On the response of a star cluster to a tidal perturbation

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ABSTRACT

We study the response of star clusters to individual tidal perturbations using controlled $N$-body simulations. We consider perturbations by a moving point mass and by a disc, and vary the duration of the perturbation as well as the cluster density profile. For fast perturbations (i.e. ‘shocks’), the cluster gains energy in agreement with theoretical predictions in the impulsive limit. For slow disc perturbations, the energy gain is lower, and this has previously been attributed to adiabatic damping. However, the energy gain due to slow perturbations by a point-mass is similar to, or larger than that due to fast shocks, which is not expected because adiabatic damping should be almost independent of the nature of the tides. We show that the geometric distortion of the cluster during slow perturbations is of comparable importance for the energy gain as adiabatic damping, and that the combined effect can qualitatively explain the results. The half-mass radius of the bound stars after a shock increases up to $\sim 7\%$ for slow, low-concentration clusters, and decreases $\sim 3\%$ for the most concentrated ones. The fractional mass loss is a non-linear function of the energy gain, and depends on the nature of the tides and most strongly on the cluster density profile, making semi-analytic model predictions for cluster lifetimes extremely sensitive to the adopted density profile.

Key words: galaxies: star clusters: general — globular clusters: general — open clusters and associations: general

1 INTRODUCTION

Theories of globular cluster (GC) formation throughout cosmic time are becoming sophisticated enough (e.g., Muratov & Gnedin 2010; Li & Gnedin 2014; Kruĳssen 2015; Li et al. 2017, 2018, 2019; Pfeffer et al. 2018; Choksi et al. 2018; Choksi & Gnedin 2019) that accurate modeling of cluster dynamics in the early evolution has become more important. GCs are expected to form in environments where the gas density is high (Elmegreen & Efremov 1997), which leads to tidal perturbations by passing molecular gas clouds soon after GC formation.

Much of our understanding of the response of a star cluster to a tidal perturbation is based on results of fast variation of the tidal field, i.e. ‘tidal shocks’ (Spitzer 1958; Ostriker et al. 1972). However, when the duration of the shock is longer than the crossing timescale of stars within the cluster – which is short for dense clusters – the response of the cluster is no longer described by the impulse approximation. During a slow perturbation, the stars conserve orbital actions, reducing the energy gain. The relevant parameter is the adiabatic parameter $x$, which is the ratio of the duration of the perturbation $\tau$ and the cluster dynamical time at the half-mass radius $t^{h}_{\text{ dyn}}$. Spitzer (1958) showed that for increasing $x$ the effect of the perturbation is adiabatically damped, leading to an exponential decrease of the energy gain with increasing $x$. In a series of important studies (Weinberg 1994a,b,c), Weinberg showed that Spitzer’s result underestimates the energy gain of slow perturbations because of resonances, leading to a power-law decrease of the energy gain with $x$. Understanding how the total energy gain of the cluster depends on $x$ makes it possible to generalize results from the impulse approximation to the adiabatic regime by introducing the concept of adiabatic correction.

Analytically, resonances in the adiabatic regime have been explored only for a star cluster passing through a one-dimensional slab that represents a galaxy disc (Weinberg 1994b,c). Similarly, adiabatic corrections have been quantified by $N$-body simulations only for disc perturbations by Gnedin & Ostriker (1999, hereafter GO99) and for extended spherical perturbers by Gnedin et al. (1999), who found that the cluster energy gain decreases as $x^{-3}$. The resulting fitting expressions are often applied to other types of perturbers (Gnedin 2003; Prieto & Gnedin 2008; Pfeffer et al. 2018), however the adiabatic corrections have not been quantified for other perturbers. There is still a large unexplored region of the parameter space, such as the nature of the perturbers and the density profile of
the cluster. In addition to disc-crossing, GCs also experience significant tidal forces from compact galactic structures such as galactic bulge or giant molecular clouds. These structures can be modeled as point mass (PM) since most encounters are distant ones such that their sizes are typically much smaller than the impact parameter to the cluster orbit. Gieles et al. (2006) showed that for a population of giant molecular clouds, most of the energy comes from penetrating encounters, for which the PM approximation does not hold. The results of such encounters can be understood from the combination of our 2 assumed models. The distinction of different perturbers is meaningful in this context. The tidal forces of a disc perturbation are compressive in the z direction, while half of the tidal forces of a passing PM are extensive, possibly leading to different forms of adiabatic corrections.

The aim of this work is to go beyond the impulse approximation and explore the adiabatic regime of tidal perturbation through a set of controlled N-body experiments. We investigate the dependence of the energy gain on the nature of the perturber and properties of the cluster. This paper is organized as follows. In Section 2 we describe the setup of the N-body models and the parameterization of the tidal perturbations. Section 3 contains our results of a study of the tidal perturbations. In Section 4 we present a discussion on the intrinsic differences between a PM perturbation and a disc perturbation, and their impact on the adiabatic corrections. Finally, we summarize our results in Section 5.

## 2 DESCRIPTION OF THE N-BODY EXPERIMENTS

### 2.1 N-body code

We use a state-of-the-art direct N-body code 

\[ \rho(Z) = \rho_p \exp(-Z^2/H^2), \]

where \( H \) is the scale height and \( \rho_p \) is the mid-plane density.

The duration of the perturbation is defined as \( \tau \equiv b/H \) for the PM case, where \( b \) is the impact parameter and distance of closest approach (which in the impulsive regime is the same as the impact parameter at infinity) and as \( \tau \equiv b/H \) for the disc shock. To achieve a certain energy gain in the impulse approximation \( \Delta E_{\text{imp}} \), the parameter \( M_p \) in N-body units is found from equation (9) of Spitzer (1958):

\[ M_p = \frac{b^3 \gamma}{2G} \sqrt{\frac{3\Delta E_{\text{imp}}}{\langle r^2 \rangle}}. \]

For the disc shock we find that \( \Delta E_{\text{imp}} = (2/3)\langle r^2 \rangle g_m \rho^2 H^2 / V^2 \) (Ostriker et al. 1972), where \( g_m = 2\pi^{3/2} G \rho_p H \) is the maximum acceleration, such that

\[ \rho_p = \frac{V}{2 \pi G \sqrt{\Delta E_{\text{imp}}}}. \]

In all simulations we adopt \( b = H = 20 \), but we note that in the tidal approximation these values are not important, because the strength of the tides are set by the density (see equations 3 and 4 below).

Each perturbation is applied to the cluster by computing the corresponding tidal tensor as a function of time. As the tidal tensor depends on the mass \( M_p \) for a PM shock, and on the density \( \rho_p \) for a disc shock, the inputs to compute the tensor are: the desired fractional energy gain in the impulsive approximation \( \Delta E_{\text{imp}} / E_0 \), the duration \( \tau \), the mean-square cluster radius \( \langle r^2 \rangle \) (which depends on the cluster model), and the type of the perturber (i.e. PM or disc). With this input, \( M_p \) and \( \rho_p \) are computed from equations (1) and (2), respectively.

Then we compute the tidal tensor of a PM shock as:

\[ T_{ij} = -\frac{\partial^2 \Phi}{\partial X_i \partial X_j} = \frac{G M_p}{R^3} \left( 3 X_i X_j - \delta_{ij} \right), \]

where \( X_i \) are the coordinates of the PM with respect to the cluster, and \( R^2 = \sum_i X_i^2 \). The time dependent values of \( X_i \) were computed for an orbit with \( b \) and \( V \) at infinity, meaning that as the orbit evolves the minimum distance (i.e. closest approach) is smaller than \( b \) and the relative velocity at closest approach is larger. For large \( x \) this effect is important.

The tidal tensor of the disc is computed as:

\[ T_{ij} = -4\pi G \rho_p \exp \left( -\frac{z^2}{H^2} \right) \delta_{ij} \delta_{xy}. \]

For the disc crossing we did not evolve the orbit but instead assumed a constant \( V \). This was done because the infinity disc has a constant acceleration for \( |Z|/H >> 1 \), which implies a strong, nonphysical correlation between the starting \( Z \) and the ratio of the initial velocity and the maximum velocity.

In terms of the tidal tensor, the expected energy change in the impulse approximation can be written as

\[ \Delta E_{\text{imp}} = \frac{1}{6} \int_{\text{tid}} \langle \tau^2 \rangle, \]

where the tidal heating parameter (Gnedin 2003)

\[ I_{\text{tid}} = \sum_{i,j} \left( \int T_{ij} dt \right)^2, \]

evaluates to

\[ I_{\text{tid}} = \frac{4}{3} \left( \frac{\rho_p^2 H^2}{V^2} \right), \]

### 2.2 Setup of the perturbers and tidal tensors

We perform experiments with two types of tidal perturbers: (i) a PM with mass \( M_p \), and (ii) an infinite disc with vertical density profile.
for the disc shock, and to
\[ I_{\text{tid}} = \frac{8G^2M_c^2}{b^4Y^2}, \]  
for the PM shock.

We define the time of impact (or closest approach to the perturber) as \( t = 0 \), such that most of the perturbation occurs between the times \(-\tau\) and \( \tau\). The values of \( \Delta E_{\text{imp}}/|E_0| \) and \( \tau \) that we consider are given in Table 1.

It is worth mentioning that the temporal resolution for the sampling of the tensors scales with the duration of the encounter; it is 0.5 for the slowest perturbations, \( \tau = 5 \), and improves up to 0.01 for the fastest shocks, \( \tau = 0.05 \).

### 2.2 Cluster setup

We explore different density profiles for the cluster, given by King (1966) models with dimensionless central concentration parameter \( W_0 = [2, 4, 6, 8] \). For these models, we find that the half-mass radius is \( r_h = [0.849, 0.827, 0.804, 0.871] \) and \( (r^2) = [1.027, 1.184, 1.693, 3.242] \) in N-body units. Each cluster model was generated using \textsc{limepy} (Gieles & Zocchi 2015), and discretized into 100,000 equal-mass particles.

For each cluster model and type of perturber we explore nine values of the shock duration, \( \tau \), from fast to slow perturbations. The regime at which the shock occurs is given by the adiabatic parameter

\[ x = \frac{\tau}{t_{\text{dyn}, h}}, \]  
where \( t_{\text{dyn}, h} \) is defined as in GO99:

\[ t_{\text{dyn}, h} \equiv \left( \frac{r_h^3}{2GM_c^2} \right)^{1/2}, \]

We find \( t_{\text{dyn}, h} = [1.74, 1.67, 1.60, 1.81] \) in N-body units, for \( W_0 = [2, 4, 6, 8] \). Note that despite different density profiles, all cluster models we consider have similar values of \( t_{\text{dyn}, h} \) as well as of a related quantity – the average density at the half-mass radius.

Fast perturbations (i.e. shocks) have \( x \ll 1 \), while slow perturbations have \( x \geq 1 \). We constructed models with different \( x \) by changing the value of \( \tau \). This combination of parameters gives us a total of 172 N-body models, summarized in Table 1. All simulations are run for \( \Delta t_{\text{imp}} = 40 N \)-body times, where \( \Delta t_{\text{imp}} = \text{int}(5\tau + 0.5) \) and the tensors are setup such that the maximum of the perturbation occurs at \( t_{\text{imp}} \). When plotting results as a function of time, we subtract \( t_{\text{imp}} \) from the \( N \)-body time, such that \( t = 0 \) corresponds to the middle of the perturbation. By starting the perturbation at \( t = -5\tau \), we miss the contribution from the tidal forces in the time interval \(-\infty < t/\tau < -5\). By integrating the analytic expressions for the tidal acceleration of Spitzer (1958) over this time interval, we estimate that it contributes approximately 1% to the total velocity change, and can therefore be safely ignored.

In order to test this estimation we have therefore repeated our slowest PM encounter (\( x = 3 \)) with a \( W_0 = 4 \) cluster, and added 40 time units to the “pre-shock” part of the original simulation. Hence, this new simulation runs for \( 40 + 10r + 40 \) time units, i.e., now the encounter is symmetric. With this new simulation we found that the originally truncated part of the encounter accounts for 0.02% of the total energy change. This result confirms that the truncated part of the encounters has a negligible effect.

### 3 RESULTS

In this section we describe the results of our N-body models and focus on the change in \( E, M \) and \( r_h \) as a result of the perturbation.

#### 3.1 Energy gain for different cluster models

First, we quantify the total energy gain of all stars (bound and unbound) in a cluster model, \( \Delta E \), due to a single PM or disc tidal perturbation.

Fig. 1 shows the fractional energy gain relative to the initial energy, \( \Delta E /|E_0| \), computed at the end of the simulation for the 72 models with the expected \( \Delta E_{\text{imp}} /|E_0| = 0.1 \). For fast shocks (\( x \leq 0.1 \)) there is a good agreement with the expected values from the impulse approximation for all models. For the disc perturbations, the energy gain due to slow perturbations (\( x \geq 1 \)) is 1-2 orders of magnitudes lower than what is expected from the impulse approximation. This reduced energy gain has been reported before and attributed to adiabatic damping.

Here it is worth mentioning that for the PM case we are using...
the values of $x$ computed at infinity; it is because as the orbit evolves, the relative velocity at closest approach is larger that the initial $V$, causing the actual $x$ value to change.

In the left panel of Fig. 1 we overlap the result of GO99 for disc perturbers and clusters with $W_0 = 4$

$$\frac{\Delta E}{|E_0|} = \frac{\Delta E_{\text{imp}}}{|E_0|} \left(1 + x^2\right)^{-3/2}. \quad (11)$$

The asymptotic behaviour $\Delta E \propto x^{-3}$ was derived by GO99 in the limit of slow perturbations, $x \gg 1$. One power of $x$ comes from the expectation of the linear perturbation theory, and two powers of $x$ come from the normalization of the shock amplitude. Together this results in the expected scaling $\Delta E \propto \tau^{-3} \propto x^{-3}$. Equation (11) matches the results of the GO99 N-body simulations for a cluster with $W_0 = 4$.

We find that equation (11) gives a fair description of the reduction in energy change for our disc $W_0 = 4$ models, however, it does not explain the results for the clusters with different $W_0$. The reduction is less for more concentrated clusters, which could be because these clusters have more stars with large periods which are in the impulsive regime. Alternatively, the larger envelopes of more concentrated models could lead to more geometrical distortion during slow perturbations. We discuss this further in Section 4.1.

However, we notice that by slightly changing this equation, we can still describe our disc perturbation models. To capture the dependence of our results on the density profile, we include an extra parameter $\epsilon$ that depends on $W_0$:

$$\frac{\Delta E}{|E_0|} = \frac{\Delta E_{\text{imp}}}{|E_0|} \left(1 + (\epsilon x)^2\right)^{-3/2}. \quad (12)$$

Fig. 2 shows the fits to our results using equation (12), where the different values of $\epsilon$ illustrate the importance of including an additional parameter to take into account the density profile of the cluster. We find $\epsilon = [1.46, 1.14, 0.74, 0.55]$ for $W_0 = [2, 4, 6, 8]$.

The results for slow PM perturbations ($x \geq 1$) are very different than those for the disc: for all cluster concentrations the energy gain is larger than what is expected from the impulsive approximation. It is also clear that none of the PM models can be described by equation (11).

Therefore we find that the difference with the impulsive prediction depends on the type of perturber and on the density profile of the cluster. We discuss possible causes of this behaviour in Section 4.2.

3.2 Energy gain for different perturbation strengths
Here we investigate a possible dependence of the cluster response on the perturbation amplitude. We apply perturbations of different strength to the $W_0 = 4$ model. For both perturbers, PM and disc, we take three values for the perturbation strength, $\Delta E_{\text{imp}}/|E_0| = [0.01, 0.03, 0.1]$.

To directly compare these cases, Fig. 3 shows the ratio $\Delta E/|E_{\text{imp}}$ as a function of $x$. For the disc case, the cluster response is almost insensitive to the perturbation strength and is well described by equation (11). In contrast, for the PM case, the cluster response depends on the amplitude of the perturbation in the regime of $x > 1$.

It is worth mentioning that GO99 tested equation (11) in the range of disc shock strength from $\sim 10^{-3}$ to $\sim 1$. Here we reproduce that result for the $W_0 = 4$ model perturbed by a disc.

3.3 Mass loss
The induced energy gain on the cluster causes some stars to escape the system, either by increasing their kinetic energy or reducing the binding energy. As a consequence, an imprint of energy to the cluster translates into a mass lost.

In our simulations $\Delta M$ comes from the particles that become unbound after the tidal perturbation. We define a particle as unbound if its specific energy $\tilde{E} = 0.5v^2 + \phi$ is positive at the end of the simulation, where $v$ is the velocity (measured in the non-rotating frame of the cluster’s center of mass) and $\phi$ the specific potential due to the other stars (bound and unbound).

In Fig. 4 we show the fractional mass loss as a function of $x$, for the disc (left) and PM (right) perturbations for $\Delta E_{\text{imp}}/|E_0| = 0.1$. Notice that the value of the mass loss depends on the cluster concentration. The fractional mass loss is a factor of 20(400) higher for the disc/PM shocks for $W_0 = 8$ clusters than for $W_0 = 2$ clusters. For stronger shocks of $\Delta E_{\text{imp}}/|E_0| = 0.5$ this difference reduces to a factor of $\sim 2$, and for $\Delta E_{\text{imp}}/|E_0| = 1$ there is no $W_0$ dependence (see Fig. 6).

To understand the $W_0$-dependence for weak shocks, we consider the fact that the maximum energy of particles in models with different $W_0$ is different: $\tilde{E}_{\text{max}} = -GM/2r_i$ and $r_i$ for $W_0 = 8$ is about 3 times larger than for a $W_0 = 2$ model. One therefore expects that a $W_0 = 2$ model requires a larger $\Delta E$ to unbind at least one particle. We need to keep in mind that the experiments were setup such that clusters have the same $\Delta E_{\text{imp}}/|E_0|$, and because $\Delta E_{\text{imp}} = \frac{1}{2} \frac{c_s^2}{\rho} \langle r^2 \rangle$ and $\langle r^2 \rangle$ is approximately 3 times smaller for $W_0 = 2$ than for $W_0 = 8$, the value of $I_{\text{dis}}$ is 3 times larger for $W_0 = 2$ models. This means that at $r_i$ the expected energy gain $(\Delta E_0 \propto I_{\text{dis}} r_i^2)$ is 3 times larger for $W_0 = 8$ models, thereby unbinding more stars that are near $r_i$.

Meanwhile, in the adiabatic regime and for disc shocks, the cluster mass loss decreases as the perturbation becomes slower, following a trend similar to the energy gain shown in Fig. 1. For the PM perturbation $|\Delta M|/M_0$ behaves in the same way as for the disc in the impulsive regime, i.e., it is nearly constant, and its value strongly depends on $W_0$. However, for slow tidal perturbations, the cluster mass loss increases, opposite to what happens with the disc.

It is worth pointing out that although fast PM and disc shocks converge to the same $\Delta E$ value (see Fig. 1), they do not converge to

\footnote{Note that we use a different symbol $\tilde{E}$ to distinguish the specific energy of an individual particle from the total energy of the whole cluster $E$.}
Therefore the maximum energy gain of individual particles, \( \Delta E \), and the evolution proceeds to the right. The dashed lines indicate the disc case. The first column shows this in Fig. 5 by showing the fractional change in mass as a function of \( t \) for a perturbation strength \( \Delta E_{\text{imp}}/|E_0| = 0.1 \).

![Figure 3](image-url) Fractional energy gain relative to the impulse approximation. Each panel shows three sets of \( N \)-body models that correspond to the shock strengths 0.1, 0.03 and 0.01, for cluster with \( W_0 = 4 \). The orange line shows the adiabatic correction fit from GO99.

![Figure 4](image-url) Fractional change in mass as a function of \( t \) for a perturbation strength \( \Delta E_{\text{imp}}/|E_0| = 0.1 \).

The same \( \Delta M \) value. Fig. 4 shows that, in the impulsive regime and for the same \( \Delta E \), the cluster systematically loses more mass under a disc shock than under a PM shock; this difference is negligible for high concentration clusters (\( W_0 = 8 \)), but becomes significant as the concentration decreases. For low concentrations (\( W_0 = 2 \)) the cluster mass loss under a fast disc shock can be more than an order of magnitude higher than under a PM fast shock, for the same energy gain.

This discrepancy is likely the result of the different nature of the tidal forces: for the disc, the tidal forces are compressive along the \( z \)-axis, whereas for the PM case, half of the tides act along the \( x \)-axis (extensive) and the other half acts along the \( z \)-axis (compressive). This means that for the disc \( \Delta v = \Delta v_z \), while for the PM \( \Delta v = \sqrt{\Delta v_x^2 + \Delta v_z^2} \) and \( \langle \Delta v_z^2 \rangle = 0.5 \langle \Delta v^2 \rangle \). Therefore the maximum energy gain of individual particles, \( \Delta E \), in the PM case is half of the maximum \( \Delta E \) in the disc case. We illustrate this in Fig. 5 by showing the \( \Delta E \) as a function of their initial energy \( E_0 \) at various moments in time for both a PM and a disc case. The first column shows \( \Delta E \) in the middle of the shock, and the evolution proceeds to the right. The dashed lines indicate the boundary between bound and unbound. Soon after the shock \((t = 1)\), the loosely bound stars in the disc case have increased more in energy than in the PM case, resulting in more unbound stars (red dots).

Fig. 6 shows the fractional mass loss as a function of the energy gain for the 64 simulations described in the second part of Table 1, i.e., four values of the perturbation strength, two values of the duration, four values of \( W_0 \), and two perturber types. We show this relation first on a linear scale because this emphasizes the large values of mass loss which are of primary interest in studies of cluster evolution. Then we show it on a log-log scale to emphasize the weak shock regime.

There is large dispersion in mass loss at low \( \Delta E/|E_0| \), which makes it difficult to find a simple relation between mass loss and energy gain. At large energy gains \( (\Delta E/|E_0| > 0.1) \), the fractional mass loss is well approximated by a linear relation

\[
\frac{|\Delta M|}{M_0} = f \frac{\Delta E}{|E_0|}, \quad f = 0.217 \pm 0.005, \quad (13)
\]

with rms of the fit \( \sigma = 0.022 \). This result is consistent with the relation found by Gieles et al. (2006) \((f = 0.22)\) for Plummer (1911) models. It is worth mentioning that, although the error in the fit is relatively small, at low values of \( \Delta E/|E_0| \) the scatter is not
random: instead, the models with $W_0 \leq 4$ systematically fall below the linear fit, and those with $W_0 \geq 6$ fall on or above the fit.

We fit the linear mass–energy change relation individually for different $W_0$ models and find the proportionality factor $f$ to vary between 0.18 and 0.22 for the disc shocks, and between 0.20 and 0.26 for the PM shocks. Interestingly, the dependence on $W_0$ is not monotonic: the $W_0 = 4$ model has the smallest slope. Most concentrated clusters have the largest mass loss, as already seen in Fig. 4. The adiabatic parameter $x$ plays a smaller role, for the range of values we considered ($x < 1$). These results also show the dependence on the perturber type. The mean values are $f = 0.195 \pm 0.006$ for disc, $0.233 \pm 0.008$ for PM, but they are still relatively close.

Fig. 6 also indicates that the relation between mass and energy change can be alternatively described as non-linear. Indeed, we fit a power-law relation

$$\frac{\Delta M}{M_0} = f_{\beta} \left( \frac{\Delta E}{|E_0|} \right) \gamma, \quad f_{\beta} \approx 0.25$$

(14)

and find the slope systematically decreasing from $\beta = 2.01 \pm 0.13$ for $W_0 = 2$ to $\beta = 0.88 \pm 0.06$ for $W_0 = 8$. The constant of proportionality of this relation is tightly constrained to be $f_{\beta} = 0.23 \pm 0.26$. These relations are shown in the bottom panel of Fig. 6. We can see again that the more concentrated cluster models show larger relative mass loss and the less steep relation.

Despite the present scatter and non-linearity for weak shocks, the relation (13) for strong shocks $(\Delta E/|E_0| > 0.1)$ can be used in analytical or sub-grid models of cluster disruption, when the energy change from tidal shocks can be calculated. In addition to specifying the information about the shocks ($I_{\text{tid}}$), this requires assumptions about the cluster profile determining $(r^2)$. To eliminate the latter assumption we can rephrase the mass loss of a cluster of a given density profile due to a perturbation with strength $I_{\text{tid}}$.

In Fig. 7 we plot the fractional mass loss as a function $I_{\text{tid}}$. The scatter is much larger than in Fig. 4, because high(low) concentration clusters have moved to the left(right). This highlights the importance of the dependence of $\Delta E$ (and therefore $\Delta M$) on $(r^2)$ of the cluster, which can vary by a factor of 3 for clusters with the same $r_v$ and almost the same $r_h$.

We find that the relation between $\Delta E$ and $I_{\text{tid}}$ is almost exactly linear for a given cluster model $W_0$ (and therefore $(r^2)$), as expected from analytical theory (equation 5): $\Delta E \propto I_{\text{tid}}^{0.98 \pm 0.01}$. However, the relation between $\Delta M$ and $I_{\text{tid}}$ is noticeably non-linear:

$$\frac{\Delta M}{M_0} \propto I_{\text{tid}}^{\gamma}$$

(15)

with the power-law index decreasing systematically with $W_0$, from $\gamma = 1.99 \pm 0.13$ for $W_0 = 2$ to $\gamma = 0.88 \pm 0.06$ for $W_0 = 8$. These slopes are very close to those for the mass-energy relation, which means that the non-linearity of the $\Delta M(I_{\text{tid}})$ relation is inherited from the $\Delta M(\Delta E)$ relation.

### 3.4 Changes in half-mass radius and density

Besides the energy change and mass loss, $r_h$ of the cluster is also modified by the perturbation. Soon after the perturbations, the particles with the largest energy change will move to larger orbits thereby increasing $r_h$. However, the most energetic particles may be unbound and leave the cluster, such that the half-mass radius of the remaining bound stars, $r_{h,\text{bound}}$, could in fact be smaller than the initial half-mass radius, $r_{h,0}$.

Fig. 8 shows the fractional change for the bound stars, $\Delta r_{h,\text{bound}}/r_{h,0}$, as a function of $x$. First note that the change in $r_{h,\text{bound}}$ is larger for the less concentrated clusters regardless of the perturber. However, the bound system expands if the cluster is not too concentrated, while for our most concentrated model ($W_0 = 8$) it contracts $-\Delta r_{h,\text{bound}}$ is negative. This is because for concentrated clusters, most of the added energy is absorbed by the distant

![Figure 5. Energy gain of particles at different times (in $N$-body units) for an impulsive shock ($x = 0.03$) due to a point-mass (top row) and a disc (bottom row). In both cases the cluster is described by a $W_0 = 2$ and the strength of the shock is $\Delta E_{\text{imp}}/|E_0| = 0.1$. Stars that have become unbound are above the dashed line and are shown in red. More stars become unbound for a disc shock because all tidal forces are exerted along a single axis (see text for details).](image-url)
Response of a star cluster to a tidal perturbation

4 DISCUSSION AND INTERPRETATION

We have shown that a star cluster responds differently to a PM tidal perturbation than to a disc perturbation. Moreover, for each perturber and for a given amplitude of the applied shock (i.e. $W_0$), the response depends also on the cluster concentration (i.e. $W_0$). Below we discuss possible reasons for this behaviour.

4.1 The dependence of $\Delta E$ on cluster concentration for slow perturbations

Fig. 1 shows that for slow perturbations with the same $\Delta E_{imp}$, $\Delta E \propto \Delta E_{imp}$ for discs, and $\Delta E > \Delta E_{imp}$ for PMs. In previous work, the reduced energy gain for discs was ascribed to adiabatic damping. We first consider this effect to see whether it could explain the correlation of $\Delta E$ with $W_0$. High $W_0$ clusters may have a larger fraction of stars in the impulsive regime, reducing the effect of damping. To test this idea, we used the discrete realisations of King models with different $W_0$ from our initial conditions, and then analogous to equation (10) we calculated an estimate of the dynamical time for each particle as $t_{dyn}(r) = [\pi^2r^3/4GM(r)]^{1/2}$, where $r$ is the radial position of the particle and $M(r)$ is the enclosed mass at that radius. The fraction of stars with the local dynamical time exceeding the shock duration for our slowest perturbation, $t_{dyn}(r) > \tau \approx 3t_{dyn,b}$ increases monotonically from 0.7% for $W_0 = 2$ to 21% for $W_0 = 8$. It is consistent with the trend of the parameter $\epsilon$ in equation (12) decreasing with $W_0$ from 1.46 to 0.55, indicating
that the transition from impulsive to adiabatic occurs at higher $x$ for larger $W_0$. However, this does not explain fully the qualitative difference between the results for the disc and PM cases shown in Fig. 1.

We therefore also look at the effect of geometrical distortions during the perturbations. For this estimate, we first consider the disc shock because of its simple one-dimensional tidal action, and the two extreme $W_0$ clusters in the suite: $W_0 = 2$ and $W_0 = 8$. We denote their respective quantities with subscript 2 and 8. Consider a slow perturbation with a given $\Delta E_{\text{imp}} = \frac{1}{2} f_{\text{tid}} (r^2)$. Because $\langle r^2 \rangle_8 = 3 \langle r^2 \rangle_2$ (Section 2.3) and $\Delta E_{\text{imp}}$ is the same for both clusters, we have $f_2 \approx 3 f_8$. The velocity change for a particle at $r \approx r_1$ is $\Delta v \propto \sqrt{f_{\text{tid}} r_1}$, and because $r_1, 8 \approx 3 r_1, 2$ (Section 2.3), we find that $\Delta v_8/\Delta v_2 \approx \sqrt{3}$. The fractional compression along $z$ direction in the first half of the perturbation is $F \approx \Delta v/\Delta r$, and therefore the ratio $F_2/F_8 \approx \sqrt{f_2}/f_8 = \sqrt{3}$. This means that, somewhat counter-intuitively, less concentrated clusters (low $W_0$) are more compressed during the perturbation than concentrated clusters. This could also explain why the final $\Delta E$ of low-concentration cluster is lower, because their $(z^2)$ are reduced more relative to the initial ones during the second half of the perturbation.

The PM results can be explained with these simple geometrical arguments: half of the tidal forces are compressive, hence the same arguments of the disc apply. However, the other half are extensive, and this should lead to a larger fractional extension for low-$W_0$ clusters, resulting in a larger $\Delta E$ for low concentration clusters. If we apply symmetry arguments, we expect the effect of compression on $\Delta E$ to cancel the effect of the extension, predicting that the energy gain for slow PM perturbations is the same as for fast shocks.

4.2 On the different response of a star cluster to a disc vs. PM perturbation

To explore in more detail the cause of the differences between a disc and a PM tidal perturbation we split the total energy gain into the kinetic ($K$) and potential ($W$) energy parts. For this exercise we look at the initial configuration and at the middle of the shock. Each of the clusters is binned in radial shells such that each shell contains the same number of particles, in this case we choose 1% of the total of particles per shell. For each shell, we compute the cumulative kinetic and potential energy of the particles contained within the shell. Then we subtract the initial $K$ and $W$ from the values at the
middle of the shock. With this procedure we obtain the cumulative \(\Delta K\) and \(\Delta W\) across the cluster.

Fig. 10 shows \(\Delta T/|E_0|\) and \(\Delta W/|E_0|\) as a function of the percentage of particles from the cluster centre. Note that the evolution of kinetic and potential energies depends strongly on the nature of the perturber. The gain in kinetic energy \(\Delta T\) for the PM perturbation decreases as the duration of the perturbation increases, but has the opposite behavior for the disc. The gain in potential energy \(\Delta W\) for the PM perturbation increases (\(W\) becomes less negative) as the perturbation enters into the adiabatic regime, while for the disc \(W\) becomes more negative.

The reason for the opposing behaviours in both \(\Delta T\) and \(\Delta W\) lies in the different directions of the tides for the two types of perturber. The PM tides have equal extensive and compressive components (Spitzer 1958), while they are fully compressive for a disc case. As shown in Fig. 1, the nature of the tide is not important for the energy gain in the impulsive regime, but it becomes important when the duration of the perturbation increases.

Fig. 10 shows that in the adiabatic regime, as \(x\) gets closer to 1, the extensive tide due to the PM has enough time to pull the stars apart, decreasing the velocity dispersion (\(\Delta T < 0\)) and reducing the binding energy (\(\Delta W > 0\)). On the other hand, the compressive tide due to the disc pushes the stars inwards, increasing the velocity dispersion (\(\Delta T > 0\)) and making the cluster more bound (\(\Delta W < 0\)).

In the formulation of tidal heating by Weinberg (1994b), a star’s energy changes when it passes through a orbital resonance with the external gravitational potential. The number and distribution of resonances is particularly important for a slow perturbation where the gradually evolving stellar orbit may go through multiple resonances. As described by Murali & Weinberg (1997), a spherical potential has a different (discrete) spectrum of resonances than the disc potential (continuous spectrum). This may be another reason for the different cluster response to the two perturbers.

4.3 The effect of the orbit

Another effect that could be important is that in the PM perturbation the impact parameter and relative velocity are time dependent, while for the disc we fix it (see Section 2.2). Here we estimate how much the PM orbit deviates from a straight line. We can do this by realising that for a straight-line orbit the kinetic energy of the orbit is much larger than its gravitational energy, i.e. \(2G M_p/(bV^2) \ll 1\). With the expression for \(M_p\) from equation (1) and the definition of \(\tau\) and \(x\) we then find that the orbit is approximately a straight line when

\[
x \leq 1.6 \sqrt{\frac{0.1}{\Delta E_{imp}/|E_0|}}.
\]

Here we used the \(\langle r^2 \rangle\) values for \(W_0 = 6\). From this we see that the 3 slowest encounters in Fig. 1 are affected by this, but the deviation from a straight-line orbit can not explain the difference between the behaviour of slow perturbations by a disc and a PM. The dependence on \(\Delta E_{imp}\) may explain why the weaker (slow) PM perturbations in Fig. 3 show a larger \(\Delta E\), their orbits are more gravitationally focused and have a smaller impact parameter \(\rho\) which leads to a larger energy gain. The maximum velocity \((V_{\text{max}})\) is higher than \(V\), but this does not compensate the smaller \(\rho\), because the energy gain depends on \(\rho^{-4}V_{\text{max}}^{-2}\).

4.4 The shape of the cluster

In Sec. 3.1 we found that not all our \(N\)-body models can be described by the analytic result for the energy gain of GO99 (equation 11). The result of GO99 is a fitting function with the theoretically expected power-law slope at large \(x\) \((x^{-2})\) and GO99 did not vary their cluster density profile to study its effect. Because of this, in this section we contemplate the possibility that the size and shape of the cluster are changing during the encounter, which could explain the disagreement between our simulations and equation (11), and between the disc and PM perturbation.

To quantify the deformation of the cluster due to the tidal perturbation we need to measure its shape. Following Zemp et al. (2011), we calculate the shape tensor, defined by the matrix components:

\[
S_{jk} = \frac{\sum_i m_i x_{ij} x_{ik}}{\sum_i m_i}.
\]

where index \(i\) runs over \(i = 1, \ldots, N\) stars, and indices \(j\) and \(k\) denote...
the three Cartesian coordinates for star $i$. The principal axes of the cluster can be interpreted as the eigenvectors of $S_{ij}$. The lengths of the primary semi-axes $a \geq b \geq c$ correspond to the square root of the eigenvalues. We compute the matrix $S_{ij}$ for all stars in our $W_0 = 4$ models under the fast and slow perturbations.

Fig. 11 shows the evolution of the semi-axes of the cluster as a function of time for the PM case and $\Delta E_{\text{imp}}/E_0 = 0.1$. The grey shaded area indicates the time interval $\Delta \tau$ where most of the perturbation occurs. We see that as the perturbation goes on, the cluster is slightly stretched along one of its axes, and compressed along the other two. Notice that the magnitude of the changes in the cluster shape are significantly larger for slow perturbations than for the fast ones.

Recall that initially the cluster is located at the origin of the coordinate system while the PM starts moving in the $y$ direction (with a closest approach of $b$ for fast shocks, see equation 1), hence, the long and middle semi-axes lie in the $x - y$ plane and their directions rotate during the flyby. Fig. 12 illustrates the changing orientation of the semi-axes as a function of time for one model. For this plot we computed the eigenvectors of the shape tensor of the cluster for snapshots before, during, and after the moment of closest approach. In the beginning of the perturbation the cluster is slightly stretched but still maintains a nearly spherical shape; then, at the moment of closest approach, the elongation of the cluster is noticeable, and keeps increasing after the PM has passed. The degree of deformation of the cluster is quantified with the axis ratio $a/b$, which increases with time. Note how the semi-major axis rotates due to the gravitational attraction exerted on the cluster by the moving PM.

On the other hand, in the case of a disc perturbation the cluster moves in the $z$ direction and crosses the disc located in the $x - y$ plane. Fig. 13 shows that during the perturbation the $x$-axis and $y$-axis of the cluster remain unchanged, while it is compressed along its $z$-axis. Again, the change in the size of the cluster is significant for slow perturbations, but negligible for the fast ones.

The main difference in cluster response to the PM and disc perturbations is in the sign of the size change. The remaining bound cluster grows in size when the perturber is a PM, while it is compressed when the perturber is a disc.

### 4.5 Expected change in cluster shape

We note that the fractional change of the semi-axis length depends not only on the perturbation strength, but also on its duration.

In this section we characterize the expected change of cluster shape as a function of $x$ and $\Delta E_{\text{imp}}/E_0$. In order to do this we use the $W_0 = 4$ subset from the first row of simulations in Table 1 (perturbation strength 0.1), together with the second and third rows, also with $W_0 = 4$ with perturbation strengths 0.03 and 0.01. Again, we compute the length of the primary semi-axes of the cluster as the square roots of the eigenvalues of the shape tensor (equation 17) at a time $\tau$ after the impact.

Fig. 14 shows the fractional change of the length of the semi-axes as a function of $x$ and different values of $\Delta E_{\text{imp}}$ for a PM shock. We notice that such fractional changes scale as a power law of $\tau/\Delta \tau_{\text{shock}}$ for a fixed shock amplitude, and as another power law of $\Delta E_{\text{imp}}/E_0$ for a fixed duration of the shock. The power indices are similar for all axes, and average close to $3/4$ and $1/2$, respectively. The scaling with the energy change is expected because $\Delta V = \Delta E_{\text{imp}}^{1/2}$ and we analyse all snapshots at the same time, such that the amount of extension/compression is proportional to $\Delta V$. The dependence of $x^{3/4}$ appears to work very well, but we do not have a physical backing for it and it resulted from trying various functional forms. The combination of these two dependencies allows us to predict the shape of the cluster after the shock as follows:

$$ l/\bar{l} = 1 + \mu_1 x^{3/4} (\Delta E_{\text{imp}}/E_0)^{1/2} \tag{18} $$

where $l = \{a, b, c\}$ denotes the three semi-axes. We find the values of constants $\mu_1$, $\mu_2$, and $\mu_3$ by a linear regression fit to the simulation results. This gives the best-fit values $\mu_1 = \{1.16, -0.28, -0.69\}$. Fig. 14 shows how well equation (18) predicts the change in cluster shape for different values of the adiabatic parameter and shock amplitude.

It is worth noting that for a PM shock the semi-major axis always grows, while the other two decrease in size. This is because the tides exerted by a PM are extensive along the line that connects the cluster with the PM ($x$-axis), and compressive in the $z$ direction. For fast shocks there is no net tidal action in the $y$ direction (Spitzer 1958), as seen in the left of Fig. 11. The slight compression in $y$ seen in the right panel is probably because the perturbation is slower, breaking the symmetry because the tides are compressive in the first half and the cluster has time to respond. The nature of the tides is reflected in the sign of the constants $\mu_2$, and well illustrated by Fig. 14, with $a/a_0 > 1$, and $b/b_0, c/c_0 < 1$.

We follow the same procedure for the disc case, putting together our $W_0 = 4$ models with shock amplitudes 0.1, 0.03, and 0.01. Fig. 15 shows the length of the cluster semi-axes as a function of shock duration and shock amplitude. Notice that in all models the semi-lengths $a$ and $b$ change very little compared with the change in the short axis $c$. Also, the change of all three semi-lengths is negative, i.e., the cluster is compressed along its three principal axes after the shock: $a/a_0, b/b_0, c/c_0 < 1$. The expected shape of the cluster after a disc shock can also be expressed by equation (18) but with different constants $\mu_2 = \{-0.081, -0.093, -1.12\}$. These three constants are negative, and reflect a larger compression in the $z$ direction, as expected for a disc crossing.

Interestingly, the change in cluster shape described above is independent of the values of $W_0$ explored in this work. The reason for this is probably because our experiments were set up such that for a certain $\Delta E_{\text{imp}}/E_0$, stars at a distance $r = \sqrt{\tau_{\text{shock}}}$ from the cluster centre receive the same velocity kick in clusters with different $W_0$. Because we then measure the shape parameter at that radius, it is expected to be similar across clusters with different $W_0$. The deformation at larger radii for clusters with larger $W_0$ is higher. Hence, either for a PM or disc shock, the expected shape of the cluster follows the same functional form (equation 18), where the sign of the constants $\mu_2$ reflects the nature of the tides along each direction.

### 4.6 Comparison with other $N$-body results

A recent study by Webb et al. (2019) investigated the mass loss of a star cluster experiencing a sequence of two tidal shocks separated by a time interval. The authors use the same version of the $\text{voron}\_\text{code}$ code to model clusters of 50,000 stars with the Plummer density profile and two King models. They consider fast shocks and low-density clusters, such that the encounters are mostly in the impulsive regime ($s < 0.25$). The clusters are subject to shocks with only one non-zero component of the tidal tensor $T_{xx} > 0$ that varies as a step function in time, i.e. a one-dimensional extensive shock. We note that the associated mass density of this tensor is negative and that realistic perturbers have several non-zero components of the tidal
Figure 11. Temporal evolution of the cluster semi-axis lengths for a fast PM shock ($x = 0.03$) and a slow PM tidal perturbation ($W_0 = 4, x = 0.62$), both with $\Delta E_{\text{imp}}/|E_0| = 0.1$. The dashed black line indicates the time of maximum approach, $t = 0$, while the grey shaded area covers the time interval $[-\tau, +\tau]$, where $\tau$ is the duration of the perturbation.

Figure 12. Temporal evolution of the cluster semi-axis direction for a PM tidal perturbation ($W_0 = 4, x = 0.62$), with $\Delta E_{\text{imp}}/|E_0| = 0.1$. The cluster is at the origin while the PM moves along the $y$ direction, which imprints a rotation to the cluster. We show the same time interval as in Fig. 11, with $t = 0$ being the moment of closest approach. We indicate the angle between the semi-major axis (red arrow) and the $x$ axis, as well as the ratio of the semi-lengths $a/b$.

Figure 13. Temporal evolution of the cluster semi-axis length for a fast disc shock ($x = 0.03$) and a slow disc tidal perturbation ($x = 0.62$), both with $\Delta E_{\text{imp}}/|E_0| = 0.1$. The dashed black line indicates the time of maximum approach, $t = 0$, while the grey shaded area covers the time interval $[-\tau, +\tau]$, where $\tau$ is the duration of the perturbation.
applying the tidal forces along a single axis. The larger fractional mass loss is likely because of their choice of the tidal tensor: applying the tidal forces along a single axis results in a larger velocity increase of individual stars than when the tides act along multiple axes (for a given energy increase, see the discussion in Section 3.3). This is why we find a higher fractional mass loss for the disc case than for the PM case. Because their tides are extensive, particles are pushed away from the cluster, which makes it easier to become unbound compared to our disc case, where particles still need to travel through the cluster. Another difference between our studies is in the cluster density profile: the Plummer model is different from the King models. However, Gieles et al. (2006) applied point-mass perturbations to Plummer models and also found $|\Delta M|/M_0 \approx 0.22 \Delta E/|E_0|$. We think that the larger fractional mass loss of Webb et al. is because of their tidal tensor. In addition, for such extensive tides the cluster may not have reached full dynamical equilibrium even after 10 crossing times and the value of $\Delta E$ may still be evolving.

More surprisingly, Webb et al. (2019) find a significantly non-linear relation $\Delta M/M_0 \propto I_{tid}^{-\gamma}$, with a shallow slope $\gamma \approx 0.6$. In contrast, our results in equation (15) point to a steeper slope $\gamma > 1$ for all but the most concentrated model. They attribute the non-linearity of this relation to a substantial escape time for newly unbound stars. While the escape time effect plays a role in decreasing mass loss in tidally limited clusters (e.g., Lee & Ostriker 1987; Baumgardt 2001), it is less likely to affect isolated clusters studied by Webb et al. (2019). We therefore conclude that the escape time is not the main cause of the non-linearity of $\Delta M(I_{tid})$. Note that in our results the $\Delta M(\Delta E)$ relation is as non-linear as $\Delta M(I_{tid})$, while $\Delta E$ is exactly linearly proportional to $I_{tid}$ as expected from the tidal theory.

We also note that in this work we considered a spherical perturber as infinitely compact PM, appropriate for dense molecular clouds or star-forming regions. More extended perturbers, such as a galactic bulge or a satellite galaxy, may lead to more significant adiabatic damping of the energy change (c.f. Gnedin et al. 1999).

### 5 SUMMARY

We studied the response of a star cluster to different tidal perturbations by using direct N-body simulations. We measured the energy gain of the cluster as it experiences a PM flyby or a disc crossing. We used different density profiles given by King (1966) models with dimensionless central concentration parameters $W_0 = [2, 4, 6, 8]$. Also, we explored the impulsive and adiabatic regimes by varying the duration of the encounter.

For fast shocks the predictions from the impulse approximation are recovered for both perturber types and all cluster concentrations. However, in the adiabatic regime we find important differences. For slow disc crossings, we find a smaller energy increase than in the impulsive regime. This has previously been attributed to the effect of adiabatic damping (Gnedin & Ostriker 1999). We show that the energy change depends on the cluster’s concentration (i.e. $W_0$), in a way that is consistent with adiabatic damping: clusters with less stars in the impulsive regime have a smaller energy increase. However, this $W_0$ dependence in the energy change can also be attributed to a geometrical distortion (i.e. compression) in the first half of the perturbation, which reduces the importance of the remaining part of the perturbation because the cluster is compressed along the $z$ direction. For similar reasons, slow PM perturbations lead to

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2 Note that Webb et al. (2019) performed the fit with $I_{tid}$ defined as the square root of our expression, and we have rescaled their slope for consistency with our definition.
much larger energy increases, comparable or even larger than what is expected in the impulsive regime. Because the PM tidal forces have an extensive component, this result can be explained by geometrical distortion. Since we see a similar correlation of $\Delta E$ with $W_0$, we conclude that adiabatic damping does play a role, of similar importance to the geometrical distortion.

We present an accurate parameterization of the expected changes in cluster shape as a function of the amplitude of the perturbation and the duration of the shock (equation 18).

These results are useful for analytical modeling of cluster disruption for a sub-grid models of cluster evolution in numerical simulations of galaxy formation. We conclude that the following steps are necessary to correctly model the impact of tidal shocks. To compute the energy gain $\Delta E$, it is important first to identify whether the perturbation is compressive (disc-like) or has an extensive component (PM-like). For a disc-like perturbation, $\Delta E$ is computed taking into account the cluster density profile as expressed in equation (12) and illustrated in Fig. 2. On the other hand, for a PM-like perturbation the value of $\Delta E$ is nearly constant for $x < 0.7$, then it increases as a function of $x$ and $W_0$.

To understand the mass evolution, the most important assumption is the cluster density profile, which affects the resulting mass loss (for a given $I_{tid}$) by more than an order of magnitude. Semi-analytic models of cluster evolution with tidal perturbations usually assume a fixed value for the $W_0$ parameter (e.g. Gieles et al. 2006; Elmegreen 2010; Pfeffer et al. 2018). To increase the predictive power of such models, we need better understanding of the evolution of the density profile of clusters evolving in a Galactic tidal field, experiencing repeated shocks, two-body relaxation and stellar evolution.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

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