Infinitesimal Paraholomorphically Projective Transformation On Cotangent Bundle With Riemannian Extension

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ABSTRACT: The main purpose of the present paper is to study some properties of infinitesimal paraholomorphically projective transformation on $T^*M$ with respect to the Levi-Civita connection of the Riemannian extension $(\nabla^\chi)$ and adapted almost paracomplex structure $J$. Moreover, if $T^*M$ be admits a non-affine infinitesimal paraholomorphically projective transformation, than $M$ and $T^*M$ are locally flat.

Keywords: Paraholomorphically projective transformation, almost paracomplex structure, Riemannian extension, adapted frame.

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INTRODUCTION

Let $M$ be an $n$-dimensional manifold and $T^*M$ its cotangent bundle. Note that in the present paper everything will be always discussed in the $C^\infty$ category, manifolds will be assumed to be connected and dimension $n > 1$. And let $\pi$ the natural projection $T^*M \rightarrow M$. The local coordinates $(U, x^j)$, $j = 1, \ldots, n$ on $M$ induces a system of local coordinates $(\pi^{-1}(U), x^j, x^\bar{j} = p_j)$, $\bar{j} = n + 1, \ldots 2n$ on $T^*M$, where $x^\bar{j} = p_j$ are the components of the covector $p$ in each cotangent space $T^*_xM$ and $x \in U$ with respect to the natural coframe $\{dx^j\}$. We denote the set of all tensor fields of type $(r, s)$, by $\mathcal{S}^r_s(M)$, $\mathcal{S}^r_s(T^*(M))$ on $M$ and $T^*M$ respectively.

The problem of determining infinitesimal holomorphically projective transformation on $M$ and $TM$ have been studied some authors, including (Hasegawa and Yamauchi, 1979; Hasegawa and Yamauchi, 2003; Hasegawa and Yamauchi, 2005; Tarakci et al., 2009; Gezer, 2011). Also, (Etayo and Gadea, 1992; Iscan and Magden, 2008), investigated some properties of infinitesimal paraholomorphically projective transformations on tangent bundle.

In this paper, we shall use the Levi-Civita connection of the Riemannian extension by using the horizontal and vertical lifts and we give definition and formulas almost paracomplex structure $J$. Then we research the infinitesimal paraholomorphically projective transformation on cotangent bundle with respect to the Levi-Civita connection of the Riemannian extension ($^R\nabla$) and adapted almost paracomplex structure.

MATERIAL AND METHODS

Let $\nabla$ be an affine connection on $M$. A vector field $V$ on $M$ is called an infinitesimal projective transformation if there exist a 1-form $\Omega$ on $M$ such that

$$(L_V \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X,$$

for any $X, Y \in \mathcal{S}^1_0(M)$, where $L_V$ is the Lie derivation with respect to $V$. In this case $\Omega$ is called the associated 1-form of $V$. Especially, if $\Omega = 0$ then $V$ is called an infinitesimal affine transformation.

An almost paracomplex manifold is an almost product manifold $(M, J)$, $J^2 = I$, such that the two eigenbundles $T^+M$ and $T^-M$ associated to the two eigenvalues $+1$ and $-1$ of $J$, respectively (Cruceanu et al., 1995; Salimov et al., 2007). $(M, J)$ be an almost paracomplex manifold with affine connection $\nabla$. A vector field $V$ on $M$ is called an infinitesimal paraholomorphically projective transformation if there exist a 1-form $\Omega$ on $M$ such that

$$(L_V \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X + \Omega(JX)Y + \Omega(JY)JX,$$

for any $X, Y \in \mathcal{S}^1_0(M)$. In this case $\Omega$ is also called the associated 1-form of $V$ (Prvanovic, 1971; Etayo and Gadea, 1992).

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions of a vector field $X$ and a covector (1-form) field $\omega$ on $M$, respectively. According to the induced coordinates the vertical lift $^V\omega$ of $\omega$, the horizontal lift $^H\!X$ and the complete lift $^C\!X$ of $X$ are obtained as follows

$$^V\omega = \omega_i \partial_i, \quad (1)$$

$$^H\!X = X^i \partial_i + p_h \Gamma^i_{lj} X^l \partial_i, \quad (2)$$

$$^C\!X = X^i \partial_i - p_h \partial_i X^h \partial_i.$$

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where $\partial_i = \frac{\partial}{\partial x^i}$ and $\Gamma^h_{ij}$ are the coefficients of symmetric (torsion-free) affine connection $\nabla$ on $M$ (Yano and Ishihara, 1973). For arbitrary $X, Y \in \mathfrak{X}_0^0(M)$ and $\theta, \omega \in \mathfrak{X}_1^0(M)$, the Lie bracket operation of vertical and horizontal vector fields on $T^*M$ is given as follows

\[
\begin{align*}
[HX, HY] &= H[X, Y] + \nabla (p \circ R(X, Y)) \\
[HX, \nu \omega] &= \nabla (\nabla_X \omega) \\
[V \theta, \nu \omega] &= 0,
\end{align*}
\]

where $R = R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature tensor of the symmetric connection $\nabla$ (Yano and Ishihara, 1973).

**The adapted frame**

The adapted frame $\{E_\alpha\} = \{E_i, E_J\}$ on each induced coordinate neighbourhood $\pi^{-1}(U)$ of $T^*M$ is given by (Yano and Ishihara, 1973)

\[
\begin{align*}
E_j &= HX(j) = \partial_j + p_a \Gamma^a_{hj} \partial_h \\
E_J &= \nu \theta(J) = \partial_J,
\end{align*}
\]

where

\[
X(j) = \frac{\partial}{\partial x^j}, \theta^i = dx^i, j = 1, \ldots, n,
\]

the indices $\alpha, \beta, \gamma, \ldots = 1, \ldots, 2n$ denote the indices according to the adapted frame. It follows from (1), (2) and (4) that

\[
\begin{align*}
\nu \omega &= \begin{pmatrix} 0 \\ \omega_j \end{pmatrix} \\
HX &= \begin{pmatrix} X^j \\ 0 \end{pmatrix}
\end{align*}
\]

according to the adapted frame $\{E_\alpha\}$.

**Lemma 1** The Lie bracket of the adapted frame of $T^*M$ satisfies the following identities (Yano and Ishihara, 1973)

\[
\begin{align*}
[E_i, E_j] &= p_s R^s_{ijl} E_l, \\
[E_i, E_J] &= -\Gamma^j_{il} E_i, \\
[E_I, E_J] &= 0,
\end{align*}
\]

where $R^s_{ijl} = \partial_i \Gamma^s_{jl} - \partial_j \Gamma^s_{il} + \Gamma^s_{ik} \Gamma^k_{jl} - \Gamma^s_{jk} \Gamma^k_{il}$ indicates the Riemannian curvature tensor of $(M, g)$.

**Lemma 2** Let $V$ be a vector field of $T^*M$ with the components $\left(\nu^h, \nu^\kappa\right)$. Then, the Lie derivatives of the adapted frame and the dual basis are obtained as follows (Bilen, 2019):

1. $L_V E_i = -(E_i \nu^h) E_k - \left(\nu^a p_s R^s_{ikl} + E_i \nu^\kappa - \nu^\alpha \Gamma^\alpha_{ik} \right) E_k$.
2. $L_V E_I = -(E_I \nu^h) E_k - \left(\nu^a \Gamma^a_{ik} + E_I \nu^\kappa \right) E_k$.
3. $L_V dx^k = (E_k \nu^h) dx^k + (E^\kappa \nu^\kappa) \delta p_k$.
4. $L_V \delta p_k = \left(\nu^a p_s R^s_{kah} + \nu^\alpha \Gamma^\alpha_{kh} + (E_k \nu^\alpha) \delta^m_h \right) dx^k + \left(\nu^a \Gamma^k_{ah} + (E_k \nu^\alpha) \delta^m_h \right) \delta p_k$.

{For more work on tangent bundles see (Hasegawa and Yamauchi, 2003; Gezer, 2011).}
Riemannian Extension

A pseudo-Riemannian metric $\nabla \in \mathfrak{S}_0^1(T^*M)$ is given by (Yano and Ishihara, 1973),

$$\nabla(C X, C Y) = -\gamma(\nabla_X Y + \nabla_Y X),$$

for any $X, Y \in \mathfrak{S}_0^1(M)$, where

$$-\gamma(\nabla_X Y + \nabla_Y X) = p_m(X^i \nabla_j Y^m + Y^i \nabla_j X^m),$$

$\nabla \in \mathfrak{S}_0^1(T^*M)$ with the following components in $\pi^{-1}(U)$

$$\nabla = (\nabla_{ij}) = \begin{pmatrix} -2p_h \Gamma^h_{ji} & \delta^i_j \\ \delta^j_i & 0 \end{pmatrix}$$

relative to the natural frame, where $\delta^i_j$ is the Kronecker delta. The analyzed tensor field defines a pseudo-Riemannian metric in $T^*M$ and a line element of the pseudo-Riemannian metric $\nabla$ is given by the formula

$$ds^2 = 2dx^i \delta p_i,$$

where

$$\delta p_i = dp_i - p_h \Gamma^h_{ji} dx^i.$$

This metric is called the Riemannian extension of the symmetric affine connection $\nabla$ (Patterson and Walker, 1952; Yano and Ishihara, 1973). Any tensor field of type (0,2) is entirely detected by its action of $\gamma X$ and $\gamma \omega$ on $T^*M$ (Yano and Ishihara, 1973). Then the Riemannian extension $\nabla$ is defined by

$$\nabla(\gamma \omega, \gamma \theta) = 0,$$

$$\nabla(\gamma \omega, \gamma X) = \gamma(\omega(X)) = (\omega(X)) \circ \pi,$$

$$\nabla(\gamma X, \gamma Y) = 0$$

for any $X, Y \in \mathfrak{S}_0^1(M)$ and $\omega, \theta \in \mathfrak{S}_0^2(M)$ (Aslanci et al., 2010).

The Levi-Civita connection of $\nabla$

$\nabla$ is the Levi-Civita connection of $\nabla$, because of $\nabla(\nabla) = 0$. ($\nabla$ is called the complete lift of $\nabla$ to $T^*M$) The Levi-Civita connection of $\nabla$ in $\pi^{-1}(U) \subset T^*M$ are given by

$$\nabla_{ij}^h = \Gamma^h_{ji},$$

$$\nabla_{ij}^h = -\Gamma^h_{ij},$$

$$\nabla_{ij}^h = \frac{1}{2} p_m(R^m_{ijh} - R^m_{ihj} + R^m_{hij}) = p_m R^m_{ihj}$$

$$\nabla^h_{ji} = \nabla^h_{ij} = \nabla^h_{ij} = \nabla^h_{ji} = \nabla^h_{ji} = 0$$

with respect to adapted frame $\{E_a\}$, where $\Gamma^h_{ij}$ denote the Christoffel symbols constructed with $g_{ij}$ on $M$ (Aslanci et al., 2010).

Let us consider a tensor field $J$ of type (1,1) on $T^*M$ defined by

$$J^h_X = -h_X \gamma \omega = \gamma \omega,$$

for any $X \in \mathfrak{S}_0^1(M)$, i.e., $J E_i = -E_i, J E_i = E_i$. Then we obtain $J^2 = I$. Therefore $J$ is an almost paracomplex structure on $T^*M$. This almost paracomplex structure is called adapted almost paracomplex structure (Etayo and Gadea, 1992).
RESULTS AND DISCUSSION

**Theorem 3** Let \((M, g)\) be a Riemannian manifold and \(T^*M\) be its cotangent bundle with the Riemannian extension and adapted almost paracomplex structure. A vector field \(V\) is an infinitesimal paraholomorphically projective transformation with associated 1-form \(\Omega\) on \(T^*M\) if and only if there exist \(B = (B^h) \in \mathcal{S}^1_0(M), D = (D_h) \in \mathcal{S}^0_1(M)\) and \(A = (A^i_i), C = (C^h_i) \in \mathcal{S}^1_1(M)\) satisfying

1. \(\left(\frac{v^k}{b^k}\right) = \left(\frac{p^s A^k_s + B^k}{D_k + p_a C^a_k + 4 \psi p_k + 2 p_a p_k \Psi^a}\right)\)
2. \(\nabla_j A^{ki} = 0, \nabla_j C^i_i = 0\)
3. \(\nabla_j \psi = 0, \nabla_j \psi^i = 0\)
4. \(A^{ia} R^s_{aij} = 0\)
5. \(A^i_a R^k_{aij} + A^k_h R^s_{si} = 0\)
6. \(\nabla_i \nabla_j \delta^k + B^{ai} R^s_{aij} = 2 \Omega_i \delta^j_k + 2 \Omega_j \delta^k_i = L_B \Gamma^k_{ij}\)
7. \(\nabla_i R^k_{jak} - \nabla_a R^k_{jki} = 0\)
8. \(R^s_{jhd} \Psi^h = 0\)
9. \(\nabla_i \nabla_j D_k + D_a R^a_{jki} = 0\)
10. \(C^h_k R^s_{jih} + C^s_A R^a_{jki} = 0\)
11. \(\Omega_j = \frac{1}{4n} \nabla_i \nabla_j B^j, \Omega_j = \Psi^j\)

where \(V = \left(\frac{v^k}{b^k}\right) = v^k E_k + v^k \overline{E_k}\), \(\Omega = \left(\Omega_j dx^j + \Omega_j \delta y^j\right)\).

**Proof.** Here we prove only the necessary condition because it is easy to prove the sufficient condition. Let \(V\) be an infinitesimal paraholomorphically projective transformation with the associated 1-form \(\Omega\) on \(T^*M\)

\[\left(L_V \nabla\right)(X, Y) = \Omega(X)Y + \Omega(Y)X + \Omega(JX)JY + \Omega(JY)JX\]

for any \(X, Y \in \mathcal{S}^1_0(M)\).

From

\[\left(L_V \nabla\right)(E_i, E_j) = \Omega(E_i)E_j + \Omega(E_j)E_i + \Omega(JE_i)J E_j + \Omega(JE_j)J E_i\]

we obtain

\[\left(L_V \nabla\right)(E_i, E_j) = 2 \left(\Omega_i \delta^j_k + \Omega_j \delta^i_k\right) E_k\] (5)

also

\[\left(L_V \nabla\right)(E_i, E_j) = \left[\partial_i \left(\partial_j v^k\right)\right] E_k + \left[\partial_j \left(\partial_i v^k\right)\right] E_k\] (6)

from (5) and (6) we obtain
\[
\partial_i \left(\partial_j v^k\right) = 0 \Rightarrow v^k = p^s A^k_s + B^k
\]

and
\[
\partial_i \left(\partial_j v^k\right) = 2 \left(\Omega_i \delta^j_k + \Omega_j \delta^i_k\right).
\] (8)

Contracting \(k\) and \(j\) in (8), we have
\[
\Omega_i = \partial_i \psi,
\] (9)
where $\psi = \frac{1}{2n+2} \partial_j v^j$. If we use the expression (9) in (8), expression (8) is rewritten as follows:

$$\partial_i \left( \partial_j v^k \right) = 2(\partial_j \psi) \delta^j_k + 2 \left( \partial_j \psi \right) \delta^j_i. \quad (10)$$

Differentiating (10) partially, we have

$$\partial_i \partial_j v^k = 2\partial_i \partial_j \psi \delta^j_k + 2\partial_i \partial_j \psi \delta^j_i$$

$$= 2\partial_i \partial_j \psi \delta^j_k + 2\partial_i \partial_j \psi \delta^j_i$$

$$= \partial_i \partial_j (4\psi \delta^j_k)$$

from here we get

$$\partial_i \partial_j (\partial_j v^k - 4\psi \delta^j_k) = 0.$$ Written here as

$$M^i_j = \partial_i (\partial_j v^k - 4\psi \delta^j_k) \quad (11)$$

and

$$C^j_k + p_a M^a_k = \partial_j v^k - 4\psi \delta^j_k, \quad (12)$$

where $C^j_k$ and $M^i_j$ are certain functions which depend only on the variables ($x^h$). Also

$$M^i_j + M^i_j = \partial_i \partial_j v^k - 4\partial_i \psi \delta^j_k + \partial_j \partial_i v^k - 4\partial_j \psi \delta^j_k.$$ Using (10) in above equation

$$M^i_j = \frac{1}{2} (M^i_j - M^j_i) = 2 \left[ (\partial_j \psi) \delta^j_k - (\partial_i \psi) \delta^j_i \right]. \quad (13)$$

Contracting $k$ and $j$ in (12), we have

$$C^j_k + p_a M^a_k = (2-2n)\psi.$$ From which

$$\psi = \frac{1}{2-2n} C^k_k + p_a \frac{1}{2-2n} M^a_k$$

and we get

$$\psi = \phi + p_a \Psi^k, \quad (14)$$

where $\phi = \frac{1}{2-2n} C^k_k$ and $\Psi^a = \frac{1}{2-2n} M^a_k$, from which we have

$$\Omega_i = \partial_i \psi = \Psi^i. \quad (15)$$

If used (13) and (14) in (12) we get

$$\partial_i \partial_j v^k = C^j_k + 4\psi \delta^j_k + 2p_a \Psi^a \delta^j_k + 2p_k \Psi^j$$

and

$$v^k = D_k + p_a C^a_k + 4\psi p_k + 2p_a p_k \Psi^a, \quad (16)$$

where $D_k$ are certain functions which depend only on ($x^h$). The coordiant transformation rule implies that $D = (D_k) \in \mathfrak{S}_1^0(M)$.

Next, from

$$(L_V \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X + \Omega(JX)JY + \Omega(JY)JX,$$

we have

$$(L_V R \nabla)(E_i, E_j) = 0$$

or

$$\Omega(1, 0, 0) = 0.$$
\[
(L^\nu R^C)(E_i, E_j) = 0
\]

from which, we get
\[
0 = [\nabla_j A^{ki}] E_k + [A^{ia} p_s R^s_{ka} + v^a R^i_{jak} + \nabla_j C^i_k + 2p_k (\nabla_j \Psi^i) + 2p_a \delta^i_k \nabla_j \Psi^a + 4 (\partial \varphi) \delta^i_k] E_k.
\]

Therefore,
\[
\nabla_j A^{ki} = 0 \tag{17}
\]

and
\[
A^{ia} p_s R^s_{ka} + v^a R^i_{jak} + \nabla_j C^i_k + 2p_s (\nabla_j \Psi^i) + 2p_a \delta^i_k \nabla_j \Psi^a + 4 (\partial \varphi) \delta^i_k = 0. \tag{18}
\]

Contracting \( k \) and \( i \) in (18), we have
\[
\begin{align*}
\nabla_j C^i_k = 0, \\
\nabla_j \varphi = 0
\end{align*}
\]

and
\[
\nabla_j \Psi^s = \frac{1}{2(n+1)} A^{ia} R^s_{aji}. \tag{20}
\]

Lastly, from
\[
(L^\nu R^C)(E_i, E_j) = (2\Omega_i \delta^k_j + 2\Omega_j \delta^k_i) E_k
\]

we obtain
\[
(2\Omega_i \delta^k_j + 2\Omega_j \delta^k_i) E_k = [\nabla_i \nabla_j v^k + v^a R^k_{aij} + A^{hk} p_s R^s_{hji}] E_k
\]

\[
+ [p_s (\nabla_i v^h) R^s_{hj} + (\nabla_j v^h) R^s_{khi} - (E^i_k v^h) R^s_{hji}]
\]

\[
+ v^a p_s (\nabla_i R^s_{jak} - \nabla_a R^s_{kji}) + (v^a R^a_{kji} + \nabla_i \nabla_j v^h) E_k
\]

from which, using (7) and (16), we obtain
\[
\nabla_i \nabla_j A^s_k + A^a R^s_{aji} + A^k R^h_{sij} = 0, \tag{21}
\]

\[
\nabla_i \nabla_j B^k + B^a R^k_{aji} = 2\Omega_i \delta^k_j + 2\Omega_j \delta^k_i = L_B \Gamma^k_j, \tag{22}
\]

\[
\nabla_i R^s_{jak} - \nabla_a R^s_{kji} = 0, \tag{23}
\]

\[
\nabla_i \nabla_j \Psi^s + R^s_{hji} \Psi^h = 0, \tag{24}
\]

\[
\nabla_i \nabla_j D^k + D^a R^a_{kji} = 0, \tag{25}
\]

\[
(\nabla_i B^a) R^s_{ka} + (\nabla_j B^a) R^s_{ka} + C^h_\alpha R^s_{kji} + C^s_{\alpha} R^a_{kji} + \nabla_i \nabla_j C^s_k = 0. \tag{26}
\]

From (26), we get
\[
K_{ij} = (\nabla_i B^a) R^s_{ka} + (\nabla_j B^a) R^s_{ka} + C^h_\alpha R^s_{kji} + C^s_{\alpha} R^a_{kji} + \nabla_i \nabla_j C^s_k = 0, \tag{27}
\]

\[
K_{ji} = (\nabla_j B^a) R^s_{ka} + (\nabla_i B^a) R^s_{ka} + C^h_\alpha R^s_{kji} + C^s_{\alpha} R^a_{kji} + \nabla_j \nabla_i C^s_k = 0.
\]

Contracting \( j \) and \( k \) in (22), we obtain
\[
\Omega_i = \frac{1}{4n} \nabla_i \nabla_j B^j. \tag{28}
\]

This completes the proof.

**Theorem 4** Let \((M, g)\) be a Riemannian manifold and \(T^*M\) be its cotangent bundle with the Riemannian...
extension and adapted almost paracomplex structure. If $T^*M$ admits a non-affine infinitesimal paraholomorphically projective transformation, than $M$ and $T^*M$ are locally flat.

**Proof.** Let $V$ be non-affine infinitesimal paraholomorphically projective transformation on $T^*M$, using (3) in the expression of theorem 3, we have $\nabla_i||\Psi||^2 = \nabla_j||\partial\phi||^2 = 0$. Hence, $||\Psi||$ and $||\partial\phi||$ are constant on $M$. Suppose that $M$ is non-locally flat, then $\Psi = \partial\phi = 0$ by virtue of (9) and (3) in the expression of theorem 3, that is, $V$ is an infinitesimal affine transformation. This is a contradiction. Therefore, $M$ is locally flat. In this case $T^*M$ is locally flat.

**Corollary 5** Let $(M, g)$ be a Riemannian manifold and $T^*M$ be its cotangent bundle with the Riemannian extension and adapted almost paracomplex structure. A vertical vector field $V$ is an infinitesimal paraholomorphically projective transformation with associated 1-form $\Omega$ on $T^*M$ if and only if there exist $D = (D_{ik}) \in \mathfrak{D}_1^1(M)$ and $C = (C^i_ka) \in \mathfrak{D}_1^1(M)$ satisfying

1. $\begin{pmatrix} v^k \\
            \nabla_l v^k \end{pmatrix} = \begin{pmatrix} 0 \\
            D_k + p_aC^a_k + 4\phi p_k + 2p_a p_k \Psi_a \end{pmatrix}$
2. $\nabla_j C^i_k = 0$
3. $\nabla_j\phi = 0, \nabla_j\psi = 0, \nabla_j\Psi_i = 0$
4. $\nabla_l\nabla_j D_k + D_a R^a_{ijkl} = 0$
5. $C^a_{}_a R^a_{ijkl} + C^h_{}R^h_{ijkl} = 0$
6. $\Psi^h R^l_{ijkl} = 0$
7. $\Psi^l R^a_{ijkl} + \Psi^s R^a_{ijkl} = 0$
8. $\Omega_j = 0, \Omega_l = \Psi^l$

where $V = \begin{pmatrix} 0 \\
            \nabla_l \end{pmatrix} = \nabla_l E^l_k, \Omega = \left( \Omega_j dx^j + \Omega_l dy^j \right)$.

**CONCLUSION**

In this article, we use the Levi-Civita connection of the Riemannian extension and we give definition and formulas almost paracomplex structure $J$. Then we research the infinitesimal paraholomorphically projective transformation on cotangent bundle with respect to the Levi-Civita connection of the Riemannian extension $ \nabla^R $ and adapted almost paracomplex structure $J$.

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