The Real Scalar Field Equation for Nariai Black Hole in the 5D Schwarzschild-de Sitter Black String Space

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The Nariai black hole, whose two horizons are lying close to each other, is an extreme and important case in the research of black hole. In this paper we study the evolution of a massless scalar field scattered around in 5D Schwarzschild-de Sitter black string space. Using the method shown by Brevik and Simonsen (2001) we solve the scalar field equation as a boundary value problem, where real boundary condition is employed. Then with convenient replacement of the 5D continuous potential by square barrier, the reflection and transmission coefficients ($R, T$) are obtained. At last, we also compare the coefficients with usual 4D counterpart.

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I. INTRODUCTION

Black hole radiation, which was proved originally by Stephen Hawking with the method of quantum field on gravitational collapsing [1][2] indicates that black holes are not perfect black but radiate thermally and eventually explode. Since then, many people have used various methods and techniques to research black hole through the particles radiating from it, such as the simple Klein-Gordon particles and Dirac particles (for some early works, see Damour and Ruffini [3] and Chandrasekhar [4] respectively). Here, scalar particles are only considered. Recently, Higuchi et al. [5] and Grispino et al. [6] gave the scalar field solution outside a Schwarzschild black hole; Brady et al. [7], Brevik et al. [8] and Tian et al. [9] studied the Schwarzschild-de Sitter (SdS) case; Guo et al. [10] made further studies in the Reissner-Nordström-de Sitter one. As for recent studies of searching evaporating black holes, one can refer to the works [11][12][13].

The idea that the world may have more than four dimensions is due to Kaluza [14] and Klein [15], who realized that a 5D manifold could be used to unify general relativity with Maxwell’s theory of electromagnetism. After that, many people focus on the robust higher dimensional space. Here, we consider the Space-Time-Matter (STM) theory presented by Wesson and co-workers [16][17]. This theory is distinguished from the classical Kaluza-Klein theory for a non-compact fifth dimension, the 4D source is induced from an empty 5D manifold. Because of this, the STM theory is also called induced matter theory and the effective 4D matter is called induced matter. That is, in STM theory, 5D manifold is Ricci-flat while 4D hypersurface is curved by the 4D induced matter. Mathematically, this approach is supported by Campbell’s theorem which states that any analytical solution of N-dimensional Einstein equations with a source can be locally embedded in an (N+1)-dimensional Ricci-flat manifold [18]. In the framework of STM, people studied many works such as Quantum Dirac Equation [19], Perihelion Problem [20], Kaluza-Klein Solitons [21], Black Hole [22][23], Solar System Tests [23] and so on.

In order to avoid interactions beyond any acceptable phenomenological limits, people assume standard model fields (such as fermions, gauge bosons, Higgs fields) are confined on a ($3 + 1$) dimensional hypersurface (3-brane) without accessing along the transverse dimensions. The branes are embedded in the higher dimensional spacetime (bulk), in which only gravitons and scalar particles without charges could propagate under standard model gauge group. There are also many works (for a review with large extra dimensions see [24]) focusing on Hawking radiation such as [25][26][27][28][29][30][31].

Cosmological constant $\Lambda$, which is a parameter with dimension $L^{-2}$ ($L$ is length), is one of focuses in Gravitation Theory. The acceleration of the universe is explained by the required repulsive force produced by a non-zero and positive cosmological constant $\Lambda$. Current SnIa observation data shows that the cosmological constant has a value of $\Lambda_0 \sim 10^{-52}m^{-2}$ [32][33][34]. Its robust non-zero magnitude engages the researching interest in the space contained cosmological constant. Especially, the black hole contained effective cosmological constant is studied widely either in higher dimensions background [26][27][28] or in usual 4D case [3][8][9][10]. Sometimes for the sake of study, $\Lambda$
is considered as a free parameter like in the works 8, 9, 10. In SdS space, the interval between black hole horizon \( r_c \) and cosmological horizon \( r_e \) becomes smaller with increase value of \( \Lambda \). If cosmological constant \( \Lambda \) reaches its maximum, Nariai black hole will be arisen. In this paper, we study how a massless scalar field evolves in this extreme case.

This paper is organized as follows: in section II, the 5D SdS black string space, the time-dependent radial equation about \( R_c(r,t) \) and the fifth dimensional equation about \( L(y) \) are restated. In section III, by a tortoise coordinate transformation, the radial equation becomes a Schrödinger wavelike one. According to the boundary condition and the tangent approximation, a full numerical solution is presented. In section VI, using the replacement of real potential barriers around black hole by square barriers, the reflection and transmission coefficients are obtained. Section V is a conclusion.

We adopt the signature \((+, -,-,-,-)\), put \( h, c, \) and \( G \) equal to unity. Lowercase Greek indices \( \mu, \nu, \ldots \) will be taken to run over \( 0, 1, 2, 3 \) as usual, while capital indices \( A, B, C \ldots \) run over all five coordinates \((0,1,2,3,4)\).

II. THE MASSLESS SCALAR FIELD IN 5D SCHWARZSCHILD-DE SITTER BLACK STRING SPACE

Within the framework of STM theory, an exact 5D solution presented by Mashhoon, Wesson and Liu 16, 37, 36 describes a 5D black hole. The line element takes the form

\[
dS^2 = \frac{\Lambda \xi^2}{3} \left[ f(r)dt^2 - \frac{1}{f(r)} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] - d\xi^2.
\]

In our case

\[
f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2,
\]

where \( \xi \) is the open non-compact extra dimension coordinate, \( \Lambda \) is the induced cosmological constant and \( M \) is the central mass. The part of this metric inside the square bracket is exactly the same line-element as the 4D SdS solution, which is bounded by two horizons — an inner horizon (black hole horizon) and an outer horizon (one may call it cosmological horizon). This metric satisfies the 5D vacuum equation \( R_{AB} = 0 \), therefore, there is no cosmological constant when viewed from 5D. However when viewed from 4D, there is an effective cosmological constant \( \Lambda \). So one can actually treat this \( \Lambda \) as a parameter which comes from the fifth dimension. This solution has been studied in many works 37 focusing mainly on the induced constant \( \Lambda \), the extra force and so on.

We redefine the fifth dimension in this model,

\[
\xi = \sqrt{\frac{3}{\Lambda}} e^{\sqrt{\frac{\Lambda}{3}} y}.
\]

Then we use 11 ~ 13 to build up a RS type brane model in which one brane is at \( y = 0 \), and the other brane is at \( y = y_1 \). Hence the fifth dimension becomes finite. It could be very small as RS I brane model 38 or very large as RS II model 39. The relation between STM theory and brane world theories, and the embedding of 5D solutions to brane models are studies in 40, 41, 42, 43. For the present brane model, when viewed from a \( (\xi \text{ or } y = \text{constant}) \) hypersurface, the 4D line-element represents exactly the SdS black hole. However, when viewed from 5D, the horizon does not form a 4D sphere — it looks like a black string lying along the fifth dimension. Usually, people call the solution to the 5D equation \( G_{AB} = \Lambda_5 (g_{AB}) \) (\( \Lambda_5 \) is the 5D cosmological constant) as the 5D SdS solution. Therefore, to distinguish with it, we call the solution 11 a black string, or more precisely, a 5D Ricci-flat SdS solution.

After redefining the fifth dimension, the metric 11 can be rewritten as

\[
dS^2 = e^{2\sqrt{\frac{\Lambda}{3}} y} \left[ f(r)dt^2 - \frac{1}{f(r)} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) - dy^2 \right],
\]

where \( y \) is the new fifth dimension. Expression 2 can be recomposed as follows

\[
f(r) = \frac{\Lambda}{3r}(r - r_c)(r - r_e) - (r - r_o).
\]

The singularity of the metric 11 is determined by \( f(r) = 0 \). Here we only consider the real solutions. The solutions to this equation are black hole event horizon \( r_c \), cosmological horizon \( r_e \) and a negative solution \( r_o = -(r_c + r_e) \). The last one has no physical significance, and \( r_c \) and \( r_e \) are given as...
is an extreme and important kind of SdS black holes. The cosmological constant in this limit is given by

\[ \Lambda = \frac{3}{32} \cos \eta, \]

\[ r_c = \frac{3}{\sqrt{\Lambda}} \cos(120^\circ - \eta), \]

where \( \eta = \frac{1}{3} \arccos(-3\sqrt{\Lambda}) \) with \( 30^\circ \leq \eta \leq 60^\circ \). The real physical solutions are accepted only if \( \Lambda \) satisfy \( \Lambda M^2 \leq \frac{1}{9} \).

Then we consider a massless scalar field \( \Phi \) in the 5D black string spacetime, obeying the Klein-Gordon equation

\[ \square \Phi = 0, \]

where \( \square = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \left( \sqrt{g} g^{AB} \frac{\partial}{\partial x} \right) \) is the 5D d'Alembertian operator. We suppose that the separable solutions to Eq. (7) are in the form

\[ \Phi = \frac{1}{\sqrt{4\pi \omega r}} R_\omega(r, t) L(y) Y_{lm}(\theta, \phi), \]

where \( R_\omega(r, t) \) is the radial time-dependent function, \( Y_{lm}(\theta, \phi) \) is the usual spherical harmonic function, and \( L(y) \) is the function about the fifth dimension. The differential equations about \( y \) and \( t, r \) are

\[ \frac{d^2 L(y)}{dy^2} + \Lambda \sqrt{\frac{\Lambda}{3}} \frac{dL(y)}{dy} + \Omega L(y) = 0, \]

\[ -\frac{1}{f(r)} r^2 \frac{\partial^2}{\partial r^2} \left( \frac{R_\omega}{r} \right) + \frac{\partial}{\partial r} \left( r^2 f(r) \frac{\partial}{\partial r} \left( \frac{R_\omega}{r} \right) \right) - \left[ \Omega r^2 + l(l+1) \right] \frac{R_\omega}{r} = 0. \]

Eq. (10) is a time-dependent radial differential equation. Eq. (9) is a differential equation about \( y \), where \( \Omega \) is a constant which is adopted to separate variables \( (t, r, \theta, \phi, y) \).

### III. THE NARIAI BLACK HOLE AND ITS BOUNDARY VALUE PROBLEM

Nariai black hole [44, 45] occurs when the cosmological horizon is very close to the black hole horizon \( r_c \rightarrow r_c \). It is an extreme and important kind of SdS black holes. The cosmological constant in this limit is given by

\[ \Lambda M^2 = \frac{1}{9}. \]

Substituting Eq. (11) into Eq. (5), we can get \( \eta = 60^\circ \) and the horizons \( r_h = r_e = r_c = 3M \). As an illustration of the accuracy, we mention that the choice \( \Lambda M^2 = 0.11 \) [8, 9] leads to \( r_c = 2.8391M \) and \( r_e = 3.1878M \). In order to simplify numerical calculation, we will put \( M = 1 \) in this paper.

#### A. The Fifth Dimensional Function \( L(y) \)

In our previous paper [24], we have introduced a massless scalar field to stabilizing this black string brane model. Considering a single mode of the scalar field, the wave function for this mode may reach its maximum value but keep smooth and finite at the brane. Hence, a steady standing wave is constructed. A suitable superposition of some of the quantized and continuous components of \( L(y) \) may provide a wave function which is very large at \( y = 0 \) and drops rapidly for \( y \neq 0 \). Naturally, a practical 3-brane is formed at the \( y = 0 \) hypersurface. According to this "standing wave" condition in the bulk, the spectrum of \( \Omega \) is broken into two parts. One is the continuous spectra below \( \frac{3}{4} \Lambda \) and the other is the discrete spectra above \( \frac{3}{4} \Lambda \). The quantum parameter \( \Omega_n \) is

\[ \Omega_n = \frac{n^2 \pi^2}{y_1^2} + \frac{3}{4} \Lambda, \]

where \( n = 1, 2, 3 \ldots \) and \( y_1 \) is the thickness of the bulk. So the solutions to Eq. (5) are

\[ L(y) = \begin{cases} 
C_1 e^{-\sqrt{\frac{3}{4} \Lambda} y} \cos \left( \frac{n\pi y}{y_1} \right), & \text{for } \Omega > \frac{3}{4} \Lambda, \\
(C_1 + C_2 y) e^{-\sqrt{\frac{3}{4} \Lambda} y}, & \text{for } \Omega = \frac{3}{4} \Lambda, \\
C_3 e^{-\sqrt{\frac{3}{4} \Lambda} 2\sqrt{\frac{3}{4} \Lambda} y} + C_4 e^{-\sqrt{\frac{3}{4} \Lambda} 3\sqrt{\frac{3}{4} \Lambda} y}, & \text{for } \Omega < \frac{3}{4} \Lambda,
\end{cases} \]

where}

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(C_1 + C_2 y) e^{-\sqrt{\frac{3}{4} \Lambda} y}, & \text{for } \Omega = \frac{3}{4} \Lambda, \\
C_3 e^{-\sqrt{\frac{3}{4} \Lambda} 2\sqrt{\frac{3}{4} \Lambda} y} + C_4 e^{-\sqrt{\frac{3}{4} \Lambda} 3\sqrt{\frac{3}{4} \Lambda} y}, & \text{for } \Omega < \frac{3}{4} \Lambda,
\end{cases} \]

where
where $\Lambda = 0.11$ is the cosmological constant, and $(C, y_0), (C_1, C_2), (C_3, C_4)$ are the three pairs of integration constants. As an illuminating example, Fig. 1 depicts the quantized states of $L(y)$. It illustrate that the eigenfunctions get the maximum on the brane $y = 0$ and get an extremum on the other brane $y = y_1$. For an exponential factor $e^{-\sqrt{3}\Lambda y}$ in the first solution of Eqs. (13), the wave function $L_n$ decays along the fifth dimension. Comparing with general result [24], extreme cosmological constant $\Lambda = 0.11$ gives a more fiercely decay solutions. As the fifth dimension becomes bigger, the probability $|L(y)|^2$ deflects from the original value ($|L(y)|^2_{y=0} = 1$) more and more larger. It gets an extremum instead of the maximum value on the other brane. In this way, two branes can be stabilized by scalar field.

### B. The Schrödinger wavelike equation

A more important aspect of scalar field is radial direction. In Eq. (10) time variable can be eliminated by the Fourier component $e^{-i\omega t}$ via

$$R_\omega(r, t) \rightarrow \Psi_{\omega n}(r)e^{-i\omega t},$$

where the subscript $n$ presents a new wave function unlike the usual 4D case $\psi_\omega$. So Eq. (10) can be rewritten as

$$\left[-f(r) \frac{d}{dr}(f(r) \frac{d}{dr}) + V(r)\right] \Psi_{\omega n}(r) = \omega^2 \Psi_{\omega n}(r),$$

where the potential function is given by

$$V(r) = f(r) \left[ \frac{1}{r} \frac{df(r)}{dr} + \frac{l(l+1)}{r^2} + \Omega \right].$$

Now we introduce the tortoise coordinate

$$x = \frac{1}{2M} \int \frac{dr}{f(r)}.$$

The tortoise coordinate can be expressed by the gravitation surface as follows

$$x = \frac{1}{2M} \left[ \frac{1}{2K_c} \ln \left( 1 - \frac{r}{r_c} \right) - \frac{1}{2K_c} \ln \left( 1 - \frac{r}{r_c} \right) + \frac{1}{2K_o} \ln \left( 1 - \frac{r}{r_o} \right) \right].$$
where
\[ K_i = \frac{1}{2} \left| \frac{d(f)}{dr} \right|_{r=r_i}. \] (19)

Explicitly, we have
\[ K_e = \frac{(r_c - r_e)(r_e - r_o)}{6r_e} \Lambda, \] (20)
\[ K_c = \frac{(r_c - r_e)(r_c - r_o)}{6r_c} \Lambda, \] (21)
\[ K_o = \frac{(r_o - r_e)(r_c - r_o)}{6r_o} \Lambda. \] (22)

So under the tortoise coordinate transformation (17), the radial equation (15) can be rewritten as
\[ \left[ -\frac{d^2}{dx^2} + 4M^2V(r) \right] \Psi_{\omega ln}(x) = 4M^2\omega^2\Psi_{\omega ln}(x), \] (23)
which likes the form of Schrödinger equation in quantum mechanics. Notice that there are two various coordinates — \( r \) and \( x \) in this equation. So people also call it Schrödinger wavelike equation. The incoming or outgoing particle flow between event horizon \( r_e \) and cosmological horizon \( r_c \) is reflected and transmitted by the potential \( V(r) \). Substituting quantum parameters \( \Omega_n \) (12) into Eq. (16), the quantum potentials are obtained as follows
\[ V_n(r) = f(r) \left[ \frac{1}{r} \frac{d(f)}{dr} + \frac{1}{r} \frac{(l+1)}{r^2} + \frac{n^2\pi^2}{y_1^2} + \frac{3}{4} \Lambda \right]. \] (24)

It is highly localizing near \( r \sim (r_e + r_c)/2 \simeq 3 \), falling off exponentially in \( x \) at both \( r = r_e \) and \( r = r_c \). Comparing with the similar case of usual 4D \( \Omega = 0 \) two additional monomials, \( n^2\pi^2/y_1^2 \) and \( 3/4\Lambda \), have appeared in the potential. The form of the potential for \( n=1, 2, 3 \) are illustrated in Fig. 2.

![FIG. 2: The potentials of Nariai black hole with \( n=1 \) (solid), \( n=2 \) (dotted), and \( n=3 \) (dashed). Here \( M=1, \Lambda = 0.11, l = 1, y_1 = 10^{3/2} \) (a very large 5th dimension). The usual 4D potential (\( \Omega = 0 \)) is also plotted with dash-dot line for comparison. The black hole horizon locates at the point A \( r_e \sim 2.8391 \) and the cosmological horizon locates at the point B. The potential tends to zero exponentially quickly as \( x \rightarrow \pm \infty \).](image)

C. The numerical solution

Near the horizons \( r_e \) and \( r_c \), \( x \rightarrow \pm \infty \). According to Eq. (14) and Eq. (24), we can get
\[ V(r_e) = V(r_c) = 0. \] (25)
FIG. 3: Variation of the field amplitude versus $x$ with $M=1$, $l=1$, $\Lambda = 0.11$, $y_1 = 10^{3/2}$ and $n=1$. The solution is close to a harmonic wave.

So Eq. (23) reduces to

\[
\left[ \frac{d^2}{dx^2} + 4M^2\omega^2 \right] \Psi_{\omega l n}(x) = 0.
\]

Its solutions are $e^{\pm \omega t}$. In this paper, we only take into account real field and choose the solution \[8\]

\[
\Psi_{\omega l n} = \cos(2M\omega x).
\]

as boundary condition near the two horizons \[6\].

In real scalar field case, there are two methods to solve Schrödinger wavelike equation (23). One is tangent approximation \[8\] and the other is polynomial approximation \[9\]. With any assigned cosmological constant $\Lambda$, we can always find an appropriate approximate method from those only by adjusting the parameters. Here the former one is adopted to analyze this model. Hence, we employ $\Lambda M^2 = 0.11$ and use the useful tangent approximation \[8\]

\[
\tilde{x}(r) = 15 \tan[b(r - d) + 5],
\]

in which $b = 2.7/(r_c - r_e)$ and $d = (r_c + r_e)/2$. Because the approximation \[28\] does not allow $|x|$ to become very large, we shorten the interval of $x$ to $[-100,100]$. So boundary condition \[27\] is rewritten as

\[
\Psi_{\omega l n}(-100) = \Psi_{\omega l n}(100) = \cos(200M\omega).
\]

Considering boundary condition \[29\] and tangent approximation \[28\], we can solve Eq. (23) numerically as a boundary value problem by Mathematica software. The variation amplitude of waves $\Psi_{\omega l}$ versus tortoise coordinate $x$ is illustrated in Fig. 3. Considering actual circumstance, we use tortoise transformation \[18\] and also plot the amplitude versus $r$ in Fig. 4 where we only give the first quantum state ($n=1$). The others can be treated by the same way.

**IV. THE REFLECTION AND TRANSMISSION**

We assume that the particle flux with energy $E$ bursts towards a square well along the positive direction of $x$ axis, where the potential is

\[
\hat{V}(x) = \begin{cases} 
V_0, & x_1 < x < x_2, \\
0, & x < x_1 \text{ or } x > x_2.
\end{cases}
\]

From the view of quantum mechanics, considering the wave behavior of the particles, this process is similar to scattering on the surface of propagation medium with thickness of $|x_2 - x_1|$. Parts of them are transmitted and parts of them are reflected back. According to statistical interpretation of wave function, whether the energy $E > V_0$ or not, there is definite probabilities to transmit or reflect by the potential. The reflection and transmission coefficients denote the magnitude of those probabilities.
FIG. 4: Variation of the field amplitude versus $r$ with $M=1$, $l=1$, $\Lambda = 0.11$, $y_1 = 10^{1/2}$ and $n=1$. The waves pile up near the horizons.

As mentioned above, it is necessary to replace the continuously varying potential barrier with a discontinuous barrier of constant height in analytical work. Therefore, the usual reflection and transmission coefficients can be obtained. With the method of [48] and [8], we suppose a scalar wave propagates from $-\infty$ to $+\infty$, which is illustrated in Fig. 6. The same denotation is cited here, namely associating $\Psi''_1$ with the incoming wave in the region $-\infty < x < x_1$, $\Psi''_2$ with the potential plateau $x_1 < x < x_2$, and $\Psi''_3$ with the outgoing wave in the region $x_2 < x < +\infty$. Hence, potential $V(x)$ in Eq. (23) reduces to

$$V(x) = \begin{cases} \hat{V}_1, & -\infty < x < x_1, \\ \hat{V}_2, & x_1 < x < x_2, \\ \hat{V}_3, & x_2 < x < +\infty. \end{cases}$$

(31)

According to square barrier (31), the solutions to Eq. (23) are

$$\Psi_{\omega n} = \begin{cases} a_1 e^{ik_1 x} + b_1 e^{-ik_1 x}, & -\infty < x < x_1, \\ a_2 e^{ik_2 x} + b_2 e^{-ik_2 x}, & x_1 < x < x_2, \\ a_3 e^{ik_3 x}, & x_2 < x < +\infty, \end{cases}$$

(32)

where $k_i = \sqrt{4M^2(\omega^2 - \hat{V}_i)}$ ($i = 1, 2, 3$) are the wave numbers; $a_i$ and $b_i$ are the undetermined coefficients to the solutions. Then we define reflection coefficients for the plane interfaces dividing two media

$$R_{ij} = \frac{(1 - Z_{ij})^2}{(1 + Z_{ij})^2},$$

(33)

where $Z_{ij} = \frac{k_j}{k_i}$ are the real impedance ratios between medium $i$ and $j$. The width of barrier is $d = x_2 - x_1$ and the height of square barrier is $H = \hat{V}_2$. So in this model reflection coefficients $R$ and transmission coefficients $T$ are given as

$$R = \left| \frac{b_1}{a_1} \right|^2 = \frac{R_{12} + R_{23} + 2\sqrt{R_{12}R_{23}} \cos(2k_2d)}{1 + R_{12}R_{23} + 2\sqrt{R_{12}R_{23}} \cos(2k_2d)},$$

$$T = \left| \frac{a_3}{a_1} \right|^2 = \frac{16}{(1 + Z_{12})^2(1 + Z_{23})^2 + 1 + R_{12}R_{23} + 2\sqrt{R_{12}R_{23}} \cos(2k_2d)}. $$

(34)

(35)

Because the same width are adopted here, we only give the functional image of $\log R$ versus height $H$ (or $\hat{V}_2$) in Fig. 5. There is no surprise that it take oscillating like forms. One can read this feature directly from Eq. (34), which contains cosine functions. Then we use tangent approximation (28) and get the replacements of the 5D continuous potentials by square barriers in Fig. 6. The different reasonable $\hat{V}_2$ are read off as the height of those square barriers. Meanwhile, we choose the incoming wave number to be $k_1 = 2 (\hat{V}_1 = 0)$ and $k_3 = 2 (\hat{V}_3 = 0)$. Substituting those parameters into Eqs. (34) (35), we can obtain reflection and transmission coefficients. Comparing those coefficients with usual 4D SdS case [8], one can see the difference clearly in Table I. Viewing from Fig. 5 we can get a relationship
FIG. 5: $\log R$ versus height $H$ (or $\hat{V}_2$) with the width $d = 40$, $M = 1$, $l = 1$, $\Lambda = 0.11$ and $y_1 = 10^{3/2}$.

FIG. 6: Replacement of real 5D SdS potential barriers around Nariai black hole by square barriers with $n=1$ (solid), $n=2$ (dotted), $n=3$ (dashed) and $\Omega = 0$ (usual 4D case with dash-dot line). We use $M=1$, $l=1$, $\Lambda = 0.11$ and $y_1 = 10^{3/2}$.

Of the four heights $H_{\Omega=0} < H_{n=1} < H_{n=2} < H_{n=3} < H_0$ (the horizontal ordinate of the first extreme point). Hence, we can say that the four modes ($\Omega = 0$, $n = 1$, $n = 2$, $n = 3$) are in the same monotone increasing space. Obviously, the reflection coefficients $R$ (or $T$, notice $R + T = 1$) of 5D SdS black string are bigger (or smaller) than 4D case. Else, $R_{|n=1} < R_{|n=2} < R_{|n=3}$.

TABLE I: The reflection and transmission coefficients

| mode     | $x_1$ | $x_2$ | $d$  | $v_2$ (or $H$) | $R$      |
|----------|-------|-------|------|----------------|---------|
| 4D SdS   | -20   | 20    | 40   | $7.3 \times 10^{-4}$ | $1.3 \times 10^{-7}$ |
| $n=1$    | -20   | 20    | 40   | $10 \times 10^{-4}$  | $2.5 \times 10^{-7}$  |
| $n=2$    | -20   | 20    | 40   | $11 \times 10^{-4}$  | $3.0 \times 10^{-7}$  |
| $n=3$    | -20   | 20    | 40   | $12.5 \times 10^{-4}$| $3.9 \times 10^{-7}$  |
V. CONCLUSION

In this paper we have solved the real scalar field $\Phi$ and obtained reflection and transmission coefficients ($R$, $T$) around the Nariai black hole in the 5D SdS black string space. We summarize what have been achieved.

1. The 5D solution presented by Mashhoon, Wesson and Liu [36] [35] [16] is exact in higher dimensional gravity theory. It satisfies the 5D Ricci-flat field equation $R_{AB} = 0$. In this paper, two branes are embedded into the bulk. One brane is at $y = 0$ where the standard matter lives. The other brane is at $y = y_1$, where $y_1$ is the thickness of bulk. It is the basal topological structure of this black string space. The usual 4D effective cosmological constant $\Lambda$ is considered to be induced from the 5D Ricci-flat space. One should notice that $\Lambda$ is considered as a free parameter. The distance between black hole horizon and cosmological horizon is shorten with bigger $\Lambda$. In this metric it has nothing to do with the value referred from current cosmological observation. So, if the value of $\Lambda$ increases, the Nariai black hole is inevitably arisen in its last fate.

2. For the well known Nariai case, we have solved the scalar field around it in the 5D SdS black string space. The fifth dimensional component $L(y)$ is presented. Taking into account the classical field theory, one know that standard model fields (such as fermions, gauge bosons, Higgs fields) are confined on a $(3 + 1)$ dimensional hypersurface (3-brane) without accessing along the transverse dimensions. In order to stabilize two branes, the scalar field is led in. According to "standing wave" condition, the fifth dimensional equation can be solved. The scalar field gets its maximum on the brane $y = 0$ and get an extremeness value on the other brane $y = y_1$. So the spectrum of parameter $\Omega$ is broken into two parties, one is quantum $\Omega_n$ and the other is continuous one. The quantum spectrum is illustrated in Fig. 1. It is clear that the extreme Nariai black hole decays more acutely than usual case [24]. Furthermore, the quantum phenomenon emerges distinctly in the waves. One can see those according to the effective potential (24), Fig. 3 and Fig. 4.

3. Because of the singular $f(r) = 0$ in the metric, potential [16] vanishes both near black hole horizon and cosmological horizon. Then the Schrödinger wavelike equation (23) reduces to a solvable one (26). Obviously, according to the real scalar field, we get the boundary conditions (27). Eq. (23) describes one dimensional transmission of waves scattering by potential barrier. In order to solve this equation, we adopt a useful tangent approximation [8] to unite radial coordinate $r$ and tortoise coordinate $x$. So Eq. (23), effective potential (24) and boundary condition (27) constitute a full boundary value problem. Because of the complicated potential (24) and fitting function (28), we only give numerical solution. By used the replacement, the continuous potential is switched into a square barrier. With the classical method of the square barrier, the reflection and transmission coefficients ($R$, $T$) are obtained naturally. The result is presented briefly in Table I.

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