COMBINATORIAL PRINCIPLES AND SOME QUESTIONS CONCERNING L-LIKE PROPERTIES AND DC_κ

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Abstract. We extend a result of Arthur Apter which answer a question of Matthew Foreman and Menachem Magidor related to mutually stationary sets. We also extend a result of Arthur Apter which answer a question of W. Hugh Woodin and prove a conjecture by Ioanna Dimitriou. Simultaneously, we study different symmetric extensions based on Levy Collapse where for a cardinal κ, DC_κ either holds or fails. Further, we observe the relationship of the fat diamond principle and other L-like properties with level by level equivalence between strong compactness and supercompactness.

1. Introduction

Serge Grigorieff proved in [Gri75] that symmetric extensions in terms of symmetric system \((\mathcal{P}, \mathcal{G}, \mathcal{F})\) are intermediate models of the form \(HOD(V[a])^{V[G]}\) as \(a\) varies over \(V[G]\). Arthur Apter constructed several symmetric inner models in terms of hereditarily definable sets. One purpose of this note is to extend a few results related to Arthur Apter’s symmetric inner model constructions, after translating the arguments to symmetric extensions in terms of symmetric system \((\mathcal{P}, \mathcal{G}, \mathcal{F})\). Simultaneously, we study different symmetric extensions based on Levy Collapse where \(DC_\kappa\) either holds or fails for a cardinal \(\kappa\). We also study a few combinatorial and model theoretical properties in those symmetric extensions.

In Lemma 1 of [Kar14], Asaf Karagila proved that if \(\mathcal{P}\) is \(\kappa\)-closed and \(\mathcal{F}\) is \(\kappa\)-complete then \(DC_{<\kappa}\) is preserved in the symmetric extension in terms of symmetric system \((\mathcal{P}, \mathcal{G}, \mathcal{F})\). Asaf Karagila and the author both observe that “\(\mathcal{P}\) is \(\kappa\)-closed” can be replaced by “\(\mathcal{P}\) has \(\kappa\)-c.c.” in Lemma 1 of [Kar14]. We note that the natural assumption that \((\mathcal{P}, \mathcal{G}, \mathcal{F})\) is a tenacious system\(^3\) is required in the proof as written by Asaf Karagila in Lemma 3.3 of [Kar].

Observation 1.1. (Lemma 3.3 of [Kar]). If \(\mathcal{P}\) has \(\kappa\)-c.c. and \(\mathcal{F}\) is \(\kappa\)-complete then \(DC_{<\kappa}\) is preserved in the symmetric extension with respect to the symmetric system \((\mathcal{P}, \mathcal{G}, \mathcal{F})\).

We apply this observation to increase the choice strength of two results. Matthew Foreman and Menachem Magidor asked in [MM01], whether it is consistent that \((S_n : 1 \leq n < \omega)\), such that each \(S_n\) is stationary on \(\aleph_n\), is mutually stationary? In [Apt04], Arthur Apter constructed a symmetric inner model preserving \(DC_\omega\), from a \(\omega\)-sequence of supercompact cardinals where if \((S_n : 1 \leq \omega)\) is mutually stationary.

Key words and phrases. Dependent choice, Mutually Stationary properties, Infinitary Chang Conjecture, Fat Diamond principle, Saturated Ideals.

1\(\mathcal{P}\) is a forcing notion, \(\mathcal{G}\) an automorphism group of \(\mathcal{P}\), and \(\mathcal{F}\) is a normal filter of subgroups over \(\mathcal{G}\).

2The author noticed this observation combining the role of \(\kappa\)-c.c. forcing notions from Lemma 2.2 of [Apt01] and the role of \(\kappa\)-completeness of \(\mathcal{F}\) from Lemma 1 of [Kar14].

3Definition 4.6 of [Kar16].
If \( n < \omega \) is a sequence of stationary sets such that \( S_n \subseteq \mathcal{R}_n \), then \( \langle S_n : 1 \leq n < \omega \rangle \) is mutually stationary. We weaken the assumption and obtain a generalized version of this phenomenon in a symmetric extension where \( DC_{<\kappa} \) is preserved applying Observation 1.1, by assuming ‘\( \kappa \) is a \( 2^\omega \)-supercompact cardinal’. In other words, we prove the following.

**Theorem 1.2.** Let \( V \) be a model of ZFC where \( \kappa \) is a \( 2^\omega \)-supercompact cardinal. Consider, an enumeration of measurable cardinals below \( \kappa \), say \( \langle \kappa_\alpha : \alpha < \kappa \rangle \). There is then a symmetric extension based on a symmetric system \( \langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle \) where \( \kappa \) remains a \( 2^\omega \)-supercompact cardinal, cardinality of each \( \kappa_\alpha \) is preserved for each \( \alpha < \kappa \) and \( DC_{<\kappa} \) is preserved. Moreover in the symmetric extension, if \( \langle \lambda_\alpha : \alpha < \beta \rangle \) is any subsequence of \( \langle \kappa_\alpha : \alpha < \kappa \rangle \) having supremum \( \lambda \), such that \( \beta < \lambda_0 \) and \( \langle S_\alpha : \alpha < \beta \rangle \) is a sequence of sets such that each \( S_\alpha \) is a stationary subset of \( \lambda_\alpha \), then \( \langle S_\alpha : \alpha < \beta \rangle \) is mutually stationary.

Secondly, W. Hugh Woodin asked if \( \kappa \) is strongly compact and GCH holds below \( \kappa \), then must GCH hold everywhere? One variant of this question is if GCH can fail at every limit cardinal less than or equal to a strongly compact cardinal \( \kappa \). In Theorem 3 of [Apt12], Arthur Apter constructed a symmetric inner model where \( \kappa \) is a regular limit cardinal and a supercompact cardinal, and GCH holds for a limit \( \delta \) if and only if \( \delta > \kappa \). In that model \( AC_\omega \) fails. Apter asked at the end of [Apt12], the following question.

**Question 1.3.** Is it possible to construct analogous of Theorem 3 in which some weak version of \( AC \) holds?

Applying Observation 1.1, we construct a symmetric extension to answer Question 1.3. Asaf Karagila helps to translate the arguments of Arthur Apter from Theorem 1 of [Apt05] in terms of symmetric extension by a symmetric system \( \langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle \). We construct a similar symmetric extension and prove the following.

**Theorem 1.4.** Let \( V \) be a model of ZFC + GCH with a supercompact cardinal \( \kappa \). There is then a symmetric extension with respect to a symmetric system \( \langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle \) where \( DC_{<\kappa} \) holds, \( \kappa \) is a regular limit cardinal and a supercompact cardinal, and GCH holds for a limit \( \delta \) if and only if \( \delta > \kappa \).

Thirdly, we note that we can slightly generalize Lemma 1 of [Kar14], since there are \( \kappa \)-strategically closed forcing notions which are not \( \kappa \)-closed. As for an example, the forcing notion \( \mathbb{P}(\kappa) \) which adds a non-reflecting stationary set of cofinality \( \omega \) ordinals in \( \kappa \) is \( \kappa \)-strategically closed but not even \( \omega_2 \)-closed. We prove that if \( \mathcal{F} \) is \( \kappa \)-complete then a \( \kappa \)-strategically closed forcing notion can also preserve \( DC_{<\kappa} \) in the symmetric extension.

**Observation 1.5.** (Lemma 4.2). If \( \mathbb{P} \) is \( \kappa \)-strategically closed and \( \mathcal{F} \) is \( \kappa \)-complete then \( DC_{<\kappa} \) is preserved in the symmetric extension with respect to the symmetric system \( \langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle \).

Fourthly, we prove a conjecture by Ioanna Dimitriou from her Ph.D. thesis regarding the failure of \( DC_\omega \) in the symmetric extension of Section 1.4. [Dim11] (this is Question 1 of Chapter 4 in [Dim11]). We prove that the Countable choice (\( AC_\omega \)) fails in the symmetric extension.

**Theorem 1.6.** Let \( V \) be a model of ZFC, \( \rho \) is an ordinal, and \( \mathcal{K} = \langle \kappa_\epsilon : 0 < \epsilon < \rho \rangle \) is a sequence of regular cardinals with a regular cardinal \( \kappa_0 \) below all the regular cardinals in \( \mathcal{K} \). There is then a symmetric extension with respect to a triple \( \langle \mathbb{P}, \mathcal{G}, I \rangle \) where \( AC_\omega \) fails.

\(^1\)\( \mathbb{P} \) is a forcing notion, \( \mathcal{G} \) an automorphism group of \( \mathbb{P} \), and \( I \) a \( \mathcal{G} \)-symmetry generator. The terminology of a \( \mathcal{G} \)-symmetry generator is adopted from [Dim11] and we discuss this in section 7.
Fifthly, we observe an infinitary Chang conjecture using Erdős-like partition property in a symmetric extension analogous to the symmetric inner model constructed in Theorem 11 of [AK06] by Arthur Apter and Peter Koepke where $\omega_1$ is singular, and thus $AC_\omega$ fails. We use the observation that it is possible to force a coherent sequence of Ramsey cardinals after performing Prikry forcing on a normal measure over a measurable cardinal $\kappa$ (Theorem 3 of [AK06]). We also use the observation that an infinitary Chang conjecture can be established in a symmetric model assuming a coherent sequence of Ramsey cardinals.

**Theorem 1.7.** Let $V$ be a model of ZFC where there is a measurable cardinal. There is then a symmetric extension with respect to a triple $(\mathbb{P}, G, T)$ where $\omega_1$ is singular and thus $AC_\omega$ fails. Moreover in the symmetric extension, an infinitary Chang conjecture holds and $\mathbb{V}_\omega$ is an almost Ramsey cardinal and a Rowbottom cardinal carrying a Rowbottom filter.

Now we start working in ZFC and study the consistency of new combinatorial principles with Level by level equivalence between strong compactness and supercompactness. For the sake of our convenience, we eliminate the phrase ‘between strong compactness and supercompactness’. In Theorem 1 of [Apt05], Arthur Apter constructed a forcing extension where level by level equivalence holds along with $\diamondsuit_\delta$ for every uncountable regular cardinal $\delta$. Moreover, in that forcing extension, he proved the existence of a stationary subset $S$ of the least supercompact cardinal where for each $\delta \in S$, $\square_\delta$ holds. At the end of [Apt05], Apter asked what new combinatorial properties can be possible in a model where level by level equivalence holds?

We observe that in the forcing extension of Theorem 1 of [Apt05], not only $\diamondsuit_\delta$ holds for each uncountable regular cardinal $\delta$, but also a stronger two cardinal diamond principle, we call $\diamondsuit_{\delta, \lambda}$ holds for every uncountable regular cardinal $\delta$, and $\lambda$ such that $\delta < \lambda$. The key observation is that the principle $\diamondsuit_{\delta, \lambda}$ can be forced by $Add(\delta, 1)$ and after forcing with $Add(\delta, 1)$, the coherent $\diamondsuit_{\delta, \lambda}$-sequence are indestructible by $\delta$-closed forcing. This is due to Lemma 20 of [BM18] by Brent Cody and Monroe Eskew.

James Cummings and Arthur Apter generalized Theorem 1 of [Apt05] in a different way in [AJ08]. In Theorem 2 of [AJ08], they constructed a forcing extension for level by level equivalence where a version of square consistent with supercompactness holds on the class of all infinite cardinals and a strong form of diamond holds on a proper class of regular cardinals. At the end of [AJ08], they asked if it is possible to generalize Theorem 2 of [AJ08] further? Using Lemma 20 of [BM18], we can similarly observe that not only $\diamondsuit_\delta$, but also $\diamondsuit_{\mu, \lambda}$ holds for every $\mu$ which is either inaccessible or the successor of a singular cardinal and $\mu < \lambda$, in the forcing extension of Theorem 2 of [AJ08].

Sy-David Friedman introduced the outer model programme in [Fri07] aiming to construct outer models containing large cardinals where $L$-like properties hold. One perspective of this paper is to

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5Saharon Shelah and Arthur Apter introduced the notion of the level by level equivalence between strong compactness and supercompactness in [AS97]. If in a model of ZFC for all regular cardinals $\delta < \lambda$, $\delta$ is $\lambda$-strongly compact if and only if $\delta$ is $\lambda$-supercompact except possibly if $\kappa$ is a measurable limit of cardinals $\delta$ which are $\lambda$-supercompact, then such a model is said to witness level by level equivalence between strong compactness and supercompactness.

6The definition is from the paragraph before Lemma 20 of [BM18].

7We give a sketch of the argument. Let $\mathbb{P}$ be the forcing extension of Theorem 3 of [Apt05]. Fix an arbitrary regular uncountable cardinal $\delta$. We can write $\mathbb{P} = \mathbb{P}_{\delta+1} * \mathbb{P}_{\delta}^{\delta^+}$. By Lemma 20 of [BM18], for $\delta < \lambda$, $\diamondsuit_{\lambda, \lambda}$ can be forced by $Add(\delta, 1)$ and the coherent $\diamondsuit_{\delta, \lambda}$-sequence are indestructible by $\delta$-closed forcing. Thus, $\diamondsuit_{\delta, \lambda}$ holds in $V^{\mathbb{P}_{\delta+1}}$ and since $\mathbb{P}_{\delta+1}$ is $\delta^+$-directed closed and thus $\delta$-closed, $\diamondsuit_{\lambda, \lambda}$ holds in $V^\mathbb{P}$. Following the proof of Theorem 1 of [Apt05], we obtain the existence of $\diamondsuit_{\delta, \lambda}$ for every uncountable regular cardinal $\delta$ and $\lambda$ such that $\delta < \lambda$. 

see a few results in this area. Specifically, we extend a few results of Arthur Apter, each of which can be thought of as a contribution to Friedman’s outer model programme.

Firstly, in [Apt09], Apter amalgamate the results of Theorem 2 of [AJ08], [AF09] and [Fri07] and constructed a model where GCH and level by level equivalence hold, along with a certain $L$-like combinatorial principles. At the end of [Apt09], Apter asked the following question.

**Question 1.8.** Which additional $L$-like principles are consistent with GCH and level by level equivalence?

Assuming $V = L$, Friedman and Kulikov proved in Section 2 of [FV15] that the fat diamond principle on $\kappa$ (we denote by $\lozenge_\kappa$) holds for every uncountable regular cardinal $\kappa$. Given $\kappa$ an uncountable regular cardinal, we prove the following facts concerning preserving $\lozenge_\kappa$ under certain forcing notions.

**Observation 1.9.** (Lemma 9.4) $\kappa$-strategically closed forcing notions can preserve $\lozenge_\kappa$.

**Observation 1.10.** (Lemma 9.6) Forcing notions that are both $\kappa$-c.c. and have cardinality $\kappa$ can preserve $\lozenge_\kappa$.

Using these preservation lemmas, and the fact that $Add(\kappa,1)$ not only introduces a $\lozenge_\kappa$-sequence but also introduces a $\bigstar_\kappa$-sequence (due to Brent Cody and Monroe Eskew) we can give a possible answer to **Question 1.8**. In particular, in the forcing extension of Theorem 1 of [Apt09] we can witness the consistency of $\bigstar_\delta$ for each Mahlo cardinal $\delta$ as well as if $\delta$ is $\aleph_1$.

**Observation 1.11.** (Generalization of Theorem 1 of [Apt09]). Let $V$ be a model of ZFC + GCH where $K \neq \emptyset$ is the class of supercompact cardinals. There is then a forcing extension $V^P$ for a forcing notion $P$ such that $V^P$ is a model of ZFC + GCH where $K$ is the class of supercompact cardinals and level by level equivalence holds. Moreover the following holds in $V^P$.

1. $\bigstar_\delta$ holds for every Mahlo cardinal $\delta$ as well as if $\delta$ is $\aleph_1$.
2. $\bigtriangleup_\delta$ holds for every successor and Mahlo cardinal $\delta$.
3. There is a stationary subset $S$ of the least supercompact cardinal $\kappa$ such that for each $\delta \in S$, $\bigstar_\kappa$ holds and $\delta$ carries a gap-1 morass.
4. $\Box_T^T$ holds for every infinite cardinal $\gamma$ where $T = Safe(\gamma)$.
5. There is a locally defined well-ordering of the universe $W$.

Secondly, in Theorem 1 of [Apt08], Arthur Apter proved the consistency of level by level equivalence with a few “inner model like” properties which are as follows.

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8We use the forcing notion to add a non-reflecting stationary set of ordinals of a certain type to $\kappa$ to prove Theorem 1.12. Since this notion of forcing is $\kappa$-strategically closed, but not $\kappa$-closed, we will see that we need to prove the preservation of $\lozenge_\kappa$ under $\kappa$-strategically closed forcing notions. Preserving $\lozenge_\kappa$ under a $\kappa$-closed forcing notion is not sufficient.

9We can mimic the whole construction of Theorem 1 of [Apt09] which starts with the forcing extension of Theorem 2 of [Apt08]. We use the fact that $Add(\delta,1)$ can force $\bigstar_\delta$ to see that $\bigstar_\delta$ is possible for every $\delta$ which is either an inaccessible cardinal or the successor of a singular cardinal in the forcing extension of Theorem 2 of [Apt08]. We use the preservation lemmas to preserve $\bigstar_\delta$ for a Mahlo cardinal $\delta$ in the final forcing extension of [Apt09]. Finally, $Add(\aleph_1,1)$ can add $\bigstar_{\aleph_1}$. 
• The class of Mahlo cardinals reflecting stationary sets is the same as the class of weakly compact cardinals.
• Every regular Jonsson cardinal is weakly compact.

Further, using Theorem 1 of [Apt05], Apter extended this result and showed these “inner model like” properties in the forcing extension of Theorem 7 of [Apt08] where the level by level equivalence hold along with L-like combinatorial principles like *Diamond* and *Square*. We can extend Theorem 7 of [Apt08] further by adding more L-like principles in such a way such that the level by level equivalence holds in the forcing extension. This result may also be classified in Woodin’s phrase as an “inner model theorem proven via forcing”. Specifically, we can prove the following.

**Theorem 1.12. (Generalization of Theorem 7 of [Apt08]).** Let $V$ be a model of ZFC where $\kappa \neq \emptyset$ is the class of supercompact cardinals. There is then a forcing extension $V^P$ for a forcing notion $P$ such that $V^P$ is a model of ZFC + GCH where $\kappa$ is the class of supercompact cardinals and level by level equivalence holds. Moreover the following holds in $V^P$.

1. ♦$_\delta$ holds for every Mahlo cardinal $\delta$ as well as if $\delta$ is $\aleph_1$.
2. ♦$_\delta$ holds for every successor and Mahlo cardinal $\delta$.
3. There is a stationary subset $S$ of the least supercompact cardinal $\kappa$ such that for each $\delta \in S$, $\Box$ holds and $\delta$ carries a gap-1 morass.
4. $\Box^*_\delta$ holds for every infinite cardinal $\gamma$ where $T = Safe(\gamma)$.
5. Mahlo cardinals reflecting stationary sets are weakly compact cardinals.
6. Every regular Jonsson cardinal is weakly compact.

Thirdly, Arthur Apter mentioned in section 3 of [Apt08] that in $L$ and higher inner models ‘the weakly compact cardinals are precisely the class of inaccessible cardinals admitting stationary reflection’. He proved in Theorem 9 of [Apt08], the consistency of such a phenomenon in the context of the level by level equivalence where L-like principles like diamond and square holds. Similarly like Theorem 1.12, we can also force more L-like principles in such a way such that the level by level equivalence holds in the forcing extension.

**Observation 1.13. (Generalization of Theorem 9 of [Apt08]).** Let $V$ be a model of ZFC where $\kappa$ is supercompact and no cardinal is supercompact up to an inaccessible cardinal. There is then a forcing extension $V^P$ for a forcing notion $P$ such that $V^P$ is a model of ZFC + GCH where $\kappa$ is supercompact and no cardinal is supercompact up to an inaccessible cardinal. In $V^P$ level by level equivalence holds. Moreover the following holds in $V^P$.

1. ♦$_\delta$ holds for every Mahlo cardinal $\delta$ as well as if $\delta$ is $\aleph_1$.
2. ♦$_\delta$ holds for every successor and Mahlo cardinal $\delta$.
3. There is a stationary subset $S$ of $\kappa$ such that for each $\delta \in S$, $\Box$ holds and $\delta$ carries a gap-1 morass.
4. $\Box^*_\delta$ holds for every infinite cardinal $\gamma$ where $S = Safe(\gamma)$.
5. Inaccessible cardinals reflecting stationary sets are precisely the weakly compact cardinals.
6. Every regular Jonsson cardinal is weakly compact.

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10 Moreover, Arthur Apter mentioned in section 3 of [Apt08] that in $L$ and higher inner models, the weakly compact cardinals are exactly the class of inaccessible cardinals admitting stationary reflection.
11 As mentioned in the paragraphs below Theorem 2 of [Apt08].
12 Since the proof is similar to the proof of Theorem 1.12, we are not going into the details.
Next, we observe that Kunen’s observations concerning saturated ideals from [Kun78] can be applied in forcing extensions of Theorem 1 of [Apt08a] and Theorem 8 of [Apt08] by Arthur Apter. In [Kun78], Kenneth Kunen proved that if $\kappa$ is assumed to be a measurable cardinal, then there is a forcing extension where $\kappa$ is inaccessible but not weakly compact and there is a $\kappa$-complete, $\kappa$-saturated ideal on $\kappa$.

There are different ways to first destroy the large cardinal property of an uncountable cardinal $\kappa$ by adding some combinatorial object by a forcing notion $S$ and then resurrect the large cardinal property of $\kappa$ by destroying the combinatorial object added by $S$ with another forcing $T$. This can happen if the combined forcing $S \ast T$ is forcing equivalent to forcing notions like $\text{Add}(\kappa, 1)$ which is the standard forcing notion for adding a Cohen subset of $\kappa$. Few situations are listed below.

1. If $\kappa$ has a large cardinal property say measurability, then the forcing $S$ which adds a $\kappa$-Suslin tree $T$ kills even the weak compactness of $\kappa$. But then after forcing with $S$, forcing $T$ which adds a branch through $T$ resurrects the measurability of $\kappa$.
2. Similarly, consider the forcing $S$ which adds a non-reflecting stationary subset $S \subset \kappa$ and the forcing $T$ which adds a club disjoint to $S$.
3. Again, if the forcing $S$ adds a $\Box_\kappa$-sequence $C$ and the forcing $T$ adds a thread through $C$.

We observe the first situation in Theorem 8 of [Apt08], where Apter proved the fact that stationary reflection can occur on a stationary subset $A$ of the least supercompact cardinal $\kappa$ composed of non-weakly compact Mahlo cardinals. By Kunen’s observation concerning saturated ideals, we can see that each $\delta \in A$ can carry a normal $\delta$-saturated ideal in the forcing extension of Theorem 8 of [Apt08]. Moreover, by Observation 14 of [BM18] for each $\delta \in A$, $\delta^+$ do not contain a $\delta$-minimal pre-saturated ideal because GCH holds in the forcing extension and $\delta$ is a regular cardinal. Further, we can force more combinatorial principles in such a way such that level by level equivalence holds in the forcing extension.

**Theorem 1.14.** (Generalization of Theorem 8 of [Apt08].) Let $V$ be a model of $\text{ZFC} + \text{GCH}$ where $K \neq \emptyset$ is the class of supercompact cardinals with a least supercompact cardinal $\kappa$. There is then a forcing extension $V^p$ for a forcing notion $\mathbb{P}$ such that $V^p$ is a model of $\text{ZFC} + \text{GCH}$ where $K$ is the class of supercompact cardinals with the least supercompact cardinal $\kappa$ and level by level equivalence holds. Moreover the following holds in $V^p$.

1. $\diamondsuit_\delta$ holds for every Mahlo cardinal $\delta$ as well as if $\delta$ is $\aleph_1$.
2. $\varnothing_\delta$ holds for every successor and Mahlo cardinal $\delta$.
3. There is a stationary subset $S$ of the least supercompact cardinal $\kappa$ such that for each $\delta \in S$, $\Box_\delta$ holds and $\delta$ carries a gap-1 morass.
4. $\Box_T^T$ holds for every infinite cardinal $\gamma$ where $T = \text{Safe}(\gamma)$.
5. There is a stationary subset $A$ of $\kappa$ composed of non-weakly compact Mahlo cardinals which reflect stationary sets and for each $\delta \in A$, $\delta$ is a $\delta$-saturated ideal but $\delta^+$ do not contain a $\delta$-minimal pre-saturated ideal.

Similarly in Theorem 1 of [Apt08a], we can obtain the existence of a stationary set $A$ where for each $\delta \in A$, $\delta$ not only reflects stationary sets but also carry a normal $\delta$-saturated ideal. We note that, for this, we need to assume the existence of a measurable cardinal $\lambda$ in place of a weakly compact cardinal $\lambda$ as done in [Apt08a].

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13We can see that $\kappa$ can be Mahlo as well.
Theorem 1.15. (Variant of Theorem 1 of [Apt08a]). Assume GCH. If $\kappa < \lambda$ are such that $\kappa$ is a strong cardinal whose strongness is indestructible under $\kappa$-strategically closed forcing and $\lambda$ is measurable, then $A = \{ \delta < \kappa : \delta$ is a non weakly compact Mahlo cardinal which reflects stationary sets and carry a normal $\delta$-saturated ideal $\}$ is unbounded in $\kappa$.

Structure of the paper.

- In section 2, we cover the basics for constructions in ZFC and in section 3, we cover the basics for constructions in ZF.
- In section 4, we prove Observation 1.5 and study a few lemmas related to preserving dependent choice in symmetric extensions inspired from Lemma 1 of [Kar14].
- In section 5, we prove Theorem 1.2 which generalize Theorem 1 of [Apt04] and study the mutually stationary property of a sequence of stationary sets in a symmetric extension.
- In section 6, we prove Theorem 1.4 which answers Question 1.3 asked by Arthur Apter.
- In section 7, we prove Theorem 1.6 and thus prove the conjecture by Ioanna Dimitriou which is Question 1 of Chapter 4 in [Dim11].
- In section 8, we prove Theorem 1.7 and observe an infinitary Chang Conjecture in a symmetric extension similar to the constructed symmetric inner model of Theorem 11 of [AK06].
- In section 9, we prove Observation 1.9, Observation 1.10 and apply these preservation lemmas of $\diamondsuit$ with respect to certain forcing notions, to prove Theorem 1.12 which extends Theorem 7 from [Apt08] by Arthur Apter. In this process, we study the relation between level by level equivalence along with fat diamond principle, gap 1 moras and a few more L-like properties. As discussed in the introduction, Observation 1.11, Theorem 1.12 and Observation 1.13 can be thought of as contributions to Friedman’s outer model programme.
- In section 10, we mainly add observations of Kunen concerning saturated ideals from [Kun78] in Arthur Apter’s works from [Apt08] and [Apt08a] and prove Theorem 1.14 and Theorem 1.15.
- Finally in section 11, we end up with some questions.

2. Basics for constructions in ZFC

2.1. Forcing and large cardinal facts. By a forcing notion $\mathbb{P}$, we mean a partially ordered set with a maximum element 1. If $G$ is $V$-generic over $\mathbb{P}$, then we will abuse the notation somewhat and use the model obtained by forcing with $\mathbb{P}$ as both $V^\mathbb{P}$ and $V[G]$.

**Definition 2.1.** Let $\gamma$ be an uncountable cardinal and $\mathbb{P}$ be a forcing notion.

- $\mathbb{P}$ has $\gamma$-c.c. ($\gamma$-chain condition) if every antichain of $\mathbb{P}$ has a size less than $\gamma$.
- $\mathbb{P}$ is $\gamma$-directed closed if every set of pairwise compatible conditions in $\mathbb{P}$ of size less than $\gamma$ has a lower bound.
- $\mathbb{P}$ is $\gamma$-closed if every decreasing sequence of conditions in $\mathbb{P}$ with length less than $\gamma$ has a lower bound.
- For an ordinal $\alpha$, consider a 2 player game $G_\alpha(\mathbb{P})$ where two players, player I and player II construct an increasing sequence of conditions $\langle p_\beta : \beta < \alpha \rangle$ where player I play odd stages and player II play even stages including the limit stages choosing the trivial condition at stage 0. Player II has a winning strategy if the game can always be continued. $\mathbb{P}$ is $\gamma$-strategically closed if player II has a winning strategy for $G_\gamma(\mathbb{P})$. 
• $\mathbb{P}$ is $\gamma$-distributive if it does not add sequences of ordinals of length less than $\gamma$.

• $\mathbb{P}$ admits a closure point at $\gamma$, if $\mathbb{P}$ can be factorized as $Q \ast \mathbb{R}$ where $Q$ is a nontrivial notion of forcing such that $|Q| \leq \gamma$ and $\mathbb{R}$ is a nontrivial notion of forcing such that $\Vdash \mathbb{R}$ is $\leq \gamma$-strategically closed.

It is well known that a $\gamma$-directed closed forcing notion is $\gamma$-closed, a $\gamma$-closed forcing notion is $\gamma$-strategically closed, a $\gamma$-strategically closed forcing notion is $< \gamma$-strategically closed, a $< \gamma$-strategically closed forcing notion is $\gamma$-distributive and a $\gamma$-distributive forcing notion add no new subsets of $\gamma$ in the forcing extension which is Theorem 15.6 of [Jec03].

We recall the definition of an inaccessible cardinal, Mahlo cardinal, weakly compact cardinal, measurable cardinal and supercompact cardinal in context of ZFC from ‘The Higher Infinite’ [Kan09] of Akihiro Kanamori. We recall that if $\mathcal{U}$ is a normal measure over $\kappa$ then $\{\alpha < \kappa : \alpha$ is Ramsey$\} \in \mathcal{U}$ (Exercise 7.19 of [Kan09]) and the fact that if $\kappa$ is $2^\kappa$-supercompact, then there is a normal measure $\mathcal{U}$ over $\kappa$ such that $\{\alpha < \kappa : \alpha$ is measurable$\} \in \mathcal{U}$ (Proposition 22.1 of [Kan09]). Following Main Theorem 3, Corollary 14, and Theorem 31 of [Ham03] it follows that if $\gamma$ is $\lambda$-supercompact in $V^\mathbb{P}$ and $\mathbb{P}$ admits a closure point at or less than the least inaccessible cardinal, then $\gamma$ must be a $\lambda$-supercompact cardinal in $V$.

As discussed in the introduction, it is possible to factor $\text{Add}(\gamma, 1)$ into $\mathbb{S} \ast \mathbb{T}$ where $\mathbb{S}$ is a forcing to add some combinatorial object and $\mathbb{T}$ is the forcing to destroy the combinatorial object added by $\mathbb{S}$. We recall the forcing notion to add a $\gamma$-Suslin tree via homogeneous trees of successor height less than $\gamma$, ordered by end extension from page 69 of [Kun78]. We recall the following fact from [Kun78].

Fact 2.2. (Section 3 of [Kun78], pages 68-71). Given $\gamma$ an inaccessible cardinal. We can factor $\text{Add}(\gamma, 1)$ as $\mathbb{R}_\gamma \ast \mathbb{T}_\gamma$ where $\mathbb{R}_\gamma$ is a $< \gamma$-strategically closed notion of forcing for adding a $\gamma$-Suslin tree $\mathbb{T}$ and $\mathbb{T}_\gamma$ is the $\gamma$-c.c. notion of forcing for adding a generic path through $\mathbb{T}$.

We recall the following fact concerning preserving a stationary subset of $\kappa$ with respect to certain forcing notions from [Jec03].

Lemma 2.3. Suppose $S$ is a stationary subset of $\kappa$ and $\mathbb{P}$ is a notion of forcing that satisfies one of the following properties.

1. $\mathbb{P}$ has $\kappa$-c.c.
2. $\mathbb{P}$ is $\kappa$-strategically closed.

Then $S$ remains stationary subset of $\kappa$ in any $\mathbb{P}$-generic extension.

Proof. (1) follows from Lemma 22.25 of [Jec03] and modifying Lemma 23.7 of [Jec03], it is possible to prove (2).

2.2. L-like combinatorial principles. Ronald Jensen analyzed the levels of constructible hierarchy in details, resulting in the fine structure theory which describes how new sets arise in the construction of $L$. He introduced the square principle, abbreviated as $\square$ and assuming $V=L$, proved

\[\text{The terminology of a forcing notion with a closure point } \gamma \text{ is due to Joel David Hamkins from Definition 12 of Ham03. We also refer to Apt09.}\]  

\[\text{We also follow Theorem 2 and the paragraph after it from Apt09.}\]
that $\square_\gamma$ holds for every cardinal $\gamma$. Given an arbitrary uncountable cardinal $\gamma$, we recall the definition of Jensen’s original square principle $\square_\gamma$ from [Apt05].

**Definition 2.4. ($\square_\gamma$-sequence, $\square_\gamma$ principle, [Apt05].)** We say $\langle C_\alpha : \alpha < \gamma^+ \text{ and } \alpha \text{ is a limit ordinal} \rangle$ is a $\square_\gamma$-sequence if we have the following.

- $C_\alpha$ is a club subset of $\alpha$.
- $C_\alpha$ has order type below $\gamma$ if $\text{cf}(\alpha) < \gamma$.
- For any limit point $\beta \in C_\alpha$, $C_\alpha \cap \beta = C_\beta$.

We say $\square_\gamma$ holds if and only if there is a $\square_\gamma$-sequence.

Along with other combinatorial and model-theoretic applications, the square principle can witness several incompactness phenomena. By *incompactness phenomenons*, we mean the existence of structures such that every substructure of a smaller cardinality has a certain property but the entire structure does not. Examples of a few incompactness phenomenons which are the applications of the square principle are as follows.

- Existence of a family of countable sets such that the entire family does not have a transversal but every subfamily of smaller cardinality has a transversal [ML12].
- Existence of a non-free abelian group of cardinality $\gamma^+$ where all smaller subgroups are free [MS94].
- Existence of non-metrizable topological space where all smaller subspaces are metrizable [ML12].

There is a cofinality preserving forcing notion to add $\square_\gamma$ (definition 6.1 of [JMM01] or [Apt05]). Intuitively, the forcing notion that adds $\square_\gamma$ consists of initial segments of a $\square_\gamma$-sequence ordered by end extension.

**Definition 2.5. (Adding $\square_\gamma$, [Apt05].)** The forcing notion $P(\gamma)$ to add a $\square_\gamma$-sequence consists of the elements of the form $\langle C_\alpha : \alpha \leq \beta < \gamma^+ \text{ and } \alpha \text{ is a limit ordinal} \rangle$ such that the following holds.

- $C_\alpha$ is a club subset of $\alpha$.
- $C_\alpha$ has order type below $\gamma$ if $\text{cf}(\alpha) < \gamma$.
- For any limit point $\delta \in C_\alpha$, $C_\alpha \cap \delta = C_\delta$.

The ordering $\leq$ is such that $p \leq q$ if and only if $p$ is a subsequence of $q$.

**Lemma 6.1 of [JMM01] states that $P(\gamma)$ is $\gamma$-strategically closed. Assuming GCH, $|P(\gamma)| = \gamma^+$. Consequently, forcing with $P(\gamma)$ over a model of GCH preserves cardinals, cofinalities, and GCH.** We recall from [Apt09] that $\square_\gamma$ is **upward absolute to a cofinality preserving generic extension**.

We introduce another weak version of the square principle, which is consistent with supercompactness, namely the square principle $\square_\kappa^{\omega_1}$. In [MM97], Matthew Foreman and Menachem Magidor proved that this sort of square principle, specifically $\square_{\kappa^{\omega_1}}^{(\kappa^{+n} : n < \omega)}$ is consistent with the supercompactness of $\kappa$.

**16**Martin Zeman generalized and proved assuming $V=K$, that $\square_\gamma$ holds for all cardinal $\gamma$ where $K$ is the Mitchell-Steel core model.
Definition 2.6. ($\square^T_\gamma$-sequence, $\square^T_\gamma$-principle, Definition 2.1 of [AJ08]). Let $\gamma$ be an infinite cardinal and $T$ be a set of regular cardinals which are less than or equal to $\gamma$. Then a $\square^T_\gamma$ sequence is a sequence $\langle C_\alpha : \alpha \in \gamma^+ \cap \text{cf}(T) \rangle$ such that,

- $C_\alpha$ is club in $\alpha$ and $Otp(C_\alpha) \leq \gamma$.
- If $\beta \in \lim(C_\alpha) \cap \lim(C_\alpha')$, then $C_\alpha \cap \beta = C_\alpha' \cap \beta$.

We say $\square^T_\gamma$ holds if and only if there is a $\square^T_\gamma$-sequence.

Definition 2.7. (Safe($\gamma$), Definition 2.2 of [AJ08]). Given an infinite cardinal $\gamma$, Safe($\gamma$) is the set of all safe regular cardinals for $\gamma$ where a regular cardinal $\mu$ is safe for $\gamma$ if and only if $\mu \leq \gamma$ and for every cardinal $\lambda \leq \gamma$, if $\lambda$ is $\gamma^+$-supercompact then $\lambda \leq \mu$.

We recall from [Apt09] that Safe($\gamma$) is upward absolute to any cofinality preserving generic extension by a forcing notion admitting a closure point at or below the least inaccessible cardinal $\kappa$. The principle $\square^{\text{Safe}}_\gamma(\gamma)$ is of our interest. We refer to [AJ08] or [Apt09], for further details concerning forcing this version of square principle.

Another L-like combinatorial principle introduced by Ronald Jensen is diamond abbreviated as $\lozenge$. We follow an excellent introduction of $\lozenge$ by Assaf Rinot from [Rin11]. Specifically he starts by stating a nonstandard formulation of Cantor’s continuum hypothesis as follows.

- There exist a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ such that for each $Z \subseteq \omega_1$, there exists two infinite ordinals $\alpha, \beta < \omega_1$ such that $Z \cap \beta = A_\alpha$.

By eliminating one closing quantifier we arrive to the following enumeration principle which is discovered by Jensen and named as diamond ($\lozenge$).

- There exist a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ such that for each $Z \subseteq \omega_1$, there exists an ordinal $\alpha < \omega_1$ such that $Z \cap \alpha = A_\alpha$.

We recall the definition of a generalised version of this principle, namely $\lozenge_\gamma$ for an uncountable cardinal $\gamma$.

Definition 2.8. ($\lozenge_\gamma$-sequence, $\lozenge_\gamma$ principle, [Apt05]). We say $\langle S_\alpha : \alpha < \gamma \rangle$ is a $\lozenge_\gamma$-sequence if the following holds.

- $S_\alpha \subseteq \alpha$.
- for every $X \subseteq \gamma$, $\{ \alpha < \gamma : X \cap \alpha = S_\alpha \}$ is a stationary subset of $\gamma$.

We say $\lozenge_\gamma$ holds if and only if there is a $\lozenge_\gamma$-sequence.

Jensen proved that $\lozenge$ holds in $L$, and using similar arguments the generalised version $\lozenge_\gamma$ also follows assuming $V = L$. Its a standard fact that forcing with $Add(\gamma, 1)$ adds $\lozenge_\gamma$ (Lemma 0.1 of [Apt05]). We recall the fact that $\lozenge_\gamma$ can be preserved by $\gamma$-strategically closed forcing notion and forcing notion which are $\gamma$-c.c. and has cardinality $\gamma$ from Fact 1.1 and Fact 1.2 of [Apt08]. Saharon Shelah showed in [She10], that under GCH, $\lozenge_\gamma$ holds for every successor cardinal $\gamma$ greater than $\aleph_1$.

\footnote{We refer to paragraph below Definition 1.2 of [Apt09].}

\footnote{$\lozenge_\omega$ is same as $\lozenge_{\omega_1}$.}
We introduce another L-like combinatorial principle, namely gap-1 morass. Morasses are another L-like combinatorial objects invented by Jensen in order to construct infinite structures from a structure of smaller cardinality which has several combinatorial as well as model-theoretic applications as follows.

- If there is a simplified \((\kappa, 1)\)-morass with linear limits, then there is \(\kappa\)-Kurepa tree with no \(\lambda\)-Aronszajn subtrees for any regular infinite \(\lambda < \kappa\) and no \(\mu\)-Cantor subtree for any infinite \(\mu < \kappa\).
- If there is a \((\kappa^+, 1)\)-morass then for every cardinal \(\lambda\), \((\lambda^{++}, \lambda) \rightarrow (\kappa^{++}, \kappa)\).
- If there is a simplified \((\kappa, 1)\)-morass with linear limits, then \(\Box_{\kappa}\) holds.

Definition of a gap-1 morass can be found in [BF09]. We recall the definition of the forcing notion \((Q_\gamma, \leq)\) for adding a gap-1 morass at \(\gamma\) from [BF09] or Theorem 10 of [Fri07].

**Definition 2.9. (Adding a gap-1 morass at \(\gamma\), Theorem 10 of [Fri07]).** A condition in \(Q_\gamma\) is a proper initial segment of a morass up to some top level, together with a map of an initial segment of this top level into \(\gamma^+\) which obeys the requirements of a morass map. To extend a condition we end extend the morass up to its top level and require that the map from the given initial segment of its top level into \(\gamma^+\) factors as the composition of a map into the top level of the stronger condition followed by the map given by the stronger condition into \(\gamma^+\).

We have the fact that \((Q_\gamma, \leq)\) is \(\gamma\)-closed and \(\gamma^+-\text{c.c.}\) from Proposition 50 and Proposition 52 of [Tay07]. We recall the fact that the existence of a gap-1 morass at \(\gamma\) is upward absolute to a cofinality preserving generic extension from [Apt09]. The theory of Morass is well developed and it is known how to construct Morass in \(L\) [Fri00].

3. Basics for constructions in ZF

3.1. Large Cardinals. In this section we define large cardinals in the context of ZF for section 4, 5, 6, 7 and 8. We recall the necessary large cardinal definitions from ‘The Higher Infinite’ [Kan09] of Akihiro Kanamori.

**Definition 3.1.** Given an uncountable cardinal \(\kappa\), we recall the following definitions.

1. \(\kappa\) is strongly inaccessible if its a regular and a strong limit cardinal.
2. \(\kappa\) is Ramsey if for all \(f : [\kappa]^{<\omega} \rightarrow 2\), there is a homogeneous set \(X \subseteq \kappa\) for \(f\) of order type \(\kappa\).
3. \(\kappa\) is almost Ramsey if for all \(\alpha < \kappa\) and \(f : [\kappa]^{<\omega} \rightarrow 2\), there is a homogeneous set \(X \subseteq \kappa\) for \(f\) having order type \(\alpha\).
4. \(\kappa\) is measurable if there is a \(\kappa\)-complete free ultrafilter on \(\kappa\). In ZF an ultrafilter \(U\) over \(\kappa\) is normal if and only if for every regressive \(f : \kappa \rightarrow \kappa\) there is an \(X \in U\) such that \(f\) is constant on \(X\) (Lemma 0.8 of [Dim11]). Thus we say a \(\kappa\)-complete ultrafilter \(U\) is normal if for every regressive \(f : \kappa \rightarrow \kappa\) there is an \(X \in U\) such that \(f\) is constant on \(X\).
5. For a set \(A\) we say \(U\) a fine measure on \(\mathcal{P}_\kappa(A)\) if \(U\) is a \(\kappa\)-complete ultrafilter and for any \(i \in A\), \(\{x \in \mathcal{P}_\kappa(A) : i \in x\} \in U\). We say \(U\) is a normal measure on \(\mathcal{P}_\kappa(A)\), if \(U\) is a fine measure and if \(f : \mathcal{P}_\kappa(A) \rightarrow A\) is such that \(f(X) \in X\) for a set in \(U\), then \(f\) is constant on a set in \(U\). \(\kappa\) is \(\lambda\)-strongly compact if there is a fine measure on \(\mathcal{P}_\kappa(\lambda)\), it is strongly compact if it is \(\lambda\)-strongly compact for all \(\kappa \leq \lambda\).
(6) \( \kappa \) is \( \lambda \)-supercompact if there is a normal measure on \( \mathcal{P}_\kappa(\lambda) \), it is supercompact if it is \( \lambda \)-supercompact for all \( \kappa \leq \lambda \).

From now on we will say strongly inaccessible cardinals as inaccessible cardinals. We recall that a limit of Ramsey cardinals is an almost Ramsey cardinal in ZF (Proposition 1 of [AP08c]).

3.2. Levy–Solovay Theorem. We state a part of Levy–Solovay Theorem (Theorem 21.2 of [Jec03]). By a small forcing extension with respect to \( \kappa \) we mean a forcing extension \( V[G] \) obtained from \( V \) after forcing with a partially ordered set of size less than \( \kappa \).

Theorem 3.2. Let \( \kappa \) be an infinite cardinal, and let \( \mathbb{P} \) be a partially ordered set of size less than \( \kappa \). Let \( G \) be a \( \mathbb{P} \)-generic filter over \( V \).

- If \( \kappa \) is Ramsey in \( V \), then \( \kappa \) is Ramsey in \( V[G] \).
- If \( \kappa \) is measurable with a \( \kappa \)-complete ultrafilter \( \mathcal{U} \) in \( V \) then \( \kappa \) is measurable with a \( \kappa \)-complete ultrafilter \( \mathcal{U}_1 = \{ X \subseteq \kappa \mid X \in V[G] \}, \exists Y \in \mathcal{U}[Y \subseteq X] \} \) defined in \( V[G] \) generated by \( \mathcal{U} \) in \( V[G] \).

Proof. Proof of preserving Ramseyness follows from Theorem 21.2 of [Jec03] and proof of preserving measurability and the fact that \( \kappa \)-complete ultrafilters in the ground model generate \( \kappa \)-complete ultrafilters in the small forcing extensions with respect to \( \kappa \) follows from the Levy–Solovay Theorem in [LS67].

We can even see that fine measures generate fine measures in small forcing extensions by Lemma 26 of [Ina13]. Consequently, by Levy–Solovay Lemma (Lemma 27 of [Ina13]) and Theorem 29 of [Ina13] we can say if given \( \gamma \geq \kappa \) and \( \kappa \) \( \gamma \)-supercompact with \( \kappa \)-complete normal ultrafilter \( \mathcal{U} \) over \( \mathcal{P}_\kappa(\gamma) \) in the ground model then \( \kappa \) remains \( \gamma \)-supercompact with \( \kappa \)-complete normal ultrafilter \( \mathcal{U} \) generated by \( \mathcal{U} \) on \( \mathcal{P}_\kappa(\gamma) \) in the small forcing extension with respect to \( \kappa \).

3.3. Symmetric extension. Symmetric extensions are symmetric submodels of the generic extension containing the ground model, where the axiom of choice can consistently fail. Let \( \mathbb{P} \) be a forcing notion, \( \mathcal{G} \) be a group of automorphisms of \( \mathbb{P} \) and \( \mathcal{F} \) be a normal filter of subgroups over \( \mathcal{G} \). We recall the following Symmetry Lemma from [Jec03].

Theorem 3.3. (Symmetry Lemma, Lemma 14.37 of [Jec03]). Let \( \mathbb{P} \) be a forcing notion, \( \varphi \) be a formula of the forcing language with \( n \) variables and let \( \sigma_1, \sigma_2, \ldots, \sigma_n \in V^\mathbb{P} \) be \( \mathbb{P} \)-names. If \( a \in \text{Aut}(\mathbb{P}) \), then \( p \models \varphi(\sigma_1, \sigma_2, \ldots, \sigma_n) \iff a(p) \models \varphi(a(\sigma_1), a(\sigma_2), \ldots, a(\sigma_n)) \).

For \( \tau \in V^\mathbb{P} \), we denote the symmetric group with respect to \( \mathcal{G} \) by \( \text{sym}^\mathcal{G}\tau = \{ g \in \mathcal{G} : g\tau = \tau \} \) and say \( \tau \) is symmetric with respect to \( \mathcal{F} \) if \( \text{sym}^\mathcal{G}\tau \in \mathcal{F} \). Let \( HS^\mathcal{F} \) be the class of all hereditary symmetric names. We define symmetric extension of \( V \) or symmetric submodel of \( V[G] \) with respect to \( \mathcal{F} \) as \( V(G)^\mathcal{F} = \{ \tau^G : \tau \in HS^\mathcal{F} \} \). For the sake of our convenience we omit the subscript \( \mathcal{F} \) sometimes and call \( V(G)^\mathcal{F} \) as \( V(G) \).

Definition 3.4. (Symmetric System, Definition 2.1 of [AY]). We say \( (\mathbb{P}, \mathcal{G}, \mathcal{F}) \) is a symmetric system if \( \mathbb{P} \) is a forcing notion, \( \mathcal{G} \) the automorphism group of \( \mathbb{P} \) and \( \mathcal{F} \) a normal filter of subgroups over \( \mathcal{G} \).
Definition 3.5. (F-tenacious system, Definition 4.6 of [Kar16]). Let \( \langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle \) be a symmetric system. A condition \( p \in \mathbb{P} \) is \( F \)-tenacious if \( \{ \pi \in \mathcal{G} : \pi(p) = p \} \in \mathcal{F} \). We say \( \mathbb{P} \) is \( F \)-tenacious if there is a dense subset of \( F \)-tenacious conditions. We say \( \langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle \) is a tenacious system if \( \mathbb{P} \) is \( F \)-tenacious.

Asaf Karagila and Yair Hayut proved in Appendix A of [Kar16] that every symmetric system is equivalent to a tenacious system. Thus, it is natural to assume tenacity and work with tenacious systems. We recall the following theorem which states that the symmetric extension \( V(G) \) is a transitive model of ZF.

Theorem 3.6. (Lemma 15.51 of [Jec03]). If \( \langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle \) is a symmetric system and \( G \) is a \( V \)-generic filter, then \( V(G) \) is a transitive model of ZF and \( V \subseteq V(G) \subseteq V[G] \).

Definition 3.7. (Weakly homogeneous forcing notion, Definition 3.15 of [Kar16]). A forcing notion \( \mathbb{P} \) is weakly homogeneous if for any \( p, q \in \mathbb{P} \), there is an automorphism \( \alpha : \mathbb{P} \to \mathbb{P} \) such that \( \alpha(p) \) and \( q \) are compatible.

We recall the fact that Easton support products of weakly homogeneous forcing notions are weakly homogeneous. A crucial feature of symmetric extensions on weakly homogeneous forcing\(^{13}\) are that they can be approximated by certain intermediate submodel where AC holds.

3.4. Failure of a weaker form of the axiom of choice. A weaker version of the axiom of choice is \( AC_\kappa \) for a cardinal \( \kappa \). We use \( AC_\kappa \) to denote the statement “Every family of \( \kappa \) non-empty sets admits a choice function”. Given a cardinal \( \kappa \), failure of \( AC_\kappa \) is possible if \( \kappa^+ \) is singular. This is due to the following well known fact.

Fact 3.8. \( AC_\kappa \implies \text{cf}(\lambda) > \kappa \) for all successor cardinal \( \lambda \).

We sketch another way of refuting \( AC_\kappa \). One of the weaker forms of AC is \( AC_A(B) \) which states that for each set \( X \) of non-empty subsets of \( B \), if there is an injection from \( X \) to \( A \) then there is a choice function for \( X \). We recall Lemma 0.2, Lemma 0.3 and Lemma 0.12 from [Dim11]. Under \( AC_A(B) \), if there is a surjection from \( B \) to \( A \), then there is an injection from \( A \) to \( B \). We recall that if \( \kappa \) is measurable with a normal measure or weakly compact and \( \alpha < \kappa \) then there is no injection \( f : \kappa \to \mathcal{P}(\alpha) \) and in ZF for every infinite cardinal \( \kappa \), there is a surjection from \( \mathcal{P}(\kappa) \) onto \( \kappa^+ \). The following lemma states that if a successor cardinal \( \kappa \) is either measurable with normal measure or weakly compact then \( AC_\kappa \) fails.

Lemma 3.9. Let \( \kappa = \alpha^+ \) be a successor cardinal. If \( \kappa \) is measurable with normal measure or weakly compact then \( AC_{\alpha^+}(\mathcal{P}(\alpha)) \) fails.

Proof. Let \( AC_{\alpha^+}(\mathcal{P}(\alpha)) \) holds. We show \( \kappa = \alpha^+ \) is neither measurable with normal measure nor weakly compact. In ZF, there is a surjection from \( \mathcal{P}(\alpha) \) onto \( \alpha^+ \). Now \( AC_{\alpha^+}(\mathcal{P}(\alpha)) \) implies there is an injection \( f \) from \( \alpha^+ \) to \( \mathcal{P}(\alpha) \) which states that \( \kappa = \alpha^+ \) is neither measurable with normal measure nor weakly compact. \( \square \)

\(^{13}\)Given infinite cardinals \( \kappa, \mu \) and a regular cardinal \( \lambda \), the Cohen forcing (\( \text{Add}(\kappa, \mu) \)), Levy collapse (\( \text{Col}(\lambda, < \kappa) \)), Shooting a club through a stationary set \( S \subseteq \lambda \) using closed and bounded subsets of \( S \) as conditions (\( \text{Club}(S) \)), Sacks forcing and Prikry forcing are some examples of weakly homogeneous forcings.
4. Preserving Dependent choice in symmetric extensions

Dependent Choice, denoted by DC or DC$_\omega$, is a weaker version of the Axiom of choice (AC) which is strictly stronger than the countable choice, denoted by AC$_\omega$. This principle is strong enough to give the basis of analysis as it is equivalent to the Baire Category Theorem which is a fundamental theorem in functional analysis. Further, DC is equivalent to other important theorems like the countable version of the Downward Löwenheim–Skolem theorem and every tree of height $\omega$ without a maximum node has an infinite branch etc. On the other hand, AC has several controversial applications like the existence of a non-Lebesgue measurable set of real numbers, Banach–Tarski Paradox and the existence of a well-ordering of real numbers whereas DC does not have such counter-intuitive consequences. Thus it is desirable to preserve dependent choice in symmetric extensions.

We denote the principle of Dependent Choice for $\kappa$ by DC$_\kappa$ for a cardinal $\kappa$. This principle states that for every non-empty set $X$, if $R$ is a binary relation such that for each ordinal $\alpha < \kappa$, and each $f : \alpha \to X$ there is some $y \in X$ such that $f R y$, then there is $f : \kappa \to X$ such that for each $\alpha < \kappa$, $f \upharpoonright \alpha R f(\alpha)$. We denote the assertion $(\forall \lambda < \kappa) DC_\lambda$ by $DC_{< \kappa}$. The axiom of choice is equivalent to $(\forall \lambda < \kappa) DC_\lambda$ and $DC_\kappa$ implies AC$_\kappa$.

Asaf Karagila proved in Lemma 1 of [Kar14], that $DC_{< \kappa}$ can be preserved in the symmetric extension in terms of the symmetric system $\langle \mathcal{P}, \mathcal{G}, \mathcal{F} \rangle$, if $\mathcal{P}$ is $\kappa$-closed and $\mathcal{F}$ is $\kappa$-complete. In Lemma 3.3 of [Kar], Asaf Karagila and the author both observed that if $\mathcal{P}$ is $\kappa$-closed” can be replaced by ”$\mathcal{P}$ has $\kappa$-c.c.” in Lemma 1 of [Kar14]. Specifically we observe the following.

‘If $\mathcal{P}$ has $\kappa$-c.c., then any antichain is of size less than $\kappa$. So by Zorn’s Lemma in the ground model, there is a maximal antichain of conditions $\mathcal{A} = \{ p_\alpha : \alpha < \gamma < \kappa \}$ extending $p$ such that for all $\alpha < \gamma$, $p_\alpha \Vdash \hat{f}(\bar{\alpha}) = \bar{t}_\alpha$ where $\bar{t}_\alpha \in HS$ and then we can follow Lemma 1 of [Kar14] to finish the proof.’

We also note that the natural assumption that $\langle \mathcal{P}, \mathcal{G}, \mathcal{F} \rangle$ is a tenacious system is required in the proof as written by Asaf Karagila in Lemma 3.3 of [Kar].

**Lemma 4.1.** (Lemma 3.3 of [Kar]). If $\mathcal{P}$ has $\kappa$-c.c. and $\mathcal{F}$ is $\kappa$-complete then $DC_{< \kappa}$ is preserved in the symmetric extension in terms of the symmetric system $\langle \mathcal{P}, \mathcal{G}, \mathcal{F} \rangle$.

We can slightly generalize Lemma 1 of [Kar14] and prove that if $\mathcal{P}$ is $\kappa$-strategically closed and $\mathcal{F}$ is $\kappa$-complete then $DC_{< \kappa}$ holds in the symmetric extension in terms of the symmetric system $\langle \mathcal{P}, \mathcal{G}, \mathcal{F} \rangle$.

**Lemma 4.2.** If $\mathcal{P}$ is $\kappa$-strategically closed and $\mathcal{F}$ is $\kappa$-complete then $DC_{< \kappa}$ holds in the symmetric extension in terms of the symmetric system $\langle \mathcal{P}, \mathcal{G}, \mathcal{F} \rangle$.

Proof. Let $G$ be a $\mathcal{P}$-generic filter over $V$. Let $\delta < \kappa$, we show $DC_\delta$ holds in $V(G)$. Let $X$ and $R$ are elements of $V(G)$ as in the assumptions of $DC_\delta$. Since $AC$ is equivalent to $\forall \kappa(\forall \alpha < \kappa)$ and $V[G]$ is a model of $AC$, using $\forall \kappa(\forall \alpha < \kappa)$ in $V[G]$, we can find a $f : \delta \to X$ in $V[G]$. We show this $f : \delta \to X$ is in $V(G)$. Let $p \Vdash f$ is a function whose domain is $\delta$ and range a subset of $V(G)$. Consider a game of length $\kappa$, between two players I and II who play at odd stages and even stages respectively such that initially II chooses a trivial condition and I chooses a condition extending $p$ and at non-limit stages

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20In Howard–Rubin’s first model (N38 in [HR98]), AC$_\omega$ holds but $DC_{\omega}$ fails.
even stages $2\alpha > 0$, II chooses a condition extending the condition of the previous stage deciding $\dot{f}(\dot{\alpha}) = \dot{t}_\alpha$ where $\dot{t}_\alpha$ is in $HS$. By $\kappa$-strategic closure of $P$, II has winning strategy. Thus, we can assume the existence of an increasing sequence of conditions $\langle p_\alpha : \alpha < \delta \rangle$ extending $p$ such that $p_\alpha \vdash \dot{f}(\dot{\alpha}) = \dot{t}_\alpha$ where $\dot{t}_\alpha$ is in $HS$ for each $\alpha < \delta$. It is enough to show that $\dot{f} = \{ \dot{t}_\beta : \beta < \delta \}$ is in $HS$ which follows using $\kappa$-completeness of $F$ as done in Lemma 1 of [Kar14]. □

4.1. An application. In [Jec68], Thomas Jech proved that $\aleph_1$ can be measurable assuming the consistency of 'ZFC + there is a measurable cardinal'. Assuming the consistency of 'ZFC + GCH + there is a measurable cardinal', we observe that any successor of a regular cardinal (for example $\aleph_1$, $\aleph_2$, $\aleph_\omega + 2$, as well as $\aleph_\omega + 2$) can be a measurable cardinal carrying an arbitrary number of normal measures.

Sy–David Friedman and Menachem Magidor proved that a measurable cardinal can be forced to carry arbitrary number of normal measures in ZFC.

Lemma 4.3. (Theorem 1 of [MD09]). Assume GCH. Suppose that $\kappa$ is a measurable cardinal and let $\alpha$ be a cardinal at most $\kappa^{++}$. In a cofinality preserving forcing extension, then $\kappa$ carries exactly $\alpha$ normal measures.

We recall the definition of a symmetric collapse from [AY].

Definition 4.4. (Symmetric Collapse, Definition 4.1 of [AY]). Let $\kappa < \lambda$ be two infinite cardinals. The symmetric collapse is the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ defined as follows.

- $\mathbb{P} = \text{Col}(\kappa, < \lambda)$.
- $\mathcal{G}$ is the group of automorphisms $\pi$ such that there is a sequence of permutations $\pi^\frown = \langle \pi_\alpha : \kappa < \alpha < \lambda \rangle$ such that $\pi_\alpha$ is a permutation of $\alpha$ satisfying $\pi \alpha(\alpha, \beta) = \pi_\alpha \alpha(\alpha, \beta)$.
- $\mathcal{F}$ is the normal filter of subgroups generated by $\text{fix}(E)$ for bounded $E \subseteq \lambda$, where $\text{fix}(E)$ is the group $\{ \pi : \forall \alpha \in E, \pi \alpha(\alpha, \beta) = \alpha(\alpha, \beta) \}$.

Lemma 4.5. Let $\kappa < \lambda$ be two infinite cardinals and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is the symmetric collapse where $\mathbb{P} = \text{Col}(\kappa, < \lambda)$. Then, $\mathcal{F}$ is $\kappa$-complete.

Proof. Fix $\gamma < \kappa$ and let, for each $\beta < \gamma$, $K_\beta \in \mathcal{F}$. There must be bounded $E_\beta \subseteq \lambda$ for each $\beta < \gamma$ such that $\text{fix}(E_\beta) \subseteq K_\beta$. Next, fix $\{ \bigcap_{\beta < \gamma} E_\beta \} \subseteq \bigcap_{\beta < \gamma} K_\beta$ and $\bigcup_{\beta < \gamma} E_\beta$ is a bounded subset of $\lambda$ implies $\bigcap_{\beta < \gamma} K_\beta \in \mathcal{F}$. □

We observe that after a symmetric collapse, the successor of a regular cardinal can be a measurable cardinal carrying an arbitrary number of normal measures assuming the consistency of a measurable cardinal. Further we can preserve dependent choice in certain cases.

Theorem 4.6. Let $V$ be a model of ZFC + GCH with a measurable cardinal $\kappa$. Let, $\lambda$ be any cardinal at most $\kappa^{++}$. There is then a symmetric extension where for a regular cardinal $\eta < \kappa$, $\kappa = \eta^+$ is a measurable cardinal carrying $\lambda$ normal measures. Moreover, $\text{AC}_\kappa$ fails and $\text{DC}<\kappa$ hold$^{21}$ in the symmetric model.

Proof. Applying Lemma 4.3, we obtain a cofinality preserving forcing extension $V'$ of $V$ where $\kappa$ is a measurable cardinal with $\lambda$ many normal measures. Let $\eta < \kappa$ be a regular cardinal in $V'$, $^{21}$If we assume $\eta > \omega$.\n
and $V'(G)$ be the symmetric extension of $V'$ obtained by the symmetric collapse $⟨P,G,F⟩$ where $P = Col(η, < κ)$ and $G$ be a $P$-generic filter over $V'$. In $V'(G)$, $κ = η^+$. We can also have the following.

- By Lemma 2.4 and Lemma 2.5 of [Apt01], $κ$ remains a measurable cardinal with $λ$ many normal measures.
- Since $κ$ is a successor as well as a measurable cardinal, $AC_κ$ fails using Lemma 3.9.
- Since $P$ is $η$-closed and the filter $F$ is $η$-complete by Lemma 4.5, $DC_{< η}$ holds using Lemma 1 of [Kar14].

5. Mutually stationary property of a sequence of stationary subsets

5.1. Mutually Stationary Sets. Let $κ$ be a cardinal. $C ⊆ κ$ is a club set if it is closed and unbounded. $S ⊆ κ$ is stationary if $S ∩ C ≠ ∅$ for every club $C$. We recall the definition of mutually stationary sets from Foreman–Magidor [MM01] and a theorem due to Foreman and Magidor.

Definition 5.1. (Mutually Stationary Sets, Definition 1.1 of [Apt04]). Let $K$ be a set of regular cardinals with supremum $λ$. Suppose $S_κ ⊆ κ$ for all $κ ∈ K$. Then $⟨S_κ : κ ∈ K⟩$ is mutually stationary if and only if for all algebras $A$ on $λ$, there is an elementary substructure $B < A$ such that for all $κ ∈ B ∩ K$, $sup(B ∩ κ) ∈ S_κ$.

Theorem 5.2. (Theorem 5.2 of [JMM]). Let $⟨κ_i : i < δ⟩$ be an increasing sequence of measurable cardinals, where $δ < κ_0$ is a regular cardinal. Let $S_i ⊆ κ_i$ be stationary for each $i < δ$. It is then the case that $⟨S_i : i < δ⟩$ is mutually stationary.

5.2. Mutually Stationary property in a Symmetric extension. It is not a theorem in ZFC, that if $K$ consists of an increasing sequence of regular cardinals and for each $κ ∈ K$, $S_κ ⊆ κ$ is stationary in $κ$, then $⟨S_κ : κ ∈ K⟩$ is mutually stationary. In particular, in $L$, by Theorem 24 of [MM01], there is a sequence of stationary sets $⟨S_n : 1 < n < ω⟩$ such that $S_n ⊆ ℵ_n$, $S_n$ is stationary and consists of points having cofinality $ℵ_1$, yet $⟨S_n : 1 < n < ω⟩$ is not mutually stationary. Foreman and Magidor asked whether it is possible to construct a model of ZFC where if $⟨S_n : 1 ≤ n < ω⟩$ is such that each $S_n$ is stationary on $ℵ_n$, then $⟨S_n : 1 ≤ n < ω⟩$ is mutually stationary. Starting from an $ω$-sequence of supercompact cardinals, Shelah constructed a model of ZFC in section 6 of [JMM], where if we define the sequence of stationary sets as follows,

$$S'_n = \{α < ℵ_n : cf(α) = ℵ_{f(n)}\} \text{ if } n > 1 \text{ and } f : ω → 2 \text{ is an arbitrary function.}$$

then the sequence $⟨S'_n : 1 < n < ω⟩$ is mutually stationary. In [Apt04], Arthur Apter gave a complete answer to the aforementioned question of Foreman and Magidor in a choiceless context. Specifically, he constructed a symmetric inner model preserving $DC_ω$ from a $ω$-sequence of supercompact cardinals where if $⟨S_n : 1 ≤ n < ω⟩$ is a sequence of stationary sets such that $S_n ⊆ ℵ_n$, then $⟨S_n : 1 ≤ n < ω⟩$ is mutually stationary. In this section we observe that we can weaken the assumption and obtain a generalized version of this mutually stationarity phenomenon in a symmetric extension where $DC_{< κ}$ is preserved by assuming ‘$κ$ is a $2^κ$-supercompact cardinal’ and prove Theorem 1.2.

Proof. (Theorem 1.2).  

\[\text{Page 290 of } [MM01].\]
Lemma 5.4. Let $\kappa$ be a model of ZFC where $\kappa$ is a $2^\kappa$-supercompact cardinal. Let $\{\kappa_\alpha : \alpha < \kappa\}$ be an enumeration of measurable cardinals below $\kappa$.

(2) **Defining symmetric system** $(P, G, F)$:
- Let $P$ be the Easton support product of $P_\alpha = Col(\kappa_\alpha^+, < \kappa_{\alpha+1})$ where $\alpha < \kappa$.
- $G$ be the Easton support product of the automorphism groups of each $P_\alpha$.
- Let $F$ be the filter generated by $\text{fix}(\alpha)$ groups for $\alpha < \kappa$, where $\text{fix}(\alpha) = \{\pi \in \Pi_{\alpha < \kappa} \text{Aut}(P_\alpha) : \pi \upharpoonright \alpha = \text{id}\}$.

(3) **Defining symmetric extension of $V$**: Let $G$ be a $P$-generic filter. We construct a model $V(G)^F$ by the symmetric system $\langle P, G, F \rangle$ defined in (2) above and call it as $V(G)$ for the sake of convenience.

Since each $P_\alpha$ is weakly homogeneous, we can have the lemma which tells that every set of ordinals in $V(G)$ is added by a bounded part of the product.

**Lemma 5.3.** If $A \in V(G)$ is a set of ordinals, then $A \in V[G \upharpoonright \alpha]$ for some $\alpha < \kappa$.

**Proof.** Let $\hat{A} = \{(p, \hat{\epsilon}) : p \Vdash \hat{\epsilon} \in \hat{A}\} \in HS$, $\hat{q} \Vdash \hat{\epsilon} \in \hat{A}$ and let $\beta$ support $\hat{\epsilon}$ and $\hat{A}$. Let, for the sake of contradiction $\hat{q} \upharpoonright \beta \not\Vdash \hat{\epsilon} \in \hat{A}$. Then, there is a $\hat{q}'$ such that $\hat{q}' \leq \hat{q} \upharpoonright \beta$ such that $\hat{q}' \Vdash \neg(\hat{\epsilon} \in \hat{A})$. Since each $P_\alpha$ is weakly homogeneous, the Easton support product is weakly homogeneous too. Thus there is a $a \in \text{fix}\beta$ such that $a(\hat{q}) \parallel \hat{q}'$. By **Symmetry Lemma**, $a(\hat{q}) \Vdash a(\hat{\epsilon}) \in a(\hat{A})$. Since $\beta$ supports $\hat{\epsilon}$ and $\hat{A}$ and $a \in \text{fix}\beta$ we get $a(\hat{q}) \Vdash \hat{\epsilon} \in \hat{A}$ which is a contradiction. Thus, $\hat{q} \upharpoonright \beta \Vdash \hat{\epsilon} \in \hat{A}$. If $\alpha = \sup\beta$ then we get that $\{(\langle \hat{q} \upharpoonright \alpha, \hat{\epsilon} \rangle : \hat{\epsilon} \in \hat{A}\}$ is a name for $A$. $\Box$

We prove that $\kappa$ remains $2^\kappa$-supercompact in $V(G)$ and we refer to **Levy Solovay Lemma** (Lemma 27 of [Ina13]) and **Theorem 29 of [Ina13]** for our purpose.

**Lemma 5.4.** In $V(G)$, $\kappa$ is $2^\kappa$-supercompact.

**Proof.** Let $\gamma = 2^\kappa$. We show if $U$ is a normal measure on $P_\kappa(\gamma)$ in $V$ (since $\kappa$ is $2^\kappa$-supercompact in $V$), then $U_1 = \{Y \subseteq P_\kappa(\gamma) : \exists X \in U(X \subseteq Y)\}$ defined in $V(G)$ generated by $U$ is a normal measure on $P_\kappa(\gamma)$ in $V(G)$.

**Ultrafilter**: Let $(X \subseteq P_\kappa(\gamma))^{V(G)}$ and $(f : P_\kappa(\gamma) \to 2)^{V(G)}$ be an indicator function of $X$. By **Lemma 5.3**, for some $\alpha < \kappa$ we get $f \in V[G \upharpoonright \alpha]$. Now we can say $G \upharpoonright \alpha$ is $P^\kappa$-generic over $V$ where $|P^\kappa| < \kappa$. By **Levy-Solovay Lemma**, we get a $Y \in U$ and $(g : Y \to 2)^{V(G)}$ such that $V[G \upharpoonright \alpha] \models f \upharpoonright Y = g$. So, $V(G) \models f \upharpoonright Y = g$. Since, $U$ is an ultrafilter on $P_\kappa(\gamma)$ in $V$ there is exactly one $i \in 2$ such that $g^{-1}(i) \in U$. Thus there exist a $Z \subseteq X$ such that $V(G) \models Z \subseteq X$ or $V(G) \models Z \subseteq P_\kappa(\gamma) \setminus X$. So exactly one of $X$ and $P_\kappa(\gamma) \setminus X$ is in $U_1$.

**$\kappa$-complete**: Let $\delta < \kappa$ and $(f : P_\kappa(\gamma) \to \delta)^{V(G)}$ be given. By **Lemma 5.3**, for some $\alpha < \kappa$ we have $f \in V[G \upharpoonright \alpha]$. Again $G \upharpoonright \alpha$ is $P^\kappa$-generic over $V$ where $|P^\kappa| < \kappa$. By **Levy-Solovay Lemma**, there exist $Y \in U$, $(g : Y \to \delta)^{V(G)}$ such that $V[G \upharpoonright \alpha] \models f \upharpoonright Y = g$. So, $V(G) \models f \upharpoonright Y = g$. Since, $U$ is $\kappa$-complete in $P_\kappa(\gamma)$ in $V$, $g^{-1}(\beta) \in U$ for some $\beta \in \delta$. Thus $f^{-1}(\beta) \in U_1$ for some $\beta \in \delta$.

**Fineness**: Let $X \in (P_\kappa(\gamma))^{V(G)}$. By **Lemma 5.3**, for some $\alpha < \kappa$ we have $X \in P_\kappa(\gamma)^{V[G[\alpha]}$. Let $U'$ be the fine measure generated by $U$ on $P_\kappa(\gamma)^{V[G[\alpha]}$ (**Theorem 29 of [Ina13]**). Now, $U \subseteq U' \subseteq U_1$ and $P_\kappa(\gamma)^{V[G[\alpha]} \subseteq P_\kappa(\gamma)^{V(G)}$ implies $U_1$ is fine.

**Choice function**: Let $V(G) \models f : P_\kappa(\gamma) \to \gamma$ and $V(G) \models \forall X \in P_\kappa(\gamma)(f(X) \in X)$. By **Lemma 5.3**, for some $\alpha < \kappa$ we get $h = f \upharpoonright (P_\kappa(\gamma))^{V[G \upharpoonright \alpha]}$, then by **Levy-Solovay Lemma** we get
Y ∈ U and (g : Y → γ)^V such that V[G | α] ⊨ h | Y = g. Now by normality of U in V we get a set x in U such that g is constant on x in V[G | α] and so h is constant on a set in U. Hence, we will get a set y in U_1 such that f is constant on y in V(G).

Lemma 5.5. In V(G), DC<κ holds.

Proof. We see that F is κ-complete. Fix γ < κ and let, for each β < γ, K_β ∈ F. There must be fixβ for each β < γ such that fixβ ⊆ K_β. Next, fix(max{β : β < γ}) ⊆ ∩β<γ fixβ ⊆ ∩β<γ K_β implies ∩β<γ K_β ∈ F. Since P is the Easton support product of the appropriate Levy collapse, P has κ-c.c. Since F is κ-complete and P has κ-c.c., we obtain DC<κ in V(G) by Lemma 4.1.

In V(G), we can see that the cardinality of each member of the sequence S = ⟨κ_α : α < κ⟩ is preserved and for each α < κ, the interval ⟨κ_α, κ_{α+1}⟩ has collapsed to κ_α. We prove that if S′ = ⟨λ_α : α < β⟩ is a subsequence of S having supremum λ in V(G), such that the following holds.

• β < λ_0.
• (S_α : α < β) is a sequence of sets such that each S_α is a stationary subset of λ_α.

Then ⟨S_α : α < β⟩ is mutually stationary.

Lemma 5.6. In V(G), if ⟨λ_α : α < β⟩ is a subsequence of ⟨κ_α : α < κ⟩ having supremum λ, such that β < λ_0 and (S_α : α < β) is a sequence of sets such that each S_α is a stationary subset of λ_α, then ⟨S_α : α < β⟩ is mutually stationary.

Proof. Suppose A is an algebra on λ in V(G) and V(G) ⊨ ⟨S_α : α < β⟩ is such that for each α < β, S_α is a stationary subset of λ_α. Since ⟨λ_α : α < β⟩ is a subsequence of ⟨κ_α : α < κ⟩, each λ_α is κ_α for some i_α < κ. Now since both A and ⟨S_α : α < β⟩ can be coded by a set of ordinals, by Lemma 5.3 there exists some γ < κ for which A and ⟨S_α : α < β⟩ ∈ V[G | γ]. We show κ_α is measurable in V[G | γ] as well as S_α is a stationary subset of κ_α in V[G | γ].

Subclaim (1): For each α < β, κ_α is measurable in V[G | γ].

Proof. Fix a α < β. Since κ_α is measurable in V, say with some κ_α-complete ultrafilter U over κ_α, we claim that U_1 = {X ⊆ κ_α : X ∈ V[G | α], ∃Y ∈ U | Y ⊆ X} generated by U defined in V[G | γ] is the corresponding κ_α-complete ultrafilter over κ_α in V[G | γ].

Ultrafilter: We note that V[G | γ] can be written as V[G_1 | G_2] where G_1 is P_1-generic over V such that |P_1| < κ_α and G_2 is P_2-generic over V[G_1] such that P_2 is κ_α-closed. Let X ⊆ κ_α be arbitrary in V[G | γ]. Since P_2 is κ_α-closed, X is not added by P_2 and hence must have been added by P_1. Now since |P_1| < κ_α, by Theorem 3.2, κ_α remains measurable in V[G_1] with the κ_α-complete ultrafilter U defined in V[G_1]. Thus X ∈ U_1 or κ_α \ X ∈ U_1 if U_1 is defined in V[G | γ].

κ_α-complete: Let γ < κ_α be arbitrary and ⟨X_δ : δ < γ⟩ be a sequence of sets in U_1 in V[G | γ]. As before, we can write V[G | γ] as V[G_1 | G_2] where G_1 is P_1-generic over V such that |P_1| < κ_α and G_2 is P_2-generic over V[G_1] where P_2 is κ_α-closed and hence does not add subsets of κ_α. Now, ⟨X_δ : δ < γ⟩ is not added by P_2 and hence must have been added by P_1. Now since |P_1| < κ_α, by Theorem 3.2, we can say that κ_α remains measurable in V[G_1] with κ_α-complete ultrafilter U defined in V[G_1] generated by U. Thus U_1 defined in V[G | γ] is κ_α-complete.
Subclaim (2): $S_\alpha$ is a stationary subset of $\kappa_{i_\alpha}$ in $V[G \upharpoonright \gamma]$.

**Proof.** Let $C$ be any club set of $\kappa_{i_\alpha}$ in $V[G \upharpoonright \gamma]$. Since the notion of club subset of $\kappa_{i_\alpha}$ is upward absolute and $V[G \upharpoonright \gamma] \subseteq V(G)$, $C$ is also a club set of $\kappa_{i_\alpha}$ in $V(G)$. Since in $V(G), S_\alpha$ is a stationary subset of $\kappa_{i_\alpha}$, we have $S_\alpha \cap C \neq \emptyset$. □

By **Subclaim (1)**, **Subclaim (2)** and Theorem 5.2 we get that $(S_\alpha : \alpha < \beta)$ is mutually stationary in $V[G \upharpoonright \gamma]$. Since $\mathfrak{A}$ is still an algebra on $\lambda$ in $V[G \upharpoonright \gamma]$, there is an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$ in $V[G \upharpoonright \gamma]$ such that for all $\alpha < \beta, sup(\mathfrak{B} \cap \kappa_{i_\alpha}) \in S_\alpha$. There is thus an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$ in $V(G)$ such that for all $\alpha < \beta, sup(\mathfrak{B} \cap \kappa_{i_\alpha}) \in S_\alpha$. Hence in $V(G)$, $(S_\alpha : \alpha < \beta)$ is mutually stationary. □

6. Failure of GCH at limit cardinals below a supercompact cardinal

In this section we prove Theorem 1.4 applying Lemma 4.1 by constructing a symmetric extension in terms of a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$. Consequently, we can answer Question 1.3 asked by Arthur Apter.

**Proof. (Theorem 1.4).**

1. **Defining ground model** $(V)$: At one stage of the construction in Theorem 3 of [Apt12], there is an enumeration $\{\kappa_i : i < \kappa\}$ of a club $C \subseteq \kappa$ of inaccessible and limit cardinals below a supercompact cardinal $\kappa$ such that $2^{\kappa_i} = \kappa_i^{++}$ holds.

2. **Defining symmetric system** $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$:

   - Let $\mathbb{P}$ be the Easton support product of $\mathbb{P}_\alpha = Col(\kappa_\alpha^{++}, \kappa^{-}), \alpha < \kappa$.
   - $\mathcal{G}$ be the Easton support product of the automorphism groups of each $\mathbb{P}_\alpha$.
   - $\mathcal{F}$ be the filter generated by $fix(\alpha)$ groups for $\alpha < \kappa$, where $fix(\alpha) = \{\pi \in \Pi_{\alpha<\kappa}Aut(\mathbb{P}_\alpha) : \pi \upharpoonright \alpha = id\}$.

3. **Defining symmetric extension** of $V$: Let $G$ be a $\mathbb{P}$-generic filter. We construct a model $V(G)^\mathcal{F}$ by the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ and call it as $V(G)$ for the sake of convenience.

As before since each $\mathbb{P}_\alpha$ is weakly homogeneous, we can have the following lemma which tells that every set of ordinals in $V(G)$ is added by a bounded part of the product.

**Lemma 6.1.** If $A \in V(G)$ is a set of ordinals, then $A \in V[G \upharpoonright \alpha]$ for some $\alpha < \kappa$.

**Proof.** Similar to Lemma 5.3. □

**Lemma 6.2.** In $V(G)$, $\kappa$ is supercompact.

**Proof.** Let $\gamma \geq \kappa$ be arbitrary. We mimic the arguments of Lemma 5.4. We show if $\mathcal{U}$ is a normal measure on $\mathcal{P}_\kappa(\gamma)$ in $V$ (since $\kappa$ is supercompact in $V$), then $\mathcal{U}_1 = \{Y \subseteq \mathcal{P}_\kappa(\gamma) \exists X \in \mathcal{U}(X \subseteq Y)\}$ defined in $V(G)$ generated by $\mathcal{U}$ is a normal measure on $\mathcal{P}_\kappa(\gamma)$ in $V(G)$.

**Ultrafilter:** Let $(X \subseteq \mathcal{P}_\kappa(\gamma))^{V(G)}$ and $(f : \mathcal{P}_\kappa(\gamma) \rightarrow 2)^{V(G)}$ be an indicator function of $X$. By Lemma 6.1 for some $\alpha < \kappa$ we get $f \in V[G \upharpoonright \alpha]$. Now we can say $G \upharpoonright \alpha$ is $\mathbb{P}^\alpha$-generic over $V$ where $|\mathbb{P}^\alpha| < \kappa$. By **Levy–Solovay Lemma** we get a $Y \in \mathcal{U}$ and $(g : Y \rightarrow 2)^V$ such that
we can obtain the result of Theorem 1 of [Apt01] starting from just one measurable cardinal. Arthur Apter constructed an analogous symmetric extension where

\[ \delta > \kappa \]

Since \( \delta \) is a limit cardinal then \( \delta > \kappa \). Consequently, there is no injection from \( V[G] \) onto \( \kappa \). We prove that in \( V[G] \) holds a limit cardinal \( \delta \) if and only if \( \delta > \kappa \). Since GCH implies AC, GCH is weakened to a form which states that there is no injection from \( \delta^{++} \) into \( P(\delta) \) in Theorem 3 of [Apt12]. We follow this weakened version of GCH in our following lemma.

**Lemma 6.3.** In \( V(G) \), \( DC_{<\kappa} \) holds.

**Proof.** Similar to Lemma 5.5. Since \( P \) has \( \kappa \)-c.c. and \( F \) is \( \kappa \)-complete, \( DC_{<\kappa} \) is preserved applying Lemma 4.1.

We prove that in \( V(G) \), GCH holds at a limit cardinal \( \delta \) if and only if \( \delta > \kappa \). Since GCH holds at any finite or infinite cardinal’ Arthur Apter constructed an analogous symmetric extension where \( DC_{<\kappa} \) holds and where \( \kappa \) can carry an arbitrary number of normal measures regardless of the specified behavior of the continuum function on sets having measure one with respect to every normal measure over \( \kappa \). We observe that we can obtain the result of Theorem 1 of [Apt01] starting from just one measurable cardinal \( \kappa \) if
we use Theorem 1 of [MD09] by Friedman and Magidor instead of passing to an inner model of Mitchell from [Mit74].

**Corollary 6.5. (of Theorem 1 of [Apt01]).** Let $V$ be a model of ZFC $+$ GCH with a measurable cardinal $\kappa$ and let $\lambda$ be a cardinal at most $\kappa^{++}$. There is then a symmetric extension with respect to a symmetric system $\langle P, G, F \rangle$ where $\kappa$ is a measurable cardinal carrying $\lambda$ many normal measures $\langle U_\alpha : \alpha < \lambda \rangle$. Moreover for each $\alpha < \lambda$, the set $\{ \delta : 2^\delta = \delta^{++}$ and $\delta$ is inaccessible $\} \in U_\alpha^{24}$ and $DC_{\kappa^+}$ holds.

**Remark 2.** Arthur Apter used analogous arguments in Lemma 2.2 of [Apt01], similar to Lemma 4.1 to preserve a certain amount of dependent choice in some symmetric models (e.g. symmetric models from Theorem 1 of [Apt01], Theorem of [Apt00] and Theorem 2 of [Apt12]).

7. **Proving Ioanna Dimitriou’s Conjecture**

Ioanna Dimitriou conjectured that $DC_{\omega_1}$ would fail in the symmetric extension constructed in Section 1.4 of [Dim11]. We observe that $AC_{\omega_1}$ fails in it and thus prove Theorem 1.6. The exposition of the work in this section is in the terminology of Ioanna Dimitriou from [Dim11] which involves the terminologies like *Approximation Lemma*, *Approximation property* and $(G, I)$-homogeneous forcing notion.

For $E \subseteq P$, let us define the pointwise stabilizer group to be $fix^*_E = \{ g \in G : \forall p \in E, g(p) = p \}$ i.e. it is the set of automorphisms which fix $E$ pointwise. Again we denote $fix^*_E$ by $fix E$ for the sake of convenience. A subset $I \subseteq P(P)$ is called $G$-symmetry generator if it is closed under unions and if for all $g \in G$ and $E \in I$, there is an $E' \in I$ s.t. $g(\text{fix} E)g^{-1} \supseteq \text{fix} E'$. It is possible to see that if $I$ is a $G$ symmetry generator, then the set $\{ \text{fix} E : E \in I \}$ generates a normal filter over $G$ (Proposition 1.23 of Chapter 1 in [Dim11]). Let $I$ be the $G$ symmetry generator generating a normal filter $F$ over $G$, we say $E \in I$ supports a name $\sigma \in HS$ if $\text{fix} E \subseteq \text{sym}(\sigma)$. Since $P, G$ and $I$ are enough to define a symmetric extension, we define a symmetric triple $\langle P, G, I \rangle$ and work with it.

**Definition 7.1. (Symmetric Triple $\langle P, G, I \rangle$).** We say $\langle P, G, I \rangle$ is a symmetric triple if $P$ is a forcing notion, $G$ an automorphism group and $I$ a $G$-symmetry generator.

Let $\langle P, G, I \rangle$ be a symmetric triple, then $I$ is projectable for the pair $(P, G)$ if for every $p \in P$ and every $E \in I$, there is a $p^* \in E$ that is minimal in the partial order and unique such that $p^* \geq p$. We call $p \upharpoonright E = p^*$ the projection of $p$ to $E$. We say that $P$ is $(G, I)$-homogeneous if for every $E \in I$, every $p \in P$ and every $q \leq p \upharpoonright E$ there is an automorphism $a \in \text{fix} E$ s.t. $a(p) \parallel q$. $(P, G, I)$ has the approximation property if for all formulae $\phi$ with $n$ free variables, names $\sigma_1, \sigma_2, \ldots, \sigma_n \in HS$ all with support $E \in I$ and for every $p \in P$, $p \models \phi(\sigma_1, \sigma_2, \ldots, \sigma_n)$ implies that $p \upharpoonright E \models \phi(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

**Lemma 7.2. (Lemma 1.27 of [Dim11]).** Let $\langle P, G, I \rangle$ be a symmetric triple. If $P$ is $(G, I)$-homogeneous, then $(P, G, I)$ has the approximation property.

**Lemma 7.3. (Approximation Lemma, Lemma 1.29 of [Dim11]).** Let $\langle P, G, I \rangle$ be a symmetric triple. If $(P, G, I)$ has the approximation property then for all set of ordinals $X \in V(G)$, there exists an $E \in I$ and an $E$ name for $X$. Thus, $X \in V[G \cap E]$.

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23 as done in the proof of Theorem 1 of [Apt01].
24 There is nothing specific about $\delta^{++}$, the continuum function can take any value.
Proof. (Theorem 1.6). Firstly, we give a description of the symmetric extension constructed in Section 1.4 of [Dim11] as follows.

1. **Defining ground model** ($V$): Let $V$ be a model of ZFC, $\rho$ is an ordinal, and $K = \langle \kappa_\epsilon : 0 < \epsilon < \rho \rangle$ is a sequence of regular cardinals with a regular cardinal $\kappa_0$ below all the regular cardinals in $K$.

2. **Defining symmetric triple** ($\mathbb{P}, \mathcal{G}, I$):
   - For each $\epsilon \in (0, \rho)$ we define the following cardinals,
     \[
     \kappa'_1 = \kappa_0, \\
     \kappa'_\epsilon = \kappa'_{\epsilon-1} \text{ if } \epsilon \text{ is a successor ordinal,} \\
     \kappa'_\epsilon = (\bigcup_{\zeta < \epsilon} \kappa_\zeta)^+ \text{ if } \epsilon \text{ is a limit ordinal and } \bigcup_{\zeta < \epsilon} \kappa_\zeta \text{ is singular,} \\
     \kappa'_\epsilon = (\bigcup_{\zeta < \epsilon} \kappa_\zeta)^{++} \text{ if } \epsilon \text{ is a limit ordinal and } \bigcup_{\zeta < \epsilon} \kappa_\zeta = \kappa_\epsilon \text{ is regular,} \\
     \kappa'_\epsilon = \bigcup_{\zeta < \epsilon} \kappa_\zeta \text{ if } \epsilon \text{ is a limit ordinal and } \bigcup_{\zeta < \epsilon} \kappa_\zeta < \kappa_\epsilon \text{ is regular.}
     \]
   - Let $\mathbb{P} = \Pi_{0 < i < \rho} Fn(\kappa'_i, \kappa_i, \kappa'_i)$ be the set support product of $Fn(\kappa'_i, \kappa_i, \kappa'_i)$ ordered componentwise where for each $0 < i < \rho$, $Fn(\kappa'_i, \kappa_i, \kappa'_i) = \{ p : \kappa'_i \rightarrow \kappa_i : |p| < \kappa'_i \}$ and $p$
is an injection\(^1\) ordered by reverse inclusion. Also \( p : \kappa'_i \to \kappa_i \) is denoted as a partial function from \( \kappa'_i \) to \( \kappa_i \).

\( \mathcal{G} = \Pi_{0 < i < \rho} \mathcal{G}_i \) where for each \( 0 < i < \rho \), \( \mathcal{G}_i \) is the full permutation group of \( \kappa_i \) that can be extended to \( \mathcal{P} \), by permuting the range of its conditions, i.e., for all \( a \in \mathcal{G}_i \) and \( p \in \mathcal{P} \), \( a(p) = \{ (\psi, a(\beta)) : (\psi, \beta) \in p \} \).

For \( m < \omega \) and \( e = \{ \alpha_i : i \leq m \} \) is a sequence of ordinals such that for each \( 1 \leq i \leq m \), there is a distinct \( \epsilon_i \in (0, \rho) \) such that \( \alpha_i \in (\kappa'_\epsilon_i, \kappa_i) \). We define \( E_e = \{ (\emptyset, \ldots, p_1 \cap (\kappa'_\epsilon_1 \times \alpha_1), \emptyset, \ldots, p_2 \cap (\kappa'_\epsilon_2 \times \alpha_2), \emptyset, \ldots, p_\epsilon \cap (\kappa'_\epsilon_\epsilon \times \alpha_\epsilon), \emptyset, \ldots, p_\epsilon_m \cap (\kappa'_\epsilon_m \times \alpha_m), \emptyset, \ldots) : p \in \mathcal{P} \} \) and \( \mathcal{T} = \{ E_e : e \in \Pi_{0 < i < \rho} (\kappa'_\epsilon_i, \kappa_i) \} \), where \( \Pi_{0 < i < \rho} (\kappa'_\epsilon_i, \kappa_i) \) is the finite support product.

(3) **Defining symmetric extension of \( \mathcal{V} \):** Clearly, \( \mathcal{T} \) is a projectable symmetry generator with projections \( \mathcal{P} \upharpoonright E_e = \{ \emptyset, \ldots, p_1 \cap (\kappa'_\epsilon_1 \times \alpha_1), \emptyset, \ldots, p_\epsilon \cap (\kappa'_\epsilon_\epsilon \times \alpha_\epsilon), \emptyset, \ldots \} \) for \( E_e \in \mathcal{T} \) by \textbf{Lemma 7.3}. Let there be distinct \( \epsilon_i \) such that \( \alpha_i \in (\kappa'_\epsilon_i, \kappa_i) \) as in \textbf{Figure 1} and let \( I = \text{max} \{ \epsilon_i : \alpha_i \in e \} \) such that \( I \) is an integer. Next let \( M = \{ i : \epsilon_i \leq I \} \) and \( M' = \{ i : \epsilon_i > I \} \). Then \( V[G \cap E_e] \) is \( \Pi_{e \in M} \text{Fn}(\kappa'_\epsilon_\epsilon, \alpha_\epsilon, \kappa_i) \times \Pi_{\epsilon \in M'} \text{Fn}(\kappa'_\epsilon_\epsilon, \alpha_\epsilon, \kappa_i) \)-generic over \( V \). By closure properties of \( \Pi_{e \in M} \text{Fn}(\kappa'_\epsilon_\epsilon, \alpha_\epsilon, \kappa_i) \), all elements of the sequence \( \langle \kappa'_\epsilon_n : n < \omega \rangle \) remain cardinals after forcing with \( \Pi_{e \in M} \text{Fn}(\kappa'_\epsilon_\epsilon, \alpha_\epsilon, \kappa_i) \). Next, since \( M \) is finite we can find \( j < \omega \) such that for all \( r \geq j \), \( |\Pi_{e \in M} \text{Fn}(\kappa'_\epsilon_\epsilon, \alpha_\epsilon, \kappa_i)| < \kappa_r \). Thus, a final segment of the sequence \( \langle \kappa'_\epsilon_n : n < \omega \rangle \) remains a sequence of cardinals in \( V[G \cap E_e] \) which is a contradiction. \( \square \)

\( \square \)

8. **Infinitary Chang conjecture from a coherent sequence of Ramsey cardinals**

8.1. **Infinitary Chang Conjecture.** We define a set of good indiscernibles, Erdős-like partition property, infinitary Chang property and state the relevant lemmas. We recall the definitions and Lemmas from Chapter 3 of [Dim11]. For the sake of our convenience we denote a structure \( \mathfrak{A} \) on domain \( A \) as \( \mathfrak{A} = \langle A, \ldots \rangle \).

**Definition 8.1.** \textbf{(Set of good indiscernibles, Definition 3.2 of [Dim11].)} For a structure \( \mathfrak{A} = \langle A, \ldots \rangle \) with \( A \subseteq \text{Ord} \), a set \( I \subseteq A \) is a set of indiscernibles if for all \( n < \omega \), all \( n \)-ary formula \( \phi \) in the language for \( \mathfrak{A} \) and every \( \alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_n \) in \( I \), if \( \alpha_1 < \ldots < \alpha_n \) and \( \alpha'_1 < \ldots < \alpha'_n \) then
\[
\mathfrak{A} \models \phi(\alpha_1, \ldots, \alpha_n) \text{ if and only if } \mathfrak{A} \models \phi(\alpha'_1, \ldots, \alpha'_n).
\]

The set \( I \) is a set of good indiscernibles if and only if it is a set of indiscernibles and we allow parameters that lie below \( \min \{ \alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_n \} \) i.e., if for all \( x_1, \ldots, x_m \in A \) such that \( x_1, \ldots, x_m \leq \min \{ \alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_n \} \) and every \( (n + m) \)-ary formula, then
\[
\mathfrak{A} \models \phi(x_1, \ldots, x_m, \alpha_1, \ldots, \alpha_n) \text{ if and only if } \mathfrak{A} \models \phi(x_1, \ldots, x_m, \alpha'_1, \ldots, \alpha'_n).
\]
Definition 8.2. (α-Erdős cardinal and Erdős-like Partition Property, Definition 3.7 of \[\text{Dim11}\]). The partition relation $\alpha \rightarrow (\beta)^2_\gamma$ for ordinals $\alpha, \beta, \gamma, \delta$ means for all $f : [\alpha]^\gamma \rightarrow \delta$ there is a $X \in [\alpha]^\beta$ such that $X$ is homogeneous for $f$. For infinite ordinal $\alpha$, the $\alpha$-Erdős cardinal $\kappa(\alpha)$ is the least $\kappa$ such that $\kappa \rightarrow (\alpha)^2_\omega$. For cardinals $\kappa > \lambda$ and ordinal $\theta < \kappa$ we mean $\kappa \rightarrow^\theta (\lambda)^{<\omega}_2$ if for every first order structure $A = (\kappa, ...)$ with a countable language, there is a set $I \in [\kappa\setminus\theta]^\lambda$ of good indiscernibles for $A$.

Definition 8.3. (Infinitary Chang Conjecture, Definition 3.10 of \[\text{Dim11}\]). Infinitary Chang conjecture is the statement $(\kappa_n)_{n \in \omega} \rightarrow (\lambda_n)_{n \in \omega}$ which means for every structure $A = (\omega, ..., \kappa_n, ...)$ there is an elementary substructure $B < A$ with domain $B$ and cardinality $\omega \land \lambda_n$ such that for every $n \in \omega$, $|B \cap \kappa_n| = \lambda_n$.

Definition 8.4. (Definition 3.14 of \[\text{Dim11}\]). Let $\langle \kappa_i : i < \omega \rangle$ and $\langle \lambda_i : 0 < i < \omega \rangle$ be two increasing sequence of cardinals such that $\kappa = \bigcup_{i \in \omega} \kappa_i$. We say $\langle \kappa_i : i < \omega \rangle$ is a coherent sequence of cardinals with the property $\kappa_{i+1} \rightarrow (\lambda_{i+1})^{<\omega}_2$ if and only if for every structure $A = (\kappa, ...)$ with a countable language there is a $\langle \lambda_i : 0 < i < \omega \rangle$-coherent sequence of good indiscernibles for $A$ with respect to $\langle \kappa_i : i < \omega \rangle$.

We recall Corollary 3.15 of \[\text{Dim11}\] which can be carried out in ZF context, proposition 3.50 and Lemma 3.52 from Chapter 3 of \[\text{Dim11}\].

Lemma 8.5. (Corollary 3.15 of \[\text{Dim11}\]). (ZF) Let $\langle \kappa_i : i < \omega \rangle$ and $\langle \lambda_i : 0 < i < \omega \rangle$ be two increasing sequence of cardinals such that $\kappa = \bigcup_{i \in \omega} \kappa_i$. If $\langle \kappa_i : i < \omega \rangle$ is a coherent sequence of cardinals with the property $\kappa_{i+1} \rightarrow (\lambda_{i+1})^{<\omega}_2$ then the Chang Conjecture $(\kappa_n)_{n \in \omega} \rightarrow (\lambda_n)_{n \in \omega}$ holds.

Lemma 8.6. (Proposition 3.50 of \[\text{Dim11}\]). Let us assume that $V = ZFC + '\kappa = \kappa(\lambda)$ exists’. $P$ is a partial order such that $|P| < \kappa$ and $Q$ is a partial order that doesn’t add subsets to $\kappa$. If $G$ is $P \times Q$ generic then for every $\theta < \kappa$, $V[G] \models \kappa \rightarrow^\theta (\lambda)^{<\omega}_2$.

Lemma 8.7. (Lemma 3.52 of \[\text{Dim11}\]). Let $\langle \kappa_i : i < \omega \rangle$ and $\langle \lambda_i : 0 < i < \omega \rangle$ be two increasing sequence of cardinals such that $\langle \kappa_i : 0 < i < \omega \rangle$ is a coherent sequence of Erdős cardinals with respect to $\langle \lambda_i : 0 < i < \omega \rangle$. If $P_1$ is a partial order of cardinality $< \kappa_1$ and $G$ is $V$-generic over $P_1$, then in $V[G]$, $\langle \kappa_n : n < \omega \rangle$ is a coherent sequence of cardinals with the property $\kappa_{n+1} \rightarrow (\lambda_{n+1})^{<\omega}_2$.

8.2. Infinitary Chang Conjecture in a symmetric model. In this section, we observe an infinitary Chang conjecture using Erdős like partition property in a symmetric extension in terms of $\langle P, G, \mathcal{I} \rangle$ triple analogous to the symmetric inner model constructed in Theorem 11 of \[AK06\] where $\omega_1$ is singular and prove Theorem 1.7.

Proof. (Theorem 1.7).

1. Defining ground model (V). Let $\kappa$ be a measurable cardinal in a model $V'$ of ZFC. By Prikry forcing it is possible to make $\kappa$ singular with cofinality $\omega$ where an end segment $\langle \kappa_i : 1 \leq i < \omega \rangle$ of the Prikry sequence $\langle \delta_i : 1 \leq i < \omega \rangle$ is a coherent sequence of Ramsey cardinals by Theorem 3 of \[Apt06\]. Now Ramsey cardinals $\kappa_i$ are exactly the $\kappa_i$-Erdős cardinals. Thus we obtain a generic extension (say $V$) where $\langle \kappa_i : 1 \leq i < \omega \rangle$ is a coherent sequence of cardinals with supremum $\kappa$ such that for all $1 \leq i < \omega$, $\kappa_i = \kappa(\kappa_i)$. Let $\kappa_0 = \aleph_\omega$.

2. Defining symmetric triple $\langle P, G, \mathcal{I} \rangle$. 


Following the arguments of [AK06] and Fact 3.8 in Lemma 8.9.

Let singular, AC be a set with full support where \( P_0 = \text{Col}(\omega, < \kappa_0) \), \( P_1 = \text{Col}(\kappa_{i+1}, < \kappa_i) \) and for each \( 1 < i < \omega \), \( P_i = \text{Col}(\kappa_i^{<\kappa_{i-1}+1}, < \kappa_i). \)

Let \( G = \Pi_{i<\omega} G_i \) where for each \( i < \omega \), \( G_i \) is the full permutation group of \( \kappa_i \) that can be extended to \( P_i \) by permuting the range of its conditions, i.e., for all \( a \in G_i \) and \( p \in P_i \), \( a(p) = \{(\psi, \alpha)(\beta) : (\psi, \beta) \in p\} \).

\( I = \left\{ E_e : e \in (\omega, \kappa_0) * (\kappa_{i+1}, \kappa_i) * \Pi_{1<i<\omega}(\kappa_i^{<\kappa_{i-1}+1}, \kappa_i) \right\} \) where for every \( e = (\alpha_0, \alpha_1, ...) \in (\omega, \kappa_0) * (\kappa_{i+1}, \kappa_i) * \Pi_{1<i<\omega}(\kappa_i^{<\kappa_{i-1}+1}, \kappa_i) \) we define \( E_e = \{ (p_0 \cap (\omega \times \alpha_0), p_1 \cap (\kappa_{i+1} \times \alpha_1), p_2 \cap (\kappa_1^{<\kappa_i+1} \times \alpha_2), \ldots p_i \cap (\kappa_{i-1}^{<\kappa_i+1} \times \alpha_i), ...) : p \in P \} \).

(3) Defining symmetric extension of \( V \). Let \( G \) be a \( P \)-generic filter. We consider the symmetric model \( V(G)^G \). We denote \( V(G)^G \) by \( V(G) \) for the sake of convenience.

Since the forcing notions involved are weakly homogeneous, we can have the following lemma which tells that every set of ordinals in \( V(G) \) is added by an intermediate submodel where \( AC \) holds.

**Lemma 8.8.** If \( A \in V(G) \) is a set of ordinals, then \( A \in V[G \cap E_e] \) for some \( E_e \in I \).

Following the arguments of [AK06] and Fact 3.8 it is possible to see that in \( V(G) \), since \( \omega_1 \) is singular, \( AC \) fails. Further in \( V(G) \), \( \kappa \) is a Rowbottom cardinal carrying a Rowbottom filter.

We prove that an infinitary Chang conjecture holds in \( V(G) \).

**Lemma 8.9.** In \( V(G) \), an infinitary Chang conjecture holds.

**Proof.** Let \( \mathfrak{A} = \langle \kappa, \ldots \rangle \) be a structure in a countable language in \( V(G) \). Let \( \left\langle \phi_n : n < \omega \right\rangle \) be an enumeration of the formulas of the language of \( \mathfrak{A} \) such that each \( \phi_n \) has \( k(n) < \omega \) and \( n \) many free variables. Define \( f : [\kappa]^{<\omega} \rightarrow 2 \) by,

\[
f(\epsilon_1, \ldots, \epsilon_n) = 1 \text{ if and only if } \mathfrak{A} \models \phi_n(\epsilon_1, \ldots, \epsilon_k(n)) \text{ and } f(\epsilon_1, \ldots, \epsilon_n) = 0 \text{ otherwise.}
\]

By Lemma 8.8, there is a \( E_e \in I \) such that \( f \in V[G \cap E_e] \). Fix an arbitrary \( 1 < i < \omega \). Since we can write \( V[G \cap E_e] = V[G_1][G_2] \) where \( G_1 \) is \( P_1 \)-generic over \( V \) where \( |P_1| < \kappa_i \) and \( G_2 \) is \( P_2 \)-generic over \( V[G_1] \) where \( G_2 \) add no subsets of \( \kappa_i \), thus by Lemma 8.6, \( \kappa_i \rightarrow \kappa_i^{<\omega} \) in \( V[G \cap E_e] \).

So, for all \( 1 \leq i < \omega \), \( \kappa_i \rightarrow \kappa_i^{\omega} \) in \( V[G \cap E_e] \).

By Definition 8.4, \( \mathfrak{A} \) has a \( \langle \kappa_i : 0 < i < \omega \rangle \)-coherent sequence of good indiscernibles \( A \) with respect to \( \langle \kappa_i : i < \omega \rangle \) and \( A \in V[G \cap E_e] \subseteq V(G) \). Therefore for all \( 1 < i < \omega \), \( \kappa_i \rightarrow \kappa_i^{\omega} \) in \( V(G) \). Using Lemma 8.5, we can obtain an infinitary Chang conjecture in \( V(G) \) as Lemma 8.5 can be proved in ZF.

We apply Proposition 1 of [AP08c], which states that a limit of Ramsey cardinals is an almost Ramsey cardinal, to prove \( \kappa_\omega \) is an almost Ramsey cardinal in \( V(G) \).

**Lemma 8.10.** In \( V(G) \), \( \kappa_\omega \) is an almost Ramsey cardinal.

**Proof.** In \( V(G) \), \( \kappa = \kappa_\omega \). We show \( \kappa \) is an almost Ramsey cardinal in \( V(G) \). Let \( f : [\kappa]^{<\omega} \rightarrow 2 \) be in \( V(G) \). Since \( f \) can be coded by a subset of \( \kappa \), by Lemma 8.8 for some \( E_e \in I \), \( f \in V[G \cap E_e] \).

We can see that \( \langle \kappa_i : i < \omega \rangle \) stays a sequence of Ramsey cardinals in \( V[G \cap E_e] \). Hence in \( V[G \cap E_e] \), \( \kappa \) being the supremum of Ramsey cardinals is an almost Ramsey cardinal by Proposition 1 of [AP08c]. Thus for all \( \beta < \kappa \), there is a set \( X_\beta \in V[G \cap E_e] \subseteq V(G) \) which is homogeneous for \( f \) and has order type at least \( \beta \). Hence, \( \kappa \) is almost Ramsey in \( V(G) \) since \( f \) was arbitrary.
9. Fat Diamond Principle and Level by Level Equivalence

Now we start working in ZFC. In this section, we prove Theorem 1.12 and study the consistency of fat diamond principle and other L-like properties with level by level equivalence.

9.1. Forcing facts concerning Fat diamond principle. We recall the definition of the combinatorial principle fat diamond on κ, denoted by ♦κ from section 2 of [FV15].

Definition 9.1. (Fat Diamond principle on κ). S ⊆ κ is called fat stationary if S is stationary and for every club C ⊆ κ and every α < κ there is a continuous increasing sequence of order type α inside C ∩ S. A sequence ⟨aα : α < κ⟩ is a fat diamond sequence at κ (denoted by ♦κ-sequence) if for every X ⊆ κ, {α : X ∩ α = aα} is fat stationary. We say that the principle ♦κ holds if there is a ♦κ-sequence.

Theorem 9.2. (Theorem in Section 2 of [FV15]). (V = L). A ♦κ-sequence exists for any uncountable regular cardinal κ.

Brent Cody communicated to us that for a regular uncountable cardinal κ, Add(κ, 1) introduces a ♦κ-sequence. We enclose those arguments in the following lemma.

Lemma 9.3. (due to Brent Cody and Monroe Eskew). If κ is a regular uncountable cardinal, then forcing with Add(κ, 1) introduces a ♦κ-sequence.

Proof. Let g : κ → 2 be a generic Cohen function on κ. Let aα = {β < α : g(α + β) = 1} for each α < κ. Let X be a name for a subset of κ, Ĉ a name for a club, ε < κ an ordinal, and q0 a condition. Construct an increasing sequence of conditions ⟨pα : α < κ⟩ extending q0 and an increasing sequence of ordinals ⟨γα : α < κ⟩ satisfying the following properties.

- The domain of each condition is an ordinal.
- pα+1 decides X ∩ dom(pα).
- pα+1 ⊨ dom(pα) < γα+1 ∈ Ĉ.
- If α is a limit, then γα = supβ<αγβ.
- If α is a limit, then letting qa = ∪β<αpβ, we have dom(pα) = α.2, pα ⊨ α = qa, and pα(α + β) = 1 if and only if qa ⊨ β ∈ X.

Let {δβ : β < ε} be the first ε limit ordinals. Then pδβ ⊨ {γδβ : β < ε} ⊆ Ĉ ∧ (∀β < ε)X ∩ γδβ = ̂δδβ.

We prove that ♦κ can be preserved by κ-strategically closed forcing notions as well as forcing notions that are both κ-c.c. and have cardinality κ.

Lemma 9.4. ♦κ is preserved by κ-strategically closed forcing notions.

Proof. Let ⟨aα : α < κ⟩ be a ♦κ-sequence and P be a κ-strategically closed notion of forcing. Let p ∈ P be any condition, Ĉ a name for a club, and X a name for a subset of κ. Consider a game of length κ where player I and player II construct an increasing sequence of conditions where initially player II chooses a trivial condition and player I chooses a condition extending p deciding 0 ∈ X and 0 ∈ Ĉ. We construct an increasing sequence of ordinals ⟨γα : α < κ⟩ such that at non limit even stages 2α > 0, player II chooses a condition forcing γα ∈ Ĉ and deciding X ∩ γα and if α is limit then we let γα = supβ<αγβ. By κ-strategic closure of P, player II has a winning strategy.
and thus we may assume the existence of an increasing sequence of conditions extending \( p \) such that the following holds.

- \( p_{\alpha+1} \) decides \( \dot{X} \cap \gamma_\alpha \).
- \( p_{\alpha+1} \Vdash \gamma_\alpha \in \dot{C} \).
- \( \gamma_\alpha = \sup_{\beta < \alpha} \gamma_\beta \), if \( \alpha \) is a limit.

Let \( Y = \{ \alpha : \exists \beta (p_\beta \Vdash \alpha \in \dot{X}) \} \) and \( D = \{ \gamma_\alpha : \alpha < \kappa \} \). For any \( \epsilon < \kappa \), there is a continuous increasing sequence \( c \) of order type \( \epsilon \) in \( \{ \alpha \in D : Y \cap \alpha = a_\alpha \} \). Consequently, \( p_\kappa \Vdash c \subseteq \{ \alpha \in \dot{C} : \dot{X} \cap \alpha = a_\alpha \} \).

**Lemma 9.5.** Fat stationary subsets of \( \kappa \) are preserved by \( \kappa \)-c.c. forcing notions.

**Proof.** Follows from the fact that if \( P \) is \( \kappa \)-c.c. then every unbounded \( A \subseteq \kappa \) in \( V^P \) has an unbounded subset in \( V \).

**Lemma 9.6.** \( \clubsuit_\kappa \) is preserved by forcing notions which are \( \kappa \)-c.c. and have cardinality \( \kappa \).

**Proof.** *(Following the sketch of the proof of Fact 1.2 of [Apt08].)* Let \( \langle a_\alpha : \alpha < \kappa \rangle \) be a \( \clubsuit_\kappa \)-sequence and \( G \) be a \( V \)-generic filter over \( P \) where \( P \) is a \( \kappa \)-c.c. forcing notion having cardinality \( \kappa \). Let \( \dot{X} \) be a name for a subset of \( \kappa \). We may assume \( \dot{X} \) is hereditarily of cardinality at most \( \kappa \) since \( |P| = \kappa \) and \( P \) satisfies \( \kappa \)-c.c. Thus let \( X^* \subseteq \kappa \) such that \( X^* \in V \) code \( \dot{X} \). Since \( \langle a_\alpha : \alpha < \kappa \rangle \) is a \( \clubsuit_\kappa \)-sequence in \( V \), \( S = \{ \alpha < \kappa : X^* \cap \alpha = a_\alpha \} \) is fat stationary in \( V \). Since \( P \) is \( \kappa \)-c.c., \( S \) remains fat stationary in \( V^P \) by **Lemma 9.5.** Thus, \( \dot{X} \) is anticipated on a fat stationary set by \( \langle B_\alpha : \alpha < \kappa \rangle \) where \( B_\alpha = A_\alpha^G \) if \( A_\alpha \) codes a \( P \)-name and \( B_\alpha = \emptyset \) otherwise. Consequently, \( \clubsuit_\kappa \) is preserved in \( V^P \).

**9.2. Adding a non-reflecting stationary set.** By *reflection principle*, we mean the principle which establish the fact that if a structure satisfies a given property, then there is a substructure of smaller cardinality which satisfies the same property. A standard reflection principle is *reflection of stationary sets.*

**Definition 9.7.** *(Reflection of stationary sets).* \( S \subseteq \gamma \) reflects at \( \alpha \) if \( \alpha \) has uncountable cofinality and \( S \cap \alpha \) is stationary at \( \alpha \). A stationary set \( S \subseteq \gamma \) reflects if it reflects at some \( \alpha < \gamma \).

Given a Mahlo cardinal \( \gamma \), we recall the forcing notion that adds a non-reflecting stationary set of ordinals of a certain type subset to \( \gamma \) from [Apt08]. We require this notion of forcing to prove **Theorem 1.12.**

**Definition 9.8.** *(Adding a non-reflecting stationary set to \( \gamma \), [Apt08].)* Given a Mahlo cardinal \( \gamma \), we say \( NR(\gamma) \) is the forcing notion that adds a non-reflecting stationary set of ordinals of a certain type subset to \( \gamma \). Formally, \( NR(\gamma) = \{ p : \text{for some } \alpha < \gamma, \, p : \alpha \rightarrow \{0, 1\} \text{ is a characteristic function of } S_p, \text{ a subset of } \alpha \text{ not stationary at its supremum nor having any initial segment which is stationary at its supremum, so that if } \beta < \sup(S_p) \text{ is inaccessible, then } S_p - (S_p \cap \beta) \text{ is composed of ordinals of cofinality at least } \beta \} \), ordered by \( q \geq p \) if and only if \( q \supseteq p \) and \( S_p = S_q \cap \sup(S_p) \).

Forcing with \( NR(\gamma) \) adds a non-reflecting stationary set of ordinals to \( \gamma \) and \( NR(\gamma) \) is \( \gamma \)-strategically closed. Further, \( |NR(\gamma)| = \gamma \).

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25Fact 2.1 of [AJJ10].
9.3. Fat Diamond principle and Level by level equivalence. In this subsection we prove Theorem 1.12. We follow the terminologies of [Apt09] and recall the part of the construction till obtaining \( V^{\mathbb{P}_2} \) from [Apt09].

**Proof. (Theorem 1.12).** We start with the forcing extension \( V \) constructed in Theorem 2 of [Aj08], as done in [Apt09]. We use the fact that \( Add(\delta, 1) \) can force \( \&_{\delta} \) to see that \( \&_{\delta} \) is possible for every \( \delta \) which is either an inaccessible cardinal or the successor of a singular cardinal in \( V \). Next we construct \( V_1 = V^{\mathbb{P}_2} \), following the terminologies of [Apt09]. We can see that \( V_1 \) is a model of ZFC + GCH. Also in \( V_1 \), \( K \) remains the class of supercompact cardinals and level by level equivalence holds. Moreover in \( V_1 \), we have the properties (1), (2), (3) and (4) of Theorem 1.12.

Adding the rest of the \( L \)-like properties (Properties 5 and 6). We construct our final forcing extension \( V_2 \) over \( V_1 \) by the forcing notion used in Theorem 1 of [Apt08] as follows.

1. **Defining ground model** (\( V_1 \)). We consider \( V_1 \) as our ground model which is a model of ZFC + GCH where \( K \) remains the class of supercompact cardinals and level by level equivalence holds. Let \( \kappa \in K \) be the least supercompact cardinal and \( V_1 \) satisfies properties (1), (2), (3) and (4) of Theorem 1.12.

2. **Defining the forcing notion** (\( \mathbb{P}_1 \)). Let \( \mathbb{P}_1 = \langle (\mathbb{P}_{1\delta}, \dot{Q}_{1\delta}) : \delta \in \text{Ord} \rangle \) be the proper class Easton support iteration which begins by forcing with \( Add(\aleph_1, 1) \) and \( \dot{Q}_{1\delta} \) names the following forcing notions.
   - NR(\( \delta \)) if \( \delta \) is Mahlo and non-Ramsey.
   - trivial forcing otherwise.

3. **Defining forcing extension of \( V_1 \).** Let \( V_2 = V_1^{\mathbb{P}_1} \) be our forcing extension.

As mentioned in [Apt08], for \( Q \) any initial proper or improper segment of \( \mathbb{P}_1 \), \( V_1^{\mathbb{P}_1} \models \text{ZFC} + \text{GCH} \) and \( V_1 \) and \( V_1^{\mathbb{P}_1} \) has same cardinals and cofinalities. Since \( \mathbb{P}_1 \) can be written as \( \mathbb{P}_{1\kappa} \ast \dot{R} \), where \( \mathbb{P}_{1\kappa} \) has \( \kappa \)-c.c. and \( V_{1\delta} \models \text{\( \mathbb{P}_{1\delta} \) is \( \kappa \)-strategically closed} \), \( S \) remains a stationary subset of \( \kappa \) in \( V_2 \) by Lemma 2.3. Since existence of gap-1 morass at \( \delta \) and \( \square_{\delta} \) are upward absolute to cofinality preserving generic extension and \( \mathbb{P}_1 \) preserve cofinalities, \( V_2 \models \square_{\delta} \), ‘\( \delta \) carries a gap-1 morass’ for each \( \delta \in S \). Following Lemma 2.1 of [Apt08], if \( \kappa < \lambda \) are such that \( \kappa \) is \( \lambda \)-supercompact and \( \lambda \) is regular in \( V_1 \), then \( \kappa \) is \( \lambda \)-supercompact in \( V_2 \). Since \( \mathbb{P}_1 \) admits a closure point at \( \aleph_1 \), by arguments similar to Lemma 2.2 and Lemma 2.3 of [Apt08], the level by level equivalence holds in \( V_2 \) and \( K \) is the class of supercompact cardinals with \( \kappa \) as the least supercompact cardinal. By Lemmas 2.4, 2.5 and 2.6 of [Apt08], Mahlo cardinals reflecting stationary sets are weakly compact and every regular Jonsson cardinal is weakly compact in \( V_2 \). Since \( \mathbb{P}_1 \) admits a closure point at \( \aleph_1 \) and forcing with \( \mathbb{P}_1 \) preserves cofinalities, we have Safe(\( \delta \)) is upward absolute to \( V_2 \) for every infinite cardinal \( \delta \). Thus, \( \square_{T} \) holds for every infinite cardinal \( \delta \) where \( T = \text{Safe}(\delta) \).

**Lemma 9.9.** In \( V_2 \), for each Mahlo cardinal \( \delta \) as well as if \( \delta \) is \( \aleph_1 \), \( \&_{\delta} \) holds.

**Proof.** Firstly, \( Add(\aleph_1, 1) \) forces \( \&_{\aleph_1} \) in \( V_2 \). Any Mahlo cardinal \( \delta \) in \( V_2 \), had to have been Mahlo in \( V_1 \). Since \( \mathbb{P}_1 \) can be written as \( \mathbb{P}_{1\delta} \ast \dot{R} \) where \( |\mathbb{P}_{1\delta}| \leq \delta \) and \( R \) is \( \delta \)-strategically closed, \( \&_{\delta} \) is preserved in \( V_2 \) applying Lemma 9.4 and Lemma 9.6. \( \square \)
Since $V_2$ is a model of GCH, $\diamond_\delta$ holds for each successor cardinal $\delta > \aleph_1$ in $V_2$ by [She10] and $Add(\aleph_1, 1)$ forces $\diamond_{\aleph_1}$ in $V_2$. Again in $V_2$, for each Mahlo cardinal $\delta$, $\diamond_\delta$ holds by arguments similar to Lemma 9.9.

10. Results concerning Saturated Ideals

10.1. Saturated Ideals. Given a set $X$, an ideal $I$ on $X$ is a non-empty family of subsets of $X$ which is closed under taking subsets and under finite unions. We recall the following preliminaries.

- If $X \notin I$, then $I$ is proper.
- Given an uncountable cardinal $\gamma$, $I$ is $\gamma$-complete if and only if $I$ is closed under unions of length less than $\gamma$.
- $I$ is said to be normal if it is closed under diagonal unions. In terms of regressive functions, $I$ is said to be normal if and only if every regressive function on a set of positive $I$-measure is constant on a set of positive $I$-measure.
- Let $I^+$ denotes the class of $I$-positive sets where $A \subseteq X$ is $I$-positive if and only if $A \notin I$. Two sets $A$ and $B$ in $I^+$ are said to be $I$-almost disjoint if and only if $A \cap B \notin I$. $I$ is $\gamma$-saturated if and only if every pairwse $I$-almost disjoint collection $F \subseteq I^+$ is of cardinality less than $\gamma$. We note that $I$ is $\gamma$-saturated if and only if $\mathcal{P}(\kappa)/I$ has $\gamma$-c.c.

We follow the definition of a presaturated ideal and a $\mu$-minimal ideal from [BM18]. Let $\kappa$ be a regular cardinal, and $I$ be a $\kappa$-complete ideal on $\kappa$. We say that $I$ is presaturated if forcing with $\mathcal{P}(\kappa)/I$ preserves $\kappa^+$. We say $I$ is $\mu$-minimal if whenever $G \subseteq \mathcal{P}(\kappa)/I$ is generic and $X \in \mathcal{P}(\mu)^{V[G]} \setminus V$, then $V[X] = V[G]$. We recall a Theorem concerning saturated ideals due to Kunen from [Kun78].

Lemma 10.1. (Kenneth Kunen [Kun78]) Let $V$ be the ground model, $\mathbb{P}$ be a $\gamma$-c.c. forcing notion and $\dot{U}$ be a $\mathbb{P}$-name for a $V$-ultrafilter on $\kappa$. Let $I = \{X \in \mathcal{P}(\kappa) \cap V : \Vdash_{\mathbb{P}} \kappa \not\in X \in \dot{U}\}$. Then, $I$ is a $\gamma$-saturated ideal. Further, ‘if $\dot{U}$ is forced to be $V$-$\kappa$-complete then $I$ is $\kappa$-complete’ and ‘if $\dot{U}$ is forced to be $V$-normal then $I$ is normal’.

10.2. Stationary set of normal saturated ideals which reflect stationary sets. In this subsection, we prove Theorem 1.15 which is a variant of Theorem 1 from [Apt08a] using Lemma 10.1.

Proof. (Theorem 1.15). We follow Lemma 2.1 of [Apt08a] and modify it as follows.

Observation 10.2. (Modification of Lemma 2.1 of [Apt08a]). If $\kappa < \lambda$ are such that $\kappa$ is a regular cardinal and $\lambda$ is measurable in $V$, then there is a $\kappa$-strategically closed notion of forcing $\mathbb{Q}$ such that $V^\mathbb{Q} \models \& \lambda$ is a non-weakly compact Mahlo cardinal which reflects stationary sets and carries a normal $\lambda$-saturated ideal”.

Proof. (Observation 10.2). We consider $V^\mathbb{Q}$ to be our final forcing extension where $\mathbb{Q}$ is the forcing notion as defined in Lemma 2.1 of [Apt08a]. Since $\lambda$ is measurable in $V$, and hence weakly compact in $V$ as well, following the proof of Lemma 2.1 of [Apt08a] it suffices to show that $\lambda$ carries a normal $\lambda$-saturated ideal in $V^\mathbb{Q}$. In $V^\mathbb{Q}$, we factor $\mathbb{Q}_\lambda$ or $Add(\lambda, 1)$ as $\mathbb{R}_\lambda * \mathbb{T}_\lambda$ where $\mathbb{R}_\lambda$ is a $<\lambda$-strategically closed notion of forcing for adding a $\lambda$-Suslin tree $\mathcal{T}$ and $\mathbb{T}_\lambda$ is a $\lambda$-c.c notion of forcing for adding a generic path through $\mathcal{T}$ (as in the proof of Lemma 2.1 of [Apt08a]).
Since $\lambda$ is measurable in $V$, let $j : V \to M$ be the elementary embedding with respect to a normal $\lambda$-complete ultrafilter on $\lambda$. We can write $j(\mathcal{P}_\lambda) = \mathcal{P}_\lambda \ast Add(\lambda, 1) \ast S$ where $S$ is $(2^\lambda)^+\text{-closed}$. Let $G$ and $H$ be the generic filters over $\mathcal{P}_\lambda$ and $Add(\lambda, 1)$ respectively. We can define a master condition $m$ and construct a generic filter $K$ over $S$ and lift $j$ to $j' : V[G \ast H] \to M[G \ast H \ast K]$. The ultrafilter with respect to $j'$ will be the corresponding $\lambda$-complete normal ultrafilter over $\lambda$ in $V[G \ast H]$.\footnote{Which is $V^{\mathcal{P}_\lambda \ast \mathcal{R}_\lambda \ast \mathcal{T}_\lambda}$.} Let $U \in V^{\mathcal{P}_\lambda \ast \mathcal{R}_\lambda \ast \mathcal{T}_\lambda}$ be a normal measure over $\lambda$ and $\mathcal{U}$ be a $\mathcal{T}_\lambda$-name for $U$. By \textbf{Lemma 10.1}, in $V^\mathcal{U} = V^{\mathcal{P}_\lambda \ast \mathcal{R}_\lambda}$, $I = \{X \subseteq \lambda : \not\not_{\mathcal{T}_\lambda} X \not\in \mathcal{U}\}$ is a normal $\lambda$-saturated ideal on $\lambda$ since $\mathcal{T}_\lambda$ is $\lambda\text{-c.c.}$ \hfill \blacksquare

Now we prove \textbf{Theorem 1.15}. We suppose $\kappa < \lambda$ are such that $\kappa$ is a strong cardinal whose strongness is indestructible under $\kappa\text{-strategically closed forcing}$ and $\lambda$ is measurable. After forcing with the forcing notion $Q$, $\lambda$ has become a non-weakly compact Mahlo cardinal which reflects stationary sets and carry a normal $\lambda$-saturated ideal following \textbf{Observation 10.2}. Further, $\kappa$ remains a strong cardinal since $Q$ is $\kappa\text{-strategically closed}$. By reflection arguments, $A = \{\delta < \kappa : \delta$ is a non-weakly compact Mahlo cardinal which reflects stationary sets and carry a normal $\delta\text{-saturated ideal}\}$ is unbounded in $\kappa$. Since $Q$ is $\kappa\text{-strategically closed}$, $A$ must be unbounded in $\kappa$ in the ground model too. \hfill \blacksquare

\textbf{10.3. Normal saturated ideals and level by level equivalence}. Once more we work with the forcing notion which adds a $\gamma\text{-Suslin tree}$ for an uncountable cardinal $\gamma$. We prove \textbf{Theorem 1.14} and observe the consistency of a stationary set of saturated ideals along with level by level equivalence. Further, we can force combinatorial principles like fat diamond principle and gap-1 morass which extends Theorem 8 of [Apt08] by Arthur Apter.

\textbf{Proof. (Theorem 1.14).} We start with the forcing extension $V$ of Theorem 2 of [Apt09] as done in [Apt09]. As argued before we can construct $V_1 = V^{\mathcal{P}_1}$ as done in [Apt09] which satisfies properties (1), (2), (3) and (4) of \textbf{Theorem 1.14}. Moreover, $\mathcal{K}$ remains the class of supercompact cardinals and level by level equivalence holds. We construct our forcing extension $V_2$ considering $V_1$ as our ground model by the forcing notion of Theorem 2 of [Apt09] as follows.

(1) \textbf{Defining ground model ($V_1$).} Let $V_1$ be our ground model (from \textbf{section 9.3}) which is a model of ZFC + GCH where $\mathcal{K}$ remains the class of supercompact cardinals and level by level equivalence holds. Let $\kappa \in \mathcal{K}$ be the least supercompact cardinal. We can see that $V_1$ satisfies properties (1), (2), (3) and (4) of \textbf{Theorem 1.14}.

- Let $A = \{\delta < \kappa : \delta$ is a measurable limit of strong cardinals having trivial Mitchell rank}. Following the proof of Theorem 2 of [Apt08], $A$ is a stationary subset of $\kappa$.
- Let $B = \{\delta < \kappa : \delta$ is a strong cardinal which is not a limit of strong cardinals\}.

(2) \textbf{Defining the forcing notion ($\mathcal{P}_1$)}. Let $\mathcal{P}_1 = (\mathcal{P}_{1, i}, \bar{Q}_{1, i})$ be the reverse Easton iteration of length $\kappa$ which begins by forcing with $Add(\mathcal{N}_{1, 1})$ and $\bar{Q}_{1,i}$ names the following.

- trivial forcing unless $\delta \in A \cup B$.
- $Add(\delta, 1)$ if $\delta \in B$.
- forcing which adds a $\delta\text{-Suslin tree}$ if $\delta \in A$.

(3) \textbf{Defining forcing extension of $V_1$}. Let $V_2 = V_1^{\mathcal{P}_1}$ be our forcing extension. Since $\mathcal{P}_1$ is $\kappa\text{-c.c.}$, $A$ remains a stationary subset of $\kappa$ after forcing with $\mathcal{P}_1$. Following \textbf{Lemma 2.7} of [Apt08] and the paragraph after that, level by level equivalence holds in $V_2$ and $\mathcal{K}$ is the class...
of supercompact cardinals with $\kappa$ as the least supercompact cardinal. As in [Apt08], GCH holds in $V_2$. Since, existence of gap-1 morass at $\delta$ and $\square_{\delta}$ are upward absolute to cofinality preserving generic extension and $P_1$ preserve cofinalities, $V_2 \models \square_{\delta}$, '$$\delta$$ carries a gap-1 morass' for each $\delta \in S$. Since, $P_1$ admits a closure point at $N_1$ and forcing with $P_1$ preserves cofinalities, we have $Safe(\delta)$ is upward absolute to $V_2$ for every infinite cardinal $\delta$. Thus, $\square^*_\delta$ holds for every infinite cardinal $\delta$ where $T = Safe(\delta)$.

**Lemma 10.3.** In $V_2$, for each Mahlo cardinal $\delta$ as well as if $\delta$ is $N_1$, $\Diamond_\delta$ holds.

**Proof.** Firstly, $Add(N_1, 1)$ forces $\Diamond_{N_1}$ in $V_2$. Any Mahlo cardinal $\delta$ in $V_2$, had to have been Mahlo in $V_1$. Since $P_1$ can be written as $P_{14} \ast R$, $\square_{\delta}$ holds in $V_2$. Since $P_1$ has $\delta$-c.c. and $R$ is $\delta$-strategically closed, $\Diamond_\delta$ is preserved in $V_2$ applying Lemma 9.4 and Lemma 9.6.

Since $V_2$ is a model of GCH, $\Diamond_\delta$ holds for each successor cardinal $\delta > N_1$ in $V_2$ by [She10] and $Add(N_1, 1)$ forces $\Diamond_{N_1}$ in $V_2$. Again in $V_2$, for each Mahlo cardinal $\delta$, $\Diamond_\delta$ holds by arguments similar to Lemma 10.3. Thus properties (1), (2), (3) and (4) are preserved in $V_2$. We show that the property (5) holds in $V_2$ as well.

**Lemma 10.4.** In $V_2$, for each $\delta \in A$, $\delta$ is a non-weakly compact Mahlo cardinal which reflect stationary sets and carry a normal $\delta$-saturated ideal but $\delta^+$ do not contain a $\delta$-minimal pre-saturated ideal.

**Proof.** Following Theorem 2 of [Apt08] for each $\delta \in A$, $\delta$ is a non-weakly compact Mahlo cardinal which reflect stationary sets. We prove that each $\delta \in A$ carries a normal $\delta$-saturated ideal. Since $P_1 = P_{14} \ast R$, the strategic closure of $\bar{P}_{\delta}^{1+1}$ states that for each $\delta \in A$, it is enough to show that $V_1^{P_{14}+1} \models \\$\delta$ carries a normal $\delta$-saturated ideal\$'. By a standard lifting argument as in the proof of Observation 10.2, since each $\delta \in A$ is measurable in $V_1$, $\delta$ stays measurable in $V_1^{P_{14} \ast Add(\delta, 1)} = V_1^{P_{14} \ast Q_{14} \ast T_{14}} = V_1^{P_{14}+1 \ast T_{14}}$. Let $U$ be a normal measure over $\delta$ in $V_1^{P_{14} \ast T_{14}}$ and $U$ be a $T_{14}$-name for $\bar{U}$. In $V_1^{P_{14}+1}$, $I = \{X : \|T_{14}\delta \setminus X \in U\}$ is a normal $\delta$-saturated ideal by Lemma 10.1 since $T_{14}$ is $\delta$-c.c. Since GCH holds in $V_2$ and each $\delta \in A$ is regular, following the arguments from Observations 14 of [BM18] we obtain that $\delta^+$ do not contain a $\delta$-minimal pre-saturated ideal for each $\delta \in A$.

11. Questions

**Question 11.1.** Can $\aleph_{\omega+1}$, $\aleph_{\omega_1+1}$ carry any number of normal measures in $ZF$?

In [Apt06] and [Apt10], Arthur Apter proved that $\aleph_{\omega+1}$ can carry $\geq \aleph_{\omega+2}$ number of normal measures and $\aleph_{\omega_1+1}$ can carry $\geq \aleph_{\omega_1+2}$ number of normal measures respectively. If it is consistent that $\kappa$ is supercompact and $\lambda > \kappa$ carry arbitrary number of normal measures then we can prove the consistency of successor of singular cardinals like $\aleph_{\omega+1}$ and $\aleph_{\omega_1+1}$ being measurable cardinals with arbitrary normal measures by methods of [Apt06] and [Apt10]. We could prove successor of regular cardinals like $\aleph_1$, $\aleph_2$, $\aleph_{\omega+2}$, as well as $\aleph_{\omega_1+2}$, can carry an arbitrary number of normal measures.
Question 11.2. Can we obtain a model of ZFC with the given requirements of Theorem 1 in [ADK16] where for an arbitrary ordinal $\rho$, $\kappa_\rho$ can carry any number of normal measures?

If the answer is affirmative then in Gitik’s symmetric model as done in [ADK16], the first measurable cardinal $\kappa_{\rho+1}$ would be the first uncountable regular cardinal with an arbitrary number of normal measures applying Lemmas 2.4 and 2.5 of [Apt01].

Question 11.3. Can we extend Corollary 6.5 as follows?

**Theorem 11.4.** Let $V$ be a model of ZFC $+$ GCH with a measurable cardinal $\kappa$ and let $\lambda$ be a cardinal at most $\kappa^{++}$. Then in a symmetric extension, $\kappa$ is a measurable cardinal carrying $\lambda$ many normal measures $\{U_\alpha^* : \alpha < \lambda\}$. Moreover for each $\alpha < \lambda$, the set $\{\delta : 2^{\delta} = \delta^{++}$ and $\delta$ is Ramsey$\} \in U_\alpha^*$ and $\text{DC}_{<\kappa}$ holds.

If $\mathcal{U}$ is a normal measure over $\kappa$ then $\{\alpha < \kappa : \alpha$ is Ramsey$\} \in \mathcal{U}$ by Exercise 7.19 of [Kan09]. Since $\mathcal{U}$ is a $\kappa$-complete and nonprincipal ultrafilter over $\kappa$, $\mathcal{U}$ is uniform. Consequently the cardinality of $\{\alpha < \kappa : \alpha$ is Ramsey$\}$ is $\kappa$. Let $\langle \delta_\alpha : \alpha < \kappa \rangle$ be an enumeration of Ramsey cardinals below $\kappa$. We can prove Theorem 11.4 if the Ramseyness of each $\delta_\alpha$ is preserved in the symmetric extension along with the fact that $2^{\delta_\alpha} = \delta_\alpha^{++}$. We ask if there is any possible way to figure out this.

Question 11.5. (Concluding Remarks, [Apt05]). Are there other combinatorial principles possible in a model where level by level equivalence between strong compactness and supercompactness holds?

**Question 11.6. (asked by Arthur Apter in [Apt09]).** Which additional $L$-like properties are consistent with level by level equivalence between strong compactness and supercompactness?

**Question 11.7. (asked by Arthur Apter in [Apt08]).** Are the non-weakly compact Mahlo cardinals admitting stationary reflection in Theorem 1.14 or Theorem 8 of [Apt08] also Jonsson cardinals?

**Question 11.8. (asked by Arthur Apter in [Apt08]).** Is it possible to extend Theorem 1.14 or Theorem 8 of [Apt08] so that there exist non-weakly compact Mahlo cardinals which reflects stationary sets above some supercompact cardinal(s)?

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