Common graphs with arbitrary connectivity and chromatic number

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Abstract

A graph $H$ is common if the number of monochromatic copies of $H$ in a 2-edge-colouring of the complete graph $K_n$ is asymptotically minimised by the random colouring. We prove that, given $k, r > 0$, there exists a $k$-connected common graph with chromatic number at least $r$. The result is built upon the recent breakthrough of Kráľ, Volec, and Wei who obtained common graphs with arbitrarily large chromatic number and answers a question of theirs.

1 Introduction

A central concept in graph Ramsey theory is the Ramsey multiplicity of a graph $H$, which counts the minimum number of monochromatic copies of $H$ in a 2-edge-colouring of the $n$-vertex complete graph $K_n$. There are some graphs $H$, the so-called common graphs, such that the number of monochromatic $H$-copies in a 2-edge-colouring of $K_n$ is asymptotically minimised by the random colouring. For example, Goodman’s formula [8] implies that a triangle $K_3$ is common, which is one of the earliest results in the area.

Partly inspired by Goodman’s formula, Erdős [5] conjectured that every complete graph is common. This was subsequently generalised by Burr and Rosta [1], who conjectured that every graph is common. In the late 1980s, both conjectures were disproved by Thomason [18] and by Sidorenko [15], respectively. Since then, there have been numerous attempts to find new common (or uncommon) graphs, e.g., [6, 11, 17]. Although the complete classification seems to be still out of reach, new common graphs have been found during the last decade by using some advances on Sidorenko’s conjecture [16] or the computer-assisted flag algebra method [14]. For more results along these lines, we refer the reader to one of the most recent results [4] on Sidorenko’s conjecture and some applications of the flag algebra method [9, 10] with references therein.

Despite all these studies on common graphs, all the known common graphs only had chromatic numbers at most four. This motivated a natural question, appearing in [2, 10], to find a common graph with arbitrarily large chromatic number. This question remained open until its very recent resolution by Kráľ, Volec, and Wei [12]. Since their construction connects a graph with high chromatic number and girth to a copy of a complete bipartite graph by a long path, they asked [12, Problem 25] if highly connected common graphs with large chromatic number exist. We answer this question in the affirmative.

Theorem 1.1. Let $k$ and $r$ be positive integers. Then there exists a $k$-connected common graph with chromatic number at least $r$.

We remark that this short follow-up note to the recent result only partially presents various in-depth studies on common graphs and relevant questions. For a modern review of a variety of results in the area, we refer the reader to recent articles [7, 12].

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2 Proof of the main theorem

A useful setting to analyse commonality of graphs is to use the modern theory of dense graph limits [13]. A graphon is a two-variable symmetric measurable function \( W : [0,1]^2 \to [0,1] \) and the homomorphism density of a graph \( H \) is defined by

\[
t(H, W) := \int \prod_{uv \in E(H)} W(x_u, x_v) \, d\mu^{V(H)},
\]

where \( \mu \) denotes the Lebesgue measure on \([0,1] \). In this language, a graph \( H \) is common if and only if \( t(H, W) + t(H, 1-W) \geq 2^{1-e(H)} \) for every graphon \( W \), where \( e(H) \) denotes the number of edges in \( H \).

The \( q \)-book \( H_{i}^{q} \) of \( H \) along an independent set \( I \subseteq V(H) \) of \( H \) is the graph obtained by taking \( q \) vertex-disjoint copies of \( H \) and identifying the corresponding vertices in \( I \). The following lemma is a straightforward consequence of Jensen’s inequality.

**Lemma 2.1.** Let \( I \) be an independent set of a graph \( H \). If \( H \) is common, then \( H_{i}^{q} \) is also common for every positive integer \( q \).

**Proof.** By a standard application of Jensen’s inequality, the inequality \( t(H_{i}^{q}, W) \geq t(H, W)^q \) holds for every graphon \( W \). Therefore,

\[
t(H_{i}^{q}, W) + t(H_{i}^{q}, 1-W) \geq t(H, W)^q + t(H, 1-W)^q
\]

\[
\geq 2 \cdot \left( \frac{t(H, W) + t(H, 1-W)}{2} \right)^q
\]

\[
\geq 2^{1-q \cdot e(H)} = 2^{1-e(H_{i}^{q})},
\]

where the second inequality is again by convexity and the last inequality uses commonality of \( H \). This proves commonality of \( H_{i}^{q} \).

To summarise, commonality is preserved under the \( q \)-book operation. Another advantage of the operation is that it preserves chromatic numbers. Indeed, a proper colouring of \( H \) can be naturally extended to \( H_{i}^{q} \) by assigning the same colour as a vertex of \( H \) to its ‘clones’ in \( H_{i}^{q} \). Our key idea is to repeatedly apply the \( q \)-book operation to a common graph \( H \), which increases connectivity while maintaining the chromatic number and commonality of \( H \).

First, enumerate the vertices in an \( r \)-vertex graph \( H \) by \( V(H) = \{v_1, v_2, \ldots, v_r\} \). Let \( H_0 := H \) and let \( H_i := (H_{i-1})_{U_i}^{q} \), the \( q \)-book of \( H_{i-1} \) along \( U_i \), where \( U_i \) is the set of all copies of \( v_i \) in \( H_{i-1} \). The \( q \)-bookpile \( H(q) \) of \( H \) is then the graph \( H(q) := H_r \) after the full \( r \)-step iteration. It is not hard to see that this graph \( H(q) \) is independent of the initial enumeration and hence well-defined. For example, if \( H = K_r \) and \( q = 2 \), then \( H(2) \) is the line graph of an \( r \)-dimensional hypercube graph. This graph in fact appeared in [3] in a different context, which partly inspired our approach.

As each \( H_i \) decomposes to \( q \) edge-disjoint copies of \( H_{i-1} \), the \( q \)-bookpile \( H(q) \) decomposes to \( q^r \) edge-disjoint copies of \( H \). To distinguish these, we say that the \( q^r \) edge-disjoint \( H \)-subgraphs of \( H(q) \) as the standard copies of \( H \) in \( H(q) \).

As already sketched, the following theorem together with the construction of connected common graphs \( H \) with arbitrarily large chromatic number in [12] implies Theorem 1.1:

**Theorem 2.2.** Let \( H \) be a connected graph. For every positive integer \( k \), there exists \( q = q(k, H) \) such that the \( q \)-bookpile \( H(q) \) of \( H \) is \( k \)-connected.
To analyse connectivity of $H(q)$, we consider an auxiliary hypergraph on $V(H(q))$ whose edge set consists of the standard copies of $H$ in $H(q)$. We shall first describe what this hypergraph looks like.

Write $[q] := \{1, 2, \ldots, q\}$ and let $\alpha$ be a variable. Let $V(q, r, \alpha)$ be the set of $r$-tuples $v = (n_1, n_2, \ldots, n_r)$, where all but exactly one entry are in $[q]$ and the one exceptional entry is $\alpha$. We call this unique entry the $\alpha$-bit of $v$. Let $H_q^r$ be the $r$-uniform hypergraph on $V(q, r, \alpha)$ with the edge set $[q]^r$, where a vertex $v = (n_1, n_2, \ldots, n_r)$ with $n_i = \alpha$ is incident to an edge $e$ if substituting $\alpha$ by an integer value in $[q]$ gives the edge $e \in [q]^r$. Note that $H_q^r$ is always a linear $r$-graph. Indeed, the codegree of a vertex pair is one if they share all the non-$\alpha$-bits and zero otherwise. In particular, if $q = 2$, then this is the line hypergraph of the $r$-dimensional hypercube graph.

**Proposition 2.3.** Let $H$ be the auxiliary $r$-graph on $V(H(q))$ whose edge set consists of the standard copies of $H$ in $H(q)$. Then $H$ is isomorphic to $H_q^r$.

**Proof.** At the $i$-th iteration of the blow-up procedure, each copy of $v_j$, $j \neq i$, is replaced by $q$ copies of it, each of which is in the edge-disjoint copies of $H_{i-1}$ glued along copies of $v_i$. By enumerating the $q$ edge-disjoint copies of $H_{i-1}$ in $H_i$, we label each copy of $v_j \in V(H)$ by a vector $(n_1, n_2, \ldots, n_r) \in V(q, r, \alpha)$, where $n_j = \alpha$ and $n_i$, $i \neq j$, indicates that the vertex is in the $n_i$-th copy of $H_{i-1}$ in $H_i$. Let $\phi : V(H) \to V(q, r, \alpha)$ be this labelling map.

We claim that this function $\phi$ is an isomorphism from $H$ to $H_q^r$. Indeed, two vertices labelled by $(n_1, n_2, \ldots, n_r)$ and $(m_1, m_2, \ldots, m_r)$, respectively, are in the same standard $H$-copy if and only if $m_i = n_i$ for all $i$ except their $\alpha$-bits. Hence, $r$ vertices in $V(H)$ form an edge if and only if their labels by $\phi$ in $V(q, r, \alpha)$ form an edge in $V(H_q^r)$, which proves the claim. \qed

From now on, we shall identify the $r$-graph $H$ with $H_q^r$. In a linear hypergraph, a path $P$ from a vertex $u$ to another vertex $v$ is an alternating sequence $v_0e_1v_1e_2 \cdots v_{\ell-1}e_\ell v_\ell$ of vertices and edges, where $v_0 = u$, $v_\ell = v$, $\{v_i, v_{i+1}\} \subseteq e_{i+1}$, and any non-consecutive edges are disjoint. Two paths $ue_1v_1e_2 \cdots v_{\ell-1}e_\ell v$ and $u'e_{i}'v'_1e'_2 \cdots v'_{\ell-1}e'_\ell v$ from $u$ to $v$ are internally vertex-disjoint if $e_1 \cap e'_1 = \{u\}$, $e_\ell \cap e'_\ell = \{v\}$, and all the other pairs $e_i$ and $e'_i$ are disjoint.

We say that the two paths are vertex-disjoint if all edges of one path are disjoint from all edges of the other. Multiple paths $P_1, P_2, \ldots, P_k$ are (internally) vertex-disjoint if they are pairwise (internally) vertex-disjoint. An $r$-graph $\mathcal{G}$ is $k$-connected if there are at least $k$ internally vertex-disjoint paths from a vertex $u$ to another vertex $v$ for all pairs of distinct vertices $u$ and $v$. We show that $H_q^r$ is highly connected in this sense for large enough $q$ in the following proposition, whose proof will be postponed for a while.

**Proposition 2.4.** For integers $k, r \geq 2$, there exists $q = q_{k, r}$ such that $H_q^r$ is $k$-connected.

Let $W_i, i \in [q]$, and $U_r$ be subsets of $V(q, r, \alpha)$ defined by

$$W_i := \{v = (n_1, \ldots, n_r) : n_r = i\} \text{ and } U_r := \{u = (m_1, \ldots, m_r) : m_r = \alpha\}.$$  

Let $H_i = H_q^r[W_i \cup U_r]$ for brevity. The $r$-extension of an $(r-1)$-graph $\mathcal{G}$ is the $r$-graph obtained by adding $e(\mathcal{G})$ extra vertices, each of which is added to a unique $(r-1)$-uniform edge in $\mathcal{G}$. Then $H_i$ is isomorphic to the $r$-extension of a copy of the $(r-1)$-graph $H_q^{r-1}$ on $W_i$ by the isomorphism that maps each vertex $v = (n_1, \ldots, n_{r-1}, i)$ in $H_i$ to $(n_1, \ldots, n_{r-1}) \in V(q, r-1, \alpha)$ and $(n_1, \ldots, n_{r-1}, \alpha)$ to the extra vertex added to extend the edge $(n_1, \ldots, n_{r-1})$.

For vertex subsets $U$ and $V$, a $U$–$V$ path is a path $v_0e_1v_1e_2 \cdots v_{\ell-1}e_\ell v_\ell$ such that $v_0 \in U$, $v_\ell \in V$, and $v_i \notin U \cup V$ for each $i$ distinct from 0 and $\ell$. 

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Lemma 2.5. For $r \geq 3$ and $1 \leq s \leq q$, let $U$ and $V$ be subsets of $U_r$ of size at least $s$. Then there exist vertex-disjoint $U$–$V$ paths $Q_1, Q_2, \ldots, Q_s$ such that each $Q_i$ is a path in $\mathcal{H}_i$.

Proof. Consider the auxiliary graph $G$ on $U_r$ such that $uv' \in E(G)$ if and only if $w$ and $w'$ share a neighbour in $\mathcal{H}_i$. That is, $uv'$ is an edge in $G$ if $w = (n_1, \ldots, n_{r-1}, \alpha)$ and $w' = (n'_1, \ldots, n'_{r-1}, \alpha)$ differ by exactly one entry, which is at the $\alpha$-bit of their common neighbour in $W_i$. Hence, this graph $G$ is isomorphic to the graph $K^{r-1}_q$ obtained by taking Cartesian product of $r-1$ copies of $K_q$ and moreover, the graph $G$ is independent of the choice of $i \in [q]$. In particular, $G$ is $(r-1)(q-1)$-connected, see, e.g., Theorem 1 in [19]. By Menger’s theorem, there are at least $s$ vertex-disjoint $U$–$V$ paths in $G$, which we denote by $P_1, P_2, \ldots, P_s$, provided $(r-1)(q-1) \geq q \geq s$.

Our goal is to construct a $U$–$V$ path $Q_i$ in $\mathcal{H}_i$ by using $P_i$. We may assume that $U$ and $V$ are disjoint, as otherwise, one may assign a trivial path at each vertex in the intersection and consider $U' := U \setminus V$ and $V' = V \setminus U$ instead of $U$ and $V$, respectively. It then suffices to find $s'$ vertex-disjoint $U'$–$V'$ paths, $s' < s$, where induction on $s$ applies.

We choose vertex-disjoint paths $P_1, P_2, \ldots, P_s$ that minimise the sum of the length of each path. Then each $P_i$ is an induced path, i.e., there are no $G$-edges on $(V(P_i) \setminus E(P_i))$. To see this, let $P_i = u_0u_1 \cdots u_k$ with $u_0 = u$ and $u_k = v$. If there is an edge $u_iu_j$ with $i + 1 < j$, then one can shorten the length of $P_i$ by replacing the path $u_iu_{i+1} \cdots u_j$ by $u_iu_j$. The internal vertices of the shorter path $P'_i$ is still non-empty as $uv \notin E(G)$ and disjoint from the internal vertices of other $P_j$’s, so we strictly reduce the sum of the $s$ vertex-disjoint paths.

Now each $U$–$V$ path $P_i = u_0u_1u_2 \cdots u_k$ yields a $U$–$V$ path $Q_i$ in $\mathcal{H}_i$. Indeed, there exists a unique edge $e_j \in \mathcal{H}_i$ containing $u_j$ such that $e_j$ and $e_{j+1}$ share a vertex $w_j$ in $W_i$ by definition of $G$. Furthermore, two non-consecutive edges $e_j$ and $e_{j'}$, $j + 1 < j'$, are always disjoint, as otherwise $u_ju_{j'} \in E(G)$. Therefore, $u_0e_1u_2e_2 \cdots u_{j-1}e_je_j$ is a path in $\mathcal{H}_i$. It then remains to check whether $Q_1, \ldots, Q_k$ in $\mathcal{H}_q$ are vertex-disjoint. The vertices in $Q_i$ and $Q_j$ are in $W_i \cup U_r$ and $W_j \cup U_r$, respectively. Indeed, the two sets $W_i$ and $W_j$ are disjoint and the vertices of $Q_i$ and $Q_j$ in $U_r$ are disjoint too, as they are exactly vertices of $P_i$ and $P_j$, respectively.

Proof of Proposition 2.4. If $r = 2$ then $\mathcal{H}^r_q$ is a copy of $K_{q,q}$, which is $q$-connected. We may hence assume that $r \geq 3$. Take $q \geq \max\{q_{r-1,k}, 3(k+1)\}$ which is a multiple of 3. Let $u, v$ be distinct vertices in $\mathcal{H}^r_q$. By induction on $r$, $\mathcal{H}^{r-1}_q$ is $k$-connected. As $\mathcal{H}_i$ is the $r$-extension of $\mathcal{H}^{r-1}_q$, there are at least $k$ internally vertex-disjoint paths in $\mathcal{H}_i$ from $u$ to $v$ if both vertices are in $W_i$.

Suppose that $u, v \in U_r$. For $1 \leq i \leq q/3$, let $P_i$ be the path $w_{i,1}w_{i,2}w_{i,3}u_i$ in $\mathcal{H}_i$, i.e., $w_i \in W_i$, $u_i \in U_r$, and $e_{i,1}$ is the only edge in $\mathcal{H}_i$ containing $u$. We may further assume that all $u_i$’s are distinct, as there are $q - 1$ neighbours of $w_j$ in $U_r$ except $u$. Analogously, take paths $P'_j = w_{j,1}w_{j,2}w_{j,3}v_j$ for $q/3 < j \leq 2q/3$ in $\mathcal{H}_j$ where $v_j$’s are all distinct. Applying Lemma 2.5 with $U = \{u_i : 1 \leq i \leq q/3\}$, $V = \{v_j : q/3 < j \leq 2q/3\}$, and $s = q/3$ gives $q/3$ vertex-disjoint $U$–$V$ paths, each of which uses a unique $\mathcal{H}_i$ for some $t > 2q/3$. Here we relabel $\mathcal{H}_i$’s if necessary. Thus, concatenating these $U$–$V$ paths with $P_i$ and $P'_j$ yields at least $k$ internally vertex-disjoint paths from $u$ to $v$.

Next, suppose that $u \in U_r$ and $v \in W_j$. For all $i \in \{q/3\} \setminus \{j\}$, we analogously collect paths $P_i$ of length two from $u$ such that each $P_i$ is in $\mathcal{H}_i$ and ends at $u_i \in U_r$, where $u_i$’s are all distinct. There are $q$ neighbours of $v$ in $U_r$, which we denote by $N(v; U_r)$. Then again by Lemma 2.5, there are at least $q/3 - 1$ vertex-disjoint $U$–$V$ paths from $U = \{u_i : i \in \{q/3\} \setminus \{j\}\}$ to $V = N(v; U_r)$, each of which uses distinct $\mathcal{H}_i$ such that $t \neq j$ and $t > q/3$. Concatenating these $U$–$V$ paths with $P_i$’s and the edges incident to $v$ gives $k$ internally vertex-disjoint paths from $u$ to $v$.

Lastly, suppose that $u \in W_i$ and $v \in W_j$ for $i \neq j$. Let $N(u; U_r)$ and $N(v; U_r)$ be neighbours of $u$ and $v$ in $U_r$, respectively. Then Lemma 2.5 gives $q$ vertex-disjoint $N(u; U_r)$–$N(v; U_r)$ paths.
After deleting those paths in $\mathcal{H}_i$ or $\mathcal{H}_j$, there are still at least $k$ paths left, which allow us to make $k$ internally vertex-disjoint path from $u$ to $v$.

Theorem 2.2 follows from the fact that internally disjoint paths in $\mathcal{H}_q^r$ translate to internally disjoint paths in $H(q)$.

**Lemma 2.6.** Let $H$ be a connected graph and let $P_1, P_2, \ldots, P_k$ be $k$ internally vertex-disjoint paths from $u$ to $v$ in $\mathcal{H}_q^r$. Then there exist internally vertex-disjoint paths $Q_1, Q_2, \ldots, Q_k$ from $u$ to $v$ in $H(q)$ such that each $Q_i$, $i \in [k]$, only uses those edges and vertices in the standard $H$-copies that correspond to edges in $P_i$.

**Proof.** Let $P_i = v_{i,0} e_{i,0} v_{i,1} e_{i,1} \cdots v_{i,\ell_i-1} e_{i,\ell_i} v_{i,\ell_i}$.

As $H$ is connected, there exists a path $Q_{i,j+1}$ from $v_{i,j}$ to $v_{i,j+1}$ in the standard copy of $H(q)$ that corresponds to the edge $e_{i,j+1}$. For paths $P$ from $x$ to $y$ and $P'$ from $y$ to $z$, we write $xPyP'z$ for the concatenation of the two paths from $x$ to $z$. For each $i \in [k]$, let $Q_i = v_{i,0}Q_{i,1}v_{i,1}Q_{i,2} \cdots v_{i,\ell_i-1}Q_{i,\ell_i}v_{i,\ell_i}$. We claim that these paths $Q_1, Q_2, \ldots, Q_k$ are internally vertex-disjoint. Indeed, $Q_{i,1}$ and $Q_{j,1}$ are in the $H$-copies that correspond to $e_{i,1}$ and $e_{j,1}$, respectively, who share the vertex $u = v_{i,0} = v_{j,0}$ only; by the same reason, $Q_{i,\ell_i}$ and $Q_{j,\ell_j}$ are also disjoint except the vertex $v = v_{i,\ell_i} = v_{j,\ell_j}$; the other $Q_{i,j}$ and $Q_{j,j'}$ are vertex-disjoint, since $e_{i,j}$ and $e_{j,j'}$ are disjoint edges in $\mathcal{H}_q^r$. \hfill \qed

### 3 Concluding remarks

After Theorem 1.1, it would be natural to ask for examples of common graphs that are even more challenging to find. We suggest to find a common graph with arbitrarily large girth, chromatic number, and connectivity.

**Question 3.1.** Let $r, k, g \geq 3$ be integers. Does there exist an $r$-chromatic $k$-connected common graph with girth at least $g$?

We believe that such a common graph exists; however, our blown-up graph $H(q)$ in Theorem 1.1 may decrease the girth of $H$, as the construction produces 4-cycles whenever $q \geq 2$ and $E(H)$ is nonempty. This suggests that solving Question 3.1 might require new ideas.

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