Note on Toda brackets

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Abstract
We provide a general definition of Toda brackets in a pointed model category, show how they serve as obstructions to rectification, and explain their relation to the classical stable operations.

Keywords Higher homotopy operations · Toda brackets · Stable homotopy

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Introduction

Toda brackets, defined by Toda in [19], play an important role in homotopy theory both for their original purpose of calculating homotopy groups in [20], and because they serve as differentials in spectral sequences (see [1,3,11]). The notion has been generalized in several ways (see, e.g., [4,9,21]), not all of which agree.

In this note we provide a definition of higher Toda brackets in a general pointed model category \( \mathcal{C} \), show how these appear as the successive obstructions to strictifying certain diagrams (namely, chain complexes in the homotopy category \( \text{ho}\mathcal{C} \)) —see Theorem 1.23 below. In Sect. 2, we explain the connection with the traditional stable
description in terms of filtered complexes. In Sect. 3, we provide two examples of settings in which higher Toda brackets occur.

1 Higher Toda brackets

We provide a variant of the definition of higher Toda brackets sketched in [9, §7] which brings out clearly their connection to rectification of linear diagrams.

Definition 1.1 Let \( C \) be a cofibrantly generated left proper pointed model category. A Toda diagram of length \( n \) for \( C \) is a diagram in the homotopy category \( \text{ho} \ C \) of the form

\[
A_0 \overset{[f_0]}{\longrightarrow} A_1 \overset{[f_1]}{\longrightarrow} A_2 \rightarrow \cdots \rightarrow A_{n-1} \overset{[f_{n-1}]}{\longrightarrow} A_n
\]

(1.2)

with \( [f_k] \circ [f_{k-1}] = 0 \) for each \( 1 \leq k < n \).

A strictification of (1.2) is a diagram in \( C \) of the form

\[
A_0' \overset{f_0'}{\longrightarrow} A_1' \overset{f_1'}{\longrightarrow} A_2' \rightarrow \cdots \rightarrow A_{n-1}' \overset{f_{n-1}'}{\longrightarrow} A_n'
\]

(1.3)

with \( f_k' \circ f_{k-1}' = 0 \) (the strict zero map) for each \( 1 \leq k < n \), which is weakly equivalent to a lift of (1.2).

To obtain a homotopy meaningful description of such strictifications, we shall need:

Definition 1.4 Let \( I_n^2 \) denote the lattice of subsets of \( \{1, 2, \ldots, n\} \), which we shall think of as an \( n \)-dimensional cube with vertices labelled by \( J = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \) with each \( \varepsilon_i \in \{0, 1\} \), so \( J \) is the characteristic function of a subset of \( \{n\} \). The cube is partially ordered in the usual way, with a unique map \( J \rightarrow J' \) whenever \( \varepsilon_i \leq \varepsilon'_i \) for all \( 1 \leq i \leq n \). Write \( J_k \) for the vertex labelled by \( k \) ones followed by \( n-k \) zeros (\( k = 0, 1, \ldots, n \)).

Note that any strictification (1.3) extends uniquely to a diagram \( D : I_n^2 \rightarrow C \), in which \( D(J_k) = A'_k \), \( D \) assigns the map \( f_k' \) to the unique map \( J_k \rightarrow J_{k+1} \) in \( I_n^2 \), and \( D \) sends all other vertices (and thus all other maps) to 0.

The category \( C_{I_n^2} \) of all “cube-shaped” diagrams in \( C \) has a cofibrantly generated model category structure, in which the weak equivalences and fibrations are defined objectwise (see [12, Theorem 11.6.1]). Any diagram \( \hat{D} : I_n^2 \rightarrow C \) weakly equivalent to the diagram \( D \) just described will be called an enhanced strictification of (1.2). In particular, when \( \hat{D} \) is a cofibrant replacement for \( D \) in the model category mentioned above, we call it a cofibrant enhanced strictification.

Remark 1.5 The cofibrations in the model category \( C_{I_n^2} \) are not easy to identify explicitly. However, because the indexing category is directed in a strong sense, the cofibrant diagrams \( B : I_n^2 \rightarrow C \) may be described inductively by filtering \( I_n^2 \) by composition length: \( F_0 \subset F_1 \subset \cdots \subset F_n = I_n^2 \). We start with \( F_0 \) consisting only of the initial object \( J_0 \); then \( F_{k+1} \) is obtained inductively from \( F_k \) by adding all indecomposable maps in \( I_n^2 \) from objects in \( F_k \), together with their targets.
We then have a recursive definition of what it means for a diagram $B$ to be cofibrant, by requiring that $B|F_k$ be cofibrant for each $k$ (for $k = 0$, this just means ensuring $B(J_0)$ is cofibrant in $C$), and then ensuring that the natural map $\text{colim} F_k B \to B(J)$ is a cofibration for each $J \in F_{k+1} \setminus F_k$ (cf. [7, §2.14]). In particular, this ensures that all morphisms in $I^n_2$ are taken by $B$ to cofibrations in $C$.

**Definition 1.6** Now let $I^n_3$ denote the $n$-dimensional cube of side length 2, with vertices labelled as before by $J$ (the remainder is $R$).

**Example 1.10** For $J = (1221122202012012)$ we have marker $M(J) = (12211222)$. Its stage is $\sigma(J) = 3$ (the length of the sequence 222 at the end of $M(J)$), and the remainder is $R(J) = 02012012$, with $r(J) = 3$. Note that any cofibrant enhanced strictification $\hat{D} : I^n_2 \to C$ has a unique extension to an extended diagram $E : I^n_3 \to C$, with the following property:

For each $1 \leq k \leq n$ and $\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon_{k+1} \ldots, \varepsilon_n \in \{0, 1, 2\}$, set:

\[
\begin{align*}
J' &= (\varepsilon_1, \ldots, \varepsilon_{k-1}, 0, \varepsilon_{k+1} \ldots, \varepsilon_n) \\
J'' &= (\varepsilon_1, \ldots, \varepsilon_{k-1}, 1, \varepsilon_{k+1} \ldots, \varepsilon_n) \\
J''' &= (\varepsilon_1, \ldots, \varepsilon_{k-1}, 2, \varepsilon_{k+1} \ldots, \varepsilon_n).
\end{align*}
\]

We then have a strict cofibration sequence:

\[
E(J') \hookrightarrow E(J'') \twoheadrightarrow E(J''').
\]

That is, $E(J') \hookrightarrow E(J''')$ is a cofibration in $C$ (see Sect. 1.18), and $E(J''')$ is the colimit of $* \hookrightarrow E(J') \twoheadrightarrow E(J'')$.

This defines $E$, by functoriality of strict cofibration sequences. In fact, (1.8) will be a strict (and also homotopy) cofibration sequence for every $1 \leq k \leq n$ and $\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon_{k+1} \ldots, \varepsilon_n \in \{0, 1, 2\}$, by the $3 \times 3$ Lemma (the cofibration sequence version of [15, XII, Lemma 3.4], which actually follows from it by working in simplicial groups). Here we use the description of $\hat{D}$ in Remark 1.5 and the fact that $C$ is left proper.

The original $(2 \times \cdots \times 2)$-cube diagram $\hat{D} = E|I^n_2$ will be called the generating cube for $E$.

**Notation 1.9** For any index $J = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ ($\varepsilon_i \in \{0, 1, 2\}$), we shall use the following terminology:

1. The longest initial segment of $J$ with $\varepsilon_i \in \{1, 2\}$ will be called the marker of $J$ (it may be empty, or all of $J$), and denoted by $M(J)$.
2. The length of the final segment of $M(J)$ consisting of digits 2 only will be called the stage of $J$, and denoted by $\sigma(J)$.
3. The complementary final segment of $J$ after $M(J)$ (necessarily starting with a 0, if nonempty) is called the remainder of $J$, and denoted by $R(J)$. The number of digits 2 in $R(J)$ will be denoted by $r(J)$.

**Example 1.10** For $J = (1221122202012012)$ we have marker $M(J) = (12211222)$. Its stage is $\sigma(J) = 3$ (the length of the sequence 222 at the end of $M(J)$), and the remainder is $R(J) = 02012012$, with $r(J) = 3$. 

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We note the following straightforward facts:

**Lemma 1.11** For any index \( J = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \), the extended diagram \( E : I^n_J \to C \) of any cofibrant enhanced strictification \( \tilde{D} : I^n_J \to C \) as above has the following properties:

(a) If \( R(J) \) has at least one digit 1, then \( E(J) \) is weakly contractible.

(b) If \( R(J) \) has no 1’s, then \( E(J) \) is weakly equivalent to \( \Sigma^r(J) E(J') \), where \( J' \) is obtained from \( J \) by replacing all 2’s in \( R(J) \) by digits 0.

(c) In particular, if \( \sigma(J) = 0 \) and \( R(J) \) has no digits 1, then \( E(J) \simeq \Sigma^r(J) A_k \), where \( k \) is the length of \( M(J) \). Thus if \( \sigma(J) = 0 \) and \( R(J) \) has only digits 0, then \( E(J) \simeq A_k \). In particular, \( E(J_k) \simeq A_k \), as in Definition 1.4. Here \( \Sigma^r(J) A \) is the \( r \)-fold suspension of \( A \) (cf. Sect. 1.9(3)).

(d) If \( \sigma(J) > 0 \) and \( R(J) \) has no digits 1, then \( E(J) \) is the cofiber of a map \( E(J_0) \to E(J_1) \), where \( \sigma(J_0) = 0 \) and \( \sigma(J_1) = \sigma(J) - 1 \). Thus we can think of \( \sigma(J) \) as the “cone length” of \( E(J) \) with respect to the objects \( A_0, \ldots, A_n \) and their suspensions.

See the diagram in Example 1.22 below for an illustration of these properties.

**Definition 1.12** We now explain how to associate to a Toda diagram of length \( n + 1 \) (Definition 1.1), with certain additional data, the Toda bracket \( \langle f_0, \ldots, f_n \rangle \), which serves as the (last) obstruction to strictifying the Toda diagram (1.2), for \( n \geq 1 \) (cf. [9, §7]).

The construction is inductive, and a necessary condition in order for it to be defined is that the Toda brackets \( \langle f_0, \ldots, f_{n-1} \rangle \) and \( \langle f_1, \ldots, f_n \rangle \), associated respectively to the initial and final segments of (1.2) of length \( n \), must vanish.

Vanishing of \( \langle f_0, \ldots, f_{n-1} \rangle \) implies that we may choose a strictification \( D : I^n_2 \to C \) of the initial segment, with the corresponding cofibrant enhanced strictification \( \tilde{D} : I^n_2 \to C \) and extended diagram \( E : I^n_3 \to C \). By Lemma 1.11(c), we may choose a representative \( f_n : E(J_n) \to A_{n+1} \) for the last term \( [f_{n+1}] \) in the diagram (1.2) (the construction does not actually depend on the choice of \( f_n \)).

The data for \( \langle f_0, \ldots, f_n \rangle \) consists of the weak homotopy type of \( D \) in \( C^{I^n_2} \), together with (the homotopy class of) a map \( \varphi_n : E(1, 2, \ldots, 2) \to A_{n+1} \) whose restriction to \( E(1, 2, \ldots, 2, 1) \simeq A_n \) represents \( [f_n] \). Note that \( E(1, 2, \ldots, 2) \) has “cone length” \( n - 1 \), by Lemma 1.11(d). The class of \( \varphi_n \) is determined by a choice of nullhomotopy for the (inductively determined) value of \( \langle f_1, \ldots, f_n \rangle \).

Since \( E(0, 2, \ldots, 2) \simeq \Sigma^{-1} A_0 \) by Lemma 1.11(c), precomposing \( \varphi_n \) with \( \alpha_n : E(0, 2, \ldots, 2) \to E(1, 2, \ldots, 2) \) yields the value in \( [\Sigma^{-1} A_0, A_{n+1}] \) of the Toda bracket (associated to the data \( \langle D, \varphi_n \rangle \)).

Formally, the Toda bracket \( \langle f_0, \ldots, f_n \rangle \) is the subset of \( [\Sigma^{-1} A_0, A_{n+1}] \) consisting of all such values. The differences between the various values together constitute the indeterminacy of \( \langle f_0, \ldots, f_n \rangle \). However, we shall not be concerned with these two notions here: essentially, we are only interested in the question of whether certain elements in \( [\Sigma^{-1} A_0, A_{n+1}] \) can be obtained as the value of some Toda bracket.

Since

\[
E(0, 2, \ldots, 2) \xrightarrow{\alpha_n} E(1, 2, \ldots, 2) \to E(2, 2, \ldots, 2)
\]

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is a (homotopy) cofibration sequence, if $\varphi_n \circ \alpha_n \sim 0$, we can choose an extension $\psi_n : E(2, 2, \ldots, 2) \to A_{n+1}$ of $\varphi_n$, up to homotopy.

See the diagrams in Examples 1.15 and 1.22 below for an illustration of these properties.

**Lemma 1.13** A choice of $\psi_n : E(2, 2, \ldots, 2) \to A_{n+1}$ extending $\varphi_n : E(1, 2, \ldots, 2) \to A_{n+1}$ as above yields a strictification for $A_0 \xrightarrow{f_0} \cdots \xrightarrow{f_n} A_{n+1}$ extending the class of $D$.

**Proof** The strictification is given by

$$E(0, 0, \ldots, 0) \to E(1, 0, \ldots, 0) \to E(2, 1, 0, \ldots, 0) \to \cdots E(2, 2, \ldots, 2) \xrightarrow{\psi_n} A_{n+1}.$$ 

Since each adjacent composition is of the form (1.8) for (1.7), it is strictly zero. The fact that $E(2, \ldots, 2, 1, 0, \ldots, 0)$ (with $k - 1$ digits 2) is weakly equivalent to $A_k$ follows from Lemma 1.11, which implies the map out of $E(2, \ldots, 2, 1, 0, \ldots, 0)$ represents $f_k$ by the original construction of $D$. \hfill $\Box$

**Remark 1.14** Note that the “middle cube” $I_{\text{mid}}^{n-1}$ of $I_3^{n-1}$, consisting of the vertices indexed by $J = (\varepsilon_1, \ldots, \varepsilon_{n-1})$ with $\varepsilon_1 = 1$, corresponds to the segment $A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n$ of (1.2) (that is, $I_{\text{mid}}^{n-1}$ is a cofibrant enhanced strictification of this segment in the sense of Sect. 1.4).

Applying Definition 1.12 to $E|_{I_{\text{mid}}^{n-1}}$ we see that in addition to the choices encoded in the given strictification of this segment, the data we obtain consists of $\varphi'_{n-1} : E(1, 1, 2, \ldots, 2) \to A_{n+1}$ whose restriction to $E(1, 1, 2, \ldots, 2, 1) \simeq A_n$ again represents $[f_n]$. The corresponding value for the length $n$ Toda bracket $\langle f_1, \ldots, f_n \rangle$ is $\varphi'_{n-1} \circ \alpha'_{n-1} : E(1, 0, 2, \ldots, 2) \simeq \Sigma^{n-2} A_1 \to A_{n+1}$. If this does not vanish, we must try other choices of the data. If this value is zero, we choose an extension $\psi'_{n-1} : E(1, 2, 2, \ldots, 2) \to A_{n+1}$ for $\varphi'_{n-1}$ in the (homotopy) cofibration sequence $E(1, 0, 2, \ldots, 2) \xrightarrow{\alpha'_{n-1}} E(1, 1, 2, \ldots, 2) \to E(1, 2, 2, \ldots, 2)$. In fact, this map $\psi'_{n-1}$ is precisely $\varphi_n$, the second piece of input needed to define our value for $\langle f_0, \ldots, f_n \rangle$.

### 1.15 The case $n = 2$

In the first step of our inductive process, for $n = 2$, there is no obstruction to strictification, and our cofibrant enhanced strictification $\widehat{D} : I_2^1 \to C$ is obtained in three steps:

(a) We choose a cofibrant replacement for $A_0$, and represent $[f_0]$ by a cofibration $f_0 : A_0 \to A_1$, and $[f_1]$ by any map $f_1 : A_1 \to A_2$.

(b) Since $[f_1] \circ [f_0] = 0$, in $\text{ho} C$, we can choose a nullhomotopy $F : CA_0 \to A_2$ for $f_1 \circ f_0$, making the outer square in the following diagram commute, where
the inner square is the (homotopy) pushout (which is the homotopy cofiber of $f_0$):

$$
\begin{array}{c}
A_0 \xleftarrow{i} \xrightarrow{f_0} C A_0 \\
\downarrow \quad \downarrow j \quad \downarrow F \\
A_1 \xleftarrow{k} \xrightarrow{f_1} \text{Cof}(f_0) \xrightarrow{\xi} A_2
\end{array}
$$

(1.16)

(c) Changing $\xi$ into a cofibration $\xi' : \text{Cof}(f_0) \hookrightarrow \tilde{A}_2$ and precomposing with the given structure maps $j$ and $k$ yields the required cofibrant $\tilde{D} : I_2^2 \to C$.

This description makes it clear that even though the homotopy classes of nullhomotopies $F$ for $f_1 \circ f_0$ are in bijection with $[\Sigma A_0, A_2]$ (see [17, §1]), the homotopy type of possible (enhanced) strictifications $\tilde{D}$ (given $[f_0]$ and $[f_1]$) is completely determined by $[\xi] \in [\text{Cof}(f_0), A_2]$. Thus, perhaps surprisingly, the homotopy classification of strictifications has less information than the choices of nullhomotopies.

The extended diagram $E : I_3^3 \to C$ is then the solid $3 \times 3$ square in

$$
\begin{array}{c}
A_0 \xleftarrow{i} \xrightarrow{f_0} C A_0 \\
\downarrow \quad \downarrow F \\
A_1 \xleftarrow{k} \xrightarrow{f_1} \text{Cof}(f_0) \xrightarrow{\xi} A_2 \\
\downarrow \quad \downarrow q \quad \quad \quad \downarrow \varphi_2 \\
C \xrightarrow{f_0} \tilde{A}_2 \xrightarrow{h_0} \text{Cof}(f_0) \xrightarrow{\alpha_2} C \xrightarrow{f_2} A_3
\end{array}
$$

(1.17)

(cf. the dual diagram in [10, (0.3)]).

Here (and later in the paper) $\tilde{\Sigma} A$ denotes the version of the suspension of $A$ obtained as the cofiber of $A \hookrightarrow C A$, as in the first row of (1.17).

**Remark 1.18** We note that more generally if we start with the pushout $P$ of two cofibrations $f : X \hookrightarrow Y$ and $k : X \hookrightarrow Z$ in a pointed model category, as in (1.16),

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the resulting extended diagram takes the form:

\[
\begin{array}{cccc}
X & f & Y & C_f \\
\downarrow h & & \downarrow i & = \\
Z & j & P & C_j \\
\downarrow & & \downarrow & \\
C_h & = & C_i & \ast
\end{array}
\]

(1.19)

If we have a further cofibration \( \xi : P \to W \), as in Sect. 1.15(c), and let \( g := \xi \circ j : Z \to W \) and \( k := \xi \circ i : Y \to W \), then taking iterated pushouts in the following solid diagram

\[
\begin{array}{cccc}
Z & j & P & W \\
\downarrow & & \downarrow \xi & \\
\ast & \downarrow & C_f & Q \\
\end{array}
\]

We see that the map \( \ell \) is again a cofibration, by cobase change, and that \( Q \) is in fact \( C_g \) (the cofiber of \( g \)), since the large rectangle in (1.20) is a also pushout. Thus from (1.19) we deduce that the induced map \( \ell : C_f \to C_g \) is a cofibration (and similarly for \( m : C_h \to C_k \)).

In summary, whenever our initial square \( D : I^2_n \to C \) is cofibrant in the model category structure of Sect. 1.5, all rows and columns in the extended diagram \( E : I^3_n \to C \) are strict cofibration sequences.

1.21 The ordinary Toda bracket

Our formalism indicates that \( \varphi_2 = f_1 \) and \( \alpha_2 = f_1 \), so the “Toda bracket \( \langle f_0, f_1 \rangle \) of length 2” is the composite \( f_1 \circ f_0 \), which is nullhomotopic by assumption. A choice of nullhomotopy determines the extension \( \psi_2 = h_0 \), and in fact setting \( \widetilde{f}_1 := q \circ f_1 : A_1 \to A_2 \), we obtain a strictification for the initial length 2 subdiagram of (1.2), as implied by Lemma 1.13.

Now we add the map \( f_2 : A_2 \to A_3 \) to the solid square in (1.17); since \( f_2 \circ f_1 \sim 0 \) and the middle row is a (homotopy) cofibration sequence, we have an induced map \( \varphi : C_{f_1} \to A_3 \), and we define the value of the Toda bracket (of length 2) associated to these choices to be the homotopy class of \( \varphi \circ g_0 : \Sigma A_0 \to A_3 \). Since the right column is again a (homotopy) cofibration sequence, \( \varphi \) extends to \( \psi \) as indicated if and only if \( \varphi \circ g_0 \sim 0 \). In this case we see that \( \widetilde{f}_2 := \psi \circ r : A_2 \to A_3 \) represents \( f_2 \), and we also have \( \widetilde{f}_2 \circ \widetilde{f}_1 = 0 \) —in other words, we can extend the above to a strictification of the initial length 3 subdiagram of (1.2) if and only if \( \varphi \circ g_0 \sim 0 \) for some choice of \( \xi \) and \( \varphi \).
1.22 The case \( n = 3 \)

Here the extended diagram is described by the solid \( 3 \times 3 \times 3 \) cube:

We may summarize the results of this section in the following

**Theorem 1.23** Assume given a Toda diagram (1.2) in a model category \( \mathcal{C} \) as in Sect. 1.1.

(a) For any representatives \( f_0 \) and \( f_1 \), a choice of nullhomotopy \( F : f_1 \circ f_0 \sim * \) yields an extended diagram \( E : I^{3}_{3} \to \mathcal{C} \) as in Sect. 1.15. One can strictify the initial segment of (1.2) of length 3 if and only there is a value of the Toda bracket which vanishes (for some choice of \( F \) and \( \varphi_2 \) in (1.17)).

(b) Any strictification of the initial segment of (1.2) of length \( k \) is equivalent (up to weak equivalence of diagrams) to a enhanced strictification \( \hat{D} : I^{k+1}_{2} \to \mathcal{C} \) as in Sect. 1.4

(c) Given such a (cofibrant) enhanced strictification, with the corresponding extended diagram \( E : I^{k+1}_{3} \to \mathcal{C} \) as in Sect. 1.6, the given strictification extends to one of the initial segments of (1.2) of length \( k+1 \) if and only if there is a choice of data for \( \langle f_0, \ldots, f_{k+1} \rangle \) for which the Toda bracket vanishes (i.e., has value 0).
Corollary 1.24 A Toda diagram of the form (1.2) can be strictified if and only if the Toda bracket \( \langle f_0, \ldots, f_{n-1} \rangle \) vanishes (for some choice of data).

2 Toda brackets and filtered objects

In stable model categories \( C \) (cf. [14]), an alternative description of Toda brackets, in terms of filtered objects in \( C \), is available. This involves continuing the extended diagram one step further in each direction.

We should remark that there is no “standard” definition of (long) Toda brackets, even stably (see [11,21] and the discussion in [16]).

2.1 Forward cubes

More formally, we assume given a Toda diagram

\[
A_{-1} \xrightarrow{[f_1]} A_0 \xrightarrow{[f_0]} \cdots \xrightarrow{[f_n]} A_n \xrightarrow{[f_n]} A_{n+1}
\]

of length \( n + 2 \) as in (1.2), together with a rectification \( D : I^n_{\text{mid}} = I^n_2 \to C \) of the central segment \( A_0 \to \cdots \to A_n \) of length \( n \), with cofibrant replacement \( \widehat{D} : I^n_2 \to C \) and corresponding extended diagram \( E : I^n_3 \to C \) as in Sect. 1.6 (see diagram 1.22 above).

Now consider the “forward” \( n \)-dimensional \( (2 \times \cdots \times 2) \)-subcube \( I^n_{\text{fwd}} \cong I^n_2 \) of \( I^n_3 \), consisting of those vertices labelled by \( J = (\varepsilon_1, \ldots, \varepsilon_n) \) with each \( \varepsilon_i \in \{1, 2\} \).

Let \( \widehat{E} : I^n_{\text{fwd}} \to C \) be a cofibrant replacement for \( E|_{I^n_{\text{fwd}}} \). We have a new forward \( (3 \times \cdots \times 3) \)-cube \( I^n_3 \), with vertices now labelled by \( J = (\varepsilon_1, \ldots, \varepsilon_n) \) with \( \varepsilon_i \in \{1, 2, 3\} \), and we see that \( \widehat{E} \) can be extended to the extended forward diagram \( F : I^n_3 \to C \) by taking strict cofibration sequences in each direction, as in Sect. 1.6.

Moreover, Lemma 1.11 extends in the obvious way to \( F \), with

\[
F(\varepsilon_1 \ldots \varepsilon_{i-1}, 3, \varepsilon_{i+1} \ldots \varepsilon_n) \simeq \Sigma E(\varepsilon_1 \ldots \varepsilon_{i-1}, 0, \varepsilon_{i+1} \ldots \varepsilon_n)
\]

for any \( \varepsilon_j \in \{1, 2, 3\} \) (\( j = 1, \ldots, i-1, i+1, \ldots, n \)).

Furthermore, if \( C \) is a stable model category (see [13, Ch. 7]), \( E \) can actually be recovered from \( F \), up to weak equivalence, by taking homotopy fibers of the forward generating cube \( F|_{I^n_{\text{fwd}}} = \widehat{E}|_{I^n_{\text{fwd}}} \) along each edge.

Now let \( J^k := (1 \ldots 12 \ldots 2) \), with \( n - k + 1 \) digits 1 followed by \( k - 1 \) digits 2 \((1 \leq k \leq n + 1)\), and write \( X_k := \widehat{E}(J^k) = F(J^k) \). Note that by Lemma 1.11 we have homotopy cofibration sequences of the form

\[
\Sigma^{k-1} A_{n-k} \simeq E(\underbrace{1 \ldots 1}_{n-k} \underbrace{2 \ldots 2}_{k-1}) \xrightarrow{g_k} E(J^k) \xrightarrow{i_k} E(J^{k+1})
\]
so for $F$ we have homotopy cofibration sequences

\[ F(J^k) \xrightarrow{i_k} F(J^{k+1}) \xrightarrow{r_k} F(1 \cdots 1 \underbrace{3 \cdots 2}_{n-k} \overbrace{2 \cdots 1}^{k-1}) \simeq \Sigma^k A_{n-k} \quad (2.4) \]

for each $0 < k \leq n$. By convention we set $F(J^0) := *$; then $F(J^1) = F(1 \ldots 1) \simeq A_n$, so (2.4) is still a homotopy cofibration sequence for $k = 0$. We write $X$ for $F(J^{n+1})$, and denote the composite cofibration $i_n \circ i_{n-1} \circ \cdots \circ i_1 : F(J^1) \hookrightarrow F(J^{n+1})$ by $j_X : A_n \to X$.

**Remark 2.5** Note that the composite $r_k \circ g_{k+1} : \tilde{\Sigma}^k A_{n-k-1} \to \tilde{\Sigma}^k A_{n-k}$ represents $\tilde{\Sigma}^k f_{n-k-1}$ (because of the functoriality of the extension of $\hat{D}$ to $E$ and then to $F$).

**Lemma 2.6** Given a Toda diagram (2.2) of length $n + 2$, the data actually needed to specify a value of the Toda bracket $\langle f_{-1}, \ldots, f_n \rangle$ of length $n + 2$ consists of:

(a) The extended forward diagram $F : \hat{I}_3^n \to C$ associated to the strictification of $A_0 \to \cdots \to A_n$;

(b) A map $\alpha_{n+1} : \Sigma^n A_{-1} \to X := F(J^{n+1})$ such that $r_n \circ \alpha_{n+1} \sim \Sigma^n f_{-1}$.

(c) A map $\varphi_{n+1} : X \to A_{n+1}$ such that $\varphi_{n+1} \circ j_X \sim f_n$.

The associated value is the class of $\varphi_{n+1} \circ \alpha_{n+1} : \Sigma^n A_{-1} \to A_{n+1}$.

**Proof** This follows from Definition 1.12, once we notice that the $n$-dimensional $(3 \times \cdots \times 3)$-cube $I_3^n$ which indexes diagram $\hat{D}$ of Sect. 2.1 is actually the middle cube $I_{\text{mid}}^n$ of length $n + 2$ with respect to the Toda diagram (2.2), and thus $X = F(2 \ldots 2)$ is in fact $E(12 \ldots 2)$ (with $n$ digits $2$) for the full $(3 \times \cdots \times 3)$-cube diagram $E : I_3^{n+1} \to C$ of (2.2). However, from the description in Sect. 1.12 we see that the only part of the enhanced strictification $E$ needed to compute the Toda bracket and not determined by $F$ is the map $\alpha_{n+1} : \tilde{\Sigma}^n A_{-1} \to E(12 \ldots 2)$. The fact that $r_n \circ \alpha_{n+1} \sim \Sigma^n f_{-1}$ follows by continuing the cofibration sequence (2.3) for $k = n+1$ one step to the right, as in (2.4).

The other ingredient needed is precisely the map $\varphi_{n+1} : E(12 \ldots 2) \to A_{n+1}$ associated to the vanishing of the right Toda bracket $\langle f_0, \ldots, f_n \rangle$. The fact that $\varphi_{n+1} \circ j_X \sim f_n$ can be read off from Lemma 1.11 and the description of $\varphi_n$ in Sect. 1.12. \hfill \Box

**Example 2.7** As Toda showed in [20, Proposition 5.6], the generator $\nu' \in \pi_6 S^3_{(2)}$ of the 2-local 3-sphere (of order 4) is one value of the Toda bracket for the length 3 ($n = 1$) diagram:

\[ S^5 \xrightarrow{\eta_4} S^4 \xrightarrow{2t_4} S^4 \xrightarrow{\eta_4} S^3, \quad (2.8) \]

where the maps $\eta_k$ are suspended Hopf maps (of order 2). There is indeterminacy of order 2, and the other values of the Toda bracket is another generator.

Thus $X_0 = *$, $X_1 = A_1 = S^4$, with $X_2$ the cofiber of $f_0 = 2t_4$, the 5-dimensional mod 2 Moore space. The fact that $\eta_4$ has order 2 implies that it extends
to a map \( \varphi_2 : X_2 \to S^3 \), while the fact that \( \eta_6 \) also has order 2 implies that it factors through \( X_2 \) in the continued cofibration sequence \( X_2 \to S^5 \to S^5 \) (since it is in the stable range), yielding \( \alpha_2 : S^6 \to X_2 \).

The composite \( S^6 = \Sigma A_0 \xrightarrow{\varphi_2 \circ \alpha_2} A_2 = S^3 \) is the required value \( \nu' \) of \( \langle \eta_3, 2\mu_4, \eta_4 \rangle \).

### 2.9 Generalized Toda brackets associated to a filtered object

We do not in fact need to be given a Toda diagram (1.2) in order to use this approach to defining Toda brackets – all we need are cofibration sequences as in (2.4).

Thus assume that for each \(-1 \leq k < \ell\) we have a homotopy cofibration sequence

\[
X_k \xrightarrow{j_k} X_{k+1} \xrightarrow{r_k} C_{k+1} \xrightarrow{\delta_k} \Sigma X_k \ldots
\]  

(2.10)

in some pointed model category \( C \), starting with \( X_{-1} = \ast \), so \( X_0 = C_0 \). We think of \( X := X_\ell \) as an object in \( C \) filtered by \( X \), with filtration quotients \( C_k \) \((k = 0, \ldots, \ell)\). Again we denote the composite cofibration \( j_{n-1} \circ \cdots \circ j_0 : X_0 \hookrightarrow X_\ell \) by \( j_X : C_0 \to X \).

The cofibration sequence (2.10) might extend to the left, presenting \( X_{k+1} \) as the homotopy cofiber of a map \( g_k : B_k \to X_k \) with \( C_{k+1} \simeq \Sigma B_k \) and \( \delta_k \sim \Sigma g_k \), as in (2.4). This need not always exist, but we shall use this notation when we have such a map \( g_k \).

We then think of the filtered space \( X \) as a template for a generalized Toda bracket of length \( \ell + 1 \). The data for the bracket consist of any two homotopy classes of maps \( \alpha_{\ell+1} : W \to X \) and \( \varphi_{\ell+1} : X \to Z \) (where \( W \) and \( Z \) are arbitrary, though often \( W \) is a sphere). The value associated to this data is the homotopy class of the composite \( \varphi_{\ell+1} \circ \alpha_{\ell+1} \) in \([W, Z]\).

**Definition 2.11** Given homotopy cofibration sequences as in (2.10), we denote \( \Sigma r_{k-1} \circ \delta_k : C_{k+1} \to \Sigma C_k \) by \( \gamma_{k+1} \). We also write \( \Sigma^{-1} C_{\ell+1} \) for \( W \) and \( \Sigma^{\ell+1} C_{-1} \) for \( Z \) as above and define \( \Sigma^{-1} r_{\ell+1} := r_{\ell} \circ \alpha_{\ell+1} \) and \( \Sigma^\ell \gamma_0 := \varphi_{\ell+1} \circ j_X \).

**Proposition 2.12** Given a filtered object \( X \) as in Sect. 2.9, the sequence

\[
\Sigma^{-1} C_{\ell+1} \xrightarrow{\Sigma^{-1} r_{\ell+1}} C_\ell \xrightarrow{\gamma_\ell} \Sigma C_{\ell-1} \xrightarrow{\Sigma \gamma_{\ell-1}} \Sigma^2 C_{\ell-2} \to \cdots \Sigma^\ell C_0 \xrightarrow{\Sigma^\ell \gamma_0} \Sigma^{\ell+1} C_{-1}
\]

(2.13)

is a Toda diagram of length \( \ell + 2 \). Furthermore, when (2.10) is obtained as in Sect. 2.1, (2.13) is the \( \ell \)-fold suspension of (1.2) (extended as above to \( A_{\ell+1} \)), with \( C_k \simeq \Sigma^k A_{\ell+1-k} \) and \( \gamma_k \sim \Sigma^k f_{\ell+1-k} \).

**Proof** We have

\[
\Sigma \gamma_k \circ \gamma_{k+1} \sim 0 \quad \text{for each } 0 < k < \ell
\]

(2.14)
since the middle factors of the composite are two successive maps from (2.10). Moreover, (2.14) also holds for $k = 0$ and $k = \ell$ (for the same reason). When (2.10) is as in Sect. 2.1, the claim follows from Lemma 2.6.

\[ \square \]

2.15 CW filtrations

If $X \in \text{Top}_*$ is an $\ell$-dimensional connected CW complex and (2.10) is its CW filtration, let $g_j : B_j \rightarrow X_j$ be the attaching map for the $j$-skeleton $X_j := X^{(j)}$ of $X$, so the cofibration sequence $B_j \xrightarrow{g_j} X_j \xrightarrow{i_j} X_{j+1}$ defines $X_{j+1}$. Thus for $j = 0, \ldots, \ell - 1$, $B_j$ will be a wedge of $j$-spheres, and in the (suspended) Toda diagram (2.13) associated to the CW filtration for $X$, each $\Sigma^j C_{\ell-j}$ is a wedge of $\ell$-spheres. Only $W = \Sigma^1 C_{\ell+1}$ and $Z = \Sigma^{\ell+1} C_{-1}$ can be arbitrary.

However, if $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ are maps between wedges of $\ell$-spheres and $g \circ f \sim 0$, this must also hold when we restrict to each wedge summand of $V_1$ and project onto each summand of $V_3$. Thus are reduced to the case where $g$, as a map from a wedge of $m$ copies of $S^\ell$ to $S^\ell$, is given by a vector of integers $(a_1, \ldots, a_m)$. Dually $f$ is also given by $(b_1, \ldots, b_m)$, with $\sum_i a_i b_i = 0$. In this case the associated Toda bracket is trivial.

This suggests the following

**Definition 2.16** A spherical filtration $X_0 \rightarrow X_1 \rightarrow \cdots X_\ell$ of length $\ell$ on a CW complex $X = X_\ell$ is a filtration as above such that for each $0 \leq k \leq \ell$:

(a) The filtration quotient $C_k$ is homotopy equivalent to a $(c_k - 1)$-connected wedge of spheres, (that is, $c_k$ is the lowest dimension of a sphere in the wedge).

(b) We never have two successive maps in (2.13) (and thus in (1.2)) between wedges of spheres where the lowest spheres have the same dimension. That is:

if $c_k + 1 = c_{k+1}$ then $c_{k-1} + 1 < c_k$ and $c_{k+1} + 1 < c_{k+2}$ (2.17)

when defined.

Let $c_{-1} - 1$ denote the connectivity of $Z$ and $c_{\ell+1} - 1$ the connectivity of $W$, and assume that (2.17) holds also for $k = 0$ and $k = \ell$. Given homotopy classes $\alpha_{\ell+1} : W \rightarrow X$ and $\varphi_{\ell+1} : X \rightarrow Z$, we then say that $\varphi_{\ell+1} \circ \alpha_{n+1} \in [W, Z]$ is a value of the spherical Toda bracket associated to the filtration on $X$.

**Remark 2.18** Using simplicial $\Pi$-algebra resolutions (see Sect. 3.6 below) and the connectivity results of [5, Proposition 4.2.2], one can show that for each $n > r \geq 2$, any $(r - 1)$-connected space $X$ has an $n$-skeleton with a spherical filtration of length $\ell \leq \lceil \frac{2(n-r+1)}{3} \rceil$. We omit the proof, since this result is not needed for our purposes.

**Example 2.19** In the stable range the 2-local spherically filtered $(k+i)$-skeleta of the $(k-1)$-connected cover of the $k$-sphere, $X = S^k(k)$ $(i = 1, \ldots, 4)$ are constructed as follows (using Toda’s calculations in [20, Ch. XIV]):
(a) We start with \( C_1 = Z_1 := S^{k+1} \vee S^{k+3} \), with the covering map \( p^{(1)} : Z_1 \to S^k \) given by the Hopf maps \( \eta_{k+1} : v_{k+1} \) (inducing a surjection in \( \pi_j \) for \( 1 \leq j \leq 4 \)). Here \( \alpha \sqcup \beta \) indicates the map induced by the coproduct structure of the wedge. We may omit \( S^{k+3} \) for \( i \leq 2 \), since \( S^{k+1} \) itself is a \((k+1)\)-skeleton for \( X \).

(b) Next, we have the cofibration sequence

\[
B_1 := S^{k+1} \vee S^{k+3} \xrightarrow{2t_{k+1} - \xi \eta_{k+4}} Z_1 \xrightarrow{j_1} Z_2 \xrightarrow{r_1} C_2 = S^{k+2} \vee S^{k+4}.
\]

Here \( \gamma_2 : C_2 \to \Sigma C_1 \) on the lowest wedge summand \( S^{k+2} \) of \( C_2 \) is the suspension of \( 2t_{k+1} \) on the \( S^{k+1} \)-summand on the right, composed with \( \text{Id} : \Sigma Z_1 \to \Sigma C_1 \). This is just the order 2 map. The covering map \( p^{(2)} : Z_2 \to S^k \) is induced from \( p^{(1)} \) by the fact that \( \eta_{k+1} \) has order 2 and \( 4v_k = \eta_k \eta_{k+1} \eta_{k+2} \).

Thus \( Z_2^{(k+i)} \) is already a \((k+i)\)-skeleton of \( X \) for \( i \leq 2 \), which may be identified with the mod 2 Moore space \( M^{k+1} \). Moreover, since

\[
\pi_i M^{k+1} \cong \begin{cases} 
\mathbb{Z}/2\langle \xi \rangle & \text{for } i = k+1 \\
\mathbb{Z}/2\langle \xi \circ \eta_{k+1} \rangle & \text{for } i = k+2 \\
\mathbb{Z}/4\langle \beta \rangle & \text{for } i = k+3 \\
\mathbb{Z}/2\langle \beta \circ \eta_{k+3} \rangle \oplus \mathbb{Z}/2\langle \xi \circ v_{k+1} \rangle & \text{for } i = k+4.
\end{cases}
\]  

(2.20)

we see that for \( 3 \leq i \leq 4 \), \( Z_2 \) is constructed by wedging \( M^{k+1} \) with \( S^{k+3} \) (yielding a \((k+3)\)-skeleton of \( X \)) and then attaching a \((k+4)\)-cell along \( 4t_{k+3} - \xi \circ \eta_{k+1} \circ \eta_{k+2} \).

(c) Noting that from (2.20) and the choice of attaching map for \( S^{k+3} \), we find that \( \pi_{k+3} Z_2 \cong \mathbb{Z}/16\langle \delta \rangle \) and \( \pi_{k+4} Z_2 \cong \mathbb{Z}/2\langle \delta \circ \eta_{k+3} \rangle \oplus \mathbb{Z}/2\langle \xi \circ v_{k+1} \rangle \).

We therefore have a cofibration sequence

\[
S^{k+3} \vee S^{k+4} \xrightarrow{88.1 \xi \circ v_{k+1}} Z_2 \xrightarrow{j_2} Z_3 \xrightarrow{r_2} C_3 = S^{k+4} \vee S^{k+5},
\]

with the covering map \( p^{(3)} : Z_3 \to S^k \) induced from \( p^{(2)} \), yielding a \((k+4)\)-skeleton for \( X \).

In this case, \( \gamma_3 \) from the lowest wedge summand \( S^{k+4} \) of \( C_3 \) is the suspension of \( 8\delta \), composed with the pinch map \( \Sigma M^{k+1} \to S^{k+3} \), which is \( \eta_{k+3} \), by (2.20).

(d) For the \((k+5)\)- and \((k+6)\)-skeleta we must kill \( \pi_{k+4} Z_3 \) and then \( \pi_{k+5} Z_4 \).
3 A sampler

Even though our version of Toda brackets can be defined in any pointed model category, the discussion in Sect. 2 indicates that in some sense they are the “correct” notion only in the stable setting. In [2], we offer an alternative definition of higher order homotopy operations, using simplicial, rather than linear, diagrams, and show that these allow a cleaner description of the higher structure of unstable homotopy groups: in fact, using such higher order operations, all elements in the homotopy groups of spheres can be generated by the Whitehead products alone (see [2, Theorem 7.15]).

Nevertheless, we offer two examples of settings in which our version of higher Toda brackets figure:

3.1 Higher order homological algebra

Consider the following Toda diagram of length three in (vertical) chain complexes (say, in degrees 0 and 1):

\[
\begin{array}{cccccc}
0 & \rightarrow & 0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & 0
\end{array}
\]

(3.2)

We may choose the map out of the cone on the leftmost chain complex:

\[
\begin{array}{cccc}
\mathbb{Z} & \rightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \rightarrow & \mathbb{Z}
\end{array}
\]

(3.3)

as a nullhomotopy for the left composite. Since the right composite is strictly zero, pinching the bottom left \( \mathbb{Z} \) in (3.3) yields the chain map \( \mathbb{Z} \rightarrow \mathbb{Z} \) (concentrated in degree 1) as the value of the Toda bracket for (3.2) (compare [4, §5]).

What we have described here, of course, is just the \( d_2 \) differential in the spectral sequence for the double complex given by the right three columns of (3.2), with the leftmost chain map defining an element in the 0th homology of the second column. For simplicity the right 2-segment of (3.2), rather than the left, has been strictified.

We may think of this double complex as representing a map of Eilenberg-Mac Lane chain complexes \( K(A, 0) \rightarrow K(B, 0) \rightarrow K(C, 1) \) (for \( A = C = \mathbb{Z}, \ B = \mathbb{Z}/p, \ f : A \rightarrow B \) the obvious surjection, and \( \phi \) a generator of \( \text{Ext}(B, C) \cong \mathbb{Z}/p \)).

From a homological algebra point of view we then have:

(a) An element \( \phi \) of \( \text{Ext}(B, C) \);
(b) A homomorphism \( f : A \rightarrow B \) with \( f^*(\phi) = 0 \) in \( \text{Ext}(A, C) \);
(c) An element \( a \in \text{Ker}(f) \),
and we may interpret the Toda bracket described above as part of the secondary structure on $\text{Ext}$ of abelian groups, given by a secondary operation

$$\langle -, - \rangle : \text{Ker}(f^*) \times \text{Ker}(f) \to C$$ (3.4)

which is linear in the second variable – that is, to each $\phi \in \text{Ker}(f^*)$ we associate a homomorphism $\langle \phi, - \rangle : \text{Ker}(f) \to C$.

In our example this homomorphism $\mathbb{Z} \to \mathbb{Z}$ is one-to-one (for any $\phi \in \text{Ker}(f^*)$). However, we may replace $C = \mathbb{Z}$ by $\mathbb{Z}/p$, and consider the length four Toda diagram:

\[
\begin{array}{ccccccccc}
0 & \to & 0 & \to & 0 & \to & \mathbb{Z} & = & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow & & p & & \downarrow \\
0 & \to & 0 & \to & \mathbb{Z} & = & \mathbb{Z} & & 0 \\
\downarrow & & \downarrow & & \downarrow & & p & & \downarrow \\
\mathbb{Z} & & p & \to & \mathbb{Z} & = & \mathbb{Z} & & 0 \\
\end{array}
\]

(3.5)

Now if $\iota$ generates $A = \mathbb{Z}$, then $\langle \phi, p \cdot \iota \rangle$ generates $C = \mathbb{Z}/p$, so $p^2 \cdot \iota$ generates the kernel of $\langle \phi, - \rangle : \text{Ker}(f) \to C$. In this case the differential $d^3$ in the spectral sequence is given by a tertiary $\text{Ext}$ operation (applied here to $p^2 \cdot \iota$, representing an element in kernel of the $d^2$ differential).

This pattern may be continued indefinitely.

### 3.6 Universal operations

Recall that a $\Pi$-algebra is a graded group $\Lambda_*$ equipped with an action of the primary homotopy operations on it, modelled on the homotopy groups of a topological space (see [18, §1]). Thus, given $\Lambda_*$, it is natural to ask whether it is isomorphic to $\pi_* X$ for some pointed connected space $X$—and if so, in how many ways. An obstruction theory for answering these questions in terms of André–Quillen cohomology classes was provided in [6].

In [8], it was shown that these cohomology classes may be represented by cycles which are described in terms of (a sequence of) higher Toda brackets. The sequence for the difference obstruction (for distinguishing between different realization of the $\Pi$-algebra $\Lambda_*$) may therefore be thought of as a collection of universal higher operations which, together with the primary structure of $\pi_* X$, determine the weak homotopy type of $X$.

The higher Toda brackets in question are obtained from an $n$-truncated simplicial space $V_\bullet$ approximating a full simplicial resolution of $Y$) by applying the Moore chains functor to yield a Toda diagram of the form

$$\nabla_{n+1} \to C_n V_\bullet \to C_{n-1} V_\bullet \to \cdots \to C_1 V_\bullet \to V_0$$

(see [8, §6.9]).
However, even though each $V_k$ is a wedge of spheres, the Moore chains objects $C_k V_*$ are not, so it is difficult to extract any meaningful information about the Toda brackets from $V_*$. On the other hand, the simplicial version of these higher operations, described in [8, §5], involve only wedges of spheres, and are thus easier to compute under favorable circumstances: see [2, §8], where for each $n \geq 1$, the Hopf map $g_n : S^{2n+1} \to \mathbb{C}P^n$ is described as the value of an explicit $n$th order operation involving only Whitehead products.

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