Chaos of a beam on a nonlinear elastic foundation under moving loads

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Abstract: Chaos of a beam on a nonlinear elastic foundation under moving vehicle loads is investigated. A nonlinear vibration equation is obtained using Galerkin method. Occurrence of chaotic motions is analyzed using the method of Melnikov. Effects of system parameters on chaotic region are studied. Poincare maps, phase trajectories and time history curves are also obtained to verify the theoretical result from Melnikov method.

1. Introduction

Pavement is always modeled by a beam on an elastic foundation. Dynamic responses of such a beam on a linear elastic foundation under moving loads [1, 2] were widely studied. Recently dynamics of a beam or a plate on a nonlinear elastic foundation was also attracted much attention. Lenci investigated chaos of a nonlinear beam on Winkler foundation under axial load using the method of Melnikov and found two different paths from bifurcation to chaos [3]. Kargarnovin obtained response of infinite beams supported by nonlinear visco-elastic foundations subjected to harmonic moving loads using a perturbation method and investigated influences of the load speed and frequency on the beam responses [4]. Kang studied the nonlinear behavior of a beam under a distributed axial load with time-dependent terms by Galerkin discretization and spectral balance method [5]. Santee investigated stability of a beam on nonlinear elastic foundation and obtained the critical boundary of system instability with the method of Melnikov [6]. Qiu analyzed the bifurcation and chaos of a circular plate on a nonlinear elastic foundation [7]. Zhang used Galerkin method and numerical integral to research nonlinear dynamics of a Timoshenko beam with damage on visco-elastic foundation [8]. Yang studied nonlinear vibration and singularities of a rectangular thin plate on nonlinear elastic foundation [9]. Xiao investigated bifurcation and chaos of rectangular moderately thick cracked plates on an elastic foundation subjected to periodic load [10]. However, these researches only considered harmonic load with fixed position. Investigation on dynamics of a beam or a plate on a nonlinear elastic foundation under moving loads is seldom found.

Chaos of a beam on a nonlinear elastic foundation under moving vehicle loads is studied in this work. A vehicle-pavement coupling system with a two-DOF nonlinear vehicle and Bernoulli-Euler beam on the nonlinear elastic foundation is built first. And a nonlinear ordinary differential equation is obtained using Galerkin method. Then the critical condition of chaotic motion is derived with

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Melnikov method and effects of system parameters on chaotic region are considered. Furthermore, the theoretical result is verified by numerical methods, such as Poincare maps, phase trajectories and time history curves.

2. Mathematic Model

A vehicle-pavement coupling system with a two-DOF nonlinear vehicle and Bernoulli-Euler beam on a nonlinear elastic foundation is built as shown in Fig. 1. The beam has two simply supported ends and the vehicle runs at a constant speed $v$ from the midpoint of the beam to right direction.

![Fig 1. The vehicle-pavement coupling model](image)

The vehicle’s equations of motion are obtained using d’Alembert principle, which reads

$$\begin{cases}
\ddot{y}_1 + K_1 (y_1 - y_0) + C_1 (y_1' - y_0') = 0 \\
\ddot{y}_2 + K_2 (y_2 - y_1) + F_c = 0
\end{cases}$$

(1)

where $m_1$, $m_2$ are masses of vehicle body and wheel respectively; $y_2$, $y_1$ are respectively body and wheel’s vertical displacements; $K_1$, $K_2$ are stiffness of tire and suspension, respectively; $C_1$ is tire’s damping coefficient. $y_0 = B_0 \sin \Omega t$ is the harmonic road roughness. $F_c$ is a nonlinear skyhook damping force of vehicle suspension. A revised Bingham model is applied here [11]:

$$F_c = C_2 y_2' + F_y \text{sgn}(y_2^2 V_0)$$

(2)

here $C_2$ is the viscous damping coefficient, $F_y$ is controlling force and $V_0$ is the velocity when MRF damping force is zero.

The vehicle loads acting on pavement can be expressed as

$$F = K_1 (y_1 - y_0) + C_1 (y_1' - y_0') - (m_1 + m_2) g = F_1 \sin \phi + \phi - F_2 \sin \phi + \theta - (m_1 + m_2) g$$

(3)

where $F_1 = \sqrt{(K_1 B_1)^2 + (C_1 B_1 \Omega)^2}$, $F_2 = \sqrt{(K_1 B_0)^2 + (C_1 B_0 \Omega)^2}$, $\theta = \tan^{-1} (C_1 \Omega / K_1)$. The deriving steps to obtain Eq.(3) and expressions of parameters in Eq.(3) are omitted here, which can be found in authors’ previous publication[12].

The vertical vibration equation of the beam on nonlinear elastic foundation under moving vehicle loads can be obtained as follows

$$EI \frac{\partial^4 y_r}{\partial x^4} + K_3 y_r + K_4 y_r^3 + C \frac{\partial y_r}{\partial t} + m \frac{\partial^2 y_r}{\partial t^2} + F \delta(x - L/2 - vt) = 0$$

(4)
where $E$, $I$, $x$, $x_1$, $K_3$, $K_4$, $C$, $L$ are the modulus of elasticity, cross-sectional moment of inertia, vehicle’s position in running direction, beam’s vertical displacement, linear foundation stiffness, nonlinear foundation stiffness, foundation damping and the beam’s length respectively.

The initial conditions and boundary conditions of the beam are

$$y_r(x,0) = y_r(0) = y_r(x_1) = y_r(L) = 0$$

(5)

Letting $y_r = U(t) \sin \pi x / L$, a nonlinear ordinary equation can be gained as follows

$$\ddot{U}(t) + \frac{C}{M} U(t) + \frac{E l^4}{M L^4} U(t) + \frac{3K_2}{4M} U(t)^3 = F \sin \frac{\pi(L/2 + vt)}{L}$$

(6)

3. The critical condition of chaos

Substituting vehicle loads (3) into Eq.(6), letting dimensionless displacement $x_r = U / B_0$ and dimensionless time $\tau = \Omega t$, one get the dimensionless system

$$\ddot{x}_r + \omega_r^2 x_r = -\omega_r \alpha_3 x^3 + (\alpha_4 \sin \tau + \alpha_5 \cos \tau + \alpha_6) \cos \gamma \tau$$

(7)

where, $w_r = \sqrt{\frac{E l^4}{M L^4}} \Omega$, $\epsilon = \frac{U}{(M \Omega)}$, $\gamma = \frac{L_0}{(2L)}$, $\alpha_3 = \frac{3K_2 B^2}{(4M \Omega^2)}$, $\alpha_4 = \frac{(F_1 \cos \phi + \theta) - F_2 \cos \theta \cos \phi}{(BM \Omega^2)}$, $\alpha_5 = \frac{(F_1 \sin \phi + \theta) - F_2 \sin \theta \cos \phi}{(BM \Omega^2)}$, $\alpha_6 = -(m_1 + m_2) g / (BM \Omega^2)$.

Eq.(7) can be simplified by triangular function transformation,

$$\ddot{x}_r + \omega_r^2 x_r = -\omega_r \alpha_3 x^3 + \alpha \cos (\gamma_1 \tau + \theta_1) + \alpha \cos (\gamma_2 \tau + \theta_2) + \alpha_6 \cos \gamma \tau$$

(8)

where $\alpha = \sqrt{\frac{\alpha_4^2 + \alpha_5^2}{2}}$, $\theta_1 = \tan^{-1}(\frac{\alpha_5}{\alpha_4} - \pi / 2)$, $\gamma_1 = 1 + \gamma$, $\gamma_2 = 1 - \gamma$.

One considers the transformation

$$\eta = x_r \sqrt{\alpha_3} / \omega_r, \xi = \omega_r \tau$$

(9)

When the nonlinear foundation stiffness $K_2$ is negative, Eq.(8) reduces to a soft spring Duffing system with three excitations,

$$\ddot{\eta} + \eta^3 + \frac{\epsilon}{\omega_r} \dot{\eta} = \frac{\alpha_1 \sqrt{\alpha_3} \cos \gamma_1 \xi + \theta_1}{\alpha_3} + \frac{\alpha_2 \sqrt{\alpha_3} \cos \gamma_2 \xi + \theta_2}{\alpha_3} + \frac{\alpha_6 \sqrt{\alpha_3} \cos \gamma \xi}{\alpha_3}$$

(10)

When $K_2$ is positive, Eq.(8) becomes a stiff spring Duffing system with three excitations,

$$\ddot{\eta} + \eta^3 + \frac{\epsilon}{\omega_r} \dot{\eta} = \frac{\alpha_1 \sqrt{\alpha_3} \cos \gamma_1 \xi + \theta_1}{\alpha_3} + \frac{\alpha_2 \sqrt{\alpha_3} \cos \gamma_2 \xi + \theta_2}{\alpha_3} + \frac{\alpha_6 \sqrt{\alpha_3} \cos \gamma \xi}{\alpha_3}$$

(11)

According to singular point analysis, system (11) has only one fixed point $O(0,0)$, while system (10) has three fixed point: $O(0,0)$, $A(1,0)$, $B(-1,0)$. In phase portrait, point $O$ is a center, and point $A$ and $B$ are saddle points. Since phase trajectories cannot cross a centre, the unperturbed system shows a stable periodic motion and the chaotic motion in the sense of Smale horseshoe cannot exist in system (11).

Hence system (10) is studied only. Eq.(10) can be rewritten to the following state equations

$$\begin{align*}
\dot{\eta}_1 &= \eta_2 \\
\dot{\eta}_2 &= -\eta_1 + \eta_3 + \epsilon [-D_2 \eta_2 + D_3 \cos(\omega r \xi) + D_4 \cos(\omega r \xi + \theta_1) + D_5 \cos(\omega r \xi + \theta_2)]
\end{align*}$$

(12)

where $D_1 = 1 / \omega_r, D_2 = \sqrt{\alpha_1 \alpha_2 / (\omega_r^2)}$, $D_3 = \sqrt{\alpha_3 \alpha_4 / (\omega_r^2)}$, $\omega = \gamma_1 / \omega_r, \omega_2 = \gamma_2 / \omega_r$.

The unperturbed system of (12) is the Hamiltonian system. In this system there are two heteroclinic orbits passing through saddle points A and B. The heteroclinic orbits satisfy

$$H = \frac{\eta_1^2}{2} + \frac{\eta_2^2}{2} - \frac{\eta_1^4}{4} = 1 / 4$$

(13)

Substituting $\dot{\eta}_1 = \eta_2$ into Eq.(13) and letting $\eta_1(0) = 0$ leads to $\eta_1$. Substituting $\eta_1$ into $\dot{\eta}_1 = \eta_2$ leads to $\eta_2$. Thus one gets heteroclinic orbits
The Melnikov’s function of the heteroclinic orbits (14) is

\[
\left\{ \begin{array}{l}
\eta_1(t) \\
\eta_2(t)
\end{array} \right\} = \begin{pmatrix}
\pm \tanh \left( \frac{1}{2} t \right) \\
\pm \frac{1}{2} \sec \left( \frac{1}{2} t \right)
\end{pmatrix}
\]

The Melnikov’s function of the heteroclinic orbits (14) is

\[
M_+(\xi_0) = \int_{-\infty}^{\infty} \eta_2(\xi)|D\eta_2(\xi) + D_2 \cos \left( \frac{\pi}{\alpha_1} (\xi + \xi_0) \right) + D_3 \cos \left( \frac{\pi}{\alpha_3} (\xi + \xi_0) + \theta_1 \right)| \, dt
\]

\[
= \frac{2\sqrt{2}}{3} D_1 \pm \sqrt{2} D_2 \sec \left( \frac{\pi}{\sqrt{2}} \right) \csc h \left( \frac{\mu_0}{\sqrt{2}} \right) \pm \sqrt{2} D_3 \sec \left( \frac{\pi}{\sqrt{2}} \right) \csc h \left( \frac{\mu_0}{\sqrt{2}} \right)
\]

The condition for onset of chaos in the sense of Smale horseshoes can be obtained:

\[
\alpha(\Omega) \geq \left[ \frac{2\sqrt{2}}{3} \frac{\sigma^2}{\alpha_3} - \sqrt{2} \frac{\mu_0}{\alpha_1} \csc \left( \frac{\pi}{\sqrt{2}} \right) \right] \pm \sqrt{2} \frac{1+\gamma}{\alpha_2} \csc h \left( \frac{\pi (1+\gamma)}{\sqrt{2} \alpha_1} \right) + \sqrt{2} \frac{1-\gamma}{\alpha_4} \csc h \left( \frac{\pi (1-\gamma)}{\sqrt{2} \alpha_4} \right) = R(\Omega)
\]

4. Numerical results

The vehicle-pavement coupling system’s parameters are

\[
m_2=10109Kg, \quad m_1=190Kg, \quad K_1=2060000N/m, \quad C_1=900Ns^2/m, \quad k_2=75000N/m, \quad C_2=9000Ns^2/m, \quad v_0=0.1, \quad F_y=1000N, \quad \varepsilon=0.15, \quad B_0=0.5 \text{ m}, \quad \Omega=2Hz, \quad L_0=20m, \quad E=1.6 \times 10^9N/m^2, \quad I=5 \times 10^4m^4, \quad K_3=48 \times 10^6N/m^2, \quad K_4=4.8 \times 10^6N/m^2, \quad C=0.3 \times 10^5Ns/m^2, \quad M=1500Kg, \quad L=120m.
\]

According to Eq.(16), Effects of \( C_1, k_1, C, B \) on chaotic region are shown in Fig.2. It can be seen from Fig.2 that,

(1) With increase of \( C_1 \) function \( \alpha(\Omega) \) decreases and \( R(\Omega) \) doesn’t change. It shows that the chaotic region between \( \alpha(\Omega) \) and \( R(\Omega) \) in parameter space shrinks with rise of \( C_1 \).

(2) Function \( \alpha(\Omega) \) increases and \( R(\Omega) \) doesn’t change when \( k_1 \) increases. Thus, the chaotic region in parameter space expands with increase of \( k_1 \).

(3) Function \( R(\Omega) \) increases and \( \alpha(\Omega) \) doesn’t change when \( C \) increases. Thus, the chaotic region in parameter space shrinks with increase of \( C \).

(4) With increase of \( B \) Function \( R(\Omega) \) decreases and \( \alpha(\Omega) \) doesn’t change. Thus, the chaotic region in parameter space expands with increase of \( B \).

In addition, effects of foundation stiffness \( K \) and pavement density \( M \) on chaotic range are also studied. It is found that these two parameters’ effects are small.

As an example, only the cases of \( B=0.008 \) and \( B=0.8 \) are investigated numerically in this work. When \( B=0.008 \), condition (16) can’t be satisfied. The system’s Lyapunov exponents are 0, 0, 0, and Lyapunov dimension is 0. When \( B=0.8 \), condition (16) is satisfied. The system’s Lyapunov exponents are 0.0219, 0, -0.0219, and Lyapunov dimension is 2.9993. Eq.(12) is integrated numerically under the initial condition (-0.8, 0) using four-order fixed step Runge-Kutta method. Fig.3 and Fig.4 show Poincare maps, phase trajectories and time history curves. From Fig.3 and Fig.4, it can be found that,

(1) In both cases, the system’s motion may be chaotic for a long time.

(2) When \( B=0.008 \) the Poincare maps go to a focus point. Phase trajectories are a series of close orbits and the time history curve decays slowly. This shows that the system’s motion is a periodic motion with gradually reduced vibration amplitude.

(3) When \( B=0.8 \) the Poincare maps go to a strange attractor. The phase trajectories are complicated and irregular, and the time history curve is not periodic. Thus the system’s motion is chaotic.
5. Conclusions

Chaos of a beam on a nonlinear elastic foundation under moving vehicle loads is investigated with the method of Galerkin and Melnikov’s function. Some numerical results are presented as well. It can be concluded from this work that:

(1) With the rise of tire damping $c_1$ and foundation damping $C$, the chaotic region will be shrunk; while with the rise of tire stiffness $k_1$ and road roughness $B$, the chaotic region will be expanded. Hence, in order to enlarge pavement’s life span one can rise tire damping $c_1$ and foundation damping $C$ or reduce tire stiffness $k_1$ and road roughness $B$.

(2) When $B=0.008m$, road’s vibration excited by moving vehicle loads changes from transient chaos to attenuated periodic motion and finally disappears.
(3) When \( B = 0.8 \text{m} \), road’s vibration excited by moving vehicle loads is a chaotic motion. This vibration may lead to pavement damage, which is harmful to pavement and vehicle.

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