Unsteady Boundary Layer Flow of a Casson Fluid past a Permeable Stretching/Shrinking Sheet: Paired Solutions and Stability Analysis

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Abstract. This paper discusses the numerical solutions of the unsteady Casson fluid flow over a permeable stretching/shrinking sheet. The suitable similarity transformations are implemented to transform the governing partial differential equations into ordinary differential equations, and those are then solved numerically by the MATLAB solver bvp4c function. The effects of the governing parameters on the velocity distributions and the reduced skin friction coefficients are given in the form of graph and examined thoroughly. The existence of paired solutions is identified, and the stability analysis revealed that the upper branch solution is stable while the lower branch solution is unstable.

1. Introduction

Casson fluid model is a generalized Newtonian fluid model which has been introduced by Casson in 1959 to explain the fluid flow of pigments and oil mixtures [1]. Conventionally, Casson fluid model as a subtype of viscoplastic fluids acts like an elastic liquid which exercised to describe the dynamics of blood flow [2]. The non-linearity constitutive equation in the Casson model epitomizes the flow curves of pigments suspension in lithographic varnishes which related in the process of preparing the printing ink [1]. Since no flow occurs with small shear stress in a Casson fluid, this model had attracted many researchers to scrutinize it [3].

The purpose of the present work is to widen the research gap in the scope of unsteady boundary layer flow in a Casson fluid by obtaining paired solutions and performing stability analysis. Previously, Mustafa et al. [4] examined the unsteady Casson fluid flow and heat transfer past an impulsively moving flat plate with a parallel free stream. Next, Mukhopadhyay et al. [5] solved the problem of unsteady boundary layer flow past a stretching surface in a Casson fluid by considering a prescribed surface temperature. Then, Mukhopadhyay [6] investigated the effects of thermal radiation in the unsteady boundary layer flow and heat transfer towards a stretching sheet in a Casson fluid. So far, paired solutions have not been discovered yet for the problem of unsteady boundary layer flow of a Casson fluid past a stretching/shrinking sheet. Therefore, the present work aims to identify the presence of paired solutions and distinguish their stability for the unsteady boundary layer flow past a stretching/shrinking sheet in a Casson fluid. The bvp4c function in the MATLAB software is used to perform the numerical computations. This study is new, and thus all the generated numerical results are original.
2. Problem formulations

This study emphasizes an unsteady two-dimensional boundary layer flow of a Casson fluid over a continuous permeable stretching/shrinking sheet. The Casson fluid flow within the region $y > 0$ propelled by a spontaneously moving stretching/shrinking sheet, which is in the $x-$ direction. Both $x$ and $y$ are Cartesian coordinates and are measured along the surface and normal to it, respectively. The sheet moves out from a slit at the origin and then moves with a velocity $U_w(x,t) = bx/(1-t)$, where $a$ and $b$ are positive constants, and $\lambda$ is a dimensionless stretching/shrinking constant where $\lambda > 0$ shows the stretching surface while $\lambda < 0$ indicates the shrinking surface.

The rheological equation of state which reflects an isotropic and incompressible flow of a Casson fluid can be expressed as follows [7]:

$$\tau_{ij} = \begin{cases} 2(\mu_B + \frac{p_y}{\sqrt{2\pi}})e_{ij}, \pi > \pi_c \\ 2(\mu_B + \frac{p_y}{\sqrt{2\pi}})e_{ij}, \pi < \pi_c \end{cases}$$ (1)

Here, $\tau_{ij}$ denotes the $(i,j)$-th component of the stress tensor, $\pi = e_{ij}e_{ij}$ and $e_{ij}$ represent the $(i,j)$-th component of the deformation rate, $\pi$ signifies the product component of deformation rate with itself, $\pi_c$ symbolizes a critical value of this product based on the non-Newtonian fluid, and $p_y$ implies the yield stress of the fluid. Next, the governing boundary layer equations for this problem are [6]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$ (2)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \left( 1 + \frac{1}{\beta} \right) \frac{\partial^2 u}{\partial y^2},$$ (3)

subject to the initial and boundary conditions

$t < 0 : \quad v = 0, u = 0, \quad \text{for any } x, y$

$t \geq 0 : \quad u = U_w(x,t), v = v_w(t), \quad \text{at } y = 0$ (4)

$$u \to 0 \quad \text{as } \quad y \to \infty,$$

where $u$ and $v$ are the velocity components along the Cartesian coordinates $x$ and $y$, $\nu$ is the kinematic viscosity of the fluid, and $\beta = \mu_B \sqrt{2\pi}/p_y$ is the non-Newtonian (Casson) parameter.

Now, we introduce the following similarity transformations which have been employed by Andersson et al. [8],

$$\psi = \sqrt{\frac{bv}{(1-\alpha t)}} x f(\eta), \quad \eta = \sqrt{\frac{b}{\nu(1-\alpha t)}} y,$$ (5)

and $\psi$ is the stream function which classically defines $u = \partial \psi/\partial y$ and $y = -\partial \psi/\partial x$. Thus, take

$$v_w(t) = -\sqrt{\frac{bv}{(1-\alpha t)}} s,$$ (6)

where $s$ is the constant wall mass transfer parameter with $s > 0$ implies suction, while $s < 0$ alludes to the state of injection. Equation (1) is identically satisfied, and substitution of Equation (5) into the Equation (3) gives the following ordinary differential equation:

$$\left( 1 + \frac{1}{\beta} \right) f''' - A \left( \frac{\eta}{2} f'' + f' \right) - (f')^2 + ff'' = 0,$$ (7)
along with the boundary conditions
\[ f(0) = s, \quad f'(0) = \lambda, \quad \text{and} \quad f'(\infty) \to 0, \quad (8) \]
where \( A = \alpha/b \) represents the unsteadiness parameter with \( A > 0 \) for an accelerating flow and \( A < 0 \) for a decelerating flow.

The associated physical quantity of interest in this study is the skin friction coefficient, \( C_f \) which can be defined as [9]
\[ C_f = \frac{\tau_w}{\rho U_w^2(x)}, \quad (9) \]
where \( \tau_w \) typifies wall shear stress along the stretching/shrinking sheet and is defined as follows [9]:
\[ \tau_w = \left( \mu_B + \frac{p}{\sqrt{2\pi}} \right) \frac{\partial u}{\partial y}, \quad (10) \]
The succeeding relation can be attained through the substitution of (10) into (9)
\[ \text{Re}_x^{1/2}C_f = \left( 1 + \frac{1}{\beta} \right) f''(0), \quad (11) \]
where \( \text{Re}_x = U_w x / \nu \) is the local Reynolds number.

3. Stability analysis
The nature of the system in (7)-(8) is to produce more than one numerical solutions. Hence, the execution of stability analysis is necessary to sort out the stable and unstable solutions [10]. Stability analysis requires the introduction of new similarity transformations which are given as
\[ \psi = \sqrt{\frac{\nu b}{2(1-\alpha t)}} x f(\eta, \tau), \quad \eta = \sqrt{\frac{b}{\nu(1-\alpha t)}} y, \quad \tau = \frac{bt}{(1-\alpha t)}. \quad (12) \]
Substitute (12) into Equation (3) and the following equation can be obtained:
\[ \left( 1 + \frac{1}{\beta} \right) \frac{\partial^3 F}{\partial \eta^3} + f \frac{\partial^2 f}{\partial \eta^2} - A \left( \frac{\eta \partial^2 f}{2 \partial \eta^2} + \frac{\partial f}{\partial \eta} \right) - \left( \frac{\partial f}{\partial \eta} \right)^2 - (1 + \tau A) \left( \frac{\partial^2 f}{\partial \eta \partial \tau} \right) = 0, \quad (13) \]
accompanied with the boundary conditions
\[ f(0, \tau) = s, \quad \frac{\partial f}{\partial \eta}(0, \tau) = \lambda, \quad \frac{\partial f}{\partial \eta}(\infty, \tau) \to 0. \quad (14) \]
It is assumed that the solutions of (13)-(14) are given in (15), where \( f = f_0 \) is the base solution which is attained by solving (7)-(8) (see [11])
\[ f(\eta, \tau) = f_0(\eta) + e^{-\gamma \tau} F(\eta, \tau), \quad (15) \]
where \( \gamma \) is an unknown eigenvalue parameter and \( F(\eta, \tau) \) is relatively small contrasted with \( f_0(\eta) \). Substitution of (15) into (13) and (14) will produce the following expression:
\[ \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial^3 F}{\partial \eta^3} \right) + \left( f_0 - \frac{A \eta}{2} \right) \left( \frac{\partial^2 F}{\partial \eta^2} \right) - \left( A + 2 \frac{\partial f_0}{\partial \eta} \right) \frac{\partial F}{\partial \eta} \]
\[ + F \left( \frac{\partial^2 f_0}{\partial \eta^2} \right) - (1 + \tau A) \frac{\partial F}{\partial \eta} + (1 + \tau A)\gamma \frac{\partial F}{\partial \eta} = 0, \quad (16) \]
subject to the boundary conditions

\[ F(0, \tau) = 0, \quad \frac{\partial F}{\partial \eta}(0, \tau) = 0, \quad \frac{\partial F}{\partial \eta}(\infty, \tau) \to 0. \]  

(17)

Further, fix \( \tau = 0 \) to test the stability of steady-state flow solution \( f_0(\eta) \) and hence \( F = F_0(\eta) \) \[11\]. Thus, the subsequent linearized eigenvalue problem is obtained:

\[ \left( 1 + \frac{1}{\beta} \right) F_0''' + F_0'' f_0 + F_0' f_0'' - \frac{\beta \eta}{2} F_0'' - A f_0' - 2 F_0' f_0' + \gamma F_0' = 0, \]  

subject to the boundary conditions

\[ F_0(0) = 0, \quad F_0'(0) = 0, \quad F_0'(\infty) \to 0. \]  

(18)

Lastly, the boundary condition \( F_0'(\infty) \to 0 \) has been replaced with a normalizing boundary condition that is \( F_0''(0) = 1 \) \[12\]. The negative or positive sign of the smallest eigenvalue \( (\gamma_1) \) will decide the stability of the steady-state flow solution.

### 4. Results and discussion

The present model (7) and (8) can be solved numerically using the Matlab solver bvp4c function. Customarily, using different initial guess values will produce dual solutions. Table 1 proves the validity of the present numerical values by showing a good agreement with some of the previous literature.

| \( A \) | Sharidan et al. \[13\] | Chamkha et al. \[14\] | Mukhopadhyay et al. \[6\] | Present study |
|---|---|---|---|---|
| 0.8 | -1.261042 | -1.261512 | -1.261479 | -1.261480 |
| 1.2 | -1.377722 | -1.378052 | -1.377850 | -1.377851 |

Figure 1 displays an increment in \( \text{Re}_{x}^{1/2} C_f \) (upper branch solution) as \( s \) increases when the sheet is shrinking. The stronger effect of suction increases the velocity gradient which eventually results in the decrement of the momentum boundary layer thickness (see Figure 2). Conversely, the lower branch solution shows \( \text{Re}_{x}^{1/2} C_f \) decreases with the increment in \( s \). This trend might suggest the situation where the shrinking sheet unable to accept more Casson fluid to penetrate it. Hence, velocity profiles of the lower branch solution in Figure 2 inform an increase in \( s \) lowers the velocity gradient which leads to thicker momentum boundary layer. Meanwhile, the unique solution which appears within the region of stretching sheet also contributes to the decrement in \( \text{Re}_{x}^{1/2} C_f \) as \( s \) increases. Since the Casson fluid flow is in the same direction as the stretching sheet, it lowers the wall shear stress, and this reduces \( \text{Re}_{x}^{1/2} C_f \) as \( s \) increases.

Figure 3 discloses the increment in \( \text{Re}_{x}^{1/2} C_f \) as \( A \) increases when the Casson fluid becomes more viscous. The decreasing values of \( \beta \) elucidate the increasing yield stress which persuades the Casson fluid to be more non-Newtonian. Further, the accelerated fluid flow expands the velocity gradient at the surface of a permeable shrinking sheet. Figure 4 presents the momentum boundary layer thickness becomes thinner with the increment in \( A \) which directly contribute to an increment in \( \text{Re}_{x}^{1/2} C_f \). Nevertheless, the lower branch solution as in Figure 3 informs \( \text{Re}_{x}^{1/2} C_f \) decreases with the increment in \( A \). Velocity profiles of lower branch solution in Figure 4 evident
this as the momentum boundary layer thickness is found to be increasing with \( A \). Perhaps the permeability limit of the shrinking sheet has been exceeded which suppressing the velocity gradient and affects \( \text{Re}^{1/2}_x C_f \) to decline. Figure 5 illustrates the increment in \( \text{Re}^{1/2}_x C_f \) as \( \lambda \) decreases when the flow is accelerated. The stronger effect of shrinking restricts Casson fluid to penetrate the sheet. Consequently, the wall shear stress along the shrinking sheet increases and leads to a rise in \( \text{Re}^{1/2}_x C_f \).

Table 2 reports the smallest eigenvalue, \( \gamma_1 \) and the upper branch solution is stable by achieving positive smallest eigenvalue whereas the lower branch solution is unstable with attaining negative smallest eigenvalues. The positive smallest eigenvalue signifies the stable solution which able to control the given disturbance in the fluid flow. The negative smallest eigenvalue proves the unstable solution to promotes the growth of the disturbance in the fluid flow.

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**Figure 1.** Variations of \( \text{Re}^{1/2}_x C_f \) against \( \lambda \) when \( A = 0.5 \), and \( \beta = 5 \).

**Figure 2.** Velocity profiles, \( f'(\eta) \) when \( A = 0.5, \lambda = -1 \) and \( \beta = 5 \) for some values of \( s \).

**Figure 3.** Variations of \( \text{Re}^{1/2}_x C_f \) against \( \beta \) when \( s = 2 \), and \( \lambda = -1 \).

**Figure 4.** Velocity profiles, \( f'(\eta) \) when \( s = 2, \lambda = -1 \) and \( \beta = 10 \) for some values of \( A \).
Table 2. Smallest eigenvalue, $\gamma_1$ for some values of $A, \beta, s$ and $\lambda$.

| $A$ | $\beta$ | $s$ | $\lambda$ | Upper branch solution, $\gamma_1$ | Lower branch solution, $\gamma_1$ |
|-----|---------|-----|-----------|-------------------------------|-------------------------------|
| 0.5 | 5       | 2   | -1.3      | 0.2713                        | -0.2493                      |
|     |         |     | -1.33     | 0.0846                        | -0.0822                      |
|     |         |     | -1.333    | 0.0266                        | -0.0261                      |
|     |         |     | -1.3334   | 0.0021                        | -0.0046                      |

Figure 5. Variations of $\text{Re}_x^{1/2}C_f$ against $A$ when $s = 2$, and $\beta = 5$.

5. Conclusions
This work is devoted to solving the problem of unsteady boundary layer flow in a Casson fluid past a permeable stretching/shrinking surface. Paired solutions are observable within some values of $\lambda, \beta$ and $A$. Stability analysis confirmed that the upper branch solution is a stable solution, and the lower branch solution is an unstable solution. An increase in $s, A$ and $\lambda$ managed to delay the flow separation. The upper branch solution showed an increment in $\text{Re}_x^{1/2}C_f$ as $s$ and $A$ increased but the lower branch solution expressed the decrement in $\text{Re}_x^{1/2}C_f$.

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