EXISTENCE OF WEAK SOLUTION FOR MEAN CURVATURE FLOW WITH TRANSPORT TERM AND FORCING TERM

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Abstract. We study the mean curvature flow with given non-smooth transport term and forcing term, in suitable Sobolev spaces. We prove the global existence of the weak solutions for the mean curvature flow with the terms, by using the modified Allen-Cahn equation that holds useful properties such as the monotonicity formula.

1. Introduction. Let $d \geq 2$ and $\Omega$ be the torus, that is, $\Omega := \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. Assume that $U_t \subset \Omega$ is an open set with a smooth boundary $M_t := \partial U_t$ for $t \geq 0$. A family $\{M_t\}_{t \geq 0}$ of hypersurfaces in $\Omega$ is called a mean curvature flow (MCF) with transport term and forcing term if the normal velocity vector $v$ of $M_t$ satisfies the following:

$$v = h + (u \cdot \nu + g)\nu \text{ on } M_t, \quad t > 0,$$

where $u : \Omega \times (0, \infty) \to \mathbb{R}^d$ and $g : \Omega \times (0, \infty) \to \mathbb{R}$ are given functions, $\cdot$ is the inner product in $\mathbb{R}^d$, $h$ and $\nu$ are the mean curvature vector and the inner unit normal vector of $M_t$, respectively. In [21, 22], they considered the MCF with transport term ($g \equiv 0$) to study the incompressible and viscous non-Newtonian two-phase fluid flow introduced by Liu and Walkington [23]. The MCF with forcing term ($u \equiv 0$) corresponds to the crystal growth (see [7, 14, 32]).

In the case of $u \equiv 0$ and $g \equiv 0$, Brakke [5] defined the general weak solution (Brakke flow) for (1) via the geometric measure theory and proved the global existence. Ilmanen [17] also showed the global existence of the Brakke flow by the phase field method. Recently, Kim and Tonegawa [20] showed the global existence of the multi-phase MCF in the sense of the Brakke flow (see also [38]). For other weak solutions, it is well-known that [8] and [12] proved the existence of the global unique solution in the sense of viscosity solutions. In addition, about the global existence of the MCF, we also mention [3, 18, 24].

In the case of $u \not\equiv 0$ or $g \not\equiv 0$, Liu, Sato and Tonegawa [21] proved the global existence of the weak solution for (1) with $g \equiv 0$ in the sense of the Brakke flow as long as the given transport term $u$ belongs to $L^P_{\text{loc}}((0, \infty); (W^{1,p}(\Omega))^d)$ for $p > (d+2)/2$ and $d = 2, 3$. Takasao and Tonegawa [36] also proved the existence for...
more general settings, that is, \( d \geq 2 \) and \( u \) belongs to \( L^{q}_{loc}(\mathbb{R}^d); (W^{1,p}(\Omega))^d \) for \( q \in (2, \infty) \) and \( p \in \left( \frac{d}{q-1}, \infty \right) \) \((p \geq 4/3 \) in addition if \( d = 2 \)). On the other hand, Mugnai and Röger [28] showed the global existence of the weak solution called \( L^2 \)-flow for (1) with \( u \in L^{q}_{loc}(\mathbb{R}^d); (L^{\infty}(\Omega))^d \) and \( g \in L^{2}_{loc}(\mathbb{R}^d; L^{\infty}(\Omega)) \) for \( d = 2,3 \) (see [28, Section 5.2]). As explained later in this section, the existence of the weak solution can be expected for \( g \) under the same conditions as [36]. One motivation in this paper is the generalization of the function space of \( g \) in the existence theorem for (1).

Let \( \varepsilon \in (0,1) \). In [17], to show the existence of the weak solution for (1) with \( u \equiv 0 \) and \( g \equiv 0 \) in the sense of the Brakke flow, the author studied the following Allen-Cahn equation [2]:

\[
\begin{cases}
\varepsilon \varphi_t^\varepsilon = \varepsilon \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon}, & (x,t) \in \Omega \times (0, \infty), \\
\varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x \in \Omega,
\end{cases}
\]

where \( W \) is the double-well potential, such as \( W(s) = (1-s^2)^2/2 \).

Set

\[
d\mu_t^\varepsilon := \frac{1}{\sigma} \left( \frac{\varepsilon |\nabla \varphi^\varepsilon(x,t)|^2}{2} + \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \right) dx
\]

and \( d\tilde{\mu}_t^\varepsilon := \frac{\varepsilon}{\sigma} |\nabla \varphi^\varepsilon(x,t)|^2 dx \), where

\[
\sigma = \int_{-1}^{1} \sqrt{2W(s)} \, ds.
\]

These measures correspond to the Hausdorff measure \( \mathcal{H}^{d-1}_{|M_t^\varepsilon|} \), where \( M_t^\varepsilon = \{ x \in \Omega | \varphi^\varepsilon(x,t) = 0 \} \). By integration by parts, we have

\[
\frac{d}{dt} \int_{\Omega} \phi \, d\mu_t^\varepsilon = \int_{\Omega} \nabla \phi \cdot h^\varepsilon - \phi |h^\varepsilon|^2 \, d\tilde{\mu}_t^\varepsilon + \int_{\Omega} \phi_t \, d\mu_t^\varepsilon, \quad \forall \, \phi \in C^1_c(\Omega \times (0, \infty); [0, \infty)),
\]

where

\[
h^\varepsilon = -\frac{\Delta \varphi^\varepsilon - W'(\varphi^\varepsilon)/\varepsilon}{|\nabla \varphi^\varepsilon|} \cdot \frac{\nabla \varphi^\varepsilon}{|\nabla \varphi^\varepsilon|}.
\]

The vector-valued function \( h^\varepsilon \) is the approximation of the mean curvature vector for \( M_t^\varepsilon \). Formally we obtain the limit \( M_t = \lim_{\varepsilon \to 0} M_t^\varepsilon \) and the following Brakke’s inequality(see [17]):

\[
\int_{M_t} \phi \, d\mathcal{H}^{d-1} \bigg|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_{M_t} \nabla \phi \cdot h - \phi |h|^2 + \phi_t \, d\mathcal{H}^{d-1} \, dt
\]

for any \( 0 \leq t_1 < t_2 < \infty \) and \( \phi \in C^1_c(\Omega \times [0, \infty); [0, \infty)) \). We remark that \( \int_{M_t} \phi |h|^2 \, d\mathcal{H}^{d-1} \leq \liminf_{\varepsilon \to 0} \int_{M_t^\varepsilon} \phi |h^\varepsilon|^2 \, d\mu_t^\varepsilon \) implies the inequality of (3). The Brakke flow is the weak solution characterized by (3). If the solution is smooth, then the definition of the Brakke flow and the MCF are equivalent (see [38, Proposition 2.1]). In addition, for any initial data \( M_0 \), there exists the trivial solution \( \{ M_t \}_{t \geq 0} \) defined by \( M_t = \emptyset \) for \( t > 0 \). Therefore, it is necessary to ensure that the weak solution obtained is non-trivial. One advantage of the existence theorem via (2) is that one can prove the existence of non-trivial solutions, since \( \{ x \in \Omega | \lim_{\varepsilon \to 0} \varphi^\varepsilon(x,t) = 1 \} \) is a \( C^4 \) function with respect to \( t \) (see [36, Proposition 8.3]).
The above discussion requires \( \lim_{\varepsilon \to 0} \mu^\varepsilon_t = \lim_{\varepsilon \to 0} \tilde{\mu}^\varepsilon_t \) as Radon measures, so the following property is important:

\[
\int_{\Omega} \left| \frac{\varepsilon |\nabla \varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \right| \, dx \to 0 \quad \text{as} \ \varepsilon \downarrow 0 \quad \text{(4)}
\]

for a.e. \( t \geq 0 \). The property (4) is called the vanishing of the discrepancy measure (see Definition 2.1 below) and is also important to show the rectifiability of the limit measure \( \lim_{\varepsilon \to 0} \mu^\varepsilon_t \) (see [17, Section 9.3]) and the existence of the \( L^2 \)-flow. To prove (4), Ilmanen [17] showed the non-positivity of the discrepancy measure, that is,

\[
\frac{\varepsilon |\nabla \varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \leq 0, \quad (x,t) \in \Omega \times [0,\infty), \quad \text{(5)}
\]

for (2) under several suitable assumptions. Using (5), one can obtain an estimate called monotonicity formula, that is,

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \rho_{y,s}(x,t) \, d\mu^\varepsilon_t(x) \leq \int_{\mathbb{R}^d} \frac{\rho_{y,s}(x,t)}{2(s-t)} \left( \frac{\varepsilon |\nabla \varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \right) \, dx \leq 0. \quad \text{(6)}
\]

Here

\[
\rho_{y,s}(x,t) := \frac{1}{(4\pi(s-t))^d} e^{-\frac{|x-y|^2}{4(s-t)}}, \quad t < s, \ x, y \in \mathbb{R}^d
\]

and, \( \varphi^\varepsilon \) and \( \mu^\varepsilon_t \) are extended periodically to \( \mathbb{R}^d \). The function \( \rho \) is called the backward heat kernel. Note that \( \rho \) converges to the Dirac delta function \( \delta_t \) for a \((d-1)\)-dimensional surface as \( t \to s \). Assume that \( D := \sup_{\varepsilon \in (0,1)} \mu^\varepsilon_t(\Omega) < \infty \). The non-positivity (5) and the monotonicity formula (6) implies that there exists \( C > 0 \) depending only on \( D \) such that

\[
\lim_{\delta \downarrow 0} \int_{\Omega} \frac{1}{s-t} \int_{\Omega} \rho_{y,s}(x,t) \left| \frac{\varepsilon |\nabla \varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \right| \, dx \, dt \leq C \quad \text{(7)}
\]

for any \((y,s) \in \mathbb{R}^d \times [0,\infty)\). Roughly speaking, if (4) does not hold, then the left hand side of (7) is unbounded for some \((y,s)\), since \( \int_{\mathbb{R}^d} \frac{1}{s-t} \, dt = \infty \). Therefore (5) is important property in this discussion. In this paper, we use the results of [29, Proposition 4.9] to obtain (4) (see Theorem 5.2 below and note that the result needs \( d = 2 \) or 3). So we do not use this argument in this paper, but (5) is still important in the case of \( d \geq 4 \), and to estimate \( \int \rho \, d\mu^\varepsilon_t \) and the upper bound of the density for the measure \( \mu^\varepsilon_t \) (see Theorem 3.1 below).

In [21, 36], to consider the MCF with additional transport term, they studied the following:

\[
\begin{aligned}
\varepsilon \varphi^\varepsilon_t &= \varepsilon \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon} - \varepsilon u^\varepsilon \cdot \nabla \varphi^\varepsilon, \quad (x,t) \in \Omega \times (0,\infty), \\
\varphi^\varepsilon(x,0) &= \varphi^\varepsilon_0(x), \quad x \in \Omega,
\end{aligned}
\quad \text{(8)}
\]

where \( u^\varepsilon \) is the smooth approximation of \( u \). In [28], they considered the following Allen-Cahn equation with forcing term:

\[
\begin{aligned}
\varepsilon \varphi^\varepsilon_t &= \varepsilon \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon} - G^\varepsilon, \quad (x,t) \in \Omega \times (0,\infty), \\
\varphi^\varepsilon(x,0) &= \varphi^\varepsilon_0(x), \quad x \in \Omega,
\end{aligned}
\quad \text{(9)}
\]

where \( G^\varepsilon \) is smooth and satisfies \( \sup_{\varepsilon > 0} \int_0^T \int_{\Omega} \varepsilon^{-1} |G^\varepsilon|^2 \, dx \, dt < \infty \). Let \( g^\varepsilon \) be the smooth approximation of \( g \). Note that substituting \( \varepsilon u^\varepsilon \cdot \nabla \varphi^\varepsilon + g^\varepsilon \sqrt{2W(\varphi^\varepsilon)} \) into \( G^\varepsilon \), we obtain (1) as \( \varepsilon \to 0 \) in the sense of \( L^2 \)-flow (see [28, Section 5.2]).
In the case of \( u^\varepsilon \neq 0 \) or \( g^\varepsilon \neq 0 \), the property (5) does not hold for (8) and (9), generally. Therefore, the proof of (4) in [17] is not applicable to (8) or (9). To prove (4), [28] used the result of [29, Proposition 4.9] (see Theorem 5.2 below). On the other hand, in [21, 36], they used weaker estimates than (5) to obtain (7) and (4). However, we can not apply the technique for the case of \( g^\varepsilon \neq 0 \) directly (see Remark 9 below). Another motivation for this paper is to propose the new phase field method that has the property (5) even when there are transport term and forcing term.

Let \( q^\varepsilon = q^\varepsilon(r) \) be a solution for

\[
\frac{\varepsilon(q^\varepsilon)^2}{2} = \frac{W(q^\varepsilon)}{\varepsilon}, \quad r \in \mathbb{R}, \quad q^\varepsilon(\pm \infty) = \pm 1, \quad q^\varepsilon(0) = 0, \quad \text{and} \quad q^\varepsilon(r) > 0, \quad r \in \mathbb{R}. \tag{10}
\]

For example, if \( W(s) = (1 - s^2)^2/2 \), then \( q^\varepsilon(r) = \tanh(r/\varepsilon) \) satisfies (10). Set \( T > 0 \). In this paper, we consider the following modified Allen-Cahn equation with transport term and forcing term:

\[
\begin{cases}
\varepsilon \varphi_t = \varepsilon \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon} - \varepsilon u^\varepsilon \cdot \nabla \varphi^\varepsilon - (g^\varepsilon + L^\varepsilon r^\varepsilon) \sqrt{2W(\varphi^\varepsilon)}, \\
\varphi^\varepsilon(x, 0) = \varphi^\varepsilon_0(x), \quad x \in \Omega,
\end{cases} \tag{11}
\]

where

\[
L^\varepsilon := 2 \sup_{(x, t) \in \Omega \times (0, T)} |\nabla u^\varepsilon(x, t)| + \sup_{(x, t) \in \Omega \times (0, T)} |\nabla g^\varepsilon(x, t)|
\]

and \( r^\varepsilon = r^\varepsilon(x, t) \) is given by \( \varphi^\varepsilon(x, t) = q^\varepsilon(r^\varepsilon(x, t)) \). Note that if there exists \( (x, t) \in \Omega \times (0, T) \) such that \( |\varphi^\varepsilon(x, t)| = 1 \), then \( r^\varepsilon \) is not well-defined. However, that case does not occur under suitable conditions (see Proposition 1 below). Define

\[
f^\varepsilon := -(u^\varepsilon \cdot \nabla r^\varepsilon) - g^\varepsilon - L^\varepsilon r^\varepsilon.
\]

We remark that by (10), the first equation of (11) is equal to

\[
\varepsilon \varphi_t^\varepsilon = \varepsilon \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon} + f^\varepsilon \sqrt{2W(\varphi^\varepsilon)}. \tag{12}
\]

By adding the forcing term \(-L^\varepsilon r^\varepsilon \sqrt{2W(\varphi^\varepsilon)}\), we can obtain (5), because if the term is added to the phase field method, then an argument similar to that in [17] (the maximum principle for \( u^\varepsilon := |\nabla r^\varepsilon|^2 - 1 \)) can be used (see Lemma 4.2 below). In addition, the additional term is very small in the framework of the phase field method under several assumptions (see Remark 10 below). Roughly speaking, the reason is that \( r^\varepsilon \approx 0 \) near the zero level set of \( \varphi^\varepsilon \). Therefore we can obtain the monotonicity formula and the convergence of the solutions for (11) to the global weak solution for (1), with \( d = 2, 3 \), and \( u \in L^1_{\text{loc}}((0, \infty); (W^{1, p}(\Omega))^d) \) and \( g \in L^q_{\text{loc}}((0, \infty); W^{1, p}(\Omega)) \), where \( q \in (2, \infty) \) and \( p \in (dq/2(q - 1), \infty) \) \((p \geq 4/3 \) in addition if \( d = 2 \). The precise statements of the main results are described in Section 3. The condition \( p \in (dq/2(q - 1), \infty) \) is natural in the following sense (same argument is mentioned in [36]). Let \( \lambda > 0 \) and consider the standard parabolic rescaling, that is, \( \tilde{x} = \frac{x}{\lambda} \) and \( \tilde{t} = \frac{t}{\lambda^2} \). The functions \( u \) and \( g \) correspond to the velocity of \( M_t \), therefore rescaled functions should be \( \tilde{u}(\tilde{x}, \tilde{t}) = \lambda u(x, t) \) and \( \tilde{g}(\tilde{x}, \tilde{t}) = \lambda g(x, t) \), since \( \frac{\tilde{t}}{\lambda^2} = \lambda \frac{t}{\lambda^2} \). We compute

\[
\left( \int_0^\infty \left( \int_{\mathbb{R}^d} |\nabla u|^p dx \right)^\frac{q}{p} dt \right)^{\frac{1}{q}} = \lambda^{\frac{q}{p} + \frac{2}{q}} \left( \int_0^\infty \left( \int_{\mathbb{R}^d} |\nabla \tilde{u}|^p d\tilde{x} \right)^\frac{q}{p} d\tilde{t} \right)^{\frac{1}{q}},
\]
where \( w = u \) or \( g \). The condition \( p \in (dq/(q-1), \infty) \) is equivalent to \( \frac{d}{p} + \frac{2}{q} - 2 < 0 \). Hence the transport term and forcing term can be regarded as perturbations.

About the phase field method for the MCF, there are a huge number of results and we mention [6, 9, 11, 14, 30, 33, 34] and references therein.

The paper is organized as follows. In Section 2, we set our notations and definitions. In Section 3, we explain the main results of this paper. In Section 4, we show the non-positivity of the discrepancy measure and the monotonicity theorems used in this paper as a supplement. In Section 5, we explain the several theorems used in this paper as a supplement.

2. Notation and definitions. Throughout this paper, we consider the case of \( \Omega = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \). For \( r > 0 \), \( k \in \mathbb{N} \) and \( y \in \mathbb{R}^k \) we define \( B^k_r(y) := \{ x \in \mathbb{R}^k \mid |x-y| < r \} \). Set \( \omega_k := \mathcal{L}^k(B^k_r(0)) \). We denote

\[
D(t) := \max \left\{ 1, \mu^+_k(\Omega), \sup_{B^k_r(x) \subset \Omega} \frac{\mu^+_k(B^k_r(x))}{\omega_{d-1}r^{d-1}} \right\}, \quad t \in [0, \infty).
\]

**Definition 2.1.** Set \( \sigma := \int_{-1}^1 \sqrt{2W(s)} \, ds \). Let \( \varphi^\varepsilon \) be a solution for (11). We define a Radon measure \( \mu^+_k \) and \( \xi^+_k \) by

\[
\mu^+_k(\phi) := \frac{1}{\sigma} \int_\Omega \phi(x) \left( \frac{\varepsilon|\nabla \varphi^\varepsilon(x,t)|^2}{2} + \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \right) \, dx
\]

and

\[
\xi^+_k(\phi) := \frac{1}{\sigma} \int_\Omega \phi(x) \left( \frac{\varepsilon|\nabla \varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \right) \, dx
\]

for any \( \phi \in C_c(\Omega) \). The measure \( \xi^+_k \) is called the discrepancy measure.

In this paper, we suppose that a function \( W \) satisfies the following:

\[
W : \mathbb{R} \to [0, \infty) \text{ is smooth and } W(\pm 1) = W'(\pm 1) = 0. \tag{13}
\]

For some \( \alpha_1 \in (-1, 1) \), \( W' < 0 \) on \((\alpha_1, 1)\) and \( W' > 0 \) on \((-1, \alpha_1)\). \tag{14}

\[ \exists \alpha_2 \in (0, 1) \text{ and } \kappa > 0 \text{ such that } W''(s) > 0 \text{ for any } \alpha_2 \leq |s| \leq 1. \tag{15} \]

\[ \exists C_1 > 0 \text{ such that } (q^{-1}(s))^2W(s) \leq C_1 \text{ for any } |s| < 1. \tag{16} \]

Here \( q \) is a solution for (10) with \( \varepsilon = 1 \) and \( q^{-1} \) is the inverse function of \( q \). For example, \( W(s) = (1-s^2)^2/2 \) satisfies (13), (14), (15), and (16). We remark that \( q(r) = \tanh r \) in the case of \( W(s) = (1-s^2)^2/2 \).

Next we recall several definitions and notations from the geometric measure theory and refer to [1, 5, 13, 15, 31, 38] for more details. For a set \( U \subset \Omega \) with finite perimeter, we denote the reduced boundary by \( \partial^*U \), and the total variation measure of the distributional derivative \( \nabla \chi_U \) is denoted by \( \|\nabla \chi_U\| \). Let \( \mu \) be a Radon measure on \( \Omega \). We call \( \mu \) \( k \)-rectifiable if \( \mu \) is represented by \( \mu = \theta \mathcal{H}^k \lceil M \), that is, \( \int_{\Omega} \eta \, d\mu = \int_M \eta \theta \, d\mathcal{H}^k \) for any \( \eta \in C_c(\Omega) \) (see [1, Section 3.5] or [31, Section 15]), where \( M \subset \Omega \) is a \( \mathcal{H}^k \)-measurable countably \( k \)-rectifiable set, and \( \theta \in L^1_{loc}(\mathcal{H}^k \lceil M) \) is a positive valued function \( \mathcal{H}^k \lceil a.e. \) on \( M \). In addition, if \( \theta \) is positive and integer-valued \( \mathcal{H}^k \lceil a.e. \) on \( M \) then we call \( \mu \) \( k \)-integral. Especially, if \( \theta \equiv 1 \), we say \( \mu \) has unit density. Let \( T \) be a hyper plane in \( \mathbb{R}^d \) with \( 0 \in T \) and \( \nu \) be the unit normal vector of \( T \). We also use \( T \) to denote the orthogonal projection \( \mathbb{R}^d \to T \), that is, \( T = \text{Id} - \nu \otimes \nu \), where \( \text{Id} \) is the identity matrix.
Assume that $M$ is a countably $(d-1)$-rectifiable and $H^{d-1}$-measurable subset of $\Omega$ and $\theta \in L^1_{\text{loc}}(H^{d-1}(M))$ is a positive function. For a Radon measure $\mu := \theta H^{d-1}|_M$, $h$ is called a generalized mean curvature vector if
\[
\int_{\Omega} \text{div}_M \Phi \, d\mu = - \int_{\Omega} h \cdot \Phi \, d\mu
\]
holds for any $\Phi \in C^1_c(\Omega; \mathbb{R}^d)$ (see [5, Section 2.9] or [31, Section 16]).

The following definition is similar to the formulation of the Brakke flow [5]:

**Definition 2.2 (L$^2$-flow [27]).** Let $T > 0$ and $\{\mu_t\}_{t \in (0, T)}$ be a family of Radon measures on $\Omega$. Set $d\mu := d\mu_t dt$. We call $\{\mu_t\}_{t \in (0, T)}$ an $L^2$-flow if the following holds:
1. $\mu_t$ is $(d-1)$-integral and has a generalized mean curvature vector $h \in L^2(\mu_t; \mathbb{R}^d)$ a.e. $t \in (0, T)$,
2. and there exist $C > 0$ and a vector $v \in L^2(0, T; (L^2(\mu_t))^d)$ such that
\[
v(x, t) \perp T_x \mu_t \quad \text{for } \mu\text{-a.e. } (x, t) \in \Omega \times (0, T)
\]
and
\[
\left| \int_0^T \int_{\Omega} (h_t + \nabla \eta \cdot v) \, d\mu_t \, dt \right| \leq C\|\eta\|_{\infty}
\]
for any $\eta \in C^1_c(\Omega \times (0, T))$. Here $T_x \mu_t$ is the approximate tangent plane of $\mu_t$ at $x$.

In addition, the above vector $v \in L^2(0, T; (L^2(\mu_t))^d)$ is called a generalized velocity vector.

**Remark 1.** If $\{\mu_t\}_{t \in (0, T)}$ is an integral Brakke flow, then it is also $L^2$-flow (see [4, Section 2.5]).

3. **Main results.** In this paper, first we show the non-positivity of the discrepancy measure and the upper bound of the density for the measure $\mu_t$.

**Theorem 3.1.** Assume that $T > 0$, $d \geq 2$, $2 < q < \infty$, $p \in [\frac{2d}{d+1}, \infty) \cap (\frac{d}{2(q-1)}, \infty)$, and
\[
0 < \gamma < \frac{1}{2}.
\]
Suppose that $\varphi^\varepsilon$ is a classical solution for (11) with $\max_{x \in \Omega} |\varphi^\varepsilon_0(x)| < 1$ and
\[
\frac{\varepsilon |\nabla \varphi^\varepsilon_0(x)|^2}{2} - \frac{W(\varphi^\varepsilon_0(x))}{\varepsilon} \leq 0, \quad x \in \Omega,
\]
and $u^\varepsilon \in (C^\infty(\Omega \times [0, T]))^d$, $g^\varepsilon \in C^\infty(\Omega \times [0, T])$ with
\[
L^\varepsilon = 2 \sup_{(x, t) \in \Omega \times (0, T)} |\nabla u^\varepsilon(x, t)| + \sup_{(x, t) \in \Omega \times (0, T)} |\nabla g^\varepsilon(x, t)| \leq \varepsilon^{-\gamma},
\]
and there exists $D_0 > 0$ such that
\[
D(0) \leq D_0.
\]
Then the following holds:
1. The non-positivity (5) holds for any $(x, t) \in \Omega \times [0, T)$. 
2. There exist $D_1 > 0$ and $\epsilon \in (0, 1)$ such that
\[
\sup_{0 \leq t \leq T} D(t) \leq D_1, \quad \epsilon \in (0, \epsilon).
\] (23)

Remark 2. Similar result about the density bound has been obtained in [21, 36]. The difficult part of the proof of the density bound is the estimate of the positive part of the discrepancy measure. Therefore, one of the advantages of this paper is that the phase field method for (1) with the non-positivity (5) was obtained. The property is also useful for obtaining the monotonicity formula and the vanishing of the discrepancy measure (see Lemma 4.5 below). In addition, in the case of $g^\epsilon \neq 0$, it will be difficult to obtain the estimate of the discrepancy measure via the phase field method without the additional term $-L^\epsilon r^\epsilon \sqrt{2W(\varphi^\epsilon)}$ (see Remark 9 below).

Remark 3. For the regularity corresponding to (20),
\[
\sup_{\Omega \times [0,T]} |u^\epsilon| \leq \epsilon^{-\gamma} \quad \text{and} \quad \sup_{\Omega \times [0,T]} |\nabla u^\epsilon| \leq \epsilon^{-(\gamma+1)}
\]
are assumed in [36], where $\gamma \in (0, \frac{1}{2})$. In Theorem 3.1, the estimate of $\sup_{\Omega \times [0,T]} |u^\epsilon|$ is not required. However, the assumption for $\sup_{\Omega \times [0,T]} |\nabla u^\epsilon|$ is stronger than that in [36].

Remark 4. The assumption (20) is used to prove that the additional term $-L^\epsilon r^\epsilon \sqrt{2W(\varphi^\epsilon)}$ converges to 0 (see Remark 10), and (21) is mainly necessary for the $L^2$-estimates of transport term and forcing term (see Lemma 4.3 and Lemma 4.6).

Set
\[
v^\epsilon = \left\{ \begin{array}{ll}
-\varphi^\epsilon_i & \text{if } |\nabla \varphi^\epsilon| \neq 0, \\
0 & \text{otherwise}.
\end{array} \right.
\]
Let $\Psi_\delta \in C_c^\infty(B_\delta(0))$ be the Dirac sequence, and $\{\delta_i\}_{i=1}^\infty$ and $\{T_i\}_{i=1}^\infty$ be positive sequences with $\delta_i \to 0$ and $T_i \to \infty$ as $i \to \infty$, respectively. For $\gamma \in (0, \frac{1}{2})$, $u \in L^q_{\text{loc}}([0, \epsilon); (W^{1,p}(\Omega))^d)$, and $g \in L^q_{\text{loc}}([\epsilon, \infty); W^{1,p}(\Omega))$, we choose a positive sequence $\{\varepsilon_i\}_{i=1}^\infty$ such that $\varepsilon_i \to 0$,
\[
\sup_{\Omega \times [0,T]} |\nabla u^{\varepsilon_i}| \leq \varepsilon_i^{-\gamma}, \quad \sup_{\Omega \times [0,T]} |\nabla g^{\varepsilon_i}| \leq \varepsilon_i^{-\gamma}
\] for any $i \geq 1$, (24)
where $u^{\varepsilon_i} := \Psi_{\delta_i} \ast u$, and $g^{\varepsilon_i} := \Psi_{\delta_i} \ast g$. Note that
\[
u^{\varepsilon_i} \to u \quad \text{in } L^q_{\text{loc}}([\epsilon, \infty); (W^{1,p}(\Omega))^d) \quad \text{and} \quad g^{\varepsilon_i} \to g \quad \text{in } L^q_{\text{loc}}([\epsilon, \infty); W^{1,p}(\Omega)).
\]
For the solution $\varphi^{\varepsilon_i}$ for (11) with $\varepsilon = \varepsilon_i$ and $T = T_i$, we define $\varphi^{\varepsilon_i}(x, t) = 1$ if $t \geq T_i$, for the following theorem. By using Theorem 3.1, we show the vanishing of the discrepancy measure and the existence of the weak solution for (1):

Theorem 3.2. Let $d = 2, 3$ and assume that $u \in L^q_{\text{loc}}([0, \epsilon); (W^{1,p}(\Omega))^d)$ and $g \in L^q_{\text{loc}}([\epsilon, \infty); W^{1,p}(\Omega))$. Let $\{\delta_i\}_{i=1}^\infty$, $\{\varepsilon_i\}_{i=1}^\infty$ and $\{T_i\}_{i=1}^\infty$ be positive sequences such that (24) holds. Assume that for any $i \geq 1$ all assumptions of Theorem 3.1 hold with $\varepsilon = \varepsilon_i$ and $T = T_i$. Then there exists a subsequence (we denote $\varepsilon_i$, by $\epsilon$ for simplicity) and the following holds:

(a) There exists a family of $(d-1)$-integral Radon measures $\{\mu_t\}_{t \in [0, \infty)}$ on $\Omega$ such that
(b1) $\mu^\epsilon \to \mu$ as Radon measures on $\Omega \times [0, \infty)$, where $d\mu = d\mu_0 dt$. 

(a2) $\mu_t^\varepsilon \to \mu_t$ as Radon measures on $\Omega$ for all $t \in [0, \infty)$.

(b) There exists $\psi \in BV_{loc}(\Omega \times [0, \infty)) \cap C^5_{loc}([0, \infty); L^1(\Omega))$ such that
    (b1) $\varphi^\varepsilon \to 2\psi - 1$ in $L^1_{loc}(\Omega \times [0, \infty))$ and a.e. pointwise.
    (b2) $\psi = 0$ or 1 a.e. on $\Omega \times [0, \infty)$.
    (b3) $\|\nabla \psi(\cdot, t)\|_2(\phi) \leq \mu_t(\phi)$ for any $t \in [0, \infty)$ and $\phi \in C_c(\Omega; [0, \infty))$.

(c) $\xi_t^\varepsilon \to 0$ as Radon measures on $\Omega$ for a.e. $t \in [0, \infty)$.

(d) For any $\Phi \in C_c(\Omega \times [0, \infty); \mathbb{R}^d)$ we have
$$\lim_{\varepsilon \to 0} \frac{1}{\sigma} \int_{\Omega \times (0, \infty)} u^\varepsilon \cdot \Phi \frac{\varepsilon}{\sqrt{2W(\varphi^\varepsilon)} \nabla \varphi^\varepsilon} \cdot dx\, dt = \int_{\Omega \times (0, \infty)} u \cdot \Phi \, d\mu.$$ 

(e) There exists a vector valued function $\tilde{g} \in L^2_{loc}(0, \infty; (L^2(\mu_t))^d)$ such that
$$\lim_{\varepsilon \to 0} \frac{1}{\sigma} \int_{\Omega \times (0, \infty)} g^\varepsilon \sqrt{2W(\varphi^\varepsilon)} \nabla \varphi^\varepsilon \cdot \Phi \, dx\, dt = \int_{\Omega \times (0, \infty)} \tilde{g} \cdot \Phi \, d\mu$$
for any $\Phi \in C_c(\Omega \times [0, \infty); \mathbb{R}^d)$.

(f) $\{\mu_t\}_{t \in (0, \infty)}$ is an $L^2$-flow with a generalized velocity vector
$$v(x, t) = h(x, t) + (Id - T_x \mu_t)u(x, t) + \tilde{g}(x, t),$$
where $h$ is the generalized mean curvature vector of $\mu_t$, $T_x \mu_t$ is the approximate tangent plane of $\mu_t$ at $x$, and
$$\lim_{\varepsilon \to 0} \int_{\Omega \times (0, \infty)} v^\varepsilon \cdot \Phi \, d\mu = \int_{\Omega \times (0, \infty)} v \cdot \Phi \, d\mu$$
for any $\Phi \in C_c(\Omega \times [0, \infty); \mathbb{R}^d)$. Moreover spt $\tilde{g} \subset \partial^* \{\psi = 1\}$ and there exists a measurable function $\theta : \partial^* \{\psi = 1\} \to \mathbb{N}$ such that
$$\tilde{g} = \int_0^\theta \nu \, d\mu \quad \mathcal{H}^d\text{-a.e. on } \partial^* \{\psi(x, t) = 1\},$$
where $\nu(\cdot, t)$ is the inner unit normal vector of $\{\psi(\cdot, t) = 1\}$ on $\partial^* \{\psi(\cdot, t) = 1\}$.

Remark 5. The assumption for $d$ comes from Theorem 5.2. In the case of $d \geq 4$, then we may need several arguments similar to that in [17, 36]. The term $(Id - T_x \mu_t)u$ corresponds to $(u \cdot \nu)\nu$ if $\mu_t$ is given by a smooth hypersurface.

Remark 6. In [28, Section 5.2], they showed the existence theorem with $u \in L_{loc}^q((0, \infty); (W^{1,p}(\Omega))^d)$ and $g \in L^2_{loc}((0, \infty); L^\infty(\Omega))$ for $d = 2, 3$. As mentioned in Section 1, natural function spaces are considered in Theorem 3.1 and Theorem 3.2.

In the case of $g \equiv 0$, the existence of the weak solution for (1) in the sense of Brakke flow with $u \in L_{loc}^q((0, \infty); (W^{1,p}(\Omega))^d)$ and $d \geq 2$ has already been proven in [36]. Here, a family of $(d - 1)$-integral Radon measures $\{\mu_t\}_{t \in [0, \infty)}$ is called a Brakke flow with transport term $u$ if
$$\int_{t_1}^{t_2} \phi \, d\mu_t \leq \int_{t_1}^{t_2} \int_{\Omega} (\nabla \phi - \phi h) \cdot (h + (\nu \cdot u)\nu) + \phi_t \, d\mu_t \, dt$$
holds for any $\phi \in C^1_c(\Omega \times [0, \infty); [0, \infty))$. Note that the regularity of the Brakke flow is also known (see [19, 37]). The main differences of the phase field methods between [36] and this paper are having or not having the proofs of the estimates of the positive part of the discrepancy measure, and the additional forcing term $-L^2r^s \sqrt{2W(\varphi^\varepsilon)}$. Because the term is very small in the sense of the Brakke flow
(see Remark 10), it is expected that same existence theorem of the Brakke flow in [36] \((d \geq 2)\) will be obtained via the phase field model (11). In addition, (5) would make it easier to prove the vanishing of the discrepancy measure than that in [36].

However, in the case of \(g \neq 0\), it is difficult to consider the weak solution for (1) in the sense of the Brakke flow, since weak convergences of \(\nu^\varepsilon\) and \(h^\varepsilon\) are insufficient to make sense of the convergence

\[
\int \phi g^\varepsilon \nu^\varepsilon \cdot h^\varepsilon \, d\tilde{\mu}_t^\varepsilon \to \int \phi g \nu \cdot h \, d\mu_t
\]

for any \(\phi \in C_c(\Omega \times [0, \infty))\),

where

\[
\nu^\varepsilon = \frac{\nabla \phi^\varepsilon}{|\nabla \phi^\varepsilon|}, \quad h^\varepsilon = -\frac{\Delta \phi^\varepsilon + W'(\phi^\varepsilon)}{|\nabla \phi^\varepsilon|} \nu^\varepsilon, \quad d\tilde{\mu}_t^\varepsilon = \frac{\varepsilon}{\sigma} |\nabla \phi^\varepsilon|^2 \, dx.
\]

In particular, when \(\mu_t\) is not a unit density measure, the treatment of the orientation of \(\nu\) is a problem. On the other hand, this problem does not occur when \(L^2\)-flow is considered, because the computation of the inner product is not necessary in the definition of the \(L^2\)-flow and the characterization of the generalized velocity (25).

**Remark 7.** Regarding energy estimates, there is no difference in the handling of transport term and forcing term. However, regarding convergence, the forcing term converges with respect to the measure \(\|\nabla \psi(\cdot, t)\|\) (see (71)). The function \(\theta\) in (27) is the inverse of the Radon-Nikodym Derivative \(d\tilde{\mu}_t^\varepsilon/d\mu_t\).

4. **Proof of main theorems.** In this section, we assume all the assumptions of Theorem 3.1. First we prove the well-posedness of the phase field model (11). Next we show the monotonicity formula via the arguments in [17] and the upper bound of the density of \(\mu_t^\varepsilon\) by using the arguments in [21, 36]. The upper bound estimates, Theorem 5.3, and standard measure theoretic arguments imply the existence theorem.

4.1. **Well-posednes of (11).** Let \(\delta \in (0, 1)\) and \(r_3^\delta : \mathbb{R} \to \mathbb{R}\) be a \(C^\infty\) function such that

\[
r_3^\delta(s) = \begin{cases} (q^\delta)^{-1}(-1 + \delta) - 1 & \text{if } s < -1, \\ (q^\delta)^{-1}(s) & \text{if } s \in [-1 + \delta, 1 - \delta], \\ (q^\delta)^{-1}(1 - \delta) + 1 & \text{if } s > 1. \end{cases}
\]

From the definition of \(r^\varepsilon\) in (11), we need the a priori estimate \(\varphi^\varepsilon(x, t) \in (-1, 1)\) for any \((x, t) \in \Omega \times [0, T)\). Therefore first we consider the following modified equation:

\[
\begin{cases}
\varepsilon \varphi_t^\varepsilon = \varepsilon \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon} - \varepsilon u^\varepsilon \cdot \nabla \varphi^\varepsilon - (g^\varepsilon + L^\varepsilon r_3^\delta(\varphi^\varepsilon)) \sqrt{2W(\varphi^\varepsilon)}, & (x, t) \in \Omega \times (0, T), \\
\varphi^\varepsilon(x, 0) = \varphi^\varepsilon_0(x), & x \in \Omega.
\end{cases}
\]

(28)

The estimate \(\varphi^\varepsilon \in (-1, 1)\) can be obtained as follows from the maximum principle.

**Lemma 4.1.** Let \(T > 0\) and \(a \in (0, 1)\). Then there exists \(\delta \in (0, 1)\) such that the following hold: Let \(\varphi^\varepsilon\) be a classical solution for (28) with \(\delta > 0\) and \(\max_{\varphi \in \Omega} |\varphi^\varepsilon_0(x)| \leq 1 - a\). Then \(\sup_{(x, t) \in \Omega \times [0, T)} |\varphi^\varepsilon(x, t)| \leq 1 - \delta\). Moreover, \(\varphi^\varepsilon\) is also a solution for (11) in \(\Omega \times [0, T)\).
Proof. Let $\varphi^\varepsilon$ be a classical solution for (28) with $\delta > 0$ and $\max_{x \in \Omega} |\varphi_0^\varepsilon(x)| \leq 1 - a$. By the definition, $r^\varepsilon_0(\varphi^\varepsilon(x, t)) = r^\varepsilon(x, t)$ if $|\varphi^\varepsilon(x, t)| \leq 1 - \delta$. So we only need to prove $\sup_{(x, t) \in \Omega \times [0, T]} |\varphi^\varepsilon(x, t)| \leq 1 - \delta$.

By the maximum principle, we obtain $\sup_{x \in \Omega, t \in [0, T]} |\varphi^\varepsilon(x, t)| \leq 1$ easily. Assume that there exists $(x, t) \in \Omega \times [0, T)$ such that $\varphi^\varepsilon(x, t) = 1$. Then $T_1 := \inf \{ t \in (0, T) \mid \varphi^\varepsilon(x, t) = 1 \text{ for some } x \in \Omega \} < T$. Note that $r^\varepsilon(x, t) = (q^\varepsilon)^{-1}(\varphi^\varepsilon(x, t))$ is well-defined for any $(x, t) \in \Omega \times [0, T_1)$.

Set $h(q) := \sqrt{2W(q)}$ for $q \in \mathbb{R}$. By (10) we obtain
\[
q^\varepsilon_r = \frac{h(q^\varepsilon)}{\varepsilon} \quad \text{and} \quad q^\varepsilon_{rr} = \frac{(h(q^\varepsilon))_r}{\varepsilon} = \frac{h_q(q^\varepsilon)}{\varepsilon} q^\varepsilon_r.
\]
By (10), (28), and (29) we have
\[
q^\varepsilon_r r^\varepsilon_t = q^\varepsilon_r \Delta r^\varepsilon + q^\varepsilon_{rr} |\nabla r^\varepsilon|^2 - q^\varepsilon_r -(u^\varepsilon \cdot \nabla r^\varepsilon)q^\varepsilon_r - (g^\varepsilon + L^\varepsilon r^\varepsilon_0)q^\varepsilon_r
\]
\[
= q^\varepsilon_r \Delta r^\varepsilon + q^\varepsilon_r \frac{h_q(q^\varepsilon)}{\varepsilon} ((|\nabla r^\varepsilon|^2 - 1) - (u^\varepsilon \cdot \nabla r^\varepsilon)q^\varepsilon_r - (g^\varepsilon + L^\varepsilon r^\varepsilon_0)q^\varepsilon_r
\]
for any $(x, t) \in \Omega \times (0, T_1)$. Thus we obtain
\[
r^\varepsilon_r = \Delta r^\varepsilon + \frac{h_q}{\varepsilon} (|\nabla r^\varepsilon|^2 - 1) - u^\varepsilon \cdot \nabla r^\varepsilon - g^\varepsilon - L^\varepsilon r^\varepsilon_0 \quad \text{in } \Omega \times (0, T_1).
\]

Set
\[
M^\varepsilon := \varepsilon^{-1} \max_{|s| \leq 1} |h_q(s)| + \sup_{x \in \Omega, t \in [0, T_1)} |g^\varepsilon(x, t)|.
\]
We remark that $\frac{h_q(q^\varepsilon)}{\varepsilon} \leq M^\varepsilon$ by $\sup_{x \in \Omega, t \in [0, T]} |\varphi^\varepsilon(x, t)| \leq 1$. From the definition, $r^\varepsilon_0 > 0$ in $U^b_T := \{(x, t) \in \Omega \times (0, T_1 - b) \mid r^\varepsilon(x, t) > 0 \}$ for $b \in (0, T_1/2)$. Therefore we have
\[
r^\varepsilon_t \leq \Delta r^\varepsilon + \left( \frac{h_q}{\varepsilon} \nabla r^\varepsilon - u^\varepsilon \right) \cdot \nabla r^\varepsilon \quad \text{in } U^b_T,
\]
where $\tilde{r}^\varepsilon := r^\varepsilon - M^\varepsilon t$. By the maximum principle, we obtain
\[
\max_{x \in \Omega, t \in [0, T_1 - b]} r^\varepsilon(x, t) \leq \max_{x \in \Omega} r^\varepsilon(x, 0) + M^\varepsilon T_1.
\]
The definition of $T_1$ implies $\lim_{b \to 0} \max_{x \in \Omega, t \in [0, T_1 - b]} r^\varepsilon(x, t) = \infty$. This contradicts (31) and $\varphi^\varepsilon(x, t) < 1$ for any $(x, t) \in \Omega \times [0, T)$. Similarly, we obtain $\varphi^\varepsilon(x, t) > -1$ for any $(x, t) \in \Omega \times [0, T)$. In addition, $\max_{x \in \Omega} |r^\varepsilon(x, 0)| \leq (q^\varepsilon)^{-1}(1 - a) \text{ imply}
\[
\max_{x \in \Omega, t \in [0, T]} |\varphi^\varepsilon(x, t)| \leq q^\varepsilon((q^\varepsilon)^{-1}(1 - a) + M^\varepsilon T) < 1.
\]
Thus $\sup_{t \in [0, T]} |\varphi^\varepsilon(x, t)| \leq 1 - \delta$ holds for sufficiently small $\delta > 0$.

By Lemma 4.1, the standard parabolic PDE theory shows

**Proposition 1.** Let $T > 0$ and $\varphi^\varepsilon$ be a smooth function on $\Omega$ with $\max_{x \in \Omega} |\varphi_0^\varepsilon(x)| < 1$. Then there exists a unique solution $\varphi^\varepsilon$ for (11) with initial data $\varphi_0^\varepsilon$ and
\[
\sup_{x \in \Omega, t \in [0, T]} |\varphi^\varepsilon(x, t)| < 1
\]
for any $t \in (0, T)$. 

4.2. Non-positivity of the discrepancy measure. Set 
\[ \xi_\varepsilon(x,t) := \frac{\varepsilon |\nabla \varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \]
for the solution \( \varphi^\varepsilon \) for (11). One of the key lemmas of this paper is the following:

Lemma 4.2. Assume that \(|\nabla r^\varepsilon(x,0)| \leq 1\) for any \( x \in \Omega \). Then we have \(|\nabla r^\varepsilon(x,t)| \leq 1\) and \( \xi_\varepsilon(x,t) \leq 0\) for any \((x,t) \in \Omega \times [0,T)\). Moreover \( \xi_\varepsilon^\varepsilon \) is a non-positive measure for \( t \in [0,T) \).

Proof. By (10) we have 
\[ \frac{\varepsilon |\nabla \varphi^\varepsilon|^2/2}{W(\varphi^\varepsilon)/\varepsilon} \leq |\nabla r^\varepsilon|^2 \text{ on } \Omega \times [0,T). \]
Therefore, if \(|\nabla r^\varepsilon| \leq 1\) then \( \xi_\varepsilon \leq 0 \) and \( \xi_\varepsilon^\varepsilon \) is a non-positive measure. Thus we only need to prove that \(|\nabla r^\varepsilon| \leq 1\) on \( \Omega \times [0,T) \).

By an argument similar to that in (30), we obtain
\[ r^\varepsilon_t = \Delta r^\varepsilon + \frac{h(q)}{\varepsilon} (|\nabla r^\varepsilon|^2 - 1) - u^\varepsilon \cdot \nabla r^\varepsilon - g^\varepsilon - L^\varepsilon r^\varepsilon, \tag{32} \]
where \( h(q) = \sqrt{2W(q)} \) for \( q \in \mathbb{R} \). We compute
\[ \nabla(-u^\varepsilon \cdot \nabla r^\varepsilon - g^\varepsilon - L^\varepsilon r^\varepsilon) \cdot \nabla r^\varepsilon \]
\[ \leq -\frac{1}{2} u^\varepsilon \cdot \nabla |\nabla r^\varepsilon|^2 + |\nabla r^\varepsilon|^2 |\nabla u^\varepsilon| + \frac{1}{2} |\nabla g^\varepsilon|(1 + |\nabla r^\varepsilon|^2) - L^\varepsilon |\nabla r^\varepsilon|^2 \leq -\frac{1}{2} u^\varepsilon \cdot \nabla |\nabla r^\varepsilon|^2 + \frac{1}{2} L^\varepsilon (1 - |\nabla r^\varepsilon|^2). \tag{33} \]
By (32) and (33), we have
\[ \partial_t |\nabla r^\varepsilon|^2 \leq \Delta |\nabla r^\varepsilon|^2 - 2 |\nabla^2 r^\varepsilon|^2 + \frac{2}{\varepsilon} \nabla r^\varepsilon \cdot \nabla h(q)(|\nabla r^\varepsilon|^2 - 1)
+ \left( \frac{2h(q)}{\varepsilon} \nabla r^\varepsilon - \frac{u^\varepsilon}{2} \right) \cdot \nabla |\nabla r^\varepsilon|^2 + \frac{1}{2} L^\varepsilon (1 - |\nabla r^\varepsilon|^2). \tag{34} \]
Set \( w^\varepsilon := |\nabla r^\varepsilon|^2 - 1 \). By (34) we obtain
\[ \partial_t w^\varepsilon \leq \Delta w^\varepsilon + \left( \frac{2h(q)}{\varepsilon} \nabla r^\varepsilon - \frac{u^\varepsilon}{2} \right) \cdot \nabla w^\varepsilon + \left( \frac{2}{\varepsilon} \nabla r^\varepsilon \cdot \nabla h(q) - \frac{1}{2} L^\varepsilon \right) w^\varepsilon. \tag{35} \]
By the assumption we have \( w^\varepsilon(0,0) = |\nabla r^\varepsilon(0,0)|^2 - 1 \leq 0 \) on \( \Omega \). Therefore by (35) and the maximum principle we obtain \( w^\varepsilon \leq 0 \) on \( \Omega \times [0,T) \). Hence we have \(|\nabla r^\varepsilon| \leq 1\) on \( \Omega \times [0,T) \). \qed

Remark 8. In the case of the volume preserving MCF, that is, \( u^\varepsilon \equiv 0, L^\varepsilon \equiv 0 \), and \( g^\varepsilon = g^\varepsilon(t) \) be a non-local term of \( \varphi^\varepsilon \), similar estimates (including the monotonicity formula below) have been proven in [35].

Remark 9. To obtain the estimate for \( \xi_\varepsilon \), a method of applying the maximum principle directly to \( \xi_\varepsilon \) with some additional term is also well known ([10, 21, 26, 36]) in the case of \( g^\varepsilon \equiv 0 \). In [36], they considered the maximum principle for
\[ \tilde{\xi}_\varepsilon := \frac{\varepsilon |\nabla \varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} - \frac{G(\varphi^\varepsilon(x,t))}{\varepsilon} \]
to show the following estimate:
\[ \frac{\varepsilon |\nabla \varphi^\varepsilon(x,t)|^2}{2} - \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \leq 10 \varepsilon^{-\beta} \text{ in } \Omega \times [0,T), \tag{36} \]
where $\varphi^\varepsilon$ is a solution for (8), $\beta \in (0, \frac{1}{2})$ and $G$ is a function such as $G(\varphi^\varepsilon) = \varepsilon\frac{1}{2}\left(1 - \frac{1}{2}(\varphi^\varepsilon - \alpha_1)^2\right)$. Clearly, (36) is weaker than (5), and the key of the proof of (36) is that $\tilde{\varphi}^\varepsilon$ satisfies

$$\partial_t \tilde{\varphi}^\varepsilon + u^\varepsilon \cdot \nabla \tilde{\varphi}^\varepsilon - \Delta \tilde{\varphi}^\varepsilon \leq F(\varepsilon, W', G', G'' \nabla \varphi^\varepsilon, \nabla u^\varepsilon)$$

for suitable $F$ (see [36, (4.32)]). However, in the case of $g \neq 0$, it is not known whether similar estimates can be obtained in this way, because we can not expect $F \leq 0$ and the control of the term $g^\varepsilon \tilde{\varphi}^\varepsilon$ is more difficult than that of the term $u^\varepsilon \cdot \nabla \tilde{\varphi}^\varepsilon$, from the viewpoint of the maximum principle.

### 4.3. $L^2$-estimates of transport term and forcing term.

The following estimate corresponds to the $L^2(\mu^\varepsilon_t)$-estimate of $f^\varepsilon$.

**Lemma 4.3.** Assume that $|\varphi^\varepsilon| < 1$ and $|\nabla r^\varepsilon| \leq 1$ in $\Omega \times [0, T)$, $p \in (2d/(d+1), \infty)$, and $0 \leq L^\varepsilon \leq \varepsilon^{-\gamma}$ for $\gamma > 0$. Then we have

$$\int_\Omega \varepsilon \left((u^\varepsilon \cdot \nabla \varphi^\varepsilon) + (g^\varepsilon + L^\varepsilon r^\varepsilon) \frac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon}\right)^2 dx = 2 \int_\Omega \varepsilon f^\varepsilon \frac{W(\varphi^\varepsilon)}{\varepsilon} dx \leq C_2(D(t)(\|u^\varepsilon(\cdot, t)|^2_{W^{1,p}(\Omega)} + \|g^\varepsilon(\cdot, t)|^2_{W^{1,p}(\Omega)}) + \varepsilon^{1-2\gamma}),$$

where $C_2 = C_2(d, p, W, |\Omega|) > 0$.

**Proof.** We compute

$$\int_\Omega \varepsilon \left((u^\varepsilon \cdot \nabla \varphi^\varepsilon) + (g^\varepsilon + L^\varepsilon r^\varepsilon) \frac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon}\right)^2 dx = 2 \int_\Omega \varepsilon f^\varepsilon \frac{W(\varphi^\varepsilon)}{\varepsilon} dx \leq C_2(D(t)(\|u^\varepsilon(\cdot, t)|^2_{W^{1,p}(\Omega)} + \|g^\varepsilon(\cdot, t)|^2_{W^{1,p}(\Omega)}) + \varepsilon^{1-2\gamma}),$$

(38)

where $|\nabla r^\varepsilon| \leq 1$ and $\nabla \varphi^\varepsilon = q^\varepsilon r^\varepsilon = \varepsilon^{-1} \sqrt{2W(\varphi^\varepsilon)} \nabla r^\varepsilon$ are used.

Next we show that there exists $C > 0$ such that

$$\int_\Omega (L^\varepsilon r^\varepsilon)^2 \frac{W(\varphi^\varepsilon)}{\varepsilon} dx \leq C \varepsilon^{1-2\gamma}. \tag{39}$$

We remark that $q^\varepsilon(r) = q(r/\varepsilon)$ and $r^\varepsilon = \varepsilon q^{-1}(\varphi^\varepsilon)$. Thus we have

$$\int_\Omega (L^\varepsilon r^\varepsilon)^2 \frac{W(\varphi^\varepsilon)}{\varepsilon} dx \leq \int_\Omega \varepsilon^{1-2\gamma}(q^{-1}(\varphi^\varepsilon))^2 W(\varphi^\varepsilon) dx \leq \varepsilon^{1-2\gamma}C_1|\Omega|,$$

where (16) is used. Hence we obtain (39).

Finally we show that there exists $C > 0$ such that

$$\int_\Omega |u^\varepsilon|^2 \frac{W(\varphi^\varepsilon)}{\varepsilon} dx \leq CD(t)\|u^\varepsilon(\cdot, t)|^2_{W^{1,p}(\Omega)}.$$

(40)

Let $\{\psi_i\}_i$ be a partition of unity on $\Omega$ with $\psi_i \in C_c^\infty(\Omega)$, $\text{diam}(\text{spt} \psi_i) \leq 1/2$ and $\|\psi_i\|_{C^2} \leq c(d)$ for any $i$. First we consider the case of $2d/(d+1) \leq p < 2$. Set
Proof. By (12) and the integration by parts, we have
\[
\int_{\Omega} |u^\varepsilon|^2 \frac{W(\varphi^\varepsilon)}{\varepsilon} \, dx \leq \left( \int_{\Omega} |u^\varepsilon|^s \, d\mu_t^\varepsilon \right)^{\frac{2}{s}} (2\mu_t^\varepsilon(\Omega))^{1-\frac{2}{s}}
\]
\[
\leq \left( \sum_i C \int_{\Omega} |\psi_i u^\varepsilon|^s \, d\mu_t^\varepsilon \right)^{\frac{2}{s}} (2\mu_t^\varepsilon(\Omega))^{1-\frac{2}{s}}
\]
\[
\leq \left( \sum_i C_{CMZ} D(t) \int_{\text{spt } \psi_i} |u^\varepsilon|^p + |\nabla u^\varepsilon|^p \, dx \right)^{\frac{2}{p}} (2D(t))^{1-\frac{2}{p}}
\]
\[
\leq CD(t)\|u^\varepsilon(\cdot, t)\|_{W^{1,p}(\Omega)}^2.
\]
For the case of \( p \geq 2 \), we compute
\[
\int_{\Omega} |u^\varepsilon|^2 \frac{W(\varphi^\varepsilon)}{\varepsilon} \, dx \leq \left( \int_{\Omega} |u^\varepsilon|^p \, d\mu_t^\varepsilon \right)^{\frac{2}{p}} (2\mu_t^\varepsilon(\Omega))^{1-\frac{2}{p}}
\]
\[
\leq \left( \sum_i C \int_{\Omega} |\psi_i u^\varepsilon|^p \, d\mu_t^\varepsilon \right)^{\frac{2}{p}} (2D(t))^{1-\frac{2}{p}}
\]
\[
\leq \left( \sum_i C_{CMZ} D(t) \int_{\text{spt } \psi_i} |u^\varepsilon|^p + |u^\varepsilon|^{p-1} |\nabla u^\varepsilon| \, dx \right)^{\frac{2}{p}} (2D(t))^{1-\frac{2}{p}}
\]
\[
\leq CD(t)\|u^\varepsilon(\cdot, t)\|_{W^{1,p}(\Omega)}^2.
\]
where (72) with \( p = 1 \) is used. By (41) and (42) we have (40). Similarly, we have
\[
\int_{\Omega} (g^\varepsilon)^{\frac{3}{2}} \frac{W(\varphi^\varepsilon)}{\varepsilon} \, dx \leq CD(t)\|g^\varepsilon(\cdot, t)\|_{W^{1,p}(\Omega)}^2.
\]
Therefore by (38), (39), (40), and (43) we obtain (37).

\[\square\]

Remark 10. The estimate (39) means that if \( \|\nabla u^\varepsilon, \nabla g^\varepsilon\|_\infty \leq \varepsilon^{-\gamma} \) for \( \gamma \in [0, 1/2) \), then the additional term \(-L\upsilon^\varepsilon \sqrt{2W(\varphi^\varepsilon)}\) vanishes as \( \varepsilon \downarrow 0 \) in the framework of the phase field method of this paper (see (62)).

4.4. Energy estimates and monotonicity formula. Next we show the standard energy estimates and the monotonicity formula for the Allen-Cahn equation (11).

Lemma 4.4. Let \( p \in [2d/(d+1), \infty) \) and \( 2 < q < \infty \). Then there exists \( C_3 = C_3(d, p, q, W, |\Omega|) > 0 \) such that for any \( 0 \leq t_1 < t_2 < T \) we have
\[
\sup_{t \in [t_1, t_2]} \mu_t^\varepsilon(\Omega) + \frac{1}{2\sigma} \int_{t_1}^{t_2} \int_{\Omega} \varepsilon \left( \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} \right)^2 \, dxdt
\]
\[
\leq \mu_0^\varepsilon(\Omega) + C_3 \left\{ \left( \|u^\varepsilon\|_{L^p([t_1, t_2]; W^{1,p}(\Omega))}^2 + \|g^\varepsilon\|_{L^q([t_1, t_2]; W^{1,q}(\Omega))}^q \right) \right\} \]
\[
\cdot \left\{ (t_2 - t_1)^{1-\frac{2}{q}} \sup_{t \in [t_1, t_2]} D(t) + (t_2 - t_1)^{1-2\gamma} \right\}.
\]

Proof. By (12) and the integration by parts, we have
\[
\frac{d}{dt} \mu_t^\varepsilon(\Omega) + \frac{1}{2\sigma} \int_{\Omega} \varepsilon \left( \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} \right)^2 \, dx \leq \frac{1}{\sigma} \int_{\Omega} |f^\varepsilon|^2 \frac{W(\varphi^\varepsilon)}{\varepsilon} \, dx.
\]
Integration of (45) over \([t_1, t_2]\) with (37) gives (44).

To localize the backward heat kernel \( \rho \), we fix a radially symmetric cut-off function
\[
\eta(x) \in C_c^\infty(B_{1/2}^d(0)) \quad \text{with} \quad \eta = 1 \quad \text{on} \quad B_{1/4}^d(0), \quad 0 \leq \eta \leq 1,
\]
and we define $\tilde{\rho}_{y,s}(x,t) := \eta(x-y)\rho_{y,s}(x,t)$. The following estimate is the monotonicity formula for the modified equation (11).

**Lemma 4.5.** Assume that $d \geq 2$, $T > 0$, $\varphi^\varepsilon$ is a solution for (11) and the initial data satisfies $|\varphi_0^\varepsilon(x)| < 1$ and $|\nabla \varphi^\varepsilon(x,0)| \leq 1$ for any $x \in \Omega$. Then

$$\frac{d}{dt} \int_{\mathbb{R}^d} \rho_{y,s}(x,t) d\mu_t^\varepsilon(x) \leq \frac{1}{2\sigma} \int_{\mathbb{R}^d} \rho_{y,s}(x,t)|f^\varepsilon(x,t)|^2 \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} dx$$

and

$$\frac{d}{dt} \int_{\mathbb{R}^d} \tilde{\rho}_{y,s}(x,t) d\mu_t^\varepsilon(x) \leq \frac{1}{2\sigma} \int_{\mathbb{R}^d} \tilde{\rho}_{y,s}(x,t)|f^\varepsilon(x,t)|^2 \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} dx + C_4 e^{2 \varepsilon^{-1} |x-y|} \mu_t^\varepsilon(B_{1/2}^d(y))$$

for any $y \in \mathbb{R}^d$, $0 \leq t < s < T$ and $\varepsilon \in (0,1)$. Here $C_4 = C_4(d) > 0$, $\mu_t^\varepsilon$ and $f^\varepsilon$ are extended periodically to $\mathbb{R}^d$.

**Proof.** In this proof, we regard all functions and measures as periodically extended on $\mathbb{R}^d$. Set $\rho = \rho_{y,s}(x,t)$. By an argument similar to that in the proof of Proposition 2.7 in [35], we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} \rho d\mu_t^\varepsilon \leq \frac{1}{2(s-t)} \int_{\mathbb{R}^d} \rho d\mu_t^\varepsilon + \frac{1}{2\sigma} \int_{\mathbb{R}^d} \rho|f^\varepsilon|^2 \frac{W(\varphi^\varepsilon)}{\varepsilon} dx. \quad (48)$$

By Lemma 4.2 and (48), we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} \rho d\mu_t^\varepsilon \leq \frac{1}{2\sigma} \int_{\mathbb{R}^d} \rho|f^\varepsilon|^2 \frac{W(\varphi^\varepsilon)}{\varepsilon} dx \leq \frac{1}{2} \int_{\mathbb{R}^d} \rho|f^\varepsilon|^2 d\mu_t^\varepsilon.$$

Therefore we have (46). In the computation (46) with $\tilde{\rho}$ instead of $\rho$, we obtain additional terms with the differentiation of $\tilde{\rho}$. Note that the integration of these terms are estimated by $c\mu_t^\varepsilon(B_{1/2}^d(y)) e^{-\varepsilon^{-1} |x-y|}$ with $c = c(d) > 0$ because $|\partial_x \tilde{\rho}, \rho| \leq c(j,d) e^{-\varepsilon^{-1} |x-y|}$ for any $x \in \Omega$ with $|x-y| > 1/4$ and $j = 0,1$. Therefore we obtain (47).

The following estimates are given in [36]. Thus we skip the proof.

**Lemma 4.6.** Let $2 < q < \infty$ and $p \in \left[ \frac{2q}{d+1}, \infty \right) \cap \left( \frac{d}{2(q-p)}, \infty \right)$. Then there exists $C_5 = C_5(d,p,q) > 0$ such that for any $0 \leq t_1 < t_2 < s < T$ we have

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \tilde{\rho}_{y,s} |u^\varepsilon|^2 d\mu_t^\varepsilon dt \leq C_5(t_2 - t_1)^{\hat{p}} \left\| u^\varepsilon \right\|_{L^q([t_1,t_2];W^{1,p}(B_{1/2}^d(y)))}^2 \sup_{t \in [t_1,t_2]} D(t) \quad (49)$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \tilde{\rho}_{y,s} |g^\varepsilon|^2 d\mu_t^\varepsilon dt \leq C_5(t_2 - t_1)^{\hat{p}} \left\| g^\varepsilon \right\|_{L^q([t_1,t_2];W^{1,p}(B_{1/2}^d(y)))}^2 \sup_{t \in [t_1,t_2]} D(t) \quad (50)$$

where $\hat{p}$ is given by $\hat{p} := \frac{2pq-2p-dq}{pq}$ when $p < d$, $\hat{p} = \frac{q-2}{q}$ when $p > d$. However $c$ depends on $\hat{p}$ in addition.
4.5. Proof of Theorem 3.1. In this section we prove the upper bound of the density of $\mu^*_t$ via the monotonicity formula. The proof is based on [21, 36].

Lemma 4.7. Assume that $2 < q < \infty$ and $p \in [\frac{2d}{d+q}, \infty) \cap (\frac{dq}{2(q-1)}, \infty)$. Then there exist $\tilde{c} \geq 2$, $c' > 0$ and $\epsilon_1 > 0$ with the following property. For $0 < t_1 < t_2 < T$ with $t_2 - t_1 < 1$, suppose $D(t_2) = \tilde{c}D(t_1)$ and $D(t) < D(t_2)$ for $t_1 \leq t < t_2$. Then for any $0 < \epsilon < \epsilon_1$, we have

$$\int (t_2 - t_1)^\tilde{p}\left(\|u^e\|_{L^q([t_1, t_2];(W^{1,p}(\Omega))^d)} + \|g^e\|_{L^q([t_1, t_2];(W^{1,p}(\Omega)))}\right) \geq c',$$

where $\tilde{p} > 0$ is as Lemma 4.6.

Proof. Set $A := \|u^e\|_{L^q([t_1, t_2];(W^{1,p}(\Omega))^d)} + \|g^e\|_{L^q([t_1, t_2];(W^{1,p}(\Omega)))}$. Let $\tilde{c} \geq 2$ and assume $D(t_2) = \tilde{c}D(t_1)$ (\(\tilde{c}\) will be chosen later). We consider the following three cases. First we consider the case of $D(t_2) = \mu^*_t(\Omega)$. By (44) we have

$$D(t_2) \leq D(t_1) + C_3\left\{(t_2 - t_1)^{1 - \frac{\tilde{p}}{2}}AD(t_2) + (t_2 - t_1)^{\epsilon_1 - 2\gamma}\right\}.$$

Therefore we obtain

$$D(t_1)\left(\tilde{c} - \tilde{c}C_3(t_2 - t_1)^\tilde{p}A - 1\right) \leq D(t_1)\left(\tilde{c} - \tilde{c}C_3(t_2 - t_1)^{1 - \frac{\tilde{p}}{2}}A - 1\right) \leq C_3\epsilon_1^{-2\gamma},$$

where $\tilde{p} \leq 1 - \frac{2}{q}$ is used. Thus, we have (51), for sufficiently large $\tilde{c} \geq 2$ and sufficiently small $\epsilon_1 > 0$.

Next we consider the case of $D(t_2) = \lim_{n \to \infty} \frac{\mu^*_t(B_{r_n}(y))}{\omega_{d-1}r_n^{d-1}}$ with $\lim_{n \to \infty} r_n \geq \frac{1}{4}$. Then there exists $n \geq 1$ such that $r_n \geq \frac{1}{5}$ and $D(t_2) - \frac{1}{100} \leq \mu^*_t(\Omega)$. Therefore we have

$$\frac{\omega_{d-1}}{5^{d-1}}D(t_2) - \frac{\omega_{d-1}}{5^{d-1}} \leq \frac{1}{100} \mu^*_t(\Omega).$$

Hence, by an argument similar to that in the first case, we obtain

$$D(t_1)\left(\tilde{c}\frac{\omega_{d-1}}{5^{d-1}} - \tilde{c}C_3(t_2 - t_1)^\tilde{p}A - 1\right) \leq C_3\epsilon_1^{-2\gamma} + \frac{\omega_{d-1}}{5^{d-1}} \cdot \frac{1}{100}.$$

Thus, we have (51), for sufficiently large $\tilde{c} \geq 2$ and sufficiently small $\epsilon_1 > 0$.

Finally we consider the case of $D(t_2) = \lim_{n \to \infty} \frac{\mu^*_t(B_{r_n}(y))}{\omega_{d-1}r_n^{d-1}}$ with $\lim_{n \to \infty} r_n < \frac{1}{4}$. Then there exists $n \geq 1$ such that $0 < r_n < \frac{1}{5}$ and

$$D(t_2) - \frac{1}{100} \leq \frac{\mu^*_t(B_{r_n}(y))}{\omega_{d-1}r_n^{d-1}}.$$

Set $R = r_n$ and $s = t_2 + \frac{R^2}{2}$. We compute that

$$\int_\mathbb{R}^d \tilde{\rho}_{\rho,s}(x, t_1) d\mu^*_t(x) \leq \frac{1}{(4\pi(s - t_1))^{d/2}} \int_\mathbb{R}^d e^{-\frac{|x|^2}{4(s - t_1)}} d\mu^*_t$$

$$= \frac{1}{(4\pi(s - t_1))^{d/2}} \int_0^1 \mu^*_t\{\{x \mid e^{-\frac{|x|^2}{4(s - t_1)}} > k\}\} dk$$

$$= \frac{1}{(4\pi(s - t_1))^{d/2}} \int_0^1 \mu^*_t\{B_{\sqrt{4(s - t_1) log k^{-1}}}(y)\} dk$$

$$\leq \frac{1}{(4\pi(s - t_1))^{d/2}} \int_0^1 D(t_1)\omega_{d-1}(\sqrt{4(s - t_1) log k^{-1}})^{d-1} dk \leq C_0D(t_1),$$

(53)
Proof of Theorem 3.1. We only need to prove (23). Choose $T_b \in (0, 1)$ such that

\[ T_b B \leq c', \tag{58} \]

where $B := \|u^\varepsilon\|_{L^\infty([0,T];W^{1,p}([0,T];\mathbb{R}^n)))} + \|g^\varepsilon\|_{L^\infty([0,T];W^{1,p}([0,T];\mathbb{R})))}$. Note that $T_b$ depends only on $d, p, q$, and $B$, by Lemma 4.7. Define

\[ D_1 := D_0 \varepsilon^{(t/T_b) + 1}, \]

where $D_1$ depends only on $d, p, q, B, T, D_0$ and $D_1 \geq 2D_0$ by $\hat{c} \geq 2$. Assume that $\varepsilon \in (0, \varepsilon_1)$. Note that we only need to check that

\[ D(t) \leq D_0 \varepsilon^{(t/T_b) + 1}, \quad t \in [0, T]. \tag{59} \]
Suppose that there exists $t' \in (0, T)$ such that $D(t') > D_0e^{(t'/T_0)+1}$. Then there exists $\tau \in (0, T)$ such that $D(t) \leq D_0e^{(t/T_0)+1} \leq D_1$ for any $t \in [0, \tau]$ and $D(\tau) = D_0e^{(\tau/T_0)+1}$. Assume $\tau \in (0, T_0)$. Then we have $D(\tau) = \tilde{c}D_0$ and $\sup_{t \in [0, \tau]} D(t) \leq \tilde{c}D_0$. Thus (51) implies $\tau^pB \geq c'$, where we used Lemma 4.7 with $t_1 = 0$ and $t_2 = \tau$.

But this contradicts $\tau < T_0$ and (58). Therefore we have $\tau > T_0$. If $\tau \in (T_0, 2T_0)$, then $D(\tau) = D_0e^2$ and $D(t) \leq D_0e$ for any $t \in [0, T_0)$. Hence there exists $\tau' \in [T_0, \tau)$ such that $D(\tau') = \tilde{c}D_0$ and $\tau - \tau' < T_0$. By Lemma 4.7 with $t_1 = \tau'$ and $t_2 = \tau$, we have $(\tau - \tau')^pB \geq c'$. But this contradicts $\tau - \tau' < T_0$ and (58) again. Repeating this argument, we obtain $\tau = T$ and (59).

4.6. Proof of Theorem 3.2. Finally, we show the existence theorem for (1) in the sense of $L^2$-flow. We can easily show the existence of a $L^2$-flow by the result of Theorem 3.1 in [28] (see Theorem 5.3). However, we need to prove $v = h + (a \cdot \nu)\nu + g\nu$ in addition.

Proof of Theorem 3.2. Fix $T > 0$. Because $T_1 > T$ for sufficiently large $i \geq 1$, so we may assume $T_i > T$ for any $i \geq 1$. By a standard argument similar to that in [36, Proposition 8.3] we obtain (b).

Set $G^\varepsilon(x, t) := f^\varepsilon(x, t)\sqrt{2W(\varphi^\varepsilon(x, t))}$. Then Lemma 4.3 and (23) imply

$$
\int_0^T \int_{\Omega} \frac{1}{\varepsilon}|G^\varepsilon|^2 \, dx \, dt = 2 \int_0^T \int_{\Omega} |f^\varepsilon|^2 \frac{W(\varphi^\varepsilon)}{\varepsilon} \, dx \, dt \\
\leq C_2(D(T)(\|u^\varepsilon\|^2_{L^2((0,T);W^{1,p}(\Omega)))} + \|g^\varepsilon\|^2_{L^2((0,T);W^{1,p}(\Omega)))}) + \varepsilon^{1-2\gamma}T, \quad \varepsilon \in (0, \epsilon).
$$

Note that the right hand side is uniformly bounded, regarding $\varepsilon \in (0, \epsilon)$. In addition, we have $\mu^\varepsilon(\Omega) \leq D_0$ for any $\varepsilon > 0$. Therefore $\mu^\varepsilon$ and $\varphi^\varepsilon$ satisfy all the assumptions of Theorem 5.3. Theorem 5.3 implies (a) and there exist $v, \tilde{G} \in L^2_{\text{loc}}(0, \infty; (L^2(\mu^\varepsilon))^d)$ such that $\{\mu^\varepsilon\}_{\varepsilon \in (0, \infty)}$ is a $L^2$-flow $v = h + \tilde{G}$ with (26), by taking a subsequence $\varepsilon \to 0$. Here $\tilde{G}$ satisfies

$$
\lim_{\varepsilon \to 0} \frac{1}{\sigma} \int_{\Omega \times (0,T)} -G^\varepsilon \nabla \varphi^\varepsilon \cdot \Phi \, dx \, dt = \int_{\Omega \times (0,T)} \tilde{G} \cdot \Phi \, d\mu
$$

for any $\Phi \in C_c(\Omega \times (0, T); \mathbb{R}^d)$. We remark that

$$
\frac{1}{\sigma} \int_{\Omega \times (0,T)} -G^\varepsilon \nabla \varphi^\varepsilon \cdot \Phi \, dx \, dt
$$

$$
= \int_0^T \int_{\Omega \cap \{\nabla \varphi^\varepsilon(x, t) \neq 0\}} \left( u^\varepsilon \cdot \frac{\nabla \varphi^\varepsilon}{|\nabla \varphi^\varepsilon|} \left( \frac{\nabla \varphi^\varepsilon}{|\nabla \varphi^\varepsilon|} \cdot \Phi \right) \right) \, d\mu^\varepsilon_t \, dt
$$

$$
+ \int_0^T \int_{\Omega \cap \{\nabla \varphi^\varepsilon(x, t) \neq 0\}} g^\varepsilon \frac{\nabla \varphi^\varepsilon}{|\nabla \varphi^\varepsilon|} \cdot \Phi \, d\mu^\varepsilon_t \, dt
$$

$$
+ \frac{1}{\sigma} \int_0^T \int_{\Omega} \nabla \varphi^\varepsilon \sqrt{2W(\varphi^\varepsilon)} \nabla \varphi^\varepsilon \cdot \Phi \, dx \, dt,
$$

where $d\mu^\varepsilon_t := \frac{\varepsilon}{\sigma}|\nabla \varphi^\varepsilon|^2 \, dx$ and $d\mu^\varepsilon_t := \frac{1}{\sigma} \sqrt{2W(\varphi^\varepsilon)}|\nabla \varphi^\varepsilon| \, dx$. We compute the third term of the right hand side. We have

$$
\left| \frac{1}{\sigma} \int_0^T \int_{\Omega} \nabla \varphi^\varepsilon \sqrt{2W(\varphi^\varepsilon)} \nabla \varphi^\varepsilon \cdot \Phi \, dx \, dt \right|
$$

$$
\leq \frac{1}{\sigma} \|\Phi\|_{\infty} \left( \int_0^T \int_{\Omega} (L^2 r^\varepsilon)^2 \frac{2W}{\varepsilon} \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega} |\nabla \varphi^\varepsilon|^2 \, dx \, dt \right)^{\frac{1}{2}}
$$
\[ \leq \frac{1}{\sigma} \| \Phi \|_{\infty} \left( \int_0^T \int_{\Omega} (L^p r^2)^2 \frac{2W}{\varepsilon} \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T 2D(t) \, dt \right)^{\frac{1}{2}}. \]

Hence, (23) and (39) imply
\[ \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega} L^p r^2 \sqrt{2W(\varphi^\varepsilon)} \nabla \varphi^\varepsilon \cdot \Phi \, dx \, dt = 0. \quad (62) \]

Now we show (c). The estimates (23) and (44) give
\[ \sup_{\varepsilon \in (0, \varepsilon)} \int_0^T \int_{\Omega} \left( \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2} \right)^2 \, dx \, dt < \infty \]
for any \( \varepsilon \in (0, \varepsilon) \). Hence Fatou’s lemma implies
\[ \liminf_{\varepsilon \to 0} \int_0^T \int_{\Omega} \left( \Delta \varphi^\varepsilon(x, t) - \frac{W'(\varphi^\varepsilon(x, t))}{\varepsilon^2} \right)^2 \, dx < \infty, \quad \text{a.e. } t \in (0, T). \]
Therefore, by Theorem 5.2, \( \xi_t^\varepsilon \to 0 \) a.e. \( t \). Thus we obtain (c).

Next we show (d). Fix \( \delta > 0 \) and \( i \geq 1 \) such that \( \| u_i \|_{L^\infty([0, T]; (W^{1, p} \Omega)^d)} \leq \delta \). Set \( \tilde{u} := u_i^\varepsilon \). For any \( \Phi \in C_c(\Omega \times [0, T]; \mathbb{R}^d) \) we have
\[ \left| \int_{\Omega \times (0, \infty)} u \cdot \Phi \, d\mu - \frac{1}{\sigma} \int_{\Omega \times (0, \infty)} u^\varepsilon \cdot \Phi \varepsilon |\nabla \varphi^\varepsilon|^2 \, dx \, dt \right| \]
\[ \leq \frac{\| \Phi \|_\infty}{\sigma} \int_{\Omega \times (0, T)} |u - \tilde{u}| \, d\mu + \frac{1}{\sigma} \int_{\Omega \times (0, \infty)} \tilde{u} \cdot \Phi \, d\mu - \frac{1}{\sigma} \int_{\Omega \times (0, T)} u^\varepsilon \cdot \Phi \varepsilon |\nabla \varphi^\varepsilon|^2 \, dx \, dt \]
\[ \leq C \delta + \frac{1}{\sigma} \int_{\Omega \times (0, T)} \tilde{u} \cdot \Phi \, d\mu - \frac{1}{\sigma} \int_{\Omega \times (0, T)} \tilde{u} \cdot \Phi \varepsilon |\nabla \varphi^\varepsilon|^2 \, dx \, dt \]
\[ + \frac{1}{\sigma} \int_{\Omega \times (0, T)} u^\varepsilon \cdot \Phi \varepsilon |\nabla \varphi^\varepsilon|^2 \, dx \, dt - \frac{1}{\sigma} \int_{\Omega \times (0, T)} u^\varepsilon \cdot \Phi \varepsilon |\nabla \varphi^\varepsilon|^2 \, dx \, dt \]
\[ =: C \delta + I_1 + I_2, \quad (63) \]
where (73) is used and \( C > 0 \) depends only on \( d, p, q, D(T), \| \Phi \|_\infty \). By \( \xi_t^\varepsilon \to 0 \) a.e. \( t, d\tilde{\mu}_t^\varepsilon := \frac{\varepsilon}{\sigma} |\nabla \varphi^\varepsilon|^2 \, dx \to d\mu_t \) a.e. \( t \). Thus \( I_1 \to 0 \) as \( \varepsilon \to 0 \). Moreover, for sufficiently small \( \varepsilon > 0 \), the Cauchy-Schwarz inequality gives \( |I_2| \leq C \delta \), where \( C > 0 \) depends only on \( D(T), \| \Phi \|_\infty \). Hence we obtain (d).

Next we prove (e). First we show that \( \tilde{\mu}_t^\varepsilon \to \mu_t \) as Radon measures for a.e. \( t \). We compute
\[ \left| \frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon} - \sqrt{2W(\varphi^\varepsilon)} |\nabla \varphi^\varepsilon| \right| \leq \left( \sqrt{\frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2}} - \sqrt{\frac{W(\varphi^\varepsilon)}{\varepsilon}} \right)^2 \]
\[ \leq \frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon}. \]
Therefore \( \xi_t^\varepsilon \to 0 \) implies \( \tilde{\mu}_t^\varepsilon \to \mu_t \) a.e. \( t \). By (23) and (73) we have
\[ \sup_{\varepsilon \in (0, \varepsilon)} \int_0^T \int_{\Omega} |g|^2 \, d\tilde{\mu}_t^\varepsilon < \infty. \quad (64) \]
Hence there exists a vector valued function \( \tilde{g} \) such that
\[ \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega \setminus \{ |\nabla \varphi^\varepsilon(\cdot, t)\| \neq 0 \}} g^\varepsilon \frac{\nabla \varphi^\varepsilon}{|\nabla \varphi^\varepsilon|} \cdot \Phi \, d\tilde{\mu}_t^\varepsilon \, dt = \int_{\Omega \times (0, T)} \tilde{g} \cdot \Phi \, d\mu \quad (65) \]
for any \( \Phi \in C_c(\Omega \times [0, T]; \mathbb{R}^d) \) (see [16, Theorem 4.4.2]). Thus we obtain (e).
Finally we show (f). By (60), (61), (62), and (65), we only need to prove (27), spt $\tilde{g} \subset \partial^* \{\psi = 1\}$, and

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega \cap \{|\nabla \varphi^\varepsilon(t)| \neq 0\}} (u^\varepsilon \cdot \nabla \varphi^\varepsilon) \left(\frac{\nabla \varphi^\varepsilon}{|\nabla \varphi^\varepsilon|} \right) \, d\mu_t^\varepsilon \, dt = \int_0^T \int_{\Omega} (\text{Id} - T_x \mu_t) w \Phi \, d\mu \quad (66)$$

for any $\Phi \in C_c(\Omega \times [0,T); \mathbb{R}^d)$. Set $\nu^\varepsilon := \frac{\nabla \varphi^\varepsilon}{|\nabla \varphi^\varepsilon|}$. We compute

$$\int_0^T \int_{\Omega \cap \{|\nabla \varphi^\varepsilon(t)| \neq 0\}} (u^\varepsilon \cdot \nu^\varepsilon)(\nu^\varepsilon \cdot \Phi) \, d\mu_t^\varepsilon \, dt$$

$$= \int_0^T \int_{\Omega \cap \{|\nabla \varphi^\varepsilon(t)| \neq 0\}} (u^\varepsilon - (\text{Id} - \nu^\varepsilon \otimes \nu^\varepsilon) u^\varepsilon) \cdot \Phi \, d\mu_t^\varepsilon \, dt.$$ 

Note that by the definition of the varifold and integrality of $\mu_t$,

$$\int (\text{Id} - \nu^\varepsilon \otimes \nu^\varepsilon) \Phi \, d\mu_t^\varepsilon \to \int T_x \mu_t \Phi \, d\mu_t$$

for any $\Psi \in C_c(\Omega \times [0,T); \mathbb{R}^d)$. By using this and an argument similar to (63), we have (66).

Set $\theta^\varepsilon := \int_0^\varepsilon \sqrt{2W(\tau)} \, d\tau$. Recall that $\psi = \lim_{\varepsilon \to 0} \frac{1}{2}(\varphi^\varepsilon + 1)$, $\varphi^\varepsilon \to \pm 1$ a.e. on $\Omega \times (0, \infty)$, and

$$\lim_{\varepsilon \to 0} k(\varphi^\varepsilon) = \lim_{\varepsilon \to 0} \int_0^\varphi^\varepsilon \sqrt{2W(s)} \, ds = \sigma \left(\psi - \frac{1}{2}\right) \quad \text{a.e. on } \Omega \times (0, \infty). \quad (67)$$

By (67), for any $\Phi \in C^1_c(\Omega; \mathbb{R}^d)$ and $t \geq 0$, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \text{div} \Phi k(\varphi^\varepsilon) \, dx = \int_{\mathbb{R}^d} \text{div} \Phi \sigma \left(\psi - \frac{1}{2}\right) \, dx = -\sigma \int_{\mathbb{R}^d} \Phi \cdot \nu d||\nabla \psi(\cdot, t)||, \quad (68)$$

where $\nu(\cdot, t)$ is the inner unit normal vector of $\{\psi(\cdot, t) = 1\}$ on $\partial^* \{\psi(\cdot, t) = 1\}$.

Fix $\delta > 0$ and $i \geq 1$ such that $\|g^i - g\|_{L^p([0,T]; W^{1,p}(\Omega))} < \delta$. Set $\tilde{g} := g^i$. For any $\Phi \in C^1_c(\Omega \times [0,T); \mathbb{R}^d)$ we have

$$\int_{\Omega \times (0,T)} \tilde{g} \cdot \Phi \, d\mu = \lim_{\varepsilon \to 0} \frac{1}{\sigma} \int_{\Omega \times (0,T)} g^\varepsilon \sqrt{2W(\varphi^\varepsilon)} \nabla \varphi^\varepsilon \cdot \Phi \, dx \, dt$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\sigma} \int_{\Omega \times (0,T)} g^\varepsilon \nabla k(\varphi^\varepsilon) \cdot \Phi \, dx \, dt = -\lim_{\varepsilon \to 0} \frac{1}{\sigma} \int_{\Omega \times (0,T)} k(\varphi^\varepsilon) \text{div} (g^\varepsilon \Phi) \, dx \, dt. \quad (69)$$

By (68), the Radon-Nikodym theorem, we have

$$-\lim_{\varepsilon \to 0} \frac{1}{\sigma} \int_{\Omega \times (0,T)} k(\varphi^\varepsilon) \text{div} (\tilde{g}\Phi) \, dx \, dt = \int_{\Omega \times (0,T)} \tilde{g} \theta \cdot \Phi \, d\mu$$

for any $\Phi \in C^1_c(\Omega \times [0,T); \mathbb{R}^d)$. Here $\theta : \text{spt} \, \mu \to \mathbb{N}$ is defined by

\[
\theta = \begin{cases} 
\left( \frac{d||\nabla \psi(\cdot, t)||}{d\mu_t} \right)^{-1} & \text{if } (x, t) \in \partial^* \{\psi = 1\}, \\
\infty & \text{otherwise},
\end{cases}
\]
where \( \frac{1}{p} = 0 \) if \( \theta = \infty \), and \( \frac{d\|\nabla \psi(\cdot , t)\|}{d\mu} \) is the Radon-Nikodym Derivative. We compute
\[
\left| \int_{\Omega \times (0, T)} g_\nu \cdot \Phi d\|\nabla \psi(\cdot , t)\|dt - \frac{1}{\sigma} \int_{\Omega \times (0, T)} g^\dagger \sqrt{2W(\varphi^\dagger)} \nabla \varphi^\dagger \cdot \Phi dxdt \right|
\]
\[
= \left| \int_{\Omega \times (0, T)} g^\dagger_\theta \nu \cdot \Phi d\mu - \frac{1}{\sigma} \int_{\Omega \times (0, T)} g^\dagger \sqrt{2W(\varphi^\dagger)} \nabla \varphi^\dagger \cdot \Phi dxdt \right|
\]
\[
\leq \|\Phi\|_\infty \int_{\Omega \times (0, T)} |g - \tilde{g}| d\mu
\]
\[
+ \left| \int_{\Omega \times (0, T)} \tilde{g} \frac{1}{\sigma} \nu \cdot \Phi d\mu - \frac{1}{\sigma} \int_{\Omega \times (0, T)} g^\dagger \sqrt{2W(\varphi^\dagger)} \nabla \varphi^\dagger \cdot \Phi dxdt \right|
\]
\[
\leq \|\Phi\|_\infty \int_{\Omega \times (0, T)} |g - \tilde{g}| d\mu + \left| \int_{\Omega \times (0, T)} \tilde{g} \frac{1}{\sigma} \nu \cdot \Phi d\mu + \frac{1}{\sigma} \int_{\Omega \times (0, T)} k(\varphi^\dagger) \text{div} (\tilde{g}\Phi) dxdt \right|
\]
\[
+ \left| - \frac{1}{\sigma} \int_{\Omega \times (0, T)} k(\varphi^\dagger) \text{div} (\tilde{g}\Phi) dxdt - \frac{1}{\sigma} \int_{\Omega \times (0, T)} g^\dagger \sqrt{2W(\varphi^\dagger)} \nabla \varphi^\dagger \cdot \Phi dxdt \right|
\]
\[
=: J_1 + J_2 + J_3.
\]
By (73) we have \( J_1 \leq C\delta \) and (70) implies \( J_2 \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \). By (73) and the integration by parts, we have
\[
J_3 \leq C\|g^{\dagger} - \tilde{g}\|_{L^q([0, T]; W^{1, p}(\Omega))} \leq C\delta + \|g^{\dagger} - g\|_{L^q([0, T]; W^{1, p}(\Omega))}^2,
\]
where \( C > 0 \) depends only on \( d, p, q, D(T), \|\Phi\|_\infty \). Therefore we obtain
\[
\int_{\Omega \times (0, T)} g_\nu \cdot \Phi d\|\nabla \psi(\cdot , t)\|dt = \int_{\Omega \times (0, T)} \frac{1}{\sigma} g^\dagger_\theta \nu \cdot \Phi d\mu
\]
\[
= \lim_{\varepsilon \rightarrow 0} \frac{1}{\sigma} \int_{\Omega \times (0, T)} g^\dagger \sqrt{2W(\varphi^\dagger)} \nabla \varphi^\dagger \cdot \Phi dxdt. \tag{71}
\]
By (69) and (71) we have (27) and \( \text{spt} \ \tilde{g} \subset \partial^* \{ \psi = 1 \} \).

\[ \square \]

5. Appendix.

5.1. **Meyers-Ziemer inequality.** Let \( \mu \) be a Radon measure on \( \mathbb{R}^d \) and \( f: \mathbb{R}^d \rightarrow \mathbb{R} \) be a given function. To define \( \mu \)-measurable \( f \) as a trace function, we use the following inequality:

**Theorem 5.1** (Meyers-Ziemer inequality). For a Radon measure \( \mu \) on \( \mathbb{R}^d \) with
\[
D = \sup_{r > 0, x \in \mathbb{R}^d} \frac{\mu(B_r^d(x))}{\omega_{d-1} r^{d-1}}
\]
and \( 1 \leq p < d \),
\[
\int_{\mathbb{R}^d} |f|^{\frac{p(d-1)}{d-p}} d\mu \leq c_{MZ} D \left( \int_{\mathbb{R}^d} |\nabla f|^p dx \right)^{\frac{d-1}{d-p}} \tag{72}
\]
for \( f \in C^1_c(\mathbb{R}^d) \). Here \( c_{MZ} = c_{MZ}(d, p) \). See [25] and [39] for \( p = 1 \).

Set
\[
\mu_t := \lim_{\varepsilon \downarrow 0} \mu_{t, \varepsilon}^t, \quad D_T := \sup_{t \in [0, T], r > 0, x \in \mathbb{R}^d} \frac{\mu_t(B_r^d(x))}{\omega_{d-1} r^{d-1}}.
\]
Note that, to make sense of the Brakke's inequality or the convergences (d), (e), and (f) in Theorem 3.2, we only need to define the transport term and forcing term as functions in $L^p_T([\mu_t \times dt])$. By Hölder inequality and (72) we have
\[
\int_{\mathbb{R}^d} |f|^2 \, d\mu_t \leq \left( \int_{\mathbb{R}^d} |f|^{\frac{2(d-p)}{p(d-1)}} \, d\mu_t \right)^\frac{p}{2(d-1)} \left( \int_{\mathbb{R}^d} |\nabla f|^p \, dx \right)^\frac{2}{p} \left( \mu_t(\text{spt } f) \right)^{\frac{p(d-1)-2d}{p(d-1)}} \quad (73)
\]
for any $f \in C^1_c(\mathbb{R}^d)$. To justify (73), we need $\frac{d(p-1)}{p^2} \geq 2$. So we need to assume
\[
p \geq \frac{2d}{d+1} \quad (74)
\]
for (73).

5.2. Existence theorem for $L^2$-flow. Let $U \subset \mathbb{R}^d$ be an open set, $\varphi^\varepsilon \in C^2(U)$ for $\varepsilon \in (0, 1)$ and $\{\varepsilon_i \}_{i=1}^\infty$ be a positive sequence with $\varepsilon_i \to 0$. Define
\[
\mu^\varepsilon(\phi) := \frac{1}{\sigma} \int_U \phi \left( \frac{|\nabla \varphi^\varepsilon|^2}{\varepsilon} + W(\varphi^\varepsilon) \right) \, dx, \quad \xi^\varepsilon(\phi) := \frac{1}{\sigma} \int_U \phi \left( \frac{|\nabla \varphi^\varepsilon|^2}{\varepsilon} - W(\varphi^\varepsilon) \right) \, dx,
\]
where $\sigma := \int_1^\infty \frac{1}{2W(s)} \, ds$. The following theorem is useful for showing the vanishing of the discrepancy measure and the integrality of the limit measure:

**Theorem 5.2** ([29]). Assume that $d = 2, 3$,
\[
\liminf_{i \to \infty} \mu^\varepsilon_i(U) < \infty, \quad \liminf_{i \to \infty} \int_U \varepsilon_i \left( \nabla \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon_i^2} \right)^2 \, dx < \infty
\]
and
\[
\mu^\varepsilon_i \to \mu \quad \text{as Radon measures.}
\]
Then the following holds:
1. $|\xi^\varepsilon_i| \to 0$ as Radon measures.
2. $\mu$ is $(d-1)$-integral.
3. $\int_U |h|^2 \, d\mu \leq \frac{1}{\varepsilon} \liminf_{i \to \infty} \int_U \varepsilon_i \left( \nabla \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon_i^2} \right)^2 \, dx$, where $h$ is the generalized mean curvature vector of $\mu$.

The following theorem is also useful for prove the existence of the weak solutions for the MCF with forcing term, in the sense of $L^2$-flow.

**Theorem 5.3** (Theorem 3.1 in [28]). Let $d = 2, 3$ and $\varphi^\varepsilon$ be a solution for the following equation:
\[
\begin{cases}
\varepsilon \varphi^\varepsilon_t = \varepsilon \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon} + G^\varepsilon, & (x, t) \in \Omega \times (0, \infty), \\
\varphi^\varepsilon(x, 0) = \varphi^0_0(x), & x \in \Omega.
\end{cases}
\]
We assume that there exists $\bar{\varepsilon} > 0$ such that
\[
\sup_{\varepsilon \in (0, \bar{\varepsilon})} \left( \mu^\varepsilon_0(\Omega) + \int_{\Omega \times (0, T)} \frac{1}{\varepsilon} (G^\varepsilon)^2 \, dx dt \right) < \infty
\]
for any $T > 0$. Then there exists a subsequence $\varepsilon \to 0$ such that the following holds:
1. There exists a family of $(d-1)$-integral Radon measures $\{\mu_t\}_{t \in (0, \infty)}$ on $\Omega$ such that
   (a) $\mu^\varepsilon \to \mu$ as Radon measures on $\Omega \times [0, \infty)$, where $d\mu = d\mu_t dt$. 

2. \( \mu_t^\varepsilon \to \mu_t \) as Radon measures on \( \Omega \) for all \( t \in [0, \infty) \).

2. There exists \( \bar{G} \in L^2_{\text{loc}}(0, \infty; (L^2(\mu_t))^d) \) such that
\[
\lim_{\varepsilon \to 0} \frac{1}{\sigma} \int_{\Omega \times (0, \infty)} -G^\varepsilon \nabla \varphi^\varepsilon \cdot \Phi \, dx \, dt = \int_{\Omega \times (0, \infty)} \bar{G} \cdot \Phi \, d\mu
\]
for any \( \Phi \in C_c(\Omega \times [0, \infty); \mathbb{R}^d) \).

3. \( \{\mu_t\}_{t \in (0, \infty)} \) is an \( L^2 \)-flow with a generalized velocity vector \( v = h + \bar{G} \) and
\[
\lim_{\varepsilon \to 0} \int_{\Omega \times (0, \infty)} v^\varepsilon \cdot \Phi \, d\mu^\varepsilon = \int_{\Omega \times (0, \infty)} v \cdot \Phi \, d\mu
\]
for any \( \Phi \in C_c(\Omega \times [0, \infty); \mathbb{R}^d) \), where \( h \) is the generalized mean curvature vector of \( \mu_t \) and
\[
v^\varepsilon = \begin{cases} 
-\varphi_t^\varepsilon & \frac{\nabla \varphi^\varepsilon}{|\nabla \varphi^\varepsilon|} & \text{if } |\nabla \varphi^\varepsilon| \neq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Remark 11.
1. The assumption for \( d \) comes from Theorem 5.2.
2. The boundary conditions of (75) of the original theorem are Neumann conditions. However, we may also obtain same results for periodic boundary conditions, with minor modification of the proof (see [27, Remark 2.3]).

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