Reconciling Inflation with Openness

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It is already understood that the increasing observational evidence for an open Universe may be reconciled with inflation if our horizon is contained inside one single huge bubble nucleated during the inflationary phase transition. In the scenario we present here, the Universe consists of infinitely many superhorizon bubbles, like our own, the distribution of which can be made to peak at \( \Omega_0 \approx 0.2 \). Therefore, unlike the existing literature, we do not have to rely upon the anthropic principle nor upon special initial conditions.

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I. INTRODUCTION

An open Universe, with \( \Omega_0 \approx 0.2 \), seems to fit most astronomical observations. In connection with CDM, it gives the best fit to the observed clustering (see e.g. Ref. [1]); a similar value is required for explaining the dynamics of bound objects on relatively small scales (see e.g. Ref. [2]); it also increases the age of the Universe, alleviating the conflict with the age of the globular clusters. Further, \( \Omega_0 \approx 0.2 \) is in better agreement with direct geometrical estimates from number counts (see e.g. Ref. [2] [3]).

However, as it is well known, inflation predicts \( \Omega_0 = 1 \). This contradiction is one of the most interesting problems in modern cosmology. It is certainly possible that the observations in favour of an open Universe are too limited to be representative of the whole Universe, and that higher values of \( \Omega_0 \) will be found at larger scales. On the other hand, it is also possible to choose the initial conditions in inflation so as to give \( \Omega_0 = 0.2 \) today, either by starting with an extremely small density parameter at the beginning of inflation, or by assuming that inflation lasted less than 60 e-foldings or so. Both possibilities, however, introduce that fine-tuning of the initial conditions that inflation itself tried to overcome; moreover, there would be also a conflict with the microwave background isotropy [4].

Two ways out of the enigma have been proposed so far. The first, the single-bubble scenario [1] [2] [3], assumes that a single giant bubble nucleated from a false vacuum (FV) state, when the Universe was already flat because of previous inflationary expansion, and inflated for about 60 e-foldings afterward: our horizon is contained inside the bubble, and appears to be locally open. The weak point of this model is the old graceful exit problem: inflation never ends outside the bubble, and there is no reason why we should live inside that infinitesimal fraction of space which nucleated out of the false vacuum, unless one invokes anthropic arguments. Other problems of the single-bubble model have been discussed in Ref. [3]. In the second proposal, the many-bubbles scenario [8], one has a two-field potential: while a field drives the inflationary slow-rolling, the second one performs a phase transition, generating bubble-like open Universes, with all possible density parameters, from zero to unity, and all possible sizes. Here the phase transition completes, all the Universe eventually nucleates out of the FV state, but again there is no reason to expect nor a preferential value of \( \Omega_0 \), nor a bubble size big enough to contain our horizon. Actually, it is difficult to avoid the conclusion that most of the volume is inside bubbles nucleated at the end of inflation, thus exponentially smaller than our observed Universe. Linde [8] argues that eventually quantum cosmology will explain why we live in a open Universe (if we really do).

The model we propose in this work implements a many-bubble scenario in which the nucleation rate varies in such a way to give a \( \Omega_0 = 0.2 \) Universe with maximal probability. In our model, the peak of the bubble nucleation can be chosen to occur early enough to have super-horizon-sized bubbles, approximating a \( \Omega_0 = 0.2 \) Universe by today, and narrow enough to consider our Universe as typical. In other words, a flat, huge (or infinite) Universe appears to be composed of locally open super-horizon-sized bubbles. It is remarkable that local observations, \( \Omega_0 \) and the amplitude of perturbations, and the “natural” assumption that our Universe is typical, put strong constraints on the theory parameters, i.e. the fundamental scales of the inflation and of the primordial phase transition.

Our model, outlined in the next Section, has been already introduced in Ref. [3] to produce large scale power out of the remnants of the primordial phase transition. All what we have to do here is to determine the parameters so to tune the nucleation peak at super-horizon scales.

II. THE MODEL

To realize our scenario we need two pre-requisites. First, we need two channels, a false vacuum channel, to drive the inflation in the parent Universe, and a true vacuum channel, to drive the shorter inflation inside the bubbles. Second, we need a tunneling rate tunable in time, so as to produce a nucleation peak at the right time. It is remarkable that the same model that we introduced in
Ref. [3] has just these features. This is certainly not the unique possibility, but we will show that it is a rather simple one.

The model works in fourth order gravity [10] and exploits two fields: one, the scalaron $R$ (i.e. the Ricci scalar) drives the slow-rolling inflation; the second, $\psi$, performs the first-order phase transition. The phase transition dynamics is governed both by the potential of $\phi$ and by its coupling to $R$; the dynamics of the slow-roll is “built-in” in the fourth order Lagrangian. We already presented our model in detail in Ref. [9]; here we sketch its main features. Our starting point is the Lagrangian density $\mathcal{L} = \mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{mat}}$, where ($c = \hbar = G = 1$)

$$\mathcal{L}_{\text{grav}} = -R + \frac{R^2}{6M^2W(\psi)},$$

(1)

and

$$\mathcal{L}_{\text{mat}} = 16\pi \left( \frac{1}{2} \psi ; \psi ; \mu - V(\psi) \right),$$

(2)

The coupling of the scalaron with $\psi$ can be thought of as a field-dependent effective mass $M_{\text{eff}}(\psi) = M W^{1/2}(\psi)$, just like in Brans-Dicke gravity the coupling is a field-dependent Planck mass. Putting $\alpha \equiv \log(a/a_{\text{in}})$, one finds in the slow-rolling regime

$$\alpha = \frac{R - R_{\text{in}}}{4M_{\text{eff}}^2},$$

(3)

which is essentially the only dynamical equation we need. In the following, the instant labelled [1] will correspond to the beginning of the last $N_T \approx 60 \epsilon$-foldings of inflation. The theory [1] can be conformally transformed [11] into canonical gravity with the new metric

$$\tilde{g}_{\alpha\beta} = e^{2\omega} g_{\alpha\beta}, \quad e^{2\omega} = \left| \frac{\partial\mathcal{L}}{\partial R} \right| = 1 - \frac{R}{3M_{\text{eff}}^2}. \quad (4)$$

Once written in the conformal frame, our model becomes indistinguishable from a ordinary gravity theory with two fields governed by a specific potential.

In the slow-rolling approximation useful relations link $\omega$, $H$ and the number $N = N_T - \alpha$ of $\epsilon$-foldings to the end of inflation:

$$\frac{4}{3} N = (e^{2\omega} - 1) = \frac{AH^2}{M_{\text{eff}}^2}. \quad (5)$$

Correspondingly, $H_{\text{in}} = M_{\text{eff}}(N_T/3)^{1/2}$. The value $N_T$ is fixed by the request that the largest observable scale, $L_h = 2H_0^{-1}$, was crossing out the horizon $N_T \epsilon$-foldings before the end of inflation, and it is close to 60 for standard cosmological values [12]. From (1) and (4) we obtain then Einstein gravity with two scalar fields $\psi$ and $\omega$, coupled by a potential given by

$$U(\psi, \omega) = e^{-4\omega} \left[ V(\psi) + \frac{3M^2}{2\pi} W(\psi)(1 - e^{2\omega})^2 \right]. \quad (6)$$

The choice of a quartic for $W$ and a mass term for $V$ realizes the two conditions discussed above:

$$W(\psi) = 1 + \frac{8\lambda}{\psi_0^4} \psi^2(\psi - \psi_0)^2, \quad V(\psi) = \frac{1}{2} \mu^2 \psi^2. \quad (7)$$

This curves in fact in [1] two parallel channels of different height, separated by a peak at $\psi_{PK} = \psi_0/2$. The degeneracy of $W(\psi)$ in $\psi = 0$ and $\psi = \psi_0$ is indeed removed by $V(\psi)$; the true vacuum (TV) channel remains at $\psi_{TV} = 0$, while the false vacuum (FV) channel is slightly displaced from $\psi_0$.

In Ref. [1] we evaluated the tunneling rate $\Gamma$ for our model, defined as

$$\Gamma = M^4 \exp(-S_E), \quad (8)$$

where $M$ is of the order of the energy of the false vacuum, and $S_E$ is the minimal Euclidean action, i.e. the action for the so-called bounce solution of the Euclidean equation of motion. The calculation of $S_E$ is simplified in the limit $\omega \gg 1$, i.e. for $N \gg 1$. In this case in fact

$$U(\psi, \omega) \approx \frac{3M^2}{32\pi} W(\psi) + V(\psi) e^{-4\omega}, \quad (9)$$

and we can directly use Coleman’s formulas [13] to evaluate $S_E$, provided we are in the thin wall limit (TWL). If $U_{\text{FW}}$ ($U_{\text{TV}}$) is the potential energy of the false vacuum (true vacuum) state, and $U_{PK}$ is the energy at the top of the barrier, the TWL is guaranteed if $U_{PK} \gg U_{\text{FW}} - U_{\text{TV}}$, i.e. if

$$\lambda \gg 8\pi \frac{m^2 \psi_0^2}{M^2}. \quad (10)$$

The result is (Ref. [1])

$$S_E = (N/N_1)^4, \quad N_1^4 = \frac{3\sqrt{3} m^3 \psi_0}{4M^2\lambda}. \quad (11)$$

The Euclidean action decreases as $N$ decreases; the bubble nucleation is then more likely to occur during the last stages of inflation than at earlier times. Finally, we can write the relevant parameter $Q = 4\pi \Gamma/9H^4$, i.e. the number of bubbles per horizon volume per Hubble time (quadrizon, for short), as

$$Q(N) = \exp \frac{N_0^4 - N^4}{N_1^4}, \quad \left( \frac{M}{M_{\text{eff}}} \right)^4 = \frac{9}{64\pi} \exp \left( \frac{N_0}{N_1} \right)^4 \quad (12)$$

where we have introduced a new parameter $N_0$ to mark the time at which there is on average one bubble per quadrizon, roughly corresponding to the end of the phase transition. It is useful to keep in mind that, as we will show later, $N_0 \approx N_T \gg N_1$. $N_0$, or $M$, in principle, can be derived in terms of the potential parameters [13]; the derivation is, however, very difficult: as customarily done in extended inflation (see e.g. [14]), we will
To summarize, our model has four characteristic quantities: the slow-rolling inflationary rate, set by $M$; the difference in energy between vacua states, set by $m$; the barrier height, set by $\lambda$; and finally the separation between the vacua states, set by $\psi_0$. These constants completely define the slow-rolling and the phase transition dynamics. In the next section we proceed to the evaluation of the tunneling probability, and show that we can tune the parameters to give $\Omega_0 = 0.2$ today with maximal probability.

### III. THE TUNNELING FUNCTION

Let the number of bubbles per horizon nucleated in the time interval $dt$ be $dn_B$, where \[ \frac{dn_B}{dt} = \Gamma a^3 V_n \exp \left[ -\frac{4\pi}{3} \int_{-\infty}^{t} dt' \Gamma(t') \left( a(t') \int_{t'}^{t} dt'' a''(t'') \right)^3 \right] \tag{13} \]
where $\Gamma = 9H^4Q/4\pi$ is the nucleation rate, and where $a^3 V_n = 4\pi/3H^3$ is the horizon volume at $N_T$. The quantity $dn_B/dt$ is proportional to the tunneling rate per volume, $\Gamma$, and to the FV volume left at the time $t$ (the volume not already occupied by bubbles). If after a certain time the exponential term in (13) decreases faster than $a^3$ increases, the FV volume fraction decreases; if this decrease is faster than $\Gamma$ increases, then $dn_B/dt$ will have a turnaround somewhere, indicating that the transition is being completed. The rest of this paper is essentially devoted to find the condition for this maximum to occur at the right time.

The $e$-folding time $N$ is defined as $N = N_T - \alpha$, where $\alpha = \log(a/a_{in}) = Ht$, so that we can put
\[ N = N_T(1 - Ht/N_T), \quad t(N_T) = 0. \tag{14} \]
Since in all what follows we have $N \approx N_T$ (i.e., the nucleation occurs around $N_T$), we can write
\[ \Gamma = \frac{9H^4}{4\pi} \exp[r_0 - r_T(1 - t/\tau)], \tag{15} \]
where
\[ r_0 = (N_0/N_1)^4, \quad r_T = (N_T/N_1)^4, \quad \tau = N_T/4H. \tag{16} \]
We will neglect the mild dependence of $H$ on $N$ compared to the exponential dependence of $\Gamma$. During inflation, $a(t) \approx a_0 \exp Ht$, so that we can integrate easily the argument of the exponential in eq. (15), and obtain finally
\[ \frac{dn_B}{dt} = A \exp[t(3H + r_T/\tau) - B \exp(r_T t/\tau)], \tag{17} \]
where $A = 3H e^{r_0 - r_T}$, and
\[ B = 3e^{r_0 - r_T} \frac{N_T}{4H} g(N_T, N_1), \tag{18} \]
and where
\[ g(N_T, N_1) = \left( 1 - \frac{3}{1 + N_T/4r_T} + \frac{3}{1 + N_T/2r_T} - \frac{1}{1 + 3N_T/4r_T} \right). \tag{19} \]
Expanding the argument of the exponential to the second order, we can approximate (13) as a Gaussian curve. The second order term in (17) should be included as well; however, for the values of interest of $N_0$ and $N_1$, it is irrelevant. The result is
\[ \frac{dn_B}{dt} = A' \exp \left[ -\frac{1}{2} \frac{(t - t_p)^2}{\sigma^2} \right], \tag{20} \]
where
\[ t_p = \frac{3H\tau^2 + r_T(1 - B)}{r_T^2 B}, \quad \sigma^2 = \tau^2/(r_T^2 B), \tag{21} \]
and where the preexponential factor is $A' = A \exp(-B + t_p^2/2\sigma^2)$. The instant $t_p$ marks the peak of the nucleation process; this will be fixed by the request to have bubbles (slightly) larger than the present horizon. We must have nucleation before $N_T$, or simply $t_p < 0$, and it is clear that a necessary condition is in any case $B > 0$.

It is now useful to use $N$ as time variable. It follows then that
\[ \left| \frac{dn_B}{dN} \right| = (A'/H) \exp \left[ -\frac{1}{2} \frac{(N - N_p)^2}{\sigma_N^2} \right], \tag{22} \]
with
\[ N_p = N_T \left( 1 - \frac{Ht_p}{N_T} \right), \quad \sigma_N^2 = \frac{N_T^2}{16r_T^2 B}, \tag{23} \]
where we require that $N_p > N_T$. We need now the relation between the present curvature $C_0 = |\Omega_0 - 1|$ and the nucleation time $N$. We know that if a bubble nucleates at $N$ its density parameter is
\[ \Omega = \frac{U_{TV}}{U_{FV}} = [1 + \gamma N^{-2}]^{-1} \approx 1 - \gamma N^{-2}, \tag{24} \]
where the constant $\gamma = 3\pi m^2 \psi_0^2/M^2$ has been taken small (compared to $N^2$) for simplicity of calculation, even if it is not necessary. Now, from the Friedmann equation, in the limit of small deviation from flatness, it is easy to see that the relation between the curvature today and the curvature during inflation is $C \approx N^{-1} e^{2(N-N_T)} C_0$. However, when the curvature deviates sensibly from zero, this relation is no longer acceptable. Neglecting the RDE phase, it is possible to derive from the standard solution of an open Universe (see e.g. [12]) the following expression
\[ C(N) \approx N^{-1} e^{2(N-N_T)} C_0 f(C_0), \]  

where \( f(C_0) = 1 + C_0^{3/2} / (1 - C_0) \). Thus, we get finally

\[ C_0 f(C_0) = \gamma e^{2(N_T-N)}/N \approx \gamma e^{2(N_T-N)}/N_T. \]  

It appears then that, as we would expect, \( C_0 \to 0 \) if the bubble nucleates at \( N \to \infty \), and \( C_0 \to 1 \) if the nucleation occurs at small \( N \). Only a value \( N \approx N_T \) would result in intermediate values of \( C_0 \). From Eq. (20), we can express the nucleation rate directly in terms of the present curvature \( C_0 \):

\[ \frac{dn_B}{dC_0} = \frac{A'}{2H} \left( \frac{1}{C_0} + \frac{f'}{f} \right) \exp \left\{ -\frac{1}{2} \left[ \frac{\log \left( \frac{C_0 f(C_0)}{C_p f(C_p)} \right)}{\sigma_C^2} \right]^2 \right\}, \]  

with \( f' = df(C_0)/dC_0 \), \( C_p f(C_p) = \gamma e^{2H_T}/N_T \), and \( \sigma_C^2 = 4\sigma_N^2 \). We can then define the probability \( P(\Omega_0) \) that we live in a \( \Omega_0 \) Universe as

\[ P(\Omega_0) \equiv \frac{dn_B}{dC_0} \frac{L(C_0)^3}{L_h^3}, \]  

where \( L(C_0) \) is the comoving size of a bubble which today has curvature \( C_0 \). As we will see shortly, \( P(\Omega_0) \) is normalized to unity. Eq. (28) is the central result of this work: it gives the probability density to live today in a bubble with given \( \Omega_0 \). As we have seen, we have obtained it using the standard tools of inflationary cosmology, without employing arguments from quantum cosmology. Most importantly, we have obtained \( P(\Omega_0) \) without assuming special initial conditions.

We now impose three observational constraints on \( P(\Omega_0) \): first, we must impose the condition that the bubbles fill the Universe, i.e. that the transition completes; second, we require the nucleation peak to occur for \( \Omega_0 = 0.2 \), the present observational value; and third, we require a narrow distribution, say \( \sigma_C \ll 1 \), to ensure our Universe is typical. This implies three conditions on our parameters \( N_0, N_1 \) and \( \gamma \). The first constraint amounts to requiring that the volume contained in bubbles nucleated from \( N = \infty \) down to \( N = 0 \) be equal to, or larger than, the present horizon:

\[ \int_0^0 \frac{dn_B}{dN} L^3 dN \geq L_h^3. \]  

where \( L(N) \) is the comoving scale of a bubble nucleated at \( N \). In terms of \( P(\Omega_0) \), Eq. (29) is simply the condition that the probability be normalized to unity (or to a value larger than unity, which simply indicates that the bubbles are too densely packed to assume spherical symmetry). Since

\[ H(N) L(N) \approx H_0 L_h \exp(N - N_T), \]  

we have the normalization condition

\[ \int_{\infty}^0 \frac{dn_B}{dN} e^{3(N-N_T)} dN \geq 1, \]  

neglecting again the time dependence of \( H \) during inflation. Using directly the form (17) where we express \( t \) in terms of \( N \) with (14) it follows

\[ \frac{3N_T e^{\sigma_N^2 N_T}}{4T} \left( 1 - e^{-B e^{2\gamma T}} \right) \approx 3g(N_T, N_1)^{-1} \geq 1. \]  

We display in Fig. 1 the region of the plane \((N_1, N_0)\) which satisfies the constraint above (with \( N_T = 60 \), for which the curves (22) are sharply peaked around \( N_p \) \((\sigma_N^2 \ll 1)\), and for which \( N_p > N_T \): as one can see, there is a vast region in which our theory is successful. Finally, the condition that \( C_p = 0.8 \) gives

\[ \gamma \approx 4N_T e^{-2H_T}. \]  

From Fig. 1 one sees that \( N_1 \) has to be roughly larger than 10, and therefore \( \gamma \approx 10^3 \) (from (21) with \( N_0 \geq 60 \)) and, by the condition (11),

\[ \left( \sqrt{3} m_{\gamma} / 4\pi \psi_0 \lambda \right)^{1/2} \geq 10. \]  

Let us summarize the constraints to which the parameters are subject. The tunneling function has to be peaked at \( N \) slightly larger than \( N_T \); the value of \( \Omega \) at that time has to be such that today \( \Omega_0 \approx 0.2 \); the peak should be narrow, so that the probability to live in a \( \Omega_0 = 0.2 \) Universe is high; and the transition should be intense enough to fill the Universe with bubbles. A further condition is that the slow roll do not generate too strong inhomogeneities; roughly, this implies \( M < 10^{-5} \) in Planck units. Finally, we have to be in the thin wall limit (11). It is remarkable that our model meets easily all these requirements. For instance, we can have \( N_1 \approx 25 \) and \( N_0 - N_T \approx 2 \), so that \( \gamma \approx 10^3 \) (from (23)); then, fixing \( M = 10^{-6} \) and \( \lambda = 10^4 \) (respecting (10)); we have \( m \approx 1 \) (the Planck scale), \( \psi_0 \approx 10^{-5} \) (near the scale of \( M \)). In Fig. 2 we plot \( P(\Omega_0) \) for this set of parameters.

IV. CONCLUSIONS

We presented a scenario in which the flat, inflationary Universe is filled by super-horizon-sized underdense bubbles, which approximate open Universes. This reconcile the astronomical observations in favour of \( \Omega_0 = 0.2 \) with inflation. Our own bubble- Universe is one of an infinite number of similar bubbles. Our model differs from the single-bubble scenario \[ \ddagger \ddagger \ddagger \ddagger \ddagger \ddagger \ddagger \] in which one must invoke anthropic arguments to explain our position, and is different also from the model presented by Linde \[ \ddagger \ddagger \ddagger \ddagger \ddagger \ddagger \ddagger \], in which there is no reason to expect preferential nucleation for any given value of \( \Omega \). As we showed in the previous Section, we can tune the parameters to achieve maximal
probability for the nucleation of $\Omega_0 = 0.2$ bubbles, without assuming special initial conditions, and satisfying all other constraints.

It is worth remarking again that the measure of $\Omega_0$ along with the assumption that the Universe had an inflationary epoch, and that our position is typical, put strong constraints on the fundamental parameters of the primordial potential. It is interesting to observe that the phase transition parameters $\lambda, \psi_0, m$ would be unobservable either if the nucleation occurred much earlier, because then the subsequent expansion would have again flattened the space, or much later, because then the very small sub-horizon bubbles would have thermalized, recovering again a $\Omega = 1$ Universe.

Inside the bubbles one has the usual mechanism of generation of inflationary perturbations [3] [6] [7]. In Ref. [9] we presented a scenario in which extra power is provided by the bubble-like structure of a primordial phase transition. It is possible that reducing the local $\Omega_0$ to 0.2 is enough to reconcile canonical CDM with large scale structure, so that no further phase transitions need to be invoked. However, evidences are increasing toward the presence of huge voids in the distribution of matter in the present Universe, and for velocity fields that are difficult to explain without a new source of strong inhomogeneities. If this is the case, the possibility of an additional primordial phase transition occurred around 50 $e$-foldings before the end of inflation should be seriously taken into account.

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FIG. 1. The shaded region indicate the values for $N_1$ and $N_0$ for which our theory is successful: underdense bubbles with a narrow peaked distribution containing our horizon’s scale today fill the whole Universe.
FIG. 2. Plot of $P(\Omega_0)$ for $N_0 = 62$ and $N_1 = 25$. Our present curvature is the most probable, and the distribution is very narrow around its mean value.