SELF-ADJOINT COMMUTING DIFFERENTIAL OPERATORS
AND COMMUTATIVE SUBALGEBRAS OF THE WEYL
ALGEBRA

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ABSTRACT. In this paper we study self-adjoint commuting ordinary differential operators. We find sufficient conditions when an operator of fourth order commuting with an operator of order 4g + 2 is self-adjoint. We introduce an equation on potentials $V(x), W(x)$ of the self-adjoint operator $L = (\partial_x^2 + V)^2 + W$ and some additional data. With the help of this equation we find the first example of commuting differential operators of rank two corresponding to a spectral curve of arbitrary genus. These operators have polynomial coefficients and define commutative subalgebras of the first Weyl algebra.

1. INTRODUCTION

The problem of finding commuting differential operators is a classical problem of differential equations (for the first results see [1]–[3]). In the case of operators of rank greater than one, this problem has not been solved until now. In this paper we study self-adjoint commuting ordinary differential operators. One of the main results of this paper is the following. We find an example of commuting differential operators of rank two corresponding to spectral curves of arbitrary genus.

If two differential operators

$$L_n = \partial_x^n + \sum_{i=0}^{n-2} u_i(x) \partial_x^i, \quad L_m = \partial_x^m + \sum_{i=0}^{m-2} v_i(x) \partial_x^i$$

commute, then there is a nonzero polynomial $R(z, w)$ such that $R(L_n, L_m) = 0$ (see [3]). The curve $\Gamma$ defined by $R(z, w) = 0$ is called the spectral curve. This curve parametrizes common eigenvalues of the operators. If

$$L_n \psi = z \psi, \quad L_m \psi = w \psi,$$

then $(z, w) \in \Gamma$. For almost all $(z, w) \in \Gamma$ the dimension of the space of common eigenfunctions $\psi$ is the same. The dimension is called the rank. The rank equals the greatest common divisor of $m$ and $n$.

In this paper we consider only commuting ordinary differential operators whose spectral curves are smooth. Commutative rings of such operators were classified by

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The ring is determined by the spectral curve and some additional spectral data. If the rank is one, then the spectral data define commuting operators by explicit formulas (see [4]). In the case of operators of rank greater than one there are the following results. Krichever and Novikov [6], [7] using the method of deformation of Tyurin parameters found operators of rank two corresponding to an elliptic spectral curves. These operators were studied in the papers [8]–[16]. Mokhov [17], using the same method found operators of rank three also corresponding to elliptic spectral curves. Besides this there are examples of operators of rank greater than one corresponding to spectral curves of genus 2, 3 and 4 (see [18]–[21]).

The main results of this paper are the following. We consider a pair \( L_4, L_{4g+2} \) of commuting differential operators of rank two whose spectral curve is a hyperelliptic curve \( \Gamma \) of genus \( g \)

\[
w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \cdots + c_0.
\]

Operators \( L_4 \) and \( L_{4g+2} \) satisfy the equation \( (L_{4g+2})^2 = F_g(L_4) \). The curve \( \Gamma \) has a holomorphic involution

\[
\sigma : \Gamma \to \Gamma, \quad \sigma(z, w) = (z, -w).
\]

Common eigenfunctions of \( L_4 \) and \( L_{4g+2} \) satisfy the second order differential equation [5]

\[
\psi''(x, P) = \chi_1(x, P)\psi'(x, P) + \chi_0(x, P)\psi(x, P).
\]

The coefficients \( \chi_0(x, P), \chi_1(x, P) \) are rational functions on \( \Gamma \) with \( 2g \) simple poles depending on \( x \), \( \chi_0 \) has also an additional simple pole at infinity. These functions satisfy Krichever’s equations (see below). To find operators \( L_4, L_{4g+2} \) it is enough to find \( \chi_0, \chi_1 \).

It is not difficult to prove that if \( \chi_1 \) is invariant under the involution \( \sigma \), then the operator \( L_4 \) is self-adjoint. S.P. Novikov has proposed the conjecture that the inverse is also true. In this paper we prove this conjecture.

**Theorem 1** The operator \( L_4 \) is self-adjoint if and only if

\[
\chi_1(x, P) = \chi_1(x, \sigma(P)).
\]

At \( g = 1 \) Theorem 1 was proved by Grinevich and Novikov [8]. Let us assume that the operator \( L_4 \) is self-adjoint

\[
L_4 = (\partial_x^2 + V(x))^2 + W(x),
\]

then the functions \( \chi_0, \chi_1 \) have simple poles at some points

\[
(\gamma_i(x), \pm \sqrt{F_g(\gamma_i(x)))}, 1 \leq i \leq g.
\]

In the next theorem we find the form of \( \chi_0(x, P), \chi_1(x, P) \).

**Theorem 2** If operator \( L_4 \) is self-adjoint, then

\[
\chi_0 = -\frac{1}{2} \frac{Q''}{Q} + \frac{w}{Q} - V, \quad \chi_1 = \frac{Q'}{Q},
\]

where \( Q = (z - \gamma_1(x)) \ldots (z - \gamma_g(x)) \). Functions \( Q, V, W \) satisfy the equation

\[
4F_g(z) = 4(z - W)Q^2 - 4V(Q')^2 + (Q'')^2 - 2Q'Q'' + 2Q(2V'Q' + 4VQ'' + Q^{(4)}),
\]
where $Q', Q'', Q^{(k)}$ mean $\partial_x Q, \partial_x^2 Q, \partial_x^k Q$.

To find self-adjoint operators $L_4, L_{4g+2}$ it is enough to solve the equation (4).

In this paper we find partial solutions of the equation for arbitrary $g$. These solutions correspond to operators with polynomial coefficients.

**Theorem 3** The operator

$$L_4^1 = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x, \quad \alpha_3 \neq 0$$

commutes with a differential operator $L_{4g+2}^1$ of order $4g + 2$. The operators $L_4^1$, $L_{4g+2}^1$ are operators of rank two. For generic values of parameters $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ the spectral curve is a nonsingular hyperelliptic curve of genus $g$.

If $g = 1$, $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, then the operators $L_4^1, L_{4g+2}^1$ coincide with the famous Dixmier operators [22] whose spectral curve is an elliptic curve. Operators $L_4^1, L_{4g+2}^1$ define commutative subalgebras in the first Weyl algebra $A_1$. Theorem 3 means that the equation $Y^2 = X^{2g+1} + c_g X^{2g} + \cdots + c_0$ has nonconstant solutions $X, Y \in A_1$ for some $c_i$. It is easy to see that the group $\text{Aut}(A_1)$ preserves the space of all such solutions. It would be very interesting to describe the orbits of $\text{Aut}(A_1)$ in the space of solutions under the action of $\text{Aut}(A_1)$. This gives a chance to compare $\text{End}(A_1)$ and $\text{Aut}(A_1)$ (the Dixmier conjecture is: $\text{End}(A_1) = \text{Aut}(A_1)$).

In Section 2 we recall the method of deformations of Tyurin parameters. In Sections 3–5 we prove Theorems 1–3.

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## 2. Operators of rank $l > 1$

Common eigenfunctions of commuting differential operators are Baker–Akhiezer functions. Let me recall the definition of the Baker–Akhiezer function at $l > 1$ [5]. We take the spectral data

$$\{\Gamma, q, k^{-1}, \gamma, \nu, \omega(x)\},$$

where $\Gamma$ is a Riemann surface of genus $g$, $q$ is a fixed point on $\Gamma$, $k^{-1}$ is a local parameter near $q$,

$$\omega(x) = (\omega_0(x), \ldots, \omega_{l-2}(x))$$

is a set of smooth functions, $\gamma = \gamma_1 + \cdots + \gamma_\ell$ is a divisor on $\Gamma$, $\nu$ is a set of vectors

$$v_1, \ldots, v_\ell, \quad v_i = (v_{i,1}, \ldots, v_{i,l-1}).$$

The pair $(\gamma, \nu)$ is called the Tyurin parameters. The Tyurin parameters define a stable holomorphic vector bundle on $\Gamma$ of rank $l$ and degree $lq$ with holomorphic sections $\eta_1, \ldots, \eta_l$. The points $\gamma_1, \ldots, \gamma_\ell$ are the points of the linear dependence

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} v_{j,i} \eta_j(\gamma_i).$$

The vector-function $\psi = (\psi_1, \ldots, \psi_l)$ is defined by the following properties.

1. In the neighbourhood of $q$ the vector-function $\psi$ has the form

$$\psi(x, P) = \left(\sum_{s=0}^{\infty} \xi_s(x) k^{-s}\right) \Psi_0(x, k),$$

where $Q', Q'', Q^{(k)}$ mean $\partial_x Q, \partial_x^2 Q, \partial_x^k Q$. 
where $\xi_0 = (1, 0, \ldots, 0), \xi_i(x) = (\xi_1^i(x), \ldots, \xi_l^i(x))$, the matrix $\Psi_0$ satisfies the equation
\[
\frac{d\Psi_0}{dx} = A\Psi_0, \quad A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
k + \omega_0 & \omega_1 & \omega_2 & \cdots & \omega_{l-2} & 0
\end{pmatrix}.
\]

2. The components of $\psi$ are meromorphic functions on $\Gamma \setminus \{q\}$ with the simple poles $\gamma_1, \ldots, \gamma_{lg}$, and
\[
\text{Res}_{\gamma_j} \psi_j = v_{i,j} \text{Res}_{\gamma_i} \psi_l, \quad 1 \leq i \leq lg, \quad 1 \leq j \leq l - 1.
\]

For the rational function $f(P)$ on $\Gamma$ with the unique pole of order $n$ at $q$ there is a linear differential operator $L(f)$ of order $ln$ such that
\[
L(f)\psi(x, P) = f(P)\psi(x, P).
\]

For two such functions $f(P), g(P)$ operators $L(f), L(g)$ commute.

The main difficulty to construct operators of rank $l > 1$ is the fact that the Baker–Akhiezer function is not found explicitly. But the operators can be found by the method of deformation of Tyurin parameters.

The common eigenfunctions of commuting differential operators of rank $l$ satisfy the linear differential equation of order $l$
\[
\psi^{(l)}(x, P) = \chi_0(x, P)\psi(x, P) + \cdots + \chi_{l-1}(x, P)\psi^{(l-1)}(x, P).
\]

Coefficients $\chi_j$ are rational functions on $\Gamma$ [5] with simple poles $P_1(x), \ldots, P_{lg}(x) \in \Gamma$, and with the following expansions in the neighbourhood of $q$
\[
\chi_0(x, P) = k + g_0(x) + O(k^{-1}),
\]
\[
\chi_j(x, P) = g_j(x) + O(k^{-1}), \quad 0 < j < l - 1,
\]
\[
\chi_{l-1}(x, P) = O(k^{-1}).
\]

Let $k - \gamma_i(x)$ be a local parameter near $P_i(x)$. Then
\[
\chi_j = \frac{c_{i,j}(x)}{k - \gamma_i(x)} + d_{i,j}(x) + O(k - \gamma_i(x)).
\]

Functions $c_{i,j}(x), d_{i,j}(x)$ satisfy the following equations [5].

**Theorem 4**

(5) $c_{i,l-1}(x) = -\gamma'_i(x)$,

(6) $d_{i,0}(x) = v_{i,0}(x)v_{i,l-2}(x) + v_{i,0}(x)d_{i,l-1}(x) - v'_{i,0}(x)$,

(7) $d_{i,j}(x) = v_{i,j}(x)v_{i,l-2}(x) - v_{i,j-1}(x) + v_{i,j}(x)d_{i,l-1}(x) - v'_{i,j}(x), j \geq 1$,

where
\[
v_{i,j}(x) = \frac{c_{i,j}(x)}{c_{i,l-1}(x)}, \quad 0 \leq j \leq l - 1, \quad 1 \leq i \leq lg.
\]

To find $\chi_i$ one should solve equations (5)–(7).
3. Proof of Theorem 1

In the case of operators of rank two the common eigenfunctions of $L_4$ and $L_{4g+2}$ satisfy equation (2). In the neighbourhood of $q$ we have the expansions

\begin{equation}
\chi_0 = \frac{1}{k} + a_0(x) + a_1(x)k + O(k^2), \quad \chi_1 = b_1(x)k + b_2(x)k^2 + O(k^3).
\end{equation}

Functions $\chi_0, \chi_1$ have $2g$ simple poles $P_1(x), \ldots, P_{2g}(x)$, and by Theorem 4

\begin{equation}
\chi_0(x, P) = \frac{-v_{i,0}(x)\gamma_i'(x)}{k - \gamma_i(x)} + d_{i,0}(x) + O(k - \gamma_i(x)),
\end{equation}

\begin{equation}
\chi_1(x, P) = \frac{-\gamma_i'(x)}{k - \gamma_i(x)} + d_{i,1}(x) + O(k - \gamma_i(x)),
\end{equation}

\begin{equation}
d_{i,0}(x) = v_{i,0}^2(x) + v_{1,0}(x)d_{1,1}(x) - v_{1,0}^4(x).
\end{equation}

Let $\Gamma$ be the hyperelliptic spectral curve (1), $q = \infty \in \Gamma$, $k = \sqrt{2}$. Let us find coefficients of the operator of order 4 corresponding to $z$, $L_4\psi = z\psi$.

**Lemma 1** The operator $L_4 = \partial_z^4 + f_2(x)\partial_x^2 + f_1(x)\partial_x + f_0(x)$ has the following coefficients:

\begin{equation}
f_0 = a_0^2 - 2a_1 - 2b_1' - a''_0, \quad f_1 = -2(b_1 + a''_0), \quad f_2 = -2a_0.
\end{equation}

Operator $L_4$ is self-adjoint if and only if $b_1 = 0$, herewith $L_4 = (\partial_z^2 + V(x))^2 + W(x)$, where $V(x) = -a_0(x)$, $W = -2a_1(x)$.

**Proof.** From (2) it follows that the fourth derivative of $\psi$ is

\begin{equation}
\psi^{(4)} = (\chi_0^2 + \chi_1\chi_0' + \chi_0(\chi_1^2 + 2\chi_1') + \chi_0'')\psi + (\chi_1^3 + 2\chi_1' + \chi_1(2\chi_0 + 3\chi_1') + \chi_1'')\psi'.
\end{equation}

With the help of (2) and the last equality we rewrite $L_4\psi = z\psi$ in the form

\begin{equation}
P_1\psi + P_2\psi' = z\psi,
\end{equation}

where

\begin{equation}
P_1 = f_0 + f_2\chi_0 + \chi_0^3 + \chi_1\chi_0' + \chi_0(\chi_1^2 + 2\chi_1') + \chi_0'',
\end{equation}

\begin{equation}
P_2 = f_1 + f_2\chi_1 + \chi_1^3 + 2\chi_1' + \chi_1(2\chi_0 + 3\chi_1') + \chi_1''.
\end{equation}

This gives

\begin{equation}
P_1 = z = \frac{1}{k^2}, \quad P_2 = 0.
\end{equation}

From (8) we have

\begin{equation}
P_1 - \frac{1}{k^2} = \frac{f_2 + 2a_0}{k} + (f_0 + a_0(f_2 + a_0) + 2(a_1 + b_1') + a''_0) + O(k) = 0,
\end{equation}

\begin{equation}
P_2 = (f_1 + 2(b_1 + a''_0)) + O(k) = 0.
\end{equation}

From here we find the coefficients of $L_4$.

Operator $L_4$ is self-adjoint if $f_1 = f''_2$, i.e. at $b_1 = 0$. Lemma 1 is proved.

If $\chi_1$ satisfies (3) then $\chi_1 = \sum_{s>1} b_{2s}k^{2s}$, hence, by Lemma 1 $L_4$ is self-adjoint.

Let us prove the inverse part of Theorem 1. We assume that $L_4$ is self-adjoint

\begin{equation}
L_4 = L_4^* = \partial_x^4 + f_2(x)\partial_x^2 + f_1'(x)\partial_x + f_0(x).
\end{equation}
If \( \psi_1, \psi_2 \in \text{Ker}(L_4 - z) \), then
\[
\psi_1 L_4 \psi_2 - \psi_2 L_4 \psi_1 = \partial_x (\psi_1 \psi''_2 - \psi_2 \psi''_1 - (\psi'_1 \psi''_2 - \psi'_2 \psi''_1)) + f_2 (\psi_1 \psi'_2 - \psi_2 \psi'_1) = 0.
\]

Hence, on the space \( \text{Ker}(L_4 - z) \) the following skew-symmetric bilinear form
\[
(.,.) : \text{Ker}(L_4 - z) \times \text{Ker}(L_4 - z) \to \mathbb{C},
\]
\[
(\psi_1, \psi_2) = \psi_1 \psi''_2 - \psi_2 \psi''_1 - (\psi'_1 \psi''_2 - \psi'_2 \psi''_1) + f_2 (\psi_1 \psi'_2 - \psi_2 \psi'_1)
\]
is defined. Let \( \psi_1(x, P), \psi_2(x, P) \) satisfy the equation (2). Using
\[
\psi''_i = (\chi_0 + \chi_1^2 + \chi_1') \psi'_i + (\chi_0 \chi_1 + \chi_0') \psi_i
\]
we get
\[
(\psi_1, \psi_2) = (\psi_1 \psi'_2 - \psi_2 \psi'_1)(f_2 + 2\chi_0 + \chi_2^2 + \chi_1^2).
\]
Since \( \psi_1, \psi_2 \) satisfy the second order differential equation (2) we have,
\[
(\psi_1, \psi_2) = e \int \chi_1(x, z) dx g_1(z, w) \left( f_2(x) + 2\chi_0(x, z, w) + \chi_1^2(x, z, w) + \chi_1'(x, z, w) \right)
\]
\[
= g_2(z, w),
\]
where \( g_1(z, w), g_2(z, w) \) are some functions on \( \Gamma \). Let us represent \( \chi_1 \) in the form
\[
\chi_1(x, z, w) = G_1(x, z) + w G_2(x, z),
\]
where \( G_1, G_2 \) are rational functions on \( \Gamma \). Let
\[
\tilde{G}_1(x, z) = \int G_1(x, z) dx, \quad \tilde{G}_2(x, z) = \int G_2(x, z) dx,
\]
then
\[
e^{\tilde{G}_1(x, z)} \left( e^{\tilde{G}_2(x, z)} \right)^w \frac{g_1(z, w)}{g_2(z, w)} = \frac{1}{f_2 + 2\chi_0 + \chi_2^2 + \chi_1^2}.
\]
From the last identity it follows that for arbitrary \( x = x_1, x = x_2 \) the function
\[
e^{\tilde{G}_1(x_1, z) - \tilde{G}_1(x_2, z)} \left( e^{\tilde{G}_2(x_1, z) - \tilde{G}_2(x_2, z)} \right)^w
\]
is a rational function on \( \Gamma \). This is possible only if
\[
\tilde{G}_2(x_1, z) - \tilde{G}_2(x_2, z) = 0,
\]
or equivalent \( G_2 = 0 \). Hence, \( \chi_1 = G_1(x, z) \). This means that \( \chi_1 \) is invariant under the involution \( \sigma \). Thus, Theorem 1 is proved.

4. Proof of Theorem 2

Assume that \( \chi_1 \) is invariant under \( \sigma \), then by (8)–(10) we have
\[
\chi_0 = \frac{H_1(x)}{z - \gamma_1(x)} + \ldots + \frac{H_g(x)}{z - \gamma_g(x)} + \frac{w(z)}{(z - \gamma_1(x)) \ldots (z - \gamma_g(x))} + \kappa(x),
\]
\[
\chi_1(x, P) = -\frac{\gamma'_1(x)}{(z - \gamma_1(x))} - \ldots - \frac{\gamma'_g(x)}{(z - \gamma_g(x))},
\]
where \( H_i(x), \kappa(x) \) are some functions. In the neighbourhood of \( q \) the function \( \chi_0 \) has the expansion
\[
\chi_0 = \frac{1}{k} + \kappa + \left( \gamma_1 + \ldots + \gamma_g + \frac{c_2 g}{2} \right) k + O(k^2).
\]

Hence, by Lemma 1
\[
V = -\kappa, \quad W = -2(\gamma_1 + \ldots + \gamma_g) - c_{2g}.
\]
Thus
\[ \chi_0 = \frac{Q_1}{Q} + \frac{w}{Q} - V(x) \]
\[ \chi_1(x, P) = \frac{Q'}{Q}. \]

Let us substitute \( \chi_0, \chi_1 \) into (12). From \( P_2 = 0 \) we get \( Q_1 = -\frac{Q'' + s}{2} \), where \( s \) is a constant. From \( P_1 = z \) we get
\[ s^2 - 4sw + 4w^2 - 4(z - W)Q^2 + 4V(Q')^2 - (Q'')^2 + 2Q'Q^{(3)} \\
- 2Q(2VQ' + 4VQ'' + Q^{(4)}) = 0. \]
The last identity is possible only if \( s = 0 \) because \( Q \) is a polynomial in \( z \). Theorem 2 is proved.

Let us differentiate (4) in \( x \) and divide the result by \( Q \). We get the following equation.

**Corollary 1** The functions \( Q, W, V \) satisfy the equation
\[ Q^{(5)} + 4VQ^3 + 2Q'(2z - 2W - V'') + 6V'Q'' - 2QW' = 0. \]

Let us substitute \( z = \gamma_j \) in (4). It gives
\[ V(x) = \left( \frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} \right) |_{z=\gamma_j} \cdot \]
We get \( g - 1 \) equations on \( \gamma_1(x), \ldots, \gamma_g(x) \).

**Corollary 2** The functions \( \gamma_1(x), \ldots, \gamma_g(x) \) satisfy the equations
\[ \left( \frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} \right) |_{z=\gamma_j} = \left( \frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} \right) |_{z=\gamma_k} \cdot \]

### 5. Proof of Theorem 3

Let
\[ \chi_0 = -\frac{1}{2} \frac{Q''}{Q} + \sqrt[\gamma_g(z)]{\frac{F_g(z)}{Q}} - (\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0), \]
\[ \chi_1 = \frac{Q'}{Q}. \]

Let us consider the equations (4) where \( V, W \) are potentials of the operator \( L_4' \)
\[ 4F_g(z) = 4(z - g(g + 1)\alpha_3 x)Q^2 - 4(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)(Q')^2 + (Q'')^2 - 2Q'Q^{(3)} \\
+ 2Q(2(3\alpha_3 x^2 + 2\alpha_2 x + \alpha_1)Q' + 4(\alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)Q'' + Q^{(4)}). \]

We prove that the nonlinear equation (16) has a polynomial solution \( Q(x, z) \) of degree \( g \) in \( z \) and degree \( g \) in \( x \) for some polynomial \( F_g(z) \). After that we prove that \( \chi_0, \chi_1 \) satisfy (11) for the curve \( w^2 = F_g(z) \). The functions \( \chi_0, \chi_1 \) have required asymptotic (8) in \( q = \infty \). From here it follows that \( L_4' \) commutes with an operator of order \( 4g + 2 \) corresponding to the rational function \( w \) on \( \Gamma \) with the unique pole of order \( 2g + 1 \) at \( q \).
Lemma 2  Equation (16) has a solution of the form
(17) \[ Q = (z - \gamma_1(x)) \cdots (z - \gamma_g(x)), \]
for some polynomial \( F_g(z) \) of degree \( 2g + 1 \).

Proof. Let us represent \( Q \)

\[ Q = (z - \gamma_1(x)) \cdots (z - \gamma_g(x)), \]

where \( \gamma_i \) are polynomials in \( x \).

From (16) we have

\[ Q'(s) + 6(3\alpha_3 s^2 + 2\alpha_2 s + \alpha_1)Q'' + 4(\alpha_2 - (6^2 + g - 3)\alpha_3 s + z)Q' \]

(18) \[ + 6(3\alpha_3 s^2 + 2\alpha_2 s + \alpha_1)Q'' - 2g(g + 1)\alpha_3 Q = 0. \]

We find a solution of (18) as a polynomial in \( z \)

\[ Q = \delta_g z^g + \cdots + \delta_1 z + \delta_0, \quad \delta_i = \delta_i(z). \]

From (18) we have

\[ \delta_s = \frac{(s + 1)}{\alpha_3(g - s)(s + g + 1)(2s + 1)}(2(\alpha_2(s + 1)^2 + z))\delta_{s+1} + \alpha_1(s + 2)(2s + 3)\delta_{s+2} \]

(20) \[ + 2\alpha_0(s + 2)(s + 3)\delta_{s+3} + 1/2(s + 2)(s + 3)(s + 4)(s + 5)\delta_{s+5}, \]

where \( 0 \leq s < g - 1 \), \( \delta_g \) is a constant, and \( \delta_s = 0 \) at \( s > g \). In particular

\[ \delta_{g-1} = \frac{\delta_g(\alpha_2 g^2 + z)}{\alpha_3(2g - 1)}. \]

From (20) it follows that \( Q \) is a polynomial of degree \( g \) in \( z \), and up to the multiplication by a constant, the polynomial \( Q \) has the form (17). The right-hand side of (16) has degree \( 2g + 1 \). Lemma 2 is proved.

Lemma 3  The polynomial \( Q \) has no multiple root in \( z \)

\[ \gamma_i \neq \gamma_j \text{ at } i \neq j. \]

Proof. Let us represent \( Q \) in the form

\[ Q = Q_H + \tilde{Q}, \]

where \( Q_H \) is a homogeneous polynomial in \( x, z \)

\[ Q_H = \tilde{\gamma}_g x^g + \tilde{\gamma}_{g-1} x^{g-1} z + \tilde{\gamma}_{g-2} x^{g-2} z^2 + \cdots + \tilde{\gamma}_0 z^g, \quad \tilde{\gamma}_0, \tilde{\gamma}_g \neq 0 \]

and \( \operatorname{deg} \tilde{Q} < g \). Since \( \tilde{\gamma}_g \neq 0 \), the polynomial \( Q \) has no constant roots (i.e. \( \gamma_i \neq \text{const} \)).

Let us note that \( Q \) has no multiple roots of order higher than 2. Indeed, if \( Q = (z - \gamma_i(x))^p \tilde{Q} \), \( p > 2 \), then from (16) \( F_g(\gamma_i(x)) = 0 \), but this is impossible.

If \( Q \) has multiple roots, then \( Q_H \) also has multiple roots. This follows from the following fact. The discriminant of \( Q \) is a polynomial \( b_N x^N + b_{N-1} x^{N-1} + \cdots + b_0 \) in \( x \). The discriminant of \( Q_H \) is \( b_N x^N \), so if the discriminant of \( Q \) is equal to zero, then the discriminant of \( Q_H \) is also zero.

From (20) it follows that

\[ \tilde{\delta}_s = \frac{2(s + 1)\delta_{s+1}}{\alpha_3(g - s)(s + g + 1)(2s + 1)}, \quad 0 \leq s \leq g - 1, \]
and that $Q_H$ satisfies the equation
\[ 2\alpha_3 x^3 Q_H^{(3)} + 2((3 - g - g^2)\alpha_3 x + z)Q_H' + 9\alpha_3 x^2 Q_H'' - g(g + 1)\alpha_3 Q_H = 0. \]

Let us multiply this equation by $Q_H$ and integrate in $x$. We get
\[ \tilde{F}_g(z) = (g(g + 1)\alpha_3 x - z)Q_H^2 + \alpha_3 x^3 (Q_H')^2 - \alpha_3 x^2 Q_H (3Q_H' + 2xQ_H'') = 0, \]
where $\tilde{F}_g(z)$ is a polynomial of degree $2g + 1$ in $z$.

From the last equation it follows that if $Q_H$ has multiple roots, then the polynomial $\tilde{F}_g(z)$ has the same roots. However, this is impossible, because all roots of $\tilde{F}_g(z)$ are constant, but $Q_H$ has no constant roots. Lemma 3 is proved.

**Lemma 4** If $(\alpha_0, \ldots, \alpha_3) \in U$, the curve $w^2 = F_g(z)$ is nonsingular, where $U \subset \mathbb{C}^4$ is some Zariski open set.

**Proof.** The idea of the proof is the following. We represent $F_g$ in the form
\[ F_g(z) = F_g^0(z) + \alpha_3 F_g^1(z) + O(\alpha_3^2), \]
and prove that $F_g^0(z) + \alpha_3 F_g^1(z)$ has no multiple roots. Therefore, $F_g(z)$ has no multiple roots for small $\alpha_3$, and consequently for $(\alpha_0, \ldots, \alpha_3) \in U$.

Let us consider (19)–(21). We put
\[ \delta_g = \alpha_3^0, \]

Moreover, from (19) it follows that $Q$ has the form
\[ Q = \alpha_3^g x^g + \cdots + \alpha_3^0 x^0 (p_0(z) + \alpha_3 q_0(z) + O(\alpha_3^2)) + \cdots + (p_0(z) + \alpha_3 q_0(z) + O(\alpha_3^2)). \]

Let us note that from (21) it follows that
\[ p_g = 1, \quad p_{g-1} = \frac{\alpha_2 g^2 + z}{2g-1}, \quad q_g = 0, \quad q_{g-1} = 0. \]

Let us substitute (22) into (16). We get
\[ F_g(z) = p_0^2(z) z + \alpha_3 p_0(z)(\alpha_1 p_1(z) + 2q_0(z)z) + O(\alpha_3^2), \]
so,
\[ F_g^0(z) = p_0^2(z) z, \quad F_g^1(z) = p_0(z)(\alpha_1 p_1(z) + 2q_0(z)z). \]

To prove Lemma 4 it is enough to prove that $p_0(z)z$ and $\alpha_1 p_1(z) + 2q_0(z)z$ have no common roots.

Let us find $p_i$ and $q_i$. For this we again substitute (22) into (18) and find the coefficients at $\alpha_3^{i+1} x^i$ and $\alpha_3^{i+2} x^i$. These coefficients must be equal to zero. It gives us
\[ p_i = \frac{2(i + 1)(\alpha_2 (i + 1)^2 + z)}{(2i + 1)(g^2 + g - i^2 - i)} p_{i+1}, \quad 0 \leq i < g - 1, \]
\[ q_i = \frac{2(i + 1)(\alpha_2 (i + 1)^2 + z)}{(g - i)(g + i + 1)(2i + 1)} q_{i+1} + \frac{\alpha_1 (i + 1) (i + 2)(2i + 3)}{(g - i)(g + i + 1)(2i + 1)} p_{i+2}, \]
where $0 \leq i < g - 2$. Hence
\[ p_i(z) = (\alpha_2 (i + 1)^2 + z) \cdots (\alpha_2 g^2 + z) A_i, \quad 0 \leq i \leq g - 1, \]
where $A_i$ is a constant. Thus to prove that $p_0(z)z$ and $\alpha_1 p_1(z) + 2q_0(z)z$ have no common roots we should prove that $z = -\alpha_2 2^2, \ldots, z = -\alpha_2 g^2$ are not roots of
Examples.

From (15) we have Lemma 4 is proved.

Let \( z \) be a local parameter near \( q \). Hence, if \( q_0(−α_2g^2) = 0 \), then \( q_{g−2}(−α_2g^2) = 0 \), but this is impossible, since \( p_g = 1 \), so \( s < g \).

Formulas (23), (24) at \( i = s − 2, i = s − 1 \) give us

\[
q_{s−2} \frac{2(s−1)(α_2(s−1)^2+z)}{(g−s+2)(g+s−1)(2s−3)} \frac{2s(α_2s^2+z)q_s + α_1s(s+1)(2s+1)p_{s+1}}{(g−s+1)(g+s)(2s−1)}
\]

\[
\frac{α_1(s−1)s(2s−1)}{(g−s+2)(g+s−1)(2s−3)} \frac{2s+1)(α_2s^2+z)}{(2s+1)(g^2+g−s^2−s)p_{s+1}} = 0.
\]

Let \( z \) be \( −α_2s^2 \). After the simplification we have

\[
g^2 + g − 3s^2 = 0.
\]

This is impossible, hence \( q_0(−α_2s^2) \) \( \neq 0 \) and \( F_g^0(−α_2s^2) \) has no multiple roots. Lemma 4 is proved.

Functions \( χ_0, χ_1 \) are rational functions on the curve \( w^2 = F_g(z) \). Let \( k = \frac{1}{√z} \) be a local parameter near \( q = ∞ \). Functions \( χ_0, χ_1 \) have asymptotic (3). By Lemma 3 \( χ_0 \) and \( χ_1 \) have simple poles \( P^±_i = (γ_i, ±√F_g(γ_i)) \). Let us choose in the neighbourhood of \( P^±_i \) the local parameter \( z − γ_i(x) \).

**Lemma 5** Functions \( χ_0, χ_1 \) satisfy the equation (11).

**Proof.** From (15) we have

\[
χ_1(x, P) = \frac{−γ_i'(x)}{z − γ_i(x)} + d_{i,1}(x) + O(z − γ_i(x))
\]

for some \( d_{i,1}(x) \). Function \( χ_1 \) has simple poles at \( γ_i(x) \), thus

\[
χ_0(x, P) = \frac{−v_{i,0}(x)γ_i'(x)}{z − γ_i(x)} + d_{i,0}(x) + O(z − γ_i(x)),
\]

for some \( v_{i,0}(x), d_{i,0}(x) \). By our construction \( χ_0, χ_1 \) satisfy (12). Let us substitute \( χ_0, χ_1 \) in (12). We get

\[
\frac{(v_{i,0}^2(x) − d_{i,0}(x) + d_{i,1}(x)v_i(x) − v_i'(x)(γ_i'(x))^2)}{(z − γ_i(x))^2} + O\left(\frac{1}{z − γ_i(x)}\right) = 0.
\]

Hence \( d_{i,0}(x), d_{i,1}(x), v_{i,0}(x) \) satisfy (11). Lemma 5 and Theorem 3 are proved.

Operator \( L^s_{4g+2} \) commuting with \( L^s_4 \) can be found from \( L^s_4L^s_{4g+2} = L^s_{4g+2}L^s_4 \). For the simplicity of the formulas we restrict ourselves to the case \( α_1 = α_2 = 0, α_3 = 1 \). Let us introduce the notations: \( H = ∂_x^2 + x^3 + α_0, \, ⟨A, B⟩ = AB + BA \).

**Examples.**
a) \( g = 2 \):
\[
L^s_{10} = H^5 + \frac{15}{2} \langle x, H^3 \rangle + 45 \langle x^2, H \rangle,
\]
\[
F_2(z) = z^5 + 27\alpha_0 z^2 + 81.
\]

b) \( g = 3 \):
\[
L^s_{14} = H^7 + 21 \langle x, H^5 \rangle + \frac{945}{2} \langle x^2, H^3 \rangle - \frac{5418}{2} \langle 113\alpha_0 + 287x^3, H \rangle - 486x,
\]
\[
F_3 = z^7 + 594\alpha_0 z^4 - 2025z^2 + 91125\alpha_0^2 z.
\]

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