Inflationary solutions with a five dimensional complex scalar field

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Abstract

We discuss inflationary solutions of the coupled Einstein-Klein-Gordon equations for a complex field in a five dimensional spacetime with a compact $x^5$-dimension. As a new feature, the scalar field contains a dependence on the extra dimension of the form $\exp(imx^5)$, corresponding to Kaluza-Klein excited modes. In a four dimensional picture, a nonzero $m$ implies the presence of a new term in the scalar field potential. An interesting feature of these solutions is the possible existence of several periods of oscillation of the scalar field around the equilibrium value at the minimum of the potential. These oscillations lead to cosmological periods of accelerated expansion of the universe.

1 Introduction

The inflationary paradigm is currently seen as a solution to the problems arising in standard hot big-bang cosmology \[1, 2\], such as flatness, horizon and structure formation. A necessary condition for inflation to occur is that the energy density of the early stage of the universe be dominated by the vacuum energy of a quasi-homogeneous scalar field, usually called the "inflaton".

After the original inflationary scenario was proposed, various models have been investigated to obtain a natural inflation. A possibility is to consider higher dimensional gravity theories of the Kaluza-Klein type coupled to a scalar field (see e.g. \[3-5\]). After reduction to four dimensions, theories originally formulated in $4 + D$ dimensions take a Jordan-Brans-Dicke form, and generically contain also scalar fields of geometrical origin, describing the size of extra dimensions. The usual Einstein gravity can be obtained using a suitable conformal transformation.

Most of the studies on this subject assume that the inflaton field represents the symmetries of the extra- $D$-dimensional compact manifold. However, the situation may be more complicated, as we point out in this letter. Considering for simplicity only one extra-dimension, we suppose that the complex inflaton field has a dependence on the $x^5$-th dimension of the form $\exp(imx^5)$. The compactness of the extra-coordinate forces the constant $m$ to take certain discrete values. However, all the observable quantities are $x^5$-independent, since the action of the generator of $x^5$ translation is equivalent to the action of a rigid phase transformation.
Also, it is well known that the presence of scalar fields in cosmology introduces the problem of the appropriate potential, a problem which has been studied extensively in the literature. We find that a $x^5$—dependence of the inflaton field will induce a new scalar potential term in the four dimensional Lagrangian, which may influence behavior of the solutions. By numerically solving the field equations we find that, for a double-well inflaton potential, there is a succession of short bursts of inflation, after a generic initial period of inflation. A nonzero $m$ generally enhances this effect.

As a new feature, we speculate about the possibility that $m$ changes its value for solutions with scalar field nodes. This leads to new qualitative features, the most interesting being the generic existence of multiple periods of accelerated expansion of the universe, for a transition to a higher $m$.

2 General framework

2.1 Action and equations of motion

We consider the following action principle in $4 + 1$ dimensions

$$S_5 = \int d^5 x \sqrt{-g(5)} \left( \frac{1}{2k^2} R_5 + L_m \right),$$

(1)

where $g_{\mu\nu}$ and $\kappa^2$ are the five-dimensional metric and gravitational constant. $L_m$ is the Lagrangian for a complex scalar field $\Psi$, with a potential $V$ depending on $|\Psi|^2 = \Psi^* \Psi$ only

$$L_m = -g_{\mu\nu} \partial_\mu \Psi^* \partial_\nu \Psi - V(\Psi).$$

(2)

Variation of the action with respect to $g^{\mu\nu}$ yields the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2 T_{\mu\nu},$$

(3)

where the energy momentum tensor is given by

$$T_{\mu\nu} = \Psi^*_{,\mu} \Psi_{,\nu} + \Psi^*_{,\nu} \Psi_{,\mu} - g_{\mu\nu} (\Psi_{,\rho} \Psi^{,\rho} + V(\Psi)).$$

(4)

The scalar field equation is

$$\left( \nabla^2 \frac{dV}{d|\Psi|^2} \right) \Psi = 0.$$ 

(5)

2.2 The ansatz

We take a five-dimensional metric ansatz with no dependence on the extra-dimension $x^5$, on the form

$$ds_5^2 = -dt^2 + a^2(t) dx^i dx^i + \Phi^2(t)(dx^5)^2,$$

(6)
where $a(t)$ is the scale factor of the three-dimensional space, $\Phi(t)$ is the scale factor of the extra-dimension and $x^i$ denotes the three space directions. Here we suppose a compact fifth dimension, with $0 \leq \chi < L$. The corresponding nonvanishing components of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ read

$$
G_i^i = -\frac{\dot{a}^2}{a^2} + 2a\ddot{a}, \quad G_t^t = -\frac{3}{a^2} (\Phi \ddot{a} + a\dot{a} \dot{\Phi}), \quad G_5^5 = -\frac{3}{a^2} (\dot{a}^2 + a\ddot{a}),
$$

where a dot denotes derivative with respect to time.

The usual scalar field ansatz considered in the literature to obtain solutions compatible with the symmetries of the line element (6) is a homogeneous one, with $\Psi = \chi(t)$. However, a dependence of the $\Psi$-field on the extra-dimension may be introduced by taking

$$
\Psi = \frac{1}{\sqrt{2}} \chi(t) e^{imx^5},
$$

where $\chi(t)$ is a real function, this matter ansatz being still compatible with the symmetries of the metric (6). Since scalar fields must be single-valued functions with respect to $x^5$, we find $\Psi(t, x^5) = \Psi(t, x^5 + L)$. From this periodicity, values of $m$ must be of the form $m = 2\pi n/L$, where $n$ is an integer, representing the winding number with respect to the extra-coordinate $x^5$. While $x^5/L$ covers the trigonometric circle once, the field winds $n$ times around.

It appears natural to consider a generalization of the simple matter ansatz (8) consisting in a superposition of $N$ individual modes

$$
\Psi = \frac{1}{\sqrt{2}} \sum_{k=0}^{N-1} \chi_k(t) e^{im_kx^5},
$$

with $\chi_k(t)$ real functions and $m_k = 2\pi k/L$. However, one can prove that this general ansatz is not compatible with the assumption that the five-dimensional line element (6) does not depend on $x^5$. A straightforward computation gives

$$
\Psi \Psi^* = \frac{1}{2} \left( \sum_{k=1}^{N-1} \chi_k^2(t) + \sum_{k,j=0}^{N-1} k \neq j \cos((m_k - m_j)x^5)\chi_k(t)\chi_j(t) \right),
$$

$$
\Psi_t \Psi_t^* = \frac{1}{2} \left( \sum_{k=1}^{N-1} \dot{\chi}_k^2(t) + \sum_{k,j=0}^{N-1} k \neq j \cos((m_k - m_j)x^5)\dot{\chi}_k(t)\dot{\chi}_j(t) \right),
$$

$$
\Psi_{x5} \Psi_{x5}^* = \frac{1}{2} \left( \sum_{k=1}^{N-1} m_k^2 \chi_k^2(t) + \sum_{k,j=0}^{N-1} k \neq j \cos((m_k - m_j)x^5)m_k m_j \chi_k(t)\chi_j(t) \right),
$$

implying from (4) a dependence of the energy momentum tensor on the extra-coordinate, which is not compatible with the Einstein tensor (7).
Therefore, rather surprising, only one single mode can be excited within the metric ansatz \( \Theta \). Thus, for the rest of this paper, we consider a scalar field \( \Psi = \frac{1}{\sqrt{2}} \chi(t) \exp(i m x^5) \), with \( m = 2 \pi n / L \) and an arbitrary integer \( n \).

With these conventions, the nonvanishing components of the energy-momentum tensor are

\[
T_i^i = p_4 = \frac{1}{2} \dot{\chi}^2 - m^2 \frac{\chi^2}{2 \Phi^2} - V(\chi),
\]

\[
-T_i^\tau = \rho = \frac{1}{2} \dot{\chi}^2 + m^2 \frac{\chi^2}{2 \Phi^2} + V(\chi),
\]

\[
T_5^5 = p_5 = \frac{1}{2} \dot{\chi}^2 + m^2 \frac{\chi^2}{2 \Phi^2} - V(\chi).
\]

One can see that a nontrivial dependence on the extra dimension increases the energy density \( \rho \) and pressure component associated with the extra-dimension \( p_5 \), while decreasing the four dimensional pressure \( p_4 = T_i^i \).

The dynamical equations for \( a, \Phi \) and \( \chi \) (with \( H = a' / a \)) are

\[
\ddot{\Phi} + 3H \dot{\Phi} = \frac{\kappa^2}{3} \left( 7m^2 \frac{\chi^2}{\Phi^2} + 4V(\chi) \right),
\]

\[
H^2 + H \frac{\dot{\Phi}}{\Phi} = \frac{\kappa^2}{3} \left( \dot{\chi}^2 + m^2 \frac{\chi^2}{\Phi^2} + 2V(\chi) \right),
\]

\[
\dddot{\chi} + (3H + \dot{\Phi}) \ddot{\chi} + m^2 \frac{\chi}{\Phi^2} + \frac{\partial V(\chi)}{\partial \chi} = 0.
\]

By using suitable combinations of the Einstein equations, we find the following useful relations

\[
\frac{\ddot{a}}{a} = H \frac{\dot{\Phi}}{\Phi} - \frac{\kappa^2}{3} \left( \chi^2 + m^2 \frac{\chi^2}{\Phi^2} \right), \quad \ddot{a} = -\frac{\kappa^2}{3} 2 a^2 p_5,
\]

\[
\frac{d^2}{dt^2} (a^3 \Phi) = \frac{\kappa^2}{3} a^3 (3m^2 \frac{\chi^2}{\Phi^2} + 8V(\chi)), \quad \frac{d}{dt} (a \Phi) = \frac{\kappa^2}{3} a \Phi \rho, \quad \frac{d}{dt} (\frac{a}{\Phi}) = \frac{\kappa^2}{3} a \Phi \left( \dot{\chi}^2 + m^2 \frac{\chi^2}{\Phi^2} \right),
\]

which can be used to predict general features of the solutions. For example, one can prove that the scale factor \( a \) of the three-dimensional space is always increasing, \( \dot{a} > 0 \). Also, for periods of accelerated expansion (\( \ddot{a} > 0 \)), the extra-dimension radius \( \Phi(t) \) is a strictly increasing function \( \dddot{\Phi} > 0 \).

### 2.3 Four dimensional reduction

With this ansatz, it is possible to reduce \( \Theta \) to a four dimensional Lagrangian. The resulting Lagrangian mimics the Lagrangian of a Jordan-Brans-Dicke theory where there is an extra scalar field (related to the size of the extra-dimension), nonminimally coupled to the four dimensional Ricci scalar.
The resulting action after the dimensional reduction for the metric ansatz (6) is
\[ S_4 = \int d^4x \sqrt{-g_4} \left( \frac{1}{2\kappa^2} \Phi R_4 - \frac{\Phi}{2} \left( \chi_\mu \chi^{\mu} + m^2 \frac{\chi^2}{\Phi^2} + 2V(\chi) \right) \right). \] (14)

One can see that a nonzero winding number implies a direct coupling between extra-dimension radius and the scalar \( \chi \), while the resulting field equations are still given by (12).

The above system admits also an alternative picture in an "Einstein frame", with a minimal coupling between the extra-dimension radius and four dimensional gravity. These are complementary pictures and the solution in one frame can be directly translated to the results in the other frame by field and time coordinate redefinitions.

If we note \( \Phi = \exp (\alpha \phi) \) (with \( \alpha = \sqrt{2\kappa^2/3} \)) and consider a five-dimensional metric of the form (6)
\[ ds_5^2 = e^{-\alpha\phi} ds_4^2 + e^{2\alpha\phi} d\chi^2 \] (15)

the reduced four dimensional Lagrangian in the Einstein frame reads
\[ S_4 = \int d^4x \sqrt{-g_4} \left( \frac{R_4}{2\kappa^2} - \frac{1}{2} \partial_\mu \phi \partial_\mu \phi - \frac{1}{2} \partial_\mu \chi \partial_\mu \chi - U(\phi, \chi) \right), \] (16)

where
\[ U(\phi, \chi) = e^{-3\alpha\phi} m^2 \chi^2 + e^{-\alpha\phi} V(\chi). \] (17)

The four dimensional metric is taken to be:
\[ ds_4^2 = -d\tau^2 + \bar{a}^2(\tau)d\vec{x}d\vec{x} \] (18)

which implies the following relations between the five- and four-dimensional functions:
\[ \tau = \int dt \ \Phi(t), \quad \bar{a} = \frac{a(t)}{\Phi(t)}. \] (19)

The field equations in Einstein frame and a line element (18) read
\[ \ddot{H} = \frac{\kappa^2}{2} (\dot{\phi}^2 + \chi^2) - U, \]
\[ \chi'' + 3H \chi' = -e^{\alpha\phi} \frac{\partial V}{\partial \chi} - e^{-3\alpha\phi} m^2 \chi, \] (20)
\[ \phi'' + 3H \phi' = -e^{\alpha\phi} \frac{\partial V}{\partial \phi} = \alpha \left( e^{\alpha\phi} V + \frac{3}{2} e^{-3\alpha\phi} m^2 \chi^2 \right), \]

where a prime denotes derivative with respect to \( \tau \) and \( \ddot{H} = \ddot{a}/\bar{a} \). If we suppose that asymptotically \( \chi \to \chi_0 \) such that \( V(\chi_0) = 0 \), while \( \phi \) takes a finite value, the term \( e^{-3\alpha\phi} m^2 \chi^2 \) will correspond to an effective four-dimensional cosmological constant.
3 Inflationary solutions

3.1 Slow roll approximation

Inflationary solutions of the equations (12) have been discussed for \( m = 0 \) by some authors in the context of inflation from higher dimensional theories (see e.g. [3]-[5]).

It can be seen immediately that an exact solution exists for \( \chi_0 \) satisfying the condition

\[
\left. \left( m^2 \frac{\chi}{\Phi^2} + \frac{\partial V}{\partial \chi} \right) \right|_{\chi=\chi_0} = 0.
\]

(21)

Therefore, if \( V(\chi_0) \neq 0 \), the scale factor \( a(t) \) expands exponentially, similar to the radius of the extra-dimensions, \( a(t) \sim e^t \), with \( c^2 = \kappa^2 V(\chi_0)/6 \). Also, it can easily be proven from [13] that there are no solutions with a constant extra-dimension radius.

The crucial ingredient of nearly all known inflationary scenarios is a period of ”slow roll” evolution of the inflaton field. During this period \( \chi \) changes very slowly, so that its kinetic energy \( \dot{\chi}^2/2 \) remains always much smaller than its potential energy.

For the situation discussed in this paper, the slow-roll approximation reads

\[
\ddot{\chi} \ll 3H \dot{\chi}, \quad \ddot{\Phi} \ll 3H \dot{\Phi},
\]

(22)

while the Kaluza-Klein modes are required to satisfy

\[
m^2 \frac{\chi}{\Phi^2} \ll 3H \dot{\chi}, \quad m^2 \frac{\chi^2}{\Phi^2} \ll V(\chi),
\]

(23)

Note that these conditions imply a number of constraints on the parameters (for example from [23], \( m \) cannot take arbitrary large values).

With these assumptions, one finds the following approximate solution

\[
\Phi = a^{2/3} + \text{const.}, \quad H = \sqrt{\frac{\kappa^2}{3} V(\chi)}, \quad t = -\frac{1}{3\kappa^2} f(\chi),
\]

(24)

where

\[
f(\chi) = \int d\chi \frac{V(\chi)}{V'(\chi)}.
\]

(25)

The end of inflation is marked by the condition

\[
\frac{1}{2} \dot{\chi}^2 + m^2 \frac{\chi^2}{2\Phi^2} \simeq V(\chi).
\]

(26)
3.2 Numerical solutions

However, the slow roll conditions are sufficient, but not necessary to maintain inflation. There is the possibility that inflation continues after the slow roll ends, during a period of fast oscillations of the inflaton field (see e.g. [7]). However, this implies a number of constrains on the inflaton potential. Here we investigate the possibility that the supplementary term in the Lagrangian induced by the inflaton field dependence on the extra dimension will lead to oscillations of the scalar fields, that translate to periods of accelerated expansion of the universe.

No analytic arguments are available in this case and the equations of motion (12) should be solved numerically. One may use the translation symmetry of the field equations to set an arbitrary value for the initial time coordinate, which is taken $t_0 = 0$. To obtain numerical solutions, one needs also to fix six initial conditions ($a(0), \dot{a}(0), \Phi(0), \dot{\Phi}(0), \chi(0), \dot{\chi}(0)$). One of the Einstein equations is a constraint equation that allows one to express one of these constants in terms of the others.

The scalar field potential considered in this letter has the usual double-well form

$$V(\chi) = \lambda^2 (\chi^2 - v^2)^2.$$  \hspace{1cm} (27)

Therefore, one should also specify the values of the constants appearing in the (27) as input parameters, as well as the value of $m$ (these are usually consider of order unity, while we take $L = 2\pi$).

Following the usual approach and using a standard ordinary differential equation solver, we evaluate the initial conditions at $t_0 = 10^{-14}$ and integrate towards $t \to \infty$, up to a maximal value $t_{\text{max}}$.

3.2.1 Solutions with fixed winding number

The case $m = 0$ has been the subject of many studies. A detailed study of the solutions of the Einstein equations with a real scalar field has been done in particular by Belinski et al. [8] (see also [9]).

Here we do not aim at finding precise quantitative features of the model presented in Section 2. Instead we are looking for qualitatively new features introduced by a nonzero value of $m$. Thus the field equations (12) have been solved for several values of $(m, \lambda, v)$ and a large set of initial conditions, looking for generic properties (for example, the solutions plotted here have $\lambda = 1$).

For all values of the parameters, the scale factor $a(t)$ is a strictly increasing function (as proven analytically). After an initial period of faster expansion, the scale function $a(t)$ becomes proportional to the function $\sqrt{t}$. We could not find configurations with a constant value of $a(t)$ for large enough values of $t$.

The scalar field behavior is also generic. Independent on the initial conditions, the scalar field always approaches asymptotically the vacuum expectation value $v$, after a number of oscillations around $v$ (see Figure 1a).

The behavior of the extra-dimension radius $\Phi(t)$ is somewhat special. Here, for some set of initial conditions we notice the existence of a local maximum of the radius of the
Figure 1. The scalar field $\chi$ and the second derivative of the scale factor $a$ are represented as a function of $t$ for typical solutions.

extra-dimension, and a more complicated behavior of this function for small values of $t$ (the function $\chi(t)$ should be a strictly increasing function in this case). However, for large enough values of $t$, $\Phi(t)$ is a strictly increasing function, $\Phi(t) \sim t$. We could not find solutions where the extra-dimension radius approaches asymptotically a constant value.

The second derivative of the function $a(t)$ (which is an indication for the existence of inflation) has an interesting behavior for $m \neq 0$. For most of the considered configurations, the oscillations of the scalar translate to a number of oscillations around zero of the function
Figure 2. The second derivative of the metric function $a$, the scalar field $\chi$ and the Hubble parameter $H = a^{-1} \frac{da}{dt}$ are represented as a function of $t$ for a typical solution with winding number transition from $m_i = 0$ to $m_f = 3$.

The behavior of $\ddot{a}$ (see Figure 1b). This implies therefore the existence of several periods of inflation separated by periods of noninflationary expansion. This "multiple inflation" behavior is a consequence of the form of the inflaton field potential. Although this effect is present also for $m = 0$, its details depend on the value of $m$. For example, the length and magnitude of $\ddot{a}$ oscillations generally strongly increase with $m$. This effect is maximized for initial value of the scalar field near the false vacuum $\chi = 0$. Also, $\ddot{a}$ vanishes always for large values of $t$, while the expansion rate $H$ is a decreasing function.

3.2.2 Solutions with winding number transitions

In the previous section we took $m$ to be a constant. If one takes $\lambda$ to be very large this is a very natural assumption, since in the limit $\lambda \to \infty$, $m$ becomes a topologically conserved charge. However, we remark that for the physical situation that we consider here with finite $\lambda$, there is no reason why the value of $m$ should be a conserved quantity. There is no charge associated with it.

It appears interesting to take $m$ as a fluctuating variable and to look for possible physical effects. A change of $m$ for an arbitrary $t$ is unacceptable, since it generally leads to discontinuities of the metric functions, which are considered unphysical [10]. This can be avoided by considering solutions with nodes, where the scalar field $\chi$ crosses the axes for some values of the time coordinate $t = t_1, t_2, ...$. Note that this requires a choice of the parameters in the problem, since nodeless solutions are present as well.

We solve the field equations for a number of configurations, starting with a configuration
with winding number $m_i$ and changing the value of $m$ from $m_i$ to $m_f$ as the scalar field $\chi$ crosses the axes for some $t$. Therefore, this does not lead to discontinuities of the relevant functions $(a, \chi, \Phi)$.

The results we have found for $m_f < m_i$ are rather similar to the case of a fixed winding number. The behavior of solutions following the changing of the winding number is very similar to the solutions with a fixed $m = m_i$, with small quantitative differences only. New qualitative features are found for transitions to a higher winding number. We find that generically this implies the existence of a second period of inflation, whose length increases with $m_f$ (see Figure 2). We can understand intuitively this effect by remarking that a transition of the scalar field to a higher (lower) winding number corresponds to injecting (extracting) extra energy in the system, as can be seen from the field equations.

The typical inflaton field behavior is presented in Figure 2 as well as the evolution of the expansion rate $H$ (which is a decreasing function). After changing the winding number, the scalar field presents very small oscillations around zero, followed by a sudden transition towards $v$ at some $t > t_1$, approaching asymptotically the vacuum expectation value. The metric functions $a, \Phi$ are strictly increasing quantities, apparently not noticing the changing of $m$.

These results are generally not sensitive to the initial conditions, but are affected by the scalar field potential parameters $\lambda, v$. However, we found that a similar behavior appears generically for a large enough difference $m_f - m_i$.

One can also imagine more complicated scenarios, with several $m$-transitions, leading to a succession of bursts of inflation.

4 Conclusions

The purpose of this work was to consider the possibility that the bulk inflaton field in a higher dimensional theory possesses a dependence on the extra-dimensions. In the simplest five-dimensional case, this introduces a new parameter in the theory, corresponding to a winding number $m$ with respect the $x^5$-direction.

After reduction to four dimensions, the dependence on the extra-dimension is apparent as a new term in the inflaton potential. We have presented numerical arguments that, for a nonzero winding number, the solutions of the Einstein-Klein-Gordon equations present a number of new qualitative features. The possibility that $m$ changes the value for solutions with scalar field nodes has been also considered.

Therefore, the inclusion of a inflaton dependence on the extra-dimensions leads to a rich model with new curious features which deserve further investigation. For example, it would be interesting to study the quantitative details as well as density fluctuations within this model. The scenario can easily be generalized for a number $D > 1$ of extra-dimensions with a torus compactification.
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