Composite-Fermion Picture for the Spin-Wave Excitation in the fractional quantum Hall system

T. Nakajima and H. Aoki

Department of Physics, University of Tokyo, Hongo, Tokyo 113, Japan

Abstract: Spin-wave excitation mode from the spin-polarized ground state in the fractional quantum Hall liquid with odd fractions ($\nu = 1/3, 1/5$) numerically obtained by the exact diagonalization of finite systems is shown to be accurately described, for wavelengths exceeding the magnetic length, in terms of the composite-fermion mean-field approximation for the spin-wave (magnon) theory formulated in the spherical geometry. This indicates that the composite picture extends to excited states, and also provides the spin stiffness in terms of peculiar exchange interactions.

PACS numbers: 73.40.Hm.
Although it is more than a decade since the fractional quantum Hall (FQH) effect was discovered and identified as a novel prototype of strongly correlated electron system, a new way of looking at the problem is now being explored in terms of the composite-fermion picture. There is a mounting evidence that the composite picture, which was first proposed by Jain as an alternative to the Haldane-Halperin hierarchy, is surprisingly adequate. The composite picture asserts that a quantum Hall liquid of electrons in the external magnetic field $B$, which corresponds to $\nu^{-1} = 2m + 1$ flux quanta per electron ($m$: an integer), is equivalent, in a mean-field sense, to a liquid of composite fermions each carrying $2m$ flux quanta in the effective magnetic field $B_{\text{eff}} = B/(2m + 1)$ that corresponds to $\nu^{-1} = 1$.

Following the Chern-Simons (CS) gauge-field theory, Halperin, Lee and Read have indeed shown that the spinless system should behave like a system of composite-fermions for which $\nu = 1/2$ corresponds to zero magnetic field. The fermionic CS approach has been extended to discuss the dynamical response and effective mass. Recent experiments by Willett et al and by Kang et al confirm the Fermi-liquid-like behaviors around $\nu = 1/2$. Du et al showed that the transport property in the FQHE state around $\nu = 1/2$ resembles the usual Shubnikov-de Haas oscillation with a well-defined energy gap $\Delta$, for which Leadley et al have subsequently estimated the effective mass from the temperature-dependence of the Shubnikov-de Haas oscillation. Goldman et al report on magnetic focusing. These results are rather surprising, since there is no apriori reason why the composite-particle picture should be a good approximation.

Now, we consider that a true test for a many-body theory is its ability to describe excited states. For odd-fraction Landau-level fillings, $\nu = 1/(2m + 1)$, the ground state (Laughlin's quantum liquid) is fully spin-polarized even when the Zeeman energy is neglected. Then the low-lying excitations are the collective spin-wave mode that restores the broken SU(2) symmetry (rotational symmetry in spin space). Hence the problem may also be regarded as looking into a two-component (spin up/down) FQH system, while the usual practice in studying the odd fractions is to just ignore the spin. The spin degrees of freedom are especially fascinating, since a most drastic effect of electron correlation in the ordinary correlated system (e.g.,
the Hubbard model) is the spin state, which is thought of as a manifestation of the exchange interaction in the appropriate basis. This usually involves the competition of the kinetic and interaction energies, which causes intriguing anomalies in the spin wave dispersion in the Hubbard model that becomes spin-polarized due to the electron correlation.\cite{16} By contrast, the Landau-level filling alone dominates the intra-Landau level excitations such as the spin wave in the FQH system due to the quenched kinetic energy. Thus a prominent question is: can we extend the the composite picture to ferromagnetic spin-wave excitation spectra?

The spin wave has recently been experimentally observed by Pinczuk et al. in which a sharp peak at the energy of the Zeeman splitting, $g \mu_B B$, detected with the inelastic light scattering at $\nu = 1/3$ is assigned to the long wavelength ($k \simeq 0$) spin-wave excitation.\cite{17}

Here we propose an application of the composite-particle picture to the spin-wave (magnon) theory for the FQH system, which is formulated in the spherical geometry to exploit the rotational symmetry.\cite{18} The result is compared with the exact spin-wave excitation numerically obtained from the diagonalization of finite FQH systems. We shall show that the spin-wave excitation spectrum for $\nu = 1/(2m + 1)$ can be explained surprisingly accurately by a composite-fermion picture with a mean-field approximation unless the wavelength is smaller than the magnetic length. One quantitative outcome is that this approach enables us to determine the spin stiffness in the FQH system, in which the magnitude of the spin-spin coupling is difficult to conceive in a conventional way.

We start from the spin-wave excitation spectrum for the flat geometry at $\nu = 1$, which has been exactly given by Kallin and Halperin\cite{13} as

$$\omega(k) - g \mu_B B = \frac{1}{2\pi} \int_0^\infty dq \, q V(q) \left[ 1 - J_0(kq\ell^2) \right] e^{-q\ell^2/2},$$  

where $V(q) = 2\pi e^2/\epsilon q$ the Fourier transform of the Coulomb interaction with $\epsilon$ being the dielectric constant, $J_0(z)$ Bessel’s function, and $\ell = \sqrt{\hbar/eB}$ the magnetic length.

If we now turn to the spherical geometry, everything can be written in terms of angular momentum quantum numbers. As we stereographically map the flat system to the spherical one, the translational symmetry is translated into the rotational symmetry, so that the wavenumber $k$ and total angular momentum $L$ are related by $k = L/R$, where $R$ is the radius of the
sphere. When the total magnetic flux going out of the sphere is $2S$ (an integer due to Dirac’s condition) times the flux quantum, the radius of the sphere becomes $R = \ell \sqrt{S}$, while the relation to $\nu$ is $2S = \nu^{-1}N$—integer with $N$ being the number of electrons. There the creation operator for the magnon with angular momentum $L$ and its $z$ component $M$ is given by $C_{LM}^{\dagger} = \sum_{j,k} (-1)^{S-k}\langle S,j;S,-k|LM\rangle a^\dagger_{j\sigma} a_{k\sigma}$, where $\langle S,j;S,-k|LM\rangle$ is the Clebsch-Gordan coefficient, $a^\dagger_{j\sigma}$ is the creation operator for $j$-th spatial orbit with spin $\sigma$.

The spin-wave excitation spectrum, $\omega_L$, which is a function of $L$ in the spherical geometry, for $\nu = 1$ requires a tedious calculation, but the result is rather elegant in that it is given in terms of Wigner’s $6j$-symbol, familiar in the nuclear physics, as

$$\omega_L - g\mu_B B = \sum_{J=0}^{2S} (2J+1)(-1)^{2S-J} V_J \left[ \frac{1}{2S+1} - (-1)^{2S-J} \begin{array}{ccc} S & S & L \\ S & S & J \end{array} \right],$$

where $L(=0,1,\ldots,2S)$ is the total angular momentum, $\{SSS\}$ is the $6j$-symbol, and $V_J$ is the Haldane pseudopotential for the relative angular momentum $2S-J$ [8, 19]. Since $\{SSS\} = (-1)^{2S-J}/(2S+1)$, the excitation energy at $k = L/R = 0$ satisfies the relation $\omega_0 = g\mu_B B$, for any inter-particle interaction ($\{V_J\}$), which guarantees Larmor’s theorem.

Now we apply Jain’s composite-fermion picture by attaching $2m$ flux quanta to each electron in the $\nu = 1$ state. When we attach $2m$ flux quanta to each electron, the relative angular momentum, $n$, between two electrons translates into the relative angular momentum $n - 2m$ between composite fermions, since an extra phase factor $e^{2mbi}$ appears in the wavefunction of the relative motion [20] (as often described in terms of the CS theory in the literature). Since the field is reduced to $B_{\text{eff}} = B/(2m+1)$, the magnetic length $\ell$ changes into $\tilde{\ell} = \sqrt{2m+1} \ell$ in a mean-field sense.

We are now in position to formulate the spin-wave excitation at $\nu = 1/(2m+1)$. The advantage of working in the spherical geometry is that the transformation into the composite-particle picture is simply given by

$$2\tilde{S} = 2S/(2m+1) = N - 1, \quad \tilde{V}_{2\tilde{S}-n}/\tilde{\epsilon} = V_{2S-n}/\epsilon \ell,$$

where $V_{2S-n}$ is the pseudopotential with the relative angular momentum $n = 2S-J$. If we plug this transformation into the ‘$6j$-formula’, eqn(3), we finally arrive at the desired expression for
the spin-wave excitation spectrum for $\nu = 1/(2m + 1)$ in the composite picture as

$$\omega_L - g\mu_B B = \sum_{J=0}^{2\tilde{S}} (2J + 1) (-1)^{2\tilde{S}-J} \tilde{V}_J \left[ \frac{1}{2\tilde{S} + 1} - (-1)^{2\tilde{S}-J} \begin{Bmatrix} \tilde{S} & \tilde{S} & L \\ \tilde{S} & \tilde{S} & J \end{Bmatrix} \right],$$

(4)

where the range of $L$ now reduces to $L = 0, 1, \cdots, 2\tilde{S}$.

We now turn to the numerical results for the low-lying excitations in the spherical geometry for a 6-electron system at $\nu = 1/3$ and for a 5-electron system at $\nu = 1/5$ in Fig.1, where we show both the one-spin-flip excitations (with the change in the total spin $\Delta S_{tot} = -1$) and the charge ($\Delta S_{tot} = 0$) excitations. In the spin-wave mode, which is the lowest branch in the excitation, we immediately recognize that the states in these finite systems appear only in the range $0 \leq L \leq 2\tilde{S} = N - 1$, while naively there is no reason why the states should not extend for $0 \leq L \leq 2S = (2m + 1)(N - 1)$. In fact, higher-energy spin excitations do indeed exist for larger $L$ in Fig.1. We attribute this truncation from $(2m + 1)(N - 1)$ to the fact that the original system at $\nu = 1/(2m + 1)$ may be mimicked by a system of composite fermions with $\nu = 1$ for the spin-wave excitation.

If we look at the spin-wave dispersion curve in Fig.1, the prediction from the composite-fermion mean-field theory (CFMFA), eqn(4), exhibits an excellent agreement with the exact result up to the wavenumber $k \sim \ell^{-1}$. The exact result starts to deviate from the CFMFA for larger $k$, which implies that the effect of fluctuations from the mean CS field becomes appreciable for large-wavenumber excitations. The fluctuations should also dominate the higher-energy spin excitations above the spin-wave mode, which have no counterparts for $\nu = 1$.

The result that both the number of discrete modes in a finite system and their dispersion agree with the prediction from the CFMFA confirms the picture that the composite-fermion picture is valid not only for the ground state but also for the spin-wave excitation in the long-wavelength region. This is the key message of the present Letter.

To look into the size dependence of the results for the spin-wave excitation, we have calculated the CFMFA result for a larger 51-electron spherical system at $\nu = 1/3$ in Fig.2. For odd fractions, there exists a theory for the spin-wave excitation spectrum by Rasolt et al[14] in the
single-mode approximation (SMA). This gives $\omega(k)$ as

$$\omega(k) - g\mu_B B = \frac{1}{2\pi} \int_0^\infty dq q V(q) [1 - J_0(kq\ell^2)] [1 - S(q)],$$  \hspace{1cm} (5)$$

with $S(q)$ being the static structure factor for the fully-polarized ground state, which agrees with numerical results obtained for finite rectangular[21] or spherical[22] systems for small wavenumbers within the finite-size correction. Thus we have also plotted the SMA result for an infinite (flat) system[23] along with the exact diagonalization result for a 5 (6)-electron system. We can see that all of the exact, CFMFA and SMA results agree with each other for wavenumbers up to $k \sim \ell^{-1}$. The exact result for the finite system for $k < \ell^{-1}$ is slightly larger than other results. This we consider comes partly from a finite-size correction: the Haldane pseudopotential becomes larger[19] for finite systems than that for an infinite system (by about 5% for the present size). Thus the agreement of the CFMFA result persists for larger systems. For $k > \ell^{-1}$ the SMA result, which is also intended for long-wavelengths,[14] the finite-system result, and the CFMFA result start to deviate from each other.

Intuitively the reason why the composite picture is good may be traced back to the `correlation hole': the probability of two repulsively-interacting electrons coming to the distance closer than $\sim \ell$ is so small that, for an electron in a long-wavelength spin wave, whether the flux is uniform or coalesced into a filament for the other electrons is unimportant.

As for the stiffness of the spin wave, this can be estimated by transforming the electron-electron interaction in the Kallin-Halperin formula, eqn (1), via eqn(3) to have

$$V(q) = 2\pi \ell^2 \sum_{n=0}^\infty 2 V_n L_n(q^2\ell^2) \rightarrow 2\pi \ell^2 \sum_{n=0}^\infty \frac{2 V_{n+2m}}{\sqrt{2m+1}} L_n(q^2\ell^2),$$  \hspace{1cm} (6)$$

where $L_n(z)$ is Laguerre’s polynomial[19]. The stiffness, $D$, of the spin-wave defined by $\omega(k)/\left(\frac{e^2}{\ell}\right) = D(k\ell)^2$ for small $k$ is thus expressed as an (asymptotic) expansion involving peculiar ‘exchange integrals’ in the angular-momentum space. Specifically, we can see the spin stiffness significantly decreases as we go from $\nu = 1$ down to 1/3, 1/5, ..., since $V(q)$ becomes progressively reduced.

As for the spin excitations other than the one-spin flips, we can show that all the low-lying excitations at $\nu = 1$ can be entirely interpreted in terms of the multiplets of weakly-interacting
magnons where the interaction is attractive at short distances. The attraction appears in the two-magnon states, for which the state having the largest possible total angular momentum has the lowest energy within a multiplet. This is consistent with Rezayi’s observation that the two-spin-flip mode has a lower energy than that of the one-spin-flip mode at \( \nu = 1 \). Sondhi et al went on to discuss the many-spin flips in analogy with a skyrmion. It is an interesting problem to ask the applicability of the composite picture to these multi-spin-flip excitations. We also notice in Fig.1 here that the roton-like charge-excitation mode exists as well, identified as a \( \Delta S_{\text{tot}} = 0 \) dispersion with an energy gap. Extension of the composite-particle picture to charge modes is also an interesting future problem.

We are grateful to Prof. D. Yoshioka and Koichi Kusakabe for valuable discussions, and also to Takahiro Mizusaki, Michio Honma and Tsutomu Sakai for helpful discussions on the numerical formulae involving \( 3j \)-symbols. The numerical calculations were done on HITAC S3800 in the Computer Centre, the University of Tokyo. This work was in part supported by a Grant-in-Aid from the Ministry of Education, Science and Culture, Japan.
References

[1] *The Quantum Hall Effect* 2nd ed., edited by R.E. Prange and S.M. Girvin (Springer, New York, 1990); T. Chakraborty and P. Pietiläinen, *The Fractional Quantum Hall Effect* (Springer, Berlin, 1988).

[2] J.K. Jain, Phys. Rev. Lett. **63**, 199 (1989); G. Dev and J.K. Jain, ibid **69**, 2843 (1992); X.G. Wu and J.K. Jain, Phys. Rev. B **49**, 7515 (1994).

[3] F.D.M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[4] B.I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984); *ibid* **52**, 2390(E) (1984).

[5] S.C. Zhang, T.H. Hansson and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989); N. Read, *ibid* **62**, 86 (1989); A. Lopez and E. Fradkin, Phys. Rev. B **44**, 5246 (1991); Z.F. Ezawa, M. Hotta and A. Iwazaki, Phys. Rev. B **46**, 7765 (1992).

[6] B.I. Halperin, P.A. Lee and N. Read, Phys. Rev. B **47**, 7312 (1993).

[7] A. Lopez and E. Fradkin, Phys. Rev. B **47**, 7080 (1993); S.H. Simon and B.I. Halperin, Phys. Rev. B **48**, 17 368 (1993).

[8] R.L. Willett et al, Phys. Rev. Lett. **71**, 3846 (1993).

[9] W. Kang et al, Phys. Rev. Lett. **71**, 3850 (1993).

[10] R.R. Du et al, Phys. Rev. Lett. **70**, 2944 (1993).

[11] D.R. Leadley et al, Phys. Rev. Lett. **72**, 1906 (1994).

[12] V.J. Goldman, B. Su and J.K. Jain, Phys. Rev. Lett. **72**, 2065 (1994).

[13] C. Kallin and B.I. Halperin, Phys. Rev. B **30**, 5655 (1984).

[14] M. Rasolt and A.H. MacDonald, Phys. Rev. B **34**, 5530 (1986).
[15] S.L. Sondhi et al, Phys. Rev. B 47, 16419 (1993) have extended the Chern-Simons-Ginzburg-Landau theory to include spin, but SMA has to be invoked to obtain the spin-wave dispersion for odd fractions; see also Yang et al, Phys. Rev. Lett. 72, 732 (1994).

[16] K. Kusakabe and H. Aoki, J. Phys. Soc. Jpn 61, 1165 (1992); Phys. Rev. Lett. 72, 144 (1994).

[17] A. Pinczuk et al, Phys. Rev. Lett. 70, 3983 (1993).

[18] T. Nakajima and H. Aoki, Proc. Int. Symposium on Frontiers in High Magnetic Fields, to be published in Physica B.

[19] F.D.M. Haldane in Ref. 1, Chapter 8; G. Fano et al, Phys. Rev. B 34, 2670 (1986).

[20] B.I. Halperin, Phys. Rev. B 25, 2185 (1982).

[21] D. Yoshioka, J. Phys. Soc. Jpn. 55, 3960 (1986).

[22] E.H. Rezayi, Phys. Rev. B 36, 5454 (1987).

[23] In the SMA calculation using the eqn(5), the static structure factor $S(q)$ for the ground state is needed. Here we have employed Girvin’s fitting to the Monte-Carlo result.

[24] E.H. Rezayi, Phys. Rev. B 43, 5944 (1991).

[25] S.M. Girvin, A.H. MacDonald and P.M. Platzman, Phys. Rev. B 33, 2481 (1986).
Figure captions

**Fig.1** The excitation spectrum for a FQH system of spin $1/2$ electrons, which comprises one-spin-flip excitations with $\Delta S_{\text{tot}} = -1$ (open circles) and charge excitations with $\Delta S_{\text{tot}} = 0$ (open squares), is shown for (a) a 6-electron system at $\nu = 1/3$ and (b) a 5-electron system at $\nu = 1/5$ (note a difference in vertical scales). The result for the spin-wave excitation in the composite-fermion mean-field approximation for the same number of electrons is shown by solid triangles. The spin-wave and roton excitations are respectively connected by a curve as a guide to the eye.

**Fig.2** The spin-wave excitation spectrum for a FQH system of spin $1/2$ electrons at $\nu = 1/3$ is shown. The exact result for a 5 (6) electron system is indicated by solid (open) circles. The result for a 51-electron system in the composite-fermion mean-field approximation in the spherical geometry (solid triangles) and the single-mode approximation for a infinite (flat) system (solid curve) are also indicated.