Research Article

A New Expanded Mixed Element Method for Convection-Dominated Sobolev Equation

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We propose and analyze a new expanded mixed element method, whose gradient belongs to the simple square integrable space instead of the classical $H(\text{div}; \Omega)$ space of Chen’s expanded mixed element method. We study the new expanded mixed element method for convection-dominated Sobolev equation, prove the existence and uniqueness for finite element solution, and introduce a new expanded mixed projection. We derive the optimal a priori error estimates in $L^2$-norm for the scalar unknown $u$ and a priori error estimates in $(L^2)^2$-norm for its gradient $\lambda$ and its flux $\sigma$. Moreover, we obtain the optimal a priori error estimates in $H^1$-norm for the scalar unknown $u$. Finally, we obtained some numerical results to illustrate efficiency of the new method.

1. Introduction

We consider the following Sobolev equation with convection term:

\[
\begin{align*}
    u_t + c(x) \cdot \nabla u - \nabla \cdot (a(x,t) \nabla u + b(x,t) \nabla u_t) &= f(x,t), \\
    (x,t) &\in \Omega \times J, \\
    u(x,t) &= 0, \quad (x,t) \in \partial \Omega \times J, \\
    u(x,0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]

where $\Omega$ is a bounded convex polygonal domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial \Omega$ and $J = (0, T]$ is the time interval with $0 < T < \infty$. $u_0(x)$ and $f(x,t)$ are given functions, coefficients $a = a(x,t)$, $b = b(x,t)$ are smooth and bounded functions, coefficient $c(x) = (c_1(x), c_2(x))$ is a bounded vector, and

\[
A_1 : 0 < a_0 \leq a(x,t) \leq a_1 < +\infty; \\
A_2 : 0 < b_0 \leq b(x,t) \leq b_1 < +\infty, \quad |b_1(x,t)| \leq d_1 < +\infty; \\
A_3 : 0 < \left( \sum_{j=1}^{2} c_j^2(x) \right)^{1/2} < +\infty,
\]

for some positive constants $a_0, a_1, b_0, b_1$, and $d_1$.

Sobolev equations are a class of important evolution partial differential equations and have a lot of applications in many physical problems, such as the porous theories concerned with percolation into rocks with cracks, the heat conduction problems in different mediums, and the transport problems of humidity in soil. In [1], the finite element method for nonlinear Sobolev equation with nonlinear boundary conditions was studied. In [2], a discontinuous Galerkin method for Sobolev equation was studied. In [3–7], some mixed finite element methods for Sobolev equations are studied and analyzed.

In 1994, Chen [8, 9] developed and studied an expanded mixed element method and proved some mathematical theories for second-order linear elliptic equation. Compared to standard mixed element methods the expanded mixed method is expanded in the sense that three variables are explicitly approximated, namely, the scalar unknown, its gradient, and its flux. From then on, the expanded mixed element method has been applied to solving other partial differential equations [10]. At the same time, many researchers proposed and studied some new numerical methods based on Chen’s expanded mixed method, such as expanded mixed hybrid methods [11], two-grid expanded mixed finite element method [12–14], expanded characteristic-mixed element method [15], expanded mixed covolume method.
ar eused. The other notations and definitions of Sobolev spaces as in such that for some partial differential equations [4,22–25]. Compared to Chen’s expanded mixed method, the gradient for the new expanded mixed method belongs to the simple square integrable space instead of the classical \( H(\text{div}; \Omega) \) space. In this paper, we will study the new expanded mixed element method for convection-dominated Sobolev equation. We will give the proof for the existence and uniqueness of the solution for semidiscrete scheme and a new expanded mixed projection and the proof of its uniqueness. We will prove the optimal a priori error estimates in \( H^1(\Omega) \) for the scalar unknown \( u \). Using Chen’s expanded mixed method, the mixed weak formulation for problem (1) is to find \( \{u_h, \lambda_h, \sigma_h\} : [0, T] \mapsto \mathbf{W}_h \times \mathbf{W}_h \times \mathbf{W}_h \) such that

\[
(a) \ (u_{h\nu}, v_h) + (c \cdot \lambda, v_h) - (\sigma_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in \mathbf{W}_h,
\]

\[
(b) \ (\lambda_h, w_h) - (\nabla u_h, w_h) = 0, \quad \forall w_h \in \mathbf{W}_h,
\]

\[
(c) \ (\sigma_h, z_h) + (a \lambda_h + b \lambda_h, z_h) = 0, \quad \forall z_h \in \mathbf{W}_h,
\]

(6)

where \( (V_h, \mathbf{W}_h) \) is chosen as the finite element pair \( P_1 - P_0^2 \) as follows:

\[V_h = \left\{ v_h \in C^0(\Omega) \cap H^1_0 | v_h \in P_1(K), \forall K \in \mathcal{K}_h \right\},\]

\[\mathbf{W}_h = \left\{ w_h = (w_{1h}, w_{2h}) \in \left( L^2(\Omega) \right)^2 \mid w_{1h}, w_{2h} \in P_0(K), \forall K \in \mathcal{K}_h \right\}.\]

From [20, 21], we find that \( (V_h, \mathbf{W}_h) \) satisfies the so-called discrete Ladyzhenskaya-Babuska-Brezzi condition.

Remark 1. Compared to Chen’s expanded mixed weak formulation (4), the gradient in the scheme (5) belongs to the simple square integrable space \( L^2(\Omega) \) instead of the classical \( H(\text{div}; \Omega) \) space. Obviously, the regularity requirements on the solution \( \lambda = \nabla u \) reduced.

Theorem 2. There exists a unique discrete solution to semidiscrete scheme (6).

Proof. Let \( \{\psi_j(x)\}_{j=1}^{n_1} \) and \( \{\varphi_j(x)\}_{j=1}^{n_2} \) be bases of \( V_h \) and \( \mathbf{W}_h \), respectively. Let

\[u_h = \sum_{j=1}^{n_1} u_j(t) \psi_j(x), \quad \lambda_h = \sum_{j=1}^{n_1} \lambda_j(t) \varphi_j(x),\]

\[\sigma_h = \sum_{k=1}^{n_2} \sigma_k(t) \varphi_k(x), \quad u_h(0) = \sum_{j=1}^{n_1} \vec{u}_j \psi_j(x),\]

\[\lambda_h(0) = \sum_{j=1}^{n_1} \vec{\lambda}_j \psi_j(x), \quad \sigma_h(0) = \sum_{k=1}^{n_2} \vec{\sigma}_k \varphi_k(x),\]

(8)

and substituting these expressions into (6) and choosing \( v_h = \psi_m, w_h = \varphi_1 \), and \( z_h = \varphi_2 \), the problems (6) can be written as follows: find \( \{\vec{u}(t), \vec{\lambda}(t), \vec{\sigma}(t)\} \) such that, for all \( t \in (0, T] \)

\[(a) \ A \vec{u}''(t) + B \vec{\lambda}(t) - C \vec{\sigma}(t) = F(t),\]

\[(b) \ D \vec{\lambda}(t) - E \vec{u}(t) = 0,\]

\[(c) \ D \vec{\sigma}(t) + H \vec{\lambda}(t) + J \vec{\lambda}(t) = 0,\]

\[(d) \ \vec{u}(0) = \vec{u}, \quad \vec{\lambda}(0) = \vec{\lambda}, \quad \vec{\sigma}(0) = \vec{\sigma},\]

(9)
In order to analyze the convergence of the method, we first introduce the new expanded mixed elliptic projection associated with our equations.

Let \((u_h, \lambda_h, \sigma_h) : [0, T] \rightarrow V_h \times W_h \times W_h\) be given by the following mixed relations:

\begin{align*}
\text{(a)} & \quad (\sigma - \overline{\sigma}_h, \nabla v_h) + \left(\mathbf{c} \cdot (\lambda - \overline{\lambda}_h), v_h\right) = 0, \quad \forall v_h \in V_h, \\
\text{(b)} & \quad (\lambda - \overline{\lambda}_h, w_h) - (\nabla (u - \overline{u}_h), w_h) = 0, \quad \forall w_h \in W_h, \\
\text{(c)} & \quad (\sigma - \overline{\sigma}_h, z_h) + \left(a (\lambda - \overline{\lambda}_h) + b (\lambda_1 - \overline{\lambda}_{1h}), z_h\right) = 0, \quad \forall z_h \in W_h. 
\end{align*}

**Theorem 3.** There exists a unique solution to the new expanded mixed elliptic projection (14).

**Proof.** Noting that mixed elliptic projection system (14) is linear, it suffices to prove the associated homogeneous system

\begin{align*}
\text{(a)} & \quad (\overline{\sigma}_h, \nabla v_h) + \left(\mathbf{c} \cdot \overline{\lambda}_h, v_h\right) = 0, \quad \forall v_h \in V_h, \\
\text{(b)} & \quad (\overline{\lambda}_h, w_h) - (\nabla \overline{u}_h, w_h) = 0, \quad \forall w_h \in W_h, \\
\text{(c)} & \quad (\overline{\sigma}_h, z_h) + \left(a \overline{\lambda}_h + b \overline{\lambda}_{1h}, z_h\right) = 0, \quad \forall z_h \in W_h
\end{align*}

has the trivial solution.

Choose \(v_h = \overline{u}_h\) in (15)(a), \(w_h = \overline{\sigma}_h\) in (15)(b), and \(z_h = \overline{\lambda}_h\) in (15)(c) to obtain

\begin{align*}
\text{(a)} & \quad (\overline{\sigma}_h, \nabla \overline{u}_h) + \left(\mathbf{c} \cdot \overline{\lambda}_h, \overline{u}_h\right) = 0, \\
\text{(b)} & \quad (\overline{\lambda}_h, \overline{\sigma}_h) - (\nabla \overline{u}_h, \overline{\sigma}_h) = 0, \\
\text{(c)} & \quad (\overline{\sigma}_h, \overline{\lambda}_h) + \left(a \overline{\lambda}_h + b \overline{\lambda}_{1h}, \overline{\lambda}_h\right) = 0.
\end{align*}

Add the three equations to get

\begin{equation}
\left\|d^{1/2} \overline{\lambda}_h\right\|^2 + \frac{d}{dt} \left\|b^{1/2} \overline{\lambda}_h\right\|^2 = \frac{1}{2} \left(b \overline{\lambda}_h, \overline{\lambda}_h\right) + \left(\mathbf{c} \cdot \overline{\lambda}_h, \overline{\sigma}_h\right).
\end{equation}

Integrate (17) with respect to time from 0 to \(t\) and use Cauchy-Schwarz inequality and Young inequality to obtain

\begin{equation}
2\alpha_0 \int_0^t \left\|\overline{\lambda}_h\right\|^2 ds + b_0 \left\|\overline{\lambda}_{1h}\right\|^2 \\
\leq C \int_0^t \left(\left\|\overline{\lambda}_h\right\|^2 + \left\|\overline{u}_h\right\|^2\right) ds.
\end{equation}

Taking \(w_h = \nabla \overline{u}_h\) in (15) and using Poincaré inequality, we obtain

\begin{equation}
\left\|\overline{u}_h\right\| \leq C \left\|\nabla \overline{u}_h\right\| \leq C \left\|\overline{\lambda}_h\right\|.
\end{equation}

Substitute (19) into (18) and use Gronwall lemma to obtain

\begin{equation}
\int_0^t \left\|\overline{\lambda}_h\right\|^2 ds + \left\|\overline{\lambda}_{1h}\right\|^2 = 0.
\end{equation}
From (20), we have
\[ \tilde{\lambda}_h = 0. \quad (21) \]
Combining (19), (21), and (15)(c), we get
\[ \tilde{u}_h = 0, \quad \tilde{\sigma}_h = 0. \quad (22) \]
Using (21) and (22), we get
\[ \tilde{u}_h = 0, \quad \tilde{\sigma}_h = 0, \quad \tilde{\lambda}_h = 0. \quad (23) \]

In the following discussion, we will give some important lemmas based on new mixed scheme.

**Lemma 4.** There exists a linear operator \( \Pi_h \) : \( (L^2(\Omega))^2 \) \( \rightarrow \) \( W_h \) such that
\[ (\sigma - \Pi_h \sigma, \nabla v_h) = 0, \quad \forall v_h \in V_h, \quad (24) \]
\[ \|\sigma - \Pi_h \sigma\|_{L^2(\Omega)} \leq Ch\|\sigma\|_{(H^1(\Omega))\lambda}. \quad (25) \]

**Lemma 5.** For the linear operator \( \Pi_h \) of Lemma 4, one has
\[ (\lambda - \Pi_h \lambda, w_h) = 0, \quad \forall w_h \in W_h, \quad (\lambda - \Pi_h \lambda\|_{L^2(\Omega)} \leq Ch\|\lambda\|_{(H^1(\Omega))\lambda}. \quad (26) \]
\[ \|\lambda - \Pi_h \lambda\|_{L^2(\Omega)} \leq Ch\|\lambda\|_{(H^1(\Omega))\lambda}. \quad (27) \]

**Lemma 6.** There exists a linear operator \( P_h : H^1_0(\Omega) \rightarrow V_h \) such that
\[ (\nabla (u - P_h u), w_h) = 0, \quad \forall w_h \in W_h, \quad \|u - P_h u\|_{L^2(\Omega)} + \|u - P_h u\|_{H^1} \leq Ch^2\|u\|_{H^1}, \quad (28) \]
\[ \|u_t - P_h u_t\|_{L^2(\Omega)} \leq Ch^2\|u_t\|_{H^1}. \quad (29) \]

From [20, 21], we can obtain the proof for Lemmas 4–6. Using the definition of \( \Pi_h \) and \( P_h \), we rewrite \( \eta, \delta, \) and \( \rho \) as
\[ \eta = u - \tilde{u}_h = u - P_h u + P_h u - \tilde{u}_h = \eta_m + \eta_c; \]
\[ \delta = \lambda - \tilde{\lambda}_h = \lambda - \Pi_h \lambda + \Pi_h \lambda - \tilde{\lambda}_h = \delta_m + \delta_c; \]
\[ \rho = \sigma - \tilde{\sigma}_h = \sigma - \Pi_h \sigma + \Pi_h \sigma - \tilde{\sigma}_h = \rho_m + \rho_c. \quad (30) \]

Since estimates of \( \eta_m, \delta_m, \) and \( \rho_m \) are known, it is enough to estimate \( \eta_c, \delta_c, \) and \( \rho_c. \) Using Lemmas 4–6, we rewrite (14) as
\[ (a) (\rho_c, \nabla \eta_c) + (c \cdot \delta_c, \eta_c) = 0, \quad \forall \eta_c \in \eta_c, \quad (b) (\delta_c, w_h) - (\nabla \eta_c, w_h) = 0, \quad \forall w_h \in W_h, \quad (31) \]
\[ (c) (\rho_c, z_h) + (a \delta_c, z_h) + (b(\delta_c), z_h) \]
\[ = - (a \delta_c, z_h) - (b(\delta_c), z_h), \quad \forall z_h \in W_h. \quad (32) \]

We discuss the following approximation properties for system (29).

**Lemma 7.** There is a constant \( C \) independent of \( h \) such that
\[ \|\delta\| \leq Ch\left(\|\lambda\|_{(H^1(\Omega))\lambda} + \int_0^t \left(\|u\|_{H^1} + \|\lambda\|_{(H^1(\Omega))\lambda} \right) \right. \]
\[ \left. + \|\lambda\|_{(H^1(\Omega))\lambda} \right) ds. \quad (33) \]

Proof. Choose \( v_h = \eta_c \) in (29)(a), \( w_h = \rho_c \) in (29)(b), and \( z_h = \delta_c \) in (29)(c) to obtain
\[ (a) (\rho_c, \nabla \eta_c) + (c \cdot \delta_c, \eta_c) = 0, \quad (34) \]
\[ (b) (\delta_c, w_h) - (\nabla \eta_c, w_h) = 0, \quad (35) \]
\[ (c) (\rho_c, z_h) + (a \delta_c, z_h) + (b(\delta_c), z_h) \]
\[ = - (a \delta_c, z_h) - (b(\delta_c), z_h), \quad \forall z_h \in W_h. \quad (36) \]

Add the three equations and use Cauchy-Schwarz inequality to get
\[ \|a^{1/2} \delta_c\|^2 + \frac{1}{2} \frac{d}{dt} \|b^{1/2} \delta_c\|^2 \]
\[ = (c \cdot \delta_c, \eta_c) + \frac{1}{2} (b \delta_c, \delta_c) - (a \delta_m, \delta_c) - (b(\delta_c), \delta_c) \]
\[ \leq C \left( \|\eta_c\|^2 + \|\delta_c\|^2 + \|\delta_m\|^2 + \|\delta_m\|^2 \right). \quad (37) \]
Integrate (35) with respect to time from 0 to \( t \) to obtain
\[
\| \delta_e \|^2 + \int_0^t \| \delta_e \|^2 \, ds \leq C \left( \int_0^t \left( \| \eta_e \|^2 + \| \delta_e \|^2 + \| \delta_m \|^2 + \| (\delta_m)_{1} \|^2 \right) \, ds \right).
\]
Using the Gronwall lemma, we have
\[
\| \delta_e \|^2 + \int_0^t \| \delta_e \|^2 \, ds \leq C \int_0^t \left( \| \eta_e \|^2 + \| \delta_m \|^2 + \| (\delta_m)_{1} \|^2 \right) \, ds.
\]
Differentiating (34)(b) and taking \( w_h = \rho_e \), we obtain
\[
\left( (\delta_e), (\rho_e) \right) - \left( (\nabla (\eta_e), \rho_e) \right) = 0.
\]
Choose \( v_h = (\eta_e) \) in (34)(a) and \( z_h = (\delta_e) \) in (34)(c) to obtain
(a) \( (\rho_e, V(\eta_e)) = -(c \cdot \delta_e, (\eta_e)) \),
(b) \( (\rho_e, (\delta_e)) = \left( ||\delta_e||^2 + ||\delta_m||^2 + ||(\delta_m)_{1}||^2 \right) \)
(c) \( (\rho_e, (\delta_e)) = -(a \delta_e, (\delta_e)) - (a \delta_m, (\delta_e)) - (b (\delta_m), (\delta_e)) \).
Combining (38) and (39), we have
\[
\left( ||\delta_e||^2 + ||\delta_m||^2 \right) \leq C \left( ||\eta_e||^2 + ||\delta_e||^2 + ||(\delta_m)_{1}||^2 \right) + \frac{b_2}{2} ||(\delta_e)_{1}||^2.
\]
Substitute (37) into (40) to obtain
\[
\| (\delta_e)_{1} \|^2 \leq C \left( ||\eta_e||^2 + ||\delta_m||^2 + ||(\delta_m)_{1}||^2 \right) + C \int_0^t \left( ||\eta_e||^2 + ||\delta_m||^2 + ||(\delta_m)_{1}||^2 \right) \, ds.
\]
Choose \( z_h = \rho_e \) in (29)(c) and use Cauchy-Schwarz inequality to obtain
\[
\| \rho_e \| \leq C \left( ||\eta_e||^2 + ||\delta_m||^2 + ||(\delta_m)_{1}||^2 \right)
\]
\[
+ C \int_0^t \left( ||\eta_e||^2 + ||\delta_m||^2 + ||(\delta_m)_{1}||^2 \right) \, ds.
\]
Choose \( w_h = V(\eta_e) \) in (29)(b) and use (37) and Cauchy-Schwarz inequality to obtain
\[
\| V(\eta_e) \| \leq C \left( ||\eta_e||^2 + ||\delta_m||^2 + ||(\delta_m)_{1}||^2 \right) \, ds.
\]
Combining (37), (42), (43), and Lemmas 4–6 and using the triangle inequality, we get the conclusion of Lemma 7.

**Lemma 8.** There is a constant \( C \) independent of \( h \) such that
\[
\| \eta_h \| \leq C h^2 ||u||_{H^2},
\]
\[
\| \eta_h \| \leq C h^2 ||u||_{H^2},
\]
\[
\| \eta_h \| \leq C \left( ||u||_{H^2} + \int_0^t \left( ||u||_{H^2} + \lambda ||(\Delta u)(\Omega)\| \right) \, ds \right).
\]

**Proof.** To estimate terms \( ||\eta||, ||\eta_h||, \) and \( ||\eta_h|| \), we consider the following auxiliary elliptic problem:
\[
- \nabla \cdot (a \nabla \chi) = \eta_h \text{ in } \Omega, \quad \chi = 0, \text{ on } \partial \Omega.
\]
Use (45) and Lemmas 4–6 to obtain
\[
\| \eta \|^2 = (u - \bar{u}, \eta) = (u - P_h u, \eta)
\]
\[
= (u - P_h u, \eta) + (P_h u - \bar{u}, \eta)
\]
\[
= (u - P_h u, \eta) + (P_h u - \bar{u}, -\nabla \cdot (a \nabla \chi))
\]
\[
= (u - P_h u, \eta) + (P_h u - \bar{u}, a \nabla \chi)
\]
\[
= (u - P_h u, \eta) + (\nabla (P_h u - \bar{u}), a \nabla \chi - \Pi_h (a \nabla \chi))
\]
\[
= (u - P_h u, \eta) \leq ||u - P_h u|| \| \eta \|
\]
\[
\leq C h^2 ||u||_{H^2} \| \eta \|.
\]
From (46), we obtain
\[
\| \eta \| \leq C h^2 ||u||_{H^2}.
\]
Using similar method to \( ||\eta||_2 \), we can obtain
\[
\| \eta_h \| \leq C h^2 ||u||_{H^2}.
\]
Combining (33) and (47), we obtain
\[
\| \eta_h \| \leq C h \left( ||u||_{H^2} + \int_0^t \left( ||u||_{H^2} + \lambda ||(\Delta u)(\Omega)\| \right) \, ds \right).
\]
Using (47)–(49), we obtain the conclusion of Lemma 8.

For a priori error estimates, we decompose the errors as
\[
u - u_h = u - \bar{u}_h + \bar{u}_h - u_h = \eta + \zeta;
\]
\[
\lambda - \lambda_h = \lambda - \bar{\lambda}_h + \bar{\lambda}_h - \lambda_h = \delta + \theta;
\]
\[
\sigma - \sigma_h = \sigma - \bar{\sigma}_h + \bar{\sigma}_h - \sigma_h = \rho + \xi.
\]
Using (5)-(6) and (14), we can get the error equations

(a) \((\zeta, v_h) + (c \cdot \theta, v_h) - (\xi, \nabla v_h) = -(\eta, v_h), \quad \forall v_h \in V_h,\)

(b) \((\theta, w_h) - (\nabla \xi, w_h) = 0, \quad \forall w_h \in W_h,\)

(c) \((\xi, z_h) + (a \theta + b \theta \xi, z_h) = 0, \quad \forall z_h \in W_h.\)  

(51)

We will prove the error estimates for semidiscrete scheme.

**Theorem 9.** Assume that \(u_h(0) = \bar{u}_h(0) \) and \(\lambda_h(0) = \bar{\lambda}_h(0)\); then one has the following estimates:

\[
\begin{align*}
\| u - u_h \| & \leq Ch^2 \left( \| u \|_{H^2} + \int_0^t \| u_t \|_{H^2} ds \right), \\
\| u_t - u_{t,h} \| & \leq Ch^2 \left( \| u_t \|_{H^2} + \int_0^t \| u_t \|_{H^2} ds \right), \\
\| \lambda - \lambda_h \| & \leq Ch \left( \| \lambda \|_{H^2(\Omega)} + \int_0^t \| \theta \| ds \right), \\
\| \lambda_t - \lambda_{t,h} \| & \leq Ch \left( \| \lambda_t \|_{H^2(\Omega)} + \int_0^t \| \theta \| ds \right), \\
\| \sigma - \sigma_h \| & \leq Ch \left( \| \sigma \|_{H^2(\Omega)} + \int_0^t \| \theta \| ds \right),
\end{align*}
\]

where \(\| \cdot \| \) denotes the Sobolev norm \(H^2(\Omega)\).

**Proof.** Choose \(v_h = \zeta\) in (51)(a), \(w_h = \xi\) in (51)(b), and \(z_h = \theta\) in (51)(c) to obtain

\[
\begin{align*}
(a) & \frac{1}{2} \frac{d}{dt} \| \zeta \|^2 - (\xi, \nabla \zeta) = -(\eta, \zeta) - (c \cdot \theta, \zeta), \\
(b) & (\theta, \xi) - (\nabla \xi, \xi) = 0, \\
(c) & (\xi, \theta) + \| a^{1/2} \theta \|^2 + \frac{1}{2} \frac{d}{dt} \| b^{1/2} \theta \|^2 = \frac{1}{2} (b_t \theta, \theta).
\end{align*}
\]

Adding the above three equations, and using Cauchy-Schwarz inequality and Young inequality, we have

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \zeta \|^2 + \| a^{1/2} \theta \|^2 + \frac{1}{2} \frac{d}{dt} \| b^{1/2} \theta \|^2 & = - (\eta, \zeta) - (c \cdot \theta, \zeta) + \frac{1}{2} (b_t \theta, \theta) \\
& \leq C \left( \| \eta \| + \| \zeta \| + \| \theta \| \right).
\end{align*}
\]

Integrate with respect to time from 0 to \(t\) to obtain

\[
\begin{align*}
\| \zeta \|^2 + \int_0^t \| \theta \|^2 ds + \| \theta \|^2 \\
& \leq C \int_0^t \left( \| \eta \|^2 + \| \zeta \|^2 + \| \theta \|^2 \right) ds.
\end{align*}
\]

Using Gronwall lemma, we obtain

\[
\| \zeta \|^2 + \int_0^t \| \theta \|^2 ds + \| \theta \|^2 \leq C \int_0^t \| \eta \|^2 ds.
\]

(56)

Take \(w_h = \xi\) in (51)(b) to have

\[
\| \nabla \xi \| \leq \| \theta \|.
\]

(57)

Differentiating (51)(b) and taking \(w_h = \xi\), we obtain

\[
(\theta_t, \xi) - (\nabla \xi, \xi) = 0.
\]

(58)

Choose \(v_h = \zeta\) in (51)(a) and \(z_h = \theta\) in (51)(c) to obtain

\[
(a) \| \zeta \|^2 - (\xi, \nabla \zeta) = -(\eta, \zeta) - (c \cdot \theta, \zeta), \\
(c) (\xi, \theta) + \| b^{1/2} \theta \|^2 = -(a \theta, \theta).
\]

(59)

Adding (58), (59)(a), and (59)(c) and using Cauchy-Schwarz inequality, Young inequality, and (56), we have

\[
\begin{align*}
\| \zeta \|^2 + \| b^{1/2} \theta \|^2 & = -(\eta, \zeta) - (c \cdot \theta, \zeta) - (a \theta, \theta) \\
& \leq C \| \eta \|^2 + \frac{1}{2} \| \zeta \|^2 + \frac{b_t}{2} \| \theta \|^2 \\
& \quad + C \int_0^t \| \eta \|^2 ds.
\end{align*}
\]

So, we have

\[
\| \zeta \|^2 + \| \theta \|^2 \leq C \left( \| \eta \|^2 + \int_0^t \| \eta \|^2 ds \right).
\]

(60)

Choosing \(z_h = \xi\) in (51)(c) and using (56) and (61), we have

\[
\begin{align*}
\frac{1}{2} \| \xi \|^2 & = -(a \theta + b \theta \xi) - \frac{1}{2} \| \xi \|^2 \\
& \leq C \left( \| \theta \|^2 + \| \theta \|^2 \right) \leq C \left( \| \eta \|^2 + \int_0^t \| \eta \|^2 ds \right).
\end{align*}
\]

(62)

Combining Lemmas 7 and 8, (56), (57), (60), (62), and the triangle inequality, we obtain the error estimate for Theorem 9. \(\square\)

4. Fully Discrete Scheme and Error Estimates

In this section, we get the error estimates of fully discrete schemes. For the backward Euler procedure, let \(0 = t_0 < t_1 < t_2 < \cdots < t_M = T\) be a given partition of the time interval \([0, T]\) with step length \(\Delta t = T / M\) and nodes \(t_n = n \Delta t\), for some positive integer \(M\). For a smooth function \(\phi\) on \([0, T]\), define \(\phi^n = \phi(t_n)\) and \(\phi^n = (\phi^n - \phi^{n-1}) / \Delta t\).

Equation (5) has the following equivalent formulation:

\[
\begin{align*}
(a) & (\partial_t u^n, v) + (c^n \cdot \lambda^n, v) - (\sigma^n, \nabla v) = (f^n + R^n_1, v), \\
& \quad \forall v \in H^1_0, \\
(b) & (\lambda^n, w) - (\nabla u^n, w) = 0, \quad \forall w \in \left( L^2(\Omega) \right)^2, \\
(c) & (\sigma^n, z) + (a^n \lambda^n + b^n \partial_t \lambda^n, z) = (b^2 R^n_2, z), \\
& \quad \forall z \in \left( L^2(\Omega) \right)^2,
\end{align*}
\]

(63)
where

\[
R_1^n = \partial_t u^n - u_n (t_n) = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} (t_{n-1} - s) u_n ds, \\
R_2^n = \partial_t \lambda^n - \lambda_n (t_n) = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} (t_{n-1} - s) \lambda_n ds.
\]

Now we formulate a completely discrete procedure. Find \((u_n^n, \lambda_n^n, \sigma_n^n) \in V_h \times W_h \times W_h, (n = 0, 1, \ldots, M)\) such that

\[
\begin{align*}
(a) \quad & \left( \partial_t u_n^n, v_h \right) + \langle c_n \cdot \lambda_n^n, v_h \rangle - \langle \sigma_n^n, \nabla v_h \rangle = \langle f^n, v_h \rangle, \\
& \forall v_h \in V_h, \\
(b) \quad & \left( \lambda_n^n, w_h \right) - \left( \nabla u_n^n, w_h \right) = 0, \quad \forall w_h \in W_h, \\
(c) \quad & \langle \sigma_n^n, z_h \rangle + \langle a^n \lambda_n^n + b^n \partial_t \lambda_n^n, z_h \rangle = 0, \\
& \forall z_h \in W_h.
\end{align*}
\]

For the fully discrete error estimates, we now split the errors

\[
\begin{align*}
& u (t_n) - u_n^n = u(t_n) - \bar{u}_n^n + \bar{u}_n^n - u_n^n = \eta^n + \varsigma^n; \\
& \lambda (t_n) - \lambda_n^n = \lambda_1^n - \bar{\lambda}_n^n + \bar{\lambda}_n^n - \lambda_n^n = \delta^n + \theta^n; \\
& \sigma (t_n) - \sigma_n^n = \sigma_1^n - \bar{\sigma}_n^n + \bar{\sigma}_n^n - \sigma_n^n = \rho^n + \xi^n.
\end{align*}
\]

We will prove the theorem for the fully discrete error estimates.

**Theorem 10.** Assume that \(u_0^n = \bar{u}_n(0)\) and \(\lambda_0^n = \bar{\lambda}_n(0)\); then there exists a positive constant \(C\) independent of \(h\) and \(\Delta t\) such that

\[
\|\sigma^n - \sigma_n^n\| \leq C \left( \|\lambda_1^n\| + \|\sigma_1^n\| + \|u_1^n\|_{L^\infty(H^2)} \right)
\]

\[
+ \Delta t \left( \|u_1^n\|_{L^\infty(H^2)} + \|u_{1,n}\|_{L^\infty(H^2)} \right)
\]

\[
+ C \Delta t \int_{t_0}^{t_f} \left( \|u_{2,n}\| + \|\lambda_{2,n}\| \right) ds.
\]

**Proof.** Using (14), (63), and (65) at \(t = t_n\), we get the error equations

\[
\begin{align*}
(a) \quad & \left( \partial_t \varsigma^n, v_h \right) + \langle c^n \cdot \theta^n, v_h \rangle - \langle \eta^n, \nabla v_h \rangle \\
& = - \left( \partial_t \eta^n, v_h \right) + \langle R_1^n, v_h \rangle, \quad \forall v_h \in V_h, \\
(b) \quad & \left( \theta^n, w_h \right) - \left( \nabla \varsigma^n, w_h \right) = 0, \quad \forall w_h \in W_h, \\
(c) \quad & \left( \xi^n, z_h \right) + \langle a^n \theta^n + b^n \partial_t \theta^n, z_h \rangle = \langle b^n R_2^n, z_h \rangle, \quad \forall z_h \in W_h.
\end{align*}
\]

Choose \(v_h = \varsigma^n\) in (68)(a), \(w_h = \xi^n\) in (68)(b), and \(z_h = \theta^n\) in (68)(c) to obtain

\[
\begin{align*}
(a) \quad & \frac{1}{\Delta t} \left( \|\varsigma^n\|^2 - \|\varsigma^{n-1}\|^2 + \|\varsigma^n - \varsigma^{n-1}\|^2 \right) \\
& + \langle c^n \cdot \theta^n, \varsigma^n \rangle - \langle \eta^n, \nabla \varsigma^n \rangle = - \left( \partial_t \eta^n, \varsigma^n \right) + \langle R_1^n, \varsigma^n \rangle, \\
(b) \quad & \left( \theta^n, \xi^n \right) - \left( \nabla \varsigma^n, \xi^n \right) = 0, \\
(c) \quad & \langle \xi^n, \theta^n \rangle + \|a^n \theta^n + b^n \partial_t \theta^n\|^2 \\
& + \frac{1}{\Delta t} \left( \|b^n \|^2 - \|b^{n-1} \|^2 + \|b^{n-1} \|^2 \right) \\
& + \frac{1}{\Delta t} \left( \|b^n \|^2 - \|b^{n-1} \|^2 + \|b^{n-1} \|^2 \right) \\
& = \left( b^n R_2^n, \theta^n \right) + \langle b^n - b^{n-1} \theta^n \rangle.
\end{align*}
\]

Adding the above three equations, we obtain

\[
\begin{align*}
& \frac{1}{\Delta t} \left( \|\varsigma^n\|^2 - \|\varsigma^{n-1}\|^2 + \|b^n \|^2 \right) \\
& - \left( \|b^{n-1} \|^2 \theta^n \right) \left| \|b^{n-1} \|^2 \theta^n \right| \\
& \leq - \left( \partial_t \eta^n, \varsigma^n \right) + \langle R_1^n, \varsigma^n \rangle - \langle c^n \cdot \theta^n, \varsigma^n \rangle \\
& + \langle b^n R_2^n, \theta^n \rangle \\
& \leq C \left( \|\partial_t \eta^n\|^2 + \|R_1^n\|^2 + \|R_2^n\|^2 \right) \\
& + C \left( \|\theta^n\|^2 + \|\theta^{n-1}\|^2 + \|\varsigma^n\|^2 \right).
\end{align*}
\]
Multiplying by $2\Delta t$ and summing (70) from $n = 1$ to $J$, the resulting equation becomes

\[(1 - C\Delta t) \| \zeta \|^2 + (b_0 - C\Delta t) \| \theta \|^2 \]
\[+ a_0 \Delta t \sum_{n=1}^{J} \| \theta^n \|^2 \leq \| \zeta^0 \|^2 + b_0 \| \theta^0 \|^2 \]
\[+ C\Delta t \sum_{n=1}^{J} \left( \| \partial_t \eta^n \|^2 + \| R_k^n \|^2 + \| R_k^2 \|^2 \right) \]
\[+ C\Delta t \sum_{n=1}^{J} \left( \| \theta^n \|^2 + \| \theta^n \|^2 \right) . \quad (71)\]

Choose $\Delta t_0$ in such a way that for $0 < \Delta t \leq \Delta t_0$, $(b_0 - C\Delta t) > 0$. Then we use Cronwall's lemma to obtain

\[\| \zeta \|^2 + \| \theta \|^2 + a_0 \Delta t \sum_{n=1}^{J} \| \theta^n \|^2 \leq C \Delta t \sum_{n=1}^{J} \left( \| \partial_t \eta^n \|^2 + \| R_k^n \|^2 + \| R_k^2 \|^2 \right) . \quad (72) \]

Note that

\[\| R_k^n \|^2 \leq C\Delta t \int_{t_n}^{t_{n+1}} \| u_{tt} \|^2 ds, \]
\[\| R_k^2 \|^2 \leq C\Delta t \int_{t_n}^{t_{n+1}} \| \lambda_{tt} \|^2 ds, \]
\[\| \partial_t \eta^n \|^2 \leq C \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \| \eta_i(s) \|^2 ds . \quad (73) \]

Substitute (73) to (72) to get

\[\| \zeta \|^2 + \| \theta \|^2 + a_0 \Delta t \sum_{n=1}^{J} \| \theta^n \|^2 \leq C \int_{t_n}^{t_{n+1}} \| \eta_i(s) \|^2 ds \]
\[+ C(\Delta t)^2 \int_{t_n}^{t_{n+1}} \left( \| u_{tt} \|^2 + \| \lambda_{tt} \|^2 \right) ds . \quad (74) \]

Taking $w_h = \nabla \zeta^n$ in (68)(b), we have

\[\| \nabla \zeta^n \| \leq \| \theta^n \|. \quad (75) \]

From (68)(b), we get

\[\left( \partial_t \theta^n, w_h \right) - \left( \nabla \partial_t \zeta^n, w_h \right) = 0 . \quad (76) \]

Choose $w_h = \xi^n$ in (76), $v_h = \partial_t \zeta^n$ in (68)(a), and $z_h = \partial_t \theta^n$ in (68)(c) to obtain

\[(a) \| \partial_t \zeta^n \|^2 + \langle \zeta^n \cdot \theta^n, \partial_t \zeta^n \rangle - \langle \zeta^n, \nabla \partial_t \zeta^n \rangle \]
\[= -\langle \partial_t \eta^n, \partial_t \zeta^n \rangle + \langle R_k^n, \partial_t \zeta^n \rangle , \]
\[(b) \left( \partial_t \theta^n, \xi^n \right) - \langle \nabla \partial_t \zeta^n, \xi^n \rangle = 0 , \quad (77) \]
\[(c) \left( \xi^n, \partial_t \theta^n \right) + \langle b^n \partial_t \theta^n, \partial_t \theta^n \rangle \]
\[= -\left( \left( a^n \right)^{1/2} \partial_t \theta^n, \partial_t \theta^n \right) + \left( b^n R_k^n, \partial_t \theta^n \right) . \]

Adding the three equations, we obtain

\[\| \partial_t \zeta^n \|^2 + \left( \left( a^n \right)^{1/2} \partial_t \theta^n \right)^2 \]
\[\leq \langle \partial_t \eta^n, \partial_t \zeta^n \rangle + \langle R_k^n, \partial_t \zeta^n \rangle - \langle \zeta^n, \nabla \partial_t \zeta^n \rangle \]
\[+ \langle b^n R_k^n, \partial_t \theta^n \rangle - \left( \left( a^n \right)^{1/2} \partial_t \theta^n, \partial_t \theta^n \right) \]
\[\leq C \left( \| \partial_t \eta^n \|^2 + \| R_k^n \|^2 + \| R_k^2 \|^2 \right) \]
\[+ \frac{a_0}{2} \| \partial_t \theta^n \|^2 + \frac{1}{2} \| \partial_t \zeta^n \|^2 + C \| \theta^n \|^2 . \quad (78) \]

Using (78) and (74), we get

\[\left( \| \partial_t \zeta^n \|^2 + a_0 \| \partial_t \theta^n \|^2 \right) \]
\[\leq \Delta t \sum_{n=0}^{J} \left( \| \partial_t \eta^n \|^2 + \| R_k^n \|^2 + \| R_k^2 \|^2 \right) \]
\[\leq C \left( \| \eta_i \|_{L_\infty(t_0,T_\eta)} + (\Delta t)^2 \left( \| u_{tt} \|_{L_\infty(L_\eta)} + \| \lambda_{tt} \|_{L_\infty(L_\eta)} \right) \right) \]
\[+ C \int_{t_0}^{T_\eta} \| \eta_i(s) \|^2 ds \]
\[+ C(\Delta t)^2 \int_{t_0}^{T_\eta} \left( \| u_{tt} \|^2 + \| \lambda_{tt} \|^2 \right) ds . \quad (79) \]

Taking $z_h = \xi^n$ in (68)(c), we get

\[\| \xi^n \|^2 = -\left( a^n \theta^n + b^n \theta^n, \xi^n \right) \]
\[\leq C \left( \| \eta_i \|^2 + \| \partial_t \theta^n \|^2 \right) + \frac{1}{2} \| \xi^n \|^2 . \quad (80) \]

Substitute (79) into (80) to get

\[\| \xi^n \|^2 \leq C \left( \| \eta_i \|_{L_\infty(t_0,T_\eta)} + (\Delta t)^2 \left( \| u_{tt} \|_{L_\infty(L_\eta)} + \| \lambda_{tt} \|_{L_\infty(L_\eta)} \right) \right) \]
\[+ C \int_{t_0}^{T_\eta} \| \eta_i(s) \|^2 ds \]
\[+ C(\Delta t)^2 \int_{t_0}^{T_\eta} \left( \| u_{tt} \|^2 + \| \lambda_{tt} \|^2 \right) ds . \quad (81) \]
5. Numerical Example

In order to illustrate the efficiency of the new expanded mixed element method, we consider the following initial-boundary value problem of 2D Sobolev equation with the convection term:

\begin{equation}
\begin{aligned}
    u_t + c(x) \cdot \nabla u - \nabla \cdot (a(x,t) \nabla u + b(x,t) \nabla u_t) &= f(x,t), \\
    u(x,t) &= 0, \quad (x,t) \in \Omega \times J,
\end{aligned}
\end{equation}

where \( \Omega = [0,1] \times [0,1] \), \( J = [0,1] \), \( a(x,t) = 1 + 2x_1^2 + x_2^2 \), \( b(x,t) = 1 + x_1^2 + 2x_2^2 \), and \( c(x) = (5,5)^T \), and \( f(x,t) \) is chosen so that the exact solution for the scalar unknown function is

\begin{equation}
    u(x,t) = e^{-t} x_1 (x_1 - 1) x_2 (x_2 - 1),
\end{equation}

the corresponding exact gradient function is

\begin{equation}
\begin{aligned}
    \lambda &= \nabla u = \left( e^{-t} (2x_1 - 1) x_2 (x_2 - 1), \\
    e^{-t} (2x_2 - 1) x_1 (x_1 - 1) \right),
\end{aligned}
\end{equation}

and its exact flux function is

\begin{equation}
\begin{aligned}
    \sigma &= -(a \nabla u + b \nabla u_t) \\
    &= \left( e^{-t} (x_2^2 - x_1^2) (2x_1 - 1) x_2 (x_2 - 1), \\
    e^{-t} (x_2^2 - x_1^2) (2x_2 - 1) x_1 (x_1 - 1) \right).
\end{aligned}
\end{equation}

We divide the domain \( \Omega \) into the triangulations of mesh size \( h \) uniformly, consider the piecewise linear space \( V_h \) with index \( k = 1 \) for the scalar unknown function \( u \) and the piecewise constant space \( W_h \) with index \( k = 0 \) for the gradient \( \lambda \) and the flux \( \sigma \), use the backward Euler procedure with uniform time step length \( \Delta t = 1/M \), and obtain some convergence results for \( \| u - u_h \|_{L^2(\Omega)} \), \( \| u - u_h \|_{H^1(\Omega)} \), \( \| \lambda - \lambda_h \|_{L^2(\Omega)} \), and \( \| \sigma - \sigma_h \|_{L^2(\Omega)} \) with \( h = 2\sqrt{\Delta t} = \sqrt{2}/8, \sqrt{2}/16, \sqrt{2}/32 \) in Table 1. At the same time, we show the exact solutions \( u, \lambda, \) and \( \sigma \) in Figures 1, 3, and 5, respectively, and the corresponding numerical solutions \( u_h, \lambda_h, \) and \( \sigma_h \) in Figures 2, 4, and 6, respectively, with \( t = 1 \), and \( h = 2\sqrt{2}\Delta t = \sqrt{2}/16 \).

It is easy to see that we obtained the optimal error estimates for \( u \) in \( L^2 \)-norm, \( H^1 \)-norm, and the error estimates for \( \lambda \) and \( \sigma \) in \( (L^2)^2 \)-norm, which confirm the theoretical results in this paper, in Table 1. The numerical results in Table 1 and Figures 1–6 show that new expanded mixed...
Figure 3: Surface for exact solution $\hat{\lambda} = (\lambda_1, \lambda_2)$.

Figure 4: Surface for numerical solution $\hat{\lambda}_h = (\lambda_{1h}, \lambda_{2h})$. 
Figure 5: Surface for exact solution $\sigma = (\sigma_1, \sigma_2)$.

Figure 6: Surface for numerical solution $\sigma_h = (\sigma_{1h}, \sigma_{2h})$. 
6. Concluding Remarks

In this paper, a new expanded mixed finite element method is proposed and studied for Sobolev equation with convection term. The proof for the existence and uniqueness of the solution for semidiscrete scheme, the new expanded mixed projection, and the proof of its uniqueness are given. The optimal a priori error estimates in $L^2$ for the scalar unknown $u$ and the a priori error estimates in $(L^2)^2$-norm for its gradient $\lambda$ and its flux $\sigma$ are proved. Especially, the optimal a priori error estimates in $H^1$-norm for the scalar unknown $u$ are derived. Finally, some numerical results are provided to confirm our theoretical analysis.

In the near future, the new expanded mixed method will be applied to other evolution equations such as evolution integrodifferential equations, hyperbolic wave equations, and nonlinear evolution equations. And the new characteristic expanded finite element method for Sobolev equation will be studied. The new expanded characteristic-mixed weak formulation is to find $\{u, \lambda, \sigma\} : [0, T] \mapsto H_0^1 \times (L^2(\Omega))^2 \times (L^2(\Omega))^2$ such that

\begin{align}
& (a) \left( \frac{\partial u}{\partial \tau}, v \right) - (\sigma, \nabla v) = (f, v), \quad \forall v \in H_0^1, \\
& (b) (\lambda, w) - (\nabla u, w) = 0, \quad \forall w \in (L^2(\Omega))^2, \\
& (c) (\sigma, z) + (a \lambda + b \lambda_1, z) = 0, \quad \forall z \in (L^2(\Omega))^2,
\end{align}

where

\[ \psi(x, t) = \left( 1 + |c(x)|^2 \right)^{1/2}, \]

\[ \frac{\partial}{\partial \tau} \psi(x, t) = \frac{1}{\psi(x, t)} \frac{\partial}{\partial t} + \frac{c}{\psi(x, t)} \cdot \nabla. \]

In another article, we will give the error estimates for the new characteristic expanded mixed finite element method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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