Origin of $\Delta I = 1/2$ Rule for Kaon Decays: QCD Infrared Fixed Point

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We replace ordinary chiral SU(3)$_L \times$ SU(3)$_R$ perturbation theory $\chi$PT$_3$ by a new theory $\chi$PT$_\sigma$ based on a low-energy expansion about an infrared fixed point $\alpha_{IR}$ for 3-flavor QCD. At $\alpha_{IR}$, the quark condensate $\langle \bar{q}q \rangle_{vac} \neq 0$ induces nine Nambu-Goldstone bosons: $\pi, K, \eta$ and a $0^{++}$ QCD dilaton $\sigma$. Physically, $\sigma$ appears as the $f_0(500)$ resonance, a pole at a complex mass with real part $\leq m_K$. The $\Delta I = 1/2$ rule for nonleptonic $K$-decays is then a consequence of $\chi$PT$_\sigma$, with a $K_S\sigma$ coupling fixed by data for $K^0_S \to \gamma\gamma$ and $\gamma\gamma \to \pi\pi$. We estimate $R_{IR} \approx 5$ for the nonperturbative Drell-Yan ratio $R = \sigma(e^+ e^- \to \text{hadrons})/\sigma(e^+ e^- \to \mu^+ \mu^-)$ at $\alpha_{IR}$.

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The precise determination of the mass and width of the $f_0(500)$ resonance [1,3] prompts us to revisit an old idea [4,5] that the chiral condensate $\langle \bar{q}q \rangle_{vac} \neq 0$ may also be a condensate for scale transformations in the chiral SU(3)$_L \times$ SU(3)$_R$ limit. This can occur in QCD if the heavy quarks $t, b, c$ are first decoupled and then the strong coupling $\alpha_s$ of the resulting 3-flavor theory runs nonperturbatively to a fixed point $\alpha_{IR}$ in the infrared limit (Fig. 1). At that point, $\beta(\alpha_{IR})$ vanishes, so the gluonic term in the strong trace anomaly [7]

$$\theta^\mu_\mu = \beta(\alpha_s) \frac{g^2}{4\alpha_s} G^{\mu\nu} G^{\mu\nu} + \left(1 - \delta(\alpha_s)\right) \sum_{q=u,d,s} m_q \bar{q}q$$  \hspace{1cm} (1)

is absent, which implies

$$\theta^\mu_\mu |_{\alpha_s = \alpha_{IR}} = \left(1 - \delta(\alpha_{IR})\right) (m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s) \to 0 \text{, SU(3)$_L \times$ SU(3)$_R$ limit} \hspace{1cm} (2)$$

and hence a $0^{++}$ QCD dilaton $\sigma$ due to quark condensation. [3] The obvious candidate for this state is the $f_0(500)$, which arises from a pole on the second sheet at a complex mass with typical value [4]

$$m_{f_0} = 441 - i \times 272 \text{ MeV}$$  \hspace{1cm} (3)

and surprisingly small errors [12]. In all estimates of this type, the real part of $m_{f_0}$ is less than $m_K$.

We begin by setting up chiral-scale perturbation theory $\chi$PT$_\sigma$ for amplitudes expanded in $\alpha_s$ about $\alpha_{IR}$. Its Lagrangian summarises soft-$\{\pi, K, \eta, \sigma\}$ meson theorems for approximate chiral $SU(3)_f \times SU(3)_R$ and scale symmetry, with results for strong interactions similar to those found originally [4,5]. Effective weak operators are then added to simulate nonleptonic $K$ decays. The main result is a simple explanation of the $\Delta I = 1/2$ rule for kaon decays: in leading order of $\chi$PT$_\sigma$, there is a dilaton pole diagram (Fig. 2) which accounts for the dominant $I = 0$ amplitude.

It is well known that dispersive analyses which include $f_0(500)$ pole amplitudes produce excellent fits to data for

$$K^0_S \to \pi\pi \text{ with couplings } g_{K^0\sigma} \text{ and } g_{\sigma\pi\pi} \text{ derived from the effective theory $\chi$PT$_\sigma$.}$$

### Notes:

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3. We have $[D_\mu, D_\nu] = ig^{\alpha\beta}_{\mu\nu} T^\alpha$, where $D_\mu$ is the covariant derivative, $\{T^\alpha\}$ generate the gauge group, $\alpha_s = g^2/4\pi$ is the strong coupling, and $\beta = \mu \partial \alpha_s / \partial \mu$ and $\delta = -\mu \partial \ln m_q / \partial \mu$ refer to a mass-independent renormalization scheme with scale $\mu$.
4. We reserve the term dilaton and notation $\sigma$ for a Nambu-Goldstone boson due to exact scale invariance in some limit. We are not talking about the $\sigma$-model, scalar gluonium [13], or “walking gauge theories” [14], where $\beta$ is small but never zero.
5. In field and string theory, it is often stated that Green’s functions are manifestly conformal invariant for $\beta = 0$. That depends on the unlikely assumption that no scale condensates are present; otherwise, conformal invariance becomes manifest only if all 4-momenta are space-like and large.
\( K \rightarrow \pi\pi \) and \( \gamma \rightarrow \pi\pi \) as well as \( \pi\pi \rightarrow \pi \pi \). The problem is that, in the context of conventional chiral SU(3)\(_f\times SU(3)\)\(_R\) perturbation theory \( \chi PT \), these corrections are too large.\(^4\) They cannot contribute to leading terms \( A_{\text{LO}} \) in \( \chi PT \) expansions

\[
A = \{ A_{\text{LO}} + A_{\text{NLO}} + \ldots \}_{\chi PT}(4)
\]

because \( f_0 \) does not belong to the \( \chi PT \) Goldstone sector \( \{ \pi, K, \eta \} \). How can this be reconciled with the success\(^5\) of \( \chi PT \) elsewhere? Where next to leading order terms \( A_{\text{NLO}} \) are corrections \( \lesssim 30\% \)? The discrepancy is extreme for \( K \rightarrow \pi\pi \): an \( f_0 \) pole in \( A_{\text{NLO}} \) producing the factor of 22 enhancement of the \( I = 0 \) amplitude would be \( \approx 70 \) times the expected 30\% correction.

Our solution to this problem is to replace \( \chi PT \) with \( \chi PT \). We include \( f_0 = \sigma \) in the Goldstone sector \( \{ \pi, K, \eta, \sigma \} \) and identify scale invariance (Eq. (3)) as the symmetry most likely to permit this. The idea works because \( \chi PT \) includes \( f_0 \) pole amplitudes in its leading order terms without upsetting successful leading order \( \chi PT \) predictions for amplitudes which do not involve the 00 channel; that is, because the \( \chi PT \) Lagrangian equals the \( \sigma \to 0 \) limit of the \( \chi PT \) Lagrangian. In next to leading order, new chiral loop diagrams involving \( \sigma \) need to be checked.

Notice that \( \sigma \) becomes a Goldstone boson only if all three quarks \( u, d, s \) become massless for \( \alpha_s \to \alpha_{\text{IR}} \).

In the limit \( m_{u,d} \to 0 \) with \( m_s \neq 0 \), we use standard chiral SU(2)\(_L\times SU(2)\)\(_R\) perturbation theory \( \chi PT \), where momenta \( \partial \delta \) are \( O(m_\tau) \) and the Goldstone sector is \( \{ \pi^+, \pi^0, \pi^- \} \). We do not include a dilaton in \( \chi PT \); \( f_0 \) belongs to the non-Goldstone sector, so it retains most of its mass and width in that limit. \( \chi PT \) is not sensitive to the behavior of \( \beta \); as a result of expanding in \( \alpha_s \) about \( \alpha_{\text{IR}} \).

The gluonic trace anomaly is represented by a term \( L_{\text{anom}} \) in an effective-scale Lagrangian

\[
L[\sigma, U, U^+] = L_{\text{inv}}^{d=4} + L_{\text{anom}}^{d=4} + L_{\text{mass}}^{d=4} .
\]

constructed from a chiral invariant QCD dilaton field \( \sigma \) and the usual chiral SU(3) field \( U = U(\pi, K, \eta) \). Both \( L_{\text{inv}} \) and \( L_{\text{anom}} \) are \( SU(3)\)\(_L\times SU(3)\)\(_R\) invariant, while \( L_{\text{mass}} \) belongs to the representation \((3, 3) \oplus (3, 3) \oplus (3, 3) \) associated with the \( \pi, K, \eta \) (mass)\(^2\) matrix \( M \). The operator dimensions of \( L_{\text{inv}} \) and \( L_{\text{mass}} \) satisfy \( d_{\text{inv}} = 4 \) and \( 1 \leq d_{\text{mass}} < 4 \), with

\[
d_{\text{mass}} = 3 + \delta(\alpha_{\text{IR}}) .
\]

As \( \alpha_s \to \alpha_{\text{IR}} \), the gluonic anomaly vanishes, so for consistency, we must suppose that terms in \( L_{\text{anom}} \) contain derivatives \( O(\partial^2) = O(M) \) or have \( O(M) \) coefficients. The result is a chiral-scale perturbation expansion \( \chi PT \) about \( \alpha_{\text{IR}} \) with QCD dilaton mass \( m_\sigma = O(m_K) \).

Note that QCD in the limit \( \alpha_s \) resembles the physical theory in the resonance region, but differs completely at high energies because it lacks asymptotic freedom; instead, Green’s functions scale with nonperturbative anomalous dimensions. All particles except \( \pi, K, \eta \) and \( \sigma \) remain massive. Strong gluon fields set the scale of the condensate \( \langle \bar{q}q \rangle_{\text{vac}} \), which then sets the scale for massive particles and resonances except (possibly) glueballs.

An explicit formula for the \( \chi PT \) Lagrangian \( L \) can be found by applying the method of Ellis \(^4\) \(^4\) \(^4\). Let \( F_\sigma \) be the coupling of \( \sigma \) to the vacuum via the energy momentum tensor \( \theta_{\mu\nu} \), improved \(^2\) \(^2\) \(^2\) when spin-0 fields are present:

\[
\langle \sigma(q)\theta_{\mu\nu}\rangle_{\text{vac}} = (F_\sigma/3)(q_\mu q_\nu - g_{\mu\nu}q^2) .
\]

The dilaton field is given a scaling property \( \sigma \to \sigma + \text{constant} \) such that \( e^{\sigma/F_\sigma} \) has dimension 1. Then the dimensions of chiral Lagrangian operators such as

\[
\mathcal{K}[U, U^+] = \frac{1}{F_\pi^2} \text{Tr}(\partial_\mu U \partial^\mu U^+) \]

and the dilaton operator \( \mathcal{K}_\sigma = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma \) can be adjusted by powers of \( e^{\sigma/F_\sigma} \) to form terms in \( \mathcal{L} \). In leading order,

\[
\mathcal{L}_{\text{inv}, \text{LO}}^{d=4} = \{ c_1 \mathcal{K} + c_2 K_\sigma + c_3 \sigma^2/F_\sigma \} e^{2\sigma/F_\sigma} ,
\]

\[
\mathcal{L}_{\text{anom}, \text{LO}}^{d=4} = \{ (1 - c_1) \mathcal{K} + (1 - c_2) K_\sigma + c_3 \sigma^2/F_\sigma \} e^{2(\beta' - \delta)/\sigma/F_\sigma} ,
\]

\[
\mathcal{L}_{\text{mass}, \text{LO}} = \text{Tr}(MU^+ + UM^+) e^{(3\delta - \beta')/\sigma/F_\sigma} ,
\]

where \( \beta' \) and \( \delta \) are the anomalous dimensions \( \beta'\) and \( \delta\) of Eqs. \(^10\) \(^10\) \(^10\) and \(^7\). The constants \( c_1 \) and \( c_2 \) are not fixed by general arguments, while \( c_3 \) and \( c_4 \)

\(^4\) Similarly \( \chi PT \) fails for \( K_L \to \pi^0\gamma \) \(^18\).

\(^5\) Except non-leptonic hyperon decays: either \( \chi PT \) for baryons or the weak sector of the Standard Model will have to be amended.
depend on how the field $\sigma$ is chosen. For the vacuum to be stable in the $\sigma$ direction at $\sigma = 0$, terms linear in $\sigma$ must cancel:

$$4c_3 + (4 + \beta')c_4 = -(3 + \delta)(\text{Tr}(MU^\dagger + UM^\dagger))_{\text{vac}}$$

$$= -(3 + \delta)F_\sigma^2(m_\chi^2 + 1/2m_\chi^2)^2.$$  \hspace{1cm} (14)

Because of our requirement $\mathcal{L}_{\text{anom}} = O(\partial^2, M)$, both $c_3$ and $c_4$ are $O(M)$.

The critical exponent $\beta'$ normalises the gluonic term in the trace of the effective energy-momentum tensor:

$\theta_\mu^\mu|_{\text{eff}} = : \beta' L^{d>4}_{\text{anom}} + (\delta - 1)L^{d<4}_{\text{mass}} :$.  \hspace{1cm} (15)

In leading order, $\mathcal{L}$ gives formulas for the $\sigma \pi \pi$ coupling

$$\mathcal{L}_{\sigma \pi \pi} = \{(2 + (1 - c_1)\beta')|\partial\pi|^2 - (3 + \delta)m_\pi^2|\pi|^2\} \sigma/(2F_\sigma)$$

and dilaton mass $m_\sigma$

$$m_\sigma^2F_\sigma^2 = F_\sigma^2(m_\chi^2 + 1/2m_\chi^2)(3 + \delta)(1 - \beta')(4 + \beta')c_4.$$  \hspace{1cm} (16)

which resemble pre-QCD results \cite{4, 8, 21, 22} but have extra gluonic terms proportional to $\beta'$. We assume that the unknown coefficient $2 + (1 - c_1)\beta'$ in Eq. (19) does not vanish accidentally. That preserves the key feature of the original work, that $\mathcal{L}_{\sigma \pi \pi}$ is mostly derivative: for soft $\pi \pi$ scattering (energies $\sim m_\pi$), the dilaton pole amplitude is negligible because the $\sigma \pi \pi$ vertex is $O(m_\pi^2)$, while the $\sigma \pi \pi$ vertex for an on-shell dilaton

$$g_{\sigma \pi \pi} = -(2 + (1 - c_1)\beta')m_\pi^2/(2F_\sigma) + O(m_\pi^2)$$

is $O(m_\pi^2)$, consistent with $\sigma$ being the broad resonance $f_0(500)$. Comparisons with data require an estimate of $F_\sigma$, most simply from $NN$ scattering and the dilaton relation

$$- F_\sigma g_{\sigma NN} \approx M_N.$$  \hspace{1cm} (19)

The data imply \cite{23} a mean value $g_{\sigma NN} \sim 9$ and hence $F_\sigma \sim -100$ MeV but with an uncertainty which is either model dependent or very large (\approx 70%). That accounts for the large uncertainty in

$$1/2 \lesssim |2 + (1 - c_1)\beta'| \lesssim 6$$  \hspace{1cm} (20)

when we compare Eq. (18) with $|g_{\sigma \pi \pi}| = 3.31^{+0.33}_{-0.15}$ GeV \cite{2} and $m_\sigma \approx 441$ MeV.

The convergence of our chiral-scale expansion can be tested by adding $\sigma$-loop diagrams to the standard analysis \cite{19, 24} for $\chi$PT\(_3\). These involve the (as yet) undetermined constants $\beta', \delta, c_1, a$: for example, corrections to $g_{\sigma \pi \pi}$ involve the $\sigma \sigma \sigma$ and $\sigma \sigma \pi$ vertices derived from Eq. (19). However, when we apply the dimensional arguments of Manohar and Georgi \cite{24} to our scheme, we find that there are two $\chi$PT\(_3\) scales $\chi_\pi = 4\pi F_\pi$ and $\chi_\sigma = 4\pi F_\sigma$, which are numerically similar ($F_\sigma \sim F_\pi$). The following points should be noted:

1. The $\alpha, \beta < 10$ scales $\chi_\pi, \chi_\sigma \approx 1$ GeV must not be confused with the scale $\Lambda_{\text{QCD}} \approx 200$ MeV for ultraviolet expansions $\alpha, \beta \gg 0$ (asymptotic freedom).

2. The small value of $F_\sigma < \chi_\sigma$ implies a $\sigma$ width

$$\Gamma_{\sigma \pi \pi} \approx \frac{|g_{\sigma \pi \pi}|^2}{16\pi m_\sigma} \sim \frac{m_\sigma^2}{16\pi F_\sigma^2} \sim 250$ MeV$$  \hspace{1cm} (21)

which is numerically misleading: $\Gamma_{\sigma \pi \pi}$ is $O(m_\sigma^2)$ and hence non-leading relative to the mass $m_\sigma$. So tree diagrams produce the leading order\(^6\) of $\chi$PT\(_3\), as in $\chi$PT\(_2\) and $\chi$PT\(_3\).

3. The technique used to obtain Eq. (13) from $\chi$PT\(_3\) also works for $O(\partial^4, M^0, M^2)$ terms in $\mathcal{L}$ (and in $\mathcal{L}_{\text{weak}}$ below). For example, a term $\sim (\text{Tr} \partial U \partial U^\dagger)^2$ appears in $\mathcal{L}_{\text{weak}}^2$ without $\sigma$-field dependence, but within $\mathcal{L}_{\text{weak}}^{d>4}$, it becomes\(^7\)

\[ \text{coefficient} \left( \text{Tr} \partial U \partial U^\dagger \right)^2 \varepsilon^{\beta' \sigma/F_\sigma}. \]

The most important feature of $\chi$PT\(_3\) is that it explains the $\Delta I = 1/2$ rule for $K^- \rightarrow \pi^- \pi^0$ decays.

In the leading order of standard $\chi$PT\(_3\), the effective weak Lagrangian

$$\mathcal{L}_{\text{weak}}|_{\sigma=0} = g_8 Q_8 + g_{27} Q_{27} + Q_M + \text{h.c.}$$  \hspace{1cm} (23)

contains an octet operator \cite{28}

$$Q_8 = J_{13}J_{21} - J_{23}J_{11}, \quad J_{ij} = (U^\dagger \partial \mu U^\dagger)_{ij}$$  \hspace{1cm} (24)

the $U$-spin triplet component \cite{29, 30} of a 27 operator

$$Q_{27} = J_{13}J_{21} + \frac{2}{3}J_{23}J_{11}$$  \hspace{1cm} (25)

and a weak mass operator \cite{31}

$$Q_M = \text{Tr}(\lambda_6 - i\lambda_7)(g_M MU^\dagger + g_M MU).$$  \hspace{1cm} (26)

Although $Q_M$ has isospin 1/2, it cannot be used to solve the $\Delta I = 1/2$ puzzle if dilatons are absent: when $Q_M$ is added to the strong mass term $\mathcal{L}_{\text{mass}}|_{\sigma=0}$, it can be removed by a chiral rotation which aligns the vacuum \cite{32} such that $\langle U \rangle_{\text{vac}} = I$ and $M = \text{real diagonal}$. Therefore the conclusion that $|g_8/g_{27}|$ is unreasonably large (\approx 22) is not avoided.

In $\chi$PT\(_3\), the outcome is entirely different. The weak mass operator's dimension $(3 + \delta_u)$ is not the same as the dimension $(3 + \delta)$ of $\mathcal{L}_{\text{mass}}$, so the $\sigma$ dependence of

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\(^6\) Beyond leading order, and in degenerate cases like the $K_L - K_S$ mass difference, methods used to estimate corrections at the $Z^0$ peak \cite{22} and the $\rho$ resonance \cite{20} may be necessary.

\(^7\) To the extent allowed by data for soft-$\sigma$ amplitudes, $O(\partial^4)$ coefficients can be predicted by saturation by non-Goldstone resonances — like $\chi$PT\(_3\) \cite{27}, but with $f_0$ excluded.
FIG. 3: Leading order diagrams for $K_S^0 \to \gamma\gamma$ in $\chi$PT$_{\sigma}$, including finite loop graphs $[32]$. The grey vertex contains $\pi^\pm$, $K^\pm$ loops as in the four $\chi$PT$_3$ diagrams to the right. An analogous set of diagrams contributes to $\gamma\gamma \to \pi^0\pi^0$.

$$Q_M e^{(3+\delta_\pi)/F_\sigma}$$ cannot be eliminated by a chiral rotation. Instead, after aligning the vacuum, we find

$$L_{\text{align}} = Q_S \sum_n g_{8n} e^{d_{8n}/F_\sigma} + g_{27} Q_{27} e^{d_{27}/F_\sigma}$$

$$+ Q_M \{ e^{(3+\delta_\pi)/F_\sigma} - e^{(3+\delta)/F_\sigma} \} + \text{h.c.} \, , \quad (27)$$

noting that $Q_S$ represents quark-gluon operators with differing dimensions at $\alpha_{IR}$. As a result, there is a residual interaction $L_{K_S^0} = g_{K_S^0} K_S^0 \sigma$ which mixes $K_S$ and $\sigma$ in leading order

$$g_{K_S^0} = (\delta_\pi - \delta) \text{Re} \{ (2m_K^2 - m_{\pi}^2)g_M - m_{\pi}^2 g_M \} F_\sigma^2 / 2F_\sigma$$

and produces the $\Delta I = 1/2$ amplitude $A_{\sigma\text{-pole}}$ of Fig. 4.

From $\gamma\gamma \to \pi^0\pi^0$ and $K_S^0 \to \gamma\gamma$ (Fig. 3), we estimate

$$|g_{K_S^0}| \approx 4.4 \times 10^3 \text{ keV}^2$$

(29)

to about 30% precision and so, to the extent that $g_{\sigma NN}$ and hence $F_\sigma$ can be determined,

$$|A_{\sigma\text{-pole}}| \approx 0.34 \text{ keV} \, . \quad (30)$$

This accounts for the large $I = 0$ $\pi\pi$ amplitude $A_0$ (31)

$$|A_0|_{\text{expt.}} = 0.33 \text{ keV} \, .$$

compared with $A_2$. So we conclude that the observed ratio $|A_0/A_2| \approx 22$ is mostly due to the dilaton-pole diagram of Fig. 2 that $g_8 = \sum_n g_{8n}$ and $g_{27}$ may have similar magnitudes as simple calculations indicate, and that only $g_{27}$ can be fixed precisely (from $K^+ \to \pi^+\pi^0$).

Consequently, the leading order of $\chi$PT$_{\sigma}$ solves the $\Delta I = 1/2$ problem for kaon decays. The chiral Ward identities which relate the on-shell $K \to 2\pi$ and $K \to \pi$ amplitudes have extra terms due to $\sigma$ bosons, but the no-tadpole theorem (30) is still valid:

$$\langle K|\mathcal{H}_{\text{weak}}|\text{vac} \rangle = O(m_\sigma^2 - m_\rho^2) \, , \quad K \text{ on shell} .$$

(32)

The presence of the $\sigma\gamma\gamma$ vertex in Fig. 3 leads us to apply the electromagnetic trace anomaly (33, 34)

$$\theta^\mu_\mu|_{\text{strong} + \epsilon\text{-mag}} = \theta^\mu_\mu + (R\alpha/6\pi) F_{\mu\nu} F^{\mu\nu} \, ,$$

$$R = \frac{\sigma(e^+ e^- \to \text{hadrons})}{\sigma(e^+ e^- \to \mu^+\mu^-)} \bigg|_{\text{high-energy}} \quad (33)$$

to QCD at the infrared fixed point $\alpha_s = \alpha_{IR}$. Here $F_{\mu\nu}$ and $\alpha$ are the electromagnetic field strength tensor and fine-structure constant.

The $\chi$PT$_{\sigma}$ result for the $\sigma\gamma\gamma$ amplitude is affected by the observation (32) that in $\gamma\gamma$ channels, charged $\pi$, $K$ loop diagrams are finite in the chiral limit. This means that they are of the same order as $\sigma$-pole diagrams: partial conservation of the dilatation current is not equivalent to simple $\sigma$-pole dominance in $\gamma\gamma$ channels. The electromagnetic trace anomaly remains the same (with $R \to R_{IR}$), but because of the extra $(\pi^\pm, K^\pm)$ loops in $(\gamma\gamma|\theta^\mu_\mu|\text{vac})$, the $\sigma\gamma\gamma$ coupling is proportional to $R_{IR} - \frac{1}{2}$, not $R_{IR}$:

$$L_{\sigma\gamma\gamma} = (R_{IR} - \frac{1}{2}) (6\pi F_\sigma)^{-1} \sigma F_{\mu\nu} F^{\mu\nu} \, .$$

(34)

Evidently, data used to estimate $|g_{K_S^0}|$ can also be used to find a phenomenological value for $R_{IR}$. In a dispersive analysis of $\gamma\gamma \to \pi\pi$, it was shown (16) that the residue of the $f_0(500)$ pole can be extracted from the Crystal Ball data (33). We use an updated value (17) of the width $\Gamma(f_0 \to \gamma\gamma) = 1.98^{+0.30}_{-0.24} \text{ keV}$. Within the large uncertainty due to that in $F_\sigma$, we find:

$$R_{IR} \approx 5 \, . \quad (35)$$

This result is a feature of the non-perturbative theory at $\alpha_{IR}$, so it has nothing to do with asymptotic freedom or the free-field formula $(N_f = 3)$

$$R(\alpha_s = 0) = \sum \{|\text{quark charges}|^2 \}^2 = 2 \, .$$

(36)

Notice that $\chi$PT$_{\sigma}$ relates amplitudes in the physical region (33) to high-energy quantities like $\delta(\alpha_{IR})$ and $R_{IR}$ characteristic of massless QCD at $\alpha_{IR}$. Does QCD simplify in that limit and, unlike QED, allow $\beta' \neq 0$ at the fixed point?

Unfortunately, our analysis does not explain the failure of chiral theory to account for non-leptonic $|\Delta S| = 1$ hyperon decays. We have shown that octet dominance is not necessary, but that makes no difference for hyperon decays: the Pati-Woo $\Delta I = 1/2$ mechanism (37) forbids all contributions from $27$ operators.

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