Filter-dependent versions of the Uniform Boundedness Principle

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Abstract
For every filter $\mathcal{F}$ on $\mathbb{N}$, we introduce and study corresponding uniform $\mathcal{F}$-boundedness principles for locally convex topological vector spaces. These principles generalise the classical uniform boundedness principles for sequences of continuous linear maps by coinciding with these principles when the filter $\mathcal{F}$ equals the Fréchet filter of cofinite subsets of $\mathbb{N}$. We determine combinatorial properties for the filter $\mathcal{F}$ which ensure that these uniform $\mathcal{F}$-boundedness principles hold for every Fréchet space. Furthermore, for several types of Fréchet spaces, we also isolate properties of $\mathcal{F}$ that are necessary for the validity of these uniform $\mathcal{F}$-boundedness principles. For every infinite-dimensional Banach space $X$, we obtain in this way exact combinatorial characterisations of those filters $\mathcal{F}$ for which the corresponding uniform $\mathcal{F}$-boundedness principles hold true for $X$.

Keywords: Uniform Boundedness Principle, Filters on $\mathbb{N}$, Fréchet spaces, Banach spaces

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1 Introduction

This paper contributes to the line of research ([2], [3], [7], [10], [12], [17], [18], ...) that is concerned with the study of filter versions of several theorems in analysis (or infinite combinatorics), whose classical proofs critically rely on Baire-categoric, measure-theoretic, or Ramsey-theoretic ideas. Before clarifying how to derive in a canonical way the statement $T_F$ for a filter version of the theorem $T$, let us quickly motivate the interest in the new statement $T_F$ (see also the introduction of [18]). As well as in generalising the theorem $T$ involved, additional motivation for the study of the filter version $T_F$ lies in gaining understanding of the combinatorial principles that are needed to prove $T$. Indeed, those filters on $\mathbb{N}$ that preserve validity of the filter versions often turn out to possess interesting combinatorial properties, closely related to the P-properties and Q-properties, that are classically studied in set theory. In the present paper, we show that this is true as well for filter versions of the uniform boundedness principle for Fréchet spaces.

One main corollary of the uniform boundedness principle entails continuity of the pointwise limits of sequences of continuous linear mappings. In [19] it was shown that one particular filter version of this consequence of the UBP holds for separable Banach spaces whenever the dual ideal has the property denoted in [19] by property (APO).

Another important consequence of the UBP (weak convergence implies boundedness in Banach spaces) was studied from the filter perspective in [7], [12] (see Remark 3.3 below).

In [3], Avilés, Kadets, Pérez and Solecki studied the Baire filters, which have the defining property that for every complete metric space $X$ and every order-reversing map

$$ f : (\mathcal{F}, \subseteq) \rightarrow (\{F \subseteq X : F \text{ nowhere dense in } X\}, \subseteq), $$

one has that

$$ \bigcup_{A \in \mathcal{F}} f(A) \neq X. $$

Given that the uniform boundedness principle comes as a consequence of the Baire category theorem, it is not surprising that the uniform $\mathcal{F}$-boundedness principle will hold for every Fréchet space whenever $\mathcal{F}$ is a (free) Baire filter (see Lemma 1.1 below). However, it was found in [3], that the notion of Baire filter is a rather restrictive one. The analytic Baire filters for instance, are all countably generated. This makes it natural to determine weaker conditions under which filter versions of the uniform boundedness principle are true.\footnote{Because of the duality between filters and ideals, one can equivalently study the corresponding ideal versions of such theorems. Several authors choose to do so.}
1.1 Terminology and notation

We now proceed to the introduction of the key concepts under consideration. Every filter \( F \) on \( \mathbb{N} \) gives rise to the generalised quantifier \( (\forall F i \in \mathbb{N}) \) defined by the following rule:

\[
(\forall F i \in \mathbb{N})(\Phi(i)) \iff \{ i \in \mathbb{N} : \Phi(i) \} \in F,
\]

for every formula \( \Phi \). We let \( C \) be the Fréchet filter consisting of the cofinite subsets of \( \mathbb{N} \) and let \( F \) be an arbitrary filter on \( \mathbb{N} \). Then, any concept which can be expressed in terms of the quantifier \( (\forall C i \in \mathbb{N}) \) has a direct \( F \)-analogue, obtained by replacing the quantifier \( (\forall C i \in \mathbb{N}) \) by the quantifier \( (\forall F i \in \mathbb{N}) \).

For example, convergence of a sequence \( (x_i)_i \) in a metric space \( (M,d) \) to \( x \in M \) can be expressed by the formula

\[
(\forall \varepsilon \in \mathbb{R} > 0)(\forall C i \in \mathbb{N}) \; d(x,x_i) < \varepsilon,
\]

and the replacement of generalised quantifiers gives the useful concept of \( F \)-convergence of the sequence \( (x_i)_i \) to \( x \):

\[
(\forall \varepsilon \in \mathbb{R} > 0)(\forall F i \in \mathbb{N}) \; d(x,x_i) < \varepsilon.
\]

In the same fashion, we now define \( F \)-analogues for two principal concepts from the theory of topological vector spaces, that of boundedness of a sequence and that of equicontinuity of a sequence of (continuous linear) maps. In what follows, all sequences are indexed by the non-negative integers, unless stated otherwise.

Let \( X \) be a topological vector space and \( F \) a filter on \( \mathbb{N} \). A sequence \( (x_i)_i \) in \( X \) is **\( F \)-bounded** if for every 0-neighbourhood \( U \) in \( X \) there exists \( t > 0 \) such that

\[
tx_i \in U \quad (\forall F i \in \mathbb{N}).
\]

Note that in a normed space \( X \), \( (x_i)_i \) is \( F \)-bounded if and only if there exists \( C > 0 \) such that

\[
\|x_i\| \leq C \quad (\forall F i \in \mathbb{N}).
\]

Also note that for general filters \( F \), \( F \)-boundedness is not necessarily preserved by rearrangements of sequences.

Given two topological vector spaces \( X,Y \), let \( \mathcal{L}(X,Y) \) denote the vector space of continuous linear maps from \( X \) to \( Y \).

A sequence \( (T_i)_i \) in \( \mathcal{L}(X,Y) \) is **\( F \)-equicontinuous** if for every 0-neighbourhood \( V \) in \( Y \) there exists a 0-neighbourhood \( U \) in \( X \) such that

\[
T_i[U] \subseteq V \quad (\forall F i \in \mathbb{N}).
\]

The reader can observe that if \( X \) and \( Y \) are both normed spaces, a sequence \( (T_i)_i \) of continuous linear maps from \( X \) to \( Y \) is \( F \)-equicontinuous precisely when it is \( F \)-bounded in the normed space \( \mathcal{L}(X,Y) \) of continuous linear operators from \( X \) to \( Y \).

Continuing in this same vein, a sequence \( (T_i)_i \) in \( \mathcal{L}(X,Y) \) is defined to be **pointwise \( F \)-bounded** when for every \( x \in X \), the sequence \( (T_i(x))_i \) is an \( F \)-bounded
sequence in $Y$. Equivalently, $(T_i)_i$ is pointwise $\mathcal{F}$-bounded when it is $\mathcal{F}$-bounded in the space $L(X,Y)$ equipped with the topology of pointwise convergence.

Given the topological vector spaces $X,Y$ and the filter $\mathcal{F}$ on $\mathbb{N}$, let $PW_\mathcal{F}(X,Y)$ and $EQ_\mathcal{F}(X,Y)$ denote respectively the set of all pointwise $\mathcal{F}$-bounded sequences in $L(X,Y)$ and the set of all $\mathcal{F}$-equicontinuous sequences in $L(X,Y)$.

It is clear that $EQ_\mathcal{F}(X,Y) \subseteq PW_\mathcal{F}(X,Y)$. The validity of the reverse inclusion is the subject of uniform $\mathcal{F}$-boundedness principles. We write $UBP_\mathcal{F}(X,Y)$ as abbreviation for the statement $PW_\mathcal{F}(X,Y) = EQ_\mathcal{F}(X,Y)$ and say that the uniform $\mathcal{F}$-boundedness principle holds for continuous linear maps from $X$ to $Y$ whenever this statement is true. We will also say that the uniform $\mathcal{F}$-boundedness principle holds for $X$ if $UBP_\mathcal{F}(X,Y)$ holds for every locally convex space $Y$.

We define $\mathcal{F}$ to be a Fréchet-UBP-filter if $UBP_\mathcal{F}(X,Y)$ holds for every Fréchet space $X$ and every locally convex space $Y$.

By the classical theorem of Banach and Steinhaus, the Fréchet filter $\mathcal{C}$ is a Fréchet-UBP-filter.

We define $\mathcal{F}$ to be a Banach-UBP-filter if $UBP_\mathcal{F}(X,Y)$ holds for every Banach space $X$ and every locally convex space $Y$.

The key concepts being introduced above, we can now state our main objectives. Before doing so, we briefly illustrate the concepts just introduced by checking that the Baire filters studied in [3] are indeed Fréchet-UBP-filters.

Lemma 1.1.
If the filter $\mathcal{F}$ (on $\mathbb{N}$) is a Baire filter containing every cofinite subset of $\mathbb{N}$, then $UBP_\mathcal{F}(X,Y)$ holds for every Fréchet space $X$ and every topological vector space $Y$.

Proof. Let $(T_i)_i$ be a pointwise $\mathcal{F}$-bounded sequence in $L(X,Y)$. We mimic the Baire category proof of the uniform boundedness principle to prove that $(T_i)_i$ is an $\mathcal{F}$-equicontinuous sequence. Let $W$ be an arbitrary 0-neighbourhood in $Y$ and choose a second 0-neighbourhood $V$ in $Y$ which is closed and balanced, and satisfies $V - V \subseteq W$.

For every $F \in \mathcal{F}$ and $n \in \mathbb{N}$, we define a corresponding set $C_{F,n} \subseteq X$ as follows:

$$C_{F,n} := \bigcap_{i \in F} T_i^{-1}[nV].$$

Then, the pointwise $\mathcal{F}$-boundedness of $(T_i)_i$ allows us to write:

$$X = \bigcup_{F \in \mathcal{F}} \bigcup_{n \in \mathbb{N}} C_{F,n} = \bigcup_{F \in \mathcal{F}} C_{F,\min(F)}.$$

Since $f : (\mathcal{F}, \subseteq) \to (P(X), \subseteq) : F \mapsto C_{F,\min(F)}$ is an order-reversing map with $C_{F,\min(F)}$ closed for any $F \in \mathcal{F}$, it follows from the defining property of Baire filters that for some $F \in \mathcal{F}$, the set $C_{F,\min(F)}$ has non-empty interior.

Therefore, there exist $x \in X$, a 0-neighbourhood $U$ in $X$ and $n \in \mathbb{N}$ such that

$$T_i[x + U] \subseteq nV \quad (\forall i \in \mathbb{N}).$$
Hence,
\[
T_i \left[ \frac{1}{n} U \right] \subseteq V - V \subseteq W \quad (\forall i \in \mathbb{N}).
\]

1.2 Aims

We will answer the following questions.

1. Is every Banach-UBP-filter a Fréchet-UBP-filter? (Theorem 5.1)

2. Does there exist a combinatorial characterisation of the Fréchet- or Banach-UBP-filters? (Theorem 5.1)

3. Do there exist Fréchet-UBP-filters that are not countably generated? (Example 5.1)

4. If \( X \) is an infinite-dimensional Banach space, does the validity of the uniform \( F \)-boundedness principle for \( X \) depend on the relationship between the space \( X \) and the filter \( F \), or solely on the filter \( F \)? (Theorem 5.1)

5. Is the existence of either Fréchet- or Banach-UBP-ultrafilters consistent with ZFC? Is the existence of such ultrafilters provable in ZFC? (Corollary 5.1)

2 Combinatorics of filters on a countable set

A filter on a set \( S \) is a non-empty \( \subseteq \)-upwards closed set \( F \subseteq P(S) \) (where \( P(S) \) denotes the power set of \( S \)) which is closed under finite intersections. Throughout this paper, we will always have \( S = \mathbb{N} := \{0, 1, 2, 3, \ldots\} \) and will only consider filters which are free, i.e. we will make the extra assumption that \( C \subseteq F \). The corresponding questions for non-free filters can be reduced to the case of free filters.

An ideal on a set \( S \) is a non-empty \( \subseteq \)-downwards closed set \( I \subseteq P(S) \) which is closed under finite unions. If \( F \) is a filter on \( S \), the set \( F^* = \{ A \subseteq S : A^c \in F \} \) is the dual ideal of \( F \). We will denote the set \( P(S) \setminus F^* \) of \( F \)-stationary sets by \( F^+ \). Note that a subset of \( S \) is \( F \)-stationary precisely when it has non-empty intersection with every element of \( F \).

For two sets \( A, B \subseteq \mathbb{N} \), we write \( A \subseteq^* B \) whenever \( A \) is contained in \( B \) modulo a finite set, i.e. \( A \setminus B \) is finite. For \( A \subseteq \mathbb{N} \) infinite, the enumerating function of \( A \) is the unique strictly increasing surjection \( \eta_A : \mathbb{N} \to A \).

We will show that the uniform boundedness principle for filters is closely related to the following combinatorial properties for filters on \( \mathbb{N} \).

**Definition 2.1.**

Let \( F \) be a filter on \( \mathbb{N} \), then \( F \) is

\[\text{\underline{\text{\textbf{\textit{\textbullet}}}}\text{\textbullet}}\text{\textbf{\textit{\textbullet}}}\]
• a weak \( \mathbf{P}^+ \)-filter if for every decreasing sequence \((A_k)_k\) of sets in \( \mathcal{F} \) and for every \( \mathcal{F} \)-stationary set \( I \subseteq \mathbb{N} \) there exists an \( \mathcal{F} \)-stationary set \( B \subseteq I \) such that \( B \subseteq^* A_k \) for every \( k \).

• a \( \mathbf{P}^+ \)-filter if for every decreasing sequence \((A_k)_k\) of \( \mathcal{F} \)-stationary sets, there exists an \( \mathcal{F} \)-stationary set \( B \) such that \( B \subseteq A_k \) for every \( k \).

• a rapid\(^+\) filter if for every \( \mathcal{F} \)-stationary set \( I \subseteq \mathbb{N} \) and every strictly increasing function \( f : \mathbb{N} \to \mathbb{N} \), there exists an \( \mathcal{F} \)-stationary set \( B \subseteq I \) such that the function enumerating \( B \) dominates \( f \), i.e., \( f \leq \eta_B \).

The rapid\(^+\) and (weak) \( \mathbf{P}^+ \)-properties of filters, along with several closely related variations, have appeared in prior literature (see [13], [15], [21], . . . ). We advise the reader to be careful while consulting this literature as the terminology used to denote combinatorial properties of filters does vary among authors. Let us point out that the study of these and related properties of filters has originated in the study of ultrafilters on \( \mathbb{N} \). For free ultrafilters the weak \( \mathbf{P}^+ \)-property, the \( \mathbf{P}^+ \)-property and several other related properties coincide and they determine a topological invariant of points in the Stone-\v{C}ech remainder \( \beta\mathbb{N} \setminus \mathbb{N} \).

In [24], this \( \mathbf{P} \)-property was used by W. Rudin to study non-homogeneity of the space \( \beta\mathbb{N} \setminus \mathbb{N} \).

In the following Lemma 2.1, Lemma 2.2 and Lemma 2.3, we formulate well-known alternative characterisations of respectively weak \( \mathbf{P}^+ \)-filters, \( \mathbf{P}^+ \)-filters and rapid\(^+\) filters, which will be of value further on. The proof of Lemma 2.2 is a straightforward modification of the proof of Lemma 2.1, so we only give the latter. See also [13] and [21] for similar proofs of these (and closely related) results.

**Lemma 2.1.**

A filter \( \mathcal{F} \) on \( \mathbb{N} \) is a weak \( \mathbf{P}^+ \)-filter if and only if for every function \( \Gamma : \mathbb{N} \to \mathbb{N} \) one of the following two statements holds:

- \( \Gamma \) is bounded on some \( \mathcal{F} \)-stationary set,
- every \( \mathcal{F} \)-stationary set \( I \) has an \( \mathcal{F} \)-stationary subset on which \( \Gamma \) is finite-to-one.

**Proof.** Suppose first that \( \mathcal{F} \) is a weak \( \mathbf{P}^+ \)-filter and that \( \Gamma : \mathbb{N} \to \mathbb{N} \) is unbounded on every \( \mathcal{F} \)-stationary set. Then, for every \( k \), the set \( A_k := \{ n \in \mathbb{N} : \Gamma(n) > k \} \) belongs to \( \mathcal{F} \). This implies that for every \( \mathcal{F} \)-stationary set \( I \), there exists an \( \mathcal{F} \)-stationary set \( B \subseteq I \) such that \( B \subseteq A_k \) for every \( k \). By consequence, \( \Gamma \) is finite-to-one on \( B \).

Suppose next that \( \mathcal{F} \) has the property of Lemma 2.1. Let \((A_k)_k\) be a decreasing sequence of sets in \( \mathcal{F} \). Let \( A_k' = A_k \cap [k, +\infty[ \). Define \( \Gamma : \mathbb{N} \to \mathbb{N} \) as follows:

\[
\begin{align*}
\Gamma(n) &= \max\{ l : n \in A_l' \} \quad \text{if} \ n \in A_0', \\
\Gamma(n) &= 0 \quad \text{if} \ n \notin A_0'.
\end{align*}
\]
Now, $\{ n \in \mathbb{N} : \Gamma(n) \leq k \} \subseteq \mathbb{N} \setminus A_{k+1}'$ cannot be $\mathcal{F}$-stationary. Hence, for every $\mathcal{F}$-stationary set $I$, there is an $\mathcal{F}$-stationary set $B \subseteq I$ on which $\Gamma$ is finite-to-one. This implies that $B \subseteq A_k' \subseteq A_k$ for every $k$. \hfill \Box

**Lemma 2.2.**

A filter $\mathcal{F}$ on $\mathbb{N}$ is a $P^+$-filter if and only if for every function $\Gamma : \mathbb{N} \to \mathbb{N}$ one of the following two statements holds:

- $\Gamma$ is bounded on some element of $\mathcal{F}$,
- $\Gamma$ is finite-to-one on some $\mathcal{F}$-stationary set.

If $I \subseteq \mathbb{N}$, we say that $I' \subseteq I$ is an interval of $I$ whenever $I' = I \cap [a, b)$ for $a, b \in \mathbb{N}$.

**Definition 2.2.**

An increasing interval-partition of $I$ is a partition of $I$ into intervals $(I_n)_n$ of $I$ such that $\max(I_n) < \min(I_m)$ for each $n < m$.

To a strictly increasing function $f : \mathbb{N} \to I$, we associate the increasing interval-partition $(I'_n)_n$ defined by $I'_n = I \cap (f(n-1), f(n)]$ (where we agree on $f(-1) = -1$ for all such $f$). Clearly, every increasing interval-partition is of the form $(I'_n)_n$ for a unique strictly increasing function $f : \mathbb{N} \to I$.

**Definition 2.3.**

Let $\mathcal{F}$ be a filter on $\mathbb{N}$ and $I \subseteq \mathbb{N}$ $\mathcal{F}$-stationary. Let $(I_n)_n$ be an increasing interval-partition of $I$. We say that $\mathcal{F}$ is slow with respect to $(I_n)_n$ if every $\mathcal{F}$-stationary $A \subseteq I$ satisfies $(\exists n \in \mathbb{N})(|A \cap I_n| > n)$.

**Lemma 2.3.**

If $\mathcal{F}$ is a filter on $\mathbb{N}$, then the following two statements are equivalent:

1. $\mathcal{F}$ is rapid$^+$.
2. No $\mathcal{F}$-stationary set $I \subseteq \mathbb{N}$ has an increasing interval-partition $(I_n)_n$ such that $\mathcal{F}$ is slow with respect to $(I_n)_n$.

Proof.

The implication 1. $\Rightarrow$ 2. is easily verified, we prove the implication 2. $\Rightarrow$ 1.

Let $I \subseteq \mathbb{N}$ be $\mathcal{F}$-stationary. Let $f : \mathbb{N} \to I$ be strictly increasing. We prove that there exists a $B \subseteq I$ $\mathcal{F}$-stationary for which $f \leq \eta_B$, given that, for each increasing interval-partition $(I_n)_n$ of $I$, $\mathcal{F}$ is not slow with respect to $(I_n)_n$.

Define $q : \mathbb{N} \to \mathbb{N} : q(n) = \frac{n(n+1)}{2}$.

Consider the following increasing interval-partition $(I_n)_n$ of $I$:

$I_0 = [0, f(q(1))] \cap I$, $I_n = [f(q(n)), f(q(n+1))] \cap I$ \quad $\forall n > 0$.

By assumption $\mathcal{F}$ is not slow with respect to $(I_n)_n$, so we can choose $B \subseteq I$ $\mathcal{F}$-stationary, for which $(\forall n \in \mathbb{N})(|B \cap I_n| \leq n)$.
It now suffices to show that for such $B \subseteq I$ $\mathcal{F}$-stationary we also have

$$(\forall n \in \mathbb{N}) \ |B \cap [0, f(n)]| \leq n.$$  

We can see this as follows. Given $n$, choose $r \in \mathbb{N}$ maximal such that $q(r) \leq n$, then $f(n) < f(q(r + 1))$, so $B \cap [0, f(n)] \subseteq \bigcup_{l \leq r} B \cap I_l$, but then

$$|B \cap [0, f(n)]| \leq \big| \bigcup_{l \leq r} B \cap I_l \big| \leq \sum_{l \leq r} l = q(r) \leq n.$$ 

Every countably generated filter $\mathcal{F}$ on $\mathbb{N}$ is both a rapid$^+$ filter and a P$^+$-filter. We now illustrate these properties with three important examples of filters. The first two examples originate in real analysis.

**Example 2.1.**

The asymptotic density filter $\mathcal{F}_d$ consists of those sets $A \subseteq \mathbb{N}$ that satisfy

$$\lim_{n \to \infty} \frac{|\{k < n : k \in A\}|}{n} = 1.$$ 

It is an elementary matter to check that $\mathcal{F}_d$ is a weak $P^+$-filter, but not a $P^+$-filter, nor a rapid$^+$ filter. The notion of convergence corresponding to this filter is also known as statistical convergence.

**Example 2.2.**

To every $f : \mathbb{N} \to \mathbb{R}_{>0}$ with $\lim_{n \to \infty} f(n) = 0$ and $\sum_{n \in \mathbb{N}} f(n) = \infty$ corresponds a filter $\mathcal{F}_{\text{sum},f}$ which is dual to the summable ideal

$$\mathcal{I}_{\text{sum},f} = \{A \subseteq \mathbb{N} : \sum_{n \in A} f(n) < \infty\}.$$ 

One checks easily that $\mathcal{F}_{\text{sum},f}$ is a $P^+$-filter which is not rapid$^+$.

**Example 2.3.**

Let $b_1, b_2 : \mathbb{N} \to \mathbb{N}$ such that the map $n \mapsto (b_1(n), b_2(n))$ is a bijection from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. The filter $\mathcal{F}_p \equiv \{\mathbb{N}\} \times \mathcal{C}$ consists of those sets $A \subseteq \mathbb{N}$ that satisfy

$$(\forall k \in \mathbb{N}) \ \{b_2(n) : n \in A \cap b_1^{-1}([k])\} \in \mathcal{C}.$$ 

By noting that a set $A \subseteq \mathbb{N}$ is $\mathcal{F}_p$-stationary if and only if there exists $k \in \mathbb{N}$ such that $A \cap b_1^{-1}([k])$ is infinite, it is easily seen that $\mathcal{F}_p$ is both a weak $P^+$-filter and a rapid$^+$ filter. However, the sequence $(A_k)_k$ with

$$A_k = \{n \in \mathbb{N} : b_1(n) \geq k\},$$

forms a decreasing sequence of $\mathcal{F}_p$-stationary sets and witnesses that $\mathcal{F}_p$ is not a $P^+$-filter.

All three filters $\mathcal{F}_d$, $\mathcal{F}_{\text{sum},f}$ and $\mathcal{F}_p$ belong to the well-studied class of analytic P-filters, see e.g. [9].
3 Sufficient conditions for Fréchet-UBP-filters

A natural weaker variation of the uniform $\mathcal{F}$-boundedness principle is given by the following stationary uniform $\mathcal{F}$-boundedness principle.

**Definition 3.1.**

Let $X, Y$ be topological vector spaces. A sequence $(T_i)_i$ in $\mathcal{L}(X, Y)$ is $\mathcal{F}$-stationary-equicontinuous if for every $\mathcal{F}$-stationary set $I \subseteq \mathbb{N}$ and every $0$-neighbourhood $V$ in $Y$, there exists a $0$-neighbourhood $U$ in $X$ such that:

$$\exists J \in \mathcal{F}^+, J \subseteq I \forall i \in J \ T_i[U] \subseteq V.$$  

We say that the **stationary uniform $\mathcal{F}$-boundedness principle** holds for continuous linear maps from $X$ to $Y$, and denote this by $\text{UBP}_{\mathcal{F}}^\text{stat}(X, Y)$, whenever every pointwise $\mathcal{F}$-bounded sequence $(T_i)_i$ in $\mathcal{L}(X, Y)$ is $\mathcal{F}$-stationary-equicontinuous. We say that the stationary uniform $\mathcal{F}$-boundedness principle holds for $X$ if $\text{UBP}_{\mathcal{F}}^\text{stat}(X, Y)$ holds for every locally convex space $Y$. We define $\mathcal{F}$ to be a **stationary Fréchet-UBP-filter** if the stationary uniform $\mathcal{F}$-boundedness principle holds for every Fréchet space $X$. **Stationary Banach-UBP-filters** are defined in the analogous way.

We will find that, in several ways, the stationary uniform $\mathcal{F}$-boundedness principle relates to the uniform $\mathcal{F}$-boundedness principle as weak $P^+$-filters relate to $P^+$-filters.

In this section, it will be proved that the previously introduced combinatorial properties for filters can be used to express sufficient conditions for the uniform $\mathcal{F}$-boundedness principles to hold for Fréchet spaces. In particular, we will prove the following theorem.

**Theorem 3.1.**

(a) Every rapid$^+$ weak $P^+$-filter is a stationary Fréchet-UBP-filter.

(b) Every rapid$^+$ $P^+$-filter is a Fréchet-UBP-filter.

The theory of uniform boundedness principles for (locally convex) topological vector spaces is traditionally phrased in terms of barrels and barrelled spaces. To stay in line with this case where $\mathcal{F}$ is equal to the Fréchet filter $\mathcal{C}$, we first introduce an $\mathcal{F}$-analogue for the concept of barrelled space. Recall that a subset $B$ of a topological vector space $X$ is a barrel when it is absorbing, balanced, closed and convex (our terminology here is in line with [28]).

**Definition 3.2.**

An $\mathcal{F}$-barrel-system in $X$ is a sequence $(B_i)_i$ consisting of barrels $B_i$ in $X$ with the property that for every $x \in X$, there is $t > 0$ such that

$$tx \in B_i \quad (\forall \mathcal{F} i \in \mathbb{N}).$$
In the following definition, we give the obvious meanings to pointwise $\mathcal{F}$-boundedness and $\mathcal{F}$-equicontinuity of a sequence of continuous seminorms.

**Definition 3.3.**
Let $X$ be a topological vector space and $(p_i)_i$ a sequence of continuous seminorms on $X$, let $(B_i)_i$ be the corresponding sequence of barrels with $B_i = \{x \in X : p_i(x) \leq 1\}$.

The sequence $(p_i)_i$ is called pointwise $\mathcal{F}$-bounded if $(B_i)_i$ is an $\mathcal{F}$-barrel-system.

The sequence $(p_i)_i$ is called $\mathcal{F}$-equicontinuous if there exists a 0-neighbourhood $U$ in $X$ such that

$$U \subseteq B_i \quad (\forall \mathcal{F}^i \in \mathbb{N}).$$

**Definition 3.4.**
A topological vector space $X$ is $\mathcal{F}$-barrelled if for every $\mathcal{F}$-barrel-system $(B_i)_i$ in $X$ there exists a 0-neighbourhood $U$ in $X$ such that

$$U \subseteq B_i \quad (\forall \mathcal{F}^i \in \mathbb{N}).$$

**Definition 3.5.**
A topological vector space $X$ is stationarily $\mathcal{F}$-barrelled if for every $\mathcal{F}$-barrel-system $(B_i)_i$ in $X$ and for every $\mathcal{F}$-stationary set $I \subseteq \mathbb{N}$ there exists a 0-neighbourhood $U$ in $X$ and an $\mathcal{F}$-stationary $J \subseteq I$ such that

$$U \subseteq B_i \quad (\forall i \in J).$$

Note that if $\mathcal{F}$ coincides with the Fréchet filter $\mathcal{C}$ of cofinite sets, the concepts of $\mathcal{F}$-barrelled space and stationarily $\mathcal{F}$-barrelled space do indeed coincide with the classical notion of barrelled space.

Just as in the case where $\mathcal{F}$ equals the Fréchet filter $\mathcal{C}$, we find that continuous linear maps from $\mathcal{F}$-barrelled spaces to locally convex spaces satisfy uniform $\mathcal{F}$-boundedness principles.

**Lemma 3.1.**
For every $\mathcal{F}$-barrelled space $X$ and every locally convex space $Y$, $\text{UBP}_\mathcal{F}(X,Y)$ holds.

For every stationarily $\mathcal{F}$-barrelled space $X$ and every locally convex space $Y$, $\text{UBP}^\text{stat}_\mathcal{F}(X,Y)$ holds.

**Proof.** We prove the first statement, the proof of the second statement is analogous, and both proofs are entirely analogous to the proof of the uniform boundedness principle for barrelled spaces.

Let $(T_i)_i$ be a pointwise $\mathcal{F}$-bounded sequence in $\mathcal{L}(X,Y)$. Let $U$ be an arbitrary 0-neighbourhood in $Y$, we may assume that $U$ is a barrel. Because every $T_i$ is...
linear and continuous, $T_i^{-1}[U]$ is a barrel for every $i$. By pointwise $\mathcal{F}$-boundedness, $(T_i^{-1}[U])_i$ is an $\mathcal{F}$-barrel-system. Using $\mathcal{F}$-barreledness, we find some 0-neighbourhood $V$ in $X$ such that $(\forall \, i \in \mathbb{N}) \, V \subseteq T_i^{-1}[U]$, so we have proved $\mathcal{F}$-equicontinuity of $(T_i)_i$.

**Remark 3.1.** It follows from the given definitions that if $X$ is a barrelled space, then $X$ is $\mathcal{F}$-barreled if and only if the uniform $\mathcal{F}$-boundedness principle for seminorms holds for $X$, i.e. every pointwise $\mathcal{F}$-bounded sequence of continuous seminorms on $X$ is $\mathcal{F}$-equicontinuous. Analogously, if $X$ is barrelled, then $X$ is stationarily $\mathcal{F}$-barreled if and only if the stationary uniform $\mathcal{F}$-boundedness principle for seminorms holds for $X$, i.e. every pointwise $\mathcal{F}$-bounded sequence of continuous seminorms on $X$ is $\mathcal{F}$-stationary-equicontinuous.

In Theorem 3.2 below, we will prove that if $\mathcal{F}$ is a rapid $+$ (weak) $P^+$-filter, then every Fréchet space is (stationarily) $\mathcal{F}$-barrelled. Our proof of Theorem 3.2 uses and extends several ideas found in the elementary proof for the uniform boundedness principle given by Sokal in [27]. In this proof, we will repeatedly make use of a general non-empty intersection principle for completely metrizable topological vector spaces which we now describe first.

Our situation of interest will be the following. Suppose given a completely metrizable topological vector space $X$ together with two countable bases $(V_n)_n$ and $(A_n)_n$ for the filter of 0-neighbourhoods of $X$. Suppose that for all integers $l \geq 0$, $n \geq 1$:

$$A_n + A_{n+1} + \ldots + A_{n+l} \subseteq V_n.$$  \hfill (L1)

We will additionally assume that all neighbourhoods $V_n$ are closed.  \hfill (L2)

Now, let $(C_n)_n$ be a sequence of subsets of $X$ and suppose that we wish to show that the intersection $\bigcap_{n \geq 1} (C_n + V_{n+1})$ is non-empty. The following lemma provides a means of deriving this conclusion, at least when the sequences $(C_n)_n$ and $(A_n)_n$ have the following property:

$$C_1 \neq \emptyset,$$

$$(\forall n \geq 2) (\forall x \in C_{n-1}) ((x + A_n) \cap C_n \neq \emptyset).$$  \hfill (L3)

We call a sequence $(C_n)_n$ with this property $(A_n)_n$-connected.

**Lemma 3.2.**

If $X$ is a completely metrizable topological vector space with $(V_n)_n$, $(A_n)_n$ two bases for the filter of 0-neighbourhoods of $X$ that satisfy (L1) and (L2), then the intersection $\bigcap_{n \geq 1} (C_n + V_{n+1})$ is non-empty for every sequence $(C_n)_n$ of subsets of $X$ that is $(A_n)_n$-connected (i.e. that satisfies (L3)).

**Proof.** We recursively construct a sequence $(x_k)_k$ in $X$.

Choose $x_1 \in C_1$ arbitrary. For $k \geq 2$, choose $x_k \in (x_{k-1} + A_k) \cap C_k$, it follows from (L3) that this is always possible. From this construction, it follows that

$$x_{k+l} - x_k \in A_{k+1} + \ldots + A_{k+l} \subseteq V_{k+1}$$

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for every $l \in \mathbb{N}$ and every $k \geq 1$.

Because $(V_k)_k$ is a 0-neighbourhood base for the complete space $X$, we find that $x_k \to x$ for some $x \in X$. Next, $(V_k)_k$ consisting of closed sets, we can take limits in $x_{k+1} - x_k \in V_{k+1}$ to find that $x \in x_k + V_{k+1}$ for every $k \geq 1$.

Hence, $x \in \bigcap_{k \geq 1} (x_k + V_{k+1}) \subseteq \bigcap_{k \geq 1} (C_k + V_{k+1})$.

We will also need the following two lemmas that both concern the local behaviour of seminorms.

**Lemma 3.3 ([27]).**

Let $X$ be a vector space and $U$ a symmetric subset (i.e., $U = -U$) of $X$.

If $p$ is a seminorm on $X$, then

$$
\sup_{y \in x + rU} p(y) \geq r \sup_{y \in U} p(y) \quad \forall x \in X, \forall r \in \mathbb{R}_{>0}.
$$

**Proof.** We can argue as follows:

$$
\sup_{y \in x + rU} p(y) \geq \frac{1}{2} \sup_{y \in U} (p(x + ry) + p(x - ry)) \\
\geq \frac{1}{2} \sup_{y \in U} (p(x + ry) - (x - ry)) = r \sup_{y \in U} p(y).
$$

If $p$ is a seminorm on a vector space $X$, the following notation will be employed to denote the closed unit seminorm-ball defined by $p$:

$$
B_p := \{ x \in X : p(x) \leq 1 \}.
$$

**Lemma 3.4.**

Let $X$ be a vector space, and $p, q_1, \ldots, q_k$ seminorms on $X$.

If $q_1, \ldots, q_k$ are all unbounded on $B_p$, then $\min(q_1, \ldots, q_k)$ is unbounded on $B_p$.

**Proof.** Because each of the seminorms $q_i$ is unbounded on $B_p$, there exists for every $i \in \{1, \ldots, k\}$ a sequence $(x^i_n)_n$ with $q_i(x^i_n) \geq 1$ and $x^i_n \xrightarrow{p} 0$ as $n \to \infty$.

Now, as an auxiliary means for the rest of the proof, fix an arbitrary free ultrafilter $\mathcal{U} \subseteq P(\mathbb{N})$. Consider for every $i$ the finite-dimensional vector space $V_i \leq \mathbb{R}^k$ defined by

$$
V_i := \{ (r_1, \ldots, r_k) \in \mathbb{R}^k : \lim_{\mathcal{U},n} q_i(r_1 x^1_n + \ldots + r_k x^k_n) = 0 \}.
$$

Since the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ (with 1 on place $i$) is not contained in $V_i$, every $V_i$ is a proper subspace of $\mathbb{R}^k$. It follows that $\mathbb{R}^k \neq \bigcup_{i=1}^k V_i$, so we can select

$$(r_1, \ldots, r_k) \in \bigcap_{i=1}^k V_i^c.$$
Set $z_n := r_1 x_n^1 + \ldots + r_k x_n^k$. From the choice of the scalars $(r_1, \ldots, r_k)$ and the definition of an ultrafilter limit, it follows that there exists $\varepsilon > 0$ and $U \in \mathcal{U}$ such that for every $i \in \{1, \ldots, k\}$ and every $n \in U$, $q_i(z_n) > \varepsilon$.

Thus, $\min(q_1, \ldots, q_k)(z_n) > \varepsilon$ for every $n \in U$, while $z_n \xrightarrow{\mathcal{B}_p} 0$ as $n \to \infty$. Hence, $\min(q_1, \ldots, q_k)$ is unbounded on $B_p$.

**Lemma 3.5.**

If $X$ is a Fréchet space, $\mathcal{F}$ a $P^+$-filter and $(q_i)_i$ a pointwise $\mathcal{F}$-bounded sequence of continuous seminorms on $X$, then there exists a 0-neighbourhood $U$ in $X$ and $F \in \mathcal{F}$ such that $q_i$ is bounded on $U$ whenever $i \in F$ (with bounds possibly depending on $i$).

**Proof.** Fix a base $(p_m)_m$ of continuous seminorms for $X$ with $p_m \leq p_{m+1}$ for every $m$. Because every $q_i$ is a continuous seminorm on $X$, there exists for every $i$ a corresponding integer $m$ such that $q_i$ is bounded on the seminorm-ball $B_{p_m}$.

As a consequence, the following function is well defined:

$$l : \mathbb{N} \to \mathbb{N} : l(i) = \min\{m \in \mathbb{N} : q_i \text{ is bounded on } B_{p_{m+1}}\}.$$  

The assertion of the lemma is established if we can show that $l$ is bounded on an element of $\mathcal{F}$. Because $\mathcal{F}$ is a $P^+$-filter, we can apply Lemma 2.2 to find that it suffices to prove that the function $l$ is not finite-to-one on any $\mathcal{F}$-stationary set.

Striving for contradiction, let’s suppose that $l$ is in fact finite-to-one on the $\mathcal{F}$-stationary set $I$. For every integer $k \geq 0$, set

$$\sigma_k := \sup\{q_i(x) : i \in I, l(i) = k \text{ and } x \in B_{p_{k+1}}\}.$$  

It follows from the definition of $l(i)$ together with $l$ being finite-to-one on $I$, that $\sigma_k < \infty$.

Next, we wish to apply Lemma 3.2 to the system $(A_k)_k$, $(V_k)_k$, $(C_k)_k$ defined by:

$$A_k = \{x \in X : p_k(x) \leq 3^{-k}\}$$
$$V_k = \{x \in X : p_k(x) \leq \frac{3}{2} 3^{-k}\}$$
$$C_k = \{x \in X : (\forall i \in I)(l(i) = k \Rightarrow q_i(x) \geq k + \frac{3}{2} \sigma_k)\}.$$  

It is easily verified that the filter bases $(A_k)_k$ and $(V_k)_k$ indeed satisfy the conditions (L1) and (L2). Before checking (L3), note that for every $k \geq 1$, the set $\{q_i : l(i) = k, i \in I\}$ contains finitely many seminorms which are all unbounded on $B_{p_k}$. Hence, it follows directly from Lemma 3.2 that $C_k$ is non-empty. We prove that in addition $(y + A_k) \cap C_k$ is non-empty for every $k \geq 2$ and every $y \in X$, so that $(C_k)_k$ is indeed $(A_k)_k$-connected. Given $y \in X$ and $k \geq 2$ it follows from a second application of Lemma 3.2 that there is $x \in \frac{1}{3^n} B_{p_k}$ such that for every $i \in I$ with $l(i) = k$ the following is true

$$q_i(x) \geq k + \frac{3}{2} \sigma_k + \max\{q_j(y) : j \in I, l(j) = k\}.$$  

Then \( x + y \) belongs to \((y + A_k) \cap C_k\). It follows that we can apply Lemma 3.2 and find that there exists \( x \in \bigcap_{k \geq 1} (C_k + V_{k+1}) \). Then for every \( i \in I \) and \( k \geq 1 \) with \( l(i) = k \), we have:

\[
q_i(x) \geq \inf \{q_i(y) : y \in C_k\} - \sup \{q_i(y) : y \in V_{k+1}\} \\
\geq k + \frac{3\sigma_k}{2} - \sup \{q_i(y) : y \in V_{k+1}\} \\
\geq k + \frac{3\sigma_k}{2} (1 - 3^{-k-1}) \to \infty,
\]

as \( k \to \infty \). Because \( l \) is also finite-to-one on \( I \), it follows that \((q_i(x))_i\) tends to \(+\infty\) on the \( F \)-stationary set \( I \) and consequently that \((q_i)_i\) can not be pointwise \( F \)-bounded.

\[\square\]

**Lemma 3.6.**

*If \( X \) is a Fréchet space, \( F \) a weak \( P^+ \)-filter and \((q_i)_i\) a pointwise \( F \)-bounded sequence of continuous seminorms on \( X \), then for every \( F \)-stationary set \( I \), there exists a \( 0 \)-neighbourhood \( U \) in \( X \) and an \( F \)-stationary subset \( J \) of \( I \) such that \( q_i \) is bounded on \( U \) whenever \( i \in J \) (with bounds possibly depending on \( i \)).*

*Proof.* As in Lemma 3.5, \( l : \mathbb{N} \to \mathbb{N} : l(i) = \min \{m \in \mathbb{N} : q_i \text{ is bounded on } B_{p_{m+1}}\} \), is well defined. Let \( I \) be an arbitrary \( F \)-stationary set. We will apply Lemma 2.1 to the mapping \( l_I : \mathbb{N} \to \mathbb{N}, \text{ defined by } l_I(i) = l(i) \text{ if } i \in I \text{ and } l_I(i) = i \text{ for every } i \in I^c \).

If \( l_I \) is bounded on an \( F \)-stationary set, then it is also bounded on an \( F \)-stationary subset of \( I \) and this would lead directly to the conclusion in the lemma. Because \( F \) is a weak \( P^+ \)-filter it suffices to prove that the function \( l_I \) is not finite-to-one on any \( F \)-stationary subset of \( I \). Suppose in desire of contradiction that \( l \) is finite-to-one on the \( F \)-stationary subset \( J \) of \( I \). Proceeding exactly as in Lemma 3.5 but with \( J \) in the role of \( I \), one uses Lemma 3.2 to produce under this assumption \( x \in X \) such that \((q_i(x))_i\) tends to \(+\infty\) on the \( F \)-stationary set \( J \). This leads to contradiction with the pointwise \( F \)-boundedness of \((q_i)_i\).

\[\square\]

We are now ready to give the proof of Theorem 3.2. Note that Theorem 3.1 then follows by combining Theorem 3.2 and Lemma 3.1.

**Theorem 3.2.**

(a) If \( X \) is a Fréchet space and \( F \) is a rapid \( P^+ \)-filter then \( X \) is \( F \)-barrelled.

(b) If \( X \) is a Fréchet space and \( F \) is a rapid \( P^+ \)-weak \( P^+ \)-filter then \( X \) is stationarily \( F \)-barrelled.

*Proof.* Let \( X \) be a Fréchet space and let \( F \) be a filter.

Fix a base \((p_m)_m\) of continuous seminorms for \( X \) with \( p_m \leq p_{m+1} \) for every \( m \). Let an arbitrary \( F \)-barrel-system \((B_i)_i \) in \( X \) be given. Denote by \( q_i \) the
Minkowski-functional corresponding to the barrel $B_i$. Because $X$ is barrelled, every $q_i$ is a continuous seminorm on $X$ and because $(B_i)_i$ forms an $\mathcal{F}$-barrel-system, the sequence $(q_i)_i$ is pointwise $\mathcal{F}$-bounded.

We first prove the following claim:

**Claim.** If $(q_i)_i$ is a pointwise $\mathcal{F}$-bounded sequence of seminorms on $X$ and $(a_k)_k$ is a sequence of non-negative integers satisfying

$$4^k \leq \sup \{q_{a_k}(x) : p_k(x) \leq 1\} < \infty$$

for every $k$, then $\{a_k : k \in \mathbb{N}\}$ cannot be $\mathcal{F}$-stationary.

**Proof of Claim.** Suppose that we could find such a sequence $(a_k)_k$ of non-negative integers with $\{a_k : k \in \mathbb{N}\}$ $\mathcal{F}$-stationary. Then, we could apply Lemma 3.2 to the system $(A_k)_k$, $(V_k)_k$, $(C_k)_k$ defined by:

- $A_k = \{x \in X : p_k(x) \leq 3^{-k}\}$
- $V_k = \{x \in X : p_k(x) \leq \frac{3}{2}3^{-k}\}$
- $C_k = \{x \in X : q_{a_k}(x) \geq \frac{2}{3^{k+1}} \sup \{q_{a_k}(x) : p_k(x) \leq 1\}\}$.

It is again easily verified that the filter bases $(A_k)_k$ and $(V_k)_k$ satisfy the conditions (L1) and (L2) from Lemma 3.2 and it follows from a direct application of Lemma 3.3 that $(C_k)_k$ is $(A_k)_k$-connected. Therefore, we can apply Lemma 3.2 to find some $x \in \bigcap_{k \geq 1} (C_k + V_{k+1})$. Then:

$$q_{a_k}(x) \geq \inf \{q_{a_k}(y) : y \in C_k\} - \sup \{q_{a_k}(y) : y \in V_{k+1}\}$$

$$\geq \frac{2}{3^{k+1}} \sup \{q_{a_k}(x) : p_k(x) \leq 1\} - \frac{3}{2}3^{-(k+1)} \sup \{q_{a_k}(x) : p_{k+1}(x) \leq 1\}$$

$$\geq \frac{1}{6}3^{-k} \sup \{q_{a_k}(x) : p_k(x) \leq 1\} \geq \frac{1}{6} \left(\frac{4}{3}\right)^k \to \infty,$$

as $k \to \infty$. Hence, $(q_i(x))_i$ tends to $+\infty$ on the $\mathcal{F}$-stationary set $\{a_k : k \in \mathbb{N}\}$, so that $(q_i)_i$ is not pointwise $\mathcal{F}$-bounded. This concludes the proof of the claim.

We proceed to prove statements (a) and (b) separately.

(a) Let the filter $\mathcal{F}$ now be a rapid $^+\mathbb{P}$-filter. To prove part (a) of the theorem, it suffices to prove that the pointwise $\mathcal{F}$-bounded sequence $(q_i)_i$ of seminorms is $\mathcal{F}$-equicontinuous. If follows from Lemma 3.3 that there is a 0-neighbourhood $U$ in $X$ and $F \in \mathcal{F}$ such that $q_i$ is bounded on $U$ whenever $i \in F$. Hence, by altering the sequence $(q_i)_i$ on a non-$\mathcal{F}$-stationary set and renumbering the seminorms $(p_m)_m$ if necessary, it can be assumed that $\sup \{q_i(x) : p_m(x) \leq 1\}$ is finite for every $i, m \in \mathbb{N}$. Suppose now that the sequence $(q_i)_i$ is not $\mathcal{F}$-equicontinuous.

Set $X_k := \{i \in \mathbb{N} : \sup \{q_i(x) : p_k(x) \leq 1\} > 4^k\}$. Then $(X_k)_k$ is a decreasing sequence of $\mathcal{F}$-stationary sets and because $\mathcal{F}$ is a $\mathbb{P}$-filter, there is an $\mathcal{F}$-stationary $B$ such that $B \subseteq^* X_k$ for every $k$. Because $\mathcal{F}$ is rapid $^+$, there exist
a_k ∈ X_k such that \{a_k : k ∈ N\} is \(\mathcal{F}\)-stationary. But then the above claim can be used to reach a contradiction with the pointwise \(\mathcal{F}\)-boundedness of the sequence \((q_i)_i\).

(b) Let the filter \(\mathcal{F}\) now be a rapid\(^+\) weak \(P^+\)-filter. Let \(I\) be an arbitrary \(\mathcal{F}\)-stationary set. To prove part (b) of the theorem, it suffices to prove that there exists an \(\mathcal{F}\)-stationary set \(J ⊆ I\) and a 0-neighbourhood \(U\) in \(X\) such that

\[
\sup\{q_j(x) : j ∈ J, x ∈ U\}
\]

is finite. Suppose (*) that such a \(J\) and \(U\) do not exist. It follows from Lemma 3.6 that we can assume that \(\sup\{q_i(x) : p_m(x) \leq 1\}\) is finite for every \(i ∈ I, m ∈ N\). Indeed, since \((q_i)\) is pointwise \(\mathcal{F}\)-bounded, this can be accomplished (without affecting generality of the proof) by replacing \(I\) by some \(\mathcal{F}\)-stationary subset of \(I\) and by renumbering the seminorms \((p_m)\), if necessary.

Set \(X_k := \{i ∈ I : \sup\{q_i(x) : p_k(x) \leq 1\} > 4^k\}\). By the assumption (*), none of the sets \(X_k^c \cap I\) can be \(\mathcal{F}\)-stationary. Then \((X_k \cup I^c)_k\) is a decreasing sequence of elements of \(\mathcal{F}\) and because \(\mathcal{F}\) is a weak \(P^+\)-filter, there is an \(\mathcal{F}\)-stationary \(B ⊆ I\) such that \(B ⊆^* X_k \cup I^c\) for every \(k\). It follows that also \(B ⊆^* X_k\) for every \(k\). Because \(\mathcal{F}\) is rapid\(^+\), there exist \(a_k ∈ X_k\) such that \(\{a_k : k ∈ N\}\) is \(\mathcal{F}\)-stationary. But then the above claim can again be used to reach a contradiction with the pointwise \(\mathcal{F}\)-boundedness of the sequence \((q_i)_i\).

\[\square\]

Remark 3.2. For the reader who is solely interested in Banach spaces instead of the more general Fréchet spaces, the previous proof can be considerably simplified. Indeed, one finds that whenever \(X\) is Banach, both Lemma 3.5 and Lemma 3.6 reduce to trivialities. Consequently, to prove that every Banach space is (stationarily) \(\mathcal{F}\)-barrelled whenever \(\mathcal{F}\) is a rapid\(^+\) (weak) \(P^+\)-filter, one can directly repeat the proof of Theorem 3.2, needing only Lemma 3.2 and Lemma 3.3 at hand.

Remark 3.3. Theorem 3.1 can be applied in similar fashion as the uniform boundedness principle to derive \(\mathcal{F}\)-dependent results in (functional) analysis. We illustrate this with two examples.

Weakly \(\mathcal{F}\)-convergent sequences

Recall the following consequence of the uniform boundedness principle which is of special value in Banach space theory:

If \(X\) is Banach, then every \(\sigma(X, X')\)-bounded sequence \((x_n)_n\) in \(X\) is norm-bounded. In particular every weakly convergent sequence \((x_n)_n\) in \(X\) is norm-bounded.

Note that since \(X\) embeds linear-isometrically in its bidual, the above statement follows indeed by a direct application of the uniform boundedness principle.

Using the same reasoning one arrives at the following consequence of Theorem 3.1.

Corollary 3.1.

Let \(X\) be a Banach space and let \(\mathcal{F}\) be a rapid\(^+\) \(P^+\)-filter, then every sequence
$(x_n)_n$ in $X$ which is $\mathcal{F}$-bounded with respect to the $\sigma(X,X')$-topology is also $\mathcal{F}$-bounded in norm. In particular, if $(x_n)_n$ is weakly $\mathcal{F}$-convergent, then $(x_n)_n$ is $\mathcal{F}$-bounded in norm.

The second conclusion of Corollary 3.1 can be read in the following way: every weakly $\mathcal{F}$-convergent sequence $(x_n)_n$ in $X$ coincides with some bounded sequence $(y_n)_n$ on a set of indices which is contained in $\mathcal{F}$.

It is instructive to note that there exist filters $\mathcal{F}$ for which this fails strongly in the following sense:

Every infinite-dimensional Banach space $X$ contains some weakly $\mathcal{F}$-convergent sequence $(x_n)_n$ with the property that no (infinite) subsequence of $(x_n)_n$ is bounded.

Indeed, an example of such a filter is the asymptotic density filter $\mathcal{F}_d$. In fact, a characterisation of the (free) filters with the above property has been established in [12, Theorem 3.2 and Remark 1], where it is shown that the following three are equivalent

(i) there exists an infinite-dimensional Banach space $X$ which contains some weakly $\mathcal{F}$-convergent sequence $(x_n)_n$ with the property that no (infinite) subsequence of $(x_n)_n$ is bounded,

(ii) every infinite-dimensional Banach space $X$ contains some weakly $\mathcal{F}$-convergent sequence $(x_n)_n$ with the property that no (infinite) subsequence of $(x_n)_n$ is bounded,

(iii) There exists a positive real sequence $(c_n)_n$ satisfying both $\lim_{n \to \infty} c_n = +\infty$ and $c_n = O(n)$, such that $\{A \subseteq \mathbb{N} : \lim_{n \to \infty} \frac{|\{k < n : k \notin A\}|}{c_n} = 0\} \subseteq \mathcal{F}$.

$\mathcal{F}$-convergence of Fourier-sums

By applying Theorem 3.1 in the Banach space $C(\mathbb{T})$ of functions continuous on the circle, equipped with the supremum norm, one proves the following strengthening of a well-known result (see e.g. [25, 5.11]) for $\mathcal{F} = C$.

**Lemma 3.7.**

For every rapid $^+$ weak $P^+$-filter $\mathcal{F}$, there exists $f \in C(\mathbb{T})$ such that the corresponding sequence

$$s_n(f) = \sum_{k=-n}^{n} \hat{f}(k)$$

of partial Fourier-sums at 0 is not $\mathcal{F}$-bounded. In particular, the series $\sum_{k \in \mathbb{Z}} \hat{f}(k)$ is not $\mathcal{F}$-convergent.

**Proof.** Note that the operator norms of the functionals $\Lambda_n : C(\mathbb{T}) \to \mathbb{R} : f \mapsto \sum_{k=-n}^{n} \hat{f}(k)$ tend to infinity as $n \to \infty$. By consequence, the sequence $(\Lambda_n)_n$ is not $\mathcal{F}$-stationary-equicontinuous, for any filter $\mathcal{F}$. The assertion follows directly by applying Theorem 3.1.

$(x_n)_n$ is defined to be weakly $\mathcal{F}$-convergent to $x \in X$ whenever $(\varphi(x_n))_n$ is $\mathcal{F}$-convergent to $\varphi(x)$ for every $\varphi \in X'$.
4 Necessary conditions for Fréchet-UBP-filters

4.1 Fréchet-UBP-filters are $P^+$-filters

We now pass to the study of those properties of $\mathcal{F}$ which are necessary for $\mathcal{F}$ to be a Fréchet-UBP-filter or a stationary Fréchet-UBP-filter. In this section we will show that every (stationary) Fréchet-UBP-filter is a (weak) $P^+$-filter.

In fact, something stronger is true: it is enough for the uniform $\mathcal{F}$-boundedness principle to hold for (at least) one arbitrary infinite-dimensional Fréchet space $X$ to imply that $\mathcal{F}$ is a $P^+$-filter. The proof is somewhat surprising in the sense that it separates the case where $X$ is normed and the case where $X$ is non-normable. In the latter case, which is treated in the following lemma, the proof is based on directly exploiting the non-normability assumption on $X$.

Lemma 4.1.

Let $X$ be an infinite-dimensional metrizable locally convex topological vector space which is not normable.

1. If the uniform $\mathcal{F}$-boundedness principle holds for $X$, then $\mathcal{F}$ is a $P^+$-filter.

2. If the stationary uniform $\mathcal{F}$-boundedness principle holds for $X$, then $\mathcal{F}$ is a weak $P^+$-filter.

Proof. We give the proof of the first statement, it is easily adapted to prove the second statement. Suppose that $\mathcal{F}$ is not a $P^+$-filter. Using Lemma 2.2, we find $\Gamma : \mathbb{N} \to \mathbb{N}$ with:

- $\Gamma$ is unbounded on every element of $\mathcal{F}$,
- $\Gamma$ is not finite-to-one on any $\mathcal{F}$-stationary set.

Let $(V_k)_k$ be a decreasing $0$-neighbourhood base in $X$. Since $X$ is not normable, none of the neighbourhoods $V_k$ is bounded. Since $X$ is locally convex, it follows from the Mackey theorem that the original topology on $X$ and the weak topology $\sigma(X, X')$ share the same bounded sets. Therefore, none of the neighbourhoods $V_k$ is $\sigma(X, X')$-bounded. It follows that we can select for each $n \in \mathbb{N}$ a functional $\varphi_n \in X'$ which is unbounded on $V_n$. We claim that the sequence $(\psi_n)_n$ with $\psi_n(x) := \frac{1}{n} \varphi_{\Gamma(n)}$ contradicts the uniform $\mathcal{F}$-boundedness principle. Indeed, it is clear that $\Gamma$ is finite-to-one on any set of the form $\{n \in \mathbb{N} : |\psi_n(x)| > C\}$, with $C \in \mathbb{R}_{>0}$ and $x \in X$. Therefore, all of these sets have to be non-$\mathcal{F}$-stationary, hence the sets $\{n \in \mathbb{N} : |\psi_n(x)| \leq C\}$ have to belong to $\mathcal{F}$ and $(\psi_n)_n$ is pointwise $\mathcal{F}$-bounded. From the uniform $\mathcal{F}$-boundedness principle it would follow that there exist $k \in \mathbb{N}$ and $A \in \mathcal{F}$ such that

$$\forall x \in V_k)(\forall n \in A) \ |\psi_n(x)| \leq 1.$$

Since $\Gamma$ is unbounded on $A$, we can choose $n \in A$ such that $\Gamma(n) > k$. Then:

$$\forall x \in V_k) \ |\varphi_{\Gamma(n)}(x)| \leq n.$$

This contradicts the fact that $\varphi_{\Gamma(n)}$ is unbounded on $V_{\Gamma(n)} \subseteq V_k$. 

\hfill \Box
In the remainder of this section we concentrate on infinite-dimensional normed spaces $X$ and show that the (stationary) uniform $\mathcal{F}$-boundedness principle cannot hold for $X$ when $\mathcal{F}$ is not a (weak) $P^+$-filter. For this purpose, we make use of the Josefson-Nissenzweig theorem which assures that the dual space of every Banach space $X$ is rich enough for counterexamples to the uniform $\mathcal{F}$-boundedness principle to exist.

**Lemma 4.2.**

*Let $X$ be an infinite-dimensional normed space, $\mathcal{F}$ a filter on $\mathbb{N}$ and $I \subseteq \mathbb{N}$ $\mathcal{F}$-stationary. If $\Gamma : \mathbb{N} \to \mathbb{N}$ is not finite-to-one on any $\mathcal{F}$-stationary subset of $I$, then there exists a pointwise $\mathcal{F}$-bounded sequence $(\varphi_l)_l$ in the dual space $X'$ such that $\|\varphi_l\| = \Gamma(l)$ for every $l \in I$.***

**Proof.** Note that by passing to the completion of $X$ if necessary, we may assume that $X$ is complete. By the Josefson-Nissenzweig theorem (see for example [4] for a complete proof), we can choose a sequence $(\psi_n)_n$ in the unit sphere of $X'$ which is $\sigma(X',X)$-null (i.e. $\psi_n(x) \to 0$ for every $x \in X$). Consider the sequence of bounded linear functionals given by

$$\varphi_l = \begin{cases} \Gamma(l)\psi_l & \text{when } l \in I, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that for every $x \in X$ and every $C \in \mathbb{R}_{>0}$,

$$\{l \in \mathbb{N} : |\varphi_l(x)| \leq C\} \in \mathcal{F}.$$  

Suppose not, then $\{l \in \mathbb{N} : |\varphi_l(x)| > C\}$ is an $\mathcal{F}$-stationary subset of $I$. Because $\Gamma$ is not finite-to-one on any $\mathcal{F}$-stationary subset of $I$, there exists $m \in \mathbb{N}$ such that $A_m := \{l \in \mathbb{N} : |\varphi_l(x)| > C \wedge \Gamma(l) = m\}$ is an infinite set. But then

$$C \leq \lim_{l \in A_m, l \to \infty} |\varphi_l(x)| = \lim_{l \in A_m, l \to \infty} m|\psi_l(x)| = 0,$$

a contradiction which justifies our claim. \hfill $\square$

**Lemma 4.3.**

*Let $X$ be an infinite-dimensional normed space and $\mathcal{F}$ a filter on $\mathbb{N}$.

(a) If the uniform $\mathcal{F}$-boundedness principle holds for $X$, then $\mathcal{F}$ is a $P^+$-filter.

(b) If the stationary uniform $\mathcal{F}$-boundedness principle holds for $X$, then $\mathcal{F}$ is a weak $P^+$-filter.

**Proof.** (a) Suppose that $\mathcal{F}$ is not a $P^+$-filter.

Using Lemma 2.2 choose $\Gamma : \mathbb{N} \to \mathbb{N}$ such that:

- $\Gamma$ is unbounded on every element of $\mathcal{F}$,
- $\Gamma$ is not finite-to-one on any $\mathcal{F}$-stationary subset of $\mathbb{N}$.
Using Lemma 4.2, we find a pointwise $\mathcal{F}$-bounded sequence $(\varphi_l)_l$ in $X'$ such that $\|\varphi_l\| = \Gamma(l)$ for every $l$. To reach a contradiction with the uniform $\mathcal{F}$-boundedness principle, it suffices to note that $(\varphi_l)_l$ cannot be $\mathcal{F}$-bounded, because $\Gamma$ is not bounded on any element of $\mathcal{F}$.

(b) Suppose that $\mathcal{F}$ is not a weak $P^+$-filter.
Using Lemma 2.1 choose $\Gamma : \mathbb{N} \rightarrow \mathbb{N}$ and an $\mathcal{F}$-stationary set $I$ such that:

- $\Gamma$ is unbounded on every $\mathcal{F}$-stationary set,
- $\Gamma$ is not finite-to-one on any $\mathcal{F}$-stationary subset of $I$.

Using Lemma 4.2, we find a pointwise $\mathcal{F}$-bounded sequence $(\varphi_l)_l$ in $X'$ such that $\|\varphi_l\| = \Gamma(l)$ for every $l \in I$. Because $\Gamma$ is unbounded on every $\mathcal{F}$-stationary set and $\|\varphi_l\| = \Gamma(l)$ for every $l \in I$, we find that $(\varphi_l)_l$ is not bounded on any $\mathcal{F}$-stationary subset of $I$. Since $(\varphi_l)_l$ is pointwise $\mathcal{F}$-bounded, this is in contradiction with the stationary uniform $\mathcal{F}$-boundedness principle.

One can note that the sequences constructed in the proofs of both Lemma 4.1 and Lemma 4.2 are not only pointwise $\mathcal{F}$-bounded but also pointwise $\mathcal{F}$-convergent to 0. By also noting that $\mathcal{F}_d$ is not a $P^+$-filter, we arrive at the following Corollary 4.1. We remark that in the case that $X$ is a Banach space, Corollary 4.1 is a direct consequence of Theorem 1 in [7].

Corollary 4.1.
For every infinite-dimensional metrizable locally convex topological vector space there exists a sequence $(\varphi_l)_l \in X'$ that is not $\mathcal{F}_d$-equicontinuous, yet pointwise converges statistically to zero.

4.2 Stationary Fréchet-UBP-filters are Rapid$^+$-filters

To conclude the characterisation of (stationary) Fréchet-UBP-filters, we prove that every stationary Banach-UBP-filter is rapid$^+$. The following more general statement holds.

Lemma 4.4.
Let $X$ be an infinite-dimensional normed space and $\mathcal{F}$ a filter on $\mathbb{N}$. If the stationary uniform $\mathcal{F}$-boundedness principle holds for $X$, then $\mathcal{F}$ is a rapid$^+$ filter.

For this purpose, we adapt an argument used in the proof of Theorem 2.6 in [2]. The main tool in this argument is Dvoretzky’s theorem on the existence of almost-euclidean sequences in normed spaces.

Let $(X, \| \cdot \|)$ now be a normed space, let $\varepsilon > 0$. Denote by $|a|_2$ the euclidean norm on $a \in \mathbb{R}^{n+1}$, $|{(a_0, \ldots, a_n)}|_2 := \sqrt{a_0^2 + \ldots + a_n^2}$. 

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We will call a finite sequence \( x_m, \ldots, x_{m+n} \) \( \varepsilon \)-almost-euclidean whenever
\[
(a_m, \ldots, a_{m+n})^2 \leq \| \sum_{i=0}^n a_{m+i} x_{m+i} \|^2 \leq (1 + \varepsilon) || (a_m, \ldots, a_{m+n}) ||^2,
\]
\((\forall (a_m, \ldots, a_{m+n}) \in \mathbb{R}^{n+1})\).

**Definition 4.1.**
Let \((I_n)_n\) be an increasing interval-partition of \( I \), let \( \varepsilon > 0 \).
A sequence \((x_n)_{n \in I}\) in \( X \) is \( \varepsilon \)-almost-euclidean on \((I_n)_n\) if the finite sequence \((x_j : j \in I_n)\) is \( \varepsilon \)-almost-euclidean for every \( n \in \mathbb{N} \).

**Lemma 4.5.**
Let \( X \) be a normed space, \( \mathcal{F} \) a filter on \( \mathbb{N} \), \( I \subseteq \mathbb{N} \) \( \mathcal{F} \)-stationary, \( \varepsilon > 0 \) and \((I_n)_n\) an increasing interval-partition of \( I \). If

(a) \( \mathcal{F} \) is slow with respect to \((I_n)_n\),
(b) \( X' \) contains a sequence \((\varphi_n)_{n \in I}\) that is \( \varepsilon \)-almost-euclidean on \((I_n)_n\),

then \( X' \) contains a pointwise \( \mathcal{F} \)-bounded sequence \((\psi_n)_n\) which is not bounded on any infinite subset of \( I \).

**Proof.** (Following closely the proof of [2, Theorem 2.6].)
Choose \( f : \mathbb{N} \to I \) strictly increasing with \((I_n)_n = (I_n^I)_n\). Because \( f \) is strictly increasing, \( f \) has a non-decreasing left-inverse \( g : I \to \mathbb{N} \). Define \( \psi_n = \sqrt{g(\varphi_n)} \) for \( n \in I \) and \( \psi_n = 0 \) otherwise. For every \( n \in I \), the vector \( \varphi_n \) is contained in an \( \varepsilon \)-almost-euclidean finite sequence and hence \( || \varphi_n || \geq 1 \). Since \( g \) is unbounded and non-decreasing, it follows that the sequence \((\varphi_n)_n\) is not bounded in \( X' \) on any infinite subset of \( I \). Next, take \( x \in X \) with \( || x || \leq 1 \) arbitrarily. We will prove that \( A_r = \{ k \in \mathbb{N} : || \psi_k(x) || \leq r \} \) belongs to \( \mathcal{F} \), for certain \( r \in \mathbb{R}_{>0} \). Let \( n \) be arbitrary, put \( B_r := \mathbb{N} \setminus A_r \) and put \( b_n = |B_r \cap I_n| \).

**Claim.** For a suitable choice of \( r \) we have that \( b_n \leq n \) for each \( n \in \mathbb{N} \).

**Proof of Claim.** Consider \( \chi_n = \sum_{i \in B_r \cap I_n} \text{sgn}(\psi_i(x)) \varphi_i \).

First note that
\[
\chi_n(x) \geq \frac{b_n r}{\sqrt{g(f(n))}} = \frac{b_n r}{\sqrt{n}}.
\]

By \((\varphi_i : i \in I_n)\) being \( \varepsilon \)-almost-euclidean in \( X' \), we have:
\[
|| \chi_n ||^2 \leq (1 + \varepsilon) b_n,
\]

hence
\[
\chi_n(x) \leq \sqrt{(1 + \varepsilon) b_n}.
\]

By combining inequalities (IE1) and (IE2) and selecting \( r = \sqrt{1 + \varepsilon} \), we find at once,
\[
\sqrt{b_n} \leq \sqrt{n}.
\]
Since $B_r \subseteq I$ and $\mathcal{F}$ is slow with respect to $(I_n)_n$, we have found $r$ such that $B_r$ is non-$\mathcal{F}$-stationary and hence $A_r \in \mathcal{F}$.

The existence of $\varepsilon$-almost-euclidean sequences in infinite-dimensional normed spaces is guaranteed by Dvoretzky’s theorem in convex geometry (for more background and a modern proof see e.g. [1]), which we use in the following form:

**Theorem 4.1** (Dvoretzky (1961) [8]).
For every $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that every normed space of dimension $N$ contains an $\varepsilon$-almost-euclidean sequence of length $k$.

**Corollary 4.2.**
Let $(I_n)_n$ be an increasing interval-partition of $I \subseteq \mathbb{N}$ and let $\varepsilon > 0$.
Every infinite-dimensional normed space contains a sequence that is $\varepsilon$-almost-euclidean on $(I_n)_n$.

This leads to the necessity of rapidity for the $\text{UBP}^{\text{stat}}$ and $\text{UBP}$-properties. We now give the proof of Lemma 4.4.

**Proof of Lemma 4.4.** If $\mathcal{F}$ is not rapid$^+$, one can (using Lemma 2.3) choose an $\mathcal{F}$-stationary set $I$ and an increasing interval-partition $(I_n)_n$ of $I$ such that $\mathcal{F}$ is slow with respect to $(I_n)_n$. By Corollary 4.2, one can choose next a sequence $(\varphi_n)_n$ in $X'$ which is $\varepsilon$-almost-euclidean on $(I_n)_n$ (where one has fixed an arbitrary $\varepsilon \in \mathbb{R}_{>0}$). By Lemma 4.5, one then finds a pointwise $\mathcal{F}$-bounded sequence $(\psi_n)_n$ in $X'$ which is not bounded on any infinite subset of $I$. This contradicts the stationary uniform $\mathcal{F}$-boundedness principle for $X$. □

5 Conclusions and additional remarks

Theorem 3.1, Lemma 4.3 and Lemma 4.4 can now be combined to obtain the following two characterisation theorems.

**Theorem 5.1.**
The following statements are equivalent for a filter $\mathcal{F}$ on $\mathbb{N}$.

1. There exists an infinite-dimensional Banach space $X$ such that the uniform $\mathcal{F}$-boundedness principle holds for $X$.
2. The uniform $\mathcal{F}$-boundedness principle holds for every Fréchet space $X$.
3. $\mathcal{F}$ is both a rapid$^+$- and a $P^+$-filter.

**Theorem 5.2.**
The following statements are equivalent for a filter $\mathcal{F}$ on $\mathbb{N}$.

1. There exists an infinite-dimensional Banach space $X$ such that the stationary uniform $\mathcal{F}$-boundedness principle holds for $X$.
2. $\mathcal{F}$ is both a rapid$^+$- and a $P^+$-filter.
2. The stationary uniform $\mathcal{F}$-boundedness principle holds for every Fréchet space $X$.

3. $\mathcal{F}$ is both a rapid$^+$- and weak $P^+$-filter.

In particular, the classes of (stationary) Fréchet-UBP-filters and (stationary) Banach-UBP-filters coincide, we will now simplify terminology and refer to these filters as (stationary) UBP-filters.

Example 5.1 ([23]).

Theorem 5.1 allows to give an elementary example of a UBP-filter $\mathcal{G}$ which is not countably generated. This example was suggested to the authors by Miroslav Repický.

Let $\mathbb{N} = \bigsqcup_{i \in \mathbb{N}} E_i$ be a partition of $\mathbb{N}$ into finite sets $E_i$. Suppose in addition that $\limsup_{i \in \mathbb{N}} |E_i| = \infty$. We consider the following filter:

$$\mathcal{G} := \{ A \subseteq \mathbb{N} : \limsup_{i \in \mathbb{N}} |E_i \setminus A| < \infty \}.$$ 

Using that $A \subseteq \mathbb{N}$ is $\mathcal{G}$-stationary if and only if

$$\limsup_{i \in \mathbb{N}} |E_i \cap A| = \infty,$$

it is easily checked that $\mathcal{G}$ is a rapid$^+$ $P^+$-filter and hence, by Theorem 5.1, also a UBP-filter. Moreover, given any sequence $(S_n)_n$ of elements of $\mathcal{G}$, it is possible to recursively construct increasing sequences of natural numbers $l_0 < l_1 < l_2 < l_3 < \ldots$ and $k_0 < k_1 < k_2 < k_3 < \ldots$ such that $k_n \in E_{l_n} \cap S_n$. Then the set $\mathbb{N} \setminus \{ k_n : n \in \mathbb{N} \}$ belongs to $\mathcal{G}$ but does not contain any of the sets $S_n$ as subset. It follows that $\mathcal{G}$ can not be countably generated.

5.1 UBP-ultrafilters

As the filter $\mathcal{G}$ in the previous example is only $F_\sigma$ in the Cantor space $P(\mathbb{N})$, this example shows that UBP-filters of uncountable character need not be complex in the sense of descriptive set theory. To study the other extreme, we now focus on the special case of UBP-ultrafilters. Adhering to our earlier convention, all ultrafilters we consider are supposed to be free. For ultrafilters, the notions of weak $P^+$-filter and $P^+$-filter coincide and one uses the term P-points to denote those ultrafilters which are also $P^+$-filters. The rapid$^+$ P-points are referred to as semi-selective ultrafilters. Semi-selective ultrafilters are also characterised by the following combinatorial property (see e.g. [6]).

Lemma 5.1.

$\mathcal{U}$ is a semi-selective ultrafilter if and only if for every sequence $(A_k)_k$ of elements of $\mathcal{U}$, there exists $\{ a_0 < a_1 < \ldots \} \in \mathcal{U}$ with $a_k \in A_k$ for every $k$.

In the special case of ultrafilters, Theorem 5.1 and Theorem 5.2 both reduce to:
Corollary 5.1.
The set of UBP-ultrafilters coincides with the set of semi-selective ultrafilters. Moreover, if $U$ is an ultrafilter and the uniform $U$-boundedness principle holds for some infinite-dimensional Banach space $X$, then $U$ is semi-selective.

This corollary extends a theorem by Benedikt ([5, Proposition 3]) which expresses in the language of nonstandard analysis that every selective ultrafilter is a Banach-UBP-filter. An ultrafilter $U$ is selective if every $\Gamma : \mathbb{N} \to \mathbb{N}$ is either constant or injective on some $A \in U$. In particular, every selective ultrafilter is semi-selective.

Since the existence of selective ultrafilters follows from Martin’s Axiom, the existence of UBP-ultrafilters is consistent with ZFC (provided the theory ZFC is itself consistent). In particular:

Theorem 5.3 ([14]).
If Martin’s Axiom for countable posets (MA(countable)) holds, then there exist $2^{\aleph_0}$-many non-isomorphic selective ultrafilters.

It follows that MA(countable) also implies the existence of $2^{\aleph_0}$-many non-isomorphic UBP-ultrafilters. MA(countable) is a proper weakening of MA and is equivalent to the statement that there is no way of writing $\mathbb{R}$ as a union $\bigcup_{\alpha < \kappa} N_\alpha$ of $\kappa < 2^{\aleph_0}$ meager sets $N_\alpha \subseteq \mathbb{R}$ (see [14]).

However, since the existence of P-points as well as the existence of rapid+ ultrafilters is independent of the axioms of ZFC-set theory, it follows that there are also no ZFC-proofs of the existence of UBP-ultrafilters (again, of course, assuming consistency of ZFC). In particular, examples of models in which there do not exist UBP-ultrafilters include models obtained by adding $\aleph_2$ random reals to a model of ZFC + GCH [20], Laver’s model for the Borel conjecture [22] and Shelah’s model for the absence of P-points [26].

5.2 Products and Quotients
Theorem 5.1 and Theorem 5.2 leave open the possibility that the uniform $F$-boundedness principle could hold for some infinite-dimensional Fréchet space without $F$ being a UBP-filter. We will now show that this can indeed occur. To this purpose, we first indicate how the validity of the uniform $F$-boundedness principle behaves under countable products and quotients of Fréchet spaces.

Let $\pi : X \to Y$ be a quotient map between locally convex spaces [28, p. 33]. Then, every pointwise $F$-bounded sequence $(T_i)_i$ of continuous linear maps from $Y$ to a locally convex space $Z$ induces the sequence $(T_i \circ \pi)_i \in \mathcal{L}(X, Z)$, which is still pointwise $F$-bounded. Moreover, $(T_i)_i$ is $F$-equicontinuous if and only if the sequence $(T_i \circ \pi)_i$ is $F$-equicontinuous. This proves the following lemma.

Lemma 5.2.
If $\pi : X \to Y$ is a quotient map between locally convex spaces and the uniform $F$-boundedness principle holds for $X$, then the uniform $F$-boundedness principle holds for $Y$ as well.
Likewise, we have:

**Lemma 5.3.**
If \( \pi : X \rightarrow Y \) is a quotient map between locally convex spaces and the stationary uniform \( \mathcal{F} \)-boundedness principle holds for \( X \), then the stationary uniform \( \mathcal{F} \)-boundedness principle holds for \( Y \) as well.

For every Fréchet space \( X \) admitting an infinite-dimensional normable quotient, we can now characterise those filters \( \mathcal{F} \) for which the (stationary) uniform \( \mathcal{F} \)-boundedness principle holds for \( X \).

**Lemma 5.4.**
Let \( X \) be a Fréchet space with an infinite-dimensional normable quotient.

- The uniform \( \mathcal{F} \)-boundedness principle holds for \( X \) if and only if \( \mathcal{F} \) is a UBP-filter.
- The stationary uniform \( \mathcal{F} \)-boundedness principle holds for \( X \) if and only if \( \mathcal{F} \) is a stationary UBP-filter.

**Proof.** The other direction being evident from the definition of (stationary) UBP-filter, it suffices to prove that \( \mathcal{F} \) is a (stationary) UBP-filter whenever the (stationary) uniform \( \mathcal{F} \)-boundedness principle holds for \( X \).

Suppose that the (stationary) uniform \( \mathcal{F} \)-boundedness principle holds for \( X \).

Then \( \mathcal{F} \) is a (weak) \( \mathcal{P}^+ \)-filter, by Lemmas 4.1 and 4.3. Because of Lemma 5.2 and Lemma 5.3, validity of the (stationary) uniform \( \mathcal{F} \)-boundedness principle will be inherited by the infinite-dimensional normable quotient of \( X \), so that it follows from Lemma 4.4 that \( \mathcal{F} \) is also rapid\(^+\). The conclusion follows from Theorem 3.1.

As long as \( \mathcal{F} \) is a (weak) \( \mathcal{P}^+ \)-filter, the (stationary) uniform \( \mathcal{F} \)-boundedness principle is also preserved under countable products of Fréchet spaces.

**Lemma 5.5.**
Let \( \mathcal{F} \) be a \( \mathcal{P}^+ \)-filter. If \( (X_i)_{i \in \mathbb{N}} \) is a sequence of (possibly finite-dimensional) Fréchet spaces and

\[
X = \prod_{i \in \mathbb{N}} X_i,
\]

then the uniform \( \mathcal{F} \)-boundedness principle holds for \( X \) if and only if the uniform \( \mathcal{F} \)-boundedness principle holds for every one of the spaces \( X_i \).

**Proof.** Let \( Y \) be an arbitrary locally convex space and let \( Q_i \) be a base of continuous seminorms for \( X_i \).

Since the projection on the \( i \)-th coordinate is a quotient map from \( X \) to \( X_i \), this direction follows from Lemma 5.2.

For the other direction, let \( (T_j)_j \) be a pointwise \( \mathcal{F} \)-bounded sequence in \( \mathcal{L}(X,Y) \). In order to show that \( (T_j)_j \) is \( \mathcal{F} \)-equicontinuous, it suffices to show that for an arbitrary continuous seminorm \( r \) on \( Y \), \( (T_j)_j \) is \( \mathcal{F} \)-equicontinuous in \( \mathcal{L}(X,(Y,r)) \). Note that, making use of the quotient mapping from \( (Y,r) \) onto
the normed space \((Y/\ker(r), r)\), we can assume that \(r\) is a norm. We will thus write \(Y\) for the normed space \((Y, r)\) in the rest of this proof. Observe that then

\[
\mathcal{L}(X, Y) = \bigoplus_{i \in \mathbb{N}} \mathcal{L}(X_i, Y),
\]

in particular, for every \(j\) we find \((P^j_i)_{0 \leq i \leq n_j}\), with \(P^j_i \in \mathcal{L}(X_i, Y)\), such that

\[
T_j((x_n)_n) = \sum_{0 \leq i \leq n_j} P^j_i(x_i), \quad \text{for all } (x_n)_n \in X.
\]

Since \(\mathcal{F}\) is a \(P^+\)-filter and \(X\) is a Fréchet space, it follows from Lemma 3.5 that there exists a 0-neighbourhood \(B\) in \(X\) and \(F \in \mathcal{F}\) such that

\[
\sup_{x \in B} r(T_j(x)) < \infty, \quad \text{for every } j \in F.
\]

Because \(B\) is a 0-neighbourhood in the product space \(X\), there exist \(N \in \mathbb{N}\), \(c > 0\) and \(q_0, \ldots, q_N\) with \(q_i \in Q_i\) such that

\[
\bigcap_{i \leq N} \{ (x_n)_n \in X : q_i(x_i) \leq c \} \subseteq B.
\]

Since every mapping \(r \circ T_j\), with \(j \in F\), is also bounded on this \(B\), one finds that \(r \circ P^j_i\) is necessarily identically zero for every \(i > N\) and every \(j \in F\). It is therefore sufficient to prove that the sequence \((P^j_i)_{j}\) is \(\mathcal{F}\)-equicontinuous for every \(i \leq N\). It follows from \((*)\) that the sequence \((P^j_i)_{i}\) is pointwise \(\mathcal{F}\)-bounded for every \(i\), so we can conclude \(\mathcal{F}\)-equicontinuity of this sequence by applying the uniform \(\mathcal{F}\)-boundedness principle for the space \(X_i\).

With an analogous argument, one proves the corresponding lemma for the stationary uniform \(\mathcal{F}\)-boundedness principle.

**Lemma 5.6.**

Let \(\mathcal{F}\) be a weak \(P^+\)-filter. If \((X_i)_{i \in \mathbb{N}}\) is a sequence of (possibly finite-dimensional) Fréchet spaces and

\[
X = \prod_{i \in \mathbb{N}} X_i,
\]

then the stationary uniform \(\mathcal{F}\)-boundedness principle holds for \(X\) if and only if the stationary uniform \(\mathcal{F}\)-boundedness principle holds for every one of the spaces \(X_i\).

As a corollary, we find an example of an infinite-dimensional Fréchet space for which the (stationary) uniform \(\mathcal{F}\)-boundedness principle can hold for filters \(\mathcal{F}\) which are not (stationary) UBP-filters.

**Corollary 5.2.**

For the sequence space \(\mathbb{R}^\mathbb{N}\), equipped with the product topology,
the uniform $F$-boundedness principle holds if and only if $F$ is a $P^+$-filter,

- the stationary uniform $F$-boundedness principle holds if and only if $F$ is a weak $P^+$-filter.

Proof. Because $\mathbb{R}^N$ is not normable, the only if directions follow from Lemma 4.1. But the (stationary) uniform $F$-boundedness principle holds trivially for the space $\mathbb{R}$, so it follows from the previous theorem that the (stationary) uniform $F$-boundedness principle will also hold for $\mathbb{R}^N$, whenever $F$ is a (weak) $P^+$-filter.

We have thus obtained a characterisation of the (weak) $P^+$-filters in terms of the topological vector space structure on $\mathbb{R}^N$.

The following Remark 5.1 and Example 5.2 each exhibit an illustration of Lemma 5.4.

Remark 5.1. Since $\mathbb{R}^N$ is a Fréchet-Montel space, Corollary 5.2 prompts the question whether every Fréchet-Montel space necessarily satisfies the uniform $F$-boundedness principle for every $P^+$-filter $F$. It follows from Lemma 5.4 that this is not the case as there exist Fréchet-Montel spaces admitting infinite dimensional Banach quotients. (See [16, 11.6.4] for an example of a Köthe sequence space that is Fréchet-Montel but surjects continuously to $\ell_1$.)

Example 5.2. By Lemma 5.4, the spaces of $m$-times continuously differentiable functions $C^m(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$ open and $m \in \mathbb{N}$ satisfy the (stationary) uniform $F$-boundedness principle if and only if $F$ is a (stationary) UBP-filter. Indeed, the Banach spaces $E^m(K)$ of Whitney differentiable functions on compact convex $K \subseteq \Omega$ appear as quotients of the spaces $C^m(\Omega)$ (cf. [11, Theorem 2.2]).

In contrast to the spaces $C^m(\Omega)$ ($m \in \mathbb{N}$) in the previous example, the space $C^\infty(\Omega)$ belongs to the class of Fréchet-Schwartz spaces (just as the space $\mathbb{R}^N$). As every Hausdorff quotient of a Schwartz space is again Schwartz and the only normable Schwartz spaces are finite-dimensional, no infinite-dimensional quotient of a Schwartz space can be normable. By consequence, if $X$ is Schwartz, Lemma 5.4 does not carry any extra information about the validity of the uniform $F$-boundedness principle for $X$.

The following problem therefore remains open.

Question 1. For which Fréchet-Schwartz spaces $X$ and which filters $F$ does the (stationary) uniform $F$-boundedness principle hold for $X$?

This question is a special case of the following more general question asking whether (stationary) UBP-filters are necessary for uniform $F$-boundedness principles in those cases left open by Lemma 5.4.
Question 2. If $X$ is a Fréchet space that does not admit an infinite-dimensional Banach quotient, for which filters $\mathcal{F}$ does the (stationary) uniform $\mathcal{F}$-bounded principle hold for $X$?

Because of Lemma 4.1, every such filter $\mathcal{F}$ should be at least a (weak) $P^+$-filter and the question comes down to determining exactly for which spaces the extra rapidity$^+$ assumption is (fully) necessary.

Finally, we point to the study of uniform $\mathcal{F}$-boundedness principles in the more general setting of (not necessarily Fréchet) barrelled topological vector spaces, which was not addressed here, but could be of interest for further research.

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