Martingale inequalities for spline sequences

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Received: 6 February 2019 / Accepted: 21 March 2019 / Published online: 30 March 2019
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Abstract
We show that D. Lépingle’s $L_1(\ell_2)$-inequality
\[ \left\| \left( \sum_n \mathbb{E}[f_n|\mathcal{F}_{n-1}]^2 \right)^{1/2} \right\|_1 \leq 2 \cdot \left\| \left( \sum_n f_n^2 \right)^{1/2} \right\|_1, \quad f_n \in \mathcal{F}_n, \]
extends to the case where we substitute the conditional expectation operators with orthogonal projection operators onto spline spaces and where we can allow that $f_n$ is contained in a suitable spline space $\mathcal{S}(\mathcal{F}_n)$. This is done provided the filtration $(\mathcal{F}_n)$ satisfies a certain regularity condition depending on the degree of smoothness of the functions contained in $\mathcal{S}(\mathcal{F}_n)$. As a by-product, we also obtain a spline version of $H_1$-BMO duality under this assumption.

Keywords Martingale inequalities · Polynomial spline spaces · Orthogonal projection operators

Mathematics Subject Classification 65D07 · 60G42 · 42C10

1 Introduction
This article is part of a series of papers that extend martingale results to polynomial spline sequences of arbitrary order (see e.g. [11,14,16–19,22]). In order to explain those martingale type results, we have to introduce a little bit of terminology: Let $k$ be a positive integer, $(\mathcal{F}_n)$ an increasing sequence of $\sigma$-algebras of sets in $[0, 1]$ where each $\mathcal{F}_n$ is generated by a finite partition of $[0, 1]$ into intervals of positive length. Moreover, define the spline space
\[ \mathcal{S}_k(\mathcal{F}_n) = \{ f \in C^{k-2}[0, 1] : f \text{ is a polynomial of order } k \text{ on each atom of } \mathcal{F}_n \} \]
and let \( P_n^{(k)} \) be the orthogonal projection operator onto \( \mathcal{S}_k(\mathcal{F}_n) \) with respect to the \( L_2 \) inner product on \([0, 1]\) with the Lebesgue measure \( | \cdot | \). The space \( \mathcal{S}_1(\mathcal{F}_n) \) consists of piecewise constant functions and \( P_n^{(1)} \) is the conditional expectation operator with respect to the \( \sigma \)-algebra \( \mathcal{F}_n \). Similarly to the definition of martingales, we introduce the following notion: let \( (f_n)_{n \geq 0} \) be a sequence of integrable functions. We call this sequence a \( k \)-martingale spline sequence (adapted to \( (\mathcal{F}_n) \)) if, for all \( n, \\
P_n^{(k)} f_{n+1} = f_n.
\]

For basic facts about martingales and conditional expectations, we refer to [15].

Classical martingale theorems such as Doob’s inequality or the martingale convergence theorem in fact carry over to \( k \)-martingale spline sequences corresponding to arbitrary filtrations \( (\mathcal{F}_n) \) of the above type, just by replacing conditional expectation operators by the projection operators \( P_n^{(k)} \). Indeed, we have

(i) (Shadrin’s theorem) there exists a constant \( C_k \) depending only on \( k \) such that

\[
\sup_n \| P_n^{(k)} : L_1 \to L_1 \| \leq C_k,
\]

(ii) (Doob’s weak type inequality for splines)

there exists a constant \( C_k \) depending only on \( k \) such that for any \( k \)-martingale spline sequence \( (f_n) \) and any \( \lambda > 0 \),

\[
|\{ \sup_n |f_n| > \lambda \}| \leq C_k \sup_n \| f_n \|_1 \frac{\lambda}{\lambda},
\]

(iii) (Doob’s \( L_p \) inequality for splines)

for all \( p \in (1, \infty] \) there exists a constant \( C_{p,k} \) depending only on \( p \) and \( k \) such that for all \( k \)-martingale spline sequences \( (f_n) \),

\[
\| \sup_n |f_n| \|_p \leq C_{p,k} \sup_n \| f_n \|_p,
\]

(iv) (Spline convergence theorem)

if \( (f_n) \) is an \( L_1 \)-bounded \( k \)-martingale spline sequence, then \( (f_n) \) converges almost surely to some \( L_1 \)-function,

(v) (Spline convergence theorem, \( L_p \)-version)

for \( 1 < p < \infty \), if \( (f_n) \) is an \( L_p \)-bounded \( k \)-martingale spline sequence, then \( (f_n) \) converges almost surely and in \( L_p \).

Property (i) was proved by Shadrin in the groundbreaking paper [22]. We also refer to the paper [25] by von Golitschek, who gives a substantially shorter proof of (i). Properties (ii) and (iii) are proved in [19] and properties (iv) and (v) in [14], but see also [18], where it is shown that, in analogy to the martingale case, the validity of (iv) and (v) for all \( k \)-martingale spline sequences with values in a Banach space \( X \) characterize the Radon–Nikodým property of \( X \) (for background information on that material, we refer to the monographs [6,20]).
Here, we continue this line of transferring martingale results to $k$-martingale spline sequences and extend Lépingle’s $L_1(\ell_2)$-inequality [12], which reads

$$\left\| \left( \sum_n \mathbb{E}[f_n | \mathcal{F}_{n-1}]^2 \right)^{1/2} \right\|_1 \leq 2 \cdot \left\| \left( \sum_n f_n^2 \right)^{1/2} \right\|_1,$$

(1.1)

provided the sequence of (real-valued) random variables $f_n$ is adapted to the filtration $(\mathcal{F}_n)$, i.e. each $f_n$ is $\mathcal{F}_n$-measurable. Different proofs of (1.1) were given by Bourgain [3, Proposition 5], Delbaen and Schachermayer [4, Lemma 1] and Müller [13, Proposition 4.1]. The spline version of inequality (1.1) is contained in Theorem 4.1.

This inequality is an $L_1$ extension of the following result for $1 < p < \infty$, proved by Stein [24], that holds for arbitrary integrable functions $f_n$:

$$\left\| \left( \sum_n \mathbb{E}[f_n | \mathcal{F}_{n-1}]^2 \right)^{1/2} \right\|_p \leq a_p \left\| \left( \sum_n f_n^2 \right)^{1/2} \right\|_p,$$

(1.2)

for some constant $a_p$ depending only on $p$. This can be seen as a dual version of Doob’s inequality $\| \sup |\mathbb{E}[f | \mathcal{F}_\ell]|\|_p \leq c_p \| f \|_p$ for $p > 1$, see [1]. Once we know Doob’s inequality for spline projections, which is point (iii) above, the same proof as in [1] works for spline projections if we use suitable positive operators $T_n$ instead of $P_n^{(k)}$ that also satisfy Doob’s inequality and dominate the operators $P_n^{(k)}$ pointwise (cf. Sects. 3.1, 3.2).

The usage of those operators $T_n$ is also necessary in the extension of inequality (1.1) to splines. Lépingle’s proof of (1.1) rests on an idea by Herz [10] of splitting $\mathbb{E}[f_n \cdot h_n]$ (for $f_n$ being $\mathcal{F}_n$-measurable) by Cauchy–Schwarz after introducing the square function $S_n^2 = \sum_{\ell \leq n} f_\ell^2$:

$$(\mathbb{E}[f_n \cdot h_n])^2 \leq \mathbb{E}[f_n^2/S_n] \cdot \mathbb{E}[S_nh_n^2]$$

(1.3)

and estimating both factors on the right hand side separately. A key point in estimating the second factor is that $S_n$ is $\mathcal{F}_n$-measurable, and therefore, $\mathbb{E}[S_n | \mathcal{F}_n] = S_n$. If we want to allow $f_n \in \mathcal{S}_k(\mathcal{F}_n)$, $S_n$ will not be contained in $\mathcal{S}_k(\mathcal{F}_n)$ in general. Under certain conditions on the filtration $(\mathcal{F}_n)$, we will show in this article how to substitute $S_n$ in estimate (1.3) by a function $g_n \in \mathcal{S}_k(\mathcal{F}_n)$ that enjoys similar properties to $S_n$ and allows us to proceed (cf. Sect. 3.4, in particular Proposition 3.4 and Theorem 3.6).

As a by-product, we obtain a spline version (Theorem 4.2) of C. Fefferman’s theorem [7] on $H^1$-BMO duality. For its martingale version, we refer to A. M. Garsia’s book [8] on Martingale Inequalities.

2 Preliminaries

In this section, we collect all tools that are needed subsequently.

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2.1 Properties of polynomials

We will need Remez’ inequality for polynomials:

**Theorem 2.1** Let $V \subset \mathbb{R}$ be a compact interval in $\mathbb{R}$ and $E \subset V$ a measurable subset. Then, for all polynomials $p$ of order $k$ (i.e. degree $k - 1$) on $V$,

$$\|p\|_{L_\infty(V)} \leq \left(4 \frac{|V|}{|E|}\right)^{k-1} \|p\|_{L_\infty(E)}.$$

Applying this theorem with the set $E = \{x \in V : |p(x)| \leq 8^{-k+1}\|p\|_{L_\infty(V)}\}$ immediately yields the following corollary:

**Corollary 2.2** Let $p$ be a polynomial of order $k$ on a compact interval $V \subset \mathbb{R}$. Then

$$\left|\{x \in V : |p(x)| \geq 8^{-k+1}\|p\|_{L_\infty(V)}\}\right| \geq |V|/2.$$

2.2 Properties of spline functions

For an interval $\sigma$-algebra $\mathcal{F}$ (i.e. $\mathcal{F}$ is generated by a finite collection of intervals having positive length), the space $\mathcal{H}_k(\mathcal{F})$ is spanned by a very special local basis $(N_i)$, the so called B-spline basis. It has the properties that each $N_i$ is non-negative and each support of $N_i$ consists of at most $k$ neighboring atoms of $\mathcal{F}$. Moreover, $(N_i)$ is a partition of unity, i.e. for all $x \in [0, 1]$, there exist at most $k$ functions $N_i$ so that $N_i(x) \neq 0$ and $\sum_i N_i(x) = 1$. In the following, we denote by $E_i$ the support of the B-spline function $N_i$. The usual ordering of the B-splines $(N_i)$—which we also employ here—is such that for all $i$, $\inf E_i \leq \inf E_{i+1}$ and $\sup E_i \leq \sup E_{i+1}$.

We write $A(t) \lesssim B(t)$ to denote the existence of a constant $C$ such that for all $t$, $A(t) \leq CB(t)$, where $t$ denote all implicit and explicit dependencies the expression $A$ and $B$ might have. If the constant $C$ additionally depends on some parameter, we will indicate this in the text. Similarly, the symbols $\gtrsim$ and $\simeq$ are used.

Another important property of B-splines is the following relation between B-spline coefficients and the $L_p$-norm of the corresponding B-spline expansions.

**Theorem 2.3** (B-spline stability, local and global) Let $1 \leq p \leq \infty$ and $g = \sum_j a_j N_j$. Then, for all $j$,

$$|a_j| \lesssim |J_j|^{-1/p} \|g\|_{L_p(J_j)}, \quad (2.1)$$

where $J_j$ is an atom of $\mathcal{F}$ contained in $E_j$ having maximal length. Additionally,

$$\|g\|_p \simeq \|(a_j|E_j|^{1/p})\|_{\ell_p}, \quad (2.2)$$

where in both (2.1) and (2.2), the implied constants depend only on the spline order $k$. 

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Observe that (2.1) implies for $g \in \mathcal{S}_k(\mathcal{F})$ and any measurable set $A \subset [0, 1]$ 

$$\|g\|_{L_\infty(A)} \lesssim \max_{j: |E_j \cap A| > 0} \|g\|_{L_\infty(J_j)}. \quad (2.3)$$

We will also need the following relation between the $B$-spline expansion of a function and its expansion using $B$-splines of a finer grid.

**Theorem 2.4** Let $\mathcal{G} \subset \mathcal{F}$ be two interval $\sigma$-algebras and denote by $(N_{G,i})_i$ the $B$-spline basis of the coarser space $\mathcal{S}_k(\mathcal{G})$ and by $(N_{F,i})_i$ the $B$-spline basis of the finer space $\mathcal{S}_k(\mathcal{F})$. Then, given $f = \sum_j a_j N_{G,j}$, we can expand $f$ in the basis $(N_{F,i})_i$

$$\sum_j a_j N_{G,j} = \sum_i b_i N_{F,i},$$

where for each $i$, $b_i$ is a convex combination of the coefficients $a_j$ with $\text{supp} N_{G,j} \supseteq \text{supp} N_{F,i}$.

For those results and more information on spline functions, in particular $B$-splines, we refer to [21] or [5].

### 2.3 Spline orthoprojectors

We now use the $B$-spline basis of $\mathcal{S}_k(\mathcal{F})$ and expand the orthogonal projection operator $P$ onto $\mathcal{S}_k(\mathcal{F})$ in the form

$$Pf = \sum_{i,j} a_{ij} \left( \int_0^1 f(x) N_i(x) \, dx \right) \cdot N_j \quad (2.4)$$

for some coefficients $(a_{ij})$. Denoting by $E_{ij}$ the smallest interval containing both supports $E_i$ and $E_j$ of the $B$-spline functions $N_i$ and $N_j$ respectively, we have the following estimate for $a_{ij}$ [19]: there exist constants $C$ and $0 < q < 1$ depending only on $k$ so that for each interval $\sigma$-algebra $\mathcal{F}$ and each $i, j$,

$$|a_{ij}| \leq C q^{q|j-i|} / |E_{ij}|. \quad (2.5)$$

### 2.4 Spline square functions

Let $(\mathcal{F}_n)$ be a sequence of increasing interval $\sigma$-algebras in $[0, 1]$ and we assume that each $\mathcal{F}_{n+1}$ is generated from $\mathcal{F}_n$ by the subdivision of exactly one atom of $\mathcal{F}_n$ into two atoms of $\mathcal{F}_{n+1}$. Let $P_n$ be the orthogonal projection operator onto $\mathcal{S}_k(\mathcal{F}_n)$. We denote $\Delta_n f = P_n f - P_{n-1} f$ and define the spline square function

$$Sf = \left( \sum_n |\Delta_n f|^2 \right)^{1/2}. $$
We have Burkholder’s inequality for the spline square function, i.e. for all $1 < p < \infty$ [16], the $L_p$-norm of the square function $Sf$ is comparable to the $L_p$-norm of $f$:

$$\|Sf\|_p \simeq \|f\|_p, \quad f \in L_p$$

with constants depending only on $p$ and $k$. Moreover, for $p = 1$, it is shown in [9] that

$$\|Sf\|_1 \simeq \sup_{\varepsilon \in \{-1,1\}^\mathbb{Z}} \| \sum_n \varepsilon_n \Delta_n f \|_1, \quad Sf \in L_1,$$

with constants depending only on $k$ and where the proof of the $\lesssim$-part only uses Khintchine’s inequality whereas the proof of the $\gtrsim$-part uses fine properties of the functions $\Delta_n f$.

### 2.5 $L_p(\ell_q)$-spaces

For $1 \leq p, q \leq \infty$, we denote by $L_p(\ell_q)$ the space of sequences of measurable functions $(f_n)$ on $[0, 1]$ so that the norm

$$\|(f_n)\|_{L_p(\ell_q)} = \left( \int_0^1 \left( \sum_n |f_n(t)|^q \right)^{p/q} \, dt \right)^{1/p}$$

is finite (with the obvious modifications if $p = \infty$ or $q = \infty$). For $1 \leq p, q < \infty$, the dual space (see [2]) of $L_p(\ell_q)$ is $L_{p'}(\ell_{q'})$ with $p' = p/(p-1), q' = q/(q-1)$ and the duality pairing

$$\langle (f_n), (g_n) \rangle = \int_0^1 \sum_n f_n(t) g_n(t) \, dt.$$

Hölder’s inequality takes the form $|\langle (f_n), (g_n) \rangle| \leq \|(f_n)\|_{L_p(\ell_q)} \|(g_n)\|_{L_{p'}(\ell_{q'})}$.

### 3 Main results

In this section, we prove our main results. Section 3.1 defines and gives properties of suitable positive operators that dominate our (non-positive) operators $P_n = P_n^{(k)}$ pointwise. In Sect. 3.2, we use those operators to give a spline version of Stein’s inequality (1.2). A useful property of conditional expectations is the tower property $E_G E_F f = E_G f$ for $G \subset F$. In this form, it extends to the operators $(P_n)$, but not to the operators $T$ from Sect. 3.1. In Sect. 3.3 we prove a version of the tower property for those operators. Section 3.4 is devoted to establishing a duality estimate using a spline square function, which is the crucial ingredient in the proofs of the spline versions of both Lépingle’s inequality (1.1) and $H_1$-BMO duality in Sect. 4.
3.1 The positive operators $T$

As above, let $\mathcal{F}$ be an interval $\sigma$-algebra on $[0, 1]$, $(N_i)$ the B-spline basis of $\mathcal{A}_k(\mathcal{F})$, $E_i$ the support of $N_i$ and $E_{ij}$ the smallest interval containing both $E_i$ and $E_j$. Moreover, let $q$ be a positive number smaller than 1. Then, we define the linear operator $T = T_{\mathcal{F}, q, k}$ by

$$Tf(x) := \sum_{i,j} q^{|i-j|} \frac{1}{|E_{ij}|} \langle f, 1_{E_i} \rangle 1_{E_j}(x) = \int_0^1 K(x, t) f(t) \, dt,$$

where the kernel $K = K_T$ is given by

$$K(x, t) = \sum_{i,j} q^{|i-j|} \frac{1}{|E_{ij}|} 1_{E_i}(t) \cdot 1_{E_j}(x).$$

We observe that the operator $T$ is selfadjoint (w.r.t the standard inner product on $L_2$) and

$$k \leq K_x := \int_0^1 K(x, t) \, dt \leq \frac{2(k+1)}{1-q}, \quad x \in [0, 1],$$

which, in particular, implies the boundedness of the operator $T$ on $L_1$ and $L_\infty$:

$$\|Tf\|_1 \leq \frac{2(k+1)}{1-q} \|f\|_1, \quad \|Tf\|_\infty \leq \frac{2(k+1)}{1-q} \|f\|_\infty.$$

Another very important property of $T$ is that it is a positive operator, i.e. it maps non-negative functions to non-negative functions and that $T$ satisfies Jensen’s inequality in the form

$$\varphi(Tf(x)) \leq K_x^{-1} T(\varphi(K_x \cdot f))(x), \quad f \in L_1, x \in [0, 1],$$

for convex functions $\varphi$. This is seen by applying the classical Jensen inequality to the probability measure $K(t, x) \, dt / K_x$.

Let $\mathcal{M} f$ denote the Hardy–Littlewood maximal function of $f \in L_1$, i.e.

$$\mathcal{M} f(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| \, dy,$$

where the supremum is taken over all subintervals of $[0, 1]$ that contain the point $x$. This operator is of weak type $(1, 1)$, i.e.

$$|\{ \mathcal{M} f > \lambda \}| \leq C \lambda^{-1} \|f\|_1, \quad f \in L_1, \lambda > 0.$$
for some constant $C$. Since trivially we have the estimate $\|\mathcal{M} f\|_\infty \leq \|f\|_\infty$, by Marcinkiewicz interpolation, for any $p > 1$, there exists a constant $C_p$ depending only on $p$ so that
\[ \|\mathcal{M} f\|_p \leq C_p \|f\|_p. \]

For those assertions about $\mathcal{M}$, we refer to (for instance) [23].

The significance of $T$ and $\mathcal{M}$ at this point is that we can use formula (2.4) and estimate (2.5) to obtain the pointwise bound
\[ |P f(x)| \leq C_1 (T |f|)(x) \leq C_2 \mathcal{M} (x), \quad f \in L_1, x \in [0, 1], \quad (3.3) \]
where $T = T_{F, q, k}$ with $q$ given by (2.5), $C_1$ is a constant that depends only on $k$ and $C_2$ is a constant that depends only on $k$ and the geometric progression $q$. But as the parameter $q < 1$ in (2.5) depends only on $k$, the constant $C_2$ will also only depend on $k$.

In other words, (3.3) tells us that the positive operator $T$ dominates the non-positive operator $P$ pointwise, but at the same time, $T$ is dominated by the Hardy–Littlewood maximal function $\mathcal{M}$ pointwise and independently of $F$.

### 3.2 Stein’s inequality for splines

We now use this pointwise dominating, positive operator $T$ to prove Stein’s inequality for spline projections. For this, let $(F_n)$ be an interval filtration on $[0, 1]$ and $P_n$ be the orthogonal projection operator onto the space $\mathcal{S}_k(F_n)$ of splines of order $k$ corresponding to $F_n$. Working with the positive operators $T_{F_n, q, k}$ instead of the non-positive operators $P_n$, the proof of Stein’s inequality (1.2) for spline projections can be carried over from the martingale case (cf. [1, 24]). For completeness, we include it here.

**Theorem 3.1** Suppose that $(f_n)$ is a sequence of arbitrary integrable functions on $[0, 1]$. Then, for $1 \leq r \leq p < \infty$ or $1 < p \leq r \leq \infty$,
\[ \| (P_n f_n) \|_{L_p(\ell_r)} \lesssim \| (f_n) \|_{L_p(\ell_r)} \quad (3.4) \]
where the implied constant depends only on $p, r$ and $k$.

**Proof** By (3.3), it suffices to prove this inequality for the operators $T_n = T_{F_n, q, k}$ with $q$ given by (2.5) instead of the operators $P_n$. First observe that for $r = p = 1$, the assertion follows from Shadrin’s theorem ((i) on page 1). Inequality (3.3) and the $L_{p'}$-boundedness of $\mathcal{M}$ for $1 < p' \leq \infty$ imply that
\[ \left\| \sup_{1 \leq n \leq N} |T_n f| \right\|_{p'} \leq C_{p', k} \|f\|_{p'}, \quad f \in L_p \quad (3.5) \]
with a constant $C_{p', k}$ depending on $p'$ and $k$. Let $1 \leq p < \infty$ and $U_N : L_p(\ell_1^N) \rightarrow L_p$ be given by $(g_1, \ldots, g_N) \mapsto \sum_{j=1}^N T_j g_j$. Inequality (3.5) implies the boundedness of
the adjoint $U_N^*: L_{p'} \to L_{p'}(\ell_1^N)$, $f \mapsto (T_j f)^N_{j=1}$ for $p' = p/(p - 1)$ by a constant independent of $N$ and therefore also the boundedness of $U_N$. Since $|T_j f| \leq T_j |f|$ by the positivity of $T_j$, letting $N \to \infty$ implies (3.4) for $T_n$ instead of $P_n$ in the case $r = 1$ and outer parameter $1 \leq p < \infty$.

If $1 < r \leq p$, we use Jensen’s inequality (3.2) and estimate (3.1) to obtain
\[
\sum_{j=1}^N |T_j g_j|^r \lesssim \sum_{j=1}^N T_j (|g_j|^r)
\]
and apply the result for $r = 1$ and the outer parameter $p/r$ to get the result for $1 \leq r < p < \infty$. The cases $1 < p \leq r < \infty$ now just follow from this result using duality and the self-adjointness of $T_j$. \hfill \Box

**3.3 Tower property of $T$**

Next, we will prove a substitute of the tower property $E_{\mathcal{G}} E_{\mathcal{F}} f = E_{\mathcal{G}} f$ ($\mathcal{G} \subset \mathcal{F}$) for conditional expectations that applies to the operators $T$.

To formulate this result, we need a suitable notion of regularity for $\sigma$-algebras which we now describe. Let $\mathcal{F}$ be an interval $\sigma$-algebra, let $(N_j)$ be the B-spline basis of $\mathcal{A}_k(\mathcal{F})$ and denote by $E_j$ the support of the function $N_j$. The k-regularity parameter $\gamma_k(\mathcal{F})$ is defined as
\[
\gamma_k(\mathcal{F}) := \max_i \max(|E_i|/|E_{i+1}|, |E_{i+1}|/|E_i|),
\]
where the first maximum is taken over all $i$ so that $E_i$ and $E_{i+1}$ are defined. The name $k$-regularity is motivated by the fact that each B-spline support $E_i$ of order $k$ consists of at most $k$ (neighboring) atoms of the $\sigma$-algebra $\mathcal{F}$.

**Proposition 3.2** (Tower property of $T$) Let $\mathcal{G} \subset \mathcal{F}$ be two interval $\sigma$-algebras on $[0, 1]$. Let $S = T_{\mathcal{G}, \sigma, k}$ and $T = T_{\mathcal{F}, \tau, k'}$ for some $\sigma, \tau \in (0, 1)$ and some positive integers $k, k'$. Then, for all $q > \max(\tau, \sigma)$, there exists a constant $C$ depending on $q, k, k'$ so that
\[
|ST f(x)| \leq C \cdot \gamma^k \cdot (T_{\mathcal{G}, q, k}|f|)(x), \quad f \in L_1, x \in [0, 1],
\]
where $\gamma = \gamma_k(\mathcal{G})$ denotes the $k$-regularity parameter of $\mathcal{G}$.

**Proof** Let $(F_i)$ be the collection of B-spline supports in $\mathcal{A}_k(\mathcal{F})$ and $(G_i)$ the collection of B-spline supports in $\mathcal{A}_k(\mathcal{G})$. Moreover, we denote by $F_{ij}$ the smallest interval containing $F_i$ and $F_j$ and by $G_{ij}$ the smallest interval containing $G_i$ and $G_j$.

We show (3.6) by showing the following inequality for the kernels $K_S$ of $S$ and $K_T$ of $T$ (cf. 3.1)
\[
\int_0^1 K_S(x, t) K_T(t, s) \, dt \leq C \gamma^k \sum_{i,j} q^{\frac{|i-j|}{|G_{ij}|}} 1_{G_i}(x) 1_{G_j}(s), \quad x, s \in [0, 1]
\]
(3.7)
for all $q > \max(\tau, \sigma)$ and some constant $C$ depending on $q, k, k'$. In order to prove this inequality, we first fix $x, s \in [0, 1]$ and choose $i$ such that $x \in G_i$ and $\ell$ such that $s \in F_\ell$. Moreover, based on $\ell$, we choose $j$ so that $s \in G_j$ and $G_j \supset F_\ell$. There are at most $\max(k, k')$ choices for each of the indices $i, \ell, j$ and without restriction, we treat those choices separately, i.e. we only have to estimate the expression

$$\sum_{m, r} \sigma^{m-i} \tau^{r-\ell} |G_m \cap F_r| / |G_{im}| |F_{\ell r}|.$$

Since, for each $r$, there are also at most $k + k' - 1$ indices $m$ so that $|G_m \cap F_r| > 0$ (recall that $\mathcal{G} \subset \mathcal{F}$), we choose one such index $m = m(r)$ and estimate

$$\Sigma = \sum_r \sigma^{m(r) - i} \tau^{r-\ell} |G_{m(r)} \cap F_r| / |G_{i, m(r)}| |F_{\ell r}|.$$

Now, observe that for any parameter choice of $r$ in the above sum,

$$G_{i, m(r)} \cup F_{\ell r} \supseteq (G_{ij} \setminus G_j) \cup G_i$$

and therefore, since also $G_{m(r)} \cap F_r \subset G_{i, m(r)} \cap F_{\ell r}$,

$$\Sigma \leq 2 \sum_r \sigma^{m(r) - i} \tau^{r-\ell} / |(G_{ij} \setminus G_j) \cup G_i| \sum_r \sigma^{m(r) - i} \tau^{r-\ell},$$

which, using the $k$-regularity parameter $\gamma = \gamma_k(\mathcal{G})$ of the $\sigma$-algebra $\mathcal{G}$ and denoting $\lambda = \max(\tau, \sigma)$, we estimate by

$$\Sigma \leq 2 \gamma k / |G_{ij}| \sum_m \lambda^{m-i} \sum_{r: m(r) = m} \lambda^{r-\ell} \lesssim \gamma k / |G_{ij}| \sum_m \lambda^{i-m+|m-j|}$$

$$\lesssim \gamma k / |G_{ij}| (|i-j| + 1) \lambda^{i-j},$$

where the implied constants depend on $\lambda, k, k'$ and the estimate $\sum_{r: m(r) = m} \lambda^{r-\ell} \lesssim \lambda^{m-j}$ used the fact that, essentially, there are more atoms of $\mathcal{F}$ between $F_r$ and $F_\ell$ (for $r$ as in the sum) than atoms of $\mathcal{G}$ between $G_m$ and $G_j$. Finally, we see that for any $q > \lambda$,

$$\Sigma \lesssim C \gamma k q^{i-j} / |G_{ij}|$$

for some constant $C$ depending on $q, k, k'$, and, as $x \in G_i$ and $s \in G_j$, this shows inequality (3.7).

As a corollary of Proposition 3.2, we have
Corollary 3.3 Let \((f_n)\) be functions in \(L_1\). We denote by \(P_n\) the orthogonal projection onto \(\mathcal{H}_k(\mathcal{F}_n)\) and by \(P'_n\) the orthogonal projection onto \(\mathcal{H}_k'(\mathcal{F}_n)\) for some positive integers \(k, k'\). Moreover, let \(T_n\) be the operator \(T_{F_n, q, k}\) from (3.3) dominating \(P_n\) pointwise.

Then, for any integer \(n\) and for any \(1 \leq p \leq \infty\),
\[
\left\| \sum_{\ell \geq n} P_n((P'_{\ell-1} f_\ell)^2) \right\|_p \lesssim \left\| \sum_{\ell \geq n} T_n((P'_{\ell-1} f_\ell)^2) \right\|_p \lesssim \gamma_k(\mathcal{F}_n)^k \cdot \left\| \sum_{\ell \geq n} f_\ell^2 \right\|_p,
\]
where the implied constants only depend on \(k\) and \(k'\).

Proof We denote by \(T_n\) the operator \(T_{F_n, q, k}\) and by \(T'_n\) the operator \(T_{F_n, q', k'}\), where the parameters \(q, q' < 1\) are given by inequality (3.3) depending on \(k\) and \(k'\) respectively. Setting \(U_n := T_{\mathcal{F}_n, \max(q, q')^{1/2}, k}\), we perform the following chain of inequalities, where we use the positivity of \(T_n\) and (3.3), Jensen’s inequality for \(T'_{\ell-1}\), the tower property for \(T_n T'_{\ell-1}\) and the \(L_p\)-boundedness of \(U_n\), respectively:
\[
\left\| \sum_{\ell \geq n} T_n((P'_{\ell-1} f_\ell)^2) \right\|_p \lesssim \left\| \sum_{\ell \geq n} T_n((T'_{\ell-1} f_\ell)^2) \right\|_p \lesssim \left\| \sum_{\ell \geq n} T_n(T'_{\ell-1} f_\ell^2) \right\|_p \leq \left\| T_n(T'_{\ell-1} f_\ell^2) \right\|_p + \left\| \sum_{\ell > n} T_n(T'_{\ell-1} f_\ell^2) \right\|_p \lesssim \left\| f_n^2 \right\|_p + \gamma_k(\mathcal{F}_n)^k \cdot \left\| \sum_{\ell > n} U_n(f_\ell^2) \right\|_p \lesssim \gamma_k(\mathcal{F}_n)^k \cdot \left\| \sum_{\ell \geq n} f_\ell^2 \right\|_p,
\]
where the implied constants only depend on \(k\) and \(k'\). \(\square\)

3.4 A duality estimate using a spline square function

In order to give the desired duality estimate contained in Theorem 3.6, we need the following construction of a function \(g_n \in \mathcal{H}_k(\mathcal{F}_n)\) based on a spline square function.

Proposition 3.4 Let \((f_n)\) be a sequence of functions with \(f_n \in \mathcal{H}_k(\mathcal{F}_n)\) for all \(n\) and set
\[
X_n := \sum_{\ell \leq n} f_\ell^2.
\]
Then, there exists a sequence of non-negative functions \( g_n \in \mathcal{S}_k(\mathcal{F}_n) \) so that for each \( n \),

(1) \( g_n \leq g_{n+1} \),
(2) \( X_n^{1/2} \leq g_n \),
(3) \( \mathbb{E} g_n \leq \mathbb{E} X_n^{1/2} \), where the implied constant depends on \( k \) and on \( \sup_{m \leq n} y_k(\mathcal{F}_m) \).

For the proof of this result, we need the following simple lemma.

**Lemma 3.5** Let \( c_1 \) be a positive constant and let \( (A_j)_{j=1}^N \) be a sequence of atoms in \( \mathcal{F}_n \). Moreover, let \( \ell : \{1, \ldots, N\} \rightarrow \{1, \ldots, n\} \) and, for each \( j \in \{1, \ldots, N\} \), let \( B_j \) be a subset of an atom \( D_j \) of \( \mathcal{F}_\ell(j) \) with

\[
|B_j| \geq c_1 \sum_{i: \ell(i) \geq \ell(j), D_i \subseteq D_j} |A_i|.	ag{3.8}
\]

Then, there exists a map \( \varphi \) on \( \{1, \ldots, N\} \) so that

(1) \( |\varphi(j)| = c_1 |A_j| \) for all \( j \),
(2) \( \varphi(j) \subseteq B_j \) for all \( j \),
(3) \( \varphi(i) \cap \varphi(j) = \emptyset \) for all \( i \neq j \).

**Proof** Without restriction, we assume that the sequence \( (A_j) \) is enumerated such that \( \ell(j+1) \leq \ell(j) \) for all \( 1 \leq j \leq N-1 \). We first choose \( \varphi(1) \) as an arbitrary (measurable) subset of \( B_1 \) with measure \( c_1 |A_1| \), which is possible by assumption (3.8). Next, we assume that for \( 1 \leq j \leq j_0 \), we have constructed \( \varphi(j) \) with the properties

(1) \( |\varphi(j)| = c_1 |A_j| \),
(2) \( \varphi(j) \subseteq B_j \),
(3) \( \varphi(i) \cap \cup_{i \leq j} \varphi(i) = \emptyset \).

Based on that, we now construct \( \varphi(j_0 + 1) \). Define the index sets \( \Gamma = \{ i : \ell(i) \geq \ell(j_0 + 1), D_i \subseteq D_{j_0+1} \} \) and \( \Lambda = \{ i : i \leq j_0 + 1, D_i \subseteq D_{j_0+1} \} \). Since we assumed that \( \ell \) is decreasing, we have \( \Lambda \subseteq \Gamma \) and by the nestedness of the \( \sigma \)-algebras \( \mathcal{F}_n \), we have for \( i \leq j_0 + 1 \) that either \( D_i \subseteq D_{j_0+1} \) or \( |D_i \cap D_{j_0+1}| = 0 \). This implies

\[
|B_{j_0+1} \setminus \bigcup_{i \leq j_0} \varphi(i)| = |B_{j_0+1}| - |B_{j_0+1} \cap \bigcup_{i \leq j_0} \varphi(i)|
\geq c_1 \sum_{i \in \Gamma} |A_i| - |D_{j_0+1} \cap \bigcup_{i \leq j_0} \varphi(i)|
\geq c_1 \sum_{i \in \Lambda} |A_i| - \bigcup_{i \in \Lambda \setminus \{j_0+1\}} \varphi(i)
\geq c_1 \sum_{i \in \Lambda} |A_i| - \sum_{i \in \Lambda \setminus \{j_0+1\}} c_1 |A_i| = c_1 |A_{j_0+1}|.
\]

Therefore, we can choose \( \varphi(j_0 + 1) \subseteq B_{j_0+1} \) that is disjoint to \( \varphi(i) \) for any \( i \leq j_0 \) and \( |\varphi(j_0 + 1)| = c_1 |A_{j_0+1}| \) which completes the proof. \( \square \)
Proof of Proposition 3.4  Fix $n$ and let $(N_{n,j})$ be the B-spline basis of $S_k(F_n)$. Moreover, for any $j$, set $E_{n,j} = \text{supp} N_{n,j}$ and $a_{n,j} := \max_{\ell \leq n} \max_{r:E_{\ell,r} \supset E_{n,j}} \|X_\ell\|_{L^\infty(E_{\ell,r})}^{1/2}$ and we define $\ell(j) \leq n$ and $r(j)$ so that $E_{\ell(j),r(j)} \supset E_{n,j}$ and $a_{n,j} = \|X_{\ell(j)}\|_{L^\infty(E_{\ell(j),r(j)})}^{1/2}$. Set

$$g_n := \sum_j a_{n,j} N_{n,j} \in \mathcal{S}_k(F_n)$$

and it will be proved subsequently that this $g_n$ has the desired properties.

PROPERTY (1): In order to show $g_n \leq g_{n+1}$, we use Theorem 2.4 to write

$$g_n = \sum_j a_{n,j} N_{n,j} = \sum_j \beta_{n,j} N_{n+1,j},$$

where $\beta_{n,j}$ is a convex combination of those $a_{n,r}$ with $E_{n+1,j} \subseteq E_{n,r}$, and thus

$$g_n \leq \sum_j \left( \max_{r:E_{n+1,j} \subseteq E_{n,r}} a_{n,r} \right) N_{n+1,j}.$$ 

By the very definition of $a_{n+1,j}$, we have

$$\max_{r:E_{n+1,j} \subseteq E_{n,r}} a_{n,r} \leq a_{n+1,j},$$

and therefore, $g_n \leq g_{n+1}$ pointwise, since the B-splines $(N_{n+1,j})_j$ are nonnegative functions.

PROPERTY (2): Now we show that $X_n^{1/2} \leq g_n$. Indeed, for any $x \in [0, 1]$,

$$g_n(x) = \sum_j a_{n,j} N_{n,j}(x) \geq \min_{j:E_{n,j} \ni x} a_{n,j} \geq \min_{j:E_{n,j} \ni x} \|X_n\|_{L^\infty(E_{n,j})}^{1/2} \geq X_n(x)^{1/2},$$

since the collection of B-splines $(N_{n,j})_j$ forms a partition of unity.

PROPERTY (3): Finally, we show $\mathbb{E}g_n \lesssim \mathbb{E}X_n^{1/2}$, where the implied constant depends only on $k$ and on $\sup_{m \leq n} \gamma_k(F_m)$. By B-spline stability (Theorem 2.3), we estimate the integral of $g_n$ by

$$\mathbb{E}g_n \lesssim \sum_j |E_{n,j}| \cdot \|X_{\ell(j)}\|_{L^\infty(E_{\ell(j),r(j)})}^{1/2}, \quad (3.9)$$

where the implied constant only depends on $k$. In order to continue the estimate, we next show the inequality

$$\|X_\ell\|_{L^\infty(E_{\ell,r})} \lesssim \max_{s:|E_{\ell,r} \cap E_{\ell,s}| > 0} \|X_\ell\|_{L^\infty(E_{\ell,s})}, \quad (3.10)$$
where by $J_{\ell,s}$ we denote an atom of $\mathcal{F}_r$ with $J_{\ell,s} \subset E_{\ell,s}$ of maximal length and the implied constant depends only on $k$. Indeed, we use Theorem 2.3 in the form of (2.3) to get $(f_m \in \mathcal{S}_k(\mathcal{F}_\ell)$ for $m \leq \ell$)

$$\|X_\ell\|_{L_\infty(E_{\ell,r})} \leq \sum_{m \leq \ell} \|f_m\|^2_{L_\infty(E_{\ell,r})} \lesssim \sum_{m \leq \ell} \sum_{s : |E_{\ell,s} \cap E_{\ell,r}| > 0} \|f_m\|^2_{L_\infty(J_{\ell,s})} = \sum_{s : |E_{\ell,s} \cap E_{\ell,r}| > 0} \sum_{m \leq \ell} \|f_m\|^2_{L_\infty(J_{\ell,s})}. \tag{3.11}$$

Now observe that for atoms $I$ of $\mathcal{F}_\ell$, due to the equivalence of $p$-norms of polynomials (cf. Corollary 2.2),

$$\sum_{m \leq \ell} \|f_m\|^2_{L_\infty(I)} \lesssim \sum_{m \leq \ell} \frac{1}{|I|} \int_I f_m^2 = \frac{1}{|I|} \int_I X_\ell \leq \|X_\ell\|_{L_\infty(I)},$$

which means that, inserting this in estimate (3.11),

$$\|X_\ell\|_{L_\infty(E_{\ell,r})} \lesssim \sum_{s : |E_{\ell,s} \cap E_{\ell,r}| > 0} \|X_\ell\|_{L_\infty(J_{\ell,s})},$$

and, since there are at most $k$ indices $s$ so that $|E_{\ell,s} \cap E_{\ell,r}| > 0$, we have established (3.10).

We define $K_{\ell,r}$ to be an interval $J_{\ell,s}$ with $|E_{\ell,r} \cap E_{\ell,s}| > 0$ so that

$$\max_{s : |E_{\ell,r} \cap E_{\ell,s}| > 0} \|X_\ell\|_{L_\infty(J_{\ell,s})} = \|X_\ell\|_{L_\infty(K_{\ell,r})}.$$ 

This means, after combining (3.9) with (3.10), we have

$$\mathbb{E}g_n \lesssim \sum_j |J_{n,j}| \cdot \|X_\ell(j)\|_{L_\infty(K_{\ell(j),r(j)})}^{1/2}. \tag{3.12}$$

We now apply Lemma 3.5 with the function $\ell$ and the choices

$$A_j = J_{n,j}, \quad D_j = K_{\ell(j),r(j)},$$

$$B_j = \left\{t \in D_j : X_\ell(j)(t) \geq 8^{-2(k-1)}\|X_\ell(j)\|_{L_\infty(D_j)} \right\}.$$

In order to see Assumption (3.8) of Lemma 3.5, fix the index $j$ and let $i$ be such that $\ell(i) \geq \ell(j)$. By definition of $D_i = K_{\ell(i),r(i)}$, the smallest interval containing $J_{n,i}$ and $D_i$ contains at most $2k - 1$ atoms of $\mathcal{F}_{\ell(i)}$ and, if $D_i \subset D_j$, the smallest interval containing $J_{n,i}$ and $D_j$ contains at most $2k - 1$ atoms of $\mathcal{F}_{\ell(j)}$. This means that, in particular, $J_{n,i}$ is a subset of the union $V$ of $4k$ atoms of $\mathcal{F}_{\ell(j)}$ with $D_j \subset V$. Since
each atom of \( F_n \) occurs at most \( k \) times in the sequence \( (A_j) \), there exists a constant \( c_1 \) depending on \( k \) and \( \sup_{u \leq \ell(j)} \gamma_k(F_u) \leq \sup_{u \leq n} \gamma_k(F_u) \) so that

\[
|D_j| \geq c_1 \sum_{i: \ell(i) \geq \ell(j) \atop D_i \subset D_j} |A_i|,
\]

which, since \( |B_j| \geq |D_j|/2 \) by Corollary 2.2, shows that the assumption of Lemma 3.5 holds true and we get a function \( \varphi \) so that \( |\varphi(j)| = c_1 |J_{n,j}|/2 \), \( \varphi(j) \subset B_j \), \( \varphi(i) \cap \varphi(j) = \emptyset \) for all \( i, j \). Using these properties of \( \varphi \), we continue the estimate in (3.12) and write

\[
\mathbb{E} g_n \lesssim \sum_j |J_{n,j}| \cdot \|X_{\ell(j)}\|_{L_\infty(D_j)}^{1/2} \leq 8^{k-1} \cdot \sum_j \frac{|J_{n,j}|}{|\varphi(j)|} \int_{\varphi(j)} X_{\ell(j)}^{1/2}
\]

\[
= \frac{2}{c_1} \cdot 8^{k-1} \cdot \sum_j \int_{\varphi(j)} X_{\ell(j)}^{1/2}
\]

\[
\lesssim \sum_j \int_{\varphi(j)} X_n^{1/2} \leq \mathbb{E} X_n^{1/2},
\]

with constants depending only on \( k \) and \( \sup_{n \leq N} \gamma_k(F_u) \).

Employing this construction of \( g_n \), we now give the following duality estimate for spline projections (for the martingale case, see for instance [8]). The martingale version of this result is the essential estimate in the proof of both Lépingle’s inequality (1.1) and the \( H^1 \)-BMO duality.

**Theorem 3.6** Let \( (\mathcal{F}_n) \) be such that \( \gamma := \sup_n \gamma_k(F_n) < \infty \) and \( (f_n)_{n \geq 1} \) a sequence of functions with \( f_n \in \mathcal{S}_k(F_n) \) for each \( n \). Additionally, let \( h_n \in L_1 \) be arbitrary. Then, for any \( N \),

\[
\sum_{n \leq N} \mathbb{E} [|f_n \cdot h_n|] \lesssim \sqrt{2} \cdot \mathbb{E} \left[ \left( \sum_{\ell \leq N} f_{\ell}^2 \right)^{1/2} \right] \cdot \sup_{n \leq N} \|P_n(\sum_{\ell=n}^N h_{\ell}^2)\|_{L_\infty}^{1/2},
\]

where the implied constant is the same constant that appears in (3) of Proposition 3.4 and therefore only depends on \( k \) and \( \gamma \).

**Proof** Let \( X_n := \sum_{\ell \leq n} f_{\ell}^2 \) and \( f_{\ell} \equiv 0 \) for \( \ell > N \) and \( \ell \leq 0 \). By Proposition 3.4, we choose an increasing sequence \( (g_n) \) of functions with \( g_0 = 0 \), \( g_n \in \mathcal{S}_k(F_n) \) and the properties \( X_n^{1/2} \leq g_n \) and \( \mathbb{E} g_n \lesssim \mathbb{E} X_n^{1/2} \) where the implied constant depends only on \( k \) and \( \gamma \). Then, apply Cauchy–Schwarz inequality by introducing the factor \( 8_n^{1/2} \) to get
\[ \sum_n \mathbb{E}[|f_n h_n|] = \sum_n \mathbb{E} \left[ \frac{f_n}{g_n^{1/2}} \cdot g_n^{1/2} h_n \right] \]
\[ \leq \left[ \sum_n \mathbb{E}[f_n^2 / g_n] \right]^{1/2} \cdot \left[ \sum_n \mathbb{E}[g_n h_n^2] \right]^{1/2}. \]

We estimate each of the factors on the right hand side separately and set
\[ \Sigma_1 := \sum_n \mathbb{E}[f_n^2 / g_n], \quad \Sigma_2 := \sum_n \mathbb{E}[g_n h_n^2]. \]

The first factor is estimated by the pointwise inequality \( X_n^{1/2} \leq g_n \):
\[ \Sigma_1 = \mathbb{E} \left[ \sum_n f_n^2 / g_n \right] \leq \mathbb{E} \left[ \sum_n f_n^2 / X_n^{1/2} \right] = \mathbb{E} \left[ \sum_n X_n - X_{n-1} / X_n^{1/2} \right] \leq 2 \mathbb{E} \sum_n (X_n^{1/2} - X_{n-1}^{1/2}) = 2 \mathbb{E} X_N^{1/2}. \]

We continue with \( \Sigma_2 \):
\[ \Sigma_2 = \mathbb{E} \left[ \sum_{\ell=1}^N g_\ell h_\ell^2 \right] = \mathbb{E} \left[ \sum_{\ell=1}^N \sum_{n=1}^\ell (g_n - g_{n-1}) h_\ell^2 \right] \]
\[ = \mathbb{E} \left[ \sum_{n=1}^N (g_n - g_{n-1}) \cdot \sum_{\ell=n}^N h_\ell^2 \right] \]
\[ = \mathbb{E} \left[ \sum_{n=1}^N P_n (g_n - g_{n-1}) \cdot \sum_{\ell=n}^N h_\ell^2 \right] \]
\[ = \mathbb{E} \left[ \sum_{n=1}^N (g_n - g_{n-1}) \cdot P_n \left( \sum_{\ell=n}^N h_\ell^2 \right) \right] \]
\[ \leq \mathbb{E} \left[ \sum_{n=1}^N (g_n - g_{n-1}) \right] \cdot \sup_{1 \leq n \leq N} \left\| P_n \left( \sum_{\ell=n}^N h_\ell^2 \right) \right\|_{\infty}, \]

where the last inequality follows from \( g_n \geq g_{n-1} \). Noting that by the properties of \( g_n \), \( \mathbb{E} \left[ \sum_{n=1}^N (g_n - g_{n-1}) \right] = \mathbb{E} g_N \lesssim \mathbb{E} X_N^{1/2} \) and combining the two parts \( \Sigma_1 \) and \( \Sigma_2 \), we obtain the conclusion.

\[ \square \]

4 Applications

We give two applications of Theorem 3.6, (i) D. Lépingle’s inequality and (ii) an analogue of C. Fefferman’s \( H_1 \)-BMO duality in the setting of splines. Once the results
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from Sect. 3 are known, the proofs of the subsequent results proceed similarly to their martingale counterparts in [8, 12] by using spline properties instead of martingale properties.

4.1 Lépingle’s inequality for splines

Theorem 4.1 Let \( k, k' \) be positive integers. Let \((\mathcal{F}_n)\) be an interval filtration with \( \sup_n \gamma_k(\mathcal{F}_n) < \infty \) and, for any \( n \), \( f_n \in S_k(\mathcal{F}_n) \) and \( P'_n \) be the orthogonal projection operator on \( S_k(\mathcal{F}_n) \). Then,

\[
\| (P'_{n-1} f_n) \|_{L_1(\ell_2)} = \left\| \left( \sum_n (P'_{n-1} f_n)^2 \right)^{1/2} \right\|_1 \lesssim \left\| \left( \sum_n f_n^2 \right)^{1/2} \right\|_1 = \| f_n \|_{L_1(\ell_2)},
\]

where the implied constant depends only on \( k, k' \) and \( \sup_n \gamma_k(\mathcal{F}_n) \).

Proof We first assume that \( f_n = 0 \) for \( n > N \) and begin by using duality

\[
\mathbb{E} \left[ \left( \sum_n (P'_{n-1} f_n)^2 \right)^{1/2} \right] = \sup_{(H_n)} \mathbb{E} \left[ \sum_n (P'_{n-1} f_n) \cdot H_n \right],
\]

where sup is taken over all \( (H_n) \in L_\infty(\ell_2) \) with \( \| (H_n) \|_{L_\infty(\ell_2)} = 1 \). By the self-adjointness of \( P'_{n-1} \),

\[
\mathbb{E}[(P'_{n-1} f_n) \cdot H_n] = \mathbb{E}[f_n \cdot (P'_{n-1} H_n)].
\]

Then we apply Theorem 3.6 for \( f_n \) and \( h_n = P'_{n-1} H_n \) to obtain (denoting by \( P_n \) the orthogonal projection operator onto \( S_k(\mathcal{F}_n) \))

\[
\sum_{n \leq N} |\mathbb{E}[f_n \cdot h_n]| \lesssim \mathbb{E} \left[ \left( \sum_{\ell \leq N} f_\ell^2 \right)^{1/2} \right] \cdot \sup_{n \leq N} \| P_n \left( \sum_{\ell=n}^N (P'_{\ell-1} H_\ell)^2 \right) \|_\infty. \tag{4.1}
\]

To estimate the right hand side, we note that for any \( n \), by Corollary 3.3,

\[
\| P_n \left( \sum_{\ell=n}^N (P'_{\ell-1} H_\ell)^2 \right) \|_\infty \lesssim \| \sum_{\ell=n}^N H_\ell^2 \|_\infty.
\]
Therefore, (4.1) implies
\[
\mathbb{E} \left[ \left( \sum_n (P'_{n-1} f'_n) \right)^{1/2} \right] = \sup_{(H_n)} \mathbb{E} \left[ \sum_n f_n \cdot (P'_{n-1} H_n) \right] \lesssim \mathbb{E} \left[ \left( \sum_{\ell \leq N} f_\ell^2 \right)^{1/2} \right],
\]
with a constant depending only on \( k, k' \) and \( \sup_{n \leq N} \gamma_k(\mathcal{P}_n) \). Letting \( N \) tend to infinity, we obtain the conclusion. \( \square \)

### 4.2 \( H_1 \)-BMO duality for splines

We fix an interval filtration \((\mathcal{F}_n)_{n=1}^\infty\), a spline order \( k \) and the orthogonal projection operators \( P_n \) onto \( \mathcal{S}_k(\mathcal{P}_n) \) and additionally, we set \( P_0 = 0 \). For \( f \in L_1 \), we introduce the notation
\[
\Delta_n f := P_n f - P_{n-1} f, \quad S_n(f) := \left( \sum_{\ell=1}^n (\Delta_\ell f)^2 \right)^{1/2}, \quad S(f) = \sup_n S_n(f).
\]

Observe that for \( \ell < n \) and \( f, g \in L_1 \),
\[
\mathbb{E}[\Delta_\ell f \cdot \Delta_n g] = \mathbb{E}[P_\ell (\Delta_\ell f) \cdot \Delta_n g] = \mathbb{E}[\Delta_\ell f \cdot P_\ell (\Delta_n g)] = 0.
\]

Let \( V \) be the \( L_1 \)-closure of \( \cup_n \mathcal{S}_k(\mathcal{P}_n) \). Then, the uniform boundedness of \( P_n \) on \( L_1 \) implies that \( P_n f \to f \) in \( L_1 \) for \( f \in V \). Next, set
\[
H_{1,k} = H_{1,k}(\mathcal{P}_n) = \{ f \in V : \mathbb{E}(S(f)) < \infty \}
\]
and equip \( H_{1,k} \) with the norm \( \| f \|_{H_{1,k}} = \mathbb{E}S(f) \). Define
\[
\text{BMO}_k = \text{BMO}_k(\mathcal{P}_n) = \left\{ f \in V : \sup_n \| \sum_{\ell \geq n} T_n ((\Delta_\ell f)^2) \|_\infty < \infty \right\}
\]
and the corresponding quasinorm
\[
\| f \|_{\text{BMO}_k} = \sup_n \| \sum_{\ell \geq n} T_n ((\Delta_\ell f)^2) \|_\infty^{1/2},
\]
where \( T_n \) is the operator from (3.3) that dominates \( P_n \) pointwise.

Let us now assume \( \sup_n \gamma_k(\mathcal{P}_n) < \infty \). In this case we identify, similarly to \( H_1 \)-BMO-duality (cf. [7,8,10]), \( \text{BMO}_k \) as the dual space of \( H_{1,k} \).
First, we use the duality estimate Theorem 3.6 and (4.2) to prove, for $f \in H_{1,k}$ and $h \in \text{BMO}_k$,

$$\left| \mathbb{E}\left[ (P_n f) \cdot (P_n h) \right] \right| \leq \sum_{\ell \leq n} \mathbb{E}\left[ |\Delta_\ell f| \cdot |\Delta_\ell h| \right] \lesssim S_n(f) \cdot \|h\|_{\text{BMO}}.$$  

This estimate also implies that the limit $\lim_n \mathbb{E}\left[ (P_n f) \cdot (P_n h) \right]$ exists and satisfies

$$\left| \lim_n \mathbb{E}\left[ (P_n f) \cdot (P_n h) \right] \right| \lesssim \|f\|_{H_{1,k}} \cdot \|h\|_{\text{BMO}}.$$  

On the other hand, similarly to the martingale case (see [8]), given a continuous linear functional $L$ on $H_{1,k}$, we extend $L$ norm-preservingly to a continuous linear functional $\Lambda$ on $L_1(\ell_2)$, which, by Sect. 2.5, has the form

$$\Lambda(\eta) = \mathbb{E}\left[ \sum_\ell \sigma_\ell \eta_\ell \right], \quad \eta \in L_1(\ell_2)$$

for some $\sigma \in L_\infty(\ell_2)$. The $k$-martingale spline sequence $h_n = \sum_{\ell \leq n} \Delta_\ell \sigma_\ell$ is bounded in $L_2$ and therefore, by the spline convergence theorem ((v) on page 2), has a limit $h \in L_2$ with $P_n h = h_n$ and which is also contained in $\text{BMO}_k$. Indeed, by using Corollary 3.3, we obtain $\|h\|_{\text{BMO}} \lesssim \|\sigma\|_{L_\infty(\ell_2)} = \|\Lambda\| = \|L\|$ with a constant depending only on $k$ and $\sup_n \gamma_k(F_n)$. Moreover, for $f \in H_{1,k}$, since $L$ is continuous on $H_{1,k}$,

$$L(f) = \lim_n L(P_n f) = \lim_n \Lambda\left( (\Delta_1 f, \ldots, \Delta_n f, 0, 0, \ldots) \right)$$

$$= \lim_n \sum_{\ell=1}^{n} \mathbb{E}[\sigma_\ell \cdot \Delta_\ell f] = \lim_n \mathbb{E}\left[ (P_n f) \cdot (P_n h) \right].$$  

This yields the following

**Theorem 4.2** If $\sup_n \gamma_k(\mathcal{F}_n) < \infty$, the linear mapping

$$u : \text{BMO}_k \to H^*_{1,k}, \quad h \mapsto \left( f \mapsto \lim_n \mathbb{E}\left[ (P_n f) \cdot (P_n h) \right] \right)$$

is bijective and satisfies

$$\|u(h)\|_{H^*_{1,k}} \simeq \|h\|_{\text{BMO}_k},$$

where the implied constants depend only on $k$ and $\sup_n \gamma_k(\mathcal{F}_n)$.

**Remark 4.3** We close with a few remarks concerning the above result and we assume that $(\mathcal{F}_n)$ is a sequence of increasing interval $\sigma$-algebras with $\sup_n \gamma_k(\mathcal{F}_n) < \infty$. 
(1) By Khintchine’s inequality, \( \|Sf\|_1 \lesssim \sup_{\varepsilon \in \{-1, 1\}} \| \sum_n \varepsilon_n \Delta_n f \|_1 \). Based on the interval filtration \((\mathcal{F}_n)\), we can generate an interval filtration \((\mathcal{G}_n)\) that contains \((\mathcal{F}_n)\) as a subsequence and each \(\mathcal{G}_{n+1}\) is generated from \(\mathcal{G}_n\) by dividing exactly one atom of \(\mathcal{G}_n\) into two atoms of \(\mathcal{G}_{n+1}\). Denoting by \(P_{\mathcal{G}}^j\) the orthogonal projection operator onto \(\mathcal{S}_k(\mathcal{G}_n)\) and \(\Delta_{\mathcal{G}}^j = P_{\mathcal{G}}^j - P_{\mathcal{G}}^{j-1}\), we can write

\[
\sum_n \varepsilon_n \Delta_n f = \sum_n \varepsilon_n \sum_{j=\alpha_n}^{\alpha_n+1-1} \Delta_{\mathcal{G}}^j f
\]

for some sequence \((\alpha_n)\). By using inequalities (2.7) and (2.6) and writing \((S_{\mathcal{G}}f)^2 = \sum_j |\Delta_{\mathcal{G}}^j f|^2\), we obtain for \(p > 1\)

\[
\|Sf\|_1 \lesssim \|S_{\mathcal{G}}f\|_1 \leq \|S_{\mathcal{G}}f\|_p \lesssim \|f\|_p.
\]

This implies \(L_p \subset H_{1,k}\) for all \(p > 1\) and, by duality, \(\text{BMO}_k \subset L_p\) for all \(p < \infty\).

(2) If \((\mathcal{F}_n)\) is of the form that each \(\mathcal{F}_{n+1}\) is generated from \(\mathcal{F}_n\) by splitting exactly one atom of \(\mathcal{F}_n\) into two atoms of \(\mathcal{F}_{n+1}\) and under the condition \(\sup_n \gamma_{k-1}(\mathcal{F}_n) < \infty\) (which is stronger than \(\sup_n \gamma_k(\mathcal{F}_n) < \infty\)), it is shown in [9] that

\[
\|Sf\|_1 \simeq \|f\|_{H_1},
\]

where \(H_1\) denotes the atomic Hardy space on \([0, 1]\), i.e. in this case, \(H_{1,k}\) coincides with \(H_1\).

Acknowledgements Open access funding provided by Austrian Science Fund (FWF). It is a pleasure to thank P. F. X. Müller for very helpful conversations during the preparation of this paper. The author is supported by the Austrian Science Fund (FWF), Project F5513-N26, which is a part of the Special Research Program “Quasi-Monte Carlo Methods: Theory and Applications”.

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