Behaviour of matter close to the event horizon

Tapas K. Das

1 Division of Astronomy, Department of Physics and Astronomy, University of California at Los Angeles, Box 951562, Los Angeles, CA 90095-1562, USA
2 Institute of Geophysics and Planetary Physics, University of California at Los Angeles, Box 951567, Los Angeles, CA 90095, USA

Accepted 2003 December 5. Received 2003 November 20; in original form 2003 September 16

ABSTRACT

Investigation of the behaviour of accreting matter close to the black hole event horizon is of fundamental importance in relativistic and high-energy astrophysics because it provides the key features of the diagnostic spectra of stellar-mass and supermassive black holes. In this paper, we examine the terminal behaviour of general relativistic matter in multi-transonic, advective black hole accretion discs. We compute, for the first time we believe, the values of various dynamical and thermodynamic multi-transonic flow variables extremely close ($\lesssim 0.01r_g$) to the event horizon, and study the dependence of these variables on fundamental accretion parameters. Our calculation is useful for a better understanding of Hawking radiation from acoustic black holes.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – relativity.

1 INTRODUCTION

Gravitational capture of surrounding fluid by massive astrophysical objects is known as accretion. There remains a major difference between black hole (BH) accretion and accretion on to other cosmic objects including neutron stars and white dwarfs. For celestial bodies other than BHs, infall of matter terminates by a direct collision either with the hard surface of the accretor or with the outer boundary of the magnetosphere, resulting in luminosity through energy release from the surface. For BH accretion, matter ultimately dives through the event horizon from where radiation is prohibited to escape according to the rule of classical general relativity (GR), and the emergence of luminosity occurs on the way towards the BH event horizon. The efficiency of the accretion process may be thought of as a measure of the fractional conversion of gravitational binding energy of matter to the emergent radiation, and is considerably high for BH accretion compared with accretion on to any other astrophysical objects. Hence accretion on to classical astrophysical BHs has been recognized as a fundamental phenomenon of increasing importance in relativistic and high-energy astrophysics (Frank, King & Raine 1992, hereafter FKR; Shapiro & Teukolsky 1983). The extraction of gravitational energy from BH accretion is believed to power the energy generation mechanism of X-ray binaries and of the most luminous objects of the Universe, the quasars and active galactic nuclei (AGN). BH accretion is thus the most appealing way through which the all-pervading power of gravity is explicitly manifested. If the infalling matter does not possess intrinsic angular momentum, the accretion flow remains spherically symmetric. However, interstellar/intergalactic fluid is always likely to possess non-vanishing rotational energy, sufficient dynamically to break the spherical symmetry, and, in almost all real physical situations, accreting matter is thrown into circular orbits around the central accretor, leading to the formation of the accretion disc around Galactic and extragalactic BHs.

If the instantaneous dynamical velocity and local acoustic velocity of the accreting fluid, moving along a space curve parametrized by $r$, are $u(r)$ and $a(r)$ respectively, then the local Mach number $M(r)$ of the fluid can be defined as $M(r) = u(r)/a(r)$. The flow will be locally subsonic or supersonic according to $M(r) < 1$ or $> 1$, i.e. according to $u(r) < a(r)$ or $u(r) > a(r)$. The flow is transonic if at any moment it crosses $M = 1$. This happens when a subsonic-to-supersonic or supersonic-to-subsonic transition takes place either continuously or discontinuously. The point(s) where such crossing takes place continuously is (are) called sonic point(s), and the points where such crossing takes place discontinuously are called shocks or discontinuities. In order to satisfy the inner boundary conditions imposed by the event horizon, accretion on to BHs exhibits transonic properties in general, which further indicates that formation of shock waves is possible in astrophysical fluid flows on to Galactic and extragalactic BHs (Das 2001, and references therein; Das, Pendharkar & Mitra 2003a, and references therein).

The study of the dynamical behaviour of such transonic accretion flows near the BH event horizon is considered to be immensely important. Owing to the strong curvature of space–time close to the BH, accreting fluid is expected to show extreme behaviour just before plunging into the event horizon; this tremendously hot, ultra-fast flow with its high-density profile is supposed to provide the key features of the diagnostic high-energy spectra of galactic and extragalactic BH candidates.

© 2004 RAS

Downloaded from https://academic.oup.com/mnras/article-abstract/349/1/375/3101619
by guest
on 27 July 2018
2.1 The energy momentum tensor

Following the pioneering contributions of Bardeen, Press & Teukolsky (1972) and Novikov & Thorne (1973, NT hereafter), a number of papers have dealt with the general relativistic (GR) BH accretion disc. The initial attempt by Fukue (1987, and references therein) and Chakrabarti (1990, 1996, hereafter C96, with errors appearing in the expression for mass and entropy accretion rate) to study the GR transonic BH accretion analytically was followed by other independent analytical works (Kafatos & Yang 1994; Yang & Kafatos 1995, YK hereafter; Pariev 1996; Peitz & Appl 1997; Lasota & Abramowicz 1997, hereafter LA; Lu et al. 1997). Although they made a number of important contributions to the study of GR advective BH accretion discs, none of the above-mentioned works investigated the flow behaviour close to the event horizon in detail. Gammie and Popham (Gammie & Popham 1998; Popham & Gammie 1998) performed the only self-consistent analytical work present in the literature so far that exploits the complete GR formalism to study various flow variables near the event horizon. Recent interesting work by Becker & Le (2003) uses post-Newtonian asymptotic analysis to study the properties of the inner region of advective accretion flows, and their results are in good agreement with the GR description. However, neither Gammie and Popham nor Becker & Le capture the multi-transonic properties of the flow which are of considerable astrophysical importance. Only the multi-transonic flow can produce shock waves in post-Newtonian BH accretion discs (Das et al. 2003a, and references therein). Shock waves in rotating flows may provide an efficient mechanism for conversion of significant amounts of gravitational energy into radiation by randomizing the directed infall motion of the accreting fluid, and the hot and dense post-shock flow is considered to be a powerful tool in understanding various important astrophysical phenomena like the spectral properties of BH candidates (Chakrabarti & Titarchuk 1995 and references therein), the formation and dynamics of accretion-powered cosmic jets (Das & Chakrabarti 1999, and references therein; Das, Rao & Vadawale 2003b), and the origin of quasi-periodic oscillations in Galactic sources (Das 2003, and references therein). One thus understands the pressing need for a self-consistent analytical model capable of studying the behaviour of a GR, multi-transonic accretion flow sufficiently close to the event horizon.

Motivated by the above-mentioned arguments, in this paper we concentrate on the stationary, axisymmetric, complete GR accretion solutions in the Schwarzschild metric which contain multiple critical points, and for such solutions we calculate all relevant dynamical and thermodynamic flow variables extremely close to the event horizon. Such a solution scheme for purely spherical GR mono-transonic BH accretion was outlined by Das (2002). We also exhaustively study the dependence of such variables (close to the event horizon) on all important initial boundary conditions governing the flow. Our generalized calculation is valid for super- as well as for sub-Edington accretion on to a BH of any mass. We thus provide a useful and self-consistent procedure to study the terminal behaviour of matter in multi-transonic, GR advective accretion discs around BHs, which has never been done in any of the existing works on BH accretion discs. Further calculations, with the ultimate ambition of modelling the most general viscous, transonic, shocked hydromagnetic accretion in Kerr–Neumann space–time, are in progress (Das, Wiita & Barai, in preparation) and will be reported elsewhere.

At this point, we would like to mention that, instead of dealing with the so-called standard disc model with significantly high angular momentum (where the presence of viscous stress allows the infall of accreting material on to the BH, i.e. outward viscous transport of angular momentum takes place to weaken the centrifugal barrier), we rather mainly concentrate on accretion flows with relatively low intrinsic angular momentum (sub-Keplerian angular momentum distribution) where substantially significant advection velocities may be obtained even for practically inviscid flow, although our formalism allows sufficient flexibility to incorporate the viscous transonic flows as well: see §4.1 for more detail. Such weakly rotating flows have not been explored much in the literature, although they are well exhibited in nature for various real physical situations like detached binary systems fed by accretion from OB stellar winds (Illarirov & Sunyaev 1975; Liang & Nolan 1984), semi-detached low-mass non-magnetic binaries (Bisikalo et al. 1998), and supermassive BHs fed by accretion from slowly rotating central stellar clusters (Illarirov & Sunyaev 1988; Ho 1999, and references therein). Even for a standard Keplerian accretion disc, turbulence may produce such low angular momentum flow (see e.g. Igumenshchev & Abramowicz 1999, and references therein).

2 FORMALISM

2.1 The energy momentum tensor

To provide the most general description of fluid flow in strong gravity, one needs to solve the equations of motion for the fluid and the Einstein equations. The problem may be made simplified by assuming the accretion to be non-self-gravitating so that the fluid dynamics may be dealt with in a metric without back-reactions. Unless otherwise mentioned for any specific reason(s), hereafter we define Schwarzschild radius \( r_g \) = \( 2GM_\text{BH}/c^2 \), where \( M_\text{BH} \) is the mass of the BH, \( G \) is the universal gravitational constant and \( c \) is the velocity of light. The radial distances and velocities are scaled in units of \( r_g \) and \( c \) respectively and all other derived quantities are scaled accordingly; \( G = c = M_\text{BH} = 1 \) is used. We use the Boyer–Lindquist coordinate with signature \( +++++ \), and an azimuthally Lorentz-boosted orthonormal tetrad basis corotating with the accreting fluid. We define \( \lambda \) to be the specific angular momentum of the flow and neglect any gravo-magneto-viscous non-alignment between \( \lambda \) and BH spin angular momentum.

Let \( v_\mu \) be the four-velocity of the (perfect) accreting fluid. The energy momentum tensor \( \mathcal{E}^{\mu\nu} \) of such a flow could be written as

\[
\mathcal{E}^{\mu\nu} = (\rho + p) v_\mu v_\nu + p g_{\mu\nu}, \quad \text{or} \quad T = (\rho + p) v^\mu v^\nu + pg. \tag{1a}
\]

A complete description of flow behaviour could be obtained by taking the covariant derivative of \( \mathcal{E}^{\mu\nu} \) and \( \rho v^\nu \) to obtain the energy momentum conservation equations and the conservation of baryonic mass.

However, at this stage, the complete solution remains analytically untenable unless we are forced to adopt a number of simplified approximations. In this paper, as already mentioned in Section 1, we would like to study the inviscid accretion of hydrodynamic fluid in the
Killing the local ‘of Bernoulli

The temporal (zeroth) component of the non-relativistic Euler equation provides

$$H(r) = 2\sqrt{\frac{3}{2}} \sqrt{\frac{p(r^3 - \lambda^2 r + \lambda^2)}{\rho}}.$$  

(2b)

It is trivial to show that, for the equation of state and the metric used in this paper,

$$p = \frac{\rho \alpha^2 (\gamma - 1)}{\gamma^2 - \gamma (1 + \alpha^2)}.$$  

(2c)

Using the above equation and the relation (NT; FKR) $a^2(r) = \gamma \kappa T(r)/\mu m_H = \Theta^2 T(r)$, where $\Theta = \sqrt{\gamma \kappa / \mu m_H}$, $\mu$ is the mean molecular weight, $m_H \sim m_p$ is the mass of the hydrogen atom and $\kappa$ is Boltzmann’s constant, we finally obtain the disc height as

$$H(r) = 5.66 \Theta \Gamma^{1/2}(r) \left( \frac{\gamma - 1}{\gamma} \right) \left[ \frac{r^3 - \lambda^2 (r - 1)}{\gamma - (1 + \Theta^2 T(r))} \right]^{1/2}.$$  

(2d)

2.3 The conserved specific energy of the accretion flow

The temporal (zeroth) component of equation (1b) leads to the conservation of specific flow energy $E$ (the relativistic analogue of Bernoulli’s constant) along each streamline as (Anderson 1989) $E = hv_i$, where $v_i$ is the four-velocity. From the normalization condition of the four-velocity ($v_{\mu}v^\mu = -1$), one writes (NT)

$$v_i^2 = \frac{4r(1 - r^2)}{(1 - a^2)(1 - 2\Omega \lambda)} \left( \frac{g_{\phi \phi}}{2} + 2\lambda g_{\phi \phi} \right).$$  

(3)
where $\Omega \equiv \omega^\phi/\omega^t$ is the angular velocity and $u$ is the radial three-velocity in the corotating fluid frame. The required metric elements are calculated as
\[ g_{tt} = -\frac{r-1}{r}, \quad g_{\phi\phi} = 4r^2, \quad g_{t\phi} = g_{t\theta} = 0 \quad (4a) \]
to obtain the expression for the angular velocity as
\[ \Omega = \left( \frac{r-1}{2r^3} \right) \lambda. \quad (4b) \]
Hence $v_t$ for a non-spinning (Schwarzschild) BH may be derived as
\[ v_t = r \sqrt{\frac{r-1}{(1-u^2)(r^3 + \lambda^2 - \lambda^2 r)}}. \quad (4c) \]
Using the value of specific flow enthalpy $h$ (defined in Section 2.1), equation (4c), and $u = aM(r) = \Theta T^{1/2}(r)M(r)$, $M(r)$ being the radial flow Mach number, we obtain the expression for the conserved specific energy (which includes the rest mass energy) as
\[ \mathcal{E} = \frac{r(\gamma-1)}{[\gamma - 1 + \Theta^2 T(r)]} \sqrt{\frac{r-1}{1 - \Theta^2 T(r)M^2(r)[r^3 - \lambda^2 r + \lambda^2]}}. \quad (4d) \]

2.4 The mass and entropy accretion rate

We obtain the mass accretion rate $\dot{M}_m$ by integrating the second part of equation (1b):
\[ \dot{M}_m = 4\pi L(r) \sqrt{\frac{r(r-1)}{1 - u^2(r) \Sigma(r)_{H(r)}}}, \quad (5a) \]
where $\Sigma(r)_{H(r)}$ is the vertically integrated surface density of the disc material. Hence one writes
\[ \Sigma(r)_{H(r)} = \int_{r_H(r)}^{r} \rho(r') \, dr', \quad (5b) \]
where $\rho(r)$ is the mean value of the radial density on the equatorial ($z = 0$) plane. We use the value of $H(r)$ from equation (2c) to perform the above integration and then substitute the value of $\Sigma(r)_{H(r)}$ in equation (5a), and use $u(r) = a(r)M(r) = \Theta T^{1/2}M(r)$ finally to obtain the mass accretion rate as
\[ \dot{M}_m = 71.05 \rho(r)\Theta T^{1/2}(r)M(r) \sqrt{\frac{r(r-1)(\gamma - 1)[r^3 - \lambda^2(r-1)]}{\gamma[1 - \Theta^2 T(r)M^2(r)][\gamma - 1 + \Theta^2 T(r)]}}. \quad (5c) \]
The entropy accretion rate $\dot{\Sigma}$ can be defined as a quasi-constant multiple of the mass accretion rate:
\[ \dot{\Sigma} = \rho^{(\gamma-1)/2} \dot{M}_m = K^{1/(\gamma-1)} \dot{M}_m. \quad (5d) \]
After substituting the value of $\dot{M}_m$ from equation (5c) with the transformation $T(r) \longrightarrow a^2(r)/\Theta^2$, $M(r) = [u(r), a(r)]$, $\dot{\Sigma}$ reads
\[ \dot{\Sigma} = 71.05 \rho^{\gamma+1} \frac{a^2}{2a^2 + M^2} \int_{r_H}^{r} \frac{r(r-1)[r^3 - \lambda^2(r-1)]}{1 - u^2} \left[ \frac{a^2(\gamma - 1)}{\gamma(1 + a^2)} \right] \frac{\lambda^2}{r^2} \, dr. \quad (5e) \]
Note that, in the absence of creation and annihilation of matter, while the mass accretion rate is an absolute constant of motion, the entropy accretion rate is not. As the expression for $\dot{\Sigma}$ contains $K(= p\rho^\gamma)$, which is a measure of the density field of the flow, $\dot{\Sigma}$ remains constant throughout the flow only if the local entropy density of the flow remains unchanged. Thus $\dot{\Sigma}$ is a constant of motion for shock-free polytropic accretion and becomes discontinuous (increases) at the shock location, if a shock forms in the accretion. Thus for a shock-free non-dissipative flow, $d\dot{\Sigma}/dr = 0$ along a streamline.

2.5 Dynamical velocity gradient and sonic quantities

We take the logarithmic differentiation (with respect to $r$) of both sides of equation (5e) to obtain
\[ \frac{d}{dr} \left( \frac{a}{\gamma + 1} \right) \left[ \frac{1}{u(a^2 - 1)} \frac{\, dr}{dr} + f(r, \lambda) \right], \quad (6a) \]
where
\[ f(r, \lambda) = \frac{1 - 2r}{2r(r-1)} + \frac{\lambda^2 - 3\lambda}{2(r^3 - \lambda^2 r + \lambda^2)}. \quad (6b) \]
We now transform \( T(r) \to a(r) \), \( M(r) \to \{ u(r), a(r) \} \) in the expression for \( \mathcal{E} \), differentiate both sides of equation (4d) and substitute the value of \( da/dr \) from equation (6a) to obtain the dynamical velocity gradient as

\[
\left( \frac{du}{dr} \right) = \left( \frac{1}{\mathcal{N}} \right) \left[ \frac{1}{\frac{\mathcal{V}^2}{\delta r^2} + \frac{\mathcal{V}^4}{\delta r^4} + \frac{\mathcal{V}^6}{\delta r^6}} \right] \to \mathcal{N} \frac{d\mathcal{V}}{dr}.
\]

(6c)

Hereafter, we use the notation \([P_3]\) for a set of values of \( \{ \mathcal{E}, \lambda, \gamma \} \), which will be our three-parameter initial boundary condition determining the flow behaviour. Since the flow is assumed to be smooth everywhere, if the denominator of equation (6c) vanishes at any radial distance \( r \), the numerator must also vanish there to maintain the continuity of the flow. One therefore arrives at the so-called ‘sonic point’ (alternatively, the ‘critical point’) conditions by simultaneously making the numerator and denominator of equation (6c) equal zero. The sonic point conditions can be expressed as

\[
u_s = \sqrt{\frac{2}{\gamma + 1} a_s} = \sqrt{\frac{1 - \frac{1}{\mathcal{V}^2}}{\frac{1}{\mathcal{V}^2} + \frac{1}{\mathcal{V}^4} + \frac{1}{\mathcal{V}^6}}}
\]

(6d)

where the subscript \( s \) indicates that the quantities are to be measured at the sonic point(s). For a fixed \([P_3]\), we substitute the values of \( u_s, a_s \) and \( r_s \) in the expression of \( \mathcal{E} \) with the substitution \( T(r) = \frac{a^2(r)}{\mathcal{V}^2} \), \( M(r) = \{ u(r), a(r) \} \), and obtain the following polynomial, the solution of which provides the sonic point(s) \( r_s \):

\[
\mathcal{E}^2 \left[ \frac{r_s}{3} - \lambda^2 (1 - r_s) \right] - \frac{r_s - 1}{1 - \Psi (r_s, \lambda)} \frac{r_s (\gamma - 1)}{\psi (r_s, \lambda)} = 0,
\]

(6e)

where

\[
\psi (r_s, \lambda) = \left[ 1 + \frac{\gamma + 1}{2} \Psi (r_s, \lambda) \right]
\]

and

\[
\Psi (r_s, \lambda) = \left[ \frac{1}{\frac{\mathcal{V}^2}{\delta r^2} + \frac{\mathcal{V}^4}{\delta r^4} + \frac{\mathcal{V}^6}{\delta r^6}} \right].
\]

To determine the behaviour of the solution near the sonic point, one needs to evaluate the value of \( (du/dr)_s \) at that point [the ‘critical velocity gradient’] by applying L’Hospital’s rule to equation (6c):

\[
\left( \frac{du}{dr} \right) = L_{r \to r_s} \left( \frac{d\mathcal{V}}{dr} \right).
\]

(6f)

We use the expression of \( \mathcal{N} \) and \( \mathcal{D} \) from equation (6c), and calculate \( L_{r \to r_s} \left( \frac{d\mathcal{V}}{dr} \right) \) and \( L_{r \to r_s} \left( \frac{d\mathcal{D}}{dr} \right) \). After a number of complicated algebraic manipulations, equation (6f) reduces to the following polynomial which could be solved to obtain the critical velocity gradient:

\[
\frac{2}{\gamma + 1} \left( \frac{3}{a_s^2} \right)^2 \left( \frac{du}{dr} \right)_s^2 + 4\xi (r_s, \lambda) \left[ \frac{1}{u_s^2} - 1 \right] \left( \frac{du}{dr} \right)_s^2,
\]

\[+ \frac{2}{\gamma + 1} \xi^2 (r_s, \lambda) \left[ \frac{r_s (3 - 1)}{\gamma + 1} - \frac{2r_s - 1}{r_s (r_s - 1)} - \frac{3r_s^2 - \lambda^2}{r_s^2 + \lambda^2 (1 - r_s)} \right],
\]

\[+ \frac{20r_s^4 - 12r_s^2 - 2\lambda^2 (3r_s - 2)}{5r_s^4 - 4r_s^2 - \lambda^2 (3r_s^2 - 4r_s + 1)} = 0,
\]

(6g)

where

\[
\xi (r_s, \lambda) = \left[ \frac{1 - 2r_s}{2r} + \left( \frac{\lambda^2 - 3r_s^2}{2r^3 + \lambda^2 (1 - r_s)} \right) \right]_{r=r_s}.
\]

3 RESULTS

3.1 Parameter space for multi-transonic accretion

For a particular \([P_3]\), if \( \mathcal{A}[P_3] \) denotes the universal set representing the entire parameter space covering all possible values of \([P_3]\), and if \( \mathcal{B}[P_3] \) represents a particular subset of \( \mathcal{A}[P_3] \) which contains only the values of \([P_3]\) providing more than one real root of equation (6e), then \( \mathcal{B}[P_3] \) can further be decomposed into two subsets \( \mathcal{C}[P_3] \) and \( \mathcal{D}[P_3] \) such that \( \mathcal{C}[P_3] \subseteq \mathcal{B}[P_3] \) for \( \mathcal{C}(r_u) > \mathcal{C}(r_{out}) \), and \( \mathcal{D}[P_3] \subseteq \mathcal{B}[P_3] \) for \( \mathcal{D}(r_u) < \mathcal{D}(r_{out}) \); then for \([P_3] \in \mathcal{C}[P_3] \), we get multi-transonic accretion where two real physical inner and outer (with respect to the BH location) X-type sonic points \( r_u \) and \( r_{out} \) encompass one O-type unphysical middle sonic point \( r_{mid} \) in between. One can obtain multi-transonic winds for \( \mathcal{C}(r_u) < \mathcal{C}(r_{out}) \), e.g. \([P_3] \in \mathcal{D}[P_3] \) provides multi-transonic winds.

© 2004 RAS, MNRAS 349, 375–384

Downloaded from https://academic.oup.com/mnras/article-abstract/349/1/375/3101619
by guest on 27 July 2018
In Fig. 1, the wedge-shaped surfaces represent \((E, \lambda) \in [P_3] \subset C[P_3] \subset B[P_3]\) for three different \(\gamma = 1.33\) (area bounded by dotted lines), 1.43 (area bounded by dashed lines) and 1.53 (area bounded by solid lines). If \(E_{\text{max}}\) be the maximum value of the energy and if \(\lambda_{\text{max}}\) and \(\lambda_{\text{min}}\) be the maximum and minimum values of the angular momentum respectively for \(C[P_3]\) for a fixed value of \(\gamma\), then \([E_{\text{max}}, \lambda_{\text{max}}, \lambda_{\text{min}}]\) non-linearly anti-correlates with \(\gamma\). It is obvious from the figure that, as the flow makes a transition from its ultra-relativistic\(^1\) to its purely non-relativistic limit, the area representing \(C[P_3]\) decreases. Using the formalism developed in Section 2, it is easy to obtain similar regions in \([P_3]\) space representing the multi-transonic winds as well. However, as our main interest is to study the behaviour of the infalling matter close to the BH, we do not concentrate on wind solutions in this paper and hence do not provide the representative surfaces for \([P_3] \in D[P_3]\).

3.2 Integral curves of motion

In Fig. 2, we show the integral curves of motion for multi-transonic accretion. While the distance from the event horizon of the central BH (scaled in units of \(r_S\) and plotted on a logarithmic scale) is plotted along the \(x\)-axis, the local Mach number of the flow is plotted along the \(y\)-axis. For values of \(C[P_3]\) shown in the figure, ABCD represents the accretion passing through the outer sonic point B, the location of which can be found by solving equation (6e). EBI represents the self-wind. Flow along GFH passes through the inner sonic point F and encompasses a middle sonic point \(r_{\text{mid}}\), the location of which is shown in the figure using an asterisk. Similar topologies could be obtained for any other \([P_3] \subset C[P_3]\). The overall scheme for obtaining the above-mentioned integral curves is as follows.

First we compute \(r_{\text{in}}, r_{\text{mid}}\) and \(r_{\text{out}}\) by solving equation (6e). Then we obtain the dynamical velocity gradient of the flow at sonic points by solving equation (6g). For a chosen \(M_{\text{in}}\) (scaled in the units of the Eddington rate \(M_{\text{Edd}}\)), we then compute the local dynamical flow velocity \(u(r)\), the local polytropic sound speed \(a(r)\), the local radial Mach number \(M(r)\), the local fluid density \(\rho(r)\) and any other related dynamical or thermodynamic quantities by solving the equations (6a)–(6g) from the outer as well as from the inner sonic point using fourth-order Runge-Kutta method. Flows passing through \(r_{\text{out}}\) may generate an amount of entropy density \(\Delta \Xi = \Xi(r_{\text{in}}) - \Xi(r_{\text{out}})\) by undergoing a steady, standing, Rankine-Hugoniot type of shock transition to produce subsonic, hotter and shock-compressed post-shock solutions (branch GFH) which dive on to the event horizon supersonically after passing through \(r_{\text{in}}\). Such shocks are time-like three-surfaces of first-order discontinuities, and are formed if the following equation is satisfied:

\[
\Pi \Sigma \left( \frac{T_{\text{\text{-}}} - T_{\text{\text{+}}}}{T_{\text{\text{+}}}} \right)^{4 \gamma} \left( \frac{1 - u_+^3}{1 - u_-^3} \right)^{\frac{3 - 2 \gamma}{4}} = 1,
\]

where \(\Sigma(=M_{\text{\text{-}}}/M_{\text{\text{+}}})\) and \(\Pi(=\Xi_{\text{\text{+}}}/\Xi_{\text{\text{-}}})\) are shock compression and the entropy enhancement ratio at the shock, and \(T(\text{-}/\text{+})\) and \(u(\text{-}/\text{+})\) are the pre-/post-shock temperature and dynamical velocities of the flow respectively. If \([P_3] \in S[P_3]\) provides real solutions of equation (7), it directly follows that \(S[P_3] \subset C[P_3]\) with \([P_3] \in N[P_3][P_3] \subset C[P_3], [P_3] \notin S[P_3]\) representing the region producing non-stationary shock solutions.

\(^1\)By the terms ‘ultra-relativistic’ and ‘purely non-relativistic’ we mean a flow with \(\gamma = 4/3\) and \(\gamma = 5/3\) respectively, according to the terminology used in FKR.
3.3 Computation of the quasi-terminal values

This section reflects the main findings of our paper. We define ‘quasi-terminal values’ (QTV) to be the value of any flow variable \( V_t \) as \( V_t^{QTV} \) at a distance \( r_f = r_e(1+\delta) \) with \( 0 < \delta \ll 1 \). The terminal value \( \delta = 0 \) of \( V_t \) diverges because of the singularity at the event horizon. For any \( [P_1] \in \mathcal{C}[P_1] \), we provide a simultaneous numerical solution of equations (4d)–(4g) to calculate various \( V_t^{QTV} \). The basic scheme of such a calculation is the following.

Consider the transonic accretion passing through the outer sonic point \( r_{\text{out}} \). We take any \( [P_1] \in \mathcal{C}[P_1] \) and integrate the flow for such \( [P_1] \) up to a distance \( \delta \) from the event horizon and calculate the values of various \( V_t^{QTV} \), like \( \{M(r), T(r), \rho(r), p(r)\}^{QTV} \) etc. at \( r_f \). We have developed an efficient numerical code, which will automatically pick up all \( [P_1] \in \mathcal{C}[P_1] \) out of the entire set \( \mathcal{C}[P_1] \), and for each \( [P_1] \in \mathcal{C}[P_1] \), it will calculate any \( V_t^{QTV} \) at \( r_f \) for any small value of \( \delta \). For \( \delta = 0.01 \) and \( [P_1] \in \mathcal{C}[P_1] \), Fig. 3 shows the QTV of the flow Mach number \( M(r) \), the temperature \( T(r) \), the rest-mass density \( \rho(r) \), and the flow pressure \( p(r) \) for shock-free accretion passing through \( r_{\text{out}} \). \( [\mathcal{E}, \lambda] \) are plotted in geometric units. Any other \( V_t^{QTV} \) for any \([P_1]\) can be obtained for \( \delta \ll 0.01 \) as well with higher computational cost. Similar calculations can be done for mono-transonic flows as well. For flows with shocks, the general profile of \( V_t^{QTV} \) as a function of \([P_1]\) remains fairly unaltered with the following difference of the numerical values of \( V_t^{QTV} \) for shocked and shock-free flows:

\[
M_{\text{Shock}} \{M_{\text{Noshock}}, \{T^{QTV}, \rho^{QTV}, p^{QTV}\}_{\text{Shock}}\},
\]

\[
\{T^{QTV}, \rho^{QTV}, p^{QTV}\}_{\text{Noshock}}.
\]

We observe that the QTV of \( u \) at 0.01\( r_e \) may become as high as 99 per cent of the velocity of light, leading to very large values of the flow Mach number near the horizon. The disc thickness in the close proximity of the event horizon turns out to be of the order of few kilometres. The figure is drawn for accretion at the Eddington rate on to a 10-M\(_{\odot}\) BH. Our generalized calculation allows us to obtain all such \( V_t^{QTV} \)s for any accretion rate on to BHs of any mass. The following trend is observed for the variation of \( V_t^{QTV} \) with \([P_1]\):

\[
M^{QTV} \propto \frac{1}{\mathcal{E}\lambda\gamma}, \quad u^{QTV} \propto \frac{\lambda}{\mathcal{E}\gamma}, \quad \rho^{QTV} \propto \frac{\lambda}{\mathcal{E}\gamma}, \quad T^{QTV} \propto \mathcal{E}\lambda\gamma,
\]

\[
\rho^{QTV} \propto \mathcal{E}\lambda\gamma, \quad H^{QTV}(r) \propto \frac{\mathcal{E}}{\lambda},
\]

where \( \propto \) indicates proportionality.

A strongly rotating purely non-relativistic flow with high energy content produces the maximum disc temperature as well as flow pressure at the event horizon, whereas a ultra-relativistic strongly rotating flow with lower energy content will produce high-density ultra-fast accretion. The disc thickness is higher for purely non-relativistic weakly rotating flows with large values of rest-mass energy at infinity, indicating the possibility of a quasi-spherical inner disc structure for multiple stellar wind-driven accretion on to BHs. For shocked accretion, the tendency of forming a quasi-spherical inner disc region is much more prominent because the entropy generated at the shock will puff up the post-shock disc structure to provide a funnel-like surface which supports the natural collimation of accretion-powered jets (Das & Chakrabarti 1999).
4 DISCUSSION

4.1 Predictions for viscous flow

In this work, viscous transport of the angular momentum is not explicitly taken into account. Even 30 years after the discovery of standard accretion disc theory (Shakura & Sunyaev 1973), exact modelling of viscous multi-transonic BH accretion, including proper heating and cooling mechanisms, is still quite an arduous task, even for post-Newtonian flow, let alone for GR accretion. Nevertheless, the extremely large radial velocity close to the BH implies that \( \tau_{\text{in}} \ll \tau_{\text{visc}} \) (\( \tau_{\text{in}} \) and \( \tau_{\text{visc}} \) are the infall and the viscous time-scales respectively), hence our assumption of inviscid flow is not at all unjustified, at least up to a few to tens of \( r_g \) or so. Far away from the BH, this may not be a very good assumption. However, one understands that one of the most significant effects of the introduction of viscosity would be the reduction of radial angular momentum. We found that the location of the sonic points anti-correlates with \( \lambda \) (weakly rotating flow makes the dynamical velocity gradient steeper), which indicates that for viscous flow the sonic points will be pushed further out and the flow would become supersonic at a larger distance for the same set of other initial boundary conditions. The terminal values of the Mach number and disc height anti-correlate, while the density and pressure correlates with \( \lambda \). Our calculations, if applied for viscous flow, would thus produce a more supersonic and quasi-spherical structure near the event horizon, while the terminal flow pressure and density would be reduced.
4.2 Spectral signature of BH spin

One of the most tantalizing issues in BH astrophysics is to study whether the spin of the BH (the Kerr parameter $a$) could be determined using any observational means – the spectral signature of the BH, for example. Our investigation of GR accretion flows in the Kerr metric may provide an important step towards a better understanding of this problem. For accretion on to the Schwarzschild BHs, we have been able to track the infalling matter up to the point extremely close to the event horizon. We can successfully perform the same procedure (with an increasing degree of mathematical complexity) for accretion on to a Kerr BH as well, and can calculate various dynamical and thermodynamic flow variables up to the point extremely close to the event horizon (as close as it could be) as a function of the Kerr parameter $a$ (Das, Wiita & Barai, in preparation). Hence the terminal behaviour of infalling matter (e.g. flow temperature, pressure, density etc. which are responsible for characterizing the observed spectra) could be studied as a function of the BH spin and the spectral signature of the BH rotation may, at least qualitatively, be determined. This will, we believe, be an important step forward towards a better understanding of BH astrophysics. However, we would like to mention here that such spectral signatures would be very difficult to detect observationally. Most of the radiations produced in such close vicinity of the event horizon will be directly swallowed by the BH itself, and the rest of the radiative signature will eventually suffer an enormous amount of gravitational redshift, and hence will remain almost inaccessible for analysis using any present-day observational techniques. Also, the temperature profile obtained from our work will mainly correspond to the extreme-ultraviolet or X-rays, while most of the observed flux for BH candidates comes from flares and jets instead of the inner region of the disc, and hence the theoretically calculated strong gravity effects would be diluted.

4.3 Hawking radiation from acoustic BHs

The pioneering attempt made by Unruh (1981) at mapping certain aspects of BH physics on to the propagation of sound waves in supersonic convergent flows leads to a number of important works (see Visser 1998, V98 hereafter, and references therein) to identify the propagation of acoustic disturbance in a polytropic inviscid fluid flow with the d’Alembertian equation of motion of a minimally coupled massless scalar field propagating in a (3+1) Lorentzian manifold. Sound propagation is described by an acoustic metric algebraically dependent on the flow density and Mach number, which is conformally related to the Painlevé–Gullstrand–Lemaître representation of Schwarzschild geometry, with a constant conformal factor close to the BH event horizon. Such an equivalence is potentially useful in tackling the tantalizing issue of Hawking radiation in terms of physical quantities governing the transonic fluid flow. In this paper we construct multiple sonic surfaces (a collection of $r_{ac}$ and $r_{ad}$ forming a 3D hypersurface on $[P^3]$ space) which is essential for a complete investigation of the formation of outer-trapped surfaces in the acoustic ergo-region and of acoustic horizons. We investigate the multi-transonic, barotropic inviscid fluid flow (accretion) using a complete GR framework and not using any Newtonian assumption. Hence we believe that our work presented in this paper may have a twofold importance. First, a GR calculation of the values of the fundamental dynamical and thermodynamic flow variables close to the BH event horizon will help in constructing diagnostic high-energy spectra, useful for the identification of Galactic and extragalactic BH candidates. Secondly, a full GR treatment of representative transonic fluid flow will take an welcome step forward towards a concrete formulation of ‘acoustic general relativity’, the term used by V98. We have also established a very important fact that GR spherical accretion on to astrophysical BHs may generate acoustic white holes, and the ratio of the Hawking temperature to the analogous sonic BH temperature is independent of the BH mass, which is a nice way out from the great debate whether astrophysical/primordial BHs can really have such a lower mass limit that the BH microphysics can really be dealt with. Details of such calculations are beyond the scope of this paper and have been presented elsewhere (Das 2004).

ACKNOWLEDGMENTS

This work is supported by grant No. NSF AST-0098670. It is my pleasure to acknowledge stimulating discussions with Paul J. Wiita, Roger D. Blandford, Robert V. Wagoner, William Unruh and Paramita Barai. I also thank the anonymous referee for useful comments.

REFERENCES

Abramowicz M. A., Czerny B., Lasota J. P., Szuszkiewicz E., 1988, ApJ, 332, 646
Anderson M., 1989, MNRAS, 239, 19
Armitage P. J., Reynolds C. S., Chiab J., 2001, ApJ, 648, 868
Artemova I. V., Björnsson G., Novikov I. D., 1996, ApJ, 461, 565
Bardeen J. M., Press W. H., Teukolsky S. A., 1972, ApJ, 178, 347
Becker P. A., Le T., 2003, ApJ, 588, 408
Bisikalo A. A., Boyarchuk V. M., Chechetkin V. M., Kuznetsov O. A., Molteni D., 1998, MNRAS, 300, 39
Chakrabarti S. K., 1990, ApJ, 350, 275
Chakrabarti S. K., 1996, ApJ, 471, 237(C96)
Chakrabarti S. K., Titarchuk L. G., 1995, ApJ, 455, 623
Chen X., Taam R. E., 2001, ApJ, 542, 254
Das T. K., 2001, A&A, 376, 697
Das T. K., 2002, MNRAS, 330, 563
Das T. K., 2003, ApJ, 588, L89
Das T. K., 2004, Phys. Rev. Lett., submitted
Das T. K., Chakrabarti S. K., 1999, Class. Quantum Grav., 16, 3879
Das T. K., Pendharkar J. K., Mitra S., 2003a, ApJ, 592, 1078
Das T. K., Rao A. R., Vadawale S. R., 2003b, MNRAS, 343, 443
Frank J., King A. R., Raine D. J., 1992, Accretion Power in Astrophysics, 2nd edn. Cambridge Univ. Press, Cambridge (FKR)
Fukue J., 1987, PASJ, 39, 309
Gammie C. F., Popham R., 1998, ApJ, 498, 313
Hawley J. F., Krolik J. H., 2001, ApJ, 548, 348
Ho L. C., 1999, in Chakrabarti S. K. ed., Observational Evidence For Black Holes in the Universe. Kluwer, Dordrecht, p. 153
Igumenshchev I. V., Abramowicz M. A., 1999, MNRAS, 303, 309
Illarionov A. F., 1988, SvA, 31, 618
Illarionov A. F., Sunyaev R. A., 1975, A&A, 39, 185
Kafatos M., Yang R. X., 1994, A&A, 268, 925
Lasota J. P., Abramowicz M. A., 1997, Class. Quantum Grav., 14, A237, (LA)
Liang E. P. T., Nolan P. L., 1984, Space Sci. Rev., 38, 353
Lu J. F., Yu K. N., Yuan F., Young E. C. M., 1997, A&A, 321, 665
Manmoto T., 2000, ApJ, 534, 734
Matsumoto R., Kato S., Fukue J., Okazaki A. T., 1984, PASJ, 36, 71
Novikov I., Thorne K. S., 1973, in De Witt C., De Witt B., eds, Black Holes. Gordon and Breach, New York, p. 342 (NT)
Paczyński B., 1987, Nat, 327, 303
Pariev V. I., 1996, MNRAS, 283, 1264
Peitz J., Appl S., 1997, MNRAS, 286, 681
Popham R., Gammie C. F., 1998, ApJ, 504, 419
Shakura N. I., Sunyaev R. A., 1973, A&A, 24, 337
Shapiro S. L., Teukolsky S. A., 1983, Black holes, white dwarfs, and neutron stars: The physics of compact objects. Wiley-Interscience, New York
Unruh W. G., 1981, Phys. Rev. Lett., 46, 1351
Visser M., 1998, Class. Quantum Grav., 15, 1767 (V98)
Weinberg S., 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. John Wiley & Sons, New York
Wiita P. J., 1999, in Iyer B. R., Bhawal B., eds, Black Holes, Gravitational Radiation, and the Universe. Kluwer, Dordrecht, p. 249
Yang R., Kafatos M., 1995, A&A, 295, 238 (YK)

This paper has been typeset from a TEX/LATEX file prepared by the author.