The 4-Adic Complexity of Interleaved Quaternary Sequences of Even Length with Optimal Autocorrelation

Xiaoyan Jing, Zhefeng Xu, Minghui Yang, and Keqin Feng

Abstract

Su et al. proposed several new classes of quaternary sequences of even length with optimal autocorrelation interleaved by twin-prime sequences pairs, GMW sequences pairs or binary cyclotomic sequences of order four in [13]. In this paper, we determine the 4-adic complexity of these quaternary sequences with period $2n$ by using correlation function and the “Gauss periods” of order four and “quadratic Gauss sums” on finite field $\mathbb{F}_n$ and valued in $\mathbb{Z}_{4^{2n-1}}^*$. Our results show that they are safe enough to resist the attack of the rational approximation algorithm.

Index Terms

quaternary sequences, 4-adic complexity, interleaved sequences, optimal autocorrelation, “quadratic Gauss sums”

I. INTRODUCTION

Binary and quaternary sequences play important roles in communication and cryptography systems. They are expected to have good autocorrelation and high complexity (linear complexity, $N$-adic complexity...
and so on) for high speed and security of communication. For two $N$-ary sequences $s = \{s_i\}_{i=0}^{n-1}$ and $t = \{t_i\}_{i=0}^{n-1}$ with period $n$, the correlation function of $s$ and $t$ is defined by

$$R_{s,t}(\tau) = \sum_{i=0}^{n-1} \xi^{s_i + \tau - t_i} \in \mathbb{Z}[\xi], \ 0 \leq \tau < n,$$

where $\xi$ is a primitive $N$-th root of unity. When $t = s$, the correlation function is called autocorrelation function of sequence $s$ and denoted by $R_s(\tau)$. The maximum out-of-phase autocorrelation magnitude of $s$ is defined as $R_{\max}(s) = \max\{|R_s(\tau)| : 1 \leq \tau < n\}$. In practice, we need the out-of-phase autocorrelation magnitude $R_{\max}(s)$ to be as small as possible. Specially, for a quaternary sequence with even period $2n$, it is called optimal autocorrelation sequence if $R_{\max}(s) = 2$ [14]. Several optimal quaternary sequences have been given by [6], [9], [5], [3]. Thereafter Su et al. [13] presented several new families of optimal quaternary sequences constructed by interleaving operator, twin-prime sequences pairs and GMW sequences pairs given by Tang and Gong in 2010 [15] or binary cyclotomic sequences of order four. For a optimal quaternary sequence, it is hoped that the 4-adic complexity be as large as possible.

For a $N$-ary sequence $s = \{s_i\}_{i=0}^{n-1}$ with period $n$, the $N$-adic complexity of $s$ is defined by

$$C_N(s) = \log_N \frac{N^n - 1}{d},$$

where $d = \gcd(S(N), N^n - 1)$, $S(N) = \sum_{i=0}^{n-1} s_i N^i \in \mathbb{Z}$. The $N$-adic complexity $C_N(s)$ measures that the smallest length of the feedback with carry shift register (FCSR) to generate an $N$-ary sequence. To resist the attack of the rational approximation algorithm, the 4-adic complexity of a quaternary sequence $s$ with period $2n$ should exceed $\frac{2n-16}{6}$ [7], [8]. The 4-adic complexity of several quaternary sequences has been computed in [10], [16], [11], [2], we will compute the 4-adic complexity of the optimal quaternary sequences given by Su [13] and show that the 4-adic complexity of such sequences is large enough to resist the attack of the rational approximation algorithm in this paper.

In Section II we briefly introduce the interleaved quaternary sequences constructed by binary sequences pair with the same period $n$, and present the optimal quaternary sequences given by Su in [13] from a pair of twin-prime sequences, GMW sequences or binary cyclotomic sequences of order four. In Section III we give the calculation of the 4-adic complexity of quaternary sequences interleaved by a pair of twin-prime sequences or GMW sequences using correlation function and give some examples to demonstrate the main results. In Section IV we present the “Gauss periods” of order four and “quadratic Gauss sums” on finite field $\mathbb{F}_n$ and valued in $\mathbb{Z}_{4^{2^m-1}}$, then use them to determine the 4-adic complexity of interleaved quaternary sequences constructed by two or three binary cyclotomic sequences of order four. In Section V we give the calculation of the 4-adic complexity of quaternary sequence of order four. We also give some examples to verify the correctness of our results. Section V is a conclusion of this paper.
II. INTERLEAVED QUATERNARY SEQUENCES

In this section, we give a brief introduction of interleaved quaternary sequences with period 2n. Let $n$ be an odd integer, $n \geq 3$, $c^0 = \{c_j^0\}_{j=0}^{n-1}$ and $c^1 = \{c_j^1\}_{j=0}^{n-1}$. Let $c^2 = \{c_j^2\}_{j=0}^{n-1}$ and $c^3 = \{c_j^3\}_{j=0}^{n-1}$ be two pairs of binary sequences with the same period $n$. Let $\lambda = \frac{n+1}{2}$, let $e = (e_0,e_1,e_2)$ be a binary sequence defined over $\mathbb{F}_2$. Considering the following two $n \times 2$ matrices over $\mathbb{F}_2 = \{0,1\}$:

$$
(a(i,l))_{n \times 2} = \begin{pmatrix}
0 & 0 \\
1 & 1 \\
\vdots & \vdots \\
0 & 1
\end{pmatrix}, \quad (b(i,l))_{n \times 2} = \begin{pmatrix}
0 & 0 \\
1 & 1 \\
\vdots & \vdots \\
0 & 1
\end{pmatrix},
$$

where $\oplus$ denotes binary addition over $\mathbb{F}_2$.

Then we have two interleaved binary sequences $a(2l+k) = a(i,l)$ and $b(2l+k) = b(i,l)$ with period 2n from the matrix $(a(i,l))_{n \times 2}$ and $(b(i,l))_{n \times 2}$ respectively, where $0 \leq l < n, \ 0 \leq k < 2$. We denote

$$a = I(c^0 \oplus L^\lambda(c^1) \oplus e_0), \quad b = I(c^2 \oplus e_1, \ L^\lambda(c^3) \oplus e_2)$$

for convenience, where $I$ is the interleaving operator and $L^\lambda(c)$ is $\lambda$-shift of the sequence $c$, namely

$$a = (c_0^0 \oplus e_0, c_1^0 \oplus e_0, \cdots, c_{n-1}^0 \oplus e_0),$$

$$b = (c_0^3 \oplus e_1, c_0^0 \oplus e_2, c_1^3 \oplus e_1, \cdots, c_{n-1}^3 \oplus e_1).$$

It is well known that the Gray mapping is defined as

$$\phi : \mathbb{F}_2 \times \mathbb{F}_2 \cong \mathbb{Z}_4,$$

$$\phi(0,0) = 0, \ \phi(0,1) = 1, \ \phi(1,0) = 2, \ \phi(1,1) = 3.$$  

Using Gray mapping $\phi$, a family of quaternary sequences $s$ with period 2n is constructed by Su [13]:

$$s = s(a,b) = \{s_i\}_{i=0}^{2n-1}, \quad s_i = \phi(a_i,b_i).$$ (1)

A. Quaternary sequences with optimal autocorrelation constructed by sequences pair

It is proved that if $c^0$ and $c^1$, $c^2$ and $c^3$ are two twin-prime sequences pairs or two GMW sequences pairs with period $n$ given in [15], the interleaved quaternary sequences $s = s(a,b)$ have optimal autocorrelation $R_s(\tau) \in \{0, -2\}$ for all $1 \leq \tau \leq 2n - 1$.

Then we have the following lemma.
Lemma 1. ([13]) Let $t_0$ and $t_1$ be the twin-prime sequences pair of length $n = p(p+2)$ or GMW sequences pair of length $n = 2^{2k} - 1$. If $e = (e_0, e_1, e_2)$ satisfies $e_0 + e_1 + e_2 \equiv 1 \pmod{2}$ and

$$(c^0, c^1, c^2, c^3) \in \{(t_0, t_1, t_0, t_1), (t_0, t_1, t_1, t_0), (t_1, t_0, t_1, t_0), (t_1, t_0, t_0, t_1)\}.$$ 

Then the quaternary sequence $s$ given by Construction (1) is an optimal autocorrelation sequence.

For a sequence $t$ with period $n$, it is called ideal autocorrelation sequences if the autocorrelation function

$$R_t(\tau) = \begin{cases} n, & \text{if } \tau = 0 \\ -1, & \text{otherwise} \end{cases}$$

(2)

And as we all know that twin-prime sequences and GMW sequences are ideal autocorrelation sequences. The autocorrelation of modified twin-prime sequences, modified GMW sequences, the cross-correlation between twin-prime sequences and modified twin-prime sequences, GMW sequences and modified GMW sequences were determined by Tang in [15], for the convenience of using in the rest part, we list the results as the following lemmas.

Lemma 2. ([15]) Let $t_0$ and $t_1$ be the twin-prime sequence and modified twin-prime sequence of length $p(p+2)$ respectively, for $0 \leq \tau < p(p+2)$, we have

$$R_{t_0, t_1}(\tau) = \begin{cases} p(p+2), & \text{if } \tau = 0 \\ -1, & \text{if } \tau = 0 \pmod{p+2} \text{ and } \tau \neq 0, \\ 3, & \text{otherwise} \end{cases}$$

And

$$R_{t_0, t_1}(\tau) = R_{t_1, t_0}(\tau) = \begin{cases} p^2, & \text{if } \tau = 0 \\ -2p - 1, & \text{if } \tau = 0 \pmod{p+2} \text{ and } \tau \neq 0, \\ 1, & \text{otherwise} \end{cases}$$

Lemma 3. ([15]) Let $t_0$ and $t_1$ be the GMW sequence and modified GMW sequence of length $2^{2k} - 1$ respectively, for $0 \leq \tau < 2^{2k} - 1$, we have

$$R_{t_0, t_1}(\tau) = R_{t_1, t_0}(\tau) = \begin{cases} 2^{2k} - 1, & \text{if } \tau = 0 \\ -1, & \text{if } \tau = 0 \pmod{2k+1} \text{ and } \tau \neq 0, \\ 3, & \text{otherwise} \end{cases}$$

And

$$R_{t_0, t_1}(\tau) = R_{t_1, t_0}(\tau) = \begin{cases} 2^{2k} - 2^{k+1} + 1, & \text{if } \tau = 0 \\ -2^{k+1} + 1, & \text{if } \tau = 0 \pmod{2k+1} \text{ and } \tau \neq 0, \\ 1, & \text{otherwise} \end{cases}$$
B. Quaternary sequences with optimal autocorrelation constructed by binary cyclotomic sequences of order four

Let \( n = 4f + 1 \) be a prime number, \( \mathbb{F}_n^* = \{0\} \), \( C = \{0^4\} \) and \( D_\gamma = \theta^\gamma C \) \((0 \leq \gamma \leq 3)\) be the cyclotomic classes of order four in \( \mathbb{F}_n \).

**Definition 1.** The cyclotomic numbers of order four in \( \mathbb{F}_n \) are defined by, for \( 0 \leq i, j \leq 3 \)

\[
(i, j) = |(D_i + 1) \cap D_j| = \neq \{ \alpha, \beta \} : \alpha \in D_i, \beta \in D_j, \alpha + 1 = \beta \}.
\]

Let \( t_i \) \((1 \leq i \leq 6)\) be six binary sequences of length \( n \) with support sets \( D_0 \cap D_1, D_0 \cap D_2, D_0 \cap D_3, D_1 \cap D_2, D_1 \cap D_3, D_2 \cap D_3 \), respectively. Namely,

\[
\begin{align*}
t_1 &= \begin{cases} 1, & i \in D_0 \cup D_1 \\ 0, & i \in D_2 \cup D_3 \cup \{0\} \end{cases}, & t_4 &= \begin{cases} 1, & i \in D_1 \cup D_2 \\ 0, & i \in D_0 \cup D_3 \cup \{0\} \end{cases}, \\
t_2 &= \begin{cases} 1, & i \in D_0 \cup D_2 \\ 0, & i \in D_1 \cup D_3 \cup \{0\} \end{cases}, & t_5 &= \begin{cases} 1, & i \in D_1 \cup D_3 \\ 0, & i \in D_0 \cup D_2 \cup \{0\} \end{cases}, \\
t_3 &= \begin{cases} 1, & i \in D_0 \cup D_3 \\ 0, & i \in D_1 \cup D_2 \cup \{0\} \end{cases}, & t_6 &= \begin{cases} 1, & i \in D_2 \cup D_3 \\ 0, & i \in D_0 \cup D_1 \cup \{0\} \end{cases}.
\end{align*}
\]

It is proved by Su in [13] that if \( c^l \in \{t_1, t_2, t_3, t_4, t_5, t_6\} \) \((l \in \{0, 1, 2, 3\})\), the interleaved quaternary sequences with optimal autocorrelation can be constructed by \( c \) and binary sequence \( e = (e_0, e_1, e_2) \). The results are shown in the following lemmas.

**Lemma 4.** ([13]) Let \( n = 4f + 1 = x^2 + 4y^2 \), \( f \) be odd and \( y = -1 \). If \( e = (e_0, e_1, e_2) \) satisfies \( e_0 + e_1 + e_2 \equiv 0 \) \((\mod 2)\) and

\[
(c^0, c^1, c^2, c^3) \in \left\{ (t_2, t_1, t_2, t_1), (t_1, t_2, t_1, t_2), (t_6, t_2, t_6, t_2), (t_2, t_6, t_2, t_6), (t_5, t_4, t_5, t_4), (t_4, t_5, t_4, t_5), (t_3, t_5, t_3, t_5), (t_5, t_3, t_5, t_3) \right\}.
\]

Then the quaternary sequence \( s \) given by Construction (1) is an optimal autocorrelation sequence.

**Lemma 5.** ([13]) Let \( n = 4f + 1 = x^2 + 4y^2 \), \( f \) be odd and \( y = -1 \). If \( e = (e_0, e_1, e_2) \) satisfies \( e_0 + e_1 + e_2 \equiv 0 \) \((\mod 2)\) and

\[
(c^0, c^1, c^2, c^3) \in \left\{ (t_1, t_2, t_1, t_2), (t_2, t_1, t_2, t_1), (t_6, t_2, t_6, t_2), (t_2, t_6, t_2, t_6), (t_4, t_5, t_4, t_5), (t_5, t_4, t_5, t_4), (t_3, t_5, t_3, t_5), (t_5, t_3, t_5, t_3) \right\}.
\]

Then the quaternary sequence \( s \) given by Construction (1) is an optimal autocorrelation sequence.
Lemma 6. ([13]) Let \( n = 4f + 1 = x^2 + 4y^2 \), \( f \) be odd and \( y = -1 \). If \( e = (e_0, e_1, e_2) \) satisfies \( e_0 + e_1 + e_2 \equiv 1 \) (mod 2) and
\[
(c^0, c^1, c^2, c^3) \in \left\{ (t_2, t_1, t_6, t_2), (t_2, t_6, t_1, t_2), (t_5, t_3, t_4, t_5), (t_5, t_4, t_3, t_5), (t_6, t_2, t_2, t_2), (t_1, t_2, t_2, t_6), (t_3, t_5, t_5, t_4), (t_4, t_5, t_5, t_3) \right\}
\]
Then the quaternary sequence \( s \) given by Construction (1) is an optimal autocorrelation sequence.

III. THE 4-ADIC COMPLEXITY OF OPTIMAL AUTOCORRELATION QUATERNARY SEQUENCE \( s \)

CONSTRUCTED BY SEQUENCES PAIR

In this section, we will compute the 4-adic complexity of the quaternary sequence \( s \) given by Lemma 1.

A. Condition I: With the notations as before, for \( t_0 \) and \( t_1 \) are the twin-prime sequences pair of length \( p(p + 2) \) or GMW sequences pair of length \( 2^{2k} - 1 \), and \( c^0 = c^2, c^1 = c^3 \).

From \( c^0 = c^2, c^1 = c^3 \), we have \( a_{2j} = c^0_j, a_{2j+1} \equiv c^1_{j+\lambda} + e_0 \) (mod 2), \( b_{2j} \equiv a_{2j} + e_1 \) (mod 2), and \( b_{2j+1} \equiv a_{2j+1} + e_2 - e_0 \) (mod 2).

With \( e = (1, 0, 0), \lambda = \frac{n+1}{2} \), from the definition of \( s \) and \( S(N) \), we have
\[
S(4) = \sum_{i=0}^{N-1} s_i 4^i \pmod{4^{2n} - 1}
\]
\[
= \sum_{i=0}^{2n-1} \phi(a_i, b_i) 4^i \pmod{4^{2n} - 1}
\]
\[
= \sum_{a_i=0, b_i=1}^{2n-1} 4^i + \sum_{a_i=1, b_i=0}^{2n-1} 2 \cdot 4^i + \sum_{a_i=1, b_i=0}^{2n-1} 3 \cdot 4^i = 2 \sum_{a_i=1}^{2n-1} 4^i + \sum_{a_i=1}^{2n-1} 4^i \pmod{4^{2n} - 1}
\]
\[
= 2 \sum_{j=0}^{n-1} 4^j \sum_{a_{2j+1}=1}^{1} + \sum_{j=0}^{n-1} 2 \cdot 4^i + \sum_{a_{2j+1}=1}^{1} \sum_{j=0}^{n-1} 4^j \pmod{4^{2n} - 1}
\]
\[
= 2 \sum_{j=0}^{n-1} 4^j + \sum_{j=0}^{n-1} 4^j \pmod{4^{2n} - 1}
\]
\[
= 2 \sum_{j=0}^{n-1} a_{2j} 4^j + \sum_{j=0}^{n-1} a_{2j+1} 4^j \pmod{4^{2n} - 1}
\]
\[
= 2 \sum_{j=0}^{n-1} c^0_j 4^j + \sum_{j=0}^{n-1} (1 - c^1_{j+\lambda}) 4^j \pmod{4^{2n} - 1}
\]
\[
= 2 \sum_{j=0}^{n-1} c^0_j 4^j + 4 \sum_{j=0}^{n-1} c^0_j 4^j + 3 \sum_{j=0}^{n-1} 4^j \pmod{4^{2n} - 1}
\]
Similarly, we obtain
\[
S(4) \equiv \begin{cases} 
2 \sum_{j=0}^{n-1} c_j^0 4^{2j} + 2 \cdot 4^n \sum_{j=0}^{n-1} c_j^1 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} & \pmod{4^{2n} - 1}, \text{if } e = (0,1,0) \\
2 \sum_{j=0}^{n-1} c_j^0 4^{2j} + 2 \cdot 4^n \sum_{j=0}^{n-1} c_j^1 4^{2j} + \sum_{j=0}^{n-1} 4^{2j+1} & \pmod{4^{2n} - 1}, \text{if } e = (0,0,1) \\
2 \sum_{j=0}^{n-1} c_j^0 4^{2j} - 2 \cdot 4^n \sum_{j=0}^{n-1} c_j^1 4^{2j} + 9 \cdot \sum_{j=0}^{n-1} 4^{2j} & \pmod{4^{2n} - 1}, \text{if } e = (1,1,1)
\end{cases}
\]

**Theorem 7.** Let \(t_0\) and \(t_1\) be the twin-prime sequences pair with period \(n = p(p+2)\), and \(s = s(a,b)\) be the optimal quaternary sequence given by Lemma [7] then for \(c^0 = c^2, c^1 = c^3\), the 4-adic complexity of the sequence \(s\) is
\[
C_4(s) = \begin{cases} 
\log_4(4^{p(p+2)} + 1)(4^{p+2} - 1), & \text{if } e = (1,0,0) \text{ or } e = (1,1,1) \\
\log_4(4^{p(p+2)} - 1)(4^{p+2} + 1), & \text{if } e = (0,1,0) \text{ or } e = (0,0,1)
\end{cases}
\]

**Proof.** Assume that \(c^0\) and \(c^1\) are the twin-prime sequences pair of length \(n = p(p+2)\), we have
\[
\sum_{j=0}^{n-1} (c_j^0 - c_j^1) 4^{2j} = \sum_{j=0}^{n-1} \varepsilon_1 \cdot 4^{2j} = \varepsilon_1 \cdot \sum_{j=0}^{n-1} 4^{2j} = \varepsilon_1 \cdot \frac{4^{2(p+2)} - 1}{4^{2(p+2)} - 1}.
\]
where
\[
\varepsilon_1 = \begin{cases} 
-1, & \text{if } c^0 \text{ is the twin-prime sequence and } c^1 \text{ is the modified twin-prime sequence} \\
1, & \text{if } c^0 \text{ is the modified twin-prime sequence and } c^1 \text{ is the twin-prime sequence}
\end{cases}
\]
For \(e = (1,0,0)\), from (3) we know that
\[
S_1(4) = 2 \sum_{j=0}^{n-1} c_j^0 4^{2j} - 2 \cdot 4^n \sum_{j=0}^{n-1} c_j^1 4^{2j} + 3 \cdot \sum_{j=0}^{n-1} 4^{2j+1} \pmod{4^{2n} - 1}
\]
\[
\equiv \begin{cases} 
2 \sum_{j=0}^{n-1} (c_j^0 - c_j^1) 4^{2j} + 3 \cdot \sum_{j=0}^{n-1} 4^{2j+1}, & \pmod{4^n - 1} \\
2 \sum_{j=0}^{n-1} (c_j^0 + c_j^1) 4^{2j} + 3 \cdot \sum_{j=0}^{n-1} 4^{2j+1}, & \pmod{4^n + 1}
\end{cases}
\]
\[
\equiv \begin{cases} 
2 \varepsilon_1 \cdot \frac{4^{2(p+2)} - 1}{4^{2(p+2)} - 1}, & \pmod{4^n - 1} \\
2 \sum_{j=0}^{n-1} (c_j^0 + c_j^1) 4^{2j} + 3 \cdot \sum_{j=0}^{n-1} 4^{2j+1}, & \pmod{4^n + 1}
\end{cases}.
\]
From \(\gcd(4^n - 1, 4^n + 1) = 1\), we have \(d = \gcd(S_1(4),4^n - 1) = \gcd(S_1(4),4^n + 1) = \gcd(S_1(4),4^n - 1) \cdot \gcd(S_1(4),4^n + 1) = \frac{4^{p(p+2)} - 1}{4^{p+2} - 1} \cdot \gcd(S_1(4),4^n + 1)\). Let \(d_+ = \gcd(S_1(4),4^n + 1)\). In the following part we will determine \(d_+\).

Let \(s_j^0 = (-1)^j, s_j^1 = (-1)^j\), that is \(c_j^0 = \frac{1}{2}(1 - s_j^0)\) and \(c_j^1 = \frac{1}{2}(1 - s_j^1)\), then we obtain
\[
S_1(4) \equiv 2 \sum_{j=0}^{n-1} (c_j^0 + c_j^1) 4^{2j} + 3 \cdot \sum_{j=0}^{n-1} 4^{2j+1} \pmod{4^n + 1}
\]
\[
\equiv \sum_{j=0}^{n-1} (2 - s_j^0 - s_j^1) 4^{2j} + 3 \cdot \sum_{j=0}^{n-1} 4^{2j+1} \pmod{4^n + 1}
\]
\[ \equiv - \sum_{j=0}^{n-1} (s_j^0 + s_j^1)4^{2j} \pmod{\frac{4^n + 1}{5}}. \]

Let \( \pi \) be a prime divisor of \( \gcd(\sum_{j=0}^{n-1} (s_j^0 + s_j^1)4^{2j}, \frac{4^{p(p+2)}+1}{5}) \). From \( \sum_{j=0}^{n-1} (s_j^0 + s_j^1)4^{2j} \equiv 0 \pmod{\pi} \), we know that

\[ \begin{align*}
0 & \equiv (\sum_{i,j=0}^{n-1} s_i s_j 4^{2i(j-i)})(\sum_{i,j=0}^{n-1} s_i s_j 4^{2i(j-i)}) \\
& \equiv \sum_{i,j=0}^{n-1} s_i^0 s_j^0 4^{2i(j-i)} + \sum_{i,j=0}^{n-1} s_i^1 s_j^1 4^{2i(j-i)} + \sum_{i,j=0}^{n-1} s_i^0 s_j^1 4^{2i(j-i)} + \sum_{i,j=0}^{n-1} s_i^1 s_j^0 4^{2i(j-i)} \pmod{\pi} \\
& \equiv \sum_{\tau,j=0}^{n-1} s_j^0 4^{2\tau} + \sum_{\tau,j=0}^{n-1} s_j^1 4^{2\tau} + \sum_{\tau,j=0}^{n-1} s_j^0 s_j^1 4^{2\tau} + \sum_{\tau,j=0}^{n-1} s_j^1 s_j^0 4^{2\tau} \pmod{\pi} \\
& \equiv \sum_{\tau=0}^{n-1} 4^{2\tau}(R_0(\tau) + R_1(\tau) + 2R_{01}(\tau)) \pmod{\pi} \\
& \equiv (2p(p+2) + 2p^2) + \sum_{\tau=1 \atop p \nmid 2|\tau}^{n-1} (-1 - 1 + 2(-2p - 1))4^{2\tau} + \sum_{\tau=1 \atop p \nmid 2|\tau}^{n-1} (-1 + 3 + 2 \cdot 1)4^{2\tau} \pmod{\pi} \\
& \quad \text{(from equation (2) and Lamme (2))}
\end{align*} \]

\[ \equiv 4p(p+1) - 4(p+1) \sum_{\tau=1 \atop p \nmid 2|\tau}^{n-1} 4^{2\tau} + 4 \sum_{\tau=1 \atop p \nmid 2|\tau}^{n-1} 4^{2\tau} \pmod{\pi} \]

\[ \equiv 4p(p+1) - 4(p+2) \sum_{\tau=1 \atop p \nmid 2|\tau}^{n-1} 4^{2\tau} + 4 \sum_{\tau=1 \atop p \nmid 2|\tau}^{n-1} 4^{2\tau} \pmod{\pi} \]

\[ \equiv 4p(p+1) - 4(p+2) \sum_{i=1}^{p-1} 4^{2i(p+2)} - 4 \pmod{\pi} \]

\[ \equiv 4p(p+1) - 4(p+2) \left( \sum_{i=0}^{p-1} 4^{2i(p+2)} - 1 \right) - 4 \pmod{\pi} \]

\[ \equiv 4(p+1)^2 - 4(p+2) \cdot \frac{4^{p(p+2)} - 1}{4^{p+2} - 1} \cdot \frac{4^{p(p+2)} + 1}{4^{p+2} + 1} \pmod{\pi}. \]  \( \quad (7) \)

If \( \pi \mid \frac{4^{p(p+2)}+1}{4^{p+2}+1} \), then \( 4(p+1)^2 \equiv 0 \pmod{\pi} \), \( \pi \mid p+1 \) and \( \pi \mid 4^{p(p+2)} + 1 \). It then follows that the order of \( 4 \pmod{\pi} \) is \( 2p(p+2) \). From Little Fermat’s Theorem we have \( 2p(p+2) \mid \pi - 1 \) is a contradiction. If \( \pi \mid \frac{4^{p+2}+1}{5} \), then we have

\[ 0 \equiv 4p(p+1) - 4(p+2) \sum_{i=1}^{p-1} 4^{2i(p+2)} - 4 \equiv 4p^2 + 4p - 4(p^2 + p - 2) - 4 = 4 \pmod{\pi}, \]
which is a contradiction. Hence, gcd\((S(4), \frac{4^n + 1}{2})\) = 1. From (5) we know that
\[
S_1(4) \equiv 2 \sum_{j=0}^{n-1} (c_j^0 + c_j^1) 4^{2j} + 3 \cdot \sum_{j=0}^{n-1} 4^{2j+1} \pmod{5}
\]
\[
\equiv 2 \sum_{j=0}^{n-1} (c_j^0 + c_j^1) + 3 \cdot 4 \cdot \sum_{j=0}^{n-1} 1 \pmod{5}
\]
\[
\equiv 2 \left( \sum_{j=0}^{n-1} (0 + 1) + \sum_{j=0}^{n-1} (1 + 1) + 2 \cdot \frac{p(p+2) - p - (p+1)}{2} + 2p(p+2) \right) \pmod{5}
\]
\[
\equiv 2(p + 2(p+1) + p(p+2) - p - (p+1) + 2p(p+2) \pmod{5}
\]
\[
\equiv 4p^2 + 2 \not\equiv 5 \pmod{5} \quad (\text{since } p^2 \equiv 1, 5, 9 \pmod{10}).
\]
To sum up, for \(e = (1, 0, 0), d_+ = \text{gcd}(S_1(4), 4^n+1) = 1, d = \text{gcd}(S_1(4), 4^{2n}-1) = \frac{4^{n(p+2)} - 1}{4^{p+1} - 1}. The 4-adic complexity of sequence \(s\) is \(C_4(s) = \log_4(4^{n(p+2)} + 1)(4^{p+2} - 1)\).

The other three cases can be proved similarly. \(\square\)

**Theorem 8.** Let \(t_0\) and \(t_1\) be the GMW sequences pair of length \(n = 2^{2k} - 1\), and \(s = s(a,b)\) be the optimal quaternary sequence given by Lemma \(7\) then for \(c^0 = c^2, c^1 = c^3\), the 4-adic complexity of the sequence \(s\) is
\[
C_4(s) = \begin{cases} 
\log_4\left(4^{2^{2k-1}+1}(4^{2^{2k+1}+1} - 1)\right), & \text{if } e = (1,0,0) \text{ or } e = (1,1,1) \\
\log_4\left(4^{2^{2k-1}+1}(4^{2^{2k+1}+1} + 1)\right), & \text{if } e = (0,1,0) \text{ or } e = (0,0,1) 
\end{cases}
\]

**Proof.** Let \(c^0\) and \(c^1\) be the GMW sequences pair of length \(n = 2^{2k} - 1\). Then we have
\[
\sum_{j=0}^{n-1} (c_j^0 - c_j^1) 4^{2j} = \sum_{j=0}^{n-1} (\varepsilon_2 \cdot 4^{2j}) = \varepsilon_2 \cdot \sum_{j=0}^{2^{2k-1}-1} 4^{2j} \cdot 4^{2^{2k+1}+j} = \varepsilon_2 \cdot \frac{4^{2^{2k+1}+1} - 1}{4^{2^{2k}+1} - 1},
\]
where
\[
\varepsilon_2 = \begin{cases} 
-1, & \text{if } c^0 \text{ is the GMW sequence and } c^1 \text{ is the modified GMW sequence} \\
1, & \text{if } c^0 \text{ is the modified GMW sequence and } c^1 \text{ is the GMW sequence}
\end{cases}
\]
For \(e = (1,0,0)\), from (5) we obtain
\[
S_2(4) \equiv 2 \sum_{j=0}^{n-1} c_j^0 4^{2j} - 2 \cdot 4^n \sum_{j=0}^{n-1} c_j^1 4^{2j} + 3 \cdot \sum_{j=0}^{n-1} 4^{2j+1} \pmod{4^n-1}
\]
\[
\equiv \begin{cases} 
\varepsilon_2 \cdot \frac{4^{2^{2k}+1} - 1}{4^{2^{2k+1}+1} - 1}, & \pmod{4^n-1} \\
2 \sum_{j=0}^{n-1} (c_j^0 + c_j^1) 4^{2j} + 3 \cdot \sum_{j=0}^{n-1} 4^{2j+1}, & \pmod{4^n+1}
\end{cases}
\]
(8)

From gcd\((4^n - 1, 4^n+1) = 1, we have d = gcd(S_2(4), 4^{2n}-1) = gcd(S_2(4), 4^n-1) \cdot gcd(S_1(4), 4^n+1) = \frac{4^{2^{2k}+1} - 1}{4^{2^{2k+1}+1} - 1} \cdot gcd(S_2(4), 4^n+1). Let d_+ = gcd(S_2(4), 4^n+1). In the following part we will determine d_+.}
Let \( s_j^0 = (-1)^{c_j^0}, s_j^1 = (-1)^{c_j^1} \), which implies \( c_j^0 = \frac{1}{\pi}(1 - s_j^0) \) and \( c_j^1 = \frac{1}{\pi}(1 - s_j^1) \), then from (8) we have

\[
S_2(4) \equiv - \sum_{j=0}^{n-1} (s_j^0 + s_j^1)4^{2j} \pmod{\frac{2^{2k+1}+1}{5}}.
\]

Let \( \pi \) be a prime divisor of \( \gcd(\sum_{j=0}^{n-1} (s_j^0 + s_j^1)4^{2j}, \frac{2^{2k+1}+1}{5}) \). From \( \sum_{j=0}^{n-1} (s_j^0 + s_j^1)4^{2j} \equiv 0 \pmod{\pi} \), we know that

\[
0 \equiv (\sum_{i=0}^{n-1} (s_i^0 + s_i^1)4^{2i})(\sum_{j=0}^{n-1} (s_j^0 + s_j^1)4^{-2j}) \equiv \sum_{i,j=0}^{n-1} (s_i^0 + s_i^1)(s_j^0 + s_j^1)4^{2(i-j)} \pmod{\pi}
\]

\[
\equiv \sum_{\tau=0}^{n-1} 4^{2\tau}(R_i(\tau) + R_j(\tau) + 2R_{\phi,c}(\tau)) \pmod{\pi}
\]

\[
\equiv 2(2^{2k} - 1) + 2(2^{2k} - 2^{k+1} + 1) + \sum_{\tau=1}^{n-1} (-1 - 1 + 2(-2^{k+1} + 1))4^{2\tau} + \sum_{\tau=1}^{n-1} (-1 + 3 + 2 \cdot 1)4^{2\tau} \pmod{\pi}
\]

\[
\equiv 2(2^{2k+1} - 2^{k+1}) - 4 \cdot 2^k \sum_{\tau=1}^{n-1} 4^{2\tau} + 4 \sum_{\tau=1}^{n-1} 4^{2\tau} \pmod{\pi}
\]

\[
\equiv 4 \cdot 2^k(2^k - 1) - 4(2^k + 1) \sum_{\tau=1}^{n-1} 4^{2\tau} + 4 \sum_{\tau=1}^{n-1} 4^{2\tau} \pmod{\pi}
\]

\[
\equiv 4 \cdot 2^k(2^k - 1) - 4(2^k + 1) \sum_{i=1}^{2^{k+2}-2} 4^{2i(2^k+1)} - 4 \equiv 4 \pmod{\pi}
\]

\[
\equiv 2^{2(k+1)} - 4(2^k + 1) \cdot \frac{4^{2^{2k+1}-1}}{4^{2(2^k+1)}} - 1 \cdot \frac{4^{2^{2k+1}-1}}{4^{2(2^k+1)}} + 1 \pmod{\pi}.
\]

Since \( \pi \) is an odd prime, it is obvious that \( \pi \mid \frac{4^{2^{2k+1}-1}}{4^{2(2^k+1)}}+1 \). If \( \pi \mid \frac{2^{2^{2k+1}-1}}{5} \), we have the contradiction that \( 0 \equiv 2^{2(k+1)} \pmod{\pi} \). Therefore we have \( \pi \mid \frac{4^{2^{2k+1}-1}}{5} \). from (9) can obtain that

\[
0 \equiv 4 \cdot 2^k(2^k - 1) - 4(2^k + 1) \sum_{i=1}^{2^{k+2}-2} 1 - 4 \equiv 4 \pmod{\pi}
\]

which is a contradiction. This implies \( \gcd(S_2(4), \frac{4^{2^{2k+1}+1}}{5}) = 1 \).

And from (8) we have

\[
S_2(4) \equiv 2 \sum_{j=0}^{n-1} (c_j^0 + c_j^1) + 2 \cdot \sum_{j=0}^{n-1} 1 \pmod{5}
\]

\[
\equiv - \sum_{j=0}^{n-1} (s_j^0 + s_j^1) + 4 \sum_{j=0}^{n-1} 1 \pmod{5}.
\]

If \( 5 \mid S_2(4) \), then we obtain

\[
n^2 \equiv (4 \cdot \sum_{j=0}^{n-1} 1)^2 \equiv \left( \sum_{j=0}^{n-1} (s_j^0 + s_j^1) \right)^2 \pmod{5}
\]

\[
\equiv \sum_{\tau=0}^{n-1} (R_i(\tau) + R_j(\tau) + 2R_{\phi,c}(\tau)) \pmod{5},
\]
that is
\[(2^{2k} - 1)^2 \equiv (2(2^{2k} - 1) + 2(2^{2k} - 2^{k+1} + 1)) + \sum_{\tau=1}^{n-1} (-1 - 1 + 2(-2^{k+1} + 1))
\]
\[+ \sum_{\tau=1}^{n-1} (-1 + 3 + 2 \cdot 1) \pmod{5}
\]
\[\equiv 2^{2k+2} - 2^{k+2} - 2^{k+2}(2^{k} - 2) + 4((2^{2k} - 2^{k} - 2) \pmod{5})
\]
\[\equiv 4 \cdot 2^{2k} \equiv -2^{2k} \pmod{5}.
\]
This shows $0 \equiv (2^{2k} - 1)^2 + 2^{2k} \equiv 4^{2k} - 4^k + 1 \equiv (-1)^{2k} - (-1)^k + 1 \equiv 2 - (-1)^k \neq 0 \pmod{5}$, which is a contradiction.

In a conclusion, $d = \gcd(S_2(4), 4^{2n} - 1) = \gcd(S_2(4), 4^n - 1) \cdot \gcd(S_2(4), 4^n + 1) = \frac{4^{2k-1} - 1}{4^{2k-1} - 1}$. The 4-adic complexity of sequence $s$ is $C_4(s) = \log_4(4^{2k} - 1)(4^{2k} + 1)$. The remaining cases can be proved in the same way. \qed

B. Condition II: With the notations as before, for $t_0$ and $t_1$ are the twin-prime sequences pair of length $p(p+2)$ or GMW sequences pair of length $2^{2k} - 1$, and $c^0 = c^3, c^1 = c^2$.

For $c^0 = c^3, c^1 = c^2$, we have $a_{2j} = c^0, a_{2j+1} = c^1 = c_{j+\lambda} + e_0 \pmod{2}, b_{2j} = c^1 + e_1 \pmod{2},$ and $b_{2j+1} = c_{j+\lambda} + e_2 \pmod{2}$.

From the definition of $s$, $S(N)$ and $\lambda = \frac{n+1}{2}$, we know that
\[
S(4) = \sum_{i=0}^{N-1} s_i 4^i \pmod{4^{2n} - 1}
\]
\[= 2(\sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j+1}) + (\sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j+1}) \pmod{4^{2n} - 1}
\]
\[= 2 \cdot \sum_{j=0}^{n-1} 4^{2j} + 2 \cdot \sum_{j=0}^{n-1} 4^{2j+1} + \sum_{j=0}^{n-1} 4^{2j} \pmod{4^{2n} - 1}
\]
\[+ \sum_{j=0}^{n-1} 4^{2j+1} \pmod{4^{2n} - 1}
\]
\[= 2 \cdot \sum_{j=0}^{n-1} 4^{2j} + 2 \cdot 4^n \cdot \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} \pmod{4^{2n} - 1}
\]
\[+ 4^n \cdot \sum_{j=0}^{n-1} 4^{2j} \pmod{4^{2n} - 1}.
\]
Let $c^0$ and $c^1$ be the twin-prime sequences pair with period $p(p + 2)$, we have

\[
c^0_i + c^1_i \equiv \begin{cases} 
0 \pmod{2}, & i \not\equiv 0 \pmod{p + 2} \\
1 \pmod{2}, & i \equiv 0 \pmod{p + 2}
\end{cases},
\]

and let $c^0$ and $c^1$ be the GMW sequences pair with period $2^{2k} - 1$, we have

\[
c^0_i + c^1_i \equiv \begin{cases} 
0 \pmod{2}, & i \not\equiv 0 \pmod{2^{k} + 1} \\
1 \pmod{2}, & i \equiv 0 \pmod{2^{k} + 1}
\end{cases}.
\]

Assume that $c$ and $c'$ is the twin-prime sequence and modified twin-prime sequence with period $n = p(p + 2)$, respectively, then we obtain

\[
\sum_{j=0}^{n-1} (c^0_j + c^1_j)4^{2j} + \sum_{j=0}^{n-1} 4^{2j} = 2 \sum_{j=0}^{n-1} c'_j4^{2j}
\]

and

\[
\sum_{j=0}^{n-1} (c^0_j - c^1_j)4^{2j} = \sum_{j=0}^{n-1} \epsilon_1 \cdot 4^{2j} = \epsilon_1 \prod_{j=0}^{p-1} 4^{2(p+2)j} = \epsilon_1 \frac{4^n - 1}{4^{2(p+2)} - 1},
\]

where

\[
\epsilon_1 = \begin{cases} 
-1, & \text{if } c^0 \text{ is the twin-prime sequence and } c^1 \text{ is the modified twin-prime sequence} \\
1, & \text{if } c^0 \text{ is the modified twin-prime sequence and } c^1 \text{ is the twin-prime sequence}
\end{cases}.
\]

Let $e = (1, 0, 0)$, and $c^0$, $c^1$ be the twin-prime sequences pair with period $p(p + 2)$, from equation (10) we know that

\[
S_2(4) = 2 \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} + 4^n \cdot \sum_{j=0}^{n-1} 4^{2j} (mod 4^{2n} - 1)
\]

\[
= 2 \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} + 4^n \cdot \sum_{j=0}^{n-1} 4^{2j} (mod 2)
\]

\[
= 2 \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j} + 4^n \cdot \sum_{j=0}^{n-1} 4^{2j} (mod 2)
\]

\[
= \left\{ \begin{array}{ll}
2 \sum_{j=0}^{n-1} c^0_j4^{2j} + \sum_{j=0}^{n-1} c^1_j4^{2j} + 3 \cdot 4^n \cdot \sum_{j=0}^{n-1} 4^{2j} - (4^n - 1) \cdot \sum_{j=0}^{n-1} 4^{2j} (mod 4^n - 1) \\
2 \sum_{j=0}^{n-1} c^0_j4^{2j} + \sum_{j=0}^{n-1} c^1_j4^{2j} + 3 \sum_{j=0}^{n-1} 4^{2j} (mod 4^n + 1)
\end{array} \right.
\]
Theorem 9. Let \( t \) and similar to the proof of Theorem 7 and 8, we came up with the following results.

\[
\begin{align*}
\text{Example 1.} & \quad \text{the optimal quaternary sequence given by Lemma 1, for } \epsilon_1=0, \text{ we have} \\
S(4) & \equiv \begin{cases} 
2\sum_{j=0}^{n-1} c'_{j} 4^{2j} + 2 \cdot \sum_{j=0}^{n-1} c_{j} 4^{2j}, & (\text{mod } 4^n - 1) \\
(2 + 2\epsilon_1) \sum_{j=0}^{n-1} 4^{2(p+2)j} + 2 \cdot 4^{p+1}, & (\text{mod } 4^n + 1)
\end{cases}.
\end{align*}
\]

Similarly, for \( e = (0,1,0) \), we have

\[
\begin{align*}
S(4) & \equiv \begin{cases} 
2\sum_{j=0}^{n-1} c'_{j} 4^{2j} + 2 \cdot \sum_{j=0}^{n-1} c_{j} 4^{2j}, & (\text{mod } 4^n - 1) \\
(2 + 2\epsilon_1) \sum_{j=0}^{n-1} 4^{2(p+2)j} + 2 \cdot 4^{p+1}, & (\text{mod } 4^n + 1)
\end{cases}.
\end{align*}
\]

For \( e = (0,0,1) \), we have

\[
\begin{align*}
S(4) & \equiv \begin{cases} 
2\sum_{j=0}^{n-1} c'_{j} 4^{2j} + 2 \cdot \sum_{j=0}^{n-1} c_{j} 4^{2j} + 4^{p+1}, & (\text{mod } 4^n - 1) \\
(2 + 2\epsilon_1) \sum_{j=0}^{n-1} 4^{2(p+2)j} - 4^{p+1}, & (\text{mod } 4^n + 1)
\end{cases}.
\end{align*}
\]

For \( e = (1,1,1) \), we obation

\[
\begin{align*}
S(4) & \equiv \begin{cases} 
2\epsilon_1 \cdot \sum_{j=0}^{n-1} 4^{2(p+2)j}, & (\text{mod } 4^n - 1) \\
4 \sum_{j=0}^{n-1} c'_{j} 4^{2j} + 2 \cdot 4^{p+1}, & (\text{mod } 4^n + 1)
\end{cases}.
\end{align*}
\]

Let \( c_0 \) and \( c_1 \) be the GMW sequences pair of length \( n = 2^k - 1 \), we can obtain \( S(4) \) in the same way.

And similar to the proof of Theorem 7 and 8 we came up with the following results.

**Theorem 9.** Let \( t_0 \) and \( t_1 \) be the twin-prime sequences pair with period \( n = p(p+2) \), and \( s = s(a,b) \) be the optimal quaternary sequence given by Lemma 1 for \( c_0 = c_3, c_1 = c_2 \), then the 4-adic complexity of the sequence \( s \) is

\[
C_4(s) = \begin{cases} 
\log_4(4^{p(p+2)} + 1)(4^{p+2} - 1), & \text{if } e = (1,0,0) \text{ or } e = (1,1,1) \\
\log_4(5)(4^{p(p+2)} - 1), & \text{if } c_0 = t_0, c_1 = t_1, e = (0,1,0) \text{ or } e = (0,0,1) \\
\log_4(4^{p(p+2)} - 1)(4^{p+2} + 1), & \text{if } c_0 = t_1, c_1 = t_0, e = (0,1,0) \text{ or } e = (0,0,1)
\end{cases}.
\]

**Theorem 10.** Let \( t_0 \) and \( t_1 \) be the GMW sequences pair of length \( n = 2^k - 1 \), and \( s = s(a,b) \) be the optimal quaternary sequence given by Lemma 1 for \( c_0 = c_3, c_1 = c_2 \), then the 4-adic complexity of the sequence \( s \) is

\[
C_4(s) = \begin{cases} 
\log_4(4^{2^{2k} - 1} + 1)(4^{2^k + 1} - 1), & \text{if } e = (1,0,0) \text{ or } e = (1,1,1) \\
\log_4(5)(4^{2^{2k} - 1} - 1), & \text{if } c_0 = t_0, c_1 = t_1, e = (0,1,0) \text{ or } e = (0,0,1) \\
\log_4(4^{2^{2k} - 1} - 1)(4^{2^k + 1} + 1), & \text{if } c_0 = t_1, c_1 = t_0, e = (0,1,0) \text{ or } e = (0,0,1)
\end{cases}.
\]

**Example 1.** Let \( p = 3 \). Then the pair of twin-prime sequences with period \( n = p(p+2) \) are given by

\[
t_0 = (000100110101111),
\]
\( t_1 = (100101110111111) \).

For \((c^0, c^1, c^2, c^3) = (t_0, t_1, t_0, t_1)\) and \((e_0, e_1, e_2) = (1, 0, 0)\), we have

\[
a = (010000100000101001110011101010),
\]

\[
b = (000101110101111100100110111111),
\]

and

\[
s = (03010121010121032301232123212321).
\]

Computing by Magma program, we obtain \(\gcd(S(4), 4^{2n} - 1) = 1049601 = 4^{15} - 1\). Then the 4-adic complexity of the sequence \(s\) is \(C_4(s) = \log_4(4^{15} + 1)(4^5 - 1)\), which is consistent with Theorem 7.

**Example 2.** Let \(k=3\). The GMW sequences pair with period \(n = 2^{2k} - 1 = 63\) are given by

\[
t_0 = (000001000011000101001111010001110010010110111011001101010111111),
\]

\[
t_1 = (100001000111000101101111010101110010110110111111001101110111111).
\]

For \((e^0, e^1, c^2, c^3) = (t_0, t_1, t_0, t_1)\) and \((e_0, e_1, e_2) = (0, 1, 0)\), we have

\[
s = (01010301032301030303230121030321232323012303030323232
     30101210103210123032312101230321232323232).
\]

Computing by Magma program, we obtain \(\gcd(S(4), 4^{2n} - 1) = 324517315723109789871420976398337 = 4^{63} - 1\). Then the 4-adic complexity of the sequence \(s\) is \(C_4(s) = \log_4(4^{63} - 1)(4^9 + 1)\), which is consistent with Theorem 8.

**IV. The 4-Adic Complexity of Optimal Autocorrelation Quaternionary Sequence \(s\) Constructed by Binary Cyclotomic Sequences of Order Four**

In this section, we will determind the 4-adic complexity of the sequences \(s\) with period \(2n\) \((n = 4f + 1 = x^2 + 4y^2\) is prime) given by Lemma 4-6. Fristly, we give the lemma about the values of the cyclotomic numbers of order four in \(\mathbb{F}_n\).

**Lemma 11.** ([11],[12]) Let \(n = 4f + 1 = x^2 + 4y^2\) be an odd prime, \(x, y \in \mathbb{Z}\) and \(f\) be an odd number. The values of the cyclotomic numbers \((i, j)\) of order four in \(\mathbb{F}_n\) are

\[
16(0, 0) = 16(2, 0) = 16(2, 2) = A = n - 7 + 2x,
\]
16(0, 1) = 16(1, 3) = 16(3, 2) = B = n + 1 + 2x - 8y,
16(0, 3) = 16(1, 2) = 16(3, 1) = B = n + 1 + 2x + 8y,
16(0, 2) = C = n + 1 - 6x,
16(1, 0) = 16(1, 1) = 16(2, 1) = 16(2, 3) = 16(3, 0) = 16(3, 3) = D = n - 3 - 2x.

In the following we give a brief introduction about “Gauss periods” of order four and “quadratic Gauss sums”.

Since \( \alpha \equiv \beta \pmod{2n} \) which implies \( 4^{2\alpha} \equiv 4^{2\beta} \pmod{4^{2n} - 1} \), we can define the following mapping

\[
f : \mathbb{Z}_n \rightarrow \mathbb{Z}_{4^n}^*, f(\alpha) = 4^{2\alpha}
\]

where \( \mathbb{Z}_n^* \) is the group of units in the ring \( \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} \) \( (m \geq 2) \). \( f \) is a homomorphism of groups from \( \mathbb{Z}_n^* \) to \( \mathbb{Z}_{4^n}^* \), and can be viewed as an additive character of finite field \( \mathbb{F}_n = \mathbb{F}_n^* \) valued in \( \mathbb{Z}_{4^n}^* \).

Then we have the “Gauss periods” of order 4

\[
\eta_\gamma = \sum_{i \in D_\gamma} 4^{2i} \pmod{4^{2n} - 1}(\gamma = 0, 1, 2, 3),
\]

where \( D_\gamma = \theta^iC \) \( (0 \leq \gamma \leq 3) \) is the cyclotomic classes of order four in \( \mathbb{F}_n^* \) and “quadratic Gauss sums”

\[
G = \sum_{i \in \mathbb{F}_n^*} 4^{2i}\chi(i) = \eta_0 - \eta_1 + \eta_2 - \eta_3 \pmod{4^{2n} - 1},
\]

where \( \chi \) is the quadratic (multiplicative) character of \( \mathbb{F}_n^* \) (the Legendre symbol). Namely, for \( i \in \mathbb{F}_n^* \),

\[
\chi(i) = \begin{cases} 
1, & \text{if } i \in D_0 \cup D_2 = \langle \theta^2 \rangle \\
-1, & \text{if } i \in D_1 \cup D_3 = \theta \langle \theta^2 \rangle 
\end{cases}.
\]

The following results show that \( \eta_\gamma \) and \( G \) have some similar properties as usual Gauss periods and Gauss sums. And the proof of the following lemmas are similar to [17] and [4], we omit them here.

**Lemma 12.** (1). \( G^2 \equiv n - \frac{4^{2n} - 1}{15} \pmod{4^{2n} - 1} \).

(2). For \( 0 \leq \gamma, \mu \leq 3 \), \( \eta_\gamma \eta_\mu \equiv \frac{n-1}{4} \delta_{\gamma, \mu+2} + \sum_{\nu=0}^{3} (\gamma - \nu + 2, \mu - \nu) \eta_\nu \pmod{4^{2n} - 1} \), where \((i, j)\) is the cyclotomic number of order four on \( \mathbb{F}_n \) and

\[
\delta_{\gamma, \mu} = \begin{cases} 
1, & \text{if } \gamma \equiv \mu \pmod{4} \\
0, & \text{otherwise}
\end{cases}.
\]

**Lemma 13.** Let \( n = 4f + 1 = x^2 + 4y^2 \) be an odd prime, \( f \) be odd and \( y = -1 \). Then

\[
16\eta_\gamma^2 = A\eta_\gamma + B\eta_{\gamma+1} + C\eta_{\gamma+2} + D\eta_{\gamma+3}
\]
Lemma 14. Let \( n \neq 1 \), then from the definition of 

\[
\begin{align*}
\gamma & \equiv \eta \pmod{4}, \\
\eta & \equiv \gamma \pmod{4}, \\
\gamma & \equiv \eta \pmod{4}, \\
\eta & \equiv \gamma \pmod{4}, \\
\end{align*}
\]

we have

\[
\begin{align*}
\gamma & = \eta + c \mod 4, \\
\eta & = \gamma + c \mod 4, \\
\gamma & = \eta + c \mod 4, \\
\eta & = \gamma + c \mod 4, \\
\end{align*}
\]

If \( e = (0, 0, 0) \), we have

\[
\begin{align*}
S(4) &= 2 \left( \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j+1} \right) + \left( \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j+1} \right) \pmod{4^n - 1} \\
&= 2 \left( \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j+1} \right) + \left( \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j+1} \right) \pmod{4^n - 1}.
\end{align*}
\]

Taking \((c^0, c^1, c^2, c^3) = (t_2, t_1, t_2, t_1)\) in Lemma [4] as an example, then we obtain

\[
\begin{align*}
S(4) &= 2 \left( \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j+1} \right) \pmod{4^n - 1} \\
&= 2 \left( \sum_{j=0}^{n-1} 4^{2j} + \sum_{j=0}^{n-1} 4^{2j+1} \right) \pmod{4^n - 1}.
\end{align*}
\]
\[= 2(\eta_0 + \eta_2) + 2 \cdot 4^n(\eta_0 + \eta_1) \pmod{4^{2n} - 1}
\]
\[= \begin{cases} 2(\eta_0 + \eta_2 + \eta_0 + \eta_1) \pmod{4^n - 1} \\ 2(\eta_0 + \eta_2 - \eta_0 - \eta_1) \pmod{4^n + 1} \end{cases}
\]
\[= \begin{cases} 2(\eta_0 + \sum_{j=1}^{n-1} 4^j - \eta_3) \pmod{4^n - 1} \\ 2(\eta_2 - \eta_1) \pmod{4^n + 1} \end{cases}
\]
\[= \begin{cases} 2(\eta_0 - \eta_3 - 1) \pmod{\frac{4^n - 1}{3}} \\ 2(\eta_2 - \eta_1) \pmod{\frac{4^n + 1}{3}} \end{cases}
\]

From the above formula, we can see that to determine \(\gcd(S(4), 4^{2n-1})\), we need to calculate \(\gcd(\eta_\gamma - \eta_{\gamma-1} - 1, \frac{4^n - 1}{3})\) and \(\gcd(\eta_\gamma - \eta_{\gamma-1}, \frac{4^n + 1}{3})\) first.

**Lemma 15.** Assume that \(d_1\) is the maximum divisor of \(\gcd(\eta_\gamma - \eta_{\gamma-1} - 1, \frac{4^n - 1}{3})\), then we have \(d_1 | n^2 + 3n + 4\).

**Proof.** For \(\gamma = 1\), on the one hand, from \(d_1 | \gcd(\eta_1 - \eta_0 - 1, \frac{4^n - 1}{3})\), we have
\[1 \equiv \eta_1 - \eta_0 \pmod{d_1}.
\]
And on the other hand,
\[G = \eta_0 - \eta_1 + \eta_2 - \eta_3 \equiv (\eta_2 - \eta_3) - 1 \pmod{d_1}.
\]
Combining with Lemmas 14 and 12 we obtain
\[G \equiv (\eta_1 - \eta_0)^2 + (\eta_2 - \eta_3)^2 \equiv 1 + (G + 1)^2 \equiv n + 2 + 2G \pmod{d_1},
\]
which shows \(-G \equiv n + 2 \pmod{d_1}, n \equiv (-G)^2 \equiv (n + 2)^2 \pmod{d_1},\) hence \(d_1 | n^2 + 3n + 4\). For the three other cases, we have the same result. \(\square\)

**Lemma 16.** Assume that \(d_2\) is the maximum divisor of \(\gcd(\eta_\gamma - \eta_{\gamma-1}, \frac{4^n + 1}{3})\), then we have \(d_2 = 1\).

**Proof.** For \(\gamma = 0\), we have
\[0 \equiv \eta_0 - \eta_3 \pmod{d_2}.
\]
\[G = \eta_0 - \eta_1 + \eta_2 - \eta_3 \equiv \eta_2 - \eta_1 \pmod{d_2}.
\]
From Lemmas 14 and 12 we obtain
\[-G \equiv (\eta_0 - \eta_3)^2 + (\eta_2 - \eta_1)^2 \equiv G^2 \equiv n \pmod{d_2},
\]
which implies \( n \equiv (-G)^2 \equiv n^2 \pmod{d_2} \), \( d_2 \mid n^2 - n = n(n-1) \), then we have \( d_2 = n \) or \( d_2 \mid n-1 \). Assume that \( \pi \) is a prime divisor of \( d_2 \), since \( d_2 \mid \frac{4^n+1}{5} \), we have the order of \( 4 \) mod \( \pi \) is \( 2n \), and \( 2n \mid \pi - 1 \), which contradicts \( \pi \mid n \) or \( \pi \mid n-1 \), therefore we arrive at \( d_2 = 1 \). For the three other cases, we have the same result.

\[ \square \]

**Theorem 17.** Let \( t_i \ (1 \leq i \leq 6) \) be six binary sequences of length \( n = 4f + 1 \) with support sets \( D_0 \cap D_1 \), \( D_0 \cap D_2 \), \( D_0 \cap D_3 \), \( D_1 \cap D_2 \), \( D_1 \cap D_3 \), \( D_2 \cap D_3 \), respectively, and \( s = s(a,b) \) be the optimal quaternary sequence given by Lemma 4, then the 4-adic complexity of the sequence \( s \) is

\[
C_4(s) = \begin{cases} 
\log_4\left(\frac{4n-1}{5d_1}\right), & \text{if } e = (0,0,0), (1,0,1) \text{ and } 3 \mid f \text{ or } e = (1,1,0), (0,1,1) \text{ and } 3 \mid f + 1 \\
\log_4\left(\frac{4n-1}{5d_1}\right), & \text{if } e = (0,0,0), (1,0,1) \text{ and } 3 \nmid f \text{ or } e = (1,1,0), (0,1,1) \text{ and } 3 \nmid f + 1
\end{cases}
\]

where \( d_1 \mid \gcd(n^2 + 3n + 4, \frac{4^n-1}{3}) \).

**Proof.** From equation (14) we know that

\[
S(4) = 2 \sum_{j=0}^{n-1} \sum_{j \in D_0 \cup D_2} 4^{2j} 2^n \cdot 4^{2j} \pmod{4^{2n} - 1}
\]

\[
= \begin{cases} 
2 \sum_{j=0}^{n-1} \sum_{j \in D_0 \cup D_2} 1 + 2 \cdot \sum_{j=0}^{n-1} 1 \pmod{3} & (3) \\
2 \sum_{j=0}^{n-1} \sum_{j \in D_0 \cup D_2} 1 - 2 \cdot \sum_{j=0}^{n-1} 1 \pmod{5} & (5)
\end{cases}
\]

\[
= 2 \cdot 2f + 2 \cdot 2f \pmod{3}
\]

\[
= 2 \cdot 2f - 2 \cdot 2f \pmod{5}
\]

\[
= 2f \pmod{3}
\]

\[
= 0 \pmod{5}
\]

Similarly, for \( e = (1,0,1) \), we have

\[
S(4) = 2 \sum_{j=0}^{n-1} \sum_{j \in D_0 \cup D_2} 4^{2j} 2^n \cdot 4^{2j} \pmod{4^{2n} - 1}
\]

\[
= \begin{cases} 
2f \pmod{3} & (3) \\
0 \pmod{5} & (5)
\end{cases}
\]

and for \( e = (1,1,0), (0,1,1) \), we obtain

\[
S(4) = 2 \sum_{j=0}^{n-1} \sum_{j \in D_0 \cup D_2} 4^{2j} 2^n \cdot 4^{2j} \pmod{4^{2n} - 1}
\]

\[
= \begin{cases} 
2 \cdot 2f + 2 \cdot 2f + (4f + 1) \pmod{3} & (3) \\
2 \cdot 2f - 2 \cdot 2f \pmod{5} & (5)
\end{cases}
\]
\[
S(4) = \begin{cases} 2f + 2 \pmod{3} \\ 0 \pmod{5} \end{cases}.
\]

Combining Lemmas 13 and 16 we prove the theorem. \quad \square

**Theorem 18.** Let \(t_i (1 \leq i \leq 6)\) be six binary sequences of length \(n = 4f + 1\) with support sets \(D_0 \cap D_1, D_0 \cap D_2, D_0 \cap D_3, D_1 \cap D_2, D_1 \cap D_3, D_2 \cap D_3,\) respectively, and \(s = s(a, b)\) be the optimal quaternary sequence given by Lemma 3 or Lemma 6 then the 4-adic complexity of the sequence \(s\) satisfies

\[
C_4(s) \geq \log_4\left(\frac{4^n + 1}{5}\right).
\]

**Proof.** Taking \(e = (0, 0, 0)\) and \((c^0, c^1, c^2, c^3) = (t_1, t_2, t_2, t_1)\) in Lemma 3 as an example, from (13) we know that

\[
S(4) = 2 \sum_{j=0}^{n-1} 4^{2j} + 2 \cdot 4^n \cdot \sum_{j=0}^{n-1} 4^{2j} + (1 + 4^n) \sum_{j=0}^{n-1} 4^{2j} \pmod{4^{2n} - 1}
\]

\[
= 2 \sum_{j=0}^{n-1} 4^{2j} + 2 \cdot 4^n \cdot \sum_{j=0}^{n-1} 4^{2j} + (1 + 4^n) \sum_{j=0}^{n-1} 4^{2j} \pmod{4^{2n} - 1} \quad (15)
\]

\[
= 2(\eta_0 + \eta_1) + 2 \cdot 4^n(\eta_0 + \eta_2) + (1 + 4^n)(\eta_1 + \eta_2) \pmod{4^{2n} - 1}
\]

\[
= \begin{cases} 2(\eta_0 + \eta_1 + \eta_0 + \eta_2 + \eta_1 + \eta_2) \pmod{4^n - 1} \\ 2(\eta_0 + \eta_1 - \eta_0 - \eta_2) \pmod{4^n + 1} \\ 4(\sum_{j=1}^{n-1} 4^{2j} - \eta_3) \pmod{4^n - 1} \\ 2(\eta_1 - \eta_2) \pmod{4^n + 1} \\ -4(\eta_3 + 1) \pmod{\frac{4^n - 1}{3}} \\ 2(\eta_1 - \eta_2) \pmod{\frac{4^n + 1}{3}} \end{cases}
\]

and by (15) we have

\[
S(4) = \begin{cases} 0 \pmod{3} \\ 0 \pmod{5} \end{cases}.
\]

Combining with Lemma 16 we have \(\gcd(S(4), 4^{2n} - 1) \leq 5 \cdot (4^n - 1),\) the rest of the cases are similar and the same conclusion can be drawn. \quad \square

**Example 3.** Let \(n = 13 = 12 + 1 = 4 \cdot 1^2 + 3^2,\) \(y = -1,\) then \(\mathbb{F}_{13}^* = \langle 2 \rangle.\) The cyclotomic classes of order 4 in \(\mathbb{F}_{13}^*\) are \(D_0 = \{1, 3, 9\}, D_1 = \{2, 5, 6\}, D_2 = \{4, 10, 12\}, D_3 = \{7, 8, 11\}.\) It is easy to see that

\[
t_1 = (0111011001000),
\]
\[ t_2 = (0101100001101), \]
\[ t_6 = (0000100110111). \]

Let \( e = (0, 0, 0), \ (c^0, c^1, c^2, c^3) = (t_2, t_1, t_2, t_1), \) then we have
\[ a = b = (00100110100000010111100111), \]
\[ s = (002002202000000020222200222). \]

Since \( \gcd(S(4), 4^{26} - 1) = \gcd(2^5 \cdot 3^2 \cdot 5 \cdot 17 \cdot 31 \cdot 67 \cdot 58184921, 3 \cdot 5 \cdot 53 \cdot 157 \cdot 1613 \cdot 2731 \cdot 8191) = 15, \)
we have \( C_s(4) = \log_4 \left( \frac{4^{26} - 1}{15} \right). \) From Theorem 17 we know that \( d_1 = \gcd(13^2 + 3 \cdot 13 + 4, \ \frac{4^{13} - 1}{3}) = 1, \) and \( f = 12, \)
\( C_s(4) = \log_4 \left( \frac{4^{13} - 1}{15d_1} \right) = \log_4 \left( \frac{4^{26} - 1}{15} \right), \) which is consistent with the direct calculation.

V. Conclusion

Quaternary sequences with optimal autocorrelation have an important role in communication and cryptography systems. Su et al. [13] constructed several new families of optimal autocorrelation quaternary sequences by using interleaved construction, sequences pairs and binary cyclotomic sequences of order four. In this paper, firstly, we determine the 4-adic complexity of quaternary sequences interleaved by a pair of twin-prime sequences or GMW sequences with correlation function of the two pairs binary sequences. Secondly, we introduce the definition of the “Gauss periods” of order four and “quadratic Gauss sums” on finite field \( \mathbb{F}_n \) and valued in \( \mathbb{Z}_{4^{26} - 1}^* \), and calculate the 4-adic complexity of interleaved quaternary sequences constructed by two or three binary cyclotomic sequences of order four. Our results show that the 4-adic complexity of these sequences is larger than \( \frac{2n - 16}{6} \) and they are safe enough to resist the attack of the rational approximation algorithm.

References

[1] Berndt B C, Evans R J, Williams K S. Gauss and Jacobi Sums. John Wiley and Sons INC, 1998
[2] Edemskiy V, Chen Z X. On the 4-adic complexity of the two-prime quaternary generator. J. Appl. Math. Comput. 2022, http://doi.org/10.1007/s12190-200-01740-z
[3] Jang J-W, Kim Y-S, Kim S-H, et al. New quaternary sequences with ideal autocorrelation constructed from binary sequences with ideal autocorrelation. In: Proceedings of IEEE International Symposium on Information Theory, Seoul, 2009. 278-281
[4] Jing X Y, Xu Z F, Yang M H, et al. On the p-Adic Complexity of the Ding-Helleseth-Martinsen binary sequences. Chin J Electron, 2021, 30: 64-71
[5] Kim Y-S, Jang J-W, Kim S-H, et al. New construction of quaternary sequences with ideal autocorrelation from Legendre sequences. In: Proceedings of the IEEE international conference on Symposium on Information Theory, Seoul, 2009. 282-285
[6] Kim Y-S, Jang J-W, Kim S-H, et al. New quaternary sequences with optimal autocorrelation. In: Proceedings of the IEEE International Conference on Symposium on Information Theory, Seoul, 2009. 286-289
[7] Klapper A. A survey of feedback with carry shift registers. In: Proceedings of Sequences and Their Applications, Seoul, 2004. 56-71
[8] Klapper A, Xu J Z. Register synthesis for algebraic feedback shift registers based on non-primes. Des Codes Cryptogr, 2004, 31: 227-250
[9] Lüke H D, Schotten H D, Hadinejad-Mahram H. Generalised Sidelnikov sequences with optimal autocorrelation properties. Electron Lett, 2000, 36: 525-527
[10] Qiang S Y, Li Y, Yang M H, et al. The 4-Adic Complexity of A Class of Quaternary Cyclotomic Sequences with Period 2p. [arXiv:2011.11875]
[11] Qiang S Y, Jing X Y, Yang M H, et al. 4-Adic Complexity of Interleaved Quaternary Sequences. [arXiv:2105.13826]
[12] Storer T. Cyclotomy and Difference Sets. Markham, Chicago, 1967
[13] Su W, Yang Y, Zhou Z C, et al. New quaternary sequences of even length with optimal auto-correlation. Sci China Inf Sci, 2018, 61: 1-13
[14] Tang X H, Ding C S. New classes of balanced quaternary and almost balanced binary sequences with optimal autocorrelation value. IEEE Trans Inf Theory, 2010, 56: 6398-6405
[15] Tang X H, Gong G. New constructions of binary sequences with optimal autocorrelation value/magnitude. IEEE Trans Inf Theory, 2010, 56: 1278-1286
[16] Yang M H, Qiang S Y, Jing X Y, et al. On the 4-Adic Complexity of Quaternary Sequences with Ideal Autocorrelation. In: Proceedings of IEEE International Symposium on Information Theory, Espoo, 2022. 528-531
[17] Zhang L L, Zhang J, Yang M H, et al. On the 2-adic complexity of the Ding-Helleseth-Martinsen binary sequences. IEEE Trans Inf Theory, 2020, 66: 4613-4620