Instability of Static Semi-Closed Worlds in Generalized Galileon Theories

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Abstract. We consider generalized Galileon theories within general relativity in four-dimensional space-time. We provide the argument showing that the generalized Galileons described by a wide class of Lagrangians do not admit stable, static, spherically symmetric semi-closed worlds. We also show that in a class of theories with $p_\perp = -\rho$ (where $p_\perp$ is transverse pressure and $\rho$ is energy density), semi-closed worlds, if exist, would be observed as objects of negative mass.

1. Introduction and summary

Models with Galileons are of interest, as they admit stable, null energy condition (NEC) violating solutions [1–6]. The property of NEC-violation makes Galileons natural candidates for fields that may support Lorentzian wormholes [7–12] and/or semi-closed worlds [13–16]. It has been shown, however, that asymptotically flat static and spherically symmetric wormholes are unstable in a class of generalized Galileon theories [17,18]. The main purpose of this paper is to extend this result to semi-closed worlds.

Figure 1. Semi-closed world.

Geometry of a static, spherically symmetric, asymptotically flat semi-closed world is schematically shown in Fig. 1. It is described by the following metric (signature
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\[ ds^2 = a^2(r)dt^2 - b^2(r)dr^2 - c^2(r)\gamma_{\alpha\beta}dx^\alpha dx^\beta \]

with asymptotics

\[
\begin{align*}
  r \to 0: & \quad a \to a_0, \ b \to b_0, \ c \to b_0r, \\
  r \to \infty: & \quad a \to a_\infty, \ b \to b_\infty, \ c \to b_\infty (r - r_*),
\end{align*}
\]

where \( \gamma_{\alpha\beta}dx^\alpha dx^\beta \) is the metric of a unit two-dimensional sphere, \( a_0, b_0, a_\infty, b_\infty \) and \( r_* \) are positive constants, see Fig. 2. A defining feature of a semi-closed world metric is the existence of a throat and hence non-vanishing \( r_* > 0 \).

![Figure 2. Behavior of \( c(r) \) for a semi-closed world.](image)

Let us summarize our findings. Spherical symmetry dictates that the non-vanishing components of the stress-energy tensor are \( T_0^0, T_r^r \) and \( T_\beta^\beta = T^\Omega_\delta^\beta \). In Galileon theories, as well as in many other scalar theories, the stress-energy tensor satisfies the relation \( T_0^0 = T^\Omega \), \( i.e. \) \( p_\perp = -\rho \), where \( p_\perp \) is transverse pressure and \( \rho \) is energy density. We show that in general relativity this property alone ensures that the mass of a semi-closed world, as seen by outside observer, is negative.

Even though negative mass objects may not be pathological in General Relativity [19–21], this result calls for more detailed analysis of concrete theories. Here we specify to a generalized Galileon theory with the Lagrangian

\[
\mathcal{L} = F(\pi, X) + K(\pi, X)\Box\pi, \tag{1}
\]

where \( \pi \) is a scalar field, \( F \) and \( K \) are arbitrary functions and the following notation is used:

\[
\begin{align*}
  X &= \nabla_\mu \pi \nabla^\mu \pi, \\
  \Box \pi &= \nabla_\mu \nabla^\mu \pi.
\end{align*}
\]

We show that there is either a ghost or a gradient instability of perturbations about any non-singular semi-closed world solution irrespectively of the forms of the Lagrangian functions \( F \) and \( K \). This is our main result: Galileons do not support static, spherically symmetric semi-closed worlds.
The paper is organized as follows. We obtain the result on the negative mass of a semi-closed world in theories with \( T_0^0 = T_\Omega \) in Sec. 2. The properties of the generalized Galileon theories with the Lagrangian (1) are discussed in Sec. 3. Sec. 4 gives the argument that static, spherically symmetric semi-closed worlds are unstable in the generalized Galileon theories.

2. Negative mass

In what follows we use the gauge
\[
b = \frac{1}{a},
\]
so that the semi-closed world metric is
\[
d s^2 = a^2 d t^2 - \frac{1}{a^2} d r^2 - c^2 \gamma_{\alpha \beta} d x^\alpha d x^\beta,
\]
where \( a(r) \) and \( c(r) \) obey the following boundary conditions:
\[
r \to 0: \ a \to a_0, \ c \to \frac{r}{a_0},
\]
\[
r \to \infty: \ a \to a_\infty, \ c \to \frac{r - r_*}{a_\infty},
\]
In this gauge the components of the Einstein tensor \( G_\nu^\mu \equiv R_\nu^\mu - \frac{1}{2} \delta_\nu^\mu R \) read
\[
G_0^0 = -2a^2 \left[ \frac{a' c'}{ac} + \frac{1}{2} \left( \left( \frac{c'}{c} \right)^2 - \frac{1}{a^2 c^2} \right) \right] - 2a^2 \frac{c''}{c},
\]
\[
G_r^r = -2a^2 \left[ \frac{a' c'}{ac} + \frac{1}{2} \left( \left( \frac{c'}{c} \right)^2 - \frac{1}{a^2 c^2} \right) \right],
\]
\[
G_\alpha^\beta = \delta_\beta^\alpha G_\Omega,
\]
\[
G_\Omega = -a^2 \left[ \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 + \left( \frac{c'}{c} \right)^2 + \frac{1}{a^2 c^2} \right].
\]
To simplify formulas below we set
\[
8\pi G = 1.
\]
Before specifying to the generalized Galileons, let us consider any theory whose stress-energy tensor, in the spherically symmetric and static case, has the property
\[
T_0^0 = T_\Omega,
\]
where \( T_\Omega \) determines the angular part, \( T_\alpha^\beta = T_\Omega \delta_\alpha^\beta \). In this situation the Einstein equations imply \( G_0^0 = G_\Omega^\Omega \), i.e.
\[
\frac{a''}{a} + \left( \frac{a'}{a} \right)^2 - \frac{c''}{c} - \frac{c'}{c} + \frac{1}{a^2 c^2} = 0.
\]
Upon the change of variables
\[ \gamma(r) = a^2(r), \quad u(r) = c^2(r), \]
the latter equation is written as
\[ \gamma'' u - u'' \gamma + 2 = 0. \]
This equation can be used to express \( \gamma(r) \) in terms of \( u(r) \):
\[ \gamma(r) = u(r) \left[ \int_r^\infty \frac{2\tilde{r} - C}{u^2(\tilde{r})} d\tilde{r} + D \right], \tag{4} \]
where \( C \) and \( D \) are constants. We now consider the asymptotic behavior of this solution and determine \( C \) and \( D \).

As \( r \) tends to 0, \( u(r) \) tends to \( r^2/a_0^2 \) and we find from (4) that
\[ \gamma(r) \equiv a^2(r) = a_0^2 \left[ 1 - \frac{C}{3} \frac{1}{r} + \mathcal{O}(r) \right]. \]
The requirement that \( \gamma(r) \) is regular at \( r = 0 \) gives
\[ C = 0. \]
As \( r \) tends to \( \infty \), \( u(r) \) tends to \( (r - r_*)^2/a_\infty^2 \) and eq. (4) gives
\[ \gamma(r) \equiv a^2(r) = a_\infty^2 \left[ 1 + \frac{2r_* - C}{3} \frac{1}{r - r_*} + D(r - r_*)^2 + o \left( \frac{1}{r - r_*} \right) \right]. \]
Since \( \gamma(r) \) should not grow as \( r \to \infty \), we have
\[ D = 0. \]
Thus, the relation between \( \gamma(r) \) and \( u(r) \) is
\[ \gamma(r) = u(r) \int_r^\infty \frac{2\tilde{r}}{u^2(\tilde{r})} d\tilde{r}. \tag{5} \]

To find the mass of a semi-closed world as seen by an outside observer, we note that the radius of distant sphere is
\[ R = \frac{r - r_*}{a_\infty}. \]
Therefore, upon the rescaling of the time coordinate
\[ T = a_\infty t, \]
eq (5) shows that the asymptotics of the metric as \( r \to \infty \) is
\[ ds^2 = \left( 1 - \frac{R_g}{R} \right) dT^2 - \frac{1}{1 - \frac{R_g}{R}} dR^2 - R^2 \gamma_{\alpha\beta} dx^\alpha dx^\beta, \]
where
\[ R_g = -\frac{2}{3} \frac{r_*}{a_\infty}. \tag{6} \]
Since \( r_* > 0 \), see Fig. 2, we conclude that in theories obeying (3), semi-closed worlds, if exist, would be observed as objects of negative mass.
3. Generalized Galileon theory

3.1. Stress-energy tensor

The stress-energy tensor for the theory (1) is

\[ T_{\mu\nu} = 2F_X \partial_\mu \pi \partial_\nu \pi + 2K_X \Box \pi \partial_\mu \pi \partial_\nu \pi - \partial_\mu K \partial_\nu \pi - \partial_\nu K \partial_\mu \pi - g_{\mu\nu} g^{\lambda\rho} \partial_\lambda K \partial_\rho \pi, \]

which in the static spherically symmetric case and in the gauge (2) becomes

\[ T_{00}^0 = -F - (a\pi')^2 K_\pi + 2a (a\pi')^2 (a\pi')' K_X, \]
\[ T_r^r = -F + (a\pi')^2 K_\pi + 2a (a\pi')^2 (a\pi')' K_X - 2 (a\pi')^2 F_X, \]
\[ T_\alpha^\beta = \delta_\alpha^\beta T_\Omega, \]
\[ T_\Omega = -F - (a\pi')^2 K_\pi + 2a (a\pi')^2 (a\pi')' K_X. \]

Note that it has the property (3), so the result of Sec. 2 applies.

3.2. Stability conditions

The perturbations about static, spherically symmetric background (\( \pi \rightarrow \pi + \chi \)) are described by the following effective quadratic Lagrangian [17] (in the gauge (2))

\[ \mathcal{L}^{(2)} = a^{-2} G^{00} \chi^2 - a^2 G^{rr} (\chi')^2 - c^{-2} G^{\Omega} \gamma^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi, \]

where the effective metric is

\[ G^{00} = F_X - K_\pi - a^2 K'_X \pi' - 2a K_X (a\pi')' - 4a^2 K_X \frac{c'}{c} \pi' - K_X^2 (a\pi')^2, \]
\[ G^{rr} = F_X - K_\pi - a^2 K'_X \pi' - 2a K_X (a\pi')' - 2a^2 K_X \frac{c'}{c} \pi' - 2a a' K_X \pi' - K_X^2 (a\pi')^2, \]
\[ G^{\Omega} = F_X - K_\pi - a^2 K'_X \pi' - 2a K_X (a\pi')' + 2a K_{XX} (a\pi')^2 (a\pi')' + 2a K_{XX} (a\pi')^3 \left( \frac{a'}{a} + 2 \frac{c'}{c} \right) + 3 K_X^2 (a\pi')^2, \]

and terms without derivatives of \( \chi \) are omitted. Hereafter \( \pi \) denotes the background field. The background is stable as long as

\[ G^{00} > 0, G^{\Omega} \geq 0, G^{rr} \geq 0. \] (7)

From now on we are interested in the first two of these conditions. Using the combination of the Einstein equations \( G_0^0 - G_r^r = T_0^0 - T_r^r \), that is

\[ -2a^2 \frac{c''}{c} = 2 (a\pi')^2 \left[ F_X - K_\pi + a K_X (a\pi')' - a^2 K_X \pi' \left( \frac{a'}{a} + 2 \frac{c'}{c} \right) \right], \]
we write
\[
\left(\pi'\right)^2 \mathcal{G}^{00} = -\left(\frac{c'}{c} + a^2 K_X (\pi')^3\right)' - \left(\frac{c'}{c} + a^2 K_X (\pi')^3\right)^2.
\]
\[
\left(\pi'\right)^2 \mathcal{G}^{\Omega} = \mathcal{G}^{00} (\pi')^2 + 2a^2 K_X (\pi')^3\left(\frac{c'}{c} - \frac{a'}{a}\right),
\]
It is now natural, following [18], to introduce the variable
\[
Q = \frac{c'}{c} + a^2 K_X (\pi')^3
\]
and rewrite eqs. (8), (9) as follows:
\[
\left(\pi'\right)^2 \mathcal{G}^{00} = -Q' - Q^2,
\]
\[
\left(\pi'\right)^2 \mathcal{G}^{\Omega} = -Q' - Q^2 + 2\left(Q - \frac{c'}{c}\right)\left(\frac{c'}{c} - \frac{a'}{a}\right).
\]
Since \(c'/c\) tends to \(1/r\) both as \(r \to 0\) and as \(r \to \infty\), it is convenient to introduce the variable
\[
q = Q - \frac{1}{r}.
\]
The expressions for \(\mathcal{G}^{00}\) and \(\mathcal{G}^{\Omega}\) in terms of \(q\) are
\[
\left(\pi'\right)^2 \mathcal{G}^{00} = -q' - \frac{2q}{r} - q^2,
\]
\[
\left(\pi'\right)^2 \mathcal{G}^{\Omega} = -q' - \frac{2q}{r} - q^2 + 2\left(q + \frac{1}{r} - \frac{c'}{c}\right)\left(\frac{c'}{c} - \frac{a'}{a}\right),
\]
4. Proof of instability

The main idea of the proof below is to show that \(q\) is negative at \(r = 0\) and positive at \(r = \infty\) and, using these properties, show that the stability conditions (7) are violated for any non-singular configuration.

4.1. The sign of \(q\) at \(r = 0\)

We are going to prove that \(q\) is negative at \(r = 0\). To this end, let us assume the opposite,
\[
q|_{r\to0} > 0.
\]
We now distinguish three types of the asymptotic behavior of \(q\) at \(r = 0\). First, let us consider the case when \(q\) grows, as \(r \to 0\), faster than \(1/r\):
\[
q \gg \frac{1}{r}.
\]
This leads to
\[
\left(\pi'\right)^2 \mathcal{G}^{00} \approx -q' - q^2 > 0.
\]
We now integrate the inequality in (12) from 0 to \( r \) and obtain
\[
\frac{1}{q} > r,
\]
which contradicts (11) under the assumption (10). This means that \( q \), if positive, cannot grow faster than \( 1/r \).

The second case to consider is \( q \sim 1/r \):
\[
q|_{r \to 0} = \frac{\alpha}{r} + \mathcal{O}\left(\frac{1}{r}\right),
\]
where \( \alpha > 0 \) because of (10). In this case we have
\[
(\pi')^2 G^{00} = -\alpha(1 + \alpha) \frac{1}{r^2} + \mathcal{O}\left(\frac{1}{r^2}\right) .
\]
The right hand side of (13) is negative, meaning that positive \( q \sim 1/r \) is also impossible.

Let us finally consider the case
\[
q|_{r \to 0} = o\left(r^{-1}\right), \tag{14}
\]
which leads to
\[
(\pi')^2 G^{00} \approx -q' - \frac{2q}{r} > 0 . \tag{15}
\]
Under the assumption of positive \( q \) this is the equivalent to
\[
- \frac{dq}{q} > 2 \frac{dr}{r} > \frac{dr}{r} . \tag{16}
\]
We may now choose \( r_1 \) and \( r_2 > r_1 \) so that \( q > 0 \) in the whole interval \([r_1, r_2]\) and integrate the inequality (16) from \( r_1 \) to \( r_2 \). We get
\[
q_{1}r_{1} > q_{2}r_{2} . \tag{17}
\]
The condition (15) under the assumption (10) requires \( q' \) to be negative, meaning that \( q \) is monotonous by decreasing, which leads to the contradiction between (17) and (14).

Thus, the arguments above prove that \( q < 0 \) near \( r = 0 \).

4.2. The sign of \( q \) at \( r = \infty \)

We needed only \( G^{00} \) to prove that \( q \) is negative near \( r = 0 \). To prove its positivity at \( r = \infty \) we also need \( G^{\Omega} \). Since \( c \to r - r_* \) as \( r \to \infty \), we have
\[
\frac{c'}{c} = \frac{1}{r} + \frac{r_*}{r^2} + o\left(\frac{1}{r^2}\right) . \tag{18}
\]
Now, we have \( a \to a_\infty \left[1 - M/(r - r_*)\right] \) as \( r \to \infty \) (with negative \( M \), see (6)), hence
\[
\frac{a'}{a} = \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{19}
\]
Using (18) and (19) we obtain
\[
(\pi')^2 G^{\Omega} = -q' - q^2 - \frac{2r_*}{r^3} + \mathcal{O}\left(\frac{q}{r^2}\right) + o\left(\frac{1}{r^3}\right)
\]
We now prove that $q_{r\to\infty} > 0$. Let us assume the opposite:

$$q_{r\to\infty} < 0. \quad (20)$$

Let us again consider the three types of asymptotic behavior of $q$.

The first case is that as $r \to \infty$

$$|q| \gg \frac{1}{r},$$

which leads to

$$(\pi')^2 \mathcal{G}^{00} \approx -q' - q^2 > 0.$$ Introducing $q'/q^2 < -1$ from $r_1$ to $r_2 > r_1$ we obtain

$$\frac{1}{q(r_2)} > \frac{1}{q(r_1)} + r_2 - r_1,$$

Since $q(r_1) < 0$ by assumption, and $r_2$ can be arbitrarily large, there is a singularity point at which $1/q(r_2) = 0$. This shows that $q < 0$ cannot tend to zero slower than $1/r$.

The second case to consider is

$$q_{r\to\infty} = \frac{\alpha}{r} + o\left(\frac{1}{r}\right),$$

where $\alpha < 0$ under the assumption (20). This leads to

$$(\pi')^2 \mathcal{G}^{\Omega} = \frac{\alpha(1 - \alpha)}{r^2} + o\left(\frac{1}{r^2}\right). \quad (21)$$

The right hand side of (21) is negative, so the background is unstable.

Finally, we consider the case

$$q_{r\to\infty} = o(r^{-1}). \quad (22)$$

Then the expression for $\mathcal{G}^{\Omega}$ is

$$(\pi')^2 \mathcal{G}^{\Omega} = -q' - q^2 - \frac{2r_s}{r^3} + o\left(\frac{1}{r^3}\right). \quad (23)$$

Under the assumptions (20) and (22) there is a region where $q' > 0$: $q$ tends to 0 from below, so that it grows at least somewhere. In this region the dominant part of $\mathcal{G}^{\Omega}$ is given by three negative terms in eq. (23), which contradicts the stability.

The arguments above show that $q$ has to be positive as $r$ tends to $\infty$. We have also shown that it has to be negative at $r = 0$. This means that there is a point $r_n$ where $q(r_n) = 0$, $q'(r_n) \geq 0$. The right hand side of eq. (10) is non-positive at $r_n$ which contradicts the stability conditions (7). This brings us to the conclusion that the stable, static, spherically symmetric semi-closed worlds do not exist in four-dimensional Galileon theories with the Lagrangians of the form (1).

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References

[1] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D 79, 064036 (2009) doi:10.1103/PhysRevD.79.064036 [arXiv:0811.2197 [hep-th]].

[2] C. Deffayet, G. Esposito-Farese and A. Vikman, Phys. Rev. D 79, 084003 (2009) doi:10.1103/PhysRevD.79.084003 [arXiv:0901.1314 [hep-th]].

[3] G. Goon, K. Hinterbichler and M. Trodden, Phys. Rev. Lett. 106, 231102 (2011) doi:10.1103/PhysRevLett.106.231102 [arXiv:1103.6029 [hep-th]].

[4] G. Goon, K. Hinterbichler and M. Trodden, JCAP 1107, 017 (2011) doi:10.1088/1475-7516/2011/07/017 [arXiv:1103.5745 [hep-th]].

[5] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Prog. Theor. Phys. 126, 511 (2011) doi:10.1143/PTP.126.511 [arXiv:1105.5723 [hep-th]].

[6] V. A. Rubakov, Phys. Usp. 57, 128 (2014) doi:10.3367/UFNe.0184.201402b.0137 [arXiv:1401.4024 [hep-th]].

[7] M. S. Morris and K. S. Thorne, Am. J. Phys. 56, 395 (1988). doi:10.1119/1.15620

[8] M. S. Morris, K. S. Thorne and U. Yurtsever, Phys. Rev. Lett. 61, 1446 (1988). doi:10.1103/PhysRevLett.61.1446

[9] M Visser“Lorentzian Wormholes: From Einstein to Hawking”, AIP Press New York, (1995)

[10] D. Hochberg and M. Visser, Phys. Rev. D 58, 044021 (1998) doi:10.1103/PhysRevD.58.044021 [gr-qc/9802046].

[11] I. D. Novikov, N. S. Kardashev and A. A. Shatskiy, Phys. Usp. 50, 965 (2007) [Usp. Fiz. Nauk 177, 1017 (2007)]. doi:10.1070/PU2007v050n09ABEH006381

[12] A. Shatskiy, I. D. Novikov and N. S. Kardashev, Phys. Usp. 51, 457 (2008) doi:10.1070/PU2008v051n05ABEH006581 [arXiv:0810.0468 [gr-qc]].

[13] V. P. Frolov, M. A. Markov and V. F. Mukhanov, Phys. Rev. D 41, 383 (1990). doi:10.1103/PhysRevD.41.383

[14] E. I. Guendelman, Int. J. Mod. Phys. D 19, 1357 (2010) doi:10.1142/S021827181001724X [arXiv:1003.3975 [gr-qc]].

[15] S. V. Chernov and V. I. Dokuchaev, Class. Quant. Grav. 25, 015004 (2008) doi:10.1088/0264-9381/25/1/015004 [arXiv:0709.0616 [gr-qc]].

[16] V. I. Dokuchaev and S. V. Chernov, JETP Lett. 85, 595 (2007) [Pisma Zh. Eksp. Teor. Fiz. 85, 727 (2007)] doi:10.1134/S0021364007120016 [arXiv:1208.5249 [gr-qc]].

[17] V. A. Rubakov, Theor. Math. Phys. 187, no. 2, 743 (2016) doi:10.1134/S004057791605010X [arXiv:1509.08808 [hep-th]].

[18] V. A. Rubakov, arXiv:1601.06566 [hep-th].

[19] H. Bondi, Rev. Mod. Phys. 29, 423 (1957). doi:10.1103/RevModPhys.29.423

[20] R. T. Hammond, Eur. J. Phys. 36, no. 2, 025005 (2015) doi:10.1088/0143-0807/36/2/025005 [arXiv:1308.2683 [gr-qc]].

[21] J. Belletête, S. Mbarek and M. B. Paranjape, Can. J. Phys. 93, no. 9, 966 (2015). doi:10.1139/cjp-2014-0540