Controlling migration of a pair of correlated particles by doubly modulated fields

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Abstract

The resonant tunneling of correlated bosons in optical lattices is investigated in the presence of doubly modulated AC-fields. The effective hopping coefficients are density-dependent. We can make use of this property to control the migration of a pair of strongly interacting particles in one- or two-dimensional uniform lattices via properly manipulating the phases, frequencies and amplitudes of the driven fields. We design a bifurcating quantum motion of the pair in contrast to the coherent quantum walk of the correlated pair in the absence of external fields.

1. Introduction

Ultracold quantum gases in optical lattices have become versatile experimental tools and robust simulators for exploring fundamental many-body systems due to their high level of controllability and cleanliness [1]. They have enabled the realization of Hubbard models [2–5] and artificial gauge fields [6,7]. Equipped with time-of-flight expansion and newly developed single-site resolved detection techniques [8,9], more insight into properties of Mott insulators in the Hubbard-regime [10] was revealed. Recently, researchers have observed Meissner effects in bosonic ladders [11,12].

The highly tunable optical lattice also provides an ideal pilot for periodically driving systems which are intriguing for the formation of various effective Hamiltonians [13]. The dynamical localization (DL) [14,15] and the photon-assisted tunneling [16] were utilized to induce the basic superfluid to Mott insulator transition [17–19]. Coherent destruction of tunneling (CDT) [20–23] was considered for quantum entanglement and quantum control of directed selective tunneling [24,25]. Other specific themes of manipulation include dipole–dipole interaction in a driven triple well [26], Landau–Zener tunneling [27] and two-particle transport control in biparticle lattices [28]. All schemes can contribute to the emulation and a better understanding of the solid state physics. By applying a periodically varied magnetic field in the Feshbach resonance, the on-site interaction can be modulated [29–31]. It leads to the density-dependent effective hopping rates, which are mirror-symmetric. In addition, double modulations (DM) of both the lattice and the on-site interaction have been studied in the high frequency regime [32]. The development in lattice control and Feshbach resonance techniques permit the DM scheme in 1D and higher dimensions [29,33]. The associated mirror-asymmetric tunneling may give rise to an insulating ground state with nontrivial orders.

In this paper, we focus on the resonant photon-assisted tunneling by the doubly driving AC-fields [16,34]. Simultaneously modulating the optical lattice and the on-site interaction leads to an unconventional effective Hubbard model which contains density-dependent hoppings. We make use of this property to control the migration of a pair of strongly correlated particles in 1D or 2D optical lattices. We demonstrate the driven model and resulting effective hoppings in section 2. Section 3 presents the 1D migration scheme and a bifurcating quantum motion. Section 4 presents the migration scheme in a 2D uniform lattice. A summary is presented in section 5.
2. Double modulation in resonant condition

We start with the periodically modulated Bose–Hubbard model (BHM) in the uniform optical lattices
\[ \hat{H} = \hat{H}_{\text{hopping}} + \hat{H}_0(t), \]
with the hopping term
\[ \hat{H}_{\text{hopping}} = -J \sum_j \left( \hat{b}_j^\dagger \hat{b}_{j+1} + \text{h.c.} \right) \]
and the time-dependent term
\[ \hat{H}_0(t) = \frac{1}{2} U(t) \sum_j \hat{n}_j (\hat{n}_j - 1) + K(t) \sum_j \hat{n}_j. \]

Here, \( \hat{b}_j^\dagger, \hat{b}_j \) is the traditional bosonic creation (annihilation) operator acting on site \( j \) and \( \hat{n}_j = \hat{b}_j^\dagger \hat{b}_j \) is the number operator. \( J \) is the hopping rate between the nearest sites. \( U(t) = U_0 + U_1 \cos(\omega_U t) \) represents the onsite interaction modulation and \( K(t) = \cos(\omega_K t) \) the lattice driving modulation. \( \omega_U \) and \( \omega_K \) are respectively the modulating frequencies of the on-site interaction and the lattice shaking.

To obtain the effective Hamiltonian, it is feasible to apply the Floquet analysis, or to follow the general formalism in [13]. Here we employ an equivalent deduction for the resonant condition: \( \omega_U = \omega_K \equiv \omega \) and \( U_0/\omega = \text{integer} \). By expanding the quantum state in the Fock basis labeled by \( |k \rangle \in \{|n_1^{(k)}, n_2^{(k)}, \cdots, n_N^{(k)}\} \), and directly applying the equation of motion [35], one gets the effective Hamiltonian
\[ H_{\text{eff}} = \sum_{(k)} \hat{b}_k^\dagger \hat{b}_k \hat{n}_j \hat{b}_j, \]
with bare hopping terms which depend on the nearest on-site occupations and break the mirror symmetry. The time-independent onsite part \( U_0 \) has been eliminated due to the photon-assisted tunneling effect, i.e., energy quanta \( \hbar = 1 \) due to the modulation can be understood as the photon, which can compensate the energy cost of multi-particle onsite occupations. Note that the photon-assisted tunneling occurs when \( \omega = U_0/\text{integer} \), otherwise, the deviation from resonant condition leads to a rapid increase of the residual onsite interaction and a rapid decrease of the effective hopping rates, thus one would expect the effect of the onsite repulsion such as an insulating state. With initial occupations \( n_j, n_{j+1} \) on sites \( j \) and \( j + 1 \), respectively, the effective hopping rates (for one particle hopping to the left or one particle to the right) are:
\[ I_1 = \frac{U_1}{\omega} (n_j - n_{j+1} + 1), \]
\[ I_2 = \frac{U_1}{\omega} (n_j - n_{j+1} - 1), \]

with \( J_{\mu,\nu} \) the \( \mu (\nu) \) th order Bessel function of the first kind. The doubly modulated model reduces to single modulated model by setting either \( K = 0 \) or \( U_1 = 0 \) [15, 16, 29–31]. As \( \omega \gg U_0 \), the hopping rates recover the previous results of the high frequency limit [32]. Unlike the situation in the high frequency regimes, the resonant modulation serves as an offset of the on-site Hubbard energy. The photon-assisted tunneling may lead to the DL if we start the modulation from the Mott insulator (MI) regime and set \( J_{\mu} (K/\omega \pm U_0/\omega) = 0 \) and \( \mu = \pm U_0/\omega = \text{integer} \), since all hopping coefficients become zero. The parameters \( K \) and \( U_1 \) provide a flexible way to control the quantum dynamics such as CDT and directed tunneling.

3. Directed migration in 1D

We first investigate the migration of a pair of correlated bosons in a 1D uniform lattice. The relevant tunneling processes include:
- \( |2, 0\rangle_{j,j+1} \leftrightarrow |1, 1\rangle_{j,j+1} \leftrightarrow |0, 2\rangle_{j,j+1} \),
- \( |0, 1\rangle_{j,j+1} \leftrightarrow |1, 0\rangle_{j,j+1} \),

where \( |n_j, n_{j+1}\rangle_{j,j+1} \) indicates a state with \( n_j \) particles at site \( j \) and \( n_{j+1} \) at site \( j + 1 \). In this paper, states with a two-particle occupation like \( |2, 0\rangle_{j,j+1} \) and \( |0, 2\rangle_{j,j+1} \) are called the doublon, whereas states with one-particle

\[ \text{See the appendix.} \]
occupation on nearest sites like \( |1, 1 \rangle_{ij} \) are called the dimer. The external driving frequency takes the first order of resonance (\( \omega = U_0 \)). The corresponding hopping rates (in units of \( f \)) are

(i) \( \mathcal{J}_i(K/\omega - U_i/\omega) |2, 0 \rangle_{ij} \leftrightarrow |1, 1 \rangle_{ij+1} \);

(ii) \( \mathcal{J}_s(K/\omega + U_i/\omega) |1, 1 \rangle_{ij+1} \leftrightarrow |0, 2 \rangle_{ij} \);

(iii) \( \mathcal{J}_0(K/\omega) |0, 1 \rangle_{ij+1} \leftrightarrow |1, 0 \rangle_{ij+1} \).

The control parameters allow for the CDT by setting some of the hopping rates to zeros. The CDT can be understood as a destructive interference effect which freezes the corresponding tunneling. Either (ii) and (iii) or (i) and (iii) tunneling channels are simultaneously frozen, with the remaining channel supporting an oscillation between the doublon and the dimer states. Namely

\( A: |2, 0 \rangle_{ij} \leftrightarrow |1, 1 \rangle_{ij+1} \);

\( B: |1, 1 \rangle_{ij+1} \leftrightarrow |0, 2 \rangle_{ij+1} \).

Here the tunneling condition \( A \) indicates \( \mathcal{J}_i(K/\omega + U_i/\omega) = \mathcal{J}_0(K/\omega) = 0 \) and \( B \) indicates \( \mathcal{J}_i(K/\omega - U_i/\omega) = \mathcal{J}_0(K/\omega) = 0 \). By properly controlling the conditions, we can arrive at the directed transport scheme in the 1D lattice:

\[ |2, 0 \rangle_{ij} \xrightarrow{A} |1, 1 \rangle_{ij+1} \xrightarrow{B} |0, 2 \rangle_{ij+1} \xrightarrow{A} |1, 1 \rangle_{ij+1,j+2}. \]

To verify the associated doublon–dimer oscillations with conditions labeled by \( A \) and \( B \), we carry out a numerical simulation on a 12-site lattice. The time evolutions are obtained by applying the Schrödinger equation with the time-dependent Hamiltonian. Under the resonant condition \( \omega = U_0 = 40/f \), which ensures photon-assisted tunneling, figures 1(a)–(c) show the expected oscillation in the density distributions with time in units of \( T = 2\pi/\omega \). The driving parameters are taken as \( K^A/\omega = 2.4048, U^A_1/\omega = 1.4269 \) in figures 1(a) and (b), which correspond to the condition \( |2, 0 \rangle_{ij} \leftrightarrow |1, 1 \rangle_{ij+1} \), and \( K^B/\omega = -2.4048, U^B_1/\omega = 1.4269 \) in figures 1(c) and (d), which correspond to the condition \( |1, 1 \rangle_{ij+1} \leftrightarrow |0, 2 \rangle_{ij+1} \). The values of the effective hopping rates for both cases are the same:

\[ \mathcal{J}_{\text{res}} = \mathcal{J}_i \left( \frac{K^A}{\omega} - \frac{U^A_1}{\omega} \right) = \mathcal{J}_s \left( \frac{K^B}{\omega} + \frac{U^B_1}{\omega} \right). \]

The analytical results of a simple two level model reveal the sinusoidal oscillation between states {doublon} and {dimer} with frequencies \( \omega_{\text{osc}}^A = \omega_{\text{osc}}^B = \sqrt{2} \mathcal{J}_{\text{res}} \) [36], as shown in figures 1(b)–(d). In general, \( \omega_{\text{osc}}^A \) and \( \omega_{\text{osc}}^B \) can be different when we set \( K/\omega \pm U_i/\omega \) or \( K/\omega \) to other zeros of their related Bessel functions.

The directed migration of the correlated particle pair can be fulfilled by combining the processes of \( A \) and \( B \) to complete a cycle of quantum transition from a doublon state to the doublon with a shift of one lattice site. Figure 2(a) displays the evolution of the density distribution of the correlated pair. We have fixed \( U_i/\omega = 1.4269 \) while periodically modulated the amplitude of the driving field, as shown in figure 3(a). By starting from a doublon state \( |2, 0 \rangle_{ij+1} (j = 6) \), we set \( U_i \) and \( K \) to meet the condition \( A \) in the time interval \( T_A = \pi/2 \omega_{\text{osc}}^A \) to ensure the full evolution into the dimer state \( |1, 1 \rangle_{ij+1} (j = 6) \). \( U_i \) and \( K \) are then switched to meet the condition \( B \) in the time interval \( T_B = \pi/2 \omega_{\text{osc}}^B \) to ensure the system to evolve into \( |2, 0 \rangle_{ij+1} (j = 7) \). We label the above controlling process as \( A(T_A)B(T_B) \). Consequently the correlated pair migrates one lattice site rightward. Repeating the two processes then gives rise to the directed motion of the correlated pair. Similarly, figure 2(b) shows a leftward motion of the pair which is achieved by exchanging the sequence of \( A \) and \( B \) processes as \( B(T_B)A(T_A) \), the corresponding modulation scheme is shown in figure 3(b). In this way, we can freely control the rightward or leftward migration of the correlated pair by properly adjust the driving parameters. Figure 2(c) displays a forth and back motion by alternatively switching from the \( B(T_B)A(T_A) \) to the \( A(T_A)B(T_B) \) at \( t = 3(T_A + T_B) \) and vice versa at \( t = 9(T_A + T_B) \).

Based on this understanding, we can further design a bifurcating quantum motion of the correlated pair by properly choosing the time point of switching conditions between \( A \) and \( B \). As the first driving process \( A \) (or \( B \)) lasts \( T_A/2 \) (or \( T_B/2 \)), the system reaches the superposition state, e.g. \( |\Psi\rangle = 1/\sqrt{2} \) (doublon) + \( 1/\sqrt{2} \) (dimer). Then the density distribution divides into two branches which propagate in the opposite directions, as shown in figure 4(a). Note that the driving modulation shown in figure 4(b) differs from that in figure 3(a) only by taking the first time interval as half of \( T_A \). Each of the branches maintains the doublon–dimer oscillation with a phase difference of \( \pi \). As one of the branches evolves into the doublon state, the other evolves into the dimer state. The strong repulsive on-site interaction makes the doublon a stable quasiparticle. The bare hopping rate is \( I_{\text{off}} = I_{\text{pair}} = (\sqrt{2}/f)^2/U_0 \). It is evidently in contrast to the free coherent quantum walk of the correlated pair in
Figure 1. Doublon–dimer oscillation of a pair of correlated particles in a 1D lattice driven by doubly modulated fields. (a) Evolution of density distribution with the lattice driving amplitude $K/\omega = 2.4048$ and the interaction modulating amplitude $U_{1\omega} = 1.4269$. These parameters allow for the transition between $|2, 0\rangle_{j+1} \leftrightarrow |1, 1\rangle_{j+1}$ while prohibits the transitions between $|1, 1\rangle_{j+1} \leftrightarrow |0, 2\rangle_{j+1}$ and between $|0, 1\rangle_{j+1} \leftrightarrow |1, 0\rangle_{j+1}$. The quantum dynamics starts from the doublon state $|2, 0\rangle_{j+1}$ with $j = 6$. (b) Evolution of the probability density of the dimer state (black) $|c_{11}\rangle^2 = |\langle \Psi | 1, 1\rangle_{j+1} + i|^2$ and the doublon state (blue) $|c_{20}\rangle^2 = |\langle \Psi | 2, 0\rangle_{j+1} + i|^2$. The real curves are numerical results and the circles indicate the analytical results. (c) and (d) are the same as in (a) and (b), respectively, which initiates from the dimer state $|1, 1\rangle_{j+1}$ with $j = 6$. The parameters are $K/\omega = -2.4048$, $U_{1\omega} = 1.4269$. In this case it allows for the transition between $|1, 1\rangle_{j+1} \leftrightarrow |0, 2\rangle_{j+1}$ while prohibits the transitions between $|0, 2\rangle_{j+1} \leftrightarrow |1, 1\rangle_{j+1}$ and between $|0, 1\rangle_{j+1} \leftrightarrow |1, 0\rangle_{j+1}$ as well. The tiny swings from numerical calculations in (b) and (d) are caused by the fast driving and modulation of the system. These micro-breadth oscillations are negligible over large time scale.

Figure 2. Directed migration of a pair of correlated particles in a 1D lattice driven by doubly modulated fields. It is achieved by the doublon–dimer–doublon scheme to move one step per driving cycle. (a) Rightward migration. The modulation of the driving field is illustrated in figure 3(a), which corresponds to the $A(T_A)B(T_B)$ process. The amplitude of modulated interaction remains $U_{1\omega} = 1.4269$. (b) Leftward migration for the $B(T_B)A(T_A)$ process. The related modulation is shown in figure 3(b). (c) A realization of a zigzag motion pattern (rightward and leftward alternatively) via combination of processes in (a) and (b). The turning points are at $t = 3(T_A + T_B)$ and $t = 9(T_A + T_B)$, respectively.
the absence of driving fields which is shown in figure 4(c). The free quantum walk exhibits a mirror-symmetric interference pattern \( \rho_i(t) = |\langle i - I, 2I_{\text{eff}} t \rangle|^2 \) (with \( i \) the lattice site and \( i_0 \) the initial site), which was recently observed by the experiment [37]. The doubly modulated Hubbard model provides new insights into the quantum engineering and the many-particle physics.

4. Directed migration in 2D

Now we turn to study the migration of the correlated pair in the 2D square lattice. It is of practical interests due to the development of the site-resolved probing techniques [10] and the rapid progress of 2D simulations in quantum computing [38, 39]. In addition to the Hubbard interaction modulation, we need to introduce two driving forces in the \( x \)- and \( y \)-direction, respectively, which have different driving frequencies. When the pair moves along the \( x \)-direction, then the hoppings in the \( y \)-direction should be damped in the mean time and vice versa. This can be realized by making use of the off-resonant driving, \( \omega_K = \omega_U = U_0/\text{integer} \). Generally, one can prove the effective hopping rate as

Figure 3. Modulation scheme of the lattice driving field for (a) the \( A(T_A)B(T_B) \) process in figures 2(a) and (b) the \( B(T_B)A(T_A) \) process in figure 3(b). The amplitude of modulated interaction is kept at \( U_0/\omega = 1.4269 \).

Figure 4. (a) Realization of bifurcating quantum motion of a pair correlated particles for a particular phase modulation of the driving fields shown in (b). The two branches result from the first incomplete period of oscillations. It is notable that the correlated pair are not broken in the quantum dynamics. If the probability of finding a doublon in one branch approaches one, then the probability in the other branch approaches zero. In contrast, (c) displays the coherent quantum walk of a pair of correlated particles in the absence of external driving fields (\( K = U_0 = 0 \)).
\[ I_{\text{eff}} = i \sum_{n,m} \left\{ J_n \left( \frac{K}{\omega_K} \right), J_m \left[ -\frac{U_0}{\omega_U}(n_j - n_{j+1} \pm 1) \right] \times, \delta_i \left[ m + n \frac{\omega_K}{\omega_U} - \frac{U_0}{\omega_U}(n_j - n_{j+1} \pm 1) \right] \right\}, \]  

where the \( \pm \) correspond to the leftward and rightward tunneling, respectively. If we take \( \omega_K = \omega_U \), then \( I_{\text{eff}} = i \sum_{n,m} \left\{ J_n \left( \frac{K}{\omega_K} \right), J_m \left[ -U_1(n_j - n_{j+1} \pm 1)/\omega_U \right] \right\} \), \( m = -n + U_0(n_j - n_{j+1} \pm 1)/\omega_K \), which will recover the resonant tunneling case. A quasi-periodic driving lattice also reveal the possibility of vanishing hopping rates \cite{40}. In our situation, when \( \omega_K/\omega_U \) takes an irrational number, e.g., \( \omega_K/\omega_U = \sqrt{2} \), we have \( I_{\text{eff}} = 0 \) for \( n = 0 \) and \( I_{\text{eff}} = J_0(K/\omega_K) J_0[-U_1(n_j - n_{j+1} \pm 1)/\omega_U] \) \( m = U_0(n_j - n_{j+1} \pm 1)/\omega_U \) for \( n = 0 \). Consequently, \( I_{\text{eff}} = 0 \) can be always achieved by setting \( J_0(K/\omega_K) = 0 \). This conclusion establishes for any value of \( U_0 \) and \( U_1 \),

In the 2D lattice, we need to manipulate the migration in one-dimension while suppressing all possible tunnelings in the other dimension. The former process can be realized via the mechanism described previously in the 1D resonant DM case. The later is implemented by the off-resonant driving. We will examine this situation explicitly. For convenience, we specify the direction with resonant (off-resonant) conditions as the \( r \) (\( o \))-direction. The resonant condition along \( r \)-direction should be satisfied, i.e., \( \omega^r_{K} = \omega^r_{U} = U_0 \). In the \( o \)-direction, the hopping rates of \( |1, 0 \rangle_{(i,j)^o} \leftrightarrow |0, 1 \rangle_{(i,j)^o} \) (where \( (i, j)^o \) indicates the nearest sites along \( o \)-direction) can be set to zero by requiring the 0th Bessel function \( J_0(K^o/\omega^o_K) = 0 \) \( (K^o/\omega^o_K = 2.4048 \) in our case). In addition, we also need to damp the transition processes \( |2, 0 \rangle_{(i,j)^o} \leftrightarrow |1, 1 \rangle_{(i,j)^o} \) and \( |0, 2 \rangle_{(i,j)^o} \leftrightarrow |1, 0 \rangle_{(i,j)^o} \). This can be achieved by switching the driving frequency of the \( o \)-direction \( \omega^o_K \) to the off-resonance: \( \omega^o_K = \omega^r_U \). Our previous analysis has proven the possibility of getting \( I_{\text{eff}} = 0 \), regardless of the values of \( U_0 \) and \( U_1 \) and the lattice driving in the \( r \)-direction. Overall, the damping conditions in the \( o \)-direction are \( J_0(K^o/\omega^o_K) = 0 \), \( \omega^o_K/\omega^r_U = p/q \) with both \( p, q \) large positive coprime integers or \( p/q \) the irrational number.

Figure 5 shows the numerical results from time-evolution for a 5 \( \times \) 5 DM lattice system, with parameters \( \omega^r_K = \omega^r_U = U_0 = 40 \), \( \omega^o_K = \sqrt{2} \omega_U, U_1/\omega_U = 1.4269 \), and \( K^o/\omega^o_K = 2.4048 \). The parameters are chosen to meet the resonant condition in the \( r \)-direction, and the suppression of tunneling in the \( o \)-axis. By arranging the driving schemes in the \( r \)-axis and properly exchanging the manipulating conditions for the \( r \) - and \( o \)-axis, we obtained the anticipated pair migration along the route from site A to site F as exhibited in figure 5(a), where the color indicate the time sequence. The evolution of the density distribution along the route of motion is shown in figure 5(b). The alternative doublon–dimer oscillation is similar to that in 1D lattice. The full data are...
exhibited in figure 5(c), where we show the evolution of density distribution at each site in the dial plate graph. These dial plates count the density evolution from $t = 0$ to $t \approx 260T$ ($T = 2\pi/\omega_U$). The density peak at $t = 0$ occurs at site A, then $t \approx 32T$ at site B, and so on.

5. Summary

In summary, we have investigated the doubly driven BHM under the resonant and the off-resonant conditions. The effective hopping rates were obtained explicitly. We proposed a method to realize the directed migration of a pair of correlated particles in the 1D and 2D uniform lattices. The 2D quantum control scheme can be immediately extended to the 3D lattice since the frozen dimension does not affect the other tunneling channels. Our results reveal the possibility of quantum control and transport in the doubly modulated lattices. The ratchet-like motion of a particle pair may form the building block of much richer physics in many particle systems. This platform also provides potential applications in exotic phenomena such as quantum entanglement.

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Appendix. Derivation of the hopping rates

The Hamiltonian is expressed as $\hat{H}(t) = \hat{H}_{\text{hopping}} + \hat{H}_0(t)$, with the hopping term and the time-dependent on-site interaction as well as the lattice driving terms defined in the main text. We expand the states in the Schrödinger picture by using the Fock bases labeled as $|k\rangle \in \{|n_j^{(1)}, n_j^{(2)}, \ldots, n_j^{(M)}\}; |\Psi(t)\rangle = \sum_{k=1}^{M} c_k|k\rangle$, with $n_j^{(k)}$ the number of particles at site $j$ in the state $|k\rangle$ and $M$ the total number of states. By employing the ansatz [35] (with $\hbar$ setting to unity)

$$c_k = a_k(t) \exp\left[-i \int_0^t \left<k|\hat{H}_0(t')|k\right> dt'\right],$$

one gets the equation of motion (EOM) for $a_k(t)$:

$$\frac{i}{\hbar} \frac{\partial a_k}{\partial t} e^{-i \int_0^t \left<k|\hat{H}_0(t')|k\right> dt'} = \sum_k a_k e^{-i \int_0^t \left<k|\hat{H}_0(t')|k\right> dt'} \times \left<k'|\hat{H}_k\right>.$$ (6)

The matrix elements of $H_0$ and $H_{\text{hopping}}$ can be expressed by

$$\langle k'|\hat{H}_0|k\rangle = \frac{1}{2} U\left(t'\right) \sum_{j=1}^{N} n_j^{(k)} (n_j^{(k)} - 1) + K\left(t'\right) \sum_{j=1}^{M} n_j^{(k)}$$ (7)

and

$$\langle k'|\hat{H}_{\text{hopping}}|k\rangle = \langle k'|\hat{H}^+|k\rangle + \langle k'|\hat{H}^-|k\rangle,$$ (8)

with $H^+ = \sum_j b_j^+ b_{j+1}^+$ and $H^- = \sum_j b_j b_{j+1}^+$, and their matrix elements

$$\langle k'|\hat{H}^+|k\rangle = -J \sum_{j=1}^{N-1} \left[ \delta_{j+1}^+ \cdot \sqrt{\left(n_j^{(k)} + 1\right)n_{j+1}^{(k)}} \right],$$ (9)

$$\langle k'|\hat{H}^-|k\rangle = -J \sum_{j=1}^{N-1} \left[ \delta_{j-1}^- \cdot \sqrt{\left(n_{j+1}^{(k)} + 1\right)n_j^{(k)}} \right],$$ (10)

where

$$\delta_{j}^{\pm} \equiv \delta_{n_j^{(k)}n_{j+1}^{(k)}\pm1} \cdot \delta_{n_{j+1}^{(k)}n_j^{(k)}\pm1} \cdot \prod_{i=j,j+1} \delta_{n_j^{(k)}n_i^{(k)}},$$ (11)

Substituting equations (7) and (8) into the EOM (6), we obtain a set of differential equations:

$$i \frac{\partial a_k(t)}{\partial t} = \sum_k \langle k'|H^+|k\rangle h_k^+ (t) a_k (t) + \sum_k \langle k'|H^-|k\rangle h_k^- (t) a_k (t),$$ (12)
where we have introduced the phase factors

\[ h_k^\pm(t) = e^{i(n_j - n_{j+1})\pm 1} \int_0^t U(r')dr' \int_0^r k(r')dr'. \]  

(13)

One gets by taking \( U(t) = U_0 + U_1 \cos(\omega t) \), \( K(t) = K \cos(\omega t) \),

\[ h_k^\pm(t) = e^{i(n_j - n_{j+1})\pm 1} \left( e^{iU_1/\omega \sin(\omega t)} - e^{-iU_1/\omega \sin(\omega t)} \right). \]  

(14)

The rapid oscillation in equation (14) indicates that an average over \( T = 2\pi/\omega \) should be considered. By making use of the formula \( e^{ix \sin(\omega t)} = \sum_{j=-\infty}^{\infty} J_j(x) e^{i j \omega t} \) and \( \frac{1}{T} \int_0^T e^{i(j \omega t - U)} = \delta(j \omega - U) \), it follows that

\[ h_k^\pm(t) = \sum_{j=-\infty}^{\infty} J_j \left( \frac{K_0}{\omega} - \frac{U_1}{\omega} (n_j^k - n_{j+1}^k + 1) \right) \delta(j \omega - U_1 (n_j^k - n_{j+1}^k + 1)). \]  

(15)

The time-average is taken over the high frequency parts while treating the lower frequency parts of the hopping terms as quasi-static. The resonance occurs when

\[ \mu = U_1 \left( n_j^k - n_{j+1}^k + 1 \right) = \text{integer}. \]  

(16)

Thus equation (12) reduce to

\[ i \frac{\partial a_k(t)}{\partial t} = \sum_k (k|J_k(Z^+)H^+|k)a_k(t) + \sum_k \langle k|J_k(Z^-)H^-|k \rangle a_k(t), \]  

(17)

where \( Z^\pm = K_0/\omega - U_1/\omega (n_j^k - n_{j+1}^k + 1) \) and \( \mu(\nu) = U_1/\omega (n_j^\nu - n_{j+1}^\nu + 1) \). The set of equation (17) is mathematically equivalent to the Shrödinger equation for \( H_{\text{hopping}} \), with renormalized tunneling rates

\[ f^\pm = J\mu(\nu) \left( \frac{K_0}{\omega} - \frac{U_1}{\omega} (n_j^k - n_{j+1}^k + 1) \right), \]  

which are represented in the main text.

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