The Dirichlet heat kernel in inner uniform domains: local results, compact domains and non-symmetric forms

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Abstract

This paper provides sharp Dirichlet heat kernel estimates in inner uniform domains, including bounded inner uniform domains, in the context of certain (possibly non-symmetric) bilinear forms resembling Dirichlet forms. For instance, the results apply to the Dirichlet heat kernel associated with a uniformly elliptic divergence form operator with symmetric second order part and bounded measurable real coefficients in inner uniform domains in \( \mathbb{R}^n \). The results are applicable to any convex domain, to the complement of any convex domain, and to more exotic examples such as the interior and exterior of the snowflake.

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1 Introduction

This paper is concerned with Dirichlet heat kernel estimates for diffusions in inner uniform domains. The monograph [12] introduced a general approach to this problem in the case of unbounded domains in strongly local Dirichlet spaces satisfying a global parabolic Harnack inequality. Sharp estimates for the heat kernel and the heat semigroup with Dirichlet boundary condition in domains have been studied by many authors. The article [5] contains seminal ideas. Varopoulos’ work [31, 32] contains definitive results for domains above the graph of a Lipschitz function. We refer the reader to [10, 14, 21, 22, 26] for related results and further pointers to the literature. The main difference between these earlier works and the present effort is twofold. First, as in [12], our results cover inner uniform domains, a class of domains that is significantly

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larger than, say, Lipschitz domains. Further, inner uniformity is an intrinsic notion that can be used in rather general metric spaces. This allows us to develop our results in the context of a large class of local Dirichlet spaces. This larger context allows us to cover many natural and interesting examples beyond elliptic operators in $\mathbb{R}^n$, for instance, sub-elliptic operators.

This paper complements the results of [12] in several significant ways. For this purpose, we rely heavily on key results contained in the companion papers [17, 16] that were developed with the applications given here in mind.

First, we treat the case of bounded inner uniform domains which is not covered by [12]. In the unbounded case, a Doob’s transform is used which involves the “harmonic profile” $h_U$ of the domain $U$, that is, a harmonic positive function in $U$ that vanishes on the boundary (in the proper sense). In the case of bounded inner uniform domains, $h_U$ must be replaced by the positive eigenfunction $\phi_U$ associated with the lowest Dirichlet eigenvalue $\lambda_U$ of the domain $U$. This requires significant adaptation of the arguments.

Second, whether the domain is bounded or not, we include a wide class of non-symmetric second order differential operators. In the case of a fixed bounded inner uniform domain, there is not much difference in the final results between the symmetric and non-symmetric cases. In the case of unbounded domains, the presence of lower order terms forces the estimates to be local in time (in a certain sense).

Third, in both the symmetric and non-symmetric cases, we relax the very global assumptions made in [12] to cover cases where the geometry of the underlying space is only controlled locally. In particular, we cover domains that are inner uniform only in a certain local sense. For instance, we treat the Dirichlet heat kernel for the Laplace-Beltrami operator in an unbounded inner uniform domain in a complete Riemannian manifold, without global curvature assumption, or under the Ricci curvature assumption $\text{Ric} \geq -\kappa g$, for some $\kappa > 0$. We also obtain some local estimates for the Dirichlet heat kernel in the interior of an unbounded convex set in $\mathbb{R}^n$. Most unbounded convex sets are not inner uniform but they are always locally inner uniform (in fact, locally uniformly).

We will work in a rather abstract setting involving the notion of (not necessarily symmetric) Dirichlet forms and the associated intrinsic distance. This setting is actually very natural for this problem because, even when treating domains in $\mathbb{R}^n$, the technique we use requires the introduction of some auxiliary abstract Dirichlet spaces in which most of the work is done. Regarding the general theory of Dirichlet spaces, we refer the reader to [4, 7] and also [18, 19]. Nevertheless, in the rest of this introduction, we illustrate the main results of this paper in the context of certain elliptic operators on a complete Riemannian manifold.

1.1 Illustrative examples

Let $(M,g)$ be a complete Riemannian manifold equipped with its Riemannian measure $\mu$ and its Riemannian distance function. Let $U$ be an inner uniform domain in $M$ (for instance, if $M = \mathbb{R}^n$, bounded convex domains are inner
uniform and the complement of any convex domain is inner uniform). Let $L$ be a second order differential operator on $M$ of the form

$$L = \Delta + X + V$$

where $\Delta$ is the Laplace-Beltrami operator on $M$, $X$ is a smooth vector field on $M$ (viewed as a differential operator acting on smooth functions $X : f \mapsto Xf = df(X)$) and $V$ is a smooth function on $M$ (viewed as a multiplication operator). This particular structure of the differential operator $L$ is chosen here for convenience and illustrative purpose. Given a domain $U$ in $M$, let $d_U$ be the inner distance in $U$ (see Section 2.2 below).

Suppose that $M$ has non-negative Ricci curvature and $X = 0, V = 0$. Suppose also that $U$ is unbounded. Then [12] provides a global space-time two-sided estimate of the Dirichlet heat kernel $h^D_U(t, x, y)$ of the form

$$C \frac{h_U(x)h_U(y)}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}h_U(x, \sqrt{t})h_U(y, \sqrt{t})} \exp \left( -c \frac{d_U(x, y)^2}{t} \right).$$

In this two-sided estimate, different constants $C, c \in (0, \infty)$ are used in the lower and upper bounds. The function $h_U$ is any fixed positive solution of $Lh = 0$ in $U$ which vanishes at the boundary (in the proper weak sense). We call this function a harmonic profile for $U$. For any $x \in U$ and $r > 0$, $x_r$ denotes a point in $U$ with the property that $d(x, x_r) \leq Ar$ and $d(\partial U, x_r) \geq ar$ where $a, A$ are independent of $x$ and $r$. The inner uniformity of $U$ ensures that there exists constants $a, A$ such that such a point $x_r$ exists for every $x \in U$ and $r > 0$.

The aim of this paper is to prove the theorems of the following type. See Theorem 7.9 and Corollary 7.10.

**Theorem 1.1.** Let $(M, g)$ be a complete Riemannian manifold with Riemannian measure $\mu$. Let $L = \Delta + X + V$ be as described above. Let $U$ be a bounded inner uniform domain in $M$. Let $A = A(U), a = a(U)$ be constants such that for any point $x$ in $U$ and any $r > 0$, there exists a point $x_r$ in $U$ at distance at most $A \min\{r, 1\}$ from $x$ and at distance at least $a \min\{r, 1\}$ from the boundary of $U$. Let $\phi$ (resp. $\phi$) be the unique positive eigenfunction associated with the lowest Dirichlet eigenvalue of $-\Delta$ (resp. $-L$) in $U$.

- There are constants $C = C(L, U), c = c(L, U)$ such that $c\phi \leq \phi \leq C\phi$, in $U$.

- There are constants $C = C(L, U)$ and $\alpha = \alpha(L, U)$ such that, for any solution $\psi$ of $L\psi = \lambda\psi$ in $U$ with Dirichlet boundary condition, we have $|\psi| \leq C(1 + |\lambda\psi|^\alpha)\phi$.

- For any fixed $T > 0$, there are constant $c_T = c_T(L, U, T) \in (0, \infty)$ such that the Dirichlet heat kernel $p^D_U(t, x, y)$ for $L$ in $U$ with respect to $\mu$ satisfies

$$p^D_U(t, x, y) \leq \frac{c_T\phi(x)\phi(y)}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}\phi(x, \sqrt{t})\phi(y, \sqrt{t})} \exp \left( -c_T \frac{d_U(x, y)^2}{t} \right)$$
and
\[ p^D_U(t, x, y) \geq \frac{c_3 \phi(x) \phi(y)}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})} \phi(x, \sqrt{t}) \phi(y, \sqrt{t})} \exp \left( -c_4 \frac{d_U(x, y)^2}{t} \right), \]
for all \((t, x, y) \in (0, T) \times U \times U.

To our knowledge, this theorem is new even when \(M = \mathbb{R}^n\) and \(L = \Delta\) is the Laplacian. Indeed, [12] does not treat bounded domains and, even in this special case, the above statement is more precise than the known intrinsic ultracontractivity results. Section 7.3 gives more detailed results in a more general context and include complementary asymptotics when \(t\) tends to infinity. In particular, Corollary 7.10 gives a refined eigenfunction estimate.

Figure 1: A polygonal domain \(\Omega\) with a slit

For very concrete examples, the reader can think of a bounded polygonal domain \(\Omega\) in \(\mathbb{R}^n\) as in Figure 1. In this context, we can consider the heat equation with Dirichlet boundary condition for the divergence form operator
\[ Lf = \sum \partial^2_i f + \sum b_i \partial_i f + \sum \partial_i (d_i f) + cf \]
where \(b_i, d_i, c\) are bounded measurable functions. Let \(\phi\) be the positive eigenfunction associated with the lowest Dirichlet eigenvalue of \(-L\) in \(\Omega\). Let \(\phi_s\) be the positive eigenfunction associated with the lowest Dirichlet eigenvalue of \(-\sum \partial^2_i\) in \(\Omega\). We show that \(\phi \asymp \phi_s\) in \(\Omega\). The function \(\phi_s\) vanishes at different rates as \(x\) tends non-tangentially to different boundary points. The rate depends on the angle at the boundary point. For instance, \(\phi\) will vanish linearly at smooth boundary points and will vanish quadratically when approaching the vertex of an interior right angle. The polygonal domain \(\Omega\) may have a vertex with interior angle of \(2\pi\) in which case the corresponding vertex is the tip of a slit. At such a vertex, \(\phi\) vanishes as the square root of the distance to the boundary. The heat kernel estimates stated above capture this in a very precise way by reducing the estimates of the Dirichlet heat kernel to the understanding of the eigenfunction \(\phi\) (equivalently, \(\phi_s\)). The case of the Koch snowflake is another good example to keep in mind.
An important special case of the results obtained in this paper arises when
the manifold $M$ has non-negative Ricci curvature (hence satisfies the parabolic
Harnack inequality at all scales) and $L = \Delta$. In this case, the results described
above hold true uniformly over the class of all inner uniform domains with
specified inner uniformity constants a stated in the following theorem.

**Theorem 1.2.** Let $(M, g)$ be a complete non-compact Riemannian manifold
with non-negative Ricci curvature. Fix constants $0 < c_u < 1 < C_u < \infty$ and let
$U$ be a bounded $(c_u, C_u)$-inner uniform domain in $M$ (see Definition 3.2). Let $\text{diam}_U$ be the inner diameter of $U$. Let $\lambda_U$ the the lowest eigenvalue of minus
the Laplacian with Dirichlet boundary condition in $U$, and let $\phi$ be the associated
positive eigenfunction normalized in $L^2(U, \mu)$. For any $x \in U$, let $x_r \in U$ be
such that $d_U(x, x_r) \leq r$ and $d(x_r, \partial U) \geq 2^{-5} c_u \min\{r, \text{diam}_U\}$. Let $p_D^U(t, x, y)$ be the Dirichlet heat kernel in $U$. There are constants $c_i \in (0, \infty)$ depending
only on $M$ and $c_u, C_u$ such that

\[
\begin{align*}
\text{• } & \quad p_D^U(t, x, y) \leq c_1 e^{-t\lambda_U} \frac{\phi(x)\phi(y)}{V(x, \sqrt{t})V(y, \sqrt{t})} \exp\left(-c_2 \frac{d_U(x, y)^2}{t}\right) \\
\text{and} \quad & \quad p_D^U(t, x, y) \geq c_3 e^{-t\lambda_U} \frac{\phi(x)\phi(y)}{V(x, \sqrt{t})V(y, \sqrt{t})} \exp\left(-c_4 \frac{d_U(x, y)^2}{t}\right),
\end{align*}
\]

for all $(t, x, y) \in (0, \infty) \times U \times U$ with $\tau = \min\{t, \text{diam}_U^2\}$.

\[
\text{• } \quad \left| t^{\frac{1}{2}}p_D^U(t, x, y) - \phi(x)\phi(y) \right| \leq c_5 e^{-c_6 t/diam^2_U},
\]

As a simple example of application of this result, let $M = \mathbb{R}^n$ be the Euclidean space. Let $\mathcal{C}(a, A)$ be the set of all convex bounded regions $U$ such $B(o, ar) \subset U \subset B(o, Ar)$ for some $o \in U$ and $r > 0$. It is not hard to see that
there are constants $c_u, C_u$, depending only on $a, A$, such that any such set is
$(c_u, C_u)$-inner uniform. The above theorem applies uniformly to all $U \in \mathcal{C}(a, A)$.

The general setting in which we will work allows us to cover many different situations including the case when the Riemannian structure used above is
replaced by a sub-Riemannian structure.

1.2 Organization of the paper

In the next section, we describe basic notation and assumptions regarding the underlying space $X$ and its geometry induced by a fixed strongly local Dirichlet. The doubling volume property and Poincaré inequalities play a key role throughout the paper.
Section 3 contains the definition of uniform and inner uniform domains as well as important local quantitative version.

Section 4 described a class of bilinear forms with dense domain in $L^2(X, \mu)$ that are adapted to the fix geometric structure carried by our space $X$. See Definition 4.2 and Assumption A. For example, if $X$ is a complete Riemannian manifold Riemannian measure $\mu$ and Dirichlet form $\int_M \nabla f_1 \cdot \nabla f_2 d\mu$ then the bilinear form

$$E(f_1, f_2) = \int_M \nabla f_1 \cdot \nabla f_2 d\mu$$

$$+ \int_M (b_1 \cdot \nabla f_1) f_2 d\mu + \int_M f_1 (b_2 \cdot \nabla f_2) f_2 d\mu + \int_M f_1 f_2 V d\mu$$

where $b_1, b_2$ are bounded vector fields on $X$ and $V$ is a bounded potential is adapted in the sense introduced in Section 4.

Section 5 discusses the notions of interior and boundary Harnack inequalities and the notion of harmonic profile of a region $U$. The harmonic profile of an unbounded domain $U$ is a positive harmonic function in $U$ satisfying the Dirichlet boundary condition along the boundary of $U$. A localized version of this definition is also introduced and the existence of harmonic profiles is discussed. Results from [17] that play an important role here are reviewed.

Section 6 provides novel variations on the notion of $h$-transform. It contains some of the key ingredients for the proof of our main Dirichlet heat kernel estimates. The main point is to understand the structure and properties of the form $E_h$ obtained via $h$-transform from our given adapted bilinear form $E$. Even if we assume that $E$ is a (non-symmetric) Dirichlet form, the form $E_h$ may not be a Dirichlet form. The precise properties of $E_h$ depend on the particular function $h$ used in the $h$-transform. We show that, for well chosen $h$, the form $E_h$ satisfies structural properties that imply the validity of a Harnack inequality (up to the boundary). See Theorem 6.12 and Theorem 6.13. This makes use of the results of [16] which were developed in part for this purpose and are the main key to obtain the result presented here.

Section 7 contains the main results obtained in this paper. It is based in an essential way on the ideas and techniques described in Section 5 and 6. Theorems 7.3–7.6 provide detailed Dirichlet heat kernel estimates covering a wide range of different hypotheses. Theorem 7.8 gives a global Harnack type estimate for weak solutions of our abstract heat equations with Dirichlet boundary condition under a range of inner uniformity conditions on the domain.

### 2 The underlying space and its geometry

#### 2.1 The intrinsic distance

Let $X$ be a connected, locally compact, separable metrizable space and let $\mu$ be a non-negative Borel measure on $X$ that is finite on compact sets and positive on non-empty open sets.
We fix a symmetric, strongly local, regular Dirichlet form \((\mathcal{E}, \mathcal{F} = D(\mathcal{E}^*)))\) on \(L^2(X, \mu)\) with energy measure \(d\Gamma\). We sometimes call this form “the model form”. By this we simply mean that this form serves to define the basic geometry of our space and the adapted forms introduced in Section 4.

Recall that \(d\Gamma\) is a measure-valued quadratic form defined by

\[
\int f d\Gamma(u, u) = \mathcal{E}^*(uf, u) - \frac{1}{2} \mathcal{E}(f, u^2) , \quad \forall f, u \in \mathcal{F} \cap L^\infty(X, \mu),
\]

and extended to unbounded functions by setting \(\Gamma(u, u) = \lim_{n \to \infty} \Gamma(u_n, u_n)\), where \(u_n = \max\{\min\{u, n\}, -n\}\). Using polarization, we obtain a bilinear form \(d\Gamma\). In particular,

\[
\mathcal{E}^*(u, v) = \int d\Gamma(u, v) , \quad \forall u, v \in \mathcal{F}.
\]

We equip the Hilbert space \(\mathcal{F}\) with its natural norm

\[
\|f\|_\mathcal{F} = \left(\int_X |f|^2 d\mu + \int d\Gamma(f, f)\right)^{1/2}.
\]

Let \(U \subset X\) be an open set. Define

\[
\mathcal{F}_{loc}(U) = \{f \in L^2_{loc}(U) : \forall \text{ compact } K \subset U, \exists f^\sharp \in \mathcal{F}, f = f^\sharp|_K \text{ a.e.}\}
\]

For \(f, g \in \mathcal{F}_{loc}(U)\) we define \(\Gamma(f, g)\) locally by \(\Gamma(f, g)|_K = \Gamma(f^\sharp, g^\sharp)|_K\), where \(K \subset U\) is open relatively compact and \(f^\sharp, g^\sharp\) are functions in \(\mathcal{F}\) such that \(f = f^\sharp\), \(g = g^\sharp\) a.e. on \(K\). Set

\[
\mathcal{F}(U) = \{u \in \mathcal{F}_{loc}(U) : \int_U |u|^2 d\mu + \int_U d\Gamma(u, u) < \infty\},
\]

\[
\mathcal{F}_c(U) = \{u \in \mathcal{F}(U) : \text{ the essential support of } u \text{ is compact in } U\}.
\]

\[
\mathcal{F}^0(U) = \text{ the closure of } \mathcal{F}_c(U) \text{ for the norm } \left(\int_U |u|^2 d\mu + \int_U d\Gamma(u, u)\right)^{1/2}.
\]

**Definition 2.1.** The intrinsic distance \(d := d_{\mathcal{E}^*}\) induced by \((\mathcal{E}^*, \mathcal{F})\) is defined as

\[
d_{\mathcal{E}^*}(x, y) := \sup \{f(x) - f(y) : f \in \mathcal{F}_{loc}(X) \cap C(X), d\Gamma(f, f) \leq d\mu\},
\]

for all \(x, y \in X\), where \(C(X)\) is the space of continuous functions on \(X\).

Throughout this paper, the spaces \(\mathcal{F}, \mathcal{F}(U), \mathcal{F}_c(U), \mathcal{F}^0(U)\) and the intrinsic distance \(d\) play an essential role. The space \(\mathcal{F}\) is the equivalent of the Sobolev space of \(L^2\) functions with gradient in \(L^2\). The distance \(d\) defines the geometry of our space and will be used to introduce fundamental assumptions.

Consider the following properties of the intrinsic distance that may or may not be satisfied. They are discussed in [29, 27].
• (A1) The intrinsic distance $d$ is finite everywhere and defines the original topology of $X$.

• (A2) The space $(X,d)$ is a complete metric space.

• (A2') $\forall x \in X, r > 0$, the open ball $B(x,r)$ is relatively compact in $(X,d)$.

Note that if (A1) holds true then, by [29, Theorem 2], (A2) is equivalent to (A2'). Moreover, (A1)-(A2) imply that $(X,d)$ is a geodesic space, i.e., any two points in $X$ can be connected by a minimal geodesic in $X$. See [29, Theorem 1]. If (A1) and (A2) hold true then the intrinsic distance is also given by (see [27, Proposition 1])

$$d(x, y) = \sup \{ f(x) - f(y) : f \in \mathcal{F} \cap C_\cdot(X), d\Gamma(f, f) \leq d\mu \}, \quad x, y \in X.$$ 

When working in an open subset $Y$ of $X$, it is sometimes sufficient to assume only (A1) and

• (A2-Y) For any ball $B(x, 2r) \subset Y$, $B(x, r)$ is relatively compact.

This is a version of property (A2') that is localized in a set $Y$ of particular interest. We will not pursue this systematically here but we will make a technical use of this fact at a later stage in the paper. In what follows we always assume that either (A1)-(A2) holds or, when justified by the context, that (A1)-(A2-Y) holds.

Example 2.2. Let $\Omega$ be a domain in Euclidean space. Consider the (symmetric) Dirichlet form $\mathcal{E}_\Omega(f, f) = \int_{\Omega} |\nabla f|^2 d\mu$ with domain $H^1_0(\Omega)$, the Sobolev space obtained by closing the space of smooth functions with compact support in $\Omega$ in the norm $(\int_{\Omega} (|f|^2 + |\nabla f|^2) dx)^{1/2}$. This form is regular on $\Omega$. The intrinsic distance is equal to the inner Euclidean distance in $\Omega$ (obtained by minimizing the length of the curves in $\Omega$ joining two points of $\Omega$, see the next section) and property (A1) is satisfied. Property (A2) is not satisfied but (A2-Y) holds true for any $Y$ with $\overline{Y} \subset \Omega$.

2.2 Inner metric

Assume (A1)-(A2) and let $\Omega$ be a non-empty domain in $X$. For any continuous path $\gamma : [0, 1] \to Y$, set

$$\text{length}(\gamma) = \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})) : n \in \mathbb{N}, 0 \leq t_0 < \ldots < t_n \leq 1 \right\}.$$

Definition 2.3. The inner metric on $\Omega$ is defined as

$$d_\Omega(x, y) = \inf \{ \text{length}(\gamma) : [0, 1] \to \Omega \text{ continuous}, \gamma(0) = x, \gamma(1) = y \}.$$ 

Let $\overline{\Omega}$ be the completion of $\Omega$ with respect to $d_\Omega$. 
Whenever we consider an inner ball $B_{\tilde{\Omega}}(x,R) = \{ y \in \tilde{\Omega} : d_{\tilde{\Omega}}(x,y) < R \}$ or $B_{\Omega}(x,R) = B_{\tilde{\Omega}}(x,R) \cap \Omega$, we assume that its radius is minimal in the sense that $B_{\tilde{\Omega}}(x,R) \neq B_{\tilde{\Omega}}(x,r)$ for all $r < R$. If $x$ is a point in $\Omega$, denote by $\delta(x) = \delta_{\tilde{\Omega}}(x) = d(x, \partial \Omega)$ the distance from $x$ to the boundary of $\Omega$. Let $\text{diam}_\Omega(\Omega)$ be the diameter of $\Omega$ in the inner metric $d_{\tilde{\Omega}}$.

**Definition 2.4.** For two open sets $V \subset \Omega$, let

$$F^0_{\text{loc}}(\Omega, V) = \left\{ f \in L^2_{\text{loc}}(\Omega) : \forall W \subset V, \text{ rel. cpt. in } \tilde{\Omega} \text{ with } d_{\tilde{\Omega}}(W, \Omega \setminus V) > 0, \exists \tilde{f} \in F^0(U) \text{ such that } f = \tilde{f} \text{ a.e. on } W \right\}.$$

**Definition 2.5.** Let $\Omega$ be a domain in $X$. For an open set $V \subset \Omega$, let $V^\dagger$ be the largest open set in $\tilde{\Omega}$ which is contained in the closure of $V$ in $\tilde{\Omega}$ and whose intersection with $\Omega$ is $V$.

**Lemma 2.6.** Let $V$ be an open set in $\Omega$. A function $g \in F_{\text{loc}}(V)$ is in $F^0_{\text{loc}}(\Omega, V)$ if and only if we have $fg \in F^0(\Omega)$ for any bounded function $f \in F(\Omega)$ with compact support in $V^\dagger$ and such that $d\Gamma(f,f)/d\mu \in L^\infty(\Omega, \mu)$.

**Proof.** See [12, Lemma 2.46].

### 2.3 The doubling property and Poincaré inequality

Let $Y \subset X$ be open and assume that the intrinsic metric $d$ satisfies (A1)-(A2) (more generally, (A1) and (A2-$Y$) suffices).

**Definition 2.7.** The form $(E^s, F)$ satisfies the volume doubling property on $Y$ if there exists a constant $D_Y \in (0, \infty)$ such that for every ball $B(x,2r) \subset Y$,

$$V(x,2r) \leq D_Y V(x,r), \quad (VD)$$

where $V(x,r) = \mu(B(x,r))$ denotes the volume of $B(x,r)$.

**Definition 2.8.** The form $(E^s, F)$ satisfies the (weak) Poincaré inequality on $Y$ if there exists a constant $P_Y \in (0, \infty)$ such that for any ball $B(x,2r) \subset Y$,

$$\forall f \in D(E), \int_{B(x,r)} |f - f_B|^2 d\mu \leq P_Y r^2 \int_{B(x,2r)} d\Gamma(f,f), \quad (PI)$$

where $f_B = \frac{1}{V(x,r)} \int_{B(x,r)} f d\mu$ is the mean of $f$ over $B(x,r)$.

The term weak refers to the fact that the ball $B(x,2r)$ is used on the right-hand side of the Poincaré inequality. It will be omitted in what follows. Under the doubling condition, strong and weak versions of the Poincaré inequality are in fact equivalent (e.g., [24]).

If $Y = X$, the properties introduced in these definitions have a very global nature as they hold uniformly at all scales and locations. It is natural to introduce a more local version of these properties.
Definition 2.9. The form \((E^*, F)\) satisfies the volume doubling property and the Poincaré inequality locally on \(Y\) if for all \(x \in Y\) there is a neighborhood \(Y(x)\) of \(x\) so that the volume doubling property and the Poincaré inequality hold in \(Y(x)\).

The form \((E^*, F)\) satisfies the volume doubling property and the Poincaré inequality up to scale \(R\) in \(Y\) if the volume doubling property and the Poincaré inequality hold in \(B(x, 2R)\) with constants independent of \(x\), for all \(x \in Y\).

Example 2.10. Let \((M, g)\) be a complete Riemannian manifold and \(Y\) an open subset of \(M\). Equip \(M\) with its Riemannian measure and the Dirichlet form \(E^*(f_1, f_2) = \int_M g(\nabla f_1, \nabla f_2) d\mu\) with its natural domain \(F\) (the first Sobolev space on \(M\)). In this case, the intrinsic distance on \(M\) equals the Riemannian distance.

• The volume doubling property and the Poincaré inequality hold locally on \(Y\).
• If \(\text{Ric} \geq -\kappa g\) on the \(2R\)-neighborhood of \(Y\) for some fixed \(\kappa > 0\) and \(R > 0\) then the volume doubling property and the Poincaré inequality hold up to scale \(R\) on \(Y\).
• If \(\text{Ric} \geq 0\) on \(Y\) then the volume doubling property and the Poincaré inequality hold on \(Y\).

Example 2.11. Let \(G\) be a unimodular Lie group equipped with its Haar measure and with a family \(\{X_1, \ldots, X_k\}\) of left invariant vector fields that, viewed as elements of the Lie algebra, generates the Lie algebra of \(G\) (this condition is often called the Hörmander condition). Consider the Dirichlet form \(E^*(f_1, f_2) = \int_G \sum_i X_i f_1 X_i f_2 d\mu\) with its natural domain \(F\), the space of functions in \(L^2(G, \mu)\) such that, for each \(i\), the distribution \(X_i f\) can be represented by an element of \(L^2(G, \mu)\). In this case, the intrinsic distance is equal to the associated sub-Riemannian distance.

• The volume doubling property and the Poincaré inequality hold up to scale \(R\) on \(G\) for any fixed \(R > 0\).
• If \(G\) has polynomial volume growth (i.e., \(\exists A, \forall r > 0, V(e, r) \leq Cr^A\)) then the volume doubling property and the Poincaré inequality hold on \(G\).

See, e.g., [24, Section 5.6] and [33].

2.4 Carré du champ and Lipschitz functions

Theorem 2.12. Suppose the form \((E^*, F)\) satisfies (A1)-(A2), and the volume doubling property holds locally on \(X\). Then for any Lipschitz function \(f\) with Lipschitz constant \(C_L\), the energy measure \(d\Gamma(f, f)\) is absolutely continuous with respect to \(d\mu\) and the Radon-Nikodym derivative \(\Upsilon(f, f) = d\Gamma(f, f)/d\mu\) satisfies

\[\Upsilon(f, f) \leq C_L^2\]

almost everywhere.
Proof. See [15, Theorem 2.1, Remark 2.1(ii)]. □

The next corollary is used to prove Proposition 6.7 and Lemma 6.10.

**Corollary 2.13.** Let $\Omega$ be a domain in $X$. Suppose the model form $(E^*,F)$ satisfies (A1)-(A2-Ω), and the volume doubling property holds locally on $\Omega$. Then any function $f$ on $\Omega$ which is Lipschitz with respect to $d_\Omega$ with Lipschitz constant $C_L$ is in $F_{loc}(\Omega)$ and satisfies

$$C_L \geq \sup_{\Omega} \sqrt{\Upsilon(f,f)}.$$

**Proof.** Follows from Theorem 2.12 and a simple adaption of the arguments in [13, Corollary 3.6], [34] or [12, Corollary 2.22]. □

3 Inner uniformity

Let $X, \mu, E^*, F, d$ be as above and assume that (A1)-(A2) are satisfied so that $(X,d)$ is a complete metric space.

3.1 Inner uniform domains

**Definition 3.1.** Fix $c \in (0,1)$, $C \in (1,\infty)$. Let $\Omega$ be a domain in $X$. Let $\gamma : [\alpha, \beta] \to \Omega$ be a rectifiable curve in $\Omega$. We say that $\gamma$ is a $(c,C)$-uniform curve in $\Omega$ if the following two conditions are satisfied:

(i) $\forall t \in [\alpha, \beta], \delta_\Omega(\gamma(t)) \geq c \min \{d(\gamma(\alpha), \gamma(t)), d(\gamma(t), \gamma(\beta))\}$

(ii) length($\gamma$) $\leq C d(\gamma(\alpha), \gamma(\beta))$.

The domain $\Omega$ is called $(c,C)$-uniform if any two points in $\Omega$ can be joined by a $(c,C)$-uniform curve in $\Omega$.

**Definition 3.2.** Fix $c \in (0,1)$, $C \in (1,\infty)$.

(i) Let $\gamma : [\alpha, \beta] \to \Omega$ be a rectifiable curve in $\Omega$. We say that $\gamma$ is a $(c,C)$-inner uniform curve in $\Omega$ if its is $(c,C)$-uniform in $\Omega$ in $(\bar{\Omega},d_\Omega)$.

(ii) We say that the domain $\Omega$ is $(c,C)$-inner uniform if $\Omega$ is $(c,C)$-uniform in $(\bar{\Omega},d_\Omega)$.

**Remark 3.3.** The notions of $(c,C)$-length-uniformity and inner-$(c,C)$-length-uniformity are defined analogously by replacing $d(\gamma(s), \gamma(t))$ by length($\gamma|_{[s,t]}$) in condition (i). The arguments used in [20, Lemma 2.7] and [12, Proposition 3.3] show that if $\gamma$ is a $(c,C)$-uniform curve in $\Omega$ joining $x$ and $y$ of length at most $R$ and if the doubling property holds in $B(x,2R)$ then there is a $(c',C')$-length uniform curve joining $x$ and $y$ in $\Omega$. For our purpose, this means that uniformity (resp. inner uniformity) and length-uniformity (resp. inner-length-uniformity) are equivalent notions.
Lemma 3.4. Let $\Omega$ be a $(c_u, C_u)$-inner uniform domain in $(X, d)$. For every ball $B = B_\Omega(x, r)$ in $(\tilde{\Omega}, d_\Omega)$ with minimal radius, there exists a point $x_r \in B$ with $d_\Omega(x, x_r) = r/4$ and $d(x_r, \tilde{\Omega} \setminus \Omega) \geq c_u r/8$.

Proof. This is immediate, see [12, Lemma 3.20].

Proving that a domain $\Omega$ is inner uniform is a difficult task. In fact, we lack a general method of constructing inner uniform domains in, say, complete metric length spaces. On the other hand, many domains in Euclidean space are inner uniform.

Example 3.5. In Euclidean space, any bounded convex domain is uniform. In addition, if $\Omega$ is convex and $B(x,aR) \subset \Omega \subset B(x,AR)$ then the uniformity constants $c_u, C_u$ depend only on $a, A$. Any bounded domain with piecewise smooth boundary with a finite number of singularities and non-zero interior angle at each of the singularities is inner uniform. The open unit ball in $\mathbb{R}^n$, $n \geq 2$, with the trace of the half-hyperplane $\{x : x_n = 0, x_{n-1} < 0\}$ deleted is inner uniform.

The interior and exterior of the Koch snowflake are inner uniform domains (in fact, uniform). The exterior of any convex set is inner uniform.

Example 3.6. Let $G = \mathbb{R}^3$ be the Heisenberg group with law

$$g_1 g_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + (1/2)(x_1 y_2 - x_2 y_1)), \quad g_i = (x_i, y_i, z_i).$$

Let $X$ and $Y$ be the left invariant vector fields on $G$ with $X(0) = \partial_x, Y(0) = \partial_y$. Let $\mathcal{E}(f, f) = \int_G (|Xf|^2 + |Yf|^2) d\mu$ where $\mu$ denotes the Haar measure on $G$ and the domain of $\mathcal{E}$ is the closure of smooth compactly functions for the norm $(\mathcal{E}(f, f) + |Xf|^2 + |Yf|^2) d\mu)^{1/2}$. Let $d$ be the corresponding intrinsic distance. Examples of uniform domains include any coordinate half-space through the origin, the coordinate unit cube in $\mathbb{R}^3$ and any metric ball $B(x, r)$ in $(G, d)$. See [5, 14] and [12] for further pointers to the literature.

3.2 Local inner uniformity

In [17], the authors derived a scale invariant boundary Harnack principle under a local version of inner uniformity which we now recall.

Definition 3.7. Fix $c_u \in (0, 1), C_u \in (1, \infty)$ and a domain $\Omega$. For a point $\xi \in \tilde{\Omega}$, let $R(\Omega, \xi) \in [0, \infty]$ be the largest $R \geq 0$ so that

(i) $8R \leq \text{diam}_\Omega(\Omega)$ (this is a non-trivial condition only when $\Omega$ is a bounded domain),

(ii) Any two points in $B_{\tilde{\Omega}}(\xi, 8R)$ can be connected by a curve that is $(c_u, C_u)$-inner uniform in $\Omega$.

Remark 3.8. It easily follows from Definition 3.7 that if $\xi$ is such that $R(\Omega, \xi) > 0$ then there exists $\eta > 0$ such that

$$d_\Omega(\xi, \xi') < R(\Omega, \xi) \implies R(\Omega, \xi') > \eta R(\Omega, \xi).$$
Consider non-empty domains $W \subset \Omega \subset X$. Let $W^\sharp$ be the largest open set in $(\widetilde{\Omega}, d_{\widetilde{\Omega}})$ whose intersection with $\Omega$ is $W$.

**Remark 3.9.** Any inner ball in $(\widetilde{W}, d_W)$ that lies in $W^\sharp$ is also an inner ball in $(\widetilde{\Omega}, d_{\widetilde{\Omega}})$. However, the metrics $d_W$ and $d_{\Omega}$ do not necessarily coincide.

**Definition 3.10.** Fix non-empty domains $W \subset \Omega \subset X$.

(i) We say that $\Omega$ is locally inner uniform near $W$ if for any point $\xi \in W^\sharp$ we have $R(\Omega, \xi) > 0$.

(ii) We say that $\Omega$ is locally inner uniform up to scale $R > 0$ near $W$ if for any point $\xi \in W^\sharp$, we have $R(\Omega, \xi) \geq R$.

**Remark 3.11.**

(i) From these definitions, it follows easily that if $\Omega$ is $(c_u, C_u)$-inner uniform then $R(\Omega, \xi) \simeq \text{diam}_\Omega(\Omega)$ for each $\xi \in \widetilde{\Omega}$. The constants implicitly contained in the notation $\simeq$ depend only on $c_u, C_u$.

(ii) By Remark 3.8 if $\Omega$ is locally inner uniform near $W$ and $\xi \in W^\sharp$, then there exists $R_\xi$ such that $\Omega$ is locally inner uniform up to scale $R_\xi$ near $B_{\Omega}(\xi, R_\xi)$.

(iii) Assume that $\Omega$ is locally $(c_u, C_u)$-inner uniform up to scale $R$ near $W$. Then for any point $\xi \in W^\sharp$ and $r \in (0, R)$ there exists a point $\xi_r \in \Omega$ such that $d_{\Omega}(\xi, \xi_r) = r/4$ and $d(\xi_r, \Omega \setminus \Omega) \geq c_u r/8$. See Lemma 3.4 and [12, Lemma 3.20].

(iv) In $\mathbb{R}^n$, any domain with smooth boundary is locally inner uniform. Many such domains (e.g., an unbounded "turnip" domain) are not locally inner uniform up to scale $R$.

### 4 Adapted forms

In this section, we introduce a large class of real bilinear forms on $L^2(X, d\mu)$ that all share a common domain $F$, the domain of our model form $E$. Further, these forms are of the type $E^s +$ lower order terms. Our goal is to pick one of these forms, $E$, and to study the Dirichlet heat kernel (and Dirichlet semigroup) associated to $E$ in a domain $U$ under the hypothesis that $U$ is inner uniform or, more generally, locally inner uniform.

#### 4.1 First and zero order parts

Given a bilinear form $E$, we set

$$E^{\text{sym}}(f, g) = \frac{1}{2}(E(f, g) + E(g, f)) \quad \text{and} \quad E^{\text{skew}}(f, g) = \frac{1}{2}(E(f, g) - E(g, f)).$$
These are, respectively, the symmetric and skew part of $\mathcal{E}$. For $f, g \in \mathcal{F}_c \cap L^\infty(X, \mu)$, we also set

$$\mathcal{L}(f, g) = \frac{1}{2} (\mathcal{E}^{\text{skew}}(fg, 1) + \mathcal{E}^{\text{skew}}(f, g)) \quad \text{and} \quad \mathcal{R}(f, g) = -\mathcal{L}(g, f).$$

Obviously,

$$\mathcal{E}^{\text{skew}}(f, g) = \mathcal{L}(f, g) + \mathcal{R}(f, g).$$

We recall the following definition taken from [10].

**Definition 4.1.** Assuming $\mathcal{E}$ is local with $D(\mathcal{E}) = \mathcal{F}$, we say that $\mathcal{E}^{\text{skew}}$ is a chain rule skew form relative to $\mathcal{F}$ if the following two properties hold:

- For any $u, v, f \in \mathcal{F} \cap \mathcal{C}_c(X)$, we have
  
  $$\mathcal{L}(uf, v) = \mathcal{L}(u, fv) + \mathcal{L}(f, uv).$$

- Let $v, u_1, u_2, \ldots, u_m \in \mathcal{F} \cap \mathcal{C}_c(X)$ and $u = (u_1, \ldots, u_m)$. If $\Phi \in C^2(\mathbb{R}^m)$, then $\Phi(u), \Phi_{x_i}(u) \in \mathcal{F}_{\text{loc}}(X) \cap L^\infty_{\text{loc}}(X, \mu)$ and
  
  $$\mathcal{L}(\Phi(u), v) = \sum_{i=1}^{m} \mathcal{L}(u_i, \Phi_{x_i}(u)v).$$

**Definition 4.2.** We say that the form $(\mathcal{E}, D(\mathcal{E}))$ is adapted to $(\mathcal{E}^*, \mathcal{F})$ if $\mathcal{E}$ is local, its domain $D(\mathcal{E})$ is $\mathcal{F}$ and:

(i) The form $\mathcal{E}$ satisfies

$$\forall f, g \in \mathcal{F}, \quad |\mathcal{E}(f, g)| \leq C \|f\|_\mathcal{F} \|g\|_\mathcal{F},$$

and, for all $f, g \in \mathcal{F}$ with $fg \in \mathcal{F}_c$,

$$|\mathcal{E}(fg, 1)| + |\mathcal{E}(1, fg)| \leq C \|f\|_\mathcal{F} \|g\|_\mathcal{F}.$$

(ii) The symmetric bilinear form $\mathcal{E}^{\text{sym}}(f, g) = \mathcal{E}^{\text{sym}}(fg, 1)$, extended by continuity to $\mathcal{F}$, is equal to the model form $\mathcal{E}^*$.

(iii) The skew part $\mathcal{E}^{\text{skew}}$ is a chain rule skew form relative to $\mathcal{F}$.

**Definition 4.3.** A symmetric bilinear form $Z$ is said to be a zero order form adapted to $\mathcal{F}$ if it is defined on $\mathcal{F}$ and satisfies

$$Z(f, g) = Z(fg, 1), \quad f, g \in \mathcal{F}, fg \in \mathcal{F}_c,$$

and

$$|Z(f, g)| \leq C \|f\|_\mathcal{F} \|g\|_\mathcal{F}.$$

Since $(\mathcal{E}^*, \mathcal{F})$ is fixed throughout, we will simply say that $(\mathcal{E}, D(\mathcal{E}))$ is an adapted form and that $Z$ is an adapted symmetric zero order form. Note that if $\mathcal{E}$ is an adapted form then its symmetric zero order part $Z$ is $\mathcal{E}^{\text{sym}}(fg, 1)$ is a zero order form adapted to $\mathcal{F}$. Further, $\mathcal{E} = \mathcal{E}^* + \mathcal{E}^{\text{skew}} + Z$.
4.2 Quantitative assumptions on the forms

We now introduce the fundamental quantitative assumptions on the bilinear forms for which we will study weak solutions of the heat equation with Dirichlet boundary condition.

**Assumption A.** The form $(E,F)$ is a bilinear form on $L^2(X,\mu)$ which is adapted to the model form $(E^*,F)$. Let $C_0 = C_0(E)$ be the constant in the sector condition $|E_{\text{skew}}(f,g)| \leq C_0 \|f\|_F \|g\|_F$. Assume further that:

(i) There are constants $C_2(E), C_3(E) \in [0,\infty)$ so that for all $f \in F$ with $f^2 \in F_c$,

$$|E_{\text{sym}}(f^2,1)| \leq 2 \left( \int f^2 d\mu \right)^{\frac{1}{2}} \left( C_2(E) \int d\Gamma(f,f) + C_3(E) \int f^2 d\mu \right)^{\frac{1}{2}}.$$  

(ii) There is a constant $C_5(E) \in [0,\infty)$ such that for all $f \in F$, $g \in F_c \cap L^\infty(X),

$$|E_{\text{skew}}(f,g^2)| \leq 2 \left( \int f^2 d\Gamma(g,g) \right)^{\frac{1}{2}} \left( C_5(E) \int f^2 g^2 d\mu \right)^{\frac{1}{2}}.$$  

Set $C_8(E) := C_2(E) + C_3(E)^{1/2} + C_5(E)$.

**Remark 4.4.** Under Assumption A the form $(E,F)$ is closed and satisfies

$$\forall f \in F, \quad E(f,f) \geq -\alpha \|f\|_2^2,$$

with $\alpha$ depending only on $C_2(E), C_3(E)$. In particular, the form $(E,F)$ induces a continuous semigroup of bounded operators $P_t$ on $L^2(X,\mu)$. We let $(L,D(L))$ denote the infinitesimal generator of this semigroup. By the results of [10], it is immediate that $P_t$ is positivity preserving.

**Remark 4.5.** For the purpose of this work, it is essential to compare Assumption A to Assumptions 0-1-2 of [16].

(i) It is plain that any form $E$ satisfying Assumption A also satisfy Assumptions 0-1-2 of [16] with respect to the model form $(E^*,F)$. Regarding Assumption 2 of [16], see [16, Remark 1.15(iv)].

(ii) Given a model form $(E^*,F)$, forms satisfying Assumption A are less general than the forms allowed by Assumptions 0-1-2 of [16]. To understand this, compare Assumption A(ii) with [16, Assumption 1(iii)] and note that Assumption A(ii) is the same as [16, Assumption 1(iii)] with $C_4 = 0$.

**Remark 4.6.** On Euclidean space, fix measurable bounded functions $a_{i,j}$, $b_i$, $d_i$, $c$, set $F = D(E) = W^1(\mathbb{R}^n)$ and

$$E(f,g) = \int \sum_{i,j=1}^n a_{i,j} \partial_i f \partial_j g \, dx + \int \sum_{i=1}^n b_i \partial_i f g \, dx + \int \sum_{i=1}^n d_i \partial_i g \, dx + \int c f g \, dx.$$
Set $\tilde{a}_{i,j} := (a_{i,j} + a_{j,i})/2$ and $\hat{a}_{i,j} = (a_{i,j} - a_{j,i})/2$. Then the symmetric part of $\mathcal{E}$ is

$$
\mathcal{E}^{\text{sym}}(f,g) = \int \sum_{i,j=1}^n \tilde{a}_{i,j} \partial_i f \partial_j g \, dx + \int \sum_{i=1}^n \frac{b_i + d_i}{2} \partial_i f \, g \, dx
$$

while the skew-symmetric part of $\mathcal{E}$ is

$$
\mathcal{E}^{\text{skew}}(f,g) = \int \sum_{i,j=1}^n \hat{a}_{i,j} \partial_i f \partial_j g \, dx + \int \sum_{i=1}^n \frac{b_i - d_i}{2} \partial_i f \, g \, dx
$$

The symmetric part $\mathcal{E}^{\text{sym}}$ can be decomposed into its strongly local part

$$
\mathcal{E}^{*}(f,g) = \sum_{i,j=1}^n \int \tilde{a}_{i,j} \partial_i f \partial_j g \, dx
$$

and its symmetric zero order part given by

$$
\mathcal{E}^{\text{sym}}(fg,1) = \int \sum_{i=1}^n \frac{b_i + d_i}{2} \partial_i (fg) \, dx + \int c \, f \, g \, dx.
$$

Assume that $(\tilde{a}_{i,j})$ is uniformly elliptic and set

$$
\mathcal{E}^{*}(f,g) = \int \sum_{i,j=1}^n \hat{a}_{i,j} \partial_i f \partial_j g \, dx, \quad f, g \in \mathcal{F}.
$$

On the one hand, under these hypotheses, the form $\mathcal{E}$ satisfies \cite{10} Assumptions 0-1-2. On the other hand, making the hypothesis that $\mathcal{E}$ is an adapted form with respect to $(\mathcal{E}^{*}, \mathcal{F})$ implies that the matrix $(a_{i,j})$ is symmetric, i.e., $(a_{i,j}) = (\hat{a}_{i,j})$.

Further, under these circumstances, the constants $C_2(\mathcal{E}), C_5(\mathcal{E})$ can be taken to be equal to 0 if $b_i = d_i = 0$ for all $i$ (i.e., if there is no drift term). The constant $C_8(\mathcal{E})$ can be taken equal to 0 if $b_i = d_i = c = 0$.

We will need the following simple Caccioppoli-type lemma. The proof is omitted.

**Lemma 4.7.** Let $(\mathcal{E}, \mathcal{F})$ be a form satisfying Assumption A. Let $u \in \mathcal{F}_{\text{loc}}$ and $\psi \in \mathcal{F}_c \cap L^\infty(X, \mu)$. For any $k_1 > 0$, we have

$$
-\mathcal{E}^{*}(u, u\psi^2) \leq 4k_1 \int u^2 d\Gamma(\psi, \psi) - \left(1 - \frac{1}{k_1}\right) \int \psi^2 d\Gamma(u, u).
$$
Moreover, for any $k_1, k_2, k_3 > 0$,

\[-E(u, w^2) \leq (4k_1 + 2k_2C_2 + k_3)\int u^2 d\Gamma(\psi, \psi) \]

\[+ \left(-1 + \frac{1}{k_1} + 2k_2C_2 \right)\int \psi^2 d\Gamma(u, u) \]

\[+ \left(\frac{1}{k_2} + k_2C_3 + \frac{C_5}{k_3} \right)\int u^2 \psi^2 d\mu.\]

4.3 Local weak solutions

Consider an adapted form $(E, F)$. Let $V$ be an open set. Recall that

\[F(V) = \{ u \in F_{\text{loc}}(V) : \int_V |u|^2 d\mu + \int_V d\Gamma(u, u) < \infty \}. \]

**Definition 4.8.** Let $V$ be open and $f \in F_c(V)'$, the dual space of $F_c(V)$ (identify $L^2(X, \mu)$ with its dual space using the scalar product). A function $u : V \rightarrow \mathbb{R}$ is a local weak solution of the Laplace equation $-Lu = f$ in $V$, if

(i) $u \in F_{\text{loc}}(V)$,

(ii) For any function $\phi \in F_c(V)$, $E(u, \phi) = \int f\phi d\mu$.

For a time interval $I$ and a Hilbert space $H$, let $L^2(I \rightarrow H)$ be the Hilbert space of those functions $v : I \rightarrow H$ such that

\[\|v\|_{L^2(I \rightarrow H)} = \left( \int_I \|v(t)\|_H^2 dt \right)^{1/2} < \infty.\]

Let $W^1(I \rightarrow H) \subset L^2(I \rightarrow H)$ be the Hilbert space of those functions $v : I \rightarrow H$ in $L^2(I \rightarrow H)$ whose distributional time derivative $v'$ can be represented by functions in $L^2(I \rightarrow H)$, equipped with the norm

\[\|v\|_{W^1(I \rightarrow H)} = \left( \int_I \|v(t)\|_H^2 + \|v'(t)\|_H^2 dt \right)^{1/2} < \infty.\]

Let

\[F(I \times X) = L^2(I \rightarrow F) \cap W^1(I \rightarrow F'),\]

where $F'$ denotes the dual space of $F$. Let

\[F_{\text{loc}}(I \times V)\]

be the set of all functions $u : I \times V \rightarrow \mathbb{R}$ such that for any open interval $J$ that is relatively compact in $I$, and any open subset $A$ relatively compact in $V$, there exists a function $u^J \in F(I \times X)$ such that $u^J = u$ a.e. in $J \times A$. Let

\[F_c(I \times V) = \{ u \in F_{\text{loc}}(I \times V) : u \text{ has compact support in } I \times V \}.\]
Definition 4.9. Let $I$ be an open interval and $V$ an open set in $X$. Set $Q = I \times V$. A function $u : Q \rightarrow \mathbb{R}$ is a local weak solution of the heat equation $\frac{\partial}{\partial t}u = Lu$ in $Q$, if

(i) $u \in \mathcal{F}_{\text{loc}}(Q),$

(ii) For any open interval $J$ relatively compact in $I$,

$$\forall \phi \in \mathcal{F}_c(Q), \int_J \int_V \frac{\partial}{\partial t}u \phi \, d\mu \, dt + \int_J \mathcal{E}(u(t, \cdot), \phi(t, \cdot)) \, dt = 0.$$ 

Remark 4.10. Assuming that the intrinsic distance satisfies (A1)-(A2), an equivalent definition of a local weak solution of $\frac{\partial}{\partial t}u = Lu$ on $Q = I \times V$ is

(i) $u \in L^2(I \rightarrow \mathcal{F}),$

(ii) For any open interval $J$ relatively compact in $I$,

$$-\int_J \int_V \frac{\partial}{\partial t}u \phi \, d\mu \, dt + \int_J \mathcal{E}(u(t, \cdot), \phi(t, \cdot)) \, dt = 0,$$

for all $\phi \in \mathcal{F}(Q)$ with compact support in $J \times V$.

See [6]. The argument uses the existence of good cut-off functions provided by (A1)-(A2).

4.4 Local weak solutions with Dirichlet boundary condition along $\partial U$

To define weak solutions with Dirichlet boundary condition, we use Definition 2.4 where the space $\mathcal{F}^0_{\text{loc}}(U, V)$ is introduced.

Definition 4.11. Let $V, U$ be open with $V \subset U$. A function $u : V \rightarrow \mathbb{R}$ is a local weak solution of the Laplace equation $-Lu = f$ in $V$ with Dirichlet boundary condition along $\partial U$ if

(i) $u$ is a local weak solution of $-Lu = f$ in $V$ and

(ii) $u \in \mathcal{F}^0_{\text{loc}}(U, V)$.

Next we fix an open interval $I$ and an open set $V$ in a domain $U$ in $X$ and define the notion of a local weak solution in $I \times V$ with Dirichlet boundary condition along the boundary of $U$. Recall that $\mathcal{F}^0(U)$ is the closure of $\mathcal{F}_c(U)$ for the norm $(\int_U |f|^2 \, d\mu + \int_U d\Gamma(f, f))^{1/2}$. Define

$$\mathcal{F}^0(I \times U) = L^2(I \rightarrow \mathcal{F}^0(U)) \cap W^1(I \rightarrow (\mathcal{F}^0(U))^\prime).$$

For $Q = I \times V$, define $\mathcal{F}^0_{\text{loc}}(U, Q)$ to be the set of all functions $v : Q \rightarrow \mathbb{R}$ such that, for any open interval $J \subset I$ relatively compact in $I$ and any open subset $W \subset V$ relatively compact in $\tilde{U}$ with $d_U(W, U \setminus V) > 0$, there exists a function $v^2$ in $\mathcal{F}^0(I \times U)$ such that $v^2 = u$ a.e. in $J \times W$. 

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Definition 4.12. Let $I$ be an open interval and $V$ an open set in $X$. Set $Q = I \times V$. We say that a function $u : Q \to \mathbb{R}$ is a local weak solution of the heat equation $\frac{\partial}{\partial t}u = Lu$ in $Q$ with Dirichlet boundary condition along $\partial U$ if

(i) $u$ is a local weak solution of the heat equation in $Q$ and

(ii) $u \in F^0_{\text{loc}}(U,Q)$.

5 Harnack inequalities

Harnack inequalities play an essential and central role in the results obtained in this paper. The next two subsections discuss interior Harnack inequalities and boundary Harnack inequalities, respectively.

In this section, we consider a fixed open subset $Y$ of $X$. We assume that the model form $(\mathcal{E}^s,\mathcal{F})$, defined in Section 2, satisfies (A1)-(A2-Y).

5.1 Interior Harnack inequalities

For any $s \in \mathbb{R}$, $\tau > 0$, $\delta \in (0,1)$ and $B(x,2r) \subset Y$, define

$$
I = (s - \tau r^2, s) \\
B = B(x,r) \\
Q = I \times B \\
Q_- = (s - (3 + \delta)\tau r^2/4, s - (3 - \delta)\tau r^2/4) \times \delta B \\
Q_+ = (s - (1 + \delta)\tau r^2/4, s) \times \delta B.
$$

Definition 5.1. Let $(\mathcal{E},\mathcal{F})$ be an adapted form.

- We say that $(\mathcal{E},\mathcal{F})$ satisfies the parabolic Harnack inequality on $Y$ if, for any $\tau > 0$, $\delta \in (0,1)$, there exists a constant $H_Y(\tau, \delta) \in (0,\infty)$ such that, for any ball $B(x,2r) \subset Y$, any $s \in \mathbb{R}$, and any positive local weak solution $u$ of the heat equation $\frac{\partial}{\partial t}u = Lu$ in $Q$, the following inequality holds.

$$
\sup_{z \in Q_-} u(z) \leq H_Y \inf_{z \in Q_+} u(z) \quad \text{(PHI)}
$$

Here both the supremum and the infimum are essential, i.e., computed up to sets of measure zero.

- We say that the parabolic Harnack inequality holds locally in $Y$ if for each $y \in Y$ there is a neighborhood $V$ of $y$ in $Y$ such that (PHI) holds in $V$ (in this case, the constant $H_Y$ may indeed depend on $V$).

- We say that the parabolic Harnack inequality holds up to scale $R$ in $Y$ there is a constant $H_Y(R)$ such that (PHI) holds in any ball $B(y,2R)$, $y \in Y$, with constant $H_{B(y,2R)}$ bounded above by $H_Y(R)$.
The parabolic Harnack inequality implies the elliptic Harnack inequality,
\[ \sup_{z \in B(x,r)} u(z) \leq H_Y' \inf_{z \in B(x,r)} u(z), \tag{EHI} \]
where \( u \) is any positive function in \( \mathcal{F}_{ws}(Q) \) with \( Lu = 0 \) weakly in \( B(x,2r) \).

Recall also that (PHI) implies the Hölder continuity of local weak solutions.

The following theorem gathers fundamental known results regarding the parabolic Harnack inequality.

**Theorem 5.2.** Let \( X,Y,\mathcal{E},\mathcal{F},d,\mu \) be as in Section 2. In particular, we assume that (A1)-(A2-Y) holds true. Let \( (\mathcal{E},\mathcal{F}) \) be a form satisfying Assumption A.

(i) The symmetric strongly local regular Dirichlet form \( (\mathcal{E}^s,\mathcal{F}) \) satisfies (PHI) on \( Y \) if and only if it satisfies the volume doubling property and the Poincaré inequality on \( Y \).

(ii) The symmetric strongly local regular Dirichlet form \( (\mathcal{E}^s,\mathcal{F}) \) satisfies (PHI) locally (resp. up to scale \( R \)) on \( Y \) if and only if it satisfies the volume doubling property and the Poincaré inequality locally (resp. up to scale \( R \)) on \( Y \).

(iii) If the model form \( (\mathcal{E}^s,\mathcal{F}) \) satisfies (PHI) locally in \( Y \) then the form \( (\mathcal{E},\mathcal{F}) \) satisfies (PHI) locally in \( Y \).

(iv) If the model form \( (\mathcal{E}^s,\mathcal{F}) \) satisfies (PHI) locally up to scale \( R < \infty \) in \( Y \) with constant \( H(\mathcal{E}^s, R) \) then \( (\mathcal{E},\mathcal{F}) \) satisfies (PHI) up to scale \( R < \infty \) in \( Y \) with constant \( H(\mathcal{E}, R) \) depending only on \( H(\mathcal{E}^s, R) \), the constants \( C_1(\mathcal{E}) - C_5(\mathcal{E}) \) and an upper bound on \( C_8(\mathcal{E})R^2 \).

**Remark 5.3.** The first two statements of this theorem are the Dirichlet form version of the characterization of the parabolic Harnack inequality by volume doubling and Poincaré inequality. See \[11, 23, 27, 28, 30\].

Statements (iii)-(iv) are variations on the key fact that the parabolic Harnack inequality for the model form \( (\mathcal{E}^s,\mathcal{F}) \) implies (PHI) for a wide variety of other forms in the spirit of the original work of Nash, Moser and Aronson and Serrin. The proof is contained in \[16, 28, 30\]. In particular, (iii)-(iv) are special cases of \[16, Theorem 2.13\] which covers a wider class of forms, namely, forms satisfying \[16, Assumptions 0-1-2\].

### 5.2 Boundary Harnack principle

Let \( (\mathcal{E},\mathcal{F}) \) be an adapted form satisfying Assumption A. Let \( U \) be a domain in \( X \). The boundary Harnack principle is concerned with positive local weak solutions of \( Lu = 0 \) with Dirichlet boundary condition along \( \partial U \) and their behavior near the boundary. We refer the reader to \[11\] for pointers to the literature.

We will use a strong version of the boundary Harnack principle which we refer to as the geometric boundary Harnack principle.
Definition 5.4. Let $X, E, F, d, \mu$ be as in Section 2. Let $W \subset U$ be non-empty domains in $X$. Let $(E, F)$ be a form satisfying Assumption A. Referring to local weak solutions of $Lu = 0$ with Dirichlet boundary condition along $\partial U$ where $L$ is the generator associated to $(E, F)$, we say that:

(i) the geometric boundary Harnack principle holds on $U$, if there exist constants $a_0, A_0, A_1 \in (0, \infty)$, depending only on $U$, with the following property. Let $\xi \in \bar{U} \setminus U$ and $r \in (0, a_0 \text{diam}_U(U))$. Then for any two positive weak solutions $u$ and $v$ of $Lu = 0$ in $B_{U}(\xi, A_0 r)$ with Dirichlet boundary condition along $\partial U$, we have
\[
\frac{u(x)}{u(x')} \leq A_1 \frac{v(x)}{v(x')}, \quad \forall x, x' \in B_{U}(\xi, r).
\]

(ii) the geometric boundary Harnack principle holds locally near $W$ if, for every compact set $K \subset W \setminus W$, there exist $A_0(K), A_1(K)$ and $R(K) > 0$ such that for any $\xi \in K$, $r \in (0, R(K))$ and any two positive weak solutions $u$ and $v$ of $Lu = 0$ in $B_{U}(\xi, A_0(K) r)$ with Dirichlet boundary condition along $\partial U$, we have
\[
\frac{u(x)}{u(x')} \leq A_1(K) \frac{v(x)}{v(x')}, \quad \forall x, x' \in B_{U}(\xi, r).
\]

(iii) the geometric boundary Harnack principle holds up to scale $R$ near $W$ if we can take $A_0(K) = A_0$, $A_1(K) = A_1$ and $R(K) = R$ in the previous statement.

The following theorem follows immediately from [17, Theorem 4.2].

Theorem 5.5. Fix $R > 0$. Let $X, E, F, d, \mu$ be as in Section 2. Let $(E, F)$ be a form satisfying Assumption A. Let $W \subset U$ be domains in $X$. Assume further that:

(i) $(E, F)$ is a (possibly non-symmetric) Dirichlet form.

(ii) The volume doubling property and the Poincaré inequality hold up to scale $R$ in $W$.

(iii) The domain $U$ is locally $(c_u, C_u)$-inner uniform up to scale $R$ near $W$.

Then there exist constants $a_0 \in (0, 1)$, $A_0, A_1 \in (1, \infty)$ such that for any $\xi \in W \setminus W$, $0 < r < a_0 R$, and any two non-negative local weak solutions $u, v$ of $Lu = 0$ in $B_{U}(\xi, A_0 r)$ with weak Dirichlet boundary condition along $\partial U$, we have
\[
\frac{u(x)}{u(x')} \leq A_1 \frac{v(x)}{v(x')},
\]
for all $x, x' \in B_{U}(\xi, r)$.
The constants $a_0, A_0$ depend only on the local inner uniformity constants $c_u, C_u$ near $W$. The constant $A_1$ depends only on the inner uniformity constants $c_u, C_u$, an upper bound on the volume doubling constant and the Poincaré inequality constant up to scale $R$ on $\overline{W}$, the constants $C_0(\mathcal{E}) - C_5(\mathcal{E})$ from Assumption A which give control over the skew-symmetric part and the killing part of the Dirichlet form $\mathcal{E}$, and an upper bound on $C_8(\mathcal{E})R^2$.

The following theorem is a direct consequence of Theorem 5.5 and the various definitions.

**Theorem 5.6.** Let $X, \mathcal{E}, \mathcal{F}, d, \mu$ be as in Section 2. Let $(\mathcal{E}, \mathcal{F})$ be a form satisfying Assumption A. Assume further that $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form. Let $U$ be a domain in $X$.

(i) Fix a domain $W \subset U$, and assume that $U$ is locally inner uniform near $W$. Assume also that the volume doubling property and the Poincaré inequality hold locally in $W$. Then the geometric boundary Harnack principle holds locally in $U$ near $W$.

(ii) Fix $R \in (0, \infty]$ and a domain $W \subset U$. Assume that $U$ is locally $(c_u, C_u)$-inner uniform up to scale $R$ near $W$ and that the volume doubling property and Poincaré inequality hold up to scale $R$ in $\overline{W}$. Then there exists $a_0 > 0$ such that the geometric boundary Harnack principle holds locally up to scale $r$ near $W$ for all $r < a_0R$ with $C_8(\mathcal{E})r^2 < \infty$.

(iii) Assume that $U$ is inner uniform and that the volume doubling property and Poincaré inequality hold in $X$. Assume further that $\mathcal{E} = \mathcal{E}^*$. Then the geometric boundary Harnack principle holds true in $U$.

### 5.3 Harmonic profiles

The main idea developed in [12] in the context of strongly local symmetric Dirichlet forms is that the Dirichlet heat kernel in a domain $U$ can be estimated in terms of the harmonic profile $h_U$. In this section we extend the notion of harmonic profiles and gather some of their key properties.

**Definition 5.7.** For an open subset $U \subset X$ and under Assumption A, consider the bilinear form $\mathcal{E}_U^D$

$$\mathcal{E}_U^D(f, g) = \mathcal{E}(f, g), \quad f, g \in \mathcal{F}^0(U),$$

where the domain $D(\mathcal{E}_U^D) = \mathcal{F}^0(U)$ is the closure of the space $\mathcal{F}_c(U)$ in the norm $(\mathcal{E}^*(f, f) + \|f\|_2)^{\frac{1}{2}}$.

Under Assumption A, the form $(\mathcal{E}_U^D, D(\mathcal{E}_U^D))$ is closed, bounded below, local and regular.

**Definition 5.8.** Let $X, \mathcal{E}, \mathcal{F}, d, \mu$ be as in Section 2. Let $(\mathcal{E}, \mathcal{F})$ be a form satisfying Assumption A and $\mathcal{E}(f, f) \geq 0$ for all $f \in \mathcal{F}$. Let $U$ be a domain in $X$. A function $h \in L^2_{\text{loc}}(U)$ is called an $\mathcal{E}$-harmonic profile in $U$ if it satisfies the following properties:
(i) \( h \) is a weak solution of \( Lu = 0 \) in \( U \);
(ii) \( h \in \mathcal{F}^0_{\text{loc}}(U) \);
(iii) \( h > 0 \) in \( U \).

Fix a domain \( W \subset U \). We say that a function \( h \) in is a \((U,W)\)-profile for \( E \) if \( h \) is defined in \( W \) and

(i) \( h \) is a weak solution of \( Lu = 0 \) in \( W \);
(ii) \( h \in \mathcal{F}^0_{\text{loc}}(U,W) \);
(iii) \( h > 0 \) in \( W \).

**Proposition 5.9.** Let \( X, \mathcal{E}, \mathcal{F}, d, \mu \) be as in Section 2. Let \((E,F)\) be a form satisfying Assumption A and which is a Dirichlet form. Fix domains \( W \subset U \). Assume that the volume doubling property and Poincaré inequality hold locally on \( U \).

(i) Assume that \( U \) is unbounded and inner uniform near \( W \). Then there exists a function \( h \) which is a local weak solution of \( Lh = 0 \) in \( U \) and is a \((U,W)\)-profile.

(ii) If \( U \) is unbounded and locally inner uniform it admits a harmonic profile \( h \).

(iii) If \( U \) is bounded, inner uniform, \( x_0 \in U \), and \( W \subset U \setminus B_U(x_0, \epsilon) \) then the Green function \( h(x) = G_U(x, x_0) \) is a \((U,W)\)-profile.

**Proof.** Note that \( G_U \) denotes the Green function in \( U \) with Dirichlet boundary condition, i.e., the Green function for the form \((E^D_U, D(E^D_U))\). The third statement follows immediately from [17, Lemma 3.9]. See also [12, Lemma 4.7] and [17, Lemma 3.10]. Note that applying the results of [17] requires assuming that the form \( E \) is a (possibly non-symmetric) Dirichlet forms. We can extend these results to the general case using the results obtained at the end of the next section.

The idea of the proof of (i)-(ii) is to construct the profile \( h \) has a limit of normalized Green functions. The details follow the same line of reasoning as in [12, Theorem 4.16] with simple adaptations using Assumption A to take care of the fact that the form \( E \) is not symmetric. \( \square \)

**Proposition 5.10.** Let \( X, \mathcal{E}, \mathcal{F}, d, \mu \) be as in Section 2. Let \((E,F)\) be a form satisfying Assumption A and which is a Dirichlet form. Fix \( R > 0 \). Let \( W \subset U \) be domains in \( X \). Assume that the volume doubling property and the Poincaré inequality hold locally up to scale \( R \) on \( W \) and that \( U \) is locally inner uniform up to scale \( R \) near \( W \). Let \( h \) be a \((U,W)\)-profile. Then there are constants \( K_0, K_1 \) such that for any inner ball \( B_U(x, r) \) with \( 0 < K_0r < R \), \( B_U(x, K_0r) \subset W \), we have

\[ \forall y \in B_U(x, r), \quad h(y) \leq K_1 h(x, r) \]
where $x_r$ is any point with $d_U(x, x_r) = r/4$ and $d(x_r, X \setminus W) \geq c_u r/8$. The constants $K_0$ depend only on the local inner uniformity constants $c_u, C_u$. The constant $K_1$ depends only on $c_u, C_u$, the doubling and Poincaré constants up to scale $R$ in $W$, the constants $C_0 - C_5$ which give control over the skew-symmetric part and the killing part of the Dirichlet form $E$ and an upper bound on $C_8(E) R^2$.

**Proof.** Compare the ratios for $h$ and an appropriately chosen Green function. The result follows as an immediate corollary of Theorem 5.5 and the Green function estimates obtained in [17, Lemmas 3.11-3.12]. See also [12, Theorem 4.17].

The next two propositions are straightforward applications of Proposition 5.10. The proofs follow the same line of reasoning as in [12, Theorem 4.17] and are omitted.

**Proposition 5.11 (Unbounded domains).** Let $X, \mathcal{E}', \mathcal{F}, d, \mu$ be as in Section 2. Let $(\mathcal{E}, \mathcal{F})$ be a form satisfying Assumption A and which is a Dirichlet form. Fix an unbounded domain $U$ in $X$ and let $h$ be an $\mathcal{E}$-harmonic profile for $U$.

(i) Assume $U$ is locally inner uniform and that the volume doubling property and the Poincaré inequality hold locally in a neighborhood $Y$ of $U$ in $X$. Then for any compact $K \subset \bar{U}$ there exist $r_K, \epsilon_K > 0$ and $C_K$ such that for any $x \in K, r \in (0, r_K)$, we have

$$\forall y \in B_U(x, r), \ h(y) \leq C_K h(x_r)$$

where $x_r \in B_U(x, r)$ is any point with $d_U(x, x_r) = r/4$ and $d(x_r, \partial U) \geq \epsilon_K r$. Further

$$V_{h,z}(x, r) = \int_{B_U(x, r)} h^2 d\mu \simeq h(x_r)^2 V(x, r).$$

(ii) Fix $R > 0$ and assume $U$ is locally $(c_u, C_u)$-inner uniform up to scale $R$ and that the volume doubling property and the Poincaré inequality hold up to scale $R$ in $\bar{U}$. Then there exists $a_0, A_1 \in (0, \infty)$ such that for any $x \in \bar{U}, r \in (0, a_0 R)$, we have

$$\forall y \in B_U(x, r), \ h(y) \leq A_1 h(x_r)$$

where $x_r \in B_U(x, r)$ is any point with $d_U(x, x_r) = r/4$ and $d(x_r, \partial U) \geq c_u r/8$. Further

$$V_{h,z}(x, r) = \int_{B_U(x, r)} h^2 d\mu \simeq h(x_r)^2 V(x, r).$$

The constant $a_0$ depends only on $(c_u, C_u)$. The constant $A_1$ depends only on $c_u, C_u$, the volume doubling and Poincaré constant up to scale $R$ in $\bar{U}$, the constants $C_0 - C_5$ which give control over the skew-symmetric part and the killing part of the Dirichlet form $E$, and an upper bound on $C_8(E) R^2$. 

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Remark 6.1. Any inner ball in $(\overline{W}, d_W)$ that lies in $W^\sharp$ is also an inner ball in $(\overline{U}, d_U)$. However, the metrics $d_W$ and $d_U$ do not necessarily coincide on $W^\sharp$.

We assume the model form $(\mathcal{E}^\ast, \mathcal{F})$ satisfies (A1)-(A2-$\overline{\mathcal{W}}$), and that the volume doubling property and the Poincaré inequality hold locally on $\overline{W}$ in $X$.

6 Some structural properties of $h$-transforms

Definition 6.2. Let $h$ be positive continuous on $W$. Let $H : f \in L^2(W, h^2 d\mu) \rightarrow L^2(W, d\mu)$, $f \mapsto Hf = hf$. Let $(\mathcal{E}^{D,W}_h, D(\mathcal{E}^{D,W}_h))$ be the form

$$\mathcal{E}^{D,W}_h(f,g) = \mathcal{E}(Hf,Hg), \quad f, g \in H^{-1}(\mathcal{F}^0(W)) = D(\mathcal{E}^{D,W}_h).$$

Remark 6.3. Dropping the reference to the Dirichlet condition and $W$ an writing $\mathcal{E}_h = \mathcal{E}^{D,W}_h$, observe that:

(i) We have

$$\mathcal{E}^{\text{sym}}_h(f,g) = \mathcal{E}^{\text{sym}}(hf, hg), \quad \mathcal{E}^{\text{skew}}_h(f,g) = \mathcal{E}^{\text{skew}}(hf, hg).$$

Further, for $f, g \in \mathcal{F}_c(W) \cap \mathcal{C}(W)$,

$$\mathcal{E}_h(f, g) = \mathcal{E}^{\text{sym}}_h(f, g) - \mathcal{E}^{\text{sym}}_h(fg, 1)$$
\[ = \mathcal{E}^*(hf,hg) - \mathcal{E}^*(hf,h) = \int h^2 d\Gamma(f,g). \]

Note that \( \mathcal{E}^*_h \) must be understood as being the symmetric strongly local part of \( \mathcal{E}_h \) which, in general, is not the same as the \( h \) transform \( (\mathcal{E}^*)_h \) of \( \mathcal{E}^* \). The two are the same exactly when \( \mathcal{E}^*(hf,g) = 0 \) for any \( f,g \in \mathcal{F}(W) \cap \mathcal{C}(W) \). This is the case when \( h \) is a \((U,W)\)-profile for \( \mathcal{E}^*_h \).

(ii) Recall that \( \mathcal{L}_h(f,g) = \frac{1}{2}(\mathcal{E}^{\text{skew}}_h(fg,1) + \mathcal{E}^{\text{skew}}_h(f,g)) \). Since \( \mathcal{E} \) is adapted, \( \mathcal{L} \) satisfies the Leibniz rule \( \mathcal{L}(uf,v) = \mathcal{L}(u,fv) + \mathcal{L}(v,fu), \) \( u,v,f \in \mathcal{F}(W) \cap \mathcal{C}(W) \). Hence, we obtain

\[
2\mathcal{L}_h(f,g) = \mathcal{L}(hf,g,h) + \mathcal{R}(hfg,h) + \mathcal{L}(hf,g) + \mathcal{R}(hf,hg)
= \mathcal{L}(hf,g,h) - \mathcal{L}(h,hfg) + \mathcal{L}(hf,g) - \mathcal{L}(hg,hf)
+ \mathcal{L}(h,hfg) + \mathcal{L}(f,h^2g) - \mathcal{L}(hg,hf)
= 2\mathcal{L}(f,h^2g).
\]

This shows that \( \mathcal{L}_h \) satisfies the appropriate Leibniz rule and chain rule as in Definition 4.1.

(iii) Under the additional assumption that \( h \) is a \((U,W)\)-profile for \( \mathcal{E}^* \), we have for \( f,g \in \mathcal{F}(W) \cap \mathcal{C}(W) \),

\[ \mathcal{E}^*_{sym}(fg,1) = \mathcal{E}^*_{sym}(hfg,h) = \mathcal{E}^*_{sym}(h,hfg) = \mathcal{E}^{\text{skew}}(hfg,h) \]

because \( \mathcal{E}(h,hfg) = 0 \) under the present condition on \( h, f, g \).

(iv) Assume that \( h \) is an \( \mathcal{E}^*\)-(\( U,W) \) profile. Then, for all \( f,g \in \mathcal{F}(W) \cap \mathcal{C}(W) \), we have \( \mathcal{E}^*(h,hfg) = 0 \). Hence, in this case,

\[ \mathcal{E}^*_{sym}(fg,1) = \mathcal{E}^*_{sym}(h,hfg) = \mathcal{E}^*_{sym}(h^2fg,1). \]

(v) Assume that \( h \in \mathcal{F}_{\text{loc}} \) is positive and continuous on \( W \) and satisfies

\[ \forall u \in \mathcal{F}(W), \quad \mathcal{E}(f,u) = \gamma \int hu \, d\mu \]

for some \( \gamma \in \mathbb{R} \). Then, for all \( f,g \in \mathcal{F}(W) \cap \mathcal{C}(W) \), we have

\[ \mathcal{E}^*_{sym}(fg,1) = \mathcal{E}^*_{sym}(h,hfg) = \mathcal{E}^{\text{skew}}(hfg,h) + \gamma \int fgh^2 \, d\mu. \]

**Definition 6.4.** For a fixed \( h, \) positive and continuous on \( W \), let

\( (\mathcal{E}^{D,W,h^2}, D(\mathcal{E}^{D,W,h^2})) \)

be the Dirichlet form on \( L^2(W,h^2d\mu) \) obtained by closing

\[ \mathcal{E}^{D,W,h^2}(f,g) = \int h^2d\Gamma(f,g), \quad f,g \in \mathcal{F}(W). \]
Let $\mathcal{F}^h = D(\mathcal{E}^{D,W,h^2})$ be the domain of this form, that is, the closure of $\mathcal{F}_c(W)$ for the norm

$$\|f\|_{\mathcal{F}^h} = \mathcal{F}^h_W = \left(\int_W |f|^2h^2d\mu + \int_W h^2d\Gamma(f,f)\right)^{1/2}.$$ 

Note that, by definition, $(\mathcal{E}^{D,W,h^2}, \mathcal{F}^h)$ is a symmetric strongly local regular Dirichlet form on $W$.

**Lemma 6.5.** Assume that $h$ is continuous positive on $W$. Then the set

$$H^{-1}(\mathcal{F}_c(W) \cap L^\infty(W,\mu))$$

is dense in the Hilbert space

$$D(\mathcal{E}_h^{D,W}) = H^{-1}(\mathcal{F}^0(W)), \quad \|f\|_{D(\mathcal{E}_h^{D,W})}^2 = \int_W d\Gamma(hf,hf) + \int_W h^2|f|^2d\mu$$

and

$$H^{-1}(\mathcal{F}_c(W) \cap L^\infty(W,\mu)) = \mathcal{F}_c(W) \cap L^\infty(W,\mu)$$

$$= \mathcal{F}_c(W) \cap L^\infty(W,h^2\mu).$$

In particular, $\mathcal{F}_c(W) \cap L^\infty(W,\mu)$ is also dense in $D(\mathcal{E}_h^{D,W}) = H^{-1}(\mathcal{F}^0(W))$.

**Proof.** We follow [12, Proposition 5.7]. The set $\mathcal{F}_c(W) \cap L^\infty(W,\mu)$ is dense in the Hilbert space $\mathcal{F}^0(W)$. Since $H$ is a unitary operator between the Hilbert spaces $D(\mathcal{E}_h)$ and $\mathcal{F}^0(W)$, it follows that $H^{-1}(\mathcal{F}_c(W) \cap L^\infty(W,\mu))$ is also dense in the Hilbert space $D(\mathcal{E}_h)$. Since $h, 1/h$ are both in $\mathcal{F}_c(W) \cap L^\infty(W,\mu)$, the equality

$$H^{-1}(\mathcal{F}_c(W) \cap L^\infty(W,\mu)) = \mathcal{F}_c(W) \cap L^\infty(W,\mu)$$

follows from the fact that $\mathcal{F}_c(W) \cap L^\infty(W,\mu)$ is an algebra. \hfill $\square$

**Lemma 6.6.** Assume that $h$ is a $(U,W)$-profile for either $\mathcal{E} + \gamma(\cdot,\cdot)$ or $\mathcal{E} + \gamma(\cdot,\cdot)$, $\gamma \in \mathbb{R}$. Then there exists $C \in (0,\infty)$ such that, for $f, g \in \mathcal{F}_c(W) \cap L^\infty(W,\mu)$, we have

(i) $\int_W d\Gamma(hf,hf) \leq C \left(\mathcal{E}_h^{D,W,h^2}(f,f) + \int_W |f|^2h^2d\mu\right)$.

(ii) $|\mathcal{E}_h^{D,W}(f,f)| \leq C \left(\mathcal{E}_h^{D,W,h^2}(f,f) + \int_W |f|^2h^2d\mu\right)$.

(iii) $\mathcal{E}_h^{D,W,h^2}(f,f) \leq C \left(\mathcal{E}_h^{D,W}(f,f) + \int_W |f|^2h^2d\mu\right)$.

In particular, $D(\mathcal{E}_h^{D,W}) = D(\mathcal{E}_h^{D,W,h^2}) = \mathcal{F}^h$.

Note that $\mathcal{F}_c(W) \cap L^\infty(W,\mu)$ is dense in the domains of both forms $\mathcal{E}_h^{D,W}$ and $\mathcal{E}_h^{D,W,h^2}$ so that the last statement follows from (ii)-(iii).
Proof. We give the proof for (ii) and (iii) when \( h \) is a \((U,W)\)-profile for \( \mathcal{E} \). The proof of (i) and the cases when \( h \) is a \((U,W)\)-profile for \( \mathcal{E} + \gamma(\cdot,\cdot) \), \( \gamma \neq 0 \) or \( \mathcal{E}^* + \gamma(\cdot,\cdot) \) are similar. To simplify notation, we set \( \mathcal{E}^{D,W,h}_2 = \mathcal{E}_h \) and \( \mathcal{E}^{D,W,h^2} = \mathcal{E}^{h^2} \) and we drop the explicit reference to the Dirichlet condition and the set \( W \).

For any \( f \in \mathcal{F}_c(W) \cap L^\infty(W,\mu) \), Assumption A(i) yields

\[
|\mathcal{E}_h(f,f)| = |\mathcal{E}^{sym}(hf,hf)| \leq \int d\Gamma(hf,hf) + |\mathcal{E}^{sym}(h^2f^2,1)| \leq C \left( \int h^2d\Gamma(f,f) + \int f^2d\Gamma(h,h) + \int h^2f^2d\mu \right).
\]

The constant \( C \) depends only on \( C_2, C_3 \). Because \( \mathcal{E}(h,hf^2) = 0 \), we get from Lemma 4.7 that for any \( k_1, k_2, k_3 > 0 \),

\[
\left( 1 - \frac{1}{k_1} - 2k_2C_2 \right) \int f^2d\Gamma(h,h) \leq (4k_1 + 2k_2C_2 + k_3) \int h^2d\Gamma(f,f) + \left( \frac{1}{k_2} + k_2C_3 + \frac{C_5}{k_3} \right) \int f^2h^2d\mu.
\]

Hence (with a different \( C \) depending only on \( C_2, C_3, C_5 \)),

\[
|\mathcal{E}_h(f,f)| \leq C \left( \mathcal{E}^{h^2}(f,f) + \int h^2f^2d\mu \right).
\]

This proves (ii).

To prove (iii), we use the fact that, for \( f \in \mathcal{F}_c(W) \cap L^\infty(W,\mu) \), \( \mathcal{E}(h,hf^2) = 0 \), and Assumption A(ii) to obtain

\[
\mathcal{E}^{h^2}(f,f) = \mathcal{E}_h(f,f) - \mathcal{E}^{sym}(h^2f^2,1) - \mathcal{E}^*(h,hf^2)
\]

\[
= \mathcal{E}_h(f,f) + \mathcal{E}^{skew}(h,hf^2)
\]

\[
\leq \mathcal{E}_h(f,f) + k_4 \int h^2d\Gamma(f,f) + \frac{C_5}{k_4} \int f^2h^2d\mu,
\]

where \( k_4 > 0 \) is arbitrary. Choosing \( k_4 = 1/2 \), we get

\[
\mathcal{E}^{h^2}(f,f) \leq 2 \left( \mathcal{E}_h(f,f) + 2C_5 \int f^2h^2d\mu \right).
\]

Proposition 6.7. Assume that \( h \) is continuous positive on \( W \) and belongs to \( \mathcal{F}^0(U,W) \). The strongly local Dirichlet form \( (\mathcal{E}^{D,W,h^2}, \mathcal{D}(\mathcal{E}^{D,W,h^2})) \) is regular on \( (W^2, h^2d\mu) \) with core \( \text{Lip}_c(W^2, dW) \).

Proof. We follow the proof of [12, Proposition 5.8]. As \( (\mathcal{E}^{D,W,h^2}, \mathcal{D}(\mathcal{E}^{D,W,h^2})) \) is regular on \( W \), \( C_c(W) \cap \mathcal{D}(\mathcal{E}^{D,W,h^2}) \) is dense in \( \mathcal{D}(\mathcal{E}^{D,W,h^2}) \). So we only need to show that \( C_c(W^2) \cap \mathcal{D}(\mathcal{E}^{D,W,h^2}) \) is dense in \( C_c(W^2) \) in the sup norm. Consider
a function \( f \in \text{Lip}_c(W^\sharp, d_W) \) with Lipschitz constant \( k \). As \( \text{Lip}_c(W^\sharp, d_W) \) is dense in \( C_c(W^\sharp) \) in sup norm, it suffices to show that \( f \in D(\mathcal{E}^{D,W,h^2}) \). In view of Lemma 6.5 it suffices to show that \( fh \in F^0(W) \). Since \( \text{Lip}_c(W^\sharp, d_W) \subset \text{Lip}(W, d_W) \), Corollary 2.13 implies that \( f \in F_{\text{loc}}(W) \) and

\[
\Upsilon(f, f) = \frac{d\Gamma(f, f)}{d\mu} \leq k^2 \quad \text{almost everywhere on } W.
\]

Since \( f \) is bounded, this shows that \( f \in F(W) \). Let \( V \subset W \) be an open set containing \( \text{supp}(f) \cap W \) and relatively compact in \( W^\sharp \) with the property that \( \text{supp}(f) \subset V^\sharp \subset W^\sharp \). Applying Lemma 2.6 with \( g = h \in F^0(W, V) \), we obtain

\[
\text{Definition 6.8. Assume that } h \text{ is continuous positive on } W \text{ and belongs to } F^0(U, W). \text{ Recall that } \mathcal{F}^h = \mathcal{F}^h_W = D(\mathcal{E}^{D,W,h^2}).
\]

For an open subset \( V \subset W^\sharp \), let

\[
\mathcal{F}^h_{\text{loc}}(V) = \{ f \in L^2_{\text{loc}}(V, h^2d\mu) : \forall \text{ compact } K \subset V, \text{ there exists } f^\sharp \in \mathcal{F}^h \text{ so that } f = f^\sharp |_K \text{ a.e.} \}.
\]

Similarly, define \( \mathcal{F}^h(V) \) and \( \mathcal{F}^h_c(V) \) in terms of \( (\mathcal{E}^{D,W,h^2}, D(\mathcal{E}^{D,W,h^2})) \).

Remark 6.9. (i) By Proposition 6.7 and Lemmas 6.5, 6.6, we have

\[
\mathcal{F}^h_W = H^{-1}(\mathcal{F}^0(W)) = D(\mathcal{E}_h).
\]

(ii) It is now plain that the symmetric strongly local regular Dirichlet form \( (\mathcal{E}^{D,W,h^2}, \mathcal{F}^h_W) \) is the strongly local part of the symmetric part of the form \( (\mathcal{E}_h, D(\mathcal{E}_h)) \). In particular, for any \( f \in \mathcal{F}^h_c(W^\sharp) \cap C(W^\sharp) \),

\[
\mathcal{E}^h(f, f) = \mathcal{E}^h_{\text{loc}}(f, f) - \mathcal{E}^h_{\text{loc}}(f^\sharp, 1) = \int h^2d\Gamma(f, f) = \mathcal{E}^{D,W,h^2}(f, f).
\]

(iii) The space \( \text{Lip}_c(W^\sharp, d_U) \) is contained in \( \text{Lip}_c(W^\sharp, d_W) \), because for any \( x, y \in W^\sharp \) it holds \( d_U(x, y) \leq d_W(x, y) \). In fact, both spaces are the same. To see this, observe that for any \( f \in \text{Lip}_c(W^\sharp, d_W) \) with Lipschitz constant \( C_W \) and any \( x, y \in W^\sharp \) with \( d_U(x, y) \) strictly less than \( d_W(x, y) \) we have

\[
|f(x) - f(y)| \leq C(d_U(x, \partial W \cap U) + d_U(y, \partial W \cap U)) \leq C d_U(x, y),
\]

where

\[
C = \max_{z \in W^\sharp} \frac{f(z)}{d_U(\text{supp}(f), \partial W \cap U)}.
\]

Hence, \( f \) is in \( \text{Lip}_c(W^\sharp, d_U) \) with Lipschitz constant \( C_U = \max\{C_W, C\} \).
Lemma 6.10. Assume that $h$ is continuous positive on $W$ and belongs to $\mathcal{F}^0(U,W)$. The metrics $d_U$, $d_W$ and $d_{E,D,W,h^2}$ coincide on any inner ball $B = B_U(a,r)$ with $B_U(a,3r) \subset W^2$.

Proof. Clearly, the inner metrics $d_W$ and $d_U$ coincide on the ball $B$, since $B$ is far away from $U \setminus W$. We follow the line of reasoning in [12, Proof of Lemma 3.32] to show that the inner metrics coincide with $d_{E,D,W,h^2}$. Fix $y, z \in B$. Then the cut-off function

$$\rho_y(x) = \max\{d_W(y, z) - d_W(y, x), 0\},$$

is a compactly supported Lipschitz function on $(W^2, d_W)$ and $\rho_y \in \mathcal{F}^h_{loc}(W^2) \cap C(W^2)$ by Proposition 6.7. Moreover, $\Upsilon(\rho_y, \rho_y) \leq 1 < \infty$ a.e. on $W^2$ by Corollary 2.13. Thus,

$$d_W(y, z) = \rho_y(y) - \rho_y(z) \leq d_{E,D,W,h^2}(y, z).$$

We now show the opposite inequality. Any two points $y, z \in B \cap W$ can be connected by a curve $\gamma = \gamma_{y,z}$ in $W$ without self-intersections. Let $A_\gamma$ be an open, relatively compact subset of $W$ that contains the curve. By [20, Theorem 3] (recall that (A1) holds on $(X, \mu, \mathcal{E}', D(\mathcal{E}))$), we have

$$\text{length}(\gamma) = \sup\{u(y) - u(z) : u \in \mathcal{F}_{loc}(A_\gamma) \cap C(A_\gamma), d\Gamma(u, u) \leq d\mu\}$$

$$= \sup\{u(y) - u(z) : u \in \mathcal{F}^h_{loc}(A_\gamma) \cap C(A_\gamma), h^2 d\Gamma(u, u) \leq h^2 d\mu\}$$

$$\geq d_{E,D,W,h^2}(y, z).$$

Hence, $d_W(y, z) = \inf\text{length}(\gamma) \geq d_{E,D,W,h^2}(y, z)$ for all $y, z \in B \cap W$. To show that $d_W$ and $d_{E,D,W,h^2}$ coincide on $B$, approximate $y$ and $z$ by points in $B \cap W$. \hfill \Box

Lemma 6.11. Assume that $h$ is a $(U,W)$-profile for either $E + \gamma(\cdot, \cdot)$ or $E^* + \gamma(\cdot, \cdot)$. Then the form $(\mathcal{E}_h, D(\mathcal{E}_h))$ satisfies Assumption A on $(W^2, h^2 d\mu)$ with respect to $(\mathcal{E}^{D,W,h^2}, \mathcal{F}^h_W)$. Further:

(i) If $h$ is a $(U,W)$-profile for $E^* + \gamma(\cdot, \cdot)$, then the sector condition constant $C_0(\mathcal{E}_h)$ and the constants $C_2(\mathcal{E}_h), C_3(\mathcal{E}_h), C_5(\mathcal{E}_h)$ for the form $\mathcal{E}_h$ on $(W^2, h^2 d\mu)$ with respect to $(\mathcal{E}^{D,W,h^2}, \mathcal{F}^h_W)$ are all bounded as follows:

$$C_0(\mathcal{E}_h) \leq C_0(\mathcal{E})(1 + |\gamma|), \quad C_2(\mathcal{E}_h) \leq C_2(\mathcal{E}),$$

$$C_3(\mathcal{E}_h) \leq C_3(\mathcal{E})|\gamma| + C_3(\mathcal{E}) + |\gamma|^2, \quad C_5(\mathcal{E}_h) \leq C_5(\mathcal{E})$$

and $C_8(\mathcal{E}_h) \leq 4(C_8(\mathcal{E}) + |\gamma|)$.

(ii) If $h$ is a $(U,W)$-profile for $E + \gamma(\cdot, \cdot)$, then the sector condition constant $C_0(\mathcal{E}_h)$ and the constants $C_2(\mathcal{E}_h), C_3(\mathcal{E}_h), C_5(\mathcal{E}_h)$ for the form $\mathcal{E}_h$ on $(W^2, h^2 d\mu)$ with respect to $(\mathcal{E}^{D,W,h^2}, \mathcal{F}^h_W)$ are all bounded in terms of an upper bound for $C_0(\mathcal{E}), C_2(\mathcal{E}), C_3(\mathcal{E}), C_5(\mathcal{E})$ and $|\gamma|$. 

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Proof. We only treat the case when $h$ is a $(U,W)$-profile for $E$ (i.e., $\gamma = 0$). The other cases are similar. We refer the reader to Remark 6.3 for various algebraic computations regarding $E_h$ that are relevant to this proof. Let $f, g \in F_h \cap C_c(W)$. We have

\[ E_h(f,g) = E_h^s(f,g) + E_h^{skew}(f,g) + E_h^{sym}(fg,1) \]

with

\[ E_h^s = E^{D,W,h_2}, \quad E_h^{skew}(f,g) = E^{skew}(hf,hg) \]

and

\[ E_h^{sym}(fg,1) = E^{sym}(hf,g,h) = E^{skew}(hf,h) \]

where the last equality follows from the fact that $E(h,hfg) = 0$ and needs to be modified appropriately when $h$ is a profile for a form different from $E$. See Remark 6.3(iii)-(iv).

Using the isometry $H : L^2(W,h^2 \mu) \to L^2(W,\mu)$ and Lemma 6.6 in an obvious way, we see that

\[ |E^{skew}(hf,hg)| \leq C \|f\|_{F_h} \|g\|_{F_h}. \]

Next, we check that

\[ |E^{skew}(hf,g,h)| \leq C \|f\|_{F_h} \|g\|_{F_h}. \]

By Assumption A(ii), we can find a constant $k$ such that the symmetric bilinear form

\[ (f,g) \mapsto E^{skew}(hf,g,h) + k \left( \int_W fgh^2d\mu + \int_W h^2d\Gamma(f,g) \right) \]

is positive definite. Applying the Cauchy-Schwarz inequality and Assumption A(ii) yield $|E^{skew}(hf,g,h)| \leq C \|f\|_{F_h} \|g\|_{F_h}$ as desired. These computations also yield $|E_h(fg,1)| + |E_h(1,fg)| \leq C \|f\|_{F_h} \|g\|_{F_h}$. Together with 6.3(ii), these estimates show that $E_h$ is adapted to $(E_h^s,F_h)$.

Next, we prove that $E_h$ satisfies Assumption A(i)–(ii). Assumption A(ii) for $E_h$ follows immediately from Assumption A(ii) for $E$. The statements about the constants are simple bookkeeping. See Remarks 6.3(ii) and 6.9(ii).

### 6.2 Properties of $h$-transforms in inner uniform domains

We continue to work under the hypotheses made at the beginning of Section 6.

**Theorem 6.12.** Assume that
The volume doubling property and the Poincaré inequality (for the model form \((E^\ast, F)\)) hold locally up to scale \(R\) on \(W\).

The domain \(U\) is locally \((c_u, C_u)\)-inner uniform up to scale \(R\) near \(W\).

Assume that \(h\) is a \((U, W)\)-profile for either \(E + \gamma\langle \cdot, \cdot \rangle\) or \(E^\ast + \gamma\langle \cdot, \cdot \rangle\). Then the following properties hold:

(i) The symmetric strongly local regular Dirichlet form \((E^{D,W,h^2}, F_W^h)\) satisfies property (A1) and (A2-B) for any inner uniform ball \(B = B_U(a, r)\) such that \(B_U(a, 3r) \subset W^2\).

(ii) There exist constants \(a_0, A_0 \in (0, \infty)\) such that, for any \(r \in (0, a_0 R)\) and any inner ball \(B = B_U(a, r)\) with \(B = B_U(a, A_0 r) \subset W^2\), we have

\[
V_{h^2}(a, 2r) \leq d(W, R)V_{h^2}(a, r)
\]

and, for any \(f \in F^h(B)\),

\[
\int_B |f - f_B|^2 h^2 d\mu \leq P(W, R)r^2 \int_B h^2 d\Gamma(f, f).
\]

The constants \(a_0, A_0\) depend only \(c_u, C_u\). If \(h\) is a \((U, W)\)-profile for \(E^\ast + \gamma\langle \cdot, \cdot \rangle\) then the constants \(D(W, R)\) and \(P(W, R)\) depend only \(c_u, C_u\), the volume doubling and Poincaré constants up to scale \(R\) in \(W^2\), and an upper bound on \(|\gamma| R^2\). If \(h\) is a \((U, W)\)-profile for \(E + \gamma\langle \cdot, \cdot \rangle\) then the constants \(D(W, R)\) and \(P(W, R)\) depend only \(c_u, C_u\), the volume doubling and Poincaré constants up to scale \(R\) in \(W^2\), and an upper bound on \(C_0(E), C_2(E), C_3(E), C_5(E), |\gamma|\) and \(R\).

The first assertion is clear by Lemma [6.10]. The proof of the second assertion is done in two stages. The first stage concerns the case when \(h\) is a profile relative to the Dirichlet form \(E^\ast\).

**Proof in the case of a \((U, W)\)-profile.** When \(h\) is a \((U, W)\)-profile, we can apply Proposition [5.10] to obtain the asserted doubling property of the volume function \(V_{h^2}\). Note that this very crucial step is based on the boundary Harnack principle for \(E^\ast\) (which has only been proved so far for Dirichlet forms). Since the volume function \(V_{h^2}\) has the doubling property, the stated Poincaré inequality follows by the line of reasoning explained in [12] Theorem 3.13. See also [12] Theorem 3.27]. One may have to change the constants \(a_0, A_0\) when passing from the volume doubling property to the Poincaré inequality. In this case, the constants \(D(W, R)\) and \(P(W, R)\) depends only on the doubling and Poincaré constants for \((E^\ast, F)\) up to scale \(R\) in \(W\) and the inner uniformity constants \(c_u, C_u\) up to scale \(R\) near \(W\).

**Proof of the case of a \((U, W)\)-profile for \(E^\ast + \gamma\langle \cdot, \cdot \rangle\).** Let \(h\) be as in the first part of the proof, that is, a \((U, W)\)-profile. Using the result proved in stage 1 together with Lemma [6.11] and [16] Theorem 2.13, it follows that there exist \(a_0, A_0\) such that the parabolic Harnack inequality holds for the form \(E^h_{h^2} + \gamma\langle h, h \rangle\)
on any inner ball $B = B_{\overline{B}}(a, r), 0 < r < a_0R$ with $B_{\overline{B}}(a, A_0r) \subset W^2$. Further, the Harnack constant depends only on $c_u, C_u$, the volume doubling and Poincaré constant for $(\mathcal{E}^*, \mathcal{F})$ up to scale $R$ on $\overline{W}$, and an upper bound on $|\gamma|R^2$.

In particular if $\hat{h}$ is a $(U, W)$-profile for $\mathcal{E}^* + \gamma(\cdot, \cdot)$ then $\hat{h}/h$ is a positive harmonic function (in the weak sense in $W$) for $\mathcal{E}_h^* + \gamma(h, h)$ and we have

$$\forall x, y \in B, \quad c_H \frac{\hat{h}(y)}{\hat{h}(x)} \leq \frac{\hat{h}(x)}{\hat{h}(y)} \leq C_H \frac{\hat{h}(y)}{\hat{h}(x)}$$

where $B$ is as above and $B_{\overline{B}}(a, A_0r) \subset W^2$. The constants $c_H, C_H$ depend only on $c_u, C_u$, the volume doubling and Poincaré constant for $(\mathcal{E}^*, \mathcal{F})$ up to scale $R$ on $\overline{W}$, and an upper bound on $|\gamma|R^2$. From this, it is clear that Theorem 6.12 also holds in the case of a $(U, W)$-profile for $\mathcal{E}^* + \gamma(\cdot, \cdot)$.

**Proof in the case of a $(U, W)$-profile for $\mathcal{E}^* + \gamma(\cdot, \cdot)$.** The proof is the same as in the case of $(U, W)$-profile for $\mathcal{E}^* + \gamma(\cdot, \cdot)$. However, in this case, the constant in the Harnack inequality for the form $\mathcal{E}_h + \gamma(h, h)$ depends on $c_u, C_u$, the volume doubling and Poincaré constant up to scale $R$ on $\overline{W}$ and an upper bound on $\mathcal{E}_0(\mathcal{E}), \mathcal{E}_2(\mathcal{E}), \mathcal{E}_3(\mathcal{E}), \mathcal{E}_5(\mathcal{E}), |\gamma|$ and $R$.

In fact, this argument proves that the forms $\mathcal{E}^* + \gamma(\cdot, \cdot, \cdot), \mathcal{E}$ and $\mathcal{E} + \gamma(\cdot, \cdot)$ all satisfy the geometric boundary Harnack principle up to scale $R$ near $W$. It follows that Theorem 5.6 still holds true without the hypothesis (i) of Theorem 5.6. Consequently, (i) and (ii) of Theorem 5.6 hold true without assuming that $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form. See Theorem 6.14 below.

As a corollary that has already been used in the proof above, we have the following very useful result.

**Theorem 6.13.** Assume that:

- The volume doubling property and the Poincaré inequality (for the model form $(\mathcal{E}^*, \mathcal{F})$) hold locally up to scale $R$ on $\overline{W}$.
- The domain $U$ is locally $(c_u, C_u)$-inner uniform up to scale $R$ near $W$.

(i) Assume that $h$ is a $(U, W)$-profile for $\mathcal{E}^* + \gamma(\cdot, \cdot)$. Then there exist constant $a_0, A_0$ such that for any inner ball $B = B_{\overline{B}}(a, r)$ with $r \in (0, a_0R)$ and $B_{\overline{B}}(a, A_0r) \subset W^2$, the parabolic Harnack inequality for $\mathcal{E}_h$ holds in $B$ up to scale $r$, with a parabolic Harnack constant which depends only on $c_u, C_u$, the volume doubling and Poincaré constant up to scale $R$ on $\overline{W}$, $\mathcal{E}_0(\mathcal{E})$, and an upper bound on $(\mathcal{E}_0(\mathcal{E}) + |\gamma|)R^2$.

(ii) Assume that $h$ is a $(U, W)$-profile for $\mathcal{E} + \gamma(\cdot, \cdot)$. Then there exist constants $a_0, A_0$ such that for any inner ball $B = B_{\overline{B}}(a, r)$ with $r \in (0, a_0R)$ and $B_{\overline{B}}(a, A_0r) \subset W^2$, the parabolic Harnack inequality for $\mathcal{E}_h$ holds in $B$ up to scale $r$, with a parabolic Harnack constant which depends only on $c_u, C_u$, the volume doubling and Poincaré constant up to scale $R$ on $\overline{W}$, $\mathcal{E}_0(\mathcal{E})$, and an upper bound on $\mathcal{E}_0(\mathcal{E}), \mathcal{E}_2(\mathcal{E}), \mathcal{E}_3(\mathcal{E}), \mathcal{E}_5(\mathcal{E}), |\gamma|$ and $R$. 

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Another useful result already mentioned and used in stages 2 and 3 of the proof of Theorem 6.13 is the following extension of Theorem 5.6 to the case when $E$ is not a Dirichlet form.

**Theorem 6.14.** Let $X, E^*, F, d, \mu$ be as in Section 2. Let $(E, F)$ be a form satisfying Assumption A. Let $U$ be a domain in $X$.

(i) Fix a domain $W \subset U$, and assume that $U$ is locally inner uniform near $W$. Assume also that the volume doubling property and the Poincaré inequality hold locally in $W$. Then the geometric boundary Harnack principle holds locally in $U$ near $W$.

(ii) Fix $R > 0$ and a domain $W \subset U$. Assume that $U$ is locally $(c_u, C_u)$-inner uniform up to scale $R$ near $W$ and that the volume doubling property and Poincaré inequality hold up to scale $R$ in $W$. Then there exists $a_0 > 0$ depending only on $c_u, C_u$ such that the geometric boundary Harnack principle holds locally up to scale $a_0 R$ near $W$ with constants depending only on $c_u, C_u$, the volume doubling and Poincaré inequality constants up to scale $R$ in $W$, $C_0(E)$, and an upper bound on $C_8(E) R^2$.

7 Estimates for the Dirichlet heat kernel

Let $(E^*, F)$ be a model form as in Section 2 and assume that it satisfies (A1)-(A2). Let $E$ be a form satisfying Assumption A. Fix a domain $U$ and consider the bilinear form $(E^{D}_U, F^0(U))$ of Definition 5.7. In this section, we derive the main results of this paper which are two-sided estimates for the kernel of the semigroup $P_{D,U,t}$ associated with $(E^{D}_U, F^0(U))$, that is, the heat kernel for $E$ with Dirichlet boundary condition along $\partial U$. For simplicity, let us assume that the volume doubling condition and the Poincaré inequality hold locally in $U$. This immediately implies that the semigroup $P_{D,U,t}$ admits a continuous positive kernel $p_{D,U}(t,x,y)$ in $U$, so that

$$P_{D,U,t} f(x) = \int p_{D,U}(t,x,y) f(y) dy.$$ 

In fact, by virtue of the local Harnack inequality, this kernel is locally Hölder continuous in $(t,x,y) \in (0, \infty) \times U \times U$.

Next, let $h$ be a positive continuous function in $U$ and consider the form $(E^{D,U}_h, H^{-1}(F^0(U)))$, where the map $H : f \mapsto h f$ is the natural isometry between $L^2(U, h^2 d\mu)$ and $L^2(U, d\mu)$ as in Definition 6.2. By construction, the form $E^{D,U}_h$ induces a semigroup $P_{D,U}^{h,t} : L^2(U, h^2 d\mu) \to L^2(U, h^2 d\mu)$ given by

$$P_{h,t}^{D,U} f = h^{-1} P_{D,U}^{h}(hf).$$

Hence, this semigroup admits a kernel $p_{h}^{D,U}(t,x,y), (t,x,y) \in (0, \infty) \times U \times U$ and we have

$$p_{h}^{D,U}(t,x,y) = h(x)h(y)p_{D,U}^{h}(t,x,y). \quad (4)$$
Clearly, to obtain good estimates for $p^D_U$, it suffices to obtain good estimates for $p^D_{h,U}$.

### 7.1 Local Dirichlet heat kernel estimates

In this subsection, we explain how to implement the strategy outlined above to obtain heat kernel estimates for $p^D_U(t, x, y)$ at two fixed points $x$ and $y$ in $U$ (these points may, in some sense, be close to the boundary).

We fix $R_x, R_y > 0$ and make the following two basic assumptions:

(i) The volume doubling property and the Poincaré inequality hold up to scale $R_x$ in $B(x, R)$ and up to scale $R_y$ in $B(y, R_y)$.

(ii) The domain $U$ is locally $(c_u, C_u)$-inner uniform up to scale $R_x$ near $B_U(x, R)$ and up to scale $R_y$ near $B_U(y, R_y)$.

Next, we pick a real $\gamma$ with the property that there exists a function $h = h_\gamma$ such that $h$ is positive continuous in $U$ and is a $(U, B_U(x, R))$-profile and a $(U, B_U(y, R))$-profile for $\mathcal{E} + \gamma \langle \cdot, \cdot \rangle$.

The following lemma provides the existence of such a pair $(\gamma, h_\gamma)$.

**Lemma 7.1.** Let $\mathcal{E}$ be a form satisfying Assumption A. Let $U$ be a domain in $X$. Set $\lambda_U = \inf \{ \mathcal{E}(f, f), f \in \mathcal{F}^0(U), \|f\|_2 = 1 \}$.

(i) If $U$ is bounded then $-\lambda_U$ is an eigenvalue for the infinitesimal generator of $P^D_{U,1}$ and the associated normalized $L^2$-eigenfunction $\phi = \phi_U \in \mathcal{F}^0(U)$ is positive in $U$.

(ii) If $U$ is unbounded and locally inner uniform, then there exists a function $h = h_U$ which is positive continuous in $U$, a local weak solution of $-Lh = \lambda_U h$ in $U$, and both a $(U, B_U(x, R_x))$-profile and a $(U, B_U(y, R_y))$-profile for $\mathcal{E} - \lambda_U \langle \cdot, \cdot \rangle$.

**Proof.** Part one follows easily from Jentzsch’s Theorem (see, e.g., [25, Theorem V.6.6]).

For part two, we consider a relatively compact increasing exhaustion $U_n$ of $U$ such that $B_U(x, R_x)$ and $B_U(y, R_y)$ are contained in $U_1$. For each $U_n$, we have an eigenfunction $\phi_{U_n}$ with eigenvalue $\lambda_{U_n}$ in $U_n$ given by (i). Fix a point $o \in U_1$ and consider the sequence $h_n = \phi_{U_n}/\phi_{U_n}(o)$. From the definitions and [16], it easily follows that these functions all satisfy local Harnack inequalities (with constants independent of $n$) in their domains and are equicontinuous. This implies that some subsequence of $(h_n)$ converges in $U$ to a function $h \in \mathcal{F}^{0*}(U)$ which is positive and a local weak solution of $-Lh = \lambda_U h$ in $U$. In addition, by Theorem 6.14, the functions $h_n$ satisfy the geometric boundary Harnack principle locally in $B_U(x, R_x)$ and $B_U(y, R_y)$, uniformly in $n$. This easily implies that the limit $h$ is a $(U, B_U(x, R_x))$-profile and a $(U, B_U(y, R_y))$-profile for $\mathcal{E} - \lambda_U \langle \cdot, \cdot \rangle$. See [12, Section 4.3.2].
Remark 7.2. (i) Recall that Assumption A implies that there exists a non-negative real $\alpha$ such that $E(f,f) \geq -\alpha \|f\|_2^2$ for all $f \in \mathcal{F}$, and that $\alpha$ is bounded above in terms of $C_2(E)$ and $C_5(E)$. Hence, $\lambda_U \geq -\alpha$.

(ii) It is not hard to modify the proof of (ii) to show that for each $\gamma \leq \lambda_U$ there exists a function $h$, which is positive continuous in $U$, a local weak solution of $-Lh = \gamma h$ in $U$, and both a $(U,B_U(x,R_x))$-profile and a $(U,B_U(y,R_y))$-profile for $E - \gamma \langle \cdot, \cdot \rangle$.

Theorem 7.3. Let $E$ be a form satisfying Assumption A. Let $U$ be a domain in $X$. Fix $x,y \in U$, $T > 0$ and $\gamma$ and assume that

(i) The volume doubling property and the Poincaré inequality hold up to scale $R_x$ in $B(x,R_x)$ and up to scale $R_y$ in $B(y,R_y)$.

(ii) $U$ is locally $(c_u,C_u)$-inner uniform up to scale $R_x$ near $B_U(x,R_x)$ and up to scale $R_y$ near $B_U(y,R_y)$.

(iii) There exists a function $h = h_\gamma$ such that $h$ is positive continuous in $U$ and both a $(U,B_U(x,R_x))$-profile and a $(U,B_U(y,R_y))$-profile for $E + \gamma \langle \cdot, \cdot \rangle$.

Then for all $t \in (0,T)$, $\xi \in B_U(x,a_0R_x)$, $\zeta \in B_U(y,a_0R_y)$, we have

$$p_u^D(t,\xi,\zeta) \leq \frac{A_1 h(\xi)h(\zeta) \exp(-a_1 d_U(\xi,\zeta)^2/t)}{\sqrt{V(\xi,r_x)V(\zeta,r_y)}h(\xi_{r_x})h(\zeta_{r_y})},$$

where $r_x = \min\{\sqrt{t},a_0R_x\}$ for $z = x,y$, and where $z_r \in U$ denotes a point such that $d_U(z,z_r) = r/4$ and $d(z_r,\partial U) \geq c_u r/8$, for $z = \xi,\zeta$ and $r = r_x,r_y$.

Further, for all $t \in (0,T)$, $\xi \in B_U(x,a_0R_x)$, $\zeta \in B_U(y,a_0R_y)$, we have

$$p_u^D(t,\xi,\zeta) \geq \frac{a_2 h(\xi)h(\zeta) \exp(-A_2 d_U(\xi,\zeta)^2/t)}{\sqrt{V(\xi,r_x)V(\zeta,r_y)}h(\xi_{r_x})h(\zeta_{r_y})}.$$

The constant $a_0$ depends only on $c_u,C_u$. The constants $a_1$ depends only on $C_0(E)-C_5(E)$. The constants $A_1,A_2,a_2$ depend only on $c_u,C_u$, the volume doubling and Poincaré constants up to scale $R_x$ (resp. $R_y$) in $B(x,R_x)$ (resp. $B(y,R_y)$), $C_0(E)-C_5(E)$ and upper bounds on $|\gamma|$ and $(C_5(E)+|\gamma|)R_x^2$, $(C_5(E)+|\gamma|)R_y^2$, $TR_x^{-2}$ and $TR_y^{-2}$.

Proof. By (i) these bounds can be deduced from similar heat kernel bounds for $p_h^{D,U}$. The desired bounds for $p_h^{D,U}$ follow from classical arguments (e.g., [24 Chapter 5] and [28, 30]) based on the validity of the parabolic Harnack inequality in $B_U(x,a_0R_x)$ and $B_U(y,a_0R_y)$ which follows from Theorem 6.13.

Remark 7.4. Theorem 7.3 holds true if we replace (iii) by the assumption that $h$ is positive continuous in $U$ and both a $(U,B_U(x,R_x))$-profile and a $(U,B_U(y,R_y))$-profile for $E^* + \gamma \langle \cdot, \cdot \rangle$. 

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7.2 Dirichlet heat kernel estimates in unbounded domains

In this section we prove two-sided Dirichlet heat kernel estimates in an unbounded domain $U$ under various hypotheses. The technique of the proof is the same as in the previous section. The form $\mathcal{E}$ is a form as in Assumption A. Our minimal assumption on the unbounded domain $U$ is that the volume doubling property and Poincaré inequality hold locally on $\bar{U}$ and that $U$ is locally inner uniform. By [12, 4.3.2], these minimal hypotheses imply the existence of a $\mathcal{E}^{*}$-harmonic profile $h$ in $\bar{U}$. Since $\mathcal{E}^{*}$ is our model form, it is natural to think of the $\mathcal{E}^{*}$-profile $h$ as a fundamental object. Hence, the heat kernel estimates given in this section are stated in terms of $h$.

The first theorem of this section provides heat kernel bounds under these minimal hypotheses. In the second theorem, these minimal hypotheses are upgraded to hypotheses that hold uniformly up to scale $R$ for some fixed $R > 0$.

**Theorem 7.5.** Let $\mathcal{E}$ be a form satisfying Assumption A. Let $U$ be a locally inner uniform unbounded domain in $X$. Assume that the volume doubling property and the Poincaré inequality hold locally on $U$. Let $h$ be a $\mathcal{E}^{*}$-harmonic profile in $U$ (extended as a $\mathcal{F}$-quasi-continuous function on $\bar{U}$). Then the Dirichlet heat kernel $p_{D}^{U}$ has the following properties:

(i) The function $(t,x,y) \mapsto p_{D}^{U}(t,x,y)$ is continuous on $(0,\infty) \times U \times U$ and the function

$$ (t,x,y) \mapsto \frac{p_{D}^{U}(t,x,y)}{h(x)h(y)} $$

is locally Hölder continuous in $(0,\infty) \times \bar{U} \times \bar{U}$.

(ii) There exist $c,C > 0$ such that for any pair of points $x, y \in \bar{U}$, there exist $r_{x}, r_{y}, A = A(x,y) \in (0,\infty)$ such that for any $t > 0$ and $\xi \in B_{U}(x, r_{x}), \zeta \in B_{U}(y, r_{y})$, we have

$$ p_{D}^{U}(t,\xi,\zeta) \leq \frac{Ah(\xi)h(\zeta)\exp(-cdU(\xi,\zeta)^{2}/t + Ct)}{\sqrt{V(\xi,\sqrt{t_{x}})V(\zeta,\sqrt{t_{y}})h(\xi,\sqrt{t_{x}})h(\zeta,\sqrt{t_{y}})}}, $$

where $t_{z} = \min\{t, r_{z}^{2}\}$ for $z = x, y$, and where $z_{r}$ is a point in $U$ at distance at most $r/4$ from $z$ and at distance at least $c_{u}r/8$ from $\partial U$ for $z = \xi, \zeta$.

**Theorem 7.6.** Let $\mathcal{E}$ be a form satisfying Assumption A. Let $U$ be an unbounded domain in $X$ that is locally $(c_{u}, C_{u})$-inner uniform up to scale $R$. Assume that the volume doubling property and the Poincaré inequality hold up to scale $R$ on $\bar{U}$. Let $h$ be a $\mathcal{E}^{*}$-harmonic profile in $U$ (extended as a $\mathcal{F}$-quasi-continuous function on $\bar{U}$). Then the Dirichlet heat kernel $p_{D}^{U}$ has the following properties:

(i) The function $(t,x,y) \mapsto p_{D}^{U}(t,x,y)$ is continuous on $(0,\infty) \times U \times U$ and there exist $\kappa, a_{0} \in (0,1)$ and a constant $A_{1}$ such that for any $r \in (0,a_{0}R)$,
Consider two points \( t, t' \in (0, \infty) \), \( x, x', y, y' \in \tilde{U} \) satisfying \( t \geq r^2 \), \( |t - t'| \leq r^2/4 \), \( d_U(x, x') \leq r \), \( d(y, y') \leq r \), we have

\[
\left| \frac{p^D_U(t, x, y)}{h(x)h(y)} - \frac{p^D_U(t', x', y')}{h(x')h(y')} \right| \leq A_1 \left( \frac{p}{r} \right) \frac{p^D_U(t + r^2, x_r, y_r)}{h(x_r)h(y_r)}.
\]

where \( \rho = \sqrt{|t - t'| + d_U(x, x') + d_U(y, y')} \) and \( z_r \) is a point in \( U \) at distance at most \( r/4 \) from \( z \) and at distance at least \( c_r r/8 \) from \( \partial U \) for \( z = x, y \).

(ii) There exist \( c, C, a_0, A_1, a_2 A_2 > 0 \) such that for any pair of points \( x, y \in \tilde{U} \) and any \( t > 0 \), we have

\[
p^D_U(t, x, y) \leq \frac{A_1h(x)h(y)\exp(-cd_U(x, y)^2/t + Ct)}{\sqrt{V(x, \sqrt{\tau})V(y, \sqrt{\tau})}h(x, \sqrt{\tau})h(y, \sqrt{\tau})}
\]

and

\[
p^D_U(t, x, y) \geq \frac{a_2h(x)h(y)\exp(-A_2d_U(x, y)^2/t - A_2t)}{\sqrt{V(x, \sqrt{\tau})V(y, \sqrt{\tau})}h(x, \sqrt{\tau})h(y, \sqrt{\tau})}
\]

where \( \tau = \min\{t, (a_0R)^2\} \).

The constant \( a_0, C \) depend only on \( c_r, C_r \). The constants \( c, C \) depend only on \( C_3(E) - C_5(E) \). The constants \( \kappa, A_1, a_2, A_2 \) depend only on \( c_r, C_r \), the volume doubling and Poincaré constant on \( \tilde{U} \) up to scale \( R \), \( C_0(E) - C_5(E) \), and on an upper bound on \( C_0(E)R^2 \).

**Proof of Theorem 7.3 (outline).** The proofs of the two theorems stated above follow well established lines of reasoning. The first (and crucial) step is to use (4) and estimate the kernel \( p^D_U \). Indeed, by Theorem 6.13 the associated form \( E_h \) satisfies a parabolic Harnack inequality. The desired bounds for \( p^D_U \) follow from classical arguments (e.g., [24, Chapter 5] and [28, 30]) based on the validity of the parabolic Harnack inequality.

**Remark 7.7.** In statement (ii) of Theorem 7.3, the denominators can be replaced by

\[ V(x, \sqrt{\tau})[h(x, \sqrt{\tau})]^2. \]

Note the lack of \( x, y \) symmetry of the resulting bounds. This is often useful in practice.

The following corollary of the Harnack inequality for \( E_h \) up to scale \( a_0 R \) in \( \tilde{U} \) is also of interest. We note that if \( u \in F^{1,0}_1(U, (0, \infty) \times U) \) is a local weak solution of the heat equation for \( E \) in \( U \), then \( u/h \in F^{1,0}_2(U, (0, \infty) \times U) \) is a local weak solution of the heat equation for \( E_h \) in \( U \). Hence \( u/h \) satisfies the Harnack inequality up to scale \( a_0 R \) in \( \tilde{U} \). This and the argument given in [24, Section 5.4.3] yield the following result.
Theorem 7.8. Let $E$ be a form satisfying Assumption A. Let $U$ be an unbounded domain in $X$ that is locally $(c_u, C_u)$-inner uniform up to scale $R$. Assume that the volume doubling property and the Poincaré inequality hold up to scale $R$ on $\bar{U}$. Let $h$ be a $E$-harmonic profile in $U$ (extended as a $\mathcal{F}$-quasi-continuous function on $\bar{U}$). Let $u$ be a positive local weak solution of the heat equation for $E$ in $U$ with Dirichlet boundary condition along $\partial U$. Then there exists a constant $A_1$ such that for all $0 < s < t < \infty$ and $x, y \in \bar{U}$, we have

$$\frac{u(s, x)}{u(t, y)} \leq A_1 \frac{h(x)}{h(y)} \exp \left( A_1 \left( 1 + \frac{t-s}{s} + \frac{t-s}{R^2} + \frac{d_U(x, y)^2}{t-s} \right) \right).$$

The constant $A_1$ depends only on $c_u, C_u$, the volume doubling and Poincaré constant on $\bar{U}$ up to scale $R$, $C_0(E) - C_5(E)$, and on an upper bound on $C_8(E)R^2$.

### 7.3 Dirichlet heat kernel estimates in bounded domains

This section focuses on estimates in bounded inner uniform domains and relates these results to refined intrinsic ultracontractivity estimates.

Very generally, consider a positivity preserving strongly continuous semigroup $P_t$ acting on $L^2(U, \mu)$, where $U$ is a bounded domain, with continuous kernel $p(t, x, y)$ such that $p(t, x, y)$ is bounded for each $t > 0$. Its adjoint $P_t^*$ (with kernel $p^*(t, x, y) = p(t, y, x)$) has the same properties. Let $\lambda_U$ be the common bottom of the $L^2$-spectrum of $-L$ and $-L^*$ where $L$ and $L^*$ are the respective infinitesimal generators. Let $\phi$ and $\phi_\lambda$ be the associated positive continuous $L^2$-normalized eigenfunctions. Following [14], we say that the pair $(P_t, P_t^*)$ is intrinsically ultracontractive if for each $t > 0$ there exists a constant $c(t)$ such that

$$p(t, x, y) \leq c(t) \phi(x) \phi_\lambda(y).$$

(5)

For selfadjoint semigroups, intrinsic ultracontractivity was introduced in [2]. Note that if $\lambda_\psi$ is an eigenvalue for $P_t$ with $L^2$-normalized eigenfunction $\psi$ then [5] implies

$$|\psi| \leq e c(1/|\lambda_\psi|)^{1/2} \phi$$

(6)

In many interesting cases, these bounds hold with $c_t = c(1 + t^{-\nu/2})e^{-\lambda_0 t}$ for some $\nu > 0$. Typically, in the literature, $U$ is a domain in $\mathbb{R}^n$ and $P_t$ is the semigroup associated with an elliptic second order differential operator (e.g., the Laplacian) with Dirichlet boundary condition along the boundary of $U$. Intrinsic ultracontractivity is then viewed as a property that depends on the regularity of the boundary of $U$. See, e.g., [2] [3]. In particular, it follows from [2] that the heat semigroup with Dirichlet boundary condition in any bounded inner uniform domain $U \subset \mathbb{R}^n$ is intrinsically ultracontractive with $c_t = c(1 + t^{-\nu/2})e^{-\lambda_U t}$ for some $c = c(U), \nu = \nu(U)$. Here, we obtain the following refined results.

Theorem 7.9. Let $E$ be a form satisfying Assumption A. Let $U$ be a bounded domain in $X$ that is locally $(c_u, C_u)$-inner uniform up to scale $R$. Assume that the volume doubling property and the Poincaré inequality hold up to scale $R$ on $\bar{U}$. Let

$$\lambda = \lambda_U = \min \{E(f, f) : f \in \mathcal{F}^0(U), \|f\|_2 = 1\},$$

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and let $\phi = \phi_U$ be the associated positive $L^2$-normalized eigenfunction (of minus the infinitesimal generator with Dirichlet boundary condition along $\partial U$). Then, for all $t \in (0, R^2)$, $x, y \in \bar{U}$, the Dirichlet heat kernel $p^D_U$ satisfies
\begin{equation}
 p^D_U(t, x, y) \leq \frac{A_1 \phi(x)\phi(y)e^{-cd_U(x, y)^2/t}}{V(x, \sqrt{t})V(y, \sqrt{t})\phi(x, \sqrt{t})\phi(y, \sqrt{t})} \tag{7}
\end{equation}
and
\begin{equation}
 p^D_U(t, x, y) \geq \frac{a_2 \phi(x)\phi(y)e^{-A_2d_U(x, y)^2/t}}{V(x, \sqrt{t})V(y, \sqrt{t})\phi(x, \sqrt{t})\phi(y, \sqrt{t})} \tag{8}
\end{equation}
Further, for $t > R^2$, we have
\begin{equation}
a_3 \leq \frac{e^{\lambda t}p^D_U(t, x, y)}{\phi(x)\phi(y)} \leq A_3 \tag{9}
\end{equation}
The constant $c$ depends only on $C_0(\mathcal{E}) - C_5(\mathcal{E})$. The constants $A_1, a_2, A_2, a_3, A_3 \in (0, \infty)$ depend only on $c_u, C_u$, the volume doubling and Poincaré constants on $\bar{U}$ up to scale $R$, $C_0(\mathcal{E}) - C_5(\mathcal{E})$, and on upper bounds on $(C_8(\mathcal{E}) + |\lambda|)R^2$ and $\text{diam}_U/R$.

**Corollary 7.10.** Referring to the notation and setting of Theorem 7.9, there exist a bounded continuous function $w$ on $U$ and a real $\omega > 0$ such that
\begin{equation}
\forall t \geq R^2, \ x, y \in U, \quad \left| \frac{e^{\lambda t}p^D_U(t, x, y)}{\phi(x)\phi(y)}w(y) - 1 \right| \leq A_4e^{-\omega t}. \tag{10}
\end{equation}
Further, $a_3 \leq w \leq A_3$,
\begin{equation}
A_4 \leq \frac{A_3}{a_3(1 - a_3/A_3)^2} \quad \text{and} \quad \omega \geq \frac{1}{R^2} \log \left( \frac{1}{1 - a_3/A_3} \right)
\end{equation}
where $a_3, A_3$ and $R$ are as in Theorem 7.9

**Proof.** By definition, the semigroup $K_t = e^{\lambda t}p^D_{\phi,t}$ with kernel
\begin{equation}
K_t(x, y) = \frac{e^{\lambda t}p^D_U(t, x, y)}{\phi(x)\phi(y)}
\end{equation}
with respect to $\phi^2d\mu$ is positivity preserving and satisfies $K_t1_U = 1_U$. It follows that its adjoint $K^*_t$ on $L^2(U, \phi^2d\mu)$ admits a positive continuous eigenfunction $w$ with eigenvalue 1. We normalize $w$ by setting $\int w\phi^2d\mu = 1$. Obviously, $w\phi^2d\mu$ is then an invariant probability measure for $K_t$ and it follows from (9) that $w$ is bounded and bounded away from 0.

In the following computation, we think of $K_t$ and $w$ as Markov operators, namely,
\begin{equation}
f \mapsto K_tf = \int K_t(\cdot, y)f(y)\phi(y)^2d\mu(y), \quad f \mapsto wf = \int fw\phi^2d\mu,
\end{equation}
acting on $L^p(U, w\phi^2d\mu)$. Note that $[3]$ implies $a_3 \leq w \leq A_3$. Hence there exists a constant $\epsilon = a_3/A_3 > 0$ such that $K_{nR^2}(x,y) \geq \epsilon w(y)$. It follows that $Q(x,y) = (1 - \epsilon)^{-1}(K_{nR^2}(x,y) - \epsilon w(y))$ is a Markov kernel on $U$ with respect to $\phi^2d\mu$ and we again denote by $Q$ the associated operator acting on $L^p(U, w\phi^2d\mu)$. Since $w\phi^2d\mu$ is an invariant probability measure for $Q$, we have $(Q - w)^n = Q^n(I - w)$. Note also that, since $Q - w = (1 - \epsilon)^{-1}(K_{nR^2} - w)$, 

$$
\sup_{x,y}|Q^n(x,y)/w(y) - 1| = \|Q^n(I - w)\|_{1\to\infty}
$$

where the right-hand side is the norm of the operator $Q^n(I - w) = Q^{n-1}(Q - w)$ from $L^1(U, w\phi^2d\mu)$ to $L^\infty(U, w\phi^2d\mu)$. We have 

$$
\|Q - w\|_{1\to\infty} \leq \epsilon^{-1}(1 - \epsilon)^{-1} \quad \text{and} \quad \|Q^{n-1}\|_{1\to1} \leq 1.
$$

Hence, we obtain 

$$
\sup_{x,y}|Q^n(x,y)/w(y) - 1| \leq \epsilon^{-1}(1 - \epsilon)^{-1}.
$$

Since $Q^n(I - w) = (Q - w)^n = (1 - \epsilon)^{-n}(K_{nR^2} - w)$, this gives 

$$
\sup_{x,y}|K_{nR^2}(x,y)/w(y) - 1| \leq \epsilon^{-1}(1 - \epsilon)^{-n}.
$$

Since $t \mapsto \sup_{x,y}|K_t(x,y)/w(y) - 1|$ is non-increasing in $t$, we obtain 

$$
\sup_{x,y}|K_t(x,y)/w(y) - 1| \leq \epsilon^{-1}(1 - \epsilon)^{-2}e^{-\omega t}, \quad \omega = -R^{-2} \log(1 - \epsilon).
$$

This is exactly the desired inequality. \hfill \square

**Remark 7.11.** Let $\phi_*$ be the positive eigenfunction associated with the bottom eigenvalue $\lambda$ for the adjoint $-L^*$ of the infinitesimal generator $-L$ of $P_{U,t}^D$. From the definitions of $\phi, \phi_*, w$, we deduce that $\phi_* = w\phi$ so that we can rewrite $[10]$ as 

$$
\forall t \geq R^2, \quad x, y \in U, \quad \left| \frac{e^{\lambda t/\phi^2(t,x,y)}}{\phi(x)/\phi_*(y)} - 1 \right| \leq A_4 e^{-\omega t}. \tag{11}
$$

Further, we have $c\phi \leq \phi_* \leq C\phi$ for some positive constants $c, C$.

**Corollary 7.12.** Referring to the notation and setting of Theorem 7.9, there exists a constant $A_5$ such that, if $\psi \neq \phi$ is an $L^2(U, \mu)$-normalized eigenfunction of $-L$ with eigenvalue $\lambda_\psi$ then $\eta = \lambda_\psi - \lambda \geq 1/(A_5 R^2)$ and 

$$
\forall x \in U, \quad |\psi(x)| \leq \frac{A_5 \phi(x)}{\sqrt{V(x, 1/\sqrt{\eta})\phi(x/\sqrt{\eta})}}. \tag{12}
$$

The constant $A_5$ depends only on $c_u, C_u, \eta$, the volume doubling and Poincaré constants on $\overline{U}$ up to scale $R$, $C_0(\mathcal{E}) - C_5(\mathcal{E})$, and on upper bounds on $(C_8(\mathcal{E}) + |\lambda|)R^2$ and $\text{diam}_U/R$. 

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Proof. By hypothesis, we have $P^{D,U}_t \psi = e^{-t \lambda \psi}$. Hence

$$e^{\lambda t} P^{D,U}_t (\psi/\phi) = e^{(\lambda - \lambda \psi) t} (\psi/\phi).$$

The previous corollary implies that

$$\lambda \psi - \lambda \geq \omega = 1/(A_5 R^2)$$

with $A_5^{-1} = \log(1 - a_3/A_3)^{-1}$. Further, for any $x \in U$ and $t \leq R^2$, (7) yields

$$\int |p^{D,U}_t (t, x, y)|^2 \phi(y)^2 d\mu(y) \leq \frac{A'_1}{V(x, \sqrt{t}) \phi(x \sqrt{t})^2}$$

where $A'_1$ depends on the same constants as $A_1$ in Theorem 7.9. Because $\int \psi^2 \phi^2 d\mu = 1$, it follows that

$$e^{(\lambda - \lambda \psi) t} \frac{|\psi(x)|}{\phi(x)} = e^{\lambda t} \left| \int p^{D,U}_t (t, x, y) \frac{\psi(y)}{\phi(y)} \phi(y)^2 d\mu(y) \right|$$

$$\leq e^{\lambda t} \left( \int |p^{D,U}_t (t, x, y)|^2 \phi(y)^2 d\mu(y) \right)^{1/2}$$

$$\leq \frac{\sqrt{A'_1} e^{\lambda t}}{\sqrt{V(x, \sqrt{t}) \phi(x \sqrt{t})}}$$

It now suffices to choose $t \simeq 1/(\lambda \psi - \lambda) = 1/\eta$ (which is, indeed, of order at most $R^2$) to obtain

$$|\psi(x)| \leq \frac{\sqrt{A'_1} e^{\lambda |R^2 \phi(x)}}{\sqrt{V(x, 1/\sqrt{\eta}) \phi(x_1/\sqrt{\eta})}}.$$

The following result provides a very useful comparison between the principal Dirichlet eigenfunction $\phi$ associated to $E$ in $U$ and the principal Dirichlet eigenfunction $\phi_s$ associated to $E_s$ in $U$. Recall that

$$\lambda = \lambda_U = \min \{ E(f, f) : f \in \mathcal{F}_0(U), \| f \|_2 = 1 \},$$

and set

$$\lambda_s = \lambda_{s,U} = \min \{ E_s(f, f) : f \in \mathcal{F}_0(U), \| f \|_2 = 1 \}.$$

Assumption A on the form $E$ implies easily that there exists a constant $A$ such that

$$\frac{1}{2} \lambda_s - A \leq \lambda \leq \lambda_s + A.$$

Further, under the assumption of Theorem 7.9 there exists a constant $A'$ such that $0 \leq \lambda_s \leq A'/R^2$. Here $A'$ depends on $c_w, C_u$ and the doubling constant up to scale $R$ on $U$.  

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Theorem 7.13. Referring to the notation and setting of Theorem 7.9, there exists a constant $A_6$ such that the principal Dirichlet eigenfunction $\phi$ associated to $\mathcal{E}$ and the principal Dirichlet eigenfunction $\phi_s$ associated to $\mathcal{E}_s$ in $U$ satisfy

$$A_6^{-1} \phi_s \leq \phi \leq A_6 \phi_s.$$ 

The constant $A_6$ depends only on $c_u, C_u$, the volume doubling and Poincaré constants on $U$ up to scale $R, C_0(\mathcal{E}) - C_5(\mathcal{E})$, and on upper bounds on $C_8(\mathcal{E})R^2$ and $\text{diam}_U/R$.

Proof. Apply Theorem 6.13(i) with $h = \phi_s$ (hence $\gamma = \lambda_s$ and $|\gamma| R^2$ is bounded above by the constant $A'$ appearing just before the theorem). Now, $\phi/\phi_s$ is a harmonic function for the form $\mathcal{E}_{\phi_s} - \lambda(\cdot, \cdot)$ and the corresponding Harnack inequality provided by Theorem 6.13(i) gives the desired result.

Remark 7.14. In Theorem 7.9, Corollary 7.10 and Corollary 7.12, consider the special case when the volume doubling property and Poincaré inequality hold globally on $(X, (\mathcal{E}', \mathcal{F}), d, \mu)$. Specialize further to the case when $\mathcal{E} = \mathcal{E}_s$. Assume that $U$ is a $(c_u, C_u)$-inner uniform domain in $(X, d)$. Then (7)-(8)-(9) and (10)-(12) hold true with $R = \text{diam}_U$ and constants $A_1, A_2, A_3, A_4, A_5$ depending only on $c_u, C_u$.

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