Performance of an Originally Massive Vector Boson in the Thermal Plasma from the Aspect of the Goldstone Equivalence Gauge

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Abstract

We applied the Goldstone equivalence gauge to calculate the thermal corrections to an originally massive vector boson. We find that part of the Goldstone is spewed out from the longitudinal polarization, and therefore one extra physical degree of freedom arises as in the originally massless vector boson case. We also show the Feynmann rules for the “external legs” of the originally massive vector boson as well as the physical Goldstone boson.

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I. INTRODUCTION

In the literature, it is well-known that the number of degrees of freedom of a photon or a gluon which is originally massless in the thermal environment is different from that in the zero temperature. Due to the collective motion of the plasma particles, the longitudinal degree of freedom arises in the form of a quasi-particle. Such an oscillation mode is called a “plasmon” and has long been studied (For the early works, see Ref. [1, 2]. For some applications in the early universe, see Ref. [3, 4], and see Ref. [5] for detecting the axion. Ref. [6] provided the systematic derivations). The performance of an originally massless vector boson in the plasma can be studied through calculating the self-energy loop diagrams. In the thermal plasma, temperature-dependent mass corrections arise for both transverse and longitudinal propagators. The dispersion relations for an “on-shell” boson in the plasma then become complicated, and are different between the transverse and longitudinal polarizations.

Transplanting these discussions directly to an originally massive vector boson faces difficulties. If we rely on the covariant \( R - \xi \) gauge, which is the most convenient in the zero temperature situation, for a general \( \xi \), the propagators cannot be simply decomposed into transverse and longitudinal polarization contributions for both on-shell and off-shell bosons separately. For the Landau gauge, the polarizations can be separated, but the relationship between the Goldstone boson and the longitudinal polarization is still indirect, disturbing our calculations and analysis very much. It is also difficult for us to figure out the Feynmann rules of the on-shell Goldstone boson as the external legs.

In some situations, such problems can be concealed if we avoid calculating the diagrams with vector bosons as the external legs. For example, the dark matter annihilation to vector bosons in the thermal plasma is equivalent to the imaginary part of the dark matter self-energy up to the two-loop level. We can resum the vector boson’s self-energy contributions in the inner propagators to avoid the external leg’s Feynmann rules. However, a reliable tree-level method can reach a better picture while avoiding the complicated and ambiguous loop calculations. Thus, we need the help of the physical gauges, for which the bilinear terms between the gauge boson and the Goldstone boson are preserved, and the Goldstone boson is also treated as a part of the extended polarization vectors. Unlike the Landau gauge situation, all the \( k^2 = 0 \) pole residues are contributed by the Goldstone part of the propagator in the physical gauge. It is then more convenient for us to observe how the
vector boson “eats” the Goldstone degree of freedom [7, 8]. With the physical gauge, the propagator can always be decomposed into the transverse and longitudinal parts for both on-shell and off-shell particles, so we can easily treat these two parts separately. This is beneficial for us to easily study the thermal performances of an originally massive vector boson, and then write down the Feynmann rules for the “on-shell” boson external legs in the momentum space.

In this paper, we adopt the “Goldstone equivalence gauge” introduced by Ref. [9]. In this gauge, the transverse polarization vector $\epsilon_\pm(k)$ is the same as in the unitary gauge, then we can reuse the existing general decomposition of the tensor structure of a vector boson’s self-energy results. We will find out that in the thermal plasma, the longitudinal polarization of the vector boson will somehow gradually “decouple” with the Goldstone boson. A massless Goldstone boson then recovers in the thermal plasma, providing an extra degree of freedom as in the case of the originally massless vector bosons.

II. LAGRANGIAN ADOPTED AND THE ZERO TEMPERATURE PROPAGATOR DECOMPOSITIONS

For simplicity, we rely on a $U(1)$ toy model with only one gauge boson $A_\mu$ and one complex Higgs boson $H$. Part of the Lagrangian is then given by

$$\mathcal{L} \supset -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu H^\dagger D^\mu H + V(H),$$

(1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu - igA_\mu$, with $g$ to be the gauge coupling constant, and $V(H)$ is the gauge-invariant potential of the scalar sector. We do not concern the details on $V(H)$, and only need to know that this induces a vacuum expectation value (vev) $v$ of the Higgs boson to break the gauge symmetry spontaneously. Therefore,

$$H = \frac{v + h + i\phi}{\sqrt{2}},$$

(2)

where $h$ is the remained Higgs boson, and $\phi$ is the Goldstone boson. Then the Lagrangian becomes

$$\mathcal{L} \supset -\frac{1}{2} \partial^\mu A^\nu \partial_{\mu\nu} A_\mu + \frac{1}{2} \partial^\mu A_\mu \partial^\nu A_\nu + \frac{1}{2} m_A^2 A_\mu A^\mu - m_A A^\mu \partial_\mu \phi + \frac{1}{2} (\partial^\mu \phi)^2,$$

(3)

where $m_A = gv$. Besides the $[\mathbf{1}]$, the vector boson might couple with other fields, which contribute to the thermal masses in the one-loop level. We just parametrize these contributions by the temperature dependent functions $\Pi_{L,T,S}(k)$, or in the hard thermal loop (HTL)
approximation, a thermal mass parameter $m_E$ in the later discussions. Since we do not study the details of the couplings, we neglect all of them in the Lagrangian.

Now we introduce the gauge $n_\mu A^\mu = 0$, where $n_\mu = (1, -\vec{k}/|\vec{k}|)$ for the Goldstone equivalence gauge\[9\] and $k$ is the four-momentum of a plain wave in the momentum space. Notice that

$$n^\mu = \sqrt{\vec{k}^2} \epsilon_{LU}^{\mu}(k) - k^\mu, \quad (4)$$

where $\epsilon_{LU}^{\mu}(k) = (|\vec{k}|, k_0 \vec{k}/|\vec{k}|)/\sqrt{\vec{k}^2}$ is the usual longitudinal polarization vector in the unitary gauge. One can easily verify that $n_\mu n^\mu = 0$, making it to be a kind of light-cone gauge to simplify the calculations.

With the aid of the gauge-fixing term

$$\mathcal{L}_{gf} = \frac{1}{2\xi} (n \cdot \nabla n \cdot A)^2, \quad (5)$$

the propagator of the vector boson together with the Goldstone part then becomes

$$\langle (A^\mu, \phi), (A^\mu, \phi) \rangle = i \frac{k^2 - m_A^2 + \text{i} \epsilon}{k^2 - m_A^2 + \text{i} \epsilon + \text{i} \epsilon_k \cdot (\vec{n} \cdot k)} \left( -\left(g^{\mu\nu} - \frac{n^\mu k^\nu + n^\nu k^\mu}{n \cdot k}\right) + \frac{m_A^2}{n \cdot k} \right) \left( -\left(g^{\mu\nu} - \frac{n^\mu k^\nu + n^\nu k^\mu}{n \cdot k}\right) + \frac{m_A^2}{n \cdot k} \right) \delta_{ij} - \frac{n^i n^j}{n \cdot k} \delta_{ij} - \frac{n^i n^j}{n \cdot k} \delta_{ij} \quad (6)$$

in the $\xi \to 0$ limit, where we have preserved the $n^2$ terms explicitly in order to compare with the general form in Ref. \[8\]. The matrix is extended from 4-dimension to 5-dimension, with an extra Goldstone degree of freedom. We use $\mu\nu\ldots$ to indicates the 4-dimensional indices, and $MN\ldots$ to express the extended indices including the Goldstone degree of freedom. Note that we adopt the symbols and the contraction rules from Ref. \[8\] for convenience.

Now we define the transverse polarization vectors $\epsilon_s^\mu$, where $s = \pm$. They satisfy

$$\epsilon_s^0 = 0, \quad k \cdot \epsilon_s = 0, \quad \epsilon_+ \cdot \epsilon_- = \epsilon_- \cdot \epsilon_+ = 0, \quad \epsilon_- \cdot \epsilon_+ = \epsilon_+ \cdot \epsilon_- = -1. \quad (7)$$

For the special $k = (k_0, 0, 0, k_3)$ case, $\epsilon_s(k) = \frac{1}{\sqrt{2}} (0, 1, \pm i, 0)$. The $\epsilon_s(k)$ of a general $k$ can be acquired by directly rotating from the z-direction case. We define the transverse projection operator $P_T^{\mu\nu} = \sum_{s=\pm} \epsilon_s^\mu \epsilon_s^\nu$. It is easy to verify that

$$P_T^{ij} = \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}, \quad P_T^{0i} = P_T^{i0} = P_T^{00} = 0, \quad (8)$$
where \( i, j \) are the space coordinates.

Extend \( \epsilon_s^\mu \) to \( \epsilon_s^M = \left( \begin{array}{c} \epsilon_s^\mu \\ 0 \end{array} \right) \), and \( P_T^{\mu \nu} \) to \( P_T^{MN} \) where the extra elements are supplemented with zero. The factors in the matrix of (6) can be decomposed of

\[
\sum_{s=\pm} \epsilon_s^M \epsilon_s^{N *} + \left( \begin{array}{cc} \frac{k^2}{(n-k)^2} n^\mu n^\nu + i \frac{m_A}{n \cdot k} n^\mu & i \frac{m_A}{n \cdot k} n^\mu \\ -i \frac{m_A}{n \cdot k} n^\mu & \frac{1}{1} \end{array} \right) = P_T + \left( \begin{array}{cc} \frac{k^2}{(n-k)^2} n^\mu n^\nu + i \frac{m_A}{n \cdot k} n^\mu & i \frac{m_A}{n \cdot k} n^\mu \\ -i \frac{m_A}{n \cdot k} n^\mu & 1 \end{array} \right). \tag{9}
\]

Near the \( k^2 = m_A^2 \) pole, the second term in (9) can be decomposed into \( \epsilon_L^M \epsilon_L^{N *} \), where \( \epsilon_L^M = \left( \begin{array}{c} -\frac{m_A}{n \cdot k} n^\mu \\ i \end{array} \right) \). This is the longitudinal polarization vector in the Goldstone equivalence gauge. However, in the off-shell case when \( k^2 \neq m_A^2 \), such a simple decomposition no longer exists. We define the longitudinal and Goldstone projector operators to be

\[
P_L = \left( \begin{array}{cc} \frac{k^2}{(n-k)^2} n^\mu n^\nu + i \frac{m_A}{n \cdot k} n^\mu & m_A^2 \frac{1}{k^2 + i \epsilon} \\ -i \frac{m_A}{n \cdot k} n^\mu & \frac{1}{1} \end{array} \right), \tag{10}
\]

\[
P_G = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & k^2 - m_A^2 + i \epsilon \\ \frac{k^2 - m_A^2 + i \epsilon}{k^2 + i \epsilon} \end{array} \right). \tag{11}
\]

Redefine

\[
\epsilon_L^M (k) = \left( \begin{array}{c} \frac{-\sqrt{k^2}}{n \cdot k} n^\mu \\ \frac{i m_A}{\sqrt{k^2}} \end{array} \right) \tag{12}
\]

as the longitudinal polarization vector for both on-shell and off-shell (at least for time-like) vector bosons, we can easily see if we neglect the \( i \epsilon \) term,

\[
P_L^{MN} = \epsilon_L^M \epsilon_L^{N *}. \tag{13}
\]

Finally, the propagator can be decomposed to

\[
\langle (A^\mu, I), (A^\nu, I) \rangle = \frac{i}{k^2 - m_A^2 + i \epsilon} (P_T + P_L + P_G), \tag{14}
\]

where the Goldstone projector denominator \( k^2 - m_A^2 + i \epsilon \) will cancel the \( k^2 = m_A^2 \) pole while contribute to another \( k^2 = 0 \) pole. This pole is then cancelled by the \( \frac{m_A^2}{k^2} \) element in the longitudinal polarization projector, leaving us no real massless degree of freedom. Therefore we can see clearly how the Goldstone boson has been “eaten” by the longitudinal polarization of the vector boson.
III. THERMAL EFFECTS ADDED

In the thermal environment, the propagator of any particle should be corrected by the distribution functions. Remember the vector bosons obey the Bose-Einstein distribution, so we define

\[ n_B(k_0) = \frac{1}{e^{\beta|k_0|} - 1}, \]

where \( \beta = \frac{1}{T} \) and \( T \) is the temperature. The tree-level thermal propagator can be written in the “diagonalized form” (See Page 204 of Ref. [10] for the corresponding details)

\[ D^{F, MN}_{ab}(k) = U_{ac}(k) \begin{pmatrix} \frac{i}{k^2 - m_A^2 + i\epsilon} & 0 \\ 0 & -\frac{i}{k^2 - m_A^2 - i\epsilon} \end{pmatrix}_{cd} U_{db}(k)(P_T + P_L + P_G)^{MN}, \]

where \( U \) is given by

\[ U(k) = \begin{pmatrix} \sqrt{1 + n_B(k_0)} & \sqrt{n_B(k_0)} \\ \sqrt{n_B(k_0)} & \sqrt{1 + n_B(k_0)} \end{pmatrix}. \]

The above propagator is calculated in the “\( \sigma = \beta/2 \)” condition. This is convenient for computing the mass shift in the “real-time formalism”, because the self-energy diagram can also be written in the “diagonalized form”

\[ -i\Pi^M_{ab}(k) = U^{-1}_{ac}(k) \begin{pmatrix} -i\Pi^M_{MN}(k) & 0 \\ 0 & (-i\Pi^M_{MN}(k)^*) \end{pmatrix}_{cd} U^{-1}_{db}(k). \]

Therefore all of the \( U(k) \) and \( U^{-1}(k) \) cancels with each other inside the “self-energy string” diagrams, leaving only those in the beginning and the end. We will then decompose \( \Pi^M_{MN}(k) \) to see its temperature dependence.

In order to calculate the full thermal corrections on \( \Pi^M_{MN}(k) \), we need by principle to compute all the self-energy diagrams in Fig. 1. Recall that at zero temperature, the first diagram contributes to the \( \delta m_1^2 A_\mu A^\mu \) operator, the second contributes to the \( \frac{1}{2} \delta m_2 A_\mu \partial_\mu \phi \) operator, and the third one contributes to the \( \delta m_3^2 \phi^2 \) operator. We expect \( \delta m_1^2 = 2m_A \delta m_2 \) and \( \delta m_3^2 = 0 \) for the self-consistence to keep the tensor structure of the (3) and (6) unchanged. These relationships are not so obvious in Fig. 1. However, if we at first rely on the model without a vacuum expectation value, and notice that many of the diagrams in Fig. 1 can be regarded as the diagrams in Fig. 2 with the vacuum expectation values inserted, we can find
the origin of such expressions. The first diagram in Fig. 2 does not correct the mass, and it only contributes to the wave function renormalization of the gauge bosons. The second diagram contribute to the $A_\mu A^\mu h^2$ operator, and correct the $g^2$. The third diagram contribute to the $A_\mu (\partial_\mu \phi) h$ operator, and correct the $g$. Therefore, we find out that $\delta m_1^2 = 2 m_A \delta m_2$ is equivalent to the $\delta g^2 = 2 g \delta g$, which is guaranteed by the gauge symmetry. The last two diagrams in Fig. 2 only correct the Goldstone boson’s mass. They are actually correcting the potential $V(H)$. More Higgs external legs might involve in some models. Since these two diagrams have nothing to do with the gauge coupling constants, we can, e.g., in the $V(H) = -m_h^2 H H^\dagger + \lambda (H H^\dagger)^2$ situation, correct both the $\delta m_h^2$ and $\delta \lambda$ to keep the Goldstone massless. This is equivalent to changing the vev in order to keep the system in a minimum. With these experiences, we then start to discuss the finite temperature case.

Define $u^\mu = (1, 0, 0, 0)$ to specify the rest frame of the system, and let $u_T^\mu = u_\mu - k_\mu \frac{u^k}{k^2}$. $\Pi^{\mu\nu}(k)$ can generally be decomposed to the following terms:[11]

$$\Pi^{\mu\nu}(k) = \Pi^T(k) P_T^{\mu\nu} + \Pi_L(k) P_L^{\mu\nu} + \Pi_S(k) S^{\mu\nu} + \Pi_U(k) P_U^{\mu\nu},$$

(19)

where

$$P_L^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} - P_T^{\mu\nu},$$

$$P_U^{\mu\nu} = \frac{k^\mu k^\nu}{k^2},$$

$$S^{\mu\nu} = \frac{1}{2|k|} (k^\mu u^\nu_T + k^\nu u^\mu_T).$$

(20)

Notice that we have slightly modified some factors compared with Ref. [11] for later practical usage. Then $\Pi^{MN}(k)$ can be written as

$$\Pi^{MN}(k) = \begin{pmatrix}
\Pi_T(k) P_T^{\mu\nu} + \Pi_L(k) P_L^{\mu\nu} + \Pi_S(k) S^{\mu\nu} + \Pi_U(k) P_U^{\mu\nu} C k^\nu + D u_T^{\mu} \\
C^* k^\nu + D^* u_T^{\mu} \\
E
\end{pmatrix}.$$
FIG. 2: Diagrams contributing to the self-energy diagrams before the $U(1)$ spontaneously breaking.

This is the general form to decompose the vector boson self-energy. We then apply the extended Ward-Takahashi identity in the broken phase $k^*_M \Pi^{MN}(k) = 0$ (See Ref. [8, 29] for some discussions.) to constrain the parameters, where $\frac{k^*_M}{m_A} = (\frac{k_\mu}{m_A}, -i)$. Comparing the tensor structures, one can acquire

$$\Pi_U(k) \frac{m_A}{C^*i} = 0, \quad (22)$$

$$\Pi_S(k) k^2 \frac{2k}{2|m_A} = D^*i = 0, \quad (23)$$

$$C \frac{k^2}{m_A} - iE = 0. \quad (24)$$

In fact, $E$ corrects the Goldstone mass term. A non zero Goldstone mass term means the departure from the minimum. Heading for a minimum looks like introducing a counter term to cancel $E$. Therefore, we will always have $E = 0$, so [24] tells us $C = 0$. Then according to [22], $\Pi_U(k)$ finally becomes zero in the new minimum. Therefore,

$$\Pi^{MN}(k) = \left( \begin{array}{cc}
\Pi_T(k) P^\mu_T + \Pi_L(k) P^\mu_L + \Pi_S(k) S^\mu_S - \frac{\Pi_S(k) k^2}{2i|k|m_A} u_T \nu & 0 \\
\frac{\Pi_S(k) k^2}{2i|k|m_A} u_T \nu & 0
\end{array} \right). \quad (25)$$

Notice that $P_L P_T = P_T P_L = P_T S = S P_T = 0 = P_L' P_T = P_T P_L' = 0$, $P_L P_L' P_L = P_L$. 

\[ P_{L}S_{P_{L}} = P_{L}, \] and when performing the calculations, e.g., the \( P_{L}^{MN}\Pi_{MN}(k) \), the “metric” \( g_{MN} = \text{diag}(1, -1, -1, -1, -1) \) is required. After some tedious calculations of summing over all the “self-energy strings”, the full propagator \((14)\) finally changes to

\[ D_{ab}^{\text{full}, MN}(k) = U_{ac}(k) \begin{pmatrix} D_{0}^{\text{full}, MN}(k) & 0 \\ 0 & D_{0}^{\text{full}*, MN}(k) \end{pmatrix}_{cd} U_{db}(k), \] (26)

where

\[ D_{0}^{\text{full}, MN}(k) = \frac{i}{k^2 - m_{A}^2 - \Pi_{T}(k) + i\epsilon} P_{T} + \frac{i}{k^2 - m_{A}^2 - \Pi_{L}(k)} + i\epsilon P_{L} + \frac{i}{k^2 + i\epsilon} \begin{bmatrix} 0_{4\times4} & 0_{4\times1} \\ 0_{1\times4} & 1 \end{bmatrix} \] (27)

As in the Ref. [11], \( \Pi_{S} \) does not contribute to the mass shifts.

Usually, when one applies the hard thermal loop approximation, \( \Pi_{T} \) and \( \Pi_{L} \) have the universal formats and are given by (Ref. [12], cited on Page 124 of Ref. [6])

\[ \Pi_{L}(k) = -2m_{E}^2 k_{0}^2 k_{0} \left( 1 - \frac{k_{0}}{|k|} Q_{0}(\frac{k_{0}}{|k|}) \right), \]
\[ \Pi_{T}(k) = \frac{1}{2}(2m_{E}^2 - \Pi_{L}(k)). \] (28)

where

\[ Q_{0}(\frac{k_{0}}{|k|}) = \frac{1}{2} \ln \frac{k_{0} + |k|}{k_{0} - |k|} - \frac{i}{2}, \] (29)

and \( m_{E} \) is the thermal mass parameter depending on the temperature \( T \) of the longitudinal polarization calculated in the Euclidean space. These arise from the vector boson’s coupling with all the particles (including itself in the non-abelian situation). The detailed calculations of the thermal masses are beyond the discussions of this paper. We therefore treat \( m_{E} \) as a parameter.

[28] is equivalent to the effective operator (See Ref. [13, 14] for early discussions. (Page 185 of Ref. [15] provides the following formalism.)

\[ \mathcal{L} \subset \frac{m_{E}^2}{2} \int d\Omega_{4} \text{Tr} \left[ \left( \frac{1}{V \cdot D} \mathcal{V}^{\alpha} F_{\alpha\mu} \right) \left( \frac{1}{V \cdot D} \mathcal{V}^{\beta} F_{\beta}^{\mu} \right) \right], \] (30)

where \( \mathcal{V} = (1, \frac{\vec{v}}{v_{0}}) \) is a light-like four-velocity, and \( D \) is the covariant derivative in the adjoint representation.

Usually, the thermal corrections introduce extra imaginary parts in the denominator of the propagators. In this paper, we are trying to figure out the “on-shell” behaviours of the
vector bosons, so we ignore these extra widths. The polarization vectors of a transverse vector boson remain unchanged, only the dispersion relation changes to \( k^2 = m_A^2 + \Pi_T(k) \).

For an on-shell longitudinal vector boson, solve the equation \( k^2 - m_A^2 - \Pi_L(k) = 0 \) to acquire the effective total mass \( k^2 = m_A^2 + \Pi_L(k) = m_A^2(k) \). Notice that the thermal mass \( m_A^2(k) \) depends on \( k \). From (10) we can acquire the polarization vector to be

\[
\epsilon'_L = \begin{pmatrix} \frac{m_A'}{m_A} n^\mu \\ \frac{i}{m_A} \end{pmatrix} .
\]

As the temperature arises, generally \( m_A' \) becomes larger, so the Goldstone component is suppressed by \( \frac{m_A}{m_A'} < 1 \). This implies that the longitudinal polarization of the vector boson is partly “spewing out” the Goldstone component.

The residue of the vector bosons are also shifted. Define

\[
Z_{T,L}(k) = \frac{2k_0}{2k_0 - \frac{\partial \Pi_{T,L}(k)}{\partial k_0}} ,
\]

as the “wave-function renormalization parameter”, then each external leg of the transverse or longitudinal vector boson should be multiplied with \( \sqrt{Z_{T,L}(k)} \).

For the \( k^2 = 0 \) pole, we still define the corresponding \( m_A^2(k) = m_A^2 + \Pi_L(k) \). Calculate the \( M = N = 4 \), or the Goldstone part of the \( D_0^{M} \) by summing over the corresponding elements in the three terms on the right-handed side of the equation (27), the pole then becomes

\[
\frac{k^2 - m_A^2 + i\epsilon + m_A^2}{k^2 - m_A^2 + i\epsilon} \approx \frac{m_A^2}{m_A^2} \frac{i}{k^2} .
\]

This means that the Goldstone \( k^2 = 0 \) pole partly resurrects to become a “physical” pole, or can become an “on-shell” state. However, all the “on-shell” Goldstone interactions receive a \( \sqrt{\frac{m_A^2 - m_E^2}{m_A^2}} \) correction. Usually \( \sqrt{\frac{m_A^2 - m_E^2}{m_A^2}} \leq 1 \), implying that only a “fraction” of the Goldstone boson recovers.

Finally, we are ready to write down the Feynmann rules of the originally massive vector bosons by the following steps

- Calculate the effective thermal potential of the Higgs boson as usual, find out the vev \( v \) for the minimum, then calculate the “original mass” of the vector boson \( m_A \).

- Calculate the \( \Pi_L(k) \) and \( \Pi_T(k) \). For the hard thermal loop approximation, these can be attributed to calculating the thermal mass for the zero-energy longitudinal vector boson \( m_E \) as usual. Remember to calculate \( Z_{T,L}(k) \) from (32) as well.
• For the vector boson/Goldstone inner propagators, directly use (27).

• For an external leg of a vector boson, the transverse polarization is the same as the zero-temperature situation. Notice that the on-shell dispersion relation should be modified to 
\[ k^2 = m_A^2 + \Pi_T(k) \]. This usually involves solving the transcendental equations. A factor of \( \sqrt{Z_T(k)} \) is also required.

• For an external leg of a longitudinal vector boson, the on-shell relation is
\[ k^2 = m_A^2 + \Pi_L(k). \]
Calculate \( m_A'^2 = m_A^2 + \Pi_L(k) \), then the polarization vector should be the form of (31). A factor of \( \sqrt{Z_L(k)} \) is required as well.

• External legs of the Goldstone boson should not be forgotten. Define 
\[ m_A'^{2}(k) = m_A^2 + \Pi_L(k), \]
there should be an extra \( \frac{\sqrt{m_A'^2 - m_A^2}}{m_A^2} \) factor for each of the external Goldstone bosons.

• Notice for each external leg, a factor \( \sqrt{n_B(k)} \) is sometimes required for each initial state vector boson (or Goldstone), and a factor \( \sqrt{1 + n_B(k)} \) is always required for each final state vector boson (or Goldstone).

IV. TRANSFERRING TO OTHER GAUGES

The physical gauges are not the main stream of the practical calculations in the literature. If we apply the usual \( R-\xi \) gauge, we can still write down the \( 5 \times 5 \) propagators like the (6), with no \( \phi-A^\mu \) “crossing-terms”. The Goldstone part is also replaced with \( \frac{k^2 - m_A^2}{k^2 - \xi m_A^2} \). However, it is difficult to find a proper decomposition like (14) except when \( \xi = 0 \) (Landau gauge).

In the Landau gauge, (6) is replaced with
\[
\langle (A^\mu, \phi), (A^\nu, \phi) \rangle = \frac{i}{k^2 - m_A^2 + i\epsilon} \begin{pmatrix}
-g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 + i\epsilon} & 0_{4\times1} \\
0_{1\times4} & \begin{pmatrix}
\frac{k^2 - m_A^2 + i\epsilon}{k^2 + i\epsilon}
\end{pmatrix}
\end{pmatrix}.
\] (34)
The propagator is decomposed to
\[
\langle (A^\mu, I), (A^\nu, I) \rangle = \frac{i}{k^2 - m_A^2 + i\epsilon}(P_T + P'_L + P_G),
\] (35)
where the \( 5 \times 5 \) longitudinal part \( P'_L \) for this situation is extended directly from the \( 4 \times 4 \) \( P'_L \) defined in (20). The following calculations are similar to the processes in our paper, and we again encounter something like (27). However, in this case, besides \( D_{ab}^{E,44} \), other components
of the matrix also contribute to the $k^2 = 0$ pole. The total residue of the $k^2 = 0$ pole can only be explicitly extracted after a complete index contraction with other fields. In fact, the vector-boson part contributes to the $k^2 = 0$ pole through the $\frac{k^n k^\nu}{k^2}$ term. If there is no thermal correction, we expect this to become $- \frac{k^2 - m_A^2 + i \epsilon}{k^2}$ ultimately to cancel with the Goldstone $k^2 = 0$ pole. However, thermal mass terms shift the $\frac{i}{k^2}$ in the numerator to $\frac{i}{k^2 - m_A^2 + i \epsilon}$ while leaving the $k^2 - m_A^2$ in the numerator intact. Notice that the Goldstone mass term again does not receive any corrections in the minimum. Summing over the $\frac{i}{k^2}$ terms, we again acquire the (33).

There is no difference on the transverse polarizations between the Landau gauge and the Goldstone equivalence gauge. However, the longitudinal polarization of a vector boson in the Landau gauge actually recovers to the usual $\epsilon_{LU\mu}(k) = (|\vec{k}|, k_0 |\vec{k}|)/\sqrt{k^2}$, with no additional Goldstone part involved. Notice that

$$\epsilon^M_L + \frac{k^M}{m_A} = \left( \begin{array}{c} \epsilon^\mu_{LU} \\ 0 \end{array} \right)_M,$$

where, $\frac{k^M}{m_A} = (\frac{k^\mu}{m_A}, -i)$. The extended Ward-Identity guarantees that $k^M$ does not contribute to the amplitude, therefore $\epsilon_L$ and $\epsilon_{LU}$ are equivalent in the practical calculations.

From the above discussions, we can see that the advantage of the physical gauge is that all the contributions to the $k^2 = 0$ pole have been transported to one single element, making it easier for us to separate the physical Goldstone terms.

Coulomb gauge is also a common selection in the literature. In this case, the gauge vector is chosen to be

$$n^\mu_C = (0, \vec{k}).$$

The longitudinal polarization vector in (12) then changes to

$$\epsilon^M_{LC}(k) = \frac{1}{\sqrt{\frac{m_A^2}{k^2} - \frac{n^2_C m_A^2}{(n^2_C k^2)}}} \left( \frac{-m_A (n^\mu_C - \frac{n^2_C k^\mu}{n^2_C k^2})}{i \left( \frac{m_A^2}{k^2} - \frac{n^2_C m_A^2}{(n^2_C k^2)} \right)} \right),$$

and $P_L$ in (10) should be replaced with

$$P^{MN}_{LC} = \epsilon^M_{LC} \epsilon^N_{LC}.$$

The following processes are similar, however it is much more complicated to acquire the (28) then the Goldstone equivalence gauge. Finally, the result is the same as in the Goldstone equivalence gauge.
V. DISCUSSIONS AND POSSIBLE APPLICATIONS

Photon and gluon are the only known massless vector bosons. Practical experiments on quark-gluon plasma could only generate the temperature of at most GeV scale. Our reliable knowledge on the cosmology does not go beyond the 1 MeV, which is the temperature scale of the big bang nucleosynthesis (For a review, see the corresponding chapters in Ref. [16]). Both of them are far below the mass threshold of a $W/Z$ boson. That is probably the reason why originally massive vector bosons in the thermal plasma have received so little attention in the literature. However, Beyond the standard model (BSM) studies involve much higher temperature scales in the earlier universe.

For the dark matter freeze-out process, the typical temperature is usually far below the mass of the dark matter mass (See Ref. [17] for a review). In fact, $T \sim \frac{m_{DM}}{26}$, where $m_{DM}$ is the dark matter mass. If, e.g., the product is the massive vector boson, with the mass $m_V \ll m_{DM}$, the final products can be regarded as the massless objects and the thermal corrections on masses do not affect the phase space integration of the final product significantly. If, on the other hand, $m_V \sim m_{DM}$, the freeze-out temperature is then much smaller than the $m_V$, so the thermal corrections are small compared with this, therefore they can still be neglected.

The feebly-interacting dark matter (FIMP) [18] is created in the higher temperature. The typical temperature for the freeze-in process is approximately of the same scale of the dark matter mass. Therefore, if the “original mass” of the massive vector boson can be compared with this scale, and such a particle participates in the freeze-in process, the thermal corrections on this vector boson might have a relatively strong effect.

Another possible application is the sterile neutrino production and decay in the early universe [19, 20]. If the mass of the sterile neutrino is comparable with the electro-weak phase transition (or strictly speaking, “cross-over” [21] in many cases) temperature $\sim 100$ GeV, $W/Z$ bosons will participate in the decay process. This is important in the sterile neutrino portal dark matter models [22–28]. Such a sterile neutrino can also induce the leptogenesis [20]. In the previous literature, people use the zero-temperature mass of the vector bosons [19], or use some ansatz method [20] to estimate these processes. Now, with the knowledge of the originally massive vector boson emotions in the thermal plasma, people can compute more precisely in the future.
VI. SUMMARY

In this paper, we have relied on a simple toy model to study the behaviour of a vector boson in a thermal environment. We applied the Goldstone equivalence gauge to separate the transverse and longitudinal polarization contributions in the propagator, and after considering the self energy diagram contributions, we can see how the longitudinal polarization vector spew out the Goldstone component, and the massless Goldstone boson pole partly recovers. Therefore, one more degree of freedom arises in the massless vector boson case. This helps us identify the external-leg Feynmann rules for these fields. It is then possible to calculate the tree-level process involving the originally massive vector bosons with these Feynmann rules.

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Appendix A: Discussions of the Extended Ward-Takahashi Identity in the Broken Phase

Ward-Takahashi identity is the result of the gauge symmetry. This can be derived from the path integral method by applying infinitesimal changes on the field parameters in the integrands at the zero temperature. For the finite temperature situation, the only difference is the time parameter integration track, and other zero-temperature results are still available. Therefore, the gauge symmetry still leads to the Ward-Takahashi identity in the thermal plasma.

In the broken phase, there is still a version of Ward-Takahashi identity. Remember that the vev is also a part of the Higgs boson, and the gauge transformation operation requires the vev to transform as well, then Ward-Takahashi identity can also be derived from the path integral method. Rather than giving a complete proof\cite{29}, we only note that in the
broken phase, the “Noether current” becomes

\[ j_\mu = j_\mu^{vi} + i(v\partial_\mu \phi - v^2 g A_\mu), \]  

(A1)

where \( j_\mu^{vi} \) are the vev independent terms. A calculation of \( \partial^\mu j_\mu \) gives \( \partial_\mu j^\mu \phi \) and \( \partial^\mu A_\mu \), and these can be replaced by the equations of motion. A direct calculation of the equations of motion shows that \( \partial^2 \phi = \frac{\partial L_{\phi-coupling \ terms}}{\partial \phi} + m_A \partial_\mu A^\mu \). Remember \( m_A = gv \), so

\[ \langle \partial^\mu j_\mu \rangle = \langle \partial^\mu j_\mu^{others} \rangle + iv \times \langle (\phi-interaction \ terms) \rangle. \]  

(A2)

Then we can follow the usual method to derive the Ward identity. Finally, \( \partial^\mu \rightarrow -ik^\mu \), and \( \phi \)-interaction terms contribute to \( m_A \) in the Goldstone component, so

\[ k_M M^{M\ldots} = 0, \]  

(A3)

where \( k_M = (k_\mu, im_A) \) for the inwards momentum.

Note that \( m_A \) originates from the vev \( gv \), therefore the \( m_A \) in the \( k_M \) does not depend on the 4-dimensional momentum values. This is important for deriving the [22, 24], [36].
[1] D. Pines and D. Bohm, Phys. Rev. 85, 338 (1952).
[2] D. Bohm and D. Pines, Phys. Rev. 92, 609 (1953).
[3] E. Braaten and D. Segel, Phys. Rev. D48, 1478 (1993), hep-ph/9302213.
[4] C. Dvorkin, T. Lin, and K. Schutz, Phys. Rev. D99, 115009 (2019), 1902.08623.
[5] F. P. Huang, K. Kadota, T. Sekiguchi, and H. Tashiro, Phys. Rev. D97, 123001 (2018), 1803.08230.
[6] M. L. Bellac, *Thermal Field Theory*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2011), ISBN 9780511885068, 9780521654777, URL http://www.cambridge.org/mw/academic/subjects/physics/theoretical-physics-and-mathematical-physics/thermal-field-theory?format=AR.
[7] Z. Kunszt and D. E. Soper, Nucl. Phys. B296, 253 (1988).
[8] J. Chen (2019), 1902.06738.
[9] J. Chen, T. Han, and B. Tweedie, JHEP 11, 093 (2017), 1611.00788.
[10] N. P. Landsman and C. G. van Weert, Phys. Rept. 145, 141 (1987).
[11] W. Buchmuller, Z. Fodor, T. Helbig, and D. Walliser, Annals Phys. 234, 260 (1994), hep-ph/9303251.
[12] H. A. Weldon, Phys. Rev. D26, 1394 (1982).
[13] J. Frenkel and J. C. Taylor, Nucl. Phys. B374, 156 (1992).
[14] E. Braaten and R. D. Pisarski, Phys. Rev. D45, R1827 (1992).
[15] M. Laine and A. Vuorinen, Lect. Notes Phys. 925, pp.1 (2016), 1701.01554.
[16] M. Tanabashi, K. Hagiwara, K. Hikasa, K. Nakamura, Y. Sumino, F. Takahashi, J. Tanaka, K. Agashe, G. Aielli, C. Amsler, et al. (Particle Data Group), Phys. Rev. D 98, 030001 (2018), URL https://link.aps.org/doi/10.1103/PhysRevD.98.030001.
[17] G. Bertone, D. Hooper, and J. Silk, Phys. Rept. 405, 279 (2005), hep-ph/0404175.
[18] L. J. Hall, K. Jedamzik, J. March-Russell, and S. M. West, JHEP 03, 080 (2010), 0911.1120.
[19] L. Lello, D. Boyanovsky, and R. D. Pisarski, Phys. Rev. D95, 043524 (2017), 1609.07647.
[20] T. Hambye and D. Teresi, Phys. Rev. Lett. 117, 091801 (2016), 1606.00017.
[21] M. Laine and M. Meyer, JCAP 1507, 035 (2015), 1503.04935.
[22] Y.-L. Tang and S.-h. Zhu (2015), [JHEP03,043(2016)], 1512.02899.
[23] Y.-L. Tang and S.-h. Zhu, JHEP **01**, 025 (2017), 1609.07841.

[24] B. Batell, T. Han, and B. Shams Es Haghi, Phys. Rev. **D97**, 095020 (2018), 1704.08708.

[25] B. Batell, T. Han, D. McKeen, and B. Shams Es Haghi, Phys. Rev. **D97**, 075016 (2018), 1709.07001.

[26] M. Escudero, N. Rius, and V. Sanz, Eur. Phys. J. **C77**, 397 (2017), 1607.02373.

[27] R. Allahverdi, Y. Gao, B. Knockel, and S. Shalgar, Phys. Rev. **D95**, 075001 (2017), 1612.03110.

[28] P. Bandyopadhyay, E. J. Chun, R. Mandal, and F. S. Queiroz, Phys. Lett. **B788**, 530 (2019), 1807.05122.

[29] M. S. Chanowitz and M. K. Gaillard, Nucl. Phys. **B261**, 379 (1985).