Two –warehouse model with preservation technology investment and advertisement dependent demand over a finite time horizon

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Abstract. In present article an optimal policy is well established for advertisement schemes with preservation technology inputs. The items are deteriorating in nature and are stored at two-warehousing system. The demand is function of advertising frequency and selling price. In stock out situation partially backlogged shortages are inculcated to make the model more general. The advertising frequency, preservation cost are treated as main decision variables in development of the optimal cost function. Whole study is done under the effect of inflation over a finite time planning horizon. This study further can be modified by incorporating other parameters of inventory control and system.

1. Introduction
In today’s worldwide market, everybody keen to be first in any kind of competition like in business, studies, research etc. The organizations, trading any kind of inventory are focused to achieve good customer relations. They want to keep them self on the top, for that they prepare a perfect inventory system. A perfect inventory control system always keep all the parameters up to the mark like demand, quality of items, storage area etc. In literature researchers have focused on these parameters in different situations like some researcher have focused on storing conditions, some about quality of items etc. In [1] author was the first who initiated the concept of two-warehouse system for storing the items. Further in [2] investigator has focused on the concept of two storage facility.

A fixed planning horizon model is put in consideration in [3], in which they have used two-warehouse system to store the inventory. Deteriorating Items are put into consideration, which deteriorate at non-instantaneous deterioration rate with promotional effort and inflation. In [4] authors have established a two-storage model for the items having deteriorating fault, in which they have focused on partially backlogged shortages.

Demand rate is another pin point which is very essential in planning of the entity especially dealing with inventory. In recent time investigators have considered various type of demand rates like stock-level related requirement function, time varying requirement function etc. Advertising related demand rate play a crucial role in generation of a perfect inventory system. The researchers of [5], have studied a fixed time horizon model for items with deterioration and requirement function varying with advertisement frequency and rate. Authors have inculcated the shortages with partial backlogged parameters in an inflationary environment.

An optimal lot sizing model is explored in [6], in that items are losing their life with time and partially backlogged shortages are also considered. The authors of the work [7], have explored a joint optimal inventory study for items having fixed life time with dynamic pricing and advertisement
policies. In similar flow but with different situations, advertisement frequency related demand is focused also in [8, 9, 10].

In literature decaying items are very first considered in the study [11]. Further this study is extended in [12], in which they have considered weibull distribution deterioration rate. In today’s technological world deterioration rate is put under control up to certain mark. So, controllable deterioration rate is possible by implementing some preservation technologies. In the sea of literature authors of [13], have considered preservation technology investment in an inventory system dealing with deteriorating items.

The researchers have explored a production model in [14], for the inventory which losses their quality as time process with time related demand and preservation technology investment. A two storage space inventory study is developed in [15] by considering preservation technologies with partially backlogged shortages. In [16], authors have explored and optimal control analysis for the entity which deal with inventory that lose its quality with time in a preserving environment.

In present theory, I have taken an initiative to put a mark for the systems dealing with items which start losing their life with time instant. I in the current article the deterioration is handled by using preservation. The demand varies with advertising frequency and the selling price. In case of stock out situation the partially backlogged shortages are entertained. The optimal policies are well presented for advertising frequency, preservation technology cost etc. The next is devoted especially for basic concepts and notation which are building blocks for the whole model.

2. Assumptions and Notations
Following are the postulation and symbolizations, which are the base of the mathematical model formulation. The subscript w and R are used for the parameters related to owned warehouse system and rented warehouse respectively.

2.1. Assumptions
• The advertisement and selling price dependent demand rate is as below
  \[ D(A, s) = A^a (a_1 - a_2s) \] where \( a, a_1, a_2 > 0 \);
• The partially backlogged shortages are allowed with backlog frequency \( B(t) = e^{-bt} \) where \( 0 \leq b \leq 1 \) is the backlog parameter;
• The time horizon is considered to be finite;
• The deteriorating rate \( \theta_i, i = w, R \) is constant which is controllable by applying preservation technologies, where \( 0 < \theta \ll 1 \);
• The preservation technologies are applied to minimize the losses due to deterioration. The preservation function and controllable deterioration rate is as follows
  \[ m_i(\xi) = \theta_i (1 - e^{-\gamma \xi}) \] where \( \gamma > 0, i = w, R \).
• The holding cost \( C_h \) per unit per timer unit, is directly related to the unit purchase cost \( C_p \) as follows:
  \[ C_{hi} = h_{i} C_{p}, \] where \( i = w, R \) such that \( 0 < h_w < 1 \) and \( h_{R} > h_w \).
• Two-warehouse system is considered for the storage, in which Owned warehouse (OW) has restricted storing area and the rented warehouse (RW) have sufficient storing space.
• The scheduling horizon \( M \) is alienated into \( k \) equal portions of length \( T = M/k \). Therefore the reorder time are \( T_{i} = iT \) \((i = 0,1,2,\ldots,k)\). The no-shortage time interval \([iT, (i+1)T]\) is \( T_i \), which is equal to \( mT \) \((0 < m < 1)\). The shortage time period starts at \( t_i = (m+i-1)T \), \((I = 1,2,\ldots,m)\) and are taken up till time \( t_i = iT \), \((I = 1,2,3,\ldots,k)\) before they are backordered.

2.2. Assumptions
• \( C_o \) The order cost per order;
• \(C_p\): The unit purchase cost;
• \(s\): The unit selling price;
• \(C_{hi}\): The unit holding cost per unit per time unit for \(i = w, R\);
• \(A^d\): The frequency of advertisement where \(\alpha > 0\);
• \(C_s\): The shortage cost per backlogging unit per cycle;
• \(C_L\): The unit lost sale cost;
• \(\theta_w\): The constant rate of deterioration at OW, \(0 < \theta_w < 1\);
• \(\theta_R\): The constant rate of deterioration at RW, \(0 < \theta_w < 1\);
• \(\tau_i\): The resultant deterioration rate \((=\theta_i - m_i(\xi))\) for \(i = w, R\);
• \(\xi\): The preservation technology cost;
• \(Q\): The order level for 2nd, 3rd ……. \(k\)th cycle;
• \(r\): The inflation rate;
• \(T\): The total cycle length;
• \(t_R\): The time at which inventory level at RW ceases up to zero level;
• \(t_1\): The time at which inventory level at OW ceases up to zero level in \(i^{th}\) cycle \((i = 1, 2, 3…. \ldots k)\);
• \(T_i - t_i\): The time period when shortages occurs \((i = 1, 2, \ldots, k)\);
• \(M\): The planning horizon;
• \(k\): The number of replenishment during planning horizon, \(k = M/T\);
• \(I_R(t)\): The instantaneous inventory level at any time in RW during \(0 \leq t \leq t_R\);
• \(I_w(t)\): The instantaneous inventory level at any time in OW during \(0 \leq t \leq t_R\);
• \(I_n(t)\): The instantaneous inventory level at any time in OW during \(t_R \leq t \leq t_1\);
• \(I_m(t)\): The instantaneous inventory level at any time in OW during \(t_1 \leq t \leq T\);
• \(W_w\): The capacity of the OW;
• \(W_R\): The maximum inventory level in RW;
• \(I_m\): The maximum inventory level per cycle;
• \(B_m\): The maximum amount of shortage demand to be backlogged;
• \(T_i\): The total time the elapsed up to and including the \(j^{th}\)replenishment cycle \((i = 1, 2, 3……….. \ldots, k\)), where \(T_0 = 0, T_1 = T_2…… T_k = M\);
• \(t\): The inventory level reaches;
• \(TC_1(A, \xi)\): The total relevant cost per time unit for first replenishment cycle;
• \(TC(A, \xi)\): The total relevant cost over finite planning horizon \(M\);

Note: The superscript * show the optimal value of respective parameter. The total relevant cost includes following costs
PC is the purchase cost; HC is the holding cost; \(AC_A\) is the advertisement cost; SC: The shortage cost; LC: The lost sale cost; \(\xi\): The preservation technology cost;

3. Mathematical model formulation
The inventory functioning is presented in the figure 1. In the beginning the stock consumption is done by using stock available at warehouse which hired at some rent policies, as the holding cost is larger than the cost of owned warehouse. At time \(t = t_R\) the inventory level ceases to zero. At the same moment inventory at owned warehouse is used for the demand satisfaction. It remains continue till the time moment \(t = t_1\), as the total inventory finished at this instant. As the result of good customer
relation, customer agreed to wait for the demand satisfaction in the zero stock level time interval that is \([t_1, T]\). The following equations are showing the total functioning of the inventory in present system.
\[
\frac{dI_R(t)}{dt} + \tau_R I_R(t) = -A^\alpha (a_1 - a_2 s); 0 \leq t \leq t_R
\] (1)
\[
\frac{dI_{1w}(t)}{dt} + \tau_w I_{1w}(t) = 0; \quad 0 \leq t \leq t_R
\] (2)
\[
\frac{dI_{2w}(t)}{dt} + \tau_w I_{2w}(t) = -A^\alpha (a_1 - a_2 s); 0 \leq t \leq t_1
\] (3)
\[
\frac{dI_{3w}(t)}{dt} = -e^{-\tau t} A^\alpha (a_1 - a_2 s); \quad t_1 \leq t \leq T
\] (4)
By applying the conditions \(I_R(t = 0) = W_R, I_{1w}(t = t_R) = I_{2w}(t = t_R), I_{2w}(t = t_1) = 0, I_{2w}(t = t_1) = 0, I_{3w}(t = t_1) = 0\) and \(I_{3w}(t = T) = B_M\) the solutions of (1) and (2) are mentioned as below
\[
I_R(t) = e^{-\tau t} R A^\alpha (a_1 - a_2 s) (t - \tau_R \frac{t^2}{2}) + W_R e^{-\tau t} R
\] (5)
\[
W_R = A^\alpha (a_1 - a_2 s) \left( t_1 - \tau_R \frac{t_1^2}{2} > t \leq \tau_R \frac{t_1^2}{2} \right)
\] (6)
\[
I_{1w}(t) = W_{we^{-\tau w}}
\] (7)
\[
I_{2w}(t) = e^{-\tau w t} A^\alpha (a_1 - a_2 s) \left( t_1 - t - \tau_w \frac{t_1^2 - t^2}{2} \right)
\] (8)
\[
I_{3w}(t) = A^\alpha (a_1 - a_2 s) \left( t_1 - t - \frac{b(t_1^2 - t^2)}{2} \right)
\] (9)
\[
W_{w} = A^\alpha (a_1 - a_2 s) \left( t_1 - t - \tau_w \frac{t_1^2 - t^2}{2} \right)
\] (10)
\[
I_{M} = W_{w} + A^\alpha (a_1 - a_2 s) \left( t_1 - \tau_R \frac{t_1^2}{2} \right)
\] (11)
\[
B_{M} = -A^\alpha (a_1 - a_2 s) \left[ T_R - \tau_R \frac{t_1^2}{2} \right]
\] (12)
\[
Q = I_{M} + B_{M}
\] (13)
Now, we will calculate different cost parameters which are included in the total cost function.
- The order cost (OC) = \(C_{\Lambda}\)
- The purchase cost (PC) = \(C_{\Lambda}[I_{M} + B_{M} e^{-\tau T}]\)
- The advertisement cost (AC\(_d\)) = \(AC\(_d\)
- The holding cost (HC) = \(HC_{w} + HC_{R}\)
\[
HC = C_{Rh} \int_{0}^{t_R} I_R (t) e^{-\tau t} dt + C_{ww} \int_{0}^{t_R} I_{1w}(t) e^{-\tau t} dt + C_{wh} \int_{t_R}^{t_1} I_{2w}(t) e^{-\tau t} dt
\] (14)
- The shortage cost (SC) = \(-C_{S} \int_{t_1}^{T} I_{3w}(t) e^{-\tau t} dt\)
\[
SC = -C_{S} A^\alpha (a_1 + a_2 s) \left[ t_1(T - t_1) \left( 1 - \frac{b t_1}{2} \right) + \left( t_1^2 - \frac{t_1^3}{3} \right) \right] + \left( t_1^3 - \frac{t_1^4}{2} \right) \left( 1 - \frac{b t_1}{2} \right) + \left( t_1^3 - \frac{t_1^4}{2} \right) + \left( t_1^3 - \frac{t_1^4}{2} \right) + \left( t_1^3 - \frac{t_1^4}{2} \right)
\] (15)
- The lost sale cost (LC) = \(C_{L} A^\alpha (a_1 - a_2 s) \left[ t_1(T - t_1) e^{-\tau t} dt \right]
\]
\[
LC = C_{L} A^\alpha (a_1 - a_2 s) b \left( \frac{T^2 - t_1^2}{2} \right)
\] (16)
Hence the total relevant cost for first replenishment cycle is as below
\[
TC_1 = [OC + PC + HC + SC + LC + AC\(_d\)]
\] (17)
Now, the total cost function is as follows:

$$TC(k, m, \xi, A) = \sum_{i=0}^{k-1} TC_1 e^{i\tau T} + C_A e^{-r M}$$

$$TC(k, m, \xi, A) = TC_1 \left( \frac{1-e^{-r M}}{1-e^{-r M/k}} \right) + C_A e^{-r M}$$

Figure 1. Pictorial representation of inventory system over a finite time horizon

The conditions for the optimality are

$$\frac{\partial TC(m, k, \xi, A)}{\partial m} = 0, \frac{\partial TC(m, k, \xi, A)}{\partial k} = 0, \frac{\partial TC(m, k, \xi, A)}{\partial \xi} = 0, \frac{\partial TC(m, k, \xi, A)}{\partial A} = 0;$$

Provided DET (H1)>0, DET (H2)>0, DET (H3)>0, DET (H4)>0 where H1, H2, H3, H4 are the principal minor of the Hessian matrix. Here is the Hessian Matrix of the total cost function.

$$
\begin{bmatrix}
\frac{\partial^2 TC(m, k, \xi, A)}{\partial m^2} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial m \partial k} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial m \partial \xi} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial m \partial A} \\
\frac{\partial^2 TC(m, k, \xi, A)}{\partial k \partial m} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial k^2} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial k \partial \xi} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial k \partial A} \\
\frac{\partial^2 TC(m, k, \xi, A)}{\partial \xi \partial m} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial \xi \partial k} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial \xi^2} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial \xi \partial A} \\
\frac{\partial^2 TC(m, k, \xi, A)}{\partial A \partial m} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial A \partial k} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial A \partial \xi} & \frac{\partial^2 TC(m, k, \xi, A)}{\partial A^2}
\end{bmatrix}
$$
Figure 2. The convexity graph of the total cost function $TC^*$ w. r. t. $m^*$ and $t_1^*$.

Figure 3. The convexity graph of the total cost function $TC^*$ w. r. t. $k^*$ and $t_R^*$.

Figure 4. The convexity graph of the total cost function $TC^*$ w. r. t. $T^*$ and $t_1^*$. 
4. Numerical Illustration

These are the values of various parameters, which are used in numerical illustration:

\[ a_1 = 110 \text{ units}, a_2 = 2 \text{ units}, \theta_w = 0.6, \theta_R = 0.2, s = 16, C_p = 6, C_A = 155, h_R = 0.35, h_w = 0.2, C_s = 7, C_L = 11, b = 0.007, \alpha = 2, C_d = 25, \gamma = 3, r = 0.23, W_w = 50 \text{ units}, M = 20 \text{ years}. \]

The optimal values are as follows:

\[ k^* = 8, m^* = 0.4568, t_1^* = 1.142, T^* = 2.5 \text{ years}, A^* = 0.087, \xi^* = 14.275, TC^* = 5798.56. \]

The convexity of total cost function is shown in figure 2, figure 3 and figure 4.

5. Sensitivity Analysis

The validity test is done by changing values of few important parameters like planning horizon M, deterioration rate \( \theta_R \), inflation rate \( r \), selling price \( s \) etc. The variations in optimal values of different parameters are given in table 1 given as follows.

| Parameter | Value | Parameter Value | Parameter Value | Parameter Value | Parameter Value | Parameter Value | Parameter Value |
|-----------|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| M         | 16    | 5               | 0.4856          | 1.5539          | 3.20            | 0.064           | 12.21           | 5465.76         |
|           | 24    | 9               | 0.4166          | 1.1123          | 2.67            | 0.073           | 14.98           | 5866.65         |
|           | 28    | 11              | 0.3866          | 0.9858          | 2.55            | 0.077           | 15.36           | 5957.67         |
| \( \theta_R \) | 0.1    | 8               | 0.4657          | 1.1643          | 2.50            | 0.082           | 13.84           | 5637.24         |
|           | 0.3    | 8               | 0.4284          | 1.0710          | 2.50            | 0.082           | 15.65           | 5837.23         |
|           | 0.5    | 8               | 0.4037          | 1.0093          | 2.50            | 0.082           | 16.63           | 5937.34         |
| R         | 0.12   | 6               | 0.4738          | 1.5793          | 3.33            | 0.089           | 15.34           | 6078.42         |
|           | 0.25   | 7               | 0.4374          | 1.2497          | 2.86            | 0.078           | 14.65           | 5897.33         |
|           | 0.30   | 9               | 0.4136          | 0.9191          | 2.22            | 0.067           | 13.42           | 4389.44         |
| b         | 0.003  | 8               | 0.4678          | 1.1695          | 2.50            | 0.087           | 14.27           | 5863.23         |
|           | 0.008  | 8               | 0.4457          | 1.1143          | 2.50            | 0.087           | 14.27           | 5635.34         |
|           | 0.0012 | 8               | 0.4533          | 1.1333          | 2.50            | 0.087           | 14.27           | 5234.86         |
| C_p       | 5      | 9               | 0.4675          | 1.0389          | 2.22            | 0.067           | 14.27           | 5627.52         |
|           | 7      | 6               | 0.4375          | 1.4583          | 3.33            | 0.089           | 14.27           | 5967.25         |
|           | 8      | 5               | 0.4075          | 1.6300          | 4.00            | 0.09            | 14.27           | 6548.64         |
| W_w       | 20     | 7               | 0.3828          | 1.0937          | 2.86            | 0.087           | 14.97           | 7657.56         |
|           | 60     | 8               | 0.4983          | 1.2458          | 2.50            | 0.087           | 16.45           | 5255.78         |
|           | 80     | 8               | 0.5436          | 1.3590          | 2.50            | 0.087           | 17.31           | 4677.58         |

The analytical discussion of above sensitivity analysis, is as follows:

- The increment in value of M outcomes in slight increase in \( m^* \), \( t_1^* \) and \( T^* \). On the other hand, as the planning horizon increases, so it very obvious that the parameters \( A^* \), \( \xi^* \) and \( TC^* \) are increased.

- The increment in value of \( \theta_R \) output is in slight increase in \( t_1^* \). On the other hand, as the deterioration rate increases, so it very obvious that the parameters \( \xi^* \) and \( TC^* \) are increased.

- The increment in value of \( r \) output is in slight increase in \( m^* \), \( t_1^* \), \( T^* \), \( A^* \), \( \xi^* \) and \( TC^* \).

- The increment in value of \( b \) output is in slight increase in \( t_1^* \), \( T^* \) and \( TC^* \) are increased.
The increment in value of $C_p$ output is in slight increase in $m^*, t_1^*$ and. On the other hand, as the purchase cost increases, so it very obvious that the parameters $T^*$, $A^*$ and $TC^*$ are increased.

The increment in $W_w$, output is in slight decrease in $T^*$, $ξ^*$ and $TC^*$. In spite of this, as the storage capability of OW increases the values of parameters $m^*$, $k^*$, $t_1^*$ and $A^*$.

6. Conclusion
The policy for advertisement scheme and proper investment, in the preserving environment with two storage facility, is well established with advertisement dependent demand with partially backlogged shortages. The preservation technologies are considered for controllable deterioration rate, which reduces the losses generated by the deteriorating nature of the items. The optimized time period, advertisement frequency, cost associated with preservation technologies and total cost function etc., are well explored to present a perfect inventory control system. Numerical verification and the sensitivity test is also carried out to sensitize the model, which shows that slight variation changes the values of optimal values of desired parameters. This study is perfect platform for the organizations dealing with fashionable products, mobile phones etc. The model further can be modifies by focusing on other parameters of inventory control and management.

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