OPTIMAL INVESTMENT WITH INSIDER INFORMATION USING SKOROKHOD & RUSSO-VALLOIS INTEGRATION

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Abstract. We study the maximization of the logarithmic utility for an insider with different anticipating techniques. Our aim is to compare the utilization of Russo-Vallois forward and Skorokhod integrals in this context. Theoretical analysis and illustrative numerical examples showcase that the Skorokhod insider outperforms the forward insider. This remarkable observation stands in contrast to the scenario involving risk-neutral traders. Furthermore, an ordinary trader could surpass both insiders if a significant negative fluctuation in the driving stochastic process leads to a sufficiently negative final value. These findings underline the intricate interplay between anticipating stochastic calculus and nonlinear utilities, which may yield non-intuitive results from the financial viewpoint.

1. Introduction

The development of investment strategies with insider information is an ongoing topic of financial mathematics ([PK96], [IPW01], [LNN03], [CIKHN04], [BØ05], [KH07], [DNØP09], [DØ15]), and it is strongly connected to advancements in the stochastic analysis theory, like Malliavin calculus ([Nua06], [NN18]), or anticipative integration and anticipative transformations ([Sko76], [Buc89], [RV93], [Buc94]). We aim to explore various interpretations of noise within the insider wealth dynamics and subsequently compare the outcomes to determine which interpretation aligns more closely with economic viability, and, in more generality, to derive consequences for financial modeling.

In this work, we compare the usage of Skorokhod [Sko76] and forward integration [RV93] in the situation that a trader has insider information about the future price of a given stock.
and desires to maximize her expected utility under a logarithmic risk aversion. She invests in a portfolio consisting of the stock and a risk-free asset until the time horizon $T$, which corresponds to the time of the privileged information. We assume the trader cannot influence the market prices.

We are inspired by the work of Escudero [Esc18], who addresses the problem of insider trading with one-period investment and without considering any utility function to model risk aversion (i.e., the traders are assumed to be risk-neutral). The author in [Esc18] compares the usage of Skorokhod and forward integration and concludes that the usage of the forward integral is more meaningful from the financial point of view because the expected utility of the insider trader under Skorokhod integration is less than that of an ordinary trader. On the contrary, the expected wealth under forward integration is bigger than that of the ordinary trader. By ordinary trader, we mean a trader who has no more information than the present and historical prices of the stock. Obviously, this work leaves open the case of a nonlinear utility; and this is precisely the question we will address herein: how the logarithmic utility interacts with both types of stochastic calculi. Surprisingly, the present results differ from those in [Esc18].

The outline of the paper is as follows. First, we develop the case in which the trader knows the exact value of the driving process of the stock price at the horizon time. Karatzas and Pikovsky [PK96] face this problem using a Brownian bridge with Itô integration and enlargement of filtration. In this work, we also start with an example of how to handle the problem using a Brownian bridge with Itô integration in Section 3.1. This case is devoted to show the consistency of our approach. Then, we continue with the usage of forward integration, based in Øksendal and Røse work [ØER17], in Section 3.2 and Skorokhod integration in Section 3.3. To handle the solution of the related stochastic differential equation in the Skorokhod scheme, we consider the anticipative Girsanov transformations studied by Buckdahn in [Buc89] and [Buc94]. We exemplify an investment assuming that a trader has insider information about the prices of the 2-Year U.S. Treasury Note Future, and conclude that with Skorokhod integration, the trader has more expected wealth. We also illustrate this fact with simulations of the Brownian paths that drive the risky asset and comparing the wealth of investments under these two types of anticipative integration.

Later on, in Section 3.6 we compare the expected differences in wealth of the investor when using forward and Skorokhod schemes in the presence of some uncertainty. We find that the expected value of the wealth under Skorokhod integration is bigger than its expected value under forward integration. Therefore, all of our current results point to the opposite direction than those in [Esc18] (with, of course, different hypotheses on risk aversion).
For the setting, we work on a probability space, \((\Omega, \mathcal{F}, \mathbb{P})\), equipped with \(\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}\), the natural filtration of the Brownian motion \(W_t, 0 \leq t \leq T\) for some \(T > 0\). The investment consists of a portfolio with a risk-free asset \(R_t\) modeled by

\[
dR_t = R_t r_t dt, \quad R_0 > 0, \quad t \in [0, T],
\]

where \(r_t\) is the risk-free instantaneous rate; and the risky asset \(S_t\), for which the trader has the mentioned information, modeled by a geometric Brownian motion

\[
dS_t = S_t \mu_t dt + S_t \sigma_t dB_t, \quad S_0 > 0, \quad t \in [0, T],
\]

where \(\mu_t\) is the appreciation rate and \(\sigma_t\) is the volatility of this risky asset, and \(B_t\) is a suitable driving process of \(S_t\), that depends on the standard Brownian motion \(W_t\). For each method we consider, we make a suitable choice for \(B_t\). In the case of the ordinary trader, we simply use \(B_t = W_t\). The alternative choice for the insider will be justified in section 3.1. We denote the proportion of the total wealth of the trader \(X_t\) invested in the stock at time \(t\) by \(\pi_t\); hereafter we will refer to this proportion as the “portfolio”. Therefore, as a consequence of the self-financing condition, the stochastic differential equation (SDE) for the wealth process of the trader is

\[
dX_t = (1 - \pi_t)X_t r_t dt + \pi_t(X_t \mu_t dt + X_t \sigma_t dB_t)
\]

\[
= X_t(\mu_t \pi_t + r_t(1 - \pi_t))dt + \sigma_t \pi_t X_t dW_t,
\]

with initial condition \(X_0 \in \mathbb{R}^+ (\mathbb{R}^+:=(0, \infty))\). We will eventually assume that no short selling is allowed, i.e. the value of \(\pi_t\) is between 0 and 1, since this condition will be necessary in order to find an interpretation of the optimal portfolio for the Skorokhod integral, although this is not needed for the forward one.

We denote by \(X_t^\pi\) the wealth of the trader under the portfolio \(\pi\). Our goal is to find the optimal portfolio \(\pi_t^*\) that maximizes the expected logarithmic terminal wealth at time \(T\),

\[
\pi_t^* := \arg \max \mathbb{E}[\log X_T^\pi].
\]

In his celebrated work [Mer69], Merton shows, with Itô integration, that if the driving process is the standard Brownian motion \(W_t\) and without more information than the historical prices, i.e. under the filtration \(\mathcal{F}_t, 0 \leq t \leq T\), the optimal value \(\pi_t^*\) that maximizes the expected logarithmic terminal wealth, \(\log X_T^\pi\), is

\[
\pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2},
\]
making the value of the optimization problem to be
\[
V_T^{\pi^*} := \mathbb{E} \left[ \log \left( \frac{X_T^{\pi^*}}{X_0} \right) \right]
\]
\[
= \mathbb{E} \int_0^T \left[ \mu_t \pi_t^* + r_t (1 - \pi_t^*) - \frac{1}{2} \sigma_t^2 \pi_t^* \right] dt
\]
under appropriate conditions on \( \pi_t, \mu_t, r_t, \) and \( \sigma_t \). This result gives us a benchmark to compare the value of the problem for an ordinary trader, who does not have privileged information, with the one of an insider trader. For the sake of simplicity, from now on we will always assume \( X_0 \in \mathbb{R}^+ \) (i.e. it is a positive constant).

2. Stochastic Analysis for Anticipative Processes

In this section, we provide the definitions and results within the field of stochastic analysis we need to apply for the portfolio optimization with anticipating information.

For a given time horizon \( T > 0 \), we work on a Wiener space \((\Omega, \mathcal{F}, \mathbb{P})\) over the sample space of continuous functions over \([0, T]\), where \( \mathcal{F} \) is the smallest Borel sigma-algebra that contains \( \Omega \), and \( \mathbb{P} \) is a Wiener measure under which the canonical process \( W_t(\omega) = \omega(t) = \omega_t, 0 \leq t \leq T \), is a standard Brownian motion. We let \( L_2(\Omega) \) denote the space of the square-integrable random variables on \( \Omega \).

2.1. The Malliavin Derivative.

Let \( \mathcal{S} \) be the space of smooth Wiener functionals in the sense that if a random variable \( F \) belongs to \( \mathcal{S} \), there exists \( n \in \mathbb{N} \) and \( n \) time points \( t_1, ..., t_n \) with \( 0 \leq t_1, ..., t_n \leq T \) and a smooth bounded function \( f \in C^\infty(\mathbb{R}^n) \) such that \( F \) is represented as \( F = f(W_{t_1}, ..., W_{t_n}) = f(w_1, ..., w_n) \).

For every smooth Wiener functional \( F \) in \( \mathcal{S} \), we define the unbounded linear operator \( D : L_2(\Omega) \rightarrow L_2([0, T] \times \Omega) \) given by
\[
D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(w_{t_1}, ..., w_{t_n}) \cdot \mathbb{I}_{[0,t_i]}(t), \ 0 \leq t_k \leq T,
\]
where \( \mathbb{I}_A(\cdot) \) is the characteristic function of set \( A \) such that \( \mathbb{I}_A(t) = 1 \) if \( t \in A \) and \( \mathbb{I}_A(t) = 0 \) otherwise. \( D_tF \) is called the Malliavin derivative of \( F \) at \( (t, \omega) \in [0, T] \times \Omega \). In general, we define the \( k \)-th derivative of \( F \), for \( k \geq 1, 0 \leq s_1, ..., s_k \leq 1 \), as:
\[
D_{s_1, ..., s_k}F = D_{s_1} ..., D_{s_k}F.
\]
The mapping $D$ is a closable unbounded linear operator from $L^2(\Omega)$ into $L^2([0, 1] \times \Omega)$ (see [Buc94]). We identify $D$ with its closed extension, and we denote its domain by $\mathbb{D}_{1,2}$. For any $k \geq 1$, $2 \leq p < \infty$, we introduce the spaces $\mathbb{D}_{k,p}$ as the closure of $\mathcal{S}$ with respect to the norm

$$\| F \|_{k,p} := \| F \|_p + \left( \int_{[0,1]^k} | D^k_x F |^2 \, dz \right)^{1/2}, F \in \mathcal{S}.$$ 

The concept of the Malliavin derivative leads us to define the Skorokhod integral in the following section.

### 2.2. The Skorokhod Integral.

Skorokhod ([Skor76]) introduces in 1976 a generalization of the Itô integral that coincides with the adjoint operator $\delta : L_2([0, T] \times \Omega) \to L_2(\Omega)$ of the derivative operator $D$ in the following sense: the domain $\text{Dom}(\delta)$ of the operator $\delta$ is the set of processes $u \in L_2([0, T] \times \Omega)$ for which there exists a random variable $G^u \in L_2(\Omega)$ satisfying the adjoint relationship

$$E[G^u F] = E \left[ \int_0^T u_s D_x F ds \right],$$

for all $F \in \mathcal{S}$. The random variable $G^u$ is uniquely determined in $L_2(\Omega)$ for every $u$, and it is called the Skorokhod integral of $u \in \text{Dom}(\delta)$, and denoted by $\delta(u) := G^u$. We also use the notation:

$$\delta(u) := \int_0^T u_s \delta W_s.$$ 

It is worth to mention that the Skorokhod integral also admits a characterization in terms of a Riemann sum approximation. See section 3.1.1 of [Nua06] for the details.

### 2.3. Anticipative Girsanov Transformations.

The anticipative Girsanov transformations allow us to solve stochastic differential equations of the form

$$X_t = X_0 + \int_0^t \hat{\mu}_s X_s ds + \int_0^t \hat{\sigma}_s X_s \delta W_s, \quad 0 \leq t \leq T,$$

where $\delta$ denotes Skorokhod integration. As always, $X_0 \in \mathbb{R}^+$. Buckdahn in [Buc89] shows that equation (2.3) with $\hat{\sigma}_t \in L_\infty([0, T])$ and $\hat{\mu}_t \in L_\infty([0, T] \times \Omega)$, $0 \leq t \leq T$, has a unique solution in the following sense:

(i) $\mathbb{1}_{[0, t]} \hat{\sigma} X \in \text{Dom}(\delta)$, and
(ii) the Skorokhod integral satisfies $\mathbb{P}$-a.s.

$$\delta(\mathbb{1}_{[0, t]} \hat{\sigma} X) = \int_0^t \hat{\sigma}_s X_s \delta W_s = X_t - X_0 - \int_0^t \hat{\mu}_s X_s ds; \quad 0 \leq t \leq T.$$
To find the solution, the author uses the anticipative Girsanov transformations, which we present hereunder.

For a deterministic process \( \hat{\sigma}_t \in L_\infty([0, T]) \), we define the family of transformations \( U_{s,t} : \Omega \rightarrow \Omega, \ 0 \leq s \leq t \leq T \), of \( \omega \in \Omega \), shifted with respect to \( \mathbb{I}_{[s,t]}(r) \cdot \hat{\sigma}_r \), given by

\[
U_{s,t} : \{ \omega_v, 0 \leq v \leq T \} \mapsto \left\{ U_{s,t} \omega_v := \omega_v - \int_0^v \mathbb{I}_{[s,t]}(r) \hat{\sigma}_r dr, \ 0 \leq v \leq T \right\}.
\]

We let \( T_{s,t} \) denote the inverse transformation of \( U_{s,t} \), given by

\[
T_{s,t} \omega := (U_{s,t} \circ T_{s,t}) \omega.
\]

for every fixed \( 0 \leq s \leq t \leq T \). For ease of notation, we write \( U_t = U_{0,t}, \ T_t = T_{0,t} \). And we use that \( T_{s}U_t \omega = U_{s,t} \omega \) for \( \hat{\sigma}_t \in L_\infty([0, T]) \), \( 0 \leq s \leq t \leq T \), since

\[
T_{s}U_t \omega. = T_{s} \left( \omega. - \int_0^t \mathbb{I}_{[0,t]} \hat{\sigma}_r dr \right) = \left[ \omega. - \int_0^t \mathbb{I}_{[0,t]} \hat{\sigma}_r dr \right] + \int_0^t \mathbb{I}_{[0,t]} \hat{\sigma}_r dr
\]

\[
= \omega. - \int_0^t \mathbb{I}_{[s,t]} \hat{\sigma}_r dr - \int_0^s \mathbb{I}_{[0,s]} \hat{\sigma}_r dr + \int_0^s \mathbb{I}_{[0,s]} \hat{\sigma}_r dr
\]

\[
= U_{s,t} \omega.
\]

The solution of (2.3), given by Buckdahn in [Buc89], is represented by

\[
X_t = X_0 \cdot \exp \left\{ \int_0^t \hat{\mu}_s(U_{s,t}) ds \right\} L_t, \ \mathbb{P} - \text{ a.s., } 0 \leq t \leq T,
\]

where \( L_t = \exp \left\{ \int_0^t \hat{\sigma}_s dW_s - \frac{1}{2} \int_0^t \hat{\sigma}_s^2 ds \right\} \).

Later on, in [Buc94], Buckdahn defines, for stochastic \( \sigma \), the transformation \( U'_{s,t} : \Omega \rightarrow \Omega, \ 0 \leq s \leq t \leq T \), of \( \omega \in \Omega \), shifted with respect to \( \mathbb{I}_{[s,t]}(r) \cdot \hat{\sigma}_r(U'_{r,t} \omega) \), given by

\[
U'_{s,t} \omega. = \omega. - \int_0^t \mathbb{I}_{[s,t]}(r) \cdot \hat{\sigma}_r(U'_{r,t} \omega.) dr,
\]

and shows that the linear SDE (2.3) with \( \hat{\sigma}_t \in L_2([0, T] \times \Omega) \) and \( \hat{\mu}_t \in L_\infty([0, T] \times \Omega) \), has the unique solution

\[
X_t = X_0 \cdot \exp \left\{ \int_0^t \hat{\mu}(U'_{s,t}) ds \right\} L'_t, \ \mathbb{P} - \text{ a.s., } 0 \leq t \leq T,
\]
where
\[
L'_t = \exp\left\{ \int_0^t \hat{\sigma}_s(U'_{s,t}) \delta W_s - \frac{1}{2} \int_0^t \hat{\sigma}_s(U'_{s,t})^2 ds - \int_0^t \int_s^t (Du) \hat{\sigma}_s(U'_{s,t}) D_s[\hat{\sigma}_u(U'_{u,t})] \, du \, ds \right\}.
\] (5)

The last expressions are closed for deterministic $\hat{\sigma}_t$, but not for stochastic $\hat{\sigma}_t$.

2.4. The Forward Integral.

Russo and Vallois define in 1993 the forward integral with respect to Brownian motion by an approximation procedure [RV93].

**Definition 2.1.** A stochastic process $\phi_t$, $t \in [0, T]$, is said to be forward integrable in the weak sense with respect to a standard Brownian motion $W_t$, if there exists another stochastic process $I_t$ such that
\[
\sup_{0 \leq t \leq T} \left| \int_0^t \phi_s \frac{W_{s+\epsilon} - W_s}{\epsilon} \, ds - I_t \right| \to 0, \quad \epsilon \to 0^+
\]
in probability. If such a process exists, we denote
\[
I_t := \int_0^t \phi_s \, d^- W_s, \quad t \in [0, T],
\]
the forward integral of $\phi_t$ with respect to $W_t$ over $[0, T]$.

The forward integral is an extension of the Itô integral. If $\phi$ is adapted to the filtration $\mathcal{F}_t$ and Itô integrable, then $\phi$ is forward integrable and its forward integral coincides with its Itô integral. The proof of this statement is in [RV93].

A forward process (with respect to $W_t$) is a stochastic process of the form
\[
X_t = x + \int_0^t u_s \, ds + \int_0^t v_s \, d^- W_s, \quad t \in [0, T],
\]
where $\int_0^T |u_t| \, ds < \infty \, a.s.$ and $v$ is a forward integrable stochastic process. A shorthand notation for this is
\[
d^- X_t = u_t \, dt + v_t \, d^- W_t.
\]

We present the Itô formula for forward integrals as stated in [DNOP09], page 136. See also [RV93] and [FR02].

**Theorem 2.2.** Let $X_t$, $t \in [0, T]$, be a forward process defined as above and let $f \in C^{1,2}([0, T] \times \mathbb{R})$. Define $Y(t) = f(t, X_t)$. Then, $Y_t$ is a forward process and
\[
d^- Y_t = \frac{\partial f}{\partial t}(t, X_t) \, dt + \frac{\partial f}{\partial x}(t, X_t) \, d^- X_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) v_t^2 \, dt.
\]
In order to have a Riemann sum interpretation of the forward integral, take a partition $J_n$ of $0 = t_0 < t_1 < ... < t_{J_n} = T$ of $[0,T]$. Assume $\varphi$ is càdlàg (i.e. left continuous with right limits) and forward integrable, and moreover a simple stochastic process, meaning that

$$\varphi(t) = \sum_{j=1}^{J_n} \varphi(t_{j-1}) \chi(t_{j-1},t_j)(t), \quad t \in [0,T];$$

then, the following identity

$$\int_0^T \varphi(s) d^-W(s) = \lim_{\Delta t \to 0} \sum_{j=1}^{J_n} \varphi(t_{j-1})(W(t_j) - W(t_{j-1}));$$

holds with convergence in probability, where $\Delta t := \max_{j=1,...,J_n}(t_j - t_{j-1}) \to 0$ as $n \to \infty$. For details, see, for instance, reference [BØ05].

Based on the previous convergence, when the integrand is adapted, the Riemann sums serve as an approximation to the Itô integral with respect to Brownian motion. Consequently, the forward integral and the Itô integral coincide. Therefore, we can view the forward integral as an extension of the Itô integral to the anticipating context. Note that this property is shared with the Skorokhod integral, although it constitutes a different extension.

**Relationship between Forward and Skorokhod Integration.**

There is a relation between the forward and Skorokhod integrals that allows to compute forward integrals in terms of the Skorokhod integrals and Malliavin derivatives. To make it explicit, we follow Chapter 8 in [DNØP09] to introduce the definition of the forward integral in the strong sense and the class of stochastic processes $D_0$. Subsequently, we state this relation along with a consequence of it.

**Definition 2.3.** The class $D_0$ consists of all measurable stochastic processes $\varphi$ such that

1) the trajectories $\varphi(\cdot,\omega) : t \to \varphi(t,\omega)$ are càdlàg a.s.
2) the random variables $\varphi(t) \in D_{1,2}$ for all $t \in [0,T]$.
3) the trajectories $t \to D_s \varphi(t)(\omega)$ are càdlàg for almost every $s \in [0,T]$ a.s.
4) the limit $D_{t+} \varphi(t) := \lim_{s \to t+} D_s \varphi(t)$ exists with convergence in $L^2(\mathbb{P})$.
5) $\varphi$ is Skorokhod integrable.

**Definition 2.4.** A stochastic process $\phi_t$, $t \in [0,T]$, is said to be forward integrable in the strong sense with respect to a standard Brownian motion $W_t$ if the limit

$$\lim_{\varepsilon \to 0^+} \int_0^T \varphi(t) W(t + \varepsilon) - W(t) dt$$

exists in $L^2(\mathbb{P})$.

The above mentioned relation reads as follows:
Theorem 2.5. Let $\varphi$ be a process in $\mathbb{D}_0$. Then, $\varphi$ is forward integrable in the strong sense and moreover
\[
\int_0^T \varphi(t) d^-W(t) = \int_0^T \varphi(t) \delta W(t) + \int_0^T D_{t+} \varphi(t) dt.
\]

By the zero-mean property of the Skorokhod integral we get the immediate consequence:

Corollary 2.6. Let $\varphi$ be a process in $\mathbb{D}_0$. Then
\[
\mathbb{E} \left[ \int_0^T \varphi(t) d^-W(t) \right] = \mathbb{E} \left[ \int_0^T D_{t+} \varphi(t) dt \right].
\]

2.5. Donsker Delta Function.

In this section, we define the space of Hida distributions, which is needed to define the Donsker delta function. To this end, we follow Chapter 6 of [DNØP09].

We start defining the Hermite polynomials $h_n(x)$ of degree $n$, which read
\[
h_n(x) := (-1)^n e^{\frac{1}{2} x^2} \frac{d^n}{dx^n} \left( e^{-\frac{1}{2} x^2} \right), \quad n = 0, 1, 2, \ldots
\]

Let $e_k$ be the $k$-th Hermite function defined by
\[
e_k(x) := \pi^{-\frac{1}{4}} ((k-1)!)^{-\frac{1}{2}} e^{-\frac{1}{2} x^2} h_{k-1} \left( \sqrt{2}x \right), \quad k = 1, 2, \ldots
\]

and additionally define
\[
\theta_k(\omega) := \langle \omega, e_k \rangle = \int_\mathbb{R} e_k(x) dW(x, \omega), \quad \omega \in \Omega.
\]

Let $\mathcal{J}$ denote the set of all finite multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m), \ m = 1, 2, \ldots$, of non-negative integers $\alpha_i$. We set
\[
H_{\alpha}(\omega) := \prod_{j=1}^m h_{\alpha_j} (\theta_j(\omega)), \quad \omega \in \Omega,
\]

and $H_0 := 1$, for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathcal{J}, \ \alpha \neq 0$.

Let $S = S(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing $C^\infty(\mathbb{R}^d)$ real functions on $\mathbb{R}^d$. We define the Hida test function space $(S)$ as the space
\[
(S) = \bigcap_{k \in \mathbb{R}} (S)_k,
\]

where $f = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in L^2(P), \ a_\alpha \in \mathbb{R}$, belongs to the Hida test function Hilbert space $(S)_k$ whenever
\[
\| f \|^2_k := \sum_{\alpha \in \mathcal{J}} a_\alpha^2 (2N)^{ak} < \infty,
\]
where
\[(2N)\alpha = \prod_{j=1}^{m}(2j)^{\alpha_j}, \text{ for } \alpha = (\alpha_1, ..., \alpha_m) \in \mathcal{J}.
\]

We finally define the Hida distribution space \((S)^*\) as the dual space of \((S)\). This is the adequate framework to introduce the Donsker delta function.

**Definition 2.7.** Let \(Y : \Omega \to \mathbb{R}\) be a random variable that belongs to the Hida distribution space \((S)^*\). Then, a continuous function \(\delta_Y(\cdot) : \mathbb{R} \to (S)^*\) is called Donsker delta function of \(Y\) if it has the property that
\[
\int_{\mathbb{R}} g(y)\delta_Y(y)dy = g(Y) \text{ a.s.}
\]
for all measurable functions \(g : \mathbb{R} \to \mathbb{R}\) such that the integral converges. Herein the integral in the left is interpreted as a Bochner integral.

For explicit representations of the Donsker delta function see, for instance, Chapter 7 in [DNOP09].

3. Anticipative Portfolio Optimization (APO)

We formulate the insider trading problem in a Black-Scholes market with two assets, as anticipated in the Introduction. We suppose the insider has additional information about the underlying noise at the horizon time, specifically that the driving process takes the value \(b \in \mathbb{R}\) at time \(T\); note that this value is assumed to be a constant rather than a random variable. In order to model the insider knowledge, we use the so called generalized Brownian bridge ending in \(b\) (which is constant but otherwise arbitrary) as the driving process. More precisely, we consider the conditioned Gaussian process
\[
(B_t | B_0 = 0, B_T = b), \quad t \in [0, T],
\]
which is characterized by its mean \(bt/T\) for each time \(t\) and its autocorrelation function \(s(1 - t/T)\) between the temporal points \(s\) and \(t\).

A simple representation of this Brownian bridge is given by (see [RW00], page 86)
\[
(6) \quad \bar{B}_t = W_t - (W_T - b)\frac{t}{T}, \quad t \in [0, T],
\]
where \(W_t\) stands for the standard Brownian motion. The SDE for (3) is
\[
(7) \quad d\bar{B}_t = dW_t - \frac{W_T - b}{T}dt, \quad t \in [0, T], \quad \bar{B}_0 = 0.
\]

We use this representation in the forward and Skorokhod schemes later on. Another representation of the generalized Brownian bridge is given by (see [Kle05], page 132)
\[
(8) \quad \tilde{B}_t = \int_{0}^{t} \frac{T - t}{T - s}dW_s + b\frac{t}{T}, \quad t \in [0, T),
\]
which we use for the first example of APO, and that satisfies the following SDE

\[ d\widehat{B}_t = dW_t - \frac{\widehat{B}_t - b}{T-t} \, dt, \quad t \in [0,T), \quad \widehat{B}_0 = 0. \]

Note that this equation has all terms adapted, unlike equation (3); in particular, these equations give rise to two stochastic processes that share the same law, but cannot be compared at the pathwise level. Although at this time both SDEs can be interpreted samplewise, given the additive nature of their noise, this fact will be crucial in the following sections. Indeed, it will allow us to compare the different notions of anticipating stochastic calculus in the present financial context.

To optimize the portfolio for each integral, we first give the driving process, then solve the resulting SDE, afterwards compute the portfolio that maximizes the value of the problem, and finally compute this value with the detected optimal portfolio. This, in particular, allows us to compare the different results that arise from the two notions of anticipating stochastic integration. We start with the simplest case, which shows that modeling the driving stochastic process as a Brownian bridge reproduces the classical results on the subject, see [PK96].

3.1. APO with Brownian Bridge.

We start with an example of how the problem can be handled using the representation of the generalized Brownian bridge given in (3). In this case, the wealth process of the insider trader is modeled by

\begin{align*}
    dX_t &= (1 - \pi_t)X_t \, r_t \, dt + \pi_t X_t (\mu_t \, dt + \sigma_t d\widehat{B}_t), \\
    X_0 &\in \mathbb{R}^+, 
\end{align*}

where we assume that \( \mu_t, r_t, \sigma_t \in L^\infty([0,T]) \), with \( \sigma_t > 0 \), are deterministic and \( \widehat{B}_t \) is given by (3).

Theorem 3.1. Let \( \pi_t \in L^2([0,T]) \) be a deterministic function of time. Then, the optimal portfolio that maximizes \( \mathbb{E}[\log(X_T/X_0)] \), where \( X_t \) solves (3.1), is

\[ \pi^*_t = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{b}{\sigma_t T}, \quad t \in [0,T], \]

and the corresponding value is

\[ V_T^{\pi^*} = \mathbb{E} \int_0^T \left[ r_t + \frac{1}{2} \left( \frac{\mu_t - r_t}{\sigma_t} + \frac{b}{T} \right)^2 \right] dt. \]
Proof. From (3) and (3.1) we find

\[ dX_t = r_t (1 - \pi_t) X_t \, dt + \pi_t X_t \left[ \mu_t \, dt + \sigma_t \left( dW_t - \frac{\hat{B}_t - b}{T - t} \, dt \right) \right] \]

\[ = r_t (1 - \pi_t) X_t \, dt + \pi_t X_t \left[ \mu_t \, dt - \sigma_t \frac{\hat{B}_t - b}{T - t} \, dt + \sigma_t dW_t \right] \]

\[ = \left[ r_t (1 - \pi_t) + \pi_t \mu_t - \pi_t \sigma_t \frac{\hat{B}_t - b}{T - t} \right] X_t \, dt + \pi_t \sigma_t X_t \, dW_t, \]

which is an Itô stochastic differential equation. We may therefore use Itô lemma for \( \log X_t \) to obtain

\[ d \log X_t = \left[ r_t (1 - \pi_t) + \pi_t \mu_t - \pi_t \sigma_t \frac{\hat{B}_t - b}{T - t} - \frac{1}{2} \pi_t^2 \sigma_t^2 \right] dt + \pi_t \sigma_t \, dW_t. \]

Taking the expectation of the integral form, we have that the value of the problem is given by

\[ \mathbb{E} [\log(\frac{X_T}{X_0})] = \mathbb{E} \left[ \int_0^T \left[ r_t (1 - \pi_t) + \pi_t \mu_t - \pi_t \sigma_t \frac{\hat{B}_t - b}{T - t} - \frac{1}{2} \pi_t^2 \sigma_t^2 \right] \, dt \right] \]

\[ + \int_0^T \pi_t \sigma_t \, dW_t \]

\[ = \int_0^T \left[ r_t (1 - \pi_t) + \pi_t \mu_t + \pi_t \sigma_t \frac{b}{T} - \frac{1}{2} \pi_t^2 \sigma_t^2 \right] \, dt, \]

since \( \mathbb{E} \left[ \int_0^T \pi_t \sigma_t \, dW_t \right] = 0 \) and \( \mathbb{E} \left[ \frac{b - \hat{B}_t}{T - t} \right] = \frac{b}{T} \). Then, we consider the maximization of the term

\[ J_t^* = r_t (1 - \pi_t) + \pi_t \mu_t + \pi_t \sigma_t \frac{b}{T} - \frac{1}{2} \pi_t^2 \sigma_t^2. \]

Considering the first derivative of \( J_t^* \), we have that the portfolio that maximizes it, is

\[ \pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{b}{\sigma_t T}, \quad t \in [0, T]. \]
To know the value of the problem, we compute

\[
J_{t}^{\pi^*} = r_{t} + (\mu_{t} - r_{t}) \pi_{t}^{*} + \pi_{t}^{*} \sigma_{t} \frac{b}{T} - \frac{1}{2} \pi_{t}^{*} \sigma_{t}^{2}
\]

\[
= r_{t} + (\mu_{t} - r_{t}) \left( \frac{\mu_{t} - r_{t}}{\sigma_{t}^{2}} + \frac{b}{\sigma_{t} T} \right) + \left( \frac{\mu_{t} - r_{t}}{\sigma_{t}^{2}} \right) \sigma_{t} \frac{b}{T}
\]

\[
- \frac{1}{2} \left( \frac{\mu_{t} - r_{t}}{\sigma_{t}^{2}} + \frac{b}{\sigma_{t} T} \right)^{2} \sigma_{t}^{2}
\]

\[
= r_{t} + \frac{(\mu_{t} - r_{t})^{2}}{\sigma_{t}^{2}} + \frac{(\mu_{t} - r_{t}) b}{\sigma_{t} T} + \frac{b^{2}}{T^{2}}
\]

\[
- \frac{1}{2} \left[ \frac{(\mu_{t} - r_{t})^{2}}{\sigma_{t}^{2}} + 2 \frac{(\mu_{t} - r_{t}) b}{\sigma_{t} T} + \frac{b^{2}}{T^{2}} \right]
\]

\[
= r_{t} + \frac{(\mu_{t} - r_{t})^{2}}{\sigma_{t}} + \frac{2(\mu_{t} - r_{t}) b}{\sigma_{t} T} + \frac{b^{2}}{T} - \frac{1}{2} \left( \frac{\mu_{t} - r_{t}}{\sigma_{t}} \right)^{2} - \frac{(\mu_{t} - r_{t}) b}{\sigma_{t} T} - \frac{1}{2} \frac{b^{2}}{T^{2}}
\]

\[
= r_{t} + \frac{1}{2} \left[ \frac{\mu_{t} - r_{t}}{\sigma_{t}} + \frac{b}{T} \right]^{2}, \text{ for every } t \in [0, T].
\]

And the value of the problem is

\[
V_{T}^{\pi^*} = \mathbb{E} \int_{0}^{T} \left[ r_{t} + \frac{1}{2} \left( \frac{\mu_{t} - r_{t}}{\sigma_{t}} + \frac{b}{T} \right)^{2} \right] dt.
\]

Since all the parameters are deterministic, the statement follows.

\[
\square
\]

**Remark 3.2.** Note that these results are fully compatible with those in [PK96].

### 3.2. APO with the Forward Integration Method

In this section, we present the portfolio optimization process with forward integration. First, we use deterministic portfolios and parameters as in the previous section.

We now suppose that the driving process \( B_{t} \) of \( S_{t} \) in (1) is given by \( \bar{B}_{t} = W_{t} - (W_{T} - b) \frac{t}{T} \), \( t \in [0, T] \), as in (3), and \( \mu_{t}, r_{t}, \sigma_{t}, \pi_{t} \) are deterministic, just like in the last case studied. If these parameters are regular enough (what will not be necessarily assumed in the following), then, by the self-financing property and Theorem 8.12 in [DNOP09], the wealth process of the insider trader is given (after substitution of the Brownian bridge as the forcing process)
by the forward process

\[
\begin{align*}
  d^-X_t &= r_t(1 - \pi_t)X_t \ dt + \pi_tX_t \left[ \mu_t \ dt + \sigma_t d^-\bar{B}_t \right], \\
  X_0 &\in \mathbb{R}^+,
\end{align*}
\]

(10)

where \(d^-\bar{B}_t = d^-W_t - \frac{W_T - b}{T} \ dt, \ t \in [0, T]\). This is the model we will use for the portfolio optimization.

**Remark 3.3.** Note that (3.2) is not an Itô SDE due to its dependence on the future value of the Brownian motion, \(W_T\), in the drift. And thus the need to use forward integration in this model.

**Theorem 3.4.** Let \(\pi_t \in L^2([0, T])\) be a deterministic function of time. Then the optimal portfolio that maximizes \(\mathbb{E}[\log(X_T/X_0)]\), where \(X_t\) solves (3.2), is

\[
\pi^*_t = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{b}{\sigma_t T}, \quad t \in [0, T],
\]

and the corresponding value is

\[
V^*_T = \mathbb{E} \left[ \int_0^T \left[ r_t + \frac{1}{2} \left( \frac{\mu_t - r_t}{\sigma_t} + \frac{b}{T} \right)^2 \right] dt \right]
\]

Taking the expectation of the integral form, we have that the value of the problem is given by

\[
\mathbb{E}[\log(X_T/X_0)] = \mathbb{E} \left[ \int_0^T \left[ r_t(1 - \pi_t) + \pi_t \mu_t - \pi_t \sigma_t \frac{W_T - b}{T} - \frac{1}{2} \pi^2_t \sigma^2_t \right] dt + \int_0^T \pi_t \sigma_t d^-W_t \right] 
\]

\[
= \int_0^T \left[ r_t(1 - \pi_t) + \pi_t \mu_t + \pi_t \sigma_t \frac{b}{T} - \frac{1}{2} \pi^2_t \sigma^2_t \right] dt,
\]

We apply Itô formula for forward integrals (as we see in Theorem 2.2) to \(\log X_t\) and find

\[
d^- \log X_t = \left[ r_t(1 - \pi_t) + \pi_t \mu_t - \pi_t \sigma_t \frac{W_T - b}{T} - \frac{1}{2} \pi^2_t \sigma^2_t \right] dt + \pi_t \sigma_t d^-W_t.
\]

Taking the expectation of the integral form, we have that the value of the problem is given by

\[
\mathbb{E}[\log(X_T/X_0)] = \mathbb{E} \left[ \int_0^T \left[ r_t(1 - \pi_t) + \pi_t \mu_t - \pi_t \sigma_t \frac{W_T - b}{T} - \frac{1}{2} \pi^2_t \sigma^2_t \right] dt + \int_0^T \pi_t \sigma_t d^-W_t \right] 
\]

\[
= \int_0^T \left[ r_t(1 - \pi_t) + \pi_t \mu_t + \pi_t \sigma_t \frac{b}{T} - \frac{1}{2} \pi^2_t \sigma^2_t \right] dt.
\]
since \( \sigma_t \) and \( \pi_t \) are deterministic, and then

\[
\mathbb{E} \left[ \int_0^T \pi_t \sigma_t dW_t \right] = \mathbb{E} \left[ \int_0^T \pi_t \sigma_t dW_t \right] = 0.
\]

In this case, we have to maximize

\[
J_t^* = r_t (1 - \pi_t) + \pi_t \mu_t + \pi_t \sigma_t b T - \frac{1}{2} \pi_t^2 \sigma_t^2.
\]

The value of \( \pi_t, t \in [0, T] \), that maximizes \( J_t^* \) is

\[
\pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{b}{\sigma_t T}, \quad t \in [0, T].
\]

By direct substitution, we obtain the value of the problem, which is

\[
V_T^\pi^* = \mathbb{E} \int_0^T \left[ r_t + \frac{1}{2} \left( \frac{\mu_t - r_t}{\sigma_t} + \frac{b}{T} \right)^2 \right] dt,
\]

and since the integrand is deterministic, the statement follows.

\[\square\]

**Remark 3.5.** Note that term \( b/T \) in (3.2) is equivalent to

\[
\mathbb{E} \left[ \frac{b - \bar{B}_t}{T - t} \right] = \frac{b - \mathbb{E}(\bar{B}_t)}{T - t}.
\]

This is a usual term we have to add to the ratio \( \frac{\mu_t - r_t}{\sigma_t} \), which represents the extra information we have at time \( t \). In the next section, allowing stochastic parameters, we will have in this term the current value of the driving process instead of its expected value.

**Remark 3.6.** Note that both Theorems 3.1 and 3.4 lead to the same results. This highlights the consistency of both approaches.

**APO with stochastic parameters**

Now we face the problem allowing \( \sigma_t, \mu_t, r_t \in L^\infty([0, T] \times \Omega) \) and \( \pi_t \in L^2([0, T] \times \Omega) \) be stochastic parameters. For that, we present a portfolio optimization that follows the procedure presented by Øksendal and Røse in [ØER17] (see also Chapter 8 in [DNØP09]).

We note that the random variable \( W_T \) obviously belongs to the Hida distribution space, and it also has a Malliavin differentiable Donsker delta function (Proposition 7.2, [DNØP09]).

We use an enlargement of filtration representing the insider information by

\[
\mathcal{G} := \{ \mathcal{G}_t : \mathcal{G}_t = \mathcal{F}_t \vee \sigma(W_T), \quad t \in [0, T], \quad T > 0 \},
\]

where \( \mathcal{F} \) is the natural filtration (the filtration of an ordinary trader). Inspired by the self-financing property, we postulate the SDE

\[
d^-X_t = r_t (1 - \pi_t) X_t \ dt + \pi_t X_t \left[ \mu_t \ dt + \sigma_t d^-\bar{B}_t \right],
\]

\[
X_0 \in \mathbb{R}^+,
\]

\[
X_T = \mathcal{G}_T.
\]
as a model of the insider wealth process $X^\pi_t$, where $\bar{B}_t$ is defined as in (3). Then, the solution of this SDE is

$$\log\left(\frac{X_T}{X_0}\right) = \int_0^t \left[ r_s(1 - \pi_s) + \pi_s \mu_s - \pi_s \sigma_s \frac{W_T - b}{T} - \frac{1}{2} \pi_s^2 \sigma_s^2 \right] ds$$

$$+ \int_0^t \pi_s \sigma_s dW_s.$$

**Theorem 3.7.** Let $\pi_t \in L^2([0, T] \times \Omega)$ be $G_t$-adapted and $\sigma_t, \mu_t, r_t \in L^\infty([0, T] \times \Omega)$ be $\mathcal{F}_t$-adapted. Moreover assume that $\sigma_t \pi_t \in \mathbb{D}_0$ in the sense of Definition 2.3 and it is forward integrable in the strong sense. Then, whenever it exists, the optimal portfolio that maximizes $\mathbb{E} [\log(\frac{X_T}{X_0})]$, where $X_t$ solves (3.2), is

$$\pi^*_t = \frac{\mu_t - r_t}{\pi_t^2} + \frac{b - \bar{B}_t}{\sigma_t(T - t)}, \quad t \in [0, T),$$

and the corresponding value is

$$V^*_{T} = \mathbb{E} \int_0^T \left[ rt + \frac{1}{2} \left( \frac{\mu_t - r_t}{\sigma_t} + \frac{b - \bar{B}_t}{T - t} \right)^2 \right] dt.$$

**Proof.** To express the value of the problem, we use the Corollary 2.6 that relates the forward integral with the Malliavin derivative, along with the tower property, to find that

$$\mathbb{E} \log(\frac{X_T}{X_0}) = \mathbb{E} \left[ \int_0^T \left( r_t + (\mu_t - r_t)\pi_t - \sigma_t \frac{W_T - b}{T} \pi_t - \frac{1}{2} \pi_t^2 \sigma_t^2 + \sigma_t D_t \pi_t \right) dt \right]$$

$$= \mathbb{E} \left[ \int_0^T \mathbb{E} \left[ r_t + (\mu_t - r_t)\pi_t - \sigma_t \frac{W_T - b}{T} \pi_t - \frac{1}{2} \pi_t^2 \sigma_t^2 + \sigma_t D_t \pi_t \big| \mathcal{F}_t \right] dt \right].$$

To proceed with the maximization with respect to $\pi_t$, we use the notation $\pi_t = f(t, Y)$, where $Y = W_T$. Then, we need to maximize

$$J(f) := \mathbb{E} \left[ (\mu_t - r_t)f(t, Y) - \sigma_t \frac{Y - b}{T} f(t, Y) - \frac{1}{2} f^2(t, Y) \sigma_t^2 \right]$$

$$+ \sigma_t D_t f(t, Y) \big| \mathcal{F}_t \right].$$

To that end, we follow [ØER17] to express $Y = W_T$ in terms of a Malliavin differentiable Donsker delta function $\delta_Y(y)$:

$$f(t, Y) = \int_0^T f(t, y) \delta_Y(y) dy,$$

$$Y f(t, Y) = \int_0^T y f(t, y) \delta_Y(y) dy,$$

$$f^2(t, Y) = \int_0^T f^2(t, y) \delta_Y(y) dy,$$
\[ D_s f(t, Y) = \int_0^T f(t, y) D_s \delta Y(y) dy. \]

We substitute these expressions in (3.2) to obtain

\[
J(f) = \mathbb{E} \left[ (\mu_t - r_t) \int_0^T f(t, y) \delta w(y) dy - \sigma_t \int_0^T \frac{y - b}{T} f(t, y) \delta w(y) dy \right. \\
- \frac{1}{2} \sigma_t^2 \int_0^T f^2(t, y) \delta w(y) dy + \sigma_t \int_0^T f(t, y) D_t \delta w(y) dy \bigg| \mathcal{F}_t \bigg]
\]

\[
= \int_0^T \left\{ (\mu_t - r_t) f(t, y) \mathbb{E} [\delta w(y) | \mathcal{F}_t] - \sigma_t \frac{y - b}{T} f(t, y) \mathbb{E} [\delta w(y) | \mathcal{F}_t] \\
- \frac{1}{2} \sigma_t^2 f^2(t, y) \mathbb{E} [\delta w(y) | \mathcal{F}_t] + \sigma_t f(t, y) \mathbb{E} [D_t \delta w(y) | \mathcal{F}_t] \right\} dy
\]

To find the value \( f^*(t, y) \) that maximizes \( J(f) \), we write

\[
(\mu_t - r_t - \sigma_t \frac{y - b}{T}) \mathbb{E} [\delta w(y) | \mathcal{F}_t] - \sigma_t^2 f^* (t, y) \mathbb{E} [\delta w(y) | \mathcal{F}_t] + \sigma_t \mathbb{E} [D_t \delta w(y) | \mathcal{F}_t] = 0,
\]

since \( J \) is a parabola on \( f \). This implies that

\[
f^*(t, y) = \frac{(\mu_t - r_t - \sigma_t \frac{y - b}{T}) \mathbb{E} [\delta w(y) | \mathcal{F}_t] + \sigma_t \mathbb{E} [D_t \delta w(y) | \mathcal{F}_t]}{\sigma_t^2 \mathbb{E} [\delta w(y) | \mathcal{F}_t]} \\
= \frac{\mu_t - r_t}{\sigma_t^2} \frac{y - b}{\sigma_t T} + \frac{\mathbb{E} [D_t \delta w(y) | \mathcal{F}_t]}{\sigma_t \mathbb{E} [\delta w(y) | \mathcal{F}_t]},
\]

and therefore,

\[
f^*(t, Y) = \frac{\mu_t - r_t}{\sigma_t^2} \frac{y - b}{\sigma_t T} \frac{W_T - W_t}{T - t} + \frac{W_T - W_t}{\sigma_t (T - t)}
\]

where we used that the quotient \( \frac{\mathbb{E} [D_t \delta w(y) | \mathcal{F}_t]}{\mathbb{E} [\delta w(y) | \mathcal{F}_t]} \) equals to \( \frac{W_T - W_t}{T - t} \) (see [AØU01]).
Then, the portfolio \( \pi_t^* \) that maximizes \( E[\ln X^\pi(T)] \) is

\[
\pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2} - \frac{W_T - b}{\sigma_t T} + \frac{W_T - W_t}{\sigma_t (T - t)}
\]

\[
= \frac{\mu_t - r_t}{\sigma_t^2} - \frac{T(W_T - b) - t(W_T - b) - T(W_T) + TW_t}{\sigma_t (T - t)}
\]

\[
= \frac{\mu_t - r_t}{\sigma_t^2} - \frac{T \bar{B}_t - Tb}{\sigma_t (T - t)};
\]

where we have used that \( T \bar{B}_t = TW_t - t(W_T - b). \) Therefore,

\[
\pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{b - \bar{B}_t}{\sigma_t (T - t)}, \quad t \in [0, T).
\]

To find the value of the portfolio, consider the equality

\[
J_t^\pi = r_t + (\mu_t - r_t) \pi_t^* + \sigma_t \pi_t^* \frac{b - W_t}{T} - \frac{1}{2} \pi_t^* \sigma_t^2 + \sigma_t D_t \pi_t^*;
\]

to compute

\[
V_T^\pi = E \int_0^T \left( J_t^\pi \right) dt.
\]

First, we write the portfolio in terms of the Wiener process

\[
\pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{b - W_t}{\sigma_t T} + \frac{W_t - W_t}{\sigma_t (T - t)};
\]

to compute its Malliavin derivative

\[
D_t \pi_t^* = \frac{1}{\sigma_t} \left( \frac{1}{T - t} - \frac{1}{T} \right) = \frac{t}{\sigma_t T (T - t)}.
\]
Substituting these expressions leads to

\[ J^\pi_t = r_t + \left( \frac{\mu_t - r_t}{\sigma_t} \right)^2 + \frac{\mu_t - r_t}{\sigma_t} \frac{b - W_t}{T} + \frac{\mu_t - r_t}{\sigma_t} \frac{W_T - W_t}{T - t} \]

\[ + \frac{\mu_t - r_t}{\sigma_t} \frac{b - W_t}{T} + \left( \frac{b - W_t}{T} \right)^2 + \frac{b - W_t}{T} \frac{W_T - W_t}{T - t} \]

\[ - \frac{1}{2} \left( \frac{\mu_t - r_t}{\sigma_t} \right)^2 - \frac{1}{2} \left( \frac{b - W_t}{T} \right)^2 - \frac{1}{2} \left( \frac{W_T - W_t}{T - t} \right)^2 \]

\[ - \frac{\mu_t - r_t}{\sigma_t} \frac{b - W_t}{T} - \frac{\mu_t - r_t}{\sigma_t} \frac{W_T - W_t}{T - t} - \frac{b - W_t}{T} \frac{W_T - W_t}{T - t} + \frac{t}{T(T - t)} \]

\[ = r_t + \frac{1}{2} \left( \frac{\mu_t - r_t}{\sigma_t} \right)^2 + \frac{\mu_t - r_t}{\sigma_t} \frac{b - W_t}{T} + \frac{1}{2} \left( \frac{b - W_t}{T} \right)^2 \]

\[ - \frac{1}{2} \left( \frac{W_T - W_t}{T - t} \right)^2 + \frac{t}{T(T - t)} \]

\[ = r_t + \frac{1}{2} \left( \frac{\mu_t - r_t}{\sigma_t} \right)^2 + \frac{\mu_t - r_t}{\sigma_t} \frac{b - W_t}{T} - \frac{1}{2} \left( \frac{W_T - W_t}{T - t} \right)^2 + \frac{t}{T(T - t)}. \]

Now, it is possible to calculate the expected value of \( J^\pi_t \):

\[ \mathbb{E}(J^\pi_t) = r_t + \frac{1}{2} \mathbb{E} \left( \frac{\mu_t - r_t}{\sigma_t} - \frac{W_t}{T} \right)^2 - \frac{1}{2} \mathbb{E} \left( \frac{W_T - W_t}{T - t} \right)^2 + \frac{t}{T(T - t)} \]

\[ = r_t + \frac{1}{2} \mathbb{E} \left( \frac{\mu_t - r_t}{\sigma_t} - \frac{W_t}{T} \right)^2 + \frac{1}{2} \left( \frac{b}{T} \right)^2 + \mathbb{E} \left( \frac{\mu_t - r_t}{\sigma_t} \right) \frac{b}{T} - \frac{1}{2(T - t)} + \frac{t}{T(T - t)} \]

\[ = r_t + \frac{1}{2} \mathbb{E} \left( \frac{\mu_t - r_t}{\sigma_t} - \frac{W_t}{T} + \frac{b}{T} + \frac{W_T - W_t}{T - t} \right)^2 \]

\[ = r_t + \frac{1}{2} \mathbb{E} \left( \frac{\mu_t - r_t}{\sigma_t} + \frac{b - b}{T - t} + \frac{W_T - W_t}{T - t} \right)^2 \]

\[ = r_t + \frac{1}{2} \mathbb{E} \left( \frac{\mu_t - r_t}{\sigma_t} + \frac{b - \hat{B}_t}{T - t} \right)^2. \]

Hence, the value of the problem is

\[ V^\pi_T = \mathbb{E} \int_0^T \left[ r_t + \frac{1}{2} \left( \frac{\mu_t - r_t}{\sigma_t} + \frac{b - \hat{B}_t}{T - t} \right)^2 \right] dt. \]
Remark 3.8. Theorem 3.7 recovers classical results of insider trading. However, it is apparently not consistent with Theorem 3.4 in the sense that, if we assumed the deterministic character of the parameters, the present results do not reduce to the previous ones. This is not the consequence of a mistaken development: simply the assumptions are different. Precisely, the current portfolio process is anticipating (it depends on a future value of Brownian motion) and the former is a deterministic function.

Remark 3.9. Note that, contrary to what happened previously, a divergence is present for $t = T$; this is the consequence of allowing stochastic portfolios.

3.3. APO with the Skorokhod Integration Method.

In this section, we present the portfolio optimization process using Skorokhod integration. To find a solution of the corresponding equation, we use anticipative Girsanov transformations, first, allowing the parameters to be stochastic and then, deterministic and finally constant to find a closed-form solution.

Again inspired by the self-financing property, we postulate the insider wealth is given by the process that solves

\begin{equation}
\delta X_t = [\mu_t \pi_t + r_t (1 - \pi_t)] X_t dt + \sigma_t \pi_t X_t \delta \bar{B}_t \\
X_0 \in \mathbb{R}^+,
\end{equation}

where

\begin{equation}
\delta \bar{B}_t = \delta W_t - \frac{W_t - b}{T} dt,
\end{equation}

\begin{equation}
\bar{B}_t = W_t - (W_t - b) \frac{t}{T}, \ t \in [0, T],
\end{equation}

and $\delta$ denotes Skorokhod integration. Then

\begin{equation}
\delta X_t = \left[ r_t (1 - \pi_t) + \pi_t \mu_t - \pi_t \sigma_t \frac{W_t - b}{T} \right] X_t dt + \pi_t \sigma_t X_t \delta W_t.
\end{equation}

To meet the assumptions in section 2.3 we need to substitute $W_T$ by $W_{T \wedge \tau}$, where $\tau$ is the stopping time $\tau = \inf\{t > 0 : |W_t| = m \sqrt{T} \}$, $m \in \mathbb{N}$, so it becomes a bounded random variable. We do this in the hope that the limit $m \to \infty$ will yield a solution to problem (3.3); we will get back to this issue below. For the model parameters, we use the assumptions in that section so that its developments can be applied.

We use equation (2.3) to find that the solution of (3.3) is

\begin{equation}
\frac{X_t}{X_0} = \exp \left\{ \int_0^t [\mu_s \pi_s + r_s (1 - \pi_s)] (U_{s,t}) ds \right\} L'_t, \ t \in [0, T],
\end{equation}

with $L'_t$ as given in (2.3).
When taking the expectation of the integral form, we have that the value of the problem is given by

\[
E[\log(X_T/X_0)] = \mathbb{E}\left[ \int_0^T \left( \mu_t\pi_t(U'_{t,T}) + r_t(1 - \pi_t)(U'_{t,T}) - \frac{1}{2}\sigma^2_t\pi^2_t(U'_{t,T}) \right) dt \right]

- \frac{W_{T\wedge \tau} - b}{T} \sigma_t\pi_t(U'_{t,T}) - \mathbb{E}\left[ \int_0^T \int_t^T (D_u\sigma_t\pi_t)(U'_{t,T}) D_t[\sigma_u\pi_u(U'_{t,T})] dudt, \right]

where we used that \(\mathbb{E}\int_0^T \sigma_t\pi_t(U'_{t,T}) dW_t = 0\).

As \(\sigma_t\) and \(\pi_t\) are adapted to the filtration \(\mathcal{F}_t\), the expected value of the term \(\int_0^T \int_t^T (D_u\sigma_t\pi_t)(U'_{t,T}) D_t[\sigma_u\pi_u(U'_{t,T})] dudt\) equals zero. Then, we compute

\[
E[\log(X_T/X_0)] = \mathbb{E}\left[ \int_0^T \mathbb{E}\left( \mu_t\pi_t(U'_{t,T}) + r_t(1 - \pi_t)(U'_{t,T}) - \frac{1}{2}\sigma^2_t\pi^2_t(U'_{t,T}) \right) \left| \mathcal{F}_t \right. dt \right]

- \frac{W_{T\wedge \tau} - b}{T} \sigma_t\pi_t(U'_{t,T}) \mathbb{E}\left[ \mathcal{F}_t \right] dt

= \mathbb{E}\left[ \int_0^T \mathbb{E}\left( \mu_t\pi_t + r_t(1 - \pi_t) - \frac{1}{2}\sigma^2_t\pi^2_t - \frac{W_{T\wedge \tau}(U'_{t,T}) - b}{T} \sigma_t\pi_t \mathbb{E}\left[ \mathcal{F}_t \right] dt \right. \right].

From this step, the problem would be similar to the one formulated with the forward integration method, but with anticipative transformations, if the term \(W_{T\wedge \tau}(U'_{t,T})\) had a closed expression; but it has not that in general. So, we need to further constraint the parameters to be constant, or at least deterministic, to achieve a closed-form solution.

**APO with deterministic parameters**

Now, we use that \(\mu_t, r_t, \sigma_t, \pi_t \in L^\infty([0, T])\) in the equation

\[
(15) \quad \delta X_t = \left( (1 - \pi_t)r_t + \pi_t\mu_t + \pi_t\frac{b - W_T}{T} \right) X_t dt + \pi_t\sigma_t X_t \delta W_t, \quad X_0 \in \mathbb{R}^+;
\]

that is, we assume our parameters to be deterministic rather than stochastic. Note that, under this assumption and in those cases in which both solution and parameters are regular enough, the equation does not need to be postulated as it follows from the self-financing assumption and Theorem 8.20 in \[DNOP09\].

To solve the above equation, we use (2.3) taking \(\hat{\sigma}_t = \sigma_t\pi_t\) and \(\hat{\mu}_t = (1 - \pi_t)r_t + \pi_t\mu_t + \pi_t\frac{b - W_T}{T}\). Since we need \(\hat{\mu}_t\) to be bounded, we consider the truncated version of the Brownian motion \(W_{T\wedge \tau}\), where \(\tau\) is the stopping time \(\tau = \inf\{t > 0 : |W_t| = m\sqrt{T}\}, m \in \mathbb{N}\).
In this way, we restrict the Brownian motion to the state space \([-m\sqrt{T}, m\sqrt{T}]\), where \(m\) represents the number of standard deviations to be considered for the random variable \(W_t, \; t \in [0, T]\). Note that this trick is the same employed in the previous section since the original equation is not solvable with the methods of \([Buc89]\); but it becomes solvable after that substitution. Contrary to what happened before, explicit solutions will become available now, and moreover the case \(m \to \infty\) will become accessible to our analysis.

In the present case we find that 
\[
U_{s,t}(W_T \wedge \tau) = W_T \wedge \tau - Z_T \wedge \tau_0 I_{[s,t]}(\pi \sigma) du,
\]
and we can use this fact to find a closed formula for the solution of (3.3), which is given by
\[
X^{(m)}_t / X_0 = \exp \left\{ \int_0^t \pi_s \sigma_s \delta W_s - \frac{1}{2} \int_0^t \pi_s^2 \sigma_s^2 ds + \int_0^t \left[ (1 - \pi_s) r_s + \pi_s \mu_s + \pi_s \sigma_s \frac{b - (W_T \wedge \tau - \int_s^T \pi_u \sigma_u du)}{T} \right] ds \right\}.
\]

Now, note that
\[
\lim_{m \to \infty} X^{(m)}_t = X_0 \exp \left\{ \int_0^t \pi_s \sigma_s \delta W_s - \frac{1}{2} \int_0^t \pi_s^2 \sigma_s^2 ds + \int_0^t \left[ (1 - \pi_s) r_s + \pi_s \mu_s + \pi_s \sigma_s \frac{b - (W_T - \int_s^T \pi_u \sigma_u du)}{T} \right] ds \right\} =: X_t,
\]
where the convergence takes place uniformly in \(t\) almost surely. Such a good behavior makes \(X_t\) a potential candidate to be the solution of the original problem; indeed, the following result shows that it is the unique solution.

**Theorem 3.10.** Let \(X_t\) be as defined above and let \(\mu_t, r_t, \sigma_t, \pi_t \in L^\infty([0, T])\) be deterministic parameters. Then, the unique solution to the linear Skorokhod stochastic differential equation (3.3) is given by \(X_t\).

**Proof.** First of all, note that the theory of \([Buc89]\) cannot be directly applied since the drift of equation (3.3) includes an unbounded random variable (the Gaussian variable \(W_T\)). However, as already noted, the perturbed equation
\[
\delta X_t = \left[ (1 - \pi_t) r_t + \pi_t \mu_t + \pi_t \sigma_t \frac{b - W_{T \wedge \tau}}{T} \right] X_t dt + \pi_t \sigma_t X_t \delta W_t, \; X_0 \in \mathbb{R}^+,
\]
falls under the hypotheses of this theory for any fixed \(m\), and therefore it follows that it possesses a unique solution, which is given by \(X^{(m)}_t\).

On the other hand, if \(\tau \geq T\), then \(W_{T \wedge \tau} = W_T\), and the same result follows. Now define
\[
\mathcal{M} := \max_{0 \leq t \leq T} W_t, \quad m := \min_{0 \leq t \leq T} W_t;
\]
from Chapter 2 in [KS91] we know that
\[ \mathbb{P}\left(\{M \geq m\sqrt{T}\}\right) = \sqrt{\frac{2}{\pi}} \int_{m}^{\infty} e^{-x^2/2} dx, \]
and by symmetry
\[ \mathbb{P}\left(\{m \leq -m\sqrt{T}\}\right) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{m} e^{-x^2/2} dx. \]
Therefore
\[ \mathbb{P}\left(\{W_{T \wedge \tau} = W_T\right) = \mathbb{P}\left(\{M \leq m\sqrt{T}\} \land \{m \geq -m\sqrt{T}\}\right) \]
\[ = 1 - \mathbb{P}\left(\{M \geq m\sqrt{T}\} \lor \{m \leq -m\sqrt{T}\}\right) \]
\[ \geq 1 - \mathbb{P}\left(\{M \geq m\sqrt{T}\}\right) - \mathbb{P}\left(\{m \leq -m\sqrt{T}\}\right) \]
\[ = 1 - 2\sqrt{\frac{2}{\pi}} \int_{m}^{\infty} e^{-x^2/2} dx. \]
Consequently, for any fixed \(m\), \(X_t^{(m)}\) is the unique solution to equation (3.3) with probability
\[ \mathbb{P}\left(\{X_t^{(m)} = X_t\}\right) = \mathbb{P}\left(\{W_{T \wedge \tau} = W_T\}\right) \geq 1 - 2\sqrt{\frac{2}{\pi}} \int_{m}^{\infty} e^{-x^2/2} dx. \]
Since \(m\) is arbitrary, take the limit \(m \to \infty\) to conclude that \(X_t\) is the unique solution to (3.3) almost surely.

Once problem (3.3) is solved, our aim is to compute the optimal portfolio for the expected logarithmic utility
\[ V_T := \mathbb{E}[\log(X_T^\pi/X_0^\pi)] \]
\[ = \mathbb{E}\left\{ \int_0^T \left[ r_t + \left( \mu_t - r_t + \frac{\sigma_t b}{T} \right) \pi_t - \frac{\sigma_t^2}{2} \pi_t^2 + \frac{\sigma_t}{T} \pi_t \left( \int_t^T \pi_s \sigma_s ds \right) \right] dt \right\} \]
\[ = \int_0^T \left[ r_t + \left( \mu_t - r_t + \frac{\sigma_t b}{T} \right) \pi_t - \frac{\sigma_t^2}{2} \pi_t^2 + \frac{\sigma_t}{T} \pi_t \left( \int_t^T \pi_s \sigma_s ds \right) \right] dt, \]
since all the parameters are deterministic. For that, we will use the calculus of variations, which methods are legitimate under the current hypotheses with the additional assumption of \(\sigma_t > 0\) (non-degeneracy of the volatility). Thus, we compute the first variation of \(V_T^\pi\) in
the direction of $\varphi$ (a perturbation of $\pi$)

$$\frac{\delta V^\pi_T}{\delta \pi} := \frac{d}{d\lambda} V[\pi + \lambda \varphi] \bigg|_{\lambda=0}$$

$$= \mathbb{E} \left\{ \int_0^T \left[ \mu_t - r_t + \frac{b}{T} \sigma_t - \sigma_t^2 \pi_t + \frac{\sigma_t}{T} \int_t^T \pi_s \sigma_s \, ds \right] \varphi_t \, dt \right. $$

$$- \frac{\sigma_t}{T} \int_t^T \pi_s \sigma_s \, ds \right \} \varphi_t \, dt + \frac{1}{T} \left( \int_0^T \pi_s \sigma_s \, ds \right) \left( \int_0^T \varphi_s \sigma_s \, ds \right) \}

$$= \int_0^T \left[ \mu_t - r_t + \frac{b}{T} \sigma_t - \sigma_t^2 \pi_t \right] \varphi_t \, dt + \frac{1}{T} \left( \int_0^T \pi_s \sigma_s \, ds \right) \left( \int_0^T \varphi_s \sigma_s \, ds \right),$$

after integration by parts. Therefore we have to solve the equation

$$\frac{\delta V_T[\pi_t]}{\delta \pi_t} = 0,$$

which admits the particular solution

$$\mu_t - r_t + \frac{b}{T} \sigma_t - \sigma_t^2 \pi_t = 0,$$

subject to the integral condition

$$(16) \quad \int_0^T \pi_s \sigma_s \, ds = 0,$$

which comes from the boundary term in the computation of the first variation. Explicitly, the solution reads

$$(17) \quad \pi_t = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{b}{\sigma_t T},$$

which comes from the algebraic equation. We substitute (3.3) in (3.3) to find

$$b = \int_0^T \frac{r_t - \mu_t}{\sigma_t} \, dt,$$

which comes from the integral condition.
Clearly, this is the global maximum of the functional whenever the integral condition is met, as can be checked from the computation

\[ V_T^\pi = \mathbb{E} \left\{ \int_0^T \left[ r_t + \left( \mu_t - r_t + \frac{\sigma_t b}{T} \right) \pi_t - \frac{\sigma_t^2}{2} \pi_t^2 \right] dt + \frac{1}{2T} \left( \int_0^T \pi_s \sigma_s ds \right)^2 \right\} \]

\[ = \mathbb{E} \left\{ \int_0^T \left[ r_t + \left( \mu_t - r_t + \frac{\sigma_t b}{T} \right) \pi_t - \frac{\sigma_t^2}{2} \pi_t^2 \right] dt \right\} \]

\[ = \int_0^T \left[ r_t + \left( \mu_t - r_t + \frac{\sigma_t b}{T} \right) \pi_t - \frac{\sigma_t^2}{2} \pi_t^2 \right] dt, \]

where we have integrated by parts in the first line and applied the integral condition in the second. Indeed, the last functional recovers the result of the previous sections, in the case of deterministic (and bounded) parameters and a fixed and concrete value of \( b \).

Substituting, we find for the portfolio

\[ \pi_t^* = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{1}{\sigma_t T} \int_0^T \frac{r_t - \mu_t}{\sigma_t} dt, \]

and for the value of the optimization problem

\[ V_T^{\pi_t^*} = \int_0^T \left[ r_t + \frac{1}{2} \left( \frac{\mu_t - r_t}{\sigma_t} + \frac{b}{T} \right)^2 \right] dt \]

\[ = \int_0^T \left[ r_t + \frac{1}{2} \left( \frac{\mu_t - r_t}{\sigma_t} + \frac{1}{T} \int_0^T \frac{r_s - \mu_s}{\sigma_s} ds \right)^2 \right] dt. \]

Again we recover the results of the previous sections, but for this particular value of \( b \). Since we have to fix this value, the developments in the present section are of limited applicability; therefore, we will use different assumptions in the following one.

**APO with constant parameters**

To partially overcome the problem of limited applicability that arose in the previous section, from now on we assume that \( \mu, r, \sigma, \) and \( \pi \) are constants (i.e. time-independent and deterministic). In this way, we plan to approach our problem without the constraint of fixing \( b \).

Therefore, we aim to solve the insider wealth equation

\[ \delta X_t = [(1 - \pi)r + \pi \mu] X_t dt + \pi \sigma X_t \delta \bar{B}_t, \]

\[ X_0 \in \mathbb{R}^+, \]

where

\[ \delta \bar{B}_t = \delta W_t + \frac{b - W_T}{T} dt, \quad t \in [0, T]. \]

Note that, in this particular case and prior to the substitution of the Brownian bridge, the equation is not a consequence of the self-financing condition, but a consequence of the
linearity of both the Skorokhod and Lebesgue integrals. Such a huge simplification comes from the assumption of the constancy of the parameters.

Substituting $\delta \bar{B}_t$ in (3.3), we get

$$
\delta X_t = \left[ (1 - \pi) r + \pi \mu + \pi \sigma \frac{b - W_T}{T} \right] X_t dt + \pi \sigma X_t \delta W_t.
$$

The solvability of this equation is guaranteed by Theorem 3.10.

Then, arguing as in the previous section, we find that the solution of (3.3) is

$$
\frac{X_t}{X_0} = \exp \left\{ \int_0^t \pi \sigma \delta W_s - \frac{1}{2} \int_0^t \pi^2 \sigma^2 ds \right. 
+ \int_0^t \left[ (1 - \pi) r + \pi \mu + \pi \sigma \left( W_T - \int_s^t \pi \sigma du \right) - \frac{1}{2} \pi^2 \sigma^2 \right] ds \left. \right\}
$$

$$
= \exp \left\{ \pi \sigma W_t + \left[ (1 - \pi) r + \pi \mu + \pi \sigma \left( W_T - \int_s^t \pi \sigma du \right) - \frac{1}{2} \pi^2 \sigma^2 \right] t + \pi^2 \sigma^2 \frac{t^2}{2T} \right\},
$$

since $U_{s,t}(W_T) = W_T - \int_0^T \mathbb{I}_{[s,t]}(r) \pi \sigma \ du = W_T - \pi \sigma(t - s)$. If we compute the expected logarithmic utility at the horizon time $t = T$, we find

$$
V_T^\pi := \mathbb{E}[\log(\frac{X_T^\pi}{X_0^\pi})]
$$

$$
= \mathbb{E} \left\{ \left[ (1 - \pi) r + \pi \mu \right] T + \pi \sigma b \right\}
$$

$$
= rT + (\mu - r)\pi T + \sigma b \pi,
$$

since $X_T = X_0 \exp \left\{ \left[ (1 - \pi) r + \pi \mu \right] T + \pi \sigma b \right\}$ is deterministic. Clearly, as the expression for $V_T^\pi$ is affine in $\pi$, there exists neither a maximum nor a minimum. To address this issue, which was not encountered in the previous sections, we now introduce the no shorting condition.

To find the value $\pi^*$ that maximizes $V_T(\pi)$, under no shorting, we consider the values of $b$ and the boundaries of $\pi$. We define $\theta := \frac{\mu - r}{\sigma}$. In consequence, we set

$$
\begin{cases}
> -\theta T, & \pi^* = 1 \text{ and } V_T(\pi^*) = \mu T + \sigma b, \\
\leq -\theta T, & \pi^* = 0 \text{ and } V_T(\pi^*) = rT.
\end{cases}
$$

Therefore, the optimal portfolio under Skorokhod integration is

$$
\pi^* = \mathbb{I}_{(b > -\theta T)},
$$
and the value of the problem in this case is

\[
V_T(\pi^*) = rT + (\theta \sigma T + \sigma b) \mathbb{I}_{\{b > -\theta T\}}.
\]

Observe that the value of the problem in (3.3) is bounded by \(rT\) and \(\mu T + \sigma b\). This value, and the general result, are in deep contrast with all the results previously obtained.

The strategy of the insider in this case consists on trading the risky asset if \(b > -\theta T\) or the risk-free asset if \(b \leq -\theta T\). The last case means that an ordinary trader might overcome the insider if there is a negative enough final value of the driving stochastic process.

3.4. Example of Performance.

With the purpose to exemplify an insider trading performance with the techniques described in this work, we simulate the situation of a trader who has privileged information and wants to use it. The features of the simulation are the following:

- **Assumptions:**
  For ease of computation, we leave out the trading costs and the difference between the bid and ask prices, and we assume there is enough liquidity to trade.

- **Stock:**
  We use the 2-Year U.S. Treasury Note Future, a marketable risky instrument of the U.S. government traded in the Chicago Mercantile Exchange.

- **Parameters:**
  For the example we use constant parameters: we compute \(\sigma\) as the monthly standard deviation of the risk asset prices; we consider an average of the U.S. 3-Month Bond Yield to compute \(r\); and we compute \(\mu\) from an ARIMA model of the log-return historic values \(\log(S_t) - \log(S_{t-1})\).

- **Dates:**
  The trader starts to invest on March 03, 2019, and the horizon time is May 30, 2019. We assume the trader has privileged information about May 30.

- **Periodicity:**
  We consider daily prices at 14:00 (GMT-5). At that time, the trader computes the proportion of her wealth that should be in the risky asset \((\pi)\) and the risk-free asset \((1-\pi)\).

We show the performance using three possible portfolios:

1. The portfolio an honest trader would use: \(\pi^{(ho)} = \frac{\mu_t - r_t}{\sigma_t^2}\).
2. The portfolio a forward trader would use: \(\pi^{(fw)} = \frac{\mu_t - r_t}{\sigma_t^2} + \frac{b}{\sigma_t T}\).
3. The portfolio a Skorokhod trader would use: \(\pi^{(sk)} = \mathbb{I}_{\{b > -\theta T\}}\).
At time zero, the insider trader computes $b$ from the equation

\[ S_T = S_0 \exp \{ (\mu - \sigma^2/2)T + \sigma b \}. \]

At time $t$ (day $t$), the investor knows the value of $S_t$ and $r_t$. The value of the wealth at that time is

\[ X_t = X_{t-1} \exp \{ (1 - \pi_{t-1})r_{t-1} + \pi_{t-1} \log(S_t/S_{t-1}) \}. \]

We show the wealth evolution $X_t$, $t \in [0, T]$ using three portfolios, the honest one, the forward one, and the Skorokhod one in Figure 3.1. We see that the wealth using the Skorokhod portfolio is bigger than using the forward one. The wealth of the honest trader is far less than the previous ones not only at the end but practically in the whole period.

\[ \text{Figure 3.1. Wealth evolution of the honest trader in yellow, the forward trader in red, and the Skorokhod trader in blue with the stock 2-Year U.S. Treasury Note Future.} \]

3.5. Simulation. In this section, we show how to perform a simulation of portfolio optimization from the point of view of both honest and insider trading. We consider two insider portfolios constructed with the forward integration approach and the Skorokhod integration one.

First, we simulate realizations of a conditioned Gaussian process

\[ (B_t \mid B_0 = 0, B_T = b), \quad t \in [0, T]. \]
We start from the given extreme points $B_0$ and $B_T$. Then, recursively, given two values $B(u)$ and $B(t)$, we simulate the value $B(s)$ for $0 < u < s < t < T$ as in [Gla03]. The random vector $[B(u)B(s)B(t)]^T$ is Gaussian with mean vector and covariance matrix:

$$
\begin{bmatrix}
B(u) \\
B(s) \\
B(t)
\end{bmatrix} \sim N
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
u & u & u \\
u & s & s \\
u & s & t
\end{pmatrix}.
$$

Thus, the conditional distribution $(B(s) \mid B(u), B(t))$ is given by

$$
N\left( \frac{(t-s)B(u) + (s-u)B(t)}{t-u}, \frac{(s-u)(t-s)}{t-u} \right),
$$

and we can simulate $B(s)$ through the expression

$$
B(s) = \frac{(t-s)B(u) + (s-u)B(t)}{t-u} + \sqrt{\frac{(s-u)(t-s)}{t-u}} Z,
$$

where $Z \sim N(0,1)$.

In Figure 3.2 we show different instances of the algorithm of Brownian bridges ending in zero.

![Brownian bridge ending in zero](image)

**Figure 3.2.** Brownian bridges ending in zero.

We apply this algorithm with different values of $b \sim N(e, 64)$ to simulate 64-day paths of a stock with initial value 100, $\mu = 0.03$ and $\sigma = 0.3$. For each path, we perform the algorithm of Section 3.4 to get the value of the problem under forward and Skorokhod integration. We repeat the process to get a distribution of the value of the problem. We compare the
distribution under these integration approaches, where we consider a risk-free rate of 0.0027. The number of days and the risk-free rate chosen are very similar to the previous example.

In Figure 3.3, we compare different mean values of the distribution of $b$. At the top we use the expected value $e = 0$, at the middle $e = 0.5$, and at the bottom $e = 1$. We see that in all cases the mean value of the problem increases if the mean of $b$ increases and that, under Skorokhod integration, the distribution of the value of the problem has a bigger mean and a lower variance than under forward integration.

3.6. Properties and comparison of methods. Now we aim to describe the properties and compare the methods we have used through this manuscript for optimizing the insider portfolio. We start with the case in which the final value of the driving Brownian process, $b$, is constant. To get close-form expressions, we consider the model parameters to be constant too.

In the previous sections, we obtained explicit expressions of optimal portfolios and values of the problem for which shorting was allowed, except in the final case (for which no-shorting permitted to interpret optimality). Therefore, in order to obtain a full comparison of the methods, we need to impose the no-shorting condition to all. We now show in detail how these expressions change when this condition is imposed.

The optimal portfolio for an honest trader is

$$\pi^{(ho)} = \left(\left(\frac{\mu - r}{\sigma^2}\right) \land 1 \right) \lor 0.$$ 

Therefore the value of the problem for this trader is

$$V_T^{(ho)} = rT + \frac{1}{2} \theta^2T \mathbb{I}_{\{\theta \in (0,\sigma)\}} + \left(\theta \sigma - \frac{1}{2} \sigma^2\right)T \mathbb{I}_{\{\theta \geq \sigma\}} = \begin{cases} rT, & \theta \leq 0, \\ rT \frac{1}{2} \theta^2T, & 0 < \theta < \sigma, \\ \mu T \frac{1}{2} \sigma^2T, & \theta \geq \sigma, \end{cases}$$

where $\theta = \frac{\mu - r}{\sigma}$.

For the forward scheme, the optimal portfolio with constant parameters is

$$\pi^{(fw)} = \left(\left(\frac{\mu - r}{\sigma^2} + \frac{b}{T\sigma}\right) \land 1 \right) \lor 0$$

(20)

$$= \frac{\theta + \alpha}{\sigma} \mathbb{I}_{\left\{\frac{\theta + \alpha}{\sigma} \in (0,1)\right\}} + \mathbb{I}_{\left\{\frac{\theta + \alpha}{\sigma} \geq 1\right\}}$$

$$= \frac{\theta + \alpha}{\sigma} \mathbb{I}_{\{b \in (-\theta T, -\theta T + \sigma T)\}} + \mathbb{I}_{\{b \geq -\theta T + \sigma T\}},$$
Figure 3.3. Histogram of $V_T^{\pi^*}(b)$ for different distributions of $b$, where $b \sim N(e, 64)$. At the top we use the expected value $e = 0$, at the middle $e = 0.5$, and at the bottom $e = 1$.

where $\alpha = \frac{b}{T}$. And the value of the problem in terms of $\pi^{(fw)}$ is

$$V_T^{(fw)} = \mathbb{E} \left[ \int_0^T \left( r + \theta \pi^* \sigma + \pi^* \sigma \frac{b}{T} - \frac{1}{2} (\pi^* \sigma)^2 \right) dt \right]$$

$$= rT + \theta \pi^* \sigma T + \pi^* \sigma b - \frac{1}{2} (\pi^* \sigma)^2 T.$$
Substituting the result (3.6), we have that
\[ V_T^{(fw)} = \begin{cases} 
    rT, & b \leq -\theta T, \\
    rT + \frac{1}{2} (\theta + \alpha)^2 T, & b \in (-\theta T, -\theta T + \sigma T], \\
    \mu T + \sigma b - \frac{1}{2} \sigma^2 T, & b > -\theta T + \sigma T, 
  \end{cases} \]
or, equivalently, using indicator functions
\[ V_T^{(fw)} = rT + \frac{1}{2} \left( \theta + \frac{b}{T} \right)^2 T \mathbb{I}_{b \in (-\theta T, -\theta T + \sigma T]} + \left( \theta \sigma T + \sigma b - \frac{1}{2} \sigma^2 T \right) \mathbb{I}_{b > -\theta T + \sigma T}. \]

For the Skorokhod scheme, we already bounded the optimal portfolio:
\[ \pi^* = \mathbb{I}_{b > -\theta T}, \]
and recall that the value of the problem is
\[ V_T^\pi (\pi^*) = rT + (\theta \sigma T + \sigma b) \mathbb{I}_{b > -\theta T}. \]

Observe that if \( b \leq -\theta T \), then \( V_T^{(sk)} = V_T^{(fw)} = rT \), and in fact it is better to invest in the risk-free asset, given that we assume \( \theta > 0 \), which is a financially meaningful condition. Let us discuss the case \( b > -\theta T \). Under this assumption, \( \mu T + \sigma b \) is bigger than \( rT + \frac{1}{2} \left( \theta + \frac{b}{T} \right)^2 T \) if \( b \leq -\theta T + \sigma T \), then \( V_T^{(sk)} > V_T^{(fw)} \), \( b \in (-\theta T, -\theta T + \sigma T] \). Finally, for the case \( b > -\theta T + \sigma T \), we also have that \( V_T^{(sk)} > V_T^{(fw)} \) since \( \mu T + \sigma b \) is bigger than \( \mu T + \sigma b - \frac{1}{2} \sigma^2 T \). Therefore, we conclude that the method involving Skorokhod integration yields a portfolio that is equally or more profitable for every value of \( b \).

As an example, we perform a numerical comparison written in Matlab software of \( V_T^\pi \) under Skorokhod and forward integration with the market parameters \( \mu = .03, \ r = .02, \ \sigma = .30 \), and \( T = 1 \) to simplify the computations.

In Figure 3.4, we show \( V_T^\pi \) with respect to \( b \) in the interval \([-\theta T, -\theta T + \sigma T]\) under Skorokhod (blue line) and forward integration (red line). We also represent the investment of an honest trader (yellow line) without anticipating information, which value is constant with respect to \( b \), and the safe investment (purple line), under the risk-free rate, which is also constant with respect to \( b \).

So far, we have been assuming the constancy of \( b \). If we considered \( b \) as a Gaussian random variable (herein assumed to be independent of the Brownian motion), the value of the problem would become a random variable depending on \( b \). In this sense, we can interpret the previous results as the conditional expectation of the insider utility given some fixed \( b = \bar{b} \): \( V_T^\pi(\bar{b}) = \mathbb{E}(\log(X_T/X_0)|b = \bar{b}) \). Then, if we no longer assume a fixed value of \( b \), we can obtain the unconditioned expectation by integrating the conditional one in the domain of \( b \), due to its independence of Brownian motion.
Figure 3.4. $V^*_T(b)$ from $-\theta T$ to $-\theta T + \sigma T$ at the top and for negative values of $b$ at the bottom.

We have performed the computation of the unconditional expectation numerically, which is the value of the problem taking into account all the possible values of $b$. In Figure 3.5, we plot $V^*_T(b)\mathbb{P}(b)$ to visualize the area under this curve, that represents the integral for the unconditional expectation $V^*_T$. We see that the curve of $V^*_T(b)\mathbb{P}(b)$ under Skorokhod integration is above the one under the forward scheme, and therefore the integral of this value is bigger under the Skorokhod scheme. We have assumed that $b \sim N(0, T)$ and have taken $T = 1$.

Finally, we show the expressions of the unconditional expectations under forward and Skorokhod integration and under the assumption that $b$ is a Gaussian random variable. For
Figure 3.5. $V_T^\pi(b) \mathbb{P}(b)$, with $b \sim \mathcal{N}(0, 1)$.

The forward scheme, we find that

$$
\mathbb{E} \left[ V_1^{(fw)} \right] = r T + \frac{1}{2} \mathbb{E} \left[ (\theta + b)^2 \mathbb{I}_{\{b \in (-\theta, -\theta + \sigma]\}} \right] \\
+ \mathbb{E} \left[ \left( \theta \sigma - \frac{1}{2} \sigma^2 + \sigma b \right) \mathbb{I}_{\{b > -\theta + \sigma\}} \right]
$$

$$
= r + \frac{1}{2} \int_{-\theta + \sigma}^{\infty} (\theta + b)^2 e^{-b^2/2} \, db \\
+ \frac{1}{2} \sqrt{2\pi} \int_{-\theta - \sigma}^{\infty} \left( \theta \sigma - \frac{1}{2} \sigma^2 + \sigma b \right) e^{-b^2/2} \, db
$$

$$
= \frac{1}{4} (\theta + 1) \text{erf} \left( \frac{\sigma - \theta}{\sqrt{2}} \right) + \frac{1}{4} (\theta + 1) \text{erf} \left( \frac{\theta}{\sqrt{2}} \right) \\
+ \frac{1}{4} \sqrt{\frac{2}{\pi}} \exp \left\{ -\frac{1}{2} (\theta^2 + \sigma^2) \right\} \left( (\theta - \sigma) \exp \{ \theta \sigma \} - \theta \exp \left\{ \frac{\sigma^2}{2} \right\} \right) \\
+ \frac{1}{4} \sigma (2\theta - \sigma) \left( \text{erf} \left( \frac{\sigma - \theta}{\sqrt{2}} \right) + 1 \right) + \frac{\sigma}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} (\theta - \sigma)^2 \right\}.
$$
Under the Skorokhod scheme, we find that
\[
E\left(V_1^{(sk)}\right) = r + E\left[\left(\theta \sigma + \sigma b\right)1\{b > -\theta\}\right]
\]
\[
= rT + \frac{1}{\sqrt{2\pi}} \int_{-\theta}^{\infty} (\theta \sigma + \sigma b)e^{-b^2/2}db
\]
\[
= r + \frac{\theta \sigma}{2} \left[ \text{erf}\left(\frac{\theta}{\sqrt{2}}\right) + 1 \right] + \frac{\sigma}{\sqrt{2\pi}} e^{-\theta^2/2}.
\]

Note that these results do not reproduce the classical ones. The reason is that, under the present assumptions, the random variable \(b\) is independent of the Brownian motion, contrary to what has been classically assumed.

4. Conclusions

In this work we have studied the role of different notions of anticipating calculus on the maximization of the logarithmic utility of an insider trader. It thus complements the previous studies in which this role was examined for risk-neutral traders. In [Esc18], [BE18], and [ERC21] it was shown that the forward integral produces intuitive results from the financial viewpoint, while the Skorokhod integral does not, in the sense that it effectively transforms the insider trader into an uninformed one in terms of performance. In particular, in all these works, the Skorokhod integral provides the insider with a wealth that is smaller than or equal to the wealth of the honest trader, and always strictly smaller than the wealth of the insider modeled with the forward integral. However, the presence of the logarithmic utility changes this situation sharply. As we have shown herein, the Skorokhod insider is the one that gets a higher value in the case of constant parameters. Even if shorting is only forbidden for the Skorokhod insider, she still gets a higher value than the forward insider. In the case of time-dependent parameters, there is one particular case that can be solved and replicates the result of the forward integral, something without precedents in the case of risk-neutral traders. Moreover, for negative enough final values of the Brownian process, the ordinary trader can overcome both Skorokhod and forward integral insiders. A related feature, that the ordinary trader can overcome the insider one for certain paths in the case of time-dependent parameters, which could also be regarded as undesirable, was already studied in [EE22], and identified as a consequence of the logarithmic utility. Now we have found that for certain driving Brownian paths, Skorokhod insiders cannot overcome ordinary traders; in particular, although the performance of Skorokhod insiders improves that of forward insiders under the logarithmic utility, it is unable to erase this feature.

Our results overall point to the fact that the interplay between stochasticity (through the introduction of a suitable stochastic integral) and nonlinearity (through the introduction
of a suitable utility function) still presents unexpected results within the realm of finance. A deeper understanding of the role of Skorokhod integration in financial modeling could go through the computation of new explicit solutions to this type of stochastic differential equations, something that has been quite elusive so far (in the present work, this translates into the necessity of assuming constant parameters and portfolios in order to fully approach the Skorokhod case); in fact, the solution to the Skorokhod differential equation present in Theorem 3.10 is new to the best of our knowledge. Also, the use of nonlinear utilities, which interacts well with classical stochastic calculus, yields new features that are not completely clear from a financial viewpoint when interrelated with anticipating calculus. Therefore, a possible future line of research is the development of a theory complementary to that of utilities and able to improve these features. Our present results along with those in [EE22] show that the uncritical use of these nonlinear utilities might shed consequences of difficult financial interpretation.

Another possible direction of future research is to establish the link between the present results and arbitrage opportunities, following for instance the line of [CIKHN04]. And finally, it is important to remark that our results rely on mathematical assumptions on the portfolios that are of technical nature and due, as already mentioned, to the difficulty of dealing with Skorokhod calculus. Consequently, it would also be interesting to approach the problem we have herein analyzed with models more closely related to the financial practice. A possible way to achieve this is the one that departs from recent results on Skorokhod nonlinear stochastic differential equations [?]. Overall, it seems that extending the present research requires both modelling and mathematical developments that allow to account for those models.

Data Availability Statement
The information used to exemplify the performance of an insider using knowledge of the 2-Year U.S. Treasury Note Future was taken from https://www.cmegroup.com/markets/interest-rates/us-treasury/2-year-us-treasury-note.html.

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