Near Optimal Routing for Small-World Networks with Augmented Local Awareness

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Abstract. In order to investigate the routing aspects of small-world networks, Kleinberg [13] proposes a network model based on a $d$-dimensional lattice with long-range links chosen at random according to the $d$-harmonic distribution. Kleinberg shows that the greedy routing algorithm by using only local information performs in $O(\log^2 n)$ expected number of hops, where $n$ denotes the number of nodes in the network. Martel and Nguyen [17] have found that the expected diameter of Kleinberg’s small-world networks is $\Theta(\log n)$. Thus a question arises naturally: Can we improve the routing algorithms to match the diameter of the networks while keeping the amount of information stored on each node as small as possible?

Existing approaches for improving the routing performance in the small-world networks include: (1) Increasing the number of long-range links [2, 15]; (2) Exploring more nodes before making routing decisions [14]; (3) Increasing the local awareness for each node [10, 17]. However, all these approaches can only achieve $O((\log n)^{1+\epsilon})$ expected number of hops, where $\epsilon > 0$ denotes a constant. We extend Kleinberg’s model and add three augmented local links for each node: two of which are connected to nodes chosen randomly and uniformly within $\log^2 n$ Mahattan distance, and the third one is connected to a node chosen randomly and uniformly within $\log n$ Mahattan distance. Our investigation shows that these augmented local connections can make small-world networks more navigable.

We show that if each node is aware of $O(\log n)$ number of neighbors via the augmented local links, there exist both non-oblivious and oblivious algorithms that can route messages between any pair of nodes in $O(\log n \log \log n)$ expected number of hops, which is a near optimal routing complexity and outperforms the other related results for routing in Kleinberg’s small-world networks. Our schemes keep only $O(\log^2 n)$ bits of routing information on each node, thus they are scalable with the network size. Our results imply that the awareness of $O(\log n)$ nodes through augmented links is more efficient for routing than via the local links [10, 17].

Besides adding new light to the studies of social networks, our results may also find applications in the design of large-scale distributed networks, such as peer-to-peer systems, in the same spirit of Symphony [15].

1 Introduction

A well-known study by Milgram in 1967 [18] shows the small-world phenomenon [9], also called “six degree of separation”, that any two people in the world can be connected by a chain of six (on the average) acquaintances, and people can deliver messages efficiently to an unknown target via their acquaintances. This study is repeated by Dodds, Muhamad, and Watts [8] recently, and the results show that it
is still true for today’s social network. The small-world phenomenon has also been shown to be pervasive in networks from nature and engineering systems, such as the World Wide Web [21, 1], peer-to-peer systems [2, 16, 15, 22], etc.

Recently, a number of network models have been proposed to study the small-world properties [19, 21, 13]. Watts and Strogatz [21] propose a random rewiring model whose diameter is a poly-logarithmic function of the size of the network. The model is constructed by adding a small number of random edges to nodes uniformly distributed on a ring, where nodes are connected densely with their near neighbors. A similar approach can also be found in Ballabás and Chung’s earlier work [6], where the poly-logarithmic diameter of the random graph is achieved by adding a random matching to the nodes of a cycle. However, these models fail to capture the algorithmic aspects of a small-world network [13]. As commented by Kleinberg in [13], the poly-logarithmic diameter of some graphs does not imply the existence of efficient routing algorithms. For example, the random graph in [6] yields a logarithmic diameter, yet any routing using only local information requires at least $\sqrt{n}$ expected number of hops (where $n$ is the size of the network) [13].

In order to incorporate routing or navigating properties into random graph models, Kleinberg [13] develops a new model based on a $d$-dimensional torus lattice with long-range links chosen randomly from the $d$-harmonic distribution, i.e., a long-range link between nodes $u$ and $v$ exists with probability proportional to $\text{Dist}(u,v)^{-d}$, where $\text{Dist}(u,v)$ denotes the Mahattan distance between nodes $u$ and $v$. Based on this model, Kleinberg then shows that routing messages between any two nodes can be achieved in $O(\log^2 n)$ \footnote{The logarithmic symbol $\log$ is with the base 2, if not otherwise specified. Also, we remove the ceiling or floor for simplicity throughout the paper.} expected number of hops by applying a simple greedy routing algorithm using only local information. This bound is tightened to $\Theta(\log^2 n)$ later by Barrière et al. [3] and Martel et al. [17]. Further research [16, 14, 17, 10] shows that in fact the $O(\log^2 n)$ bound of the original greedy routing algorithm can be improved by putting some extra information in each message holder. Manku, Naor, and Wieder [16] show that if each message holder at a routing step takes its own neighbors’ neighbors into account for making routing decisions, the bound of routing complexity can be improved to $O\left(\frac{\log^2 n}{q \log q}\right)$, where $q$ denotes the number of long-range contacts for each node. Lebah and Schabanel [14] propose a routing algorithm for 1-dimensional Kleinberg’s model, which visits $O\left(\frac{\log^2 n}{\log^2(1+q)}\right)$ nodes on expectation before routing the message, and they show that a routing path with expected length of $O\left(\frac{\log n (\log \log n)^2}{\log^2(1+q)}\right)$ can be found. Two research groups, Fraigniaud et al. [10], and Martel and Nguyen [17], independently report that if each node is aware of its $O(\log n)$ closest local neighbors, the routing complexity in $d$-dimensional Kleinberg’s small-world networks can be improved to $O(\log n \log^{1+1/d} n)$ expected number of hops. The difference is that [17] requires keeping additional state information, while [10] uses an oblivious greedy routing algorithm. Fraigniaud et al. [10] also show that $\Theta(\log^2 n)$ bits of topological aware-
ness per node is optimal for their oblivious routing scheme. In [17], Martel and Nguyen show that the expected diameter $^2$ of a $d$-dimensional Kleinberg network is $\Theta(\log n)$. As such, there is still some room for reducing the routing complexity, which motivates our work.

Other small-world models have also been studied. In their recent paper [20], Nguyen and Martel study the diameters of variants of Kleinberg’s small-world models, and provide a general framework for constructing classes of small-world networks with $\Theta(\log n)$ expected diameter. Aspnes, Diamadi, and Shah [2] find that the greedy routing algorithms in directed rings with a constant number of random extra links given in any distribution requires at least $\Omega(\log^2 n / \log \log n)$ expected number of hops. Another related models are the small-world percolation models [16, 4, 7, 5]. The diameters of these models are studied by Benjamin et al. [4], Coppersmith et al. [7] and Biskup [5]. The routing aspects of the percolation models, such as the lower bound and upper bound of greedy routing algorithms with 1-lookahead, are studied in [16].

Applications of small-world phenomenon in computer networks include efficient lookup in peer-to-peer systems [16, 2, 15, 22], gossip protocol in a communication network [12], flooding routing in ad-hoc networks [11], and the study of diameter of World Wide Web [1], etc.

1.1 Our Contributions

We extend Kleinberg’s structures of small-world models with slight change. Besides having long-range and local links on the grid lattice, each node is augmented with three extra links, two of which are connected to nodes chosen randomly and uniformly within $\log^2 n$ Mahattan distance, and the third one is connected to a node chosen randomly and uniformly within $\log n$ Mahattan distance. Based on this extended model, we present near optimal algorithms for decentralized routing with $O(\log n)$ augmented awareness. We show that if each node is aware of $O(\log n)$ number of nodes via the augmented neighborhood, there exist both non-oblivious and oblivious routing algorithms that perform in $O(\log n \log \log n)$ expected number of hops (see Theorem 1 and Theorem 2). Our investigation constructively show that the augmented local connections can make small-world networks more navigable.

A comparison of our algorithm with the other existing schemes is shown in Table 1. Our decentralized routing algorithms assume that each node can compute a shortest path among a poly-logarithmic number of known nodes. Such an assumption is reasonable since each node in a computer network is normally a processor and can carry out such a simple computation. Our schemes keep $O(\log^2 n)$ bits of routing information stored on each node, thus they are scalable with the increase of network size. Our investigation shows that the awareness of $O(\log n)$

$^2$ Although the authors in [17] only consider the expected distance between two most distant nodes, the $\Theta(\log n)$ expected diameter still holds for the expected distance between any two nodes in the network.
nodes through the augmented links is more efficient for routing than via the local links [10, 17].

We note that besides adding new light to the studies of social networks such as Milgram’s experiment [18], our results may also find applications in the design of large-scale distributed networks, such as peer-to-peer systems, in the same spirit of Symphony [15]. Since the links in our extended model are randomly constructed according to the probabilistic distribution, the network may be less vulnerable to adversarial attacks, and thus provide good fault tolerance.

1.2 Organization

The rest of the paper is organized as follows. Section 2 gives notations for Kleinberg’s small-world model and its extended version with augmented local connections. Section 3 gives some preliminary notations for decentralized routing. In Section 3, we propose both non-oblivious and oblivious routing algorithms with near optimal routing complexity in our extended model. Section 5 briefly concludes the paper.

2 Definitions of Small-World Models

In this section, we will give definition of Kleinberg’s small-world model and its extended version in which each node has extra links. For simplicity, we only consider the one-dimensional model with one long-range contact for each node. The results for multiple-dimensional models could be easily obtained by using similar methods. In addition, we assume that all links are directed, which is consistent with the real-world observation, for example, person x knows person y, but y may not know x.

**Definition 1. (Kleinberg’s Small-World Network (KSWN) [13])** A Kleinberg’s Small-World Network, denoted as $\mathcal{K}$, is based on a one-dimensional torus (or ring) $[n] = \{0, 1, \ldots, n\}$. Each node $u$ has a directed local link to its next neighbor $(u + 1) \mod n$ on the ring. We refer to this local link as **Ring-link** (or **R-link** for short), and refer to node $(u + 1) \mod n$ as the **R-neighbor** of node $u$. In addition,
each node has one long-range link to another node chosen randomly according to the 1-harmonic distribution, that is, the probability that node \( u \) sends a long-range link to node \( v \) is \( \Pr[u \rightarrow v] = \frac{1}{Z_u \cdot \text{Dist}(u,v)} \), where \( \text{Dist}(u,v) \) denotes the ring distance \(^3\) from \( u \) to \( v \), and \( Z_u = \sum_{z \neq u} \frac{1}{\text{Dist}(u,z)} \). We refer to this long-range link as the **Kleinberg-link** (or **K-link** for short), and refer to node \( v \) as a **K-neighbor** of node \( u \) if a K-link exists from \( u \) to \( v \).

Our extended structure introduces several extra links for each node. Its definition is given below.

**Definition 2. (KSWN with Augmented Local Connections (KSWN*))** A Kleinberg’s Small-World Network with Augmented Local Connections, denoted as \( K^* \), has the same structure of KSWN, except that each node \( u \) in \( K^* \) has three extra links. First, node \( u \) has two links to nodes chosen randomly and uniformly from the interval \((u, u + \log^2 n]\). We refer to these two links as the **primary augmented local links** (or **PAL-links** for short), and refer to node \( v \) as a **PAL-neighbor** of node \( u \) if a PAL-link exists from \( u \) to \( v \). In addition, each node \( u \) has one link to a node chosen randomly and uniformly from the interval \((u, u + \log n]\). We refer to this link as the **secondary augmented local link** (or **SAL-link** for short), and refer to node \( v \) as a **SAL-neighbor** of node \( u \) if a SAL-link exists from \( u \) to \( v \).

There are in total five links for each node in a KSWN*: one R-link, one K-link, two PAL-links and one SAL-link. We refer to all nodes linked directly by node \( u \) as the **immediate neighbors** of \( u \). Our extended structure retains the same \( O(1) \) order of node degree as that of Kleinberg’s original model.

### 3 Decentralized Routing Algorithms

Based on the original model, Kleinberg presents a class of decentralized routing algorithms, in which each node makes routing decisions by using local information and in a greedy fashion. In other words, the message holder forward the message to its immediate neighboring node, including its K-neighbor, which is closest to the destination in terms of the Mahattan distance. Kleinberg shows that such a simple greedy algorithm performs in \( O(\log^2 n) \) expected number of hops. The other existing decentralized routing algorithms \([2, 15, 14, 10, 17, 16]\) mainly rely on three approaches to improve routing performance: (1) Increasing the number of long-range links \([2, 15]\); (2) Exploring more nodes before making routing decisions \([14]\); (3) Increasing the local awareness for each node \([10, 17]\). However, so far using these approaches can only achieve \( O((\log n)^{1+\epsilon}) \) expected number of hops in routing, where \( \epsilon > 0 \). Although the scheme in \([16]\), where each node makes routing decision by looking ahead its neighbors’s neighbors, can achieve an optimal \( O(\log n / \log \log n) \) bound, their result depends on the fact that each node has at least \( \Omega(\log n) \) number of K-links.

There are normally two approaches for decentralized routing: oblivious and non-oblivious schemes \([10]\). A routing protocol is **oblivious** if the message holder

\(^3\) or Mahattan distance for multi-dimensional models.
makes routing decisions only by its local information and the target node, and independently of the previous routing history. On the other hand, if the message holder need to consider certain information of the previous routing history to make routing decisions, the protocol is referred to as non-oblivious. The non-oblivious protocol is often implemented by adding a header segment to the message packet so that the downstream nodes can learn the routing decisions of upstream nodes by reading the message header information. The scheme in [10] is oblivious, while the schemes in [14] and [17] are non-oblivious.

We refer to the message holder as the current node. For the current node $x$, we define a sequence of node sets $T_0, T_1, \cdots, T_i, \cdots$, where $T_0 = \{x\}$, $T_1 = \{u$’s PAL-neighbors, $\forall u \in T_0\}$, $T_2 = \{u$’s PAL-neighbors, $\forall u \in T_1\}$, and so on. We refer to $T_i$ as the set of nodes in the $i$th level of PAL neighborhood, and let $H_i = \bigcup_{j \leq i} T_j$ denote the set of all nodes in the first $i$ levels of PAL neighborhood. At a certain level $i$ of PAL neighborhood, we may also refer to $H_{i-1}$ as the set of previously known nodes. Let $L_i = T_i - H_{i-1}$ denote the set of new nodes discovered during the $i$th level of PAL neighborhood. Let $A_x(k) = H_k$ denote the primary augmented local awareness (or PAL awareness for short) of a given node in a KSWN*, where each node is aware of the first $k$ levels of its PAL neighborhood.

In Section 4, we will show that there exists a sufficiently large constant $\sigma$ such that $|A_x(\log \log n)| \geq \log n/\sigma$, based on which we propose both non-oblivious and oblivious routing algorithms running in $O(\log n \log \log n)$ expected number of hops and requiring $O(\log^2 n)$ bits of information on each node.

Our near optimal $O(\log n \log \log n)$ bound on the routing complexity outperforms the other related results for Kleinberg’s small-world networks. The number of $\log \log n$ levels of PAL neighborhood in our schemes is small compared with that in [10, 17], which requires $(\log n)^{1/d}$ levels of local neighborhood for a $d$-dimensional model.

4 Near Optimal Routing with $O(\log n)$ Awareness

4.1 Primary Augmented Local Awareness of $O(\log n)$

In this subsection, we will show that $|A_x(\log \log n)|$, the number of distinct nodes that node $x$ is aware of via the first $\log \log n$ levels of PAL neighborhood, is not less than $\log n/\sigma$ for a constant $\sigma$, which, as will be shown in Lemma 3, is sufficiently large to guarantee that $A_x(\log \log n)$ contains a K-link that jumps over half distance (Suppose that the destination node is at a certain large distance from the current node). These results are useful for the subsequent analysis of our oblivious and non-oblivious routing schemes.

**Lemma 1.** Let $A_x(\log \log n)$ denote the PAL awareness of node $x$ in a KSWN* $\mathcal{K}^*$, where each node is aware of $\log \log n$ levels of PAL-neighbors. Then

$$\Pr[|A_x(\log \log n)| \geq \frac{\log n}{\sigma}] > \psi,$$

where $\sigma$ denotes a sufficiently large constant and $\psi$ denotes a positive constant.
Proof: Throughout the proof, we assume that $|H_i| < \frac{\log n}{\sigma}$ for all $1 \leq i \leq \log \log n$, otherwise, the lemma already holds, since $|A_x(\log \log n)| = |H_{\log \log n}| > \log n/\sigma$. We will show that at each level of PAL neighborhood, the probability that each PAL-link points to previously known nodes is small so that a large number of distinct nodes will be found via the first $\log \log n$ levels of PAL neighborhood.

Consider the construction of a PAL-link for the current node $x$. By definition of KSWN*, each PAL-link of $x$ is connected to a node randomly and uniformly chosen from the interval $(x, x + \log^2 n]$, that is, each PAL-link of $x$ points to a node in the interval $(x, x + \log^2 n]$ with probability $(\log n)^{-2}$. By assumption, there could be no more than $\log n/\sigma$ previously known nodes in the interval $(x, x + \log^2 n]$. Thus, the probability for a PAL-link of a given node to point to a previously known node is at most $(\log n/\sigma) \cdot (\log n)^{-2} = (\sigma \log n)^{-1}$. Thus, the probability for a PAL-link of $x$ to point to a new node is at least $1 - (\sigma \log n)^{-1}$. There are in total at most $2 \cdot |H_{\log \log n}| \leq 2 \log n/\sigma$ number of PAL-links, so the probability for all PAL-links to point to new nodes is at least $(1 - (\sigma \log n)^{-1})^{2 \log n/\sigma} \geq 1 - \frac{4}{\sigma^2}$ for sufficiently large $n$. Here we use the fact $(1 + x)^a \geq 1 + ax$ for $x > -1$ and $a \geq 1$. When $\sigma$ is a sufficiently large constant, we have $\Pr[|A_x| \geq \frac{\log n}{\sigma}] > \psi$ for a positive constant $\psi = 1 - \frac{4}{\sigma^2} > 0$. Thus, the proof of Lemma 1 is completed.

4.2 Non-Oblivious Decentralized Routing

Our non-oblivious routing algorithm is given as follows: Initially the source node $s$ finds in its PAL awareness $A_s(\log \log n)$ an intermediate node $z$ that is closest to the destination, and then computes a shortest path $\pi$ from $s$ to $z$ in $A_s(\log \log n)$. Before routing the message, $s$ adds the information about shortest path $\pi$ to the message header. Once the message passes a node on the shortest path $\pi$, the next stop is read off the header stack. When the message reaches node $z$, node $z$ can tell that it is an intermediate target by reading the message header and then route the message to its K-neighbor. Such processes are repeated until the message reaches a certain distance to the destination node. After that, Kleinberg’s plain greedy algorithm can be used to route the message effectively to the target node. Given a message $M$, a source node $s$ and a target node $t$ in a KSWN* $K^*$, the pseudocodes of our non-oblivious algorithm running on the current node $x$ are given in Algorithm 1.

Next we will analyze the performance of the Algorithm 1. We first give a basic lemma, which provide a lower bound and an upper bound on the probability of the existence of a K-link in Kleinberg’s small-world networks. Its proof can be found in Appendix A.

Lemma 2. Let $\Pr[u \xrightarrow{K} v]$ denote the probability that node $u$ sends a K-link to node $v$ in a KSWN* $K^*$. Suppose that $a \leq \text{Dist}(u, v) \leq b$, then $\frac{c_1}{\alpha \log n} \leq \Pr[u \xrightarrow{K} v] \leq \frac{c_2}{\lambda \log n}$, where $c_1$ and $c_2$ are constants independent of $n$.

In Lemma 1, we have shown that $\Pr[|A_x(\log \log n)| \geq \log n/\sigma]$ is at least a positive constant for a sufficiently large constant $\sigma$. Based on this result, Lemma 3
Algorithm 1
Input: the source $s$, the target $t$ and the message $M$.

Initialization:
- $x \leftarrow s$.
- Set the header stack of the message $M$ to be empty.

while $\text{Dist}(x, t) \geq (\log n)^2 \log \log n$ do
  if the header stack of the message $M$ is empty then
    Route the message $M$ to $x$'s K-neighbor $y$.
    Find an intermediate node $z$ in $A_y(\log \log n)$ whose K-neighbor is closest to $t$ (ties are broken arbitrarily).
    Compute a shortest path $\pi : x_0 = y, x_1, \ldots, x_l = z$ from $y$ to $z$, and push the shortest path information
    $\pi : x_1, \ldots, x_l = z$ into the header stack of the message $M$.
  else
    Pop up the first node $x_i$ from the header stack and route the message $M$ to node $x_i$.
  end if
end while

Final phase (Kleinberg’s greedy algorithm):
- Route the message $M$ to an immediate neighbor of $x$ that is closest to the target $t$, until it reaches $t$.

shows that the probability for $A_x(\log \log n)$ to contain a K-link jumping over half distance is at least a positive constant.

Lemma 3. Suppose that the distance between the current node $x$ and the target node $t$ in a $\text{KSWN}^* \text{K}^*$ is $\text{Dist}(x, t) \geq \log^2 n \log \log n$. Then with probability at least a positive constant, node $x$’s PAL awareness $A_x(\log \log n)$ contains a K-neighbor within $\text{Dist}(x, t)/2$ distance to the target node $t$.

Proof: Let $A$ denote the event that $|A_x(\log \log n)| \geq \frac{\log n}{\sigma}$. By Lemma 1, we have $\Pr[A] > \psi$ for a constant $\psi > 0$.

Let $B_l(t)$ denote the set of all nodes within $l$ ring distance to $t$. Let $\Pr[x \xrightarrow{K} B_l(t)]$ denote the probability that $x$’s K-neighbor is inside the ball $B_l(t)$.

Let $m = \text{Dist}(x, t)$. By Lemma 2, the probability for a K-link to point to a given node inside the ball $B_{l/2}(t)$ is at least $\frac{c_1}{m \log n}$, so we have

$$\Pr[x \xrightarrow{K} B_{l/2}(t)] \geq |B_{l/2}(t)| \cdot \frac{c_1}{m \log n} = \frac{m}{2} \cdot \frac{c_1}{m \log n} \geq \frac{c_3}{\log n},$$

where $c_3$ is a constant.

Since $\text{Dist}(x, t) \geq \log^2 n \log \log n$ and each PAL-link spans a distance no more than $\log^2 n$, the nodes in PAL awareness $A_x(\log \log n)$ are all between the current node $x$ and the target node $t$. Let $\Pr[A_x \xrightarrow{K} B_{l/2}(t)]$ denote the probability that at least one node in $A_x(\log \log n)$ has a K-neighbor in $B_{l/2}(t)$. Then we have

$$\Pr[A_x(\log \log n) \xrightarrow{K} B_{l/2}(t)] \geq \Pr[A_x(\log \log n) \xrightarrow{K} B_{l/2}(t) | A] \cdot \Pr[A]$$

$$\geq (1 - (1 - \frac{c_3}{\log n})^{\log n}) \cdot \psi$$

$$\geq \psi(1 - e^{-\frac{c_3}{\log n}}),$$

which is larger than a positive constant. At the last step, we obtain $(1 - \frac{c_3}{\log n})^{\log n} \leq e^{-\frac{c_3}{\sigma}}$ by using the fact that $(1 + \frac{b}{2})^x \leq e^b$ for $b \in \mathbb{R}$ and $x > 0$. ■
Lemma 4. Suppose that the distance between the current node $x$ and the target node $t$ in a $KSWN^*$ is $\text{Dist}(x,t) \geq \log^2 n \log \log n$. Then after at most $O(\log n \log \log n)$ expected number of hops, Algorithm 1 will reduce the distance to within $\log^2 n \log \log n$.

**Proof:** Since $\text{Dist}(x,t) \geq \log^2 n \log \log n$, all known nodes in $x$’s PAL awareness $A_x(\log \log n)$ are between the current node $x$ and the target node $t$. We can apply the result in Lemma 3 to analyze Algorithm 1.

We refer to the routing steps from a given node $x$ to its intermediate node $z$ in $A_x(\log \log n)$ as an indirect phase. By Lemma 3, the probability that node $x$’s PAL awareness $A_x(\log \log n)$ contains a K-neighbor within $\text{Dist}(x,t)/2$ distance to the target node $t$ is at least a positive constant, so after at most $O(1)$ expected number of indirect phases, Algorithm 1 will find an intermediate node whose K-link jumps over half distance. Since each indirect phase takes at most $\log \log n$ hops and the maximum distance is $n$, after at most $O(\log n \log \log n)$ expected number of hops, the message will reach a node within $\log^2 n \log \log n$ distance to the target node $t$.

Lemma 5. Suppose that the distance between the current node $x$ and the target node $t$ in a $KSWN^*$ is $\text{Dist}(x,t) \leq \log^2 n \log \log n$. Then using the final phase of Algorithm 1 (i.e. using Kleinberg’s greedy algorithm) can route the message to the target node $t$ in $O(\log n)$ expected number of hops.

**Proof:** See Appendix B for details.

Combining the above lemmas, it is not difficult for us to obtain the routing complexity of Algorithm 1.

**Theorem 1.** In a $KSWN^* K^*$, Algorithm 1 performs in $O(\log n \log \log n)$ expected number of hops.

### 4.3 Oblivious Decentralized Routing

In our oblivious scheme, when the distance is large, the current node $x$ first finds in $A_x(\log \log n)$ whether there is an intermediate node $z$, which contains a K-neighbor within $\text{Dist}(x,t)/2$ distance to the target node, and is closest to node $x$ in terms of PAL-links (any possible tie is broken arbitrarily). Next, node $x$ computes a shortest path $\pi$ from $x$ to $z$ among the PAL awareness $A_x(\log \log n)$, and then routes the message to its next PAL-neighbor on the shortest path $\pi$. When the distance is small, Kleinberg’s plain greedy algorithm is applied.

Given a message $M$, a source $s$ and a target $t$ in a $KSWN^* K^*$, the pseudocodes of our oblivious algorithm running on the current node $x$ are given in Algorithm 2.

**Lemma 6.** Suppose that the distance between the current node $x$ and the target node $t$ in a $KSWN^* K^*$ is $\text{Dist}(x,t) \geq c(\log n)^2 \log \log n$, where $c$ is a sufficiently large constant. Then after at most $O(\log \log n)$ expected number of hops, Algorithm 2 will reduce the distance to within $\text{Dist}(x,t)/2$. 
Algorithm 2

Input: the source $s$, the target $t$ and the message $M$.

Initialization:

$x \leftarrow s$.

while $\text{Dist}(x, t) \geq c \log n$ ($c$ is a sufficiently large constant and will be given later) do

$z \leftarrow$ a node in $A_x(\log \log n)$ that contains a K-neighbor within $\text{Dist}(x, t)/2$ distance to $t$, and is closest to node $x$ in terms of PAL-links (ties are broken arbitrarily).

if node $z$ does not exist then

Route the message $M$ to an immediate neighbor closest to node $t$.

else

Compute a shortest path $\pi$ from $x$ to $z$ among $A_x(\log \log n)$.

if $\pi$ consists of only node $x$ itself then

Route the message $M$ to the K-neighbor.

else

Route the message $M$ to the next PAL-neighbor on the shortest path $\pi$.

end if

end if

end while

Final phase (Kleinberg’s greedy algorithm):

Route the message $M$ to an immediate neighbor of $x$ that is closest to the target $t$, until it reaches $t$.

Proof: As shown in Figure 1, node $r$ is the midpoint of $xt$, and node $r'$ is between $r$ and $t$ such that $\text{Dist}(r, r') = \log^2 n \log \log n$. Let $z$ be an intermediate node in $A_x(\log \log n)$ that contains a K-neighbor between $r$ and $t$, and is closest to $x$ in terms of PAL-links. We refer to a node $z$ in $x$’s PAL awareness $A_x(\log \log n)$ as a good intermediate node if it satisfies the following two conditions: (1) has a K-neighbor within $\text{Dist}(x, t)/2$ to the target node; (2) is closest to node $x$ in terms of PAL-links. Let $\pi : x_0 = x, x_1, \ldots, x_t = z$ denote a shortest path that $x$ finds from itself to $z$ among the PAL awareness $A_x(\log \log n)$. We divide the next routing into two cases according to the different locations of $z$’s K-neighbor.

In the first case, $z$’s K-neighbor is within $r't$. Since the distance between $x$ and the right most node in $A_x(\log \log n)$ is at most $\log^2 n \log \log n$, $z$’s K-neighbor is also within $\text{Dist}(x_i, t)/2$ to the target node for every $x_i$ on the shortest path $\pi$, that is, node $z$ always satisfies the first condition of a good intermediate node for every node $x_i$. Also, if $z$ is an intermediate node closest to $x$, it is also a closest intermediate node to every $x_i$ on the shortest path $\pi$, that is, $z$ also satisfies the second condition of a good intermediate node for every node $x_i$. Therefore, node $z$
will become a fixed good intermediate node for all nodes $x_i$ on the shortest path. When this case happens, Algorithm 2 will route the message along a shortest path from $x$ to $z$ in an oblivious routing fashion. Thus, in this case, after at most $\log \log n$ number of hops, the message will reach a good intermediate node and the routing distance will be reduced by half. In the second case, $z$’s K-neighbor is within $\overrightarrow{rt}$. When this happens, the intermediate node $z$ may change for each $x_i$ along the shortest path $\pi: x_0 = x, x_1, \cdots, x_t = z$, and the message may not be routed along the shortest path as expected by the previous node $x$. However, we will show that the latter case will not happen very likely, since the length of $\overrightarrow{rt}$ is relatively small when $\text{Dist}(x, t) \geq c(\log n)^2 \log \log n$ for a sufficiently large constant $c$.

Let $F_1$ denote the event that $A_x(\log \log n)$ contains a K-neighbor in $\overrightarrow{rt}$. By using a similar technique in Lemma 3, we can easily obtain that $F_1$ occurs with probability at least a positive constant.

Let $F_2$ denote the event that $A_x(\log \log n)$ contains a K-neighbor in $\overrightarrow{rt}$. For any node $y$ in $A_x(\log \log n)$, we have $\text{Dist}(y, r) \geq \frac{1}{3}c(\log n)^2 \log \log n$ when $c$ is a sufficiently large constant. By Lemma 2, the probability for a node $y$ in $A_x(\log \log n)$ to send a K-link to a node in $\overrightarrow{rt}$ is at most $\frac{3c_2^2}{c(\log n)^2(\log \log n) \log n}$. Because there are in total $\log^2 n \log \log n$ nodes in $\overrightarrow{rt}$, a node in $A_x(\log \log n)$ has a K-neighbor in $\overrightarrow{rt}$ with probability at most $\frac{3c_2^2}{c(\log n)^2(\log \log n) \log n} \cdot \log^2 n \log \log n = \frac{3c_2^2}{c \log n}$. Since $|A_x(\log \log n)| \leq 1 + 2 + 2^2 + \cdots + 2^{\log \log n} \leq 2 \log n$, the event $F_2$, i.e., $A_x(\log \log n)$ has a K-neighbor in $\overrightarrow{rt}$, occurs with probability at most $\frac{3c_2^2}{c \log n} \cdot 2 \log n = \frac{6c_2}{c}$, which is smaller than a certain constant when $c$ is a sufficiently large constant. Thus, we have $\Pr[\neg F_2] > \gamma$ for a constant $\gamma > 0$, if we choose a sufficiently large constant $c$.

Therefore, $\Pr[\neg F_2 \cap F_1]$ is larger than a positive constant, if we choose a sufficiently large constant $c$. Thus, after at most $c' \log \log n$ expected number of hops for a constant $c'$, the event $\neg F_2 \cap F_1$ will occur, that is, a message will be routed to a node $x$ whose PAL awareness $A_x(\log \log n)$ contains a K-neighbor in $\overrightarrow{rt}$, but no K-neighbor in $\overrightarrow{rt}$. When such a node $x$ is reached, the intermediate node $z$ is fixed for every node $x_i$ on a shortest path $\pi: x_0 = x, x_1, \cdots, x_t = z$ in an oblivious routing fashion. Then after at most $\log \log n$ number of hops, the message will be routed to the fixed intermediate node $z$, which has a K-link jumping over half distance.

Therefore, after at most $c' \log \log n + \log \log n = O(\log \log n)$ expected number of hops, the distance will be reduced by half. \[\blacksquare\]

**Lemma 7.** Suppose that the distance between the current node $x$ and the target node $t$ in a KSWN* $K^*$ is $\text{Dist}(x, t) \geq c \log^2 n \log \log n$, where $c$ is a sufficiently large constant.
large constant. Then after at most \(O(\log n \log \log n)\) expected number of hops, Algorithm 2 will reduce the distance to within \(c \log^2 n \log \log n\).

**Proof:** The proof is similar to that of Lemma 4, and hence is omitted here.

**Lemma 8.** Suppose that the distance between current node \(x\) and the target node \(t\) in a KSWN* \(K^*\) is \(m < c(\log n)^2 \log \log n\), where \(c\) is a sufficiently large constant. Then using the final phase of Algorithm 2 (i.e. using Kleinberg’s greedy algorithm) can route the message to the target node \(t\) in \(O(\log n)\) expected number of hops.

**Proof:** The proof is similar to that of Lemma 5, and hence is omitted here.

Combining the above lemmas, we can easily obtain the following theorem.

**Theorem 2.** In a KSWN* \(K^*\), Algorithm 2 performs in \(O(\log n \log \log n)\) expected number of hops.

## 5 Conclusion

We extend Kleinberg’s small-world network with augmented local links, and show that if each node participating in routing is aware of \(O(\log n)\) neighbors via augmented links, there exist both non-oblivious and oblivious decentralized algorithms that can finish routing in \(O(\log n \log \log n)\) expected number of hops, which is a near optimal routing complexity. Our investigation shows that the awareness of \(O(\log n)\) nodes through the augmented links will be more efficient for routing than via the local links [10, 17].

Our extended model may provide an important supplement for the modelling of small-world phenomenon, and may better approximate the real-world observation. For example, each person in a human society is very likely to increase his/her activities randomly within some certain communities, and thus is aware of certain levels of “augmented” acquaintances. This augmented awareness would surely help delivery the message to an unknown target in the society.

Our results may also find applications in the design of large-scale distributed networks, such as distributed storage systems. Unlike most existing deterministic frameworks for distributed systems, our extended small-world networks may provide good fault tolerance, since the links in the networks are constructed probabilistically and less vulnerable to adversarial attacks.

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### Appendix A. Proof of Lemma 2

**Lemma 2.** Let $\Pr[u \xrightarrow{K} v]$ denote the probability that node $u$ sends a $K$-link to node $v$ in a KSWN* $\mathcal{K}^*$. Suppose that $a \leq \text{Dist}(u,v) \leq b$, then $\frac{c_1}{\log n} \leq \Pr[u \xrightarrow{K} v] \leq \frac{c_2}{a \log n}$, where $c_1$ and $c_2$ are constants independent of $n$.

**Proof:** The probability that node $v$ is a K-neighbor of node $u$ is $\Pr[u \xrightarrow{K} v] = \frac{1}{\text{Dist}(u,v) Z_v}$, where $\text{Dist}(u,v)$ is the ring distance between nodes $u$ and $v$, and $Z_v = \sum_{z \neq v} \frac{1}{\text{Dist}(v,z)}$.

Observe that $Z_v = \sum_{i=1}^{n} \frac{|U_i|}{i}$, where $|U_i|$ is the set of all nodes at distance $i$ away to node $v$. Since $|U_i| = \Theta(1)$, we have $Z_v = \sum_{i=1}^{n} \frac{\Theta(1)}{i} = \Theta(\log n)$.

Since $a \leq \text{Dist}(u,v) \leq b$, we have $\frac{c_1}{\log n} < \Pr[u \xrightarrow{K} v] < \frac{c_2}{a \log n}$, for some constants $c_1$ and $c_2$ independent of $n$. Thus the lemma follows. ■
Appendix B. Proof of Lemma 5

Lemma 5. Suppose that the distance between the current node $x$ and the target node $t$ in a $^{*}$KSWN$^{*}$ is $\text{Dist}(x, t) \leq \log^2 n \log \log n$. Then using the final phase of Algorithm 1 (i.e. using Kleinberg’s greedy algorithm) can route the message to the target node $t$ in $O(\log n)$ expected number of hops.

Proof: When the distance $\text{Dist}(x, t) \leq \log^2 n \log \log n$, the final phase in Algorithm 1 is executed.

First, since each PAL-neighbor is chosen uniformly and randomly from the interval $[x, x + \log^2 n]$, the probability for a PAL-link to jump over $\log^2 n/2$ distance is $1/2$, that is, after $O(1)$ number of hops, the routing distance will be reduced by $\log^2 n/2$. Thus, after at most $O(\log \log n)$ expected number of hops, the routing distance will be reduced to within $\log^2 n$. Next, by applying a similar method, we can also show that after $O(\log n)$ expected number of hops via secondary augmented local links (SAL-links), the routing distance will be reduced to within $\log n$. Finally, using the simple greedy algorithm via the R-links can route the message to the target node in $\log n$ hops. Therefore, the greedy routing algorithm only by using the information of immediate neighbors can finish the routing in $O(\log n)$ expected number of hops. \qed