Propagators of isochronous an-harmonic oscillators and Mehler formula for the x-Hermite polynomials

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It is shown that fundamental solutions $K^\sigma(x, y; t) = \langle x | e^{-iH^\sigma t} | y \rangle$ of the non-stationary Schrödinger equation (Green functions, or propagators) for the rational extensions of the Harmonic oscillator $H^\sigma = H_{osc} + \Delta V^\sigma$ are expressed in terms of elementary functions only. An algorithm to calculate explicitly $K^\sigma$ for an arbitrary $\sigma \in \mathbb{N}$ is given, compact expressions for $K^{(1,2)}$ and $K^{(2,3)}$ are presented. A generalization of the Mehler’s formula to the case of exceptional Hermite polynomials is given.

Introduction

In this work we present new examples of exactly-solved propagators in one-dimensional quantum mechanics. We consider rationally extended Harmonic oscillators [1]. In this case the evolution of wave packets is periodic, we have so called isochronous anharmonic oscillators [2]. This periodicity is related with the quasi-equidistant structure of the spectrum of the Hamiltonian [3].

The simplest rational extension is given by the potential [4]

$$V^{(1,2)}[x] = \frac{x^2}{4} + 2 \left(1 + \frac{1}{2} \left(\frac{x^2 - 1}{(x^2 + 1)^2}\right)\right),$$  \hspace{1cm} (1)

which leads to the quasi-equidistant spectrum [3], $E_n = n + \frac{1}{2}$, where $n \in \mathbb{N}_0 \setminus \{1, 2\}$, for the stationary Schrödinger equation.

Another example is the two-well perturbation of the oscillator

$$V^{(2,3)}[x] = \frac{x^2}{4} + 2 \left(1 + \frac{x^2}{(x^2 + 3)^2}\right)$$  \hspace{1cm} (2)

with the quasi-equidistant spectrum, $E_n = n + \frac{1}{2}$, $n \in \mathbb{N}_0 \setminus \{2, 3\}$.

Note that each rational extension $V^\sigma$ of the Harmonic oscillator is defined by a sequence of levels $\sigma \in \mathbb{N}$ which are deleted from the spectrum by Darboux-Crum transformations [1]. Darboux transformations represent a powerful tool to manipulate physical properties of one-dimensional quantum systems [3]. In the case of shape-invariant potentials, relations between propagators were studied in [6, 7]. A more general approach for the calculations of propagators for potentials generated by an arbitrary chain of Darboux transformations were developed in [8–11].

As a particular example, propagators $K^{(k,k+1)}$ for the $V^{(k,k+1)}$ family were defined through a generating function $S(x, y; t|J)$ which contains the error-function [11]. Here we will further simplify and extend this result. First, we will show that the propagator $K^\sigma$ for an arbitrary potential $V^\sigma$ is expressed by elementary functions only. In the case of potentials [11] and [2] we get the following propagators

$$K^{(1,2)}(x, y; t) = e^{-2it} K_{osc}(x, y; t) \left(1 - \frac{4i \sin t \left[xy - e^{it}\right]}{(1 + x^2)(1 + y^2)}\right),$$  \hspace{1cm} (3)

$$K^{(2,3)}(x, y; t) = e^{-2it} K_{osc}(x, y; t) \left(1 - \frac{8i \sin t \left[xy(x^2 + 3) - 3(x^2 + y^2) \cos t - 3i(x^2y^2 + 1) \sin t\right]}{(3 + x^4)(3 + y^4)}\right),$$  \hspace{1cm} (4)

where the propagator of the Harmonic oscillator [12] is used

$$K_{osc}(x, y; t) = \frac{1}{\sqrt{4\pi i \sin t}} e^{-i(x^2 + y^2) \cos t - 3i(x^2y^2 + 1) \sin t}. $$  \hspace{1cm} (5)

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Second, we will define a rational anzatz to compute propagators $K^\sigma$. In the general case, propagators for the rationally extended oscillators have the following structure

$$ K^\sigma(x, y; t) = K_{osc}(x, y; t) \frac{\sigma[[-1]+1] \sum_{k=0}^{\sigma[[-1]+1]} Q_k^\sigma(x, y)e^{-ikt}}{\sum_{k=0}^{\sigma[[-1]+1]} Q_k^\sigma(x, y)} $$

(6)

where $Q_k^\sigma(x, y) = Q_k^\sigma(y, x)$ are some polynomials that can be determined iteratively, which is more efficient than the method based on the generating function. The polynomials $Q_k^\sigma(x, y)$ allow also calculate Green functions $G_\sigma(x, y; E)$ and generalize Melcher’s formula for the x-Hermite polynomials.

I. HARMONIC OSCILLATOR, HERMITE POLYNOMIALS AND POTENTIALS WITH QUASI-EQUIDISTANT SPECTRUM

Consider the hamiltonian of the Harmonic oscillator

$$ H_{osc} = -\frac{\partial^2}{\partial x^2} + \frac{x^2}{4}, $$

(7)

with eigen-functions defined in terms of probabilistic Hermite polynomials

$$ \psi_n(x) = p_n H_n(x)e^{-\frac{x^2}{4}}, \quad p_n = \left(\frac{\pi n!}{\sqrt{2}}\right)^{-\frac{1}{2}}. $$

(8)

Rational extensions are defined as the following perturbations of the Harmonic oscillator

$$ H^\sigma = -\frac{\partial^2}{\partial x^2} + \frac{x^2}{4} - 2\partial^2_{xx}(\ln \text{Wr}[\psi_\sigma(x), x]) $$

(9)

where $\sigma$,

$$ \sigma = \{k_1, k_1 + 1, \ldots, k_M, k_M + 1\}, \quad |\sigma| = 2M, $$

(10)

is a strictly increasing sequence of natural numbers, or a Krein-Adle sequence.

Following standard notations of the Mathematica program language we denote by $\sigma[[i]]$ and by $\sigma[[-1]] i$-th element and the last element of this sequence, respectively. A sequence of natural numbers appearing as a subscript, for instance, $\psi_{\sigma}(x)$, implies a set of elements,

$$ \psi_{\sigma}(x) = \psi_{\{n_1, n_2, \ldots, n_{2M}\}}(x) = \{\psi_{n_1}(x), \psi_{n_2}(x), \ldots, \psi_{n_{2M}}(x)\}, $$

that is, if $(A_n)_{n>0}$ is a sequence of elements, then $A_{\{n_1, \ldots, n_m\}}$ is a set of elements. This agreement allows us to write Wronskians in a compact form

$$ \text{Wr}[\psi_{\sigma}(x), x] = \begin{vmatrix} \psi_{\sigma[[1]]}(x) & \psi_{\sigma[[2]]}(x) & \cdots & \psi_{\sigma[[1]-1]}(x) \\ \psi_{\sigma[[1]]'}(x) & \psi_{\sigma[[2]]'}(x) & \cdots & \psi_{\sigma[[1]-1]'}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{\sigma[[1]]'}(x) & \psi_{\sigma[[2]]'}(x) & \cdots & \psi_{\sigma[[1]-1]'}(x) \end{vmatrix}. $$

(11)

Hamiltonians $H_{osc}$ and $H^\sigma$ can be embedded into a polynomial SUSY algebra

$$ LH_{osc} = H^\sigma L, \quad L^+L = \prod_{j=1}^{2M} (H_{osc} - \sigma[[j]]), \quad LL^+ = \prod_{j=1}^{2M} (H^\sigma - \sigma[[j]]), $$

(12)

where $L$ is a differential operator of $2M$-th order.

$$ Lf(x) = \frac{\text{Wr}[\psi_{\sigma}(x), \{f(x)\}, x]}{\text{Wr}[\psi_{\sigma}(x), x]}. $$

(13)

Operator $L$ maps the oscillator eigen-functions to the eigen-functions of the rationally extended oscillator

$$ \psi_{\sigma}^n(x) = N_n L\psi_n(x), $$

(14)
where a normalization factor is taken into account

\[
N_n = \begin{cases} 
\left( \frac{2M}{\prod_{j=1}^{2M} (n - \sigma([j]))} \right)^{\frac{1}{2}}, & n \notin \sigma, \\
0, & n \in \sigma.
\end{cases} 
\]  

(15)

Using explicit form of oscillator eigen-functions and the following identities

\[
W_r[\psi_\sigma(x), x] = e^{-\frac{Mx^2}{2}} \prod_{n=1}^{2M} \sigma[[n]]
\]

\[
\ln W_r[\psi_\sigma(x), x] = -\frac{Mx^2}{2} + \ln \prod_{n=1}^{2M} \sigma[[n]] + \ln W_r[He_\sigma(x), x]
\]

we can express rationally extended Harmonic oscillators through the Wronskian of probabilistic Hermite polynomials only

\[
H_\sigma = -\partial_{xx}^2 + \frac{x^2}{4} - 2\partial_{xx}^2 (\ln W_r[He_\sigma(x), x]) + 2M.
\]  

(16)

Note (see for instance [1]) that in the chosen class of \( \sigma \) is the polynomial of \( x^2 \) and

\[
\deg_x W_r[He_\sigma(x), x] = \sum_{n=1}^{2M} (\sigma[[n]] - n + 1).
\]

It is convenient to introduce normalized polynomials

\[
h_n(x) = p_n He_n(x),
\]  

(17)

and the corresponding normalized exceptional Hermite polynomials

\[
h_n^\sigma(x) = N_n W_r[h_\sigma \cup h_n, x]
\]  

(18)

Finally, we define compact notations

\[
W(x) = W_r[\psi_\sigma(x), x], \quad \hat{W}(x) = W_r[He_\sigma(x), x],
\]  

(19)

\[
W_n(x) = W_r[\psi_\sigma(\{\sigma[[n]]\}), x], \quad \hat{W}_n(x) = W_r[He_\sigma(\{\sigma[[n]]\}), x],
\]  

(20)

\[
\hat{L} f = \frac{W_r[He_\sigma(x) \cup \{ f \}, x]}{W_r[He_\sigma(x), x]}.
\]  

(21)

Note that

\[
W(x) = e^{-\frac{Mx^2}{2}} \hat{W}(x) \prod_{n=1}^{2M} \sigma[[n]], \quad W_n(x) = e^{-\frac{(2M+1)x^2}{2}} \hat{W}_n(x) \prod_{n=1}^{2M} \sigma[[n]],
\]  

(22)

\[
Le^{-\frac{x^2}{2}} f = e^{-\frac{x^2}{2}} \hat{L} f.
\]  

(23)

II. PROPAGATORS OF RATIONALLY EXTENDED HARMONIC OSCILLATORS

A. Generating function formalism

The Schrödinger equation for the Green function reads

\[
(i \partial_t - H)K(x, y; t) = 0, \quad K(x, y, 0) = \delta(x - y).
\]  

(24)
If two Hamiltonians $H_0$ and $H_N$ are related by $N$-th order Darboux transformation which remove $N$ levels from the spectrum of $H_0$, then corresponding propagators $K_0$ and $K_N$ are related as follows [11],

$$K_N(x, y; t) = L_x \sum_{n=1}^{N} (-1)^n \frac{W_n(y)}{W(y)} \int_y^b K_0(x, z; t) u_n(z) dz.$$  \hspace{1cm} (25)

In the case of rationally extended Harmonic oscillators $b = \infty$, $N = |\sigma|$, and the transformation solutions coincide with Harmonic oscillator eigenfunctions $u_n = \psi_{\sigma[n]}(z)$.

Using [22] we can replace Wronskians of wave functions by Wronskians of Hermite polynomials

$$K^\sigma(x, y; t) = 2^M \sum_{n=1}^{2M} (-1)^n e^{\frac{x^2}{2}} \frac{W_n(y)}{W(y)} L_x \int_y^\infty K_{osc}(x, z; t) e^{-\frac{z^2}{2}} H_{\sigma[n]}(z) dz.$$  \hspace{1cm} (26)

The occurring integrals $\int_y^\infty K_{osc}(x, z; t) e^{-\frac{z^2}{2}} H_{\sigma[n]}(z) dz$ can be represented as derivatives of the generating function with respect to the auxiliary current $J$

$$\int_y^\infty K_{osc}(x, z; t) e^{-\frac{z^2}{2}} H_{\sigma[n]}(z) dz = [H_{\sigma[n]}(\partial_J) S(J)]_{J=0} = \left[ \sum_{k=0}^{\sigma[n]} h_{\sigma[n], k} \frac{\partial^k S(J)}{\partial J^k} \right]_{J=0},$$

where $h_{m,k}$ are coefficients of the $H_m$. The generating function reads

$$S(J|x, y, t) = \frac{1}{2} e^{\left(\frac{y^2}{4} - \frac{x^2}{4}\right)} R \left[ i J \sqrt{2i \sin t} e^{-\frac{t}{4}}, \frac{xe^{-\frac{t}{2i}}}{i \sqrt{2i \sin t}} \right] E[J, x, y, t]$$  \hspace{1cm} (27)

where

$$R \left[ i J \sqrt{2i \sin t} e^{-\frac{t}{4}}, \frac{xe^{-\frac{t}{2i}}}{i \sqrt{2i \sin t}} \right] = \exp \left( J(i J \sin t + x) \exp(-it) \right),$$

$$E[J, x, y, t] = \left( 1 + \text{erf} \left[ -J \sqrt{2i \sin t} e^{-\frac{t}{4}} \right] - \frac{i \sqrt{1}}{2 \sqrt{\sin t}} \left( ye^{-\frac{t}{2i}} - xe^{-\frac{t}{4i}} \right) \right),$$

$$\text{erf} (z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt.$$  

Function $R$ is the generating function of rescaled Hermite polynomials (see also Appendix A)

$$R[z \sqrt{\alpha}, \frac{x}{\sqrt{\alpha}}] = \exp \left( xz - \frac{\alpha z^2}{2} \right) = \sum_{n=0}^{\infty} H_{n}^{[\alpha]}(x) \frac{z^n}{n!}.$$  

For the compact writing we define

$$\alpha = -2i \sin t e^{-it} = 1 - e^{2it}, \hspace{1cm} e^{-\frac{t}{2i}} \sqrt{\sin t} = \frac{1}{\sqrt{2}} \left( 1 - e^{-2it} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (28)

In what follows we need derivatives of generating functions $R$, $E$ and

$$E_0[x, y; t] = E[0, x, y, t] = \left( 1 + \text{erf} \left[ \frac{i \sqrt{1}}{2 \sqrt{\sin t}} \left( xe^{-\frac{t}{2i}} - ye^{\frac{t}{2i}} \right) \right] \right),$$

1. $J$-derivatives of $R$-function

$$\frac{k!}{(k-m)!} \left( \partial_J^{k-m} R[J \sqrt{\alpha} e^{-it}, \frac{x}{\sqrt{\alpha}}] \right)_{J=0} = \frac{k!}{(k-m)!} \frac{\partial^m}{\partial x^m} H_{k-m}^{[\alpha]}(x) e^{-i(k-m)t} = e^{-i(k-m)t} \frac{d^m}{dx^m} H_{k-m}^{[\alpha]}(x).$$  \hspace{1cm} (29)
2. J-derivatives of E-function

\[
\frac{\partial^{m+1} E[J]}{\partial J^{m+1}} \bigg|_{J=0} = \mathcal{K}_{\text{osc}}(x, y; t) e^{\frac{x^2 + y^2}{2}} e^{-\frac{1}{2i} \sin t} \sum_{j=0}^{m} q_{j, m+1}(x, y) e^{-itj} \tag{30}
\]

3. x-derivatives of \(E_0\)-function

\[
\frac{\partial^{k+1} E_0[x, y, t]}{\partial x^{k+1}} = \mathcal{K}_{\text{osc}}(x, y; t) e^{\frac{x^2 - u^2}{2}} e^{-\frac{1}{2i} \sin t} \left( \frac{1}{2i} \right)^{\frac{k}{2}} \sum_{j=0}^{k} w_{j, m+1}(x, y) e^{-itj} \tag{31}
\]

4. Mixed derivatives of E-function

\[
\frac{\partial^k}{\partial x^k} \left( \frac{\partial^{m+1} E[J]}{\partial J^{m+1}} \right) \bigg|_{J=0} = \left( \frac{1}{2i} \right)^\frac{k-1}{2} e^{\frac{x^2 - u^2}{2}} \sum_{m=0}^{k} C_k^m \left( \frac{\partial^{k-m} R[J \sqrt{\alpha e^{-it}}, \frac{x}{\sqrt{\alpha}}]}{\partial J^n E[J, x, y, t]} \right) \bigg|_{J=0} \tag{32}
\]

In the above expressions, \(q_{j, m}\) and \(w_{k, m}\) are some polynomials.

Now we will substitute these derivatives to calculate

\[
\left[ \mathcal{H}_{\sigma[j]} (\partial J) S(J) \right] \bigg|_{J=0} = \frac{e^{-it\beta^2 - \frac{x^2}{2}}}{2} \left[ \sum_{k=0}^{\sigma[j]} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \left( \frac{\partial^{k-m} R[J \sqrt{\alpha e^{-it}}, \frac{x}{\sqrt{\alpha}}]}{\partial J^n E[J, x, y, t]} \right) \right] \bigg|_{J=0} \tag{33}
\]

Consider first the following double sum

\[
\sum_{m=0}^{\sigma[j]} \frac{1}{m!} (\partial J^n E[J, x, y, t]) \sum_{k=m}^{\sigma[j]} h_{\sigma[j], k} e^{-i(k-m)t} \frac{d^m}{dx^m} \mathcal{H}_k^{[\alpha]}(x) \bigg|_{J=0} \tag{34}
\]

We will change the order of summation \((k, m \rightarrow m, k)\), that is we will fix \(m\) and calculate first sum by \(k \in \{\sigma[j] - m, \sigma[j]\}\).

\[
\sum_{m=0}^{\sigma[j]} \sum_{k=m}^{\sigma[j]} h_{\sigma[j], k} e^{-i(k-m)t} \frac{d^m}{dx^m} \mathcal{H}_k^{[\alpha]}(x) \bigg|_{J=0} \tag{35}
\]

In the last expression we can change the inferior limit \(k = m\) to \(k = 0\) since \(\frac{d^m}{dx^m} \mathcal{H}_k^{[\alpha]}(x) = 0\) when \(k < m\),

\[
e^{-i\sigma[j]t} \sum_{m=0}^{\sigma[j]} \frac{1}{m!} (\partial J^n E[J, x, y, t]) e^{imt} \sum_{k=0}^{\sigma[j]} h_{\sigma[j], k} (e^{2it})^{\frac{\sigma[j] - k}{2}} \mathcal{H}_k^{[\alpha]}(x) \bigg|_{J=0} \tag{36}
\]

Afterwards we note that \(h_{\sigma[j], k}(e^{2it})^{\frac{\sigma[j] - k}{2}} = h_{\sigma[j], k}^{[1-\alpha]}\) (see Appendix A, (55)),

\[
e^{-i\sigma[j]t} \sum_{m=0}^{\sigma[j]} \frac{1}{m!} (\partial J^n E[J, x, y, t]) e^{imt} \sum_{k=0}^{\sigma[j]} h_{\sigma[j], k}^{[1-\alpha]} \mathcal{H}_k^{[\alpha]}(x) \bigg|_{J=0} \tag{37}
\]

The sum by \(k\) represent the umbral composition (57) for the generalized Hermite polynomials \(\mathcal{H}_k^{[\alpha]}(x)\) (16), which yields

\[
e^{-i\sigma[j]t} \sum_{m=0}^{\sigma[j]} \frac{1}{m!} (\partial J^n E[J, x, y, t]) e^{imt} \sum_{k=0}^{\sigma[j]} h_{\sigma[j], k}^{[1-\alpha]} \mathcal{H}_k^{[\alpha]}(x) \bigg|_{J=0} \tag{38}
\]
As a result we obtain the following intermediate expression for the propagator

\[ K^\sigma = \frac{e^{-it}}{2} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \sum_{m=0}^{\sigma[j]} \frac{1}{m!} (\partial_j^m E[J, x, y, t])_{j=0} e^{-i(\sigma[j]-m)t} \frac{dm}{dx^m} \sigma_{x}[j] (x) \cdot \]

We split the last expression in two terms \( K^\sigma = K_E + K_R \), where the first term contains error function in \( E_0(x, y; t) \) whereas the second term contains elementary functions only

\[ K_E = \frac{e^{-it}}{2} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \sum_{m=0}^{\sigma[j]} \frac{1}{m!} (\partial_j^m E[J, x, y, t])_{j=0} e^{-i(\sigma[j]-m)t} \frac{dm}{dx^m} \sigma_{x}[j] (x) , \]

\[ K_R = \frac{e^{-it}}{2} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \sum_{m=0}^{\sigma[j]} \frac{1}{m!} (\partial_j^m E[J, x, y, t])_{j=0} e^{-i(\sigma[j]-m)t} \frac{dm}{dx^m} \sigma_{x}[j] (x) . \]

Changing operator \( \hat{L} \) by \( \hat{L} \) with the aids of (22) we get

\[ K_E = \frac{e^{-it}}{2} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \hat{L} L_0[x, y, t] \sigma_{x}[j] (x) , \]

\[ K_R = \frac{e^{-it}}{2} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \hat{L} L_0[x, y, t] \sigma_{x}[j] (x) + \frac{2M}{2} \sum_{j=1}^{2M} A_n, j(x) \sigma^2 \sigma_{x}[j] (x) + \frac{2M}{2} \sum_{j=1}^{2M} A_n, j(x) \sigma^2 \sigma_{x}[j] (x) . \]

Let us consider the first term \( K_E \). Though the function \( E_0 \) contains non-elementary error function \( \text{erf} \) it can be seen that \( K_E(x, y; t) \) is expressed in the terms of elementary functions only. The error function will be cancelled due to the operator \( \hat{L} \) as follows

\[ K_E = \frac{e^{-it}}{2} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \hat{L} L_0[x, y, t] \sigma_{x}[j] (x) = \]

\[ \frac{e^{-it}}{2} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \left( E_0[x, y, t] \hat{L} L_0[x, y, t] \sigma_{x}[j] (x) + \frac{2M}{2} \sum_{j=1}^{2M} A_n, j(x) \sigma^2 \sigma_{x}[j] (x) \right) , \]

where \( A_n, j(x) \) are some polynomials in \( x \). By the definition (22) of the operator \( \hat{L} \) we have \( \hat{L} \sigma_{x}[j] (x) = 0 \), therefore

\[ K_E = \frac{e^{-it}}{2} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \sum_{m=0}^{\sigma[j]} A_n, j(x) \sigma^m E_0(x, y; t) = \]

\[ \frac{e^{-it}}{2} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \sum_{m=0}^{\sigma[j]} A_n, j(x) \sigma^m E_0(x, y; t) = \]

\[ \frac{e^{-it}}{2} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \sum_{m=0}^{\sigma[j]} A_n, j(x) \sigma^m E_0(x, y; t) = \]

Thus we proved that \( K_E \) contains elementary functions only. Denote \( e^{-it} \) by \( \lambda \). We can further simplify \( K_E \) writing it as a product of \( K_{\text{osc}} \) and a rational function of \( x, y \) and \( \lambda \),

\[ K_E = \frac{K_{\text{osc}}(x, y; t)}{W(y)W(x)} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \sum_{m=0}^{\sigma[j]} A_n, j(x) \sigma^m E_0(x, y; t) = \]

\[ \frac{K_{\text{osc}}(x, y; t)}{W(y)W(x)} \sum_{j=1}^{2M} (-1)^j e^{\frac{x^2}{2} \hat{W}_j(y)} W(y) \sum_{m=0}^{\sigma[j]} A_n, j(x) \sigma^m E_0(x, y; t) = \]
\[
\begin{align*}
K_{\text{osc}}(x, y; t) &= \frac{\sum_{n=1}^{4M+\sigma[[-1]]-1} \tilde{Q}_n(x, y)\lambda^n}{W(y)W(x)} = \frac{\sum_{n=0}^{4M+\sigma[[-1]]-1} \hat{Q}_n(x, y)\lambda^n}{(1 - \lambda^2)^{2M-1}},
\end{align*}
\]

where \(\tilde{Q}_n(x, y)\) are some polynomials.

Following the same line we will consider the second term. It can be seen that structures of \(K_E\) and \(K_R\) coincide, and \(K_R\) reads
\[
K_R = \frac{K_{\text{osc}}(x, y; t)}{W(y)W(x)} \frac{\sum_{n=0}^{4M+\sigma[[-1]]-1} \hat{Q}_n(x, y)\lambda^n}{(1 - \lambda^2)^{2M-1}}.
\]

Combining now \(K_E\) and \(K_R\) we obtain the following expression
\[
K = \frac{K_{\text{osc}}(x, y; t)}{W(y)W(x)} \frac{\sum_{n=0}^{\sigma[-1]+1+4M-2} \hat{Q}_n(x, y)\lambda^n}{(1 - \lambda^2)^{2M-1}} = \frac{K_{\text{osc}}(x, y; t)}{W[h_\sigma(x), x][W(h_\sigma(y), y)]} \frac{\sum_{n=0}^{\sigma[-1]+1+4M-2} \hat{Q}_n(x, y)\lambda^n}{(1 - \lambda^2)^{2M-1}} = \sum_{n=0}^{\sigma[-1]+1} Q_n^\sigma(x, y)\lambda^n
\]

where some constant multiplier \(W[h_\sigma(x), x] = CW(x)\) is absorbed by redefinition of polynomials \(\hat{Q}\).

Consider the limit \(t \to 0\), where \(K^\sigma \to \delta(x - y)\) and \(K_{\text{osc}} \to \delta(x - y)\). From here it follows that the rational \(\lambda\)-depending factor has no pole, therefore
\[
\sum_{n=0}^{\sigma[-1]+1+4M-2} \hat{Q}_n^\sigma(x, y)\lambda^n = \frac{(1 - \lambda^2)^{2M-1}}{(1 - \lambda^2)^{2M-1}} \sum_{n=0}^{\sigma[-1]+1} Q_n^\sigma(x, y)\lambda^n = \sum_{n=0}^{\sigma[-1]+1} Q_n^\sigma(x, y)\lambda^n
\]

and
\[
\sum_{k=0}^{\sigma[-1]+1} Q_k^\sigma(x, y) = \text{Wr}[h_\sigma(x), x][W(h_\sigma(y), y)].
\]

We finally get a rational anzatz for the propagators
\[
K^\sigma(x, y; t) = \frac{K_{\text{osc}}(x, y; t)}{W[h_\sigma(x), x][W(h_\sigma(y), y)]} \frac{\sum_{k=0}^{\sigma[-1]+1} Q_k^\sigma(x, y)e^{-ikt}}{\sum_{k=0}^{\sigma[-1]+1} Q_k^\sigma(x, y)\lambda^n}
\]

where \(Q_k^\sigma(x, y) = Q_k^\sigma(y, x)\) are some polynomials to be determined.

### B. Rational anzatz for the propagators

Substituting into the rational anzatz \([35]\) expansions of propagators in terms of eigen-functions
\[
\begin{align*}
K_{\text{osc}}(x, y; t) &= e^{-\frac{x^2+y^2}{2}} \sum_{n=0}^{\infty} h_n(x)h_n(y)\lambda^n, \quad \lambda = e^{-it},
K_\sigma(x, y; t) &= e^{-\frac{x^2+y^2}{2}} \sum_{n \in \mathbb{N}\sigma} h_n^\sigma(x)h_n^\sigma(y)\lambda^n
\end{align*}
\]

we obtain a system of equations for the polynomial coefficients \(Q_k^\sigma(x, y)\). The solution of this system is given by a recursive procedure
\[
Q_k^\sigma(x, y) = \frac{1}{h_0(x)h_0(y)} \left( h_k^\sigma(x)h_k^\sigma(y) - \sum_{j=1}^{k} Q_{k-j}^\sigma(x, y)h_j(x)h_j(y) \right), \quad 0 \leq k \leq \sigma[-1] + 1.
\]
1. Nonlinear connection Lemma for the exceptional Hermite polynomials

Lemma. Given a Krein-Adler sequence $\sigma = \{k_1, k_1 + 1, \ldots, k_M, k_M + 1\}$, the corresponding family of (formally normalized) polynomials $h^\sigma_n(x)$ obeys the following relation

$$\sum_{k=0}^{\sigma[[-1]]+1} h_{m-k}(x)h_{m-k}(y)Q^\sigma_k(x,y) = h^\sigma_m(x)h^\sigma_m(y),$$

(39)

where polynomials $Q^\sigma_k$ are given by (38).

Proof. The proof follows from (35), (36), (37) and (38) \(\square\)

Relation (39) can also be written in terms of wave functions

$$\sum_{k=0}^{\sigma[[-1]]+1} \psi_{m-k}(x)\psi_{m-k}(y)Q^\sigma_k(x,y) = \psi^\sigma_m(x)\psi^\sigma_m(y).$$

(40)

There are following properties of the $Q^\sigma$-polynomials:

1) Symmetry

$Q^\sigma_k(x,y) = Q^\sigma_k(y,x)$

2) Parity

$Q^\sigma_0(-x,-y) = (-1)^{\deg h^\sigma_0}Q^\sigma_0(x,y)$

$Q^\sigma_{2k}(-x,-y) = (-1)^{\deg h^\sigma_0}Q^\sigma_{2k}(x,y)$, $Q^\sigma_{2k+1}(-x,-y) = (-1)^{\deg h^\sigma_0}Q^\sigma_{2k+1}(x,y)$

where $\deg h^\sigma_0 = \sum_{j=1}^{\sigma} (\sigma[j] - j + 1) - |\sigma|$.

In the Appendix B we present an example of a non-Krein-Adler sequence $\sigma = \{1\}$ when the nonlinear connection lemma is also holds. We suppose that (39) holds for an arbitrary $\sigma$.

2. x-Mehler formula

We first recall the Mehler formula

$$\sum_{n=0}^{\infty} H_{\text{He}}(x)H_{\text{He}}(y) \frac{\lambda^n}{n!} = \frac{1}{\sqrt{1 - \lambda^2}} e^{-\frac{\lambda^2 x^2 + y^2 + 2\lambda xy}{2(1 - \lambda^2)}}.$$

(41)

Using the nonlinear connection lemma (39) we obtain the following generalization of the Mehler formula to the case of exceptional Hermite polynomials

$$\sum_{n=0}^{\infty} h^\sigma_n(x)h^\sigma_n(y)\lambda^n = \frac{1}{\sqrt{2\pi(1 - \lambda^2)}} e^{-\frac{\lambda^2 x^2 + y^2 + 2\lambda xy}{2(1 - \lambda^2)}} \sum_{j=0}^{\sigma[[-1]]+1} Q^\sigma_j(x,y)\lambda^j.$$

(42)

3. Alternative form of the rationally extended Harmonic oscillators

Consider integral kernels of Hamiltonian operators

$$H_{\text{osc}}(x, y) = [-\partial^2_{xx} + V_{\text{osc}}(x)] \delta(x - y) = \sum_{n=0}^{\infty} n\psi_n(x)\psi_n(y),$$

(43)

$$H^\sigma(x, y) = [-\partial^2_{xx} + V^\sigma(x)] \delta(x - y) = \sum_{n=0}^{\infty} n\psi^\sigma_n(x)\psi^\sigma_n(y),$$

(44)
using (40) we can represent the second kernel as follows

\[
H^\sigma (x, y) = \left[ -\partial^2_{xx} + V_{osc}(x) - \sum_{k=0}^{\sigma[-1]+1} \frac{k Q^\sigma_k(x, y)}{\sigma[-1]+1} \right] \delta(x - y). \tag{45}
\]

From here it follows that

\[
\Delta V^\sigma(x) = -2\partial_{xx} \log [\text{Wr}[\psi_\sigma(x), x]] = -\sum_{k=0}^{\sigma[-1]+1} \frac{k Q^\sigma_k(x, x)}{\sigma[-1]+1} Q^\sigma_j(x, x).
\]

4. Green functions

Consider the Green function (resolvent kernel of the Hamiltonian operator)

\[
G(x, y; E) = i \int_0^\infty K(x, y; t) e^{iEt} dt
\]

The transformation formula for the propagators implies also the following relation for the Green functions

\[
G^\sigma(x, y; E) = \frac{1}{W[h^\sigma(x), x]W[h^\sigma(y), y]} \sum_{k=0}^{\sigma[-1]+1} Q_k(x, y) G_{osc}(x, y; E - k) \tag{46}
\]

C. Examples

Using first and second excited states of Harmonic oscillator

\[
\psi_1(x) = p_1 x e^{-x^2/4} \quad \psi_2(x) = p_2(x^2 - 1)e^{-x^2/4}
\]

we obtain a perturbed harmonic oscillator potential [17]

\[
V_{(1,2)}[x] = \frac{x^2}{4} + 2 \left( 1 + 2 \frac{(x^2 - 1)}{(x^2 + 1)^2} \right) \tag{47}
\]

Connection polynomials read

\[
Q_{(0,1,2,3)}^{(1,2)} = \left\{ \frac{1}{2\pi}, -\frac{xy}{2\pi}, \frac{x^2y^2 + x^2 + y^2 - 1}{4\pi}, \frac{xy}{2\pi} \right\}.
\]

The propagator for the Schrödinger equation with Hamiltonian \(K^{(1,2)} = -\partial^2_x + V^{(1,2)}(x)\) has the following compact expression

\[
K^{(1,2)}(x, y; t) = e^{-2it} K_{osc}(x, y; t) \left( 1 - \frac{4i \sin t [xy - e^{it}]}{(1 + x^2)(1 + y^2)} \right), \tag{48}
\]

Next simple expression for the propagator we can obtain using second and third excited states of harmonic oscillator

\[
\psi_2(x) = p_2(x^2 - 1)e^{-x^2/4} \quad \psi_3(x) = p_3x(x^2 - 3)e^{-x^2/4}.
\]

In this case we obtain two-well perturbed harmonic oscillator potential (potentials \(V_{(k,k+1)}\) have \(k\) shallow minima at their bottom)

\[
V_{(2,3)}[x] = \frac{x^2}{4} + 2 \left( 1 + 4x^2 \frac{x^4 - 9}{(x^4 + 3)^2} \right). \tag{49}
\]
Connection polynomials read
\[ Q_{\{2,3\}}^{\{0,1,2,3,4\}} = \left\{ \frac{(x^2 + 1)(y^2 + 1)}{4\pi}, \frac{xy(3 - x^2 y^2)}{6\pi}, \frac{x^4 y^4 + 3x^4 + 3y^4 - 12x^2 y^2 - 3}{24\pi}, \frac{xy(x^2 y^2 - 3)}{6\pi}, \frac{(x^2 - 1)(y^2 - 1)}{4\pi} \right\}. \]

The propagator for the Schrödinger equation with Hamiltonian \( H^{\{2,3\}} = \frac{-\partial^2}{\partial x^2} + V^{\{2,3\}}(x) \) reads
\[ K^{\{2,3\}}(x, y; t) = e^{-\frac{2}{i}t} K_{osc}(x, y; t) \left( 1 - \frac{8i \sin t [xy(x^2 y^2 - 3) - 3(x^2 + y^2) \cos t - 3i (x^2 y^2 + 1) \sin t]}{(3 + x^4)(3 + y^4)} \right), \]

### III. CONCLUSIONS

Propagators
\[ K^\sigma(x, y; t) = K_{osc}(x, y; t) \frac{\sum_{k=0}^{\sigma[-1]+1} Q_k^\sigma(x, y) e^{-ikt}}{\sum_{k=0}^{\sigma[-1]+1} Q_k^\sigma(x, y)} \]

present a new example of Feynman path integrals that can be calculated analytically [18]. The key formula [38]
\[ Q_k^\sigma(x, y) = \frac{1}{h_0(x) h_0(y)} \left( h_k^\sigma(x) h_k^\sigma(y) - \sum_{j=1}^{k} Q_{k-j}^\sigma(x, y) h_j(x) h_j(y) \right), \quad 0 \leq k \leq \sigma[-1] + 1, \]

which define nonlinear connection between x-Hermite and Hermite polynomials can be easily realized in any computer algebra system. One can use these propagators to test various approximations for the corresponding path integrals. Using these propagators wave-packet dynamics in multi-well potentials can be studied analytically.

### Appendix A. Rescaled Hermite polynomials, Appell sequences and umbral composition [16]

The generating function of the probabilistic Hermite polynomials reads
\[ R[z, x] = \exp \left( \frac{xz - z^2}{2} \right) = \sum_{n=0}^{\infty} \text{He}_n(x) \frac{z^n}{n!}. \]

\[ \text{He}_n(x) = \sum_{k=0}^{n} h_{n,k} x^k, \]

Rescaled Hermite polynomials
\[ \text{He}_n^{[\alpha]}(x) = \alpha^{\frac{n}{2}} \text{He}_n \left( \frac{x}{\sqrt{\alpha}} \right), \]
\[ h_{n,k}^{[\alpha]} = \alpha^{(n-k)} h_{n,k}, \]

with the following generating function
\[ R[z \sqrt{\alpha}, \frac{x}{\sqrt{\alpha}}] = \exp \left( \frac{xz - \alpha z^2}{2} \right) = \sum_{n=0}^{\infty} \text{He}_n^{[\alpha]}(x) \frac{z^n}{n!}. \]

form an Appell sequence of polynomials.

Let \( A_n(x) \) and \( B_n(x) \) be two Appell sequences of polynomials [19] generated by functions \( S[x, g], R[x, g] \),
\[ S[x, g] = s(g) e^{xg} = \sum_{n=0}^{\infty} A_n(x) \frac{g^n}{n!}, \]
\[ R[x,g] = r(g)e^{xg} = \sum_{n=0}^{\infty} B_n(x) \frac{g^n}{n!}, \]

where \( A_n(x) = \sum_{k=0}^{n} a_{n,k} x^k, B_n(x) = \sum_{k=0}^{n} b_{n,k} x^k. \)

Define the umbral composition by the following formula

\[ (A_n \circ B)(x) = \sum_{k=0}^{n} a_{n,k} B_k(x) = \sum_{k=0}^{n} \sum_{j=0}^{k} a_{n,k} b_{k,j} x^j. \] (56)

Then the sequence \( C_n(x) = (A_n \circ B)(x) \) also is an Appell sequence. Its generating function, denoted by \( U[x,g] = S[x,g] \circ R[x,g], \) has the following form

\[ U[x,g] = s(g)r(g)e^{xg} = \sum_{n=0}^{\infty} C_n(x) \frac{g^n}{n!}. \]

We can apply these facts to the sequences of generalized Hermite polynomials \( He_n^{[\alpha]}(x) \)

\[ \exp \left( x - \frac{\alpha z^2}{2} \right) \circ \exp \left( x - \frac{\beta z^2}{2} \right) = \exp \left( x - \frac{(\alpha + \beta)z^2}{2} \right). \]

As a result we get the umbral composition of rescaled Hermite polynomials

\[ \left( He_n^{[\alpha]} \circ He^{[\beta]} \right)(x) = \sum_{k=0}^{n} h_n^{[\alpha]} H_k^{[\beta]}(x) = He_n^{[\alpha+\beta]}(x). \] (57)

Appendix B. Q-polynomials for non-Krein-Adler sequence \( \sigma \)

Here we consider an example of the polynomial connection when \( \sigma = \{1\} \), which is a non-Krein-Adler sequence \([1]\), and hence, when polynomials \( h_n^1 \) do not represent a sequence of orthogonal polynomials. Nevertheless, by direct calculations we can verify that the Non-linear connection lemma is valid in this case. Consider formally normalized product of two wronskians

\[ h_n^{[1]}(x) h_m^{[1]}(y) = \frac{1}{\sqrt{2\pi(1-m)}} \left| \begin{array}{ll} x & y \\ h_m(x) & h_m(y) \\ \end{array} \right| \left| \begin{array}{ll} 1 \\ h'_m(x) \\ 1 \\ h'_m(y) \end{array} \right| \]

\[ = \frac{1}{\sqrt{2\pi(1-m)}} \left( x \sqrt{m} h_{m-1}(x) - h_m(x) \right) \left( y \sqrt{m} h_{m-1}(y) - h_m(y) \right). \]

Using the recurrence relation for the normalized probabilistic Hermite polynomials

\[ x \sqrt{m} h_{m-1}(x) = m h_m(x) + \sqrt{m(m-1)} h_{m-2}(x) \]

we get

\[ h_m^{[1]}(x) h_m^{[1]}(y) = \frac{1}{\sqrt{2\pi(1-m)}} \left( \left[ m-1 \right] h_m(x) + \sqrt{m(m-1)} h_{m-2}(x) \right) \left( \left[ m-1 \right] h_m(y) + \sqrt{m(m-1)} h_{m-2}(y) \right) \]

\[ = -\frac{1}{\sqrt{2\pi}} \left( h_m(x) + \frac{\sqrt{m}}{\sqrt{(m-1)^2}} h_{m-2}(x) \right) \left( \left[ m-1 \right] h_m(y) + \sqrt{m(m-1)} h_{m-2}(y) \right) \]

\[ = -\frac{1}{\sqrt{2\pi}} \left( -h_m(x) h_m(y) + m h_m(x) h_m(y) + \sqrt{m(m-1)}(h_m(x) h_m(y) - h_m(x) h_m(y)) + h_m(x) h_m(y) \right) \]

\[ = \frac{1}{\sqrt{2\pi}} h_m(x) h_m(y) - \frac{1}{\sqrt{2\pi}} \left( x h_{m-1}(x) - \sqrt{(m-1)^2} h_{m-2}(x) \right) \left( y h_{m-1}(y) - \sqrt{(m-1)^2} h_{m-2}(y) \right) \]

\[ -\frac{1}{\sqrt{2\pi}} \sqrt{(m-1)} \left( \left( x h_{m-1}(x) - \sqrt{(m-1)^2} h_{m-2}(x) \right) h_{m-2}(y) + \left( y h_{m-1}(y) - \sqrt{(m-1)^2} h_{m-2}(y) \right) h_{m-2}(x) \right) \]
\[\begin{align*}
&-\frac{1}{\sqrt{2\pi}} mh_{m-2}(x) h_{m-2}(y) \\
&= \frac{1}{\sqrt{2\pi}} h_m(x) h_m(y) - \frac{1}{\sqrt{2\pi}} (x h_{m-1}(x) y h_{m-1}(y) + (m - 1) h_{m-2}(x) h_{m-2}(y)) \\
&+ \frac{1}{\sqrt{2\pi}} 2(m - 1) h_{m-2}(x) h_{m-2}(y) \\
&- \frac{1}{\sqrt{2\pi}} mh_{m-2}(x) h_{m-2}(y) \\
&= \frac{1}{\sqrt{2\pi}} (h_m(x) h_m(y) - xy h_{m-1}(x) h_{m-1}(y) - h_{m-2}(x) h_{m-2}(y)) \quad (m \geq 2) \\
&= \frac{1}{\sqrt{2\pi}} (h_m(x) h_m(y) - x y h_{m-1}(x) h_{m-1}(y) - h_{m-2}(x) h_{m-2}(y)) + \frac{1}{\sqrt{2\pi}} (m - 1) h_{m-1}(x) h_{m-1}(y) \quad (m \geq 3)
\end{align*}\]

That is, polynomials \(h_m^{(1)}(x) h_m^{(1)}(y)\) satisfy to the following 3-term representation

\[h_m^{(1)}(x) h_m^{(1)}(y) = \sum_{k=0}^{2} h_{m-k}(x) h_{m-k}(y) Q_k^{(1)}(x, y),\]

where

\[Q_0^{(1)}(x, y) = \frac{1}{\sqrt{2\pi}} ,\]

\[Q_1^{(1)}(x, y) = -\frac{xy}{\sqrt{2\pi}} ,\]

\[Q_2^{(1)}(x, y) = -\frac{1}{\sqrt{2\pi}} .\]

We also verified by computer algebra that the nonlinear connection Lemma is valid for arbitrary \(\sigma\) with \(\sigma[[-1]] < 5\).
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[20] We assume that $V_\sigma$ is defined for all $x \in \mathbb{R}$, which restricts possible choice of $\sigma$, in particular, $|\sigma| = 2M$. See for the details [1].