G_2-HOLONOMY METRICS CONNECTED WITH A 3-SASAKIAN MANIFOLD

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Abstract. We construct complete noncompact Riemannian metrics with \( G_2 \)-holonomy on noncompact orbifolds that are \( \mathbb{R}^3 \)-bundles with the twistor space \( Z \) as a spherical fiber.

1. Introduction

This article addressing \( G_2 \)-holonomy metrics is a natural continuation of the study of \( \text{Spin}(7) \)-holonomy metrics which was started in [1]. We consider an arbitrary 7-dimensional compact 3-Sasakian manifold \( M \) and discuss the existence of a smooth resolution of the conic metric over the twistor space \( Z \) associated with \( M \).

Briefly speaking, a manifold \( M \) is 3-Sasakian if and only if the standard metric on the cone over \( M \) is hyper-Kahler. Each manifold of this kind \( M \) is closely related to the twistor space \( Z \) which is an orbifold with a Kahler–Einstein metric. We consider the metrics that are natural resolutions of the standard conic metric over \( Z \):

\[
\bar{g} = dt^2 + A(t)^2(\eta_2^2 + \eta_3^2) + B(t)^2(\eta_4^2 + \eta_5^2) + C(t)^2(\eta_6^2 + \eta_7^2),
\]

where \( \eta_2 \) and \( \eta_3 \) are the characteristic 1-forms of \( M \), \( \eta_4, \eta_5, \eta_6, \) and \( \eta_7 \) are the forms that annul the 3-Sasakian foliation on \( M \), and \( A, B, \) and \( C \) are real functions.

One of the main results of the article is the construction (in the case when \( M/SU(2) \) is Kahler) of a \( G_2 \)-structure which is parallel with respect to (**) if and only if the following system of ordinary differential equations is satisfied:

\[
\begin{align*}
A' &= \frac{2A^2 - B^2 - C^2}{B^2 - C^2 - 2A^2}, \\
B' &= \frac{C^2 - 2A^2 - B^2}{A^2 - B^2}, \\
C' &= \frac{C^2 - 2A^2 - B^2}{A^2 - B^2}.
\end{align*}
\]

In case (**) we thus see that (**) has holonomy \( G_2 \); hence, (**) is Ricci-flat. The system of equations (**) was previously obtained in [2] in the particular case \( M = SU(3)/S^1 \).

For a solution to (**) to be defined on some orbifold or manifold, some additional boundary conditions are required at \( t_0 \) that we will state them later. These conditions cannot be satisfied unless \( B = C \), which leads us to the functions that give rise to the solutions found originally in [3] when \( M = S^7 \) and \( M = SU(3)/S^1 \). If \( B = C \) then (**) is defined on the total space of an \( \mathbb{R}^3 \)-bundle \( N \) over a quaternionic-Kahler orbifold \( O \). In general, \( N \) is an orbifold except in the event that \( M = S^7 \) and \( M = SU(3)/S^1 \). Note that it is unnecessary for \( O \) to be Kahler in case \( B = C \).
2. Construction of a Parallel $G_2$-Structure

The definition of 3-Sasakian manifolds, their basic properties, and further references can be found in [1]. We mainly take our notation from [1].

Let $M$ be a 7-dimensional compact 3-Sasakian manifold with characteristic fields $\xi^1$, $\xi^2$, and $\xi^3$ and characteristic 1-forms $\eta_1$, $\eta_2$, and $\eta_3$. Consider the principal bundle $\pi: \tilde{M} \to \mathcal{O}$ with the structure group $Sp(1) \times SO(3)$ over the quaternionic-Kahler orbifold $\mathcal{O}$ associated with $M$. We are interested in the special case when $\mathcal{O}$ additionally possesses a Kahler structure.

The field $\xi^1$ generates a locally free action of the circle $S^1$ on $M$, and the metric on the twistor space $Z = M/S^1$ is a Kahler–Einstein metric. It is obvious that $Z$ is topologically a bundle over $\mathcal{O}$ with fiber $S^2 = Sp(1)/S^1$ (or $S^2 = SO(3)/S^1$) associated with $\pi$. Consider the obvious action of $SO(3)$ on $\mathbb{R}^3$. The two-fold cover $Sp(1) \to SO(3)$ determines the action of $Sp(1)$ on $\mathbb{R}^3$, too. Now, let $\tilde{N}$ be a bundle over $\mathcal{O}$ with fiber $\mathbb{R}^3$ associated with $\pi$. It is easy to see that $\mathcal{O}$ is embedded in $\tilde{N}$ as the zero section, and $Z$ is embedded in $\tilde{N}$ as a spherical section. The space $\tilde{N}\setminus\mathcal{O}$ is diffeomorphic to the product $Z \times (0, \infty)$. Note that $\tilde{N}$ can be assumed to be the projectivization of the bundle $\mathcal{M}_1 \to \mathcal{O}$ of [1]. In general, $\tilde{N}$ is a 7-dimensional orbifold; however, if $M$ is a regular 3-Sasakian space then $\tilde{N}$ is a 7-dimensional manifold.

Let $\{e^i\}, i = 0, 2, 3, \ldots, 7$, be an orthonormal basis of 1-forms on the standard Euclidean space $\mathbb{R}^7$ (the numeration here is chosen so as to emphasize the connection with the constructions of [1] and to keep the original notation wherever possible). Putting $\Psi_0 = \sum_{ijk} \epsilon_{ijk} e^i \wedge e^j \wedge e^k$, consider the following 3-form $\Psi_0$ on $\mathbb{R}^7$:

$$
\Psi_0 = -e^{023} - e^{045} + e^{067} + e^{346} - e^{375} - e^{247} + e^{256}.
$$

A differential 3-form $\Psi$ on an oriented 7-dimensional Riemannian manifold $N$ defines a $G_2$-structure if, for each $p \in N$, there exists an orientation-preserving isometry $\phi_p : T_pN \to \mathbb{R}^7$ defined in a neighborhood of $p$ such that $\phi_p^* \Psi_0 = \Psi |_p$. In this case the form $\Psi$ defines the unique metric $g_\Psi$ such that $g_\Psi(v, w) = \langle \phi_p v, \phi_p w \rangle$ for $v, w \in T_pN$ [3]. If the form $\Psi$ is parallel ($\nabla \Psi = 0$) then the holonomy group of the Riemannian manifold $N$ lies in $G_2$. The parallelness of the form $\Psi$ is equivalent to its closeness and co-closeness [3]:

$$
d\Psi = 0, \quad d^* \Psi = 0. \tag{1}
$$

Note that the form $\Psi_0 = e^1 \wedge \Psi_0 - * \Psi_0$, where $*$ is the Hodge operator in $\mathbb{R}^7$, determines a $Spin(7)$-structure on $\mathbb{R}^8$ with the orthonormal basis $\{e^i\}_{i=0,1,2,\ldots,7}$.

Locally choose an orthonormal system $\eta_4, \eta_5, \eta_6, \eta_7$ that generates the annihilator of the vertical subbundle $\mathcal{V}$ so that

$$
\omega_1 = 2(\eta_4 \wedge \eta_5 - \eta_6 \wedge \eta_7), \quad \omega_2 = 2(\eta_4 \wedge \eta_6 - \eta_7 \wedge \eta_5), \quad \omega_3 = 2(\eta_4 \wedge \eta_7 - \eta_5 \wedge \eta_6),
$$

where the forms $\omega_i$ correspond to the quaternionic-Kahler structure on $\mathcal{O}$. It is clear that $\eta_2, \eta_3, \ldots, \eta_7$ is an orthonormal basis for $\mathcal{M}$ annulling the one-dimensional
foliation generated by \( \xi^1 \); therefore, we can consider the metric of the following form on \((0, \infty) \times Z\):
\[
\tilde{g} = dt^2 + A(t)^2(\eta_2^2 + \eta_3^2) + B(t)^2(\eta_4^2 + \eta_5^2) + C(t)^2(\eta_6^2 + \eta_7^2).
\] (2)
Here \( A(t) \), \( B(t) \), and \( C(t) \) are defined on the interval \((0, \infty)\).

We suppose that \( \mathcal{O} \) is a Kahler orbifold; therefore, \( \mathcal{O} \) has the closed Kahler form that can be lifted to the horizontal subbundle \( \mathcal{H} \) as a closed form \( \omega \). Without loss of generality we can assume that we locally have
\[
\omega = 2(\eta_4 \wedge \eta_5 + \eta_6 \wedge \eta_7).
\]
If we now put
\[
e^0 = dt, \quad e^i = A\eta_i, \quad i = 2, 3, \quad e^j = B\eta_j, \quad j = 4, 5, \quad e^k = C\eta_k, \quad k = 6, 7,
\]
then the forms \( \Psi_0 \) and \( \ast \Psi_0 \) become
\[
\Psi_1 = -e^{023} - \frac{B^2 + C^2}{4}e^0 \wedge \omega_1 - \frac{B^2 - C^2}{4}e^0 \wedge \omega + \frac{BC}{2}e^3 \wedge \omega_2 - \frac{BC}{2}e^2 \wedge \omega_1,
\]
\[
\Psi_2 = C^2B^2\Omega - \frac{B^2 + C^2}{4}e^{23} \wedge \omega_1 - \frac{B^2 - C^2}{4}e^{23} \wedge \omega + \frac{BC}{2}e^{02} \wedge \omega_2 + \frac{BC}{2}e^{03} \wedge \omega_3,
\]
where \( \Omega = \eta_4 \wedge \eta_5 \wedge \eta_6 \wedge \eta_7 = -\frac{1}{8}\omega_1 \wedge \omega_1 = -\frac{1}{8}\omega_2 \wedge \omega_2 = -\frac{1}{8}\omega_3 \wedge \omega_3. \)

It is now obvious that \( \Psi_1 \) and \( \Psi_2 \) are defined globally and independently of the local choice of \( \eta_i \); consequently, they uniquely define the metric \( \tilde{g} \) given locally by (2). Then the condition (1) that the holonomy group lies in \( G_2 \) is equivalent to the equation
\[
d\Psi_1 = d\Psi_2 = 0. \tag{3}
\]

**Theorem.** If \( \mathcal{O} \) possesses a Kahler structure then (2) on \( \mathcal{N} \) is a smooth metric with holonomy \( G_2 \) given by the form \( \Psi_1 \) if and only if the functions \( A \), \( B \), and \( C \) defined on the interval \([t_0, \infty)\) satisfy the system of ordinary differential equations
\[
A' = \frac{2A^2 - B^2 - C^2}{BC}, \quad B' = \frac{B^2 - C^2 - 2A^2}{CA}, \quad C' = \frac{C^2 - 2A^2 - B^2}{AB} \tag{4}
\]
with the initial conditions
1. \( A(0) = 0 \) and \( |A'(0)| = 2; \)
2. \( B(0), C(0) \neq 0 \), and \( B'(0) = C'(0) = 0; \)
3. the functions \( A \), \( B \), and \( C \) have fixed sign on the interval \((t_0, \infty)\).

**Proof.**
In [1] the following relations were obtained, closing the algebra of forms:
\[
de e^0 = 0,
\]
\[
de e^i = \frac{A'_i}{A_i}e^0 \wedge e^i + A_i \omega_i - \frac{2A_i}{A_{i+1}A_{i+2}}e^{i+1} \wedge e^{i+2}, \quad i = 1, 2, 3 \mod 3,
\]
\[
d \omega_i = \frac{2}{A_{i+1}}\omega_{i+1} \wedge e^{i+2} - \frac{2}{A_{i+2}}e^{i+1} \wedge \omega_{i+2}, \quad i = 1, 2, 3 \mod 3.
\]
By adding the relation \( d\omega = 0 \) and carrying out some calculations to be omitted here, we obtain the sought system.

The smoothness conditions for the metric at \( t_0 \) are proven by analogy with the case of holonomy \( Spin(7) \) which was elaborated in [1]. We only note that, taking
the quotient of the unit sphere $S^3$ by the Hopf action of the circle, we obtain the sphere of radius $1/2$, which explains the condition $|A'(0)| = 2$.

In case $B = C$ the system reduces to the pair of equations
\[ A' = 2 \left( \frac{A^2}{B^2} - 1 \right), \quad B' = -2 \frac{A}{B} \]
whose solution gives the metric
\[ \bar{g} = \frac{dr^2}{1 - r_0^2/r^4} + r^2 \left( 1 - \frac{r_0^4}{r^4} \right) \left( \eta_2^2 + \eta_3^2 \right) + 2r^2 \left( \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 \right). \]
The regularity conditions hold. This smooth metric was originally found in [3] in the event that $M = SU(3)/S^1$ and $M = S^7$ (observe that we need not require $O$ to be Kahler when $B = C$).

In the general case $B \neq C$ system (4) can also be integrated [2]. However, the resulting solutions do not enjoy the regularity conditions.

### 3. Examples

Some interesting family of examples arises when we consider the 7-dimensional biquotients of the Lie group $SU(3)$ as 3-Sasakian manifolds. Namely, let $p_1$, $p_2$, and $p_3$ be pairwise coprime positive integers. Consider the following action of $S^1$ on the Lie group $SU(3)$:
\[ z \in S^1: A \mapsto \text{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \cdot A \cdot \text{diag}(1, 1, z^{-p_1-p_2-p_3}). \]
This action is free; moreover, it was demonstrated in [4] that there is a 3-Sasakian structure on the orbit space $S = S_{p_1,p_2,p_3}$. Moreover, the action of $SU(2)$ on $SU(3)$ by right translations
\[ B \in SU(2): A \mapsto A \cdot \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \]
commutes with the action of $S^1$ and can be pushed forward to the orbit space $S$. The corresponding Killing fields will be the characteristic fields $\xi_i$ on $S$. Therefore, the corresponding twistor space $Z = Z_{p_1,p_2,p_3}$ is the orbit space of the following action of the torus $T^2$ on $SU(3)$:
\[ (z, u) \in T^2: A \mapsto \text{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \cdot A \cdot \text{diag}(u, u^{-1}, z^{-p_1-p_2-p_3}). \quad (5) \]

**Lemma.** The space $Z_{p_1,p_2,p_3}$ is diffeomorphic to the orbit space of $U(3)$ with respect to the following action of $T^3$:
\[ (z, u, v) \in T^3: A \mapsto \text{diag}(z^{-p_2-p_3}, z^{-p_1-p_3}, z^{-p_1-p_2}) \cdot A \cdot \text{diag}(u, v, 1). \quad (6) \]

It suffices to verify that each $T^3$-orbit in $U(3)$ exactly cuts out an orbit of the $T^2$-action (5) in $SU(3) \subset U(3)$.

Action (6) makes it possible to describe the topology of $Z$ and, consequently, the topology of $N$ clearly. Here we use the construction of [3]. Consider the submanifold $E = \{(u,[v]) \mid u \perp v\} \subset S^5 \times CP^2$. It is obvious that $E$ is diffeomorphic to $U(3)/S^1 \times S^1$ (the "right" part of (6)) and is the projectivization of the $\mathbb{C}^2$-bundle
\( \bar{E} = \{(u,v) \mid u \perp v\} \subset S^5 \times \mathbb{C}^3 \) over \( S^5 \). By adding the trivial one-dimensional complex bundle over \( \bar{E} \), we obtain the trivial bundle \( S^5 \times \mathbb{C}^3 \) over \( S^5 \).

The group \( S^1 \) acts from the left by the automorphisms of the vector bundle \( \bar{E} \), and \( \mathcal{Z} = S^1 \backslash \bar{E} \) is the projectivization of the \( \mathbb{C}^2 \)-bundle \( S^1 \backslash \bar{E} \) over the weighted complex projective space \( \mathcal{O} = \mathbb{C}P^2(q_1, q_2, q_3) = S^1 \backslash S^5 \), where \( q_i = (p_{i+1} + p_{i+2})/2 \) for \( p_i \) all odd and \( q_i = (p_{i+1} + p_{i+2})/2 \) otherwise.

The above implies that the bundle \( S^1 \backslash \bar{E} \) is stably equivalent to the bundle \( S^1 \backslash (S^5 \times \mathbb{C}^3) \) over \( \mathcal{O} \). The last bundle splits obviously into the Whitney sum \( \sum_{i=1}^{3} \xi^{q_i} \), where \( \xi \) is an analog of the one-dimensional universal bundle of \( \mathcal{O} \).

**Corollary.** The twistor space \( \mathcal{Z} \) is diffeomorphic to the projectivization of a two-dimensional complex bundle over \( \mathbb{C}P^2(q_1, q_2, q_3) \) which is stably equivalent to \( \xi^{q_1} \oplus \xi^{q_2} \oplus \xi^{q_3} \).

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