The Quenching Problem in the Nonlinear Heat Equations

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Abstract

In this paper we study the quenching problem in nonlinear heat equations with power nonlinearities. For nonlinearities of power \( p < 0 \) and for an open set of slowly varying initial conditions we prove that the solutions will collapse in a finite time. We find the collapse profile and estimate the remainder.

1 Introduction

In this paper we study the problem of collapse of positive solutions for the nonlinear heat equation

\[
\begin{align*}
\partial_t u &= \partial_x^2 u + |\text{sign}(p-1)||u|^{p-1}u \\
u(x,0) &= u_0(x).
\end{align*}
\] (1)

For \( p > 1 \) and for suitable initial conditions \( u_0 \) the solutions of (1) blowup in finite time (see [3, 12, 11, 15, 6, 20, 29, 7, 8, 9, 2, 22, 23, 24]). When \( p < 1 \) one expects the solution to collapse in finite time for certain initial conditions. It is the second case which is of the interest in this paper. The \( p < 1 \) problem arises in the study of the quenching problem in combustion theory [14, 5] and vortex reconnection [21]. Moreover it presents a simple mathematical model for the neck pinching problem of mean curvature flow and Ricci flow [27, 26, 28, 16, 17, 4] and for collapse in the Keller-Segal problem of chemotaxis [1].

We say that a positive solution \( u \) collapses at time \( t^* \) if \( u(\cdot, t) \geq c(t) > 0 \) for \( t < t^* \) and some positive scalar function \( c(t) \) and \( u \to 0 \) as \( t \to t^* \) on some set \( x \in S \subset \mathbb{R} \). Because of its application in combustion theory the problem of collapse of solutions is referred to as the quenching problem. In this paper, for technical reason, we limit ourselves to the case \( p < 0 \) and \( u_0 \geq c_0 > 0 \) for some constant \( c_0 \).

The problem of quenching for (1) with \( p = -1 \) on bounded domain was studied first in [18], where a sufficient condition for collapsing is found. Later Huisken [17] proved that if \( \partial_x^2 u_0(x) - u_0^p(x) \leq 0 \), then the solution collapses in finite time. Merle and Zaag proved in [21] that there exists initial condition \( u_0(x) \) such

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that the solution $u(x,t)$ collapses in finite time $t^*$ and
\[
\lim_{t \to t^*} (t^* - t)^{-\frac{2}{1-p}} u(x((t^* - t)\ln(t^* - t)))^{1/2}, t) - (1 - \frac{(1-p)^2}{4p}x^2)^{\frac{1}{1-p}} \|_{\infty} = 0. \tag{2}
\]
For the neck pinching problem in mean curvature flow, Huisken proved in \[17\] a result weaker than \[2\] holds on a bounded space domain. For other related works we refer to \[14, 5, 28, 16\]. The starting point in these works is to study the rescaled function $(t^* - t)^{-\frac{1}{2}} u(x(t^* - t)^{\frac{1}{2}}, t)$ using the technique of Sturm Liouville theorem for linear parabolic equations, as used in \[10, 12\].

The origin of scaling and asymptotic in \[2\] lies in the following key properties of Equation \[1\]:

1. \[1\] is invariant with respect to the scaling transformation,
\[
u(x,t) \to \lambda^{\frac{2}{1-p}} u(\lambda^{-1}x, \lambda^{-2}t) \tag{3}
\]
for any constant $\lambda > 0$, i.e. if $u(x,t)$ is a solution, so is $\lambda^{\frac{2}{1-p}} u(\lambda^{-1}x, \lambda^{-2}t)$.

2. \[1\] has $x-$independent (homogeneous) solutions:
\[
u_{hom} = [u_0^{-p+1} - |p - 1|t]^{-\frac{1}{1-p}} \tag{4}
\]
These solutions collapse (blow up) in finite time $t^* = (|p - 1|u_0^{2-p} - 1)^{-1}$.

In what follows we use the notation $f \lesssim g$ for two functions $f$ and $g$ satisfying $f \leq Cg$ for some universal constant $C$. We will also deal, without specifying it, with weak solutions of Equation \[1\] in some appropriate sense. These solutions can be shown to be classical for $t > 0$.

In our paper we consider Equation \[1\] in $\mathbb{R}$ with $p < 0$ and with the initial conditions $u_0$ even, bounded below by a positive constant and having a local minimum at 0 modulo small fluctuation. We prove that there exists a time $t^* < \infty$ such that $u(x,t)$ collapses at time $t^*$; moreover there exist $C^3$ functions $\lambda(t), b(t), c(t)$ and $\eta(x,t)$ such that
\[
u(x,t) = \lambda^{\frac{2}{1-p}}(t)[(1 - \frac{p + b(t)\lambda^{-2}(t)x^2}{2c(t)} + \eta(x,t)] \tag{5}
\]
with
\[
\|\lambda^{-1}(t)x^{-\frac{3}{2}}\eta(x,t)\|_{\infty} \lesssim b^{3/2}(t). \tag{6}
\]
Furthermore the scalar functions $\lambda(t), b(t)$ and $c(t)$ satisfy the estimates
\[
\lambda(t) \quad = \quad \lambda(0)(t^* - t)^{\frac{1}{2}}(1 + o(1)), \quad \lambda(0) = (2\lambda_0 + \frac{2}{1-p}b_0)^{-1/2};
\]
\[
b(t) \quad = \quad \frac{(p - 1)^2}{4p(1 + O(\frac{1}{|\lambda(t)|^{1/2}}))}; \tag{7}
\]
\[
c(t) \quad = \quad \frac{1}{2} + \frac{1-p}{4p}\lambda(t)^{-1}(1 + O(\frac{1}{|\lambda(t)|^{1/2}}))
\]
where $o(1) \to 0$ as $t \to t^*$. 


Before stating the main theorem we define a function \( g(y, b_0) \) as
\[
g(y, b_0) := \begin{cases} 
\left( \frac{1-p}{2} \right)^{\frac{1}{2-p}} & \text{if } b_0 y^2 \leq 4(1-p) \\
\left( \frac{2(1-p)}{3} \right)^{\frac{1}{2-p}} & \text{if } b_0 y^2 > 4(1-p)
\end{cases}
\]
and define the constant \( q \) which will be used throughout the rest of paper
\[
q := \min \{ \frac{4}{1-p}, \frac{2(2-p)}{(1-p)^2}, 1 \}.
\]
The following is the main result of our paper.

**Theorem 1.1.** Assume the initial datum \( u_0(x) \) in (17) is even and satisfies the estimates
\[
\| (x)^{-n}[u_0(x) - \left( \frac{1-p+b_0x^2}{2c_0} \right)^{\frac{1}{2-p}}] \|_{\infty} \leq \delta_0 b_0^\frac{2}{p} \tag{8}
\]
n = 2, 3, \( u_0(x) \in (x)^{\frac{2}{p}}L^\infty \) and \( u_0(x) \geq (2c_0 + \frac{2b_0}{(1-p)})^{\frac{1}{2-p}} g((2c_0 + \frac{2b_0}{1-p})^{1/2} x, b_0) \) for some \( 1/2 \leq c_0 \leq 2 \), and (3) with \( n = q \) if \( p < -1 \). Then there exists a constant \( \delta \) such that if \( \delta_0, b_0 \leq \delta \) then there exists a finite time \( 0 < t^* < \infty \) such that \( \| \frac{1}{u(\cdot, t)} \|_{\infty} < \infty \) if \( t < t^* \) and
\[
\| \frac{1}{u(\cdot, t)} \|_{\infty} \to \infty \text{ as } t \to t^*.
\]
Moreover there exist \( C^1 \) functions \( \lambda(t), b(t), c(t) \) and \( \eta(x, t) \) such that \( u(x,t) \) satisfies the estimates in (30)-(33).

The proof of this theorem is given in Section 8. This theorem shows the collapse at 0 for a certain neighborhood of the homogeneous solution, (4), and it provides a detailed description of the leading term and an estimate of the remainder in \( (x)^3L^\infty \). In fact, we have not only the asymptotic expressions for the parameters \( b \) and \( c \) determining the leading term and the size of the remainder, but also dynamical equations for these parameters:
\[
b_\tau = \frac{4p}{(p-1)^2} b^2 + c^{-1} c_\tau b + \mathcal{R}_b(\eta, b, c), \tag{9}
\]
\[
c^{-1} c_\tau = 2 \left( 1 - c \right) - \frac{2}{p-1} b + \mathcal{R}_c(\eta, b, c), \tag{10}
\]
where \( \tau \) is a ‘collapse’ time related to the original time \( t \) as \( \tau(t) := \int_0^t \lambda^{-2}(s) ds \) and the remainders have the estimates
\[
\mathcal{R}_b(\eta, b, c), \mathcal{R}_c(\eta, b, c) = O \left( b^3 + \| c - \frac{1}{2} \|_X + \| c_\tau \|_X b^2 + \| \eta(\cdot, t) \|_X + \| \eta(\cdot, t) \|_{X^p} \right) \tag{11}
\]

with the norm \( \| \eta(\cdot, t) \|_X := \| (\lambda^{-1}(t)x)^{-3} \eta(\cdot, t) \|_{\infty} \).

This paper is organized as follows. In Section 4 we prove the local well-posedness of Equation (1). In Sections 5-6 we present some preliminary derivations and motivations for our analysis. In Section 7 we formulate a priori bounds on solutions to (11) which are proven in Sections 9-15. We use these bounds and a lower bound proved in Section 7 in Section 8 to prove our main result, Theorem 1.1. In Section 9 we lay the ground work for the proof of the a priori bounds of Section 7 in particular by using a Lyapunov-Schmidt-type argument we derive equations for the parameters \( a, b \) and \( c \) and fluctuation \( \eta \).
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2 Local Well-posedness of \((12)\)

In this section we prove the local well-posedness of \((1)\) for \(p < 0\) in spaces used in this paper. Since in what follows we are dealing with \(p < 0\) and \(u_0(x) > 0\) we restate \((1)\) as

\[
\frac{\partial_t u}{u(x, 0)} = \frac{\partial^2_x u - u^p}{u_0(x) > 0}, \quad p < 0
\]

\[(12)\]

**Theorem 2.1.** If \(u_0(x) \in \langle x \rangle^{\frac{2}{1-p}} L^\infty\) and \(u_0(x) \geq \kappa_0 > 0\), then there exist a function \(f(x, t) \in \langle x \rangle^{\frac{2}{1-p}} L^\infty\) for any time \(t < \infty\) and \(\delta(\kappa_0, \|\langle x \rangle^{-\frac{2}{1-p}} u_0\|_\infty)\) such that for any time \(t_0 \leq t \leq t_0 + \delta\), \((12)\) has a unique solution \(u(x, t) = u_0(x)\) and \(f(x, t) \geq u(x, t) \geq \frac{1}{2} \kappa_0\).

Moreover, if \(t_*\) is the supremum of such \(\delta(\kappa_0, \|\langle x \rangle^{-\frac{2}{1-p}} u_0\|_\infty)\) then either \(t_* = \infty\) or \(\|\langle x \rangle^{\frac{2}{1-p}} u(x, t)\|_\infty \to \infty\) as \(t \to t_*\).

**Proof.** We transform \((12)\) as

\[
u(x, t) = e^{t \partial^2_x} u_0(x) - \int_0^t e^{(t-s) \partial^2_x} u^p(x, s) ds.
\]

Define a new function \(u_1(x, t)\) by

\[
u_1(x, t) := u(x, t) - f(x, t)
\]

with \(f(x, t) := e^{t \partial^2_x} u_0(x)\). Then \((13)\) becomes

\[
u_1(x, t) = - \int_0^t e^{(t-s) \partial^2_x} [f(x, s) + u_1(x, s)]^p ds.
\]

In the next we use the fix point argument to prove the existence and the uniqueness of the solution \(\nu_1\) to \((14)\), hence those of \(u\) to complete the proof.

We start with proving \(f(x, t) \geq \kappa_0 > 0\) and \(\langle x \rangle^{-\frac{2}{1-p}} f(\cdot, t) \in L^\infty\). It is well known that the integral kernel of \(e^{t \partial^2_x}\) is \(\frac{1}{\sqrt{4\pi t}} e^{-\frac{\|x-y\|^2}{4t}} > 0\), consequently

\[
f(x, t) \geq e^{t \partial^2_x} \kappa_0 = \kappa_0.
\]

Moreover we claim that \(\langle x \rangle^{-\kappa} e^{t \partial^2_x} \langle x \rangle^\kappa < \infty\) for any \(2 \geq \kappa \geq 0\). Hence by the fact \(2 > \frac{2}{1-p} > 0\) we have \(f(\cdot, t) \in \langle x \rangle^{\frac{2}{1-p}} L^\infty\). In the following we prove the claim for \(\kappa = 0\) and \(\kappa = 2\), the general case follows.
by interpolation between them. It is easy to prove $\kappa = 0$ by the fact $e^{t\partial_x^2}1 = 1$; for $\kappa = 2$ the fact $y^2 \leq 2(x - y)^2 + 2x^2$ yields
\[\langle x \rangle^{-2} e^{t\partial_x^2} x^2 = \langle x \rangle^{-2} \int \frac{1}{\sqrt{\pi t}} e^{-\frac{(x-y)^2}{t}} y^2 dy \leq 2\langle x \rangle^{-2} \int \frac{1}{\sqrt{\pi t}} e^{-\frac{(x-y)^2}{t}} [(x-y)^2 + x^2] dy = \langle x \rangle^{-2} [2x^2 + t\delta] < \infty\]
with the constant $\delta_0 := \frac{2}{\sqrt{\pi}} \int e^{-x^2} x^2 dx$. Thus we finish proving the properties of $f(\cdot, t)$.

By the integral kernel we have $f(x, t) \to u_0(x)$ as $t \to 0^+$. Moreover by using the contraction lemma on (13) it is not hard to prove that there exists a time $\delta(\delta_0, \|\langle x \rangle^{-\frac{1}{2}} u_0\|_{\infty})$ such that for time $t \in [0, \delta]$ there exists a unique bounded negative solution $u_1$ such that
\[\|u_1(\cdot, t)\|_{\infty} \leq \frac{\delta_0}{2}. \tag{16}\]

Thus $u(x, t) = f(x, t) + u_1(x, t) \geq \frac{\delta_0}{2}$ has all the properties in the proposition, thus the proof is complete. $\square$

3 Blow-Up Variables and Almost Solutions

In this section we pass from the original variables $x$ and $t$ to the blowup variables $y := \lambda^{-1}(t)(x - x_0(t))$ and $\tau := \int_0^t \lambda^{-2}(s) ds$. The point here is that we do not fix $\lambda(t)$ and $x_0(t)$ but consider them as free parameters to be found from the evolution of (12). Assume for simplicity that $u_0$ is even with respect to $x = 0$. In this case $x_0$ can be taken to be 0. Suppose $u(x, t)$ is a solution to (12) with an initial condition $u_0(x)$. We define the new function
\[v(y, \tau) := \lambda^{2\tau} \lambda^{-1}(t) u(x, t) \tag{17}\]
with $y := \lambda^{-1}(t)x$ and $\tau := \int_0^t \lambda^{-2}(s) ds$. The function $v$ satisfies the equation
\[v_{\tau} = (\partial_y^2 - ay\partial_y) + \frac{2a}{1 - p}) v - v^p. \tag{18}\]
where $a := -\lambda \partial_\lambda \lambda$. The initial condition is $v(y, 0) = \lambda_0^{\frac{2}{1 - p}} u_0(\lambda_0 y)$, where $\lambda_0$ is the initial condition for the scaling parameter $\lambda$.

If the parameter $a$ is a constant, then (18) has the following homogeneous, static (i.e. $y$ and $\tau$-independent) solutions
\[v_a := \left(\frac{1 - p}{2a}\right)^{\frac{1}{1 - p}} \tag{19}\]
In the original variables $t$ and $x$, this family of solutions corresponds to the homogeneous solution (4) of the nonlinear heat equation with the parabolic scaling $\lambda^2 = 2a(T - t)$, where the collapsing time, $T := \left[u_0^{p-1}(1 - p)\right]^{-1}$, is dependent on $u_0$, the initial value of the homogeneous solution $u_{hom}(t)$.

If the parameter $a$ is $\tau$ dependent but $|a_\tau|$ is small, then the above solutions are good approximations to the exact solutions. Another approximation is the solution of $ayv_y + \frac{2a}{1 - p} v + v^p = 0$, obtained from (18) by neglecting the $\tau$ derivative and second order derivative in $y$. This equation has the general solution
\[v_{ab} := \left(\frac{1 - p + by^2}{2a}\right)^{\frac{1}{1 - p}}. \tag{20}\]
for all $b \in \mathbb{R}$. In what follows we take $b \geq 0$ so that $v_{ab}$ is nonsingular. Note that $v_{a,0} = v_a$.

4 “Gauge” Transform

We assume that the parameter $a$ depends slowly on $\tau$ and treat $|a_\tau|$ as a small parameter in a perturbation theory for Equation (18). In order to convert the global non-self-adjoint operator $ay\partial_y$ appearing in this equation into a more tractable local and self-adjoint operator we perform a gauge transform. Let

$w(y, \tau) := e^{-\frac{a_\tau^2}{4}} v(y, \tau)$.  

Then $w$ satisfies the equation

$$w_\tau = (\partial_y^2 - \frac{1}{4} \omega^2 y^2 - (\frac{2}{p-1} - \frac{1}{2})a)w - e^{\frac{1}{4}(p-1)y^2} w^p,$$

where $\omega^2 = a^2 + a_\tau$. The approximate solution $v_{ab}$ to (18) transforms to $v_{abc}$ where

$v_{abc} := (\frac{1}{2} - p + by)^2 \frac{1}{2c} e^{-\frac{a_\tau^2}{4}}$.

5 Re-parametrization of Solutions

In this section we split solutions to (22) into the leading term - the almost solution $v_{abc}$ - and a fluctuation $\xi$ around it. More precisely, we would like to parametrize a solution by a point on the manifold $M_{as} := \{v_{abc} | a, b, c \in \mathbb{R}, b \leq \epsilon, a = a(b, c)\}$ of almost solutions and the fluctuation orthogonal to this manifold (large slow moving and small fast moving parts of the solution). Here $a = a(b, c)$ is a twice differentiable function of $b$ and $c$. For technical reasons, it is more convenient to require the fluctuation to be almost orthogonal to the manifold $M_{as}$. More precisely, we require $\xi$ to be orthogonal to the vectors $\phi_{0a} := (\frac{\alpha}{2\pi})^\frac{1}{4} e^{-\frac{1}{4}y^2}$ and $\phi_{2a} := (\frac{\alpha}{8\pi})^\frac{1}{4} (1 - ay^2) e^{-\frac{1}{4}y^2}$ which are almost tangent vectors to the above manifold, provided $b$ is sufficiently small. Note that $\xi$ is already orthogonal to $\phi_{1a} := (\frac{\alpha}{2\pi})^\frac{3}{4} \sqrt{ay} e^{-\frac{1}{4}y^2}$ since our initial conditions, and therefore, the solutions are even in $x$.

In what follows we fix the relation between $a$ and $c$ as

$$2c(\tau) = a(\tau) + \frac{1}{2}.$$

Define a new function $V_{a,b} := (\frac{1-p+by^2}{a+\frac{1}{2}})^{-\frac{1}{4}}$ and a neighborhood $U_{\epsilon_0}$:

$$U_{\epsilon_0} := \{v \in L^\infty(\mathbb{R}) | \|e^{-\frac{1}{4}y^2} (v - V_{ab})\|_\infty = o(b) \text{ for some } a \in [1/4, 1], b \in [0, \epsilon_0] \}.$$

Proposition 5.1. There exist an $\epsilon_0 > 0$ and a unique $C^1$ functional $g : U_{\epsilon_0} \to \mathbb{R}^+ \times \mathbb{R}^+$, such that any function $v \in U_{\epsilon_0}$ can be uniquely written in the form

$$v = V_g(v) + \eta,$$  

(24)
with \( \eta \perp e^{-\frac{a}{2}y^2}\phi_{0a}, e^{-\frac{a}{2}y^2}\phi_{2a} \) in \( L^2(\mathbb{R}) \), \((a, b) = g(v)\). Moreover if \( \|e^{-\frac{a}{2}y^2}(v - V_{a_0}b_0)\|_\infty = o(b_0) \) for some \( a_0 \) and \( b_0 \) then

\[
|g(v) - (a_0, b_0)| \lesssim \|e^{-\frac{a}{2}y^2}(v - V_{a_0}b_0)\|_\infty.
\]

(25)

**Proof.** The orthogonality conditions on the fluctuation can be written as \( G(\mu, v) = 0 \), where \( \mu = (a, b) \) and \( G : \mathbb{R}^+ \times \mathbb{R}^+ \times L^\infty(\mathbb{R}) \to \mathbb{R}^2 \) is defined as

\[
G(\mu, v) := \left( \begin{array}{c} \langle V_\mu - v, (\frac{a}{2\pi})^{-\frac{1}{4}}e^{-\frac{a}{4}y^2}\phi_{0a} \rangle \\ \langle V_\mu - v, (\frac{a}{8\pi})^{-\frac{1}{4}}e^{-\frac{a}{4}y^2}\phi_{2a} \rangle \end{array} \right).
\]

Here and in what follows, all inner products are \( L^2 \) inner products. Using the implicit function theorem we will prove that for any \( \mu_0 := (a_0, b_0) \in [\frac{1}{4}, 1] \times (0, \epsilon_0) \) there exists a unique \( C^1 \) function \( g : L^\infty \to \mathbb{R}^+ \times \mathbb{R}^+ \) defined in a neighborhood \( U_{\epsilon_0} \subset L^\infty \) of \( V_{\mu_0} \) such that \( G(g(v), v) = 0 \) for all \( v \in U_{\epsilon_0} \).

Note first that the mapping \( G \) is \( C^1 \) and \( G(\mu_0, V_{\mu_0}) = 0 \) for all \( \mu_0 \). We claim that the linear map \( \partial_\mu G(\mu_0, V_{\mu_0}) \) is invertible. Indeed, we compute

\[
\partial_\mu G(\mu, v)|_{\mu = \mu_0} = A_1(\mu) + A_2(\mu, v)|_{\mu = \mu_0}
\]

where

\[
A_3 := \left( \begin{array}{cc} \langle \partial_a V_\mu, e^{-\frac{a}{2}y^2} \rangle & \langle \partial_b V_\mu, e^{-\frac{a}{2}y^2} \rangle \\ \langle \partial_a V_\mu, (1 - ay^2)e^{-\frac{a}{2}y^2} \rangle & \langle \partial_b V_\mu, (1 - ay^2)e^{-\frac{a}{2}y^2} \rangle \end{array} \right)
\]

and

\[
A_2 := -\frac{1}{2} \left( \begin{array}{cc} 0 & \langle V_\mu - v, (5 - ay^2)y^2e^{-\frac{a}{2}y^2} \rangle \\ 0 & 0 \end{array} \right).
\]

By the condition in the proposition, we have for \( A_2 \) that

\[
\|A_2(\mu_0, v)\| \lesssim |b - b_0| + |a - a_0| + |b_0|.
\]

For \( b > 0 \) and small, we expand the matrix \( A_1 \) in \( b \) to get \( A_1 = G_1G_2 + o(b) \), where the matrices \( G_1 \) and \( G_2 \) are defined as

\[
G_1 := \left( \begin{array}{cc} \langle -y^2e^{-\frac{a}{2}y^2}, e^{-\frac{a}{2}y^2} \rangle & \frac{1}{a^2 + \frac{3}{2}} \langle e^{-\frac{a}{2}y^2}, e^{-\frac{a}{2}y^2} \rangle \\ \langle -y^2e^{-\frac{a}{2}y^2}, (1 - ay^2)e^{-\frac{a}{2}y^2} \rangle & 0 \end{array} \right)
\]

and

\[
G_2 := \left( \begin{array}{cc} 1 - p \frac{1}{a + 1/2} \frac{1}{p - 1} & \frac{1}{p - 1} \frac{1}{p - 1} \frac{1}{1} \\ \frac{1}{p - 1} \frac{1}{p - 1} \frac{1}{1} & 0 \end{array} \right).
\]

Obviously the matrices \( G_1 \) and \( G_2 \) have uniformly (if \( a \in [\frac{1}{4}, 1] \)) bounded inverses. We claim that this observation implies our proposition. Indeed, expand \( G(\mu, v) \) as

\[
G(\mu, v) = G(\mu_0, v) + \partial_\mu G(\mu_0, v) \left( \begin{array}{cc} a - a_0 \\ b - b_0 \end{array} \right) + O(|b - b_0|^2 + |a - a_0|^2),
\]

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provided that \( b, b_0 > 0 \). By the assumption in the proposition we have that

\[
G(\mu_0, v) = o(b_0) \text{ if } a_0 \in \left[ \frac{1}{4}, 1 \right].
\]

Moreover as we have seen above, the matrix \( \partial_{\mu} G(\mu_0, v) \) for \( v \in U_{a_0} \) has uniformly bounded inverse. Hence we have that \( G(\mu, v) = 0 \) has a unique solution \( g(v) \) satisfying

\[
|g(v) - (a_0, b_0)| \lesssim |G(\mu_0, v)| \lesssim \|e^{-\frac{3}{4}v^2} (v - V_{a_0, b_0})\|_{\infty}
\]

which is (25).

Recall that \( q = \min\left\{ \frac{4}{1-p}, \frac{2(2-p)}{(1-p)^2}, 1 \right\} \).

**Proposition 5.2.** In the notation of Proposition [57] if \( \|\langle y \rangle^{-n}(v - V_{\mu_0})\|_{\infty} \leq \delta_0 b_0^{\frac{n}{2}} \) with \( n = 2, 3 \), and \( n = q \) if \( p < -1 \), with \( \delta_0, b_0 \) small, then

\[
|g(v) - \mu_0| \lesssim \|\langle y \rangle^{-3}(v - V_{\mu_0})\|_{\infty} \quad (27)
\]

\[
\|\langle y \rangle^{-3}(v - V_{g(v)})\|_{\infty} \lesssim \|\langle y \rangle^{-3}(v - V_{\mu_0})\|_{\infty} \quad (28)
\]

\[
\|\langle y \rangle^{-2}(v - V_{g(v)})\|_{\infty} \lesssim \delta_0 b_0 + \delta_0 b_0^{3/2} \quad (29)
\]

and if \( p < -1 \) then

\[
\|\langle y \rangle^{-q}(v - V_{g(v)})\|_{\infty} \lesssim \delta_0 b_0^{3/2} + \delta_0 b_0^{\frac{1+q}{2}} \quad (30)
\]

**Proof.** Equation (25) implies that

\[
|g(v) - \mu_0| \lesssim \|\langle y \rangle^{-3}(v - V_{\mu_0})\|_{\infty} \quad (31)
\]

with \( \mu_0 := (a_0, b_0) \). This together with the fact \( \|\langle y \rangle^{-3}(v - V_{\mu_0})\|_{\infty} \leq \delta_0 b_0^{3/2} \) yields (27). Moreover

\[
\|\langle y \rangle^{-3}(v - V_{g(v)})\|_{\infty} \lesssim \|\langle y \rangle^{-3}(v - V_{\mu_0})\|_{\infty} + \|\langle y \rangle^{-3}(V_{g(v)} - V_{\mu_0})\|_{\infty}
\]

\[
\lesssim \|\langle y \rangle^{-3}(v - V_{\mu_0})\|_{\infty} + |\mu_0 - g(v)|
\]

\[
\lesssim \|\langle y \rangle^{-3}(v - V_{\mu_0})\|_{\infty}
\]

which is (28).

To prove Equation (29), we write

\[
\|\langle y \rangle^{-2}(v - V_{g(v)})\|_{\infty} \leq \|\langle y \rangle^{-2}(v - V_{\mu_0})\|_{\infty} + \|\langle y \rangle^{-2}(V_{g(v)} - V_{\mu_0})\|_{\infty}.
\]

By (31), we have \( \|\langle y \rangle^{-2}(V_{g(v)} - V_{\mu_0})\|_{\infty} \lesssim |g(v) - \mu_0| \lesssim \delta_0 b_0^{3/2} \), hence

\[
\|\langle y \rangle^{-2}(V_{g(v)} - V_{\mu_0})\|_{\infty} \lesssim |g(v) - \mu_0| \lesssim \delta_0 b_0^{3/2}.
\]

This together with the fact \( \|\langle y \rangle^{-2}(v - V_{\mu_0})\|_{\infty} \leq \delta_0 b_0 \) completes the proof of (29).

Finally for Equation (30) we have

\[
\|\langle y \rangle^{-q}(v - V_{g(v)})\|_{\infty} \leq \|\langle y \rangle^{-q}(v - V_{\mu_0})\|_{\infty} + \|\langle y \rangle^{-q}(V_{g(v)} - V_{\mu_0})\|_{\infty}.
\]
By its definition we have $1 \geq q > \frac{2}{1-p}$ which together with the definition of $V_{a,b}$ yields

$$\|\langle y \rangle^{-q}(V_{g(v)} - V_{\mu_0})\|_\infty \lesssim |a - a_0| + \|\frac{|b - b_0|^q}{|1-p| b_0^{q-1}}\|_\infty$$

$$\lesssim |a - a_0| + |b - b_0| b_0^{q-1}.$$ 

By the estimate of $|a - a_0| + |b - b_0|$ above we have

$$\|\langle y \rangle^{-q}(V_{g(v)} - V_{\mu_0})\|_\infty \lesssim \delta_0 b_0^{\frac{q+q}{2}}$$

which together with $\|\langle y \rangle^{-q}(\nu - V_{\mu_0})\|_\infty \leq \delta_0 b_0^{\frac{q}{2}}$ in the proposition implies (30).

6 A priori Estimates

In this section we assume that (12) has a unique solution, $u(x,t), 0 \leq t \leq t_*$, such that $v(y,\tau) = \lambda^{-\frac{2}{p}}(t)u(x,t)$, where $y = \lambda^{-1}x$ and $\tau(t) := \int^t \lambda^{-2}(s)ds$, is in the neighborhood $U_{e_0}$ determined in Proposition 5.1. Then by Proposition 5.1 there exist $C^1$ functions $a(\tau)$ and $b(\tau)$ such that $v(y,\tau)$ can be represented as

$$v(y,\tau) = \left(\frac{1-p + b_0^q}{a + \frac{1}{2}}\right)^{\frac{1}{1-p}} + e^{\frac{aq}{p}} \xi(y,\tau)$$

where $\xi(\cdot, \tau) \perp \phi_0, \phi_2$ (see (24)). Now we set

$$-\lambda(t) \partial_t \lambda(t) = a(\tau(t)).$$

In the following we define some estimating functions to control $\xi$, $a$ and $b$.

$$M_1(\tau) := \max_{s \leq \tau} \beta^{-3/2}(s) \|\langle y \rangle^{-3} e^{\frac{aq}{p}} \xi(s)\|_\infty,$$

$$M_2(\tau) := \max_{s \leq \tau} \beta^{-1}(s) \|\langle y \rangle^{-2} e^{\frac{aq}{p}} \xi(s)\|_\infty,$$

$$A(\tau) := \max_{s \leq \tau} \beta^{-2}(s)|a(s) - \frac{1}{2} + \frac{2}{1-p} b(s)|$$

$$B(\tau) := \max_{s \leq \tau} \beta^{-\frac{1}{2}}(s)|b(s) - \beta(s)|$$

moreover if $p < -1$ we define

$$M_q(\tau) := \max_{s \leq \tau} \beta^{-\frac{q}{2}}(s) \|\langle y \rangle^{-q} e^{\frac{aq}{p}} \xi(s)\|_\infty$$

and recall that $q := \min\{\frac{4}{1-p}, \frac{2(2-p)}{(p-1)^2}\}$. The function $\beta(\tau)$ is defined as

$$\beta(\tau) := \frac{1}{\beta(0) - \frac{4p}{(p-1)^2} \tau}.$$ (34)

In the next section, we present the a priori bounds on the fluctuation $\xi$ which are proved in later sections.
Proposition 6.1. Suppose in (12) the datum $u_0(x)$ satisfies all the conditions in Theorem 1.1. Let the parameters $a(\tau), b(\tau)$ and the functions $v$ and $\xi$ the same as in Proposition 6.2. Then they satisfy the following estimates: if

$$M_1(\tau) \leq \frac{1}{8}(1-p)\frac{1}{1-p} - \frac{2}{3}, B(\tau) \leq \beta^{-1/4}(\tau)$$

in some interval $\tau \in [0,T]$ then for any $\tau \in [0,T]$ we have

(1) if $0 < p \leq -1$ then

$$B(\tau) \lesssim 1 + M_1(\tau)A(\tau) + M_1^{2-p}(\tau) + A(\tau),$$

$$A(\tau) \lesssim A(0) + 1 + \beta^{1/2}(0)[1 + M_1^{2-p}(\tau) + A(\tau)M_1(\tau)],$$

$$M_1(\tau) \lesssim M_1(0) + M_1(\tau)M_2(\tau) + \beta^{1/4}(0)[1 + M_1^{2-p}(\tau) + A(\tau)M_1(\tau)],$$

$$M_2(\tau) \lesssim M_2(0) + M_1(\tau) + M_2^2(\tau) + \beta^{1/4}(0)[1 + M_1^{2-p}(\tau) + A(\tau)M_1(\tau)];$$

(2) if $p < -1$ then (35) and (36) still hold, and

$$M_1(\tau) \lesssim M_1(0) + \beta^{1/4}(0)[1 + M_1^{2-p}(\tau) + A(\tau)M_1(\tau)] + M_1(\tau)[M_1^{2-p}(\tau) + M_q(\tau)],$$

$$M_2(\tau) \lesssim M_2(0) + M_1(\tau) + M_2(\tau)[M_1^{2-p}(\tau) + M_q(\tau)] + \beta^{1/4}(0)[1 + M_1^{2-p}(\tau) + M_2(\tau) + A(\tau)M_1(\tau)],$$

$$M_q(\tau) \lesssim M_q(0) + M_2(\tau) + M_q^{2-p}(\tau) + M_q^2(\tau) + \beta^{1/4}(0)[1 + M_q(\tau) + M_1^{2-p}(\tau) + M_1(\tau)A(\tau)]$$

where the function $\beta$, the constant $q$ and the estimating functions are defined above.

We prove Equations (35) and (36) in Section 10. (37) and (38) in Section 13. (39) and (40) in Section 14. (41) in Section 15.

7 The Lower Bound of $v$

In this section we prove a lower bound for $v$ defined in (17). The main tool is a generalized form of maximum principle.

Lemma 7.1. Suppose $u(y, \tau)$ is a smooth function having the following properties: there exist smooth, bounded functions $a_1, a_2, d$ such that if $|y| \geq c(\tau) \geq 0$ then

$$u_\tau - u_{yy} - [a_1(y, \tau) + d(\tau)]u_y - a_2(y, \tau)u \leq 0;$$

$$\langle y \rangle^{-l}u(y, \tau) \in L^\infty \text{ with } l \geq 0;$$

$$u(y, 0) \leq 0 \text{ if } |y| \geq c(0), \text{ and } u(c(\tau), \tau) \leq 0 \text{ if } \tau \leq T;$$

then we have

$$u(y, \tau) \leq 0 \text{ if } |y| \geq c(\tau), \tau \leq T.$$
Proof. In order to use the standard maximum principle we first transform the function \( u \). Define a new function \( w \) by
\[
\langle z \rangle^I w(z, \tau) := u(y, \tau)
\]
with \( z := ye^{\int_0^\tau d(s)ds} \). Then \( w \) is a smooth, bounded function satisfying the inequality
\[
w_\tau - w_{zz} - a_3(z, \tau)w_z - a_4(z, \tau)w \leq 0
\]
for some bounded, smooth functions \( a_3, a_4 \); moreover
\[
w(z, 0) \leq 0 \text{ if } |z| \geq c(0), \text{ and } w(c(\tau)e^{\int_0^\tau d(s)ds}, \tau) \leq 0 \text{ if } \tau \leq T.
\]
By the standard maximum principle (see [19]) we have
\[
w(z, \tau) \leq 0 \text{ if } \tau \leq T, \ |z| \geq c(\tau)e^{\int_0^\tau d(s)ds}.
\]
By the definition of \( w \) in (42) we complete the proof.

Recall the definition of the function \( g(y, \beta) \) as
\[
g(y, \beta) := \begin{cases} 
\frac{(1-p)}{2} \frac{1}{\tau^{p+1}} & \text{if } \beta y^2 \leq 4(1-p) \\
(2(1-p)) \frac{1}{\tau^{p+1}} & \text{if } \beta y^2 > 4(1-p)
\end{cases}
\]
The following lemma states an observation important for our analysis.

**Lemma 7.2.** Let \( v \) be as in (17). Suppose that for time \( \tau \leq \tau_1 \), \( M_1(\tau) \leq \frac{1}{8}(1-p)\frac{1}{\tau^{p+1}} \frac{1}{2} \), \( A(\tau), B(\tau) \leq \beta^{-1/4}(\tau) \) and
\[
\langle y \rangle^{-\frac{3}{p+1}} v(y, \tau) \in L^\infty \text{ and } v(y, \tau) \geq c(\tau)
\]
for some \( c(\tau) > 0 \). Then we have
\[
v(y, \tau) \geq (2c_0 + \frac{2b_0}{1-p}) \frac{1}{\tau^{p+1}} g((2c_0 + \frac{2b_0}{1-p})^{1/2} y, \beta(y)).
\]

Proof. By the scaling invariance of (12), without losing generality we assume that \( 2c_0 + \frac{2b_0}{1-p} = 1 \). By the assumption on the datum we have that
\[
v(y, 0) \geq g(y, \beta(0)).
\]
Recall that \( p < 0 \) and
\[
v(y, \tau) = \left( \frac{1-p + b(\tau)y^2}{a(\tau) + \frac{1}{2}} \right)^{\frac{1}{p+1}} + e^{\frac{a(y)}{p+1}} \xi(y, \tau).
\]
The assumption \( M_1(\tau) \leq \frac{1}{8}(1-p)^{\frac{1}{p+1}} \frac{1}{2} \) yields \( e^{\frac{a(y)}{p+1}} \xi(y, \tau) \leq (\frac{1-p}{10})^{\frac{1}{p+1}} \beta^{3/2}(\tau)\langle y \rangle^3 \). Moreover by the assumption on \( A \) we have \( a(\tau) \in [\frac{1}{4}, \frac{3}{4}] \). Consequently
\[
v(y, \tau) \geq (\frac{1-p}{2})^{\frac{1}{p+1}} \text{ when } \beta(\tau)y^2 \leq 4(1-p)
\]
and
\[
v(y, \tau) \geq (2(1-p))^{\frac{1}{p+1}} \text{ when } \beta(\tau)y^2 = 4(1-p)
\]
\[
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\]
provided that $\beta(0)$ is sufficiently small.

For the region $\beta(\tau)y^2 \geq 4(1 - p)$, by the fact $p < 0$ we have that

$$H((2(1 - p))^{\frac{1}{1-p}}) \leq 0 \text{ and } H(v) = 0$$

where the map $H(g)$ is defined as

$$H(g) = g_t - g_{yy} + g^p + ay\partial_y g - \frac{2a}{1 - p}g.$$

Equations (45)-(48) and the assumption (43) enable us to use Lemma 7.1 on the equation for $(2(1 - p))^{\frac{1}{1-p}} - v$ to obtain

$$v(y, \tau) \geq g(y, \beta(\tau)) \text{ if } \beta(\tau)y^2 \geq 4(1 - p).$$

This together with the analysis on the region $\beta(\tau)y^2 \leq 4(1 - p)$ completes the proof.

8 Proof of Main Theorem 1.1

In the next lemma we show that restriction (8) on the initial conditions involving two parameters can be rescaled into a condition involving one parameter. Recall that $q = \min\{\frac{4}{1 - p}, \frac{2(2 - p)}{1 - p}\}$.

**Lemma 8.1.** Let $u_0$ satisfy the condition (8). Then there exist some scalars $k_0$, $\delta_1$, $\beta > 0$ such that

$$\|\langle k_0x \rangle^{-n}[k_0^{\frac{2}{p}}u_0(k_0x) - \left(\frac{1 - p + \beta x^2}{1 - \frac{2}{1-p}\beta}\right)^{\frac{1}{1-p}}]\|_{\infty} \leq \delta_1 \beta^\frac{2}{p},$$

$$k_0^{\frac{2}{p}}u_0(k_0x) \geq g(x, \beta)$$

for $n = 2, 3$, and if $p < -1$ (50) with $n = q$.

**Proof.** Define $k_0 := (2c_0 + \frac{2}{1-p}b_0)^{-1/2}$, $\delta_1 := \delta_0 k_0^{\frac{2}{p}}$ and $\beta := b_0 k_0^2$. It is straightforward to verify that the function $k_0^{\frac{2}{p}}u_0(k_0x)$ has all the properties above.

By this lemma in what follows we only study the case

$$\|\langle x \rangle^{-n}[u_0(x) - \left(\frac{1 - p + b_0 x^2}{1 - \frac{2}{1-p}b_0}\right)^{\frac{1}{1-p}}]\|_{\infty} \leq \delta_0 b_0^\frac{2}{p},$$

$$u_0(x) \geq g(x, b_0)$$

for $n = 2, 3$, and if $p < -1$, (51) with $n = q$.

By Proposition 2.1 there exists $\infty \geq t_* > 0$ such that Equation (12) has a unique solution $u(x, t)$ for $0 \leq t < t_*$, but has no solutions on a larger time interval. Moreover, if $t_* < \infty$, then $\|u(x, t)\|_{\infty} \to \infty$ as $t \to t_*$. Recall that the solutions $u(x, t)$, $v(y, \tau)$ and $w(y, \tau)$ and the corresponding initial conditions are
related by the scaling and gauge transformations (see (17) and (21)). Take \( \lambda(0) = 1 \). Then we have that 

\[
u_0(x) = v_0(y).
\]

Choose \( b_0 \) so that \( \delta_0 b_0^{3/2} \leq \frac{1}{\epsilon} \epsilon_0 \) with \( \delta_0 \) the same as in (51) and with \( \epsilon_0 \) given in Proposition 5.1. Then \( v_0 \in U_{\frac{1}{2} \epsilon_0} \), by the condition (51) with \( n = 3 \) on the initial conditions. By continuity there is a (maximal) time \( t_\# \leq t_\# \) such that \( v \in U_{\epsilon_0} \) for \( t < t_\# \). For this time interval Propositions 5.1 and 5.2 hold for \( v \) and in particular we have the splitting (32). Recall that we assume \( a = 2c - \frac{1}{2} \) in the decomposition (32). In particular, this implies that the initial condition can be written in the form

\[
v_0(y) = V_{a(0), b(0)}(y) + e^{-\frac{\|a(0)\|^2}{4}} \xi_0(y),
\]

where \((a(0), b(0)) = g(v_0)\) and \( \xi_0 \perp e^{-\frac{\|a(0)\|^2}{4}}, (1 - a(0)^2)e^{-\frac{\|a(0)\|^2}{4}}. \) Moreover \( M_n(0) \lesssim \delta_0, \ n = 1, 2 \) and especially \( q \) for \( p < -1 \).

Before starting proof, we note that the definition of \( g \) in Lemma 7.2 implies

\[
g(y, \tau) \geq \left( \frac{1 - p}{2} \right)^{\frac{1}{1 - p}}
\]

for any time \( \tau \).

By the conditions of Theorem 1.1 on the datum, we have that \( b(0) \) is small, \( M_1(0), M_2(0) \lesssim \delta_0 \) for some small \( \delta_0, A(0) \) and \( B(0) \) are bounded, and

\[
u_0 \in \langle x \rangle^{\frac{2}{1 - p}} L^\infty \text{ and } u_0 \geq \left( \frac{1 - p}{2} \right)^{\frac{1}{1 - p}}.
\]

Then by the local well posedness Theorem 2.1 and the splitting Proposition 5.2 there exists a time interval \([0, T]\) such that for any \( \tau \in [0, T] \),

\[
M_1(\tau) \leq \frac{1}{8} (1 - p)^{\frac{1}{1 - p}} - \frac{3}{2}, \quad B(\tau), \quad A(\tau) \leq \beta^{-1/4}(\tau)
\]

and \( v(y, \tau) \geq \frac{1}{2} \left( \frac{1 - p}{2} \right)^{\frac{1}{1 - p}}. \) Thus we have Equations (33)-(34) which together with the fact that \( \beta(0) \) is small imply that

\[
M_1(\tau), M_2(\tau), M_3(\tau) \lesssim \delta_0, \quad B(\tau), \quad A(\tau) \lesssim 1 \ll \beta^{-1/4}(\tau)
\]

if \( \tau \in [0, T] \). In the next we only show how to use (30)-(38) to get the estimate for \( A, B, M_1 \) and \( M_2 \), the proof of \( M_3 \) is similar. Indeed, since \( M_1(\tau) \leq 1 \), we can solve (36) for \( A(\tau) \). We substitute the result into Equations (37)-(38), and substitute the estimate for \( M_1(\tau) \) into the right hand side of (39), to obtain inequalities involving only the estimating functions \( M_1(\tau) \) and \( M_2(\tau) \). Consider the resulting inequality for \( M_2(\tau) \). The only terms on the r.h.s., which do not contain \( \beta(0) \) to a power at least 1/4 as a factor, are \( M_2^2(\tau) \) and \( M_1(\tau)M_2(\tau) \). Hence for \( M_2(0) \ll 1 \) this inequality implies that \( M_2(\tau) \ll M_2(0) + M_1(0) + \beta^\frac{1}{4}(0) \). Substituting this result into the inequality for \( M_1(\tau) \) we obtain that \( M_1(\tau) \ll M_1(0) + \beta^\frac{1}{4}(0) \) as well. The last two inequalities together with (35) and (36) imply the desired estimates on \( A(\tau) \) and \( B(\tau) \).

Moreover by Lemma 7.2 we obtain

\[
v(\tau, \tau) \in \langle y \rangle^{\frac{2}{1 - p}} L^\infty \text{ and } v(y, \tau) \geq g(y, \tau) \geq \left( \frac{1 - p}{2} \right)^{\frac{1}{1 - p}}.
\]
By using the procedure above recursively we have that
\[ M_1(\tau), \ M_2(\tau), M_q(\tau) \lesssim \delta_0, \ B(\tau), \ A(\tau) \lesssim 1 \]
for any \( \tau \).

By the definitions of \( A(\tau) \) and \( B(\tau) \) in (53) and the facts that \( A(\tau), B(\tau) \lesssim 1 \) proved above, we have
\[
a - \frac{1}{2} = - \frac{2b}{1 - p} + O(\beta^2), \quad b(\tau) = \beta(\tau) + O(\beta^{3/2}). \tag{53}
\]
Hence \( a - \frac{1}{2} = O(\beta(\tau)) \), where, recall the definition of \( \beta(\tau) \) from (34).

The facts \( \lambda(0) = 1 \) and \( a = -\lambda \frac{d}{dt} \lambda \) after Equation (18) yield
\[
\lambda(t) = [1 - 2 \int_0^t a(\tau(s))ds]^{1/2}. \tag{54}
\]

Since \( |a(\tau(t)) - \frac{1}{2}| = O(b(\tau(t))) \) there exists a time \( t^* \) such that \( 1 = 2 \int_{t^*}^t a(\tau(s))ds \), i.e. \( \lambda(t) \to 0 \) as \( t \to t^* \) with the latter defined in (33). For any time \( t \leq t^* \) (12) is well posed and can be split by the fact that \( M_1(\tau(t)), \ M_2(\tau(t)), M_q(\tau(t)) \lesssim \delta_0 \) and \( A(\tau(t)) \) and \( B(\tau(t)) \) are bounded. Obviously \( t^* \leq t_\ast \). Moreover by the definition of \( \tau \) and the property of \( a \) we have that \( \tau(t) \to \infty \) as \( t \to t^* \). Equation (53) implies \( b(\tau(t)) \to 0 \) and \( a(\tau(t)) \to \frac{1}{2} \) as \( t \to t^* \), where, recall the definition of \( \beta(\tau) \) from (34). By the analysis above and the definitions of \( a, \tau \) and \( \beta \) we have
\[
\lambda(t) = (t^* - t)^{1/2}(1 + o(1)), \quad \tau(t) = -\ln|t^* - t|(1 + o(1))
\]
with \( o(1) \to 0 \) as \( t \to t^* \) and consequently
\[
\beta(\tau(t)) = \frac{(p - 1)^2}{4p|\ln|t^* - t||} (1 + O(\frac{1}{|\ln|t^* - t||^{3/2}}))
\]

By (53) we have
\[
b(\tau(t)) = \frac{(p - 1)^2}{4p|\ln|t^* - t||} (1 + O(\frac{1}{|\ln|t^* - t||^{3/2}}))
\]
and
\[
a(\tau(t)) = \frac{1}{2} + \frac{1 - p}{2p|\ln|t^* - t||} (1 + O(\frac{1}{|\ln|t^* - t||})).
\]

If we define \( 2e(t) = a(\tau(t)) + \frac{1}{2} \) in Theorem 1.1 then \( c(t) \) has the same estimate as (7).

By the fact \( M_1 \ll 1 \) we have
\[
\|\langle y\rangle^{-3} e^{\frac{e\tau}{2}} \xi(\cdot, \tau(t))\|_\infty \ll \beta^{3/2}(\tau(t))
\]
thus
\[
u(0, t) = \lambda^{\frac{p-1}{2}}(t)[a_{\nu b} + e^{\frac{\tau}{2}} \xi(y, \tau)]|_{y=0} \leq \lambda^{\frac{p-1}{2}}(t)[\left(\frac{1-p}{a(\tau(t)) + \frac{1}{2}}\right)^{1/2} + C\beta^{3/2}(\tau(t))]
\]

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for some constant $C > 0$, hence $\frac{1}{u(t)} \to \infty$ as $t \to t^*$. Moreover by the facts $v(y, \tau) \geq (\frac{1-p}{2})^{1-p}$ proved above and $u(x,t) = \lambda^{\frac{1}{\tau_p}}(t)v(y,\tau)$ we have that for any $t < t^*$

$$\|\frac{1}{u(\cdot, t)}\|_\infty \leq \lambda^{\frac{1}{\tau_p}}(t)(\frac{2}{1-p})^{1-p} < \infty.$$ 

Those facts together with the estimates of $\lambda(t)$, $b(\tau(t))$ and $a(\tau(t))$ imply that $\frac{1}{u(t)}$ collapses at time $t^*$ and $t^* = t_*$. Thus we finish proving the first argument of Theorem 1.1.

Collecting the facts above we complete the proof.

9 Parametrization and the Linearized Operator

Substitute (52) into (18) we have the following equation for $\xi$

$$\frac{d}{d\tau} \xi(y, \tau) = -L(a,b)\xi + F(a,b) + N(a,b,\xi) \tag{55}$$

where the linear operator $L(a,b)$ is defined as

$$L(a,b) := -\partial^2_y + \frac{a^2 + a\tau}{4} y^2 - \frac{a}{2} \frac{2a}{1-p} + p \frac{1+a}{1-p + by^2},$$

and the function $F(a,b)$ is defined as

$$F(a,b) := \left( \frac{1-p+by^2}{a+\frac{1}{2}} \right)^{1-p} \frac{1}{1-p} \left[ \frac{a^2}{2} [\Gamma_1 + \Gamma_2] \frac{y^2}{1-p + by^2} + F_1 \right] \tag{56}$$

with

$$\Gamma_1 := \frac{a\tau}{a+\frac{1}{2}} + a - \frac{1}{2} + \frac{2b}{1-p},$$

$$\Gamma_2 := -b\tau - b( a - \frac{1}{2} + \frac{2b}{1-p}) + \frac{4p}{(p-1)^2} b^2,$$

$$F_1 := \frac{p}{(1-p)^2} \frac{4b^3 y^4}{(1-p+by^2)^2}; \tag{57}$$

and the nonlinear term $N$ is

$$N(a,b,\xi) := -v^p e^{-\frac{a\xi^2}{4}} + \left( \frac{a+\frac{1}{2}}{1-p+by^2} \right)^{1-p} e^{-\frac{a\xi^2}{4}} + p \frac{a+\frac{1}{2}}{1-p + by^2} \xi,$$

where, recall the definition of $v$ in (17).

In the next we derive various estimates for $\Gamma_1$, $\Gamma_2$ and $N(a,b,\xi)$. 

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Lemma 9.1. Suppose that $A(\tau), B(\tau) \leq \beta^{-1/4}(\tau)$, $v(y, \tau) \geq \frac{1}{2}(\frac{1-p}{2})^{1/\tau}$ and $b(0) = \beta(0)$ is small. Then for $0 > p \geq -1$ we have

$$|N(a, b, \xi)| \lesssim e^{-\frac{ax^2}{4}}(\frac{1}{1+\beta y^2})^{\frac{2}{1-p}}|e^{\frac{ax^2}{4}}\xi|^2;$$

(58)

and for $p < -1$

$$|N(a, b, \xi)| \lesssim e^{-\frac{ax^2}{4}}[(\frac{1}{1+\beta y^2})^{\frac{2}{1-p}}|e^{\frac{ax^2}{4}}\xi|^2|e^{\frac{ax^2}{4}}\xi|^2];$$

(59)

Proof. By direct computation and the assumption on $v$ we have

$$|N(a, b, \xi)| \lesssim e^{-\frac{ax^2}{4}}v^p||(1+\phi)^{-p} - 1 + p\phi|||\phi||((1+\phi)^{-p} - 1)|$$

and

$$|N(a, b, \xi)| \lesssim e^{-\frac{ax^2}{4}}v^p||((1+\phi)^{-p} - 1 + p\phi|||\phi||((1+\phi)^{-p} - 1)|$$

where the function $\phi$ is defined as $\phi := e^{\frac{ax^2}{4}}\xi V_{a,b}^{-1}$ with $V_{a,b} := (\frac{(1-p+by^2)}{1+a})^\frac{1}{1-p}.$

(A) When $|\phi| \geq 1/2$ we have that if $0 > p \geq -1$ then

$$1, |(1+\phi)^{-p}| \lesssim \phi^2, \text{ thus } |N(a, b, \xi)| \lesssim e^{-\frac{ax^2}{4}}\phi^2;$$

if $p < -1$ then

$$1, |(1+\phi)^{-p}|, |\phi| \lesssim |\phi|^2-p, \text{ thus } |N(a, b, \xi)| \lesssim e^{-\frac{ax^2}{4}}|\phi|^2-p;$$

(B) when $|\phi| < 1/2$ by the remainder estimate of the Tylor expansion we have

$$(1+\phi)^{-p} - 1 + p\phi, |\phi||((1+\phi)^{-p} - 1| \lesssim \phi^2.$$ Collecting the estimate above we have that if $0 > p \geq -1$ then

$$|N(a, b, \xi)| \lesssim e^{-\frac{ax^2}{4}}|\phi|^2;$$

if $p < -1$ then

$$|N(a, b, \xi)| \lesssim e^{-\frac{ax^2}{4}}(|\phi|^2 + |\phi|^2-p).$$

Moreover the assumptions $A(\tau), B(\tau) \leq \beta^{-1/4}(\tau)$ imply that $b = \beta + o(\beta)$ and $a = \frac{1}{2} + O(\beta)$, thus $V_{a,b} \gtrsim (1 + \beta(\tau)y^2)^\frac{1}{1-p}$, i.e.

$$|\phi| \lesssim (\frac{1}{1+\beta y^2})^{\frac{1}{1-p}}e^{\frac{ax^2}{4}}\xi$$

which together with the definition of $\phi$ and the estimates of $N$ above implies (58). \qed

In the following proposition we establish some estimates for $\Gamma_1$ and $\Gamma_2$.

Proposition 9.2. Suppose that $M_1(\tau) \leq \frac{1}{2}(1-p)^{\frac{1}{1-p}-\frac{3}{2}}$, $A(\tau), B(\tau) \leq \beta^{-1/4}(\tau)$ and $v(y, \tau) \geq \frac{1}{2}(\frac{1-p}{2})^{1/\tau}$ and $b(0) = \beta(0)$ is small. Then we have

$$|\Gamma_1|, |\Gamma_2| \lesssim \beta^{5/2}(1 + M_1^{2-p} + AM_1).$$

(60)
Proof. Taking inner products on (55) with the functions \( \phi_{0,a} := (\frac{a}{2\pi})^{\frac{1}{2}} e^{-\frac{ay^2}{2}} \) and \( \phi_{2,a} := (\frac{a}{2\pi})^{\frac{1}{2}} (ay^2 - 1)e^{-\frac{ay^2}{2}} \) and using the facts \( \xi \perp \phi_{0,a}, \phi_{2,a} \) in (52), we have

\[
|\langle F(a,b), \phi_{0,a} \rangle| \leq G_1, \quad |\langle F(a,b), \phi_{2,a} \rangle| \leq G_2
\]

(61)

where the functions \( G_1, G_2 \) are defined as

\[
G_1 := |\langle by^2 \xi, \phi_{0,a} \rangle| + |\langle a_\tau y^2 \xi, \phi_{0,a} \rangle| + |\langle N(a,b,\xi), \phi_{0,a} \rangle|,
\]

and

\[
G_2 := |\langle by^2 \xi, \phi_{2,a} \rangle| + |\langle a_\tau y^2 \xi, \phi_{2,a} \rangle| + |\langle N(a,b,\xi), \phi_{2,a} \rangle|.
\]

The estimate of \( N \) in (59), the assumption on \( B(\tau) \) and the definition of \( M_1 \) yield

\[
G_1, G_2 \lesssim \beta^{5/2}[M_1 + M_1^2 + M_1^{2-p}] + |a_\tau|\beta^{3/2}M_1.
\]

Now we study the right hand sides of (61). We rewrite the function \( F(a,b) \) in (56) as

\[
F(a,b) = \chi(a,b)[\Gamma_1 + \Gamma_2 \frac{1}{a[1-p+by^2]} + \Gamma_3 \frac{ay^2 - 1}{a[1-p+by^2]} + F_1].
\]

where, recall the definition of \( F_1 \) in (57), and the term \( \chi(a,b) \) is defined as

\[
\chi(a,b) := (\frac{1-p+by^2}{a + \frac{1}{2}})^{\frac{1}{2}} \frac{1}{1-p} e^{-\frac{ay^2}{2}}.
\]

In the \( L^\infty \) space we expand \( F(a,b) \) as

\[
F(a,b) = (\frac{1}{1-p})^{\frac{1}{2}} (a + \frac{1}{2})^{\frac{1}{2}} e^{-\frac{ay^2}{2}} [\Gamma_1 + \frac{1}{a(1-p)} \Gamma_2 + \frac{ay^2 - 1}{a(1-p)} \Gamma_3] + O(b^3 + |b||\Gamma_1| + |\Gamma_2|)
\]

where we use the observation \( \|\chi(a,b)F_1\|_\infty = O(b^3) \) and the fact \( a \geq \frac{1}{4} \) implied by the assumption on \( A \).

By using the fact that \( \phi_{0,a} \perp \phi_{2,a} \) we have

\[
|\langle F(a,b), \phi_{0,a} \rangle| \gtrsim |\Gamma_1 + \frac{1}{a(1-p)} \Gamma_2| - |b(|\Gamma_1| + |\Gamma_2|) - |b|^3
\]

and

\[
|\langle F(a,b), \phi_{2,a} \rangle| \gtrsim |\Gamma_2| - |b(|\Gamma_1| + |\Gamma_2|) - |b|^3,
\]

which together with (61) and the assumption on \( B \) implies that

\[
|\Gamma_1|, |\Gamma_2| \lesssim \beta^{5/2}(1 + M_1^2 + M_1^{2-p}) + |a_\tau|\beta^2 M_1.
\]

(62)

By the definitions of \( \Gamma_1 \) and \( A, a_\tau \) has the bound

\[
|a_\tau| \lesssim |\Gamma_1| + \beta^{3/2}(\tau)A(\tau).
\]

Consequently after using \( M_1(\tau) \leq \frac{1}{(1-p)^{1/2}} \) and some manipulation on (62) we obtain

\[
|\Gamma_1|, |\Gamma_2| \lesssim \beta^{5/2}(1 + M_1^{2-p} + \beta^{1/2}AM_1).
\]

The proof is complete. □
Recall that $q := \min\left\{ \frac{4}{1-p}, \frac{2(2-p)}{1-p} \right\}$. The estimates on $\Gamma_1$ and $\Gamma_2$ imply

**Corollary 9.3.**

\[ \| \langle y \rangle^{-n} e^{\frac{ay^2}{2}} F \|_{\infty} \lesssim \beta^{\frac{n}{2} + 1} [1 + M_1^{2-p} + AM_1] \] (63)

with $n = 2, 3$ and especially $n = q$ if $p < -1$.

**Proof.** Recall that $p < 0$ and the definition of $F$ in Equation (56) as

\[ e^{\frac{ay^2}{2}} F(a, b) = \left( \frac{1 - p + by^2}{a + \frac{1}{2}} \right)^{\frac{1}{1-p}} \frac{1}{1-p} [\Gamma_1 + \Gamma_2 \frac{y^2}{1-p+by^2}] + F_1]. \]

The estimates on $\Gamma_1$ and $\Gamma_2$ in (60) implies that

\[ \| \langle y \rangle^{-n} (\frac{1 - p + by^2}{a + \frac{1}{2}})^{\frac{1}{1-p}} [\Gamma_1 + \Gamma_2 \frac{y^2}{1-p+by^2}] \|_{\infty} \lesssim |\Gamma_1| + |\Gamma_2| \lesssim \beta^{5/2} (1 + M_1^{2-p} + AM_1) \]

with $n = 2, 3$. Moreover for $p < -1$ we have that $1 \geq q > \frac{2}{1-p}$ hence

\[ \frac{|y|^{2-q}}{(1-p+by^2)^{1+\frac{1}{1-p}}} \leq \frac{|y|^{2-q}}{(1-p+by^2)^{\frac{1}{1-p}}} \leq b^\frac{1}{2}. \]

consequently

\[ \| \langle y \rangle^{-q} (\frac{1 - p + by^2}{a + \frac{1}{2}})^{\frac{1}{1-p}} [\Gamma_1 + \Gamma_2 \frac{y^2}{1-p+by^2}] \|_{\infty} \lesssim \beta^{\frac{5}{2} + 1} (1 + M_1^{2-p} + AM_1) \]

For the term $F_1$ by straightforward computation we have

\[ \| \langle y \rangle^{-2} F_1 (\frac{1 - p + by^2}{a + \frac{1}{2}})^{\frac{1}{1-p}} \|_{\infty} \lesssim \beta^2, \quad \| \langle y \rangle^{-3} F_1 (\frac{1 - p + by^2}{a + \frac{1}{2}})^{\frac{1}{1-p}} \|_{\infty} \lesssim \beta^{5/2} (\tau), \]

using the fact $1 \geq q > \frac{2}{1-p}$ for $p < -1$ again

\[ \| \langle y \rangle^{-q} F_1 (\frac{1 - p + by^2}{a + \frac{1}{2}})^{\frac{1}{1-p}} \|_{\infty} \lesssim \beta^{1+\frac{1}{2}}. \]

Collecting the estimates above we complete the proof. \Box

**10 Proof of Estimates (35)-(36)**

**Lemma 10.1.** If $B(\tau) \leq \beta^{-1/4}(\tau)$ for $\tau \in [0, T]$, then Equation (35) holds.
Proof. We rewrite the estimate of $\Gamma_2$ in Equation (60) as

$$|b_\tau - \frac{4p}{(p-1)^2} b^2| \lesssim b|\frac{1}{2} - a - \frac{2b}{p-1}| + \beta^{5/2}(1 + M_1^{2-p} + AM_1).$$

The first term on the right hand side is bounded by $b\beta A \lesssim \beta^{5/2} A$ by the definition of $A$, consequently

$$|b_\tau - \frac{4p}{(p-1)^2} b^2| \lesssim \beta^{5/2}(1 + M_1 A + M_2^{2-p} + M_2^2).$$

We divide both sides by $b^2$ and use the inequality $b \lesssim \beta$ implied by the assumption on $B$ to obtain the estimate

$$\left|\partial_\tau \frac{1}{b} + \frac{4p}{(p-1)^2} \right| \lesssim \beta^{1/2}(1 + M_1 A + M_2^{2-p} + A). \tag{64}$$

Since $\beta$ is a solution to $\partial_\tau \beta^{-1} + 4p(p-1)^{-2} = 0$, $\beta(0) = b(0)$, Equation (64) implies that

$$|\partial_\tau (\frac{1}{b} - \frac{1}{\beta})| \lesssim \beta^{1/2}(1 + M_1 A + M_2^{2-p} + A).$$

Integrating this equation over $[0, \tau]$, multiplying the result by $\beta^{1/2}$ and using that $b \lesssim \beta$ give the estimate

$$\beta^{-\frac{3}{2}}|\beta - b| \lesssim \beta^{\frac{1}{2}} \int_0^\tau \beta^{1/2}(1 + M_1 A + M_2^{2-p} + A) ds.$$

By the definitions of $\beta$ we have $\beta^{1/2}(\tau) \int_0^\tau \beta^{1/2}(s)(1 + M_1 A + M_2^{2-p} + A) ds \lesssim 1$ which together with the definition of $B$ gives (65).

Lemma 10.2. If $A(\tau), B(\tau) \leq \beta^{-1/4}(\tau)$ for $\tau \in [0, T]$, then Equation (60) holds.

Proof. Define the quantity $\Gamma := a - \frac{1}{2} + \frac{2}{p-1} b$. We prove the proposition by integrating a differential inequality. Differentiating $\Gamma$ with respect to $\tau$ and substituting for $b_\tau$ and $a_\tau$ in Equation (60) we obtain

$$\partial_\tau \Gamma + (a + \frac{1}{2} - \frac{2}{p-1} b) \Gamma = -\frac{8p}{(p-1)^3} b^2 + R_b,$$

where $R_b$ has the bound

$$|R_b| \leq \beta^{5/2}(1 + M_1^{2-p} + AM_1).$$

Let $\mu = \exp \int_0^\tau a(s) + \frac{1}{2} - \frac{2}{p-1} b(s) ds$. Then the above equation implies that

$$\mu \Gamma - \Gamma(0) = -\frac{8p}{(p-1)^3} \int_0^\tau \mu b^2 ds + \int_0^\tau \mu R_b ds.$$ 

We now use the inequality $b \lesssim \beta$ and the estimate of $R_b$ to simplify the bound of $\Gamma$ as

$$|\Gamma| \lesssim \mu^{-1} \Gamma(0) + \mu^{-1} \int_0^\tau \mu b^2 ds + \mu^{-1} \int_0^\tau \mu \beta^{5/2}(1 + M_1^{2-p} + AM_1) ds.$$
For our purpose, it is sufficient to use the less sharp inequality

\[ |\Gamma| \lesssim \mu^{-1} \Gamma(0) + \mu^{-1} \int_0^t \mu \beta^2 ds (1 + \beta^{3/2} (0) [M_1^{2-p} + AM_1]). \]

The assumption that \( A(\tau), B(\tau) \leq \beta^{-1/4}(\tau) \) implies that \( a + \frac{1}{2} - \frac{2}{p-1} b \geq \frac{1}{2} \). Thus it is not difficult to show that \( \beta^{-2} \mu^{-1} \Gamma(0) \leq A(0) \); and by the slow decay of \( \beta \) we have that if \( s \leq \tau \) then \( e^{-\frac{\tau-s}{\lambda}} \beta^2(s) \lesssim \beta^2(\tau), \)

consequently

\[ \beta^{-2}(\tau) \mu^{-1}(\tau) \int_0^\tau \mu(s) \beta^2(s) ds \leq \beta^{-2}(\tau) \int_0^\tau e^{-\frac{s-\tau}{\lambda}} \beta^2(s) ds \lesssim \int_0^\tau e^{-\frac{s-\tau}{\lambda}} ds \lesssim 1. \]

Collecting the estimates above we have

\[ \beta^{-2} |\Gamma| \lesssim 1 + A(0) + \beta^{1/2}(0) (1 + M_1^{2-p} + AM_1) \]

which together with the definition of \( A \) implies (56). Thus the proof is complete. \( \square \)

### 11 Rescaling of Fluctuations on a Fixed Time Interval

We return to our key equation (53). In this section we re-parameterize the unknown function \( \xi(y, \tau) \) in such a way that the \( y^2 \)-term in the linear part of the new equation has a time-independent coefficient (cf [3]).

Let \( \tau(t) \) be the inverse function to \( \tau(t) \), where \( \tau(t) = \int_0^t \lambda^{-2}(s) ds \) for any \( \tau \geq 0 \). Pick \( T > 0 \) and approximate \( \lambda(t(\tau)) \) on the interval \( 0 \leq \tau \leq T \) by the new trajectory, \( \lambda_1(t(\tau)) \), tangent to \( \lambda(\tau) \) at the point \( \tau = T : \lambda_1(t(T)) = \lambda(t(T)) \), and \( \alpha := -\lambda_1(t(\tau)) \partial_1 \lambda_1(t(\tau)) = a(T) \) where, recall \( a(\tau) := -\lambda(t(\tau)) \partial_1 \lambda(t(\tau)) \). Now we introduce the new independent variables \( z \) and \( \sigma \) as \( z(x, t) := \lambda_1^{-1}(t)x \) and \( \sigma(t) := \int_0^t \lambda_1^{-2}(s) ds \) and the new unknown function \( \eta(z, \sigma) \) as

\[ \lambda_1^{\frac{1}{2}}(t)e^{\frac{z^2}{2}} \eta(z, \sigma) := \lambda^{\frac{1}{2}}(t)e^{\frac{z^2}{2}} \xi(y, \tau). \]

In this relation one has to think of the variables \( z \) and \( y, \sigma, \tau \) and \( t \) as related by \( z = \frac{\lambda(t)}{\lambda_1(t)} y, \sigma(t) := \int_0^t \lambda_1^{-2}(s) ds \) and \( \tau = \int_0^t \lambda^{-2}(s) ds \), and moreover \( a(\tau) = -\lambda(t(\tau)) \partial_1 \lambda(t(\tau)) \) and \( \alpha = a(T) \).

For any \( \tau = \int_0^{t(\tau)} \lambda^{-2}(s) ds \) with \( t(\tau) \leq t(T) \) (or equivalently \( \tau \leq T \)) we define a new function \( \sigma(\tau) := \int_0^{t(\tau)} \lambda_1^{-2}(s) ds \). Observe the function \( \sigma \) is invertible, we denote by \( \tau(\sigma) \) as its inverse. Especially we define

\[ S := \int_0^{t(T)} \lambda_1^{-2}(s) ds. \]

The new function \( \eta \) satisfies the equation

\[ \frac{d}{d\sigma} \eta(\sigma) = -\mathcal{L}_{\alpha, \beta} \eta(\sigma) + \mathcal{W} \eta(\sigma) + \mathcal{F}(a, b)(\sigma) + \mathcal{N}_1(a, b, \alpha, \eta)(\sigma), \]

with the operators

\[ \mathcal{L}_{\alpha, \beta} := L_\alpha + V, \]

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\[ L_\alpha := -\frac{d^2}{dz^2} + \frac{\alpha^2}{4}z^2 - \frac{1}{2}\alpha - \frac{2\alpha}{1 - p}, \]

\[ V := \frac{2p\alpha}{1 - p + \beta(z)^2}, \]

\[ W := \frac{p\lambda^2}{\lambda_1^2(1 - p + b(z)^2)} - \frac{p}{1 - p + \beta(z)^2}, \]

the function

\[ F(a, b) := e^{\frac{\alpha}{\lambda_1}} e^{-\frac{\alpha^2}{\lambda_1} - \frac{\beta}{\lambda_1} - \frac{p}{\lambda_1} - \frac{1}{\lambda_1}} F(a, b), \]

the nonlinear term

\[ N_1(a(\tau(\sigma)), b(\tau(\sigma)), \alpha, \eta(\sigma)) := \frac{\lambda_1}{\lambda}(t(\tau(\sigma))) - 1 \]

where, recall the definitions of \( F \) and \( N \) after (56).

In the next we prove that the new trajectory is a good approximation of the old one.

**Proposition 11.1.** For any \( \tau \leq T \) we have that if \( A(\tau) \leq \beta^{-1/4}(\tau) \) then

\[ |\frac{\lambda}{\lambda_1}(t(\tau)) - 1| \lesssim \beta(\tau) \quad (69) \]

for some constant \( c \) independent of \( \tau \).

**Proof.** By the properties of \( \lambda \) and \( \lambda_1 \) we have

\[ \frac{d}{d\tau} \left| \frac{\lambda}{\lambda_1}(t(\tau)) - 1 \right| = 2\alpha(\frac{\lambda}{\lambda_1}(t(\tau)) - 1) + G(\tau) \quad (70) \]

with

\[ G := |a - \alpha - a + (\alpha - a)(\frac{\lambda}{\lambda_1})^2 + \frac{\lambda}{\lambda_1} + 1| + a(\frac{\lambda}{\lambda_1} - 1)^2 [\frac{\lambda}{\lambda_1} + 2]. \]

By the definition of \( A(\tau) \) we have that in the time interval \( \tau \in [0, T] \), if \( A(\tau) \leq \beta^{-1/4}(\tau) \) then

\[ |a(\tau) - a|, |a(\tau) - \frac{1}{2}| \lesssim \beta(\tau). \quad (71) \]

Thus

\[ |G| \lesssim \beta + (\frac{\lambda}{\lambda_1} - 1)^2 + |\frac{\lambda}{\lambda_1} - 1|^3 + \beta|\frac{\lambda}{\lambda_1} - 1|. \quad (72) \]

Observe that \( \frac{\lambda}{\lambda_1}(t(\tau)) - 1 = 0 \) when \( \tau = T \). Thus Equations (70) can be rewritten as

\[ \frac{\lambda}{\lambda_1}(t(\tau)) - 1 = -\int_T^\tau e^{\int_s^\tau 2a(\tau)dt} G(s)ds. \quad (73) \]

We claim that Equations (71) and (72) imply (69). Indeed, define an estimating function \( \Lambda(\tau) \) as

\[ \Lambda(\tau) := \sup_{\tau \leq s \leq T} \beta^{-1}(s)|\frac{\lambda}{\lambda_1}(t(s)) - 1|. \]
Then (73) and the assumption $A(\tau), B(\tau) \leq \beta^{-1/4}(\tau)$ implies
\[
\left| \frac{\Lambda}{\Lambda_1}(t(\tau)) - 1 \right| \lesssim \int_{\tau}^{T} e^{-\frac{1}{2}(T-\tau)} [\beta(s) + \beta^2(s)\Lambda^2(\tau) + \beta^3(s)\Lambda(\tau)] ds
\lesssim \beta(\tau) + \beta^2(\tau)\Lambda^2(\tau) + \beta^3(\tau)\Lambda^3(\tau) + \beta(\tau)\Lambda(\tau),
\]
or equivalently
\[
\beta^{-1}(\tau)\left| \frac{\Lambda}{\Lambda_1}(t(\tau)) - 1 \right| \lesssim 1 + \beta(\tau)\Lambda^2(\tau) + \beta^2(\tau)\Lambda^3(\tau) + \beta(\tau)\Lambda(\tau).
\]
Consequently by the fact that $\beta(\tau)$ and $\Lambda(\tau)$ are decreasing functions we have
\[
\Lambda(\tau) \lesssim 1 + \beta(\tau)\Lambda^2(\tau) + \beta^2(\tau)\Lambda^3(\tau) + \beta(\tau)\Lambda(\tau)
\]
which together with $\Lambda(T) = 0$ implies $\Lambda(\tau) \lesssim 1$ for any time $\tau \in [0, T]$. Thus we prove the claim by the definition of $\Lambda(\tau)$.

**Lemma 11.2.** For any $c_1, c_2 > 0$ there exists a constant $c(c_1, c_2)$ such that
\[
\int_{0}^{S} e^{-c_1(S-s)\beta^{c_2}(\tau(s))} ds \leq c(c_1, c_2)\beta^{c_2}(T). \tag{74}
\]

**Proof.** By the definition of $\tau(\sigma)$ we have that $\sigma = \int_{0}^{\tau(\sigma)} \lambda^{-2}(k) dk$ and $\tau = \int_{0}^{\tau} \lambda^{-2}(k) dk$. By Proposition 11.1 we have that $\frac{1}{2} \leq \frac{\Lambda}{\Lambda_1} \leq 2$, thus
\[
\frac{1}{4} \sigma \geq \tau(\sigma) \geq 4\sigma
\]
which implies $\frac{1}{\tau(\sigma)} \leq \frac{4}{\tau(\sigma)}$. Moreover this yields
\[
\int_{0}^{S} e^{-c_1(S-s)\beta^{c_2}(\tau(s))} ds \leq c(c_1, c_2)\frac{1}{(\frac{1}{\tau(\sigma)} + S)^{c_2}}. \tag{76}
\]
Using (76) again to get $4S \geq \tau(S) = T \geq \frac{1}{2}S$ which together with (76) implies that
\[
\int_{0}^{S} e^{-c_1(S-s)\beta^{c_2}(\tau(s))} ds \leq c(c_1, c_2)\frac{1}{(\frac{1}{\tau(\sigma)} + T)^{c_2}} \lesssim c(c_1, c_2)\beta^{c_2}(T).
\]
Hence
\[
\int_{0}^{S} e^{-c_1(S-s)\beta^{c_2}(\tau(s))} ds \leq c(c_1, c_2)\beta^{c_2}(T)
\]
which is (74). \qed
12 Propagator Estimates

In the next we present the decay estimates of the propagators generated by \(-L_\alpha\) and \(-L_{\alpha,\beta}\) which play essential roles in proving the estimates for \(M_1, M_2\) and \(M_3\) below.

We start with analyzing the spectrum of the linearized operator \(L_{\alpha,\beta}\) and \(L_\alpha\). Due to the quadratic term \(\frac{1}{4}\alpha^2 z^2\), the operators have discrete spectrum. Then \(L_\alpha + \frac{2\alpha^2}{1-p}\) and \(L_\alpha\) approximate \(L_{\alpha,\beta}\) near zero and at infinity respectively. The spectrum of the operator \(L_\alpha\) is

\[
\sigma(L_\alpha) = \left\{ n\alpha - \frac{2\alpha}{1-p} \mid n = 0, 1, 2, \ldots \right\}.
\]  

(77)

The first three normalized eigenvectors of \(L_\alpha\), which are used below, are

\[
\phi_{0\alpha} := \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha}{4} z^2}, \quad \phi_{1\alpha} := \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{4}} \sqrt{\alpha} z e^{-\frac{\alpha}{4} z^2}, \quad \phi_{2\alpha} := \left(\frac{\alpha}{8\pi}\right)^{\frac{1}{4}} (1 - \alpha z^2) e^{-\frac{\alpha}{4} z^2}.
\]  

(78)

Denote the integral kernel of \(e^{-\frac{\alpha^2}{4} z^2} e^{-L_\alpha} e^{-\frac{\alpha^2}{4} z^2}\) by \(U_0(x, y)\). By a standard formula (see [25]) we have

\[
e^{-L_\alpha} (x, y) = 4\pi (1 - e^{-2\alpha \sigma})^{-1/2} \sqrt{\alpha e^{-\alpha \sigma}} e^{-\frac{\alpha}{2(1 - e^{-2\alpha \sigma})}}.
\]

In what follows we need the following result.

**Lemma 12.1.** For any function \(g\) and \(\sigma > 0\) we have that

\[
\| \langle z \rangle^{-n} e^{-\frac{\alpha z^2}{4}} e^{-L_\alpha} g \|_\infty \lesssim e^{\frac{2\alpha}{1-p} \sigma} \| \langle z \rangle^{-n} e^{-\frac{\alpha z^2}{4}} g \|_\infty, \quad n = 0, 1, 2,
\]  

(79)

or equivalently

\[
e^{-\frac{\alpha z^2}{4}} \int (x)^{-2} U_0(x, y) e^{-\frac{\alpha}{4} y^2} \langle y \rangle^2 \, dy \lesssim e^{\frac{2\alpha}{1-p} \sigma}.
\]  

(80)

**Proof.** Note that the first three eigenvectors of \(L_\alpha\) are \(e^{-\frac{\alpha z^2}{4}}, \ z e^{-\frac{\alpha z^2}{4}}, (\alpha z^2 - 1)e^{-\frac{\alpha z^2}{4}}\) with the eigenvalues \(-\frac{2\alpha}{1-p}, -\frac{2\alpha}{1-p} + \alpha, -\frac{2\alpha}{1-p}\) (see (77)). Using that the integral kernel of \(e^{-\sigma L_\alpha}\) is positive and therefore \(\|e^{-\sigma L_\alpha} g\|_\infty \leq \|f^{-1} g\|_\infty \|e^{-\sigma L_\alpha} f\|_\infty\) for any \(f > 0\) and using that \(e^{-\sigma L_\alpha} e^{-\frac{\alpha z^2}{4}} = e^{2\alpha \sigma} e^{-\frac{\alpha z^2}{4}}\) and \(e^{-\sigma L_\alpha} (\alpha z^2 - 1)e^{-\frac{\alpha z^2}{4}} = e^{2\alpha \sigma} (\alpha z^2 - 1)e^{-\frac{\alpha z^2}{4}}\), we find that

\[
\|\langle z \rangle^{-2} e^{-\frac{\alpha z^2}{4}} e^{-\sigma L_\alpha} g\|_\infty \leq \|\langle z \rangle^{-2} e^{-\frac{\alpha z^2}{4}} e^{-\sigma L_\alpha} e^{-\frac{\alpha z^2}{4}} (z^2 + 1)\|_\infty \|\langle z \rangle^{-2} e^{-\frac{\alpha z^2}{4}} g\|_\infty
\]  

\[
= \|\langle z \rangle^{-2} [e^{2\alpha \sigma} \frac{1}{\alpha} + e^{2\alpha \sigma} (z^2 - 1)]\|_\infty \|\langle z \rangle^{-2} e^{-\frac{\alpha z^2}{4}} g\|_\infty
\]  

\[
\leq 2(\frac{1}{\alpha} + 1)e^{2\alpha \sigma} \|\langle z \rangle^{-2} e^{-\frac{\alpha z^2}{4}} g\|_\infty
\]

and

\[
\|e^{-\frac{\alpha z^2}{4}} e^{-\sigma L_\alpha} g\|_\infty \leq \|e^{-\frac{\alpha z^2}{4}} e^{-\sigma L_\alpha} e^{-\frac{\alpha z^2}{4}} \|_\infty \|e^{-\frac{\alpha z^2}{4}} g\|_\infty
\]  

\[
= e^{2\alpha \sigma} \|e^{-\frac{\alpha z^2}{4}} g\|_\infty
\]

which are the case \(n = 0, 2\) of (79). The case \(n = 1\) follows from the interpolation between \(n = 0\) and \(n = 2\). Moreover this together with the definition of \(U_0(x, y)\) after (78) and setting \(g = \langle z \rangle^2 e^{-\frac{\alpha z^2}{4}}\) implies (80).  

\(\square\)
We define $P_n^\alpha = 1 - P_n^\alpha$ and define $P_n^\alpha$, $n = 1, 2, 3$, as the projection onto the space spanned by the first $n$ eigenvectors of $L_\alpha$ with the form

$$
\overline{P_n^\alpha} = \sum_{m=0}^{n-1} |\phi_{m,\alpha}\rangle \langle \phi_{m,\alpha}|,
$$

where, recall the definitions of $\phi_{m,\alpha}$ in (78).

**Proposition 12.2.** Let $P_n^\alpha$ be the projection defined above. Then for any function $g$ and time $\sigma \geq 0$ we have

$$
\|\langle \tau \rangle^{-2} e^{-L_\alpha \sigma} P_2^\alpha g\|_\infty \lesssim e^{\frac{2\alpha^2}{4}\sigma} \|\langle \tau \rangle^{-2} e^{-\frac{\alpha^2}{4}} g\|_\infty,
$$

(82)

$$
\|\langle \tau \rangle^{-k} e^{-\frac{\alpha^2}{4}} e^{-L_\alpha \sigma} P_1^\alpha g\|_\infty \lesssim e^{\left(\frac{1}{2} - k\right)\sigma} \|\langle \tau \rangle^{-k} e^{-\frac{\alpha^2}{4}} g\|_\infty
$$

(83)

for any $k \in [0,1]$; and there exists constant $c_0 > 0$ such that for any $\tau \geq \sigma \geq 0$

$$
\|\langle \tau \rangle^{-2} e^{-\frac{\alpha^2}{8}} P_3^\alpha U(\tau, \sigma) P_3^\alpha g\|_\infty \lesssim e^{-c_0(\tau - \sigma)} \|\langle \tau \rangle^{-2} e^{-\frac{\alpha^2}{4}} g\|_\infty
$$

(84)

where $U(\tau, \sigma)$ denotes the propagator generated by the operator $-P_3^\alpha L_{\alpha,\beta} P_3^\alpha$.

**Proof.** (84) is proved in [3].

Now we prove (82). Define a new function $f := e^{-\frac{\alpha^2}{4}} P_2^\alpha g$. The definition of $U_0(x, y)$ after (78) implies

$$
e^{-L_\alpha \sigma} P_2^\alpha g = \int e^{\frac{\alpha^2}{4}} U_0(x, y) f(y) dy.
$$

(85)

Integrate by parts on the right hand side of (85) to obtain

$$
e^{-L_\alpha \sigma} P_2^\alpha g = e^{\frac{\alpha^2}{4}} \int \partial_y^2 U_0(x, y) f(-2)(y) dy
$$

(86)

where $f(-m-1)(x) := \int_{-\infty}^{x} f(-m)(y) dy$ and $f(0) := f$. Now we estimate the right hand side of Equation (86).

(A) By the facts that $f = e^{-\frac{\alpha^2}{4}} P_2^\alpha g$ and $P_2^\alpha g \perp y^0 e^{-\frac{\alpha^2}{4}}$, $n = 0, 1$, we have that $f \perp 1, y$. Therefore by integration by parts we have

$$
f(-m)(y) = \int_{-\infty}^{y} f(-m+1)(x) dx = -\int_{y}^{\infty} f(-m+1)(x) dx, \quad m = 1, 2
$$

which together with the definition of $f(-m)$ yields

$$
|f(-2)(y)| \lesssim e^{-\frac{\alpha^2}{8} y^2} \|\langle y \rangle^{-2} e^{-\frac{\alpha^2}{4}} P_2^\alpha g\|_\infty.
$$

(B) Using the explicit formula for $U_0(x, y)$ given above we find

$$
|\partial_y^2 U_0(x, y)| \lesssim \frac{e^{-2\alpha \sigma}}{(1 - e^{-2\alpha \sigma})^2 (|x| + |y|)^2} U_0(x, y).
$$
Collecting the estimates (A)-(B) above and using Equation (80), we have
\[
(x)^2 e^{-\frac{\alpha^2}{4}} |e^{-L_\alpha }P_3 g(x)| \\
\lesssim \frac{e^{-\frac{\alpha^2}{2}}}{1-e^{-\frac{\alpha^2}{2}}}|x|^{-2} e^{-\frac{\alpha^2}{4}} \int (|x| + |y|)^2 U_0(x,y) |f(-2)(y)| dy \\
\lesssim \frac{e^{-\frac{\alpha^2}{2}}}{1-e^{-\frac{\alpha^2}{2}}} e^{-\frac{\alpha^2}{4}} \int (x^2 - U_0(x,y) e^{-\frac{\alpha^2}{2}} |y| dy \|y\|^{-2} e^{-\frac{\alpha^2}{2}} P_3 g \|\infty \\
\lesssim \frac{e^{-\frac{\alpha^2}{2}}}{1-e^{-\frac{\alpha^2}{2}}} e^{-\frac{\alpha^2}{4}} \int (x^2 - U_0(x,y) e^{-\frac{\alpha^2}{2}} |y| dy \|y\|^{-2} e^{-\frac{\alpha^2}{2}} g \|\infty
\]
where in the last step we used the explicit form of $P_3$ in (31). This together with the estimate (80) gives the estimate (82) when $\sigma > 1$. If $\sigma \leq 1$ we use (79) to remove the singularity at $\sigma = 0$.

When $k = 1$ the proof of (83) is almost the same to the proof of (82) and, thus omitted; when $k = 0$ we have
\[
\|e^{a^2 \tau / 2} e^{-L_\alpha } P_1 g \|\infty \lesssim e^{a^2 \tau / 2 \sigma} \|e^{a^2 \tau / 2} P_1 g \|\infty \lesssim \|e^{a^2 \tau / 2} g \|\infty
\]
by using (79) and the observation that $\|P_1 g \|\infty \lesssim \|g \|\infty$. The general case follows from the interpolation between $k = 1$ and $k = 0$.

Thus the proof is complete.

\section{Estimate of $M_1$}

In this subsection we derive an estimate for $M_1$ in Equations (33).

Given any time $\tau$, choose $T = \tau$ and do the estimates as in Proposition 11.1. We start from estimating $\eta$ in Equation (67). Observe that the function $\eta$ is not orthogonal to the first three eigenvectors of the operator $L_\alpha$, defined in (67), thus we put projections on both sides of Equation (67) to get
\[
\frac{d}{d\sigma} P_3 \eta = -P_3 L_\alpha P_3 \eta + \sum_{n=1}^{4} P_3 D_n \tag{87}
\]
where the functions $D_n$, $n = 1, 2, 3, 4$, are defined as
\[
D_1 := -P_3 V \eta + P_3 V P_3 \eta, \quad D_2 := V \eta, \quad D_3 := F(a,b), \quad D_4 := N_1(a,b,\alpha,\eta),
\]
where, recall the definitions of the operators $F$, $N_1$ and the operators $V$, $V \eta$, after Equation (67).

Now we start with estimating the terms $D_n$, $n = 1, 2, 3, 4$, on the right hand side of (87).

\begin{lemma}
Suppose that $M_1(\sigma) \leq \frac{1}{2} (1 - p)^{\frac{\beta - \frac{1}{2}}{2}}$, $A(\tau)$, $B(\tau) \leq \beta^{-1/4}(\tau)$ and the function $v(y,\tau)$ defined in (17) satisfies the estimate $v(y,\tau) \geq \frac{1}{2} (1 - p)^{-\frac{1}{2}}$. Then we have
\[
\|\langle z \rangle^{-\frac{\alpha^2}{4}} D_1(\sigma) ||\| \lesssim \beta^2(\sigma) M_1(T), \tag{88}
\]
\[
\|\langle z \rangle^{-\frac{\alpha^2}{4}} D_2(\sigma) ||\| \lesssim \beta^{7/4}(\tau) M_1(T), \tag{89}
\]
\end{lemma}
In what follows we implicitly use (69) and the assumptions on $M_1, A, B$.

By the relation between $\xi$ and $\eta$ in (65) and the fact that $y = \frac{\lambda_1}{\lambda} z$, we have

$$
\|\langle z \rangle^{-3} e^{\frac{a_2}{2} T} D_3(\sigma) \|_\infty \lesssim \beta^{3/2}(\sigma(\tau)) [1 + M_1^{2-p}(T) + A(T)M_1(T)],
$$

if $-1 \leq p < 0$ then

$$
\|\langle z \rangle^{-3} e^{\frac{a_2}{2} T} D_4(\sigma) \|_\infty \lesssim \beta^{3/2}(\sigma(\tau)) M_1(T)M_2(T).
$$

if $p < -1$ then

$$
\|\langle z \rangle^{-3} e^{\frac{a_2}{2} T} D_4(\sigma) \|_\infty \lesssim \beta^{3/2}(\sigma(\tau)) M_1(T)[M_1^{1-p}(T) + M_2(T)]
$$

where, recall that $q = \min\{\frac{2}{1-p}, \frac{2(2-p)}{p-1}, 1\}$.

**Proof.** In what follows we implicitly use

$$
\frac{\lambda_1}{\lambda}(t(\tau)) - 1 = O(\beta(\tau)), \text{ thus } \frac{\lambda_1}{\lambda}(t(\tau)), \frac{\lambda}{\lambda_1}(t(\tau)) \leq 2
$$

implied by (69) and the assumptions on $M_1, A, B$.

If $n = 2, 3$ and $n = q$ for $p < -1$. By the definition of $M_1, M_2$ and the fact that $\tau(\sigma) \leq T$ we have

$$
\|\langle z \rangle^{-n} e^{\frac{a_2}{2} T} \eta(\sigma) \|_\infty \lesssim \beta^{3/2}(\sigma(\tau)) M_{k_2}(T)
$$

with $n = 2, 3$, especially $n = q$ for $p < -1$, and $k_2 = 2$, $k_3 = 3$, $k_q = q$. This proves (88) as a special case.

We rewrite $D_1 = P_3^\alpha D_1$ as

$$
P_3^\alpha D_1(\sigma) = -P_3^\alpha p \frac{\alpha + \frac{1}{2}}{p - 1 + b(\tau(\sigma))z^2} b(\tau(\sigma))z^2(1 - P_3^\alpha) \eta(\sigma)
$$

which admits the estimate

$$
\|\langle z \rangle^{-3} e^{\frac{a_2}{2} T} P_3^\alpha D_1(\sigma) \|_\infty \lesssim \|\langle z \rangle^{-1} b(\tau(\sigma))z^2 \|_\infty \|\langle z \rangle^{-3} e^{\frac{a_2}{2} T} (1 - P_3^\alpha) \eta(\sigma) \|_\infty \lesssim b^{1/2}(\tau(\sigma)) \|\langle z \rangle^{-3} e^{\frac{a_2}{2} T} \eta(\sigma) \|_\infty \lesssim \beta^{1/2}(\tau(\sigma)) \|\langle z \rangle^{-3} e^{\frac{a_2}{2} T} \eta(\sigma) \|_\infty
$$

where we use that $b(\tau) \lesssim \beta(\tau)$ implied by $B(\tau) \leq \beta^{-1/4}(\tau)$ and the fact that

$$
\|\langle z \rangle^{-2} e^{\frac{a_2}{2} T} (1 - P_3^\alpha) g \|_\infty \lesssim \|\langle z \rangle^{-3} e^{\frac{a_2}{2} T} g \|_\infty
$$

from the explicit form of $1 - P_3^\alpha$ in (81). This estimate together with (94) implies (88).

Now we prove (89). Recall that $y = \frac{\lambda_1}{\lambda} z$. After some manipulation on the expression of $W$ we have

$$
|D_2(\sigma)| \lesssim \left(\frac{\lambda}{\lambda_1} - 1\right) + |a(\tau(\sigma)) - \alpha| + \left|\frac{b(\tau(\sigma)) - \beta(\tau(\sigma))}{\beta(\tau(\sigma))}\right| |\eta(\sigma)|.
$$
Equations (69) and (71) imply that \(|\frac{1}{\beta} - 1| + |a(\tau(\sigma)) - \alpha| \lesssim \beta(\tau(\sigma))\); the assumption on \(B\) and its definition imply \(\frac{|\beta(\tau(\sigma)) - \beta(\tau(\sigma))|}{\beta(\tau(\sigma))} \lesssim \beta^{1/4}(\tau(\sigma))\). Consequently

\[
\|\langle z \rangle^{-3} e^{\frac{a\beta}{2}} D_2(\sigma)\| \lesssim \beta^{1/4}(\tau(\sigma))\|\langle z \rangle^{-3} e^{\frac{a\beta}{2}} \eta(\sigma)\|_{\infty}
\]

which together with (94) implies (91).

For (90), by the relation between \(D_4, F, F\) in (67), (68) and Equation (82) we have

\[
\|\langle z \rangle^{-3} e^{\frac{a\beta}{2}} D_3(\sigma)\|_{\infty} \lesssim \|\langle y \rangle^{-3} e^{\frac{a(\tau(\sigma))y^2}{4}} F(a(\tau(\sigma)), b(\tau(\sigma)))\|_{\infty}
\]

which together with the estimate of \(F(a, b)\) in (83) implies (90).

When \(0 > p \geq -1\), by the relation between \(D_4, N_1(a, b, \alpha, \eta)\) and \(N(a, b, \xi)\) in (67), (68) and the estimate (88) we have

\[
|e^{\frac{a\beta}{2}} D_4(\sigma)| \lesssim \left(\frac{1}{1 + \beta(\tau(\sigma))y^2}\right)^{\frac{3}{2}} e^{\frac{a\beta}{2}} |\xi|^2 \leq \frac{1}{\beta(\tau(\sigma))} (y^{-2} e^{\frac{a\beta}{2}} |\xi|^2,
\]

which together with (94) implies (91).

When \(p < -1\), by (69) and the definition of \(q\) we have

\[
|e^{\frac{a\beta}{2}} D_4(\sigma)| \lesssim \left(\frac{1}{1 + \beta(\tau(\sigma))}\right)^{\frac{3}{2}} e^{\frac{a\beta}{2}} |\xi|^2 - p \left(\frac{1}{1 + \beta(\tau(\sigma))}\right)^{\frac{3}{2}} e^{\frac{a\beta}{2}} |\xi|^2 \\
\lesssim \beta^{\frac{3}{2}} (y^{q(p-1)} e^{\frac{a\beta}{2}} |\xi|^2 - p + \beta^{-\frac{3}{2}} y^{-q} e^{\frac{a\beta}{2}} |\xi|^2)
\]

which together with (94) implies (92).

Hence the proof is complete. \(\square\)

### 13.1 Proof of Equations (37) and (39)

By Duhamel principle we rewrite Equation (87) as

\[
P_3^0 \eta(S) = U(S, 0) P_3^0 \eta(0) + \sum_{n=1}^{4} \int_0^S U(S, \sigma) P_3^0 D_n(\sigma) d\sigma,
\]

where, recall, \(U(t, s)\) is the propagator generated by the operator \(-P_3^0 L_{\alpha, \beta} P_3^0\). Use (83) to get

\[
\beta^{-3/2}(T) \|\langle z \rangle^{-3} e^{\frac{a\beta}{2}} P_3^0 \eta(S)\|_{\infty} \lesssim X_1 + X_2
\]

with

\[
X_1 := e^{-\epsilon_0 S} \beta^{-3/2}(T) \|\langle z \rangle^{-3} e^{\frac{a\beta}{2}} \eta(0)\|_{\infty},
\]

\[
X_2 := \beta^{-3/2}(T) \int_0^S e^{-\epsilon_0 (S - \sigma)} \sum_{n=1}^{4} \|\langle z \rangle^{-3} e^{\frac{a\beta}{2}} D_n(\sigma)\|_{\infty} d\sigma.
\]

Now we estimate each term on the right hand side.
(A) The slow decay of $\beta(\tau)$ implies $e^{-c_0 S} \beta^{-3/2}(T) \lesssim \beta^{-3/2}(0)$. Then we use Equation (94) to obtain

$$X_1 \lesssim \beta^{-3/2}(0) \|z^{-3} e^{\frac{ap^2}{4}} \eta(0)\|_\infty \lesssim M_1(0).$$

(98)

(B) By the integral estimate (74), the estimates of $D_n$, $n = 1, 2, 3, 4$, in (88)-(91) and the fact $\beta(\tau) \leq \beta(0)$ we obtain

$$X_2 \lesssim \beta^{-3/2}(T) \int_0^S e^{-c_0(S-\sigma)} \beta^{3/2}(\tau(\sigma)) d\sigma X^{(p)} \lesssim X^{(p)}$$

with

$$X^{(p)} := \begin{cases} \beta^{1/4}(0)[1 + M_2^{2-p}(T) + A(T)M_1(T)] + M_1(T)M_2(T), & \text{if } 0 > p \geq -1; \\ \beta^{1/4}(0)[1 + M_2^{2-p}(T) + A(T)M_1(T)] + M_1(T)[M_1^{2-p}(T) + M_2(T)], & \text{if } p < -1. \end{cases}$$

In Equation (93) we define $\lambda_1(t(T)) = \lambda(t(T))$, $\xi(\cdot,T) = \eta(\cdot,S)$ and $\alpha = a(T)$, thus $z = y$ and $P_3^\alpha \eta(S) = \xi(T)$, consequently

$$\|z^{-3} e^{\frac{ap^2}{4}} P_3^\alpha \eta(S)\|_\infty = \|y^{-3} e^{\frac{ap^2}{4}} \xi(T)\|_\infty$$

which together with Equations (97)-(99) implies

$$\beta^{-3/2}(T)\|y^{-3} e^{\frac{ap^2}{4}} \xi(T)\|_\infty \lesssim M_1(0) + X^{(p)} + \beta^{1/4}(0)[1 + M_1^{2-p}(T) + A(T)M_1(T)].$$

By the definition of $M_1$ in (89), we obtain

$$M_1(T) \lesssim M_1(0) + X^{(p)} + \beta^{1/4}(0)[1 + M_1^{2-p}(T) + A(T)M_1(T)].$$

This together with the fact that $T$ is arbitrary implies Equation (87) and (89).

□

14 Proof of Equations (38) and (40)

We rewrite Equation (67) as

$$\frac{d}{d\sigma} \eta(\sigma) = -L_\alpha \eta(\sigma) - V \eta + \sum_{n=2}^4 D_n$$

where, recall the definitions of the operators $L_\alpha$, $-V$ in (67), the definition of $D_n$, $n = 2, 3, 4$, in (87).

Lemma 14.1. Suppose that $M_1(\tau) \leq \frac{1}{8}(1-p)^{\frac{1-p}{2}}$, $A(\tau)$, $B(\tau) \leq \beta^{-1/4}(\tau)$ and the function $v(y, \tau)$ defined in (77) has the lower bound $v(y, \tau) \geq \frac{1}{2}(\frac{1-p}{2})^{\frac{1-p}{2}}$. Then we have

$$\|z^{-2} e^{\frac{ap^2}{4}} V \eta(\sigma)\|_\infty \lesssim \beta(\tau(\sigma)) M_1(T),$$

(101)

$$\|z^{-2} e^{\frac{ap^2}{4}} D_2(\sigma)\|_\infty \lesssim \beta^{5/4}(\tau(\sigma)) M_2(T),$$

(102)

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By the estimates of $V\eta$, $D\eta$, the proof is easier than that of Lemma 13.1, thus is omitted.

Rewrite (100) to have

$$P_2^\alpha\eta(S) = e^{-L_s}P_2^\alpha\eta(0) + \int_0^S e^{-L_s(S-s)}P_2^\alpha[-V\eta(s) + \sum_{n=2}^4 D_n]ds,$$

where, recall the definition of $S$ in (99). The propagator estimate of $e^{-L_s}P_2^\alpha$ in (52) yields

$$\beta^{-1}(T)\|z\|^{-2} e^{2\alpha^2} P_2^\alpha\eta(0)|_\infty \lesssim K_0 + K_1$$

where $K_n$'s are given by

$$K_0 := e^{2\alpha^2} \beta^{-1}(T)\|z\|^{-2} e^{2\alpha^2} \eta(0)|_\infty,$$

$$K_1 := \beta^{-1}(T) \int_0^S e^{2\alpha^2} (S-s)\|z\|^{-2} e^{2\alpha^2} V\eta(s)|_\infty + \sum_{n=2}^4 \|z\|^{-2} e^{2\alpha^2} D_n(s)|_\infty ds,$$

where, recall that $p < 0$.

In the next we estimate $K_n$'s, $n = 0, 1$.

(K0) First, $K_0$ has the bound

$$K_0 \lesssim \beta^{-1}(T)e^{2\alpha^2} \|z\|^{-2} e^{2\alpha^2} \eta(0)|_\infty$$

The slow decay of $\beta$ and Equation (94) yield

$$K_0 \lesssim \beta^{-1}(0)\|z\|^{-2} e^{2\alpha^2} \eta(0)|_\infty \lesssim M_2(0).$$

(K1) By the estimates of $V\eta$, $D_2$, $D_3$, $D_4$, in Equations (101)-(105) and the integral estimate (74), we have that if $0 > p \geq -1$ then

$$K_2 \lesssim K^{(p)}$$

with the constant

$$K^{(p)} := M_1(T) + M_2^2(T) + \beta^{1/4}(0)|1 + M_1^{2-p}(T) + A(T)M_1(T)|$$

if $0 > p \geq -1$;

and

$$K^{(p)} := M_1(T) + M_2^2(T)(M_q^1(T) + M_q(T)) + \beta^{1/4}(0)|1 + M_1^{2-p}(T) + M_2(T) + A(T)M_1(T)|$$

if $p < -1$. 

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Collecting the estimates \((106)-(108)\) we have
\[
\beta^{-3/2}(T)\|\langle z \rangle^{-2} e^{\frac{az^2}{1+\beta(\tau)\eta}}\|_\infty \lesssim M_2(0) + K^{(p)}.
\]
(109)

Recall the definition of \(T\) in \((65)\). By the relation between \(\xi\) and \(\eta\) in Equation \((65)\), we have \(\xi(T) = \eta(S)\), \(\alpha = a(T)\), and \(y = z\), hence \(\alpha = 0\). The proof is complete.

\section{Proof of Equation \((41)\)}

In the following lemma we present the estimates for \(D_n\), \(n = 2, 3, 4\) in \((110)\). Recall that \(q = \min\{\frac{4}{1-p}, \frac{2(2-p)^2}{(1-p)^2}, 1\}\).

\begin{lemma}
Suppose that for \(\tau \leq T\), \(M_1(\tau) \leq \frac{1}{4} \left(1 - p\right)^{\frac{1}{1+p}}\), \(A(\tau)\), \(B(\tau) \leq \beta^{-1/4}(\tau)\) and the function \(v(y, \tau)\) defined in \((17)\) has the lower bound \(v(y, \tau) \geq \frac{1}{2} \left(\frac{1-p}{1-p}\right)^{\frac{1}{1+p}}\). Then we have
\[
\|\langle z \rangle^{-q} e^{\frac{az^2}{1+\beta(\tau)\eta}} V \eta(\sigma)\|_\infty \lesssim \beta^\frac{2}{(1-p)} M_2(T),
\]
(110)
\[
\|\langle z \rangle^{-q} e^{\frac{az^2}{1+\beta(\tau)\eta}} D_2(\sigma)\|_\infty \lesssim \beta^\frac{4}{(1-p)} M_4(T),
\]
(111)
\[
\|\langle z \rangle^{-q} e^{\frac{az^2}{1+\beta(\tau)\eta}} D_3(\sigma)\|_\infty \lesssim \beta^\frac{4}{(1-p)} M_2(0) + \tau^{-1/4}(\tau) M_4(T),
\]
(112)
\[
\|\langle z \rangle^{-q} e^{\frac{az^2}{1+\beta(\tau)\eta}} D_4(\sigma)\|_\infty \lesssim \beta^\frac{4}{(1-p)} M_4(T)[M_4(0) + \tau^{-1/4}(\tau) M_4(T)].
\]
(113)
\end{lemma}

\begin{proof}
The proofs of \((111)-(113)\) are easier than that of Lemma \((13)\) thus is omitted. For \((110)\) by the fact \(q \leq 1\) and the arguments between \((93)\) and \((94)\) we have
\[
\|\langle z \rangle^{-q} e^{\frac{az^2}{1+\beta(\tau)\eta}} V \eta(\sigma)\|_\infty \lesssim \|\langle y \rangle^{-q} e^{\frac{a(y, \tau)^2}{1+\beta(\tau)\eta}} V \eta(\sigma)\|_\infty
\]
\[
\leq \beta^{-\frac{2(q-1)}{1-q}}(\tau)\|\langle y \rangle^{-2} e^{\frac{a(y, \tau)^2}{1+\beta(\tau)\eta}} V \eta(\sigma)\|_\infty
\]
\[
\leq \beta^\frac{2}{(1-p)}(\tau)\|\langle z \rangle^{-2} e^{\frac{az^2}{1+\beta(\tau)\eta}} V \eta(\sigma)\|_\infty
\]
which is \((110)\).

The proof is complete. \(\square\)

Rewrite \((110)\) to have
\[
P_1^\alpha \eta(S) = e^{-L_\alpha S} P_1^\alpha \eta(0) + \int_0^S e^{-L_\alpha(S-\sigma)} P_1^\alpha [-V \eta(\sigma) + \sum_{n=2}^4 D_n]d\sigma,
\]
30
where, recall the definition of $S$ in (66). The propagator estimate of $e^{-L_\alpha \sigma} P_1^\alpha$ in (83) yields
\[ \| \langle z \rangle - q e^{\frac{a^2}{4}} P_1^\alpha \eta(S) \|_\infty \lesssim J_0 + J_1 \] (114)
where $J_n$'s are given by
\[
J_0 := \beta^{-\frac{a}{2}}(T) e^{\frac{a^2}{4p} - q} \| \langle z \rangle - q e^{\frac{a^2}{4}} \eta(0) \|_\infty,
\]
\[
J_1 := \beta^{-\frac{a}{2}}(T) \int_0^S e^{\frac{a^2}{4p} - q} e^{\frac{a^2}{4}} V \eta(\sigma) \|_\infty + \sum_{n=2}^4 \| \langle z \rangle - q e^{\frac{a^2}{4}} D_n(\sigma) \|^2 d\sigma.
\]
We observe that $\frac{a^2}{4p} - q < 0$ by the definition of $q$ and the fact that $p < -1$.

By the estimates (110)-(113) and similar procedures as in Sections 13 and 14 we have
\[
J_0 \lesssim M_q(0)
\]
\[
J_1 \lesssim M_2(T) + \beta^{1/4}(0)[1 + M_q(T) + M_2(T) - p(T) + M_1(T)A(T)] + M_2(T) + M_2(T).
\]
These together with the fact that $\| \langle z \rangle - q e^{\frac{a^2}{4}} P_1^\alpha \eta(S) \|_\infty = \| \langle y \rangle - q e^{\frac{a^2}{4}} \eta(T) \|_\infty$ and the definition of $M_q$ yield (11) when $\tau = T$. Since $T$ is arbitrary we have (11).

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