A NOTE ON THE BUCHSBAUM-RIM FUNCTION OF A PARAMETER MODULE

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Abstract. In this article, we prove that the Buchsbaum-Rim function $\ell_A(S_{\nu+1}/N_{\nu+1})$ of a parameter module $N$ in $F$ is bounded above by $e(F/N)(d+r-1)$ for every integer $\nu \geq 0$. Moreover, it turns out that the base ring $A$ is Cohen-Macaulay once the equality holds for some integer $\nu$. As a direct consequence, we observe that the first Buchsbaum-Rim coefficient $e_1(F/N)$ of a parameter module $N$ is always non-positive.

1. Introduction

Let $(A, m)$ be a Noetherian local ring of dimension $d$. Let $F = A^r$ be a free module of rank $r > 0$, and let $S = S_A(F)$ be the symmetric algebra of $F$, which is a polynomial ring over $A$. For a submodule $M$ of $F$, let $R(M)$ denote the image of the natural homomorphism $S_A(M) \to S_A(F)$, which is a standard graded subalgebra of $S$. Assume that the quotient $F/M$ has finite length and $M \subseteq mF$. Then we can consider the function

$$\lambda : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} : \nu \mapsto \ell_A(S_{\nu+1}/M_{\nu+1})$$

where $S_{\nu}$ and $M_{\nu}$ denote the homogeneous components of degree $\nu$ of $S$ and $R(M)$, respectively. Buchsbaum and Rim studied this function in [4] in order to generalize the notion of the usual Hilbert-Samuel multiplicity of an $m$-primary ideal. They proved that $\lambda(\nu)$ eventually coincides with a polynomial $P(\nu)$ of degree $d + r - 1$. This polynomial can then be written in the form

$$P(\nu) = \sum_{i=0}^{d+r-1} (-1)^i e_i(F/M) \binom{\nu + d + r - 1 - i}{d + r - 1 - i}$$

with integer coefficients $e_i(F/M)$. The coefficients $e_i(F/M)$ are called the Buchsbaum-Rim coefficients of $F/M$. The Buchsbaum-Rim multiplicity...
of $F/M$, denoted by $e(F/M)$, is now defined to be the leading coefficient $e_0(F/M)$.

In their article Buchsbaum and Rim also introduced the notion of a parameter module (matrix), which generalizes the notion of a parameter ideal (system of parameters). The module $N$ in $F$ is said to be a parameter module in $F$, if the following three conditions are satisfied: (i) $F/N$ has finite length, (ii) $N \subseteq mF$, and (iii) $\mu_A(N) = d + r - 1$, where $\mu_A(N)$ is the minimal number of generators of $N$.

A starting point of this article is the characterization of the Cohen-Macaulay property of $A$ given in [4, Corollary 4.5] by means of the equality $\ell_A(F/N) = e(F/N)$ for every parameter module $N$ of rank $r$ in $F = A^r$. Brennan, Ulrich and Vasconcelos observed in [1, Theorem 3.4] that if $A$ is Cohen-Macaulay, then in fact $\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\begin{pmatrix} \nu + d + r - 1 \\ d + r - 1 \end{pmatrix}$ for all integers $\nu \geq 0$. Our main result is now as follows:

Theorem 1.1. Let $(A, m)$ be a Noetherian local ring of dimension $d > 0$.

(1) For any rank $r > 0$, the inequality

$$\ell_A(S_{\nu+1}/N^{\nu+1}) \geq e(F/N)\begin{pmatrix} \nu + d + r - 1 \\ d + r - 1 \end{pmatrix}$$

always holds true for every parameter module $N$ in $F = A^r$ and for every integer $\nu \geq 0$.

(2) The following statements are equivalent:

(i) $A$ is a Cohen-Macaulay local ring;

(ii) There exists an integer $r > 0$ and a parameter module $N$ of rank $r$ in $F = A^r$ such that the equality

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\begin{pmatrix} \nu + d + r - 1 \\ d + r - 1 \end{pmatrix}$$

holds true for some integer $\nu \geq 0$.

This extends our previous result [10, Theorem 1.3] where we assumed that $\nu = 0$.

Concerning not only parameter modules but also their ”powers”, Theorem [11] generalizes in two directions the classical result saying that the inequality $\ell_A(A/Q) \geq e(A/Q)$ holds true for any parameter ideal $Q$ in a local ring $A$ with equality for some parameter ideal if and only if $A$ is
Cohen-Macaulay. Theorem 1.1 seems to contain some new information even in the ideal case. Indeed, the equivalence of (i) and (ii) in (2) improves a recent observation that the ring \( A \) is Cohen-Macaulay if there exists a parameter ideal \( Q \) in \( A \) such that \( \ell_A(A/Q^{\nu+1}) = e(A/Q)^{(\nu+d)} \) for all \( \nu \gg 0 \) (see [8, 11]). Moreover, as a direct consequence of (1), we have the non-positivity of the first Buchsbaum-Rim coefficient of a parameter module.

**Corollary 1.2.** For any rank \( r > 0 \), the inequality

\[
e_1(F/N) \leq 0
\]

always holds true for every parameter module \( N \) in \( F = A^r \).

Mandal and Verma have recently proved that \( e_1(A/Q) \leq 0 \) for any parameter ideal \( Q \) in \( A \) (see [14], and also [8]). Corollary 1.2 can be viewed as the module version of this fact. However, our proof based on the inequality in Theorem 1.1 (1) is completely different from theirs and is considerably more simple.

The proof of Theorem 1.1 will be completed in section 3. It utilizes the fact that the Buchsbaum-Rim multiplicity of a parameter module can be determined as the Euler-Poincaré characteristic of the corresponding Eagon-Northcott complex. The next section is of preliminary character. In section 3, we will obtain Theorem 1.1 as a corollary of a more general result (Theorem 3.1).

2. Preliminaries

Let \( (A, \mathfrak{m}) \) be a Noetherian local ring of dimension \( d \). Let \( F = A^r \) be a free module of rank \( r > 0 \). Let \( S = S_A(F) \) be the symmetric algebra of \( F \). Let \( N \) be a parameter module in \( F \), that is, \( N \) is a submodule of \( F \) satisfying the conditions: (i) \( \ell_A(F/N) < \infty \), (ii) \( N \subseteq \mathfrak{m}F \), and (iii) \( \mu_A(N) = d + r - 1 \). We put \( n = d + r - 1 \). Let \( N^\nu \) be the homogeneous component of degree \( \nu \) of the graded subalgebra \( \mathcal{R}(N) = \operatorname{Im}(S_A(N) \to S) \) of \( S \). Let \( \tilde{N} = (c_{ij}) \) be the matrix associated to a minimal free presentation

\[
A^n \xrightarrow{\tilde{N}} F \to F/N \to 0
\]

of \( F/N \). Let \( I(N) \) be the 0-th Fitting ideal of \( F/N \), which is the ideal generated by the maximal minors of \( \tilde{N} \). Let \( X = (X_{ij}) \) be a generic matrix of the same size \( r \times n \). We denote by \( I_s(X) \) the ideal in the polynomial ring \( A[X] = A[X_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n] \) generated by the
s-minors of $X$. Let $B = A[X]_{(m,X)}$ be the ring localized at the graded maximal ideal $(m, X)$ of $A[X]$. The substitution map $A[X] \to A$ where $X_{ij} \mapsto c_{ij}$ now induces a map $\varphi : B \to A$. We consider the ring $A$ as a $B$-algebra via the map $\varphi$. Let

$$b = \text{Ker } \varphi = (X_{ij} - c_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n)B.$$ 

Set $G = B^r$, and let $L$ denote the submodule $\text{Im}(B^n \xrightarrow{X} G)$ of $G$. Let $G_\nu$ and $L^\nu$ be the homogeneous components of degree $\nu$ of the graded algebras $S_B(G)$ and $R(L)$, respectively. In the sequel we will utilize the exact sequences

$$(*) \quad 0 \to L^\nu G_t/L^{\nu+1}G_{t-1} \to G_{\nu+t}/L^{\nu+1}G_{t-1} \to G_{\nu+t}/L^\nu G_t \to 0$$

where $\nu, t \geq 0$. Here $LG_{t-1} = I_t(X)B$ and $NS_{t-1} = I(N)$.

We recall the following fact from [9]:

**Proposition 2.1.** For any integer $t \geq 0$, the $B$-module $L^\nu G_t/L^{\nu+1}G_{t-1}$ is isomorphic to the direct sum of $(\nu + n - 1) \choose (n - 1)$ copies of $G_t/LG_{t-1}$ for all $\nu \geq 0$. That is, we have for all $\nu, t \geq 0$ an isomorphism of $B$-modules

$$L^\nu G_t/L^{\nu+1}G_{t-1} \cong (G_t/LG_{t-1})^{(\nu + n - 1) \choose (n - 1)}.$$

**Proof.** See [9, Proposition 3.1].

**Lemma 2.2.** For any integers $\nu, t \geq 0$, we have the following:

1. $(G_{\nu+t}/L^{\nu+1}G_{t-1}) \otimes_B (B/b) \cong S_{\nu+t}/N^{\nu+1}S_{t-1}$;
2. $\text{Supp}_B(G_{\nu+t}/L^{\nu+1}G_{t-1}) = \text{Supp}_B(B/I_r(X)B)$;
3. The ideal $b$ is generated by a system of parameters of the module $G_{\nu+t}/L^{\nu+1}G_{t-1}$.

**Proof.** The first assertion is easy to see. Let us then verify the second one. It is well-known that $\sqrt{\text{Ann}_B(G/L)} = \sqrt{\text{Fitt}_0(G/L)}$ (see [2, (16.2) Proposition]). Since $\text{Fitt}_0(G/L) = I_r(X)B$, we have $\text{Supp}_B(G/L) = \text{Supp}_B(B/I_r(X)B)$. An easy localization argument gives

$$\text{Supp}_B(G_{\nu+t}/L^{\nu+1}G_{t-1}) = \text{Supp}_B(G/L)$$

for all $\nu \geq 0$ and $t \geq 1$. It therefore remains to show that

$$\text{Supp}_B(G_{\nu}/I_r(X)L^{\nu}) = \text{Supp}_B(B/I_r(X)B),$$

but this is easily checked by using the exact sequence (4) in the case $t = 0$ combined with Proposition 2.1. Thus the assertion (2) follows. In order to prove the third assertion, recall first that $\dim B/I_r(X)B = 4$. 

\[ d + (n + 1)(r - 1) = rn \] (see [2] (5.12) Corollary). The assertion (3) then follows from (1), (2) and the fact that \( b \) is generated by \( rn \) elements. \( \square \)

**Lemma 2.3.** For any integer \( \nu \geq 0 \), we have

1. \( G_{\nu}/I_{\nu}(XB) \) and \( G_{\nu+1}/L_{\nu+1} \) are perfect \( B \)-modules of grade \( d \);
2. \( G_{\nu+1}/L_{\nu+1}G_{t-1} \) has finite projective dimension for all \( t \geq 0 \).

**Proof.** The claim concerning \( G_{\nu+1}/L_{\nu+1} \) in (1) is already known by [3, Corollary 3.2] (see also [12, Proposition 3.3]). Consider the exact sequence \( \ast \) with \( t = 0 \). Since \( B/I_{\nu}(XB) \) is a perfect \( B \)-module of grade \( d \) (see [2] (2.8) Corollary), Proposition 2.1 implies that so is \( L_{\nu}/I_{\nu}(XB) \). It thus follows that \( G_{\nu}/I_{\nu}(XB) \) is a perfect \( B \)-module of grade \( d \) for all \( \nu \geq 0 \). This proves (1). We can then prove (2) by induction on \( t \) using the exact sequence \( \ast \) and Proposition 2.1. \( \square \)

**Proposition 2.4.** For any \( p \in \text{Min}_B(B/I_{\nu}(XB)) \), the equality

\[ \ell_{B_p}((G_{\nu+1}/L_{\nu+1}G_{t-1})_p) = \ell_{B_p}((B/I_{\nu}(XB))_p) \left( \frac{\nu + d + r - 1}{d + r - 1} \right) \]

holds true for all integers \( \nu \geq 0 \) and \( t \geq 0 \).

**Proof.** Take \( p \in \text{Min}_B(B/I_{\nu}(XB)) \) and fix an integer \( \nu \geq 0 \). We put \( I_p = I_{\nu}(XB)p \). We start with the case \( t = 1 \). By Lemma 2.3 (1), \( \text{grade}_B p = d \), because perfect modules are grade unmixed. It now follows that \( I_{\nu-1}(XB) \not\subseteq p \). Indeed, if \( I_{\nu-1}(XB) \subseteq p \), then

\[
\begin{align*}
d &= \text{grade}_B p \\
&\geq \text{grade}_B I_{\nu-1}(XB) \\
&= (n - (r - 1) + 1)(n - (n - 1) + 1) \\
&= 2(d + 1),
\end{align*}
\]

which is a contradiction. Consider now the following free presentation

\[
B^n_p \xrightarrow{X} G_p \xrightarrow{G_p/L_p} 0
\]

of \( G_p/L_p \). Since \( I_{\nu-1}(XB) \not\subseteq p \), we may assume that after elementary row and column transformations over \( B_p \) the matrix \( X \) has the form

\[
\begin{pmatrix}
E_{\nu-1} & O \\
0 & \begin{pmatrix}
a_1 & \cdots & a_d \\
0 & \cdots & 0
\end{pmatrix}
\end{pmatrix},
\]

...
where $E_{r-1}$ is the identity matrix of size $r - 1$. Let us fix a free basis \{t_1, \ldots, t_r\} for $G_p$. Then we have $I_p = (a_1, \ldots, a_d)B_p$ and $R(L_p) \cong B_p[t_1, \ldots, t_{r-1}, I_p t_r]$. Therefore, we get the following isomorphisms:

\[
(G_{\nu + 1}/L^{\nu + 1})_p \cong (G_{\nu + 1}/(L_p)^{\nu + 1}) \\
\cong B_p[t_1, \ldots, t_{r-1}, t_r]_{\nu + 1}/B_p[t_1, \ldots, t_{r-1}, I_p t_r]_{\nu + 1} \\
\cong \bigoplus_{i_1, \ldots, i_r \geq 0, i_1 + \cdots + i_r = \nu + 1} (B_p/I_p^{i_1} t_1^{i_1} \cdots t_r^{i_r}) \\
= \nu + 1 \bigoplus_{i_1, \ldots, i_{r-1} \geq 0, i_1 + \cdots + i_{r-1} = \nu + 1 - i} (B_p/I_p^{i_1} t_1^{i_1} \cdots t_r^{i_r-1}) t_r.
\]

Notice that the ring $B_p$ is Cohen-Macaulay and the system of generators \{a_1, \ldots, a_d\} of $I_p$ forms a regular sequence on $B_p$. Hence we can compute the length in question as follows:

\[
\ell_{B_p}((G_{\nu + 1}/L^{\nu + 1})_p) = \sum_{i=1}^{\nu + 1} \text{rank}_{B_p}(B_p[t_1, \ldots, t_{r-1}]_{\nu + 1 - i}) \cdot \ell_{B_p}(B_p/I_p^i) \\
= \ell_{B_p}(B_p/I_p) \sum_{i=1}^{\nu} \binom{\nu - i + r - 1}{r - 2} \binom{i + d - 1}{d} \\
= \ell_{B_p}(B_p/I_p) \sum_{i=0}^{\nu} \binom{\nu - i + r - 2}{r - 2} \binom{i + d}{d} \\
= \ell_{B_p}(B_p/I_p) \binom{\nu + d + r - 1}{d + r - 1}.
\]

We will next prove the case $t = 0$. Consider the exact sequence localized at $p$. By Proposition 2.1 and the case $t = 1$, we get

\[
\ell_{B_p}((G_{\nu}/I_r(X)L^\nu)_p) = \ell_{B_p}((G_{\nu}/L^\nu)_p) + \ell_{B_p}((L^\nu/I_r(X)L^\nu)_p) \\
= \ell_{B_p}(B_p/I_p) \left\{ \binom{\nu - 1 + n}{n} + \binom{\nu + n - 1}{n - 1} \right\} \\
= \ell_{B_p}(B_p/I_p) \binom{\nu + n}{n}.
\]

The cases $t = 0, 1$ have been thus proven, we will now proceed by induction on $t$. Again, look at the exact sequence localized at $p$. By
Proposition 2.1 and the induction hypothesis, we then have
\[
\ell_B \left( \left( \frac{G_{\nu/t}}{L^{\nu+1}G_{t-1}} \right)_p \right) \\
= \ell_B \left( \left( \frac{G_{\nu+1}}{L^{\nu+2}G_{t-2}} \right)_p \right) - \ell_B \left( \left( \frac{L^{\nu+1}G_{t-1}}{L^{\nu+2}G_{t-2}} \right)_p \right) \\
= \ell_B(\mathcal{B}_p/\mathcal{I}_p) \left( \frac{\nu + 1 + n}{n} \right) - \ell_B \left( \left( \frac{G_{t-1}/LG_{t-2}}{L^{\nu+2}G_{t-2}} \right)_p \right) \left( \frac{\nu + n}{n - 1} \right) \\
= \ell_B(\mathcal{B}_p/\mathcal{I}_p) \left( \frac{\nu + n}{n} \right)
\]
as desired. \( □ \)

3. Proof of Theorem 1.1

Theorem 1.1 will be a consequence of the following more general result:

**Theorem 3.1.** Let \((A, \mathfrak{m})\) be a Noetherian local ring of dimension \(d \geq 0\).

1. For any rank \(r > 0\), the inequality

\[
\ell_A \left( \frac{S_{\nu+t}}{N^{\nu+1}S_{t-1}} \right) \geq e(F/N) \left( \frac{\nu + d + r - 1}{d + r - 1} \right)
\]

always holds true for every parameter module \(N\) in \(F = A^r\) and all integers \(\nu, t \geq 0\), where \(NS_{t-1} = I(N)\).

2. The following statements are equivalent:

   (i) \(A\) is a Cohen-Macaulay local ring;

   (ii) There exists an integer \(r > 0\) and a parameter module \(N\) of rank \(r\) in \(F = A^r\) such that the equality

\[
\ell_A \left( \frac{S_{\nu+t}}{N^{\nu+1}S_{t-1}} \right) = e(F/N) \left( \frac{\nu + d + r - 1}{d + r - 1} \right)
\]

holds true for some integers \(0 \leq t \leq d\) and \(\nu \geq 0\).

In order to prove Theorem 3.1, we need to introduce more notation. For any matrix \(a\) of size \(r \times n\) over an arbitrary ring, we denote by \(K_\bullet(a)\) its Eagon-Northcott complex \([6]\). When \(r = 1\), the complex \(K_\bullet(a)\) is just the ordinary Koszul complex of the sequence \(a\). See \([7, \text{Appendix A2}]\) for the definition and more details of complexes of this type. Recall in particular that if \(N\) is a parameter module in a free module \(F\) as in section 2, then

\[
e(F/N) = \chi(K_\bullet(\tilde{N}))
\]
where $\chi(K_\bullet(\tilde{N}))$ denotes the Euler-Poincaré characteristic of the complex $K_\bullet(\tilde{N})$ (see [4] and [13]).

**Lemma 3.2.** Using the setting and notation of section 2, we have

$$
\chi(K_\bullet(b) \otimes_B (B/I_r(X)B)) = \chi(K_\bullet(\tilde{N})).
$$

**Proof.** We use an idea from [5]. Set $I = I_r(X)$. Since the Eagon-Northcott complex is compatible with the base change, $K_\bullet(X) \otimes_B A \cong K_\bullet(\tilde{N})$. The complex $K_\bullet(X)$ is a $B$-free resolution of $B/IB$ and hence, by tensoring with $A$ and taking the homology, we obtain

$$
H_p(K_\bullet(\tilde{N})) \cong H_p(K_\bullet(X) \otimes_B A) \cong \text{Tor}_p^B(B/IB, A)
$$

for all $p \geq 0$. On the other hand, since the ideal $b$ is generated by a regular sequence of length $rn$ in $B$, the ordinary Koszul complex $K_\bullet(b)$ associated to a system of generators of $b$ is a $B$-free resolution of $A$. Hence, by tensoring with $B/IB$, we can compute the Tor as follows:

$$
\text{Tor}_p^B(B/IB, A) \cong H_p(K_\bullet(b) \otimes_B (B/IB)).
$$

Therefore, for any $p \geq 0$,

$$
H_p(K_\bullet(\tilde{N})) \cong H_p(K_\bullet(b) \otimes_B (B/IB)).
$$

Thus $\chi(K_\bullet(b) \otimes_B (B/IB)) = \chi(K_\bullet(\tilde{N}))$ as wanted. \qed

Now we can give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** We use the same notation as in section 2. Put $I = I_r(X)$.
(1): Fix integers $\nu \geq 0$ and $t \geq 0$. The ideal $b$ being generated by a system of parameters of the module $G_{\nu+t}/L^{\nu+1}G_{t-1}$, we get

$$\ell_A(S_{\nu+t}/N^{\nu+1}S_{t-1}) = \ell_B((G_{\nu+t}/L^{\nu+1}G_{t-1}) \otimes_B (B/b)) \geq e(b; G_{\nu+t}/L^{\nu+1}G_{t-1})$$

$$= \sum_{p \in \text{Assh}_B(G_{\nu+t}/L^{\nu+1}G_{t-1})} e(b; B/p) \cdot \ell_{B_p}((G_{\nu+t}/L^{\nu+1}G_{t-1})_p)$$

$$= \sum_{p \in \text{Assh}_B(B/IB)} e(b; B/p) \cdot \ell_{B_p}((B/IB)_p) \left( \frac{\nu + d + r - 1}{d + r - 1} \right)$$

$$= e(b; B/IB) \left( \frac{\nu + d + r - 1}{d + r - 1} \right)$$

$$= \chi(K_\bullet(b) \otimes_B (B/IB)) \left( \frac{\nu + d + r - 1}{d + r - 1} \right)$$

$$= \chi(K_\bullet(N)) \left( \frac{\nu + d + r - 1}{d + r - 1} \right)$$

$$= e(F/N) \left( \frac{\nu + d + r - 1}{d + r - 1} \right)$$

as desired, where $e(b; \ast)$ denotes the multiplicity of $\ast$ with respect to $b$.

(2): The other implication being clear, by the ideal case, for example, it is enough to show that (ii) implies (i). Assume thus that

$$\ell_A(S_{\nu+t}/N^{\nu+1}S_{t-1}) = e(F/N) \left( \frac{\nu + d + r - 1}{d + r - 1} \right)$$

for some $\nu \geq 0$ and $t \geq 0$. The above argument then gives

$$\ell_B((G_{\nu+t}/L^{\nu+1}G_{t-1}) \otimes_B (B/b)) = e(b; G_{\nu+t}/L^{\nu+1}G_{t-1}).$$

It follows that $G_{\nu+t}/L^{\nu+1}G_{t-1}$ is a Cohen-Macaulay $B$-module of dimension $rn$. By Lemma 2.3, $G_{\nu+t}/L^{\nu+1}G_{t-1}$ is a $B$-module with finite projective dimension. Thus, by the Auslander-Buchsbaum formula,

$$\text{depth } B = \text{depth}_B(G_{\nu+t}/L^{\nu+1}G_{t-1}) + \text{pd}_B(G_{\nu+t}/L^{\nu+1}G_{t-1})$$

$$\geq \dim_B(G_{\nu+t}/L^{\nu+1}G_{t-1}) + \text{grade}_B(G_{\nu+t}/L^{\nu+1}G_{t-1})$$

$$= rn + d$$

$$= \dim B.$$ 

Therefore $B$ is Cohen-Macaulay so that $A$ is Cohen-Macaulay, too. $\square$
Taking $t = 1$ in Theorem 3.1 now readily gives Theorem 1.1.

Remarks 3.3. Suppose that $A$ is Cohen-Macaulay. Because of Lemma 2.3 (1) the above argument shows that the equality

$$\ell_A(S_\nu/I(N)N^\nu) = \ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\left(\frac{\nu + d + r - 1}{d + r - 1}\right)$$

holds true for all $\nu \geq 0$. When $t \geq 2$, we do not know whether the Cohen-Macaulayness of $A$ implies the equality

$$\ell_A(S_{\nu+t}/N^\nu S_{t-1}) = e(F/N)\left(\frac{\nu + d + r - 1}{d + r - 1}\right),$$

except in the following cases:

1. When $0 \leq t \leq d$, the equality $\ell_A(S_t/NS_{t-1}) = e(F/N)$ holds true by [10, Theorem 4.1].
2. When $d = 2$, we know by [9, Theorem 4.1] that $G_{\nu+2}/L^{\nu+1}G_1$ is a perfect $B$-module of grade two for all $\nu \geq 0$. The argument in the proof of Theorem 3.1 then gives

$$\ell_A(S_{\nu+2}/N^{\nu+1}S_1) = e(F/N)\left(\frac{\nu + r + 1}{r + 1}\right)$$

for all $\nu \geq 0$.
3. If $t \geq d+1$, then $pd_B(G_t/LG_{t-1}) \geq d+1$ (see [2] (2.19) Remarks], for instance). So the equality does not hold in this case.

References

[1] J. Brennan, B. Ulrich and W. V. Vasconcelos, The Buchsbaum-Rim polynomial of a module, J. Algebra 241 (2001), 379–392
[2] W. Bruns and U. Vetter, Determinantal Rings, Lecture Notes in Math. 1327, Springer-Verlag Berlin Heidelberg, 1988
[3] D. A. Buchsbaum and D. Eisenbud, Generic free resolutions and a family of generically perfect ideals, Adv. in Math. 18 (1975), 245–301
[4] D. A. Buchsbaum and D. S. Rim, A generalized Koszul complex. II. Depth and multiplicity, Trans. Amer. Math. Soc. 111 (1964), 197–224
[5] D. A. Buchsbaum and D. S. Rim, A generalized Koszul complex. III. A Remark on Generic Acyclicity, Proc. Amer. Math. Soc. 16 (1965), 555–558
[6] J. A. Eagon and D. G. Northcott, Ideals defined by matrices and a certain complex associated with them, Proc. Roy. Soc. Ser. A 269 (1962), 188–204
[7] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry, Graduate Texts in Mathematics, 150, Springer-Verlag, New York, 1995
[8] L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T. T. Phuong, and W. V. Vasconcelos, Cohen-Macaulayness versus the vanishing of the first Hilbert coefficient of parameter ideals, to appear in J. London Math. Soc.
[9] F. Hayasaka and E. Hyry, A family of graded modules associated to a module, Communications in Algebra, Volume 36 (2008), Issue 11, 4201–4217
[10] F. Hayasaka and E. Hyry, A note on the Buchsbaum-Rim multiplicity of a parameter module, Proc. Amer. Math. Soc. 138 (2010), 545–551
[11] Y. Kamoi, Remark on the polynomial type Poincaré series, Proceedings of the Second Japan-Vietnam Joint Seminar on Commutative Algebra (2006), 162–168
[12] D. Katz and C. Naudé, Prime ideals associated to symmetric powers of a module, Comm. Algebra 23 (1995), no. 12, 4549–4555
[13] D. Kirby, On the Buchsbaum-Rim multiplicity associated with a matrix, J. London Math. Soc. (2) 32 (1985), no. 1, 57–61
[14] M. Mandal, B. Singh and J. K. Verma, On some conjectures about the Chern number of filtrations, Preprint 2010, arXiv:1001.2822

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