Some notes on the vector-valued extension of Littlewood–Paley–Rubio de Francia inequality for Walsh functions

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Abstract

J. L. Rubio de Francia proved the one-sided Littlewood–Paley inequality for arbitrary intervals in $L^p$, $2 \leq p < \infty$ and later N. N. Osipov proved the similar inequality for Walsh functions. In this paper we investigate some properties of Banach spaces $X$ such that the latter inequality holds for $X$-valued functions.

1 Introduction

Let $\{I_m\}$ be a sequence of disjoint intervals in $\mathbb{Z}$. In the paper [13] Rubio de Francia showed that the following inequality holds for any function $f \in L^p(\mathbb{T})$, $2 \leq p < \infty$:

$$\left\| \left( \sum_m \left| \widehat{f 1_{I_m}} \right|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}. \quad (1.1)$$

We use the notation "$\lesssim$" to indicate that the left hand side does not exceed some positive constant times the right hand side. Here this constant does not depend on $f$ or the intervals $I_m$. It is worth noting that Rubio de Francia worked with functions defined on $\mathbb{R}$ but it is easy to transfer the results of the paper [13] to the circle $\mathbb{T}$.

Later, in the paper [10] N. Osipov proved the same inequality (in the "dual" form) in the context of Walsh functions. Since we will use Walsh functions in what follows, we remind the reader the basic notions concerning them.

The Rademacher functions are defined as $r_k(x) = \text{sign} \sin 2^k \pi x$. For $n = \sum_{k=1}^{m} 2^k$ where $k_1 > k_2 > \ldots > k_m \geq 0$ we define the $n$th Walsh function as $w_n = r_{k_1+1}r_{k_2+1}\ldots r_{k_m+1}$. It is well-known that Walsh functions form an orthonormal basis in $L^2[0,1]$. For any function $f$ defined on $[0,1]$ we denote by $\widehat{f}$ the sequence of its Walsh coefficients, that is, $\widehat{f}(n) = (f, w_n) = \int f w_n$.

Suppose that $n_1 = \sum_{k=0}^{m} \alpha_k 2^k$ and $n_2 = \sum_{j=0}^{m} \beta_j 2^j$.

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are two integers and $\alpha_k$ and $\beta_j$ are their binary digits. We are going to use the following notation:

$$n_1 \uplus n_2 = \sum_{k=0}^{m} ((\alpha_k + \beta_k) \mod 2) 2^k.$$ 

Then it is easy to see that the following formula holds true:

$$w_{n_1} w_{n_2} = w_{n_1 \uplus n_2}.$$ 

The Walsh functions are closely related to dyadic martingales. We denote by $\mathcal{F}_k$ the $\sigma$-algebra generated by dyadic intervals of length $2^{-k}$. It is easy to see that the conditional expectation of a function $f$ with respect to $\mathcal{F}_k$ may be written as

$$E(f | \mathcal{F}_k) = \sum_{n=0}^{2^k-1} (f, w_n) w_n.$$ 

We will use the notation $E_k f$ instead of $E(f | \mathcal{F}_k)$ for simplicity. If we denote by $\delta_k$ the interval $[2^{-k}, 2^{-k}-1]$ in $\mathbb{Z}^+$, then the martingale differences may be written in the following way:

$$\Delta_k f = E_k f - E_{k-1} f = \sum_{n \in \delta_k} (f, w_n) w_n, \quad k \geq 1.$$ 

We also put $\delta_0 = \{0\}$ and $\Delta_0 f = (f, w_0) w_0$.

For any set $A \subset \mathbb{Z}^+$ we define the following orthogonal projection on $L^2[0, 1]$:

$$P_A f = \sum_{n \in A} (f, w_n) w_n = (\chi_A \hat{f})^\vee.$$ 

If $\{I_s\}$ is a collection of pairwise disjoint intervals in $\mathbb{Z}^+$, then the following analogue of Rubio de Francia’s inequality (1.1) holds for any function $f \in L^p[0, 1], 2 \leq p < \infty$:

$$\left\| \left( \sum_s |P_s f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}. \quad (1.2)$$

As we mentioned before, the proof of this inequality (in the dual form) may be found in the paper [10]. However, we will present a different proof which may be generalized to the vector-valued functions $f$. Since all functions we consider are defined on $[0, 1]$, we will omit this fact in our notation.

We denote by $\varepsilon_s$ a sequence of Rademacher functions (they should not be confused with $r_k$; this is another copy of a sequence of Rademacher functions and we may assume that they are defined on some other probability space $\Omega$). If $X$ is a Banach space and $f$ is an $X$-valued function, then the analogue of inequality (1.2) has the following form:

$$\left\| \sum_s \varepsilon_s P_s f \right\|_{L^p(\text{Rad} X)} \lesssim \|f\|_{L^p(X)}. \quad (1.3)$$

We denote by Rad$X$ the closure in $L^p(\Omega; X)$ of $X$-valued functions of the form

$$\sum_{j=1}^{\infty} \varepsilon_j(\omega)x_j, \quad x_j \in X.$$
It is not difficult to see that Khintchine–Kahane inequality (see for instance [3, p. 191]) implies that this definition does not depend on $p$ for $1 \leq p < \infty$. Also, for $X = \mathbb{R}$ Khintchine’s inequality implies that the inequalities (1.2) and (1.3) are equivalent.

We introduce the following definition.

**Definition.** We say that Banach space $X$ has the LPR$_w^p$ property if the inequality (1.3) holds for any function $f \in L^p(X)$.

This is an analogue of the LPR$_p^p$ property, which was axiomatised in [1], for Walsh functions. The LPR$_p^p$ property (which deals with the Fourier transform and the original inequality from the paper [13] in the vector-valued setting) was studied in the papers [5], [12] and others. In this paper we prove the results of the paper [12] for our setting of Walsh functions. Now we pass to the exact statements of our main results.

**Theorem 1.** If $X$ is a Banach lattice such that its $2$-concavification $X(2)$ is a UMD Banach lattice then $X$ is a space with LPR$_w^p$ property for $2 < p < \infty$.

**Theorem 2.** If $X$ is a Banach space with LPR$_w^p$ property for some $p \geq 2$, then $X$ is also a Banach space with LPR$_q^q$ property for any $q > p$.

These are the analogues of the results of the paper [12] for LPR$_p^p$ property. All necessary definitions regarding Banach lattices may be found in the book [8].

## 2 LPR$_w^p$ property in Banach lattices

In this section we are going to prove Theorem 1. We note that for UMD Banach lattices the inequality (1.3) can be rewritten in the same form as the inequality for scalar-valued functions:

$$\left\| \left( \sum_s |P_s f|^2 \right)^{1/2} \right\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}.$$

This fact follows from the Khintchine–Maurey inequality (see for instance [4, p.86]; note that UMD property implies finite cotype).

We start with the presentation of the proof of the desired inequality for scalar-valued functions which is different from the proof in the paper [10] and then generalize it to the case of $X$-valued functions $f$.

### 2.1 A combinatorial construction

Suppose that $I_s = [a_s, b_s)$. We are going to use the decomposition of these intervals which was constructed in the paper [10]:

$$I_s = \{a_s\} \cup \bigcup_{j \in \Theta_s} J_{js} \cup \bigcup_{i \in \widetilde{\Theta}_s} \widetilde{J}_{is}, \quad \text{where} \quad \Theta_s, \widetilde{\Theta}_s \subset \mathbb{N}. \quad (2.1)$$

The intervals $J_{js}$ and $\widetilde{J}_{is}$ are such that the following relations hold:

$$a_s + J_{js} = \delta_j \quad \text{and} \quad b_s + \widetilde{J}_{is} = \delta_i. \quad (2.2)$$
We briefly describe the construction of such decomposition (and refer the reader to the paper \[10\] for details).

Let us omit the index \(s\) and construct the decomposition of an interval \(I = [a, b]\).

Consider the binary expansion of the numbers \(a\) and \(b\):

\[
a = \sum_{k=0}^{N} \alpha_k 2^k, \quad b = \sum_{k=0}^{N} \beta_k 2^k.
\]

First, we decompose the interval \([0, b)\). We number all digits in the binary expansion of \(b\) which are equal to 1:

\[
\beta_{k_1} = \beta_{k_2} = \ldots = \beta_{k_l} = 1, \quad k_1 > k_2 > \ldots > k_l.
\]

Now we take the following intervals:

\[
\tilde{J}_{k_i+1} = \left( \sum_{r=1}^{i-1} 2^{k_r}, \sum_{t=1}^{i} 2^{k_t} \right), \quad i = 1, 2, \ldots, l.
\]

It is not difficult to see that \(b + \tilde{J}_{k_i+1} = \delta_{k_i+1}\).

One of these segments contains the number \(a\). Suppose that \(a \in \tilde{J}_{k_m+1}\). It means that \(\alpha_j = \beta_j\) for \(j > k_m\) and \(\alpha_{k_m} = 0\). Now we need to decompose the interval

\[
(a, \sum_{t=1}^{m} 2^{k_t})
\]

It may be done in a similar way. We number the binary digits of \(a\) which are equal to 0:

\[
\alpha_{k_m} = \alpha_{\kappa_h} = \ldots = \alpha_{\kappa_1} = 0, \quad k_m > \kappa_h > \ldots > \kappa_1.
\]

Now we take the following intervals:

\[
J_{\kappa_i+1} = \left( a + \sum_{j=1}^{i-1} 2^{\kappa_j} + 1, a + \sum_{g=0}^{i} 2^{\kappa_g} \right), \quad i = 1, 2, \ldots, h.
\]

It is not difficult to see that \(a + J_{\kappa_i+1} = \delta_{\kappa_i+1}\) and therefore we get the desired decomposition.

2.2 The main argument for scalar-valued functions

The inequality (1.2) will now follow from the following three inequalities:

\[
\left\| \left( \sum_{s} |P_{[a_s]} f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p},
\]

\[
\left\| \left( \sum_{s} \sum_{j \in \Theta_s} P_{j_s} f \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p},
\]

\[
\left\| \left( \sum_{s} \sum_{i \in \tilde{\Theta}_s} P_{i_s} f \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}.
\]

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The first inequality follows easily from orthogonality arguments (because \( \sum_s |P_{(a_s)}f|^2 \leq \|f\|_{L^2}^2 \)). Therefore, we should prove the second inequality (since the third one has the similar form and can be proved in the same way). We introduce the following notation:

\[
J_s = \bigcup_{j \in \Theta_s} J_{js}.
\]

We have:

\[
P_{J_s}f = wa_s \sum_{j \in \Theta_s} \Delta_j (wa_s f).
\]

This identity is an easy consequence of the formulas (2.2). Therefore, we can rewrite our inequality in the following form:

\[
\left( \sum_s \left| \sum_{j \in \Theta_s} \Delta_j (wa_s f) \right|^2 \right)^{1/2} \lesssim \|f\|_{L^p}.
\]

We denote by \( G \) the following operator from \( L^p \) to the space of \( \ell^2 \)-valued functions:

\[
f \mapsto \left( \sum_{j \in \Theta_s} \Delta_j (wa_s f) \right)_s.
\] (2.3)

We need to show that \( G \) is a bounded operator from \( L^p \) to \( L^p(\ell^2) \). For \( \ell^2 \)-valued function \( g \) define the (dyadic) sharp maximal function in a usual way:

\[
g^\#(x) = \sup_{I \ni x} \left( \frac{1}{|I|} \int_I \|g(t) - g_I\|_{\ell^2}^2 \, dt \right)^{1/2}, \quad \text{where} \quad g_I = \frac{1}{|I|} \int_I g(s) \, ds \in \ell^2.
\]

The supremum here is taken over all dyadic intervals which contain the point \( x \).

We also need the following dyadic maximal function (at this time it is defined for scalar-valued functions \( f \)):

\[
M_2 f(x) = \sup_{I \ni x} \left( \frac{1}{|I|} \int_I |f|^2 \right)^{1/2}.
\]

Here the supremum is also taken over all dyadic intervals which contain the point \( x \).

Our goal is to prove the following pointwise estimate:

\[
(G f)^\#(x) \lesssim M_2 f(x).
\] (2.4)

This estimate will finish the proof of the theorem for scalar-valued functions \( f \) because we may write:

\[
\|G f\|_{L^p(\ell^2)} \lesssim \|G f\|^\#_{L^p} \lesssim \|M_2 f\|_{L^p} \lesssim \|f\|_{L^p}.
\]

The first inequality here follows from the fact that the inequality

\[
\|g\|_{L^p(\ell^2)} \lesssim \|g^\#\|_{L^p}
\]

holds for any \( \ell^2 \)-valued function \( g \) such that \( \int_0^1 g = 0 \). Note that \( f_0^1 G f = 0 \in \ell^2 \) because \( \int_0^1 \Delta_j (h) = 0 \) for any function \( h \) and any number \( j > 0 \).
In order to prove the inequality (2.4), it is enough to show that the following estimate holds:
\[
\left( \frac{1}{|I|} \int_I \left| \sum_{j \in \Theta_s} \Delta_j(w_{as}f) - h_s \right|^2 \right)^{1/2} \lesssim \left( \frac{1}{|I|} \int_I |f|^2 \right)^{1/2}.
\] (2.5)

where
\[
h_s = \frac{1}{|I|} \int_I \sum_{j \in \Theta_s} \Delta_j(w_{as}f).
\]

Here \(I\) is a dyadic interval.

We note that for any function \(g\) the integral \(\int_I \Delta_j(g)\) is equal to 0 whenever \(2^{j-1} \geq |I|^{-1}\) — this is true because for any such interval \(\int_I E_j g = \int_I E_{j-1} g = f g\). Therefore, the expression for \(h_s\) can be rewritten in the following way:
\[
h_s = \frac{1}{|I|} \int_I \sum_{2^j \leq |I|^{-1}} \Delta_j(w_{as}f).
\]

On the other hand, for any function \(g\) the function \(\Delta_j(g)\) is constant on the interval \(I\) whenever \(|I| \leq 2^{-j}\) (because both functions \(E_j g\) and \(E_{j-1} g\) are constants on \(I\)). Hence, the left-hand side in the formula (2.5) equals to the following expression:
\[
\left( \frac{1}{|I|} \int_I \sum_{2^{j-1} \geq |I|^{-1}} \Delta_j(w_{as}f) \right)^{1/2}.
\]

We consider the restriction of the function \(f\) to the interval \(I\): \(\tilde{f} = f|_I\). Suppose that \(|I| = 2^{-m}\) and
\[
a_s = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_t}
\]
is the dyadic expansion of one of the numbers \(a_s\), where
\[
k_1 < \ldots < k_t, \quad k_t \leq m - 1, \quad k_{t+1} \geq m.
\]
We denote by \(\tilde{a}_s\) the number
\[
2^{k_t+1-m} + \ldots + 2^{k_t-m}.
\]

Also, we denote by \(\tilde{w}_m\) the Walsh functions “scaled” to the interval \(I\) (i.e., they form the orthonormal basis in the space \(L^2(I; |I|^{-1} dx)\)). Furthermore, we use the notation \(\Delta_j\) for the dyadic martingale differences on \(I\). Then the function \(\Delta_j(w_{as}f)\) coincides on \(I\) with the function \(\pm \tilde{\Delta}_{j-m}(\tilde{w}_{as}\tilde{f})\) (note that the functions \(r_{k_1+1}, \ldots, r_{k_t+1}\) are constants equivalent to \(\pm 1\) on \(I\); also, note that the operators \(\Delta_j\) are “local” in a sense that the value of a function \(\Delta_j g\) on \(I\) depends only on the restriction of \(g\) to the interval \(I\) if \(2^{j-1} \geq |I|\)).

Now it is not difficult to see that the fact that
\[
\{a_s + \delta_j\}_{j \in \Theta_s}
\]
are pairwise disjoint sets for different values of \(s\) and \(j\) (which is true because \(a_s + \delta_j = J_{js}\); see the formula (2.2)) implies that the sets
\[
\{\tilde{a}_s + \delta_{j-m}\}_{j \in \Theta_s, j \geq m+1}
\]
are also pairwise disjoint.

Therefore, simply using Plancherel theorem, we see that

\[ \left( \frac{1}{|I|} \int_I \sum_s \left| \sum_{j \in \Theta_s} \Delta_j(w_{as}, f) \right|^2 \right)^{1/2} \leq \| \tilde{f} \|_{L^2(I; |f|^{-1} \, dx)} = \left( \frac{1}{|I|} \int_I |f|^2 \right)^{1/2}, \]

and the proof of the inequality (2.4) is finished.

### 2.3 The inequality for vector-valued functions

Once the estimate (2.4) is proved, we may complete the proof of Theorem 1 in a similar way as it is done in the paper [12] (for the usual LPR property).

Suppose that \( X \) is a Banach lattice. We may assume that \( X \) is a Köthe function space defined on some probability space \( (\Omega, \Sigma, \mu) \) (see [8, p.25]). Indeed, the UMD property for the lattice \( X(2) \) implies the UMD property for the lattice \( X \) (see [14, Theorem 4]) and hence \( X \) is reflexive (see [17] page 183); besides that, we may assume that \( X \) is separable and that it contains a weak unit.

Recall that 2-concavification of \( X \) is a Banach lattice \( X(2) \) with the norm

\[ \| x \|_{X(2)} = \| |x|^{1/2} \|_{X}. \]

The space \( X(2) \) is a Banach lattice if and only if \( X \) is 2-convex, that is, the following inequality holds for any \( x_1, \ldots, x_n \in X \):

\[ \left\| \left( \sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_X \leq \left( \sum_{j=1}^n \| x_j \|_X^2 \right)^{1/2}. \]

It is worth noting that the overview [14] is an excellent reference for many facts about Banach lattices with UMD property which we are going to use.

As we obtained before (see the beginning of Subsection 2.2), Theorem 1 would follow once we prove the following three inequalities (this time for \( X \)-valued functions \( f \)):

\[ \left\| \left( \sum_s \left| P_{\{a_s\}} f \right|^2 \right)^{1/2} \right\|_{L^p(X)} \lesssim \| f \|_{L^p(X)}, \] \hspace{1cm} (2.6)

\[ \left\| \left( \sum_{j \in \Theta_s} \left| \sum_s P_{J_s} f \right|^2 \right)^{1/2} \right\|_{L^p(X)} \lesssim \| f \|_{L^p(X)}, \] \hspace{1cm} (2.7)

\[ \left\| \left( \sum_i \left| \sum_{i \in \Theta_s} P_{I_s} f \right|^2 \right)^{1/2} \right\|_{L^p(X)} \lesssim \| f \|_{L^p(X)}. \] \hspace{1cm} (2.8)

The estimate (2.6) is again easy. Note that for any fixed \( \omega \in \Omega \) Plancherel theorem implies the inequality

\[ \left( \sum_s \left| (f, w_{as}) \right|^2 \right)^{1/2} (\omega) \lesssim \left( \int_0^1 |f|^2 \right)^{1/2} (\omega). \]
Therefore, we may write:

\[
\left\| \left( \sum_s |(f, w_s)|^2 \right)^{1/2} \right\|_X \leq \left\| \left( \int_0^1 |f|^2 \right)^{1/2} \right\|_X \leq \left( \int_0^1 \|f\|_X^2 \right)^{1/2} \leq \left( \int_0^1 \|f\|_{V_p}^{1/p} \right) = \|f\|_{L_p(X)}.
\]

We used 2-convexity of \( X \) in the second inequality (note that we may assume that \( f \) is a finite linear combination of Walsh functions and therefore it is constant on the dyadic intervals of length \( 2^{-n} \) for sufficiently large value of \( n \); hence, we can write the sum instead of the integral and use 2-convexity).

Now we pass to the proof of the estimates (2.7) and (2.8). We only show how to prove the inequality (2.7) since the proof of (2.8) is similar.

Recall that the space \( X(\ell^2) \) is defined as the space of sequences \( x = (x_j) \subset X \) such that

\[
\|x\|_{X(\ell^2)} = \sup_{n \in \mathbb{N}} \left\| \left( \sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_X < \infty.
\]

We define the operator \( G \) in the same way as we did for scalar-valued functions (see the formula (2.3)). This time, our goal is to prove that \( G \) is a bounded operator from \( L^p(X) \) to \( L^p(X(\ell^2)) \).

For an \( X \)-valued function \( f \) we define the maximal function \( M_2f \) in the following way:

\[
M_2f(x) = \sup_{I \ni x} \left( \frac{1}{|I|} \int_I |f|^2 \right)^{1/2}.
\]

Note that this is a function of two variables: for each fixed \( \omega \in \Omega \) this is a maximal function of \( f(\cdot, \omega) \).

For an \( X(\ell^2) \)-valued function \( g \) we define the dyadic sharp maximal function \( g^\# \) in a similar way: for each fixed \( \omega \in \Omega \) we apply the usual sharp maximal function (which we already defined in the previous Subsection) to the \( \ell^2 \)-valued function \( g(\cdot, \omega) \). The formula which defines \( g^\# \) for \( X(\ell^2) \)-valued function \( g = (g_1, g_2, \ldots) \) has the following form:

\[
g^\#(x) = \sup_{I \ni x} \left( \frac{1}{|I|} \int_I \sum_{j=1}^\infty |g_j(t) - (g_I)_j|^2 dt \right)^{1/2}, \quad \text{where} \quad g_I = \frac{1}{|I|} \int_I g(s) \, ds \in X(\ell^2).
\]

The inequality (2.3) for scalar valued functions implies the following estimate in our context:

\[
G(f(\cdot, \omega))^\# \lesssim M_2(f(\cdot, \omega)), \quad \text{a.e.} \ \omega \in \Omega.
\]

Therefore, we have the following estimate:

\[
\|(Gf)^\#\|_{L_p(X)} \lesssim \|M_2f\|_{L_p(X)}.
\]

Hence, the proof of Theorem 1 is finished once we prove the following two estimates:

\[
\|g\|_{L_p(X(\ell^2))} \lesssim \|g^\#\|_{L_p(X)} \quad \text{and} \quad \|M_2f\|_{L_p(X)} \lesssim \|f\|_{L_p(X)} \quad (2.9)
\]
for an arbitrary $X(\ell^2)$-valued function $g$ such that $\int g = 0$ (recall that in the previous subsection we proved that $\int Gf = 0$) and $X$-valued function $f$. The second inequality follows immediately from the boundedness of Hardy–Littlewood maximal operator in $L^p(X)$ for UMD Banach lattices $X$ (which is proved in [2]. See also [14] Theorem 3), we simply need to apply it to $X$ for an arbitrary $g$ function $u$ and $\ell$ usual, there is no difference with $S$.

Here $\text{Proof.}$

present its proof here.

In the last inequality we used the following simple statement (we note that if $X$ is a UMD Banach lattice then $X^*$ is also a Banach lattice with UMD property).

Lemma. If $Y$ is a UMD Banach lattice and $h = (h_1, h_2, \ldots)$ is a $Y(\ell^2)$-valued function then for any $q$, $1 < q < \infty$, we have the following inequality: $\|S h\|_{L^q(Y)} \lesssim \|h\|_{L^q(Y(\ell^2))}$.

Since it turned out to be difficult to find an exact reference for this statement, we present its proof here.

Proof. We again suppose that $Y$ is a Banach function space defined on some measure space $(\Omega', \Sigma', \mu')$. Denote by $d_j$ the operators of martingale difference with respect to Haar filtration. Using Khintchine’s inequality and then Minkowski’s integral inequality we can write the following estimate:

$$S h(x, \omega') = \left( \sum_n \sum_j |d_j h_n(x, \omega')|^2 \right)^{1/2} \lesssim \left( \sum_n \left[ \mathbb{E} \sum_j \varepsilon_j d_j h_n(x, \omega') \right]^2 \right)^{1/2}$$

$$\leq \mathbb{E} \left( \left[ \sum_n \left( \sum_j d_j h_n(x, \omega') \right)^2 \right]^{1/2} \right).$$
Using this inequality, we estimate the norm of the square function in the following way:

\[ \|Sh\|_{L^q(Y)}^q \lesssim \int \mathbb{E}\left( \left( \sum_n \left( \sum_j \varepsilon_j d_j h_n(x) \right)^2 \right)^{1/2} \right)^q dx \]

\[ \leq \int \mathbb{E}\left( \left( \sum_n \left( \sum_j \varepsilon_j d_j h_n(x) \right)^2 \right)^{1/2} \right)^q dx = \mathbb{E}\left( \left\| \sum_j \varepsilon_j d_j h \right\|_{L^q(Y)^2}^q \right) \lesssim \|h\|_{L^q(Y)^2}^q. \]

Here the last inequality follows from the definition of UMD property for the space \( Y(\ell^2) \) (which follows from the UMD property of \( Y \), see the overview [14]).

Now we take the supremum in the estimate (2.10) over all \( h \in L^p(X^*(\ell^2)) \) with the unit norm and get the desired bound.

3 \( LPR^w_p \) implies \( LPR^w_q \) for \( q > p \)

Now we pass to the proof of Theorem 2. Suppose that \( X \) is a Banach space with \( LPR^w_p \) property (recall that it means that the inequality (1.3) holds for any \( f \in L^p(X) \)). We note that it implies that \( X \) is a UMD space (since we may take the “family” \( \{I_s\} \) consisting of one interval \( [0, N] \) and therefore get the uniform boundedness of projections \( P_{[0,N]} \) which already implies UMD property; see for instance [11 p.254]).

Our goal is to prove that for any family of disjoint intervals \( \{I_s\} \) the following inequality holds for \( f \in L^q(X) \):

\[ \left\| \sum_s \varepsilon_s P_{I_s} f \right\|_{L^q(X)} \lesssim \|f\|_{L^q(X)}, \quad q > p. \]

We again use the combinatorial construction from the paper [10] (see (2.1) and (2.2)). We introduce the notation \( J_0 = a_s \). It is enough to prove the following two inequalities:

\[ \left\| \sum_s \varepsilon_s \sum_{j \in \Theta \cup \{0\}} P_{J_s} f \right\|_{L^q(X)} \lesssim \|f\|_{L^q(X)}, \]

\[ \left\| \sum_s \varepsilon_s \sum_{i \in \Theta} P_{J_s} f \right\|_{L^q(X)} \lesssim \|f\|_{L^q(X)}. \]

Since the proofs of these inequalities are similar, we only show how to prove the first of them. Recall that \( P_{J_s} f = w_{a_s} \Delta_j(w_{a_s} f) \) and therefore our inequality may be rewritten in the following form:

\[ \left\| \sum_s \varepsilon_s w_{a_s} \sum_{j \in \Theta \cup \{0\}} \Delta_j(w_{a_s} f) \right\|_{L^q(X)} \lesssim \|f\|_{L^q(X)}. \]

Now we use the standard Kahane’s contraction principle (see [3 p.181]) and conclude that our inequality is equivalent to the following:

\[ \left\| \sum_s \varepsilon_s \sum_{j \in \Theta \cup \{0\}} \Delta_j(w_{a_s} f) \right\|_{L^q(X)} \lesssim \|f\|_{L^q(X)}. \]
Consider the operator $T$ which maps the function $f \in L^p(X)$ to
\[
\sum_s \varepsilon_s \sum_{j \in \Theta_s \cup \{0\}} \Delta_j(w_{a_s} f).
\]
We see that LPR$_p^w$ property implies that $T$ is a bounded linear operator from $L^p(X)$ to $L^p(\text{Rad}X)$ (here it is important to note that the set $\cup_{j \in \Theta_s \cup \{0\}} J_s$ is a segment in $\mathbb{Z}_+$) and we need to prove that $T$ is bounded from $L^q(X)$ to $L^q(\text{Rad}X)$.

We consider the operator $T^*: L^p(\text{Rad}X^*) \to L^p(X^*)$ (note that $(\text{Rad}X)^* \sim \text{Rad}X^*$, see Section 3 in [6]). The direct computation shows that $T^*$ has the following form:
\[
T^*\left(\sum_s \varepsilon_s g_s\right) = \sum_s w_{a_s} \sum_{j \in \Theta_s \cup \{0\}} \Delta_j g_s.
\]
Here $g = \sum_s \varepsilon_s g_s$ is a RadX$^*$-valued function. In order to prove that $T^*$ is a bounded operator from $L^p(\text{Rad}X^*)$ to $L^p(X^*)$ we use the following vector-valued version of Calderón–Zygmund decomposition from the paper [7].

**Fact 1.** Suppose that $E$ is a Banach space, $g$ is a simple $E$-valued function (which means that $g = \mathbb{E}_k g$ for sufficiently large values of $k$) and $\lambda$ is an arbitrary positive number. Then there exist simple functions $b$ and $h$ such that $g = b + h$ and the following conditions hold.
1. $\|h\|_{L^\infty(Y)} \lesssim \lambda$ and $\|h\|_{L^1(Y)} \lesssim \|g\|_{L^1(Y)}$.
2. $\int_0^1 b = 0$ and for every $n \geq 1$ we have $\Delta_n b = 1_{e_n} \Delta_n b$, where $e_n \in F_{n-1}$ and $\bigcup_{n \geq 1} e_n \lesssim \lambda^{-1}\|g\|_{L^1(Y)}$.

Our goal is to prove that the operator $T^*$ is of weak type $(1, 1)$ because in this case the boundedness from $L^q(\text{Rad}X^*)$ to $L^q(X^*)$ would follow by Marcinkiewicz interpolation theorem. We fix a number $\lambda > 0$ and apply Fact 1 to $Y = \text{Rad}X^*$ and our function $g$. Note that we may consider only simple functions $g$ since by density we may assume that the collection $\{I_s\}$ is finite and each function $g_s$ is a finite linear combination of Walsh functions. We have:
\[
|\{x : \| (T^*g)(x) \|_{X^*} > \lambda\}| \leq |\{x : \| (T*b)(x) \|_{X^*} > \lambda/2\}| + |\{x : \| (T^*h)(x) \|_{X^*} > \lambda/2\}|. \tag{3.2}
\]
In order to estimate the second summand in this formula, we use the boundedness of $T^*$ from $L^p(\text{Rad}X^*)$ to $L^p(X^*)$:
\[
|\{x : \| (T^*h)(x) \|_{X^*} > \lambda/2\}| \leq \left(\frac{\lambda}{2}\right)^{-p'} \|T^*h\|_{L^p(\text{Rad}X^*)}^{p'} \lesssim \lambda^{-p'} \|h\|_{L^p(\text{Rad}X^*)}^{p'} \lesssim \lambda^{-2p'} \|h\|_{L^\infty(\text{Rad}X^*)}^{p'} \|h\|_{L^1(\text{Rad}X^*)} \lesssim \lambda^{-1}\|g\|_{L^1(\text{Rad}X^*)}.
\]

Now we estimate the first summand from the formula (3.2). It is easy to see that the formula (3.1) and the property of $b$ from Fact 1 imply that the supports of the function $T*b$ is contained in the set $\bigcup_{n \geq 1} e_n$. Therefore, we have:
\[
|\{x : \| (T*b)(x) \|_{X^*} > \lambda/2\}| \leq |\{x : (T*b)(x) \neq 0\}| \leq \bigcup_{n \geq 1} e_n \lesssim \lambda^{-1}\|g\|_{L^1(\text{Rad}X^*)},
\]
and the proof of Theorem 2 is finished.
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