Duality functors for \( n \)-fold vector bundles\(^*\)

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Abstract

Double vector bundles may be dualized in two distinct ways and these duals are themselves dual. These two dualizations generate a group, denoted \( \mathcal{DF}_2 \), which is the symmetric group \( S_3 \) on three symbols. In the case of triple vector bundles the authors proved in a previous paper that the corresponding group \( \mathcal{DF}_3 \) is an extension of \( S_4 \) by the Klein four-group. In this paper we show that the group \( \mathcal{DF}_n \), for \( n \)-fold vector bundles, \( n \geq 3 \), is an extension of \( S_{n+1} \) by a certain product of groups of order 2, and show that the centre is nontrivial if and only if \( n \) is a multiple of 4. The methods employ an interpretation of duality operations in terms of certain graphs on \( (n+1) \) vertices.

1 Introduction

In a previous paper \cite{5}, the authors showed that the group of duality functors of triple vector bundles has order 96, and is an extension of the symmetric group \( S_4 \) by the Klein four-group. This followed work by one of us on the duality of double vector bundles \cite{7,8}.

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Duality for double and multiple vector bundles originated in Poisson geometry; see [13, 11] and references there. In [7] one of us applied Pradines’ duality for vector bundle objects in the category of groupoids [13] to double vector bundles and showed that the two duals of a double vector bundle are themselves dual. This result was so unexpected that it was natural to investigate the triple and higher cases.

Double vector bundles have also been used in treatments of connection theory and of theoretical mechanics for many years [2, 1, 14], though without any consideration of duality. Voronov [15, §6] has begun the study of bracket structures on multiple vector bundles, using super techniques. In the present paper we are only concerned with duality for unstructured multiple vector bundles, and we do not consider bracket structures or geometric applications.

So far as we know, these groups have not appeared before; in particular, they do not seem to be a modern formulation of a classical construction. Their significance is not fully clear, but the relation (11) below which defines the duality group for $n = 2$ is the key to the compatibility of Lie algebroid structures on a double vector bundle [10]. Whether there are corresponding results for bracket structures on multiple vector bundles for other $n$ will be investigated elsewhere. For the moment, we only remark that the sequence of groups for $n = 2, 3, 4$ shows new features at each term and this is sufficient reason to investigate whether the sequence becomes regular.

Before describing the main results of the paper, we recall the results of the double case from [5]. For double and perhaps triple vector bundles, the notation used is cumbersome, but it is designed to handle the $n$-fold case, which is the main concern of the paper.

A **double vector bundle** is a manifold $E_{1,2}$ with two vector bundle structures, over bases $E_1$ and $E_2$, each of which is a vector bundle on a manifold $M$, such that the structure maps of $E_{1,2} \rightarrow E_1$ (the bundle projection, the addition, the scalar multiplication, the zero section) are morphisms of vector bundles with respect to the other structure. We write $E$ to denote the entire structure, and sometimes for the total space. See Figure 1(a),

Concisely, a double vector bundle is a vector bundle object in the category of vector bundles. A definition in full detail is given in [9, Chap. 9].

![Diagram of double vector bundles](attachment:double_vector_bundles.png)
The core \([12]\) of a double vector bundle \(E\) is the set of elements \(e \in E_{1,2}\) which project to zero in both \(E_1\) and \(E_2\). The core, which we denote \(E_{12}\) (without the comma), is closed under both additions in \(E\) and these additions coincide, giving \(E_{12}\) a natural structure of vector bundle over \(M\).

Dualizing \(E_{1,2}\) over its base \(E_1\) leads to a double vector bundle as in Figure 1(b), in which \(E_2\) has been ‘replaced’ by the dual of the core. We write the base over which the dualization takes place on the line to avoid multiple superscripts, and use the symbol \(\dagger\) to avoid confusion with other uses of stars and asterisks.

Likewise dualizing \(E_{1,2}\) over \(E_2\) leads to the double vector bundle shown in Figure 1(c). For brevity, denote the action of dualizing vertically and horizontally by \(E^X\) and \(E^Y\). Repeating these operations leads to the double vector bundles shown in Figure 2.

In particular \(E^{XYX}\) has side bundles \(E_1\) and \(E_2\) but these have been interchanged. Note that \(E^{XYX}\) is not canonically isomorphic to the flip of \(E\), by which we mean the double vector bundle obtained by interchanging the two structures on \(E\), as for the canonical involution on a double tangent bundle. Roughly speaking, although \(E^{XYX}\) has side bundles \(E_1\) and \(E_2\), its core has been reversed; see the discussion following Prop. 2.10 in [5]. However \(E^{YXY}\) is canonically isomorphic to \(E^{XYX}\) and we therefore write

\[
XYX = YXY. \tag{1}
\]

Together with \(X^2 = Y^2 = 1\) it follows that the dualizations of a double vector bundle form the symmetric group on three symbols, and every sequence of dualization operations applied to \(E\) results in a double vector bundle canonically isomorphic to one of those in Figure 2.

As [5] showed, it is crucial to regard the operations \(X\) and \(Y\) as functors on appropriate categories. We denote the group of these dualization functors by \(\mathcal{DF}_2\).

Write \(E_0 = E_{12}\) so that the two side bundles and the dual of the core of \(E\) are \(E_1, E_2, E_0\). The core of \(E^X\) is \(E_2^*\) and so the side bundles and the dual of the core of \(E^X\) are
$E_1, E_0, E_2$. The side bundles and the dual of the core of $E^\gamma$ are $E_0, E_2, E_1$. Thus $D\mathcal{F}_2$ can be regarded as the symmetric group on the bundles $E_1, E_2, E_0$.

The notation for a triple vector bundle, as used in Figure 3(a), is shown in Figure 3(a). The total space is denoted by $E_{1,2,3}$, the double vector bundles which form the lower faces are denoted by $E_{i,j}$, and we write $E_i$ for the vector bundles which form the edges abutting $M$. In all figures we read oblique arrows as coming out of the page.

![Diagram](image)

(a) A triple vector bundle ... (b) ... and its core structure. (c) The dual of $E$ over $E_{2,3}$

Figure 3(b) shows the core structure of $E$. The core of $E_{i,j}$ is denoted $E_{ij}$ without the comma; it is a vector bundle over $M$. The core of the top face of $E$ is denoted $E_{3,12}$; it is a vector bundle over $E_3$. The vector bundle structure of $E_{1,2,3}$ with base $E_{1,2}$ restricts to $E_{3,12}$ and gives it also the structure of a vector bundle over $E_{12}$; with these two structures $E_{3,12}$ is a double vector bundle with side bundles $E_3$ and $E_{12}$.

The cores of the rear and left faces are denoted $E_{2,31}$ and $E_{1,23}$ and are double vector bundles in a similar way.

These three core double vector bundles have the same core, called the ultracore; this is denoted $E_{123}$ and is a vector bundle over $M$. It is the set of all elements of $E_{1,2,3}$ which project to the double zero element in each of $E_{1,2}$, $E_{2,3}$ and $E_{3,1}$.

For a triple vector bundle one may dualize in three directions, which we denote for the moment by $X$, $Y$, and $Z$. Take the dual over $E_{2,3}$, as in Figure 3(c), to be $E^X$ and write $E_0 = E^*_{123}$. Then $X$ exchanges $E_1$ and $E_0$ and leaves $E_2$ and $E_3$ fixed. In terms of the effect which $X$ has on the four bundles $E_1$, $E_2$, $E_3$ and $E_0$, we can regard $X$ as the transposition $(01)$ in $S_4$. Likewise $Y$ acts as $(02)$ and $Z$ as $(03)$. These three transpositions generate $S_4$, so for every $\sigma \in S_4$ there is a word $W$ in $X$, $Y$, and $Z$ which acts as $\sigma$.

To express this more precisely, write $D\mathcal{F}_3^1$ for the group on $X, Y, Z$ subject to $X^2 = Y^2 = Z^2 = 1$ and to

$$ (XY)^3 = (YZ)^3 = (ZX)^3 = 1. \quad (2) $$

The group $D\mathcal{F}_3^1$ should be thought of as ‘version 1’ of the duality functor group for triple vector bundles. The discussion above shows that there is a surjective morphism from $D\mathcal{F}_3^1$ to the symmetric group $S_4$.
In particular, the word \((XYXZ)^2\) is mapped to the identity of \(S_4\). It is shown in [5] that \((XYXZ)^2\) has order 2 in \(DF_3^1\), and that the surjection \(DF_3^1 \rightarrow S_4\) has for kernel a Klein four–group \(K_4\), consisting of \((XYXZ)^2\) together with its conjugates and the identity.

Now define \(DF_3^2\) to be the quotient of \(DF_3^1\) over the relations
\[
(XYZ)^4 = (YZYX)^4 = (ZXZY)^4 = 1. \tag{3}
\]

Then the above discussion may be formulated as the statement that \(DF_3^2\) is the duality functor group \(DF_3\).

This completes a brief review of the main results in the triple case. In the present paper we first show that, for all \(n \geq 4\), there is again a short exact sequence
\[
1 \longrightarrow K_{n+1} \longrightarrow DF_n \longrightarrow S_{n+1} \longrightarrow 1. \tag{4}
\]

where \(K_{n+1}\) is a direct product of copies of \(C_2\), the cyclic group of order 2. To calculate the number of copies we introduce in [5] an equivalent description of \(K_{n+1}\) in terms of certain graphs on \(n+1\) vertices. This interpretation was not needed in the case \(n = 3\) but is the key to the cases \(n \geq 4\). We find in Corollary [5.3] that \(K_{n+1}\) is the direct product of \(\frac{1}{2}(n+1)(n-2)\) copies of \(C_2\).

The \(DF_n\) are thus unexpectedly large and, as far as we know, have no precedent in earlier or classical work. It is tempting to look for smaller groups which embody equivalent information, but we see no prospect of this. The relationship between \(DF_n\) and \(K_{n+1}\) is reminiscent of the relationship between braids and pure braids, or between gauge transformations and pure gauges. In these theories, it is generally sufficient to concentrate on the pure case, but that is not so here; one cannot focus just on \(K_{n+1}\). This is already clear from the double and triple cases.

We show in [7] using the description of the kernel in terms of graphs, that (4) splits for \(n = 4\) and for \(n = 2 \mod 4\); we do not know what the situation is for \(n = 8\). We also show that for \(n = 0 \mod 4\) the centre of \(DF_n\) has order 2, and is trivial for all other \(n\).

A complete description of \(DF_n\) for general \(n\) will probably require a set of relations in terms of the dualization operators. For double and triple vector bundles the relations are as given in (2) and (3). At the end of the paper we find that the relations for \(DF_4\), in addition to those corresponding to (2) and (3), require words of length 24 and 32. It will be interesting if these have geometric interpretations like that of ‘cornering’ for the relation in the double case [8].

The paper divides into two parts. In the first, consisting of Section 2 and Section 3, we are concerned to set up the notation and terminology needed to work with \(n\)-fold vector bundles. This takes some time, but it is necessary to have a systematic notation for the spaces associated with dualizing an \(n\)-fold vector bundle before introducing the notation for the appropriate concept of automorphism. As we emphasized in [5], the problem — when are the results of two sequences of dualization operations canonically isomorphic?
— is itself difficult to formulate effectively, and that is what makes the material of Section 2 and Section 3 necessary.

The second part concerns the actual calculations for the duality groups. Many readers may prefer to go directly to Section 4 and refer back as needed. As much as possible, we have avoided repeating material from [5].

2 \textit{n-fold vector bundles}

In order to work with \textit{n-fold vector bundles}, we need an effective notation for the various side bundles and cores. The actual definition of an \textit{n-fold vector bundle} is in 2.1.

We first extend the notation used in Figures 1 and 3. The top space of an \textit{n-fold vector bundle} will generally be denoted by \(E_{1,...,n}\). The various faces will be \(k\)-fold vector bundles for \(k \leq n\), indexed by \(k\)-element subsets of \(\{1,\ldots,n\}\) and the cores of the various faces will be denoted by removing appropriate commas from the suffices. Thus we need a notation which makes clear what is denoted by, for example, \(E_{1,2,34,5}\). Further, when considering duals we will need to start with index sets other than \(\{1,\ldots,n\}\).

Later in the paper we will need to use other sets of sets of integers to denote certain maps of \textit{n-fold vector bundles}, and for this reason we use a distinctive terminology, ‘hops and runs’, for the sets which index the faces and cores.

\textbf{Hops and runs}

Let \(A\) be a finite set. A \textit{hop} across \(A\) is a set of non-empty, disjoint subsets of \(A\). The \textit{length} of a hop is the number of subsets it contains.

For instance, if \(A = \{1,2,3,4,5\}\), then \(\{\{1,3\},\{4\}\}\) is a hop across \(A\) of length 2. When there is no ambiguity, we may omit the curly braces; in that case, we will separate different subsets by commas, and we will not use a comma to separate elements in the same subset. For instance, instead of \(\{\{1,3\},\{4\}\}\) we may write \(\{13,4\}\) or just \(13,4\). When we are using hops as indexes, in particular, we will simply write \(E_{13,4}\) instead of \(E_{\{1,3\},\{4\}}\). (In this paper we will not consider examples with \(n > 9\).)

For any positive integer \(n\), we denote the set \(\{1,\ldots,n\}\) by \(\underline{n}\). Given our abuses of notation, we will also write \(\underline{n}\) for the hop that has \(n\) elements of size 1, namely \(\{\{1\},\ldots,\{n\}\}\).

A \textit{pure hop} is a hop all of whose elements are of size 1. A \textit{run} is a hop with a single element. (That is, one ‘runs’ so long as there is no comma, and has to ‘hop’ over any comma.) For instance, \(\{\{1\},\{3\},\{4\}\}\) is a pure hop, and \(\{1,3,4\}\) is a run. Abusing notation, we will regard a run both as a subset of \(A\) and as a set with one element (which is a subset of \(A\)). If \(A\) is a set with size \(n\), then there are \(2^n\) pure hops and \(2^n - 1\) runs on \(A\). Given a hop \(H\), there is a natural way to make it into a pure hop, which we denote
Given a set $A$, a hop $H$ across $A$, and a subset $I$ of $A$, the notation $I \in H$ has the usual meaning. If $k \in A$, we will use the notation $k \in H$ to mean that there exists $I \in H$ such that $k \in I$. The negation of $k \in H$ will be written as $k \notin H$. Given two elements $i, j \in H$, we say that $i$ and $j$ are together in $H$ if there exists $I \in H$ such that $i, j \in I$; otherwise we say that $i$ and $j$ are separate in $H$. For instance, $13 \in \{13, 4\}$, $3 \notin \{13, 4\}$, $3 \in \{13, 4\}$, $2 \notin \{13, 4\}$; 1 and 3 are together in $\{13, 4\}$; 1 and 4 are separate in $\{13, 4\}$. A hop $H$ across $A$ is called complete if $j \in H$ for all $j \in A$. Thus a complete hop across $A$ is a partition of $A$.

Two hops $H_1$ and $H_2$ are called disjoint if there is no element $i$ such that $i \in H_1$ and $i \in H_2$. Given two such hops we denote the hop $H_1 \cup H_2$ by $H_1, H_2$ (inserting a comma).

Given two runs $R_1$ and $R_2$ which are disjoint, we define $R := R_1 R_2$ as the run $R$ whose only element is the union of the only elements in $R_1$ and $R_2$. Compare: if $R_1 = \{1\}$ and $R_2 = \{2, 3\}$, then $R_1, R_2 = \{1, \{2, 3\}\} = 123$.

Let $H$ be a hop in $A$ and let $i$ be any element of $A$ such that $i \in A$. We define $H \setminus i$ as the hop obtained from $H$ when the subset $I \in H$ such that $i \in I$ has been replaced with $I \setminus \{i\}$, or removed if $I = \{i\}$. We define $H \setminus i, j$ as $(H \setminus i) \setminus j$. For instance, if $H = \{13, 4\}$, then $H \setminus 3 = \{1, 4\}$, whereas $H \setminus 4 = \{13\}$.

**Definition of $n$-fold vector bundle**

**Definition 2.1.** Let $n$ be a non-negative integer. An $n$-fold vector bundle consists of a smooth manifold $E_H$ for every pure hop $H$ across $\underline{n}$, together with a vector bundle structure on $q_H^{H,i}: E_{H,i} \to E_H$ for every pure hop $H$ and $i \notin H$, such that

$$
\begin{array}{ccc}
E_{H,i,j} & \longrightarrow & E_{H,j} \\
\downarrow & & \downarrow \\
E_{H,i} & \longrightarrow & E_H
\end{array}
$$

is a double vector bundle for every pure hop $H$, and every $i, j \notin H$.

We refer to the whole structure (that is, to the $n$-fold vector bundle) as $E$. The total space of $E$ is $E_{\underline{n}}$, and $M := E_{\emptyset}$ is the final base.

Later in the section we will extend this definition to allow index sets other than $\underline{n}$.

**Remarks 2.2.** (i) The commas in the subscripts are important: we are using pure hops (not runs) as indices. The various cores associated with the structure will be labeled by hops which are not pure.
(ii) The concept of “$n$-fold vector bundle” differs from the concept of “$n$-vector bundle” that appears in category theory. We refer to $n$-fold vector bundles generically as multiple structures to distinguish them from “higher vector bundles”.

(iii) It might seem that Definition 2.1 should include more compatibility conditions. For instance, we could require that certain maps in the structure of an $n$-fold vector bundle form a morphism of $k$-fold vector bundles for $k < n$. Such conditions are all implied by the current definition.

(iv) In [5] we added a further condition to the definition, namely that a certain combination of the bundle projections is a surjective submersion. This turns out to be implied by the rest of the definition, as is discussed after Definition 3.1 below.

(v) A 0-fold vector bundle is just a manifold. A 1-fold vector bundle is a vector bundle in the usual sense. According to Definition 2.1, a 2-fold vector bundle is precisely a double vector bundle as defined in [5] and references there. Applying Definition 2.1 with $n = 3$, a 3-fold vector bundle consists of a commutative diagram as in Figure 3(a), such that every two-dimensional face forms a double vector bundle. Again, this definition of 3-fold vector bundle agrees with the definition of triple vector bundle [8, 5]. In general, the $2^n$ manifolds that constitute an $n$-fold vector bundle can be arranged as the vertices of an $n$-dimensional cube in a commutative diagram. We refer to this as the outline of the $n$-fold vector bundle.

**Example 2.3.** Let $E$ be an $n$-fold vector bundle. Then $TE$ has a natural structure of an $(n+1)$-fold vector bundle. To specify this, first apply the tangent functor to every map in the outline of $E$; this produces an $n$-fold vector bundle with final base $TM$. Next, for every pure hop $H$ across $\underline{n}$, define $(TE)_{H,n+1} := T(E_H)$ with its structure as tangent bundle of $E_H$. The result is an $(n+1)$-fold vector bundle, called the tangent prolongation of $E$.

**Example 2.4.** Let $M$ be a manifold and let $n$ be a positive integer. Suppose given, for every run $R$ across $\underline{n}$, a vector bundle $E_R \to M$. From this data we will construct an $n$-fold vector bundle.

For every pure hop $H$ across $\underline{n}$ write $E_H$ for the pullback manifold $\bigstar_R E_R$ where the pullback is taken over all runs $R$ such that $R \subseteq r(H)$. Suppose that $H_1$ and $H_2$ are two pure hops related by $H_1 = H_2, i$ for some $i \in \underline{n}$. Form the Whitney sum vector bundle $W := \oplus_S E_S$ on $M$, where the $\oplus$ is over all runs $S$ across $\underline{n}$ with $i \in S$. Now the inverse image vector bundle of $W \to M \rightarrow$ across the projection $E_{H_2} \to M$ gives $E_H$, a vector bundle structure on base $E_{H_2}$.

In this way we have constructed an $n$-fold vector bundle with total space the manifold

$$E_{\underline{n}} = \bigstar \{E_R \mid R \text{ a run across } \underline{n}\}.$$
This is the decomposed $n$-fold vector bundle constructed from the $E_R$. Although $E_n$ may be considered the Whitney sum of all the $E_R$, this is not part of the structure of the $n$-fold vector bundle, and is usually not relevant.

**Definition 2.5.** Let $E$ and $F$ be two $n$-fold vector bundles. A morphism of $n$-fold vector bundles $\varphi: E \to F$ consists of a set of smooth maps $\varphi_H: E_H \to F_H$ for every pure hop $H$ across $n$, such that, for every pure hop $H$ and every $i \notin H$, the following

$$
\begin{array}{ccc}
E_{H,i} & \xrightarrow{\varphi_{H,i}} & F_{H,i} \\
\downarrow & & \downarrow \\
E_H & \xrightarrow{\varphi_H} & F_H
\end{array}
$$

is a morphism of vector bundles. An isomorphism of $n$-fold vector bundles is a morphism which is a diffeomorphism.

Note that (5) is a diagram of a morphism, not of a double structure.

Various alternative formulations of Definition 2.1 are possible. Consider an $n$-fold vector bundle $E$ and let $j \in n$. Write $E^{(j)}$ for the $(n-1)$-fold vector bundle consisting of the manifolds $E_H$ for pure hops $H$ such that $j \in H$ and the maps between them. In a similar way, write $E^{(j)}_\emptyset$ for the $(n-1)$-fold vector bundle consisting of the manifolds $E_H$ for pure hops $H$ such that $j \notin H$ and the maps between them. The rest of the structure on the $n$-fold vector bundle $E$ can be described as a morphism of $(n-1)$-fold vector bundles from $E^{(j)}$ to $E^{(j)}_\emptyset$.

By reversing this description, an $n$-fold vector bundle can be recursively defined as a “vector bundle object in the category of $(n-1)$-fold vector bundles”.

**The cores of an $n$-fold vector bundle**

The core structure of a triple vector bundle was recalled in the Introduction; see Figure 3(b). The approach used there may be extended to $n$-fold vector bundles for any $n$, and we outline it very briefly. Consider an $n$-fold vector bundle $E$, and any $1 \leq i \neq j \leq n$. Write $H = n \setminus i,j$. Then $E^{(2)} = E_{i,j,H}$ is a double vector bundle with side bundles $E_{H,i}$ and $E_{H,j}$ and final base $E_H$. As such it has a core, which we denote $E_{ij,H}$, and which is a vector bundle on base $E_H$. Further, for each $k \in n$, $k \neq i,j$, the vector bundle structure on $E^{(2)} \to E^{(2)}_{\setminus k}$ restricts to give $E_{ij,H}$ a vector bundle structure on base $E_{ij,H \setminus k}$. Thus $E_{ij,H}$ is an $(n-1)$-fold vector bundle.

This process continues inductively until we reach the ultracore, which we need throughout the rest of the paper.

**Definition 2.6.** Let $E$ be an $n$-fold vector bundle. The ultracore of $E$ is the set of elements $e \in E$ such that, for every $i, j \in n$, with $i \neq j$, the element $q_{n \setminus i,j}^n(e)$ is a zero of the vector bundle $E_{n \setminus i} \to E_{n \setminus i,j}$. 

9
We denote the ultracore of \( E \) by \( E_{r(n)} \) or by \( C(E) \). As in the triple case, the \( n \) vector bundle structures on \( E_{r(n)} \) coincide and make the ultracore a vector bundle on \( M \).

We will also need the ultracores of various substructures of \( E \). Let \( R \) be a run across \( n \) and write \( H = p(R) \). Assume that \( H \) is not \( n \) (we have already considered that case), and that it contains more than one element. If \( H \) has \( k \) elements, \( 1 < k < n \), then \( E_H \) is a \( k \)-fold vector bundle with respect to the structure induced from \( E \). Write \( E_R \) for the ultracore of \( E_H \).

We now have, for each run \( R \) across \( n \), a vector bundle \( E_R \) on base \( M \), which is the ultracore of \( E_{p(R)} \). To avoid trouble with extreme cases, we define the ultracore of a 1-fold vector bundle to be the vector bundle itself.

**Remark 2.7.** There is a complicated system of cores of substructures lying between the \( E_R \) just defined, and the cores of the double vector bundles which have \( E_n \) as total space. We describe some of these in the next subsection.

**Definition 2.8.** Let \( E \) be a \( n \)-fold vector bundle. The **building bundles** of \( E \) are the vector bundles \( E_R \to M \) for all runs \( R \) across \( n \). The set of building bundles is denoted \( E \bullet \).

In particular the side bundles \( E_i \), \( 1 \leq i \leq n \), with base \( M \), are building bundles. Altogether there are \( 2^n - 1 \) building bundles.

Applying the construction of Example 2.4 to the building bundles yields an \( n \)-fold vector bundle which we call the **decomposed form** of \( E \) and denote by \( \overline{E} \).

**Definition 2.9.** Let \( E \) and \( F \) be two \( n \)-fold vector bundles which have the same building bundles. A **statomorphism** \( \varphi : E \to F \) is a morphism of \( n \)-fold vector bundle which induces the identity on all building bundles.

A statomorphism is necessarily an isomorphism; indeed any morphism of \( n \)-fold vector bundles which induces an isomorphism of vector bundles on all building bundles is an isomorphism of \( n \)-fold vector bundles.

Statomorphisms and building bundles are essential for the calculation in Section 4 of the duality functor groups.

**The duals of an \( n \)-fold vector bundle**

Let \( E \) be an \( n \)-fold vector bundle. The total space \( E_n \) has \( n \) distinct structures of vector bundle. Take \( i \in n \) and consider the dualization of \( E_n \) as a vector bundle over \( E_{n\setminus i} \). We must show that dualization does lead to another \( n \)-fold vector bundle. There are two aspects to the problem.

First, for each \( j \neq i \), \( j \in n \), there is a double vector bundle for which \( E_n \) is the total space and \( E_{n\setminus i} \) is a side bundle. See Figure 4(a). We denote the core by \( E_{H,ij} \) where
\( H = n \setminus i, j \). Recall that for the core of a double vector bundle which is contained within a multiple structure, our rule for notation is to combine the two indices which distinguish the side bundles, leaving unchanged the hop which indexes the final base of the double vector bundle.

\[
\begin{align*}
E_n & \rightarrow E_n \setminus j & E_n \uparrow E_H \rightarrow E_{H, ij} \uparrow E_H \\
E_n \setminus i & \rightarrow E_n \setminus j & E_H \rightarrow E_H
\end{align*}
\]

\( (a) \quad (b) \)

Figure 4.

The dual of Figure 4(a) over \( E_n \setminus i \) is shown in Figure 4(b). A specific example, with \( n = 4, i = 1, j = 2 \), is shown in Figure 5.

\[
\begin{align*}
E_{1,2,3,4} & \rightarrow E_{1,3,4} & E_{1,2,3,4} \uparrow E_{2,3,4} & \rightarrow E_{12,3,4} \uparrow E_{3,4} \\
E_{2,3,4} & \rightarrow E_{3,4} & E_{2,3,4} & \rightarrow E_{3,4}
\end{align*}
\]

\( (a) \quad (b) \)

Figure 5

In general there will be \((n - 2)\) further double vector bundles to consider. In the case \( n = 4 \) these are shown with their duals over \( E_{2,3,4} \) in Figures 6 and 7.

\[
\begin{align*}
E_{1,2,3,4} & \rightarrow E_{1,2,4} & E_{1,2,3,4} \uparrow E_{2,3,4} & \rightarrow E_{13,2,4} \uparrow E_{2,4} \\
E_{2,3,4} & \rightarrow E_{2,4} & E_{2,3,4} & \rightarrow E_{2,4}
\end{align*}
\]

\( (a) \quad (b) \)

Figure 6

We now have four vector bundle structures on \( E_{1,2,3,4} \uparrow E_{2,3,4} \), with bases respectively \( E_{2,3,4} \), \( E_{12,3,4} \uparrow E_{3,4} \), \( E_{13,2,4} \uparrow E_{2,4} \) and \( E_{14,2,3} \uparrow E_{2,3} \). The triple vector bundle structure of the first remains unchanged in this dualization. We need to show that the other three have natural triple vector bundle structures and that these ‘fit together’ correctly.
First consider $E_{12,3,4,1} \times| E_{3,4}$. The bundle of which this is the dual is a triple vector bundle in a natural way. Namely, $E_{12,3,4,1}$ is the core of the double vector bundle in Figure 5(a). The two other vector bundle structures on $E_{12,3,4,1}$, on bases $E_{12,3}$ and $E_{12,4}$, restrict to give vector bundle structures on $E_{12,3,4,1}$ over bases $E_{12,3}$ and $E_{12,4}$, and these form the triple vector bundle shown in Figure 8(a). The cores of the upper faces of this triple vector bundle are (left) $E_{12,3,4}$, (rear) $E_{12,4}$ and (top) $E_{12,3,4}$. The ultracore is $E_{12,3,4}$.

The dual over $E_{3,4}$ is shown in Figure 8(b). In the same way we obtain the two triple vector bundles shown in Figure 9.

We can now complete the outline of the 4-fold vector bundle towards which we are working. This is shown in Figure 10.
Figure 10
The two double vector bundles in the centre of Figure 10 have been found from their total spaces and the arrangement of the triple vector bundle $E_{2,3,4}$ which is left unchanged by the dualization. Their structure as triple vector bundles can be seen to be consistent with those in Figure 9, which were obtained as duals of triple vector bundle structures on the cores of double vector bundles with total space $E_{1,2,3,4}$.

There is one ‘new’ triple vector bundle in Figure 10, namely that obtained by deleting the triple vector bundle $E_{2,3,4}$. The proof that this is a triple vector bundle follows the same pattern as for the corresponding result in the dual case [8, 5.4].

In the case of general $n$ we would at this stage have reached a collection of $(n-2)$-fold vector bundles and can proceed by induction. At each stage we are taking the duals of double vector bundles, as described in [5] and elsewhere. The only task is to ensure consistency.

We evidently have an $n$-fold vector bundle but it is not presented as Definition 2.1 requires. For this we need to allow index sets other than $n$. Write $[n] := \{0, 1, \ldots, n\}$ and for $i \in [n]$ write $[n, i]$ for the set $[n] \setminus \{i\}$.

The bundles appearing in the structure of $E_{n} \uparrow E_{n} \setminus i$ are of the form $E_{R,H} \uparrow E_{H}$ where $R$ is a run across $n$ and $H$ is a hop across $n$. Let $R'$ denote the run across $[n]$ such that $R, H, R'$ is a complete hop across $[n]$ and write

$$E_{R',H} := E_{R,H} \uparrow E_{H}$$

(we write $E_{R,H}$ or $E_{H,R'}$ as convenient). For example, for the space $E_{13,2,4} \uparrow E_{2,4}$ in Figure 10 we have $R' = 0$ and so $E_{13,2,4} \uparrow E_{2,4} = E_{0,2,4}$.

With this renaming, $E_{n} \uparrow E_{n} \setminus i$ satisfies Definition 2.1 with the index set now $[n, i]$. In particular, applying this notation to Figure 10 shows that $E_{1,2,3,4} \uparrow E_{2,3,4}$ is a 4-fold vector bundle with index set $\{0, 2, 3, 4\}$.

Equation (6) effectively defines $E_{H'}$ for every hop $H'$ across $[n]$. Namely, write $H' = R'$, $H$ where $R'$ is the element of $H'$ with $0 \in R'$ (if there is no such element, then $E_{H'}$ is already defined) and $H$ is a hop across $n$. Let $R'$ denote the run across $[n]$ such that $R, H, R'$ is a complete hop across $[n]$ and write

$$E_{R,H} := E_{R,H} \uparrow E_{H}$$

(we write $E_{R,H}$ or $E_{H,R'}$ as convenient). For example, for the space $E_{13,2,4} \uparrow E_{2,4}$ in Figure 10 we have $R' = 0$ and so $E_{13,2,4} \uparrow E_{2,4} = E_{0,2,4}$.

Let $S_{n+1}$ be the group of permutations of $[n]$. Write $\lambda_i$ for the transposition $(0 i)$ in $S_{n+1}$. Then $\lambda_i$ defines a bijection between $[n, 0] = n$ and $[n, i]$, and therefore induces a bijection between the hops across $n$ and the hops across $[n, i]$; we denote this bijection also by $\lambda_i$.

Define an $n$-fold vector bundle $E^{X_i}$ by the formula

$$(E^{X_i})_H := E_{\lambda_i(H)}$$

for every pure hop $H$ across $n$. To summarize:

**Theorem 2.10.** $E^{X_i}$ is a well-defined $n$-fold vector bundle.
The cores of $E^{X_i}$ are given by the same formula $(E^{X_i})_H = E_{\lambda_i(H)}$ where now $H$ is any hop across $n$.

We call $E^{X_i}$ the $X_i$-dual of $E$ or the $i$-dual of $E$.

Notice that if we only want to describe the manifolds that appear in the definition of $E^{X_i}$, then they are entirely determined by the permutation $\lambda_i = (0 \ i) \in S_{n+1}$. In particular $\lambda_i$ permutes the vector bundles $E_1, \ldots, E_n$ and the dual $E_0$ of the ultracore.

3 The duality functor groups

We have defined the action of dualization on multiple vector bundles. The reader will appreciate however, that composing several dualizations using this formulation leads to unwieldy expressions which are difficult to recognize. Further, as [5] showed, it is actually not possible to tell from diagrams of the outlines whether two multiple vector bundles are canonically isomorphic (as defined below). We therefore need to extend the techniques of [5] to the $n$-fold case. The key idea is to consider the effect of dualization not only on the multiple vector bundles but on the maps between them. This is a simple idea, but it is worth drawing attention to, since the role it plays does not arise for duality of ordinary vector bundles.

To be able to dualize, in any direction, a map between two $n$-fold vector bundles, it is necessary that the dualizations take place over maps which are isomorphisms. We ensure this by considering only statomorphisms.

Statomorphism categories

Let $C_n$ be the category whose objects are $n$-fold vector bundles and whose morphisms are statomorphisms of $n$-fold vector bundles. Let the opposite category be denoted by $C_n^{op}$ or $C_n^{-1}$. The dualization operators $X_k$ defined above extend in a natural way to functors $C_n \rightarrow C_n^{op}$, also denoted $X_k$. Since every morphism in $C_n$ is invertible, we can also consider them as functors $X_k: C_n^{op} \rightarrow C_n$. It is therefore possible to compose such functors. Write $\mathcal{W}_n$ for the group generated by the $X_1, \ldots, X_n$.

As noted after Theorem [2.10], every dualization functor produces a permutation of the vector bundles $E_1, \ldots, E_n, E_0$. Thus there is a surjective group homomorphism $\pi: \mathcal{W}_n \rightarrow S_{n+1}$.

Decomposed $n$-fold vector bundles

Recall that, for an $n$-fold vector bundle $E$, the decomposed form of $E$, constructed as in Definition [2.8] is denoted $\overline{E}$.
Definition 3.1. A decomposition of an \(n\)-fold vector bundle \(E\) is a statomorphism onto its decomposed form \(E \rightarrow \overline{E}\).

Grabowski and Rotkiewicz [4] proved that every \(n\)-fold vector bundle has a decomposition. Rather than consider duality operators on an arbitrary \(n\)-fold vector bundle, it is therefore sufficient to consider the decomposed case.

This situation is analogous to that of trivializations of an ordinary vector bundle. If a vector bundle is, say, flat and on a simply-connected base, then a trivialization exists, but is not unique (assuming that the rank is positive). Rather than consider the group of automorphisms of the vector bundle itself, one considers the group of automorphisms of a trivial vector bundle of the same rank. For multiple vector bundles a decomposition always exists, but is not unique, and we consider automorphisms not of the given multiple vector bundle, but of its decomposed form.

Write elements \(e \in \overline{E}\) in the form \(e = e_I\) where \(I\) ranges over all non-empty subsets of \(n\).

A statomorphism \(\overline{E} \rightarrow \overline{E}\) consists of a set of multilinear maps, which we now describe. For ease of notation, we write their domains as tensor products, but their arguments as strings of vectors.

Throughout the rest of the paper, ‘statomorphism’ always refers to a statomorphism which is an automorphism of a decomposed \(n\)-fold vector bundle. Recall the case \(n = 3\) from [5].

Example 3.2. Let \(E\) be a triple vector bundle. The elements of \(\overline{E}\) are strings \((e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123})\) with \(e_I \in E_I\) for each nonempty \(I \subseteq \{1, 2, 3\}\).

A statomorphism \(\varphi: \overline{E} \rightarrow \overline{E}\) is of the form

\[
\varphi(e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123}) = (e_1, e_2, e_3,
\begin{align*}
e_{12} + \varphi_{12}(e_1, e_2), & \quad e_{23} + \varphi_{23}(e_2, e_3), \\
& e_{31} + \varphi_{31}(e_3, e_1), \\
e_{123} + \varphi_{123}(e_{12}, e_3) + \varphi_{231}(e_{23}, e_1) + \varphi_{312}(e_{31}, e_2) + \varphi_{123}(e_1, e_2, e_3)
\end{align*}
\]

where \(\varphi_{ij}: E_i \otimes E_j \rightarrow E_{ij}\), \(\varphi_{ij,k}: E_{ij} \otimes E_k \rightarrow E_{ijk}\), and \(\varphi_{123}: E_1 \otimes E_2 \otimes E_3 \rightarrow E_{123}\) are multilinear maps.

These \(\varphi_I\) can also be written as elements of \(E_1 \otimes E_2 \otimes E_{12}^*, E_2 \otimes E_3 \otimes E_{23}^*, \ldots, E_1 \otimes E_2 \otimes E_3 \otimes E_{123}^*\).

From (6) it follows that for any \(I \subseteq \underline{n}\), we have \(E_I^* := E_{I^C}\), where \(I^C := [n] \setminus I\).

The seven tensor products can therefore be written as \(E_1 \otimes E_2 \otimes E_{03}, E_2 \otimes E_3 \otimes E_{01}, \ldots, E_1 \otimes E_2 \otimes E_3 \otimes E_0\). These correspond to the seven partitions of the set \(\{0, 1, 2, 3\}\) into three or more subsets.

In general, a statomorphism adds a term to \(e_I\) for each partition of \(I\) into two or more nonempty sets. We now formalize this statement.
A partition of \([n]\) is the same as a complete run across \([n]\), and we will continue to use the notation set up in §2 for runs. For every positive integer \(n\), denote by \(\mathcal{P}_n\) the set of all partitions of \([n]\) into three or more subsets. The number of ways in which \([n]\) can be partitioned into \(k \geq 2\) subsets is the Stirling number of the second kind [6],

\[
\begin{aligned}
\binom{n+1}{k} &= \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^{n+1}.
\end{aligned}
\]

Thus the number of components in a statomorphism of \(\overline{E}\) is \(\sum_{k=3}^{n+1} \binom{n+1}{k}\). The values for \(n = 3, 4, 5, 6\) are 7, 36, 171 and 813.

For any \(P \in \mathcal{P}_n\), define

\[
V_P := \bigotimes_{I \in P} E_I^*
\]

Elements of a given \(V_P\) can be interpreted in various ways. For example, consider \(\varphi \in V_{012,3,4}\). This \(\varphi\) is a section of \(E_{012} \otimes E_3^* \otimes E_4^*\), or a map from \(E_{012} \otimes E_3 \otimes E_4\) to the real line bundle. However, since \(E_{012} = E_{34}^*\), we can also regard \(\varphi\) as a map from \(E_3 \otimes E_4\) to \(E_{34}\). Similarly, we can also think of \(\varphi\) as a map from \(E_3\) to \(E_{34} \otimes E_{0123}\), and so on. We will use the same letter to refer to all these maps, according to which is most convenient at a particular time. For instance, given elements \(e_I \in E_I\) for each run \(I\), if we write \(\varphi(e_3, e_4)\), it should be understood that we are thinking of \(\varphi\) as \(E_3 \otimes E_4 \rightarrow E_{34}\) and that \(\varphi(e_3, e_4) \in E_{34}\).

**Definition 3.3.** The parameter space \(\mathcal{L}_{E^*}\) of the set of building bundles \(E^*\) is defined as

\[
\mathcal{L}_{E^*} := \bigoplus_{P \in \mathcal{P}_n} V_P
\]

We will see shortly that \(\mathcal{L}_{E^*}\) can be canonically identified with the group of statomorphisms \(\text{Dec} \overline{E}\) and, moreover, with the group of statomorphisms \(\text{Dec} \overline{E}^W\) for any dualization functor \(W\).

Take a word \(W\) in \(X_1, \ldots, X_n\); that is, \(W \in \mathcal{W}_n\) and write \(k := \pi(W)(0)\), where \(\pi\) is the surjection \(\mathcal{W}_n \rightarrow S_{n+1}\). Then

\[
\overline{E}^W = \overline{E}^W = \bigoplus_{R \text{ a run across } [n,k]} E_R
\]

Write elements \(e \in \overline{E}^W\) as tuples \((e_R)\) where \(e_R \in E_R\) for each run \(R\). Now define a map

\[
\Omega^k : \mathcal{L}_{E^*} \rightarrow \text{Dec} \overline{E}^W.
\]
Take \( \varphi \in \mathcal{L}_{E^*} \) and write \( \varphi = (\varphi_P) \) where \( P \) runs through \( \mathcal{P}_n \). Take \( e \in E^W \) and define \( f := \Omega^k(\varphi)(e) \) by the equations:

\[
 f_R = e_R + \sum \varphi_P(e_{J_1}, \ldots, e_{J_m})
\]

(10)

for each run \( R \) across \( [n, k] \). The sum is over all \( P \in \mathcal{P}_n \) such that \( R^c \in P \) and the \( J_i \) are the remaining elements of \( P \); that is, \( P = \{R^c, J_1, \ldots, J_m\} \). The proof of the following is now a matter of unwinding the statement.

**Theorem 3.4.** Let \( W \in \mathcal{W}_n \). Then the map \( \Omega^k \) defined in (10) is a well-defined, canonical bijection between the set \( \mathcal{L}_{E^*} \) and the group \( \text{Dec} E^W \).

For the case \( n = 3 \), see Section 4 of [5]. The treatment here corrects some problems with the account in [5]. In what follows we will not need the group structures on the various \( \text{Dec} E^W \).

The basic theorem

We now give the result which is the foundation of the subsequent calculations, **Theorem 3.6** an element of \( \mathcal{W}_n \) which induces the identity element of \( S_{n+1} \) is naturally isomorphic to the identity under statomorphisms if and only if it acts trivially on \( \mathcal{L}_{E^*} \). This provides a concrete and explicit calculation that determines when two dualization functors are naturally isomorphic.

To begin, for each run \( R \subseteq [n] \) let \( E_R \rightarrow M \) be a vector bundle, subject to the conditions \( E_\emptyset = M \) and \( E_{R^c} = E_R^* \). We fix this data for the rest of the section.

Denote by \( E_\bullet \) the set of bundles \( \{E_I \mid I \subseteq [n] \} \) and pick an \( n \)-fold vector bundle \( E \) which has \( E_\bullet \) as its building bundles. As usual we denote by \( \overline{E} \) the decomposed form of \( E \). (Of course \( \overline{E} \) can be constructed from \( E_\bullet \) and vice versa.) Take \( W \in \mathcal{W}_n \) and write \( k := \pi(W)(0) \). The decomposed form of \( E^W \) is \( \overline{E^W} = E^W \). We will denote the building bundles of \( E^W \) by \( E^W_\bullet \). Notice that, as a set, \( E^W_\bullet = \{E_I \mid I \subseteq [n, k] \} \). Finally, let \( \varepsilon_W \) be +1 or −1 depending on the parity of \( \pi(W) \).

We now define the crucial action. Let \( \varphi \in \mathcal{L}_{E^*} \) and pick a decomposition \( S_1 \) of \( E \). Then \( S_2 := S_1 \circ \Omega^0(\varphi) \) is another decomposition of \( E \), with \( \Omega^0 \) as defined in (10). Notice that \( (S^W_1)^{\varepsilon_W} \) and \( (S^W_2)^{\varepsilon_W} \) are decompositions of \( E^W \). Hence there is a unique \( \psi \in \mathcal{L}_{E^*} \) such that \( S^W_2 = \Omega^k(\psi) \circ S^W_1 \). We now have the situation shown in Figure 11.

Then we define the map \( \theta^E_W : \mathcal{L}_{E^*} \rightarrow \mathcal{L}_{E^*} \) by \( \theta^E_W(\varphi) := \psi \). We have the following results:

**Theorem 3.5.**

1. The map \( \theta^E_W \) is well defined. Specifically, \( \theta^E_W(\varphi) \) depends only on the building bundles \( E_\bullet \) on \( W \) and on \( \varphi \in \mathcal{L}_{E^*} \); it does not depend on the choice of \( n \)-fold vector bundle \( E \) or on the choice of decomposition \( S_1 \).
2. If $W_1, W_2 \in \mathcal{W}_n$, then
\[
\theta^{(E^W_1)} - \theta^{E_1} = \theta^{E_2}.
\] (11)

The proof follows the same pattern as for the triple case [5]. The first part is proved by a direct computation in Section 4. The second part follows from the definition, keeping in mind that Dec_{E} and Dec_{E}^W are groups.

Equation (11) can be interpreted as a groupoid action, but we will not do so here.

Next, notice that the set of building bundles $E^W_\bullet$ depends only on $E_\bullet$ and $\pi(W)$. In particular, if $\pi(W)$ is the identity, then $E^W_\bullet = E_\bullet$. Define
\[
\tilde{K}_{n+1} := \{W \in \mathcal{W}_n \mid \pi(X) = \text{id}\}.
\] (12)

We can therefore define an action
\[
\theta: \tilde{K}_{n+1} \times L_{E_\bullet} \rightarrow L_{E_\bullet} \quad \text{by} \quad \theta_W(\varphi) := \theta^{E_\bullet}_W(\varphi).
\]

This action, as anticipated, is faithful, as given by the next theorem.

**Theorem 3.6.** Let $W_1, W_2 \in \mathcal{W}_n$ such that $\pi(W_1) = \pi(W_2)$. Then the functors $W_1$ and $W_2$ are naturally isomorphic to each other if and only if $\theta^{E_\bullet}_{W_1} = \theta^{E_\bullet}_{W_2}$.

The proof of this theorem is identical to the case of double and triple vector bundles (see [5 2.7]), so we omit it. We can finally define the group which is the subject of the paper.

**Definition 3.7.** The group $D_{\mathcal{F}_n}$, called the dualization functor group for $n$-fold vector bundles, is the quotient of $\mathcal{W}_n$ over natural isomorphism in $\mathcal{C}_n$.

From now on, we write $X_i$ for the element of $D_{\mathcal{F}_n}$ corresponding to the dualization operator $X_i$, and so on.

The morphism $\pi$ descends to a surjective morphism $D_{\mathcal{F}_n} \rightarrow S_{n+1}$, also denoted $\pi$. The kernel is the quotient of $\tilde{K}_{n+1}$ over natural isomorphism, and we denote it $K_{n+1}$.

**Corollary 3.8.** The restriction of $\theta$ to the group $K_{n+1}$ is faithful on the set $L_{E_\bullet}$.
The corollary allows us to identify $K_{n+1}$ with its action on $L_{E^*}$, that is, with a subgroup of the symmetric group on $L_{E^*}$.

We need one final lemma preparatory to our calculation of $K_{n+1}$. There is an evident short exact sequence,

$$1 \longrightarrow K_{n+1} \longrightarrow \mathcal{D} \mathcal{F}_n \xrightarrow{\pi} S_{n+1} \longrightarrow 1. \quad (13)$$

and we know that $\mathcal{D} \mathcal{F}_n$ is generated by $X_1, \ldots, X_n$. We also know that $\pi(X_k) = (0k)$ and that the $(0k)$ generate $S_{n+1}$. The following lemma is an easy exercise in algebra.

**Lemma 3.9.** Let $\pi: G \to S$ be a surjective group homomorphism. Let $g_1, \ldots, g_n$ be a set of generators of $G$ and write $\sigma_i := f(g_i)$. Let $\{R_j(\sigma_1, \ldots, \sigma_n) \mid j = 1, \ldots, m\}$ be a set of relations for a presentation of $S$ with generators $\sigma_1, \ldots, \sigma_n$. Then the kernel of $\pi$ is the normal subgroup of $G$ generated by $\{R_j(g_1, \ldots, g_n) \mid j = 1, \ldots, m\}$.

We use the standard presentation of $S_{n+1}$ with generators $\sigma_k := (0k)$ by relations $\sigma_i^2$, $(\sigma_i \sigma_j)^3$, and $(\sigma_i \sigma_j \sigma_k)^2$ with $i, j, k \in n$ distinct. Hence, $K_{n+1}$ is the normal subgroup of $\mathcal{D} \mathcal{F}_n$ generated by

$$X_i^2, (X_i X_j)^3, (X_i X_j X_i X_k)^2 \quad (14)$$

for all distinct values $i, j, k \in n$. From ordinary duality and the case $n = 2$ we know that $X_i^2 = (X_i X_j)^3 = 1$. In [3] we will find the order of $K_{n+1}$.

### 4 Calculation of the action $\theta$ of $K_{n+1}$ on $L_E$.

In this section we calculate the action of $K_{n+1}$ on $L_E$. The first step is to compute $\theta_k := \theta_{X_k}^E$ for $k \in n$. We begin by recalling the action in the triple case.

**Example 4.1.** The statomorphism $\varphi$ in Example 3.2 consists of seven maps; in accordance with the notation of [3], denote these by

$$(1, 2, 03), (2, 3, 01), (3, 1, 02), (12, 3, 0), (23, 1, 0), (31, 2, 0), (1, 2, 3, 0). \quad (15)$$

Here we are suppressing $\varphi$, rather as if we were to denote the entries of a matrix $(a_{ij})$ just by $ij$. In [3] these were denoted, respectively, by $\gamma, \alpha, \beta, \nu, \lambda, \mu, \rho$.

We found $\theta_X$ of these to be, respectively,

$$-(2, 13, 0), (2, 3, 01), -(3, 12, 0), -(02, 3), -(23, 1, 0), (31, 2, 0),$$

$$-(0, 2, 3, 1) + (1, 2, 03)(0, 3, 12) + (1, 3, 02)(0, 2, 13).$$

For clarity, we introduce a neologism and call an element of a $V_P$ for some $P \in \mathcal{P}_n$, a tomo. A tomo is a statomorphism in its own right and so $\theta_X$ acts upon it. Further, for
\( \mu \in V_P \) where \( P \) is a partition into \( m \) subsets, call \( \mu \) an \((m - 1)\)-tomo. Thus \((m - 1)\) is the number of commas in the explicit expression of \( \mu \).

Example 4.1 illustrates two general features: for a 2-tomo \( \mu \), any \( \theta_X(\mu) \) is \( \pm \mu' \) for another 2-tomo \( \mu' \). Secondly, for a \( m \)-tomo \( \mu \) with \( m \geq 3 \), the formula for \( \theta_X(\mu) \) is a signed sum of products of tomos. We now explain this product, which we will denote \( \star \).

**Definition 4.2.** Let \( P, Q \in \mathcal{P}_n \) be two partitions and let \( I \subseteq [n] \). We say that \( P \) and \( Q \) are compatible through \( I \) if \( I \in P \) and \( I^C \in Q \).

Write \( P = \{ I_1, I_2, \ldots, I_r \} \) and \( Q = \{ I^C, J_1, \ldots, J_s \} \). Then we define a new partition \( P \star Q := \{ I_1, \ldots, I_r, J_1, \ldots, J_s \} \).

If \( P \) and \( Q \) are compatible, then there is a unique \( I \) satisfying the definition, and so \( P \star Q \) is well-defined.

Let \( P \) and \( Q \) be compatible partitions. Then we can define a product \( V_P \times V_Q \xrightarrow{} V_{P \star Q} \), also denoted \( \star \), as follows. If \( e_K \in E_K \) for \( K = I_1, \ldots, I_r, J_1, \ldots, J_s \), then

\[
(\varphi \star \psi)(e_{I_1}, \ldots, e_{I_r}, e_{J_1}, \ldots, e_{J_s}) := \langle \varphi(e_{I_1}, \ldots, e_{I_r}) \mid \psi(e_{J_1}, \ldots, e_{J_s}) \rangle
\]  

(16)

where \( \langle \mid \rangle \) is the pairing of the bundles \( E_{I^C} \) and \( E_I \), which are dual to each other.

**Example 4.3.** Consider \( \varphi \star \psi \) where \( \varphi = (1, 2, 03) \) and \( \psi = (0, 3, 12) \) as in Example 4.1. Here \( I = \{0, 3\} \). Writing \( P \) and \( Q \) for the partitions, we have \( P \star Q = \{1, 2, 3, 0\} \). Now

\[
(\varphi \star \psi)(e_1, e_2, e_3, e_0) = \langle \varphi(e_1, e_2) \mid \psi(e_3, e_0) \rangle,
\]

where the pairing is \( E_{12} \times_M E_{30} \rightarrow \mathbb{R} \).

We can finally describe \( \theta^{E_{X^*}}_{X_k} \).

**Lemma 4.4.** Let \( \varphi = (\varphi_P)_{P \in \mathcal{P}_n} \) and let \( \psi = (\psi_P)_{P \in \mathcal{P}_n} := \theta^{E_{X^*}}_{X_k}(\varphi) \). Then

\[
\psi_P = \varphi_P
\]

(17)

if 0 and \( k \) are together in \( P \), and

\[
\psi_P = \sum_{j=1}^{n-1} (-1)^j \sum_{P=Q_1 \star \ldots \star Q_j} \varphi_{Q_1} \star \ldots \star \varphi_{Q_j}
\]

(18)

otherwise. The last sum in (18) is taken over all \( j \)-tuples of partitions \( Q_1, \ldots, Q_j \in \mathcal{P}_n \) such that 0 and \( k \) are separate in \( Q_i \) for all \( i \); \( Q_i \) and \( Q_{i+1} \) are compatible through \( I_i \) for all \( i = 1, \ldots, j - 1 \); \( I_i \neq I_{i+1}^C \) (otherwise the iterated composition does not make sense); and \( P = Q_1 \star \ldots \star Q_j \).
Proof. Consider Figure 11 with $W = X_k$. Let $A := E_{n \setminus k}$. Recall that $E$ and $E^{X_k}$ are dual vector bundles over $\overline{A}$. Let $e \in E$ and let $e' \in E^{X_k}$ be elements on fibers over the same $a \in A$. Then

$$\langle e \mid e' \rangle = \langle \Omega^0(\varphi)(e) \mid \Omega^k(\psi)(e') \rangle,$$

where $\langle \mid \rangle$ is the pairing of the dual bundles over $\overline{A}$. Now we substitute (9) and (10) in (19) and we obtain (17) and (18).

Example 4.5. In the case $n = 4$, $k = 1$, a typical action on a 3-tomo is

$$\theta_{X_1}(1, 2, 3, 04) = -(1, 2, 3, 04) + (04, 2, 13) \ast (1, 3, 024) + (04, 3, 12) \ast (1, 2, 034).$$

For the unique 4-tomo, $\theta_{X_4}(0, 1, 2, 3, 4)$ is given by

$$-(1, 2, 3, 4, 0) + (0, 2, 134) \ast (1, 3, 4, 02) + (0, 3, 124) \ast (1, 2, 4, 03) + (0, 4, 123) \ast (1, 2, 3, 04) + (14, 2, 3, 0) \ast (1, 4, 023) - (0, 2, 134) \ast (14, 3, 02) \ast (1, 4, 023) - (0, 3, 124) \ast (14, 2, 03) \ast (1, 4, 023) + (13, 2, 4, 0) \ast (1, 3, 024) - (0, 2, 134) \ast (13, 4, 02) \ast (1, 3, 024) - (0, 4, 123) \ast (13, 2, 04) \ast (1, 3, 024) + (12, 3, 4, 0) \ast (1, 2, 034) - (0, 3, 124) \ast (12, 4, 03) \ast (1, 2, 034) - (0, 4, 123) \ast (12, 3, 04) \ast (1, 2, 034).$$

Using Lemma 4.4 and Equation (11), it is in principle possible to obtain the explicit form of $\theta_W^x$ for any $W \in \mathcal{D}_n$. This is a computation of intimidating length, which we will not attempt here. It will be sufficient to consider certain special cases.

Let $\varphi$ be a 2-tomo. Since there are no factorizations of the partition, $\theta_{X_k}(\varphi)$, for each $X_k$ and hence each $W \in \mathcal{D}_n$, must be $\pm \varphi'$ for another 2-tomo $\varphi'$. For $W \in K_{n+1}$ the integers are unchanged. This provides a direct proof of the following corollary of Lemma 4.4.

Corollary 4.6. For $W \in K_{n+1}$ and $\varphi$ a 2-tomo, $\theta_W(\varphi) = \pm \varphi$.

It follows that every nonidentity element of $K_{n+1}$ has order 2, and so $K_{n+1}$ is abelian and a direct product of $C_2$'s.

Returning to the discussion following (11), since by Corollary 3.8 the restriction of $\theta$ to $K_{n+1}$ is faithful, we can drop the notation $\theta$. As noted already, we have $X_i^2 = 1$ from standard duality, and $(X_iX_j)^3 = 1$ from the double case.

Proposition 4.7. $K_{n+1}$ is generated by the words $(X_iX_jX_k)^2$, for $i, j, k$ distinct, each of which has order 2, and their conjugates.

In the triple case, the conjugates of $(X_iX_jX_k)^2$ are of the same form; for $n \geq 4$ this is no longer so.

In the next section we obtain a precise combinatorial description of the group $K_{n+1}$. 

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To calculate the order of $K_{n+1}$, we identify it with a set of graphs.

Denote the set of subgraphs of the complete graph on $n+1$ vertices, labeled $0, 1, \ldots, n$, by $\overline{K}_{n+1}$. Under symmetric difference, $\overline{K}_{n+1}$ is a group. Every nonidentity element is of order 2, and $\overline{K}_{n+1}$ is isomorphic to the direct product of $\left(\frac{n+1}{2}\right)$ copies of $C_2$.

Denote by $c_{ij}$ the element of $\overline{K}_{n+1}$ for which the only edge is the edge joining vertices $i$ and $j$. The \{ $c_{ij}$ | $0 \leq i < j \leq n$ \} are a set of generators for $\overline{K}_{n+1}$.

There is a natural action of $S_{n+1}$ on $\overline{K}_{n+1}$ by permuting the vertices, which makes $\overline{K}_{n+1}$ into a $S_{n+1}$-module. There is also an action of $S_{n+1}$ on $\overline{K}_{n+1}$ arising from the exact sequence (13); since $\overline{K}_{n+1}$ is abelian, the conjugation action of $D \mathcal{F}_n$ quotients to an action of $S_{n+1}$.

We define an action $\bar{\theta}: \overline{K}_{n+1} \times \mathcal{L}_E \rightarrow \mathcal{L}_E$. It is enough to describe the action of the generators $c_{ij}$ of $\overline{K}_{n+1}$. Let $\varphi = (\varphi_P)_{P \in \mathcal{P}_n}$ and let $\psi = (\psi_P)_{P \in \mathcal{P}_n} := \bar{\theta}_{c_{ij}}(\varphi)$. Then $\psi_P = \varphi_P$ if $i$ and $j$ are together in $P$, and $\psi_P = -\varphi_P$ otherwise.

This gives us an identification of $K_{n+1}$ with a subgroup of $\overline{K}_{n+1}$, and an embedding of $D \mathcal{F}_n$ in a semidirect product.

**Theorem 5.1.** For every $n$, there is an injective group homomorphism $\Psi: \mathcal{D}\mathcal{F}_n \rightarrow \overline{K}_{n+1} \rtimes S_{n+1}$ defined by

$$\Psi(X_i) = (c_{0i}, (0 \ i))$$

for $i = 1, \ldots, n$.

**Proof.** That $\Psi(X_i)^2 = (c_{0i}, (0 \ i))^2$ is the identity is clear. For $i \neq j$ we have

$$(c_{0i}, (0 \ i)) (c_{0j}, (0 \ i)) (c_{0i}, (0 \ i)) = (c_{0i} + c_{ij} + c_{0j}, (i \ j))$$

and the square of this is again the identity. Lastly, for $i, j, k$ distinct we have,

$$(c_{0i}, (0 \ i)) (c_{0j}, (0 \ j)) (c_{0i}, (0 \ i)) (c_{0k}, (0 \ k)) = (c_{kj} + c_{ij} + c_{ki} + c_{0k}, (i \ j)(0 \ k)).$$

Squaring this, the $S_{n+1}$ component vanishes and the $\overline{K}_{n+1}$ component is $c_{0i} + c_{ik} + c_{kj} + c_{0j}$, which is of order two. Thus $\Psi$ is well-defined and defines an injective homomorphism $\mathcal{D}\mathcal{F}_n \rightarrow \overline{K}_{n+1} \rtimes S_{n+1}$.

Using $\Psi$, define $\psi: K_{n+1} \rightarrow \overline{K}_{n+1}$ by

$$\psi((X_i X_j X_i X_k)^2) = c_{0i} + c_{0j} + c_{ki} + c_{kj} = \begin{vmatrix} 0 & -i \\ j & -k \end{vmatrix}$$

(20)
**Theorem 5.2.** The map $\psi$ is an injective morphism of $S_{n+1}$–modules, and commutes with the actions of $K_{n+1}$ and $\overline{K}_{n+1}$ on $L_{E^*}$. That is,

$$\overline{\theta}_{\psi(W)} = \theta_W$$

for all $W \in K_{n+1}$. The image $\psi(K_{n+1})$ is the subgroup of $\overline{K}_{n+1}$ consisting of graphs such that

- each vertex has even valency, and
- the total number of edges is even.

**Proof.** Recall that we are identifying the group $K_{n+1}$ with its action on $L_{E^*}$. Since $K_{n+1}$ is an abelian group in which every non-identity element has order 2, we only need check that $\psi$ is $S_{n+1}$–equivariant and that (21) holds. Injectivity follows from the faithfulness of $\theta$. \hfill \Box

In the following, we will abuse notation and think of $K_{n+1}$ both as a subgroup of $\overline{K}_{n+1}$ and as a subgroup of $\mathcal{D}\mathcal{F}_{n+1}$. By counting the number of graphs satisfying the two conditions at the end of Theorem 5.2 we obtain

**Corollary 5.3.** As a group, $K_{n+1}$ is isomorphic to the direct product of $\frac{1}{2}(n+1)(n-2)$ copies of $C_2$. In particular

$$|\mathcal{D}\mathcal{F}_n| = 2^{\frac{1}{2}(n+1)(n-2)}(n+1)!$$

## 6 The kernel for $n = 4$

To give an example in detail which displays all the features described in §5, consider the case $n = 4$. The computations here were done in part by hand and in part using a Java program written by Ms. Diksha Rajen, a graduate student in Computer Science at Sheffield, under the supervision of Dr. Mike Stannett.

The group $K_5$ contains 12 elements of the form $(ijik)^2$, where we abbreviate each $X_i$ to $i$. We give these arbitrary labels as follows:

$$A := (1213)^2, \quad B := (1312)^2, \quad C := (2321)^2, \quad D := (1214)^2, \quad E := (1412)^2, \quad F := (2421)^2, \quad K := (1413)^2, \quad L := (1314)^2, \quad M := (4341)^2, \quad P := (4243)^2, \quad Q := (4342)^2, \quad R := (2324)^2.$$ 

Of course $A = (2123)^2 = (3121)^2 = (3212)^2$ also, and likewise for the other elements. The action of these on the 25 2-tomos is given in Table 1 at the end of the paper.

To express the products of these elements we introduce three more labels,

$$T := AD, \quad U := BF, \quad V := AQ.$$
We also encounter products which coincide with one of these 15 elements on the 2-tomos of the form \(2 + 2 + 1\) but have reversed signs on the 2-tomos of the form \(3 + 1 + 1\). We therefore define an element \(i \in K_5\) in terms of its action: \(i\) preserves each 2-tomo of the form \(2 + 2 + 1\) and reverses the signs on the 2-tomos of the form \(3 + 1 + 1\).

Write \(a := Ai, b := Bi, \ldots, v := Vi\) and finally write \(I\) for the identity element of \(K_5\). This completes the description of the 32 elements of \(K_5\) in terms of their action on 2-tomos. The multiplication table is given in Table 2.

The elements of \(\overline{K}_5\) corresponding to the 12 elements \(A, \ldots, R\) are of the type shown in (20). Calculating symmetric differences, we obtain Figure 12 for \(T, U\) and \(V\).

![Figure 12. Elements of \(\overline{K}_5\) corresponding to \(T, U\) and \(V\).](image)

The graph for \(i\) is the complete graph on the 5 vertices, and that for \(I\) is the graph with no edges. Multiplication by \(i\) converts a graph to its complement.

From Figure 12(a) we see that \(T = 1Q1\), the conjugate of \(Q\) by \(X_1\). Likewise, it is equal to 2M2, 3D3 and 4A4. In this way arbitrary conjugates can be calculated.

### 7 Description of \(\mathcal{DF}_n\)

Given that \(\mathcal{DF}_n\) is an extension (13) of \(S_{n+1}\) by an abelian group, a natural question to ask is whether the extension splits; that is, whether \(\mathcal{DF}_n\) is isomorphic to the semidirect product \(K_{n+1} \rtimes S_{n+1}\). We proved in [5] that this is not the case for \(n = 3\).

**Proposition 7.1.** The extension (13) is split for \(n = 4\).

**Proof.** Using the presentation in Proposition 7.4 below, GAP [3] shows that \(\mathcal{DF}_4\) has subgroups isomorphic to \(S_5\). Let \(S\) be any such subgroup. Then if the restriction of \(\mathcal{DF}_4 \to S_{n+1}\) to \(S \to S_5\) is not an isomorphism, it must have kernel \(A_5\) or \(S\) itself. But the kernel of \(\mathcal{DF}_4 \to S_5\) is a product of \(C_2\)s. \(\square\)

**Proposition 7.2.** If \(n = 2 \pmod{4}\), then the extension (13) splits. Specifically, let \(\gamma_{ij} \in K_{n+1}\) correspond to the complete graph on the vertices \([n] \setminus \{i, j\}\). Then there is an isomorphism \(\Phi: \mathcal{DF}_n \to K_{n+1} \rtimes S_{n+1}\) defined by

\[
\Phi(X_j) = (\gamma_{0j}, (0, j))
\]

for \(j = 1, \ldots, n\).
Proof. First, we notice that for $\gamma_{0j}$ to be an element of $K_{n+1}$ it needs to have an even number of edges and only even-valency vertices. This happens exactly when $n = 2 \pmod{4}$. To check that $\Phi$ is a homomorphism, we need to check the three conditions:

\[ \Phi(X_i)^2 = 1, \quad (\Phi(X_i)\Phi(X_j))^3 = 1, \quad (\Phi(X_i)\Phi(X_j)\Phi(X_k))^2 = \frac{0-i}{j-k}, \]

for all distinct $i, j, k$. The first is straightforward. For the second we have

\[ (\gamma_{0i}, (0 i)) (\gamma_{0j}, (0 j)) (\gamma_{0i}, (0 i)) = (\gamma_{0i} + \gamma_{ij} + \gamma_{0j}, (i j)). \quad (23) \]

Squaring this gives the identity, since $\gamma_{0i} + \gamma_{ij} + \gamma_{0j}$ is preserved by $(i j)$.

Now write $\beta_{i,0,j} = \gamma_{0i} + \gamma_{ij} + \gamma_{0j}$. This consists of the null graph on $\{i, 0, j\}$ and the full graph on the complementary set of vertices $[n] \setminus \{0, i, j\}$, with each of $0, i, j$ joined to each of the vertices in $[n] \setminus \{0, i, j\}$. We have

\[ (\gamma_{0i}, (0 i)) (\gamma_{0j}, (0 j)) (\gamma_{0k}, (0 k)) = (\beta_{i,0,j} + \gamma_{0k}, (i j) (0 k)). \]

Squaring this, the kernel term is $\beta_{i,0,j} + \beta_{i,j,k}$ and this is the graph in (20).

We do not know whether (13) splits for $n \geq 8$ a multiple of 4, or what the situation is for odd values $\geq 5$. In particular we withdraw the announcement at the end of Section 4 of [5], that (13) splits if and only if $n$ is even.

Although $\mathcal{DF}_n$ is not always a semidirect product, we can always see it as a subgroup of a semidirect product, as Theorem 5.1 demonstrates. Indeed Theorem 5.1 provides perhaps the most enlightening model for $\mathcal{DF}_n$. For a dualization operation $W \in \mathcal{DF}_n$ with $\Psi(W) = (\gamma, \lambda)$, the permutation $\lambda \in S_{n+1}$ gives the action of the functor $W$ on the building bundles, and the graph $\gamma \in \overline{K}_{n+1}$ tells us the action of $W$ on the “set of changes of decompositions” $\mathcal{L}_E$. One may regard $\gamma$ as measuring to what extent $W$ fails to be merely a rearrangement of the building bundles.

Equivalently, if $\Psi(W_1) = (\gamma_1, \lambda_1)$ and $\Psi(W_2) = (\gamma_2, \lambda_2)$ have $\lambda_1 = \lambda_2$, then $\gamma_1 - \gamma_2$ measures the failure of $W_1$ and $W_2$ to be naturally isomorphic functors.

As a final general result, we compute the centre of $\mathcal{DF}_n$. Let $n \geq 2$. A central element needs to be in $K_{n+1}$, since $S_{n+1}$ has trivial centre, and it needs to be invariant under the $S_{n+1}$-action. The only options are the empty graph, which is the identity, and the complete graph on $n+1$ vertices. The complete graph is only an element of $K_{n+1}$ when $n = 0 \pmod{4}$. Hence we conclude:

**Proposition 7.3.** If $n$ is a multiple of 4, then $|Z(\mathcal{DF}_n)| = 2$. Otherwise, $\mathcal{DF}_n$ has trivial centre.

A remaining question is to describe $\mathcal{DF}_n$ in terms of relations in the $X_i$. From Proposition 4.7 we know that in $\mathcal{DF}_n$ for any $n \geq 3$ the relations

\[ X_i^2, \quad (X_iX_j)^3, \quad (X_iX_jX_iX_k)^4, \quad (24) \]
where $i, j, k$ are distinct, hold. For $n = 4$ it is easy to verify with GAP [3] that the group defined by these relations is infinite (and has Mathieu groups as quotients). From Table 2 we see that $AK = P$ and that $AD = MQ$. These give the relations

\[
12131213.1413141.4243424, \quad 12131213.12141214.24342434.14341434
\]  

(25)

We would like to have an interpretation of these relations comparable to the interpretations of $(ij)^3 = 1$ in terms of the duality of doubles, or the notion of cornering in [8].

**Proposition 7.4.** The group $D \mathcal{F}_4$ is the group on generators $X_i$, $1 \leq i \leq 4$, subject to the relations (24) and (25).

This is again a straightforward calculation in GAP [3]. The general case is the subject of ongoing work.

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| A    | B    | C    | D   | E   | F   | K   | L   | M   | P   | Q   | R   | T   | U   | V   |
|------|------|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| (12, 0, 34) | +    | −    | −   | −   | −   | −   | +   | −   | −   | +   | −   | +   | +   | +   |
| (13, 0, 24) | −    | +    | −   | −   | +   | −   | −   | −   | −   | +   | −   | −   | +   | +   |
| (14, 0, 23) | −    | −    | +   | −   | −   | −   | +   | −   | −   | −   | +   | −   | +   | +   |
| (02, 1, 34) | −    | +    | −   | −   | +   | −   | −   | +   | +   | +   | −   | −   | +   | +   |
| (03, 1, 24) | +    | −    | −   | −   | +   | +   | −   | −   | −   | −   | −   | +   | −   | −   |
| (04, 1, 23) | −    | −    | +   | −   | −   | −   | +   | +   | +   | −   | −   | −   | +   | +   |
| (01, 2, 34) | −    | −    | −   | +   | +   | −   | +   | −   | −   | +   | −   | −   | +   | +   |
| (03, 14, 2) | +    | −    | −   | −   | +   | +   | +   | +   | −   | −   | −   | −   | +   | +   |
| (04, 13, 2) | −    | +    | −   | −   | +   | +   | −   | +   | +   | −   | −   | −   | +   | +   |
| (01, 24, 3) | −    | −    | +   | +   | +   | −   | −   | −   | +   | −   | −   | −   | +   | +   |
| (02, 14, 3) | −    | +    | −   | +   | +   | +   | −   | +   | +   | −   | −   | −   | +   | +   |
| (04, 12, 3) | +    | −    | +   | +   | +   | +   | −   | −   | +   | −   | −   | −   | +   | +   |
| (01, 23, 4) | +    | +    | −   | −   | +   | −   | +   | −   | −   | −   | −   | −   | +   | +   |
| (02, 13, 4) | +    | +    | −   | +   | −   | −   | −   | −   | −   | −   | −   | −   | +   | +   |
| (03, 12, 4) | +    | +    | +   | −   | −   | −   | +   | −   | −   | −   | −   | −   | +   | +   |
| (012, 3, 4) | +    | +    | +   | +   | +   | +   | −   | +   | +   | +   | +   | +   | +   | +   |
| (013, 2, 4) | +    | +    | +   | −   | +   | +   | +   | −   | +   | +   | −   | −   | +   | +   |
| (014, 2, 3) | −    | −    | +   | +   | +   | +   | +   | −   | −   | −   | −   | −   | +   | +   |
| (023, 1, 4) | +    | +    | −   | −   | +   | −   | −   | +   | +   | +   | −   | −   | +   | +   |
| (024, 1, 3) | −    | +    | +   | +   | −   | −   | +   | +   | +   | −   | −   | −   | +   | +   |
| (034, 1, 2) | +    | −    | +   | −   | −   | −   | +   | +   | +   | +   | +   | +   | +   | +   |
| (123, 0, 4) | +    | +    | +   | −   | −   | −   | +   | −   | −   | −   | −   | −   | −   | −   |
| (124, 0, 3) | +    | −    | −   | +   | +   | +   | −   | −   | −   | −   | −   | −   | −   | −   |
| (134, 0, 2) | −    | +    | −   | +   | +   | −   | +   | −   | −   | −   | −   | −   | −   | −   |
| (234, 0, 1) | −    | −    | −   | +   | −   | −   | +   | −   | −   | −   | −   | −   | −   | −   |

Table 1. See §6
Table 2. See \[6\]

References

[1] A. L. Besse. *Manifolds all of whose geodesics are closed*, volume 93 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin, 1978. With appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J. L. Kazdan.

[2] J. Dieudonné. *Treatise on analysis. Vol. III*. Academic Press, New York, 1972. Translated from the French by I. G. MacDonald, Pure and Applied Mathematics, Vol. 10-III.

[3] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4.9*, 2006. [http://www.gap-system.org](http://www.gap-system.org).

[4] J. Grabowski and M. Rotkiewicz. Higher vector bundles and multi-graded symplectic manifolds. *J. Geom. Phys.*, 59(9):1285–1305, 2009.

[5] A. Gracia-Saz and K. C. H. Mackenzie. Duality functors for triple vector bundles. *Lett. Math. Phys.*, 90(1-3):175–200, 2009.

[6] D. E. Knuth. *The art of computer programming*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, third edition, 1997. Volume 1: Fundamental algorithms, Addison-Wesley Series in Computer Science and Information Processing.

[7] K. C. H. Mackenzie. On symplectic double groupoids and the duality of Poisson groupoids. *Internat. J. Math.*, 10(4):435–456, 1999.
[8] K. C. H. Mackenzie. Duality and triple structures. In The breadth of symplectic and Poisson geometry, volume 232 of Progr. Math., pages 455–481. Birkhäuser Boston, Boston, MA, 2005.

[9] K. C. H. Mackenzie. General theory of Lie groupoids and Lie algebroids, volume 213 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2005.

[10] K. C. H. Mackenzie. Ehresmann doubles and Drinfel’d doubles for Lie algebroids and Lie bialgebroids. J. Reine Angew. Math., 658:193–245, 2011.

[11] K. C. H. Mackenzie and P. Xu. Lie bialgebroids and Poisson groupoids. Duke Math. J., 73(2):415–452, 1994.

[12] J. Pradines. Fibrés vectoriels doubles et calcul des jets non holonomes. Notes poly-copiées, Amiens, 1974.

[13] J. Pradines. Remarque sur le groupoïde cotangent de Weinstein-Dazord. C. R. Acad. Sci. Paris Sér. I Math., 306(13):557–560, 1988.

[14] W. M. Tulczyjew. A symplectic formulation of particle dynamics. In Differential geometric methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975), pages 457–463. Lecture Notes in Math., Vol. 570. Springer, Berlin, 1977.

[15] Th. Th. Voronov. Q-manifolds and Mackenzie theory. Comm. Math. Phys. (to appear).