THE GÁLVEZ–KOCK–TONKS CONJECTURE FOR LOCALY DISCRETE DECOMPOSITION SPACES

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Abstract

Gálvez-Carrillo, Kock, and Tonks [15] constructed a decomposition space $U$ of all Möbius intervals, as a recipient of Lawvere’s interval construction for Möbius categories, and conjectured that $U$ enjoys a certain universal property: for every Möbius decomposition space $X$, the space of culf functors from $X$ to $U$ is contractible. In this paper, we work at the level of homotopy 1-types to prove the first case of the conjecture, namely for locally discrete decomposition spaces. This provides also the first substantial evidence for the general conjecture.

This case is general enough to cover all locally finite posets, Cartier–Foata monoids, Möbius categories and strict (directed) restriction species. The proof is 2-categorical. First, we construct a local strict model of $U$, which is then used to show by hand that the Lawvere interval construction, considered as a natural transformation, does not admit other self-modifications than the identity.

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Introduction

Incidence algebras and Möbius inversion form a cornerstone of combinatorics. It has important applications in many areas of mathematics. Beyond the original applications in number theory (see Hardy and Wright [19]) and group theory ([36] and [18]), one can cite applications in probability theory [32] and algebraic topology [20], and it is also closely related to Hopf-algebraic renormalisation [25] in quantum field theory.

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Since Rota formalised the theory [33], [22] (on the grounds of previous contributions [36], [18]) the standard framework for the theory has been that of posets, but the theory has also been used in the context of monoids [7], and in the more general framework of certain categories called Möbius categories, introduced by Leroux [30]. The uniform appearance of the Möbius inversion formula across all application areas led Lawvere [28] in the 1980s to discover that there is a universal Möbius function which induces all other Möbius functions. It is an ‘arithmetic function’ on a certain Hopf algebra of Möbius intervals. A category is an interval if it has an initial and a terminal object [27], and the Möbius condition is a certain finiteness condition. This Hopf algebra has the property that it receives a canonical coalgebra homomorphism from every incidence coalgebra of a Möbius category. This includes all locally finite posets and all the monoids considered in [7]. Lawvere’s work remained unpublished for some decades, but it is cited in influential texts from that time, such as Joyal [24] and Joni–Rota [22]. Independently, Ehrenborg [10] constructed a closely related Hopf algebra, but less universal. It only accounts for intervals in posets. In both cases, the universal object can be interpreted as the colimit of all incidence coalgebras of intervals. The possibility of this is closely related to the local nature of coalgebras, expressed for example in the well-known fact that every coalgebra is the colimit of its finite-dimensional subcoalgebras, see Sweedler [35].

Lawvere’s discovery did not appear in print until Lawvere–Menni [29] in 2010. In that work the authors took an important step towards explaining the universal property by lifting the construction of the Hopf algebra of Möbius intervals to the objective level. This means that its comultiplication is realised as something called a pro-comonoidal structure on certain extensive categories. The original Hopf algebra is exhibited as being only a numerical shadow of this categorical construction. There are at least two precursors to the idea of a more objective approach to incidence algebras. One is given by Joyal [24]. In his foundational paper on species, there is a final section where he considers certain decomposition structures on categories (that final section has little to do with species). Another is in the work of Dür [8] who constructed incidence coalgebras of certain categorical and simplicial structures.

However,

many coalgebras, bialgebras and Hopf algebras in combinatorics are not of incidence type,

meaning that they cannot arise directly as the incidence coalgebra of any Möbius category. In fact the Lawvere–Menni Hopf algebra is not of incidence type. This gives the somewhat unsatisfactory situation that the universal object is not of the same type as the objects it is universal for.

A solution to this problem was found by Gálvez, Kock, and Tonks [13, 14, 15]. They discovered that the incidence coalgebra construction and Möbius inversion make sense for objects more general than Möbius categories (recall that Möbius categories include locally finite posets and Cartier–Foata monoids). These are completely new objects in this context which they call decomposition spaces. They are certain simplicial objects subject to an axiom that expresses decomposition, in the same way as the Segal condition (which characterises categories among simplicial sets) expresses composition. Decomposition spaces are the same thing as the 2-Segal spaces of Dyckerhoff and Kapranov [9] (see [11] for the last piece of this equivalence). It seems likely that all combinatorial coalgebras, bialgebras and Hopf algebras (with nonnegative section coefficients) arise from the incidence coalgebra construction of decomposition spaces. This has been shown for most of Schmitt’s examples [34] (restriction species in Gálvez–Kock–Tonks [12] and hereditary species in Carlier [5]). Gálvez–Kock–Tonks (as also Dyckerhoff–Kapranov) work in the fully homotopical setting of simplicial
∞-groupoids, but already the discrete case of the notion is very rich, as exemplified by work of Bergner et al. [2] and Kock–Spivak [26], who relate the notion to constructions in algebraic topology and category theory.

Gálvez, Kock and Tonks [15] showed that the Lawvere–Menni Hopf algebra is the incidence coalgebra of a decomposition space $U$. With this discovery the universal property could be stated, showing its nature as a moduli space: For any decomposition space $X$ the mapping space $\text{map}(X, U)$ is contractible. This statement is the Gálvez–Kock–Tonks conjecture, which is the objective of the present paper. The mapping space is the space of culf maps, as detailed further below. Culf maps were identified to play a key role already in the work of Lawvere and Menni [29].

Lawvere’s original work (suitably upgraded to the new context) shows that $\text{map}(X, U)$ is not empty: it contains $I: X \to U$, which is essentially Lawvere’s interval construction. Gálvez, Kock and Tonks [15] were able to establish one further ingredient of the conjecture, namely that $\text{map}(X, U)$ is connected, meaning that every map is homotopy equivalent to $I$.

The finer property of being contractible is the full homotopy uniqueness statement, that not only is every map equivalent to $I$: it is so uniquely (in a coherent homotopy sense).

The homotopy content was one of the reasons for Gálvez, Kock and Tonks to develop the whole theory in a homotopy setting: decomposition spaces are defined to be certain simplicial ∞-groupoids, and everything is fully homotopy invariant. It is an important insight of higher category theory (see for example Lurie [31]) that a universal object cannot exist in any truncated situation. Most famous is the fact that the topos of sets (0-types) contains a classifier for monomorphisms ((−1)-types) but cannot contain a classifier for sets (0-types), and that for these to be classified one needs the 2-topos of groupoids (1-types), and to classify 1-types one needs to 3-topos of 2-types, and so on. Only in the limit is it possible to find a classifier for general homotopy types (∞-groupoids) in the ∞-topos of ∞-groupoids.

At the moment, the technical difficulties of the general Gálvez–Kock–Tonks conjecture are too big.

In the present paper, the first case of the conjecture is proved. Working at the level of 1-types, we define the simplicial groupoid $U$ of discrete intervals (i.e. intervals that are simplicial sets rather than simplicial spaces), and show that:

Theorem. (Theorem 6.7.) $\text{map}(X, U)$ is a contractible 1-groupoid for every 1-truncated locally discrete decomposition space $X$.

This is the first substantial evidence for the full conjecture. According to the discussion above, the expected generality for $X$ is that of discrete decomposition spaces, but in fact (as kindly pointed out by the anonymous referee) the proofs work the same for a broader class of decomposition spaces, namely those 1-truncated decomposition spaces with the property that all their intervals are discrete. Clearly, discrete decomposition spaces have this property, so the level of generality already covers all the classical theory of incidence algebras and Möbius inversion in combinatorics, since locally finite posets, Cartier–Foata monoids, Möbius categories, and Schmitt’s examples are all 0-truncated simplicial spaces. In particular it gives finally a firm formalisation of Lawvere’s intuition that the interval construction should be universal in some sense. As a particular case it establishes also the universal property of the Ehrenborg Hopf algebra.

The idea of the proof is the following. The theorem, namely the contractibility of the 1-groupoids $\text{map}(X, U)$, is a 2-categorical statement. The proof we give is based on 2-category theory. However, a direct verification of the statement seems intractable, due to coherence problems. The difficulty is that $U: \Delta^{op} \to \text{Grpd}$ is only a pseudo-simplicial groupoid.
Jardine [21] has identified all the 2-cell structure and the 17 coherence conditions for pseudosimplicial groupoids. The definition of modification in this context requires compatibility with all that. The strategy to overcome this difficulty is to build a local strict model, a kind of neighbourhood $U_X \subset U$ around the intervals of a given locally discrete decomposition space $X$. The bulk of the paper is concerned with setting up this local model and show that it is strict. To construct this, we introduce a stricter algebraic notion of interval, where the initial and terminal objects are not just given as properties of a discrete decomposition space, but are carried around as data, in the notion of chosen initial and terminal objects. This focus is inspired by the work in another context of Batanin and Markl on operadic categories [1]. This is quite technical, but the benefit is to achieve a strict local model $U_X$ which is shown to be a strict simplicial groupoid and a complete decomposition groupoid, and to receive a strict version of the interval construction. With this strict local model in place, the local version of the contractibility of map($X, U_X$) can be established with 2-category theory by showing that $I: X \rightarrow U_X$, interpreted as a natural transformation, does not admit other self-modifications than the identity modification. In the end this check is not so difficult.

At this point is it natural to ask whether the techniques developed here can be applied or refined to prove the conjecture in full generality. Unfortunately this is not very likely, or it would require new conceptual simplifications and new technical tools. The point is that the proof relies on explicit strictification through strict models constructed through explicit data standing in for universal properties (initial and terminal objects, at the level of the objects involved, and strict simplicial objects at the higher level). The level of 2-categories is in practice the highest level where this kind of technique can be applied, and already at this level it is quite tricky to find the balance between properties and property-like structures. The next level, which would be the contractibility of the 2-groupoid map($X, U$) for $X$ a 1-truncated decomposition space, and $U$ the universal simplicial 2-groupoid of 1-intervals, seems out of reach. It seems more promising to pass directly to the homotopical setting of the full conjecture, aiming at using the theory of ($\infty, 2$)-categories. But it seems quite daunting to carry over the explicit strictification strategies to that setting.

In conclusion, the present contribution may be seen as only a small step towards the full conjecture, but it is nevertheless an important step (and actually the first step ever carried out), and enough to cover all the cases envisioned by Lawvere, which includes essentially all the examples from classical combinatorics. It is also already a striking example of how higher category theory (in this case 2-categories) serves to solve problems even in discrete mathematics.

**Organisation of the paper**

We begin in Section 1 with a brief review of basic notions and some results on homotopy pullbacks of groupoids. In 1.4, we recall from [13] some basic notions and results of the theory of decomposition groupoids. In 1.11 we review the notion of incidence coalgebra of a decomposition space. In 1.14, we briefly explain the cufc condition for a simplicial map. In 1.17, we review the notion of decalage.

In Section 2, we introduce some necessary material relating to the notion of slice and coslice of decomposition groupoids. Furthermore, we give the definition of interval (2.12).

In Section 3, we identify the level of generality. In principle what we need to impose is that all the intervals of $X$ are discrete, but for technical reasons we also impose some strictness. To be more precise we will work with strict simplicial groupoids such that all active-inert squares are strict pullbacks and such that $d_1$ is a discrete isofibration. (It follows
that all the strict pullbacks are also homotopy pullbacks.) For short we shall call such decomposition groupoids rigid (3.1). Furthermore, we explain the concept of chosen initial and chosen terminal object (3.5). Also, the notion of discrete algebraic interval (3.7) and some results for discrete algebraic intervals are given, in particular a lifting property (3.17 and 3.19).

In Section 4, we construct the stretched-culf factorisation system in the category of discrete algebraic intervals. Furthermore, we introduce important working tools (4.5 and 4.7) that will be useful in next sections.

In Section 5, we define the decomposition groupoid of all discrete algebraic intervals $U$ [15]. In 5.2, we construct a strict simplicial groupoid $U_X$ (5.7) that only contains the information about the discrete algebraic intervals of a fixed rigid decomposition groupoid $X$ and prove that $U_X$ is a complete decomposition groupoid (5.11 and 5.12). Furthermore, we define a simplicial map $I: X \to U_X$ and prove that $I$ is culf (5.14). In 5.20, we explain the interval construction of and interval. Furthermore, we compare $U_X$ with a strictification $\tilde{U}$ of $U$ suggested by the referee in 5.25.

In Section 6, we come to the Gálvez–Kock–Tonks conjecture formulated in [15], and we prove a partial result (6.1) about the connectedness of the mapping space $\mathrm{map}_{\mathrm{cDcmp}}(X, U)$ in the case of rigid decomposition groupoids. In 6.2, we use the concept of modification (6.4) to prove a truncated version of the conjecture, the case of rigid decomposition groupoids. We first show that $\mathrm{map}_{\mathrm{cDcmp}}(X, U_X)$ is contractible (6.6) and from this we deduce that the groupoid map $\mathrm{map}_{\mathrm{cDcmp}}(X, U)$ is contractible (6.7). This is the version of the Gálvez–Kock–Tonks conjecture that is the main theorem of this paper.

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1 Preliminaries

For the convenience of the reader, this section recalls a few background facts and establishes notation. These results are not new.

Homotopy pullbacks are important to the theory of decomposition spaces. They are examples of homotopy limits, and as such are defined only up to equivalence. The most used result for homotopy pullbacks is the prism lemma:

\textbf{Lemma 1.1.} Consider a diagram

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,0) node[midway, below] {$\downarrow$} -- (2,0) node[midway, below] {$\downarrow$} -- (3,0);
\draw (0,1) -- (1,1) node[midway, below] {$\downarrow$} -- (2,1) node[midway, below] {$\downarrow$} -- (3,1);\end{tikzpicture}
\end{center}

where the right square is a homotopy pullback. Then the left square is a homotopy pullback if and only if the outer diagram is a homotopy pullback.
A particular case of homotopy pullbacks is given by the homotopy fibres. Given a map of groupoids \( p: X \to S \) and an object \( s \in S \), the homotopy fibre \( X_s \) of \( p \) over \( s \) is the homotopy pullback

\[
\begin{array}{ccc}
X_s & \longrightarrow & X \\
\downarrow & & \downarrow^p \\
1 & \longrightarrow & S \\
\end{array}
\]

We use the following standard lemma many times.

**Lemma 1.2.** [6] A square of groupoids

\[
\begin{array}{ccc}
P & \xrightarrow{u} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & S \\
\end{array}
\]

is a homotopy pullback if and only if for each \( x \in X \) the induced comparison map \( u_x: P_x \to Y_{f(x)} \) is an equivalence.

Since homotopy pullback is defined up to equivalence, for some calculations it is important to work with a specific model. Let’s look at one of the models to be used: the homotopy fibre product of a pair of functors \( f: A \to C \) and \( g: B \to C \) between groupoids is the groupoid \( H \) whose objects are triples \((a, \theta, b)\) consisting of objects \( a \in A \), \( b \in B \) and an isomorphism \( \theta: f(a) \to g(b) \) in \( C \), and whose arrows \((\alpha, \beta): (a, \theta, b) \to (a', \theta', b')\) consist of arrows \( \alpha: a \to a' \in A \) and \( \beta: b \to b' \in B \) such that \( g(\beta) \circ \theta = \theta' \circ f(\alpha) \). The groupoid \( H \) fits into a homotopy commutative square

\[
\begin{array}{ccc}
H & \xrightarrow{\pi_A} & A \\
\pi_B & \downarrow & \downarrow f \\
B & \xrightarrow{g} & C \\
\end{array}
\]

where \( \pi_A: H \to A \) and \( \pi_B: H \to B \) are the canonical projections, and the components of the natural isomorphism is given by \( \theta \) itself. Note that the projections always are isofibrations [6].

Another model is possible when one of the two legs \( f \) and \( g \) is an isofibration. In that case, the strict pullback is also a homotopy pullback [23, Theorem 1].

We will use the following variation of the prism lemma in Section 2.

**Lemma 1.3.** [6] Consider a diagram

\[
\begin{array}{ccc}
\vdots & \longrightarrow & \vdots \\
\downarrow & & \downarrow \\
\vdots & \longrightarrow & \vdots \\
\end{array}
\]

where the right square is a homotopy fibre product. Then the left square is a strict pullback if and only if the outer diagram is a homotopy fibre product.

A map of groupoids \( f: X \to Y \) is a monomorphism when it is fully faithful. Equivalently, its homotopy fibres are \((-1)\)-groupoids, that is, are either empty or contractible.
1.4. Decomposition groupoids

This paper is concerned with a truncated case of the Gálvez–Kock–Tonks conjecture. For that we only have to deal with simplicial groupoids rather than simplicial spaces. For this reason, we prefer to use the word decomposition groupoid rather than decomposition space in all the paper.

The simplex category $\Delta$ is the category whose objects are the nonempty finite ordinals and whose morphisms are the monotone maps. These are generated by coface maps $d^i: [n-1] \to [n]$, which are the monotone injective functions for which $i \in [n]$ is not in the image, and codegeneracy maps $s^i: [n+1] \to [n]$, which are monotone surjective functions for which $i \in [n]$ has a double preimage. We write $d^0 := d^\perp$ and $d^n := d^\top$ for the outer coface maps.

An arrow of $\Delta$ is termed active, and written $g: [m] \to [n]$, if it preserves end-points, $g(0) = 0$ and $g(m) = n$. An arrow is termed inert, and written $f: [m] \to [n]$, if it is distance preserving, $f(i + 1) = f(i) + 1$ for $0 \leq i < m$.

Definition 1.5. [13, Definition 3.1] A decomposition groupoid is a simplicial groupoid

$$X: \Delta^{\text{op}} \to \text{Grpd}$$

such that the image of any pushout diagram in $\Delta$ of an active map $g$ along an inert map $f$ is a homotopy pullback of groupoids,

$$
\begin{array}{ccc}
X([p]) & \xrightarrow{g^*} & X([m]) \\
\downarrow{f^*} & & \downarrow{f^*} \\
X([q]) & \xrightarrow{g^*} & X([n])
\end{array}
\Rightarrow
\begin{array}{ccc}
X_p & \xrightarrow{g^*} & X_m \\
\downarrow{f^*} & & \downarrow{f^*} \\
X_q & \xrightarrow{g^*} & X_n.
\end{array}
$$

This is equivalent [13, Proposition 3.5] to requiring that the following diagrams are homotopy pullbacks for all $0 < i < n$:

$$
\begin{array}{ccc}
X_{n+1} & \xrightarrow{d_{i+1}} & X_n \\
\downarrow{d_\perp} & & \downarrow{d_\perp} \\
X_n & \xrightarrow{d_i} & X_{n-1}
\end{array}
\Rightarrow
\begin{array}{ccc}
X_{n+1} & \xrightarrow{d_i} & X_n \\
\downarrow{d_\top} & & \downarrow{d_\top} \\
X_n & \xrightarrow{d_i} & X_{n-1}.
\end{array}
$$

Definition 1.6. [13, §2.9] A simplicial groupoid $X: \Delta^{\text{op}} \to \text{Grpd}$ is called a Segal groupoid if it satisfies the Segal condition,

$$X_n \xrightarrow{\simeq} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \text{ for all } n \geq 0.$$ 

This is equivalent [13, Lemma 2.10] to requiring that for each $n > 0$ the following diagram is a homotopy pullback

$$
\begin{array}{ccc}
X_{n+1} & \xrightarrow{d_\top} & X_n \\
\downarrow{d_\perp} & & \downarrow{d_\perp} \\
X_n & \xrightarrow{d_\top} & X_{n-1}.
\end{array}
$$

Proposition 1.7. [13, Proposition 3.7] Any Segal groupoid is a decomposition groupoid.
Example 1.8. The decomposition groupoid of rooted trees $\mathbf{RT}$ is defined as follows [13]. Recall that a forest is a disjoint union of rooted trees. An admissible cut of a rooted tree is a splitting of the set of nodes into two subsets such that the second forms a subtree containing the root node or is the empty forest. $\mathbf{RT}_1$ denotes the groupoid of isoclasses of forests, and $\mathbf{RT}_2$ denotes the groupoid of isoclasses of forests with an admissible cut. More generally, $\mathbf{RT}_0$ is defined to be a point, and $\mathbf{RT}_k$ is the groupoid of isoclasses of forests with $k-1$ compatible admissible cuts. These form a simplicial groupoid in which the inner face maps forget a cut, and the outer face maps project away stuff; $d_\perp$ deletes the crown and $d_\top$ deletes the bottom layer. It is readily seen that $\mathbf{RT}$ is not a Segal groupoid: a tree with a cut cannot be reconstructed from its crown and its bottom tree, which is to say that $\mathbf{RT}_2$ is not equivalent to $\mathbf{RT}_1 \times_{\mathbf{RT}_0} \mathbf{RT}_1$. It is straightforward to check that it is a decomposition groupoid [13].

Recall that a simplicial map $F: X \to Y$ is cartesian on an arrow $[n] \to [k]$ in $\Delta$, if the naturality square for $F$ with respect to this arrow is a homotopy pullback. A simplicial map $F: X \to Y$ is called a right fibration if it is cartesian on all bottom coface maps $d_\perp$. Similarly, $F$ is called a left fibration if it is cartesian on $d_\top$.

Lemma 1.9. Let $Y$ be a Segal groupoid and let $F: X \to Y$ be a simplicial map that is a left or a right fibration, then also $X$ is a Segal groupoid.

Certain pullbacks in $\Delta^{op}$ are preserved by general decomposition groupoids, which is the content of the following result.

Lemma 1.10. [13, Lemma 3.10] Let $X$ be a decomposition groupoid. For all $0 < i < j < n$, the following squares of active face and degeneracy maps are homotopy pullbacks
\[
\begin{array}{ccc}
X_{n+1} & \xrightarrow{d_i} & X_n \\
\downarrow{d_{j+1}} & \downarrow{d_j} & \downarrow{s_j} \\
X_n & \xrightarrow{d_i} & X_{n-1} \\
\end{array}
\quad
\begin{array}{ccc}
X_{n-3} & \xrightarrow{s_{i-1}} & X_{n-2} \\
\downarrow{s_{j-2}} & \downarrow{s_j} & \downarrow{s_{j-1}} \\
X_{n-2} & \xrightarrow{s_{i-1}} & X_{n-1} \\
\end{array}
\]

A decomposition groupoid $X$ is complete when $s_0: X_0 \to X_1$ is a monomorphism (i.e. is $(-1)$-truncated). It follows from the decomposition groupoid axiom that in this case all degeneracy maps are monomorphisms [14, Lemma 2.5].

1.11. The incidence coalgebra of a decomposition groupoid

The span
\[
X_1 \xleftarrow{d_0} X_2 \xrightarrow{(d_2, d_0)} X_1 \times_{X_0} X_1
\]
defines a linear functor, the comultiplication
\[
\Delta: \text{Grpd}_{/X_1} \to \text{Grpd}_{/X_1 \times_{X_0} X_1}
\]
\[
f \mapsto (d_2, d_0)_! \circ d'_1(f).
\]
Likewise, the span
\[
X_1 \xleftarrow{s_0} X_0 \xrightarrow{f} 1
\]
defines a linear functor, the counit

$$\delta : \text{Grpd}_{/X_1} \to \text{Grpd}$$

$$f \mapsto t_1 \circ s_0^*(f).$$

The decomposition groupoid axioms serve to ensure that \(\Delta\) is coassociative with counit \(\delta\), up to coherent homotopy [13, §5.3]. This coalgebra \((\text{Grpd}_{/X_1}, \Delta, \delta)\) is called the incidence coalgebra. The classical notion of incidence coalgebras in vector spaces is obtained by taking homotopy cardinality; see [14]. A monoidal structure on \(X\) gives furthermore a bialgebra structure. This is not needed in this paper, except that it will be mentioned in some examples.

**Example 1.12.** The Butcher–Connes–Kreimer Hopf algebra of rooted trees is the free commutative algebra on the set of isomorphism classes of rooted trees, with comultiplication defined by summing over certain admissible cuts \(c\):

$$\Delta(T) = \sum_{c \in \text{admi.cuts}(T)} P_c \otimes R_c.$$

Recall that an admissible cut \(c\) is a splitting of the set of nodes into two subsets, such that the second forms a subtree \(R_c\) containing the root node (or is the empty forest); the first subset, the complement crown, then forms a subforest \(P_c\). The Butcher–Connes–Kreimer Hopf algebra is in fact the incidence bialgebra of the decomposition groupoid of rooted trees of Example 1.8 [13].

**Example 1.13.** [13, §5.1] If \(X\) is the nerve of a category (for example, a poset) then \(X_2\) is the set of all composable pairs of arrows. The comultiplication is then defined by:

$$\Delta(f) = \sum_{boa = f} a \otimes b,$$

and the counit sends identity arrows to 1 and other arrows to 0.

**1.14. Culf maps**

**Definition 1.15.** [13, §4] A simplicial map \(F : X \to Y\) is called culf if \(F\) is cartesian on each active map.

Culf stands for ‘conservative’ and ‘unique lifting factorisations’ where conservative means cartesian on all codegeneracy maps, and unique lifting factorisations means cartesian on all coface maps. The culf condition can be seen as an abstraction of coalgebra homomorphism: the conservative condition corresponds to counit preservation, and ulf corresponds to comultiplicativity.

**Proposition 1.16.** [13, Lemma 4.3] A simplicial map between decomposition groupoids is culf if and only if it is cartesian on \(d_1 : [1] \to [2]\).

**1.17. Decalage**

Given a simplicial groupoid \(X\), the lower dec \(\text{Dec}_\perp X\) is a new simplicial groupoid obtained by deleting \(X_0\) and shifting everything one place down, deleting also all \(d_0\) face maps and all \(s_0\) degeneracy maps. It comes equipped with a simplicial map, called the lower dec map, \(d_\perp : \text{Dec}_\perp X \to X\) given by the original \(d_0\). Similarly, the upper dec \(\text{Dec}_\top X\) is obtained by instead deleting, in each degree, the top face map \(d_\top\) and the top degeneracy map \(s_\top\). The deleted top face maps becomes the upper dec map \(d_\top : \text{Dec}_\top X \to X\).
Proposition 1.18. [13, Proposition 4.9] If $X$ is a decomposition groupoid then the dec maps $d_T : \text{Dec}_T X \to X$ and $d_\perp : \text{Dec}_\perp X \to X$ are cuf.

The decomposition property can be characterised in terms of decalage:

Theorem 1.19. [9, 11, 13] For a simplicial groupoid $X : \Delta^{op} \to \text{Grpd}$, the following are equivalent

1. $X$ is a decomposition groupoid
2. both $\text{Dec}_\perp X$ and $\text{Dec}_T X$ are Segal groupoids.

Throughout we write $\delta A : \Delta^{op} \to \text{Grpd}$ for the constant simplicial groupoid on a groupoid $A$. We have a natural transformation $\pi_{\text{last}} : \text{Dec}_T X \to \delta(X_0)$ defined as follows: the map $\pi_{\text{last}} : \text{Dec}_T X \to \delta(X_0)$ sends an $n$-simplex $\lambda$ in $\text{Dec}_T X$ to $d_\perp^{n+1}(\lambda)$ in $X_0$ and an arrow $\alpha : \lambda \to \eta$ in $(\text{Dec}_T X)_n$ to $d_\perp^{n+1}(\alpha)$ in $X_0$. We denote by $\Delta^i$ the category whose objects are finite linear orders with a top element, and whose arrows are the maps that preserve the order and the top element. Since $[0]$ is terminal in $(\Delta^i)^{op}$, the map $\pi_{\text{last}}$ is a simplicial map.

Lemma 1.20. The natural transformation $\pi_{\text{last}}$ is cartesian on right fibrations. That is, given a right fibration $p : X \to Y$, the square

\[
\begin{array}{ccc}
\text{Dec}_T X & \xrightarrow{\pi_{\text{last}}} & \delta(X_0) \\
\text{Dec}_T p & \downarrow & \downarrow \delta(p_0) \\
\text{Dec}_T Y & \xrightarrow{\pi_{\text{last}}} & \delta(Y_0)
\end{array}
\]

is a homotopy pullback.

Proof. The pullback property can be checked level-wise. Note that $(\text{Dec}_T X)_n = X_{n+1}$. In level $n \geq 0$, the square (1) is

\[
\begin{array}{ccc}
X_{n+1} & \xrightarrow{d_\perp^{n+1}} & X_0 \\
p_{n+1} & \downarrow & \downarrow p_0 \\
Y_{n+1} & \xrightarrow{d_\perp^{n+1}} & Y_0,
\end{array}
\]

which is a homotopy pullback since $p$ is a right fibration.

We also have a natural transformation $\pi_{\text{first}} : \text{Dec}_\perp X \to \delta(X_0)$ defined as follows: the simplicial map $\pi_{\text{first}} : \text{Dec}_\perp X \to \delta(X_0)$ sends an $n$-simplex $\lambda$ in $\text{Dec}_\perp X$ to $d_T^{n+1}(\lambda)$ in $X_0$ and an arrow $\alpha : \lambda \to \eta$ in $(\text{Dec}_\perp X)_n$ to $d_T^{n}(\alpha)$ in $X_0$. The proof of the following result is analogous to that of Lemma 1.20.

Lemma 1.21. The natural transformation $\pi_{\text{first}}$ is cartesian on left fibrations.

2 Slices and intervals

In this section, we introduce some constructions with slice and coslice of decomposition groupoids required to introduce the concept of interval.

Lemma 2.1. [11, Proposition 2.1] Let $X$ be a decomposition groupoid. For all $0 \leq i \leq n$ the following squares are homotopy pullbacks:
The pullbacks of Lemma 2.1 are called the upper and lower unital condition.

**Definition 2.2.** Let $X$ be a decomposition groupoid. For an object $y$ in $X_0$, the slice $X/y$ is defined as the homotopy pullback

$$
\begin{array}{ccc}
X/y & \rightarrow & 1 \\
\downarrow \scriptstyle u & & \downarrow \scriptstyle \gamma^y \\
\Dec X & \rightarrow & \delta(X_0).
\end{array}
$$

**Remark 2.3.** Taking the upper decalage construction of $X$ gives a simplicial object starting in $X_1$, but equipped with an augmentation $d_0: X_1 \to X_0$. Pulling back this simplicial object along $\gamma^y: 1 \to X_0$, yields a new simplicial object which is $X/y$. The map $u$ is cartesian since $1 \to \delta(X_0)$ is cartesian and cartesian maps are stable under pullback. Therefore, $u$ is a right fibration, and as a consequence, $X/y$ is Segal by 1.9.

**Definition 2.4.** Let $X$ be a Segal groupoid. An object $b \in X_0$ is called terminal if the map $d_\top \circ u_{X/b} \to X$ is a levelwise equivalence, where $d_\top: \Dec X \to X$ is the upper dec map.

**Proposition 2.5.** Let $X$ be a decomposition groupoid. Then for an object $y$ in $X_0$, the object $s_0(y)$ is terminal in $X/y$.

**Proof.** In the diagram

$$
\begin{array}{ccc}
(X/y)_{/s_0(y)} & \rightarrow & 1 \\
\downarrow \scriptstyle u' & & \downarrow \scriptstyle \gamma_{s_0y} \\
\Dec X/y & \rightarrow & \delta((X/y)_0) \\
\downarrow \scriptstyle \Dec u & & \downarrow \scriptstyle \delta(u_0) \\
\Dec \Dec X & \rightarrow & \delta((\Dec X)_0),
\end{array}
$$

the square (1) is a homotopy pullback by definition of $(X/y)_{/s_0(y)}$. Since $u: X/y \to \Dec X$ is a right fibration, we have that (2) is a homotopy pullback by Lemma 1.20. Therefore, the outer diagram is a homotopy pullback. Furthermore, note that $\delta(u_0)(s_0(y)) = s_0(y)$. This means that $(X/y)_{/s_0(y)}$ is the homotopy pullback of $\pi_{\text{last}}$ along $\gamma_{s_0y}: 1 \to \delta((\Dec X)_0)$. Note that in the diagram

$$
\begin{array}{ccc}
X/y & \rightarrow & 1 \\
\downarrow \scriptstyle u & & \downarrow \scriptstyle \gamma^y \\
\Dec X & \rightarrow & \delta(X_0) \\
\downarrow \scriptstyle H & & \downarrow \scriptstyle \delta(s_0) \\
\Dec \Dec X & \rightarrow & \delta((\Dec X)_0)
\end{array}
$$

the square (3) is a homotopy pullback by definition of $X/y$. The map $H: \Dec X \to \Dec \Dec X$ is defined by $H((\Dec X)_n) = s_{n+1}(X_{n+1})$. The square (4) is a pullback as a consequence of Lemma 2.1 and the definition of $H$. 

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Proof. We write \( s \) as an object in \( d \) and is therefore Segal too by Lemma 2.1. The proof is analogous to that of Proposition 2.8. Furthermore, note that \( \delta(s_0)(y) = s_0(y) \). This means that \( X/y \) is the homotopy pullback of \( \pi_{\text{last}} \) along \( \gamma s_0(y) : 1 \to \delta((\text{Dec}_\top X)_0) \). Since \( X/y \) and \((\text{Dec}_\top X)_0 \) are homotopy pullbacks over the same diagram, we get a canonical identification \( (X/y)/s_0(y) \cong X/y \). Furthermore, this identification is given by the canonical projection map \( \text{pr}: (X/y)/s_0(y) \to X/y \) since \( H \circ u \circ \text{pr} = u' \circ \text{Dec}_\top u \).

**Definition 2.6.** Let \( X \) be a decomposition groupoid. For an object \( x \) in \( X_0 \), the coslice \( X_{x/} \) is defined as the homotopy pullback

\[
\begin{array}{ccc}
X_{x/} & \rightarrow & 1 \\
\downarrow v & & \downarrow \gamma_x \\
\text{Dec}_\perp X & \xrightarrow{\pi_{\text{first}}} & \delta(X_0).
\end{array}
\]

We write \( v: X_{x/} \to \text{Dec}_\perp X \) for the canonical map of Definition 2.6. Note that for each \( x \) in \( X_0 \), the coslice \( X_{x/} \) is Segal. Indeed, \( \text{Dec}_\perp X \) is Segal and \( X_{x/} \) is a left fibration over \( \text{Dec}_\perp X \), and is therefore Segal too by Lemma 1.9.

**Definition 2.7.** Let \( X \) be a Segal groupoid. An object \( a \in X_0 \) is called **initial** if the map \( d_\perp \circ v: X_{a/} \to X \) is a levelwise equivalence, where \( d_\perp : \text{Dec}_\perp X \to X \) is the lower dec map.

**Proposition 2.8.** Let \( X \) be a decomposition groupoid. For an object \( x \) in \( X_0 \), the object \( s_0(x) \) is an initial object in \( X_{x/} \).

**Proof.** The proof is analogous to that of Proposition 2.5.

**Lemma 2.9.** Let \( \mathcal{C} \) be a Segal groupoid with an initial object \( \perp \mathcal{C} \). Then for each object \( y \) in \( \mathcal{C} \), the slice \( \mathcal{C}_{/y} \) has an initial object.

**Proof.** Since \( \perp \mathcal{C} \) is an initial object, we have a map \( f_{\perp \mathcal{C}} : \perp \mathcal{C} \to y \). This map can be regarded as an object in \( \mathcal{C}_{/y} \) or in \( \mathcal{C}_{\perp y} \), and after two pullbacks of \( \text{Dec}_\top \text{Dec}_\perp = \text{Dec}_\perp \text{Dec}_\top \) we get the natural identification \( (\mathcal{C}_{/y})_{f_{\perp \mathcal{C}}} \cong (\mathcal{C}_{\perp \mathcal{C}})_{f_{\perp \mathcal{C}}} \). Furthermore, in the diagram

\[
\begin{array}{ccc}
(\mathcal{C}_{\perp \mathcal{C}})_{f_{\perp \mathcal{C}}} & \rightarrow & 1 \\
\downarrow u & & \downarrow \gamma_{f_{\perp \mathcal{C}}} \\
\text{Dec}_\top \mathcal{C}_{\perp \mathcal{C}} & \xrightarrow{\pi_{\text{last}}} & \delta((\mathcal{C}_{\perp \mathcal{C}})_0) \\
\downarrow \pi_{\text{last}} & & \downarrow \delta(d_\perp) \\
\text{Dec}_\top \mathcal{C} & \xrightarrow{\pi_{\text{last}}} & \delta(\mathcal{C}_0)
\end{array}
\]

the square (1) is a homotopy pullback by definition of \( (\mathcal{C}_{\perp \mathcal{C}})_{f_{\perp \mathcal{C}}} \). Since \( \perp \mathcal{C} \) is an initial object, we have that \( d_\perp : \mathcal{C}_{\perp y} \to \mathcal{C} \) is a levelwise equivalence. This implies that (2) is a homotopy pullback. Combining (1) and (2), we have that the outer diagram is a homotopy pullback. Furthermore, note that \( d_\perp(f_{\perp \mathcal{C}}) = y \). This means that \( (\mathcal{C}_{\perp \mathcal{C}})_{f_{\perp \mathcal{C}}} \) is the homotopy pullback of \( \pi_{\text{last}} \) along \( \gamma y : 1 \to \delta(\mathcal{C}_0) \). But this is precisely the definition of \( \mathcal{C}_{/y} \). This implies that \( (\mathcal{C}_{\perp \mathcal{C}})_{f_{\perp \mathcal{C}}} \cong \mathcal{C}_{/y} \) and therefore \( (\mathcal{C}_{/y})_{f_{\perp \mathcal{C}}} \cong \mathcal{C}_{/y} \). Furthermore, this isomorphism is given by the canonical projection map \( \text{pr}: (\mathcal{C}_{/y})_{f_{\perp \mathcal{C}}} \to \mathcal{C}_{/y} \) since \( u'' \circ \text{pr} = \text{Dec}_\top d_\perp \circ u \), where \( u'' : \mathcal{C}_{/y} \to \text{Dec}_\top \mathcal{C} \) denotes the canonical map of Definition 2.2.

**Lemma 2.10.** Let \( \mathcal{C} \) be a Segal groupoid with a terminal object. Then for each object \( x \) in \( \mathcal{C} \), the coslice \( \mathcal{C}_{x/} \) has a terminal object.
Proof. The proof is analogous to that of Lemma 2.9.

Let \( X \) be a decomposition groupoid. For \( \lambda: \Delta^n \to X \), we denote by \( \text{long}(\lambda) \) the 1-simplex \( \Delta^1 \to \Delta^n \xrightarrow{\lambda} X \). Applying lower and upper decalage to \( X \), we obtain a new decomposition groupoid \( \text{Dec}\top \text{Dec}\bot X \) and a map \( \epsilon: \text{Dec}\top \text{Dec}\bot X \to X \) which is culf by Proposition 1.18. Furthermore, we have a natural transformation \( \pi_{\text{long}}: \text{Dec}\top \text{Dec}\bot X \to \delta(X_1) \) defined as follows: the map \( \pi_{\text{long}}: \text{Dec}\top \text{Dec}\bot X \to \delta(X_1) \) sends an \( n \)-simplex \( \lambda \) in \( \text{Dec}\top \text{Dec}\bot X \) to \( \text{long}(\lambda) \) in \( X_1 \) and an arrow \( \alpha: \lambda \to \eta \) in \( (\text{Dec}\top \text{Dec}\bot X)_n \) to \( \text{long}(\alpha) \) in \( X_1 \). The category \( \Delta^{\text{act}} \) is the subcategory of \( \Delta \) whose objects are the nonempty finite ordinals and whose morphisms are the active maps. Since \([1]\) is terminal in \((\Delta^{\text{act}})^{\text{op}}\), the map \( \pi_{\text{long}} \) is a simplicial map.

Lemma 2.11. The natural transformation \( \pi_{\text{long}} \) is cartesian on culf maps. That is, given a culf map \( F: X \to Y \) between decomposition groupoids, the square

\[
\begin{array}{ccc}
\text{Dec}\top \text{Dec}\bot X & \xrightarrow{\pi_{\text{long}}} & \delta(X_1) \\
\downarrow^{\text{Dec}\top \text{Dec}\bot F} & & \downarrow^{\delta(F_1)} \\
\text{Dec}\top \text{Dec}\bot Y & \xrightarrow{\pi_{\text{long}}} & \delta(Y_1)
\end{array}
\]

is a homotopy pullback.

Proof. The pullback property can be checked levelwise. Note that \((\text{Dec}\top \text{Dec}\bot X)_n = X_{n+2}\). For \( n \geq 0 \), the square

\[
\begin{array}{ccc}
X_{n+2} & \xrightarrow{d_{n+1}} & X_1 \\
\downarrow^{F_{n+2}} & & \downarrow^{F_1} \\
Y_{n+2} & \xrightarrow{d_{n+1}} & Y_1
\end{array}
\]

is a homotopy pullback since \( F \) is culf. \( \square \)

Definition 2.12. Let \( X \) be a decomposition groupoid and let \( f \) be an object in \( X_1 \). The Segal groupoid \( I_f \) is defined as the homotopy pullback, called the interval of \( f \),

\[
\begin{array}{ccc}
I_f & \xrightarrow{w} & 1 \\
\downarrow^{\text{Dec}\top \text{Dec}\bot} & & \downarrow^{\epsilon \text{Dec}\bot} \\
\text{Dec}\top \text{Dec}\bot X & \xrightarrow{\pi_{\text{long}}} & \delta(X_1)
\end{array}
\]

We write \( w: I_f \to \text{Dec}\top \text{Dec}\bot X \) for the simplicial map obtained in this way. From its construction as a pullback of a map between constant simplicial groupoids, it is clear that \( w \) is culf. The double decalage construction induces a culf map \( M_f: I_f \to X \), defined by the composition of \( w \) and the canonical map \( \epsilon: \text{Dec}\top \text{Dec}\bot X \to X \).

Remark 2.13. When \( X \) is the ordinary nerve of a category, the description of \( I_f \) is due to Lawvere [29]: the objects of \( I_f \) are two-step factorisations of \( f \). The 1-cells are arrows between such factorisations, or equivalently 3-step factorisations, and so on. More generally, let \( X \) be a decomposition set and \( f \in X_1 \). The Segal set \( I_f \) is described as follows:

1. An object of \( I_f \) is any \( \sigma \in X_2 \) such that \( d_1(\sigma) = f \).
2. Given two objects \( \sigma \) and \( \sigma' \) in \( I_f \), a morphism \( \gamma: \sigma \to \sigma' \) in \( I_f \) is any object \( \gamma \in X_3 \), such that \( d_2(\gamma) = \sigma \) and \( d_1(\gamma) = \sigma' \).

3. Given two morphisms \( \gamma: \sigma \to \sigma' \) and \( \gamma': \sigma' \to \sigma'' \) of \( I_f \), the composition is defined by \( \gamma' \circ \gamma = d_2(\eta) \), where \( \eta \in X_4 \) satisfies that \( d_1(\eta) = \gamma' \) and \( d_3(\eta) = \gamma \). The unique existence of \( \eta \) is a consequence of the decomposition-groupoid axioms in the form of Lemma 1.10. Associativity also follows by Lemma 1.10.

Applying lower and upper decalage to \( X \) generate two sections on \( \text{Dec}_\top \text{Dec}_\bot X \). The first one is induced by \( s_\bot: X \to \text{Dec}_\bot X \) and the other is induced by \( s_\top: X \to \text{Dec}_\top X \). We shall see later that \( s_\bot(f) \) is an initial object and \( s_\top(f) \) is a terminal object in \( I_f \). Recall that we write \( u: X/y \to \text{Dec}_\top X \) for the canonical map of Definition 2.2 and \( v: X_x/ \to \text{Dec}_\bot X \) for the canonical map of Definition 2.6. When further (co)slicing is used we decorate the \( u \) or \( v \) with a prime.

**Lemma 2.14.** Let \( X \) be a decomposition groupoid. For \( f \in X_1 \), put \( x = d_1(f) \) and \( d_0(f) = y \). There are canonical equivalences \( (X_{x/})_f/ \to I_f \) and \( (X/y)_f/ \to I_f \) such that the following diagram commutes up to isomorphism

\[
\begin{align*}
(X_{x/})_f/ & \xrightarrow{\simeq} I_f & (X/y)_f/ \xleftarrow{\simeq} I_f \\
\xrightarrow{d_\top \circ u} & (1) & \xrightarrow{d_\bot \circ v'} (2) \\
X_{x/} & \xrightarrow{d_\bot \circ v} X & \xleftarrow{d_\top \circ u'} X/y.
\end{align*}
\]

**Proof.** In the diagram

\[
\begin{align*}
(X_{x/})_f/ & \xrightarrow{1} 1 \\
\xrightarrow{u} & \xrightarrow{\delta(\pi_{\text{last}}(X_{x/}))} \xrightarrow{\delta(\pi_{\text{long}})} \delta(X_1) \\
\text{Dec}_\top X_{x/} \xrightarrow{\pi_{\text{last}}} & \xrightarrow{\text{Dec}_\top \pi} \text{Dec}_\bot X \xrightarrow{\pi_{\text{long}}} \text{Dec}_\bot X \xrightarrow{\gamma} X
\end{align*}
\]

the square (3) is a homotopy pullback by construction of \( (X_{x/})_f/ \). Since \( v \) is a right fibration the square (4) is a homotopy pullback by Lemma 1.21. Therefore, the outer diagram is a homotopy pullback. Note that \( \delta(v_0)(f) = y \). This implies that \( (X_{x/})_f/ \) is the homotopy pullback of \( \pi_{\text{long}} \) along \( \gamma \): \( 1 \to \delta(X_1) \). But this is precisely the definition of \( I_f \). This gives us an equivalence \( G: (X_{x/})_f/ \to I_f \) such that \( w \circ G \simeq \text{Dec}_\top v \circ u \), which is the upper square in the diagram

\[
\begin{align*}
(X_{x/})_f/ & \xrightarrow{G} I_f \\
\xrightarrow{u} & \xrightarrow{\text{Dec}_\top \pi} \text{Dec}_\top X \xrightarrow{\text{Dec}_\bot \pi} \text{Dec}_\bot X \xrightarrow{M_f} X
\end{align*}
\]

Since the other regions in the diagram commute strictly (by functoriality of upper decalage and by definition of \( \epsilon \) and \( M_f \)), we get a natural isomorphism for the outer square, which is precisely (1). By analogous arguments, (2) commutes up to isomorphism. \( \square \)
When $X$ is the ordinary nerve of a category, Lemma 2.14 is the same as Lemma 3.2 in [29].

**Lemma 2.15.** Let $X$ be a Segal groupoid with an initial object $\perp$ and a terminal object $\top$. Let $h: \perp \to \top$ be a map from $\perp$ to $\top$, then $X \simeq I_h$.

**Proof.** Applying Lemma 2.14 to $h$, we have that $(X_{\perp})/h \simeq I_h$. Applying Lemma 2.9 to $\top$, it follows that $X/\top \simeq (X_{\perp})/h$. Furthermore, $X/\top \simeq X$ since $\top$ is a terminal object. Combining these equivalences, we get that $X \simeq I_h$. \hfill $\square$

When $X$ is the ordinary nerve of a category, Lemma 2.15 is the same as Lemma 3.3 in [29].

**Proposition 2.16.** Let $X$ be a complete decomposition groupoid. Then for each $f \in X_1$, the Segal groupoid $I_f$ is complete in the sense of decomposition groupoids, meaning that $s_0: (I_f)_0 \to (I_f)_1$ is a monomorphism.

**Proof.** By construction of $I_f$, we have the following diagram

\[
\begin{array}{ccc}
(I_f)_0 & \xrightarrow{s_0} & (I_f)_1 \\
\downarrow w_0 & & \downarrow d_0 \\
X_1 & \xrightarrow{s_1} & X_2
\end{array}
\]

Since $X$ is complete, the map $s_1: X_1 \to X_2$ is a monomorphism, and therefore also its pullback $s_0: (I_f)_0 \to (I_f)_1$ is a monomorphism, which is to say that $I_f$ is complete. \hfill $\square$

### 3 Algebraic intervals and rigid decomposition groupoids

To study the first case of the Gálvez–Kock–Tonks conjecture, the obvious level of generality would be discrete decomposition groupoids, but the proofs to be presented in this section work for locally discrete decomposition groupoids of the kind featured in the following definition:

**Definition 3.1.** A **rigid decomposition groupoid** is a strict simplicial groupoid $X$ such that $d_1: X_2 \to X_1$ is a discrete fibration, $s_0: X_0 \to X_1$ is a monomorphism and the active-inert squares are strict pullbacks.

The point, as we shall see, is for a rigid decomposition groupoid $X$, we have that for all $f \in X_1$, the Segal groupoid $I_f$ (2.12) is discrete. Note that every discrete decomposition groupoid is a rigid decomposition groupoid. This means that the rigid decomposition groupoids already cover locally finite posets, Cartier–Foata monoids and Möbius categories. The importance of locally discrete is to cover also strict (directed) restriction species as shown in the following example:

**Example 3.2.** Let $\mathbb{I}$ be the category of finite sets and injections. Let $\mathbb{C}$ be the category of finite posets and convex monotone injections. A restriction species [34] is a functor $R: \mathbb{I}^{\text{op}} \to \text{Set}$ and a directed restriction species [12] is a functor $R: \mathbb{C}^{\text{op}} \to \text{Set}$. Note that any restriction species is a directed restriction species. An element of $R[P]$, where $P \in \mathbb{C}$, is called an $R$-structure on $P$. An $R$-structure on a poset $P$ thus restricts to any convex subposet $Q \subseteq P$. A directed restriction species $R$ induces a decomposition groupoid $R$ [12], where $R_n$ is the groupoid of $R$ structures with an $n$-layering of the underlying poset $P$, that
is a monotone map $P \to \mathbb{n}$, the linear order with $n$ elements. The map $d_1: \mathbb{R}_2 \to \mathbb{R}_1$ forgets the layering and is clearly a discrete fibration. Altogether, the decomposition groupoid $\mathbb{R}$ is rigid.

**Example 3.3.** Recall that for the decomposition groupoid of rooted trees $\mathbb{RT}$ of Example 1.8, $\mathbb{RT}_2$ is the groupoid of forests with an admissible cut, $\mathbb{RT}_1$ is the groupoid of forests and the map $d_1: \mathbb{RT}_2 \to \mathbb{RT}_1$ forgets the admissible cut. It is straightforward to see that $d_1$ is a discrete fibration. Therefore, $\mathbb{RT}$ is a rigid decomposition groupoid.

Diri [8] gave an incidence-coalgebra construction of the Butcher–Connes–Kreimer coalgebra by starting with the category of forests and root-preserving inclusions, generating a coalgebra and imposing the equivalence relation that identifies two root-preserving forest inclusions if their complement crowns are isomorphic forests.

Consider the rigid decomposition groupoid of rooted trees $\mathbb{RT}$. We can consider a tree $T$ as an object in $\mathbb{RT}_1$. The interval $I_T$ can be described as follows: $(I_T)_0$ is the set of all isoclasses of admissible cuts of $T$, and $(I_T)_k$ is the set of isoclasses of all $k + 1$ compatible admissible cuts of $T$.

We can relate the construction of Diri with the interval construction of a rooted tree as follows: note that admissible cuts are essentially the same thing as root-preserving forest inclusions: then the cut is interpreted as the division between the included forest and the forest induced on the nodes in its complement. In this way we see that $(I_T)_k$ is the discrete groupoid of $k + 1$ consecutive root-preserving inclusions ending in $T$.

**Remark 3.4.** In a decomposition groupoid $X$, every active face map is a pullback of $d_1: X_2 \to X_1$ [14, Lemma 1.10]. Therefore, in the case where $X$ is rigid we have that $\pi_{\text{long}}$ is a levelwise discrete fibration since in each level it is the long-edge map, which is a composition of active face maps, and these are all discrete fibrations. Therefore, the strict pullback of $\pi_{\text{long}}$ along $\delta: X_1 \to \delta(X_1)$ is also a homotopy pullback. Furthermore, for every $f \in X_1$, the Segal groupoid $I_f$ is discrete.

**Definition 3.5.** Let $X$ be a discrete Segal groupoid. A chosen terminal object is the choice of a terminal object $\bot$. A chosen initial is the choice of an initial object $\top$.

**Example 3.6.** Let $X$ be a discrete decomposition groupoid. Let $y$ be an object in $X_0$. We already know from Proposition 2.5 that $s_0(y)$ is a terminal object in $X_{/y}$, which we take as the chosen one. In this way $X_{/y}$ acquires a canonical chosen terminal. Similarly, $s_0(y)$ is an initial object in $X_{y/}$ (Proposition 2.8), which we take as the chosen one. In this way $X_{x/}$ acquires a canonical chosen initial.

**Definition 3.7.** A discrete algebraic interval is a discrete Segal groupoid $\mathcal{C}$ with a chosen initial object $\bot$ and a chosen terminal object $\top$. We denote the map from the chosen initial to the chosen terminal by $\varpi: \bot \to \top$.

**Remark 3.8.** In the case of the nondiscrete algebraic intervals, further structure is required in the notion of chosen terminal, namely the choice of a section $s: X \to X_{/b}$ for the canonical map $d_\top u: X_{/b} \to X$. This will not be needed in the present paper, but the interested reader can find the theory of these worked out in Version 1 of this paper on arXiv.

**Remark 3.9.** Batanin and Markl [1] used the notion of a category with chosen local terminal objects, meaning a category which in each connected component is provided with a chosen terminal object. This notion plays an important role in their theory of operadic categories. Garner, Kock and Weber [17] observed that the structure of chosen local terminal objects is
precisely to be a coalgebra for the upper-Dec comonad. This in turn amounts to having an extra top degeneracy map for the nerve of the category. When we insist on having a chosen terminal object, it is inspired by this decalage viewpoint on chosen terminals. Similarly of course, the notion of chosen local initial object amounts to coalgebra structure for the lower-Dec comonad, via extra bottom degeneracy maps, as the chosen initial object in our definition. Finally, the main point here is the combination of the two ideas. A discrete algebraic interval structure is in particular a coalgebra for the two-sided-Dec comonad. This is very much in line with the notion of flanking of Galvez–Kock–Tonks [15, §1].

Definition 2.12 can be rewritten in terms of rigid decomposition groupoid as follows:

**Definition 3.10.** Let $X$ be a rigid decomposition groupoid and let $f$ be an object in $X_1$. The Segal groupoid $I_f$ is defined as the strict pullback

$$
\begin{array}{ccc}
I_f & \rightarrow & 1 \\
\downarrow w & & \downarrow \epsilon_f \\
\operatorname{Dec}_\top \operatorname{Dec}_\perp X & \rightarrow & \delta(X_1).
\end{array}
$$

In fact this strict pullback is also a homotopy pullback since $\pi_{\text{long}}$ is a discrete fibration as a consequence of the fact that $d_1$ is a discrete fibration.

We write $w: I_f \rightarrow \operatorname{Dec}_\top \operatorname{Dec}_\perp X$ for the simplicial map obtained in this way. The double decalage construction induces a culf map $M_f: I_f \rightarrow X$, defined by the composition of $w$ and the canonical map $\epsilon: \operatorname{Dec}_\top \operatorname{Dec}_\perp X \rightarrow X$.

**Lemma 3.11.** Let $X$ be a rigid decomposition groupoid and $f \in X_1$. The Segal groupoid $I_f$ has a canonical structure of an algebraic interval, where the chosen initial object is $s_0(f)$ and the chosen terminal object is $s_1(f)$.

**Proof.** The object $s_1(f)$ is a terminal object in $I_f$ as a consequence of Lemma 2.5, which we take as the chosen one. On the other hand, the object $s_0(f)$ is an initial object in $I_f$ as a consequence of Lemma 2.8, which we take as the chosen initial.

A simplicial map $F: X \rightarrow Y$ between rigid decomposition groupoids is called strict culf if the naturality square for $F$ with respect any active map $[n] \rightarrow [k]$ in $\Delta$ is a strict pullback. Since the active maps are fibrations in rigid decomposition groupoids, it follows that the strict pullbacks of a strict culf map are also homotopy pullbacks, so that strict culf map is in fact culf in the usual homotopy invariant sense. Lemma 2.11 can be rewritten in terms of the strict condition as follows:

**Lemma 3.12.** The natural transformation $\pi_{\text{long}}$ is cartesian on strict culf maps. That is, given a strict culf map $F: X \rightarrow Y$ between rigid decomposition groupoids, the square

$$
\begin{array}{ccc}
\operatorname{Dec}_\top \operatorname{Dec}_\perp X & \rightarrow & \delta(X_1) \\
\downarrow \operatorname{Dec}_\top \operatorname{Dec}_\perp F & & \downarrow \delta(F_1) \\
\operatorname{Dec}_\top \operatorname{Dec}_\perp Y & \rightarrow & \delta(Y_1)
\end{array}
$$

is a strict pullback.
Proof. The proof is analogous to that of Lemma 2.11. Furthermore, the strict pullback is also a homotopy pullback since \( \pi_{\text{long}} \) is a discrete fibration.

The culf maps preserve the algebraic structure of a decomposition groupoid, but do not necessarily preserve the chosen initial and chosen terminal objects for a discrete algebraic interval. The maps that preserve this structure is the content of the following definition:

**Definition 3.13.** A simplicial map between discrete algebraic intervals is termed stretched, and written \( \mathcal{C} \to \mathcal{D} \), if it preserves the chosen initial object \( \perp_C \) and the chosen terminal object \( \top_C \).

**Lemma 3.14.** Let \( X \) be a rigid decomposition groupoid and let \( f \) be a 1-simplex in \( X \). The unique stretched map \( \varpi: \Delta^1 \to I_f \) is compatible with \( M_f \), meaning that we have a commutative triangle

\[
\Delta^1 \xrightarrow{\varpi} I_f \\
\downarrow f \quad \downarrow M_f \\
X.
\]

**Proof.** Put \( x := d_{\top}(f) \) and \( y := d_{\perp}(f) \) (the domain and codomain of \( f \)). Recall that the objects of \( I_f \) are 2-simplices with long edge \( f \). The arrows in \( I_f \) are 3-simplices with long edge \( f \). We know that the (chosen) initial object is \( s_{\perp}(f) \) (which can be thought of as the triangle with short sides \( \text{id}_x \) and \( f \)) and the (chosen) terminal object is \( s_{\top}(f) \) (which can be thought of as the triangle with short sides \( f \) and \( \text{id}_y \)). The unique arrow \( \varpi \) from the initial to the terminal is the tetrahedron \( s_{\perp}s_{\top}(f) \) (which we can think of as the tetrahedron with short sides \( \text{id}_x, f, \) and \( \text{id}_y \)). By definition \( M_f = d_{\top}d_{\perp}w \). Since \( I_f \) is a discrete algebraic interval the map \( w: I_f \to \text{Dec}_\top \text{Dec}_\perp X \) is level-wise injective on objects. So what \( M_f \) does is that it applies \( d_{\top}d_{\perp} \). In conclusion we have \( M_f(\varpi) = d_{\top}d_{\perp}s_{\top}s_{\perp}(f) = f \), which is what we wanted to prove.

**Example 3.15.** In the situation of Lemma 3.14, if \( X \) is already an interval and \( f \) is its long edge, we see from the argument in the proof that \( M_f \) is stretched in this case we will see in 4.4 that \( M_f \) is actually invertible in this case.

**Remark 3.16.** \( \Delta^n \) is an algebraic interval for each \( n \). The stretched maps \( \Delta^m \to \Delta^n \) are precisely the active maps. Every algebraic interval \( A \) receives a unique stretched map from \( \Delta^1 \). A simplicial map between algebraic intervals \( A \to B \) is stretched if and only if it commutes with the stretched maps from \( \Delta^1 \).

Let \( \mathcal{C} \) be a discrete algebraic interval. Let \( \mathcal{C}_n \) denote the set of \( n \)-simplices in \( \mathcal{C} \) and let \( \mathcal{C}^{\text{str}}_n \) be the subset of stretched \((n + 2)\)-simplices in \( \mathcal{C} \).

**Lemma 3.17.** The map \( d_{\top}d_{\perp}: \mathcal{C}^{\text{str}}_{n+2} \to \mathcal{C}_n \) is a bijection.

**Proof.** We will construct an inverse \( t: \mathcal{C}_n \to \mathcal{C}^{\text{str}}_{n+2} \) of \( d_{\top}d_{\perp} \) as follows. Let \( \lambda \) be an object in \( \mathcal{C}_n \), put \( a := d_{\top}(\text{long}(\lambda)) \) and \( b := d_{\perp}(\text{long}(\lambda)) \). Since \( \mathcal{C} \) is a discrete algebraic interval, we have a chosen edge \( f_{\perp}: \Delta^1 \to \mathcal{C} \) such that \( d_{\top}(f_{\perp}) = \perp_{\mathcal{C}} \) and \( d_{\perp}(f_{\perp}) = a \). By the same argument, we have a chosen edge \( f_{\top}: \Delta^1 \to \mathcal{C} \) such that \( d_{\perp}(f_{\top}) = \top_{\mathcal{C}} \) and \( d_{\top}(f_{\top}) = b \). In the diagram
the squares (2) and (3) are strict pullbacks since \( \mathcal{C} \) is a discrete Segal groupoid. Therefore, the outer rectangle is a strict pullback. By the pullback property of \( \mathcal{C}_{n+1} \), there exists a unique map \( \mu: 1 \rightarrow \mathcal{C}_{n+1} \) such that the diagram commutes. Since the square (2) commutes and \( d_{\perp} \mu = \lambda \), we have that \( d_{\perp} \text{long}(\mu) = b \). Furthermore, in the diagram

\[
\begin{array}{c}
1 \\
\mu \\
\mathcal{C}_{n+1} \downarrow \mathcal{C}_n \downarrow \mathcal{C}_0 \\
\quad \downarrow \mathcal{C}_1 \downarrow \mathcal{C}_0 \\
\lambda \\
\mathcal{C}_n \\
\end{array}
\]

the squares (6) and (7) are strict pullbacks since \( \mathcal{C} \) is a discrete Segal groupoid. Therefore, the outer rectangle is a strict pullback. By the pullback property of \( \mathcal{C}_{n+2} \), there exists a unique map \( \eta: 1 \rightarrow \mathcal{C}_{n+2} \) such that the diagram commutes. Note that \( d_{\top}(\text{long}(\eta)) = \perp_{\mathcal{C}} \) since (4) commutes and \( d_{\top}(f_{\perp}) = \perp_{\mathcal{C}} \). Since (8) commutes and \( d_{\perp}(f_{\top}) = \top_{\mathcal{C}} \), we have that \( d_{\perp}(\text{long}(\eta)) = \top_{\mathcal{C}} \). This together with \( d_{\top}(\text{long}(\eta)) = \perp_{\mathcal{C}} \) implies that \( \text{long}(\eta) = \psi_{\mathcal{C}} \) since \( \mathcal{C} \) is a discrete algebraic interval. We define

\[
t(\lambda) := \eta.
\]

Combining (1) and (5), we have that \( d_{\perp} d_{\top} \eta = \lambda \). This means that \( d_{\top} d_{\perp} (t(\lambda)) = \lambda \). Now we will check that \( t \circ d_{\top} d_{\perp} = \text{id}_{\mathcal{C}_{n+2}} \). Let \( \psi \) be an object in \( \mathcal{C}_{n+2}^{\text{str}} \). Since \( \mathcal{C} \) is a discrete algebraic interval, we have a chosen edge \( f'_{\perp}: \Delta^1 \rightarrow \mathcal{C} \) such that \( d_{\top}(f'_{\perp}) = \perp_{\mathcal{C}} \) and \( d_{\perp}(f'_{\perp}) = d_{\perp}(\text{long}(d_{\top}d_{\perp} \psi)) \). By the same argument, we have a chosen edge \( f'_{\top}: \Delta^1 \rightarrow \mathcal{C} \) such that \( d_{\perp}(f'_{\top}) = \top_{\mathcal{C}} \) and \( d_{\top}(f'_{\top}) = d_{\top}(\text{long}(d_{\top}d_{\perp} \psi)) \). The commutative diagrams

\[
\begin{align*}
1 & \quad \mu' \\
\mathcal{C}_{n+1} & \quad \mathcal{C}_n \\
\quad & \quad \mathcal{C}_0 \\
\mathcal{C}_1 & \quad \mathcal{C}_0 \\
\end{align*}
\]

are given by the construction of \( t \). Furthermore, \( t(d_{\top}d_{\perp} \psi) = \eta' \). If we substitute \( d_{\top} \psi \) by \( \mu' \), we have that the left diagram commutes. By the uniqueness of \( \mu' \), it follows that
µ′ = d⊤ψ. This together with the stretched condition of ψ implies that the right diagram commutes if we substitute ψ by η′. Therefore, by the uniqueness of η′, we have that η′ = ψ. This means that t(d⊤d⊥(ψ)) = ψ. □

Suppose X is a rigid decomposition groupoid. For an object f: x → y, we have a canonical projection πm: (Xx)f → X defined as the composite

\[
(Xx)f \xrightarrow{\psi} \text{Dec}_T Xx \xrightarrow{\text{Dec}_T \psi} \text{Dec}_T \text{Dec}_⊥ X \xrightarrow{\epsilon} X.
\]

Lemma 3.18. Let C be a discrete algebraic interval. The canonical projection πm: (C⊥)/πc → C has an inverse L: C → (C⊥)/πc.

Proof. Since C has a chosen initial object, the projection d⊥: C⊥ → C is an equivalence, and therefore an isomorphism since C and C⊥ are discrete. The map p⊥: C → C⊥ denotes the inverse of d⊥. The object πc is terminal in C⊥ since it is chosen terminal in C. This implies that the projection d⊥: (C⊥)/πc → C⊥ has an inverse pπc: C⊥ → (C⊥)/πc since d⊥ is an equivalence between discrete algebraic intervals. So we define L as the composite

\[
C \xrightarrow{p⊥} C⊥ \xrightarrow{pπc} (C⊥)/πc.
\]

Since C is a discrete algebraic interval, u and v are level-wise injective on objects. This implies that πm is equal to d⊥ ◦ d⊥. Since pπc and p⊥ are inverse of d⊥ and d⊥, it follows that L ◦ πm = id(C⊥)/πc and πm ◦ L = idC.

Recall that the class of culf maps can also be characterised as the class right orthogonal to the active maps (between representables). That is, X → Y is culf if and only if for every active map p: [m] → [n] and every commutative square

\[
\begin{array}{ccc}
\Delta^m & \xrightarrow{p} & X \\
\downarrow & \equiv & \downarrow \\
\Delta^n & \xrightarrow{f} & Y
\end{array}
\]

there is a unique filler. Usually this is about homotopy commutative squares and a contractible space of lifts, but for strict culf, it is actually about strictly commutative squares and truly unique lifts.

Proposition 3.19. For any n-simplex λ: Δn → X with long edge f, there is a unique lift φλ for the square

\[
\begin{array}{ccc}
\Delta^1 & \xrightarrow{\phi_λ} & I_f \\
\downarrow & \equiv & \downarrow \\
\Delta^n & \xrightarrow{\lambda} & X
\end{array}
\]

Proof. The square commutes by Lemma 3.14. Indeed the composite Δ1 → Δn → X is equal to f since f was defined as the long edge, and Δ1 → I_f → X is equal to f by Lemma 3.14. Furthermore, since Δ1 → Δn is active and M_f is strict culf, we have a unique filler which we denote as φλ. □
Lemma 3.20. Let $G: X \to Y$ be a simplicial map between rigid decomposition groupoids. For $f \in X_1$, there is a unique stretched map $G_f$ fitting into the commutative diagram

$$
\begin{array}{ccc}
I_f & \xrightarrow{G_f} & I_{G_f} \\
\downarrow{M_f} & & \downarrow{M_{G_f}} \\
X & \xrightarrow{G} & Y.
\end{array}
$$

If $G$ is strict culf then $G_f$ is an isomorphism.

Proof. In the diagram

$$
\begin{array}{ccc}
I_f & \xrightarrow{\gamma_f} & \delta(X_1) \\
\downarrow{w} & & \downarrow{\pi_{\text{long}}} \\
\text{Dec}_\top \text{Dec}_\perp G & \xrightarrow{\delta(G_1)} & \delta(Y_1) \\
\downarrow{\epsilon} & & \downarrow{\pi_{\text{long}}} \\
X & \xrightarrow{G} & Y
\end{array}
$$

the square (2) is a strict pullback by construction of $I_f$, and (3) commutes since $\pi_{\text{long}}$ is a natural transformation. Combining (2) and (3), we have that the outer diagram commutes. By the pullback property of $I_{G_f}$, we have a unique map $G_f: I_f \to I_{G_f}$ fitting into a commutative diagram

$$
\begin{array}{ccc}
I_f & \xrightarrow{G_f} & I_{G_f} \\
\downarrow{w} & & \downarrow{\pi_{\text{long}}} \\
\text{Dec}_\top \text{Dec}_\perp G & \xrightarrow{\delta(G_1)} & \delta(Y_1) \\
\downarrow{\epsilon} & & \downarrow{\pi_{\text{long}}} \\
X & \xrightarrow{G} & Y
\end{array}
$$

Combining (4) and (5), we get that (1) commutes. This implies that $M_{G_f} G_f(\varpi_f) = G_f$. Since $X$ and $Y$ are rigid, the maps $w$ and $w'$ are level-wise injective on objects. The functor $G_f$ is described as follows: for an $n$-simplex $\lambda$ in $I_f$, we have that $(G_f)_n(\lambda) = G_{n+2}(\lambda)$. This description is possible since we work with strict pullbacks, $w$ is level-wise injective on objects and (1) commutes. This implies that $w(\lambda)$ is the same $\lambda$ but interpreted as an $(n+2)$-simplex in $X$. Using this description of $I_f$ it is immediate to see that $G_f(s_0(f)) = s_0(G_f)$ and $G_f(s_1(f)) = s_1(G_f)$, this means that $G_f$ sends the chosen initial and terminal objects of $I_f$ to the chosen initial and terminal objects of $I_{G_f}$. In other words, $G_f$ is stretched.

Recall that the chosen edge $\varpi_f: \top_{I_{G_f}} \to \top_{I_f}$ of $I_f$, satisfying that $M_{G_f} \varpi_f = G_f$. Applying Proposition 3.19 to the map $G_f$, we have a unique stretched map $\phi_{G_f}$ satisfies $M_{G_f} \phi_{G_f} = G_f$. But as shown above, $G_f(\varpi_f)$ and $\varpi_{G_f}$ also satisfy this condition. This implies that $G_f(\varpi_f) = \varpi_{G_f}$. Furthermore, if $G$ is strict culf, (3) is a strict pullback by Lemma 3.12. Therefore, combining (2) and (3), we have that $I_f$ is the strict pullback of $\pi_{\text{long}}: \text{Dec}_\top \text{Dec}_\perp \delta(Y_1)$ along $\gamma G_f: 1 \to \delta(Y_1)$. But this is precisely the definition of $I_{G_f}$. Since $I_f$ and $I_{G_f}$ are pullbacks over the same diagram, it follows that $I_f \cong I_{G_f}$. Furthermore, this isomorphism is given by $G_f$ since the squares (3) and (4) commute. \qed
Remark 3.21. The uniqueness of Lemma 3.20 immediately implies the following ‘transitivity’ property of the construction $G \mapsto G_f$: Given

$$
\begin{array}{c}
I_f \\
\downarrow M_f
\end{array}
\xymatrix{X \ar[r]^G & Y \ar[r]^H & Z,}
\quad
\begin{array}{c}
I_{Gf} \\
\downarrow M_{Gf}
\end{array}
\xymatrix{I_G \ar[r]^{Gf} & I_H \ar[r]^{HGf} & I_{HGf},}
\quad
we have

$$(H \circ G)_f = H_{Gf} \circ G_f.$$

4 Stretched-culf factorisation system

A factorisation system in a category $\mathcal{D}$ consists of two classes $E$ and $F$ of maps, that we shall depict as $\rightarrow$ and $\Rightarrow$, such that

1. The class $F$ is closed under isomorphisms.

2. The classes $E$ and $F$ are orthogonal, $E \perp F$. That is, given $e \in E$ and $f \in F$, for every solid square

$$
\begin{array}{c}
\downarrow e
\end{array}
\xymatrix{e \ar[r] & f}
$$

there is a unique filler.

3. Every map $h$ admits a factorisation

$$
\begin{array}{c}
\downarrow e
\end{array}
\xymatrix{e \ar[r] & f \ar[r]^h & .}
$$

with $e \in E$ and $f \in F$.

Remark 4.1. The classical notion of orthogonal factorisation system requires that $E$ be closed under isomorphism. In our case it is not required. In case $E$ is not closed under isomorphism we can always saturate it.

Let $\text{Ar}^E(\mathcal{D}) \subset \text{Ar}(\mathcal{D})$ denote the full subcategory spanned by the arrows in the left-hand class $E$.

Lemma 4.2. [15, Lemma 1.3] The domain projection $\text{Ar}^E(\mathcal{D}) \to \mathcal{D}$ is a cartesian fibration. The cartesian arrows in $\text{Ar}^E(\mathcal{D})$ are given by squares of the form

$$
\begin{array}{c}
\downarrow
\end{array}
\xymatrix{. \ar[r] & .}
$$

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Let \( \text{aInt} \) be the category whose objects are discrete algebraic intervals and whose morphisms are functors. We need some preliminary results to prove that the stretched functors as left-hand class and the culf functors as right-hand class form a factorisation system in \( \text{aInt} \).

**Lemma 4.3.** Consider the following commutative diagram of simplicial maps

\[
\begin{array}{ccc}
A & \xrightarrow{F} & C \\
S & \downarrow & G \\
B & \rightarrow & E.
\end{array}
\]

where \( A, B \) and \( C \) are discrete algebraic intervals, and \( S \) is stretched. Then \( F \) is stretched if and only if \( G \) is stretched.

**Proof.** Let \( \varpi_A : \bot_A \to \top_A \) be the unique map from \( \bot_A \) to \( \top_A \). Let \( \varpi_B : \bot_B \to \top_B \) and \( \varpi_C : \bot_C \to \top_C \) be the unique maps in \( B \) and \( C \). It is obvious that \( F \) is stretched when \( G \) is stretched. For the other direction, suppose \( F \) stretched. We have that

\[
G(\varpi_B) = G(S(\varpi_A)) = F(\varpi_A) = \varpi_C.
\]

This means that \( G \) is stretched. \( \square \)

**Lemma 4.4.** Let \( C \) be a discrete algebraic interval with long edge \( \varpi \). The simplicial map \( M_\varpi : I_\varpi \to C \) has an inverse \( W : C \to I_\varpi \).

**Proof.** Since \( C \) is a discrete algebraic interval, we have a map \( L : C \to (C_{\bot C}/)_{/\varpi} \) by Lemma 3.18. Recall that for an \( n \)-simplex \( \lambda \) in \( C_n \), the \( n \)-simplex \( L(\lambda) \) satisfies that long\((L(\lambda)) = s_\top s_\bot \varpi_C \) and \( d_\top d_\bot L(\lambda) = \lambda \). Consider the canonical projections \( u : (C_{\bot C}/)_{/\varpi} \to \text{Dec}_\top C_{\bot C} \) and \( v : C_{\bot C}/ \to \text{Dec}_\bot C_{/\varpi} \). Since \( C \) is a discrete algebraic interval, the canonical projections are level-wise injective on objects. So it is straightforward to check that \( \pi_{\text{long}}(\text{Dec}_\top v \circ u \circ L(\lambda)) = \varpi_C \). Therefore, the outer diagram

\[
\begin{array}{ccc}
C & \xrightarrow{W} & I_\varpi \\
\text{Dec}_\top v \circ u \circ L & \downarrow \pi_{\text{long}} & 1 \\
\text{Dec}_\top C_{/\varpi} & \downarrow \varpi_{\text{long}} & \delta(C_1)
\end{array}
\]

commutes. By the pullback property of \( I_\varpi \), we have a unique map \( W : C \to I_\varpi \) such that the diagram commutes. Informally, for an \( n \)-simplex \( \lambda \) in \( C_n \), the map \( W \) only adds to \( \lambda \) the chosen initial edge \( \bot_C \to d_\top (\lambda) \) by precomposing and the chosen terminal edge \( d_\bot (\lambda) \to \top_C \) by postcomposing. Since \( w \circ W = \text{Dec}_\top v \circ u \circ L \) and \( d_\top d_\bot L(\lambda) = \lambda \), we have that \( M_{\varpi} \circ W(\lambda) = \lambda \). By analogous arguments we have that \( W \circ M_{\varpi} = \text{id}_{I_\varpi} \). \( \square \)

**Lemma 4.5.** Let \( A \) and \( B \) be discrete algebraic intervals, and let \( X \) be a rigid decomposition groupoid. Suppose we have a fork diagram

\[
\begin{array}{ccc}
A & \xrightarrow{V} & B \\
W & \longleftarrow & F \\
& \xrightarrow{X} &
\end{array}
\]
(meaning $FV = FW$) where $V$ and $W$ are stretched and $F$ is strict culf. Then already $V = W$.

**Proof.** Let $\omega_A$ denote the long edge of $A$ and $\omega_B$ the long edge of $B$, as usual. Since $V$ and $W$ are stretched, we have $V(\omega_A) = \omega_B = W(\omega_A)$, so the following diagram is well formed from applying the construction of Lemma 3.20 (for composable maps as in Remark 3.21):

$$
\begin{array}{ccc}
I_{\omega_A} & \xrightarrow{V_{\omega_A}} & I_{\omega_B} \\
M_{\omega_A} & \downarrow & M_{\omega_B} \\
A & \xrightarrow{V} & B
\end{array}
\quad
\begin{array}{ccc}
& & I_{F_{\omega_B}} \\
& & M_{F_{\omega_B}} \\
& & F
\end{array}
\quad
\begin{array}{ccc}
& & I_{F_{\omega_B}} \\
& & M_{F_{\omega_B}} \\
& & F
\end{array}
\quad
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
$$

Since $FV = FW$, we also have $F_{\omega_B}V_{\omega_A} = F_{\omega_B}W_{\omega_A}$. This is a consequence of the uniqueness statement in Lemma 3.20 as in Remark 3.21. But since $F$ is strict culf, the map $F_{\omega_B}$ is an isomorphism by Lemma 3.20. It follows that $V(\omega_A) = W(\omega_A)$. Finally, since $A$ and $B$ are discrete algebraic intervals and $\omega_A$ and $\omega_B$ are their long edges, it follows from Lemma 4.4 that the two vertical maps $M_{\omega_A}$ and $M_{\omega_B}$ are isomorphisms, and this implies that $V = W$.

**Lemma 4.6.** Let $\mathcal{C}$ and $\mathcal{D}$ be discrete algebraic intervals. Let $F : \mathcal{C} \to \mathcal{D}$ be a simplicial map. Then $F$ admits a stretched-culf factorisation.

**Proof.** Let $\omega_c$ be the long 1-simplex of the interval $\mathcal{C}$. By Lemma 3.20, we have a stretched map $F_{\omega_c} : I_{\omega_c} \to I_{F_{\omega_c}}$ fitting into the commutative diagram

$$
\begin{array}{ccc}
I_{\omega_c} & \xrightarrow{F_{\omega_c}} & I_{F_{\omega_c}} \\
M_{\omega_c} & \downarrow & M_{F_{\omega_c}} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
$$

Recall that the vertical arrows are strict culf. The map $M_{\omega_c}$ is stretched by Example 3.15. Since $\mathcal{C}$ is a discrete algebraic interval, we have that $M_{\omega_c}$ is invertible by Lemma 4.4. From the diagram

$$
\begin{array}{ccc}
I_{\omega_c} & \xrightarrow{id_{\omega_c}} & I_{\omega_c} \\
M_{\omega_c} & \downarrow & M_{\omega_c}^{-1} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
$$

it follows that $M_{\omega_c}^{-1}$ is also stretched, by Lemma 4.3. So altogether, the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
$$

commutes, where the map $F_{\omega_c} \circ M_{\omega_c}^{-1}$ is stretched and $M_{F_{\omega_c}}$ is culf.

**Lemma 4.7.** Let $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{C}$ be discrete algebraic intervals. Let $X$ be a rigid decomposition groupoid. For the commutative square of simplicial maps
where \( S: \mathcal{E} \to \mathcal{E}' \) is stretched and \( F: \mathcal{C} \to X \) is strict culf, there is a unique filler.

**Proof.** We will first construct a filler \( L: \mathcal{E}' \to \mathcal{C} \) and then prove it is unique. For each \( n \)-simplex \( \lambda: \Delta^n \to \mathcal{E}' \), Lemma 3.17 gives an \((n + 2)\)-simplex \( \eta_\lambda: \Delta^{n+2} \to \mathcal{E}' \) such that

\[
d_\perp d_\top (\eta_\lambda) = \lambda.
\]

and

\[
\text{long}(\eta_\lambda) = \varpi_{\mathcal{E}'}
\]

We assumed that \( S \) is stretched, so \( S(\varpi_{\mathcal{E}}) = \varpi_{\mathcal{E}'} \). This together with the equation \( HS = FG \) and the stretched condition of \( S \) are used in the following calculation:

\[
\text{long}(H(\eta_\lambda)) = H(\text{long}(\eta_\lambda)) = H(\varpi_{\mathcal{E}'}) = H(S(\varpi_{\mathcal{E}})) = F(G(\varpi_{\mathcal{E}})).
\]

In other words, the outer diagram

\[
\begin{array}{ccc}
1 & \xrightarrow{id} & 1 \\
\downarrow{H}\eta_\lambda & - & \downarrow{FG}\varpi_{\mathcal{E}} \\
X_n+2 & \xrightarrow{w_n'} & X_1 \\
\uparrow{H}(\eta_\lambda) & \xleftarrow{d_\perp d_\top} & \uparrow{\varpi_{\mathcal{E}'}} \\
\end{array}
\]

commutes. The pullback property of \( IFG\varpi_{\mathcal{E}} \) gives the dotted map \( H\eta_\lambda: \Delta^n \to IFG\varpi_{\mathcal{E}} \) such that the diagram commutes. We define the map \( V: \mathcal{E}' \to IFG\varpi_{\mathcal{E}} \) by \( V(\lambda) = H(\eta_\lambda) \), for each \( n \)-simplex \( \lambda: \Delta^n \to \mathcal{E}' \). It is straightforward to check that \( V \) is a simplicial map. Furthermore,

\[
M_{FG\varpi_{\mathcal{E}}} V(\lambda) = d_\perp d_\top w_n' H(\eta_\lambda) = d_\perp d_\top H(\eta_\lambda) = H d_\perp d_\top (\eta_\lambda) = H(\lambda).
\]

This means that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{E}' & \xrightarrow{H} & X \\
\downarrow{V} & \xleftarrow{M_{FG\varpi_{\mathcal{E}}}} & \downarrow{M_{FG\varpi_{\mathcal{E}}}} \\
I_{FG\varpi_{\mathcal{E}}} & \xleftarrow{I_{FG\varpi_{\mathcal{E}}}} & I_{FG\varpi_{\mathcal{E}}} \\
\end{array}
\]

Since \( F \) is strict culf, Lemma 3.20 gives an isomorphism \( K: I_{G\varpi_{\mathcal{E}}} \to I_{FG\varpi_{\mathcal{E}}} \) fitting into the commutative diagram

\[
\begin{array}{ccc}
I_{G\varpi_{\mathcal{E}}} & \xrightarrow{M_{G\varpi_{\mathcal{E}}}} & \mathcal{C} \\
\downarrow{K^{-1}} & \xleftarrow{S} & \downarrow{F} \\
I_{FG\varpi_{\mathcal{E}}} & \xleftarrow{M_{FG\varpi_{\mathcal{E}}}} & X \\
\end{array}
\]

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Combining the commutativity of (4) and (5) gives $H = F \circ M_{G \pi E} \circ K^{-1} \circ V$, and the hypothesis that $H \circ S = F \circ G$, we have that

$$F \circ M_{G \pi E} \circ K^{-1} \circ V \circ S = H \circ S = F \circ G. \quad (6)$$

Lemma 4.5 says that it is possible to cancel $F$ in Eq. (6) if $F$ is strict culf, which is given by hypothesis. Therefore, the diagram

$$\begin{array}{c}
\mathcal{E}' \\
\downarrow S \\
\mathcal{E}
\end{array} \xrightarrow{V} \begin{array}{c}
\mathcal{E} \\
\downarrow G \\
\mathcal{C}
\end{array} \xrightarrow{K^{-1}} \begin{array}{c}
\mathcal{E} \\
\downarrow M_{G \pi E} \\
\mathcal{C}
\end{array} \quad (7)
$$

commutes. The pieces now fit together to form the commutative diagram

$$\begin{array}{c}
\mathcal{E}' \\
\downarrow S \\
\mathcal{E}
\end{array} \xrightarrow{V} \begin{array}{c}
\mathcal{E} \\
\downarrow G \\
\mathcal{C}
\end{array} \xrightarrow{M_{G \pi E}} \begin{array}{c}
\mathcal{E} \\
\downarrow M_{G \pi E} \\
\mathcal{C}
\end{array} \quad (7)
$$

We define $L: \mathcal{E}' \to \mathcal{C}$ as $L := M_{G \pi E} \circ K^{-1} \circ V$. Finally we establish uniqueness, exploiting that we already have existence given by the functor $L$. Suppose we have two fillers

$$\begin{array}{c}
\mathcal{E}' \\
\downarrow H \\
\mathcal{E}
\end{array} \xrightarrow{G} \begin{array}{c}
\mathcal{E} \\
\downarrow S \\
\mathcal{C}
\end{array} \xrightarrow{F} \begin{array}{c}
\mathcal{C} \\
\downarrow L_1 \\
\mathcal{X}
\end{array} \quad L_2
$$

Now factor $G$ as a stretched map $G'$ followed by a strict culf map $C$,

$$\begin{array}{c}
\mathcal{E} \\
\downarrow S \\
\mathcal{E}'
\end{array} \xrightarrow{G'} \begin{array}{c}
\mathcal{E}' \\
\downarrow H \\
\mathcal{C}
\end{array} \xrightarrow{C} \begin{array}{c}
\mathcal{C} \\
\downarrow C \\
\mathcal{C}
\end{array}
$$

which is possible by Lemma 4.6. Now we can invoke existence of lifts to the situation

$$\begin{array}{c}
\mathcal{E}' \\
\downarrow S \\
\mathcal{E}'
\end{array} \xrightarrow{G'} \begin{array}{c}
\mathcal{E}' \\
\downarrow H \\
\mathcal{C}
\end{array} \xrightarrow{C} \begin{array}{c}
\mathcal{C} \\
\downarrow C \\
\mathcal{C}
\end{array}
$$
since $S$ is stretched and $C$ is culf. This gives the existence of $L'_1$ and $L'_2$ as indicated, and they are stretched by Lemma 4.3 since both $S$ and $G'$ are stretched. But now we are in position to apply Lemma 4.5: Since we have $FL_1 = FL_2$ (as both are equal to $H$), we also have $FCL'_1 = FCL'_2$. Furthermore, since $FC$ is culf and $L'_1$ and $L'_2$ are stretched, we conclude by Lemma 4.5 that already $L'_1 = L'_2$, and therefore also $L_1 = L_2$. 

**Remark 4.8.** In Lemma 4.7, we have that the diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{G} & \mathcal{C} \\
\downarrow S & & \downarrow F \\
\mathcal{E}' & \xrightarrow{L} & \mathcal{X} \end{array}
\]

commutes. By hypothesis $F$ is culf. Therefore, $L$ is a culf if and only if $H$ is culf. On the other hand, by hypothesis $S$ is stretched and applying Lemma 4.3, we have that $L$ is stretched if and only if $G$ is stretched.

**Remark 4.9.** If we had required $X$ to be a discrete algebraic interval, then Lemma 4.7 would say that the stretched and strict culf maps are orthogonal classes of maps in the category of discrete algebraic intervals and simplicial maps, as exploited in the following proposition. It will be important later in 5.2 that we allow $X$ to be more general than just a discrete algebraic interval.

**Proposition 4.10.** The stretched maps as left-hand class and the strict culf functors as right-hand class form a factorisation system in $\mathbf{aInt}$.

**Proof.** The strict culf maps are closed under isomorphism. We have that every simplicial map $F$ in $\mathbf{aInt}$ admits a stretched-culf factorisation by Lemma 4.6. Therefore, we only have to prove that the classes are orthogonal, which follows from Lemma 4.7. 

## 5 The decomposition groupoid $\mathcal{U}$

In Section 4, the stretched-culf factorisation system was defined in $\mathbf{aInt}$, which we can use to define a fibration that encodes the pseudo-simplicial groupoid of discrete algebraic intervals.

Let $\text{Ar}^s(\mathbf{aInt}) \subset \text{Ar}(\mathbf{aInt})$ denote the full subcategory spanned by the stretched functors. $\text{Ar}^s(\mathbf{aInt})$ is a cartesian fibration over $\mathbf{aInt}$ via the domain projection by Lemma 4.2. We now restrict this cartesian fibration to $\Delta \subset \mathbf{aInt}$

\[
\text{Ar}^s(\mathbf{aInt})_{|\Delta} \xrightarrow{\text{f.f.}} \text{Ar}^s(\mathbf{aInt})}
\]

We put

\[
\mathcal{U} := \text{Ar}^s(\mathbf{aInt})_{|\Delta}.
\]

$\mathcal{U} \to \Delta$ is the cartesian fibration of subdivided algebraic discrete intervals. By Lemma 4.2, the cartesian maps in $\mathcal{U}$ are squares

\[
\begin{array}{ccc}
\Delta^k & \xrightarrow{\text{f.culf}} & \Delta^n \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{\text{culf}} & \mathcal{D}.
\end{array}
\]
The cartesian fibration $U \to \Delta$ determines a right fibration $U^{\text{cart}} \to \Delta$, and hence by straightening [4, Theorem 8.3.1] a simplicial groupoid

$$U: \Delta^{\text{op}} \to \hat{\text{Grpd}}$$

where $\hat{\text{Grpd}}$ is the 2-category of large groupoids, functors and natural transformations.

The following result is due to Gálvez–Kock–Tonks [13, Theorem 4.8], who prove it in the more general setting of $\infty$-groupoids.

**Theorem 5.1.** The simplicial groupoid $U: \Delta^{\text{op}} \to \hat{\text{Grpd}}$ is a complete decomposition groupoid.

**5.2. The complete decomposition groupoid $U_X$**

The decomposition groupoid $U: \Delta^{\text{op}} \to \hat{\text{Grpd}}$ is not a strict simplicial object but only a pseudo-simplicial object. In a famous paper [21], Jardine figured out all the 2-cell data and 17 coherence laws for pseudo-simplicial objects in terms of face and degeneracy maps. We overcome the difficulty of working with these coherence laws by building a local strict model, a kind of neighbourhood $U_X \subset U$ around the discrete algebraic intervals of a given rigid decomposition groupoid $X$.

From the viewpoint of cartesian fibrations, the problem with $U \to \Delta$ is that it is not split. It is not possible to define a coherent global choice of cartesian lifts of arrows in $\Delta$. To fix this, we restrict to a full subcategory $U_X$ consisting only of the (subdivided) intervals of $X$ (and not even including isomorphic intervals).

**Definition 5.3.** Let $U_X$ denote the category whose objects are pairs $(f, \phi_{\lambda}: \Delta^n \to I_f)$, with $f \in X_1$ and $\phi_{\lambda}: \Delta^n \to I_f$ an $n$-subdivision of the interval $I_f$. Here $\lambda \in X_n$ and $\text{long}(\lambda) = f$. The morphisms $F: (\phi_{\lambda}, f) \to (\phi_{\lambda'}, f')$ are culf maps $F: I_f \to I_{f'}$ such that $F(\phi_{\lambda}) = \phi_{\lambda'}$. To simplify the notation, we will write the objects of $U_X$ as $(\phi_{\lambda}, I_f)$.

The benefit is that when everything is inside $X$, we can make canonical choices of cartesian lifts. They are given by the following lemma.

**Lemma 5.4.** Let $X$ be a rigid decomposition groupoid, and let $p: \Delta^{n'} \to \Delta^n$ be a map in $\Delta$. For any $n$-simplex $\lambda: \Delta^n \to X$, the commutative triangle

$$\begin{array}{c}
\Delta^{n'} \\
\downarrow^p \\
\Delta^n
\end{array}
\xymatrix{
\Delta^n \\
\ar[r]_\lambda \\
X}
\xymatrix{
\Delta^n \\
\ar[r]_\lambda \\
X}
$$

gives the standard factorisations (3.19) as in the solid square

$$\begin{array}{c}
\Delta^{n'} \\
\downarrow^p \\
\Delta^n
\end{array}
\xymatrix{
\Delta^{n'} \\
\phi_{\lambda'} \ar[r]_{\phi_{\lambda}} \\
I_{f'} \ar[r]_{M_{f'}} \\
I_f \ar[r]_{M_f} \\
X}
\xymatrix{
\Delta^n \\
\ar[r]_\lambda \\
X}
\xymatrix{
\Delta^n \\
\ar[r]_\lambda \\
X}
$$

The statement is that there is a unique filler $c_{\lambda}^p$ as indicated with the dotted arrow, and this map is strict culf.
Proof. Since $\phi_{\lambda'}$ is stretched and $M_f$ is strict culf, the required map $c^\lambda_{\lambda'}$ is given by Lemma 4.7, and it is strict culf by Remark 4.8.

Notice how the ambient $X$ is crucially exploited to characterise the lift uniquely. We also spell out how this choice of lifts act on isomorphisms:

**Lemma 5.5.** Let $X$ be a rigid decomposition groupoid, and consider an isomorphism $F: (I_f, \phi_{\lambda'}) \cong (I_g, \phi_{\mu'})$ in $(U_X)_n$, as on the right in the following diagram. For any map $p: \Delta^n' \to \Delta^n$ in $\Delta$, there is induced an isomorphism $F': (I_f', \phi_{\lambda'}) \cong (I_g', \phi_{\mu'})$ in $(U_X)_{n'}$, as indicated with the dotted arrow:

This $F'$ is characterised as the unique isomorphism in $(U_X)_{n'}$ compatible with the canonical interval inclusions $c^\lambda_{\lambda'}$ and $c^\mu_{\mu'}$ (that is, unique making the whole diagram commute).

Let us explain the notation. The domain and codomain of $F$ are objects in $(U_X)_n$: as usual, the notation refers to an $n$-simplex $\lambda: \Delta^n \to X$ with long edge $f := \text{long}(\lambda)$ and another $n$-simplex $\mu: \Delta^n \to X$ with long edge $g := \text{long}(\mu)$, and $F: I_f \cong I_g$ is an isomorphism of intervals compatible with the subdivisions $\phi_{\lambda}: \Delta^n \to I_f$ and $\phi_{\mu}: \Delta^n \to I_g$ provided by Proposition 3.19.

The map $p: \Delta^n' \to \Delta^n$ gives rise to $n'$-simplices $\lambda'$ and $\mu'$ in $X$:

and induced interval inclusions (strict culf maps)

as in Lemma 5.4.

Proof. Rearranging the bottom and left part of the diagram as

we see that $F'$ is the unique lift existing by Lemma 4.7 since $\phi_{\lambda'}$ is stretched and $c_{\mu'}^{\mu}$ is strict culf.

**Remark 5.6.** In Lemma 5.5, the diagram
commutes. When $p$ is active, we have that $f' = f$ and $g' = g$. Furthermore, if we substitute $F'$ by $F$, the diagram also commutes. By Lemma 4.7, we have that $F' = F$. Therefore, when we work with an active map, we will use $F$ instead of $F'$.

With these preparations, we can establish that $\mathcal{U}_X$ is split:

**Proposition 5.7.** The cartesian fibration $\mathcal{U}_X \to \Delta$ is split. The splitting is given by the cartesian arrows chosen in Lemma 5.4.

**Proof.** That this choice of lifts constitutes a splitting means that it is functorial: composites of chosen lifts are lifts of composites, and lift of identity arrows are identity arrows. For composition: given the solid diagram

there are induced $c^q_\lambda$ and $c^p_\lambda$ making the whole diagram commute. Now by the uniqueness characterisation of $c$-maps, the composite $c^p_\lambda \circ c^q_\lambda$ must be equal to $c^{pq}_\lambda$, as required.

Knowing that the $c$-maps provide a splitting for $\mathcal{U}_X \to \Delta$, there is now induced a strict functor

$U_X : \Delta^{\text{op}} \to \text{Grpd}$

(groupoid-valued functor corresponding to the associated right fibration). We can now simply spell out explicitly what this simplicial groupoid is. On objects, we simply have to describe the fibres: $(U_X)_n$ is thus the groupoid whose objects are subdivided intervals of $X$, say $\phi_\lambda : \Delta^n \to I_f$ (for some $\lambda \in X_n$ with long edge $f$), and whose arrows are the vertical arrows in $\mathcal{U}_X$, namely strictly commutative triangles

Note that since $\mathcal{U}_X$ was defined as full inside $\mathcal{U}$, there are no compatibility requirement with the ‘inclusions’ $M_f : I_f \to X$ and $M_g : I_g \to X$.

The simplicial operators act via cartesian lifts: the formula for $p : \Delta^{n'} \to \Delta^n$ is

$p^*(\Delta^n \xrightarrow{\phi_\lambda} I_f) = (\Delta^{n'} \xrightarrow{\phi^\lambda_{n'}} I_{f'})$
with reference to the chosen cartesian arrow

$$
\Delta^n' \xrightarrow{p} \Delta^n \\
\phi_\lambda \downarrow \\
I_{f'} \xrightarrow{\phi_\lambda} I_f.
$$

(1)

The action of the simplicial operator on an isomorphism in \((U_X)_n\), say \(F : (I_f, \phi_\lambda) \cong (I_g, \phi_\mu)\), is given by Lemma 5.5. Indeed, this lemma is nothing but the standard description of how a vertical isomorphism is transported along a cartesian lift.

(Note that the construction of the isomorphism \(F'\), which in Lemma 5.5 was given using the stretched-culf factorisation system, can also be regarded as the argument why general arrows in a cartesian fibration factor uniquely through cartesian arrows. Indeed the stretched-culf factorisation system is the abstract reason why we have a cartesian fibration.)

**Lemma 5.8.** Let \(p : [n] \to [m] \) be an active map in \(\Delta\). Then \(p^* : (U_X)_m \to (U_X)_n\) is a discrete fibration.

**Proof.** Let \((I_g, \phi_\mu)\) be an object in \((U_X)_m\) and let \(F : (I_f, \phi_\lambda) \to p^*(I_g, \phi_\mu)\) be a morphism in \((U_X)_n\). To provide a lift is to use the same underlying \(F\) (by 5.6, since \(p\) is active), but the compatibility which characterises morphisms in \((U_X)_m\) is now with the \(\phi\) maps from \(\Delta^m\) instead of from \(\Delta^n\). In other words, we need to find the dashed arrow in the diagram

$$
\Delta^m \xrightarrow{\phi_\mu} I_g \\
p \downarrow \\
\Delta^n \xrightarrow{\phi_\lambda} I_f \\
\phi_\eta \downarrow \\
F \to I_g,
$$

which is possible since \(F\) is invertible, in fact \(\eta = M_F F^{-1}(\phi_\lambda)\). Therefore \(p^*\) is a discrete fibration. \(\square\)

**Example 5.9.** In general, the image of an inert map of \(\Delta^{op}\) under \(U_X\) is not a discrete fibration. Let \(\mathcal{C}\) be the category pictured by the following commutative diagram

$$
\begin{array}{ccc}
x & \xrightarrow{f} & z \\
\downarrow{a} & & \downarrow{g} \\
y & \xrightarrow{b} & w \\
\downarrow{a'} & & \downarrow{b'} \\
y' & \xrightarrow{f} & w
\end{array}
$$

Since \(x\) is an initial object and \(w\) is a terminal object, we have that \(N(\mathcal{C}) \simeq I_{gf}\) by Lemma 4.4. Let \(\phi_{gf}\) be the 2-simplex induced by the morphisms \(f\) and \(g\) in \((N(\mathcal{C}))_2\). Let \((N(\mathcal{C}), \phi_{gf}) \in (U_N(\mathcal{C}))_2\) be the interval construction of \(\phi_{gf}\). Applying \(d_0\) to \(\phi_{gf}\), we have that \(d_0(N(\mathcal{C}), \phi_{gf}) = (I_g, \phi_g)\). Let \(id_{I_g} : (I_g, \phi_g) \to (I_g, \phi_g)\) be the identity morphism in \((U_N(\mathcal{C}))_1\).

We can construct two lifts of \(id_{I_g}\) in \((U_N(\mathcal{C}))_2\). Let \(F : (N(\mathcal{C}), \phi_{gf}) \to (N(\mathcal{C}), \phi_{gf})\) be the functor that fixes all the objects in \(\mathcal{C}\) except \(y\) and \(y'\). It is easy to check that \(d_0 F = id_{I_g}\). On the other hand it is straightforward to see that the identity morphism \(id_{I_{gf}}\) satisfies \(d_0 id_{I_{gf}} = id_{I_g}\). Therefore, \(F\) and \(id_{I_{gf}}\) are two different lifts of \(id_{I_g}\).

When \(S\) is a simplicial groupoid, we have a simplicial set induced by the **object functor** \(\text{Obj} : \text{Grpd} \to \text{Set}\), which is defined as forgetting the morphisms. We denote \(\text{Obj} \circ S\) as \(S^0\).
Proposition 5.10. Let $X: \Delta^{\text{op}} \to \text{Grpd}$ be a rigid decomposition groupoid. Then $U_X^0 \cong X^0$.

Proof. The proof is easily deduced from the fact that every object $(I_f, \phi_\lambda)$ in $(U_X^0)_n$ corresponds to some $\lambda \in X_n^0$ by definition of $U_X$.

\[ \begin{array}{l}
\text{Lemma 5.11.} \text{ Let } X \text{ be a rigid decomposition groupoid. The simplicial groupoid } U_X: \Delta^{\text{op}} \to \text{Grpd} \text{ is a decomposition groupoid.} \\
\text{Proof.} \text{ We need to show that for an active-inert pullback square in } \Delta^{\text{op}}, \text{ the image under } U_X \text{ is a homotopy pullback}
\end{array} \]

\[
\begin{array}{ccc}
(U_X)_m & \xrightarrow{g} & (U_X)_n \\
\downarrow h & & \downarrow \overline{\rho} \\
(U_X)_k & \xrightarrow{\overline{\gamma}} & (U_X)_s.
\end{array}
\]

Here $g$ and $\overline{\gamma}$ are active maps, $h$ and $\overline{\rho}$ are inert maps. Since $g$ and $\overline{\gamma}$ are active maps, they are discrete fibrations by Lemma 5.8. Therefore, we can work with strict fibres. By Lemma 1.2, the previous square is a homotopy pullback if and only if for each object $(I_f, \phi_\lambda)$ in $(U_X)_n$, corresponding to some $\lambda \in X_n$, the morphism $h^\prime: \text{Fib}_{(I_f, \phi_\lambda)}(g) \to \text{Fib}_{\overline{\lambda}(I_f, \phi_\lambda)}(\overline{\gamma})$, induced by the morphism $h$, is an equivalence. Here $\text{Fib}_{(I_f, \phi_\lambda)}(g)$ is the strict fibre of $g$ over $(I_f, \lambda)$ and $\text{Fib}_{\overline{\lambda}(I_f, \phi_\lambda)}(\overline{\gamma})$ is the strict fibre of $\overline{\gamma}$ over $\overline{\lambda}(I_f, \phi_\lambda)$.

The fibres $\text{Fib}_{(I_f, \phi_\lambda)}(g)$ and $\text{Fib}_{\overline{\lambda}(I_f, \phi_\lambda)}(\overline{\gamma})$ are discrete groupoids since $g$ and $g'$ are discrete fibrations. Furthermore, as a consequence of Proposition 5.10, we have a bijection between the objects of $X$ and $U_X$. This implies that the diagram

\[
\begin{array}{ccc}
\text{Fib}_{\lambda}(g) & \xrightarrow{h^\prime} & \text{Fib}_{\overline{\lambda}(\overline{\gamma})} \\
\downarrow & & \downarrow \\
\text{Fib}_{(I_f, \phi_\lambda)}(g) & \xrightarrow{h^\prime} & \text{Fib}_{\overline{\lambda}(I_f, \phi_\lambda)}(\overline{\gamma})
\end{array}
\]

commutes. Here $h^\prime: \text{Fib}_{\lambda}(g) \to \text{Fib}_{\overline{\lambda}(\overline{\gamma})}$ is the morphism induced by $h: X_m \to X_k$. Since $X$ is rigid, the morphism $h^\prime$ is an equivalence. Since $\text{Fib}_{(I_f, \phi_\lambda)}(g)$ and $\text{Fib}_{\overline{\lambda}(I_f, \phi_\lambda)}(\overline{\gamma})$ are discrete groupoids, the vertical maps are equivalences by Proposition 5.10. Hence, the map $h^\prime: \text{Fib}_{(I_f, \phi_\lambda)}(g) \to \text{Fib}_{\overline{\lambda}(I_f, \phi_\lambda)}$ is an equivalence.

\[ \begin{array}{l}
\text{Lemma 5.12.} \text{ Let } X \text{ be a rigid decomposition groupoid. Then the decomposition groupoid } U_X \text{ is a complete.} \\
\text{Proof.} \text{ To establish that } U_X \text{ is complete, we need to check that the map } s_0: (U_X)_0 \to (U_X)_1 \text{ is a monomorphism. This means that we need to show that the fibre is either empty or singleton. Remember that the objects in } (U_X)_0 \text{ are given by 0-simplices of } X. \text{ Combining this with the fact that the long edge of a 0-simplex is } s_0(x), \text{ we have that the objects in } (U_X)_0 \text{ are of the form } (I_{s_0(x)}, \phi_x). \text{ Since } s_0 \text{ is active, we have that } s_0 \text{ is a discrete fibration by Lemma 5.8. Therefore, we will consider strict fibres. For } f \in X_1 \text{ denote by } \phi_f: \Delta^1 \to I_f \text{ the unique stretched map. The strict fibre over } (I_f, \phi_f) \in (U_X)_1 \text{ is given by the strict pullback}
\end{array} \]

\[
\begin{array}{ccc}
\text{Fib}_{(I_f, \phi_f)}(s_0) & \xrightarrow{\iota} & (U_X)_0 \\
\downarrow & & \downarrow s_0 \\
1 & \xrightarrow{\iota_{(I_f, \phi_f)}} & (U_X)_1.
\end{array}
\]

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Unless \( f \) is degenerate, the strict fibre is empty. In the degenerate case, consider \((I_{s_0(x)}, \phi_{s_0(x)})\) and \((I_{s_0(y)}, \phi_{s_0(y)})\) two objects in \(\text{Fib}(I_f, \phi_I)(s_0)\) such that \((I_{s_0(x)}, \phi_{s_0(x)}) = (I_{s_0(y)}, \phi_{s_0(y)})\). This means that \(\phi_{s_0(x)} = \phi_{s_0(y)}\). This together with the rigid condition of \(X\) (the map \(s_0: X_0 \to X_1\) is a monomorphism) implies that \(x = y\).

To construct a map from \(X\) to \(U_X\), the following result is necessary:

**Lemma 5.13.** Given an isomorphism \(\alpha: \lambda \to \mu\) in \(X_n\), there is induced an isomorphism\

\[
\begin{array}{c}
\phi_\lambda \\
\Delta^n \\
I_f \\
F_\alpha \\
I_g \\
\phi_\mu
\end{array}
\]

As usual, \(f = \text{long}(\lambda)\) and \(g = \text{long}(\mu)\).

**Proof.** The main point is to prove it just for 1-simplices: given \(\text{long}(\alpha): f \to g\) in \(X_1\), the interval \(I_f\) is the fibre over \(f \in \delta(X_1)\) of the whole simplicial groupoid \(\text{Dec}\_\top \text{Dec}\_\bot X \to \delta(X_1)\), and \(I_g\) is the fibre over \(g\). This is a level-wise fibration over \(\delta(X_1)\), since it is formed entirely of active maps. But in a fibration, any isomorphism \(f \cong g\) between two objects in the base induces an isomorphism \(F_\alpha: I_f \to I_g\) between the fibres. Recall the objects of \(I_f\) are the 2-simplices with long edge \(f\), and the objects of \(I_g\) are the 2-simplices with long edge \(g\). So the isomorphism \(F_\alpha\) sends a 2-simplex with long edge \(f\) to a 2-simplex with long edge \(g\). This forces \(F_\alpha s_0(f) = s_0(g)\) and \(F_\alpha s_1(f) = s_1(g)\), which is equivalent to saying that \(F_\alpha\) is stretched and therefore \(F_\alpha \circ I_f = \circ I_g\).

Coming back to the general case, \(\lambda \cong \mu\): we have the solid outer square:

\[
\begin{array}{c}
\Delta^1 \\
\phi_\lambda \\
I_f \\
F_\alpha \\
I_g \\
\phi_\mu \\
\Delta^n \\
\mu \\
X
\end{array}
\]

The curved triangle commutes by the 1-simplex case already treated. The dotted arrows then exist individually by Proposition 3.19. The triangle that these two dotted arrows form with \(I_f \simeq I_g\) is now forced to commute, since \(\Delta^1 \to \Delta^n\) is active and \(M_g\) is strict culf.

We define a simplicial map \(I: X \to U_X\), using the interval construction:

- the map \(I\) sends an object \(\lambda \in X_n\) to the pair \((I_f, \phi_\lambda)\) where \(f = \text{long}(\lambda)\) and \(\phi_\lambda\) is the \(n\)-simplex induced by \(\lambda\) of Proposition 3.19.

- the map \(I\) sends an arrow \(\alpha: \lambda \to \mu\) in \(X_n\) to the isomorphism \(F_\alpha: (I_f, \phi_\lambda) \to (I_g, \phi_\mu)\) induced by \(\alpha\) of Lemma 5.13. (As usual, \(f = \text{long}(\lambda)\) and \(g = \text{long}(\mu)\).)

**Proposition 5.14.** Let \(X\) be a rigid decomposition groupoid. The simplicial map \(I: X \to U_X\) is strict culf.

**Proof.** Since \(d_1\) is active, we have that \(d_1\) is a discrete fibration by Lemma 5.8. Therefore, as a consequence of Proposition 1.16, to prove that \(I\) is culf it is enough to prove that the following diagram is a strict pullback
which is equivalent to proving that the functor $G: X_2 \to (U_X)_2 \times (U_X)_1 X_1$ induced by the pullback property is an isomorphism. For each $\sigma \in X_2$, the object $G(\sigma)$ is equal to $(I_d(\sigma), \phi(\sigma), d_1(\sigma))$ where $\phi(\sigma)$ is given by Proposition 3.19. For a morphism $\alpha: \sigma \to \sigma'$ put $f = d_1(\sigma)$ and $g = d_1(\sigma')$, the morphism $G(\sigma)$ is equal to $(H, d(\sigma))$. Here $H: (I_f, \phi(\sigma)) \to (I_g, \phi(\sigma'))$ is the isomorphism given by Lemma 5.13.

Recall that $d_1$ is a discrete fibration, this together with Proposition 3.19 allows to construct a functor $R: (U_X)_2 \times (U_X)_1 \to X_2$. For an object $(\phi(\sigma), f)$, the object $R(\phi(\sigma), f)$ is defined as $M_f \phi(\sigma)$ in $X_2$. For a morphism $(H, \pi: f \to g)$, the morphism $R(H, \pi)$ is defined as the morphism $\alpha: M_f \phi(\sigma) \to M_g \phi(\sigma')$ which is the lifting of the arrow $\pi: f \to g$ with respect to $M_f \phi(\sigma)$ and $M_g \phi(\sigma')$. The lift is unique since $d_1$ is a discrete fibration. It is straightforward to verify that $R$ is the inverse of the functor $G$. Note that the diagram is also a homotopy pullback since it is a strict pullback and $d_1$ is a discrete fibration.

5.15. Compatibility of M-maps and subdivided intervals

Given a simplicial map $G: X \to Y$, there will be natural relationships between intervals in $X$ and intervals in $Y$, but to compare them we need to step out to the global $U$, leaving the realms of $U_X$ and $U_Y$.

Lemma 3.20 can be proved in an alternative way as follows:

Lemma 5.16. For any simplicial map between rigid decomposition groupoids $G: X \to Y$, there is a unique stretched map $G_f: I_f \to I_{Gf}$, compatible with M-maps. This means that the diagram

$$
\begin{array}{ccc}
I_f & \xrightarrow{G_f} & I_{Gf} \\
\downarrow & & \downarrow \\
X & \xrightarrow{G} & Y
\end{array}
$$

commutes. If $G$ is culf then $G_f$ is invertible.

Proof. The unique stretched map is given by Lemma 4.7:

$$
\begin{array}{ccc}
\Delta^1 & \xrightarrow{I_{Gf}} & I_{Gf} \\
\downarrow & & \downarrow \\
I_f & \xrightarrow{M_f} & X \\
\end{array}
$$

If $G$ is culf, then the dotted arrow is culf too by Remark 4.8, and since it is both culf and stretched, it is invertible as a consequence of Proposition 4.10.

Lemma 5.17. For any simplicial map between rigid decomposition groupoids $G: X \to Y$, there is a unique stretched map $G_f: I_f \to I_{Gf}$, compatible with subdivision: if we start with
$\lambda : \Delta^n \to X$ (with long edge $f$), then the triangle

\[
\begin{array}{ccc}
\phi_\lambda & \Delta^n & \phi_{G\lambda} \\
I_f & \downarrow & \downarrow \\
G_f & I_{Gf} & \downarrow \\
M_f & \downarrow & \downarrow M_{Gf} \\
X & \downarrow & Y
\end{array}
\]

commutes.

Proof. Orthogonality (Lemma 4.7) for the square

\[
\begin{array}{ccc}
\Delta^1 & \Delta^n & \phi_{G\lambda} \\
I_f & \downarrow & \downarrow \\
G_f & I_{Gf} & \downarrow \\
X & \downarrow & Y
\end{array}
\]

gives a unique filler, which has to be $G_f$, since it is also a filler for the square starting at $\Delta^1$.

Finally we need to establish also the corresponding result for isomorphisms in $X_1$: given $\lambda \cong \mu$ in $X_n$, Lemma 5.13 gives isomorphisms of (subdivided) intervals

\[
\begin{array}{ccc}
\phi_\lambda & \Delta^n & \phi_\mu \\
I_f & \sim & I_g \\
G_f & \sim & I_{Gf} \\
X & \sim & Y
\end{array}
\]

Lemma 5.18. Let $G : X \to Y$ be a simplicial map between rigid decomposition groupoids. For any $f \cong g$ in $X_1$, the diagram

\[
\begin{array}{ccc}
I_f & \xrightarrow{G_f} & I_{Gf} \\
\sim & \downarrow & \sim \\
I_g & \xrightarrow{G_g} & I_{Gg}
\end{array}
\]

commutes. Here the horizontal arrows are given by Lemma 5.16 and the vertical arrows by Lemma 5.13.

Proof. The diagram

\[
\begin{array}{ccc}
\Delta^n & \phi_{G\lambda} & I_{Gf} \\
\phi_\lambda & \phi_\mu & \phi_{G\mu} \\
I_f & \sim & I_g \\
M_f & \sim & M_g \\
X & \sim & Y
\end{array}
\]

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commutes: the middle pentagon region is (1), and the triangles are (2). Inside the outer square we have the following two dotted maps:

\[
\begin{array}{c}
\Delta^n \xrightarrow{\phi_{G\lambda}} I_{Gf} \sim I_{Gg} \\
\phi_\lambda \downarrow \sim I_f \xrightarrow{M_g} X \xrightarrow{G} Y.
\end{array}
\]

The two triangle-shaped regions with dotted arrows also commute: the leftmost triangle is the triangle part of (1) for \(\lambda\), and the rightmost ‘triangle’ is the square part of (1) for \(\mu\). The dotted parallelogram is now forced to commute, since both composites in it are fillers for the outer square, and by orthogonality (Lemma 4.7) only one filler can exist as \(\phi_\lambda\) is stretched and \(M_{Gg}\) is strict culf.

So now we completely control the \(G\)-maps in each simplicial degree individually.

We shall also establish the naturality in simplicial operators: We have seen (in Lemma 5.4) that for any \(p: \Delta^{n'} \to \Delta^n\), there is induced a canonical culf map \(c^p_\lambda: I_{f'} \to I_f\) compatible like this:

\[
\begin{array}{c}
\Delta^{n'} \xrightarrow{p} \Delta^n \\
\phi_{\lambda'} \downarrow \phi_\lambda \\
I_{f'} \xrightarrow{c^p_\lambda} I_f.
\end{array}
\]

(4)

The following lemma shows that these functorialities are compatible.

**Lemma 5.19.** Let \(G: X \to Y\) be a simplicial map between rigid decomposition groupoids. From the situation

\[
\begin{array}{c}
\Delta^{n'} \xrightarrow{p} \Delta^n \\
\phi_{\lambda'} \downarrow \phi_\lambda \\
I_{f'} \xrightarrow{c^p_\lambda} I_f \\
\xrightarrow{G} Gf \\
I_{Gf'} \xrightarrow{c^p_{G\lambda}} I_{Gf}
\end{array}
\]

we get a commutative square

\[
\begin{array}{c}
I_{f'} \xrightarrow{c^p_\lambda} I_f \\
\xrightarrow{G_f} Gf \\
I_{Gf'} \xrightarrow{c^p_{G\lambda}} I_{Gf}
\end{array}
\]

involving the maps from the previous functorialities.

**Proof.** The diagram

\[
\begin{array}{c}
\Delta^{n'} \xrightarrow{p} \Delta^n \xrightarrow{\phi_\lambda} I_f \xrightarrow{G_f} I_{Gf} \\
\phi_{\lambda'} \downarrow \sim I_{f'} \xrightarrow{G_{f'}} I_{Gf'} \xrightarrow{M_{Gf'}} I_{Gf} \\
\xrightarrow{M_g} X \xrightarrow{G} Y.
\end{array}
\]
commutes: the pentagon by construction of the $\phi$-maps (Proposition 3.19), and the two squares by Equation (1).

The outer square has the following two dotted $c$-maps:

The two triangle-shaped regions with dotted arrows commute by construction of the $c$-maps (Lemma 5.4). The dotted parallelogram is now forced to commute, since both composites in it are fillers for the outer square, and only one filler can exist, as $\phi_N$ is stretched and $M_G$ is strict culf.

5.20. Interval construction of an interval

Let $A$ be a discrete algebraic interval (simplicial set), and consider a subdivision of it, $a: \Delta^n \to A$. This whole data describes an $n$-simplex in $U$, which we denote $a: \Delta^n \to U$. Note that the long edge of $a$ is $A$ itself.

We can now apply Proposition 3.19 to $a$ (as an $n$-simplex in $U$) to get

Lemma 5.21. There is a canonical isomorphism $A \simeq I^U_A$ compatible with the subdivision:

Proof. There is a canonical simplicial map $A \to \text{Dec}_\perp \text{Dec}_\top U$, given by sending an $n$-simplex $\lambda: \Delta^n \to A$ to the corresponding stretched $(n+2)$-simplex $\overline{\lambda}: \Delta^{n+2} \to A$, interpreted as an $(n+2)$-simplex in $U$. This simplicial map clearly factors through $I^U_A \to \text{Dec}_\perp \text{Dec}_\top U$. We claim that the induced simplicial map $A \to I^U_A$ is an isomorphism. Indeed, $(I^U_A)_n$ is by definition the strict pullback

which is to say that it is the groupoid of stretched maps $\Delta^{n+2} \to A$, in turn isomorphic to the groupoid of general maps $\Delta^n \to A$, which is the groupoid $A_n$.

Lemma 5.22. For any stretched isomorphism of intervals $A \cong B$, we get from Lemma 5.13 an isomorphism $I_A \cong I_B$. The statement is that these isos are compatible, meaning that the diagram

$$
\begin{array}{ccc}
A & \to & I_A \\
\downarrow & & \downarrow \\
B & \to & I_B \\
\end{array}
$$
commutes.

Proof. This follows since the isomorphisms involved

$$(I^n_A)_n \simeq \text{map}^{\text{str}}(\Delta^{n+2}, A) \simeq \text{map}(\Delta^n, A) = A_n$$

are all natural in stretched isomorphisms $A \simeq B$. \hfill \Box

A simplicial map $G : X \to Y$ between decomposition groupoids is full and faithful if for all objects $x, y \in X$ it induces an equivalence on the mapping groupoids

$$G_{x,y} : \text{map}_X(x, y) \to \text{map}_Y(Gx, Gy).$$

By the way it was defined $U_X$, we have a canonical simplicial map $\jmath : U_X \to U$, defined by $\jmath(I_f, \phi_\lambda) = (I_f, \phi_\lambda)$ for $(I_f, \phi_\lambda) \in (U_X)_n$ and $\jmath F = F$ for $F \in (U_X)_n$. It is straightforward to prove the following result.

**Lemma 5.23.** Let $X$ be a rigid decomposition groupoid. Then the simplicial map $\jmath : U_X \to U$ is full and faithful.

**Remark 5.24.** Gálvez-Carrillo, Kock and Tonks [15] defined the culf classifying map $I' : \Delta/I_X \to \mathfrak{U}$. It takes an $n$-simplex $\lambda : \Delta^n \to X$ to an $n$-subdivided interval $\phi_\lambda : \Delta^n \to I_f$ in $\mathfrak{U}$ (or to the pair $(I_f, \phi_\lambda)$ in $U$). Here $f = \text{long}(\lambda)$ and $\Delta/I_X$ denotes the Grothendieck construction of $X$. In the present paper $I'$ is the map $(j \circ I) : X \to U$, since for each $\lambda \in X_n$, we have that $(j \circ I)(\lambda) = (I_f, \phi_\lambda)$ which is the same as $I'(\lambda)$. We will abuse notation and denote $I'$ as $I$ in Section 6.

**5.25. Comparison with a strictification of $U$**

In this section, we briefly compare our local strict model $U_X$ with a global strictification $\check{U}$ of $U$, proposed by the referee.

There is a well-known construction that replaces a pseudo-functor into $\text{Grpd}$ with a strict functor (see for example [16, §6.4]). In the present case there is a very explicit combinatorial description of such a strictification. An inert map from $\Delta^k$ to $\Delta^n$ is completely determined by the values of 0 and $k$. So we will denote an inert map $\Delta^k \to \Delta^n$ as a pair $(i, j) : \Delta^k \to \Delta^n$ such that $0 \mapsto i$ and $k \mapsto j$. We denote by $P_n$ the poset of inert faces of $\Delta^n$. We define $\check{U}_n$ to be the groupoid of liftings

$$\begin{array}{ccc}
U & \to & \Delta \\
\downarrow \text{dom} & & \\
P_n & \to & \\
\end{array}$$

This gives a whole family of stretched maps $\Delta^k \to \mathfrak{C}$, one for each $(i, j) \in P_n$, and squares

$$\begin{array}{ccc}
\Delta^k & \to & \Delta^{k'} \\
\downarrow & & \downarrow \\
\mathfrak{C}_{ij} & \to & \mathfrak{C}_{ij'} \\
\end{array}$$
for each map \((i, j) \mapsto (i', j')\). Here \(C_{ij}\) and \(C_{i'j'}\) are discrete algebraic intervals. For example an object in \(\hat{U}_2\) is pictured as follows

\[
\begin{array}{cccc}
\Delta^0 & \Delta^1 & \Delta^2 \\
\downarrow & \downarrow & \downarrow \\
C_{22} & C_{11} & C_{12} \\
\end{array}
\]

It is possible to define face and degeneracy maps between the groupoids \(\hat{U}_n\) to assemble them into a strict simplicial groupoid \(\hat{U}\). Informally, the face map \(d_i\) acts by ‘erasing’ all stretched maps containing an index \(i\). The degeneracy maps \(s_i\) repeat the \(i\)th row and the \(i\)th column. We have a canonical equivalence \(\pi_{\text{endvertex}}: \hat{U} \to U\) that on objects erases all the stretched maps except the last one. In case we consider only intervals that come from a fixed strict decomposition groupoid \(X\), we have a strict simplicial groupoid \(\sim U_X\) and a canonical equivalence \(\pi'_{\text{endvertex}}: \sim U_X \to U_X\).

The interval construction \(I: X \to U_X\) from [15] can easily be factored through \(\hat{U}_X\) to give a refined interval construction \(\tilde{I}: X \to \hat{U}_X\) that sends an \(n\)-simplex \(\lambda: \Delta^n \to X\) to \((I_{\lambda c}, \phi_{\lambda c})\) for each \(c: \Delta^k \to \Delta^n \in P_n\). Here \(\phi_{\lambda c}\) is given by Proposition 3.19. For example, for a 1-simplex \(f: \Delta^1 \to X\), the object \(\tilde{I}(f)\) in \((\hat{U}_X)_1\) is given by the following diagram

\[
\begin{array}{ccc}
\Delta^0 & \Delta^1 & I_{f_{d^1}} \\
\downarrow & \downarrow & \downarrow \\
\phi_{f_{d^1}} & \phi_f & \phi_f \\
\end{array}
\]

Note that since \(U_X\) is already strict, all this refined data is redundant.

The four versions of \(U\) and the four interval constructions are compatible, as indicated in the commutative diagram

\[
\begin{array}{ccc}
\hat{U}_X & \sim U & U_X \\
\tilde{I} & \downarrow \pi_{\text{endvertex}} & \downarrow \pi_{\text{endvertex}} \\
X & \sim U & U \\
\end{array}
\]

The original \(U\) is hard to work with, as it is pseudo-simplicial instead of strict simplicial. Both \(\hat{U}_X\) and \(U_X\) are practical because they are strict. \((\hat{U}_X\) is strict but is too redundant.) In this paper we prefer to work with \(U_X\) since in any case most of the arguments are carried out locally at \(X\), and in this situation it is the most direct approach.
6 Gálvez–Kock–Tonks Conjecture

Let $\mathbf{cDcmp}$ denote the $\infty$-category of complete decomposition spaces and culf maps. The construction of the complete decomposition groupoid $U$ was motivated by the following statement:

**Gálvez–Kock–Tonks Conjecture** [15, §5.4] For each decomposition space $X$, the space $\text{map}(\mathbf{cDcmp}(X, U)$ is contractible.

A partial result states that $\text{map}(X, U)$ is connected. An $\infty$-version of this result is Theorem 5.5 in [15]. We include a proof here for two reasons. Firstly, we need to be more precise regarding strictness conditions, and secondly, the proof in [15] does not actually give any argument for naturality in inert maps. As we shall see, this is a subtle issue, and the lack of argument in [15] may be considered a gap in that proof.

In our setting of rigid decomposition spaces, the relevant maps are the strict culf maps. We are now concerned with culf maps to $U$. Recall that $U: \Delta^{\text{op}} \to \text{Grpd}$ is only pseudo-simplicial, but that it is actually strict on active maps. Furthermore, for active $[n'] \to [n]$, the corresponding $U_n \to U_{n'}$ is a fibration. The notion of strict culf map $J: X \to U$ is therefore still meaningful: we do allow pseudo-simplicial maps, but they are still required to be strict on the active part, and the naturality squares on active maps are required to be strict pullbacks. This implies in particular that for the unique active map $\text{long}: [1] \to [n]$, and for every $n$-simplex $\lambda \in X_n$ with long edge $f = \text{long}(\lambda)$, we have a strict equality

$$\text{long}(J_n(\lambda)) = J_1(f).$$  \hspace{1cm} (1)

For general $p: [n'] \to [n]$ in $\Delta$ (not necessarily active) it follows that $J$ takes a strict triangle

$$\begin{array}{ccc}
\Delta^n & \xrightarrow{p} & \Delta^{n'} \\
\downarrow \lambda & & \downarrow \lambda' \\
X & & X
\end{array}$$

to a commutative square of the form

$$\begin{array}{ccc}
\Delta^n & \xrightarrow{p} & \Delta^{n'} \\
\downarrow J_n(\lambda) & & \downarrow J_{n'}(\lambda') \\
J_1(f) & \xrightarrow{e} & J_1(f')
\end{array}$$  \hspace{1cm} (2)

with $e$ culf. (This is to say, it is a cartesian morphism for the right fibration $\mathcal{U}^{\text{cart}} \to \Delta$.)

**Theorem 6.1.** For any rigid decomposition groupoid $X$, the groupoid $\text{map}(X, U)$ of strict culf maps from $X$ to $U$ is connected. More precisely, for any strict culf map $J: X \to U$, there is a natural transformation (actually a modification) $\Gamma: J \Rightarrow I$.

**Proof.** Recall that $I$ was defined in Remark 5.24. There are three steps in the proof: Step 1 is to establish a canonical isomorphism $J_1(f) \cong I_1(f)$ for each $f \in X_1$, and show that this is natural in arrows in $X_1$. Step 2 is to exploit culfness to extend this isomorphism canonically to $J_n(\lambda) \cong I_n(\lambda)$ for each $\lambda \in X_n$ (again naturally in $\lambda$). The construction in Step 2 actually shows that these isomorphisms are natural in active maps in $\Delta$. But in any case, Step 3
consists in showing that the isomorphisms are natural in all maps in $\Delta$, meaning that for any $p: [n'] \to [n]$ in $\Delta$, the naturality square

\[
\begin{array}{ccc}
\Delta^{n'} & \rightarrow & \Delta^n \\
\downarrow & & \downarrow \\
J_1(f') & \rightarrow & J_1(f) \\
\sim & & \sim \\
I_1(f') & \rightarrow & I_1(f)
\end{array}
\]

commutes (as will be detailed).

**Step 1.** Given $f \in X_1$, we construct isomorphisms

$$I^X_f \cong I^{U}_{J_1(f)} \cong J_1(f).$$

Here the first isomorphism is an instance of Lemma 5.4, where $J: X \to U$ plays the role of $G: X \to Y$. The second isomorphism is an instance of Lemma 5.21, where $J_1(f)$ plays the role of $A$.

We should now argue why these isomorphisms are natural in arrows in $X_1$: given $f \cong g$, we need to check that this naturality square commutes:

$$\begin{array}{ccc}
J_1(f) & \rightarrow & I_1(f) \\
\downarrow & & \downarrow \\
J_1(g) & \rightarrow & I_1(g)
\end{array}$$

Since the vertical isomorphisms are composites of isos from Lemma 5.16 and from Lemma 5.21, the naturality in maps inside $X_1$ follows from the naturality expressed by Lemma 5.18 and Lemma 5.22.

**Step 2.** We now show that these isomorphisms $J_1(f) \cong I_1(f)$ extend to isomorphisms $J_n(\lambda) \cong I_n(\lambda)$ for each $n$, using that both $I$ and $J$ are strict culf. We have

$$\begin{array}{ccc}
X_1 & \rightarrow & X_n \\
\downarrow & & \downarrow \\
(U_X)_1 & \rightarrow & (U_X)_n
\end{array}$$

Since these horizontal maps are fibrations, we can describe the objects in $(U_X)_n$ as follows. To give an object $J_n(\lambda)$ in $(U_X)_n$ is to give the underlying interval $J_1(f)$ and an object in the fibre over $J_1(f)$. Since the square is a strict pullback, to give an object in the fibre of the bottom horizontal map is the same as giving an object in the fibre over $f$ of the top horizontal maps, i.e. a subdivision, i.e. an object $\lambda \in X_n$. This same description holds for $I$. So to give, for a fixed $\lambda \in X_n$, an isomorphism $J_n(\lambda) \cong I_n(\lambda)$ is to give an isomorphism $J_1(f) \cong I_1(f)$, and keep the $\lambda$ in the fibres fixed.

As in Step 1, we should now argue why these isomorphisms are natural in arrows in $X_n$: given $\lambda \cong \mu$ in $X_n$, we need to check that this naturality square commutes:

$$\begin{array}{ccc}
J_n(\lambda) & \rightarrow & I_n(\lambda) \\
\downarrow & & \downarrow \\
J_n(\mu) & \rightarrow & I_n(\mu)
\end{array}$$
The argument is the same as that given in degree 1, but invoking now Lemma 5.17 instead of Lemma 5.16.

Note that the isomorphisms are natural in all active maps $[n] \to [1]$ by construction, and therefore, by the standard prism-lemma argument, are also natural in all active maps.

**Step 3.** The final step is to show that the isomorphisms are also natural in inert maps, and in fact we prove uniformly that they are natural in all maps $p : [n'] \to [n]$ in $\Delta$. Given $\lambda : \Delta^n \to X$ (with long edge $f$) and a map $p : \Delta^{n'} \to \Delta^n$, put $\lambda' := \lambda \circ p$ (with long edge $f'$), so that we have

\[
\begin{array}{ccc}
\Delta^{n'} & \xrightarrow{p} & \Delta^n \\
\alpha' \downarrow & & \downarrow \alpha \\
J_1(f') & \rightleftharpoons & J_1(f)
\end{array}
\]

which is sent by $J$ to

\[
\begin{array}{ccc}
\Delta^{n'} & \xrightarrow{p} & \Delta^n \\
\alpha' \downarrow & & \downarrow \alpha \\
J_1(f') & \rightleftharpoons & J_1(f)
\end{array}
\tag{3}
\]

By Step 1 we have isomorphisms in each simplicial degree, which are strictly compatible with the subdivisions by Step 2, to give commutative triangles

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{a} & J_1(f') \\
\downarrow J_1(f) & & \downarrow J_1(f) \\
I_1(f') & \xrightarrow{\sim} & I_1(f)
\end{array}
\]

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{a} & J_1(f) \\
\downarrow J_1(f) & & \downarrow J_1(f) \\
I_1(f) & \xrightarrow{\sim} & I_1(f)
\end{array}
\]

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{a} & J_1(f') \\
\downarrow J_1(f) & & \downarrow J_1(f) \\
I_1(f') & \xrightarrow{\sim} & I_1(f)
\end{array}
\tag{4}
\]

These diagrams together with Lemma 5.4 ensure that the following outer rectangle commutes:

\[
\begin{array}{ccc}
\Delta^{n'} & \xrightarrow{p} & \Delta^n \\
\alpha' \downarrow & & \downarrow \alpha \\
J_1(f') & \rightleftharpoons & J_1(f) \\
\downarrow \sim & & \downarrow \sim \\
I_1(f') & \rightleftharpoons & I_1(f) \\
\downarrow & & \downarrow \\
X & & X
\end{array}
\]

We furthermore have the two diagonal dotted arrows indicated. The leftmost triangle-shaped region is (3), and the right-most triangle is given in Lemma 5.4. The composed dotted parallelogram is now forced to commute, since the composites in it are fillers for the outer square, and only one filler can exist since $a'$ is stretched and $M_f$ is strict culf.

The dotted arrows are the cartesian lifts of $p$ to $J_1(f)$ and $I_1(f)$, and the commutativity (by Lemma 5.19) of

\[
\begin{array}{ccc}
\Delta^{n'} & \xrightarrow{p} & \Delta^n \\
\alpha' \downarrow & & \downarrow \alpha \\
J_1(f') & \xrightarrow{\sim} & J_1(f) \\
\downarrow \sim & & \downarrow \sim \\
I_1(f') & \xrightarrow{\sim} & I_1(f)
\end{array}
\]

now shows that the isomorphisms $J_n \rightleftharpoons I_n$ are natural in $p$ (and thereby with the whole simplicial structure).
6.2. Modifications

Theorem 6.1 implies that every natural transformation from $X$ to $U$ is isomorphic to $I$. Therefore, to prove the conjecture, we only need to prove that $I$ does not admit other self-modifications than the identity. Thus, we will introduce the notion of modification in the context in which we need it.

A modification between two natural transformations is a family of 2-cells in the 2-category of (small) categories that satisfies some coherence conditions as indicated in the following definition:

**Definition 6.3.** [3, Definition 7.3.1] Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories. Let $F, G: \mathcal{C} \to \mathcal{D}$ be two functors and $\alpha, \beta: F \Rightarrow G$ be two natural transformations from $F$ to $G$. A modification $\Gamma: \alpha \Rightarrow \beta$ assigns to each object $x$ in $\mathcal{C}$ a 2-cell $\Gamma_x: \alpha_x \Rightarrow \beta_x$ of $\mathcal{D}$ compatibly with the 2-cell components of $F$ and $G$ in the sense of the equation

$$F(x) \xrightarrow{\Gamma_x} G(x) \quad F(y) \xrightarrow{\Gamma_y} G(y).$$

We are interested in the case where $\mathcal{C} = \Delta$ and $\mathcal{D} = \text{Grpd}$, and where $F = X$ and $G = U_X$, and where $\alpha$ and $\beta$ are both equal to $I$. In this case, Definition 6.3 can be written as follows.

**Definition 6.4.** A modification $\Gamma: I \to I$ assigns to each $[n]$ in $\Delta$ a natural transformation $\Gamma_n: I_n \to I_n$ in $\text{Grpd}$ such that for each $n \geq 1$ the following equations hold for each $0 \leq i \leq n$ and $0 \leq j < n$:

$$X_n \xrightarrow{\Gamma_n} (U_X)_n \quad X_{n-1} \xrightarrow{\Gamma_{n-1}} (U_X)_{n-1}$$

$$X_n \xrightarrow{\Gamma_n} (U_X)_n \quad X_{n-1} \xrightarrow{\Gamma_{n-1}} (U_X)_{n-1}$$

$$X_n \xrightarrow{\Gamma_n} (U_X)_n \quad X_{n-1} \xrightarrow{\Gamma_{n-1}} (U_X)_{n-1}$$

**Remark 6.5.** We can define a modification $\Gamma: I \to I$ level by level, so let $\Gamma_n: I_n \to I_n$ be a component of the modification $\Gamma$. Given $\lambda \in X_n$, let $\phi_\lambda$ be the $n$-simplex induced by $\lambda$ constructed in Proposition 3.19 and $f = \text{long}(\lambda)$. The modification $\Gamma$ assigns to $\lambda$ an invertible morphism $\Gamma^\lambda_n: (I_f, \phi_\lambda) \to (I_f, \phi_\lambda)$ in $(U_X)_n$. The morphism $\Gamma^\lambda_n$ has associated an underlying map $\Gamma^\lambda_n: I_f \to I_f$.

Let $p: [m] \to [n]$ be an active map. By Remark 5.6, we have that $p^* \Gamma_n = \Gamma^\lambda_n$. This implies that

$$\Gamma_m^p = \Gamma_n^\lambda$$

where $\Gamma_m^p: I_f \to I_f$ is the underlying map of $\Gamma_m^p$. The difference between $\Gamma_n^\lambda$ and $\Gamma_m^p$ is that the first one respects the $n$-subdivision $\phi_\lambda$ and the other respects the $m$-subdivision $\phi_{\lambda p}$.
Lemma 6.6. Let $X$ be a rigid decomposition groupoid. The mapping groupoid $\text{map}_{\text{cDcmp}}(X, U_X)$ is contractible.

Proof. Theorem 6.1 shows that we only have to prove that $I$ does not admit other self-modifications $\Gamma$ than the identity. Let $\lambda$ be an $n$-simplex in $X$ and put $f = \text{long}(\lambda)$. Let $\Gamma$ a modification, with components $\Gamma_n: I_n \rightarrow I_n$ and let $\Gamma^\lambda_n: (I_f, \phi_\lambda) \rightarrow (I_f, \phi_\lambda)$ of Remark 6.5.

Since $\text{long}: [1] \rightarrow [n]$ is an active map in $\Delta$, by Remark 6.5, we have that $\Gamma^\lambda_n = \Gamma^f I$ where $\Gamma^f I: (I_f, \phi_f) \rightarrow (I_f, \phi_f)$. On the other hand, given a morphism $\alpha: \sigma \rightarrow \sigma$ in $I_f$, Lemma 3.17 gives a stretched 3-simplex $\eta_\alpha: \Delta^3 \rightarrow [n]$ such that $d_\bot d_\top \eta_\alpha = \alpha$.

The modification $\Gamma$ assigns to $M^\alpha_\lambda f \eta_\alpha$ an invertible map $\Gamma^\eta_\alpha_3: (I_f, \eta_\alpha) \rightarrow (I_f, \eta_\alpha)$ such that $\Gamma^\eta_\alpha_3 \eta_\alpha = \eta_\alpha$. Furthermore, $\Gamma^\eta_\alpha_3(\alpha) = \Gamma^\eta_\alpha_3 d_\bot d_\top \eta_\alpha$.

By Definition 6.4, we have the equality

$$X_3 \xrightarrow{I_3} (U_X)_3 \xrightarrow{d_1 d_4} (U_X)_1 = X_3 \xrightarrow{d_1 d_4} X_1 \xrightarrow{I_1} (U_X)_1.$$ 

This equation implies that $d_1 d_4(\Gamma^\eta_\alpha_3) = \Gamma^f I$. Since $d_1 d_4$ is active, we have that $\Gamma^\eta_\alpha_3 = \Gamma^f I$ by Remark 6.5. Hence altogether, for each $\alpha \in I_f$

$$\Gamma^\lambda_n(\alpha) = \Gamma^f I(\alpha) = \Gamma^\eta_\alpha_3(\alpha) = \alpha.$$ 

Since $\Gamma^\lambda_n$ is the identity arrow for each $\lambda \in X_n$, we have that $\Gamma$ is the identity modification.

Theorem 6.7. Let $X$ be a rigid decomposition groupoid. The mapping groupoid $\text{map}_{\text{cDcmp}}(X, U)$ is contractible.

Proof. Each natural transformation from $X$ to $U$ is isomorphic to $I$ by Theorem 6.1. We can factor $I: X \rightarrow U$ as

$$X \xrightarrow{I} U \xrightarrow{j} U_X.$$ 

Since $j: U_X \rightarrow U$ is full and faithful (5.23), we have that $j: \text{map}_{\text{cDcmp}}(X, U_X) \rightarrow \text{map}_{\text{cDcmp}}(X, U)$ is also full and faithful. Since $\text{map}_{\text{cDcmp}}(X, U_X)$ is contractible (6.6) and $\text{map}_{\text{cDcmp}}(X, U)$ is connected (6.1), it follows that $\text{map}_{\text{cDcmp}}(X, U)$ is contractible.

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References

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