ITERATED INDEX AND THE MEAN EULER CHARACTERISTIC

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ABSTRACT. The aim of the paper is three-fold. We begin by proving a formula, both global and local versions, relating the number of periodic orbits of an iterated map and the Lefschetz numbers, or indices in the local case, of its iterations. This formula is then used to express the mean Euler characteristic (MEC) of a contact manifold in terms of local, purely topological, invariants of closed Reeb orbits, without any non-degeneracy assumption on the orbits. Finally, turning to applications of the local MEC formula to dynamics, we use it to reprove a theorem asserting the existence of at least two closed Reeb orbits on the standard contact $S^3$ (by Cristofaro-Gardiner and Hutchings in the most general form) and the existence of at least two closed geodesics for a Finsler metric on $S^2$ (Bangert and Long).

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1. INTRODUCTION

In this paper we prove an elementary formula relating the number of periodic orbits of an iterated map and the Lefschetz numbers of the iterations. We also establish a local version of this formula which connects the number of periodic orbits, suitably defined, of a germ at an isolated fixed point and the indices of its iterations. The latter result is then used to express the mean Euler characteristic

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of a contact manifold with finitely many simple closed Reeb orbits in terms of local, purely topological, invariants of closed Reeb orbits, when the orbits are not required to be non-degenerate. Finally, this expression is utilized to reprove a theorem asserting the existence of at least two Reeb orbits on the standard $S^3$ (see [CGH, GHHM, LL]) and the existence of at least two closed geodesics for a Finsler metric, not necessarily symmetric, on $S^2$ (see [BL, CGH]).

Let us now describe the results in some more detail.

Consider first a smooth map $F: M \to M$, where $M$ is a closed manifold. We show that the number $I_\kappa(F)$ of $\kappa$-periodic orbits of $F$, once the orbits of a certain type are discounted, can be expressed via the Lefschetz numbers $\sum F^d$ for $d|\kappa$ (see Theorem 2.2), and hence $I_\kappa(F)$ is homotopy invariant. In a similar vein, given a germ of a smooth map at a fixed point $x$, isolated for all iterations, one can associate to it a certain invariant $I_\kappa(F)$, an iterated index, which counts $\kappa$-periodic orbits of a small perturbation of $F$ near $x$, with again some orbits being discounted. The iterated index can also be expressed via the indices of the iterations $F^d$ for $d|\kappa$ (see Theorem 2.5), and hence $I_\kappa(F)$ is again a homotopy invariant of $F$ as long as $x$ remains uniformly isolated for $F^\kappa$. We further investigate the properties of the iterated index in Section 2, which is essentially independent of the rest of the paper. The results in this section, although rather elementary, are new to the best of our knowledge; see however [CMPY].

Next, we apply the iterated index to calculation of the mean Euler characteristic (MEC) of contact manifolds. The MEC of a contact manifold, an invariant introduced in [VK] (cf. [Ra]), is the mean alternating sum of the dimensions of contact homology. It was observed in [GK] that when a Reeb flow has finitely many simple periodic orbits and these orbits are totally non-degenerate, the MEC can be expressed via certain local invariants of the closed orbits, computable in terms of the linearized flow. (Here an orbit is called totally non-degenerate if all its iterations are non-degenerate.) This expression for the MEC generalizes a resonance relation for the mean indices of the closed Reeb orbits on the sphere, proved in [Vi1], and is further generalized to certain cases where there are infinitely many orbits in [Es], including the Morse-Bott setting. The results of this type have been used for calculations of the MEC (see, e.g., [Es]) and also in applications to dynamics; [GK, GG2, Ra]. It is this latter aspect of the MEC formula that we are interested in here.

Versions of the MEC formula where the Reeb orbits are allowed to degenerate are established in [HM, LLW] with applications to dynamics in mind. In these formulas, however, the contributions of closed orbits are expressed in terms of certain local homology groups associated with the orbits and are contact-geometrical in nature. Here, in Theorem 3.6, we extend the result from [GK] to the degenerate case with the orbit contributions computable via the linearized flow and a certain purely topological invariant of the Poincaré return map. The latter invariant is essentially the mean iterated index.

Two remarks concerning this result are due now.

First of all, it should be noted that the authors are not aware of any example of a Reeb flow with finitely many closed Reeb orbits such that not all of them are totally non-degenerate. (It is conceivable that such Reeb flows do not exist, although proving this is likely to be absolutely out reach at the moment.) Thus the degenerate case of the local MEC formula (Theorem 3.6) appears to be of
little interest for the MEC calculations for specific manifolds. However, it does have applications to dynamics. For instance, it allows one to rule out certain orbit patterns and, as a consequence, obtain lower bounds on the number of closed orbits (cf. [HM, LL, WHL]); see Section 4 and a discussion below.

Secondly, our local MEC formula is identical to the one established in [HM] and also, apparently, to the one proved for the sphere in [LLW]. The difference lies in the interpretation or the definition of the terms in the formula. Although our local MEC formula could be directly derived from [HM, Theorem 1.5] (see Remark 3.13), we give, for the sake of completeness, a rather short proof of the formula, still relying, however, on some of the results from [HM]; cf. [GHHM]. In Section 3, we also discuss in detail the definition of the MEC, examples, the local and filtered contact homology and other ingredients of the proof of Theorem 3.6, and state a variant of the asymptotic Morse inequalities for contact homology (Theorem 3.14) generalizing some results from [GK, HM].

Finally, in the last section, we turn to applications of Theorem 3.6. We give a new proof of the theorem that every Reeb flow on the standard contact $S^3$ has at least two closed Reeb orbits; see [CGH, GHHM, LL]. (In fact, a more general result is now known to be true. Namely, the assertion holds for any contact three-manifold. This fact, proved in [CGH] using the machinery of embedded contact homology, is out of reach of our methods.) The proofs in [GHHM, LL] both rely on an analogue of the “degenerate case of the Conley conjecture” for contact forms, asserting that the presence of one closed Reeb orbit of a particular type (a so-called symplectically degenerate minimum) implies the existence of infinitely many closed Reeb orbits. This result holds in all dimensions; see [GHHM]. Another non-trivial (and strictly three-dimensional) ingredient in the argument in [GHHM] comes from the theory of finite energy foliations (see [HWZ1, HWZ2]), while the argument in [LL] uses, also in a non-trivial way, the variant of the local MEC formula from [LLW] for degenerate Reeb flows on the standard contact sphere. In this paper, we bypass the results from the theory of finite energy foliations and give a very simple proof of Theorem 4.1 utilizing Theorem 3.6 and the “Conley conjecture” type result mentioned above. The advantage of this approach is that it minimizes the 3-dimensional counterparts of the proof and, we hope, is the first step towards higher-dimensional results. We also reprove, in slightly more general form, a theorem from [BL] that every Finsler metric on $S^2$ has at least two closed geodesics. (Clearly, this also follows from [CGH].)

A word is due on the degree of rigor in this paper, which varies considerably between its different parts. Section 2, dealing with the iterated index, is of course completely rigorous. The rest of the paper, just as [GHHM], heavily relies on the machinery of contact homology (see, e.g., [Bou2, EGH] and references therein), which is yet to be fully put on a rigorous basis (see [HWZ3, HWZ4]). Note however that, to get around the foundational difficulties, one can replace here, following, e.g., [CDvK, FSvK], the linearized contact homology by the equivariant symplectic homology, which carries essentially the same information (see [BO]), at the expense of proofs getting somewhat more involved; cf. Remarks 3.1 and 3.16, and [HM] vs. [McL] or [Es] vs. [FSvK].

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2. Iterated index

In this section, we establish a few elementary results concerning the count, with multiplicity and signs, of periodic orbits of smooth maps. Although we feel that these results must be known in some form, we are not aware of any reference; see however [CMPY, Do], and also [JM, Chap. 3] and [Sm, Sect. I.4], for related arguments. Throughout the section, all maps are assumed to be at least $C^1$-smooth unless explicitly stated otherwise.

2.1. Iterated index of a map. As a model situation, consider a $C^1$-smooth map $F: M \to M$, where $M$ is a closed manifold. We are interested in an algebraic count of periodic orbits of $F$. To state our main results, let us first review some standard definitions and facts.

Denote by $\text{Fix}(F)$ the set of the fixed points of $F$. Recall that $x$ is a $\kappa$-periodic point of $F$ if $F^\kappa(x) = x$, i.e., $x \in \text{Fix}(F^\kappa)$, and that $\tau$ is the minimal positive integer such that $F^\tau(x) = x$. The $\kappa$-periodic orbit containing $x$ is the collection $O = \{x, F(x), \ldots, F^{\kappa-1}(x)\}$ of not necessarily distinct points (naturally parametrized by $\mathbb{Z}/\kappa\mathbb{Z}$). The minimal period is then the length (i.e., the cardinality) of the orbit or, more precisely, of its image in $M$. Note that necessarily $\tau|\kappa$.

The index $I(F, x)$ of an isolated fixed point $x$ of $F$ is the degree of the map $S^{n-1} \to S^{n-1}$ given, in a local chart containing $x$, by

$$z \mapsto \frac{z - F(z)}{\|z - F(z)\|}$$

where $z$ belongs to a small sphere $S^{n-1}$ centered at $x$. It is not hard to see that all points in a $\kappa$-periodic orbit $O$ have the same index as fixed points of $F^\kappa$. Thus, setting $I(F^\kappa, O) := I(F^\kappa, x)$ for any $x \in O$, we have the index assigned to a $\kappa$-periodic orbit.

Furthermore, recall that a fixed point $x$ of $F$ is said to be non-degenerate if 1 is not an eigenvalue of the linearization $DF_x: T_xM \to T_xM$ and that $F$ is called non-degenerate if all its fixed points are non-degenerate. To proceed, let us first assume that all periodic points of $F$ are non-degenerate, i.e., all iterations $F^\kappa$ are non-degenerate. (As follows from (a part of) the Kupka–Smale theorem, this is a $C^\infty$-generic condition; see, e.g., [AR, Sm].) Then the index $I(F^\kappa, x)$ is equal to $(-1)^m$, where $m$ is the number of real eigenvalues of $DF_x^\kappa$ in the range $(1, \infty)$.

**Definition 2.1.** Let $x$ be a periodic point of $F$ with minimal period $\tau$. We say that $x$ is even (odd) if the number of real eigenvalues of $DF_x^\tau: T_xM \to T_xM$ in the interval $(-\infty, -1]$ is even (odd). A $\kappa$-periodic point $x$ with minimal period $\tau$ is said to be bad if it is an even iteration of an odd point, i.e., $x$ is odd and the ratio $\kappa/\tau$ is even. Otherwise, $x$ is said to be good. A $\kappa$-periodic orbit $O$ is bad (good) if one, or equivalently all, periodic points in $O$ are bad (good).

Alternatively, the difference between even and odd (or good and bad) periodic points can be seen as follows. Let $x$ be a periodic point with minimal period $\tau$. We can also view $x$ as a $\kappa$-periodic point for any positive integer $\kappa$ divisible by $\tau$. Then $I(F^\kappa, x) = I(F^\tau, x)$ when $x$ is even and $I(F^\kappa, x) = (-1)^{1+\kappa/\tau}I(F^\tau, x)$ when $x$ is odd. Finally, $x$, viewed as a $\kappa$-periodic point, is good or bad depending on whether $I(F^\kappa, x) = I(F^\tau, x)$ or not. Since all periodic points in a periodic orbit have the same index, this definition extends to periodic orbits.
The terminology we use here is borrowed from the theory of contact homology (see, e.g., [Bou2, EGH]). In dynamics, odd periodic orbits are sometimes also referred to, at least for flows, as M"obius orbits (see [CMPY]) or flip orbits. Furthermore, note that the above discussion relies heavily on the fact that no root of unity is an eigenvalue of $DF_k$ due to the non-degeneracy assumption.

As is well known, the Lefschetz number $I(F) := \sum_{x \in \text{Fix}(F)} I(F, x)$ is a homotopy invariant of $F$; see, e.g., [Fr, JM]. (In particular, $I(F)$ can be extended “by continuity” to all, not necessarily non-degenerate, maps $F$.) This, of course, applies to $F^K$ as well, and hence $I(F^K) = \sum_{x \in \text{Fix}(F^K)} I(F^K, x)$, the number of periodic points counted with signs, is also homotopy invariant. However, the number of $\kappa$-periodic orbits, taken again with signs, is not homotopy invariant, i.e., $\sum_{\mathcal{O}} I(F^K, \mathcal{O})$, where the summation extends to all (not necessarily simple) $\kappa$-periodic orbits, can vary under a deformation of $F$. (An example is, for instance, the second iteration of the period doubling bifurcation map in dimension one, starting with an attracting fixed point with an eigenvalue in $(-1, 0)$ and ending with an odd repelling fixed point and an (even) attracting orbit of period two, with the sum for $\kappa = 2$ changing from 1 before the bifurcation to 0 afterwards; cf. [YA].) However, this sum is very close to being homotopy invariant and it becomes such once the summation is restricted to good orbits only. Namely, assuming as above that all periodic points of $F$ are non-degenerate, set

$$I_\kappa(F) := \sum_{\text{good } \mathcal{O}} I(F^K, \mathcal{O}),$$

where the sum is now taken over all good $\kappa$-periodic orbits $\mathcal{O}$ of $F$, not necessarily with minimal period $\kappa$. We call $I_\kappa(F)$ the \textit{iterated index} of $F$. In the example of a period doubling bifurcation mentioned above, we have $I_2(F) = 1$ before and after the bifurcation. We emphasize that $I_\kappa(F)$ depends in general not only on $F^K$, but separately on $\kappa$ and $F$.

Let $\varphi$ be the Euler function, i.e., $\varphi(\kappa)$ is the number of positive integers which are smaller than $\kappa$ and relatively prime with $\kappa$. By definition, $\varphi(1) = 1$.

\textbf{Theorem 2.2.} Assume, as above, that $F^K$ is non-degenerate. Then

$$I_\kappa(F) = \frac{1}{\kappa} \sum_{d | \kappa} \varphi(\kappa/d) I(F^d). \quad (2.1)$$

Since the right hand side of (2.1) is obviously homotopy invariant, so is $I_\kappa(F)$. Furthermore, it extends by continuity to all $F$, not necessarily meeting the non-degeneracy requirement. Namely, let $\tilde{F}$ be a small perturbation of $F$ such that all $\kappa$-periodic points of $\tilde{F}$ are non-degenerate. Then $I_\kappa(F) := I_\kappa(\tilde{F})$ is well defined, i.e., independent of the perturbation $\tilde{F}$.

\textit{Example 2.3.} Assume that $F$ is homotopic to the identity. Then, so is $F^d$ for all $d$, and, by Euler’s formula (see (2.2) below), we have $I(F^K) = I(F) = I_\kappa(F)$ for all $\kappa \geq 1$.

\textit{Proof of Theorem 2.2.} A $\kappa$-periodic orbit $\mathcal{O}$ with minimal period $\tau = |\mathcal{O}|$ contributes to $I_\kappa(F)$ only when $\tau | \kappa$, and to the individual terms on the right hand side of (2.1) when $\tau | d$. Set $\kappa = \tau \kappa'$ and $d = \tau d'$. 

When $\mathcal{O}$ is good, its contribution to $I_\kappa(F)$ is equal to $I(F^\kappa, \mathcal{O}) = I(F^\tau, \mathcal{O})$. On the other hand, its contribution to the right hand side of (2.1) is

$$\frac{1}{\kappa} \sum_{\tau \mid d|\kappa} \varphi(\kappa/d) I(F^d, \mathcal{O}) \tau = \frac{1}{\kappa'} \sum_{d'|\kappa} \varphi'(d') I(F^\tau, \mathcal{O}) = I(F^\tau, \mathcal{O}),$$

where we use Euler’s formula

$$\sum_{l \mid r} \varphi(l) = r; \tag{2.2}$$

see, e.g., [HW, Theorem 63].

Assume next that $\mathcal{O}$ is bad. Then $\kappa' = \kappa/\tau$ is even and odd, and $I(F^d, \mathcal{O}) = (-1)^{1+d/\tau} I(F^\tau, \mathcal{O})$. Thus the contribution of $\mathcal{O}$ to $I_\kappa(F)$ is zero and its contribution to the right hand side is

$$\frac{1}{\kappa} \sum_{\tau \mid d|\kappa} \varphi(\kappa/d) I(F^d, \mathcal{O}) \tau = - \frac{1}{\kappa'} \sum_{d'|\kappa} \varphi'(d') (-1)^{d'} I(F^\tau, \mathcal{O}) = 0,$$

where we now use the fact that

$$\sum_{l \mid r} (-1)^l \varphi(r/l) = 0 \tag{2.3}$$

for any positive even integer $r$.

Finally, (2.3) can be proved, for instance, as follows. Set $r = 2^mc$, where $c$ is odd and $m \geq 1$ since $r$ is even. Then, regrouping the terms on the left hand side of (2.3), we have

$$\sum_{l \mid r} (-1)^l \varphi(r/l) = \sum_{a \mid c} P(a),$$

where

$$P(a) = (-1)^{2^ma} \varphi(c/a) + (-1)^{2^{m-1}a} \varphi(2c/a) + \ldots + (-1)^a \varphi(2^mc/a),$$

and it suffices to show that $P(a) = 0$. Set $b = c/a$ and recall that $\varphi(2^jb) = 2^{j-1} \varphi(b)$ since $\varphi$ is multiplicative and $b$ is odd; [HW]. When $m = 1$, we clearly have $P(a) = \varphi(b) - \varphi(b) = 0$. Likewise, as is easy to see,

$$P(a) = \varphi(b) + \varphi(b) + \ldots + 2^{m-2} \varphi(b) - 2^{m-1} \varphi(b) = 0$$

when $m \geq 2$. This completes the proof of the theorem. \hfill \square

**Remark 2.4.** The fact that the iterated index is homotopy invariant enables one to extend the definition of $I_\kappa(F)$ to continuous maps $F: M \to M$, when $M$ is still a smooth manifold, by setting $I_\kappa(F) = I_\kappa(\tilde{F})$ where $\tilde{F}$ is a smooth approximation of $F$. Clearly, Theorem 2.2 still holds in this case.

**2.2. Iterated index of a germ.** Consider now a germ of a $C^1$-smooth map $F$ at a fixed point $x$, which is assumed to be isolated for all iterations $F^\kappa$ but not necessarily non-degenerate. Thus the index $I(F^\kappa, x)$ is defined and homotopy invariant as long as $x$ remains *uniformly isolated*: $I(F^\kappa, x) = \text{const}$ when $F^\kappa$ varies smoothly or continuously with parameter $s$ and there exists a neighborhood $U$ of $x$ such that $x$ is the only fixed point of $F^\kappa_s$ in $U$ for all $s$.

For a given $\kappa$, consider a sufficiently small perturbation $\tilde{F}$ such that all $\kappa$-periodic points of $\tilde{F}$ are non-degenerate and set

$$I_\kappa(F, x) := \sum_{\text{good } \mathcal{O}} I(\tilde{F}^\kappa, \mathcal{O}).$$
where the sum is again taken over all good $\kappa$-periodic orbits $O$ of $\tilde{F}$. For instance, when $F$ and all its iterations are non-degenerate, we have $I_\kappa(F, x) = I(F, x)$ when $x$ is even or when $x$ is odd and $\kappa$ is odd, and $I_\kappa(F, x) = 0$ otherwise. In what follows, when the point $x$ is clear from the context, we will use the notation $I(F)$ and $I_\kappa(F)$.

**Theorem 2.5.** We have

$$I_\kappa(F) = \frac{1}{\kappa} \sum_{d|\kappa} \varphi(\kappa/d) I(F^d). \quad (2.4)$$

We omit the proof of this theorem; for it is word-for-word identical to the proof of Theorem 2.2. An immediate consequence is

**Corollary 2.6.** The iterated index $I_\kappa(F, x)$ is well defined and homotopy invariant as long as $x$ is uniformly isolated fixed point of $F^\kappa$.

**Remark 2.7.** As in the case of maps of smooth manifolds (see Remark 2.4), the definition of the iterated index $I_\kappa(F, x)$ extends to continuous maps $F$ and Theorem 2.5 remains valid. In what follows, however, the requirement that $F$ is at least $C^1$ becomes essential; cf. [SS].

Assuming that $x$ is isolated for all iterations $F^\kappa$, consider the sequence of indices $i_\kappa = I(F^\kappa)$. This sequence is bounded (see [SS]) and in fact periodic; see [CMPY] and also, e.g., [JM] and references therein. (It is essential here that $F$ is at least $C^1$-smooth.) Moreover, there exists a finite collection of positive integers $N$ with the following properties:

(i) $1 \in N$;
(ii) for any two elements $q$ and $q'$ in $N$, the least common multiple $\text{lcm}(q, q')$ is also an element of $N$;
(iii) for any $\kappa$, we have $i_\kappa = i_{q(\kappa)}$, where $q(\kappa)$ is the largest element in $N$ dividing $\kappa$.

Thus the index sequence $i_\kappa$ is $q_{\text{max}}$-periodic, where $q_{\text{max}} = \text{max} N$, and takes values in the set \{ $i_q$ $|$ $q \in N$ \}. The collection $N = N(F)$ is generated in the obvious sense by the degrees of the roots of unity occurring among the eigenvalues of the linearization $DF_x$ and, in addition, $2 \in N$ when $x$ is odd. (For instance, if all eigenvalues of $DF_x$ are equal to 1, we have $N = \{ 1 \}$ and the sequence $i_\kappa$ is constant; when none of the eigenvalues is a root of unity, i.e., $x$ is non-degenerate for all iterations, we have either $N = \{ 1 \}$ or $N = \{ 1, 2 \}$ depending on whether $x$ is even or odd.)

Condition (iii) relating the sequence and the set $N$ satisfying (i) and (ii) is particularly important, and we say that a sequence $a_\kappa$ is subordinated to $N$ when (iii) holds: for any $\kappa$, we have $a_\kappa = a_{q(\kappa)}$, where $q(\kappa)$ is the largest element in $N$ dividing $\kappa$. (Note that (iii) is an easy consequence of the fact, implicitly contained in the proof of the Shub–Sullivan theorem [SS], that $I(F^\kappa) = I(F)$ wherever $\kappa$ is relatively prime with all $q \in N$; cf. [CMPY, GG3].)

**Theorem 2.8.** The sequence $I_\kappa(F)$ is subordinated to $N$.

In particular, it follows that $I_\kappa(F)$ is also periodic with period $q_{\text{max}}$.

**Proof.** The result is a formal consequence of Theorem 2.5. Namely, let $a_\kappa$ be any sequence subordinated to a finite set $N$ satisfying (i) and (ii), and let $b_\kappa$ be obtained...
from $\alpha_\kappa$ via (2.4):

$$b_\kappa = \frac{1}{\kappa} \sum_{d|\kappa} \varphi(\kappa/d) a_d.$$ 

Then we claim that $b_\kappa$ is also subordinated to $\mathcal{N}$.

At this point the fact that the original sequences are integer-valued becomes inessential, and it is more convenient to assume that the sequences in question are real. We can view the transform $\{a_\kappa\} \mapsto \{b_\kappa\}$ as a map $\Phi$ from the vector space $\mathbb{R}^\mathcal{N}$ of all real sequences subordinated to $\mathcal{N}$ to the vector space of all sequences. Our goal is to show that $\Phi$ actually sends $\mathbb{R}^\mathcal{N}$ to itself.

Consider the basis $\delta(q)$, $q \in \mathcal{N}$, of $\mathbb{R}^\mathcal{N}$, where $\delta(q)_\kappa = 1$ when $q|\kappa$ and $\delta(q)_\kappa = 0$ otherwise. Then, as we will prove shortly, $\Phi$ is diagonal in this basis with entries $1/q$, i.e., $\Phi(\delta(q)) = \delta(q)/q$. In particular, $\Phi \colon \mathbb{R}^\mathcal{N} \to \mathbb{R}^\mathcal{N}$ and the theorem follows.

By definition,

$$\Phi(\delta(q))_\kappa = \frac{1}{\kappa} \sum_{d|\kappa} \varphi(\kappa/d) \delta(q)_d.$$ 

The only non-zero terms on the right hand side are those with $q/d$. Thus, when $\kappa$ is not divisible by $q$, the right hand side is zero and hence $\Phi(\delta(q))_\kappa = 0$. When $q|\kappa$, set $\kappa = q\kappa'$ and $d = q\kappa''$ as in the proof of Theorem 2.2. Then

$$\Phi(\delta(q))_\kappa = \frac{1}{q\kappa'} \sum_{\kappa''|\kappa'} \varphi(\kappa'/\kappa'') = \frac{1}{q},$$

where we again used (2.2). This completes the proof of the theorem. \hfill \square

**Remark 2.9 (Integrality).** In addition to being subordinated to $\mathcal{N}$, the sequence $\iota_\kappa \equiv I(F^\kappa)$ is known to satisfy some further conditions. Namely, [CMPY, Theorem 2.2] asserts that, in the notation from the proof of Theorem 2.8, the sequence $\iota_\kappa$ is an integral linear combination of the sequences $q\delta(q)$, where $q \in \mathcal{N}$. (Furthermore, the coefficients must meet certain “sign-reversing” constraints if $x$ is odd.) Moreover, it follows that the mean index (the $\varphi$-index in the terminology of [CMPY]) is an integer:

$$\bar{\iota} = \lim_{N \to \infty} \frac{1}{N} \sum_{\kappa=1}^{N} \iota_\kappa = \frac{1}{q_{\max}} \sum_{\kappa=1}^{q_{\max}} \iota_\kappa \in \mathbb{Z};$$

see [CMPY, Corollary 2.3].

Arguing as in the proof of Theorem 2.8, one can derive [CMPY, Theorem 2.2] from the fact that the right hand side of (2.4) is an integer by the definition of $I_\kappa(F)$, and conversely [CMPY, Theorem 2.2] implies that the right hand side of (2.4) is an integer since $\Phi(q\delta(q)) = \delta(q)$.

**Example 2.10.** Assume that $2n = 2$ and $F$ is elliptic with eigenvalues $e^{\pm 2\pi i \alpha}$, where $\alpha$ is rational: $\alpha = p/q$ with $p$ and $q$ relatively prime and $q \geq 2$. Then it follows from [CMPY, Theorem 2.2] or our Theorem 2.8 that $I(F^\kappa) = 1 - rq$ when $q|\kappa$ and $I(F^\kappa) = 1$ otherwise. Thus $I_\kappa(F) = 1 - r$ when $q|\kappa$ and $I_\kappa(F) = 1$ otherwise. In addition, when $F$ is area preserving, $I(F^\kappa) \leq 1$ (see [Ni, Si]), and hence $r \geq 0$ in this case. There are no other restrictions on the index sequence when $2n = 2$: all values of $r$ and $q$ do occur. For instance, in the area preserving case, one can take as $F$ the composition of the rotation $e^{2\pi i/q}$ with the flow of the “monkey saddle” Hamiltonian $\Re(z^q)$. One immediate consequence of this result, otherwise entirely non-obvious, is that $I(F^\kappa) \neq 0$ for all $\kappa$, when $I(F) \neq 0$. (Remarkably,
under some additional assumptions, these facts remain true for homeomorphisms; see [LeC, LeCY] and references therein.)

3. Mean Euler characteristic

3.1. Notation and conventions. Let \((M^{2n-1}, \xi)\) be a closed contact manifold strongly fillable by an exact symplectically aspherical manifold. In other words, we require \((M, \xi)\) to admit a contact form \(\alpha\) such that there exists an exact symplectic manifold \((W, \omega = d\alpha_W)\) with \(M = \partial W\) (with orientations) and \(\alpha_W|_M = \alpha\) and \(c_1(TW) = 0\). Then the linearized contact homology \(HC_\ast(M, \xi)\) is defined and independent of \(\alpha\); see, e.g., [Bon2, EGH]. Although \(HC_\ast(M, \xi)\) depends in general on the filling \((W, \omega)\), we suppress this dependence in the notation. Moreover, when \(M\) is clear from the context, we will simply write \(HC_\ast\).

When \(\alpha\) is non-degenerate, \(HC_\ast(\xi)\) is the homology of a complex \(CC_\ast(\alpha)\) generated over \(\mathbb{Q}\) by the good closed, not necessarily simple, Reeb orbits \(x\) of \(\alpha\), where an orbit is said to be good/bad depending on whether the corresponding fixed point of the Poincaré return map is good/bad. The complex is graded by the Conley–Zehnder index up to a shift of degree by \(n - 3\), i.e., \([x] = \mu_{CZ}(x) + n - 3\). The exact nature of the differential on \(CC_\ast(\alpha)\) is inessential for our considerations.

The complex \(CC_\ast(\alpha)\) further breaks down into a direct sum of sub-complexes \(CC_\ast(\alpha; \gamma)\) generated by the closed Reeb orbits in the free homotopy class \(\gamma\) of loops in \(W\). When \(\gamma \neq 0\), fixing grading (or, equivalently, a way to evaluate the Conley–Zehnder index of \(x\)) requires fixing an extra structure. A convenient choice of such an extra structure in our setting is a non-vanishing section \(s\) of the linearized contact homology \(HC_\ast(M, \xi)\). We refer the reader to [Es] for a very detailed discussion of the mean index in this context and for further references, and also to [Lo, SZ]. Here we only mention that the mean index \(\Delta(x)\) measures the total rotation of certain eigenvalues on the unit circle of the linearized Poincaré return map of the Reeb flow at \(x\). In general, the grading does depend on the choice of \(s\).

Finally, let us also fix a collection \(\Gamma\) of free homotopy classes \(\gamma\) generated by the closed Reeb orbits in the free homotopy class \(\gamma\) of loops in \(W\). When \(\gamma \neq 0\), fixing grading (or, equivalently, a way to evaluate the Conley–Zehnder index of \(x\)) requires fixing an extra structure. A convenient choice of such an extra structure in our setting is a non-vanishing section \(s\) of the linearized contact homology \(HC_\ast(M, \xi)\). We again suppress the dependence of the homology on \(s\) and \(\Gamma\) (and on \(W\)) in the notation and \(HC_\ast\).

Furthermore, assume that

\(\text{(CH) there are two integers } l_+ \text{ and } l_- \text{ such that the space } HC_l(\xi) \text{ is finite-dimensional for } l \geq l_+ \text{ and } l \leq l_-\).

In all examples considered here the contact homology is finite dimensional in all degrees and this condition is automatically met. Set

\[
\chi^\pm(\xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{l = l_\pm}^N (-1)^l \dim HC_{\pm l}(\xi), \quad (3.1)
\]
provided that the limits exist. We call $\chi^{\pm}(\xi)$ the positive/negative mean Euler characteristic (MEC) of $\xi$. (Invariants of this type for contact manifolds are originally introduced and studied in [VK, Section 11.1.3]; see also [Es, GK, Ra].) Note that $\chi^{\pm}(\xi)$ depends on $\Gamma$ and also, when $\Gamma \neq \{0\}$, on $s$. (However, when, say, $M$ is simply connected, it is not hard to see that $\chi^{\pm}(\xi)$ is independent of the filling under some natural additional conditions on $\xi$; cf. [CDvK, KvK, OV].)

Assume now that the dimensions of the contact homology spaces remain bounded as $l \to \pm\infty$:

$$\dim_{\mathbb{Q}} \text{HC}_l(\xi) \leq \text{const} \quad \text{when} \quad |l| \geq l_{\pm}. \quad (3.2)$$

Then, although the limit in (3.1) need not exist, one can still define $\chi^{\pm}(\xi)$ as, e.g., following [CDvK],

$$\chi^{\pm}(\xi) = \frac{1}{2} \left[ \limsup_{N \to \infty} \frac{1}{N} \sum_{l \leq l_{\pm}} (-1)^l \dim \text{HC}_{\pm l}(\xi) + \liminf_{N \to \infty} \frac{1}{N} \sum_{l \leq l_{\pm}} (-1)^l \dim \text{HC}_{\pm l}(\xi) \right].$$

Finally, let us also point out that the machinery of contact homology used in this section is yet to be fully put on a rigorous basis and the foundations of the theory is still a work in progress; see [HWZ3, HWZ4].

Remark 3.1. One can also use the cylindrical contact homology, when it exists, to define the MEC of a contact manifold $(M, \xi)$. The construction is similar to the one for the linearized contact homology with obvious modifications. For instance, in the cylindrical case, $s$ is a non-vanishing section of the line bundle $(\wedge_C \xi)^{\otimes 2}$ and the collection $\Gamma$ is formed by free homotopy classes of loops in $M$. Of course, the two definitions give the same result when the cylindrical and contact homology groups are equal or when the dimension of the contact complex is bounded as a function of the degree and the MEC can be evaluated using the complex rather than the homology. This is the case, for example, in the setting of Theorem 3.6.

Alternatively, the MEC can be defined using the positive equivariant symplectic homology (see [CDvK, FSvK]), resulting in an invariant (somewhat hypothetically) equal to the one obtained using the linearized contact homology; [BO]. Note that this approach bypasses the foundational difficulties related to contact homology but usually results in somewhat more involved proofs and calculations; cf. Remark 3.16. Finally, variants of the MEC exist and have been used for “classical” homology theories; see, e.g., [LLW, Ra, WHL].

3.2. Examples. In this section we briefly review, omitting the proofs, some examples where the MEC is not difficult to determine.

Example 3.2 (The Standard Sphere). For the standard contact structure $\xi_0$ on $S^{2n-1}$, the contact homology is one-dimensional in every even degree starting with $2n - 2$; see, e.g., [Bou2]. Thus, in this case, $\chi^+ = 1/2$ and $\chi^- = 0$. Alternatively, one can use here the Morse–Bott approach as in Example 3.4; see [Es].

Example 3.3 (The Ustilovsky Spheres). In [Us], Ustilovsky considers a family of contact structures $\xi_p$ on $S^{2n-1}$ for odd $n$ and positive $p \equiv \pm 1 \mod 8$. For a fixed $n$, the contact structures $\xi_p$ fall within a finite number of homotopy classes, including the class of the standard structure $\xi_0$. The contact homology $\text{HC}_*(\xi_p)$ is computed in [Us], and it is not hard to see that in this case

$$\chi^+(\xi_p) = \frac{1}{2} \left( \frac{p(n-1) + 1}{p(n-2) + 2} \right). \quad (3.3)$$
and \( \chi^-(\xi_p) = 0 \); see [KvK, VK] and also [Es, GK] for a different approach. The right-hand side of (3.3) is a strictly increasing function of \( p > 0 \). Hence, \( \chi^+ \) distinguishes the structures \( \xi_p \) with \( p > 0 \). Note also that \( \chi^+(\xi_p) > \chi^+(\xi_0) = 1/2 \) when \( p > 1 \) and \( \chi^+(\xi_1) = 1/2 \). In particular, \( \chi^+ \) distinguishes \( \xi_p \) with \( p > 1 \) from the standard structure \( \xi_0 \).

**Example 3.4 (Pre-quantization Circle Bundles).** Let \( \pi: M^{2n-1} \to B \) be a prequantization circle bundle over a closed strictly monotone symplectic manifold \((B, \omega)\). In other words, \( M \) is an \( S^1 \)-bundle over \( B \) with \( c_1 = [\omega] \), i.e., we have \( \pi^* \omega = d\alpha \), where \( \alpha \) is a connection form on \( M \) and we use a suitable identification of the Lie algebra of \( S^1 \) and \( \mathbb{R} \); see, e.g., [GGK, Appendix A]. As is well known, \( \alpha \) is a contact form. Assume first for the sake of simplicity that the fiber \( x \) of \( \pi \) is contractible.

Then, for \( \Gamma = \{0\} \), we have \( \chi^-(\xi) = 0 \) and

\[
\chi^+(\xi) = \frac{\chi(B)}{2} \langle c_1(TB), u \rangle,
\]

where \( u \in \pi_2(B) \) is the image of a disk bounded by \( x \) in \( M \) and \( \chi(B) \) is the ordinary Euler characteristic of the base \( B \). (Here we are using the cylindrical contact homology of \((M, \xi)\) rather than the linearized contact homology because the natural filling of \( M \) by the disk bundle is not exact or even symplectically aspherical.) This is an easy consequence, for instance, of the Morse–Bott version of (3.5) and of the fact that, essentially by definition, the denominator is \( \Delta(x) \); see [Es, Example 8.2]. Furthermore, since \( x \) is contractible, \( \langle \omega, u \rangle = 1 \) and \( B \) is monotone, \( \langle c_1(TB), u \rangle \) is the minimal Chern number \( N \) of \( B \); cf. [Bou1, p. 100].

To summarize, we have

\[
\chi^+(\xi) = \frac{\chi(B)}{2N}.
\]

(3.4)

If \( x \) is not contractible, the above calculation still holds for \( \Gamma = \{0\} \). More generally, when the order \( r \) of the class \([x]\) in \( \pi_1(M) \) is finite, for the collection \( \Gamma \) generated by \([x]\) and any \( s \), we have \( \chi^+(\xi) = r\chi(B)/(2N) \). When \( B \) is negative monotone, the roles of \( \chi^+ \) and \( \chi^- \) are interchanged. (See [CDvK] for far reaching generalizations of this calculation.)

**Example 3.5 (The Unit Cotangent Bundle of \( S^n \)).** Let \((M, \xi)\) be the unit cotangent bundle \( ST^* S^n \) with the standard contact structure \( \xi \). A Riemannian metric on \( S^n \) gives rise to a contact form \( \alpha \) on \( M \) and, for the standard round metric, \( \alpha \) is a prequantization contact form as in Example 3.4, where \( B = \text{Gr}^+(2, n+1) \) is the real “oriented” Grassmannian. (The closed geodesics are the oriented great circles on \( S^n \), i.e., the intersections of \( S^n \) with the oriented 2-planes in \( \mathbb{R}^{n+1} \).) It is not hard to see that \( \chi(B) = 2[(n+1)/2] \). Furthermore, since the minimal Chern number \( N \) of a (simply connected) hypersurface of degree \( d \) in \( \mathbb{C}P^{m+1} \) is \( m + 2 - d \) and since \( B \) is a complex quadric in \( \mathbb{C}P^n \), we have \( N = n - 1 \); cf. [LM, p. 88] and [MS, Example 4.27 and Exercise 6.20]. Thus \( \chi^-(\xi) = 0 \) and \( \chi^+(\xi) = 1/2 + 1/(n-1) \) if \( n \geq 3 \) is odd and \( \chi^+(\xi) = 1/2 + 1/2(n-1) \) when \( n \geq 2 \) is even. When \( n = 2 \) we have \( \chi^+(\xi) = 1/2 \) for \( \Gamma = \{0\} \) and \( \chi^+(\xi) = 1 \) for \( \Gamma = \pi_1(M) = \mathbb{Z}_2 \). Alternatively, \( \chi^x(ST^* S^n) \) can be calculated directly via contact homology; see [VK, KvK]. Note that here one can use either the cylindrical contact homology or the linearized contact homology with the filling \( W \) of \( M \) by the unit ball bundle in \( T^* S^n \). (The only modification that is needed in the latter case is that when \( n = 2 \), the fiber \( x \) is contractible in \( W \) and \( \chi^+(\xi) = 1 \) for \( \Gamma = \{0\} \).)
We refer the reader to [CDvK, Es, VK, KvK] for further examples.

3.3. Local formula for the MEC. Let now $x$ be a (simple) closed orbit of the Reeb flow of $\alpha$, which we assume to be isolated for all iterations, i.e., all iterated orbits $x^\kappa$ are isolated. Note that these orbits are not required to be non-degenerate. The Poincaré return map $F$ of $x$ is a germ of a smooth map with an isolated fixed point which we also denote by $x$. Clearly, the Poincaré return map of the iterated orbit $x^\kappa$ is just $F^\kappa$.

Since the fixed point $x$ of $F$ is isolated for all iterations, the iterated indices $I_\kappa(F, x)$ are defined for all $\kappa$ and the sequence $I_\kappa(F, x)$ is periodic with period $q_{\text{max}}$; see Section 2.2.

Set the mean iterated index of $x$ to be

$$\sigma(x) = \frac{1}{q_{\text{max}}} \sum_{\kappa=1}^{q_{\text{max}}} I_\kappa(F, x) = \lim_{N \to \infty} \frac{1}{N} \sum_{\kappa=1}^{N} I_\kappa(F, x).$$

This is a diffeomorphism (and even homeomorphism) invariant of the flow near the orbit $x$. When $x$ and all its iterations are non-degenerate, we have $\sigma(x) = I(F, x)$ when $x$ is even and $\sigma(x) = I(F, x)/2$ when $x$ is odd. (A simple closed Reeb orbit is said to be even/odd depending on whether the fixed point $x$ is even/odd for $F$.)

Recall that $(M^{2n-1}, \xi)$ is a closed contact manifold strongly fillable by an exact symplectically aspherical manifold and that a filling $(W, \omega)$, the homotopy class of the section $s$ and the collection $\Gamma$ are fixed, and hence, in particular, we have the graded space $\text{HC}_*(\xi)$ and the mean indices of the orbits unambiguously defined.

**Theorem 3.6.** Assume that the Reeb flow of $\alpha$ on $(M^{2n-1}, \xi)$ has only finitely many simple periodic orbits $x_1, \ldots, x_r$, not necessarily non-degenerate, in the collection $\Gamma$. Then the conditions (CH) and (3.2) are satisfied with $l_+ = 2n - 3$ and $l_- = 3$. Furthermore, the limit in (3.1) exists for both the positive and negative MEC of $(M, \xi)$, and

$$\chi^\pm(\xi) = \sum \frac{\sigma(x_i)}{\Delta(x_i)}.$$

where $\sum^\pm$ stands for the sum taken over the orbits $x_i$ with positive/negative mean index $\Delta(x_i)$.

This result generalizes the non-degenerate case of (3.5) proved in [GK] and inspired by [Vil] where such a formula was established for $(S^{2n-1}, \xi_0)$. A variant of (3.5) for convex hypersurfaces in $\mathbb{R}^n$ is proved in [WHL]. The general form of (3.5) given here is literally identical to a MEC formula obtained in [HM] (see [HM, Theorem 1.5]), and probably to the one established in [LLW] for $(S^{2n-1}, \xi_0)$, as can be seen by comparing the proofs. The main difference lies in the definitions of the numerators: in [HM], $\sigma(x_i)$ is defined as the MEC of the local contact homology of $x_i$, which is a contact invariant, while here $\sigma(x_i)$ is defined purely topologically as a diffeomorphism and even homeomorphism invariant of the Poincaré return map.

(Likewise, in [LLW], the numerators are defined via certain local homology associated with $x_i$.) Finally note that the Morse–Bott version of (3.5) and its variant for the so-called asymptotically finite contact manifolds are proved in [Es]. (The Morse–Bott version for the geodesic flows is originally established in [Ra].)

**Remark 3.7.** The condition of Theorem 3.6 that $\omega$ is exact on $W$ can be slightly relaxed. Namely, when $\Gamma = \{0\}$, it suffices to assume that $\omega|_{\pi_2(W)} = 0$. In general,
when $\Gamma \neq \{0\}$, it is enough to require $\omega$ to be atoroidal; cf. [GHHM]. However, the assumption that $c_1(TW) = 0$ or at least that $c_1(TW)|_{\pi_2(W)} = 0$ appears to be essential.

3.4. Preliminaries and the proof. Our goal in this section is to prove Theorem 3.6. To this end, we need first to recall several definitions and results concerning filtered and local contact homology following mainly [HM] and [GHHM]. Throughout the section, we continue to work in the setting of Section 3.1. In particular, the contact manifold $(M, \xi)$ is fixed as are the background structures used in the definition of the contact homology (the filling $W$, the section $s$ and the collection $\Gamma$). A contact form, $\alpha$, is always assumed to support $\xi$, i.e., $\ker \alpha = \xi$.

The proof of Theorem 3.6 has non-trivial overlaps with the proof of its counterpart in [HM] – after all, the difference between the two results lies mainly in the interpretation of (3.5) – although our argument is rather more concise. Moreover, our proof is not entirely self contained and depends on [HM] at two essential points. One is the construction of the local contact homology and the other is one of its properties. This is (LC3) stated in Section 3.4.2; see also [GHHM, Theorem 3 and Corollary 1]. We further elaborate on the relation between the two theorems and show how, once some preliminary work is done, our Theorem 3.13 can be derived from [HM, Theorem 1.5] in Remark 3.13.

3.4.1. Filtered contact homology. Let us first assume that the contact form $\alpha$ is non-degenerate. The complex $CC^\ast(\alpha)$ is filtered by the action

$A_\alpha(x) = \int_x \alpha,$

i.e., its subspace $CC^a(\alpha)$, where $a \in \mathbb{R}$, generated by the orbits $x$ with $A_\alpha(x) \leq a$ is a subcomplex. We refer to the homology $HC^a_\ast(\alpha)$ of this complex as the filtered contact homology; see, e.g., [GHHM, HM]. Note that, in contrast with $HC_\ast(\xi)$, these spaces depend on $\alpha$. Whenever $a \leq b$, we have a natural map

$HC^a_\ast(\alpha) \to HC^b_\ast(\alpha)$

induced by the inclusions of the complexes and, since homology commutes with direct limits,

$\lim_{a \to \infty} HC^a_\ast(\alpha) = HC_\ast(\xi).$ (3.6)

This definition extends to degenerate forms by continuity. Below we will discuss this definition in detail; for the construction is essential for what follows and it involves certain non-obvious (to us) nuances. Let us start, however, by recalling some standard facts and definitions.

The action spectrum $S(\alpha)$ of $\alpha$ is the collection of action values $A_\alpha(x)$, where $x$ ranges through the set of closed Reeb orbits of $\alpha$ (in the class $\Gamma$). This is a closed zero-measure subset of $\mathbb{R}$.

For any two contact forms $\alpha$ and $\alpha'$, giving rise to the same contact structure $\xi$, write $\alpha' > \alpha$ when $\alpha'/\alpha > 1$, i.e., $\alpha' = f\alpha$ with $f > 1$. This is clearly a partial order, and, when both forms are non-degenerate, we have a homomorphism $HC^a_\ast(\alpha') \to HC^a_\ast(\alpha)$ induced by the natural cobordism in the symplectization of $\xi$ between $\alpha$ and $\alpha'$.

Furthermore, we will need the following invariance property of filtered contact homology, stated in a slightly different form in [GHHM, Proposition 5]: Let $\alpha_s,$
s ∈ [0, 1], be a family of contact forms such that α₀ and α₁ are non-degenerate and \( a \not\in S(\alpha_s) \) for all s. Then the contact homology spaces \( \text{HC}^a_*(\alpha_0) \) and \( \text{HC}^a_*(\alpha_1) \) are isomorphic and the isomorphism is independent of the family \( \alpha_s \) as long as \( a \not\in S(\alpha_s) \). Furthermore, assume that the family \( \alpha_s \) is decreasing, i.e., \( \alpha_{s'} > \alpha_s \) when \( s' < s \). Then the natural map \( \text{HC}^a_*(\alpha_0) \to \text{HC}^a_*(\alpha_1) \) is an isomorphism. This proposition can be proved in exactly the same way as its Floer homological counterparts; see, e.g., [BPS, Gi1, Vi2].

Let now \( \alpha \) be a possibly degenerate contact form and \( a \not\in S(\alpha) \). Set
\[
\text{HC}^a_*(\alpha) := \text{HC}^a_*(\tilde{\alpha}),
\]
where \( \tilde{\alpha} \) is a non-degenerate \( C^\infty \)-small perturbation of \( \alpha \), i.e., \( \tilde{\alpha} \) is non-degenerate and \( C^\infty \)-close to \( \alpha \). By the invariance property of the filtered contact homology, for any two non-degenerate perturbations sufficiently close to \( \alpha \) the homology groups on the right hand side are canonically isomorphic, and hence the left hand side is well defined. Alternatively, we could have set
\[
\text{HC}^a_*(\alpha) := \lim_{\tilde{\alpha} > \alpha} \text{HC}^a_*(\tilde{\alpha}),
\]
where the limit is taken over all non-degenerate forms \( \tilde{\alpha} > \alpha \). These two definitions are obviously equivalent. In contrast with \( \text{HC}_*(\xi) \), the graded vector spaces \( \text{HC}^a_*(\alpha) \) are automatically finite dimensional: \( \sum_i \dim \text{HC}^a_i(\alpha) < \infty \).

With this definition, we have a well-defined map \( \text{HC}^a_*(\alpha') \to \text{HC}^a_*(\alpha) \) for any two, not necessarily non-degenerate, forms \( \alpha' > \alpha \), and hence the non-degeneracy requirement on \( \alpha_0 \) and \( \alpha_1 \) in the invariance property can be dropped. Now, however, once \( \alpha \) is not assumed to be non-degenerate, \( (3.6) \) requires a proof. Implicitly, this result is already contained in [HM].

**Lemma 3.8.** For any, not necessarily non-degenerate form \( \alpha \), \( (3.6) \) holds as \( a \to \infty \) through \( \mathbb{R} \setminus S(\alpha) \).

**Proof.** Fix a sequence \( a_i \to \infty \) with \( a_i \not\in S(\alpha) \) and consider a decreasing sequence of non-degenerate forms \( \alpha_j \) \( C^\infty \)-converging to \( \alpha \) from above, i.e., we have
\[
\alpha_1 > \alpha_2 > \cdots > \alpha.
\]
These sequences give rise to the maps
\[
\ldots \to \text{HC}^{a_i}_*(\alpha_j) \to \text{HC}^{a_i}_*(\alpha_{j+1}) \to \ldots \quad (3.7)
\]
In addition, of course, we also have the homomorphisms
\[
\ldots \to \text{HC}^{a_i}_*(\alpha_j) \to \text{HC}^{a_{i+1}}_*(\alpha_j) \to \ldots \quad (3.8)
\]
commuting with the maps \( (3.7) \). After passing to the limit as \( i \to \infty \), the maps \( (3.7) \) induce the identity map on \( \text{HC}_*(\xi) \).

Set
\[
L = \lim_{a_i \to \infty} \text{HC}^{a_i}_*(\alpha_i)
\]
with respect to the “diagonal” maps
\[
\delta_i: \text{HC}^{a_i}_*(\alpha_i) \to \text{HC}^{a_{i+1}}_*(\alpha_{i+1}).
\]
To prove the lemma, it suffices to show that \( L \cong \text{HC}_*(\xi) \).

Let \( u \in \text{HC}_*(\xi) \). By \( (3.6) \), there exists \( i(u) \) such that \( u \) is the image of \( u_{i(u)} \in \text{HC}^{a_{i(u)}}_*(\alpha_1) \). Applying the maps \( (3.7) \) and \( (3.8) \) to \( u_{i(u)} \), we obtain a double sequence \( u_{ij} \in \text{HC}^{a_i}_*(\alpha_j) \) where \( i \geq i(u) \). Set \( v_i = u_{ii} \), \( i \geq i(u) \). Clearly, \( \delta_i(v_i) = \ldots \).
$v_{i+1}$, and thus the sequence $v_i$ gives rise to an element $v \in L$. As a result, we have constructed a homomorphism

$$\Phi : HC_*(\xi) \to L, \quad \Phi(u) = v.$$  

Conversely, consider a sequence $v_i \in HC^n_*(\alpha_i)$ with $i \geq i(v)$ such that $\delta_i(v_i) = v_{i+1}$. The image $u$ of $v_i$ in $HC_*(\xi)$ is independent of choice of $v_i$, and we have a map

$$\Psi : L \to HC_*(\xi), \quad \Psi(v) = u.$$  

Essentially by the definition, $\Psi \Phi = id$. \hfill $\Box$

Finally let $HC^{(a,b)}_*(\alpha)$, where $a < b$, be the homology of the quotient complex $CC^\xi_*(\alpha)/CC^\xi_*(\alpha)$, provided that $\alpha$ is non-degenerate. When $a < b$ are outside $S(\alpha)$ this definition again extends to all $\alpha$ by continuity. In any event, we obviously have the exact sequence

$$\ldots \to HC^n_*(\alpha) \to HC^b_*(\alpha) \to HC^{(a,b)}_*(\alpha) \to \ldots.$$  

3.4.2. Local contact homology. Let $x$ be an isolated, but not necessarily simple or non-degenerate, closed Reeb orbit of $\alpha$. Consider a small non-degenerate perturbation $\tilde{\alpha}$ of $\alpha$. Under this perturbation $x$ splits into, in general, several non-degenerate orbits $\tilde{x}_1, \ldots, \tilde{x}_r$ with nearly the same period (i.e., action) and mean index as $x$. The vector space $CC_*(\tilde{\alpha}, x)$ generated over $\mathbb{Q}$ by the good orbits in this collection can be equipped with a differential. The resulting homology, denoted by $HC_*(x)$, is independent of the perturbation $\tilde{\alpha}$; see [HM]. We call $HC_*(x)$ the local contact homology of $x$. These complexes and the homology spaces are graded just as the ordinary contact homology, i.e., $|\tilde{x}_i| = \mu_{c2}(\tilde{x}_i) + n - 3$.

Example 3.9. Assume that $x$ is non-degenerate. Then $HC_*(x)$ is equal to $\mathbb{Q}$ and concentrated in degree $|x|$ when $x$ is good and $HC_*(x) = 0$ when $x$ is bad.

Example 3.10. Assume that $x$ is simple. Then $HC_*(x)$ is isomorphic up to a shift of degree to the local Floer homology $HF_*(F)$ of the Poincaré return map $F$ of $x$. See [GHHM, HM] for the proof of this fact, which can also be established by repeating word-for-word the proof of [EKP, Proposition 4.30], and, e.g., [Gi2, GG3, McL] for a detailed discussion of local Floer homology.

Example 3.11. When $x$ is not simple, the relation between the local contact homology and the Floer homology is more involved. Namely, $HC_{n+n-3}(y^\kappa) = HF^\mathbb{Z}_*(F^\kappa)$, where $F$ is the Poincaré return map of a simple orbit $y$ and the right hand side stands for the $\mathbb{Z}_\kappa$-equivariant local Floer homology; [GHHM, Theorem 3]. The proof of this theorem depends on the machinery of multivalued perturbations and is only outlined in [GHHM]. Although we find this relation illuminating, the present paper does not rely on this result.

The proof of Theorem 3.6 makes use of several properties of local contact homology, which we now recall following [GHHM]:

(LC1) $HC_*(x)$ is finite dimensional and supported in the range of degrees $[\Delta(x) - 2, \Delta(x) + 2n - 4]$, i.e., $HC_*(x)$ vanishes when $\star$ is outside this range.

(LC2) Assume that $x = y^\kappa$, where $y$ is a simple closed Reeb orbit, and let $F$ be the Poincaré return map of $y$. Then, in the notation of Section 2.2,

$$\sum (-1)^i \dim HC_i(x) = I_\kappa(F, y).$$
(LC3) For all iterations $\kappa$ of a simple orbit $x$, the (total) dimension of the graded vector space $HC_s(x^\kappa)$ is bounded by a constant independent of $\kappa$, provided that $x^\kappa$ remains isolated.

(LC4) Assume that $c$ is the only point of $S(\alpha)$ in the interval $[a, b]$ and that all closed Reeb orbits with action $c$ are isolated. (Hence, there are only finitely many such orbits.) Then

$$HC_s^{(a, b)}(\alpha) = \bigoplus_{A_s(x) = c} HC_s(x).$$

Here only (LC3) is not straightforward to prove. The assertion (LC1) follows from the facts that

$$|\mu_{cZ}(x) - \Delta(x)| \leq n - 1$$

for any closed Reeb orbit on a $(2n - 1)$-dimensional contact manifold and that $\Delta(x)$ depends continuously on $x$; see, e.g., [Lo, SZ]. The property (LC2) is an immediate consequence of the definition, and (LC4) follows from the fact that a holomorphic curve in the symplectization with zero $\omega$-energy must be the cylinder over a Reeb orbit; see, e.g., [BEHWZ, Lemma 5.4]. Finally, (LC3) is a far reaching generalization a theorem of Gromoll and Meyer, [GM]. This result is established in [HM, Section 6] as a consequence of a theorem from [GG3], asserting a similar upper bound for local Floer homology. (Note that (LC3) also follows from [GHHM, Theorem 3 and Corollary 1] stated in Example 3.11 above. An analogue of (LC3) for (non-equivariant) symplectic homology is proved in [McL].)

3.4.3. Proof of Theorem 3.6. In the proof, we will focus on the case of $\chi^+$; for its negative counterpart, $\chi^-$, can be handled in a similar fashion. Thus let $x_1, \ldots, x_r$ be the orbits of $\alpha$ with positive mean index: $\Delta(x_i) > 0$. Set $l_+ = 2n - 3$. By (LC1), this is the lowest degree for which the orbits with $\Delta = 0$ cannot contribute to the homology.

Our first goal is to prove (3.2) and (CH). To this end, observe that we have the following version of the Morse inequalities:

$$\dim HC_l(\alpha) \leq \sum \dim HC_t(x), \quad (3.9)$$

where, when $l \geq l_+$, the sum is taken over all orbits $x = x^\kappa_t$ with $\Delta(x) > 0$. This is a consequence of (LC4) and the long exact sequence for filtered contact homology. Since the mean index is homogeneous, i.e., $\Delta(x) = \kappa \Delta(x_i)$, we see that when $\kappa$ is large the orbits $x^\kappa_t$ do not contribute to $HC_l(\alpha)$ or, to be more precise, to the right hand side of (3.9), due to (LC1) again. (It suffices to take $\kappa > (l + 2)/\Delta(x_i)$ here.) In particular, the right hand side of (3.9) is finite when $l \geq l_+$. Furthermore, by (LC3), the contribution of $x^\kappa_t$ to $HC_l(\alpha)$ is bounded from above and the bound is independent of $l$. This proves (3.2): $\dim HC_l(\alpha) \leq \text{const}$, where the constant is independent of $l \geq l_+$.

Let us now prove (3.5) and, particular, the fact that $\chi^+(\xi)$ is defined. Set

$$\chi(x) = \sum_l (-1)^l \dim HC_l(x).$$

Thus, in the notation of (LC2), $\chi(x) = I_\alpha(F)$ where $x = y^\kappa$ and $F$ is the Poincaré return map of $y$. Next note that for any $\alpha \notin A(\alpha)$, we have

$$\sum_l (-1)^l \dim HC_l^\alpha(\alpha) = \sum_{A_s(x) < c} \chi(x), \quad (3.10)$$
where the sum is now over all closed Reeb orbits with action less than \( a \), but not as in (3.9) over only the iterations of the orbits \( x_1 \). This is again an immediate consequence of (LC2) and (LC4) and the long exact sequence for filtered contact homology.

When the summation is restricted to the degrees \( l_+ \leq l \leq N \), a variant of (3.10) still holds up to a bounded error:

\[
\left| \sum_{l=1}^{N} (-1)^l \dim \text{HC}_l^\alpha(\alpha) - \sum \chi(x_\kappa) \right| \leq \text{const}, \tag{3.11}
\]

where \( \text{const} \) on the right is independent of \( a \) and \( N \) and the second sum is taken now over the orbits \( x_\kappa \) with \( \mathcal{A}_\kappa(x_\kappa) < a \) and \( \Delta(x_\kappa) \leq N \). Just as (3.9) and (3.10), this readily follows from (LC2) and (LC4) and the long exact sequence for filtered contact homology.

Let us divide (3.11) by \( N \) and let \( a \to \infty \) and then \( N \to \infty \). By Lemma 3.8, the first sum will then converge to \( \chi^+(\xi) \), and hence

\[
\chi^+(\xi) = \sum_1 \lim_{N \to \infty} \frac{1}{N} \sum_{\Delta(x_\kappa) \leq N} \chi(x_\kappa),
\]

provided that the limit on the right hand side exists. Since \( \Delta(x_\kappa) = \kappa \Delta(x_1) \), we also have

\[
\sum_{\Delta(x_\kappa) \leq N} \chi(x_\kappa) = \frac{1}{\Delta(x_1)} \sum_{\kappa=1}^{N} \chi(x_\kappa) + O(1),
\]

as \( N \to \infty \). As a result, by the definition of \( \sigma(x_1) \) and (LC2),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{\Delta(x_\kappa) \leq N} \chi(x_\kappa) = \frac{\sigma(x_1)}{\Delta(x_1)},
\]

which concludes the proof of the theorem. \( \square \)

**Remark 3.12.** To guarantee the existence of the positive/negative MEC and prove (3.2) for positive/negative range, it suffices to assume only that the collection of simple Reeb orbits with positive/negative mean index is finite.

**Remark 3.13.** By (LC2) and Theorem 2.8, the function \( \kappa \to \chi(x_\kappa) \) is subordinated to the collection \( \mathcal{N}(F) \) associated with the Poincaré return map \( F \) of \( x \) as in Section 2.2. Furthermore, recall that the local MEC of \( x \) is defined in [HM] as

\[
\hat{\chi}(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{\kappa=1}^{N} \chi(x_\kappa).
\]

By (LC2), we have \( \hat{\chi}(x) = \sigma(x) \). (In particular, the limit in the definition of \( \hat{\chi}(x) \) exists.) With these observations in mind, our Theorem 3.6 can be easily obtained as a consequence of [HM, Theorem 1.5].

### 3.5. Asymptotic Morse inequalities.

The argument from the previous section lends itself readily to a proof of the asymptotic Morse inequalities. Namely, in the setting of Section 3.1, assume that \( \xi \) satisfies (HC) and set

\[
\beta^\pm(\xi) = \limsup_{N \to \infty} \frac{1}{N} \sum_{l=1}^{N} \dim \text{HC}_{\pm l}(\xi).
\]
Likewise, when $x$ is a simple closed Reeb orbit such that all iterations $x^k$ are isolated, set
\[ \beta(x) = \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \dim \text{HC}_*(x^k). \]
This is finite number by (LC3); in fact, one can expect that in this case the limit exists; cf. Remark 3.15.

**Theorem 3.14.** Assume that the Reeb flow of $\alpha$ on $M^{2n-1}$ has only finitely many simple periodic orbits $x_1, \ldots, x_r$, not necessarily non-degenerate, in the collection $\Gamma$. Then $\beta^\pm(\xi)$ is finite and moreover
\[ \beta^\pm(\xi) \leq \sum \pm \frac{\beta(x_i)}{\Delta(x_i)}, \]
where $\sum \pm$ stands for the sum taken over the orbits $x_i$ with positive/negative mean index $\Delta(x_i)$.

This theorem can be proved exactly in the same way as Theorem 3.6. The non-degenerate case of this result is pointed out in [GK, Remark 1.10]. Note that, as an immediate consequence of Theorem 3.14, we obtain a theorem from [HM] (cf. [GM, McL]) asserting that $\alpha$ must have infinitely many closed Reeb orbits when $\dim \text{HC}_l(\xi)$ is unbounded as a function of $\pm l \geq l_\pm$.

**Remark 3.15.** At this stage, very little is known about the sequence $\dim \text{HC}_*(x^k)$, except that this sequence is bounded, or about the “mean Betti number” $\beta(x)$. For instance, recall that the sequence $\chi(x^k) = I_k(F)$ is periodic by Theorem 2.8 and (LC2), and hence $\sigma(x)$ is rational. Moreover, a similar result holds for the local Floer homology. In fact, as is easy to see from the proof of [GG3, Theorem 1.1], the sequence $\dim \text{HF}_*(x^k)$, where $x$ is an isolated periodic orbit of a Hamiltonian diffeomorphism, is still subordinated to $N$. We conjecture that this is also true for $\dim \text{HC}_*(x^k)$, and hence $\beta(x) \in Q$. However, neither of these facts have been proved yet.

**Remark 3.16 (Variations).** Variants of Theorems 3.6 and 3.14 hold for cylindrical contact homology, when the latter is defined, with straightforward modifications to the setting and virtually the same proofs; cf. Remark 3.1.

One can also expect the analogues of these theorems to hold for, say, the (positive) equivariant symplectic homology or for the $S^1$-equivariant Morse homology of the energy functional on the space of loops on Riemannian or Finsler manifold, or for the equivariant Morse-type homology associated with certain other functionals on loop spaces; see [CDvK, FSvK, LLW, Ra, WHL] for some relevant results. However, now the situation is more delicate because the corresponding complexes are not directly, by definition, generated by the (good) orbits and, strictly speaking, even in the non-degenerate case an extra step in the proof is needed to relate the homology and the orbits. For instance, in the non-degenerate analogue of Theorem 3.6 for either of these “homology theories”, the contribution of an orbit $x^k$ is equal to the Euler characteristic of the infinite lens space $S^{2n-1}/\mathbb{Z}_k$ with respect to a certain local coefficient system; cf. [Bot, Lecture 3]. This contribution is trivial when $x^k$ is bad and equal to $\pm 1$ when the orbit is good. With this in mind, the proof of Theorem 3.6 should go through with mainly notational changes except for an analogue of (LC3), which now requires a proof; cf. [HM] and [McL].
Our goal in this section is to reprove the following

**Theorem 4.1** ([CGH, GHHM, LL]). Let $\alpha$ be a contact form on $S^3$ such that $\ker \alpha$ is the standard contact structure. Then the Reeb flow of $\alpha$ has at least two closed orbits.

In fact, the result proved in [CGH] is much more general and holds for all closed contact 3-manifolds; the proof uses the machinery of embedded contact homology. The proofs in [GHHM, LL] both rely on a theorem establishing the “degenerate case of the Conley conjecture” for contact forms and more specifically asserting that the presence of one closed Reeb orbit of a particular type (the so-called SDM) implies the existence of infinitely many closed Reeb orbits. This result holds in all dimensions and is stated explicitly below. Another non-trivial, and in this case strictly three-dimensional, counterpart of the argument in [GHHM] comes from the theory of finite energy foliations (see [HWZ1, HWZ2]), while the argument in [LL] uses, also in a non-trivial way, a variant of the resonance relation formula from [LLW]. Here, we give a very simple proof of Theorem 4.1 establishing it as a consequence of our Theorem 3.6 and the “Conley conjecture” type result mentioned above.

To state this result, we need to recall some definitions. Let $F_t, t \in [0, 1]$, be a family of germs of Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ fixing the origin $0 \in \mathbb{R}^{2n}$ and generated by the germs of Hamiltonians $H_t$ depending on $t \in S^1$. Set $F = F_1$ and let us assume that the origin is an isolated fixed point of $F$.

Denote by $\Delta(F) \in \mathbb{R}$ and $\text{HF}_*(F)$ the mean index and, respectively, the local Floer homology of $F$ over $\mathbb{Q}$. Again, we refer the reader to [Gi2, GG3, McL] for the definition of the latter. Here we only note that both of these invariants depend actually on the homotopy type (with fixed end points) of the entire family $F_t$. However, changing this family to a non-homotopic one, results in a shift of $\Delta(F)$ and of the grading of $\text{HF}_*(F)$ by the same even number. Furthermore, $\text{HF}_*(F)$ is supported in the range $[\Delta(F) - n, \Delta(F) + n]$, i.e., $\text{HF}_*(F)$ vanishes for degrees outside this range. When $2n = 2$, $\text{HF}_*(F)$ is supported in at most only one degree; [GG3]. Finally, analogously to (LC2) and essentially by definition, we have

$$I(F) = (-1)^n \sum (-1)^l \dim \text{HF}_l(F).$$

The fixed point of $F$, the origin, is said to be a *symplectically degenerate maximum* (SDM) of $F$ if $\Delta(x) \in \mathbb{Z}$ and $\text{HF}_*(F)$ does not vanish at the upper boundary of its possible range, i.e., $\text{HF}_{\Delta(x) + n}(F) \neq 0$. This is clearly a feature of the time-one map $F$, independent of the homotopy type of the family $F_t$. Note also that in this case $F$ is necessarily totally degenerate, i.e., all eigenvalues of $DF$ are equal to 1, although $DF$ need not be the identity. (Clearly, $\Delta(F) \in \mathbb{Z}$ when $F$ is totally degenerate; the converse, however, is not true without homological requirements.) Furthermore, then $\text{HF}_{\Delta(x) + n}(F) = \mathbb{Q}$ and $\text{HF}_*(F) = 0$ in all other degrees. (See [Gi2, GG3] for this and other results and a detailed discussion of SDM’s.)

**Example 4.2.** Assume that $H$ on $\mathbb{R}^{2n}$ is autonomous, has an isolated critical point at the origin and this critical point is a local maximum, and that the Hessian $d^2H$ at the origin is identically zero. (More generally, it suffices to assume that the eigenvalues of $d^2H$ with respect to the symplectic form are all zero.) Then the origin is an SDM of the time-one map $F$ generated by $H$. 

Returning to Reeb flows, we say that an isolated closed Reeb orbit \( x \) is an SDM if the corresponding fixed point of the Poincaré return map \( F \) of \( x \) is an SDM. (This is equivalent to requiring that \( \Delta(x) \in \mathbb{Z} \) and \( \text{HC}_{l}(x) = \mathbb{Q} \) when \( l = \Delta(x) + 2n - 4 \) and zero otherwise; see [GHHM] for this and other related results.) Note that, although the grading of \( \text{HF}^{*}(x) \) depends on some extra choices, e.g., a trivialization of \( \xi \) along \( x \), the notion of SDM is independent of these choices.

In the setting of Section 3.1, we have

**Theorem 4.3 ([GHHM]).** Let \((M^{2n-1}, \ker \alpha)\) be a contact manifold admitting a strong exact symplectic filling \((W, \omega)\) such that \( c_{1}(TW) = 0 \). Assume that the Reeb flow of \( \alpha \) has an isolated simple closed Reeb orbit \( x \) which is an SDM. Then the Reeb flow of \( \alpha \) has infinitely many periodic orbits.

**Remark 4.4.** The proof of Theorem 4.3 is a straightforward, although lengthy and cumbersome, adaptation of the proof of the degenerate case of the Conley conjecture in [GG1]. In fact, in [GHHM], the theorem is established under somewhat less restrictive conditions. Namely, it suffices to require that \( \omega|_{\pi_{2}(W)} = 0 \), when \( x \) is contractible (rather than that \( \omega \) is exact) and that \( \omega \) is atoroidal in general; cf. Remark 3.7.

Replacing the upper boundary of the range by the lower boundary in the definition of an SDM, we arrive at the notion of a symplectically degenerate minimum (SDMin). Thus \( F \) has an SDMin at the origin if and only if \( \Delta(F) \in \mathbb{Z} \) and \( \text{HF}_{\Delta(F)-n}(F) \neq 0 \). This notion naturally arises in dynamics (see [He] and also [GHHM, Remark 1]), and Theorem 4.3 holds with an SDM replaced by an SDMin.

Just as the local Floer homology in dimension two, the local contact homology of an isolated closed Reeb orbit \( x \) on a 3-manifold is concentrated in at most one degree. Hence, in this case we have the following mutually exclusive possibilities when \( x \) is isolated and degenerate, but not necessarily simple:

- SDM: \( \text{HC}_{\Delta(x)}(x) \neq 0 \) and \( x \) is an SDM,
- SDMin: \( \text{HC}_{\Delta(x)-2}(x) \neq 0 \) and \( x \) is an SDMin,
- Saddle or “Monkey Saddle”: \( \text{HC}_{\Delta(x)-1}(x) \neq 0 \),
- Homologically Trivial: \( \text{HC}^{*}(x) = 0 \).

All of these cases do occur. Furthermore, for any isolated iterated closed Reeb orbit \( y^{\kappa} \) (not necessarily non-degenerate), we have

\[ (-1)^{l} \dim \text{HC}_{l}(y^{\kappa}) = I_{\kappa}(F), \quad (4.1) \]

where \( F \) is the Poincaré return map of \( y \) and \( l \) is the degree where the cohomology of \( y^{\kappa} \) is supported. (Note that \( \text{HC}_{*}(y^{\kappa}) = 0 \) if and only if \( I_{\kappa}(F) = 0 \).) This is an immediate consequence of (LC2) and again the fact that the cohomology is supported in only one degree.

**Proof of Theorem 4.1.** Arguing by contradiction, assume that the Reeb flow of \( \alpha \) on \((S^{3}, \xi_{0})\) has only one closed simple orbit \( x \). We will prove that then \( x \) is an SDM, and hence the flow has infinitely many periodic orbits by Theorem 4.3. Applying Theorem 3.6 and Example 3.2, we have

\[ \frac{\sigma(x)}{\Delta(x)} = \frac{1}{2}, \quad (4.2) \]

where \( \Delta(x) > 0 \). Therefore, \( \sigma(x) > 0 \).
Furthermore, recall that $HC_l(\xi_0) = \mathbb{Q}$ when $l \geq 2$ is even and $HC_l(\xi_0) = 0$ otherwise; see, e.g., [Bon2].

The orbit $x$ is necessarily elliptic: all Floquet multipliers, i.e., eigenvalues of $DF$, lie on the unit circle. Indeed, if $x$ is hyperbolic and even, we have $\sigma(x) = -1$, which is impossible. If $x$ is hyperbolic and odd, we have $\sigma(x) = 1/2$ and $\Delta(x) = 1$.

Furthermore, all orbits $x^{\kappa}$ are non-degenerate and $\mu_{cz}(x^{\kappa}) = \kappa$. Since the even iterations of $x$ are bad orbits, the complex $CC_*(\alpha)$ is generated by the orbits $x^{\kappa}$, where $\kappa$ is odd, with $|x^{\kappa}| = \kappa - 1$, which is also impossible since then $HC_0(\xi_0) = \mathbb{Q}$.

With $x$ elliptic, write $\Delta(x) = 2m + 2\alpha$, where $m$ is a non-negative integer and $\alpha \in [0, 1)$. By the definition of the mean index, $e^{\pm 2\pi i \alpha}$ is an eigenvalue of $F$. From (4.2) we see that $\alpha$ is necessarily rational. Set $\alpha = p/q$, where $q \geq 2$ and $q > p \geq 0$, and $p$ and $q$ are mutually prime. (In particular, $x^q$ is degenerate.) To summarize, we are in the setting of Example 2.10: $I_\kappa(F) = 1 - r$ for some $r \geq 0$ when $q|\kappa$, and $I_\kappa(F) = 1$ otherwise. We conclude that

$$\sigma(x) = 1 - \frac{r}{q} \quad \text{and} \quad \Delta(x) = 2m + \frac{2p}{q},$$

and (4.2) amounts to

$$1 - \frac{r}{q} = m + \frac{p}{q},$$

where $m$, $r$ and $p$ are non-negative and $q \geq 2$. This condition can only be satisfied when $m = 0$ or $m = 1$.

Let us first examine the case $m = 0$, which is exactly the point where in [GHHM] we had to rely on the results from [HWZ1, HWZ2]. Then $p > 0$ (for otherwise $\Delta(x) = 0$) and $p + r = q$. In particular, $r < q$.

To rule out this case, observe first that $p = 1$. Indeed, we have $|x^{\kappa}| = 0$ as long as $kp/q < 1$. If $p \geq 2$, the first degree jump occurs when $kp/q$ becomes greater than 1 and the degree increases to 2. The next jumps occur when $kp/q$ moves over the subsequent integers, while $x^{\kappa}$ is still nondegenerate, or when $\kappa = q$. In the latter case, $x^q$ is degenerate, but $\Delta(x^q) \geq 4$ since $\kappa \geq 2$ and $p \geq 2$, and $x^q$ cannot contribute to the homology in degree 0. We conclude that again $HC_0(\alpha) \neq 0$ when $p > 1$, which is impossible.

Thus $p = 1$. The degrees of $x^{\kappa}$ form a sequence

$$|x^{\kappa}| = 0, \ldots, 0, *, 2, \ldots, 2, *, \ldots,$$

where the undefined degrees of the degenerate iterations $|x^{n}|$ are entered as the asterisks. It is clear, however, that $HC_*(x^q)$ must be concentrated in degree one, to cancel the contribution of previous iterations to degree zero, and hence $r \geq 1$.

It follows from (4.1) that $HC_*(x^{\kappa}) = \mathbb{Q}^{r-1}$, concentrated in degree $2\kappa/q - 1$, when $q|\kappa$ and that $HC_*(x^{\kappa}) = \mathbb{Q}$, concentrated in degree $2|\kappa/q|$, otherwise, i.e., when $x^{\kappa}$ is non-degenerate. Hence, all iterates $x^{\kappa}$ with $\kappa \leq q - 1$ contribute to degree 0 while the iterates $x^{\kappa}$ with $\kappa > q$ contribute to the degrees greater than or equal to 2:

$$HC_*(x^{\kappa}) = \mathbb{Q}_{q-1}^{0}, \ldots, \mathbb{Q}_{q-1}, \mathbb{Q}_{2}^{r-1}, \ldots, \mathbb{Q}_{2}, \mathbb{Q}_{q}^{r-1}, \ldots,$$
where the subscripts on the right hand side indicate the degree. To ensure that
\( HC_0(\alpha) = 0 \), we must have \( q - 1 \leq r - 1 \), by the long exact sequence for filtered
contact homology, i.e., \( q \leq r \), which is impossible since \( r < q \) as stated above.

It follows now that \( m = 1 \) and \( p = 0 = r \). Therefore, the orbit \( x \) itself is
degenerate and \( \Delta(x) = 2 \). Clearly, since \( HC_2(x) \) is concentrated in at most one
degree, the fact that \( HC_2(\xi_0) = Q \) implies that \( HC_2(x) = Q \), and hence \( x \) is an
SDM. This concludes the proof of the theorem.

\[ \square \]

**Remark 4.5.** Let \( \alpha \) be a contact form on \( (S^3, \xi_0) \) invariant under a contact involution
\( \tau \). (For instance, this is the case when \( \alpha \) comes from a symmetric star-shaped
embedding of \( S^3 \) into \( \mathbb{R}^4 \) and \( \tau \) is the antipodal involution or when \( \alpha \) is the lift of
a contact form on \( \mathbb{R}P^3 \) supporting the standard contact structure.) Then the Reeb
flow of \( \alpha \) cannot have exactly two simple Reeb orbits “swapped” by \( \tau \). Indeed,
assume the contrary and denote the orbits by \( x \) and \( \tau(x) \). Then these orbits have
the same local invariants: the same mean index, the same mean iterated index and
the same local contact homology. Now, arguing as in the proof of Theorem 4.1,
but without relying on Theorem 4.3, it is not hard to show that this is impossible.
However, as the example of an irrational ellipsoid in \( \mathbb{R}^4 \) shows, \( \alpha \) can have exactly
two simple orbits each of which is invariant under \( \tau \). Projecting the irrational
ellipsoid contact form to \( \mathbb{R}P^3 = ST^*S^2 \), we obtain an asymmetric Finsler metric
on \( S^2 \) with exactly two geodesics; see \([Ka, Zi]\) and also \([Gü]\).

As an easy application of Theorem 4.1, we have

**Corollary 4.6.** Let \( M \) be a fiberwise star-shaped (with respect to the zero section)
hypersurface in \( T^*S^2 \). Then \( M \) carries at least two closed characteristics.

When \( M \) is fiberwise convex, the Reeb flow is just the geodesic flow of a (not
necessarily symmetric) Finsler metric on \( S^2 \). Thus, in particular, it follows that
any Finsler metric on \( S^2 \) has at least two simple closed geodesics. This result is
originally proved in \([BL]\). Of course, Corollary 4.6 also immediately follows from
the main theorem of \([CGH]\).

**Proof.** The hypersurface \( M \) is of contact type and contactomorphic to \( \mathbb{R}P^3 =
ST^*S^2 \) with the standard contact structure \( \xi_0 \) and some contact form \( \alpha \). Arguing
by contradiction, assume that \((M, \alpha)\) carries only one simple closed Reeb orbit
\( x \). We claim that \( x \) is not contractible.

This is a standard fact which can be established in a variety of ways. For
instance, this is a consequence of Remark 4.5. Alternatively, this follows from
the fact that the Reeb flow of any such form \( \alpha \) must have non-contractible orbits
because the cylindrical contact homology of \( ST^*S^2 \) is non-zero for the non-trivial
free homotopy type of loops in \( \mathbb{R}P^3 \). The latter assertion can be easily proved by
examining the indices of the closed orbits for the form coming from an irrational
ellipsoid in \( \mathbb{R}^4 \) or by using the Morse–Bott techniques; see, e.g., \([Bou2, VK, KvK]\).
(For Finsler metrics on \( S^2 \) this is, of course, a well-known result in the standard
calculus of variations: the orbit in question is the energy minimizer within the fixed
non-trivial free homotopy class; cf. \([Bot]\).)

Finally, since \( x \) is not contractible, it lifts to one orbit on \( S^3 \) and we obtain
a contact form on \( S^3 \) with only one closed Reeb orbit, which is impossible by
Theorem 4.1. \[ \square \]
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