NEUMANN PROBLEM FOR A CLASS OF FULLY NONLINEAR EQUATIONS

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Abstract. In this paper, we establish a priori estimates for a class of fully nonlinear equations with Neumann boundary conditions. By the continuity method, we have obtained the existence theorem for the Neumann problem.

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1. Introduction

In this paper, we consider a priori estimates for the following Neumann problem for a class of fully nonlinear elliptic type equations,

\[
\begin{aligned}
\sigma_k(D^2u) + \alpha(x)\sigma_{k-1}(D^2u) &= \sum_{l=0}^{k-2} \alpha_l(x)\sigma_l(D^2u) & \text{in } \Omega, \\
D_\nu u &= \varphi(x, u) & \text{on } \partial\Omega.
\end{aligned}
\]  

Here \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial\Omega \) and \( D^2u \) is the Hessian matrix of the function \( u \). \( \nu \) is outer normal vector of \( \partial\Omega \). \( \alpha_l(x) > 0 \) with \( l = 0, 1, \cdots, k - 2 \) and \( \alpha(x) \) are given smooth functions in \( \Omega \). \( \sigma_m(A) \) denotes the \( m \)-th elementary symmetric function of an \( n \times n \) symmetric matrix \( A \) given by

\[
\sigma_m(A) = \sigma_m(\lambda(A)) = \sum_{i_1 < i_2 < \cdots < i_m} \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_m},
\]

where \( \lambda(A) = (\lambda_1, \lambda_2, \cdots, \lambda_n) \) denotes the eigenvalues of the matrix \( A \) for \( 1 \leq m \leq n \) and \( \sigma_0(A) = 1 \). Specially, it is the Hessian equation corresponds to the case that \( \alpha(x) = \alpha_l(x) \equiv 0 \). Let \( \Gamma_k \) be an open convex cone in \( \mathbb{R}^n \):

\[
\Gamma_k = \{ \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n | \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0 \}.
\]
This kind of equations is motivated from the study of many important geometric problems. For example, the problem of prescribing convex combination of area measures was proposed in [14]. Another important example is Fu-Yau equation arising from the study of the Hull-Strominger system in theoretical physics, which is an equation that can be written as the linear combination of the first and the second elementary symmetric functions in [4, 5].

A general notion of fully nonlinear elliptic equations was considered by Krylov in [10]. He considered Dirichlet problem of following degenerate equations,

$$\sigma_k(D^2u) = \sum_{l=0}^{k-1} \alpha_l(x)\sigma_l(D^2u)$$

with $\alpha_l(x) \geq 0$ for $0 \leq l \leq k - 1$. And he observed that the natural admissible cone to make equation elliptic is also the $\Gamma_k$, which is the same as the Hessian equations case. Recently, Guan-Zhang in [6] don’t require the sign of $\alpha(x)$ and prove that the admissible solution in the sense that $\lambda(D^2u) \in \Gamma_{k-1}$. They also study the Dirichlet problem of the corresponding degenerate equations as an extension of the equations studied by Krylov. In this paper, we consider the Neumann boundary problem for this kind of equations in [6], and we state our main theorems as follows.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a $C^3$ bounded domain. Suppose $u \in C^2(\bar{\Omega}) \cap C^3(\Omega)$ with $\lambda(D^2u) \in \Gamma_k$ is a solution to the Neumann boundary problem (1.1). Here $|u| \leq M_0$, $\varphi$ is given function defined on $\bar{\Omega} \times [-M_0, M_0]$, and $|\varphi(x, u)|_{C^3(\bar{\Omega} \times [-M_0, M_0])} \leq L_1$. Then there exists a positive constant $M_1$ depending on $n$, $k$, $\Omega$, $L_1$, $M_0$, $\inf \alpha_l(x)$, $\alpha_l(x)$ such that

$$\sup_{\Omega} |Du| \leq M_1.$$  

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^n$ be a $C^4$ bounded uniformly convex domain in $\mathbb{R}^n$. Suppose $u \in C^3(\bar{\Omega}) \cap C^4(\Omega)$ is a solution to the Neumann boundary problem (1.1), with $\lambda(D^2u) \in \Gamma_k$. Here $\varphi$ is given function defined on $\bar{\Omega} \times [-M_0, M_0]$, $|\varphi(x, u)|_{C^3(\bar{\Omega} \times [-M_0, M_0])} \leq L_1$. Then there exists a positive constant $M_2$ depending on $n$, $k$, $\Omega$, $\inf \alpha_l(x)$, $\alpha_l(x)$, $\alpha_l(x)$, $M_0$, $M_1$, $L_1$, such that

$$\sup_{\Omega} |D^2u| \leq M_2.$$
To guarantee the existence of the classical solution, it is necessary to assume the structure condition:

\( \varphi_z \equiv \frac{\partial \varphi(x, z)}{\partial z} \leq -\gamma_0 < 0. \)

**Theorem 1.3.** Let \( \Omega \) be a \( C^4 \) bounded uniformly convex domain in \( \mathbb{R}^n \) and \( \alpha(x) < 0, \inf_l \alpha_l > 0 \). Here \( \varphi \) is given function defined on \( \bar{\Omega} \times [-M_0, M_0] \), \( |\varphi(x, u)|_{C^3} \leq L_1 \), and (1.2) holds. Then there exists a unique solution \( u \in C^{3,\alpha}(\bar{\Omega}) \) with \( \lambda(D^2u) \in \Gamma_k \) for the Neumann problem (1.1).

The Neumann or oblique problems of partial differential equations were widely studied. Lions-Trudinger-Urbas in [12] treated the elliptic Neumann boundary problem for the equation of Monge-Ampère type by using the convexity of the domain in the second order derivative estimates. Urbas [17] studied oblique boundary value problems for Hessian equations in two dimension. For the two-dimensional curvature equations, Urbas [18] which is a sequel to [17] studied nonlinear oblique boundary value problems for curvature equations, and obtained the existence of smooth solutions with certain strong structural hypotheses on the boundary condition. The semilinear Neumann boundary is the special case and the two dimensionality played a crucial role for the second derivative estimates in his paper. For Hessian equations when \( 2 \leq k \leq n - 1 \), Trudinger [20] established the existence theorem in a ball. Recently, Ma-Qiu [13] have proved the existence of a classical solution to a Neumann boundary problem for Hessian equations in uniformly convex domain. Chen-Zhang [2] obtain some important inequalities of Hessian quotient operators, and establish the existence theorem. Jiang-Trudinger [7, 8, 9], studied the general oblique boundary problem for augmented Hessian equations with some regular conditions and concavity conditions.

The rest of this paper is organized as follows. In Section 2, we give some definitions and important lemmas. In Section 3, we prove \( C^0 \) and \( C^1 \) estimates. In Section 4, we shall derive global and boundary estimates for second order derivative of \( k \)-convex solutions. In Section 5, we give the proof for the existence in some special case.

2. Preliminaries

In this section, we introduce some notations and key lemmas which will be used later, and omit some details of the proof for lemmas. We refer the readers to see the details in [2, 6].
For the convenience of notations, we denote
\begin{equation}
G_k(\lambda) = \frac{\sigma_k}{\sigma_{k-1}}(\lambda), \quad G_l(\lambda) = -\frac{\sigma_l}{\sigma_{k-1}}(\lambda) \quad \text{for} \ 0 \leq l \leq k - 2,
\end{equation}
then by the equation (1.1), we denote
\begin{equation}
G(D^2 u) := G_k(D^2 u) + \sum_{l=0}^{k-2} \alpha_l(x) G_l(D^2 u) = -\alpha(x).
\end{equation}

**Lemma 2.1.** If $u$ is a $C^2(\Omega)$ function with $\lambda(D^2 u) \in \Gamma_{k-1}$, then the operator
\begin{equation}
G(\lambda(D^2 u)) := \frac{\sigma_k(\lambda(D^2 u))}{\sigma_{k-1}(\lambda(D^2 u))} - \sum_{l=0}^{k-2} \alpha_l(x) \frac{\sigma_l(\lambda(D^2 u))}{\sigma_{k-1}(\lambda(D^2 u))}
\end{equation}
is elliptic and concave. (See the proof in [6])

At any $x_0 \in \Omega$, by differentiating equation (1.1) twice, we have,
\begin{equation}
G^{ii} u_{iip} = (-\alpha)_p,
\end{equation}
and
\begin{equation}
G^{ij,rs} u_{ijp} u_{rsp} + G^{ii} u_{iipp} = (-\alpha)_{pp} - 2 \sum_{l=0}^{k-2} (\alpha_l)_p G^{ii} u_{iip} - \sum_{l=0}^{k-2} (\alpha_l)_{pp} G_l.
\end{equation}
Moreover, by Lemma 2.1 the operator $G$ is elliptic and concave in $\Gamma_{k-1}$ cone.

**Lemma 2.2.** If $u$ is a $C^2(\Omega)$ function with $\lambda(D^2 u) \in \Gamma_{k-1}$, then
\begin{equation}
\sum_{i=1}^{n} G^{ii} \lambda_i = G_k - \sum_{l=0}^{k-2} (k-l-1) \alpha_l(x) G_l
\end{equation}
\begin{equation}
= -\alpha(x) + \sum_{l=0}^{k-2} (k-l) \alpha_l(x) \frac{\sigma_l}{\sigma_{k-1}}.
\end{equation}
Moreover, there is a positive constant $L_2$ depending on $n$, $k$, $\|\alpha\|_{C^0}$, $\|\alpha_l\|_{C^0}$, $\inf_l \alpha_l$ such that $|\sum_{i=1}^{n} G^{ii} \lambda_i| \leq L_2$.

**Proof.** Since $G_k$ is homogeneous of degree one and $G_l$ is homogeneous of degree $-(k-1-l)$ for $0 \leq l \leq k-2$, we can obtain the first equality in (2.6), then by the equation in (1.1) we have the second equalities in (2.6). Next we need to bound $|G^{ii} \lambda_i|$ from above.
Recall that $\lambda(D^2u) \in \Gamma_{k-1}$, and so we have either $\sigma_k(\lambda) \geq 0$ or $\sigma_k(\lambda) \leq 0$. If $\sigma_k(\lambda) \leq 0$, then we have $\frac{\sigma_i}{\sigma_{k-1}} \leq \frac{\alpha(x)}{\inf \alpha(x)}$. So we are done. Next, if $\sigma_k(\lambda) \geq 0$, we shall discuss into two cases. We note that if there is a constant $N$ such that

**Case 1.** If $\frac{\sigma_k}{\sigma_{k-1}} \leq N$, then we get,

$$\frac{\sigma_i}{\sigma_{k-1}} \leq \frac{1}{\inf \alpha_l} (\alpha(x) + \frac{\sigma_k}{\sigma_{k-1}}) \leq \frac{1}{\inf \alpha_l} (\alpha(x) + N).$$

**Case 2.** If $\frac{\sigma_k}{\sigma_{k-1}} \geq N$, i.e. $\frac{\sigma_{k-1}}{\sigma_k} \leq \frac{1}{N}$. Note that by the Newton-MacLaurin’s inequality,

$$\frac{\sigma_i}{\sigma_{k-1}} \leq \tilde{C}(\frac{\sigma_{k-1}}{\sigma_k})^{k-1-l}.$$ 

Thus

$$|\sum_{i=1}^{n} G^{ii} \lambda_i| \leq |\alpha| + \sum_{l=0}^{k-2} \tilde{C} |\alpha_l| (k-l) (\frac{1}{N})^{k-1-l}.$$ 

Therefore, we have obtained $|\sum_{i=1}^{n} G^{ii} \lambda_i| \leq L_2$. 

**Lemma 2.3.** If $u$ is a $C^2(\Omega)$ function with $\lambda(D^2u) \in \Gamma_{k-1}$, then

$$\sum_{i=1}^{n} G^{ii} = \frac{n-k+1}{k} + \sum_{l=0}^{k-2} \alpha_l \{ (n-k+2) \frac{\sigma_l \sigma_{k-2}}{\sigma_{k-1}^2} - (n-l+1) \frac{\sigma_{l-1} \sigma_{k-1}}{\sigma_{k-1}^2} \}. \quad (2.7)$$

Moreover, there is a positive constant $L_3$ depending on $n$, $k$, $\|\alpha\|_{C^0}$, $\|\alpha_l\|_{C^0}$ and $\inf_\Omega \alpha_l(x)$, such that $\sum_{i=1}^{n} G^{ii} \leq L_3$.

**Proof.** First, we get (2.1) by direct computation. Then we shall use the non-degenerate assumption. Without loss of generality we may assume that $\alpha_l \geq \hat{c}_0 > 0$ to control the leading term $\frac{\sigma_i}{\sigma_{k-1}}$, when $0 \leq l \leq k-2$. Note that by equation (1.1),

$$\sigma_k + \alpha(x) \sigma_{k-1} \geq \alpha_l(x) \sigma_l \geq \hat{c}_0 \sigma_l.$$ 

Recall that $\lambda(D^2u) \in \Gamma_{k-1}$, thus we have $\sigma_k(D^2u) \geq 0$ or $\sigma_k(D^2u) \leq 0$. If $\sigma_k(D^2u) \leq 0$, then $\frac{\sigma_i}{\sigma_{k-1}} \leq \frac{|\alpha(x)|}{\alpha_0}$. Next for $\sigma_k(D^2u) \geq 0$, we can divide into two cases similar to the proof in Lemma [2.2] then we obtain the upper bound for $\sum_{i=1}^{n} G^{ii}$.

**Lemma 2.4.** If $u$ is a smooth function with $\lambda(D^2u) \in \Gamma_{k-1}$, then

$$\sum_{i=1}^{n} G^{ii} \geq \frac{n-k+1}{k} + \sum_{l=0}^{k-2} C(n, k, l) \alpha_l(x) \frac{\sigma_l \sigma_{k-2}}{\sigma_{k-1}^2}.$$
The exact calculation for Lemma 2.4 is in [6]. The following lemmas play an important role in the proof of derivative estimates. The idea of the proof for these lemmas comes from the paper in [2].

**Lemma 2.5.** Suppose \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Gamma_k, k \geq 2, \) and \( \lambda_1 < 0. \) Then we have

\[
\frac{\partial G}{\partial \lambda_1} \geq \frac{n}{k} \frac{1}{(n-k+2)^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i}.
\]

**Proof.** From the Lemma 2.5 in [2], we have

\[
\frac{\partial G}{\partial \lambda_1} \geq \frac{n}{k} \frac{1}{(n-k+1)^2} \sum_{i=1}^{n} \frac{\partial G_k}{\partial \lambda_i},
\]

and

\[
\frac{\partial \left[ \frac{\sigma_{k-1}}{\sigma_l} \right]}{\partial \lambda_1} \geq \frac{n(k-1-l)}{(k-1)(n-l)(n-k+2)} \sum_{i=1}^{n} \frac{\partial \left[ \frac{\sigma_{k-1}}{\sigma_l} \right]}{\partial \lambda_i}.
\]

Thus

\[
\frac{\partial G}{\partial \lambda_1} \geq \frac{n}{k} \frac{1}{(n-k+2)^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i} + \frac{1}{k} \left[ \sum_{i=1}^{n} \frac{\partial G_k}{\partial \lambda_i} + \sum_{l=0}^{k-2} \alpha_l(x) \frac{\partial \left[ \frac{\sigma_{k-1}}{\sigma_l} \right]}{\partial \lambda_i} \right] \frac{n(k-1-l)}{(k-1)(n-l)(n-k+2)} \sum_{i=1}^{n} \frac{\partial \left[ \frac{\sigma_{k-1}}{\sigma_l} \right]}{\partial \lambda_i}.
\]

\[
= \frac{n}{k} \frac{1}{(n-k+2)^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i} + \frac{n}{k} \frac{1}{(n-k+1)^2} \sum_{i=1}^{n} \frac{\partial G_k}{\partial \lambda_i} + \sum_{l=0}^{k-2} \alpha_l(x) \frac{n(k-1-l)}{(k-1)(n-l)(n-k+2)} \sum_{i=1}^{n} \frac{\partial G_i}{\partial \lambda_i}.
\]

\[
\geq \frac{n}{k} \frac{1}{(n-k+2)^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i} + \sum_{l=0}^{k-2} \alpha_l(x) \frac{\partial G_k}{\partial \lambda_i}.
\]

\[
= \frac{n}{k} \frac{1}{(n-k+2)^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i}.
\]

\[\square\]

**Lemma 2.6.** Suppose \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Gamma_k, k \geq 2, \) and \( \lambda_2 \geq \cdots \geq \lambda_n. \) If \( \lambda_1 > 0, \lambda_n < 0, \lambda_1 \geq \delta \lambda_2, \) and \( -\lambda_n \geq \epsilon \lambda_1 \) for small positive constant \( \delta \) and \( \epsilon, \) then
we have

\[(2.12) \quad \sigma_m(\lambda|1) \geq c_1 \sigma_m(\lambda), \quad \text{for } m = 0, 1, \ldots, k-1,\]

where \(c_1 = \min \left\{ \frac{e^2 \delta^2}{2(n-2)(n-1)}, \frac{e^2}{4(n-1)} \right\} \). Moreover, we have

\[(2.13) \quad \frac{\partial G}{\partial \lambda_1} \geq c_2 \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i},\]

where \(c_2 = \frac{n c_1^2}{k (n-k+2)^2}\).

**Proof.** In the paper [2], inequality (2.12) has been proved. Here we omit the details of the proof and just prove the (2.13) based on (2.12). We know from Lemma 2.6 in [2],

\[(2.14) \quad \frac{\partial G}{\partial \lambda_1} \geq \frac{n}{k(n-k+1)^2} \frac{1}{c_1^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i}.\]

\[(2.15) \quad \frac{\partial \frac{\sigma_{k-1}}{\sigma_i}}{\partial \lambda_1} \geq \frac{n(k-l-1)c_1^2}{(n-l)(k-1)(n-k+2)} \sum_{i=1}^{n} \frac{\partial \frac{\sigma_{k-1}}{\sigma_i}}{\partial \lambda_i}.\]

\[(2.16) \quad \frac{\partial G}{\partial \lambda_1} \geq \frac{n}{k(n-k+1)^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i} + \sum_{l=0}^{k-2} \alpha_l(x) \frac{\sigma_i^2}{\sigma_{k-1}^2} \frac{\sigma_{k-1}}{\sigma_i} \sum_{i=1}^{n} \frac{\partial \frac{\sigma_{k-1}}{\sigma_i}}{\partial \lambda_i}\]

\[\geq n \frac{c_1^2}{k(n-k+1)^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i} + \sum_{l=0}^{k-2} \alpha_l(x) \frac{n(k-l-1)c_1^2}{(n-l)(k-1)(n-k+2)} \sigma_i^2 \sum_{i=1}^{n} \frac{\partial \frac{\sigma_{k-1}}{\sigma_i}}{\partial \lambda_i}\]

\[\geq n \frac{c_1^2}{k(n-k+1)^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i} + \sum_{l=0}^{k-2} \alpha_l(x) \sum_{i=1}^{n} \frac{\partial G_i}{\partial \lambda_i}\]

\[\geq n \frac{c_1^2}{k(n-k+2)^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i} + \sum_{l=0}^{k-2} \alpha_l(x) \frac{n c_1^2}{(k-1)(n-k+2)^2} \sum_{i=1}^{n} \frac{\partial G_i}{\partial \lambda_i}\]

\[= n \frac{c_1^2}{k(n-k+2)^2} \sum_{i=1}^{n} \frac{\partial G}{\partial \lambda_i}.\]
We set the distance function of $\Omega$,
\[ d(x) = \text{dist}(x, \partial\Omega), \]
and
\[ \Omega_\mu = \{ x \in \Omega : d(x) < \mu \}. \]
It is well known that there is a small positive constant $0 < \mu \leq 1$ such that $d(x) \in C^4(\tilde{\Omega}_\mu)$ when $\Omega$ is a $C^4$ domain. We have $\nu = -Dd$ in $\Omega_\mu$. We denote
\[ e^{ij} = \delta_{ij} - \nu^i \nu^j, \quad \Omega_\mu, \]
\[ |D'u|^2 = \sum_{1 \leq i, j \leq n} e^{ij} u_i u_j, \]
and
\[ h(x) = -d(x) + d^2(x). \]
We need the following lemma to prove boundary second order derivatives estimates similar to the Lemma 4.4 in [2].

**Lemma 2.7.** If $\Omega$ is a $C^4$ uniformly convex domain, and $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is a solution of equation (1.1), with $\lambda(D^2u) \in \Gamma_k$. Then
\[ \sum G^{ij} h_{ij} \geq c_0(1 + \sum G^{ii}), \quad \text{in } \Omega_\mu, \]
where $c_0$ is a positive constant depending only on $n$, $k$, $\Omega$.

**Proof.** The distance function $d$ is $C^4$ in $\Omega_\mu$ for some constant $\mu \in (0, \frac{1}{10})$ small. It holds
\[ |Dd| = 1, \quad \text{in } \tilde{\Omega}_\mu; \quad -Dd = \nu, \quad \text{on } \partial\Omega_\mu. \]
For any $x_0 \in \Omega_\mu$, there is a $y_0 \in \partial\Omega$ such that $d(x_0) = |x_0 - y_0|$. We have
\[ -Dd(x_0) = \nu(y_0) = (0, \cdots, 0, 1); \]
\[ -D^2d(x_0) = \text{diag}\{ \frac{\kappa_1(y_0)}{1 - \kappa_1(y_0)d(x_0)}, \cdots, \frac{\kappa_{n-1}(y_0)}{1 - \kappa_{n-1}(y_0)d(x_0)} \}, 0\}, \]
where $\kappa_1(y_0), \cdots, \kappa_{n-1}(y_0)$ are the principal curvature of $\partial\Omega$ at $y_0$.

Since $\Omega$ is uniformly convex domain, then there exist two positive constants $\kappa_{\min} < 1$ and $\kappa_{\max}$ depending only on $\Omega$ and $\mu$ such that
\[ \kappa_{\min} \text{diag}\{1, \cdots, 1, 0, \cdots, 0\} \leq -D^2d(x_0) \leq \kappa_{\max} \text{diag}\{1, \cdots, 1, 0\}, \]
in the principal coordinate system. Hence

\begin{equation}
(2.22) \quad \kappa_{\min} \text{diag}\{1, \cdots, 1, 0, \cdots, 0, 1\} \leq D^2 h(x_0) \leq (1 + \kappa_{\max}) \text{diag}\{1, \cdots, 1, 1\},
\end{equation}

in the principal coordinate system. If \( D^2 u(x_0) \) is diagonal and denote \( \lambda = (\lambda_1, \cdots, \lambda_n) \) with \( \lambda_i = u_{ii} \). We also assume \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). We know in [11] \( \sigma_{k-1}(\lambda|i) \geq c(n, k)\sigma_{k-1}(\lambda) \) when \( i \geq k \), for some positive constant \( c(n, k) \) depending only on \( n, k \). From Lemma 4.4 in [2],

\begin{equation}
(2.23) \quad \sum_{i=1}^{n} G_{ki}^{ii} h_{ii} \geq \kappa_{\min} \frac{n}{k(n-k+1)^2} c^2(n, k) \sum_{i=1}^{n} \frac{\partial G_k}{\partial \lambda_i}.
\end{equation}

\begin{align}
G_{k}^{ii} h_{ii} &= \sum_{i=1}^{n} \sigma_{k-2}(\lambda|i) \sigma_1(\lambda) - \sigma_{k-1}(\lambda) \sigma_{l-1}(\lambda|i) \frac{h_{ii}}{\sigma_{k-1}^2(\lambda)}

&= \sum_{i=1}^{n} \sigma_{k-2}(\lambda|i) \sigma_1(\lambda) - \sigma_{l-1}(\lambda|i) \sigma_{k-1}(\lambda) \frac{h_{ii}}{\sigma_{k-1}^2(\lambda)}

&\geq \sum_{i=1}^{n} \left[ 1 - \frac{C_{k-1}^{l-1}}{C_{n-1}^{l-1}} \sigma_{k-2}(\lambda|i) \sigma_1(\lambda) \sigma_{k-1}(\lambda|i) \right] h_{ii}

&\geq \kappa_{\min} \left[ 1 - \frac{C_{k-1}^{l-1}}{C_{n-1}^{l-1}} \sigma_{k-2}(\lambda|i) \sigma_1(\lambda) \right] c(n, k) c(l, k) \frac{1}{n-k+2} \sum_{i=1}^{n} \frac{\partial G_k}{\partial \lambda_i}

&= \kappa_{\min} c(n, k) \min_{l} c(l, k) \frac{n(n-k-1)}{(n-l)(n-k+1)(n-k+2)} \sum_{i=1}^{n} \frac{\partial G_k}{\partial \lambda_i}.
\end{align}

By Lemma 2.1 we can get

\begin{equation}
(2.25) \quad \sum_{i=1}^{n} G_{k}^{ii} h_{ii} = \sum_{i=1}^{n} \left[ C_{k}^{ii} h_{ii} + \sum_{l=0}^{k-2} \alpha_l(x) C_{l}^{ii} h_{ii} \right]

\geq \frac{n}{k(n-k+2)^2} c(n, k) \min_{l} c(n, k) \min_{l} c(l, k) \sum_{i=1}^{n} G_{k}^{ii}.
\end{equation}

Furthermore by Lemma 2.4, we can obtain (2.17).

\[ \square \]

3. Maximum and Gradient estimates

3.1. Maximum estimates. For the completeness, we introduce a simple proof here following the idea in [12].
Theorem 3.1. Suppose $\Omega$ is a $C^1$ bounded domain, and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution with $\lambda(D^2u) \in \Gamma_{k-1}$ of equation (1.1). Then we have

$$\sup_{\overline{\Omega}} |u| \leq M_0,$$

where $M_0$ depends on $n$, $k$, $\Omega$, $L_1$, $\|\alpha\|_{C^0}$, $\|\alpha_l\|_{C^0}$, $\inf_l \alpha_l$, $\gamma_0$.

Proof. Firstly, since $u$ is subharmonic, the maximum of $u$ is attained at some boundary point $x_0 \in \partial \Omega$. Without loss of generality, we can assume $u(x_0) > 0$. Then we get

$$0 \leq D_{\nu} u(x_0)$$

$$= \varphi(x_0, u)$$

$$= \varphi(x_0, u) - \varphi(x_0, 0) + \varphi(x_0, 0)$$

$$= \varphi_u(x_0, \xi(x_0)) u(x_0) + \varphi(x_0, 0)$$

$$\leq -\gamma_0 u(x_0) + \varphi(x_0, 0).$$

Thus $u(x_0) \leq \frac{\varphi(x_0, 0)}{\gamma_0}$.

On the other hand, we assume $0 \in \Omega$ and we know $u - \hat{A}|x|^2$ attains its minimum at some boundary point $\tilde{x}_0 \in \partial \Omega$. In fact, by Lemma 2.2 and Lemma 2.4, we have

$$\sum_{i=1}^{n} G^{ii}[u - \hat{A}|x|^2]_{ii} = \sum_{i=1}^{n} G^{ii} u_{ii} - 2\hat{A} \sum_{i=1}^{n} G^{ii}$$

$$\leq L_2 - 2\hat{A} \frac{n - k + 1}{k} < 0,$$

by taking $\hat{A}$ big enough. Without loss of generality, we can assume $u(\tilde{x}_0) < 0$. Then

$$0 \geq D_{\nu}(u - \hat{A}|x|^2)(\tilde{x}_0)$$

$$= \varphi(\tilde{x}_0, u(\tilde{x}_0)) - 2\hat{A} \cdot diam(\Omega)$$

$$= \varphi(\tilde{x}_0, u(\tilde{x}_0)) - \varphi(\tilde{x}_0, 0) + \varphi(\tilde{x}_0, 0) - 2\hat{A} \cdot diam(\Omega)$$

$$= \varphi_u(\tilde{x}_0, \xi(\tilde{x}_0)) u(\tilde{x}_0) + \varphi(\tilde{x}_0, 0) - 2\hat{A} \cdot diam(\Omega)$$

$$\geq -\gamma_0 u(\tilde{x}_0) + \varphi(\tilde{x}_0, 0) - 2\hat{A} \cdot diam(\Omega).$$

$$\min_{\overline{\Omega}} u \geq \min_{\overline{\Omega}} (u - \hat{A}|x|^2)$$

$$= u(\tilde{x}_0) - \hat{A}|\tilde{x}_0|^2$$

$$\geq \frac{\varphi(\tilde{x}_0, 0) - 2\hat{A} \cdot diam(\Omega)}{\gamma_0} - \hat{A} \cdot diam(\Omega)^2,$$

where the last inequality is attained according to (3.4). \qed
3.2. Gradient estimates. In this subsection, we divide gradient estimates into two parts. The first part is interior gradient estimates and the second part is near boundary gradient estimates. To prove Theorem 1.1, we give interior gradient estimates in $\Omega/\Omega_\mu$, and then we establish near boundary gradient estimates in $\Omega_\mu$.

3.2.1. Interior gradient estimates. In this subsection we follow the idea in [3, 1] to derive interior gradient estimates for admissible solutions of the following equation

\[
G(\lambda(D^2u)) = \frac{\sigma_k(\lambda(D^2u))}{\sigma_{k-1}(\lambda(D^2u))} - \sum_{l=0}^{k-2} \alpha_l(x) \frac{\sigma_l(\lambda(D^2u))}{\sigma_{k-1}(\lambda(D^2u))} = -\alpha(x).
\]

**Theorem 3.2.** Let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ is a solution of equation (3.6), with $\lambda(D^2u) \in \Gamma_k$. And $\Omega = B_r(0)$, $\alpha(x) \in C^1(\Omega)$ and $0 < \alpha_l(x) \in C^1(\Omega)$ with $l = 0, 1, \ldots, k - 2$ are given functions in $\Omega$. Then we have

\[
|Du(0)| \leq C,
\]

where $C$ depends on $n, k, r, \sup |u|, \|\alpha(x)\|_{C^1(\Omega)}, \|\alpha_l(x)\|_{C^1(\Omega)}, \inf \alpha_l$.

**Proof.** Let

\[
W(x, \xi) = u_\xi(x)\phi(u)\rho(x),
\]

where $\rho(x) = (1 - \frac{|x|^2}{r^2})^+$, $\phi(u) = \frac{1}{\sqrt{M - u}}$ and $M = 4\sup |u|$. Suppose $W$ attains its maximum at $x = x_0$ and $\xi = e_1$. Then at $x_0$,

\[
0 = W_i = u_{i1}\phi\rho + u_1u_i\phi'\rho + u_1\phi\rho_i,
\]

i.e.

\[
u_{i1} = -\frac{u_1}{\phi\rho}(u_i\phi'\rho + \phi\rho_i),
\]

\[
0 \geq W_{ij} = u_{ij}\phi\rho + u_1u_{ij}\phi'\rho + u_1u_iu_j\phi''\rho + u_1\phi\rho_{ij} + (u_1u_j + u_{ij}u_i)\phi'\rho + \phi(u_{ij}\rho_i + u_{i1}\rho_j) + u_1\phi'(u_i\rho_j + u_j\rho_i).
\]

By differentiating equation (3.6) we have

\[
G^{ij}u_{ij1} = (-\alpha)_1.
\]
Then, we have
\[ 0 \geq G_{ij} W_{ij} = (-\alpha)_1 \phi \rho + u_1 G_{ij} u_{ij} \phi' \rho + u_1 G_{ij} u_i u_j \phi'' \rho + u_1 \phi G_{ij} \rho_{ij} + 2 G_{ij} u_1 (u_j \phi' \rho + \phi \rho_j) + u_1 \phi' G_{ij} (u_i \rho_j + u_j \rho_i) \]
\[ = (-\alpha)_1 \phi \rho + u_1 G_{ij} u_{ij} \phi' \rho + G_{ij} u_i u_j (\phi'' - \frac{2\phi'^2}{\phi}) u_1 \rho + u_1 \phi G_{ij} \rho_{ij} \]
\[ - u_1 \phi' G_{ij} (u_i \rho_j + u_j \rho_i) - \frac{2u_1 \phi}{\rho} G_{ij} \rho_{ij}. \]

By Lemma 2.2 we have
\[ 0 \geq (-\alpha)_1 \phi \rho M^{5/2} + \frac{1}{16} \rho G_{11} u_1^3 - \phi' \rho L_2 u_1 - \sum G_{ii} (\frac{Cu_1}{r^2} + \frac{Cu_2}{r} + \frac{Cu_1}{\rho r^2}), \]
where \( C \) is independent of \( r \).

From Lemma 2.5 and Lemma 2.4 we have \( G_{11} \geq \frac{n}{k} \sum_{i=1}^{n} G_{ii} \geq \frac{n(n-k+1)}{k^2} \frac{1}{(n-k+2)^2} \), the rest proof is similar to those in [3, 1], we omit the details here.

### 3.2.2. Near boundary gradient estimates

In this subsection we follow the idea in [13] to derive near boundary gradient estimates for problem (1.1).

**Theorem 3.3.** Let \( \Omega \subset \mathbb{R}^n \) be a \( C^3 \) bounded domain, and \( \nu \) is the outer unit normal vector of \( \partial \Omega \). Suppose \( u \in C^2(\Omega) \cap C^3(\Omega) \) is a solution with \( \lambda(D^2 u) \in \Gamma_k \) to the boundary problems (1.1). Here \( |u| \leq M_0, \phi \) is given function defined on \( \bar{\Omega} \times [-M_0, M_0] \), and \( |\varphi(x, u)|_{C^3(\bar{\Omega} \times [-M_0, M_0])} \leq L_1 \). Then there exists a small positive constant \( \mu_0 \) which depends only on \( n, k, \Omega, M_0, L_1, \alpha(x), \alpha_t(x), \inf_l \alpha_l \) such that
\[ \sup_{\Omega_{\mu_0}} |Du| \leq C. \]

**Proof.** We consider the following test function,
\[ H = \log |Dw|^2 + \hat{h}(u) + g(d), \]
where
\[ \hat{h}(u) = -\log(1 + 4M_0 - u); \tag{3.8} \]
\[ w(x) = u(x) + \varphi(x, u)d(x); \tag{3.9} \]
and
\[ g(d) = Ad(x), \tag{3.10} \]
in which $A$ large to be chosen later.

By (3.8),
\begin{align*}
- \log(1 + 5M_0) &\leq \hat{h} \leq - \log(1 + 3M_0), \\
\frac{1}{1 + 5M_0} &\leq \hat{h}' \leq \frac{1}{1 + 3M_0}, \\
\frac{1}{(1 + 5M_0)^2} &\leq \hat{h}'' \leq \frac{1}{(1 + 3M_0)^2}.
\end{align*}

By (3.9), we have
\begin{equation}
(3.11) \quad w_i = u_i + (\varphi_i + \varphi_u u_i)d + \varphi d_i.
\end{equation}

If we assume that $|Du|$ large enough and $d$ small enough, it follows from (3.11) that
\begin{equation}
(3.12) \quad \frac{1}{4} |Du| \leq |Dw| \leq 2 |Du|.
\end{equation}

We assume that $H(x)$ attains its maximum at $x_0 \in \Omega_{\mu_0}$. Based on the position of $x_0$, we can divide the proof into three cases. The first case is $x_0 \in \partial \Omega$, we shall use the Hopf Lemma to bound $H(x_0)$. The second case is $x_0 \in \Omega_{\mu_0}$, we shall use the maximum principle to get the bound when $\mu_0$ is small enough. The last case is $x_0 \in \partial \Omega_{\mu_0} \cap \Omega$, we shall use Theorem 3.2 to get the bound.

Case I. If the maximum of $H$ is attained at $x_0$ on the boundary $\partial \Omega$. Here we don’t need the equations for boundary estimates and only use the boundary condition, the proof is similar to those in [13]. For completeness we contain its proof in this section.

By the Hopf Lemma at the maximum point we have
\[ 0 \leq H_\nu = \frac{|Dw|_{i}^{2} \nu^i}{|Dw|^2} - g' + \hat{h}'u_\nu. \]

Since $w_\nu = u_\nu + D_\nu \varphi d - \varphi = 0$, we have
\begin{equation}
(3.13) \quad |Dw|_{i}^{2} \nu^i = c^{ij}_{m} w_{i} w_{j} \nu^{m} + 2 c^{ij}_{m} w_{im} w_{j} \nu^{m} + 2 w_{i} D_{m} w_{\nu} \nu^{m} \\
= c^{ij}_{m} w_{i} w_{j} \nu^{m} + 2 c^{ij}_{m} (u_{im} + D_{im} \varphi d + D_{i} \varphi d_{m} + D_{m} \varphi d_{i} + \varphi d_{im}) w_{j} \nu^{m} \\
= c^{ij}_{m} w_{i} w_{j} \nu^{m} + 2 c^{ij}_{m} u_{im} w_{j} - 2 c^{ij}_{m} D_{i} \varphi w_{j} + 2 c^{ij}_{m} D_{m} \varphi w^{m} d_{i} w_{j} + 2 c^{ij}_{m} \varphi d_{im} w_{j} \nu^{m}.
\end{equation}

By the boundary condition, we have
\[ c^{ij}_{m} u_{j\nu} + c^{ij}_{m} u_{m} D_{j} \nu^{m} = c^{ij}_{m} D_{j} \varphi. \]

Then the second derivative of $u$ can be replaced by the first derivative term.
\[ |Dw|^{2} \nu^{i} \leq C_{1} |Dw|^{2} + C_{2} |Du|. \]
Therefore,

\[ 0 \leq H_{\nu} \leq -A + C_1 + \frac{C_2}{|Dw|} + \frac{L_1}{1 + 3M_0} \leq -\frac{A}{2} + \frac{C_2}{|Dw|}. \]

So we have the upper bound of $|Dw(x_0)|$, then we get the upper bound for $H(x_0)$.

Case II. If $H$ attains its maximum in $\Omega_{\mu_0}$, we take differentiating the auxiliary function twice at $x_0$,

\[ 0 = H_i = \frac{2w_kw_{ki}}{|Dw|^2} + AD_i d + \hat{h}'u_i, \]

\[ H_{ij} = \frac{2w_kw_{kj} + 2w_kw_{kij}}{|Dw|^2} \frac{4w_kw_{km}w_{mj}}{|Dw|^4} + AD_{ij} d + \hat{h}''u_i u_j + \hat{h}'u_{ij}. \]

By the definition of $w = u + \varphi d$, we have

\[ w_{ij} = u_{ij} + (\varphi_{ij} + \varphi_{u_i} u_j + \varphi_{u_j} u_i + \varphi_{uu} u_i u_j + \varphi_{u_2} u_{ij})d \]

\[ + \varphi_i D_j d + \varphi_j D_i d + \varphi_{u_i} D_i d + \varphi_{u_j} D_j d + \varphi D_{ij} d, \]

\[ w_{ijk} = u_{ijk} + (\varphi_{ijk} + \varphi_{iju} u_k + \varphi_{iu} u_k u_j + \varphi_{iu} u_k u_j + \varphi_{iuk} u_{ij} + \varphi_{iu} u_{kj} + \varphi_{ui} u_{kj} \]

\[ + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} \]

\[ + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} \]

\[ + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} + \varphi_{u_3} u_{ijk} \]

\[ (1 + \varphi_d)u_{ij} - C\mu_0|Du|^2 - C|Du| - C \leq w_{ij} \leq (1 + \varphi_d)u_{ij} + C\mu_0|Du|^2 + C|Du| + C. \]

Now we choose a coordinate at $x_0$ such that $|Dw| = w_1$ and $(u_{ij})_{2\leq i,j\leq n}$ is diagonal. Thus we have

\[ u_1 = \frac{w_1 - \varphi_1 d - \varphi D_1 d}{1 + \varphi_d}, \]

\[ w_{11} = -\frac{1}{2}(AD_1 d + \hat{h}'u_1)w_1, \]
and for \(2 \leq i \leq n\),
\[
    u_i = \frac{-\varphi_i d - \varphi D_i d}{1 + \varphi u d},
\]
(3.18)
\[
    w_{1i} = -\frac{1}{2}(AD_i d + \hat{h}'u_i)w_1.
\]
By taking \(|Du(x_0)|\) big enough such that for \(i \geq 2\), we have
\[
    |u_i| \leq \frac{1}{9n}|Du(x_0)|,
\]
(3.19)  

By (3.12) and (3.19), we have
\[
    u_1 = \sqrt{|Du|^2 - \sum_{i=2}^{n} u_i^2} \geq \frac{|Du|}{2} \geq \frac{w_1}{4}.
\]
(3.20)

Further, we can assume \(\mu_0\) small and \(|w_1(x_0)|\) big enough such that at \(x_0\), we can get the key fact
\[
    u_{11}(x_0) = \frac{1}{1 + \varphi u d}[-\frac{1}{2}(AD_1 d + \hat{h}'u_1)w_1 - (\varphi_{11} + 2\varphi_{u1} + \varphi_{uu}u_1^2)d
    - 2\varphi_1 D_1 d - 2\varphi_1 u_1 D_1 d - \varphi D_{11} d]
    \leq -\frac{\hat{h}'w_1^2}{128} < 0.
\]
(3.21)

At the same time, for \(i \geq 2\) we have
\[
    |u_{1i}| = \frac{1}{1 + \varphi u d}[-\frac{1}{2}(AD_i d + \hat{h}'u_i)w_1 - (\varphi_{1i} + \varphi_{ui} u_1 + \varphi_{u1} u_i)d
    - (\varphi_1 D_i d + \varphi_1 D_1 d + \varphi_1 u_1 D_i d + \varphi u_1 D_1 d + \varphi D_{1i} d)
    \leq C w_1^2.
\]

Then
\[
    0 \geq G^{ij} H_{ij} = \frac{2G^{ij} w_{ki} w_{kij}}{|Du|^2} + \frac{2G^{ij} w_{ki} w_{kij}}{|Du|^2} - \frac{4G^{ij} w_{ki} w_{mi} w_{mj}}{|Du|^4}
    + AG^{ij} D_{ij} d + \hat{h}''G^{ij} u_i u_j + \hat{h}'G^{ij} u_{ij}
\]
(3.22)
\[
    \geq \frac{2G^{ij} w_{1ii}}{w_1} - \frac{2G^{ij} w_{1ii}}{w_1^2} + AG^{ij} D_{ij} d + \hat{h}''G^{ij} u_i u_j + \hat{h}'G^{ij} u_{ij}
    \geq \frac{2G^{ij} w_{1ii}}{w_1} + G^{ij}[(\hat{h}' - \frac{1}{2}\hat{h}'')u_i u_j - A\hat{h}'d_i u_j + Ad_{ij} - \frac{1}{2} Ad_i d_j] + \hat{h}'G^{ij} u_{ij}.
\]
We know
\[ G^{ij}[(\hat{h}'' - \frac{1}{2}\hat{h}'^2)u_i u_j - Ah'd_i u_j + Ad_{ij} - \frac{1}{2}Ad_i d_j] \]
(3.23)
\[
\geq \frac{\hat{h}^2}{2}[G^{ii}u_i u_i - \hat{h}'Du] - AC\sum_{i=1}^{n}G^{ii} - AC\sum_{i=1}^{n}G^{ii},
\]
\[ G^{ij}u_{1ij} \]
\[
= \frac{2}{w_1}G^{ij}\{u_{ij1} + (\varphi_{ij1} + \varphi_{iju} u_1 + \varphi_{iu1} u_j + \varphi_{iun} u_1 u_j + \varphi_{iu} u_{j1})
+ \varphi_{uji1} u_1 + \varphi_{uuu} u_1 u_i + \varphi_{uu} u_{ij1} + \varphi_{uu1} u_i u_j + \varphi_{uu} u_{u1j} u_1 + \varphi_{uu} u_{u11} u_j
+ \varphi_{uu} u_{u1i} u_j + \varphi_{uu} u_{u1j} u_i + \varphi_{uu} u_{u1} u_{ij} + \varphi_{uu} u_{u1} u_{ij1}) + \frac{1}{w_1} Du - ACw_1 \sum_{i=1}^{n}G^{ii} - AC\sum_{i=1}^{n}G^{ii},
\]
(3.24)
\[
\geq - \frac{C}{w_1} - Cw_1^2 \mu_0 \sum_{i=1}^{n}G^{ii} - Cw_1 \sum_{i=1}^{n}G^{ii}.
\]
By inserting (3.23) and (3.24) into (3.22), we have
\[
0 \geq G^{ij}H_{ij}
\]
(3.25)
\[
\geq \frac{\hat{h}^2}{32} G^{ii} w_1^2 - ACw_1 \sum_{i=1}^{n}G^{ii} - AC \sum_{i=1}^{n}G^{ii}
- Cw_1^2 \mu_0 \sum_{i=1}^{n}G^{ii} - Cw_1 \sum_{i=1}^{n}G^{ii} - \hat{h}'L_2,
\]
using Lemma [2.5], we have \[ G^{ii} \geq \frac{n}{\mu} \sum_{i=1}^{n} G^{ii}, \] so we can obtain the bound of \[ w_1(x_0) \] by taking \[ \mu_0 \] small enough. Furthermore we get the estimate of \[ H(x_0). \]

\[ \square \]

4. Second derivatives estimates

In this section we give a priori estimates for the global and the boundary second order derivatives following the idea of Lions-Trudinger-Urbas [12], Ma-Qiu [13].
Theorem 4.1. Let \( \Omega \subset \mathbb{R}^n \) is a \( C^4 \) uniformly convex domain, \( u \in C^4(\Omega) \cap C^2(\overline{\Omega}) \) is a solution of equation (1.1), with \( \lambda(D^2u) \in \Gamma_k \). Then we have

\[
\sup_{\overline{\Omega}} |D^2u| \leq M_2,
\]

where \( M_2 \) depends on \( n, k, \Omega, \alpha(x), \alpha_l(x), \inf_l \alpha_l, L_1, M_0 \) and \( M_1 \).

Following the idea of Lions-Trudinger-Urbas [12], we divide the proof of Theorem 4.1 into two steps. For the first step, we reduce global second order derivatives estimates to double normal second derivatives estimates on the boundary, then we prove double normal second order derivatives estimates on the boundary.

4.1. Reduce global derivatives estimates to double normal second derivatives estimates on the boundary.

Theorem 4.2. Let \( \Omega \subset \mathbb{R}^n \) is a \( C^4 \) uniformly convex, \( u \in C^4(\Omega) \cap C^2(\overline{\Omega}) \) is a solution of equation (1.1), with \( \lambda(D^2u) \in \Gamma_k \). Then we have

\[
\sup_{\Omega} |D^2u| \leq C(1 + \max_{\partial \Omega} |u_{\nu\nu}|),
\]

where \( C \) depends on \( n, k, \Omega, \alpha(x), \alpha_l(x), \inf_l \alpha_l, L_1, M_0 \) and \( M_1 \).

Proof. We assume \( 0 \in \Omega \), and consider the auxiliary function

\[
v(x, \xi) = u_{\xi\xi} - v'(x, \xi) + K|x|^2 + |Du|^2,
\]

where

\[
v'(x, \xi) = 2(\xi \cdot \nu)\xi'(D\varphi - u_lD\nu^l) = a_l'u_l + b,
\]

where \( \xi' = \xi - (\xi \cdot \nu)\nu, a'_l = 2(\xi \cdot \nu)(\xi'^l\varphi_z - \xi^lD_i\nu^l), \) and \( b = 2(\xi \cdot \nu)\xi'^l\varphi_{x_l}. \) We have

\[
v_i = u_{\xi\xi} - D_i a_l'u_l - a'_l u_l - D_i b + 2Kx_i + 2u_lu_{li},
\]

\[
v_{ij} = u_{\xi\xi} - D_{ij} a_l'u_l - D_i a'_l u_{li} - a'_l u_{li} - D_{ij} b + 2K\delta_{ij} + 2u_lu_{li} + 2u_{li}u_{ij}.
\]
By (2.4) and (2.5), we have
\[ G_{ij}v_{ij} = G_{ij}u_{\xi ij} - G_{ij}D_ia^lu_l - 2G_{ij}D_ia^lu_{ij} - G_{ij}a^lu_{iiij} - G_{ij}D_ib \\
+ 2KG_{ij}\delta_{ij} + 2G_{ij}u_{iij} + 2u_{i}G_{ij}u_{ij} \\
= G_{ij}u_{\xi ij} - G_{ij}D_ia^lu_l - 2G_{ij}D_ia^lu_{ij} - a^l(-\alpha)_l - G_{ij}D_ib \\
+ 2KG_{ij}\delta_{ij} + 2G_{ij}u_{iij} + 2u_{i}(-\alpha)_l \\
= (-\alpha)_{\xi\xi} - 2 \sum_{l=0}^{k-2} (\alpha_l)_{\xi}G_{ij}^l u_{ii\xi} - \sum_{l=0}^{k-2} (\alpha_l)_{\xi\xi}G_l - G_{ij,rs}^l u_{ij\xi}u_{rs\xi} \\
- G_{ij}D_ia^lu_l - 2G_{ij}D_ia^lu_{ij} - a^l(-\alpha)_l - G_{ij}D_ib \\
+ 2KG_{ij}\delta_{ij} + 2G_{ij}u_{iij} + 2u_{i}(-\alpha)_l \\
\geq -2 \sum_{l=0}^{k-2} (\alpha_l)_{\xi}G_{ij}^l u_{ii\xi} - \sum_{l=0}^{k-2} (\alpha_l)_{\xi\xi}G_l - G_{ij,rs}^l u_{ij\xi}u_{rs\xi} - C(1 + \sum G_{ij}^l) \\
- 2G_{ij}D_ia^lu_l - G_{ij}D_ib + 2KG_{ij}\delta_{ij} + 2G_{ij}u_{iij}u_{ij}. \\
\]

Using the concavity of \( G_k = \frac{\sigma_k}{\sigma_{k-1}}, \) that is
\[ G_{ij,rs}^k X_{ij} X_{rs} \leq 0, \text{ for any symmetric matrix } (X_{ij}) \in \mathbb{R}^{n \times n}, \]
we obtain
\[ G_{ij}v_{ij} \geq -C(1 + \sum G_{ij}^l) - 2G_{ij}D_ia^lu_{ij} - G_{ij}D_ib + 2KG_{ij}\delta_{ij} + 2G_{ij}u_{iij}u_{ij} \\
- \sum_{l=0}^{k-2} a_l G_{ij,rs}^l u_{ij\xi}u_{rs\xi} - 2 \sum_{l=0}^{k-2} (\alpha_l)_{\xi}G_{ij}^l u_{ii\xi} - \sum_{l=0}^{k-2} (\alpha_l)_{\xi\xi}G_l \\
\geq -C(1 + \sum G_{ij}^l) - G_{ij}D_ia^lu_{ij} - G_{ij}D_ib + 2KG_{ij}\delta_{ij} + 2G_{ij}u_{iij}u_{ij} \\
- \sum_{l=0}^{k-2} a_l G_{ij,rs}^l u_{ij\xi}u_{rs\xi} - 2 \sum_{l=0}^{k-2} (\alpha_l)_{\xi}G_{ij}^l u_{ii\xi} - \sum_{l=0}^{k-2} (\alpha_l)_{\xi\xi}G_l \\
\geq -C(1 + \sum G_{ij}^l) + 2KG_{ij}\delta_{ij} \\
- \sum_{l=0}^{k-2} a_l G_{ij,rs}^l u_{ij\xi}u_{rs\xi} - 2 \sum_{l=0}^{k-2} (\alpha_l)_{\xi}G_{ij}^l u_{ii\xi} - \sum_{l=0}^{k-2} (\alpha_l)_{\xi\xi}G_l \\
\]

By Krylov in [10], the operator \( \left( \frac{\sigma_k}{\sigma_{k-1}} \right)^{\frac{1}{k-1}} \) is concave for \( 0 \leq l \leq k - 2. \) It follows that \( \left( \frac{\sigma_k}{\sigma_{k-1}} \right)^{\frac{1}{k-1}} \) is a concave operator for \( l = 0, \cdots, k - 2. \) As a consequence we have
\[ -G_{ij,rs}^l u_{ij1}u_{rs1} \geq -\left( 1 + \frac{1}{k - 1 - l} \right)G_{ij}^{-1}G_{ij}^l G_{ij,rs}^l u_{ij1}u_{rs1}. \]
Using this inequality we can use the estimate in [6]

\begin{equation}
- \sum_{l=0}^{k-2} \alpha_l G_l^{ijsr} u_{ij1} u_{rs1} - 2 \sum_{l=0}^{k-2} (\alpha_l) \xi G_l^{ii} u_{ii1} \xi \geq \sum_{l=0}^{k-2} \frac{(k-l-1) (\alpha_l)^2}{k-l} \alpha_l G_l.
\end{equation}

Then

\begin{equation}
G^{ij} v_{ij} \geq -C(1 + \sum G^{ii}) + 2 K G^{ij} \delta_{ij} + \sum_{l=0}^{k-2} \frac{(k-l-1) (\alpha_l)^2}{k-l} \alpha_l - (\alpha_l) \xi \xi G_l.
\end{equation}

By Lemma 2.4, we shall discuss the term \( \sum_{l=0}^{k-2} \frac{(k-l-1) (\alpha_l)^2}{k-l} \alpha_l - (\alpha_l) \xi \xi G_l \) into two cases.

Case 1. We note that if there is a constant \( N_2 \) such that

\[ \sigma_l \sigma_{k-1} \leq N_2, \quad l = 0, \ldots, k-2, \]

then we obtain

\begin{equation}
\frac{K}{8} \sum_{l=0}^{k-2} (k-l-1) (\alpha_l)^2 \alpha_l - (\alpha_l) \xi \xi G_l \geq \frac{K n - k + 1}{8} - C N_2 \geq 0,
\end{equation}

when we take \( K \) big enough which is depending on \( \inf \alpha_l \).

Case 2. When \( \frac{\sigma_l}{\sigma_{k-1}} > N_2 \), using the same trick in [6], we obtain

\[ \frac{\sigma_{k-2}}{\sigma_{k-1}} \geq \left( \frac{\sigma_l}{\sigma_{k-1}} \right)^{1-k} \geq N_2^{k-1}. \]

Then, we obtain

\begin{align*}
\frac{K}{8} G^{ii} &+ \sum_{l=0}^{k-2} \frac{(k-l-1) (\alpha_l)^2}{k-l} \alpha_l - (\alpha_l)_{11} G_l \\
&\geq \frac{K n - k + 1}{8} + \sum_{l=0}^{k-2} \frac{K C(n, k, l)}{8} \alpha_l(\sigma_l \sigma_{k-2} \sigma_{k-1} - (\alpha_l) \xi \xi G_l) \\
&\geq \sum_{l=0}^{k-2} \frac{K C(n, k, l) \inf \alpha_l(\sigma_{k-2} \sigma_{k-1} - (\alpha_l) \xi \xi G_l)}{8} - C \frac{\alpha_l}{\inf \alpha_l} \frac{\sigma_l}{\sigma_{k-1}} \sigma_{k-1} \xi \xi G_l \\
&\geq \sum_{l=0}^{k-2} \frac{K C(n, k, l) \inf \alpha_l(\sigma_{k-2} \sigma_{k-1} - (\alpha_l) \xi \xi G_l)}{8} - C \frac{\alpha_l}{\inf \alpha_l} \frac{\sigma_l}{\sigma_{k-1}} \sigma_{k-1} \xi \xi G_l \\
&\geq 0,
\end{align*}

by taking \( K \) big enough. Thus we have

\[ \sum G^{ij} v_{ij} > 0. \]
So $v(x, \xi)$ attains its maximum on $\partial \Omega$. We can assume $\max_{\Omega \times S^{n-1}}$ attains at $(x_0, \xi_0) \in \partial \Omega \times S^{n-1}$.

Case a. $\xi_0$ is tangential to $\partial \Omega$ at $x_0$. We shall take tangential derivative twice to the boundary condition. We can obtain

$$u_{\xi\xi\nu} = -2\xi^p \xi^i u_{\nu i} D_p \nu^j - u_\nu \xi^p D_{ip} \nu^j \xi^i + u_{\nu\nu} \sum_{i=1}^{n} \xi^p D_p \nu^j \xi^i$$

$$- \sum_{i=1}^{n} \xi^p \xi^i \nu^j D_p \nu^j \varphi + \varphi_u u_{\xi\xi} + \xi^p \xi^i \varphi_{ip} + \varphi_{uu} u_{\xi}^2 + 2u_\xi \xi^i \varphi_{iu}$$

$$\leq -2\xi^p \xi^i u_{\nu i} D_p \nu^j + \varphi_u u_{\xi\xi} + C + C|u_{\nu\nu}|.$$ 

If we assume $\xi = e_1$, it is easy to get the bound for $|u_{\nu i}(x_0)| \leq C$ for $i \neq 1$ from the maximum of $v(x, \xi)$ in the $\xi$ direction. We can find the detail in [13]. On the other hand, by $D_1 \nu \geq \kappa_{\min} > 0$, we have

$$u_{\xi\xi\nu} \leq -2\kappa_{\min} u_{\xi\xi} + C(1 + |u_{\nu\nu}|).$$

By the Hopf Lemma,

$$0 \leq v_{\nu}$$

$$= u_{\xi\xi\nu} - D_{\nu} a' u_t - a' u_{\nu\nu} - b_{\nu} + 2K(x \cdot \nu) + 2u_\nu u_{\nu\nu}$$

$$\leq -2\kappa_{\min} u_{\xi\xi} + C(1 + |u_{\nu\nu}|).$$

Therefore we have

$$u_{\xi\xi} \leq C(1 + |u_{\nu\nu}(x_0)|).$$

Case b. $\xi_0$ is non-tangential to $\partial \Omega$ at $x_0$. We write $\xi = \hat{\alpha} \tau + \hat{\beta} \nu$, where $\hat{\alpha} = \xi \cdot \tau$, $\tau \cdot \nu = 0$, $|\tau| = 1$, $\hat{\beta} = \xi \cdot \nu \neq 0$ and $\hat{\alpha}^2 + \hat{\beta}^2 = 1$,

$$u_{\xi\xi} = \hat{\alpha}^2 u_{\tau\tau} + \hat{\beta}^2 u_{\nu\nu} + 2\hat{\alpha}\hat{\beta} u_{\tau\nu}.$$ 

Then

$$v(x_0, \xi) = \hat{\alpha}^2 v(x_0, \tau) + \hat{\beta}^2 v(x_0, \nu) \leq \hat{\alpha}^2 v(x_0, \xi) + \hat{\beta}^2 v(x_0, \nu),$$

hence

$$v(x_0, \xi) \leq v(x_0, \nu).$$ 

So we can reduce $C^2$ boundary estimates to the pure normal case. □
4.2. Lower estimates of double normal second derivatives on boundary.

**Lemma 4.3.** If $\Omega$ is a $C^4$ uniformly convex domain, and $u \in C^4(\Omega) \cap C^2(\overline{\Omega})$ is a solution of equation (1.1), with $\lambda(D^2u) \in \Gamma_k$. Then

\[(4.16) \quad \min_{\partial \Omega} u_{\nu\nu} \geq -C,\]

where $C$ is a constant depending on $n, k, \Omega, M_0, M_1, L_1, \alpha(x), \alpha_i(x), \inf_l \alpha_l$.

**Proof.** Without loss of generality, we can assume $\min_{\partial \Omega} u_{\nu\nu} < 0$, otherwise we have (4.16). Also if $-\min_{\partial \Omega} u_{\nu\nu} < \max_{\partial \Omega} u_{\nu\nu}$, that is $\max_{\partial \Omega} |u_{\nu\nu}| = \max_{\partial \Omega} u_{\nu\nu}$, we shall deal this case in the next subsection. Thus we shall assume $-\min_{\partial \Omega} u_{\nu\nu} \geq \max_{\partial \Omega} u_{\nu\nu}$, that is $\max_{\partial \Omega} |u_{\nu\nu}| = -\min_{\partial \Omega} u_{\nu\nu}$. Denote $M = -\min_{\partial \Omega} u_{\nu\nu} > 0$ and $\bar{y}_0 \in \partial \Omega$ such that $\min_{\partial \Omega} u_{\nu\nu} = u_{\nu\nu}(\bar{y}_0)$.

We take the test function

\[(4.17) \quad P(x) = (1 + \beta d)[Du \cdot (-Dd) - \varphi(x, u)] + (B + \frac{1}{2} M)h(x),\]

where $\beta$ and $B$ are positive constants to be chosen later.

On $\partial \Omega$, $P(x) = 0$, and on $\partial \Omega_\mu/\partial \Omega$, we have $d = \mu$ and

\[
P(x) \leq (1 + \beta \mu)[|Du| + |\varphi(x, u)|] + (B + \frac{1}{2} M)[-\mu + \mu^2] \leq 0,
\]

since we take $B$ big enough. So on $\partial \Omega_\mu$, we have $P \leq 0$.

Next to prove $P$ attains its maximum only on $\partial \Omega$ by contradiction, we assume $P$ attains its maximum at some point $\bar{x}_0 \in \Omega_\mu$. Rotating the coordinates $D^2u(\bar{x}_0)$ is diagonal. In the following, all the calculations are at $\bar{x}_0$. We have

\[
0 = P_i(\bar{x}_0) = \beta d_i[-u_j d_j - \varphi(x, u)] + (1 + \beta d)[-u_{ji} d_i + u_j d_{ji}] - \varphi_i - \varphi_u u_i + (B + \frac{1}{2} M)h_i
\]

\[(4.18) \quad \beta d_i[-u_j d_j - \varphi(x, u)] + (1 + \beta d)[-u_{ii} d_i + u_j d_{ji}] - \varphi_i - \varphi_u u_i + (B + \frac{1}{2} M)h_i
\]
and

\[0 \geq P_{ii}(\bar{x}_0)
= \beta d_i[-u_jd_j - \varphi(x,u)] + 2\beta d_i[-(u_{ji}d_j + u_jd_{ji}) - \varphi_i - \varphi_\mu u_i]
+ (1 + \beta d)[-u_{ji}d_j + 2u_{ij}d_{ji} + u_jd_{jii}) - (\varphi_{ii} + 2\varphi_{iu}u_i + \varphi_{uu}u_i^2 + \varphi_u u_i)]
+ (B + \frac{1}{2}M)h_{ii}
\]

\[(4.19)\]

\[= \beta d_i[-u_jd_j - \varphi(x,u)] + 2\beta d_i[-u_{ji}d_j - \varphi_i - \varphi_u u_i]
+ (1 + \beta d)[-u_{ji}d_j + 2u_{ij}d_{ji} + u_jd_{jii}) - (\varphi_{ii} + 2\varphi_{iu}u_i + \varphi_{uu}u_i^2 + \varphi_u u_i)]
+ (B + \frac{1}{2}M)h_{ii}
\]

\[\geq - 2\beta u_{ii}d_i^2 + (1 + \beta d)[-u_{ji}d_j + 2u_{ij}d_{ji} - \varphi_u u_{ii}] + (B + \frac{1}{2}M)h_{ii} - C.
\]

Hence by (2.4) and Lemma 2.2, we have

\[0 \geq G^{ii} P_{ii}(\bar{x}_0)
\]

\[\geq - 2\beta G^{ii} u_{ii}d_i^2 + (1 + \beta d)[-G^{ii}u_{ji}d_j - 2G^{ii}u_{ii}d_{ii} - \varphi_u G^{ii} u_{ii}]
+ (B + \frac{1}{2}M)G^{ii} h_{ii} - CG^{ii}
\]

\[(4.20)\]

\[\geq - 2\beta G^{ii} u_{ii}d_i^2 - 2(1 + \beta d)G^{ii} u_{ii}d_{ii} + [(B + \frac{1}{2}M)c_0 - C](\sum_i G^{ii} + 1).
\]

From (4.18), we have

\[(4.21)\]

\[u_{ii} = -\frac{1}{2} - \frac{2d_i}{1 + \beta d}(B + \frac{1}{2}M) + \frac{\beta[-u_jd_j - \varphi(x,u)]}{1 + \beta d} + \frac{-u_{ji}d_j - \varphi_i - \varphi_u u_i}{d_i}.
\]

Denote \(E = \{i : \beta d_i^2 < \frac{1}{n}, 1 \leq i \leq n\}\) and \(F = \{i : \beta d_i^2 \geq \frac{1}{n}, 1 \leq i \leq n\}\). Let \(\beta \geq \frac{1}{\mu} > 1\), then

\[d_i^2 < \frac{1}{n} = \frac{1}{n}|Dd|^2, \ i \in E.
\]

Thus we have \(\sum_{i \in E} d_i^2 \leq 1 = |Dd|^2\) and \(F\) is not empty.

We choose \(B\) large such that for \(i \in F,\)

\[(4.22)\]

\[\frac{\beta[-u_jd_j - \varphi(x,u)]}{1 + \beta d} + \frac{-u_{ji}d_j - \varphi_i - \varphi_u u_i}{d_i} \leq \frac{B}{5},
\]

thus

\[(4.23)\]

\[-\frac{6B}{5} - \frac{M}{2} \leq u_{ii} \leq -\frac{B + M}{5}, \text{ for } i \in F.
\]
There must be an \( i_0 \in F \) such that

\[
(4.24) \quad d_{i_0}^2 \geq \frac{|Dd|^2}{n} = \frac{1}{n}.
\]

By (4.20), we get

\[
(4.25) \quad 0 \geq G^{ii} P_{ii}(\bar{x}_0) \\
\geq -2\beta \sum_{i \in F} G^{ii} u_{ii} d_i^2 - 2\beta \sum_{i \in E} G^{ii} u_{ii} d_i^2 - 2(1 + \beta d) \sum_{u_{ii} < 0} G^{ii} u_{ii} d_{ii} \\
+ [(B + \frac{1}{2} M) c_0 - C] (\sum_i G^{ii} + 1) \\
\geq -2\beta \sum_{i \in F} G^{ii} u_{ii} d_i^2 - 2\beta \sum_{i \in E} G^{ii} u_{ii} d_i^2 + 4 \kappa_{\max} \sum_{u_{ii} < 0} G^{ii} u_{ii} \\
+ [(B + \frac{1}{2} M) c_0 - C] (\sum_i G^{ii} + 1).
\]

Direct calculations yield

\[
(4.26) \quad -2\beta \sum_{i \in F} G^{ii} u_{ii} d_i^2 \geq -2\beta G^{i_0 i_0} u_{i_0 i_0} d_{i_0}^2 \geq -\frac{2\beta}{n} G^{i_0 i_0} u_{i_0 i_0},
\]

and

\[
-2\beta \sum_{i \in E} G^{ii} u_{ii} d_i^2 \geq -2\beta \sum_{i \in E, u_{ii} > 0} G^{ii} u_{ii} d_i^2 \\
\geq -\frac{2}{n} \sum_{i \in E, u_{ii} > 0} G^{ii} u_{ii} \\
\geq -\frac{2}{n} \sum_{u_{ii} > 0} G^{ii} u_{ii} \\
\geq -\frac{2}{n} \left[ \sum G^{ii} u_{ii} - \sum_{u_{ii} < 0} G^{ii} u_{ii} \right] \\
\geq -\frac{2}{n} L_2 + \frac{2}{n} \sum_{u_{ii} < 0} G^{ii} u_{ii},
\]

where the last inequality is according to Lemma 2.2.
Therefore, by the key Lemma 2.5 we have

\(0 \geq G^{ii} P_{ii}(\bar{x}_0)\)

\(\geq -\frac{2\beta}{n} G^{ii} u_{i0i0} + \left(\frac{2}{n} + 4\kappa_{\text{max}}\right) \sum_{u_{ii} < 0} G^{ii} u_{ii}\)

\(\geq \left[\left(B + \frac{1}{2} M\right)c_0 - C\right]\left(\sum_i G^{ii} + 1\right)\)

\(\geq \frac{2\beta}{k}\left(B + M\right) \frac{5}{5} \sum_{i=1}^n G^{ii} + \left(\frac{2}{n} + 4\kappa_{\text{max}}\right) \sum_{u_{ii} < 0} G^{ii} u_{ii}\)

\(\geq \frac{2\beta}{k}\left(B + M\right) \frac{5}{5} \sum_{i=1}^n G^{ii} - \left(\frac{2}{n} + 4\kappa_{\text{max}}\right) C(1 + M) \sum_{i=1}^n G^{ii}\)

\(> 0,\)

by taking \(\beta\) big enough. This is a contradiction. So \(P\) attains its maximum only on \(\partial\Omega\).

Finally, we can get

\(0 \leq P_\nu(\bar{x}_0)\)

\(= \left[u_{\nu\nu}(\bar{x}_0) - u_j d_{j\nu} - D_\nu \varphi(x, u)\right] + \left(B + \frac{1}{2} M\right)\)

\(\leq \min_{\partial\Omega} u_{\nu\nu} + C + \left(B + \frac{1}{2} M\right)\)

hence \((4.16)\) holds. \(\Box\)

4.3. Upper estimates of double normal second derivatives on boundary.

**Lemma 4.4.** If \(\Omega\) is a \(C^4\) uniformly convex domain, and \(u \in C^4(\Omega) \cap C^2(\overline{\Omega})\) is an admissible solution of equation \((1.1)\), with \(\lambda(D^2 u) \in \Gamma_k\). Then

\(\max_{\partial\Omega} u_{\nu\nu} \leq C,\)

where \(C\) is a constant depending on \(n, k, \Omega, \alpha(x), \alpha_l(x), \inf_1 \alpha_l, L_1, M_0\) and \(M_1\).

**Proof.** Without loss of generality, we can assume \(\max_{\partial\Omega} u_{\nu\nu} > 0\), otherwise we have \((4.16)\). Also if \(-\min_{\partial\Omega} u_{\nu\nu} > \max_{\partial\Omega} u_{\nu\nu}\), that is \(\max_{\partial\Omega} |u_{\nu\nu}| = -\min_{\partial\Omega} u_{\nu\nu}\), by Lemma 4.3 we have

\(\max_{\partial\Omega} u_{\nu\nu} < -\min_{\partial\Omega} u_{\nu\nu} < C.\)

Thus we shall assume \(-\min_{\partial\Omega} u_{\nu\nu} \leq \max_{\partial\Omega} u_{\nu\nu}\), that is \(\max_{\partial\Omega} |u_{\nu\nu}| = \max_{\partial\Omega} u_{\nu\nu}\). Denote \(M = \max u_{\nu\nu} > 0\) and \(\hat{z}_0 \in \partial\Omega\) such that \(\max_{\partial\Omega} u_{\nu\nu} = u_{\nu\nu}(\hat{z}_0)\).
We take the test function

\[ \hat{P}(x) = (1 + \beta d)[Du \cdot (-Dd) - \varphi(x, u)] - (B + \frac{1}{2} M)h(x), \]

where \( \beta \) and \( B \) are positive constants to be chosen later.

On \( \partial \Omega \), \( \hat{P}(x) = 0 \), and on \( \partial \Omega_\mu/\partial \Omega \), we have \( d = \mu \) and

\[
\hat{P}(x) \geq -(1 + \beta \mu)[|Du| + |\varphi(x, u)|] - (B + \frac{1}{2} M)[-\mu + \mu^2]
\]

\[
\geq 0,
\]

since we take \( B \) big enough. So on \( \partial \Omega_\mu \), we have \( \hat{P} \geq 0 \).

Next to prove \( \hat{P} \) attains its minimum only on \( \partial \Omega \) by contradiction, we assume \( \hat{P} \) attains its minimum at some point \( \hat{x}_0 \in \Omega_\mu \). Rotating the coordinates \( D^2 u(\hat{x}_0) \) is diagonal. In the following, all the calculations are at \( \hat{x}_0 \).

We have

\[
0 = \hat{P}_i(\hat{x}_0) = \beta d_i[-u_jd_j - \varphi(x, u)] + (1 + \beta d)[-u_jd_j] - \varphi_i - \varphi u_i
\]

\[
- (B + \frac{1}{2} M)h_i
\]

\[
(4.32)
\]

\[
= \beta d_i[-u_jd_j - \varphi(x, u)] + (1 + \beta d)[-u_i d_i + u_jd_j] - \varphi_i - \varphi u_i
\]

\[
- (B + \frac{1}{2} M)h_i
\]

and

\[
0 \leq \hat{P}_{ii}(\hat{x}_0) = \beta d_{ii}[-u_jd_j - \varphi(x, u)] + 2\beta d_i[-u_jd_j] - \varphi_i - \varphi u_i
\]

\[
+ (1 + \beta d)[-u_{jii}d_j + 2u_{ji}d_{ji} + u_jd_{jii}] - (\varphi_{ii} + 2\varphi_{iu}u_i + \varphi_{uu}u_i^2 + \varphi_{uu}u_i)
\]

\[
- (B + \frac{1}{2} M)h_{ii}
\]

\[
(4.33)
\]

\[
= \beta d_{ii}[-u_jd_j - \varphi(x, u)] + 2\beta d_i[-u_i d_i + u_jd_j] - \varphi_i - \varphi u_i
\]

\[
+ (1 + \beta d)[-u_{jii}d_j + 2u_{ji}d_{ii} + u_jd_{jii}] - (\varphi_{ii} + 2\varphi_{iu}u_i + \varphi_{uu}u_i^2 + \varphi_{uu}u_i)
\]

\[
- (B + \frac{1}{2} M)h_{ii}
\]

\[
\leq -2\beta u_{iii}d_i^2 + (1 + \beta d)[-u_{jii}d_j + 2u_{ji}d_{ii}] - \varphi u_{ii} - (B + \frac{1}{2} M)h_{ii} + C.
\]
Hence by (2.4) and Lemma 2.2 we have

\[
0 \leq G^{ii} \hat{P}_{ii}(\hat{x}_0) \\
\leq -2\beta G^{ii}u_{ii}d_i^2 + (1 + \beta d)[-G^{ii}u_{jj}d_j - 2G^{ii}u_{ii}d_{ii} - \varphi_uG^{ii}u_{ii}] \\
- (B + \frac{1}{2}M)G^{ii}h_{ii} + CG^{ii} \\
\leq -2\beta G^{ii}u_{ii}d_i^2 - 2(1 + \beta d)G^{ii}u_{ii}d_{ii} - [(B + \frac{1}{2}M)c_0 - C](\sum_i G^{ii} + 1).
\]

From (4.32), we have

\[
u_{ii} = 1 - \frac{2d_i}{1 + \beta d}(B + \frac{1}{2}M) + \frac{\beta[-u_jd_j - \varphi(x, u)]}{1 + \beta d} + \frac{-u_jd_{ji} - \varphi_i - \varphi_uu_i}{d_i}.
\]

Denote \( E = \{i : \beta d_i^2 < \frac{1}{n}, 1 \leq i \leq n\} \) and \( F = \{i : \beta d_i^2 \geq \frac{1}{n}, 1 \leq i \leq n\} \). Let \( \beta \geq \frac{1}{\mu} > 1 \), then

\[d_i^2 < \frac{1}{n} = \frac{1}{n}|Dd|^2, \quad i \in E.\]

Thus we have \( \sum_{i \in E} d_i^2 \leq 1 = |Dd|^2 \) and \( F \) is not empty. We choose \( B \) large such that for \( i \in F \),

\[
\frac{\beta[-u_jd_j - \varphi(x, u)]}{1 + \beta d} + \frac{-u_jd_{ji} - \varphi_i - \varphi_uu_i}{d_i} \leq \frac{B}{5},
\]

thus

\[
\frac{3B}{5} + \frac{2M}{5} \leq u_{ii} \leq \frac{6B}{5} + \frac{M}{2}, \quad \text{for } i \in F.
\]

There must be an \( i_0 \in F \) such that

\[
d_{i_0}^2 \geq \frac{|Dd|^2}{n} = \frac{1}{n}.
\]
By (4.34), we get

\begin{align}
0 & \leq G^{ii} \hat{\tilde{P}}_{ii}(\hat{x}_0) \\
& \leq -2\beta \sum_{i \in F} G^{ii} u_{ii} d_i^2 - 2\beta \sum_{i \in E} G^{ii} u_{ii} d_i^2 \\
& \quad - 2(1 + \beta d) \sum_{u_{ii} > 0} G^{ii} u_{ii} d_{ii} - 2(1 + \beta d) \sum_{u_{ii} < 0} G^{ii} u_{ii} d_{ii} \\
& \quad + [-(B + \frac{1}{2} M)c_0 + C](\sum_i G^{ii} + 1) \\
& \leq -2\beta \sum_{i \in F} G^{ii} u_{ii} d_i^2 - 2\beta \sum_{i \in E} G^{ii} u_{ii} d_i^2 + 4\kappa_{\max} \sum_{u_{ii} > 0} G^{ii} u_{ii} \\
& \quad + [-(B + \frac{1}{2} M)c_0 + C](\sum_i G^{ii} + 1).
\end{align}

(4.39)

Direct calculations yield

\begin{align}
-2\beta \sum_{i \in F} G^{ii} u_{ii} d_i^2 & \leq -2\beta G^{\sigma \sigma} u_{\sigma \sigma} d_{\sigma}^2 \leq -\frac{2\beta}{n} G^{\sigma \sigma} u_{\sigma \sigma}, \\
\sum_{i \in F} G^{ii} u_{ii} d_i^2 & \leq -\frac{2\beta}{n} \sum_{i \in E, u_{ii} < 0} G^{ii} u_{ii}
\end{align}

(4.40)

\begin{align}
-2\beta \sum_{i \in E} G^{ii} u_{ii} d_i^2 & \leq -\frac{2\beta}{n} \sum_{i \in E, u_{ii} < 0} G^{ii} u_{ii} \\
& \leq -\frac{2}{n} \sum_{i \in E, u_{ii} < 0} G^{ii} u_{ii} \\
& = -\frac{2}{n} \sum_{u_{ii} > 0} G^{ii} u_{ii} - \sum_{u_{ii} > 0} G^{ii} u_{ii} \\
& \leq \frac{2}{n} \sum_{u_{ii} > 0} G^{ii} u_{ii} + \frac{2}{n} L_2,
\end{align}

(4.41)

where the last inequality is according to Lemma 2.2.
Therefore,
\begin{equation}
0 \leq G^{ii} \hat{P}_{ii}(\hat{x}_0) \\
\leq -\frac{2\beta}{n} G^{i0i0} u_{i0i0} + \left(4\kappa_{\max} + \frac{2}{n}\right) \sum_{u_{ii} > 0} G^{ii} u_{ii} \\
+ \left[-(B + \frac{1}{2} M) c_0 + C \right]\left(\sum_i G^{ii} + 1\right).
\end{equation}

We divide into three cases to prove the result. Without loss of generality, we can assume that $i_0 = 1 \in F$, and $u_{22} \geq \cdots \geq u_{nn}$.

Case I. $u_{nn} > 0$.
In this case, we have by lemma 2.2
\begin{equation}
0 \leq G^{ii} \hat{P}_{ii}(\hat{x}_0) \\
\leq -\frac{2\beta}{n} G^{i0i0} u_{i0i0} + \left(4\kappa_{\max} + \frac{2}{n}\right) \sum_{u_{ii} > 0} G^{ii} u_{ii} + \left[-(B + \frac{1}{2} M) c_0 + C \right]\left(\sum_i G^{ii} + 1\right)
\leq (4\kappa_{\max} + \frac{2}{n}) \sum_{i=1}^{n} G^{ii} u_{ii} + \left[-(B + \frac{1}{2} M) c_0 + C \right]\left(\sum_i G^{ii} + 1\right)
\leq C + \left[-(B + \frac{1}{2} M) c_0 + C \right]\left(\sum_i G^{ii} + 1\right)
\leq 0,
\end{equation}
by taking $B$ large enough. This is a contradiction.

Case II. $u_{nn} < 0$ and $-u_{nn} < \frac{c_0}{10(4\kappa_{\max} + 4)} u_{11}$.
\begin{equation}
\begin{aligned}
(4\kappa_{\max} + \frac{2}{n}) \sum_{u_{ii} > 0} G^{ii} u_{ii} \\
= (4\kappa_{\max} + \frac{2}{n}) \left[\sum_{i=1}^{n} G^{ii} u_{ii} - \sum_{u_{ii} < 0} G^{ii} u_{ii}\right]
\leq (4\kappa_{\max} + \frac{2}{n}) \left[\sum_{i=1}^{n} G^{ii} u_{ii} - u_{nn} \sum_{i=1}^{n} G^{ii}\right]
\leq C + \frac{c_0}{10} u_{11} \sum_{i=1}^{n} G^{ii}
\leq C + \frac{c_0}{10} \left(\frac{6B}{5} + \frac{M}{2}\right) \sum_{i=1}^{n} G^{ii}.
\end{aligned}
\end{equation}
Hence combining (4.42) and (4.44), we have

\[ 0 \leq G^{ii} \hat{P}_{ii}(\hat{x}_0) \leq (4\kappa_{\text{max}} + \frac{2}{n}) \sum_{u_{ii} > 0} G^{ii} u_{ii} + \left[-(B + \frac{1}{2} M) c_0 + C\right] \left(\sum_i G^{ii} + 1\right) \]

(4.45)

\[ \leq C + \frac{c_0}{10} \left(\frac{6B}{5} + \frac{M}{2}\right) \sum_{i=1}^n G^{ii} + \left[-(B + \frac{1}{2} M) c_0 + C\right] \left(\sum_i G^{ii} + 1\right) \]

\[ < 0, \]

by taking \( B \) large enough. This is a contradiction.

Case III. \( u_{nn} < 0 \) and \( -u_{nn} \geq \frac{c_0}{10(4\kappa_{\text{max}} + \frac{2}{n})} u_{11} \).

We have \( u_{11} \geq \frac{3B}{5} + \frac{2M}{5} \) and \( u_{22} \leq C(1 + M) \). So \( u_{11} \geq \frac{2}{5} C u_{22} \). Let \( \delta = \frac{2}{5C} \) and \( \varepsilon = \frac{c_0}{10(4\kappa_{\text{max}} + \frac{2}{n})} \), by Lemma 2.6, we have

(4.46)

\[ G^{11} \geq c_2 \sum_{i=1}^n G^{ii}. \]

Hence from (4.42) and (4.46), we have

(4.47)

\[ 0 \leq G^{ii} \hat{P}_{ii}(\hat{x}_0) \leq -\frac{2\beta}{n} G^{11} u_{11} + \left(4\kappa_{\text{max}} + \frac{2}{n}\right) \sum_{u_{ii} > 0} G^{ii} u_{ii} + \left[-(B + \frac{1}{2} M) c_0 + C\right] \left(\sum_i G^{ii} + 1\right) \]

\[ \leq -\frac{2\beta}{n} c_2 \left(\frac{3B}{5} + \frac{2M}{5}\right) \sum_{i=1}^n G^{ii} + \left(4\kappa_{\text{max}} + \frac{2}{n}\right) C(1 + M) \sum_{i=1}^n G^{ii} \]

\[ < 0, \]

by taking \( \beta \) big enough. This is a contradiction. So \( \hat{P} \) attain its minimum only on \( \partial \Omega \). Finally, we can get

\[ 0 \geq P_\nu(\hat{x}_0) \]

(4.48)

\[ = \left[u_{\nu\nu}(\hat{x}_0) - u_j d_{j\nu} - D_\nu \varphi(x, u)\right] - (B + \frac{1}{2} M) \]

\[ \geq \max_{\partial \Omega} u_{\nu\nu} - C - (B + \frac{1}{2} M), \]

hence (4.30) holds. \( \square \)
5. Existence

In this section, we complete the proof of Theorem 1.3. Combining Theorem 3.1, Theorem 1.1, and Theorem 1.2 with the global second order derivative Hölder estimates we get the estimates

\begin{equation}
\|u\|_{C^{2,\beta}} \leq C,
\end{equation}

for a solution with \(\lambda(D^2u) \in \Gamma_k\), and \(\alpha < 0\), where \(C\) and \(\beta\) depending on \(n, k, \Omega, |u|_{C^2(\overline{\Omega})}, \varphi, \alpha(x), \alpha_l(x), \inf \alpha_l(x)\). Then applying the continuity method we complete the proof of Theorem 1.3.

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