Modulational Instability for Nonlinear Schrödinger Equations with a Periodic Potential.

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Abstract

We study the stability properties of periodic solutions to the Nonlinear Schrödinger (NLS) equation with a periodic potential. We exploit the symmetries of the problem, in particular the Hamiltonian structure and the $\mathbb{U}(1)$ symmetry. We develop a simple sufficient criterion that guarantees the existence of a modulational instability spectrum along the imaginary axis. In the case of small amplitude solutions that bifurcate from the band edges of the linear problem this criterion becomes especially simple. We find that the small amplitude solutions corresponding to the band edges alternate stability, with the first band edge being modulationally unstable in the focusing case, the second band edge being modulationally unstable in the defocusing case, and so on. This small amplitude result has a nice physical interpretation in terms of the effective mass of a particle in the periodic potential. We also consider, in somewhat less detail, some sideband instabilities in the small amplitude limit. We find that, depending on the Krein signature of the collision, these can be of one of two types. Finally we illustrate this with some exact calculations in the case where $V(x)$ is an elliptic function, where all of the relevant calculations can be done explicitly.

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1 Introduction

In this paper we consider the stability of standing wave solutions to the NLS equation with an external potential:

\[ i\psi_t = -\frac{1}{2}\psi_{xx} \pm \epsilon|\psi|^2\psi + V(x)\psi, \]

(1)

where the potential \( V(x) \) has period 1. We assume that we have a standing wave solution \( \psi(x,t,\epsilon) = \exp(-i\omega(\epsilon)t)\phi(x,\epsilon) \), where \( \phi(x,\epsilon) \) is real valued and either periodic \( (\phi(x + 1,\epsilon) = \phi(x,\epsilon)) \) or antiperiodic \( (\phi(x + 1,\epsilon) = -\phi(x,\epsilon)) \). For certain perturbative results we will also assume that the standing wave bifurcates continuously from the Floquet-Bloch eigenfunctions in the usual way. In other words we assume that \( \phi(x,0) \) is an eigenfunction of the periodic Schrödinger operator

\[ \mu\phi = \omega(0)\phi = -\frac{1}{2}\phi_{xx} + V(x)\phi \]

(2)

corresponding to a band edge, and \( \phi(x,\epsilon) \in C^2([0,1] \times (-\delta,\delta)) \).

We find that the combination of the Hamiltonian structure and the \( U(1) \) symmetry dramatically simplifies the structure and possible bifurcations of the spectrum of the linearized operator. We find a simple condition on the \( \mathbf{L}_+ \) operator which guarantees the existence of a modulational instability. In the case of weakly nonlinear standing waves, when \( \epsilon \) is small, we apply a perturbation argument to show that the lower band edges are modulationally unstable in the focusing case, while the upper band edges are modulationally unstable in the defocusing case.

We also study, in somewhat less detail, some bifurcations which occur for non-zero values of \( \mu \) when a certain eigenvalue degeneracy condition is met. We find that this bifurcation generically leads to the emergence of complex eigenval-
ees, and thus instability. Depending on the Krein signature of the unperturbed eigenvalues this may or may not lead to the opening of a gap along the real axis.

2 Fundamentals of Systems with Symplectic Structure and Periodic Coefficients

In this section we review some results of Floquet theory for the special case in which the equations admit a Hamiltonian formulation. For more details see the text of Yakubovich and Starzhinskii [18].

Throughout this paper we will apply the following notation. $U(x)$ will denote the solution operator of the periodic ODE governing stability. The matrix $M$ will denote the monodromy, or period, map $U(1)$. We will have occasion to consider the spectral properties of the second order operators $L_{\pm}$ individually. The matrix $m$ will denote the monodromy matrix of the second order problem associated to $L_{\pm}$. For each of these quantities the dependence on the parameters $\mu, \epsilon$ will be generally be suppressed unless it is necessary for clarity. Similarly $K_{\pm}(\mu)$ will denote the Floquet discriminants of the full stability problem, while $k(\mu)$ will denote the Floquet discriminant of the second order problem associated with $L_{\pm}$.

2.1 The Lyapunov-Poincare theorem

To begin we assume a Floquet problem of the following form:

$$U_x = JH(x)U \quad U(0) = I_{2N \times 2N} \quad H(x + 1) = H(x) \quad H' = H$$  \hspace{1cm} (3)

where, for simplicity, we have set the period to one. Here $I_{2N \times 2N}$ is the $2N \times 2N$ identity matrix and $J$ is a skew-symmetric matrix. We will assume that $J^2 = -I$. 


− I since the above problem with a non-singular J can always be mapped to one of that form. The monodromy matrix M is defined to be the period map $M = U(1)$. It is easy to see that $U(x)$, and thus $M$, satisfies the relation

$$U^t J U = J,$$  \hspace{1cm} (4)

so that $U(x)$ is a symplectic\(^3\) matrix. From this it follows that $\det(U) = 1$, and more generally that the characteristic polynomial $P(\lambda) = \det(U - \lambda I)$ satisfies

$$\det(U - \lambda I) = \det(U^t - \lambda I) = \det(J(U^t - \lambda I)J^t)$$  \hspace{1cm} (5)

$$= \det(U^{-1} - \lambda I) = \lambda^{2N} \det(U^{-1}) \det(U - \lambda^{-1} I)$$  \hspace{1cm} (6)

$$= \lambda^{2N} P(\lambda^{-1})$$  \hspace{1cm} (7)

thus the polynomial is palindromic - if $P(\lambda) = \sum_{0}^{2N} a_j \lambda^j$ then $a_{2N-j} = a_j$. This result is known as the Lyapunov-Poincare theorem [18].

This symmetry implies that the problem of finding the roots of the $2N^{th}$ degree polynomial $P(\lambda)$ can be reduced to finding the roots of an $N^{th}$ degree polynomial $\tilde{P}(z)$ by means of the transformation $z = \lambda + \lambda^{-1}$. For the case of the stability problem for standing wave solutions to the nonlinear Schrodinger equation the monodromy matrix is $4 \times 4$ ($N = 2$) In the case the characteristic polynomial of the monodromy matrix takes the form

$$P(\lambda) = 1 + a\lambda + b\lambda^2 + a\lambda^3 + \lambda^4 = 0.$$  \hspace{1cm} (8)

Under the conformal map the roots of this polynomial are mapped to the roots of the second degree polynomial

$$\tilde{P}(z) = z^2 + az + (b - 2) = 0$$  \hspace{1cm} (9)

\(^3\)Sometimes called J-unitary
with $\lambda = \frac{x \pm \sqrt{x^2 - 1}}{2}$, and the characteristic polynomial admits an explicit factorization into two second degree polynomials,

$$P(\lambda) = (1 - K_+ \lambda + \lambda^2)(1 - K_- \lambda + \lambda^2)$$

(10)

where the $K_\pm$ are the following algebraic functions of $a, b$:

$$K_\pm = \frac{-a \pm \sqrt{a^2 - 4b + 8}}{2}.$$  

(11)

The functions $K_\pm$ will be referred to as the Floquet discriminants. Since the conformal mapping $z = \lambda + \frac{1}{\lambda}$ takes the unit circle to the real interval $[-2, 2]$ it follows that the monodromy matrix has two eigenvalues (counted by algebraic multiplicity) on the unit circle if $K_+$ lies in the real interval $[-2, 2]$, and two more if $K_- \in [-2, 2]$.

It is convenient to express the $K_\pm$ in terms of the invariants of $M$. Since $a = -\text{Tr}(M), b = -\frac{1}{2}\text{Tr}(M^2) + \frac{1}{2}\text{Tr}(M)^2$ the Floquet discriminants are given by

$$K_\pm = \frac{\text{Tr}(M) \pm \sqrt{-(\text{Tr}(M))^2 + 2\text{Tr}(M^2) + 8}}{2}.$$  

(12)

This leads to our first observation

**Lemma 1** The possible (algebraic) multiplicities of the eigenvalues of a $4 \times 4$ monodromy matrix are $(1, 1, 1, 1)$, $(1, 1, 2)$, $(2, 2)$ and $4$. The conditions on the monodromy matrix $M$ which produce eigenvalues of higher multiplicity are as follows:
### Condition on $K_{\pm}$ vs. Condition on $M$ vs. Root Multiplicities

| Condition on $K_{\pm}$ | Condition on $M$ | Root Multiplicities |
|-------------------------|------------------|----------------------|
| $K_+ = \pm 2$ or $K_- = \pm 2$ | $\text{Tr}(M^2) = (\text{Tr}(M) \pm 2)^2$ | $(1,1,2)$ |
| Simple Band Edge       |                  |                      |
| $K_+ = \pm 2$ and $K_- = \mp 2$ | $\text{Tr}(M) = 0$ | $(2,2)$ |
| Repeated Band Edge     | $\text{Tr}(M^2) = 4$ |                      |
| $K_+ = K_- \neq \pm 2$ | $2\text{Tr}(M^2) = (\text{Tr}(M))^2 - 8$ | $(2,2)$ |
| ‘Accidental’ Degeneracy| $\text{Tr}(M) \neq \pm 4$ |                      |
| $K_+ = K_- = \pm 2$   | $\text{Tr}(M) = \pm 4$ | $4$ |
| Double Band Edge       | $\text{Tr}(M^2) = 4$ |                      |

**Proof:** A straightforward calculation from the explicit factorization given in (10).

It is worthwhile to consider the codimensions of the above possibilities. Since we have a one parameter family of monodromy matrices, parameterized by the spectral parameter $\mu$, we should generically expect to see possibilities 1 and 3, which require only one condition, but not possibilities 2 and 4, which require two independent conditions be satisfied. We shall see that possibility 4 always happens at $\mu = 0$ for small amplitude waves, and is forced by symmetries. It is this degeneracy which leads to the modulational instability. It is this situation that we will consider in most detail. We will also consider possibility 3, in somewhat less detail. We will not consider possibility 2, since it is non-generic, nor possibility 1, since it is not very interesting - one expects that the band edges will typically move under perturbation.
2.2 Krein Signature

There is an extensive theory of the structural stability of symplectic matrices due to Krein and collaborators. We shall give only a brief summary of this theory here. It is clear that, if a symplectic matrix $M$ has $k$ distinct eigenvalues on the unit circle it follows from continuity that a nearby symplectic matrix $\tilde{M}$ must also have $k$ distinct eigenvalues on the unit circle. Thus eigenvalues of symplectic matrices can only leave the unit circle via collisions. What is less clear is that only certain collisions can lead to pairs of eigenvalues leaving the unit circle. The quantity which distinguishes such “dangerous” collisions is that of the Krein signature. The Krein sign $\eta$ of an eigenvector $\vec{v}$ of a symplectic matrix $M$ is defined to be

$$\eta = \text{sgn}(\langle \vec{v} J \vec{v} \rangle),$$

(13)

while the signature $(p, q)$ of an $r$–dimensional eigenspace is defined as follows: $p$ (resp. $q$) is the number of linearly independent eigenvectors with positive (resp. negative) Krein sign.

The fundamental stability result for symplectic matrices says that a symplectic matrix is structurally stable (i.e. the number of eigenvalues on the unit circle does not change under perturbation \footnote{Here perturbations are always assumed to preserve the symplectic nature of the matrix.}) if all eigenspaces corresponding to eigenvalues on the unit circle have a definite Krein signature. Further this result is tight: if an eigenspace is of indefinite Krein signature then a generic perturbation will cause roots to move off of the unit circle. The first result is known as Krein’s theorem, while the second is known as the Krein-Gelfand-Lidskii strong stability theorem. For precise statements, as well as proof, see Yakubovich and Starzhinskii\cite{[18]}.
2.3 Hill’s Equation

In this section we state some properties of the Hill’s equation

\[ H\psi = -\psi_{xx} + Q(x)\psi = \mu\psi \]  \hspace{1cm} (14)

that will be useful in the sequel. We will also define a quantity that will be useful in the analysis to follow.

In the remainder of this section \( m \) is the \((2 \times 2)\) monodromy matrix associated to the Hill’s equation (14):

\[ m = \begin{pmatrix}
\psi_1(1, \mu) & \psi_2(1, \mu) \\
\psi_1'(1, \mu) & \psi_2'(1, \mu)
\end{pmatrix}, \]  \hspace{1cm} (15)

with \( k(\mu) = \text{Tr}(m) \), and \( j \) is the standard skew-symmetric form

\[ j = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}. \]  \hspace{1cm} (16)

We begin with some well-known facts. The spectrum of the above eigenvalue problem consists of the union of the set of intervals \( \text{spec}(H) = [\mu_0, \mu'_1] \cup [\mu'_2, \mu_1] \cup [\mu_2, \mu'_3] \ldots \), where the \( \mu_i \) are the roots of \( k(\mu) = 2 \) and the \( \mu'_i \) are the roots of \( k(\mu) = -2 \). The points \( \{\mu_i\} \) are the periodic eigenvalues, while the points \( \{\mu'_i\} \) are the anti-periodic eigenvalues. Together these comprise the band edges. It is always true that \( \mu_{2i} < \mu_{2i+1}' \) and \( \mu'_{2i} < \mu_{2i-1} \), so the bands are always nontrivial. It can happen that \( \mu'_{2i-1} = \mu'_{2i} \) or \( \mu_{2i-1} = \mu_{2i} \) corresponding to a closed gap. This is commonly referred to as a double point. In the interior of each band there are two quasi-periodic solutions, while at the band edge there is one periodic or antiperiodic solution and one solution which grows linearly unless the band edge is a double point, in which case there are two periodic or antiperiodic solutions. The derivative of the Floquet discriminant is non-zero in
the bands, has exactly one zero in each gap, and is nonzero on the band edges
unless the band edge is a double point, in which case the derivative of Floquet
discriminant has a simple zero. Proofs of these facts can be found in Magnus
and Winkler\cite{11}.

In the first lemma we introduce a quantity which will play an important
role in the perturbation analysis. The sign of this quantity will be important in
determining the modulational stability of the standing wave.

**Lemma 2** Define the quantity $\sigma(\mu)$ as follows:

$$
\sigma(\mu) = \text{Tr}(jm(\mu))
$$

Then the following hold

- $\sigma(\mu)$ is non-zero on the interior of each band.

- $\sigma(\mu)$ vanishes at a band edge iff the band edge is a double point.

- $\text{sgn}(\sigma(\mu)) = -\text{sgn}(k'(\mu)) = -\text{sgn}(\text{Tr}(\frac{\partial m}{\partial \mu}))$ if $\mu \in \text{spec}(H)$.

Proof: On the interior of a band it holds

$$
k^2(\mu) - 4 < 0.
$$

If we set $\eta_1 = \psi_1(1,\mu)$, $\eta_2 = \psi_2(1,\mu)$, $\eta'_1 = \psi'_1(1,\mu)$, and $\eta'_2 = \psi'_2(1,\mu)$, then

$$
k^2(\mu) - 4 = (\eta_1 - \eta'_2)^2 + 4\eta'_1\eta_2,
$$

which implies that $\eta'_1\eta_2 \leq 0$. By its definition, $\sigma(\mu) = -\eta'_1 + \eta_2 = -\text{sgn}(\eta'_1)(|\eta'_1| + |\eta_2|)$. But (18) implies that $\eta'_1 \neq 0$ and $\eta_2 \neq 0$. Thus it follows that $\sigma(\mu)$ is nonzero in the interior of each band.

Let a band edge be a double point. It follows that $k^2(\mu) - 4 = 0$ and moreover

$$
k'(\mu) = \text{sgn}(\eta'_1)(|\eta'_1| \int \psi_2^2 + |\eta_2| \int \psi_1^2 \pm 2 \sqrt{|\eta'_1| \sqrt{|\eta_2|} \int \psi_1\psi_2}),
$$

(19)
where $\psi_1, \psi_2$ are the two linearly independent solutions of

$$\psi_{xx} + (\mu - Q(x))\psi = 0,$$

(20)

that satisfy the initial conditions $\psi_1(0, \mu) = 1$, $\psi'_1(0, \mu) = 0$, $\psi_2(0, \mu) = 0$, and $\psi'_2(0, \mu) = 1$. We used again (18) and, in particular, the inequality $\eta'_1 \eta_2 \leq 0$. By Cauchy-Schwartz it follows that $\text{sgn}(\eta'_1) = 0$, thus $\eta'_1 = 0$. But then, also $\eta_1 - \eta'_2 = 0$, which in turn forces $\eta_2 = 0$.

The other implication is proved as follows: $\sigma(\mu) = 0$ implies $0 = k^2(\mu) - 4 = (\eta_1 - \eta'_2)^2 + 4 \eta_2^2$, so $\eta_2 = \eta'_1 = 0$ and $\eta_1 - \eta'_2 = 0$. But then clearly $k'(\mu) = 0$.

Finally, again in [11] it was shown that if $\eta'_1 \neq 0$ then $k'(\mu)$ and $\eta'_1$ have the same sign. The result follows, since from the first part of this lemma, $\sigma(\mu)$ and $\eta'_1$ have opposite signs.

This quantity is related to the Krein sign, and actually agrees with the Krein sign of the eigenvalue of the monodromy matrix in the upper half plane. However the main utility of this quantity is that it allows one to compute the sign of the off-diagonal piece of the Jordan normal form at a band edge, as in the lemma below.

**Lemma 3** At a band edge the monodromy matrix $m$ has the following Jordan normal form:

$$m = o^t \begin{pmatrix} \pm 1 & \sigma(\mu) \\ 0 & \pm 1 \end{pmatrix} o$$

(21)

where $o$ is a proper orthogonal matrix: $oo^t = I$, $\det(o) = +1$

**Proof:** It is known that at a band edge that is not a double point $\pm 1$ is an eigenvalue of algebraic multiplicity two, and geometric multiplicity 1. The
Jordan normal form implies that

\[
\mathbf{m} = \mathbf{o}^t \begin{pmatrix} 
\pm 1 & K \\
0 & \pm 1 
\end{pmatrix} \mathbf{o} = \mathbf{o}^t \tilde{\mathbf{m}} \mathbf{o} \tag{22}
\]

where \( \mathbf{o} \) is orthogonal and can be chosen to have determinant \(+1\). It remains to be checked that \( K = \sigma(\mu) \). One easily observes that \( \mathbf{j} \) is a rotation and thus commutes with \( \mathbf{o} \). This implies that \( \sigma = \text{Tr}(\mathbf{j}\mathbf{m}) = \text{Tr}(\mathbf{O}\mathbf{j}\mathbf{o}^t \tilde{\mathbf{m}}) = K \). If \( \mathbf{o} \) is chosen to be on the other connected component of the orthogonal group \( \mathbf{j} \) and \( \mathbf{o} \) anti-commute, and the sign of the off-diagonal element is reversed.

The value of \( \sigma \) at a band edge, in particular the sign, is important in determining the stability of small amplitude standing wave solutions. Note that the Krein signature of an eigenvalue of the monodromy matrix in the upper half-plane is equal to the sign of \( \sigma \), and thus has the opposite sign from the derivative of the Floquet discriminant.

3 Modulational Instability of Standing Waves

3.1 General Results

We assume that there exists a \( C^2 \) family of standing wave solutions \( \psi = e^{-i\omega(x)\epsilon} \phi_{\text{stand}}(x, \epsilon) \) to the NLS equation with a periodic potential

\[
i\psi_t = -\frac{1}{2}\psi_{xx} + \epsilon|\psi|^2\psi + V(x)\psi \quad V(x + 1) = V(x),
\]

which is either periodic \( \phi(x + 1) = \phi(x) \) or anti-periodic \( \phi(x + 1) = -\phi(x) \). The eigenvalue problem governing the linearized stability is given by

\[
\mathbf{L}_{+p} = -\mu q,
\]

\[
\mathbf{L}_{-q} = \mu p
\]
where the operators $L_{\pm}$ are given by

$$
L_- = -\frac{1}{2} \partial_{xx} + V(x) + \epsilon |\phi_{stand}|^2(x, \epsilon) - \omega(\epsilon)
$$

$$
L_+ = -\frac{1}{2} \partial_{xx} + V(x) + 3\epsilon |\phi_{stand}|^2(x, \epsilon) - \omega(\epsilon)
$$

The above eigenvalue problem has a Hamiltonian formulation for real $\mu$. The Lyapunov-Poincare theorem of the previous section implies that the characteristic polynomial of the monodromy matrix is palindromic, $P(\lambda) = \lambda^4 P(\lambda^{-1})$, for $\mu$ on the real axis. Also note that for arbitrary values of $\epsilon$ the spectrum is symmetric about the real and imaginary axes, since the eigenvalue problem is invariant under the transformations $\mu \to -\mu$, $p \to -p$, $q \to q$ and $\mu \to \bar{\mu}$, $p \to \bar{p}$, $q \to \bar{q}$.

We begin by noting a few simple properties of this eigenvalue problem, and the associated monodromy matrix:

**Proposition 1** The Floquet stability problem for an NLS standing wave has the following properties:

- $\mu = 0$ is a periodic (anti-periodic) eigenvalue.
- $M(\mu)$ is an entire function of $\mu$, of order $\frac{1}{2}$.
- The monodromy matrix $M$ is symplectic for all $\mu \in \mathbb{C}$.
- The Floquet discriminants $k_{\pm}(\mu)$ are analytic functions of $\mu$ away from the branch points where $2 \text{Tr}(M^2) - \text{Tr}(M)^2 = 8$.
- At $\mu = 0$ the characteristic polynomial of the monodromy matrix has the following special form:

$$
P(\lambda)|_{\mu=0} = 1 - \text{Tr}(M)\lambda + (2 \text{Tr}(M) - 2)\lambda^2 - \text{Tr}(M)\lambda^3 + \lambda^4
$$

(24)
Proof:

That $\mu = 0$ is always an eigenvalue follows from Noether’s theorem and the phase invariance of NLS. The corresponding eigenvector is $p = \phi_{\text{stand}}(x), q = 0$. The fact that $M$ is entire follows from standard arguments - see for example the text of Sibuya[17]. Note that $2 \text{Tr}(M^2) - \text{Tr}(M)^2 = 8$ is also an entire function of fractional order, and thus must have a countable number of zeros. The fact that $M$ is symplectic for real $\mu$ follows from the results cited in the previous section. To see that this in fact holds for all $\mu \in \mathbb{C}$ simply note that $J - M^t J M$ is an entire function that is zero on the real axis, and thus must be identically zero. A simple division shows that the polynomial $\lambda - 1$ divides the polynomial $\lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + 1$ if and only if $b + 2a = 2 = 0$, which proves the last part. Note that this same calculation shows that if $\lambda = 1$ is a root it is necessarily of multiplicity 2 or 4.

Lemma 4 If 0 is not a periodic eigenvalue of the $L_+$ operator the Floquet discriminants $K_\pm(\mu)$ are analytic in a neighborhood of $\mu = 0$.

Proof: For $\mu = 0$ the stability problem decouples, and the monodromy matrix takes the block diagonal form

$$M = \begin{pmatrix} m_+ & 0 \\ 0 & m_- \end{pmatrix},$$

(25)

where $m_\pm$ are the monodromy matrices associated with $L_\pm$. It follows from the previous lemma that $\lambda = 1$ is an eigenvalue of $m_-$ with multiplicity 2. A short calculation using the fact that $2 \times 2$ matrices satisfy $\text{Tr}(m^2) = \text{Tr}(m)^2 - 2 \det(m)$ shows that $2 \text{Tr}(M^2) - \text{Tr}(M)^2 + 8 = (\text{Tr}(m_-) - 2)^2$. If 0 is not a periodic eigenvalue of $L_+$ then $2 \text{Tr}(M^2) - \text{Tr}(M)^2 + 8 \neq 0$ and $K_\pm$ are analytic in a neighborhood of $\mu = 0$. 

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Theorem 1 For general $\epsilon$ a sufficient condition for the existence of a modulational instability spectrum is that $\mu = 0$ is in the interior of a band of the $L_+$ operator.

Proof: From the previous lemma if $\mu = 0$ is in the interior of a band of the $L_+$ then the Floquet discriminants $K_{\pm}(\mu)$ are analytic in a neighborhood of the origin. From the fact that the coefficients of the characteristic polynomial are invariant under the transformation $\mu \rightarrow -\mu$ it follows that $K_{\pm}(\mu)$ are even, and thus are real on an segment of the imaginary axis containing the origin. Since $K_-(0) \in (-2, 2)$ it follows that $K_-(\mu)$ is real and $\in (-2, 2)$ on some segment of the imaginary axis containing the origin, and thus there is a modulational instability.

Remark: The same argument guarantees the existence of a modulational instability if either $K(0) = +2$ and $K''(0) \geq 0$ or $K(0) = -2$ and $K''(0) \leq 0$. The calculation of the sign of the second derivative appears to require a difficult second order perturbation calculation.

3.2 Perturbative Results for weak nonlinearity

In this section we study the Floquet spectrum for small amplitude standing waves of the nonlinear Schrodinger equation with a periodic potential, with a particular emphasis on the behavior near $\mu = 0$. We shall see that the $U(1)$ phase invariance of the NLS forces a four-fold degeneracy of the eigenvalues of the unperturbed monodromy matrix at $\mu = 0$. Under perturbation this degeneracy can lead to the birth of a spine of continuous spectrum lying along the imaginary axis. Whether or not such a spine is born is determined by the relative sign of the nonlinearity and the quantity $\sigma$ defined in the previous
section, and the length of the spine is determined by the magnitude of $\sigma$.

When $\epsilon = 0$ the operators $L_{\pm}$ are equal and are given by

$$L_{-}(0) = L_{+}(0) = -\frac{1}{2} \partial_{xx} + V(x) - \omega(0)$$

(26)

and the resulting eigenvalue problem is self-adjoint. It is easy to see that in this case the spectrum of the stability problem is given by $\text{spec}(L_{-}) \cup \text{spec}(-L_{-})$. In this case it is also straightforward to calculate that the monodromy matrix $M$ of the full stability problem takes the block diagonal form

$$M = \begin{pmatrix} m(\mu) & 0 \\ 0 & m(-\mu) \end{pmatrix}$$

(27)

where $m(\pm \mu)$ is the monodromy matrix for the problem

$$L_{-}(0) \psi = \pm \mu \psi.$$  

(28)

From this block diagonal form it follows that for $\epsilon = 0$ the invariants of the full monodromy matrix can be expressed in terms of the Floquet discriminants $k(\pm \mu)$ of the second order problem via

$$\text{Tr}(M(\mu)) = \text{Tr}(m(\mu)) + \text{Tr}(m(-\mu)) = k(\mu) + k(-\mu)$$

$$\text{Tr}(M^2(\mu)) = \text{Tr}(m^2(\mu)) + \text{Tr}(m^2(-\mu)) = k^2(\mu) + k^2(-\mu) - 4.$$  

Here we have used the fact that $2 \times 2$ matrices satisfy the identity $\text{Tr}(m^2) = \text{Tr}(m)^2 - 2 \det(m)$. From this it follows that the Floquet discriminants $K_{\pm}(\mu)$ for the full problem can be written in terms of the Floquet discriminant of $L_{-}$ via

$$K_{\pm}(\mu) = \frac{k(\mu) + k(-\mu) \pm \sqrt{(k(\mu) - k(-\mu))^2}}{2}.$$  

(29)

Obviously this could be simplified to $K_{\pm}(\mu) = k(\pm \mu)$ however we do not do this, since it obscures the degeneracy at the points where $k(\mu) = k(-\mu)$.  

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The U(1) symmetry of the NLS equation implies that $\mu = 0$ is a band-edge eigenvalue of $L_{-}$. Thus for $\epsilon = 0, \mu = 0$ the monodromy matrix has has a single eigenvalue $\pm 1$ of multiplicity four. This corresponds to the last entry in the table in figure 1. This seemingly non-generic bifurcation is forced by the phase-invariance symmetry along with the symplectic nature of the monodromy matrix, and can give rise to the modulational instability at non-zero wave amplitudes.

In the next lemma we present a normal form calculation for the Floquet discriminants in a neighborhood of $\epsilon = 0, \mu = 0$. The calculation is particularly simple due to some additional symmetries, which dramatically reduce the number of coefficients which need to be computed. We present the calculation for periodic band edges: the calculation for the anti-periodic bands edges is identical except for some sign changes. In this latter case we merely state the final result.

**Lemma 5** The Floquet discriminants of the monodromy matrix have the following normal form in a neighborhood of $\epsilon = 0, \mu = 0$:

$$K_{\pm}(\mu) = 2 + \frac{k'(0)}{2} \mu^2 + 2\epsilon \sigma <\phi_1^4> + E_1 \pm \sqrt{(k'(0)\mu)^2 + (2\sigma \epsilon <\phi_1^4>)^2} + E_2$$

$$E_1 = o(\epsilon, \mu^2)$$

$$E_2 = o(\epsilon^2, \mu^2)$$

where $k(\mu)$ is the Floquet discriminant for the $L_{-}(0)$ operator.

**Proof:** This perturbation calculation is somewhat tedious but can be made simpler by the use of some of the previously derived identities. From the last part of Lemma 4 we can express the $\epsilon$ derivatives of the coefficients of the characteristic polynomial at the origin in terms of $\text{Tr}(M|\mu = 0)$. To compute
the $\mu$ derivatives of the coefficients of the characteristic polynomial at the origin we use the fact that $a(\mu, 0) = -(k(\mu) + k(-\mu)), b(\mu, 0) = k(\mu)k(-\mu) + 2$. The mixed partial $\frac{\partial^2}{\partial \mu \partial \epsilon}$ vanishes at the origin since the $a, b$ coefficients are even functions of $\mu$ for all $\epsilon$. Note that, because of the square root branch point, it is necessary to carry the expansion out to second order to get what amounts to a first order result - the local normal form is a cone.

We present the calculation of $\frac{\partial \text{Tr}(M|_{\mu=0})}{\partial \epsilon}$ first. The problem $L_- \psi = 0$ has a periodic solution that we denote by $\psi_1 (= \psi_{\text{stand}})$, and a linearly growing solution that we denote by $\psi_2$. We choose $\psi_{1,2}$ to form an orthogonal right-handed coordinate system:

$$
\psi_1(0) = \cos(\theta) \quad \psi_1'(0) = \sin(\theta)
$$
$$
\psi_2(0) = -\sin(\theta) \quad \psi_2'(0) = \cos(\theta)
$$

Since $\mu = 0$ is a band edge for the $L_-$ operator we have the following expressions for the period map:

$$
\psi_2(1) = \psi_2(0) + \sigma \psi_1(0) = \sigma \cos(\theta) - \sin(\theta)
$$
$$
\psi_2'(1) = \sigma \sin(\theta) + \cos(\theta).
$$

Note that $\sigma$ is non-zero as long as the band edge is not a double point. It is convenient to define a second set of solutions $\phi_{1,2}$ which satisfy $L_- \phi = 0$ along with the initial conditions $\phi_1(0) = 1, \phi_1'(0) = 0$ and $\phi_2(0) = 0, \phi_2'(0) = 1$. These are obviously related to the $\psi_{1,2}$ by $\phi_1 = \cos(\theta)\psi_1 - \sin(\theta)\psi_2$ and $\phi_2 = \sin(\theta)\psi_1 + \cos(\theta)\psi_2$. These functions form a natural basis in which to do perturbation theory on the $L_+$ operator. As is usual in Floquet theory we must construct a fundamental set of solutions $\tilde{\phi}_{1,2}$ to

$$
L_+ \tilde{\phi}_{1,2} = L_- \tilde{\phi}_{1,2} + 2\epsilon|\psi|^2 \tilde{\phi}_{1,2} = 0
$$

(30)
that satisfy the boundary conditions
\[
\begin{align*}
\tilde{\phi}_1(0) &= 1 & \tilde{\phi}'_1(0) &= 0 \\
\tilde{\phi}_2(0) &= 0 & \tilde{\phi}'_2(0) &= 1
\end{align*}
\]

A straightforward perturbation calculation gives the following expressions for
the fundamental set of solutions to \(L + \tilde{\phi} = 0\):
\[
\begin{align*}
\tilde{\phi}_1(x) &= \phi_1 - 4\epsilon \left( \psi_1 \int \psi_1^2 \psi_2 \phi_1 - \psi_2 \int \psi_1^3 \phi_1 \right) \\
\tilde{\phi}_2(x) &= \phi_2 - 4\epsilon \left( \psi_1 \int \psi_1^2 \psi_2 \phi_2 - \psi_2 \int \psi_1^3 \phi_2 \right)
\end{align*}
\]

Using the fact that \(\psi_2(1) = \psi_2(0) + \sigma \psi_1(0)\) gives the following expression for
the trace of the monodromy matrix to the leading order in \(\epsilon\):
\[
\begin{align*}
\tilde{\phi}_1(1) &= \phi_1(1) - 4\epsilon \left( \cos^2(\theta)(<\psi_1^3 \psi_2> - \sigma <\psi_1^4>) \\
&\quad + \sin \theta \cos(\theta)(\sigma <\psi_1^3 \psi_2> + <\psi_1^4> - <\psi_1^4 \psi_2^2>) \\
&\quad - \sin^2(\theta) <\psi_1^3 \psi_2>
\right)
\]
\[
\tilde{\phi}_2(1) &= \phi_2(1) - 4\epsilon (-\cos^2(\theta) <\psi_1^3 \psi_2>) \\
&\quad + \sin(\theta) \cos(\theta) <\psi_1^3 \psi_2^2> - \sigma <\psi_1^4 > \psi_2 > - <\psi_1^4>) \\
&\quad + \sin^2(\theta) (<\psi_1^3 \psi_2 > - \sigma <\psi_1^4>)
\]

where \(<f> = \int_0^1 f(y) dy\). From this it follows that the monodromy matrix in
leading order of \(\epsilon\) is given by
\[
\text{Tr}(M|\mu = 0) = \tilde{\phi}_1(1) + \tilde{\phi}_2(1) + \phi_1(1) + \phi_2(1) = 2(\phi_1(1) + \phi_2(1)) + 4\epsilon \sigma <\psi_1^4> + O(\epsilon^2)
\]

It is easy to calculate that \(a(\mu, 0) = -4 - k''(0)\mu^2 + o(\mu^2)\), and that \(b(\mu, 0) = 6 + (2k''(0) - (k'(0))^2)\mu^2 + o(\mu^2)\). This, together with the vanishing of the mixed
partial and the explicit formula gives the above result.
**Remark** From the proof of the preceeding lemma, and the use of the time-invariant Hamiltonian energy functional

\[
\mathcal{H} = \int \left( \frac{1}{2} |\psi_x|^2 \pm \frac{\epsilon}{2} |\psi|^4 - V(x)|\psi|^2 \right) dx,
\]  

(32)

it follows that

\[
\left. \frac{\partial \text{Tr} M}{\partial \epsilon} \right|_{(0,0)} = 8\sigma \left. \frac{\partial \mathcal{H}}{\partial \epsilon} \right|_{\epsilon=0}.
\]  

(33)

Thus, the instability criterion can be expressed in terms of the signs of \(\sigma\) and of the derivative of the energy functional. This is a feature that appears regularly in the study of stability of nonlinear waves: that the sign of the derivative of some conserved quantity is a proxy for the index of some linearized operator. For some results of a similar flavor in a variety of different contexts (both conservative and dissipative) see [13, 6, 7, 12, 8].

**Theorem 2** For \(\epsilon\) small and positive (focusing NLS) and solutions bifurcating from a lower band edge that is not a double point, or for \(\epsilon\) small and negative and solutions bifurcating from an upper band edge that is not a double point there exists a band of spectrum along the imaginary axis.

**Proof:** The proof follows from the preceeding lemma and the same analyticity argument as the first theorem. From the explicit factorization of the characteristic polynomial it follows that, for \(\epsilon\) fixed, the Floquet discriminants \(K_{\pm}(\mu)\) are analytic functions in the cut plane with branch points at the points where

\[
2 \text{Tr}(M^2) - \text{Tr}(M)^2 + 8 = 0
\]  

(34)

From the preceeding Lemma it follows that at \(\mu = 0\) this quantity reduces to

\[
(2 \text{Tr}(M^2) - \text{Tr}(M)^2 + 8) \big|_{\mu=0} = (\text{Tr}(M)|_{\mu=0} - 4)^2 = 16\sigma^2 \epsilon^2 <\psi_1^4>^2 + o(\epsilon^2)
\]  

(35)
and thus this quantity is nonzero for small but nonzero ϵ: the fact that the band
edge is not a double point guarantees that σ ≠ 0. It follows that the Floquet
discriminants \( K_\pm \) are analytic in a neighborhood of \( \mu = 0 \) and satisfy

\[
K_+(0) = 2 + 4\epsilon\sigma < \psi_1^4 > + O(\epsilon^2) \\
K_-(0) = 2.
\]

As in the first theorem the fact that \( K_+(\mu) \) is an even function implies that
\( \mu = 0 \) is a critical point, and \( K_+ \) is real on a segment of the imaginary axis
containing the origin. Recall that for odd numbered bands the periodic eigen-
value is the lower band edge, and \( \sigma \) is positive, and for even numbered bands
the periodic eigenvalue is the upper band edge, and \( \sigma \) is negative. For \( \epsilon < 0 \)
(defocusing) and odd numbered bands, or for \( \epsilon > 0 \) and even numbered bands
the perturbation result implies that \( K_+(0) \in (-2, 2) \), so there exists an interval
on the imaginary axis on which \( K_+(\mu) \) is real and \( \in (-2, 2) \). The analogous
calculation for solutions bifurcating from the anti-periodic eigenvalues shows
that for \( \epsilon < 0 \) (defocusing) and even numbered bands, for \( \epsilon > 0 \) and odd num-
ered bands \( K_+(0) \in (-2, 2) \), and there exists a spine of spectrum along the
imaginary axis. Thus, for \( \epsilon \) small and defocusing nonlinearity standing wave sol-
lutions which bifurcate from the lower band edge are modulationally unstable,
while for focusing nonlinearity standing wave solutions which bifurcate from the
upper band edge are modulationally unstable. Said more simply the solutions
which bifurcate from the lower band edges are modulationally unstable in the
defocusing case, and the solutions which bifurcate from the upper band edges
are modulationally unstable in the focusing case.

Note that if \( K_\pm''(0) = 0 \) there exist additional arcs of spectrum emerging
from the origin into the complex plane. For \( \epsilon \) sufficiently small the local normal
form calculation guarantees that the second derivative is non-vanishing, and 
this does not occur. It is a possibility for larger $\epsilon$, however.

4 Sideband Instabilities

In this section we consider the possibility of sideband type instabilities. This 
case is somewhat more difficult than the case of modulational instabilities, since 
there are fewer symmetries, and our results are somewhat less detailed. In 
addition to the analyticity arguments of the previous section a major tool will 
be the Krein signature.

First we note the following lemma

Lemma 6 The spectrum of the stability problem consists of a union of contin-
uous curves in the complex plane. The possible endpoints of the curves are band 
edges $K_{\pm}(\mu) = \pm 2$, branch points $a^2(\mu) - 4b(\mu) + 8 = 0$ (or $K_{\pm}(\mu) = K_{\mp}(\mu)$) 
or critical points $K_{\pm}'(\mu) = 0$.

Proof: As before $K_{\pm}(\mu)$ are analytic away from the (isolated) points where 
$a^2(\mu) - 4b(\mu) + 8 = 0$. Suppose $\mu$ is a point in the spectrum that is not a 
branch point, band edge or critical point. We assume for the sake of argument 
that $K_{+}(\mu) \in (-2, 2)$. Since $K_{+}$ is analytic in a neighborhood of $\mu$ and $\mu$ is 
not a critical point of $K_{+}$ an appeal to the implicit function theorem shows the 
existence of a unique curve through $\mu$ along which $\Re(K_{+}) = 0$. Since $\mu$ is not 
a band edge continuity implies that $\Re(K_{+}) \in (-2, 2)$ in some neighborhood of 
$\mu$.

We now consider the possibility of sideband instabilities arising from points 
where $K_{+}(\mu) = K_{-}(\mu)$. First note that in the small amplitude limit, when $\epsilon = 0$, 

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we have $a^2 - 4b + 8 = (k(\mu) - k(-\mu))^2$ so the branch points are degenerate. Under perturbation this degeneracy is expected to break, and generically it can do so in one of two ways, which are generally referred to as the avoided collision and the open gap. These possibilities are illustrated in figure 5. It is rather tedious to compute the normal form for this case, since one lacks the even symmetry, but consideration of the Krein-Gelfand-Lidskii theorem can reduce the possibilities.

In the case of a collision of two eigenvalues with the same Krein sign the open gap is forbidden, since that corresponds to eigenvalues leaving the unit circle. In the case of two eigenvalues with the opposite Krein sign one expects that under perturbation the eigenvalues should leave the unit circle. We formalize this observation as a theorem.

**Theorem 3** In the neighborhood of an accidental degeneracy $(\mu = \mu^*, \epsilon = 0)$ the Floquet discriminants have the following normal form

$$K_{\pm}(\mu) = k(\mu^*) + (k'(\mu^*) - k'(\mu^*))(\mu - \mu^*) + \alpha \epsilon + E_1 \pm$$

$$\sqrt{(k'(\mu^*) + k'(\mu^*))^2 (\mu - \mu^*)^2 + \gamma \epsilon^2 + E_2}$$

$$E_1 = O(\epsilon^2, (\mu - \mu^*)^2, \epsilon(\mu - \mu^*))$$

$$E_2 = O(\epsilon^3, (\mu - \mu^*)^3, \epsilon(\mu - \mu^*), \epsilon^2(\mu - \mu^*))$$

Further, if $\text{sgn}(k'(\mu^*)) = \text{sgn}(k'(\mu^*))$ the quadratic form $(k'(\mu^*) + k'(\mu^*))^2 (\mu - \mu^*)^2 + \beta \epsilon(\mu - \mu^*) + \gamma \epsilon^2$ is nonnegative.

A straightforward, though slightly tedious, perturbation argument shows that $\frac{\partial}{\partial \epsilon}(a^2 - 4b + 8)|_{\epsilon=0} = 0$, so that this quantity is locally quadratic, as well as having a vanishing mixed partial. The Krein signs of the unperturbed eigenvalues are the same as the signs of $k'(\mu)$ and $k'(-\mu)$ respectively. In the case where these signs are the same the Krein-Gelfand-Lidskii theorem guarantees
that, under small perturbations, the eigenvalues of $M$ remain on the unit circle.

It is clear that a necessary condition for the eigenvalues to remain on the unit circle is non-negativity of the quadratic form, thus the sign condition guarantees non-negativity of this form. Note that this theorem does not preclude the following possibilities: in the case of like Krein sign collisions it is possible that the quadratic form has a zero eigenvalue, or that the intersection persists under perturbation. In the case of collisions of opposite Krein sign a naive application of the Krein-Gelfand-Lidskii theorem gives no information. We conjecture that, in this case, the signature of the quadratic form is equal to the signature of the subspace. This would imply that that intersections of opposite Krein signs always open to a gap, while intersections of the same Krein sign always open to an avoided collision. Very preliminary numerical evidence has supports this conjecture, but unfortunately it appears that the only way to verify this analytically is via a very tedious perturbation argument.

An obvious corollary of this is the following:

**Corollary 1** If $\text{sgn}(k'(\mu^*)) = \text{sgn}(k'(-\mu^*))$ and $\gamma > 0$, or if $\text{sgn}(k'(\mu^*)) = -\text{sgn}(k'(-\mu^*))$ and $\gamma < 0$ then there exists a band of spectrum off of the real axis.

From the local normal form it is an easy calculation that, in either case, for sufficiently small $\epsilon$ the Floquet discriminants $K_{\pm}$ must have a critical point in the neighborhood of $\mu^*$, and that $a^2 - 4b + 8 \neq 0$ at this critical point. From the analyticity arguments of the previous section this guarantees that $K_{\pm}$ are real and $\in (-2, 2)$ in some neighborhood of the critical point. Again one expects that in the case of a like Krein sign collision one should generically have $\gamma > 0$, leading to the avoided collision and a loop of spectrum opening into
the complex plane, but there seems to way to show this in any particular case without actually calculating $\gamma$.

5 Explicit Examples and Numerics

In this section we present some examples. We will primarily be working with known exact elliptic function solutions. We consider the nonlinear Schrödinger equation

$$i\psi_t = -\frac{1}{2}\psi_{xx} + V_0 \text{sn}^2(x, k)\psi \pm |\psi|^2\psi$$  \hspace{1cm} (36)$$

This equation has a one-parameter family of exact solutions given by

$$\psi(x, t) = r(x)e^{-\omega t + i\theta(x)}$$  \hspace{1cm} (37)$$

$$r^2(x) = A\text{sn}^2(x, k) + B$$  \hspace{1cm} (38)$$

$$\theta(x) = c \int \frac{dx'}{r^2(x')}$$  \hspace{1cm} (39)$$

$$A = -(V_0 + k^2)$$  \hspace{1cm} (40)$$

$$c^2 = B \left(\frac{B}{V_0 + k^2 - Bk^2}\right)(V_0 + k^2 - Bk^2)$$  \hspace{1cm} (41)$$

Where $B \in (-\infty, -k^2) \cup \left(\frac{V_0 + k^2}{k^2}, V_0 + k^2\right)$ for the focusing sign and $B \in \left(-(V_0 + k^2), -\frac{V_0 + k^2}{k^2}\right) \cup (-k^2, \infty)$ for the defocusing sign. These solutions represent nonlinear stationary states which bifurcate from the linear Bloch states.

We are primarily interested in the solutions which bifurcate from the band edges. These correspond to the boundaries of the above regions of validity. In the focusing case we have

$$r_0(x) = \sqrt{\frac{V_0 + k^2}{k^2}} \text{dn}(x, k) \quad \omega = -1 - \frac{V_0}{k^2} + \frac{k^2}{2} \quad V_0 + k^2 > 0$$  \hspace{1cm} (42)$$

$$r_1(x) = \sqrt{\frac{V_0 + k^2}{k^2}} \text{cn}(x, k) \quad \omega = \frac{1}{2} - V_0 - k^2 \quad v_0 + k^2 > 0$$  \hspace{1cm} (43)$$

$$r_2(x) = \sqrt{-(V_0 + k^2)} \text{sn}(x, k) \quad \omega = \frac{1 + k^2}{2} \quad V_0 + k^2 < 0,$$  \hspace{1cm} (44)$$
while in the defocusing case we have the analogous solutions

\[
\begin{align*}
    r_0(x) &= \frac{\sqrt{V_0 + k^2}}{k^2} \text{dn}(x, k) \quad \omega = -1 - \frac{V_0}{k^2} + \frac{k^2}{2} \quad V_0 + k^2 < 0 \quad (45) \\
    r_1(x) &= \sqrt{V_0 + k^2} \text{cn}(x, k) \quad \omega = \frac{1}{2} - V_0 - k^2 \quad v_0 + k^2 < 0 \quad (46) \\
    r_2(x) &= \sqrt{-(V_0 + k^2)} \text{sn}(x, k) \quad \omega = \frac{1 + k^2}{2} \quad V_0 + k^2 > 0. \quad (47)
\end{align*}
\]

The solutions all bifurcate from the the linear Bloch states at \( V_0 + k^2 = 0 \), with the \( \text{dn} \), \( \text{cn} \) solutions existing on one side of the bifurcation and the \( \text{sn} \) solution on the other side. It is easy to check that the spectrum of the \( L_- \) operator in each of these cases is the 1-gap Lamé operator plus some constant which differs in each case. The ground state of \( L_- \) is \( \text{dn}(x, k) \), while the next two antiperiodic eigenfunctions are given by \( \text{cn}(x, k), \text{sn}(x, k) \). In what follows the parameter \( \epsilon = V_0 + k^2 \). There is an obvious rescaling which transforms the above form of the NLS equation to the form considered earlier.

5.1 Focusing Case:

It follows from Theorem (2) that for \( V_0 + k^2 = \epsilon \) sufficiently small that the \( \text{dn} \) and \( \text{sn} \) solutions, as lower band edges, are modationally unstable. Moreover, from Theorem (1) we are guaranteed a modulational instability as long as the \( L_+ \) operator is in the interior of a band at \( \mu = 0 \). Thus in each case we have an open interval \( (0, \epsilon^*) \) in which we are guaranteed a modulational instability, where \( \epsilon^* \) is the smallest positive value of \( \epsilon \) such that \( \mu = 0 \) is a band edge of the \( L_+ \) operator.

We will focus on the \( \text{sn} \) solution, since the instability of the \( \text{dn} \) solution follows from more elementary arguments[]. We have found numerically that \( \mu = 0 \) is in the interior of a band for \( \epsilon \in (-1.33, 0) \cup (-3.0, -1.56) \cup (-6.7, -5.6) \ldots \), which shows instability for \( \epsilon \) in these intervals. The numerical evidence further shows that in this particular case the second derivative of the branch of the Floquet
discriminant passing through $-2$ is always negative, which implies that these solutions are always unstable. We do not currently have a proof of this. It is also unclear whether this feature is special to the elliptic function solutions, or if it holds in greater generality. It is interesting to note that the sn$(x, k)$ solution also has a side-band type instability which appears at arbitrarily small positive amplitude. This is illustrated in Figure 1. When $\epsilon = 0$ the Floquet discriminants of the unperturbed problem $k(\pm \mu)$ intersect near $\mu \approx 0.45$. This is a collision of opposite Krein sign, since the discriminants have the same sign. This intersection opens into a gap under perturbation.

The stability of the cn type solutions is extremely interesting. For small $\epsilon$ the perturbation result guarantees that $\mu = 0$ is in a gap for the $L_+$ operator, and numerically we find no modulational instability for small $\epsilon$. As before we expect that for some critical value of $\epsilon \mu = 0$ will enter a band of $L_+$ and an instability will develop. In this case the critical value of $\epsilon$ can actually be computed explicitly. When $V_0 = 0(\epsilon = k^2)$ we find that $L_+$ is the 2-gap Lamé operator, with $\mu = 0$ is a band edge. It is also straightforward to show using the Sturm oscillation theorem and monotonicity of the band edges in $\epsilon$ that this is the smallest value of $\epsilon$ for which $L_+$ has a band edge at $\mu = 0$. Thus

Figure 1: The Floquet Discriminants for the sn$(x, k)$ solution in the focusing case with $\epsilon = 0.1$
we are guaranteed a modulational instability for some interval of beginning at $V_0 = 0 (\epsilon = k^2)$. It is worth remarking that in this critical case (the integrable one) all of the information about the spectrum of the linearization, as well as a great deal more, can be obtained by the methods of algebro-geometric tools. See, in particular, Chapter 4 of the text by Belokolos, Bobenko, Enol’skii, Its and Matveev. The birth of a modulational instability at $V_0 = 0$ supports a intuition based on physical reasoning that was put forth in [3] which suggested that such solutions should go unstable at $V_0 = 0$.

Interestingly it appears that there exists a finite amplitude side-band type instability that sets in before the modulational instability. This is illustrated in figure 2 which shows the Floquet discriminants for the linearized operator of the focusing NLS about a cn type solution with modulus $k^2 = 1/2$ and a sequence of different values of $\epsilon$. Near $V_0 \approx -0.23$ the Floquet discriminants $K_{\pm}$ cross, causing a gap to open and a loop of spectrum to emerge into the complex plane. This instability disproves a guess made in [3], based on numerical experiments, that the cn type solutions should be stable for $V_0 < 0$. As $V_0$ increases the neighborhood about the origin in which the Floquet discriminants are real shrinks, until at $V_0 = 0$ the discriminants are only real at the origin. For positive $V_0$ the discriminants again become real in a neighborhood of the origin.

6 Conclusions

We have established a sufficient condition for the modulational instability of a periodic standing wave solution to the NLS equation, which can be easily checked. In the case of weak nonlinearity this reduces to a physically reasonable
Figure 2: The Floquet discriminants for the focusing cn($x,k$) solution for $V_0 = -1/2, -1/4, -0.228, -1$.

criterion based on the effective mass of a particle in the periodic potential. We have also made some preliminary progress into studying the birth of side-band type instabilities. In the case of like Krein-sign collisions the gap-opening bifurcation is forbidden. Very preliminary numerical experiments suggest that, in the case of opposite Krein-sign collisions the “avoided collision” bifurcation is forbidden, but we do not currently have a proof of this.

Acknowledgements: JCB is partially supported by nsf-dms 0203938 and an NSF FRG grant.

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Figure 3: A perturbed Floquet spectrum where the unperturbed problem has degenerate eigenvalues of opposite Krein signs. The graphs show the Floquet discriminants along the real $\mu$ axis for $\epsilon = 0, 0.025, 0.05, -0.05$. Note the opening of a small gap in the neighborhood of the intersection.
Figure 4: A perturbed Floquet spectrum where the unperturbed problem has degenerate eigenvalues of the same Krein sign for a particular value of $\mu$ along the real axis. The first figure shows the Floquet discriminants along the real $\mu$ axis (note the avoided collision) and the second the imaginary part of the Floquet discriminant in the complex plane. The imaginary part of the discriminant is negative inside the eye and positive outside.

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