Shear viscosity and electrical conductivity of relativistic fluid in presence of magnetic field: a massless case

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We have explored the shear viscosity and electrical conductivity calculations for bosonic and fermionic medium by extending from without to with magnetic field picture. We have found new expressions of four shear viscosity components, which might be considered as general case as its strong field limit gives us the expressions, existing in earlier references. Realizing different components of shear viscosity and electrical conductivity as two main components - normal and Hall-type coefficients, we have seen their temperature and magnetic field dependence for massless bosonic and fermionic medium. Along with collisional time scale, a magnetic time scale for cyclotron motion of charge particle in the medium due to external magnetic field basically creates anisotropic transportation in the medium, for which many components are found. Similar to a temperature dependent collisional time, for which shear viscosity to entropy density ratio touch its lower bound, we have obtained a new analytic expression of temperature and magnetic field dependent collisional time by restricting the same bound.

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I. INTRODUCTION

Heavy-ion collisions have been the subject of intensive research to extract information about the nuclear properties of matter in extreme conditions like high temperature and high external magnetic fields. This field covers many interesting phenomena such as magnetic catalysis [1], chiral magnetic effect [2,3], inverse magnetic catalysis [4,5] etc. A detail discussion on the effects of magnetic field in quantum field theory has been addressed in Ref. [6]. A verification of these anticipated results is possible by studying QCD matter under the influence of electromagnetic fields. A review of the phase structure of QCD in the presence of magnetic field has been given in Ref. [7].

Ref. [8] shows analytically that fields produced in RHIC and LHC can reach up to $m_T^2$ and $10m_T^2$ respectively after collision. Possible space time evolution of electromagnetic fields, generated in heavy-ion collisions are well discussed in Refs. [10,12]. Applying that space time evolution of magnetic field information to the hydrodynamical expansion of quark-gluon plasma will construct a detail expansion dynamics, which is commonly known as magneto hydrodynamic (MHD). Under the influence of a strong magnetic field the properties of quark-gluon plasma in heavy-ion collisions have been studied under the generalized framework of Bjorken flow [13,14]. In the limit of ideal magnetohydrodynamics, i.e., for infinite conductivity, and irrespective of the strength of the initial magnetization, the decay of the fluid energy density with proper time is the same as for the Bjorken flow without magnetic field. It has been found in Ref. [14] that under the influence of magnetic field the energy density and temperature decay slowly because of magnetic field. But taking into consideration the magnetic field produced in heavy-ion collisions the decay is suppressed. Numerical approaches to magnetohydrodynamics and its application to heavy-ion collisions have been studied [15,16]. Impact of external magnetic field through transport simulation can be noticed in Ref. [17]. For dissipative picture of MHD or transport simulation, field dependent transport coefficients might be required as inputs and therefore, a parallel microscopic calculation of transport coefficients in presence of magnetic field [18] is an important topic in heavy ion Physics community. In recent time transport coefficients in presence of magnetic field are investigated in Refs. [19,20], where shear viscosity [19,22], electrical conductivity [23,33], bulk viscosity [24,38] for light quark sector as well as field impact in heavy quark sector [30,40] are studied. A calculation of the viscous pressure tensor of QGP in strong magnetic field in Ref. [19] shows the anisotropic character of the viscous pressure tensor generates the asymmetric transverse flow. Anisotropic character of electrical conductivity of QGP in magnetic field has been discussed in the context of a warm neutron star crust [28] in the presence of magnetic field. Using the Kubo formalism [21,22] electrical conductivity of quark matter under external strong magnetic field has also been calculated. Transport coefficients at finite magnetic field in the direction of gauge gravity duality is also studied in Refs. [41,42]. In present work, we also have gone through a microscopic investigations of shear viscosity and electrical conductivity of medium in presence of magnetic field, where we have gone through a detail derivation of four components of shear viscosity, whose expressions are completely new and general. We have seen that in strong field limit, expressions are modified to well existing expressions, obtained in earlier references.
The article is organized as follows. Sec. (Ⅲ) covers a detail derivation of shear viscosity components at finite magnetic field along with brief description of electrical conductivity. Next in Sec. (Ⅲ) we have explored the massless case results, where analytic expressions of different thermodynamical quantities and transport coefficients are obtained. Then in Sec. (Ⅳ), we have focused on shear viscosity to entropy density ratio of bosonic and fermionic medium and searched their lower bound possibilities in different ranges of temperature and magnetic field. At the end, Sec. (Ⅵ) has summarized the investigations and few calculation gap are addressed in Sec. (Ⅵ).

II. GENERAL EXPRESSION OF SHEAR VISCOSITY AND ELECTRICAL CONDUCTIVITY IN PRESENCE OF MAGNETIC FIELD

Keeping in the mind about the tiny fluid-like matter, produced in heavy ion collision experiments like RHIC and LHC, the present section is going to describe the formalism of shear viscosity of that matter. After nucleus-nucleus collision at RHIC or LHC energy, they are passed through each other but their huge kinetic energy, invested in mid-rapidity region, is sufficient to produce the quark gluon plasma (QGP), which will expand and cool down. After a few fm/c (~ 10^{-20} s), the QGP will be converted to hadronic matter, which further expands and freezes out at certain temperature. Dissipative relativistic hydrodynamic can describe this expansion video of femtoscopic magnetic field along with brief description of electrical conductivity, which will also be addressed below.

Reminding the final expression of shear viscosity for without magnetic field picture [43][44],

$$\eta = \frac{g_\beta}{15} \int \frac{d^3p}{(2\pi)^3} \frac{p^4}{\omega} \tau_c f_0 (1 - a f_0)$$

and its background steps in relaxation time approximation (RTA) method [43][44], we will proceed for calculation of shear viscosity in presence of magnetic field. Here, we will follow same steps. Only our starting point will be little different. We can find multiple tensor components \(C_{\alpha\beta}^n\) (\(n\) denotes different components) instead of a single tensor component. Hence, we start with [15][17]

$$\Delta T_{\alpha\beta} = \sum_{n=0}^{4} \eta_n C_{\alpha\beta}^n$$

and also from microscopic theory we can write

$$\Delta T_{\alpha\beta} = - \int \frac{d^3k}{(2\pi)^3} v_\alpha v_\beta \omega \delta f$$

Now, relativistic Boltzmann’s equation in presence of magnetic field will take form,

$$\frac{\omega}{T} v_\alpha v_\beta V_{\alpha\beta} f_0 (1 \mp f_0) = - \frac{qB}{\omega} b_{\alpha\beta} v_\beta \frac{\partial}{\partial \nu_a} (\delta f) + I(\delta f)$$

where, \(I(\delta f)\) stand for linearized collision integral, \(V_{\alpha\beta} = \frac{1}{2} (\frac{\partial n}{\partial \epsilon_{\alpha\beta}} + \frac{\partial \epsilon_{\alpha\beta}}{\partial n})\), \(B\) is magnetic field strength and unit vector is \(b_\alpha\) and \(q\) represent total charge. Also, \(b_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} b_\gamma\).

Now, form of the solution of eq. (3) can be [15][17],

$$\delta f = \delta f^1 = \sum_{n=0}^{4} g_n C_{\gamma\delta}^n v_\gamma v_\delta$$

now putting \(\delta f\) in eq. (3) and comparing with eq. (2) we have,

$$\eta_n = - \frac{1}{15} \int \frac{d^3k}{(2\pi)^3} \omega g_n v^4$$

where we used the relation

$$\langle v_\alpha v_\beta v_\gamma v_\delta \rangle = \frac{1}{15} v^4 (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$$

Now to solve for \(g_n\) we have to put \(\delta f\) from eq. (5) in eq. (4). And using RTA, \(I(\delta f^1) = - \frac{\delta f^1}{\tau_c}\) and also we know, \(\frac{1}{\tau_B} = \tau_B\), where, \(\tau_c\) is thermal relaxation time and \(\tau_B\) is magnetic relaxation time or inverse of synchrotron frequency. Former is mainly controlled by randomness of the medium at particular temperature \(T\), but it may also have dependence with magnetic field. Although in present work, we will consider it as free parameter. The \(\tau_B\) is completely originated from external magnetic field and inversely depends on it.

So from Eq. (10), we get

$$\frac{\omega}{T} v_\alpha v_\beta V_{\alpha\beta} f_0 (1 \mp f_0) = - \frac{1}{\tau_B} b_{\alpha\beta} v_\beta \frac{\partial}{\partial \nu_a} \left( \sum_{n=0}^{4} g_n C_{\gamma\delta}^n v_\gamma v_\delta \right)$$

and

$$\Rightarrow \frac{\omega}{T} v_\alpha v_\beta V_{\alpha\beta} f_0 (1 \mp f_0) = - \frac{1}{\tau_B} b_{\alpha\beta} v_\beta 2 \times \left( \sum_{n=0}^{4} g_n C_{\alpha\gamma}^n v_\gamma \right)$$

Now, in the presence of magnetic field \(C_{\alpha\beta}^0\) would be of the form,

$$C_{\alpha\beta}^0 = (3 b_{\alpha\beta} - \delta_{\alpha\beta}) (b_\gamma b_\delta V_{\gamma\delta} - \frac{1}{3} \nabla \cdot v),$$
\[ C_{\alpha\beta}^4 = 2V_{\alpha\beta} + \delta_{\alpha\beta}V_{\gamma\delta}b_{\gamma}b_{\delta} - 2V_{\alpha\gamma}b_{\gamma}b_{\beta} - 2V_{\beta\gamma}b_{\beta}b_{\alpha} (b_{\alpha}b_{\beta} - \delta_{\alpha\beta})\nabla\cdot \mathbf{v} + b_{\alpha}b_{\beta}V_{\gamma\delta}b_{\gamma}b_{\delta}, \]
\[ C_{\alpha\beta}^2 = 2(V_{\gamma\delta}b_{\gamma}b_{\beta} + V_{\beta\gamma}b_{\beta}b_{\gamma} - 2b_{\gamma}b_{\beta}V_{\gamma\delta}b_{\delta}), \]
\[ C_{\alpha\beta}^3 = V_{\alpha\gamma}b_{\gamma} + V_{\beta\gamma}b_{\beta} - V_{\gamma}\delta b_{\gamma}b_{\beta} - V_{\delta}\beta b_{\beta}b_{\delta}, \]
\[ C_{\alpha\beta}^0 = 2(V_{\gamma\delta}b_{\gamma}b_{\delta} + V_{\beta\gamma}b_{\beta}b_{\delta}) \quad (9) \]

Also using the following conditions,
\[ \nabla\cdot \mathbf{V}_{\alpha\beta} = V_{\alpha\alpha} = 0, V_{\alpha\beta}b_{\alpha\beta} = 0, \]
\[ b_{\alpha\beta}b_{\alpha\beta} = 0, b_{\alpha\beta}v_{\alpha\beta} = 0, \]
\[ b_{\alpha}b_{\beta} = b^2, b_{\alpha\beta} = -b_{\beta\alpha} \]
So we get, \[ C_{\alpha\beta}^0 = 0. \]

Now using the above conditions, eq. (8) takes the form,
\[
\frac{\omega}{T}v_{\alpha\beta}V_{\alpha\beta}f_0(1 \mp f_0) = -\frac{g_1}{\tau_c} [g_1 \{2V_{\alpha\gamma}b_{\alpha\beta}v_{\alpha\gamma}\} - 2V_{\alpha\beta}b_{\alpha\beta}v_{\alpha\beta}(b\cdot v)]
+ g_2 \{2V_{\alpha\beta}v_{\beta}b_{\beta}(b\cdot v)\}
+ g_3 \{2V_{\alpha\beta}v_{\alpha\beta}v_{\gamma}(v^2 + (b\cdot v))\}
+ g_4 \{4V_{\alpha\beta}v_{\alpha\beta}v_{\gamma}(v\cdot b)\}.
\] (10)

Now, comparing same tensor structure on both side we get,
\[ g_1 = \frac{\omega}{2T} \frac{\tau_c}{4\{\frac{1}{4} + (\frac{\tau_c}{\tau_B})^2\}} f_0(1 \mp f_0) \quad (11) \]
\[ g_2 = \frac{\omega}{2T} \frac{\tau_c}{4\{\frac{1}{4} + (\frac{\tau_c}{\tau_B})^2\}} f_0(1 \mp f_0) \quad (12) \]
\[ g_3 = \frac{\omega}{2T} \frac{\tau_c}{2\{\frac{1}{4} + (\frac{\tau_c}{\tau_B})^2\}} f_0(1 \mp f_0) \quad (13) \]
\[ g_4 = \frac{\omega}{2T} \frac{\tau_c}{2\{\frac{1}{4} + (\frac{\tau_c}{\tau_B})^2\}} f_0(1 \mp f_0) \quad (14) \]

Now using these expression in eq. (8) we get,
\[ \eta_1 = \frac{g_\beta}{15} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\omega} \left[ f_0(1 \mp f_0) \right] \quad (15) \]
\[ \eta_2 = \frac{g_\beta}{15} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\omega} \left[ f_0(1 \mp f_0) \right] \quad (16) \]
\[ \eta_3 = \frac{g_\beta}{15} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\omega} \left[ f_0(1 \mp f_0) \right] \quad (17) \]
\[ \eta_4 = \frac{g_\beta}{15} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\omega} \left[ f_0(1 \mp f_0) \right] \quad (18) \]

![FIG. 1: Ratio of conductivity (a,b) and viscosity (c,d) with and without magnetic field along \( \frac{\pi}{4} \) axis, which is classified into two: \( \frac{\tau_c}{\tau_B} < 1 \) (a,c) and \( \frac{\tau_c}{\tau_B} > 1 \) (b,d).](image)

One can find similar kind of detail derivation in Ref. [21], which has not considered the Bose enhancement/Pauli suppression part. These four different viscosity components can be treated as more general expressions, which can be tuned with the expressions of strong field limit, addressed in Refs. [12, 13, 19]. The limiting cases are discussed below.

For \( B \to 0, \tau_B \to \infty \), we get
\[ \eta_2 = \eta_1 \quad (19) \]
\[ \eta_4 = 2\eta_3 \quad (20) \]

One can identify \( \eta_1, \eta_2 \) as normal shear viscosity as they merge to \( \eta \) as \( B \to 0 \). Seeing the vanishing values of \( \eta_3, \eta_4 \) in absence of magnetic field, one can realize them as Hall-type shear viscosity, as they completely originate from \( B \).

If we take strong field limit \( B \to \infty \), for which \( \tau_B \to 0 \), then we will get
\[ \eta_2 = 4\eta_1 \quad (21) \]
\[ \eta_4 = 2\eta_3 \quad (22) \]
which are exactly same as obtained in Ref. [19].

Next, we come to the expression of electrical conductivity in presence of magnetic field. Here, again we can first recall without magnetic field expression of the electrical conductivity,

\[
\sigma = e^2 g \beta \frac{1}{3} \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{\omega} \tau_c f_0 (1 - f_0),
\]

(23)

and then, instead of repeating its formalism, given in Refs. [23], let us come directly to final expressions

\[
\sigma_n = e^2 g \beta \frac{1}{3} \int \frac{d^3 p}{(2\pi)^3} \frac{p^2}{\omega} \tau_c (\tau_c/\tau_B)^n f_0 (1 - f_0),
\]

(24)

where \(\sigma_0\) is normal conductivity along xx or yy direction, \(\sigma_1\) is Hall conductivity along xy or yx direction and \(\sigma_{zz} = \sigma_0 + \sigma_2 = \sigma\) is longitudinal conductivity if \(B\) is applied along z-direction. Here also, if we take the limit \(B \rightarrow 0\), then \(\sigma_0 \rightarrow \sigma, \sigma_1 \rightarrow 0\). In strong field limit \(B \rightarrow \infty\), we will get similar kind of expressions:

\[
\sigma_0 = \frac{g \beta e^2}{3} \int \frac{d^3 k}{(2\pi)^3} \frac{\tau_B^2}{\tau_c} \left( \frac{k^2}{\omega} \right)^2 [f_0 (1 + f_0)]
\]

(25)

\[
\sigma_1 = \frac{g \beta e^2}{3} \int \frac{d^3 k}{(2\pi)^3} \frac{\tau_B}{\tau_c} \left( \frac{k^2}{\omega} \right)^2 [f_0 (1 + f_0)]
\]

(26)

Analytic outcomes of \(\eta_n\) and \(\sigma_n\) for two opposite limits are drawn in Fig. 1(a-d) for getting better visualization. \(B \rightarrow 0\) and \(B \rightarrow \infty\) can be alternatively realized by \(\tau_c/\tau_B \rightarrow 0\) and \(\tau_c/\tau_B \rightarrow \infty\) or \(\tau_c/\tau_B \approx 1\) in numerical point of view. In Fig. 1(a) and (b), \(\sigma_{0.1}\) and \(\eta_{1.2,3,4}\) are plotted against \(\tau_c/\tau_B\) from 0 to 1, where we can find the merging \(\sigma_0\) to \(\sigma\) and \(\eta_{1.2}\) to \(\eta\) at \(\tau_c/\tau_B = 0\). It means that we can get back the normal conductivity of shear viscosity in absence of magnetic field just by putting \(B \rightarrow 0\) or \(\tau_c/\tau_B \rightarrow 0\). On the other hand, Hall conductivity \(\sigma_1\) and shear viscosity components \(\eta_{3,4}\) is disappeared at \(B \rightarrow 0\) or \(\tau_c/\tau_B \rightarrow 0\) as it is completely magnetic field induced phenomena. Next, in Fig. 1(c) and (d), \(\sigma_{0.1,2}\) are extended for \(\tau_c/\tau_B > 1\) with same line-style curves. By using approximation \(\tau_c/\tau_B \gg 1\), we have already got strong field limit expressions of \(\sigma_{0.1,2}\), which are renamed as \(\sigma_{N,H}\) in Eqs. (25), (26) and \(\eta_{N,H}\) in Eqs. (21), (22). Plotting them against \(\tau_c/\tau_B\)-axis, we notice that \(\sigma_{0.1,2}\) are merging to \(\sigma_{N,H,}\) around and beyond \(\tau_c/\tau_B = 4\). It means that we can safely use strong field approximated expressions for \(\tau_c/\tau_B \geq 4\) but they can not be used for \(\tau_c/\tau_B < 4\).

### III. FOR MASSLESS QUARK MATTER

#### A. Thermodynamics for \(B = 0\)

Here, we will address the analytic forms of different thermodynamical quantities like energy density \(\epsilon\), pressure \(P\), entropy density \(s\) for massless fermionic and bosonic medium, where \(s\) is our main required quantity to estimate \(\eta/s\) in Sec. (VI A).

In terms of distribution function \(f_0\), the energy density and pressure of any medium can be expressed as

\[
\epsilon = g \int_0^\infty \frac{d^3 p}{(2\pi)^3} \frac{\omega}{\tau_c} f_0 ,
\]

\[
P = g \int_0^\infty \frac{d^3 p}{(2\pi)^3} \frac{p^2}{\tau_c} f_0 ,
\]

(27)

which are connected as \(P = \frac{1}{3} \epsilon\) for massless case. So, entropy density of the system is

\[
s = \frac{\epsilon + P}{T} = \frac{4\epsilon}{3T}
\]

(28)

We can take a general form thermal distribution function as

\[
f_0 = \frac{1}{e^{\beta \omega} + a},
\]

(29)

which becomes Maxwell-Boltzmann (MB) for \(a = 0\), Fermi-Dirac (FD) for \(a = +1\) and Bose-Einstein (BE) for \(a = -1\).

Solving Eq. (28) with \(p = \omega\) as for massless case, we get (See Sec. (VIA) in Appendix)

\[
s = \frac{4g}{\pi^2} T^3 \text{ for MB}
\]

(30)

\[
= \frac{4g}{\pi^2} \zeta(4) T^3 = \frac{4g \pi^2}{90} T^3 \text{ for BE}
\]

\[
= \left( \frac{7}{8} \right) \frac{4g}{\pi^2} \zeta(4) T^3 = \frac{7g \pi^2}{180} T^3 \text{ for FD },
\]

(30)

where

\[
\zeta(4) = \frac{1}{\Gamma(4)} \int_0^\infty x^3/(e^x - 1)
\]

\[
= \sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.
\]

(31)

#### B. Shear viscosity and electrical conductivity for \(B = 0\)

Now, let us come to shear viscosity and electrical conductivity expressions for massless and \(B = 0\) case.
By solving Eq. (1) for massless case, one can get (See Sec. (VI B) in Appendix):

\[
\eta = \frac{4 g \tau_c}{5 \pi^2} T^4 \quad \text{for MB}
\]
\[
\eta = \frac{4 g \tau_c \zeta(4)}{5 \pi^2} T^4 \quad \text{for BE}
\]
\[
\eta = \left(\frac{7}{8}\right) \frac{4 g \tau_c}{5 \pi^2} \zeta(4) T^4 \quad \text{for FD}
\]

\[\zeta(2) = \frac{1}{\Gamma(2)} \int_0^\infty \frac{x^3}{e^x - 1} dx\]

and solving Eq. (23) for massless case, we get:

\[
\sigma = \frac{g e^2 \tau_c}{3 \pi^2} T^2 \quad \text{for MB}
\]
\[
\sigma = \frac{g e^2 \tau_c \zeta(2)}{6 \pi^2} T^2 = \frac{g e^2 \tau_c}{36} T^2 \quad \text{for BE}
\]
\[
\sigma = \frac{g e^2 \tau_c \zeta(2)}{3 \pi^2} T^2 = \frac{g e^2 \tau_c}{18} T^2 \quad \text{for FD}, \quad (33)
\]
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \] 

From Eqs. (32) and (33), we can understand that \( \eta/(\tau_c T^4) \) and \( \sigma/(\tau_c T^2) \) are constant values, whose magnitude are different for different distribution function as shown by straight horizontal lines (solid line for MB, dash line for BE and dotted line for FD) in Fig. 2(a), (b). Here we did not take any degeneracy factor, i.e. we keep \( g = 1 \).

So the constant value of \( \eta/(\tau_c T^4) \) and \( \sigma/(\tau_c T^2) \) means that shear viscosity and electrical conductivity component of massless quark matter are proportional to fourth power and second power of temperature respectively. Another thing is that both transport coefficients \( \eta \) and \( \sigma \) are proportional to relaxation time in \( B = 0 \) picture, which will be modified at \( B \neq 0 \) case. We will see it in next subsection.

### C. Shear viscosity and electrical conductivity for \( B \neq 0 \)

Let us come to \( B \neq 0 \) picture and applying massless case in the expressions of normal and Hall-type shear viscosity and electrical conductivity components, given in Eqs. (16), (18) and (21). Using massless relation \( p = \omega \) in Eqs. (16) and (18), we get

\[ \eta_2 = \eta(B = 0) = \frac{4g \tau_c}{1 + (\tau_c/\tau_B)^2} T^4 \quad \text{for MB} \]

\[ \eta_2 = \eta(B = 0) = \frac{4g \tau_c}{1 + (\tau_c/\tau_B)^2} T^4 \quad \text{for BE} \]

\[ \eta_2 = \eta(B = 0) = \frac{7g \tau_c}{1 + (\tau_c/\tau_B)^2} T^4 \quad \text{for FD} \] (35)

and

\[ \eta_4 = \frac{\eta(B = 0)(\tau_c/\tau_B)}{1 + (\tau_c/\tau_B)^2} = \frac{4g \tau_c}{1 + (\tau_c/\tau_B)^2} \quad \text{for MB} \]

\[ \eta_4 = \frac{\eta(B = 0)(\tau_c/\tau_B)}{1 + (\tau_c/\tau_B)^2} = \frac{4g \tau_c}{1 + (\tau_c/\tau_B)^2} \quad \text{for BE} \]

\[ \eta_4 = \frac{\eta(B = 0)(\tau_c/\tau_B)}{1 + (\tau_c/\tau_B)^2} = \frac{7g \tau_c}{1 + (\tau_c/\tau_B)^2} \quad \text{for FD} \] (36)

For calculation simplification, we have considered average energy in \( \tau_B \) (See Sec. (VI C) in Appendix)

\[ \tau_B = \frac{\omega_{av} / eB}{3T} = \frac{3T}{eB} \quad \text{for MB} \]

\[ \tau_B = \frac{\omega_{av} / eB}{3T} = \frac{3T}{eB} \quad \text{for BE} \]

\[ \tau_B = \frac{\omega_{av} / eB}{3T} = \frac{3T}{eB} \quad \text{for FD} \] (37)

Solving Eq. (21) for massless case, we get

\[ \sigma_n = \frac{\sigma(B = 0)(\tau_c/\tau_B)^n}{1 + (\tau_c/\tau_B)^2} = \frac{\omega_{av}^2 \tau_c}{35} T^2 (\tau_c/\tau_B)^n \quad \text{for MB} \]

\[ \sigma_n = \frac{\sigma(B = 0)(\tau_c/\tau_B)^n}{1 + (\tau_c/\tau_B)^2} = \frac{\omega_{av}^2 \tau_c}{35} T^2 (\tau_c/\tau_B)^n \quad \text{for BE} \]

\[ \sigma_n = \frac{\sigma(B = 0)(\tau_c/\tau_B)^n}{1 + (\tau_c/\tau_B)^2} = \frac{\omega_{av}^2 \tau_c}{35} T^2 (\tau_c/\tau_B)^n \quad \text{for FD} \] (38)

Using the Eqs. (35), (36), we have plotted \( \eta_{2,4}/(\tau_c T^4) \) as a function of \( T \) and \( eB/m_\pi^2 \) in Fig. 2(a) and 3(a). In similar way, by putting \( n = 0, 1 \) in Eq. (24), we can estimate \( \sigma_{0,1}/(\tau_c T^4) \), whose \( T \) and \( eB/m_\pi^2 \) dependence are shown in Fig. 2(b) and 3(b). The normal shear viscosity component \( \eta_2 \) and electrical conductivity component \( \sigma_0 \) increases with \( T \) and decreases with \( B \) due to increasing and decreasing of anisotropic factor

\[ A_N = 1/[1 + (\tau_c/\tau_B)^2] \quad , \] (39)

where \( \tau_B \propto T/B \). However, this monotonic trend cannot be obtained for Hall-type shear viscosity component \( \eta_4 \) and electrical conductivity component \( \sigma_1 \) because their anisotropic factor

\[ A_H = (\tau_c/\tau_B)/[1 + (\tau_c/\tau_B)^2] \quad . \] (40)

follow a non-monotonic \( T, B \) dependence. For constant value of \( \tau_c, \frac{\tau_B}{\tau_c} \propto \frac{T}{B} \) increases by increasing \( T \) and/or decreasing \( B \). Hence, the anisotropic factor \( A_H \) will increase first in the domain of \( \frac{\tau_B}{\tau_c} < 1 \) and then decrease in the domain of \( \frac{\tau_B}{\tau_c} > 1 \). Those domains can be seen in Figs. 2(c) and 3(c), where \( \tau_B \) is plotted against \( T \) and \( B \) axes and compared with \( \tau_c = 2 \) fm value.

To get a simplified analytic form of \( \eta_{2,4} \) and \( \sigma_{0,1} \), as given in Eqs. (35), (36) and (21) we have considered momentum/energy independent expression of \( \tau_B = \omega_{av}/(eB) \), given in Eq. (41). However, one can obtain numerical values of \( \eta_{2,4} \) from Eqs. (16), (18), and (21) from Eq. (21) by using massless relation \( p = \omega \). These numerical and analytical results are plotted by black dash and blue solid lines in Fig. 1, where their values are quite separated but well merged at low \( T \) and/or high \( B \) and \( \tau_c \). Their merging zone is basically strong field domain, where \( \tau_c/\tau_B >> 1 \) will be achieved. The expressions of \( \eta_{2,4} \) and \( \sigma_{0,1} \) for strong field case are already discussed in Eqs. (21), (22), (25) and (20), where their anisotropic factors are transformed as

\[ A_N = 1/[1 + (\tau_c/\tau_B)^2] \rightarrow (\tau_B/\tau_c)^2 \]

\[ A_H = (\tau_c/\tau_B)/[1 + (\tau_c/\tau_B)^2] \rightarrow (\tau_B/\tau_c) \] . (41)

So, in this strong field limit, the transformed anisotropic factors can be brought outside the integration of Eqs. (21), (22), (25) and (20) by considering \( \tau_B = \omega_{av}/(eB) \). Since the qualitative nature of numerical and analytic expressions for \( \eta_{2,4}/(T, B, \tau_c) \) and \( \sigma_{0,1}/(T, B, \tau_c) \) are same in entire domain, so we will continue our further discussion still with analytic forms.
FIG. 4: Numerical values of $\eta_2 A/(\tau_c T^4)$ from Eqs. (10), (18), and $\phi_0 A/(\tau_c T^4)$ from Eq. (24) are plotted by black dash line. Corresponding values from analytic Eqs. (55), (35) and (38) are plotted by blue solid line.

IV. ON PERFECT FLUID CASE FOR $B = 0$ AND $B \neq 0$

Considering $T$, $B$ and $\tau_c$ as free parameters, let us analysis the domain, where perfect fluid nature of bosonic and fermionic medium will be built. We know that $\eta/s$ is very important quantity which measures fluidity of the medium. In classical case, we can imagine a perfect fluid, having $\eta/s = 0$ but in quantum case, we get a lower bound of $\eta/s$, which is also known as KSS bound [16]. For massless medium, using $\eta$ from Eq. (32) and $s$ from Eq. (30), we can get $\eta/s = \tau_c T/5$, which is interestingly same for MB, BE and FD cases, although their individual $\eta$ and $s$ expressions are different. If at $\tau_c = \tau_c^0$, lower bound of $\eta/s$ for massless medium is achieved, then we get

$$\frac{\eta}{s} = \frac{c_0 T}{5} = \frac{1}{4\pi} \Rightarrow \tau_c^0 = \frac{5}{4\pi T},$$

which is quite standard well-known results.

Now let us find similar kind of analytic results, when massless bosonic and fermionic medium face an external magnetic field $B$. Using normal viscosity component $\eta_2$ from Eq. (35) and $s$ from Eq. (30), we will get $\eta_2/s$, which will be now different for MB, BE and FD cases as it contains $\tau_B$, whose average values are different for different cases. By restricting $\eta_2/s = 1/4\pi$, we can get quadratic equation of $\tau_c$:

$$\tau_c^2 - 4\pi^2 T^2 \eta(B = 0) + \tau_B^2 = 0$$

$$\Rightarrow \tau_c^2 = \left(\frac{4\pi T \tau_B^2}{\eta(B = 0)}\right)\Rightarrow \tau_c + \tau_B^2 = 0$$

The solution of above equation is

$$\tau_c = \tau_c^{2\pm} = \frac{\tau_B^2}{2\tau_c^0} \left[1 \pm \sqrt{1 - 4\left(\frac{\tau_B^0}{\tau_B}\right)^2}\right].$$

So far from our best knowledge, we are first time addressing an analytic expressions of $\tau_c(T, B)$, where massless bosonic/fermionic matter in presence of magnetic field reach the KSS bound. To get a physical solution of Eq. (44), we need

$$1 - 4\left(\frac{\tau_B^0}{\tau_B}\right)^2 \geq 0$$

$$\Rightarrow \tau_B \geq 2\tau_c^0.$$  \hspace{1cm} (45)

Using MB relation $\tau_B = \frac{3T}{eB}$ in above inequality, we have

$$\frac{3T}{eB} \geq \frac{5}{4\pi T} T \geq \left(\frac{5eB}{6\pi}\right)^{1/2}. $$

$$\Rightarrow T \geq \left[\frac{5eB}{6\pi}\right]^{1/2}.$$ \hspace{1cm} (46)

Corresponding FD and BE relations from Eq. (37) will give

$$T \geq \left[\frac{5eB}{6\pi\zeta(3/4)}\right]^{1/2}$$

$$T \geq \left[\frac{5eB}{6\pi\zeta(4)}\right]^{1/2}$$

$$T \geq \left[\frac{2\zeta(3/4)}{6\pi}\right]^{1/2}$$

$$T \geq \left[\frac{5eB}{6\pi\zeta(4)}\right]^{1/2}$$

Drawing $T - eB$ curves of Eqs. (46) and (47) in Fig. 5(a), one can identify upper allowed domain, where KSS bound can be achieved while lower domain is forbidden zone if we believe that $\eta_2/s$ never goes below $1/4\pi$. To explore the fact, we have drawn straight horizontal (green solid) line at $T = 0.170$ GeV, where we have chosen points $eB = 3m_c^2$ and $eB = 7m_c^2$ in allowed and forbidden zone for bosonic medium. Similar points are $eB = 15m_c^2$ and $eB = 20m_c^2$ for fermionic medium. Generating $\eta_2/s$ vs $\tau_c$ curves for bosonic medium (b) and fermionic (c) medium at those points, we can see that $\eta_2/s$ always remain below KSS value at the points of forbidden zone. At the point of allowed zone, we can get solution of Eq. (44) at $\tau_c =$
The zone can cross the KSS line (red horizontal line in Figs. 5(b), (c), (d), (e)). (f), (g), (h), (i), (j) are same as (a), (b), (c), (d), (e) for Hall viscosity component.

FIG. 5: (a) $T \propto \sqrt{eB}$ curves for MB (black solid line), BE (red dash line), FD (blue dotted line) cases, below which $\eta_2/s$ never touch KSS line as shown by pink dash-dotted line in (b), (c) for bosonic and (d), (e) for fermionic medium. $\eta_2/s$ at the points of upper allowed zone can cross the KSS line (red horizontal line in b, c, d, e). (f), (g), (h), (i), (j) are same as (a), (b), (c), (d), (e) for Hall viscosity component.

$\tau_{c}^{\pm}$, which can be identified by crossing points of $\eta_2/s$ with KSS line (red horizontal line) in Figs. 5(b) and (c). We have also drawn $\eta/s \propto \tau_c$ (dotted line) curves in Figs. 5(b) and (c). Due to simple proportional nature, $\eta/s$ will cross KSS line at one point ($\tau_c = \tau_c^0$), but $\eta_2/s$ cross the KSS line in two points ($\tau_c = \tau_c^{2\pm}$) because of non-monotonic relation $\eta_2/s \propto 1/\sqrt{(\tau_c/\tau_B)}$. We notice that $\tau_c^0$ and $\tau_c^{2\pm}$ are very close in numerical values and both signify the lower values of $\tau_c$, for which viscosity to entropy density ratio touch its KSS bound. Interestingly, we are getting an upper value of $\tau_c (\tau_c^{+})$, where $\eta_2/s$ again reach its KSS bound. This fact is completely new fact, appeared in the picture of finite magnetic field.

Now, let us come to the picture when Hall viscosity component $\eta_A/s$ will reach KSS bound. Using Eq. (35) for $\eta_A$ and Eq. (30) for $s$, we will get $\eta_A/s$, and then restricting $\eta_A/s = 1/4\pi$, we can get another quadratic equation of $\tau_c$:

$$\tau_c^2 \left( \frac{\tau_B}{\tau_c^0} - 1 \right) = \tau_B^2$$

$$\Rightarrow \tau_c = \pm \frac{\tau_B}{\sqrt{\tau_B/\tau_c^0} - 1}. \quad (48)$$

Ignoring the impossible negative values of $\tau_c$, let us focus on $\tau_c = \sqrt{\tau_B/\tau_c^0}$ (say). Here again, we will get a physical solution of Eq. (48), when

$$\tau_B/\tau_c^0 - 1 \geq 0 \Rightarrow \tau_B \geq \tau_c^0. \quad (49)$$

Using MB relation $\tau_B = 3T/eB$ in above inequality, we have

$$\frac{3T}{eB} \geq \frac{5T}{4\pi} \Rightarrow T \geq \left( \frac{5eB}{12\pi} \right)^{1/2}. \quad (50)$$

Corresponding FD and BE relations from Eq. (37) will give

$$T \geq \left[ \left( \frac{\zeta(3)}{\zeta(4)} \right) \frac{5eB}{12\pi} \right]^{1/2} \text{ for BE}$$

$$T \geq \left[ \left( \frac{2\zeta(3)}{7\zeta(4)} \right) \frac{5eB}{12\pi} \right]^{1/2} \text{ for FD}. \quad (51)$$

So with respect to normal viscosity component, Hall viscosity $T - eB$ curves of Eqs. (50) and (51) are $\sqrt{2}$ times shifted down as shown in Fig 5(d). To understand the allowed and forbidden zones, here, we have drawn the straight horizontal (green solid) line at $T = 0.120$ GeV, where we have chosen same points at $eB/m_\pi^2$ for bosonic and fermionic medium. Generating $\eta_A/s$ vs $\tau_c$ curves at $T = 0.120$ GeV for bosonic medium and fermionic medium in Figs. 5(e), (f) respectively, we find that their $\eta_A/s$ can cross KSS line for $eB = 3m_\pi^2$, $eB = 15m_\pi^2$ but the possibilities become forbidden for $eB = 7m_\pi^2$, $eB = 20m_\pi^2$. 

\[ \begin{align*}
T &= \frac{\sqrt{eB}}{m_\pi^2} \text{ in the picture of finite magnetic field.}
\end{align*} \]
Above graphical discussion give us an idea about allowed/forbidden $T$-$B$ domain, where $\tau_c$ parameter can/can’t tune $\eta_{2/s}$ to $1/(4\pi)$. Now, let us draw $\tau_c^{2/3}(T, B)$ and $\tau_c^4(T, B)$ along with $\tau_c^6(T)$ in Fig. 6(a) by using Eqs. (44), (48) and (42). We have filtered out unphysical points of $\tau_c^{2/3}(T, B)$ and $\tau_c^4(T, B)$. Fig. 6(a), (d) shows $\tau_c^{2/3}$ increases with $T$ and decreases with $B$, whereas $\tau_c^4$ follow a mild decrement (increment) with $T$ ($B$) as show in Fig. (b), (c). The $\tau_c^4$ in Fig. (c), (f) shows mild decrement with both $T$ and $B$ with a rapid blowing up tendency near unphysical points. At the end, Fig. (g) has captured $B$ dependence of different time scales like $\tau_c^{2/3}$, $\tau_c^4$ and $\tau_B$ at initial and freeze-out temperatures $T_i = 0.400$ GeV and $T_f = 0.100$ GeV of expanding RHIC/LHC matter. We have drawn all curves for u quark only. At initial temperature $T_i = 0.400$ GeV, RHIC/LHC matter might face $eB = 10-20m^2_c$ magnetic field, where $\tau_c$ parameter of u quark can be bounded from $\tau_c^{-2} \approx 0.2$ fm to $\tau_c^{-2} = 100-20$ fm to follow the inequality $\eta_{2/s} \geq 1/(4\pi)$. After expansion of RHIC/LHC matter, temperature will be reduced and at certain temperature, called freeze-out temperature ($T_f = 0.100$ GeV), beyond which it loses its medium identity. At $T_f = 0.100$ GeV, we noticed that $\eta_{2/s} \geq 1/(4\pi)$ is possible between $\tau_c^{-2} (eB = 0 - 6m^2_c) \approx \infty - 1.5$ fm and $\tau_c^2 (eB = 0 - 6m^2_c) \approx 0.8 - 1.5$ fm. At $T_f = 0.100$ GeV, $eB > 6m^2_c$ will be treated as forbidden zone of $\tau_c$, as $\eta_{2/s}$ goes down KSS value. Although, there is a possibility of decaying magnetic field so fast that before freeze-out temperature, we can get $B = 0$. Again, we may not be worry about $\tau_c^2$, as life time of the RHIC/LHC matter ($\sim 10$ fm) can be considered as upper limit of $\tau_c$. So roughly $\tau_c$ of RHIC/LHC matter have lower limit path from $\tau_c^2 \approx (T_i = 0.400$ GeV, $eB = 20m^2_c) = 0.2$ fm to $\tau_c^2 \approx (T_f = 0.100$ GeV, $eB = 0) = 4$ fm and upper limit 10 fm. This path has been shown by orange arrow in Fig. (g). With respect to curves of $\tau_B(T, B)$ and $\tau_B(T_f, B)$, the orange arrow or $\tau_c$ path of RHIC/LHC matter remain quite lower, which roughly indicate about weak-moderate field ($\frac{\tau_c}{\tau_B} \leq 1$) scenario.

V. SUMMARY

In summary, we have focused on the shear viscosity and electrical conductivity calculations for bosonic and fermionic medium, facing an external magnetic field. For electrical conductivity, we have adopted existing expressions, but we have gone through a detail derivation shear viscosity and found a new expressions of four shear viscosity components. We have identified those expressions as general case, since its strong field limit give us the expressions, existing in earlier references. We have also checked that the zero magnetic field limit of those anisotropic expressions merge to a single isotropic expression, obtained in the absence of magnetic field. We have realized different components of shear viscosity and electrical conductivity as two main components - normal and Hall-type coefficients, which depend on three parameters - temperature, magnetic field and collisional relaxation time.

As a special case, we have chosen massless bosonic and fermionic matter, controlled by their own distribution functions - Bose-Einstein and Fermi-Dirac. We have also included the case of Maxwell-Boltzmann dis-
tribution functions. In external magnetic field, a magnetic time scale for cyclotron motion of charge particle in the medium is introduced along with collisional time scale. The competition between these two time scale basically create anisotropic transportation in the medium, for which many components are found. Taking average energy approximation in magnetic time scale, we get analytic expressions for all components of shear viscosity and electrical conductivity, which are also graphically studied. Normal component always reduce with magnetic field and enhance with temperature, while Hall component follow a non-monotonic trend with temperature and magnetic field. Interestingly, normal component follow non-monotonic dependence with collisional time and a monotonically increasing function is observed in Hall component, which approaches toward a saturate value at high collisional time scale. Normal component exhibit proportional and inverse-proportional relation with collisional time at two extreme limits - weak and strong field limits.

We have rigorously explored on fluidity of medium, which is quantified by shear viscosity to entropy density ratio. The expression of inverse temperature dependence of collisional time is very standard and well-known expression for massless matter in absence of magnetic field to achieve its perfect fluid nature. In presence of magnetic field, we have explored the possibility of perfect fluid establishment in normal and Hall component viscosity, where we have obtained a new analytic expression of temperature and magnetic field dependence collisional time, required to build perfect fluidity in massless matter.

Our calculation is completely based on classical picture, although we have considered Fermi-Dirac and Bose-Einstein distribution, which might be considered as quantum aspect of statistical mechanics. So we might call this description as semi-classical framework. However, complete quantum mechanical description by considering Landau levels of charge particle of medium can be considered as its immediate extension of present framework, which we are planning for future studies.

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**VI. APPENDIX**

**A. Thermodynamics of massless quark at \( B = 0 \)**

Here we have shown explicit calculation of energy density, pressure and entropy density for quasi particle system considering mass \( m = 0 \) for MB, BE, FD at \( B = 0 \).

Thermal distribution function can be written in a general way in the form

\[
f = \frac{1}{e^{\beta E} + a}
\]

where, \( a = 0 \) for MB, \( a = 1 \) for FD and \( a = -1 \) for BE statistics.

**BE:** Energy density for bosons is given by

\[
\epsilon = g \int_0^\infty \frac{d^3p}{(2\pi)^3} \frac{E}{e^{\beta E} - 1}
\]

Here \( E = p \) which gives us

\[
dp = dE
\]

and the integral becomes

\[
\epsilon = g \int_0^\infty \frac{4\pi E^3}{(2\pi)^3} \frac{1}{e^{\beta E} - 1}
\]

\[
= g \int_0^\infty \frac{E^3}{2\pi^2} \frac{1}{e^{\beta E} - 1}
\]

Substituting \( \beta E = x \) giving

\[
dE = \frac{dx}{\beta}
\]

gives us

\[
\epsilon = \frac{g}{2\pi^2\beta^4} \int_0^\infty \frac{x^3}{e^x - 1} dx
\]

This integral can be converted into a \( \zeta(s) \) function by using

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx
\]

where \( \Gamma(s) \) is the gamma function.

\[
\epsilon = g \times \frac{(k_B T)^4}{2\pi^2} \zeta(4) \Gamma(4)
\]

\[
= g \frac{\pi^2}{30} T^4
\]

Using \( \zeta(4) = \frac{\pi^4}{90} \) and \( \Gamma(4) = 3! \) The corresponding pressure density is

\[
P = \frac{\epsilon}{3} = g T^4 \frac{\pi^2}{90}
\]

The entropy density is

\[
s = \frac{\epsilon + P}{T} = 4 g \frac{\pi^2}{90} T^3
\]

**FD:** Energy density for fermion is

\[
\epsilon = g \int \frac{d^3p}{(2\pi)^3} \frac{E}{e^{\beta E} + 1}
\]

\( E = p \) which gives the integral

\[
\epsilon = \frac{g}{2\pi^2} \int_0^\infty \frac{E^3 dE}{e^{\beta E} + 1}
\]
Substituting $\beta E = x$ we get
\[ dE = \frac{dx}{\beta} \]
\[ \epsilon = \frac{g}{2 \pi^2} T^4 \Gamma(4) \left( 1 - \frac{1}{24 - 1} \right) \zeta(4) \]
\[ = \frac{7}{8} \frac{\pi^2}{30} T^4 \] (61)

Pressure density of fermions is given by
\[ P = \epsilon + \frac{\epsilon}{T} = \frac{7}{8} \frac{\pi^2}{90} T^4 \] (62)

The entropy density is given by
\[ s = \frac{\epsilon + P}{T} = \frac{7}{8} \frac{\pi^2}{90} T^3 \]

MB: For the Maxwell Boltzmann distribution the energy density is given by
\[ \epsilon = g \int_0^{\infty} \frac{d^3p}{(2\pi)^3} e^{\beta E} \] (63)

Since $E = p$ the energy density calculated by changing the integral to a beta function is
\[ \epsilon = \frac{3gT^4}{\pi^2} \] (64)

The pressure density is $P = \frac{4}{3}$
\[ P = \frac{9}{\pi^2} T^4 \] (65)

The entropy density is given by
\[ s = \frac{\epsilon + P}{T} = \frac{4g}{\pi^2} T^3 \] (66)

B. Shear viscosity and electrical conductivity of massless quark at $B = 0$

Here we have shown details derivation of $\eta(T)$ and $\sigma(T)$ for MB, BE, FD at $B = 0$ for massless quark system. The shear viscosity $\eta$ for bosons is given by
\[ \eta = \frac{g\beta}{15} \int \frac{d^3p}{(2\pi)^3} p^4 \tau_c f_0(1 + f_0) \] (67)

Here $E = p$ which gives us
\[ \eta = \frac{g\beta}{15} \frac{\tau_c}{2\pi^2} \int_0^{\infty} p^4 dp f_0(1 + f_0) \] (68)

The shear viscosity for fermions is given by
\[ \eta = \frac{g\beta}{15} \int \frac{4\pi p^2 dp}{(2\pi)^3} p^2 \tau_c f_0(1 + f_0) \]
\[ = \frac{g\beta \tau_c}{15} \int_0^{\infty} p^4 dp f_0(1 + f_0) \] (69)

The above 2 expressions are written as
\[ \eta = A \int_0^{\infty} p^4 dp f_0(1 + f_0) = AI_1 \] (70)

for fermions and
\[ \eta = A \int_0^{\infty} p^4 dp f_0(1 + f_0) = AI_2 \] (71)

for bosons, where the constant $A$ is the substitution for $A = \frac{g\beta}{30\pi^2} \tau_c$ (72)

The $I_1$ integral is evaluated as follows
\[ I = \int_0^{\infty} p^4 dp f_0(1 - f_0) \] (73)

where $f_0 = \frac{1}{e^{\beta E} + 1} = \frac{1}{e^{\beta p} + 1}$ is the distribution function for fermions.

\[ I_1 = \int_0^{\infty} p^4 dp \frac{1}{e^{\beta p} + 1} \left( 1 - \frac{1}{e^{\beta p} + 1} \right) \]
\[ = \int_0^{\infty} p^4 dp \frac{e^{\beta p}}{(e^{\beta p} + 1)^2} \]
\[ = - \frac{\partial}{\partial \beta} \int_0^{\infty} \frac{p^3}{e^{\beta p} + 1} dp \] (74)

By using the definition of $d(s)$ and $\zeta(s)$ function we solve the above integral as

\[ I_1 = - \frac{\partial}{\partial \beta} \frac{\Gamma(4)}{\beta^4} \left( 1 - \frac{1}{2^4} \right) \zeta(4) \]
\[ = - \frac{\partial}{\partial \beta} \frac{3!}{\beta^4} \frac{7}{8} \zeta(4) \]
\[ = A \frac{4!}{\beta^5} \zeta(4) \frac{7}{8} \] (75)

Thus
\[ \eta_2|_{\text{fermions}} = A \frac{4!}{\beta^5} \zeta(4) \frac{7}{8} \]

The integral for Bosons is
\[ I_2 = \int_0^{\infty} p^4 dp \frac{1}{e^{\beta p} - 1} \left( 1 + \frac{1}{e^{\beta p} - 1} \right) \]
\[ = \int_0^{\infty} p^4 dp \frac{e^{\beta p}}{(e^{\beta p} - 1)^2} \]
\[ = - \frac{\partial}{\partial \beta} \int_0^{\infty} \frac{p^3}{e^{\beta p} - 1} dp \] (76)
Substituting $\beta p = x$ we get

$$I_2 = -\frac{\partial}{\partial \beta} \left( \int_0^\infty dx \frac{x^3}{\beta^4(e^x - 1)} \right)$$  \hspace{1cm} (77)

Using the definition of $\zeta(s)$ the integral is calculated as

$$I_2 = \frac{4}{\beta^5} \Gamma(4) \zeta(4)$$  \hspace{1cm} (78)

$\eta|_{\text{Bosons}}$ is obtained as

$$\eta|_{\text{Bosons}} = A \frac{4}{\beta^5} \Gamma(4) \zeta(4)$$  \hspace{1cm} (79)

For Maxwell-Boltzmann distribution the shear viscosity is obtained as follows

$$\eta = \int_0^\infty \frac{dp}{(2\pi)^3} f_0 \tau_c$$

where $f_0 = e^{-\beta E}$ and here $E = p$.

$$\eta = \frac{g\beta}{15} \int_0^\infty \frac{d^3p}{(2\pi)^3} p^2 f_0 \tau_c$$

$$\eta = \frac{A}{\beta^5} \int_0^\infty e^{-x} x^4 dx$$

$$\eta = \frac{A}{\beta^5} \Gamma(5)$$  \hspace{1cm} (81)

where $A = \frac{3^3}{3!} \tau_c$ and $\Gamma(5) = 4!$

**Electrical Conductivity** $\sigma$ for different distributions is calculated as follows:

**MB:** For Maxwell-Boltzmann distribution electrical conductivity is

$$\sigma = \frac{q^2 g\beta}{3} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{E^2 \tau_c} f_0$$

The distribution function is $f_0 = e^{-\beta p}$ for Maxwell-Boltzmann distribution.

$$\sigma = \frac{q^2 g\beta}{3(2\pi)^2} \int_0^\infty p^2 dp [\tau_c e^{-\beta p}]$$

Substituting $\beta p = x$ we get

$$\sigma = \frac{q^2 g\beta}{3(2\pi)^2} \tau_c \int_0^\infty \frac{dx}{\beta^3 x^2 e^{-x}}$$

$$\sigma = \frac{q^2 g\beta}{3(2\pi)^2} \frac{2!}{\beta^3} \tau_c$$

$$\sigma = \frac{q^2 g}{3\pi^2 \beta^2} \tau_c$$  \hspace{1cm} (84)

**FD:** The electrical conductivity of fermions is given by

$$\sigma = \frac{q^2 g\beta}{3} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{E^2 \tau_c} f_0(1 - f_0)$$  \hspace{1cm} (85)

where the distribution function of fermions is given by

$$f_0 = \frac{1}{e^{\beta p} + 1}$$

$$\sigma = \frac{q^2 g\beta}{3 \times 2\pi^2} \int_0^\infty \frac{p^2 dp \tau_c}{(e^{\beta p} + 1)^2}$$  \hspace{1cm} (86)

Using the definition of $\zeta(s)$ the integral is evaluated to be

$$\sigma = \frac{q^2 g - 1}{2 \times 2\pi^2} \frac{\Gamma(4)}{4} \zeta(3) \tau_c$$  \hspace{1cm} (87)

where $\Gamma(4) = 3!$

**BE:** For bosons $\sigma$ is

$$\sigma = \frac{q^2 g\beta}{3 \times 2\pi^2} \int_0^\infty \frac{p^2 dp}{(e^{\beta p} - 1)^2} \tau_c$$  \hspace{1cm} (88)

where the distribution function of bosons is given by

$$f_0 = \frac{1}{e^{\beta p} - 1}$$

$$\sigma = \frac{q^2 g\beta}{3 \times 2\pi^2} \int \frac{p^2 dp \tau_c}{(e^{\beta p} - 1)^2}$$  \hspace{1cm} (89)

where the integral $I$ is given by

$$I = \int_0^\infty \frac{p^2 e^{\beta p}}{(e^{\beta p} - 1)^2} dp$$  \hspace{1cm} (90)

is solved as follows

$$I = -\frac{\partial}{\partial \beta} \int_0^\infty \frac{p}{e^{\beta p} - 1} dp$$

$$= -\frac{\partial}{\partial \beta} \left( \frac{1}{\beta^3} \right) \int_0^\infty \frac{x}{e^x - 1} dx$$

$$= \frac{2}{\beta^3} \int_0^\infty \frac{x}{e^x - 1} dx$$

$$= \frac{2}{\beta^3} \zeta(2) \Gamma(2) = \frac{2}{\beta^3} \zeta(2)$$  \hspace{1cm} (91)

Putting this in the expression for conductivity we get

$$\sigma = \frac{q^2 g\beta}{3 \times 2\pi^2} \frac{2}{\beta^3} \zeta(2) \tau_c$$  \hspace{1cm} (92)
C. Thermal average Energy

The expressions of viscosity and conductivity contain magnetic relaxation time $\tau_B$ which is $\tau_B = \frac{e^2}{qB}$. But for simplicity of calculation we will consider average energy for $\tau_B$ calculation. So, $\tau_B = \frac{d}{qB}$.

**FD:** Average energy for fermions is

$$\langle E \rangle = \frac{\int d^3p \langle E \rangle e^{-\beta E}}{\int d^3p e^{-\beta E}}$$

$$= \int_0^\infty p^3 dp e^{-\beta p}$$

$$= \frac{6}{\beta} = \frac{3}{\beta} \tag{93}$$

**MB:** Average energy for Maxwell Boltzmann distribution with $E = p$

$$\langle E \rangle = \frac{\int d^3p \langle E \rangle e^{-\beta E}}{\int d^3p e^{-\beta E}}$$

$$= \int_0^\infty p^3 dp e^{-\beta p}$$

$$= \frac{6}{\beta} = \frac{3}{\beta} \tag{94}$$

Using the definition of $\zeta(s)$ and substituting $\beta p = x$ the above integral is evaluated as

$$\langle E \rangle = \frac{\Gamma(4)}{\beta^4} \left( 1 - \frac{1}{2\pi^2} \right) \zeta(4)$$

$$\frac{\Gamma(3)}{\beta^3} \left( 1 - \frac{1}{2\pi^2} \right) \zeta(3)$$

$$= \frac{7}{2} \frac{\zeta(4)}{\zeta(3)} \tag{95}$$

**BE:** Average energy of Bosons with $E = p$

$$\langle E \rangle = \frac{\int d^3p \langle E \rangle e^{-\beta E}}{\int d^3p e^{-\beta E}}$$

$$= \int_0^\infty p^3 dp e^{-\beta p}$$

$$= \frac{6}{\beta^3} \tag{96}$$

Using the definition of $\zeta(s)$ and the substitution $\beta p = x$ the integral can be solved as

$$\langle E \rangle = \frac{1}{\beta} \frac{\zeta(4)}{\zeta(3)} \tag{97}$$

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