Counting divisorial contractions with centre a $cA_n$-singularity

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Abstract

First, we simplify the existing classification due to Kawakita and Yamamoto of 3-dimensional divisorial contractions with centre a $cA_n$-singularity, also called compound $A_n$ singularity. Next, we describe the global algebraic divisorial contractions corresponding to a given local analytic equivalence class of divisorial contractions with centre a point. Finally, we consider divisorial contractions of discrepancy at least 2 to a fixed variety with centre a $cA_n$-singularity. We show that if there exists one such divisorial contraction, then there exist uncountably many such divisorial contractions.

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1 Introduction

The minimal model program and the Sarkisov program give a general framework for the birational classification of algebraic varieties, a central problem in algebraic geometry. Morphisms called divisorial contractions play a major role in both the minimal model program and the Sarkisov program. Therefore, classifying divisorial contractions is a fundamental problem.
A divisorial contraction is a proper birational morphism \( \varphi : Y \to X \) between terminal algebraic varieties such that the exceptional locus of \( \varphi \) is a prime divisor and \( -K_Y \) is \( \varphi \)-ample. The explicit classification of 3-dimensional divisorial contractions where the centre is a point has been completed, except when the centre is a \( cD_n \) or a \( cE_n \)-singularity and the discrepancy is 1, for which there are only unpublished manuscripts [Haya, Hayb]. The case where the centre is a non-Gorenstein point has been done in [Hay99, Hay00, Hay05, Kaw05, Kaw12, Kaw96] and the Gorenstein case in [Kaw01, Kaw02, Kaw03], [Kaw05, Theorem 1.2] and [Yam18].

Above, the divisorial contractions are classified up to local analytic equivalence, meaning that we allow local analytic changes on \( X \) around \( P \) and on \( Y \) around the exceptional locus. Since the local analytic germ of a \( \mathbb{Q} \)-factorial variety can be non-factorial, the classification is given more generally for \( \mathbb{Q} \)-Gorenstein varieties with terminal singularities without requiring \( \mathbb{Q} \)-factoriality. If a morphism \( \varphi : Y \to X \) is a divisorial contraction in this sense, without requiring \( \mathbb{Q} \)-factoriality, and if \( X \) is \( \mathbb{Q} \)-factorial, then we automatically find that \( Y \) is \( \mathbb{Q} \)-factorial, since the prime exceptional divisor is Cartier.

In this paper we focus on \( cA_n \)-singularities, also called compound \( A_n \) singularities, meaning that a general section through the point defines the surface \( A_n \)-singularity, see Definition 3.7. Compound \( A_n \) singularities are the simplest 3-dimensional terminal hypersurface singularities. There is an on-going project with the goal of showing that all smooth Fano 3-folds are obtainable by deformations from singular toric 3-folds with \( cA_n \)-singularities, [CR].

The classification due to Hayakawa, Kawakita, Kawamata and Yamamoto gives a list of weighted blowups such that every divisorial contraction is locally analytically equivalent to at least one member of the list. One way to improve the classification is to find which members of the classification lists give locally analytically equivalent blowups:

**Problem 1.1.** Describe the local analytic equivalence classes of 3-dimensional divisorial contractions with centre a point.

This is roughly what was asked in [Cor00, Problem 3.8].

We have solved Problem 1.1 for \( cA_n \)-singularities in Theorem 6.1 and Lemma 6.2. This can drastically simplify the classification, as the complicated family in Theorem 3.10(3) reduces to just one simple case Theorem 6.1(3).

The next step is to classify divisorial contractions globally algebraically:

**Problem 1.2.** Describe all global algebraic blowups up to isomorphism over the base that are locally analytically equivalent to a given weighted blowup.

We have solved Problem 1.2 completely in Corollary 5.6. The global algebraic classification has applications in birational rigidity, finding birational relations and computing Sarkisov links, see [AK16, AZ16, Oka14, Oka18, Oka20, Pae20].

To prove that a given morphism is a divisorial contraction of a certain type, for example when computing Sarkisov links, it is best to have a classification list where the conditions are as mild and as easy to check as possible. One way to phrase this:

**Problem 1.3.** Describe an algorithm to determine whether a given weighted blowup is locally analytically equivalent to a given member of the classification list.

We have solved Problem 1.3 for \( cA_n \)-singularities in Theorem 6.1 and Lemma 6.2. It is straight-forward to determine the weight of a power series and it is algorithmic to check the singularity type of a simple singularity (Definition 3.6). To check whether a given
singularity is of type $A_n$, $D_n$, $E_6$, $E_7$ or $E_8$, see [AGZV85, 16.2 The determinator 1.−9k.] or [GLS07, Theorems I.2.48, I.2.51 and I.2.53]. It can be computed using a computer algebra system, for example Singular [DGPS22].

On the other hand, to prove local properties or local inequalities such as [KOPP24, Theorem 1.2], it helps to have a list which is as specific as possible, containing only few members. Even though the classification lists in the literature contain uncountable families of weighted blowups, a countably infinite list or even a finite list in certain cases may suffice.

**Problem 1.4.** Given a variety $X$ and a point $P$, determine whether there exist finitely many, countably many or uncountably many divisorial contractions to $X$ with centre $P$, depending on the singularity type of $P$, where the counting is up to local analytic equivalence and up to global algebraic isomorphism over $X$.

We have solved Problem 1.4 for $cA_n$-singularities in Theorem 6.5. We have also included the case of smooth points in Table 1. By the proof of Theorem 6.5 the cardinalities up to local biholomorphism around the exceptional loci over the base are the same as up to global algebraic isomorphism over the base in the case of $cA_n$-points and smooth points.

| singularity          | isomorphism over base | local analytic equivalence |
|----------------------|-----------------------|---------------------------|
| smooth point         | uncountable           | countable                 |
| $cA_n$, only discr 1 | $n$                   | $\lceil n/2 \rceil$      |
| $cA_n$, admits discr > 1 | uncountable           | finite                    |

As an example application of such results, [Oka20] uses the specific counts of divisorial contractions of a given type described in [Hay99] (such as [Hay99, Theorem 6.4]) to prove birational birigidity of varieties.

Table 1 raises the following questions:

**Question 1.5.** Let $X$ be a 3-dimensional variety and $P \in X$ a singular point. Do there exist only finitely many divisorial contractions to $X$ with centre $P$ up to local analytic equivalence?

**Question 1.6.** Let $X$ be a 3-dimensional variety and $P \in X$ a point. Is it true that there exist uncountably many divisorial contractions to $X$ with centre $P$ up to isomorphism over $X$ if and only if there exists a divisorial contraction to $X$ with centre $P$ with discrepancy greater than 1?

Regarding Question 1.6, it is known that there exist only finitely many divisorial contractions of discrepancy at most 1 to a fixed variety, see Proposition 6.4. By [Pae20, §6] we expect there to be only finitely many divisorial contractions of ordinary type that are $(r_1, r_2, a, 1)$-blowups with centre a $cA_n$-singularity even if the discrepancy $a$ is greater than 1 as long as the inequalities $a \leq r_1 \leq r_2$ are satisfied. This does not answer Question 1.6 negatively, see Theorem 6.5.

The proofs in this paper rely on the concept of *weight-respecting maps*, see Definition 4.1, which is comparable to the equivalence relation $\sim'$ defined in [Hay99, 3.7 Weighted valuations] for 3-dimensional index $\geq 2$ terminal singularities embedded as hypersurfaces in orbifolds.
2 Meaning of classification

The classification due to Hayakawa, Kawakita, Kawamata and Yamamoto is a classification list, as defined in Definition 2.1, except that it does not satisfy Item (1) if the discrepancy of $\varphi$ is 1 and the centre is either a $cD$ or a $cE$ point.

**Definition 2.1.** A set $L$ is called a **classification list** if it consists of pairs $(w,Z)$, where $w := (w_1, \ldots, w_5) \in ((1/m)\mathbb{Z})^5$ is a vector of positive rational numbers and $Z$ is a codimension 2 complete intersection complex analytic space with an isolated singularity at the origin $0$ inside an orbifold $\mathbb{C}^5/\mathbb{Z}_m$, such that

1. for every 3-dimensional divisorial contraction $\varphi$ with centre a point: $\varphi$ is locally analytically equivalent (Definition 3.3) to the $w$-blowup of $Z$ for some $(w,Z)$ in $L$,
2. for every pair $(w,Z)$ in $L$: there exists a $\mathbb{Q}$-Gorenstein variety $X$ with terminal singularities and a point $P \in X$ such that $(X,P)$ is locally biholomorphic to $(Z,0)$, and
3. for every $\mathbb{Q}$-Gorenstein variety $X$ with terminal singularities and point $P \in X$: if $(X,P)$ is locally biholomorphic to $(Z,0)$ for some $(w,Z)$ in $L$, then there exists a divisorial contraction to $X$ which is locally analytically equivalent to the $w$-blowup of $Z$.

The divisorial contraction in Item (3) can be constructed using either Proposition 5.1 or Corollary 5.6.

Given a classification list $L$ and two pairs $(w,Z)$ and $(w',Z')$, Problem 1.1 asks to determine when the weighted blowups of $Z$ and $Z'$ are locally analytically equivalent. For $cA_n$-points, we prove their local analytic equivalence if $w = w'$ in Lemma 6.2. It should not be difficult to prove in the case where there are only finitely many such divisorial contractions, which happens for example when the discrepancy is at most 1, see Proposition 6.4. See [Hay99, Hay00] for explicit descriptions and counts of minimal discrepancy divisorial contractions with centre a non-Gorenstein point.

**Definition 2.2.** We say a classification list $L$ is a **nice classification list** if for every two pairs $(w,Z)$ and $(w',Z')$ in $L$: the $w$-blowup of $Z$ and the $w'$-blowup of $Z'$ are locally analytically equivalent if and only if $Z$ and $Z'$ are biholomorphic around the origins and $w = w'$.

Finding a nice classification list, if it exists, would solve Problem 1.1.

**Question 2.3.** Does there exist a nice classification list?

By Theorem 6.1 and Lemma 6.2, the answer to Question 2.3 is ‘yes’ in the case of $cA_n$-points. We give two nice classification lists $L$ and $L'$ for $cA_n$-singularities and smooth points, corresponding to columns 1, 2, 3 and columns 1, 2, 4, 5 of Table 2, respectively. The final column ‘disc’ in Table 2 gives the discrepancy of the divisorial contraction.

For $cA_n$-singularities, the classification list contains only weighted blowups such that $X$ is embedded locally analytically as a hypersurface $V(f)$ in $\mathbb{C}^4$. We can also embed it as a codimension 2 complete intersection $V(f,x_5)$ in $\mathbb{C}^5$ choosing the weight $w_5$ to be any positive integer. In such cases we write only the first four variables $x,y,z,t$ and their weights $w_1, w_2, w_3, w_4$. Similarly $\mathbb{C}^3$ can be embedded in $\mathbb{C}^5$ by $V(x_4,x_5)$ with any positive integer weights $w_4, w_5$ for $x_4, x_5$, so we give only the first three weights.
The first nice classification list for $cA_n$-singularities and smooth points is given by

$$L := L_1 \cup L_2 \cup L_3 \cup L_4,$$

where

\begin{align*}
L_1 &:= \{((1, a, b), \mathbb{C}^3) \mid a \text{ and } b \text{ are coprime positive integers, } a \leq b\} \\
L_2 &:= \{((r_1, r_2, a, 1), \mathbb{V}(xy + g(z, t))) \mid \text{wt } g = r_1 + r_2\} \\
L_3 &:= \{((1, 5, 3, 2), \mathbb{V}(xy + z^2 + t^3))\} \\
L_4 &:= \{((4, 3, 2, 1), \mathbb{V}(x^2 + y^2 + z^3 + xt^2))\},
\end{align*}

where in $L_2$ the convergent power series $g \in \mathbb{C}\{z, t\}$ defines an isolated singularity at the origin and $r_1, r_2$ and $a$ are positive integers that satisfy $r_1 \leq r_2$, $a$ divides $r_1 + r_2$, $a$ is coprime to both $r_1$ and $r_2$, and $a(n + 1) = r_1 + r_2$. The polynomial $xy + g(z, t)$ with wt $g = r_1 + r_2$ appears in [Kaw02, Theorem 1.1] and the polynomial $xy + z^2 + t^3$ appears in [Kaw03, Theorem 1.13], whereas the complicated condition of [Yam18, Theorem 2.6] is simplified in Theorem 6.1 to $x^2 + y^2 + z^3 + xt^2$.

The second nice classification list for $cA_n$-singularities and smooth points is given by

$$L' := L_1 \cup L_2' \cup L_3' \cup L_4',$$

where

\begin{align*}
L_2' &:= \{((r_1, r_2, a, 1), \mathbb{V}(f)) \mid (\mathbb{V}(f), 0) \text{ is a } cA_n\text{-singularity, } \text{wt } f = r_1 + r_2\} \\
L_3' &:= \{((1, 5, 3, 2), \mathbb{V}(f)) \mid (\mathbb{V}(f), 0) \text{ is an } A_2\text{-singularity, } \text{wt } f = 6\} \\
L_4' &:= \{((4, 3, 2, 1), \mathbb{V}(f)) \mid (\mathbb{V}(f), 0) \text{ is an } E_6\text{-singularity, } \text{wt } f = 6\},
\end{align*}

where the convergent power series $f \in \mathbb{C}\{x, y, z, t\}$ defines an isolated singularity at the origin and in $L_2$ the positive integers $r_1, r_2$ and $a$ satisfy the same conditions as for the first classification list. The singularities $cA_n$, $A_2$ and $E_6$ are defined in Definitions 3.6 and 3.7.

Table 2: Local analytic equivalence classes of divisorial contractions, $cA_n$-singularities

| $w$    | conditions                        | $f$      | wt $f$ | sing | discr |
|--------|-----------------------------------|----------|--------|------|-------|
| $(1, a, b)$ | $(a, b) = 1, \ a \leq b$     |          |        | sm   | $a + b$ |
| $(r_1, r_2, a, 1)$ | $r_1 \leq r_2, \ a \mid r_1 + r_2,$    | $xy + g(z, t),$ | $r_1 + r_2$ | $cA_n$ | $a$   |
|         | $(r_1, a) = (r_2, a) = 1,$       |          |        |      |       |
|         | $a(n + 1) = r_1 + r_2$          |          |        |      |       |
| $(1, 5, 3, 2)$ |                                  | $xy + z^2 + t^3$ | $6$ | $A_2$ | $4$   |
| $(4, 3, 2, 1)$ |                                  | $x^2 + y^2 + z^3 + xt^2$ | $6$ | $E_6$ | $3$   |

3 Preliminaries

Notation 3.1. Let $\mathbb{C}$ denote the complex numbers. A variety, short for algebraic variety, is defined to be an integral separated scheme of finite type over $\mathbb{C}$. All morphisms of varieties are defined over $\mathbb{C}$. The $\mathbb{C}$-algebra of power series that are absolutely convergent
in a neighbourhood of the origin is denoted \( \mathbb{C}\{x\} \), short for \( \mathbb{C}\{x_1, \ldots, x_n\} \). The complex space, short for complex analytic space, corresponding to a variety \( X \) is denoted \( X^{an} \). A singularity is defined to be a complex space germ (see [GLS07, Definition I.1.47]). If \( I \) is an ideal of regular functions on a variety or an ideal of holomorphic functions on a complex space, then \( \mathbb{V}(I) \) denotes the zero locus of \( I \). If \( I \) is an ideal of holomorphic function germs on a complex space germ \((X,P)\), then \((\mathbb{V}(I),P)\) denotes the (possibly non-reduced) subgerm defined by \( I \) (see [GLS07, §I.1.4]).

Given a convergent power series \( f \in \mathbb{C}\{x\} := \mathbb{C}\{x_1, \ldots, x_n\} \) we define the multiplicity of \( f \), denoted \( \text{mult} \ f \), by

\[
\text{mult} \ f := \min \{i_1 + \cdots + i_n \mid x_1^{i_1} \cdots x_n^{i_n} \text{ has non-zero coefficient in } f \}.
\]

Given positive integer weights \( w_1, \ldots, w_n \) for variables \( x_1, \ldots, x_n \) we define the weight of \( f \), denoted \( \text{wt} \ f \), by

\[
\text{wt} \ f := \min \{w_1i_1 + \cdots + w_n i_n \mid x_1^{i_1} \cdots x_n^{i_n} \text{ has non-zero coefficient in } f \}
\]

if \( f \) is non-zero, and we define \( \text{wt} 0 = \infty \) otherwise. We denote the quasihomogeneous weight \( d \) part of \( f \) by \( f_{\text{wt}=d} \). The quadratic part of \( f \) is defined to be the homogeneous degree 2 part of \( f \). The quadratic rank of \( f \) is defined to be the rank of the symmetric matrix \( M \) with complex coefficients such that the quadratic part of \( f \) is equal to \( x^T M x \) where \( x \) is the vector \((x_1, \ldots, x_n)\).

We remind that the Jacobian ideal \((\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \subseteq \mathbb{C}\{x\}\) of \( f \) by \( j(f) \). We say that a set \( S \) of monomials of \( \mathbb{C}[x] := \mathbb{C}\{x_1, \ldots, x_n\} \) is a monomial spanning set for a \( \mathbb{C}\{x\}\)-algebra \( \mathbb{C}\{x\}/J \) if the set \( \{s + J \mid s \in S\} \) generates the \( \mathbb{C}\)-vector space \( \mathbb{C}\{x\}/J \), and we say \( S \) is a monomial basis for the \( \mathbb{C}\)-algebra \( \mathbb{C}\{x\}/J \) if \( \{s + J \mid s \in S\} \) is a basis for the \( \mathbb{C}\)-vector space \( \mathbb{C}\{x\}/J \). By Zorn’s Lemma every \( \mathbb{C}\)-algebra \( \mathbb{C}\{x\}/J \) has a (possibly infinite) monomial basis.

**Definition 3.2.** A divisorial contraction is a proper birational morphism \( \varphi: Y \to X \) between normal \( \mathbb{Q}\)-Gorenstein varieties with terminal singularities such that

1. the exceptional locus of \( \varphi \) is a prime divisor and
2. \( -K_Y \) is \( \varphi \)-ample.

**Definition 3.3** ([Pae21, Definition 2.14]). Let \( \varphi: Y \to X \) and \( \varphi': Y' \to X' \) be birational morphisms of varieties (or bimeromorphic holomorphisms of complex analytic spaces). We say that an isomorphism \( X \to X' \) lifts if there exists an isomorphism \( Y \cong Y' \) such that the diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & Y' \\
\downarrow \varphi & & \downarrow \varphi' \\
X & \longrightarrow & X'
\end{array}
\]

commutes. We say that \( \varphi \) and \( \varphi' \) are equivalent if there exist isomorphic open subsets \( U \subseteq X \) and \( U' \subseteq X' \) containing the centres of the morphisms \( \varphi \) and \( \varphi' \) such that the restrictions \( \varphi|_{\varphi^{-1}U}: \varphi^{-1}U \to U \) and \( \varphi'|_{\varphi'^{-1}U'}: \varphi'^{-1}U' \to U' \) are equivalent.

If we consider the complex space corresponding to a variety or when we wish to emphasize that we are working in the category of complex spaces, then we say analytically equivalent or locally analytically equivalent.
Definition 3.4. Let $n$ be a positive integer and let $w = (w_1, \ldots, w_n)$ be positive integers, called the weights of the blowup. Define a $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ by $\lambda \cdot (u, x_1, \ldots, x_n) = (\lambda^{-1}u, \lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n)$ and define $T$ by the geometric quotient $(\mathbb{C}^{n+1} \setminus \mathcal{V}(x_1, \ldots, x_n))/\mathbb{C}^*$ (or its analytification). Then the map $\varphi: T \to \mathbb{C}^n$, $[u, x_1, \ldots, x_n] \mapsto (u^{w_1}x_1, \ldots, u^{w_n}x_n)$ is called the $w$-blowup of $\mathbb{C}^n$. If $Z \subseteq \mathbb{C}^n$ is a closed subvariety (or a closed complex subspace $Z \subseteq D$ where $D \subseteq \mathbb{C}^n$ is open) and $\bar{Z}$ is the closure of $\varphi^{-1}(Z \setminus \{0\})$ in $T$ (in $\varphi^{-1}D$), then the restriction $\varphi|_{\bar{Z}}: \bar{Z} \to Z$ is called the $w$-blowup of $Z$. Let $\rho: Y \to X$ be a surjective birational morphism of varieties (or a surjective bimeromorphic holomorphism of complex spaces). Given an open subset $U \subseteq X$ containing the centre of $\rho$ and an isomorphism $U \cong X' \subseteq \mathbb{C}^n$ taking a point $P \in X$ to the origin $0$, the map $\rho$ is called the $w$-blowup of $X$ at $P$ if the restriction $\rho|_{\rho^{-1}U}: \rho^{-1}U \to U$ is equivalent, through the given isomorphism $U \cong X'$, to the $w$-blowup of $X'$.

Remark 3.5. (a) A weighted blowup crucially depends both on the isomorphism $U \cong X'$ and a choice of coordinates $x_1, \ldots, x_n$, even though it is not explicit in the notation.

(b) Replacing $w$ by $(w_1/g, \ldots, w_n/g)$ in Definition 3.4, where $g$ is the greatest common divisor of $w_1, \ldots, w_n$, gives an isomorphic blowup over $X$.

(c) By [CLS11, Theorem 5.1.11], the weighted blowup of an affine space in Definition 3.4 coincides with the toric description of subdividing a cone in [KM92, Proposition-Definition 10.3].

Definition 3.6. A simple hypersurface singularity, also known as an ADE-singularity, is a complex space germ $(X, P)$ isomorphic to $(\mathbb{V}(f), 0)$ where $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ is one of the following:

- $A_k$: $x_1^{k+1} + x_2^2 + \ldots + x_n^2$, $k \geq 1$,
- $D_k$: $x_1(x_2^2 + x_1^{k-2}) + x_3^2 + \ldots + x_n^2$, $k \geq 4$,
- $E_6$: $x_1^3 + x_2^4 + \ldots + x_n^2$,
- $E_7$: $x_1(x_1^2 + x_3^2) + x_3^3 + \ldots + x_n^2$,
- $E_8$: $x_1^3 + x_2^4 + \ldots + x_n^2$,

where $n$ is at least 1 in case $A_k$ and at least 2 in all other cases.

Definition 3.7. Let $k$ be a positive integer. A compound $A_k$-singularity, denoted $cA_k$, is a complex space germ isomorphic to $(\mathbb{V}(xy + g), 0) \subseteq (\mathbb{C}^4, 0)$, where $g \in \mathbb{C}\{z, t\}$ has multiplicity $k + 1$.

We state the known classification of divisorial contractions to both smooth points and $cA_n$-points.

Theorem 3.8 ([Kaw01, Theorem 1.1]). Let $P$ be a smooth point of a $3$-dimensional $\mathbb{Q}$-Gorenstein variety $X$ with terminal singularities. Let $\varphi: Y \to X$ be a surjective birational morphism with centre $P$. Then $\varphi$ is a divisorial contraction if and only if $\varphi$ is locally analytically equivalent to the $(1, a, b)$-blowup of $\mathbb{A}^3$, where $a$ and $b$ are coprime positive integers.

Remark 3.9. By [CLS11, Lemma 11.4.10] the discrepancy of the $(1, a, b)$-blowup of $\mathbb{A}^3$ is $a + b$.

Theorem 3.10. Let $n$ be a positive integer. Let $P$ be a $cA_n$-point of a $\mathbb{Q}$-Gorenstein variety with terminal singularities. Let $\varphi: Y \to X$ be a surjective birational morphism with centre $P$. Then $\varphi$ is a divisorial contraction if and only if one of the following holds:
Remark
The main tools we use in this paper are weight-respecting maps (see Definition 4.1) and Theorem 1.13 and [Yam18, Theorem 2.6] except for the small difference in that the valuations for 3-dimensional index $\theta$ ([Pae21, Corollary 4.4])

Lemma 4.2

(1) $\varphi$ is locally analytically equivalent to the $(r_1, r_2, a, 1)$-blowup of $\mathbb{V}(f)$ at $0$ where $f \in \mathbb{C}\{x, y, z, t\}$ is such that

(1a) $r_1, r_2$ and $a$ are positive integers such that $r_1 \leq r_2$, $a(n + 1) = r_1 + r_2$, $a$ divides $r_1 + r_2$, $a$ is coprime to both $r_1$ and $r_2$ and

(1b) $f = xy + g(z, t)$ where $\text{wt } g = r_1 + r_2$,

(2) $n = 1$ and $\varphi$ is locally analytically equivalent to the $(1, 5, 3, 2)$-blowup of $\mathbb{V}(f)$ at $0$ where $f \in \mathbb{C}\{x, y, z, t\}$ is such that

(2a) $f = xy + z^2 + t^3$

(3) $n = 2$ and $\varphi$ is locally analytically equivalent to the $(4, 3, 2, 1)$-blowup of $\mathbb{V}(f)$ at $0$ where $f \in \mathbb{C}\{x, y, z, t\}$ is such that

(3a) $f = x^2 + y^2 + 2cyx + 2xp(z, t) + 2cyp_{\text{wt}=3}(z, t) + z^3 + g(z, t)$ where $c \in \mathbb{C}\{-1, 1\}$, $\text{wt } g \geq 6$, the power series $p$ contains only monomials of weight 2 and 3 for the weights $(4, 3, 2, 1)$, the coefficient of $t^2$ is non-zero in $p$ and $\text{deg } g(z, 1) \leq 2$.

Conditions (1b), (2a) and (3a) are the same as in [Kaw02, Theorem 1.1], [Kaw03, Theorem 1.13] and [Yam18, Theorem 2.6] except for the small difference in that the condition that $z^{(r_1 + r_2)/a}$ has a non-zero coefficient in $f$ is replaced by the equivalent condition $a(n + 1) = r_1 + r_2$. We simplify Theorem 3.10 in Theorem 6.1. In particular, we show that we can replace condition (3a) with the much simpler condition $f = x^2 + y^2 + z^3 + xt^2$. This polynomial appears in [Kaw03, Example 6.8].

Remark 3.11. By [Hay99, §3.9] the discrepancy in Theorem 3.10 in cases (1), (2) and (3) is respectively $a$, 4 and 3.

4 Weight-respecting maps

The main tools we use in this paper are weight-respecting maps (see Definition 4.1) and some classical theorems from singularity theory in weight-respecting form (see Lemmas 4.3 and 4.5 and Corollary 4.7).

For Definition 4.1 and Lemma 4.2, let $n$ and $m$ be positive integers. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ denote the coordinates on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Choose positive integer weights for $x$ and $y$.

Definition 4.1 ([Pae21, Definition 4.1]). Let $X \subseteq \mathbb{C}^n$ and $X' \subseteq \mathbb{C}^m$ be complex analytic spaces. We say that a biholomorphic map $\psi: X \to X'$ taking $0$ to $0$ is weight-respecting if denoting its inverse by $\theta$, we can locally analytically around the origins write $\psi = (\psi_1, \ldots, \psi_m)$ and $\theta = (\theta_1, \ldots, \theta_n)$ where for all $i$ and $j$, the power series $\psi_j \in \mathbb{C}\{x\}$ and $\theta_i \in \mathbb{C}\{y\}$ satisfy $\text{wt}(\psi_j) \geq \text{wt}(y_j)$ and $\text{wt}(\theta_i) \geq \text{wt}(x_i)$.

Compare Definition 4.1 with the equivalence relation ‘$\sim$’ defined in [Hay99, 3.7 Weighted valuations] for 3-dimensional index $\geq 2$ terminal singularities embedded as hypersurfaces in orbifolds.

Lemma 4.2 ([Pae21, Corollary 4.4]). If a biholomorphism from $X \subseteq \mathbb{C}^n$ to $X' \subseteq \mathbb{C}^m$ taking $0$ to $0$ is weight-respecting, then it lifts to the weighted blown-up spaces.
exists an automorphism singularity at the origin with Milnor number.

Applying Nakayama Lemma ([GLS07, Proposition B.3.6]) to Equation (4.3.2) gives Equation (4.3.1). Since the interval $[0, 1]$ is compact, there exist finitely many biholomorphic map germs $\psi_1, \ldots, \psi_k$ such that the composition $\psi := \psi_k \circ \ldots \circ \psi_1$ satisfies $f \circ \psi = f + h$ and $x_i \circ \psi - x_i \in m^{N+1-\mu}$.

Finally, choosing $h$ to be the negative of the sum of the degree $> N$ parts of $f$ and choosing $\Psi$ to be the precomposition by $\psi$ proves the lemma. \qed
The lemma in [AGZV85, §12.6] is useful for computing normal forms of singularities. Here we give a weight-respecting version.

**Lemma 4.4.** Let \( n \) be a positive integer and let \( w := (w_1, \ldots, w_n) \) be positive integer weights for the variables \( x_1, \ldots, x_n \). Let \( f \in \mathbb{C}\{x\} \) define an isolated singularity at the origin. Let \( f_0 \) denote the least weight non-zero quasihomogeneous part of \( f \). Choose a monomial spanning set \( S \subseteq \mathbb{C}[x] \) for the Milnor algebra \( \mathbb{C}\{x\}/j(f_0) \) of \( f_0 \). Then there exists an automorphism \( \Psi \) of \( \mathbb{C}\{x\} \) of the form \( \forall i: \Psi(x_i) = x_i + g_i \), where each \( g_i \in \mathbb{C}\{x\} \) is either zero or satisfies \( \text{wt} g_i > \text{wt} x_i \), such that every monomial of \( \mathbb{C}[x] \) with weight greater than \( \text{wt} f_0 \) that does not belong to \( S \) has coefficient zero in \( \Psi(f) \).

**Proof.** We find an automorphism \( \Psi' \) of \( \mathbb{C}[[x]] \) satisfying the conditions of the lemma following the proof in [AGZV85, §12.6]. Next we define an automorphism \( \hat{\Psi} \) of \( \mathbb{C}\{x\} \) by letting \( \hat{\Psi}(x_i) \) be the truncation of \( \Psi'(x_i) \) up to some high enough degree \( N \). The automorphism \( \hat{\Psi} \) satisfies the conditions of the lemma except that there might be monomials of weight greater than \( N \) that have a non-zero coefficient in \( \hat{\Psi}(f) \). Now using Lemma 4.3 we find a suitable \( \Psi \).

**Lemma 4.5.** Let \( n \geq 2 \) be an integer and let \( w = (w_1, \ldots, w_n) \) be positive integer weights for variables \( x = (x_1, \ldots, x_n) \). Let \( f \in \mathbb{C}\{x\} \) be such that the coefficient of \( x_1x_2 \) is non-zero in \( f \) and \( \text{wt} x_1x_2 = \text{wt} f \). Then there exists a weight-respecting automorphism \( \Psi \) of \( \mathbb{C}\{x\} \) such that the only monomial that belongs to the ideal \( (x_1, x_2) \) and has non-zero coefficient in \( \Psi(f) \) is \( x_1x_2 \).

**Proof.** See the proof of [Pae21, Proposition 4.6].

**Remark 4.6.** In the case where \( f \) defines an isolated singularity at the origin, Lemma 4.5 follows also from Lemma 4.4.

One easy corollary of Lemma 4.4 is the following:

**Corollary 4.7.** Let the variables \( x = (x_1, \ldots, x_n) \) have positive integer weights \( w = (w_1, \ldots, w_n) \). Let the least weight non-zero quasihomogeneous part \( f_0 \) of \( f \in \mathbb{C}\{x\} \) be one of the five forms described in Definition 3.6. Then there is a weight-respecting automorphism \( \Psi \) of \( \mathbb{C}\{x\} \) such that \( \Psi(f) = f_0 \).

**Proof.** Use Lemma 4.4 with a set \( S \) that does not contain any elements of weight greater than \( \text{wt} f \).

## 5 From analytic to algebraic category

In Proposition 5.1, we show how to extend blowups along points with possibly non-reduced structure (equivalently, blowups along coherent ideal sheaves with cosupport a point) from the analytic category to the algebraic. Proposition 5.1 was explained to me by Masayuki Kawakita.

**Proposition 5.1.** Let \( X \) be a variety and \( \mathcal{J} \) a coherent \( \mathcal{O}_{X^{an}} \)-ideal sheaf with cosupport a point, where \( X^{an} \) is the analytification of \( X \). Then there exists a coherent \( \mathcal{O}_X \)-ideal sheaf \( \mathcal{I} \) such that its analytification is \( \mathcal{J} \).

**Proof.** Since the cosupport of \( \mathcal{J} \) is a point \( P \), there exists a positive integer \( k \) such that the \( k \)-th power of the maximal ideal of \( \mathcal{O}_{X^{an}, P} \) is in the stalk \( \mathcal{J}_P \). The proposition follows.

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We give an alternative construction in Corollary 5.6(a) which describes the divisorial contraction as a weighted blowup. Corollary 5.6(a) is a modification of [Pae21, Lemma 4.9] which was used for explicitly constructing weighted blowups of affine hypersurfaces with a cAn-point.

**Construction 5.2.** Let U be an affine variety Spec(ℂ[x₁, ..., xₙ]/I) containing the point \( \mathcal{V}(x₁, ..., xₙ) \), for some ideal \( I \subseteq ℂ[x₁, ..., xₙ] \). Assign positive integer weights \( w₁, ..., wₘ \) to the variables \( y₁, ..., yₘ \) of \( ℂ^m \) and assign weights \( 1, ..., 1 \) to the variables \( x₁, ..., xₙ \).

Let \( ψ: (U^{an}, 0) \rightarrow (Z, 0) \) be a local biholomorphism to a complex space \( Z \subseteq ℂ^m \) containing the origin. Define the variety \( \hat{U} \) by

\[
\hat{U} : \mathcal{V}(I + (ψ^{<w₁}_1 - y₁, ..., ψ^{<wₘ}_m - yₘ)) \subseteq ℂ^{n+m} := \text{Spec} ℂ[x₁, ..., xₙ, y₁, ..., yₘ],
\]

where \( ψ^{<w_j}_j \in ℂ[x₁, ..., xₙ] \) denotes the truncation of the \( j \)-th coordinate power series of \( ψ \) up to order \( w_j - 1 \). Note that \( \hat{U} \) is isomorphic to \( U \).

**Proposition 5.3.** In Construction 5.2, the local biholomorphism \( (\hat{U}^{an}, 0) \rightarrow (Z, 0) \) given by the composition of \( y_j \mapsto y_j + ψ^{<w_j}_j - ψ_j \) and the projection to \( ℂ^m \) is weight-respecting.

**Proof.** The local biholomorphism \( y_j \mapsto y_j + ψ^{<w_j}_j - ψ_j \) with inverse \( y_j \mapsto y_j - ψ^{<w_j}_j + ψ_j \) is clearly weight-respecting. The projection to \( ℂ^m \) is given by

\[
(x₁, ..., xₙ, y₁, ..., yₘ) \mapsto (y₁, ..., yₘ)
\]

with inverse

\[
(θ₁, ..., θₙ, y₁, ..., yₘ) \mapsto (y₁, ..., yₘ),
\]

where \( θ_i \in ℂ\{y₁, ..., yₘ\} \) are convergent power series with constant term zero. We see that for all \( i \), either \( wt_θ_i \geq wt xᵢ = 1 \) or \( θ_i = 0 \). This shows that the projection to \( ℂ^m \) is weight-respecting.

**Remark 5.4.** If any of the weights \( w_j \) was zero in Construction 5.2, then the truncation \( ψ^{<w_j}_j \) might not be a polynomial.

**Lemma 5.5.** Let \( Y₁ \rightarrow X \) and \( Y₂ \rightarrow X \) be birational morphisms of varieties. Then \( Y₁ \) and \( Y₂ \) are isomorphic over \( X \) if and only if the analytifications \( Y₁^{an} \) and \( Y₂^{an} \) are locally biholomorphic over \( X^{an} \) around the exceptional loci.

**Proof.** \( \Rightarrow \). The isomorphism \( Y₁ \rightarrow Y₂ \) induces a biholomorphism \( Y₁^{an} \rightarrow Y₂^{an} \).

\( \Leftarrow \). The local biholomorphism extends to a unique biholomorphism \( Y₁^{an} \rightarrow Y₂^{an} \) over \( X^{an} \). Now, suffices to show that if \( f \) is a rational map of varieties such that its analytification is holomorphic, then \( f \) is a morphism of varieties. For this, it suffices to show that if \( f \) is a rational function on an affine variety \( Z = \text{Spec} A \) such that its analytification \( f^{an} \) is holomorphic, then \( f \in A \). For this, we follow the argument in [JS19].

First, we show that \( f \) is integral over \( A \). Let \( \hat{A} \) be the integral closure of \( A \) in its field of fractions. Using the inclusions \( \text{Frac} A \rightarrow \text{Frac} \hat{A} \) and \( \mathcal{O}_{(\text{Spec} A)^{an}} \rightarrow \mathcal{O}_{(\text{Spec} \hat{A})^{an}} \), we see that \( f \) is a rational function on \( \text{Spec} \hat{A} \) and \( f^{an} \) is a holomorphic function on \( (\text{Spec} \hat{A})^{an} \). Therefore, \( f^{an} \) is bounded on every small analytic neighbourhood of any point of \( (\text{Spec} \hat{A})^{an} \). Therefore, the order of vanishing of \( f \) along every prime divisor \( D \) of \( \text{Spec} \hat{A} \) is non-negative. Since \( \text{Spec} \hat{A} \) is normal, we find \( f \in A \).

By [JK20, Proposition 2.2], \( A \) is integrally closed in \( \mathcal{O}_{Z^{an}}(Z^{an}) \). Since \( f \) is holomorphic, we have \( f \in \mathcal{O}_{Z^{an}}(Z^{an}) \), and since \( f \) is integral over \( A \), we find \( f \in A \).
Corollary 5.6. Let $X$ be a variety, $P \in X$ a closed point and $U \subseteq X$ an affine open containing $P$. Let $W \rightarrow Z$ be a weighted blowup of complex spaces with centre a point $Q \in Z$ such that $(X^{an}, P)$ is locally biholomorphic to $(Z, Q)$. Then:

(a) The construction in Proposition 5.3 gives a weighted blowup $Y \rightarrow X$ that is locally analytically equivalent to $W \rightarrow Z$.

(b) Every blowup $Y \rightarrow X$ that is locally analytically equivalent to $W \rightarrow Z$ is isomorphic over $X$ to a blowup $\hat{Y} \rightarrow X$ given in Construction 5.2 for some $\psi$.

Proof. (a) Suffices to consider the case where $Z$ is a complex subspace of $\mathbb{C}^n$ and $Q$ is the origin. Using Proposition 5.3, we find an isomorphism $U \rightarrow \hat{U} \subseteq \mathbb{A}^{n+m}$ and a choice of weights for the variables $x_1, \ldots, x_n, y_1, \ldots, y_m$ of $\mathbb{A}^{n+m}$ such that the weighted blowup of $\hat{U} \subseteq \mathbb{A}^{n+m}$ is locally analytically equivalent to $W \rightarrow Z$ by Lemma 4.2. By gluing, we find a weighted blowup $Y \rightarrow X$ which is locally analytically equivalent to $W \rightarrow Z$.

(b) Let $X^{an} \rightarrow Z$ be local biholomorphism that lifts to the blown-up spaces. The construction in Proposition 5.3 gives a weighted blowup $\hat{Y} \rightarrow X$, an isomorphism $\hat{X} \rightarrow X$ and a weight-respecting local biholomorphism $\hat{X}^{an} \rightarrow Z$. Since both $X^{an} \rightarrow Z$ and $\hat{X}^{an} \rightarrow Z$ lift to the blown-up spaces, $\hat{X}^{an} \rightarrow X^{an}$ locally lifts to blown-up spaces. By Lemma 5.5, the isomorphism $\hat{X} \rightarrow X$ lifts to the blown-up spaces. \qed

Example 5.7 shows that Proposition 5.1 and Corollary 5.6(a) cannot always be true when we blow up a positive-dimensional closed complex subspace.

Example 5.7. Let $X$ be a $\mathbb{Q}$-factorial 3-fold with a unique singular point $P$ which is an ordinary double point, meaning a singularity isomorphic to $(V(xy + zt), 0) \subseteq (\mathbb{C}^4, 0)$. Then locally analytically there exists a small resolution $\varphi$, the blowup of the divisor $V(x, z) \cap V(xy + zt)$ with exceptional locus a curve. On the other hand, since $X$ is $\mathbb{Q}$-factorial, there is no proper birational morphism $Y \rightarrow X$ from a smooth variety $Y$ which is locally analytically equivalent to $\varphi$.

6 Counting divisorial contractions

We show that we can simplify Theorem 3.10.

Theorem 6.1. Theorem 3.10 remains true if we

(1) replace condition (1b) with “$\text{wt } f = r_1 + r_2$”,

(2) replace condition (2a) with “$(V(f), 0)$ is an $A_2$-singularity and $\text{wt } f = 6$” and

(3) replace condition (3a) with either “$f = x^2 + y^2 + z^3 + xt^2$” or with “$(V(f), 0)$ is an $E_6$-singularity and $\text{wt } f = 6$”.

Proof. (1) Follows from [Pae21, Proposition 4.6].

(2) Let $f \in \mathbb{C}\{x, y, z, t\}$ be such that $\text{wt } f = 6$ and $(V(f), 0)$ is an $A_2$-singularity.

If the coefficient of $yt$ is non-zero and the coefficient of $xy$ is zero in $f$, then after a suitable coordinate change of the form $t \mapsto by + cz$, where $b$ and $c$ are complex numbers, the coefficients of $y^2$ and $yz$ will be zero in $f$. This coordinate change is weight-respecting. Since $f$ has quadratic rank 3, after scaling, the quadratic part will be $yt + z^2$. Now $(V(f), 0)$ is an $A_2$-singularity if and only if the coefficient of $x^3$ is non-zero, which cannot happen since $\text{wt } f = 6$.

Therefore, since the quadratic rank of $f$ is 3, the coefficient of $xy$ is non-zero. After a suitable coordinate change of the form $x \mapsto x + by + cz + dt$, where $a, b, c, d$ are complex
numbers, the coefficients of $y^2$, $yz$ and $yt$ will be zero. This coordinate change is weight-respecting. Now the coefficient of $z^2$ must be non-zero. After scaling, the quadratic part of $f$ will be $xy + z^2$. We see that $(\forall(f), 0)$ is an $A_2$-singularity if and only if the coefficient of $t^3$ is non-zero.

After scaling, the least weight non-zero quasihomogeneous part of $f$ with respect to weights $w = (3, 3, 3, 2)$ will be $xy + z^2 + t^3$. Corollary 4.7 gives a weight-respecting automorphism $\Psi$ such that $(\Psi(f), 0)$ is an $E_6$-singularity at the origin.

(3) To begin, we show that $f \in \mathbb{C}\{x, y, z, t\}$ satisfying condition (3a) of Theorem 3.10 defines an $E_6$-singularity at the origin. Let $\Psi$ be the automorphism of $\mathbb{C}\{x, y, z, t\}$ given by composing $x \mapsto x/\sqrt{1-c^2}$ with $y \mapsto y - c(x + p_{wt=3})$. Defining $p', g' \in \mathbb{C}\{z, t\}$ by $p = 1/(\sqrt{1-c^2})p + c^2p_{wt=3}$ and $g = g' + c^2 + p^2_{wt=3} - z^3$, we find that $\Psi(f)$ is equal to $x^2 + y^2 + xp' + g'$, where wt $p'$ is 2, wt $g'$ is 6 and all monomials of weight greater than 3 have coefficient zero in $p'$. Let $\Phi$ be the coordinate change $x \mapsto x - p'/2$ composed with a suitable scaling of the variable $t$. Then the least weight non-zero quasihomogeneous part of $\Phi(\Psi(f))$ will be $x^2 + y^2 + z^3 + t^4$ under the weights $w = (6, 6, 4, 3)$. Using the lemma in [AGZV85, §12.6] or Corollary 4.7 we find that $\Phi(\Psi(f))$ is right equivalent to $x^2 + y^2 + z^3 + t^4$, proving that $f$ defines an $E_6$-singularity at the origin.

Now let $f \in \mathbb{C}\{x, y, z, t\}$ be such that wt $f = 6$ and $(\forall(f), 0)$ is an $E_6$-singularity.

First, we show that the coefficient of $xz$ in $f$ is zero. We remind that the quadratic rank of a convergent power series defining a 3-dimensional $E_6$-singularity is 2. If the coefficient of $xz$ is non-zero, since $f$ has quadratic rank 2, the coefficient of $y^2$ must be zero. After a suitable coordinate change of the form $z \mapsto ax + by + cz$, where $a, b$ and $c$ are complex numbers and $c$ is non-zero, the quadratic part of $f$ will be $xz$. By Lemma 4.5 after a weight-respecting coordinate change $f$ will be of the form $xz + h$ where $h \in \mathbb{C}\{y, t\}$. If the coefficient of $y^2t$ is non-zero, then by [GLS07, Theorem I.2.51] $(\forall(f), 0)$ is either a $D_3$-singularity or a non-isolated singularity, a contradiction. So the coefficient of $y^2t$ is zero. Since $(\forall(f), 0)$ is a $cA_2$-singularity, after scaling the 3-jet of $f$ will be $xz + y^3$. If the coefficient of $yt^3$ is non-zero, then Corollary 4.7 with $w := (9, 6, 9, 4)$ and $f_0 := xz + y^3 + dyt^3$ implies that $(\forall(f), 0)$ is an $E_7$-singularity, a contradiction. Therefore, the coefficient of $yt^3$ is zero. Now [GLS07, Theorem I.2.55(2)] shows that $(\forall(f), 0)$ is not a simple singularity, a contradiction.

Second, we show that the coefficient of $y^2$ is non-zero. If the coefficient of $y^2$ is zero, then the coefficient of $xy$ must be non-zero. After a suitable coordinate change of the form $y \mapsto ax + y + h$, where $a \in \mathbb{C}$ and $h \in \mathbb{C}\{x, y, z, t\}$ has multiplicity at least 2 or is zero, the only monomial that is divisible by $x$ and has non-zero coefficient in $f$ will be $xy$. Now the weight of $f$ is at least 5 and the monomials of weight 5 that have a non-zero coefficient in $f$ are in the set $\{zt^3, t^5\}$. After a suitable coordinate change of the form $x \mapsto x + h'$, where $h' \in \mathbb{C}\{y, z, t\}$ has multiplicity at least 2 or is zero, the only monomial in the ideal $(x, y)$ that has non-zero coefficient in $f$ will be $xy$. The weight of $f$ is still at least 5 and the monomials of weight 5 are still in the set $\{zt^3, t^5\}$. Since $(\forall(f), 0)$ is a $cA_2$-singularity, the coefficient of $z^3$ is non-zero. If the coefficient of $zt^3$ is non-zero, then Corollary 4.7 with $w := (9, 9, 6, 4)$ and $f_0 := axy + bxz + czt^3$, where $a, b, c \in \mathbb{C}$ are non-zero, shows that $(\forall(f), 0)$ is an $E_7$-singularity, a contradiction. So the coefficient of $zt^3$ is zero. If the coefficient of $t^5$ is non-zero, then Corollary 4.7 with $w := (15, 15, 5, 3)$ and $f_0 := axy + bz^3 + ct^5$, where $a, b, c \in \mathbb{C}$ are non-zero, shows that $(\forall(f), 0)$ is an $E_8$-singularity, a contradiction. Therefore, $f - xy$ belongs to the ideal $(z, t^2)$ of $\mathbb{C}\{z, t\}$.

By [GLS07, Theorem I.2.55(2)] $(\forall(f), 0)$ is not a simple singularity, a contradiction.

Next, we show that the coefficient of $xt^2$ is non-zero. If the coefficient of $xt^2$ is zero,
then after a suitable linear weight-respecting coordinate change the quadratic part of $f$ will be $xy + y^2$. Now after a suitable linear weight-respecting coordinate change of the form $y \mapsto y + h$, where $h \in \mathbb{C}\{x, y, z, t\}$ has multiplicity at least 2 or is non-zero, followed by an application of Lemma 4.3, the only monomial with non-zero coefficient in $f$ that is divisible by $x$ will be $xy$. After a suitable coordinate change of the form $x \mapsto x + h'$, where $h' \in \mathbb{C}\{x, y, z, t\}$, the only monomial in the ideal that has non-zero coefficient in $f$ will be $xy$ and the weight of $f$ will still be 6. By [GLS07, Theorem I.2.55(2)] $(V(f), 0)$ is not a simple singularity, a contradiction.

Now, after a suitable linear weight-respecting coordinate change, the quadratic part of $f$ will be $x^2 + y^2$. Using a suitable weight-respecting coordinate change of the form $x \mapsto x + h$ and $y \mapsto y + h'$, where $h, h' \in \mathbb{C}\{x, y, z, t\}$, followed by an application of Lemma 4.3 the power series $f$ will have the form

$$f = x^2 + y^2 + xp + g,$$

where $p \in \mathbb{C}\{z, t\}$ has only monomials of weight 2 and 3, the coefficient of $t^2$ in $p$ is 1 and the coefficient of $z^3$ in $g \in \mathbb{C}\{z, t\}$ is 1.

Finally, we show that there exists a weight-respecting automorphism $\Psi$ of $\mathbb{C}\{x, y, z, t\}$ such that $\Psi(f) = x^2 + y^2 + z^3 + xt^2$, where $f$ is given by Equation (6.1.1). The least weight non-zero quasihomogeneous part of $g - p^2/4$ under the weights $(4, 3)$ is $z^3 - t^4/4$. By Corollary 4.7 there exists an automorphism $\Phi$ of $\mathbb{C}\{z, t\}$ such that $\Phi(g - p^2/4)$ is equal to $z^3 - t^4/4$, $\Phi(z) - z$ is in the ideal $(z, t^2)$ and $\Phi(t) - t$ is in the ideal $(z, t^2)$. So $\Phi$ is weight-respecting with respect to weights $(2, 1)$. We find that

$$(\Phi(p) - t^2)(\Phi(p) + t^2) = 4(\Phi(g) - z^3)$$

is either zero or has weight at least 6. The term $\Phi(p) + t^2$ has weight 2 since the coefficient of $t^2$ is 2. Therefore, $\Phi(p) - t^2$ is either zero or has weight at least 4. Applying $\Phi^{-1}$ to Equation (6.1.2), we find that $\Phi^{-1}(t^2) - p$ is also either zero or has weight at least 4. Now suffices to choose $\Psi$ to be

$$\Psi(x) := x + \frac{t^2 - \Phi(p)}{2}, \quad \Psi(y) := y, \quad \Psi(z) := \Phi(z), \quad \Psi(t) := \Phi(t).$$

$\square$

**Lemma 6.2.** Let $n$ be a positive integer. Let $P$ be a $cA_n$-point of a $\mathbb{Q}$-Gorenstein variety $X$ with terminal singularities. Then any two divisorial contractions to $X$ with centre $P$ are locally analytically equivalent if they are either

1. both of type (1) with the same weights $(r_1, r_2, a, 1)$,
2. both of type (2) or
3. both of type (3).

**Proof.** Case (1) is [Pae21, Proposition 4.7], case (2) is clear and case (3) follows from Theorem 6.1(3). $\square$

We describe conditions for the existence of divisorial contractions to $X$ with centre $P$ of types (1), (2) and (3) of Theorem 3.10.

**Lemma 6.3.** Let $P$ be a $cA_n$-point of a $\mathbb{Q}$-Gorenstein variety $X$ with terminal singularities.

(a) If there exists a divisorial contraction of type (1) to $X$ with center $P$ which is an $(r_1, r_2, a, 1)$-blowup, then for all $a' \in \{1, \ldots, a\}$ and for all $r_1' \in \{1, \ldots, a'(n+1) - 1\}$ such that $a'$ is coprime to both $r_1'$ and $r_2' := a'(n+1) - r_1'$ there exists a divisorial contraction of type (1) which is an $(r_1', r_2', a', 1)$-blowup.
(b) There is a positive integer \( N \) such that there is no divisorial contraction of type (1) to \( X \) with center \( P \) which is an \((r_1, r_2, a, 1)\)-blowup where \( a > N \).

(c) If \( n = 1 \), then there exists a divisorial contraction of type (1) which is an \((r_1, r_2, a, 1)\)-blowup if and only if \((X^\text{an}, P)\) is an \(A_k\)-singularity where \( k \geq a \).

(d) If \( n = 1 \), then there exists a divisorial contraction of type (2) if and only if \((X^\text{an}, P)\) is the \(A_2\)-singularity.

(e) If \( n = 2 \), then there exists a divisorial contraction of type (1) with \( a = 2 \) if and only if \((X^\text{an}, P)\) is not a simple singularity.

(f) If \( n = 2 \), then there exists a divisorial contraction of type (3) if and only if \((X^\text{an}, P)\) is an \(E_6\)-singularity.

Proof. (a) If \( f \) is of the form \( xy + g \) and the weight of \( g \in \mathbb{C}\{z,t\} \) is \( r_1 + r_2 \) with respect to the weights \((r_1, r_2, a, 1)\), then the weight of \( g \) is also \( r_1' + r_2' \) with respect to the weights \((r_1', r_2', a', 1)\).

(b) By [GLS07, Corollary I.2.18] or [AGZV85, §12.2] the Milnor number of \( xy + g_{\text{wt}=r_1+r_2} \) is at least \( n/(n+1)a - 1 \). On the other hand, the isolated singularity \((X^\text{an}, P)\) has finite Milnor number.

(c) By [GLS07, Theorem I.2.55(2)] a \(cA_2\)-singularity \((V(f), 0)\), where \( f \) is in \( \mathbb{C}\{x, y, z, t\} \), is not contact simple if and only if there is an automorphism \( \Psi \) of \( \mathbb{C}\{x, y, z, t\} \) such that \( \Psi(f) = xy + g(z, t) \), where \( g \) is in the ideal \( z, t^2 \) of \( \mathbb{C}\{z, t\} \).

Parts (c), (d) and (f) follow from the definition of simple singularities (Definition 3.6). \( \square \)

It is known that there are only finitely many divisorial contractions with discrepancy at most 1, see [Kaw05, below Theorem 1.2]. I have added a proof here since I have not found a proof in the literature. The precise statement is as follows:

**Proposition 6.4.** Let \( X \) be a \(\mathbb{Q}\)-Gorenstein variety with terminal singularities. Then there are only finitely many divisorial contractions to \( X \) with discrepancy at most 1.

Proof. Let \( f : Y \to X \) be a resolution of singularities with exceptional locus of pure codimension 1. Let \( v \) be the valuation on the function field \( \mathbb{C}(X) \) given by the exceptional divisor of a divisorial contraction to \( X \). Then \( v \) is equal to the valuation given by a prime divisor \( D \) on a normal variety \( Z \) with a proper birational morphism \( Z \to Y \). The centre of \( D \) on \( Y \) is necessarily contained in an exceptional prime divisor of \( f \). We see that if the discrepancy of \( D \) is at most 1, then the centre of \( D \) on \( Y \) necessarily coincides with an exceptional prime divisor of \( f \). So \( v \) is equal to the valuation given by one of the finitely many exceptional prime divisors of \( f \). The proposition follows from the fact that any two divisorial contractions whose exceptional divisors define the same valuation are isomorphic over \( X \), see [Kaw01, Lemma 3.4]. \( \square \)

**Theorem 6.5.** Let \( n \) be a positive integer. Let \( P \) be a point of a \(\mathbb{Q}\)-Gorenstein variety \( X \) with terminal singularities. We count the number of divisorial contractions to \( X \) with centre \( P \).

(a) If \((X^\text{an}, P)\) is smooth, then there are uncountably many divisorial contractions up to isomorphism over \( X \) and countably many up to local analytic equivalence.

(b) If \((X^\text{an}, P)\) is a \(cA_n\)-singularity that admits only discrepancy 1 divisorial contractions, then there are exactly \( n \) divisorial contractions up to isomorphism over \( X \) and exactly \( \lceil n/2 \rceil \) up to local analytic equivalence, where \( \lceil r \rceil \) denotes the smallest integer greater than or equal to the real number \( r \).
(c) If $(X^a, P)$ is a $\text{cA}_n$-singularity that admits a divisorial contraction with discrepancy $\geq 2$, then there are uncountably many divisorial contractions up to isomorphism over $X$ and finitely many up to local analytic equivalence.

Proof. (a) By Theorem 3.8 there are countably many divisorial contractions up to local analytic equivalence. Since the automorphism $\Psi$ of $\mathbb{C}\{x, y, z\}$ given by $z \mapsto z + ax$, where $a \in \mathbb{C}$ is non-zero, does not lift to an isomorphism of the blown-up spaces when performing a $(1, 1, 2)$-blowup, there are uncountably many divisorial contractions up to isomorphism over $X$.

(b) Similarly to the proof of [Hay99, Theorem 6.4] we can show that there are exactly $n$ local analytic germs of divisorial contractions up to isomorphism over $X^a$. Note that the last sentence in the statement of [Hay99, Theorem 6.4] contains a typo, it should say: “Furthermore, there are exactly $k$ divisors with discrepancies $1/m$ over $X$” (the symbol $k$ was missing). The global algebraic divisorial contractions are constructed using Proposition 5.1 or Corollary 5.6. To see that there are exactly $[n/2]$ divisorial contractions up to local analytic equivalence, note that $(x, y, z, t) \mapsto (y, x, z, t)$ is weight-respecting with respect to the weights $(r_1, r_2, a, 1)$ and $(r_2, r_1, a, 1)$.

(c) It follows from Lemma 6.3(b) and Lemma 6.2 that there are only finitely many divisorial contractions up to local analytic equivalence.

If $(X^a, P)$ is not an $E_6$-singularity, then there exists a divisorial contraction of type (1) of Theorem 6.1 with $a > 1$. By Lemma 6.3(a) there exists a divisorial contraction with $r_1 = 1$. Let $f \in \mathbb{C}\{x, y, z, t\}$ be as in condition (1b). For any $c \in \mathbb{C}$ there exists an automorphism $\Phi_c$ of $\mathbb{C}\{x, y, z, t\}$ that fixes $f$ given by

$$\Phi_c(x) := x, \quad \Phi_c(y) := y + h, \quad \Phi_c(z) := z + cx, \quad \Phi_c(t) := t,$$

where $h \in \mathbb{C}\{x, y, z, t\}$ depends on $f$. Each automorphism $\Phi_c$ defines a divisorial contraction of the analytic germ $(X^a, P)$, naming composing the divisorial contraction to $X$ with the precomposition with $\Phi_c$. The composition $\Phi_{c^2} \circ \Phi_{c^{-1}}$ is weight-respecting with respect to weights $(1, r_2, a, 1)$ if and only if $c = c'$. We can check on the affine patch $t \neq 0$ of the $(1, r_2, a, 1)$-blown-up space that the biholomorphic map germ corresponding to $\Phi_{c^2} \circ \Phi_{c^{-1}}$ lifts to an isomorphism of the blown-up spaces if and only if $c = c'$. Thus there are uncountably many analytic germs of $(1, r_2, a, 1)$-blowups to $X$ with centre $P$. By Proposition 5.1 or Corollary 5.6 each such analytic germ extends to a divisorial contraction to $X$ with centre $P$.

If $(X^a, P)$ is an $E_6$-singularity, then for any complex number $v \in \mathbb{C}$, any square root $w$ of $1 - v^2$ and any $u \in \{-1, 1\}$, the automorphism $\Psi_{u,v,w}$ of $\mathbb{C}\{x, y, z, t\}$ given by

$$\Psi_{u,v,w}(x) := vx + wy + (v - 1)t^2/2, \quad \Psi_{u,v,w}(z) := z$$

$$\Psi_{u,v,w}(y) := uwx - uwy + ut^2/2, \quad \Psi_{u,v,w}(t) := t$$

fixes $x^2 + y^2 + z^2 + xt^2$. Note that $\Psi_{u',v',w'} \circ \Psi_{u,v,w}^{-1}$ is weight-respecting with respect to weights $(4, 3, 2, 1)$ if and only if $v' = v$ and $w' = uw$. We can check that the biholomorphic map germ corresponding to $\Psi_{u',v',w'} \circ \Psi_{u,v,w}^{-1}$ lifts to an isomorphism of the blown-up spaces if and only if $v' = v$ and $w' = uw$. Similarly to the previous case, this shows that there are uncountably many divisorial contractions of type (3) to $X$ with center $P$. 

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