INTERSECTIONS OF HIGHER-WEIGHT CYCLES
OVER QUATERNIONIC MODULAR SURFACES
AND MODULAR FORMS OF NEBENTYPUS

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In 1976 Hirzebruch and Zagier [6] computed the pairwise intersection multiplicities for a family of algebraic cycles \((T^n)_{n \in \mathbb{N}}\) in the Hilbert modular surface associated to \(\mathbb{Q}(\sqrt{p})\), where \(p\) is a prime congruent to 1 mod 4, and showed that the generating function for those intersection multiplicities
\[
\sum_{n=0}^{\infty} (T^n \cdot T^n) e[nt]
\]
was an elliptic modular form of weight 2 and Nebentypus for \(\Gamma_0(p)\). Shortly afterwards Zagier [22] observed that if certain weighting factors were attached to the intersection numbers, then the new generating function was again an elliptic modular form of the same level and Nebentypus but now of higher weight. Thus he was led to ask if these weighted intersection numbers could also be realized as the ordinary geometric intersection multiplicities of some algebraic cycles in some appropriate homology theory for the Hilbert modular surface. The purpose of this announcement is to describe an answer to Zagier’s question for quaternionic modular surfaces. By combining some ideas of Millson [14] about higher-weight cycles in torus bundles over locally symmetric spaces with the notion of an algebraically defined subspace in the cohomology of a Kuga fiber variety (cf. [4]), we are able to associate to each Hirzebruch-Zagier cycle \(T^n\) in a quaternionic modular surface \(\mathcal{X}\) an algebraic cycle \(T^n\) in a family of abelian varieties \(\mathcal{A}\) over \(\mathcal{X}\) such that: (a) The pairwise intersection multiplicities \((T_m \cdot T_n)\) of these cycles have precisely the same form as Zagier’s weighted intersection numbers (Theorem 1), and (b) the generating function for intersection numbers
\[
\sum_{n=0}^{\infty} (T^n \cdot T^n) e[nt]
\]
is an elliptic modular form of higher weight and Nebentypus for an appropriate \(\Gamma_0(N) \subset \text{SL}_2(\mathbb{Z})\) (Theorem 2).

Tong [20] also looked at Zagier’s question. In 1979 his response was to associate to each Hirzebruch-Zagier cycle a current in the cohomology of the Hilbert modular surface with coefficients in a complex vector bundle and show, using the theory of [19], that the intersection multiplicities of these currents coincided with Zagier’s weighted intersection numbers. Now it is no coincidence that the cohomology classes represented by the cycles \(T^n\) live in a subspace \(H^\bullet(\mathcal{M})\) of \(H^\bullet(\mathcal{A}, \mathbb{Q})\) which is isomorphic to the vector-bundle-valued cohomology that Tong worked with (cf. Lemma 1 below), and that under such an isomorphism the cycles \(T^n\) correspond to his currents. The point is that the whole picture can be realized algebraically inside of the variety \(\mathcal{A}\) (Proposition 1), so that one could start a priori with the “motive” \(H^\bullet(\mathcal{M})\). Moreover, the

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Eichler-Shimura isomorphism identifies this $H^\bullet(M)$ with a space of cusp forms of higher weight for an arithmetic subgroup of $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ (cf. [13 and 11], Chapter II). Thus the present work could also be viewed as an algebraic-geometric formulation of the kind of lifting of automorphic forms which arises from restricting the Weil representation to the groups of a dual reductive pair; see [15 and 9], for example.

I would like to thank Zagier for suggesting this problem to me as well as for his continuing interest and encouragement. I also profited from some conversations with Kudla.

1. Let $V$ be a 4-dimensional vector space over $\mathbb{Q}$ equipped with an anisotropic quadratic form $q$ of signature $(2, 2)$, and let $L$ be a lattice in $V$ on which $q$ is integer-valued. Assume that the discriminant $D$ of $L$ is not a square. Then the even Clifford algebra $C^+(V)$ is a totally indefinite division quaternion algebra over the real quadratic field $\mathbb{Q}(\sqrt{D})$; its reduced discriminant is generated by the product of those rational primes at which $q$ is anisotropic; and it contains the even Clifford algebra $C^+(L)$ as an order. Moreover, for 4-dimensional $V$ the spin group $G := \text{Spin}(V)$ is the group of norm 1 units in $C^+(V)$. Thus $G$ is simple over $\mathbb{Q}$ while $G_{\mathbb{R}} \simeq \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$. Let $\psi: G \to \text{SO}(V)$ denote the natural representation of $G$ on $V$, and let $\lambda$ denote the left regular representation of $G$ on $C^+(V)$. Next choose an integer $k \geq 0$ and let $M$ be a free $\mathbb{Z}$-module of rank $k$. We include the case $k = 0$ for completeness and take $M = \{0\}$ when $k = 0$. Then $\Lambda := C^+(L) \otimes_{\mathbb{Z}} M$ is a lattice in $W := C^+(V) \otimes_{\mathbb{Z}} M$, and the representation $\sigma := \lambda \otimes 1$ of $G$ on $W$ is equivalent over $\mathbb{Q}$ to $k$ copies of the spin representation of $G$.

Now let $\Gamma$ be any torsion-free normal subgroup of finite index in $C^+(L) \cap G$. (Such groups exist by [1].) Then $\psi(\Gamma)$ preserves $L$ and $\sigma(\Gamma)$ preserves $\Lambda$. Denote the symmetric domain associated to $G$ by $\mathcal{D}$, i.e. $\mathcal{D} \simeq G_{\mathbb{R}}/K$ for a maximal compact subgroup $K$ of $G_{\mathbb{R}}$. Then it is well known (cf. [7]) that $\mathcal{X} := \Gamma \backslash \mathcal{D}$ can be embedded as a complex projective algebraic surface. What’s more, it follows from the work of Kuga [11] and Satake [17, 18] that the torus bundle $\mathcal{A} := \Gamma \backslash (\mathcal{D} \times (W_{\mathbb{R}}/\Lambda))$ over $\mathcal{X}$ can also be embedded as a complex projective algebraic variety with the structure of a family of polarized abelian varieties parameterized by $\mathcal{X}$.

**Lemma 1.** Let $(\rho, E)$ be the absolutely irreducible representation of $G$ defined over $\mathbb{Q}$ of highest weight $(2k, 2k)$. Then there exists a unique subspace $H^{4k+2}(M) \subseteq H^{4k+2}(\mathcal{A})$ such that $H^{4k+2}(M) \simeq H^2(\Gamma, E)$.

Here we take cohomology with coefficients in $\mathbb{Q}$ unless otherwise indicated. With the observation that $\rho$ occurs with multiplicity one in $\bigwedge^{4k} \sigma$ (cf. [11, Lemma IV–2–1]), this lemma follows, for example, from [12, (1.3.3)]. Then the following proposition identifies $H^{4k+2}(M)$, defined by the lemma, as being an algebraically defined subspace of $H^{4k+2}(\mathcal{A})$, that is, essentially the Betti realization of a motive (cf. [2, §0]).

**Proposition 1.** There exists an algebraic cycle $\mathcal{P} \subseteq \mathcal{A} \times \mathcal{A}$ of (complex) dimension $4k + 2$ such that the projection from $H^{4k+2}(\mathcal{A})$ to $H^{4k+2}(M)$ can be obtained by lifting a class from $H^{4k+2}(\mathcal{A})$ to $H^{4k+2}(\mathcal{A} \times \mathcal{A})$ via the first
projection from $A \times A$ to $A$, then taking the cup product with the class that $P$ represents, $[P] \in H^{4k+2}(A \times A)$, and then taking the image in $H^{4k+2}(A)$ under the Gysin homomorphism associated to the second projection from $A \times A$ to $A$. In particular, algebraic classes in $H^{4k+2}(A)$ project to algebraic classes in $H^{4k+2}(M)$, and a class in $H^{4k+2}(M)$ is algebraic if and only if it is algebraic as a class in $H^{4k+2}(A)$.

This proposition is a special case of [4, Theorem 1].

2. Let $(\cdot, \cdot)$ denote the symmetric bilinear form on $V$ defined by $(u, v) := q(u + v) - q(u) - q(v)$. For nonzero $v \in L$, let $V_v$ denote the orthogonal complement of $Q_v$ in $V$, and let $L_v := L \cap V_v$. The following lemma is actually true for any quadratic lattice over a principal ideal domain so long as $v$ is not isotropic.

**Lemma 2.** Let $n_L(v) := \gcd\{\langle v, w \rangle | w \in L\}$. Then the discriminant $D_v$ of $L_v$ is related to the discriminant $D$ of $L$ by $D_v = n_L(v)^2(v, v)D$.

Now let $L^+ := \{v \in L | q(v) > 0\}$. Then for $v \in L^+$ we can construct an algebraic family of abelian varieties $A_v$ associated to $V_v$ and $L_v$ just as before: The even Clifford algebra $C^+(V_v)$ is an indefinite division quaternion algebra over $\mathbb{Q}$ contained in $C^+(V)$; it contains $C^+(L_v)$ as an order; $G_v$ can be defined as $G \cap G^+(V_v)$, or as Spin$(V_v)$, or as the stabilizer of $Q_v$ in $G$; let $W_v := C^+(V_v) \otimes \mathbb{Z}M$ and $A_v := C^+(L_v) \otimes \mathbb{Z}M$; let $\Gamma_v := \Gamma \cap G_v$; and let $D_v$ be the hermitian symmetric domain associated to $G_v$. Then as before $A_v := \Gamma_v \setminus (D_v \times (W_v \otimes \mathbb{R}/A_v))$ can be embedded as a complex projective variety with the structure of a family of polarized abelian varieties parameterized by $X_v := \Gamma_v \setminus D_v$. Moreover, the natural inclusions of $D_v$ and $D_v \times W_v \otimes \mathbb{R}$ in $D$ and $D \times W_v \otimes \mathbb{R}$, respectively, induce holomorphic immersions $\iota_v := X_v \to X$ and $h_v := A_v \to A$ compatible with the fiber structure of $A_v$ and $A$.

**Definition.** (a) For $v \in L^+$, let

$$\tau_v := n_L(v)^2 p_{2^*}(\rho \cdot (h_v(A_v) \times A)),$$

where $\rho$ is an algebraic cycle in $A \times A$ with the properties described in Proposition 1, and $(\cdot)$ denotes the intersection product (in the sense of rational homology) in $A \times A$, and $p_{2^*}$ is the map on cycles induced by the second projection from $A \times A$ to $A$, and $n_L(v)$ is defined as in Lemma 2. Then $\tau_v$ is an algebraic cycle of higher weight which represents a well-defined class in $H^{4k+2}(A)$. In fact, $[\tau_v] = P(n_L(v)^2[h_v(A_v)])$, where $P$ denotes the projection from $H^{4k+2}(A)$ to $H^{4k+2}(M)$.

(b) For $m \in \mathbb{Z}_+$, let

$$\tau_m := \sum_{v \in \Gamma \setminus L(m)} \tau_v,$$

where $L(m) := \{v \in L | q(v) = m\}$. We call $\tau_m$ an arithmetic cycle of higher weight, as its definition depends on the arithmetic of $L$.

(c) Let $\tau_0$ be defined formally by

$$[\tau_0] := \begin{cases} 2^{-1}c_1 & \text{if } k = 0, \\ 0 & \text{if } k > 0, \end{cases}$$

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where $c_1$ is the first Chern class on $\mathcal{X}$.

The arithmetic cycles of higher weight are the analogs of the Hirzebruch-Zagier cycles in the present case. Of course if $k = 0$ then $\mathcal{T}_m$ is not of higher weight at all, rather it coincides with the Hirzebruch-Zagier cycle $T_m$ in $\mathcal{X}$. On the other hand, when $k > 0$ then $T_m$ lives over $T_m$ in the sense that

$$ T_m = \phi_\ast \left( \sum_{v \in \Gamma \setminus L(m)} h_v(A_v) \right) = \sum_{v \in \Gamma \setminus L(m)} i_v(X_v), $$

where $\phi_\ast$ is the map on cycles associated to the natural projection $\phi: A \to X$.

3. Let $(\mathcal{T}_m \cdot \mathcal{T}_n)$ denote the intersection multiplicity of $\mathcal{T}_m$ with $\mathcal{T}_n$ in the sense of rational homology.

**Theorem 1.**

$$(\mathcal{T}_m \cdot \mathcal{T}_n) = \sum_{(v, w) \in \Gamma \setminus (L(m) \times L(n))} P_{2k}(v, w) + \sum_{(v, w) \in \Gamma \setminus (L(m) \times L(n))} E(X_v)P_{2k}(v, w)$$

where $D(v, w) := (v, w)^2 - 4q(v)q(w)$,

$$P_{2k}(v, w) := \sum_{j=0}^{k} (-1)^{j} C_{2k}^{j} q(v)^{j} q(w)^{j} (v, w)^{2k-2j}$$

and $E(X_v)$ is the Euler volume of $X_v$.

Some remarks concerning this theorem might be in order. First of all, notice that $D(v, w)$ is the discriminant of $q$ restricted to $Zv + Zw$. So when $v$ or $w$ is in $L^+$, then $D(v, w) < 0$ if and only if this binary quadratic form is positive definite. On the other hand, for nonzero $v$ and $w$ then $D(v, w) = 0$ if and only if $Qv = Qw$, in which case $h_v(A_v) = h_w(A_w)$ and $T_v$ is a multiple of $T_w$. Thus the second term in the expression for $(\mathcal{T}_m \cdot \mathcal{T}_n)$ comes from the self-intersection multiplicities of the common components of $\mathcal{T}_m$ and $\mathcal{T}_n$—this is why the Euler number appears. In any case the second term vanishes unless $mn$ is a square.

Secondly, note that the polynomial $P_{2k}(v, w)$ is homogeneous of degree $2k$ in $v$ and $w$ separately. In fact $P_{2k}(v, w) = q(v)^k q(w)^k C_{2k}^{1}(v', w')$, where $C_{2k}^{1}$ is the Gegenbauer, or ultraspherical, polynomial (cf. [21, Chapter IX], or your favorite text on orthogonal polynomials) and $u' := (u, u)^{-1/2} u \in V_\mathbb{R}$ is the unit vector in the $u$-direction for any $u \in L^+$.

As for the proof of Theorem 1, if $k = 0$ then the methods of [8] suffice. On the other hand, when $k > 0$ we can write down the harmonic differential form $\eta_m$ on $A$ which represents $[\mathcal{T}_m]$, and then integrate $\eta_m$ over $\mathcal{T}_n$. Unfortunately this is a somewhat lengthy business, as it must begin with choosing coordinates on $A$. Eventually, however, the computations on which the proof depends are just like Zagier’s [22, Theorem 6 and 23, Theorem 2].

4. **Theorem 2.** The Fourier series

$$F_m(\tau) := \sum_{n=0}^{\infty} \langle \mathcal{T}_m \cdot \mathcal{T}_n \rangle e^{\pi i n \tau}$$
is an elliptic modular form of weight $2k + 2$ on $\Gamma_0(N)$ with character $\chi$, where $N$ and $\chi$ are the level and character, respectively, of $L$ (in the sense of [5, §4]). If $k > 0$ then $F_m(\tau)$ is a cusp form.

The proof of Theorem 2 from Theorem 1 is quite amusing. The starting point is the following proposition, which can be proved by combinatorial arguments in the spirit of [8] or can be deduced from [10]. Let $Z := (\tau \ z)\ z \tau \ z \tau$ denote an element of the Siegel upper half-plane of genus 2.

**Proposition 2.** The Fourier series

$$
\Phi(Z) := 2^{-1} E(X) + \sum_{(v, w) \in \Gamma \setminus (L^+ \times L^+)} \frac{e[\varepsilon(v)\tau + (v, w)z + q(w)\tau']}{D(v, w) < 0} 
$$

$$
+ \sum_{(v, w) \in \Gamma \setminus (L^+ \times L^+)} \frac{E(X_w)e[\varepsilon(v)\tau + (v, w)z + q(w)\tau']}{D(v, w) = 0}
$$

is a Siegel modular form of genus 2, weight 2 and character $\chi$ for $\Gamma_0(2)(N) \subset \text{Sp}_4(Z)$.

In fact $\Phi(Z)$ is the theta function for $\Gamma$-inequivalent representations of positive semidefinite binary quadratic forms in $L$.

So if we expand $\Phi(Z)$ in a Fourier series as a function of $\tau'$, then it follows immediately from the proposition that each Fourier coefficient $\phi_m(\tau, z)$, for $m \in \mathbb{N}$, is a Jacobi form of weight 2 and index $m$, in the terminology of [3], for $\Gamma_0(N) \subset \text{SL}_2(\mathbb{Z})$ with character $\chi$ (cf. [3, Theorem 6.1, or 16]). And now suddenly Theorem 2 follows as a special case of [3, Theorem 3.1]! For it is readily checked that $F_m(\tau)$ is the $2k$th “development coefficient” of $\phi_m(\tau, z)$, meaning that it is derived from the $2k$th Taylor coefficient of $\phi_m(\tau, z)$ at $z = 0$ in such a way that it becomes an elliptic modular form of weight $2k + 2$ also for $\Gamma_0(N)$ with character $\chi$.

**Remark.** It should be noted that more notations but no new ideas are needed to generalize the methods and results described in this note to the case where $V$ is a 4-dimensional vector space over a totally real number field $F$ and $q$ is an anisotropic quadratic form on $V$ which has signature $(2, 2)$ at some of the real places of $F$ and $(4, 0)$ at the rest.

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