A new Combined Sieve algorithm is presented with cost proportional to the number of enumerated factors over a series of intervals. This algorithm achieves a significant speedup, over a traditional sieve, when handling many ($[10^4, 10^7]$) intervals concurrently. The speedup comes from a space-time tradeoff and a novel solution to a modular equation. In real world tests, this new algorithm regularly runs 10,000x faster. This faster sieve paired with higher sieving limits eliminates more composites and accelerates the search for large prime gaps by 30-70%. During the development and testing of this new algorithm, two top-10 record merit prime gaps were discovered.

**Keywords** primes, prime differences, prime gaps, parallel sieve

1 Introduction

A prime gap is the difference between successive primes. A record is kept of the smallest known prime for each gap. For example the smallest prime followed by a gap of 4 is 7; a gap of 6 first follows 23; and a gap of 96 first follows 360,653. Until his death in 2019, Thomas R. Nicely maintained a list of record prime gaps, first in print and later on his faculty website [1]. Since 2019, a group from MersenneForum.org, including the author, has maintained the records on GitHub[2].

The search for prime gaps precedes computers but the digital era accelerated the search. Computer searches for prime gaps start circa 1930 when D.H. Lehmer constructed a physical computer from bicycle chains, The Lehmer Sieve, and used it to construct a table of the primes less than 37 million [3]. In 1961 Baker, Gruenberger, And Armerding added to the search in *Statistics on the First Six Million Prime Numbers* [4]. Both listed first occurrences of prime gaps. Lander and Parkin using a CDC 6600 extended the search and published a paper exclusively on prime differences in 1967 [5]. At least a dozen papers from as many authors have followed since.

This paper outlines a new algorithm that speeds up searches for large gaps by combining the sieving of many nonadjacent but regularly spaced intervals at a substantial speedup. With a faster sieve more small factors can be found reducing the number of candidates tested per interval. Ready-to-run code, benchmarks, and validation can be found on GitHub1.

1.1 New Prime Gaps with Merit 38.048 & 37.051

Large gaps become more common as the density of primes decrease. The merit of a prime gap, $g_k$, following the prime, $p_k$, is defined as $\frac{g_k}{\log(p_k)}$. It is the ratio of the gap to the “average” gap near that point. Appendix 3 shows the merit of each record prime for gaps under $100,000$. Gaps with merit > 35 are exceedingly rare, only 30 are known, so finding two using this new algorithm is both an accomplishment and involved quite a bit of luck. The two new prime gaps are:

A prime gap of 35,308 follows 100054841 $953\#_{210}$ — 9670 with merit = 38.048 this is the 4th largest known merit2.

A prime gap of 36,702 follows 164065661 $1019\#_{210}$ — 18566 with merit = 37.051 this is the 9th largest know merit.

---

1 https://github.com/sethtroisi/prime-gap
2 https://primegap-list-project.github.io/lists/top20-overall-merits/
2 Searching for Prime Gaps

An exhaustive search for the first occurring prime gap has been performed up to $2^{64}$ [6]. Thomas Nicely explains the search process

No general method more sophisticated than an exhaustive search is known for the determination of first occurrences and maximal prime gaps. As in the present study, this is most efficiently done by sieving successive blocks of positive integers for primes, recording the successive differences, and thus determining directly the first occurrences and maximal gaps. [1]

The largest known first occurring prime gap is the gap of 1550 following 18361375334787046697 with merit $\log(1.836e19) = 34.944$ found by Bertil Nyman in 2014 and independently confirmed in 2018 [2]. A history of the upper bound for exhaustive searches is found on Thomas Nicely’s page [7] and an updated version on GitHub [6].

Larger Gaps The natural approach to search for larger gaps is similar. First, an interval around a test number, $N$, is sieved using small primes to mark numbers with small factors as composite. Then a primality test is run on the remaining numbers to find the “surrounding” primes, the next and previous prime. The prime gap containing $N$ is the difference between the two surrounding primes. An existing respected program PGsurround.pl [8] use this approach with the additional optimization of searching only $N$ of a certain forms known to produce larger gaps.

The speed a program using this approach can find the primes surrounding $N$ is determined by the speed of the primality test and how many composite numbers are removed by the sieve. Generally less than 2% of execution time is spent on the sieve stage; the Combined Sieve doesn’t change this dynamic, even with the deepest sieves, sieving up to $10^{11}$, and smallest $N$, $N \sim 3,000$ bits, sieving takes less than 10% of amortized runtime (see Appendix 2).

Primorials It’s possible to construct arbitrarily large gaps by looking near primorials.

Let $K = P\# = \prod_{p \in P} p$

\[
\begin{align*}
2 & \mid (K + 2) \implies K + 2 \notin \mathbb{P} \\
3 & \mid (K + 3) \implies K + 3 \notin \mathbb{P} \\
2 & \mid (K + 4) \implies K + 4 \notin \mathbb{P} \\
& \vdots \\
(K + X) & \\
\vdots \\
P & \mid (K + P) \implies K + P \notin \mathbb{P}
\end{align*}
\]

Thus there are no primes in the interval $[K + 2, K + P]$ and the gap from $\text{prev\_prime}(K + 2)$ to $\text{next\_prime}(K + 2)$ is $\geq P$. If both $K \pm 1$ are composite (by far the average case) then the gap is $\geq 2P + 3$. Note that as the gap grows larger so does $\log(K)$ so merit doesn’t grow arbitrarily large.

The average gap can be increased by choose a small high composite divisor $d$ (e.g. 11#) and looking near $K = \frac{P\#}{d}$. To illustrate why, consider $K = \frac{P\#}{2}$ all odd $X$, even those $> P$, are multiples of two ($K$ is odd so $K + X$ is even) and multiples of small primes eliminate all even $X \leq 2P$ that aren’t powers of two.

Because of the way primorials align their prime factors a greater share of nearby numbers are composite, meaning larger gaps. For this reason nearly all of the records ($\approx 97\%$ of the 95,000+ record gaps with gap $> 2000$) are of the form $m\frac{P\#}{a} - a$, $m$ being a small multiple, $a$ being the offset to the first prime below $m\frac{P\#}{a}$.

2.1 Sieving

For each $N$, an interval (typically a few thousand numbers corresponding to average or optimistic prime gap) is sieved around $N$ and numbers without any factors, heading away from $N$, are tested for primality till the surrounding primes are found.

The traditional sieve calculates, for all small primes $p$, $N \equiv r \mod p$ and uses that to mark $N - r \pm i \times p$ as composite. E.g. $N \equiv 5 \mod 73 \implies 73 | N - 5$, $73 | N - 73$, and $73 | N - 5 + 73$. 


2.2 Probabilistic Primality Testing

The best known deterministic primality tests are much slower than their probabilistic counterparts with respectively runtime of $\mathcal{O}(\log(n)^6)$ and $\mathcal{O}(\log(n)^2)$. As the probability of a composite being marked as prime can be reduced to an arbitrary level (BPSW has no known counterexamples) and no primes is ever marked as composite, it’s advantageous to use a probabilistic primality test. Notable gaps, with high merit, often have their endpoints later certified as prime with a deterministic test.

GMP version 6.2.0 [9] mpz_probab_prime_p uses a BPSW primality test. GMP is well optimized for numbers of all sizes and is a natural choice for primality testing. Some testing was done with PrimeForm/GW which is faster for large numbers (1000+ digits) but more difficult to work with.

3 Combined Sieve Algorithm

For reasons described above, the majority of prime gap searches are centered around numbers of the form $N = mP_#d$. Often millions of values of $m$ are tested for the same $K = \frac{P_#}{d}$. The Combined Sieve algorithm sieves $M$ sequential intervals simultaneously at a dramatically speedup to sieving $M$ intervals serially. The algorithm works on any arithmetic sequence but for the purpose of prime gap it’s beneficial to focus on numbers of this deficient primorial form.

![Figure 1: Speedup for various values of $M_c$ with $K = \frac{1511#}{7#}, X = 18,000$, and $P_{limt} = 10^{11}$](image)

Note that $m$’s that share a factor with $d$ are skipped partially to avoid duplicate testing (e.g. $m = 100$, $d = 35$ would be a repeat of $m = 20$, $d = 7$) and partially because small $d$ tend to generate lower merit gaps. Figure 1 accounts for this by counting the number of $m$ coprime to $d$ less than $M$, $M_c = |\{m : 1 \leq m \leq M \land (m,d) = 1\}|$

3.1 Space–Time Tradeoff

By caching $K \mod p$, $N \mod p$ is reduced to a machine word sized modulo.

\[
N \mod p = m\frac{P_#}{d} \mod p = mK \mod p \\
\equiv (m \mod p)(K \mod p) \mod p \\
\text{<64bits} \quad \text{500-20,000 bits}
\]

After this trade-off only the first $m$ is slow for the other $M_c - 1$ values of $m$ each $N \mod p$ can be computed with a memory lookup, a 64 bit multiply and a 64 bit modulo irregradarlly of the size of $K$. This means the max speedup is $M_c$ since $K \mod p$ is still calculated once for each $p$. This upper bound is easily seen in Figure 1 up to $M_c \sim 10^5$.

There are two easy ways to cache $K \mod p$. $K \mod p$ can be caching for all primes than two nested for-loops iterate over $m$, $p$ this requires storing the modulo for all primes in memory. The other method inverts the order and iterating over $p$, $m$ this requires storing the list of coprime $m$ and a very large boolean array in memory of the composite status of all offsets for all $M_c$ intervals.

---

3Some of the inspiration for this paper came from analysis done while improving GMP’s next_prime code
3.2 Large Prime Optimization

This section details a complicated trick that allows the algorithm to avoid calculating \( mK \mod p \) for each \( m \) and instead directly compute the set of \( m \) where \( p \) divides a number in the intervals around \( N = mK \).

Assume \( X \) numbers before and after \( N \) are sieved for an interval of size \( S = 2X + 1 \), on average \( \frac{S}{2} \) numbers will be divisible by 2, \( \frac{S}{3} \) by 3, \( \frac{S}{5} \) by 5. In general, each prime \( p \) will divide \( \lfloor \frac{S}{p} \rfloor \) or \( \lceil \frac{S}{p} \rceil \) numbers in the interval around \( N \). When \( p \gg S \) most primes will not divide a number in the interval. Mertens’ second theorem states

\[
\sum_{p \leq X} \frac{1}{p} = \log(\log(n)) + M_m + o(1) \tag{2}
\]

Where \( M_m \approx 0.2615 \) is the Meissel-Mertens constant.

With \( X = 100000, N = \frac{503\times 10^7}{210} \), all of the 17,984 primes \( 2 \leq p \leq S \) divide one or more number in the interval. \( \frac{24744}{60514} = 41\% \) of the primes \( S < p \leq 10^6 \) divide a number in the interval. \( \frac{81355}{50769036} = 0.16\% \) of the primes \( 10^6 < p \leq 10^9 \) divide a number in the interval. This means that 99.84% of large primes for any given \( m \) don’t divide a number in the interval but still require processing. As the sieve limit increases, to \( 10^{10}, 10^{11}, 10^{12}, \ldots \), the fraction of primes dividing a number in the interval decreases further.

3.2.1 Testing many intervals

With a fixed prime \( p \) and fixed \( K \) and 1 \( \leq m \leq M \), sieving can be restated as trying to finding \( m \)’s where

\[-X \leq mK \leq X \mod p \]

\[\equiv 0 \leq mK + X \leq 2X \mod p \] \tag{3}

For each \( p \) larger than some threshold (naively \( p > 10S \) i.e. \( p \) divides fewer than 1 in 10 intervals), \( K \mod p \) is calculated, and the smallest \( m \) satisfying Equation 3 is found. \( mK \mod p \) is computed and one of \( \lfloor \frac{mK}{p} \rfloor \) or \( \lceil \frac{mK}{p} \rceil \) is marked as a composite. The algorithm continues finding larger \( m \) and marking off composites till \( m > M \).

Because we’re only interested in \( m \) which are coprime to \( d \), work is wasted when \( m \) shares a factor with \( d \) and the search advances to the next \( m \) without marking a composite. The smaller and more numerous the factors of \( d \), the more unnecessary work. Very large \( d \) are uncommon so a reasonable worst case is \( d = 17\# \) where 81.9% of \( m \) share a factor with \( d \).

Each solution to Equation 3 (ignoring \( m \) which share a factor with \( d \) and the last solution with \( m > M \)) is another factor of a number in one of the interval. This is a very powerful transformation, by solving Equation 3 the Combined Sieve algorithm can enumerate \( m \)'s where \( p \) divides a number in the interval \([mK - X, mK + X]\). Each prime divides an average \( \frac{MS}{P} \) numbers because \( p \gg S \) this is a large reduction from testing all \( M_e \) intervals individually. With \( S = 60,001 \) and \( P \sim 10^{10} \) only 1 in 160,000 \( m \) will need to calculate \( mK \mod p \).

Equation 2 gives an estimate of the number of primes \( p \leq P_{\text{limit}} \) that divide a number in each interval as \( S (\log(\log(P_{\text{limit}})) + o(1)) \). This means that all prime factors \( 10S \leq p \leq n \) for all \( M \) intervals can be enumerated by solving Equation 3 \( \approx M \times S (\log(\log(P_{\text{limit}})) - \log(\log(10S))) + \text{PrimePi}(P_{\text{limit}}) - \text{PrimePi}(10S) \) times\(^4\).

3.3 Solving Equation 3

Let \( G(p, K, L, R) = m \) be the smallest solution to \( L \leq mK \leq R \mod p \) \tag{4}

\(^4\)This is one of the many theoretical results verified during execution. The count of all \( m \) tested over all \( p \) generally matches to \( \approx 0.05\% \).
This solution assumes \( p \) and \( K \) are coprime which is guaranteed by the definition of \( K \) as a primorial and \( p \) a large prime (at minimum \( p > S > L, R \)). \( p \) and \( K \) coprime also guarantees the existence of a solution \((L \times K^{-1}) \mod p\).

A handwavy proof that 5 returns the minimal answer can be constructed by inspecting 4 with and without the \( \mod p \) condition. If Equation 4 has a solution without considering \( \mod p \), the minimal solution is the first multiple of \( K \geq L \Rightarrow m = \lceil L/K \rceil \), otherwise \( L \leq Km - ip \leq R \) for some \( i \geq 1 \).

\[
\begin{align*}
L &\leq mK - ip \leq R \\
L - mK &\leq -i \times p \leq R - mK
\end{align*}
\]

consider \( \mod K \)

\[
i = G(K, -p \mod K, L \mod K, R \mod K) \\
m = \lceil \frac{L + pG(K, -p \mod K, L \mod K, R \mod K)}{K} \rceil
\]

5d which reduces \( K > \frac{p}{2} \) to \( K < \frac{p}{2} \) is the result of some rearrangement.

Equation 5 either directly returns \( m \) (5a, 5c) or reduces to a new sub-problem with \( K < \frac{p}{2} \) (5d) or \( p < K \) (5e). It follows that \( G \) is \( \mathcal{O}(\log(p)) \).

Given \( p < 2^{64} \) and \( K \mod p \) is cached, this runs quickly. In practice it takes \( 2 \log_2(\frac{p}{\pi - \pi}) \leq 2 \log_2(\frac{10^{13}}{10000}) \leq 60 \) steps. On an i7-2600K with \( p \) a 50 bit prime and \( R - L = S = 100,000 \) an optimized implementation takes 100-200 ns."
| $P_{\text{limit}}$ | primes   | theoretical | experimental |
|------------------|----------|-------------|--------------|
| $10^4$           | 1229     | 0.060960    | 0.06088      |
| $10^5$           | 9592     | 0.048768    | 0.04875      |
| $10^6$           | 78498    | 0.040640    | 0.040638     |
| $10^8$           | 5761455  | 0.030480    | 0.030480     |
| $4 \times 10^9$ | 189961812| 0.025394    | 0.025394     |
| $10^{10}$        | 455052511| 0.024384    | 0.024384     |
| $10^{11}$        | $4.1 \times 10^9$ | 0.022167 |            |
| $10^{12}$        | $3.8 \times 10^{10}$ | 0.020320 |            |
| $10^{13}$        | $3.5 \times 10^{11}$ | 0.018757 |            |

Joined with the fact that a number near $N$ is prime $\approx \frac{1}{\log(n)}$ of the time, the expected number of primality tests that will need to be performed to find a prime after removing numbers with factors less than $P_{\text{limit}}$ is

$$E(N, P_{\text{limit}}) \approx \frac{\log(N)}{\log(P_{\text{limit}}) e^\gamma}$$

A sieve limit can be chosen such that the quasi-linear cost of calculating $K \mod p$ and solving Equation 3 multiplied by the expected number of primality tests reduced is near the cost of running the avoided primality tests.

In practice combined_sieve computes the derivative of Equation 7 at regular intervals giving an estimate of primality tests avoided per second of additional sieving. When that rate falls below the benchmarked rate of primality tests per second, combined_sieve advises the user to stop sieving. This allows for better control of sieve limit based on CPU and memory characteristics.

A counter-intuitively observation is the optimal sieve limits can decrease as $N$ increases. The optimal limit strongly depends on $M$ and for large $N$ it doesn’t make sense to sieve a lot of intervals when only a few can be tested per day. Anecdotally setting $P_{\text{limit}}$ much higher than $5SM$ is sub-optimal as the ratio of time spent performing $K \mod p$ starts to dwarf the time spent solving Equation 3. Some values used during exploratory searches are listed in Appendix 1.

### 4.2 Sieve Interval

For any $N$, if both of the surrounding primes is within $X$ of $N$ then time was wasted finding factors of numbers further away. If either of the surrounding primes is further away than $X$ additional traditional sieving will need to be done. Choosing $X$ in $[6 \log(N), 10 \log(N)]$ is reasonable. In practice combined_sieve performs a small test sieve of a few $m$ and issues a warning if the estimated probability of the surrounding primes exceeding $X$ is greater than 0.5%.

### 5 Runtime

Let

$$S = 2X + 1 = \text{the sieve interval}$$
$$M = \text{number of combined sieves to run simultaneously}$$
$$M_c = \text{number of } m \text{ coprime to } d$$
$$C_{\text{mult}} = \text{small prime multiplier constant } \approx 10 \frac{M}{M_c} \approx 50$$
$$P_{\text{limit}} = \text{sieve limit}$$
$$|P| = \text{PrimePi}(P_{\text{limit}}) \approx \frac{P_{\text{limit}}}{\log(P_{\text{limit}})}$$
$$|P_{\text{small}}| = \text{PrimePi}(C_{\text{mult}} S) \approx \frac{C_{\text{mult}} S}{\log(C_{\text{mult}} S)}$$
$$|P_{\text{large}}| = |P| - |P_{\text{small}}|$$

**Note:** This section ignores the small one-time cost of finding primes up to $P_{\text{limit}}$. 
Sieving each coprime \( m \) without the **Space-Time Tradeoff** and **Large Prime Optimization** involves computing \((m \times K) \mod p\) then marking off an average of \( \frac{S}{p} \) composites. \( N \mod p \) can be computed in \( \log(N) \) time,

\[
|P| \log(N) + S(\log(\log(P_{\text{limit}})) + M_m)
\]

The Combined Sieve algorithm computes \((K \mod p)\) once and reused this for each \( m \). Small primes, which are likely to divide a number in the sieve interval (e.g. \( \frac{P}{S} < C_{mult} \)) are handled similarly to the traditional sieve but use cached \( K \mod p \) to calculating \( N \mod p \) with 64 bit integers. For larger primes the \((m \times K) \mod p\) calculation is avoided for the vast majority of \( m \) at the amortized cost of \( \frac{MS}{p} + 1 \) calls to Equation 3. The cost for \( M \) intervals is

\[
\frac{|P| \log(K)}{K \mod p \text{ for all primes}} + M_c S (\log(\log(P_{\text{limit}})) + M_m) +
\]

\[
\frac{M_c |P_{\text{small}}|}{m \times K \mod p \text{ for small primes}} + (MS(\log(\log(P_{\text{limit}})) - \log(\log(C_{mult}S))) + |P_{\text{large}}|) \times \log_2\left(\frac{P_{\text{limit}}}{S}\right)
\]

With

\[
K = \frac{8887\#}{13\#}
\]

\[
\log_2(K) = 12,651 \quad (\log(P\#) \approx \log(e^P) \approx P)
\]

\[
M = 10,000
\]

\[
M_c = 1,917
\]

\[
P_{\text{limit}} = 10^{10}
\]

\[
|P| = 455,052,511
\]

Optimizing the searching above and below for up to 10 merit

\[
S = 2X + 1 = 2 \times 10 \times 9000 + 1 = 180,001
\]

\[
C_{eq3} = 30 \text{ experimental constant for speed of Equation 3}
\]

\[
C_{mod} = \frac{1}{16} \text{ experimental constant for speed of modulo}^6
\]

---

^6On a i7-2600K @3.4 GHz, GMP computes \( N \mod p \) in \( \approx \frac{4}{11} \log_2(N) \) cycles, Equation 3 takes \( \approx 30 \log_2(\frac{P}{X}) \) cycles to solve.
The experimental speedup of the Combined Sieve over the traditional sieve is

$$|P| \log_2(K)C_{mod} + M_c S(\log(\log(P_{limit})) + M_m) +$$

$$M_c |P_{small}| + (MS(\log(\log(P_{limit})) - \log \log(C_{multi}S)) + |P_{large}|)C_{eq3} \log_2\left(\frac{P_{limit}}{S}\right)$$

$$\approx 455,052,511 \frac{12,651}{16} + 1,917 \times 180,001(3.14 + 0.26) +$$

$$1,917 \times 602,492 + (10,000 \times 180,001 \times (3.14 - 2.86) + 452,618,854) \times 30 \log_2\left(\frac{10^{10}}{400,001}\right)$$

$$\approx 3.60 \times 10^{11} + 1.17 \times 10^{10} + 1.15 \times 10^{10} + 5.24 \times 10^{11}$$

$$\approx 8.86 \times 10^{11}$$

$$\text{vs}$$

$$M_c (|P| \log_2(K)C_{mod} + S(\log(\log(P_{limit})) + M_m))$$

$$\approx 1,917 \times (455,052,511 \frac{12,651}{16} + 180,001(3.14 + 0.26))$$

$$\approx 6.90 \times 10^{14}$$

$$\frac{6.90 \times 10^{14}}{8.86 \times 10^{11}} = 778x \text{ speedup}$$

(10)

This result matches experimental observations of speedups. In general the speedup mostly depends on the magnitude of $M$ (The speedup is bounded by $M_c$, 1,917 in this example), moderately on $P_{limit}$, and a small amount on the magnitude of $N$.

6 Results

Traditionally sieving with primes twice as large, means finding half as many factors in the same amount of time. Using the Combined Sieve algorithm $K \mod p$ takes the same amount of time irrespective of $p$ but larger primes require solving Equation 3 fewer times. As long as a reasonable percentage of time is spent solving Equation 3 vs calculating $K \mod p$, the sieve speeds up as it processes larger primes. Anecdotally it takes less than twice as long to sieve to a 10x higher sieve limit (e.g. if it takes 2 hours to handle primes up to 50 million, all primes up to 500 million can be handle in less than 4 hours).

Use of the Combined Sieve allows for substantial higher sieve limits which reduces the number of primality tests that need to be perform. Appendix 1 has estimates of new optimal sieve limits. Calculating the resulting speedups compared to the sieve limits used by PGSurround.p1 the overall speedup from using the Combined Sieve is 30-70%.

Real world searching  A number of complex relationships govern overall speed. To maximize the number of records found a wide search strategy is utilized, various combinations of $\frac{P_{#}}{M}$ are sieved with $M = 2,000$. For each $m_n$ an auxiliary program computes $Prob_n = P(\text{record gap}|m_n \times K, \text{sieved interval})$, for the top 5% $\{Prob_n, m_n, K, \text{sieved interval}\}$ is stored in a database. Several background threads read from the database testing the highest likelihood intervals. If $\frac{Prob}{amp}$ for a $\frac{P_{#}}{M}$ is above average, combined_sieve is run with the largest $M$ that fits in memory and a higher sieve limit. A number of additional metrics and stats are collected to verify assumptions and integrity during this process; see Appendix 4 for plots of theoretical and experimental distributions.

6.1 Future work

combined_sieve contains a number of other significant speedups not discussed in this paper. Section 6 briefly describes an optimization involving directly computing the probability of a record gap for each interval using the remaining numbers in an interval. This can be utilized to only test the most likely 5-30% of $m$ or to avoid testing the high sides if previous prime is close to $N$.

Acknowledgments  The author would like to thank Craig Citro for advice on publication and Iulia Turc for regular check-ins on progress.
## References

[1] Thomas R. Nicely. “New maximal prime gaps and first occurrences”. In: *Math. Comput.* 68 (1999). An updated copy is available at [https://faculty.lynchburg.edu/~nicely/gaps/gaps.html](https://faculty.lynchburg.edu/~nicely/gaps/gaps.html), pp. 1311–1315.

[2] Graham Higgins. *Prime gap list*. GitHub repository. [https://primegap-list-project.github.io/lists/top20-overall-merits/](https://primegap-list-project.github.io/lists/top20-overall-merits/) and [https://github.com/primegap-list-project/prime-gap-list](https://github.com/primegap-list-project/prime-gap-list). 2020.

[3] J. Selfridge and D. H. Lehmer. “Tables Concerning the Distribution of Primes up to 37 Millions”. In: *Mathematics of Computation* 21 (1967), pp. 483–488.

[4] F. Gruenberger and G. Armerding. *Statistics on the First Six Million Prime Numbers*. Paper. Rand Corporation, 1961.

[5] L. J. Lander and T. R. Parkin. “On First Appearance of Prime Differences”. In: *Mathematics of Computation* 21 (1967), pp. 488–489.

[6] Graham Higgins. *Exhaustively analyzed gaps*. GitHub repository. [https://primegap-list-project.github.io/fully-analyzed/](https://primegap-list-project.github.io/fully-analyzed/). 2020.

[7] Thomas R. Nicely. *Some Results of Research in Computational Number Theory*. URL: [https://faculty.lynchburg.edu/~nicely/](https://faculty.lynchburg.edu/~nicely/).

[8] Dana Jacobsen. “PGsurround.pl”. A historical version can be found at [http://sti15.com/misc/tgaps5-30.txt](http://sti15.com/misc/tgaps5-30.txt). A more recent version was provided by Robert W. Smith and is available on GitHub [https://gist.github.com/sethtroisi/8f0ade362690c6bd416f025bc38df2be](https://gist.github.com/sethtroisi/8f0ade362690c6bd416f025bc38df2be). A second version of unknown date is available on GitHub [https://gist.github.com/sethtroisi/ac50b89b8041b2c78f3ddbc1ff9cc0116](https://gist.github.com/sethtroisi/ac50b89b8041b2c78f3ddbc1ff9cc0116).

[9] Torbjörn Granlund and the GMP development team. *GNU MP: The GNU Multiple Precision Arithmetic Library*. 6.2.0. [http://gmplib.org/](http://gmplib.org/). 2020.

## Appendix

### A.1: Experimental Sieve Limits

| P   | log(P#) | sieve limit  | expected primality tests | PGsurround.pl sieve limit | Theoretical Speedup |
|-----|---------|--------------|--------------------------|--------------------------|---------------------|
| 1511| 1469    | 250 \times 10^9 | 31.5                     | 5.0 \times 10^9           | 70.2%               |
| 2221| 2054    | 800 \times 10^9   | 40.9                     | 1.4 \times 10^7           | 71.8%               |
| 4441| 4353    | 2000 \times 10^9 | 86.3                     | 1.3 \times 10^8           | 51.4%               |
| 5333| 5228    | 2000 \times 10^9 | 103.7                    | 2.3 \times 10^8           | 47.0%               |
| 8887| 8779    | 1000 \times 10^9 | 178.4                    | 1.1 \times 10^9           | 32.5%               |

### A.2: Combined Sieve Amortized Runtime (time per coprime m)

| M       | d     | X      | M       | sieve limit | Combined Sieve(s) | Testing(s) |
|---------|-------|--------|---------|-------------|-------------------|------------|
| 1511    | 7\#   | 16,000 | 10,000,000 | 100 \times 10^9 | 0.0112           | 0.134      |
| 1511    | 7\#   | -      | -       | 250 \times 10^9 | 0.0129           | 0.127      |
| 2221    | 5\#   | 23,000 | 10,000,000 | 800 \times 10^9 | 0.0252           | 0.568      |
| 4441    | 11\#  | 47,000 | 10,000,000 | 2000 \times 10^9 | 0.0605           | 6.04       |
| 5333    | 11\#  | 65,000 | 1,000,000  | 2000 \times 10^9 | 0.158            | 11.23      |
| 8887    | 17\#  | 80,000 | 100,000  | 1000 \times 10^9 | 0.709            | 72.16      |
A.3: Merit of record gap for gaps up to 100,000

A.4: Stats and distribution for $K = \frac{3001 #}{2190} M = 1 + [0, 100000)$

Top row: Probability of gap $X$, Distribution of expected gap, CDF of expected gap

Middle row: Sum of probability of merit 12+ gap after $M$ tests, histogram of probabilities, CDF of probability

Bottom row: Sum of probability of record gap after $M$ tests, histogram of probabilities, CDF of probability