A SUPERCONVERGENT HDG METHOD FOR THE INCOMPRESSIBLE
NAVIER-STOKES EQUATIONS ON GENERAL POLYHEDRAL MESHES

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Abstract. We present a superconvergent hybridizable discontinuous Galerkin (HDG) method for the steady-state incompressible Navier-Stokes equations on general polyhedral meshes. For arbitrary conforming polyhedral mesh, we use polynomials of degree \( k + 1 \), \( k \), \( k \) to approximate the velocity, velocity gradient and pressure, respectively. In contrast, we only use polynomials of degree \( k \) to approximate the numerical trace of the velocity on the interfaces. Since the numerical trace of the velocity field is the only globally coupled unknown, this scheme allows a very efficient implementation of the method. For the stationary case, and under the usual smallness condition for the source term, we prove that the method is well defined and that the global \( L^2 \)-norm of the error in each of the above-mentioned variables and the discrete \( H^1 \)-norm of the error in the velocity converge with the order of \( k + 1 \) for \( k \geq 0 \). We also show that for \( k \geq 1 \), the global \( L^2 \)-norm of the error in velocity converges with the order of \( k + 2 \). From the point of view of degrees of freedom of the globally coupled unknown: numerical trace, this method achieves optimal convergence for all the above-mentioned variables in \( L^2 \)-norm for \( k \geq 0 \), superconvergence for the velocity in the discrete \( H^1 \)-norm without postprocessing for \( k \geq 0 \), and superconvergence for the velocity in \( L^2 \)-norm without postprocessing for \( k \geq 1 \).

1. Introduction

In this paper, we consider a new hybridizable discontinuous Galerkin (HDG) method for the steady-state incompressible Navier-Stokes equations, which can be written as the following first order system:

\[
\begin{align*}
L &= \nabla u \quad \text{in } \Omega, \\
-\nu \nabla \cdot L + \nabla \cdot (u \otimes u) + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
u \nabla \cdot u &= 0 \quad \text{on } \partial \Omega, \\
\int_{\Omega} p &= 0,
\end{align*}
\]

where the unknowns are the velocity \( u \), the pressure \( p \), and the gradient of the velocity \( L \). \( \nu \) is the kinematic viscosity and \( f \in L^2(\Omega) \) is the external body force. The domain \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \) is polygonal (\( d = 2 \)) or polyhedral (\( d = 3 \)).

To define the HDG method, we adopt the notations and norms used in [6]. We consider conforming triangulation \( T_h \) of \( \Omega \) made of shape-regular polyhedral elements which can be non-convex. We denote by \( E_h \) the set of all faces \( F \) of all elements \( K \in T_h \) and set \( \partial T_h := \{ \partial K : K \in T_h \} \). For scalar-valued functions \( \phi \) and \( \psi \), we write

\[
(\phi, \psi)_{T_h} := \sum_{K \in T_h} (\phi, \psi)_K, \quad \langle \phi, \psi \rangle_{\partial T_h} := \sum_{K \in T_h} \langle \phi, \psi \rangle_{\partial K}.
\]

Here \( (\cdot, \cdot)_D \) denotes the integral over the domain \( D \subset \mathbb{R}^d \), and \( \langle \cdot, \cdot \rangle_D \) denotes the integral over \( D \subset \mathbb{R}^{d-1} \). For vector-valued functions and matrix-valued functions, a similar notation is taken. For example, for vector-valued functions, we write \( (\phi, \psi)_{T_h} := \sum_{i=1}^{n} (\phi_i, \psi_i)_{T_h} \). For matrix-valued functions, we write \( (\phi, \psi)_{T_h} := \sum_{1 \leq i, j \leq n} (\phi_{ij}, \psi_{ij})_{T_h} \).

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Like all other HDG schemes, to define the HDG method for the problem, we introduce an additional unknown *numerical trace* which is the approximation of the velocity on the skeleton of the mesh. Namely, our HDG method seeks an approximation \((L_h, u_h, p_h, \hat{u}_h) \in G_h \times V_h \times Q_h \times M_h^0\) to the exact solution \((L|\mathcal{T}_h, u|\mathcal{T}_h, p|\mathcal{T}_h, u|\mathcal{E}_h)\) where the finite dimensional spaces are defined as:

\[
G_h := \{ G \in L^2(\Omega) : \quad G|_K \in P_k(K), \quad \forall K \in \mathcal{T}_h \},
\]

\[
V_h := \{ v \in L^2(\Omega) : \quad v|_K \in P_{k+1}(K), \quad \forall K \in \mathcal{T}_h \},
\]

\[
Q_h := \{ p \in L^2(\Omega) : \quad p|_K \in P_k(K), \quad \forall K \in \mathcal{T}_h \},
\]

\[
M_h := \{ \mu \in L^2(\mathcal{E}_h) : \quad \mu|_F \in P_k(F), \quad \forall F \in \mathcal{E}_h \},
\]

\[
M_h^0 := \{ \mu \in M_h : \quad \mu|_{\partial \Omega} = 0 \}.
\]

Here \(P_l(D)\) denotes the set of polynomials of total degree at most \(l \geq 0\) defined on \(D\), \(P_k(D)\) denotes the set of vector-valued functions whose \(d\) components lie in \(P_k(D)\), \(P_k(\Omega)\) denotes the set of square matrix-valued functions whose \(d \times d\) entries also lie in \(P_k(\Omega)\), and \(L^2(\Omega) = \{ p \in L^2(\Omega) : \int_\Omega p = 0 \}\).

The method determines the approximate solution by requiring that it solves the following weak formulation:

\[
(L_h, G)_{\mathcal{T}_h} + (u_h, \nabla \cdot G)_{\mathcal{T}_h} - (\hat{u}_h, G n)|_{\partial \Omega} = 0,
\]

\[
(\nu L_h, \nabla v)_{\mathcal{T}_h} - (u_h \otimes u_h, \nabla v)_{\mathcal{T}_h} - (p_h, \nabla \cdot v)_{\mathcal{T}_h} - (\nu \hat{L}_h n - \hat{p}_h n - (u_h \otimes u_h) n, v)_{\mathcal{T}_h}
\]

\[
-\left(\frac{1}{2} \nabla \cdot u_h, v\right)_{\mathcal{T}_h} + \left(\frac{1}{2} \nabla \cdot (u_h - \hat{u}_h), v\right)_{\mathcal{T}_h} = (f, v)_{\mathcal{T}_h},
\]

\[
-
\left(u_h, \nabla n\right)_{\mathcal{T}_h} + \left(\hat{u}_h \cdot n, q\right)_{\mathcal{T}_h} = 0,
\]

\[
(\nu \hat{L}_h n - \hat{p}_h n - (u_h \otimes u_h) n, \mu)_{\mathcal{T}_h} = 0,
\]

for all \((G, v, q, \mu) \in G_h \times V_h \times Q_h \times M_h^0\). Here,

\[
(\nu \hat{L}_h - \hat{p}_h) n := \nu L_h n - p_h n - \frac{\nu}{h} (\Pi_M u_h - \hat{u}_h) - \tau_C(\hat{u}_h)(u_h - \hat{u}_h) \quad \text{on } \partial \mathcal{T}_h,
\]

\[
\tau_C(\hat{u}_h) := \max(\hat{u}_h \cdot n, 0)
\]

on each \(F \in \partial \mathcal{T}_h\). Here \(\Pi_M\) is the \(L^2\)–projection onto \(M_h\). The goal of this paper is to consider the analytical aspects of the method including the rigorous proof of the uniqueness and existence of the solution of the above system and the error estimates for all unknowns. The computational aspects of the method will be discussed in a separate paper.

Like other HDG schemes, the HDG method \((1.2)\) uses the numerical trace of the primary variable \(\hat{u}_h\) as the only globally-coupled variable. Our formulation is close to that of the HDG method in \([6, 23]\), in which they have the same global degrees of freedom \(\hat{u}_h\). Nevertheless, there are three crucial differences which lead to special properties of our HDG method. Firstly, our method uses \(P_{k+1}(\mathcal{T}_h)\) to approximate the primary variable, which is the velocity, on each element while methods in \([6, 23]\) use \(P_k(\mathcal{T}_h)\) instead. Secondly, the stabilization part of the stabilization function \(\frac{2}{h}(\Pi_M u_h - \hat{u}_h)\) in \((1.2a)\) is totally different from those used in \([6, 23]\). Finally, motivated by the work in \([20]\) and \([8]\), we insert two terms \(-\frac{1}{2} \nabla \cdot u_h\) and \((\frac{1}{2} \nabla \cdot (u_h - \hat{u}_h)) n, v)_{\partial \mathcal{T}_h}\) in \((1.2)\). As a consequence of the above modifications, the new HDG method allows to use general polygonal mesh with optimal approximations for all unknowns. In addition, from the implementation point of view, like many other numerical methods for Navier-Stokes equation, we apply the classical Picard iteration to obtain the numerical solution. In each iteration we need to solve an Oseen equation. Due to our modification in \((1.2)\), we can use the convection field obtained from previous step directly without the use of postprocessing. We notice that by adding these two terms in \((1.2)\), the HDG method \((1.2)\) is not locally conservative (see \([8]\) for detailed explanation). On the other hand, since the HDG method \((1.2)\) has high order accuracy for all variables, lack of being locally conservative is only a minor issue.

It is worth to mention that the HDG method using an enhanced space for the primary variable and the special stabilization function like \(\frac{2}{h}(\Pi_M u_h - \hat{u}_h)\) was first introduced by Lehrenfeld in Remark 1.2.4 for diffusion problem in \([24]\). He numerically showed that the methods provide optimal order of convergence for all unknowns without analysis. In \([24]\), we gave rigorous analysis for this approach for linear elasticity problems. Optimal order of convergence for all unknowns is obtained for both equations. In \([24]\), Oikawa analyzed a
HDG method for diffusion problem which uses the same polynomial spaces as in [20], with a different choice of the numerical flux, he proved the optimality of the method for all unknowns. Since the polynomial order of the numerical trace of these HDG methods [20, 25, 24] is one less than that of the approximation space of the primary variable, from the point of view of degrees of freedom of the globally coupled unknown, they obtain superconvergence for the primary variable without postprocessing. In addition, all these methods work on general polyhedral meshes. However, the standard stability analysis for these methods can only provide the upper bound of

\[ \left( \| \nabla u_h \|_{L_2(h)}^2 + h^{-1} \| M_{\Omega} u_h - \hat{u}_h \|_{H^1(\partial T_h)} \right)^{1/2}, \]

which can not control the standard discrete \( H^1 \)-norm of \( u_h \). We would like to emphasize that the control of the standard discrete \( H^1 \)-norm of \( u_h \) is essentially necessary in the proof of the HDG method (1.2) having a unique solution. In [26], roughly speaking, we prove that

\[ (\| \nabla u_h \|_{L_2(h)}^2 + h^{-1} \| u_h - \hat{u}_h \|_{H^1(\partial T_h)})^{1/2} \leq C \left( \| \nabla u_h \|_{L_2(h)}^2 + h^{-1} \| M_{\Omega} u_h - \hat{u}_h \|_{H^1(\partial T_h)} \right)^{1/2}. \]

The inequality (1.3) enables us to control the standard discrete \( H^1 \)-norm of \( u_h \) of the HDG method (1.2) such that we can show our method has a unique solution under the usual smallness condition for the source term.

In literature, there are many existing mixed and DG methods designed for Navier-Stokes equations. See the classic mixed methods [13, 2, 12], the stabilized methods proposed in [17, 16, 19], and the DG methods [11, 18, 2, 14, 25, 27, 5, 8, 9, 13, 21]. An IP-like method and a compact discontinuous Galerkin (CDG) method were introduced in [22]. Recently, In [3], a mixed method is developed based on the pseudostress-velocity formulation. This method uses row-wise \( k \)-th order RT space and \( P_k(T_h) \) to approximate the pseudostress and velocity, respectively. The method provides the convergence of both variables in \( L^3 \) norm of order \( k + 1 - \frac{d}{2} \). In [4], a modified method is introduced and it approximates the pressure directly with same convergence rate as in [3]. Later in [13], a new mixed finite element method is introduced in which the stress is the primary variable. This method uses \( P_k(T_h) \), row-wise \( k \)-th order RT space and \( P_k(T_h) \) to approximate the velocity gradient, stress and velocity, respectively. The convergence of the velocity gradient and velocity in \( L^2 \)-norm and the stress in \( H((\text{div})\)-norm is of order \( k \). More recently, in 2015, Cockburn et al [6] gave an error analysis of the HDG method developed in [25] which is close to method in this paper. Our method may be criticized by the fact that with the modification of the scheme, we no longer have the local conservation of the momentum. Nevertheless, our approach has several advantages comparing with the one in [6, 23]. For instance, the analysis in [6, 23] is only valid for simplicial meshes and it needs a postprocessing procedure to obtain superconvergent approximation to the velocity. The method in [6, 23] does not have superconvergent approximation to the velocity in the discrete \( H^1 \)-norm if \( k = 0 \), while our method does even this the lowest order case. From the implementation point of view, in each iteration, the scheme in [6, 23] needs to solve a Oseen equation using a postprocessed convection field from the previous iteration while in our scheme we can use the velocity field directly from the previous iteration without any postprocessing.

In this paper, we prove that the discrete \( H^1 \)-norm of the error in the velocity, the \( L^2 \)-norm of the error in the velocity, the pressure and even in the velocity gradient converge with the order \( k + 1 \) for any \( k \geq 0 \); and that the velocity, for \( k \geq 1 \), converges with order \( k + 2 \). Notice that as a built-in feature of HDG methods, see [11], the degrees of freedom of the globally-coupled unknown comes from the numerical trace of the velocity on the mesh skeleton. From the point of view of the global degrees of freedom, the method provides optimal convergent approximations to the velocity, velocity gradient and pressure in \( L^2 \)-norm for \( k \geq 0 \), superconvergent approximation to the velocity in the discrete \( H^1 \)-norm without postprocessing for \( k \geq 1 \), and superconvergent approximation to the velocity in \( L^2 \) norm without postprocessing for \( k \geq 1 \). In addition, the analysis of our method is valid for general polyhedral meshes. To the best of our knowledge, no other known finite element method for the Navier-Stokes equations has all of these properties.

The rest of paper is organized as follows. In Section 2, we introduce our HDG method for the problem and present the main a priori error estimates. In Section 3, we present some preliminary inequalities and stability estimates. In Section 4, we prove the existence and uniqueness of the numerical solution. In Section 5, we provide the detailed proof of the main results.
2. Main Results

In this section, we present the main error estimates results. To state the main results, we need to introduce

for vector- and matrix-valued functions \( \phi \) and \( \Phi \), we use \( \| \phi \|_{\ell,p,D} = \sum_{i=1}^{d} \| \phi_i \|_{\ell,p,D} \), and \( \| \Phi \|_{\ell,p,D} = \sum_{i,j=1}^{d} \| \Phi_{ij} \|_{\ell,p,D} \). Moreover, when \( p = 2 \) and \( \ell < \infty \), we denote \( W^{\ell,2}(D) \) by \( H^\ell(D) \) and \( \| \cdot \|_{\ell,2,D} \), by \( \| \cdot \|_{\ell,D} \). When \( \ell = 0 \), we denote \( W^{0,p}(D) \) by \( L^p(D) \) and the norm by \( \| \cdot \|_{L^p(D)} \), when \( \ell = 0 \) and \( p = 2 \), we denote the \( L^2(D) \) norm by \( \| \cdot \|_{D} \).

We also introduce the following norms and seminorms:

\[
\begin{align*}
\| (v, \mu) \|_{0,h} &= \left( \| v \|_{T_h}^2 + \| h_K^{1/2} \mu \|_{\partial T_h}^2 + \| h_K^{-1/2} (v - \mu) \|_{\partial T_h}^2 \right)^{1/2} \quad \forall (v, \mu) \text{ in } H^1(T_h) \times L^2(\mathcal{E}_h), \\
\| (v, \mu) \|_{1,h} &= \left( \| \nabla v \|_{T_h}^2 + \| h_K^{-1/2} (v - \mu) \|_{\partial T_h}^2 \right)^{1/2} \quad \forall (v, \mu) \text{ in } H^1(T_h) \times L^2(\mathcal{E}_h), \\
\| (v, \mu) \|_{\infty,h} &= \| v \|_{L^\infty(\Omega)} + \| \mu \|_{L^\infty(\mathcal{E}_h)} \quad \forall (v, \mu) \text{ in } L^\infty(\Omega) \times L^\infty(\mathcal{E}_h).
\end{align*}
\]

Here \( \| \cdot \|_{\partial T_h} := \left( \sum_{K \in T_h} \| \cdot \|_{\partial K}^2 \right)^{1/2} \). We also set

\[
\| v \|_{0,h} := \| v \|_{L^2(\Omega)} \quad \| v \|_{1,h} := \| (v, \nabla v) \|_{1,h},
\]

where the average of \( v, \nabla v \), is defined as follows: On an interior face \( F = \partial K^- \cap \partial K^+ \), we have \( \| v \| := \frac{1}{2} (v^+ + v^-) \), where \( v^\pm \) denote the trace of \( v \) from the interior of \( K^\pm \) and \( n^\pm \) is the outward unit normal to \( K^\pm \). On a boundary face \( F \subset \partial K^- \cap \partial \Omega \), we formally set \( v^+ := v \) such that \( \| v \| = v \) on \( \partial \Omega \). We note that \( \| \cdot \|_{1,h} \) is the standard discrete \( H^1 \)-seminorm.

We are now ready to state our first main result on the existence and uniqueness of the numerical solution.

**Theorem 2.1 (Existence, uniqueness and stability).** If \( |\mathbf{f}|_{\Omega} \) is small enough, the HDG method \([1,2]\) has a unique solution \((L, u_h, p_h, \mathbf{u}_h) \in G_h \times V_h \times Q_h \times M_h^1\). Furthermore, the following stability bound is satisfied

\[
\| (u_h, \mathbf{u}_h) \|_{1,h} \leq C \nu^{-1} \| \mathbf{f} \|_{\Omega},
\]

for some constant \( C \) independent of \( \nu \), the discretization parameters and the exact solution.

Next we present the error estimates result for all unknowns. In order to have optimal \( L^2 \)-error estimate for the velocity, we need some regularity assumption of the following dual problem. Consider the problem of seeking \((\phi, \psi)\) such that

\[
\begin{align*}
(2.2a) & \quad \Phi - \nabla \phi = 0 \quad \text{in } \Omega, \\
(2.2b) & \quad -\nu \nabla \cdot \Phi - \nabla \cdot (\phi \otimes u) - \nabla \psi - \frac{1}{2} (\nabla \phi)^\top u + \frac{1}{2} (\nabla u)^\top \phi = \theta \quad \text{in } \Omega, \\
(2.2c) & \quad \nabla \cdot \phi = 0 \quad \text{in } \Omega, \\
(2.2d) & \quad \phi = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Assume that the solution to the dual problem satisfies the following regularity estimate:

\[
\| \Phi \|_{1,\Omega} + \| \phi \|_{2,\Omega} + \| \psi \|_{1,\Omega} \leq C \| \theta \|_{\Omega}.
\]

**Remark 2.2.** If \( \| u \|_{H^1(\Omega)} \) is small enough compared with the diffusion coefficient \( \nu \), the dual problem \([2.2]\) has a unique solution \((\phi, \psi) \in H^1_0(\Omega) \times H^1(\Omega) \). In fact, when we use the standard energy argument, we need to have

\[
\frac{1}{2} \| (\nabla u)^\top \phi - (\nabla \phi)^\top u, \phi \|_{\Omega} \leq \nu \| \nabla \phi \|_{\Omega}^2
\]

to obtain energy estimate of \( \phi \). We notice that \( \frac{1}{2} \| (\nabla u)^\top \phi - (\nabla \phi)^\top u, \phi \|_{\Omega} \leq C \| u \|_{H^1(\Omega)} \| \phi \|_{H^1(\Omega)}^2 \). It is easy to see that \([2.4]\) holds if \( \| u \|_{H^1(\Omega)} \) is small enough compared with the diffusion coefficient \( \nu \). This completes the proof of the above claim. If we further assume \( u \in W^{1,2}(\Omega) \cap L^\infty(\Omega) \), then, the regularity assumption \([2.3]\) comes from a standard regularity estimate \([13]\) for the Stokes equations.
3. Preliminary estimates

In this section, we present some preliminary inequalities for the proof of our main results. First, we would like to recall an important inequality which was introduced in [26]. Here we write it in a slightly general way. Though our results in this section and the following ones are valid for conforming meshes with shape regular assumption, we assume the meshes are quasi-uniform for the sake of simplicity.

**Lemma 3.1.** For any given function \((L, v, \mu) \in G_h \times V_h \times M_h\) satisfying (1.2a), then we have

\[
\|(v, \mu)\|_{1,h} \leq C_{HDC}(\|L\|_\Omega + h^{-\frac{d}{2}} \|M v - \mu\|_{\partial \Omega_h}).
\]

For the proof of the above result, we refer the Lemma 3.2 in [26]. In addition, we also need the following basic inequalities:

**Lemma 3.2.** For \(1 \leq q < \infty (d = 2), 1 \leq q \leq 4 (d = 3)\), there exist positive constant \(C_q\) such that

\[
\begin{align*}
(3.1a) \quad &\|v\|_{L^q(\Omega)} \leq C_q\|v\|_{1,h}, &\forall v \in V(h), \\
(3.1b) \quad &\|v\|_{L^q(\Omega)} \leq C_q\|(v, \mu)\|_{1,h}, &\forall (v, \mu) \in V(h) \times M_h^0.
\end{align*}
\]

Here \(V(h) := H^1_0(\Omega) + V_h\). In addition, we have a trace inequality:

\[
\begin{align*}
(3.1c) \quad &\|v\|_{L^q(\partial \Omega_h)} \leq C_{h}h^{-\frac{d}{2}}\|v\|_{1,h} \leq C_{h}h^{-\frac{d}{2}}\|(v, \mu)\|_{1,h}, &\forall (v, \mu) \in V(h) \times M_h^0.
\end{align*}
\]

The proofs of (3.1a)-(3.1c) are provided in Proposition A.2 in [6], Proposition 4.5 and (7.7) in [18]. To simplify our notations, we group all the nonlinear terms in our formulation as the following operator:

**Definition 3.3.** For any \((w, \hat{w}), (u, \hat{u}), (v, \hat{v}) \in H^1(T_h) \times L^2(\mathcal{E}_h)\), we define the operator:

\[
\mathcal{O}((w, \hat{w}); (u, \hat{u}), (v, \hat{v})) := -(u \otimes w, \nabla v)_{T_h} - \left(\frac{1}{2} (\nabla \cdot w) u, v\right)_{T_h} + \left(\frac{1}{2} u \otimes (w - \hat{w}) n, v\right)_{\partial T_h}
\]

\[
+ \left\langle \tau C(\hat{w})(u - \hat{u}), v - \hat{v}\right\rangle_{\partial T_h} + \left\langle (\hat{u} \otimes \hat{w}) n, v - \hat{v}\right\rangle_{\partial T_h}.
\]

The above operator plays a crucial role in the analysis. It has the following coercive property:

**Proposition 3.4.** For any \((w, \hat{w}), (u, \hat{u}) \in H^1(T_h) \times L^2(\mathcal{E}_h)\), if \(\hat{u}|_{\partial \Omega} = 0\), then we have

\[
\mathcal{O}((w, \hat{w}); (u, \hat{u}), (v, \hat{v})) = \left\langle (\tau C(\hat{w}) - \frac{1}{2} \hat{w} \cdot n)(u - \hat{u}), u - \hat{u}\right\rangle_{\partial T_h} \geq 0.
\]

**Proof.** By Definition 3.3 we have

\[
\mathcal{O}((w, \hat{w}); (u, \hat{u}), (u, \hat{u})) := -(u \otimes w, \nabla u)_{T_h} - \left(\frac{1}{2} (\nabla \cdot w) u, u\right)_{T_h} + \left(\frac{1}{2} u \otimes (w - \hat{w}) n, u\right)_{\partial T_h}
\]

\[
+ \left\langle \tau C(\hat{w})(u - \hat{u}), u - \hat{u}\right\rangle_{\partial T_h} + \left\langle (\hat{u} \otimes \hat{w}) n, u - \hat{u}\right\rangle_{\partial T_h}.
\]

Applying integration by parts for the first term, we have

\[
(u \otimes w, \nabla u)_{T_h} = -\left(\nabla \cdot (u \otimes w), u\right)_{T_h} + \left((u \otimes w) n, u\right)_{\partial T_h}
\]

\[
= -\left((\nabla \cdot w) u, u\right)_{T_h} - (u \otimes w, \nabla u)_{T_h} + \left((u \otimes w) n, u\right)_{\partial T_h}.
\]

This implies that

\[
-(u \otimes w, \nabla u)_{T_h} - \left(\frac{1}{2} (\nabla \cdot w) u, u\right)_{T_h} + \left(\frac{1}{2} u \otimes (w - \hat{w}) n, u\right)_{\partial T_h} = 0.
\]
Inserting above identity into $O((\mathbf{w}, \hat{\mathbf{w}}); (\mathbf{u}, \hat{\mathbf{u}}), (\mathbf{u}, \hat{\mathbf{u}}))$, we have

$$O((\mathbf{w}, \hat{\mathbf{w}}); (\mathbf{u}, \hat{\mathbf{u}}), (\mathbf{u}, \hat{\mathbf{u}})) = \langle \tau_{C}(\hat{\mathbf{w}})(\mathbf{u} - \hat{\mathbf{u}}), \mathbf{u} - \hat{\mathbf{u}} \rangle_{\Gamma_{h}} + \langle (\hat{\mathbf{u}} \otimes \hat{\mathbf{w}}) \mathbf{n}, \mathbf{u} - \hat{\mathbf{u}} \rangle_{\partial \Gamma_{h}} - \frac{1}{2} \langle \hat{\mathbf{w}} \otimes \hat{\mathbf{w}} \rangle_{\partial \Gamma_{h}}$$

$$= \langle \tau_{C}(\hat{\mathbf{w}}) - \frac{1}{2} \hat{\mathbf{w}} \cdot \mathbf{n}, \mathbf{u} - \hat{\mathbf{u}} \rangle_{\partial \Gamma_{h}} - \frac{1}{2} \langle \hat{\mathbf{w}} \cdot \mathbf{n}, \mathbf{u} - \hat{\mathbf{u}} \rangle_{\partial \Gamma_{h}}$$

$$= \langle \tau_{C}(\hat{\mathbf{w}}) - \frac{1}{2} \hat{\mathbf{w}} \cdot \mathbf{n}, \mathbf{u} - \hat{\mathbf{u}} \rangle_{\partial \Gamma_{h}} - \frac{1}{2} \langle \hat{\mathbf{w}} \cdot \mathbf{n}, \mathbf{u} - \hat{\mathbf{u}} \rangle_{\partial \Gamma_{h}} \geq 0.$$

The last step is due to the fact that $\hat{\mathbf{u}}$ is single valued on $\mathcal{E}_{h}$ and $\hat{\mathbf{u}}_{\partial \Omega} = 0$. □

Next, we present a continuity result for the nonlinear operator $O$ that we will use throughout the analysis. We first define the following space:

$$\widetilde{H}_{0}^{1}(\Omega) := \{ (\mathbf{w}, \hat{\mathbf{w}}) \in H_{0}^{1}(\Omega) \times L^{2}(\mathcal{E}_{h}) | \mathbf{w}|_{\mathcal{E}_{h}} = \hat{\mathbf{w}} \}.$$

The above space is the graph space of the trace mapping from $H^{1}(\Omega)$ onto $L^{2}(\mathcal{E}_{h})$. We are ready to state the following result:

**Lemma 3.5.** There is a positive constant $C_{O}$ such that

$$(3.2)$$

$$|O((\mathbf{w}_{1}, \hat{\mathbf{w}}_{1}); (\mathbf{u}, \hat{\mathbf{u}}), (\mathbf{v}, \hat{\mathbf{v}})) - O((\mathbf{w}_{2}, \hat{\mathbf{w}}_{2}); (\mathbf{u}, \hat{\mathbf{u}}), (\mathbf{v}, \hat{\mathbf{v}}))| \leq C_{O} \| (\mathbf{w}_{1}, \hat{\mathbf{w}}_{1}) - (\mathbf{w}_{2}, \hat{\mathbf{w}}_{2}) \|_{1,h} \| (\mathbf{u}, \hat{\mathbf{u}}) \|_{1,h} \| (\mathbf{v}, \hat{\mathbf{v}}) \|_{1,h},$$

for all $(\mathbf{w}_{1}, \hat{\mathbf{w}}_{1}), (\mathbf{w}_{2}, \hat{\mathbf{w}}_{2}), (\mathbf{u}, \hat{\mathbf{u}}) \in \widetilde{H}_{0}^{1}(\Omega)$ and any $(\mathbf{v}, \hat{\mathbf{v}}) \in \mathbf{V}_{h} \times M_{h}^{b}$. 

**Proof.** Setting $\delta_{\mathbf{w}} := \mathbf{w}_{1} - \mathbf{w}_{2}, \delta_{\hat{\mathbf{w}}} := \hat{\mathbf{w}}_{1} - \hat{\mathbf{w}}_{2},$ by the definition of the operator $O$, we have

$$O((\mathbf{w}_{1}, \hat{\mathbf{w}}_{1}); (\mathbf{u}, \hat{\mathbf{u}}), (\mathbf{v}, \hat{\mathbf{v}})) - O((\mathbf{w}_{2}, \hat{\mathbf{w}}_{2}); (\mathbf{u}, \hat{\mathbf{u}}), (\mathbf{v}, \hat{\mathbf{v}})) =$$

$$- \langle \mathbf{u} \otimes \delta_{\mathbf{w}}, \nabla \mathbf{v} \rangle_{\Gamma_{h}} - \langle \frac{1}{2} (\nabla \cdot \delta_{\mathbf{w}}) \mathbf{u}, \mathbf{v} \rangle_{\Gamma_{h}} + \frac{1}{2} \langle \mathbf{u} \otimes (\delta_{\mathbf{w}} - \delta_{\hat{\mathbf{w}}}) \mathbf{n}, \mathbf{v} \rangle_{\partial \Gamma_{h}}$$

$$+ \langle \tau_{C}(\hat{\mathbf{w}}_{1}) - \tau_{C}(\hat{\mathbf{w}}_{2}) (\mathbf{u} - \hat{\mathbf{u}}), \mathbf{v} - \hat{\mathbf{v}} \rangle_{\partial \Gamma_{h}} + \langle (\hat{\mathbf{u}} \otimes \delta_{\mathbf{w}}) \mathbf{n}, \mathbf{v} - \hat{\mathbf{v}} \rangle_{\partial \Gamma_{h}},$$

applying integration by parts in the first term, rearranging the terms, we have

$$= \langle \frac{1}{2} (\nabla \cdot \delta_{\mathbf{w}}) \mathbf{u}, \mathbf{v} \rangle_{\Gamma_{h}} + \langle \mathbf{v} \otimes \delta_{\mathbf{w}}, \nabla \mathbf{u} \rangle_{\Gamma_{h}} + \frac{1}{2} \langle \mathbf{u} \otimes (\delta_{\mathbf{w}} - \delta_{\hat{\mathbf{w}}}) \mathbf{n}, \mathbf{v} \rangle_{\partial \Gamma_{h}}$$

$$+ \langle \tau_{C}(\hat{\mathbf{w}}_{1}) - \tau_{C}(\hat{\mathbf{w}}_{2}) (\mathbf{u} - \hat{\mathbf{u}}), \mathbf{v} - \hat{\mathbf{v}} \rangle_{\partial \Gamma_{h}} + \langle (\hat{\mathbf{u}} \otimes \delta_{\mathbf{w}} - \mathbf{u} \otimes \delta_{\mathbf{w}}) \mathbf{n}, \mathbf{v} - \hat{\mathbf{v}} \rangle_{\partial \Gamma_{h}},$$

$$= T_{1} + T_{2} + T_{3} + T_{4} + T_{5}.$$

Next we estimate each $T_{i}$.

For $T_{1}$, we apply the generalized Hölder’s inequality,

$$T_{1} \leq \| \nabla \cdot \delta_{\mathbf{w}} \|_{\Gamma_{h}} \| \mathbf{u} \|_{L^{4}(\Omega)} \| \mathbf{v} \|_{L^{4}(\Omega)} \leq C \| (\delta_{\mathbf{w}}, \delta_{\mathbf{w}}) \|_{1,h} \| (\mathbf{u}, \hat{\mathbf{u}}) \|_{1,h} \| (\mathbf{v}, \hat{\mathbf{v}}) \|_{1,h},$$

the second inequality is due to (3.11). $T_{2}$ can be bounded in a similar way.

For $T_{3}$, we apply the generalized Hölder’s inequality,

$$T_{3} \leq h^{-\frac{3}{4}} \| \delta_{\mathbf{w}} - \delta_{\hat{\mathbf{w}}} \|_{L^{2}(\partial \Gamma_{h})} h^{\frac{1}{2}} \| \mathbf{u} \|_{L^{4}(\partial \Gamma_{h})} h^{\frac{1}{2}} \| \mathbf{v} \|_{L^{4}(\partial \Gamma_{h})} \leq C \| (\delta_{\mathbf{w}}, \delta_{\mathbf{w}}) \|_{1,h} \| (\mathbf{u}, \hat{\mathbf{u}}) \|_{1,h} \| (\mathbf{v}, \hat{\mathbf{v}}) \|_{1,h},$$

the second inequality is by (3.11).

For $T_{4}$, by the generalized Hölder’s inequality we have:

$$T_{4} \leq C \| \tau_{C}(\hat{\mathbf{w}}_{1}) - \tau_{C}(\hat{\mathbf{w}}_{2}) \|_{L^{4}(\partial \Gamma_{h})} \| \mathbf{u} - \hat{\mathbf{u}} \|_{L^{4}(\partial \Gamma_{h})} \| \mathbf{v} - \hat{\mathbf{v}} \|_{L^{4}(\partial \Gamma_{h})},$$

by the fact that the function max($\mathbf{w} \cdot \mathbf{n}, 0$) is Lipschitz,

$$\leq C \| \delta_{\mathbf{w}} \|_{L^{4}(\partial \Gamma_{h})} \| \mathbf{u} - \hat{\mathbf{u}} \|_{L^{2}(\partial \Gamma_{h})} \| \mathbf{v} - \hat{\mathbf{v}} \|_{L^{4}(\partial \Gamma_{h})}$$

$$\leq C h^{\frac{1}{4}} (\| \delta_{\mathbf{w}} - \delta_{\hat{\mathbf{w}}} \|_{L^{4}(\partial \Gamma_{h})} + \| \delta_{\mathbf{w}} \|_{L^{4}(\partial \Gamma_{h})}) \| (\mathbf{u}, \hat{\mathbf{u}}) \|_{1,h} \| \mathbf{v} - \hat{\mathbf{v}} \|_{L^{4}(\partial \Gamma_{h})}.$$
Notice here if \((\delta_w, \delta_h) \in \tilde{H}_0^2(\Omega) + (V_h \times M_h^0)\), then \(\delta_w - \delta_h \in V_h|_{T_h}\). Hence, we can apply inverse inequality on \(\|\delta_w - \delta_h\|_{L^2(\partial T_h)}\), \(\|v - \delta_h\|_{L^2(\partial T_h)}\) to have
\[
C h^2 \left( h^{-\frac{d}{2}} (\|\delta_w - \delta_h\|_{L^2(\partial T_h)} + \|\delta_w\|_{L^2(\partial T_h)}) \right) \|u, \delta_h\|_{1,h} \|v, \delta_h\|_{1,h} \leq C h^3 \left( h^{-\frac{d}{2}} (\|\delta_w - \delta_h\|_{L^2(\partial T_h)} + \|\delta_w\|_{L^2(\partial T_h)}) \right) \|u, \delta_h\|_{1,h} \|v, \delta_h\|_{1,h}.
\]
by Lemma 3.2 and the fact that \(d = 2, 3\).

Finally, for \(T_5\) we first break it into two terms:
\[
T_5 = -((\delta_w - \delta_h) n, v)_{\partial T_h} - (((u - \delta_h) n, v)_{\partial T_h}.
\]
For the first term, by the specialized H"older's inequality, we have:
\[
(C h^2 \left( h^{-\frac{d}{2}} (\|\delta_w - \delta_h\|_{L^2(\partial T_h)} + \|\delta_w\|_{L^2(\partial T_h)}) \right) \|u, \delta_h\|_{1,h} \|v, \delta_h\|_{1,h} \leq C h^2 \left( h^{-\frac{d}{2}} (\|\delta_w - \delta_h\|_{L^2(\partial T_h)} + \|\delta_w\|_{L^2(\partial T_h)}) \right) \|u, \delta_h\|_{1,h} \|v, \delta_h\|_{1,h}.
\]
In the last step we used the same argument as in \(T_4\) for the term \(\|u - \delta_h\|_{L^2(\partial T_h)}\) and Lemma 3.2. The second term can be estimated in a similar way. We complete the proof by combining the estimates of \(T_1\). □

4. Uniqueness and existence of the numerical solution.

We will apply the Picard fixed point theorem to show the existence and uniqueness of the solution of (\ref{eq:1.2}). To this end, we begin by rewriting the method into a more compact and appropriate form for the proof. If we add \(\ref{eq:1.2a} - \ref{eq:1.2d}\), the method can be written as: Find \((L_h, u_h, p_h, \hat{u}_h) \in G_h \times V_h \times Q_h \times M_h^0\) such that
\[
S(L_h, u_h, p_h, \hat{u}_h, G, v, q, \mu) + \mathcal{O}((u_h, \hat{u}_h); (u_h, \hat{u}_h), (v, \mu)) = (f, v)_{T_h},
\]
for all \((G, v, q, \mu) \in G_h \times V_h \times Q_h \times M_h^0\).

We also define a mapping \(F\) as follows: for any \((w, \tilde{w}) \in H^1(T_h) \times L^2(\mathcal{E}_h), (u_h, \hat{u}_h) \in V_h \times M_h\) is part of the solution \((L_h, u_h, p, \hat{u}_h) \in G_h \times V_h \times Q_h \times M_h^0\) of
\[
S((L_h, u_h, p_h, \hat{u}_h), G, v, q, \mu) + \mathcal{O}((w, \tilde{w}); (u_h, \hat{u}_h), (v, \mu)) = (f, v)_{T_h}.
\]
for all \((G, v, q, \mu) \in G_h \times V_h \times Q_h \times M_h^0\). It is worth to mention that when \(w \in H(\text{div}, \Omega), \nabla \cdot w = 0\) and \(\tilde{w} = w|_{\mathcal{E}_h}\), the above system is a HDG scheme for the Oseen equation. Clearly, \((u_h, \hat{u}_h)\) is a solution of (\ref{eq:1.2}) if and only if it is a fixed point of the mapping \(F\). Next we present a stability result for the above scheme.

**Lemma 4.1.** If \((L_h, u_h, p_h, \hat{u}_h) \in G_h \times V_h \times Q_h \times M_h^0\) is a solution of (\ref{eq:1.2}), then there exists a constant \(C\) that solely depends on the constants \(C_{\text{HDG}}\) and \(C_2\) in Lemma 3.1 such that\[
\|(u_h, \hat{u}_h)\|_{1,h} \leq C \nu^{-1} \|f\|_{\Omega}.
\]

**Proof.** Taking \((G, v, q, \mu) = (\nu L_h, u_h, p_h, \hat{u}_h)\) in (\ref{eq:1.2}), after some algebraic manipulation, we have a simplified equation:
\[
\nu\|L_h\|_{\Omega}^2 + \frac{1}{h} (\|M u_h - \hat{u}_h\|_{\Omega}^2 + \|\mu\|_{\Omega}^2)
\]
Therefore, by Lemma 3.1 Young’s inequality and Proposition 3.4 we have
\[
\nu\|(u_h, \hat{u}_h)\|_{1,h}^2 \leq 2C_{\text{HDG}}^2 (\nu\|L_h\|_{\Omega} + \frac{1}{h} \|M u_h - \hat{u}_h\|_{\Omega}^2 + \|\mu\|_{\Omega}^2)
\]
\[
\leq 2C_{\text{HDG}}^2 (\nu\|L_h\|_{\Omega} + \frac{1}{h} \|M u_h - \hat{u}_h\|_{\Omega} + \|\mu\|_{\Omega}) \leq 2C_{\text{HDG}}^2 \|(u_h, \hat{u}_h)\|_{1,h} \|f\|_{\Omega}.
\]
The last step is by Lemma 3.2 with \(q = 2\). This completes the proof with \(C = 2C_{\text{HDG}}^2\). □
Inspired by the above stability result, we define a subspace of \( V_h \times M_h^0 \):
\[
\mathcal{K}_h := \{(v, \mu) \in V_h \times M_h^0 : \| (v, \mu) \|_{1,h} \leq C_2 \| f \|_{\Omega} \}.
\]

We are now ready to give the proof of the existence and uniqueness result for the HDG scheme (1.2)/(4.1).

**Proof of Theorem 2.1**

Clearly, \( F \) maps \( V_h \times M_h^0 \) into \( \mathcal{K}_h \) hence it maps \( \mathcal{K}_h \) into itself. In order to show the existence and uniqueness of the solution of (1.2)/(4.1), it suffices to show that \( F \) is a contraction on the subspace \( \mathcal{K}_h \). To this end, let \( (w_1, \hat{w}_1), (w_2, \hat{w}_2) \in \mathcal{K}_h \) and \( (L_1, u_1, p_1, \hat{u}_1) \) are the solutions of the problem (4.2) with \( (\hat{w}, \hat{\nu}) = (w_i, \hat{w}_i) \), \( (i = 1, 2) \). So we have \( (u_1, \hat{u}_1) := F(w_1, \hat{w}_1) \) and \( (u_2, \hat{u}_2) := F(w_2, \hat{w}_2) \). If we set \( \delta_L := L_1 - L_2, \delta_u := u_1 - u_2, \delta_p := p_1 - p_2 \) and \( \delta_{\hat{u}} := \hat{u}_1 - \hat{u}_2 \), due to the linearity of the operator \( S \), we have
\[
S(\delta_L, \delta_u, \delta_p, \delta_{\hat{u}}) = S((w_1, \hat{w}_1); (u_1, \hat{u}_1), (v, \mu)) - O((w_2, \hat{w}_2); (u_2, \hat{u}_2), (v, \mu)) = 0,
\]
for all \( (G, v, q, \mu) \in G_h \times V_h \times Q_h \times M_h^0 \). Taking \( (G, v, q, \mu) = (\nu \delta_L, \delta_u, \delta_p, \delta_{\hat{u}}) \) into the above identity, after some algebraic manipulations, we obtain
\[
\nu\| \delta_L \|^2 + \left( \frac{\nu}{h} \| P_M \delta_u - \delta_{\hat{u}} \|_{\partial \Omega} + \| P_M \delta_u - \delta_{\hat{u}} \|_{\partial \Omega} \right) = -O((w_1, \hat{w}_1); (u_1, \hat{u}_1), (v, \mu)) \leq O((w_2, \hat{w}_2); (u_2, \hat{u}_2), (v, \mu)).
\]

By Lemma 3.3, Young's inequality and Proposition 3.4, we have
\[
\nu\| \delta_u, \delta_{\hat{u}} \|^2 \leq 2C^2 \nu \| (w_1 - w_2, \hat{w}_1 - \hat{w}_2) \|^2 + \nu \| (\hat{w}_1 - w_2, \hat{w}_1 - \hat{w}_2) \|^2 + \| (L_1 - L_2, u_1 - u_2, p_1 - p_2, \hat{u}_1 - \hat{u}_2) \|^2
\]

Therefore, we have shown that
\[
\| (\delta_u, \delta_{\hat{u}}) \|^2 \leq 2C_2 \nu \| (w_1 - w_2, \hat{w}_1 - \hat{w}_2) \|^2 + \| (L_1 - L_2, u_1 - u_2, p_1 - p_2, \hat{u}_1 - \hat{u}_2) \|^2.
\]

By the fixed point theorem, there is a unique fixed point \( (u_h, \hat{u}_h) \in \mathcal{K}_h \) of the mapping \( F \). It is also the unique solution of the system (1.2). This completes the proof.

5. **Proof of the error estimates**

In this section, we provide the detailed proof of the main error estimates for all unknowns. We proceed in several steps.

**Step 1: Error equations.** We begin by introducing the error equations that we are going to use in the analysis. For convention, we introduce the following notations for the errors:
\[
eq_L := L - L_G L,
\]

\[
\delta_L := L - L_G L,
\]

\[
\delta_u := u - L_G u,
\]

\[
\delta_p := p - P_M p,
\]

\[
\delta_{\hat{u}} := \hat{u} - P_M \hat{u}.
\]
Here $P_l^G, P_l^V, P_l^Q, P_l^M$ are the $L^2-$projections onto $G_h, V_h, Q_h, M_h$ respectively. In addition to the basic inequalities listed in Lemma 3.2, we will frequently use the following basic inequalities as well:

\[(5.1a) \quad \|q\|_F \leq Ch_K^{2\over 3} \|q\|_{K}, \quad \text{for all } q \in P_l(K), \quad l \geq 0,\]

\[(5.1b) \quad \|D^m(q - P_lq)\|_K \leq Ch_K^{l+1-m} \|q\|_{l+1,K}, \quad \text{for all } q \in H^{l+1}(K), \quad 0 \leq m \leq l,\]

\[(5.1c) \quad \|q - P_lq\|_F \leq Ch_K^{l+\frac{2}{3}} \|q\|_{l+1,K}, \quad \text{for all } q \in H^{l+1}(K), \quad 0 \leq m \leq l,\]

\[(5.1d) \quad \|q - P_l^Mq\|_F \leq Ch^{k+\frac{2}{3}} \|q\|_{k+1,K}, \quad \text{for all } q \in H^{l+1}(K),\]

Here $P_l$ denotes the $L^2-$projection onto $P_l(K)$, $F$ denotes any face of $K$. In addition, we have the following estimate for the projections under the triple norm $\| \cdot \|_{1,h}$:

**Proposition 5.1.** For any $u \in H^3(\Omega)$, we have

\[(5.2) \quad \|(P_l^V u, P_l^M u)\|_{1,h} \leq C\|u\|_{1,\Omega}.\]

**Proof.** By the definition of the norm $\| \cdot \|_{1,h}$, we have

$$
\|(P_l^V u, P_l^M u)\|_{1,h} \leq 2(\|\nabla P_l^V u\|_{0,h} + h^{-\frac{1}{2}} \|P_l^V u - P_l^M u\|_{0,h}).
$$

We are going to bound each of the above terms by $\|u\|_{1,\Omega}$. For the first term, we have

$$
\|\nabla P_l^V u\|_{0,h} = \|\nabla (P_l^V u - \bar{u})\|_{0,h} \leq Ch^{-1}\|P_l^V u - \bar{u}\|_{0,h},
$$

here $\bar{u}$ denotes the average of $u$ within each element $K \in T_h$, the inequality is by the inverse inequality of the polynomial spaces.

$$
\|\nabla P_l^V u\|_{0,h} \leq C\|u\|_{1,h} = C\|u\|_{1,\Omega},
$$

by the Poincaré inequality for each $K \in T_h$.

For the second term, applying a triangle inequality we have

$$
h^{-\frac{1}{2}} \|P_l^V u - P_l^M u\|_{0,h} \leq h^{-\frac{1}{2}}(\|u - P_l^V u\|_{0,h} + \|u - P_l^M u\|_{0,h}) \leq 2h^{-\frac{1}{2}}\|u - P_l^M u\|_{0,h},
$$

here $P_l$ denotes the $L^2-$projection onto $P_l(K)$ for each $K \in T_h$,

$$
\leq C(\|\nabla (u - P_l^M u)\|_{0,h} + h^{-1}\|u - P_l^M u\|_{0,h}) \quad \text{by the trace inequality},
$$

$$
\leq C\|u\|_{1,\Omega},
$$

the last step is by a similar argument as for the first term and (5.1). This completes the proof. \qed

It is not hard to verify that the exact solution $(u, L, p, u|_{\partial \Omega})$ satisfies the following equation:

$$
S((L, u, p, \bar{u}), ((G, v, q, \mu)) + O((u, \bar{u}), (\bar{u}, \bar{u}), (v, \mu)) = (f, v)_{\Omega} + (\nu h \delta_u u, v - \mu)_{\partial \Omega},
$$

for all $(G, v, q, \mu) \in G_h \times V_h \times Q_h \times M^0_h$. Subtracting (5.1b), we have

$$
S((L, u, p, \bar{u}), ((G, v, q, \mu)) - S((L, u, p, \bar{u}), ((G, v, q, \mu)))
$$

$$
+ O((u, u), (u, (v, \mu)) - O((u, \bar{u}), (\bar{u}, (v, \mu))) = (\nu h \delta_u u, v - \mu)_{\partial \Omega},
$$

by the linearity of the first operator $S$, we have

$$
S((e_L, e_u, e_p, e_u), ((G, v, q, \mu))) + O((u, u), (u, (v, \mu)) - O((u, \bar{u}), (\bar{u}, (v, \mu))) =
$$

$$
- S((\delta_u L, \delta_u u, \delta_u p, e_u), ((G, v, q, \mu))) + (\nu h \delta_u u, v - \mu)_{\partial \Omega}.
$$

Finally, by the definition of the operator $S$ and the orthogonality property of the $L^2-$projections we have the error equation:

$$
S((e_L, e_u, e_p, e_u), ((G, v, q, \mu))) + O((u, u), (u, (v, \mu)) - O((u, \bar{u}), (\bar{u}, \bar{u}), (v, \mu)) =
$$

$$
(\nu \delta_u L - \delta_u p - \nu h P_l^M u, v - \mu)_{\partial \Omega},
$$

for all $(G, v, q, \mu) \in G_h \times V_h \times Q_h \times M^0_h$.\]
Step 2: Estimates for $e_L$, $e_u$. We first apply an energy argument to bound the errors $e_L$, $e_u$, which can be stated as follows:

**Lemma 5.2.** If the exact solution $u, L, p$ is smooth enough, and $\|u\|_{1, \Omega}$ is small enough, we have

$$\|e_u\|_{1, \Omega} + \|(e_u, e_u)\|_{1, h} \leq C_{HDG}(\|e_L\|_{\Omega} + h^{-\frac{1}{2}}\|\Pi_M e_u - e_u\|_{\partial \Omega}) \leq C h^{-1}.$$  

Here the constant $C$ depends on $\|u\|_{k+2, \Omega}$, $\|u\|_{\infty, \Omega}$, $\|p\|_{k+1, \Omega}$, $\nu$ and $k$ but independent of $h$.

**Proof.** Taking $(G, v, q, \mu) = (e_L, e_u, e_v, e_v)$ in the error equation, the resulting equation can be simplified as

$$\nu\|e_L\|_{\Omega}^2 + \frac{\nu}{h}\|\Pi_M e_u - e_u\|_{\partial \Omega}^2 = \mathcal{O}((u, u); (u, u), (e_u, e_u)) + \mathcal{O}((u, \hat{u}_h); (u, \hat{u}_h), (e_u, e_u)) + \langle \nu \delta_L n - \delta_p n - \frac{\nu}{h}\Pi_M \delta_u, e_u - e_u \rangle_{\partial \Omega},$$

or

$$\nu\|e_L\|_{\Omega}^2 + \frac{\nu}{h}\|\Pi_M e_u - e_u\|_{\partial \Omega}^2 = \mathcal{O}((u, u); (u, u), (e_u, e_u)) + \mathcal{O}((u, \hat{u}_h); (u, \hat{u}_h), (e_u, e_u)) + \langle \nu \delta_L n - \delta_p n - \frac{\nu}{h}\Pi_M \delta_u, e_u - e_u \rangle_{\partial \Omega}.$$  

(5.4)

Let us first estimate the last term of the above equation. Applying the Cauchy-Schwarz inequality, we have

$$\langle \nu \delta_L n - \delta_p n - \frac{\nu}{h}\Pi_M \delta_u, e_u - e_u \rangle_{\partial \Omega} \leq h^\frac{1}{2}(\nu\|\delta_L n\|_{\partial \Omega} + \|\delta_p\|_{\partial \Omega} + \nu h^{-1}\|\Pi_M \delta_u\|_{\partial \Omega}) h^{-\frac{1}{2}}\|e_u - e_u\|_{\partial \Omega}$$

$$\leq C(\|\delta_L n\|_{\Omega} + \|\delta_p\|_{\Omega} + h^{-1}\|\delta_u\|_{\Omega}) \|(e_u, e_u)\|_{1, h} \text{ by (5.1a),}$$

$$\leq Ch^{-1}(\|L\|_{k+1, \Omega} + \|p\|_{k+1, \Omega} + \|u\|_{k+2, \Omega}) \|(e_u, e_u)\|_{1, h} \text{ by (5.1b).}$$

The main effort in the analysis is to have an optimal estimate for the nonlinear terms, we rewrite these two terms into four parts:

$$\mathcal{O}((u, u); (u, u), (e_u, e_u)) = \mathcal{O}((u, \hat{u}_h); (u, \hat{u}_h), (e_u, e_u)) + \mathcal{O}((u, \hat{u}_h); (u, \hat{u}_h), (e_u, e_u)) + \mathcal{O}((u, \hat{u}_h); (u, \hat{u}_h), (e_u, e_u)) + \mathcal{O}((u, \hat{u}_h); (u, \hat{u}_h), (e_u, e_u))$$

Notice that the operator $\mathcal{O}$ is linear with respect to the last two components, so we have

$$T_1 = -\mathcal{O}((u, \hat{u}_h); (u, \hat{u}_h), (e_u, e_u)) \leq 0,$$

by Proposition 3.3. For $T_2$, we apply the estimate $\|\Pi_M u\|_{k, \Omega}$ in Lemma 3.5, we have

$$T_2 \leq C_{\mathcal{O}}(\|\Pi_M u\|_{k, \Omega}) \|(e_u, e_u)\|_{1, h} \leq C_{\mathcal{O}}(\|u\|_{1, \Omega} \|(e_u, e_u)\|_{1, h}^2.$$  

For $T_3, T_4$, if we directly apply the continuity property of the operator $\mathcal{O}$, we will only obtain suboptimal order of convergence. We can recover optimal convergence rate by a refined argument for each term. We start with $T_3$, by the linearity of $\mathcal{O}$ for the last two components, we have

$$T_4 = -\mathcal{O}((u, u); (\delta_u, \delta_u), (e_u, e_u)) = (\delta_u \otimes u, \nabla e_u)_{\Omega} - \langle \tau_C(u) (\delta_u - \delta_u), e_u - e_u \rangle_{\partial \Omega} - \langle (\delta_u \otimes u) n, e_u - e_u \rangle_{\partial \Omega} := T_{41} + T_{42} + T_{43}.$$  

We now apply the generalized Hölder’s inequality to bound these three terms as follows:

$$T_{41} \leq \sum_{K \in \mathcal{T}_h} \|\Pi_M u\|_{k, \Omega} \|\delta_u\|_{k} \|\nabla e_u\|_{k} \leq C h^{k+2} \|\Pi_M u\|_{k, \Omega} \|(e_u, e_u)\|_{1, h},$$

$$T_{42} \leq \sum_{K \in \mathcal{T}_h} \|\Pi_M u\|_{k, \Omega} \|\delta_u\|_{k} \|\nabla e_u\|_{k} \leq C \|u \cdot n\|_{\infty, \mathcal{T}_h} h^{k+1} \|u\|_{h+1, \Omega} \|(e_u, e_u)\|_{1, h},$$

$$T_{43} \leq C \|u \cdot n\|_{\infty, \mathcal{T}_h} h^{k+1} \|u\|_{h+1, \Omega} \|(e_u, e_u)\|_{1, h}.$$
For $T_3$, by the definition of $\mathcal{O}$, we have

$$
T_3 = (\Pi_Y u \otimes \delta_u, \nabla e_u)_{\mathcal{T}_h} + \frac{1}{2}(\nabla \cdot \delta_u) \Pi_Y u, e_u)_{\mathcal{T}_h} - \frac{1}{2}(\Pi_Y u \otimes (\delta_u - \delta_{\tilde{u}})) n, e_u - e_{\tilde{u}})_{\partial \mathcal{T}_h}
+ \langle (\tau C(\Pi_M u) - \tau C(u))(\Pi_Y u - \Pi_M u), e_u - e_{\tilde{u}} \rangle_{\partial \mathcal{T}_h} - \langle (\Pi_M u \otimes \delta_u) n, e_u - e_{\tilde{u}} \rangle_{\partial \mathcal{T}_h}.
$$

Among the five terms in the above expression, the third term needs some special treatment in order to obtain optimal convergence rates. For the others, we can bound them in a similar way as for $T_{41}, T_{42}, T_{43}$. For the sake of simplicity, here we show how to bound the last term and then focus on the third term. By applying the generalized Hölder's inequality on the last term, we have

$$
\langle (\Pi_M u \otimes \delta_u) n, e_u - e_{\tilde{u}} \rangle_{\partial \mathcal{T}_h} \leq \sum_{K \in \mathcal{T}_h} \| \Pi_M u \|_{\infty, \partial K} h^{\frac{1}{2}} \| \delta_u \|_{\partial K} h^{-\frac{1}{2}} \| e_u - e_{\tilde{u}} \|_{\partial K}
\leq C h^{k+1} \| u \|_{\infty, \mathcal{E}_h} \| u \|_{k+1, \Omega} \| (e_u, e_{\tilde{u}}) \|_{1,h}.
$$

In the last step we applied the inequality:

$$
\| \Pi_M u \|_{\infty, \partial K} \leq C \| u \|_{\infty, \partial K}.
$$

This result can be obtained by a simple scaling argument. Finally, let us focus on the third term. We rewrite the term as follows,

$$
\langle \frac{1}{2}(\Pi_Y u \otimes (\delta_u - \delta_{\tilde{u}})) n, e_u - e_{\tilde{u}} \rangle_{\partial \mathcal{T}_h} = \langle \frac{1}{2}(\Pi_Y u \otimes (\delta_u - \delta_{\tilde{u}})) n, e_u - e_{\tilde{u}} \rangle_{\partial \mathcal{T}_h} + \langle \frac{1}{2}(\Pi_Y u \otimes \delta_u) n, e_u \rangle_{\partial \mathcal{T}_h}
- \langle \frac{1}{2}(\Pi_Y u \otimes \delta_u) n, e_{\tilde{u}} \rangle_{\partial \mathcal{T}_h} = T_{31} + T_{32} + T_{33}.
$$

For $T_{31}$, we apply the generalized Hölder's inequality,

$$
T_{31} \leq \sum_{K \in \mathcal{T}_h} \| \Pi_Y u \|_{\infty, \partial K} h^{\frac{1}{2}} \| \delta_u - \delta_{\tilde{u}} \|_{\partial K} h^{-\frac{1}{2}} \| e_u - e_{\tilde{u}} \|_{\partial K} \leq C h^{k+1} \| u \|_{\infty, \Omega} \| u \|_{k+1, \Omega} \| (e_u, e_{\tilde{u}}) \|_{1,h}.
$$

For $T_{32}$, we have

$$
T_{32} \leq \sum_{K \in \mathcal{T}_h} \| \Pi_Y u \|_{\infty, \partial K} h^{\frac{1}{2}} \| \delta_u \|_{\partial K} \| e_{\tilde{u}} \|_{\partial K} \leq C h^{k+1} \| u \|_{\infty, \Omega} \| u \|_{k+2, \Omega} (h^{\frac{1}{2}} \| e_{\tilde{u}} \|_{\partial \mathcal{T}_h}).
$$

For $T_{33}$, inserting a zero term $\langle \frac{1}{2}(u \otimes \delta_{\tilde{u}}) n, e_{\tilde{u}} \rangle_{\partial \mathcal{T}_h}$ into $T_{33}$ we obtain:

$$
T_{33} = \langle \frac{1}{2}(\delta_u \otimes \delta_{\tilde{u}}) n, e_{\tilde{u}} \rangle_{\partial \mathcal{T}_h} \leq \sum_{K \in \mathcal{T}_h} \| \delta_u \|_{\infty, \partial K} h^{-\frac{1}{2}} \| \delta_u \|_{\partial K} (h^{\frac{1}{2}} \| e_{\tilde{u}} \|_{\partial K}) \leq C h^{k+1} \| u \|_{\infty, \Omega} \| u \|_{k+2, \Omega} (h^{\frac{1}{2}} \| e_{\tilde{u}} \|_{\partial \mathcal{T}_h}).
$$

The last step is to show that

$$
h^{\frac{1}{2}} \| e \|_{\partial \mathcal{T}_h} \leq C \| (e_u, e_{\tilde{u}}) \|_{1,h}.
$$

To this end, we apply a triangle inequality and (3.10),

$$
h^{\frac{1}{2}} \| e \|_{\partial \mathcal{T}_h} \leq h^{\frac{1}{2}} \| e \|_{\partial \mathcal{T}_h} + h^{\frac{1}{2}} \| e_u - e_{\tilde{u}} \|_{\partial \mathcal{T}_h} \leq C (\| e_u \|_{\Omega} + h^{\frac{1}{2}} \| e_u, e_{\tilde{u}} \|_{1,h}) \leq C \| (e_u, e_{\tilde{u}}) \|_{1,h}.
$$

This completes the estimate for $T_3$. Finally, if we combine the estimates for $T_1 - T_4$ and Lemma 3.1 we obtain the result stated in the Lemma 5.2.

**Step 3: Estimates for $e_p$.** Next we present the optimal error estimate for $e_p$. As usual, we bound the pressure error via an inf-sup argument. It is well-known [2] that the following inf-sup condition holds for a polygonal domain $\Omega$:

$$
\sup_{w \in H^1_0(\Omega) \setminus \{0\}} \frac{\langle \nabla \cdot w, q \rangle_{\Omega}}{\| w \|_{1, \Omega}} \geq \kappa \| q \|_{\Omega}, \quad \text{for all } q \in L^2_{\Omega}(\Omega).
$$

Here $\kappa > 0$ is independent of $w, p$. We can bound $e_p$ using the above result.

**Lemma 5.3.** Under the same assumption as in Lemma 5.2, we have

$$
\| e_p \|_{\Omega} \leq C h^{k+1}.
$$

Here the constant $C$ depends on $\| u \|_{k+2, \Omega}, \| u \|_{\infty, \Omega}, \| p \|_{k+1, \Omega}, \nu, k$ and $\kappa$ but independent of $h$. 

\]
We bound

\[ (\nabla \cdot w, e_p)_{\Omega} = (e_p, \nabla \cdot \Pi V w)_{\Omega} + \langle (w - \Pi V w) \cdot n, e_p \rangle_{\partial T_h} = \langle e_p, \nabla \cdot \Pi V w \rangle_{\Omega} - \langle (\Pi V w - \Pi M w) \cdot n, e_p \rangle_{\partial T_h}. \]

If we take \((G, \pi, q, \mu) = (0, \Pi V w, 0, \Pi M w)\) in the error equation \((5.3)\), we obtain:

\[
\langle e_p, \nabla \cdot \Pi V w \rangle_{\Omega} - \langle (\Pi V w - \Pi M w) \cdot n, e_p \rangle_{\partial T_h}
\]

\[
= \langle \nu e_L, \nabla \Pi V w \rangle_{\Omega} - \langle \nu e_L n - \nu h (\Pi M e_u - e_u), \Pi V w - \Pi M w \rangle_{\partial T_h}
\]

\[
+ \langle \nu \delta_l n - \delta_l n - \nu h (\Pi M \delta u, \Pi V w - \Pi M w) \rangle_{\partial T_h}
\]

\[
+ \left( O((u, u), (\Pi V w, \Pi M w)) - O((u_h, 0), (\Pi V w, \Pi M w)) \right)
\]

\[
:= T_1 + T_2 + T_3 + T_4.
\]

Next we show that

\[ T_1 + T_2 + T_3 + T_4 \leq Ch^{k+1} \| w \|_{1,\Omega}. \]

For \( T_1 \) we have

\[ T_1 \leq \nu \| e_L \|_{\Omega} \| \nabla \Pi V w \|_{\partial T_h} \leq Ch^{k+1} \| w \|_{1,\Omega}, \]

by Lemma \[5.2\] and \[5.1\]. For \( T_2 \), by the Cauchy-Schwarz inequality, we have

\[ T_2 \leq \nu (h^{\frac{1}{2}} \| e_L \|_{\partial T_h} + h^{-\frac{1}{2}} \| \Pi M e_u - e_u \|_{\partial T_h}) (h^{-\frac{1}{2}} \| \Pi V w - \Pi M w \|_{\partial T_h}) \]

\[ \leq Ch^{k+1} \| w \|_{1,\Omega}. \]

\( T_3 \) can be estimated similarly as \( T_2 \). Finally, for the last term, we split into three terms as:

\[ T_4 = O((u, u), (u, u), (\Pi V w, \Pi M w)) - O((u, u), (\Pi V u, \Pi M u), (\Pi V w, \Pi M w)) \]

\[ + O((u, u), (\Pi V u, \Pi M u), (\Pi V w, \Pi M w)) - O((u_h, 0), (\Pi V u, \Pi M u), (\Pi V w, \Pi M w)) \]

\[ + O((u_h, 0); (\Pi V u, \Pi M u), (\Pi V w, \Pi M w)) - O((u_h, 0), (\Pi V u, \Pi M u), (\Pi V w, \Pi M w)) \]

\[ = T_{41} + T_{42} + T_{43}. \]

We bound \( T_{41} \) separately.

\[ T_{41} = O((u, u); (\delta_u, \delta_u), (\Pi V w, \Pi M w)) \]

\[ = (\delta_u \otimes u, \Pi V w)_{\Omega} + \langle \tau C(u) (\delta_u - \delta_u), \Pi V w - \Pi M w \rangle_{\partial T_h} + \langle (\delta_u \otimes u) n, \Pi V w - \Pi M w \rangle_{\partial T_h} \]

\[ \leq \| u \|_{\infty, \Omega} \| \delta_u \|_{\Omega} \| \Pi V w \|_{\Omega} + \| u \|_{\infty, \Omega} h^{\frac{1}{2}} (\| \delta_u - \delta_u \|_{\partial T_h} + \| \delta_u \|_{\partial T_h}) h^{-\frac{1}{2}} \| \Pi V w - \Pi M w \|_{\partial T_h} \]

\[ \leq Ch^{k+1} \| u \|_{\infty, \Omega} \| u \|_{k+1, \Omega} (\| \Pi V w, \Pi M w \|_{1, h} \leq Ch^{k+1} \| u \|_{\infty, \Omega} \| w \|_{1, \Omega}. \]

the last step is by the approximation properties of the projections \[5.1, 5.14\].

For \( T_{43} \), due to the linearity of \( O \) on the last two components and \[8.2\], we have

\[ T_{43} = O((u_h, 0); (e_u, e_u), (\Pi V w, \Pi M w)) \leq C \| (u_h, 0) \|_{1, h} \| (e_u, e_u) \|_{1, h} \| (\Pi V w, \Pi M w) \|_{1, h} \]

\[ \leq C (\| (\Pi V w, \Pi M w) \|_{1, h} + \| (e_u, e_u) \|_{1, h}) \leq C \| (\Pi V w, \Pi M w) \|_{1, h} \]

\[ \leq Ch^{k+1} \| u \|_{1, \Omega} \| (\Pi V w, \Pi M w) \|_{1, h} \leq Ch^{k+1} \| w \|_{1, \Omega}. \]

by \[5.1\] and Lemma \[5.2\].

For \( T_{42} \), if we directly apply \[5.4\], we will only obtain suboptimal convergence rates. Alternatively, we need a refined analysis for this term. First, we let \( E_u := u - u_h = e_u + \delta_u \) and \( E_u := u - \tilde{u} = e_u + \tilde{\delta_u} \), Next, by the definition of \( O \), we can write \( T_{42} \) as

\[ T_{42} = - (\Pi V u \otimes E_u, \nabla \Pi V w)_{\Omega} - \left( \frac{1}{2} (\nabla \cdot E_u) \Pi V u, \Pi V w \right)_{\Omega} + \left( \frac{1}{2} (\Pi V u \otimes (E_u - E_u)) n, \Pi V w \right)_{\partial T_h} \]

\[ + \left( (\tau C(u) - \tau C(\tilde{u})) (\Pi V u - \Pi M u, \Pi V w - \Pi M w) \right)_{\partial T_h} - \left( (\Pi M u \otimes E_u) n, \Pi V w - \Pi M w \right)_{\partial T_h} = S_1 + \cdots + S_5. \]
Notice that by Lemma 5.2 and (5.1b) we have
\[ (5.7a) \quad \| E_u \|_{1,h} \leq \| e_u \|_{1,h} + \| \delta u \|_{1,h} \leq Ch^{k+1} \| u \|_{k+1,\Omega}, \]
\[ (5.7b) \quad \| E_u \|_{\partial \Omega} \leq \| \delta u \|_{\partial \Omega} + \| e_u \|_{\partial \Omega} \leq Ch^{k+\frac{1}{2}}, \]
by (5.1d), Lemma 5.2 Now we bound each of $S_1$. By the generalized Hölder’s inequality, we have
\[ S_1 \leq \| \Pi_V u \|_{\infty,\Omega} \| E_u \|_{\Omega} \| \nabla \Pi_V w \|_{\Omega} \leq Ch^{k+1} \| u \|_{\infty,\Omega} \| w \|_{1,\Omega}. \]
By a similar argument, we can bound $S_2$ as:
\[ S_2 \leq Ch^{k+1} \| u \|_{\infty,\Omega} \| w \|_{1,\Omega}. \]
For $S_4$, we apply the generalized Hölder’s inequality to have
\[ S_4 \leq h^{\frac{1}{2}} \| \tau_C(u) - \tau_C(\hat{u}_h) \|_{\partial \Omega} \| \Pi_V u - \Pi_M u \|_{\infty,\partial \Omega} \| \Pi_V w - \Pi_M w \|_{\partial \Omega} \]
\[ \leq Ch^{\frac{1}{2}} \| u - \hat{u}_h \|_{\partial \Omega} \| u \|_{\infty,\Omega} h^{-\frac{1}{2}} \| \Pi_V w - \Pi_M w \|_{\partial \Omega} \]
\[ \leq Ch^{k+1} \| u \|_{\infty,\Omega} \| w \|_{1,\Omega}, \]
by the estimate (5.7b) and (5.1).
By a similar argument we can bound $S_5$ as
\[ S_5 \leq Ch^{k+1} \| u \|_{\infty,\Omega} \| w \|_{1,\Omega}. \]
For the last term $S_3$, if we apply similar estimate as the others, we will only obtain suboptimal order convergence rates. Therefore, we need a refined estimate for this term. We rewrite $S_3$ as follows:
\[ S_3 = \left( \frac{1}{2} \langle \Pi_V u \otimes (e_u - \hat{e}_u) \rangle n, \Pi_V w \rangle_{\partial \Omega} \right) + \left( \frac{1}{2} \langle \Pi_V u \otimes \delta u \rangle n, \Pi_V w \rangle_{\partial \Omega} \right) - \left( \frac{1}{2} \langle \Pi_V u \otimes \delta u \rangle n, \Pi_V w \rangle_{\partial \Omega} \right) \]
\[ \leq h^{\frac{1}{2}} \| \Pi_V w \|_{\partial \Omega} \| \Pi_V u \|_{\infty,\partial \Omega} h^{-\frac{1}{2}} \| \Pi_V w - \Pi_M w \|_{\partial \Omega} - \frac{1}{2} \langle \Pi_V u \otimes \delta u \rangle n, \Pi_V w \rangle_{\partial \Omega} \]
\[ \leq Ch^{k+1} - \frac{1}{2} \langle \Pi_V u \otimes \delta u \rangle n, \Pi_V w \rangle_{\partial \Omega}, \]
by (5.1d), Lemma 5.2 For the last term, we further split it into two terms as:
\[ \langle \Pi_V u \otimes \delta u \rangle n, \Pi_V w \rangle_{\partial \Omega} = \langle \Pi_V u \otimes \delta u \rangle n, \Pi_V w \rangle_{\partial \Omega} - \langle \Pi_V u \otimes \delta u \rangle n, \Pi_V w \rangle_{\partial \Omega} \]
\[ = -\langle \Pi_V u \otimes \delta u \rangle n, \Pi_V w \rangle_{\partial \Omega} - \langle \Pi_V u \otimes \delta u \rangle n, \Pi_V w \rangle_{\partial \Omega}, \]
where the last step is obtained by inserting a zero term $\langle \Pi_V u \otimes \delta u \rangle n, \Pi_V w \rangle_{\partial \Omega} = 0$.
\[ \leq h^{\frac{1}{2}} \| \Pi_V u \|_{L^1(\partial \Omega)} h^{\frac{1}{2}} \| \Pi_V w \|_{L^1(\partial \Omega)} \| \Pi_V u \|_{\infty,\partial \Omega} \| \Pi_V w \|_{\partial \Omega} \]
\[ \leq Ch^{k+1} \| \Pi_V u \|_{1,h} \Pi_V \|_{1,\Omega} + Ch^{k+1} \| u \|_{\infty,\Omega} \| w \|_{1,\Omega} \]
\[ \leq Ch^{k+1} \| u \|_{2,\Omega} + \| u \|_{\infty,\Omega} \| w \|_{1,\Omega}, \]
where in the last step we used the inequalities (5.1c), (5.1b), (5.1d). The proof is complete if we combine all the above estimates.

\[ \square \]

**Step 4: Optimal estimate for $e_u$.** Notice that Lemma 5.2 provides an optimal estimate for $e_L$ but only suboptimal estimate for $e_u$. This is due to the fact that we use a $P_{k+1}$ polynomial space for the unknown $u$. To obtain an optimal convergence estimate for $e_u$ we will use the adjoint problem (2.2) to apply a dual argument. We begin by the following identity for the error $e_u$: 
Lemma 5.4. Let $(\phi, \psi)$ be the solution of the dual problem (2.2) with the source term $\theta = e_u$, then we have

$$
\|e_u\|_{T_h}^2 = -\langle e_u - e_u, \nu \delta_u n + \delta_\psi n \rangle_{T_h}
- \langle \nu \Pi_M e_u - e_u, \Pi_V \phi - \Pi_M \phi \rangle_{T_h}
+ \langle \nu \delta_L n - \delta_u n - \nu \Pi_M \delta_u, \Pi_V \phi - \Pi_M \phi \rangle_{T_h}
$$

$$
- \left( \langle e_u, \nabla \cdot (\phi \otimes u) \rangle_{T_h} + O((u, u); (e_u, e_u), (\Pi_V \phi, \Pi_M \phi)) \right)
- \langle (u, u); (\delta_u, \delta_u), (\Pi_V \phi, \Pi_M \phi) \rangle
+ \left( O((u, u); (u_h, u_h), (\delta_\phi, \delta_\phi)) - O((u, u); (u_h, u_h), (\delta_\phi, \delta_\phi)) \right)
$$

$$
+ \langle (u, u); (\delta_\phi, \delta_\phi), (\phi, \phi) \rangle - O((u, u); (u_h, u_h), (\phi, \phi)) - (e_u, Y)_{T_h}
$$

$$
+ \langle \nu e_L, \nabla \delta_\phi \rangle_{T_h} - (e_p, \nabla \cdot \delta_\phi)_{T_h}
$$

$$
:= T_1 + \cdots + T_8.
$$

Here $Y := \frac{1}{2}(\nabla \phi)^T u - \frac{1}{2}(\nabla u)^T \phi$ and

$$
\delta_\phi = \Phi - \Pi_G \Phi, \quad \delta_\phi := \phi - \Pi_V \phi, \quad \delta_\psi := \psi - \Pi_Q \psi, \quad \delta_\phi := \phi - \Pi_M \phi.
$$

Proof. By the adjoint problem (2.4) - (2.5) we have

$$
\|e_u\|_{T_h}^2 = -\nu(e_u, \nabla \cdot \Phi)_{T_h} - (e_u, \nabla \cdot (\phi \otimes u))_{T_h} - (e_u, \nabla \psi)_{T_h} - (e_u, Y)_{T_h}
$$

$$
- \langle \nu e_L, \Phi \rangle_{T_h} + \langle \nu e_L, \nabla \phi \rangle_{T_h}
$$

$$
- (e_p, \nabla \cdot \phi)_{T_h}
$$

rearranging the terms, we have

$$
= -\nu(e_u, \nabla \cdot \Phi)_{T_h} - (\nu e_L, \Phi)_{T_h}
$$

$$
- (e_u, \nabla \psi)_{T_h}
$$

$$
- (e_u, \nabla \cdot (\phi \otimes u))_{T_h} + \langle \nu e_L, \nabla \phi \rangle_{T_h} - (e_u, \nabla \phi)_{T_h} - (e_u, Y)_{T_h}
$$

$$
= -\nu(e_u, \nabla \cdot \Pi_G \Phi)_{T_h} - (\nu e_L, \Pi_G \Phi)_{T_h} - \nu(e_u, \nabla \delta_\phi)_{T_h}
$$

$$
- (e_u, \nabla \Pi_Q \psi)_{T_h} - (e_u, \nabla \delta_\psi)_{T_h}
$$

$$
- (e_u, \nabla \cdot (\phi \otimes u))_{T_h} + \langle \nu e_L, \nabla \phi \rangle_{T_h} - (e_u, \nabla \phi)_{T_h} - (e_u, Y)_{T_h}
$$

taking $(G, v, q, \mu) = (\nu \Pi_G \Phi, 0, \Pi_Q \psi, 0)$ in the error equation (5.3), inserting the resulting identity into the above expression and simplifying, we have

$$
= -\langle e_u, \nu \Pi_G \Phi \rangle_{T_h} - \nu(e_u, \nabla \cdot \delta_\phi)_{T_h}
$$

$$
- \langle e_u, \Pi_Q \psi \rangle_{T_h} - (e_u, \nabla \delta_\phi)_{T_h}
$$

$$
- \langle e_u, \nabla \cdot (\phi \otimes u) \rangle_{T_h} + \langle \nu e_L, \nabla \phi \rangle_{T_h} - (e_u, \nabla \phi)_{T_h} - (e_u, Y)_{T_h}
$$

inserting two zero terms: $\langle e_u, \nu \Phi \rangle_{T_h} = \langle e_u, \psi \rangle_{T_h}$ and integrating by parts in the first two lines to obtain

$$
= -\langle e_u - e_u, \nu \delta_\phi n + \delta_\psi n \rangle_{T_h}
$$

$$
- (e_u, \nabla \cdot (\phi \otimes u))_{T_h} + \langle \nu e_L, \nabla \phi \rangle_{T_h} - (e_u, \nabla \phi)_{T_h} - (e_u, Y)_{T_h}
$$

Next we work on the last line in the above expression. We first insert the projection of $\phi$ to have

$$
-\langle e_u, \nabla \cdot (\phi \otimes u) \rangle_{T_h} + \langle \nu e_L, \nabla \phi \rangle_{T_h} - (e_u, \nabla \phi)_{T_h} - (e_u, Y)_{T_h}
$$

$$
= \langle \nu e_L, \nabla \Pi_V \phi \rangle_{T_h} - (e_p, \nabla \cdot \Pi_V \phi)_{T_h} - (e_u, \nabla \cdot (\phi \otimes u))_{T_h} - (e_u, Y)_{T_h}
$$

$$
+ \langle \nu e_L, \nabla \delta_\phi \rangle_{T_h} - (e_p, \nabla \cdot \delta_\phi)_{T_h}
$$
taking \((G, v, q, \mu) = (0, \Pi_V \phi, 0, \Pi_M \phi)\) in the error equation \(5.3\), intergrating by parts for the last two terms in the above expression and simplifying, we have,

\[
-\frac{\nu}{h} (\Pi_M e_u - e_u) + \Pi_V \phi - \Pi_M \phi)_{\partial \Omega} + \left(\nu \delta_t n - \delta_p n - \frac{\nu}{h} \Pi_M \delta u, \Pi_V \phi - \Pi_M \phi)_{\partial \Omega} + O((u, u; (u, u), (\Pi_V \phi, \Pi_M \phi)) + O((u, u; (u, \hat{u}), (\Pi_V \phi, \Pi_M \phi))
\]

\[
- (e_u, \nabla \cdot (\phi \otimes u)_{\Omega} - (e_u, Y)_{\Omega} + \left(\nu \delta_t n - \delta_p n - \frac{\nu}{h} \Pi_M \delta u, \Pi_V \phi - \Pi_M \phi)_{\partial \Omega}
\]

\[
- O((u, u; (\delta_u, \delta_u), (\Pi_V \phi, \Pi_M \phi)) + O((u, u; (u, \hat{u}), (\Pi_V \phi, \Pi_M \phi)) - O((u, u; (u, \hat{u}), (\Pi_V \phi, \Pi_M \phi)) - (e_u, Y)_{\Omega} + \left(\nu \delta_t n - \delta_p n - \frac{\nu}{h} \Pi_M \delta u, \Pi_V \phi - \Pi_M \phi)_{\partial \Omega}.
\]

We can obtain the expression in the Lemma by inserting \((\phi, \phi)\) in the two \(O\) terms in the above identity. This completes the proof. \(\square\)

Now we are ready to prove our last result:

**Lemma 5.5.** Under the same assumption as in Lemma \(5.2\) in addition we assume the full \(H^2\)-regularity of the adjoint problem \(2.12\) holds and \(k \geq 1\), then we have

\[
\|e_u\|_\Omega \leq C h^{k+2},
\]

Here the constant \(C\) depends on \(\|u\|_{k+2, \Omega}, \|u\|_{W^{1, \infty}(\Omega)}, \|p\|_{k+1, \Omega}, \nu \) and \(k\) but independent of \(h\).

**Proof.** By identity in Lemma \(5.4\) it suffice to estimate \(T_1 - T_8\).

For \(T_1\), we apply Cauchy-Schwarz inequality, Lemma \(5.2\) \((5.13)\) and the regularity inequality \(2.3\) to have

\[
T_1 \leq \frac{1}{h} \|u\|_{\partial \Omega} h^2 \|\nu \delta_t n + \delta_p n\|_{\partial \Omega} \leq C h^{k+1} \cdot h^2 h^2 \|\phi\|_{1, \Omega} \leq C h^{k+2} \|e_u\|_\Omega.
\]

Similarly, for \(T_2\) we have

\[
T_2 \leq \frac{1}{h^2} \|\Pi_M e_u - e_u\|_{\partial \Omega} h^2 \|\Pi_V \phi - \Pi_M \phi\|_{\partial \Omega} \leq C h^{k+1} \cdot \frac{1}{h^2} h^2 \|\phi\|_{2, \Omega} \leq C h^{k+2} \|e_u\|_\Omega.
\]

Using Cauchy-Schwarz inequality, \((5.12), (5.13)\) and \((2.3)\), we can bound \(T_3\) as

\[
T_3 \leq C h^{k+2} (\|1\|_{k+1, \Omega} + \|p\|_{k+1, \Omega} + \|u\|_{k+2, \Omega} h^2 \|\phi\|_{2, \Omega} \leq C h^{k+2} \|e_u\|_\Omega.
\]

For \(T_8\), we simply apply the Cauchy-Schwarz inequality, \((5.1b), (5.2)\), Lemma \(5.3\) and the regularity inequality \(2.3\) to have

\[
T_8 \leq \nu \|e_L\|_\Omega + \|e_p\|_\Omega \|\nabla \phi\|_{\Omega} \leq C (\nu \|e_L\|_\Omega + \|e_p\|_\Omega) \|\phi\|_{2, \Omega} \leq C h^{k+2} \|e_u\|_\Omega.
\]

For \(T_5\), we explicitly write this term:

\[
T_5 = - (\delta_u \otimes u, \nabla \Pi_V \phi)_{\Omega} + \left(\tauC(u) (\Pi_M u - u, \Pi_V u) + \Pi_V \phi - \Pi_M \phi)_{\partial \Omega} + ((u \otimes \delta_u)n, \Pi_V \phi - \Pi_M \phi)_{\partial \Omega}
\]

\[
\leq \|u\|_{\infty, \Omega} (\|\delta_u\|_{\Omega} \|\nabla \Pi_V \phi\|_{\Omega} + \|\Pi_M u - u, \Pi_V u)_{\partial \Omega} \|\Pi_V \phi - \Pi_M \phi\|_{\partial \Omega} \|\delta_u\|_{\partial \Omega} \|\Pi_M \phi - \Pi_V \phi\|_{\partial \Omega} \)
\]

\[
\leq C \|u\|_{\infty, \Omega} (h^{k+2} \|u\|_{k+2, \Omega} \|\phi\|_{1, \Omega} + h^{k+2} \|u\|_{k+1, \Omega} \|\phi\|_{2, \Omega}) \quad \text{by} \quad (5.1b), (5.1c) \quad \text{and} \quad (5.1d)
\]

\[
\leq C h^{k+2} \|e_u\|_\Omega,
\]

by the regularity assumption \(2.3\).
For $T_4$, we first expand the term as:

$$T_4 = - (e_u \cdot (\nabla \cdot u) \phi)_{\partial T_h} - (e_u \otimes u, \nabla \phi)_{\partial T_h} + (e_u \otimes u, \nabla \Pi V \phi)_{\partial T_h}$$

$$- \left< \tau_C(u)(e_u - e_\mathcal{A}), \Pi V \phi - \Pi M \phi \right|_{\partial T_h} - \left< (e_u \otimes u) n, \Pi V \phi - \Pi M \phi \right|_{\partial T_h}$$

$$= -(e_u \otimes u, \nabla \delta \phi)_{\partial T_h} - \left< \tau_C(u)(e_u - e_\mathcal{A}), \Pi V \phi - \Pi M \phi \right|_{\partial T_h} - \left< (e_u \otimes u) n, \Pi V \phi - \Pi M \phi \right|_{\partial T_h}$$

$$\leq C \|u\|_\infty \Omega \|e_u\|_\Omega \|\nabla \delta \phi\|_{\partial T_h} + h^{-\frac{1}{2}} \|e_u - e_\mathcal{A}\|_{\partial T_h} h^{\frac{1}{2}} \|\Pi V \phi - \Pi M \phi\|_{\partial T_h} + \|e_u\|_{\partial T_h} \|\Pi V \phi - \Pi M \phi\|_{\partial T_h}$$

$$\leq C h^{k+\frac{1}{2}} \|e_u\|_\Omega + C h^{\frac{1}{2}} \|e_u\|_\Omega \|e_u\|_{\partial T_h},$$

by Lemma 5.2, 5.11b, 5.11c and 5.11d. By a triangle inequality we have

$$\|e_u\|_{\partial T_h} \leq \|e_u - e_\mathcal{A}\|_{\partial T_h} + \|e_u\|_{\partial T_h} \leq C (h^{k+\frac{1}{2}} + h^{k+\frac{1}{2}}).$$

Inserting this inequality into the estimate for $T_4$ we obtain:

$$T_4 \leq C h^{k+\frac{1}{2}} \|e_u\|_\Omega.$$

To bound $T_6$, we first derive some useful inequalities, we first bound $\|u_h\|_\infty \Omega$:

$$\|u_h\|_\infty \Omega \leq \|e_u\|_\infty + \|\Pi V u\|_\infty \Omega \leq C (h^{-\frac{1}{2}} \|e_u\|_\Omega + \|u\|_\infty \Omega).$$

Next by a triangle inequality, we have

$$\|u_h - \tilde{u}_h\|_{\partial T_h} \leq \|u_h - e_\mathcal{A}\|_{\partial T_h} + \|\Pi V u - \Pi M u\|_{\partial T_h} \leq C (h^{k+\frac{1}{2}} + h^{k+\frac{1}{2}}) \leq C h^{k+\frac{1}{2}}.$$

Consequently, we have

$$\|\tilde{u}_h\|_{\infty, \partial T_h} \leq \|u_h - \tilde{u}_h\|_{\infty, \partial T_h} + \|u_h\|_{\infty, \partial T_h} \leq C h^{k+\frac{1}{2}} + C (h^{k+\frac{1}{2}} \|e_u\|_\Omega + \|u\|_\infty \Omega).$$

The last step we applied a scaling argument for the polynomials on $\partial T_h$. Finally, applying triangle inequality we obtain the following estimates:

$$\|u - u_h\|_\Omega \leq \|e_u\|_\Omega + \|\delta_u\|_\Omega \leq C h^{k+1},$$

$$\|\nabla u - u_h\|_{\partial T_h} \leq \|\nabla e_u\|_{\partial T_h} \leq h^{k+1}.$$

Now we are ready to present the estimate for $T_6$. If we expand $T_6$ using the definition of $O$, we obtain:

$$T_6 = -(u_h \otimes (u - u_h), \nabla \delta \phi) - (\frac{1}{2} \left( \nabla \cdot (u - u_h) \right) u_h, \delta \phi)_{\partial T_h} + (\frac{1}{2} (u_h \otimes (u - u_h)) n, \delta \phi)_{\partial T_h}$$

$$+ \left< (\tau_C(u) - \tau_C(\tilde{u}_h))(u_h - u_h), \delta \phi - \delta \tilde{u}_h \right|_{\partial T_h} + \left< (u_h \otimes (u - u_h)) n, \delta \phi - \delta \tilde{u}_h \right|_{\partial T_h}$$

applying the generalized Hölder’s inequality for each term, we have

$$\leq \|u_h\|_\infty \Omega \|u - u_h\|_\Omega \|\nabla \delta \phi\|_{\partial T_h} + \|\nabla \cdot (u - u_h)\|_{\partial T_h} \|\delta \phi\|_{\partial T_h} + \|u_h\|_\infty \Omega \|u - u_h\|_{\partial T_h} \|\delta \phi\|_{\partial T_h}$$

$$+ \|u_h\|_\infty \|u_h - \tilde{u}_h\|_{\infty, \partial T_h} \|\delta \phi - \delta \tilde{u}_h\|_{\partial T_h} + \|u_h\|_\infty \|u_h - \tilde{u}_h\|_{\infty, \partial T_h} \|\delta \phi - \delta \tilde{u}_h\|_{\partial T_h}$$

now if we apply the inequalities 5.8, 5.11a, 5.11b, 5.11c and 5.11d, we have

$$\leq C h^{k+\frac{1}{2}} (h^{k+\frac{1}{2}} + 1) \|e_u\|_\Omega \leq C h^{k+\frac{1}{2}} \|e_u\|_\Omega.$$

Finally, we need to estimate $T_7$ which is more involved than the previous terms. To this end, we begin by expanding the nonlinear operator $O$:

$$T_7 = (u_h \otimes (u - u_h), \nabla \phi)_{\partial T_h} + (\frac{1}{2} \nabla \cdot (u - u_h) u_h, \phi)_{\partial T_h} + (\frac{1}{2} (u_h \otimes (u - u_h)) n, \phi)_{\partial T_h} - (e_u, Y)_{\partial T_h},$$

integrating by parts the second term, we have

$$= (u_h \otimes (u - u_h), \nabla \phi)_{\partial T_h} + (\frac{1}{2} (u_h \otimes (u - u_h)) n, \phi)_{\partial T_h} - \frac{1}{2} (u_h \otimes (u - u_h), \nabla \phi)_{\partial T_h}$$

$$- \frac{1}{2} (\phi \otimes (u - u_h), \nabla u_h)_{\partial T_h} + (\frac{1}{2} (u_h \otimes (u - u_h)) n, \phi)_{\partial T_h} - (e_u, Y)_{\partial T_h}$$

$$= (\frac{1}{2} (u_h \otimes (u - u_h), \nabla \phi)_{\partial T_h} + (\frac{1}{2} (u_h \otimes (u - u_h)) n, \phi)_{\partial T_h} - (\frac{1}{2} \phi \otimes (u - u_h), \nabla u_h)_{\partial T_h} - (e_u, Y)_{\partial T_h},$$

$$- \frac{1}{2} (\phi \otimes (u - u_h), \nabla u_h)_{\partial T_h} + (\frac{1}{2} (u_h \otimes (u - u_h)) n, \phi)_{\partial T_h} - (e_u, Y)_{\partial T_h}.$$
inserting the zero term $-\frac{1}{2}(u \otimes (u - \hat{u}_h)) n, \phi)_{\partial T_h} = 0$ into above expression,

\[
-\frac{1}{2}((u - \hat{u}_h) \otimes (u - \hat{u}_h)) n, \phi)_{\partial T_h} + \frac{1}{2} u_h \otimes (u - u_h), \nabla \phi)_{\partial T_h} - \frac{1}{2} \phi \otimes (u - u_h), \nabla u_h)_{\partial T_h} - (e_u, Y)_{T_h}
\]

\[
-\frac{1}{2}((u - \hat{u}_h) \otimes (u - \hat{u}_h)) n, \phi)_{\partial T_h} - \frac{1}{2} (u - u_h) \otimes (u - u_h), \nabla \phi)_{\partial T_h} + \frac{1}{2} \phi \otimes (u - u_h), \nabla (u - u_h))_{T_h}
\]

\[
+ \frac{1}{2} u \otimes (u - u_h), \nabla \phi)_{T_h} - \frac{1}{2} \phi \otimes (u - u_h), \nabla u)_{T_h} - (e_u, Y)_{T_h},
\]

by the definition of $Y = \frac{1}{2}(\nabla \phi)^{\top} u - \frac{1}{2}(\nabla u)^{\top} \phi$, we obtain:

\[
-\frac{1}{2}((u - \hat{u}_h) \otimes (u - \hat{u}_h)) n, \phi)_{\partial T_h} - \frac{1}{2} (u - u_h) \otimes (u - u_h), \nabla \phi)_{\partial T_h} + \frac{1}{2} \phi \otimes (u - u_h), \nabla (u - u_h))_{T_h}
\]

\[
+ \frac{1}{2} u \otimes (u - u_h), \nabla \phi)_{T_h} - \frac{1}{2} \phi \otimes (u - u_h), \nabla u)_{T_h} = T_{T_1} + \cdots + T_{T_5}.
\]

We are going to estimate each of the above terms. For $T_{T_1}$ we apply the generalized Hölder’s inequality, (3.11), (5.11), and (2.3),

\[
T_{T_1} \leq \|u - u_h\|_{L^4(\partial T_h)} \|\nabla \phi\|_{L^4(\partial T_h)} \leq C h^{-\frac{2}{5}} \|u - u_h\|_{1,h} \|\phi\|_{1,h} \leq C h^{2k+1} \|e_u\|_{\Omega}.
\]

For $T_{T_2}$, we apply the generalized Hölder’s inequality, (3.11), (5.11) and (2.3) to get:

\[
T_{T_2} \leq \|u - u_h\|_{L^4(\Omega)} \|\nabla \phi\|_{\Omega} \leq C \|u - u_h\|_{1,h} \|\phi\|_{1,\Omega} \leq C h^{k+2} \|e_u\|_{\Omega}.
\]

Similarly, we can bound $T_{T_3}$ as

\[
T_{T_3} \leq \|\phi\|_{L^4(\Omega)} \|u - u_h\|_{L^4(\Omega)} \|\nabla (u - u_h)\|_{T_h} \leq C h^{2k+2} \|e_u\|_{\Omega}.
\]

For $T_{T_4}, T_{T_5}$ we apply the generalized Hölder’s inequality as

\[
T_{T_4} \leq \|u\|_{\infty, \Omega} \|\delta u\|_{\Omega} \|\nabla \phi\|_{\Omega} \leq C \|u\|_{\infty, \Omega} h^{k+2} \|e_u\|_{\Omega},
\]

\[
T_{T_5} \leq \|\nabla u\|_{\infty, \Omega} \|\phi\|_{\Omega} \|\delta u\|_{\Omega} \leq C \|\nabla u\|_{\infty, \Omega} h^{k+2} \|e_u\|_{\Omega}.
\]

The proof is complete by combining all the estimates for $T_1 - T_5$.

6. Concluding remarks

In this paper, we introduced and analyzed a new HDG method for the Navier-Stokes equations. The work can be seen as a continuation of our previous work on HDG methods for linear problems, see [25, 26]. Comparing with the original HDG method for Navier-Stokes equation [6, 23], our method uses an enriched polynomial space for the velocity in each element, a modified numerical flux and a modified HDG formulation. As a consequence, we obtained optimal order of convergence for all unknowns and superconvergence for the velocity without postprocessing. In addition, similar as in [25, 26], the analysis in this paper is valid for general polygonal meshes.

Numerical study of the method as well as other computational aspects including the characterization of the scheme, implementation of the method using Picard iteration and other related issues will be extensively discussed in a separate paper.

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