Polar gravitational perturbations and quasinormal modes of LQBH

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Abstract

In this work, we investigate the polar gravitational quasinormal modes (QNMs) for a simplified black hole model, that arises in the context of Loop Quantum Gravity (LQG), known as Loop Quantum Black Hole (LQBH). The LQBH metric is a quantum corrected version of the Schwarzschild black hole (SchBH). To find the characteristic modes of LQBH, we consider the WKB approach. It is found that the quasinormal frequencies related to polar gravitational perturbation are strongly dependent on the LQG parameters. From the obtained results, the LQBH has been shown to be stable under polar gravitational perturbations.

Keywords: Quasinormal modes, Polar gravitational perturbation, WKB method, Black holes, Loop Quantum Gravity.
I. INTRODUCTION

One of the most exceptional predictions of General Relativity (GR) \[1\] is the existence of the black holes (BHs), that are objects from which nothing (even light signals) can escape after crossing their horizons. The interest in BHs goes far beyond astrophysics. In fact, they appeared as possible objects that can help us to understand one of the most intriguing problem in theoretical physics today, that is the nature of the quantum gravity. This is because, in the presence of a very strong gravitational field the quantum properties of spacetime must become relevant.

Interest in black hole physics research has been boosted in the last years because of gravitational wave observations since 2015 \[2\], which was originated from a binary black hole merging, and since then this class of events has been observed with increasing precision through the LIGO and Virgo collaborations \[3\].

In the LQG realm, it is possible to find interesting theoretical models that provide an insight into the quantum characteristics of spacetimes revealed by BHs. Actually, in the context of this theory, an interesting BH scenario corresponds to the Loop Quantum Black Hole (LQBH), or self-dual black hole solution \[4\], which corresponds to a quantum version of the SchBH and has the interesting property of self-duality. From such duality, the BH singularity is removed and replaced by an asymptotically flat region, which is an expected effect in the quantum gravity regime.

A very important fact is that we cannot find a BH completely isolated in nature, i.e., they are always interacting with other structures in its neighborhood (for example, stars and even other black holes). However, even if we suppose that nothing exists near the black hole, they will interact with the surrounding vacuum, creating pairs of particles, and consequently evaporating due to Hawking phenomena \[5\]. Therefore, real black holes are always in a perturbed state and emit gravitational waves as a response to those perturbations. Such gravitational waves are characterized by some complex eigenvalues of the wave equations known as quasinormal modes. In this context, the real part of the gravitational modes gives us the oscillation frequencies, while the imaginary part determines the damping rates of the frequencies. These eigenvalues are known as quasinormal modes and, they depend only on BHs parameters and not on the source that causes the perturbations.

The QNMs are of considerable interest in different contexts. For example, in the context of the AdS/CFT, they are studied because of the possibility of observing quasinormal ringing of astrophysical BHs, when one considers the thermodynamic properties of black holes \[6\ \[7\].
Moreover, the quasinormal modes of near extremal black branes has also been investigated in [8]. In recent years, it has been suggested that the QNMs of BHs might play a role in quantum gravity, mainly in approaches like string theory and Loop Quantum Gravity (LQG). Especially in the context of LQG, it has been also suggested that the QNMs can be used to fix the Immirzi parameter, a parameter measuring the quantum of the spacetime [9]. Nevertheless, this is a fundamental issue that remains open in this field. Inspired by these considerations, quasinormal modes of self-dual black hole have been computed in [10] where the authors have considered scalar perturbations and also in [11] where the present authors considered axial gravitational perturbations.

The present work has as its main goal to examine the QNMs of an LQBH [4], by taking a model of a gravitational wave obtained from polar gravitational perturbations. To compute the frequencies for quasinormal modes we shall use the WKB method, and then, we shall compare them with calculations in the classical limit [12]. In addition, we will analyze the corrections induced by polymeric parameter $P$ to the QNMs.

The paper is organized as follows. In Section II we briefly review the LQBH solution and discuss their self-dual property. In Section III we derive a Schrödinger-type equation, by considering the polar gravitational perturbations. In Section IV we shall calculate the QNMs through the WKB method. We summarize our results and draw concluding remarks in Section V. Throughout this work, we have used natural units $\hbar = c = G = 1$.

II. THE METRIC OF LOOP QUANTUM BLACK HOLE

Some efforts have been previously considered [4, 13] in order to find out BHs solutions in LQG. In this section, we will briefly introduce the self-dual black hole solution that arises from a simplified model of LQG consisting of a symmetry reduced model corresponding to homogeneous spacetimes [4].

The structure of the LQBH corresponds to a quantum version of the SchBH, and is described by the metric below:

$$ds^2 = -G(r)dt^2 + \frac{dr^2}{F(r)} + H(r) \left[ d\theta^2 + \sin^2(\theta)d\phi^2 \right],$$  

where the functions $G(r)$, $F(r)$ and $H(r)$ are defined by

$$G(r) = \frac{(r-r_+)(r-r_-)(r+r_*)^2}{r^4 + a_0^2}, \quad F(r) = \frac{(r-r_+)(r-r_-)r^4}{(r+r_*)^2(r^4 + a_0^2)}, \quad H(r) = \left(r^2 + \frac{a_0^2}{r^2}\right).$$  

In the functions \[\text{II.2}\], we have the presence of an external horizon localized in $r_+ = 2m$, an intermediate in $r_* = \sqrt{r_+r_-}$ and a Cauchy horizon localized at $r_- = 2mP^2$, where the polymeric...
function $P$, is given by

$$P = \frac{\sqrt{1 + \epsilon^2} - 1}{\sqrt{1 + \epsilon^2} + 1}. \quad (\text{II.3})$$

In the Eq. (II.3) we have the parameter $\epsilon = \gamma \delta_b$, where $\gamma$ is the Barbero-Immirzi parameter, and $\delta_b$ is polymeric parameter used for the quantization in LQG. Also, in the functions in the Eq. (II.2) appear the parameter $a_0$ defined by

$$a_0 = \frac{A_{\text{min}}}{8\pi}, \quad (\text{II.4})$$

where $A_{\text{min}}$ is minimal area in LQG.

It is important to notice, that the Eq. (II.1) is written in terms of the LQBH mass, where $m$ is associated with the ADM mass as follows

$$M = m(1 + P)^2. \quad (\text{II.5})$$

Furthermore, it is important to see that the function $H(r)$ defined in the Eq. (II.2) is equal to $r^2$ only in asymptotic limit. In this way, a new physical radial coordinate that measures the circumference distance is given by

$$R = \sqrt{r^2 + \frac{a_0^2}{r^2}}. \quad (\text{II.6})$$

From Eq. (II.6), we can see an important characteristic of the internal structure of the LQBH. When $r$ decreases from infinity to zero, the $R$ coordinate decreases from infinity to $R = \sqrt{2a_0}$ in $r = \sqrt{a_0}$, and then increases again to infinity. Considering the Eq. (II.6) in the external event horizon, i.e., in $r = r_+$, we get

$$R_+ = \sqrt{H(r_+)} = \sqrt{(2m)^2 + \left(\frac{a_0}{2m}\right)^2}. \quad (\text{II.7})$$

A very interesting characteristic of this scenery is the self-duality of metric in Eq. (II.1). The self-duality means that, if we introduce new coordinates, $\tilde{r} = a_0/r$ and $\tilde{t} = t r^2/a_0$, the form of metric is preserved. The dual radial coordinate is given by $\tilde{r} = \sqrt{a_0}$, and corresponds to a minimal element of surface. Furthermore, Eq. (II.6) can be written in the form $R = \sqrt{r^2 + a_0^2}$ that clearly shows an asymptotically flat space, that is, a Schwarzschild region in the place of singularity in the limit as $r$ goes to zero. This region corresponds to a wormhole with size of the order of the Planck length. The Carter-Penrose diagram for the LQBH is shown in Fig. 1.

We can interpret the LQBH metric of the Eq. (II.1) as a solution of the Einstein equation [4], given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}^{\text{eff}}, \quad (\text{II.8})$$
where $T_{\mu\nu}^{\text{eff}}$ is associated with an effective matter fluid that simulates the LQG corrections, given by

$$T_{\mu\nu}^{\text{eff}} = \text{diag} (-\rho, p_r, p_\theta, p_\theta) ,$$

and the components of the effective stress-energy tensor are: $\rho = -G_{tt}/8\pi$, $p_r = G_{rr}/8\pi$ and $p_\theta = G_{\theta\theta}/8\pi$. So, in the limit where $P$ and $a_0$ go to zero, the LQBH metric of Eq. (II.1) gives us the classical Schwarzschild solution ($g_{\mu\nu}^{\text{Sch}}$) that satisfies $G_{\mu\nu}(g_{\mu\nu}^{\text{Sch}}) \equiv 0$.

### III. EFFECTIVE POTENTIAL OF POLAR GRAVITATIONAL PERTURBATION IN LQBH SPACETIME

In this section, we will use the method used by Zerilli for finding a master equation considering the polar gravitational perturbation [14] [15]. So, if small perturbations are introduced, the resulting spacetime metric can be written in the form:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} ,$$

where $g_{\mu\nu}$ is the background metric (II.1) and $h_{\mu\nu}$ are the spacetime perturbations, where $|h_{\mu\nu}| \ll |g_{\mu\nu}|$.

Considering the spherical symmetry of a black hole metric, the perturbation of the metric $h_{\mu\nu}$ can be composed by a sum of two parts, where the first is characterized by axial perturbations...
$h_{\mu \nu}^{\text{axial}}$ (which has been analyzed in [11] by considering the LQBH). The second part is the polar perturbation $h_{\mu \nu}^{\text{polar}}$, that we can still separate in two independent terms, one depending on time and radial coordinates, and the second depending on angular coordinates through spherical harmonics.

In this context, the polar gravitational perturbation in the Regge-Wheeler gauge [15], is given by

$$h_{\mu \nu}^{\text{pol}} = \begin{bmatrix} G(r)H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & F(r)^{-1}H_2(r) & 0 & 0 \\ 0 & 0 & H(r)K(r) & 0 \\ 0 & 0 & 0 & H(r)\sin^2(\theta)K(r) \end{bmatrix} e^{-i\omega \tau} P_l(\cos \theta) , \quad (\text{III.2})$$

where $H_0(r)$, $H_1(r)$, $H_2(r)$ and $K(r)$ are unknown functions for the polar perturbations. Thus, substituting the Eqs. (\text{III.1}) and (\text{III.2}) in Eq. (\text{III.8}) we get a set of equations, given by

$$\frac{dK(r)}{dr} = -\frac{l(l+1)}{2 F(r)} K(r) + \frac{i\omega}{G(r)} H_1(r) - \frac{G''(r)}{G(r)} Q(r) + \frac{1}{2} \frac{G''(r)}{G(r)} H_1(r) , \quad (\text{III.3})$$

$$\frac{dQ(r)}{dr} = \frac{dK(r)}{dr} - \frac{i\omega}{G(r)} H_1(r) - \frac{G''(r)}{G(r)} Q(r) , \quad (\text{III.4})$$

$$\frac{dH_1(r)}{dr} = -\frac{1}{2} \frac{F'(r)}{F(r)} H_1(r) - \frac{1}{2} \frac{G'(r)}{G(r)} H_1(r) - \frac{i\omega}{F(r)} Q(r) - \frac{i\omega}{F(r)} K(r) , \quad (\text{III.5})$$

where we have taken $H_0(r) = H_2(r) \equiv Q(r)$ and the prime means the derivative with respect to the radial coordinate. In addition to the equations above, the functions $K(r)$, $Q(r)$ and $H_1(r)$ also should satisfy the following algebraic identity:

$$\Lambda_0 Q(r) + \Lambda_1 K(r) + \Lambda_2 H_1(r) = 0 , \quad (\text{III.6})$$

where

$$\Lambda_0 = -\frac{1}{2} \frac{F(r)H''(r)}{G(r)} - \frac{1}{4} \frac{F'(r)H'(r)}{G(r)} - \frac{l(l+1)}{2} \frac{G''(r)F'(r)H(r)}{G(r)} - \frac{1}{2} \frac{G''(r)}{G(r)} F(r)H(r)$$

$$+ \frac{1}{4} \frac{G'(r)^2}{G(r)^2} F(r)H(r) , \quad (\text{III.7})$$

$$\Lambda_1 = \frac{\omega^2}{G(r)} \frac{H(r)}{G(r)} + \frac{1}{2} \frac{F(r)H''(r)}{G(r)} - \frac{1}{2} \frac{l(l+1)}{2} \frac{F'(r)H'(r)}{G(r)} + \frac{1}{4} \frac{G'(r)^2 F(r)H(r)}{G(r)^2} , \quad (\text{III.8})$$
and

\[ \Lambda_2 = \frac{-1}{2} \frac{i \omega F(r)H'(r)}{G(r)} - \frac{1}{2} \frac{i F(r)^2 H(r) G'''(r)}{\omega G(r)} - \frac{3}{8} \frac{i F(r)^2 H(r) G'(r)^3}{\omega G(r)^3} \]

\[ + \frac{1}{2} \frac{i F(r)^2 G''(r)^2 H'(r)}{\omega G(r)^2} - \frac{1}{2} \frac{i F(r)^2 G'(r) H''(r)}{\omega G(r)} + \frac{3}{4} \frac{i F(r)^2 H(r) G'(r) G''(r)}{\omega G^2(r)} \]

\[ - \frac{i F(r)^2 H'(r) G''(r)}{\omega G(r)} - \frac{3}{4} \frac{i F(r) H(r) F'(r) G''(r)}{\omega G(r)} - \frac{1}{4} \frac{i F(r) H(r) G'(r) F''(r)}{\omega G(r)} \]

\[ + \frac{3}{8} \frac{i F(r) H(r) G'(r)^2 F'(r)}{\omega G(r)^2} - \frac{3}{4} \frac{i F(r) H'(r) G'(r)^2 F'(r)}{\omega G(r)} + \frac{1}{4} \frac{i F(r) G'(r)}{\omega G(r)} l(l + 1) . \]

(III.9)

At this point, we define a new function \( R(r) \equiv H_1(r)/\omega \). Thus, the Eq. (III.6) can be rewritten as

\[ Q(r) = \left[ \alpha(r) + \beta(r) \omega^2 \right] K(r) + \left[ \gamma(r) + \delta(r) \omega^2 \right] R(r) , \]

(III.10)

where

\[ \alpha(r) = \left[ - F(r) H''(r) - \frac{1}{2} F'(r) H'(r) + l(l + 1) - \frac{1}{2} \frac{G'(r) F'(r) H(r)}{G(r)} - \frac{G''(r)}{G(r)} F(r) \right] \]

\[ \times H(r) + \frac{1}{2} \frac{G'(r)^2 F(r) H(r)}{G(r)^2} \right]^{-1} \left[ - F(r) H''(r) + l(l + 1) - \frac{1}{2} F'(r) H'(r) \right] \]

(III.11)

\[ \beta(r) = \left[ - F(r) H''(r) - \frac{1}{2} F'(r) H'(r) + l(l + 1) - \frac{1}{2} \frac{G'(r) F'(r) H(r)}{G(r)} - \frac{G''(r)}{G(r)} F(r) \right] \]

\[ \times H(r) + \frac{1}{2} \frac{G'(r)^2 F(r) H(r)}{G(r)^2} \right]^{-1} \left[ - \frac{H(r)}{G(r)} \right] , \]

(III.12)

\[ \gamma(r) = \left[ - F(r) H''(r) - \frac{1}{2} F'(r) H'(r) + l(l + 1) - \frac{1}{2} \frac{G'(r) F'(r) H(r)}{G(r)} - \frac{G''(r)}{G(r)} F(r) \right] \]

\[ \times H(r) + \frac{1}{2} \frac{G'(r)^2 F(r) H(r)}{G(r)^2} \right]^{-1} \left[ \frac{i F(r)^2 H(r) G'''(r)}{G(r)} + \frac{3}{4} \frac{i F(r)^2 H(r) G'(r) G''(r)}{G(r)^3} \right] \]

\[ - \frac{i F(r)^2 G''(r)^2 H'(r)}{G(r)^2} + \frac{i F(r)^2 G'(r) H''(r)}{G(r)} - \frac{3}{2} \frac{i F(r)^2 H(r) G'(r) G''(r)}{G(r)^2} + 2 \]

\[ \times \frac{i F(r)^2 H'(r) G''(r)}{G(r)} + \frac{3}{2} \frac{i F(r) H'(r) F'(r) G''(r)}{G(r)} + \frac{1}{2} \frac{i F(r) H(r) G'(r) F''(r)}{G(r)} \]

\[ - \frac{3}{4} \frac{i F(r) H(r) G'(r)^2 F'(r)}{G(r)^2} + \frac{3}{2} \frac{i F(r) H'(r) G'(r)^2 F'(r)}{G(r)} - \frac{1}{2} \frac{i F(r) G'(r)}{G(r)} l(l + 1) \]

(III.13)
and
\[
\delta(r) = \left[ -F(r)H''(r) - \frac{1}{2} F'(r)H'(r) + l(l + 1) - \frac{1}{2} \frac{G'(r)F'(r)H(r)}{G(r)} - \frac{G''(r)}{G(r)} F(r) \times H(r) + \frac{1}{2} \frac{G'(r)^2}{G(r)^2} F(r)H(r) \right]^{-1} \left[ \frac{F(r)H'(r)}{G(r)} \right].
\]

(III.14)

In addition, by substituting in the Eq. (III.3), we get
\[
\frac{dK(r)}{dr} = -\left[ \alpha_0(r) + \alpha_2(r) \omega^2 \right] K(r) - \left[ \beta_0(r) + \beta_2(r) \omega^2 \right] R(r),
\]

(III.15)

where
\[
\alpha_0(r) = \frac{1}{2} \left[ \frac{H'(r)}{H(r)} - \frac{G'(r)}{G(r)} \right] - \frac{1}{2} \frac{H'(r)}{H(r)} \alpha(r), \quad \alpha_2(r) = -\frac{1}{2} \frac{H'(r)}{H(r)} \beta(r),
\]

(III.16)

\[
\beta_0(r) = -\frac{1}{2} \frac{H'(r)}{H(r)} \gamma(r) - \frac{iF(r)}{2} \left[ \frac{l(l + 1)}{F(r)H(r)} - \frac{1}{2} \frac{G'(r)F'(r)}{G(r)F(r)} - \frac{G''(r)}{G(r)} \frac{1}{2} \frac{G'(r)^2}{G(r)^2} - \frac{G'(r)F'(r)}{G(r)H(r)} \right], \quad \beta_2(r) = -\frac{1}{2} \frac{H'(r)}{H(r)} \delta(r).
\]

(III.17)

Now for the Eq. (III.5), we get
\[
\frac{dR(r)}{dr} = -\left[ \gamma_0(r) + \gamma_2(r) \omega^2 \right] K(r) - \left[ \delta_0(r) + \delta_2(r) \omega^2 \right] R(r),
\]

(III.18)

where
\[
\gamma_0(r) = \frac{i}{F(r)} \left[ 1 + \alpha(r) \right], \quad \gamma_2(r) = \frac{i}{F(r)} \beta(r),
\]

(III.19)

\[
\delta_0(r) = \frac{i}{F(r)} \gamma(r) + \frac{1}{2} \frac{F'(r)}{F(r)} \left[ \frac{G'(r)}{G(r)} \right], \quad \delta_2(r) = \frac{i}{F(r)} \delta(r).
\]

(III.20)

Finally, to obtain a Schrödinger-type equation, we will perform a change of variable, given by
\[
K(r) = f(r) \hat{K}(x) + g(r) \hat{R}(x), \quad R(r) = h(r) \hat{K}(x) + t(r) \hat{R}(x),
\]

(III.21)

where
\[
g(r) = 1, \quad t(r) = -\frac{\alpha_2(r)}{\beta_2(r)}, \quad f(r) = -\sqrt{G(r)F(r)} \left[ \alpha_0(r) - \beta_0(r) \frac{\alpha_2(r)}{\beta_2(r)} \right]
\]

(III.22)

and
\[
h(r) = \frac{1}{\beta_2(r)} \left\{ \frac{1}{\sqrt{G(r)F(r)}} + \sqrt{G(r)F(r)} \left[ \alpha_0(r) \alpha_2(r) - \beta_0(r) \frac{\alpha_2(r)^2}{\beta_2(r)} \right] \right\}.
\]

(III.23)
Together with the Eqs. (III.21), we still need to impose:

\[
\frac{d\hat{K}(x)}{dx} = \hat{R}(x) \quad \text{and} \quad \frac{d\hat{R}(x)}{dx} = \left[ V_{\text{eff}}(r) - \omega^2 \right] \hat{K}(x), \tag{III.24}
\]

where the variable \( x \) is called tortoise coordinate and is given by

\[
\frac{dr}{dx} = \sqrt{G(r)F(r)}. \tag{III.25}
\]

Consequently, from Eq. (III.24) we get a Schrödinger-type equation given by

\[
\frac{d^2\hat{K}(x)}{dx^2} + \left[ \omega^2 - V_{\text{eff}}(r(x)) \right] \hat{K}(x) = 0, \tag{III.26}
\]

where the effective potential, \( V_{\text{eff}}(r) \), is given by

\[
V_{\text{eff}}(r) = -\sqrt{G(r)F(r)} \left[ \alpha_0(r)f(r) + \beta_0(r)h(r) + \frac{df(r)}{dr} \right]. \tag{III.27}
\]

The behavior of the effective potential of the Eq. (III.27) is shown in the Fig. 2 considering different values for the polymeric parameter \( P \) and for the multipole number \( l \). We can see that, when \( P \) and the minimal area of LQG tends to zero, the effective potential tends to the classical Zerilli potential for the Schwarzschild black hole [14].

![Fig. 2](image_url)

**FIG. 2:** The graphics of the effective potential for polar gravitational perturbations. In these plots, we consider the black hole mass \( m = 1 \) and \( a_0 = \sqrt{3}/2 \). In the plot (a) we consider the multipole value \( l = 2 \) and vary the polymeric parameter as \( P = 0.0, 0.1, 0.2 \) and \( 0.3 \). Here, the polymeric parameter value \( P = 0.0 \) (and also \( a_0 = 0 \)) is associated with the classical SchBH. In the plot (b) we consider the fixed value \( P = 0.1 \) and vary the multipole values as \( l = 2, 3 \) and \( 4 \).
IV. QUASINORMAL MODES FROM SELF-DUAL BLACK HOLE

In this section, our focus will be the calculus of quasinormal modes of black hole described by the metric of the Eq. (II.1). As we saw in the last section, the perturbation equations of a black hole can be reduced to a Schrödinger-type wave equation given by Eq. (III.26). The effective potential $V_{\text{eff}}(x)$, that appears in Eq. (III.27) is constant in the event horizon ($x = -\infty$) and at infinity ($x = \infty$) and has a maximum at some intermediate ($x = x_0$).

Thus, we can make a direct analogy with the problem of scattering near the peak of the barrier potential in Quantum Mechanics, where $\omega^2$ in Eq. (III.26) plays the role of the energy. To obtain black hole quasinormal modes, several methods have been developed. However, we chose an approximated method, that is the well-known WKB approach introduced by Schutz and Will [16]. This treatment was later improved to the 3rd order by Iyer and Will [17] and is the approximation that we shall consider below. For a review of the available QNMs techniques, we suggest [8] and [18–20] for further readings.

In this way, with the objective of following the WKB method, we have supposed that $\hat{K}(x(r))$ had a harmonic asymptotic behavior in $t$ coordinate, $\hat{K}(x) \sim e^{-i\omega(t+x)}$ and $V_{\text{eff}}(r(x)) \to 0$ as $x \to \pm \infty$ in Eq. (III.26). So, the frequencies $\omega_n$ that appear in Eq. (III.26) are determined (up to 3rd order of the WKB approximation), by the following equation:

$$\omega_n = \sqrt{(V_0 + \Delta) - i \left( n + \frac{1}{2} \right) \sqrt{-2V''_0 (1 + \Omega) }}, \quad \text{(IV.1)}$$

where

$$\Delta = \frac{1}{8} \left( \frac{V^{(4)}_0}{V'_0} \right) \left( \frac{1}{4} + \alpha^2 \right) - \frac{1}{288} \left( \frac{V''''_0}{V''_0} \right)^2 \left( 7 + 60\alpha^2 \right), \quad \text{(IV.2)}$$

$$\Omega = -\frac{1}{2V''_0} \left\{ \frac{5}{6912} \left( \frac{V''''_0}{V''_0} \right)^4 \left( 77 + 188\alpha^2 \right) - \frac{1}{384} \left[ \left( \frac{V''''_0}{V''_0} \right)^2 \left( \frac{V^{(4)}_0}{V'_0} \right)^3 \right] (51 + 100\alpha^2) \\ + \frac{1}{2304} \left( \frac{V^{(4)}_0}{V''_0} \right)^2 (65 + 68\alpha^2) + \frac{1}{288} \left( \frac{V''''_0}{V''_0} \right) \left( V''''_0 \right) (19 + 28\alpha^2) \right\} \quad \text{(IV.3)}$$

In the relations above, $\alpha = n + \frac{1}{2}$ and $V^{(n)}_0$ denotes the $n$-order derivative of the potential on the maximum $x_0$ of the effective potential. Thus, using the effective potential of Eq. (III.27) and Eq. (IV.1), we can calculate the quasinormal modes, $\omega_n$ for the LQBH, which are shown in the tables I, II and III for different values of LQBH parameter. For the case where the polymeric...
$P$ and minimal area $a_0$ parameters go to zero, the results approach to those of the Schwarzschild black hole [12].

| $P$ | $\omega_0$ | $\omega_1$ | $\omega_2$ |
|-----|-------------|-------------|-------------|
| 0.1 | 0.4588080 - 0.0943415i | 0.4350910 - 0.2893620i | 0.3976210 - 0.4945600i |
| 0.2 | 0.5294850 - 0.0964412i | 0.5129000 - 0.2947210i | 0.4883820 - 0.5020030i |
| 0.3 | 0.5893080 - 0.0945579i | 0.5760760 - 0.2862250i | 0.5534050 - 0.4828290i |
| 0.4 | 0.6373510 - 0.0912246i | 0.6285820 - 0.2768290i | 0.6158650 - 0.4686530i |
| 0.5 | 0.6724460 - 0.0848729i | 0.6658430 - 0.2560750i | 0.6546990 - 0.4304320i |
| 0.6 | 0.6928030 - 0.0769778i | 0.6881010 - 0.2318850i | 0.6796620 - 0.3898860i |
| 0.7 | 0.6926800 - 0.0677585i | 0.6926040 - 0.2040160i | 0.6861650 - 0.3420780i |
| 0.8 | 0.6804520 - 0.0584345i | 0.6768360 - 0.1776110i | 0.6724440 - 0.3025970i |
| 0.9 | 0.6441710 - 0.0499027i | 0.6394600 - 0.1522590i | 0.6328630 - 0.2612890i |

**TABLE I:** First quasinormal modes to LQBH considering $a_0 = \sqrt{3}/2$ and $l = 2$.

| $P$ | $\omega_0$ | $\omega_1$ | $\omega_2$ |
|-----|-------------|-------------|-------------|
| 0.1 | 0.7103050 - 0.0954021i | 0.6957200 - 0.2872270i | 0.6675730 - 0.4815990i |
| 0.2 | 0.8001510 - 0.0968958i | 0.7892090 - 0.2921900i | 0.7692650 - 0.4909660i |
| 0.3 | 0.8777030 - 0.0953165i | 0.8693880 - 0.2869460i | 0.8539780 - 0.4809750i |
| 0.4 | 0.9407610 - 0.0915475i | 0.9347890 - 0.2760760i | 0.9246220 - 0.4641020i |
| 0.5 | 0.9870080 - 0.0852833i | 0.9826610 - 0.2565230i | 0.9741900 - 0.4294660i |
| 0.6 | 1.0134100 - 0.0773285i | 1.0102700 - 0.2325430i | 1.0045700 - 0.3892210i |
| 0.7 | 1.0165400 - 0.0679873i | 1.0140600 - 0.2042550i | 1.0093900 - 0.3431380i |
| 0.8 | 0.9926870 - 0.0581686i | 0.9901220 - 0.1744740i | 0.9849480 - 0.2906990i |
| 0.9 | 0.9395680 - 0.0497044i | 0.9362930 - 0.1496610i | 0.9301570 - 0.2511700i |

**TABLE II:** First quasinormal modes to LQBH considering $a_0 = \sqrt{3}/2$ and $l = 3$.

| $P$ | $\omega_0$ | $\omega_1$ | $\omega_2$ |
|-----|-------------|-------------|-------------|
| 0.1 | 0.9472730 - 0.0967026i | 0.9372570 - 0.2922060i | 0.9198500 - 0.4928240i |
| 0.2 | 1.0587100 - 0.0973907i | 1.0507900 - 0.2929440i | 1.0357700 - 0.4904810i |
| 0.3 | 1.1554600 - 0.0956521i | 1.1493700 - 0.2872840i | 1.1375000 - 0.4797910i |
| 0.4 | 1.2344300 - 0.0918630i | 1.2298300 - 0.2768920i | 1.2219900 - 0.4653630i |
| 0.5 | 1.2925000 - 0.0855016i | 1.2892200 - 0.2568970i | 1.2830100 - 0.4293580i |
| 0.6 | 1.3254200 - 0.0774855i | 1.3230500 - 0.2327260i | 1.3185400 - 0.3887160i |
| 0.7 | 1.3285600 - 0.0681213i | 1.3267200 - 0.2044680i | 1.3231200 - 0.3410980i |
| 0.8 | 1.2969100 - 0.0583603i | 1.2949200 - 0.1753403i | 1.2911200 - 0.2930790i |
| 0.9 | 1.2275200 - 0.0496775i | 1.2252700 - 0.1487480i | 1.2205700 - 0.2469960i |

**TABLE III:** First quasinormal modes to LQBH considering $a_0 = \sqrt{3}/2$ and $l = 4$.  

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For a best visualization of effects of the quantum gravity corrections present in the LQG scenario for the black hole quasinormal spectrum, we have shown the behavior of the frequencies by graphs in Figs. 3, 4 and 5 as a function of $n$, for different values of the polymeric function $P$. It has been shown the real and imaginary parts of $\omega$ for the $l = 2, 3$ and 4, by consideration of the following values of the polymeric function: $P = 0.1, 0.2$ and 0.6. In addition, for effect of comparison, it has been plotted together the behavior of the quasinormal spectrum for the Schwarzschild black hole ($P = 0$ and also $a_0 = 0$).

FIG. 3: Graphs for the behavior of the quasinormal modes considering fixed $l = 2$ and varying $P = 0.0, 0.1, 0.2$ and 0.3. In plot (a) is shown the real part of frequencies, while the imaginary part is shown in (b).

FIG. 4: Graphs for the behavior of the quasinormal modes considering fixed $l = 3$ and varying $P = 0.0, 0.1, 0.2$ and 0.3. In plot (a) is shown the real part of frequencies, while the imaginary part is shown in (b).
FIG. 5: Graphs for the behavior of the quasinormal modes considering fixed \( l = 4 \) and varying \( P = 0.0, 0.1, 0.2 \) and 0.3. In plot (a) is shown the real part of frequencies, while the imaginary part is shown in (b).

V. CONCLUDING REMARKS

The gravitational wave observations have opened a new window to gravitational physics research. In this framework, black holes offer a great scenario in order to test the predictions of candidates to quantum gravity theories. With this in mind, we have studied the black hole perturbation and quasinormal mode spectrum in the quantized version of the Schwarzschild black hole, which is known as self-dual black hole and is a Loop Quantum Gravity solution.

In this work, we have considered polar gravitational small perturbations of Eq. (III.2), which linearizes the Einstein equation and provides a Schrödinger-type equation with an effective potential given by Eq. (III.27). Next, by using the WKB approach we found the quasinormal modes given in the tables I, II and III. In Figs. 3, 4 and 5 we have plotted the graphs of the frequency spectrum for better visualization of the quasinormal modes behavior with \( P \) and \( l \) parameters.

From the obtained results, we verify that the quasinormal modes strongly depend on the LQG parameters. For very large masses, the quantum contributions become irrelevant, so in this limit, the Schwarzschild black hole results should be recovered. In fact, this aspect is in accord with the recent gravitational wave observations [21,23], where for the macroscopic black holes the gravitational wave spectrum agrees with General Relativity predictions.

Our results also show that as the parameter \( P \) grows, the real part of the QNMs suffers an initial increase and then starts to decrease, while the magnitude of the imaginary part decreases for a fixed parameter \( a_0 \). This characteristic reveals that the damping of polar perturbations
in the self-dual black hole is slower and the oscillations are faster or slower depending on the value of $P$. Thus, based on these results, we conclude that the self-dual black hole has a stable behavior under polar gravitational perturbations.
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