On the increments of the principal value of Brownian local time

Endre Csáki
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, P.O.B. 127, H-1364, Hungary. E-mail: csaki@renyi.hu

Yueyun Hu
Département de Mathématiques, Institut Galilée (L.A.G.A. UMR 7539) Université Paris XIII, 99 Avenue J-B Clément, 93430 Villetaneuse, France. E-mail: yueyun@math.univ-paris13.fr

Summary. Let $W$ be a one-dimensional Brownian motion starting from 0. Define $Y(t) = \int_0^t \frac{ds}{W(s)} := \lim_{\epsilon \to 0} \int_0^t 1_{|W(s)| > \epsilon} \frac{ds}{W(s)}$ as Cauchy’s principal value related to local time. We prove limsup and liminf results for the increments of $Y$.

Running title. Principal value increments.

Keywords. Brownian motion, local time, principal value, large increments.

2000 Mathematics Subject Classification. 60J65 60J55 60F15

---

1 Research supported by the Hungarian National Foundation for Scientific Research, Grant No. T 037886 and T 043037.
1. Introduction

Let \( \{W(t); t \geq 0\} \) be a one-dimensional standard Brownian motion with \( W(0) = 0 \), and let \( \{L(t, x); t \geq 0, x \in \mathbb{R}\} \) denote its jointly continuous local time process. That is, for any Borel function \( f \geq 0 \),
\[
\int_0^t f(W(s)) \, ds = \int_{-\infty}^\infty f(x)L(t, x) \, dx, \quad t \geq 0.
\]

We are interested in the process
\[
(1.1) \quad Y(t) := \int_0^t \frac{ds}{W(s)}, \quad t \geq 0.
\]

Rigorously speaking, the integral \( \int_0^t ds/W(s) \) should be considered in the sense of Cauchy’s principal value, i.e., \( Y(t) \) is defined by
\[
(1.2) \quad Y(t) := \lim_{\epsilon \to 0^+} \int_0^t \frac{ds}{W(s)} \mathbb{1}_{\{|W(s)| \geq \epsilon\}} = \int_0^\infty \frac{L(t, x) - L(t, -x)}{x} \, dx.
\]

Since \( x \mapsto L(t, x) \) is Hölder continuous of order \( \nu \), for any \( \nu < 1/2 \), the integral on the extreme right in (1.2) is almost surely absolutely convergent for all \( t > 0 \). The process \( \{Y(t), t \geq 0\} \) is called the principal value of Brownian local time.

It is easily seen that \( Y(\cdot) \) inherits a scaling property from Brownian motion, namely, for any fixed \( a > 0 \), \( t \mapsto a^{-1/2}Y(at) \) has the same law as \( t \mapsto Y(t) \). Although some properties distinguish \( Y(\cdot) \) from Brownian motion (in particular, \( Y(\cdot) \) is not a semimartingale), it is a kind of folklore that \( Y \) behaves somewhat like a Brownian motion. For detailed studies and surveys on principal value, and relation to Hilbert transform see Biane and Yor [4], Fitzsimmons and Getoor [13], Bertoin [2], [3], Yamada [20], Boufoussi et al. [5], Ait Ouahra and Eddahbi [1], Csáki et al. [11] and a collection of papers [22] together with their references. Biane and Yor [4] presented a detailed study on \( Y \) and determined a number of distributions for principal values and related processes.

Concerning almost sure limit theorems for \( Y \) and its increments, we summarize the relevant results in the literature. It was shown in [17] that the following law of the iterated logarithm holds:

**Theorem A.** (Hu and Shi [17])
\[
(1.3) \quad \limsup_{T \to \infty} \frac{Y(T)}{\sqrt{T \log \log T}} = \sqrt{8}, \quad \text{a.s.}
\]

This was extended in [10] to a Strassen-type [18] functional law of the iterated logarithm.

**Theorem B.** (Csáki et al. [10]) With probability one the set
\[
(1.4) \quad \left\{ \frac{Y(xT)}{\sqrt{8T \log \log T}} : 0 \leq x \leq 1 \right\}_{T \geq 3}
\]
is relatively compact in $C[0,1]$ with limit set equal to

$$
S := \left\{ f \in C[0,1] : f(0) = 0, \text{ } f \text{ is absolutely continuous and } \int_0^1 (f'(x))^2 \, dx \leq 1 \right\}.
$$

Concerning Chung-type law of the iterated logarithm, we have the following result:

**Theorem C.** (Hu [16])

$$
\liminf_{T \to \infty} \sqrt{\frac{\log \log T}{T}} \sup_{0 \leq s \leq T} |Y(s)| = K_1, \quad \text{a.s.}
$$

with some (unknown) constant $K_1 > 0$.

The large increments were studied in [7] and [8]:

**Theorem D.** (Csáki et al. [7]) Under the conditions

$$
\begin{align*}
0 < a_T & \leq T, \\
T \to a_T \text{ and } T \to T/a_T & \text{ are both non-decreasing}, \\
\lim_{T \to \infty} \frac{\log(T/a_T)}{\log \log T} & = \infty,
\end{align*}
$$

we have

$$
\lim_{T \to \infty} \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t+s) - Y(t)| \leq 2
$$
a.s.

Wen [19] studied the lag increments of $Y$ and among others proved the following results.

**Theorem E.** (Wen [19])

$$
\limsup_{T \to \infty} \sup_{0 \leq t \leq T} \sup_{t \leq s \leq T} \frac{|Y(s) - Y(s-t)|}{\sqrt{a_T \log(T/a_T)}} = 2, \quad \text{a.s.}
$$

Under the conditions $0 < a_T \leq T$, $a_T \to \infty$ as $T \to \infty$, we have

$$
\limsup_{T \to \infty} \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} \frac{|Y(t+s) - Y(t)|}{\sqrt{a_T (\log((t/a_T) + 2 \log \log a_T)}} \leq 2, \quad \text{a.s.}
$$

If $a_T$ is onto, then we have equality in (1.10).

In this note our aim is to investigate further limsup and liminf behaviors of the increments of $Y$. 

- 3 -
Theorem 1.1. Assume that $T \mapsto a_T$ is a function such that $0 < a_T \leq T$, and both $a_T$ and $T/a_T$ are non-decreasing. Then

(i)
\[
\limsup_{T \to \infty} \frac{\sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{a_T (\log \sqrt{T/a_T} + \log \log T)}} = \sqrt{8}, \quad \text{a.s.}
\]

(ii) If $a_T > T(\log T)^{-\alpha}$ for some $\alpha < 2$, then
\[
\liminf_{T \to \infty} \sqrt{\frac{\log \log T}{a_T}} \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| = K_2, \quad \text{a.s.}
\]

(iib) If $a_T \leq T(\log T)^{-\alpha}$ for some $\alpha > 2$, then
\[
\liminf_{T \to \infty} \frac{\sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} = K_3, \quad \text{a.s.}
\]

with some positive constants $K_2, K_3$. If, moreover,
\[
\lim_{T \to \infty} \frac{\log(T/a_T)}{\log \log T} = \infty,
\]
then $K_3 = 2$.

Theorem 1.2. Assume that $T \mapsto a_T$ is a function such that $0 < a_T \leq T$, and both $a_T$ and $T/a_T$ are non-decreasing. Then

(i)
\[
\liminf_{T \to \infty} \frac{\sqrt{T \log \log T}}{a_T} \inf_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| = K_4, \quad \text{a.s.}
\]

with some positive constant $K_4$. If, $\lim_{T \to \infty} (a_T/T) = 0$, then $K_4 = 1/\sqrt{2}$.

(ii) If $0 < \rho \leq 1$, then
\[
\limsup_{T \to \infty} \frac{\inf_{0 \leq t \leq T-\rho T} \sup_{0 \leq s \leq \rho T} |Y(t + s) - Y(t)|}{\sqrt{T \log \log T}} = \rho \sqrt{8}, \quad \text{a.s.}
\]

(iib) If
\[
\lim_{T \to \infty} \frac{a_T(\log \log T)^2}{T} = 0,
\]
then
\[
\limsup_{T \to \infty} \frac{\sqrt{T}}{a_T \sqrt{\log \log T}} \inf_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| = K_5, \quad \text{a.s.}
\]

with some positive constant $K_5$. 

- 4 -
Remark 1. The exact values of the constants $K_i, i = 2, 3, 4, 5$ are unknown. It seems difficult to determine the exact values of these constants. In the proofs we establish upper and lower bounds with possibly different constants. It follows however by 0-1 law for Brownian motion that the limsup’s and liminf’s considered here are non-random constants.

Remark 2. Plainly we recover some previous results on the path properties of $Y$ by considering particular cases of Theorems 1.1 and 1.2. For instance, Theorems A and C follow from (1.11) and (1.12) respectively by taking $a_T = T$, and (1.8) follows from (1.11) combining with (1.13). However in Theorem 1.1(ii) and Theorem 1.2(ii) there are still small gaps in $a_T$.

The organization of the paper is as follows: In Section 2 some facts are presented needed in the proofs. Section 3 contains the necessary probability estimates. Theorem 1.1(i) and Theorem 1.1(iia,b) are proved in Sections 4 and 5, resp., while Theorem 1.2(i) and Theorem 1.2(iia,b) are proved in Sections 6 and 7, resp.

Throughout the paper, the letter $K$ with subscripts will denote some important but unknown finite positive constants, while the letter $c$ with subscripts denotes some finite and positive universal constants not important in our investigations. When the constants depend on a parameter, say $\delta$, they are denoted by $c(\delta)$ with subscripts.

2. Facts

Let $\{W(t), t \geq 0\}$ be a standard Brownian motion and define the following objects:

\begin{align}
(2.1) & & g &:= \sup \{t : t \leq 1, W(t) = 0\} \\
(2.2) & & B(s) &:= \frac{W(sg)}{\sqrt{g}}, \quad 0 \leq s \leq 1, \\
(2.3) & & m(s) &:= \frac{|W(g + s(1-g))|}{\sqrt{1-g}}, \quad 0 \leq s \leq 1.
\end{align}

Here we summarize some well-known facts needed in our proofs.

Fact 2.1. (Biane and Yor [4])

\begin{align}
(2.4) & & \mathbb{P}(Y(1) \in dx) &= \sqrt{\frac{2}{\pi^3}} \sum_{k=0}^{\infty} (-1)^k \exp \left( -\frac{(2k+1)^2 x^2}{8} \right), \quad x \in \mathbb{R}.
\end{align}

Consequently we have the estimate: for $\delta > 0$

\begin{align}
(2.5) & & c_1 \exp \left( -\frac{z^2}{8(1-\delta)} \right) &\leq \mathbb{P}(Y(1) \geq z) \leq \exp \left( -\frac{z^2}{8} \right), \quad z \geq 1.
\end{align}
with some positive constant $c_1 = c_1(\delta)$. Moreover, $g$, $\{B(s), 0 \leq s \leq 1\}$ and $\{m(s), 0 \leq s \leq 1\}$ are independent, $g$ has arcsine distribution, $B$ is a Brownian bridge and $m$ is a Brownian meander.

\[
\mathbb{P}\left(\int_0^1 \frac{dv}{m(v)} < z \mid m(1) = 0\right) = \sum_{k=-\infty}^{\infty} (1 - k^2z^2) \exp\left(-\frac{k^2z^2}{2}\right) = \frac{8\pi^2\sqrt{2\pi}}{z^3} \sum_{k=1}^{\infty} \exp\left(-\frac{2k^2\pi^2}{z^2}\right), \quad z > 0.
\]

(2.7) \[\mathbb{P}(m(1) > x) = e^{-x^2/2}, \quad x > 0.\]

Fact 2.2. (Yor [21, Exercise 3.4 and pp. 44]) Let $Q^\delta_{x \to 0}$ be the law of square of Bessel bridge from $x$ to 0 of dimension $\delta > 0$ during time interval $[0,1]$. The process $(m^2(1-v), 0 \leq v \leq 1)$ conditioned on $\{m^2(1) = x\}$ is distributed as $Q^3_{x \to 0}$. Furthermore, we have

(2.8) \[Q^\delta_{x \to 0} = Q^0_{x \to 0} * Q^0_{x \to 0}, \quad \forall \delta > 0, \ x > 0,\]

where * denotes convolution operator. Consequently, for any $x > 0$

(2.9) \[\mathbb{P}\left(\int_0^1 \frac{dv}{m(v)} < z \mid m(1) = x\right) \leq \mathbb{P}\left(\int_0^1 \frac{dv}{m(v)} < z \mid m(1) = 0\right).\]

Fact 2.3. (Hu [16]) For $0 < z \leq 1$

(2.10) \[c_2 \exp\left(-\frac{c_3}{z^2}\right) \leq \mathbb{P}(\sup_{0 \leq s \leq 1} |Y(s)| < z) \leq c_4 \exp\left(-\frac{c_5}{z^2}\right)\]

with some positive constants $c_2, c_3, c_4, c_5$.

Fact 2.4. (Csörgő and Révész [12]) Assume that $T \mapsto a_T$ is a function such that $0 < a_T \leq T$, and both $a_T$ and $T/a_T$ are non-decreasing. Then

(2.11) \[
\limsup_{T \to \infty} \frac{\sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |W(t+s)-W(t)|}{\sqrt{a_T (\log(T/a_T) + \log \log T)}} = \sqrt{2}, \quad \text{a.s.}
\]

Fact 2.5. (Strassen [18]) If $f \in S$ defined by (1.5), then for any partition $x_0 = 0 < x_1 < \ldots < x_k < x_{k+1} = 1$ we have

(2.12) \[
\sum_{i=1}^{k+1} \frac{(f(x_i) - f(x_{i-1}))^2}{x_i - x_{i-1}} \leq 1.
\]

Fact 2.6. (Chung [6])

(2.13) \[
\liminf_{t \to \infty} \frac{\log \log t}{t} \sup_{0 \leq s \leq t}|W(s)| = \frac{\pi}{\sqrt{8}}, \quad \text{a.s.}
\]

Define $g(T) := \max\{s \leq T : W(s) = 0\}$. A joint lower class result for $g(T)$ and $M(T) := \sup_{0 \leq s \leq T}|W(s)|$ reads as follows.
Fact 2.7. (Grill [15]) Let \( \beta(t), \gamma(t) \) be positive functions slowly varying at infinity, such that
\[ 0 < \beta(t) \leq 1, 0 < \gamma(t) \leq 1, \beta(t) \text{ is non-increasing}, \beta(t)\sqrt{t} \uparrow \infty, \gamma(t) \text{ is monotone}, \gamma(t)t \uparrow \infty, \gamma(t)/\beta^2(t) \text{ is monotone}. \]
Then
\[ \mathbb{P}\left( M(T) \leq \beta(T)\sqrt{T}, g(T) \leq \gamma(T)T \text{ i.o.} \right) = 0 \text{ or } 1 \]
according as \( I(\beta, \gamma) < \infty \) or \( = \infty \), where
\[ I(\beta, \gamma) = \int_1^\infty \frac{1}{t^2 \beta^2(t)} \left( 1 + \frac{\beta^2(t)}{\gamma(t)} \right)^{-1/2} \exp\left( -\frac{4 - 3\gamma(t)\pi^2}{8\beta^2(t)} \right) \, dt. \]

Now define \( d(T) := \min \{ s \geq T : W(s) = 0 \} \). Since \( \{ d(T) > t \} = \{ g(t) < T \} \), we deduce from Fact 2.7 the following estimate on \( d(T) \) when \( T \to \infty \).

Fact 2.8. With probability 1
\[ d(T) = O(T(\log T)^3), \quad T \to \infty. \]

3. Probability estimates

Lemma 3.1. For \( T \geq 1, \delta, z > 0 \) we have
\[
\mathbb{P}\left( \sup_{0 \leq t \leq T-1} \sup_{0 \leq s \leq 1} |Y(t + s) - Y(t)| > z \right)
\leq c_6 \left( \sqrt{T} \exp\left( -\frac{z^2}{8(1 + \delta)} \right) + T \exp\left( -\frac{z^2}{2(1 + \delta)} \right) \right)
\]
with some positive constant \( c_6 = c_6(\delta) \).

For the proof see Csáki et al. [7], Lemma 2.8.

Lemma 3.2. For \( T > 1, 0 < \delta < 1/2, z > 1 \) we have
\[
\mathbb{P}\left( \sup_{0 \leq t \leq T-1} (Y(t + 1) - Y(t)) \geq z \right)
\geq \min \left( \frac{1}{2}, \frac{c\sqrt{T-1}}{z} \exp\left( -\frac{z^2}{8(1 - \delta)} \right) \right) - \exp\left( -z^2 \right)
\]
with some positive constant \( c_7 = c_7(\delta) > 0 \).

Proof. Let us construct an increasing sequence of stopping times by \( \eta_0 := 0 \) and
\[ \eta_{k+1} := \inf \{ t > \eta_k + 1 : W(t) = 0 \}, \quad k = 0, 1, 2, \ldots \]
Let
\[ \nu_t := \min \{ i \geq 1 : \eta_i > t \} \]
\[ Z_i := Y(\eta_i - 1) - Y(\eta_i - 1), \quad i = 1, 2, \ldots \]

Then \((Z_i, \eta_i - \eta_i - 1)_{i \geq 1}\) are i.i.d. random vectors with
\[ \eta_i - \eta_i - 1 \overset{\text{law}}{=} 1 + \tau^2, \quad Z_i \overset{\text{law}}{=} Y(1), \]

where \(\tau\) has Cauchy distribution. Clearly, for \(t > 0\),
\[ \sup_{0 \leq s \leq t} (Y(s + 1) - Y(s)) \geq \max_{1 \leq i \leq \nu_t} Z_i = \overline{Z}_{\nu_t}, \]

with \(\overline{Z}_k := \max_{1 \leq i \leq k} Z_i\). First consider the Laplace transform \((\lambda > 0)\):
\[
\lambda \int_0^\infty e^{-\lambda u} \mathbb{P}(\overline{Z}_{\nu_u} < z) \, du \\
= \lambda \sum_{k=1}^\infty \mathbb{E} \int_0^\infty e^{-\lambda u} 1_{\{\eta_{k-1} \leq u < \eta_k\}} 1_{\{\overline{Z}_k < z\}} \, du \\
= \sum_{k=1}^\infty \mathbb{E} \left( \left[ e^{-\lambda \eta_{k-1}} - e^{-\lambda \eta_k} \right] 1_{\{\overline{Z}_k < z\}} \right) \\
= \sum_{k=1}^\infty \mathbb{E} \left( \left[ e^{-\lambda \eta_{k-1}} \right] - \mathbb{E} \left[ 1_{\{\overline{Z}_k < z\}} e^{-\lambda \eta_k} \right] \right) \\
= \sum_{k=1}^\infty \mathbb{E} \left[ 1_{\{\overline{Z}_k < z\}} e^{-\lambda \eta_{k-1}} \right] - \mathbb{E} \left[ 1_{\{\overline{Z}_k < z, Z_k \geq z\}} e^{-\lambda \eta_{k-1}} \right] - \mathbb{E} \left[ 1_{\{\overline{Z}_k < z\}} e^{-\lambda \eta_k} \right] \\
= 1 - \sum_{k=1}^\infty \mathbb{E} \left[ 1_{\{\overline{Z}_{k-1} < z, Z_k \geq z\}} e^{-\lambda \eta_{k-1}} \right] \\
= 1 - \sum_{k=1}^\infty \mathbb{E} \left[ 1_{\{\overline{Z}_{k-1} < z\}} e^{-\lambda \eta_{k-1}} \right] \mathbb{P}(Y(1) \geq z) \\
= 1 - \sum_{k=1}^\infty \left( \mathbb{E} \left[ 1_{\{Z_1 < z\}} e^{-\lambda \eta_1} \right] \right)^{k-1} \mathbb{P}(Y(1) \geq z) \\
= 1 - \frac{\mathbb{P}(Y(1) \geq z)}{1 - \mathbb{E} \left[ 1_{\{Z_1 < z\}} e^{-\lambda \eta_1} \right]}, \]

i.e.,
\[
(3.3) \quad \lambda \int_0^\infty e^{-\lambda u} \mathbb{P}(\overline{Z}_{\nu_u} \geq z) \, du = \frac{\mathbb{P}(Y(1) \geq z)}{1 - \mathbb{E} \left[ 1_{\{Z_1 < z\}} e^{-\lambda \eta_1} \right]}. \]

But (recalling that \(Z_1 = Y(1)\))
\[
1 - \mathbb{E} \left[ 1_{\{Z_1 < z\}} e^{-\lambda \eta_1} \right] \leq 1 - \mathbb{E}(e^{-\lambda \eta_1}) + \mathbb{P}(Y(1) \geq z) 
\]
and (cf. [14], 3.466/1)

\[1 - E e^{-\lambda \eta} = 1 - \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\lambda (1 + x^2)} \, dx = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\lambda}} e^{-x^2} \, dx \leq 2 \sqrt{\lambda},\]

hence

\[
\lambda \int_0^\infty e^{-\lambda u} \mathbb{P}(Z_{\nu_u} \geq z) \, du \geq \frac{\mathbb{P}(Y(1) \geq z)}{2 \sqrt{\lambda} + \mathbb{P}(Y(1) \geq z)}.
\]

On the other hand, for any \(u_0 > 0\) we have

\[
\lambda \int_0^\infty e^{-\lambda u} \mathbb{P}(Z_{\nu_u} \geq z) \, du = \lambda \int_0^{u_0} e^{-\lambda u} \mathbb{P}(Z_{\nu_u} \geq z) \, du + \lambda \int_{u_0}^\infty e^{-\lambda u} \mathbb{P}(Z_{\nu_u} \geq z) \, du \leq \mathbb{P}(Z_{\nu_{u_0}} \geq z) + e^{-\lambda u_0}.
\]

It turns out that

\[
\mathbb{P}(Z_{\nu_{u_0}} \geq z) \geq \frac{\mathbb{P}(Y(1) \geq z)}{2 \sqrt{\lambda} + \mathbb{P}(Y(1) \geq z)} - e^{-\lambda u_0} \geq \min \left( \frac{1}{2}, \frac{\mathbb{P}(Y(1) \geq z)}{4 \sqrt{\lambda}} \right) - e^{-\lambda u_0},
\]

where the inequality

\[
\frac{x}{y + x} \geq \min \left( \frac{1}{2}, \frac{x}{2y} \right), \quad x > 0, \ y > 0
\]

was used. Choosing \(u_0 = T - 1, \lambda = z^2/u_0\), and applying (2.5) of Fact 2.1, we finally get

\[
(3.4) \quad \mathbb{P} \left( \sup_{0 \leq t < T} (Y(t + 1) - Y(t)) \geq z \right) \geq \min \left( \frac{1}{2}, \frac{c_8(\delta) \sqrt{T - 1}}{z} \exp \left( -z^2 \frac{8(1 - \delta)}{8(1 - \delta)} \right) \right) - \exp \left( -z^2 \right).
\]

This proves Lemma 3.2. \( \square \)

**Lemma 3.3.** For \(T \geq 2, \ 0 \leq \kappa < 1\) and \(\delta, z > 0\) we have

\[
(3.6) \quad \mathbb{P} \left( \sup_{0 \leq t < T} (Y(t + 1) - Y(t)) < z \right) \leq \frac{5}{T^{\kappa/2}} + \exp \left( -c_9 T^{(1 - \kappa)/2} e^{-(1 + \delta) z^2/8} \right)
\]

with some positive constant \(c_9 = c_9(\delta)\).

See Csáki et al. [7], Lemma 3.1.

**Lemma 3.4.** For \(T > 1, \ 0 < z \leq 1/2\) we have

\[
\mathbb{P} \left( \sup_{0 \leq t < T} \sup_{0 \leq s \leq 1} |Y(t + s) - Y(t)| < z \right) \geq \frac{c_{10}}{\sqrt{T}} \exp \left( -\frac{c_{11}}{z^2} \right)
\]

- 9 -
with some positive constants $c_{10}, c_{11}$.

**Proof.** Define the events

$$A := \left\{ \sup_{0 \leq s \leq 1} |Y(s)| < \frac{z}{4}, \ W(1) \geq \frac{4}{z}, \inf_{1 \leq u \leq T} W(u) \geq \frac{2}{z} \right\}$$

and

$$\tilde{A} := \left\{ \sup_{0 \leq t \leq T-1} \sup_{0 \leq s \leq 1} |Y(t+s) - Y(t)| < z \right\}.$$

Then $A \subset \tilde{A}$, since if $A$ occurs and $t < 1$, $t + s \leq 1$, then

$$|Y(t+s) - Y(t)| \leq 2 \sup_{0 \leq s \leq 1} |Y(s)| \leq \frac{z}{2} < z.$$

If $A$ occurs and $t < 1$, $s \leq 1$, $1 < t + s \leq T$, then

$$|Y(t+s) - Y(t)| \leq Y(t+s) - Y(1) + |Y(t) - Y(1)| \leq \int_1^{t+s} \frac{du}{W(u)} + \frac{z}{2} < z.$$

Moreover, if $A$ occurs and $1 \leq t$, $s \leq 1$, $t + s \leq T$, then

$$|Y(t+s) - Y(t)| = \int_t^{t+s} \frac{du}{W(u)} \leq \frac{z}{2} < z.$$

Hence $A \subset \tilde{A}$ as claimed. But by the Markov property of $W$,

$$(3.8) \quad \mathbb{P}(A) = \int_{1/z}^{\infty} \mathbb{P}\left( \sup_{0 \leq s \leq 1} |Y(s)| < \frac{z}{4} \bigg| W(1) = x \right) \mathbb{P}\left( \inf_{1 \leq u \leq T} W(u) \geq \frac{2}{z} \bigg| W(1) = x \right) \varphi(x) \, dx,$$

where $\varphi$ denotes the standard normal density function.

Using reflection principle and $x \geq 4/z$, $z \leq 1/2$, we get

$$(3.9) \quad \mathbb{P}\left( \inf_{1 \leq u \leq T} W(u) \geq \frac{2}{z} \bigg| W(1) = x \right) = 2\Phi\left( \frac{x-2/z}{\sqrt{T-1}} \right) - 1 \geq 2\Phi\left( \frac{4}{\sqrt{T}} \right) - 1 \geq \frac{c_{12}}{\sqrt{T}},$$

with some constant $c > 0$, where $\Phi(\cdot)$ is the standard normal distribution function. Hence

$$(3.10) \quad \mathbb{P}(\tilde{A}) \geq \mathbb{P}(A) \geq \frac{c_{12}}{\sqrt{T}} \mathbb{P}\left( \sup_{0 \leq s \leq 1} |Y(s)| \leq \frac{z}{4}, \ W(1) \geq \frac{4}{z} \right).$$

To get a lower bound of the probability on the right-hand side, define $g$, $(m(v), 0 \leq v \leq 1)$, $(B(u), 0 \leq u \leq 1)$ by (2.1), (2.2) and (2.3), respectively. Recall (see Fact 2.1) that these three objects are independent, $g$ has arc sine distribution, $m$ is a Brownian meander and $B$ is a Brownian
bridge. Moreover, \((g, m, B)\) are independent of \(\text{sgn}(W(1))\) which is a Bernoulli variable. Observe that

\[
\sup_{0 \leq s \leq g} |Y(s)| = \sqrt{g} \sup_{0 \leq s \leq 1} \left| \int_0^s \frac{du}{B(u)} \right|,
\]

\[
\sup_{g \leq s \leq 1} |Y(s)| = |Y(1) - Y(g)| = \sqrt{1 - g} \int_0^1 \frac{dv}{m(v)},
\]

\[
|W(1)| = \sqrt{1 - g} m(1).
\]

Then

\[
\mathbb{P} \left( \sup_{0 \leq s \leq 1} |Y(s)| \leq \frac{z}{4}, W(1) \geq \frac{4}{z} \right) \geq \mathbb{P} \left( \sup_{0 \leq s \leq g} |Y(s)| \leq \frac{z}{8}, Y(1) - Y(g) \leq \frac{z}{8}, W(1) \geq \frac{4}{z} \right)
\]

\[
\geq \mathbb{P} \left( \sqrt{g} \sup_{0 \leq s \leq 1} \left| \int_0^s \frac{du}{B(u)} \right| \leq \frac{z}{8}, \sqrt{1 - g} \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, \sqrt{1 - g} m(1) \geq \frac{4}{z}, W(1) > 0, g < z^2 \right)
\]

\[
\geq \mathbb{P} \left( \sup_{0 \leq s \leq 1} \left| \int_0^s \frac{du}{B(u)} \right| \leq \frac{1}{8}, \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, m(1) \geq \frac{4}{z \sqrt{1 - z^2}}, W(1) > 0, g < z^2 \right)
\]

\[
= c_{13} z \mathbb{P} \left( \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8}, m(1) \geq \frac{4}{z \sqrt{1 - z^2}} \right)
\]

\[
= c_{13} z \int_{4/(z \sqrt{1 - z^2})}^{\infty} \mathbb{P} \left( \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8} \right) m(1) = x \right) \mathbb{P}(m(1) \in dx).
\]

It follows from Facts 2.1 and 2.2 that for \(x > 0, z > 0\)

\[
(3.11) \quad \mathbb{P} \left( \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8} \right) m(1) = x \right) \geq \mathbb{P} \left( \int_0^1 \frac{dv}{m(v)} \leq \frac{z}{8} \right) m(1) = 0 \right) \geq \frac{c_{14}}{z^3} \exp \left( -\frac{c_{15}}{z^2} \right)
\]

and

\[
(3.12) \quad \mathbb{P} \left( m(1) > \frac{4}{z \sqrt{1 - z^2}} \right) = \exp \left( -\frac{8}{z^2 (1 - z^2)} \right).
\]

Putting (3.10), (3.11), (3.12) together, we get (3.7). \(\Box\)

**Lemma 3.5.** For \(T > 1, 0 < z \leq 1/2, 0 < \delta \leq 1/2\) we have

\[
\mathbb{P} \left( \inf_{0 \leq t \leq T - 1} \sup_{0 \leq s \leq 1} |Y(t + s) - Y(t)| < z \right)
\]

\[
\leq c_{16} \left( \exp \left( -\frac{(1 - \delta)^2}{2(1 + \delta)^2 z^2 T} \right) + \exp \left( -\frac{c_5 \delta}{4(1 + \delta)^2 z^2} \right) + \exp \left( c_{17} \frac{z^2}{z^2 T} e^{c_{19}/z^2} \right) \right)
\]

\[-11-\]
with some positive constants $c_{16}$, $c_{17} = c_{17}(\delta)$, $c_{18} = c_{18}(\delta)$, $c_{19} = c_{19}(\delta)$.

**Proof.** Consider a positive integer $N$ to be given later, $h = (T - 1)/N$, $t_k = kh$, $k = 0, 1, 2, \ldots, N$. Then for $0 < \delta \leq 1/2$ we have

$$
P \left( \inf_{0 \leq t \leq T - 1} \sup_{0 \leq s \leq 1} |Y(t + s) - Y(t)| < z \right) \leq P \left( \inf_{0 \leq k \leq N} \sup_{0 \leq s \leq 1} |Y(t_k + s) - Y(t_k)| \leq (1 + \delta)z \right) + P \left( \sup_{0 \leq t \leq T - 1} \sup_{0 \leq s \leq h} |Y(t + s) - Y(t)| > \delta z \right) =: P_1 + P_2.
$$

By scaling and Lemma 3.1

$$
P_2 = P \left( \sup_{0 \leq t \leq (T - 1)/h} \sup_{0 \leq s \leq 1} |Y(t + s) - Y(t)| > \frac{\delta z}{\sqrt{h}} \right) \leq c_6 \left( \sqrt{\frac{T - 1}{h}} + 1 \exp \left( -\frac{\delta^2 z^2}{8h(1 + \delta)} \right) + \frac{1}{h} \exp \left( -\frac{\delta^2 z^2}{2h(1 + \delta)} \right) \right) \leq 2c_6(N + 1) \exp \left( -\frac{\delta^2 z^2}{8h(1 + \delta)} \right).
$$

To bound $P_1$, we denote by $d(t) := \inf\{s \geq t : W(s) = 0\}$ the first zero of $W$ after $t$. Consider those $k$ for which $\sup_{0 \leq s \leq 1} |Y(t_k + s) - Y(t_k)| \leq (1 + \delta)z$. If, moreover, $d(t_k) \geq t_k + 1 - \delta$, which means that the Brownian motion $W$ does not change sign over $[t_k, t_k + 1 - \delta)$, then

$$(1 + \delta)z \geq |Y(t_k + 1 - \delta) - Y(t_k)| = \int_0^{1-\delta} \frac{ds}{|W(t_k + s)|} \geq \frac{1 - \delta}{\sup_{0 \leq s \leq T} |W(s)|},$$

and it follows that

$$P_1 \leq P \left( \sup_{0 \leq s \leq T} |W(s)| > \frac{(1 - \delta)}{z(1 + \delta)} \right) + P \left( \exists k \leq N : \sup_{0 \leq s \leq 1} |Y(t_k + s) - Y(t_k)| \leq (1 + \delta)z; d(t_k) < t_k + 1 - \delta \right) \leq 4 \exp \left( -\frac{(1 - \delta)^2}{2(1 + \delta)^2 z^2 T} \right) + \sum_{k=0}^{N} P \left( \sup_{0 \leq s \leq 1} |Y(t_k + s) - Y(t_k)| \leq (1 + \delta)z; d(t_k) < t_k + 1 - \delta \right).
$$

Let $\hat{W}(s) = W(d(t_k) + s)$ for $s \geq 0$ and $\hat{Y}(s)$ be the associated principal values. Observe that on $\{\sup_{0 \leq s \leq 1} |Y(t_k + s) - Y(t_k)| \leq (1 + \delta)z; d(t_k) < t_k + 1 - \delta\}$, we have $\sup_{0 \leq u \leq \delta} |\hat{Y}(u) + (Y(d(t_k)) - Y(t_k))| < (1 + \delta)z$, and $|Y(d(t_k)) - Y(t_k)| \leq (1 + \delta)z$ which implies that

$$\sup_{0 \leq u \leq \delta} |\hat{Y}(u)| < 2(1 + \delta)z.$$
By scaling and Fact 2.3 we have
\[
P \left( \sup_{0 \leq u \leq \delta} |\hat{Y}(u)| < 2(1 + \delta)z \right) \leq c_4 \exp \left( -\frac{c_5\delta}{4(1 + \delta)^2 z^2} \right).
\]
Therefore, we obtain:
\[
P_1 \leq 4 \exp \left( -\frac{(1 - \delta)^2}{2(1 + \delta)^2 z^2 T} \right) + c_4 (N + 1) \exp \left( -\frac{c_5\delta}{4(1 + \delta)^2 z^2} \right).
\]
Hence
\[
P_1 + P_2 \leq 4 \exp \left( -\frac{(1 - \delta)^2}{2(1 + \delta)^2 z^2 T} \right) + c_4 (N + 1) \exp \left( -\frac{c_5\delta}{4(1 + \delta)^2 z^2} \right) + 2c_6 (N + 1) \exp \left( -\frac{\delta^2 z^2}{8h(1 + \delta)} \right).
\]
By taking \( N = \lceil e^{c_5\delta/(4(1 + \delta)^2 z^2)} \rceil + 1 \), we get
\[
P_1 + P_2 \leq c_{16} \left( \exp \left( -\frac{(1 - \delta)^2}{2(1 + \delta)^2 z^2 T} \right) + \exp \left( -\frac{c_5\delta}{4(1 + \delta)^2 z^2} \right) + \exp \left( \frac{c_{17}}{z^2} - \frac{c_{18} z^2}{T} e^{c_{19}/z^2} \right) \right)
\]
with relevant constants \( c_{16}, c_{17}, c_{18}, c_{19} \), proving (3.13).

\[\square\]

4. Proof of Theorem 1.1(i)

The upper estimation, i.e.
\[
\limsup_{T \to \infty} \frac{\sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{8a_T \left( \log \sqrt{T/a_T} + \log \log T \right)}} \leq 1, \quad \text{a.s.}
\]
follows easily from Wen's Theorem E.

Now we prove the lower bound, i.e.
\[
\liminf_{T \to \infty} \frac{\sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{8a_T \left( \log \sqrt{T/a_T} + \log \log T \right)}} \geq 1, \quad \text{a.s.}
\]
In the case when \( a_T = T \), (4.2) follows from the law of the iterated logarithm (1.3) of Theorem A. Now we assume that \( a_T/T \leq \rho < 1 \), with some constant \( \rho \) for all \( T > 0 \).

By scaling, (3.2) of Lemma 3.2 is equivalent to
\[
P \left( \sup_{0 \leq t \leq T - a_T} (Y(t + a) - Y(t)) \geq z\sqrt{a} \right)
\]
\[
\geq \min \left( \frac{1}{2}, \frac{c_7 \sqrt{T/a - 1}}{z} \exp \left( -\frac{2}{8(1 - \delta)} \right) \right) - \exp \left( -z^2 \right)
\]

- 13 -
for $0 < a < T$, $0 < \delta < 1/2$, $z > 1$.

Define the sequences

\begin{equation}
 t_k := e^{7k \log k}, \quad k = 1, 2, \ldots
\end{equation}

and $\theta_0 := 0$,

\begin{equation}
 \theta_k := \inf\{t > T_k : W(t) = 0\}, \quad k = 1, 2, \ldots
\end{equation}

where $T_k := \theta_{k-1} + t_k$. For $0 < \delta < \min(1/2, 1 - \rho)$ define the events

\[ A_k := \left\{ \sup_{0 \leq t \leq t_k(1-\delta) - a t_k} (Y(\theta_{k-1} + t + a t_k) - Y(\theta_{k-1} + t)) \geq (1-\delta)\beta_k \right\}, \quad k = 1, 2, \ldots \]

with

\[ \beta_k := \sqrt{8 a t_k \left( \log \frac{t_k}{a t_k} + \log \log t_k \right)} . \]

Applying (4.3) with $T = t_k(1 - \delta)$, $a = a t_k$, $z = (1 - \delta)\sqrt{8(\log \sqrt{t_k/a t_k} + \log \log t_k)}$, we have for $k$ large

\[ \mathbb{P}(A_k) = \mathbb{P} \left( \sup_{0 \leq t \leq t_k(1-\delta) - a t_k} (Y(t + a t_k) - Y(t)) \geq (1-\delta)\beta_k \right) \]

\[ \geq \min \left\{ \frac{1}{2}, \frac{b_k}{(\log t_k)^{1-\delta}} \right\} - \frac{1}{(\log t_k)^{8(1-\delta)^2}} \]

with

\[ b_k = \frac{c_7 \sqrt{t_k(1-\delta)/a t_k - 1}}{(t_k/a t_k)^{(1-\delta)/2} \sqrt{\log \sqrt{t_k/a t_k} + \log \log t_k}} \geq \frac{c_{20}}{\sqrt{\log k}} . \]

Hence $\sum_k \mathbb{P}(A_k) = \infty$ and since $A_k$ are independent, Borel-Cantelli lemma yields

\[ \mathbb{P}(A_k \text{ i.o.}) = 1. \]

It follows that

\begin{equation}
 \limsup_{k \to \infty} \frac{\sup_{0 \leq t \leq t_k(1-\delta) - a t_k} (Y(\theta_{k-1} + t + a t_k) - Y(\theta_{k-1} + t))}{\sqrt{8 a t_k \left( \log \frac{t_k}{a t_k} + \log \log t_k \right)}} \geq 1 - \delta, \quad \text{a.s.}
\end{equation}

It can be seen (cf. [9]) that we have almost surely for large enough $k$ \[ t_k \leq T_k \leq t_k \left( 1 + \frac{1}{k} \right), \]
consequently

\[ \lim_{k \to \infty} \frac{t_k}{T_k} = 1, \quad \text{a.s.} \]

Since by our assumptions

\[ \frac{t_k}{T_k} \leq \frac{a_{t_k}}{a_{T_k}} \leq 1, \]

we have also

\[ \lim_{k \to \infty} \frac{a_{t_k}}{a_{T_k}} = 1, \quad \text{a.s.} \]

On the other hand, for any \( \delta > 0 \) small enough we have almost surely for large \( k \)

\[ a_{T_k} \leq (1 + \delta)a_{t_k} \leq t_k\delta + a_{t_k}, \]

thus

\[ T_k - a_{T_k} \geq T_k - t_k\delta - a_{t_k}, \]

consequently

\[ \sup_{0 \leq t \leq T_k - a_{T_k}} \sup_{0 \leq s \leq a_{T_k}} |Y(t + s) - Y(t)| \geq \sup_{0 \leq t \leq t_k(1-\delta) - a_{t_k}} (Y(\theta_{k-1} + t + a_{t_k}) - Y(\theta_{k-1} + t)), \]

hence we have also

\[ \limsup_{k \to \infty} \frac{\sup_{0 \leq t \leq T_k - a_{T_k}} \sup_{0 \leq s \leq a_{T_k}} |Y(t + s) - Y(t)|}{\sqrt{8a_{t_k} \left( \log \sqrt{\frac{t_k}{a_{t_k}}} + \log \log t_k \right)}} \geq 1 - \delta, \quad \text{a.s.} \]

and since \( \delta > 0 \) can be arbitrary small, (4.2) follows by combining (4.7), (4.8), (4.9) and (4.10). \( \Box \)

5. Proof of Theorem 1.1(ii)

First assume that

\[ a_T > \frac{T}{(\log T)^\alpha} \quad \text{for some} \quad \alpha < 2. \]

By Theorem C,

\[ \liminf_{T \to \infty} \sqrt{\log \log \frac{T}{a_T}} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| \]

\[ \geq \liminf_{T \to \infty} \sqrt{\log \log a_T} \sup_{0 \leq s \leq a_T} |Y(s)| \geq K_1, \quad \text{a.s.} \]
proving the lower bound in (1.12).

To get an upper bound, note that by scaling, (3.7) of Lemma 3.4 is equivalent to

\[
P \left( \sup_{0 \leq t \leq T-a} \sup_{0 \leq s \leq a} |Y(s + t) - Y(t)| < z \sqrt{a} \right) \geq c_{10} \sqrt{\frac{a}{T}} \exp \left( -\frac{c_{11}}{z^2} \right)
\]

for \( T \geq a, \ 0 < z \leq 1/2. \)

Let \( t_k \) and \( \theta_k \) be defined by (4.4) and (4.5), resp., as in the proof of Theorem 1.1(i) and for any \( \varepsilon > 0 \) and for \( \delta > 0 \) such that \( \alpha/2 + c_{11}/\delta^2 < 1 \), define the events

\[
E_k := \left\{ \sup_{0 \leq t \leq (1+\varepsilon)t_k - a_{t_k}(1+\varepsilon)} \sup_{0 \leq s \leq a_{t_k}(1+\varepsilon)} |Y(\theta_k^{-1} + t + s) - Y(\theta_k^{-1} + t)| \leq \delta \sqrt{\frac{a_{t_k}}{\log \log t_k}} \right\}.
\]

Then putting \( T = (1 + \varepsilon)t_k, \ a = a_{(1+\varepsilon)t_k}, \ z = \delta/\sqrt{\log \log t_k}, \) into (5.3), we get

\[
P(E_k) = P \left( \sup_{0 \leq t \leq (1+\varepsilon)t_k - a_{t_k}(1+\varepsilon)} \sup_{0 \leq s \leq a_{t_k}(1+\varepsilon)} |Y(t + s) - Y(t)| \leq \delta \sqrt{\frac{a_{t_k}}{\log \log t_k}} \right) \geq c_{10} \sqrt{\frac{a_{t_k}}{t_k}} \exp(- (c_{11}/\delta^2) \log \log((1 + \varepsilon)t_k)) \geq \frac{c_{10}}{(\log t_k)^{\alpha/2 + c_{11}/\delta^2}},
\]

hence \( \sum_k P(E_k) = \infty \), and since \( E_k \) are independent, we have \( P(E_k \text{ i.o.}) = 1 \), i.e.

\[
\liminf_{k \to \infty} \sqrt{\frac{\log \log t_k}{a_{t_k}}} \sup_{0 \leq t \leq (1+\varepsilon)t_k - a_{t_k}(1+\varepsilon)} \sup_{0 \leq s \leq a_{t_k}(1+\varepsilon)} |Y(\theta_k^{-1} + t + s) - Y(\theta_k^{-1} + t)| \leq \delta, \quad \text{a.s.}
\]

for any \( \varepsilon \). Put, as before, \( T_k = \theta_k^{-1} + t_k. \) For large enough \( k \) by (4.7) and (4.8) we have \( a_{T_k} \leq (1 + \varepsilon)a_{t_k}, \ a.s. \) and \( T_k - a_{T_k} \leq \theta_k^{-1} + (1 + \varepsilon)t_k - (1 + \varepsilon)a_{t_k}, \ a.s. \) Thus given any \( \varepsilon > 0 \), we have for large \( k \)

\[
\sup_{0 \leq t \leq T_k - a_{T_k}} \sup_{0 \leq s \leq a_{T_k}} |Y(t + s) - Y(t)| \leq 2 \sup_{0 \leq t \leq \theta_k^{-1}} |Y(t)| + \sup_{0 \leq t \leq (1+\varepsilon)t_k - a_{t_k}(1+\varepsilon)} \sup_{0 \leq s \leq a_{t_k}(1+\varepsilon)} |Y(\theta_k^{-1} + t + s) - Y(\theta_k^{-1} + t)|.
\]

By Theorem A, Fact 2.8, (4.7), (5.1) and simple calculation,

\[
\sup_{0 \leq t \leq \theta_k^{-1}} |Y(t)| = O(\theta_k^{-1} \log \log \theta_k^{-1})^{1/2}
\]

\[
= O(t_k^{-1}(\log t_k^{-1})^3 \log \log t_k^{-1})^{1/2} = o \left( \frac{a_{t_k}}{\log \log t_k} \right)^{1/2}, \quad \text{a.s.}
\]

as \( k \to \infty. \) Assembling (5.4), (5.5) and (5.6), we get

\[
\liminf_{k \to \infty} \sqrt{\frac{\log \log t_k}{a_{t_k}}} \sup_{0 \leq t \leq T_k - a_{T_k}} \sup_{0 \leq s \leq a_{T_k}} |Y(t + s) - Y(t)|
\]
\[
\liminf_{k \to \infty} \sqrt{\frac{\log \log T_k}{aT_k}} \sup_{0 \leq t \leq T_k - aT_k} \sup_{0 \leq s \leq aT_k} |Y(t + s) - Y(t)| \leq \delta, \quad \text{a.s.}
\]

which together with (5.2) yields (1.12).

Now assume that

(5.7) \[ a_T \leq \frac{T}{(\log T)^\alpha} \quad \text{for some} \quad \alpha > 2. \]

By Theorem 1.1(i),

\[
\begin{align*}
\liminf_{T \to \infty} & \frac{\sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} \\
& \leq \limsup_{T \to \infty} \frac{\sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} \\
& \leq \limsup_{T \to \infty} \frac{\sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{\frac{2a_T}{\alpha + 2} \left( \log \sqrt{T/a_T} + \log \log T \right)}} \leq 2 \sqrt{\frac{\alpha + 2}{\alpha}} ,
\end{align*}
\]

i.e., an upper bound in (1.13) follows.

To get a lower bound under (5.7), observe that by scaling, (3.6) of Lemma 3.3 is equivalent to

\[
\mathbb{P} \left( \sup_{0 \leq t \leq T - a} (Y(t + a) - Y(t)) < z\sqrt{a} \right) \leq 5 \left( \frac{a}{T} \right)^{\kappa/2} + \exp \left( -c_9 \left( \frac{T}{a} \right)^{(1-\kappa)/2} e^{-(1+\delta)z^2/8} \right)
\]

for \( a \leq T, \ 0 \leq \kappa < 1, \ 0 < \delta, \ 0 < z. \) Using (5.7) we get further

\[
\begin{align*}
\mathbb{P} \left( \sup_{0 \leq t \leq T - a} (Y(t + a) - Y(t)) < z\sqrt{a} \right) \\
& \leq \frac{5}{(\log T)^{\alpha \kappa/2}} + \exp \left( -c_9 \left( \log T \right)^{\alpha(1-\kappa)/2} e^{-(1+\delta)z^2/8} \right).
\end{align*}
\]

In the case when (1.7) holds, (1.13) was proved in [7]. In other cases the proof is similar. Let \( T_k = e^k \) and define the events

\[
F_k = \left\{ \sup_{0 \leq t \leq T_k - aT_k} (Y(t + aT_k) - Y(t)) \leq C_1 \sqrt{aT_k \log \frac{T_k}{aT_k}} \right\}
\]

with some constant \( C_1 \) to be given later. By (5.9)

\[
\mathbb{P}(F_k) \leq \frac{5}{k^{\alpha \kappa/2}} + \exp \left( -c_9 k^{\alpha(1-\kappa)/2 - (1+\delta)C_1^2/8} \right).
\]

For given \( \alpha > 2 \), choose small \( \varepsilon > 0, \ \kappa = 2/\alpha + \varepsilon, \)

\[
C_1 = 2 \sqrt{\frac{\alpha - 2 - 2\varepsilon(1+\alpha)}{(1+\varepsilon)\alpha}}.
\]
One can easily see that with these choices \( \sum_k \mathbb{P}(F_k) < \infty \), consequently

\[
\liminf_{k \to \infty} \frac{\sup_{0 \leq t \leq T_k - a_T} (Y(t + a_T) - Y(t))}{\sqrt{a_T k \log \frac{T_k}{a_T}}} \geq C_1, \quad \text{a.s.,}
\]

implying also

\[
\liminf_{k \to \infty} \frac{\sup_{0 \leq t \leq T_k - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)|}{\sqrt{a_T k \log \frac{T_k}{a_T}}} \geq 2 \sqrt{\frac{\alpha - 2}{\alpha}}, \quad \text{a.s.,}
\]

for \( \epsilon \) can be chosen arbitrary small.

Since \( \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| \) is increasing in \( T \), we obtain a lower bound in (1.13). This together with the 0-1 law for Brownian motion complete the proof of Theorem 1.1(ii).

\[\square\]

6. Proof of Theorem 1.2(i)

If \( a_T = T \), then (1.14) is equivalent to Theorem C. Now assume that \( \rho := \lim_{T \to \infty} a_T/T < 1 \).

First we prove the lower bound, i.e.

\[
(6.1) \quad \liminf_{T \to \infty} \frac{\sqrt{T \log \log T}}{a_T} \inf_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| \geq c, \quad \text{a.s.}
\]

By scaling, (3.13) of Lemma 3.5 is equivalent to

\[
(6.2) \quad \mathbb{P} \left( \inf_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| < z \right) \leq c_{16} \left( \exp \left( -\frac{a(1 - \delta)^2}{2(1 + \delta)^2 z^2 T} \right) + \exp \left( -\frac{c_5 \delta}{4(1 + \delta)^2 z^2} \right) + \exp \left( \frac{c_{17}}{z^2} - \frac{c_{18} a z^2}{T e^{c_{19}/z^2}} \right) \right)
\]

for \( a < T \), \( 0 \leq z \leq 1/2 \), \( 0 < \delta \leq 1/2 \).

Define the events

\[
G_k = \left\{ \inf_{0 \leq t \leq T_{k+1} - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| < z_k \right\} \quad k = 1, 2, \ldots
\]

Let \( T_k = e^k \) and put \( T = T_{k+1} \), \( a = a_T \),

\[
z = z_k = C_2 \sqrt{\frac{a_T k \log \log T_k}{T_{k+1} \log \log T_{k+1}}}
\]

into (6.2). The constant \( C_2 \) will be chosen later. Denoting the terms on the right-hand side of (6.2) by \( I_1, I_2, I_3 \), resp., we have

\[
\mathbb{P}(G_k) \leq c_{16}(I_1^{(k)} + I_2^{(k)} + I_3^{(k)}),
\]
where

\[ I_1^{(k)} = \exp \left( -\frac{c_{21}}{C_2^2} \log \log T_{k+1} \right), \]

\[ I_2^{(k)} = \exp \left( -\frac{c_{22} T_k}{C_2^2 a T_k} \log T_{k+1} \right), \]

\[ I_3^{(k)} = \exp \left( \frac{c_{23} T_k \log \log T_{k+1}}{C_2^2 a T_k} - \frac{c_{24} C_2^2 a T_k}{T_k^2 \log \log T_{k+1}} (\log T_{k+1} \frac{c_{25} T_k}{C_2^2 a T_k}) \right) \]

with some constants \( c_{21} = c_{21}(\delta) \), \( c_{22} = c_{22}(\delta) \), \( c_{23}, c_{24}, c_{25} \).

One can see easily that for any choice of positive \( C_2 \) and for all possible \( a T \) (satisfying our conditions) we have \( \sum_k I_3^{(k)} < \infty \). So we show that for appropriate choice of \( C_2 \) we have also \( \sum_k I_j^{(k)} < \infty \), \( j = 1, 2 \).

First consider the case \( 0 < \rho > 0 \). Choosing a positive \( \delta \) one can select \( C_2 < \min(\sqrt{c_{21}}, \sqrt{c_{22}} \rho) \) and it is easy to verify that \( \sum_k I_j^{(k)} < \infty \), \( j = 1, 2 \), hence also \( \sum_k \mathbb{P}(G_k) < \infty \).

In the case \( \rho = 0 \) choose \( C_2 < (1 - \delta)/(1 + \delta) \sqrt{2} \). With this choice we have \( \sum_k I_1^{(k)} < \infty \) for arbitrary \( \delta > 0 \). Since \( \lim_{k \to \infty} (T_k/a T_k) = \infty \), we have also \( \sum_k I_2^{(k)} < \infty \) and \( \sum_k \mathbb{P}(G_k) < \infty \). Borell-Cantelli lemma and interpolation between \( T_k \)'s finish the proof of (6.1). We have also verified that in the case \( \rho = 0 \) one can choose \( C_2 = 1/\sqrt{2} \), since \( \delta \) can be chosen arbitrary small.

Now we turn to the proof of the upper bound, i.e.

\[
\liminf_{T \to \infty} \frac{\sqrt{T \log \log T}}{a T} \inf_{0 \leq t \leq T-a T} \sup_{0 \leq s \leq a T} |Y(t + s) - Y(t)| \leq C_3 \quad \text{a.s.}
\]

with some constant \( C_3 \).

If \( \rho > 0 \), then

\[
\inf_{0 \leq t \leq T-a T} \sup_{0 \leq s \leq a T} |Y(t + s) - Y(t)| \leq \sup_{0 \leq s \leq a T} |Y(s)| \leq \sup_{0 \leq s \leq T} |Y(s)|
\]

and hence (6.3) with some positive constant \( C_3 \) follows from Theorem C.

If \( \rho = 0 \), then let for any \( \varepsilon > 0 \)

\[
\lambda_T := \inf \{ t : |W(t)| = \sup_{0 \leq s \leq T(1-\varepsilon)} |W(s)| \}.
\]

According to the law of the iterated logarithm, with probability one there exists a sequence \( \{ T_i, i \geq 1 \} \) such that \( \lim_{i \to \infty} T_i = \infty \) and

\[
|W(\lambda_{T_i})| \geq \sqrt{2 T_i(1-\varepsilon) \log \log T_i}.
\]
But Fact 2.4 implies that for $\varepsilon > 0$

$$|W(\lambda T_i) - W(s)| \leq \sqrt{2(1 + \varepsilon)|\varepsilon|T_i \log \log T_i}, \quad \lambda T_i \leq s \leq \lambda T_i + \varepsilon T_i, \quad i \geq 1. \quad (6.6)$$

Now assume that $W(\lambda T_i) > 0$. The case when $W(\lambda T_i) < 0$ is similar. Then (6.5) and (6.6) imply

$$W(s) \geq (\sqrt{1 - \varepsilon} - \sqrt{\varepsilon(1 + \varepsilon)}) \sqrt{2T_i \log \log T_i}, \quad \lambda T_i \leq s \leq \lambda T_i + \varepsilon T_i. \quad (6.7)$$

$\rho = 0$ implies that $a_T \leq \varepsilon T$ for any $\varepsilon > 0$ and large enough $T$, hence we have from (6.7) for large $i$

$$\sup_{0 \leq s \leq a_T} (Y(\lambda T_i + s) - Y(\lambda T_i)) = Y(\lambda T_i + a_T_i) - Y(\lambda T_i) = \int_{\lambda T_i}^{\lambda T_i + a_T_i} \frac{ds}{W(s)} \leq \frac{\sqrt{2T_i \log \log T_i}}{\sqrt{1 - \varepsilon} - \sqrt{\varepsilon(1 + \varepsilon)}}.$$

Since $\varepsilon > 0$ is arbitrary, (6.3) follows with $C_3 = 1/\sqrt{2}$. This completes the proof of Theorem 1.2(i).

\[ \square \]

7. Proof of Theorem 1.2(ii)

If $\rho = 1$, then (1.15) is equivalent to (1.3) of Theorem A. So we may assume that $0 < \rho < 1$.

First we prove the upper bound

$$\limsup_{T \to \infty} \inf_{0 \leq t \leq T - \rho T} \sup_{0 \leq s \leq \rho T} |Y(t + s) - Y(t)| \leq \rho, \quad \text{a.s.} \quad (7.1)$$

Let $k$ be the largest integer for which $k \rho < 1$ and put $x_i = i \rho$, $i = 0, 1, \ldots, k$, $x_{k+1} = 1$. It suffices to show that if $f \in \mathcal{S}$ defined by (1.5), then

$$\min_{1 \leq i \leq k+1} |f(x_i) - f(x_{i-1})| \leq \rho.$$

Assume on the contrary that

$$|f(x_i) - f(x_{i-1})| > \rho, \quad \forall i = 1, 2, \ldots, k + 1.$$

Then

$$\sum_{i=1}^{k+1} \frac{(f(x_i) - f(x_{i-1}))^2}{x_i - x_{i-1}} > \sum_{i=1}^{k} \frac{\rho^2}{\rho} + \frac{\rho^2}{1 - k\rho} = k\rho + \frac{\rho^2}{1 - k\rho} \geq 1,$$

contradicting (2.12) of Fact 2.5. This proves (7.1).
The lower bound

\[
(7.2) \quad \limsup_{T \to \infty} \inf_{0 \leq t \leq T - \rho T} \sup_{0 \leq s \leq \rho T} \frac{|Y(t + s) - Y(t)|}{\sqrt{8T \log \log T}} \geq \rho, \quad \text{a.s.}
\]

follows from the fact that by Theorem B the function \( f(x) = x, \ 0 \leq x \leq 1 \) is a limit point of

\[
\frac{Y(xt)}{\sqrt{8T \log \log T}}
\]

and for this function

\[
\min_{0 \leq x \leq 1 - \rho} |f(x + \rho) - f(x)| = \rho.
\]

This completes the proof of Theorem 1.2(iia). \( \square \)

Now assume that

\[
(7.3) \quad \lim_{T \to \infty} \frac{a_T \log \log T^2}{T} = 0.
\]

Define \( \lambda_T \) as in (6.4). Then according to Chung’s LIL (cf. Fact 2.6)

\[
(7.4) \quad |W(\lambda_T)| \geq \frac{\pi}{\sqrt{8}} (1 - \varepsilon) \sqrt{\frac{T}{\log \log T}}
\]

for every \( T \) sufficiently large. But according to Fact 2.4,

\[
\sup_{0 \leq s \leq a_T} |W(\lambda_T + s) - W(\lambda_T)|
\]

\[
\leq \sqrt{(2 + \varepsilon)a_T (\log(T/a_T) + \log \log T)} \leq \sqrt{\frac{(2 + \varepsilon)\varepsilon T}{\log \log T}}.
\]

Assuming \( W(\lambda_T) > 0 \), we get

\[
W(\lambda_T + s) \geq W(\lambda_T) - \sqrt{\frac{(2 + \varepsilon)\varepsilon T}{\log \log T}} \geq c \sqrt{\frac{T}{\log \log T}}.
\]

Hence

\[
\inf_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |Y(t + s) - Y(t)| \leq Y(\lambda_T + a_T) - Y(\lambda_T)
\]

\[
= \int_{\lambda_T}^{\lambda_T + a_T} \frac{ds}{W(\lambda_T + s)} \leq \frac{a_T}{c} \sqrt{\frac{\log \log T}{T}}
\]

for all large \( T \).

The case when \( W(\lambda_T) < 0 \) is similar. This shows the upper bound in (1.16).
For the lower bound we use Fact 2.6: with probability one

\begin{equation}
(7.5) \quad g_T \leq \frac{T}{(\log \log T)^2}, \quad \max_{0 \leq u \leq T} |W(u)| \leq \frac{\pi}{\sqrt{2}} \sqrt{\frac{T}{\log \log T}} \text{ i.o.}
\end{equation}

According to Theorem 1.2(i) for every large $T$ we have for any $\varepsilon > 0$ and sufficiently large $T$

\begin{equation}
(7.6) \quad \inf_{0 \leq t \leq T} \sup_{0 \leq s \leq aT} |Y(t + s) - Y(t)| \geq \frac{(K_4 - \varepsilon)a_T}{\sqrt{\left(\frac{T}{(\log \log T)^2} + a_T\right) \log \log T}} \leq \frac{(K_4 - \varepsilon)a_T}{\sqrt{(1 + \varepsilon)T \log \log T}}.
\end{equation}

On the other hand, if $T(\log \log T)^{-2} \leq t \leq T - aT$, then by (7.5)

\[ |Y(t + aT) - Y(t)| = \int_t^{t + aT} |W(s)| \geq \frac{a_T \sqrt{\log \log T}}{\pi \sqrt{T}}. \]

Combining (7.6) and (7.7) we get for $\varepsilon > 0$ and all large $T$

\[ \inf_{0 \leq t \leq T - aT} \sup_{0 \leq s \leq aT} |Y(t + s) - Y(t)| \geq \min\left(\frac{K_4 - \varepsilon}{\sqrt{1 + \varepsilon}}, \frac{\sqrt{2}}{\pi}\right) \frac{a_T \sqrt{\log \log T}}{T}. \]

This shows the lower bound in (1.16). The proof of Theorem 1.2(iib) is complete by applying the 0-1 law for Brownian motion. \qed

**Acknowledgements**

The authors are indebted to Marc Yor for useful remarks. Cooperation between the authors was supported by the joint French–Hungarian Intergovernmental Grant ”Balaton” (grant no. F-39/00).

**References**

[1] Ait Ouahra, M. and Eddahbi, M.: Théorèmes limites pour certaines fonctionnelles associées aux processus stables sur l’espace de Hölder. *Publ. Mat.* 45 (2001), 371–386.

[2] Bertoin, J.: On the Hilbert transform of the local times of a Lévy process. *Bull. Sci. Math.* 119 (1995), 147–156.

[3] Bertoin, J.: Cauchy’s principal value of local times of Lévy processes with no negative jumps via continuous branching processes. *Electronic J. Probab.* 2 (1997), Paper No. 6, 1–12.

[4] Biane, P. and Yor, M.: Valeurs principales associées aux temps locaux browniens. *Bull. Sci. Math.* 111 (1987), 23–101.
[5] Boufoussi, B., Eddahbi, M. and Kamont, A.: Sur la dérivée fractionnaire du temps local brownien. *Probab. Math. Statist.* **17** (1997), 311–319.

[6] Chung, K.L.: On the maximum partial sums of sequences of independent random variables. *Trans. Amer. Math. Soc.* **64** (1948), 205–233.

[7] Csáki, E., Csörgő, M. Földes, A. and Shi, Z.: Increment sizes of the principal value of Brownian local time. *Probab. Th. Rel. Fields* **117** (2000), 515–531.

[8] Csáki, E., Csörgő, M. Földes, A. and Shi, Z.: Path properties of Cauchy’s principal values related to local time. *Studia Sci. Math. Hungar.* **38** (2001), 149–169.

[9] Csáki, E. and Földes, A.: A note on the stability of the local time of a Wiener process. *Stoch. Process. Appl.* **25** (1987), 203–213.

[10] Csáki, E., Földes, A. and Shi, Z.: A joint functional law for the Wiener process and principal value. *Studia Sci. Math. Hungar.* **40** (2003), 213–241.

[11] Csáki, E., Shi, Z. and Yor, M.: Fractional Brownian motions as “higher-order” fractional derivatives of Brownian local times. In: *Limit Theorems in Probability and Statistics* (I. Berkes et al., eds.) Vol. I, pp. 365–387. János Bolyai Mathematical Society, Budapest, 2002.

[12] Csörgő, M. and Révész, P.: *Strong Approximations in Probability and Statistics.* Academic Press, New York, 1981.

[13] Fitzsimmons, P.J. and Getoor, R.K.: On the distribution of the Hilbert transform of the local time of a symmetric Lévy process. *Ann. Probab.* **20** (1992), 1484–1497.

[14] Gradshteyn, I.S. and Ryzhik, I.M.: *Table of Integrals, Series, and Products.* Sixth ed. Academic Press, San Diego, CA, 2000.

[15] Grill, K.: On the last zero of a Wiener process. In: *Mathematical Statistics and Probability Theory* (M.L. Puri et al., eds.) Vol. A, pp. 99–104. D. Reidel, Dordrecht, 1987.

[16] Hu, Y.: The laws of Chung and Hirsch for Cauchy’s principal values related to Brownian local times. *Electronic J. Probab.* **5** (2000), Paper No. 10, 1–16.

[17] Hu, Y. and Shi, Z.: An iterated logarithm law for Cauchy’s principal value of Brownian local times. In: *Exponential Functionals and Principal Values Related to Brownian Motion* (M. Yor, ed.), pp. 131–154. Biblioteca de la Revista Matemática Iberoamericana, Madrid, 1997.

[18] Strassen, V.: An invariance principle for the law of the iterated logarithm. *Z. Wahrsch. verw. Gebiete* **3** (1964), 211–226.

[19] Wen, Jiwei: Some results on lag increments of the principal value of Brownian local time. *Appl. Math. J. Chinese Univ. Ser. B* **17** (2002), 199–207.

[20] Yamada, T.: Principal values of Brownian local times and their related topics. In: *Itô’s Stochastic Calculus and Probability Theory* (N. Ikeda et al., eds.), pp. 413–422. Springer, Tokyo, 1996.

[21] Yor, M.: *Some Aspects of Brownian Motion. Part 1: Some Special Functionals.* ETH Zürich Lectures in Mathematics. Birkhäuser, Basel, 1992.

[22] Yor, M., editor: *Exponential Functionals and Principal Values Related to Brownian Motion.* Biblioteca de la Revista Matemática Iberoamericana, Madrid, 1997.