On extension of partial orders to total preorders with prescribed symmetric part

Dmitry V. Akopian and Valentin V. Gorokhovik

Institute of Mathematics,
The National Academy of Sciences of Belarus,
Surganova st., 11, Minsk 220072, Belarus
e-mail: gorokh@im.bas-net.by

Abstract For a partial order $\preceq$ on a set $X$ and an equivalency relation $S$ defined on the same set $X$ we derive a necessary and sufficient condition for the existence of such a total preorder on $X$ whose asymmetric part contains the asymmetric part of the given partial order $\preceq$ and whose symmetric part coincides with the given equivalence relation $S$. This result generalizes the classical Szpilrajn theorem on extension of a partial order to a perfect (linear) order.

Key words partial order, preorder, extension, Szpilrajn theorem, Dushnik-Miller theorem.

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Let $X$ be an arbitrary nonempty set and let $G \subset X \times X$ be a binary relation on $X$. A binary relation $G \subset X \times X$ is called a preorder if it is reflexive ($(x, x) \in G \forall x \in X$) and transitive ($(x, y) \in G, (y, z) \in G \Rightarrow (x, z) \in G \forall x, y, z \in X$). If in addition a preorder $G \in X \times X$ is antisymmetric ($(x, y) \in G, (y, x) \in G \Rightarrow x = y \forall x, y \in X$), then it is called a partial order. A total partial order is called a perfect (or linear) order. (A binary relation $G \subset X \times X$ is total if for any $x, y \in X$ either $(x, y) \in G$ or $(y, x) \in G$ holds.) In the sequel, a preorder will be preferably denoted by the symbol $\preceq$ whereas a partial order as well as a perfect order by the symbol $\preceq$.

One of the key results of the theory of ordered sets is the following theorem proved by E. Szpilrajn in 1930 [1].

**Theorem 1** (E. Szpilrajn [1]) For every partial order $\preceq \subset X \times X$ there exists a perfect extension, i.e., there exists a perfect order $\preceq' \subset X \times X$ such that $\preceq \subset \preceq'$. Moreover, for any pair of elements $a, b \in X$ such that $(a, b) \not\in \preceq$ and $(b, a) \not\in \preceq$ a perfect extension $\preceq' \subset X \times X$ for the partial order $\preceq$ can be chosen in such a way that $(a, b) \in \preceq'$.

In 1941 E. Dushnik and B. Miller proved the following strengthening of the Szpilrajn theorem.

**Theorem 2** (E. Dushnik, B. Miller [2]) Every partial order $\preceq \subset X \times X$ is the intersection of all its perfect extensions.

In the recent literature the Szpilrajn theorem and the Dushnik–Miller theorem and their proofs can be found in the monographs [3,4]. The generalizations of the Szpilrajn theorem to the case when partial orders and perfect orders extending them are defined on groups, rings and some other algebraic systems and are compatible with their algebraic operations are presented in the monograph of L. Fuchs [5]. Due to the duality between compatible perfect orders defined on a real vector space $X$ and semispaces of $X$ (the cones of positive elements of compatible perfect orders are complements of semispaces at zero) it follows from the results of V. Klee devoted to semispaces [6] that any compatible partial order defined on a real vector space $X$ can be extended to a compatible perfect order.

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order. For relations defined on topological spaces the conditions under which there exist continuous total preorders extending partial orders were obtained by G. Bosi and G. Herden [7, 8]. The results of the studies devoted to the existence of utility functions for partial orders (see [3, 9] as well the monographs [10, 11] and bibliography cited there) can also be considered as generalizations of the Szpilrajn theorem.

Every binary relation $G$ on $X$ can be presented as the disjoint union $G = P_G \cup S_G$ ($P_G \cap S_G = \emptyset$) of its asymmetric part $P_G := \{(x, y) \in G \mid (y, x) \notin G\}$ and its symmetric part $S_G := \{(x, y) \in G \mid (y, x) \in G\}$. If $G$ is a preorder then its symmetric part $S_G$ is reflexive, symmetric and transitive and, consequently, in that case $S_G$ is an equivalency relation on $X$, which is reduced to the equality relation when $G$ is a partial order. The asymmetric part of a preorder (and, in particular, the asymmetric part of a partial order) is an asymmetric and transitive binary relation. On the other hand, the union of any asymmetric and transitive binary relation with the equality relation is a partial order. Thus, there exists the one-to-one correspondence between partial orders and asymmetric and transitive binary relations. Note that different preorders can have the same asymmetric part.

Let $\preceq$ be a partial order on $X$ and $S$ an equivalency relation defined on the same set $X$.

A total preorder $\preceq \subset X \times X$ will be referred to as a total preorder $S$–extension of the partial order $\preceq$ if the asymmetric part of $\preceq$ contains the asymmetric part of the given partial order $\preceq$ and the symmetric part of $\preceq$ coincides with the equivalency relation $S$, that is, if $P_{\preceq} \subset P_{\preceq}$ and $S_{\preceq} = S$.

The main purpose of this paper is to derive for a given partial order $\preceq$ and a given equivalency relation $S$ a necessary and sufficient condition for the existence of a total preorder $S$–extension of $\preceq$.

In the case when $S$ is the equality relation on $X$, i.e., when $S = E := \{(x, y) \in X \times X \mid x = y\}$, due to the Szpilrajn theorem, such an extension exists for any partial order $\preceq$. As it will be shown below in the general case the required extension exists if and only if the partial order $\preceq$ and the equivalency relation $S$ are compatible in some way. Thus the main results of the paper can be considered as a generalization of the Szpilrajn theorem.

Let us begin with consideration of a particular case. Assume that a partial order $\preceq$ and an equivalency relation $S$ hold the additional condition

$$P_{\preceq} \circ S = S \circ P_{\preceq} = P_{\preceq}$$

(the symbol $\circ$ denotes the composition of binary relations).

It immediately follows from (1) that the union $\preceq \cup S$ is a preorder on $X$ the symmetric part of which coincides with $S$. Let $X/S$ be the quotient of $X$ with respect to the equivalency relation $S$ and let $T$ be the quotient of the preorder $\preceq \cup S$ with respect to $S$. Since $T$ is a partial order on $X/S$, due to the Szpilrajn theorem, $T$ can be extended to a perfect order $Q$ on $X/S$. Setting $x T y \iff [x]_S Q [y]_S$ (here $[x]_S$ and $[y]_S$ stands for the equivalency classes of $S$ containing $x$ and $y$, respectively) we obtain the total preorder $\preceq$ on $X$ which is a total preorder $S$–extension of $\preceq$.

Thus the following theorem generalizing both the Szpilrajn theorem and the Dushnik–Miller theorem is true.

**Theorem 3** Let $\preceq$ be a partial order on $X$. For any equivalency relation $S$ on $X$ which satisfies condition (1), there exists a total preorder $S$–extension of the partial order $\preceq$.

Moreover, for any pair of points $a, b \in X$ such that $(a, b) \notin \preceq \cup S$ and $(b, a) \notin \preceq \cup S$, there exists a total preorder $\preceq$, which is a total preorder $S$–extension of the partial order $\preceq$ and $(a, b) \in \preceq$.

The intersection of all total preorder $S$–extensions of a preorder $\preceq$ coincides with the preorder $\preceq \cup S$.

Along with each relation of preorder $\preceq$ we will consider the indifference relation $I_{\preceq} := \{(x, y) \in X \times X \mid (x, y) \notin P_{\preceq}, (y, x) \notin P_{\preceq}\}$ corresponding to $\preceq$. In the general case the indifference relation
It is not hard to verify that \( R \) is defined by negatively transitive (it means that the negation of \( P \) and only one of the following three alternatives: \((x, y) \in P \), \((y, x) \in P \) and \((x, y) \in I \).

Another binary relation on \( X \) generated by a preorder \( \preceq \) is the equipotency relation \( R_{\preceq} \), which is defined by

\[
(x, y) \in R_{\preceq} \iff \{ z \in X \mid (x, z) \in I \} = \{ z \in X \mid (y, z) \in I \}.
\]

It is not hard to verify that \( R_{\preceq} \) is an equipotency relation on \( X \) with \( R_{\preceq} \subseteq I \). The equipotency relation \( R_{\preceq} \) is equal to the indifference relation \( I \), i.e. \( R_{\preceq} = I \), if and only if \( P \) is negatively transitive (or, equivalently, if and only if \( I \) is transitive).

**Proposition 1** Let \( \preceq \) be a preorder on \( X \). An equivalency relation \( S \subseteq X \times X \) holds the equalities \( P \circ S = S \circ P = P \) if and only if \( S \subseteq R_{\preceq} \).

**Proof** Assume that an equivalency relation \( S \) satisfies the equalities \( P \circ S = S \circ P = P \) and let \((x, y) \in S\). The alternative \((x, y) \in P \) is impossible, because otherwise it would follow from \((y, x) \in S\) and from the equality \( P \circ S = P \) that \((x, x) \in P \), but it contradicts the asymmetric property of \( P \). Similarly we can show that the alternative \((y, x) \in P \) is also impossible. Hence, \((x, y) \in I \).

Let us prove that in fact \((x, y) \in R_{\preceq} \). Choose an arbitrary element \( z \in X \) such that \((x, z) \in I \) and consider the ordered pair \((y, z) \in X \times X \). The alternatives \((y, z) \in P \) and \((z, y) \in P \) are impossible, because otherwise it would follow from \((x, y) \in S \) and \( P \circ S = P \) that \((x, x) \in P \), which contradicts the choice of \( z \). Hence, \((y, z) \in I \) and, consequently, \( \{ z \in X \mid (x, z) \in I \} \subseteq \{ z \in X \mid (y, z) \in I \} \). The converse inclusion is proved in the similar way. Thus, \( \{ z \in X \mid (x, z) \in I \} \) and we conclude from the definition of \( R_{\preceq} \) that \((x, y) \in R_{\preceq} \).

To prove the converse statement we note that the inclusions \( P \subseteq P \circ S \) and \( P \subseteq S \circ P \) follow from the reflexivity of the relation \( S \). So we need to prove the converse inclusions. Let \((x, y) \in S \circ P \). Then there exists an element \( z \in X \) such that \((x, z) \in S \) and \((z, y) \in P \). Assume that \((y, x) \in P \). Due to the transitivity of \( P \) we conclude from \((z, y) \in P \) that \((z, x) \in P \), which contradicts \((x, z) \in S \subseteq R_{\preceq} \subseteq I \). Consequently, \((y, x) \notin P \). The assumption \((x, y) \in I \) also leads to a contradiction. Indeed, for \((x, z) \in S \subseteq R_{\preceq} \) we have due to the definition of \( R_{\preceq} \) that \((x, y) \in I \) implies \((z, y) \in I \), which contradicts \((z, y) \in P \). Hence, \((x, y) \notin I \) and, consequently, the alternative \((x, y) \in P \) is uniquely possible. Thus, \( S \circ P \subseteq P \).

The inclusion \( P \circ S \subseteq P \) is proved in the similar way.

**Corollary** For every partial order \( \preceq \) on the set \( X \) and every equivalency relation \( S \) on the same set \( X \) such that \( S \subseteq R_{\preceq} \) there exists a total preorder \( S-\) extension of \( \preceq \).

Let us consider now the general case, that is the case when a partial order \( \preceq \) and an equivalency relation \( S \) do not necessarily satisfy equalities \( [1] \). We begin with the following (evident) necessary condition for the existence of a total preorder \( S-\) extension of a partial order \( \preceq \).

**Theorem 4** Let \( S \) be an equivalency relation on a set \( X \). If for a partial order \( \preceq \subseteq X \times X \) there exists a total preorder \( S-\) extension then \( S \subseteq I \).

**Proof** Let \((x, y) \in S \). If \((x, y) \in P \) or \((y, x) \in P \), then for any total preorder \( S-\) extension \( \preceq \) of a partial order \( \preceq \) we would have \((x, y) \in P \) or \((y, x) \in P \), respectively. However, since \( P \cap S = \emptyset \), the both alternatives are impossible and, hence, \((x, y) \in I \).

**Proposition 2** Let \( S \) be an equivalency relation on a set \( X \) and \( \preceq \) a partial order defined on
the same set $X$. Then $S \subseteq I_\preceq$ if and only if the composition $S \circ P_\preceq \circ S$ is irreflexive.

Proof Recall that the irreflexivity of the binary relation $S \circ P_\preceq \circ S$ means that $(x, x) \notin S \circ P_\preceq \circ S$ for all $x \in X$.

Let $S \subseteq I_\preceq$. Assume that contrary to the assertion of the proposition the composition $S \circ P_\preceq \circ S$ is not irreflexive. The latter means that $(x, x) \in S \circ P_\preceq \circ S$ for some $x \in X$. Due to the definition of the composition we can find such elements $y, z \in X$ that $(x, y) \in S$, $(y, z) \in P_\preceq$, and $(z, x) \in S$.

Since $S$ is transitive, it follows from $(z, x) \in S$ and $(x, y) \in S$ that $(z, y) \in S$. Hence, since $S$ is symmetric, $(y, z) \in S \cap P_\preceq$, which contradicts $I_\preceq \cap P_\preceq = \emptyset$. This proves that $S \subseteq I_\preceq$.

Assume now that the composition $S \circ P_\preceq \circ S$ is irreflexive, but the inclusion $S \subseteq I_\preceq$ is not the case. Then there exists $(x, y) \in S$ such that $(x, y) \notin I_\preceq$ and, consequently, either $(x, y) \notin P_\preceq$, or $(x, y) \in P_\preceq$.

If $(x, y) \in P_\preceq$, it follows from $(x, y) \in S$, $(x, y) \in P_\preceq$, and $(y, y) \in S$ that $(y, y) \in S \circ P_\preceq \circ S$, but it is impossible since $S \circ P_\preceq \circ S$ is irreflexive. Using the similar argument, we conclude that the case $(y, x) \in P_\preceq$ is also impossible. It proves that $S$ is a subset of $I_\preceq$. □

Recall that a binary relation $G \subseteq X \times X$ is said to be acyclic if for any finite collection of elements $x_1, x_2, \ldots, x_n \in X$ it follows from $(x_i, x_{i+1}) \in G$, $i = 1, \ldots, n - 1$, that $(x_n, x_1) \notin G$.

The transitive hull of a binary relation $G$ is the smallest transitive relation $TH(G)$ containing $G$. There holds the equality $TH(G) = \cup\{G^n \mid n \in \mathbb{N}\}$, where $\mathbb{N}$ stands for the set of natural numbers and $G^n := G \circ G \circ \cdots \circ G$. It immediately follows from the latter equality that a binary relation $G$ is acyclic if and only if its transitive hull $TH(G)$ is irreflexive (or, equivalently, if and only if $TH(G)$ is asymmetric).

Theorem 5 Let $S$ be an equivalency relation on a set $X$ and $\preceq$ a partial order defined on the same set $X$. Then the following statements are equivalent:

(i) there exists a total preorder $-\text{extension of } \preceq$;

(ii) the composition $S \circ P_\preceq \circ S$ is acyclic.

Proof (i) $\Rightarrow$ (ii) Let a total preorder $\preceq$ be a total preorder $-\text{extension of a partial order } \preceq$. Assume that the composition $S \circ P_\preceq \circ S$ is not acyclic and let the collection $x_1, x_2, \ldots, x_m \in X$, $m \geq 2$, hold $(x_i, x_{i+1}) \in S \circ P_\preceq \circ S$, $i = 1, \ldots, m - 1$, and $(x_m, x_1) \in S \circ P_\preceq \circ S$. Then there exist collections $y_1, y_2, \ldots, y_m \in X$ and $z_1, z_2, \ldots, z_m \in X$ such that $(x_i, y_i) \in S$, $(y_i, z_i) \in P_\preceq$, $(z_i, x_{i+1}) \in S$, $i = 1, 2, \ldots, m - 1$, and $(x_m, y_m) \in S$, $(y_m, z_m) \in P_\preceq$, $(z_m, x_1) \in S$. The inclusion $P_\preceq \subseteq P_\preceq$ implies that $(y_i, z_i) \in P_\preceq$, $i = 1, \ldots, m$. Since $S = R_\preceq$, we conclude from Proposition 1 that $P_\preceq \circ S = S \circ P_\preceq = P_\preceq$. Hence, it follows from $(x_i, y_i) \in S$, $(y_i, z_i) \in P_\preceq$, $(z_i, x_{i+1}) \in S$, $i = 1, 2, \ldots, m - 1$, that $(x_i, x_{i+1}) \in P_\preceq$, $i = 1, 2, \ldots, m - 1$, whence, due to the transitivity of $P_\preceq$, we obtain $(x_1, x_m) \in P_\preceq$. On the other hand, from $(x_m, y_m) \in S$, $(y_m, z_m) \in P_\preceq$, $(z_m, x_1) \in S$, using the equalities $P_\preceq \circ S = S \circ P_\preceq = P_\preceq$, we deduce $(x_m, x_1) \in P_\preceq$. This is a contradiction because $P_\preceq$ is asymmetric. It proves that $S \circ P_\preceq \circ S$ should be acyclic.

(ii) $\Rightarrow$ (i) The relation $S \circ P_\preceq \circ S$ is acyclic if and only if its transitive hull $TH(S \circ P_\preceq \circ S)$ is asymmetric. Using the equality $TH(S \circ P_\preceq \circ S) = \cup\{(S \circ P_\preceq \circ S)^n \mid n \in \mathbb{N}\}$ it is not difficult to verify that $S \circ TH(S \circ P_\preceq \circ S) = TH(S \circ P_\preceq \circ S) \circ S = TH(S \circ P_\preceq \circ S)$. Hence, due to Theorem 3 there exists a total preorder $\preceq$ on $X$ which is a total preorder $-\text{extension of a partial order } E \cup TH(S \circ P_\preceq \circ S)$ (recall that $E := \{(x, x) \in X \times X \mid x \in X\}$ is the equality relation on $X$).

Since $P_\preceq \subseteq TH(S \circ P_\preceq \circ S)$, the preorder $\preceq$ is also a total preorder $-\text{extension of a partial order } \preceq$. □

Theorem 6 Let $\preceq$ be a partial order on a set $X$ and $S$ an equivalency relation defined on the same set $X$. If the composition $S \circ P_\preceq \circ S$ is acyclic then the intersection of all total preorder $-\text{extension of the partial order } \preceq$ is the preorder $S \cup TH(S \circ P_\preceq \circ S)$, that is the preorder whose asymmetric part is the transitive hull of $S \circ P_\preceq \circ S$ and whose symmetric part coincides with $S$.  

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Proof Since the asymmetric part of every total preorder $S^-$ extension of the partial order $\preceq$ is transitive and contains $S \circ P \circ S$, it also contains the transitive hull of $S \circ P \circ S$. Hence, every total preorder $S^-$ extension of the partial order $\preceq$ is at the same time a total preorder $S^-$ extension of the partial order $E \cup TH(S \circ P \circ S)$. Conversely, it follows from $P \subseteq S \circ P \circ S$ that every total preorder $S^-$ extension of the partial order $E \cup TH(S \circ P \circ S)$ is a total preorder $S^-$ extension of the partial order $\preceq$. Since $S \circ TH(S \circ P \circ S) = TH(S \circ P \circ S) \circ S = TH(S \circ P \circ S)$, we conclude from the second statement of Theorem 3 that the intersection of all total preorder $S^-$ extension of the partial order $\preceq$ coincides with the preorder $S \cup TH(S \circ P \circ S)$. □

Given a partial order $\preceq$ on $X$, by the symbol $\Sigma(\preceq)$ (respectively, $\Sigma^*(\preceq)$) we denote the collection consisting of all equivalence relations $S$ defined on $X$ such that the composition $S \circ P \circ S$ is reflexive (respectively, $S \circ P \circ S$ is acyclic). Clearly, $\Sigma^*(\preceq)$ is a subcollection of the collection $\Sigma(\preceq)$. It also follows from Proposition 2 and Theorem 5 that $S \in \Sigma(\preceq)$ if and only if $S \subseteq I_{\preceq}$ and $S \in \Sigma^*(\preceq)$ is equivalent to the existence of a total preorder $S^-$ extension of the partial order $\preceq$.

**Theorem 7** Let $S$ be an equivalence relation defined on a set $X$. A partial order $\preceq$ defined on the same set $X$ has a unique total preorder $S^-$ extension if and only if $S$ is maximal (in inclusion) in the subcollection $\Sigma^*(\preceq)$ and the transitive hull of the composition $S \circ P \circ S$ is negatively transitive.

Proof Let $\preceq^*$ be a unique total preorder $S^-$ extension of the partial order $\preceq$. Suppose to the contrary that $S$ is not maximal (in inclusion) in the subcollection $\Sigma^*(\preceq)$. Then there exists an equivalence relation $S'$ in $\Sigma^*(\preceq)$ such that $S \subseteq S'$, $S \neq S'$. Let $\preceq'$ be an arbitrary total preorder $S'$-extension of the partial order $\preceq$. Suppose that $P_{\preceq'} \not\subseteq P_{\preceq}$. Denote by $\preceq^*$ an arbitrary total preorder $S^-$ extension of the partial order $P_{\preceq'} \cup E$. The existence of $\preceq^*$ follows from the inclusion $S \subseteq S' = R_{\preceq'}$ and Corollary 6. Since $P_{\preceq'} \subseteq P_{\preceq} \subseteq P_{\preceq'}$, then $\preceq^*$ is also a total preorder $S^-$ extension of the initial partial order $\preceq$. It follows from the assumption $P_{\preceq'} \not\subseteq P_{\preceq}$ and the inclusion $P_{\preceq'} \subset P_{\preceq^*}$ that $P_{\preceq^*} \not\subseteq P_{\preceq}$. Hence $\preceq^* \not\subseteq \preceq$. Since it contradicts the uniqueness of a total preorder $S^-$ extension of the partial order $\preceq$, then the inclusion $P_{\preceq^*} \not\subseteq P_{\preceq}$ is impossible and, consequently, we have $P_{\preceq^*} \subseteq P_{\preceq}$. In this case we define on $X$ the relation $\preceq^0 := P_{\preceq^*} \cup ((P_{\preceq}^{-1} \cap S') \cup S)$. It is not difficult to verify that $\preceq^0$ is a total preorder, which differs from $\preceq$ only on the equivalence classes of $S'$, where it coincides with the converse relation of $\preceq$. Since $P_{\preceq^*} \subseteq P_{\preceq} \subseteq P_{\preceq^*} := P_{\preceq^*} \cup ((P_{\preceq}^{-1} \cap S') \cup S)$ and $S_{\preceq^0} = S$, then $\preceq^0$ is a total preorder $S^-$ extension of the partial order $\preceq$, which differs from $\preceq^*$. Again we get the contradiction to the uniqueness of a total preorder $S^-$ extension of the partial order $\preceq$. This completes the proof that $S$ is maximal (in inclusion) in the subcollection $\Sigma^*(\preceq)$.

It remains to prove that $TH(S \circ P \circ S)$ is negatively transitive. Since $\preceq$ is the unique total preorder $S^-$ extension of the partial order $\preceq$, then due to Theorem 6 we conclude that $P_{\preceq} = TH(S \circ P \circ S)$. Notice now that $P_{\preceq}$ is the asymmetric part of the total preorder $\preceq$ and therefore it is negatively transitive. Hence, $\bar{TH}(S \circ P \circ S)$ is negatively transitive too.

For the converse, notice that the assumption $S \in \Sigma^*(\preceq)$ is equivalent to the asymmetry property of the transitive hull $TH(S \circ P \circ S)$. It implies that the relation $\preceq := S \cup TH(S \circ P \circ S)$ is the preorder. Since $TH(S \circ P \circ S)$ is negatively transitive, the indifference relation $I_{\preceq}$ corresponding to $\preceq$ is transitive and hence $I_{\preceq}$ is an equivalence relation. Then the relation $I_{\preceq} \cup TH(S \circ P \circ S)$ is a total preorder and, moreover, it follows from $P_{\preceq} \subseteq TH(S \circ P \circ S)$ that $I_{\preceq} \subseteq \Sigma^*(\preceq)$. Thus, the preorder $\preceq := S \cup TH(S \circ P \circ S)$ is total and consequently it is a total preorder $S^-$ extension of the partial order $\preceq$. From Theorem 6 we conclude that there are no other total preorder $S^-$ extensions of the partial order $\preceq$. □

**Theorem 8** Let $\preceq$ be a partial order on $X$ and $S$ an equivalence relation defined on the same set $X$. The relation $S \cup (S \circ P \circ S)$ is the unique total preorder $S^-$ extension of the partial order $\preceq$ if and only if $S$ belongs to the subcollection $\Sigma^*(\preceq)$ and is maximal (by inclusion) in the collection $\Sigma(\preceq)$. 

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Proof Assume that an equivalence relation $S$ belongs to $\Sigma^*(\preceq)$ and is maximal (by inclusion) in $\Sigma(\preceq)$. First we prove that for any $x, y \in X$ there holds exactly one alternative of the following three ones:

$$(x, y) \in S, \quad (x, y) \in S \circ P_\preceq \circ S, \quad (y, x) \in S \circ P_\preceq \circ S.$$ 

Choose $x, y \in X$ with $(x, y) \notin S$. If $(x, y) \in P_\preceq$ or $(y, x) \in P_\preceq$, then $(x, y) \in S \circ P_\preceq \circ S$ or $(y, x) \in S \circ P_\preceq \circ S$, respectively. Let $(x, y) \in I_\preceq \setminus S$ and let $[x]_S$ and $[y]_S$ be the equivalency classes of $S$ containing $x$ and $y$, respectively. Since $S$ is maximal (in inclusion) in $\Sigma(\preceq)$, there exist $x_1 \in [x]_S$ and $y_1 \in [y]_S$ such that either $(x_1, y_1) \in P_\preceq$, or $(y_1, x_1) \in P_\preceq$. Indeed, if $(x_1, y_1) \notin P_\preceq$ and $(y_1, x_1) \notin P_\preceq$ for any $x_1 \in [x]_S$ and $y_1 \in [y]_S$, then the equivalency relation $S'$ defined on $X$ by

$$(u, v) \in S' \iff \text{either } (u, v) \in S \text{ or } u, v \in [x]_S \cup [y]_S.$$ 

satisfies $S' \subset I_\preceq$. Since $S \subset S'$, $S \neq S'$, it contradicts maximality (by inclusion) of $S$ in $\Sigma(\preceq)$.

Thus, for any pair $(x, y) \in I_\preceq \setminus S$ there exist $x_1 \in [x]_S$ and $y_1 \in [y]_S$ such that either $(x_1, y_1) \in P_\preceq$, or $(y_1, x_1) \in P_\preceq$.

If $(x_1, y_1) \in P_\preceq$ is the case then it follows from $(x_1, x_1) \in S$, $(x_1, y_1) \in P_\preceq$, $(y_1, y) \in S$ that $(x, y) \in S \circ P_\preceq \circ S$. Similarly, in the case when $(y_1, x_1) \in P_\preceq$ we get from $(y, y_1) \in S$, $(y_1, x_1) \in P_\preceq$, $(x_1, x) \in S$ that $(y, x) \in S \circ P_\preceq \circ S$.

The fact that for any $x, y \in X$ there holds exactly one of the three possible alternatives follows from the assumption $S \in \Sigma^*(\preceq)$ or, equivalently from the acyclicity of $S \circ P_\preceq \circ S$.

Assume now that $\preceq \subset X \times X$ is an arbitrary total preorder $S$–extension of the partial order $\preceq$ (the existence of total preorder $S$–extensions for $\preceq$ is guaranteed by the assumption that $S \in \Sigma^*(\preceq)$.) It follows from $\preceq \subset S \cup P_\preceq \subset S \circ P_\preceq \circ S = S$ that $S \circ P_\preceq \circ S \subset P_\preceq \circ S = P_\preceq$. To prove the converse inclusion let us consider a pair $(x, y) \in P_\preceq$. Then $(x, y) \notin S$ and by the assertion proved above we have either $(x, y) \in S \circ P_\preceq \circ S$, or $(y, x) \in S \circ P_\preceq \circ S$. The latter is impossible because it contradicts the asymmetry property of $P_\preceq$. Hence, $(x, y) \in S \circ P_\preceq \circ S$ and we get $P_\preceq = S \circ P_\preceq \circ S$. Since $\preceq$ is an arbitrary total preorder $S$–extension of $\preceq$ we conclude that $S \cup (S \circ P_\preceq \circ S)$ is the unique total preorder $S$–extension of the partial preorder $\preceq$.

To verify the converse, assume that $S \cup (S \circ P_\preceq \circ S)$ is a total preorder $S$–extension of a partial order $\preceq$. Obviously, $S \in \Sigma^*(\preceq)$. Suppose that $S \subset S'$ for some $S' \in \Sigma(\preceq)$. Then $S \circ P_\preceq \circ S \subset S' \circ P_\preceq \circ S'$ and $S' \cap (S' \circ P_\preceq \circ S') = \emptyset$. Since the preorder $S \cup (S \circ P_\preceq \circ S)$ is total, we get that $S' \subset S$. Hence, $S = S'$ and it proves that $S$ is maximal in $\Sigma(\preceq)$.

\[\square\]

Remark Let $X$ be a real vector space and let a partial order $\preceq$ and an equivalency relation $S$ defined on $X$ be compatible with algebraic operations on $X$. Then the condition $S \cap P_\preceq = \emptyset$ is both necessary and sufficient for the existence of a compatible total preorder $S$–extension of $\preceq$. This criterion follows from the Kakutani–Tukey theorem on separation of convex sets by halfspaces (see, for instance, [12, Theorem 1.9.1, p. 12]) and from the duality between compatible total preorders and conic halfspaces [15–16].

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