ON THE COMPLEX ASYMPTOTICS OF THE HCIZ AND BGW INTEGRALS

JONATHAN NOVAK

There seems to be a connection between large $N$ and the permutation groups. — Stuart Samuel, 1980

CONTENTS

1. Introduction 2
  1.1. Objective 2
  1.2. Results 3
  1.3. Context 6
2. Exact Formulas 7
  2.1. Symmetric polynomials 7
  2.2. Basic integrals 8
  2.3. Character expansions 10
  2.4. String expansions 12
  2.5. Basic bounds 14
3. Stable Asymptotics 15
  3.1. String coefficients 15
  3.2. Stable integrals 18
  3.3. Topological expansion 20
  3.4. Topological factorization 21
4. Functional Asymptotics 22
  4.1. Analytic candidates 22
  4.2. Polynomial approximation 24
  4.3. Feynman extension 25
  4.4. Topological bound 26
  4.5. Analytic error functions 27
  4.6. Reduction to uniform boundedness 28
  4.7. Proof of uniform boundedness 30
References 31
1. Introduction

1.1. Objective. The purpose of this paper is to prove a longstanding conjecture on the $N \to \infty$ asymptotic behavior of the Harish-Chandra/Itzykson-Zuber (HCIZ) integral,

$$I_N = \int_{U(N)} e^{zN \text{Tr} AUBU^{-1}} dU,$$

and its additive counterpart, the Brézin-Gross-Witten (BGW) integral,

$$J_N = \int_{U(N)} e^{zN (\text{Tr} AU + \text{Tr} BU - 1)} dU.$$

These are integrals over $N \times N$ unitary matrices against unit mass Haar measure, the integrands of which depend on a complex parameter $z$ and a pair of $N \times N$ complex matrices $A$ and $B$. The conjecture we prove emerged from a cluster of 1980 theoretical physics papers on the large $N$ limit of $U(N)$ lattice gauge theory \[6, 17, 55, 80, 85\], and has been of perennial interest in physics ever since; see the reviews \[18, 68, 88\]. It entered mathematics in the early 2000s along with growing interest in random matrices, and was precisely formulated in work of Collins \[20, Section 5\], Guionnet \[47, Section 4.3\], and Zelditch \[87, Section 4\]. The conjecture has since attained the status of an outstanding open problem in asymptotic analysis, and has become perhaps the most prominent question at the confluence of random matrix theory and representation theory; see e.g. \[12\] for a recent perspective. It may be stated as follows.

Given a Young diagram $\alpha$ with $d$ cells, $\ell(\alpha)$ rows, and $\alpha_i$ cells in the $i$th row, let

$$p_{\alpha}(x_1, \ldots, x_N) = \prod_{i=1}^{\ell(\alpha)} \sum_{j=1}^{\alpha_i} x_j^{\alpha_i}$$

be the corresponding Newton power sum symmetric polynomial in $N$ variables.

**Conjecture 1.1.** Given any $M \geq 0$, there exists a corresponding $\varepsilon_M > 0$ such that, for any integer $k \geq 0$,

$$I_N = e^{\sum_{g=0}^{k} \frac{N^{2-2g} F^g_N}{g!} + o(N^{2-2k})} \quad \text{and} \quad J_N = e^{\sum_{g=0}^{k} \frac{N^{2-2g} G^g_N}{g!} + o(N^{2-2k})}$$

as $N \to \infty$, where the error term is uniform over complex numbers $z$ of modulus at most $\varepsilon_M$ and complex matrices $A, B$ of spectral radius at most $M$, and

$$F^g_N = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta} \frac{p_{\alpha}(a_1, \ldots, a_N) p_{\beta}(b_1, \ldots, b_N)}{N^{\ell(\alpha)}} F_g(\alpha, \beta),$$

$$G^g_N = \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\beta} \frac{p_{\beta}(c_1, \ldots, c_N)}{N^{2\ell(\beta)}} G_g(\beta),$$

are analytic functions of $z$, the eigenvalues $a_1, \ldots, a_N$ of $A$, the eigenvalues $b_1, \ldots, b_N$ of $B$, and the eigenvalues $c_1, \ldots, c_N$ of $C = AB$. Moreover, the coefficients $F_g(\alpha, \beta)$ and $G_g(\beta)$ are integers.
The main result of this paper is a proof of Conjecture [1.1]. Before outlining our argument, let us briefly unpack the conjecture’s meaning. Its salient feature is the claim that \( I_N \) and \( J_N \) admit what physicists call “strong coupling expansions” — their logarithms have complete \( N \to \infty \) asymptotic expansions on the scale \( N^{2-2g} \) provided the “coupling constant” \( z \) is sufficiently small and the “external fields” \( A \) and \( B \) are uniformly bounded (our parameter \( z \) is inversely proportional to the physical coupling constant, so that small \(|z|\) corresponds to strong coupling). Without loss in generality, we may take \( M = 1 \) as the uniform bound on the spectral radii of \( A \) and \( B \). The conjecture then asserts the existence of \( \varepsilon > 0 \) such that, for any given \( k \geq 0, \kappa > 0 \), there is a corresponding \( N(k, \kappa) \) with \( N \geq N(k, \kappa) \) implying

\[
\left| \log I_N - \sum_{g=0}^{k} N^{2-2g} F_N^{(g)}(\varepsilon) \right| \leq \kappa N^{2-2k} \quad \text{and} \quad \left| \log J_N - \sum_{g=0}^{k} N^{2-2g} G_N^{(g)}(\varepsilon) \right| \leq \kappa N^{2-2k}
\]

for all complex numbers \( z \) of modulus at most \( \varepsilon \) and all complex matrices \( A, B \) with eigenvalues of modulus at most 1, where “log” denotes the principal branch of the complex logarithm. The coefficients of these purported asymptotic expansions — the “free energies” \( F_N^{(g)}(\varepsilon) \) and \( G_N^{(g)}(\varepsilon) \) — are themselves dependent on \( N \), and hence could conceivably interact with the asymptotic scale. The conjecture addresses this by further claiming that \( F_N^{(g)}(\varepsilon) \) and \( G_N^{(g)}(\varepsilon) \) are analytically determined by the data \((z, A, B)\) in a manner which precludes this possibility: it implies the bounds

\[
|F_N^{(g)}(\varepsilon)| \leq \sum_{d=1}^{\infty} \frac{\varepsilon^d}{d!} \sum_{\alpha,\beta} |F_g(\alpha, \beta)| \quad \text{and} \quad |G_N^{(g)}(\varepsilon)| \leq \sum_{d=1}^{\infty} \frac{\varepsilon^{2d}}{d!} \sum_{\beta} |G_g(\beta)|,
\]

which are finite and depend only on \( \varepsilon \) and \( g \). Finally, the conjecture asserts that the universal coefficients \( F_g(\alpha, \beta) \) and \( G_g(\beta) \), which determine \( F_N^{(g)}(\varepsilon) \) and \( G_N^{(g)}(\varepsilon) \) but do not depend on the data \((z, A, B)\), are integers. This claim is rooted in the notion of “topological expansion,” a fundamental but analytically non-rigorous principle in quantum field theory which generalizes the apparatus of Feynman diagrams to matrix integrals [1, 11, 14, 26, 55, 86], and beyond [34, 62]. This principle predicts that the structure constants \( F_g(\alpha, \beta) \) and \( G_g(\beta) \) are combinatorial invariants of compact connected genus \( g \) Riemann surfaces.

1.2. Results. The main result of this paper is a proof of Conjecture [1.1]. Our argument proceeds in three stages: exact formulas, stable asymptotics, and functional asymptotics.

1.2.1. Exact formulas. Our point of departure is a pair of novel absolutely convergent series expansions of \( I_N \) and \( J_N \) which are amenable to large \( N \) analysis.

**Theorem 1.2.** For any \( N \in \mathbb{N} \), we have

\[
I_N = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \mathbb{P} (\text{LIS}_d \leq N) \sum_{\alpha,\beta} p_{\alpha}(a_1, \ldots, a_N) p_{\beta}(b_1, \ldots, b_N) \langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle,
\]

\[
J_N = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} N^d \mathbb{P} (\text{LIS}_d \leq N) \sum_{\beta} p_{\beta}(c_1, \ldots, c_N) \langle \Omega_N^{-1} \omega_\beta \rangle,
\]
where $P(\text{LIS}_d \leq N)$ is the probability that a uniformly random permutation from the symmetric group $S(d)$ has no increasing subsequence of length $N+1$, and $\langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle$ is the Plancherel expectation of a certain natural observable of Young diagrams with $d$ cells and at most $N$ rows. These series converge absolutely and uniformly on compact subsets of $\mathbb{C}^{2N+1}$ and $\mathbb{C}^{N+1}$, respectively.

We call these series the “string expansions” of $I_N$ and $J_N$; this terminology is explained in Section 2 below. In the absence of external fields, the string expansion of the BGW integral reduces to the beautiful formula

$$\int_{U(N)} e^{z \text{Tr}(U U^{-1})} dU = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} N^{2d} P(\text{LIS}_d \leq N),$$

which is due to Rains [78] and equivalent to a result of Gessel [35]. The Gessel-Rains identity was the starting point of Baik, Deift, and Johansson [8] in their seminal work showing that the $d \to \infty$ fluctuations of LIS$_d$ around its asymptotic mean value of $2\sqrt{d}$ are governed by the Tracy-Widom distribution. Informative expositions of this landmark result may be found in [2, 79, 82]. The existence of a connection between the HCIZ integral and increasing subsequences appears to have been previously unknown. Since the Fourier transform of any unitarily invariant random matrix is a mixture of HCIZ integrals, the HCIZ-LIS connection exposes a new and very direct link between random matrices and random permutations.

1.2.2. Stable asymptotics. In Section 3 we analyze the $N \to \infty$ asymptotics of each fixed string coefficient of $I_N$ and $J_N$, i.e. the large $N$ asymptotics of the Plancherel expectation $\langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle$ with fixed $\alpha, \beta \vdash d$. We show that $\langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle$ admits a convergent asymptotic expansion on the scale $1/N$, and that this expansion is a generating function for “monotone” walks on the Cayley graph of the symmetric group $S(d)$ with boundary conditions $\alpha, \beta$. Monotone walks are self-interacting trajectories: the future of a monotone walk depends on its past. It is a fundamental fact, discovered in [70] and further developed in [64], that these trajectories play the role of Feynman diagrams for integration against Haar measure on the unitary group.

For any fixed $N \in \mathbb{N}$, one can replace the first $N$ string coefficients of $I_N$ and $J_N$ with their $1/N$ expansions, but not so for higher terms. The issue is conceptually similar to that faced when studying the homotopy groups of $U(N)$, which behave regularly at first but eventually become wild. Topologists see past this by studying the stable unitary group $U$, an $N = \infty$ version of $U(N)$ which does not suffer from this defect [10]. The price paid is that $U$ is not a Lie group, but an infinite-dimensional manifold which is not locally compact. In Section 3 we introduce the stable HCIZ and BGW integrals, $I$ and $J$, which are $N = \infty$ versions of $I_N$ and $J_N$. Conceptually, these objects are the integrals

$$I = \int_U e^{\frac{\hbar}{2} \text{Tr} A U B U^{-1}} dU \quad \text{and} \quad J = \int_U e^{\frac{\hbar}{2} \text{Tr}(A U + B U^{-1})} dU,$$

with $\hbar$ an infinitely small parameter, $A$ and $B$ infinitely large matrices, and $dU$ the non-existent Haar measure on the stable unitary group $U$. Like the homotopy groups of $U$, the topological expansions of $I$ and $J$ can be completely understood; the price paid is that $I$ and $J$ are not analytic functions, but formal power series in infinitely many variables which are not convergent.
Theorem 1.3. We have

\[ I = e^{\sum_{g=0}^{\infty} \frac{\hbar^{2g}}{2} F(g)} \quad \text{and} \quad J = e^{\sum_{g=0}^{\infty} \frac{\hbar^{2g}}{2} G(g)}, \]

where the stable free energies \( F(g) \) and \( G(g) \) are generating functions for the genus \( g \) monotone double and single Hurwitz numbers, respectively.

Hurwitz theory, familiar to algebraic geometers as the prototypical enumerative theory of maps from curves to curves, plays a prominent role in contemporary enumerative geometry; see [31, 42, 59, 74], and [29] for a recent overview. Monotone Hurwitz theory [37, 38, 39, 40, 41] is a desymmetrized version of classical Hurwitz theory which, rather surprisingly, is exactly solvable to exactly the same extent. Just as there are explicit formulas for classical Hurwitz numbers in genus zero and one [54, 84], there are explicit formulas for monotone Hurwitz numbers in genus zero and one [37, 38], and the two sets of formulas are structurally analogous. Monotone Hurwitz numbers manifest versions of polynomiality [38] and integrability [40] which mirror the polynomiality [31] and integrability [73] of their classical counterparts. The consonance between the classical and monotone theories is to some extent explained by the fact that both are governed by the Eynard-Orantin topological recursion formalism — the two theories are structurally identical, but are generated by different spectral curves [13, 27].

Monotone Hurwitz theory has proved to be a useful tool with diverse applications [10, 24, 36, 67, 72], and its discovery has sparked a surge of interest in combinatorial deformations of classical Hurwitz numbers [3, 4, 19, 27, 28, 30, 52]. Although the subject has taken on a life of its own, monotone Hurwitz numbers were originally summoned from the void as a weapon with which to attack Conjecture 1.1. In this paper, they fulfill their initial purpose.

1.2.3. Functional asymptotics. In order to prove Conjecture 1.1 we must descend from the stable world of \( N = \infty \) to the unstable world of finite \( N \). To navigate this passage, we must address the questions of convergence and approximation which are the analytic substance of Conjecture 1.1. Prior knowledge of the stable limit, which comprises the combinatorial substance of Conjecture 1.1, is extremely useful in this regard — since we know what the answer is supposed to be, the analysis becomes a task of verification rather than discovery.

More precisely, if Conjecture 1.1 is true then the free energies \( F_N(g) \) and \( G_N(g) \) must be generating functions for monotone Hurwitz numbers of genus \( g \). Remarkably, monotone Hurwitz theory guarantees that the stable free energies \( F(g) \) and \( G(g) \) remain stable at finite \( N \): replacing the formal parameter \( \hbar \) with \( N^{-1} \) and the formal alphabets \( A, B, C \) with the spectra of uniformly bounded \( N \times N \) complex matrices yields absolutely summable power series. Even better, the radius of convergence of these series is bounded below by a positive constant \( \delta \) independent of both \( N \) and \( g \). We thus have explicit analytic candidates for \( F^g_N \) and \( G^g_N \), with a stable domain of holomorphy.

The stable analyticity of \( F^g_N \) and \( G^g_N \) does not mean that one can deduce Conjecture 1.1 from Theorem 1.3 simply by replacing \( \hbar \) with \( N^{-1} \) — this fails because the series

\[ F_N = \sum_{g=0}^{\infty} N^{2g} F^g_N \quad \text{and} \quad G_N = \sum_{g=0}^{\infty} N^{2g} G^g_N, \]
are not uniformly convergent on any nondegenerate polydisc for any finite \( N \). This is typical of generating functions associated with 2D quantum gravity \[26, 86\], and one sees similar phenomena in the world of maps on surfaces and Hermitian matrix integrals \[33, 66\]. The divergence of these series forces the introduction of a cutoff at fixed genus \( g = k \), and an ensuing analysis of the holomorphic discrepancy functions

\[
1 - \frac{I_N}{e^{\sum_{g=0}^{k} N^{2-2g} F^g_N}} \text{ and } 1 - \frac{J_N}{e^{\sum_{g=0}^{k} N^{2-2g} G^g_N}}.
\]

It is here that knowledge of the full string expansions of \( I_N \) and \( J_N \) at finite \( N \) is essential: it leads to a “topological bound” which controls the moduli of the discrepancy functions on small polydiscs by a quantity of order \( N^{2-2k} \). Complex analytic tools may then be utilized to convert the topological bound into a topological approximation, replacing a uniform \( O \)-term with a uniform \( o \)-term at the logarithmic scale. The upshot of this analysis is our main theorem, which proves Conjecture 1.1.

**Theorem 1.4.** Conjecture 1.1 is true, and the structure constants \( F_g(\alpha, \beta) \) and \( G_g(\beta) \) are given by

\[
F_g(\alpha, \beta) = (-1)^{l(\alpha)+l(\beta)} \tilde{H}_g(\alpha, \beta) \quad \text{and} \quad G_g(\beta) = (-1)^{d+l(\beta)} \tilde{H}_g(\beta),
\]

where \( \tilde{H}_g(\alpha, \beta) \) and \( \tilde{H}_g(\beta) \) are the monotone double and single Hurwitz numbers of genus \( g \).

1.3. **Context.** Conjecture 1.1 is the subject of a large literature, and many powerful and impressive results have previously been obtained. For the HCIZ integral, the main highlight is Guionnet and Zeitouni’s large deviation theory proof \[51\] of Matytsin’s heuristics \[65\], which characterize the leading asymptotics of \( I_N \) in terms of the flow of a compressible fluid. For the BGW integral, one has Johansson’s Toeplitz determinant proof \[57\] of Gross and Witten’s explicit formula \[46\] for the leading asymptotics of \( J_N \) in the absence of external fields, a result which set the stage for the breakthrough work \[8\]. Another powerful technique is the use of Schwinger-Dyson “loop” equations \[48\] to obtain both the leading \[21\] and sub-leading \[50\] asymptotics of a large class of unitary matrix integrals containing the HCIZ and BGW integrals as prototypes.

The common limitation of these prior works is that they are restricted to real asymptotics: they are obtained under the additional hypothesis that both the coupling constant and the eigenvalues of the external fields are real. This assumption is required in order to force the integrands of \( I_N \) and \( J_N \) to be positive functions on \( U(N) \), so that probabilistic methods can be applied. Indeed, all previous approaches to Conjecture 1.1 are, ultimately, elaborations of the classical Laplace method for the asymptotic evaluation of real integrals depending on a large real parameter. As soon as complex parameters are allowed, \( I_N \) and \( J_N \) become oscillatory integrals. The failure of previous works to treat the complex asymptotics of \( I_N \) and \( J_N \) is not just a technical limitation: many if not most situations in which one would like to invoke the conclusion of Conjecture 1.1 involve complex parameters in an essential way. For example, in order to analyze the spectral asymptotics of random matrices using characteristic functions, one needs the asymptotics of the orbital integral \( I_N \) with complex coupling \( z = i \), which were previously inaccessible, except in certain degenerate scaling limits \[49, 76\]. This is the sole reason that Fourier analysis has
not been a viable technique in the asymptotic spectral analysis of random matrices. For exactly the same reason, it has not been possible to make direct use of the Harish-Chandra/Kirillov formula \[53, 61\] in asymptotic representation theory. The results of this paper clear the way for a direct and unified approach to asymptotic random matrix theory and asymptotic representation theory based on Fourier analysis. Our results can moreover be applied to analyze certain asymptotic problems of physical problems interest which have been mired in confusion for some time \[?\].

We have taken a conceptual as opposed to computational approach to the asymptotics of the HCIZ and BGW integrals by first constructing and understanding their \(N = \infty\) stable limits and then using this insight to build \(N \to \infty\) approximations. A first pass at this was made in \[39\], where Goulden, Guay-Paquet and the author succeeded in obtaining complete asymptotics for each fixed HCIZ string coefficient, but failed to understand the full string series at finite \(N\) and its remarkable connection with longest increasing subsequences and Plancherel measure. Consequently, \[39\] failed to bridge the infinitely large gap between \(N = \infty\) and \(N \to \infty\). Moreover, the fundamental fact that the relationship between the HCIZ and BGW integrals is precisely the relationship between double and single Hurwitz numbers was not perceived in \[39\], where the BGW integral was not considered. Indeed, prior to the present work, no matrix model for monotone single Hurwitz numbers was known, and it was an open question to find one \[4, 27\]. Given that the known matrix model \[13\] for classical single Hurwitz numbers is somewhat contrived, it is remarkable that its counterpart for monotone single Hurwitz numbers is given by none other than the BGW integral, the basic special function of lattice gauge theory.

2. Exact Formulas

In this section, we prove Theorem 1.2 which is the starting point of our analysis.

2.1. Symmetric polynomials. Given a Young diagram \(\alpha\), the associated Newton power sum symmetric polynomial \(p_\alpha\) in commuting variables \(x_1, \ldots, x_N\) is

\[ p_\alpha(x_1, \ldots, x_N) = \prod_{i=1}^{\ell(\alpha)} \sum_{j=1}^{N} x_j^{a_i}. \]

It is a classical result of Newton (see \[63, 81\]) that the polynomials

\[ p_\alpha(x_1, \ldots, x_N), \quad \alpha \vdash d, \]

span the space \(\Lambda^d_N\) of homogeneous degree \(d\) symmetric polynomials in \(x_1, \ldots, x_N\). The Newton polynomials interface naturally with analysis: if \(a_1, \ldots, a_N\) is a point configuration in \(\mathbb{C}\), then normalized power sums evaluated on these points are products of moments of the corresponding empirical probability measure \(\mu\). That is, we have

\[ \frac{p_\alpha(a_1, \ldots, a_N)}{N^{\ell(\alpha)}} = \prod_{i=1}^{\ell(\alpha)} \int_{\mathbb{C}} \zeta^{a_i} \mu(d\zeta). \]

In particular, the normalized power sums which appear in Conjecture 1.4 are products of moments of the empirical eigenvalue distributions of the matrices \(A, B,\) and \(C\). The power sums are the preferred basis for coupling expansions in lattice gauge theory, where they are referred to as “string states” \[5\].
There is another family of symmetric polynomials which play a role in what follows: the Schur polynomials. Given a Young diagram $\lambda$ with $d$ cells, let $(V^\lambda, R^\lambda)$ denote the corresponding irreducible complex representation of the symmetric group $S(d)$, and set

$$\chi_\alpha(\lambda) = \text{Tr } R^\lambda(\pi),$$

where $\pi \in S(d)$ belongs to the conjugacy class $C_\alpha$ of permutations of cycle type $\alpha$. The Schur polynomials,

$$s_\lambda(x_1, \ldots, x_N) = \frac{1}{d!} \sum_{\alpha | C_\alpha} |C_\alpha| \chi_\alpha(\lambda)p_\alpha(a_1, \ldots, a_N), \quad \lambda \vdash d, \ell(\lambda) \leq N,$$

form a basis of $\Lambda^d_N$, and the expansion of a given Newton polynomial in the Schur basis is

$$p_\alpha(x_1, \ldots, x_N) = \sum_{\lambda \vdash d, \ell(\lambda) \leq N} \chi_\alpha(\lambda) s_\lambda(x_1, \ldots, x_N).$$

Evaluations of Schur polynomials at complex points also have representation-theoretic meaning: they are irreducible characters of the general linear group $\text{GL}_N(\mathbb{C})$. More precisely, given a Young diagram $\lambda$ with at most $N$ rows, let $(W^\lambda, S^\lambda)$ denote the corresponding irreducible polynomial representation of $\text{GL}_N(\mathbb{C})$. Then

$$\text{Tr } S^\lambda(A) = s_\lambda(a_1, \ldots, a_N)$$

for any $A \in \text{GL}_N(\mathbb{C})$ with eigenvalues $a_1, \ldots, a_N$.

2.2. Basic integrals. Given a symmetric polynomial $f$ in $N$ variables and an $N \times N$ matrix matrix $A$, write $f(A)$ for the evaluation of $f$ on the spectrum of $A$. We shall need the following basic integration formulas, which are well-known manifestations of Schur orthogonality, see e.g. [63]. We provide a proof for the sake of completeness.

**Lemma 2.1.** For any Young diagrams $\lambda, \mu$ and matrices $A, B \in \text{Mat}_N(\mathbb{C})$, we have

$$\int_{U(N)} s_\lambda(AUBU^{-1})dU = \frac{s_\lambda(A)s_\lambda(B)}{\dim W^\lambda}$$

and

$$\int_{U(N)} s_\lambda(AU)s_\mu(BU^{-1})dU = \delta_{\lambda\mu} \frac{s_\lambda(AB)}{\dim W^\lambda}.$$

**Proof.** Suppose first that $A, B \in \text{GL}_N(\mathbb{C})$. Then $AUBU^{-1} \in \text{GL}_N(\mathbb{C})$, and we have
\[ s_{\lambda}(AU B U^{-1}) = \text{Tr} S_{\lambda}(AU B U^{-1}) \]
\[ = \text{Tr} S_{\lambda}(A) S_{\lambda}(U) S_{\lambda}(B) S_{\lambda}(U^{-1}) \]
\[ = \sum_{i,j,k,l=1}^{N} S_{\lambda}(A)_{ij} S_{\lambda}(U)_{jk} S_{\lambda}(B)_{kl} S_{\lambda}(U^{-1})_{li}. \]

Thus
\[ \int_{U(N)} s_{\lambda}(AU B U^{-1}) dU = \sum_{i,j,k,l=1}^{N} S_{\lambda}(A)_{ij} S_{\lambda}(B)_{kl} \int_{U(N)} S_{\lambda}(U)_{jk} S_{\lambda}(U^{-1})_{li} dU. \]

By Schur orthogonality for the matrix elements of an irreducible representation, we have
\[ \int_{U(N)} S_{\lambda}(U)_{jk} S_{\lambda}(U^{-1})_{li} dU = \frac{\delta_{ij}\delta_{kl}}{\dim W_{\lambda}}, \]
and hence
\[ \int_{U(N)} s_{\lambda}(AU B U^{-1}) dU = \frac{1}{\dim W_{\lambda}} \sum_{i=1}^{N} S_{\lambda}(A)_{ii} \sum_{k=1}^{N} S_{\lambda}(B)_{kk} \]
\[ = \frac{\text{Tr} S_{\lambda}(A) \text{Tr} S_{\lambda}(B)}{\dim W_{\lambda}} \]
\[ = \frac{s_{\lambda}(A) s_{\lambda}(B)}{\dim W_{\lambda}}. \]

Similarly, if \( A, B \in \text{GL}_N(\mathbb{C}) \), then \( AU, B U^{-1} \in \text{GL}_N(\mathbb{C}) \), and we have
\[ s_{\lambda}(AU) = \text{Tr} S_{\lambda}(A) S_{\lambda}(U) = \sum_{i,j=1}^{N} S_{\lambda}(A)_{ij} S_{\lambda}(U)_{ji} \]
\[ s_{\mu}(BU^{-1}) = \text{Tr} S_{\mu}(B) S_{\mu}(U^{-1}) = \sum_{k,l=1}^{N} S_{\mu}(B)_{kl} S_{\mu}(U^{-1})_{lk}. \]

Thus
\[ \int_{U(N)} s_{\lambda}(AU) s_{\mu}(BU^{-1}) dU = \sum_{i,j,k,l=1}^{N} S_{\lambda}(A)_{ij} S_{\mu}(B)_{kl} \int_{U(N)} S_{\lambda}(U)_{ji} S_{\mu}(U^{-1})_{lk} dU. \]

By Schur orthogonality for the matrix elements of different irreducible representations,
\[ \int_{U(N)} S_{\lambda}(U)_{ji} S_{\mu}(U^{-1})_{lk} dU = \frac{\delta_{\lambda \mu} \delta_{ij} \delta_{lk}}{\dim W_{\lambda}}, \]
and we conclude that
\[
\int_{U(N)} s_\lambda(AU)s_\mu(BU^{-1})dU = \delta_{\lambda\mu} \frac{\dim W^\lambda}{\dim W^\mu} \sum_{i,j=1}^{N} S^\lambda(A)_{ij} S^\mu(B)_{ji} = \delta_{\lambda\mu} \frac{s_\lambda(AB)}{\dim W^\lambda}.
\]

That these integral evaluations remain valid for arbitrary complex matrices \(A\) and \(B\) can be seen by taking limits. Let \((A_n)_{n=1}^\infty\) and \((B_n)_{n=1}^\infty\) be sequences in \(\text{GL}_N(\mathbb{C})\) such that
\[
\lim_{n \to \infty} A_n = A \quad \text{and} \quad \lim_{n \to \infty} B_n = B,
\]
and apply the Dominated Convergence Theorem to obtain
\[
\int_{U(N)} s_\lambda(AUBU^{-1})dU = \int_{U(N)} \lim_{n \to \infty} s_\lambda(A_n B_n U^{-1})dU
\]
\[
= \lim_{n \to \infty} \int_{U(N)} s_\lambda(A_n B_n U^{-1})dU
\]
\[
= \lim_{n \to \infty} \frac{s_\lambda(A_n)s_\lambda(B_n)}{\dim W^\lambda}
\]
\[
= \frac{s_\lambda(A)s_\lambda(B)}{\dim W^\lambda},
\]
and
\[
\int_{U(N)} s_\lambda(AU)s_\mu(BU^{-1})dU = \int_{U(N)} \lim_{n \to \infty} s_\lambda(A_n U)s_\mu(B_n U^{-1})dU
\]
\[
= \lim_{n \to \infty} \int_{U(N)} s_\lambda(A_n U)s_\mu(B_n U^{-1})dU
\]
\[
= \lim_{n \to \infty} \delta_{\lambda\mu} \frac{s_\lambda(A_n B_n)}{\dim W^\lambda}
\]
\[
= \delta_{\lambda\mu} \frac{s_\lambda(AB)}{\dim W^\lambda}.
\]

2.3. Character expansions. Lemma [2.1] leads to the following series representations of \(I_N\) and \(J_N\) in terms of Schur polynomials. Expansions of this sort appear in various forms in the physics literature, and were perhaps first utilized in work of James [56] in multivariate statistics, where \(I_N\) and \(J_N\) are treated as hypergeometric functions with matrix arguments.

**Theorem 2.2.** For any \(z \in \mathbb{C}\), and any \(A, B \in \text{Mat}_N(\mathbb{C})\), we have
\[
\int_{U(N)} e^{z N \text{Tr} AUBU^{-1}}dU = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} N^d \sum_{\ell(\lambda) \leq N} \frac{s_\lambda(a_1, \ldots, a_N)s_\lambda(b_1, \ldots, b_N)}{\dim W^\lambda}
\]
\[
\int_{U(N)} e^{z N \text{Tr}(AU+BU^{-1})}dU = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!^2} N^{2d} \sum_{\ell(\lambda) \leq N} \frac{s_\lambda(c_1, \ldots, c_N)(\dim V^\lambda)^2}{\dim W^\lambda}.
\]
where $a_1, \ldots, a_N$ are the eigenvalues of $A$, $b_1, \ldots, b_N$ are the eigenvalues of $B$, and $c_1, \ldots, c_N$ are the eigenvalues of $C = AB$. These series converge absolutely and uniformly on compact subsets of $\mathbb{C}^{2N+1}$ and $\mathbb{C}^{N+1}$, respectively.

**Proof.** Consider first the HCIZ integral. Differentiating under the integral sign, the Maclaurin series of $I_N$ as an entire function of $z$ is

$$I_N = \int_{U(N)} e^{z N p_1 (A B U^{-1})} dU$$

$$= 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} N^d \int_{U(N)} p_1(A B U^{-1}) dU$$

$$= 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} N^d \sum_{\lambda \vdash d \leq N} \frac{(\dim V^\lambda)}{\ell(\lambda)} s_\lambda(A B U^{-1}) dU$$

$$= 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} N^d \sum_{\lambda \vdash d \leq N} s_\lambda(A) s_\lambda(B) \frac{(\dim V^\lambda)}{\dim W^\lambda},$$

by Lemma 2.1.

For the BGW integral, we have

$$J_N = \int_{U(N)} e^{z N p_1(A U) e^{z N p_1(B U^{-1})}} dU$$

$$= 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} N^{2d} \sum_{\lambda \vdash d \leq N} \sum_{\mu \vdash d \leq N} \frac{(\dim V^\lambda)(\dim V^\mu)}{\ell(\lambda) \ell(\mu)} s_\lambda(A U) s_\mu(B U^{-1}) dU$$

$$= 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} N^{2d} \sum_{\lambda \vdash d \leq N} s_\lambda(A B) \frac{(\dim V^\lambda)^2}{\dim W^\lambda},$$

by Lemma 2.1.

Let us perform a consistency check by examining these formulas in the absence of external fields, i.e. when both $A$ and $B$ are the identity matrix. For the HCIZ integral, we see directly from the definition that $I_N = e^{z N^2}$ when $A$ and $B$ are the identity. In this case the character expansion of $I_N$ becomes

$$I_N = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} N^d \sum_{\lambda \vdash d \leq N} (\dim V^\lambda)(\dim V^\lambda) = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} N^{2d},$$

where we have used the isotypic decomposition of the space of $N$-dimensional tensors of rank $d$ as an $S(d) \times \text{GL}_N(\mathbb{C})$ module,

$$(\mathbb{C}^N) \otimes^d \simeq \bigoplus_{\ell(\lambda) \leq N} V^\lambda \otimes W^\lambda.$$

For the BGW integral, in the case $AB = I$ the character expansion becomes
\[ J_N = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d! \cdot N^{2d}} \sum_{\lambda \vdash d, \ell(\lambda) \leq N} (\dim V^\lambda)^2 = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} N^{2d} p(LIS_d \leq N), \]

where in the final equality we used the Robinson-Schensted correspondence (see below). This is exactly the Gessel-Rains identity. Generalizations of the Gessel-Rains identity to integrals over truncated unitary matrices were obtained in [69, 71], and analogues for the other classical groups may be found in [9, 78].

2.4. String expansions. In order to address Conjecture 1.1, we want expansions of \( I_N \) and \( J_N \) in terms of Newton polynomials rather than Schur polynomials — string expansions rather than character expansions. We will now obtain the string expansions of \( I_N \) and \( J_N \) from their character expansions.

As is well-known [63, 81], the dimension of \( V^\lambda \) is equal to the number of standard Young tableaux of shape \( \lambda \). Thus, by the Robinson-Schensted correspondence [81], we have

\[ \sum_{\lambda \vdash d, \ell(\lambda) \leq N} (\dim V^\lambda)^2 = |S_N(d)|, \]

where \( S_N(d) \subseteq S(d) \) is the set of permutations with no increasing subsequence of length \( N + 1 \). It follows that

\[ \lambda \mapsto \frac{(\dim V^\lambda)^2}{|S_N(d)|} \]

is the mass function of a probability measure on the set of Young diagrams with \( d \) cells an at most \( N \) rows. This probability measure is known as the (row-restricted) Plancherel measure, see [60, 79]. We denote expectation with respect to Plancherel measure by angled brackets:

\[ \langle f \rangle = \sum_{\lambda \vdash d, \ell(\lambda) \leq N} f(\lambda) \frac{(\dim V^\lambda)^2}{|S_N(d)|}. \]

Note that if \( N \geq d \) then the restriction on number of rows is vacuous, and the Plancherel measure is a probability measure on the full set of Young diagrams with \( d \) cells whose normalization constant is \( |S(d)| = d! \).

For a Young diagram \( \alpha \vdash d \), let us identify the conjugacy class \( C_\alpha \subseteq S(d) \) with the formal sum of its elements, so that \( C_\alpha \) becomes a central element in the group algebra \( CS(d) \). By Schur’s Lemma, \( C_\alpha \) acts as a scalar operator in any irreducible representation \( (V^\lambda, R^\lambda) \) of \( CS(d) \), i.e.

\[ R^\lambda(C_\alpha) = \omega_\alpha(\lambda) I_{V^\lambda} \]

where

\[ \omega_\alpha(\lambda) = \frac{|C_\alpha| \chi_\alpha(\lambda)}{\dim V^\lambda} \]

and \( I_{V^\lambda} \in \text{End} V^\lambda \) is the identity operator.

Let us introduce the positive function \( \Omega_N \) on Young diagrams with \( d \) cells and at most \( N \) rows defined by
\[ \Omega_N(\lambda) = \frac{d!}{N^d} \frac{\dim V^\lambda}{\dim W^\lambda} = \prod_{\square \in \lambda} \left(1 + \frac{c(\square)}{N}\right). \]

Here we have used the dimension formulas \[63, 81\]
\[ \dim V^\lambda = \frac{d!}{\prod_{\square \in \lambda} h(\square)} \quad \text{and} \quad \dim W^\lambda = \frac{\prod_{\square \in \lambda} N + c(\square)}{h(\square)}, \]
where \( h(\square) \) is the hook length of a given cell \( \square \in \lambda \) (number of cells to the right of \( \square \) plus number of cells below \( \square \) plus one) and \( c(\square) \) is its content (column index less row index), to render \( \Omega_N(\lambda) \) as an explicit product. Note that \( \Omega_N^{-1}(\lambda) = \prod_{\square \in \lambda} \frac{1}{1 + \frac{c(\square)}{N}} \)

is well-defined and positive since \( \ell(\lambda) \leq N \). The functions \( \Omega_N^{\pm 1} \) seem to be closely related to the “\( \Omega \)-points” considered by physicists in the context of gauge/string dualities \[5, 23, 44\], but which seem not to have been fully understood in that context. In terms of \( \Omega_N \), the Schur function expansions of \( I_N \) and \( J_N \) are

\[ I_N = 1 + \sum_{d=1}^{\infty} z^d \sum_{\lambda \vdash d, \ell(\lambda) \leq N} s_\lambda(A)s_\lambda(B)\Omega_N^{-1}(\lambda) \]
\[ J_N = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} N^d \sum_{\lambda \vdash d, \ell(\lambda) \leq N} s_\lambda(C)\Omega_N^{-1}(\lambda) \dim V^\lambda. \]

We now prove Theorem 1.2, which we restate here using the notation just established.

**Theorem 2.3.** For any \( z \in \mathbb{C} \) and any \( A, B \in \text{Mat}_N(\mathbb{C}) \), we have

\[ \int_{U(N)} e^{zN \text{Tr}(AU+BU^{-1})} \ dU = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \mathbb{P}(\text{LIS}_d \leq N) \sum_{\alpha, \beta \vdash d} p_\alpha(A)p_\beta(B)\langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle, \]
\[ \int_{U(N)} e^{zN \text{Tr}(AU+BU^{-1})} \ dU = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} N^d \mathbb{P}(\text{LIS}_d \leq N) \sum_{\beta \vdash d} p_\beta(C)\langle \Omega_N^{-1} \omega_\beta \rangle, \]
where \( C = AB \).

**Proof.** For the HCIZ integral, we have
Thus we have

\[ I_N = 1 + \sum_{d=1}^{\infty} z^d \sum_{\ell(\lambda) \leq N} \frac{|C_{\gamma} \chi_{\alpha}(\lambda)|}{d!} p_{\alpha}(A) \left( \sum_{\beta \neq d} \frac{|C_{\beta} \chi_{\beta}(\lambda)|}{d!} p_{\beta}(B) \right) \Omega_N^{-1}(\lambda) \]

\[ = 1 + \sum_{d=1}^{\infty} z^d \sum_{\ell(\lambda) \leq N} \sum_{\alpha \neq d} \frac{|C_{\gamma} \chi_{\alpha}(\lambda)|}{d!} p_{\alpha}(A) \sum_{\beta \neq d} \frac{|C_{\beta} \chi_{\beta}(\lambda)|}{d!} p_{\beta}(B) \Omega_N^{-1}(\lambda) \]

\[ = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \frac{S_N(d)}{d!} \sum_{\alpha \neq d} \sum_{\alpha \neq d} \frac{|C_{\gamma} \chi_{\alpha}(\lambda)|}{d!} p_{\alpha}(A) \Omega_N^{-1}(\lambda) \frac{|C_{\beta} \chi_{\beta}(\lambda)|}{d!} p_{\beta}(B) \Omega_N^{-1}(\lambda) \frac{(\dim V^\lambda)^2}{|S_N(d)|} \]

\[ = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \mathbb{P}(\text{LIS}_d \leq N) \sum_{\alpha \neq d} \sum_{\beta \neq d} \frac{p_{\gamma}(A) p_{\beta}(B)}{(\omega_{\alpha} \Omega_N^{-1} \omega_{\beta})}. \]

For the BGW integral, we have

\[ J_N = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \frac{N^d}{d!} \sum_{\ell(\lambda) \leq N} \frac{|C_{\gamma} \chi_{\alpha}(\lambda)|}{d!} p_{\alpha}(A) \Omega_N^{-1}(\lambda) \frac{(\dim V^\lambda)^2}{|S_N(d)|} \]

\[ = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \frac{N^d}{d!} \sum_{\ell(\lambda) \leq N} \sum_{\alpha \neq d} \frac{|C_{\gamma} \chi_{\alpha}(\lambda)|}{d!} p_{\alpha}(A) \Omega_N^{-1}(\lambda) \frac{(\dim V^\lambda)^2}{|S_N(d)|} \]

\[ = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \frac{N^d}{d!} \mathbb{P}(\text{LIS}_d \leq N) \sum_{\alpha \neq d} \sum_{\beta \neq d} \frac{p_{\gamma}(A) p_{\beta}(B)}{(\omega_{\alpha} \Omega_N^{-1} \omega_{\beta})}. \]

\[ \square \]

2.5. Basic bounds. Let us write the string expansions of \( I_N \) and \( J_N \) in the form

\[ I_N = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} I_N(d) \quad \text{and} \quad J_N = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} J_N(d). \]

Thus \( I_N(d) \) and \( J_N(d) \) are the symmetric polynomials

\[ I_N(d) = N^d \sum_{\ell(\lambda) \leq N} \frac{\dim V^\lambda}{\dim W^\lambda} \]

\[ = \mathbb{P}(\text{LIS}_d \leq N) \sum_{\alpha, \beta \neq d} \frac{p_{\gamma}(A) p_{\beta}(B)}{(\omega_{\alpha} \Omega_N^{-1} \omega_{\beta})} \]

and
\[ J_N(d) = \frac{N^{2d}}{d!} \sum_{\lambda \vdash d, \ell(\lambda) \leq N} s_\lambda(c_1, \ldots, c_N) \frac{(\dim V_\lambda)^2}{\dim W_\lambda} \]
\[ = N^d \mathbb{P}(\text{LIS}_d \leq N) \sum_{\beta \vdash d} p_\beta(c_1, \ldots, c_N) \langle \Omega_N^{-1} \omega_\beta \rangle. \]

The following bounds—which say that \( I_N(d) \) and \( J_N(d) \) have maximum modulus in the case of trivial external fields—will be needed in Section 4.

**Proposition 2.4.** For any \( d, N \in \mathbb{N} \) and any \( a_1, \ldots, a_N, b_1, \ldots, b_N, c_1, \ldots, c_N \in \mathbb{C} \) of modulus at most one, we have
\[ |I_N(d)| \leq N^{2d} \text{ and } |J_N(d)| \leq \mathbb{P}(\text{LIS}_d \leq N) N^{2d}. \]

**Proof.** This follows from the fact that the Schur polynomials are monomial positive. \( \square \)

3. Stable Asymptotics

In this section, we analyze the \( N \to \infty \) asymptotics of the string coefficients of \( I_N \) and \( J_N \). We obtain a convergent \( N \to \infty \) asymptotic expansion for each fixed string coefficient, the coefficients of which count monotone walks on the symmetric groups with prescribed length and boundary conditions. These expansions are grouped together to form the stable HCIZ and BGW integrals \( I \) and \( J \), which are formal power series. The stable integrals \( I \) and \( J \) satisfy a formal power series version of Conjecture 1.1, the form of which points the way to an analytic solution.

3.1. String coefficients. Our present goal is to determine the \( N \to \infty \) asymptotics of the string coefficients of \( I_N \) and \( J_N \),

\[ \mathbb{P}(\text{LIS}_d \leq N)(\omega_\alpha \Omega_N^{-1} \omega_\beta), \]

in the regime where \( \alpha, \beta \vdash d \) are fixed and \( N \to \infty \). In this regime we may assume \( N \geq d \), so that the string coefficients are pure Plancherel expectations:

\[ \langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle = \sum_{\lambda \vdash d} \omega_\alpha(\lambda) \Omega_N^{-1}(\lambda) \omega_\beta(\lambda) \frac{(\dim V_\lambda)^2}{d!}. \]

Since

\[ \lim_{N \to \infty} \Omega_N^{-1}(\lambda) = 1 \]

for any fixed \( \lambda \), these Plancherel expectations are deformations of the usual inner product on the center of the group algebra \( \mathbb{C}S(d) \), with respect to which the functions \( \omega_\alpha \) form an orthogonal basis:

\[ \langle \omega_\alpha \omega_\beta \rangle = \sum_{\lambda \vdash d} \omega_\alpha(\lambda) \omega_\beta(\lambda) \frac{(\dim V_\lambda)^2}{d!} = \delta_{\alpha\beta} |C_\alpha|. \]

We thus have

\[ \langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle = \delta_{\alpha\beta} |C_\alpha| + o(1) \]
as \( N \to \infty \), simply because of the orthogonality of irreducible characters. We will now quantify the error term in this approximation.

Let \( \hbar \) be a complex parameter, and consider the function on Young diagrams \( \lambda \) defined by

\[
\Psi_{\hbar}(\lambda) = \prod_{\Box \in \lambda} (1 - \hbar c(\Box)).
\]

This is a polynomial function of \( \hbar \) whose roots are the reciprocals of the contents of the off-diagonal cells of \( \lambda \). Explicitly, this polynomial is given by

\[
\Psi_{\hbar}(\lambda) = \sum_{r=0}^{d} (-\hbar)^r e_r(\lambda),
\]

where \( e_r(\lambda) \) is the degree \( r \) elementary symmetric polynomial in \( d \) variables,

\[
e_r = \sum_{i: [r] \to [d]} x_{i(1)} \cdots x_{i(r)},
\]

evaluated on the contents of the diagram \( \lambda \). Note that \( e_r(\lambda) \) is a shifted symmetric function of \( \lambda \); see [74] for a discussion of shifted symmetric functions.

The function \( \Psi_{\hbar}^{-1}(\lambda) = \frac{1}{\Psi_{\hbar}(\lambda)} \) is a nonvanishing rational function of \( \hbar \) whose poles are the roots of \( \Psi_{\hbar}(\lambda) \). In particular, for any diagram \( \lambda \), the function \( \Psi_{\hbar}^{-1}(\lambda) \) is analytic on the disc

\[
|h| < \frac{1}{\max(\lambda_1 - 1, \ell(\lambda) - 1)}
\]

with Maclaurin series

\[
\Psi_{\hbar}^{-1}(\lambda) = \sum_{r=0}^{\infty} h^r f_r(\lambda),
\]

where \( f_r(\lambda) \) is the degree \( r \) complete symmetric polynomial in \( d \) variables,

\[
f_r = \sum_{i: [r] \to [d]} x_{i(1)} \cdots x_{i(r)},
\]

evaluated on the contents of \( \lambda \).

The functions \( \Omega_{\hbar}^{\pm 1} \) are recovered from the functions \( \Psi_{\hbar}^{\pm 1} \) by setting

\[
\hbar = -\frac{1}{N}.
\]

In particular, the \( N \to \infty \) asymptotics of \( \langle \omega_{\alpha} \Omega_{\hbar}^{-1} \omega_{\beta} \rangle \) may be obtained from the \( \hbar \to 0 \) asymptotics of \( \langle \omega_{\alpha} \Psi_{\hbar}^{-1} \omega_{\beta} \rangle \), i.e. from its Taylor expansion around \( \hbar = 0 \) as derived above. We will now give a diagrammatic interpretation of this Maclaurin series.

For a given pair of Young diagrams \( \alpha, \beta \vdash d \), we have the Taylor series
\[ \langle \omega_{\alpha} \Psi^{-1}_{\hbar} \omega_{\beta} \rangle = \sum_{r=0}^{\infty} \hbar^r \langle \omega_{\alpha} f_r \omega_{\beta} \rangle, \]

which is absolutely convergent for \(|\hbar| < \frac{1}{d-1}\). For any \(\lambda \vdash d\), the observable \(\omega_{\alpha}(\lambda) f_r(\lambda) \omega_{\beta}(\lambda)\) is the eigenvalue of the central element \(C_{\alpha} f_r(X_1, \ldots, X_d) C_{\beta}\) acting in the irreducible representation \(V^\lambda\) of the group algebra \(CS(d)\) corresponding to \(\lambda\), where \(X_1, \ldots, X_d \in CS(d)\) are the Jucys-Murphy elements \([25, 75]\):

\[ X_j = \sum_{i<j} (i, j), \quad 1 \leq j \leq d. \]

Thus, by the Fourier isomorphism,

\[ CS(d) \cong \bigoplus_{\lambda \vdash d} \text{End} V^\lambda, \]

the Plancherel expectation \(\langle \omega_{\alpha} f_r \omega_{\beta} \rangle\) is the normalized character of the central element \(C_{\alpha} f_r(X_1, \ldots, X_d) C_{\beta}\) in the regular representation of \(CS(d)\), i.e. the coefficient of the identity permutation in the sum

\[ \sum_{\rho \in C_{\alpha}, \sigma \in C_{\beta}} \sum_{i,j : [r] \to [d]} \rho(i(1), j(1)) \ldots (i(r), j(r)) \sigma. \]

Adopting the convention that permutations are multiplied from left to right, this number may be visualized as follows.

Identify the symmetric group \(S(d)\) with its right Cayley graph, as generated by the conjugacy class of transpositions. Introduce an edge labeling on this graph by marking each edge corresponding to the transposition \((i, j)\) with \(j\), the larger of the two elements interchanged. Thus, emanating from each vertex of \(S(d)\), one sees a single 2-edge, two 3-edges, three 4-edges, etc. Figure 4 shows \(S(4)\) equipped with this edge labeling. A walk on \(S(d)\) is said to be monotone if the labels of the edges it traverses form a weakly increasing sequence. Given Young diagrams \(\alpha, \beta \vdash d\) and a nonnegative integer \(r\), let \(\tilde{W}^r(\alpha, \beta)\) denote the number of \(r\)-step monotone walks on \(S(d)\) which begin at a point of \(C_{\alpha}\) and end at a point of \(C_{\beta}\). Then from the calculation above we have the identity

\[ \langle \omega_{\alpha} f_r \omega_{\beta} \rangle = \tilde{W}^r(\alpha, \beta). \]

Equivalently,

\[ \langle \omega_{\alpha} \Psi^{-1}_{\hbar} \omega_{\beta} \rangle = \sum_{r=0}^{\infty} \hbar^r \tilde{W}^r(\alpha, \beta), \]

the generating function for monotone walks on \(S(d)\) with boundary conditions \(\alpha, \beta\), the sum being absolutely convergent for \(|\hbar| < \frac{1}{d-1}\). In the special case \(\alpha = (1^d)\), we have

\[ \langle \Psi^{-1}_{\hbar} \omega_{\beta} \rangle = \sum_{r=0}^{\infty} \hbar^r \tilde{W}^r(\beta), \]
where \( \bar{W}^r(\beta) = \bar{W}^r(1^d, \beta) \) is the number of \( r \)-step monotone walks on \( S(d) \) which begin at the identity permutation and end at a permutation of cycle type \( \beta \). We may thus conclude the following.

**Theorem 3.1.** For any positive integers \( 1 \leq d \leq N \) and any Young diagrams \( \alpha, \beta \vdash d \), we have

\[
\omega_{\alpha} \Omega^{-1}_{\beta} \approx \sum_{r=0}^{\infty} (-1)^r \frac{\bar{W}^r(\alpha, \beta)}{N^r}
\]

and

\[
\Omega^{-1}_{\beta} \approx \sum_{r=0}^{\infty} (-1)^r \frac{\bar{W}^r(\beta)}{N^r}
\]

and the series are absolutely convergent.

Note that the \( 1/N \) expansions in Theorem 3.1 are not actually alternating series: their nonzero terms are either all negative or all positive.

Theorem 3.1 gives a convergent \( N \to \infty \) asymptotic expansion of the Plancherel expectation \( \langle \omega_{\alpha} \Omega^{-1}_{\beta} \rangle \) wherein monotone walks play the role of Feynman diagrams. As a consistency check, observe that

\[
\bar{W}^0(\alpha, \beta) = \delta_{\alpha, \beta} |C_{\alpha}|,
\]

corresponding to the fact that there exists a zero-step walk from \( C_{\alpha} \) to \( C_{\beta} \) if and only if these otherwise disjoint sets are equal, and in this case the number of such walks is just the cardinality of \( C_{\alpha} \).

3.2. **Stable integrals.** For any fixed \( N \in \mathbb{N} \), Theorem 3.1 describes the first \( N \) nonconstant terms in the string expansions of \( I_N \) and \( J_N \); we have
ON THE COMPLEX ASYMPTOTICS OF THE HCIZ AND BGW INTEGRALS

\[ I_N = 1 + \sum_{d=1}^{N} \frac{z^d}{d!} \sum_{\alpha, \beta, r} p_\alpha(a_1, \ldots, a_N) p_\beta(b_1, \ldots, b_N) \sum_{r=0}^{\infty} (-1)^r \frac{\tilde{W}^r(\alpha, \beta)}{N^r} + \text{higher terms}, \]

and

\[ J_N = 1 + \sum_{d=1}^{N} \frac{z^{2d}}{d!} N^d \sum_{\beta} p_\beta(b_1, \ldots, b_N) \sum_{r=0}^{\infty} (-1)^r \frac{\tilde{W}^r(\beta)}{N^r} + \text{higher terms}. \]

This description suggests that, as \( N \to \infty \), the integrals \( I_N \) and \( J_N \) approximate generating functions for monotone walks on all of the symmetric groups, of all possible lengths and boundary conditions. Unfortunately, for any finite \( N \), almost all terms of the string expansion are “higher terms.”

To see past this analytic limitation, let us view \( z \) as a formal variable, and replace the number \( -\frac{1}{N} \) with a formal semiclassical parameter \( \hbar \). Furthermore, let us replace the eigenvalues of the matrices \( A, B, C = AB \) with countably infinite alphabets of commuting indeterminates, these being formal stand-ins for the eigenvalues infinite-dimensional matrices. Let \( \Lambda_A, \Lambda_B, \Lambda_C \) be the affiliated algebras of symmetric functions, i.e. the polynomial algebras

\[ \Lambda_A = \mathbb{C}[p_1(A), p_2(A), \ldots], \quad \Lambda_B = \mathbb{C}[p_1(B), p_2(B), \ldots], \quad \Lambda_C = \mathbb{C}[p_1(C), p_2(C), \ldots], \]

where

\[ p_k(A) = \sum_{a \in A} a^k, \quad p_k(B) = \sum_{b \in B} b^k, \quad p_k(C) = \sum_{c \in C} c^k, \quad k \in \mathbb{N}, \]

are the pure power sums over these alphabets. Set \( \Lambda_{A,B} = \Lambda_A \otimes \Lambda_B \).

We define the stable HCIZ integral to be the formal power series

\[ I = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta, r} p_\alpha(A) p_\beta(B) \sum_{r=0}^{\infty} \hbar^r \tilde{W}^r(\alpha, \beta), \]

which is an element of the ring \( \Lambda_{A,B}[[z, \hbar]] \). Similarly, we define the stable BGW integral to be the formal power series

\[ J = 1 + \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\beta, r} (-1)^d \hbar^{-d} p_\beta(B) \sum_{r=0}^{\infty} \hbar^r \tilde{W}^r(\beta), \]

which is an element of \( \Lambda_C[[z, \hbar^{\pm 1}]] \). Thus \( I \) and \( J \) are “grand canonical” partition functions enumerating monotone walks of all possible lengths and boundary conditions, over all symmetric groups.

**Theorem 3.2.** We have

\[ I = e^F \quad \text{and} \quad J = e^G, \]

where

\[ F = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta, r} p_\alpha(A) p_\beta(B) \sum_{r=0}^{\infty} \hbar^r \tilde{H}^r(\alpha, \beta) \]
and \( \vec{H}^r(\alpha, \beta) \) is the number of monotone \( r \)-step walks on \( S(d) \) which begin at a permutation of cycle type \( \alpha \), end at a permutation of cycle type \( \beta \), and have the property that their steps and endpoints together generate a transitive subgroup of \( S(d) \), and

\[
G = \sum_{d=1}^{\infty} \frac{z^d}{d!} (-1)^d \sum_{\beta \vdash d} p_{\beta}(C) \sum_{r=0}^{\infty} h^r \vec{H}^r(\beta)
\]

with \( \vec{H}^r(\beta) = \vec{H}^r(1^d, \beta) \).

Theorem 3.2 follows from a fundamental result in enumerative combinatorics, the Exponential Formula [81, Chapter 5], according to which the exponential of a generating function for a class of “connected” combinatorial structures is a generating function for possibly “disconnected” structures of the same type. For walks on groups, the role of connectedness is played by transitivity. For a careful justification of the use of the Exponential Formula in the context of monotone walks on symmetric groups, see [37, 38, 40].

3.3. Topological expansion. The numbers \( \vec{H}^r(\alpha, \beta) \) and \( \vec{H}^r(\beta) \) appearing in Theorem 3.2 are known as the monotone double and single Hurwitz numbers, respectively. These enumerative quantities, introduced in [37, 38, 39] and studied in numerous articles since, are a combinatorial variant of the classical double and single Hurwitz numbers \( H^r(\alpha, \beta) \) and \( H^r(\beta) = H^r(1^d, \beta) \), which count transitive \( r \)-step walks \( C_\alpha \to C_\beta \) without the monotonicity constraint. Clearly, \( \vec{H}^r(\alpha, \beta) \leq H^r(\alpha, \beta) \), and in a sense monotone Hurwitz numbers are a “desymmetrized” version of classical Hurwitz numbers; see [37, 38].

Reversing a classical construction due to Hurwitz [54] (see [32] for a modern treatment), the number \( H^r(\alpha, \beta) \) may alternatively be interpreted as the number of isomorphism classes of degree \( d \) branched covers of the Riemann sphere \( \mathbb{P}^1(\mathbb{C}) \) by a compact, connected Riemann surface \( S \) which have profiles \( \alpha, \beta \vdash d \) over \( 0, \infty \in \mathbb{P}^1(\mathbb{C}) \) and the simplest nontrivial branching over the \( r \)th roots of unity on the sphere. The monotone double Hurwitz number \( \vec{H}^r(\alpha, \beta) \) is a signed enumeration of the same class of covers, see [3, 64]. The genus \( g \) of \( S \) is determined by the data \( d, r, \alpha, \beta \) according to the Riemann-Hurwitz formula,

\[
g = r + 2 - c_\alpha - c_\beta,
\]

with the understanding that \( H^r(\alpha, \beta) = 0 \) unless this formula returns a nonnegative integer. In particular, one may parameterize nonzero (classical and monotone) Hurwitz numbers by genus, setting \( H_g(\alpha, \beta) := H^{2g-2+c_\alpha+c_\beta}(\alpha, \beta) \) and \( \vec{H}_g(\alpha, \beta) := \vec{H}^{2g-2+c_\alpha+c_\beta}(\alpha, \beta) \). In the genus parameterization, Theorem 3.2 becomes the following topological expansion of the stable HCIZ and BGW integrals.

**Theorem 3.3.** We have

\[
I = e^\sum_{g=0}^{\infty} h^{2g-2} F(g) \quad \text{and} \quad J = e^\sum_{g=0}^{\infty} h^{2g-2} G(g),
\]

where

\[
F(g) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} h^{c_\alpha+c_\beta} p_\alpha(A)p_\beta(B) \vec{H}_g(\alpha, \beta).
\]
Define the disconnected monotone double and single Hurwitz numbers by component has genus at least \(k\)  
\[
G(g) = \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} (-1)^d \sum_{\beta^+d} h^{(\beta)} p_\beta(C) \bar{H}_g(\beta).
\]

**Proof.** Applying the Riemann-Hurwitz formula, we have  
\[
F = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta^+d} p_\alpha(A) p_\beta(B) \sum_{g=0}^{\infty} h^{2g-2+\ell(\alpha)+\ell(\beta)} \bar{H}_g(\alpha, \beta)
\]
\[
= \sum_{g=0}^{\infty} h^{2g-2} \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta^+d} h^{\ell(\alpha)+\ell(\beta)} p_\alpha(A) p_\beta(B) \bar{H}_g(\alpha, \beta)
\]
and  
\[
G = \sum_{d=1}^{\infty} \frac{z^d}{d!} (-1)^d h^{-d} \sum_{\beta^+d} p_\beta(B) \sum_{g=0}^{\infty} h^{2g-2+d+\ell(\beta)} \bar{H}_g(\beta)
\]
\[
= \sum_{g=0}^{\infty} h^{2g-2} \sum_{d=1}^{\infty} \frac{z^d}{d!} (-1)^d \sum_{\beta^+d} h^{\ell(\beta)} p_\beta(C) \bar{H}_g(\beta).
\]

\(\square\)

### 3.4. Topological factorization

For any nonnegative integer \(k\), the topological expansions of \(I\) and \(J\) given by Theorem 3.3 can be split into two corresponding factors,  
\[
I = e^{\sum_{g=0}^{k} h^{2g-2} F^{(s)}} e^{\sum_{g=k+1}^{\infty} h^{2g-2} F^{(s)}} \quad \text{and} \quad J = e^{\sum_{g=0}^{k} h^{2g-2} G^{(s)}} e^{\sum_{g=k+1}^{\infty} h^{2g-2} G^{(s)}}.
\]

These factorizations have a clear enumerative meaning: the first factor is a generating function enumerating possibly disconnected covers/walks in which each connected component has genus at most \(k\), while the second factor is a generating function enumerating possibly disconnected covers/walks in which each connected component has genus at least \(k+1\). This may be equivalently stated as follows. Define the disconnected monotone double and single Hurwitz numbers by  
\[
\bar{H}_g^{\bullet}(\alpha, \beta) = \bar{W}^{r_g(\alpha, \beta)}(\alpha, \beta) \quad \text{and} \quad \bar{H}_g^{\bullet}(\beta) = \bar{H}_g^{\bullet}(1^d, \beta),
\]
where \(g \in \mathbb{Z}\) and \(r_g(\alpha, \beta) = 2g - 2 + \ell(\alpha) + \ell(\beta)\). In particular, for disconnected Hurwitz numbers the genus \(g\) may be negative (corresponding to the fact that the Euler characteristic is additive), but \(\bar{H}_g^{\bullet}(\alpha, \beta)\) vanishes unless \(r_g(\alpha, \beta) \geq 0\). In terms of disconnected monotone Hurwitz numbers, the above factorization identities may be stated as follows.

**Theorem 3.4.** For any \(k \in \mathbb{N} \cup \{0\}\)  
\[
\frac{I}{e^{\sum_{g=0}^{k} h^{2g-2} F^{(s)}}} = 1 + \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta^+d} h^{\ell(\alpha)+\ell(\beta)} p_\alpha(A) p_\beta(B) \sum_{g=k+1}^{\infty} h^{2g-2} \bar{H}_g^{\bullet}(\alpha, \beta)
\]
and
\[
J = \frac{1}{e^{\sum_{g=0}^\infty g^2 \ell g} G(g)} = 1 + \sum_{d=1}^\infty \sum_{a_i, \beta} \frac{(-1)^d h^{l(\beta)} p_\beta(B)}{d!} \sum_{g=k+1}^{\infty} h^{2g-2} \hat{H}_g^\bullet(\beta).
\]

As a corollary of Theorem 3.4, we obtain the following pair of “topological bounds,” which are significant since they indicate what sorts of bounds we may expect to be valid for the entire functions \(I_N\) and \(J_N\) at finite \(N\). Let us introduce the following formal order notation: given a formal power series \(Z \in \Lambda_{A,B}[[z,\hbar]]\) and a nonnegative integer \(n\), we write

\[
Z = O(\hbar^n)
\]

if \(Z\) belongs to the principal ideal generated by \(\hbar^n\). We use the analogous order notation in \(\Lambda_{C}[[z,\hbar^{\pm 1}]]\).

**Corollary 3.5.** For each \(k \in \mathbb{N} \cup \{0\}\),

\[
1 - \frac{I}{e^{\sum_{g=0}^\infty g^2 \ell g} G(g)} = O(\hbar^{2k}) \quad \text{and} \quad 1 - \frac{J}{e^{\sum_{g=0}^\infty g^2 \ell g} G(g)} = O(\hbar^{2k}).
\]

### 4. Functional Asymptotics

In this Section, we prove our main result, Theorem 1.4. To achieve this, we must bridge the gap between the \(N < \infty\) string expansions of the HCIZ and BGW integrals and their \(N = \infty\) stable topological expansions. It is here that the mollifying effect of the LIS distribution plays a critical role: it controls the tail of the finite \(N\) string expansions of \(I_N\) and \(J_N\), effectively truncating them to polynomials of degree \(O(N^2)\) for small \(z\). The existence of this quadratic cutoff is a key feature of \(I_N\) and \(J_N\) that has not previously been recognized.

#### 4.1. Analytic candidates.

Throughout this section, we will use the following notation for complex polydiscs. Given a real number \(\rho\) and a positive integer \(N\), we will ambiguously write \(\mathbb{D}_\rho\) to mean either of the closed polydiscs

\[
\mathbb{D}_\rho \times \mathbb{D}_1^N \times \mathbb{D}_1^N \quad \text{or} \quad \mathbb{D}_\rho \times \mathbb{D}_1^N,
\]

where \(\mathbb{D}_\rho\) is the closed origin-centred disc of radius \(\rho\) in the complex plane. Although the first of these domains lives in \(\mathbb{C}^{2N+1}\) and the second lives in \(\mathbb{C}^{N+1}\), which of the two domains \(\mathbb{D}_\rho^N\) is intended to represent will be clear from context. Let \(\| \cdot \|_\rho\) denote the sup norm on \(\mathbb{D}_\rho^N\). Note that this is really a sequence of norms defined on a sequence of domains of growing dimension.

Let \(N \in \mathbb{N}\) be a positive integer, and let \(a_1, \ldots, a_N, b_1, \ldots, b_N, c_1, \ldots, c_N\) be any points sampled from the closed unit disc in \(\mathbb{C}\). Consider the corresponding specializations

\[
\Lambda_{A,B}[[z,\hbar]] \rightarrow \mathbb{C}[[z]] \quad \text{and} \quad \Lambda_{C}[[z,\hbar]] \rightarrow \mathbb{C}[[z]]
\]

defined by setting \(\hbar = -1/N\) and

\[
A = \{a_1, \ldots, a_N\}, B = \{b_1, \ldots, b_N\}, C = \{c_1, \ldots, c_N\},
\]

and let
be the images of $F^{(g)}$ and $G^{(g)}$ under these specializations. A priori, $F^{(g)}_N$ and $G^{(g)}_N$ are only formal power series. However, they are in fact absolutely summable, and hence define analytic functions. This follows from an established result on the convergence of generating functions for monotone Hurwitz numbers \[41\].

**Theorem 4.1.** For each $g \in \mathbb{N} \cup \{0\}$, the power series

\[
\vec{H}^{\text{simple}}_g = \sum_{d=1}^{\infty} \frac{z^d}{d!} \vec{H}_g(1^d, 1^d),
\]

\[
\vec{H}^{\text{single}}_g = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\beta \vdash d} \vec{H}_g(1^d, \beta),
\]

\[
\vec{H}^{\text{double}}_g = \sum_{d=1}^{\infty} \frac{z^{2d}}{d!} \sum_{\alpha, \beta \vdash d} \vec{H}_g(\alpha, \beta)
\]

have radii of convergence exactly $2/27$, at least $1/27$, and at least $1/54$, respectively.

The exact computation of the radius of convergence of the generating function for monotone simple Hurwitz numbers follows from a rational parameterization of this series in terms of the Gauss hypergeometric function, see \[38, 41\]. A combinatorial argument based on sorting transpositions (a variant of the Hurwitz braid action) then shows that the radius of convergence drops by at most a factor of two for each new branch point added, see \[41\] for details. The author believes that the radius of convergence is in fact exactly $2/27$ in all three cases, but this has not been proved.

It was pointed out to the author by Philippe Di Francesco that the number of isomorphism classes of finite groups of order $p^N$, with $p$ prime, is known \[77\] to be asymptotically $p^{\frac{2}{27} N^3}$ as $N \to \infty$. The author has no explanation for this numerical coincidence. For another interesting appearance of the number $2/27$, see \[58\].

**Theorem 4.2.** There exists $\delta > 0$ such that the series $F^{(g)}_N$ and $G^{(g)}_N$ converge absolutely and uniformly on $\mathbb{D}^N_\delta$, for all $g \geq 0$ and $N \geq 1$.

**Proof.** For any Young diagrams $\alpha, \beta$, we have

\[
\left| \frac{p_\alpha(a_1, \ldots, a_N)}{N^{\ell(\alpha)}} \right|, \left| \frac{p_\beta(b_1, \ldots, b_N)}{N^{\ell(\beta)}} \right|, \left| \frac{p_\beta(c_1, \ldots, c_N)}{N^{\ell(\beta)}} \right| \leq 1,
\]

on $\mathbb{D}^N_\delta$. We thus have
\[ |F_N^{(g)}| \leq \sum_{d=1}^{n} \frac{\delta^d}{d!} \sum_{\alpha, \beta \vdash d} H_g(\alpha, \beta) \]

\[ |G_N^{(g)}| \leq \sum_{d=1}^{n} \frac{\delta^{2d}}{d!} \sum_{\beta \vdash d} H_g(\beta) \]

uniformly on \( \mathbb{D}_N^{\delta} \) for any \( n \in \mathbb{N} \), and the claim thus follows from Theorem 4.1. \( \square \)

Let us fix \( \delta > 0 \) so that \( F_N^{(g)} \) and \( G_N^{(g)} \) converge to define analytic functions on \( \mathbb{D}_N^{\delta} \), for all \( N \in \mathbb{N} \). Then, these functions are uniformly bounded in the following sense.

**Corollary 4.3.** We have

\[ \sup_{N \in \mathbb{N}} \|F_N^{(g)}\|_{\delta} < \infty \quad \text{and} \quad \sup_{N \in \mathbb{N}} \|G_N^{(g)}\|_{\delta} < \infty. \]

4.2. **Polynomial approximation.** We now consider the behavior of the full string expansions of \( I_N \) and \( J_N \) given by Theorem 1.2 with \( N \) large but finite. In this regime, the factor \( \mathbb{P}(\text{LIS}_d \leq N) \) has a dramatic effect on the string expansions — it effectively truncates them to polynomials of degree \( O(N^2) \). The mechanism behind this cutoff is the law of large numbers for longest increasing subsequences in random permutations, which is due to Vershik and Kerov [60]: we have

\[ \lim_{d \to \infty} \frac{\text{LIS}_d}{\sqrt{d}} = 2, \]

where the convergence is in probability. A detailed exposition of this LLN is given in [79], which also presents the corresponding central limit theorem of Baik-Deift-Johansson [8], which asserts Tracy-Widom fluctuations of \( \text{LIS}_d \) around \( 2\sqrt{d} \) on the scale \( d^{1/6} \). In particular, the distribution of the longest increasing subsequence in large uniformly random permutation is strongly concentrated around its mean. This implies that, for large \( N \) we have the approximate step function behavior

\[ \mathbb{P}(\text{LIS}_d \leq N) \approx \begin{cases} 1, & 1 \leq d \leq \frac{1}{4} N^2 \\ 0, & d > \frac{1}{4} N^2 \end{cases}. \]

Consequently, for \( |z| \) small and \( N \) large, \( I_N \) and \( J_N \) are well-approximated by their “string polynomials”

\[ \tilde{I}_N = 1 + \sum_{d=1}^{\left\lfloor \frac{1}{4} N^2 \right\rfloor} z^d \frac{1}{d!} \sum_{\alpha, \beta \vdash d} p_{\alpha}(a_1, \ldots, a_N) p_{\beta}(b_1, \ldots, b_N) \langle \omega_\alpha \Omega^{-1} \omega_\beta \rangle \]

and

\[ \tilde{J}_N = 1 + \sum_{d=1}^{\left\lfloor \frac{1}{4} N^2 \right\rfloor} z^{2d} \frac{1}{N^d} \sum_{\beta \vdash d} p_{\beta}(c_1, \ldots, c_N) \langle \Omega^{-1} \omega_\beta \rangle, \]

which are obtained from the string expansions of \( I_N \) and \( J_N \) as given by Theorem 1.2 by replacing the factor \( \mathbb{P}(\text{LIS}_d \leq N) \) with the above step function.
4.3. Feynman extension. The polynomial approximations \( \tilde{I}_N \) and \( \tilde{J}_N \) of \( I_N \) and \( J_N \) are only useful insofar as we are able to understand the Plancherel expectations

\[
\langle \omega_{\alpha} \Omega^{-1}_N \omega_{\beta} \rangle = \sum_{\lambda, \delta, \ell(\lambda) \leq N} \omega_{\alpha}(\lambda) \Omega^{-1}_N(\lambda) \omega_{\beta}(\lambda) \frac{(\dim V^\lambda)^2}{|S_N(d)|}
\]

in the range \( 1 \leq d \leq \frac{1}{4}N^2 \). This means that we must extend Theorem 3.1 which gives the convergent \( 1/N \) expansion

\[
\langle \omega_{\alpha} \Omega^{-1}_N \omega_{\beta} \rangle = \sum_{r=0}^{\infty} \left( -\frac{1}{N} \right)^r W^r(\alpha, \beta) = \frac{(-1)^{\ell(\alpha)+\ell(\beta)}}{N^{\ell(\alpha)+\ell(\beta)}} \sum_{g=-\infty}^{2g \leq \ell(\alpha)+\ell(\beta)} N^{2-2g} \tilde{H}^*_g(\alpha, \beta)
\]

in the linear range \( 1 \leq d \leq N \), to the range where \( d \) may be as large as \( \frac{1}{4}N^2 \). This may be done as follows.

For any \( d, N \in \mathbb{N} \) we may rewrite the expectation \( \langle \omega_{\alpha} \Omega^{-1}_N \omega_{\beta} \rangle \) as a conditional expectation against the unrestricted Plancherel measure: we have

\[
\langle \omega_{\alpha} \Omega^{-1}_N \omega_{\beta} \rangle = \frac{1}{P(\text{LIS}_d \leq N)} \sum_{\lambda, \delta, \ell(\lambda) \leq N} \omega_{\alpha}(\lambda) \Omega^{-1}_N(\lambda) \omega_{\beta}(\lambda) \frac{(\dim V^\lambda)^2}{d!}
\]

Let us split this conditional expectation into two pieces: we write

\[
\langle \omega_{\alpha} \Omega^{-1}_N \omega_{\beta} \rangle = \langle \omega_{\alpha} \Omega^{-1}_N \omega_{\beta} \rangle_1 + \langle \omega_{\alpha} \Omega^{-1}_N \omega_{\beta} \rangle_2,
\]

where

\[
\langle \omega_{\alpha} \Omega^{-1}_N \omega_{\beta} \rangle_1 = \frac{1}{P(\text{LIS}_d \leq N)} \sum_{\lambda, \delta, \ell(\lambda) \leq N} \omega_{\alpha}(\lambda) \Omega^{-1}_N(\lambda) \omega_{\beta}(\lambda) \frac{(\dim V^\lambda)^2}{d!}
\]

and

\[
\langle \omega_{\alpha} \Omega^{-1}_N \omega_{\beta} \rangle_2 = \frac{1}{P(\text{LIS}_d \leq N)} \sum_{\lambda, \delta, \ell(\lambda) \leq N} \omega_{\alpha}(\lambda) \Omega^{-1}_N(\lambda) \omega_{\beta}(\lambda) \frac{(\dim V^\lambda)^2}{d!}.
\]

In the range \( 1 \leq d \leq N \), the second component of this decomposition vanishes. In the extended range \( N < d \leq \frac{1}{4}N^2 \), when \( N \) is large, the first component of this decomposition is virtually equal to \( \langle \omega_{\alpha} \Omega^{-1}_N \omega_{\beta} \rangle_1 \), while the second is negligible. Indeed, it follows from the Vershik-Kerov limit shape theorem \([60, 79]\) that for \( N < d_N \leq \frac{1}{4}N^2 \), a Plancherel-random Young diagram with \( d_N \) cells is contained in the \( N \times N \) rectangular diagram \( R(N, N) \) with overwhelming probability.

Observe now that the massive component \( \langle \omega_{\alpha} \Omega N \omega_{\beta} \rangle_1 \) of \( \langle \omega_{\alpha} \Omega N \omega_{\beta} \rangle \) admits an absolutely convergent \( 1/N \) expansion. Indeed, for any \( \lambda \subseteq R(N, N) \), we have the absolutely convergent expansion

\[
\Omega^{-1}_N(\lambda) = \prod_{\square \in \lambda} \frac{1}{1 + \frac{\omega_{\alpha}(\square)}{N}} = \sum_{r=0}^{\infty} \left( -\frac{1}{N} \right)^r f_r(\lambda),
\]

so that
\[
\langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle_1 = \sum_{r=0}^{\infty} \left( -\frac{1}{N} \right)^r \tilde{W}_N^r(\alpha, \beta),
\]
where
\[
\tilde{W}_N^r(\alpha, \beta) = \sum_{\lambda \vdash d} \omega_\alpha(\lambda) f_r(\lambda) \omega_\beta(\lambda) \frac{(\dim \mathcal{V}^\lambda)^2}{d!}.
\]

agrees with \( \tilde{W}^r(\alpha, \beta) \) up to an exponentially small error. Thus, for any fixed but arbitrary \( s \in \mathbb{N} \cup \{0\} \), we can replace the first \( s \) coefficients of the massive component \( \langle \omega_\alpha \Omega_N \omega_\beta \rangle_1 \) with their \( N \)-independent counterparts up to an exponentially small error. Ignoring the negligible component \( \langle \omega_\alpha \Omega_N \omega_\beta \rangle_2 \), this gives the \( N \to \infty \) asymptotic approximation
\[
\langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle = \sum_{r=0}^{s} \left( -\frac{1}{N} \right)^r \tilde{W}^r(\alpha, \beta) + O \left( \frac{1}{N^{s+1}} \right),
\]
which extends Theorem 3.1 to the range \( 1 \leq d \leq \frac{1}{4} N^2 \). Note that this expansion implies the sharper estimate
\[
\langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle = \sum_{r=0}^{s} \left( -\frac{1}{N} \right)^r \tilde{W}^r(\alpha, \beta) + O \left( \frac{1}{N^{s+2}} \right),
\]
since the numbers \( \tilde{W}^r(\alpha, \beta) \) which are nonzero correspond to either \( r \) even, or \( r \) odd. In particular, for any \( k \in \mathbb{N} \cup \{0\} \), we have that
\[
\langle \omega_\alpha \Omega_N^{-1} \omega_\beta \rangle = \frac{(-1)^{\ell(\alpha)+\ell(\beta)}}{N^{\ell(\alpha)+\ell(\beta)}} \sum_{g=-\infty}^{k} N^{2-2g} \tilde{H}_g^*(\alpha, \beta) + O \left( N^{-2k} \right).
\]

4.4. **Topological bound.** The following topological bound bridges the gap between formal asymptotics and functional asymptotics. Conceptually, this result is the unstable analytic shadow of the stable topological bounds appearing in Corollary 3.5.

**Theorem 4.4.** There exists \( \gamma > 0 \) such that, for each fixed \( k \in \mathbb{N} \cup \{0\} \), we have
\[
\left\| 1 - \frac{I_N}{e^{\sum_{g=0}^{k} N^{2-2g} F_N^{(g)}}} \right\|_{\gamma} = O(N^{2-2k}) \quad \text{and} \quad \left\| 1 - \frac{J_N}{e^{\sum_{g=0}^{k} N^{2-2g} G_N^{(g)}}} \right\| = O(N^{2-2k})
\]
as \( N \to \infty \).

**Proof.** We give the proof for the HCIZ integral; the argument for the BGW integral is essentially the same.

With \( \delta \) as in Theorem 4.2, take \( \gamma \leq \delta \) sufficiently small so that \( I_N \) can be replaced with \( \tilde{I}_N \) as \( N \to \infty \). Replacing the coefficients of \( \tilde{I}_N \) with their asymptotic expansions to order \( k + 1 \) and applying Theorem 3.4, we obtain
\[ 1 - \frac{I_N}{e^{2k - 2g F_N^{(g)}}} = \sum_{d=1}^{\lfloor \frac{1}{4} N^2 \rfloor} \sum_{\alpha, \beta} \frac{p_\alpha(a_1, \ldots, a_N) p_\beta(b_1, \ldots, b_N)}{N^{\ell(\alpha)}} \left( N^{-2k} \tilde{H}_{k+1}^*(\alpha, \beta) + O(N^{-2k-2}) \right) \\
+ O(z^{\lfloor \frac{1}{4} N^2 \rfloor + 1}). \]

On \( \mathbb{D}_N^\gamma \), we have the estimate

\[ \left| \sum_{d=1}^{\lfloor \frac{1}{4} N^2 \rfloor} \frac{\zeta^d}{d!} \sum_{\alpha, \beta} \frac{p_\alpha(a_1, \ldots, a_N) p_\beta(b_1, \ldots, b_N)}{N^{\ell(\alpha)}} \left( N^{-2k} \tilde{H}_{k+1}^*(\alpha, \beta) + O(N^{-2k-2}) \right) \right| \leq N^{-2k} \sum_{d=1}^{\lfloor \frac{1}{4} N^2 \rfloor} \frac{z^d}{d!} \sum_{\alpha, \beta} \left( \tilde{H}_{k+1}^*(\alpha, \beta) + O(N^{-2}) \right) \]

\[ = O(N^{-2k}). \]

□

Corollary 4.5. For \( N \) sufficiently large, the integrals \( I_N \) and \( J_N \) are non-vanishing on \( \mathbb{D}_N^\gamma \). In particular, for \( N \) sufficiently large, \( \log I_N \) and \( \log J_N \) are defined and analytic on \( \mathbb{D}_N^\gamma \).

Proof. This follows from the \( k = 2 \) case of Theorem 4.4, which implies that

\[ \left| 1 - \frac{I_N}{e^{N^2 F_N^{(0)} + F_N^{(1)} + N^{-2} F_N^{(2)}}} \right| < 1 \quad \text{and} \quad \left| 1 - \frac{J_N}{e^{N^2 G_N^{(0)} + G_N^{(1)} + N^{-2} G_N^{(2)}}} \right| < 1 \]

on \( \mathbb{D}_N^\gamma \) for \( N \) sufficiently large. These inequalities in turn imply the non-vanishing of

\[ \frac{I_N}{e^{N^2 F_N^{(0)} + F_N^{(1)} + N^{-2} F_N^{(2)}}} \quad \text{and} \quad \frac{J_N}{e^{N^2 G_N^{(0)} + G_N^{(1)} + N^{-2} G_N^{(2)}}} \]

on \( \mathbb{D}_N^\gamma \), from which we conclude the nonvanishing of \( I_N \) and \( J_N \) on this polydisc. □

4.5. Analytic error functions. Set \( \xi = \min(\gamma, \delta) \), where \( \gamma \) is the positive constant in Theorem 4.4 and \( \delta \) is the positive constant in Theorem 4.2. We may define an array of analytic functions on \( \mathbb{D}_\xi^N \) by

\[ \Delta_N^{(0)} = N^{-2} \log I_N - F_N^{(0)} \]
\[ \Delta_N^{(k)} = N^2 \Delta_N^{(k-1)} - F_N^{(k)}, \quad k \in \mathbb{N} \]

Explicitly, we have

\[ \Delta_N^{(k)} = N^{2k-2} \left( \log I_N - \sum_{g=0}^k N^{2-2g} F_N^{(g)} \right), \quad k \in \mathbb{N} \cup \{0\}. \]
We could also have defined $\Delta_N^{(k)}$ using $J_N$ in place of $I_N$, and $G_N^{(g)}$ in place of $F_N^{(g)}$, and in what follows $\Delta_N^{(k)}$ may equally well be replaced with this function instead. Our main result, Theorem 1.4, is an immediate consequence of the following convergence theorem, the proof of which occupies the remainder of the paper.

**Theorem 4.6.** For any $0 < \varepsilon < \xi$ we have $\lim_{N \to \infty} \|\Delta_N^{(k)}\|_{\varepsilon} = 0$ for each $k \in \mathbb{N}$.

4.6. **Reduction to uniform boundedness.** By virtue of its definition, the function $\Delta_N^{(k)}$ admits the string expansion

$$\Delta_N^{(k)}(\alpha, \beta) = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} \frac{p_\alpha(a_1, \ldots, a_N) p_\beta(b_1, \ldots, b_N)}{N^{\ell(\alpha)} N^{\ell(\beta)}} \Delta_N^{(k)}(\alpha, \beta),$$

the coefficients of which are given by

$$\Delta_N^{(k)}(\alpha, \beta) = N^{2k-2} \left( L_N(\alpha, \beta) - \sum_{g=0}^{k} \frac{H_g(\alpha, \beta)}{N^{2g}} \right),$$

where

$$\log I_N = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} \frac{p_\alpha(a_1, \ldots, a_N) p_\beta(b_1, \ldots, b_N)}{N^{\ell(\alpha)} N^{\ell(\beta)}} L_N(\alpha, \beta),$$

and both series converge absolutely on $D^N_{\varepsilon}$. From Section 3 we know that, for each $k \in \mathbb{N} \cup \{0\}$, each fixed string coefficient of $\Delta_N^{(k)}$ converges to zero as $N \to \infty$,

$$\lim_{N \to \infty} \Delta_N^{(k)}(\alpha, \beta) = 0.$$

In fact, asymptotic vanishing of the string coefficients of $\Delta_N^{(k)}$ implies uniform asymptotic vanishing of string series provided we have uniform boundedness.

Let $m \in \mathbb{N}$ be an arbitrary positive integer. By a “normalized string series” on $D^N_{\varepsilon} \times \mathbb{D}^m_1$, we mean a power series of the form

$$\Delta_N = \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha_1, \ldots, \alpha_m \vdash d} \Delta_N(\alpha_1, \ldots, \alpha_m)$$

which converges absolutely on $D^N_{\varepsilon}$. In order to prove Theorem 4.6, we will use the fact that, in the presence of uniform boundedness, uniform convergence of $\Delta_N$ on any closed proper subset of $D^N_{\varepsilon}$ follows from the convergence of each of its string coefficients $\Delta_N(\alpha_1, \ldots, \alpha_m)$.

**Lemma 4.7.** If $\sup_{N \in \mathbb{N}} \|\Delta_N\|_{\Xi} < \infty$ and

$$\lim_{N \to \infty} \Delta_N(\alpha_1, \ldots, \alpha_m) = 0$$

for any $d \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \vdash d$, then

$$\lim_{N \to \infty} \|\Delta_N\|_\varepsilon = 0$$

for any $0 < \varepsilon < \Xi$. 

28 JONATHAN NOVAK
Proof. Fix $\varepsilon < \xi$. Let $\kappa > 0$ be given. For any $n, N \in \mathbb{N}$, we have

$$
\|\Delta_N\|_\varepsilon \leq \sum_{d=1}^{n} \frac{\varepsilon^d}{d!} \sum_{\alpha^1, \ldots, \alpha^m \vdash d} |\Delta_N(\alpha^1, \ldots, \alpha^m)|
$$

$$
+ \sum_{d=n+1}^{\infty} \frac{\varepsilon^d}{d!} \sum_{\alpha^1, \ldots, \alpha^m \vdash d} |\Delta_N(\alpha^1, \ldots, \alpha^m) \prod_{i=1}^{m} p_{\alpha^i}(a_1, \ldots, a_N) N^{\ell(\alpha^i)}|,
$$

by the triangle inequality. By Cauchy’s estimate,

$$
\frac{1}{d!} \sum_{\alpha^1, \ldots, \alpha^m \vdash d} |\Delta_N(\alpha^1, \ldots, \alpha^m) \prod_{i=1}^{m} p_{\alpha^i}(a_1, \ldots, a_N) N^{\ell(\alpha^i)}| \leq \frac{\|\Delta_N\|_\xi}{\xi^d}.
$$

Thus

$$
\|\Delta_N\|_\varepsilon \leq \sum_{d=1}^{n} \frac{\varepsilon^d}{d!} \sum_{\alpha^1, \ldots, \alpha^m \vdash d} |\Delta_N(\alpha^1, \ldots, \alpha^m)| + \left(\frac{\varepsilon}{\xi}\right)^n K,
$$

where

$$
K = \sup_{N \in \mathbb{N}} \frac{\|\Delta_N\|_\xi}{1 - \frac{\xi}{\varepsilon}}
$$

is a constant. Since

$$
\lim_{n \to \infty} \left(\frac{\varepsilon}{\xi}\right)^n = 0,
$$

we can choose $n_0$ sufficiently large so that

$$
\left(\frac{\varepsilon}{\xi}\right)^n K < \frac{\kappa}{2}.
$$

Then, since

$$
\lim_{N \to \infty} |\Delta_N(\alpha^1, \ldots, \alpha^m)| = 0
$$

for each $d \in \mathbb{N}$ and all $\alpha^1, \ldots, \alpha^m \vdash d$, we can choose $N_0$ sufficiently large so that $N \geq N_0$ implies

$$
\sum_{d=1}^{n_0} \frac{\varepsilon^d}{d!} \sum_{\alpha^1, \ldots, \alpha^m \vdash d} |\Delta_N(\alpha^1, \ldots, \alpha^m)| < \frac{\kappa}{2}.
$$

We conclude that $N \geq N_0$ implies

$$
\|\Delta_N\|_\varepsilon < \kappa,
$$

as required. \qed
4.7. Proof of uniform boundedness. In view of Lemma 4.7, the following result completes the proof of Theorem 4.6 and hence also of Theorem 1.4, which proves Conjecture 1.1.

**Theorem 4.8.** For any \( \varepsilon < \xi \), we have

\[
\sup_{N \in \mathbb{N}} \| \Delta_N^{(k)} \|_\varepsilon < \infty
\]

for each \( k \in \mathbb{N} \cup \{0\} \).

*Proof.* Let \( (\varepsilon_k)_{k=0}^\infty \) be a strictly decreasing sequence in the interval \((\varepsilon, \xi)\). Then, for each \( N \in \mathbb{N} \), we have a corresponding sequence of nested closed polydiscs,

\[
\mathbb{B}_{\varepsilon}^N \supset \mathbb{B}_{\varepsilon_0}^N \supset \mathbb{B}_{\varepsilon_1}^N \supset \cdots \supset \mathbb{B}_{\varepsilon}^N.
\]

Let \( k \in \mathbb{N} \cup \{0\} \) be fixed. Observe that

\[
\frac{I_N}{e^{\sum_{n=0}^{N} N^2-2g N^2}} = e^{N^2-2k} \Delta_N^{(k)}.
\]

Thus, by Theorem 4.4, we have

\[
\left| 1 - e^{N^2-2k} \Delta_N^{(k)} \right| \leq c_k N^{2-2k}
\]

on \( \mathbb{D}_{\varepsilon_k}^N \) for \( N \) sufficiently large, where \( c_k \) is a positive constant depending only on \( k \). This in turn implies

\[
\left| e^{N^2-2k} \Delta_N^{(k)} \right| \leq 1 + c_k N^{2-2k}
\]

on \( \mathbb{D}_{\varepsilon_k}^N \) for \( N \) sufficiently large. Thus

\[
N^{2-2k} \Re \Delta_N^{(k)} \leq \log \left( 1 + c_k N^{2-2k} \right).
\]

on \( \mathbb{D}_{\varepsilon_k}^N \) for \( N \) sufficiently large. If \( k \neq 1 \), this yields

\[
N^{2-2k} \Re \Delta_N^{(k)} \leq \log \left( 1 + c_k N^{2-2k} \right) \leq c_k N^{2-2k}
\]

on \( \mathbb{D}_{\varepsilon_k}^N \) for \( N \) sufficiently large, so that

\[
\Re \Delta_N^{(k)} \leq c_k
\]

on \( \mathbb{D}_{\varepsilon_k}^N \) for \( N \) sufficiently large. If \( k = 1 \) we obtain instead

\[
\Re \Delta_N^{(k)} \leq \log (1 + c_k)
\]

on \( \mathbb{D}_{\varepsilon_k}^N \) for \( N \) sufficiently large. Thus in all cases we have a uniform bound on the real part of \( \Delta_N^{(k)} \), i.e. a bound of the form

\[
\Re \Delta_N^{(k)} \leq \hat{c}_k
\]

for some positive constant \( \hat{c}_k \). In order to leverage this into a bound on the modulus, we apply the Borel-Carathéodory inequality (see e.g. [83]), which bounds the sup norm of an analytic function on a closed disc in terms of the supremum of its real
part on a larger closed disc. Applying the Borel-Carathéodory inequality, we obtain the bound
\[ \| \Delta_N^{(k)} \|_{\varepsilon_{k+1}} \leq \frac{2\varepsilon_{k+1}}{\varepsilon_k - \varepsilon_{k+1}} \sup_{D_{\varepsilon_k}} \text{Re} \Delta_N^{(k)} \leq \frac{2\varepsilon_{k+1}}{\varepsilon_k - \varepsilon_{k+1}} \tilde{c}_k. \]
Since \( \varepsilon < \varepsilon_k \), we have
\[ \| \Delta_N^{(k)} \|_{\varepsilon} \leq \| \Delta_N^{(k)} \|_{\varepsilon_{k+1}} \leq \frac{2\varepsilon_{k+1}}{\varepsilon_k - \varepsilon_{k+1}} \tilde{c}_k, \]
as required.

\[ \square \]

References

1. G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. B 72 (1974), 461-473.
2. D. Aldous, P. Diaconis, Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem, Bull. Amer. Math. Soc. 36 (1999), 413-432.
3. A. Alexandrov, G. Chapuy, B. Eynard, J. Harnad, Fermionic approach to weighted Hurwitz numbers and topological recursion, Commun. Math. Phys. 360 (2018), 777-826.
4. A. Alexandrov, D. Lewanski, S. Shadrin, Ramifications of Hurwitz theory, KP integrability and quantum curves, J. High Energy Phys. 5 (2016), 1-30.
5. J. Baez, W. Taylor, Strings and two-dimensional QCD for finite \( N \), Nucl. Phys. B 246—(1994), 53-70.
6. I. Bars, U(\( N \)) integral for the generating functional in lattice gauge theory, J. Math. Phys. 21 (1980), 2678-2881.
7. I. Bars, F. Green, Complete integration of U(\( N \)) lattice gauge theories in a large-\( N \) limit, Phys. Rev. D 20 (1979), 3311-3330.
8. J. Baik, P. Deift, K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, J. Amer. Math. Soc. 12 (1999), 1119-1178.
9. J. Baik, E. Rains, Algebraic aspects of increasing subsequences, Duke Math. J. 109 (2001), 1-65.
10. G. Berkolaiko, J. Kuipers, Combinatorial theory of the semiclassical evolution of transport moments I: equivalence with the random matrix approach, J. Math. Phys. 54 (2013), 112103.
11. D. Bessis, C. Itzykson, J. B. Zuber, Quantum field theory techniques in graphical enumeration, Adv. Appl. Math. 1 (1980), 109-157.
12. S. Belinschi, A. Guionnet, J. Huang, Large deviation principles via spherical integrals, arXiv:2004.07117v1 [math.PR] 15 April 2020.
13. G. Borot, B. Eynard, M. Mulase, B. Safnuk, A matrix model for simple Hurwitz numbers, and topological recursion, J. Geom. Phys. 61 (2011), 522-540.
14. E. Brézin, C. Itzykson, G. Parisi, J. B. Zuber, Planar diagrams, Commun. Math. Phys. 59 (1978), 35-51.
15. G. Borot, N. Do, M. Karev, D. Lewanski, E. Moskovsky, Double Hurwitz numbers: polynomiality, topological recursion, and intersection theory, arXiv:2002.00900v1, Jan. 31 2020.
16. R. Bott, The stable homotopy of the classical groups, Ann. Math. 70 (1959), 313-337.
17. E. Brézin, D. Gross, The external field problem in the large \( N \) limit of QCD, Phys. Lett. 97 (1980), 120-124.
18. J. Bun, J. P. Bouchaud, S. Majumdar, M. Potters, Instanton approach to large \( N \) Harish-Chandra-Itzykson-Zuber integrals, Phys. Rev. Lett. 113 (2014), 070201.
19. G. Chapuy, M. Dolega, Non-orientable branched coverings, b-Hurwitz numbers, and positivity for multiparametric Jack expansions, arXiv preprint.
20. B. Collins, Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability, Int. Math. Res. Not. IMRN 17 (2003), 953-982.
21. B. Collins, A. Guionnet, E. Maurel-Segala, Asymptotics of unitary and orthogonal matrix integrals, Adv. Math. 222 (2009), 172-215.
22. B. Collins, S. Matsumoto, J. Novak, An Invitation to the Weingarten Calculus, book in preparation.
23. S. Cordes, G. Moore, S. Ramgoolam, Large N 2D Yang-Mills theory and topological string theory, Comm. Math. Phys. 185 (1997), 543-619.
24. F. Cunden, A. Dahlqvist, N. O’Connell, Integer moments of complex Wishart matrices and Hurwitz numbers.
25. P. Diaconis, C. Greene, Applications of Murphy’s elements, Technical Report No. 335 (1989), Department of Statistics, Stanford University.
26. P. Di Francesco, P. Ginsparg, J. Zinn-Justin, 2D gravity and random matrices, Phy. Reports 254 (1995), 1-133.
27. N. Do, A. Dyer, D. V. Mathews, Topological recursion and a quantum curve for monotone Hurwitz numbers, J. Geom. Phys. 120 (2017), 19-36.
28. N. Do, M. Karev, Monotone orbifold Hurwitz numbers, J. Math. Sci. 226 (2017), 568-587.
29. B. Dubrovin, D. Yang, D. Zagier, Classical Hurwitz numbers and related combinatorics, Moscow Math. J. 17 (2017), 601-633.
30. P. Dunin-Barkowski, R. Kramer, A. Popolitov, S. Shadrin, Cut-and-join equation for monotone Hurwitz numbers revisited, J. Geom. Phys. 137 (2019), 1-6.
31. T. Ekedahl, S. K. Lando, M. Shapiro, A. Vainshtein, Hurwitz numbers and intersections on moduli spaces of curves, Invent. Math. 146 (2001), 297-327.
32. D. Eisenbud, N. Elkies, J. Harris, R. Speiser, On the Hurwitz scheme and its monodromy, Compositio Math. 77 (1991), 95-117.
33. N. M. Ercolani, K. D. T.-R. Mclaughlin, Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration, Int. Math. Res. Not. 14 (2003), 755-820.
34. B. Eynard, N. Orantin, Invariants of algebraic curves and topological expansion, Communications in Num. Theory and Phys. 1 (2007), 347-452.
35. I. M. Gessel, Symmetric functions and P-recursiveness, J. Combin. Theory Ser. A 53 (1990), 257-285.
36. M. Gissoni, T. Grava, G. Ruzza, Laguerre ensemble: correlators, Hurwitz numbers, and Hodge integrals, arXiv:1912.00525v2, 2020.
37. I. P. Goulden, M. Guay-Paquet, J. Novak, Monotone Hurwitz numbers in genus zero, Canad. J. Math. 65 (2013), 1020-1042.
38. I. P. Goulden, M. Guay-Paquet, J. Novak, Monotone Hurwitz numbers in higher genera, Adv. Math. 238 (2013), 1-23.
39. I. P. Goulden, M. Guay-Paquet, J. Novak, Monotone Hurwitz numbers and the HCIZ integral, Ann. Math. Blaise Pascal 21 (2014), 71-99.
40. I. P. Goulden, M. Guay-Paquet, J. Novak, Toda equations and piecewise polynomiality for mixed Hurwitz numbers, SIGMA Symmetry Integrability Geom. Methods Appl. 12 (2016), 40-50.
41. I. P. Goulden, M. Guay-Paquet, J. Novak, On the convergence of monotone Hurwitz generating functions, Ann. Comb. 21 (2017), 73-81.
42. I. P. Goulden, D. M. Jackson, R. Vakil, Towards the geometry of double Hurwitz numbers, Adv. Math. 198 (2005), 43-92.
43. D. Gross, Two-dimensional QCD as a string theory, Nucl. Phy. B 400 (1993), 161-180.
44. D. Gross, W. Taylor IV, Two-dimensional QCD is a string theory, Nucl. Phys. B 400 (1993), 181-208.
45. D. Gross, W. Taylor IV, Twists and Wilson loops in the string theory of two-dimensional QCD, Nucl. Phys. B 403 (1993), 395-449.
46. D. Gross, E. Witten, Possible third-order phase transition in the large N lattice gauge theory, Phy. Rev. D 2 (1980), 446-453.
47. A. Guionnet, Large deviations and stochastic calculus for large random matrices, Prob. Surveys 1 (2004), 72-172.
48. A. Guionnet, Asymptotics of Random Matrices and Related Models: The Uses of Dyson-Schwinger Equations, AMS Regional Conference Series in Mathematics 130, 2019.
49. A. Guionnet, M. Maida, A Fourier view on the R-transform and related asymptotics of spherical integrals, J. Funct. Anal. 222 (2005), 435-490.
50. A. Guionnet, J. Novak, Asymptotics of unitary multimatrix models: Schwinger-Dyson equations and topological recursion, J. Funct. Anal. 268 (2015), 2851-2905.
51. A. Guionnet, O. Zeitouni, Large deviations asymptotics for spherical integrals, J. Funct. Anal. 188 (2002), 461-515.
52. M. Hahn, R. Kramer, D. Lewanski, Wall-crossing formulae and strong piecewise polynomiality for mixed Grothendieck dessins d’enfant, monotone, and double simple Hurwitz numbers, Adv. Math. 336 (2018), 38-69.
53. Harish-Chandra, Differential operators on a semisimple Lie algebra, Amer. J. Math. 79 (1957), 87-120.
54. A. Hurwitz, Über die Anzahl der Riemann’schen Flächen mit gegebenen Verzweigungspunkten, Math. Ann. 55 (1902), 53-66.
55. C. Itzykson, J.-B. Zuber, The planar approximation. II, J. Math. Phys. 21 (1980), 411-421.
56. A. T. James, Distributions of matrix variates and latent roots derived from normal samples, Ann. Math. Statist. 2 (1942), 475-501.
57. K. Johansson, The longest increasing subsequence in a random permutation and a unitary random matrix model, Math. Res. Lett. 5 (1998), 63-82.
58. D. Kane, T. Tao, A bound on partitioning clusters, Electron. J. Combinatorics 24(2) (2017), #P2.31.
59. M. E. Kazarian, S. K. Lando, An algebro-geometric proof of Witten’s conjecture, J. Amer. Math. Soc. 20 (2007), 1079-1089.
60. S. V. Kerov, Asymptotic Representation Theory of the Symmetric Group and its Applications in Analysis, AMS Translations of Mathematical Monographs, Volume 219, 2003. Translated from the Russian by N. V. Tsilevich.
61. A. Kirillov, Lectures on the Orbit Method, AMS Graduate Studies in Mathematics 64, 2004.
62. M. Kontsevich, Y. Soibelman, Airy structures and symplectic geometry of topological recursion, arXiv:1701.09137v2 [math.AG] 9 Mar 2017.
63. I. G. Macdonald, Symmetric Functions and Hall Polynomials. Second Edition, Oxford Science Publications, 1995.
64. S. Matsumoto, J. Novak, Jucys-Murphy elements and unitary matrix integrals, Int. Math. Res. Not. IMRN 2 (2013), 362-397.
65. A. Matytsin, On the large $N$ limit of the Itzykson-Zuber integral, Nucl. Phys. B 411 (1994), 805-820.
66. E. Maurel-Segala, High order expansion of matrix models and enumeration of maps, arXiv:math/0608192v1 2006.
67. A. M. Montanaro, Weak multiplicativity for random quantum channels, Commun. Math. Phys. 319 (2013), 535-555.
68. A. Morozov, Unitary integrals and related matrix models, Oxford Handbook of Random Matrix Theory.
69. J. Novak, Truncations of random unitary matrices and Young tableaux, Electron. J. Combin. 14 (2007), #R21.
70. J. Novak, Jucys-Murphy elements and the Weingarten function, Banach Cent. Publ. 89 (2010), 231-235.
71. J. Novak, Vicious walkers and random contraction matrices, Int. Math. Res. Not. IMRN 17 (2009), 3310-3327.
72. J. Novak, Lozenge tilings and Hurwitz numbers, J. Stat. Phys. 161 (2015), 509-517.
73. A. Okounkov, Toda equations for Hurwitz numbers, Math. Res. Lett. 7 (2000), 447-453.
74. A. Okounkov, R. Pandharipande, Gromov-Witten theory, Hurwitz theory, and completed cycles, Ann. Math. 163 (2006), 517-560.
75. A. Okounkov, A. Vershik, A new approach to representation theory of symmetric groups, Selecta Math. 2 (1996), 581-605.
76. G. Olshanski, A. Vershik, Ergodic unitarily invariant measures on the space of infinite Hermitian matrices, Amer. Math. Transl. Ser. 2 175 (1996), 137-175.
77. L. Pyber, Enumerating finite groups of given order, Ann. Math. 137 (1993), 203-220.
78. E. M. Rains, Increasing subsequences and the classical groups, 5 (1998), #R12.
79. D. Romik, The Surprising Mathematics of Longest Increasing Subsequences, Cambridge University Press, 2015.
80. S. Samuel, $U(N)$ integrals, $1/N$, and the De Wit - ’t Hooft anomalies, J. Math. Phys. 21 (1980), 2695-2703.
81. R. P. Stanley, Enumerative Combinatorics. Vol. 2. Cambridge University Press, New York, 1999.
82. R. P. Stanley, *Increasing and decreasing subsequences and their variants*, Proceedings of the ICM 2006, Volume I, 545-579.
83. E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1938.
84. R. Vakil, *Genus 0 and 1 Hurwitz numbers: recursions, formulas, and graph-theoretic interpretations*, Trans. Amer. Math. Soc. 353 (2001), 4025-4038.
85. S. R. Wadia, *N = ∞ phase transition in a class of exactly soluble model lattice gauge theories*, Phys. Lett. B 93 (1980), 403-410.
86. E. Witten, *Two-dimensional quantum gravity and intersection theory on moduli space*, Surv. Diff. Geo. 1 (1991), 243-310.
87. S. Zelditch, *Macdonald’s identities and the large N limit of YM_2 on the cylinder*, Commun. Math. Phys. 245 (2004), 611-626.
88. J.-B. Zuber, P. Zinn-Justin, *On some integrals over the U(N) unitary group and their large N limits*, J. Phys. A: Math. Gen. 36 (2003), 3173-3193.

Department of Mathematics, University of California, San Diego, USA
E-mail address: jinovak@uucsd.edu