ADAPTED METRICS FOR CODIMENSION ONE SINGULAR HYPERBOLIC FLOWS

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Abstract. For a partially hyperbolic splitting $T_\Gamma M = E \oplus F$ of $\Gamma$, a $C^1$ vector field $X$ on a $m$-manifold, we obtain singular-hyperbolicity using only the tangent map $DX$ of $X$ and its derivative $DX_\Gamma$ whether $E$ is one-dimensional subspace. We show the existence of adapted metrics for singular hyperbolic set $\Gamma$ for $C^1$ vector fields if $\Gamma$ has a partially hyperbolic splitting $T_\Gamma M = E \oplus F$ where $F$ is volume expanding, $E$ is uniformly contracted and a one-dimensional subspace.

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1. Introduction

Let $M$ be a connected compact finite $m$-dimensional manifold, $m \geq 3$, with or without boundary. We consider a vector field $X$, such that $X$ is inwardly transverse to the boundary $\partial M$, if $\partial M \neq \emptyset$. The flow generated by $X$ is denoted by $X_t$.

A hyperbolic set for a flow $X_t$ on a finite dimensional Riemannian manifold $M$ is a compact invariant set $\Gamma$ with a continuous splitting of the tangent bundle, $T_\Gamma M = E^s \oplus E^u$.
$E^X \oplus E^u$, where $E^X$ is the direction of the vector field, for which the subbundles are invariant under the derivative $DX_t$ of the flow $X_t$

$$DX_t \cdot E^*_x = E^*_x(x), \quad x \in \Gamma, \quad t \in \mathbb{R}, \quad *=s,X,u; \quad (1)$$

and $E^s$ is uniformly contracted by $DX_t$ and $E^u$ is likewise expanded: there are $K, \lambda > 0$ so that

$$\|DX_t|_{E^s_x}\| \leq Ke^{-\lambda t}, \quad \|(DX_t|_{E^s_x})^{-1}\| \leq Ke^{-\lambda t}, \quad x \in \Gamma, \quad t \in \mathbb{R}. \quad (2)$$

Very strong properties can be deduced from the existence of such hyperbolic structure; see for instance [12, 13, 28, 19, 26].

An important feature of hyperbolic structures is that it does not depend on the metric on the ambient manifold (see [16]). We recall that a metric is said to be adapted to the hyperbolic structure if we can take $K = 1$ in equation (2).

Weaker notions of hyperbolicity (e.g. dominated splitting, partial hyperbolicity, volume hyperbolicity, sectional hyperbolicity, singular hyperbolicity) have been developed to encompass larger classes of systems beyond the uniformly hyperbolic ones; see [11] and specifically [30, 5, 9] for singular hyperbolicity and Lorenz-like attractors.

In the same work [16], Hirsch, Pugh and Shub asked about adapted metrics for dominated splittings. The positive answer was given by Gourmelon [15] in 2007, where it is given adapted metrics to dominated splittings for both diffeomorphisms and flows, and he also gives an adapted metric for partially hyperbolic splittings as well.

Proving the existence of some hyperbolic structure is, in general, a non-trivial matter, even in its weaker forms.

In [20], Lewowicz stated that a diffeomorphism on a compact riemannian manifold is Anosov if and only if its derivative admits a nondegenerate Lyapunov quadratic function.

An example of application of the adapted metric from [15] is contained in [6], where the first author jointly with V. Araújo, following the spirit of Lewowicz’s result, construct quadratic forms which characterize partially hyperbolic and singular hyperbolic structures on a trapping region for flows.

In [7], the first author and V. Araújo provided an alternative way to obtain singular hyperbolicity for three-dimensional flows using the same expression as in Proposition 2.1 applied to the infinitesimal generator of the exterior square $\wedge^2 DX_t$ of the cocycle $DX_t$. This infinitesimal generator can be explicitly calculated through the infinitesimal generator $DX$ of the linear multiplicative cocycle $DX_t$ associated to the vector field $X$.

Here, we provide a similar result as above for $m$-dimensional flows if this admits a partially hyperbolic splitting for which one of the invariant subbundles is one-dimensional.

Moreover, we show the existence of adapted metrics for a singular hyperbolic set $\Gamma$ for $C^1$ vector fields if $\Gamma$ has a partially hyperbolic splitting $T_m = E \oplus F$, where $F$ is volume expanding, $E$ is uniformly contracted and one-dimensional subbundle.

The paper is organized as follow. In Section 2 we provide definitions and statement of results. In Section 3 we provide some auxiliary results. Finally, in Section 4 are given the proofs of our theorems.
We now present preliminary definitions and results.

We recall that a trapping region $U$ for a flow $X_t$ is an open subset of the manifold $M$ which satisfies: $X_t(U)$ is contained in $U$ for all $t > 0$, and there exists $T > 0$ such that $X_t(U)$ is contained in the interior of $U$ for all $t > T$. We define $\Gamma(U) = \cap_{t>0} X_t(U)$ to be the maximal positive invariant subset in the trapping region $U$.

A singularity for the vector field $X$ is a point $\sigma \in M$ such that $X(\sigma) = \vec{0}$ or, equivalently, $X_t(\sigma) = \sigma$ for all $t \in \mathbb{R}$. The set formed by singularities is the singular set of $X$ denoted $\text{Sing}(X)$. We say that a singularity is hyperbolic if the eigenvalues of the derivative $DX(\sigma)$ of the vector field at the singularity $\sigma$ have nonzero real part.

Definition 1. A dominated splitting over a compact invariant set $\Lambda$ of $X$ is a continuous $DX_t$-invariant splitting $T\Lambda M = E \oplus F$ with $E_x \neq \{0\}$, $F_x \neq \{0\}$ for every $x \in \Lambda$ and such that there are positive constants $K, \lambda$ satisfying

$$\|DX_t|_{E_x}\| \cdot \|DX_t|_{F_x(\sigma)}\| < Ke^{-\lambda t}, \text{ for all } x \in \Lambda, \text{ and all } t > 0.$$ (3)

A compact invariant set $\Lambda$ is said to be partially hyperbolic if it exhibits a dominated splitting $T\Lambda M = E \oplus F$ such that subbundle $E$ is uniformly contracted, i.e., there exists $C > 0$ and $\lambda > 0$ such that $\|DX_t|_{E_x}\| \leq Ce^{-\lambda t}$ for $t \geq 0$. In this case $F$ is the central subbundle of $\Lambda$. Or else, we may replace uniform contraction along $E$ by uniform expansion along $F$ (the right hand side condition in (2).

We say that a $DX_t$-invariant subbundle $F \subset T\Lambda M$ is a sectionally expanding subbundle if $\dim F_x \geq 2$ is constant for $x \in \Lambda$ and there are positive constants $C, \lambda$ such that for every $x \in \Lambda$ and every two-dimensional linear subspace $L_x \subset F_x$ one has

$$|\det(DX_t|_{L_x})| > Ce^{\lambda t}, \text{ for all } t > 0.$$ (4)

Definition 2. [21, Definition 2.7] A sectional-hyperbolic set is a partially hyperbolic set whose central subbundle is sectionally expanding.

This is a particular case of the so called singular hyperbolicity whose definition we recall now. A $DX_t$-invariant subbundle $F \subset T\Lambda M$ is said to be a volume expanding if in the above condition (3), we may write

$$|\det(DX_t|_{F_x})| > Ce^{\lambda t}, \text{ for all } t > 0.$$ (5)

Definition 3. [22, Definition 1] A singular hyperbolic set is a partially hyperbolic set whose central subbundle is volume expanding.

Clearly, in the three-dimensional case, these notions are equivalent.

This is a feature of the Lorenz attractor as proved in [29] and also a notion that extends hyperbolicity for singular flows, because sectional hyperbolic sets without singularities are hyperbolic; see [23, 5].

We assume that coordinates are chosen locally adapted to $J$ in such a way that $J(v) = \langle J_x(v), v \rangle, v \in T_x M, x \in U$, and $J_x : T_x M \to \mathbb{R}$ is a self-adjoint linear operator having diagonal matrix with $\pm 1$ entries along the diagonal.
We say that a $C^1$ family $\mathcal{J}$ of indefinite and non-degenerate quadratic forms is compatible with a continuous splitting $E_\Gamma \oplus F_\Gamma = E_\Gamma$ of a vector bundle over some compact subset $\Gamma$ if $E_x$ is a $\mathcal{J}$-negative subspace and $F_x$ is a $\mathcal{J}$-positive subspace for all $x \in \Gamma$.

**Proposition 2.1.** [6, Proposition 1.3] A $\mathcal{J}$-non-negative vector field $X$ on $U$ is strictly $\mathcal{J}$-separated if, and only if, there exists a compatible family $\mathcal{J}_0$ of forms and there exists a function $\delta : U \to \mathbb{R}$ such that the operator $\tilde{J}_{0,x} := J_0 \cdot DX(x) + DX(x)^* \cdot J_0$ satisfies

$$\tilde{J}_{0,x} - \delta(x)J_0 \quad \text{is positive definite,} \quad x \in U,$$

where $DX(x)^*$ is the adjoint of $DX(x)$ with respect to the adapted inner product.

**Remark 2.2.** The expression for $\tilde{J}_{0,x}$ in terms of $J_0$ and the infinitesimal generator of $DX_t$ is, in fact, the time derivative of $\mathcal{J}_0$ along the flow direction at the point $x$, which we denote $\partial_t J_0$; see item 1 of Proposition 3.1. We keep this notation in what follows.

Let $A : G \times \mathbb{R} \to G$ be a smooth map given by a collection of linear bijections

$$A_t(x) : G_x \to G_{X_t(x)}, \quad x \in \Gamma, t \in \mathbb{R},$$

where $\Gamma$ is the base space of the finite dimensional vector bundle $G$, satisfying the cocycle property

$$A_0(x) = Id, \quad A_{t+s}(x) = A_t(X_s(x)) \circ A_s(x), \quad x \in \Gamma, t, s \in \mathbb{R},$$

with $\{X_t\}_{t \in \mathbb{R}}$ a complete smooth flow over $M \supset \Gamma$. We note that for each fixed $t > 0$ the map $A_t : G \to G, v_x \in G_x \mapsto A_t(x) \cdot v_x \in G_{X_t(x)}$ is an automorphism of the vector bundle $G$.

The natural example of a linear multiplicative cocycle over a smooth flow $X_t$ on a manifold is the derivative cocycle $A_t(x) = DX_t(x)$ on the tangent bundle $G = TM$ of a finite dimensional compact manifold $M$. Another example is given by the exterior power $A_t(x) = \wedge^k DX_t$ of $DX_t$ acting on $G = \wedge^k TM$, the family of all $k$-vectors on the tangent spaces of $M$, for some fixed $1 \leq k \leq \dim G$.

It is well-known that the exterior power of a inner product space has a naturally induced inner product and thus a norm. Thus $G = \wedge^k TM$ has an induced norm from the Riemannian metric of $M$. For more details see e.g. [10].

In what follows we assume that the vector bundle $G$ has a smoothly defined inner product in each fiber $G_x$ which induces a corresponding norm $\| \cdot \|_x, x \in \Gamma$.

**Definition 4.** A continuous splitting $G = E \oplus F$ of the vector bundle $G$ into a pair of subbundles is dominated (with respect to the automorphism $A$ over $\Gamma$) if

- the splitting is invariant: $A_t(x) \cdot E_x = E_{X_t(x)}$ and $A_t(x) \cdot F_x = F_{X_t(x)}$ for all $x \in \Gamma$ and $t \in \mathbb{R}$; and
- there are positive constants $K, \lambda$ satisfying

$$\|A_t|_{E_x}\| \cdot \|A_t|_{F_{X_t(x)}}\| < Ke^{-\lambda t}, \quad \text{for all} \quad x \in \Gamma, \quad \text{and all} \quad t > 0. \quad (6)$$

We say that the splitting $G = E \oplus F$ is partially hyperbolic if it is dominated and the subbundle $E$ is uniformly contracted: $\|A_t \cdot E_x\| \leq Ce^{-\mu t}$ for all $t > 0$ and suitable constants $C, \mu > 0$. 
Let $E_U$ be a finite dimensional vector bundle with inner product $\langle \cdot, \cdot \rangle$ and base given by the trapping region $U \subset M$. Let $\mathcal{J} : E_U \to \mathbb{R}$ be a continuous family of quadratic forms $\mathcal{J}_x : E_x \to \mathbb{R}$ which are non-degenerate and have index $0 < q < \dim(E) = n$. The index $q$ of $\mathcal{J}$ means that the maximal dimension of subspaces of non-positive vectors is $q$. Using the inner product, we can represent $\mathcal{J}$ by a family of self-adjoint operators $J_{x} : E_{x} \rightleftharpoons$ as $\mathcal{J}_x(v) = \langle J_{x}(v), v \rangle, v \in E_x, x \in U$.

We also assume that $(\mathcal{J}_x)_{x \in U}$ is continuously differentiable along the flow. The continuity assumption on $\mathcal{J}$ means that for every continuous section $Z$ of $E_U$ the map $U \ni x \mapsto \mathcal{J}(Z(x)) \in \mathbb{R}$ is continuous. The $C^1$ assumption on $\mathcal{J}$ along the flow means that the map $\mathbb{R} \ni t \mapsto \mathcal{J}_x(T(X_t(x))) \in \mathbb{R}$ is continuously differentiable for all $x \in U$ and each $C^1$ section $Z$ of $E_U$.

Using Lagrange diagonalization of a quadratic form, it is easy to see that the choice of basis to diagonalize $\mathcal{J}_y$ depends smoothly on $y$ if the family $(\mathcal{J}_x)_{x \in U}$ is smooth, for all $y$ close enough to a given $x$. Therefore, choosing a basis for $T_x$ adapted to $\mathcal{J}_x$ at each $x \in U$, we can assume that locally our forms are given by $\langle J_{x}(v), v \rangle$ with $J_x$ a diagonal matrix whose entries belong to $\{ \pm 1 \}$, $J^*_x = J_x$, $J^2_x = I$ and the basis vectors depend as smooth on $x$ as the family of forms $(\mathcal{J}_x)_x$.

We let $\mathcal{C}_\pm = \{C_{\pm}(x)\}_{x \in U}$ be the family of positive and negative cones associated to $\mathcal{J}$

$$C_{\pm}(x) := \{0\} \cup \{v \in E_x : \pm \mathcal{J}_x(v) > 0\}, \quad x \in U,$$

and also let $\mathcal{C}_0 = \{C_0(x)\}_{x \in U}$ be the corresponding family of zero vectors $C_0(x) = \mathcal{J}_x^{-1}\{0\}$ for all $x \in U$.

Let $A_t : E \times \mathbb{R} \to E$ be a linear multiplicative cocycle on the vector bundle $E$ over the flow $X_t$. The following definitions are fundamental to state our results.

**Definition 5.** Given a continuous field of non-degenerate quadratic forms $\mathcal{J}$ with constant index on the positively invariant open subset $U$ for the flow $X_t$, we say that the cocycle $A_t(x)$ over $X_t$ is

- **$\mathcal{J}$-separated** if $A_t(x)(C_+(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$ (simple cone invariance);
- **strictly $\mathcal{J}$-separated** if $A_t(x)(C_+(x) \cup C_0(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$ (strict cone invariance);
- **$\mathcal{J}$-monotone** if $\mathcal{J}_{X_t(x)}(DX_t(x)v) \geq \mathcal{J}_x(v)$, for each $v \in T_xM \setminus \{0\}$ and $t > 0$;
- **strictly $\mathcal{J}$-monotone** if $\partial_t(\mathcal{J}_{X_t(x)}(DX_t(x)v)) \big|_{t=0} > 0$, for all $v \in T_xM \setminus \{0\}$, $t > 0$ and $x \in U$;
- **$\mathcal{J}$-isometry** if $\mathcal{J}_{X_t(x)}(DX_t(x)v) = \mathcal{J}_x(v)$, for each $v \in T_xM$ and $x \in U$.

We say that the flow $X_t$ is (strictly) $\mathcal{J}$-separated on $U$ if $DX_t(x)$ is (strictly) $\mathcal{J}$-separated on $T_vU$. Analogously, the flow of $X$ on $U$ is (strictly) $\mathcal{J}$-monotone if $DX_t(x)$ is (strictly) $\mathcal{J}$-monotone.

**Remark 2.3.** If a flow is strictly $\mathcal{J}$-separated, then for $v \in T_xM$ such that $\mathcal{J}_x(v) \leq 0$ we have $\mathcal{J}_{X_{-t}(x)}(DX_{-t}(v)) < 0$, for all $t > 0$, and $x$ such that $X_{-s}(x) \in U$ for every $s \in [-t, 0]$. Indeed, otherwise $\mathcal{J}_{X_{-t}(x)}(DX_{-t}(v)) \geq 0$ would imply $\mathcal{J}_x(v) = \mathcal{J}_x(DX_t(DX_{-t}(v))) > 0$, contradicting the assumption that $v$ was a non-positive vector.
This means that a flow $X_t$ is strictly $\mathcal{J}$-separated if, and only if, its time reversal $X_{-t}$ is strictly $(-\mathcal{J})$-separated.

A vector field $X$ is $\mathcal{J}$-non-negative on $U$ if $\mathcal{J}(X(x)) \geq 0$ for all $x \in U$, and $\mathcal{J}$-non-positive on $U$ if $\mathcal{J}(X(x)) \leq 0$ for all $x \in U$. When the quadratic form used in the context is clear, we will simply say that $X$ is non-negative or non-positive.

A characterization of dominated splittings, via quadratic forms is given in [6] (see also [33]) as follow.

**Theorem 2.4.** [6] Theorem 2.13 The cocycle $A_t(x)$ is strictly $\mathcal{J}$-separated if, and only if, $E_U$ admits a dominated splitting $F_{-} \oplus F_{+}$ with respect to $A_t(x)$ on the maximal invariant subset $\Lambda$ of $U$, with constant dimensions $\dim F_{-} = q, \dim F_{+} = p, \dim M = p + q$.

This is an algebraic/geometrical way to prove the existence of dominated splittings. In fact, we have an analogous result about partial hyperbolic splittings, as follow.

We say that a compact invariant subset $\Lambda$ is non-trivial if
- either $\Lambda$ does not contain singularities;
- or $\Lambda$ contains at most finitely many singularities, $\Lambda$ contains some regular orbit and is connected.

**Theorem 2.5.** [6] Theorem A A non-trivial compact invariant subset $\Gamma$ is a partially hyperbolic set for a flow $X_t$ if, and only if, there is a $C^1$ field $\mathcal{J}$ of non-degenerate and indefinite quadratic forms with constant index, equal to the dimension of the stable subspace of $\Gamma$, such that $X_t$ is a non-negative strictly $\mathcal{J}$-separated flow on a neighborhood $U$ of $\Gamma$.

Moreover $E$ is a negative subspace, $F$ a positive subspace and the splitting can be made almost orthogonal.

Here strict $\mathcal{J}$-separation corresponds to strict cone invariance under the action of $DX_t$ and $\langle \cdot, \cdot \rangle$ is a Riemannian inner product in the ambient manifold. We recall that the index of a field quadratic forms $\mathcal{J}$ on a set $\Gamma$ is the dimension of the $\mathcal{J}$-negative space at every tangent space $T_xM$ for $x \in U$. Moreover, we say that the splitting $T_{\Gamma}M = E \oplus F$ is almost orthogonal if, given $\varepsilon > 0$, there exists a smooth inner product $\langle \cdot, \cdot \rangle$ on $T_{\Gamma}M$ so that $|\langle u, v \rangle| < \varepsilon$, for all $u \in E, v \in F$, with $\|u\| = 1 = \|v\|$.

We note that the condition stated in Theorem 2.5 allows us to obtain partial hyperbolicity checking a condition at every point of the compact invariant set that depends only on the tangent map $DX$ to the vector field $X$ together with a family $\mathcal{J}$ of quadratic forms without using the flow $X_t$ or its derivative $DX_t$. This is akin to checking the stability of singularity of a vector field using a Lyapunov function.

### 2.1. Exterior powers.

We note that if $E \oplus F$ is a $DX_t$-invariant splitting of $T_{\Gamma}M$, with $\{e_1, \ldots, e_{\ell}\}$ a family of basis for $E$ and $\{f_1, \ldots, f_h\}$ a family of basis for $F$, then $\bar{F} = \wedge^\ell F$ generated by $\{f_{i_1} \wedge \cdots \wedge f_{i_h}\}_{1 \leq i_1, \ldots, i_h \leq h}$ is naturally $\wedge^k DX_t$-invariant by construction. In addition, $\bar{E}$ generated by $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \leq i_1, \ldots, i_h \leq \ell}$ together with all the exterior products of $i$ basis elements of $E$ with $j$ basis elements of $F$, where $i + j = k$ and $i, j \geq 1$, is also $\wedge^k DX_t$-invariant and, moreover, $\bar{E} \oplus \bar{F}$ gives a splitting of the $k$th exterior power $\wedge^k T_{\Gamma}M$ of the
subbundle $T\Gamma M$. Let $T\Gamma M = E\Gamma \oplus F\Gamma$ be a $DX_t$-invariant splitting over the compact $X\Gamma$-invariant subset $\Gamma$ such that $\dim F = k \geq 2$. Let $\widetilde{F} = \wedge^k F$ be the $\wedge^k DX_t$-invariant subspace generated by the vectors of $F$ and $\widetilde{E}$ be the $\wedge^k DX_t$-invariant subspace such that $\widetilde{E} \oplus \widetilde{F}$ is a splitting of the $k$th exterior power $\wedge^k T\Gamma M$ of the subbundle $T\Gamma M$.

We consider the action of the cocycle $DX_t(x)$ on $k$-vector that is the $k$-exterior $\wedge^k DX_t$ of the cocycle acting on $\wedge^k T\Gamma M$.

We denote by $\| \cdot \|$ the standard norm on $k$-vectors induced by the Riemannian norm of $M$, see [10].

**Remark 2.6.** Let $V$ vectorial space of dimension $N$.

(i) The dimension of space $\wedge^r V$ is $\dim \wedge^r V = \binom{N}{r}$. If $\{e_1, ..., e_N\}$ is a basis of $V$, so the set $\{e_{k_1} \wedge ... \wedge e_{k_r} : 1 \leq k_1 < ... < k_r \leq N\}$ is a basis in $\wedge^r V$ with $\binom{N}{r}$ elements.

(ii) If $V$ has the inner product $\langle \cdot, \cdot \rangle$, then the bilinear extension of $\langle u_1 \wedge ... \wedge u_r, v_1 \wedge ... \wedge v_r \rangle := \det(\langle u_i, v_j \rangle)_{r \times r}$ defines an inner product in $\wedge^r V$. In particular, $\|u_1 \wedge ... \wedge u_r\| = \sqrt{\det(\langle u_i, u_j \rangle)_{r \times r}}$ is the volume of $r$-dimensional parallelepiped $H$ spanned by $u_1, ..., u_r$, we write $\text{vol}(u_1, ..., u_r) = \text{vol}(H) = \det(H) = |\det(u_1, ..., u_r)|$.

(iii) If $A : V \rightarrow V$ is a linear operator then the linear extension of $\wedge^r A(u_1, ..., u_r) = A(u_1) \wedge ... \wedge A(u_r)$ defines a linear operator $\wedge^r A$ on $\wedge^r V$.

(iv) Let $A : V \rightarrow V$, and $\wedge^r A : \wedge^r V \rightarrow \wedge^r V$ linear operators with $G$ spanned by $v_1, ..., v_s \in V$. Define $H := A|_G$, then $H$ is spanned by $A(v_1), ..., A(v_s)$. So $|\det A|_G| = \text{vol}(A|_G) = \text{vol}(H) = \text{vol}(A(v_1), ..., A(v_s)) = |\|A(v_1) \wedge ... \wedge A(v_s)\|| = |\| \wedge^r A(v_1 \wedge ... \wedge v_s)\||$.

When $DX_t(u_i) = v_i(t) = v_i$, where $G$ is spanned by $u_1, ..., u_r \in T\Gamma M$, and $H$ is spanned by $v_1, ..., v_r$, we have $H = DX_t(G) = DX_t|_G$. Thus,

$$|\det(DX_t|_G)| = \text{vol}(DX_t(u_1), ..., DX_t(u_r)) = \|DX_t(u_1) \wedge ... \wedge DX_t(u_r)\| = \|\wedge^r DX_t(u_1 \wedge ... \wedge u_r)\|.$$

It is natural to consider the linear multiplicative cocycle $\wedge^k DX_t$ over the flow $X_t$ of $X$ on $U$, that is, for any $k$ choice, $u_1, u_2, ..., u_k$ of vectors in $T_x U$, $x \in U$ and $t \in \mathbb{R}$ such that $X_t(x) \in U$ we set

$$(\wedge^k DX_t) \cdot (u_1 \wedge u_2 \wedge ... \wedge u_k) = (DX_t \cdot u_1) \wedge (DX_t \cdot u_2) \wedge ... \wedge (DX_t \cdot u_k)$$

see [10] Chapter 3, Section 2.3 or [31] for more details and standard results on exterior algebra and exterior products of linear operator.

In [7], the authors proved the following relation between a dominated splitting and its exterior power.
Theorem 2.7. [7, Theorem A] The splitting $T_\Gamma M = E \oplus F$ is dominated for $DX_t$ if, and only if, $\wedge^k T_\Gamma M = \tilde{E} \oplus \tilde{F}$ is a dominated splitting for $\wedge^k DX_t$.

Hence, the existence of a dominated splitting $T_\Gamma M = E_\Gamma \oplus F_\Gamma$ over the compact $X_t$-invariant subset $\Gamma$, is equivalent to the bundle $\wedge^k T_\Gamma M$ admits a dominated splitting with respect to $\wedge^k DX_t: \wedge^k T_\Gamma M \to \wedge^k T_\Gamma M$.

As a consequence, they obtain the next characterization of three-dimensional singular sets.

Corollary 2.8. [7, Corollary 1.5] Assume that $M$ has dimension $3$, $E$ is uniformly contracted by $DX_t$, and that $k = 2$. Then $E \oplus F$ is a singular-hyperbolic splitting for $DX_t$ if, and only if, $\tilde{E} \oplus \tilde{F}$ is partially hyperbolic splitting for $\wedge^2 DX_t$ such that $\tilde{F}$ is uniformly expanded by $\wedge^2 DX_t$.

The main idea in [7] was to give a characterization of sectional hyperbolicity following the well known algebraic feature of cross product.

The absolute value of the cross product (also called vector product) on a 3-dimensional vector space $V$, denote by $w = u \times v$, provides the length of the vector $w$. It is very useful to calculate the area expansion of the parallelogram generated by $u, v$, under the action of a linear operator.

Following this way, in [7], the first author and V. Araujo proved the result below.

Theorem 2.9. [7, Theorem B] Suppose that $X$ is 3-dimensional vector field on $M$ which is non-negative strictly $J$-separated over a non-trivial subset $\Gamma$, where $J$ has index $1$. Then

1. $\wedge^2 DX_t$ is strictly $(-J)$-separated;
2. $\Gamma$ is a singular hyperbolic set if either one of the following properties is true
   a. $\wedge^b_i(x) \to -\infty$ for all $x \in \Gamma$.
   b. $\wedge - 2 \text{tr}(DX)J > 0$ on $\Gamma$.

Here, we generalized this result to $m$ and $k = m - 1$, as follows.

2.2. Statements of results.

If $\wedge^k DX_t$ is strictly separated with respect to some family $J$ of quadratic forms, then there exists the function $\delta_k$ as stated in Proposition 2.1 with respect to the cocyle $\wedge^k DX_t$.

We set

$$\tilde{\Delta}^b_a(x) := \int_a^b \delta_k(X_s(x)) ds$$

the area under the function $\delta_k: U \to \mathbb{R}$ given by Proposition 2.1 with respect to $\wedge^k DX_t$ and its infinitesimal generator.

If $k = m - 1$, it is not difficult to see that this function is related to $X$ and $\delta$ as follows: let $\delta: \Gamma \to \mathbb{R}$ be the function associated to $J$ and $DX_t$, as given by Proposition 2.1 then $\delta_k = 2 \text{tr}(DX) - \delta$, where $\text{tr}(DX)$ represents the trace of the linear operator $DX_x: T_x M \to \mathbb{R}$, $x \in M$.

We recall that $\wedge = \partial_t J$ is the time derivative of $J$ along the flow; see Remark 2.2.
Theorem A. Suppose that $X$ is $m$-dimensional vector field on $M$ which is non-negative strictly $J$-separated over a non-trivial subset $\Gamma$, where $J$ has index 1. Then

1. $\wedge^{(m-1)}DX_t$ is strictly $(-J)$-separated;
2. $\Gamma$ is a singular hyperbolic set if either one of the following properties is true
   a. $\tilde{\Delta}_0^t(x) \xrightarrow{t \to +\infty} -\infty$ for all $x \in \Gamma$.  
   b. $\tilde{J} - 2 \text{tr}(DX)J > 0$ on $\Gamma$.

We work with exterior products of codimension one. See [14] for more details.

This result provides useful sufficient conditions for a $m$-dimensional vector field to be singular hyperbolic if $k = m - 1$, using only one family of quadratic forms $J$ and its space derivative $DX$, avoiding the need to check cone invariance and contraction/expansion conditions for the flow $X_t$ generated by $X$ on a neighborhood of $\Gamma$.

Definition 6. We say a Riemannian metric $\langle \cdot, \cdot \rangle$ adapted to a singular hyperbolic splitting $TT = E \oplus F$ if it induces a norm $| \cdot |$ such that there exists $\lambda > 0$ satisfying for all $x \in \Gamma$ and $t > 0$ simultaneously

$$ |DX_t|_{E_x} \cdot |(DX_t|_{F_x})^{-1}| \leq e^{-\lambda t}, \quad |DX_t|_{E_x} \leq e^{-\lambda t} \quad \text{and} \quad |\det(DX_t|_{F_x})| \geq e^{\lambda t}. $$

We call it singular adapted metric, for simplicity.

In [7], the first author and V. Araujo proved the following.

Theorem 2.10. [7, Theorem C] Let $\Gamma$ be a singular-hyperbolic set for a $C^1$ three-dimensional vector field $X$. Then $\Gamma$ admits a singular adapted metric.

Here, we generalize this result for any codimension one singular hyperbolic flow in higher dimensional manifolds. Consider a partially hyperbolic splitting $T\Gamma = E \oplus F$ where $E$ is uniformly contracted and $F$ is volume expanding. We show that for $C^1$ flows having a singular-hyperbolic set $\Gamma$ such that $E$ is one-dimensional subspace there exists a metric adapted to the partial hyperbolicity and the area expansion, as follows.

Theorem B. Let $\Gamma$ be a singular-hyperbolic set of codimension one for a $C^1$ $m$-dimensional vector field $X$. partially hyperbolic splitting satisfying (1) together with uniform contraction along $E$ and volume expanding along $F$ such that $\dim E = 1$. Then $\Gamma$ admits a singular adapted metric.

Remark 2.11. Using the Theorem 2.9, the first author and V. Araujo provided in [7] a proof of the Theorem 2.10. Analogously, we can show the Theorem B using the Theorem A. Here, we’ll present an alternative proof for Theorem B that is independent of the Theorem A.

We briefly present these results in what follows with the relevant definitions.

3. Auxiliary results

3.1. Properties of $J$-separated linear multiplicative cocycles.
Proposition 3.1. [6, Proposition 2.7] Let $A_t(x)$ be a cocycle over $X_t$ defined on an open subset $U$ and $D(x)$ its infinitesimal generator. Then

1. $\mathcal{J}(v) = \partial_t A_t(x)v = \langle J_{X_t}(x)A_t(x)v, A_t(x)v \rangle$ for all $v \in E_x$ and $x \in U$, where
   \[ J_x := J \cdot D(x) + D(x)^* \cdot J \]
   and $D(x)^*$ denotes the adjoint of the linear map $D(x) : E_x \to E_x$ with respect to the adapted inner product at $x$;

2. the cocycle $A_t(x)$ is $\mathcal{J}$-separated if, and only if, there exists a neighborhood $V$ of $\Lambda$, $V \subset U$ and a function $\delta : V \to \mathbb{R}$ such that
   \[ \mathcal{J}_x \geq \delta(x) \mathcal{J}_x \quad \text{for all} \quad x \in V. \]  
   In particular we get $\partial_t \log |\mathcal{J}(A_t(x)v)| \geq \delta(X_t(x)), v \in E_x, x \in V, t \geq 0$;

3. if the inequalities in the previous item are strict, then the cocycle $A_t(x)$ is strictly $\mathcal{J}$-separated. Reciprocally, if $A_t(x)$ is strictly $\mathcal{J}$-separated, then there exists a compatible family $\mathcal{J}_0$ of forms on $V$ satisfying the strict inequalities of item (2).

4. For a $\mathcal{J}$-separated cocycle $A_t(x)$, we have $|\mathcal{J}(A_t(x)v)| \geq \exp \Delta_{t_2}^{t_1}(x) v \in E_x$ and reals $t_1 < t_2$ so that $\mathcal{J}(A_t(x)v) \neq 0$ for all $t_1 \leq t \leq t_2$, where $\Delta_{t_1}^{t_2}(x)$ was defined in [5].

5. we can bound $\delta$ at every $x \in \Gamma$ by $\inf_{v \in C_+(x)} \frac{\mathcal{J}(v)}{|\mathcal{J}(v)|} \leq \delta(x) \leq \sup_{v \in C_-(x)} \frac{\mathcal{J}(v)}{|\mathcal{J}(v)|}$.

Remark 3.2. We stress that the necessary and sufficient condition in items (2-3) of Proposition 3.1 for (strict) $\mathcal{J}$-separation, shows that a cocycle $A_t(x)$ is (strictly) $\mathcal{J}$-separated if, and only if, its inverse $A_{-t}(x)$ is (strictly) $(-\mathcal{J})$-separated.

Remark 3.3. Item (2) above of Proposition 3.1 shows that $\delta$ is a measure of the “minimal instantaneous expansion rate” of $|\mathcal{J} \circ A_t(x)|$.

Proposition 3.4. [6, Theorem 2.23] Let $\Gamma$ be a compact invariant set for $X_t$ admitting a dominated splitting $E_{\Gamma} = F_- \oplus F_+$ for $A_t(x)$, a linear multiplicative cocycle over $\Gamma$ with
values in $E$. Let $\mathcal{J}$ be a $C^1$ family of indefinite quadratic forms such that $A_t(x)$ is strictly $\mathcal{J}$-separated. Then

1. $F_- \oplus F_+$ is partially hyperbolic with $F_+$ uniformly expanding if $\Delta^0_+ (x) \xrightarrow{t \to +\infty} +\infty$ for all $x \in \Gamma$.
2. $F_- \oplus F_+$ is partially hyperbolic with $F_-$ uniformly contracting if $\Delta^0_- (x) \xrightarrow{t \to +\infty} -\infty$ for all $x \in \Gamma$.
3. $F_- \oplus F_+$ is uniformly hyperbolic if, and only if, there exists a compatible family $\mathcal{J}_0$ of quadratic forms in a neighborhood of $\Gamma$ such that $\mathcal{J}_0'(v) > 0$ for all $v \in E_x$ and all $x \in \Gamma$.

For the proof and more details about the Proposition 3.4, see [6].

Above we write $\tilde{\mathcal{J}}(v) = \langle \tilde{J}_x v, v \rangle$, where $\tilde{J}_x$ is given in Proposition 2.1, that is, $\tilde{\mathcal{J}}(v)$ is the time derivative of $\mathcal{J}$ under the action of the flow.

We use Proposition 3.4 to obtain sufficient conditions for a flow $X_t$ on $m$-manifold to have a $\wedge^{m-1}DX_t$-invariant one-dimensional uniformly expanding direction orthogonal to the $(m-1)$-dimensional center-unstable bundle.

First, we present some properties about exterior products and the main lemma to prove the theorem [A].

Let $V$ a $m$-dimensional vector space, we denote $V$ by $V^m$, consider $\wedge^k V^m$ where $2 \leq k \leq m$. Let $\mathcal{B} = \{e_1, \ldots, e_m\}$ a basis of $V^m$. So $\{e_j_1 \wedge \ldots \wedge e_j_k : 1 \leq j_1 < \ldots < j_k \leq m\}$ is a basis of $\wedge^k V^m$, and $J := \{(j_1, \ldots, j_k) \in \mathbb{N}^k : 1 \leq j_1 < \ldots < j_k \leq m\}$. Let $l = \binom{m}{k}$, so we have $l$ combination of $k$ vectors in $\{e_1, \ldots, e_m\}$, and $|J| = l$.

Take $u_1, u_2, \ldots, u_k \in V^m$ where $u_j = (u_j^1, u_j^2, \ldots, u_j^m)_{\mathcal{B}}$ for all $j \in \{1, \ldots, k\}$. Define

$$C := \begin{pmatrix} u_1^1 & \cdots & u_k^1 \\ \vdots & \ddots & \vdots \\ u_1^m & \cdots & u_k^m \end{pmatrix}_{m \times k}$$

(11)

For $(j_1, \ldots, j_k) \in J$, consider

$$C_{j_1, \ldots, j_k} := \begin{pmatrix} u_{j_1}^1 & \cdots & u_k^1 \\ \vdots & \ddots & \vdots \\ u_{j_1}^k & \cdots & u_k^k \end{pmatrix}_{k \times k}$$

(12)

The following result holds

$$u_1 \wedge \cdots \wedge u_k = \sum_{(j_1, \ldots, j_k) \in J} \det(C_{j_1, \ldots, j_k})(e_{j_1} \wedge \cdots \wedge e_{j_k}).$$

(13)

Let $A : V^m \to V^m$ a linear operator with matrix in basis $\mathcal{B}$ given by

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \ddots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}_{(m \times m)}$$

(14)
We will denote this matrix by $A$ too.

Consider $\wedge^k A : \wedge^k V^m \rightarrow \wedge^k V^m$, note that $A(u_1) \wedge \ldots \wedge A(u_k) = \wedge^k A(u_1 \wedge \ldots \wedge u_k)$, by (13) and the linearity of $\wedge^k A$, we have that

$$A(u_1) \wedge \ldots \wedge A(u_k) = \sum_{(j_1, \ldots, j_k) \in J} \det(C^{j_1, \ldots, j_k}) \wedge^k A(e_{j_1} \wedge \ldots \wedge e_{j_k})$$  \hspace{1cm} (15)

Define $A_j := A(e_j)$, so $A_j$ is the $j$-th column of $A$, i.e., $A(e_j) = A_{j} = (a_{1j}, \ldots, a_{mj})^T$, so $A(e_j) = [a_{ij}]_{m \times 1}$. Let $A_{j_1 \ldots j_k} := (A_{j_1} \ldots A_{j_k})_{m \times k}$ where $(j_1, \ldots, j_k) \in J$. For each $(i_1 \ldots i_k), (j_1 \ldots j_k) \in J$ consider

$$A_{j_1 \ldots j_k}^{i_1 \ldots i_k} := \begin{pmatrix} a_{i_1 j_1} & \ldots & a_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ a_{i_k j_1} & \ldots & a_{i_k j_k} \end{pmatrix}_{k \times k}$$  \hspace{1cm} (16)

Using that $\wedge^k A(e_{j_1} \wedge \ldots \wedge e_{j_k}) = A(e_{j_1}) \wedge \ldots \wedge A(e_{j_k})$ with matrix $A_{j_1 \ldots j_k} := (A_{j_1} \ldots A_{j_k})_{m \times k}$, by (13) we obtain that

$$A(e_{j_1}) \wedge \ldots \wedge A(e_{j_k}) = \sum_{(i_1, \ldots, i_k) \in J} \det(A_{j_1 \ldots j_k}^{i_1 \ldots i_k})(e_{i_1} \wedge \ldots \wedge e_{i_k}).$$  \hspace{1cm} (17)

**Lemma 3.5.** Let $V$ a $m$-dimensional manifold and $A : T_x V \rightarrow T_y V$ a linear operator then $\wedge^{(m-1)} A = \det(A) \cdot (A^{-1})^*$.  

**Proof.** Consider $k = m - 1$.

We use the following identification between $\wedge^{(m-1)} T_x M$ and $T_x M$. For each $(j_1, \ldots, j_{m-1}) \in J$, we identify $e_{j_1} \wedge \ldots \wedge e_{j_k}$ in $\wedge^{(m-1)} T_x M$ by $\delta_p e_p$ in $T_x M$, where $p \notin \{j_1, \ldots, j_{m-1}\}$, $\delta_p = 1$ if $p$ is odd, and $\delta_p = -1$ if $p$ is even.

We must show that for each $(j_1, \ldots, j_{m-1}) \in J$ the exterior product $\wedge^{(m-1)} A(e_{j_1} \wedge \ldots \wedge e_{j_k})$ corresponds to the $\det(A) \cdot (A^{-1})^* (\delta_p e_p)$, where $\delta_p e_p$ is given as above.

Define $S := \det(A) \cdot (A^{-1})^*$, using that $A^{-1} = \frac{1}{\det(A)} \text{Adj}(A)$, we obtain that $S = \text{cof}(A)$ where $\text{cof}(A) = [(-1)^{i+j} M_{ij}]_{m \times m}$ and $M_{ij}$ is the determinant of the submatrix formed by deleting the $i$-th row and $j$-th column. We have that $M_{ij} = \det(A_{s_1 \ldots s_k})$ where $i \notin r_1, \ldots, r_k$ and $j \notin s_1, \ldots, s_k$.

Note that

$$\text{cof}(A)(\delta_p e_p) = \delta_p \text{cof}(A)(e_p) = \delta_p ((-1)^{i+p} M_{1p}, (-1)^{2+p} M_{2p}, \ldots, (-1)^{m+p} M_{mp})_B.$$

In case $p$ is odd, $\delta_p = 1$ and $\text{cof}(A)(\delta_p e_p) = (M_{1p}, -M_{2p}, \ldots, (-1)^{m+p} M_{mp})_B$.

We obtain that

$$\text{cof}(A)(\delta_p e_p) = M_{1p} e_1 + M_{2p} e_2 + \ldots + M_{mp}(-1)^{m+p} = M_{1p}(e_1 \delta_1) + M_{2p}(e_2 \delta_2) + \ldots + M_{mp}(e_m \delta_m).$$
Using that
\[ A(e_{j_1}) \wedge \ldots \wedge A(e_{j_k}) = \sum_{(i_1, \ldots, i_k) \in J} \det(A_{j_1 \ldots j_k}^{i_1 \ldots i_k})(e_{i_1} \wedge \ldots \wedge e_{i_k}) \]
and \( M_{ij} = \det(A_{s_1 \ldots s_k}^{r_1 \ldots r_k}) \) where \( i \notin r_1, \ldots, r_k \) and \( j \notin s_1, \ldots, s_k \), we have that \( \text{cof}(A)(\delta_p e_p) \simeq A(e_{j_1}) \wedge \ldots \wedge A(e_{j_k}) \).

This concludes the proof. \( \square \)

We generalize the corollary 2.8 to arbitrary \( n \) and \( k \).

We just need to prove the following result:

**Lemma 3.6.** The subbundle \( F_1 \) is volume expanding by \( DX_t \) if, and only if, \( \tilde{F} \) is uniformly expanded by \( \wedge^k DX_t \).

In particular, \( E \oplus F \) is a singular hyperbolic splitting, where \( F \) is volume expanding for \( DX_t \) if, and only if, \( \tilde{E} \oplus \tilde{F} \) is partially hyperbolic splitting for \( \wedge^k DX_t \) such that \( \tilde{F} \) is uniformly expanded by \( \wedge^k DX_t \).

**Proof.** We consider the action of the cocycle \( DX_t(x) \) on \( k \)-vector that is the \( k \)-exterior \( \wedge^k DX \) of the cocycle acting on \( \wedge^k T_x M \).

Denote by \( || \cdot || \) the standard norm on \( k \)-vectors induced by the Riemannian norm of \( M \); see, e.g. [10]. We write \( m = \dim M \).

Suppose that \( T_x M \) admits a splitting \( E \oplus F \) with \( \dim E = m - k \) and \( \dim F = k \).

We note that if \( E \oplus F \) is a \( DX_t \)-invariant splitting of \( T_x M \), with \( \{e_1, \ldots, e_i\} \) a family of basis for \( E \) and \( \{f_1, \ldots, f_k\} \) a family of basis for \( F \), then \( \tilde{F} = \wedge^k F \) generated by \( \{f_1 \wedge \ldots \wedge f_k\}_{1 \leq i_1 < \ldots < i_k \leq k} \) is naturally \( \wedge^k DX_t \)-invariant by construction. Then, the dimension of \( \tilde{F} \) is one with basis given by the vector \( f_1 \wedge \ldots \wedge f_k \).

Assume that \( F_1 \) is volume expanding by \( DX_t \). We must show that there exist constants \( C \) and \( \lambda > 0 \) such that \( || \wedge^k DX_t(x) || \geq C e^{\lambda t} \) for all \( t > 0 \), where \( P \) is spanned by \( f_1 \wedge \ldots \wedge f_k \).

Note that
\[ || \wedge^k DX_t(x) || = || \wedge^k DX_t(f_1 \wedge \ldots \wedge f_k) || = || DX_t(f_1) \wedge \ldots \wedge DX_t(f_k) ||. \]

But \( f_1, \ldots, f_k \) is a basis for \( F_1 \), by hypothesis there exist constants \( C \) and \( \lambda > 0 \) such that \( || \det(DX_t(x)) || \geq C e^{\lambda t} \) for all \( t > 0 \). So,
\[ || \det(DX_t(x)) || = \text{vol}(DX_t(f_1), \ldots, DX_t(f_k)) = || DX_t(f_1) \wedge \ldots \wedge DX_t(f_k) ||. \]

The reciprocal statement is straightforward.

Given a basis \( \{f_1, \ldots, f_k\} \) of \( F_1 \), we have that
\[ || \det(DX_t(x)) || = \text{vol}(DX_t(f_1), \ldots, DX_t(f_k)) = || DX_t(f_1) \wedge \ldots \wedge DX_t(f_k) || = \frac{1}{|| \wedge^k DX_t(x) ||}. \]

where \( P \) is spanned by \( f_1 \wedge \ldots \wedge f_k \).

However, by hypothesis, there exist \( C \) and \( \lambda > 0 \) such that \( || \wedge^k DX_t(x) || \geq C e^{\lambda t} \), for all \( t > 0 \). \( \square \)
Corollary 3.7. Assume that $E$ is uniformly contracted by $DX_t$. $E \oplus F$ is a singular-hyperbolic splitting for $DX_t$ if, and only if, $\tilde{E} \oplus \tilde{F}$ is partially hyperbolic splitting for $\wedge^k DX_t$ such that $\tilde{F}$ is uniformly expanded by $\wedge^k DX_t$.

Let $M$ Riemannian manifold $m$-dimensional with $\langle \cdot, \cdot \rangle$ inner product in $T\Gamma M$, and $\langle \cdot, \cdot \rangle_*$ the inner product in $\wedge^k T\Gamma M$ induced by $\langle \cdot, \cdot \rangle$ where $\wedge^k T\Gamma M = \bigcup_{x \in \Gamma} \wedge^k T_x M$. So for $x \in \Gamma$, we have that $\langle \cdot, \cdot \rangle$ is acting on $T_x M$, and $\langle \cdot, \cdot \rangle_*$ is acting on $\wedge^k T_x M$.

Lemma 3.8. Let $M$ be a riemannian $m$-dimensional manifold. Then, for all $\langle \cdot, \cdot \rangle_*$ inner product in $\wedge^{(m-1)} T\Gamma M$ there exists a inner product $\langle \cdot, \cdot \rangle$ on $T\Gamma M$ such that $\langle \cdot, \cdot \rangle_*$ is induced by $\langle \cdot, \cdot \rangle$.

Proof. Let $M$ be a riemannian manifold $m$-dimensional with $\langle \cdot, \cdot \rangle$ an inner product in $T\Gamma M$, and $\langle \cdot, \cdot \rangle_*$ the inner product in $\wedge^{(m-1)} T\Gamma M$ induced by $\langle \cdot, \cdot \rangle$.

Take $\langle \cdot, \cdot \rangle_{**}$ an arbitrary inner product in $\wedge^{(m-1)} T\Gamma M$. Using that $\langle \cdot, \cdot \rangle_{**}$ and $\langle \cdot, \cdot \rangle_*$ are inner products in $\wedge^{(m-1)} T\Gamma M$ there exists $J : \wedge^{(m-1)} T\Gamma M \to \wedge^{(m-1)} T\Gamma M$ isomorphism linear such that $\langle u, v \rangle_{**} = \langle J(u), J(v) \rangle_*$.

Define $\varphi : GL(T\Gamma M) \to GL(\wedge^{(m-1)} T\Gamma M)$ given by $A \mapsto \wedge^{(m-1)} A$.

Note that $\varphi$ is an injective linear homomorphism, and due to the dimensions of the spaces, $\varphi$ is a linear isomorphism.

Hence, there exists $A \in GL(T\Gamma M)$ such that $\wedge^{(m-1)} A = J$.

Consider $\langle x, y \rangle := \langle A(x), A(y) \rangle$ for $x, y \in T\Gamma M$, then $\langle u, v \rangle_* = \det(\langle u_i, v_j \rangle)_{(m-1) \times (m-1)}$, where $u = u_1 \wedge \ldots \wedge u_{m-1}$ and $v = v_1 \wedge \ldots \wedge v_{m-1}$.

We have that

$$\langle u, v \rangle_* = \det(\langle A(u_i), A(v_j) \rangle)_{(m-1) \times (m-1)}.$$  

On the other hand,

$$\langle u, v \rangle_{**} = \langle \wedge^{(m-1)} A(u), \wedge^{(m-1)} A(v) \rangle_* = \det(\langle A(u_i), A(v_j) \rangle)_{(m-1) \times (m-1)}.$$  

Therefore, $\langle \cdot, \cdot \rangle_* = \langle \cdot, \cdot \rangle_{**}$, and we are done.

4. Proofs of main results

4.1. Proof of Theorem A

Proof. Consider $M$ is a $m$-manifold and $\Gamma$ is a compact $X_t$-invariant subset having a singular-hyperbolic splitting $T\Gamma M = E_\Gamma \oplus F_\Gamma$. By Theorem 2.7 we have a $\wedge^{(m-1)} DX_t$-invariant partial hyperbolic splitting $\wedge^{(m-1)} T\Gamma M = \tilde{E} \oplus \tilde{F}$ with $\dim \tilde{F} = 1$ and $\tilde{F}$ uniformly expanded. Following the proof of Theorem 2.7 if we write $e$ for a unit vector in $E_x$ and $\{u_1, u_2, \ldots, u_{m-1}\}$ an orthonormal base for $F_x$, $x \in \Gamma$, then $\tilde{E}_x$ is a $(m-1)$-dimensional vector space spanned by set $\{e \wedge u_{i_1} \wedge u_{i_2} \wedge \ldots \wedge u_{i_{m-2}}\}$ with $i_1, \ldots, i_{m-2} \in \{1, \ldots, m-1\}$.

From Theorem 2.5 and the existence of adapted metrics (see e.g. [15]), there exists a field $\mathcal{J}$ of quadratic forms so that $X$ is $\mathcal{J}$-non-negative, $DX_t$ is strictly $\mathcal{J}$-separated on a neighborhood $U$ of $\Gamma$, $E_\Gamma$ is a negative subbundle, $F_\Gamma$ is a positive subbundle and these subspaces are almost orthogonal. In other words, there exists a function $\delta : \Gamma \to \mathbb{R}$ such that $\mathcal{J}_x - \delta(x)\mathcal{J}_x > 0, x \in \Gamma$ and we can locally write $\mathcal{J}(v) = \langle J(v), v \rangle$ where $J =$
diagonal $\{\pm1,0,...,1\}$ with respect to the basis $\{e,u_1,...,u_{m-1}\}$ and $\langle \cdot, \cdot \rangle$ is the adapted inner product; see [6].

By lemma 3.5, $\Lambda^{(m-1)} A = \det(A) \cdot (A^{-1})^*$ with respect to the adapted inner product which trivializes $\mathfrak{g}$, for any linear transformation $A : T_{\tilde{t}} M \rightarrow T_{\tilde{t}} M$. Hence $\Lambda^{(m-1)} DX_t(x) = \det(DX_t(x)) \cdot (DX_t \circ X_t)^*$ and a straightforward calculation shows that the infinitesimal generator $D^{(m-1)}(x)$ of $\Lambda^{(m-1)} DX_t$ equals $\text{tr}(DX_t(x)) \cdot \text{Id}_x - DX_t(x)^*$.

Therefore, using the identification between $\Lambda^{(m-1)} T_{\tilde{t}} M$ and $T_{\tilde{t}} M$ through the adapted inner product and Proposition 3.1

$$\hat{\mathfrak{g}}_x = \partial_t(-\mathfrak{g})(\Lambda^{(m-1)} DX_t \cdot v)_{|t=0} = \langle -(J \cdot D^{(m-1)}(x) + D^{(m-1)}(x)^* \cdot J)v, v \rangle$$

$$= \langle [\mathfrak{g} \cdot DX_t(x) + DX_t(x)^* \cdot \mathfrak{g}] - 2 \text{tr}(DX_t(x)) \mathfrak{g} ]v, v \rangle$$

$$= (\hat{\mathfrak{g}} - 2 \text{tr}(DX_t(x)) \mathfrak{g})(v). \ (18)$$

To obtain strict $(-\mathfrak{g})$-separation of $\Lambda^{(m-1)} DX_t$, we search a function $\delta_{(m-1)} : \Gamma \rightarrow \mathbb{R}$ so that

$$\langle (\hat{\mathfrak{g}} - 2 \text{tr}(DX_t(x)) \mathfrak{g}) - \delta_{(m-1)}(-\mathfrak{g}) \rangle > 0 \quad \text{or} \quad \hat{\mathfrak{g}} - (2 \text{tr}(DX_t(x)) - \delta_{(m-1)}) \mathfrak{g} > 0.$$

Hence it is enough to make $\delta_{(m-1)} = 2 \text{tr}(DX_t(x)) - \delta$. This shows that in our setting $\Lambda^{(m-1)} DX_t$ is always strictly $(-\mathfrak{g})$-separated.

Finally, according to Theorem 3.4, to obtain the partial hyperbolic splitting of $\Lambda^{(m-1)} DX_t$ which ensures singular-hyperbolicity, it is sufficient that either $\tilde{\Delta}^b_k(x) = \int_a^b \delta_{(m-1)}(X_s(x)) \, ds$ satisfies item (1) of Proposition 3.4 or $\hat{\mathfrak{g}}_x$ is positive definite, for all $x \in \Gamma$. This amounts precisely to the sufficient condition in the statement of Theorem A and we are done. \hfill \square

### 4.2. Proof of Theorem [B]

**Proof.** Let singular-hyperbolic set $\Gamma$ for vector field with a splitting $E \oplus F$ where $E$ is uniformly contracted and $F$ is volume expanding.

Suppose that $T_{\tilde{t}} M$ admits a splitting $E_{\Gamma} \oplus F_{\Gamma}$ with $\dim E_{\Gamma} = 1$ and $\dim F_{\Gamma} = k = m - 1$.

We note that if $E \oplus F$ is a $DX_t$-invariant splitting of $T_{\tilde{t}} M$, with $\{e_1\}$ a basis for $E$ and $\{f_1,\ldots,f_k\}$ a family of basis for $F$, then $\tilde{F} = \wedge^k F$ generated by $\{f_1 \wedge \cdots \wedge f_k\}_{1 \leq i_1 < \cdots < i_k \leq k}$ is naturally $\wedge^k DX_t$-invariant by construction. Then, the dimension of $\tilde{F}$ is one with basis given by the vector $f_1 \wedge \cdots \wedge f_k$.

By corollary 3.7, we have a partially hyperbolic splitting $\tilde{E} \oplus \tilde{F}$ for $\wedge^k DX_t$ such that $\tilde{F}$ is uniformly expanded by $\wedge^k DX_t$. Hence, from [13] Theorem 1, there exists an adapted inner product $\langle \cdot, \cdot \rangle_*$ for $\wedge^k DX_t$. There exists $\lambda > 0$ satisfying for all $x \in \Gamma$ and $t > 0$ such that $\| \wedge^k DX_t \|_{\tilde{F}_x} \| \geq e^{\lambda t}$ for all $t > 0$.

By Lemma 3.8, $\langle \cdot, \cdot \rangle_*$ is induced by an inner product $\langle \cdot, \cdot \rangle$ in $T_{\tilde{t}} M$. So, we have a partially hyperbolic splitting $\tilde{E} \oplus \tilde{F}$ for $\wedge^k DX_t$ such that $\tilde{F}$ is uniformly expanded by $\wedge^k DX_t$. By Theorem 2.7, we have that $E \oplus F$ is a dominated splitting for $DX_t$. From Theorem 2.4, there exists $C^1$ field of quadratic $\mathfrak{g}$ such that $DX_t$ is strictly $\mathfrak{g}$-separated.

But $DX_t$ is strictly $\mathfrak{g}$-separated, this ensures, in particular, that the norm

$$|w| = |\xi \sqrt{\mathfrak{g}(w_E)^2 + \mathfrak{g}(w_F)^2}|$$

is adapted to the dominated splitting $E \oplus F$ for the cocycle $DX_t$, where $w = w_E + w_F \in E_x \oplus F_x$, $x \in \Gamma$, and $\xi$ is an arbitrary positive constant; see [6].
Section 4.1. This means that there exists $\mu > 0$ such that $|DX_t|_{E_x} |\cdot DX_{-t} |_{F_{X_t(x)}} | \leq e^{-\mu t}$ for all $t > 0$.

Moreover, from the definition of the inner product and $\wedge$, it follows that 

$$| \det(DX_t |_{F_x})| = \|(\wedge^k DX_t)(u_1 \wedge ... \wedge u_k)\| = \|(\wedge^k DX_t) |_{\tilde{F}} \| \geq e^{\lambda t}$$

for all $t > 0$, so $| \cdot |$ is adapted to the volume expanding along $F$.

To conclude, we are left to show that $E$ admits a constant $\omega > 0$ such that $|DX_t|_E | \leq e^{-\omega t}$ for all $t > 0$.

But since $E$ is uniformly contracted, we know that $X(x) \in F_x$ for all $x \in \Lambda$.

**Lemma 4.1.** Let $\Gamma$ be a compact invariant set for a flow $X$ of a $C^1$ vector field $X$ on $M$. Given a continuous splitting $T\Gamma M = E \oplus F$ such that $E$ is uniformly contracted, then $X(x) \in F_x$ for all $x \in \Lambda$.

See [4, Lemma 5.1] and [6, Lemma 3.3].

On the one hand, on each non-singular point $x$ of $\Gamma$ we obtain for each $w \in E_x$

$$e^{-\mu t} \geq \frac{|DX_t \cdot w|}{|DX_t \cdot X(x)|} = \frac{|DX_t \cdot w|}{|X(X_t(x))|} \geq \frac{|DX_t \cdot w|}{\sup\{|X(z)| : z \in \Gamma\}} \geq |DX_t \cdot w|,$$

since we can always choose a small enough constant $\xi > 0$ in such a way that $\sup\{|X(z)| : z \in \Gamma\} \leq 1$. We note that the choice of the positive constant $\xi$ does not change any of the previous relations involving $| \cdot |$.

On the other hand, for $\sigma \in \Gamma$ such that $X(\sigma) = 0$, we fix $t > 0$ and, since $\Gamma$ is a non-trivial invariant set, we can find a sequence $x_n \to \sigma$ of regular points of $\Gamma$. The continuity of the derivative cocycle ensures $|DX_t |_{E_\sigma} | = \lim_{n \to \infty} |DX_t |_{E_{x_n}} | \leq e^{-\lambda t}$. Since $t > 0$ was arbitrarily chosen, we see that $| \cdot |$ is adapted for the contraction along $E_\sigma$.

This shows that $\lambda = \mu$ and completes the proof. $\square$

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