \mathcal{A}_1 \), for every \( i \in I_2 \) there exists a unique \( j \in I_1 \) such that \( A_{i,2} \subseteq A_{j,1} \). Notice that \( U_{i,2} \cap B_{j,1} \) is a functionally ambiguous set of the class \( \alpha \) and applying Lemma [9] we obtain a functionally ambiguous set \( B_{i,2} \) of the class \( \alpha \) such that \( A_{i,2} \subseteq B_{i,2} \subseteq U_{i,2} \cap B_{j,1} \). Proceeding in this way we obtain a sequence \( ((B_{i,k} : i \in I_k))_{k=1}^{n} \) of families of subsets of \( X \) such that
\begin{itemize}
  \item $B_{i,k} \subseteq U_{i,k}$ for every $i \in I_k$;
  \item $B_{i,k}$ is a functionally ambiguous set of the class $\alpha$ for every $i \in I_k$;
  \item for every $k < n$ and for every $i \in I_{k+1}$ there exists a unique $j \in I_k$ such that
    \begin{equation}
      A_{i,k+1} \subseteq A_{j,k},
    \end{equation}
    \begin{equation}
      A_{i,k+1} \subseteq B_{i,k+1} \subseteq B_{j,k}.
    \end{equation}
    for all $k = 1, \ldots, n$. Observe that for every $k$ the set
    \begin{equation*}
      B_k = \bigcup_{i \in I_k} B_{i,k}
    \end{equation*}
    is functionally ambiguous of the class $\alpha$ according to Corollary 1.
  \end{itemize}

We take any points $y_0 \in f(X)$ and $y_{i,k} \in f(A_{i,k})$ for every $k$ and $i \in I_k$. For all $x \in X$ we put
\begin{equation*}
  g_0(x) = y_0.
\end{equation*}
Assume that for some $k < n$ we have already defined mappings $g_1, \ldots, g_k$ from $\Sigma'_\alpha(X,Y)$ such that
\begin{equation}
  g_k(x) = \begin{cases} 
    g_{k-1}(x), & \text{if } x \in X \setminus B_k, \\
    y_{i,k}, & \text{if } x \in B_{i,k} \text{ for some } i \in I_k.
  \end{cases}
\end{equation}
We put
\begin{equation*}
  g_{k+1}(x) = \begin{cases} 
    g_k(x), & \text{if } x \in X \setminus B_{k+1}, \\
    y_{i,k+1}, & \text{if } x \in B_{i,k+1} \text{ for some } i \in I_{k+1}.
  \end{cases}
\end{equation*}
Then $g_{k+1} \in \Sigma'_\alpha(X,Y)$ by Lemma 8. Repeating inductively this process we obtain mappings $g_1, \ldots, g_n$ from $\Sigma'_\alpha(X,Y)$ each of which satisfies (7).

Now we prove that
\begin{equation}
  d(g_{k+1}(x), g_k(x)) < \frac{1}{2^{k+2}}
\end{equation}
for all $0 \leq k < n$ and $x \in X$. Indeed, if $x \in X \setminus B_{k+1}$, then $g_{k+1}(x) = g_k(x)$ and $d(g_{k+1}(x), g_k(x)) = 0$. Assume that $x \in B_{i,k+1}$ for some $i \in I_{k+1}$ and choose $j \in I_k$ such that (5) and (6) holds. Then $g_{k+1}(x) = y_{i,k+1}$ and $g_k(x) = y_{j,k}$. Since $f(A_{i,k+1}) \subseteq f(A_{j,k})$, $y_{i,k+1} \in f(A_{j,k})$. Hence, $d(g_{k+1}(x), g_k(x)) \leq \text{diam}(f(A_{j,k})) < \frac{1}{2^{k+2}}$.

We put $g = g_n$ and show that (11) holds. Let $1 \leq k \leq n$ and $x \in \cup A_k$. Then $x \in A_{i,k} \subseteq B_{i,k}$ for some $i \in I_k$. It follows that $g_k(x) = y_{i,k}$ and consequently
\begin{equation*}
  d(f(x), g_k(x)) \leq \text{diam}(f(A_{i,k})) < \frac{1}{2^{k+2}}.
\end{equation*}
Taking into account (8) we obtain that
\begin{equation*}
  d(f(x), g_n(x)) \leq d(f(x), g_k(x)) + \sum_{i=k}^{n-1} d(g_i(x), g_{i+1}(x)) < \frac{1}{2^{k+2}} + \frac{1}{2^{k+1}} < \frac{1}{2^k}.
\end{equation*}
5. A generalization of Banach’s theorem

Let $X, Y$ be topological spaces and $A \subseteq Y^X$. By $\overline{A}^p$ we denote the set of all pointwise limits of sequences of mappings from $A$.

We put

$$\Lambda_1(X, Y) = \Sigma^f_1(X, Y)$$

and for all $1 < \alpha < \omega_1$ let

$$\Lambda_\alpha(X, Y) = \bigcup_{\beta < \alpha} \Lambda_\beta(X, Y)^p.$$

Clearly,

$$\Lambda_\beta(X, Y) \subseteq \Lambda_\alpha(X, Y),$$

if $\beta \leq \alpha$. Moreover,

$$\Lambda_{\alpha+1}(X, Y) = \Lambda_\alpha(X, Y)^p,$$

and if $\alpha = \sup \alpha_n$ is a limit ordinal, then

$$\Lambda_\alpha(X, Y) = \bigcup_{n=1}^{\infty} \Lambda_{\alpha_n}(X, Y)^p.$$

Theorem 17. Let $X$ be a topological space, $(Y, d)$ be a metric space. Then

(i) $\Sigma^f_{\alpha}(X, Y) \subseteq \Lambda_\alpha(X, Y)$, if $1 \leq \alpha \leq \omega_0$;

(ii) $\Sigma^f_{\alpha+1}(X, Y) \subseteq \Lambda_\alpha(X, Y)$, if $\omega_0 \leq \alpha < \omega_1$.

Proof. The proposition is obvious for $\alpha = 1$. Assume it is true for all $1 \leq \beta < \alpha$ and prove the proposition for $\beta = \alpha$.

Let $\alpha < \omega_0$ and let $f$ be a mapping from $\Sigma^f_\alpha(X, Y) = \Sigma^f_{\alpha-1}(X, Y)$ with a $\sigma$-sfd base $B$ which consists of sets of the $(\alpha - 1)$’th functionally multiplicative class in $X$. For every $k \in \mathbb{N}$ we consider a covering $U_k$ of $Y$ by open balls of diameters $< \frac{1}{2^k}$ and put

$$B_k = \{ B \in B : \exists U \in U_k \mid B \subseteq f^{-1}(U) \}.$$ 

Then $B_k$ is a $\sigma$-sfd family and $X = \bigcup B_k$ for every $k$. By Lemma 13 for every $k \in \mathbb{N}$ there exists a sequence $(B_{k,n})_{n=1}^{\infty}$ of sfd families of sets of the $(\alpha - 1)$’th functionally multiplicative class in $X$ such that $\bigcup_{n=1}^{\infty} B_{k,n} < B_k$, $B_{k,n} \sim B_{k,n+1}$ and $\bigcup_{n=1}^{\infty} B_{k,n} = X$ for every $n \in \mathbb{N}$. For all $k, n \in \mathbb{N}$ we put

$$F_{k,n} = \{ B_1 \cap \cdots \cap B_k : B_m \in B_{m,n}, 1 \leq m \leq k \}.$$ 

Notice that every family $F_{k,n}$ is strongly functionally discrete, consists of sets of the $(\alpha - 1)$’th functionally multiplicative class and

(a) $F_{k+1,n} \prec F_{k,n}$,

(b) $F_{k,n} \prec F_{k,n+1}$,

(c) $\bigcup_{n=1}^{\infty} F_{k,n} = X$.

For every $n \in \mathbb{N}$ we apply Lemma 16 to the mapping $f$ and to the families $F_{1,n}$, $F_{2,n}, \ldots, F_{n,n}$ and obtain a sequence of mappings $g_n \in \Sigma^f_{\alpha-1}(X, Y)$ such that the inclusion $x \in \bigcup F_{k,n}$ for $k \leq n$ implies the inequality

$$d(f(x), g_n(x)) < \frac{1}{2^k}.$$
We prove that \( g_n \to f \) pointwise on \( X \). Fix \( x \in X \) and \( \varepsilon > 0 \). Let \( k \in \mathbb{N} \) be a number such that \( \frac{1}{2^k} < \varepsilon \). Conditions (b) and (c) imply that there exists \( n_0 \geq k \) such that \( x \in \mathcal{F}_{k,n} \) for all \( n \geq n_0 \). Then for all \( n \geq n_0 \) we have \( d(f(x), g_n(x)) < \frac{1}{2^k} < \varepsilon \).

Since \( g_n \in \Lambda_{\alpha-1}(X, Y) \) for every \( n \) by the inductive assumption, \( f \in \Lambda_{\alpha}(X, Y) \). Therefore, the proposition (i) is proved for all \( \alpha < \omega_0 \).

Now let \( \alpha \geq \omega_0 \) be a limit ordinal and \( f \in \Sigma^f_{\alpha+1}(X, Y) \). Lemma \([13]\) implies that there exists a sequence of mappings \( f_n \in \Sigma^f_{\alpha}(X, Y) \) which converges uniformly to \( f \) on \( X \). Without loss of generality we may assume that
\[
d(f_{n+1}(x), f_n(x)) < \frac{1}{2^n}
\]
for all \( x \in X \) and \( n \in \mathbb{N} \). For every \( n \in \mathbb{N} \) there exists a sequence of mappings \( f_{n,m} \in \Sigma^f_{\alpha}(X, Y) \) such that
\[
f_{n,m} \xrightarrow{\text{st}} f_n
\]
on \( X \). For every \( x \in X \) we put \( h_{0,m}(x) = f_{0,m}(x) \) for every \( m \in \mathbb{N} \). Suppose that we have already defined sequences of mappings \( (h_{k,m})_{m=1}^{\infty} \) for some \( n \in \mathbb{N} \) and for every \( 0 \leq k \leq n \) such that
(a) \( h_{k,m} \xrightarrow{\text{st}} f_k \) for all \( 0 \leq k \leq n \);
(b) \( h_{k,m} \in \Sigma^f_{\alpha}(X, Y) \) for all \( 0 \leq k \leq n \) and \( m \in \mathbb{N} \);
(c) \( d(h_{k+1,m}(x), h_{k,m}(x)) < \frac{1}{2^n} \) for all \( 0 \leq k < n \), \( m \in \mathbb{N} \) and \( x \in X \).

For every \( m \in \mathbb{N} \) we consider the set
\[
A_m = \left\{ x \in X : d(f_{n+1,m}(x), h_{n,m}(x)) < \frac{1}{2^n} \right\}
\]
and put
\[
h_{n+1,m}(x) = \begin{cases} f_{n+1,m}(x), & \text{if } x \in A_m, \\ h_{n,m}(x), & \text{if } x \notin A_m. \end{cases}
\]

We check the condition (a) for the sequence \( (h_{n+1,m})_{m=1}^{\infty} \). If \( x \in X \), then there exists a number \( m_0 \) such that \( f_{n+1,m}(x) = f_{n+1}(x) \) and \( f_{n,m}(x) = f_n(x) \) for all \( m \geq m_0 \). Then
\[
d(f_{n+1,m}(x), h_{n,m}(x)) = d(f_{n+1}(x), f_n(x)) < \frac{1}{2^n}
\]
for all \( m \geq m_0 \). Hence, \( x \in A_m \) for all \( m \geq m_0 \). Consequently, \( h_{n+1,m}(x) = f_{n+1,m}(x) \), which implies that
\[
h_{n+1,m} \xrightarrow{\text{st}} f_{n+1}
\]
on \( X \).

Now we check the condition (b). For every \( m \in \mathbb{N} \) the mapping \( \varphi(x) = d(f_{n+1,m}(x), h_{n,m}(x)) \) belongs to a class \( \Sigma_{\alpha}(X, \mathbb{R}) \) according to Proposition \([7]\). Hence, every set \( A_{m} = \varphi^{-1}((-\infty, \frac{1}{2^n})) \) is functionally ambiguous of a class \( < \alpha \). Thus, \( h_{n+1,m} \in \Sigma^f_{\alpha}(X, Y) \) by Proposition \([8]\).

Finally, we check the condition (c). Let \( x \in X \) and \( m \in \mathbb{N} \). If \( x \in A_m \), then \( h_{n+1,m}(x) = f_{n+1,m}(x) \), and if \( x \notin A_m \), then \( h_{n+1,m}(x) = h_{n,m}(x) \). In both cases
\[
d(h_{n+1,m}(x), h_{n,m}(x)) < \frac{1}{2^n}.
\]

Therefore, we have constructed sequences of mappings \( (h_{n,m})_{m=1}^{\infty} \) which satisfy (a)–(c) for every \( n \in \mathbb{N} \).
We prove that \( h_{n,n} \xrightarrow{n \to \infty} f \) pointwise on \( X \). Fix \( x \in X \) and \( \varepsilon > 0 \). Choose \( n_0 \in \mathbb{N} \) such that
\[
d(f(x), f_{n_0}(x)) < \frac{1}{2n_0} < \frac{\varepsilon}{2}.
\]

There exists a number \( n_1 \geq n_0 \) such that
\[
h_{n_0,n}(x) = f_{n_0}(x)
\]
for all \( n \geq n_1 \). Hence, for all \( n \geq n_1 \) we have
\[
d(f(x), h_{n,n}(x)) \leq d(f(x), f_{n_0}(x)) + \sum_{k=n_0}^{n-1} d(h_{k,n}(x), h_{k+1,n}(x)) < \frac{1}{2n_0} + \frac{1}{2n_0} < \varepsilon.
\]

By the inductive assumption \( h_{n,n} \in \Lambda_{<\alpha}(X,Y) \) for every \( n \in \mathbb{N} \). Hence, \( f \in \Lambda_{\alpha}(X,Y) \). Consequently, the proposition (i) is proved for all \( \alpha \leq \omega_0 \) and the proposition (ii) is proved for all limit ordinals \( \alpha \).

It \( \alpha = \beta + m \), where \( \beta \) is a limit ordinal and \( m \in \mathbb{N} \), and \( f \in \Sigma^f_{\alpha+1}(X,Y) \), then, by the same method as in proof of (i), one can show that there exists a sequence of mappings \( g_n \in \Sigma^f_{\beta+m}(X,Y) \) which is pointwise convergent to \( f \) on \( X \). By the inductive assumption, \( g_n \in \Lambda_{\beta+m-1}(X,Y) \), and hence \( f \in \Lambda_{\alpha}(X,Y) \).

**Theorem 18.** Let \( X \) be a topological space, \( Y \) be a space with a \( \sigma \)-disjoint base. Then the class \( \Sigma^f(X,Y) \) is closed under pointwise limits.

**Proof.** Let \( (f_n)_{n=1}^\infty \) be a sequence of mappings \( f_n \in \Sigma^f(X,Y) \) which converges pointwise to a mapping \( f : X \to Y \). We show that \( f \in \Sigma^f(X,Y) \).

Consider a \( \sigma \)-disjoint base \( \mathcal{V} = \bigcup_{m=1}^\infty \mathcal{V}_m \) of \( Y \) and a \( \sigma \)-sfd base \( \mathcal{B}_n = \bigcup_{m=1}^\infty \mathcal{B}_{n,m} \) for \( f_n \), \( n \in \mathbb{N} \).

Denote \( \mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n \) and for every \( B \in \mathcal{B} \) we put
\[
\tilde{\mathcal{V}}_B = (V \in \mathcal{V} : \exists n \ | \ f_n(B) \subseteq V).
\]

Notice that the family \( \tilde{\mathcal{V}}_B \) is at most countable, since every family \( \mathcal{V}_m \) is disjoint. Enumerate the family \( \tilde{\mathcal{V}}_B \) in a sequence \( (V_{B,k} : k \in \mathbb{N}) \). For all \( n, m, k \in \mathbb{N} \) we put
\[
\mathcal{B}_{n,m,k} = (B \cap f^{-1}(V_{B,k}) : B \in \mathcal{B}_{n,m}).
\]

Clearly, \( \mathcal{B}_{n,m,k} \) is an sfd-family. It remains to prove that the family
\[
\tilde{\mathcal{B}} = \bigcup_{n,m,k} \mathcal{B}_{n,m,k}
\]
is a base for \( f \). Let \( V \in \mathcal{V} \) and \( x \in f^{-1}(V) \). Take a number \( n \) such that \( f_n(x) \in V \). Since \( \mathcal{B}_n \) is a base for \( f_n \), there are \( m \in \mathbb{N} \) and \( B \in \mathcal{B}_{n,m} \) such that \( x \in B \subseteq f_n^{-1}(V) \). Then \( V \in \tilde{\mathcal{V}}_B \). Hence, \( x \in B \cap f^{-1}(V_{B,k}) \subseteq f^{-1}(V) \).

**Theorem 19.** Let \( X \) be a topological space, \( Y \) be a perfectly normal space, \( 0 \leq \alpha < \omega_1 \) and let \( (f_n)_{n=1}^\infty \) be a sequence of mappings \( f_n \in K_{\alpha}(X,Y) \) which converges pointwise to a mapping \( f : X \to Y \). Then \( f \in K_{\alpha+1}(X,Y) \).
Proof. Let $F$ be a closed subset of $Y$ and $(G_n)_{n=1}^\infty$ be a decreasing sequence of open subsets of $Y$ such that

$$F = \bigcap_{n=1}^\infty G_n = \bigcap_{n=1}^\infty G_n.$$  

It follows from the equality $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in A$ that

$$f^{-1}(F) = \bigcap_{n=1}^\infty \bigcup_{k=n+1}^\infty f_k^{-1}(G_n).$$

Since $f_k \in K_\alpha(X, Y)$, for every $n$ the set $f_k^{-1}(G_n)$ belongs to the $\alpha$’th functionally additive class, hence, $f^{-1}(F)$ is a set of the $(\alpha + 1)$’th functionally multiplicative class in $X$.

**Theorem 20.** Let $X$ be a topological space, $Y$ be a perfectly normal space with a $\sigma$-disjoint base and $0 < \alpha < \omega_1$. Then

(i) $\Lambda_\alpha(X, Y) \subseteq \Sigma^f_\alpha(X, Y)$, if $\alpha < \omega_0$,

(ii) $\Lambda_\alpha(X, Y) \subseteq \Sigma^f_{\alpha+1}(X, Y)$, if $\alpha \geq \omega_0$.

Proof. The theorem is obvious for $\alpha = 1$. Assume it is true for all $1 \leq \beta < \alpha$ and prove it for $\beta = \alpha$. Let $f \in \Lambda_\alpha(X, Y)$.

We consider the case $\alpha = \beta + 1$ and take a sequence $(f_n)_{n=1}^\infty$ of mappings $f_n \in \Lambda_\beta(X, Y)$ which is pointwise convergent to $f$ on $X$. By the assumption $f_n \in \Sigma^f_\beta(X, Y)$ in case $\alpha$ is finite, or $f_n \in \Sigma_\alpha(X, Y)$ in case $\alpha$ is infinite. Then, respectively, $f_n \in K_\beta(X, Y) \cap \Sigma^f(X, Y)$ or $f_n \in K_\alpha(X, Y) \cap \Sigma^f(X, Y)$ by Theorem 6. Applying Theorems 18 and 19 we obtain that $f \in K_\alpha(X, Y) \cap \Sigma^f(X, Y) = \Sigma^f_\alpha(X, Y)$ if $\alpha < \omega_0$, or $f \in K_{\alpha+1}(X, Y) \cap \Sigma^f(X, Y) = \Sigma^f_{\alpha+1}(X, Y)$, if $\alpha \geq \omega_0$.

Now we suppose that $\alpha = \sup \alpha_n$ is a limit ordinal and let $(f_n)_{n=1}^\infty$ be a sequence of mappings $f_n \in \Lambda_{\alpha_n}(X, Y)$ which converges pointwise to $f$ on $X$. By the assumption $f_n \in \Sigma^f_{\alpha_n+1}(X, Y) \subseteq \Sigma^f_\alpha(X, Y)$ for every $n$. Theorems 6, 18 and 19 imply that $f \in K^f_{\alpha+1}(X, Y) \cap \Sigma^f(X, Y) = \Sigma^f_{\alpha+1}(X, Y)$.

Combining Theorems 17 and 20 we get the following result.

**Theorem 21.** Let $X$ be a topological space, $Y$ be a metric space and $0 < \alpha < \omega_1$. Then

(i) $\Lambda_\alpha(X, Y) = \Sigma^f_\alpha(X, Y)$, if $\alpha < \omega_0$,

(ii) $\Lambda_\alpha(X, Y) = \Sigma^f_{\alpha+1}(X, Y)$, if $\alpha \geq \omega_0$.

Finally, we show that the condition on $X$ to be perfect in Theorem 13 is essential.

**Example 1.** There exists a normal space $X$ such that $H_2(X, \mathbb{R}) \neq \Phi_2(X, \mathbb{R})$.

Proof. Let $A$ be a $G_{\delta_\sigma}$-set which is not an $F_{\sigma_\delta}$-set in $\mathbb{R}$, and let $X = (\mathbb{R}, \tau)$ be the real line with a topology $\tau$ such that a basic neighborhood of $x \in \mathbb{R} \setminus A$ is the set $\{x\}$ and a basic neighborhood of $x \in A$ is an interval $(x - \varepsilon, x + \varepsilon)$, $\varepsilon > 0$. The normality of $X$ follows from [2, Example 5.1.22].

Since every open set in $\mathbb{R}$ is open in $X$, $A$ is $G_{\delta_\sigma}$ in $X$. Moreover, $A$ is closed in $X$, therefore, $A$ is an ambiguous set of the second class in $X$. Hence, the characteristic function $f = \chi_A : X \to \mathbb{R}$ of $A$ belongs to the class $H_2(X, \mathbb{R})$.

Notice that the normality of $X$ implies the equality $H_1(X, \mathbb{R}) = K_1(X, \mathbb{R})$ (see [2, Proposition 1.8]). Then $\Phi_2(X, \mathbb{R}) = \Lambda_2(X, \mathbb{R})$. 
We prove that \( f \not\in \Phi_2(X, \mathbb{R}) \). To obtain a contradiction, suppose that \( f \in \Lambda_2(X, \mathbb{R}) \). By Theorem 21 there exists a sequence of functions \( f_n \in \Sigma_1^f(X, \mathbb{R}) = K_1(X, \mathbb{R}) \) which converges pointwise to \( f \) on \( X \). Notice that every function \( f_n : X \to \mathbb{R} \) is of the first Baire class (see for instance [7]). Hence, \( f : X \to \mathbb{R} \) is the function of the second Baire class. Let \( (f_{n,m})_{n,m=1}^{\infty} \) be a sequence of continuous functions on \( X \) such that \( f(x) = \lim_{n \to \infty} \lim_{m \to \infty} f_{n,m}(x) \) for every \( x \in X \).

The definition of \( \tau \) implies that for all \( n, m \) the set \( A \) is contained in the set \( C(f_{n,m}) \) of all points of continuity of \( f_{n,m} \) on \( \mathbb{R} \). It is well-known that \( C(f_{n,m}) \) is a \( G_\delta \)-subset of \( \mathbb{R} \). We put \( B = \bigcap_{n,m=1}^{\infty} C(f_{n,m}) \). Then \( B \) is a \( G_\delta \)-set in \( \mathbb{R} \) which contains \( A \) and the restriction \( f|_B \) belongs to the second Baire class in the Euclidean topology. Since \( A = (f|_B)^{-1}((0, 1]) \), \( A \) is an \( F_{\sigma\delta} \)-subset of \( B \) and, consequently, is an \( F_{\sigma\delta} \)-subset of \( \mathbb{R} \), a contradiction.

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