The standard formulation of quantum theory relies on a fixed space-time metric determining the localisation and causal order of events. In general relativity, the metric is influenced by matter, and it is expected to become indefinite when matter behaves quantum mechanically. Here we explore the problem of operationally defining events and their localisation in the presence of gravitating quantum systems. We develop a framework for “time reference frames,” in which events are defined in terms of quantum operations with respect to a quantum clock. We find that, when clocks and quantum systems interact gravitationally, the temporal localisability of events becomes relative, depending on the time reference frame. We argue that the impossibility to find a reference frame in which all events are localised is a signature of an indefinite metric, which might yield an indefinite causal order of events. Even if the metric is indefinite, for any event we can find a time reference frame with respect to which the event is localised in time, while other events may remain delocalised. In this frame, time evolution takes its standard (Schrödinger) form, including the unitary dilation of the quantum operation defining the event. In addition, this form is preserved when moving from the frame localising one event to the frame localising another one, thereby implementing a form of covariance with respect to quantum reference frame transformations.

I. INTRODUCTION

Quantum theory allows us to predict the probabilities of obtaining certain outcomes when we perform operations on a physical system. These operations are specified by laboratory procedures, including preparations, transformations and measurements. Apart from these operational elements, quantum theory relies on a definite space-time metric to make empirical predictions. Indeed, the metric is often implicit in the definition of an operation, and it determines the causal structure of the space-time where the operations are performed.

According to general relativity, the space-time metric is obtained by solving Einstein’s equations. Generically, a solution to these equations depends on the matter distribution, which is assumed to be classical. Understanding the consequences of replacing gravitating classical matter by gravitating quantum systems in general relativity is at the heart of the problem of quantum gravity [1, 2]. At the moment, a fully satisfactory and broadly accepted theory of quantum gravity is lacking, and it is far from clear how, if at all, the essential notions of quantum mechanics and general relativity will be modified in the more fundamental theory [3, 4].

In the absence of such a theory, making quantum mechanical predictions when quantum systems act as gravitational sources is an important challenge. This is because gravitating quantum systems can lead to an indefinite spacetime metric, giving rise to well known difficulties [5–11]. These difficulties include the following:

i) The dynamical law of quantum theory (the Schrödinger equation) relies on a time parameter to specify the evolution of quantum systems. What plays the role of such time parameter in the absence of a definite metric?

ii) If the space-time metric is indefinite, the causal order between different operations is also indefinite. In this case, how are we supposed to apply the quantum mechanical rules for calculating probabilities?

iii) The physical realisation of an operation on a quantum system typically relies on “background” degrees of freedom, which are crucial for defining the operation. For example, the pointer position of a clock can define the time when the operation is applied [12]. If the metric field is indefinite, the clock does not “know” how fast to tick, due to an uncertainty in its time dilation [13]. (More precisely, the clock gets entangled with the gravitating degrees of freedom.) In this situation, how are we supposed to define the operation in the first place?

In this work, we propose a method for tackling some aspects of the above difficulties. We develop a framework for “time reference frames,” which are (quantum) reference frames associated to quantum clocks. The following paragraphs summarise our findings in the context of the difficulties mentioned above:

i) We construct a “timeless” approach [14–18] for multiple quantum clocks, and consider cases in which the space-time metric is indefinite due to gravitating quantum systems in a superposition of energy or position eigenstates. In our approach, a notion of time evolution arises with respect to each of these clocks, even in the presence of gravitating quantum systems. Moreover, the Schrödinger equation in one time reference frame is covariant (form invariant)
under the transformation into a different time reference frame. This fact constitutes a concrete implementation of the covariance of physical laws in quantum reference frames advocated for in Ref. [19].

ii) We introduce a local, operational definition of an “event,” in which an operation is applied to a quantum system conditioned on a clock reading a specific “time” [15, 18]. We study the temporal localisation of events with respect to different time reference frames, and find that, when quantum clocks interact with gravitating quantum systems, the temporal localisability of an event becomes relative, depending on the time reference frame. This result provides a concrete physical realisation, in terms of quantum systems which are sources of the gravitational field, of the claim reported in Refs. [20, 21], that the localisability of events is observer-dependent, and that operations might be performed in time-delocalised subsystems. We argue that this relativity characterises an indefinite metric, which can lead to an indefinite causal structure of events. We illustrate this fact by using our framework to analyse the gravitational quantum switch [12].

iii) Naively, one might expect that the time evolution with respect to a gravitationally interacting clock $A$ is “noisy” or decoherent. After all, $A$ is expected to get entangled due to its interaction, and its quantum state is expected to become mixed. We show, however, that this view is (quantum) reference frame dependent, and find a transformation to the time reference frame of $A$ where time evolution is unitary. This means, in particular, that if we define an “event” by applying an operation to a quantum system when clock $A$ shows a specific “time,” this operation will be represented, in $A$’s time reference frame, by its unitary dilation, like in ordinary quantum mechanics with a fixed metric. Therefore, by “jumping” into a suitable time reference frame, quantum operations take their usual form, even if the metric is indefinite. This result resonates with the concept of the “quantum equivalence principle” advocated by Hardy [9, 10], and can be seen as another example of covariance of physical laws in quantum reference frames [19], in a case where the space-time metric is indefinite.

Sections III, IV and V of this work end each with a subsection containing remarks. These remarks point out links with other works, limitations to our approach, or follow lines of thought which are somehow separated from the main argument of the paper. We include them separately so that the reader is not distracted from the main line of argument. In order for the reader to easily follow the discussion without losing track of the terminology, we have marked out important terms in italics in the part of the paper where we define the term.

II. REVIEW OF THE TIMELESS APPROACH TO QUANTUM MECHANICS

Let us start by reviewing the basic formalism underlying our work. In this Section we focus mainly on the mathematical aspects of the framework, leaving its physical interpretation for the next section, where we introduce the idea of time reference frames. The “timeless” approach, by which we mean formulations of quantum mechanics such as the Page and Wootters mechanism [14, 15], and covariant quantum mechanics [17, 18], has as a basic starting point the idea that time evolution “emerges” from relational degrees of freedom of quantum systems. Although classical dynamics admits a timeless formulation as well [16, 22], here we will focus on the quantum description, which is the one relevant to our work.

In the timeless formulation, one typically considers a total system composed of a clock, $A$, and the system of interest (the rest), $R$. The joint quantum system formed by $A$ and $R$ has a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_R$, where $\mathcal{H}_A$ and $\mathcal{H}_R$, denote the Hilbert spaces of $A$ and $R$, respectively. The time evolution of $R$ with respect to $A$ is encoded in the “history state,” $|\Psi\rangle$, which contains all the information about the correlations between $A$ and $R$, thus defining the dynamics of $R$ with respect to the clock $A$. In the timeless approach, $|\Psi\rangle$ is subjected to a constraint, $\hat{C}$, acting on $\mathcal{H}$:

$$\hat{C} |\Psi\rangle = 0.$$  \hspace{1cm} (1)

The space of all states satisfying Eq. (1) is called the physical Hilbert space. Strictly speaking, it is not in general a subspace of the kinematical Hilbert space $\mathcal{H}$, because the inner product of the two spaces may differ. A convenient way for obtaining the history state, which we will use later on, is by “projecting” onto the physical Hilbert space, (see, e.g., [23])

$$|\Psi\rangle = \int d\alpha \, e^{-i\alpha \hat{C}} |\varphi\rangle.$$  \hspace{1cm} (2)

Eq. (2) gives a solution to Eq. (1) for any given $|\varphi\rangle \in \mathcal{H}$. $\hat{P} := \int d\alpha \, e^{-i\alpha \hat{C}}$ is not a projector in the mathematical sense, hence the quotation marks. In this paper, all integrals without specified limits are to be understood as integrals over $\mathbb{R}$.

In order to explain how time evolution emerges from Eq. (1), we analyse the simplest case, in which reading time $t_A$ on clock $A$ corresponds to measuring the eigenvalue $t_A$ of an observable $\hat{T}_A$, acting on $\mathcal{H}_A$ in the same way the...
The position operator acts in ordinary quantum mechanics. The joint system $A + R$ obeys the constraint

\[ (\hat{H}_A + \hat{H}_R) |\Psi\rangle = 0, \]

where the Hamiltonian of the clock, $\hat{H}_A$, is canonically conjugate to $\hat{T}_A$, $[\hat{T}_A, \hat{H}_A] = i$, and $\hat{H}_R$ acts on $\mathcal{H}_R$. For simplicity of notation, we have left the tensor products implicit in Eq. (3) and written, for example, $\hat{H}_A \otimes 1_{\mathcal{H}_R}$ simply as $\hat{H}_A$. We will follow this notation throughout the paper.

The ket $|\psi(t)\rangle := \langle t |_A \cdot |\Psi\rangle$ is called the reduced state, and has the physical interpretation of “the state of system $R$ when clock $A$ shows time $t$.” It is easy to see that, by acting on Eq. (3) with $|t\rangle_A\cdot$, the Schrödinger equation for $|\psi(t)\rangle$ with respect to time $t$ follows $[15, 24]$. (Alternatively, we can find $|\Psi\rangle$ directly by means of Eq. (2) and show that the reduced state undergoes a unitary time evolution.) It is in this sense that time evolution is recovered from the “timeless” condition given by Eq. (1). Eq. (3) describes a situation in which time evolution is specified by a “perfect clock.” By definition, a perfect clock is a system satisfying the following: i) It is associated with an operator $\hat{T}_A$, acting on an infinite dimensional Hilbert space $\mathcal{H}_A$. ii) Its Hamiltonian, $\hat{H}_A$, is canonically conjugate to $\hat{T}_A$. Although not realistic (its Hamiltonian is unbounded from below), a perfect clock is a convenient idealisation for the purposes of this paper. It can be considered (although it need not be) as an approximation of an $n$-level finite-dimensional system in the limit where $n$ is very large $[25]$.

Describing measurements performed at multiple times in the timeless framework is a subtle issue. Naively applying the projection postulate of quantum mechanics to $|\Psi\rangle$ would, in general, produce a post-measurement state that violates Eq. (1). However, as noted in $[15, 18]$, a consistent way of describing multiple measurements in a timeless framework is to purify them. This means dividing the subsystem $R$ into the system of interest, $S$, and a set of ancillary systems $\{a_i\}$. The ancilla $a_i$ acts as measurement device that records the information about the $i$th measurement performed on $S$. The purification consists in modelling the measurement on $S$ by explicitly including an interaction Hamiltonian between $S$ and the ancillary systems $a_i$ in the constraint $C$. After the interaction, the result of the measurement is revealed by independently measuring in the state of the ancillas. For example, a simple model of a multiple-time measurement, happening at times $t_1$ and $t_2$, with $t_1 < t_2$, corresponds to the constraint equation

\[ (\hat{H}_A + \hat{H}_S + \delta(\hat{T}_A - t_1)\hat{K}^{(1)} + \delta(\hat{T}_A - t_2)\hat{K}^{(2)}) |\Psi\rangle = 0. \]

Here, $\hat{H}_S$ is the free Hamiltonian of the system of interest, $S$, $\delta$ denotes the Dirac delta distribution, and $K^{(i)}$, for $i = 1, 2$, is an operator acting on $\mathcal{H}_S \otimes \mathcal{H}_{a_i}$, where $\mathcal{H}_{a_i}$ is the Hilbert space associated to the $i$th ancilla. The history state describes measurement interactions between the system $S$ and the ancillae $a_i$. These interactions are sharply localised at times $t_1$ and $t_2$. At times where the interactions are turned off, $S$ evolves freely under the Hamiltonian $\mathcal{H}_S$ $[15]$.

The probabilities for the results of measurements are given by projecting the history state only once. For example, the probability for obtaining a result $a_1$, corresponding to the measurement performed at $t_1$, and a result $a_2$, corresponding to the measurement performed at $t_2$, given that clock $A$ shows a time $t > t_2$, is $[15]$

\[ p(a_1, a_2| t > t_2) = |\langle t |_A \otimes \langle a_1 | \otimes \langle a_2 | \cdot |\Psi\rangle |^2. \]

By assumption, the state $|\psi(t)\rangle := \langle t |\Psi\rangle$ is normalised to one. Note that there is nothing special about the choice of time $t > t_2$. If one wishes, one can also compute the probabilities for the outcome of the first measurement only, by projecting at a time $t_1 < t < t_2$. Moreover, if one is interested in measurements at times $t_1'$ and $t_2'$, different from $t_1$ and $t_2$, one can simply replace $t_1$ and $t_2$ by $t_1'$ and $t_2'$ in Eq. (4), and compute the desired probabilities by means of Eq. (5).

### III. REFERENCE FRAMES FOR TIME EVOLUTION

It is a well known fact that time is not absolute in special and general relativity. In these theories, time is specified by the reading of a clock moving along a specific world-line. For this reason, there is not a single time but rather there are multiple times—one for each clock. Motivated by this fact, in this Section we extend the timeless approach of the previous Section to include multiple clocks into the formalism. (Note that, recently, clocks at the interplay between quantum mechanics and general relativity have gained significant attention $[26–30]$.)

Notwithstanding our motivation, our approach departs from the usual case of special and general relativity, because we do not assume that the space-time metric is fixed. Because a space-time is defined by a manifold and a metric, our clocks and systems are not, strictly speaking, embedded in space-time. Nevertheless, we will insist on the operational definition of events with respect to quantum clocks, associating to each clock a reference frame by which we will define “evolution” and “temporal localisation.” We will call each of these frames a time reference frame. In Subsection
III A, we explain the operational interpretation of our framework. Importantly, we assume that events take place, in principle, on an indefinite causal structure. In an indefinite causal structure, whether an event is in the causal past, causal future or is causally disconnected from another event is not a factual property of the world. In Subsection III B, we study the time evolution and the time localisation of events with respect to two different (non-interacting) clocks.

A. Operational meaning of the framework

Consider the experimental setup shown in Fig. 1. Two observers, $A$ and $B$, perform experiments on a quantum system $S$. $A$ and $B$ have each a quantum clock at their disposal, and we will use the same notation to refer to the clock and the observer in cases where no confusion arises. $A$ and $B$ do not assume that the experiments take place in a specific, well-defined causal structure. This could be because of practical reasons—they might have only a probabilistic knowledge about the space-time geometry where the experiments take place—or because of fundamental reasons—they might do experiments involving gravitating quantum systems, which can lead to situations with indefinite causal structure [12]. However, $A$ and $B$ know that they can make quantum mechanical predictions by using the timeless approach to quantum mechanics, and are able to verify which is the constraint equation and the history state corresponding to their experiments.

The experiment has two stages, “preparation” and “detection.” In the preparation stage, $A$ and $B$ prepare the states of their clocks, ancillas and systems. They can, for example, choose to set their clocks at $t = 0$, and set $S$ in a particular state. $S$ can be a composite system with subsystems accessible to $A$ and $B$, and the observers can prepare $S$ in an entangled state in the partition defined by these subsystems. In the detection stage of the experiment, $A$ and $B$ analyse the measurement results by looking at the outcome of their clocks and ancillas. For example, they might be interested in checking whether the ancillas are in a certain state at a given time, as defined by one of the clocks. Alternatively, they might project one of the clocks, say $A$, in a specific basis, which need not be the time basis $\{\ket{t_A}_A\}$. By assumption, the choices made by $A$ and $B$ in the preparation stage do not depend on the results obtained in the detection stage, even if the experiments take place in an indefinite causal structure.

The clocks and ancillas allow $A$ and $B$ to produce events localised in time, as defined by the reading of their own clocks. For the purposes of this work, an event consists in any quantum operation performed on a quantum system, conditioned on one of the clocks being in a state which is sharply localised around a specific time. For example, $A$ can perform an experiment such that, when her clock shows time $t_A^*$, an entangling unitary $\hat{U}$ between $S$ and $a$
occurs, recording information of $S$ in $a$. If the outcome of $a$ reveals information about $S$ in the detection stage of the experiment, the event corresponds to a measurement of $S$ by means of $a$. An event is not necessarily a measurement. For example, the application of a unitary operation on $S$ alone is also an event. However, in order for this event to be defined operationally, the information that such unitary took place has to be recorded in a physical system (a counter) to which $A$ and $B$ have access in the detection stage. For example, the counter can be a two-level system which shows 1 if the unitary has been applied and shows 0 otherwise. Our definition of “event” is adapted to the features of general relativity, where diffeomorphism invariance means that points in space-time have no physical meaning on their own, and events have to be defined via coincidences of physical fields [8, 31]. In this work we insist on such an operational definition of events even in the case where no space-time background is assumed.

After repeating the experiment many times, $A$ and $B$ will be able to track the time, relative to their respective clocks, at which the events occur. For example, $B$ can find that the aforementioned unitary $\hat{U}$ takes place sharply when his clock shows the time $t_B^*$. In the case where the event corresponds to a measurement, $B$ can identify such event in terms of a statement like “at time $t_B^*$ a measurement was performed on $S$ yielding the result ‘up’.” By keeping track of the different events in the experiment, and comparing the times at measurement, $B$ when his clock shows the time $t_B^*$, he can define an event by means of an interaction between the $A$ and $B$ clocks, at which the events occur. For example, $B$ can identify such event in terms of a statement like “at time $t_B^*$ a measurement was performed on $S$ yielding the result ‘up’.” By keeping track of the different events in the experiment, and comparing the times at which these events occur with respect to their own clocks, $A$ and $B$ can assign a time coordinate to each event. In this way, they can define a mapping of the set of events into “space-time,” as shown in Fig. 1. In the case depicted in Fig. 1, the two events are sharply localised in both $A$ and $B$’s time reference frames. This feature is consistent with a fixed space-time background, where the time localisation of events is absolute. As we will see in Sections IV and V, there are situations involving gravitating quantum systems which lead to an indefinite metric background. In such backgrounds, whether an event is sharply localisable in time or not depends on the time reference frame.

Although in this paper we focus on the time localisation of events, it is natural to assume that $A$ and $B$ have access to an additional set of spatial degrees of freedom, which allows them to operationally localise events in space as well. In the present setup, the inclusion of such (quantum) degrees of freedom can be carried out by the methods developed in Ref. [19]. In the remaining of this work, whenever we mention the notion of space or distance, we will do so with this context in mind.

B. Non interacting clocks

Let us now illustrate the above ideas by means of a simple example. Imagine that $A$ and $B$ have two clocks, which interact neither with each other nor with anything else in the experiment. Each clock is associated with an operator $\hat{T}_i$, for $i = A, B$. We assume that $A$ and $B$ are perfect clocks, in the sense of the discussion below Eq. (3). Suppose that $A$ “sets up” an event by means of an interaction between the $S$ and her ancilla, $a$. The event is produced when $A$’s clock is in a sharp state corresponding to $t_A^* > 0$. Suppose that a similar setting holds for $B$, with a corresponding time $t_B^* > 0$ and an ancilla $b$. Under these conditions, the constraint equation that describes the experiment is

$$ (\hat{H}_A + \hat{H}_B + \hat{f}_A(\hat{T}_A) + \hat{f}_B(\hat{T}_B))|\Psi\rangle = 0. \quad (6) $$

Here, $\hat{f}_A(\hat{T}_A) = \delta(\hat{T}_A - t_A^*)\hat{K}^{(A)}$, where $\hat{K}^{(A)}$ is a hermitian operator on $\mathcal{H}_S \otimes \mathcal{H}_a$. A similar statement holds for $\hat{f}_B(\hat{T}_B) = \delta(\hat{T}_B - t_B^*)\hat{K}^{(B)}$ and the ancilla $b$. For simplicity, we have assumed that the free Hamiltonian for the system is trivial, $\hat{H}_S = 0$. In this paper we focus on the case where the events are triggered when the state of the clocks are sharply peaked around a given time. However, more general models are possible by suitably choosing the hermitian-operator-valued functions $\hat{f}_I(\hat{T}_I)$, for $I = A, B$.

In order to find the state $|\Psi\rangle$, we plug Eq. (6) into Eq. (2). After some manipulations, we get (see Appendix A)

$$ |\Psi\rangle = \int d\alpha e^{-i\alpha(\hat{H}_A + \hat{H}_B)}Te^{-i\int_0^\alpha ds(\hat{f}_A(s+\hat{T}_A) + \hat{f}_B(s+\hat{T}_B))}|\varphi\rangle. \quad (7) $$

Here, $T$ denotes the usual time ordering operator, defined by $T\hat{f}(s_1)\hat{f}(s_2) = \Theta(s_2 - s_1)\hat{f}(s_2)\hat{f}(s_1) + \Theta(s_1 - s_2)\hat{f}(s_1)\hat{f}(s_2)$, for any operator-valued function $\hat{f}$ of $s$. $\Theta(s)$ is the usual Heaviside function, equal to 1 if $s > 0$, 1/2 if $s = 0$, and 0 otherwise.

The history state in the form of Eq. (7) is neither in the time reference frame of $A$ nor in that of $B$. Rather, it is written in a “perspective neutral” form [32], which we have introduced here in view of the calculations done in Sections IV and V. The physical interpretation of Eq. (7) becomes more transparent once we “jump” into the time reference frame of one of the parties, say $A$. In order to do so, we first expand the state $|\varphi\rangle$ in Eq. (7) in the basis $|t'_A,t'_B\rangle$. Then, after acting with the operators on the clocks, we define the coordinate $t_A = \alpha + t'_A$ and make a change of coordinates to eliminate $\alpha$ in favour of $t_A$. After some manipulations, (see Appendix C) the history state in the
time reference frame of $A$ reads

$$|\Psi\rangle = \int dt_A |t_A\rangle_A \otimes e^{-it_A H_B} T e^{-i\int_0^{t_A} ds \left( f_A(s) + f_B(s + \hat{T}_B) \right)} |\psi_A(0)\rangle_A.$$  

(8)

Here, $|\psi_A(0)\rangle_A = \int dt'_B \phi(t'_B) |t'_B\rangle_B \otimes |\phi\rangle_R$ is the state of all systems, except clock $A$, conditioned on clock $A$ being in the state $|t = 0\rangle_A$. In this work, $I$, for $I = A, B, \ldots$, denotes all subsystems except for subsystem $I$. We will also use primed time variables to refer to the initial state of the clocks. The relation between $|\psi_A(0)\rangle_A$ and $|\phi\rangle$ is given explicitly in Appendix C. It is physically meaningful to assume that the support of the wave packets of clock $B$ does not overlap with the time defining any of the the events. In Appendix C we explicitly construct a wave-packet with this feature.

As noted in the previous Section, the state $|\psi_A(t_A)\rangle\bar{A} = \langle t | A : |\Psi\rangle$ represents the state of all systems except clock $A$ conditioned on such clock being in state $|t\rangle_A$. Because Eq. (8) contains all such conditional states for each value of $t_A$, the history state $|\Psi\rangle$ contains all the information regarding the time evolution and measurements carried out on system $A$. Importantly, the history state, represented in the time reference frame $A$, describes the subsystem $A$ evolving unitarily, from the initial state $|\psi_A(0)\rangle_A$, with the evolution operator $U(t_A)\bar{A} = \exp(-it_A H_B) T \exp(-i\int_0^{t_A} ds \left( f_A(s) + \hat{f}_B(s + \hat{T}_B) \right))$. By unitarity, the normalisation of $|\psi_A(0)\rangle_A$ is preserved in $t_A$.

Note that, in Eq. (8), the argument of $f_A$ is a c-number. This means that, in $A$’s time reference frame, the time at which operation $A$ is applied is always sharp — it happens at the well-defined time $t_A$. By contrast, the argument of $\hat{f}_B$ in Eq. (8) depends on the operator $\hat{T}_B$. When $\hat{f}_B(\hat{T}_B)$ acts on the initial state $|\psi_A(0)\rangle\bar{A}$, the argument of $\hat{f}_B$ will become dependent on $t'_B$, and will therefore be modulated by the wave packet $\phi(t'_B)$. If we assume $\phi(t'_B)$ to be not sharply peaked but rather to have a finite width $\sigma$, this effect will lead to an uncertainty, from the point of view of $A$, as to when the operation triggered by clock $B$ is applied. The larger the width $\sigma$, the greater the uncertainty. This effect is easily seen if we write down explicitly the conditional state $|\psi_A(t_A)\rangle\bar{A} = \langle t | A : |\Psi\rangle$. For simplicity, suppose that $t'_B < t_A < t'_A$, so that we do not need to discuss the operation triggered by clock $A$, which we already know to be sharply localised in $A$’s time reference frame. A simple calculation yields,

$$|\psi_A(t_A)\rangle\bar{A} = \int dt'_B \phi(t'_B) T e^{-i\int_0^{t'_A} ds \hat{f}_B(s + t'_B)} |t_A + t'_B\rangle_B \otimes |\phi\rangle_R.$$  

(9)

Eq. (9) is nothing but a coherent superposition of quantum states, each of them depending on $t'_B$. The amplitudes of the superposition are given by $\phi(t'_B)$. For each of these amplitudes, the (operator valued) function $\hat{F}_B(t_A, t'_B) = \int_0^{t'_A} ds \hat{f}_B(s + t'_B) = \Theta(t_A + t'_B - t'_A) \hat{K}(t'_B)$ takes different values. In $A$’s reference frame, $B$’s operation will already be applied at a given time $t_A$ only for those amplitudes corresponding to a $t'_B$ such that $t_A + t'_B > t'_B$. Because, given a time $t_A$, $B$’s operation will already be applied only for some amplitudes, this analysis clearly shows that the time localisation of $B$’s operation is uncertain with respect to $A$. In conclusion, $A$’s description of the experiment features two events: one is sharply defined at time $t'_A$, and the other one, which $A$ describes as triggered by $B$’s clock, is uncertain, due to the uncertainty of $B$’s clock (see Fig. 2 (left)).

How does the experiment look like from the point of view of $B$? As can be easily checked by applying the general methods developed in Appendix B, the history state in $B$’s time reference frame is exactly the same as that in $A$’s time reference frame, provided that the wave-packet $\phi(t'_B)$ is symmetric under the operation $t \rightarrow -t$. This is not surprising, given the symmetry of Eq. (6). The important point is that $B$ describes the initial state of $A$’s clock by the same wave packet $\phi(t'_A)$. As a consequence of the finite wave-packet width, $\sigma$, the localisation of $A$’s operation in time, as defined by $B$, will have an uncertainty modulated by $\phi(t'_A)$ (see Fig. 2 (right)). In contrast, the operation triggered by clock $B$ will be always sharp in $B$’s time reference frame. To summarise, from the point of view of a given time reference frame $I$, a measurement triggered by clock $I$ is always localised in time, while the measurements triggered by the other clocks are, in general, delocalised with respect to $t_I$, the local time of clock $I$.

As we will see in the following Sections, this relativity of localisation in time is not only a feature of a “poorly-chosen” initial state. Rather, it is an unavoidable effect when clocks experience time dilation due to their interaction with gravitating quantum systems.

C. Remarks

We conclude this Section by making some remarks:

Perfect clocks. – In this work we use the concept of a perfect clock as a convenient abstraction, which will help us to capture the essential features of the time-ordering of events and time evolution in the presence of gravitating quantum systems. By using this abstraction, we can circumvent the difficulties associated to defining the time localisation of
events with realistic clocks [33], and focus on the status of the notion of an event and its “space-time” localisation in the absence of a definite gravitational field.

**Distinction between clocks and local operations.**— It is important to distinguish two different types of terms playing different roles in Eq. (6). On the one hand, we have the term $\hat{H}_L := \hat{H}_A + \hat{H}_B$, acting only on the clocks degrees of freedom. The clocks allow $A$ and $B$ to acquire information about “when” the operations take place. Roughly speaking, these degrees of freedom (together with the spatial ones [19]) correspond to the relative “spatiotemporal localisation of the local laboratories” in which events take place. On the other hand, the term $\hat{H}_M := \int_A (\hat{T}_A) + \int_B (\hat{T}_B)$ describes the operations on the system located at the time defined by the clocks. Roughly speaking, they correspond to the “local operations” happening inside of the local laboratories. Motivated by the framework of process matrices [11], in Appendix D we make this correspondence explicit. That is, we write down the Born rule for the probability $p$, Eq. (5), in the form

$$p = \text{Tr} WM,$$

where $W$ depends only on the spatiotemporal part, corresponding to $\hat{H}_L$, and $M$ encodes the information about the operations inside the local laboratories, corresponding to $\hat{H}_M$. $M$ can depend on the spatiotemporal part as well. As shown in Appendix D, Eq. (10) is valid for a general $\hat{C} = \hat{H}_L + \hat{H}_M$. In general, the matrix elements of $W$ are given by

$$W_{\alpha\alpha'} = e^{-i\alpha \hat{H}_L} |\varphi\rangle \langle \varphi| e^{i\alpha' \hat{H}_L},$$

whereas

$$M_{\alpha\alpha'} = V^\dagger_{\alpha'} E V_{\alpha},$$

where $E$ is a measurement operator acting on the clocks and the ancillas, and $V_{\alpha} = e^{-i\alpha (\hat{H}_L + \hat{H}_M)} e^{i\alpha \hat{H}_L}$.

**Methods for transforming between time reference frames.**— As our final remark of this Section, we note that we have found the solution to Eq. (6) by group averaging —using Eq. (2). This led to the “perspective neutral” representation of $|\Psi\rangle$, Eq. (7), which we used to “jump” into the time reference frame of $A$. The perspective neutral representation offers an approach for changing from the time reference frame of $A$ to that of $B$. Moreover, it underscores the
interesting fact that “jumping” into a (classical or quantum) reference frame amounts to fixing the redundant degrees of freedom imposed by the constraint \( \hat{C} \) (see Refs. [29, 32, 34]). It may happen that one wishes to change from the time reference frame of \( A \) to that of \( B \) without explicitly computing the perspective neutral representation of \( |\Psi\rangle \). In Appendix B, we develop an alternative method which achieves this goal.

IV. GRAVITATING QUANTUM CLOCKS

In the previous Section we have seen how to describe the dynamical evolution and events of a quantum experiment with respect to different quantum clocks, \( A \) and \( B \). In our analysis, we made the important assumption that the clocks do not interact with each other. However, practical reasons aside, this assumption must fundamentally break down once the gravitational effects of \( A \) and \( B \) become significant—gravity is universal and cannot be shielded. Furthermore, the situation becomes all the more radical when we consider the quantum properties of \( A \) and \( B \): Any clock, say \( B \), must run in a superposition of different energies of its Hamiltonian, \( \hat{H}_B \) [35]. Because energy-momentum determines the metric field via Einstein’s equations, each of these energies corresponds to a different metric background. Therefore, the fact that the state of \( B \) is indefinite with respect to the observable \( \hat{H}_B \) means that the metric background, determined by \( B \), is indefinite, too [13]. In this situation, how would another clock, say \( A \), describe the time evolution and events of a quantum experiment?

In this Section we aim to answer the above question from the point of view of time reference frames. In Subsection IV A we will show that, even in the lack of a definite space-time background, an operational notion of time evolution can be defined if we “jump” into the time reference frame of a given clock, say \( A \). Importantly, \( A \) can interact gravitationally and get entangled with another clock, say \( B \), from the point of view of the observer \( C \). Because the state of \( A \) is mixed in \( C \)’s frame, one may naively think that the time evolution with respect to \( A \) is “noisy” or decoherent. However, we will show that this needs not be the case. We will present a model where, even if \( C \) “sees” clock \( A \) entangled with clock \( B \) due to their gravitational interaction, a consistent notion of “local time” exists in the time reference frame of \( A \), with respect to which time evolution is unitary. This fact is closely related to the result reported in Ref. [19], that entanglement and superposition are (quantum) reference frame dependent features. Afterwards, in Subsection IV B, we study the time localisation of events with respect to different time reference frames. We will find that, in general, the time localisation of events becomes uncertain due to the gravitational coupling between the clocks. Moreover, we will show that this uncertainty is time reference frame dependent, and argue that the impossibility to localise all events with respect to a single quantum clock in an experiment is a general feature of an indefinite metric background.

A. Time evolution with respect to gravitationally interacting clocks

Let us consider three clocks, \( A \), \( B \), and \( C \), subjected to each other’s gravitational influence. Suppose that the clocks obey the constraint

\[
\left( \sum_I \hat{H}_I + \frac{1}{2} \sum_{I \neq J} \lambda_{IJ} \hat{H}_I \hat{H}_J \right) |\Psi\rangle = 0.
\]

Eq. (13) describes, to the first order in \( 1/c^2 \), clocks interacting with each other via their gravitational field. Note that, in principle, the relative distances between the clocks, \( x_{IJ} \), and therefore the \( \lambda_{IJ} \)s are quantum operators as well. However, in this work we assume, for simplicity, that the \( x_{IJ} \)s are c-numbers and that they are time independent (with respect to any clock). This assumption means that the clocks follow semiclassical trajectories and remain at the same relative distance with respect to each other. Under this condition, the \( \lambda_{IJ} \)s are c-numbers as well. (This coupling between quantum clocks was introduced in [13] and was considered in [28] under a different physical interpretation.)

Let us analyse how time evolution emerges from the point of view of \( C \). In order to do this, we act on Eq. (13) with \( \langle \tau |_C \). We obtain

\[
i \left( 1 + \lambda_{AC} \hat{H}_A + \lambda_{BC} \hat{H}_B \right) \frac{d}{d\tau} |\psi(\tau)\rangle_C = \left( \hat{H}_A + \hat{H}_B + \lambda_{AB} \hat{H}_A \hat{H}_B \right) |\psi(\tau)\rangle_C.
\]

Note that Eq. (14) is not of the form of the Schrödinger equation, and is not the description of time evolution with respect to the proper time of clock \( C \), due to the extra term \( i(\lambda_{AC} \hat{H}_A + \lambda_{BC} \hat{H}_B) \partial_\tau \) on the left hand side. Before we
complete the “jump” into $C$’s time reference frame, it is interesting to discuss heuristically the physical meaning of Eq. (14). Consider first the simpler situation of a single quantum system evolving on a fixed space-time background given by the weak field metric $ds^2 = -(1 + 2\Phi(x)/c^2)dx^2 + dx \cdot dx$. For a static observer, the state of the system $|\psi\rangle$ evolves under the Schrödinger equation $i\hbar \frac{d}{dt} |\psi\rangle = \sqrt{-g_{00}}H_R |\psi\rangle$, where $g_{00} = -(1 + 2\Phi(x)/c^2)$ and $H_R$ is the Hamiltonian in the reference frame where the system is at rest \[36, 37\]. Note that, for this observer $\sqrt{-g_{00}} = \tau$, where $\tau$ denotes the proper time of the system and the dot denotes derivative with respect to time $t$. Defining $H_F = \sqrt{-g_{00}}H_R$ as the Hamiltonian in the laboratory reference frame, and using the fact that, for a static observer, $\partial_t = \tau \partial_\tau$, we can rewrite the evolution of the system as

$$i\sqrt{-g_{00}} \frac{d}{d\tau} |\psi\rangle = H_F |\psi\rangle.$$  

(15)

Under the approximation $\sqrt{-g_{00}} \approx 1 + \Phi(x)/c^2$, we see that Eq. (15) is precisely of the form of Eq. (14), for the case where the gravitational potential $\Phi$ is sourced by a gravitating classical body. We can therefore interpret Eq. (14) as a generalisation of Eq. (15) for the case where the “gravitational potential” is sourced by a gravitating quantum system. With this intuition, the operator in brackets on the left hand side of Eq. (14) can be interpreted as a “redshift” operator-valued coordinate time $\tilde{\tau}$. Then, explicitly,

$$\frac{d}{dt} := \left(1 + \lambda_{AC} \hat{H}_A + \lambda_{BC} \hat{H}_B\right) \frac{d}{d\tau}.$$  

(16)

(For a rigorous definition of a derivative with respect to an operator, see \[38\] ) This interpretation is tightly related to the idea of a quantum reference frame \[19\], where the set of symmetry transformations is generalised to include reference frames associated to quantum systems. Such transformations are obtained by promoting the parameter associated to the transformation to an operator acting on an additional Hilbert space.

We now complete the process of moving into $C$’s time reference frame by formally dividing by the redshift factor operator in Eq. (14), with the assumption that the energies of the state of the system $\bar{C}$ are small enough such that no divergences occur. Under these assumptions, we obtain the Schrödinger Equation in the time reference frame of $C$:

$$i\hbar \frac{d}{d\tau} |\psi(\tau)\rangle_{\bar{C}} = \left(\hat{H}_A + \hat{H}_B + \lambda_{AB} \hat{H}_A \hat{H}_B\right) |\psi(\tau)\rangle_{\bar{C}} / \left(1 + \lambda_{AC} \hat{H}_A + \lambda_{BC} \hat{H}_B\right).$$  

(17)

Eq. (17) shows that, in the time reference frame of $C$, the evolution of clocks $A$ and $B$ is unitary, despite the fact that there is a non-negligible interaction term between $C$ and the clocks $A$ and $B$ in Eq. (13). Importantly, in $C$’s reference frame, there is an interaction between $A$ and $B$, leading in general to entanglement between $A$ and $B$ in the view of $C$. In order to make this point clearer, let us assume that all $\lambda_{IJS}$ are small. We obtain, to first order in $\lambda_{IJ}$,

$$i\hbar \frac{d}{d\tau} |\psi(\tau)\rangle_{C} = (\hat{H}_A + \hat{H}_B + \lambda_{AB} \hat{H}_A \hat{H}_B) |\psi(\tau)\rangle_{C},$$  

(18)

where $\hat{H}_I = \hat{H}_I(1 - \lambda_{IC}\hat{H}_I)$, for $I = A, B$, and $\lambda_{AB} = \lambda_{AB} - \lambda_{AC} - \lambda_{BC}$. Eq. (18) is precisely the Schrödinger equation for two gravitationally interacting clocks, $A$ and $B$, to the first order in $1/c^2$. Note that, because $C$ is not infinitely far-away from $A$ and $B$, the Hamiltonians of $A$ and $B$ in $C$’s frame are blue-shifted from the original Hamiltonians appearing in Eq. (13). In $C$’s frame, the gravitational coupling between $A$ and $B$ in Eq. (18) is shifted as well. Taking the limits $\lambda_{IC} \to 0$, for $I = A, B$, recovers the Hamiltonians and couplings of Eq. (13). This limit corresponds to the case where $C$ is infinitely far away from the gravitating clocks $A$ and $B$ and therefore does not get entangled with them. This no-entanglement condition was achieved in Ref. \[13\] under the additional assumptions that the accuracy of time measurement, the sharpness of the state of the clock pointer, and the duration of the experiment were all finite, and was used to derive Eq. (18). Here we have shown these assumptions are not necessary in the context of time reference frames.

The important point is, however, that nothing in principle prevents us from deriving an analogous equation from the perspective of either $A$ or $B$. Suppose we were to “jump” into the time reference frame of $A$. Clearly, under similar assumptions as for the case of $C$, we would obtain an evolution equation of the form of Eq. (17) (or Eq. (18)) but with the indices $A$ and $C$ interchanged. Because Eq. (17) is a Schrödinger equation, the evolution of $B$ and $C$ in $A$’s time reference frame is also unitary, even if $A$ gets entangled with $B$ in $C$’s reference frame. Therefore, in this approach, each quantum clock constitutes a legitimate temporal (quantum) reference frame for which a notion of “evolution with respect to time $\tau$” is available. Importantly, this evolution is unitary, regardless of whether the clock is located far away or not from gravitating quantum systems, or whether the clock gets entangled with such
B. Events with respect to interacting clocks

As discussed in the previous Subsection, each clock defines a time reference frame with respect to which a notion of time evolution can be defined. This is true even in the presence of gravitating quantum systems that lead to an indefinite metric background. What is the difference, in terms of the time localisation of events relative to an observer, between a situation with a definite metric background from one where such background is not fixed? In this Subsection we address this question by studying how the localisation of operationally defined events depends on the observer, between a situation with a definite metric background from one where such background is not fixed? In this situation we have in mind is described by the constraint equation

$$\left(\hat{H}_A + \hat{H}_B + \hat{H}_C + \lambda \hat{H}_A \hat{H}_B + \int (\hat{T}_A) (1 + \lambda \hat{H}_B) \right) |\Psi\rangle = 0, \tag{19}$$

where $\lambda = -G/(c^4 x_{AB})$. Here, as in the previous Section, $\int (\hat{T}_A)$ denotes a hermitian-operator-valued function modelling the interaction between the system of interest, $S$, and an ancilla $a$, recording the result of a measurement on $S$. The presence of the term $(1 + \lambda \hat{H}_B)$, multiplying $\int (\hat{T}_A)$, is due to the fact that making $S$ and $a$ interact requires some energy, which will necessarily couple to $B$ due to the universality of gravity. Let us first analyse the history state in the time reference of $C$. As shown in Appendices A and C, this state is

$$|\Psi\rangle = \int d\tau_C |\tau_C\rangle_C \otimes e^{-i\tau_C (\hat{H}_A + \hat{H}_B + \lambda \hat{H}_A \hat{H}_B + \int (\hat{T}_A))} T e^{-i \int_0^{\tau_C} ds (1 + \lambda \hat{H}_B) f(s(1 + \lambda \hat{H}_B) + \tau_A) |\psi_C(0)\rangle_C}. \tag{20}$$

Here, we have used the notation $\tau_C$ to emphasise that this is the time measured with respect to clock $C$, i.e., its proper time.

Eq. (20) shows that, with respect to $C$, the event corresponding to the function $\int$ is not sharply defined in time, due to the presence of the operators in the argument of $\int$ in Eq. (20). Indeed, even if the initial state with respect to $C$ is such that clock $A$ is sharply defined at $t_A' = 0$, the presence of the operator $\hat{H}_B$ in the argument of $\int$ will lead to an uncertainty in the time localisability of the event. More precisely, suppose that the initial state in $C$’s frame is given by $|\psi_C(0)\rangle_C = |t_A' = 0\rangle_A \otimes \int_0^{\tau_C} d\tau_B \varphi_B(t_B') |t_B'\rangle_B \otimes |\phi_R\rangle (R$ denotes the subsystem formed by $S$ and $a)$. Because $B$ is a clock, and therefore cannot be sharp in $\hat{H}_B$, $\varphi_B(t_B')$ must have a finite width $\sigma$. In particular, this implies that the Fourier transform $\tilde{\varphi}_B(\omega_B)$ of $\varphi_B(t_B')$ will have a non-zero width $1/\sigma$. As a consequence, the history state of Eq. (20) will be a coherent superposition, modulated by $\tilde{\varphi}_B(\omega_B)$, of terms containing $\int f(s(1 + \lambda \omega_B))$ for different values of $\omega_B$. This case is mathematically similar to the case studied in Subsection III B, with $\omega_B$ playing the role of $t_B'$. Because the trigger of the measurement depends on $\omega_B$, different values of $\omega_B$ correspond to different times at which the event happens (as described by $C$). Therefore, there will be an uncertainty of the order of $1/\sigma$ in the time localisation of the event (with respect to $C$). In fact, there is a type of uncertainty relation between the accuracy of clock $B$ and the temporal localisability of events defined by clock $A$: The sharper clock $B$ is localised in $\hat{T}_B$, the “fuzzier” the events defined by clock $A$ appear from the point of view of $C$. This effect is depicted in Fig. 3.

We now change to the time reference frame corresponding to $A$. First, we insert the expression for $|\psi_C(0)\rangle_C$ into Eq. (20). Then, we define the coordinate $\tau_A(\tau_C) := t_A' + \tau_C(1 + \lambda \omega_B)$ and make a change of variables to eliminate $\tau_C$ in favour of $\tau_A$. Following the steps outlined in Appendix C, the history state in the reference frame of $A$ reads

$$|\Psi\rangle = \int d\tau_A |\tau_A\rangle_A \otimes T e^{-i \int_0^{\tau_A} ds \left( \frac{\hat{H}_A + \hat{H}_B + f(s)}{1 + \lambda \omega_B} \right) |\psi_A(0)\rangle_A}. \tag{21}$$

where the relation between the initial state with respect to $A$, $|\psi_A(0)\rangle_A$, and that with respect to $C$, $|\psi_C(0)\rangle_C$, can be found by the methods of Appendix B. As in the non-interacting case, we assume that the support of the wave packets of clocks $B$ and $C$ do not overlap with the time defining the event.
FIG. 3. Gravitating quantum clocks from the point of view of C. In a thought experiment, A (below) sets up an event, consisting in an interaction between S (blue ball) and a (red ball), when her clock shows a certain time \( t_A^* \). A’s clock is influenced by a gravitational field sourced by the energy of B’s clock (above). The initial quantum state of B’s clock (depicted by the green Gaussian) has a characteristic width \( \sigma \), which specifies its accuracy (the smaller \( \sigma \), the higher the accuracy). As a consequence, the energy of B’s clock is not well defined —it has an uncertainty of \( 1/\sigma \). Therefore, the gravitational field sourced by B is not well defined either. As a consequence, the time dilation of clock A becomes uncertain from the point of view of C. This is shown by the “fuzzy” red wave packets representing A’s clock state. By Eq. (20), this uncertainty leads to an indefiniteness in the localisation of A’s event, as depicted by the “fuzzy” red stars on the bottom of the wave packets.

Importantly, as we can see from Eq. (21), the description of the history state in the time reference frame of A is such that event is always sharp in \( \tau_A - \hat{f} \) is a function of the c-number \( s \) only. By contrast, Eq. (20) shows that the same event is not sharply localised from the point of view of C. This comparison shows that, when clocks interact gravitationally, the localisability of events in time is relative, and depends on the time reference frame which defines the events. Because it is not possible to “shield” gravity, this result suggests that the relativity of event localisability is a general feature of an operational definition of events in experiments with gravitating quantum systems. Moreover, if the interaction between the clocks were turned off, it would be possible to sharply define the time localisation of any event with respect to every time reference frame. The fact that we cannot do that, practical reasons aside, is precisely because we have an indefinite metric background. This discussion suggests that, in this context, we can characterise a definite metric if we can find a time reference frame such that, with respect to it, every event (defined sharply with respect to some clock as we do in this paper) is sharply localised. This fact is deeply related to the result in Ref. [21], where it is shown that the localisability of local operations in a causal reference frame is not absolute if and only if the corresponding process matrix is causally non-separable.

C. Remarks

We end this section by making some of remarks:

**Quantum coordinates.** It is interesting to note that we can understand the application of operations in Eq. (20) as being done with respect to a “quantum coordinate.” Specifically, following the definition of an integral with respect to an operator given in non commutative analysis [38], we note that the integral part of the exponent in Eq. (20) can be written as \( \int_0^{\tau_C} ds(1 + \lambda \hat{H}_B) \hat{f}(s(1 + \lambda \hat{H}_B) + \hat{T}_A) = \int_0^{\tau_C} d\hat{\tau}_A \hat{f}(\hat{\tau}_A) \), where \( \hat{\tau}_A = s(1 + \lambda \hat{H}_B) + \hat{T}_A \) is the “quantum coordinate” upon which the integral is defined. This is the integral version of the “derivative” with respect to a quantum coordinate expressed by Eq. (16).

**Quantum-controlled change of coordinates.** Now we comment on the change of coordinates that eliminates \( \tau_C \) in favour of \( \tau_A \), leading from Eq. (20) to Eq. (21). Although mathematically very simple, this change of variables has an important physical interpretation. Because it associates a different \( \tau_A \) to different amplitudes of the state of clock B, this change of coordinates is quantum-controlled by the state of B. Note that B is a source of the gravitational field, and is in a state which contains different amplitudes corresponding to different energies. Importantly, each of these energies corresponds to a different metric field, and therefore, to a different space-time. Then, the change of coordinates that goes from C’s to A’s frame associates different values of \( \tau_A \) to different metric fields, each of
which corresponds to a different amplitude in B’s state. In this sense, it is more general than the usual coordinate transformations in general relativity, where there is a fixed metric and therefore a single amplitude. We will discuss this issue in more detail in the following Section, where we analyse the gravitational quantum switch [12].

No-divergence condition. – Finally, as in the previous Subsection, we assume that the wave packet of B’s clock is such that no divergences occur in the denominator $1 + \lambda \dot{H}_B$. It is easy to check that the condition for no divergences implies that $\xi x_{AB} > \ell_P^2$, where $\xi = c \sigma$ is the characteristic with of the wave packet $\varphi_B$ in units of length, and $\ell_P^2 = \hbar G/c^3$ is the Planck area. The fact that the divergences occur at the Planck length is consistent with the widely-held view that, at this scale, typical quantum gravity effects become predominant.

V. GRAVITATIONAL QUANTUM SWITCH

In this Section we analyse the gravitational quantum switch [12] from the perspective of time reference frames. The gravitational switch is a thought-experiment where a gravitating body put in a quantum superposition of positions leads to an indefinite causal order of events. The experimental set up is as follows (see Fig. 4). Two parties, A and B, perform operations on a quantum system S (one operation each party). Each operation happens when the corresponding local clock shows time $t^*$. Apart from A and B’s clocks, there is a mass M, prepared in a superposition of two different position eigenstates, labeled by L (left) and R (right), as described in the reference frame of C. We assume that the superposition has the same weight for M and B, respectively.

The experimental set up is such that, for the amplitude corresponding to (for the mass on the left) and (for the mass on the right). The experimental set up is such that, for the amplitude corresponding to $L$ (R), M is closer to A (B). Because of gravitational time dilation, in the spacetime $\mathcal{M}_L^{(C)}$ ($\mathcal{M}_R^{(C)}$), the event where A’s clock shows $t^*$ and A acts on S is in the past (future) light cone of the event where B’s clock shows $t^*$ and B acts on S. Thus, the gravitational switch leads to a superposition of causal orders. For simplicity, we assume that the distance between M and A in the configuration given by L is the same as the distance between M and B in the configuration given by R, and the distance between A and B is constant.

We now use the framework of time reference frames to analyse and give a geometric description of the gravitational switch. We proceed by writing down the constraint describing the experiment and then comparing the history state in the reference frame of C to that in the reference frame of A (the case of B is analogous to that of A). The Hilbert space of the whole system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_S \otimes \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_M$, where A, B, and C denote the clocks of the three different time reference frames, S is the system on which A and B perform operations, and a and b are the ancillas that record the measurement outcomes of A and B, respectively. M denotes the relative degrees of freedom.
between the massive system and the clocks. For simplicity, we focus on the subspace generated by relative position eigenstates, labeled by \(Q = L, R\), denoting “mass close to A” and “mass close to B,” respectively. In the framework of time reference frames, this process is described by the constraint equation

\[
\left( \sum_I \hat{H}_I(1 + \hat{\Phi}_I) + \sum_I \hat{f}_I(T_I)(1 + \hat{\Phi}_I) \right) |\Psi\rangle = 0,
\]

(22)

where \(I\) takes the values \(A, B\) and \(C\). As in the previous Sections, we consider sharply localised measurements, \(\hat{f}_A(T_A) = \delta(T_A - t^*) \hat{R}_{SA}^A\), with an analogous definition for \(B\). By assumption, \(\hat{f}_C = 0\). Here, \(\hat{\Phi}_I = -GM/\left(\epsilon^2 \hat{x}_{IM}\right)\) denotes the gravitational potential, \(M\) is the mass of the system put in a superposition of locations, and \(\hat{x}_{IM}\) is the relative distance operator between the mass and clock \(I\). By definition, the operator \(\hat{\Phi}_I\) acts on \(\mathcal{H}_M\) as \(\hat{\Phi}_I |Q\rangle_M = \Phi_{IC}^Q |Q\rangle_M\). Note that \(\mathcal{H}_M\) is generated by the eigenstates of the operators \(\hat{x}_{IM}\), for \(I = A, B, C\). The states \(|Q\rangle_M\) are therefore eigenstates of each \(\hat{x}_{IM}\), for \(I = A, B, C\). For simplicity, we assume that \(C\) is located equidistant from both locations of \(M\), so that \(\Phi_{IC}^Q = \Phi_{CI}^Q = \Phi_C\).

Let us now find the history state, \(|\Psi\rangle\), of Eq. (22) in the reference frames of \(A\) and \(C\). In this case it is instructive to find first \(|\Psi\rangle\) in a “perspective neutral” representation, and then use this representation to move to the time reference frames of \(A\) and \(C\). By inserting the constraint \(\hat{C}\) of Eq. (22) into Eq. (2) we find (see Appendix A1)

\[
|\Psi\rangle = \int d\alpha e^{-i \sum_{I=A,B,C} \hat{H}_I(1 + \hat{\Phi}_I)} \hat{T} e^{-i \sum_{I=A,B} \int_0^\infty ds (1 + \hat{\Phi}_I) \hat{f}_I(s(1 + \hat{\Phi}_I) + \hat{T}_I)} |\varphi\rangle.
\]

(23)

The description of the quantum switch given in Eq. (23) can be roughly interpreted as the quantum state seen by a distant observer who uses the coordinate \(\alpha\) as parameter time. However, this interpretation is not physically rigorous, because the constraint (22) implies that there is no external time parameter according to which the systems evolve. For this reason, we write down the history state from the perspective of \(C\) in order to do so, we expand \(|\varphi\rangle\) in the \(|t'_{A,C}, t'_B,\rangle_{ABC} \otimes |Q\rangle_M \otimes |\bar{\chi}\rangle_{Sab}\) basis and follow the steps outlined in Appendix C. The result is

\[
|\Psi\rangle = \int d\tau_C |\tau_C\rangle_C \otimes T e^{-i \sum_I \int_0^\infty ds (\hat{\Delta}(I,C)(\hat{R}_I + \hat{f}_I(s(\hat{R}_I + \hat{T}_I)) |\psi_C(0)\rangle}_C,
\]

(24)

where the sum is over \(I = A, B\). We have defined \(\hat{\Delta}(I,J) := (1 + \hat{\Phi}_I)/(1 + \hat{\Phi}_J)\), for \(I, J = A, B, C\). Note that \(\Delta(I,J)\) is the operator version of the usual redshift factor, formed by the ratio of the proper time of clock \(I\) to that of clock \(J\). We denote the eigenvalues of \(\Delta(I,J)\) by \(\Delta^Q(I,J)\), where \(\Delta(I,J) |Q\rangle_M = \Delta^Q(I,J) |Q\rangle_M\). In this way,
\( \Delta^Q(I, J) \) is the ratio of the proper time of clock \( I \) to that of clock \( J \) in the mass configuration \( Q \). As in previous cases, we assume that the initial state, \( |\psi_C(0)\rangle_C \), has both clocks \( A \) and \( B \) sharply localised around \( t_A = t_B = 0 \). The reason for this assumption is simply that we are more interested in the effects due to the gravitating quantum system \( M \) than in those due to the “fuzziness” of the initial state. The relation between \( |\varphi\rangle \) and the initial state \( |\psi_C(0)\rangle_C \) is given in Appendix B in general form.

The fact that \( \Delta(I, C) \) is an operator acting on \( \mathcal{H}_M \) has important consequences for the localisation of events in the reference frame of \( C \). Indeed, the time ordering operator in Eq. (24) enforces that the operations of \( A \) and \( B \) are applied in different orders for the amplitudes corresponding to \( Q = L \) and \( Q = R \). Specifically, for \( Q = L \) (\( Q = R \)) we have \( \Delta^L(A, C) < \Delta^L(B, C) \) (\( \Delta^L(A, C) > \Delta^L(B, C) \)), so that \( A \)'s operation is applied after (before) \( B \)'s. Note that, for simplicity, we are working in an approximation where a given event is in the causal past of another one if the time associated to the former is earlier than the time associated to the latter. In Appendix C we comment on a possible way of taking a more complete approach, in terms of the light cones emanating from each event. In the present case, our simplification just means that, by assumption, the event triggered by clock \( A \) in the configuration \( Q = L \) (\( Q = R \)) is in the future light cone (past light cone) of the event triggered by clock \( B \).

We now compute the reduced state of system \( \bar{C} \), \( |\psi(\tau_f)\rangle_{\bar{C}} := \langle \tau_f | \cdot |\Psi\rangle \), at a time \( \tau_f > t^*/\Delta^R(A, C) = t^*/\Delta^L(B, C) \), when both operations have already occurred in \( C \)'s time reference frame. From Eq. (24), we have

\[
|\psi(\tau_f)\rangle_{\bar{C}} = \frac{1}{\sqrt{2}} \int \frac{dt_A' dt_B' dt_C'}{1 + \Phi_C} \varphi(t_A', t_B', t_C') \left( |\phi_L\rangle_{AB} \otimes |L\rangle_M \otimes U_{aS}^A(t^*/\Delta^L(A, C)) U_{bS}^B(t^*/\Delta^L(B, C)) + |\phi_R\rangle_{AB} \otimes |R\rangle_M \otimes U_{bS}^B(t^*/\Delta^R(B, C)) U_{aS}^A(t^*/\Delta^R(A, C)) \right) |\chi\rangle_R,
\]

where \( |\phi_Q\rangle_{AB} = \otimes_{I=A,B} |t'_I + \Delta^Q(I, C)(\tau_f - t'_C)\rangle_I \), for \( Q = L, R \). Here, the operator \( U_{aS}^A = \exp(-iK^A) \) represents the interaction by which the outcome of the event “the system is measured when clock \( A \) shows \( t = t^* \)” is recorded in the ancilla \( a \). Although this operator is time independent, we have written in both amplitudes the time, in \( C \)'s reference frame, at which it was applied. This is in order to emphasise the timing labelling of events in this reference frame. Similar remarks hold for the case of \( B \). The state of Eq. (25) is nothing but the familiar description of the quantum switch [40] after the local operations of \( A \) and \( B \) have been performed and before the control system, formed in this case by the mass and the clocks of \( A \) and \( B \), is recombined.

The description of the experiment according to the time reference frame of \( C \) is depicted in Fig. 5, where we have used two different manifolds to depict the two different space-times \( \mathcal{M}_L^C \) and \( \mathcal{M}_R^C \). As the figure shows, no event happens sharply localised in time according to \( C \). Rather, the event corresponding to \( A \) occurs in a superposition in time between \( \tau_C = t^*/\Delta^R(A, C) \) (early) and \( \tau_C = t^*/\Delta^L(A, C) \) (late), with a similar statement applying to the case of \( B \). Note that, according to our framework and its physical interpretation, the claim that the events in the switch experiment “involve 4 spacetime points” is a frame dependent statement —valid for \( C \) in this case. This is the natural picture of the gravitational switch that emerges if one imagines the experiment as seen by an observer far-away from the mass. However, note that, in our analysis, no assumptions regarding the distance from \( C \) to \( M \) were needed in order to obtain this description. Of course, we work with states on \( \mathcal{H}_M \) such that no divergences in the redshift factors \( \Delta^Q(I, J) \) occur. This is ensured as long as \( x_{IM} > GM/c^2 \) (the order of magnitude of the Schwarzschild radius corresponding to the mass \( M \)).

Let us now turn to the description of the experiment in the time reference frame of \( A \) (the case for \( B \) being completely analogous). From Eq. (23), the quantum controlled change of coordinates \( \alpha \rightarrow \tau_A := t_A + \alpha(1 + \phi^Q_A) \) gives the history state in \( A \)-representation

\[
|\Psi\rangle = \int d\tau_A \ |\tau_A\rangle_A \otimes e^{-i \int \Delta(1, A)(\bar{H}_1 + \bar{H}_2 + \bar{H}_3)} |\psi_A(0)\rangle_A,
\]

where the index \( I \) in the exponent takes values \( I = B, C \). The relation between \( |\psi_A(0)\rangle_A \) and the state \( |\varphi\rangle \) is found generically following in Appendix B.

Note that the argument of \( \bar{f}_A \) in Eq. (26) does not depend on any redshift factor, classical or quantum. This means that, according to \( A \), her operation is always localised in time. This fact can be seen clearly when writing down the conditional state \( |\psi(\tau_f)\rangle_A := \langle \tau_f | \cdot |\Psi\rangle \) for a time \( \tau_f > t^*/\Delta^R(B, A) \).

\[
|\psi(\tau_f)\rangle_A = \frac{1}{\sqrt{2}} \int \frac{dt_A' dt_B' dt_C'}{1 + \Phi_A} \varphi(t_A', t_B', t_C') |\phi_C\rangle \otimes (|\phi_L\rangle_B \otimes |L\rangle_M \otimes U_{aS}^A(t^*/\Delta^L(B, A)) U_{bS}^B(t^*/\Delta^L(B, A)) + |\phi_R\rangle_B \otimes |R\rangle_M \otimes U_{bS}^B(t^*/\Delta^R(B, A)) U_{aS}^A(t^*)) |\chi\rangle_R.
\]

For simplicity of notation, we have written \( \tau_f \) to refer to the “final” time when analysing the experiment both from \( A \)'s (Eq. (27)) and \( C \)'s (Eq. (25)) perspective. However, these two times need not be the same. Note that here
\[ |\phi\rangle_C = |t' + \Delta(C, A)(t_f - t'_A)|_C \] factors out from the mass degrees of freedom due to the assumption \( \Phi^L_C = \Phi^R_C \).

On the other hand, the state of clock \( B \) is entangled with the mass. The important point is that, in Eq. (27), the operation \( U^A_{aS} \) depends only on \( t^* \) in both amplitudes \( Q = L, R \). As noted before, this means that the operation happens sharply at \( t^* \) in the reference frame of \( A \), independent of where the mass is. However, the operation \( U^B_{aS} \) occurs before (after) \( U^A_{aS} \) for the configuration \( Q = L (Q = R) \). Therefore, for \( A \), events in the vicinity of her clock are always well defined in time, whereas events outside this vicinity are “spread out” in her time [21]. As noted in the previous Section, this is a signature of an indefinite metric field. An indefinite metric field can lead to an indefinite causal order of events if, like in this case, the events are suitably chosen.

The situation described by Eqs. (26) and (27) is depicted in Fig. 6, where a geometric description of the experiment from the point of view of \( A \) is given. As noted before, \( A \)'s operation takes place at time \( t^* \) in both amplitudes.

![FIG. 6. The gravitational switch thought experiment as described in the time reference frame of \( A \) (local observer). The event in which \( A \) acts on \( S \) is depicted by a red star. This event is perfectly localised in time from \( A \)'s point of view. However, in the previous case, in the spacetime \( M^C_R (M^C_L) \), the mass \( M \) is on the right (left), implying that \( A \) acts before (after) \( B \). The causal order is conserved, for each spacetime amplitude, after the change of perspective. In both \( M^C_R \) and \( M^C_L \), \( A \)'s action happens at time \( t^* \), whereas \( B \) acts at \( t^*/\Delta(B, A) \approx t^* + \epsilon \) (resp. \( t^*/\Delta(L, A) \approx t^* - \epsilon \)) in \( M^C_R (M^C_L) \), for \( \epsilon = (\Phi^L_B - \Phi^L_A)/t^* \). Note that the dashed red line joining \( A \)'s event in both spacetimes is now parallel to the dashed black line joining \( t^* \) in both spacetimes. This means that we are in the time reference frame where the event happens at a precise, sharp time. In contrast, the dashed green line joining \( B \)'s event in both spacetimes is not parallel to the black dashed one, showing that the time localisation of \( B \)'s event is not sharp in \( A \)'s time reference frame.](image)

A few (speculative) remarks are in order:

**Reference frames associated with a set of manifolds.**—The geometric picture of the gravitational switch shown in Figs. 5 and 6 suggests that a time reference frame should not be considered as “attached” to a single spacetime manifold. Rather, a time reference frame should be defined “transversally,” by “piercing” through the different manifolds. This resonates with Hardy’s idea [10] of quantum coordinate systems as an identification of points along different manifolds (this is depicted by the black, dashed lines in Figs. 5 and 6, which identify the time \( t^* \) in the two different space-times).

**Hypersurfaces of constant t.**—The previous remark suggests that, when writing a history state in the form \( |\Psi\rangle = \int dt \otimes |\psi(t)\rangle \), the ket \( |\psi(t)\rangle \) should not be interpreted as a state living on (the Hilbert space corresponding to a spatial slice of) a single spacetime, but rather on the hypersurface of “constant \( t \)” that “pierces” through the set of manifolds. The state on a single manifold is obtained by restricting \( |\psi(t)\rangle \) to a single spacetime. (For the gravitational switch, this restriction would correspond to, roughly speaking, “projecting” into the subspace corresponding to the spacetime where the mass is, say, on the right.)

**Signature of an indefinite metric and robustness of the global causal structure.**—We emphasise that, in experiments involving gravitating quantum systems, the fact that events are delocalised in time with respect to some observers is a signature of an indefinite metric field —ignoring uncertainties due to “fuzzy” clock pointers and measurement interactions extended in time. More specifically, if the metric field is indefinite, one cannot, in general, change to a
time reference frame where all events are localised in time. For example, in the gravitational switch, localising the event of A means that the event of B is uncertain in time. However, time reference frame transformations cannot change the global causal structure of the events in an experiment (either form A’s or C’s perspective, there is an amplitude where A’s event is in the past of B’s and an amplitude where A’s event is in the future of B’s). In other words, the localisability of a single event in time is a relative concept, whereas the global causal structure of events is an absolute one. This is reminiscent of what happens in the case of process matrices, where continuous and reversible transformations do not modify the global causal order of a process matrix [41].

Geometric “arena” for superpositions of semiclassical states.– We point out that the above geometric picture, with multiple manifolds identified by time reference frames “piercing” through them, suggests a new suitable geometric “arena” in which phenomena with “superpositions of spacetimes” can be studied mathematically, at least in the case where the gravitating quantum systems are in a quantum superposition of semiclassical states. It would be interesting to investigate if an extension of this geometric picture can be useful beyond this case as well.

VI. DISCUSSION AND OUTLOOK

We have constructed a framework for temporal reference frames which associates a quantum clock to each time reference frame. Within this framework, we have shown that one can consistently define a meaningful notion of quantum operation, time evolution and localisation of events in time with respect to different quantum clocks. Importantly, this is true even in the case where these clocks interact gravitationally and the space-time metric is indefinite. We have studied how these notions change when we go from one time reference frame to another and found situations where the physical description of time evolution is covariant (form invariant) with respect to transformations between time reference frames, in line with Ref. [19]. Our definition of an event allowed us to show that there is a time reference frame where the description of a given event assumes its familiar form in ordinary quantum mechanics (in terms of a unitary dilation of a quantum operation), even in cases where the metric field is indefinite. In the cases studied, the operations which are not localised with respect to the time reference frame acquire the form of a “quantum controlled unitary dilation,” where the time parameter is replaced by an operator. It would be interesting to know exactly how our approach is related to Hardy’s implementation of a “quantum equivalence principle” [10], where he proposes that the causal order can be made definite locally, around any given point.

We have used throughout the concept of a perfect clock. Although unrealistic, perfect clocks allowed us to explore the concepts of time evolution and time localisation of events without the difficulties associated with more realistic clock models. In experiments, these difficulties would clearly have to be taken into account, but the conceptual framework laid out here is independent of such problems. In fact, we consider our results as a basis of a more complete framework to describe phenomena in absence of a well-defined space-time metric. In this respect, the formulation of the process matrix framework in the form of Ref. [21] leads one to speculate that such formulation might not even need the explicit consideration of clocks.

An important effect occurs when quantum clocks interact gravitationally. In this case, we have shown that whether an event is localised in time or not depends on the time reference frame chosen. We have argued that the impossibility to find a time reference frame in which all events are local characterises, within our framework, a situation where the metric field is indefinite. This is relevant in the context of in Refs. [20, 21], where a connection between causal reference frames (or time delocalised subsystems) and pure process matrices [42] was established. It would be interesting to find out exactly how the frameworks of Refs. [20, 21] relate to our work, especially with the goal of understanding if a quantum violation of causal inequalities [11] can be achieved with gravitating quantum systems.

In this work, we have assumed that the spatial degrees of freedom of the gravitating quantum systems were in a semiclassical state, so that we could treat them classically. Although this simplification is useful, it would be important to extend our framework beyond this approximation. In a similar vein, it would also be interesting to generalise our framework to cases where terms of higher order in $1/c^2$ are included in the gravitational interaction.

Finally, on a more speculative note, our geometrical description of the gravitational quantum switch points to the fact that that, at least in this simplified case, time reference frame transformations can be thought of as transformations on a space formed by a collection of manifolds, one per each classical solution arising from a definite location of the mass $M$. This observation suggests a new, convenient mathematical “arena,” suitable for the analysis of physics involving gravitating quantum systems. It is commonly agreed that, when quantum mechanics and general relativity are put together, the concept of space-time would have to yield to “something else.” We hope that, using operational reasoning along the lines of the one presented in this paper, we can gain some insight into what this “something else” might be.
ACKNOWLEDGEMENTS

We thank P. Allard-Guérin, L. Hardy, P. A. Hoehn, I. Kull, L. Maccone, O. Oreshkov, W. Wieland and M. P. Woods for interesting discussions. We acknowledge support from the research platform Testing Quantum and Gravity Interface with Single Photons (TURIS), the Austrian Science Fund (FWF) through the Projects No. I-2526-N27, No. I- 2906 and BeyondC (F7113-N48), the ÖAW Innovationsfonds-Projekt “Quantum Regime of Gravitational Source Masses,” and the doctoral program Complex Quantum Systems (CoQuS) under Project No. W1210-N25. We also acknowledge financial support from the EU Collaborative Project TEQ (Grant Agreement No. 766900). A.B. is supported by H2020 through the MSCA IF pERFEcTO (Grant Agreement nr. 795782). This work was funded by a grant from the Foundational Questions Institute (FQXi) Fund. This publication was made possible through the support of a grant from the John Templeton Foundation (Project No. 60609). The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

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Appendix A: Trotter formulas

In this Appendix we derive the Trotter formulas used to find the history states in the main text.

1. Basic formula

Consider a Hilbert space $\mathcal{H}$ with a tensor factorisation of the form $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$. Let $\hat{T}_I$ and $\hat{H}_I$ be operators on $\mathcal{H}_I$ satisfying $[\hat{T}_I, \hat{H}_I] = i$, and let $\hat{f}(s)$ be an operator-valued function of the parameter $s$ acting on $\mathcal{H}_R$. If we replace the argument in $\hat{f}(s)$ by the operator $\hat{T}_I$, we obtain an operator valued function, $\hat{f}(\hat{T}_I)$, of the operator $\hat{T}_I$ acting on $\mathcal{H}$. In this setting, the basic Trotter formula reads

$$ e^{-i\alpha(\hat{H}+\hat{f}(\hat{T}_I))} = e^{-i\alpha\hat{H}}Te^{-i\int_0^\alpha ds \hat{f}(s+\hat{T}_I)}, \quad (A1) $$

where $T$ denotes the time-ordering operator, which acts as $T\hat{f}(s_1)\hat{f}(s_2) = \Theta(s_2-s_1)\hat{f}(s_2)\hat{f}(s_1) + \Theta(s_1-s_2)\hat{f}(s_1)\hat{f}(s_2)$ for any operator-valued function $\hat{f}$ of $s$.

In order to prove Eq. (A1), we use Trotter formula,

$$ e^{-i\alpha(\hat{H}+\hat{f}(\hat{T}_I))} = \lim_{N \to \infty} \left( e^{-i\frac{\alpha}{N} \hat{H}} e^{-i\frac{\alpha}{N} \hat{f}(\hat{T}_I)} \right)^N, \quad (A2) $$

applied to the generic basis vector $|t, r\rangle \in \mathcal{H}$, satisfying $\hat{T}_I |t\rangle = t |t\rangle$ and $|r\rangle \in \mathcal{H}_R$. After successive application of $L = e^{-i\frac{\alpha}{N} \hat{H}} e^{-i\frac{\alpha}{N} \hat{f}(\hat{T}_I)}$ to $|t, r\rangle$, one finds

$$ L^N |t, r\rangle = e^{-i\alpha\hat{H}} \prod_{k=1}^{N} e^{-i\frac{\alpha}{N} \hat{f}(\hat{T}_I+(k-1)\frac{\alpha}{N})} |t, r\rangle. \quad (A3) $$

Note that the right hand side (rhs) of Eq. (A3) can be seen as a (\Pi-ordered) Riemann sum, which, in the limit $N \to \infty$ gives

$$ \lim_{N \to \infty} L^N |t, r\rangle = e^{-i\alpha\hat{H}} \prod_{k=1}^{N} e^{-i\frac{\alpha}{N} \hat{f}(\hat{T}_I+(k-1)\frac{\alpha}{N})} |t, r\rangle, \quad (A4) $$

The change of integration variable $s \to s - t$ gives Eq. (A1) applied to the basis vector $|t, r\rangle$. At this point, we can substitute the parameter $t$ for the operator $\hat{T}_I$. By linearity, Eq. (A1) holds.

2. Variation 1: gravitating quantum clocks

Consider a Hilbert space of the form $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$, with operators $\hat{T}_A$ and $\hat{H}_A$ on $\mathcal{H}_A$ satisfying $[\hat{T}_A, \hat{H}_A] = i$, for $I = A, B$, and an operator-valued function $\hat{f}(\hat{T}_A)$, where $\hat{f}(s)$ is an operator on $\mathcal{H}_R$ for all $s$. A variation of formula (A1), relevant to the case considered in Sec. IV A, is

$$ e^{-i\alpha(\hat{H}_A+\hat{H}_B+\lambda \hat{H}_A\hat{H}_B+(1+\lambda \hat{H}_B)\hat{f}(\hat{T}_A))} = e^{-i\alpha(\hat{H}_A+\hat{H}_B+\lambda \hat{H}_A\hat{H}_B)}Te^{-i\int_0^\alpha ds (1+\lambda \hat{H}_B) \hat{f}(s(1+\lambda \hat{H}_B)+\hat{T}_A)}, \quad (A5) $$

In order to prove this formula, we proceed by dividing the $[0, \alpha]$ interval into $N$ equal parts, as in the previous case, and applying successively the operator $L = e^{-i\frac{\alpha}{N}(\hat{H}_A+\hat{H}_B+\lambda \hat{H}_A\hat{H}_B)} e^{-i\frac{\alpha}{N} \hat{f}(\hat{T}_A)(1+\lambda \hat{H}_B)}$ to the basis vector $|t, \omega, r\rangle$, with $\hat{T}_A |t\rangle = t |t\rangle$, $\hat{H}_B |\omega\rangle = \omega |\omega\rangle$ and $|r\rangle \in \mathcal{H}_R$. Following the same steps as in the previous case, we obtain, in the limit $N \to \infty$,

$$ \lim_{N \to \infty} L^N |t, \omega, r\rangle = e^{-i\alpha(\hat{H}_A+\hat{H}_B+\lambda \hat{H}_A\hat{H}_B)}Te^{-i\int_t^{t+\alpha(1+\lambda \omega)} ds \hat{f}(s)} |t, \omega, r\rangle. \quad (A6) $$

Eq. (A5) results after the change of integration variable $s \to (s-t)/(1+\lambda \omega)$ and the substitution of the parameters $t$ and $\omega$ for the operators $\hat{T}_A$ and $\hat{H}_B$. 
3. Variation 2: gravitational quantum switch

In this case, the Hilbert space is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_M \otimes \mathcal{H}_R$, with operators $\hat{T}_I$ and $\hat{H}_I$ on $\mathcal{H}_I$ satisfying $[\hat{T}_I, \hat{H}_I] = 0$, for $I = A, B, C$ and operator-valued functions $f_I(s)$ an operator on $\mathcal{H}_R$ for all $s$ and for $I = A, B, C$. Consider, in addition, operators $\hat{\phi}_I$ on $\mathcal{H}_M$, with mutual eigenvectors $|Q\rangle \in \mathcal{H}_R$ satisfying $\hat{\phi}_I |Q\rangle = \phi^Q_I |Q\rangle$ for $I = A, B, C$. The relevant formula for the case of the gravitational quantum switch is

$$e^{-i\Delta t(\hat{H}_I(1+\hat{\phi}_I)+\sum_{I=A,B,C} f_I(\hat{T}_I)(1+\hat{\phi}_I))} = e^{-i\Delta t(\hat{H}_I(1+\hat{\phi}_I)+\sum_{I=A,B} f_I(\hat{T}_I)(1+\hat{\phi}_I))} e^{-i\int_0^1 \frac{ds}{(1+\hat{\phi}_I)} f_I(s(1+\hat{\phi}_I) + \hat{T}_I). \ (A7)}$$

We can prove formula (A7) by following the same steps as in the two previous formulas. This time, we apply successively the operator $L = e^{-i\frac{1}{2} \sum_{I=A,B,C} H_I(1+\hat{\phi}_I)} e^{-i\frac{1}{2} \sum_{I=A,B} f_I(T_I)(1+\hat{\phi}_I)}$ to the basis vector $|t_A, t_B, t_C, Q, R\rangle$, where $\hat{T}_I |t_I\rangle = t_I |t_I\rangle$ for $I = A, B, C$.

Appendix B: General method for changing time reference frames

Here we develop a method for computing the history state for one observer, say $A$, given the history state with respect to another observer, say $C$. In this method, we can go directly from the reference frame of $C$ to the reference frame of $A$ without passing through the “perspective neutral” representation. As noted in the main text, the timeless approach is based on a constraint equation of the form

$$\hat{C} |\Psi\rangle = 0. \ (B1)$$

We can obtain $|\Psi\rangle$ by “projecting” an arbitrary state $|\varphi\rangle$ onto the space of solutions of Eq. (B1), that is

$$|\Psi\rangle = \hat{P} |\varphi\rangle, \ (B2)$$

where $\hat{P} = \int d\alpha \exp(-i\alpha \hat{C})$. Strictly speaking, $\hat{P}$ is not a projector: it maps the kinematical Hilbert space $\mathcal{H}$ onto a physical Hilbert space formed by solutions of Eq. (B1). As mentioned before, the physical Hilbert space is, in general, not isomorphic to the kinematical one, because the inner products of each space might be different to each other.

We can extract operationally meaningful physical predictions from $|\Psi\rangle$ by expressing it in a specific time reference frame (say $A$ or $C$, for concreteness). As we have seen in the main text, there are interesting cases where the time evolution with respect to either $A$ or $C$ is unitary. In these cases, we have

$$\langle t_I \hat{P} \cdot t' \rangle_I = \bar{U}_I(t - t'), \ (B3)$$

where $I$ can be either $A$ or $C$ (the time reference frames analysed in the main text), and $\bar{U}_I(t - t')$ is the evolution operator in the time reference frame of $I$. This operator is unitary and acts on all degrees of freedom except for $I$’s. Like all evolution operators, it satisfies the composition property: $\bar{U}_I(t - t') = \bar{U}_I(t) \bar{U}_I(-t') = \bar{U}_I(t) \bar{U}_I^T(t')$. We will focus on the cases where Eq. (B3) holds.

We can express $|\Psi\rangle$ in the time reference frame of $C$ by plugging $|\varphi\rangle = \int dC' |t'_{C'}\rangle_C \otimes |\psi_C(t'_{C'})\rangle_C$ into Eq. (B2), and then inserting a resolution of the identity, $\mathbb{1} = \int dC |t_C\rangle_C \langle t_C|_C$ on the right hand side of $\hat{P}$. Using Eq. (B3), we obtain

$$|\Psi\rangle = \int dC |t_C\rangle_C \otimes \bar{U}_C(t_C) |\psi_C(0)\rangle_C, \ (B4)$$

where

$$|\psi_C(0)\rangle_C = \int dC' \bar{U}_C^T(t_C') |\psi_C(t_C')\rangle_C. \ (B5)$$

Note that we have used the composition property of $\bar{U}_C$ in order to write down Eq. (B4) and Eq. (B5). Note also that, when introducing the resolution of identity, we have assumed that, in the physical Hilbert space, the integration measure in the position (or rather time) representation is given by $dt_C$. This is the case in all instances considered in this work and we will assume it in the following. In more general cases, one simply has to consider the specific form of the integration measure when inserting resolutions of identity in the time representation. In general, because the state $|\psi_I(0)\rangle_I$ evolves unitarily with respect to an arbitrary clock $I$, its normalisation is preserved in time with respect to this arbitrary frame.
We can obtain \(|\Psi\rangle\) in the time reference frame of \(C\) in different ways. For example, we can obtain first \(|\Psi\rangle\) in a “perspective neutral” representation, by computing explicitly the integral with respect to \(\alpha\) in Eq. (B2), and afterwards, by using the methods of Refs.\[29, 32, 34\], we can “jump” into the reference frame of \(C\). Alternatively, we can carry out explicitly the computations leading from Eq. (B2) to Eq. (B4), or we can act with \((t_C | C)\cdot\) on Eq. (B1) and solve the resulting differential equation. In any case, suppose we are given \(|\Psi\rangle\) in the time reference frame of \(C\), that is, in the form of Eq. (B4), and we would like to change the representation to the time reference frame of \(A\). We can do so directly by noting that

\[
\hat{P} = \int \! dt_C \, dt'_C \, |t'_C|_C \otimes \hat{U}_C(t_C - t'_C).
\]  

(B6)

Therefore, by Eq. (B3), we have

\[
\hat{U}_A(t_A - t'_A) = \int \! dt_C \, dt'_C \, |t'_C|_C \otimes \langle t_A | A \rangle \hat{U}_C(t_C - t'_C) |t'_A\rangle_A.
\]  

(B7)

Note that Eq. (B7) allows us to obtain the (unitary) evolution operator in the time reference frame of \(A\) directly from the evolution operator in the reference frame of \(C\). All we need to obtain \(|\Psi\rangle\) in the time reference frame of \(A\) is an equation for the initial state with respect to \(A\), \(|\psi_A(0)\rangle\rangle_A\), in terms of that of \(C\). But this is easy because, by definition, \(|\psi_A(0)\rangle\rangle_A = \langle t_A = 0 | A \cdot |\Psi\rangle\). Then, by Eq. (B4), we have

\[
|\psi_A(0)\rangle\rangle_A = \int \! dt_C \, |t_c|_C \otimes \langle t_A = 0 | A \cdot \hat{U}_C(t_c) |\psi_C(0)\rangle\rangle_C.
\]  

(B8)

Eqs. (B7) and (B8) are all we need to change the time reference frame representation of \(|\Psi\rangle\). Note that, in principle, \(\hat{S}_{AC} = \int \! dt_C \, |t_c|_C \otimes \langle t_A = 0 | A \cdot \hat{U}_C(t_c)\rangle\), which transforms between the initial states of \(C\) and \(A\) in Eq. (B8), need not be unitary. In the simplest case, where \(\hat{C} = \hat{H}_A + \hat{H}_C + \hat{H}_S\), we have

\[
\hat{S}_{AC} = \hat{P}_{AC} e^{i \hat{T}_A \hat{H}_S},
\]  

(B9)

where \(\hat{P}_{AC} := \int \! dt_C |t_c|_C \cdot (-t_C | A\rangle\) is the parity-swap operator between \(C\) and \(A\). Note that this transformation matches exactly the transformation introduced in Ref. \[19\], for transforming between two spatial quantum reference frames.

Finally, if we wish to change from the reduced state at time \(t_C\) in \(C\)’s frame, \(|\psi_C(t_c)\rangle\rangle_C = \langle t_c | C \cdot |\Psi\rangle\), to the reduced state at time \(t_A\) in \(A\)’s frame, \(|\psi_A(t_A)\rangle\rangle_A = \langle t_A | C \cdot |\Psi\rangle\), we simply have to put together Eqs. (B4) and (B8):

\[
|\psi_A(t_A)\rangle\rangle_A = \hat{U}_A(t_A) \hat{S}_{AC} \hat{U}_A^\dagger(t_c) |\psi_C(t_c)\rangle\rangle_C.
\]  

(B10)

In general, this transformation is not unitary, because \(\hat{S}_{AC}\) is not always unitary.

Appendix C: Derivation of history states with respect to different time reference frames

In this Appendix we will show in detail how to derive Eq. (8) from Section III, Eqs. (20) and (21) from Section IV, and Eqs. (24) and (26) from Section V in the main text. We will proceed by using Eq. (2) in order to obtain the “perspective neutral” representation of \(|\Psi\rangle\) and then changing coordinates to a specific time reference frame \[29, 34\].

1. History state for non-interacting clocks

We begin by analysing the simplest case, that is, the one discussed in Section III. The constraint is

\[
\hat{C} = \hat{H}_A + \hat{H}_B + \hat{f}_A(\hat{T}_A) + \hat{f}_A(\hat{T}_A)^\dagger
\]  

(C1)

First we obtain the history state \(|\Psi\rangle\), satisfying \(\hat{C} |\Psi\rangle = 0\), by using the results of Appendix A. By slightly generalising Eq. (A1), we obtain Eq. (7):

\[
|\Psi\rangle = \int \! d\alpha e^{-i\alpha(\hat{H}_A + \hat{H}_B)} T e^{-i \int_0^\alpha \! ds (\hat{f}_A(s + \hat{T}_A) + \hat{f}_A(s + \hat{T}_A)^\dagger)} |\varphi\rangle.
\]  

(C2)
Let us now write $|\varphi\rangle$ explicitly as

$$|\varphi\rangle = \int dt'_A |t'_A\rangle \otimes |\varphi(t'_A, t'_B)\rangle |t'_A, t'_B\rangle_{AB} \otimes |\chi\rangle_R \quad (C3)$$

and insert Eq. (C3) into Eq. (C2). Next, we define the variable $t_A(\alpha) = t'_A + \alpha$ and change variables to eliminate $\alpha$ in favour of $t_A$. Then, the history state reads

$$|\Psi\rangle = \int dt'_A dt'_B \varphi(t'_A, t'_B) T e^{-i \int_0^{t_A} ds (f_A(s+t_A) + f_B(s+t_B))} |t_A\rangle_A \otimes |t_A + t'_B - t'_A\rangle_B \otimes |\chi\rangle_R \quad (C4)$$

Now, in the integral in $s$, we make the change of variable $s \rightarrow s - t'_A$. We next divide the resulting integral in the exponent into two integrals, one from 0 to $t_A$ and another one from $t'_A$ to 0. After rearranging terms, turning the necessary c-numbers into operators, and using the fact that the time ordering operator $T$ allows us to commute terms acted upon by it, we obtain

$$|\Psi\rangle = \int dt'_A |t_A\rangle_A \otimes e^{-i t_A B B} T e^{-i \int_0^{t_A} ds (f_A(s) + f_B(s+t_B))} |\psi(0)\rangle \quad (C5)$$

where

$$|\psi(0)\rangle = \int dt'_A dt'_B \varphi(t'_A, t'_B) e^{i \int_0^{t_A} ds (f_A(s) + f_B(s+t_B))} |t'_A, t'_B\rangle |t'_B - t'_A\rangle_B \otimes |\chi\rangle_R \quad (C6)$$

Eq. (C5) is precisely Eq. (8) from the main text. Clearly, following the same steps, we can compute the history state and the initial state from the point of view of $B$. Moreover, by symmetry, it is easy to check that $|\psi_B(0)\rangle_B$ can be obtained from Eq. (C6) by swapping the labels $A$ and $B$ everywhere but in the argument of $\varphi$. Now, as mentioned in Sec. III, it is physically meaningful to view $B$ from the point of view of $A$, as well as the initial state of clock $A$, from the point of view of $A$, are both centred around times that are strictly smaller than the times $t'_A$, at which $S$ and the ancillas interact. Both conditions can be achieved by demanding that $\varphi(t'_A, t'_B)$ be supported in a region $D_{\varphi}^2$, where $D_{\varphi} \subset \{ s, \min_{s = A,B} |t'_{s}| \}$, with $\epsilon > 0$. Moreover, the state $|\psi_A(0)\rangle_A$ is normalised if and only if $|\psi_B(0)\rangle_B$ is normalised as well. In the case where $\varphi(t'_A, t'_B) = \delta(t'_A)\varphi(t'_B)$, where $\varphi(t'_B)$ is a symmetric function, centred around $t'_B = 0$ and supported on $D_{\varphi}$, $|\psi_A(0)\rangle_A$ and $|\psi_B(0)\rangle_B$ have exactly the same form, as can be seen by inserting this choice of $\varphi$ into Eq. (C6) and using the symmetry of $\varphi(t'_B)$ together with the fact that the initial states of $A$ and $B$ are related by a change of labels. This is the choice of initial state that we used for our discussion in Section III.

### 2. History state for gravitating quantum clocks

We will now derive explicitly Eq. (21). Eq. (20) is easily obtained following the same logic. Consider the constraint

$$\hat{C} = \hat{H}_A + \hat{H}_B + \lambda \hat{H}_A \hat{H}_B + f(\hat{T}_A)(1 + \lambda \hat{H}_B) \quad (C7)$$

By using Eq. (A5) and writing down explicitly

$$|\varphi\rangle = \int dt'_A dw_B dt'_C \varphi(t'_A, t'_B, \omega_B, t'_C) |t'_A, t'_B, t'_C\rangle_{ABC} \otimes |\chi\rangle_R \quad (C8)$$

in Eq. (2), we obtain, after some manipulation

$$|\Psi\rangle = \int d\omega_d dt'_A dw_B dt'_C \varphi(t'_A, \omega_B, t'_C)e^{-i \omega_B} T e^{-i \int_0^{\tau_A} ds (1 + 1 + \lambda \omega_B) f(1 + \lambda \omega_B) + t'_A} |t'_A + \alpha(1 + \lambda \omega_B), \omega_B, t'_C + \alpha\rangle_{ABC} \otimes |\chi\rangle_R \quad (C9)$$

We now define the new variable $\tau_A(\alpha) := t'_A + \alpha(1 + \lambda \omega_B)$ and change variables to eliminate $\alpha$ in favour of $\tau_A$. Note that, by doing this, we change the integration measure concerning the variable $\omega_B$, which now reads $d\omega(\omega_B) = d\omega_B/(1 + \lambda \omega_B)$. We assume that $\varphi(t'_A, \omega_B, dt'_C)$ is such that no divergencies in the new integration measure occur. Following the same steps leading to Eq. (C5) in Subsection C1, we also do the change of integration variable $s \rightarrow (s - t'_A)/(1 + \lambda \omega_B)$. After some manipulations analogous to those in Subsection C1, we arrive at

$$|\Psi\rangle = \int d\tau_A |\tau_A\rangle \otimes T e^{-i \int_0^{\tau_A} ds \left( \frac{n_B + n_C}{1 + \lambda \omega_B} f(s) \right)} |\psi_A(0)\rangle_A \quad (C10)$$
where the initial state is given by

\[ |\psi_A(0)\rangle_A = \int dt_A' d\mu(\omega_B) dt' B |\varphi(t' A, \omega_B) \rangle e^{i \int_0^{t_A} ds(\partial_B + \partial_C + f(s))} |\omega_B, t' C\rangle_B \otimes |\chi\rangle_R. \]  

(C11)

Eq. (C10) is precisely Eq. (21) in the main text.

Eq. (20) is obtained following the same steps leading to Eq. (21) but with the changes of variables \( \tau_C(\alpha) := t'_C + \alpha \) and \( s \to s - t'_C \). The initial state is calculated in exactly the same way as that in Eq. (C11).

3. History state for the gravitational quantum switch

Let us now show how to derive Eq. (26), with Eq. (24) being completely analogous. Consider the constraint

\[ \hat{C} = \sum_I \hat{H}_I (1 + \Phi_I) + \sum I f_I (\hat{T}_I)(1 + \hat{\Phi}_I). \]  

(C12)

For the state

\[ |\varphi\rangle = \sum Q \int dt' A dt' B dt' C |\varphi_Q(t' A, t' B, t' C)\rangle |t' A, t' B, t' C\rangle_{ABC} \otimes |Q\rangle_M \otimes |\chi\rangle_{Sab}. \]  

(C13)

Eqs. (2) and (A7) lead to

\[ |\Psi\rangle = \sum Q \int \Pi_I dt_I \varphi_Q(\{t_I\}) T e^{-\sum I \int_0^{t_A} ds(\partial_I + f(s(1 + \Phi_I)^2 + t'_I))} \otimes \int |t'_ I + \alpha(1 + \Phi_I)^2\rangle_I \otimes |Q\rangle_M \otimes |\chi\rangle_{MSab}. \]  

(C14)

where

\[ \varphi_Q(\{t_I\}) = \prod_I \varphi_Q(\{t' I\}) T e^{i \sum I \int_0^{t_A} ds(f(s(t'_ I + \hat{H}_I + f_I s(\hat{H}_I + \hat{T}_I))))} |\psi_A(0)\rangle_A, \]  

(C15)

\[ |\psi_A(0)\rangle_A = \sum Q \int \frac{\Pi_I dt_I}{1 + \Phi^2_I} \varphi_Q(\{t' I\}) T e^{i \sum I \int_0^{t_A} ds(f(s(t'_ I + \hat{H}_I + f_I s(\hat{H}_I + \hat{T}_I))))} |t'_ I\rangle_I \otimes |Q\rangle_M \otimes |\chi\rangle_{MSab}. \]  

(C16)

Note that in Eq. (C16) (Eq. (26) in the main text) as well as in Eq. (24), the order in which the operations are applied, in A’s or C’s reference frame, is controlled only by the time parameter corresponding to A’s or C’s clock. Therefore, in this approximation, A’s operation will be in the causal past of B’s operation if the time when A’s operation is applied (in the frame of A or C) is smaller that the time when B’s operation is applied. A more complete approach would have to take into account not only the times but also the light cones at each event. Then, the commutation relations between the operations applied should depend on whether the events are time-like or space-like separated. In principle, such a situation can be modelled with a family of time-dependent operators \( \hat{K}_{iS}(\hat{T}_I) \), for the clocks \( I = A, B \) and for the ancillas \( i = a, b \). These operators would commute if the corresponding events are outside of each other’s light cones (defined with respect to each position of the mass) and not commute otherwise.
Appendix D: Separating “knob settings” and “outside”

In this appendix we rewrite the probability rule, Eq. (5) in a way that suggest a separation between “knob settings,” corresponding to interventions performed by an agent on a quantum system in a controlled laboratory setting, and free evolution, describing the physics “outside” the laboratory. This separation is explained in Section III. We consider a constraint of the form \( \hat{C} = \hat{H}_L + \hat{H}_M \), where \( \hat{H}_L \) is a function of the (possibly interacting) Hamiltonian of the clocks and \( \hat{H}_M \) includes the interventions made on the system. Dropping, for simplicity of notation, the dependence on \( x, y, ..., z \) and \( a, b, ..., c \), Eq. (5) reads

\[
p = |E|\Psi|^2 = \text{Tr} E^\dagger E |\Psi\rangle\langle \Psi| = \text{Tr} E^\dagger E \int d\alpha d\alpha' e^{-i\alpha(\hat{H}_L+\hat{H}_M)}|\varphi\rangle\langle \varphi| e^{i\alpha' (\hat{H}_L+\hat{H}_M)} = \text{Tr} E^\dagger E \int d\alpha d\alpha' V_{\alpha} e^{-i\alpha \hat{H}_L} |\varphi\rangle\langle \varphi| e^{i\alpha' \hat{H}_L} V_{\alpha}^\dagger = \int d\alpha d\alpha' \text{Tr} e^{-i\alpha \hat{H}_L} |\varphi\rangle\langle \varphi| e^{i\alpha' \hat{H}_L} V_{\alpha}^\dagger E E \alpha = \text{Tr} W M, (D1)
\]

where \( W_{\alpha,\alpha'} = e^{-i\alpha \hat{H}_L} |\varphi\rangle\langle \varphi| e^{i\alpha' \hat{H}_L} \) and \( M_{\alpha',\alpha} = V_{\alpha}^\dagger E E \alpha \), and \( V_{\alpha} e^{-i\alpha \hat{H}_L} = e^{-i\alpha (\hat{H}_L+\hat{H}_M)} \).