THE HRT CONJECTURE FOR A SPECIAL CONFIGURATION

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Abstract. The following work explores a subcase of the Heil-Ramanathan-Topiwala (HRT) conjecture, which proposes that a set of any finite time-frequency shifts of a nonzero square-integrable function is linearly independent. We identify and discuss certain sufficient conditions, focusing on the rational dimension of a particular vector space and the size of the zero set of the Zak transform under which the conjecture remains valid. A notable implication of our main result is the successful resolution of a case of the HRT Subconjecture, originally proposed by Chris Heil [9].

1. Introduction

For a nonzero function \( f \in L^2(\mathbb{R}) \), a time-frequency shift of \( f \) is a function of the type

\[
M_y T_x f (t) = e^{-2\pi iyt} f (t - x) \quad (x, y) \in \mathbb{R}^2.
\]

The operator \( M_y \) is called a modulation or frequency-shift operator, and \( T_x \) is an operator that acts by shifting \( f \) in time domain. Generally, modulation and translation operators do not commute, and in fact, these two families of unitary operators generate a non-commutative group called the Heisenberg group. The ubiquitous nature of the Heisenberg group and its relevance across numerous subjects within harmonic analysis is a fact that has been extensively documented in the literature [13]. For instance, the Heisenberg group is fundamental to the foundation of time-frequency or Gabor analysis and is a source of important examples in frame theory [7, 11]. For a historical development, as well as an in-depth treatment of Gabor analysis and connection with topics such as frame theory, we refer the reader to [5, 6, 7, 10].

We recall that given a countable subset \( \Lambda \) of \( \mathbb{R}^2 \), the collection of functions

\[
\mathcal{G} (f, \Lambda) = \{ M_y T_x f : (x, y) \in \Lambda \}
\]

is called a Gabor system. For comparison, the system obtained by substituting modulation for dilation operators is often called a wavelet system. The underlying groups for these systems the Heisenberg group and the affine group share many striking similarities, as documented in [17, 16]. However, there are also several instances in which these systems behave in drastically different ways. For example, it is well-known that there exist finite systems

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produced by time-dilation operators which are linearly dependent. This property guarantees
the existence of a scaling function as a solution to a suitable functional equation—a central
ingredient in the construction of multiresolution orthonormal wavelets [11]. However, as of
current date, the extent to which an analogous statement holds for finite systems of time-
frequency shifts remains unknown. In fact, most progress made in this direction suggests
that, any finite system of time-frequency shifted copies of a nonzero function is linearly in-
dependent. This conjecture, initially posited by Heil, Ramanathan, and Topiwala, has come
to be known as the HRT Conjecture [12]. It can be stated as follows:

**Conjecture 1.** [12] (HRT Conjecture) Given a fixed finite set \( \Lambda \subset \mathbb{R}^2 \), and a nonzero
function \( f \in L^2(\mathbb{R}) \), the associated finite Gabor system, \( \mathcal{G}(f, \Lambda) \) is linearly independent.

For the specific case where all the points lie on the same line (which is equivalent to stating
that all modulation parameters are identical or all the translation parameters are identical),
a straightforward application of the Fourier transform verifies the conjecture. Generally,
however, the problem is much more challenging than one might initially anticipate. As
of the current date, the most general result is attributed to Linnell [14], who was able to
corroborate the conjecture for the case where \( \Lambda \) is a shifted copy of a finite subset of a
discrete subgroup of \( \mathbb{R}^{2d} \), \( d \in \mathbb{N} \).

In terms of the number of points, the smallest case not addressed by Linnell’s result can
only occur for specific configurations of a set \( \Lambda \) containing four points. To date, the status
of the HRT conjecture for sets of four points can be summarized as follows.

**Proposition 1.** Let \( \Lambda \subset \mathbb{R}^2 \) be such that \( \# \Lambda = 4 \), and let \( 0 \neq f \in L^2(\mathbb{R}) \). The finite Gabor
system \( \mathcal{G}(f, \Lambda) \) is linearly independent in each of the following cases:

1. \( \Lambda \) is a \((2, 2)\)– configuration, that is, two of the points are on a line and the other two
   on a parallel line, and \( 0 \neq f \in L^2(\mathbb{R}) \) is arbitrary, \[3, 4\];
2. \( \Lambda \) is a \((1, 3)\)– configuration, that is, three of the points are on a line and the fourth
   point is off that line, and \( 0 \neq f \in S(\mathbb{R}) \) \[3\];
3. \( \Lambda \) is an arbitrary set of four points, but extra restrictions are imposed on \( f \), e.g.,
   \[1, 2, 15\].

Nevertheless, the general problem posed by the HRT conjecture for sets of four points
remains unresolved. The primary objective of this paper is to introduce a new result in this
direction. To state the first result, we introduce the Wiener amalgam space \( W(\mathbb{R}) \) and refer
the reader to [7, Section 6.1] for further details.

A measurable function \( f \in W(\mathbb{R}) \) is said to belong to the Wiener amalgam space if

\[
\| f \|_{W(\mathbb{R})} = \sum_{n \in \mathbb{Z}} \text{esssup}_{x \in [0, 1]} | f(x + n) | < \infty.
\]
Denoting $C(\mathbb{R})$ as the space of continuous functions on $\mathbb{R}$, we define $W_0(\mathbb{R}) = C(\mathbb{R}) \cap W(\mathbb{R})$ as the space of continuous functions within the Wiener amalgam space. If $S(\mathbb{R})$ denotes the class of Schwartz functions, it can be proved that

$$S(\mathbb{R}) \subset W_0(\mathbb{R}) \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

The main aim of the following work is to establish the following:

**Theorem 2.** Consider a finite set $\{(x_k, y_k) : 1 \leq k \leq N\} \subset \mathbb{Z}^2$, and let $(\alpha, \beta)$ be a pair in $\mathbb{R}^2$ but not in $\mathbb{Z}^2$. Assume that a function $f$ either belongs to $W_0(\mathbb{R})$ or is a Schwartz function, and it satisfies the equation

$$\sum_{k=1}^{N} c_k M_{y_k} T_{x_k} f = M_\alpha T_\beta f$$

where $c_1, \ldots, c_N$ are nonzero complex coefficients. Under these conditions, the following statements hold:

1. Let $\alpha_{(-\beta, \alpha)} : [0, 1)^2 \to [0, 1)^2$ be the bijective map given by $\alpha_{(-\beta, \alpha)}(z) = (z + (-\beta, \alpha)) \mod \mathbb{Z}^2$. Then the zero set of the Zak transform of $f$ is necessarily $\alpha_{(-\beta, \alpha)}$-invariant.
2. If the dimension of the vector space over the rationals $\mathbb{Q}$, spanned by $1, \alpha, \text{ and } \beta$, is 2 (i.e., $\dim_\mathbb{Q}(\mathbb{Q} + \mathbb{Q} \alpha + \mathbb{Q} \beta) = 2$), then the zero set of the Zak transform of $f$ cannot be finite.
3. If the dimension of this space is 3 (i.e., $\dim_\mathbb{Q}(\mathbb{Q} + \mathbb{Q} \alpha + \mathbb{Q} \beta) = 3$), then it necessarily follows that $f$ must be the zero function.

One important aspect of Theorem 2 is that it helps solve a particular case of the HRT Subconjecture, which Chris Heil described in [9, Subconjecture 9.2].

2. **Proof of Theorem 2**

To prove Theorem 2, we proceed by contradiction and assume that there exists a nonzero function $f \in W_0(\mathbb{R})$ such that

$$\sum_{k=1}^{N} c_k M_{y_k} T_{x_k} f = M_\alpha T_\beta f$$

for some nonzero complex coefficients $c_1, \ldots, c_N$. Note that any finite set of time-frequency operators parametrized by a collection of finite points on an integer lattice in $\mathbb{R}^2$ forms a collection of pairwise commuting operators. As such, the Spectral Theorem [8] guarantees the existence of a unitary operator that diagonalizes the operator

$$J = \sum_{k=1}^{N} c_k M_{y_k} T_{x_k}.$$
In our case, this unitary operator is the Zak transform \( Z \) : \( L^2(\mathbb{R}) \rightarrow L^2([0,1]^2) \), defined formally by

\[
Zf(t, \omega) = \sum_{k \in \mathbb{Z}} f(t + k) e^{-2\pi ik\omega i}
\]

The following section contains a summary of the main properties of the Zak transform, and we refer the reader to [7] for a complete introduction to this transform.

**Lemma 1.** For \( \varphi \in L^2(\mathbb{R}) \) and \((t, \omega) \in [0,1]^2\), we have:

- \([ZT \varphi](t, \omega) = e^{-2\pi i\omega i}[Z\varphi](t, \omega)\).
- \([ZM \varphi](t, \omega) = e^{-2\pi it}[Z\varphi](t, \omega)\).
- For \(\alpha, \beta > 0\), \([ZM_{\alpha}T_{\beta} \varphi](t, \omega) = e^{-2\pi i\omega_1}(Z\varphi)(t - \beta, \omega + \alpha)\).
- For any integer \(j\), \([ZT \varphi](t + j, \omega) = e^{2\pi i\omega j}[Z\varphi](t, \omega)\).
- For any integer \(j\), \([ZT \varphi](t, \omega + j) = [Z\varphi](t, \omega)\).
- If \(\varphi \in W_0(\mathbb{R})\) or \(\varphi \in S(\mathbb{R})\), then \(Z\varphi\) is continuous on \(\mathbb{R}^2\); see [7, Lemma 8.2.1; Theorem 8.2.5].
- If \(\varphi\) is such that \(Z\varphi\) is continuous on \(\mathbb{R}^2\), then \(Z\varphi\) has a zero in \([0,1]^2\); see [7, Lemma 8.4.2].

Applying the Zak transform to the equation \(Jf = M_{\alpha}T_{\beta}f\) (see (3) and (4)) and letting \(F = Zf\), for almost every \((t, \omega) \in [0,1]^2\), we derive:

\[
\sum_{k=1}^{N} c_k e^{-2\pi i y_k t} e^{-2\pi i x_k \omega} \cdot F(t, \omega) = e^{-2\pi i \alpha t} \cdot F(t - \beta, \omega + \alpha),
\]

or equivalently,

\[
p(t, \omega) \cdot F(t, \omega) = e^{-2\pi i \alpha t} \cdot F(t - \beta, \omega + \alpha),
\]

where \(p : \mathbb{R}^2 \rightarrow \mathbb{C}\) is the non-zero trigonometric polynomial given by

\[
p(t, \omega) = \sum_{k=1}^{N} c_k e^{-2\pi i y_k t} e^{-2\pi i x_k \omega} = \sum_{k=1}^{N} c_k e^{-2\pi i \langle(t,\omega),(y_k,x_k)\rangle}.
\]

It follows that for almost every \((t, \omega) \in [0,1]^2\),

\[
|F((t, \omega) + \gamma)| = |p(t, \omega)| \cdot |F(t, \omega)|
\]

where, for simplicity in notation, we set \(\gamma = (-\beta, \alpha)\) and \(z = (t, \omega)\). Next, by iterating Equation (6), we see that for almost every \(z \in [0,1]^2\) and for any positive integer \(n \geq 1\),

\[
|F(z + n\gamma)| = \prod_{j=0}^{n-1} |p(z + j\gamma)| \cdot |F(z)|.
\]
Using Lemma 1 and the assumption that \( F = Zf \) is continuous on \( \mathbb{R}^2 \), we conclude that there exists \( \lambda \in [0,1]^2 \) such that \( F(\lambda) = Zf(\lambda) = 0 \). Then for each \( n \geq 1 \), we have

\[
|F(\lambda + n\gamma)| = \left| \prod_{j=0}^{n-1} p(\lambda + j\gamma) \right| \cdot |F(\lambda)| = 0.
\]

Let zero \((F)\) be the zero set of the Zak transform of \( f \). Next, let \( \alpha_{\gamma} : [0,1)^2 \to [0,1)^2 \) be the bijective map given by \( \alpha_{\gamma}(z) = (z + \gamma) \mod \mathbb{Z}^2 \). By virtue of the fact that

\[
|F(\alpha_{\gamma}(z))| = |p(z)| \cdot |F(z)|
\]

it follows that zero \((F)\) is \( \alpha_{\gamma} \)-invariant and consequently

\[
F \left( \bigcup_{n \in \mathbb{N}} \alpha_{\gamma}^n \left( \text{zero} \ (F) \right) \right) = \{0\}.
\]

In light of this, the following is immediate:

- Suppose that \( \dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q} \alpha + \mathbb{Q} \beta) = 2 \). By assumption, zero \((F)\) is non-empty and consequently, necessarily not finite.
- Suppose that \( \dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q} \alpha + \mathbb{Q} \beta) = 3 \). By assumption, zero \((F)\) is non-empty and consequently necessarily dense in \([0,1)^2\). Since \( F \) is continuous and vanishing on a set dense subset of \([0,1)^2\) then must be everywhere vanishing. Next, appealing to the fact that the Zak transform is unitary, it follows that \( f \) must be the zero vector in \( L^2(\mathbb{R}) \).

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