Liouvillian Solutions of Schrödinger Equation with Polynomial Potentials using Gröbner Basis

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Abstract. The main aim of this paper is the presentation of a new methodology to obtain Liouvillian solutions of Schrödinger equation with quasi-solvable polynomial potentials through the using of differential Galois theory and Gröbner basis. We illustrate these results by the computing of polynomial potentials of degree 4, 6, 8, 10, 12, 14. We start the paper with the analysis of some transformations for polynomial and differential equations. The paper ends with the appendix that contains some tables to illustrate the completing squares in polynomials of degree 4, 6, 8, 10, 12 and 14.

Keywords: Differential Galois theory, Gröbner basis, quasi-solvable potentials, Liouvillian solutions, Schrödinger Equation.

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Introduction

Differential Galois theory is the Galois theory in the context of linear differential equations, see [10, 11, 15] for theoretical aspects and see [1, 2, 14] for applications of Differential Galois theory in classical and quantum mechanics.

Quasi-solvable models in quantum mechanics is a recent research topic for people working in mathematical physics, see [3, 5, 7, 9, 13]. This paper is a significant improvement of [16], which was inspired in [1, 2]. The theoretical background used in this paper correspond to [4, 6, 8, 12].

In this paper we present a new computational method to obtain the explicit values of energy and wave functions in an Schrodinger equation with polynomial potential of degree even. The method includes the using of Gröbner basis.

1. Theoretical Background

In this section we will discuss some important concepts on differential Galois theory in order to understand what does a Liouvillian function means.
A derivation on a field $K$ is a map $': K \to K$ satisfying the following properties, for all $a, b \in K$:

1. $(a + b)' = a' + b'$ and
2. $(ab)' = ab' + a'b$.

A field $K$ equipped with a derivation is named a differential field. A differential field $F \supset K$ is a differential extension of the differential field $K$ if the derivation of $F$ restricts to a derivation in $K$. Throughout this work we will only consider fields of characteristic zero.

**Definition 1.1.** Let $K$ be a differential field. An element $c \in K$ is called a constant if $\partial(c) = 0$

The set of all constants of a field $K$ is a subfield of $K$, this field is called the field of constants and we will denote by $C_K$.

There exist an analogue concept of the splitting field of a polynomial on the differential Galois theory, and it is called the Picard-Vessiot extension of a homogeneous linear differential equation (HLDE). Given an HLDE $L(y) = 0$ of order $n$ over a differential field $K$, a differential extension $F$ is a Picard-Vessiot extension if:

1. $C_F = C_K$,
2. The $C_F$-vector space $V$ of solutions of $L(y) = 0$ has dimension $n$,
3. $F$ is generated over $K$ by the solutions of $L(y) = 0$.

Finally, we are able to describe a Liouvillian function. Let $(K,')$ be a differential field. An extension $L/K$ is said Liouvillian if $C_K = C_L$ and there exist a tower of differential fields $K = K_0 \subset K_1 \subset \cdots \subset K_n = L$ such that $K_i = K_{i-1}(t_i), i \in \{0, 1, \ldots, n\}$ where

1. $t_i \in K_{i-1}$, i.e., $t_i$ is an integral, or
2. $t_i \neq 0$ and $\frac{t_i}{t_i} \in K_{i-1}$, i.e., $t_i$ is an exponential, or
3. $t_i$ is an algebraic element over $K_{i-1}$.

**Definition 1.2.** Let $F/K$ be the Picard-Vessiot extension associated to the HLDE $L(y) = 0$. We say that the solutions of $L(y) = 0$ are Liouvillian if there exist a Liouvillian extension $L/K$ such that $K \subset F \subset L$.

**1.1. Polynomial Transformations**

**Exercise 1.3.** Every polynomial $f(x) = ax^2 + bx + c$ can be written in the form $q(\tau) = \tau^2 + k$

**Solution:**
Let $f(x)$ and $q(\tau)$ be the polynomials on the assumption. Now substitute $\tau = \epsilon x + \mu$ into $q(\tau)$ to get

\[(\epsilon x + \mu)^2 + k = \epsilon^2 x^2 + 2\epsilon \mu x + \mu^2 + k.\]  \(1\)

Comparing the right-hand side of the above equation with $f(x)$, the coefficients $a, b, c$ could be found.

$\epsilon = \pm \sqrt{a}$. Suppose $\epsilon = \sqrt{a}$, the other case is similar. Then $\mu = \frac{b}{2\sqrt{a}}$ and $k = c - \mu^2 = c - \frac{b^2}{4a} = -\frac{4ac-b^2}{4a}$. Thus $f(x) = ax^2 + bx + c$ can be written $q(\tau) = \tau^2 - \frac{4ac-b^2}{4a}$. 
This exercise can be generalized in order to transform any polynomial of degree $n$ into a polynomial of degree $n$ without the $(n-1)$-th term, this fact will be prove in next proposition.

**Proposition 1.4.** A polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ can be written in the form $q(\tau) = \tau^n + \sum_{j=0}^{n-2} b_j \tau^j$

**Proof.** Let $\tau = \epsilon x + \mu$ be. Then $q(\tau) = (\epsilon x + \mu)^n + \sum_{j=0}^{n-2} b_j (\epsilon x + \mu)^j$. Using the Binomial theorem $q(\tau) = \sum_{i=0}^{n} \binom{n}{i} (\epsilon x)^{n-i} \mu^i + \sum_{j=0}^{n-2} b_j \sum_{k=0}^{j} (\epsilon x)^{j-k} \mu^k$, if we expand (1), we get

$$\sum_{i=0}^{n} \binom{n}{i} (\epsilon x)^{n-i} \mu^i = \left(\sum_{i=0}^{n} \binom{n}{i} (\epsilon x)^{n-i} \mu^i\right) + \sum_{j=0}^{n-2} b_j \left(\sum_{k=0}^{j} (\epsilon x)^{j-k} \mu^k\right)$$

Grouping and factorizing $b_j$ we obtain.

$j = 0 \rightarrow b_0$

$j = 1 \rightarrow b_1 (\binom{1}{0} \epsilon x + \binom{1}{1} \mu)$

$j = 2 \rightarrow b_2 (\binom{2}{0} \epsilon x)^2 + (\binom{2}{1} \epsilon x \mu + (\binom{2}{2} \mu^2)$

$j = 3 \rightarrow b_3 (\binom{3}{0} \epsilon x)^3 + (\binom{3}{1} \epsilon x)^2 \mu + (\binom{3}{2} \epsilon x \mu^2 + (\binom{3}{3} \mu^3$

\[j = n - 4 \rightarrow b_{n-4}(\binom{n-4}{0} \epsilon x)^{n-4} + (\binom{n-4}{1} \epsilon x)^{n-5} \mu + \cdots + (\binom{n-4}{n-8} \epsilon x)^{n-8} \mu^{n-8} + (\binom{n-4}{n-5} \epsilon x^3 \mu^{n-7} + (\binom{n-4}{n-6} \epsilon x)^2 \mu^{n-6} + (\binom{n-4}{n-5} \epsilon x) \mu^{n-5} + (\binom{n-4}{n-4} \mu^{n-4})$

\[j = n - 3 \rightarrow b_{n-3}(\binom{n-3}{0} \epsilon x)^{n-3} + (\binom{n-3}{1} \epsilon x)^{n-4} \mu + (\binom{n-3}{2} \epsilon x)^{n-5} \mu^2 + \cdots + (\binom{n-3}{n-7} \epsilon x)^4 \mu^{n-7} + (\binom{n-3}{n-6} \epsilon x)^3 \mu^{n-6} + (\binom{n-3}{n-5} \epsilon x)^2 \mu^{n-5} + (\binom{n-3}{n-4} \epsilon x) \mu^{n-4} + (\binom{n-3}{n-3} \mu^{n-3})$

\[j = n - 2 \rightarrow b_{n-2}(\binom{n-2}{0} \epsilon x)^{n-2} + (\binom{n-2}{1} \epsilon x)^{n-3} \mu + (\binom{n-2}{2} \epsilon x)^{n-4} \mu^2 + (\binom{n-2}{3} \epsilon x)^{n-5} \mu^3 + \cdots + (\binom{n-2}{n-6} \epsilon x)^4 \mu^{n-6} + (\binom{n-2}{n-5} \epsilon x)^3 \mu^{n-5} + (\binom{n-2}{n-4} \epsilon x)^2 \mu^{n-4} + (\binom{n-2}{n-3} \epsilon x) \mu^{n-3} + (\binom{n-2}{n-2} \mu^{n-2})

The coefficient of each $x^k$ in $q(\epsilon x + \mu)$ can be obtained from the above computation. Then comparing it with $f(x)$, we find $e^n = a_n$ and $\mu = \frac{a_{n-1}}{(n-1)! ^{n-1}}$ as well as $b_0 = a_0 - (\sum_{i=1}^{n-2} b_i \mu^i + \mu^n) \nu b_{n-2} = \frac{a_{n-2}}{e^n-2} - \mu^2(\frac{n}{2})$. The other coefficient could be find by the
formula:

\[ b_{n-k} = \frac{a_{n-k}}{c^{n-k}} - \sum_{i=1}^{k-2} \binom{n-k+i}{i} \mu^i b_{n-k+i} - \binom{n}{k} \mu^k, \text{ for } k = 3, 4, \ldots, n - 1. \]  

(2)

\[ P_{2n}(x) = x^{2n} + \sum_{k=0}^{2n-1} a_k x^k = \left( x^n + \sum_{k=0}^{n-1} b_k x^k \right)^2 + \sum_{k=0}^{n-1} c_k x^k. \]  

(3)

Proof. First, note:

\[ (x^n + \sum_{k=0}^{n-1} b_k x^k)^2 + \sum_{k=0}^{n-1} c_k x^k = x^{2n} + 2 \sum_{k=0}^{n-1} b_k x^{k+n} + \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} b_k b_{i+k} + \sum_{k=0}^{n-1} c_k x^k. \]  

(4)

Comparing the right-hand side of above equation with \( x^{2n} + \sum_{k=0}^{2n-1} a_k x^k \) we will get the coefficients, which allow us rewrite \( P_{2n}(x) \), namely:

\[ b_{n-1} = \frac{a_{2n-1}}{2}, b_{n-2} = \frac{a_{2n-2} - b_0^2}{2}, b_{n-3} = \frac{a_{2n-3} - 2b_{n-1} b_{n-2}}{2}, \text{ in general for } 2 \leq k \leq n \text{ and } i, j \in \mathbb{N}, i, j \leq n - 1 \text{ we have the next formulas} \]

\[ b_{n-k} = \frac{a_{2n-k} - \sum_{i,j \leq n-1, i+j=2n+k} b_i b_j}{2}, \]  

(5)

and \( c_0 = a_0 - b_0^2, c_1 = a_1 - 2b_0 b_1, b_2 = a_2 - 2b_0 b_2 - 2b_1^2, \ldots, c_k = a_k - \sum_{i,j \leq n-1, i+j=k} b_i b_j \) for \( k = 0, 1, 2, \ldots, n - 1 \).

\[ \text{Remark 1.6. If } a_{2n-1} \text{ in } P_{2n}(x) \text{ is null the formulas will have a bit change on index } i \text{ and } j, \text{ since these should be strictly smaller than } n - 1 \]

1.2. Transformations on Differential Equation

Lemma 1.7. Let \( f \) and \( g \) analytic functions over \( \mathbb{C} \). Then

\[ (f g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} \]  

(6)

Proof. Let \( A = \{ n \in \mathbb{N} : (f g)^{(n)} = \sum_{k=0}^{n} f^{(n-k)} g^{(k)} \} \) be. It is clear that \( 0 \in A \) and for \( n = 1 \) the Leibniz’s rule is satisfied. Hence \( 1 \in A \). Now, suppose the assumption is true for \( n = m \) and let’s prove it is true for \( n = m + 1 \). Indeed
\[(fg)^{(m+1)} = \partial((fg)^{(m)}) = \partial\left( \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)} g^{(k)} \right) \]

\[
= \sum_{k=0}^{m} \binom{m}{k} (f^{(m-k+1)} g^{(k)} + f^{(m-k)} g^{(k+1)}) \\
= \sum_{k=0}^{m} \binom{m}{k} f^{(m-k+1)} g^{(k)} + \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)} g^{(k+1)} \\
= f^{(m+1)} g + \sum_{k=1}^{m} \binom{m}{k} f^{(m-k+1)} g^{(k)} + \sum_{k=0}^{m-1} \binom{m}{k} f^{(m-k)} g^{(k+1)} + f g^{(m+1)} \\
= f^{(m+1)} g + \sum_{k=1}^{m} \binom{m}{k} f^{(m-k+1)} g^{(k)} + \sum_{k=1}^{m-1} \binom{m}{k-1} f^{(m-k)} g^{(k)} + f g^{(m+1)} \\
= f^{(m+1)} g + \sum_{k=1}^{m} \binom{m+1}{k} f^{(m-k)} g^{(k)} + f g^{(m+1)} = \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(m+1-k)} g^{(k)} \\
\]

Then \(m + 1 \in A\). We conclude that \(A = \mathbb{N}\) and therefore \((fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}\) for every \(n \in \mathbb{N}\) 

\[\textbf{Theorem 1.8.} \textit{The equation } z^{(n)} + a_{n-1} z^{(n-1)} + \ldots + a_1 z' + a_0 z = 0 \textit{ with } a_i \in \mathbb{C}(x) \textit{ can be written in the form } y^{(n)} + b_{n-2} z^{(n-2)} + \ldots + b_1 y' + b_0 y = 0 \textit{ for all } b_i \in \mathbb{C}(x). \]

\[\textit{Proof.} \textit{Let } z^{(n)} + a_{n-1} z^{(n-1)} + \ldots + a_1 z' + a_0 z = 0 \textit{ and } y^{(n)} + b_{n-2} z^{(n-2)} + \ldots + b_1 y' + b_0 y = 0 \textit{ be such as in the assumption. Moreover let } y = z v \textit{ be. Then we get } (z v)^{(n)} + \sum_{i=0}^{n-2} b_i (z v)^{(i)} = 0 \textit{ and by lemma above}
\]

\[
\sum_{j=0}^{n} \binom{n}{j} z^{(n-j)} v^{(j)} + \sum_{i=0}^{n-2} b_i \sum_{k=0}^{i} \binom{i}{k} z^{(i-k)} v^{(k)} = 0 \\
\frac{1}{v} \sum_{j=0}^{n} \binom{n}{j} z^{(n-j)} v^{(j)} + \frac{1}{v} \sum_{i=0}^{n-2} b_i \sum_{k=0}^{i} \binom{i}{k} z^{(i-k)} v^{(k)} = 0.
\]

Expanding (1), we get

\[
\frac{1}{v} \sum_{j=0}^{n} \binom{n}{j} z^{(n-j)} v^{(j)} = \frac{n}{v} z^{(n-1)} v' + \frac{n}{v} z^{(n-2)} v'' + \frac{n}{v} z^{(n-3)} v''' + \frac{n}{v} z^{(n-4)} v'''' + \\
\ldots + \frac{n}{v} z^{(4)} v^{(4)} + \frac{n}{v} z^{(3)} v^{(n-3)} v'' + \frac{n}{v} z^{(2)} v^{(n-2)} v''' + \frac{n}{v} z^{(1)} v^{(n-1)} v'' + \frac{n}{v} z^{(n)} v^{(n)}.
\]
Now in (2) we search the factors by $b_i$ for $i = 0, 1, 2, \ldots, n - 2$

$j = 0 \rightarrow b_0 z$

$j = 1 \rightarrow b_1 \frac{1}{v}(z'v + vz')$

$j = 2 \rightarrow b_2 \frac{1}{v}(z''v + \left(\frac{2}{1}\right) z'v' + vz'')$

$j = 3 \rightarrow b_3 \frac{1}{v}(z'''v + \left(\frac{3}{1}\right) z''v' + zv''')$

\[\vdots\]

\[j = n - 3 \rightarrow b_{n-3} \frac{1}{v}(z^{(n-3)}v + \left(\frac{n-3}{1}\right) z^{(n-4)}v' + \left(\frac{n-4}{2}\right) z^{(n-5)}v'' + \ldots + \left(\frac{n-3}{n-6}\right) z^{(n-6)}v^{(n-6)} + \left(\frac{n-3}{n-5}\right) z''v^{(n-5)} + \left(\frac{n-4}{n-3}\right) z'v^{(n-4)} + zv^{(n-3)}\]

\[j = n - 2 \rightarrow b_{n-2} \frac{1}{v}(z^{(n-2)}v + \left(\frac{n-2}{1}\right) z^{(n-3)}v' + \left(\frac{n-3}{2}\right) z^{(n-4)}v'' + \ldots + \left(\frac{n-2}{n-5}\right) z^{(n-5)}v^{(n-5)} + \left(\frac{n-2}{n-4}\right) z''v^{(n-4)} + \left(\frac{n-3}{n-3}\right) z'v^{(n-3)} + zv^{(n-2)}\]

Now as in the procedure of Theorem 2.1.1, we compare $(zv)^{(n)} + \sum_{i=0}^{n-2} b_i(zv)^{(i)} = 0$

with $z^{(n)} + a_{n-1}z^{(n-1)} + \ldots + a_1z' + a_0z = 0$ and find $\frac{v'}{v} = \frac{a_{n-1}}{a_n} = \frac{1}{n}a_{n-1}$. Hence $v = e^{\frac{x}{n}}a_{n-1}$ and $b_j, 0 \leq j \leq n - 2$ are:

\[b_{n-2} = a_{n-2} - \left(\frac{n}{2}\right) \frac{v'}{v}\]

\[b_{n-3} = a_{n-3} - \left(\frac{n-2}{1}\right) \frac{v'}{v} b_{n-2} - \left(\frac{n}{3}\right) \frac{v''}{v}\]

\[\vdots\]

\[b_{n-k} = a_{n-k} - \sum_{i=1}^{k-2} \left(\frac{n-k+i}{i}\right) \frac{v^{(i)}}{v} b_{n-k+i} - \left(\frac{n}{k}\right) \frac{v^{(k)}}{v}\]

\[b_0 = -\sum_{i=1}^{n-2} b_i \frac{v^{(i)}}{v} = \frac{v^{(n)}}{v}.\]

\[\Box\]

**Proposition 1.9** (Hard algebraization). The second order differential equation with coefficients in $\mathbb{C}(z)$

\[\partial^2_z y + a(z)\partial_z y + b(z)y = 0, \quad a(z), b(z) \in \mathbb{C}(z) \quad (7)\]

is algebraizable by the change $z = \theta(x)$ and its algebraic form is

\[\partial^2_{\theta} r + (a(\theta(x))\partial_{\theta} \theta(x) - \partial_{\theta}^2 \theta(x))\partial_{\theta} r + b(\theta(x))(\partial_{\theta} \theta(x))^2 r = 0, \quad (8)\]

where $r = y \circ z$.

**Proof.** Let $z = \theta(x)$ be. By the change of variable we have $\partial_z y \partial_z \theta = \partial_r r$ that is the
same as \( \partial_z y = \frac{\partial x}{\partial \theta} \). Then differentiating again \( \partial_z^2 y \partial_x \theta = \partial_x (\frac{\partial x}{\partial \theta}) \), therefore
\[
\partial_z^2 y = \frac{1}{\partial_x \theta} \left( \frac{\partial_x^2 r \partial_x \theta - \partial_x r \partial_x^2 \theta}{(\partial_x \theta)^2} \right) = \frac{\partial_x^2 r}{(\partial_x \theta)^2} - \partial_x r \frac{\partial_x^2 \theta}{(\partial_x \theta)^3}. \tag{9}
\]

Replacing \( \partial_z^2 y \) and \( \partial_z y \) into (7) we get
\[
\frac{\partial_x^2 r}{(\partial_x \theta)^2} - \left( \frac{a(\theta)}{\partial_x \theta} - \frac{\partial_x^2 \theta}{(\partial_x \theta)^3} \right) \partial_x r + b(\theta) r = 0. \tag{10}
\]

Now if we multiply by \((\partial_x \theta)^2\) we will get
\[
\partial_x^2 r + (a(\theta) \partial_x \theta - \frac{\partial_x^2 \theta}{\partial_x \theta}) \partial_x r + b(\theta) (\partial_x \theta)^2 r = 0. \tag{11}
\]
with \( r = y \circ z \).

**Example 1.10.** Consider the equation
\[
\partial_x^2 r - \frac{1}{x(\ln x + 1)} \partial_x r - (\ln x + 1)^2 = 0. \tag{12}
\]

Let’s find the algebraic form of this equation, indeed, let \((\partial_x \theta)^2 = (\ln x + 1)^2\) be. Hence
\[ z = \theta(x) = \int (\ln x + 1) dx = \int \ln x dx + \int dx = x \ln x \] moreover we can see that
\[ b(\theta) = -1. \] Let’s see what is \( -\frac{\partial_x^2 \theta}{\partial_x \theta} \). First at all \( \partial_x^2 \theta = \partial_x (\ln x + 1) = \frac{1}{x} \). It follows \( -\frac{\partial_x^2 \theta}{\partial_x \theta} = \frac{1}{x(\ln x+1)} \). Now compare the coefficient of \( \partial_x r \) into equation (12) with the respective into equation (8) we shall get
\[ a(\theta)(\ln x + 1) - \frac{1}{x(\ln x + 1)} = \frac{1}{x(\ln x + 1)}. \]

It follows \( a(\theta) = 0 \) and the algebraic form of (12) is
\[ \partial_x^2 y - y = 0. \]

Whose fundamental system of solutions is \( \langle e^x, e^{-x} \rangle \), then, the fundamental system of solutions of equation (12) is \( \langle e^{x \ln x}, e^{-x \ln x} \rangle \), or \( \langle x^x, x^{-x} \rangle \) likewise.

Due to Theorem 1.9 is possible to study the behavior of equation (7) when \( z = \infty \). By making the change of variable \( z = \frac{1}{x} \) and analyzing the behavior of (8) when \( x = 0 \). In this way, \( z = \infty \) is a singular regular point of (7), if \( x = 0 \) is a singular regular point of (8).
1.2.1. Hamiltonian algebrization

**Definition 1.11.** A change \( z = z(x) \) is said to be hamiltonian if \((z(x), \partial_x z(x))\) is a solution curve of the system (hamiltonian)

\[
\begin{align*}
\partial_z z &= \partial_w H \\
\partial_x w &= -\partial_z H
\end{align*}
\]

where \( H = H(z, w) = \frac{w^2}{2} + V(z), \ V(z) \in \mathbb{C}(z) \)

**Proposition 1.12.** The equation

\[
\partial_x^2 r = q(x) r \tag{13}
\]

is algebrizable through a Hamiltonian change \( z = z(x) \) if and only if there exist \( f, \alpha \) such that

\[
\frac{\partial \alpha}{\alpha}, \frac{f}{\alpha} \in \mathbb{C}(z),
\]

where \( f(z(x)) = q(x) \) and \( \alpha(z) = 2(H - V(z)) = (\partial_x z)^2 \). Furthermore, the algebraic form of the equation \( \partial_x^2 r = q(x) r \) is

\[
\partial_y^2 + \frac{1}{2} \frac{\partial \alpha}{\alpha} \partial_y - \frac{f}{\alpha} y = 0.
\]

**Proof.** Let \( z = z(x) \) be a hamiltonian change of variable for the equation \( \partial_x^2 r = q(x) r \). Then

\[
\partial_x z = \partial_r H = r,
\]

therefore

\[
\partial_x^2 z = \partial_x r = -\partial_x V(z).
\]

There also exist \( f, \alpha \) such that \( f(z(x)) = q(x) \) and \( \alpha(z) = 2(H - V(z)) = r^2 = (\partial_x z)^2 \). Now applying hard algebrization we get \( -f(z) = b(z)(\partial_x z)^2 \) which is equivalent to \( b(z) = \frac{-f(z)}{(\partial_x z)^2} = \frac{-f(z)}{\alpha(z)} \). Also \( a(z)\partial_x z - \frac{\partial^2 z}{\partial z} = 0 \) or \( a(z) = \frac{\partial^2 z}{(\partial_x z)^2} \). Since \( \partial_x \alpha(z) = -2\partial_x V(z) = 2\partial_x^2 z \) hence \( a(z) = \frac{1}{2} \frac{\partial \alpha(z)}{\alpha} \). Now, since \( \frac{\partial \alpha(z)}{\alpha}, \frac{-f(z)}{\alpha(z)} \in \mathbb{C}(x) \) and \( f(z(x)) = q(x) \) the algebraic form of (13) is

\[
\partial_y^2 + \frac{1}{2} \frac{\partial \alpha}{\alpha} \partial_y - \frac{f}{\alpha} y = 0.
\]

\[\square\]

**Example 1.13.** Consider the differential equation

\[
\partial_x^2 r = \sqrt{1 + x^2 + x^2} r. \tag{14}
\]
Let \( z(x) = \sqrt{1 + x^2} \) be, so \( \partial_x z = \frac{x}{\sqrt{1+x^2}} \). Then \((\partial_x z)^2 = \frac{z^2-1}{z^2} = \alpha(z)\), hence \( z(x) = \sqrt{1 + x^2} \) is a hamiltonian change of variable. Now \( f(z(x)) = \frac{z^2-1+z}{z^2-1} \in \mathbb{C}(z) \) (15)

on the other hand

\[
\partial_z \ln \alpha(z) = \frac{\partial_z \alpha(z)}{\alpha(z)} = \partial_z(\ln(z^2 - 1) - \ln z^2) = \frac{2z}{z^2-1} - \frac{2}{z} \in \mathbb{C}(z) \quad (16)
\]

Then the algebraic form of \( \partial_x^2 r = \frac{\sqrt{1+x^2}+x^2}{1+x^2}r \) is

\[
\partial_x^2 y + \left( \frac{z}{z^2-1} - \frac{1}{z} \right) \partial_x y - \frac{z^2-1+z}{z^2-1} y = 0 \quad (17)
\]

**Proposition 1.14.** Consider \( g(z_1, \ldots, z_n) \) where \( z_i = e^{\lambda_i x}, \lambda_i \in \mathbb{C} \). The equation \( \partial_x r = g(x)r \) is algebrizable if and only if \( \frac{\lambda_i}{\lambda_j} \in \mathbb{Q}^* \) with \( 1 \leq i, j \leq n \), and \( g \in \mathbb{C}(z) \). Furthermore \( \lambda_i = c_i \lambda \) where \( \lambda \in \mathbb{C}^* \) and \( c_i \in \mathbb{Q}^* \) by the hamiltonian change

\[
z = e^{\lambda x} \text{ where } c_i = \frac{p_i}{q_i}, p_i, q_i \in \mathbb{Z}^*, (p_i, q_i) = 1 \text{ and } [q_1, q_2, \cdots, q_n] = q.
\]

The algebraic form of \( \partial_x r = g(x)r \) is

\[
\partial_x^2 y + \frac{1}{z} \partial_x y - q^2 g(z^{m_1 \cdots m_n}) = 0 \text{ with } m_i = \frac{p_i}{q_i}, f(z(x)) = r(x).
\]

**Proof.** Suppose \( \frac{\lambda_i}{\lambda_j} \in \mathbb{Q}^* \). There exist \( \lambda \in \mathbb{C}^* \) and \( c_i \in \mathbb{Q}^* \) such that \( \lambda_i = c_i \lambda \)

\[
e^{\lambda_i x} = e^{c_i \lambda x} = e^{\frac{p_i}{q_i} \lambda x} \quad \text{where } p_i, q_i \in \mathbb{Z}^*, (p_i, q_i) = 1 \text{ and } [q_1, q_2, \cdots, q_n] = q
\]

\( \frac{p_i}{q_i} \in \mathbb{Z} \) since \( [q_1, q_2, \cdots, q_n] = q \) hence we can choose \( z = z(x) = e^{\lambda x} \) as our Hamiltonian change of variable. We see \( f(z(x)) = g(z^{m_1}, \ldots, z^{m_n}) \) where every \( m_i = q_i \), further \( \alpha(z) = (\partial_x z)^2 = \frac{\lambda^2 z^2}{q^2} \). Then

\[
\partial_z \ln \alpha(z) = \frac{\partial_z \alpha(z)}{\alpha(z)} = \partial_z(\ln \frac{\lambda^2}{q^2} + \ln z^2) = \frac{2}{z} \quad (18)
\]

Then the algebraic form of \( \partial_x r = g(x)r \) is given by

\[
\partial_x^2 y + \frac{1}{z} \partial_x y - q^2 g(z^{m_1 \cdots m_n}) \frac{\lambda^2 z^2}{q^2} y = 0 \quad (19)
\]
Example 1.15. Consider the differential equation $\partial^2_x r = b(\sin^2(x) + 1)r$. Since $-e^{ix} = -\cos x - i \sin x$ and $e^{-ix} = \cos x - i \sin x$, adding the above two equations, we get

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$  \hfill (21)

We identify $\lambda_1 = i$ and $\lambda_2 = -i$. Since $\frac{i}{-i} = -1 \in \mathbb{Q}$, the differential equation $\partial^2_x r = b(\sin^2(x) + 1)r$ is algebrizable by the change $z(x) = e^{ix}$, note that $\lambda = i$ and $q = 1$. Now $f(z(x)) = g(x) = b(\sin^2(x) + 1) = b\left(\frac{(z-\zbar)^2}{4} + 1\right)$. Then

$$f(z(x))/z^2 = b\left(\frac{(z-\zbar)^2}{4} + 1\right) = b\left(\frac{(z-\zbar)^2 - 4}{-4z}\right).$$ \hfill (22)

Then the algebraic form of $\partial^2_x r = b(\sin^2(x) + 1)r$ is

$$\partial^2_y z + \frac{1}{z} \partial_z y + \frac{b[(z-\zbar)^2 - 4]}{-4z} y = 0$$ \hfill (23)

Corollary 1.16. The differential equation

$$\partial^2_x r = \left(\sum_{k=0}^{n} c_k x^k\right)r, \quad c_k \in \mathbb{C}$$  \hfill (24)

is algebrizable by the change $z(x) = \mu x$ with $\mu \in \mathbb{C}$.

Proof. Let $z(x) = \mu x$ be, $\mu \in \mathbb{C}$. Then $\partial_x z = \mu$, it follows immediately that $\alpha(z) = (\partial_z z)^2 = \mu^2 \in \mathbb{C}(z)$, therefore $z = \mu x$ is a hamiltonian change and the algebraic form of (24) and is given by

$$\partial^2_y z - \sum_{k=0}^{n} \frac{c_k x^k}{\mu^2} y = \partial^2_y z - \sum_{k=0}^{n} \frac{c_k x^k}{\mu^2} y = 0$$ \hfill (25)

then, for $\mu = \sqrt[n]{\sqrt{\mu}}$.

$$\partial^2_y y - \left(\sum_{k=0}^{n-1} \frac{c_k x^k}{\mu^{k+2}}\right) y = 0.$$ \hfill (26)

\hfill $\Box$

1.3. Schrödinger Equation

The Schrödinger Equation is a differential equation that describes the time-evolution of the system through the wave function. This equation can be obtained in this way:

In Quantum mechanics, the corresponding hamiltonian operator to the energy of a system usually is expressed in the form:
\[ H = T + V(\vec{x}) \]  

(27)

where \( V(\vec{x}) \) is the potential energy and \( T = -\frac{\hbar^2}{2m} \nabla^2 \) the operator associated with the kinetic energy. Hence the Hamiltonian operator in non-relativistic terms of a particle of mass \( m \) is given by

\[ H = -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{x}) \]  

(28)

And The Schrödinger Equation is \( H\psi = E\psi \), where \( \psi \) is the wave function and \( E \) is the level of energy.

Henceforth, we will consider the one-dimensional time-independent Schrödinger Equation. Furthermore, for purposes of this work we will only consider polynomial potentials which we will write in the form:

\[ V(x) = \sum_{i=0}^{n} a_i x^i \]  

(29)

**Proposition 1.17.** The equation

\[ -\hbar^2 \frac{\partial^2}{2m} \psi + V(x)\psi = E\psi \]  

(30)

where \( \tilde{\psi}(z(x)) = \psi(x) \), \( \tilde{V}(z(x)) = V(x) \), \( \tilde{V}(z) = z^n + q_{n-1}z^{n-1} + \cdots + q_0 \), \( q_i = \frac{a_i^2}{\epsilon + \epsilon^2} \), 

\[ \epsilon = \sqrt{n+2/\frac{a_{2n}}{m^2}} \]  

and \( \lambda = \frac{E^2}{m^2} \).

**Corollary 1.18.** The equation (30) can be written as

\[ \pounds^2 \phi = (w^n + \beta_{n-2}w^{n-2} + \cdots + \beta_0 - \lambda)\phi \]  

(31)

where \( w = z + \mu, \mu \in \mathbb{C}, \phi(w(z(x))) = \psi(x) \) and \( \beta_i (i = 0, 1, 2, ..., n-2) \) are the coefficients found by applying the Proposition 1.4 of polynomial transformations.

### 1.3.1. Solvability of potential

**Definition 1.19.** We named the set of all eigenvalues which the equation (30) is integrable, algebraic spectrum and we denote by \( \Lambda \subseteq \mathbb{C} \).

**Definition 1.20.** We say that a potential \( V(x) \in \mathbb{C}[x] \) is:

1. Algebraically solvable if \( \Lambda \) is infinite, or
2. Algebraically quasi-solvable if \( \Lambda \) is finite, or
3. Algebraically non-solvable if \( \Lambda = \emptyset \)

**Example 1.21.**

1. \( V(x) = 0 \), then \( \Lambda = \mathbb{C} \), consequently \( V(x) \) is a potential algebraically solvable.
2. \( V(x) = x^n + 3x^2 \), then \( \Lambda = \{0\} \), consequently \( V(x) \) is a potential algebraically quasi-solvable.
3. \( V(x) = x \), then \( \Lambda = \emptyset \), consequently \( V(x) \) is a potential algebraically non-solvable.

Next theorem is the main tool in this work and it finally allows us make a characterization of quasi-solvable polynomial potential

**Theorem 1.22** (Galoisian characterization of polynomial potentials). *Let us consider the Schrödinger equation (30), with \( V(x) \in \mathbb{C}[x] \) a polynomial of degree \( k > 0 \). Then, its differential Galois group \( DGal(L_\lambda/K) \) falls in one of the following cases:

1. \( DGal(L_\lambda/K) = SL(2, \mathbb{C}) \),
2. \( DGal(L_\lambda/K) = \mathbb{B} \).

Furthermore, \( DGal(L_\lambda/K) = \mathbb{B} \) if and only if the following conditions hold:

1. \( V(x) - \lambda \) is a polynomial of degree \( k = 2n \) writing in its completing square form
2. \( c_{n-1} - n \sigma - c_{n-1} - n \) is a positive even number \( 2s, s \in \mathbb{Z}_+ \).
3. There exists a monic polynomial \( P_s \) of degree \( s \), satisfying:

\[
\begin{align*}
\partial_x^2 P_s + 2(nx^n + \sum_{k=0}^{n-1} b_k x^k)\partial_x P_s + (nx^n + \sum_{k=0}^{n-2} b_k x^k) (k+1) b_k x^k - \\
\sum_{k=0}^{n-1} c_k x^k) P_s &= 0, \\
\partial_x^2 P_s - 2(nx^n + \sum_{k=0}^{n-1} b_k x^k)\partial_x P_s - (nx^n + \sum_{k=0}^{n-2} b_k x^k) (k+1) b_k x^k + \\
\sum_{k=0}^{n-1} c_k x^k) P_s &= 0.
\end{align*}
\]

In such cases, the only possibilities for eigenfunctions with polynomial potentials are given by

\[
\psi_\lambda = P_s e^{f(x)}, \quad \text{or} \quad \Psi_\lambda = P_s e^{-f(x)}, \quad \text{where} \quad f(x) = \frac{x^{n+1}}{n+1} + \sum_{k=0}^{n-1} \frac{b_k x^{k+1}}{k+1}.
\]

**Corollary 1.23.** Assume that \( V(x) \) is an algebraically solvable polynomial potential. Then, \( V(x) \) is a polynomial of degree 2.

**Corollary 1.24.** Suppose \( V(x) \) a polynomial potential of odd degree. Then (30) is not integrable.

### 1.4. Gröbner Basis

In this section we will briefly describe Gröbner basis and why are they useful for our main study. Given a polynomial system \( f_i = 0 \), we can consider the ideal generated by this set of polynomial, the idea is to find a better set of polynomial generating the same ideal (which means that they will have the same set of solutions, even whit multiplicity) but in a triangular form. In order to give a formal definition of Gröbner basis, we have to consider the following definitions.

**Definition 1.25.** Let \( I \subseteq \mathbb{k}[x_1, \ldots, x_n] \) be a nontrivial ideal, and fix a monomial ordering on the ring of polynomials \( \mathbb{k}[x_1, \ldots, x_n] \). Then we denote by \( LT(I) \) the set of leading terms of nonzero elements of \( I \) and \( \langle LT(I) \rangle \) the ideal generated by the elements in \( LT(I) \).
The following proposition will guarantee the existence of Gröbner basis, this proposition is also useful to demonstrate the Hilbert basis theorem which states that every ideal \( I \subseteq k[x_1, \ldots, x_n] \) has a finite generating set (This is the so called ideal description problem).

**Proposition 1.26.** Let \( I \subseteq k[x_1, \ldots, x_n] \) be an ideal other than \( \{0\} \). Then

1. \( \langle LT(I) \rangle \) is a monomial ideal.
2. there exist \( g_1, \ldots, g_s \in I \) such that \( \langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_s) \rangle \).

Not all bases as described in last proposition have the same behavior. This one receive a special name

**Definition 1.27.** Fix a monomial order on the polynomial ring \( k[x_1, \ldots, x_n] \). A finite subset \( G \{g_1, \ldots, g_s\} \) of an ideal \( I \subseteq k[x_1, \ldots, x_n] \) different from zero ideal is said to be a Gröbner basis if \( \langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_s) \rangle \).

Now, the most important fact (for our propose) is that a Gröbner basis is indeed a basis.

**Proposition 1.28.** Fix a monomial order. Then every ideal \( I \subseteq k[x_1, \ldots, x_n] \) has a Gröbner basis. Furthermore, any Gröbner basis for an ideal \( I \) is a basis of \( I \).

It could happen that \( G = \{1\} \), this means that the ideal \( I \) is indeed the polynomial ring \( k[x_1, \ldots, x_n] \) and obviously we won’t have any solution for our system. This will be very useful to determine nonintegrability in the next section. More information about Gröbner basis can be found in [8].

## 2. Quasi-Solvable Polynomial Potentials

Our goal in this section is to show how useful Gröbner basis are to find the solution of the problem of Liouvillian integrability for polynomial potentials, we present a simple algorithm that together to theorem 1.22 reduce our initial problem to a linear algebra.

**Algorithm 2.1.**

1. Set the potential and write it in completing the square form.
2. Set a value for \( s \).
3. Formulate the equations described on step three on theorem 1.22.
4. Set a polynomial with unknown coefficients.
5. Replace polynomial described in last step in one of the equations formulated in step 3.
6. Set an homogeneous polynomial system with the coefficients of the polynomial obtained in last step.
7. Compute the Gröbner basis of the ideal generated by the system obtained in the last step.
2.1. Quartic Potential

Let us consider the potential \( V(x) = x^4 + 4x^3 + 2x^2 - \mu x \), so applying completing the square (see figure 2) we can write this potential as follows,

\[
V(x) = (x^2 + 2x - 1)^2 + (4 - \mu)x - 1.
\] (32)

Now, by virtue of theorem 1.22 we have \( \pm (4 - \mu) - 2 = 2s \), where \( s \in \mathbb{Z}_+ \). Consequently, \( \mu \) is a discrete parameter that can be \( 2 - 2s \) or either \( 6 + 2s \). As a third step in Galoisian characterization theorem, in order to determinate Liouvillian integrability for the Schrödinger equation associated to potential (32), it must exist a monic polynomial \( P_s \) of degree \( s \) satisfying at least one of the following equations

\[
P_s'' + 2(x^2 + 2x - 1)P_s' + ((\mu - 2)x + 3 + \lambda)P_s = 0,
\] (33)

\[
P_s'' - 2(x^2 + 2x - 1)P_s' + ((\mu - 6)x - 1 + \lambda)P_s = 0.
\] (34)
Table 1. The spectral polynomial of potential (32) for the case $\mu = 2 - 2s$.

| $s$ | $\mu = 2 - 2s$ | $T(s, \lambda)$ |
|-----|----------------|-----------------|
| 0   | 2              | $\lambda - 1$  |
| 1   | 0              | $\lambda^2 + 10\lambda + 17$ |
| 2   | $-2$           | $\lambda^3 + 21\lambda^2 + 115\lambda + 135$ |
| 3   | $-4$           | $\lambda^4 + 36\lambda^3 + 406\lambda^2 + 1572\lambda + 1521$ |
| 4   | $-6$           | $\lambda^5 + 55\lambda^4 + 1050\lambda^3 + 8366\lambda^2 + 26613\lambda + 27659$ |
| 5   | $-8$           | $\lambda^6 + 78\lambda^5 + 2255\lambda^4 + 30276\lambda^3 + 196015\lambda^2 + 596046\lambda + 777825$ |

Table 2. The spectral polynomial of potential (32) for the case $\mu = 6 + 2s$.

| $s$ | $\mu = 6 + 2s$ | $T(s, \lambda)$ |
|-----|----------------|-----------------|
| 0   | 6              | $\lambda - 1$  |
| 1   | 8              | $\lambda^2 - 6\lambda + 1$ |
| 2   | 12             | $\lambda^3 - 15\lambda^2 + 43\lambda + 51$ |
| 3   | 14             | $\lambda^4 - 28\lambda^3 + 214\lambda^2 - 156\lambda - 1615$ |
| 4   | 16             | $\lambda^5 - 45\lambda^4 + 650\lambda^3 - 2634\lambda^2 - 8027\lambda + 41799$ |
| 5   | 18             | $\lambda^6 - 66\lambda^5 + 1535\lambda^4 - 13404\lambda^3 + 3343\lambda^2 + 428670\lambda - 984879$ |

Some values in $\Lambda$ are easily computable, for example:

For $\mu = 2 - 2s$ we have,

| $s$ | $\Lambda$ |
|-----|-----------|
| 0   | $\{1\}$  |
| 1   | $\{-2\sqrt{2} - 5, 2\sqrt{2} - 5\}$ |

or for $\mu = 6 + 2s$ we have,

| $s$ | $\Lambda$ |
|-----|-----------|
| 0   | $\{1\}$  |
| 1   | $\{3 - 2\sqrt{2}, 2\sqrt{2} + 3\}$ |

in addition, the solutions of the Schrödinger equation is given by

$$\psi_\lambda(x) = \begin{cases} P_s e^{\frac{3}{2}x^2 - x} & \text{if } \mu = 2 - 2s, \\ P_s e^{-\frac{3}{2}x^2 + x} & \text{if } \mu = 6 + 2s. \end{cases}$$ (35)
2.2. Sextic Potential

Let us consider the nonsingular turbiner potential $x^6 - (4J + 1)x^2$, where $J$ is a nonnegative integer, this potential has been studied by Bender and Dunne in [3], they showed that there exist a correspondence between the solutions of the Schrödinger equation associated to this potential and sets of orthogonal polynomials $P_s$.

We shall distinguish two cases:

**Case 1:** Let us set

$$
\psi''(x) = (x^6 - (4J - 1)x^2 - \lambda)\psi(x),
$$

(36)

as our object of debate. Due to step two in theorem 1.22, we can conclude $\mp(4J - 1) - 3 = 2s$ where $s \in \mathbb{Z}_+$. If $-(4J - 1) - 3 = 2s$, then $J = -\frac{s+1}{2}$ which is not possible because $J$ is a nonnegative integer. So $(4J - 1) - 3 = 2s$, i.e., $J = \frac{s+2}{2}$, therefore $s$ takes nonnegative even values.

Then, in order to achieve Liouvillian integrability, there must exist a monic polynomial $P_s(x)$ satisfying the equation

$$
P''_s - 2x^3P'_s - (3x^2 - \lambda - (4J - 1)x^2)P_s = 0.
$$

(37)

Algorithm 2.1, give us a tool to find spectral polynomials to equation (37)

| $s$ | $J$ | $T(s, \lambda)$ |
|-----|-----|----------------|
| 0   | 1   | $\lambda$     |
| 2   | 2   | $\lambda^2 - 8$ |
| 4   | 3   | $\lambda^4 - 64\lambda$ |
| 6   | 4   | $\lambda^6 - 240\lambda^4 + 880$ |
| 8   | 5   | $\lambda^8 - 640\lambda^4 + 47104\lambda$ |
| 10  | 6   | $\lambda^{10} - 1400\lambda^6 + 331456\lambda^2 + 5184000$ |

Table 3. Spectral polynomials of equation (37).

on the other hand, the solutions of equations (36) and (37) are easily calculable once the zeroes of $T(s, \lambda)$ are known, for example: The Gröbner basis generated by replace $P_6 = ax^5 + bx^4 + cx^3 + dx^2 + ex + g + x^6$ in (37) is,

$$
\{2880 - 240\lambda^2 + \lambda^4, 384g - 216\lambda + \lambda^3, e, 120 + 32d - \lambda^2, c, 4b + \lambda, a\}
$$

we conclude that the coefficients $g$, $d$, and $b$ are polynomials in $\lambda$ and that any other coefficients is zero. In addition, the solution to the Schrödinger equation is given by

$$
\psi_\lambda(x) = P_{s,\lambda}(x)e^{x^2/4}.
$$

(38)

| $s$ | $\Lambda$ | $P_{s,\lambda}(x)$ | $\psi_\lambda(x)$ |
|-----|-----------|------------------|------------------|
| 0   | $\{0\}$  | 1                | $exp(\frac{x^2}{4})$ |
| 2   | $\{\mp2\sqrt{2}\}$ | $x^2 \pm \frac{1}{\sqrt{2}}$ | $(x^2 \pm \frac{1}{\sqrt{2}})exp(\frac{x^2}{4})$ |
| 4   | $\{0, \mp8\}$ | $x^4 - \frac{3}{2}, x^4 \pm 2x^2 + \frac{1}{2}$ | $(x^4 - \frac{3}{2})exp(\frac{x^2}{4}), (x^4 \pm 2x^2 + \frac{1}{2})exp(\frac{x^2}{4})$ |

Table 4. Solutions to (36) and (37) for small values of $s$. 

**Case 2:** In this case, let us set
\[
\psi''(x) = (x^6 - (4J + 1)x^2 - \lambda)\psi(x),
\]
(39)
as our object of study. In a similar way to the above case we can conclude that \(J = \frac{s+1}{2}\), therefore \(s\) takes nonnegative odd values. And by theorem 1.22, there must exist a monic polynomial \(P_s(x)\) satisfying the equation,
\[
P''_s - 2x^3 P'_s - (3x^2 - \lambda - (4J + 1)x^2)P_s = 0.
\]
(40)
A simple application of algorithm 2.1, give us a list of the spectral polynomial for each value of \(s\).

| \(s\) | \(J\) | \(T(s, \lambda)\) |
|----|----|----------------|
| 1  | 1  | \(\lambda\) |
| 3  | 2  | \(\lambda^2 - 24\) |
| 5  | 3  | \(\lambda^3 - 128\lambda\) |
| 7  | 4  | \(\lambda^4 - 400\lambda^2 + 12096\) |
| 9  | 5  | \(\lambda^5 - 960\lambda^3 + 129024\lambda\) |

Table 5. Spectral polynomials of equation (40).

In this case, the solution to the Schrödinger equation (39) is given by
\[
\psi_\lambda(x) = P_{s, \lambda}e^{-\frac{x^4}{4}}.
\]
(41)

| \(s\) | \(\Lambda\) | \(P_{s, \lambda}\) |
|----|----|----------------|
| 1  | \{0\} | \(x\) |
| 3  | \{\pm 2\sqrt{6}\} | \(x^3 \mp \frac{\sqrt{6}}{2}x\) |
| 5  | \{0, \pm 8\sqrt{2}\} | \(x^3 - \frac{8\sqrt{2}}{2}x, x^3 \mp 2\sqrt{2}x^3 + \frac{3}{2}x\) |

Table 6. Solutions to equation (40) for small values of \(s\).

2.3. Octic Potential
Let us consider the potential \(V(x) = x^8 + (2\delta + 4)x^4 + \mu x^3 + \delta^2 + 4\delta + 4\), this potential can be written in the following form via completing the square (see figure 4):
\[
V(x) = (x^4 + \delta + 2)^2 + \mu x^3
\]
(42)
Now, we can use theorem 1.22 to determine conditions over the parameters \(\delta\) and \(\mu\) in order to achieve Liouvillian integrability. First, one can say that \(\mu\) is a discrete parameter that can be \(2s + 4\) or either \(-2s - 4\) where \(s\) is an nonnegative integer. Secondly, there must exist a monic polynomial \(P_s\) of degree \(s\) satisfying at least one of the following equations:
\[
P''_s + 2(x^4 + \delta + 2)P'_s + (-2sx^3 + \lambda)P_s = 0
\]
(43)
\[
P''_s - 2(x^4 + \delta + 2)P'_s - (-2sx^3 - \lambda)P_s = 0
\]
(44)
Algorithm 2.1 provide us a tool to calculate (if they exist) the polynomial solutions of above equations.
Table 7. Suitable parameters and solution for equation (43).

| $s$ | $\mu$ | $\delta$ | $\Lambda$ | $P_s$ |
|-----|-------|----------|----------|------|
| 0   | 4     | $\mathbb{C}$ | 0        | 1    |
| 1   | 6     | -2       | 0        | $x$  |
| 2   | Not integrable |         |          |      |
| 3   | Not integrable |         |          |      |
| 4   | Not integrable |         |          |      |
| 5   | 14    | -2       | 0        | $x^5 + 2$ |

Table 8. Suitable parameters and solution for equation (44).

| $s$ | $\mu$ | $\delta$ | $\Lambda$ | $P_s$ |
|-----|-------|----------|----------|------|
| 0   | -4    | $\mathbb{C}$ | 0        | 1    |
| 1   | -6    | -2       | 0        | $x$  |
| 2   | Not integrable |         |          |      |
| 3   | Not integrable |         |          |      |
| 4   | Not integrable |         |          |      |
| 5   | -14   | -2       | 0        | $x^5 - 2$ |

In addition, the solutions to the Schrödinger equation associated to the potential (42) are given by:

$$\psi_\lambda(x) = \begin{cases} 
P_s e^{\frac{\lambda}{x} x^{\frac{1}{2} + (\delta + 2)x}} & \text{if } \mu = 2s + 4, \\
P_s e^{-\frac{\lambda}{x} x^{\frac{1}{2} - (\delta + 2)x}} & \text{if } \mu = -2s - 4. 
\end{cases}$$  \hspace{1cm} (45)

We can conclude that algorithm 2.1 is also a useful tool to determine non-integrability of certain systems.

2.4. Decatic Potential

In this section let us consider the potential $V(x) = x^{10} - x^8 + x^6 + \delta x^4 + \epsilon x^2$, which include the specific potentials studied by Chaudhuri and Mondal in [7], there is a approach developed by D. Brandon and N. Saad in [5] using asymptotic iteration method but in this work we will apply the theorem 1.22 in order to make a Galoisian approach.

Above decatic potential can be written in the following way via completing the square (see figure 5):

$$V(x) = \left(x^5 - \frac{x^3}{2} + \frac{3x}{8}\right)^2 + \left(\delta + \frac{3}{8}\right) x^4 + \left(\epsilon - \frac{9}{64}\right) x^2,$$ \hspace{1cm} (46)

in virtue of theorem 1.22, in order to determine Liouvillian integrability, there must exist a monic polynomial $P_s$ of degree $s$, satisfying one of the following equations

$$P_s'' + 2 \left(x^5 - \frac{x^3}{2} + \frac{3x}{8}\right) P_s' + \left(\frac{37}{8} - \delta\right) x^4 + \left(-\epsilon - \frac{87}{64}\right) x^2 + \frac{3}{8} + \lambda \right) P_s = 0 \hspace{1cm} (47)$$

$$P_s'' - 2 \left(x^5 - \frac{x^3}{2} + \frac{3x}{8}\right) P_s - \left(\delta + \frac{43}{8}\right) x^4 + \left(\epsilon - \frac{105}{64}\right) x^2 + \frac{3}{8} - \lambda \right) P_s = 0 \hspace{1cm} (48)$$

Remark 2.2. It is also clear from theorem 1.22 that $\delta$ is a number of the form $2s + \frac{37}{8}$ or $-2s - \frac{43}{8}$ where $s \in \mathbb{Z}_+$. 
An application of algorithm 2.1, give us values of parameters of the potential (46),

Table 9. Suitable parameters for equation (47).

| s   | δ     | $M_s(\epsilon)$                                      | λ                  |
|-----|-------|-------------------------------------------------------|--------------------|
| 0   | $\frac{37}{8}$ | $-\epsilon - \frac{51}{64}$                          | $-\frac{3}{8}$     |
| 1   | $\frac{53}{8}$ | $64\epsilon + 151$                                   | $-\frac{3}{8}$     |
| 2   | $\frac{69}{8}$ | $262144\epsilon^3 + 2117632\epsilon^2 + 6925504\epsilon + 17694023$ | $-4096\epsilon^2 - 19328\epsilon - 49425$ |
| 3   | $\frac{85}{8}$ | $262144\epsilon^3 + 2904064\epsilon^2 + 11947200\epsilon + 43776519$ | $-4096\epsilon^2 - 27520\epsilon - 85137$ |

Table 10. Suitable parameters for equation (48).

| s   | δ    | $M_s(\epsilon)$                                      | λ                  |
|-----|------|-------------------------------------------------------|--------------------|
| 0   | $\frac{43}{8}$ | $-\epsilon + \frac{105}{64}$                         | $\frac{3}{8}$      |
| 1   | $\frac{59}{8}$ | $64\epsilon - 169$                                   | $\frac{3}{8}$      |
| 2   | $\frac{75}{8}$ | $262144\epsilon^3 - 2338816\epsilon^2 + 8178880\epsilon - 3037945$ | $4096\epsilon^2 - 21632\epsilon + 55185$ |
| 3   | $\frac{91}{8}$ | $262144\epsilon^3 - 3125248\epsilon^2 + 13642944\epsilon + 2959431$ | $4096\epsilon^2 - 29824\epsilon + 93201$ |

Remark 2.3. The zeroes of polynomial $M_s(\epsilon)$ are the suitable values of parameter $\epsilon$ of potential (46) in order to obtain Liouvillian integrability.

The solutions of the equation (47) and (48) are easily computable once the value of $\epsilon$ sets.

Table 11. Polynomial solutions $P_s$.

| s   | Eq. (47) | Eq. (48) |
|-----|----------|----------|
| 0   | $1$      | $1$      |
| 1   | $x$      | $x$      |
| 2   | $x^2 - \frac{64\epsilon + 215}{256}$ | $x^2 + \frac{64\epsilon - 231}{256}$ |
| 3   | $x^3 - \frac{64\epsilon + 279}{256}x$ | $x^3 + \frac{64\epsilon - 297}{256}x$ |

The solutions of the Schrödinger equation $\psi_\lambda$ generated by the equation (48) give us bound states of the form

$$\psi_\lambda(x) = P_{s, \lambda}e^{\frac{6}{8}x^6 + \frac{a^4}{8x^4} - \frac{a^2}{8x^2}}.$$ (49)

In the other hand, we never obtain bound states from equation (47) but the solution $\psi_\lambda$ is given by,

$$\psi_\lambda(x) = P_{s, \lambda}e^{\frac{6}{8}x^6 + \frac{a^4}{8x^4} + \frac{a^2}{16}}.$$ (50)

2.5. Dodecatic Potential

Consider the potential $V(x) = x^{12} + \kappa x^6 + \mu x^5$, this potential can be written in the following form using completing the square (see figure 6):

$$V(x) = \left(x^6 + \frac{\kappa}{2}\right)^2 + \mu x^5 - \frac{\kappa^2}{4}.$$ (51)

By step two in theorem 1.22 we have that $\mu$ is a discrete parameter that can be $2s + 6$ or either $-2s - 6$. Now, by third step in theorem 1.22, there must exist a monic polynomial.
$P_s$ of degree $s$ that satisfies at least one of the following equations:

\begin{align*}
    P_s'' + (2x^6 + \kappa)P_s' + \left( (6 - \mu) x^5 + \frac{\kappa^2}{4} + \lambda \right) P_s &= 0 \quad (52) \\
    P_s'' - (2x^6 + \kappa)P_s' - \left( (6 + \mu) x^5 - \frac{\kappa^2}{4} - \lambda \right) P_s &= 0 \quad (53)
\end{align*}

A direct application of algorithm 2.1 gives us suitable values of parameters of the potential (51) in order to determine Liouvillian integrability or establish nonintegrability.

| $s$ | $\mu = 2s + 6$ | $\mu = -2s - 6$ | $\kappa$ | $\lambda$ | $P_s$ |
|-----|----------------|----------------|---------|---------|-------|
| 0   | 6              | -6             | $\mathbb{C}$ | $-\frac{\kappa^2}{4}$ | 1     |
| 1   | 8              | -8             | 0       | 0       | $x$   |
| 2   | Not integrable |                |         |         |       |
| 3   | Not integrable |                |         |         |       |
| 4   | Not integrable |                |         |         |       |

Additionally, the solution to the Schrödinger equation associated to the potential (51) is given by:

\[
    \psi_\lambda(x) = \begin{cases} 
    P_s e^{\frac{x^7 + \delta}{2}} & \text{if } \mu = 2s + 6, \\
    P_s e^{-\frac{x^7 - \delta}{2}} & \text{if } \mu = -2s - 6. 
\end{cases} \quad (54)
\]

### 2.6. Tetrakaidecatic Potential

Consider the potential $V(x) = (x^7 + \delta + 2)^2 + \mu x^6 + \kappa x^2$, this potential can be written in the following form using completing the square (see figure 7):

\[
    V(x) = (x^7 + \delta + 2)^2 + \mu x^6 + \kappa x^2 \quad (55)
\]

It is clear that $\mu$ is a discrete parameter of the form $2s + 7$ or either $-2s - 7$. Now, in order to determine Liouvillian integrability, there must exist a monic polynomial $P_s$ of degree $s$ which satisfies some of the following equations:

\begin{align*}
    P_s'' + 2(x^7 + \delta + 2)P_s' + (-2sx^6 - \kappa x^2 + \lambda) &= 0 \quad (56) \\
    P_s'' - 2(x^7 + \delta + 2)P_s' - (-2sx^6 + \kappa x^2 - \lambda) &= 0 \quad (57)
\end{align*}

Applying algorithm 2.1 to these equations, we will obtain suitable parameters for potential (55) or we will determine nonintegrability for the associated Schrödinger equation.

| $s$ | $\mu = 2s + 7$ | $\mu = -2s - 7$ | $\delta$ | $\kappa$ | $\lambda$ | $P_s$ |
|-----|----------------|----------------|---------|---------|---------|-------|
| 0   | 7              | -7             | $\mathbb{C}$ | 0       | 0       | 1     |
| 1   | 9              | -9             | -2      | 0       | 0       | $x$   |
| 2   | Not integrable |                |         |         |         |       |
| 3   | Not integrable |                |         |         |         |       |
| 4   | Not integrable |                |         |         |         |       |
In addition, the solution to the Schrödinger equation associated to the tetrakaidecatic potential (55) is given by:

\[
\psi_{\lambda}(x) = \begin{cases} 
    P_s e^{\frac{x^8}{8} + (\delta + 2)x} & \text{if } \mu = 2s + 7, \\
    P_s e^{\frac{x^8}{8} - (\delta + 2)x} & \text{if } \mu = -2s - 7.
\end{cases}
\] (58)

3. Appendix A

Figure 2. Completing the square for degree 4

| n = 2  | \(a_3 \neq 0\) | \(a_3 = 0\) |
|--------|-----------------|--------------|
| \(b_3\) | \(a_3/2\)        | 0            |
| \(b_0\) | \((a_2 - b_1^2)/2\) | \(a_2/2\)    |
| \(c_1\) | \(a_1 - 2b_0b_3\) |              |
| \(c_0\) | \(a_0 - b_0^2\)  |              |

Figure 3. Completing the square for degree 6

| n = 3  | \(a_5 \neq 0\) | \(a_5 = 0\) |
|--------|-----------------|--------------|
| \(b_3\) | \(a_5/2\)        | 0            |
| \(b_1\) | \((a_4 - b_3^2)/2\) | \(a_4/2\)    |
| \(b_0\) | \((a_3 - 2b_2b_3)/2\) | \(a_3/2\)    |
| \(c_2\) | \(a_2 - 2b_0b_2 - b_1^2\) |              |
| \(c_1\) | \(a_1 - 2b_0b_3\)  |              |
| \(c_0\) | \(a_0 - b_0^2\)  |              |

Figure 4. Completing the square for degree 8

| n = 4  | \(a_7 \neq 0\) | \(a_7 = 0\) |
|--------|-----------------|--------------|
| \(b_3\) | \(a_7/2\)        | 0            |
| \(b_2\) | \((a_6 - b_3^2)/2\) | \(a_6/2\)    |
| \(b_1\) | \((a_5 - 2b_2b_3)/2\) | \(a_5/2\)    |
| \(b_0\) | \((a_4 - 2b_3b_1 - b_2^2)/2\) | \((a_4 - b_3^2)/2\) |
| \(c_3\) | \(a_3 - 2b_0b_2 - 2b_1b_2\) |              |
| \(c_2\) | \(a_2 - 2b_0b_2 - b_1^2\)  |              |
| \(c_1\) | \(a_1 - 2b_0b_3\)  |              |
| \(c_0\) | \(a_0 - b_0^2\)  |              |
Figure 5. Completing the square for degree 10

| $n = 5$ | $a_{11} \neq 0$ | $a_{11} = 0$ |
|---|---|---|
| $b_4$ | $a_9/2$ | 0 |
| $b_3$ | $(a_8 - b_1^2)/2$ | $a_8/2$ |
| $b_2$ | $(a_7 - 2b_2b_3)/2$ | $a_7/2$ |
| $b_1$ | $(a_6 - 2b_4b_2 - b_1^2)/2$ | $(a_6 - b_1^2)/2$ |
| $b_0$ | $(a_5 - 2b_2b_3 - 2b_4b_1)/2$ | $(a_5 - 2b_2b_3)/2$ |
| $c_4$ | $a_4 - 2b_1b_3 - 2b_0b_4 - 2b_2^2$ | |
| $c_3$ | $a_3 - 2b_0b_3 - 2b_1b_2$ | |
| $c_2$ | $a_2 - 2b_0b_2 - b_1^2$ | |
| $c_1$ | $a_1 - 2b_0b_1$ | |
| $c_0$ | $a_0 - b_0^2$ | |

Figure 6. Completing the square for degree 12

| $n = 6$ | $a_{11} \neq 0$ | $a_{11} = 0$ |
|---|---|---|
| $b_5$ | $a_{11}/2$ | 0 |
| $b_4$ | $(a_{10} - b_2^2)/2$ | $a_{10}/2$ |
| $b_3$ | $(a_9 - 2b_2b_3)/2$ | $a_9/2$ |
| $b_2$ | $(a_8 - 2b_2b_3 - b_1^2)/2$ | $(a_8 - b_1^2)/2$ |
| $b_1$ | $(a_7 - 2b_3b_4 - 2b_2b_1)/2$ | $(a_7 - 2b_3b_4)/2$ |
| $b_0$ | $(a_6 - 2b_3b_4 - 2b_2b_1 - b_1^2)/2$ | $(a_6 - 2b_3b_4 - b_1^2)/2$ |
| $c_5$ | $a_5 - 2b_2b_5 - 2b_1b_4 - 2b_2b_3$ | |
| $c_4$ | $a_4 - 2b_3b_4 - 2b_1b_3 - b_2^2$ | |
| $c_3$ | $a_3 - 2b_1b_3 - 2b_1b_2$ | |
| $c_2$ | $a_2 - 2b_0b_2 - b_1^2$ | |
| $c_1$ | $a_1 - 2b_0b_1$ | |
| $c_0$ | $a_0 - b_0^2$ | |
Figure 7. Completing the square for degree 14

| $n = 7$ | $a_{13} \neq 0$ | $a_{13} = 0$ |
|---------|-----------------|--------------|
| $b_6$   | $a_{13}/2$      | 0            |
| $b_5$   | $(a_{12} - b_5^2)/2$ | $a_{12}/2$ |
| $b_4$   | $(a_{11} - 2b_6b_5)/2$ | $a_{11}/2$ |
| $b_3$   | $(a_{10} - 2b_6b_3 - b_5^2)/2$ | $(a_{10} - b_5^2)/2$ |
| $b_2$   | $(a_9 - 2b_6b_3 - 2b_4b_5)/2$ | $(a_9 - 2b_4b_3)/2$ |
| $b_1$   | $(a_8 - 2b_6b_1 - 2b_4b_5 - b_3^2)/2$ | $(a_8 - 2b_3b_5 - b_3^2)/2$ |
| $b_0$   | $(a_7 - 2b_6b_1 - 2b_5b_2 - 2b_3b_4)/2$ | $(a_7 - 2b_5b_2 - 2b_3b_4)/2$ |
| $c_6$   | $a_6 - 2b_6b_5 - 2b_5b_1 - 2b_3b_4 - b_3^2$ | |
| $c_5$   | $a_5 - 2b_5b_2 - 2b_4b_3 - 2b_3b_3$ | |
| $c_4$   | $a_4 - 2b_4b_3 - 2b_3b_4 - b_2^2$ | |
| $c_3$   | $a_3 - 2b_3b_2 - 2b_1b_2$ | |
| $c_2$   | $a_2 - 2b_2b_1 - b_1^2$ | |
| $c_1$   | $a_1 - 2b_1b_1$ | |
| $c_0$   | $a_0 - b_0^2$ | |

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