Baxter $d$-permutations and other pattern avoiding classes

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Permutations and diagrams

A permutation $\sigma = \sigma(1), \ldots, \sigma(n) \in S_n$ is a bijection from $[n] := \{1, 2, \ldots, n\}$ to itself.
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Example: the diagram of $\sigma = 413526$
Permutations and diagrams

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**Example:** the diagram of $\sigma = 413526$

A **diagram** (of size $n$) is a point set on $[n] \times [n]$ with exactly 1 point per row and 1 point per column. $S_n$ the set of permutations of size $n$. $|S_n| = n!$
The Starting Question

- Permutations ⇔ Diagrams
- Diagrams are 2D objects
- What could be a ”3D” diagram?
- ⇒ ”3D” Permutations?
**d-Diagrams and d-Permutations**

A **3-diagram** of size \(n\) is a point set on \([n]^3\) such that
A **3-diagram** of size $n$ is a point set on $[n]^3$ such that each plane of the grid (orthogonal to $x, y$ or $z$) contains exactly 1 point.

**Points:**
- $(1,2,6)$
- $(2,5,5)$
- $(3,4,4)$
- $(4,1,3)$
- $(5,4,2)$
- $(6,6,1)$

A **$d$-diagram** is a point set of size $n$ on $[n]^d$ such that each **hyperplane** orthogonal $x_i = j$ with $i \in [d]$ and $j \in [n]$ contains exactly 1 point.
A 3-permutation $\sigma := (\sigma_y, \sigma_z)$ is a pair of permutations. $S_{n}^{2} := \{\text{3-permutations of size } n\}$. $|S_{n}^{2}| = n!^2$.

A $d$-permutation of size $n$, $\sigma := (\sigma_1, \ldots, \sigma_{d-1})$ is a sequence of $d-1$ permutations of size $n$. $S_{n}^{d-1} := \{\text{$d$-permutations of size } n\}$. $|S_{n}^{d-1}| = n!^{d-1}$.

Points:
(1,2,6)
(2,5,5)
(3,4,4)
(4,1,3)
(5,4,2)
(6,6,1)

(253146, 654321)
Projections

\[ \text{proj}_{xy}(\sigma) = \sigma_y, \]
\[ \text{proj}_{xz}(\sigma) = \sigma_z, \]
\[ \text{proj}_{yz}(\sigma) = \sigma_z \sigma_y^{-1}. \]
\[ \text{proj}_{yz}((253146, 654321)) = 264251. \]
\[ \text{proj}_{y,x}(\sigma) = \sigma_y^{-1}. \]

Let \( \overline{\sigma} := (Id_n, \sigma_1, \ldots, \sigma_{d-1}) \). , the projection on \( i \) of \( d \)-permutation \( \sigma \) is the \( d' \)-permutation \( \text{proj}_i(\sigma) := \overline{\sigma}_{i_2} \overline{\sigma}_{i_1}^{-1}, \overline{\sigma}_{i_3} \overline{\sigma}_{i_1}^{-1}, \ldots, \overline{\sigma}_{i_{d'}} \overline{\sigma}_{i_1}^{-1} \). \( d' \) is the dimension of the projection.
Projections

\[ \text{proj}_{xy}(\sigma) = \sigma_y, \]
\[ \text{proj}_{xz}(\sigma) = \sigma_z, \]
\[ \text{proj}_{yz}(\sigma) = \sigma_z \sigma_y^{-1}. \]
\[ \text{proj}_{yz}((253146, 654321)) = 264251. \]
\[ \text{proj}_{y,x}(\sigma) = \sigma_y^{-1}. \]

\( i := i_1, \ldots, i_{d'} \in [d]^{d'}, \) the projection \( \text{proj}_i \) is **direct** if \( i_1 < i_2 < \cdots < i_{d'} \).

Let \( \overline{\sigma} := (\text{Id}_n, \sigma_1, \ldots, \sigma_{d-1}). \) , the **projection** on \( i \) of \( d \)-permutation \( \sigma \) is the \( d' \)-permutation \( \text{proj}_i(\sigma) := \overline{\sigma}_{i_2} \overline{\sigma}_{i_1}^{-1} \overline{\sigma}_{i_3} \overline{\sigma}_{i_1}^{-1} \cdots \overline{\sigma}_{i_{d'}} \overline{\sigma}_{i_1}^{-1}. \) \( d' \) is the **dimension** of the projection.
A permutation $\sigma$ **contains** a permutation (or a **pattern**) $\pi = \pi(1), \ldots, \pi(k) \in S_k$ if there exist indices $c_1 < \cdots < c_k$ such that $\sigma(c_1) \cdots \sigma(c_k)$ is order-isomorphic to $\pi$.

The set of points of indices $c_1, \cdots, c_k$ is an **occurrence** of the $\pi$.

$\sigma = 413526$ contains [several occurrences of] the pattern $\pi = 213$. 

\[ \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 3 & 5 & 2 & 6 & 4 \\
\end{array} \]
Pattern (classic)

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$$S_n(\pi) := \text{the set of permutations that avoids } \pi.$$  
$$S_n(\pi_1, \ldots, \pi_k) := \text{ ... avoids all } \pi_1, \ldots, \pi_k.$$
Pattern avoidance classes

\[ S_n(21) = \{Id_n\} \]
Pattern avoidance classes

\( S_n(21) = \{ Id_n \} \)

[Knuth 73]: \(|S_n(312)| = \frac{1}{n+1} \binom{2n}{n} = 1, 2, 5, 14, 42, 132, \ldots \)
$S_n(21) = \{Id_n\}$

[Knuth 73]: $|S_n(312)| = \frac{1}{n+1} \binom{2n}{n} = 1, 2, 5, 14, 42, 132, \ldots$

$\pi$ and $\tau$ are **Wilf-equivalent** of $|S_n(\pi)| = |S_n(\tau)|$.

$|S_n(21)| = |S_n(12)| = 1$
Pattern avoidance classes

\[ S_n(21) = \{Id_n\} \]

[Knuth 73]: \[ |S_n(312)| = \frac{1}{n+1} \binom{2n}{n} = 1, 2, 5, 14, 42, 132, \ldots \]

\( \pi \) and \( \tau \) are **Wilf-equivalent** if \( |S_n(\pi)| = |S_n(\tau)| \).

\[ |S_n(21)| = |S_n(12)| = 1 \]

\( \pi \) and \( \tau \) are **trivially Wilf-equivalent** if there is a symmetry \( s \) of the square such that \( \forall \sigma, \sigma \in S(\pi) \) iif \( s(\sigma) \in S(s(\tau)) \).
Pattern avoidance classes

| Patterns      | (1) | Sequence                               | Comment       |
|---------------|-----|----------------------------------------|---------------|
| 12            | 2   | 1, 1, 1, 1, 1, 1, 1, 1, 1, ...         |               |
| 12, 21        | 1   | 1, 0, 0, 0, 0, 0, 0, 0, ...            |               |
| 312           | 4   | $\frac{1}{n+1}\binom{2n}{n} = 1, 2, 5, 14, 42, 132, ...$ | [Knuth 73]    |
| 123           | 2   | $\frac{1}{n+1}\binom{2n}{n} = 1, 2, 5, 14, 42, 132, ...$ | [Knuth 73]    |
| 123, 321      | 1   | 1, 2, 4, 4, 0, 0, 0, ...               | [Simion 85]   |
| 213, 321      | 4   | $1 + \frac{n(n-1)}{2} = 1, 2, 4, 7, 11, 16, 22, ...$ | [Simion 85]   |
| 312, 231      | 2   | $2^{n-1} = 1, 2, 4, 8, 16, 32, 64, ...$ | [Simion 85]   |
| 231, 132      | 4   | $2^{n-1} = 1, 2, 4, 8, 16, 32, 64, ...$ | [Simion 85]   |
| 312, 321      | 4   | $2^{n-1} = 1, 2, 4, 8, 16, 32, 64, ...$ | [Simion 85]   |
| 213, 132, 123 | 2   | 1, 2, 3, 5, 8, 13, 21, ...             | [Simion 85]   |
| 231, 213, 321 | 8   | $n = 1, 2, 3, 4, 5, 6, 7, ...$         | [Simion 85]   |
| 312, 132, 213 | 4   | $n = 1, 2, 3, 4, 5, 6, 7, ...$         | [Simion 85]   |
| 312, 321, 123 | 4   | 1, 2, 3, 1, 0, 0, 0, ...               |               |
| 321, 213, 123 | 4   | 1, 2, 3, 1, 0, 0, 0, ...               |               |
| 321, 213, 132 | 2   | $n = 1, 2, 3, 4, 5, 6, 7, ...$         | [Simion 85]   |

(1): Number of trivially Wilf-Equivalent patterns.
patterns and $d$-permutations

$s$ contains a pattern $\pi$ if

$(1432, 3124)$ contains the pattern $(132, 213)$. 
patterns and $d$-permutations

\[ \sigma \text{ contains a pattern } \pi \text{ if there exists a subset of points of } \sigma \text{ that is equal (once standardized) to } \pi. \]

\[ (1432, 3124) \text{ contains the pattern } (132, 213). \]
patterns and $d$-permutations

$\sigma$ contains a pattern $\pi$ if there exists a subset of points of $\sigma$ that is equal (once standardized) to $\pi$.

$\textbf{(1432, 3124)}$ contains the pattern $\textbf{(132, 213)}$.

$\sigma$ contains a pattern $\pi_1$ if $\text{proj}_{xy}, \text{proj}_{xz}$ or $\text{proj}_{yz}$ contains $\pi_1$. 
patterns and $d$-permutations

Let $\sigma \in S_{n}^{d-1}$ and $\pi \in S_{k}^{d'-1}$ with $k \leq n$. Then $\sigma$ contains the pattern $\pi$, if there exist a direct projection $\sigma' = \text{proj}_{i}(\sigma)$ of dimension $d'$ that contains $\pi$.

$(1432, 3124)$ contains the pattern $(132, 213)$ and the pattern $231$.

$\sigma$ contains the pattern $\pi$ if there are indices $c_1 < \cdots < c_k$ such that $\sigma'_i(c_1) \cdots \sigma'_i(c_k)$ is order-isomorphic to $\pi_i$ for all $i \in [d']$. 
patterns and $d$-permutations

Let $\sigma \in S_{n}^{d-1}$ and $\pi \in S_{k}^{d'-1}$ with $k \leq n$. Then $\sigma$ contains the pattern $\pi$, if there exist a direct projection $\sigma' = \text{proj}_i(\sigma)$ of dimension $d'$ that contains $\pi$.

$(1432, 3124)$ contains the pattern $(132, 213)$ and the pattern 231.

Remark 1: $(132, 312)$ doesn’t contain $(12, 12)$ but 132 and 312 both contain the pattern 12 (but on different positions).

$\sigma$ contains the pattern $\pi$ if there are indices $c_1 < \cdots < c_k$ such that $\sigma'_i(c_1) \cdots \sigma'_{i}(c_k)$ is order-isomorphic to $\pi_i$ for all $i \in [d']$. 
## 3-Pattern avoidance classes

| Patterns                  | (1) | Sequence                               | Comment                  |
|---------------------------|-----|----------------------------------------|--------------------------|
| (12, 12)                 | 4   | 1, 3, 17, 151, 1899, 31711, ⋯          | weak-Bruhat intervals    |
| (12, 12), (12, 21)       | 6   | n! = 1, 2, 6, 24, 120 ⋯               | σ₁ ⇒ σ₂                   |
| (12, 12), (12, 21),      | 4   | 1, 1, 1, 1, 1, 1, ⋯                   | 1 diagonal               |
| (21, 12)                 |     |                                        |                          |
| (12, 12), (12, 21),      | 1   | 1, 0, 0, 0, 0, 0, ⋯                   |                          |
| (21, 12), (21, 21)       |     |                                        |                          |
| (123, 123)               | 4   | 1, 4, 35, 524, 11774, 366352, 14953983, ⋯ | new                      |
| (123, 132)               | 24  | 1, 4, 35, 524, 11768, 365558, 14871439, ⋯ | new                      |
| (132, 213)               | 8   | 1, 4, 35, 524, 11759, 364372, 14748525, ⋯ | new                      |
| (12, 12), (132, 312)    | 48  | (n + 1)^{n-1} = 1, 3, 16, 125, 1296 ⋯ | [Atkinson et al. 93]     |
| (12, 12), (123, 321)    | 12  | 1, 3, 16, 124, 1262, 15898, ⋯         | distributive lattice     |
| (12, 12), (231, 312)    | 8   | 1, 3, 16, 122, 1188, 13844, ⋯         | A295928?                 |

(1): Number of trivially Wilf-Equivalent patterns.
2-Pattern avoidance classes

| Patterns | (1) | Sequence | Comment |
|----------|-----|----------|---------|
| 12       | 1   | 1, 0, 0, 0, 0, 0, ··· | unavoidable pattern |
| 21       | 1   | 1, 1, 1, 1, 1, ··· | 1 diagonal |
| 123      | 1   | 1, 4, 20, 100, 410, 1224, 2232, ··· | new |
| 132      | 2   | 1, 4, 21, 116, 646, 3596, 19981, ··· | new |
| 231      | 2   | 1, 4, 21, 123, 767, 4994, 33584, ··· | new |
| 321      | 1   | 1, 4, 21, 128, 850, 5956, 43235, ··· | new |
| 123, 132 | 2   | 1, 4, 8, 8, 0, 0, 0, ··· | |
| 123, 231 | 2   | 1, 4, 9, 6, 0, 0, 0, ··· | |
| 123, 321 | 1   | 1, 4, 8, 0, 0, 0, 0, ··· | |
| 132, 213 | 1   | 1, 4, 12, 28, 58, 114, 220, ··· | new |
| 132, 231 | 4   | 1, 4, 12, 32, 80, 192, 448, ··· | A001787? |
| 132, 321 | 2   | 1, 4, 12, 27, 51, 86, 134, ··· | A047732? |
| 231, 312 | 1   | 1, 4, 10, 28, 76, 208, 568, ··· | A026150? |
| 231, 321 | 2   | 1, 4, 12, 36, 108, 324, 972, ··· | A003946? |

(1): Number of trivially Wilf-Equivalent patterns.
### 1— and 2-Patterns avoidance classes

| Patterns           | (1) | Sequence                        | Comment          |
|--------------------|-----|---------------------------------|------------------|
| 12, (12, 12)       | 1   | 1, 0, 0, 0, 0, 0, ...           | 12               |
| 12, (21, 12)       | 3   | 1, 0, 0, 0, 0, 0, ...           | 12               |
| 21, (12, 12)       | 1   | 1, 0, 0, 0, 0, 0, ...           |                  |
| 21, (21, 12)       | 3   | 1, 1, 1, 1, 1, 1, ...           | 21               |
| 123, (12, 12)      | 1   | 1, 3, 14, 70, 288, 822, 1260, ...| *new*            |
| 123, (12, 21)      | 3   | 1, 3, 6, 6, 0, 0, 0, ...        |                  |
| 132, (12, 12)      | 2   | 1, 3, 11, 41, 153, 573, 2157, ...| A0281593?        |
| 132, (12, 21)      | 6   | 1, 3, 11, 43, 173, 707, 2917, ...| A026671?         |
| 231, (12, 12)      | 2   | 1, 3, 9, 26, 72, 192, 496, ...   | A072863?         |
| 231, (12, 21)      | 4   | 1, 3, 11, 44, 186, 818, 3706, ...| *new*            |
| 231, (21, 12)      | 2   | 1, 3, 12, 55, 273, 1428, 7752, ...| A001764?         |
| 321, (12, 12)      | 1   | 1, 3, 2, 0, 0, 0, 0, ...        |                  |
| 321, (12, 21)      | 3   | 1, 3, 11, 47, 221, 1113, 5903, ...| A217216?         |

(1): Number of trivially Wilf-Equivalent patterns.
Separable permutations $Sep_n$

**direct sum** $\sigma \oplus \pi$: add $\pi$ in the top right corner of $\sigma$.

**skew sum** $\sigma \ominus \pi$: add $\pi$ in the bottom right corner of $\sigma$.

**separable**: size 1 or a direct/skew sum separable permutations.

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$\sigma$ and $\pi$ two permutations respectively of size $n$ and $k$.

$\sigma \oplus \pi := \sigma(1), \ldots, \sigma(n), \pi(1) + k, \ldots, \pi(k) + n$ and

$\sigma \ominus \pi := \sigma(1) + k, \ldots, \sigma(n) + k, \pi(1), \ldots, \pi(k)$. 

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[Brightwell 92]:

$|Sep_n| = n - 1^n - X_{k=0}^{n-1} n - 1^k + 1 \left(1 - \frac{1}{n - k - 1}\right)$.

[Bose Buss Lubiw 98]:

$Sep_n = S_n(2413, 3142)$.
Separable permutations $Sep_n$

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On the left the separable permutation $643512 = 1 \ominus ((1 \ominus 1) \oplus 1) \ominus (1 \oplus 1)$. 

[Graphs illustrating separable permutations]
Separable permutations $Sep_n$

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$|Sep_n| = n - 1 \cdot n - 2 \cdot X_{k=0}^{n-1} n - 1 \cdot n - 2 \cdot (n - k - 1)$.

$Sep_n = S_n(2413, 3142)$ \cite{BoseBussLubiw98}
Separable permutations $Sep_n$

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[Brightwell 92]:

$$|Sep_n| = \frac{1}{n-1} \sum_{k=0}^{n-2} \binom{n-1}{k} \binom{n-1}{k+1} (-1)^{n-k-1}.$$
Separable permutations $Sep_n$

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$Sep_n = S_n(2413, 3142)$ [Bose Buss Lubiw 98]
**Separable $d$-permutations** [Atkinson Mansour 10]

**separable $d$-permutation:** size 1 or a $d$-sum separable permutations

\[ p_1 = (132, 132) = (1, 1) \oplus (++) ((1, 1) \oplus (--) (1, 1)) \]

Let $\sigma$ and $\pi$ two $d$-permutations and $\text{dir} \in \{+, -\}^d$. The $d$-sum with respect to direction $\text{dir}$ is:

\[
\sigma \oplus_{\text{dir}} \pi := \sigma_2 \oplus_{\text{dir}} \pi_2, \ldots, \sigma_d \oplus_{\text{dir}} \pi_d,
\]

where $\oplus_{\text{dir}}^i$ is $\oplus$ if $\text{dir}_i = +$ and $\ominus$ if $\text{dir}_i = -$. 
Separable $d$-permutations

\[\begin{align*}
&\text{[[1, 3, 2], [2, 3, 1]]} \\
&\text{[[3, 1, 2], [2, 3, 1]]} \\
&\text{[[2, 1, 3], [1, 3, 2]]} \\
&\text{[[2, 1, 3], [3, 1, 2]]} \\
&\text{[[1, 3, 2], [2, 1, 3]]} \\
&\text{[[3, 1, 2], [2, 1, 3]]} \\
&\text{[[2, 3, 1], [3, 1, 2]]} \\
&\text{[[2, 3, 1], [1, 3, 2]]}
\end{align*}\]
Separable $d$-permutations

\[ Sep_{n-1}^d = S_{n-1}^d(Sym((132, 213)), 2413, 3142) \]
Separable $d$-permutations

\[ Sep_{n}^{d-1} = S_{n}^{d-1}(\text{Sym}((132, 213)), 2413, 3142) \]

\[ |Sep_{n}^{d-1}| = \frac{1}{n-1} \sum_{k=0}^{n-2} \binom{n-1}{k} \binom{n-1}{k+1} (2^{d-1} - 1)^{k} (2^{d-1})^{n-k-1}. \]
Vincular Patterns and Baxter permutations

**vincular pattern** : a pattern where some entries must be consecutive in the permutation (*adjacencies*). Ex: $2413|_2$ and $3142|_2$
Vincular Patterns and Baxter permutations

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Vincular Patterns and Baxter permutations

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**Baxter permutations**: $B_n := S_n(2413|_2, 3142|_2)$. 
Generalized vincular Patterns

**generalized vincular pattern**: horizontal and/or vertical adjacencies. Ex: $2413|_{2,2}$ and $3142|_{2,2}$.

\[ B_n = S_n(2413|_{2,2}, 3142|_{2,2}) . \]
Generalized vincular Patterns

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\[
B_n = S_n(2413|_{2,2}, 3142|_{2,2}).
\]

Natural extension of generalized vincular patterns to $d$-permuations...
Generalized vincular Patterns

generalized vincular pattern: horizontal and/or vertical adjacencies. Ex: $2413\ |_2^2$ and $3142\ |_2^2$.

\[ B_n = S_n(2413\ |_2^2, 3142\ |_2^2). \]
Natural extension of generalized vincular patterns to $d$-permutations...
But what could be a Baxter $d$-permutation...
Well sliced permutations

A slice is a rectangle defined by two adjacent points. type: horizontal or vertical. The direction of a slice is: $+$ or $-$. 
Well sliced permutations

A slice is a rectangle defined by two adjacent points. **type:** horizontal or vertical. The **direction** of a slice is: + or -.

well-sliced: each slice intersects exactly 1 slice of each type and two intersecting slices share the same direction.
Well sliced permutations

A slice is a rectangle defined by two adjacent points. type: horizontal or vertical. The direction of a slice is: + or -.

well-sliced: each slice intersects exactly 1 slice of each type and two intersecting slices share the same direction.

Proposition: Baxter $\equiv$ well-sliced
Well-sliced $d$-permutations

The **direction** of a slice is: ++, +- , -+, ++. The **type** of a slice is: x, y or z.

A **Baxter $d$-permutation** is a $d$-permutation such that each of its $d' \leq d$ projection is well-sliced.
Well-sliced $d$-permutations vs Baxter $d$-permutation

well-sliced but not Baxter.
Baxter $d$-permutation

well-sliced but not Baxter.
Theorem

\[ B_{n}^{d-1} = S_{n}^{d-1} (Sym(2413|_{2,2}), Sym((312, 213)|_{1,2,.}), Sym((3412, 1432)|_{2,2,.}), Sym((2143, 1423)|_{2,2,.})). \]
Baxter $d$-permutation enumeration

| $|B_{n}^{d-1}|$ |
|------------------|
| $n/d$            | 2   | 3   | 4   | 5   |
| 1                | 1   | 1   | 1   | 1   |
| 2                | 2   | 4   | 8   | 16  |
| 3                | 6   | 28  | 120 | 496 |
| 4                | 22  | 260 | 2440| 20816|
| 5                | 92  | 2872| 59312| 1035616|
| 6                | 422 | 35620|
| 7                | 2074| 479508|

**open problem:** Enumeration formula?
[Bonichon Bousquet Fusy 10] There is a bijection between Baxter permutations and plane bipolar orientations.
Maps ?
conclusion/perspectives

- nice framework
- nice generalisation of Baxter permutation
- lot of open problems.
- implementation available
  plmlab.math.cnrs.fr/bonichon/multipermutation

Have fun!
conclusion/perspectives

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