Linear Logic, the $\pi$-calculus, and their Metatheory: A Recipe for Proofs as Processes

FABRIZIO MONTESI, University of Southern Denmark, Denmark
MARCO PERESSOTTI, University of Southern Denmark, Denmark

The Curry-Howard correspondence between natural deduction and the $\lambda$-calculus has provided a canonical foundation for the study of typed functional languages. Initiated by Abramsky [1994], the Proofs as Processes agenda is to establish a similar foundation for the study of concurrent languages, by researching the connection between linear logic [Girard 1987] and the $\pi$-calculus [Milner et al. 1992].

To date, Proofs as Processes is still a partial success. Caires and Pfenning [2010] and Wadler [2014] showed that linear propositions correspond to session types: structured communication protocols that prescribe the observable behaviour of processes [Honda 1993]. Further, Carbone et al. [2018b], Montesi and Peressotti [2018], and Kokke et al. [2019] demonstrated that adopting devices from hypersequents [Avron 1991] shapes proofs (typing derivations) such that they correspond to the expected syntactic structure of processes in the $\pi$-calculus. What remains is reconstructing the expected metatheory of session types and the $\pi$-calculus. In particular, the hallmark of session types is session fidelity: an operational correspondence between the observable behaviours of processes and their session types, formulated via labelled transition systems. To date, this result still has to be reconstructed for Proofs as Processes, because it is unclear how a transition system for session types can arise from linear logic.

In this article, we present $\Pi LL$, a new process calculus rooted in linear logic. The key novelty of $\Pi LL$ is that it comes with a carefully formulated design recipe for Proofs as Processes, which is based on a dialgebraic view of labelled transition systems and their homomorphisms [Ciancia 2013]. Thanks to our recipe, $\Pi LL$ offers the first reconstruction of the expected labelled transition systems of session types based on linear logic, which we use to establish session fidelity for Proofs as Processes. In general, we use $\Pi LL$ to carry out the first thorough investigation of the metatheoretical properties enforced by linear logic on the observable behaviour of processes, uncovering connections with standard relations such as similarity and bisimilarity. We also prove that $\Pi LL$ and our recipe offer a robust basis for the further exploration of Proofs as Processes, by considering different features and extensions: polymorphism, higher-order communications (code mobility), and recursion. Our higher-order extension of $\Pi LL$, called HO$\Pi LL$, operates as a higher-order linear logic that enforces linear usage of process variables.

Additional Key Words and Phrases: Session Types, Linear Logic, Propositions as Types, Behavioural Theory, Higher-Order Processes

1 INTRODUCTION

“What matters most in mobile interactive systems is not values, but connectivity and mobility of processes. With or without types, the unifying feature is behaviour, and what it means to say that two different processes behave the same.”

—Robin Milner, foreword for “The $\pi$-calculus: A Theory of Mobile Processes” [Sangiorgi and Walker 2001].

Background. The $\pi$-calculus is the most influential theory of mobile processes, whereby concurrent processes can restructure networks by communicating over (channel) names [Milner et al. 1992]. Its conception has been the origin of a prolific line of research on the metatheory of mobile processes, which studies the communication behaviours that can be expressed in the $\pi$-calculus and its variants [Sangiorgi and Walker 2001]. Two concerns have been particularly relevant.
• **Behavioural theory**, which investigates techniques for comparing processes.
• **Type systems**, which syntactically check process definitions to guarantee some desirable properties about their behaviours.

On the side of behavioural theory, the most accredited notion is bisimilarity. Bisimilarity is widely accepted as the finest desirable notion of equivalence between behaviours, and its solidity has been confirmed repeatedly over more than three decades of research in both concurrency theory and other fields [Sangiorgi 2011]. In fact, bisimilarity is so robust that researchers often use it to evaluate how “good” the design of a process calculus is: the operators of the calculus should preserve bisimilarity. Having design principles like this is especially valuable because new process calculi are still developed regularly.

On the side of type systems, the situation is more unstable. The benchmark for solid foundations in this area is the Propositions as Types correspondence between the $\lambda$-calculus and natural deduction uncovered by Curry and Howard, which to date is still a major source of inspiration for the development of functional programming languages and proof assistants [Wadler 2015]. In 1994, Abramsky launched the quest for finding a similar guiding light for the future development of typed process calculi. The core of this agenda is to develop a Proofs as Processes correspondence, where proofs in Girard’s linear logic correspond to processes in the $\pi$-calculus. While achieving Proofs as Processes in a way that is fully satisfactory on the side of processes proved to be elusive [Bellin and Scott 1994], linear logic has already been tremendously influential in the development of typed process calculi. For example, it inspired the typing discipline of session types, where types prescribe the actions that a process will perform on a channel to participate in a protocol [Honda 1993]. Session types grew to be one of the most popular approaches to typing processes to date [Ancona et al. 2016; Hüttel et al. 2016], but the link to linear logic was purely inspirational and not rooted in a formal correspondence.

In 2010, Caires and Pfenning kickstarted a renaissance of the exploration of Proofs as Processes, by discovering that propositions in Intuitionistic Linear Logic can be interpreted as session types. Using sequents from Intuitionistic Linear Logic introduces a distinction between “required” and “provided” channels that is usually not present in the $\pi$-calculus. Wadler [2014] addressed the issue by revisiting Caires and Pfenning’s approach in Classical Linear Logic (CLL), which removes the distinction.

The works by Caires and Pfenning [2010] and Wadler [2014] lacked a convincing interpretation of the parallel operator, the most fundamental structural operator of process calculi: no rule in the logic corresponds directly to constructing a parallel composition. This shortcoming undermines the reconstruction of the expected labelled transition system (lts) semantics of the $\pi$-calculus and, consequently, its behavioural theory. A series of works that link Proofs as Processes to choreographies (“Alice and Bob” security protocol notation) provided a preliminary answer [Carbone et al. 2018a,b; Montesi 2013]: the parallel operator corresponds to composing hypersequents—collections of sequents, previously studied by [Avron 1991]. Inspired by this idea, Montesi and Peressotti [2018] introduced a device to classical linear logic for typing compositions of independent processes: if two processes $P$ and $Q$ implement respectively the types in the typing environments $\Gamma$ and $\Delta$, then we can type the parallel composition $P | Q$ with the hyperenvironment $\Gamma | \Delta$; this indicates that all names in $\Gamma$ are used in parallel to those in $\Delta$. The rule for composing hyperenvironments in parallel corresponds to the term constructor for composing processes in parallel, and supports proof transformations that can be used to construct an lts semantics [Montesi and Peressotti 2018]. Kokke et al. [2019] later used hyperenvironments to define a conservative extension of linear logic.

\(^1\)The term hyperenvironment was introduced later, in [Kokke et al. 2019].
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that supports an lts semantics for processes with non-blocking I/O, and proved that the behavioural theory of the resulting calculus can be explored using the standard toolbox of bisimilarity.

The results developed so far on Proofs as Processes are promising. However, they do not match what should be expected of a convincing foundation for the session-typed $\pi$-calculus yet. The transitions of a well-typed process should guarantee session fidelity: the property that all observable actions performed by the process are allowed precisely by its session types [Honda et al. 2016; Montesi and Yoshida 2013]. By contrast, it is still unclear how this result could be reconstructed for transition systems built on linear logic [Kokke et al. 2019; Montesi and Peressotti 2018]. In general, there is still no clear recipe for building calculi based on linear logic that present session fidelity and the standard principles of behavioural theory found in the $\pi$-calculus.

This is unfortunate: while previous studies of important extensions for the original $\pi$-calculus, such as polymorphism and higher-order processes, could rely on a solid core calculus equipped with a behavioural theory that carries clear principles for “good” design (e.g., that bisimilarity is a congruence), Proofs as Processes has not reached this position yet. Indeed, while many calculi based on linear logic resemble each other, there is still no version that can claim to be a minimal and satisfactory foundation. For example, the behavioural theory in [Kokke et al. 2019] assumes non-blocking I/O, which is not a basic feature of process calculi but rather an extension [Merro and Sangiorgi 2004], and some of the transition rules therein are not structural: instead of inspecting only the outer-most term constructor, they have side-conditions that traverse the whole syntactic structure of the process. These side-conditions are not present in standard presentations of the $\pi$-calculus, and go against the philosophy of the Structural Operational Semantics approach by Plotkin [2004]. Furthermore, Kokke et al. [2019] sacrificed polymorphism, which was instead present in previous studies [Montesi and Peressotti 2018; Wadler 2014].

In summary, while previous results suggest that transition systems based on hyperenvironments could play an important role in the relationship between linear logic and process calculi, it is still unclear whether they can form a solid foundation that Proofs as Processes can stand upon.

This article. We present a design recipe for the development of session-typed process calculi based on linear logic, which addresses the shortcomings of previous approaches and ties up their loose ends.

Our recipe is unifying: it is carefully formulated to support a series of new metatheoretical results for Proofs as Processes that justify our lts semantics and explain the role of hyperenvironments, both from the point of view of behavioural theory and that of type systems. In particular, we achieve session fidelity and discover new connections between the metatheory of derivations and behavioural theory.

Our recipe is also robust: starting from a core calculus, we test the recipe with different features, including polymorphism and code mobility. These extensions preserve our properties, and similarity and bisimilarity are congruences in all the related calculi.

The main contributions presented in this article are:

1. We present our recipe by developing $\piLL$: a new process calculus rooted in linear logic and hyperenvironments. In $\piLL$, proofs (typing derivations) are processes [Abramsky 1994; Bellin and Scott 1994], propositions are session types [Caires and Pfenning 2010; Wadler 2014] and parallel composition of processes corresponds to the parallel composition of hyperenvironments [Carbone et al. 2018b; Kokke et al. 2019; Montesi and Peressotti 2018]. Inspired by [Montesi and Peressotti 2018], the semantics of $\piLL$ is based on hyperenvironments and a labelled transition system (lts) for typing derivations that follows the Structural Operational Semantics approach by Plotkin [2004].
The key novelty is our recipe. By adopting a dialgebraic view of labelled transition systems and their homomorphisms [Ciancia 2013], which we apply for the first time in this context, we systematically extract from the lts of derivations an lts for untyped processes and an lts for typing environments. The former models the expected process dynamics of a session-typed π-calculus. We show that well-typed processes enjoy erasure and preservation of typability under execution.

Our lts for typing environments is the first observable semantics of session types rooted in linear logic. Session type dynamics give an abstract view of the execution of protocols. Thanks to this transition system, we can finally prove session fidelity for Proofs as Processes. πLL is also the first process calculus rooted in linear logic to support an lts semantics and behavioural polymorphism.

(2) By investigating the metatheory of process dynamics in πLL, we perform the first thorough investigation of the properties enforced by first-order linear logic and hyperenvironments on the observable behaviour of processes. Bisimilarity and similarity for πLL are defined as expected. We prove that they are, respectively, a congruence and a precongruence.

Our new lts of typing environments and similarity yield an elegant characterisation of session fidelity: well-typed processes are simulated by their types. For well-typed processes, similarity actually characterises type inhabitancy: if a well-typed process is simulated by some typing environment, then the process necessarily inhabits it (this is derivable in the type system). Furthermore, bisimilarity is a sound type equivalence check: for any two well-typed processes, if the two processes are bisimilar then they inhabit exactly the same types. Parallelism has no side-effects. If a process is ready to perform actions on different channels, then the order in which these actions are executed does not matter: they can be executed simultaneously or serially in any order.

We also uncover the first behavioural interpretation of parallel composition of hyperenvironments, by establishing a correspondence between the parallel operators for hyperenvironments and processes in terms of strong bisimilarity. Namely, if a process is typed by the parallel composition of two hyperenvironments, then it can be rewritten as a strongly-bisimilar parallel composition of two processes. As we are going to see, this rewriting turns out to be a useful principle for proving other results, including readiness (a generalisation of progress) and the soundness of our proof theory wrt classical linear logic.

(3) We develop HOπLL, an extension of πLL to code mobility. The proof theory of HOπLL introduces rules for assuming that a sequent can be proven and providing a witness for such an assumption. On the process calculus side, these rules correspond to receiving and sending a process, respectively. Differently from previous work on higher-order process calculi, HOπLL introduces a resource interpretation of code, from which we gain control on how many times transmitted code can be used.

Sangiorgi [1993] established that that code mobility can be simulated by channel mobility in the π-calculus. The same holds for HOπLL and πLL. Interestingly, our encoding from HOπLL to πLL leverages the translation of proofs from πLL to CLL, in particular the rewriting of processes within strong bisimilarity that makes parallelism syntactically manifest.

(4) We show that adding recursion and infinite protocols to πLL does not require extending its syntax or semantics. Rather, the addition of a typing rule for expanding types based on type equations is sufficient, because it introduces the feature of having self-calling servers. This feature is powerful enough for modelling recursive procedures and the standard divergent process Ω from process calculi.
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We believe that our results demonstrate that πLL and our recipe form a solid candidate foundation for the Proofs as Processes agenda, by finally putting in harmony the two metatheoretical concerns mentioned at the beginning: behavioural theory and type systems.

Structure of the paper. Section 2 presents πMLL: a fragment of πLL that corresponds to the multiplicative fragment of linear logic. πMLL suffices to present the key ideas of our development. We extend πMLL to full linear logic in Section 4. The resulting calculus, called πLL, can express choices (associatives), replicated processes (exponentials), and polymorphism (first-order quantifiers). In Section 5, we present HOπLL, the calculus that extends πLL to code mobility, and the translation from HOπLL to πLL. We then introduce how our theory can be extended to (potentially infinite) recursive protocols in Section 6. We discuss related work in Section 7 and conclude in Section 8.

2 MULTIPLICATIVE FRAGMENT OF πLL

We start our study from the multiplicative fragment of πLL, πMLL for short (following the nomenclature used for fragments of linear logic). This is the fragment of πLL that corresponds to the multiplicative fragment of linear logic. πMLL is minimalistic, yet it suffices to explain our development.

2.1 Processes and Typing

Processes. Programs in πMLL are processes \((P, Q, R)\). Processes communicate by using names \((x, y, z)\), which represent endpoints of sessions. Our typing discipline will enforce that sessions are binary, i.e., each session involves two endpoints. We assume an infinite set of names, and that equality of names is decidable. The process terms of πMLL are defined by the following grammar.

\[
P, Q ::= x[y].P \quad \text{output } y \text{ on } x \text{ and continue as } P
\]

\[
| x(y).P \quad \text{input } y \text{ on } x \text{ and continue as } P
\]

\[
| x[].P \quad \text{output (empty message) on } x \text{ and continue as } P
\]

\[
| x().P \quad \text{input (empty message) on } x \text{ and continue as } P
\]

\[
| (\nu xy) P \quad \text{name restriction, “cut”}
\]

\[
| P \mid Q \quad \text{parallel composition of processes } P \text{ and } Q
\]

\[
| 0 \quad \text{terminated process}
\]

In defining the terms for communication actions, we follow Wadler’s convention of indicating output with square brackets “[−]” and input with round parentheses “(−)”. Term \(x[y].P\) allocates a fresh name \(y\), outputs \(y\) over \(x\) and then proceeds as \(P\). Dually, term \(x(y).P\) inputs a name \(y\) over \(x\) and then proceeds as \(P\). Names that are sent or received are bound (\(y\) in our syntax for output and input). Terms \(x[].P\) and \(x().P\) respectively model output and input with no content. A restriction term \((\nu xy) P\) connects the endpoints \(x\) and \(y\) of \(P\) to form a session, thus enabling communications from \(x\) to \(y\) and vice versa. Restriction hides the endpoints \(x\) and \(y\) from the context. Term \(P \mid Q\) is the parallel composition of two processes \(P\) and \(Q\). Term \(0\) is the terminated process, which also acts as the unit of parallel composition. We often omit trailing 0s in examples.

In both the communication terms \(x[y].P\) and \(x(y).P\), the name \(y\) is bound to the continuation \(P\). In the restriction term \((\nu xy) P\), both \(x\) and \(y\) are bound to \(P\). This gives rise to the expected notions of free names, bound names, and α-renaming for processes [Sangiorgi and Walker 2001]. We write \(fn(P)\) and \(bn(P)\) for, respectively, the free and bound names of a process \(P\).

Example 2.1. The following process, \(\text{Latch}_{xyz}\), models a simple linear latch that can concurrently receive signals on the endpoints \(x\) and \(y\). When they are both received, it signals the environment of the event through endpoint \(z\).

\[
\text{Latch}_{xyz} \equiv (\nu x_1 x_2)(\nu y_1 y_2)(x().x_1[] \mid y().y_1[] \mid x_2().y_2().z[])\]
Remark 2.2. \(\pi\text{MLL}\) is essentially a fragment of the internal \(\pi\)-calculus, the variant of the \(\pi\)-calculus where input and output are symmetric [Sangiorgi 1996]. Having symmetric input and output primitives gives a more elegant behavioural theory without sacrificing expressivity: just as for the internal \(\pi\)-calculus, we shall see in Example 4.1 how the typical “output of a free name” primitive can be recovered by using forwarders.

Remark 2.3. In the original \(\pi\)-calculus, two processes in parallel can communicate by using the same name \(x\) on their input and output actions, and restriction has the form \((\nu x)P\). Here we use the restriction term \((\nu x y)P\), which was originally proposed by Vasconcelos [2012] for session-typed calculi and then later adopted in some presentations of process calculi typed with linear logic [Carbone et al. 2016; Kokke et al. 2019]. We are going to come back to the reason behind this choice when we present the typing rules of \(\pi\text{MLL}\).

Types. Types in \(\pi\text{MLL}\) \((A, B, C, \ldots)\) are the propositions of the multiplicative fragment of classical linear logic (CLL) [Girard 1987]. A type specifies how an endpoint is used. We recall the syntax of propositions in multiplicative CLL below, and describe how they are interpreted as types—the interpretation is the same as in [Carbone et al. 2016].

\[
A, B \coloneqq A \otimes B \quad \text{send } A, \text{ proceed as } B \\
| A \otimes B \quad \text{receive } A, \text{ proceed as } B \\
| 1 \quad \text{empty output, unit for } \otimes \\
| \bot \quad \text{empty input, unit for } \otimes
\]

Recall that, in CLL, each proposition has a dual. We write \(A^\perp\) for the dual of type \(A\). We will use the notion of duality to check that two endpoints are used in compatible ways. Duality is defined inductively as follows.

\[(A \otimes B)^\perp = A^\perp \otimes B^\perp \quad (A \otimes B)^\perp = A^\perp \otimes B^\perp \quad 1^\perp = \bot \quad \bot^\perp = 1\]

According to the behavioural interpretation of types, duality essentially checks that each send action has a corresponding receive action. As usual, duality is an involution: \((A^\perp)^\perp = A\).

Environments. An environment \((\Gamma, \Delta, \ldots)\) associates names to types. We write environments as lists; for example, \(\Gamma = x_1 : A_1, \ldots, x_n : A_n\) associates each \(x_i\) to its respective type \(A_i\), for \(1 \leq i \leq n\). All names in an environment are distinct. Environments allow for exchange, that is, order in environments is ignored. Environments can be combined when they do not share names: assuming that \(\Gamma\) and \(\Delta\) do not share names, \(\Gamma, \Delta\) is the environment that consists exactly of all the associations in \(\Gamma\) and those in \(\Delta\). Environments carry the structure of a (partial) commutative monoid with “\(\cdot\)”, acting as sum and the empty environment \(\bullet\) as unit: we equate environments according to the following rules, for all \(\Gamma, A, \Delta, \Xi\).

\[
\Gamma, \bullet = \Gamma \quad \Gamma, A = A, \Gamma \quad (\Gamma, \Delta), \Xi = \Gamma, (\Delta, \Xi)
\]

Environments specify how names are used, but they do not specify whether they are used independently. The next ingredient, hyperenvironments, deals exactly with this aspect.

Hyperenvironments. A hyperenvironment \((G, H, \ldots)\) is a collection of environments, which are composed by the parallel operator “\(\parallel\)”; for example, given some environments \(\Gamma_1, \ldots, \Gamma_n\), then \(G = \Gamma_1 \parallel \cdots \parallel \Gamma_n\) is a hyperenvironment. All names in a hyperenvironment are distinct. Like environments, hyperenvironments can be combined when they do not share names: assuming that the names in \(G\) and \(H\) are all different, \(G \parallel H\) is the hyperenvironment containing exactly all the environments in \(G\) and those in \(H\). We write \(\emptyset\) for the empty hyperenvironment, i.e., the hyperenvironment containing no environments. Hyperenvironments carry the structure of a (partial) commutative
monoid with “|” acting as sum and \( \emptyset \) as unit: we equate hyperenvironments according to the following rules, for all \( G, H, \) and \( I \).

\[
G | \emptyset = G \quad G | H = H | G \quad (G | H) | I = G | (H | I)
\]

As we are going to see, separation of (hyper)environments by “|” has meanings related to independence of derivability and process behaviour.

**Judgements.** A judgement \( \vdash P : G \) states that process \( P \) uses its free names according to \( G \). Judgements can be derived by using the inference rules displayed in Figure 1.

The only axiom is **rule** \( \text{mix}_0 \), which types the inert process \( 0 \) with the empty hyperenvironment. **Rule mix** types the parallel composition of two processes, by composing their respective hyperenvironments. The information that the two premises are proven independently is recorded explicitly by the use of the parallel operator in the hyperenvironment of the conclusion. This information allows us to reformulate rules of CLL that require independent premises as rules that require a single premise with independent environments (denoted by the presence of “|”). For example, the standard cut rule of CLL becomes **rule** \( \text{cut}_\text{sc} \). The rule types a restriction \( (vxy)P \) by checking that the endpoints \( x \) and \( y \) are used by parallel components in \( P \) (the environments in the premise are separated by “|”) in a dual way (as usual in CLL).

Moving to the logical rules, **rule** \( \otimes \) is the standard rule of CLL, but reformulated to use hyperenvironments following the same intuition that we discussed for **rule** \( \text{cut}_\text{sc} \). When typing an output \( x[y].P \), we require that the names \( x \) and \( y \) are used independently (in parallel) in \( P \). This is standard [Caires and Pfenning 2010; Wadler 2014], and it is going to be important when reasoning about progress. **Rule 1** types an empty output, requiring the continuation to have no free names.\(^2\) **Rules** \( \otimes \) and \( \bot \) are exactly as in CLL, and respectively type the input of a name and empty input.

We range over typing derivations with the letters \( D, E, \) and \( F \). Also, we write \( \vdash \hat{P} : G \) whenever a derivation \( D \) has the judgement \( \vdash P : G \) as conclusion.

In the remainder, we say that a process \( P \) is well-typed if there exist some \( D \) and \( G \) such that \( \vdash D \vdash P : G \).

\(^2\)For now, this means that the continuation of an empty output is necessarily term \( 0 \), but this is not going to be the case when we extend \( \pi \text{LL} \) to recursive types in Section 6.
Example 2.4. The process \textit{Latch}_{xyz} from Example 2.1 has the typing \( \vdash \text{Latch}_{xyz} :: x : \perp, y : \perp, z : 1 \), as shown by the following derivation. For precision, we report trailing 0s.

\[
\begin{array}{c}
\vdash 0 :: \perp \quad \text{MIX}_0 \\
\vdash x_1 :: x_1 : 1 \quad \text{MIX}_0 \\
\vdash x().x_1 :: y_1 : 1, x_1 : \perp \\
\vdash y().y_1 :: y_1 : 1, y : \perp \\
\vdash y().y_1 :: x_2().y_2().z :: 1, y : \perp \\
\vdash x().x_1 :: y().y_1 :: x_2().y_2().z :: 1, x : \perp \\
\vdash (\forall y_1.x_2() (x().x_1 :: y().y_1 :: x_2().y_2().z :: 1), x : \perp \\
\end{array}
\]

2.2 Operational Semantics of Derivations

We define a labelled transition system (LTS) for derivations by following the approach originally proposed by Montesi and Peressotti [2018], which applies the Structural Operational Semantics (SOS) style by Plotkin [2004]. Specifically, we view:

- inference rules as operations of a (sorted) signature;
- derivations as terms generated by this signature;
- (labelled) transformations of derivations as (labelled) transitions;
- and a specification of rules for deriving transformations of derivations as an SOS specification.

In Section 2.3, we will use the LTS for derivations to obtain corresponding semantics for processes and types.

As an example, consider a derivation that ends with an application of rule \( \perp \).

\[
\begin{array}{c}
\vdash P :: \Gamma \\
\vdash x().P :: \Gamma, x : \perp \\
\end{array}
\]

We can view rule \( \perp \) as the outermost operation used in the derivation: the subderivation \( \mathcal{D} \) of the premise \( \vdash P :: \Gamma \) is the only argument of the operation and \( x \) is a parameter (operations are on derivations). Notice that this operation corresponds to the term constructor \( x().(\cdot) \) in the syntax of processes, which in this example takes the continuation \( P \) (corresponding to the derivation \( \mathcal{D} \)) as argument. In the \( \pi \)-calculus, such terms define observable actions that come with corresponding transition rules, where the target (a.k.a. \textit{derivative}) is the operation argument and the transition label represents the applied operation. Similarly, in our setting, we can define the transition axiom below—for readability, we \( \boxed{} \) proofs.

\[
\begin{array}{c}
\vdash P :: \Gamma \\
\vdash x().P :: \Gamma, x : \perp \\
\end{array}
\Rightarrow
\begin{array}{c}
\vdash \mathcal{D} \\
\vdash P :: \Gamma \\
\end{array}
\]

The label \( x() \) represents unambiguously the rule application that we consume in the transition (for convenience, we use as labels the same syntactic form of the corresponding action prefix).

Following the same methodology outlined for rule \( \perp \), we obtain the following three axioms for the semantics of rules 1, \( \otimes \) and \( \oslash \):
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The labels for the transitions discussed so far correspond to the action prefixes in πMLL and form the set $\text{Act} = \{x[], x(), x[y], x(y) \mid x, y \text{ names}\}$.

For rule rule mix, we define three rules: two for executions that involve only either the left or the right component (rules PAR1 and PAR2) and one where the two components interact (rule SYN).

**Rules PAR1 and PAR2** allows for a parallel component to perform an action independently of the other, provided the usual hygienic condition on the bound names of the action from one component being different from the free names in the other parallel component. This condition arises from the requirement of having distinct names in hyperenvironments. Pleasantly, this condition is standard in the internal π-calculus (cf. [Sangiorgi 1993]), so our development can be seen as a logical justification of it. Rule SYN synchronises two actions $l$ and $l'$ performed by parallel components, yielding a transition labelled by the parallel composition ($l \parallel l'$). When writing ($l \parallel l'$), we require that $l$ and $l'$ are not pairs themselves (i.e., $l, l' \in \text{Act}$), as interactions in πMLL have two parties. Also, the order in such pairs of labels is ignored: for all $l$ and $l'$, $(l \parallel l')$ and $(l' \parallel l)$ are considered the same. The condition on disjointness of bound names in rule SYN arises from the well-formedness of the resulting hyperenvironments.

As in [Vasconcelos 2012], communication in πMLL takes place when two compatible actions can be performed over endpoints connected by a restriction term. This is obtained by transition rules that simplify applications of rule cut. The rule below simplifies a cut on units, resulting in
an internal transition with the standard label $\tau$.

\[
\frac{\Downarrow}{\vdash P : G \mid x : 1 \mid \Gamma, y : \bot} \quad (x[\cdot]y(\cdot))}
\]

\[
\frac{\Downarrow}{\vdash P : G \mid x : 1 \mid \Gamma, y : \bot} \quad \text{CUT}
\]

Similarly, the following rule simplifies a cut between a $\otimes$ and a $\ltimes$.

\[
\frac{\Downarrow}{\vdash P : G \mid \Gamma, \Delta, x : A \otimes B \mid \Xi, y : A^\bot \otimes B^\bot} \quad \Gamma \quad \text{CUT}
\]

\[
(\forall x') P' : G \mid \Gamma, x : A \mid x' : A \mid \Xi, y : B^\bot, \ y' : A^\bot
\]

Both rules for simplifying an application of rule CUT assume a transition that deconstructs dual propositions associated to the names connected by the cut. These transformations do not interact with the context nor have any effect on the types of the conclusions, as expected by transitions labelled with $\tau$ in typed process calculi. These rules are the last ones to introduce labels for $\pi$MLL. Formally, the set $\text{Lbl}$ of labels for $\pi$MLL is $\{\tau, !, (l \parallel l') \mid l, l' \in \text{Act}\}$.

Lastly, the following rule performs the standard propagation of unrestricted actions, as found in the $\pi$-calculus.

\[
\frac{\Downarrow}{\vdash P : G \mid \Gamma, x : A \mid \Delta, y : A^\bot} \quad \text{RES}
\]

\[
\frac{\Downarrow}{\vdash P : G \mid \Gamma, x : A \mid \Delta, y : A^\bot} \quad 
\]

\[
\frac{\Downarrow}{\vdash P : G \mid \Gamma, x : A \mid \Delta, y : A^\bot} \quad \text{CUT}
\]

The rule below ensures that $\alpha$-convertible derivations support the same transformations.

\[
\frac{\Downarrow}{P =_{\alpha} Q \quad \vdash Q : G} \quad \text{MIX}
\]

\[
\frac{\Downarrow}{\vdash Q : G} \quad \text{CUT}
\]

Definition 2.5. The lts of derivations for $\pi$MLL, denoted $\text{lt}_{\text{dis}}$, is the triple $(\text{Der}, \text{Lbl}, \rightarrow)$, where:

- The set $\text{Der}$ is the set of typing derivations for $\pi$MLL.
- The set $\text{Lbl}$ is the set of transition labels: $\text{Lbl} = \{\tau, !, (l \parallel l') \mid l, l' \in \text{Act}\}$.
- The relation $\rightarrow \subseteq \text{Der} \times \text{Lbl} \times \text{Der}$ is the least relation closed under the SOS rules stated in this subsection.

Example 2.6. Consider the following derivation $\mathcal{D}$:

\[
\mathcal{D} = \quad \Downarrow \quad \text{MIX}
\]

\[
\Downarrow \quad \text{CUT}
\]

\[
\frac{\Downarrow}{\vdash x[x'].Q : \Gamma, \Gamma', x : A \otimes B \mid y(y').z().R : \Delta, y : A^\bot \otimes B^\bot, z : \bot}
\]

\[
\frac{\Downarrow}{\vdash (vxy)(x[x'].Q | y(y').z().R) : \Gamma, \Gamma', \Delta, z : \bot}
\]
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where

$$
\mathcal{E} = \frac{\vdash Q : \Gamma, x : A | \Gamma', x' : B}{\vdash x[x']Q : \Gamma, x : A \otimes B}
$$

and

$$
\mathcal{F} = \frac{\vdash R : \Delta, y : A^\perp, y' : B^\perp}{\vdash z().R : \Delta, y : A^\perp, y' : B^\perp, z : \bot}
$$

for some $\mathcal{E}'$ and $\mathcal{F}'$. Then, we have the following transitions.

$$
\frac{\frac{\vdash Q : \Gamma, x : A | \Gamma', x' : B}{\vdash Q | z().R : \Gamma, x : A | \Gamma', x' : B | \Delta, y : A^\perp, y' : B^\perp, z : \bot}}{\frac{\vdash (vxy')(Q | z().R) : \Gamma, x : A | \Gamma', \Delta, y : z : \bot}{\frac{\vdash (vxy)(vxy')(Q | z().R) : \Gamma, \Gamma', \Delta, z : \bot}}}
$$

(1)

3. From Transitions for Derivations to Processes and Environments

Process calculi, like the $\pi$-calculus, typically come with a semantics for processes that does not depend on typing. The point of session types is then to guarantee that observable actions performed by a (well-typed) process match what has been promised in its types. This property is known as session fidelity [Honda et al. 2016; Montesi and Yoshida 2013]. In this subsection, we show a recipe for obtaining all these ingredients and their properties directly from our semantics of derivations.

2.3.1 Semantics of Processes. Observe that the transitions in trace 1 make sense even if we completely ignore types, and they resemble the expected ones in an untyped calculus. Specifically, the following transitions for processes are taken by reading off the red parts in the conclusions of the derivations in the transitions from trace 1.

$$
(vxy)(x[x']Q | y(y').z().R) \xrightarrow{\tau} (vxy)(vxy')(Q | z().R) \xrightarrow{z()} (vxy)(vxy')(Q | R)
$$

We formalise the intuitive idea of "erasing typing from derivations" as a function $\text{proc}$ that goes from the set of derivation $\text{Der}$ to the set of process terms $\text{Proc}$. This function is defined by recursion on the structure of derivations: it just homomorphically maps the application of a rule to the term constructor introduced in its conclusion, as shown below in the case of rule $\bot$.

$$
\text{proc} \left( \frac{\vdash P : \Gamma}{\vdash x().P : \Gamma, x : \bot} \right) \triangleq x().\text{proc} \left( \frac{\vdash P : \Gamma}{\vdash P : \Gamma} \right)
$$
Because the structure of a well-typed πMLL term drives the structure of its derivation, this definition is equivalent to taking a derivation to the process term in its conclusion:

\[ \text{proc} \left( \frac{D}{P \vdash G} \right) = P. \]

In defining a semantics for process terms we want to ensure that it agrees with the semantics of derivations for well-types terms. That is, for any derivation \( D \in \text{Der} \) and \( l \in \text{Lbl} \):

- if \( D \xrightarrow{l} D' \) then \( \text{proc}(D) \xrightarrow{l} \text{proc}(D') \), and
- if \( \text{proc}(D) \xrightarrow{l} P' \) then \( D \xrightarrow{l} D' \) for some \( \text{proc}(D') = P' \).

We call this property of a semantics for processes erasure.

As we will see in Section 3, erasure plays a crucial role in the metatheory of πMLL and it is reasonable to expect the same from extensions of the language that want to preserve its metatheory. Because many of these extensions introduce changes to the syntax of processes, typing judgements, and semantics (cf. Sections 4 and 5), we need a uniform definition that abstracts from these aspects. To this end, we observe that the characterisation of erasure given above is equivalent to the requirement that the function \( \text{proc} : \text{Der} \to \text{Proc} \) carries a homomorphism of labelled transition systems: from the lts of derivations, \( \text{lts}_D \), to the lts of processes, say \( \text{lts}_P \). The function \( \text{proc} \) carries such homomorphism if it makes the diagram below commute where \( \mathcal{P}(\quad) \) and \( \mathcal{L}(\quad) \) are the functors over the category Set of sets and functions defined aside (\( X \) stands for a set and \( f : X \to Y \) for a function from \( X \) to \( Y \)).

\[
\begin{array}{ccc}
\mathcal{L}(\text{Der}) & \xrightarrow{\mathcal{L}(\text{proc})} & \mathcal{L}(\text{Proc}) \\
\text{lts}_D & & \text{lts}_P \\
\mathcal{P}(\text{Der}) & \xrightarrow{\mathcal{P}(\text{proc})} & \mathcal{P}(\text{Proc})
\end{array}
\]

The presentation of labelled transition systems used in the diagram above is equivalent to the one based on triples we used before. Inf act, one has the following bijection (we use a generic set of labels \( L \) instead of \( \text{Lbl} \) in the definition of \( \mathcal{L}(\quad) \)):

- a triple \((S, L, \rightarrow)\) corresponds to the function mapping \((s, l) \in S \times L\) to \(\{s' | s \xrightarrow{l} s'\} \in \mathcal{P}(S)\);
- a function \(\text{lts} : S \times L \to \mathcal{P}(S)\) corresponds to the triple \((S, L, \rightarrow)\) such that \(s \xrightarrow{l} s'\) iff \(s' \in \text{lts}(s, l)\).

In the sequel, we will use these two representations interchangeably.

**Definition 2.7 (Erasure).** Let \( \mathcal{L}(\quad) \) be a functor over Set and fix an type erasure function \( \text{proc} : \text{Der} \to \text{Proc} \). A labelled transition system \( \text{lts}_1 : \mathcal{L}(\text{Proc}) \to \mathcal{P}(\text{Proc}) \) enjoys erasure wrt \( \text{lts}_2 : \mathcal{L}(\text{Der}) \to \mathcal{P}(\text{Der}) \) if the function \( \text{proc} \) is a homomorphism from \( \text{lts}_2 \) to \( \text{lts}_1 \).

The principle of “erasing types” can be applied also to the SOS specification for \( \text{lts}_d \) to derive an SOS specification for processes that is independent from typing. For every rule

\[
\begin{array}{c}
D_1 \xrightarrow{l_1} D'_1 \quad \ldots \quad D_n \xrightarrow{l_n} D'_n \quad \text{condition on names}
\end{array}
\]

\[
D \xrightarrow{l} D'
\]

We adopt a dialgebraic interpretation of labelled transition systems and their homomorphisms [Ciancia 2013] instead of the more common coalgebraic interpretation \((\text{Der} \to \mathcal{P}(\text{Der})^{\text{Lbl}})\) because it simplifies the treatment of higher-order behaviours in Section 5.
in the SOS specification for derivations, we introduce a corresponding rule

\[ \text{proc}(D_1) \xrightarrow{I_1} \text{proc}(D'_1) \quad \ldots \quad \text{proc}(D_n) \xrightarrow{I_n} \text{proc}(D'_n) \quad \text{condition on names} \]

\[ \text{proc}(D) \xrightarrow{I} \text{proc}(D') \]

to the SOS specification for processes, obtained by point-wise application of \text{proc}. The resulting SOS specification is displayed in Figure 2.

**Definition 2.8.** The lts of processes for \(\pi\)-MLL, denoted \(\text{Lts}_p\), is the triple \((\text{Proc}, \text{Lbl}, \longrightarrow)\) where

- \(\text{Proc}\) is the set of process terms for \(\pi\)-MLL,
- \(\text{Lbl}\) is the set of labels for \(\pi\)-MLL (see Definition 2.5),
- \(\longrightarrow \subseteq \text{Proc} \times \text{Lbl} \times \text{Proc}\) is the least relation closed under the SOS rules in Figure 2.

The lts of processes enjoys erasure (wrt \(\text{Lts}_d\)). The proof of this result will become clear later, when we present the metatheory of \(\pi\)-MLL.

**Theorem 2.9 (Erasure).** \(\text{Lts}_p\) enjoys erasure wrt \(\text{Lts}_d\).

**Remark 2.10.** Erasure constrains only the semantics of well-typed processes and allows us to attribute any meaning to ill-typed processes like \(x().x[].0\). By selecting a semantics for processes that is based on the SOS specification of derivations we obtain some level of “uniformity” and a familiar semantics also for ill-typed processes. This decision allows us to generalise some of the metatheory of \(\pi\)LL (e.g., Theorem 3.5) to all processes and not just well-typed ones. Indeed, these results depend only on properties of the SOS specification for processes and not on their typing.

From erasure, we get the standard result of typability preservation as a corollary.

**Corollary 2.11 (Typability Preservation).** Let \(P\) be well-typed. Then, \(P \xrightarrow{I} P'\) implies that \(P'\) is well-typed.

**Example 2.12.** Recall the process \(\text{Latch}_{xyz}\) from Example 2.1.

\[ \text{Latch}_{xyz} \triangleq (\langle x_1, x_2 \rangle (\langle y_1, y_2 \rangle (x().x[] | y().y[] | x_2().y_2().z[]))) \]
The following transitions are valid, showing the concurrency informally described in Example 2.1.

\[
\begin{align*}
\text{Latch}_{xyz} & \quad x() \rightarrow y() \rightarrow \tau \rightarrow z[] \rightarrow 0 \\
\text{Latch}_{xyz} & \quad x() \rightarrow y() \rightarrow \tau \rightarrow z[] \rightarrow 0
\end{align*}
\]

2.3.2 **Semantics of Typing Environments.** Recall again the derivation transitions in trace 1, and observe that \(\tau\)-transitions (transitions labelled with \(\tau\)) do not alter environments, while transitions with observable actions do: in the second transition the typing of name \(z\) disappears. In the area of session types, the safety of these modifications to types by observable actions is known as session fidelity.

Previous work not based on linear logic formalised session fidelity as an operational correspondence between processes and their environments [Honda et al. 2016; Montesi and Yoshida 2013]. So far, this result has not been reproduced in the research line of Proofs as Processes, because it was unclear how the lts of environments could be justified by the proof theory of linear logic.

As it sometimes happens in works inspired by the Curry-Howard correspondence, the answer is obvious as soon as it is shown. Just like we can read off the red part of derivation transitions to obtain transitions of (hyper)environments, the transitions below are obtained by reading the blue part of \(\text{trace 1}\).

\[
\begin{align*}
\Gamma, \Gamma', \Delta, z: \bot & \rightarrow \tau \rightarrow \Delta, \Gamma', \Gamma, z: \bot \rightarrow z() \\
& \rightarrow \Gamma, \Gamma', \Delta
\end{align*}
\]

The hyperenvironment of a well-typed process represents its interface to the context. Hence, transitions of hyperenvironments give us an immediate way to formulate what “going wrong” means for a well-typed process: to perform an observable action that cannot be matched by a transition of its hyperenvironment.

To obtain a semantics for hyperenvironments, we follow the same approach shown for the semantics of process terms. Let \(\text{env}\) be the function from the set of derivations \(\text{Der}\) to the set of hyperenvironments \(\text{Env}\) that maps a derivation to the hyperenvironment in its conclusion:

\[
\text{env} \left( \vdash P :: \mathcal{G} \right) = \mathcal{G}.
\]

We use the semantics of hyperenvironments to define a notion of safety that captures session fidelity. Specifically, the notion of session fidelity wrt a semantics for environments \(\text{lts}_e = (\text{Env}, \text{Lbl}, \rightarrow)\) requires that, given \(\vdash P :: \mathcal{G}\), \(P \xrightarrow{l} P'\) for any \(l\) and \(P'\) implies \(\mathcal{G} \xrightarrow{l} \mathcal{G}'\) for some \(\mathcal{G}'\) such that \(\vdash P' :: \mathcal{G}'\). Like erasure, session fidelity can be generalised using lts-homomorphisms: the key observation is that the span of functions (\(\text{proc}, \text{env}\)) identifies all \(P\) and \(\mathcal{G}\) such that \(\vdash P :: \mathcal{G}\) as well as every witness of this judgement (every \(\mathcal{D}\) such that \(\text{proc}(\mathcal{D}) = P\) and \(\text{env}(\mathcal{D}) = \mathcal{G}\)).

Consider for a moment the lts of derivations (by erasure, this is equivalent to considering well-typed processes). Then, the notion of session fidelity above corresponds to the requirement that if \(\mathcal{D} \xrightarrow{l} \mathcal{D}'\) then \(\text{env}(\mathcal{D}) \xrightarrow{l} \text{env}(\mathcal{D}')\) or, in terms of homomorphisms, that the function \(\text{env}: \text{Der} \rightarrow \text{Env}\) carries a lax homomorphism of labelled transition systems from \(\text{lts}_d\) to \(\text{lts}_e\), as shown by the diagram below (\(\mathcal{L}(\cdot)\) and \(\mathcal{P}(\cdot)\) are the functors over Set defined above).

\[
\begin{array}{ccc}
\mathcal{L}(\text{Der}) & \xrightarrow{\mathcal{L}(\text{env})} & \mathcal{L}(\text{Env}) \\
\text{lts}_d & \subseteq & \text{lts}_e \\
\mathcal{P}(\text{Der}) & \xrightarrow{\mathcal{P}(\text{env})} & \mathcal{P}(\text{Env})
\end{array}
\]
Then, to capture session fidelity for processes we only need the second condition of erasure, which states that if \( \text{proc}(D) \xrightarrow{\Gamma} P' \) then \( D \xrightarrow{\Gamma} D' \) for some \( D' \) such that \( \text{proc}(D') = P' \). This condition is equivalent to the requirement that \( \text{proc} \) carries an oplax homomorphism.

**Definition 2.13 (Session fidelity).** Let \( \mathcal{L}(\cdot) \) be a functor over Set and fix the erasure functions \( \text{proc} : \text{Der} \to \text{Proc} \) and \( \text{env} : \text{Der} \to \text{Env} \). A labelled transition system \( \text{lts}_1 : \mathcal{L}((\text{Proc}) \to \mathcal{P}((\text{Proc})) \) enjoys session fidelity wrt \( \text{lts}_2 : \mathcal{L}((\text{Der}) \to \mathcal{P}((\text{Der})) \) iff the diagram below holds for some \( \text{lts}_3 \).

\[
\begin{array}{ccc}
\mathcal{L}((\text{Proc}) & \subseteq & \mathcal{L}(\text{Der})
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{L}(\text{proc}) & \subseteq & \mathcal{L}(\text{env})
\end{array}
\]

To obtain a semantics for (hyper)environments, we apply the same principle of “erasing processes” to the SOS specification of \( \pi \text{MLL} \), which gives us the specification in Figure 3.

**Definition 2.14.** The lts of environments for \( \pi \text{MLL} \), denoted \( \text{lts}_e \), is the triple \( (\text{Env}, Lbl, \rightarrow) \) where

- \( \text{Env} \) is the set of hyperenvironments for \( \pi \text{MLL} \) (\( G, H, \ldots \)),
- \( Lbl \) is the set of labels for \( \pi \text{MLL} \) (see Definition 2.5),
- \( \rightarrow \subseteq \text{Env} \times Lbl \times \text{Env} \) is the least relation closed under the SOS rules in Figure 3.

Well-typed processes never go wrong, as stated by the following result. The proof of this result depends on some properties about the behavioural theory of \( \pi \text{MLL} \), which are presented in the next subsection and make session fidelity straightforward.

**Theorem 2.15 (Session fidelity).** \( \text{lts}_p \) enjoys session fidelity wrt \( \text{lts}_e \).

**Example 2.16.** Recall from Example 2.4 that \( \vdash \text{Latch}_{xyz} :: x : \bot, y : \bot, z : 1 \). The process \( \text{Latch}_{xyz} \) can perform a transition with label \( x() \) or another with label \( y() \). In either case, its typing hyperenvironment can mimic it. We show the first case.

\[
\begin{align*}
\text{Latch}_{xyz} & \xrightarrow{x()} (v x_1 x_2) (v y_1 y_2) (x_1 [] | y().y_1[] | x_2().y_2().z[]) \\
& \xrightarrow{x()} y : \bot, z : 1
\end{align*}
\]

The derivative of the process is typed by the derivative of the hyperenvironment, as expected by session fidelity. The former can now perform either a transition with label \( y() \) or another with
label \( r \) (by resolving its internal communication on \( x_1 \) and \( x_2 \)). In the second case we obtain the following, and the derivatives remain related by typing.

\[
(vx_1;x_2) (vy_1;y_2) (x_1[] | y().y_1[] | x_2().y_2().z[]) \xrightarrow{r} (vy_1;y_2) (y().y_1[] | y_2().z[]) \\
y: \bot, z: 1 \xrightarrow{r} y: \bot, z: 1
\]

3 METATHEORY

We now move to the metatheoretical study of \( \pi \text{MLL} \).

We start from behavioural theory, showing that the transition system of \( \pi \text{MLL} \) gives rise to the expected notions of similarity and bisimilarity. Then, we leverage these notions to prove that \( \pi \text{MLL} \) enjoys erasure and session fidelity, by establishing appropriate (bi)simulations between processes, derivations, and environments.

Afterwards, we focus on studying the metatheoretical implications of hyperenvironments. We find that well-typed processes enjoy serialisation and non-interference for independent actions, and that independence is guaranteed by separation (parallel composition) in hyperenvironments.

Interestingly, our investigation uncovers that standard bisimilarity laws underpin the definition of a deterministic function akin to previous formulations of a transformation called “disentanglement” [Kokke et al. 2018], which rewrites a process into a parallel composition of independent components. Formulating disentanglement based on bisimilarity laws leads to important benefits, including: it reveals that it does not alter behaviour (strong bisimilarity), and it simplifies the proof of readiness for \( \pi \text{MLL} \) (a property that generalises progress and deadlock-freedom) compared to previous works.

We end this subsection by showing that \( \pi \text{MLL} \) is in a strict relationship with classical linear logic.

3.1 Behavioural Theory

The standard definitions of simulations and bisimulations for the \( \pi \)-calculus can be adopted for \( \pi \text{MLL} \) and its family of semantics.

Similarly to erasure and session fidelity, we state simulations and bisimulations in terms of homomorphisms of labelled transition systems abstracting from the specific \( \mathcal{L}(\cdot) \) and set of labels of \( \pi \text{MLL} \)—our definitions are straightforward consequences of [Ciancia 2013; Hasuo 2006].

**Definition 3.1 (Strong Bisimilarity).** Let \( \mathcal{L}(\cdot) \) be a functor over the category of sets and functions, \( \text{lts}_1 : \mathcal{L}(S_1) \rightarrow \mathcal{P}(S_1) \) and \( \text{lts}_2 : \mathcal{L}(S_2) \rightarrow \mathcal{P}(S_2) \) two labelled transition systems, \( \mathcal{R} \) a relation between their state-spaces \( S_1 \) and \( S_2 \), and \( \text{fst} : \mathcal{R} \rightarrow S_1 \) and \( \text{snd} : \mathcal{R} \rightarrow S_2 \) the associated canonical projections.

- \( \mathcal{R} \) is a **strong forward simulation** from \( \text{lts}_1 \) to \( \text{lts}_2 \) if the diagram below holds for some \( \text{lts}_r \).

\[
\begin{array}{c}
\mathcal{L}(S_1) \xrightarrow{\mathcal{L}(	ext{fst})} \mathcal{L}(\mathcal{R}) \xrightarrow{\mathcal{L}(	ext{snd})} \mathcal{L}(S_2) \\
\text{lts}_1 \downarrow \quad \subseteq \quad \text{lts}_r \downarrow \quad \subseteq \quad \text{lts}_2 \\
\mathcal{P}(S_1) \xleftarrow{\mathcal{P}(	ext{fst})} \mathcal{P}(\mathcal{R}) \xrightarrow{\mathcal{P}(	ext{snd})} \mathcal{P}(S_2)
\end{array}
\]

Strong forward similarity is the largest relation \( \subseteq \) that is a strong forward simulation.
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- $R$ is a strong backward simulation from $lts_1$ to $lts_2$ if its symmetric $R^{-1}$ is a strong forward simulation from $lts_2$ to $lts_1$ i.e. if the diagram below holds for some $lts_r$.

\[
\begin{array}{c}
\mathcal{L}(S_1) & \xrightarrow{\mathcal{L}(f)} & \mathcal{L}(R) & \xrightarrow{\mathcal{L}(\text{snd})} & \mathcal{L}(S_2) \\
lts_1 & \geq & lts_r & \geq & lts_2 \\
\mathcal{P}(S_1) & \xleftarrow{\mathcal{P}(f)} & \mathcal{P}(R) & \xleftarrow{\mathcal{P}(\text{snd})} & \mathcal{P}(S_2) \\
\end{array}
\]

Strong backward similarity is the largest relation $\geq$ that is a strong backward simulation.

- $R$ is a strong bisimulation for $lts_1$ and $lts_2$ if it is both a strong forward and backward simulation from $lts_1$ to $lts_2$ i.e. if the diagram below commutes for some $lts_r$.

\[
\begin{array}{c}
\mathcal{L}(S_1) & \xrightarrow{\mathcal{L}(f)} & \mathcal{L}(R) & \xrightarrow{\mathcal{L}(\text{snd})} & \mathcal{L}(S_2) \\
lts_1 & \xrightarrow{l} & lts_r & \xrightarrow{l} & lts_2 \\
\mathcal{P}(S_1) & \xleftarrow{\mathcal{P}(f)} & \mathcal{P}(R) & \xleftarrow{\mathcal{P}(\text{snd})} & \mathcal{P}(S_2) \\
\end{array}
\]

Strong bisimilarity is the largest relation $\sim$ that is a strong bisimulation.

To instantiate Definition 3.1 on $\pi$MLL we simply have to use $\mathcal{L}(\_)$ defined in (2) i.e.

$$\mathcal{L}(X) = X \times \text{Lbl} \quad \mathcal{L}(f) = \lambda(x,l).(f(x),l)$$

where $X$ is a set and $f: X \to Y$ a function. When we unfold the definition of homomorphism, we obtain the familiar phrasing of simulation and bisimulations for the internal $\pi$-calculus (here $lts_1 = (S_1, \text{Lbl}, \to_1)$ and $lts_2 = (S_2, \text{Lbl}, \to_2)$).

- $R$ is a strong forward simulation from $lts_1$ to $lts_2$ if $s_1 R s_2$ implies that if $s_1 \xrightarrow{l_1} s'_1$ then $s_2 \xrightarrow{l_2} s'_2$ for some $s'_2$ such that $s'_1 R s'_2$.

- $R$ is a strong backward simulation from $lts_1$ to $lts_2$ if $s_1 R s_2$ implies that if $s_2 \xrightarrow{l} s'_2$ for some $s'_2$ such that $s'_1 R s'_2$.

- $R$ is a strong bisimulation for $lts_1$ and $lts_2$ if $s_1 R s_2$ implies that

  - if $s_1 \xrightarrow{l_1} s'_1$ then $s_2 \xrightarrow{l_2} s'_2$ for some $s'_2$ such that $s'_1 R s'_2$; and that
  - if $s_2 \xrightarrow{l} s'_2$ then $s_1 \xrightarrow{l_1} s'_1$ for some $s'_1$ such that $s'_1 R s'_2$.

**Fact 3.2.** As in standard process algebras, parallel composition and $0$ obey the laws of abelian monoids under (strong) bisimilarity. Formally, for any $P$, $Q$, and $R$:

$$P \parallel 0 \sim P \quad P \parallel Q \sim Q \parallel P \quad P \parallel (Q \parallel R) \sim (P \parallel Q) \parallel R$$

Restriction distributes over parallel composition and restriction, provided that they do not depend on the restricted channel. For $x, y \notin (\text{fn}(Q))$:

$$(vxy)(P \parallel Q) \sim (vxy)(P) \parallel Q \quad (vxy)(vx'y')P \sim (vx'y')(vxy)P$$
Strong behavioural relations can discriminate processes whose behaviour differs only by \(\tau\)-labelled transitions. The standard approach for calculi in the \(\pi\)-calculus family is to define (bi)-simulations in terms of “saturated” transitions [Milner 1989; Sangiorgi and Walker 2001]. Formally, the saturation of an lts \((S, L, \rightarrow)\) with \(\tau \in L\) is the lts \((S, L, \Rightarrow)\) where \(\Rightarrow\) is the smallest relation such that: \(s \Rightarrow s\) for all \(s \in S\); and if \(s_1 \xrightarrow{r} s_2, s_2 \xrightarrow{t} s_3\), and \(s_3 \xrightarrow{r} s_4\), then \(s_1 \xrightarrow{r} s_4\).

Definition 3.3 (Bisimilarity). Let \(lts_1\) and \(lts_2\) be two labelled transition systems over the same set of labels. A relation \(R\) between their statespaces is a simulation (resp. bisimulation) from \(lts_1\) to \(lts_2\) whenever it is a strong simulation (resp. bisimulation) from the saturation of \(lts_1\) to the saturation of \(lts_2\). Similarity (resp. bisimilarity) is the largest relation \(\subseteq\) (resp. \(\sim\)) that is a simulation (resp. bisimulation).

It follows from the inclusion of rule \(\tau\) in the SOS specifications of \(\pi\)MLL that \(\tau\) \(\subseteq\) \(\sim\). Also, from the definition of saturation, it follows that \(\sim\) \(\subseteq\) \(\sim\) \(\subseteq\) \(\sim\) \(\subseteq\) \(\sim\) [Brewos et al. 2015].

The behavioural relations that we have presented are congruences for the lts of processes, so they allow for local reasoning.

Definition 3.4 (Congruence). An equivalence (resp. preorder) relation \(\equiv\) over processes is a congruence (resp. precongruence) if it is closed under all syntactic constructs of the language.

For \(\pi\)MLL, \(\equiv\) is a (pre)congruence if \(P \equiv Q\) implies that:

1. \(P \parallel R \equiv Q \parallel R\) for any \(R\) (and the symmetric);
2. \(\pi.P \equiv \pi.Q\) for any prefix \(\pi\);
3. \((\forall x.y) P \equiv (\forall x.y) Q\) for any \(x, y\).

Theorem 3.5 (Congruence). On the lts of processes, \(\subseteq\) and \(\equiv\) are precongruences and \(\sim\) and \(\equiv\) are congruences.

Similar results hold for the lts of derivations and the lts of hyperenvironments: replacing a subderivation \(E\) in \(D\) with \(E' \equiv E\) yields a derivation \(D' \equiv D\). This claim can be proved with minor adaptations to the proof of Theorem 3.5.

The standard tools and techniques of behavioural theory are central to the developments in this work. As a first instance, we observe that erasure and session fidelity (Definitions 2.7 and 2.13) can be stated in terms of strong (bi)simulations. This observation holds also for extensions of the language thanks to our formulation in terms of homomorphisms.

Lemma 3.6 (Erasure, Behaviourally). Let \(lts_p' = (Proc, Lbl, \rightarrow)\) and \(lts_d' = (Der, Lbl, \rightarrow)\). The following statements are equivalent.

- \(lts_p'\) enjoys erasure wrt \(lts_d'\).
- \(\{ (\text{proc}(D), D) \mid D \in Der \}\) is a strong bisimulation for \(lts_p'\) and \(lts_d'\).

Lemma 3.7 (Session Fidelity, Behaviourally). Let \(lts_p' = (Proc, Lbl, \rightarrow)\) and \(lts_d' = (Env, Lbl, \rightarrow)\). The following statements are equivalent.

- \(lts_p'\) enjoys session fidelity wrt \(lts_d'\).
- \(\{ (\text{proc}(D), \text{env}(D)) \mid D \in Der \}\) is a strong forward simulation from \(lts_p'\) to \(lts_d'\).

The behavioural characterisations of erasure and session fidelity give us a proof strategy for obtaining these properties.

Theorem 3.8. The relation \(\{ (\text{proc}(D), D) \mid D \in Der \}\) is a strong bisimulation for \(lts_p\) and \(lts_d\).

Theorem 3.9. The relation \(\{ (\text{proc}(D), \text{env}(D)) \mid D \in Der \}\) is a strong forward simulation from \(lts_p\) to \(lts_d\).
Theorem 3.9 holds, granted some mild assumptions: if a hyperenvironment simulates a process and does not contain more names, then it can be used to type that process.

**Lemma 3.10.** Let \( P \) be well-typed and \( \mathcal{G} \) such that \( \text{fn}(P) = \text{cn}(\mathcal{G}) \). If \( P \sqsubseteq \mathcal{G} \), then \( \vdash P :: \mathcal{G} \).

In general, the fact that a process is simulated by a hyperenvironment does not imply that it can also be typed by the same environment if the process is ill-typed. Consider the process \( P = \sigma(x).(.(\nu x)y)0 \), we have that \( P \not\sqsubseteq x : 1 \) and that \( P \) is ill-typed.

Combining the last two results we have that on well-typed processes similarity identifies all types inhabited by a given process.

**Theorem 3.11.** For \( P \) well-typed and \( \mathcal{G} \) such that \( \text{fn}(P) = \text{cn}(\mathcal{G}) \), \( P \not\sqsubseteq \mathcal{G} \) iff \( \vdash P :: \mathcal{G} \).

Because composing simulations with bisimulations yields simulations, it follows that bisimilarity of well-typed processes implies type equivalence. This result positions any procedure for checking bisimilarity as a sound procedure for checking type equivalence.

**Corollary 3.12.** For \( P \) and \( Q \) well-typed, if \( P \approx Q \) and \( \vdash Q :: \mathcal{G} \), then \( \vdash P :: \mathcal{G} \).

### 3.2 Metatheory of Parallelism

In the following, we write \( l_1 \neq l_2 \) to denote that the labels \( l_1 \) and \( l_2 \) do not share free names, i.e., \( \text{fn}(l_1) \cap \text{fn}(l_2) = \emptyset \).

**Diamond Property.** We say that an lts enjoys the diamond property when concurrent actions on distinct names can be interleaved in any order without affecting the trace final state.

**Definition 3.13 (Diamond Property).** An lts \((S, Lbl, \longrightarrow)\) enjoys the diamond property provided that for any \( s \in S \) and \( l_1 \neq l_2 \) if \( s \xrightarrow{l_1} s_1 \) and \( s \xrightarrow{l_2} s_2 \), then there exists \( s_3 \) such that \( s_1 \xrightarrow{l_1} s_3 \) and \( s_1 \xrightarrow{l_2} s_3 \).

Saturation preserves the diamond property.

**Lemma 3.14.** If \((S, Lbl, \longrightarrow)\) enjoys the diamond property, then its saturation \((S, Lbl, \equiv)\) enjoys the diamond property.

**Theorem 3.15.** The lts of derivations for \( \pi \text{MLL} \) enjoys the diamond property.

By erasure, well-typed processes and their transitions enjoy the diamonad property.

**Corollary 3.16.** Let \( P \) be well-typed and \( l_1 \neq l_2 \).

- If \( P \xrightarrow{l_1} P_1 \) and \( P \xrightarrow{l_2} P_2 \), then there exists \( P_3 \) such that \( P_1 \xrightarrow{l_2} P_3 \) and \( P_2 \xrightarrow{l_1} P_3 \).
- If \( P \equiv P_1 \) and \( P \equiv P_2 \), then there exists \( P_3 \) such that \( P_1 \equiv P_3 \) and \( P_2 \equiv P_3 \).

The lts of processes does not enjoy the diamonad property because there are processes that violate the property e.g. \( x[], y[][], 0 | x[][], z[][], 0 \). Any such process is necessarily ill-typed.

A consequence of the diamond property is that \( r \)-labelled transitions do not affect the possible interactions with the environment: any interaction available before an internal step will still be available afterwards.

**Lemma 3.17.** Let \((S, L \longrightarrow)\) enjoy the diamond property. If \( s \xrightarrow{r} s' \), then \( s \approx s' \).
Serialisation. We say that an lts enjoys serialisation if whenever a state can perform two actions in parallel (i.e., it is the source of a transition with a label like \((l_1 \parallel l_2)\)) it can also be perform the same actions sequentially.

**Definition 3.18 (Serialisation).** An lts \((S, \text{Lbl}, \rightarrow)\) enjoys **serialisation** provided that for any \(s \in S\) and \(l_1 \neq l_2\) if \(s \xrightarrow{(l_1\parallel l_2)} s'\), then \(s \xrightarrow{l_1} \xrightarrow{l_2} s'\).

Saturation preserves serialisation.

**Lemma 3.19.** If \((S, \text{Lbl}, \rightarrow)\) enjoys serialisation, then its saturation \((S, \text{Lbl}, \implies)\) enjoys serialisation.

**Theorem 3.20.** The lts of derivations for \(\pi\text{MLL}\) enjoys serialisation.

It follows by erasure that well typed processes and their transitions enjoy the diamond property.

**Corollary 3.21.** Let \(P\) be well-typed.

- If \(P \xrightarrow{(l_1\parallel l_2)} P'\), then \(P \xrightarrow{l_1} \xrightarrow{l_2} P'\).
- If \(P \xrightarrow{l_1\parallel l_2} P'\), then \(P \xrightarrow{l_1} \xrightarrow{l_2} P'\).

Non-interference. We say that an lts enjoys non-interference if when a state is ready to perform two actions on different names, then these can be parallelised, i.e., they do not interfere.

**Definition 3.22 (Non-interference).** An lts \((S, \text{Lbl}, \rightarrow)\) enjoys **non-interference** provided that for any \(s \in S\) and \(l_1, l_2 \in \text{Act}\) if \(l_1 \neq l_2\), \(s \xrightarrow{l_1}\), and \(s \xrightarrow{l_2}\), then \(s \xrightarrow{(l_1\parallel l_2)}\).

Saturation preserves non-interference.

**Lemma 3.23.** If \((S, \text{Lbl}, \rightarrow)\) enjoys non-interference, then its saturation \((S, \text{Lbl}, \implies)\) enjoys non-interference.

**Theorem 3.24.** The lts of derivations for \(\pi\text{MLL}\) enjoys the **non-interference**.

It follows by erasure that well typed processes and their transitions enjoy non-interference.

**Corollary 3.25.** Let \(P\) be well-typed.

- If \(P \xrightarrow{l_1}\) and \(P \xrightarrow{l_2}\) if and only if \(P \xrightarrow{(l_1\parallel l_2)}\).
- If \(P \xrightarrow{l_1\parallel l_2}\) if and only if \(P \xrightarrow{(l_1\parallel l_2)}\).

Dissentangement. Kokke et al. [2019, Lemma 2.6] observed that applications of rule \textsc{mix} can be moved until either they reach the top level or become attached to cuts or the formation of tensors. They use this “dissentangement” result to show that if a hyperenvironment is derivable in their system, then any of its environments is derivable in CLL using a proof closely related to some sub-derivation rooted under a top-level mix in the disentangled derivation.

Differently here, we show that dissentangement can be seen as a rewriting relation derived from standard laws of strong bisimilarity (cf. Fact 3.2). Our definition does not rely on types: it can be applied to any process, even ill-typed ones. Also, it is a deterministic procedure for computing disentangled processes (hence derivations). Thanks to its semantic foundation and constructive definition, our notion of dissentangement has applications that go beyond relating the type theory of our calculi to CLL. Notably, we use it later in this subsection to prove that \(\pi\text{MLL}\) enjoys readiness (a generalisation of progress) and in Section 5 to encode higher-order communications into first-order communications.
Definition 3.26 (Disentanglement). The disentanglement dis(P) of a well-typed process P is the process obtained by recursively rewriting a processes using the laws:

1. \( P \vdash 0 \sim P \);
2. \( (vxy)(P \mid Q) \sim (vxy)P \mid Q \) if \( x, y \notin \text{fn}(Q) \);
3. \( (vxy)(P \mid Q) \sim P \mid (vxy)Q \) if \( x, y \notin \text{fn}(P) \);

read from left to right until it is no longer possible.

Example 3.27. Consider the following process, which is a subterm of the latch in Example 2.1.

\[(vxy_1y_2)(x().x_1[][] | y().y_1[][] | x_2().y_2().z[])\]

The following process is the disentanglement of the term above.

\[x().x_1[][] | (vxy_1y_2)(y().y_1[][] | x_2().y_2().z[]).\]

This definition is equivalent to the procedure defined below by recursion on the structure of processes (for restriction we need to apply several applications of the binary law above at once).

Lemma 3.28. The disentanglement dis(P) of a process P is the process recursively defined as follows:

\[
dis(\pi.P) = \pi.\text{dis}(P) \quad \text{dis}(0) = 0 \quad \text{dis}(P \mid Q) = \begin{cases} 
\text{dis}(P) & \text{if } Q = 0 \\
\text{dis}(P) \mid \text{dis}(Q) & \text{otherwise}
\end{cases}
\]

\[\text{dis}((vxy)P) = (vxy)(P_1 \mid P_j) \mid (\text{dis}(P) \setminus P_1, P_j)\]

where \( \text{dis}(P) = P_1 \mid \cdots \mid P_n \), \( x \in \text{fn}(P_1) \), \( y \in \text{fn}(P_j) \)

Disentanglement preserves the semantics of processes.

Lemma 3.29. \( P \sim \text{dis}(P) \).

When applied to well-typed processes, disentanglement preserves typing. Furthermore, each parallel component of the disentangled process corresponds to and is typed by a parallel component of the hyperenvironment that types the initial process.

Lemma 3.30. Let \( \vdash P : \Gamma_1 \mid \cdots \mid \Gamma_n \). Then, \( \text{dis}(P) = P_1 \mid \cdots \mid P_n \) s.t. \( \vdash P_i : \Gamma_i \) for each \( i \in [1, n] \).

Readiness. A well-typed process whose hyperenvironment consists of a single environment is ready to perform a saturated transition on at least one of its free names. In the following, we write \( l_x \) for a label with \( x \) free \( (x[], x[y], \ldots) \).

Lemma 3.31. If \( \vdash P : \Gamma \), then there exist \( x \in \text{cn}(\Gamma) \) and \( l \) such that \( x \in \text{fn}(l) \) and \( P \overset{l}{\Rightarrow} \).

It thus follow from disentanglement, that any well-typed process is always ready on at least one name for each component of its hyperenvironment.

Theorem 3.32 (Readiness). Let \( \vdash P : \Gamma_1 \mid \cdots \mid \Gamma_n \). For every \( i \in [1, n] \), there exist \( x \in \text{cn}(\Gamma_i) \) and \( l \) such that \( x \in \text{fn}(l) \) and \( P \overset{l}{\Rightarrow} \).

Example 3.33. Consider the typing environment \( \Gamma = x: \bot, y : \bot, z: 1 \). From readiness we know that in any process \( P \) typed by \( \Gamma \) must be ready on at least one name in \( \Gamma \) and by session fidelity we know that this name cannot be \( z \) (its type is 1). Process \( \text{Latch}_{xyz} \) from Example 2.4 is typed by \( \Gamma \) and indeed it is ready on both \( x \) and \( y \). Other examples of processes typed by \( \Gamma \) are \( P = x().y().z[].\text{nil} \) which is ready only on \( x \) and \( Q = y().x().z[].\text{nil} \) which is ready only on \( y \). Any other process \( R \) typed by \( \Gamma \) can only differ from \( \text{Latch}_{xyz} \), \( P \) or \( Q \) on its internal behaviour (the type of \( x, y, \) and \( z \) prescribe their use and any other action must be on a bound name). It follows that \( Q \) is necessarily bisimilar to one of the three processes above.
3.3 Relation to Linear Logic and Classical Processes

The theory of πLL is strongly connected to CLL and its corresponding process calculus of Classical Processes (CP) [Wadler 2014]. In particular, this allows for lifting existing proof search techniques for CLL to πLL. Here we present the connection between πMLL and MCLL and MCP, the multiplicative fragments of CLL and CP. We will later extend this connection to the full theories of πLL, CLL, and CP in Section 4.

3.3.1 Multiplicative Classical Processes (MCP). Process terms in the multiplicative fragment of CP are defined by the grammar below (we follow the presentation of CP by Carbone et al. [2016], which is based on endpoints like πLL).

\[
\begin{array}{l}
P, Q ::= x[y].(P | Q) \quad \text{output on } x \\
| x(y).P \quad \text{input on } x \\
| x[] \quad \text{output (empty message) on } x \\
| x().P \quad \text{input (empty message) on } x \\
| (\forall xy)(P | Q) \quad \text{name restriction, “cut”}
\end{array}
\]

The typing discipline of MCP corresponds to MCLL and is shown in Figure 4. Rule \(\text{cut}^\text{CP}\) corresponds to the standard cut rule of CPL. The cut rule of CPL requires two separate premises, which in CP corresponds to two processes. As a result, rule \(\text{cut}^\text{CP}\) is intensional: it must inspect the syntactic structure of the process under the restriction to check that there are two parallel components. In other words, the rule requires treating restriction and parallel as two separate operators, as usual in process calculi. Similarly, to cut, the original rule \(\otimes\) from CLL has two separate premises and the corresponding rule in CP, \(\otimes^\text{CP}\), is also intensional (whereas rule \(\otimes\) in πMLL is not). From the viewpoint of process calculi, the fact that rule \(\otimes\) has a single premise is key to reconstructing the typical output term of the internal π-calculus, which has a single continuation [Sangiorgi 1996]. The remaining rules in Figure 4 all have an equivalent rule (or literally the same rule) in πMLL.

Derivations in MCP can be interpreted as derivations in πMLL where the rules \(\text{cut}^\text{CP}\), \(\otimes^\text{CP}\) and \(\perp^\text{CP}\) are derivable in terms of rules \(\text{cut}\), \(\otimes\), \(\text{mix}\), \(1\), and \(\text{mix}_0\). In terms of processes, this corresponds to regarding the term constructors \((\forall xy)(- | -), x[y].(- | -), \) and \(x[]\) of multiplicative CP as syntactic sugar for πMLL.

**Proposition 3.34.** If \(\vdash_{\text{CP}} P : \Gamma\) in MCP then \(\vdash P : \Gamma\) in πMLL.

**Remark 3.35.** Wadler [2014] proposed a different rule for typing parallel composition, inspired by the mix rule by Girard [1987].

\[
\begin{array}{c}
\text{MIX}^\text{CP} \\
\hline
\vdash_{\text{CP}} P : \Gamma \\
\vdash_{\text{CP}} Q : \Delta \\
\hline
\vdash_{\text{CP}} P | Q : \Gamma, \Delta
\end{array}
\]

**Fig. 4.** Multiplicative CP, typing rules.
Unfortunately, combining the environments from the premises by using "", makes the rule forget that the names in $\Gamma$ are implemented in parallel to the names in $\Delta$. This prevents $P$ and $Q$ from ever communicating in CP, since we cannot be sure that connecting them with a restriction term will not introduce a cycle.

3.3.2 From $\pi$MLL to MCP, and Back. There are well-typed $\pi$MLL processes that are not valid processes in CP, for example $(\forall x y) (P | Q | R)$. However, if a $\pi$MLL process can be typed using a single environment, then its disentanglement can be typed in CP with the same environment.

**Proposition 3.36.** If $\vdash P :: \Gamma$ in $\pi$MLL then $\vdash_{\text{CP}} \text{dis}(P) :: \Gamma$ in MCP.

It follows that the parallel components of the disentanglement of any well-typed $\pi$LL process are well-typed processes.

**Lemma 3.37.** Let $\vdash P :: \Gamma_1 | \cdots | \Gamma_n$. Then, dis$(P) = P_1 | \cdots | P_n$ and $\vdash_{\text{CP}} P_i :: \Gamma_i$ in CP for all $i \in [1, n]$.

The result in Lemma 3.37 inspires the exploration of the more general question:

Given any $\pi$MLL process $P$, can we systematically construct a CP process $P'$ that is well-typed if and only if $P$ is?

Kokke et al. [2018, 2019] provide a partial answer to this question at the level of typing environments in their calculi: for any $\mathcal{G}$, there is $\Gamma$ such that $\mathcal{G}$ is inhabited iff $\Gamma$ is inhabited in CP. We extend the development presented therein to processes.

In CLL, it is well known that any environment $\Gamma$ can be internalised as the type $\otimes \Gamma$, meaning that $\Gamma$ is derivable iff $\otimes \Gamma$ is. This type can be defined as follows.

$$\Gamma \cdot \Downarrow \bot \quad \otimes \Gamma, A \Downarrow A \otimes \otimes \Gamma$$

Kokke et al. [2018] observed that like "," can be internalised as "$\otimes$", so can "$|$" be internalised as "$\otimes$". Formally, a hyperenvironment $\mathcal{G}$ is internalised as the type $\otimes \mathcal{G}$, defined as follows.

$$\otimes \Downarrow \otimes \Gamma \Downarrow \otimes \Gamma \otimes \otimes \mathcal{G}$$

Thus, $\mathcal{G}$ is inhabited iff $x : \otimes \mathcal{G}$ is inhabited [Kokke et al. 2018, Theorem 4.11].

The same result holds for $\pi$MLL. Additionally, we extend this recipe to processes by providing an encoding of $\pi$MLL into CP. At the core of our encoding are disentanglement and multi-hole contexts, inspired by the internalisation of "", and "$|$" recalled above. We call these contexts packing contexts, because of their effect on process interfaces and the role that they play in the encoding of higher-order into first-order processes that we will present Section 5. Below, we write $\square$ to denote a hole in the syntactic tree of a process that can be replaced with a process term [Sangiorgi and Walker 2001].

**Definition 3.38 (Packing context).** The packing context for $\Gamma$ and $y \notin \text{cn}(\Gamma)$ is the 1-holed context pack$^\mathcal{G}_y(\Gamma)$ defined below:

$$\text{pack}^\mathcal{G}_y(\bullet) \Downarrow y() . \square \quad \text{pack}^\mathcal{G}_y(\Gamma, z : A) \Downarrow y(z) . \text{pack}^\mathcal{G}_y(\Gamma)$$

The packing context for $\mathcal{G} = \Gamma_1 | \cdots | \Gamma_n$ and $x \notin \text{cn}(\mathcal{G})$ is the $n$-holed context pack$^\mathcal{G}_x(\mathcal{G})$, which is defined as follows.

$$\text{pack}^\mathcal{G}_x(\varnothing) \Downarrow x[.]_0 \quad \text{pack}^\mathcal{G}_x(\mathcal{G} | \Gamma) \Downarrow x[y] . (\text{pack}^\mathcal{G}_x(\mathcal{G}) | \text{pack}^\mathcal{G}_y(\Gamma))$$

The choice of $x$ and $y$ in the definition above is immaterial as long as they do not occur in the (hyper)environment associated to the packing context. In the remainder we will often omit them, and simply call pack$^\mathcal{G}_x(\mathcal{G})$ and pack$^\mathcal{G}_y(\Gamma)$ packing contexts for $\mathcal{G}$ and for $\Gamma$, respectively.
We write pack$^x_\otimes(\Gamma)[P]$ for the process obtained by replacing the hole $\Box$ in pack$^\otimes_\Box(\Gamma)$ with $P$, and pack$^\otimes_\Diamond(\mathcal{G})[P_1, \ldots, P_n]$ for the process obtained by replacing each hole in pack$^\otimes_\Box(\mathcal{G})$ with the respective process $P_i$ (from left to right).

As the name suggest, a packing context “packs” the interface of a process and exposes it over the given channel.

**Lemma 3.39.** Let $P, P_1, \ldots, P_n$ be processes.

- If $\vdash P : \Gamma$ and $x \notin \text{cn}(\Gamma)$, then $\vdash \text{pack}^\otimes_\Box(\Gamma)[P] : x : \bigotimes \Gamma$.
- If $\vdash P_1 \mid \cdots \mid P_n : \mathcal{G}$ and $x \notin \text{cn}(\mathcal{G})$, then $\vdash \text{pack}^\otimes_\Box(\mathcal{G})[P_1, \ldots, P_n] : x : \bigotimes \mathcal{G}$.

To pack an arbitrary well-typed process in $\pi\text{MLL}$, we simply disentangle its parallel components and plug them into the packing context for its typing hyperenvironment.

**Definition 3.40 (Process Packing).** Let $\vdash P : \mathcal{G}$ and $x \notin \text{fn}(P)$. The packing of $P$, denoted pack$^\otimes_\Box(x)(\vdash P : \mathcal{G})$, is the process pack$^\otimes_\Box(\mathcal{G})[P_1, \ldots, P_n]$ where $\text{dis}(P) = P_1 \mid \cdots \mid P_n$.

Process packing allows us to demonstrate that the multiplicative fragment of CP is a “complete” theory for $\pi\text{MLL}$, in the sense of the following theorem.

**Theorem 3.41.** If $\vdash P : \mathcal{G}$ in $\pi\text{MLL}$, then $\vdash_{\text{CP}} \text{pack}^\otimes_\Box(x)(\vdash P : \mathcal{G}) : x : \bigotimes \mathcal{G}$ in MCP.

The converse of Theorem 3.41 holds as well. To prove this claim, we define a transformation for CP derivations of packed processes. By Proposition 3.36 we can transform any derivation of $\vdash_{\text{CP}} \text{pack}^\otimes_\Box(x)(\vdash P : \mathcal{G}) : x : \bigotimes \mathcal{G}$ into a derivation $\mathcal{D}$ of $\vdash_{\text{CP}} \text{pack}^\otimes_\Box(x)(\vdash P : \mathcal{G}) : x : \bigotimes \mathcal{G}$ in the theory of $\pi\text{MLL}$. The derivation $\mathcal{D}$ has the following form (for $\mathcal{G} = \Gamma_1 \mid \cdots \mid \Gamma_n$).

$$
\begin{align*}
\vdash P_1 : x_1 : \Gamma_1 & \quad \vdash P_n : x_n : \Gamma_n \\
\vdash \text{pack}^\otimes_\Box(\Gamma_1)[P_1] : x : \bigotimes \Gamma_1 & \quad \vdash \text{pack}^\otimes_\Box(\Gamma_n)[P_n] : x : \bigotimes \Gamma_n \\
\vdash \text{pack}^\otimes_\Box(\mathcal{G})[P_1, \ldots, P_n] : \bigotimes \mathcal{G} \\
\end{align*}
$$

By repeatedly applying rule $\text{mix}$ to the premises $\mathcal{E}_1, \ldots, \mathcal{E}_n$, we obtain the following derivation of $\vdash P_1 \mid \cdots \mid P_n : \mathcal{G}$ in $\pi\text{MLL}$.

$$
\begin{align*}
\vdash P_1 : x_1 : \Gamma_1 & \quad \vdash P_n : x_n : \Gamma_n \\
\vdash \text{pack}^\otimes_\Box(\Gamma_1)[P_1] & \quad \vdash \text{pack}^\otimes_\Box(\Gamma_n)[P_n] \\
\vdash \text{pack}^\otimes_\Box(\mathcal{G})[P_1, \ldots, P_n] : \bigotimes \mathcal{G} \\
\end{align*}
$$

Let $\psi$ denote this transformation of derivations, which essentially replaces the applications of rules $\otimes$ and $\otimes$ for typing a packing context with applications of rule $\text{mix}$. We use it to prove the following lemma.

**Lemma 3.42.** If $\vdash_{\text{CP}} \text{pack}^\otimes_\Box(x)(\vdash P : \mathcal{G}) : x : \bigotimes \mathcal{G}$ in MCP, then $\vdash P : \mathcal{G}$ in $\pi\text{MLL}$.

**Lemma 3.42** is quite inconvenient if our aim is to reuse proof search methods for CLL in the context of $\pi\text{MLL}$, because we have to guess $P$ and pack it: what if we wanted to find out whether a generic $\mathcal{G}$ is derivable or not in $\pi\text{MLL}$? To reach a more general procedure, we first observe that, when we consider the cut-free fragment of CP, any process that can be typed by the internalisation of a hyperenvironment is necessarily a packing.

**Lemma 3.43.** Let $\mathcal{G} = \Gamma_1 \mid \cdots \mid \Gamma_n$. If $\vdash_{\text{CP}} P : x : \bigotimes \mathcal{G}$ and $P$ is cut-free, then there are $Q_1, \ldots, Q_n$ such that $P = \text{pack}^\otimes_\Box(\mathcal{G})[Q_1, \ldots, Q_n]$ and $\vdash_{\text{CP}} Q_i : x_i : \Gamma_i$ for $i \in [1, n]$. 


It follows that given a cut-free derivation $D$ for $\vdash_{CP} P \Rightarrow x : \otimes G$, we can exploit the transformation $\psi$ defined above to obtain the derivation $\psi(D)$ such that $env(\psi(D)) = G$. In other words, any proof search strategy for CLL can be the base for a proof search strategy for $\pi MLL$: we simply search for a proof in CLL that concludes the internalisation of the hyperenvironment of interest, and then apply $\psi$ to the resulting derivation.

**Corollary 3.44.** If $D$ is a cut-free derivation of $\otimes G$ in the multiplicative fragment of CLL, then $\psi(D)$ is a cut-free derivation of $G$ in $\pi MLL$.

Since cut is admissible in CLL, Corollary 4.23 implies in general that proof search in $\pi MLL$ can be reduced to proof search in CLL.

4 **$\pi LL$**

In this section, we present $\pi LL$, a calculus in correspondence with classical linear logic with first-order quantifiers (CLL$_{01}$, herein simply CLL). Compared to $\pi MLL$/MCLL (Section 2), $\pi LL$ introduces terms in correspondence with additives, exponentials, and first-order quantifiers.

4.1 **Processes and Typing**

Processes. Processes in $\pi LL$ are given by the following grammar. We note on the right-hand side which terms have already been discussed in $\pi MLL$.

$$P, Q ::= x[y].P \quad \text{output y on x and continue as P} \quad (\pi MLL)$$

$$| \ x(y).P \quad \text{input y on x and continue as P} \quad (\pi MLL)$$

$$| \ x[].P \quad \text{output (empty message) on x and continue as P} \quad (\pi MLL)$$

$$| \ x().P \quad \text{input (empty message) on x and continue as P} \quad (\pi MLL)$$

$$| \ x \in \mathsf{inl}.P \quad \text{select left on x and continue as P}$$

$$| \ x \in \mathsf{inr}.P \quad \text{select right on x and continue as P}$$

$$| \ x \Rightarrow \{ \mathsf{inl} \colon P; \mathsf{inr} \colon Q \} \quad \text{offer a binary choice between P (left) or Q (right) on x}$$

$$| \ x[A].P \quad \text{output type A on x and continue as P} \quad (\pi MLL)$$

$$| \ x(X).P \quad \text{input a type as X on x and continue as P}$$

$$| \ !x(y).P \quad \text{offer a service}$$

$$| \ ?x[y].P \quad \text{consume a service}$$

$$| \ ?x[x_1, x_2].P \quad \text{duplicate a service} \quad (\pi MLL)$$

$$| \ ?x[].P \quad \text{dispose of a service}$$

$$| \ (\forall x y) P \quad \text{name restriction, “cut”}$$

$$| \ P \parallel Q \quad \text{parallel composition of processes P and Q} \quad (\pi MLL)$$

$$| \ 0 \quad \text{terminated process}$$

$$| \ x \leftrightarrow y \quad \text{link x and y}$$

We describe only the new terms (the others have already been discussed in Section 2).

Terms $x \in \mathsf{inl}.P$ and $x \in \mathsf{inr}.P$ respectively send on $x$ the selection of the left or right branch of a (binary) offer available on the other end of the channel before proceeding as $P$. Dually, term $x \Rightarrow \{ \mathsf{inl} \colon P; \mathsf{inr} \colon Q \}$ offers on $x$ a choice between proceeding as $P$ (left branch) or $Q$ (right branch). Terms $x[A].P$ and $x(X).P$ enable polymorphism: term $x[A].P$ sends type $A$ over $x$ and proceeds as $P$; term $x(X).P$ receives a type over $x$ abstracted by the type variable $X$ and then proceeds as $P$, where $X$ is bound in $P$. Term $!x(y).P$ is a server that offers on $x$ a replicable process $P$, where $y$ is bound in $P$. A server can be used by clients any number of times. Accordingly, we have three client terms to interact with a server. The client term $?x[y].P$ requests exactly one process by the server on $x$, and then proceeds by communicating with the process on channel $y$. The client term $?x[].P$ disposes of the server on $x$—the server is used zero times. The client term $?x[x_1, x_2].P$ requests that
the server on x is duplicated, and that the two resulting servers be available on the new channels \( x_1 \) and \( x_2 \), respectively. Term \( x \mapsto y \) links the endpoints x and y and has the effect of merging their sessions; it can be intuitively though of as a forwarding proxy.

In the remainder, we use \( \pi \) to range over term prefixes: \( x[x'], x(x'), x[], x() \), \( x \cdot \text{inl}, x \cdot \text{inr}, !x(y) \), \( ?x[y], ?x[x_1, x_2] \), and \( ?x[] \).

Free and bound names of processes and prefixes are defined as expected, as well as \( \alpha \)-conversion. In particular, all names that appear inside of round parentheses “(−)” or square brackets “[−]” in a term prefix are bound to the continuation of the prefix. As in \( \pi \text{MLL} \), a restriction (\( \forall xy \)) \( P \) binds x and y to P. All other names are free.

**Example 4.1 (Free output).** Forwarders allow for recovering the usual output primitive for sending free names found in the original \( \pi \)-calculus, as syntactic sugar [Atkey et al. 2016].

\[
x(y).P \triangleq x[z].(y \mapsto z | P)
\]

Similar considerations apply to polyadic communications (terms that send multiple names) [Sangiorgi and Walker 2001].

**Example 4.2 (Bit Operations [Atkey et al. 2016; Kokke et al. 2019]).** We report how to write a server that computes the logical AND of two bits in calculi based on linear logic. We use selections to model sending bits. The example uses the following syntactic sugar.

\[
x[0].P \triangleq x[x'].x' \cdot \text{inl}.x'[].P \quad x[1].P \triangleq x[x'].x' \cdot \text{inr}.x'[].P
\]

\[
\text{x} \triangleright \{ 0 \mapsto P; 1 \mapsto Q \} \triangleq \text{x} \triangleright \{ \text{inl}().P; \text{inr}().Q \}
\]

With these abbreviations, we can write a server that offers a service for computing logical AND.

\[
\text{Server}_{y} \triangleq !y(y').y'(p).y'(q).p \triangleright \{ 0 \mapsto q \triangleright \{ 0 \mapsto y'[0].y'[.].0; 1 \mapsto y'[0].y'[.].0 \} \}
\]

\[
1 \mapsto q \triangleright \{ 0 \mapsto y'[0].y'[.].0; 1 \mapsto y'[1].y'[.].0 \}
\]

We now define a compatible client, \( \text{Client}_{x_1, x_2}^{b_1, b_2} \), which sends bits \( b_1 \) and \( b_2 \) (0 or 1) to a server that accepts two bits on x (the client abstracts from the concrete operation that the server computes). The client uses the result to decide whether to select left or right on another channel z.

\[
\text{Client}_{x_1, x_2}^{b_1, b_2} = ?x[x'].x'[b_1].x'[b_2].x' \triangleright \{ 0 \mapsto x'().z \cdot \text{inl}.z[].0; 1 \mapsto x'().z \cdot \text{inr}.z[].0 \}
\]

**Example 4.3 (Polymorphic API Gateway).** In the software paradigm of microservices [Dragoni et al. 2017], an API gateway (Application Programming Interface gateway) is a polymorphic proxy server that offers a single endpoint through which clients can access other servers [Montesi and Weber 2016]. We can model this behaviour as the following process.

\[
\text{Gateway}_{x} \triangleq x(x_l).x(x_r).x(x_l).x(x_r).!x(x').x' \triangleright \{ \text{inl}(): ?x_l[].?x_l[x'[]].x' \leftrightarrow x' \}
\]

\[
\text{inr}(): ?x_r[].?x_r[x'[]].x' \leftrightarrow x'
\]

Process \( \text{Gateway}_{x} \) models a gateway for two other servers. First, it receives on x the types of the two servers that clients will be able to use, abstracted by \( X_l \) and \( X_r \), along with the endpoints over which these servers are available. Then, it waits to receive on x client invocations, which subsequently decide whether they want to use the first or the second API. The gateway then sets up a connection to the right server by using appropriate client requests and a forwarder.

Let us see an example of how \( \text{Gateway}_{x} \) can be used. Let \( S_l \) and \( S_r \) be servers that offer services at the endpoints \( y_l \) and \( y_r \), respectively. We can aggregate them using \( \text{Gateway}_{x} \) to offer both services on x as follows, where \( A_l \) and \( A_r \) stand for the respective types of the behaviours offered on \( y_l \) and \( y_r \).

\[
S_{l+r} \triangleq (\forall xy)(\text{Gateway}_{x} | y[A_l].y[A_r].y[y_l].)(S_l | y[y_r].)(S_r | y \mapsto z)))
\]
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\[
\text{Structural rules}
\]

\[
\begin{array}{c}
\Gamma \vdash x \rightsarrow y = x : A^\perp, y : A & \text{AX} \\
\vdash P \Rightarrow \Gamma, x : A \wedge y : A^\perp & \text{CUT} \\
\vdash P \Rightarrow \Gamma, A^\perp & \text{MIX} \\
\vdash 0 \Rightarrow \emptyset & \text{MIX}_0
\end{array}
\]

\[
\text{Logical rules}
\]

\[
\begin{array}{c}
\vdash P \Rightarrow \Gamma, y : A \wedge \Delta, x : B & \text{\Theta}_1 \\
\vdash P \Rightarrow \emptyset & \text{\emptyset} \\
\vdash x \oplus P \Rightarrow \Gamma, x : A \oplus B & \text{\oplus}_2 \\
\vdash x \otimes P \Rightarrow \Gamma, x : A \otimes B & \text{\otimes}_2 \\
\vdash P \Rightarrow \Gamma, x : A & \text{\Gamma} \\
\vdash P \Rightarrow \Gamma, x : A & \text{\Gamma} \\
\vdash P \Rightarrow \Gamma, x : A \wedge B & \text{\Gamma} \\
\vdash P \Rightarrow \Gamma, x : \langle A | X \rangle & \text{\Gamma} \\
\vdash P \Rightarrow \Gamma, x : X & \text{\Gamma} \\
\vdash P \Rightarrow \Gamma, x : \exists X . A & \text{\exists X} \\
\vdash P \Rightarrow \Gamma, x : \forall X . B & \text{\forall X} \\
\vdash P \Rightarrow ? \Gamma, x : ! A & \text{\Gamma} \\
\vdash P ? \Rightarrow ? \Gamma, x : ? A & \text{\Gamma} \\
\vdash P \Rightarrow ? \Gamma, x : ? A & \text{\Gamma} \\
\vdash P \Rightarrow ? \Gamma, x : ? A & \text{\Gamma} \\
\vdash P \Rightarrow ? \Gamma, x : ? A & \text{\Gamma}
\end{array}
\]

Fig. 5. πLL, typing rules.

To use one of the services behind the gateway, a client simply needs to first send a left or right selection. For example, \( ? \Gamma, x : ! A \) is a client for \( S_{t_to} \) that uses the service offered by \( S_I \) through the gateway.

\[
(\forall xz) \ (\forall x' : ?A, x'' : ?A)
\]

Typing. Types in πLL extend those of πMLL to include also the additive and exponential propositions of CLL.

\[
A, B ::= A \otimes B \quad \text{send } A, \text{ proceed as } B
\]

(\text{πMLL})

| A \otimes B | receive A, proceed as B
| 1 | empty output, unit for \( \otimes \)
| \( \bot \) | empty input, unit for \( \otimes \)
| A \otimes B | select A or B
| A & B | offer A or B
| X | type variable
| X^\perp | dual of type variable
| \( \exists X . A \) | existential, type output
| \( \forall X . A \) | universal, type input
| ?A | client request
| !A | server accept

Duality is extended to additives and exponentials exactly as in CLL.

\[
\begin{array}{c}
(A \otimes B)^\perp = A^\perp \otimes B^\perp & \ (A \otimes B)^\perp = A^\perp \otimes B^\perp & 1^\perp = \bot & \bot^\perp = 1 \\
(A \oplus B)^\perp = A^\perp \oplus B^\perp & \ (A \otimes B)^\perp = A^\perp \otimes B^\perp & (?A)^\perp = !A^\perp & \ (A^\perp)^\perp = ?A^\perp \\
(X)^\perp = X^\perp & \ (X^\perp)^\perp = X & \ (\exists X . A)^\perp = \forall X . A^\perp & \ (\forall X . A)^\perp = \exists X . A^\perp
\end{array}
\]

Environments and hyperenvironments are defined as in πMLL (but contain types in the extended syntax). Typing judgements have the same form, \( \vdash P \Rightarrow \Gamma \), as well.

The inference rules for deriving typing judgements in πLL are displayed in Figure 5.

Rules \text{CUT}, \text{MIX}, \text{MIX}_0, \otimes, 1 and \( \bot \) correspond to πMLL and are covered in detail in Section 2. We discuss the new rules, which cover the additive, exponential, and first-order quantification fragments of CLL.
Rule $\text{AX}$ types a link (or “forwarder”) between $x$ and $y$ by requiring that the types of $x$ and $y$ are the dual of each other. This ensures that any message on $x$ can be safely forwarded to $y$, and vice versa. Rules $\exists$ and $\forall$ type polymorphic session behaviour by using quantifiers.

Rules $\oplus_1$ and $\oplus_2$ type, respectively, the choice of the left or the right branch of an alternative behaviour offered on the other end of the session. Dually, rule $\&$ types the offering of a choice between two behaviours.

All rules for typing channels enforce linear usage aside for client requests (typed with the exponential connective $?$), for which contraction and weakening are allowed. Specifically, contraction (rule $c$) allows for having multiple client requests for the same server endpoint, and weakening (rule $w$) allows for having a client that does not use a server. Rule $!$ types a server, where $?\Gamma$ denotes that all session types in $\Gamma$ must be client requests, i.e., $?\Gamma := x_1 : ?A_1, \ldots, x_n : ?A_n$. A server must be executable any number of times, since it does not know how many client requests it will have to support. This is guaranteed by requiring that all resources used by the server are acquired by communicating with the client (according to protocol $A$) and with other servers ($?\Gamma$).

**Example 4.2**

When translated to theories like $\pi$LL, the proof of admissibility of the axiom for general propositions corresponds to an $\eta$-expansion procedure: for every $A$, $x$, and $y$, we can construct a process $\text{Fwd}^A_{x,y}$ that implements the forwarding of protocol $A$ from $x$ to $y$. Below we show some illustrative cases (others are similar, see [Carbone et al. 2016]).

$$
\begin{align*}
\vdash & P :: \Gamma, x : B \\
\vdash & x(y).P :: \Gamma, y : A^\bot, x : A \otimes B
\end{align*}
$$

Example 4.4. The syntactic sugar for free output shown in Example 4.1 can be typed with a derivable rule, as shown below.

$$
\vdash \gamma : \Gamma, x : B \\
\vdash x(y).P :: \Gamma, y : A^\bot, x : A \otimes B
$$

Example 4.5. Define the types for sending and receiving a bit, respectively.

$$
\text{Bit} \triangleq 1 + 1 \quad \text{send a bit} \\
\text{Bit}^\bot \triangleq \bot \otimes \bot \quad \text{receive a bit}
$$

Thus, by rules $\text{Cut}$ and $\text{MIX}$ we can type their composition for all distinct names $x$, $y$, and $z$, and any bits $b_1$ and $b_2$, e.g., to compute the logical AND of 0 and 1:

$$
\vdash (vxy)(\text{Client}_{xz}^{b_1,b_2} \mid \text{Server}_y) :: z : \text{Bit}.
$$

Remark 4.6. Rule $\text{AX}$ is the standard axiom of CLL. It is well known that the axiom is admissible in the presence of a restricted form of it, as shown below, which accepts only atomic propositions (type variables).

$$
\vdash x \leftrightarrow y :: x : X^\bot, y : X
$$

When translated to theories like $\pi$LL, the proof of admissibility of the axiom for general propositions corresponds to an $\eta$-expansion procedure: for every $A$, $x$, and $y$, we can construct a process $\text{Fwd}^A_{x,y}$ that implements the forwarding of protocol $A$ from $x$ to $y$. Below we show some illustrative cases (others are similar, see [Carbone et al. 2016]).

$$
\begin{align*}
\vdash & \text{Fwd}^1_{x,y} :: x : \bot, y : 1 \\
\vdash & 0 :: \bot^1 \quad \text{MIX}_0 \\
\vdash & y[\bot].0 :: x : \bot, y : 1
\end{align*}
$$

$$
\begin{align*}
\vdash & \exists X \quad X \quad \text{Fwd}^\exists X_{x,y} :: x : \forall X.X^\bot, y : \exists X.X \\
\vdash & x(X).y[X] \leftrightarrow y :: x : \forall X.X^\bot, y : \exists X.X
\end{align*}
$$
4.2 Operational Semantics of Derivations

We present the SOS specification for the new ingredients of πLL compared to πMLL.

**Additives.** The derivation rules for selection (⊕₁, ⊕₂) and choice (⊗) are given below. There are left and right rules for actions and communications. They are all symmetric.

**Quantifiers.** Polymorphism is achieved by communicating types, according to the following transition axioms and rule.

**Axiom.** For derivations that consist of an application of rule `AX`, we have two (symmetric) transition axioms.
When the transition of an axiom interacts with a related cut, the cut disappears and a substitution is performed. This rule mimics the typical simplification of cuts applied to axioms found in linear logic [Wadler 2014].

\[ \vdash P : \Gamma, x : A \rightarrow \vdash P : \Gamma, x : \Delta \rightarrow \vdash P : \Gamma, x : A \]

Exponentials. The semantics of exponentials models interactions between clients and servers. A clients can use a server in three ways: request a single instance of the process provided by the server, duplicate the server, or dispose of the server.

Instance requests are modelled by the following rules, for the prefixes and their interaction.

\[ \vdash (\forall x) P : \Gamma, x : \Delta, y : !A \rightarrow \vdash P : \Gamma, x : A \]

The following rules model server duplication. A technicality: since a server can depend on other servers (the \( ? \Gamma \) in rule !), duplicating a server requires in turn to duplicate its dependencies. In the following, we use \( \sigma \) to denote a name substitution. We write \( P, \sigma, \Gamma, \Delta, x \sigma \) for the application of a substitution \( \sigma \) to a process \( P \), an environment \( \Gamma \), a derivation \( D \), and a name \( x \sigma \), respectively.

\[ \vdash P : \Gamma, x : A \rightarrow \vdash x(x') : P : \Gamma, x : A \]

The following rules model server duplication. A technicality: since a server can depend on other servers (the \( ? \Gamma \) in rule !), duplicating a server requires in turn to duplicate its dependencies. In the following, we use \( \sigma \) to denote a name substitution. We write \( P, \sigma, \Gamma, \Delta, x \sigma \) for the application of a substitution \( \sigma \) to a process \( P \), an environment \( \Gamma \), a derivation \( D \), and a name \( x \sigma \), respectively.
The disposal of a server is modelled by the following rules, again for the prefixes and their interaction. Disposing of a server triggers the disposal of all its dependencies.

\[
\begin{align*}
\frac{\mathcal{D} \vdash P \vdash \Gamma}{\not\vdash \exists x[P].P \vdash \Gamma, x : ?A} & \quad \frac{\vdash x[P].P \vdash \Gamma, x : \bot}{\vdash ?x[P].P \vdash \Gamma, x : \bot} \\
\frac{\vdash P \vdash ?\Gamma, x^?: A}{\not\vdash !x(P) \vdash ?\Gamma, x : !A} & \quad \frac{\vdash x(P) \vdash ?\Gamma, x : !A}{\vdash ?z_1[\ldots]z_n[P].\bot \vdash ?\Gamma, x : \bot}
\end{align*}
\]

\[\text{fn}(P) \setminus \{x'\} = \{z_1, \ldots, z_n\} \]

Let \(\mathcal{D}\) be a derivation.

1. \(\frac{\vdash P \vdash \Gamma, x : ?A | \Delta, y : !A}{\not\vdash (\forall x y)P \vdash \Gamma, \Delta} \quad \frac{\vdash P \vdash \Gamma, x : ?A | \Delta, y : !A}{\not\vdash (\exists x y)P \vdash \Gamma, \Delta} \quad \frac{\vdash P \vdash \Gamma, x : ?A | \Delta, y : !A}{\not\vdash (\forall x)P \vdash \Gamma, \Delta} \quad \frac{\vdash P \vdash \Gamma, x : ?A | \Delta, y : !A}{\not\vdash (\exists x)P \vdash \Gamma, \Delta} \quad \frac{\vdash P \vdash \Gamma, x : ?A | \Delta, y : !A}{\not\vdash (\forall y)P \vdash \Gamma, \Delta} \quad \frac{\vdash P \vdash \Gamma, x : ?A | \Delta, y : !A}{\not\vdash (\exists y)P \vdash \Gamma, \Delta}
\]

\(\mathcal{D}'\) is the least relation closed under the SOS rules in \(\mathcal{D}\). The resulting specifications are in \(\mathcal{D}'\) and \(\mathcal{D}\).

\(\mathcal{D}\) is the least relation closed under the SOS rules forming the specificication of \(\pi\text{MLL}\) (see Section 2.2) and the ones stated in this subsection.

**Definition 4.7.** The lts of derivations for \(\pi\text{LL}\), denoted \(\text{lts}_d\), is the triple \((\text{Der}, \text{Lbl}, \rightarrow)\), where:

- The set \(\text{Der}\) is the set of typing derivations for \(\pi\text{LL}\).
- The set \(\text{Lbl}\) is the set of transition labels.
- The relation \(\rightarrow \subseteq \text{Der} \times \text{Lbl} \times \text{Der}\) is the least relation closed under the SOS rules forming the specificication of \(\pi\text{MLL}\) (see Section 2.2) and the ones stated in this subsection.

### 4.3 Operational Semantics of Processes and Environments

To define the semantics for processes and typing environments of \(\pi\text{LL}\), we follow the recipe introduced in Section 2.3.

The first step is to specify how derivations are projected to process terms and typing environments. Let \(\text{Proc}\) and \(\text{Env}\) denote the sets of all process terms and typing environments of \(\pi\text{LL}\). The projections \(\text{proc} : \text{Der} \rightarrow \text{Proc}\) and \(\text{env} : \text{Der} \rightarrow \text{Env}\) are defined as in Section 2.3:

\[
\text{proc} \left( \frac{\vdash P \vdash \mathcal{G}}{P} \right) = P \quad \text{env} \left( \frac{\vdash P \vdash \mathcal{G}}{P} \right) = \mathcal{G}.
\]

The second step of the recipe is to formalise the notions of erasure and session fidelity. Thanks to our characterisation in terms of homomorphisms, we only need to instantiate Definitions 2.7 and 2.13 with the \(\text{Lbl}, \text{lts}_d, \text{proc}, \text{env}\) defined above.

The third step is to apply the principles of “erasing types” and “erasing processes” to the SOS for \(\pi\text{LL}\) derivations. The resulting specifications are in Figures 6 and 7, respectively.

**Definition 4.8.** The lts of processes for \(\pi\text{LL}\), denoted \(\text{lts}_p\), is the triple \((\text{Proc}, \text{Lbl}, \rightarrow)\) where

- \(\text{Proc}\) is the set of process terms for \(\pi\text{LL}\),
- \(\text{Lbl}\) is the set of labels for \(\pi\text{LL}\),
- \(\rightarrow \subseteq \text{Proc} \times \text{Lbl} \times \text{Proc}\) is the least relation closed under the SOS rules in Figure 6.
Definition 4.9. The LTS of environments for π LL, denoted \( \text{LTS}_e \), is the triple \( (\text{Env}, \text{Lb}l, \rightarrow) \) where:

- \( \text{Env} \) is the set of typing hyperenvironments for \( \pi \text{ LL} \),

\[
\begin{align*}
\text{LTS}_e &= (\text{Env}, \text{Lb}l, \rightarrow) \\
\text{Env} &= \{ \emptyset, \Gamma \} \\
\text{Lb}l &= \{ \\
\vdash &\text{type} \} \\
\rightarrow &= \{ \}
\end{align*}
\]
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- $Lbl$ is the set of labels for $\pi$LL,
- $\rightarrow \subseteq Env \times Lbl \times Env$ is the least relation closed under the SOS rules in Figure 7.

The fourth and final step of the recipe is to verify that semantics of processes and of typing environments enjoy erasure and session fidelity.

**Theorem 4.10 (Erasure).** $lts_p$ enjoys erasure wrt $lts_d$.

**Theorem 4.11 (Session Fidelity).** $lts_p$ enjoys session fidelity wrt $lts_e$.

As for $\pi$MLL, erasure entails typability preservation.

**Corollary 4.12 (Typability Preservation).** Let $P$ be well-typed. Then, $P \rightarrow P'$ implies that $P'$ is well-typed.

## 4.4 Metatheory

All definitions and results presented in Section 3 apply to $\pi$LL. Here we present explicitly the ones that are most relevant or that will be needed later.

### 4.4.1 Behavioural Theory. The standard definitions of behavioural equivalences and preorders (Definitions 3.1 and 3.3), their laws (Fact 3.2), and definition of congruence (Definition 3.4) apply to $\pi$LL without modification. When we unfold Definition 3.4 using the grammar of $\pi$LL we obtain that a relation on processes $\equiv$ is a (pre)congruence if $P \equiv Q$ implies that:

1. $P | R \equiv Q | R$ for any $R$ (and the symmetric);
2. $\pi.P \equiv \pi.Q$ for any prefix $\pi$;
3. $\forall x.y P \equiv (\forall x.y)Q$ for any $x, y$;
4. $x\{\text{inl} : R; \text{inr} : P\} \equiv x\{\text{inl} : R; \text{inr} : Q\}$ for any $x$ and $R$ (and the symmetric).

The only difference with respect to $\pi$MLL is the new case for external choice (Item 4) which is the only new term constructor introduced by $\pi$LL that is not a prefix.

**Theorem 4.13 (Congruence).** On the lts of processes, $\preceq$ and $\subseteq$ are precongruences and $\sim$ and $\approx$ are congruences.

The type checking and similarity checking for $\pi$LL coincide.

**Theorem 4.14.** For $P$ well-typed and $\mathcal{G}$ such that $\text{fn}(P) = \text{cn}(\mathcal{G})$, $P \subseteq \mathcal{G}$ iff $\vdash P :: \mathcal{G}$.

It follows that bisimilarity is sound with respect to type equivalence.

**Corollary 4.15.** For $P$ and $Q$ well-typed, if $P \approx Q$ and $\vdash Q :: \mathcal{G}$, then $\vdash P :: \mathcal{G}$.

**Remark 4.16 (Soundness of $\eta$-expansion).** For the first time, we can use bisimilarity to observe that $\eta$-expansion is behaviourally sound. Specifically, under the ”linking” semantics of $\pi$LL for forwarders, $x\rightarrow y$ is indistinguishable from its expansion (recall Remark 4.6) for any well-typed process, in the following sense.

$$(\forall x.y) (P | y \rightarrow z) \approx (\forall x.y) (P | F w d^A_{y,z})$$

for any $A$ and $\vdash P :: \mathcal{G} | \Gamma, x :: A$.

### 4.4.2 Metatheory of Parallelism. $\pi$LL enjoys the diamond property, serialisation, non-interference, and readiness.

**Theorem 4.17 (Diamond Property).** The lts of derivations enjoys the diamond property.

**Theorem 4.18 (Serialisation).** The lts of derivations enjoys serialisation.

**Theorem 4.19 (Non-Interference).** The lts of derivations enjoys the non-interference.
The definition of disentanglement (Lemma 3.28) applies to πLL without modification. As for πMLL, we can compute disentanglement directly using a procedure given by recursion on the structure of processes. The definition of this procedure follows the one given in Lemma 4.20 for πMLL. We only need to add two cases, for \( x \rightarrow y \) and \( x \rightarrow \{ \text{inl}: P; \text{inr}: Q \} \), which are immediate.

**Lemma 4.20.** The disentanglement \( \text{dis}(P) \) of a process \( P \) is the process recursively defined as follows:

\[
\text{dis}(\pi P) = \pi \cdot \text{dis}(P) \\
\text{dis}(x \rightarrow y) = x \rightarrow y \\
\text{dis}(0) = 0 \\
\text{dis}(P \mid Q) = \begin{cases} 
\text{dis}(P) & \text{if } Q = 0 \\
\text{dis}(P) \mid \text{dis}(Q) & \text{otherwise}
\end{cases}
\]

\[
\text{dis}((\forall xy) P) = (\forall xy) (P_1 \mid P_j) \mid (\text{dis}(P) \setminus P_1, P_j) \text{ where } \text{dis}(P) = P_1 \mid \cdots \mid P_n, x \in \text{fn}(P_i), y \in \text{fn}(P_j)
\]

**Theorem 4.21 (Readiness).** Let \( \vdash P : \Gamma_1 \mid \cdots \mid \Gamma_n. \) For every \( i \in [1, n] \), there exist \( x \in \text{cn}(\Gamma_i) \) and \( l \) such that \( x \in \text{fn}(l) \) and \( P \Rightarrow l \).

**4.4.3 Relation with Linear Logic and Classical Processes.** All the ingredients we used in Section 3.3 to relate πMLL to the multiplicative fragment of CLL and CP extend without modifications to relate πLL to CLL and CP: the internalisation of “\( \& \)" and “\( \mid \)" and the definition of packing rely only on the multiplicative fragment of πLL and both are unaffected by the introduction of the axiom, additives, exponentials, and quantifiers.

**Theorem 4.22.** \( \vdash P : G \text{ in } \pi LL \text{ iff } \\vdash_{\text{CP}} \text{pack}_x (\vdash P : G) : x : \otimes G \text{ in } \text{CP}. \)

**Corollary 4.23.** If \( D \) is a cut-free derivation of \( \otimes G \) in CLL, then \( \psi(D) \) is a cut-free derivation of \( G \) in \( \pi LL \).

## 5 HIGHER-ORDER πLL

In this section, we extend our approach to process mobility, by enhancing πLL with higher-order communication primitives. In particular, we investigate rules that (i) type new process terms that capture the feature of process mobility, and (ii) enable proof transformations that yield their expected semantics. We call this language Higher-Order πLL (abbreviated as HOπLL).

### 5.1 Processes and Typing

**Processes.** To obtain process terms that enable process mobility, we can get direct inspiration from the higher-order π-calculus (HOπ) [Sangiorgi and Walker 2001]. We need three basic features:

- A term for sending process code over a channel.
- A term for receiving process code over a channel and storing it in a process variable.
- A term for running the process code stored in a variable.

We thus extend the syntax of πLL as follows, where \( p, q, r, \ldots \) range over process variables.

\[
P, Q ::= \cdots \mid x[(\rho)P].Q \quad \text{output abstraction (\( \rho \)\( P \) on \( x \) and continue as \( Q \)}
\mid x(p).P \quad \text{input an abstraction as } p \text{ on } x \text{ and continue as } P
\mid p^G(\rho) \quad \text{run abstraction } p \text{ instantiated with } \rho
\mid P[q := (\rho)Q] \quad \text{explicit process variable substitution, “chop”}
\]

We describe the new terms.
The higher-order output term \( x[(\rho)P].Q \) sends the abstraction \( (\rho)P \) over channel \( x \) and then continues as \( Q \). An abstraction is a parameterised process term, enabling the reuse of process code in different contexts. The concept of abstraction is standard in the literature of HO\(\pi\) [Sangiorgi and Walker 2001]; here, we apply a slight twist and use named formal parameters \( \rho \) rather than just a parameter list, to make the invocation of abstractions not dependent on the order in which actual parameters are passed. This plays well with the typing contexts of linear logic, used for typing process terms in \( \pi\mathbb{L} \), since they are order-independent as well (exchange is allowed). We use \( f \) to range over formal parameter names. These are constants, and are thus not affected by \( \alpha \)-renaming. Formally, \( \rho \) is a record that maps parameter names to channels used inside \( P \), i.e., \( \rho = \{f_i = x_i\}_{i \in I} \) (where \( I \) is finite). We omit curly brackets for records in the remainder when they are clear from the context, e.g., as in \( p^{\Theta}(f_1 = x, f_2 = y) \). Given a record \( \rho = \{f_i = x_i\}_{i \in I} \), we call the set \( \{f_i\}_{i \in I} \) the preimage of \( \rho \) and the set \( \{x_i\}_{i \in I} \) the image of \( \rho \). An abstraction \( (\rho)P \) binds all names in the image of \( \rho \) in \( P \). We require abstractions to be closed with respect to channels, in the sense that the image of \( \rho \) needs to be exactly the set \( \text{fn}(P) \) of all free channel names in \( P \).

Dual to higher-order output, the higher-order input term \( x(\rho).Q \) receives an abstraction over channel \( x \) and stores it in the process variable \( \rho \), which can be used in the continuation \( Q \).

Abstractions stored in a process variable \( \rho \) can be invoked (we also say run) using term \( p^{\Theta}(\rho) \), where \( \rho \) are the actual named parameters to be used by the process.

**Remark 5.1.** The choice of having (named) parameters and abstractions is justified by the desire to use received processes to implement behaviours. For example, we may want to receive a channel and then use a process to implement the necessary behaviour on that channel: \( x(y).x(\rho).p^{\Theta}(f = y) \).

A different way of achieving the same result would be to support the communication of processes with free names (not abstractions), and then allow invoke terms to dynamically rename free names of received processes. For example, if we knew from typing that any process received for \( \rho \) above has \( z \) as a free name, we could write \( x(y).x(\rho).p^{\Theta}(y/z) \).

Such a dynamic binding mechanism would make our syntax simpler, since invocations would simply be \( p^{\Theta}(\sigma) \) (\( \sigma \) is a name substitution) and the higher-order output term would be \( x[P].Q \). However, dynamic binding is undesirable in a programming model, since the scope of free names can change due to higher-order communications. The reader unfamiliar with dynamic binding may consult [Sangiorgi and Walker 2001, p. 376] for a discussion of this issue in HO\(\pi\).

**Types.** Types in HO\(\pi\mathbb{L} \) extend those of \( \pi\mathbb{L} \) to include types for higher-order outputs and inputs. These types are inspired to [Montesi 2018], the only difference the explicit treatment of parallelism with the use of hyperenvironments instead of environments.

\[
A, B ::= \ldots \quad \text{all types from } \pi\mathbb{L} \\
| \langle G \rangle \quad \text{higher-order output} \\
| \langle G \rangle^\perp \quad \text{higher-order input}
\]

From the perspective of typing derivations, the new type constructor \( \langle G \rangle \) can be interpreted as "assumes a derivation of \( G^\perp \)" and, dually, \( \langle G \rangle \) as "provides a derivation of \( G \)."

Duality is as in \( \pi\mathbb{L} \) and extended the new higher-order types.

\[
\langle G \rangle^\perp = \langle G \rangle \quad \langle G \rangle = \langle G \rangle^\perp
\]

**Typing Environments.** Environments and hyperenvironments are defined as in \( \pi\mathbb{M} \) (but contain types in the extended syntax).

A process environment (\( \Theta, \Pi, \ldots \)) associates process symbols to hyperenvironments. We write process environments as lists; for example, \( \Theta = p_1: G_1, \ldots, p_n: G_n \) associates each \( p_i \) to its respective hyperenvironment \( G_i \), for \( 1 \leq i \leq n \). Notice, that since each \( p_i \) is going to be instantiated by
an abstraction, we are not typing channels in each $G_i$, but the names of formal parameters ($f$). We write $\text{pv}(\Theta)$ for the set of process symbols associated by $\Theta$. All process symbols in an environment are distinct. Process environments allow for exchange, that is, order in environments is ignored. Process environments can be combined when they do not share process symbols: assuming that $\Theta$ and $\Pi$ do not share names ($\text{pv}(\Theta) \cap \text{pv}(\Pi) = \emptyset$), $\Theta, \Pi$ is the process environment that consists exactly of all the associations in $\Theta$ and those in $\Pi$. Combining processes is a commutative, associative partial operations with $\cdot$, the empty process environment, acting as unit.

**Typing Judgements.** Typing judgements in $\text{HO}\pi\text{LL}$ have the form $\Theta \vdash P : G$ and state that process $P$ uses its free names according to $G$ and its free process variables represent processes that use their names according to the corresponding type in $\Theta$. Judgements can be derived by using the inference rules displayed in Figure 8.

Each typing rule, has a distinct process environment for each of its premises and these are combined into the in the rule conclusion. The only exception in this process is rule $\uparrow$ which requires an empty process environment to ensure the linear usage of process symbols. We obtained the remaining rules for typing terms forming $\pi\text{LL}$ (i.e., all rules Figure 8 except rules $\text{CHOP}, \text{ID}, \uparrow, [\cdot]$ and $()$) by applying this discipline to the typing rules of $\pi\text{LL}$. We refer to this process as “lifting”.

Rule $\text{ID}$ says that if we run $p$ by providing all the name parameters (collected in $\rho$) required by the abstraction that it represents, then we implement exactly the session types that type the code that may be associated with in $p$ (modulo the renaming of the formal parameters performed with $\rho$, hence the $G_\rho$). Note that we use all our available “resources”—the endpoint names in the image of $\rho$—to run $p$, passing them as parameters. This ensures linearity. Rule $()$ says that if we receive a process of type $G_\rho$ over endpoint $x$ and associate it with $p$, we can use $p$ later in the continuation $Q$ assuming that it implements $G$ hence the $x$ is $(G)$. Rule $[\cdot]$ says that if we send a process of type $P$ over channel $x$ and $P$ implements $G$, then $x$ has type $[G]$. Note that we require $P$ to use all the process symbols available in the context $(\Theta)$. The association of process symbols and abstractions is maintained by the term for explicit substitution and this is typed by rule $\text{CHOP}$. This rule says
that we can replace \( p \) with \((\rho)P\) in \( Q\), provided that \( P\) and \( Q\) have compatible typing (up to the formal named parameters). The idea is that a variable \( p \) stands for a “hole” in a proof, which has to be filled as expected by the type for \( p \).

The typing rules for sending and receiving processes is close to the ones for multiplicative units in their use of endpoints (say, instead of the primitives for sending and receiving channels): the endpoint is discarded in the continuation. This is design is aimed at keeping the language minimal in the sense that these variations can be recovered as syntax sugar.

**Remark 5.2 (Higher-order I/O with continuations).** We can derive constructs for sending and receiving processes over channels and then allow us to continue using that channel. We distinguish these sugared counterparts from our output and input primitives by using bold brackets.

\[
P, Q ::= \ldots \mid x[(\rho)P], Q \quad \text{output a process and continue} \mid x(\rho)P \quad \text{input a process and continue.}
\]

These constructs are desugared as follows (we show directly the proofs).

\[
\begin{align*}
\Theta \vdash P : \mathcal{G}\rho & \quad \Pi \vdash Q : \Gamma, x : A \\
\Theta, \Pi \vdash x[(\rho)P], Q : \Gamma, x : \langle \mathcal{G} \rangle \otimes A & \quad \triangleq \\
\Theta \vdash P : \mathcal{G}\rho & \quad \Gamma \vdash 0 : \emptyset \\
\Theta, (\rho)P, y, 0 \vdash y : \langle \mathcal{G} \rangle & \quad \Pi \vdash Q : \Gamma, x : A \\
\Theta, \Pi \vdash y[(\rho)P], 0 \mid Q : \langle \mathcal{G} \rangle & \mid \Gamma, x : A \\
\Theta, \Pi \vdash x[y].(y[(\rho)P], 0) : \langle \mathcal{G} \rangle \otimes A & \quad \otimes \\
\Theta \vdash x(y).y[(\rho)P] : \Gamma, x : \langle \mathcal{G} \rangle \otimes A & \quad \otimes \\
\Theta \vdash x(\rho)P : \Gamma, x : \langle \mathcal{G} \rangle & \quad \langle \\
\Theta \vdash x(\rho)P : \Gamma, x : \langle \mathcal{G} \rangle \otimes A & \quad \otimes
\end{align*}
\]

**Remark 5.3 (Procedures).** If the abstraction that we send over a channel does not refer to any free process variable, then we can always replicate it as many times as we wish. Here is the proof.

\[
\begin{align*}
\triangleright P : \mathcal{G}\rho & \quad \vdash 0 : \emptyset \\
\triangleright y[(\rho)P], 0 \vdash y : \langle \mathcal{G} \rangle & \quad ! \\
\triangleright !x(y).y[(\rho)P], 0 \vdash x : \langle \mathcal{G} \rangle & \quad ! \\
\end{align*}
\]

We use this property to build a notion of procedures that can be used at will. We denote procedure names with \( K \), for readability. We will later use it as a channel name in our desugaring.

\[
P, Q ::= \ldots \mid \text{def } K := (\rho)P \text{ in } Q \quad \text{procedure definition} \mid K(\rho) \quad \text{procedure invocation}
\]

A term \( \text{def } K := (\rho)P \) in \( Q \) defines procedure \( K \) as \((\rho)P\) in the scope of \( Q \), and a term \( K(\rho) \) invokes procedure \( K \) by passing the parameters \( \rho \).

Below is the desugaring of both constructs. For simplicity of presentation, we assume that \( K \) in term \( \text{def } K := (\rho)P \) in \( Q \) is used at least once in \( Q \). The generalisation to the case where \( Q \) does not use \( K \) at all is straightforward (thanks to rule \text{cut} \( w \)).

\[
\begin{align*}
\vdash P : \Delta\rho & \quad \Theta \vdash Q : \mathcal{G} \mid \Gamma, K : ?(\Delta) \\
\Theta \vdash \text{def } K := (\rho)P \text{ in } Q : \mathcal{G} \mid \Gamma & \quad \triangleq \\
\vdash P : \Delta\rho & \quad \Theta \vdash 0 : \emptyset \\
\vdash y[(\rho)P], 0 \vdash y : \emptyset & \quad ! \\
\vdash !x(y).y[(\rho)P], 0 \vdash x : ?(\Delta) & \quad ! \\
\Theta \vdash x(y).y[(\rho)P] : \mathcal{G} \mid \Gamma, K : ?(\Delta) & \quad \Gamma \vdash Q \quad \Theta \vdash \mathcal{G} \mid \Gamma, K : ?(\Delta) & \quad \otimes \\
\Theta \vdash (v x K) : (\mathcal{G} \mid \Gamma, K : ?(\Delta)) & \quad \vdash (v x K) : (\mathcal{G} \mid \Gamma, K : ?(\Delta)) & \quad \otimes
\end{align*}
\]
It is possible to derive a more general rule for procedure definitions that allows the type of the
body $P$ to have any number of sequents (instead of exactly one). The same generalisation is not
possible for procedure calls since rule $\triangleright$ expects exactly one sequent.

Observe that even if procedures can be used at will, typing ensures that each usage respects
linearity (i.e., every usage "consumes" the necessary linear resources available in the context). Note also that self-invocations are not supported, as typing forbids them (types must be finite).

**Remark 5.4 (Higher-order parameters).** We chose not to make abstractions parametric on pro-
cess variables for economy of the calculus. This feature can be reconstructed with the following
syntactic sugar.

$$P, Q ::= \ldots \mid x\lambda q.P \quad \text{named higher-order parameter} \mid P(x = (\rho)Q) \quad \text{application}$$

The desugaring is simple, interpreting a named higher-order parameter as a channel on which we
perform a single higher-order input.

$$\Theta, p : \Delta \vdash P : \Gamma \quad \Rightarrow \quad \Theta, p : \Delta \vdash P \vdash \Gamma$$

The desugaring of $P(x = (\rho)Q)$ yields a process that allows for reductions to happen in $P$ before
the application takes place, since the corresponding higher-order named parameter term may be
nested inside of $P$. Also, this desugaring cannot be implemented by merely using an application
of rule $\triangleright$, since $p$ is bound to $P$ in term $x\lambda p.P$ (as can be observed by its desugaring), and rule $\triangleright$ acts on free process variables.

### 5.2 Example: A cloud server

We illustrate the expressiveness of HOπLL by implementing a cloud server for running applica-
tions that require a database (this example is adapted from the technical report [Montesi 2018]). The idea is that clients are able to choose between two options: run both the application and
the database it needs in the server, or run just the application in the server and connect it to an
externally-provided database (which we could imagine is run somewhere else in the cloud).

A cloud server. The process for the cloud server follows. We assume that $A$ is the protocol (left
unspecified) that applications have to use in order to communicate with databases, and $f$ is the
named parameter used by both abstractions to represent this shared connection (the parameter
names may be different, we do this only for a simpler presentation). We also let applications and
databases access some external services, e.g., loggers, through the parameters $\overline{f}$ and $\overline{g}$, respectively.
The types for our cloud server follows the intuition that we initially discussed for this example. And this is the proof for the right branch.

\[ \text{Let } \Gamma = (f : A_1), \Delta = (g : B_1), \rho = \{ f = z, f = u \}, \text{ and } \rho' = \{ f = w, g = v \}. \]

For readability, we first type the left and right branches in the choice offered through \( x \). Here is the proof for the left branch.

\[
\begin{align*}
\text{app: } & (\Delta, f : A) \vdash \text{app}(\rho) : ?\Gamma, \rho, z : A \\
\text{db: } & (?\Delta, f : A^+) \vdash db(\rho') : ?\Delta, \rho', w : A^+ \\
\text{app: } & (?\Delta, f : A^+) \vdash \text{db}(\rho') : ?\Gamma, \rho, z : A \\
\text{app: } & (?\Delta, f : A^+) \vdash \text{app}(\rho) \parallel db(\rho') \\
\end{align*}
\]

\[
\cdot \vdash x(\text{app}). x'(db). (\text{vzw}) (\text{app}(\rho)) \parallel db(\rho') : ?\Gamma, ?\Delta, x' : (?\Delta, f : A^+), x : (?\Delta, f : A^+) \parallel (?\Gamma, f : A)
\]

And this is the proof for the right branch.

\[
\begin{align*}
\text{app: } & (\Delta, f : A) \vdash \text{app}(\rho) : ?\Gamma, \rho, z : A \\
\text{extdb: } & extdb \vdash extdb : A, w : A^+ \\
\text{app: } & (?\Delta, f : A^+) \vdash \text{app}(\rho) \parallel extdb \vdash extdb : A, w : A^+ \\
\end{align*}
\]

\[
\cdot \vdash x(\text{app}). (\text{vzw})(\text{app}(\rho)) \parallel extdb \vdash extdb : A, x : (?\Gamma, f : A) \\
\cdot \vdash x(\text{app}). (\text{vzw})(\text{app}(\rho)) \parallel extdb \vdash extdb : ?\Gamma, ?\Delta, x : (?\Delta, f : A^+) \parallel (?\Gamma, f : A)
\]

Now that we have proofs for the two branches, we can use them to type the entire cloud server. Let \( \mathcal{L} \) and \( \mathcal{R} \) be the two proofs above, respectively, and \( P_L \) and \( P_R \) the processes that they type (the left and right branches in the cloud server). Then, we type the cloud server as follows.

\[
\begin{align*}
\mathcal{L} & \cdot \vdash \text{extdb} : w \parallel \text{extdb} \vdash \text{extdb} : A, w : A^+ \\
\mathcal{R} & \cdot \vdash \text{extdb} \vdash \text{extdb} : A, w : A^+ \\
\end{align*}
\]

\[
\cdot \vdash !x : (?\Delta, f : A^+) \parallel (?\Gamma, f : A) \parallel (?\Delta, f : A^+) \parallel (?\Gamma, f : A)
\]

The types for our cloud server follows the intuition that we initially discussed for this example. The typing derivation also illustrates the interplay between our new features (process mobility and the usage of process variables) with the other features of the calculus—in this example: channel mobility, exponentials (replicated services), choices, and links.

**A generic improvement.** The code of our cloud server implementation would not depend on how clients and database communicate, were it not for the hardcoded protocol \( A \). We can get rid of the hardcoded \( A \) and reach a generic implementation using polymorphism. Here is the improved

implementation, where we underline the improvements.

\[
!cs(x).x(X).x! \begin{cases}
\text{inl}: x(x').x(app).x'(db).vzw \left( app(f = z, f' = u) \mid db(f = w, g = v) \right) \\
\text{inr}: x(extendb).x(app).vzw \left( app(f = w, f' = u) \mid \text{extendb} \leftrightarrow w \right)
\end{cases}
\]

In the improved cloud server, the client must also send us the protocol \( X \) that the application will use to communicate with the database. The cloud server is thus now generic, and the type of channel \( cs \) is the following.

\[
!(\forall X. \left( \left( \left( \forall \Delta, f : X' \right) \Leftrightarrow \left( \forall \Gamma, f : X \right) \right) \land \left( X \Leftrightarrow \left( \forall \Gamma, f : X \right) \right) \right))
\]

Here is the proof.

\[
\begin{align*}
&!x[]{\text{inl}: P_L; \text{inr}: P_R} \vdash \tau L, \tau R', x : \left( \left( \forall \Delta, f : X' \right) \Leftrightarrow \left( \forall \Gamma, f : X \right) \right) \land \left( X \Leftrightarrow \left( \forall \Gamma, f : X \right) \right) \\
&!x(x).x!{\text{inl}: P_L; \text{inr}: P_R} \vdash \tau L, \tau R', \forall X. \left( \left( \forall \Delta, f : X' \right) \Leftrightarrow \left( \forall \Gamma, f : X \right) \right) \land \left( X \Leftrightarrow \left( \forall \Gamma, f : X \right) \right) \vdash\text{cut}
\end{align*}
\]

The proofs \( \mathcal{L}(X/A) \) and \( \mathcal{R}(X/A) \) used above are as \( \mathcal{L} \) and \( \mathcal{R} \) respectively, but wherever we had \( A \) we now have \( X \). The interplay between type variables and process variables merits illustration, so we show \( \mathcal{R}(X/A) \) in full.

\[
\begin{align*}
\text{id} & : x!{\text{app}: \left( \forall \Delta, f : X' \right) \Leftrightarrow \left( \forall \Gamma, f : X \right) \land \left( X \Leftrightarrow \left( \forall \Gamma, f : X \right) \right)} \\
\text{cut} & : x{\text{app}: \left( \forall \Delta, f : X' \right) \Leftrightarrow \left( \forall \Gamma, f : X \right) \land \left( X \Leftrightarrow \left( \forall \Gamma, f : X \right) \right)}
\end{align*}
\]

Observe the application of rule \( \otimes \) in the proof. If we read it bottom-up, we are moving the type variable \( X \) from the typing of a channel in the conclusion—\( x!{X \Leftrightarrow \left( \forall \Gamma, f : X \right)} \)—to the typing of a process in the premise—\( \text{app}: \left( \forall \Delta, f : X' \right) \Leftrightarrow \left( \forall \Gamma, f : X \right) \land \left( X \Leftrightarrow \left( \forall \Gamma, f : X \right) \right) \). This is then carried over to the application of rule \( \text{id} \), which is thus able to type the usage of a process variable that is generic on the behaviour that will be enacted.

5.3 Operational Semantics of Derivations

We present the SOS specification for the new ingredients of HO\(\pi\)LL compared to \(\pi\)LL. The full SOS specification can be found in Appendix A.

The semantics of the fragment corresponding to \(\pi\)LL does not interact with the process environment (\(\Theta\)) in any significant way and can thus be defined by lifting the SOS specification of \(\pi\)LL (Section 4.2). Intuitively, this amounts to adding process environments following the same receptacle we used to lift the typing rules of \(\pi\)LL. For instance, the semantics of rule \( \bot \) is given by the following axiom.

\[
\begin{array}{c}
\Theta \vdash P \equiv \Gamma \\
\Theta \vdash x().P \equiv \Gamma, x : \bot
\end{array}
\]

Process mobility. The derivation rules for higher-order output ([|]) and input ([|]) are similar to the rules of other action prefixes and are given below.
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The derivation rule for higher-order communication synchronises a send and receive over connected endpoints and records the association between the process abstraction from the sender and the process symbol at the receiver with an explicit substitution (CHOP). This is similar to how in endpoint communication, the association between the output and input endpoints is recorded in a new restriction term (CUT) by rule $\otimes \emptyset$.

$$
\frac{D}{\Theta, \Pi \vdash Q : G | x : [H] | \Gamma, y : \langle H \rangle} \quad \frac{(x | (\rho) E) \mid y(p)}{\Theta, p : H \vdash Q' : G | \Gamma}
$$

$[\emptyset \emptyset]$

Explicit substitution. To invoke the process bound to a symbol variable we need to obtain this process from the context of the invocation, i.e., the closest substitution term binding the process symbol (which can then be discarded thanks to the linear use of process symbols). This mechanism is captured by the derivation rules below.

$$
\frac{D}{p : G \vdash \rho^G(\sigma) \bowtie G\sigma} \quad \frac{[\rho^G := (\rho) D]}{\Pi \vdash P(\sigma \bowtie \rho^{-1}) \bowtie G\sigma}
$$

$$
\frac{D}{\Theta, p : H \vdash Q : G} \quad \frac{[\rho^H := (\rho) E]}{\Theta, \Pi \vdash Q' : G}
$$

Explicit substitution.

$$
\frac{\Pi \vdash P \bowtie H\rho \quad \Theta, p : H \vdash Q \bowtie G}{\Theta, \Pi \vdash Q[p := (\rho) P] \bowtie G} \quad \frac{\mathcal{E}}{\Theta, \Pi \vdash Q' \bowtie G} \quad \frac{\mathcal{E}}{\Theta, p : H \vdash Q' \bowtie G}
$$

$$
\frac{\Pi \vdash P \bowtie H\rho \quad \Theta, p : H \vdash Q \bowtie G}{\Theta, \Pi \vdash Q[p := (\rho) P] \bowtie G} \quad \frac{\mathcal{E}}{\Theta, p : H \vdash Q' \bowtie G'}
$$

The following derivation rule captures scope extrusion for process symbols that occur in the payload of a higher-order output (rule $[\cdot]$ allows for free process symbols in abstractions).

$$
\frac{\mathcal{D}}{\Theta, p : H \vdash Q \bowtie G} \quad \frac{x[(\sigma) \mathcal{F}]}{\Theta' \vdash Q' \bowtie G'} \quad \frac{\mathcal{E}}{\mathcal{F}'} \quad \frac{\mathcal{F}}{\Xi, \Pi \vdash R \bowtie I} \quad \frac{\mathcal{E}}{\Xi, \Pi \vdash R[p := (\rho) P] \bowtie I} \quad \frac{p \in \text{fpv}(R)}{\Xi, \Pi \vdash R'} \quad \frac{\mathcal{E}}{\Xi, \Pi \vdash R'} \quad \frac{\mathcal{E}}{\Xi, \Pi \vdash R'}
$$

Similarly to rule RES, the following rule performs the propagation (or lifting) of actions that do not involve the process symbol bound by the explicit substitution term.

$$
\frac{\mathcal{E}}{\Theta, p : H \vdash Q \bowtie G} \quad \frac{\mathcal{D}}{\Theta', p : H' \vdash Q' \bowtie G'} \quad \frac{\mathcal{E}}{\Xi, \Pi \vdash R \bowtie I} \quad \frac{\mathcal{E}}{\Xi, \Pi \vdash R[p := (\rho) P] \bowtie I} \quad \frac{p \notin \text{fpv}(l)}{\Theta', p : H' \vdash Q' \bowtie G'}
$$

LTS of Derivations. We extend the sets $\text{Act}$ of action labels and $\text{Lbl}$ of all labels defined in Section 4.2 to include the labels introduced in this section. To avoid “leaking” derivations to the labels used by the transition systems of processes and environments we parametrise the definition...
of these sets in a $S$ of higher-order payloads–for derivations $S = \text{Der}$, for processes $S = \text{Proc}$, etc.

$\text{Act}(S) \triangleq \{ x[], x(), x[y], x(y), x \leftarrow \text{inl} x \rightarrow \text{inr} x \rightarrow \text{inr}, \quad x, y, z \text{ names, } A \text{ type, } \\
\quad x[y], \text{(ofy)}, x[y, z], \text{(ofy,z)}, x[], !x(), \quad p \text{ process variable, } \\
\quad x[A], x(A), x[(\sigma)s], x(p) \quad (\sigma)s \text{ abstraction for } s \in S \}$

$\text{Lbl}(S) \triangleq \{ \tau, x \leftrightarrow y, l, (l \parallel l' ), [p^G := (\sigma)s] \quad l, l' \in \text{Act}(S), x, y \text{ names, } p \text{ process variable, } \\
\quad \theta \in \text{Env}, (\sigma)s \text{ abstraction for } s \in S \}$

Every function $f : S \rightarrow S'$ induces a function $\text{Lbl}(f) : \text{Lbl}(S) \rightarrow \text{Lbl}(S')$ that replaces payloads from $S$ with payloads from $S'$. This function acts as the identity on every label except for higher-order output and explicit substitution where it acts as follows:

$$\text{Lbl}(f)([p^G := (\sigma)s]) \triangleq [p^G := (\sigma)f(s)] \quad \text{Lbl}(f)(x[(\sigma)s]) \triangleq x[(\sigma)f(s)].$$

**Definition 5.5.** The $\text{Lts}_d$ of derivations for $\text{HO}_\pi\text{LL}$, denoted $\text{Lts}_d$, is the triple $(\text{Der}, \text{Lbl}(\text{Der}), \rightarrow)$:

- The set $\text{Der}$ is the set of typing derivations for $\pi\text{LL}$.
- The set $\text{Lbl}(\text{Der})$ is the set of transition labels.
- The relation $\rightarrow \subseteq \text{Der} \times \text{Lbl}(\text{Der}) \times \text{Der}$ is the least relation closed under the SOS rules forming the specification of $\text{HO}_\pi\text{LL}$ (see ??).

### 5.4 Operational Semantics of Processes and Environments

To define the semantics for processes and typing environments of $\text{HO}_\pi\text{LL}$, we follow the recipe introduced in Section 2.3.

The first step is to specify how derivations are projected to process terms and typing environments. Let $\text{Proc}$ and $\text{Env}$ denote the sets of all process terms and typing environments of $\text{HO}_\pi\text{LL}$. The projections $\text{proc} : \text{Der} \rightarrow \text{Proc}$ and $\text{env} : \text{Der} \rightarrow \text{Env}$ are defined as in Section 2.3 (typing environments now include process environments):

$$\text{proc} \left( \Theta \vdash P : \mathcal{G} \right) = P \quad \text{env} \left( \Theta \vdash P : \mathcal{G} \right) = (\Theta; \mathcal{G}).$$

The second step of the recipe is to formalise the notions of erasure and session fidelity. Because the set labels is parametrised in the set of payloads for higher-order operations, we need slightly update the definition of $\mathcal{L}(-)$ to account for this parameter (which is exactly the state space of the Lts):

$$\mathcal{L}(X) = X \times \text{Lbl}(X) \quad \mathcal{L}(f)(x, l) = (f(x), \text{Lbl}(f)(l)).$$

Then, we can instantiate Definitions 2.7 and 2.13 with the Lts, proc, env, and $\mathcal{L}(-)$ defined above. The resulting notion of erasure for $\text{HO}_\pi\text{LL}$ states that for any derivation $\mathcal{D}$ and $l \in \text{Lbl}(\mathcal{D})$:

- if $\mathcal{D} \xrightarrow{l} \mathcal{D}'$, then $\text{proc}(\mathcal{D}) \xrightarrow{\text{Lbl}(\text{proc})(l)} \text{proc}(\mathcal{D}')$, and
- if $\text{proc}(\mathcal{D}) \xrightarrow{\text{Lbl}(\text{proc})(l)} P'$, then $\mathcal{D} \xrightarrow{l} \mathcal{D}'$ for some $\text{proc}(\mathcal{D}') = P'$.

Observe that the second clause ignores higher-order operations with ill-typed payloads (if $l$ has a payload, then it is a derivation hence proc yields a well-typed process). This kind of restriction is standard for higher-order calculi and many typed process calculi e.g., it can be found in the subject reduction theorems of the Simply Typed $\pi$-calculus and $\text{HO}$ [Sangiorgi and Walker 2001, p. 245,p. 377]. Session fidelity for $\text{HO}_\pi\text{LL}$ states that given $\Theta \vdash P : \mathcal{G}$ and $l \in \text{Lbl}(\mathcal{D})$ if $P \xrightarrow{\text{Lbl}(\text{proc})(l)} P'$ for some $P'$ then $(\Theta; \mathcal{G}) \xrightarrow{\text{Lbl}(\text{env})(l)} (\Theta'; \mathcal{G}')$ for some $\Theta'$ and $\mathcal{G}'$ such that $\Theta' \vdash P' : \mathcal{G}'$.

The third step is to apply the principles of “erasing types” and “erasing processes” to the SOS for $\pi\text{LL}$ derivations. The resulting specifications are in Figures 6 and 7, respectively—for processes we include only the rules that are new wrt $\pi\text{LL}$. 

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Definition 5.6. The lts of processes for HOπLL, denoted $\text{Lts}_p$, is the triple $(\text{Proc}, \text{Lbl}(\text{Proc}), \rightarrow)$ where

- $\text{Proc}$ is the set of process terms for HOπLL,
- $\text{Lbl}(\text{Proc})$ is the set of labels,
- $\rightarrow \subseteq \text{Proc} \times \text{Lbl}(\text{Proc}) \times \text{Proc}$ is the least relation closed under the SOS rules in Figures 6 and 9.

Definition 5.7. The lts of environments for HOπLL, denoted $\text{Lts}_e$, is the triple $(\text{Env}, \text{Lbl}(\text{Env}), \rightarrow)$ where

- $\text{Env}$ is the set of typing environments for HOπLL,
- $\text{Lbl}(\text{EnvSet})$ is the set of labels,
- $\rightarrow \subseteq \text{Env} \times \text{Lbl}(\text{EnvSet}) \times \text{Env}$ is the least relation closed under the SOS rules in Figure 10.

The fourth and final step of the recipe is to verify that semantics of processes and of typing environments enjoy erasure and session fidelity.
Theorem 5.8 (Erasure). \( lts_p \) enjoys erasure wrt \( lts_d \).

Theorem 5.9 (Safety). \( lts_p \) enjoys session fidelity wrt \( lts_e \).

As for \( \pi MLL \), erasure entails typability preservation.

Corollary 5.10 (Typability Preservation). Let \( P \) be well-typed. Then, \( P \xrightarrow{l} P' \) implies that \( P' \) is well-typed.

5.5 Metatheory

5.5.1 Behavioural Theory. The standard definitions of behavioural equivalences and preorders (Definitions 3.1 and 3.3), their laws (Fact 3.2), and definition of congruence (Definition 3.4) apply to \( H\pi LL \) without modification save for minor changes to handle the parametric definitions of labels.

When we consider higher-order parameters in labels and instantiate Definition 3.1 we obtain the following definition. Let \( lts_1 = (S_1, Lbl(S_1), \rightarrow_1) \) and \( lts_2 = (S_2, Lbl(S_2), \rightarrow_2) \) be two labelled transition systems and let \( R \) be a relation between \( S_1 \) and \( S_2 \).

- \( R \) is a strong forward simulation from \( lts_1 \) to \( lts_2 \) if \( s_1 R s_2 \) implies that
  \[
  s_1 \xrightarrow{s([x])s''} s_1' \text{ then } s_2 \xrightarrow{s([x])s''} s_2' \text{ for some } s_1', s_2', s_2 \text{ such that } s_1 R s_2 \text{ and } s_1'' R s_2'',
  \]
  - if \( s_1 \xrightarrow{l} s_1' \) then \( s_2 \xrightarrow{l} s_2' \) for some \( s_1', s_2', s_2 \) such that \( s_1' R s_2' \) and \( s_1'' R s_2'' \),
  - otherwise if \( s_1 \xrightarrow{l} s_1' \) then \( s_2 \xrightarrow{l} s_2' \) and \( s_1' R s_2' \).

- \( R \) is a strong backward simulation from \( lts_1 \) to \( lts_2 \) if its symmetric \( R^{-1} \) is a strong forward simulation from \( lts_2 \) to \( lts_1 \).

When we unfold Definition 3.4 using the grammar of \( H\pi LL \) we obtain that a relation on processes \( \approx \) is a (pre)congruence if \( P \approx Q \) implies that:

1. \( P \upharpoonright R \approx Q \upharpoonright R \) for any \( R \) (and the symmetric);
2. \( \pi P \approx \pi Q \) for any prefix \( \pi \);
3. \( (\varepsilon x y) P \approx (\varepsilon x y) Q \) for any \( x, y \);
4. \( x \cdot \{ \text{inl} : R; \text{inr} : P \} \approx x \cdot \{ \text{inl} : R; \text{inr} : Q \} \) for any \( x \) and \( R \) (and the symmetric);
5. \( P[p := (\rho)R] \approx Q[p := (\rho)R] \) for any \( p \) and \( \rho \);
6. \( R[p := (\rho)P] \approx R[p := (\rho)Q] \) for any \( p, \rho \), and \( \sigma \);
7. \( x[(\rho)P].R \approx x[(\sigma)Q].R \) for any \( x, \rho, \) and \( \sigma \).

The only difference with respect to \( \pi LL \) is the new case for explicit substitution and higher-order output (Items 5 to 7) which are the only new term constructors with a process that is not their continuation.

Theorem 5.11 (Congruence). On the lts of processes, \( \leq \) and \( \preceq \) are precongruences and \( \approx \) are congruences.

Type checking and similarity checking for \( H\pi LL \) coincide.

Theorem 5.12. For \( P \) well-typed, \( \Theta \), and \( G \) such that \( \text{fn}(P) = \text{cn}(G), \text{fpv}(P) = \text{pv}(\Theta), P \preceq (\Theta; G) \) iff \( \Theta \vdash P :: G \).

It follows that bisimilarity is sound with respect to type equivalence.

Corollary 5.13. For \( P \) and \( Q \) well-typed, if \( P \approx Q \) and \( \Theta \vdash Q :: G \), then \( \Theta \vdash P :: G \).
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5.5.2 Metatheory of Parallelism. Like $\pi$LL, HOPiLL enjoys the diamond property, serialisation, non-interference, and readiness.

Theorem 5.14 (Diamond Property). The $\llbracket\cdot\|\cdot\rrbracket$ of derivations enjoys the diamond property.

Theorem 5.15 (Serialisation). The $\llbracket\cdot\|\cdot\rrbracket$ of derivations enjoys serialisation.

Theorem 5.16 (Non-Interference). The $\llbracket\cdot\|\cdot\rrbracket$ of derivations enjoys the non-interference.

If a HOPiLL process does not have free process variables, then it enjoys the same notion of readiness of $\pi$LL.

Theorem 5.17 (First-Order Readiness). Let $\cdot \vdash P \coloneqq \Gamma_1 | \cdots | \Gamma_n$. For every $i \in [1, n]$, there exist $x \in \text{cn}(\Gamma_i)$ and $l$ such that $x \in \text{fn}(l)$ and $P \xRightarrow{l}$.

If a HOPiLL process has free process variables, then it either enjoys the same notion of readiness of $\pi$LL or is ready to invoke a process variable.

Theorem 5.18 (Higher-Order Readiness). Let $\Theta \vdash P \coloneqq \Gamma_1 | \cdots | \Gamma_n$. Either of the following statements holds.
- For every $i \in [1, n]$, there exist $x \in \text{cn}(\Gamma_i)$ and $l$ such that $x \in \text{fn}(l)$ and $P \xRightarrow{l}$.
- There exist $p : \mathcal{G} \in \Theta$ and $l$ such that $l = [p^\Theta := (\sigma)Q]$ and $P \xRightarrow{l}$.

5.5.3 Relation with $\pi$LL. $\pi$LL can be regarded as the first-order fragment of HOPiLL (we simply have to require all process environments in the relevant typing rules to be empty).

Is HOPiLL strictly more expressive than its first-order fragment? To answer this question we present a fully-abstract translation $\llbracket\cdot\rrbracket$ of HOPiLL to $\pi$LL.

The translation of session types acts as the identity in all cases except for provide and assume where we first need to apply “packing” (cf. Section 3) as follows.

\[
\llbracket[G]\rrbracket \triangleq [\otimes[G]] \quad \llbracket(G)\rrbracket \triangleq [\otimes[G]^\perp]
\]

The translation of (hyper)environments is given by point-wise extension of the translation of session types.

\[
\llbracket\Gamma_1 | \cdots | \Gamma_n\rrbracket \triangleq \llbracket\Gamma_1\rrbracket | \cdots | \llbracket\Gamma_n\rrbracket \quad \llbracket x_1 : A_1, \ldots, x_n : A_n\rrbracket \triangleq x_1 : [A_1], \ldots, x_n : [A_n]
\]

The translation of processes requires packing which in turn depends on typing information. For simplicity, we define the translation of well-typed processes by recursion on their typing derivation. For conciseness, let $\mathcal{G} = \Gamma_1 | \cdots | \Gamma_n$ and assume $x^\rho_i$ fresh as needed.

\[
\begin{align*}
\llbracket\Theta \vdash P \coloneqq \mathcal{G}\rho \cdot \vdash Q \coloneqq \emptyset\rrbracket & \triangleq \text{pack}^\emptyset_{x^\rho_i}(\mathcal{G}\rho)\llbracket P_1, \ldots, P_n\rrbracket \mid \llbracket\cdot\vdash Q \coloneqq \emptyset\rrbracket \\
& \quad \text{where } \text{dis}(\llbracket\Theta \vdash P \coloneqq \mathcal{G}\rho\rrbracket) = P_1 | \cdots | P_n \\
\llbracket\Theta, \cdot \vdash x(p) : P \coloneqq \Gamma\rrbracket & \triangleq x(x^\rho_1) \cdots x(x^\rho_n).x().\llbracket\Theta, \cdot \vdash P \coloneqq \Gamma\rrbracket \\
\llbracket p : \mathcal{G} \vdash p(\rho) \coloneqq \mathcal{G}\rho\rrbracket & \triangleq \text{unpack}_{x^\rho_i}(\llbracket\Gamma_i\rrbracket) | \cdots | \text{unpack}_{x^\rho_n}(\llbracket\Gamma_n\rrbracket) \\
& \quad \text{where } \text{unpack}_{x^\rho_i}(\cdot) \triangleq x[x]_{\emptyset} \\
& \quad \text{and } \text{unpack}_{x^\rho_i}(\Gamma, y : A) \triangleq x(\cdot).\text{unpack}_{x^\rho_i}(\Gamma) \\
\llbracket\Pi \vdash P \coloneqq \mathcal{G}\rho \quad \Theta, \cdot \vdash Q[P \coloneqq (\rho)P] \coloneqq \mathcal{H}\rrbracket & \triangleq (\nu x^\rho_1 y^\rho_1) \cdots (\nu x^\rho_n y^\rho_n)(Q_1 | \cdots | Q_n | \llbracket\Theta, \cdot \vdash P \coloneqq Q \coloneqq \mathcal{H}\rrbracket)
\end{align*}
\]
where \( Q_l = \text{pack}_{\chi^l}^\Sigma(\bigotimes [\Gamma_i \rho]) [P_l] \) and
\[
\text{dis}(\prod P : \mathcal{G} \rho) = P_1 \mid \cdots \mid P_n
\]

Remaining cases are translated homomorphically.

The “unpacking” processes \( \text{unpack}_x(\Gamma) \) introduced above has type \( \Gamma, x : (\bigotimes \Gamma)^\perp \) and in can be regarded as symmetric to the packing of \( \Gamma \) (hence the name) in the following sense.

**Lemma 5.19.** Let \( \Theta \vdash P : \Gamma \).

- \( \Theta \vdash (\nu xy) (\text{pack}_x^\Sigma(\Gamma)[P] \mid \text{unpack}_y(\Gamma)) : \Gamma \) and
- \( (\nu xy) (\text{pack}_x^\Sigma(\Gamma)[P] \mid \text{unpack}_y(\Gamma)) \approx P. \)

If a processes has no free process symbols, say \( \cdot \vdash P : \mathcal{G} \), then its translation \( \llbracket \cdot \vdash P : \mathcal{G} \rrbracket \) has type \( \llbracket \mathcal{G} \rrbracket \) in \( \pi LL \). A similar result holds for processes with free process variables, the only difference is the introduction of new channels \( x_i^\rho \) for representing process variables. The distribution of these names in the type depends on the structure of the process. For instance, \( \llbracket p : \mathcal{G} \vdash p(\rho) : \mathcal{G} \rho \rrbracket \) has type \( \llbracket \Gamma_1, x_1^\rho : (\bigotimes [\Gamma_1])^\perp \mid \cdots \mid \Gamma_n, x_n^\rho : (\bigotimes [\Gamma_n])^\perp \mid \mathcal{G} = \Gamma_1 \mid \cdots \mid \Gamma_n \).

The translation of \( \text{HO}\pi LL \) to \( \pi LL \) is fully abstract in the sense it is invariant under bisimilarity.

**Theorem 5.20 (Full Abstraction).** For \( \Theta \vdash P, Q : \mathcal{G}, P \approx Q \iff \llbracket \Theta \vdash P : \mathcal{G} \rrbracket \approx \llbracket \Theta \vdash Q : \mathcal{G} \rrbracket. \)

### 6 RECURSION

We briefly discuss how recursion could be added to \( \pi LL \) and \( \text{HO}\pi LL \). For conciseness we use \( \text{HO}\pi LL \) for our discussion.

Finiteness of types in \( \text{HO}\pi LL \) prevents well-typed processes from exhibiting infinite behaviours. To extend \( \text{HO}\pi LL \) with recursion and infinite (regular) behaviours we extend the type theory of \( \text{HO}\pi LL \) with equi-recursive types (similarly for \( \pi LL \)).

We formalise recursive types as equi-recursive types instead of iso-recursive ones because the former allows us to introduce recursion in \( \pi LL \) changing neither the syntax nor the semantics of the language (e.g. iso-recursive types require explicit folding/unfolding of recursive types).

We write \( \Sigma \) for a set of (recursive) type definitions of form \( T := E \) where \( T \) is a type name and \( E \) is a type expression obtained extending the grammar of \( \pi LL \) types with type names. We implicitly assume \( \Sigma \) is well-formed in the sense that all type names in its expressions are defined and that every defined name has a unique definition. The dual \( T^\perp \) of a recursive type given by \( T := E \) is defined by the expression \( T^\perp := E^\perp \). We write \( A \equiv_\Sigma B \) to signify that types \( A \) and \( B \) are definitionally equal given the set of definitions \( \Sigma \) (\( \Sigma \)-equivalent, for short). For instance, given \( \Sigma = \{ T := 1 \otimes T^\perp \} \) we have that \( T^\perp \equiv_\Sigma 1 \otimes (\perp \otimes T) \).

Typing judgements in \( \pi LL \) with recursive types have form \( \Theta \vdash_\Sigma P : \mathcal{G} \) and which reads “process \( P \) uses process variables according to \( \Theta \) and channels according to \( \mathcal{G} \) given the recursive type definitions in \( \Sigma \)”. Typing rules are exactly those of \( \pi LL \) but in derivations \( \Sigma \)-equivalent types can be arbitrarily replaced. For readability, we elicit the use of \( \Sigma \)-equivalence by means of rule \( \Sigma\text{-EQ} \) below where an instance of type \( B \) in the premise is replaced with one of type \( A \) provided they are definitionally equivalent under \( \Sigma \).

\[
\frac{\Theta \vdash_\Sigma P : \mathcal{G}[B]}{\Theta \vdash_\Sigma P : \mathcal{G}[A]} \quad \Sigma\text{-EQ}
\]
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For simplicity, we also adopt the following rule that allow us introduce local (i.e. limited to the premises of the rule) recursive type definitions in $\Sigma$.

$$
\Theta \vdash_{\Sigma, \Sigma'} P : G \\
\Theta \vdash_{\Sigma} P : G \\
\Sigma \text{-S}
$$

Recursive procedures. Recursive types allows us to send over a channel abstractions that refer to a channel able to provide the same abstraction (or any abstraction of the same type). Here is a proof.

$$
\begin{align*}
\vdash_{\Sigma} P : \Gamma, K : T \\
\vdash_{\Sigma} (\rho)P : \Theta \vdash_{\Sigma, \Sigma'} \Gamma, K : T \\
\vdash_{\Sigma} \lambda y.(\rho)P . 0 : y : [\Gamma, K : T] \\
\vdash_{\Sigma} \lambda y. y[(\rho)P].0 : x : [\Gamma, K : T] \\
\vdash_{\Sigma} \lambda y. y[(\rho)P].0 : x : T^+ \\
\vdash_{\Sigma} \lambda y. y[(\rho)P].0 : x : T^+ \\
\vdash_{\Sigma} \lambda y. y[(\rho)P].0 : x : T^+ \\
\end{align*}
$$

We use this property to extend the notion of procedures introduced in Remark 5.3 to support recursion.

$$
P, Q ::= \ldots \\
\mid \text{rec } K := (\rho)P \text{ in } Q \quad \text{recursive procedure definition} \\
\mid K(\rho) \quad \text{procedure invocation}
$$

A term rec $K := (\rho)P$ in $Q$ defines a recursive procedure $K$ as $(\rho)P$ in the scope of $Q$, and a term $K(\rho)$ invokes procedure $K$ by passing the parameters $\rho$.

Below are (derivable) rules for typing recursive procedure definitions and their invocations. As in the non-recursive case, in term proc$K(\rho)PQ$ we assume that $K$ is used at least once in $Q$. The generalisation to the case where $Q$ does not use $K$ at all is straightforward (thanks to rule w).

$$
\begin{align*}
\vdash_{\Sigma, \Sigma'} \Gamma, K : T &\quad \Theta \vdash_{\Sigma, \Sigma'} Q : G \mid \Gamma, K : T \\
\vdash_{\Sigma} \text{rec } K := (\rho)P \text{ in } Q &\quad \vdash_{\Sigma} \Theta (\rho)PQ \mid \Gamma, K : T \\
\end{align*}
$$

Below is the desugaring of rec $K := (\rho)P$ in $Q$ as well as the proof that the corresponding typing rule is derivable ($\mathcal{D}$ stands for the derivation in (3)).

$$
\begin{align*}
\vdash_{\Sigma, \Sigma'} \Gamma, K : T &\quad \Theta \vdash_{\Sigma, \Sigma'} \Gamma, K : T \\
\vdash_{\Sigma} \text{rec } K := (\rho)P \text{ in } Q &\quad \Theta \vdash_{\Sigma} (\rho)PQ \mid \Gamma, K : T \\
\end{align*}
$$

Below is the desugaring of $K(\rho)$ as well as the proof that the corresponding typing rule is derivable.

$$
\begin{align*}
\vdash_{\Sigma} \rho : \Gamma, K : T &\quad \vdash_{\Sigma} \rho : \Theta (\rho)PQ \mid \Gamma, K : T \\
\end{align*}
$$

Divergence. Recursive types and $\pi$LL are enough to write divergent processes. For instance, we can mimic the diverging $\lambda$-term $\Omega = (\lambda x.x)(\lambda x.x)$ using exponentials:

$$
\begin{align*}
\omega_1 &\triangleq x[x_1, x_2] . x_1[w_1]. w_1(x_2) . w_1[1] . 0 \\
\omega_1 &\triangleq y(z). z(x). z() . \omega_2 \\
\Omega &\triangleq (\nu y)(\omega_1 | \omega_2)
\end{align*}
$$
Process $\Omega$ does not terminate since every sequence of transitions eventually recreates $\Omega$ as shown below (for each transition we indicate the relevant cut-elimination rule):

$\Omega$

\[ \Gamma \]
\[ (\nu x_1 y_1) (y_1[y_2].(\omega_1 y_1/y) \mid \omega_1 y_2/y) \mid x_1(x_2).x_1[w_1].w_1(x_2).w_1[.].0) \]

\[ \Gamma \]
\[ (\nu x_1 y_1) (\nu x_2 y_2) (\omega_1 y_1/y) \mid \omega_1 y_2/y \mid x_1[w_1].w_1(x_2).w_1[.]0 \]

\[ \Gamma \]
\[ (\nu z_1 w_1) (\nu x_2 y_2) (z_1(x).z_1().\omega_1 \mid \omega_1 y_2/y) \mid w_1(x_2).w_1[.]0 \]

\[ \Gamma \]
\[ (\nu z_1 w_1) (\nu w_2 x) (\nu w_2 y_2) (z_1().\omega_1 \mid \omega_1 y_2/y) \mid w_2 \leftrightarrow x_2 \mid w_1[.]0 \]

\[ \Gamma \]
\[ (\nu x_2 y_2) (\omega_1 x_2/x) \mid \omega_1 y_2/y) =_\alpha \Omega \]

All other executions for $\Omega$ differ from the one above exclusively for the interleaving of transitions derived using rules $\bot \bot$ and $\text{AXCUT}$.

Let $\Sigma = \{ T = !(\bot \otimes T^\bot) \}$ and $D$ be the following derivation.

\[ \Gamma \]
\[ \Sigma \vdash w_2 \leftrightarrow x_2 : w_2:T \mid x_2:T^\bot \quad \text{AX} \]
\[ \Gamma \]
\[ \Sigma \vdash w_1[.]0 : w_1:1 \quad \text{MIX} \]
\[ \Gamma \]
\[ \Sigma \vdash w_2 \leftrightarrow x_2 \mid w_1[.]0 : w_1:1 \mid w_2:T \mid x_2:T^\bot \quad \otimes \]
\[ \Gamma \]
\[ \Sigma \vdash x_1[w_1].w_1[w_2].(w_2 \leftrightarrow x_2 \mid w_1[.]0) : x_1:!(\otimes T) \mid x_2:T^\bot \quad \Sigma \text{-EQ} \]
\[ \Gamma \]
\[ \Sigma \vdash x_1[x_1,x_2].x_1[x_1,w_1].w_1[w_2].(w_2 \leftrightarrow x_2 \mid w_1[.]0) : x:!(1 \otimes T) \quad \Sigma \text{-EQ} \]

Process $\Omega$ is well-typed.

\[ \Gamma \]
\[ \Sigma \vdash \omega_1 : x:T^\bot \]
\[ \Gamma \]
\[ \Sigma \vdash z().\omega_1 : x:T^\bot \mid z: \bot \quad \bot \]
\[ \Gamma \]
\[ \Sigma \vdash z(x).\omega_1 : z: \bot \otimes T^\bot \quad \otimes \]
\[ \Gamma \]
\[ \Sigma \vdash !y(z).z(x).\omega_1 : y:!(\bot \otimes T^\bot) \quad \Sigma \text{-EQ} \]
\[ \Gamma \]
\[ \Sigma \vdash !y(z).z(x).\omega_1 : y:T \quad \text{MIX} \]
\[ \Gamma \]
\[ \Sigma \vdash (vx)y(!y(z).z(x).\omega_1,\omega_1) : \bot \quad \text{CUT} \]

Observe that the empty sequent $\bullet$ is not inhabited by any process written in $\pi$-LL without recursive types. Indeed, $\vdash \bullet$ is not derivable in $\text{CLL}$.

7 RELATED WORK

Our work stands on previous efforts in the research line of Proofs as Processes. Abramsky [1994] and Bellin and Scott [1994] took the first steps in setting up this line, by connecting proofs in linear logic to processes in the $\pi$-calculus. Even though the Proofs as Processes agenda has met many challenges along the way, the idea of adopting linearity in types for processes spawned interesting research lines, like the seminal theories of linear types for the $\pi$-calculus [Kobayashi et al. 1999] and session types [Honda et al. 1998]. Even though these theories do not use exactly linear logic,
Dardha et al. [2017] showed that the underlying link is still strong enough that session types can be encoded into linear types.

Caires and Pfenning [2010] established that propositions in Intuitionistic Linear Logic (ILL) can be interpreted as session types. Adopting ILL means having to distinguish between “required” and “provided” endpoints, depending on whether an endpoint appears on the left- or the right-hand side of a sequent. Wadler [2014] removed this distinction by reformulating the correspondence with session types in Classical Linear Logic (CLL), yielding a more standard presentation of session types.

The usage of hyperenvironments to capture the parallel operator of the π-calculus originates from Carbone et al. [2018b], who used the composition of hypersequents that share a name to model connected processes. Hypersequents were originally investigated by Avron [1991]. Later, Montesi and Peressotti [2018] and Kokke et al. [2019] revisited the approach by Carbone et al. [2018b] to formulate hyperenvironments and a labelled transition system. The calculus by Kokke et al. [2019] is based on non-blocking I/O, which introduces equivalences that are not present in the canonical π-calculus, such as \( x().y().P \sim y().x().P \). Similar observations hold for previous calculi based on linear logic, including [Caires and Pfenning 2010] and [Wadler 2014]. πLL dispenses with these additions for the sake of minimality, and in particular for showing that the expected observable behaviour of the π-calculus can be reconstructed in Proofs as Processes. Furthermore, the non-blocking I/O in [Kokke et al. 2019] breaks the property that separation of hyperenvironments guarantees independence (formally, for example, the diamond property does not hold). If wanted, non-blocking I/O can be added to πLL by extending the logical typing rules to allow for additional environments in the hyperenvironment of interest, as in [Kokke et al. 2019].

Our transition rules for communications evoke cut reductions in CLL: the way in which types are matched and deconstructed is similar. The key difference is that we do not need to permute cuts in derivations (commuting conversions) until they reach the rule applications that formed the types being deconstructed. This is because we can observe the action corresponding to the deconstruction of a type within a subderivation from our transition labels, rather than having to inspect intensionally the structure of the derivation for the premises of our transition rules. Commuting conversions give rise to non-standard reductions in [Wadler 2014] as described above, so observing actions is an important component for achieving the standard process dynamics of the π-calculus.

To the best of our knowledge, πLL is the first calculus that comes with a logical reconstruction of session fidelity in terms of the expected transition systems, as in standard presentations of session types [Honda et al. 2016; Kouzapas and Yoshida 2014; Montesi and Yoshida 2013]. Bartoletti et al. [2014] suggested a new behavioural relation for session types that recalls our behavioural characterisation of session fidelity, but in the case of πLL our characterisation uses standard similarity.

Our recipe for constructing πLL is inspired by dialgebraic interpretations of labelled transition systems (LTS) [Ciancia 2013]. Previous languages with transition systems rooted in linear logic do not feature polymorphism, nor higher-order communication [Kokke et al. 2019; Montesi and Peressotti 2018].

Other process calculi based on linear logic have been extended with primitives for moving processes by relying on a functional layer that can encapsulate process terms as values [Toninho et al. 2013; Toninho and Yoshida 2018]. Instead, our HOπLL offers the first logical reconstruction of (a linear variant of) HOπ [Sangiorgi 1993], where mobile code is just processes, instead of functions (or values as intended in the λ-calculus). There are two key differences between HOπLL and the calculi by Toninho et al. [2013] and Toninho and Yoshida [2018]. First, HOπLL treats process variables linearly, following the intuition that it is a higher-order linear logic. This gives control on how process variables are used. Second, thanks to our usage of hyperenvironments, the syntax
and semantics of HO\(\pi\)LL are nearer to those expected for the \(\pi\)-calculus: restriction and parallel are independent operators, rather than combined with other term constructors; and our extracted operational semantics is the expected one—the proof of subject reduction for the systems in Toninho et al. [2013]; Toninho and Yoshida [2018] comes from Caires et al. [2016], which requires term rewritings in order to align the result of proof transformations with the expected result in the \(\pi\)-calculus. Generally speaking, the ideas of process abstractions and functions are similar. Our formulation based on process abstractions could be arguably seen as more direct, making the theory of HO\(\pi\)LL simpler. For example, we do not require the additional asymmetric connectives in session types used in [Toninho et al. 2013; Toninho and Yoshida 2018] for communicating processes (\(\tau \supset A\) and \(\tau \land A\)). The “send a process and continue over channel \(x\)” primitive found in [Mostrous and Yoshida 2015; Toninho et al. 2013; Toninho and Yoshida 2018] can be encoded in HO\(\pi\)LL, as shown in Remark 5.2.

The proof theory of the higher-order extension of \(\pi\)LL, HO\(\pi\)LL, generalises linear logic by allowing to assume that some judgements can be proven, and to provide evidence for resolving these assumptions. Similar ideas have been used in the past in different contexts, for example for modal logic [Nanevski et al. 2008] and logical frameworks [Bock and Schürmann 2015].

Process mobility in the context of the \(\pi\)-calculus has been the subject of deep study, starting from the inception of the Higher-Order \(\pi\)-calculus (HO\(\pi\)) by Sangiorgi [1993], who was also the first one to provide a fully-abstract encoding from HO\(\pi\) to the (first-order) \(\pi\)-calculus. Lanese et al. [2008] showed that channel passing, restriction, and having continuations in the term for higher-order output are not necessary to achieve Turing-completeness in (untyped) HO\(\pi\). Mostrous and Yoshida [2015] proposed a session-typed asynchronous HO\(\pi\) that is based on the original theory of session types [Honda et al. 1998], rather than the Proofs as Processes correspondence. Their system does not support behavioural polymorphism like HO\(\pi\)LL. In this context, Kouzapas et al. [2016] discovered that name passing and recursion can be simulated by higher-order I/O in the session-typed HO\(\pi\). Their result is based on an encoding that requires open abstractions (abstractions that include free names), whereas all abstractions in HO\(\pi\)LL are closed. Hence the same technique does not apply to our calculus. Indeed, it would be surprising that adding closed abstractions as in HO\(\pi\)LL would add behaviours that cannot be simulated with first-order features—the curious reader may consult other studies on the behavioural theory and expressivity of higher-order calculi, e.g., [Fu 2013, 2017; Lanese et al. 2011]. An interesting direction to generalise abstractions in HO\(\pi\)LL might be to adopt a notion of sharing for linear logic, along the lines of the work by Balzer and Pfenning [2017].

Our explicit substitutions are inspired by those originally developed to formalise execution strategies for the \(\lambda\)-calculus that are more amenable to efficient implementations [Abadi et al. 1991].

Our treatment of recursion is different than in previous work. Lindley and Morris [2016] extended CP with (co)recursive types, following the formulation of fixed points in classical linear logic by Baelde [2012]. Our approach is more similar to the more recent works by Balzer and Pfenning [2017]; Balzer et al. [2018, 2019], where typing judgements depend on a pre-populated signature \(\Sigma\) of recursive procedure definitions. In our case, \(\Sigma\) needs to contain only (recursive) type definitions, since we can use higher-order primitives to capture procedures as derivable constructs.

8 CONCLUSIONS

Since its inception, linear logic has been described as the logic of concurrency [Girard 1987]. The Proofs as Processes agenda aims at grasping this potential for the study of concurrent processes.

In this article, we have presented a new calculus rooted in linear logic, \(\pi\)LL, along with a robust recipe for its construction and extension based on dialgebras. Thanks to our construction, for the
first time, Proofs as Processes is in harmony with the expected metatheory of session types and the $\pi$-calculus, and we have gained substantial understanding on the implications of hyperenvironments about the behavioural theory of processes. We have also extended the recent developments on hyperenvironments to deal with polymorphism, code mobility (higher-order communication), and recursion. The higher-order variant of $\pi$LL, HO$\pi$LL, extends linear logic to higher-order reasoning, viewing proofs as linear "resources" that can be used to assume premises in other proofs.

Our development provides guidelines for the future extension of the Proofs as Processes correspondence, recalling the Curry-Howard correspondence. We hope that $\pi$LL and its accompanying recipe can serve as an inspiration for the uniform integration of different results in the field of session types.

Interesting future directions include the integration of HO$\pi$LL with multiparty session types—session types that consider more than two endpoints [Honda et al. 2016]—and choreographic programming—where choreographic descriptions of multiparty computations are translated to process implementations [Montesi 2013]. These topics have been studied in linear logic before [Caires and Pérez 2016; Carbone et al. 2016, 2018b, 2017; Ciobanu and Horne 2015], but we still lack (i) an understanding of higher-order communication rooted in linear logic for multiparty session types, and (ii) a choreographic programming model that supports code mobility at all. Mezzina and Pérez [2017] augmented multiparty session types with higher-order I/O outside of the Proofs as Processes correspondence: a successful integration of HO$\pi$LL with the results by Carbone et al. [2017] might root this feature in linear logic for the first time.

Process mobility is the underlying concept behind the emerging interest on runtime adaptation, a mechanism by which processes can receive updates to their internal code at runtime. Different attempts at formalising programming disciplines for runtime adaptations have been made, e.g., by Bono et al. [2017]; Dalla Preda et al. [2017]; Di Giusto and Pérez [2015], but none are rooted in a propositions as types correspondence and all offer different features and properties. Leveraging HO$\pi$LL to formulate runtime adaptation based on Proofs as Processes represent another interesting direction for future work.

The translation of HO$\pi$LL into $\pi$LL shows how our higher-order processes that use mobile code can be simulated by using reference passing instead. This result is distinct from the original one by Sangiorgi [1993], because we are operating in a typed setting. Understanding what a calculus with behavioural types, such as session types, can express—i.e., what well-typed terms can model—is a nontrivial challenge in general [Pérez 2016]. In practice, our translation has the usual implications: the translation gives us the possibility to write programs that use code mobility and then choose later whether we should really use code mobility or translate it to an implementation based on reference passing. This choice depends on the application case. If we are modelling the transmission of an application to be run somewhere else (as in cloud computing), then code mobility is necessary. Otherwise, if we are in a situation where we can choose freely, then we should choose whichever implementation is more efficient. For example, code mobility is useful if two processes, say a client and a server, are operating on a slow connection; then, instead of performing many communications over the slow connection, the client may send an application to the server such that the server can communicate with the application locally, and then send to the client only the final result. Lastly, if we are using code mobility in an environment where communications are implemented in local memory (as in many object-oriented language implementations or other emerging languages, like Go), then the translation gives us a compilation technique towards a simpler language without code mobility ($\pi$LL), which we can use to simplify runtime implementations.
ACKNOWLEDGMENTS
The authors thank Luís Cruz-Filipe and Davide Sangiorgi for useful discussions.
This work was partially sponsored by Villum Fonden, grant no. 29518, and by Independent Research Fund Denmark, grant no. 0135-00219.

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A. HOπLL, COMPLETE SPECIFICATION

A.1 SOS of derivations

Here we report the complete SOS specification for the lts of derivations of HOπLL.

A.1.1 Actions.
A.1.2 Structural.
Linear Logic, the π-calculus, and their Metatheory: A Recipe for Proofs as Processes
A.1.3 Communication.
Linear Logic, the π-calculus, and their Metatheory: A Recipe for Proofs as Processes

A.2 SOS of processes

**Actions**

\[
\begin{array}{ll}
\text{x} \mapsto \text{y} & : \text{y} \\
\text{x} \mapsto \text{y} & : \text{y} \\
\text{y} \mapsto \text{x} & : \text{x} \\
\text{y} \mapsto \text{x} & : \text{x} \\
\text{p} \mapsto \text{q} & : \text{q} \\
\text{p} \mapsto \text{q} & : \text{q} \\
\end{array}
\]

**Structural**

\[
\begin{array}{ll}
\text{P} \vdash \text{Q} & : \text{Q} \\
\text{P} \vdash \text{Q} & : \text{Q} \\
\end{array}
\]

\[
\begin{array}{ll}
\text{\vdash} & \text{\vdash} \\
\text{\vdash} & \text{\vdash} \\
\end{array}
\]

**Communications**

\[
\begin{array}{ll}
\text{P} \vdash \text{Q} & : \text{Q} \\
\text{P} \vdash \text{Q} & : \text{Q} \\
\text{P} \vdash \text{Q} & : \text{Q} \\
\end{array}
\]
