ON A SPECTRAL SEQUENCE FOR TWISTED COHOMOLOGIES

WEIPING LI, XIUGUI LIU, AND HE WANG

Abstract. Let $(\Omega^\ast(M), d)$ be the de Rham cochain complex for a smooth compact closed manifolds $M$ of dimension $n$. For an odd-degree closed form $H$, there are a twisted de Rham cochain complex $(\Omega^\ast(M), d + H \wedge)$ and its associated twisted de Rham cohomology $H^\ast(M, H)$. We show that there exists a spectral sequence $\{E^p,q_{r}, d_{r}\}$ derived from the filtration $F_{p}(\Omega^\ast(M)) = \bigoplus_{i \geq p} \Omega^{i}(M)$ of $\Omega^\ast(M)$, which converges to the twisted de Rham cohomology $H^\ast(M, H)$. We also show that the differentials in the spectral sequence can be given in terms of cup products and specific elements of Massey products as well, which generalizes a result of Atiyah and Segal. Some results about the indeterminacy of differentials are also given in this paper.

1. Introduction

Let $M$ be a smooth compact closed manifold of dimension $n$, and $\Omega^\ast(M)$ the space of smooth differential forms over $\mathbb{R}$ on $M$. We have the de Rham cochain complex $(\Omega^\ast(M), d)$, where $d: \Omega^{p}(M) \to \Omega^{p+1}(M)$ is the exterior differentiation, and its cohomology $H^\ast(M)$ (the de Rham cohomology). The de Rham cohomology with coefficients in a flat vector bundle is an extension of the de Rham cohomology.

The twisted de Rham cohomology was first studied by Rohm and Witten [14] for the antisymmetric field in superstring theory. By analyzing the massless fermion states in the string sector, Rohm and Witten obtained the twisted de Rham cochain complex $(\Omega^\ast(M), d + H_{3})$ for a closed 3-form $H_{3}$, and mentioned the possible generalization to a sum of odd closed forms. A key feature in the twisted de Rham cohomology is that the theory is not integer graded but (like K-theory) is filtered with the grading mod 2. This has a close relation with the twisted K-theory and the Atiyah-Hirzebruch spectral sequence (see [1]).

Let $H$ be $\sum_{i=1}^{\frac{n-1}{2}} H_{2i+1}$, where $H_{2i+1}$ is a closed $(2i + 1)$-form. Then one can define a new operator $D = d + H$ on $\Omega^\ast(M)$, where $H$ is understood as an operator acting by exterior multiplication (for any differential form $w$, $H(w) = H \wedge w$). As in [4], there is a filtration on $(\Omega^\ast(M), D)$:

$$K_{p} = F_{p}(\Omega^\ast(M)) = \bigoplus_{i \geq p} \Omega^{i}(M).$$

This filtration gives rise to a spectral sequence

$$\{E^{p,q}_{r}, d_{r}\}$$

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converging to the twisted de Rham cohomology $H^*(M, H)$ with

\begin{equation}
E^{p,q}_2 \equiv \begin{cases} 
H^p(M) & q \text{ even}, \\
0 & q \text{ odd}.
\end{cases}
\end{equation}

For convenience, we first fix some notations in this paper. The notation $[r]$ denotes the greatest integer part of $r \in \mathbb{R}$. In the spectral sequence (1.2) $E^{p,q}_2$, for any $[y_p]_k \in E^{p,q}_k, [y_p]_{k+1}$ represents its class to which $[y_p]_k$ survives in $E^{p,q}_{k+1}$. In particular, as in Proposition 3.1 for $x_p \in E^{p,q}_1, [x_p]_2 = [x_p]_3 \in E^{p,q}_2 = E^{p,q}_3$ represents the de Rham cohomology class $[x_p]$. $d_r[x_p]$ represents a class in $E^{p+r,q-r+1}_{2}$ which survives to $d_r[x_p]_r \in E^{p+r,q-r+1}_{2}$.

In the appendix I of [14], Rohm and Witten first gave a description of the differentials $d_2$ and $d_3$ for the case when $D = d + H_3$. Atiyah and Segal [1] showed a method about how to construct the differentials in terms of Massey products, and gave a generalization of Rohm and Witten’s result: the iterated Massey products give (up to sign) all the higher differentials of the spectral sequence for the twisted cohomology (see [1] Proposition 6.1). Mathai and Wu in [9, p. 5] considered the general case that $H = \sum_{i=1}^{n} H_{2i+1}$ and claimed, without proof, that $d_2 = d_4 = \cdots = 0$, while $d_3, d_5, \cdots$ are given by the cup products with $H_3, H_5, \cdots$ and by the higher Massey products with them. Motivated by the method in [1], we give an explicit description of the differentials in the spectral sequence (1.2) in terms of Massey products.

We now describe our main results. Let $A$ denote a defining system for the $n$-fold Massey product $\langle x_1, x_2, \cdots, x_n \rangle$ and $c(A)$ its related cocycle (see Definition 5.1). Then

\begin{equation}
\langle x_1, x_2, \cdots, x_n \rangle = \{ c(A) | A \text{ is a defining system for} \langle x_1, x_2, \cdots, x_n \rangle \}
\end{equation}

by Definition 5.3. To obtain our desired theorems by specific elements of Massey products, we restrict the allowable choices of defining systems for Massey products (cf. [15]). By Theorems 4.1 and 4.3 in this paper, there are defining systems for the two Massey products we need (see Lemma 5.5). The notation $\langle H_{3}, \cdots, H_{3}, x_p \rangle_A$ in Theorem 1.1 below denotes a cohomology class in $H^*(M)$ represented by $c(A)$, where $A$ is a defining system obtained by Theorem 4.1 (see Definition 5.6). Similarly, the notation $\langle H_{2i+1}, \cdots, H_{2i+1}, x_p \rangle_A$ in Theorem 1.2 below denotes a cohomology class in $H^*(M)$ represented by $c(A)$, where $A$ is a defining system obtained by Theorem 4.3 (see Definition 5.6).

**Theorem 1.1.** For $H = \sum_{i=1}^{n} H_{2i+1}$ and $[x_p]_{2i+3} \in E^{p,q}_{2i+3}$, the differential of the spectral sequence (1.2) $d_{2i+3} : E^{p,q}_{2i+3} \to E^{p+2i+3, q-2i-2}_{2i+3}$ is given by

$$d_{2i+3}[x_p]_{2i+3} = (-1)^t \langle(H_{3}, \cdots, H_{3}, x_p)_{A}\rangle_{2i+3},$$

and $\langle(H_{3}, \cdots, H_{3}, x_p)_{A}\rangle_{2i+3}$ is independent of the choice of the defining system $A$ obtained from Theorem 4.1.
Specializing Theorem 1.1 to the case in which \( H = H_{2s+1} \) \((s \geq 2)\), we obtain
\[
(1.5) \quad d_{2t+3}[x_p]_{2t+3} = (-1)^t[(0, \ldots, 0, x_p)_A]_{2t+3}.
\]

Obviously, much information has been concealed in the expression above. In particular, we give a more explicit expression of differentials for this special case which is compatible with Theorem 1.1 (see Remark 5.14).

**Theorem 1.2.** For \( H = H_{2s+1} \) \((s \geq 1)\) only and \([x_p]_{2t+3} \in E_{2t+3}^{p,q} \) \((t \geq 1)\), the differential of the spectral sequence (1.2) \( d_{2t+3} : E_{2t+3}^{p,q} \to E_{2t+3}^{p+2t+3,q-2t-2} \) is given by
\[
d_{2t+3}[x_p]_{2t+3} = \begin{cases} 
[H_{2s+1} \wedge x_p]_{2t+3} & t = s - 1, \\
(-1)^t \cdot ([H_{2s+1} \wedge \cdots \wedge H_{2s+1}], x_p)_B]_{2t+3} & t = ls - 1 \ (l \geq 2), \\
0 & \text{otherwise},
\end{cases}
\]
and \([H_{2s+1} \wedge \cdots \wedge H_{2s+1}, x_p]_B]_{2t+3} \) is independent of the choice of the defining system \( B \) obtained from Theorem 4.3.

Atiyah and Segal in [1] gave the differential expression in terms of Massey products when \( H = H_3 \) (see [1] Proposition 6.1)). Obviously, the result of Atiyah and Segal is a special case of Theorem 1.2.

Theorem 1.1 is essentially Theorem 5.8, and Theorem 1.2 is Theorem 5.13. Some of the results above are known to experts in this field, but there is a lack of mathematical proof in the literature.

This paper is organized as follows. In Section 2, we recall some backgrounds about the twisted de Rham cohomology. In Section 3, we consider the structure of the spectral sequence converging to the twisted de Rham cohomology, and give the differentials \( d_i \) \((1 \leq i \leq 3)\) and \( d_{2k} \) \((k \geq 1)\). With the formulas of the differentials in \( E_{2t+3}^{p,q} \) in Section 4, Theorems 1.1 and 1.2 (i.e., Theorems 5.8 and 5.13) are shown in Section 5. In Section 6, we discuss the indeterminacy of differentials for the spectral sequence (1.2).

## 2. Twisted de Rham Cohomology

For completeness, in this section we recall some knowledge about the twisted de Rham cohomology. Let \( M \) be a smooth compact closed manifold of dimension \( n \), and \( \Omega^*(M) \) the space of smooth differential forms on \( M \). We have the de Rham cochain complex \((\Omega^*(M), d)\) with the exterior differentiation \( d : \Omega^p(M) \to \Omega^{p+1}(M) \), and its cohomology \( H^*(M) \) (the de Rham cohomology).

Let \( H \) denote \( \sum_{i=1}^{[n-1]} H_{2i+1} \), where \( H_{2i+1} \) is a closed \((2i+1)\)-form. Define a new operator \( D = d + H \) on \( \Omega^*(M) \), where \( H \) is understood as an operator acting by exterior multiplication (for any differential form \( w \), \( H(w) = H \wedge w \), also denoted by \( H_\wedge \)). It is easy to show that
\[
D^2 = (d + H)^2 = d^2 + dH + Hd + H^2 = 0.
\]

However \( D \) is not homogeneous on the space of smooth differential forms \( \Omega^*(M) = \bigoplus_{i \geq 0} \Omega^i(M) \).
Define $\Omega^*(M)$ a new (mod 2) grading

\begin{equation}
\Omega^*(M) = \Omega^o(M) \oplus \Omega^e(M),
\end{equation}

where

\begin{equation}
\Omega^o(M) = \bigoplus_{i \equiv 1 \pmod{2}} \Omega^i(M) \quad \text{and} \quad \Omega^e(M) = \bigoplus_{i \equiv 0 \pmod{2}} \Omega^i(M).
\end{equation}

Then $D$ is homogenous for this new (mod 2) grading:

\[ \Omega^o(M) \xrightarrow{D} \Omega^e(M) \xrightarrow{D} \Omega^o(M). \]

Define the twisted de Rham cohomology groups of $M$:

\begin{equation}
H^o(M, H) = \ker[D : \Omega^o(M) \to \Omega^o(M)] / \text{im}[D : \Omega^e(M) \to \Omega^o(M)]
\end{equation}

and

\begin{equation}
H^e(M, H) = \ker[D : \Omega^e(M) \to \Omega^o(M)] / \text{im}[D : \Omega^o(M) \to \Omega^e(M)].
\end{equation}

**Remark 2.1.** (i) The twisted de Rham cohomology groups $H^*(M, H)$ ($* = o, e$) depend on the closed form $H$ and not just on its cohomology class. If $H$ and $H'$ are cohomologous, then $H^*(M, H) \cong H^*(M, H')$ (see [1, §6]).

(ii) The twisted de Rham cohomology is also an important homotopy invariant (see [1, §1.4]).

Let $E$ be a flat vector bundle over $M$ and $\Omega^i(M, E)$ be the space of smooth differential $i$-forms on $M$ with values in $E$. A flat connection on $E$ gives a linear map

\[ \nabla^E : \Omega^i(M, E) \to \Omega^{i+1}(M, E) \]

such that, for any smooth function $f$ on $M$ and any $\omega \in \Omega^i(M, E)$,

\[ \nabla^E(f \omega) = df \wedge \omega + f \cdot \nabla^E \omega, \quad \nabla^E \circ \nabla^E = 0. \]

Similarly, define $\Omega^*(M, E)$ a new (mod 2) grading

\begin{equation}
\Omega^*(M, E) = \Omega^o(M, E) \oplus \Omega^e(M, E),
\end{equation}

where

\begin{equation}
\Omega^o(M, E) = \bigoplus_{i \equiv 1 \pmod{2}} \Omega^i(M, E) \quad \text{and} \quad \Omega^e(M, E) = \bigoplus_{i \equiv 0 \pmod{2}} \Omega^i(M, E).
\end{equation}

Then $D^E = \nabla^E + H_A$ is homogenous for the new (mod 2) grading:

\[ \Omega^o(M, E) \xrightarrow{D^E} \Omega^e(M, E) \xrightarrow{D^E} \Omega^o(M, E). \]

Define the twisted de Rham cohomology groups of $E$:

\begin{equation}
H^o(M, E, H) = \ker[D^E : \Omega^o(M, E) \to \Omega^e(M, E)] / \text{im}[D^E : \Omega^e(M, E) \to \Omega^o(M, E)]
\end{equation}

and

\begin{equation}
H^e(M, E, H) = \ker[D^E : \Omega^e(M, E) \to \Omega^o(M, E)] / \text{im}[D^E : \Omega^o(M, E) \to \Omega^e(M, E)].
\end{equation}

Results proved in this paper are also true for the twisted de Rham cohomology groups $H^*(M, E, H)$ ($* = o, e$) with twisted coefficients in $E$ without any change.
3. A spectral sequence for twisted de Rham cohomology and its differentials $d_i$ ($1 \leq i \leq 3$), $d_{2k}$ ($k \geq 1$)

Recall that $D = d + H$ and $H = \sum_{i=1}^{n-1} H_{2i+1}$, where $H_{2i+1}$ is a closed $(2i + 1)$-form. Define the usual filtration on the graded vector space $\Omega^*(M)$ to be

$$K_p = F_p(\Omega^*(M)) = \bigoplus_{i \geq p} \Omega^i(M),$$

and $K = K_0 = \Omega^*(M)$. The filtration is bounded and complete,

$$(3.1) \quad K \equiv K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_n \supset K_{n+1} = \{0\}.$$ 

We have $D(K_p) \subset K_p$ and $D(K_p) \subset K_{p+1}$. The differential $D(= d + H)$ does not preserve the grading of the de Rham complex. However, it does preserve the filtration $\{K_p\}_{p \geq 0}$. The filtration $\{K_p\}_{p \geq 0}$ gives an exact couple (with bidegree) (see [13]). For each $p$, $K_p$ is a graded vector space with $K_p = (K_p \cap \Omega^p(M)) \oplus (K_p \cap \Omega^p(M)) = K^o_p \oplus K^e_p$, where $K^o_p = K_p \cap \Omega^p(M)$ and $K^e_p = K_p \cap \Omega^p(M)$. The cochain complex $(K_p, D)$ is induced by $D : \Omega^*(M) \rightarrow \Omega^*(M)$. Similar to [23], there are two well-defined cohomology groups $H^p_D(K_p)$ and $H^e_D(K_p)$. Note that a cochain complex with grading $K_p/K_{p+1} = (K^o_p/K_{p+1}) \oplus (K^e_p/K_{p+1})$ derives cohomology groups $H^p_D(K_p/K_{p+1})$ and $H^e_D(K_p/K_{p+1})$. Since $D(K_p) \subset K_{p+1}$, we have $D = 0$ in the cochain complex $(K_p/K_{p+1}, D)$.

**Lemma 3.1.** For the cochain complex $(K_p/K_{p+1}, D)$, we have

$$H^p_D(K_p/K_{p+1}) \cong \begin{cases} \Omega^p(M) & p \text{ is odd,} \\ 0 & p \text{ is even.} \end{cases}$$

and

$$H^e_D(K_p/K_{p+1}) \cong \begin{cases} \Omega^p(M) & p \text{ is even,} \\ 0 & p \text{ is odd.} \end{cases}$$

**Proof.** If $p$ is odd, then

$$K_p \cap \Omega^p(M) = K_{p+1} \cap \Omega^p(M) \text{ and } (K_p \cap \Omega^p(M)) / (K_{p+1} \cap \Omega^p(M)) = 0.$$ 

Also

$$(K_p \cap \Omega^p(M)) / (K_{p+1} \cap \Omega^p(M)) = K^o_p / K^o_{p+1} \cong \Omega^p(M),$$

and

$$H^p_D(K_p/K_{p+1}) \cong \Omega^p(M) \text{ and } H^e_D(K_p/K_{p+1}) = 0.$$ 

Similarly for even $p$, we have

$$H^e_D(K_p/K_{p+1}) \cong \Omega^p(M) \text{ and } H^e_D(K_p/K_{p+1}) = 0.$$ 

By the filtration [5.4], we obtain a short exact sequence of cochain complexes

$$(3.2) \quad 0 \rightarrow K_{p+1} \xrightarrow{i} K_p \xrightarrow{j} K_p/K_{p+1} \rightarrow 0,$$
which gives rise to a long exact sequence of cohomology groups

\[
\cdots \rightarrow H^{p+q}_D(K_{p+1}) \overset{i^*}{\rightarrow} H^{p+q}_D(K_p) \overset{j^*}{\rightarrow} H^{p+q}_D(K_p/K_{p+1}) \overset{\delta}{\rightarrow} H^{p+q+1}_D(K_{p+1}) \overset{i^*}{\rightarrow} H^{p+q+1}_D(K_p) \overset{j^*}{\rightarrow} \cdots.
\]

Note that in the exact sequence above,

\[
H^D(D, K_p) = \begin{cases} 
H^D_p(K_p) & \text{if } i \text{ is even}, \\
H^D_p(K_p) & \text{if } i \text{ is odd}.
\end{cases}
\]

and

\[
H^D(D, K_{p+1}) = \begin{cases} 
H^D_p(K_p/K_{p+1}) & \text{if } i \text{ is even}, \\
H^D_p(K_p/K_{p+1}) & \text{if } i \text{ is odd}.
\end{cases}
\]

Let

\[
E_1^{p,q} = H^{p+q}_D(K_p/K_{p+1}), \quad D_1^{p,q} = H^{p+q}_D(K_p),
\]

\[i_1 = i^*, \quad j_1 = j^*, \quad \text{and } k_1 = \delta.
\]

We get an exact couple from the long exact sequence (3.3)

\[
\begin{array}{c}
D_1^{*,*} \\
\downarrow i_1 \\
E_1^{*,*}
\end{array}
\]

\[
\begin{array}{c}
D_1^{*,*} \\
\downarrow k_1 \\
E_1^{*,*}
\end{array}
\]

\[
\begin{array}{c}
D_1^{*,*} \\
\downarrow j_1 \\
E_1^{*,*}
\end{array}
\]

with \(i_1\) of bidegree \((-1,1)\), \(j_1\) of bidegree \((0,0)\) and \(k_1\) of bidegree \((1,0)\).

We have \(d_1 = j_1k_1 : E_1^{*,*} \rightarrow E_1^{*,*}\) with bidegree \((1,0)\), and \(d_1^2 = j_1k_1j_1k_1 = 0\).

By (3.3), we have the derived couple

\[
\begin{array}{c}
D_2^{*,*} \\
\downarrow i_2 \\
E_2^{*,*}
\end{array}
\]

\[
\begin{array}{c}
D_2^{*,*} \\
\downarrow k_2 \\
E_2^{*,*}
\end{array}
\]

\[
\begin{array}{c}
D_2^{*,*} \\
\downarrow j_2 \\
E_2^{*,*}
\end{array}
\]

by the following:

1. \(D_2^{*,*} = i_1D_1^{*,*}, \ E_2^{*,*} = H_{d_1}(E_1^{*,*})\).
2. \(i_2 = i_1|_{D_2^{*,*}}\), also denoted by \(i_1\).
3. If \(a_2 = i_1a_1 \in D_2^{*,*}\), define \(j_2(a_2) = [j_1a_1]_{d_1}\), where \([\ ]_{d_1}\) denotes the cohomology class in \(H_{d_1}(E_1^{*,*})\).
4. For \([b]_{d_1} \in E_2^{*,*} = H_{d_1}(E_1^{*,*})\), define \(k_2([b]_{d_1}) = k_1b \in D_2^{*,*}\).

The derived couple (3.6) is also an exact couple, and \(j_2\) and \(k_2\) are well-defined (see [2] [3]).

**Proposition 3.2.**

(i) There exists a spectral sequence \((E_\infty^{p,q}, d_r)\) derived from the filtration \(\{K_n\}_{n \geq 0}\), where \(E_1^{p,q} = H^{p+q}_D(K_p/K_{p+1})\) and \(d_1 = j_1k_1\), and \(E_2^{p,q} = H_{d_1}(E_1^{p,q})\) and \(d_2 = j_2k_2\). The bidegree of \(d_r\) is \((r, 1 - r)\).

(ii) The spectral sequence \(\{E_r^{p,q}, d_r\}\) converges to the twisted de Rham cohomology

\[
\bigoplus_{p+q=1} E_\infty^{p,q} \cong H^*(M, H) \quad \text{and} \quad \bigoplus_{p+q=0} E_\infty^{p,q} \cong H^*(M, H).
\]
Proof. Since the filtration is bounded and complete, the proof follows from the standard algebraic topology method (see [13]). □

Remark 3.3. (1) Note that

\[ H^i_D(K_p) = \begin{cases} H^e_D(K_p) & \text{if } i \text{ is even,} \\ H^o_D(K_p) & \text{if } i \text{ is odd.} \end{cases} \]

and

\[ H^i_D(K_p/K_{p+1}) = \begin{cases} H^e_D(K_p/K_{p+1}) & \text{if } i \text{ is even,} \\ H^o_D(K_p/K_{p+1}) & \text{if } i \text{ is odd.} \end{cases} \]

Then we have that \( H^i_D(K_p) \) and \( H^i_D(K_p/K_{p+1}) \) are 2-periodic on \( i \). Consequently, the spectral sequence \( \{E^{p,q}_r, d_r\} \) is 2-periodic on \( q \).

(2) There is also a spectral sequence converging to the twisted cohomology \( H^*(M,E,H) \) for a flat vector bundle \( E \) over \( M \).

**Proposition 3.4.** For the spectral sequence in Proposition 3.2,

(i) The \( E^{p,q}_1 \)-term is given by

\[ E^{p,q}_1 = H^{p+q}_D(K_p/K_{p+1}) \cong \begin{cases} \Omega^p(M) & q \text{ is even,} \\ 0 & q \text{ is odd.} \end{cases} \]

and \( d_1 x_p = dx_p \) for any \( x_p \in E^{p,q}_1 \).

(ii) The \( E^{p,q}_2 \)-term is given by

\[ E^{p,q}_2 = H_{d_1}(E^{p,q}_1) \cong \begin{cases} H^p(M) & q \text{ is even,} \\ 0 & q \text{ is odd.} \end{cases} \]

and \( d_2 = 0 \).

(iii) \( E^{p,q}_3 = E^{p,q}_2 \) and \( d_3[x_p] = [H_3 \wedge x_p] \) for \( [x_p]_3 \in E^{p,q}_3 \).

Proof. (i) By Lemma 3.1 we have the \( E^{p,q}_1 \)-term as desired, and by definition we obtain \( d_1 = j_1 k_1 : E^{p,q}_1 \to E^{p+1,q}_1 \). We only need to consider the case when \( q \) is even, otherwise \( d_1 = 0 \). By (3.2) for odd \( p \) (the case when \( p \) is even, is similar), we have a large commutative diagram

\[ (3.8) \]

where the rows are exact and the columns are cochain complexes.
Let \( x_p \in \Omega^p(M) \cong H^{p+q}_D(K_p/K_{p+1}) \cong E_2^{p,q} \) and

\[
x = \sum_{i=0}^{\lfloor \frac{n-p}{2} \rfloor} x_{p+2i}
\]

be an (inhomogeneous) form, where \( x_{p+2i} \) is a \((p+2i)\)-form \((0 \leq i \leq \lfloor \frac{n-p}{2} \rfloor)\). Then \( x \in K_p^p \), \( jx = x_p \) and \( Dx \in K_{p+1}^p \). Also \( Dx \in K_{p+1}^p \). By the definition of the homomorphism \( \delta \) in (3.3), we have

\[
k_1x_p = [Dx]_D,
\]

where \([ \ ]_D\) is the cohomology class in \( H^{p+q}_D(K_{p+1}) \). The class \([Dx]_D\) is well defined and independent of the choices of \( x_{p+2i} \) \((1 \leq i \leq \lfloor \frac{n-p}{2} \rfloor)\) (see [3, p. 116]).

Choose \( x_{p+2i} = 0 \) \((1 \leq i \leq \lfloor \frac{n-p}{2} \rfloor)\). Then we have

\[
k_1x_p = [Dx]_D
= [dx_p + H \wedge x_p]_D
= [dx_p + \sum_{l=1}^{\lfloor \frac{n-p}{2} \rfloor} H_{2l+1} \wedge x_p]_D \in H^{p+q+1}_D(K_{p+1}).
\]

Thus one obtains

\[
d_1x_p = (j_1k_1)x_p
= j_1(k_1(x_p))
= j_1(dx_p + \sum_{l=1}^{\lfloor \frac{n-p}{2} \rfloor} H_{2l+1} \wedge x_p)_D
= dx_p.
\]

(ii) By the definition of the spectral sequence and (i), one obtains that \( E_2^{p,q} \cong H^p(M) \) when \( q \) is even, and \( E_2^{p,q} = 0 \) when \( q \) is odd. Note that \( d_2 : E_2^{p,q} \to E_2^{p+2,q-1} \). It follows that \( d_2 = 0 \) by degree reasons.

(iii) Note that \([x_p]_3 \in E_3^{p,q}\) implies \( dx_p = 0 \). Choose \( x_{p+2i} = 0 \) for \( 1 \leq i \leq \lfloor \frac{n-p}{2} \rfloor\), and we get

\[
[Dx]_D = [H \wedge x_p]_D = \left[ \sum_{l=1}^{\lfloor \frac{n-p}{2} \rfloor} H_{2l+1} \wedge x_p \right]_D \in H^{p+q+1}_D(K_{p+1}),
\]

where \( x \) is given in the proof of (i). Note that

\[
H^{p+q+1}_D(K_{p+1}) \xleftarrow{i_1^*} H^{p+q+1}_D(K_{p+3}) \xrightarrow{j_3} H^{p+q+1}_D(K_{p+3}/K_{p+4})
\]

\[
[Dx]_D \xrightarrow{(i_1^{-1})^2} [Dx]_D \xrightarrow{j_1} H_3 \wedge x_p.
\]

It follows that

\[
d_3[x_p]_3 = j_3k_3[x_p]_3
= j_3(k_1x_p)
= j_3[Dx]_D
= [j_1((i_1^{-1})^2[Dx]_D)]_3
= [H_3 \wedge x_p]_3,
\]

\[
(i_1^{-1})^2
\]

\[
(j_1)
\]

\[
(k_1)
\]

\[
(i_1)
\]

\[
(j_3)
\]

\[
k_3
\]

\[
x_p
\]

\[
H_3 \wedge x_p
\]
where the first, second and fourth identities follow from the definitions of $d_3$, $k_3$ and $j_3$ respectively, and the third and the last identities follow from (3.10) and (3.11).

By (ii), $d_2 = 0$, so $E_2^{p,q} = E_2^{p,q}$. Then we have

$$d_3[x_p] = [H_3 \wedge x_p].$$

\[ \square \]

**Corollary 3.5.** $d_{2k} = 0$ for $k \geq 1$. Therefore, for $k \geq 1$,

$$E_{2k+1}^{p,q} = E_{2k}^{p,q}. \tag{3.13}$$

**Proof.** Note that $d_{2k} : E_{2k}^{p,q} \to E_{2k+2k,q+1-2k}^{p,q}$. By Proposition 3.4 (ii), if $q$ is odd, then $E_{2k}^{p,q} = 0$ which implies that $E_{2k+1}^{p,q} = 0$. By degree reasons, we have $d_{2k} = 0$ and $E_{2k+1}^{p,q} = E_{2k}^{p,q}$ for $k \geq 1$. \[ \square \]

The differential $d_3$ for the case in which $H = H_3$ is shown in [1, §6], and the $E_2^{p,q}$-term is also known.

### 4. Differential $d_{2t+3}$ ($t \geq 1$) in terms of cup products

In this section, we will show that the differentials $d_{2t+3}$ ($t \geq 1$) can be given in terms of cup products.

We first consider the general case that $H = \sum_{i=1}^{[n/2]} H_{2i+1}$. For $[x_p]_{2i+3} \in E_{2i+3}^{p,q}$, we let $x = \sum_{j=0}^{[n/2]} x_{p+2j} \in F_p(\Omega^*(M))$. Then we have

$$\begin{align*}
Dx &= (d + \sum_{i=1}^{[n/2]} H_{2i+1})(\sum_{j=0}^{[n/2]} x_{p+2j}) \\
&= dx_p + \sum_{j=0}^{[n/2]} (dx_{p+2j} + \sum_{i=1}^{j+1} H_{2i+1} \wedge x_{p+2(j-i)+2}).
\end{align*} \tag{4.1}
$$

Denote $y = Dx = \sum_{j=0}^{[n/2]} y_{p+2j+1}$, where

$$\begin{align*}
y_{p+1} &= dx_p, \\
y_{p+2j+3} &= dx_{p+2j+2} + \sum_{i=1}^{j+1} H_{2i+1} \wedge x_{p+2(j-i)+2} \ (0 \leq j \leq \lceil \frac{n-2}{2} \rceil - 1).
\end{align*} \tag{4.2}
$$

**Theorem 4.1.** For $[x_p]_{2i+3} \in E_{2t+3}^{p,q}$ ($t \geq 1$), there exist $x_{p+2i} = x_{p+2i}^{(t)}$ ($1 \leq i \leq t$) such that $y_{p+2j+1} = 0$ ($0 \leq j \leq t$) and

$$d_{2t+3}[x_p]_{2t+3} = \sum_{i=1}^{t} H_{2i+1} \wedge x_{p+2(i-1)+2}^{(t)} + H_{2t+3} \wedge x_p^{(t)}_{2t+3},$$

where the $(p + 2i)$-form $x_{p+2i}^{(t)}$ depends on $t$.

**Proof.** The theorem is shown by mathematical induction on $t$.

When $t = 1$, $[x_p]_{2t+3} = [x_p]_5$. $[x_p]_5 \in E_5^{p,q}$ implies that $dx_p = 0$ and $d_3[x_p] = [H_3 \wedge x_p] = 0$ by Proposition 3.4. Thus there exists a $(p + 2)$-form $v_1$ such that $H_3 \wedge x_p = d(-v_1)$. We can choose $x_{p+2}^{(1)} = v_1$ to get $y_{p+3} = dx_{p+2}^{(1)} + H_3 \wedge x_p = \ldots$
\( d\nu_1 + H_3 \wedge x_p = 0 \) from (4.2). Note that

\[
\begin{align*}
H_D^{p+q+1}(K_{p+1}) &\xrightarrow{i_1^4} H_D^{p+q+1}(K_{p+5}) \xrightarrow{j_1^l} H_D^{p+q+1}(K_{p+5}/K_{p+6}) \\
\end{align*}
\]

we obtain

\[
\begin{align*}
d_5[x_p]_5 &= j_5 k_5[x_p]_5 \\
&= j_5(k_1 x_p) \\
&= J_5[Dx]_D \\
&= \left[j_1(i_1^{-1})^4[Dx]_D\right]_5 \\
&= [y_{p+5}]_5.
\end{align*}
\]

The reasons for the identities in (4.4) are similar to those of (3.12). Thus we have

\[
\begin{align*}
d_5[x_p]_5 &= [dx_{p+4} + H_3 \wedge x_{p+2} + H_5 \wedge x_p]_5 \\
&= [H_3 \wedge x_{p+2}^1 + H_5 \wedge x_p]_5,
\end{align*}
\]

where the first identity follows from (4.4) and the definition of \( y_{p+5} \) in (4.2), and the second one follows from that \( dx_{p+4} \) vanishes in \( E_5^{*,*}. \) Hence the result holds for \( t = 1. \)

Suppose the result holds for \( t \leq m - 1. \) Now we show that the theorem also holds for \( t = m. \)

From \( [x_p]_{2m+3} \in E_{2m+3}^{p,q}, \) we have \( [x_p]_{2m+1} \in E_{2m+1}^{p,q} \) and \( d_{2m+1}[x_p]_{2m+1} = 0. \) By induction, there exist \( x_{p+2i}^{(m-1)} \) \( (1 \leq i \leq m - 1) \) such that

\[
\begin{align*}
\begin{cases}
y_{p+1}[x_p] = dx_p = 0, \\
y_{p+3}[x_p] = dx_{p+2} = dx_{p+2}^{(m-1)} + H_3 \wedge x_p = 0, \\
y_{p+2i+1}[x_p] = dx_{p+2i}^{(m-1)} + \sum_{j=1}^{i-1} H_{2j+1} \wedge x_{p+2(i-j)}^{(m-1)} + H_{2i+1} \wedge x_p \\
&= 0 \quad (2 \leq i \leq m - 1), \\
d_{2m+1}[x_p]_{2m+1} &= \left[ \sum_{i=1}^{m-1} H_{2i+1} \wedge x_{p+2(i+1)}^{(m-1)} + H_{2m+1} \wedge x_p \right]_{2m+1} = 0.
\end{cases}
\end{align*}
\]

By \( d_{2m} = 0 \) and the last equation in (4.5), there exists a \( (p+2) \)-form \( w_{p+2} \) such that

\[
\begin{align*}
\sum_{i=1}^{m-1} H_{2i+1} \wedge x_{p+2(i+1)}^{(m-1)} + H_{2m+1} \wedge x_p]_{2m+1} = d_{2m-1}[w_{p+2}]_{2m-1}.
\end{align*}
\]

By induction and \( [w_{p+2}]_{2m-1} \in E_{2m-1}^{p+2,q-2}, \) there exist \( w_{p+2(i+1)}^{(m-2)} \) \( (1 \leq i \leq m - 2) \) such that
Note that
\[
(4.8)
\]
\[
d_2 m-2 \{ w_{p+2} \} \sum_{i=1}^{m-2} H_{2i+1} \wedge (x_{p+2i}^{(m-1)} - w_{p+2(m-i)}) + H_{2m-1} \wedge (x_{p+2}^{(m-1)} - w_{p+2}) + H_{2m+1} \wedge x_{p} \} = 0.
\]

By (4.9), the last equation in (4.7), we obtain
\[
[ \sum_{i=1}^{m-2} H_{2i+1} \wedge (x_{p+2i}^{(m-1)} - w_{p+2(m-i)}) + H_{2m-1} \wedge (x_{p+2}^{(m-1)} - w_{p+2}) + H_{2m+1} \wedge x_{p} ] \sum_{i=1}^{m-2} = 0.
\]

By induction and \[ w_{p+2(m-2)} \] \[ \sum_{i=1}^{m-2} H_{2i+1} \wedge (x_{p+2i}^{(m-1)} - w_{p+2(m-i)}) + H_{2m-1} \wedge (x_{p+2}^{(m-1)} - w_{p+2}) + H_{2m+1} \wedge x_{p} \] \[ \sum_{i=1}^{m-2} = 0.\]
and \( y_{p+2m-1}^{(0)}(w_{p+2(m-1)}) = dw_{p+2(m-1)} = 0 \). Thus there exists a \((p + 2m)\)-form \( w_{p+2m} \) such that

\[
\sum_{i=1}^{m-1} \left( H_{2i+1} \wedge (x_{p+2(m-i)}^{(m-1)}) - \sum_{j=1}^{m-i-1} u_{p+2(m-i)}^{(m-1-j)} - w_{p+2(m-i)} \right) + H_{2m+1} \wedge x_p = dw_{p+2m}.
\]

By comparing (4.10) with (4.2), we choose at this time

\[
\begin{align*}
 x_{p+2} &= x_{p+2}^{(m)} = x_{p+2}^{(m-1)} - w_{p+2}, \\
 x_{p+2i} &= x_{p+2i}^{(m)} = x_{p+2i}^{(m-1)} - \sum_{j=1}^{m-1} u_{p+2i}^{(m-1-j)} - w_{p+2i} \\
 x_{p+2m} &= x_{p+2m}^{(m)} = -w_{p+2m}.
\end{align*}
\]

From (4.2), by a direct computation we have

\[
\begin{align*}
 y_{p+1} &= y_{p+1}^{(m-1)}(x_p) = 0, \\
 y_{p+2i-1} &= y_{p+2i-1}^{(m-1)}(x_p) - \sum_{j=1}^{i-1} y_{p+2i-1}^{(m-1-j)}(w_{p+2j}) = 0 \\
 y_{p+2m+1} &= 0.
\end{align*}
\]

Note that

\[
H_{D}^{p+q+1}(K_{p+1}) \xrightarrow{(i-1)^{2i(m-1)+1}} H_{D}^{p+q+1}(K_{p+2m+3}) \xrightarrow{j_1} H_{D}^{p+q+1}(K_{p+2m+3}/K_{p+2m+4})
\]

\[
[Dx]_D \xrightarrow{(i-1)^{2i(m-1)+1}} [Dx]_D \xrightarrow{j_1} y_{p+2m+3}.
\]

By the similar reasons as in (4.12), the following identities hold.

\[
d_{2m+3}[x_p]_{2m+3} = j_{2m+3}k_{2m+3}[x_p]_{2m+3}
\]

\[
= j_{2m+3}(k_1 x_p)
\]

\[
= j_{2m+3}[Dx]_D
\]

\[
= [j_1(l_1^{-1})^{2(m+1)}][Dx]_D]_{2m+3}
\]

\[
= [y_{p+2m+3}]_{2m+3}.
\]

So we have

\[
d_{2m+3}[x_p]_{2m+3} = [y_{p+2m+3}]_{2m+3}
\]

\[
= [dx_{p+2m+2} + \sum_{i=1}^{m} H_{2i+1} \wedge x_{p+2(m-i+1)}^{(m)} + H_{2m+3} \wedge x_p]_{2m+3} \text{ by (4.2)}
\]

\[
= [\sum_{i=1}^{m} H_{2i+1} \wedge x_{p+2(m-i+1)}^{(m)} + H_{2m+3} \wedge x_p]_{2m+3},
\]

showing that the result also holds for \( t = m \).

The proof of the theorem is finished.

\( \square \)

Remark 4.2. Note that \( x_{p+2i}^{(t)} \) \((1 \leq i \leq t)\) depends on \( t \), and \( x_{p+2i}^{(t_1)} \neq x_{p+2i}^{(t_2)} \) on the condition that \( t_1 \neq t_2 \) generally. \( x_{p+2i}^{(t)} \) \((1 \leq i \leq t)\) are related to \( x_{p+2j}^{(t-1)} \) \((1 \leq j \leq t-1, \ j \leq i)\).

Now we consider the special case in which \( H = H_{2s+1} \) \((s \geq 1)\) only. For this special case, we will give a more explicit result which is stronger than Theorem 4.1.
For $x = \sum_{j=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} x_{p+2j}$, we have

$$Dx = (d + H_{2s+1}) \left( \sum_{j=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} x_{p+2j} \right) = \sum_{j=0}^{s-1} dx_{p+2j} + \sum_{j=s}^{\left\lfloor \frac{n-2}{2} \right\rfloor} (dx_{p+2j} + H_{2s+1} \wedge x_{p+2(j-s)}) \tag{4.15}$$

Denote

$$y_{p+2j+1} = dx_{p+2j} \quad (0 \leq j \leq s-1),$$
$$y_{p+2j+3} = dx_{p+2j+2} + H_{2s+1} \wedge x_{p+2(j-s)+2} \quad (s-1 \leq j \leq \left\lfloor \frac{n-2}{2} \right\rfloor - 1).$$

Then $Dx = \sum_{j=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} y_{p+2j+1}$.

**Theorem 4.3.** For $H = H_{2s+1}$ ($s \geq 1$) only and $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$ ($t \geq 1$), there exist $x_{p+2is} = x_p^{(\left\lfloor \frac{t}{s} \right\rfloor)}$, $x_{p+2(i-1)s+2j} = 0$ and $x_{p+2\left\lfloor \frac{t}{s} \right\rfloor} = 0$ for $1 \leq i \leq \left\lfloor \frac{t}{s} \right\rfloor$, $1 \leq j \leq s-1$ and $1 \leq k \leq t - \left\lfloor \frac{t}{s} \right\rfloor$ such that $y_{p+2u+1} = 0$ ($0 \leq u \leq t$) and

$$d_{2t+3 + x_{p+2is}} \begin{cases} 
[H_{2s+1} \wedge x_p]_{2t+1} & t = s-1, \\
[H_{2s+1} \wedge x_{p+2(l-1)s}]_{2t+3} & t = ls-1 (l \geq 2), \\
0 & \text{otherwise},
\end{cases}$$

where the $(p+2is)$-form $x_{p+2is}^{(\left\lfloor \frac{t}{s} \right\rfloor)}$ depends on $\left\lfloor \frac{t}{s} \right\rfloor$.

**Proof.** The proof of the theorem is by mathematical induction on $s$.

When $s = 1$, the result follows from Theorem 4.1.

When $s \geq 2$, we prove the result by mathematical induction on $t$. We first show that the result holds for $t = 1$. Note that $[x_p]_5 \in E_5^{p,q}$ implies $y_{p+1} = dx_p = 0$. Choose $x_{p+2} = 0$ and make $y_{p+3} = 0$.

(i). When $s = 2$, by (4.13) we have

$$d_5[x_p]_5 = [y_{p+5}]_5 = dx_{p+4} + H_5 \wedge x_p]_5 = [H_5 \wedge x_p]_5.$$

(ii). When $s \geq 3$, by (4.14) we have

$$d_5[x_p]_5 = [y_{p+5}]_5 = [dx_{p+4}]_5 = 0.$$

Combining (i) and (ii), we have that the theorem holds for $t = 1$.

Suppose the theorem holds for $t \leq m - 1$. Now we show that the theorem also holds for $t = m$.

**Case 1.** $2 \leq m \leq s - 1$.

By induction, the theorem holds for $1 \leq t \leq m - 1$. Choose $x_{p+2i} = 0$ ($1 \leq i \leq m$), and from (4.15) we easily get that $y_{p+2j+1} = 0$ ($0 \leq j \leq m$). By (4.13) and (4.14), we have

$$d_{2m+3}[x_p]_{2m+3} = [y_{p+2m+3}]_{2m+3} = [dx_{p+2(m+1)}]_{2m+3} = 0 \quad \text{for} \quad 2 \leq m \leq s - 2, \quad m = s - 1,$$

$$[H_{2s+1} \wedge x_p]_{2m+1} = m = s - 1.$$

**Case 2.** $m = ls - 1$ ($l \geq 2$).

By induction, the theorem holds for $t = m - 1 = ls - 2$. Thus, there exist $x_{p+2is} = x_p^{(\left\lfloor \frac{t}{s} \right\rfloor)}$, $x_{p+2(i-1)s+2j} = 0$ and $x_{p+2(l-1)s+2k} = 0$ for $1 \leq i \leq$
l−1, 1≤j≤s−1 and 1≤k≤s−2 such that yp+2u+1 = 0 (0 ≤ u ≤ ls − 2).
Choose xp+2(t−1) = 0, and by (4.15) we get

\[
y_{p+2(t−1)+1} = dx_{p+2(t−1)} + H_{2s+1} \wedge x_{p+2(t−1)s−2} \\
= 0 + H_{2s+1} \wedge 0 \\
= 0.
\]

Then we have

\[
d_{2(t−1)+3}[x_p]_{2(t−1)+3} = [y_{p+2ls+1}]_{2(t−1)+3} \text{ by (4.14)} \\
= [dx_{p+2ls} + H_{2s+1} \wedge x_{p+2(l−1)s−2}]_{2(t−1)+3} \text{ by (4.15)} \\
= [H_{2s+1} \wedge x_{p+2(l−1)s}2(t−1)+3].
\]

**Case 3.** m = ls (l ≥ 1).

By induction, there exist \(x_{p+2i} = x_p^{(l−1)}\) such that \(x_{p+2i} = x_p^{(l−1)}\), \(x_{p+2(l−1)i+2j} = 0\) for \(1 ≤ i ≤ l−1, 1 ≤ j ≤ s−1\) and \(1 ≤ k ≤ s−1\) such that \(y_{p+2u+1} = 0 (0 ≤ u ≤ ls−1)\). By the same method as in Theorem 4.1 one has that there exist \(x_{p+2is} = x_p^{(l)}\), \(x_{p+2(l−1)s+2j} = 0\) and \(x_{p+2(l−1)s+2k} = 0\) for \(1 ≤ i ≤ l−1, 1 ≤ j ≤ s−1\) and \(1 ≤ k ≤ s−1\) such that \(y_{p+2u+1} = 0 (0 ≤ u ≤ ls−1)\). By (4.14), (4.15) and \(x_{p+2ls−2s+2} = 0\), we have

\[
d_{2ls+3}[x_p]_{2ls+3} = [y_{p+2ls+3}]_{2ls+3} \\
= [dx_{p+2ls+2} + H_{2s+1} \wedge x_{p+2ls−2s+2}]_{2ls+3} \\
= 0.
\]

**Case 4.** ls < m < (l + 1)s − 1 (l ≥ 1).

By induction, there exist \(x_{p+2is} = x_p^{(l−1)}\) such that \(x_{p+2is} = x_p^{(l)}\), \(x_{p+2(l−1)s+2j} = 0\) and \(x_{p+2ls+2k} = 0\) for \(1 ≤ i ≤ l, 1 ≤ j ≤ s−1\) and \(1 ≤ k ≤ m−ls−1\) such that \(y_{p+2u+1} = 0 (0 ≤ u ≤ m−1)\). Choose \(x_{p+2m} = 0\) and make \(y_{p+2m+1} = 0\). By (4.14), (4.15) and \(x_{p+2m−2s+2} = 0\), we have

\[
d_{2m+3}[x_p]_{2m+3} = [y_{p+2m+3}]_{2m+3} \\
= [dx_{p+2m+2} + H_{2s+1} \wedge x_{p+2m−2s+2}]_{2m+3} \\
= 0.
\]

Combining Cases 1-4, we have that the result holds for \(t = m\) and the proof is completed.

**Remark 4.4.**

(1) Theorems 4.1 and 4.3 show that the differentials in the spectral sequence (1.2) can be computed in terms of cup products with \(H_{2i+1}\)'s.

The existence of \(x_p^{(l)}\) and \(x_p^{(l−1)}\) in Theorems 4.1 and 4.3 plays an essential role in proving Theorems 5.8 and 5.13 respectively. Theorems 4.1 and 4.3 give a description of the differentials at the level of \(E_2^{pq}\) for the spectral sequence (1.2), which was ignored in the previous studies of the twisted de Rham cohomology in [11, 9].

(2) Note that Theorem 4.3 is not a corollary of Theorem 4.1 and it can not be obtained from Theorem 4.1 directly.

5. DIFFERENTIALS \(d_{2t+3} (t ≥ 1)\) IN TERMS OF MASSEY PRODUCTS

The Massey product is a cohomology operation of higher order introduced in [8], which generalizes the cup product. In [10], May showed that the differentials in the Eilenberg-Moore spectral sequence associated with the path-loop fibration
of a path connected, simply connected space are completely determined by higher order Massey products. Kraines and Schochet [5] also described the differentials in Eilenberg-Moore spectral sequence by Massey products. In order to describe the differentials $d_{2t+3}$ ($t \geq 1$) in terms of Massey products, we first recall briefly the definition of Massey products (see [4, 10, 11, 13]). Then the main theorems in this paper will be shown.

Because of different conventions in the literature used to define Massey products, we present the following definitions. If $x \in \Omega^t(M)$, the symbol $\bar{x}$ will denote $(-1)^{1+\text{deg}x}x = (-1)^{1+\text{deg}x}x$. We first define the Massey triple product.

Let $x_1, x_2, x_3$ be closed differential forms on $M$ of degrees $r_1, r_2, r_3$, with $[x_1][x_2] = 0$ and $[x_2][x_3] = 0$, where $[\cdot]$ denotes the de Rham cohomology class. Thus, there are differential forms $\bar{v}_1$ of degree $r_1 + r_2 - 1$ and $\bar{v}_2$ of degree $r_2 + r_3 - 1$ such that $d\bar{v}_1 = \bar{x}_1 \wedge x_2$ and $d\bar{v}_2 = \bar{x}_2 \wedge x_3$. Define the $(r_1 + r_2 + r_3 - 1)$-form

\[
\omega = \bar{v}_1 \wedge x_3 + \bar{x}_1 \wedge \bar{v}_2.
\]

Then $\omega$ satisfies

\[
d(\omega) = (-1)^{r_1 + r_2}d\bar{v}_1 \wedge x_3 + (-1)^{r_1}\bar{x}_1 \wedge d\bar{v}_2 = (-1)^{r_1 + r_2}d\bar{x}_1 \wedge x_2 \wedge x_3 + (-1)^{r_1 + r_2 + 2}d\bar{x}_1 \wedge x_2 \wedge x_3 = 0.
\]

Hence a set of all the cohomology classes $[\omega]$ obtained by the above procedure is defined to be the Massey triple product $\langle x_1, x_2, x_3 \rangle$ of $x_1, x_2$ and $x_3$. Due to the ambiguity of $v_i, i = 1, 2$, the Massey triple product $\langle x_1, x_2, x_3 \rangle$ is a representative of the quotient group

\[H^{r_1 + r_2 + r_3 - 1}(M)/([x_1]H^{r_2 + r_3 - 1}(M) + H^{r_1 + r_2 - 1}(M)[x_3]).\]

**Definition 5.1.** Let $(\Omega^*(M), d)$ be de Rham complex, and $x_1, x_2, \cdots, x_n$ closed differential forms on $M$ with $[x_i] \in H^{r_i}(M)$. A collection of forms, $A = (a_{i,j})$, for $1 \leq i \leq j \leq k$ and $(i, j) \neq (1, n)$ is said to be a defining system for the $n$-fold Massey product $\langle x_1, x_2, \cdots, x_n \rangle$ if

1. $a_{i,j} \in \Omega^{r_i + r_j - 1}(M)$,
2. $a_{i,i} = x_i$ for $i = 1, 2, \cdots, k$,
3. $d(a_{i,j}) = \sum_{r=i}^{j-1} a_{i,r} \wedge a_{r+1,j}$.

The $(r_1 + \cdots + r_n - n + 2)$-dimensional cocycle, $c(A)$, defined by

\[
c(A) = \sum_{r=1}^{n-1} a_{1,r} \wedge a_{r+1,n} \in \Omega^{r_1 + \cdots + r_n - n + 2}(M)
\]

is called the related cocycle of the defining system $A$.

**Remark 5.2.** There is a unique matrix associated to each defining system $A$ as follows.

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-2} & a_{1,n-1} \\
a_{2,2} & a_{2,3} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\
a_{3,3} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\
a_{n-1,n-1} & a_{n-1,n-1} & a_{n-1,n} \\
a_{n,n} & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}_{n \times n}.
\]
**Definition 5.3.** The \( n \)-fold Massey product \( \langle x_1, x_2, \cdots, x_n \rangle \) is said to be defined if there is a defining system for it. If it is defined, then \( \langle x_1, x_2, \cdots, x_n \rangle \) consists of all classes \( w \in H^{r_1+\cdots+r_{n-1}+n+2}(M) \) for which there exists a defining system \( A \) such that \( c(A) \) represents \( w \).

**Remark 5.4.** There is an inherent ambiguity in the definition of the Massey product arising from the choices of defining systems. In general, the \( n \)-fold Massey product may or may not be a coset of a subgroup, but its indeterminacy is a subset of a matrix Massey product (see [16, §2]).

Based on Theorems 4.4 and 4.5, we have the following lemma on defining systems for the two Massey products we consider in this paper.

**Lemma 5.5.** (1) For \( [x_p]_{2l+3} \in E_{2l+3}^{p,q} \) \((l \geq 1)\), there are defining systems for \( \langle H_3, \cdots, H_{2l+3} \rangle \), obtained from Theorem 4.4.

(2) For \( [x_p]_{2l+3} \in E_{2l+3}^{p,q} \), when \( t = ls - 1 \) \((l \geq 2)\), there are defining systems for \( \langle H_{2s+1}, \cdots, H_{2l+3} \rangle \), obtained from Theorem 4.5.

**Proof.** (1) From Theorem 4.4 there exist \( x_{p+2j}^{(t)} \) \((1 \leq j \leq t)\) such that \( y_{p+2i+1} = 0 \) \((0 \leq i \leq t)\) and \( d_{2l+3}[x_p]_{2l+3} = \bigl( \sum_{i=1}^{t} H_{2i+1} \wedge x_{p+2(t-i+1)}^{(t)} \bigr) + H_{2l+3} \wedge x_{p}^{(t)} \). By Theorem 4.4 and (4.2), there exists a defining system \( A = (a_{i,j}) \) for \( \langle H_3, \cdots, H_{2l+3} \rangle \), obtained from Theorem 4.5.

\[
\begin{align*}
\begin{bmatrix}
a_{t+2,t+2} = x_p, \\
a_{i+i+k} = (-1)^k H_{2i+3}, \\
a_{i,t+2} = (-1)^{(l+2-t-j)} x_{p+2(t+2-i)}^{(t)}
\end{bmatrix} & \quad \text{for} \quad 1 \leq i \leq t + 1 - k, \quad 0 \leq k < t,
\end{align*}
\]

(2) By Theorem 4.5 there exist \( x_{p+2s} = x_{p+2s}, x_{p+2(s-1)} \)

\[
\begin{align*}
(1) & \quad 0 \leq i \leq t, 1 \leq j \leq s-1, \quad \text{and} \quad 1 \leq k \leq s-1, \quad \text{such that} \quad y_{p+2i+1} = 0 \quad (0 \leq i \leq t)
\end{align*}
\]

and \( d_{2l+3}[x_p]_{2l+3} = [H_{2s+1} \wedge x_{p+2(t-1)s+2k}]_{2l+3} \). By Theorem 4.5, there also
exists a defining system \(A = \langle a_{i,j} \rangle\) for \(\langle H_{2s+1}, \ldots, H_{2s+1}, x_p \rangle\) as follows:

\[
\begin{align*}
\begin{cases}
a_{i,j} = 0 & (1 \leq i < j \leq l), \\
a_{i,i} = H_{2s+1} & (1 \leq i \leq l), \\
a_{i+1,j+1} = x_p, \\
a_{i,i+1} = (-1)^{l+1-i}x_{p+2(l+1-i)s} & (2 \leq i \leq l),
\end{cases}
\end{align*}
\]

(5.5)

to which the matrix associated is given by

\[
\begin{pmatrix}
H_{2s+1} & 0 & 0 & \cdots & 0 & 0 \\
0 & H_{2s+1} & 0 & \cdots & 0 & 0 \\
0 & 0 & H_{2s+1} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & H_{2s+1} & 0 \\
0 & 0 & 0 & \cdots & 0 & H_{2s+1}
\end{pmatrix} 
+ (l+1) \times (l+1).
\]

The desired result follows.

\[
□
\]

To obtain our desired theorems by specific elements of Massey products, we restrict the allowable choices of defining systems for the two Massey products in Lemma \ref{lem:5.5} (cf. \[15\]). By Lemma \ref{lem:5.5} we give the following definitions.

**Definition 5.6.** (1) Given a class \([x_p]_{2t+3} \in E_{2t+3}^{p,q} (t \geq 1)\), a specific element of \((t + 2)\)-fold Massey product \(\langle H_3, \ldots, H_3, x_p \rangle\), denoted by \(\langle H_3, \ldots, H_3, x_p \rangle_A\), is a class in \(H^{p+2t+3}(M)\) representing by \(c(A)\), where \(A\) is a defining system obtained from Theorem 4.1. We define the \((t + 2)\)-fold allowable Massey product \(\langle H_3, \ldots, H_3, x_p \rangle_*\) to be the set of all the cohomology classes \(w \in H^{p+2t+3}(M)\) for which there exists a defining system \(A\) obtained from Theorem 4.1 such that \(c(A)\) represents \(w\).

(2) Similarly, given a class \([x_p]_{2t+3} \in E_{2t+3}^{p,q} (t \geq 1)\), when \(t = ls - 1 \ (l \geq 2)\) we define the specific element of \((l + 3)\)-fold Massey product \(\langle H_{2s+1}, \ldots, H_{2s+1}, x_p \rangle\) and the \((l + 1)\)-fold allowable Massey product \(\langle H_{2s+1}, \ldots, H_{2s+1}, x_p \rangle_*\) by replacing \(t+1\) by Theorem 4.1 in (1).

**Remark 5.7.** (1) From Definition 5.6 we can get the following

\[
\langle H_3, \ldots, H_3, x_p \rangle_* \subseteq \langle H_3, \ldots, H_3, x_p \rangle, \langle H_3, \ldots, H_3, x_p \rangle_* \subseteq \langle H_3, \ldots, H_3, x_p \rangle.
\]

(2) The allowable Massey product \(\langle H_3, \ldots, H_3, x_p \rangle_*\) is less ambiguous than the general Massey product \(\langle H_3, \ldots, H_3, x_p \rangle\). Take \(\langle H_3, H_3, x_p \rangle_*\) in Definition 5.6 for
example. Suppose $H = \sum_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$. By Theorem 4.1 and 4.2, there exist $x^{(i)}_{p+2j}$ such that $y_{p+2j} = 0$ ($0 \leq i \leq 1$) and $d_5[x_p] = [H_3 \wedge x^{(1)}_{p+2} + H_5 \wedge x_p]$. By Lemma 5.5 we get a defining system $A$ for $(H_3, H_3, x_p)$ and its related cocycle $c(A) = -H_3 \wedge x^{(1)}_{p+2} - H_5 \wedge x_p$. Thus, we have

\begin{equation}
\langle H_3, H_3, x_p \rangle_A = [-H_3 \wedge x^{(1)}_{p+2} - H_5 \wedge x_p].
\end{equation}

Obviously, the indeterminacy of the allowable Massey product $\langle H_3, H_3, x_p \rangle$, is $[H_3]H^{p+2}(M)$. However, in the general case, the indeterminacy of the Massey product $\langle H_3, H_3, x_p \rangle$ is $[H_3]H^{p+2}(M) + H^5(M)[x_p]$.

Similarly, the allowable Massey product $\langle H_{2+1}, \ldots, H_{2+1}, x_p \rangle$, is less ambiguous than the general Massey product $\langle H_{2+1}, \ldots, H_{2+1}, x_p \rangle$.

Now we begin to show our main theorems.

**Theorem 5.8.** For $H = \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} H_{2i+1} \text{ and } [x_p]_{2i+3} \in E^{p,q}_{2i+3}$, the differential of the spectral sequence $E^{p,q}_{2i+3} : E^{p,q}_{2i+3} \rightarrow E^{p+2,3,q-2t-2}_{2i+3}$ is given by

\begin{equation}
d_{2i+3}[x_p]_{2i+3} = (-1)^t [\langle H_3, \ldots, H_3, x_p \rangle_{A_{2i+3}}, t+1],
\end{equation}

and $\langle H_3, \ldots, H_3, x_p \rangle_{A_{2i+3}}$ is independent of the choice of the defining system $A$ obtained from Theorem 4.1.

**Proof.** By Lemma 5.5 (1), there exist defining systems for $(H_3, \ldots, H_3, x_p)$ given by Theorem 4.1. For any defining system $A = (a_{i,j})$ given by Theorem 4.1, by (5.4) we have

\begin{equation}
c(A) = (-1)^t \sum_{i=1}^{t} H_{2i+1} \wedge x^{(i)}_{p+2(t-i+1)} + H_{2t+3} \wedge x_p.
\end{equation}

By Definition 5.6 we have

\begin{equation}
\langle H_3, \ldots, H_3, x_p \rangle_{A_{t+1}} = [c(A)].
\end{equation}

Then by Theorem 4.1, we have

\begin{equation}
d_{2i+3}[x_p]_{2i+3} = \sum_{i=1}^{t} H_{2i+1} \wedge x^{(i)}_{p+2(t-i+1)} + H_{2t+3} \wedge x_p]_{2t+3} = (-1)^t [\langle H_3, \ldots, H_3, x_p \rangle_{A_{2i+3}}, t+1].
\end{equation}

Thus, we have $d_{2i+3}[x_p]_{2t+3} = (-1)^t [\langle H_3, \ldots, H_3, x_p \rangle_{A_{2i+3}}, t+1]$. By the arbitrariness of $A$, we have $\langle (H_3, \ldots, H_3, x_p)_{A_{t+1}} \rangle$ is independent of the choice of the defining system $A$ obtained from Theorem 4.1. \qed
Example 5.9. For formal manifolds which are manifolds with vanishing Massey products, it is easy to get

\[ E_4^{p,q} \cong E_\infty^{p,q} \]

by Theorem 5.8. Note that simply connected compact Kähler manifolds are an important class of formal manifolds (see [2]).

Remark 5.10. (1) From the proof of the theorem above, we have that the specific element \( \langle H_3, \cdots, H_3, x_p \rangle_A \) represents a class in \( E_\infty^{*,*} \). For two different defining systems \( A_1 \) and \( A_2 \) given by Theorem 4.1, we have

\[ \langle H_3, \cdots, H_3, x_p \rangle_{A_1} \neq \langle H_3, \cdots, H_3, x_p \rangle_{A_2} \]

generally. However, in the spectral sequence (1.2) we have

\[ \langle H_3, \cdots, H_3, x_p \rangle_{A_1} \cong \langle H_3, \cdots, H_3, x_p \rangle_{A_2} \]

which is expressed only by \( H_3 \) and \( x_p \). From the proof of Theorem 5.8, we know that the expression above conceals some information, because the other \( H_{2t+1} \)'s affect the result implicitly.

We have the following corollary (see [1, Proposition 6.1]).

Corollary 5.11. For \( H = H_3 \) only and \( [x_p]_{2t+3} \in E_\infty^{p,q} \) \((t \geq 1)\), we have that in the spectral sequence (1.2),

\[ d_{2t+3}[x_p]_{2t+3} = (-1)^t \langle H_3, \cdots, H_3, x_p \rangle_{A} \]

and \( \langle H_3, \cdots, H_3, x_p \rangle_{A} \) is independent of the choice of the defining system \( A \) obtained from Theorem 4.1.

Remark 5.12. (1) Because the definition of Massey products is different from the definition in [1], the expression of differentials in Corollary 5.11 differs from the one in [1, Proposition 6.1].

(2) The two specific elements of \( \langle H_3, \cdots, H_3, x_p \rangle \) in Theorem 5.8 and Corollary 5.11 are completely different, and equal \([c(A_1)]\) and \([c(A_2)]\) respectively, where \( c(A_i) \) \((i = 1, 2)\) are related cocycles of the defining systems \( A_i \) \((i = 1, 2)\) obtained from Theorem 4.1. The matrices associated to the two
defining systems are given by

\[
\begin{pmatrix}
H_3 & -H_5 & H_7 & \cdots & (1)^{t-1}H_{2t+1} & (1)^tH_{2t+3} \\
H_3 & -H_5 & \cdots & (1)^{t-2}H_{2t-1} & (1)^{t-1}H_{2t+1} & (1)^t \langle x_p \rangle_{p+2t} \\
H_3 & \cdots & (1)^{t-3}H_{2t-3} & (1)^{t-2}H_{2t-1} & (1)^{t-1} \langle x_p \rangle_{p+2t-2} & \\
& \vdots & & \vdots & & \\
& & & & & \\
H_3 & -H_5 & \cdots & (1)^2 \langle x_p \rangle_{p+4} & (1)^t \langle x_p \rangle_{p+2} & x_p \\
& & & & & (t+2) \times (t+2)
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
H_3 & 0 & 0 & \cdots & 0 & 0 \\
H_3 & 0 & \cdots & 0 & 0 & (1)^t \langle x_p \rangle_{p+2t} \\
H_3 & \cdots & 0 & 0 & (1)^{t-1} \langle x_p \rangle_{p+2t-2} & \\
& \vdots & \vdots & \vdots & \vdots & \\
& & & & & \\
H_3 & 0 & (1)^2 \langle x_p \rangle_{p+4} & \\
& & (1)^t \langle x_p \rangle_{p+2} & x_p \\
& & & & & (t+2) \times (t+2),
\end{pmatrix}
\]

respectively. Here \(x_p \rangle_{p+2t} (1 \leq i \leq t)\) in the first matrix are different from those in the second one.

For \(H = H_{2s+1} (s \geq 2)\) only (i.e., in the case \(H_i = 0, i \neq 2s + 1\) and \([x_p]_{2t+3} \in E^{p,q}_{2t+3} (t \geq 1)\), we make use of Theorem 4.3 to get that

\[
d_{2t+3}[x_p]_{2t+3} = (1)^t \langle[0, \cdots, 0, x_p, \cdots]_{t+1}\rangle_{2t+3}.
\]

Obviously, some information has been concealed in the expression above. Now we give another description of the differentials for this special case.

**Theorem 5.13.** For \(H = H_{2s+1} (s \geq 1)\) only and \([x_p]_{2t+3} \in E^{p,q}_{2t+3} (t \geq 1)\), the differential of the spectral sequence \(E^{p,q}_{2t+3} \rightarrow E^{p+2,3,q-2t-2}_{2t+3}\) is given by

\[
d_{2t+3}[x_p]_{2t+3} = \begin{cases} \\
[H_{2s+1} \wedge x_p]_{2t+3} & i = s - 1, \\
(1)^{t-1} \langle[H_{2s+1}, \cdots, H_{2s+1}, x_p]_B\rangle_{2t+3} & t = ls - 1 (l \geq 2), \\
0 & \text{otherwise,}
\end{cases}
\]

and \([H_{2s+1}, \cdots, H_{2s+1}, x_p]_B\rangle_{2t+3}\) is independent of the choice of the defining system \(B\) obtained from \(4.3\).

**Proof.** When \(t = s - 1\), the result follows from Theorem 4.3.

When \(t = ls - 1 (l \geq 2)\), from Lemma 5.5 (2) we know that there exist defining systems for \([H_{2s+1}, \cdots, H_{2s+1}, x_p]_B\rangle_{2t+3}\) obtained from Theorem 4.3. For any defining system \(B\) given by Theorem 4.3 by (5.6), we get \(c(B) = (1)^{t-1} H_{2s+1} \wedge x_p^{(l-1)}_{p+2(l-1)s}\).
By Definition 5.6

(5.10) \[
\langle H_{2s+1}, \ldots, H_{2s+1}, x_p \rangle_B^t = [c(B)].
\]

Then by Theorem 4.3 we have

\[
d_{2t+3}[x_p]_{2t+3} = [H_{2s+1} \wedge x_{t+1}]_{2t+3} = (-1)^{l-1} \left[ \langle H_{2s+1}, \ldots, H_{2s+1}, x_p \rangle_B \right]_{2t+3}.
\]

Thus

\[
d_{2t+3}[x_p]_{2t+3} = (-1)^{l-1} \left[ \langle H_{2s+1}, \ldots, H_{2s+1}, x_p \rangle_B \right]_{2t+3}.
\]

By the arbitrariness of \(B\), we have \(\langle H_{2s+1}, \ldots, H_{2s+1}, x_p \rangle_B^{2t+3}\) is independent of the choice of the defining system obtained from Theorem 4.3.

For the rest cases of \(t\), the results follows from Theorem 4.3.

The proof of this theorem is completed.

Remark 5.14. We now use the special case that \(H = H_5\) and \(d_9[x_p]_9\) to illustrate the compatibility between Theorems 5.8 and 5.13 for \(s = 2\) and \(t = 3\).

Note that in this case \(H_3 = 0\) and \(H_i = 0\) for \(i > 5\). By Theorem 5.8 we get the corresponding matrix associated to the defining system \(A\) for \(\langle 0, 0, 0, 0, x_p \rangle^t\) is

(5.11) \[
\begin{pmatrix}
0 & -H_5 & 0 & 0 \\
0 & -H_5 & 0 & -x_{p+6}^{(3)} \\
0 & -H_5 & x_{p+4}^{(3)} & -x_{p+4}^{(3)} \\
0 & -H_5 & x_{p+1}^{(3)} & x_p
\end{pmatrix}_{5 \times 5}
\]

and

(5.12) \[
\tilde{d}_9[x_p]_9 = -[\langle 0, 0, 0, 0, x_p \rangle^t_A]_9.
\]

By Theorem 5.13 in this case the matrix associated to the defining system \(B\) for \(\langle H_5, H_5, x_p \rangle_B^t\) is

(5.13) \[
\begin{pmatrix}
H_5 & 0 \\
H_5 & -x_{p+4}^{(1)} \\
x_p
\end{pmatrix}_{3 \times 3}.
\]

and

(5.14) \[
\tilde{d}_9[x_p]_9 = -[\langle H_5, H_5, x_p \rangle_B^t]_9.
\]

We claim that \(\langle H_5, H_5, x_p \rangle^t = \langle 0, 0, 0, 0, x_p \rangle^t\). For any defining system \(B\) above, there is a defining system \(\tilde{B}\)

\[
\begin{pmatrix}
0 & -H_5 & 0 & 0 \\
0 & -H_5 & 0 & 0 \\
0 & -H_5 & x_{p+4}^{(1)} & x_p
\end{pmatrix}_{5 \times 5}
\]
for \( \langle 0, 0, 0, 0, x_p \rangle \) which can be obtained from Theorem 5.1 such that

\[
\langle 0, 0, 0, 0, x_p \rangle_A = \langle H_5, H_5, x_p \rangle_B.
\]

Hence \( \langle H_5, H_5, x_p \rangle_* \subseteq \langle 0, 0, 0, 0, [x_p] \rangle_* \). On the other hand, for any defining system \( A \) above there is also a defining system \( A \)

\[
\begin{pmatrix}
H_5 & 0 \\
H_5 & -x_{p+4}^{(3)} \\
x_p & 0
\end{pmatrix}
\]

\(3 \times 3\)

for \( \langle H_5, H_5, x_p \rangle \) which can be obtained from Theorem 4.9 such that

\[
\langle H_5, H_5, x_p \rangle_A = \langle 0, 0, 0, 0, x_p \rangle_A.
\]

Therefore \( \langle 0, 0, 0, 0, x_p \rangle_* \subseteq \langle H_5, H_5, x_p \rangle_* \), and the claim follows.

By Theorem 5.8 and Remark 5.7 we have that \( \tilde{d}_5[y_p]_5 = -[\langle 0, 0, y_p \rangle_A]_5 = -[-H_5 \wedge y_p]_5 = [H_5 \wedge y_p]_5 \). By Theorem 5.13 \( \tilde{d}_5[y_p]_5 = [H_5 \wedge y_p]_5 \). By Proposition 3.4 \( \tilde{d}_1 = \tilde{d}_4 = d \) and \( \tilde{d}_3 = \tilde{d}_5 = 0 \). It follows that \( \tilde{d}_5 = \tilde{d}_5 \).

By Theorems 5.8 and 4.1 we have that \( \tilde{d}_7[z_p]_7 = \langle 0, 0, 0, z_p \rangle_A \rangle_7 = [-H_5 \wedge z_p]_7 \). By Proposition 5.10 (2), we take \( z_{p+2}^{(2)} \) is an arbitrary \( (p + 2) \)-form satisfying \( d(z_{p+2}^{(2)}) = 0 \). By Remark 5.10 (2), we take \( z_{p+2}^{(2)} = 0 \). Then we have \( \tilde{d}_7[z_p]_7 = 0 \), i.e., \( \tilde{d}_7 = 0 \). At the same time, we also have \( \tilde{d}_7 = 0 \) from Theorem 5.13. Thus \( \tilde{d}_7 = \tilde{d}_7 = 0 \).

By \( E_1^{p,q} = E_1^{p,q} \), \( \tilde{d}_i = \tilde{d}_i \) for \( 1 \leq i \leq 7 \) and \( \langle H_5, H_5, x_p \rangle_* \langle 0, 0, 0, 0, x_p \rangle_* \), we can conclude that \( \tilde{d}_9 = \tilde{d}_9 \) from (5.12) and (5.14).

6. The indeterminacy of differentials in the spectral sequence (1.2)

Let \([x_p] \in E_\infty^{p,q}\). The indeterminacy of \([x_p] \) is a normal subgroup \( G \) of \( H^*(M) \), which means that if there is another element \([y_p] \in H^p(M) \) which also represents the class \([x_p] \in E_\infty^{p,q}\), then \([y_p] - [x_p] \in G\).

In this section, we will show that for \( H = \sum_{t=1}^{r-1} H_{2t+1} \) and \([x_p]_{2r+3}\), the indeterminacy of the differential \( d_{2r+3} [x_p] \in E_2^{p+2r+3,q-2t-2} \) is a normal subgroup of \( H^*(M) \).
From the long exact sequence (3.3), we have a commutative diagram (6.1)

\[
\begin{array}{ccccccc}
\vdots & & & & & & \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \\
\cdots & \to & H_D^{p+q}(K_{p+1}) & \to & H_D^{p+q}(K_{p+1}/K_{p+2}) & \to & H_D^{p+q+1}(K_{p+2}) & \to \cdots \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \\
\cdots & \to & H_D^{p+q}(K_p) & \to & H_D^{p+q}(K_{p}/K_{p+1}) & \to & H_D^{p+q+1}(K_{p+1}) & \to \cdots \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \\
\cdots & \to & H_D^{p+q}(K_{p-1}) & \to & H_D^{p+q}(K_{p-1}/K_{p}) & \to & H_D^{p+q+1}(K_p) & \to \cdots \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \\
\cdots & & & & & & \\
\end{array}
\]

in which any sequence consisting of a vertical map \(i^*\) followed by two horizontal maps \(j^*\) and \(\delta\) and then a vertical map \(i^*\) followed again by \(j^*\) and \(\delta\) and iteration of this is exact. From this diagram there is obtained a spectral sequence in which \(E_1^{p,q} = H_D^{p+q}(K_{p}/K_{p+1})\) and for \(r \geq 2\), \(E_r^{p,q}\) is defined to be the quotient \(Z_r^{p,q}/B_r^{p,q}\), where

\[
E_1^{p,q} = \delta^{-1}(i^{*r-1}H_D^{p+q+1}(K_{p+r})),
\]

\[
B_r^{p,q} = j^*(\ker[i^{*r-1} : H_D^{p+q}(K_{p}) \to H_D^{p+q}(K_{p-r+1}))].
\]

We also have a sequence of inclusions

\[
E_r^{p,q} \subset \cdots \subset E_r^{p,q} \subset E_{r+1}^{p,q} \subset \cdots \subset Z_{r+1}^{p,q} \subset \cdots \subset Z_{\infty}^{p,q}.
\]

By [6, 7], the \(E_r^{p,q}\) term defined above is the same as the one in the spectral sequence (1.2). A similar argument about a homology spectral sequence is given in [16] p. 472-473.

**Theorem 6.1.** Let \(H = \sum_{n=1}^{n-1} H_{2n+1}\) and \([x_p] \in E_r^{p,q} \ (r \geq 3)\), then the indeterminacy of \([x_p] \in E_2^{p,q} \cong H^p(M)\) is the following normal subgroup of \(H^p(M)\)

\[
\frac{\text{im}[\delta : H_D^{p+q-1}(K_{p-r+1}/K_{p}) \to H_D^{p+q}(K_{p}/K_{p+1})]}{\text{im}[d : \Omega^{p-1}(M) \to \Omega^p(M)]},
\]

where \(d\) is just the exterior differentiation and \(\delta\) is the connecting homomorphism induced by the short exact sequence of cochain complexes

\[
0 \to K_{p}/K_{p+1} \to K_{p-r+1}/K_{p+1} \to K_{p-r+1}/K_{p} \to 0.
\]

**Proof.** From the tower (6.3) above, we get a tower of subgroups of \(E_2^{p,q}\)

\[
B_3^{p,q}/B_2^{p,q} \subset \cdots \subset B_r^{p,q}/B_{r-1}^{p,q} \subset \cdots \subset Z_r^{p,q}/B_{r-1}^{p,q} \subset \cdots \subset Z_3^{p,q}/B_2^{p,q} \subset Z_2^{p,q}/B_2^{p,q} = E_2^{p,q}.
\]

Note that \(E_2^{p,q} \cong (Z_3^{p,q}/B_3^{p,q})/(B_r^{p,q}/B_{r-1}^{p,q})\). It follows that the indeterminacy of \([x_p]\) is the normal subgroup \(B_3^{p,q}/B_2^{p,q}\) of \(H^p(M)\).
From the short exact sequences of cochain complexes
\[
0 \to K_p \xrightarrow{i} K_{p-r+1} \xrightarrow{j} K_{p-r+1}/K_p \to 0,
0 \to K_p/K_{p+1} \xrightarrow{i} K_{p-r+1}/K_{p+1} \xrightarrow{j} K_{p-r+1}/K_p \to 0,
\]
we can get the following long exact sequence of cohomology groups
\[
\cdots \xrightarrow{\delta'} H^q_D(K_p) \xrightarrow{\iota^*} H^q_D(K_{p-r+1}) \xrightarrow{j^*} H^q_D(K_{p-r+1}/K_p) \xrightarrow{\delta'} \cdots,
\]
\[
\cdots \xrightarrow{\delta} H^q_D(K_p/K_{p+1}) \xrightarrow{i^*} H^q_D(K_{p-r+1}/K_{p+1}) \xrightarrow{j^*} H^q_D(K_{p-r+1}/K_p) \xrightarrow{\delta} \cdots,
\]
where \(\delta\) and \(\delta'\) are the connecting homomorphisms.

Combining (3.3) and (6.4), we have the following commutative diagram of long exact sequences
\[
\begin{array}{ccc}
H^{p+q-1}_D(K_{p-r+1}/K_p) & \xrightarrow{\delta'} & H^{p+q}_D(K_{p-r+1}) \\
\downarrow{\delta} & & \downarrow{j^*} \\
H^{p+q}_D(K_p/K_{p+1}) & \xrightarrow{i^*} & H^{p+q}_D(K_{p-r+1}/K_p) \\
\end{array}
\]
\[
\begin{array}{ccc}
H^{p+q}_D(K_{p-r+1}/K_p) & \xrightarrow{\delta} & H^{p+q}_D(K_{p-r+1}/K_p) \\
\downarrow{j^*} & & \downarrow{i^*} \\
H^{p+q+1}_D(K_{p+1}) & \xrightarrow{\delta} & H^{p+q}_D(K_{p-r+1}/K_p) \\
\end{array}
\]

Using the commutative diagram above and the fact that \(i^{*r-1} = i^*\), we have
\[
\begin{align*}
B^p_r & = j^*([\ker[i^{*r-1}], H^{p+q}_D(K_p) \to H^{p+q}_D(K_{p-r+1})]) \\
& = j^*([\ker[i^{*r}], H^{p+q}_D(K_p) \to H^{p+q}_D(K_{p-r+1})]) \\
& \cong j^*([\ker[i^{*r}], H^{p+q-1}_D(K_{p-r+1}/K_p) \to H^{p+q}_D(K_p)]) \\
& \cong \ker[\delta \colon H^{p+q-1}_D(K_{p-r+1}/K_p) \to H^{p+q}_D(K_{p-r+1}/K_p)].
\end{align*}
\]

When \(r = 2\), from (6.5) we have that
\[
\delta = \delta' j^* : H^{p+q-1}_D(K_{p-1}/K_p) \to H^{p+q}_D(K_{p}/K_{p+1}).
\]

From (3.4), it follows that \(\delta = d_1\). By Proposition (3.4) \(\delta = d\). Thus we have
\[
\begin{align*}
B^p_2 & \cong \ker[\delta \colon H^{p+q-1}_D(K_{p-1}/K_p) \to H^{p+q}_D(K_{p}/K_{p+1})] \\
& \cong \ker[d : \Omega^{p-1}(M) \to \Omega^p(M)].
\end{align*}
\]

The desired result follows. \(\square\)

By Theorem 6.1 we obtain the following corollary.

**Corollary 6.2.** In Theorem 5.8 for \(d_{2t+3}[x_p]_{2t+3} \in E_{2t+3}^{p+2t+3,q-2t-2}\) we have the indeterminacy of \(d_{2t+3}[x_p]\) is a normal subgroup of \(H^{p+2t+3}(M)\)
\[
\begin{align*}
\ker[\delta : H^{p+q}_D(K_{p+2t+3}/K_{p+2t+4}) \to H^{p+q+1}_D(K_{p+2t+3}/K_{p+2t+4})] & \\
\ker[d : \Omega^{p+2t+2}(M) \to \Omega^{p+2t+3}(M)].
\end{align*}
\]

where \(d\) is just the exterior differentiation and \(\delta\) is the connecting homomorphism of the long exact sequence induced by the short exact sequence of cochain complexes
\[
0 \to K_{p+2t+3}/K_{p+2t+4} \xrightarrow{i} K_{p+1}/K_{p+2t+4} \xrightarrow{j} K_{p+1}/K_{p+2t+3} \to 0.
\]

**Proof.** In Theorem 6.1 \(r, p\) and \(q\) are replaced by \(2t+3, p+2t+3\) and \(q-2t-2\), then the desired result follows. \(\square\)
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Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, U.S.A

E-mail address: wli@math.okstate.edu

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P.R.China

E-mail address: xgliu@nankai.edu.cn

School of Mathematical Sciences, Nankai University, Tianjin 300071, P.R.China

E-mail address: wanghe85@yahoo.com.cn