On the oscillatory integration of some ordinary differential equations

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Abstract Conditions are given for a class of nonlinear ordinary differential equations $x'' + a(t)w(x) = 0$, $t \geq t_0 \geq 1$, which includes the linear equation to possess solutions $x(t)$ with prescribed oblique asymptote that have an oscillatory pseudo-wronskian $x'(t) - \frac{x(t)}{t}$.

2000 Mathematics Subject Classification: 34A30 (Primary), 34E05, 34K25
Keywords: Ordinary differential equation; Asymptotic integration; Prescribed asymptote; Non-oscillation of solutions

1 Introduction

A certain interest has been shown recently in studying the existence of bounded and positive solutions to a large class of elliptic partial differential equations which can be displayed as

$$\Delta u + f(x, u) + g(|x|) x \cdot \nabla u = 0, \quad x \in G_R,$$

where $G_R = \{x \in \mathbb{R}^n : |x| > R\}$ for any $R \geq 0$ and $n \geq 2$. We would like to mention the contributions [1], [3], [8] – [11], [13, 14], [18] and their references in this respect.

It has been established, see [8, 9], that it is sufficient for the functions $f$, $g$ to be Hölder continuous, respectively continuously differentiable in order to analyze the asymptotic behavior of the solutions to (1) by the comparison
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In fact, given $\zeta > 0$, let us assume that there exist a continuous function $A : [R, +\infty) \to [0, +\infty)$ and a nondecreasing, continuously differentiable function $W : [0, \zeta] \to [0, +\infty)$ such that

$$0 \leq f(x, u) \leq A(|x|)W(u) \quad \text{for all } x \in G_R, \ u \in [0, \zeta]$$

and $W(u) > 0$ when $u > 0$. Then we are interested in the positive solutions $U = U(|x|)$ of the elliptic partial differential equation

$$\Delta U + A(|x|)W(U) = 0, \quad x \in G_R,$$

for the rôle of super-solutions to (1).

M. Ehrnström [13] noticed that, by imposing the restriction $x \cdot \nabla U(x) \leq 0, \ x \in G_R,$ upon the super-solutions $U$, an improvement of the conclusions from the literature is achieved for the special subclass of equations (1) where $g$ takes only nonnegative values. Further developments of Ehrnström’s idea are given in [1, 3, 11, 14].

Translated into the language of ordinary differential equations, the research about $U$ reads as follows: given $c_1, c_2 \geq 0$, find (if any) a positive solution $x(t)$ of the nonlinear differential equation

$$x'' + a(t)w(x) = 0, \quad t \geq t_0 \geq 1,$$

where the coefficient $a : [t_0, +\infty) \to \mathbb{R}$ and the nonlinearity $w : \mathbb{R} \to \mathbb{R}$ are continuous and given by means of $A, W$, such that

$$x(t) = c_1 t + c_2 + o(1) \quad \text{when } t \to +\infty \quad (3)$$

and

$$\mathcal{W}(x, t) = \frac{1}{t} \left| \begin{array}{cc} x'(t) & 1 \\ x(t) & t \end{array} \right| = \frac{x'(t)}{t} - \frac{x(t)}{t} < 0, \quad t > t_0. \quad (4)$$

The symbol $o(f)$ for a given functional quantity $f$ has here its standard meaning. In particular, by $o(1)$ we refer to a function of $t$ that decreases to 0 as $t$ increases to $+\infty$.

The papers [2, 3, 20, 22, 23] present various properties of the functional quantity $\mathcal{W}$, which shall be called pseudo-wronskian in the sequel. Our aim in this note is to complete their conclusions by giving some sufficient conditions upon $a$ and $w$ which lead to the existence of a solution $x$ to (2) that verifies (3) while having an oscillatory pseudo-wronskian (this means that there exist the unbounded from above sequences $(t_n^\pm)_{n \geq 1}$ and $(t_n^0)_{n \geq 1}$ such that $t_{2n-1}^0 < t_n^+ < t_{2n}^0 < t_n^- < t_{2n+1}^0$ and $\mathcal{W}(t_n^+) > \mathcal{W}(t_n^-) = 0 > \mathcal{W}(t_n^0)$ for all $n \geq 1$). We answer thus to a question raised in [3, p. 371], see also the comment in [2, pp. 46–47].
2 The sign of $\mathcal{W}$

Let us start the discussion with a simple condition to settle the sign issue of the pseudo-wronskian.

**Lemma 1** Given $x \in C^2([t_0, +\infty), \mathbb{R})$, suppose that $x''(t) \leq 0$ for all $t \geq t_0$. Then $\mathcal{W}(x, \cdot)$ can change from being nonnegative-valued to being negative-valued at most once in $[t_0, +\infty)$. In fact, its set of zeros is an interval (possibly degenerate).

**Proof.** Notice that
\[
\frac{d^2}{dt^2}[x(t)] = \frac{1}{t} \cdot \frac{d}{dt}[t\mathcal{W}(x, t)], \quad t \geq t_0.
\]

The function $t \mapsto t\mathcal{W}(x, t)$ being nonincreasing, it is clear that, if it has zeros, it has either a unique zero or an interval of zeros. □

The result has an obvious counterpart.

**Lemma 2** Given $x \in C^2([t_0, +\infty), \mathbb{R})$, suppose that $x''(t) \geq 0$ for all $t \geq t_0$. Then, $\mathcal{W}(x, \cdot)$ can change from being nonpositive-valued to being positive-valued at most once in $[t_0, +\infty)$. Again, its set of zeros is an interval (possibly reduced to one point).

Consider that $x$ is a positive solution of equation (2) in the case where $a(t) \geq 0$ in $[t_0, +\infty)$ and $w(u) > 0$ for all $u > 0$. Then, we have
\[
\frac{d\mathcal{W}}{dt} = -\frac{\mathcal{W}}{t} - a(t)w(x(t)), \quad t \geq t_0,
\]
which leads to
\[
\mathcal{W}(x, t) = \frac{1}{t} \left[ t_0\mathcal{W}_0 - \int_{t_0}^{t} sa(s)w(x(s))ds \right], \quad \mathcal{W}_0 = \mathcal{W}(x, t_0), \quad (5)
\]
throughout $[t_0, +\infty)$ by means of Lagrange’s variation of constants formula.

The integrand in (5) being nonnegative-valued, we regain the conclusion of Lemma 1. In fact, if $T \in [t_0, +\infty)$ is a zero of $\mathcal{W}(x, \cdot)$ then it is a solution of the equation
\[
t_0\mathcal{W}_0 = \int_{t_0}^{T} sa(s)w(x(s))ds. \quad (6)
\]
On the other hand, if the pseudo-wronskian of $x$ is positive-valued throughout $[t_0, +\infty)$ then it is necessary to have
\[
(t_0 \mathcal{W}_0 \geq 0) \quad \int_{t_0}^{+\infty} sa(s) w(x(s)) ds < +\infty.
\]

It has become clear at this point that whenever the equation (2) has a positive solution $x$ such that $\mathcal{W}_0 \leq 0$, the functional coefficient $a$ is nonnegative-valued and has at most isolated zeros and $w(u) > 0$ for all $u > 0$, the pseudo-wronskian $\mathcal{W}$ satisfies the restriction (4). Now, returning to the problem stated in the Introduction, we can evaluate the main difficulty of the investigation: if the positive solution $x$ has prescribed asymptotic behavior, see formula (3) or a similar development, then we cannot decide upfront whether or not $\mathcal{W}_0 \leq 0$. The formula (6) shows that there are also certain difficulties to estimate the zeros of the pseudo-wronskian.

3 The behavior of $\mathcal{W}$

Let us survey in this section some of the recent results regarding the pseudo-wronskian.

It has been established that its presence in the structure of a nonlinear differential equation
\[
x'' + f(t, x, x') = 0, \quad t \geq t_0 \geq 1,
\]
where the nonlinearity $f : [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, allows for a remarkable flexibility of the hypotheses when searching for solutions with the asymptotic development (3) (or similar).

**Theorem 1** ([20, p. 177]) Assume that there exist the nonnegative-valued, continuous functions $a(t)$ and $g(s)$ such that $g(s) > 0$ for all $s > 0$ and $xg(s) \leq g(x^{1-\alpha}s)$, where $x \geq t_0$ and $s \geq 0$, for a certain $\alpha \in (0, 1)$. Suppose further that
\[
|f(t, x, x')| \leq a(t)g\left(\left|x' - \frac{x}{t}\right|\right) \quad \text{and} \quad \int_{t_0}^{+\infty} \frac{a(s)}{s^\alpha} ds < \int_{c+|\mathcal{W}_0|t_0^{1-\alpha}}^{+\infty} \frac{du}{g(u)}.
\]

Then the solution of equation (8) given by (2) exists throughout $[t_0, +\infty)$ and has the asymptotic behavior
\[
x(t) = c \cdot t + o(t), \quad x'(t) = c + o(1) \quad \text{when } t \to +\infty \quad \text{(9)}
\]
for some $c = c(x) \in \mathbb{R}$.  

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To compare this result with the standard conditions in asymptotic integration theory regarding the development (9), see the papers [2, 3, 24] and the monograph [19].

Another result is concerned with the presence of the pseudo-wronskian in the function space $L^1((t_0, +\infty), \mathbb{R})$.

**Theorem 2** ([3, p. 371]) Assume that $f$ does not depend explicitly of $x'$ and there exists the continuous function $F : [t_0, +\infty) \times [0, +\infty) \to [0, +\infty)$, which is nondecreasing with respect to the second variable, such that

$$|f(t, x)| \leq F\left(t, \frac{|x|}{t}\right) \text{ and } \int_{t_0}^{+\infty} t \left[1 + \ln\left(\frac{t}{t_0}\right)\right] F\left(t, |c| + \frac{\varepsilon}{t_0}\right) dt < \varepsilon$$

for certain numbers $c \neq 0$ and $\varepsilon > 0$. Then there exists a solution $x(t)$ of equation (8) defined in $[t_0, +\infty)$ such that

$$x(t) = c \cdot t + o(1) \text{ when } t \to +\infty \quad \text{and} \quad \mathcal{W}(x, \cdot) \in L^1.$$

The effect of perturbations upon the pseudo-wronskian is investigated in the papers [2, 20, 22].

**Theorem 3** ([20, p. 183]) Consider the nonlinear differential equation

$$x'' + f(t, x, x') = p(t), \quad t \geq t_0 \geq 1,$$

where the functions $f : [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ and $p : [t_0, +\infty) \to \mathbb{R}$ are continuous and verify the hypotheses

$$|f(t, x, x')| \leq a(t) \left|x' - \frac{x}{t}\right|, \quad \int_{t_0}^{+\infty} ta(t) dt < +\infty$$

and

$$\lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} sp(s) ds = C \in \mathbb{R} - \{0\}.$$

Then, given $x_0 \in \mathbb{R}$, there exists a solution $x(t)$ of equation (10) defined in $[t_0, +\infty)$ such that

$$x(t_0) = x_0 \quad \text{and} \quad \lim_{t \to +\infty} \mathcal{W}(x, t) = C.$$

In particular,

$$\lim_{t \to +\infty} \frac{x(t)}{t \ln t} = C.$$
A slight modification of the discussion in [22, Remark 3], see [2] p. 47, leads to the next result.

**Theorem 4** Assume that $f$ in (10) does not depend explicitly of $x'$ and there exists the continuous function $F : [t_0, +\infty) \times [0, +\infty) \to [0, +\infty)$, which is nondecreasing with respect to the second variable, such that

$$|f(t, x)| \leq F(t, |x|) \quad \text{and} \quad \int_t^{+\infty} sF\left(s, |P(s)| + \sup_{\tau \geq s} \{q(\tau)\}\right) ds \leq q(t), \quad t \geq t_0,$$

for a certain positive-valued, continuous function $q(t)$ possibly decaying to 0 as $t \to +\infty$. Here, $P$ is the twice continuously differentiable antiderivative of $p$, that is $P''(t) = p(t)$ for all $t \geq t_0$. Suppose further that

$$\limsup_{t \to +\infty} \left[ tW(P, t) \right] > 1 \quad \text{and} \quad \liminf_{t \to +\infty} \left[ tW(P, t) \right] < -1.$$

Then equation (10) has a solution $x(t)$ throughout $[t_0, +\infty)$ such that

$$x(t) = P(t) + o(1) \quad \text{when} \quad t \to +\infty$$

and $W(x, \cdot)$ oscillates.

Finally, the presence of the pseudo-wronskian in the structure of a non-linear differential equation can lead to multiplicity when searching for solutions with the asymptotic development [3].

**Theorem 5** ([23, Theorem 1]) Given the numbers $x_0, x_1, c \in \mathbb{R}$, with $c \neq 0$, and $t_0 \geq 1$ such that $t_0 x_1 - x_0 = c$, consider the Cauchy problem

$$(11) \quad \begin{cases} x'' = \frac{1}{t} g(tx' - x), & t \geq t_0 \geq 1, \\ x(t_0) = x_0, & x'(t_0) = x_1, \end{cases}$$

where the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $g(c) = g(3c) = 0$ and $g(u) > 0$ for all $u \neq c$. Assume further that

$$\int_{2c}^{2c+} \frac{du}{g(u)} < +\infty \quad \text{and} \quad \int_{2c}^{(3c) -} \frac{du}{g(u)} = +\infty.$$

Then problem (11) has an infinity of solutions $x(t)$ defined in $[t_0, +\infty)$ and developable as

$$x(t) = c_1 t + c_2 + o(1) \quad \text{when} \quad t \to +\infty$$

for some $c_1 = c_1(x)$ and $c_2 = c_2(x) \in \mathbb{R}$. 

The asymptotic analysis of certain functional quantities attached to the solutions of equations (2), (8) and (10), as in our case the pseudo-wronskian, might lead to some surprising consequences. Among the functional quantities that gave the impetus to spectacular developments in the qualitative theory of linear/nonlinear ordinary differential equations we would like to refer to

\[ K(x(t)) = x(t)x'(t), \quad t \geq t_0, \]

employed in the theory of Knöres-solutions, see the papers [6, 7] for the linear and respectively the nonlinear case and the monograph [19], and

\[ HW(x) = \int_{t_0}^{+\infty} x(s)w(x(s))ds. \]

The latter quantity is the core of the nonlinear version of Hermann Weyl’s limit-point/limit-circle classification designed for equation (2), see the well-documented monograph [5] and the paper [21].

4 The negative values of \( \mathcal{W} \)

We shall assume in the sequel that the nonlinearity \( w \) of equation (2) verifies some of the hypotheses listed below:

\[ |w(x) - w(y)| \leq k|x - y|, \quad \text{where } k > 0, \quad (12) \]

and

\[ w(0) = 0, \quad w(x) > 0 \text{ when } x > 0, \quad |w(xy)| \leq w(|x|)w(|y|) \quad (13) \]

for all \( x, y \in \mathbb{R} \). We notice that restriction (13) implies the existence of a majorizing function \( F \), as in Theorem 2, given by the estimates

\[ |f(t, x)| = |a(t)w(x)| \leq |a(t)| \cdot w(t) w \left( \frac{|x|}{t} \right) = F \left( t, \frac{|x|}{t} \right). \]

We can now use the paper [24] to recall the main conclusions of an asymptotic integration of equation (2). It has been established that whenever

\[ \int_{t_0}^{+\infty} tw(t)|a(t)|dt < +\infty, \]

all the solutions of (2) have asymptotes (3) and their first derivatives are developable as

\[ x'(t) = c_1 + o \left( t^{-1} \right) \quad \text{when } t \to +\infty. \quad (14) \]

Consequently, \( \mathcal{W}(x, t) = -c_2 t^{-1} + o(t^{-1}) \) for all large \( t \)'s. In this case (the functional coefficient \( a \) has varying sign), when dealing with the sign of the
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pseudo-wronskian, of interest would be the subcase where \( c_2 = 0 \). Here, the asymptotic development does not even ensure that \( \mathcal{W} \) is eventually negative. Enlarging the family of coefficients to the ones subjected to the restriction \( \int_{t_0}^{+\infty} \mathcal{F} w(t) |a(t)| dt < +\infty \), where \( \varepsilon \in [0,1) \), the developments (3), (14) become

\[
x(t) = ct + o(t^{1-\varepsilon}) , \quad x'(t) = c + o\left(t^{-\varepsilon}\right) , \quad c \in \mathbb{R},
\]

yielding the less precise estimate \( \mathcal{W}(x, t) = o(t^{-\varepsilon}) \) when \( t \to +\infty \). We have again a lack of precision in the asymptotic development of \( \mathcal{W}(x, \cdot) \) with respect to the sign issue. We also deduce on the basis of (3), (15) that some of the coefficients \( a \) in these classes verify (7), a fact that complicates the discussion.

The next result establishes the existence of a positive solution to (2) subjected to (4), (15) for the largest class of functional coefficients: \( \varepsilon = 0 \). By taking into account Lemmas 1, 2 and the non-oscillatory character of equation (2) when the nonlinearity \( w \) verifies (13), we conclude that for an investigation within this class of coefficients \( a \) of the solutions with oscillatory pseudo-wronskian it is necessary that \( a \) itself oscillates. Also, when \( a \) is non-negative valued we recall that the condition

\[
\int_{t_0}^{+\infty} a(t) dt < +\infty
\]

is necessary for the linear case of equation (2) to be non-oscillatory, see [16], while in the case given by \( w(x) = x^\lambda \), \( x \in \mathbb{R} \), with \( \lambda > 1 \) (such an equation is usually called an Emden-Fowler equation, see the monograph [19]) the condition

\[
\int_{t_0}^{+\infty} t a(t) dt = +\infty
\]

is necessary and sufficient for oscillation, see [4]. In the case of Emden-Fowler equations with \( \lambda \in (0,1) \) and a continuously differentiable coefficient \( a \) such that \( a(t) \geq 0 \) and \( a'(t) \leq 0 \) throughout \( [t_0, +\infty) \), another result establishes that equation (2) has no oscillatory solutions provided that condition (16) fails, see [17].

Regardless of the oscillation of \( a \), it is known [3, p. 360] that the linear case of equation (2) has bounded and positive solutions with eventually negative pseudo-wronskian.

**Theorem 6** Assume that the nonlinearity \( w \) verifies hypothesis (13) and is nondecreasing. Given \( c, d > 0 \), suppose that the functional coefficient \( a \) is
nonnegative-valued, with eventual isolated zeros, and
\[
\int_{t_0}^{+\infty} w(t)a(t)dt \leq \frac{d}{w(c+d)}.
\]
Then, the equation (2) has a solution \(x\) such that \(W_0 = 0\),
\[
c - d \leq x'(t) < \frac{x(t)}{t} \leq c + d \quad \text{for all } t > t_0
\]
and
\[
\lim_{t \to +\infty} x'(t) = \lim_{t \to +\infty} \frac{x(t)}{t} = c.
\]

**Proof.** We introduce the set \(D\) given by
\[
D = \{ u \in C([t_0, +\infty), \mathbb{R}) : ct \leq u(t) \leq (c + d)t \text{ for every } t \geq t_0 \}.
\]

A partial order on \(D\) is provided by the usual pointwise order “\(\leq\)”, that is, we say that \(v_1 \leq v_2\) if and only if \(v_1(t) \leq v_2(t)\) for all \(t \geq t_1\), where \(v_1, v_2 \in D\). It is not hard to see that \((D, \leq)\) is a complete lattice.

For the operator \(V : D \to C([t_0, +\infty), \mathbb{R})\) with the formula
\[
V(u)(t) = t \left\{ c + \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau a(\tau)w(u(\tau))d\tau ds \right\}, \quad u \in D, t \geq t_0,
\]
the next estimates hold
\[
c \leq \frac{V(u)(t)}{t} = c + \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau a(\tau) \cdot w(\tau)w\left(\frac{u(\tau)}{\tau}\right) d\tau ds
\leq c + \sup_{\xi \in [0,c+d]} \{ w(\xi) \} \cdot \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau w(\tau)a(\tau)d\tau ds
= c + w(c+d) \left[ \frac{1}{t} \int_0^t \tau w(\tau)a(\tau)d\tau + \int_t^{+\infty} \tau w(\tau)a(\tau)d\tau \right]
\leq c + w(c+d) \int_{t_0}^{+\infty} \tau w(\tau)a(\tau)d\tau \leq c + d
\]
by means of (13). These imply that \(V(D) \subseteq D\).

Since \(c \cdot t \leq V(c \cdot t)\) for all \(t \geq t_0\), by applying the Knaster-Tarski fixed point theorem [12] p. 14, we deduce that the operator \(V\) has a fixed point \(u_0\) in \(D\). This is the pointwise limit of the sequence of functions \((V^n(c \cdot Id_I))_{n \geq 1}\), where \(V^1 = V, V^{n+1} = V^n \circ V\) and \(I = [t_0, +\infty)\).

We deduce that
\[
u_0'(t) = [V(u_0)]'(t) = \frac{u_0(t)}{t} - \frac{1}{t} \int_0^t \tau a(\tau)w(u_0(\tau))d\tau < \frac{u_0(t)}{t},
\]
when \(t > t_0\), and thus (17), (18) hold true.

The proof is complete. □
5 The oscillatory integration of equation (2)

Let the continuous functional coefficient $a$ with varying sign satisfy the restriction

$$\int_{t_0}^{+\infty} t^2|a(t)|dt < +\infty.$$ 

We call the problem studied in the sequel an oscillatory (asymptotic) integration of equation (2).

Theorem 7 Assume that $w$ verifies (12), $w(0) = 0$ and there exists $c > 0$ such that

$$L^c_+ > 0 > L^c_-,$$

where

$$L^c_+ = \limsup_{t \to +\infty} \frac{t \int_t^{+\infty} sw(cs)a(s)ds}{\int_t^{+\infty} s^2|a(s)|ds}, \quad L^c_- = \liminf_{t \to +\infty} \frac{t \int_t^{+\infty} sw(cs)a(s)ds}{\int_t^{+\infty} s^2|a(s)|ds}.$$ 

Then the equation (2) has a solution $x(t)$ with oscillatory pseudo-Wronskian such that

$$x(t) = c \cdot t + o(1) \quad \text{when} \quad t \to +\infty.$$ 

**Proof.** There exist $\eta > 0$ such that $L^c_+ > \eta$, $L^c_- < -\eta$ and two increasing, unbounded from above sequences $(t_n)_{n \geq 1}$, $(t^n)_{n \geq 1}$ of numbers from $(t_0, +\infty)$ such that $t^n \in (t_n, t_{n+1})$ and

$$t_n \int_{t_n}^{+\infty} sw(cs)a(s)ds + k\eta \int_{t_n}^{+\infty} s^2|a(s)|ds < 0$$

and

$$t^n \int_{t^n}^{+\infty} sw(cs)a(s)ds - k\eta \int_{t^n}^{+\infty} s^2|a(s)|ds > 0$$

for all $n \geq 1$.

Assume further that

$$\int_{t_0}^{+\infty} \tau^2|a(\tau)|d\tau \leq \frac{\eta}{k(c + \eta)}.$$
and introduce the complete metric space $S = (D, \delta)$ given by

$$D = \{ y \in C([t_0, +\infty), \mathbb{R}) : t |y(t)| \leq \eta \text{ for every } t \geq t_0 \}$$

and

$$\delta(y_1, y_2) = \sup_{t \geq t_0} \{ t |y_1(t) - y_2(t)| \}, \quad y_1, y_2 \in D.$$ 

For the operator $V : D \to C([t_0, +\infty), \mathbb{R})$ with the formula

$$V(y)(t) = \frac{1}{t} \int_t^{+\infty} sa(s)w \left( s \left[ c - \int_s^{+\infty} \frac{y(\tau)}{\tau} d\tau \right] \right) ds, \quad y \in D, \ t \geq t_0,$$

the next estimates hold (notice that $|w(x)| \leq k|x|$ for all $x \in \mathbb{R}$)

$$t |V(y)(t)| \leq k \int_t^{+\infty} s^2 |a(s)| \left[ c + \eta \int_s^{+\infty} \frac{d\tau}{\tau^2} \right] ds \leq \eta \quad (23)$$

and

$$t |V(y_2)(t) - V(y_1)(t)| \leq k \int_t^{+\infty} s^2 |a(s)| \left( \int_s^{+\infty} \frac{d\tau}{\tau^2} \right) ds \cdot \delta(y_1, y_2) \leq \frac{k}{t_0} \int_t^{+\infty} s^2 |a(s)| ds \leq \frac{\eta}{c + \eta} \cdot \delta(y_1, y_2).$$

These imply that $V(D) \subseteq D$ and thus $V : S \to S$ is a contraction.

From the formula of operator $V$ we notice also that

$$\lim_{t \to +\infty} tV(y)(t) = 0 \quad \text{for all } y \in D. \quad (24)$$

Given $y_0 \in D$ the unique fixed point of $V$, one of the solutions to (2) has the formula $x_0(t) = t \left[ c - \int_t^{+\infty} \frac{y_0(s)}{s} ds \right]$ for all $t \geq t_0$. Via (24) and L’Hospital’s rule, we provide also an asymptotic development for this solution, namely

$$\lim_{t \to +\infty} \left[ x_0(t) - c \cdot t \right] = - \lim_{t \to +\infty} t \int_t^{+\infty} \frac{y_0(s)}{s} ds = - \lim_{t \to +\infty} ty_0(t) = - \lim_{t \to +\infty} tV(y_0)(t) = 0.$$ 

The estimate

$$\left| ty_0(t) - \int_t^{+\infty} sw(cs)a(s)ds \right| \leq k \int_t^{+\infty} s^2 |a(s)| \left[ \int_s^{+\infty} \frac{|y_0(\tau)|}{\tau} d\tau \right] ds \leq k\eta \cdot \frac{1}{t} \int_t^{+\infty} s^2 |a(s)| ds, \quad t \geq t_0,$$
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accompanied by (21), (22), leads to

\[ y_0(t_n) = W(x_0, t_n) < 0 \quad \text{and} \quad y_0(t^n) = W(x_0, t^n) > 0. \] (25)

The proof is complete. \( \square \)

**Remark 1** When equation (2) is linear, that is \( w(x) = x \) for all \( x \in \mathbb{R} \), the formula (19) can be recast as

\[ L_+ = \limsup_{t \to +\infty} \frac{t \int_t^{+\infty} s^2 a(s) ds}{\int_t^{+\infty} \frac{1}{s^2} |a(s)| ds} > 0 > \liminf_{t \to +\infty} \frac{t \int_t^{+\infty} s^2 a(s) ds}{\int_t^{+\infty} \frac{1}{s^2} |a(s)| ds} = L_- . \]

We claim that for all \( c \neq 0 \) there exists a solution \( x(t) \) with oscillatory pseudo-wronskian which verifies (27). In fact, replace \( c \) with \( c_0 \) in the formulas (21), (22) for a certain \( c_0 \) subjected to the inequality \( \min \{L_+, -L_-\} > \frac{\eta}{c_0} \). It is obvious that, when \( L_+ = -L_- = +\infty \), formulas (21), (22) hold for all \( c_0, \eta > 0 \). Given \( c \in \mathbb{R} - \{0\} \), there exists \( \lambda \neq 0 \) such that \( c = \lambda c_0 \). The solution of equation (2) that we are looking for has the formula

\[ x(t) = \lambda \cdot x_0 \]

where \( x_0(t) = t \left[ c_0 - \int_t^{+\infty} \frac{y_0(s)}{s} ds \right] \) for all \( t \geq t_0 \) and \( y_0 \) is the fixed point of operator \( V \) in \( D \). Its pseudo-wronskian oscillates as a consequence of the obvious identity

\[ \lambda \cdot W(x_0, t) = W(x, t), \quad t \geq t_0. \]

**Example 1** An immediate example of functional coefficient \( a \) for the problem of linear oscillatory integration is given by \( a(t) = t^{-2} e^{-t} \cos t \), where \( t \geq 1 \).

We have

\[ \int_t^{+\infty} \frac{1}{s^2} \cos \left( t + \frac{\pi}{4} \right) e^{-t} \quad \text{and} \quad \int_t^{+\infty} \frac{1}{s^2} |a(s)| ds \leq e^{-t} \]

throughout \([1, +\infty)\) which yields \( L_+ = +\infty, L_- = -\infty \).

Sufficient conditions are provided now for an oscillatory pseudo-wronskian to be in \( L^p((t_0, +\infty), \mathbb{R}) \), where \( p > 0 \). Since \( \lim_{t \to +\infty} \frac{W(x, t)}{x} = 0 \) for any solution \( x(t) \) of equation (3) with the asymptotic development (20), (14), we are interested in the case \( p \in (0, 1) \).

**Theorem 8** Assume that, in the hypotheses of Theorem 7, the coefficient \( a \) verifies the condition

\[ \int_{t_0}^{+\infty} \left[ \frac{t}{\int_t^{+\infty} \frac{1}{s^2} |a(s)| ds} \right]^{1-p} t^2 |a(t)| dt < +\infty \quad \text{for some } p \in (0, 1). \] (26)

Then the equation (2) has a solution \( x(t) \) with an oscillatory pseudo-wronskian in \( L^p \) and the asymptotic expansion (20).
Proof. Recall that \( y_0 \) is the fixed point of operator \( V \). Then, formula (23) implies that

\[
|y_0(t)| \leq k(c + \eta) \cdot \frac{1}{t} \int_{t}^{\infty} s^2 |a(s)| ds, \quad t \geq t_0.
\]

Via an integration by parts, we have

\[
\frac{1}{|k(c + \eta)|^p} \int_{t}^{T} |y_0(s)|^p ds \\
\leq \frac{T^{1-p}}{1-p} \left[ \int_{T}^{\infty} s^2 |a(s)| ds \right]^{p} + \frac{p}{1-p} \int_{t}^{T} \left[ \int_{s}^{\infty} \frac{s}{\tau^2 |a(\tau)|} d\tau \right]^{1-p} s^2 |a(s)| ds
\]

for all \( T \geq t \geq t_0 \).

The estimates

\[
\frac{T^{1-p}}{1-p} \left[ \int_{T}^{\infty} s^2 |a(s)| ds \right]^{p} = \frac{T^{1-p}}{1-p} \int_{T}^{\infty} \left[ \int_{s}^{\infty} \frac{1}{\tau^2 |a(\tau)|} d\tau \right]^{1-p} s^2 |a(s)| ds
\]

allow us to establish that

\[
\frac{1}{|k(c + \eta)|^p} \int_{t}^{T} |y_0(s)|^p ds \leq \frac{1+p}{1-p} \int_{t}^{\infty} \left[ \int_{s}^{\infty} \frac{s}{\tau^2 |a(\tau)|} d\tau \right]^{1-p} s^2 |a(s)| ds.
\]

The conclusion follows by letting \( T \to +\infty \).

The proof is complete. \( \square \)

Example 2
An example of functional coefficient \( a \) in the linear case that verifies the hypotheses of Theorem 8 is given by the formula

\[
t^2 a(t) = b(t) = \begin{cases} 
  a_k(t - 9k), & t \in [9k, 9k + 1], \\
  a_k(9k + 2 - t), & t \in [9k + 1, 9k + 3], \\
  a_k(t - 9k - 4), & t \in [9k + 3, 9k + 4], \\
  a_k(9k + 4 - t), & t \in [9k + 4, 9k + 5], \\
  a_k(t - 9k - 6), & t \in [9k + 5, 9k + 7], \\
  a_k(9k + 8 - t), & t \in [9k + 7, 9k + 8], \\
  0, & t \in [9k + 8, 9(k + 1)],
\end{cases} \\
\]

for \( k \geq 1 \).

Here, we take \( a_k = k^{-\alpha} - (k + 1)^{-\alpha} \) for a certain integer \( \alpha > \frac{2-p}{p} \).
Oscillatory integration

To help the computations, the $k$-th "cell" of the function $b$ can be visualized next.

It is easy to observe that

$$\int_{9k}^{9k+4} b(t) dt = \int_{9k+4}^{9k+8} b(t) dt = 0 \quad \text{for all } k \geq 1.$$  

We have

$$\int_{9k+2}^{+\infty} b(t) dt = \int_{9k+2}^{9k+4} b(t) dt = -a_k, \quad \int_{9k+6}^{+\infty} b(t) dt = \int_{9k+6}^{9k+8} b(t) dt = a_k$$

and respectively

$$\int_{9k+2}^{+\infty} |b(t)| dt = 3a_k + 4 \sum_{m=k+1}^{+\infty} a_m, \quad \int_{9k+6}^{+\infty} |b(t)| dt = a_k + 4 \sum_{m=k+1}^{+\infty} a_m.$$

By noticing that

$$L_+ = \lim_{k \to +\infty} \frac{(9k+6) \int_{9k+6}^{+\infty} b(t) dt}{\int_{9k+6}^{+\infty} |b(t)| dt}, \quad L_- = \lim_{k \to +\infty} \frac{(9k+2) \int_{9k+2}^{+\infty} b(t) dt}{\int_{9k+2}^{+\infty} |b(t)| dt},$$

we obtain $L_+ = \frac{9 \alpha}{4}$ and $L_- = -\frac{9 \alpha}{4}$.

To verify the condition (26), notice first that

$$I_k = \int_{9k}^{9(k+1)} \left[ \frac{t}{\int_{t}^{+\infty} |b(s)| ds} \right]^{1-p} t^2 |a(t)| dt \leq \int_{9k}^{9(k+1)} \left[ \frac{9(k+1)}{\int_{9(k+1)}^{+\infty} |b(s)| ds} \right]^{1-p} a_k dt, \quad k \geq 1.$$

The elementary inequality $a_k \leq (2^\alpha - 1)(k+1)^{-\alpha}$ implies that

$$I_k \leq \frac{c_\alpha}{(k+1)^{(1+\alpha)p-1}}, \quad \text{where } c_\alpha = 9 \left( \frac{9}{4} \right)^{1-p} (2^\alpha - 1),$$

and the conclusion follows from the convergence of the series $\sum_{k \geq 1} (k+1)^{1-(1+\alpha)p}$.
Acknowledgement The author is indebted to Professor Ondrej Došly and to a referee for valuable comments leading to an improvement of the initial version of the manuscript. The author was financed during this research by the Romanian AT Grant 97GR/25.05.2007 with the CNCSIS code 100.

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