The Casimir Effect in the Presence of a Minimal Length

U. Harbach
Institut für Theoretische Physik
J. W. Goethe-Universität
Robert-Mayer-Str. 10
60054 Frankfurt am Main, Germany

S. Hossenfelder
Department of Physics
University of Arizona
1118 East 4th Street
Tucson, AZ 85721, USA

(Dated: April 6, 2018)

Large extra dimensions lower the Planck scale to values soon accessible. Not only is the Planck scale the energy scale at which effects of modified gravity become important. The Planck length also acts as a minimal length in nature, providing a natural ultraviolet cutoff and a limit to the possible resolution of spacetime.

In this paper we examine the influence of the minimal length on the Casimir energy between two plates.

I. EXTRA DIMENSIONS

The study of models with Large eXtra Dimensions (LXD) has recently received a great deal of attention. These models, which are motivated by string theory\cite{1, 2, 3}, provide us with an extension to the standard model (SM) in which observables can be computed and predictions for tests beyond the SM can be addressed. This in turn might help us to extract knowledge about the underlying theory. The models of LXD successfully fill the gap between theoretical conclusions and experimental possibilities as the extra hidden dimensions may have radii large enough to make them accessible to experiments. The need to look beyond the SM infected many experimental groups to search for such SM violating processes, for a summary see e.g. \cite{4}.

Arkani-Hamed, Dimopoulos and Dvali \cite{5, 6} proposed a solution to the hierarchy problem by the introduction of \(d\) additional compactified spacelike dimensions in which only the gravitons can propagate. The SM particles are bound to our 4-dimensional sub-manifold, often called our 3-brane. Due to its higher dimensional character, the gravitational force at small distances then is much stronger as it goes in the radial distance \(r\) with the power \(-d-1\) instead of the usual \(-1\). This results in a lowering of the Planck scale to a new fundamental scale, \(M_f\), and gives rise to the exciting possibility of TeV scale GUTs \cite{7}. The radius \(R\) of the extra dimension lies in the range \(10^3\) to \(10^7\) fm for \(d\) from 2 to 7, or the inverse radius \(1/R\) lies in energy range \(eV\) to \(MeV\), resp. Throughout this paper the new fundamental scale is fixed to be \(M_f = 1\) TeV as a representative value. For recent constraints see e.g. \cite{8}.

II. THE MINIMAL LENGTH

Even if a full description of quantum gravity is not yet available, there are some general features that seem to go hand in hand with all promising candidates for such a theory. One of them is the need for a higher dimensional spacetime, one other the existence of a minimal length scale.

As the success of string theory arises from the fact that interactions are spread out on the world-sheet and do no longer take place at one singular point, the finite extension of the string has to become important at small distances or high energies, respectively. Now, that we are discussing the possibility of a lowered fundamental scale, we want to examine the modifications arising from this as they might get observable soon. If we do so, we should clearly take into account the minimal length effects.

In perturbative string theory\cite{9, 10}, the feature of a fundamental minimal length scale arises from the fact that strings cannot probe distances smaller than the string scale. If the energy of a string reaches this scale \(M_s = \sqrt{\alpha'}\), excitations of the string can occur and increase its extension\cite{11}. In particular, an examination of the spacetime picture of high-energy string scattering shows, that the extension of the string grows proportional to its energy\cite{9} in every order of perturbation theory. Due to this, uncertainty in position measurement can never become arbitrarily small.

Motivations for the occurrence of a minimal length are manifold. A minimal length can not only be found in string theory but also in loop quantum gravity and non-commutative geometries. It can be derived from various studies of thought-experiments, from the Lie-
algebraic stabilisation of the Heisenberg-Poincaré algebra [51], from black hole physics, the holographic principle and further more. Perhaps the most convincing argument, however, is that there seems to be no self-consistent way to avoid the occurrence of a minimal length scale. For reviews on this topic see e.g. [12].

Instead of finding evidence for the minimal scale as has been done in numerous studies, on can use its existence as a postulate and derive extensions to quantum theories with the purpose to examine the arising properties in an effective model.

Naturally, the minimum length uncertainty is related to a modification of the standard commutation relations between position and momentum [13]. With the Planck scale as high as $10^{16}$ TeV, applications of this are of high interest mainly for quantum fluctuations in the early universe and for inflation processes and have been examined closely. Now, in the presence of extra dimensions, we have not only a lowered fundamental scale but also a raised minimal length.

In [14, 15] we used a model for the effects of the minimal length by modifying the relation between the wave vector $k$ and the momentum $p$. We assume, no matter how much we increase the momentum $p$ of a particle, we can never decrease its wavelength below some minimal length $L_t$ or, equivalently, we can never increase its wave-vector $k$ above $M_t = 1/L_t$ [50]. Thus, the relation between the momentum $p$ and the wave vector $k$ is no longer linear $p = k$ but a function $f(k) = k(p)$.

This function $k(p)$ has to fulfill the following properties:

a) For energies much smaller than the new scale we reproduce the linear relation: for $p \ll M_t$ we have $p \approx k$.

b) It is an an uneven function (because of parity) and $k \parallel p$.

c) The function asymptotically approaches the upper bound $M_t$.

The quantization in this scenario is straight forward and follows the usual procedure. The commutators between the corresponding operators $\hat{k}$ and $\hat{x}$ remain in the standard form. Using the well known commutation relations

\[ [\hat{x}_i, \hat{k}_j] = i\hbar \delta_{ij} \]  

and inserting the functional relation between the wave vector and the momentum then yields the modified commutator for the momentum

\[ [\hat{x}_i, \hat{p}_j] = +i \frac{\partial p_i}{\partial k_j} \]  

which reflects the fact that by construction it is not possible anymore to resolve space-time distances arbitrarily good. Since $k(p)$ gets asymptotically constant, its derivative $\partial k/\partial p$ drops to zero and the uncertainty in Eq. (3) increases for high energies. Thus, the introduction of the minimal length reproduces the limiting high energy behavior found in string theory [9].

The arising physical modifications - as investigated in [14, 16] - can be traced back to an effective replacement of the usual momentum measure by a measure which is suppressed at high momentum

\[ \frac{d^3k}{(2\pi)^3} \rightarrow \frac{d^3p}{(2\pi)^3} \left| \frac{\partial k}{\partial p} \right| , \]  

where the absolute value of the partial derivative denotes the Jacobian determinant of $k(p)$.

In the following, we will use the specific relation for $k(p)$ by choosing

\[ k_\mu(p) = \epsilon_\mu \int_0^p e^{-\epsilon p^2} , \]  

where $\epsilon_\mu$ is the unit vector in $\mu$ direction, $p^2 = \vec{p} \cdot \vec{p}$ and $\epsilon = L_t^2 \pi/4$ (the factor $\pi/4$ is included to assure, that the limiting value is indeed $1/L_t$). Is is easily verified that this expression fulfills the requirements (a) - (c).

The Jacobian determinant of the function $k(p)$ is best computed by adopting spherical coordinates and can be approximated for $p \sim M_t$ with

\[ \left| \frac{\partial k}{\partial p} \right| \approx e^{-\epsilon p^2} . \]  

With this parametrization of the minimal length effects the modifications read

\[ \Delta p_i \Delta x_i \geq \frac{1}{2} \epsilon^2 + \epsilon p^2 \]  

\[ \frac{d^3k}{(2\pi)^3} \rightarrow \frac{d^3p}{(2\pi)^3} e^{-\epsilon p^2} . \]  

In field theory [54], one imposes the commutation relations Eq. (11) and (12) on the field $\phi$ and its conjugate momentum $\Pi$. Its Fourier expansion leads to the annihilation and creation operators which must obey

\[ \hat{a}_k, \hat{a}^\dagger_k = \delta(k - k') \]  

\[ \hat{a}_k, \hat{a}^\dagger_k = -i \left[ \hat{a}_{k'}, \hat{\Pi}^\dagger_{k'} \right] \]  

\[ \hat{a}_p, \hat{a}^\dagger_{p'} = e^{-\epsilon p^2} \delta(p - p') \]  

Note, that it is not necessary for our field to propagate into the extra dimensions to experience the consequences of the minimal length scale. In particular, we will assume that the field is bound on our submanifold to exclude the additional presence of KK-excitations. The existence of the extra dimensions is important for the case under discussion only by lowering the Planck scale and raising the minimal length.
III. THE CASIMIR EFFECT

Zero-point fluctuations of any quantum field give rise to observable Casimir forces if boundaries are present \[17\]. The Casimir effect is our experimental grip to the elusive manifestations of vacuum energy. Its importance for the understanding of the fundamental laws of quantum field theory lies in the direct connection to the problem of renormalization. Vacuum energies in quantum field theories are divergent. The presence of infinities in physics always signals that we have missed some crucial point in our mathematical treatment.

The Casimir effect has received great attention also in the context of extra dimensions and has been extensively studied in a wide variety of topics in those and related scenarios:

- The question how vacuum fluctuations affect the stability of extra dimensions has been explored in \[15\], \[16\], \[20\], \[21\], \[22\], \[23\], \[24\], \[48\], \[49\]. Especially the detailed studies in the Randall-Sundrum model have shown the major contribution of the Casimir effect to stabilize the radion \[24\], \[25\], \[26\], \[27\].

- Cosmological aspects like the cosmological constant as a manifestation of the Casimir energy or effects of Casimir energy during the primordial cosmic inflation have been analyzed \[28\], \[29\], \[30\], \[31\], \[32\], \[33\], \[34\], \[35\], \[36\], \[40\].

- The Casimir effect in the context of string theory has been investigated in \[33\], \[35\], \[38\], \[40\].

- The Casimir effect in a model with minimal length based on the assumption of Path Integral Duality \[38\], \[52\] has been studied in \[52\].

- It has been shown \[41\], \[42\] that the Casimir effect provides an analogy to the Hawking radiation of a black hole. The presence of Large eXtra Dimensions allows black hole creation in colliders \[42\] and the understanding of the evaporation properties is crucial for the interpretation of the signatures.

As one might expect, the introduction of a minimal length scale yields an ultraviolet cut off for the quantum theory which renders the occurring infinities finite.

Using the above framework, in the presence of a minimal length the operator for the field energy density is now given by

\[
\hat{H} = \frac{1}{2} \sum p \left( \hat{a}_p^\dagger \hat{a}_p + \hat{a}_p \hat{a}_p^\dagger \right) E ,
\]  

where \( E \) is the energy of a mode with momentum \( p \). With Eq. \[41\] and \( \hat{a}_p |0\rangle = 0 \) this results in the expectation value for the vacuum energy density

\[
\langle 0 | \hat{H} | 0 \rangle = \frac{1}{2} \sum p e^{-cp^2} E .
\]  

For Minkowski space in \( 3+1 \) dimensions without boundaries, this energy density now is finite due to the squeezed momentum space at high energies and given by

\[
\epsilon_{\text{Mink}} = \langle 0 | \hat{H} | 0 \rangle = \frac{16}{\pi} \frac{M_f}{L_f^3} .
\]  

We will now consider the case of two conducting parallel plates in a distance \( a \) in direction \( z \). We will neglect effects arising from surface corrections and finite plate width. We will further assume that the plates are perfect conductors and infinitely extended in the longitudinal directions \( x \) and \( y \), such that no boundaries effects are present.

The quantization of the wavelengths between the plates in the \( z \)-direction yields the condition \( k_l = l/a \).

Since the wavelengths can no longer get arbitrarily small, the smallest wavelength possible belongs to a finite number of nodes \( l_{\text{max}} = |a/L_f| \), where the brackets denote the next smaller integer. Resulting from this, momenta come in steps \( p_l = p(k_l) \) which are no longer equidistant, \( \Delta p_l = p_l - p_{l-1} \). Then

\[
\epsilon_{\text{Plates}} = \pi \sum_{l=-l_{\text{max}}}^{l_{\text{max}}} \Delta p_l \int_0^\infty dp\| e^{-cp^2}\| e^{-cp^2} E \| p\| ,
\]  

where \( p\|^2 = p_x^2 + p_y^2 \) and \( E^2 = p_x^2 + p_y^2 \).

Experiments do not measure absolute energy values but only differences. Therefore, the difference between the inside and the outside region has to be taken, i.e. Eq. \[41\] has to be subtracted from Eq. \[41\]. This then yields the Casimir energy accessibility by experiment through the induced pressure which results in a force acting on the plates. For the case of two parallel plates, the pressure is negative in the inside, or the force is attractive, respectively.

In the limit of large \( M_f \), i.e. of small \( L_f \), the renormalized standard result is obtained. This can be seen directly from taking the difference between the outside and inside region, that is Eq. \[43\] and Eq. \[43\] and applying the Abel-Plana-formula \[44\]. In this expression, the integral over the directions parallel to the plates is the same in both terms and may thus be taken conjointly:

\[
\lim_{L_f \to 0} \int_0^\infty dp\| \left( \sum_{l=-l_{\text{max}}}^{l_{\text{max}}} \Delta p_l e^{-cp^2}\| E \| p\| - \int_\infty^\infty dp e^{-cp^2}\| E \| p\| \right) e^{-cp^2}\| = \lim_{L_f \to 0} \int_0^\infty dp\| \left( \sum_{l=-\infty}^{l_{\text{max}}} \Delta p_l e^{-cp^2}\| E \| p\| - \int_\infty^\infty dp e^{-cp^2}\| E \| p\| - 2 \sum_{l=l_{\text{max}}}^{l_{\text{max}}} \Delta p_l e^{-cp^2}\| E \| p\| \right) e^{-cp^2}\| .
\]  

Taking the limit \( L_f \to \infty \) we have \( \Delta p_l \to 1/a \) and \( l_{\text{max}} \to \infty \). Then, the last term vanishes, while the first
terms are the same that appear in the classical calculation of the Casimir energy. Since the exponential, which acts as a dampening function, is holomorphic, the Abel-Plana-formula can be used to evaluate the difference. The obtained integral is uniformly convergent, and one can perform the limit before the integration. This then yields the classical expression:

$$\frac{1}{a} \int_{0}^{\infty} dp \sum_{l=-\infty}^{\infty} E \left| p \right| = \int_{0}^{\infty} dp \int_{-\infty}^{\infty} dp \ E \left| p \right|. \quad (17)$$

These computations shows very nicely, how the minimal length acts as a natural regulator in calculating the Casimir energy.

The result of our computation from a numerical analysis is shown in Fig. 1. As can be seen, the slope of the curve changes every time another mode fits between the plates. Although the slope (and thus the Casimir force) is singular at these points, the plot clearly shows that a finite energy is sufficient to surmount them and thus the result is physical. Also, the singularities seem to stem from the assumption of two strictly localised plates and should be cured in a full theory by the minimal length uncertainty on their position.

If the distance eventually drops below the minimal length, the energy density, and thus the pressure acting on the plates, becomes constant. This is to be contrasted with the standard result in which the curve diverges towards minus infinity for small distances.

Though the here discussed minimal length is some orders of magnitude out of range for experimentally measuring the modifications of the Casimir pressure, this result is interesting not only from a theoretical point of view: As mentioned before, the analogy to the black hole’s temperature is an important application. We can state that towards small black hole sizes the temperature does not increase according to the Hawking evaporation but is severely modified close to the new fundamental scale and eventually gets constant. Since the time evolution of the temperature is mostly ignored for the event generation of black hole decays (see e.g. [43]), the here presented result justifies this treatment.

IV. CONCLUSION

We have discussed the existence of a minimal length scale and used an effective model to include it into today’s quantum theory. Such a minimal scale would affect experimental measurements in the presence of Large eXtra Dimensions and yield to interesting phenomenological implications. The introduced minimal length acts as a natural ultraviolet regulator of the theory. We applied our model to the calculation of the Casimir energy and gave a numerical evaluation of the resulting expression. Furthermore, we showed how the minimal scale provides a physical motivation for the dampening function method used in the classical calculation of the Casimir energy via the Abel-Plana formula. Using the analogy to the black hole evaporation characteristics we showed that the time evolution of the system can be ignored close to the new fundamental scale.

Acknowledgments

This work was supported by the German Academic Exchange Service (DAAD), NSF PHY/0301998, DFG and the Frankfurt Institute for Advanced Studies (FIAS). We want to thank Marcus Bleicher, Achim Kempf and Jörg Ruppert for helpful discussions.

[1] I. Antoniadis, Phys. Lett. B 246, 377 (1990);
[2] I. Antoniadis and M. Quiros, Phys. Lett. B 392, 61 (1997).
[3] K. R. Dienes, E. Dudas and T. Gherghetta, Nucl. Phys. B 537, 47 (1999).
[4] G. Azuelos et al., arXiv:hep-ph/0204031.
[5] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 429, 263 (1998);
[6] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 436, 257 (1998); N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Rev. D 59, 086004 (1999).
[7] K. R. Dienes, E. Dudas and T. Gherghetta, Phys. Lett. B 436, 55 (1998).
[8] K. Cheung, arXiv:hep-ph/0409028; G. Landsberg [CDF and D0 - Run II Collaboration], arXiv:hep-ex/0412028.
[9] D. J. Gross and P. F. Mende, Nucl. Phys. B 303, 407 (1988).
[10] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B 216 (1989) 41.
[11] E. Witten, Phys. Today 50N5, 28 (1997).
[12] L. J. Garay, Int. J. Mod. Phys. A 10, 145 (1995); Y. J. Ng, Mod. Phys. Lett. A 18, 1073 (2003); S. Hossenfelder, Mod. Phys. Lett. A 19, 2727 (2004); A. Kempf, arXiv:hep-th/9810215.
[13] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52 (1995) 1108; A. Kempf and G. Mangano, Phys. Rev. D 55 (1997) 7909.
[14] L. J. Garay, Int. J. Mod. Phys. A 10, 145 (1995); Y. J. Ng, Mod. Phys. Lett. A 18, 1073 (2003); S. Hossenfelder, Mod. Phys. Lett. A 19, 2727 (2004); A. Kempf, arXiv:hep-th/9810215.
[15] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52 (1995) 1108; A. Kempf and G. Mangano, Phys. Rev. D 55 (1997) 7909.
[16] L. J. Garay, Int. J. Mod. Phys. A 10, 145 (1995); Y. J. Ng, Mod. Phys. Lett. A 18, 1073 (2003); S. Hossenfelder, Mod. Phys. Lett. A 19, 2727 (2004); A. Kempf, arXiv:hep-th/9810215.
[17] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52 (1995) 1108; A. Kempf and G. Mangano, Phys. Rev. D 55 (1997) 7909.
[18] L. J. Garay, Int. J. Mod. Phys. A 10, 145 (1995); Y. J. Ng, Mod. Phys. Lett. A 18, 1073 (2003); S. Hossenfelder, Mod. Phys. Lett. A 19, 2727 (2004); A. Kempf, arXiv:hep-th/9810215.
[19] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52 (1995) 1108; A. Kempf and G. Mangano, Phys. Rev. D 55 (1997) 7909.
[20] L. J. Garay, Int. J. Mod. Phys. A 10, 145 (1995); Y. J. Ng, Mod. Phys. Lett. A 18, 1073 (2003); S. Hossenfelder, Mod. Phys. Lett. A 19, 2727 (2004); A. Kempf, arXiv:hep-th/9810215.
[21] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52 (1995) 1108; A. Kempf and G. Mangano, Phys. Rev. D 55 (1997) 7909.
[22] L. J. Garay, Int. J. Mod. Phys. A 10, 145 (1995); Y. J. Ng, Mod. Phys. Lett. A 18, 1073 (2003); S. Hossenfelder, Mod. Phys. Lett. A 19, 2727 (2004); A. Kempf, arXiv:hep-th/9810215.
[23] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52 (1995) 1108; A. Kempf and G. Mangano, Phys. Rev. D 55 (1997) 7909.
[24] L. J. Garay, Int. J. Mod. Phys. A 10, 145 (1995); Y. J. Ng, Mod. Phys. Lett. A 18, 1073 (2003); S. Hossenfelder, Mod. Phys. Lett. A 19, 2727 (2004); A. Kempf, arXiv:hep-th/9810215.