On surfaces with a prescribed curvilinear projection of one field of principal directions

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Abstract

A class of surfaces-graphs in a Riemannian 3-space with a prescribed projection of one field of principal directions onto a surface Π is considered. A problem of determination of such surfaces when both principal curvatures are given over a line in Π is formulated and studied. The geometric problem is reduced to the Cauchy problem for quasilinear PDE’s which, under certain conditions for data, are hyperbolic and admit a unique solution. It is shown that the parallel curved (PC) surfaces in space forms provide a special class of global solutions to the geometrical problem with weaker regularity assumptions. Such solutions may be found by an iteration function sequence.

Keywords and Phrases: Riemannian space, surface, principal curvature/direction, hyperbolic PDE’s

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Introduction

The surfaces possessing nontrivial deformations which preserve principal curvatures and directions (or, equivalently, the shape operator) were investigated by several authors, see [4], [10] and review with bibliography in [6]. It is known that surfaces with one family of principal curves being geodesic (as for parallel curved (PC) surfaces recently studied in [2] – [3]) represent degenerate case in studying immersions of simply connected surfaces with a prescribed shape operator. Recent studying of reconstruction of surfaces by their partially given principal curvatures and directions may be useful for applications of differential geometry to computer graphics, the wavefront analysis in applied optics, etc.

In what follows, (\(\bar{M}^3, \bar{g}\)) denotes a \(C^3\)-regular Riemannian 3-space with coordinates \(x_1, x_2, x_3\) (\(|x_i| \leq a_i\)) for some \(a_i \in \mathbb{R}\), \(\Pi = \{x_3 = 0\}\) a \(C^3\)-regular surface, \(\gamma = \{x_2 = x_3 = 0\}\) the coordinate curve, and \(\pi(x_1, x_2, x_3) = (x_1, x_2, 0)\) the curvilinear projection.
In the paper we consider surfaces-graphs \( M^2 \subset \tilde{M}^3 \) with prescribed curvilinear projection (onto \( \Pi \)) of one field of principal directions. We show (see Theorem 1) that such surfaces depend on two arbitrary functions of one variable, namely, the principal curvatures over \( \gamma \) which are assumed close enough to corresponding values for \( \Pi \). More precisely, we study the following.

**Problem 1** Given \((\tilde{M}^3, \bar{g})\), a vector field \( l \) transversal to \( \gamma \) on \( \Pi \), functions \( \bar{k}_1 \) and \( \bar{k}_2 \) of class \( C^0(\gamma) \), find a function \( f \) of class \( C^2(\Pi) \), whose graph \( M^2 : x_3 = f(x_1, x_2) \) in \( M^3 \) satisfies the conditions:

(1) *the projection* \((\pi)\) *onto* \( \Pi \) *of the field* \( \partial_1 \) *of principal directions corresponding to the principal curvature* \( k_1 \) *is coincides with* \( l \),

(2) *the principal curvatures* \( k_i \) *over* \( \gamma \) *coincide with* \( \bar{k}_i \; k_i|_\gamma = \bar{k}_i \),

(3) *the values of* \( f \) *and* \( df \) *at the point* \((0,0,0)\) *of* \( \gamma \) *are given.

Our approach is based on reducing the **Problem 1** to the Cauchy problem for a quasilinear system of PDE’s which, under certain conditions for data, is hyperbolic and admits a unique local smooth solution. The PC surfaces in \( \mathbb{R}^3 \) represent a special class of solutions when a family of curvature lines projects onto \( \Pi \) as parallel lines or concentric circles. Such surfaces are recovered by an iteration function sequence and using the reconstruction of two planar curves by their curvature (see Theorem 2 and Proposition 3). Notice that the space of PC surfaces free of umbilics and having the same shape operator depends on one arbitrary function of one variable, see [4].

The structure of the work is the following. **Section 1** represents main results (Theorems 1 – 2 and Corollary 1 for \( M^2 \subset \mathbb{R}^3 \)). **Section 2** contains proofs. **Section 3** contains necessary facts on PC surfaces and examples.

### 1 Main results

We shall use the following notation: \( u = (u_1, \ldots, u_n) \in \mathbb{R}^n; \| u \|_\infty = \max_{1 \leq i \leq n} |u_i|, C^0(D) \) the linear space of bounded continuous functions \( u : D \rightarrow \mathbb{R}^m \) \((D \) is a domain in \( \mathbb{R}^n); \| u \|_D = \sup_{x \in D} \| u(x) \|_\infty \) the norm in \( C^0(D); C^k(D) \) is the set of functions \( u : D \rightarrow \mathbb{R}^m \), having in \( D \) continuous partial derivatives of order \( k \). For short, we omit \( m \) from the above notations.

For simplicity, we assume in what follows that \( \Pi \) is a totally umbilical surface with the normal curvature \( \lambda \) (if \( \lambda \equiv 0 \) then \( \Pi \) is totally geodesic).

The main result of the paper is the following.

**Theorem 1** Let \( \bar{k}_1, \bar{k}_2 \) be functions of class \( C^1(\gamma) \), \( l \) a vector field of class \( C^2(\Pi) \), that is transversal but not orthogonal to \( \gamma \). If \( \| \bar{k}_i - \lambda |_\gamma \|_\gamma \) are small enough, then **Problem 1** admits in \((\tilde{M}^3, \bar{g})\) a unique local solution. Namely, there are \( \Delta, K > 0 \) such that if \( \| \bar{k}_i - \lambda |_\gamma \|_\gamma < \Delta \; (i = 1, 2) \), then for some \( \varepsilon \in (0, a_2] \) there exists a function \( f \) of class \( C^3 \) on \( \Pi_{K,\varepsilon} = \{ |x_1| + Kx_2 \leq a_1, 0 \leq x_2 \leq \varepsilon, x_3 = 0 \} \) with the properties: the principal curvatures \( k_i \) of \( M^2 : x_3 = f(x_1, x_2) \) satisfy \( k_i|_\gamma = \bar{k}_i \), \( l \) is tangent to the
\( \pi \)-projection onto \( \Pi_{K, \varepsilon} \) of \( k_1 \)-curvature lines, and \( f(0,0) = df(0,0) = 0 \). Moreover, there is \( r \in (0, a_3] \) such that the solution \( f \) is unique in the class of \( C^3 \)-regular functions satisfying \( \|(f, f_{x_1}, f_{x_2})\|_{\Pi_{K, \varepsilon}} \leq r \).

One may apply Theorem 1 to surfaces in 3-space forms (see also Section 3.2). We illustrate this for \( \mathbb{R}^3 \) with cartesian coordinates.

**Corollary 1** Let \( \alpha \neq 0 \) be a function of class \( C^2 \) on a rectangle \( \Pi = \{|x| \leq a_1, |y| \leq a_2, z = 0\} \) in \( \mathbb{R}^3 \) (with cartesian coordinates), and \( k_1, k_2 \) functions of class \( C^1 \) on the segment \( \gamma = \{|x| \leq a_1, y = z = 0\} \). If \( \|\vec{k}_i\|_\gamma \) are small enough, then Problem 1 admits a unique local solution. Namely, there are \( \Delta, K > 0 \) such that if \( \|\vec{k}_i\|_\gamma < \Delta \), then for some \( \varepsilon \in (0, a_2] \) there exists a function \( f \) of class \( C^3 \) on \( \Pi_{K, \varepsilon} = \{|x| + K y \leq a_1, 0 \leq y \leq \varepsilon, z = 0\} \) with the properties: the principal curvatures \( k_i \) of the graph \( M^2 : z = f(x,y) \) in \( \mathbb{R}^3 \) satisfy \( k_i(x,0) = \vec{k}_i(x) \), the vector field \( \alpha \partial_x + \partial_y \) is tangent to the projection onto \( \Pi_{K, \varepsilon} \) of \( k_1 \)-curvature lines, and \( f(0,0) = df(0,0) = 0 \). Moreover, there is \( r > 0 \) such that the solution \( f \) is unique in the class of \( C^3 \)-regular functions satisfying \( \|(f, f_{x_1}, f_{y})\|_{\Pi_{K, \varepsilon}} \leq r \).

**Remark 1** (a) The condition that \( \Pi \) is totally umbilical can be dropped. In this case, \( k_i \) should be close enough to corresponding principal curvatures of \( \Pi \) along \( \gamma \), and \( l \) close enough to one of principal directions on \( \Pi \). The values of \( f(0,0) \) and \( df(0,0) \) can be taken small enough (see Proposition 5), in the present text for simplicity we assume them zero.

If \( \alpha = \text{const} \neq 0 \) in Corollary 1 then \( M^2 \) is a PC surface, i.e., the planes \( \{x - \alpha y = c\} \) intersect \( M^2 \) by curvature lines, see Section 3.1. PC surfaces in spherical coordinates, see also Section 3.1 illustrate Theorem 1.

(b) A normal geodesic graph in \( (M^3, g) \) of a function \( f : \Pi \to \mathbb{R}, \ |f| < r_f(\Pi) \) (\( r_f(\Pi) \) is the focal radius of \( \Pi \)) is a surface \( M^2 = \bigcup_{x \in \Pi} \{\gamma_x(f(x))\} \subset \bar{M}^3 \), where \( \gamma_x(t) (x \in \Pi) \) is a unit speed geodesic normal to \( \Pi \).

The semi-geodesic coordinates \( (x_1, x_2) \) on \( \Pi \) with the base curve \( \{x_2 = 0\} \); have the metric is \( g_2 = \xi^2(x_1, x_2) dx_1^2 + dx_2^2 \), where \( \xi_{x_2} + K(x_1, x_2)\xi = 0, \xi(x_1,0) = 1, \xi_2(x_1,0) = 0 \) (see [9]), and \( K \) is the gaussian curvature of \( \Pi \).

Let \( \bar{M}^3 \) has the coordinates \( x_1, x_2, x_3 \) \( |x_1| \leq a_i \) such that
- \((x_1, x_2)\) are semi-geodesic coordinates on a surface \( \Pi = \{x_3 = 0\} \),
- the curve \( \gamma = \{x_2 = x_3 = 0\} \) is a simple geodesic in \( \Pi \), and
- \( x_3 \) is the signed distance to \( \Pi \) (hence \( \bar{g}_{33} = \delta_3 \) and \( \bar{\Gamma}^k_{33} = 0 \)).

One may obtain corollary of Theorem 1 where \( \pi : \bar{M}^3 \to \Pi \) means “the nearest point” in \( \Pi \) (for \( \mathbb{R}^3 \) we again have Corollary 1).

We will formulate for \( \bar{M}^3 \) (and study in the paper for \( \mathbb{R}^3 \)) the problem that plays essential role in solving Problem 1 for PC surfaces.

**Problem 2** Given a vector field \( l = l(x_1, x_2) \) on \( \Pi \subset \bar{M}^3 \), an integral curve \( \gamma_1 \subset \Pi \) of \( l \) through \( O \in \Pi \), a curve \( \gamma_2 \subset \Pi \) transversal to \( \gamma_1 \) through \( O \) and functions
\( \bar{k}_i \in C^0(\gamma_i) \) on \( \gamma_i \) \((i = 1, 2)\), find a function \( f \) of class \( C^2(\Pi) \), whose graph \( M^2 : x_3 = f(x_1, x_2) \) in \( \bar{M}^3 \) satisfies the conditions:

(i) the projection \((\pi)\) onto \( \Pi \) of the field \( \partial_1 \) of principal directions corresponding to the principal curvature \( k_1 \) of \( M^2 \), coincides with \( l \),

(ii) the principal curvatures \( k_i \) of \( M^2 \) coincide with \( \bar{k}_i \) over \( \gamma_i \): \( k_i|_{\gamma_i} = \bar{k}_i \),

(iii) the values of \( f \) and \( df \) at the point \( O \) are given.

Define a rectangle \( \Pi(a) = \{(x, y) : |\alpha x + y| \leq a, |x - \alpha y| \leq a\} \) in the \( xy \)-plane of \( \mathbb{R}^3 \), where \( a, \alpha \in \mathbb{R} \) are positive. The PC surfaces represent a special class of global solutions (i.e., on the domains \( \Pi(a) \) with an arbitrary \( a < a_1 \)) to Problems 1 and 2 with weaker regularity assumptions.

**Theorem 2** Let \( a_1, \alpha > 0 \) be real, \( \bar{k}_1 \in C^1([-a_1, a_1]) \) and \( \bar{k}_2 \in C^0([-a_1, a_1]) \). Then for any \( a \in (0, a_1) \) one can choose \( \delta > 0 \) such that if \( \|\bar{k}_i\|_{[-a_1, a_1]} < \delta \) \((i = 1, 2)\), then there exists a unique PC surface \( M^2 \subset \mathbb{R}^3 : z = f(x, y) \), where \( f \in C^2(\Pi(a)) \), with the properties

- the \( k_1 \)-principal direction \( \partial_1 \) on \( M^2 \) is parallel to the plane \( x - \alpha y = 0 \),
- the principal curvatures of \( M^2 \) satisfy \( k_i(x, 0) = \bar{k}_i(x) \) \(|x| \leq a, i = 1, 2\).

Theorem 2 is based on the following result concerning existence of a solution to Problem 2 in the class of PC surfaces.

**Proposition 1** Given \( a_1, \alpha > 0 \), let \( \gamma_1 = \{(u, v) : |u| \leq \frac{a_1}{\sqrt{\alpha^2 + 1}}\} \) and \( \gamma_2 = \{(x, 0) : |x| \leq a_1\} \) be line segments in \( xy \)-plane of \( \mathbb{R}^3 \). Let \( \bar{k}_1 \in C^1(\gamma_1) \) and \( \bar{k}_2 \in C^0(\gamma_2) \). Then for any \( a \in (0, a_1) \) there is \( \delta > 0 \) such that if \( \bar{k} = \max \{ \|\bar{k}(x, \alpha/\sqrt{\alpha^2 + 1})\|, |\bar{k}_2(x)| \} < \delta \), then there exists a unique PC surface \( M^2 : z = f(x, y) \) in \( \mathbb{R}^3 \), where \( f \in C^2(\Pi(a)) \), with the properties

- the principal direction \( \partial_1 \) is orthogonal to the vector \( e_1 - \alpha e_2 \),
- \( \bar{k} \) is the principal curvature of \( M^2 \) over \( \gamma_1 \) corresponding to \( \partial_1 \), and
- \( \bar{k}_2 \) is the principal curvature of \( M^2 \) over \( \gamma_2 \cap \Pi(a) \) corresponding to the second principal direction \( \partial_2 \).

The solution can be found using reconstruction of two planar curves by their curvature functions.

## 2 Proofs

In Section 2.1 we prove Theorem 1 and its Corollary 1. Based on the Euler formula for the principal curvatures we deduce a system of PDE’s (Proposition 2). Using compatibility conditions, we transform above equations to equivalent quasi-linear system (Proposition 3), for which we formulate the Cauchy Problem. The initial values are analyzed in Lemma 1 where \( M^2 \) is recovered over \( \gamma \). Section 2.2 shows that the PC surfaces in space forms (i.e., a family of curvature lines projects onto
The coefficients of the first and the second fundamental forms of $M_C$ and necessary facts on hyperbolic PDE's. Section 2.3 contains auxiliary lemmas and necessary facts on hyperbolic PDE's.

2.1 Proof of Theorem 1 and its corollary

A surface-graph $M^2$ in $M^3$ is defined by equation $x_3 = f(x_1, x_2)$. Let $\tilde{e}_i = \frac{\partial}{\partial x_i}$ be coordinate vector fields on $\tilde{M}^3$, $\hat{e}_i$ their restrictions on $M^2$, and $e_1, e_2$ the coordinate vector fields (lifts under $\pi$ of $\tilde{e}_1, \tilde{e}_2$ from $\Pi$) on $M^2$. For simplicity assume in what follows that $\tilde{g}_{13} = \tilde{g}_{23} = 0$ (see also Remark 1). We have $e_1 = \hat{e}_1 + p \hat{e}_3$ and $e_2 = \hat{e}_2 + q \hat{e}_3$, where $p = f_x$, and $q = f_{x_3}$. The metric on $M^2$ is given by $g_{ij} = \tilde{g}(e_i, e_j)$.

The coefficients of the first and the second fundamental forms of $M^2$ are denoted by

$$E = g_{11}(x_1, x_2), \quad F = g_{12}(x_1, x_2) = g_{21}(x_1, x_2), \quad G = g_{22}(x_1, x_2),$$

$$L = b_{11}(x_1, x_2), \quad M = b_{12}(x_1, x_2) = b_{21}(x_1, x_2), \quad N = b_{22}(x_1, x_2).$$

Suppose that $\partial_1 = \alpha e_1 + e_2$, $\partial_2 = \beta e_1 + \beta_2 e_2$ are the principal directions on $M^2$. We compute the product $0 = \hat{g}(\partial_1, \partial_2) = \beta_1(\alpha E + F) + \beta_2(\alpha E + G)$. Hence $\beta_1 : \beta_2 = -(\alpha F + G) : (\alpha E + F)$. Denote by $\hat{g}_{ij} = \hat{g}(\hat{e}_i, \hat{e}_j)$.

From above and Lemma 3 (Section 2.3) it follows

$$\alpha^2 E + 2 \alpha F + G = g(\partial_1, \partial_1) > 0,$$

$$\alpha E + F = \hat{g}_{33}(\alpha p + q)p + \alpha \hat{g}_{11} + \hat{g}_{12},$$

$$EG - F^2 = \hat{g}_{33}(\hat{g}_{22}p^2 + \hat{g}_{11}q^2 - 2\hat{g}_{12}pq) + \hat{g}_{11}\hat{g}_{22} - \hat{g}_{12}^2 \geq \hat{g}_{11}\hat{g}_{22} - \hat{g}_{12}^2. \tag{1}$$

Define the functions $H_{ij} = H_{ij}^{(1)} k_1 + H_{ij}^{(2)} k_2 + H_{ij}^{(0)} (i, j = 1, 2)$, where

$$H_{11}^{(1)} = \frac{\delta(\alpha E + F)^2}{\alpha^2 E + 2 \alpha F + G}, \quad H_{11}^{(2)} = \frac{\delta(EG - F^2)}{\alpha^2 E + 2 \alpha F + G}, \quad H_{11}^{(1)} = \frac{\delta(\alpha E + F)(\alpha F + G)}{\alpha^2 E + 2 \alpha F + G},$$

$$H_{12}^{(1)} = -\frac{\delta(EG - F^2)}{\alpha^2 E + 2 \alpha F + G}, \quad H_{22}^{(1)} = \frac{\delta(\alpha E + F)(\alpha F + G)}{\alpha^2 E + 2 \alpha F + G}, \quad H_{22}^{(2)} = \frac{\delta\alpha^3(\alpha E + F)^2}{\alpha^2 E + 2 \alpha F + G}, \tag{2}$$

$$\delta = \sqrt{(EG - F^2)/\det \hat{g}} \geq 1/\sqrt{\hat{g}_{33}}$$

For $\mathbb{R}^3$ we get $H_{ij} = H_{ij}^{(1)} k_1 + H_{ij}^{(2)} k_2$, where $H_{21}^{(1)} = H_{12}^{(1)}$

$$H_{11}^{(1)} = \frac{\delta(\alpha E + F)^2}{\alpha^2 E + 2 \alpha F + G}, \quad H_{11}^{(2)} = \frac{\delta(EG - F^2)}{\alpha^2 E + 2 \alpha F + G}, \quad H_{11}^{(1)} = \frac{\delta(\alpha E + F)(\alpha F + G)}{\alpha^2 E + 2 \alpha F + G},$$

$$H_{12}^{(1)} = \frac{\delta(EG - F^2)}{\alpha^2 E + 2 \alpha F + G}, \quad H_{22}^{(1)} = \frac{\delta(\alpha E + F)(\alpha F + G)}{\alpha^2 E + 2 \alpha F + G}, \quad H_{22}^{(2)} = \frac{\delta\alpha^3(\alpha E + F)^2}{\alpha^2 E + 2 \alpha F + G}, \tag{3}$$

and $\delta_1 = (\alpha p + q)^2 + \alpha^2 + 1 \geq 1$. 

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Example 1 Let \( M^2 = \Pi = \{ x_3 = 0 \} \), hence \( f = p = q = 0 \). By Lemma \( \text{[2],[3]} \), the coefficients of the 1-st and the 2-nd fundamental forms are

\[
\hat{E} = \hat{g}_{11}, \quad \hat{F} = \hat{g}_{12}, \quad \hat{G} = \hat{g}_{22}, \quad \hat{L} = \hat{\Gamma}_1^3 \sqrt{g_{33}}, \quad \hat{M} = \hat{\Gamma}_2^3 \sqrt{g_{33}}, \quad \hat{N} = \hat{\Gamma}_2^3 \sqrt{g_{33}}.
\]

The functions \( \text{[2]} \) on \( \Pi \) have the following form:

\[
\hat{H}_{11}^{(1)} = \delta \frac{(\alpha g_{11} + \beta g_{12})^2}{\alpha g_{11} + \beta g_{12} + g_{22}}, \quad \hat{H}_{11}^{(2)} = \delta \frac{\alpha g_{11} + \beta g_{12} - g_{22}^2}{\alpha g_{11} + \beta g_{12} + g_{22}}, \quad \hat{H}_{12}^{(1)} = \delta \frac{\alpha g_{11} + \beta g_{12}}{\alpha g_{12} + g_{22}^2}, \quad \hat{H}_{12}^{(2)} = \delta \frac{(\alpha g_{11} + \beta g_{12}) (\alpha g_{12} + g_{22})}{\alpha g_{12} + g_{22}^2},
\]

where \( \delta = 1/\sqrt{g_{33}} \). The function \( \alpha \) (of the principal direction of \( \Pi \)) satisfies

\[
(\hat{g}_{11} \hat{\Gamma}_1^3 - \hat{g}_{12} \hat{\Gamma}_1^3) \alpha^2 + (\hat{g}_{11} \hat{\Gamma}_2^3 - \hat{g}_{22} \hat{\Gamma}_1^3) \alpha + (\hat{g}_{12} \hat{\Gamma}_2^3 - \hat{g}_{22} \hat{\Gamma}_1^3) = 0. \tag{4}
\]

The principal curvatures \( \hat{k}_i \) of \( \Pi \) are solutions to the quadratic equation

\[
(\hat{g}_{11} \hat{g}_{22} - \hat{g}_{12}^2) k^2 - 2\sqrt{g_{33}(\hat{g}_{11} \hat{\Gamma}_1^3 + \hat{g}_{22} \hat{\Gamma}_1^3 - 2\hat{g}_{12} \hat{\Gamma}_1^3)k + g_{33} (\hat{\Gamma}_1^3)^2 - (\hat{\Gamma}_1^3)^2} = 0. \tag{5}
\]

Since \( \Pi \) is totally umbilical, \( \hat{k}_1 = \hat{k}_2 = \lambda \) are the roots of \( \text{[5]} \), hence

\[
\hat{\Gamma}_1^3_{ij} = -\lambda \hat{g}_{ij}/\hat{g}_{33} \quad (1 \leq i, j \leq 3). \tag{6}
\]

In this case, \( \text{[4]} \) is satisfied by any \( \alpha \).

First, we will prove Propositions \( \text{[2],[3]} \) and Lemma \( \text{[1]} \).

Proposition 2 Let \( M^2 \subset \tilde{M}^3 \) be the graph of \( f \in C^2(\Pi) \). Then \( k_1, k_2 \in C^0(\Pi) \) are the principal curvatures and \( l = \alpha \hat{e}_1 + \hat{e}_2 \) (with \( \alpha \in C^1(\Pi) \)) is the projection onto \( \Pi \) of \( k_1 \)-principal direction of \( M^2 \) if and only if

\[
p_{x_1} = H_{11}(x_1, x_2, f, p, q, k_1, k_2), \quad q_{x_1} = H_{12}(x_1, x_2, f, p, q, k_1, k_2), \quad f_{x_1} = p \tag{7a}
\]

\[
p_{x_2} = H_{21}(x_1, x_2, f, p, q, k_1, k_2), \quad q_{x_2} = H_{22}(x_1, x_2, f, p, q, k_1, k_2), \quad f_{x_2} = q. \tag{7b}
\]

If \( f \in C^3(\Pi) \), \( k_1, k_2 \in C^1(\Pi) \) and \( \alpha E + F \neq 0 \), then the compatibility conditions for \( \text{[7a] b} \) are reduced to PDE’s

\[
k_{1, x_2} - \frac{\alpha E + F}{\alpha E + F} k_{1, x_1} = \Psi_1, \quad k_{2, x_2} + \alpha k_{2, x_1} = \Psi_2, \tag{8}
\]

where \( \Psi_i(x_1, x_2, f, p, q, k_1, k_2) \) are known functions (see the proof). The characteristics of \( \text{[3]} \) are the projections onto \( \Pi \) of curvature lines of \( M^2 \).

Proof. Let \( V = \mu_1 e_1 + \mu_2 e_2 \) be a vector on \( M^2 \). The functions \( k_1, k_2 \in C^0(\Pi) \) are the principal curvatures of \( M^2 \), and \( l = \alpha e_1 + e_2 \) is the projection onto \( \Pi \)
Notice that if \( \partial_1 = \alpha e_1 + e_2 \) is the principal direction on \( M^2 \) then \( \partial_2 = -(\alpha F + G)e_1 + (\alpha E + F)e_2 \) is the second principal direction. We have

\[
I(\partial_1) = \alpha^2 E + 2\alpha F + G, \quad I(\partial_2) = (EG - F^2)(\alpha^2 E + 2\alpha F + G),
\]

\[
g(V, \partial_1) = \alpha \mu E + (\alpha \nu_2 + \mu_1) F + \mu_2 G = (\alpha E + F) \mu_1 + (\alpha F + G) \mu_2,
\]

\[
g(V, \partial_2) = (EG - F^2)(-\mu_1 + \alpha \mu_2), \quad I(V) = E \mu_1^2 + 2F \mu_1 \mu_2 + G \mu_2^2.
\]

Since \( V \) is arbitrary, (9) is equivalent to the system

\[
\delta L = H^{(1)}_{11} k_1 + H^{(2)}_{11} k_2, \quad \delta M = H^{(1)}_{12} k_1 + H^{(2)}_{12} k_2, \quad \delta N = H^{(2)}_{22} k_1 + H^{(2)}_{22} k_2,
\]

which, in view of \( p_{x_1} = \delta L - L_1, p_{x_2} = \delta N - N_1 \) and \( p_{x_2} = q_{x_1} = \delta M - M_1 \) (see Lemma 3 in Section 2.3), yields (7a,b).

If \( f \in C^3(\Pi) \) and \( k_1, k_2 \in C^1(\Pi) \), then the compatibility conditions for (7a,b), i.e., \( (p_{x_1})_{x_2} = (p_{x_2})_{x_1}, (q_{x_1})_{x_2} = (q_{x_2})_{x_1} \), take a form

\[
\begin{align*}
H_{11,x_1 + H_{11},p} H_{12} + H_{11,q} H_{22} + H_{11,f} q &= H_{12,x_1} + H_{12,p} H_{11} + H_{12,q} H_{12} + H_{12,f} p, \\
H_{12,x_2 + H_{12},p} H_{12} + H_{12,q} H_{22} + H_{12,f} q &= H_{22,x_1} + H_{22,p} H_{11} + H_{22,q} H_{12} + H_{22,f} p.
\end{align*}
\]

(10)

Substituting (7a,b) into (10), we obtain PDE’s for \( k = (k_1, k_2) \)

\[
A(x_1, x_2, f, p, q) k_{x_2} + B(x_1, x_2, f, p, q) k_{x_1} + b(x_1, x_2, f, p, q, k) = 0,
\]

(11)

where the matrices \( A = \begin{pmatrix} -H_{11}^{(1)} & -H_{11}^{(2)} \\ H_{12}^{(1)} & H_{12}^{(2)} \end{pmatrix} \), \( B = \begin{pmatrix} H_{12}^{(1)} & H_{12}^{(2)} \\ -H_{22}^{(1)} & -H_{22}^{(2)} \end{pmatrix} \) are \( C^1 \)-regular, and the components of the vector \( b = (b_1, b_2) \) are

\[
b_1 = \left( H_{12,x_1} - H_{11,x_2} \right) k_1 + \left( H_{12,x_1} - H_{11,x_2} \right) k_2 \\
+ H_{12,p} H_{11} + H_{12,q} H_{12} - H_{11,p} H_{12} + H_{11,q} H_{22} + H_{12,f} p - H_{11,f} q,
\]

\[
b_2 = \left( H_{12,x_2} - H_{22,x_1} \right) k_1 + \left( H_{12,x_2} - H_{22,x_1} \right) k_2 \\
+ H_{12,p} H_{12} + H_{12,q} H_{22} - H_{22,p} H_{11} - H_{22,q} H_{12} + H_{22,f} p - H_{12,f} q.
\]

Notice that \( \det A = \delta^2 (EG - F^2)(\alpha E + F) \neq 0 \) when \( \alpha E + F \neq 0 \). The direct computation shows that \( A^{-1} B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \), where \( \lambda_1 = -\frac{\alpha E + F}{\alpha E + F} \) and \( \lambda_2 = \alpha \). Hence (11) is equivalent to the system (8) with \( (\Psi_1, \Psi_2)^T = A^{-1} b \). 

**Remark 2** Let \( M^2 \subset \mathbb{R}^3 \) be a \( C^2 \)-regular surface without umbilical points parameterized by coordinates \( u, v \) of the curvature lines, \( r(u, v) \). Hence \( F = M = 0 \). Let
Proposition 3 Let the functions $f, p, q, k_1, k_2, \alpha$ of class $C^1$ on $\Pi$ (in particular, on $\Pi_{K, x}$) satisfy (7b).

(i) If (7a) holds in $\Pi$, then $f \in C^3(\Pi)$ and (10) holds in $\Pi$.

(ii) If (7a) holds for $x_2 = 0$ and (10) holds in $\Pi$, then (7a) holds in $\Pi$.

Proof. (i) If (7a) is satisfied in $\Pi$, then, since $H_{ij}$ are of class $C^1$, equations (7a-b) imply that $p, q$ are of class $C^2$. Hence $f \in C^3(\Pi)$ and, by commutativity of partial derivatives, (10) holds in $\Pi$.

(ii) Denote for short $x_1 = u, x_2 = v$. By (7a) with $v = 0$ and (7b), we conclude that $p, q$ and $f$ satisfy in $\Pi$ the integral equations

\[ p(u, v) = p(0, 0) + \int_0^u H_{12}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) \, d\eta + \int_0^u H_{11}(\xi, 0, f(u, \eta), p(\xi, 0), q(\xi, 0)) \, d\xi, \]

\[ q(u, v) = q(0, 0) + \int_0^u H_{22}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) \, d\eta + \int_0^u H_{12}(\xi, 0, f(u, \eta), p(\xi, 0), q(\xi, 0)) \, d\xi, \]

\[ f(u, v) = f(0, 0) + \int_0^v q(u, \eta) \, d\eta + \int_0^u p(\xi, 0) \, d\xi \]  

(for short we omit the variables $k_1$ and $k_2$ in $H_{ij}$). By conditions imposed on $p, q, f$ and $k_1$, one may differentiate by $u$ the first integrand in (12a). Using (7b) and \( \frac{d}{du} H_{11} = H_{11,2} + H_{11,p} p_v(x, y) + H_{11,q} q_v(u, v) + H_{11,f} f_v(u, v) \), we get

\[ p_u(u, v) = H_{11}(u, 0, f(u, 0), p(u, 0), q(u, 0)) \]

\[ + \int_0^v \left[ H_{12,u}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) \right] H_{11}(u, 0, f(u, 0), p(u, 0), q(u, 0)) \, d\eta \]

\[ + H_{12,p}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) p_u(u, \eta) \]

\[ + H_{12,q}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) q_u(u, \eta) + H_{12,f} f_u(u, \eta) \]  

\[ = H_{11}(u, 0, f(u, 0), p(u, 0), q(u, 0)) \]

\[ + \int_0^v \frac{d}{du} H_{11}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) \, d\eta \]

\[ + \int_0^v \left[ H_{12,p}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) q_u(u, \eta) + H_{12,f} f_u(u, \eta) \right] \, d\eta. \]

Define the functions $\Theta_1 = p_u - H_{11}(u, v, f, p, q), \, \Theta_2 = q_u - H_{12}(u, v, f, p, q), \, \text{and} \, \Theta_3 = f_u - p.$ By (10), from (13) it follows

\[ \Theta_1(u, v) = \int_0^v \left[ H_{12,p}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) \right] \Theta_1(u, \eta) \]

\[ + H_{12,q}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) \Theta_2(u, \eta) \]

\[ + H_{12,f}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) \Theta_3(u, \eta) \]  

(14a)
Similarly, differentiating (12b) by \( u \), and using (11b) and (10b), we obtain
\[
\Theta_2(u, v) = \int_0^v [H_{22,p}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) \Theta_1(u, \eta) + H_{22,q}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) \Theta_2(u, \eta) + H_{22,f}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta)) \Theta_3(u, \eta)] \, d\eta.
\]

Differentiating (12c) by \( u \) and using the first equation in (11b), we obtain
\[
f_u(u, v) = p(u, 0) + \int_0^v p_v(u, \eta) \, d\eta + \int_0^v [q_u(u, \eta) - H_{12}(u, \eta, f(u, \eta), p(u, \eta), q(u, \eta))] \, d\eta.
\]

Hence the following equation is satisfied:
\[
\Theta_3(u, v) = \int_0^v \Theta_2(u, \eta) \, d\eta.
\]

For each \( u \in [-a_1, a_1] \) the system of integral equations (14a-c) is equivalent to Cauchy problem for linear homogeneous ODE’s with initial conditions \( \Theta_i|_{v=0} = 0 \) (\( i = 1, 2, 3 \)). Hence \( \Theta_1 \equiv \Theta_2 \equiv \Theta_3 \equiv 0 \), and \( (1a) \) are satisfied.

First, we will recover the graph \( M^2 \) of \( f : \Pi \to [-a_3, a_3] \) infinitesimally along \( \gamma \), i.e., to solve (16) for \( f_0 \) and \( p_0, q_0 \). Define the quantity
\[
\bar{k}_0 = \max\{\|\bar{k}_1 - \lambda|_{\gamma}\|_{\gamma}, \|\bar{k}_2 - \lambda|_{\gamma}\|_{\gamma}\}.
\]

**Lemma 1** Let \( \alpha_0 = \alpha|_{\gamma} \in C^1(\gamma) \). Then for any \( r \in (0, a_3] \) there is \( \Delta \in (0, r] \) such that for \( \bar{k}_1, \bar{k}_2 \in C^0(\gamma) \) satisfying \( \|\bar{k}_i - \lambda|_{\gamma}\|_{\gamma} < \Delta \) (\( i = 1, 2 \)), the Cauchy problem
\[
\begin{align*}
\frac{df_0}{dx_1} &= p_0, \\
\frac{dp_0}{dx_1} &= H_{0,11}^{(1)} \bar{k}_1 + H_{0,11}^{(2)} \bar{k}_2 - L_0, \\
\frac{dq_0}{dx_1} &= H_{0,12}^{(1)} \bar{k}_1 + H_{0,12}^{(2)} \bar{k}_2 - M_0, \\
f_0(0) &= p_0(0) = q_0(0) = 0
\end{align*}
\]
has on \( \gamma \) a unique \( C^1 \)-regular solution \((f_0, p_0, q_0)\) satisfying \( \|(f_0, p_0, q_0)\|_{\gamma} < r \).

**Proof.** Denote \( \Delta_i = \bar{k}_i - \lambda|_{\gamma} \) for \( i = 1, 2 \). Substituting \( \bar{k}_i = \lambda|_{\gamma} + \Delta_i \) into (first two equations of) (16) and using (3) gives us along \( \gamma \)
\[
\begin{align*}
\frac{df_0}{dx_1} &= p_0, \\
\frac{dp_0}{dx_1} &= \delta_0 \lambda|_{\gamma} E_0 + L_0 + H_{0,11}^{(1)} \Delta_1 + H_{0,11}^{(2)} \Delta_2, \\
\frac{dq_0}{dx_1} &= \delta_0 \lambda|_{\gamma} \alpha_0 F_0 + M_0 + H_{0,12}^{(1)} \Delta_1 + H_{0,12}^{(2)} \Delta_2, \\
f_0(0) &= p_0(0) = q_0(0) = 0
\end{align*}
\]
with \( E_0 = E|_{\gamma}, F_0 = F|_{\gamma}, G_0 = G|_{\gamma}, \) and \( \delta_0 = \left(\frac{E_0 G_0 - F_0^2}{\det g_{\gamma}}\right)^{1/2} \). Due to Example 1 \( \delta_0 \lambda|_{\gamma} E_0 + L_0 = \delta_0 \lambda|_{\gamma} \alpha_0 F_0 + M_0 = 0 \) on \( \Pi \), see (9). Hence \( f_0 = p_0 = q_0 \equiv 0 \) is the
solution to (17) with $\Delta_i \equiv 0$. Let us take $r \in (0, a_3]$. We claim that if $\bar{k}_i$ are close enough to $\lambda_{i\gamma}$, then the Cauchy problem (17) has on $\gamma$ a unique smooth solution satisfying $\|\langle f_0, p_0, q_0 \rangle\|_\gamma < r$. In view of (18), (19) and Proposition 5, this solution satisfies in $\bar{\Omega}$ they satisfy in $\bar{\Omega}$.

Since the functions $P$ and $Q$ have continuous partial derivatives w.r. to $f_0, p_0, q_0$, they satisfy in $\bar{\Omega}_r = \{ |x_1| \leq a_1, |f_0| \leq r, |p_0| \leq r, |q_0| \leq r \}$ the Lipschitz condition (for $f_0, p_0, q_0$) with some $\bar{L} = \bar{L}(r) > 0$. By (11), $\min_{\Omega_r}(\alpha^2E + 2\alpha F + G) > 0$, and by (2), there is $C(r) > 0$ such that $|H^{(k)}_{01ij}| \leq C(r)$ on $\bar{\Omega}_r$. Hence,

$$\|Q - P\|_{\Omega_r} \leq 2\bar{k}_0 C(r), \tag{18}$$

where $\bar{k}_0$ is defined by (15). Assume that

$$\bar{k}_0 < \Delta(r) := \min \left\{ \frac{r}{4a_1C(r)} e^{-\bar{L}(r)a_1}, r \right\}. \tag{19}$$

By the theory of ODE’s, there is a maximal interval $-\varepsilon_1 \leq t \leq \varepsilon_2$ ($\varepsilon_i \in (0, a_1]$), in which (17) admits a unique solution with the property

$$|f_0(x_1)| \leq r/2, \quad |p_0(x_1)| \leq r/2, \quad |q_0(x_1)| \leq r/2. \tag{20}$$

On the other hand, in view of (18), (19) and Proposition 5 this solution satisfies in $[-\varepsilon_1, \varepsilon_2]$ strong inequalities $|p_0(x_1)| < r/2$, $|q_0(x_1)| < r/2$ and $|f_0(x_1)| < r/2$. If $\varepsilon_1 < 1$ or $\varepsilon_2 < 1$, due to the theory of ODE’s the solution can be extended on a larger interval with the property (20). Hence, $\varepsilon_1 = \varepsilon_2 = a_1$. \hfill $\Box$

**Proof of Theorem 1** Since $l$ is transversal to $\gamma$, one may assume $l = \alpha(x_1, x_2)\bar{e}_1 + \bar{e}_2$ for some function $\alpha$ of class $C^2$ in a neighborhood of $\gamma$ in $\Pi$. Since $l$ is not orthogonal to $\gamma$, we have $\alpha \bar{g}_{11} + \bar{g}_{12} \neq 0$ along $\gamma$. There is $r_1 \in (0, a_3]$ such that $\alpha \bar{g}_{11} + \bar{g}_{12} \neq 0$ over $\gamma$ for $|f| \leq r_1$. Define the functions $E, F, G$ and $L, M, N$ by (43c) in what follows. In view of $|\langle \alpha E + F \rangle - (\alpha \bar{g}_{11} + \bar{g}_{12})| \leq \bar{g}_{33}|(\alpha p + q)p|$, see (11), we get $\alpha E + F \neq 0$ on the set $\{ (x_1, 0, f, p, q) : |x_1| \leq a_1, \| (f, p, q) \|_\infty \leq r \}$ for some $r \in (0, r_1]$.

Restricting (12a) on $\gamma$ and denoting $\alpha_0 = \alpha|_\gamma$, $M_0 = M|_\gamma$, $L_0 = L|_\gamma$ and $H^{(k)}_{ij} := H^{(k)}_{ij}|_\gamma$, yields the system (13) for the functions $f_0$ and $p_0$, $q_0$. By Lemma 1 there exist $\Delta \in (0, r]$ such that if $\|\bar{k}_i - \lambda_i\|_\gamma \leq \Delta$ ($i = 1, 2$), the Cauchy problem (16) has on $[-a_1, a_1]$ a unique solution $(f_0, p_0, q_0)$ of class $C^1$, satisfying $\|\langle f_0, p_0, q_0 \rangle\|_\gamma < r$. By Propositions 2 and 3 the Problem 1 is reduced to the Cauchy problem, see (7b) and (8),

$$f_{x_2} = q(x_1, x_2),$$
\[ p_{x_2} = H_{12}^{(1)} k_1 + H_{12}^{(2)} k_2 - M_1, \]
\[ q_{x_2} = H_{22}^{(1)} k_1 + H_{22}^{(2)} k_2 - N_1, \]
\[ k_{1,x_2} - \frac{\alpha F + G}{\alpha E + F} k_{1,x_1} = \Psi_1(x_1, x_2, f, p, q, k_1, k_2), \]
\[ k_{2,x_2} + \alpha k_{2,x_1} = \Psi_2(x_1, x_2, f, p, q, k_1, k_2), \]
with the initial conditions for the functions \( f, p, q \) and \( k_1, k_2, \)
\[ f(\cdot, 0) = f_0, \quad p(\cdot, 0) = p_0, \quad q(\cdot, 0) = q_0, \]
\[ k_i(\cdot, 0) = \bar{k}_i, \quad (i = 1, 2). \]

Notice that there exists \( \rho \in (0, a_2] \) such that \( \alpha E + F \neq 0 \) if \( |x_1| \leq a_1, |x_2| \leq \rho, \| (f, p, q) \|_\infty \leq r, \) and (21) is non-singular. Since \( \alpha \in C^2 \), from the definition of \( \Psi_i \) (see the proof of Proposition 2) it follows that the functions \( \Psi_i(x_1, x_2, f, p, q, k_1, k_2) \in C^1(\Omega_{\rho,r}) \), where
\[
\Omega_{\rho,r} := \{(x_1, x_2, f, p, q, k_1, k_2) : |x_1| \leq a_1, |x_2| \leq \rho, \| (f, p, q) \|_\infty \leq r\}.
\]

The normal curvature \( \lambda \) of a \( C^3 \)-surface \( \Pi \) is \( C^1 \)-regular. From above it follows that functions in right hand side of (21) belong to class \( C^1(\Omega_{\rho,r}) \). The system (21) for the functions \( f, p, q, k_1, k_2 \) is hyperbolic, and it has a diagonal form in its main part (containing the derivatives of unknown functions). First three families of characteristics of (21) are lines \( \{x_1 = c\} \) and the last two families of characteristics are integral curves of ODE’s \( \frac{dx_1}{dx_2} = -\frac{\alpha E + G}{\alpha E + F} \) and \( \frac{dx_1}{dx_2} = \alpha \). Denote \( K := \max\{\|\frac{\alpha F + G}{\alpha E + F}\|_\Omega, \|\alpha\|_\Omega\} \), where \( \bar{\Omega} \) is the projection of \( \Omega_{\rho,r} \) onto the space of variables \( f, p, q \). By Theorem A (and remark after it, with \( c_1 > \|\bar{k}_i\|_\gamma \) for \( k_i \)) there is \( \varepsilon \in (0, \rho] \) such that the Cauchy problem (21), (22a,b) admits a unique solution \( \{f, p, q, k_1, k_2\} \in C^1(\Pi_{K,\varepsilon}) \) with \( \| (f, p, q) \|_{\Pi_{K,\varepsilon}} \leq r \). By Proposition 3, equations (7a,b) are valid in \( \Pi_{K,\varepsilon} \) and \( f \in C^3(\Pi_{K,\varepsilon}) \). Furthermore, by Proposition 2 the functions \( k_i \ (i = 1, 2) \) are the principal curvatures of the graph \( M^2 : x_3 = f(x_1, x_2) \), and, in view of (22b), the conditions \( k_i, k_i, = \bar{k}_i \ (i = 1, 2) \) are satisfied. Thus, the surface \( M^2 \) represents a solution of Problem 1.

Suppose that a \( C^3 \)-regular surface-graph \( M^2 : x_3 = f(x_1, x_2) \) over \( \Pi_{K,\varepsilon} \) is a solution of Problem 1 (with principal curvatures \( k_i, \ i = 1, 2 \)) satisfying the condition \( \| (f, f_{x_1}, f_{x_2}) \|_{\Pi_{K,\varepsilon}} \leq r, \) where \( K, \varepsilon \) and \( r \) have been chosen above. By Propositions 2 and 3, \( (f, f_{x_1}, f_{x_2}, k_1, k_2) \) is a solution to (21) and (22a,b), in which \( (f_0, p_0, q_0) \) is a solution to (16). Since solutions of these Cauchy problems are unique, the solution of Problem 1 is also unique in the class of functions under consideration. □

**Proof of Corollary 1**. Let \( M^2 \) in \( \mathbb{R}^3 \) (with cartesian coordinates) be a surface-graph of \( f \in C^2(\Pi) \) defined on \( \Pi = \{|x| \leq a_1, |y| \leq a_2, z = 0\} \). By Lemma 3 or directly, we find that \( n = \frac{f_{x_1}}{\sqrt{1 + p^2 + q^2}} [-p, -q, 1] \) is the unit normal to \( M^2 \), and the 1-st and the 2-nd fundamental forms of \( M^2 \) are \( E = 1 + p^2, \ F = pq, \ G = 1 + q^2, \) and \( L = \frac{f_{xx}}{\sqrt{1 + p^2 + q^2}}, \ M = \frac{f_{xy}}{\sqrt{1 + p^2 + q^2}}, \ N = \frac{f_{yy}}{\sqrt{1 + p^2 + q^2}} \). From Proposition 2 it follows that
\[ k_1, k_2 \in C^0(\Pi) \] are the principal curvatures and the vector \( l = (\alpha(x, y), 1) \) (where \( \alpha \in C^1(\Pi) \)) is the projection onto \( \Pi \) of \( k_1 \)-principal direction of \( M^2 \) if and only if

\[
\begin{align*}
p_x &= H_{11}(x, y, p, q, k_1, k_2), \quad q_x = H_{12}(x, y, p, q, k_1, k_2), \\
p_y &= H_{21}(x, y, p, q, k_1, k_2), \quad q_y = H_{22}(x, y, p, q, k_1, k_2).
\end{align*}
\tag{23a,b}
\]

For \( \mathbb{R}^3(x, y, z) \), the system \((7b) - (8)\) with \( \lambda = 0 \) has a form (see also \([4]\))

\[
\begin{align*}
p_y &= H_{11}^{(1)} k_1 + H_{12}^{(2)} k_2, \\
q_y &= H_{12}^{(1)} k_1 + H_{22}^{(2)} k_2, \\
k_{1,y} - \frac{\alpha pq + q^2 + 1}{\alpha (p^2 + 1) + pq} k_{1,x} + (c_1 + c_3 k_1)(k_2 - k_1) &= 0, \\
k_{2,y} + \alpha k_{2,x} + c_2(k_2 - k_1) &= 0,
\end{align*}
\tag{24}
\]

where \( c_1 = \frac{(p^2 + q^2 + 1)(\alpha a_x + \alpha_y)}{\delta_1 \delta_2 (p^2 + 1) + pq}, \)
\( c_2 = \frac{(\alpha (p^2 + 1) + pq) \alpha_y - (\alpha pq + q^2 + 1) \alpha}{\delta_1}, \)
\( c_3 = \frac{\sqrt{p^2 + q^2 + 1}}{\alpha (p^2 + 1) + pq}. \)

If \( f \in C^3(\Pi) \) and \( k_1, k_2 \in C^1(\Pi) \), then \((24)\) are compatibility conditions for \((23a, b)\), i.e., \((p_y)_y = (p_y)_x, (q_y)_y = (q_y)_x. \)

Similarly to the proof of Theorem \([1]\) one may show that there are \( r, K > 0 \) and \( \Delta \in (0, r) \) such that if \( \|\hat{k}_i\|_{\gamma} < \Delta \), then for some \( \varepsilon \in (0, a_2) \) the Cauchy problem \((24)\) with \( p(\cdot, 0) = p_0, \ q(\cdot, 0) = q_0, \ k_i(\cdot, 0) = \hat{k}_i, \ (i = 1, 2) \) is non-singular and admits a unique solution \( p, q, k_1, k_2 \) of class \( C^1(\Pi_{K,\varepsilon}) \) with \( \|(p, q, k_1, k_2)\|_{\Pi_{K,\varepsilon}} \leq r. \)

Moreover, there is a unique function \( f \) of class \( C^2(\Pi_{K,\varepsilon}) \) such that \( f(x) = p, \ f(y) = q \) and \( f(0, 0) = df(0, 0) = 0. \) As in the proof of Theorem \([1]\) one may show that \( f \in C^3(\Pi_{K,\varepsilon}) \) and the surface-graph \( M^2 : z = f(x, y) \) represents a unique solution of Problem \([1]\) in the class of functions from the formulation of the corollary. \( \Box \)

### 2.2 Proof of Theorem \([2]\) and Proposition \([1]\)

A surface \( M^2 \) in \( \mathbb{R}^3(k) \) is called parallel curved (PC) if there is a totally umbilical (or totally geodesic) surface \( \beta \subset \mathbb{R}^3(k) \) such that at each point \( x \in M^2 \) there is a principal direction tangent to \( \beta \) (a plane surface to \( \beta \) on the distance \( d \)). Surfaces of revolution and cylinders in Euclidean space \( \mathbb{R}^3 \) provide examples of PC surfaces (see \([2]\) and \([3]\)). We study PC surfaces in space forms \( \mathbb{R}^3(k) \) (see also Section \([3.2]\)) in relation to Problems \([1]\) and \([2]\).

A PC surface \( M^2 \) in \( \mathbb{R}^3(k) \) (a Riemannian 3-space of constant curvature \( k \)) can be recovered locally by exact procedure. We will prove the claim for PC surfaces in \( \mathbb{R}^3 \). In Proposition \([1]\) we solve Problem \([2]\) using twice the reconstruction a plane curve by its curvature.

**Proof of Proposition \([1]\)** will be divided into three steps.

1. We apply geometrical construction of Proposition \([6]\) when \( \beta \subset \mathbb{R}^3 \) is a plane with the normal \( \alpha e_2 - e_1 \). In this case, \( l = \alpha e_1 + e_2 \) is a constant vector field (its trajectories are parallel lines). The planes parallel to \( \beta \) intersect \( M^2 \) transversally by curvature lines, Fig. \([1](a)\).
Let $\pi_1 : M^2 \to \beta_0$ be the orthogonal projection onto the plane $\beta_0 = \{x - \alpha y = 0\}$. Let $\gamma_0(t) \subset \beta_0$ be a $k_1$-curvature line on $M^2$ through the origin $O(0,0,0)$. The normals to $\gamma_0$ (lines in $\beta_0$) and parallel curves (of constant distance) to $\gamma_0$ form a semi-geodesic net on $\beta_0$ on a neighborhood of $\gamma_0$. Let us parameterize $\gamma_0$

$$v = v_0(u), \quad v_0(0) = v'_0(0) = 0 \quad (|u| \leq \varepsilon \text{ for some } \varepsilon > 0)$$

in $(u,v)$-coordinates of $\beta_0$. Notice that $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ $(\tilde{e}_3 = (\alpha e_1 - e_2)/\sqrt{\alpha^2 + 1}$ is a unit normal to $\beta_0$) is the orthonormal frame of $\mathbb{R}^3$. The curvature of $\gamma_0$ is

$$\tilde{k}(u) = v''_0(u)/(1 + v'^2_0(u))^{3/2}. \quad (26)$$

From (25) and (26) it follows the inequality $|v'_0(u)| \leq \tilde{k} f_0 [1 + v'^2_0(\varepsilon)]^{3/2} ds$. By Lemma 2 if $\tilde{k} \leq \sqrt{\frac{\alpha^2 + 1}{a_1}}$, then $v'_0(u) \leq \tilde{k} u/(1 - (\tilde{k} u)^2)^{1/2}$ (a’priori estimate) for $|u| \leq \frac{a_1 \alpha}{\sqrt{\alpha^2 + 1}}$. If we take $\tilde{k} \leq \sqrt{\frac{\alpha^2 + 1}{2a_1}}$, then a unique solution $v_0(u)$ to (25), (26), defined for $|u| \leq \frac{a_1 \alpha}{\sqrt{\alpha^2 + 1}}$, satisfies the inequalities

$$|v_0| \leq \sqrt{2} \frac{a_1 \tilde{k} \alpha}{\alpha^2 + 1}, \quad |v'_0| \leq \sqrt{2} \frac{a_1 \tilde{k} \alpha}{\sqrt{\alpha^2 + 1}}, \quad |v''_0| \leq \sqrt{8} \tilde{k}. \quad (27)$$

Since $\tilde{k}(u)$ is $C^1$-regular, from (26) it follows that $v_0(u)$ is $C^3$-regular. Hence, the curve $\gamma_\rho(u) = [u - \frac{v'_0(u)}{|1 + v^2_0(u)|^{1/2}}, v_0(u) + \frac{\rho}{|1 + v^2_0(u)|^{1/2}}]$ (on the distance $\rho$ to $\gamma_0$ in the plane $\beta_0$) is $C^2$-regular for small enough $\rho$, its curvature is $k_\rho(u) = \frac{-\tilde{k}(u)}{1 - k(u)\rho}$, see 9.

2. Assume now that $\rho = \rho(h)$ (the function of $h$) and translate $\gamma_\rho(h)$ on the height $h$ in the normal direction to $\beta_0$. Then, the obtain $(u,h)$-parametrization of $M^2$ near $\gamma_0$ with $\beta_0$-level curves $\gamma_\rho$,

$$U = u - \frac{v'_0(u) \rho(h)}{(1 + v^2_0(u))^{1/2}}, \quad V = v_0(u) + \frac{\rho(h)}{(1 + v^2_0(u))^{1/2}}, \quad W = h. \quad (28)$$

Using the equation for principal curvatures, $(EG - F^2)k^2 - (EN + GL - 2FM)k + (LN - M^2) = 0$, since $F = M = 0$, we find

$$k_1(u, h) = \frac{\tilde{k}(u)}{1 - \tilde{k}(u)\rho(h)} \cdot \frac{1}{(1 + \rho^2(h))^{1/2}}, \quad k_2(u, h) = \frac{\rho''(h)}{(1 + \rho^2(h))^{3/2}}. \quad (29)$$

Notice that $k_1(u, h)$ is the curvature of $\gamma_\rho$ multiplied by $\cos \varphi$, see (29), where $\varphi$ is the angle between $\beta_0$ and the tangent plane to $M^2$ through the intersection point. The 2-nd principal curvature, $k_2(u, h)$ (does not depend on $u$) is simply the curvature of the curve $r(h) = [h, \rho(h)]$ in the vertical plane $\beta^\perp = \{\alpha x + y = 0\}$ with the $\rho$-axis $Oz$. One may recover a $C^2$-regular function $\rho(h)$ for small enough $h$ from (29), solving the BVP

$$\rho''(h)(1 + \rho^2(h))^{3/2} = \tilde{k}_2(h\sqrt{\alpha^2 + 1}), \quad \rho(0) = \rho'(0) = 0. \quad (30)$$
If \( \bar{k} \leq \frac{\sqrt{\alpha^2 + 1}}{\sqrt{2a_1}} \), then by Lemma 2, a unique solution \( \rho(h) \) to Cauchy problem (30) exists for \( |h| \leq \frac{a_1}{\sqrt{\alpha^2 + 1}} \), and satisfies the inequalities

\[
|\rho| \leq \sqrt{2} \frac{a_1^2 \bar{k}}{\alpha^2 + 1}, \quad |\rho'| \leq \sqrt{2} \frac{a_1 \bar{k}}{\sqrt{\alpha^2 + 1}}, \quad |\rho''| \leq \sqrt{8} \bar{k}.
\] (31)

One may recover \( \tilde{k}(u) \) from (29), for \( \rho = 0 \) with known \( k_1(u,0) \). Finally, \( v_0(u) \) is a unique \( C^2 \)-regular solution to (26) with initial values (25). It was shown that if \( \bar{k} \leq \frac{\sqrt{\alpha^2 + 1}}{\sqrt{2a_1}} \min\{1, 1/\alpha\} \), then the parametrization (28) defines a regular surface over the rectangle of parameters \((u,h)\)

\[
\tilde{\Pi}(a_1) = \{(u,h) \in \mathbb{R}^2 : |u| \leq a_1\alpha/\sqrt{\alpha^2 + 1}, \quad |h| \leq a_1/\sqrt{\alpha^2 + 1}\}.
\]

Equations (28) of \( M^2 \) in cartesian coordinates \((x,y,z)\) take a form

\[
X = \frac{h + \alpha U(u,h)}{\sqrt{\alpha^2 + 1}}, \quad Y = \frac{U(u,h) - \alpha h}{\sqrt{\alpha^2 + 1}}, \quad Z = V(u,h). \tag{32}
\]

In view of (28), (32), \( C^3 \)-regularity of \( v_0(u) \) and \( C^2 \)-regularity of \( \rho(h) \), the surface \( M^2 \) is \( C^2 \)-regular.

3. Along \( \gamma_1 \) (i.e., \( \rho = 0 \)) we get \([X_h,h, Z_h]_{(u,0)} = [\frac{1}{\sqrt{\alpha^2 + 1}}, \frac{-\alpha}{\sqrt{\alpha^2 + 1}}] \), and \([X_u, Y, Z]_{(u,0)} = [\frac{2}{\sqrt{\alpha^2 + 1}}, \frac{1}{\sqrt{\alpha^2 + 1}}, v_0'(u)]\). Hence \( \det \begin{pmatrix} X_h(u,0) & Y_h(u,0) & Z_h(u,0) \\ X_u(u,0) & Y_u(u,0) & Z_u(u,0) \end{pmatrix} = 1 > 0 \), and \( M^2 \) regularly projects onto a neighborhood of \( \gamma_1 \) in \( xy \)-plane. It is not difficult to see that \( M^2 \) is \( C^2 \)-regular. Let us show that for any \( a \in (0,a_1) \) there is \( \delta > 0 \) such that if \( \bar{k} < \delta \), the surface \( M^2 \) regularly projects onto \( \Pi(a) \). Based on (28), we write (32), in the equivalent form \( u = \frac{a X + Y}{\sqrt{\alpha^2 + 1}}, \quad h = \frac{X - \alpha Y}{\sqrt{\alpha^2 + 1}}. \)

Consider the mapping \( T : (u,h) \rightarrow \frac{\alpha x + y}{\sqrt{\alpha^2 + 1}} + \frac{v_0'(u) \rho(h)}{(1 + v_0'^2)^{1/2}}, \quad h = \frac{x - \alpha y}{\sqrt{\alpha^2 + 1}} \) associated with above system. We will show that for any \((x,y) \in \Pi(a)\) the system

\[
\begin{align*}
\frac{\alpha x + y}{\sqrt{\alpha^2 + 1}} + \frac{v_0'(u) \rho(h)}{(1 + v_0'^2)^{1/2}}, \quad h = \frac{x - \alpha y}{\sqrt{\alpha^2 + 1}}.
\end{align*}
\]

admits a unique solution \((u,h)\) in \( \tilde{\Pi}(a_1) \). Notice \( \tilde{\Pi}(a) \) is the image of \( \Pi(a) \) under the linear mapping \( u = \frac{a x + y}{\sqrt{\alpha^2 + 1}}, \quad h = \frac{x - \alpha y}{\sqrt{\alpha^2 + 1}}. \)

Assume that the metric in \( \tilde{\Pi}(a_1) \) is induced by the norm \( \|(u,h)\|_{\infty} = \max\{|u|, |h|\} \). In order to apply the Banach fixed point theorem, we will show that for a small enough \( \bar{k} \) the mapping \( T \) maps \( \tilde{\Pi}(a_1) \) into itself, and that \( T \) is a contraction. If \( \bar{k} \leq \frac{(a_1 - a) a \sqrt{\alpha^2 + 1}}{\sqrt{2a_1^2}} \), then using (31), we obtain for \((x,y) \in \Pi(a)\) and \((u,h) \in \tilde{\Pi}(a)\) that \(|x - \alpha y| \leq a_1\) and

\[
\left| \frac{\alpha x + y}{\sqrt{\alpha^2 + 1}} + \frac{v_0'(u) \rho(h)}{(1 + v_0'^2)^{1/2}} \right| \leq \frac{\alpha a}{\sqrt{\alpha^2 + 1}} + |\rho(h)| \leq \frac{\alpha a_1}{\sqrt{\alpha^2 + 1}}.
\]
Under above condition for \( \bar{k} \), \( T \) maps \( \bar{\Pi}(a_1) \) into itself. To show that \( T : \bar{\Pi}(a_1) \to \bar{\Pi}(a_1) \) is a contraction, we find the differential of \( T \) at \( (u, h) \in \bar{\Pi}(a_1) \):

\[
d T(u, h) \left( \frac{\Delta u}{\Delta h} \right) = \left( -\frac{\rho(h) v'_0(u)}{(1 + v'_0(u))^3/2} \Delta u + \frac{\rho'(h) v'_0(u)}{(1 + v'_0(u))^{3/2}} \Delta h \right).
\]

Using (27), (31), we obtain the following estimates:

\[
\left| \frac{\rho(h) v''_0(u)}{(1 + v'_0)^{3/2}} \right| \leq \frac{4\sqrt{2}a_1^2}{\alpha^2 + 1} \bar{k}^2, \quad \left| \frac{\rho'(h) v'_0(u)}{(1 + v'_0)^{1/2}} \right| \leq \frac{\sqrt{2}a_1}{\alpha^2 + 1} \bar{k}.
\]

Hence, for a small enough \( \bar{k} \) the norm \( \|dT(u, h)\|_\infty \) is less than 1 for any \( (u, h) \in \bar{\Pi}(a_1) \), that is the mapping \( T \) is a contraction in \( \bar{\Pi}(a_1) \). Thus, for small enough \( \bar{k} \), the projection \( \text{proj}_{xy} \) (the orthogonal projection onto the \( xy \)-plane) realizes a bijection between the surface \( M^2 \cap \text{proj}_{xy}^{-1}(\Pi(a)) \) and \( \Pi(a) \). Since the linear operator \( \text{id} - dT \) is invertible for any \( (u, h) \in \bar{\Pi}(a_1) \), by the Implicit Function Theorem this projection is regular.

**Proof of Theorem 2** The condition \( Y(u, h) \equiv 0 \) determines the intersection curve of \( M^2 \) (see (32)) with \( xz \)-plane of the form \( (X(u), 0, Z(u)) \), where \( Z(u) := V(u, \frac{X(u)}{\sqrt{\alpha^2 + 1}}) \). Now, \( X(u) \) and \( v_0(u) \) are functions to be found using the boundary values of the principal curvatures. Denote \( w = v'_0(u) \). Consider the system

\[
X = \frac{\sqrt{\alpha^2 + 1}}{\alpha} \left( u - \frac{\tilde{\rho}(X) w}{(1 + w^2)^{3/2}} \right), \quad w = \int_0^u \frac{\tilde{k}_1(X)(1 + w^2)^{3/2} d\eta}{\tilde{\rho}(X)\tilde{k}_1(X) + \Phi(X)},
\]

where \( \tilde{\rho}(x) := \rho(\frac{x}{\sqrt{\alpha^2 + 1}}), \Phi(x) := \tilde{\Phi}(\frac{x}{\sqrt{\alpha^2 + 1}}), \tilde{\Phi}(h) = \frac{1}{(1 + \tilde{\rho}(h)^2)^{1/2}}, \tilde{k}_1(x) \) is a continuous in \([-a, a] \) function, and \( X = X(\eta), w = w(\eta) \) in the integrand.

Due to the proof of Proposition 1 we are looking for the parametric form (32) of \( M^2 \), where \( U, V, W \) (the functions of \( u, h \)) are given in (23), and \( v_0(u) \) satisfies (26) with unknown function \( \tilde{k}(u) \). By Proposition 1 the principal curvature \( k_2 \) of \( M^2 \) satisfies \( k_2(x, 0) = \tilde{k}_2(x) \). It is sufficient to show that for small enough \( \bar{k}_i \) there exists a unique pair \( X(u), v_0(u) \) (and hence \( M^2 \)) such that \( k_1(x, 0) = \tilde{k}_1(x) \) for the principal curvature \( k_1 \) of \( M^2 \). From (32) with \( Y = 0 \) we conclude that the pair \( X = (X(u), v_0(u)) \) satisfies the equation \( \frac{a}{\sqrt{\alpha^2 + 1}} X = u - \frac{\tilde{\rho}(X) v'_0(u)}{(1 + v'_0)^{1/2}} \) for \( u \in I = [-\frac{a\alpha}{\sqrt{\alpha^2 + 1}}, \frac{a\alpha}{\sqrt{\alpha^2 + 1}}] \quad (a \in (0, a_1)) \). In order to satisfy \( k_1(x, 0) = \tilde{k}_1(x) \), we rewrite (29) as

\[
\frac{\tilde{k}(u)}{1 - \tilde{k}(u)\tilde{\rho}(X)} \Phi(X) = \tilde{k}_1(X) \Leftrightarrow \tilde{k}(u) = \frac{\tilde{k}_1(X)}{\tilde{\rho}(X)\tilde{k}_1(X) + \Phi(X)}.
\]

Substituting \( \tilde{k}(u) \) of (26) into (34), we get the integral equation

\[
v'_0(u) = \int_0^u \frac{\tilde{k}_1(X(\eta))(1 + v'_0(\eta)^2)^{3/2}}{\tilde{\rho}(X(\eta))\tilde{k}_1(X(\eta)) + \Phi(X(\eta))} d\eta \quad (u \in I).
\]
This leads to the system \((\text{33})\) for the functions \(X(u)\) and \(w(u) := v'_0\).

We claim that for small enough \(\bar{k}_1\) and \(\bar{k}_2\), \((\text{33})\) admits a unique solution in the product space \(B := B(0, a_1) \times C^0(I)\), where \(B(\phi, r)\) is the closed ball of radius \(r > 0\) in \(C^0(I)\) centered at the function \(\phi\). First we will investigate \((\text{33})_1\). Set \(\bar{k} := \max\{|\bar{k}_1(x)|, |\bar{k}_2(x)|\}\). By the proof of Proposition 1 if \(\bar{k} \leq \frac{\sqrt{\alpha^2 + 1}}{\sqrt{2}a_1}\), the Cauchy problem \((\text{30})\) admits a unique solution \(\rho(h)\) defined for \(|h| \leq a_1/\sqrt{\alpha^2 + 1}\) and the estimates \((\text{31})\) are valid. Denote

\[
F(X, u, w) := \frac{\sqrt{\alpha^2 + 1}}{\alpha} \left( u - \frac{\bar{\rho}(X) w}{(1 + w^2)^{1/2}} \right).
\]

From \((\text{31})_{1,2}\) we get that if \(\bar{k} \leq \min\{\frac{\sqrt{\alpha^2 + 1}}{\sqrt{2}a_1}, \frac{\alpha \sqrt{2}}{\alpha^2 + 1}\}\). Hence \(|F| \leq a + \frac{\sqrt{\alpha^2 + 1}}{\alpha} |\bar{\rho}(X)| \leq a + \frac{\sqrt{2}a_1}{\alpha \sqrt{\alpha^2 + 1}} \bar{k} \leq a_1\) for any \((X, u, w) \in [-a_1, a_1] \times I \times \mathbb{R}\), that is

\[
\forall (u, w) \in I \times \mathbb{R} : \quad F(\cdot, u, w) : [-a_1, a_1] \to [-a_1, a_1],
\]

\[
|\partial_X F| \leq \frac{\sqrt{\alpha^2 + 1}}{\alpha} |\bar{\rho}'(X)| \leq \frac{\sqrt{2}a_1}{\alpha \sqrt{\alpha^2 + 1}} \bar{k} \leq \frac{1}{\sqrt{2}}. \tag{36}
\]

Hence \(\partial_X(X - F(X, u, w)) > 0\) for any \((X, u, w) \in [-a_1, a_1] \times I \times \mathbb{R}\). We conclude that \((\text{33})_1\) admits a unique solution \(X = \tilde{X}(u, w) \in [-a_1, a_1]\) for any \((u, w) \in I \times \mathbb{R}\). Since the function \(F\) is \(C^1\)-regular in \([-a_1, a_1] \times I \times \mathbb{R}\), by the Implicit Function Theorem, the function \(\tilde{X}\) is \(C^1\)-regular in \(I \times \mathbb{R}\). Let us substitute \(X = \tilde{X}(u, w)\) into \((\text{33})_2\), which is equivalent to Cauchy problem

\[
\frac{dw}{du} = \frac{\bar{k}_1(\tilde{X})}{\bar{\rho}(\tilde{X})\bar{k}_1(\tilde{X}) + \Phi(\tilde{X})} \quad w(0) = 0. \tag{37}
\]

We will show that \((\text{37})\) admits a unique solution \(w(u)\) in \(C^0(I)\). By estimates \((\text{31})_{1,2}\) and definition of function \(\Phi\), we have

\[
|\bar{\rho}'(\tilde{X}) \bar{k}_1(\tilde{X}) + \Phi(\tilde{X})| \geq \sqrt{2} - \sqrt{2} \frac{a_1^2 \bar{k}}{\alpha^2 + 1} \geq \frac{1}{\sqrt{2}} \tag{38}
\]

for \(\bar{k} \leq \frac{\alpha^2 + 1}{2a_1^2}\). Since \(\bar{k}_1(X)\), \(\bar{\rho}(X)\) and \(\Phi(X)\) are \(C^1\)-regular in \([-a_1, a_1]\), the ODE \((\text{37})\) satisfies the conditions required for local existence and uniqueness of a solution to Cauchy problem. In order to show that the solution to \((\text{37})\) does not blow up in \(I\), we need an a’priori estimate of a solution to \((\text{33})_2\) with \(X = \tilde{X}(u, w)\). Let \(w = w(u)\) be a continuous solution to this equation in \([-c, c] \subseteq I\). From \((\text{33})_2\) and \((\text{38})\) it follows

\[
|w(u)| \leq \sqrt{2} \bar{k} \text{sign}(u) \int_0^u (1 + w^2(\eta))^{3/2} \, d\eta \quad (u \in [-c, c]).
\]
By Lemma 2, the estimate \(|w(u)| \leq \frac{\sqrt{2}k|u|}{\sqrt{1-2k^2u^2}} \leq 1\) is valid in \([-c, c]\), if \(\bar{k} \leq \frac{\sqrt{a^2+1}}{2\alpha a_1}\).

Since the bound for \(|w(u)|\) in \([-c, c]\) does not depend on \(c\), the solution \(w(u)\) exists and is continuous on \(I\). So, we have proved the claim \((X(u) = \tilde{X}(u, w(u)))\) in the product space \(\tilde{B}\), moreover, by above a priori estimate, the solution belongs to the smaller space \(\tilde{B} := B(0, a_1) \times B(0, 1)\).

The desired \(C^2\)-regular surface \(M^2\) is given by \((32), (28)\), where \(v_0(u) = \int_0^u w(\eta) \, d\eta\). Using \((34)\), we obtain that for any \(a \in (0, a_1)\) there exists \(\delta > 0\) such that if \(\bar{k} < \delta\), then \(M^2\) projects regularly onto \(\Pi(a)\) and \(M^2\) is a unique surface with the properties indicated in the theorem. \(\square\)

**Proposition 4** The solution in Theorem 2 can be represented in the form \((28), (32)\), where \(v_0(u) = \int_0^u w(\eta) \, d\eta\) and \(w(u)\) is the second component of the solution \((X, w)\) to \((33)\). Moreover, \((X, w)\) is the limit of the iterated function sequence for the operator \(S\) in \(C^0(\mathbb{R}) \times C^0(\mathbb{R})\)

\[
S : (X, w) \rightarrow \left[\frac{\sqrt{\alpha^2+1}}{\alpha} \left(u - \frac{\tilde{\rho}(X) w}{(1 + w^2)^{1/2}}\right), \int_0^u \tilde{k}_1(X)(1 + w^2)^{3/2} \, d\eta \right]
\]

with the starting point \((\frac{\sqrt{\alpha^2+1}}{\alpha} u, 0)\).

**Proof.** Indeed, a fixed point of \(S\) is a solution to \((33)\). Let us prove that for a small enough \(\bar{k}\) the solution \((X(u), w(u))\) to \((33)\) (that determines the surface \(M^2\)) can be found by an iterative process. First we will show that for a small enough \(\bar{k}\) the operator \(S\) maps \(\tilde{B} = B(0, a_1) \times B(0, 1)\) into itself. Denote \(G(X, w) := \frac{\tilde{k}_1(X)(1 + w^2)^{3/2}}{\tilde{\rho}(X)k_1(X) + \Phi(X)}\). By \((31)\), \(\tilde{k}_1(X)\) is a contraction w. r. to some metric on \(\tilde{B}\) for a small enough \(\bar{k}\). Let us compute the differential of the second component \(S_2\) (of \(S\)) at a point \((X(u), w(u)) \in \tilde{B}\):

\[
ds_2(X, w)\left(\frac{\Delta X}{\Delta w}\right) = \int_0^u (\partial_X G(X(\eta), w(\eta)) \Delta X(\eta) + \partial_{w} G(X(\eta), w(\eta)) \Delta w(\eta)) \, d\eta
\]

where \(\partial_X G = \frac{1}{\tilde{\rho}(X)k_1(X) + \Phi(X)} \left[\frac{(1 + w^2)^{3/2} \tilde{k}_1(X)(\rho'(X)k_1(X) + k_1(X)\tilde{\rho}(X)\Phi'(X))}{(\rho(X)k_1(X) + \Phi(X))^2} + \frac{(1 + w^2)^{3/2} \tilde{k}_1(X)}{\tilde{\rho}(X)k_1(X) + \Phi(X)}\right]\) and \(\partial_{w} G = \frac{3 \tilde{k}_1(X) w(1 + w^2)^{1/2}}{\tilde{\rho}(X)k_1(X) + \Phi(X)}\). Using \(\Phi' = -\frac{\rho''}{(1 + \rho^2)^{3/2}}\) and \((31), (38)\), yields that the function \(G\)
satisfies the Lipschitz condition w.r. to \(X, w\) in \([-a_1, a_1] \times [-1, 1]\) with a Lipschitz constant \(L_2 > 0\) for a small enough \(\bar{k}\). Notice that \(L_2\) is not arbitrary small for a small enough \(\bar{k}\), because the expression for \(dS_2\) contains the derivative \(\bar{k}_1\) that is not assumed to be small.

In aim to show that \(S\) is a contraction, let us define the following metric in the second component \(\tilde{B}(0, 1)\) of the product space \(\mathcal{B}\): \(d\tilde{T}(w_1, w_2) = \max_{u \in I} e^{-T|u|} |w_1(u) - w_2(u)|\) (see \([8]\)), where \(T > 0\) will be chosen in the sequel. Clearly, this metric is equivalent to the original \(C^0\)-metric \(d_{\infty}(w_1, w_2) = \max_{u \in I} |w_1(u) - w_2(u)|\) in \(B(0, 1)\). The metric of \(\mathcal{B}\) is

\[
\tilde{d}^T((X_1, w_1), (X_2, w_2)) = \max\{d_{\infty}(X_1, X_2), d^T(w_1, w_2)\}.
\]

Let us estimate for \((X_i, x_i) \in \mathcal{B} (i = 1, 2)\), \(u \in I\), \(u \geq 0\):

\[
|e^{-Tu}(S_2(X_1, w_1)(u) - S_2(X_2, w_2)(u))| = |\int_0^u e^{-T(u-n)} e^{-T\eta} \max\{|X_1(\eta) - X_2(\eta)|, |w_1(\eta) - w_2(\eta)|\}| d\eta| \leq L_2 \int_0^u e^{-T(u-n)} d\eta \tilde{d}^T((X_1, w_1), (X_2, w_2)) = (L_2/T) \tilde{d}^T((X_1, w_1), (X_2, w_2)).
\]

A similar estimate is valid for \(u < 0\). Thus,

\[
d^T(S_2(X_1, w_1), S_2(X_2, w_2)) \leq (L_2/T) \tilde{d}^T((X_1, w_1), (X_2, w_2)).
\]

Above arguments imply that if \(T > L_2\) and \(\bar{k}\) is small enough, then \(S\) is a contraction in \(\mathcal{B}\) w.r. to the metric defined above. By the Banach fixed point theorem, for any \(\psi = (X_1(u), w_1(u))\) in \(\mathcal{B}\) the iterated function sequence \(\psi, S(\psi), S(S(\psi)), \ldots\) converges uniformly on \(I\) to the unique fixed point \((X(u), w = v_0(u))\) of \(S\) in \(\mathcal{B}\). Since \(\psi = (X(0)/\alpha, 0) \in \mathcal{B}\), this point can be chosen as starting one in the iterative process. \(\square\)

### 2.3 Auxiliary results

We consider a first order quasilinear system of PDE’s, \(n\) equations in \(n\) unknown functions \(u = (u_1, \ldots, u_n)\) and two variables \(x, y \in \mathbb{R}\),

\[
du/dy + A(x, y, u) du/dx = b(x, y, u),
\]

where \(A = (a_{ij}(x, y, u))\) is an \(n \times n\) matrix, \(b = (b_i(x, y, u))\) is an \(n\)-vector.

The **Cauchy problem** for (39) is the problem of finding \(u\) such that (39) and \(u(x, 0) = u_0(x)\) are satisfied, where \(u_0\) is given. When the coefficient matrix \(A\) and the vector \(b\) are functions of \(x\) and \(y\) only, the system is **linear**. When \(A\) and \(b\) are functions of \(x\), \(y\) and \(u\), the system is **quasilinear**.

The system (39) is called **hyperbolic** in the \(y\)-direction at \((x, y, u)\) (in an appropriate domain of the arguments of \(A\)) if the (right) eigenvectors of \(A\) are real and span \(\mathbb{R}^n\). In this case, let \(R = [r_1, \ldots, r_n]\) be the matrix of the (right) eigenvectors
Let the quasi-linear system of PDE’s be such that (i) it is hyperbolic in the y-direction in \( \Omega = \{ |x| \leq a, 0 \leq y \leq s, \|u\|_\infty \leq r \} \) for some \( s, r > 0 \); (ii) the matrices \( A, R \) and the vector \( b \) are \( C^1 \)-regular in \( \Omega \); (iii) it is satisfied an initial condition

\[ u(x, 0) = u_0(x), \quad -a \leq x \leq a \quad (40) \]

for which \( \|u_0\|_{[-a,a]} < r \) and \( u_0 \) is \( C^1 \)-regular in \( [-a,a] \).

Then there is \( \varepsilon \in (0, s) \) such that (39) and (40) admit a unique \( C^1 \)-regular solution \( u(x, y) \) in the trapeze \( \Pi_{K, \varepsilon} = \{ (x, y) : |x| + Ky \leq a, 0 \leq y \leq \varepsilon \} \), where \( K = \max\{ |\lambda_i(x, y, u)| : (x, y, u) \in \Omega, 1 \leq i \leq n \} \).

In Theorem A, one may use the norm \( \|u\|_{c, \infty} = \max_{1 \leq i \leq n} c_i |u_i| \) for \( c_i > 0 \). To show this one should replace unknown functions \( v_i = u_i / c_i \) (hence \( \|v\|_\infty = \|u\|_{c, \infty} \)) to reduce to original Theorem A for \( v \). We use Theorem A in the proof of Theorem \( \|u\|_{c, \infty} \) for diagonal matrices \( A = R \).

The next proposition is known. For convenience of a reader we prove it.

**Proposition 5** Let vector functions \( P \) and \( Q \) satisfy the Lipschitz condition

\[ \|P(t, u) - P(t, v)\|_\infty \leq \bar{L}\|u - v\|_\infty, \quad \|Q(t, u) - Q(t, v)\|_\infty \leq \bar{L}\|u - v\|_\infty, \]

(with the same \( \bar{L} \)) for \( t \in [0, h] \) and \( u, v \in \Omega \subset \mathbb{R}^n \) (\( \Omega \) a domain). Let \( y(t) \) \( (y(0) = y_0 \in \Omega) \) and \( z(t) \) \( (z(0) = z_0 \in \Omega) \) are solutions to ODE’s \( y'(t) = P(t, y(t)) \), and \( z'(t) = Q(t, z(t)) \), resp., where \( t \in [0, h] \) and \( y(t), z(t) \in \Omega \). Then \( \|y - z\|_{[0, h]} \leq (mh + \|y_0 - z_0\|_\infty) e^{\bar{L}h} \), where \( m = \|P - Q\|_{[0,h] \times \Omega} \).

**Proof.** We present BVP equivalently in the integral form

\[ (y - z) - (y_0 - z_0) = \int_0^t [P(x, y(x)) - P(x, z(x)) + P(x, z(x)) - Q(x, z(x))] \, dx. \]

Hence \( \|y - z\|_\infty \leq (\|y_0 - z_0\|_\infty + mh) + \bar{L} \int_0^t \|y - z\|_\infty \, dx. \) From the Gronwall-Bellmann inequality

\[ u(t) \leq A + \int_0^t u(x)v(x) \, dx \quad (u, v > 0, \quad A \geq 0) \quad \Rightarrow \quad u(t) \leq Ae^{\int_0^t v(x) \, dx}, \]

with \( A = \|y_0 - z_0\|_\infty, \quad u = \|y - z\|_\infty \) and \( v = \bar{L} \) it follows the claim. \( \square \)

**Lemma 2** Let a function \( u \geq 0 \) of class \( C^0([0, a]) \) obeys the inequality \( u(y) \leq A \int_0^y (1 + u^2(\eta))^{3/2} \, d\eta \) with \( A \in (0, 1/a) \). Then \( u(y) \leq \frac{Ay}{(1 - Ay^2)^{1/2}} \) for \( y \in [0, a] \).
Proof. Denote \( f(u) = A(1+u^2)^{3/2} \). Then \( U := \frac{Au}{(1-A^2y^2)^{3/2}} \) is the solution to the ODE \( \frac{du}{dy} = f(u) \) with the initial condition \( U(0) = 0 \). Since \( A \in (0,1/a) \), \( U \) is continuous in \([0,a]\). Clearly, \( U(y) \) satisfies in \([0,a]\) the integral equation
\[
U(y) = \int_0^y f(U(\eta)) \, d\eta. \tag{41}
\]
Denote by \( Y(y) = U(y) - u(y) \) and \( \Lambda(y) = \int_0^y \partial_u f(tU(y) + (1-t)u(y)) \, dt \). Hence \( f(U(y)) - f(u(y)) = \Lambda(y)Y(y) \). Since \( \partial_u f(u) \) is non-negative and continuous in \([0,\infty)\), the function \( \Lambda(y) \) is also non-negative and continuous in \([0,a]\). Furthermore, by conditions of the lemma and (41), their limit \( Y(y) \) is also non-negative, that is the desired estimate is valid. \( \square \)

The covariant derivative of a \((0,1)\)-tensor \((\mu_i)\) in \((M^3,\bar{g})\) is defined by
\[
\nabla_i \mu_j = \mu_{,ix_j} - \sum_k \tilde{\Gamma}^k_{ij} \mu_k \tag{42}
\]
where \( \tilde{\Gamma}^k_{ij} = \sum_s \bar{g}^{sk}(\bar{g}_{is,x_j} + \bar{g}_{js,x_i} - \bar{g}_{ij,x_s}) \) are Christoffel symbols.

Lemma 3 Let \( M^2 : x_3 = f(x_1,x_2) \) be the graph of a function \( f \in C^2(\Pi) \) in \((M^3,\bar{g})\). Then the unit normal \((n_i)\) to \( M^2 \), the coefficients of the 1-st and the 2-nd fundamental forms of \( M^2 \) are
\[
n_1 = \frac{(\hat{g}_{12} - \hat{g}_{22}p)}{\sqrt{EG - F^2/\det \bar{g}}} \hat{g}_{33}, \quad n_2 = \frac{(\hat{g}_{12}p - \hat{g}_{11}q)}{\sqrt{EG - F^2/\det \bar{g}}} \hat{g}_{33}, \quad n_3 = \frac{1}{\delta \hat{g}_{33}}, \tag{43a}
\]
\[
E = \hat{g}_{11} + \hat{g}_{33}p^2, \quad F = \hat{g}_{12} + \hat{g}_{33}pq, \quad G = \hat{g}_{22} + \hat{g}_{33}q^2, \tag{43b}
\]
\[
L = (f_{x_1x_1} + L_1)/\delta, \quad M = (f_{x_1x_2} + M_1)/\delta, \quad N = (f_{x_2x_2} + N_1)/\delta, \tag{43c}
\]
where \( f_{x_1} = p, \ f_{x_2} = q, \ \delta = \sqrt{(EG - F^2)/\det \bar{g}} \geq 1/\sqrt{g_{33}} \), and
\[
L_1 = \hat{\Gamma}^3_{12} + \hat{\Gamma}^3_{13}p + \hat{\Gamma}^3_{33}p^2 + \delta \sum_{i,j,l} \hat{g}_{ij}n_l(\hat{\Gamma}^i_{11} + 2\hat{\Gamma}^i_{13}p + \hat{\Gamma}^i_{33}p^2) - \Gamma^1_{11}p - \Gamma^3_{11}q, \tag{44}
\]
\[
M_1 = \hat{\Gamma}^3_{12} + \hat{\Gamma}^3_{23}p + \hat{\Gamma}^3_{33}p^2 + \delta \sum_{i,j,l} \hat{g}_{ij}n_l(\hat{\Gamma}^i_{12} + \hat{\Gamma}^i_{23}p + \hat{\Gamma}^i_{33}p^2) - \Gamma^1_{12}p - \Gamma^3_{12}q, \tag{44}
\]
\[
N_1 = \hat{\Gamma}^3_{22} + 2\hat{\Gamma}^3_{23}q + \hat{\Gamma}^3_{33}q^2 + \delta \sum_{i,j,l} \hat{g}_{ij}n_l(\hat{\Gamma}^i_{22} + 2\hat{\Gamma}^i_{23}q + \hat{\Gamma}^i_{33}q^2) - \Gamma^3_{22}p - \Gamma^3_{22}q. \tag{44}
\]

The proof of Lemma 3 is based on the following

Proposition A (see [1]). The equations \( x_i = \hat{f}_i(u_1,u_2), (i = 1,2,3) \) define a regular surface \( M^2 \) in \((M^3,\bar{g})\) if and only if \( \hat{f}_i \) are regular (of class \( C^2 \)), and the
rank of \((\tilde{f}, \tilde{x})\) is equal to 2. The first \((g_{ij})\) and the second \((b_{ij})\) fundamental forms of \(M^2\) with the unit normal \((n_3)\) are given by

\[
g_{ij} = \sum_{\mu,\nu} \frac{\partial}{\partial x^i} \tilde{g}_{\mu,\nu} \frac{\partial}{\partial x^j}, \quad b_{ij} = \sum_{\mu,\nu} \Gamma^s_{\mu,\nu} \frac{\partial}{\partial x^i} \tilde{g}_{\mu,\nu} \frac{\partial}{\partial x^j},
\]

where \(\Gamma^s_{\mu,\nu}\) are Christoffel symbols of the 2-nd kind on \(M^3\).

Proof of Lemma 3 From the definition \(E = \hat{g}(e_1, e_1), F = \hat{g}(e_1, e_2), G = \hat{g}(e_2, e_2)\) it follows (43b). Let \(n = n_1 \hat{e}_1 + n_2 \hat{e}_2 + n_3 \hat{e}_3\) be a unit normal to \(M^2\). We find \(n_3\) from \(n_3 \hat{g}_{33} = \tilde{g}(\hat{e}_3, n) = \frac{\det(\hat{e}_1, \hat{e}_2, \hat{e}_3)}{\det(\hat{e}_1, \hat{e}_2, n)} = \frac{\hat{g}(\hat{e}_1, \hat{e}_2, \hat{e}_3)}{\sqrt{EG-F^2}} = \frac{1}{3}. \) Here \(\delta^2 = \frac{EG-F^2}{\det g} \geq \frac{1}{g_{33}},\) see (3).

The expressions for \(n_1\) and \(n_2\) of (43a) follow from the linear system

\[
g(e_1, n) = n_1 \hat{g}_{11} + n_2 \hat{g}_{12} + n_3 \hat{g}_{33} p = 0, \quad g(e_2, n) = n_1 \hat{g}_{12} + n_2 \hat{g}_{22} + n_3 \hat{g}_{33} q = 0.
\]

From (15) and \(\tilde{g}_{13} = \tilde{g}_{23} = f_{a,ij} = 0\) \((a = 1, 2), \sum_{a,b} \hat{g}_{ab} n_a n_b = 1,\) we have

\[
b_{ij} = \sum_{a,b} \hat{g}_{ab} n_b (b_{ij} n_a) = \sum_{a,b} \hat{g}_{ab} n_b (f_{a,ij} + \sum_{\mu,\nu} \hat{\Gamma}^a_{\mu,\nu} \frac{\partial}{\partial x^i} \hat{g}_{\mu,\nu} \frac{\partial}{\partial x^j})
\]

Hence, the coefficients \(L = b_{11}, M = b_{12} = b_{21}\) and \(N = b_{22}\) of II are given by

\[
\delta L = f_{11} + \hat{\Gamma}^3_{11} + 2 \hat{\Gamma}^3_{13} p + \hat{\Gamma}^3_{33} q^2 + \delta \sum_{i,j \leq 2} \hat{g}_{ij} n_j (\hat{\Gamma}^i_{11} + 2 \hat{\Gamma}^i_{13} p + \hat{\Gamma}^i_{33} q),
\]

\[
\delta N = f_{22} + \hat{\Gamma}^3_{22} + 2 \hat{\Gamma}^3_{23} q + \hat{\Gamma}^3_{33} q^2 + \delta \sum_{i,j \leq 2} \hat{g}_{ij} n_j (\hat{\Gamma}^i_{22} + 2 \hat{\Gamma}^i_{23} q + \hat{\Gamma}^i_{33} q^2),
\]

\[
\delta M = f_{12} + \hat{\Gamma}^3_{12} + \hat{\Gamma}^3_{13} p + \hat{\Gamma}^3_{13} q + \hat{\Gamma}^3_{33} q p + \delta \sum_{i,j \leq 2} \hat{g}_{ij} n_j (\hat{\Gamma}^i_{12} + \hat{\Gamma}^i_{33} q p + \hat{\Gamma}^i_{23} q + \hat{\Gamma}^i_{33} q^2)
\]

where \(f_{,ij} = f_{x,x} - \Gamma^i_{,i} p - \Gamma^3_{,ij} q\) are the covariant derivatives, see (42).

\(\square\)

3 Appendix: Parallel curved surfaces

We survey basic properties of PC surfaces in aim to illustrate that the PC surfaces provide a special class of solutions to the geometrical problem.

3.1 PC surfaces in \(\mathbb{R}^3(k)\)

(a) For \(\alpha = \text{const} > 0\), a solution to (22a,b), (24) is a PC surface in \(\mathbb{R}^3\). If \(k_2 = 0\), we get a cylinder \(M^2 : z = \sqrt{1/k^2 - (y + \alpha x)^2/(1 + \alpha^2)}\) of radius \(1/k_1\) with the axis \(\omega = (-1, \alpha, 0)\). We will build a PC surface with \(c_3 \neq \text{const}\), see Corollary III. Let \(M_1 : X^2 + Z^2 = R^2(Y)\) be a surface of revolution in \(\mathbb{R}^3\), where \(R \geq 0\) is an increasing \(C^1\)-regular function. Revolving about z-axis, \(X = \frac{\alpha x + y}{\sqrt{1 + \alpha^2}}, Y = \frac{\alpha y - x}{\sqrt{1 + \alpha^2}}, Z = z,\)
and replacing the function $R(t) = \frac{r(t)}{\sqrt{1 + \alpha^2}}$, we obtain $z = \sqrt{R^2(Y) - X^2} = (1 + \alpha^2)^{-1} \sqrt{r^2(\alpha y - x) - (ax + y)^2}$. A parallel $\{Y = c\}$ of above $M^2$ lies in the plane $\alpha y - x = c$, and projects onto $xy$-plane as a line segment.

(b) Consider spherical coordinates $(\rho, \varphi, \theta)$ in the domain $U = \{|\rho - 1| \leq a_1, |\varphi| \leq a_2, |\theta - \pi/2| \leq a_3\}$ of $\mathbb{R}^3$, where $0 < a_1 < 1$ and $0 < a_2 < \pi$ and $0 < a_3 < \pi/2$. The curvilinear projection onto $\Pi = \{|\rho - 1| \leq a_1, |\varphi| \leq a_2, \theta = \pi/2\}$ (with $\lambda = 0$) is given by $\pi(\rho, \varphi, \theta) = (\rho, \varphi, \pi/2)$.

(b) Denote $\gamma = \{|\rho - 1| \leq a_1, \varphi = 0, \theta = \pi/2\}$ the line segment in $\Pi$. Let $\bar{k}_1, \bar{k}_2$ be the functions of class $C^1([-a_1, a_1])$ and $l$ a vector field of class $C^2(\Pi)$ that is transversal but not orthogonal to $\gamma$. By Theorem 1 if $\bar{k}_i$ are small enough in the $C^0$-norm, then Problem 1 admits a unique smooth solution $M^2 : \theta = f(\varphi, \rho)$ on $\Pi_{K\varepsilon} = \{|\rho - 1| + K\varphi \leq a_1, 0 \leq \varphi \leq \varepsilon, \theta = \pi/2\}$.

(b) Assume that $M^2$ is a PC surface relative to the sphere $\beta = \{\rho = \rho_0\}$. Take $\gamma = \{|\rho - 1| \leq a_1, \varphi = b(\rho - 1), \theta = \frac{\pi}{2}\}$ for some $b \in (0, \frac{a_2}{1-a_1})$. The $k_1$-curvature lines of $M^2$ project onto concentric circles $\{\rho = c\}$ on $\Pi$, hence $l = \partial \varphi$ is transversal but not orthogonal to $\gamma$ (Theorem 1 is applicable). Now let $\bar{k}_1 \in C^1([-a_1, a_1])$ and $\bar{k}_2 \in C^0([-a_1, a_1])$ are small enough in the $C^0$-norm. Follow the proof of Theorem 2 (Section 2.2), one may show that $M^2 : \theta = f(\varphi, \rho)$ can be recovered over a curvilinear rectangle $\Pi(a_1)$.

The geometric construction of a PC surface $M^2$ is as follows. The spheres $S^2(c) = \{\rho = c\}$ intersect $M^2$ transversally by $k_1$-curvature lines. Let $\pi_1 : \mathbb{R}^3 \setminus \{0\} \to S^2(1)$ be the radial projection onto the unit sphere, i.e., $\pi_1(x) = x/\|x\|$. Let $\gamma(t)$ be a $k_1$-curvature line on $M^2$, and $\gamma(t)$ belongs to $S^2(c)$ for some $c$. The curve $\gamma_0 = \pi_1(\gamma)$ is homothetic to $\gamma$ (the coefficient of homothety is $1/c$). The great circles on $S^2(1)$ orthogonal to $\gamma_0$ and the curves of constant distance to $\gamma_0$ form a semi-geodesic net on $S^2(1)$ near $\gamma_0$, see Lemma 4.

A 1-parameter family of geodesics and their orthogonal curves on $\beta$ is called a semi-geodesic net (it is uniquely determined by the base curve $\gamma_0$).

**Lemma 4** There are (locally) four types of semi-geodesic nets on $(\beta, g_k)$:

(a) **cartesian net**, $k = 0, -1$: $\gamma_0$ is a line for $k = 0$, (horocycle for $k = -1$),
(b) **polar net**, $k = 0 \pm 1$: $\gamma_0$ is a circle,
(c) **evolvent net**, $k = 0, \pm 1$: normals to $\gamma_0$ are tangent to a curve $\gamma_1$,
(d) **super-parallel net**, $k = -1$: $\gamma_0$ is a line.

The cartesian and polar nets correspond to cylindrical surfaces ($k_2 = 0$) and surfaces of revolution ($k_1 = \text{const}$ along $\mathcal{F}_1$-curves, the axis is orthogonal to $\beta$), resp. (Case (c) appears on PC surfaces illustrated in Fig. 1(b)).

**Proof.** It is known that the normals to a regular curve in $\mathbb{R}^2(k)$ (locally) form one of four families: (super-)parallel lines, lines through a point and enveloping a smooth curve $\gamma_1$. \(\square\)
Figure 1: (a) Projection of a PC surface $M^2$. (b) PC surface in $\mathbb{R}^3$ of type (c).

**Proposition 6** Let $M^2 \subset \mathbb{R}^3(k)$ be a PC surface-graph related to a totally umbilical surface $\beta$, and $\mathcal{F}_i$ ($i = 1, 2$) the $k_i$-curvature lines. Then

(i) the principal curvature $k_2$ is constant along the curves of $\mathcal{F}_1$,
(ii) the curves of $\mathcal{F}_2$ are geodesics, they belong to planes orthogonal to $\beta$,
(iii) both families of curves project onto $\beta$ as a semi-geodesic net.

**Proof.** The curves of $\mathcal{F}_1$ belong to totally umbilical surfaces $\beta_d$ (on the distance $d$ to $\beta$), and the curves are parallel on $M$. Hence, $\mathcal{F}_2$ (that is orthogonal to $\mathcal{F}_1$) consists of geodesics of $M^2$. Let $X_i$ ($i = 1, 2$) be unit vector fields tangent to $\mathcal{F}_i$, $n$ a unit normal to $M^2$, $\partial_t$ a unit normal to $\beta$. Then $X_1$ is orthogonal to $n$ and $\partial_t$. By (2) of Lemma 5 (see Section 3.2) and Rodrigues theorem (see [9]),

$$X_1((n, \partial_t)) = \langle \nabla X_1 n, \partial_t \rangle + \langle n, \nabla X_1 \partial_t \rangle = \langle k_1 X_1, \partial_t \rangle + \langle n, (\log \phi)' X_1 \rangle = 0$$

where $\nabla$ is the covariant derivative. Hence, the angle between surfaces $M^2$ and $\beta_d$ along the curvature lines $\mathcal{F}_1$ (the intersection) is constant. The projections of $\mathcal{F}_1$ onto $\beta$ are parallel curves $\bar{\mathcal{F}}_1$, hence their orthogonal trajectories $\bar{\mathcal{F}}_2$ are geodesics on $\beta$. Thus $(\bar{\mathcal{F}}_1, \bar{\mathcal{F}}_2)$ is a semi-geodesic net on $\beta$. In coordinates of curvature lines we have $k_{2,1} = (k_1 - k_2) g^{22}_{2g22}$, see Remark 2. Since $g_{22} = 1$ ($\mathcal{F}_2$-curves are unit speed geodesics), we obtain $k_{2,1} = 0$, hence $k_2 = \text{const}$ along $\mathcal{F}_1$-curves. One may show that (as in $\mathbb{R}^3$, see [2], [3] and (29) in what follows) the curves of $\mathcal{F}_2$ are congruent in $M^3(k)$ each to another, and lie in planes through geodesics $\bar{\mathcal{F}}_2 \subset \beta$ and orthogonal to $\beta$. \hfill $\Box$

### 3.2 PC surfaces in a Riemannian warped product 3-space

Let $(S, g_k)$ be a Riemannian 2-space of constant curvature $k$, and $\psi : I \to \mathbb{R}_+$. The **Riemannian warped product 3-space** is $\mathbb{R}^3(k, \psi) = (I \times S, g^k_\psi)$ where $g^k_\psi = dt^2 + \psi^2(t) g_k$. $\mathbb{R}^3(k, \psi)$ contains no open subsets of constant curvature if and only if $(\log \psi)'' + k/\psi^2 \neq 0$ on $I$, [5]. A surface $S(t) = \{t\} \times S$ is a slice of $\mathbb{R}^3(k, \psi)$. The
mean curvature vector of $M^2 \subset \mathbb{R}^3(k, \psi)$ is defined by $H = (\text{tr} \, h)/2$, where $h$ the
second fundamental form of $M^2$. A surface $M^2$ is
- totally geodesic if $h = 0$;
- totally umbilical if $h(X, Y) = g^k_{\psi}(X, Y)H \quad (X, Y \in TM)$;
- $\mathcal{H}$-surface if the vector field $\partial_t$ is tangent to $M^2$ at each point on $M^2$.

We decompose a vector field $v$ on $\mathbb{R}^3(k, \psi)$ into a sum $V = \phi_V \partial_t + \tilde{V}$, where $\phi_V = g(V, \partial_t)$ and $\tilde{V}$ (a vertical component) is orthogonal to $\partial_t$.

**Lemma 5 ([5])** The connection and the curvature of $\mathbb{R}^3(k, \psi)$ satisfy

1. $\nabla_{\partial_t} \partial_t = 0$,
2. $\nabla_{\partial_t} X = \nabla_X \partial_t = (\log \psi)'X$,
3. $g(\nabla_X Y, \partial_t) = -g(X, Y)(\log \psi)'$,
4. $\nabla_X Y$ is the lift of $\nabla^S_X Y$ on $S$,
5. $R(\partial_t, X)\partial_t = (\psi''/\psi)X$, $R(X, \partial_t)Y = \langle X, Y \rangle(\psi''/\psi)\partial_t$, $R(X, Y)\partial_t = 0$,
   $R(X, Y)Z = (k - (\psi')^2/\psi^2)\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\}$ for $X, Y, Z \in TS$.

**Lemma 6** An $\mathcal{H}$-surface $\Pi_\gamma = I \times \{\gamma\}$ over a smooth curve $\gamma \subset S$ is

(i) a ruled surface with rulings $I \times \{s\}$ ($s \in \gamma$),
(ii) a totally geodesic in $\mathbb{R}^3(k, \psi)$ if and only if $\gamma$ is a geodesic in $S$.

**Proof.** Let $h$ be the second fundamental form of $\Pi_\gamma$. Denote $X$ the (unit)
velocity field of a geodesic $\gamma$. Using Lemma 5 we have on $\Pi_\gamma$:

by (1): $h(\partial_t, \partial_t) = 0$. Hence $I \times \{s\}$ are rulings (geodesics in $\mathbb{R}^3(k, \psi)$);
by (2): $\nabla_{\partial_t} X \in TM^2$, hence $h(\partial_t, X) = 0$;
by (4): $h(X, X) = 0$ if and only if $\gamma$ is a geodesic in $S$.

We conclude that $h = 0$ when $\gamma$ is a geodesic in $S$. On the other hand, by (3)
and (4) of Lemma 5 $\nabla^S_X X = 0$ if and only if $h(X, X) = 0$. Hence, if $h = 0$ then
$\nabla^S_X X = 0$, that is $\gamma$ is a geodesic in $S$. $\blacksquare$

An $\mathcal{H}$-surface $\Pi_\gamma = I \times \{\gamma\}$ is totally umbilical with $\nabla^\perp H = 0$ if and only if
$\Pi_\gamma$ is totally geodesic, see [5]. Any such $\Pi_\gamma$ over an $S$-geodesic $\gamma$ will be named
$\mathcal{H}$-plane. By Lemma 5 the gaussian curvature of $\Pi_\gamma$ is $K = \psi''/\psi$.

A surface $M^2 \subset \mathbb{R}^3(k, \psi)$ is called parallel curved (PC) relative to $S$ if it does not
belong to a slice, and at each point $x \in M^2$ at least one principal direction is tangent
to $S(t)$ passing through $x$. A PC surface is regular if such principal directions form
a 1-dimensional foliation $(\mathcal{F}_1)$.

Proposition 7 can be extended as follows

**Proposition 7** Let $M^2 \subset \mathbb{R}^3(k, \psi)$ be a regular PC surface-graph over domain in $S$.
Then the 2-nd family of curvature lines $(\mathcal{F}_2)$ consists of geodesics on $M^2$ which lie in
$\mathcal{H}$-planes. Two families $(\mathcal{F}_1$ and $\mathcal{F}_2)$ of curves project onto $S$ as a semi-geodesic net.

Hence PC surfaces in $\mathbb{R}^3(k, \psi)$ represent a special class of solutions to Problem 1
for graphs over domains of $\mathcal{H}$-plane $\Pi_\gamma$ with $\gamma$ transversal to slices.
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