Computation of the Locations of the Libration Points in the Relativistic Restricted Three-Body Problem

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Abstract: Problem statement: In this study, the effects of Relativistic Restricted three-Body Problem (in brief RRTBP) on the equilibrium points of both triangular and collinear is considered. The approximate locations of the collinear and triangular points are determined. Series expansions are used to develop in $\mu$ and $1/c^2$ as small parameters. To check the validity of our solution, when ignoring $1/c^2$ terms we get directly the classical results of the restricted three-body problem. Conclusion/Recommendations: A MATHEMATICA program is constructed to give a numerical application of the relativistic perturbations in the locations of the equilibrium points of the three body problem.

Key words: Relativistic restricted three body problem, location of the libration points, classical limit

INTRODUCTION

The three-body problem considers three mutually interacting masses $m_1$, $m_2$ and $m_3$. In the restricted three-body problem, $m_3$ is taken to be small enough so that it does not influence the motion of $m_1$ and $m_2$, which are assumed to be in circular orbits about their center of mass. The orbits of the three masses are further assumed to all lie in a common plane. If $m_1$ and $m_2$ are in elliptical instead of circular orbits, the problem is variously known as the “elliptic restricted problem” or “pseudo restricted problem” Szebehely (1967).

The history of the restricted three-body problem began with Euler and Lagrange continues with Jacobi (1836), Hill (1878), Poincare (1957) and Birkhoff (1915). Euler and Courvoisier (1980) first introduced a synodic (rotating) coordinate system, the use of which led to an integral of the equations of motion, known today as the Jacobian integral. Euler himself did not discover the Jacobi integral which was first given by Jacobi (1836) who, as Wintner remarks, “rediscovered” the synodic system. The actual situation is somewhat complex since Jacobi published his integral in a sidereal (fixed) system in which its significance is definitely less than in the synodic system.

Many authors hope to investigate the relativistic effects in this problem. But unfortunately, the Einstein field equations are nonlinear and therefore cannot in general be solved exactly. By imposing the symmetry requirements of time independence and spatial isotropy we are able to find one useful exact solution, the Schwarzschild metric, but we cannot actually make use of the full content of this solution, because in fact the solar system is not static and isotropic.

Indeed, the Newtonian effects of the planet’s gravitational fields are an order of magnitude greater than the first corrections due to general relativity and completely swamp the higher corrections that are in principle provided by the exact Schwarzschild solution. It is worth noting to highlight some important articles in this field.

Computed the post-Newtonian deviations of the triangular Lagrangian points from their classical positions in a fixed frame of reference for the first time, but without explicitly stating the equations of motion. Treated the relativistic (RTBP) in rotating coordinates. He derived the Lagrangian of the system and the deviations of the triangular points as well. Weinberg (1972) calculated the components of the metric tensor by using the post-Newtonian approximation in order to obtain the (RTBP) problem equations of motion. Soffel (1989) obtained the angular frequency $\omega$ of the rotating frame for the relativistic two-body problem. Brumberg (1972; 1991) studied the problem in more details and collected most of the important results on relativistic celestial mechanics.

In this study, the approximate positions of the collinear and triangular points are determined using the
equations of motion of the relativistic restricted three-body problem. The formulas are expanded in terms of $\mu$ and $\frac{2}{c^2}$ as small parameters.

**Equations of motion:** The equations of motion of the infinitesimal mass in the (RRTBP) in a synodic frame of reference $(\xi, \eta)$, in which the primary coordinates on the $\xi$-axis (-$\mu$, 0), (1-$\mu$, 0) are kept fixed and the origin at the center of mass, is given by Brumberg (1991).

From Bhatnagar and Hallan (1997) Eq. 1:

$$
\begin{align*}
\xi - 2n \eta &= \frac{\partial U}{\partial \xi} - \frac{d}{dt} \left( \frac{\partial U}{\partial \xi} \right) \\
\eta + 2n \xi &= \frac{\partial U}{\partial \eta} - \frac{d}{dt} \left( \frac{\partial U}{\partial \eta} \right)
\end{align*}
$$

where, $U$ is the potential-like function of the (RRTBP), which can be written as composed of two components, namely the potential of the classical (RTBP) potential $U_c$ and the relativistic correction $U_r$ Eq. 2:

$$
U = U_c + U_r
$$

Eq. 2:

$$
U_c = \frac{r^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}
$$

And Eq. 4:

$$
U_r = \frac{r^2}{2c^2}(\mu(1-\mu)-3) + \frac{1}{8c^2}(\dot{\xi}^2 + \dot{\eta}^2)
$$

$$
+ (\eta - \xi) \ddot{\xi} + \frac{3}{2c^2} \left( \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right)
$$

$$
(\dot{\xi}^2 + \dot{\eta}^2) - \frac{1}{2c^2} \left( \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right)
$$

$$
- \frac{\mu(1-\mu)}{2c^2} \left[ \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right] (1-3\mu-7\xi-8\eta) + \eta^2
$$

$$
- \frac{(\dot{\xi} + \dot{\eta})^2}{c^2} \left[ \frac{\mu}{r_1} + \frac{1-\mu}{r_2} \right]
$$

with:

$$
n = 1 + \frac{1}{2c^2}(\mu(1-\mu)-3), \quad r = \sqrt{\dot{\xi}^2 + \dot{\eta}^2}
$$

$$
r_1 = \sqrt{\dot{\xi} + \mu}^2 + \eta^2, \quad r_2 = \sqrt{\dot{\xi} + \mu - 1}^2 + \eta^2
$$

Remark: All five libration points lie in the $\xi$-$\eta$ plane, i.e., in the plane of motion.

The libration points are obtained from equations of motion after setting $\dot{\xi} = \eta = \dot{\xi} = 0$. These points represent particular solutions of equations of motion Eq. 5:

$$
\frac{\partial U_c}{\partial \xi} - \frac{\partial U_r}{\partial \xi} = 0, \quad \frac{\partial U_c}{\partial \eta} - \frac{\partial U_r}{\partial \eta} = 0
$$

The explicit formulas are (Remembering that $U = U_c + U_r$) Eq. 6:
\[
\frac{\partial U}{\partial \xi} = \xi \left( \frac{1 - \mu}{r_1^3} - \frac{\mu(\xi + \mu)}{r_2^3} \right) + \frac{1}{c^2} \left( \mu - \mu^2 - 3\xi \right) + \frac{1}{2 \left( \eta^2 + \xi^2 \right)} + \frac{3}{2} \left( \frac{1 - \mu}{r_1^3} + \frac{\mu(\xi + \mu - 1)}{r_2^3} \right) (\eta^2 + \xi^2) \]
\[
\frac{\partial U}{\partial \eta} = \eta \left( \frac{1 - \mu}{r_1^3} - \frac{\mu(\xi + \mu)}{r_2^3} \right) + \frac{1}{c^2} \left( \mu - \mu^2 - 3\eta \right) + \frac{1}{2 \left( \eta^2 + \xi^2 \right)} + \frac{1}{2 \left( \eta^2 + \xi^2 \right)} \left( \frac{1 - \mu}{r_1^3} + \frac{\mu(\xi + \mu - 1)}{r_2^3} \right) \eta \]

\[
\frac{\partial U}{\partial \eta} = \eta \left( \frac{1 - \mu}{r_1^3} - \frac{\mu(\xi + \mu)}{r_2^3} \right) + \frac{1}{c^2} \left( \mu - \mu^2 - 3\eta \right) + \frac{1}{2 \left( \eta^2 + \xi^2 \right)} + \frac{1}{2 \left( \eta^2 + \xi^2 \right)} \left( \frac{1 - \mu}{r_1^3} + \frac{\mu(\xi + \mu - 1)}{r_2^3} \right) \eta \]

Location of collinear libration points:

Location of \( L_1 \).

The collinear points must, by definition, have \( \xi = \eta = 0 \), since \( 1/c^2 \ll 1 \) and the solution of the classical (RTBP) is (Fig. 2) Eq. 8:

\[
r_1 + r_2 = 1, \quad r_1 = \xi + \mu, \quad r_2 = 1 - \mu - \xi, \quad \frac{\partial r_1}{\partial \xi} = \frac{\partial r_2}{\partial \xi} = 1
\]
\[
\frac{\partial U}{\partial \xi} = 1 - \mu - b_i + \epsilon_i - \frac{1 - \mu}{(1 - b_i)^2} \left( 1 - \frac{2 \epsilon_i}{1 - b_i} \right) \\
+ \frac{\mu}{b_i} \left( 1 + \frac{2 \epsilon_i}{b_i} \right) + \frac{1}{c^2} \left[ \frac{1 - \mu - b_i}{(1 - b_i)^2} \left( 1 - \frac{2 \epsilon_i}{1 - b_i} \right) \right] \\
- \left( \frac{1 - \mu}{(1 - b_i)^2} - \frac{\mu}{b_i} \right) \left( 1 - \frac{1}{(1 - b_i)^2} \right) + \frac{1}{2} \left( 1 - \frac{1}{(1 - b_i)^2} \right) (1 - \mu - b_i)^2 \\
- \frac{3}{2} \left( \frac{1 - \mu}{(1 - b_i)^2} - \frac{\mu}{b_i} \right) \left( 1 - \frac{1}{(1 - b_i)^2} \right) (1 - \mu - b_i)^2 \\
+ 3 \left( \frac{1 - \mu}{(1 - b_i)^2} + \frac{\mu}{b_i} \right) (1 - \mu - b_i) \\
- \frac{1}{2} \mu \left( 1 - \mu \right) \left[ - \frac{1}{(1 - b_i)^2} - 7 \left( \frac{1}{(1 - b_i)^2} - \frac{1}{b_i} \right) \right] \\
+ \left( \frac{1}{(1 - b_i)^2} + \frac{1}{b_i^2} \right) \left( 6 - 7b_i - 4\mu \right) = 0 \tag{11}
\]

Setting,

\[
\left( \frac{2(1 - \mu)}{(1 - b_i)^2} \right)^2 = d_1, \quad \frac{1}{b_i} = e_i, \\
\frac{1}{b_i^2} = f_i, \quad \frac{1}{1 - b_i} = g_i, \quad \frac{1}{b_i} = h_i
\]

Eq. 11 -13 can be solved for \( \epsilon_i \) to yield:

\[
\epsilon_i = d_1 \left( -1 + b_i + \mu + (1 - \mu) h_i - \mu f_i \right) \\
- \frac{1}{c^2} \left( (\mu - 2\mu - 3)(1 - \mu - b_i) + (1 - \mu) g_i + \mu e_i \right) (1 - \mu) h_i - \mu f_i \right) \right) \\
+ \frac{1}{2} \left( 1 - \mu - b_i \right)^2 - \frac{3}{2} \left( (1 - \mu) h_i - \mu f_i \right) (1 - \mu - b_i)^2 \\
+ 3 \left( (1 - \mu) g_i + \mu e_i \right) (1 - \mu - b_i) \\
- \frac{1}{2} \mu \left( -h_i - 7 \left( g_i + e_i \right) \right) + \left( h_i + f_i \right) (6 - 7b_i - 4\mu) \right) \tag{12}
\]

Expanding \( b_1, d_1, e_1, f_1, g_1 \) and \( h_1 \) in order of \( \mu \) retaining terms up to order \( \left( \frac{\mu}{3} \right)^4 \). Then substituting back into Eq.s 10 and 8 we get the location for the first libration point \( \xi_{oL1} \)

\[
\xi_{0L_1} = 1 - \left( \frac{\mu}{3} \right)^3 + \frac{1}{3} \left( \frac{\mu}{3} \right)^5 - \frac{26}{9} \left( \frac{\mu}{3} \right)^7 \\
+ \frac{58}{81} \left( \frac{\mu}{3} \right)^9 + \frac{11}{243} \left( \frac{\mu}{3} \right)^{11} - \frac{4}{9} \left( \frac{\mu}{3} \right)^{13} \\
- \frac{3544}{6561} \left( \frac{\mu}{3} \right)^{15} - \frac{34030}{6561} \left( \frac{\mu}{3} \right)^{17} - \frac{2561}{729} \left( \frac{\mu}{3} \right)^{19} + \cdots \\
\frac{1}{c^2} \left( - \frac{\mu}{3} \right)^{15} + \frac{5}{9} \left( \frac{\mu}{3} \right)^{17} - \frac{4}{9} \left( \frac{\mu}{3} \right)^{19} + \frac{2425}{486} \left( \frac{\mu}{3} \right)^{21} \\
+ \frac{1729}{486} \left( \frac{\mu}{3} \right)^{23} + \frac{6395}{2187} \left( \frac{\mu}{3} \right)^{25} + \frac{398335}{336366} \left( \frac{\mu}{3} \right)^{27} \\
+ \frac{422957}{59049} \left( \frac{\mu}{3} \right)^{29} - \frac{837450}{354294} \left( \frac{\mu}{3} \right)^{31} + \cdots \right)
\tag{13}
\]

In this equation terms that are not factored by \( 1/c^2 \) represent the location of \( \xi_{oL_1} \) in the classical RTBP while the terms that factored by \( 1/c^2 \) represent the correction due to the inclusion of the relativistic terms.

**Location of L2:** The L2 point locates outside the small massive primary of mass \( \mu \). We now drive an approximate location for this point. At the L2 point we have (see Fig. 3) Eq. 14:

\[
\begin{align*}
\eta - r_1 &= 1, \quad \eta = \xi + \mu, \\
\eta - r_2 &= \xi + \mu - 1, \quad \frac{\partial \eta}{\partial \xi} = \frac{\partial r_2}{\partial \xi} = 1 \tag{14}
\end{align*}
\]

The procedure is similar to Eq. 10 with little modification according to the location of \( \xi_{oL_2} \) as:

\[
\begin{align*}
r_1 &= a_1 + \epsilon_2, \quad r_2 = b_2 + \epsilon_2, \quad a_2 = 1 + b_2
\end{align*}
\]

With \( a_2 \) and \( b_2 \) are the unperturbed positions of \( r_1 \) and \( r_2 \) respectively, where \( b_2 \) is by:

\[
\text{Fig. 3: Shows the Location of L2}
\]

\[
\xi_{0L_2} = 1 - \left( \frac{\mu}{3} \right)^3 + \frac{1}{3} \left( \frac{\mu}{3} \right)^5 - \frac{26}{9} \left( \frac{\mu}{3} \right)^7 \\
+ \frac{58}{81} \left( \frac{\mu}{3} \right)^9 + \frac{11}{243} \left( \frac{\mu}{3} \right)^{11} - \frac{4}{9} \left( \frac{\mu}{3} \right)^{13} \\
- \frac{3544}{6561} \left( \frac{\mu}{3} \right)^{15} - \frac{34030}{6561} \left( \frac{\mu}{3} \right)^{17} - \frac{2561}{729} \left( \frac{\mu}{3} \right)^{19} + \cdots \\
\frac{1}{c^2} \left( - \frac{\mu}{3} \right)^{15} + \frac{5}{9} \left( \frac{\mu}{3} \right)^{17} - \frac{4}{9} \left( \frac{\mu}{3} \right)^{19} + \frac{2425}{486} \left( \frac{\mu}{3} \right)^{21} \\
+ \frac{1729}{486} \left( \frac{\mu}{3} \right)^{23} + \frac{6395}{2187} \left( \frac{\mu}{3} \right)^{25} + \frac{398335}{336366} \left( \frac{\mu}{3} \right)^{27} \\
+ \frac{422957}{59049} \left( \frac{\mu}{3} \right)^{29} - \frac{837450}{354294} \left( \frac{\mu}{3} \right)^{31} + \cdots \right)
\]

662
As is done in the calculation of the location of \( \xi_{0,L1} \), we can similarly calculate the location of \( \xi_{0,L2} \) as Eq. 15:

\[
\xi_{0,L2} = 1 + \left( \frac{\mu}{3} + \frac{\mu}{3} - \frac{28}{9} \frac{\mu}{3} \right) + \frac{50}{81} \left( \frac{\mu}{3} \right)^{2} + \frac{34}{243} \left( \frac{\mu}{3} \right)^{3} - \frac{40}{81} \left( \frac{\mu}{3} \right)^{4} + \frac{5542}{6561} \left( \frac{\mu}{3} \right)^{5} - \frac{7505}{2187} \left( \frac{\mu}{3} \right)^{6} + \frac{162625}{59049} \left( \frac{\mu}{3} \right)^{7} + \ldots
\]

(15)

**Location of \( L_3 \):** The \( L_3 \) point lies on the negative \( \xi \)-axis. We now derive an approximate location for this point. At the \( L_3 \) point we have (Fig. 4) Eq. 16:

\[
r_2 - r_1 = 1, \quad r_1 = -\xi - \mu, \\
\frac{\partial r_2}{\partial \xi} = \frac{\partial r_1}{\partial \xi} = -1
\]

(16)

The procedure is similar to Eq. 10 with little modification according to the location of \( \xi_{0,L2} \) as:

\[
r_1 = a_3 + \epsilon_3, \quad r_2 = b_3 + \epsilon_3, \quad a_3 = b_3 = 1
\]

where \( a_3 \) and \( b_3 \) are the unperturbed values of \( r_1 \) and \( r_2 \) respectively and \( b_3 \) is given after some successive approximation by:

\[
b_3 = 2 - \frac{7}{12} \left( \frac{23}{144} + \frac{25921}{298594} \right)
\]

As is done in the calculation of the location of \( \xi_{0,L1} \), we can similarly calculate the location of \( \xi_{0,L3} \) as Eq. 17:

\[
\xi_{0,L3} = 1 - \frac{5}{12} \mu + \frac{1127}{20736} \mu^3 - \frac{7889}{248832} \mu^5 + \ldots
\]

\[
= \frac{698005}{3981312} \mu^5 + \ldots - \frac{1}{c^2} \left( \frac{3}{4} \mu + \frac{7}{16} \mu^3 \right) + \frac{3227}{41472} \mu^5 + \ldots
\]

(17)

**Location of the Triangular Points \( L_4,5 \):** Since \( 1/c^2 << 1 \) and the solution of the classical restricted three-body problem is \( r_1 = r_2 = 1 \), then it may be reasonable in our case to assume that the positions of the equilibrium points \( L_{4,5} \) are the same as given by classical restricted three-body problem but perturbed due to the inclusion of the relativistic correction by quantities \( (\epsilon_3 = O(1/c^2)) \) Eq. 18:

\[
\xi = (1 + \epsilon_1), \quad r_1 = (1 + \epsilon_2)
\]

(18)

Substituting in the second set of (7) and (8) and solving for \( \xi \) and \( \eta \) up to the first order in the involved small quantities \( \epsilon_1 \) and \( \epsilon_2 \), we get:

\[
\xi = (\epsilon_1 - \epsilon_2 + \frac{1 - 2\mu}{2}), \quad \eta = \left[ \sqrt{\frac{3}{2}} + \frac{\sqrt{3}}{3} (\epsilon_1 + \epsilon_2) \right]
\]

(19)

Substituting the values of \( r_1 \), \( r_2 \), \( \xi \) and \( \eta \) into equations:

\[
\frac{\partial U}{\partial \xi} = \frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta} = 0, \quad \frac{\partial U}{\partial \eta} = \frac{\partial U}{\partial \eta} + \frac{\partial U}{\partial \eta} = 0
\]

Evaluating the included partial derivatives and retaining the terms up to the first order in the small parameters \( \epsilon_1 \), \( \epsilon_2 \) and also the first order terms in the relativistic correction, we obtained Eq. 20:

\[
\begin{array}{l}
(1-\mu)\epsilon_1 - \mu \epsilon_2 - \frac{3}{8c^2} \mu (1-2\mu) \\
(1-\mu) = 0, (1-\mu) = 0 + \mu \epsilon_2 \\
- \frac{7}{8c^2} \mu (1-\mu) = 0.
\end{array}
\]

(20)
Which represent two simultaneous equations in \( \epsilon_1 \) and \( \epsilon_2 \). Their solutions are being Eq. 21:

\[
\begin{align*}
\epsilon_1 &= -\frac{1}{8c^2}(2+3\mu) \\
\epsilon_2 &= -\frac{1}{8c^2}(1-\mu)(5-3\mu)
\end{align*}
\]  

Substituting the values of \( \epsilon_1 \) and \( \epsilon_2 \) into Eq. 19, yields the coordinates of the triangular points Eq. 22:

\[
\begin{align*}
\xi_{0,L_{4,5}} &= \frac{(1-2\mu)}{2}(1+\frac{5}{4c^2}) \\
\eta_{0,L_{4,5}} &= \pm\frac{\sqrt{3}}{2}(1-\frac{1}{12c^2}(6\mu^2 - 6\mu + 5))
\end{align*}
\]  

The classical limit: To check our solution, when ignoring \( 1/c^2 \) terms we get directly the classical results of the restricted three-body problem as:

\[
\begin{align*}
\xi_{0,L_1} &= 1 - \left\{ \frac{\mu}{3} + \frac{\mu^2}{3} - \frac{26\mu}{9} - \frac{58}{81} \right\} \\
\xi_{0,L_2} &= 1 + \left\{ \frac{\mu}{3} + \frac{\mu^2}{3} - \frac{28\mu}{9} \right\} \\
\xi_{0,L_3} &= 1 - \left\{ \frac{\mu}{3} + \frac{\mu^2}{3} - \frac{1127\mu}{20736} \right\} \\
\xi_{0,L_{4,5}} &= \frac{(1-2\mu)}{2}, \quad \eta_{0,L_{4,5}} = \pm\frac{\sqrt{3}}{2}
\end{align*}
\]

Numerical results and concluding remarks: We have used the above mentioned analysis and the explicit formulas obtained to design a computer program using MATHEMATICA software. We have plotted the locations of the equilibrium points \( L_1, L_2, L_3 \), versus the whole range of the mass ratio \( \mu \in [0,0.5] \).

In Fig. 5-8, \( [L_{ic}, i = 1, 2, 3] \) denotes for the position of the equilibrium points without the relativistic contribution, i.e. within the classical problem of the restricted three bodies, while \( [L_{ir}, i = 1, 2, 3] \) denotes for the position of the equilibrium points including the relativistic contribution.
In Fig. 5 the difference $L_{3R}-L_{3c}$ seems to be constant beyond $\mu \geq 0.05$. In Fig. 6 the difference $L_{2R} - L_{2c}$ increases with increasing the mass ratio $\mu$. In Fig. 7 the difference $L_{3R} - L_{3c}$ increases with increasing the mass ratio $\mu$ within the domain $0 \leq \mu \leq 0.35$ and decreases with increasing $\mu$ within the domain $0.35 \leq \mu \leq 0.5$. Fig. 8 represents an assembly plot of the all collinear equilibrium points.

We may see that, in most of these cases, the positions of $L_3$ and $L_2$ are much more affected by the influence of the relativistic terms than that of the equilibrium point, $L_1$.

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