A GROUP-THEORIST’S PERSPECTIVE ON SYMMETRY GROUPS IN PHYSICS

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Abstract. There are many Lie groups used in physics, including the Lorentz group of special relativity, the spin groups (relativistic and non-relativistic) and the gauge groups of quantum electrodynamics and the weak and strong nuclear forces. Various grand unified theories use larger Lie groups in different attempts to unify some of these groups into something more fundamental. There are also a number of finite symmetry groups that are related to the finite number of distinct elementary particle types. I offer a group-theorist’s perspective on these groups, and suggest some ways in which a deeper use of group theory might in principle be useful. These suggestions include a number of options that seem not to be under active investigation at present. I leave open the question of whether they can be implemented in physical theories.

1. Introduction

1.1. The status quo. The most important Lie group in macroscopic physics is surely the Lorentz group \(SO(3, 1)\), that describes the coordinate transformations on spacetime that are necessary to incorporate the experimental fact that the speed of light in a vacuum is independent of the relative velocity of the source and the observer. This group is the cornerstone of the theory of special relativity [1], and modern physics is almost inconceivable without it. The Lorentz group is also used as a foundation for the theory of general relativity [2, 3, 4], but it is also possible to use the larger group \(SL(4, \mathbb{R})\), that is the special linear group of degree 4, consisting of all \(4 \times 4\) real matrices with determinant 1. Either this group or the general linear group \(GL(4, \mathbb{R})\) is used in various attempts to quantise the theory of gravity [5, 6]. Of particular importance in all these cases is the representation of the group on the anti-symmetric square of spacetime. In special relativity, this representation holds the electromagnetic field, as a complex 3-dimensional vector field. The Lorentz group acts in this representation as the complex orthogonal group \(SO(3, \mathbb{C})\). Again, it is possible to extend to \(SL(4, \mathbb{R})\) acting as the orthogonal group \(SO(3, 3)\). The Lie groups used in quantum physics are rather different. The most important is surely the spin group \(SU(2)\), that describes the spin of an electron, and is necessary in order to incorporate the experimental fact that the electron has spin 1/2. The spin group is isomorphic to a double cover of the orthogonal group \(SO(3)\), and it can be extended to the relativistic spin group \(SL(2, \mathbb{C})\) that is isomorphic to a double cover of the restricted Lorentz group \(SO(3, 1)^+\). Then there are the gauge groups \(U(1)\) of electromagnetism, \(SU(2)\) of the weak force [9] and \(SU(3)\) of the strong force [10]. The weak gauge group is isomorphic to the spin group, but is certainly not equal to it. Therefore we want to maintain a stricter than usual distinction between equality of groups, and isomorphism of groups.
Various grand unified theories have attempted to combine these three gauge groups into a larger Lie group. Two important cases are the Georgi–Glashow model \cite{11} built on $SU(5)$, and the Pati–Salam model \cite{12} built on $SU(2) \times SU(2) \times SU(4)$. These theories are not generally accepted, since they appear to predict proton decay, which is essentially ruled out by experiment. However, they continue to be influential, and still appear to contain some important insights into the nature of elementary particles. Many larger Lie groups have also been used \cite{13,14,15}, but all essentially suffer from the same problem, and/or other problems \cite{16}.

It is also important to mention the finite symmetries, which in the standard model are not fundamental, but are derived from the Lie groups. For example, the weak gauge group has a 2-dimensional complex representation, which is used to describe the discrete ‘symmetries’ between certain pairs of particles, such as electron and neutrino, or proton and neutron, or up and down quark. This group also has a 3-dimensional complex representation, which is used to describe the three intermediate vector bosons, that is the neutral $Z$ boson and the two charged $W^+$ and $W^-$ bosons. One goal of the grand unified theories is to unify all the fundamental bosons (force mediators) into a single representation, and similarly unify all the fundamental fermions. In the standard model, there are 13 fundamental bosons, and 45 fundamental fermions. The list of 45 fermions is almost certainly complete, but the list of 13 bosons does not include any mediators for the force of gravity, so its completeness is perhaps still open to question.

1.2. Possible developments. One thing that immediately strikes a group-theorist faced with this list of groups is that macroscopic physics is mostly described by real and/or orthogonal Lie groups, while quantum physics is mostly described by complex and/or unitary Lie groups. A unified viewpoint must somehow reconcile these two types of groups. The standard identification of the relativistic spin group with a double cover of $SO(3,1)^+$ goes some way towards this, but does not deal with the gauge groups. It is easy, of course, to identify $U(1)$ with $SO(2)$, and the weak $SU(2)$ is again isomorphic to a double cover of $SO(3)$, but the strong $SU(3)$ cannot be related to any real or orthogonal group in such a way. Conversely, the orthogonal group $SO(3,3)$ cannot be related in this way to a unitary group.

In other words, the standard model appears to present a broadly consistent approach across the range of theories from quantum electrodynamics and the weak force, to macroscopic electromagnetism and special relativity. But it cannot incorporate either the strong force or general relativity into the same group-theoretical system. There may of course be very good physical reasons for this, and a consistent unification of all four fundamental forces may not be possible. On the other hand, if a new approach to unification can be found, different from all those that have been tried before, then it may be possible to make progress.

In this paper I consider what mathematical changes might be necessary in order to produce such a unification. I pay relatively little attention to whether these mathematical changes are consistent with experimental reality, although I do point out a few experimental results that I think are relevant to deciding between different options. The two main questions to decide are (a) whether to try to transplant the macroscopic real/orthogonal groups to quantum physics, or the quantum complex/unitary groups to macroscopic physics, and (b) how to relate the finite groups to the Lie groups.
2. From large to small

2.1. Assembling the gauge groups. I start by considering the possibility of using the macroscopic group $SL(4, \mathbb{R})$ in quantum physics. This group is not itself an orthogonal group, but is isomorphic to the spin group $Spin(3, 3)$, a double cover of the simple orthogonal group $SO(3, 3)^+$. In other words, it can be used as a container for all the spin-type groups, including $SL(2, \mathbb{C})$ and the electroweak gauge group $U(2)$. It does not, of course, contain $SU(3)$, so that some modification to quantum chromodynamics (QCD) would certainly be required in order to carry this programme through. Since direct (rather than indirect) evidence for QCD is hard to come by experimentally, this may not be too serious a problem. It is certainly possible to embed $SL(3, \mathbb{R})$ in $SL(4, \mathbb{R})$, and $SL(3, \mathbb{R})$ is just a different real form of $SU(3)$, specifically the split real form rather than the compact real form. It is therefore plausible to model exactly the same calculations as are done in QCD, by the simple expedient of introducing a few judicious factors of $i$ into the calculations at strategic points.

With this important proviso, therefore, $SL(4, \mathbb{R})$ contains all the necessary subgroups. But are they in the right places? That is, is it possible to embed these subgroups in such a way that all the required relationships between them are correct? The first, obvious, point to make is that $SL(4, \mathbb{R})$ does not contain the direct product of all the specified subgroups. It is therefore necessary to examine the assumption that these groups all commute with each other.

In fact, this assumption does not hold even in the standard model. The gauge group $U(1)$ of quantum electrodynamics does not in fact commute with the weak gauge group $SU(2)$, but only with a subgroup $U(1)$ of $SU(2)$. Similarly, to maintain the fiction that the weak $SU(2)$ commutes with the strong $SU(3)$ it is necessary to implement a whole raft of mixing angles in the Cabibbo–Kobayashi–Maskawa (CKM) matrix [17, 18] and the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix [19, 20]. In reality, these mixing angles are proof that the groups in practice (rather than in theory) do not commute with each other. If further proof is needed, the direct product of $SU(2)$ and $SU(3)$ (or any other real forms) cannot be embedded in a simple Lie group of dimension less than 24, at which point there are too many gauge bosons for the model to be consistent with experiment.

It is surely necessary, at least, for every pair of the three groups to be disjoint. For the weak force, we need a copy of $SU(2)$, although in the standard model this is the complex form $SL(2, \mathbb{C})$, rather than a real form. For quantum electrodynamics (QED), we need a copy of $U(1)$, disjoint from $SU(2)$, and probably disjoint from $SL(2, \mathbb{C})$. This is easy to arrange, since $U(1)$ defines a complex structure on the underlying 4-dimensional real space, and there is a subgroup $SL(2, \mathbb{C})$ of $SL(4, \mathbb{R})$ preserving this complex structure. Both $U(1)$ and $SL(2, \mathbb{C})$ are necessarily disjoint from $SL(3, \mathbb{R})$, since every element of $SL(3, \mathbb{R})$ fixes a non-zero vector, while only the identity elements of $U(1)$ and $SL(2, \mathbb{C})$ fix any non-zero vector.

2.2. The Lorentz group. In macroscopic physics, the Lorentz group is a subgroup of $SO(3, 1)$ of $SL(4, \mathbb{R})$. It is therefore not possible, in the scenario envisaged here, to identify the Lorentz group with the relativistic spin group $SL(2, \mathbb{C})$. The latter group can still exist in the model, of course, and calculations can be done with it in exactly the same way as in the standard model. It is just that the relationship to macroscopic spacetime cannot work in quite the same way.
On the face of it, this looks like a proof that this proposed strategy cannot work. However, it is also possible that it shows why the relationship between quantum physics and classical physics is so problematical, and why the measurement problem is still unsolved. If the isomorphism between the Lorentz group and the central quotient of the relativistic spin group is a mathematical accident (since there is only one complex Lie algebra of rank 1), rather than a fundamental physical principle, then the foundations of quantum mechanics appear in a different light.

Indeed, the same problem exists already in non-relativistic quantum mechanics, in which the relevant groups are the spin group $SU(2)$ and the rotation group $SO(3)$. If these groups are related by being disjoint subgroups of $SL(4, \mathbb{R})$, then they can generate the compact subgroup $SO(4)$ of $SL(4, \mathbb{R})$. Since $SO(4)$ contains two normal subgroups isomorphic to $SU(2)$, there are two canonical quotient maps from $SO(4)$ onto $SO(3)$. Each of these canonical quotients restricts to a canonical isomorphism between the rotation group $SO(3)$ and the spin quotient $SO(3)$. Hence the requirement for the spin group to be canonically isomorphic to a double cover of the rotation group is still fulfilled. It is only that this canonical isomorphism cannot be mathematically represented as an equality, although it has exactly the same effect in physics as if it were an equality.

Finally, it is worth remarking that the usual assumption that the gauge groups commute with the Lorentz group is the conclusion of the Coleman–Mandula theorem [21]. But there is no guarantee that the Coleman–Mandula theorem applies in the more general situation that I am considering here. The hypotheses of the theorem are quite restrictive, and only apply to a quite narrow class of potential theories. In particular, the hypotheses include the identification of the Lorentz group with a quotient of the relativistic spin group $SL(2, \mathbb{C})$, and this hypothesis does not hold in any potential model of the type being considered here.

To conclude, it is possible to transplant the macroscopic groups into quantum mechanics, provided only that (a) QCD can be implemented with the split real form $SL(3, \mathbb{R})$ rather than the compact real form $SU(3)$ of the gauge group, and (b) the foundations of quantum mechanics can be implemented with a canonical quotient (or pair of canonical quotients) from the spin group $SU(2)$ to the rotation group $SO(3)$, rather than assuming strict equality of groups. In group-theoretical terms, I believe this is a strong foundation on which to try to build a unified model.

2.3. The Clifford algebra. The relativistic spin group $Spin(3,1)$ can be constructed from a Clifford algebra with either signature $(3,1)$ or $(1,3)$. In the former case, the algebra $Cl(3,1)$ is isomorphic as an algebra to the algebra of all real $4 \times 4$ matrices. On the other hand, $Cl(1,3)$ requires $4 \times 4$ complex matrices. In the standard model, these two algebras are combined into the algebra of all $4 \times 4$ complex matrices. However well it may work in practice, this process is mathematically unnatural, and destroys the Clifford algebra property.

It is fairly clear that both $Cl(3,1)$ and $Cl(1,3)$ are required in the standard model. Essentially, $Cl(3,1)$ is used for QED, and $Cl(1,3)$ for the weak interaction, although the ‘mixing’ of the two forces means that the modelling is rather more complicated than this. In the standard model each Clifford algebra is obtained from the other by multiplying the generators by $i$. But there are other, more natural, ways to combine the two algebras, for example by embedding them both in $Cl(3,3)$. The structure of this Clifford algebra is discussed in detail in [22].
2.4. **Grand unified theories.** If it is true that some of the unitary groups in the standard model of particle physics might be better replaced by orthogonal groups, then the same is likely to be true for some of the grand unified theories. In particular, the Georgi–Glashow model based on $SU(5)$ might be less in conflict with experiment if it were based on an orthogonal group $SO(5, \mathbb{C})$ instead, or indeed on some real form thereof. The obvious real form to choose is $SO(2, 3)$, since this exhibits a symmetry-breaking into $2 + 3$ that is a clear feature of the standard model. The corresponding spin group is also known as the real symplectic group $Sp(4, \mathbb{R})$.

We have now reduced from a 24-dimensional Lie group, with too many gauge bosons, to a 10-dimensional one, with too few. The proposal made earlier to use the 15-dimensional group $SL(4, \mathbb{R})$ instead seems like the best compromise, as far as potential matching to experiment is concerned. An alternative is to consider replacing the unitary groups, not by orthogonal groups such as real or complex forms of $SO(5)$ or $SO(6)$, but by finite groups of permutations on 5 or 6 letters.

Indeed, if the theory is required to describe a finite number of distinct particles, then group theory is quite clear that a finite permutation group is required, not a Lie group. Moreover, finite groups are much more varied and subtle than Lie groups, and can describe many types of symmetries that Lie groups simply cannot describe. There are therefore many possibilities for generation symmetries, fermion-boson supersymmetries, and other types of symmetries that cannot be implemented in any grand unified theory based on Lie groups.

3. **From small to large**

3.1. **Two complex dimensions.** Let us now consider instead the possibility that the standard model is essentially correct in all its details, including the universal use of unitary groups for the gauge groups, and look at the options for using unitary groups as symmetry groups on a large scale. The group $U(1)$ already is so used, essentially by multiplying the time coordinate of spacetime by $i$ so that the electromagnetic field has a 3-dimensional complex structure. This group does not act on spacetime, however, but only on the electromagnetic field.

The spin group $SU(2)$ could in principle be implemented as a symmetry group of spacetime on any scale. It would impose a quaternionic structure on spacetime, such that time lies in the real part and $i, j, k$ represent three perpendicular space directions. As far as I can see, this is not inconsistent with any fundamental principles of physics. The natural metric on (non-relativistic) spacetime would then be Euclidean rather than Lorentzian (Minkowskian), but in a non-relativistic context that seems perfectly reasonable.

The extension to $SL(2, \mathbb{C})$ is different, however, since it requires a choice of complex subalgebra in the quaternions, and therefore a choice of a direction in space. As long as this direction remains internal to the particle, and unobservable, there is no problem. But as soon as it becomes a macroscopic direction in space, it becomes observable and must be identified. In an inertial frame, there is no preferred direction in space, and this group cannot therefore have a sensible macroscopic meaning in an inertial frame. But we do not live in an inertial frame, and experiments are only rarely done in an inertial frame. In particular, expensive elementary particle experiments are never done in an inertial frame.
Therefore there are plenty of meaningful directions in macroscopic space that could potentially be used to define particular copies of \( SL(2, \mathbb{C}) \) for use in particular contexts. In the case of the relativistic spin group, this macroscopic direction would appear to be local to the experiment, and unrelated to larger scales. However, experiments on the spin of entangled electrons can be on quite large scales, and it may be necessary to pay particular attention to how ‘local’ is ‘local’.

In the case of the weak gauge group, on the other hand, there is a distinct possibility that the observed symmetry-breaking of the weak force may indeed be related to a large-scale direction defined by the motion of the Earth’s surface, to which the experiments are attached. If true, this would be a startling development, and call into question a number of basic assumptions about the nature of elementary particles. On the other hand, testing such a proposal experimentally would be very challenging, and therefore I cannot see any good mathematical or physical reason for eliminating this possibility from consideration at this stage.

3.2. Three complex dimensions. The remaining group that we should consider is \( SU(3) \). It must presumably act on a complex 3-space. One obvious macroscopic complex 3-space is the electromagnetic field. Is there any prospect that the symmetry group of the electromagnetic field might be better presented as \( SU(3) \) rather than \( SO(3, \mathbb{C}) \)? This seems highly unlikely, given the phenomenal success of special relativity, but is it possible? Indeed, a complex 3-space in physics would normally be assumed automatically to be a unitary space. The fact that the electromagnetic field is treated instead as an orthogonal space should therefore make one sit up and ask questions. I merely ask the question. I do not profess to provide any answers.

Another obvious complex 3-space in electrodynamics is the momentum-current vector. Since the rest mass and charge are regarded as fixed and immutable, the symmetry group in classical electrodynamics is \( SO(3) \). But is it possible that in general mass and charge can be converted into each other, and the real symmetry group is \( SU(3) \)? Experimental evidence is that rest mass is not conserved, but charge is, which makes it difficult to see how \( SU(3) \) could act in this way. Nevertheless, is it possible that there is a more subtle way of interpreting the complex 3-space, which might allow an \( SU(3) \) symmetry group? Again, I merely ask the question, expecting the answer ‘no’.

Overall, I think my conclusion is that transplanting the complex/unitary groups into a macroscopic setting is less likely to be successful than transplanting the real/orthogonal groups in the opposite direction. The conflict with special relativity just seems insurmountable. But I cannot rule it out completely. The possibilities that general relativity (or some other theory of gravity) is better described by \( SU(4) \), or perhaps by \( SU(3, 1) \), than by \( SL(4, \mathbb{R}) \), remain options, I believe.

4. FROM INFINITE TO FINITE

4.1. Finite symmetries of elementary particles. The second thing that strikes a group theorist as rather odd about the way that particle physicists use group theory, is the way that Lie groups are used to describe finite symmetries. If it works, of course, then there is no reason to change it. But it seems to me that it doesn’t always work very well. There are certain places in the standard model where the number of experimentally observed particles of a given type does not match the dimension of the representation that is used to hold them.
The most obvious example concerns the kaons, which in theory live in two 2-dimensional representations of $SU(2)$, but which experimentally are five (or possibly even six) distinct particles. It may well be that the mathematics that is used to deal with this situation describes kaons correctly. But this mathematics is not group theory, and the continued discovery of new anomalies (i.e. disagreements between theory and experiment) for kaons [23, 24, 25] suggests that there may be a problem with the standard model in this area.

4.2. Finite permutation groups. The numbers of fundamental fermions of various types (6 leptons and $6 \times 3$ quarks, consisting of 45 individual Weyl spinors, 15 in each of 3 generations) suggest that the relevant group is closely related to the symmetric group $Sym(6)$ on 6 letters, and the corresponding simple group $Alt(6)$. The latter is the most interesting and unusual of all the alternating groups [26, 27], since in addition to the extension to $Sym(6)$, and the double cover $2.Alt(6)$, that come from generic constructions applicable to all alternating groups, there is a further outer automorphism of $Sym(6)$ (a potential ‘supersymmetry’?) and a triple cover $3.Alt(6)$.

The primitive permutation representations of $Alt(6)$ are on 6, 15, 10, 15 and 6 points. The numbers 6 and 10 are the same as the numbers of anti-symmetric and symmetric rank 2 tensors on 4-dimensional spacetime, while 15 is the dimension of the adjoint representation of $SL(4, \mathbb{R})$, or of the compact real form $SU(4)$. More specifically, the Clifford algebra $Cl(3,3)$ or $Cl(6)$ splits into representations of dimensions $1 + 6 + 15 + 10 + 10 + 15 + 6 + 1$, such that, at least in the compact case, $Alt(6)$ can act as permutations on these representations.

Hence $Alt(6)$ can plausibly model finite symmetries on these various tensors and the Clifford algebra. The difference in signature between the compact real forms $SU(4)$ and $SO(6)$ on the one hand, and the split real forms $SL(4, \mathbb{R})$ and $SO(3,3)$ on the other, suggests that there is an essential symmetry-breaking from $Alt(6)$ down to the subgroup that fixes a splitting of the 6 letters into $3 + 3$. The further splitting into $1 + 2 + 3$ required by the standard model may arise from fixing a particular one of the 15 fermions, which in the usual interpretation would be the right-handed electron. A different splitting as $2 + 1 + 3$ would permit an identification of the fixed fermion as the left-handed neutrino instead.

There is a (unique) transitive permutation representation on 45 points, that might correspond to the standard model fundamental fermions. Various subgroups of $Alt(6)$ can be used to describe the breaking of the symmetries between different particles. Of particular interest here are two conjugacy classes each of the groups $Alt(5)$, $Sym(4)$ and $Sym(3)$, as well as the subgroup of order 36 that fixes a partition of the six letters into two disjoint triples. For example, the subgroup $Sym(3)$ acts as $1 + 2 + 3 + 3 + 6$ on one of the sets of 15 points, which look exactly like the 15 fermions in a single generation, and group together under $Alt(5)$ as $(2 + 3) + (1 + 3 + 6)$, exactly as the Georgi–Glashow model proposes.

It is possible, of course, that the coincidences of the various small numbers mentioned here are just that, coincidences. But there is quite a limited choice of finite groups that can act on sets of these sizes, and if there is a model of quantum physics based on finite groups, then there is little choice but to use the groups I have suggested.
4.3. **Finite matrix groups.** A further aspect of the finite perspective is the existence of finite representations, that is representations over finite fields \([28]\). The group \(\text{Sym}(6)\) is isomorphic to the 5-dimensional orthogonal group \(SO(5, 2)\) over the field of two elements. This can be seen in the representation on all even-weight bit-strings of length 6. There is a fixed bit-string, 111111, so that the representation is not irreducible, but there is an irreducible representation on the even bit-strings modulo complementation.

One of the subgroups \(\text{Alt}(5)\) acts on this 4-space as a 4-dimensional orthogonal group, and can be used to describe the Georgi–Glashow allocation of 15 fermions. For example, on the splitting \(A+BC+DEF\) of 6 letters we can put the neutrino in \(AB\), the left-handed electron in \(AC\) and the right-handed electron in \(BC\), three colours of right-handed down quarks in \(AD, AE, AF\) and up quarks in \(DE, DF, EF\), with the left-handed down quarks in \(CD, CE, CF\) and up quarks in \(BD, BE, BF\). Then the splitting \(ABC+DEF\) reveals the leptons as a 2-dimensional subspace of the 4-space of even bit-strings modulo complementation, and the splitting \(BC+DEF\) reveals a 3-dimensional subspace containing all the right-handed particles. Thus the linear structure of the representation encodes certain fundamental physical properties, that are less easily visible from the Lie group perspective.

Moreover, one sees the transposition \((B, C)\) acting on the left-handed particles only, swapping electron with neutrino, and up with down quarks, to create the weak doublets of the standard model. Similarly, the permutations of \(D, E, F\) act to permute the three colours. It is noticeable that the right-handed up quarks appear to be different from the other quarks, in that they have two colours rather than one in this model. But colour confinement implies that this is equivalent to the standard model.

A different finite representation is provided by the isomorphism of the double cover \(2.\text{Alt}(6)\) with the group \(\text{SL}(2, 9)\) of all \(2 \times 2\) matrices with determinant 1 over the field of order 9. This field is most easily obtained from the integers modulo 3 (where \(1 + 1 = -1\)) by adjoining a square root of \(-1\), so forms a finite analogue of the complex numbers. Hence \(\text{SL}(2, 9)\) shares some of the properties of \(\text{SL}(2, \mathbb{C})\), that plays a fundamental role in the theory. Moreover, the subgroup \(2.\text{Alt}(5)\) is isomorphic to \(\text{SU}(2, 5)\), and \(2.\text{Alt}(4)\) is isomorphic to \(\text{SU}(2, 3)\), so there is a choice of various finite analogues of \(\text{SU}(2)\) to use as required.

4.4. **Three generations.** The triple cover \(3.\text{Alt}(6)\) has a 3-dimensional representation over the field of order 4, that is linear but not unitary. Here the field consists of elements 0, 1, \(v, w\) such that \(1 + 1 = 0, vw = 1\) and \(1 + v + w = 0\). The representation becomes unitary on restriction to the subgroup of index 10 that fixes a given splitting of 6 letters into 3 + 3. This subgroup is isomorphic to a subgroup of index 2 in \(\text{SU}(3, 2)\). The full group \(\text{SU}(3, 2)\) lies inside the larger group, sometimes called \(3.M_{10}\), obtained by adjoining to \(3.\text{Alt}(6)\) the outer automorphism that does not invert the scalars. The group \(3.\text{Alt}(6)\) acts on the 63 non-zero vectors as \(45 + 18\), so it may be possible to model the 45 standard model fermions in this representation.

To see how this might work, take the following matrices as generators for \(3.\text{Alt}(6)\), mapping on to the permutations \((A, B, C), (D, E, F), (B, C)(E, F)\) and \((A, D)(E, F)\) respectively:

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & w & 0 \\
0 & 0 & v
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]
Scalar multiplication implements the generation symmetry, so that ignoring the scalars gives back the usual fermions of a single generation, just as in the Georgi–Glashow model. With the same ordering of the letters as before, we can take following types of particles:

\[
\begin{align*}
\nu &= (1, 0, 0), & e_L &= (0, 1, 0), & d_R &= (1, 1, 0), \\
\epsilon_R &= (0, 0, 1), & u_R &= (1, 1, 1), & u_L &= (1, 0, 1), & d_L &= (0, 1, 1).
\end{align*}
\]

Colours and generations are obtained by multiplying the coordinates by \(v\) and \(w\) in suitable ways. The two rows give the 5 + 10 splitting of the Georgi–Glashow model, acted on by a subgroup \(\text{Alt}(5) \cong \text{SL}(2, 4)\), acting on the first two coordinates. The standard model is obtained by restricting to a suitable copy of \(\text{SL}(2, 2)\).

Two things are worth noting about this scheme. First, that the permutation \((D, E, F)\) that in the Georgi–Glashow model acts on colours only, here also acts on the generations of leptons. This is an unavoidable fact about this model, which cannot therefore implement a complete separation of the concepts of colour and generation as in the standard model. It is a matter of opinion whether one regards this as proof that this finite model cannot work, or instead as a hint as to how the three generations can be incorporated in a natural way into the standard model, rather than bolted on as an afterthought.

Second, that the permutation \((B, C)\), that one might want to use to describe weak doublets, is odd, and therefore inverts the scalars \(v\) and \(w\) that are used for the triplet generation symmetry. To model a weak force that fixes the three generations of leptons it is necessary to use instead an even permutation, such as \((B, C)(E, F)\). Doing so creates an unavoidable mixing of the weak and strong forces, as they are described by the standard model. The standard model does, of course, contain a mixing of the weak and strong forces, described by the CKM matrix.

5. FROMFINITE TO INFINITE

5.1. Complex representations. In order to relate these finite groups to Lie groups, we need to look at complex (and also real and quaternionic) representations. There are representations of \(3.\text{Alt}(6)\) inside \(\text{SU}(3)\), that restrict to representations of \(\text{Alt}(5)\) inside \(\text{SO}(3)\), and there are representations of \(2.\text{Alt}(6)\) inside \(\text{Sp}(4)\) inside \(\text{SU}(4)\) that restrict to representations of \(2.\text{Alt}(5)\) inside \(\text{SU}(2)\) inside \(\text{SU}(2) \times \text{SU}(2)\). Hence several of the important ingredients of the standard model can be found in finite form in the theory of \(\text{Alt}(6)\).

5.2. The double cover. Let us look first at the representations of \(2.\text{Alt}(6)\). There are two 4-dimensional complex representations, that are really 2-dimensional quaternionic representations, and are therefore suitable for modelling spin. They both extend to \(2.\text{Sym}(6)\), and are swapped by the outer automorphism of \(\text{Sym}(6)\). Let us consider the possibility of using one of them for the Dirac spinors. Its antisymmetric square breaks up as \(1 + 5\), and its symmetric square is an irreducible 10. To obtain the required structure of the complexified Clifford algebra as used in the standard model, it is necessary to restrict to a subgroup \(2.\text{Alt}(5)\), such that the representation breaks up as the sum of two 1-dimensional quaternionic representations, so that the relevant Lie group is now \(\text{SU}(2) \times \text{SU}(2)\). Then 5 restricts as \(1 + 4\), while 10 restricts as \(3a + 3b + 4\), so that the Clifford algebra structure appears as \(1 + 4 + 3a + 3b + 4 + 1\) for the group \(\text{SO}(4)\).
Of course, this is a complex representation, so that one can change the signature of the orthogonal group at will. Hence all the required structure of the Dirac spinors and the Clifford algebra arises from these representations. The additional feature that is present in this approach, that is not present in the standard model, is the finite symmetry group $2.\text{Alt}(5)$.

The other 2-dimensional quaternionic representation of $2.\text{Alt}(6)$ remains irreducible on restriction to $2.\text{Alt}(5)$. Its anti-symmetric square breaks up as $1 + 5$, and its symmetric square as $1 + 4 + 5$. These representations do not obviously appear in the standard model, but may possibly be of use in some extension to include quantum gravity. The 5-dimensional representation is the spin 2 representation of $SU(2)$, which certainly relates to the fact that in certain prospective models of quantum gravity, gravitons have spin 2. A possible alternative interpretation, invoking the hypothetical ‘supersymmetry’ between fermions and bosons, is that these representations describe massive bosons, in which case the 4 perhaps represents the charged pions and kaons, and $1 + 5$ the neutral pions, kaons and eta meson, and possibly also the eta-prime meson.

A third representation that may be of interest is the tensor product of the two 2-dimensional quaternionic representations. This is a real 16-dimensional representation that breaks up as $8a + 8b$ for $\text{Alt}(6)$, and further as $3a + 5 + 3b + 5$ for $\text{Alt}(5)$. We shall see shortly how the 8-dimensional representations can be regarded as copies of the adjoint representation of $SU(3)$, and therefore as embodying the strong force.

### 5.3. The triple cover.

Now let us look at the representations of $3.\text{Alt}(6)$. There are four distinct 3-dimensional unitary representations, coming in dual pairs $3A/3A^*$ and $3B/3B^*$. The representations $3A$ and $3B$ differ in the sign of $\sqrt{5}$. The tensor product of any of these representations with its dual (complex conjugate) breaks up as $1 + 8$, which permits the identification of these 8-spaces with the adjoint representations of two distinct copies of $SU(3)$. Hence one or both of them can be used in the standard model to represent the strong force.

The tensor product of $3A$ with $3B^*$ is an irreducible real 9-dimensional representation, and is isomorphic to the tensor product of $3B$ with $3A^*$. This representation does not appear in the standard model, but if we restrict to $\text{Alt}(5)$, so that $3A$ and $3B^*$ become real orthogonal representations, then we obtain the spin $(1,1)$ representation of $SO(4)$. This is simply the compact version of the spin $(1,1)$ representation of $SO(3,1)$ that is used in general relativity for the Einstein tensor, among other things. So there is a potential application of this representation in a theory of quantum gravity.

Of course, this is highly speculative, but the results of this section show that the representation theory of the exceptional sextuple cover $6.\text{Alt}(6)$ of the alternating group $\text{Alt}(6)$ can give rise to all the necessary mathematical structures needed for all the four fundamental forces of nature. I believe, therefore, that this group is a strong contender for a universal symmetry group for elementary particles. The double cover $2.\text{Alt}(6)$ contains all the structures necessary for the electroweak forces, and the triple cover $3.\text{Alt}(6)$ contains what is necessary for the strong force and the three generations of fermions. The group $\text{Alt}(6)$ itself contains all that is necessary for large-scale physics, in which individual elementary particle spins and colours are no longer relevant, but combine, in a way that must ultimately be described by the representation theory, to create a macroscopic concept of mass.
6. Conclusions

In this paper I have looked at three different ways in which the group theory that is used in fundamental physics might in principle be developed to a more sophisticated level. I have not tried to address in detail the question as to whether these suggested developments can in fact be applied to physics, although I have tried to evaluate at a basic philosophical level the relative merits of the different possible approaches. The three approaches can be summarised as

1. try to use the Lie groups from general relativity in particle physics;
2. try to use the Lie groups from particle physics in (quantum) gravity;
3. use finite groups in particle physics, and derive the Lie groups from representations of the finite groups.

My initial conclusions are that the second option is unlikely to be viable, but that both the first and the last options show considerable promise. The first option looks likely to offer minor changes to the standard model that would permit a more thorough unification of the electromagnetic, weak and strong forces. The last option offers a tantalising glimpse of a possible quantum theory of gravity, that is potentially quite different from existing approaches [29, 30, 31, 32, 33].

References

[1] T. Dray (2012), The geometry of special relativity, A K Peters.
[2] A. Einstein (1916), Die Grundlage der allgemeinen Relativitätstheorie, Annalen der Physik 49 (7), 769–822.
[3] A. Einstein (1955), The meaning of relativity, 5th ed., Princeton UP.
[4] G. ’t Hooft (2001), Introduction to general relativity, Rinton.
[5] M. Leclerc (2006), The Higgs sector of gravitational gauge theories, Annals of Physics 321, 708–743.
[6] D. Ivanenko and G. Sardanashvily (1983), The gauge treatment of gravity, Physics Reports 94, 1–45.
[7] D. Griffiths (2008), Introduction to elementary particles, 2nd ed, Wiley.
[8] A. Zee (2016), Group theory in a nutshell for physicists, Princeton University Press.
[9] S. Weinberg (1967), A model of leptons, Phys. Rev. Lett. 19, 1264–66.
[10] M. Gell-Mann (1961), The eightfold way: a theory of strong interaction symmetry, Synchrotron Lab. Report CTSL-20, Cal. Tech.
[11] H. Georgi and S. Glashow (1974), Unity of all elementary-particle forces, Physical Review Letters 32 (8), 438.
[12] J. C. Pati and A. Salam (1974), Lepton number as the fourth ‘color’, Phys. Rev. D 10 (1), 275–289.
[13] A. Hartanto and L. T. Handoko (2005), Grand unified theory based on the $SU(6)$ symmetry, Physical Review D 71 (9), 095013.
[14] C. A. Manogue and T. Dray (2010), Octonions, $E_8$, and particle physics, J. Phys: Conf. Ser. 254, 012005.
[15] A. G. Lisi (2007), An exceptionally simple theory of everything, arXiv:0711.0770.
[16] J. Distler and S. Garibaldi (2010), There is no $E_8$ theory of everything, Communications in Math. Phys. 298 (2), 419–436.
[17] N. Cabibbo (1963), Unitary symmetry and lepton decays, Physical Review Letters 10 (12), 531–533.
[18] M. Kobayashi and T. Maskawa (1973), CP-violation in the renormalizable theory of weak interaction, Progress of Theoretical Physics 49 (2), 652–657.
[19] B. Pontecorvo (1958), Inverse beta processes and non-conservation of lepton charge, Soviet Physics JETP 7, 172.
[20] Z. Maki, M. Nakagawa and S. Sakata (1962), Remarks on the unified model of elementary particles, Progress of Theoretical Physics 28 (5), 870.
[21] S. Coleman and J. Mandula (1967), All possible symmetries of the S matrix, *Physical Review* **159** (5), 1251.
[22] R. A. Wilson (2020), Potential uses of representations of $SL(4, \mathbb{R})$ in particle physics. Preprint 19014, Isaac Newton Institute, Cambridge.
[23] J. H. Christenson, J. W. Cronin, V. L. Fitch and R. Turlay (1964), Evidence for the $2\pi$ decay of the $K^0$ meson, *Phys. Rev. Lett.* **13**, 138.
[24] S. Shinohara (2019), Search for the rare decay $K_L \rightarrow \pi^0\nu\bar{\nu}$ at JPARC-KOTO experiment, KAON2019, Perugia, Italy.
[25] T. Kitahara, T. Okui, G. Perez, Y. Soreq and K. Tobioka (2020), New physics implications of recent search for $K_L \rightarrow \pi^0\nu\bar{\nu}$ at KOTO, *Phys. Rev. Lett.* **124**, 071801.
[26] R. A. Wilson (2009), *The finite simple groups*, Springer.
[27] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson (1985), *An Atlas of finite groups*, Oxford UP.
[28] Ch. Jansen, K. Lux, R. A. Parker and R. A. Wilson (1995), *An Atlas of Brauer characters*, Oxford UP.
[29] K. Becker, M. Becker and J. Schwarz (2007), *String theory and M-theory: a modern introduction*, Cambridge UP.
[30] M. B. Green, J. H. Schwarz and E. Witten (1987) *Superstring theory*, Cambridge UP.
[31] C. Rovelli (2004), *Quantum gravity*, Cambridge Univ. Press.
[32] E. Verlinde (2017), Emergent gravity and the dark universe, *SciPost Phys.* **2**, 016. [arXiv:1611.02269](https://arxiv.org/abs/1611.02269)
[33] S. Hossenfelder (2017), A covariant version of Verlinde’s emergent gravity, *Phys. Rev. D* **95**, 124018. [arXiv:1703.01415](https://arxiv.org/abs/1703.01415)