On Twisted Zeta-Functions at $s = 0$

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1 Introduction

Let $K/k$ be an abelian extension of number fields with Galois group $G$ and for each $\sigma \in G$ let $\zeta(s;\sigma)$ denote the associated partial zeta-function. Letting $\sigma$ vary, we obtain a meromorphic, group-ring-valued function $\Theta_{K/k}: \mathbb{C} \to \mathbb{C}G$ by setting $\Theta_{K/k}(s) := \sum_{\sigma \in G} \zeta(s;\sigma)^{\sigma^{-1}}$ (for more details of this construction, see Section 2). We first note the rationality property satisfied by the value at $s = 0$, namely $\Theta_{K/k}(0) \in \mathbb{Q}G$. This follows from work of Siegel [Si] and Klingen (or of Shintani [Sh, Cor. to Thm. 1]). Next, let $\mu(K)$ denote the group of roots of unity in $K$ and let $\text{ann}_{\mathbb{Z}G}(\mu(K))$ be its annihilator ideal as a module for the group-ring $\mathbb{Z}G$. Deligne-Ribet and Pi. Cassou-Noguès proved the following integrality property concerning $\Theta_{K/k}(0)$ (see [D-R] and [CN]):

$$\text{ann}_{\mathbb{Z}G}(\mu(K))\Theta_{K/k}(0) \subset \mathbb{Z}G \quad (1)$$

In particular, if $w_K = |\mu(K)|$ then $w_K\Theta_{K/k}(0)$ lies in $\mathbb{Z}G$. One can show that $\Theta_{K/k}(0)$ is essentially always zero unless $k$ is totally real and $K$ is totally complex and that in this case $(1 + c)\Theta_{K/k}(0)$ vanishes for every complex conjugation $c \in G$. (We say that $\Theta_{K/k}(0)$ lies in the ‘minus part’ of $\mathbb{Q}G$ w.r.t. such $c$). In fact, one loses little generality in assuming that $K$ is a CM field with whose unique complex conjugation $c$ lies in $G$ (see Remark 3.1(ii)). In this context, the Brumer-Stark Conjecture takes the form of a conjectural generalisation of Stickelberger’s Theorem and may be stated as follows. If $a$ is any fractional ideal of $K$, then $a^{w_K\Theta_{K/k}(0)}$ is a principal ideal generated by some element $a$ in the minus part of $K^\times$ w.r.t. $c$. (One also imposes other conditions on $a$. See [Gr] for more details and an account of recent work demonstrating this conjecture in many cases.)

This paper initiates the study at $s = 0$ of a function $\Phi_{K/k}(s)$ related to $\Theta_{K/k}(s)$ and in which, roughly speaking, the partial zeta-functions are replaced by the twisted zeta-functions introduced and studied in [So2] – [So4] and [R-S] (see ibid. and Section 2). We shall establish what, in the broadest terms, may be described as a ‘rationality property’ and an ‘integrality property’ for $\Phi_{K/k}(0)$ and also formulate a new ‘Stark-type conjecture’ related to it. The
statements of these properties and our conjecture are, however, far from being obvious variants of the corresponding statements for $\Theta_{K/k}(0)$ mentioned above. In particular they are all fundamentally $p$-adic in nature.

Here is a slightly more detailed summary of this paper. Section 2 contains more on the definitions of the functions $\Theta_{K/k}(s)$ and $\Phi_{K/k}(s)$ and gives two relations between them. The first is a ‘functional equation’ relating $\Theta_{K/k}(s)$ to $\Phi_{K/k}(1-s)$ (Theorem 2.1). It means that one could equally well regard this paper as initiating the study of the function $\Theta_{K/k}(s)$ at $s = 1$ (but see Remark 3.1(iv)). Our second relation, Theorem 2.2, is, roughly speaking, a formula linking $\Phi_{K/k}(s)$ at to the functions $\tilde{\Theta}_{K/k}(s)$ as $\tilde{K}/k$ runs through certain sub-extensions of $K/k$.

As in the case of $\Theta_{K/k}(0)$, our study of $\Phi_{K/k}(0)$ loses very little from the assumption that $k$ is totally real which we shall therefore make for the rest of this summary. The ‘rationality property’ mentioned above is Proposition 3.4. In order to state it, we first define a $p$-adic group-ring-valued regulator $R_{K/k,p}$ on the group $U_p(K)$ of $p$-semilocal units of $K$. Then we show that the product $\sqrt{d_k} \Phi_{K/k}(0) * R_{K/k,p}(\theta)$ lies in $\mathbb{Q}_p G$ for any $\theta \in \bigwedge^d_{\mathbb{Z} G} U_p(K)$, where $d_k$ denotes the absolute discriminant of $k$ and other notations will be defined later. If we denote this product by $s_{K/k}(\theta)$ then we have defined for each prime $p$ a map $s_{K/k}$ from $\bigwedge^d_{\mathbb{Z} G} U_p(K)$ to $\mathbb{Q}_p G$. The image $\mathfrak{S}_{K/k}$ of $s_{K/k}$ is a $\mathbb{Z}_p G$ submodule of $\mathbb{Q}_p G$ which is somewhat analogous to the generalised Stickelberger ideal $\text{ann}_{\mathbb{Z} G}(\mu(K)) \Theta_{K/k}(0)$. Unlike $\Theta_{K/k}(s)$ however, $\Phi_{K/k}(s)$ has no ‘trivial zeroes’ at $s = 0$. Consequently $\mathfrak{S}_{K/k}$ has the pleasing property of spanning the entire minus part of $\mathbb{Q}_p G$ (w.r.t all complex conjugations. See Remark 3.3(ii) for more details).

Section 4 contains our main result. This is Theorem 4.1 – the ‘integrality property’ for $\mathfrak{S}_{K/k}$ – which implies in particular that $\mathfrak{S}_{K/k}$ is contained in $\mathbb{Z}_p G$, provided that certain hypotheses are satisfied (principally that $p$ is odd and splits completely in $k$). It is not yet clear to what extent these hypotheses are necessary for the conclusion of Theorem 4.1. They do however figure prominently in its proof which is by far the most substantial in this paper and draws on two different sources. On the one hand it borrows ideas from Coleman’s method in [Cole] for studying the the dual of the image of the local logarithm using one-variable $p$-adic formal power-series. On the other, it uses Shintani’s method (with improvements from Colmez) for generating twisted zeta-values by means of cone decompositions and multivariable formal power series. The algebraic properties of the two sets of power-series marry together very naturally and actually lead to a neat formula for $s_{K/k}(\theta)$ under our hypotheses.

The final section of this paper contains further discussion of integrality questions and three conjectures. The last of these, Conjecture 5.3, is ‘of Stark-type’ and was motivated by the results of [Se2]. Assuming that $p \neq 2$ splits in $k$, that $K$ contains the $p^{n+1}$th roots of unity and that $\mathfrak{S}_{K/k} \subset \mathbb{Z}_p G$, it proposes certain congruences for $s_{K/k}(\theta)$ modulo $p^{n+1}$ in terms of Hilbert symbols and the Rubin-Stark units of $K^+/k$, where $K^+$ denotes the maximal real subfield of $K$.

In addition to those introduced above, the following notations and conventions will be used throughout this paper. All number fields will be considered as finite extensions of $\mathbb{Q}$ within the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. Concerning the ‘base field’ $k$, we shall write
\( \mathcal{O} = \mathcal{O}_k \) for its ring of integers, \( E(k) = \mathcal{O}^\times \) for its unit group and \( S_\infty = S_\infty(k) \) for the set of its infinite places, \( r_1(k) \) of which we shall initially assume to be real and \( r_2(k) \) complex so that \( r_1(k) + 2r_2(k) = d := [k : \mathbb{Q}] \). We also fix once and for all elements \( \tau_1, \ldots, \tau_d \) of \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) extending the \( d \) distinct embeddings \( k \to \bar{\mathbb{Q}} \).

A cycle for \( k \) will be a formal product \( f \mathfrak{j} \) where \( f \) is a non-zero ideal of \( \mathcal{O} \) and \( \mathfrak{j} \) is (the formal product of) a subset of the real places of \( k \). The ray-class group and the ray-class field of \( k \) modulo \( \mathfrak{m} \) will be denoted \( \text{Cl}_m(k) \) and \( k(\mathfrak{m}) \) respectively. If \( G_m \) denotes \( \text{Gal}(k(\mathfrak{m})/k) \) then the Artin isomorphism \( \text{Cl}_m(k) \to G_m \) takes a ray-class \( \mathfrak{c} \) to \( \sigma_{\mathfrak{c}} = \sigma_{\mathfrak{c}, \mathfrak{m}} \). If \( \mathfrak{a} \) is a fractional ideal of \( k \) prime to \( \mathfrak{f} \) then its class in \( \text{Cl}_m(k) \) will be denoted \( [\mathfrak{a}]_m \) and we shall sometimes write \( \sigma_{\mathfrak{a}, \mathfrak{m}} \) in place of \( \sigma_{[\mathfrak{a}]_m, \mathfrak{m}} \). If \( \mathfrak{m} \) is a cycle dividing \( \mathfrak{m} \), then \( k(\mathfrak{m}) \) is contained in \( k(\mathfrak{m}) \) and the restriction homomorphism \( G_m \to G_{\mathfrak{m}} \) corresponds by the Artin maps to the homomorphism \( \text{Cl}_m(k) \to \text{Cl}_{\mathfrak{m}}(k) \) taking \( [\mathfrak{a}]_m \) to \( [\mathfrak{a}]_{\mathfrak{m}} \). We shall write \( \pi_{\mathfrak{m}, \mathfrak{m}} \) for either homomorphism or indeed for the \( R \)-linear extension of \( \pi_{\mathfrak{m}, \mathfrak{m}} \) to a homomorphism of group rings \( RG_m \to RG_{\mathfrak{m}} \) for any commutative ring \( R \).

If \( K \) is any abelian extension of \( k \) we shall, by a slight abuse of notation, write \( S_{\text{ram}}(K/k) \) for the set of finite places of \( k \) which ramify in \( K \) together with \( \textit{all those in } S_\infty \). For any place \( v \) of \( k \), whether finite or infinite, \( D_v(K/k) \) will denote the decomposition subgroup of \( G \) associated to some (hence any) place of \( K \) above \( v \). If \( \bar{K}/k \) is a subextension of \( K/k \) then \( \pi_{K, \bar{K}} \) will denote the restriction \( \text{Gal}(K/k) \to \text{Gal}(\bar{K}/k) \) (linearly extended to group rings where appropriate). We shall write \( \mathfrak{m}(K) = \mathfrak{f}(K)\mathfrak{j}(K) \) for the \textit{conductor} of \( K \) over \( k \), namely the minimal cycle \( \mathfrak{m} \) such that \( K \subset k(\mathfrak{m}) \). The support of \( \mathfrak{m}(K) \) consists of the places ramified in \( K/k \) (finite or infinite) and for any fractional ideal \( \mathfrak{a} \) of \( k \) whose support is disjoint from this set, we write \( \sigma_{\mathfrak{a}, K} \) for the corresponding element of \( G \) under the Artin map, \( \textit{i.e.} \sigma_{\mathfrak{a}, \mathfrak{m}} = \pi_{k(\mathfrak{m}(K))/k}(\sigma_{\mathfrak{a}, \mathfrak{m}(K)}) \).

For each prime number \( p \) we denote by \( \mathbb{C}_p \) the completion of a fixed algebraic closure \( \bar{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \) with respect to the \( p \)-adic metric. We denote by \( | \cdot |_p \) the unique absolute value on \( \mathbb{C}_p \) normalised such that \( |p|^p = p^{-1} \). Finally, for any positive integer \( f \), we shall write \( \mu_f \) for the group of \( f \)-th roots of unity, whether in \( \mathbb{C} \), or in \( \mathbb{C}_p \) for some \( p \).

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## 2 The Functions \( \Theta_{K/k}(s) \) and \( \Phi_{K/k}(s) \)

We first record some basic facts about these functions, referring to \cite{Tate} Ch. IV and to \cite{So2}, \cite{So3}, \cite{So4} for more details on \( \Theta_{K/k}(s) \) and \( \Phi_{K/k}(s) \) respectively. If \( \mathfrak{m} = \mathfrak{f} \mathfrak{j} \) is any cycle for \( k \) and \( \mathfrak{c} \) is any ideal class in \( \text{Cl}_m(k) \), we define the corresponding partial zeta-function of a complex variable \( s \) by the following Dirichlet series (absolutely convergent for \( \text{Re}(s) > 1 \))

\[
\zeta(s; \mathfrak{c}) = \zeta_m(s; \mathfrak{c}) := \sum_a N\mathfrak{a}^{-s}
\]
where the sum runs over the set of integral ideals \( a \) (prime to \( f \)) in \( \mathcal{C} \). This extends to a meromorphic function on \( \mathbb{C} \) having only a simple pole at \( s = 1 \) and we define a \( \mathbb{C}G_m \)-valued meromorphic function \( \Theta_m \) on \( \mathbb{C} \) by setting
\[
\Theta_m(s) := \sum_{c \in \text{Cl}_m(k)} \zeta(s; c)\sigma_{c,m}^{-1} = \prod_{p \mid f} (1 - Np^{-s}\sigma_{p,m}^{-1})^{-1}
\]
(2)
where the Euler product (over primes ideals of \( k \) not dividing \( f \)) converges for \( \text{Re}(s) > 1 \). If \( \tilde{m} = \tilde{f}_3 \) is another cycle with \( \tilde{m} \mid m \) then clearly
\[
\pi_{m,\tilde{m}}(\Theta_m(s)) = \prod_{p \mid f, \ p \nmid \tilde{f}} (1 - Np^{-s}\sigma_{p,m}^{-1})\Theta_{\tilde{m}}(s)
\]
(3)
For any (finite) abelian extension \( K \) of \( k \) with group \( G \) we set
\[
\Theta_{K/k}(s) := \pi_{(m(K)),k}(\Theta_m(K))(s)
\]
(4)
(a meromorphic function of \( s \in \mathbb{C} \) with values in \( \mathbb{C}G \)). We thus have the following expressions intrinsic to \( K \)
\[
\Theta_{K/k}(s) = \prod_{p \notin S_{\text{ram}}(K/k)} (1 - Np^{-s}\sigma_{p,K}^{-1})^{-1} = \sum_{\sigma \in G} \zeta(s; \sigma)\sigma^{-1}
\]
(5)
as in the introduction, where, for \( \text{Re}(s) > 1 \), the partial zeta-function \( \zeta(s; \sigma) = \zeta_{K/k}(s; \sigma) \) equals the sum \( \sum Na^{-s} \) as \( a \) runs through the ideals of \( \mathcal{O} \) with support disjoint from \( S_{\text{ram}}(K/k) \) and such that \( \sigma_{a,K} = \sigma \). If \( K/k \) is a sub-extension of \( K/k \) then clearly
\[
\pi_{K,\tilde{K}}(\Theta_{K/k}(s)) = \prod_{p \in S_{\text{ram}}(K/k)} (1 - Np^{-s}\sigma_{p,K}^{-1})\Theta_{\tilde{K}/k}(s)
\]
(6)
**Remark 2.1** Note that \( \Theta_m \) and \( \Theta_{K/k} \) are the functions \( \Theta_{k(m)/k,\mathcal{S} \cup S_{f}} \) and \( \Theta_{K/k,\mathcal{S} \cup S_{\text{ram}}(K/k)} \) of [Tate, Ch. IV], where \( S_{f} := \{ p : p \mid f \} \). The containment \( S_{f} \supset S_{\text{ram}}(k(m)/k) \) may be strict for a general cycle \( m \) (though not if \( m \) is a conductor) in which case it follows that \( \Theta_m(s) \) is not equal to \( \Theta_{k(m)/k}(s) \) but is obtained from it by multiplying by \( (1 - Np^{-s}\sigma_{p,k(m)}^{-1}) \) for all \( p \in S_{f} \setminus S_{\text{ram}}(k(m)/k) \). In this case \( \zeta_m(s; c) \) is not equal to \( \zeta_{k(m)/k}(s; \sigma_{c,m}) \), for \( c \in \text{Cl}_m(k) \).

For any cycle \( m = f_3 \), we defined in [So2] a finite set denoted \( \mathfrak{W}_m \) and consisting, in brief, of the \( 3 \)-equivalence classes of those additive characters of fractional ideals of \( k \) whose precise \( \mathcal{O} \)-annihilator is \( f \). We equipped \( \mathfrak{W}_m \) with a distinguished element, denoted \( \mathfrak{w}_m^0 \) and a free, transitive action of \( \text{Cl}_m(k) \). For each \( \mathfrak{w} \in \mathfrak{W}_m \) we shall here write simply \( Z(s; \mathfrak{w}) \) for the case \( T = \emptyset \) (the empty set) of the twisted zeta-function \( Z_T(s; \mathfrak{w}) \). We refer to [So2] for the precise definition of this latter Dirichlet series (see also the proofs of Theorem 2.2 and Lemma 4.14 and Example 3.1 for a special case). As in [So2], we set
\[
\Phi_m(s) := \sum_{c \in \text{Cl}_m(k)} Z(s; c \cdot \mathfrak{w}_m^0)\sigma_c^{-1}
\]
(7)
to get a meromorphic, $\mathbb{C}G_m$-valued function $\Phi_m$ with at most a simple pole at $s = 1$. (In fact, $\Phi_m$ is holomorphic if $f \neq \mathcal{O}$). For $m|\mathfrak{m}$, Theorem 3.2 of [So2] gives the following analogue of (3)

$$\pi_{m,\mathfrak{m}}(\Phi_m(s)) = \left(\frac{Nf}{Nf}\right)^{1-s} \prod_{p|f, p\nmid \mathfrak{f}} (1 - Np^{s-1}\sigma^{-1}_{p, \mathfrak{m}})\Phi_{\mathfrak{m}}(s)$$  \hfill (8)

For any abelian extension $K/k$ as above, we now define a meromorphic, $\mathbb{C}G$-valued function by setting

$$\Phi_{K/k}(s) := (|d_k|Nf(K))^{s-1}\pi_{k,(m(K)),K}(\Phi_{m(K)}(s))$$  \hfill (9)

(This agrees with the case $T = 0$ of equation [So4, eq. 4] although there $K$ was assumed to be totally real). Note in particular the factor $(|d_k|Nf(K))^{s-1}$ in this definition which, when combined with (8) gives the following analogue of (3) in the same situation

$$\pi_{K,\hat{K}}(\Phi_{K/k}(s)) = \prod_{p \in \text{Ram}(K/k), p \nmid Nf} (1 - Np^{s-1}\sigma^{-1}_{p, \hat{K}})\Phi_{K/k}(s)$$  \hfill (10)

Unlike $\Theta_m(s)$ and $\Theta_{K/k}(s)$, however, there is no Euler product for $\Phi_m(s)$ or $\Phi_{K/k}(s)$ (but see Thm. 2.3). The coefficient of $\sigma^{-1}$ in $\Phi_{K/k}(s)$ may be denoted $Z_{K/k}(s; \sigma)$ and loosely called ‘a twisted zeta-function of $K/k$’.

Now let $\chi : G \rightarrow \mathbb{C}^\times$ be any (irreducible) character of $G$ which may also be regarded as a character of $G_m(K)$ and hence of $\text{Cl}_m(K)(\hat{K})$. We write $m(\chi) = f(\chi)\hat{\chi}(\chi)$ for the conductor of the extension $K^{\ker(\chi)}/K$ cut out by $\chi$. Then $\chi = \hat{\chi} \circ \pi_{m(K),m(\chi)}$ where $\hat{\chi}$ is the primitive character associated to $\chi$, as defined on $\text{Cl}_m(\chi)(k)$, hence on $G_m(\chi)$ and also on the group of fractional ideals prime to $f(\chi)$. The corresponding $L$-function is $L(s, \hat{\chi})$ which equals $\prod_{p|f(\chi)} (1 - Np^{-s}\hat{\chi}(p))^{-1}$ for $\text{Re}(s) > 1$. Using this notation and extending $\chi$ linearly to $\mathbb{C}G$, equation (5) gives

$$\chi(\Theta_{K/k}(s)) = \prod_{p \nmid f(\chi)} (1 - Np^{-s}\hat{\chi}^{-1}(p))L(s, \hat{\chi}^{-1})$$  \hfill (11)

An analogous equation for $\chi(\Phi_{K/k}(s))$ follows from [So2, Thm. 3.3] and (9) above, namely

$$\chi(\Phi_{K/k}(s)) = g(\hat{\chi})(|d_k|Nf(\chi))^{s-1} \prod_{p \nmid f(K), p \nmid f(\chi)} (1 - Np^{s-1}\hat{\chi}^{-1}(p))L(s, \hat{\chi})$$  \hfill (12)

where $g(\hat{\chi}) = g_m(\chi)(\hat{\chi}) \in \hat{\mathbb{Q}}^\times$ is the Gauss sum attached to $\hat{\chi}$ as a character of $\text{Cl}_m(\chi)(k)$ or $G_m(\chi)$. (For a definition, see [So2] §6.4 or our Remark 2.2(ii)). The last two equations give alternative definitions of $\Theta_{K/k}(s)$ and $\Phi_{K/k}(s)$ respectively since for any $x \in \mathbb{C}G$ we have $x = \sum \chi(x)e_\chi$ where $e_\chi := |G|^{-1}\sum_{g \in G} \chi(g)g^{-1}$ is the idempotent of $\mathbb{C}G$ associated to $\chi$. However, for many purposes the ‘equivariant’ definitions (2) and (11) are more helpful.

The function $\Theta_{K/k}$ – and in particular, its rôle in the Stark Conjectures – is far better known than $\Phi_{K/k}$ so we now give two relations between these functions. For each place
is trivial. We also define a function $C_v : \mathbb{C} \to \mathbb{C} G$ by

$$C_v(s) = \begin{cases} e^{i\pi s} - e^{-i\pi s}c_v = 2i\sin(\pi s) & \text{if } v \text{ is complex} \\ e^{i\pi s/2} + e^{-i\pi s/2}c_v & \text{if } v \text{ is real} \end{cases}$$

**Theorem 2.1** If $K$ is any abelian extension of $k$ then, with the above notations,

$$i^{r_2(k)}\sqrt{|d_k|}\Phi_{K/k}(1 - s) = ((2\pi)^{-s}\Gamma(s))^d \left( \prod_{v \in S_\infty} C_v(s) \right) \Theta_{K/k}(s)$$

as meromorphic, $\mathbb{C} G$-valued functions of $s \in \mathbb{C}$. (Note that $i^{r_2(k)}\sqrt{|d_k|}$ is a square-root of $d_k$).

**Proof.** It suffices to prove the $\chi$-part of this equation for each character $\chi$. By means of equations (11) and (12) we are reduced to showing

$$g(\hat{\chi})(|d_k|N\hat{\chi}(\chi))^{-s}L(1 - s, \hat{\chi})/L(s, \hat{\chi}^{-1}) = ((2\pi)^{-s}\Gamma(s))^d_i^{r_2(k)}\sqrt{|d_k|}^{-1} \prod_{v \in S_\infty} \chi(C_v(s)) \quad (13)$$

This is in fact the functional equation for the $L$-function. To put it into a more familiar form we may use the identities $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$ and $\Gamma(z)\Gamma(z + \frac{1}{2}) = (2\pi)^{z+1/2}2^{1-z}\Gamma(2z)$ to rewrite $\chi(C_v(s))$ as $2\pi i/((\Gamma(s)\Gamma(1 - s))$ if $v$ is complex, as $(2\pi)^{z+1/2}\Gamma(s)^{-1}((\Gamma(\frac{s+1}{2})/\Gamma(1 - \frac{s}{2}))$ if $v$ is real and $v \not\mid \mathfrak{f}(\chi)$ (i.e. $\chi(c_v) = -1$) and as $(2\pi)^{z+1/2}\Gamma(s)^{-1}((\Gamma(\frac{s+1}{2})/\Gamma(1 - \frac{s+1}{2}))$ if $v$ is real and $v \mid \mathfrak{f}(\chi)$ (i.e. $\chi(c_v) = 1$). Equation (13) then becomes

$$g(\hat{\chi})(|d_k|N\hat{\chi}(\chi))^{-s}L(1 - s, \hat{\chi})/L(s, \hat{\chi}^{-1}) = 2^{r_2(k)(1-2s)}\pi^{-d(s-\frac{1}{2})}\sqrt{|d_k|}^{-1} i^q \times (\Gamma(s)/\Gamma(1 - s))^{r_2(k)}((s + 1)/2)/((2 - s)/2))^{q}((s/2)/((1 - s)/2))^{q_1(k)-q}$$

(where $q = |\mathfrak{f}(\chi)| = |\mathfrak{f}(\chi^{-1})|$) and this, after rearranging, is the functional equation in the form of [Latu, Eq. (5)], with ‘$\chi$’ and ‘$\hat{\chi}$’ replaced by our $\hat{\chi}^{-1}$ and $\hat{\chi}$ respectively. (The Gauss sum which would be denoted $F(\hat{\chi}^{-1})$ in [Latu] is our $g(\hat{\chi})$ by [So2, Rem. 6.3].) 

**Remark 2.2** One could of course use Theorem 2.1 to define $\Phi_{K/k}(s)$ in terms of $\Theta_{K/k}(s)$ but the definition given via $\Phi_m(s)$ and the twisted zeta-functions seems a little more natural. In any case, $\Phi_m(s)$ has several important properties which we shall exploit in our study of $\Phi_{K/k}(s)$, especially in Theorem 4.1.

For any abelian group $H$ and commutative ring $R$ we define an involutive automorphism $\star$ of $RH$ by setting $(\sum a_i h)^\star = \sum a_i h^{-1}$. Our second relation expresses $\Phi_m(s)^\star$ in terms of the functions $\Theta_{\mathfrak{g}}(s)$ as $\mathfrak{g}$ runs over the ideals dividing $\mathfrak{f}$. For the rest of this section we
shall use the notation $\mathfrak{n}$ generically to represent the cycle $\mathfrak{g}_{\mathfrak{n}}$ for an ideal $\mathfrak{g}$ dividing $\mathfrak{f}$. Let $\tilde{\nu}_{n,m} : \mathbb{C}\mathfrak{G}_n \to \mathbb{C}\mathfrak{G}_m$ be the linear extension of the map sending an element of $\mathfrak{G}_n$ to the sum of its pre-images under $\pi_{m,n}$. Thus $\nu_{n,m} := |\ker(\pi_{m,n})|^{-1}\tilde{\nu}_{n,m}$ is a ring homomorphism right inverse to $\pi_{m,n}$. As in $[\text{So2}]$, we write $k_n^\times$ for the group of elements of $k$ which are congruent to $1 \mod \mathfrak{n}$ and $E_n$ for $E(k) \cap k_n^\times$. For any fractional ideal $I$ we denote by $T(\mathfrak{g}, I)$ the set of precise $\mathfrak{g}$-torsion classes in the $\mathfrak{O}$-module $k/I$ (in particular $T(\mathfrak{g}, I) \subset \mathfrak{g}^{-1}I/I$).

These sets parametrize $\text{Cl}(k)$ as explained in $[\text{So2}]$ Prop./Def. 6.1. In particular, each $y \in T(\mathfrak{g}, \mathfrak{g}J^{-1})$ gives a well-defined class $[y; J]_n \in \text{Cl}(k)$ lying in the fibre of $\pi_{n,3}$ over $[J]_3$ (which we denote $\pi_{n,3}^{-1}([J]_3)$) and defined by $[y; J]_n := [bJ]_n$ for any $b$ in $y \cap k_3^\times$. We write $e(z)$ for $\exp(2\pi iz)$, $\text{Tr}$ for $\text{Tr}_{k/Q}$ and $\mathcal{D}_k$ for the different of $k/Q$. Thus the map $u \mapsto e(\text{Tr}(u))$ is a well-defined (additive) character of $k/D_k^{-1}$, taking $\mathfrak{g}^{-1}D_k^{-1}/D_k^{-1}$ onto $\mu_f$, where $f \in \mathbb{Z}$ denotes the positive generator of the ideal $\mathfrak{g} \cap \mathbb{Z}$ of $\mathfrak{Z}$.

Let

$$A_n := \sum_{u \in T(\mathfrak{g}, \mathcal{D}_k^{-1})} e(\text{Tr}(u)) \sigma_{[u; \mathfrak{g}\mathcal{D}_k]_n} \in \mathbb{Z}[\mu_f]G_n$$

**Theorem 2.2** With notations and hypotheses as above,

$$\Phi_m(s)^* = \sum_{\mathfrak{g} \mid \mathfrak{f}} \left( \frac{N\mathfrak{f}}{N\mathfrak{g}} \right)^{-s} [E_n : E_m] \tilde{\nu}_{n,m}(A_n\Theta_n(s))$$

$$= \sum_{\mathfrak{g} \mid \mathfrak{f}} \left( \frac{N\mathfrak{f}}{N\mathfrak{g}} \right)^{1-s} \left( \prod_{p \mid \mathfrak{f}, \mathfrak{f} \nmid \mathfrak{g}} (1 - Np^{-1}) \right) \nu_{n,m}(A_n\Theta_n(s))$$

**Remark 2.3**

(i) If $\mathfrak{f} = \emptyset$ then $[-u; \mathfrak{g}\mathcal{D}_k]_n = [u; \mathfrak{g}\mathcal{D}_k]_n$. It follows that the coefficients of $A_n$ are real in this case.

(ii) For any character $\chi : \mathbb{G}_m \to \mathbb{C}^\times$ the Gauss sum $g(\chi) = g_m(\chi)$ equals $\chi^{-1}(A_m)$.

(iii) A rather complicated ‘functional equation for $\Theta$’ might be obtained by combining Theorem 2.2 (with $1 - s$ for $s$) and Theorem 2.1.

**Proof of Thm. 1.2** By meromorphic continuation we can assume $\text{Re}(s) > 1$ so that all the sums below are absolutely convergent. The definition of the twisted zeta-function $Z_{\emptyset}$ in $[\text{So2}]$ together with Lemma 6.1 of *ibid.* with $H = E_m$, $T = \emptyset$ yields the expression

$$\Phi_m(s)^* = [E_\mathfrak{f} : E_m]^{-1} \sum_{J \in \mathcal{J}} \sum_{y \in T(\mathfrak{f}, J^{-1})} \left( \sum_{a \in (f^{-1}D_k^{-1}J \cap k_3^\times)/E_m} e(\text{Tr}(ay)) N(a\mathfrak{f}\mathcal{D}_kJ^{-1})^{-s} \right) \sigma_{[y; J]_m}$$

where $\mathcal{J}$ is any set of fractional ideals representing $\text{Cl}(k)$ and $a$ runs over a set of representatives for the action of $E_m$ on $f^{-1}D_k^{-1}J \cap k_3^\times$. Now, for fixed $y$, the value of $e(\text{Tr}(ay))$ depends only on the class $w$ of $a$ in the $\mathcal{O}$-module $f^{-1}D_k^{-1}J/D_k^{-1}J$. Writing $f^{-1}D_k^{-1}J$ as the
union of such classes $w$ and grouping them according to their $O$-annihilator $g$ (which must divide $f$) we see that the sum over $a$ in the last equation may be written as

$$\sum_{g \mid f} N(g^{-1})^{-s} \sum_{w \in T(g, D_k^{-1})} e(\text{Tr}(wy)) \sum_{(a \in w \cap k, \Delta)/E_m} N(a g D_k J^{-1})^{-s} =$$

$$\sum_{g \mid f} N(g^{-1})^{-s} \sum_{w \in T(g, D_k^{-1})} e(\text{Tr}(wy))[E_n : E_m] \zeta_n(s, [w; g D_k J^{-1}]_n)$$

by part (iv) of So2 Prop./Def. 6.1 applied to $w$ in place of $y$, *mutatis mutandis*. Hence

$$\Phi_m(s)^* = \sum_{g \mid f} N(g^{-1})^{-s}[E_n : E_m]^{-1} \times$$

$$\sum_{J \in J} \left[ \sum_{w \in T(g, D_k^{-1})} \zeta_n(s, [w; g D_k J^{-1}]_n) \sum_{y \in T(f, f J^{-1})} e(\text{Tr}(wy)) \sigma_{[y; J]_m} \right] \quad (16)$$

Now for fixed $g$ and $J$, the reduction map $\delta : J^{-1}/f J^{-1} \to J^{-1}/g J^{-1}$ takes $T(f, f J^{-1})$ onto $T(g, g J^{-1})$. Let us gather up the terms in the last sum of (16) according to the value of $z = \delta(y)$. We note that $e(\text{Tr}(wy)) = e(\text{Tr}(wz))$ and we claim that for each $z \in T(g, g J^{-1}),$

$$\sum_{y \in T(f, f J^{-1})} \sigma_{[y; J]_m} = [E_n : E_m] \tilde{\nu}_{n,m}(\sigma_{[z; J]_n}) \quad (17)$$

(proof deferred). Furthermore, if $z$ is fixed, then as $w$ ranges through $T(g, D_k^{-1})$ so $u := wz$ ranges exactly once through $T(g, D_k^{-1})$ and $[w; g D_k]_n = [w; g D_k J^{-1}]_n [z; J]_n$ (see Props. 6.1 and 6.2 (i) of So2). Therefore the term in square brackets in (16) can be rewritten as

$$\sum_{w \in T(g, D_k^{-1})} \zeta_n(s, [w; g D_k J^{-1}]_n) \sum_{z \in T(g, g J^{-1})} [E_n : E_m] e(\text{Tr}(wz)) \tilde{\nu}_{n,m}(\sigma_{[z; J]_n}) =$$

$$[E_n : E_m] \tilde{\nu}_{n,m} \left( \sum_{z \in T(g, g J^{-1})} \left\{ \sum_{w \in T(g, D_k^{-1})} e(\text{Tr}(w)) \zeta_n(s, [w; g D_k]_n [z; J]_n^{-1}) \right\} \sigma_{[z; J]_n} \right)$$

Notice that the term in braces is simply the coefficient of $\sigma_{[z; J]_n}$ in $A_n \Theta_n(s)$. Note also as $J$ runs through $J$ and $z$ through $T(g, g J^{-1})$ for each $J$, so $\sigma_{[z; J]_n}$ runs exactly $[E_n : E_m]$ times through $G_n$ (by So2 Prop./Def. 6.1) again). Therefore, substituting the R.H.S. of the last equation for the term in square brackets in (16), the factor $[E_n : E_m]^{-1}$ is cancelled, giving (17). Equation (15) follows easily from the latter and the equality $[E_n : E_m]| \ker(\pi_{m,n})| = |(O/f)^x|/(|O/g|^x)$ which follows in turn from the exact sequence

$$1 \to E_n/E_m \to (O/f)^x \to Cl_m(k) \to Cl_n(k) \to 1 \quad (18)$$

and a similar one with $n$ and $g$ in place of $m$ and $f$. It only remains to establish equation (17). But this follows from the commutativity of the square (19) of surjective maps (with $\alpha(y) = [y; J]_m$, $\beta(z) = [z; J]_n$) together with the following lemma.
Lemma 2.1  Given \( z \in \mathcal{T}(g, gJ^{-1}) \) and \( c \in \pi_{m,n}^{-1}([J]_3) \) such that \( \beta(z) = \pi_{m,n}(c) \), there exists precisely one \( E_n \)-orbit of elements \( y \in \mathcal{T}(f, fJ^{-1}) \) such that \( \delta(y) = z \) and \( \alpha(y) = c \). In particular, there are exactly \([E_n : E_m]\) such elements.

This lemma is in turn easily deduced from [So2, Prop./Def. 6.1] with some diagram chasing. □

Now suppose that \( K/k \) is an abelian extension with group \( G \) such that \( m(K) = m \). Then we can use Theorem 2.2 to obtain rather complicated expressions for \( \Phi_{K/k}(s)^* \) in terms of the \( \Theta_{\bar{K}/k}(s) \) as \( \bar{K}/k \) runs over certain sub-extensions of \( K/k \). As an example we record the specialisation at \( s = 0 \) of one such expression which will be useful in the next section and, potentially, for computation. For each \( n|m \) as in Theorem 2.2, we set \( K_n = K \cap k(n) \) and for any \( \bar{K} \) with \( K \supset \bar{K} \supset k \) we define \( \tilde{\nu}_{\bar{K},K} : \mathbb{C}Gal(\bar{K}/k) \to \mathbb{C}G \) in a manner entirely analogous to \( \tilde{\nu}_{n,m} \). It is easy to see that

\[
\pi_k(m) \circ \tilde{\nu}_{n,m} = [k(m) : Kk(n)] \tilde{\nu}_{K[n],K} \circ \pi_k(n)K[n]
\]

as maps from \( \mathbb{C}G_n \) to \( \mathbb{C}G \). Therefore, equations (14) and (9) give

\[
\Phi_{K/k}(0)^* = \frac{1}{|d_k|Nf} \sum_{\ell|f} [E_n : E_m][k(m) : Kk(n)] \tilde{\nu}_{K[n],K}(\pi_k(n)K[n](A_n\Theta_n(0)))
\]

Since \( m(K[n]) \) divides – but is not in general equal to – \( n \), we use (3) and (4) to calculate \( \pi_k(n)K[n](\Theta_n(0)) \) giving

Corollary 2.1  Suppose \( m(K) = m \) then with the above notation (\( m = g_3 \) etc.) we have

\[
\Phi_{K/k}(0)^* = \frac{1}{|d_k|Nf} \sum_{\ell|f} \tilde{\nu}_{K[n],K}(B_n\Theta_{K[n]/k}(0))
\]

where, for each \( g|f \), the element \( B_n \) of \( \mathbb{Z}[\mu_f]\)Gal(\( K[n]/k \)) is given by

\[
B_n = [E_n : E_m][k(m) : Kk(n)]\pi_k(n)K[n](A_n) \prod_{p|m(K[n]) \mid p \not| n} (1 - \sigma_p^{-1})
\]

□
3 The Behaviour of $\Phi_{K/k}(0)$

The main object of interest in this paper is the value of $\Phi_{K/k}(s)$ at $s = 0$, particularly from a $p$-adic viewpoint. We shall suppose to start with that $K/k$ is any abelian extension with notation as in the previous section. For any character $\chi : G \to \mathbb{C}^\times$, equation (12) shows that the order of vanishing of $\chi(\Phi_{K/k}(s))$ at $s = 0$ is the same as that of $L(s, \chi)$. One knows (e.g. by the functional equation) that for $\chi \neq \chi_0$ (the trivial character of $G$) the latter order equals the number of places $v \in S_\infty$ for which $D_v(K/k) \subset \ker(\chi)$ while if $\chi = \chi_0$ then it equals $|S_\infty| - 1$. A first consequence is that $\Phi_{K/k}(0)$ vanishes unless $k$ is either a totally real or an imaginary quadratic field. Moreover, in the latter case it is given as an explicit rational multiple of $\zeta_k(0)e_{\chi_0} = -(h_k/w_k)e_{\chi_0}$ by (12) and is of no great interest in the present context. We shall therefore assume henceforth the following

**Hypothesis 3.1** The base field $k$ is totally real.

This condition implies that $d_k$ is a positive integer. Siegel-Klingen and Shintani’s results mentioned in the introduction, imply that $T_{L/k}(m)$ has rational coefficients for any abelian $L/k$ and any $m \in \mathbb{Z}_{\leq 0}$. It follows from Cor. 2.1 that $\Phi_{K/k}(0)$ lies in $\mathbb{Q}(\mu_{f(K)})G$ where $f(K)$ is the positive generator of $f(K) \cap \mathbb{Z}$ (see also [So3, Lemma 3.3]). Let $e_{K/k}$ denote the idempotent $\prod_{v \in S_\infty} (\frac{1}{2}(1 - c_v))$ of $\mathbb{Q}G$.

**Proposition 3.1** If $k \neq \mathbb{Q}$ then $\Phi_{K/k}(0)$ is a generator of the ideal $e_{K/k} \mathbb{Q}(\mu_{f(K)})G$ of $\mathbb{Q}(\mu_{f(K)})G$. For $k = \mathbb{Q}$ the same is true of $\Phi_{K/\mathbb{Q}}(0) + \frac{1}{2} \prod_{q \mid f(K)} (1 - q^{-1}) e_{\chi_0}$.

**Proof** Except in the case $k = \mathbb{Q}$ and $\chi = \chi_0$, the previous discussion gives the equivalences $\chi(\Phi_{K/k}(0)) \neq 0 \iff \chi(c_v) \neq 1 \forall v \iff \chi(c_v) = -1 \forall v$. The Proposition follows easily, using equation (12) to show that $\chi_0(\Phi_{K/\mathbb{Q}}(0)) = -\frac{1}{2} \prod_{q \mid f(K)} (1 - q^{-1})$.

**Remark 3.1**

(i) Let $H^+$ and $H^-$ be the subgroups of $G$ generated by the sets $\{c_v : v \in S_\infty\}$ and $\{c_v c_{v'} : v, v' \in S_\infty\}$ respectively. Thus $G \supset H^+ \supset H^-$ and the index $|H^+ : H^-|$ is 1 or 2. Now $c_v e_{K/k}^- = e_{K/k}^- \forall v \in S_\infty$ and it follows that $H^-$ fixes $e_{K/k}^-$. Thus, if $H^+ = H^-$ $(\Leftrightarrow 1$ is the product of an odd number of $c_v$’s) then $e_{K/k}^-$ vanishes and so will $\Phi_{K/k}(0)$, by Proposition 3.1 unless $k = \mathbb{Q}$. Thus one loses very little in assuming that $|H^+ : H^-| = 2$. This condition is equivalent to the statement that $K$ contains a $CM$ subfield and so, in particular, is totally complex. Indeed the unique maximal CM subfield is $K^- := K^{H^-}$ and $K^+ := K^{H^+}$ is its maximal real subfield. In this case, one can show that $e_{K/k}^-$ is non-zero. The Proposition therefore implies that $\Phi_{K/k}(0)$ is non-zero and fixed by $H^-$.

The Proposition therefore implies that $\Phi_{K/k}(0)$ is non-zero and fixed by $H^-$. It follows from (10) (with $\tilde{K} = K^-$) that in this case

$$\Phi_{K/k}(0) = |H^-|^{-1} \nu_{K^-, K}(\prod_{\text{primes } p \in S_{\text{ram}}(K/k)} (1 - Np^{-1}\sigma_{p, K^-}^{-1}) \Phi_{K^-/k}(0))$$

(21)

Furthermore, $|H^-|$ divides $2^{d-1}$. For certain purposes, this formula allows one to further reduce to the case where $K$ is a CM field.
(ii) Using equation (11) in place of (12) one can show similarly that Θ_{K/k}(0) = 0 unless k is totally real or k is imaginary quadratic and K/k is unramified. Assuming the former, one finds that Θ_{K/k}(0) lies in e_{K/k}^{-1}QG (unless K = k = Q) and the analogue of (21) holds with Θ in place of Φ and the product replaced by \prod (1 - \sigma_p^{-1}K). Since the latter lies in ZG, the reduction to the CM case is even easier than for Φ_{K/k}(0) (cf. [Tate, Thm. IV.5.2]).

(iii) In general, Θ_{K/k}(s) ‘has more zeroes’ at s = 0 than Φ_{K/k}(s) i.e. Θ_{K/k}(0) may not generate the whole of e_{K/k}^{-1}QG over QG. More precisely, a comparison of equations (11) and (12) shows that \chi(Θ_{K/k}(0)) vanishes not only when \chi(Φ_{K/k}(0)) does but also when D_p(K/k) \subset \ker(\chi) for some finite prime p \in S_{ram}(K/k). This also explains the necessity of using not only Θ_{K/k}(0) but also Θ_{K/k}^{n.t}(0) (for subfields \tilde{K} of K) to express Φ_{K/k}(0), as in e.g. Cor. 2.1.

(iv) On the other hand, Theorem 2.1 gives a simple relation between Φ_{K/k}(0) and Θ_{K/k}(s) at s = 1. To avoid some relatively unimportant complications coming from the pole of the latter, we write Θ_{K/k}^{n.t}(s) for the function (1 − e_{0})Θ_{K/k}(s) which is regular at s = 1. Then Theorem 2.1 gives

$$\sqrt{d}e_{K/k}(1 - e_{0})\Phi_{K/k}(0) = (i/\pi)^de_{K/k}^{-1}\Theta_{K/k}^{n.t}(1)$$

In particular, the coefficients of e_{K/k}^{-1}\Theta_{K/k}^{n.t}(1) are algebraic numbers multiplied by π^d. By contrast, the coefficients of (1 − e_{K/k})Θ_{K/k}^{n.t}(1) are expected to contain other transcendental terms (as well as powers of π) which encode arithmetic information on K. Indeed, if \chi is neither totally odd nor trivial then the Stark Conjectures predict factors in \chi(Θ_{K/k}(1)) coming from logarithms of absolute values of units of K. (See [Tate], particularly Conjecture I.8.2 and Lemme I.8.7).

Every element α of Gal(\bar{Q}/Q) induces an automorphism of Q(μ_{f(k)})G by its action on coefficients. To describe its effect on Φ_{K/k}(0), we let Q^{ab} and k^{ab} denote the maximal abelian extensions of Q and k respectively inside \bar{Q} and we write Ver for the transfer homomorphism from Gal(Q^{ab}/Q) (identified with the abelianisation Gal(\bar{Q}/Q^{ab})) to Gal(k^{ab}/k) (identified with Gal(\bar{Q}/k)^{ab}). If F is any extension of k within k^{ab} we compose Ver with the restriction to get a homomorphism V_F : Gal(Q^{ab}/Q) → Gal(F/k). For α as above, it follows from [SoX Prop. 3.1] that α(Φ_{m(K)}(0)) = V_{k(m(K))}(\alpha(Q^{ab})Φ_{m(K)}(0) (product in Q(μ_{f(K)})G_{m(K)}) on the R.H.S.). Applying (d_kN(\hat{K}))^{−1}\pi_{k(m(K))} to both sides gives

**Proposition 3.2** For each α ∈ Gal(\bar{Q}/Q) we have α(Φ_{K/k}(0)) = V_{K}(\alpha(Q^{ab})Φ_{K/k}(0) (product in Q(μ_{f(K)})G).

We now turn to the p-adic properties of Φ_{K/k}(0) where p is a prime number, fixed throughout this paper. We fix also an embedding j : Q → \bar{Q}_p which extends coefficientwise to a homomorphism of group rings j : QH → \bar{Q}_pH for any group H. In particular, for any abelian extension K of k, j(Φ_{K/k}(0)) is an element of j(Q(μ_{f(K)}))G ⊂ \bar{Q}_p(μ_{f(K)})G.

Let us define the p-adic regulator R_{K/k,p} = R_{K/k,p}^{(j)} mentioned in the introduction. We use the natural topological ring isomorphism to identify Q_p ⊗ Q K with the product \prod_{P_\mathfrak{p}} K_{\mathfrak{p}} of the completions of K at the primes \mathfrak{P} of \mathfrak{O}_K dividing p. This allows us to regard each K_{\mathfrak{p}}
as a subgroup of \((\mathbb{Q}_p \otimes \mathbb{Q} K) \times\) and identifies \((\mathbb{Z}_p \otimes \mathbb{Z} \mathcal{O}_K) \times\) (resp. its Sylow pro-\(p\) subgroup) with the \(p\)-semilocal unit group \(\mathcal{U}_p(K) := \prod_{\mathfrak{P} | p} U(\mathcal{P}_p)\) (resp. with \(\mathcal{U}_p^1(K) := \prod_{\mathfrak{P} | p} U^1(\mathcal{P}_p)\)). Each \(g \in G\) acts by \(1 \otimes g\) on \((\mathbb{Q}_p \otimes \mathbb{Q} K) \times\), sending the subgroup \(K_p^\times\) isomorphically onto \(K_{\mathfrak{P}_p}^\times\) for each \(\mathfrak{P} \mid p\) and making \(\mathcal{U}_p^1(K)\) into a natural \(\mathbb{Z}_p G\)-module.

For each \(i = 1, \ldots, n\), we have \(\mathfrak{P}_i\) for the prime of \(K\) above \(p\) corresponding to the embedding \(j_{\mathfrak{P}_i} := j \circ \tau_i : K \to \mathbb{Q}_p\). namely \(\mathfrak{P}_i = \{a \in \mathcal{O}_K : |j_{\mathfrak{P}_i}(a)|_p < 1\}\) and we write \(p_i\) for \(\mathfrak{P}_i \cap \mathcal{O}\). (Note that \(\{p_1, \ldots, p_d\} = \{p_i\mid p\}\) but in general we may have \(p_{i, i'} = p_{i', i}\), and even \(\mathfrak{P}_i = \mathfrak{P}_{i'}\) for \(i \neq i'\).) The the embedding \(j_{\mathfrak{P}_i}\) extends to a homomorphism \(\mathbb{Q}_p \otimes \mathbb{Q} K \to \mathbb{Q}_p\) which is the composite of the projection onto \(K_{\mathfrak{P}_i}\) with the natural isomorphism \(K_{\mathfrak{P}_i} \to \overline{j_{\mathfrak{P}_i}(K)}\) (topological closure in \(\mathbb{C}_p\)) induced by \(j_{\mathfrak{P}_i}\). Both this extension of \(j_{\mathfrak{P}_i}\) and this isomorphism will, by abuse, also be denoted \(j_{\mathfrak{P}_i}\). We define a \(\mathbb{Z}_p G\)-homomorphism \(\lambda_{i,p} : (\mathbb{Q}_p \otimes \mathbb{Q} K)^\times \to \mathbb{C}_p G\) by setting \(\lambda_{i,p}(a) = \sum_{g \in G} \log_p(j_{\mathfrak{P}_i}(ga))g^{-1}\) where \(\log_p\) denotes Iwasawa’s \(p\)-adic logarithm. Thus \(\lambda_{i,p}\) factors through the projection onto \(\prod_{\mathfrak{P} \mid p} K_{\mathfrak{P}}^\times\) and actually takes values in \(\overline{j_{\mathfrak{P}_i}(K)} G\). Furthermore, its restriction to \(U_p(K)\) also factors through the natural projection of this group onto \(U_p^1(K)\) and its restriction to the latter is \(\mathbb{Z}_p G\)-linear. There is therefore a unique \(\mathbb{Z}_p G\)-homomorphism \(R_{K/k,p}^{(j)} : \bigwedge_{\mathbb{Z}_p G}^d U_p(K) \to \mathbb{Q}_p G\) taking \(u_1 \wedge \ldots \wedge u_d\) to \(\det(\lambda_{i,p}(u_i))^{d}_{i,t=1}\). Moreover \(R_{K/k,p}^{(j)}\) is the composite of the natural map \(\bigwedge_{\mathbb{Z}_p G}^d U_p(K) \to \bigwedge_{\mathbb{Z}_p G}^d U_p^1(K)\) with a \(\mathbb{Z}_p G\)-homomorphism \(\bigwedge_{\mathbb{Z}_p G}^d U^1_p(K) \to \mathbb{Q}_p G\) which is also denoted \(R_{K/k,p}^{(j)}\) and is determined by the same formula.

**Remark 3.2** The precise kernel and image of \(R_{K/k,p}^{(j)}\) are hard to determine but it is not difficult to see that the former is finite and the latter spans \(\mathbb{Q}_p G\) over \(\mathbb{Q}_p\). We sketch a proof. For the image, it obviously suffices to construct \(\theta_0 \in \bigwedge_{\mathbb{Z}_p G}^d U_p^1(K)\) such that \(R_{K/k,p}^{(j)}(\theta_0)\) lies in \(\mathbb{Q}_p G^\times\). Now, by means of the the \(p\)-adic exponential function in each completion \(K_{\mathfrak{P}}\), one can construct \(\text{Exp}_p : p\mathcal{O}_K \to U_p^1(K)\) such that \(\lambda_{i,p}(\text{Exp}_p(x)) = \rho_i(x) := \sum_{g \in G} j_{\mathfrak{P}_i}(gx)g^{-1}\) (a sort of resolvent of \(x\)). Let \(\theta_0 = u_1 \wedge \ldots \wedge u_d\) where \(u_t = \text{Exp}_p(y_t \alpha), \{y_1, \ldots, y_d\}\) is any \(\mathbb{Q}\)-basis of \(k\) and \(\alpha\) generates a normal basis for \(K\) over \(k\) (i.e. \(K = kG\alpha\)), these being chosen so that \(y_t \alpha \in p\mathcal{O}_K \forall t\). Then \(R_{K/k,p}^{(j)}(\theta_0) = \det(\rho_i(y_t \alpha))^{d}_{t=1}\). That this lies in \(\mathbb{Q}_p G^\times\) follows from a simple calculation, the fact that \(\sum_{g \in G} g(\alpha)g^{-1}\) lies in \(KG^\times\), see [FT] Prop. I.4.1 etc. For the kernel, one shows easily that \(\text{Exp}_p\) extends to a \(p\mathcal{O}_K \otimes \mathbb{Z} \mathcal{Z}_p\) and then that the \(u_t\) for \(t = 1, \ldots, d\) span a free \(\mathbb{Z}_p G\)-submodule of rank \(d\) whose index in \(U_p^1(K)\) is finite. The finitude of \(\ker R_{K/k,p}^{(j)}\) now follows from the fact that \(R_{K/k,p}^{(j)}(\theta_0)\) is a unit.

The dependence of \(R_{K/k,p}^{(j)}\) on \(j\) is determined as follows.

**Proposition 3.3** For any \(\alpha \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) and \(\theta \in \bigwedge_{\mathbb{Z}_p G}^d U_p(K)\), we have

\[
(j \circ \alpha)(\sqrt{d_k})R_{K/k,p}^{(j_{\alpha})}(\theta) = \mathcal{V}_K(\alpha|_{\mathbb{Q}^{ab}})j(\sqrt{d_k})R_{K/k,p}^{(j)}(\theta)
\]

Moreover \(j(\sqrt{d_k})R_{K/k,p}^{(j)}(\theta)\) lies in \(\mathbb{Q}_p(\mu_{f(K)}) G\).
Prop. 3.4 which may be deduced from it as follows: take \( x \) actually has coefficients in \( \tau \) and/or ordering of the coset representatives

Propositions 3.2 and 3.3 show that the image of \( s \) is replaced by \( \beta \) or to \( e \) is replaced by \( d \) is run through \( \Phi(\bar{K}/k) \) and is independent of the choice of the embedding \( j \).

Proof Propositions 3.2 and 3.3 show that \( j(\sqrt{d_k\Phi_{K/k}(0)^*}) R^{(j)}_{K/k,p}(\theta) \) is the cycle \( \beta(\sqrt{d_k\Phi_{K/k}(0)^*}) R^{(j)}_{K/k,p}(\theta) = j(\sqrt{d_k\Phi_{K/k}(0)^*}) R^{(j)}_{K/k,p}(\theta) \) by the second statement. Now let \( \beta \) run through \( \text{Gal}(\bar{Q}_p/Q_p) \).

As a consequence, the following makes sense and is independent of the choice of \( j \) above:

Definition 3.1 For \( K/k \) as above, we define \( s_{K/k} = s_{K/k,p} : \bigwedge_{\text{ZG}}^d U_p(K) \to Q_p G \) by setting \( s_{K/k}(\theta) := j(\sqrt{d_k\Phi_{K/k}(0)^*}) R^{(j)}_{K/k,p}(\theta) \) for every \( \theta \in \bigwedge_{\text{ZG}}^d U_p(K) \). We write \( \mathfrak{S}_{K/k} = \mathfrak{S}_{K/k,p} \) for the image of \( s_{K/k} \).

Thus \( s_{K/k} \) is \( \text{ZG} \)-linear and factors through a \( \text{Z}_p G \)-linear map on \( \bigwedge_{\text{ZG}}^d U_p^1(K) \). Consequently its image \( \mathfrak{S}_{K/k} \) is a (finitely generated) \( \text{Z}_p G \)-submodule of \( Q_p G \).

Remark 3.3

(i) Slight variants of the proofs of Props. 3.3 and 3.4 give the following: If \( x_i \in K \) for \( t = 1, \ldots, d \) then \( \sqrt{d_k\Phi_{K/k}(0)^*} \det(\sum_{g \in G} \tau_i g(x_i) g^{-1})_{i,t} \) lies in \( Q G \). This result illuminates Prop. 3.3 which may be deduced from it as follows: take \( x_i \) to be the \( N \)th truncation of the logarithmic series evaluated at an element \( u_i \) of \( K \cap U^1_p(K) \) (which is dense in \( U^1_p(K) \)). Then apply \( j \) to the result and let \( N \to \infty \).

(ii) Proposition 3.1 and Remark 3.2 imply that the \( Q_p \)-span of \( \mathfrak{S}_{K/k} \) is equal to \( e_{K/k} \text{Q}_p G \) or to \( e_{K/k} \text{Q}_p G + Q_p e_{x_0} \) according as \( k \neq Q \) or \( k = Q \). Similarly they determine the kernel of \( s_{K/k} \) up to finite index.

(iii) We note that the maps \( \lambda^{(j)}_{K/k,i,p} \), hence also \( R^{(j)}_{K/k,p} \) and \( s_{K/k} \), depend on the choice and/or ordering of the coset representatives \( \tau_i, \ldots, \tau_d \). However, changing these choices clearly only multiplies these maps by an element of \( G \) and/or \( \pm 1 \) and this does \textit{not} affect \( \mathfrak{S}_{K/k} \) at all.

Example 3.1 The Case \( k = Q \). Suppose \( k = Q \) and \( m \) is the cycle \( f \mathbb{Z}.\mathbb{Z} \) for some integer \( f > 1 \). Thus \( k(m) \) is the field \( Q(\mu_f) \). We may identify \( \text{Cl}_m(Q) \) with \( (\mathbb{Z}/f \mathbb{Z})^\times \) by
\[(a)_m \leftrightarrow \bar{a} \text{ for integers } a > 0 \text{ with } (a, f) = 1. \text{ The Artin map then sends } \bar{a} \text{ to } \sigma_a \in G_m \text{ where } \sigma_a(\zeta) = \zeta^a \text{ for every } \zeta \in \mu_f. \text{ Working through the definitions in this case and setting } \xi_f = e(1/f), \text{ equation (7) becomes}
\[
\Phi_m(s) = \sum_{a=1}^{f} \frac{Z(s; a, f)\sigma_a^{-1}}{(a, f)_1} \text{ where } Z(s; a, f) := \frac{\zeta_{an}}{n^s} \text{ for } \text{Re}(s) > 1
\]

Standard methods of analytic continuation (e.g. a very simple case of [Sh, Prop. 1]) give
\[Z(0; a, f) = \xi_f/(1 - \xi_f).\] For any complex abelian extension \(K\) of \(\mathbb{Q}\) with \(m(K) = m\) it follows that
\[
\Phi_{K/Q}(0) = \frac{1}{f} \sum_{a=1}^{f} \left( \frac{\xi_f}{1 - \xi_f} \right) \sigma_a^{-1}|_K \tag{22}
\]
To simplify, we now suppose that the prime number \(p\) is odd and we take \(K = \mathbb{Q}(\mu_{p^{n+1}})\) for some \(n \geq 0\) in the above. Thus \(m = m(K) = p^{n+1}\mathbb{Z}_{\infty}\) and \(\mathbb{Q}(m)\) equals \(K\) which has a unique, totally ramified prime \(\mathfrak{p} = \mathfrak{P}_1\) above \(p\). If we also take \(\tau_1\) to be the identity then \(j = j\tau_1\) defines an isomorphism from \(K_{\mathfrak{p}}\) to \(j(K) = \mathbb{Q}_p(\mu_{p^{n+1}})\) and by total ramification we have an isomorphism \(G \to \text{Gal}(\mathbb{Q}_p(\mu_{p^{n+1}})/\mathbb{Q}_p)\) sending \(g\) to \(\hat{g}\) such that \(jg(x) = \hat{g}j(x)\) for all \(x \in K_{\mathfrak{p}}\). Thus if \(u\) is any element of \(\bigwedge^1_{\mathbb{Z}_{p}G} U^1_{\mathbb{Q}}(K) = U^1_{\mathbb{Q}}(K) = U^1(K_{\mathfrak{p}})\) then, in this notation,
\[
R^{(j)}_{K/Q, \mathfrak{p}}(u) = \lambda_{1, \mathfrak{p}}^{(j)}(u) = \sum_{b=1}^{p^{n+1}} \log_p(j\sigma_b(u))\sigma_b^{-1} = \sum_{b=1}^{p^{n+1}} \hat{\sigma}_b(\log_p(j(\sigma_b(u))))\sigma_b^{-1}
\]
On the other hand, if we let \(\zeta_n := j(\xi_{p^{n+1}})\) then equation (22) gives
\[
j(\sqrt{d_Q}\Phi_{K/Q}(0)^*) = \frac{1}{p^{n+1}} \sum_{a=1}^{p^{n+1}} \left( \frac{\zeta_n^a}{1 - \zeta_n^a} \right) \sigma_a = \frac{1}{p^{n+1}} \sum_{a=1}^{p^{n+1}} \hat{\sigma}_a \left( \frac{\zeta_n}{1 - \zeta_n} \right) \sigma_a
\]
If we write \(\alpha_c\) for the coefficient of \(\sigma_c^{-1}\) in \(s_{K/Q}(u)\) then the last two equations imply that
\[
\alpha_c = \frac{1}{p^{n+1}} \text{Tr}_{\mathbb{Q}_p(\mu_{p^{n+1}})/\mathbb{Q}_p} \left( \frac{\zeta_n}{1 - \zeta_n} \log_p(j\sigma_c(u)) \right)
\]
where, of course, \(j\sigma_c(u)\) lies in \(U^1(\mathbb{Q}_p(\mu_{p^{n+1}}))\). The right-hand side is familiar from the explicit reciprocity law proved by Artin and Hasse in [A-H] (see also [A-T, Thm. 10, Ch. 12] for the case \(n = 0\)). More precisely, their law states firstly (and implicitly) that
\[
\alpha_c \in \mathbb{Z}_p \quad \text{for all } c \text{ and all } u \in U^1(K_{\mathfrak{p}}) \tag{23}
\]
which is not \textit{a priori} obvious, and secondly that
\[
\zeta_n^{c^*} = ((1 - \zeta_n)^{-1}, j\sigma_c(u))_{p^{n+1}} \quad \text{for all } c \text{ and all } u \in U^1(K_{\mathfrak{p}}) \tag{24}
\]
where \((\cdot, \cdot)_{p^{n+1}} : \mathbb{Q}_p(\mu_{p^{n+1}})^\times \times \mathbb{Q}_p(\mu_{p^{n+1}})^\times \to \mu_{p^{n+1}}\) is the Hilbert symbol with values in \(\mu_{p^{n+1}}\). (Thus \((\alpha, \beta)_{p^{n+1}} = \left(\frac{\alpha}{\beta}\right)\) in the notation of [A-H].) Of course, (24) is simply the statement that \(\mathcal{S}_{K/q}\) is contained in \(\mathbb{Z}_pG\). This motivates the more general question of the \(p\)-integrality of \(\mathcal{S}_{K/k}\) which we shall address in the next section and in Subsection 5.2.

Equation (24) amounts to a congruence for \(s_{K/q}(u)\) modulo \(p^{n+1}\) of which we shall give a conjectural generalisation in Subsection 5.3.

4 Integrality Properties of \(\mathcal{S}_{K/k}\)

We shall investigate an integrality condition on \(\mathcal{S}_{K/k}\) as a \(\mathbb{Z}_pG\)-submodule of \(\mathbb{Q}_pG\). Largely for the sake of simplicity, we shall assume henceforth

Hypothesis 4.1 \(p\) is odd.

and we introduce the notation \(\delta(K/k) = \delta_p(K/k) := |\{i : 1 \leq i \leq d, p_i \not\in S_{\text{ram}}(K/k)\}|\). A preliminary result is

**Proposition 4.1** If \(e_{K/k}^- = 0\) then \(\mathcal{S}_{K/k} \subset p^{\delta(K/k)}\mathbb{Z}_pG\).

If \(k \neq \mathbb{Q}\) then Prop. 3.1 gives \(\Phi_{K/k}(0) = 0\) and the result is trivial. On the other hand, \(\Phi_{K/q}(0) = -\frac{1}{2} \prod_{q \mid f(K)} (1 - q^{-1})e_{\chi_0}\). The result follows from this formula, properties of \(\log_p\) and the fact that \(N_{K/q}(U_p^1(K)) = N_{Q(\mu_{f(K)})/q(U_p^1(Q(\mu_{f(K)})))} \subset 1 + f(K)\mathbb{Z}_p\). We omit the details since Cor 4.1 is much more general.

Equation (11) implies that \(\Theta_{K/k}(0)\) has coefficients in \(w_K^{-1}\mathbb{Z}\). Equation (20) (and the obvious fact that \(w_{K|n}|w_K\) for all \(n\)) therefore implies

\[\Phi_{K/k}(0) \in (w_Kd_kN\mathfrak{f}(K))^{-1}\mathbb{Z}[\mu_{f(K)}]G,\]

(25)

Now suppose that \(p\) is unramified in \(K/\mathbb{Q}\). Then \(p \nmid w_Kd_kN\mathfrak{f}(K)\) (recall: \(p \neq 2\)) so equation (25) implies \(j(\Phi_{K/k}(0))^{\star} \in \bar{Z}_pG\) where \(\bar{Z}_p\) denotes the integral closure of \(Z_p\) in \(\bar{Q}_p\).

Furthermore, in this case, \(j\tau_i(K)\) is an unramified extension of \(Q_p\) for each \(i\). This implies \(\lambda_{i,p}(u) \in p\bar{Z}_pG\) for all \(i\) and \(u \in U_p^1(K)\) and hence \(R_{k/k,p}^{(j)}(\theta) \in \mathbb{Z}_p^d\bar{Z}_pG\) for all \(\theta \in U^d_{ZG} U_p^1(K)\).

From these considerations, we deduce easily

**Proposition 4.2** If \(p\) is unramified in \(K/\mathbb{Q}\) then \(\mathcal{S}_{K/k} \subset p^{\delta(K/k)}\mathbb{Z}_pG\).

Returning to the case in which \(p\) may ramify in \(K/\mathbb{Q}\), the above argument makes it clear that we can expect non-\(p\)-integral coefficients both in \(\Phi_{K/k}(0)^\star\) (coming from \(w_K^{-1}, d_k^{-1}\) and/or \(N\mathfrak{f}(K)^{-1}\)) and in \(R_{K/k,p}^{(j)}(\theta)\) (coming from \(\log_p(x), x \in U_1(j\tau_i(K))\)). The proof of the following result is therefore much more delicate than the above and occupies the rest of this section.
Theorem 4.1 Suppose that $p$ splits completely in $k$ and also that $K$ satisfies at least one of the following two conditions:

there exists $q \in S_{\text{ram}}(K/k)$ not dividing $p$, or \hspace{1cm} (26)

$$p \nmid w_K$$ \hspace{1cm} (27)

Then $\mathfrak{S}_{K/k} \subset p^{\delta(K/k)}\mathbb{Z}_pG$.

Corollary 4.1 If $k = \mathbb{Q}$ then $\mathfrak{S}_{K/\mathbb{Q}} \subset p^{\delta(K/\mathbb{Q})}\mathbb{Z}_pG$. □

Proof of Corollary 4.1 If Condition (27) fails then $\mathbb{Q}(\mu_p) \subset K$. In particular, $K$ is complex so if Condition (26) also fails then $m(K) = p^{n+1}\infty$ for some $n \geq 0$, hence $K \subset \mathbb{Q}(m(K)) = \mathbb{Q}(\mu_{p^{n+1}})$. The last two inclusions – and the minimality of $n$ in the second – imply $K = \mathbb{Q}(\mu_{p^{n+1}})$ and this case was dealt with explicitly in Example 3.1. □

Proof of Theorem 4.1 Since $p$ splits in $k$, the primes $p_1, \ldots, p_d$ are distinct and we can uniquely write

$$m(K) = \mathfrak{f}(K)\mathfrak{f}(K) = \mathfrak{f}(K)p_1^{n_1(K)+1} \cdots p_d^{n_d(K)+1}\mathfrak{f}(K)$$

where $\mathfrak{f}(K)$ is prime to $p$ and $n_i(K) \geq -1$ for $i = 1, \ldots, d$. Condition (26) holds if and only if $\mathfrak{f}(K) \neq \mathcal{O}$. If it fails, we proceed as follows. Lemme IV.1.1 of [Tate] implies that $w_K = \gcd\{Nq - 1 : q \in \mathcal{Q}\}$ where $\mathcal{Q}$ denotes the set of of all prime ideals of $k$ such that $q \nmid pw_K$, $q \not\in S_{\text{ram}}(K/k)$ and $\sigma_{q,K/k} = 1$. By assumption, Condition (27) must hold, so we can find $q_0 \in \mathcal{Q}$ such that $p \nmid Nq_0 - 1$. Now set

$$\mathfrak{f}' = \begin{cases} \mathfrak{f}(K) & \text{if Condition (26) holds} \\ q_0 & \text{if not.} \end{cases}$$

Thus $\mathfrak{f}' \neq \mathcal{O}$ and $(\mathfrak{f}', p) = 1$ in both cases and we define an ideal $\mathfrak{f}$ and a cycle $m$ by

$$m := \mathfrak{f}\infty = \mathfrak{f}'p_1^{n_1+1} \cdots p_d^{n_d+1}\infty$$ \hspace{1cm} (28)

where $n_i = \max\{n_i(K), 0\}$ and $\infty$ denotes the ‘product’ of all the infinite (real) places of $k$. Since $m(K)|m$, we have $k(m) \supset K$. (In fact, it is not hard to see that $m = m(K(\mu_p))$ or $q_0m(K(\mu_p))$ according as Condition (26) holds or fails. In the second case, $m$ is not necessarily a conductor but in either case we actually have $k(m) \supset K(\mu_p)$). Theorem 4.1 is now a consequence of the two following statements.

Claim 4.1 $k(m)/K$ is at most tamely ramified at all primes dividing $p$.

Claim 4.2 $j(N\mathfrak{f}_m(0)^*)R_{k(m)/k,p}(\theta)$ lies in $\bar{\mathbb{Z}}_pG_m$ for every $\theta \in \bigwedge^d_{\mathbb{Z}_pG_m}U_p^{1}(k(m))$. 

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Let us see how Claims 4.1 and 4.2 imply the Theorem. Using equation (8) we find

$$\pi_{k(m),K}(N_f^{-1}\Phi_m(0)^*) = \pi_{k(m),K} \circ \pi_{m,m(K)}(N_f^{-1}\Phi_m(0)^*) = p^{-\delta_p(K/k)}(d) \sqrt{d_K} \Phi_{K/k}(0)^*$$

where $A$ is either $\sqrt{d_K} \prod_{p | p(K)} (p - \sigma_p K/k)$ or this element multiplied by $1 - N_q^{-1}0$ according as Condition (23) holds or fails. In any case it is easy to see that $j(A)$ lies in $\mathbb{Z}_p G^\times$. (The fact that $p$ splits in $k$ implies $j(\sqrt{d_K}) \in \mathbb{Z}_p G^\times$.) Now Claim 4.1 implies that the norm $N_K := N_{k(m)/K}$ defines a surjective map from $U_1^d(k(m))$ to $U_1^d(K)$ and hence a unique surjection from $\bigwedge_{\mathcal{Z}_m} U_1^d(k(m))$ to $\bigwedge_{\mathcal{Z}_m} U_1^d(K)$ - also denoted $N_K$ - sending $u_1 \wedge \ldots u_d$ to $N_K u_1 \wedge \ldots N_K u_d$. It is easy to check that $B_{K/k,p}^{(j)} \circ N_K = \pi_{k(m),K} \circ R_{k(m)/k,p}^{(j)}$. Thus given any $\phi \in \bigwedge_{\mathcal{Z}_m} U_1^d(K)$ we may choose $\theta \in \bigwedge_{\mathcal{Z}_m} U_1^d(k(m))$ such that $\phi = N_K(\theta)$ and we find

$$p^{-\delta_p(K/k)} \phi_{K/k}(\phi) = p^{-\delta_p(K/k)}(\sqrt{d_K} \Phi_{K/k}(0)^*) R_{k/k,p}^{(j)}(\phi) = j(A)^{-1} \pi_{k(m),K} (j(\sqrt{d_K} \Phi_{K/k}(0)^*) R_{k(m)/k,p}^{(j)}(\theta))$$

which lies in $\mathbb{Z}_p G$ by Claim 4.2 and also in $\mathbb{Q}_p G$ hence in $\mathbb{Z}_p G$. This gives the Theorem.

Some preliminary notation and lemmas will establish Claim 4.1 and pave the way for the proof of the harder Claim 4.2. Suppose that $L/F$ is a finite, abelian field extension. If $L$ and $F$ are local fields and $l \geq -1$ is an integer, we write $G(L/F)^l$ for the $l$th ramification group in the ‘upper numbering’ and if $L$ and $F$ are number fields and $q$ a prime of $O_F$, then $G(L/F)^l_q$ will denote the $l$th ramification group at any prime $\mathfrak{Q}$ of $O_L$ above $q$, naturally identified with $G(L_{\mathfrak{Q}}/F_q)^l$. Taking $F$ to be $k$, local and global class field theory give the formula

$$\text{ord}_q(j(L)) = \min\{l \in \mathbb{N} : G(L/k)^l_q = \{1\}\} \quad \text{for all primes } q \text{ of } \mathcal{O}$$

(29)

For any integer $r \geq -1$ we shall use the abbreviation $\mu(r)$ for $\mu_{p^r+1}$, either as a subgroup of $\mathbb{Q}_p^\times$ or of $\mathbb{Q}_p^\times$. Similarly, $\zeta_r$ will denote either $e(1/p^{r+1})$ or its embedding in $\mathbb{Q}_p$ under $j$. It is well known that, for any $l \geq 0$

$$G(\mathbb{Q}_p(\zeta_r)/\mathbb{Q}_p)^l$$

equals Gal(\mathbb{Q}_p(\zeta_r)/\mathbb{Q}_p(\zeta_{l-1})) \text{ if } l \leq r, \{1\} \text{ if not}$$

(30) (see [9, Ch. IV]). It follows easily from the above that $m(k(\zeta_r)) = p^{r+1} \mathcal{O}_{\infty}$. Given any $i = 1, \ldots, d$ and a cycle $n = g^i$ we shall write $n^{(i)}$ for its prime-to-$p_i$ part so that $n = p_i^{r_i+1} n^{(i)}$ where $r_i + 1 = \text{ord}_{p_i}(g)$. We also write $D_{n,i}$ for $D_{p_i}(k(n)/k) = G(k(n)/k)^{-1}_{p_i}$ and $T_{n,i}$ for $T_{p_i}(k(n)/k) = G(k(n)/k)^0_{p_i}$ so that $T_{n,i} \subset D_{n,i} \subset G_n$ and $T_{n,i} = \text{Gal}(k(n)/k(n^{0})) = \ker(\pi_{n,n^{(i)})}$. By mapping the sequence (18) for $n = n$ onto the one for $n^{(i)}$, we see that $T_{n,i}$ is a quotient of $(\mathcal{O}/p_i^{r_i+1} \mathcal{O}_{\infty})$ and since $p_i$ splits in $k$ we obtain

**Lemma 4.1** The inertia group $T_{n,i}$ is a quotient of $(\mathbb{Z}/p^{r_i+1} \mathbb{Z})^\times$ where $r_i + 1 = \text{ord}_{p_i}(n)$. □

We also set $f_{n,i} := |D_{n,i} : T_{n,i}| = |D_{n^{(i)},i}|$ (the residual degree of $k(n)/k$ above $p_i$). Now let $n = \max\{n_1, \ldots, n_d\} = \max\{0, n_1(K), \ldots, n_d(K)\}$ and let $\tilde{m}$ be the cycle $f_p n^{0}\mathcal{O}_{\infty}$, so that $m(K)|m|\tilde{m}$ and $K \subset k(m(K)) \subset k(\tilde{m})$. 

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Lemma 4.2 For each \( i = 1, \ldots, d \) we have \( k(\tilde{m}) = k(\tilde{m}^{(i)})k(\zeta_n) \) and \( k(\tilde{m}^{(i)}) \cap k(\zeta_n) = k \) so that \( \text{Gal}(k(\tilde{m})/k) = T_{\tilde{m},i} \times \text{Gal}(k(\tilde{m})/k(\zeta_n)) \) and the restriction maps \( \text{Gal}(k(\tilde{m})/k(\zeta_n)) \to G_{\tilde{m}^{(i)}} \) and \( T_{\tilde{m},i} \to \text{Gal}(k(\zeta_n)/k) \) are isomorphisms.

**Proof** Since \( m(k(\zeta_n)) = p^{n+1}O_{\infty} \) divides \( \tilde{m} \) it follows that \( k(\zeta_n) \subset k(\tilde{m}) \) and \( k(\zeta_n)/k \) is totally ramified at each \( p_i \) since \( p \) splits in \( k \). Hence the reduction map \( T_{\tilde{m},i} \to \text{Gal}(k(\zeta_n)/k) \) is onto. But the previous lemma shows that \( T_{\tilde{m},i} \) is a quotient of \( (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times \cong \text{Gal}(k(\zeta_n)/k) \) so this map must be an isomorphism. We have already observed that \( T_{\tilde{m},i} = \text{Gal}(k(\tilde{m})/k(\tilde{m}^{(i)})) \) so the result follows. \( \square \)

The lemma shows that the action of \( T_{\tilde{m},i} \) on \( \mu(n) \) defines an isomorphism onto \( (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times \). Its inverse gives rise to a surjection \( \tilde{\sigma}_i : \mathbb{Z}_p^\times \to T_{\tilde{m},i} \) fitting into the exact sequence

\[ 1 \to (1 + p^{n+1}\mathbb{Z}_p) \to \mathbb{Z}_p^\times \xrightarrow{\tilde{\sigma}_i} T_{\tilde{m},i} \to 1 \]

The isomorphism \( T_{\tilde{m},i} \to \text{Gal}(k(\zeta_n)/k) \) respects higher ramification groups in the upper numbering, so \([3]\) implies

**Lemma 4.3** \( \tilde{\sigma}_i(1 + p^l\mathbb{Z}_p) = G(k(\tilde{m})/k)^{1}_{p_i} \) for all \( l \geq 1 \) and \( i = 1, \ldots, d \). In particular, if \( 1 \leq l \leq n \) then the unique maximal subgroup of \( G(k(\tilde{m})/k)^{1}_{p_i} \) equals \( G(k(\tilde{m})/k)^{n+1}_{p_i} \) and is of index \( p \).

**Lemma 4.4** Suppose \( n_i(K) \geq 0 \) for some \( i \) so that \( n_i = n_i(K) \) (by definition). Then

\[ G(k(\tilde{m})/K)^{1}_{p_i} = G(k(\tilde{m})/k(\zeta_n)^{1})_{p_i} = G(k(\tilde{m})/k)^{n_i+1}_{p_i} \quad (31) \]

**Proof** The inclusions \( K \subset k(m(K)) \subset k(\tilde{m}) \) give \( G(k(\tilde{m})/K)^{1}_{p_i} \supset G(k(\tilde{m})/k(m(K)))^{1}_{p_i} \supset G(k(\tilde{m})/k(\zeta_n))^{1}_{p_i} \) (take Sylow \( p \)-subgroups of inertia). Lemma \([4.3]\) shows that \( G(k(\tilde{m})/k(\zeta_n))^{1}_{p_i} \) is a \( p \)-subgroup of index at most \( p^{n_i} \) in \( G(k(\tilde{m})/k)^{1}_{p_i} \), so Lemma \([4.3]\) gives the inclusion \( G(k(\tilde{m})/k)^{1}_{p_i} \supset G(k(\tilde{m})/k)^{n_i+1}_{p_i} \) (which also follows from \( G(k(\tilde{m})/k)^{n_i+1}_{p_i} = \{1\} \)). If the resulting inclusion \( G(k(\tilde{m})/K)^{1}_{p_i} \supset G(k(\tilde{m})/k)^{n_i+1}_{p_i} \) were strict then we would have \( n_i \geq 1 \) and \( G(k(\tilde{m})/K)^{1}_{p_i} \supset G(k(\tilde{m})/k)^{n_i}_{p_i} \), by Lemma \([4.3]\) again. This would imply \( G(K/k)^{n_i}_{p_i} = 1 \) so that \( n_i(K) + 1 = \text{ord}_{p}(m(K)) \) could be at most \( n_i \), by \([2.9]\), contradicting our assumption. We conclude that \( G(k(\tilde{m})/K)^{1}_{p_i} = G(k(\tilde{m})/k)^{n_i+1}_{p_i} \) and the result follows. \( \square \)

**Proof of Claim** \([4.1]\) The equality of the first and the third groups in \([31]\) show that if \( n_i(K) \geq 0 \) then the extension \( k(\tilde{m})/K \) is at most tamely ramified above \( p_i \). On the other hand, if \( n_i(K) = -1 \) then \( n_i = 0 \) so the same holds of the whole extension \( k(m)/k \) (e.g. by Lemma \([4.4]\) with \( n = m \)). \( \square \)

From now on we can forget the fields \( K \) and \( k(m(K)) \) and concentrate on \( k(m) \) and \( k(\tilde{m}) \). Since \( m(k(\zeta_n)) = pO_{\infty} \) which divides \( m \), we have \( k(\zeta_n) \subset k(m) \). In particular, \( p - 1 \) divides \( e_{p_i}(k(m)/k) \) for all \( i \) so Lemma \([4.4]\) implies that the tame ramification indices above \( p \) in \( k(\tilde{m})/k(m) \) are all 1. Thus

\[ \ker(\pi_{\tilde{m},m} : T_{\tilde{m},i} \to T_{m,i}) = G(k(\tilde{m})/k(m))^{0}_{p_i} = G(k(\tilde{m})/k(m))^{1}_{p_i} = G(k(\tilde{m})/k)^{n_i+1}_{p_i} \]
Hence by Lemma 4.3 we may deduce the exactness of the sequence

$$1 \to (1 + p^{n_i+1}\mathbb{Z}_p) \to \mathbb{Z}_p^{\times} \to \pi_{m,i} \to 1$$  \hspace{1cm} (32)

Now Lemma 4.2 implies that for each $i = 1, \ldots, d$ there exists a unique element $\tilde{\phi}_i$ of $D_{m,i}$ which restricts to the Frobenius $\sigma_{p_i, m(i)} \in D_{m(i),i}$ on $k(m(i))$ and fixes $\mu(n)$.

**Lemma 4.5** Fix $i \in \{1, \ldots, d\}$, let $Z$ be any set of representatives of $\mathbb{Z}_p^\times$ modulo $1 + p^{n_i+1}\mathbb{Z}_p$ and let $L$ any set of representatives of $\mathbb{Z}$ modulo $f_{m,i}\mathbb{Z}$. Then the map $Z \times L \to D_{m,i}$ sending $(z, l)$ to $\pi_{m,m}(\tilde{\sigma}_i(z)\tilde{\phi}_i^l)$ is bijective.

**Proof** The exact sequence (32) shows that $|Z \times L| = |T_{m,i}|f_{m,i} = |D_{m,i}|$ so it suffices to prove injectivity. But if $\pi_{m,m}(\sigma_i(z')\tilde{\phi}_i^l) = \pi_{m,m}(\tilde{\sigma}_i(z)\tilde{\phi}_i^l)$ then applying $\pi_{m,m(i)}$, we find $\sigma_{p_i, m(i)} = \sigma_{p_i, m(i)}^l$ so $l' \equiv l \pmod{f_{m,i}}$ so $l = l'$ and $\pi_{m,m}(\sigma_i(z')) = \pi_{m,m}(\tilde{\sigma}_i(z))$ which implies $z' = z$ by (32) again. \hfill \Box

We next introduce some notation and preparatory lemmas in the local situation. We shall write $\mathfrak{p}_{m,i}$ and $\mathfrak{p}_{m,i}$ for those primes above $p_i$ in $k(\tilde{m})$ and $k(m)$ respectively which are induced by $j \tau_i$. We set $M_{m,i} := j \tau_i(k(\tilde{m}))$ which contains $M_{m,i} := j \tau_i(k(m))$, isomorphic via $j \tau_i$ to the completions $k(\mathfrak{p}_{m,i})$ and $k(\mathfrak{p}_{m,i})$, respectively. Lemma 4.2 and the splitting hypothesis show that $M_{m,i} = H_{f_{m,i}}(\zeta_n)$ and where $H_f$ denotes the unique unramified extension of $\mathbb{Q}_p$ of degree $f$ inside $\mathbb{Q}_p$. The group $D_{m,i}$ is isomorphic to $\text{Gal}(M_{m,i}/\mathbb{Q}_p) = \text{Gal}(M_{m,i}/H_{f_{m,i}}) \times \text{Gal}(M_{m,i}/\mathbb{Q}_p(\zeta_n))$ by the map $g \mapsto \tilde{g}$ where $\tilde{g} \circ (j \tau_i) = (j \tau_i) \circ g$ on $k(m)$. Now choose an unramified extension $H$ of $\mathbb{Q}_p$ which contains $H_{f_{m,i}}$ for $i = 1, \ldots, d$. Let $\mu(\infty) := \bigcup_{m \geq 1} \mu(n)$ be the group of all $p$-power roots of unity and write $H(\zeta_{\infty})$ for $H(\mu(\infty))$ etc. Thus $\text{Gal}(H(\zeta_{\infty})/\mathbb{Q}_p) = \text{Gal}(H(\zeta_{\infty})/H) \times \text{Gal}(H(\zeta_{\infty})/\mathbb{Q}_p(\zeta_{\infty}))$. The action on $\mu(\infty)$ defines an isomorphism $\tilde{\sigma} : \mathbb{Z}_p^\times \to \text{Gal}(H(\zeta_{\infty})/H)$ and shall write $\tilde{\phi}$ for the unique lift to $\text{Gal}(H(\zeta_{\infty})/\mathbb{Q}_p(\zeta_{\infty}))$ of the local Frobenius element $\tilde{\phi} \in \text{Gal}(H/\mathbb{Q}_p)$. These notations clearly link up with the global ones as follows

$$\tilde{\sigma}(z)|_{M_{m,i}} = \tilde{\sigma}_i(z)\text{ and } \tilde{\phi}|_{M_{m,i}} = \tilde{\phi}_i \text{ for all } z \in \mathbb{Z}_p^\times \text{ and } i = 1, \ldots, d$$ \hspace{1cm} (33)

**Lemma 4.6** For each $i$ we have inclusions $M_{m,i} \subset H_{f_{m,i}}(\zeta_n) \subset H(\zeta_n)$. Moreover the extension $H(\zeta_n)/M_{m,i}$ is unramified.

**Proof** Since $e(H(\zeta_n)/\mathbb{Q}_p) = (p-1)p^{n_i} = e(M_{m,i}/\mathbb{Q}_p)$ by the exact sequence (32), it suffices to prove the first of the two inclusions. Now $M_{m,i} \subset M_{m,i} = H_{f_{m,i}}(\zeta_n)$ so let $\sigma$ be any element of $\text{Gal}(H_{f_{m,i}}(\zeta_n)/H_{f_{m,i}}(\zeta_n))$. Clearly, $\sigma$ may be lifted to $\tilde{\sigma}(z) \in \text{Gal}(H(\zeta_{\infty})/H)$ for some $z \in 1 + p^{n_i+1}\mathbb{Z}_p$. The sequence (32) shows that $\tilde{\sigma}_i(z)$ fixes $k(m)$ and it follows from (33) that $\tilde{\sigma}(z)$ fixes $M_{m,i}$ i.e. $\sigma \in \text{Gal}(H_{f_{m,i}}(\zeta_n)/M_{m,i})$. The result follows \hfill \Box

**Remark 4.1** There is in general no reason to suppose that $M_{m,i}$ is contained in $H_{f_{m,i}}(\zeta_n)$, even though these two fields have the same residual degree over $\mathbb{Q}_p$. Since their absolute
ramification indices are also equal, such an inclusion would actually imply equality and occurs if and only if $\zeta_{n_i} \in M_{m_i}$.

We define a unique action (denoted ‘·’) of $\text{Gal}(H(\zeta_\infty)/\mathbb{Q}_p)$ on the power series ring $H[[X]]$ by setting

$$\hat{\sigma}(z) \cdot F(X) = F((1 + X)^{z} - 1) \quad \text{and} \quad \hat{\phi} \cdot F(X) = \hat{\phi}(F(X)) \quad \text{for all } z \in \mathbb{Z}_p^* \text{ and } F \in H[[X]]$$

(where $\hat{\phi}(\sum_{m \geq 0} a_i X^i)$ means $\sum_{m \geq 0} \hat{\phi}(a_i) X^i$). For any $\sigma \in \text{Gal}(H(\zeta_\infty)/\mathbb{Q}_p)$ the map $F \mapsto \sigma \cdot F$ is a ring automorphism preserving $O_H[[X]]$. Moreover, if $F$ lies in $O_H[[X]]$ then $F(\zeta - 1)$ converges for all $\zeta \in \mu(\infty)$ and $\sigma(F(\zeta - 1)) = (\sigma \cdot F)(\zeta - 1)$ (which also equals $F(\sigma(\zeta - 1))$ if $\sigma \in \text{Gal}(H(\zeta_\infty)/H)$). The following result, which is a key ingredient in the proof of Claim 4.2, has its origins in Coleman’s proof of [Cole1, Thm. 26].

**Lemma 4.7** Suppose $i \in \{1, \ldots, d\}$ and $\hat{u} \in U^1(M_{m_i})$ are given. Then there exists a power series $g \in H[[X]]$ with the following properties

(i). $g \in O_H[[X]]$,

(ii). $g(0) = 0$,

(iii). $g(\zeta - 1) \in M_{m_i} \forall \zeta \in \mu(n_i)$ and

(iv). $\log_p(\hat{u}) = \sum_{t=0}^{n_i} \hat{\phi}^t(g(\zeta_{n_i-t} - 1))$.

**Proof** Since $H(\zeta_{n_i})/M_{m_i}$ is unramified by Lemma 4.6, we may choose $\tilde{u} \in U^1(H(\zeta_{n_i}))$ such that $\hat{u} = N_{H(\zeta_{n_i})/M_{m_i}}(\tilde{u})$. Since $\zeta_{n_i} - 1$ is a uniformizer for $H(\zeta_{n_i})$, we may clearly choose $\tilde{h} \in 1 + XO_{H[[X]]}$ such that $\hat{u} = \hat{h}(\zeta_{n_i} - 1)$. Choose also a set $\Sigma \subseteq \text{Gal}(H(\zeta_\infty)/\mathbb{Q}_p)$ of representatives modulo $\text{Gal}(H(\zeta_\infty)/M_{m_i})$ and set $h(X) = \prod_{\sigma \in \Sigma} (\sigma \cdot \tilde{h}(X))$. Thus $h \in 1 + XO_{H[[X]]}$ and

$$h(\zeta - 1) = \prod_{\sigma \in \Sigma} (\sigma \cdot \tilde{h}(X)) \in U^1(M_{m_i}) \forall \zeta \in \mu(n_i)$$

In particular, $h(\zeta_{n_i} - 1) = \hat{u}$. Let

$$g = \log(h(X)) - \frac{\hat{\phi}((\log(1 + X)^p - 1))}{p} = \frac{1}{p} \log(h(X)^p/\hat{\phi}(h((1 + X)^p - 1)))$$

where $\log : 1 + XH[[X]] \to H[[X]]$ is the homomorphism sending the power series $a(X)$ to $\sum_{m=1}^{\infty} (-1)^{m-1} a(X) - (1)^m/m$. Since $h(X)^p$ and $\hat{\phi}(h((1 + X)^p - 1))$ lie in the multiplicative group $1 + XO_{H[[X]]}$, so does their quotient and reducing modulo $p$ it is easy to see that the latter actually lies in $1 + pXO_{H[[X]]}$. Applying $\log$, it follows by standard estimates that

$$g(\zeta - 1) = \log_p(h(\zeta - 1)) - \frac{\hat{\phi}((\log_p(h(\zeta^p - 1)))}{p}$$

(35)
(Identities such as \( \log(h)(\zeta_n - 1) = \log_p(h(\zeta_n, 1)) \) etc. may be rigorously treated using e.g. the one-variable analogue of \( \text{[So1] Prop. 5.3} \). Property (iii) follows by combining (35) and (34) as does Property (iv) after a short computation, noting that \( h(0) = 1 \).

**Proof of Claim 4.2** First note that \( U^1_p(k(m)) \) is generated as a \( \mathbb{Z}G_m \)-module by the subgroups \( U^1(k(m))_{\mathfrak{P}_{m,i}} \) for \( i = 1, \ldots, d \) so we only need to treat the case \( \theta = u_1 \wedge \ldots \wedge u_d \) where each \( u_i \) lies in \( U^1(k(m))_{\mathfrak{P}_{m,i}} \) for some map \( s : \{1, \ldots, d\} \rightarrow \{1, \ldots, d\} \). Moreover \( \lambda_i \) vanishes on \( U^1(k(m))_{\mathfrak{P}} \) for \( \mathfrak{P} = \mathfrak{P}_{m,i} \). It follows from the splitting hypothesis that the \( \mathfrak{P}_{m,i} \) are distinct, hence that \( R_{k(m)/k,\mathfrak{P}}^{(j)}(\theta) = 0 \) unless \( s \) is surjective, hence a permutation. Thus we may assume w.l.o.g. that \( u_i \in U^1(k(m))_{\mathfrak{P}_{m,i}} \). For the rest of the proof, we fix such a choice of \( u_i, i = 1, \ldots, d \). It is easy to see that \( R_{k(m)/k,\mathfrak{P}}^{(j)}(u_1 \wedge \ldots \wedge u_d) = \lambda_{1,\mathfrak{P}}(u_1) \ldots \lambda_{d,\mathfrak{P}}(u_d) \) and, moreover, that for each \( i \)

\[
\lambda_{i,\mathfrak{P}}(u_i) = \sum_{g \in G_m} \log_p(j\tau_i(gu_i))^i - 1 = \sum_{\delta \in D_m} \log_p(j\tau_i(\delta u_i))^i \delta^{-1} = \sum_{\delta \in D_m} \tilde{\delta}(\log_p(\hat{u}_i))^i \delta^{-1}
\]

where we set \( \hat{u}_i = j\tau_i(u_i) \in U^1(M_{m,i}) \) and \( \tilde{\delta} \) is any lift of \( \delta \) to \( D_m \). Furthermore, \( N\tilde{\delta}^{-1} = N\tilde{\delta}^{-1} \) where \( N := n_1 + \ldots + n_d + d \) and \( N\tilde{\delta} \in \mathbb{Z}_p^\times \). So, if we define \( a_h \) in \( \mathbb{Q}_p \) (actually in \( \mathbb{Q}_p \)) by

\[
p^{-N} j(\Phi_m(0)^*) \sum_{\delta \in D_m} \tilde{\delta}(\log_p(\hat{u}_i))^i \delta^{-1} \sum_{\delta \in D_m} \tilde{\delta}(\log_p(\hat{u}_d))^i \delta^{-1} = \sum_{g \in G_m} a_h g \tag{36}
\]

then it suffices to show that \( a_h \in \mathbb{Z}_p \) for all \( h \in G_m \). Now the definition (17) gives \( j(\Phi_m(0)^*) = \sum_{g \in G_m} j(Z(0; g \cdot \mathfrak{m}_m^0))g \) where we let \( G_m \) act on \( \mathfrak{m}_m^0 \) via the Artin isomorphism with \( \text{Cl}_m(k) \) (see below). Furthermore, for each \( i \) we can choose a power series \( g_i \in H[[X]] \) satisfying the four properties of Lemma 4.7 with respect to \( \hat{u} = \hat{u}_i \). Substituting for \( \log_p(\hat{u}_i) \) into (36) and expanding gives

\[
a_h = p^{-N} \sum_{\delta \in D_m, i} \left( j(Z(0; \delta_1 \ldots \delta_d \cdot \mathfrak{m}_m^0)) \prod_{i=1}^d \tilde{\delta}_i \left( \sum_{t_i=0}^{n_i} \phi^{t_i} (g_i(\zeta_{n_i-t_i} - 1)) \right) \right) = p^{-N} \sum_{0 \leq t \leq n} a_{h,\mathfrak{P}} \tag{37}
\]

say, where, in an obvious notation, the last sum ranges over all \( t = (t_1, \ldots, t_d) \) such that \( 0 \leq t_i \leq n_i \) \( \forall i \) and

\[
a_{h,\mathfrak{P}} := p^{-\sum_{i=1}^d t_i} \sum_{\delta \in D_m, i} \left( j(Z(0; \delta_1 \ldots \delta_d \cdot \mathfrak{m}_m^0)) \prod_{i=1}^d \tilde{\delta}_i \phi^{t_i} (g_i(\zeta_{n_i-t_i} - 1)) \right)
\]

(which makes sense, by Lemma 4.7(iii)). For each \( i = 1, \ldots, d \) we fix once and for all a set \( Z_i \) of representatives of \( \mathbb{Z}_p^\times \) modulo \( 1 + p^{n_i+1}\mathbb{Z}_p \) and a set \( L_i \) of representatives of \( \mathbb{Z} \) modulo \( f_{m,i}\mathbb{Z} \). If \( t \) is any integer then, as \( t_i \) runs through \( L_i \), so \( t_i - t \) runs through another set of representatives of \( \mathbb{Z} \) modulo \( f_{m,i}\mathbb{Z} \). Hence, by Lemma 4.3 if \( t_i \) is fixed then each element \( \delta_i \)
of $D_{a,i}$ maybe written $\pi_{m,m}(\tilde{\sigma}_i(z_i)\delta_l^{l_i-t_i})$ for some unique $z_i \in Z_i$ and $l_i \in L_i$. Thus, taking $\tilde{\sigma}_i$ to be $\tilde{\sigma}_i(z_i)\delta_l^{l_i-t_i}$ and applying (38) we get

$$a_{h,i} := p^{-l_1-\ldots-l_d} \sum_{z_i \in Z_i} \sum_{l_i \in L_i} \left( j(Z(0); \prod_{i=1}^d \pi_{m,m}(\tilde{\sigma}_i(z_i)\delta_l^{l_i-t_i})) h \cdot w_0^m \right) \times \prod_{i=1}^d \tilde{\sigma}_i(z_i)\delta_l^{l_i} (g_i(\zeta_{n_i-t_i}^{-1})) \right)$$

(38)

Our aim is to rewrite this expression for $a_{h,i}$ in terms of certain $d$-variable power series with coefficients in a finite extension of $H$. This requires an interlude during which we recall some facts about $W_m$ and give some more auxiliary results: An element of $W_m$ is an equivalence class $\{\xi, I\}_{m}$ where the pair $(\xi, I)$ consists of a fractional ideal $I$ of $k$ and a character of finite order $\xi : I \to \mathbb{C}^\times$ such that $\text{ann}_\mathcal{O}(\xi) := \{x \in \mathcal{O} : \xi(xI) = 0\}$ equals $f$. (See [So2] for more details here and in the following). The equivalence relation is given (for $j = \infty$) by

$$\{\xi, I\}_{m} \equiv \{\xi', I'\}_{m} \iff \exists c \in k_\infty^\times \text{ such that } I = cI' \text{ and } \xi(c\alpha') = \xi'(\alpha') \forall \alpha' \in I'$$

(39)

(recall that $k_\infty^\times$ is the group of totally positive elements of $k$). Suppose $w \in W_m$ is represented by $(\xi, I)$ (i.e. $w = \{\xi, I\}_{m}$) and $g = \sigma_{a,m} \in G_m$ for some integral ideal $a$ of $\mathcal{O}$ prime to $f$. Then our definitions and those of [So2] give

$$g \cdot w = [a]_m \cdot w := \{\xi|_{a\alpha}, aI\}_{m}$$

One checks that this depends only on the class $[a]_m \in Cl_m(k)$ and gives a well-defined action of $G_m$, via $Cl_m(k)$, on $W_m$. This action turns out to be free and transitive; moreover $W_m$ contains a distinguished element $w_0^m$ (see [So2]).

Now suppose that we are given a fractional ideal $J$ of $k$. For simplicity we shall assume that $J$ is prime to $p$ which incurs no real loss of generality in the sequel. Let $\rho : J \to \mathbb{C}^\times$ be a character of finite order with $\text{ann}_\mathcal{O}(\rho) = f'$. This means that the image of $\rho$ is precisely $\mu_{f'}$ where, as previously, $f' \in \mathbb{Z}$ denotes the positive generator of $f' \cap \mathbb{Z}$. But $p \nmid f'$ so $\mu_{f'}$ is uniquely $p$-divisible and we may extend $\rho$ uniquely to a character

$$\rho : \mathbb{Z}[1/p]J = \bigcup_{m \in \mathbb{Z}^d} p^mJ \longrightarrow \mu_{f'} \subset \mathbb{C}^\times$$

where, for any $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, the shorthand $\langle p^m \rangle$ denotes the fractional ideal $p_1^{m_1} \ldots p_d^{m_d}$. For any such $m$ and any $s = (s_1, \ldots, s_d) \in \mathbb{Z}^d$, we define a character $\xi_{\rho, s, m} : p^m J \to \mathbb{C}^\times$ by

$$\xi_{\rho, s, m}(a) := \rho(a)\zeta_{m_1}^{\sum_{i=1}^d s_i p^{-m_i j_{\tau_i}(a)}}$$

(40)

(The assumptions $a \in p^m J$ and $(J, p) = 1$ imply $p^{-m_i j_{\tau_i}(a)} \in \mathbb{Z}_p$ for all $i$). Let $S(a)$ denote the set $\{s \in \mathbb{Z}^d : \text{ord}_{p}(s_i) = n - n_i, \forall i\}$. By writing characters as the products of their $p$- and prime-to-$p$-part, the reader should have no difficulty in proving the following
Lemma 4.8 Suppose \( J \) and \( m \in \mathbb{Z}^d \) are as above. For any \( \rho : J \to \mathbb{C}^\times \) with \( \text{ann}_\mathcal{O}(\rho) = f' \) and \( z \in \mathcal{S}(\mathfrak{m}) \) we have \( \text{ann}_\mathcal{O}(\xi_{\rho,z,m}) = f \). Conversely, if \( \xi : \mathfrak{m}mJ \to \mathbb{C}^\times \) is any character with \( \text{ann}_\mathcal{O}(\xi) = f \) then \( \xi = \xi_{\rho,z,m} \) for some \( \rho : J \to \mathbb{C}^\times \) with \( \text{ann}_\mathcal{O}(\rho) = f' \) and \( z \in \mathcal{S}(\mathfrak{m}) \). Moreover, \( \rho \) is unique and \( z \) is unique modulo \( p^n\mathbb{Z}_p \).

Given \( \rho, J, m \) and \( z \) as in the Lemma, we write \( \mathfrak{w}_{\rho,z,m} \) for the class \( \{\xi_{\rho,z,m}, \mathfrak{m}mJ\} \in \mathfrak{M}_m \).

Lemma 4.9 Suppose given \( \rho : J \to \mathbb{C}^\times \) with \( \text{ann}_\mathcal{O}(\rho) = f' \), \( m \in \mathbb{Z}^d \) and \( z \in \mathcal{S}(\mathfrak{m}) \). Then for any \( l = (l_1, \ldots, l_d) \in \mathbb{Z}^d \) and \( z = (z_1, \ldots, z_d) \in (\mathbb{Z}_p^\times)^d \), we have

\[
\prod_{i=1}^d \pi_{m,m}(\tilde{\sigma}_i(z_i)\tilde{\phi}_i) \cdot \mathfrak{w}_{\rho,z,m} = \mathfrak{w}_{\rho,z+\mathfrak{l}+m}
\]

where \( z+ \) denotes \((z_1s_1, \ldots, z_ds_d) \in \mathcal{S}(\mathfrak{m}) \).

Proof Since \( \cdot \) is an action, it suffices to treat the case \( l = (0, \ldots, 0, 1, 0, \ldots, 0) \) ('1' in the \( i \)th position) and \( z = (1, \ldots, 1, z_1, \ldots, 1) \) \((z \in \mathbb{Z}_p^\times \) in the \( i \)th position) for some \( i \). To simplify notation, we shall further assume \( i = 1 \) (the other \( i \) being treated identically). In this case, we choose an integral ideal \( \mathfrak{a} \) of \( \rho \) such that \( \sigma_{a,m} = \tilde{\sigma}_1(z)\tilde{\phi}_1 \in D_{m,1} \subset G_m \) so that \( \pi_{m,m}(\tilde{\sigma}_1(z)\tilde{\phi}_1) = \sigma_{a,m} \). The definitions of \( \tilde{\phi}_1 \) and \( \tilde{\sigma}_1(z) \) mean that

\[
a = xp_1 \text{ for some } x \in k^\times \text{ with } x \equiv 1 \mod \mathfrak{m}^{(1)}
\]  

and \( \sigma_{a,m} \) acts on \( \mu(n) \) by \( \zeta \mapsto \zeta^z \). By explicit class field theory over \( \mathbb{Q} \), this latter condition translates as

\[
Na \equiv z \mod p^{n+1}\mathbb{Z}_p
\]

We need to show that \( \{\xi_{\rho,z,m}|_{\mathfrak{a}mJ}, \mathfrak{a}mJ\}_m = \{\xi_{\rho,z,m'}, \mathfrak{a}m'J\}_m \) where \( s' := (zs_1, s_2, \ldots, s_d) \) and \( m' = (1 + m_1, m_2, \ldots, m_d) \) so that \( p^{m'}J = p\mathfrak{a}mJ \). Now \( xp^{m'}J = \mathfrak{a}mJ \subset p^{m'}J \) so we may define \( \xi' : p^{m'}J \to \mathbb{C}^\times \) by

\[
\xi'(a) = \xi_{\rho,z,m}(xa) \text{ for all } a \in \mathfrak{a}mJ
\]

But \( (11) \) implies \( x \in k^\times \), so \( (39) \) gives \( \{\xi_{\rho,z,m}|_{\mathfrak{a}mJ}, \mathfrak{a}mJ\}_m = \{\xi', p^{m'}J\}_m \). It therefore suffices to show that \( \xi' = \xi_{\rho,z,m'} \), i.e. that

\[
\rho(xa)\zeta_{\sum_{i=1}^d s_ip^{-m_1j_1r_1}(a)} = \rho(a)\zeta_{\sum_{i=1}^d s_ip^{-m_1j_1r_1}(a) + \sum_{i=2}^d s_ip^{-m_1j_1r_1}(a)} \quad \forall a \in \mathfrak{a}m'J
\]

Now \( px \in \mathfrak{a} \subset \mathcal{O} \) and \( (11) \) implies \( px \equiv p \mod f' \) so that \( \rho(xa)^p = \rho(pxa) = \rho(p) = \rho(a)^p \) in \( \mu_{f'} \), hence

\[
\rho(xa) = \rho(a)
\]

equation \( (41) \) also says that if \( i \geq 2 \) then \( \text{ord}_p(x - 1) \geq n + 1 \) so that

\[
j_1r_i(x) \in 1 + p^{n+1}\mathbb{Z}_p \quad \forall i \geq 2
\]

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and since $x \in k^X$ equation (43) gives
\[ z \equiv p|N_{k/q}x| = pN_{k/q}x = pj_1(x)j_2(x) \ldots j_{d}(x) \equiv pj_1(x) \mod p^{n+1}\mathbb{Z}_p \tag{46} \]
eq \text{ onto the subspace of } C = (C = \text{C}) \text{ should be the identity}. \text{ For any } c \in \mathbb{R}_0^+ \text{ we define the norm } || \cdot ||_c \text{ by setting}
\[ ||F||_c := \sup \{ |a_m|_pc^{m_1+\ldots+m_d} : m \geq 0 \} \in \mathbb{R}_0^+ \cup \{ \infty \} \]
Then the set $B(n) := \{ F \in C_p[[X]] : ||F||_c < \infty \text{ for some } c > p^{-1/(p-1)p^n} \}$ is precisely the $C_p$-algebra of those power series converging and bounded on some ‘open’ polydisc about $0 \in \mathbb{C}^d_p$ containing $\mu(n)^d$. The latter group acts on $B(n)$ by setting
\[ \zeta \cdot F(X) := F(\zeta^{(1)}(X_1+1) - 1, \ldots, \zeta^{(d)}(X_d+1) - 1) \forall \zeta = (\zeta^{(1)}, \ldots, \zeta^{(d)}) \in \mu(n)^d, F \in B(n) \]
(A rigorous treatment and discussion of the case $d = 2$ is given in [S61, Sec. 3.2] and the case of general $d$ may be treated in the same way, \textit{mutatis mutandis}). For any $\underline{r} = (r_1, \ldots, r_d) \in \mathbb{Z}_0^d$ such that $r_i \leq n + 1 \forall i$ we write $\mu(\underline{r} - 1)$ for the subgroup $\mu(r_1 - 1) \times \ldots \times \mu(r_d - 1)$ of $\mu(n)^d$ and define a $C_p$-linear projection operator $V_{\underline{r}}$ onto the subspace of $B(n)$ fixed by $\mu(\underline{r} - 1)$ by acting with the corresponding idempotent:
\[ V_{\underline{r}} : B(n) \longrightarrow B(n) \]
\[ F(X) \longmapsto \frac{1}{p^{r_1+\ldots+r_d}} \sum_{\zeta \in \mu(\underline{r} - 1)} \zeta \cdot F(X) \]
(The notation has been chosen in order that $V_{\underline{0}}$ should be the identity). For any $\underline{z} = (z_1, \ldots, z_d) \in \mathbb{Q}_p^d$ we shall use the abbreviation $(1 + X)^{\underline{z}}$ for the product of the binomial series $(1 + X_1)^{z_1} \ldots (1 + X_d)^{z_d} \in \mathbb{Q}_p[[X]]$ which lies in $\mathbb{Z}_p[[X]]$ (hence in $B(n)$) if $\underline{z} \in \mathbb{Z}_p^d$.

\textbf{Lemma 4.10}

(i) If $\underline{z} \in \mathbb{Z}_p$ then
\[ V_{\underline{z}}(1 + X)^{\underline{z}} = \begin{cases} (1 + X)^{\underline{z}} & \text{if } p^{r_i}|z_i \text{ for all } i \text{ and} \\ 0 & \text{otherwise} \end{cases} \]

(ii) If $M$ is any (topologically) closed subfield of $\mathbb{C}_p$ with ring of valuation integers $\mathcal{O}_M$ then $V_{\underline{z}}\mathcal{O}_M[[X]] \subset \mathcal{O}_M[[X]]$.

\textbf{Proof} The group $\mu(n)^d$ acts on $(1 + X)^{\underline{z}}$ by multiplication by the character sending $\zeta$ to $\zeta^{(1)z_1} \ldots \zeta^{(d)z_d}$. This character is trivial on $\mu(\underline{r} - 1)$ if and only if $p^{r_i}|z_i$ for all $i$. Part (i) follows as does the statement of part (ii) with $\mathcal{O}_N[[X]]$ replaced by $\mathcal{O}_N[X]$ (since the latter
is clearly spanned over $\mathcal{O}_n$ by $\{(1 + X)^m : m \in \mathbb{Z}^d_0\}$. Now if $F$ lies in $\mathcal{O}_n[\{X]\}$ then $\|F\|_c < \infty$ for any $c$ with $1 > c > p^{-1/(p-1)p^n}$. Moreover, for such $c$ we can find a sequence of polynomials $F_l \in \mathcal{O}_n[X]$ tending to $F$ w.r.t. $\|\cdot\|_c$ as $l \to \infty$ (e.g. a sequence of truncations of $F$). On the other hand, the $\mu(n)^d$ action preserves the norm $\|\cdot\|_c$ (see e.g. [10] Sec. 3.2) from which it follows that $V\mathcal{H}$ is $\|\cdot\|_c$-continuous. Thus $V\mathcal{H}F_l$ tends to $V\mathcal{H}F$ with respect to $\|\cdot\|_c$, hence also coefficientwise. But $V\mathcal{H}F_l$ lies in $\mathcal{O}_n[X]$ for all $l$ so $V\mathcal{H}F$ lies in $\mathcal{O}_n[\{X]\}$.

The following Proposition is crucial to the proof of Claim 4.2 and of some interest in its own right. For ease of reading we shall defer its proof to the end of this section.

**Proposition 4.3** Suppose given $\rho : J \to \mathbb{C}^\times$ with $\operatorname{ann}_\mathcal{O}(\rho) = \mathfrak{f}$ as in Lemma 4.9. Then there exists a family $\{F_m : m \in \mathbb{Z}^d\}$ of $d$-variable, formal $p$-adic power series (depending on $\rho$ and $j$) such that

(i). $F_m(X)$ lies in $\mathbb{Z}_p[\mu_p]\{X\}$ for all $m \in \mathbb{Z}^d$,

(ii). $V\mathcal{H}F_m(X) = F_{m+s}(1 + X_1)^{p^{s_1}} - 1, \ldots, (1 + X_d)^{p^{s_d}} - 1)$ for all $m \in \mathbb{Z}^d$ and $s = (s_1, \ldots, s_d) \in \mathbb{Z}^d_{\geq 0}$ with $s_i \leq n + 1 \forall i$, and

(iii). $j(Z(0; w_{\rho,s,m})) = F_m(\zeta_n^{s_1} - 1, \ldots, \zeta_n^{s_d} - 1)$ for all $m \in \mathbb{Z}^d$ and $s \in S(n)$.

Returning now to equation (38) we choose a fractional ideal $J$ of $k$ and a character of finite order $\xi : J \to \mathbb{C}^\times$ with $\operatorname{ann}_\mathcal{O}(\xi) = \mathfrak{f}$ and such that $h \cdot w_m = (\xi, J)_m$ in $\mathfrak{W}_m$. Weak approximation and the equivalence relation (39) allow us to assume that $J$ is prime to $p$ so that, by Lemma 4.8 we have $\xi = \xi_{p,s,0}$ for some $\rho : J \to \mathbb{C}^\times$ with $\operatorname{ann}_\mathcal{O}(\rho) = \mathfrak{f}$ and $s \in S(n)$. Thus $h \cdot w_m = w_{\rho,s,0}$. Let $\{F_m : m \in \mathbb{Z}^d\}$ be a family of power series satisfying in Proposition 4.3 w.r.t. $J$ and $\rho$. Using also Lemma 4.9 etc., equation (38) becomes

$$a_{h,\mathfrak{L}} = p^{-t_1 - \ldots - t_d} \sum_{z_i \in \mathfrak{L}_i \atop i = 1, \ldots, d} \left( j(Z(0; w_{\rho,s,z_i})) \prod_{i=1}^d \hat{\sigma}_i(z_i) \hat{\phi}_i(g_i(\zeta_n^{s_i} - 1)) \right)$$

$$= p^{-t_1 - \ldots - t_d} \sum_{z_i \in \mathfrak{L}_i \atop i = 1, \ldots, d} \left( F_{z_i}(\zeta_n^{s_i} - 1, \ldots, \zeta_n^{s_d} - 1) \prod_{i=1}^d \hat{\phi}_i(g_i(\zeta_n^{z_i} - 1)) \right)$$

Since $t_i$ is strictly less than $n_i + 1$ it follows that, for each $i$, $\mu(t_i - 1)$ acts (multiplicatively and freely) on the difference set $\mu(n_i) \setminus \mu(n_i - 1)$ of primitive $p^{n_i+1}$th roots of unity and we choose a set $\Gamma(n_i, t_i)$ of orbit representatives for this action for each $i$. Fixing $i$ we see that $\zeta_n^{z_i}$ lies in $\mu(n_i) \setminus \mu(n_i - 1)$ for every $z_i \in \mathfrak{L}_i$ and so may be written

$$\zeta_n^{z_i} = \gamma_i \nu_i$$

for some unique $\gamma_i \in \Gamma(n_i, t_i)$ and $\nu_i \in \mu(t_i - 1)$ (depending on $z_i$)

Thus $\zeta_n^{z_i} = \zeta_n^{z_i'} = \gamma_i^{s_i'} t_i^{s_i'}$ where we write $s_i'$ for $p^{n_i - n_i} s_i' \in \mathbb{Z}_p$. Moreover, as $z_i$ runs once through $\mathfrak{L}_i$, so $\zeta_n^{z_i}$ runs once through $\mu(n_i) \setminus \mu(n_i - 1)$, hence the pair $(\gamma_i, \nu_i)$ runs once through $\Gamma(n_i, t_i) \times \mu(t_i - 1)$, and the same must obviously be true of the pair $(\gamma_i, \nu_i'')$. On
the other hand \( \zeta_{n_i-t_i}^i = (\zeta_{11}^i)^{p_i} = \gamma_i^{p_i} \) which is independent of \( \nu_i \) and clearly runs exactly once through \( \mu(n_i - t_i) \setminus \mu(n_i - t_i - 1) \) as \( \gamma_i \) runs through \( \Gamma(n_i, t_i) \). Putting this together and using Proposition 4.3(ii) the last equation gives

\[
\sum_{t_i \in L_i} \sum_{\gamma_i \in \Gamma(n_i, t_i)} \sum_{\nu_i \in \mu(n_i-t_i-1)} \left( p^{-t_1-\cdots-t_d} F_{k-2}(\gamma_1^{s_1} \nu_1^{s_1} - 1, \ldots, \gamma_d^{s_d} \nu_d^{s_d} - 1) \prod_{i=1}^d \phi_i(g_i(\gamma_i^{p_i} - 1)) \right)
\]

Substituting this equation into (37) and noting that \( g_i(0) = 0 \) (by Lemma 4.7(ii)) we get

\[
a_h = p^{-N} \sum_{\zeta^{(i)} \in \mu(n_i)} \sum_{t_i \in L_i} \left( F_{k}(\zeta^{(1)} s_1^i - 1, \ldots, \zeta^{(d)} s_d^i - 1) \prod_{i=1}^d \phi_i(g_i(\zeta^{(i)} - 1)) \right)
\]

where we have set \( n + 1 := (n_1 + 1, \ldots, n_d + 1) \) and

\[
F(X) := \sum_{t_i \in L_i} \left( F_{k}(1 + X_1 s_1^i - 1, \ldots, 1 + X_d s_d^i - 1) \prod_{i=1}^d \phi_i \cdot g_i(X_i) \right)
\]

The power series \( (1 + X_i)^{s_i^i} - 1 \) has coefficients in \( \mathbb{Z}_p \) for all \( i \) so it follows from Lemma 4.7(i) and Proposition 4.3(iii) that \( F(X) \) lies in \( \mathcal{O}_H[\mu_{f'}][[X]] \). As well as \( j, \rho \) and \( s \), the power series \( F \) depends on the choices of the \( L_i \), the \( F_{k}(X) \) for \( l \in L_1 \times \ldots \times L_d \) and the \( g_i \). Consequently \( a_h \) lies in \( \mathcal{O}_H[\mu_{f'}] \subset \mathbb{Z}_p \) (hence actually in \( \mathbb{Z}_p \)) by Lemma 4.10(ii). Thus Claim 4.2 will be established once we have proved Proposition 4.3.

We shall shortly see that the existence of the power series \( F_{m}(X) \) appearing in Proposition 4.3 is a very natural consequence of the method of Shintani [Sh] (with refinements from [Colm]) for evaluating Dirichlet series like \( Z(s; \mathfrak{m}_{F_{m}}) \) by means of cone decompositions of fundamental domains for the action of units on \( k^\times \). Such decompositions are best
visualised by using the embedding \( \tau : k \to \mathbb{R}^d \) where \( \tau(a) \) is defined to be \((\tau_1(a), \ldots, \tau_d(a))\). This extends to an isomorphism \( k \otimes \mathbb{Q} \mathbb{R} \cong \mathbb{R}^d \) and sends \( \tau(k_{\infty}) \) into \( \mathbb{R}^d_0 \). Similarly, we define an embedding \( j\tau : k \to \mathbb{Q}_p^d \) by \( j\tau(a) \) for any \( m \in \mathbb{Z}^d \) we shall write \( \tau(mj\tau(a)) \) for the \( \tau \)-tuple \((\tau_1(a), \ldots, \tau_d(a))\) (which extends to an isomorphism \( k \otimes \mathbb{Q}_p \cong \mathbb{Q}_p^d \)). For any \( \mathbf{m} \in \mathbb{Z}^d \) we shall write \( p^{-\mathbf{m}j\tau(a)} \) for the \( d \)-tuple \((p^{-m_1j\tau_1(a)}, \ldots, p^{-m_dj\tau_d(a)})\). Recall also the notation \( \mathbf{p}^m \) for \( \mathbf{p}_1^{m_1} \cdots \mathbf{p}_d^{m_d} \) and the fact that the character \( \rho \) appearing in Proposition 4.3 has been uniquely extended to a homomorphism from \( \mathbb{Z}_{(p)}^1J = \bigcup_{m \in \mathbb{Z}^d} \mathbf{p}^mJ \) onto \( \mu_f' \subset \mathbb{C}^\times \). We shall write simply \( j\rho \) for the composite \( j \circ \rho \). Since we are assuming that \( J \) is prime to \( p \) and \( j\tau_i \) induces \( \mathbf{p}_i \) for each \( i \), it follows that

\[
\text{If } a \in \mathbb{Z}_{(p)}^1J \text{ and } \mathbf{m} \in \mathbb{Z}^d \text{ then } a \in \mathbf{p}^mJ \Leftrightarrow j\tau(a) \in \prod_{i=1}^d (p^{m_i} \mathbb{Z}_p) \Leftrightarrow p^{-\mathbf{m}j\tau(a)} \in \mathbb{Z}_p^d \quad (49)
\]

Now, for any \( d' \)-tuple \( \mathbf{v} = (v_1, \ldots, v_{d'}) \) of \( \mathbb{Q} \)-linearly independent elements of \( k_{\infty}' \) (with \( 1 \leq d' \leq d \)) we write \( C(\mathbf{v}) \) for the open simplicial cone in \( \mathbb{R}^d_{>0} \) spanned by \( \tau(v_1), \ldots, \tau(v_{d'}) \), namely \( C(\mathbf{v}) = \sum_{i=1}^{d'} \mathbb{R}_{>0} \tau(v_i) \). Given \( \rho \), we make the following informal pseudo-definition:

\[
" F_{m,\mathbf{v}}(X) = \sum_{a \in \mathbb{Z}^d(\mathbf{m}J) \cap C(\mathbf{v})} \left. \rho(a)(1 + X)^{p^{-\mathbf{m}j\tau(a)}} \right. \text{ for every } \mathbf{m} \in \mathbb{Z}^d \quad (50)\]

The R.H.S. of (50) is an infinite sum of formal power series in \( \mathbb{C}_p[[X]] \) each of whose constant terms is a root of unity. Since there is no obvious sense in which it converges, the value of equation (50) is largely conceptual. To give a meaningful definition of \( F_{m,\mathbf{v}}(X) \) we note first that, for any \( \mathbf{m} \in \mathbb{Z}^d \) and \( l \in \{1, \ldots, d'\} \), the intersection \( \mathbb{Q}_{>0}v_l \cap \mathbf{p}^mJ \) equals \( \mathbb{Z}_{>0}w_{l,\mathbf{m}} \) for a unique element \( w_{l,\mathbf{m}} \) of \( k_{\infty}' \). Now choose any element \( \mathbf{w} = (w_1, \ldots, w_{d'}) \) of \( (k_{\infty}')^d \) such that \( w_l \in \mathbb{Q}_{>0}v_l \cap \mathbf{p}^mJ \) (for instance \( \mathbf{w} = w_{\mathbf{m}}^0 := (w_{0,\mathbf{m},1}, \ldots, w_{0,\mathbf{m},d'}) \)). Let us denote by \( P(\mathbf{w}) \) the half-open parallelipiped \( \sum_{l=1}^{d'} (0,1] \tau(w_l) \subset C(\mathbf{v}) \) and define \( F_{m,\mathbf{v}}(X) \) to be \( G_{m,\mathbf{w}}(X)/H_{m,\mathbf{w}}(X) \) where

\[
G_{m,\mathbf{w}}(X) := \sum_{a \in \mathbb{Z}^d(\mathbf{m}J) \cap P(\mathbf{w})} \left. \rho(a)(1 + X)^{p^{-\mathbf{m}j\tau(a)}} \right.
\]

and

\[
H_{m,\mathbf{w}}(X) := \prod_{l=1}^{d'} \left( 1 - j\rho(w_l)(1 + X)^{p^{-\mathbf{m}j\tau(w_l)}} \right)
\]

The sum defining \( G_{m,\mathbf{w}}(X) \) is finite since \( P(\mathbf{w}) \) has compact closure and \( \tau(\mathbf{p}^mJ) \) is discrete in \( \mathbb{R}^d \). Since \( H_{m,\mathbf{w}}(X) \) is non-zero, we may certainly consider \( F_{m,\mathbf{v}}(X) \) as an element of the fraction field of \( \mathbb{C}_p[[X]] \) which might, \textit{a priori}, depend on the choice of \( \mathbf{w} \). In fact:

\textbf{Lemma 4.11} For \( \rho, \mathbf{v}, \mathbf{m} \) and \( \mathbf{w} \) as above

(i) \( G_{m,\mathbf{w}}(X) \) and \( H_{m,\mathbf{w}}(X) \) lie in \( \mathbb{Z}_p[\mu_f'][[[X]] \).

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(ii). $F_{\bar{m},v}(X)$ is independent of the choice of $w$ (given $\rho$, $v$ and $\bar{m}$).

(iii). If $\mathbb{Q}v_l \cap p^{m_l}J \not\subset \ker(\rho)$ for $l = 1, \ldots, d'$ then $F_{\bar{m},v}(X)$ lies in $\mathbb{Z}_p[\mu_{f'}][[X]]$.

**Proof** Part (i) follows from equation (49). Let us write $w_l$ as $q_l w_{m_l}^0$ with $q_l \in \mathbb{Z}_{>0}$ for $l = 1, \ldots, d'$ and set $R(X) = \prod_{l=1}^{d'} \left( \sum_{r_l=0}^{q_l-1} j \rho(r_l w_{m_l}^0)(1 + X)^{p^{-m_l}r_l(r_l w_{m_l}^0)} \right)$. Then clearly, $H_{\bar{m},w}(X) = H_{\bar{m},w_m^0}(X) R(X)$. Furthermore, $G_{m,w}(X) = G_{m,w_m^0}(X) R(X)$ because $w_{m_l}^0$ lies in $p^{m_l}J$ for all $l$ and so the set $\{ a \in p^{m}J : z(a) \in \frac{P(w)}{v} \}$ is the disjoint union of the translates $\{ a \in p^{m}J : \tau(a) \in P(w_m^0) \} + (r_1 w_{m_1}^0 + \ldots + r_d w_{m_d}^0)$ as $r_l$ runs from 0 to $q_l - 1$ for all $l$. Thus $G_{\bar{m},w}(X)/H_{\bar{m},w}(X) = G_{m,w_m^0}(X)/H_{\bar{m},w_m^0}(X)$ which proves (ii). Finally, if $\mathbb{Q}v_l \cap p^{m_l}J \not\subset \ker(\rho)$ for each $l$ we may take $w_l \in \mathbb{Q}_{>0}v_l \cap p^{m_l}J$ such that $j \rho(w_l)$ is a non-trivial $f'$-th root of unity in $\mathbb{C}_p$. Since $p \not\mid f'$, the constant term $\prod_{l=1}^{d'} (1 - j \rho(w_l))$ of $H_{\bar{m},w}(X)$ is then a unit of $\mathbb{Z}_p[\mu_{f'}]$ so that $H_{\bar{m},w}(X)$ is a unit of $\mathbb{Z}_p[\mu_{f'}][[X]]$ and part (iii) follows from part (i).

**Remark 4.2** The pseudo-formula (50) can be seen as the result of an informal limiting process in which the integers $q_l$ in the above proof tend to infinity in such a way that $H_{\bar{m},w}(X)$ tends to 1.

In view of Lemma 4.11(iii) we introduce the following hypothesis which may or may not be verified by a character $\rho : J \rightarrow \mathbb{C}^\times$ and a $d'$-tuple $v$ as above:

**Hypothesis** For each $l = 1, \ldots, d'$ we have $\mathbb{Q}v_l \cap J \not\subset \ker(\rho)$.

If $m \in \mathbb{Z}^d$ then $p^N(\mathbb{Q}v_l \cap J) \subset \mathbb{Q}v_l \cap p^{m_l}J \subset p^{-N}(\mathbb{Q}v_l \cap J)$ for some $N >> 0$ and since $\rho$ takes values in $\mu_{f'}$ and $p \not\mid f'$, it follows that $H(\rho,v)$ is equivalent to $\mathbb{Q}v_l \cap p^{m_l}J \not\subset \ker(\rho)$ for all $l = 1, \ldots, d'$ and all $m \in \mathbb{Z}^d$ (which in turn is equivalent to $\rho(w_{m_l}^0) \neq 1$ for all $l$ and $m$).

**Lemma 4.12** Suppose $H(\rho,v)$ holds. Then $F_{\bar{m},v}(X) \in \mathbb{Z}_p[\mu_{f'}][[X]] \forall m \in \mathbb{Z}^d$. Moreover, for any $\underline{r} = (r_1, \ldots, r_d) \in \mathbb{Z}_{>0}^d$ with $r_i \leq n + 1 \forall i$, we have $V_{\underline{r}} F_{\bar{m},v}(X) = F_{\bar{m}+\underline{r},v}((1 + X_1)^{\rho r_1} - 1, \ldots, (1 + X_d)^{\rho r_d} - 1) \forall \bar{m} \in \mathbb{Z}^d$.

**Proof** The first statement follows from Lemma 4.11(iii) and the above discussion. For the second, choose $w$ such that $w_l$ lies in $\mathbb{Q}_{>0}v_l \cap p^{m_l+\bar{m}}J$ for all $l = 1, \ldots, d'$. Then $G_{m,w}, H_{\bar{m},w}, G_{\bar{m}+\underline{r},w}$ and $H_{\bar{m}+\underline{r},w}$ are all defined and lie in $\mathbb{Z}_p[\mu_{f'}][[X]]$. Moreover, it follows from (49) (with $\bar{m}+\underline{r}$ in place of $m$) that $p^{-m_l}j \tau_l(w_l) \in \mathbb{Z}_p^\times$ for all $l$ and $i$, with the result that $\mu(\underline{r} - 1)$ acts trivially on $H_{\bar{m},w}(X)$. Therefore

$$H_{\bar{m},w}(X) V_{\underline{r}} F_{\bar{m},v}(X) = V_{\underline{r}}(H_{\bar{m},w}(X) F_{\bar{m},v}(X)) = V_{\underline{r}} G_{m,w}(X)$$

(51)

If $a \in p^{m_l}J$ then Lemma 4.10(i) combined with (49) shows that $V_{\underline{r}}(1 + X)^{p^{-m_l}r_l(a)}$ equals $(1 + X)^{p^{-m_l}r_l(a)}$ if $a \in p^{m_l}J$ and 0 otherwise. It follows easily from the definitions of $G_{m,w}(X)$ and $H_{\bar{m},w}(X)$ that

$$V_{\underline{r}} G_{m,w}(X) = G_{\bar{m}+\underline{r},w}((1 + X_1)^{\rho r_1} - 1, \ldots, (1 + X_d)^{\rho r_d} - 1)$$

$$= H_{\bar{m}+\underline{r},w}((1 + X_1)^{\rho r_1} - 1, \ldots, (1 + X_d)^{\rho r_d} - 1) \times$$

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\[ F_{z+m,w}(1 + X_1)^{p^m} - 1, \ldots, (1 + X_d)^{p^m} - 1 \]
\[ = H_{m,w}(X) F_{z+m,w}(1 + X_1)^{p^m} - 1, \ldots, (1 + X_d)^{p^m} - 1 \]

Combining the last equation with (51) and cancelling \( H_{m,w}(X) \) gives the result. \( \square \)

**Remark 4.3** The inequalities \( r_i \leq n + 1 \) in the statement of the Lemma are unnatural and unnecessary. Indeed, if \( n \) is allowed to tend to infinity, we get an action of \( \mu(\infty) \) on the subring \( \bigcap_{n \geq 0} B(n) \) of \( \mathbb{C}_p[[X]] \) (namely, those power series converging on the ‘open’ polydisc radius 1 about 0 \( \in \mathbb{C}_p^n \)) which contains \( \mathbb{Z}_p[[X]] \). The argument then goes through for any \( r \in \mathbb{Z}^d_\geq 0 \). Furthermore, the second statement of the Lemma is easily seen to hold even when \( H(\rho,v) \) fails, provided we use a natural extension of the \( \mu(\infty) \) action to the fraction field of \( \mathbb{Z}_p[[X]] \).

Recall that for any \( \rho : J \to \mathbb{C}^\times \), \( z \in \mathcal{S}(n) \) and \( m \in \mathbb{Z}^d \) as in Propostion 4.3 we have defined a character \( \xi_{\rho,z,m} : \mathbb{Z}^m/J \to \mathbb{C}^\times \) by (10). This in turn defines a complex Dirichlet series \( Z_{v}(s, \xi_{\rho,z,m}, p^mJ) \) by the following formula.

\[
Z_{v}(s, \xi_{\rho,z,m}, p^mJ) := \sum_{a \in p^mJ \atop \tau(a) \in C(v)} \frac{\xi_{\rho,z,m}(a)}{(\tau_1(a) \ldots \tau_d(a))^s} = \sum_{a \in p^mJ \atop \tau(a) \in C(v)} \xi_{\rho,z,m}(a)N_{k/Q}(a)^{-s}
\]

**Lemma 4.13** Suppose \( H(\rho,v) \) holds for some \( d' \)-tuple \( v \) as above. Then for any \( m \in \mathbb{Z}^d \) and \( z \in \mathcal{S}(n) \) the function \( Z_{v}(s, \xi, I) \) converges absolutely for \( \text{Re}(s) > d'/d \) and possesses a meromorphic continuation to \( \mathbb{C} \). Moreover the latter is regular at zero with value in \( Q(\mu_{r^d} \alpha^{\mu_{r^{d}+1}}) \).

**Proof** This all follows from the results of [Sh], the particularly simple form of (52) being a consequence our Hypothesis \( H(\rho,v) \). We sketch the derivation. Let us fix \( m, z \) and choose \( w = (w_1, \ldots, w_d) \) with \( w_l \in \mathbb{Q}_{>0}v_l \cap p^mJ \) for \( l = 1, \ldots, d' \) and \( \rho(w_l) \neq 1 \forall l \) (by \( H(\rho,v) \)). The set \( \{a \in p^mJ : \tau(a) \in C(v)\} \) is the disjoint union of the translates \( \{a \in p^mJ : \tau(a) \in P(w)\} + (r_1w_1 + \ldots + r_dw_d) \) as \( r_l \) runs from 0 to \( \infty \) for each \( l \). It follows that

\[
Z_{v}(s, \xi_{\rho,z,m}, p^mJ) = \sum_{a \in p^mJ} \xi_{\rho,z,m}(a)Z_{w}(s, a, \xi_{\rho,z,m})
\]

where \( a \) runs through the finite set \( \{a \in p^mJ : \tau(a) \in P(w)\} \) and

\[
Z_{w}(s, a, \xi_{\rho,z,m}) := \sum_{r_1, \ldots, r_d \in \mathbb{Z}_{\geq 0}} \frac{\xi_{\rho,z,m}(w_1)^{r_1} \ldots \xi_{\rho,z,m}(w_d)^{r_d}}{\prod_{l=1}^{d} \tau_l(a + (r_1w_1 + \ldots + r_dw_d))^s}
\]

The last two equations are formal until the absolute convergence of \( Z_{w}(s, a, \xi_{\rho,z,m}) \) is established. But this series is just \( \zeta(s, A, x, \chi) \) in the notation of [Sh] p. 396 where Shintani’s \( (n, r) \) is our \( (d', d') \), his \( A \) is the matrix \( (a_{l_i}) := (\tau(w_l)) \) with \( 1 \leq l \leq d', 1 \leq i \leq d \), his \( x \) is the \( d' \)-tuple \( (x_1, \ldots, x_{d'}) \in (\mathbb{Q} \cap (0, 1])^{d'} \) such that \( a = x_1w_1 + \ldots + x_{d'}w_{d'} \) and his \( \chi \) is the
By meromorphic continuation, it suffices to prove the equality for Re$(s) > 1$. Now by definition, \( \mathfrak{w}_{\rho, s, m} = \{\xi_{\rho, s, m}, p^m J\}_m \in \mathfrak{M}_m \). Therefore, the definitions, notations and equations of \([So2]\) pp. 15,16] give

\[
Z(s, \mathfrak{w}_{\rho, s, m}) = \frac{1}{|E_m : E|} \sum_{\nu \in C} Z_\nu(s, \xi_{\rho, s, m}, p^m J) Z_\nu \quad = \quad \frac{N(p^m J)^s}{|E_m : E|} \sum_{a \in \mathcal{R}} \xi_{\rho, s, m}(a) N_{k/\mathbb{Q}}(a)^{-s}
\]

where \( B_1(A,1-x,\chi)^{(i)} \) denotes the constant term of a certain Laurent series. But since \( \rho(w_l) \) is a non-trivial \( f' \)th root of unity and \( p \nmid f' \), it follows from \([10]\) that \( \chi_l \neq 1 \forall l \) which means that these Laurent series are actually all power series in Shintani's variables \( u \) and \( t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_d \). Setting these all equal to zero we deduce that \( B_1(A,1-x,\chi)^{(i)} \) is simply equal to \((\prod_{l=1}^d (1-\chi_l))^{-1} \) independently of \( i \) and \( a \), which is therefore also the value of \( Z_w(0,a,\xi_{\rho, s, m}) \) for all \( a \). Thus

\[
Z_v(0, \xi_{\rho, s, m}, p^m J) = \frac{\sum_{a \in \mathbb{Z}^{m}} \xi_{\rho, s, m}(a) \prod_{l=1}^d (1 - \xi_{\rho, s, m}(w_l))}{\prod_{l=1}^d (1 - \xi_{\rho, s, m}(w_l))}
\]

The result follows from the definition of \( F_{m,v} \) on applying \( j \) to both sides of the above equation, noting that if \( a \in \mathbb{Z}^{m} \) then \( j(\xi_{\rho, s, m}(a)) \) is the value of the power series \( j \rho(a)(1 + X) p^{-m_j(a)} \) at \( X = (\zeta_{s_1} - 1, \ldots, \zeta_{s_d} - 1) \), by \([10]\). (Recall that “\( j(\zeta_n) = \zeta_n \)” in our notation.) \( \square \)

The group \( E_\infty = E_\infty(k) \) of totally positive units of \( k \) acts on \( \mathbb{R}^d_0 \) by setting \( \varepsilon(x_1, \ldots, x_d) := (\tau_1(x_1), \ldots, \tau_d(x_d)) \) for all \( \varepsilon \in E_\infty \) and \( (x_1, \ldots, x_d) \in \mathbb{R}^d_0 \). Given any subgroup \( E \) of finite index in \( E_\infty \), a \( k \)-rational cone decomposition of a fundamental domain for \( E \) acting on \( \mathbb{R}^d_0 \) (called just a cone decomposition for \( E \) for short) is a finite set \( C \) of \( d \)-tuples \( \mathfrak{v} \) of \( \mathbb{Q} \)-linearly independent elements of \( k^\times \) (with \( d' \) depending on \( \mathfrak{v} \) and varying between 1 and \( d \)) such that the cones \( \varepsilon C(\mathfrak{v}) = C(\varepsilon \mathfrak{v}) \) for \( \varepsilon \in E \) and \( \varepsilon \in E \) are pairwise disjoint and their union is \( \mathbb{R}^d_0 \). Shintani proved in \([Sh]\) that cone decompositions exist for any such \( E \). Their relevance to our situation comes from a corresponding decomposition of twisted zeta-functions. Recall that for \( \rho : J \rightarrow \mathbb{C}^\times \), \( s \in \mathcal{S}(n) \) and \( m \in \mathbb{Z}^d \) as above, Lemma \([4,3]\) implies that \( \text{ann}_C(\xi_{\rho, s, m}) = \mathfrak{f} \) where \( \mathfrak{m} = f \infty \). In this situation:

**Lemma 4.14** For any subgroup \( E \) of finite index in \( E_m \) (hence in \( E_\infty \)) and any cone decomposition \( C \) for \( E \) we have the following equality (as meromorphic functions of \( s \in \mathbb{C} \))

\[
Z(s, \mathfrak{w}_{\rho, s, m}) = \frac{N(p^m J)^s}{|E_m : E|} \sum_{\nu \in C} Z_\nu(s, \xi_{\rho, s, m}, p^m J)
\]

**Proof** By meromorphic continuation, it suffices to prove the equality for Re\((s) > 1\). Now by definition, \( \mathfrak{w}_{\rho, s, m} = \{\xi_{\rho, s, m}, p^m J\}_m \in \mathfrak{M}_m \). Therefore, the definitions, notations and equations of \([So2]\) pp. 15,16] give

\[
Z(s, \mathfrak{w}_{\rho, s, m}) = \frac{1}{|E_m : E|} \sum_{a \in \mathcal{R}} \xi_{\rho, s, m}(a) N_{k/\mathbb{Q}}(a)^{-s}
\]

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for all \( s \in \mathbb{C}, \text{Re}(s) > 1 \), where \( \mathcal{R} \) is any set of orbit representatives for the (multiplicative) action of \( E \) on \( k^\times \cap p\mathcal{L}_J \). We may clearly take \( \mathcal{R} \) to be the set \( \{ a \in p\mathcal{L}_J : \tau(a) \in \bigcup_{v \in E} C(v) \}\) (a disjoint union). The Lemma therefore follows by absolute convergence from the definition of \( Z_v(s, \xi_{\mathcal{R},p}, p\mathcal{L}_J) \).

The final ingredient in proving Proposition 4.3 concerns the existence of subgroups of \( E_m \) and cone decompositions with the properties we require.

**Lemma 4.15** Suppose given \( \rho : J \to \mathbb{C}^\times \) with \( \text{ann}_O(\rho) = \mathcal{O}' \). Then there exists a subgroup \( E \) of finite index in \( E_m \) and a cone decomposition \( \mathcal{C} \) for \( E \) satisfying

\[
p \nmid [E_m : E]
\]

and

\[
H(\rho, v) \text{ holds for all } v \in \mathcal{C}
\]

**Proof** We shall use Colmez’ construction of a cone decomposition in \([\text{Colm}, \S 2]\), with two small modifications. We shall harmonise our set-up with his by suppressing \( \tau \) and regarding \( k \) as embedded in \( \mathbb{R}^d \). A direct application of Colmez’ Lemme 2.1 loc. cit. (with \( F = k, n = d, V = E_m \)) would give rise to elements \( \varepsilon_1, \ldots, \varepsilon_{d-1} \) of \( E_m \) satisfying Colmez’ hypothesis (H). However, to ensure that they generate a subgroup \( E \) satisfying Condition (53), we need to modify their construction in the proof of Lemma 2.1 to which we now refer. Specifically, we recall that Colmez chooses a positive real number \( r(V) = r(E_m) \) such that \( \log(E_m) \) has non-empty intersection with the ball \( B(l_t(M), r(E_m)) \) for each \( t \) and each \( M > 0 \). (We follow him in using the sup-norm \( || \cdot || \) on \( \mathbb{R}^d \).) Let us choose a \( \mathbb{Z} \)-basis \( \eta_1, \ldots, \eta_{d-1} \) of \( E_m \) and set \( r'(E_m) = r(E_m) + (p-1)(d-1) \max \{|\log(\eta_i)| : 1 \leq t \leq d-1\} \). It is easy to see that the larger ball \( B(l_t(M), r'(E_m)) \) must contain a complete set of representatives for \( \log(E_m) \) modulo \( p \log(E_m) = \log(E_m^p) \). Therefore, for any \( M > 0 \) we may choose \( \varepsilon_t \in E_m, t = 1, \ldots, d-1 \) such that \( \log(\varepsilon_t) \in B(l_t(M), r'(E_m)) \forall t \) and such that the matrix \( A \) in \( M_{d-1}(\mathbb{Z}) \) representing the \( \varepsilon_t \)'s in the basis of the \( \eta_t \)'s has any required reduction \( \bar{A} \) in \( M_{d-1}(\mathbb{Z}/p\mathbb{Z}) \). In particular we may insist that \( \det(\bar{A}) \neq 0 \) so that \( p \nmid |\det(A)| = |E_m : \langle \varepsilon_1, \ldots, \varepsilon_{d-1} \rangle| \). By this means, we produce units \( \varepsilon_1, \ldots, \varepsilon_{d-1} \) such that Condition (53) holds with \( E := \langle \varepsilon_1, \ldots, \varepsilon_{d-1} \rangle \). Moreover, if we substitute \( r'(E_m) \) for \( r(E_m) \) throughout the proof of Lemma 2.1, the reader may easily check that its validity (with a new, corresponding choice of \( M \)) is unaffected. Consequently the units \( \varepsilon_1, \ldots, \varepsilon_{d-1} \) produced will also satisfy Colmez’ hypothesis (H). Now, for each \( i \in \{1, \ldots, d\} \) and permutation \( \sigma \in S_{d-1} \) Colmez defines \( f_{i, \sigma} := \prod_{1 \leq t < i} \varepsilon_{\sigma(t)} \in E \) (with \( f_{1, \sigma} := 1 \forall \sigma \)). For any non-empty \( I \in \mathcal{P}\{1, \ldots, d\}\) (the power set) and \( \sigma \in S_{d-1} \) we shall write \( v_{(\sigma, I)} \) for the \( |I| \)-tuple consisting of the \( f_{i, \sigma} \) ordered by increasing \( i \in I \). Then Lemme 2.2 of \([\text{Colm}]\) says that a cone decomposition \( \hat{\mathcal{C}} \) for \( E \) may be obtained by setting \( \hat{\mathcal{C}} := \{ v_{(\sigma, I)} : (\sigma, I) \in \mathcal{T} \} \) where \( \mathcal{T} \) is any set of representatives of equivalence classes for \( S_{d-1} \times \mathcal{P}\{1, \ldots, d\}\) modulo a certain natural equivalence relation. Finally, since \( \mathcal{O}' \neq \mathcal{O} \), the character \( \rho \) is non-trivial, so there exists \( b \in J \) with \( \rho(b) \neq 1 \). Adding to \( b \) a large positive element of \( \mathcal{O}' J \cap \mathbb{Z} \) if necessary, we may assume that it also lies in \( k^\times \). Now take \( \mathcal{C} \) to be the ‘multiplicative translate of \( \hat{\mathcal{C}} \) by \( b \). In other words, \( \mathcal{C} := \{ b v_{(\sigma, I)} : (\sigma, I) \in \mathcal{T} \} \) where \( b v_{(\sigma, I)} \)
is the $|I|$-tuple consisting of the $b_{f_{i,\sigma}}$ ordered by increasing $i \in I$. It is easy to see that $C$ is a cone decomposition for $E$ (since $\tilde{C}$ is). Moreover, since $f_{i,\sigma}$ lies in $E_m$, we have $b_{f_{i,\sigma}} \in J$ and $b_{f_{i,\sigma}} - b \in f'J$ for each $i$ and $\sigma$. Hence $\rho(b_{f_{i,\sigma}}) = \rho(b) \neq 1 \quad \forall i, \sigma$ which implies Condition (54) and thus completes the proof of the Lemma.

**Remark 4.4**

(i) The above proof clarifies the purpose of Conditions (26) and (27): they allow us to take $f'$ and hence $\rho$ to be non-trivial so that the useful Condition (54) can be satisfied.

(ii) In practice, the construction of Lemma 4.15 may yield a subgroup $E$ whose index in $E_m$ is far larger than the minimum required to satisfy Conditions (53) and (54) for some $C$.

Indeed, in the interesting case $d = 2$ ($k$ real quadratic) we can always take $E$ to be $E_m$ itself (so, explicitly, $C = \{(b), (b, b\varepsilon)\}$ where $E_m = \langle \varepsilon \rangle$ and $\rho(b) \neq 1$).

**Proof of Prop. 4.3**

Lemma 4.15 gives a cone decomposition $C$ for $E \subseteq E_m$ such that $p \nmid [E_m : E]$ and $H(\rho, v)$ holds for the given character $\rho$ and all $v \in C$. We set

$$F_m(X) := \frac{1}{|E_m : E|} \sum_{v \in C} F_v(X)$$

for each $m \in \mathbb{Z}^d$.

Then Condition $[i]$ of the Proposition follows from the first statement of Lemma 4.12 while Condition $[ii]$ follows from the second statement and the linearity of $V$. Finally, Lemmas 4.13 and 4.14 imply that $j(Z(0; m_{\rho, \pm m}))$ lies in $\mathbb{Q}(\mu_{f', p, 1})$ and that, for any $m \in \mathbb{Z}^d$ and $s \in S(n)$,

$$j(Z(0; m_{\rho, \pm m})) = \frac{1}{|E_m : E|} \sum_{v \in C} j(Z(v, \xi_{\rho, \pm m}, p^n J))$$

$$= \frac{1}{|E_m : E|} \sum_{v \in C} F_m(v(\zeta_1^{s_1} - 1, \ldots, \zeta_d^{s_d} - 1)) = F_m(\zeta_1^{s_1} - 1, \ldots, \zeta_d^{s_d} - 1)$$

which establishes Condition $[iii]$. This completes the proof of Proposition 4.3 hence also of Theorem 4.1.

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**5 Remarks and Conjectures**

**5.1 The $p$-Integrality of $\Phi_m(0)$**

Suppose that $m = f \infty$ but $p$ does not necessarily split completely in $k$. If $p \nmid w_k(m)$ then (11) implies that $\Theta_{m(k(m))}(0) = \Theta_{k(m)/k}(0)$ lies in $\mathbb{Z}_p G$ and so therefore must $\Theta_m(0)$ (which is a $\mathbb{Z}G$-multiple of $\Theta_{m(k(m))}(0)$ by (3)). It follows from (14) that

$$j(\Phi_m(0)) \in \hat{\mathbb{Z}}_p G$$

whenever $p \nmid w_k(m)$. Suppose on the other hand that $p \neq 2$ splits completely in $k$ and that $m$ is as in (28) (in particular, the prime-to-$p$ part $f'$ of $f$ is non-trivial). Then we may deduce
Conjecture 5.1 Suppose \( k \) is totally real and \( p \neq 2 \). Then \((55)\) holds if either \( \mathfrak{f}' \neq \mathcal{O} \) or \( p \nmid w_{k(m)} \).

This conjecture might be established by adapting the techniques used in the proof of Theorem 4.1 to the case where \( p \) does not split completely in \( k \). On the other hand, even when it does, \((55)\) will not hold without some condition on \( \mathfrak{f}' \) and/or \( w_{k(m)} \). (Consider the case \( k = \mathbb{Q} \), \( f = p^{n+1}\mathbb{Z} \), cf. Example 3.1). The condition \( p \neq 2 \) may however be unnecessary.

5.2 Theorem 4.1 Revisited

The question arises as to whether the \( p \)-integrality property stated in Theorem 4.1 really is a new phenomenon or whether it too might follow from a combination of (1) and Theorem 2.2 (presumably allowing us to dispose of the splitting hypothesis). In support of the first alternative we now suppose that \( p \) splits in \( k \) as \( p_1 \ldots p_d \) and examine the case where \( K = k(m) \) and \( \mathfrak{m} = \mathfrak{f}\infty \) is a conductor (for simplicity). In this case, \( G = G_m \), so Definition 3.1 (9) and (15) give (suppressing \( j \) from the notation)

\[
g_K/k(\theta) = \sum_{\mathfrak{g} \mid \mathfrak{f}} \frac{1}{\sqrt{d_K N \mathfrak{g}}} \left( \prod_{p \mid \mathfrak{f}, \mathfrak{f} \mid \mathfrak{g}} (1 - Np^{-1}) \right) \nu_{n,m}(A_n \Theta_n(0)) R_{K/k,p}(\theta)
\]

for any \( \theta \in \bigwedge_{d} U_p(K) \). (Recall that \( n \) generically denotes the cycle \( g\infty \)). Let us write \( \Upsilon(\mathfrak{m}, g, \theta) \) for the term corresponding to \( g \) in this sum. It lies \textit{a priori} in \( \bar{\mathbb{Q}}_p G \). The most obvious hope is to use (11) to show that \( \Upsilon(\mathfrak{m}, g, \theta) \) actually lies in \( \bar{\mathbb{Z}}_p G \) for all \( g \) and thus deduce Theorem 4.1. We shall now show that in fact the coefficients of \( \Upsilon(\mathfrak{m}, g, \theta) \) may be arbitrarily large in \( p \)-adic absolute value (and thus for fixed \( k \) and \( g = \mathcal{O} \)). To do so, we fix also an ideal \( a \) of \( \mathcal{O} \) prime to \( p \) such that \( a\infty \) is a conductor. Then \( m_n := p^{n+1}a\infty \) is also a conductor for all \( n \geq 0 \) (that of \( k(a\infty)(\mu_{p^{n+1}}) \)). Write also \( K_n \) for \( k(m_n) \) and \( G_n \) for \( \text{Gal}(K_n/k) \). Then, taking \( \mathfrak{m} = m_n \) and \( \theta_n \in \bigwedge_{d} U_p^1(K_n) \), we have \( A_{\infty} = 1 \) and so

\[
\Upsilon(m_n, \mathcal{O}, \theta_n) = cv_{m_n,m_n}(\Theta_{\infty}(0)) R_{K_n/k,p}(\theta_n)
= cv_{m_n,m_n}(\Theta_{\infty}(0) \pi_{m_n,\infty}(R_{K_n/k,p}(\theta_n)))
= cv_{m_n,m_n}(\Theta_{\infty}(0) R_{k(\infty)/k,p}(N_{k(m_n)}/k^{(\infty)} \theta_n))
= c[k(m_n)] k(\infty)^{-1} \nu_{m_n,m_n}(\Theta_{\infty}(0) R_{k(\infty)/k,p}(N_{k(m_n)}/k^{(\infty)} \theta_n))
\]

where \( N_{k(m_n)/k^{(\infty)}} : \bigwedge_{d} U_p^1(K_n) \to \bigwedge_{d} U_p^1(k^{(\infty)}) \) is the norm map previously described and \( c \in \mathbb{Q}^* \) is independent of \( n \geq 0 \). The idea is now to choose a sequence \( \{\theta_n\}_{n \geq 0} \) such
that \( N_{k(m_n)/k(\infty)}\theta_n \) is a fixed element \( \theta \in \bigwedge_{\mathbb{Z}[G_{\infty}]} \mathbb{Z}_p(\theta) \) independent of \( n \) and such that \( \Theta_{\infty}(0)R_{k(\infty)/k,p}(\theta) \neq 0 \). The desired result will then follow, since \( k(m_n) \) contains \( k(\mu_{p^n+1}) \) which implies that \( p^n \) divides \( [k(m_n) : k(\infty)] \). The existence of such sequences is particularly easy to establish under the following additional assumptions

(i). Leopoldt’s Conjecture holds for \( p \),

(ii). \( k(\infty) \) is a totally complex quadratic extension of \( k \) and

(iii). for each \( i = 1, \ldots, d \) there is a unique prime \( \mathfrak{p}_i \) of \( k(\infty) \) lying above \( \mathfrak{p}_i \).

These assumptions – as well as the splitting hypothesis – are valid when \( k = \mathbb{Q}(\sqrt{7}) \) and \( p = 3 \), for example. Conditions (ii) and (iii) imply that \( k(\infty)_{\mathfrak{p}_i}/\mathbb{Q}_p \) is unramified of degree 2 for each \( i \) so that \( U^1(k(\infty)_{\mathfrak{p}_i}) \cong \mathbb{Z}_p^2 \) while Condition (i) implies that the quotient of \( \lim_{\leftarrow}G_n \) by a finite group is isomorphic to \( \mathbb{Z}_p \). It follows from local class field theory that for each \( i \) there exists \( u_i \in U^1(k(\infty)_{\mathfrak{p}_i}) \setminus \{1\} \) which is the (local) norm of a principal local unit of \( k(m_n) \) (completed at any prime above \( \mathfrak{p}_i \)) for every \( n \geq 0 \). Setting \( \theta := u_1 \wedge \ldots \wedge u_d \in \bigwedge_{\mathbb{Z}[G_{\infty}]} \mathbb{Z}_p(k(\infty)) \) we see that \( \theta \) equals \( N_{k(m_n)/k(\infty)}\theta_n \) for some \( \theta_n \in \bigwedge_{\mathbb{Z}[G_{\infty}]} \mathbb{Z}_p(k(m_n)) \) for every \( n \geq 0 \). It only remains to check that \( \Theta_{\infty}(0)R_{k(\infty)/k,p}(\theta) = \Theta_{\infty}(0)\lambda_{1,p}(u_1) \ldots \lambda_{d,p}(u_d) \) is non-zero. Suppose this were not the case. Condition (ii) implies that the unique non-trivial character \( \chi \) of \( G_{\infty} \) is totally odd so \( \chi(\Theta_{\infty}(0)) = L(0,\chi) \neq 0 \) and we would have \( \chi(\lambda_{i,p}(u_i)) = 0 \) for some \( i \). On the other hand \( N_{k(\infty)_{\mathfrak{p}_i}/k,p,p_i}u_i \) is, by construction, a norm from \( \mathbb{Q}(\mu_{p^n+1}) \) (the completion of \( k(\mu_{p^n+1}) \subset k(m_n) \)) for every \( n \geq 0 \). Since \( k_{\mathfrak{p}_i} \cong \mathbb{Q}_p \) it follows from local class field theory that \( N_{k(\infty)_{\mathfrak{p}_i}/k,p,p_i}u_i = 1 \). Therefore the trivial character also vanishes on \( \lambda_{i,p}(u_i) \), so \( \lambda_{i,p}(u_i) = 0 \) and in particular \( \log_p(j\tau_i(u_i)) = 0 \). Since \( \mathfrak{p}_i/p \) is unramified, this contradicts the fact that \( u_i \in U^1(k(\infty)_{\mathfrak{p}_i}) \setminus \{1\} \) and the desired result follows.

We note also a particular feature of our proof of Theorem 4.1. By means equations (47) and (48) it gives a neat and more-or-less explicit expression for the coefficients of \( s_{K/k}(u_1 \wedge \ldots \wedge u_d) \) where \( u_i \in U^1(K_{\mathfrak{p}_i}) \) for \( i = 1, \ldots, d \). This assumes of course that the \( g_i \)’s and the \( F_i \)’s are known explicitly. In practice, one might first choose a power series ‘\( h \)’ as in Lemma 4.7 to determine each \( u_i \) as well as the corresponding series \( g_i \). Furthermore, the proof of Proposition 4.2 would give a quite explicit choice for \( F_i \) once a suitable subgroup \( E \) and cone decomposition \( C \) had been found (e.g. by Remark 4.4 for \( d = 2 \)).

Finally, on the basis of Theorem 4.1 and Corollary 4.1 we hazard the

**Conjecture 5.2** Suppose that \( k \) is totally real and that \( p \) is odd and splits completely in \( k \). Then \( \mathfrak{S}_{K/k} \subset p^{\delta(K/k)}\mathbb{Z}_pG \) for any \( K \).

Proposition 4.2 might be taken as a (rather weak) hint that the splitting hypothesis can be dropped from the above conjecture but we prefer to keep it pending further evidence.
5.3 Hilbert Symbols and Rubin-Stark Units

We assume from now on that the hypotheses of Conjecture 5.2 hold and that $K$ contains $\mu_{\rho_{n+1}}$ for some $n \geq 0$ so that $K$ is totally complex and $\delta(K/k) = 0$. We shall also assume that the conclusions of Conjecture 5.2 are valid (for instance, if (26) holds, by Theorem 4.4). In this situation we shall set out some conjectural congruences for $s_{K/k}$ modulo $p^{n+1}$ which were foreshadowed in Example 3.1 as generalisations of those coming from (24) in the case $k = \mathbb{Q}$, $K = \mathbb{Q}(\mu_{p^{n+1}})$.

First, we need to define some pairings coming from local Hilbert symbols. Let $K_p$ denote $\mathbb{Q}_p \otimes \mathbb{Q} K = \prod_{\mathfrak{p}|p} K_{\mathfrak{p}}$ so that, as in Section 3, the composite $j_{\tau_i}$ extends to a map $K_p \to \mathbb{Q}_p$ for $i = 1, \ldots, d$. The field $j(\tau_i(K_p))$ being a finite (abelian) extension of $\mathbb{Q}_p$ containing $\mu_{\rho_{n+1}}$, the Hilbert symbol $(\cdot, \cdot)_{\rho_{n+1}}$ gives a well-defined, $\mu_{\rho_{n+1}}$-valued, bilinear pairing on its multiplicative group. Thus for each $i$ we obtain a unique $\mathbb{Z}$-bilinear pairing $[\cdot, \cdot]_{i,n} : K_p^* \times K_p^* \to \mathbb{Z}/\rho_{n+1}^* \mathbb{Z}$ satisfying

$$\zeta_n^{[\alpha, \beta]_{i,n}} = (j_{\tau_i}(\alpha), j_{\tau_i}(\beta))_{\rho_{n+1}} \quad \text{for all } \alpha, \beta \in K_p^*$$

This exhibits the following equivariance property with respect to the decomposition group $D_{pi} = G(K/k)_{\rho_{n+1}} \subset G$}

$$[\delta \alpha, \delta \beta]_{i,n} = \kappa_n(\delta) [\alpha, \beta]_{i,n} \quad \text{for all } \delta \in D_{pi}, \quad (56)$$

where $\kappa_n : G \to (\mathbb{Z}/\rho_{n+1}^* \mathbb{Z})^\times$ is the character defined by $\sigma(\zeta) = \zeta_n^{\kappa_n(\sigma)}$ for $\zeta \in \mu_{\rho_{n+1}}$, $\sigma \in G$. Choose a set $R_i$ of coset representatives for $D_{pi}$ in $G$ and define a pairing $[\cdot, \cdot]_{i,n}$ on $K_p^*$ with values in the group-ring $(\mathbb{Z}/\rho_{n+1}^* \mathbb{Z})G$ by setting

$$[\alpha, \beta]_{i,n}^G := \sum_{\rho \in R_i} \kappa_n(\rho)^{-1} \sum_{\sigma \in G} [\rho \alpha, \sigma \beta]_{i,n} \sigma^{-1} \rho \quad \text{for all } \alpha, \beta \in K_p^*$$

It follows from property (56) that $[\cdot, \cdot]_{i,n}^G$ is independent of the choice of $R_i$. Moreover it is $\mathbb{Z}G$-(semi)linear in each variable as follows. For any $\alpha, \beta \in K_p^*$ we have

$$[x \alpha, \beta]_{i,n}^G = \kappa_n^* (x) [\alpha, \beta]_{i,n}^G \quad \text{and} \quad [\alpha, x \beta]_{i,n}^G = x [\alpha, \beta]_{i,n}^G \quad \text{for all } x \in \mathbb{Z}G, \quad (57)$$

where $\kappa_n^* : \mathbb{Z}G \to (\mathbb{Z}/\rho_{n+1}^* \mathbb{Z})G$ is the ring homomorphism sending $\sum a_g \gamma$ to $\sum a_g \kappa_n(g) \gamma^{-1}$. (We shall summarise property (57) by saying that the pairing $[\cdot, \cdot]_{i,n}^G$ is $(\kappa_n^*, 1)$-bilinear.) Summing over $i$ we get a pairing $[\cdot, \cdot]^G : K_p^* \times K_p^* \to (\mathbb{Z}/\rho_{n+1}^* \mathbb{Z})G$, namely $[\alpha, \beta]^G := \sum_{i=1}^d [\alpha, \beta]_{i,n}^G$, which is also $(\kappa_n^*, 1)$-bilinear. Finally, by regarding both $K_p^*$ and $U_p^1(K)$ as submodules of $K_p^*$, it follows that there is a unique pairing $\mathcal{H}_n$ from $\bigwedge_{\mathbb{Z}G}^d K_p^* \times \bigwedge_{\mathbb{Z}pG}^d U_p^1(K)$ to $(\mathbb{Z}/\rho_{n+1}^* \mathbb{Z})G$ which is $(\kappa_n^*, 1)$-bilinear and satisfies

$$\mathcal{H}_n(x_1 \wedge \ldots \wedge x_d, u_1 \wedge \ldots \wedge u_d) = \det([x_i, u^G_{i,t}]_{i,t})$$

We would now like to construct an element $\eta \in \bigwedge_{\mathbb{Z}G}^d K_p^*$ such that $\mathcal{H}_n(\eta, \cdot)$ plays essentially the same rôle as $((1 - \zeta_n)^{-1}, \cdot)_{\rho_{n+1}}$ plaed in equation (24) of Example 3.1 (where $d = 1$).
The best tool we currently have for this construction is Rubin’s restatement of the Stark Conjecture for $K^+/k$ where $K^+$ is the maximal real subfield of $K$. Not only is this conjecture itself unproven in almost all the relevant cases but, as we shall see, its very nature makes our conjectural congruences for $s_{K/k}$ vaguer and more awkward than Example 3.3 might reasonably lead one to expect. With these caveats, let us recall Rubin’s formulation using the higher derivatives of the function $\Theta_{K^+/k}(s)$ at $s = 0$, referring to [Ru] and [So2, §4] for more details and relevant remarks. (See also [So2, §5] and [So3] for a reformulation using $\Phi_{K^+/k}(s)$ at $s = 1$, essentially by Theorem 2.1). Let us write $S_0$ for $S_{\text{ram}}(K/k)$ and $U_{S_0}(K^+)$ for the group of (global) $S_0$-units of $K^+$. Our assumptions force $p_i \in S_0 \forall i$. This means that $|S_0| \geq 2d$ (and also that $U_{S_0}(K^+)$ is not contained in the product of local units, $U_p(K^+)$, despite the notation). Set $G^+ := \text{Gal}(K^+/k)$ and define the archimedean analogue of $\lambda_{K^+/k,i,p}$ by $\lambda_{K^+/k,i,p}(a) = \sum_{g \in G^+} \log |\tau_i(ga)| g^{-1} \in \mathbb{R}G^+$ for any $a \in K^+$. We obtain a unique, archimedean regulator $R_{K^+/k} : \mathbb{Q} \otimes \bigwedge^d_{\mathbb{Z}G^+} U_{S_0}(K^+) \to \mathbb{R}G^+$ taking $u_1 \wedge \ldots \wedge u_d$ to $\det((\lambda_{K^+/k,i}(u_t))_{t=1}^d)$. On the other hand, we may define

$$\Theta_{K^+/k,S_0}(s) := \prod_{p \in S_0 \setminus S_{\text{ram}}(K^+/k)} (1 - Np^{-s} \sigma_{p,K^+}^{-1}) \Theta_{K^+/k}(s) = \pi_{K,K^+}(\Theta_{K/k}(s))$$

(This is the function $\Theta_{K^+,S_0,0}(s)$ of [Ru]). Since $S_0$ contains at least $d+1$ places of $k$, and the $d$ real ones split completely in $K^+$, it follows that $\Theta_{K^+/k,S_0}(s)$ has at least a $d$-fold zero at $s = 0$. We set $\Theta_{K^+/k,S_0}^{(d)}(0) := \lim_{s \to 0} s^{-d} \Theta_{K^+/k,S_0}(s) = \mathbb{C}G^+$ and write $X(S_0,d,G^+)$ for the set of complex irreducible characters $\chi$ of $G^+$ for which $\chi(\Theta_{K^+/k,S_0}^{(d)}(0)) \neq 0$. Let us take the extension ‘K/k of [Ru] to be $K^+/k$, S’ to be $S_0$, ‘T’ to be $\emptyset$, ‘r’ to be d and the chosen places ‘$w_1, \ldots, w_d$’ of $K^+$ to be the real ones defined by $\tau_1, \ldots, \tau_d$. The torsion subgroup of $U_{S_0}$ is $\{\pm 1\}$ so the conditions of Rubin’s Conjectures B and B’ over $\mathbb{Z}$ (see [Ru Hyp. 2.1]) are actually not quite met with $T = \emptyset$. However, we shall only need his Conjecture A’ over $\mathbb{Q}$, for which the choice of $T$ is irrelevant. The latter is in fact equivalent to Stark’s conjecture for each $\chi \in X(S_0,d,G^+)$ (see [Tate, Conjecture I.5.1]) and states

**Conjecture 5.3 (Conjecture A’ of [Ru])**

There exists an element $\eta_{K^+/k,S_0}$ of $\mathbb{Q} \otimes \bigwedge^d_{\mathbb{Z}G^+} U_{S_0}(K^+)$ such that

(i). $\Theta_{K^+/k,S_0}^{(d)}(0) = R_{K^+/k}(\eta_{K^+/k,S_0})$ and

(ii). $e_\chi \eta_{K^+/k,S_0} = 0$ in $\mathbb{C} \otimes \bigwedge^d_{\mathbb{Z}G^+} U_{S_0}(K^+)$ for every character $\chi$ of $G^+$ not in $X(S_0,d,G^+)$. We assume henceforth that $\eta_{K^+/k,S_0}$ satisfies the two conditions of this conjecture. This actually makes $\eta_{K^+/k,S_0}$ unique and we define $\eta_{K/k}^+$ to be its image under the natural map $\mathbb{Q} \otimes \bigwedge^d_{\mathbb{Z}G^+} U_{S_0}(K^+) \to \mathbb{Q} \otimes \bigwedge^d_{\mathbb{Z}G^+} K^\times$. Consider the case $K/k = \mathbb{Q}(\zeta_n+1)/\mathbb{Q}$. We have $d = 1$, $S_0 = \{\infty, p\}$ and one can check that Conjecture 5.3 always holds with $\eta_{K^+/k,S_0} = -\frac{1}{2} \otimes (1 - \zeta_n)(1 - \zeta_n^{-1})$ (cf. [Tate, §III.5] and [So3, §3.5]). In fact, $\Theta_{K^+/k,S_0}(s) = \Theta_{K^+/k,S_0}(s)$ except in the trivial case $p^{n+1} = 3$. It follows that $\eta_{K/k} = 1 \otimes (1 - \zeta_n)^{-1} \in \mathbb{Q} \otimes \bigwedge^1_{\mathbb{Z}G^+} K^\times = \mathbb{Q} \otimes K^\times$. 36
Moreover the reader may check that if we take $\tau_1$ to be $1 \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ then (24) is precisely the statement that $\overline{s_{K/k}} = \mathcal{H}_n(1-\zeta_n,1,u)$ in $(\mathbb{Z}/p^n+1\mathbb{Z})G$ for all $u \in U^1_p(K) = \bigwedge \mathbb{Z}_{pG}^1 U_p^1(K)$. Unfortunately, for general $K^+/k$, Rubin’s conjecture does not require that $\eta_{K/k}$ should be of the form $1 \otimes \tilde{\eta}$ for some $\tilde{\eta} \in \bigwedge^d_{\mathbb{Z}G} K^\times$. The author knows of no case satisfying our hypotheses where this condition actually fails. On the other hand, [Ru, Prop. 4.4] shows that in a very different case, the counterpart of the element $\eta_{K^+/k,K}$ satisfying Rubin’s general conjecture need not even lie in the counterpart of $\mathbb{Z}_{pG}^1 U_{S_n}(K^+)$. It therefore seems safer to proceed as follows. We define an ideal $\mathcal{I}(\eta_{K/k})$ of finite index in $\mathbb{Z}G$ by

$$\mathcal{I}(\eta_{K/k}) := \{ x \in \mathbb{Z}G : x\eta_{K/k} = 1 \otimes \tilde{\eta} \text{ for some } \tilde{\eta} \in \bigwedge^d_{\mathbb{Z}G} K^\times \}$$

and formulate the

**Conjecture 5.4** Suppose that $p$ is odd and splits completely in $k$ and that other hypotheses and notations are as above. Then for every $x \in \mathcal{I}(\eta_{K/k})$ there exists $\tilde{\eta}_x \in \bigwedge^d_{\mathbb{Z}G} K^\times$ such that

(i). $x\eta_{K/k} = 1 \otimes \tilde{\eta}_x$ and

(ii). $\kappa_n(x)\overline{s_{K/k}(\theta)} = \mathcal{H}_n(\tilde{\eta}_x,\theta)$ in $(\mathbb{Z}/p^n+1\mathbb{Z})G$ for all $\theta \in \bigwedge^d_{\mathbb{Z}G} U_p^1(K)$.

It clearly suffices to check this conjecture for a set of elements $x$ generating $\mathcal{I}(\eta_{K/k})$ over $\mathbb{Z}G$. Note that condition (i) only determines $\tilde{\eta}_x$ up to $\mathbb{Z}$-torsion in $\bigwedge^d_{\mathbb{Z}G} K^\times$, which does not lie in the kernel of $\mathcal{H}_n(\cdot,\theta)$ for all $\theta$, even in the case $K/k = \mathbb{Q}(\zeta_{p^n+1})$. This means firstly that condition (i) cannot be expected to hold for all lifts $\tilde{\eta}_x$ satisfying condition (ii) and secondly that one could weaken the conjecture by allowing $\tilde{\eta}_x$ satisfying the equality in (ii) to depend on $\theta$ as well as $x$ (subject to (i)).

We briefly explain the two pieces of evidence that motivate Conjecture 5.4, postponing a more detailed account for a future paper. Firstly, if $k = \mathbb{Q}$ then the conjecture holds with $\mathcal{I}(\eta_{K/k}) = \mathbb{Z}G$ and $\tilde{\eta}_1 = N_{\mathbb{Q}(\xi_{f(K)})/\mathbb{Q}}(1-\xi_{f(K)})^{-1}$ where $\xi_{f(K)} = e(1/f(K))$. This may be deduced from equation (22) and a generalisation of Artin-Hasse’s explicit reciprocity law to $\mathbb{Q}_p(\mu_{f(K)})$ which is proved by Coleman in [Cole2]. Secondly, Conjecture 5.4 actually follows from the same result of Coleman together with those of [So4] provided that it is weakened by using a slightly smaller ideal than $\mathcal{I}(\eta_{K/k})$ and that certain extra hypotheses are satisfied. The most important of these are: (a) that $K = L(\mu_{p^n+1})$ for some totally real or CM extension $L$ of $k$ which is unramified above $p$ and (b) that a certain strongly norm-coherent form of Conjecture 5.3 over $k$, together with its $p$-adic analogue, holds in the cyclotomic $\mathbb{Z}_p$-tower over $K^+$. For a more precise form of hypothesis (b) we refer to properties $P1(K^+/k,p)$, and e.g. $P2(K^+/k,p)$ of [So4] §3]. By using Example 3.2 of ibid., one can construct a large infinite class of extensions $K/k$ with $K$ absolutely abelian and $k \neq \mathbb{Q}$ which satisfy all the above hypotheses and hence also the weakened form of Conjecture 5.4.

Finally, we mention that methods from e.g. [R-S] could be used to test Conjecture 5.4 numerically and also the necessity of the condition that $p$ split completely in $k$. 

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