Finite-time convergence of solutions of Hamilton-Jacobi equations

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Abstract

Suppose that $H(x, u, p)$ is strictly decreasing in $u$ and satisfies Tonelli conditions in $p$. We show that each viscosity solution of $H(x, u, u_x) = 0$ can be reached by many viscosity solutions of

$$w_t + H(x, w, w_x) = 0,$$

in a finite time.

Keywords. Hamilton-Jacobi equations, viscosity solutions, weak KAM theory

1 Introduction

Let $M$ be a smooth, connected, compact Riemannian manifold without boundary, and $T^*M$ denote the cotangent bundle of $M$. Assume $H : T^*M \times \mathbb{R} \to \mathbb{R}$, $H = H(x, u, p)$, is a $C^3$ function satisfying: (H1) the Hessian $\frac{\partial^2 H}{\partial p^2}(x, u, p)$ is positive definite for all $(x, u, p) \in T^*M \times \mathbb{R}$; (H2) for every $(x, u) \in M \times \mathbb{R}$, $H(x, u, p)$ is superlinear in $p$; (H3) there are constants $K_1 > 0$ and $K_2 > 0$ such that

$$-K_1 \leq \frac{\partial H}{\partial u}(x, u, p) \leq -K_2, \quad \forall (x, u, p) \in T^*M \times \mathbb{R}.$$

Here, for convenience, we denote $(x, p) \in T^*M$, $u \in \mathbb{R}$, by $(x, u, p) \in T^*M \times \mathbb{R}$.

The notion of viscosity solutions of scalar nonlinear first order Hamilton-Jacobi equations was introduced by Crandall, Evans and Lions [3], [4]. In this paper we aim to understand the long-time behavior of viscosity solutions of

$$\begin{cases}
w_t + H(x, w, w_x) = 0, & (x, t) \in M \times (0, +\infty), \\
w(x, 0) = \varphi(x), & x \in M,
\end{cases}$$

(CP)

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where $\varphi \in C(M, \mathbb{R})$ is the initial data. More precisely, for each viscosity solution $u$ of

$$H(x, u, u_x) = 0, \quad x \in M,$$

we are interested in whether there is $\varphi \in C(M, \mathbb{R})$ different from $u$ such that the unique viscosity solution $w_\varphi$ of (CP) converges to $u$ in a finite time. There have been various literatures dealing with long-time behavior of viscosity solutions of evolutionary Hamilton-Jacobi equations, where convergence results like

$$\lim_{t \to +\infty} \left( w_\varphi(x, t) + ct \right) = u(x) \quad \text{for some constant } c,$$

were proved under various different assumptions, see for instance [7] and the references therein. In contrast, we are interested in whether there is $\varphi \in C(M, \mathbb{R})$ such that

$$\inf_{\gamma} \left\{ \varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), T^{-}_\tau \varphi(\gamma(\tau)), \dot{\gamma}(\tau))d\tau \right\},$$

where the infimum is taken among the absolutely continuous curves $\gamma : [0, t] \to M$ with $\gamma(t) = x$. It was also proved in [9] that $\{ T^{-}_t \}_{t \geq 0}$ is a semigroup of operators and the function $(x, t) \mapsto T^{-}_t \varphi(x)$ is a viscosity solution of (CP). Thus, we call $\{ T^{-}_t \}_{t \geq 0}$ the backward solution semigroup. Similarly, one can define another semigroup of operators $\{ T^+_t \}_{t \geq 0}$, called the forward solution semigroup, by

$$T^+_t \varphi(x) = \sup_{\gamma} \left\{ \varphi(\gamma(t)) - \int_0^t L(\gamma(\tau), T^+_\tau \varphi(\gamma(\tau)), \dot{\gamma}(\tau))d\tau \right\},$$

where the supremum is taken among the absolutely continuous curves $\gamma : [0, t] \to M$ with $\gamma(0) = x$. $T^{-}_t \varphi(x)$ and $T^+_t \varphi(x)$ can be represented by [9]:

$$T^{-}_t \varphi(x) = \inf_{y \in M} h_y,\varphi(y)(x, t), \quad T^+_t \varphi(x) = \sup_{y \in M} h^y,\varphi(y)(x, t), \quad (x, t) \in M \times (0, +\infty),$$

respectively. Here, the continuous functions $h_\cdot,\cdot : M \times \mathbb{R} \times M \times (0, +\infty) \to \mathbb{R}$, $(x_0, u_0, x, t) \mapsto h_{x_0, u_0}(x, t)$ and $h^\cdot,\cdot : M \times \mathbb{R} \times M \times (0, +\infty) \to \mathbb{R}$, $(x_0, u_0, x, t) \mapsto h^{x_0, u_0}(x, t)$ were introduced in [8], called forward and backward implicit action functions respectively. For any $(x_0, u_0, u, x, t) \in M \times \mathbb{R} \times \mathbb{R} \times M \times (0, +\infty)$, the following relation

$$h_{x_0, u_0}(x, t) = u \quad \text{if and only if} \quad h_{x, u}(x_0, t) = u_0$$

holds true. Given $x_0 \in M$, $u_1, u_2 \in \mathbb{R}$, if $u_1 < u_2$, then $h_{x_0, u_1}(x, t) < h_{x_0, u_2}(x, t)$ and $h^{x_0, u_1}(x, t) < h^{x_0, u_2}(x, t)$, for all $(x, t) \in M \times (0, +\infty)$. See [8], [9], [10] for more properties of implicit action functions.

1.b. – Weak KAM solutions. Following Fathi [5, 6], one can define backward and forward weak KAM solutions of equation (HJ), and prove that backward weak KAM solutions and viscosity solutions are the same under assumptions imposed in this paper. Moreover, $u$ is a backward weak KAM solution if and only if $T^{-}_t u = u$ for all $t \geq 0$, and $u$ is a forward weak KAM solution if and only if $T^+_t u = u$ for all $t \geq 0$. See [10], [11] for more details.
Weak KAM solutions of Hamilton-Jacobi equations

1.c. – Solvability. Let \( F(x, u, p) := H(x, -u, -p) \). Then \( F \) satisfies Tonelli conditions in \( p \) and is strictly increasing in \( u \). It is a well known fact that

\[
F(x, u, Du) = 0 
\]

has a unique viscosity solution (or equivalently, backward weak KAM solution). Moreover, \((1.1)\) admits at least a forward weak KAM solution. By the relation of weak KAM solutions of \((1.1)\) and \((HJ)\): \( u \) is a backward (resp. forward) weak KAM solution of \((HJ)\) if and only if \(-u\) is a forward (resp. backward) weak KAM solution of \((1.1)\), it is clear that the set \( S \) of all viscosity solutions (or equivalently, backward weak KAM solutions) of \((HJ)\) is non-empty, and the forward weak KAM solution of \((HJ)\) is unique, denoted by \( u^+ \). Readers can find all the above results in [10].

1.d. – Decompositions of \( C(M, \mathbb{R}) \).

\[
\begin{align*}
A & := \{ \varphi \in C(M, \mathbb{R}) : \min_{x \in M} (\varphi(x) - u^+(x)) = 0 \}, \\
A_+ & := \{ \varphi \in C(M, \mathbb{R}) : \min_{x \in M} (\varphi(x) - u^+(x)) > 0 \}, \\
A_- & := \{ \varphi \in C(M, \mathbb{R}) : \min_{x \in M} (\varphi(x) - u^+(x)) < 0 \}.
\end{align*}
\]

It is obvious that \( C(M, \mathbb{R}) = A \cup A_+ \cup A_- \).

Under assumptions (H1)-(H3), it was proved in [11, Main Result 2 (1)] that

\[
T_t^- A \subset A, \quad T_t^- A_+ \subset A_+, \quad T_t^- A_- \subset A_-, \quad \forall t \geq 0.
\]

Moreover, one can deduce that

- \( \varphi \in A \) if and only if \( T_t^- \varphi(x) \) is bounded on \( M \times [0, +\infty) \);
- \( \varphi \in A_+ \) if and only if \( \lim_{t \to +\infty} T_t^- \varphi(x) = +\infty \) uniformly in \( x \in M \);
- \( \varphi \in A_- \) if and only if \( \lim_{t \to +\infty} T_t^- \varphi(x) = -\infty \) uniformly in \( x \in M \).

In view of the above arguments, it is clear that

\[
T_t^- A \subset T_s^- A, \quad \forall s \leq t.
\]

Let

\[
A_t := T_t^- A, \quad \forall t \geq 0,
\]

and

\[
A_\infty := \bigcap_{t \geq 0} T_t^- A.
\]

In view of \( S = \{ u \in C(M, \mathbb{R}) : T_t^- u = u, \ \forall t \geq 0 \} \), we get that \( S \subset A_\infty \).

1.e. – Aubry sets. For any \( \varphi \in A \), define

\[
I_\varphi := \{ x \in M : \varphi(x) = u^+(x) \},
\]

where \( u^+ \) is the aforementioned unique forward weak KAM solution of \((HJ)\). Let \( u \in S \). Then \( T_t^- u = u \) for all \( t \geq 0 \). Thus, one can deduce that \( u \in A \). By [10, Theorem 1.2], we have

\[
u^+ \leq u^- \leq u \quad \text{everywhere},
\]
where \( u_- \) is the smallest viscosity solution of (HJ) in the sense of

\[
u_-(x) = \min_{u \in S} u(x), \quad \forall x \in M.
\]

So, it is clear that

\[
I_u \subset I_{u_-},
\]

where \( I_{u_-} \) was called the projected Aubry set in [10]. For any \( x \in I_{u_-} \), there is a global calibrated curve passing through it.

For any \( u \in S \), let

\[
A_u := \{ \varphi \in A : I_u \subset I_\varphi \}.
\]

It is easy to see that \( u_- \in A_u \) for any \( u \in S \).

1.f. – Main results. Now we are in a position to state our first main result.

**Theorem 1.1.** Let \( u \in S \) and \( \varphi \in A_u \). For any \( \epsilon > 0 \), there is \( \varphi_\epsilon \in A_u \) with \( \| \varphi_\epsilon - \varphi \|_\infty < \epsilon \) such that

\[
w_{\varphi_\epsilon}(\cdot, t) = u(\cdot), \quad \forall t \geq t_0,
\]

where \( \varphi_\epsilon \) depends on \( u, \varphi, \epsilon \), and \( t_0 > 0 \) is a constant depending on \( u, \varphi, \epsilon \) and \( \varphi_\epsilon \).

As pointed out in Theorem 1.1, the finite time \( t_0 \) depends on the initial data. We can also provide the following result where the first reach time is uniform with respect to initial data.

**Theorem 1.2.** Let \( K_2 \) be as in (H3). For any \( \epsilon > 0 \),

\[
S \subset A_\infty \subset T^-_t(B_\epsilon(u_+)), \quad \forall t \geq \max \left\{ \frac{1}{K_2} \ln \frac{C_1 + 1 + \| u_+ \|_\infty}{\epsilon}, 1 \right\},
\]

where \( B_\epsilon(u_+) := \{ u \in A, \| u - u_+ \|_\infty < \epsilon \} \), and the constant \( C_1 > 0 \) depends only on \( u_+ \).

This result means that each viscosity solution of (HJ) can be reached by \( T^-_t(\cdot) \) from a neighbourhood of the unique forward weak KAM solution \( u_+ \) in a uniform finite time \( T_0 \), where \( T_0 \) depends only on \( u_+ \) and the neighbourhood.

Our tools come from some dynamical results on the Aubry-Mather theory and the weak KAM theory for contact Hamiltonian systems [8, 9, 10, 11], where variational principles [8, 2, 1] played essential roles.

1.g. – List of symbols.
- \( C(M, \mathbb{R}) \): space of continuous functions on \( M \)
- \( \| \cdot \|_\infty \): the supremum norm on \( C(M, \mathbb{R}) \)
- \( u_+ \): the unique forward weak KAM solution of (HJ)
- \( S \): the set of all viscosity solutions (or equivalently, backward weak KAM solutions) of (HJ)
- \( \{ T^\pm_t \} \): forward and backward solution semigroups associated with \( H \)
- \( w_\varphi \): the unique viscosity solution of (CP)

The rest of this paper is organized as follows. We prove Theorem 1.1 in Section 2. The proof of Theorem 1.2 is given in Section 3.
2 Finite-time convergence

This section is devoted to the proof of Theorem 1.1.

Proof of Theorem 1.1: Let $\varphi \in A_u$. Since $u \geq u_+$ everywhere, by the definition of $I_u$, for any $\epsilon > 0$, there are an open neighbourhood $O_\epsilon$ of $I_u$ and $\varphi_\epsilon \in A$, such that: (i) $\varphi_\epsilon(x) = u(x), \forall x \in O_\epsilon$; (ii) $\varphi_\epsilon(x) > u_+(x)$, $\forall x \in M \setminus O_\epsilon$; (iii) $\|\varphi_\epsilon - \varphi\|_{\infty} < \epsilon$. Here, $\|\cdot\|_{\infty}$ denotes the supremum norm. Note that $I_u = I_{\varphi_\epsilon}$ and thus $\varphi_\epsilon \in A_u$.

Step 1: We aim to show that there is $t_1 > 0$ such that

$$T^-_t \varphi_\epsilon(x) = \inf_{y \in O_\epsilon} h_{y,\varphi_\epsilon(y)}(x,t), \quad (x,t) \in M \times [t_1, +\infty). \quad (2.1)$$

Let $\delta := \min_{x \in M \setminus O_\epsilon} (\varphi_\epsilon(x) - u_+(x))$. Then by (ii) $\delta > 0$ is well defined. Let $u_\delta := u_+ + \delta$. Then $\varphi_\epsilon(x) \geq u_\delta(x)$ for all $x \in M \setminus O_\epsilon$. Recall that $\varphi_\epsilon \in A_u \subset A$. Thus, there is a constant $K > 0$ depending on $\varphi_\epsilon$ such that

$$|T^-_t \varphi_\epsilon(x)| \leq K, \quad \forall t \geq 0, \forall x \in M.$$

Since $u_\delta \in A_+$, then

$$\lim_{t \to +\infty} T^-_t u_\delta(x) = +\infty, \quad \text{uniformly in } x \in M.$$ 

So, there is $t_1 > 0$ such that

$$T^-_t u_\delta(x) \geq K + 1, \quad \forall t \geq t_1, \forall x \in M,$$

where $t_1$ depends on $\epsilon, u, \varphi$ and $\varphi_\epsilon$. Hence, for any $t \geq t_1$, any $x \in M$, we get that

$$\inf_{y \in M \setminus O_\epsilon} h_{y,\varphi_\epsilon(y)}(x,t) \geq \inf_{y \in M} h_{y,\varphi_\epsilon(y)}(x,t) \geq \inf_{y \in M} h_{y,u_\delta(y)}(x,t) = T^-_t u_\delta(x) \geq K + 1. \quad (2.2)$$

The second inequality in (2.2) comes from the monotonicity property of implicit action functions: $v_1 \leq v_2$ implies $h_{x,v_1}(y,t) \leq h_{x,v_2}(y,t)$ for all $(x,y,t) \in M \times M \times (0, +\infty)$. Hence, for any $t \geq t_1$, any $x \in M$, by (2.2), we have

$$T^-_t \varphi_\epsilon(x) = \inf_{y \in M} h_{y,\varphi_\epsilon(y)}(x,t) = \inf_{y \in O_\epsilon} h_{y,\varphi_\epsilon(y)}(x,t), \quad \forall x \in M.$$ 

Thus, (2.1) holds true.

Step 2: Next we show that for above $O_\epsilon$, there is $t_2 > 0$ such that

$$u(x) = \inf_{y \in O_\epsilon} h_{y,u(y)}(x,t), \quad \forall t \geq t_2, \forall x \in M. \quad (2.3)$$

Let $\sigma := \min_{x \in M \setminus O_\epsilon} (u(x) - u_+(x))$. Then by the definition of $I_u$, $\sigma > 0$ is well defined. Let $u_\sigma := u_+ + \sigma$. Then $u_\sigma \in A_+$ and $u(x) \geq u_\sigma(x)$, $\forall x \in M \setminus O_\epsilon$. Thus,

$$\lim_{t \to +\infty} T^-_t u_\sigma(x) = +\infty, \quad \text{uniformly in } x \in M.$$ 

Hence, there is $t_2 > 0$ such that

$$T^-_t u_\sigma(x) \geq \|u\|_{\infty} + 1, \quad \forall t \geq t_2, \forall x \in M,$$
where \( t_2 > 0 \) depends on \( u \) and \( \epsilon \). Thus, for any \( t \geq t_2 \) and any \( x \in M \), we get

\[
\inf_{y \in M \setminus O_e} h_{y,u}(x,t) \geq \inf_{y \in M \setminus O_e} h_{y,u_\sigma(y)}(x,t) \geq \inf_{y \in M} h_{y,u_\sigma(y)}(x,t) = T_t^- u_\sigma(x) \geq \|u\|_{\infty} + 1. \tag{2.4}
\]

Since \( T_t^- u = u \) for all \( t \geq 0 \), then

\[
u(x) = T_t^- u(x) = \inf_{y \in M} h_{y,u}(x,t), \quad \forall t \geq 0, \forall x \in M.
\]

Hence, for any \( t \geq t_2 \) and any \( x \in M \), by (2.4), we get

\[
u(x) = \inf_{y \in M} h_{y,u}(x,t) = \min \left\{ \inf_{y \in O_e} h_{y,u}(x,t), \inf_{y \in M \setminus O_e} h_{y,u}(x,t) \right\} = \inf_{y \in O_e} h_{y,u}(x,t). \tag{2.5}
\]

**Step 3:** Let \( t_0 := \max\{t_1, t_2\} \). Then by (2.1) and (2.5), we obtain that

\[
T_t^- \varphi_{\epsilon}(x) = \inf_{y \in O_e} h_{y,\varphi_{\epsilon}}(x,t) = \inf_{y \in O_e} h_{y,u}(x,t) = \nu(x),
\]

for all \( t \geq t_0 \) and all \( x \in M \). \( \square \)

### 3 Uniform finite-time convergence

**Lemma 3.1.** For each \( x, x_0 \in M, t > 0, u, v \in \mathbb{R} \), there holds

\[
|h_{x_0,u}(x,t) - h_{x_0,v}(x,t)| \geq e^{K_2 t} |u - v|.
\]

**Proof.** By the monotonicity property of \( h_{x_0,u}(x,t) \) with respect to \( u \), if \( u < v \), then \( h_{x_0,u}(x,t) < h_{x_0,v}(x,t) \).

Let \( \gamma_u \) be a minimizer of \( h_{x_0,u}(x,t) \) with \( \gamma_u(0) = x_0 \) and \( \gamma_u(t) = x \). Then, for any \( s \in [0, t] \),

\[
h_{x_0,u}(\gamma_u(s), s) \leq h_{x_0,u}(\gamma_u(s), s). \tag{3.1}
\]

In terms of the definition of \( h_{x_0,v}(x,t) \) and (H3), we have

\[
h_{x_0,v}(\gamma_v(s), s) - h_{x_0,u}(\gamma_u(s), s)
\]

\[
\geq v - u + \int_0^s L(\gamma_v(\tau), h_{x_0,v}(\gamma_v(\tau), \tau), \dot{\gamma}_v(\tau)) - L(\gamma_u(\tau), h_{x_0,u}(\gamma_u(\tau), \tau), \dot{\gamma}_u(\tau)) d\tau
\]

\[
\geq v - u + K_2 \int_0^s h_{x_0,u}(\gamma_u(\tau), \tau) - h_{x_0,u}(\gamma_v(\tau), \tau) d\tau
\]

Let \( F(\tau) := h_{x_0,v}(\gamma_v(\tau), \tau) - h_{x_0,u}(\gamma_u(\tau), \tau) \). It follows from (3.1) that \( F(\tau) > 0 \) for any \( \tau \in (0, t] \). Hence, we have

\[
F(s) \geq v - u + K_2 \int_0^s F(\tau) d\tau, \quad s \in [0, t].
\]

It yields \( F(t) \geq e^{K_2 t}(v - u) \).

Changing the roles of \( u \) and \( v \), a quite similar argument completes the proof. \( \square \)

**Corollary 3.2.** Let \( \varphi, \psi \in C(M, \mathbb{R}) \). If \( \varphi > \psi \) everywhere, then

\[
T_t^- \varphi(x) - T_t^- \psi(x) \geq e^{K_2 t} \min_{y \in M} \{ \varphi(y) - \psi(y) \}, \quad \forall (x, t) \in M \times (0, +\infty).
\]
Proof. Recall that for each $t > 0$ and each $x \in M$, we have
\[ T_t^- \varphi(x) = \inf_{y \in M} h_{y, \varphi(y)}(x, t), \quad T_t^- \psi(x) = \inf_{y \in M} h_{y, \psi(y)}(x, t). \tag{3.2} \]
Note that $h_{y, \varphi(y)}(x, t)$ is continuous in $y$. By the compactness of $M$,
\[ T_t^- \varphi(x) = h_{y_0, \varphi(y_0)}(x, t) \]
for some $y_0 \in M$. By Lemma 3.1 and (3.2), for each $t > 0$ and each $x \in M$, we have
\[ T_t^- \varphi(x) - T_t^- \psi(x) \geq h_{y_0, \varphi(y_0)}(x, t) - h_{y_0, \psi(y_0)}(x, t) = e^{K_2 t} \left( \varphi(y_0) - \psi(y_0) \right) \geq e^{K_2 t} \min_{y \in M} \{ \varphi(y) - \psi(y) \}. \]
The proof is complete. \hfill \qed

Lemma 3.3. For each $t \geq 0$, $T_t^- u_+ \geq u_+$ everywhere.

Proof. It is clear that $T_0^- u_+ = u_+$. For $t > 0$, we have
\[ T_t^- u_+(x) = \inf_{y \in M} h_{y, u_+(y)}(x, t), \quad \forall x \in M. \]
Thus, in order to prove $T_t^- u_+ \geq u_+$ everywhere, it is sufficient to show that for each $y \in M$, $h_{y, u_+(y)}(x, t) \geq u_+(x)$ for all $(x, t) \in M \times (0, +\infty)$. For any given $(x, t) \in M \times (0, +\infty)$, let $v(y) := h_{y, u_+(y)}(x, t)$ for all $y \in M$. Then $u_+(y) = h^{x, v(y)}(y, t)$. Since
\[ u_+(y) = T_t^+ u_+(y) \leq \sup_{z \in M} h^{x, u_+(z)}(y, t), \]
which implies $u_+(y) \geq h^{x, u_+(x)}(y, t)$, i.e., $h^{x, v(y)}(y, t) \geq h^{x, u_+(x)}(y, t)$. By the monotonicity of backward implicit action functions, we have $v(y) \geq u_+(x)$ for all $y \in M$, i.e., $h_{y, u_+(y)}(x, t) \geq u_+(x)$ for all $y \in M$. \hfill \qed

Lemma 3.4. For any given $t > 0$,
\[ C_t := \sup_{\varphi \in A} \| T_t^- \varphi \|_{\infty} < +\infty \]
i.e., $A_t$ is bounded by $C_t$.

Proof. Since $\varphi \in A$, then $\varphi \geq u_+$ and thus $T_t^- \varphi \geq T_t^- u_+ \geq u_+$ everywhere by Lemma 3.3. On the other hand, recall that $I_\varphi = \{ x : \varphi(x) = u_+(x) \}$. Then
\[ T_t^- \varphi(x) = \inf_{y \in M} h_{y, \varphi(y)}(x, t) \leq \inf_{y \in I_\varphi} h_{y, u_+(y)}(x, t) \leq \sup_{y \in I_\varphi} h_{y, u_+(y)}(x, t) \leq \sup_{y \in M} h_{y, u_+(y)}(x, t), \quad \forall x \in M. \]
Since the function $(x_0, u_0, x, s) \mapsto h_{x_0, u_0}(x, s)$ is continuous on $M \times \mathbb{R} \times M \times (0, +\infty)$, then it is bounded on $M \times [-\| u_+ \|_{\infty}, \| u_+ \|_{\infty}] \times M \times \{ t \}$, and thus
\[ C_t := \sup_{\psi \in A_t} \| \psi \|_{\infty} = \sup_{\varphi \in A} \| T_t^- \varphi \|_{\infty} < +\infty, \]
where $C_t$ depends only on $t$ and $u_+$. \hfill \qed
Proof of Theorem 1.2. For any $\varphi \in A_1$ and any $\epsilon > 0$, define $\varphi_\epsilon \in B_\epsilon(u_+)$ by

$$\varphi_\epsilon(x) = \begin{cases} 
\varphi(x), & x \in O_\epsilon, \\
u_+(x) + \epsilon, & x \in M \setminus O_\epsilon,
\end{cases}$$

where $O_\epsilon := \{x \in M : \varphi(x) < u_+(x) + \epsilon\}$. By definition, we get

$$\varphi(x)|_{M\setminus O_\epsilon} \geq \varphi_\epsilon(x)|_{M\setminus O_\epsilon} = u_+(x)|_{M\setminus O_\epsilon} + \epsilon.$$ 

Define $u_\epsilon(x) := u_+(x) + \epsilon$ for all $x \in M$. By Corollary 3.2, we have

$$|T_t^- u_\epsilon(x) - T_t^- u_+(x)| \geq e^{K_2 t} \min_{y \in M} \{u_\epsilon(y) - u_+(y)\} = e^{K_2 t} \epsilon, \quad \forall t > 0, \forall x \in M.$$ 

Since $T_t^- u_\epsilon > T_t^- u_+$ everywhere, we have

$$T_t^- u_\epsilon(x) \geq e^{K_2 t} \epsilon + T_t^- u_+(x), \quad \forall t > 0, \forall x \in M.$$

Set

$$T_0 := \max \left\{ \frac{1}{K_2} \ln \frac{C_1 + 1 + \|u_+\|_\infty}{\epsilon}, 1 \right\},$$

where $C_1$ is as in Lemma 3.4. For any $t > 0$, by the monotonicity property of implicit action functions and the definition of $\varphi_\epsilon$, we get

$$\inf_{y \in M\setminus O_\epsilon} h_{y,\varphi(y)}(x, t) \geq \inf_{y \in M\setminus O_\epsilon} h_{y,\varphi_\epsilon(y)}(x, t) = \inf_{y \in M\setminus O_\epsilon} h_{y,u_\epsilon(y)}(x, t) \geq \inf_{y \in M} h_{y,u_\epsilon(y)}(x, t)$$

(3.3)

For any $t > T_0 > 1$, we have

$$\inf_{y \in M} h_{y,u_\epsilon(y)}(x, t) = T_t^- u_\epsilon(x) \geq T_t^- u_+(x) + e^{K_2 T_0 \epsilon}$$

$$\geq u_+(x) + C_1 + 1 + \|u_+\|_\infty \geq \sup_{\varphi \in A} \|T_t^- \varphi\|_\infty + 1$$

(3.4)

Note that $T_t^- \varphi, T_t^- \varphi_\epsilon \in A_t$. Then by (3.3) and (3.4),

$$\inf_{y \in M\setminus O_\epsilon} h_{y,\varphi(y)}(x, t) \geq C_t + 1 \geq \|T_t^- \varphi\|_\infty + 1 \quad \text{and} \quad \inf_{y \in M\setminus O_\epsilon} h_{y,\varphi_\epsilon(y)}(x, t) \geq C_t + 1 \geq \|T_t^- \varphi_\epsilon\|_\infty + 1.$$ 

Hence, we get

$$T_t^- \varphi_\epsilon(x) = \inf_{y \in O_\epsilon} h_{y,\varphi_\epsilon(y)}(x, t) = \inf_{y \in O_\epsilon} h_{y,\varphi(y)}(x, t)$$

$$= \inf_{y \in O_\epsilon} h_{y,\varphi(y)}(x, t) = \inf_{y \in M} h_{y,\varphi(y)}(x, t) = T_t^- \varphi(x), \quad \forall t \geq T_0,$$

which implies that

$$A_\infty \subset T_t^- A_1 \subset T_t^- (B_\epsilon(u_+)), \quad \forall t \geq T_0.$$
Example 3.5. Consider the following Hamiltonian

\[ H(x, u, p) = -2u + p^2, \quad x \in \mathbb{S}, \ p \in \mathbb{R}, \ u \in \mathbb{R}. \]

Here, \( \mathbb{S} := (-\frac{1}{2}, \frac{1}{2}] \) denotes the unit circle. The corresponding ergodic Hamilton-Jacobi equation reads

\[ -2u + (u')^2 = 0, \quad x \in \mathbb{S}. \tag{3.5} \]

Let \( u_1 \) be the even 1-periodic extension of \( \frac{1}{2}x^2 \) in \( [0, \frac{1}{2}] \). Then it is clear that \( u_1 \) is a viscosity solution of (3.5). Note that \( u = 0 \) is a viscosity solution of

\[ 2u + (u')^2 = 0. \]

In view of the uniqueness of viscosity solutions of the above equation, \( u_+ = 0 \) is the unique forward weak KAM solution of (3.5). Thus, \( I_{u_1} = \{0\} \).

Let \( \varphi \) be the even 1-periodic extension of \( \frac{1}{2}x^2 + x \) in \( [0, \frac{1}{2}] \). Then one can deduce that \( \varphi \in A_{u_1} \). For any given small \( 0 < \epsilon \), define \( \varphi_\epsilon \) as the even 1-periodic extension of

\[ \varphi_\epsilon(x) = \begin{cases} 
\frac{1}{2}x^2, & x \in [0, \epsilon], \\
\frac{1}{2}\epsilon^2 + (\frac{3}{2}\epsilon + 2)(x - \epsilon), & x \in [\epsilon, 2\epsilon], \\
\frac{1}{2}\epsilon^2 + x, & x \in [2\epsilon, \frac{1}{2}].
\end{cases} \]

Then by the proof of Theorem 1.1 there is a finite time \( t_0 > 0 \) such that for all \( t \geq t_0 \),

\[ w_{\varphi_\epsilon}(x, t) = u_1(x), \quad \forall x \in M. \]

Moreover, for \( \varphi \in \text{Lip}(M, \mathbb{R}) \) similarly with the proof of Theorem 1.2 one obtains the following estimation of a finite time in Theorem 1.1

\[ t_0 := \frac{1}{K_2} \ln \frac{M_0 + 1 + \|u_+\|_\infty}{f(\epsilon)}, \]

where \( M_0 := \sup_{\psi \in \mathbb{S}} \|\psi\|_\infty \) and

\[ f(\epsilon) = \min_{\text{dist}(x, I_u) = \epsilon} \frac{\min \{u(x) - u_+(x)\}}{\max(\text{Lip}(\varphi), \text{Lip}(u))}. \]

In this example (3.5), \( \|u_+\|_\infty = 0 \), \( M_0 = \frac{1}{3} \), \( K_2 = 4 \) and \( f(\epsilon) = \frac{2}{9}\epsilon^2 \). Hence \( t_0 = \frac{1}{2} \ln \frac{9}{4\epsilon} \) for given small \( \epsilon \).

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