Comparison of two numerical approaches for natural convection in cavities with energy sources

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Abstract. A numerical investigation of two-dimensional natural convection in an enclosure with a local square energy source has been carried out. The presented problem has been solved by two numerical methods – first of all by the finite difference method using the non-primitive variables “stream function – vorticity” and by the lattice Boltzmann method. As a result the distributions of temperature fields and stream functions and also velocity fields have been obtained. A comparison of the obtained data and the evaluation of the effectiveness of methods for solving of this class of problems have been performed.

1. Introduction
In recent times, numerical investigations have begun to play a huge role in the most diverse branches of science and technology. At this point in time, the data obtained both experimentally and as a result of a numerical experiment are especially valuable in science. In addition, numerical modeling is actively used in problems of chemistry, biology, medicine and many other branches of science [1-4]. As a result of such intensive development of numerical modeling, various commercial computational codes began to be actively developed for solving technical problems. Thus, due to the increasing use of numerical modeling, it is necessary to develop and improve the existing computational methods.

Recently, there has been a significant amount of interest in Lattice Boltzmann Method, this method gained great popularity because of the ease of implementation and the multi-zone applications. Many different scientists and researchers from various countries actively use this approach for numerical research in different branches of science and technology [1-9]. It is necessary to note a set of advantages of this method in comparison with known and more thoroughly studied methods. Such advantages of LBM are an ability to model the complex geometries, multiphase flows, change phase materials, flow in porous media, low-density flow, and a number of other advantages [7]. For another thing, for time-consuming task calculations it is necessary to use the parallelization algorithm for various processes (both on the central processor and on the graphic card processors) - in the lattice Boltzmann method, such processes will be more natural, since the calculations for the two main steps of the method — collisions and particle transport are local [8].

In this paper we conducted numerical investigation of natural convection in a square cavity with local energy source by the finite difference method and lattice Boltzmann method. The benchmark assessment of the results gained by considered numerical methods has been performed.
2. Physical and mathematical model

In this work a numerical simulation of natural convective heat and mass transfer in a square cavity, schematically represented in Figure 1, has been performed. Inside the field of exploration, a heat source of square shape has been located. The side vertical walls are cooled and this temperature is constant and minimum inside the cavity. The top and bottom boundary walls are insulated. The cavity is filled with the incompressible Newtonian fluid satisfying the Boussinesq approximation. The thermophysical properties of the medium are considered to be constant.

![Figure 1. Considered domain of interest.](image)

The finite difference method (FDM) and the lattice Boltzmann method (LBM) as the numerical method for solving the problem have been chosen. The governing equations in the dimensionless non-primitive variables “stream function – vorticity” for FDM can be written as follows [10-11]:

\[
\frac{\partial \Omega}{\partial \tau} + \frac{\partial \Psi}{\partial Y} \frac{\partial \Omega}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \Omega}{\partial Y} = \frac{Pr}{Ra} \left( \frac{\partial^2 \Omega}{\partial X^2} + \frac{\partial^2 \Omega}{\partial Y^2} \right) + \frac{\partial \Theta}{\partial X} 
\]

\[
\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = -\Omega 
\]

\[
\frac{\partial \Theta}{\partial \tau} + \frac{\partial \Psi}{\partial Y} \frac{\partial \Theta}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \Theta}{\partial Y} = \frac{1}{Pr \cdot Ra} \left( \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} \right) 
\]

The non-dimensionalization of these governing equations with the help of the following parameters has been carried out:

\[
X = x/L, \quad Y = y/L, \quad \tau = t \sqrt{g \beta \Delta T L}, \quad U = u \sqrt{g \beta \Delta T L}, \quad V = v \sqrt{g \beta \Delta T L},
\]

\[
\Theta = (T - T_c)/\Delta T, \quad \Psi = \sqrt{g \beta \Delta T L} \psi, \quad \Omega = \omega \sqrt{L/g \beta \Delta T}, \quad \Delta T = T_h - T_c
\]

Here \(X, Y\) are the dimensionless Cartesian coordinates; \(\tau\) is the dimensionless time; \(\Theta\) is the dimensionless temperature; \(\Psi\) is the dimensionless stream function; \(U, V\) are the dimensionless component of velocity vector in the projection of axis coordinates \(X, Y\) accordingly; \(\Omega\) is the dimensionless vorticity vector, \(Ra = g \beta \Delta T L \sqrt{\nu \alpha_f}\) is the Rayleigh number (\(\beta\) is the thermal
expansion coefficient; $\nu$ is kinematic viscosity, $g$ is acceleration of gravity, $L$ is length of the cavity, $\Delta T$ is temperature difference, $\alpha_f$ is thermal diffusivity of fluid). $Pr = \frac{\nu}{\alpha_f}$ is the Prandtl number.

**Initial conditions**

$$\Psi(X, Y, 0) = \Omega(X, Y, 0) = \Theta(X, Y, 0) = 0;$$

Inside the heat source: $\Theta = 1$;

**Boundary conditions**

On vertical walls $X = 0, X = 1$:

$$\Psi = 0, \frac{\partial \Psi}{\partial X} = 0, \Omega = -\frac{\partial^2 \Psi}{\partial X^2}, \Theta = 0$$

On horizontal walls $Y = 0, Y = 1$:

$$\Psi = 0, \frac{\partial \Psi}{\partial Y} = 0, \Omega = -\frac{\partial^2 \Psi}{\partial Y^2}, \frac{\partial \Theta}{\partial Y} = 0$$

On heat source surface:

$$\Psi = 0, \Theta = 1$$

The governing equation for the LBM can be written as follows:

$$f_i(x + c_i \Delta t + \Delta t) - f_i(x, t) = -\frac{\Delta t}{\tau}[f_i(x, t) - f_i^{eq}(x, t)] + \Delta t F_i$$

Here $f_i^{eq}(x, t) = w_i \rho \left( 1 + \frac{u \cdot c_i}{c_s^2} + \frac{(u \cdot c_i)^2}{2c_s^4} - \frac{u \cdot u}{2c_s^2} \right)$ is the equilibrium density distribution function, $w_i$ are the weight functions $w_0 = \frac{4}{9}, w_{2.4} = \frac{1}{9}, w_{2.9} = \frac{1}{36}, c_s^2 = \frac{\Delta X^2}{3 \Delta t^2}$ is the sound speed, $F_i = 3w_i \rho g \beta (T - T_{avg})c_i$ is the force of gravity.

During calculation of the non-isothermal problems, it is necessary to add the distribution function for energy $g_i$:

$$g_i(r + c_i \Delta t, t + \Delta t) = g_i(r, t) - \frac{\Delta t}{\tau^e}(g_i - g_i^{eq})$$

The distribution function of equilibrium for energy is

$$g_i^{eq} = w_i T \left( 1 + 3 \frac{c_i u}{c_s^2} \right)$$

The macroscopic functions are determined through distribution functions as follows:

$$\rho = \sum_{i=0}^{8} f_i$$

$$\rho u = \sum_{i=0}^{8} f_i c_i$$

$$T = \sum_{i=0}^{8} g_i$$

Using (9) we can write the relation of relaxation time $\tau_f$ with kinematic viscosity $\nu$ for hydrodynamic processes:

$$\nu = c_s^2 \left( \frac{\tau_f}{2} - \frac{\Delta t}{2} \right)$$
and the relation of relaxation time $\tau_g$ with thermal diffusivity coefficient:

$$\alpha = c_s^2 \left( \frac{\tau_g}{2} - \frac{\Delta t}{2} \right)$$  \hspace{1cm} (11)

The lattice speed can be described using the following expression:

$$c_i = \begin{cases} (0, 0) & \text{if } i = 0 \\ (\cos[(i-1)\pi/2], \sin[(i-1)\pi/2])c & \text{if } i = 1, \ldots, 4 \\ \sqrt{2}(\cos[(i-5)\pi/2 + \pi/4], \sin[(i-5)\pi/2 + \pi/4])c & \text{if } i = 5, \ldots, 8 \end{cases}$$  \hspace{1cm} (12)

![Figure 2](image-url)

**Figure 2.** The comparison of results obtained by LBM (dashed line) and FDM (solid line). a – streamlines, b – isotherms, c – velocity field obtained by LBM, d – velocity field obtained by FDM, e – comparison of the average temperatures inside the cavity.

The developed numerical model for the classical de Vahl Davis natural convection problem within the differentially-heated chamber has been tested [12]. As compared parameters we chose the distribution of stream function and temperature isolines, as well as the velocity field and the average temperature in the cavity under consideration (see Figure 2 a-e). As it can be seen in the figure, a fairly good agreement of the results has been obtained.

3. Results and discussions

The calculations were conducted at $Pr = 0.7$, $Ra = 10^5$. In consequence of performed investigation the stream function isolines have been obtained. Two symmetric convective cells are formed above the local energy source, due to the fact that the air flow above the source moves up as a result of heating from the energy source, then the air flow is lowered near the cooling walls. In this way, 2 different direction vortexes are formed inside the cavity. Figure 3 b illustrates temperature fields. It can be observed that the heat plume is formed above the energy source. Figures 3 c and d depict the velocity fields obtained by LBM and FDM, accordingly. For all represented results there is a good match for the considered methods. Figure 3 e gives a good view of full agreement of average temperature in the cavity at the time of the study. The difference in the gained data for the average temperature at the steady-state moment is 0.014%. With almost absolute agreement of the results obtained by two fundamentally different methods, it is necessary to notice an important parameter of the predominance
of LBM over FDM: it is the speed of calculation, the advantage in calculation time reaches more than 2 times.

Figure 3. The comparison of results obtained by LBM (dashed line) and FDM (solid line).

Conclusions
As a result of comparison between the obtained fields for LBM and FDM, and also as a result of evaluating the calculations for the average temperature, we can conclude about the effectiveness of using LBM to solve the problems of natural convection in cavities in the presence or absence of local heat sources. In the work presented here a numerical study of natural convection in a cavity with a local energy source has been carried out. In this way the stream function fields, the temperature fields and the velocity fields have been obtained. As a result of comparing the obtained fields, and also as a result of evaluating the calculations for the average temperature, we can emphasize the effectiveness of using LBM to solve problems of natural convection in cavities in the presence or absence of local heat sources.

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