A General Method for Common Intervals
Ismael Belghiti *and M. Habib †

May 22, 2014

Abstract

Given an elementary chain of vertex set \( V \), seen as a labelling of \( V \) by the set \( \{ 1, \ldots, n = |V| \} \), and another discrete structure over \( V \), say a graph \( G \), the problem of common intervals is to compute the induced subgraphs \( G[I] \), such that \( I \) is an interval of \( [1, n] \) and \( G[I] \) satisfies some property \( \Pi \) (as for example \( \Pi = \text{"being connected"}. \) This kind of problems comes from comparative genomic in bioinformatics, mainly when the graph \( G \) is a chain or a tree [11, 10, 2].

When the family of intervals is closed under intersection, we present here the combination of two approaches, namely the idea of potential beginning developed in [13, 6] and the notion of generator as defined in [4]. This yields a very simple generic algorithm to compute all common intervals, which gives optimal algorithms in various applications. For example in the case where \( G \) is a tree, our framework yields the first linear time algorithms for the two properties: "being connected" and "being a path". In the case where \( G \) is a chain, the problem is known as: common intervals of two permutations [13], our algorithm provides not only the set of all common intervals but also with some easy modifications a tree structure that represents this set.

Keywords: connected intervals, common intervals, graph algorithms.

1 Introduction

All the graphs considered here are supposed to be finite, undirected, simple and loopless. For such a graph \( G \), we denote by \( V(G) \) and \( E(G) \) its vertex and edge sets respectively. Furthermore if \( U \) is a subset of \( V(G) \), we denote by \( G[U] \) the induced subgraph of \( G \).

The problem of finding the common connected components of two graphs was defined in [5], as follows: let \( G_1, G_2 \) be two graphs on the same vertex set \( V \), find the maximal partition \( \mathcal{Q} = \{V_1, \ldots, V_k\} \) of \( V \) such for that for every \( i \in [1,k], G_1[V_i] \text{ and } G_2[V_i] \text{ are connected.} \) Of course this problem is polynomially tractable and some subcases are solvable in linear time (see [7]).

In this paper we are particularly interested in the close problem of finding all the common connected subsets of two graphs: let \( G_1, G_2 \) be two graphs on the same vertex set \( V \), find all the subsets \( U \subset V \) such that \( G_1[U] \text{ and } G_2[U] \) are connected.

More precisely we mainly study the particular case where \( G_1 \) is a elementary chain, seen as a labelling of \( V \) by \( \{ 1, \ldots, n = |V| \} \), and \( G_2 \) is a graph \( G \) with \( V(G) = \{ 1, \ldots, n \} \), the previous problem becomes the problem of common intervals. That is to compute the induced subgraphs \( G[I] \), such that \( I \) is an interval of \( [1, n] \) and \( G[I] \) satisfies some property \( \Pi \) (as for example \( \Pi = \text{"being connected"}. \) This kind of problems appears in Biology from comparative genomic, in a more specific case when the graph \( G \) is a chain or a tree [11, 10, 2].

Combining two approaches namely the idea of potential beginning developed in [13, 6] and the notion of generator as defined in [4], we succeed to obtain a very simple generic algorithm which yields optimal algorithms in various applications. For example in the case where \( G \) is a tree, our

*École Normale Supérieure de Paris, France. Email: ismael.alouii.belghiti@ens.fr
†LIAFA, CNRS & Université Paris Diderot - Paris 7, France. Email: habib@liafa.univ-paris-diderot.fr
framework yields the first linear time algorithms for the two properties: "being connected" and "being a path".

Furthermore in the particular case where $G$ is a chain, we deal with common intervals of two permutations, although some linear time algorithms already exist \[13, 6, 4\], our framework yields very simple linear time algorithms that compute not only the common intervals, but also the associated tree decomposition.

In this paper we will first present the general framework, which deals with families of intervals closed by intersection and then describe how the generic algorithms can be specialized for some applications. Due to space constraints we will not develop in details all these applications.

2 General Framework for families closed under intersection

In the sequel, we only consider families of intervals closed under intersection. In other words, we will consider families $\mathcal{F}$ of intervals such that: if two intervals $I_1, I_2 \in \mathcal{F}$ intersect then their intersection $I_1 \cap I_2$ is also in $\mathcal{F}$. For example, in the cases where $G$ is a tree and $\Pi = "being connected"$ or $\Pi = "being a path"$, the resulting families are closed by intersection. But it is not always true, as for the case where $G$ is a graph and $\Pi = "being connected"$, and to manage this case we need to extend the framework presented here, see [3].

In the whole section, we assume by convention that the considered families of intervals contain all the singletons of their ground set. Let us now describe a generic algorithm to compute a convenient representation for these families and another one to enumerate their elements. These algorithms will be specialized different ways in the section 3, according to the particular combinatorial structures we consider.

2.1 Representation by a generator

[4] introduced the notion of generator to represent in linear space families of intervals closed under intersection:

**Definition 2.1 (Generator).** A generator of a family $\mathcal{F}$ of intervals over $\{1, \ldots, n\}$ closed under intersection is a couple of vectors $(L, R)$ such that:

- $\forall x \in \{1, \ldots, n\}, R[x] \geq x$
- $\forall y \in \{1, \ldots, n\}, L[y] \leq y$
- $[x, y] \in \mathcal{F} \iff R[x] \geq y$ and $L[y] \leq x$

The following lemma shows that the families of intervals closed under intersection do admit such a representation.

**Lemma 2.2 ([4] Existence of a representation by generator).** Let $\mathcal{F}$ be a family of intervals closed under intersection. There exists a generator that represents the family $\mathcal{F}$.

**Proof.** Let $\text{maxEnd}[x]$ be the maximum end of an interval of $\mathcal{F}$ starting at $x$ and let $\text{minBeg}[y]$ be the minimum beginning of an interval of $\mathcal{F}$ ending at $y$. $(\text{maxEnd}, \text{minBeg})$ is a generator of $\mathcal{F}$. \qed

Notice that this representation is particularly useful when we want to consider the intersection of several families of intervals closed under intersection:

**Lemma 2.3 ([4] Generators and intersection of families).** If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two families of intervals closed under intersection, $\mathcal{F}_1$ being represented by the generator $(L_1, R_1)$ and $\mathcal{F}_2$ by the generator $(L_2, R_2)$, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is represented by the generator $(L, R)$ defined by:

1. $\forall x \in \{1, \ldots, n\}, R[x] = \text{min}(R_1[x], R_2[x])$
2. $\forall y \in \{1, \ldots, n\}, L[y] = \text{max}(L_1[y], L_2[y])$
2.2 Potential Beginnings and Right-Splitters

Let $F$ be a family of intervals over $\{1, \ldots, n\}$ closed under intersection. An element of $\{1, \ldots, n\}$ will be called a vertex. In our algorithms, we will consider the vertices $y$ in increasing order and we will be interested in the beginnings of the intervals of $F$ ending at $y$. When we have considered the vertices $\{1, \ldots, y\}$, it is natural to only keep the vertices $x$ that could be the beginning of an interval of $F$ ending after $y$. To capture this idea let us introduce a notion of potential beginning (with respect to $y$), that relies on a simple property such $x$ have to satisfy, and symmetrically a notion of potential end such that: $[x,y]$ is in $F$ iff $x$ is a potential beginning of $y$ and $y$ is a potential end of $x$.

**Example 2.1.** Assume that we are given a permutation $P$ over $\{1, \ldots, n\}$ and that we are interested in the family $F$ of intervals $[x, y] \subseteq \{1, \ldots, n\}$ such that $P([x, y]) = [x, y]$ (remark that $F$ is closed under intersection). Defining the potential beginnings of $y$ as the $x \leq y$ such that $x \leq \min P([x, y])$ and the potentials ends of $x$ as the $y \geq x$ such that $y \geq \max P([x, y])$, it is straightforward to check that $[x,y]$ is in $F$ iff $x$ is a potential beginning of $y$ and $y$ is a potential end of $x$.

We thus introduce the following definition:

**Definition 2.4.** A couple of potentiality for the family $F$ is a couple $(\text{potBeg}, \text{potEnd})$ such that:

- $\forall x, y \in \{1, \ldots, n\}, \text{potBeg}(y) \subseteq \{1, \ldots, y\}$ and $\text{potEnd}(x) \subseteq \{x, \ldots, n\}$
- $\forall y \in \{2, \ldots, n\}, \text{potBeg}(y) \subseteq \text{potBeg}(y - 1) \cup \{y\}$.
- $\forall x \in \{1, \ldots, n - 1\}, \text{potEnd}(x) \subseteq \text{potEnd}(x + 1) \cup \{x\}$.
- $\forall 1 \leq x \leq y \leq n, [x, y] \in F \Leftrightarrow (x \in \text{potBeg}(y) \text{ and } y \in \text{potEnd}(x))$.

The elements of $\text{potBeg}(y)$ are called the potential beginnings of $y$ and the elements of $\text{potEnd}(x)$ are called the potential ends of $x$.

Remark that we want the notion of potential beginning to be such that, when we consider the vertices $y$ in increasing order, each vertex $x$ will be a potential beginning during a certain time until it loses its potentiality.

**Definition 2.5.** We define the right-splitter of $x$, denoted $\text{RSplitter}[x]$, as the minimum $y > x$ such that $x \notin \text{potBeg}(y)$ (if such an index $y$ does not exist, we set $\text{RSplitter}[x] = \infty$). Symmetrically, we define the left-splitter of $y$, denoted $\text{LSplitter}[y]$, as the maximum $x < y$ such that $y \notin \text{potEnd}(x)$ (if such an $x$ does not exist, we set $\text{LSplitter}[y] = -\infty$).

From the above definitions, we have a straightforward link between the notion of potential beginning and right-splitter:
Proposition 2.6 (Link between potential beginnings and right-splitters). For \( x, y \in \{1, \ldots, n\} \) with \( x \leq y \), \( x \) is a potential beginning of \( y \) iff \( y < R\text{Splitter}[x] \). Symmetrically: \( y \) is a potential end of \( x \) iff \( x > L\text{Splitter}[y] \).

Hence, we have that \((L\text{Splitter} + 1, R\text{Splitter} - 1)\) is a generator of \( F \).

Left and right splitters and potential extremities are somehow dual and give two different points of view of the same structural properties. During a sweep taking the vertices \( y \) in increasing order, we will maintain the set \( \text{PotBeg} \) of the potential beginnings of \( y \) in a data structure depending on the application.

Remark 2.7. Each vertex will be pushed once in this structure and thus removed at most once. The right-splitter of a vertex \( x \) corresponds to the \( y \) we are considering when we remove \( x \) from the set \( \text{PotBeg} \).

Proposition 2.8 (Suffix property). The set of the beginnings of the intervals of \( F \) ending at \( y \) forms a suffix of \( \text{potBeg}(y) \). More precisely, this set is equal to \( \text{potBeg}(y) \cap [L\text{Splitter}(y) + 1, y] \).

This last property will be used in particular in the enumeration of \( F \) (see Algorithm 2). Moreover, it will be also useful in the computation of the tree-decomposition of the common intervals of two permutations (see section 4).

2.3 Generic algorithms

We assume that we have exhibited a notion of potential beginning and a notion potential end as defined above. Under this assumption, we here give very general method to deal with the family \( F \). In the section 3 we will explicit how theses approaches can be applied to specific combinatorial structures.

We first describe a generic algorithm, using two symmetrical sweeps, to compute a representation by generator for \( F \). Then we describe another one that enumerates all the elements of \( F \).

2.3.1 A generic algorithm to compute a generator

We give here a very simple algorithm to compute a generator representing \( F \). Our algorithms proceed in two sweeps to compute the couple of vectors \((L\text{Splitter}, R\text{Splitter})\). Recall that this computation answers the problem since \((L\text{Splitter} + 1, R\text{Splitter} - 1)\) is a generator of \( F \). During the first sweep we consider the vertices in increasing order and we maintain the set of potential beginnings (of the current \( y \)) in order to compute the vector \( L\text{Splitter} \). The second sweep is symmetrical: we consider the vertices in decreasing order and maintain the set of potential ends in order to compute the vector \( R\text{Splitter} \).

\begin{verbatim}
Function Computation of the generator
    ComputeRightSplitter()
    ComputeLeftSplitter()

Using these 2 vectors one can check in constant time if a given interval is in \( F \) using the following function:

Function isInFamily(x,y)
    Return (x > LSplitter[y] and y < RSplitter[x])
\end{verbatim}

For simplicity, we will only explicit the computation of \( R\text{Splitter} \) \((L\text{Splitter} \text{ being obtained symmetrically})\). To do this computation, we consider the vertices \( y \) in increasing order and we maintain the set \( \text{PotBeg} \) of the potential beginnings of the current \( y \).

When we have considered the vertices \( \{1, \ldots, y - 1\} \) and computed the set \( \text{potBeg}(y - 1) \), the update to \( \text{potBeg}(y) \) can be done by removing from \( \text{PotBeg} \) the vertices \( x \) that are not a potential
Each time we remove a vertex $x$, we set $RSplitter[x]$ to $y$. After this sequence of removals, we add $y$ to our set $PotBeg$.

**Algorithm 1: Generic Computation of $RSplitter$**

$$
RSplitter \leftarrow [\infty, \ldots, \infty] \\
PotBeg \leftarrow \text{EmptySet} \\
\text{for } y \text{ from } 1 \text{ to } n \text{ do} \\
\quad \text{foreach } x \text{ in } PotBeg \text{ that is not a potential beginning of } y \text{ do} \\
\qquad RSplitter[x] \leftarrow y \\
\qquad \text{Remove } x \text{ from } PotBeg \\
\quad \text{Add } y \text{ to } PotBeg
$$

**Proposition 2.9.** Since each vertex is removed at most once, if we can perform each removal in time $O(f(n))$ then the whole algorithm is in $O(nf(n))$.

In all the specific cases studied in this paper, the set $PotBeg$ has a simple behaviour (for example, it behaves like a stack in the case of the connected intervals of a tree) and we can perform each removal in time $O(1)$.

### 2.3.2 A generic algorithm to enumerate the intervals

According to the Suffix property (proposition 2.8), we can easily enumerate for each $y$ the beginnings of the intervals of $F$ ending at $y$. To do this, we consider the elements of $PotBeg$ from right to left, until we find one that is not such a beginning. We will therefore assume that we have a primitive function $PotBeg.left$ that permits to go from one element of $PotBeg$ to the one directly on its left. Remark that, whatever the data-structure we use for $PotBeg$, it is always possible to use a supplementary doubly-linked list $L$ with an external array indexed, from 1 to $n$, that indicates for each $x \in \{1, \ldots, n\}$ the corresponding node (when it exists) in $L$. This additional structure permits to compute $PotBeg.left$ in constant time. It should be noticed that in all the cases studied in this paper, we do not need to do this, since the set $PotBeg$ has a very simple behaviour in all these examples. In this algorithm, we assume that the while condition can be checked easily ($O(1)$).

**Algorithm 2: Enumeration of the intervals**

$$
PotBeg \leftarrow \text{EmptySet} \\
\text{for } y \text{ from } 1 \text{ to } n \text{ do} \\
\quad \text{foreach } x \text{ in } PotBeg \text{ that is not a potential beginning of } y \text{ do} \\
\qquad \text{Remove } x \text{ from } PotBeg \\
\qquad \text{Add } y \text{ to } PotBeg \\
\qquad \text{Let } x \text{ be the right-most vertex of } PotBeg \\
\quad \text{while } [x, y] \text{ is an interval of } F \text{ do} \\
\qquad \text{Output}([x, y]) \\
\qquad x \leftarrow PotBeg.left(x)
$$

Under the assumption that we have exhibited a notion of potential beginning and a notion of potential end satisfying the above conditions, we can applied the two algorithms. To achieve a good complexity, we have to do efficiently the updates of the data structures.

Notice that, to test the while condition in Algorithm 2, we can first precompute the vector $LSplitter$ and then test in constant time the condition $x > LSplitter[y]$ (indeed, since $x$ is in $potBeg(y)$, we only have to test if $y$ is in $potEnd(x)$). Consequently:

**Proposition 2.10.** If we have a $O(f(n))$ implementation of Algorithm 2, we can derive from it a $O(f(n) + |F|)$ algorithm to enumerate all the elements of $F$. 

5
Corollary 2.11. If we can perform the removals from PotBeg and PotEnd in time $O(1)$, then we obtain optimal algorithms both for the computation of a generator and for the enumeration of $F$.

3 Applications

In this section, we exhibit a non exhaustive list of applications of the framework introduced above. Recall that these methods can be applied when considering a family of intervals closed under intersection for which we have defined a couple of potentiality. To apply the generic algorithms, we just have to specify how to update the data structure PotBeg (resp. PotEnd).

In particular we introduce the first algorithms to compute the intervals corresponding to subsets of nodes in a tree $T$ for the two properties: “being connected” and “being a path”.

As we will see, in all these applications, PotBeg has a very simple behaviour and all the obtained algorithms are optimal.

3.1 Connected Intervals of a Tree

Let $T$ be a tree on vertex set $V = \{1, \ldots, n\}$. We denote $T_{\geq x}$ the subgraph of $T$ induced by \{x, x + 1, \ldots, n\}, $T_{\leq y}$ the subgraph induced by \{1, 2, \ldots, y\}, and $T[x, y]$ the one induced by $[x, y] = \{x, x + 1, \ldots, y\}$. We say that $[x, y] \subset V$ is a connected interval when $T[x, y]$ is connected and we denote $I$ the set of connected intervals of $T$.

The problem of finding the connected intervals of a tree is both a generalization of the one of finding the common intervals of two permutations and a special case of the ”Common Intervals of Tree” problem. [10] defined the common intervals of two trees $T_1$ and $T_2$ on vertex set $V$ as the subsets of $X \subset V$ such that both $T_1[X]$ and $T_2[X]$ are connected.

Although there exists linear-time algorithms to compute the common intervals of two permutations, only $O(n^2)$ algorithms are known to compute a satisfactory representation of the common connected subsets of two trees (notice that there could be an exponential number of such subsets). When one of the tree is a path, the problem becomes equivalent to find the connected intervals of a tree (by renumbering the vertices in the order of the path).

We here address this special case where one of the tree is a path. First remark that $I$ is a family of intervals closed under intersection. Using the general methods described above, we give the first $O(n)$ algorithm that computes a convenient representation for $I$ and the first $O(n + |I|)$ algorithm that outputs all the intervals of $I$.

We can introduce for this problem a very simple notion of potential beginning (resp. potential end).

Definition 3.1 (Potential beginning). For a given $y \in V$, we define the potential beginnings for the end $y$ as the $x \leq y$ such that, in $T_{\geq x}$, $x$ accesses all the elements of $\{x, \ldots, y\}$. The potential ends of a vertex $x$ are defined symmetrically.

Theorem 3.2 (Characterization of the connected intervals). $[x, y] \subset V$ is a connected interval iff $x \in$ potBeg(y) and $y \in$ potEnd(x).

Proof. If $[x, y] \in I$ it is clear from the previous definition that $x \in$ potBeg(y) and $y \in$ potEnd(x). Reciprocally assume that $x \in$ potBeg(y) and $y \in$ potEnd(x). First remark that the path $P$ between $x$ and $y$ has all its values in $[x, y]$ since it is both in $T_{\geq x}$ and $T_{\leq y}$. Moreover for all $z \in [x, y]$, the path joining $z$ to $P$ is the intersection of the path between $z$ and $x$ and the one between $z$ and $y$ therefore it has all its values in $[x, y]$. From this we derive that $[x, y]$ is a connected interval.

The corresponding notions of right-splitter and left-splitter are then:

Proposition 3.3 (Splitters). For a given $x$, the right-splitter of $x$ is the minimum $z > x$ such that there exists a vertex $x' < x$ on the path between $x$ and $z$. The notion of left-splitter is symmetrical.

In this special case, PotBeg has a simple behaviour.
Proposition 3.4 (Stack behaviour of PotBeg). In this context, PotBeg behaves like a stack (when we consider the \(y\) in increasing order).

More formally: If \(x < x'\) are in \(\text{potBeg}(y - 1)\) then, if \(x'\) is still in \(\text{potBeg}(y)\) then so is \(x\).

Proof. If \(x'\) is still in \(\text{potBeg}(y)\) then, since \(x\) accesses \(x'\) in \(T_{\geq x}\) and \(x'\) accesses \(y\) in \(T_{\geq x'}\), so a fortiori in \(T_{\geq x}\), we have that \(x\) accesses \(y\) in \(T_{\geq x}\). We conclude that \(x\) is a potential beginning for \(y\).

This behaviour of PotBeg will be really useful in our algorithm since, as we will see, it is easy to maintain this stack. \(RSplitter[x]\) is then simply obtained as the \(y\) we are considering when \(x\) is popped from the stack.

If \(x\) is a potential beginning of \(y - 1\), we can test if it is still one for \(y\) by checking that \(x\) accesses \(y\) in \(T_{\geq x}\). In other words, we only have to check that the path between \(x\) and \(y\) does not contain values less than \(x\).

To do this test in constant time, we will ask for the minimum on the path between \(x\) and \(y\) to check if it is less than \(x\) or not. We can get this minimum by computing the lowest common ancestor (LCA) of \(x\) and \(y\) in the Cartesian Tree of \(T\). In the RAM model, this Cartesian Tree can be computed in linear time and the LCA queries can be answered in constant time with a linear precomputation. For further details on these data structures, please refer to [14]. We thus assume that we have a function \(\text{MinOnPath}(x, y)\) that outputs in time \(O(1)\) the minimum on the path between \(x\) and \(y\).

From these remarks, we obtained the following \(O(n)\) specialization of Algorithm 1:

\begin{algorithm}
\textbf{Algorithm 3: ComputeRightSplitter}
\begin{algorithmic}
\State \(RSplitter \leftarrow [\infty, \ldots, \infty]\)
\State PotBeg.push(1)
\For {\(y\) from 2 to \(n\)}
\State \textbf{while} \(\text{MinOnPath(PotBeg.top(), y)) \neq PotBeg.top()} \textbf{do}
\State \(RSplitter[PotBeg.top()] \leftarrow y\)
\State PotBeg.pop()
\State PotBeg.push(y)
\EndFor
\end{algorithmic}
\end{algorithm}

Since we can test the pop condition in constant time, we can also implement Algorithm 2 to run in time \(O(n + |I|)\).

3.2 Paths in a Tree

We have previously seen how to compute in linear time the intervals \(I\) such that \(T[I]\) is connected. We here give a linear time algorithm to compute the intervals \(I\) such that \(T[I]\) is a path. We denote \(P\) this latter family.

The notion of potential beginnings we will use is:

**Definition 3.5.** \(x\) is a potential beginning of \(y\) when both the following conditions are satisfied:

1. \(x\) accesses all vertices of \([x, y]\) in \(T_{\geq x}\)
2. \(T[x, y]\) is contained in a path of \(T\).

We describe now the data structure we use to update efficiently PotBeg. When we have computed \(\text{potBeg}(y - 1)\), we have to remove from PotBeg all the \(x\) that are not a potential beginning for \(y\). We proceed in two steps:

1. We remove from PotBeg all the \(x\) that do not satisfy the former condition
2. Then we remove the remaining ones that do not satisfy the latter one.
Remark that since the first condition is identical to the one of section 3.1, we can perform the first step exactly as in Algorithm 3 therefore we will only describe the second step. The second condition will be called the alignment condition.

Since PotBeg is contained in a path, we can consider the doubly-linked list L that contains the vertices of PotBeg in the order in which they appear in the path. During the first step, we will just do some removals from L, each vertex being removed in constant time.

**Proposition 3.6.** The set of vertices removed from PotBeg during the second step form a prefix of PotBeg.

**Proof.** If \([x, y]\) is not contained in a path then neither is \([x', y]\) for \(x' < x\).

During this second step, we will remove the minimum element of PotBeg while the alignment condition is not respected. To check this, we can test if the set of three vertices composed of the two ends of \(L\) and \(y\) are aligned or not (this can be done in constant time using a fixed number of LCA computations). The insertion of \(y\) into \(L\) can also be done easily.

We can thus perform each removal in constant time. Therefore we can build a generator for \(\mathcal{P}\) in linear time and also enumerate \(\mathcal{P}\) in optimal time.

### 3.3 Closed intervals of a DAG

Let \(G\) be a directed acyclic graph (DAG) with vertex set \(V(G) = \{1, \ldots, n\}\) and \(m\) arcs. A **closed interval** is an integer interval \([x, y] \subset V(G)\) such that all accessible vertices from a \(z \in [x, y]\) are in \([x, y]\). It is easy to prove that this family of intervals is closed under intersection. Given an interval \([x, y] \subset V(G)\), we denote \(Cl([x, y])\) the set of vertices reachable from at least one of the vertices of \([x, y]\).

![Figure 2: An example of DAG on vertex set \{1, \ldots, 9\}. \([2, 5]\) is a closed interval whereas \([4, 9]\) is not.](image)

In this context, it is natural to define the notion of potential beginning as follows:

**Definition 3.7** (Potential beginning). For a given \(y\), we say that \(x \in \{1, \ldots, y\}\) is a potential beginning of \(y\) when \(x \leq \min Cl([x, y])\). (The notion of potential end is defined symmetrically).

As in the case of the connected intervals of a tree, we have the following properties:

**Lemma 3.8.** With this notion of potential beginning:

1. PotBeg behaves like a stack when we consider the \(y\) in increasing order
2. \([x, y]\) is a closed interval iff \(x \in \text{potBeg}(y)\) and \(y \in \text{potEnd}(x)\).

**Proof.** We first show that PotBeg behaves like a stack. Consider \(x < x'\) two potential beginnings of \(y - 1\). If \(x\) is not a beginning for \(y\) then there exists \(z < x\) reachable from \(y\). Since \(z < x'\), \(x'\) is not a potential beginning for \(y\).

We now show the second property. If \([x, y]\) is a closed interval then \(Cl([x, y]) = [x, y]\), therefore \(x\) is a potential beginning of \(y\) and \(y\) is a potential end of \(x\). Reciprocally, if \(x\) is a potential
beginning of \( y \) and \( y \) is a potential end of \( x \) then for all \( z \in Cl([x, y]) \), we have that \( x \leq z \leq y \) thus \( Cl([x, y]) = [x, y] \) and \([x, y]\) is a closed interval.

Let us show how to compute the vector \( RSplitter \). The only difference with the connected intervals of a tree is the pop condition. In this context, it is easy to check if the top \( x \) of the stack is still a potential beginning for \( y \) since we just have to test if the minimum label reachable from \( y \), denoted \( MinBelow(y) \) is less than \( x \) or not. To perform this test, we first pre-compute, using dynamic programming, the vector \( MinBelow \) in time \( O(n + m) \).

**Algorithm 4:** Computation of \( RSplitter \) for Closed Interval of a DAG

\[
\begin{align*}
&\text{RSplitter} \leftarrow [\infty, \ldots, \infty] \\
&\text{PotBeg.push}(1) \\
&\text{for } y \text{ from } 2 \text{ to } n \text{ do} \\
&\quad \text{while } MinBelow(y) \not< \text{PotBeg.top()} \text{ do} \\
&\quad \quad \text{RSplitter[PotBeg.top()]} \leftarrow y \\
&\quad \quad \text{PotBeg.pop()} \\
&\quad \text{PotBeg.push}(y)
\end{align*}
\]

We thus have a linear-time algorithm that computes a generator representing the family of closed intervals of a DAG (and also a linear time algorithm to enumerate all these intervals).

### 3.4 Other examples

Let us present here a non-exhaustive list of problems which can be solved using the above framework. In the following, \( P \) will always denote a permutation, \( T \) a tree, and \( D \) a DAG (Direct Acyclic Graph):

- **A**: intervals \( I \) such that \( P(I) \) is an interval
- **B**: intervals \( I \) such that \( P(I) = I \)
- **C**: intervals \([x, y]\) such that \( P([x, y]) \subset [P(x), P(y)] \)
- **D**: intervals \([x, y]\) such that \( P([x, y]) = [P(x), P(y)] \)
- **E**: intervals \( I \) such that \( T[I] \) is connected.
- **F**: intervals \( I \) such that \( T[I] \) is contained in a path
- **G**: intervals \( I \) such that \( T[I] \) is a path
- **H**: intervals \( I \) such that \( D[I] \) is closed

All of them describe classes that can be represented by a generator. The class \( D \) has played a central role in comparative genomics. [9] introduced it to compare the genomes of cabbage and turnip, the intervals of this class are called **hurdles**.

| Class | Potential-beginning of \( y \) | Behaviour of PotBeg |
|-------|-------------------------------|---------------------|
| A     | see Definition [4,1]          | Stack               |
| B     | \( x \) s.t. \( \min P([x, y]) \geq x \) | Stack               |
| C     | \( x \) s.t. \( \min P[x, y] \geq P(x) \) | Stack               |
| D     | intersection of \( A \) and \( C \) | Stack               |
| E     | \( x \) s.t. no \( x' < x < z \leq y \) with \( x' \) between \( x \) and \( z \) | Stack               |
| F     | \( x \) s.t. \( [x, y] \) aligned in \( T \) | Queue               |
| G     | intersection of \( E \) and \( F \) | Queue-Stack         |
| H     | \( x \) s.t. \( \min Cl([x, y]) \geq x \) | Stack               |
In all the previous cases, we have obtained optimal algorithms both for the computation of a generator and the enumeration. Furthermore for the classes B and D, all the algorithms of the section can be adapted to these families. In particular, we can compute their decomposition tree in linear time.

4 Tree Decomposition of common intervals of permutations

In this section, we address the problem of computing the connected intervals of a path. This problem is equivalent to the problem of finding the common intervals of two permutations: given two permutations $P_1, P_2$ of $\{1, \ldots, n\}$, compute the subsets of $\{1, \ldots, n\}$ that appear consecutively both in $P_1$ and $P_2$. First remark that we can assume that $P_2 = Id_n = (1, 2, \ldots, n)$ (by renumbering the elements). The problem then becomes: given a permutation $P$ of $\{1, \ldots, n\}$, compute the integer intervals $I$ such that $P(I)$ is an integer interval. These integer intervals are exactly the connected intervals of a path over $\{1, \ldots, n\}$ that visits the vertices in the order given by $P$.

In the literature, the prevalent formulation of the problem is the computation of the common intervals of two permutations. It appears especially in comparative genomic: if the genomes of two species are close, then we expect that important parts coincides up to some reordering of the genes. It also models the notion of gene cluster: several genes that present functional associations are expected to appear consecutively.

This problem of finding the common intervals of two permutations was introduced by [13] in 2000. They propose an optimal, but complex, algorithm that enumerates the $K$ common intervals in time $O(n + K)$. [11] introduced the notion of irreducible intervals and obtained an $O(kn + K)$ algorithm that outputs all $K$ common intervals a $k$ permutations. [6] introduced the tree structure of this family of intervals and presented a linear time algorithm to compute this tree. [4] presented a simplest linear time algorithm, introducing the notion of generator that we use to represent the connected intervals of a tree.

In all the section, we consider the permutation $P$ given by the order in which the path visits the vertices (since this order is defined up to a reversal, we arbitrarily choose one of the two possible directions). The connected intervals $I$ of our path are exactly the intervals $I$ such that $P(I)$ is an interval. Representing the permutation $P$ in two dimensions [3], these connected intervals are represented by "squares."

![Figure 3: A representation in two dimensions of the permutation (3, 8, 1, 5, 7, 4, 6, 2). The connected interval [4, 7] is represented with a square.](image)

In the tree case, $I$ is closed under union and intersection of its overlapping members. When $G$ is a path, $I$ is also closed under the difference of its overlapping members. A family closed under union, intersection and difference of its overlapping members is called weakly partitive and admits
a canonical tree of decomposition [8]. Since \( I \) is moreover a family of intervals, this tree has a particularly simple structure that we will explicit.

In the previous section, we presented a linear time algorithm to compute a generator representing the family \( I \) when \( G \) is a tree. We can use this algorithm to compute a generator for the case of a path and then use the procedure described in [?3] to create from a generator the tree representing a weakly partitive family of intervals. However, using the specific properties of this special case, we can obtain a simplest linear time algorithm that only uses basic data structures like stacks and directly builds the tree. We will use, like in the tree case, the notion of potential beginning and potential end.

We will first address the problem of checking the simplicity of a permutation. A permutation is called simple (or prime) when all its corresponding connected intervals are trivial (i.e. of length 1 or \( n \)). Simple permutations are of first interests in the combinatorial study of permutation classes. In particular, [11] if the simplicity of a permutation could be checked in linear time. Of course, the computation of the tree decomposition of \( I \) answers this question. However, we present a very simple linear time algorithm to answer it. We afterwards extend this algorithm to build the decomposition tree of \( I \).

## 4.1 A very simple algorithm to test the simplicity of a permutation

Given the permutation \( P \), we present a very simple linear time algorithm that computes a non trivial connected interval when there exists one. This algorithms will cover the main ideas we will use to compute the decomposition tree of \( I \).

We will, as in the tree case, consider the elements one by one and maintain the potential beginnings. For convenience, we redefine the notion of potential beginning in this new context (even if it coincides with the one given inherited from the tree case).

### Definition 4.1 (Potential beginning)

Given an end \( y \), we say that \( x \) is a potential beginning for \( y \) when the following conditions are both satisfied:

- \( \exists z_1 < x < z_2 \leq y, P(x) < P(z_1) < P(z_2) \)
- \( \exists z_1 < x < z_2 \leq y, P(x) > P(z_1) > P(z_2) \)

Only the elements of \( \text{PotBeg}(y) \) have a chance to be the beginning of a connected interval ending after \( y \). Indeed, if there exists for example \( z_1 < x < z_2 \) with \( P(x) < P(z_1) < P(z_2) \), then for \( y' \geq y \), \( P([x, y']) \) contains \( P(x) \) and \( P(z_2) \) but not \( P(z_1) \) that is between \( P(x) \) and \( P(z_2) \) so \( [x, y'] \notin I \)

For the more general case of a tree, we have shown that \( \text{PotBeg} \) behaves like a stack (when we consider the \( y \) in increasing order) and that the beginnings of the connected intervals ending at \( y \) form a suffix of this stack. So for a given \( y \), we will just have to check the head of \( \text{PotBeg} \) to detect if there is a non trivial connected interval ending at \( y \).

### 4.1.1 Maintaining the stack

To maintain the stack, i.e.; update \( \text{potBeg}(y - 1) \) to \( \text{potBeg}(y) \), we just have to pop the top \( x \) of the stack while there exists \( z < x \) such that \( P(x) < P(z) < P(y) \) or \( P(x) > P(z) > P(y) \).

In order to check this condition in constant time, we precompute for each \( x \) the values \( \text{minGreaterOnLeft}[x] = \min\{P(z) | z < x, P(z) > x \} \) and \( \text{maxSmallerOnLeft}[x] = \max\{P(z) | z < x, P(z) < x \} \). This precomputation is a classic one and can be done easily in linear time with a stack.

Precisely, we have to pop \( x \) when:

- \( \text{minGreaterOnLeft}[x] < P(y) \)
- \( \text{maxSmallerOnLeft}[x] > P(y) \)

We can thus check in constant time (using simplest data structures than in the tree case), if the top of the stack has to be popped.
4.1.2 Detection of a non trivial connected interval

Recall that to detect if there exists a non trivial connected interval ending at \( y \), we just have to check if the greater potential beginning \( x < y \) of \( \text{potBeg}(y) \) is the beginning of a connected interval ending at \( y \).

Denoting \( \maxi(x, y) = \max \{ P(z) \mid x \leq z \leq y \} \) and \( \mini(x, y) = \min \{ P(z) \mid x \leq z \leq y \} \), we can check if \( [x, y] \in \mathcal{I} \) by testing if \( \maxi(x, y) - \mini(x, y) = y - x \). In order to compute \( \maxi(x, y) \) and \( \mini(x, y) \) when we want to perform this test, we have to maintain the maximum and minimum between each pair of consecutive potential beginnings in the stack \( \text{PotBeg} \) (as shown in the algorithm).

**Algorithm 5:** Detection of a non trivial connected interval

```plaintext
for y from 1 to n do
    mini ← P(y)
    maxi ← P(y)
    while PotBeg.size()≠0 and (minGreaterOnLeft[PotBeg.top()] ≠ P(y) or maxSmallerOnLeft[PotBeg.top()] ≠ P(y)) do
        mini ← min(mini, minBefore.top())
        maxi ← max(maxi, maxBefore.top())
        PotBeg.pop(), minBefore.pop(), maxBefore.pop()
    x ← PotBeg.top()
    PotBeg.push(y), minBefore.push(mini), maxBefore.push(maxi)
    if max(maxi, perm[x]) - min(mini, perm[x]) = y-x then
        Return [x,y]
```

4.2 A very simple algorithm to enumerate the connected intervals

Notice that, since the beginnings of the connected intervals ending at \( y \) form a suffix of the stack \( \text{PotBeg} \), we can enumerate all the \( K \) connected intervals in time \( O(n + K) \) by replacing the last if-statement of Algorithm 5 by:

**Function** Enumeration of the beginnings

```plaintext
iPotBeg ← PotBeg.size()-1
x ← PotBeg[iPotBeg]
mini ← P(y)
maxi ← P(y)
while iPotBeg ≥ 0 and (max(maxi, perm[x]) - min(mini, perm[x]) = y-x) do
    Output([x,y])
    mini ← min(mini, minBefore[iPotBeg])
    maxi ← max(maxi, maxBefore[iPotBeg])
    iPotBeg ← iPotBeg - 1
if iPotBeg ≥ 0 then
    x ← PotBeg[iDeb]
```

This algorithm is much simpler than the one given by Uno and Yagiura [13].

4.3 The tree representation of the family

Recall that two sets overlap when they intersect without inclusion. Each time we consider a family \( \mathcal{F} \) on a ground set \( V \), we assume that \( \mathcal{F} \) contains \( V \) and all the singletons of \( V \). A weakly partitive family is a family \( \mathcal{F} \) such that if \( A, B \) are two members of \( \mathcal{F} \) that overlap then \( A \cup B, A \cap B, A \setminus B, B \setminus A \) are in \( \mathcal{F} \). It is easy to check that, when \( G \) is a path, \( \mathcal{I} \) is a weakly partitive family of intervals. [8] showed that a weakly partitive family admits a tree representation (of linear size).
We will here explicit this tree in the case of the family of the connected intervals of a permutation $P$. The nodes of the tree are given by the overlap-free members of the family: a member is overlap-free when it does not overlap any other. The family $\mathcal{L}$ of overlap-free members of $\mathcal{I}$ is laminar by definition and then can be represented with a tree $T_L$ where the parent of $u \in \mathcal{L}$ is the smallest $v \in \mathcal{L}$ that strictly contains $u$. This tree will be labelled in order to represent the whole family $\mathcal{I}$.

**Lemma 4.2.** If $X \in \mathcal{I} \setminus V$ then $X$ is an union of children of the smallest overlap-free member of $\mathcal{I}$ that contains it.

**Definition 4.3** (Quotient Family). Let $u$ be a node of $T_L$. We denote $\text{children}(u)$ the list of the children of $u$ in $T_L$ given from left to right. If $v_1$ and $v_2$ are two distinct children of $u$, $P(v_1)$ and $P(v_2)$ are two disjoint intervals. Denoting $P(v_1) \preceq P(v_2)$ when $\max(P(v_1)) < \min(P(v_2))$, we thus obtain a total order. The quotient family $Q(u)$ of $u$ is defined as the permutation given by $\preceq$ on $\text{children}(u)$.

**Theorem 4.4** (Description of quotients). Let $u$ be an internal node of $T_L$ having $k$ children. Exactly one of the following assertions holds:

1. $Q(u)$ is the increasing permutation of $\{1, \ldots, k\}$.
2. $Q(u)$ is the decreasing permutation of $\{1, \ldots, k\}$
3. $k \geq 3$ and $Q(u)$ is simple.

This theorem shows that we can label each internal node of $T_L$ either as Prime, Increasing or Decreasing. From the obtained labeled tree, we can easily derive the whole family $\mathcal{I}$.

4.4 Computation of the tree

4.4.1 General description

Extending the previous algorithm, we present an algorithm that computes the decomposition tree of $\mathcal{I}$. Roughly speaking, we can just process as before and contract the connected intervals found on the fly.

Each time we contract a connected interval, we will manipulate it like a singleton. More generally we will speak about nodes. PotBeg is now a stack of nodes and all the variables used in the previous algorithms can be adapted to support this notion of node.

We have defined above the decomposition tree of a permutation. In fact, we can define the decomposition forest of any injection from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$ by considering the overlap-free connected intervals.
The general idea is still to consider the $y$ in increasing order. At each step, we maintain the decomposition forest $F_y$ of the restriction of $P$ over $\{1, \ldots, y\}$. We will see how to update $F_{y-1}$ to $F_y$. To this update, we will have two operations $\text{tryExtension}$ and $\text{tryPrimeCreation}$.

Algorithm 6: Computation of the tree

```plaintext
for y from 1 to n do
    S.push(\text{Node}(y))
    stable ← false
    while not stable do
        stable ← true
        if $\text{tryExtension}()$ or $\text{tryPrimeCreation}()$ then
            stable ← false
    Remark: the $\text{or}$ is lazy so that the priority is given to the extension case.
```

4.4.2 Maintaining the forest of decomposition

Assume that we have considered $\{1, \ldots, y-1\}$ and computed the forest $F_{y-1}$ of the overlap-free connected intervals in the restriction of $P$ over $\{1, \ldots, y-1\}$. Denoting $A_1, \ldots, A_p$ the trees of this forest (numbered from left to right), when we consider $y$ we build the node $A$ corresponding to the singleton $y$ and we represent this configuration by the notation $< A_1, \ldots, A_p | A >$. This configuration will evolve until the update to $F_y$ is achieved, $A$ representing the tree decomposition of the restriction of $P$ on its support.

$S = A_1, \ldots, A_p$ can be seen as a stack (remark that the stack of potential beginnings is a substack of $S$) and we will consider operations considering $A$ and a suffix of $S$. Metaphorically speaking, we can consider that $A$ will “eat” head terms of $S$. More precisely, we consider two kinds of operations.

Operation 1: Monotonic extension ($\text{tryExtension}$) Let us describe the case of an increasing extension (the case of a decreasing one being symmetrical). Consider $I_1$ the connected interval corresponding to the root of $A_p$ and $I_2$ the one corresponding to the root of $A$. When $I_1$ is an increasing node and $\max P(I_1) + 1 = \min P(I_2)$, we do the following:

- Pop $A_p$ from the stack $S$.
- $A$ becomes $\text{Add}(A_p, A)$.

where $\text{Add}(T_1, T_2)$ denotes the operation that returns the tree obtained by appending $T_2$ at the end of the list of children of the root of $T_1$.

Operation 2: Prime super-node creation ($\text{tryPrimeCreation}$) This operation can be performed when there exists $0 \leq i \leq p$ such that $A_i, \ldots, A_p, A$ form a prime quotient (if it exists, such an $i$ is unique and is the top of $\text{PotBeg}$). In this case:

- Pop $A_i, \ldots, A_p$ from the stack $S$.
- $A$ becomes the prime node whose children are $(A_i, \ldots, A_p, A)$

As long as possible, we use this two operations with priority given to the first one. Since $p$ always decreases, this process terminates.

4.4.3 Proof of the correctness

In the following, when we denote $A_i$ a tree whose leaves are an interval of integers, $S_i$ will denote this interval (the support of $A_i$).

The correction of this algorithm mainly comes from the following lemma:
Lemma 4.5. Assume that we have a forest \( A_1, \ldots, A_m \) such that:

- for all \( 1 \leq i \leq m \), \( A_i \) is the decomposition tree of the restriction of \( P \) to \( S_i \)
- there are no \( 1 \leq i < j \leq m \) such that \( S_i \cup \ldots \cup S_j \) is a connected interval

then this forest is the decomposition forest.

Proof. Assume that the forest \( A_1, \ldots, A_m \) satisfies the above conditions.

First, we show that a connected interval does not overlap any of the supports \( S_1, \ldots, S_p \).
Assume towards contradiction that a connected interval \( I \) overlaps \( S_i \) and we choose \( I \) minimal for inclusion. Without loss of generality, we assume that \( I \) overlaps \( S_i \) on the right (i.e.; the beginning of \( S_i \) is not in \( I \)). Let \( S_i, \ldots, S_j \) be the supports that \( I \) intersects. If \( j = i + 1 \) then we have that \( S_i \cup S_j \) is a connected interval, that contradicts the assumptions. We thus have \( j > i + 1 \). \( I \) does not overlap \( S_j \) (if not we would contradict its minimality by considering \( I \setminus S_j \), so \( S_j \subset I \). From this we derive that \( I \setminus S_i = S_{i+1} \cup \ldots \cup S_j \) is a connected interval, contradiction.

From this property we have that a connected interval is contained in one of the supports \( S_1, \ldots, S_m \). Consequently, every overlap-free connected interval is a node of one of the trees \( A_1, \ldots, A_m \), since each \( A_i \) is the decomposition tree of its support. We hence have the decomposition forest. \( \square \)

Theorem 4.6. If \( A_1, \ldots, A_m \) are the trees of the decomposition forest of \( P \) restricted to \( \{1, \ldots, k-1\} \), then \( A_1, \ldots, A_p, A \), obtained with the previous algorithm when adding \( k \), is the decomposition forest of \( P \) restricted to \( \{1, \ldots, k\} \).

Proof. From the previous lemma, if we assume that \( A \) is the decomposition tree on its support then, since \( A_1, \ldots, A_p \) are the decomposition trees on their support and no prime creation is possible, we obtain the decomposition forest of \( P \) restricted to \( \{1, \ldots, k\} \).

We then have to demonstrate that \( A \) is the decomposition tree on its support. Recall that we begin with configuration \( < A_1, \ldots, A_p | A > \) where \( A \) is the tree whose only node is the singleton \( k \). Recall moreover that while we can perform a monotonic extension or a prime creation, we process it giving priority to the monotonic extension. \( p \) and \( A \) thus evolves until we obtain the final configuration.

First, we show that we have the two following invariants:

- (i) If \( A_p \) is increasing and we can do an increasing extension, then \( A \) is not increasing.
- (ii) If \( A_p \) is decreasing and we can do a decreasing extension, then \( A \) is not decreasing.

We show only (i), ((ii) being symmetrical). Assume that \( A_p \) and \( A \) are increasing nodes and that we can do an increasing extension. Let \( I \) denotes the rightmost child of the root of \( A_p \), and let \( I' \) be the first of the root of \( A \). \( I \cup I' \) would be a connected interval that overlaps the support of \( A_p \), it contradicts the fact that the root of \( A_p \) was overlap-free in the decomposition of the restriction of \( P \) to \( \{1, \ldots, k-1\} \).

We now show the following invariant: \( A \) is the tree decomposition on its support.

\( \triangleright \) We demonstrate first that a monotonic extension preserves this property. As before, we only consider the increasing case. We consider the node \( A \) just after the increasing extension and let \( F_1, \ldots, F_s \) be its children (\( F_s \) is so the old value of \( A \)). We now only consider the restriction of \( P \) to the support of \( A \). Assume towards contradiction that one of the nodes of \( A \) is not overlap-free. We obtain that there exists a connected interval that overlaps \( F_s \) but it would contradict the fact that \( F_s \) is not increasing (according to (i)). Hence the nodes of \( A \) all correspond to overlap-free connected intervals. Reciprocally, it is straightforward that all overlap-free connected intervals of the restriction of \( P \) on the support of \( A \) are represented by a node of \( A \).

\( \triangleright \) Eventually we demonstrate that a prime super-node creation, when there is no possible monotonic extension, preserves the invariant too. Let \( F_1, \ldots, F_s \) be the children of the prime
node created ($F_s$ is the old value of $A$). Each $F_i$ is the tree decomposition on its support. If a connected interval would overlap the support of $F_i$, then it would also intersect $F_s$ and we could have processed a monotonic extension. Moreover, there is no connected interval different from the support of $A$ that are an union of several $F_i$. From lemma 4.5 we have have the result.

4.4.4 Complexity

The linearity of the algorithm comes from the $O(1)$ detection of the monotonic extensions and Prime super-nodes creations using the stack of potential beginnings. Moreover, these two kinds of operations take a constant time to be performed. Since each of these two kinds of updates create at least one arc in the final decomposition tree, there is at most $n - 1$ such updates. Hence the whole complexity is $O(n)$.

Conclusion

The framework presented here not only simplify existing algorithms, but it allows to solve optimally new problems as developed in section 3. We are convinced that this framework can also be applied to improve some algorithms dealing with permutations avoiding some patterns as defined in [12]. In a companion paper [3] we have studied the case where $G$ is a graph and $\Pi = \text{"being connected"}$ and develop a variation of this framework using more sophisticated data structures.

References

[1] M. H. Albert, M. D. Atkinson, and M. Klazar. The enumeration of simple permutations. J. Integer Seq, 6, 2003.

[2] Marie-Pierre Béal, Anne Bergeron, Sylvie Corteel, and Mathieu Raffinot. An algorithmic view of gene teams. Theor. Comput. Sci., 320(2-3):395–418, 2004.

[3] Ismael Belghiti and Michel Habib. Connected intervals of a graph. Technical report, LIAFA, University Paris Diderot, 2013.

[4] Anne Bergeron, Cedric Chauve, Fabien de Montgolfier, and Mathieu Raffinot. Computing common intervals of $k$ permutations, with applications to modular decomposition of graphs. SIAM J. Discrete Math., 22(3):1022–1039, 2008.

[5] Frédéric Boyer, Anne Morgat, Laurent Labarre, Joël Pothier, and Alain Viairi. Syntons, metabolons and interactons: an exact graph-theoretical approach for exploring neighbourhood between genomic and functional data. Bioinformatics, 21(23):4209–4215, 2005.

[6] Binh-Minh Bui-Xuan, Michel Habib, and Christophe Paul. Revisiting T. Uno and M. Yagiura’s algorithm. In ISAAC, pages 146–155, 2005.

[7] Binh-Minh Bui-Xuan, Michel Habib, and Christophe Paul. Competitive graph searches. Theor. Comput. Sci., 393(1-3):72–80, 2008.

[8] M. Chein, Michel Habib, and M. C. Maurer. Partitive hypergraphs. Discrete Mathematics, 37(1):35–50, 1981.

[9] Sridhar Hannenhalli and Pavel A. Pevzner. Transforming cabbage into turnip: Polynomial algorithm for sorting signed permutations by reversals. J. ACM, 46(1):1–27, 1999.

[10] Steffen Heber and Carla D. Savage. Common intervals of trees. Inf. Process. Lett., 93(2):69–74, 2005.

[11] Steffen Heber and Jens Stoye. Finding all common intervals of $k$ permutations. In CPM, pages 207–218, 2001.
[12] Adeline Pierrot. *Combinatoire et algorithmique dans les classes de permutations*. PhD thesis, University Paris Diderot, 2013.

[13] Takeaki Uno and Mutsunori Yagiura. Fast algorithms to enumerate all common intervals of two permutations. *Algorithmica*, 26(2):290–309, 2000.

[14] Jean Vuillemin. A unifying look at data structures. *Commun. ACM*, 23(4):229–239, 1980.