Coupling of Brans-Dicke scalar field with Horava-Lifshitz Gravity

Joohan Lee
Department of Physics, University of Seoul, Seoul 130-743 Korea

Tae Hoon Lee
Department of Physics and Institute of Natural Sciences, Soongsil University, Seoul 156-743 Korea

Phillial Oh
Department of Physics and Institute of Basic Science, Sungkyunkwan University, Suwon 440-746 Korea

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We look for a Brans-Dicke type of generalization of Horava-Lifshitz gravity. It is shown that such a generalization is possible within the detailed balance condition. The resulting theory reduces in the IR limit to the usual Brans-Dicke theory with a negative cosmological constant for certain values of parameters. We then consider homogeneous and isotropic cosmological situation in the context of this generalized theory, and find some interesting features of the Brans-Dicke scalar field in determining the behavior of the universe.

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Recently, a new theory of gravity has been proposed by Horava[1–3]. This theory, being based on anisotropic scaling of space and time, breaks the spacetime symmetry. It has a much better UV behavior than the theories with spacetime diffeomorphism symmetry such as general relativity, but reduces to Einstein’s gravity in the infrared limit, thereby recovering spacetime diffeomorphism symmetry. Physical constants such as the speed of light, Newton’s constant, and cosmological constants all emerge from the relevant deformation of the non-relativistic theory at short distance. These interesting features as well as related findings have received a great deal of attention[4].

On the other hand, even if we consider only the low energy limit of the gravity there are many alternative theories and extensions of the Einstein theory. In particular, in the context of cosmology various models with a scalar field have been considered and a possible role of the scalar field in explaining the behavior of the universe in the early inflationary stage as well as the late stage has been investigated[5, 6]. Therefore, it would be interesting to see if Horava’s theory can be extended in such a way that in the infrared limit it reduces to those alternative theories. In this regard, of particular interest for us is the one with a non-minimally coupled scalar field because minimally coupled scalar source had already been investigated[7, 8]. Typical examples would be the Brans-Dicke theory[8] and the the gravity with a dilaton field[9] arising for instance in the string theory.

In this paper, we extend the Horava-Lifshitz gravity to include the Brans-Dicke field as a concrete example of the non-minimally coupled scalar field. It turns out that such an extension is possible within the context of the detailed balance condition and in the IR limit reduces to the four-dimensional Brans-Dicke theory with negative cosmological constant when parameters of the theory satisfy certain conditions.

We then study cosmological implication of the theory assuming homogeneity and isotropy. Without the negative cosmological constant and the dark radiation term[10] the equations are those of the Brans-Dicke theory. So, we concentrate only on their effects on cosmology. Still, we find several interesting features. In the early universe limit there exists a solution where the scale factor $a(t)$ grows like $t^{1/2}$, which corresponds to the behavior of the universe in the presence of the normal radiation. Furthermore, in the large universe limit we find a solution which increases exponentially in spite of the existence of a negative cosmological term. This is contrary to the usual expectation that the exponential solution is possible only for a positive cosmological constant[11, 12]. Both these aspects are possible because of the Brans-Dicke field.

Let us consider the four-dimensional Brans-Dicke theory[8], where the action is given by

$$S = \int d^4x \sqrt{-g} \left( \phi R - \omega \phi^{-1} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right). \tag{1}$$
Decomposition of this action into (3 + 1) form, including the speed of light, c, yields\(^1\)

\[
\sqrt{-g} \phi R \simeq N \sqrt{q} \phi \left( R + c^{-2} (K_{ab} K^{ab} - K^2) \right) - 2N \sqrt{q} c^{-2} K \pi - 2N \sqrt{q} D^2 \phi, \quad (2)
\]

\[
-\sqrt{-g} \omega \phi^{-1} q^{\mu \nu} \partial_\mu \phi \partial_\nu \phi = N \sqrt{q} \omega \phi^{-1} c^{-2} \pi^2 - N \sqrt{q} \omega \phi^{-1} D^a \phi D_a \phi, \quad (3)
\]

where the four metric \( g \) is decomposed into the lapse function \( N \), the shift vector \( N^a \) and the three metric \( q_{ab} \), and the corresponding three-dimensional covariant derivative and its scalar curvature are denoted respectively by \( D_a \), \( R \). The Brans-Dicke parameter is assumed positive, \( \omega > 0 \). In the first equation an irrelevant total divergence term was dropped. The time derivatives of the three-metric and the scalar field are encoded in the following quantities;

\[
K_{ab} \equiv \frac{1}{2N} (\dot{g}_{ab} - D_a N_b - D_b N_a), \quad (4)
\]

\[
\pi \equiv \frac{1}{N} (\dot{\phi} - N^a \partial_a \phi). \quad (5)
\]

Using the above result the Brans-Dicke action can be split into the two parts \( S_{BD} = S_{BD}^K + S_{BD}^V \), where the kinetic and potential parts are

\[
S_{BD}^K = c^{-1} \int dt d^3 x N \sqrt{q} \left( \phi(K_{ab} K^{ab} - K^2) - 2K \pi + \omega \phi^{-1} \pi^2 \right), \quad (6)
\]

\[
S_{BD}^V = c \int dt d^3 x N \sqrt{q} \left( \phi R - 2D^2 \phi - \omega \phi^{-1} D^a \phi D_a \phi \right). \quad (7)
\]

Re-scaling the scalar field \( \phi \) and the corresponding field \( \pi \), we find

\[
S_{BD}^K = \int dt d^3 x N \sqrt{q} \left( \phi(K_{ab} K^{ab} - K^2) - 2K \pi + \omega \phi^{-1} \pi^2 \right), \quad (8)
\]

\[
S_{BD}^V = c^2 \int dt d^3 x N \sqrt{q} \left( \phi R - 2D^2 \phi - \omega \phi^{-1} D^a \phi D_a \phi \right). \quad (9)
\]

Note that the factor of \( c^2 \) appears in front of the potential term. For the later purpose regarding the detailed balance it is important to express the kinetic part in the following matrix form;

\[
S_{BD}^K = \int dt d^3 x N \sqrt{q} \left( K_{ab} \pi \right) \left( \phi G_{abcd}^{abcd} q^{abcd} - q^{abcd} \omega \phi^{-1} \right) \left( K_{cd} \pi \right), \quad (10)
\]

where

\[
G_{abcd} = \frac{1}{2} \left( q^{ac} q^{bd} + q^{ad} q^{bc} \right) - q^{ab} q^{cd}. \quad (11)
\]

Note that the matrix in the middle of the kinetic part of the action can be regarded as the supermetric on the space of \( (q_{ab}, \phi) \), naturally extending the DeWitt metric on the space of three-metrics.

We intend to construct a Brans-Dicke type extension of Horava-Lifshitz gravity with the detailed balance condition. So, we choose the action of the form, \( S_{HLBD} = S_{HLBD}^K + S_{HLBD}^V \), where the kinetic part is

\[
S_{HLBD}^K = \int dt d^3 x N \sqrt{q} \left( K_{ab} \pi \right) \left( \phi G_{abcd}^{abcd} (\lambda) - q^{ab} \omega \phi^{-1} \right) \left( K_{cd} \pi \right), \quad (12)
\]

and the potential part is of the form

\[
S_{HLBD}^V = - \int dt d^3 x N \sqrt{q} \left( \frac{\delta W}{\delta q_{ab}} \frac{1}{2} \frac{\delta W}{\delta \phi} \right) \left( \phi G_{abcd}^{abcd} (\lambda) - q^{ab} \omega \phi^{-1} \right)^{-1} \left( \frac{\delta W}{\delta \phi} \right) \quad (13)
\]

for some suitable choice of function \( W(q, \phi) \). The supermetric \( G_{abcd}^{abcd}(\lambda) \) was slightly deformed compared to the Eq. \(^{11}\) to include the parameter \( \lambda \) as usual,

\[
G_{abcd}^{abcd}(\lambda) = \frac{1}{2} \left( q^{ac} q^{bd} + q^{ad} q^{bc} \right) - \lambda q^{ab} q^{cd}. \quad (14)
\]

\(^1\) This result was considered in the context of conformal gravity in Ref. \(^{13}\).
Such a choice of the action is a natural generalization of the Horava-Lifshitz gravity in the context of the detailed balance condition. The factor of two was inserted in front of the variation of \( W \) with respect to \( \phi \) to compensate for different normalization in time derivatives in Eqs.(4) and (5). It is a straightforward matter to calculate the inverse supermetric. It comes out to be of form

\[
\begin{pmatrix}
\phi^{-1} G_{abcd} & -A q_{ab} \\
-A q_{cd} & B \phi
\end{pmatrix},
\]

where

\[
G_{abcd} = \frac{1}{2} (q_{ac}q_{bd} + q_{ad}q_{bc}) - \bar{\lambda} q_{ab} q_{cd},
\]

with

\[
A = \frac{1}{\omega(3\lambda - 1) + 3},
\]

\[
B = \frac{3\lambda - 1}{\omega(3\lambda - 1) + 3},
\]

\[
\bar{\lambda} = \frac{1 + \omega \lambda}{\omega(3\lambda - 1) + 3}.
\]

Note that this inverse supermetric is well-defined even for \( \lambda = 1/3 \) contrary to the pure gravity case and becomes singular instead when \( \lambda = 1 \) and \( \omega = -3/2 \) (\( \omega > 0 \) is assumed in this work and it is nonsingular if \( \lambda > 1/3 \)). The singular case corresponds to the conformal scalar. If we take the limit of \( \omega \to \infty \), \( A \) and \( B \) vanish and \( \lambda = 1/(3\lambda - 1) \), reproducing the pure gravity case.

We choose

\[
W = c_1 \int d^3 x \sqrt{q} (R - 2\Lambda_b) - c_2 \int d^3 x \sqrt{q} \omega^{-1} D^a \phi D_a \phi.
\]

In general all possible marginal and relevant terms can be included. The above choice of \( W \) corresponds to keeping only terms important in the infrared limit. Then, from

\[
\frac{\delta W}{\delta q_{ab}} = -c_1 \Lambda_b \phi q_{ab} + Q_{ab},
\]

\[
\frac{1}{2} \frac{\delta W}{\delta \phi} = -c_1 \Lambda_b + Q,
\]

where

\[
Q_{ab} \equiv c_1 (-\phi G_{ab} + D^a D^b \phi - q_{ab} D^2 \phi),
\]

\[
Q \equiv c_1 \frac{R}{2} - c_2 \left( -\omega \phi^{-1} D^2 \phi + \frac{\omega}{2} \phi^{-2} D^a \phi D_a \phi \right),
\]

with \( G_{ab} \) being the Einstein tensor constructed with the three-dimensional metric, we find after a straightforward calculation that

\[
S_{BDHL}^V = \int dt d^3 x N \sqrt{q} \left\{ \alpha \phi + \beta (\phi R - \frac{c_2}{c_1} \omega \phi^{-1} D^a \phi D_a \phi) + \gamma (-2D^2 \phi) \right\}
\]

\[- \int dt d^3 x N \sqrt{q} (Q_{ab} \phi^{-1} G_{abcd} Q_{cd} - 2A Q_{ab} g_{ab} Q + B \phi Q^2),
\]

where

\[
\alpha = (c_1 \Lambda_b)^2 \frac{3\omega + 7 - 3\lambda}{\omega(3\lambda - 1) + 3},
\]

\[
\beta = -(c_1)^2 \Lambda_b \frac{\omega + 5 - 3\lambda}{\omega(3\lambda - 1) + 3},
\]

\[
\gamma = -(c_1)^2 \Lambda_b \frac{2(\omega + 1) - \frac{c_2}{c_1} \omega(4 - 3\lambda)}{\omega(3\lambda - 1) + 3}.
\]
where the matter is assumed, for consistency, to satisfy the usual form of the continuity equation;

\[ S_{BDH}^{V,IR} = -(c_1)^2 \Lambda_b \frac{\omega + 2}{2\omega + 3} \int \! dt \! d^3 x N \sqrt{q} \left( \phi(R - 2\Lambda) - 2D^2 \phi - \omega \phi^{-1} D^a \phi D_a \phi \right), \quad (29) \]

where

\[ \Lambda = \frac{3\omega + 4}{2(\omega + 2)} \Lambda_b. \quad (30) \]

This expression coincides with that of the Brans-Dicke theory except that the cosmological constant term is present.

Comparison with the kinetic part yields the speed of light

\[ c^2 = -(c_1)^2 \Lambda_b \frac{\omega + 2}{2\omega + 3}. \quad (31) \]

As in the case of the Horava gravity the constant \( \Lambda_b \) must be negative, consequently allowing only negative cosmological constant \( \Lambda \). The Newton constant is related to the expectation value of the scalar field \( \langle \phi \rangle \) as follows,

\[ G_N = \frac{c^2}{16\pi \langle \phi \rangle}. \quad (32) \]

Now, we consider the homogeneous, isotropic cosmology. We will restrict ourself to the case of \( \lambda = 1 \), \( c_1 = c_2 \), and set the speed of light to unity, i.e., \( c = 1 \). We choose vanishing shift vector \( N^a = 0 \), the three-metric to be the usual maximally symmetric ones with curvature constant \( k = -1, 0, +1 \),

\[ ds^2 = a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + \sin^2 \theta d\phi^2 \right). \quad (33) \]

In this case the higher derivative terms become greatly simplified due to homogeneity and isotropy,

\[ Q^{ab} = kc_1 \frac{\phi}{a^2} \delta^{ab}, \quad Q = 6kc_1 \frac{1}{a^2}. \quad (34) \]

Substituting this result into the action, (25), yields the following mini-superspace action,

\[ S_{BDHL} = \int \! dt \! dt^4 \left[ \frac{1}{N} \left( -6\phi \left( \frac{\dot{a}}{a} \right)^2 - 6\dot{a} \phi + \omega \phi^{-1} \phi^2 \right) + N \left( \phi \left( \frac{6k}{a^2} - 2\Lambda \right) + \frac{3\omega}{2(2\omega + 3)} (kc_1)^2 \phi \right) \right], \quad (35) \]

where the lapse field is set \( N = 1 \) after deriving the field equation. The field equations (including the matter) become

\[ 3\left( \frac{\dot{a}}{a} \right)^2 \phi + 3\frac{\ddot{a}}{a} \phi - \frac{1}{2} \omega \phi^{-1} \phi^2 + \left( \frac{3k}{a^2} - \Lambda \right) \phi + \frac{3\omega}{2(2\omega + 3)} (kc_1)^2 \phi = \frac{1}{2} \rho_m, \quad (36) \]

\[ -2\frac{\dddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 \phi - \ddot{a} \phi - 2\ddot{a} \phi + \frac{1}{2} \omega \phi^{-1} \phi^2 - \left( \frac{k}{a^2} - \Lambda \right) \phi + \frac{3\omega}{6(2\omega + 3)} (kc_1)^2 \phi = \frac{1}{2} \rho_m, \quad (37) \]

\[ \omega \dddot{\phi} - \frac{1}{2} \omega \phi^{-1} \phi^2 + 3k \frac{\dot{a}}{a} \phi - 3\frac{\dddot{a}}{a} \phi - 3\frac{\dddot{a}}{a} \phi - \phi \left( \frac{3k}{a^2} - \Lambda \right) \phi - \frac{3\omega}{2(2\omega + 3)} (kc_1)^2 \phi = 0, \quad (38) \]

where the matter is assumed, for consistency, to satisfy the usual form of the continuity equation;

\[ \rho_m + 3\frac{\dot{a}}{a} (\rho_m + p_m) = 0. \quad (39) \]

Only two equations are independent and can be chosen to be

\[ 3H^2 + 3H \frac{\dot{\phi}}{\phi} - \frac{1}{2} \omega \left( \frac{\dot{\phi}}{\phi} \right)^2 = \frac{1}{2} \phi^{-1} \rho_m - \left( \frac{3k}{a^2} - \Lambda \right) - \frac{1}{2} \left( \frac{B^2}{a^4} \right), \quad (40) \]

\[ (2\omega + 3) \left( \frac{\dot{\phi}}{\phi} + 3H \frac{\dot{\phi}}{\phi} \right) = \frac{1}{2} \phi^{-1} (\rho_m - 3p_m) + 2\Lambda + \frac{B^2}{a^4}, \]
where $H \equiv (\dot{a}/a)$ is the Hubble constant and

$$B^2 = \frac{3\omega}{2\omega + 3} (kc_1)^2 = \frac{3\omega(3\omega + 4)}{2(\omega + 2)} \frac{k^2}{(-\Lambda)}. \tag{41}$$

The first in Eq. (40) is the Friedmann equation of the Brans-Dicke theory with a negative cosmological term and the dark radiation term included. In the absence of those two terms the equations simply become those of the usual Brans-Dicke theory\[14\]. Therefore, we restrict our attention to the new effects resulting from those two terms.

Consider two limiting cases for vacuum, $\rho_m = p_m = 0$. First, for small $a$ the dark radiation term dominates, so

$$3H^2 + 3H \frac{\dot{\phi}}{\phi} - \frac{1}{2}\omega \left(\frac{\dot{\phi}}{\phi}\right)^2 = -\frac{1}{2} \frac{B^2}{a^2}, \tag{42}$$

$$(2\omega + 3) \left(\frac{\ddot{\phi}}{\phi} + 3H \frac{\dot{\phi}}{\phi}\right) = B^2 \frac{a^2}{a^4}. \tag{43}$$

These equations can be solved by

$$H = \frac{h}{a^2}, \tag{44}$$

$$\frac{\dot{\phi}}{\phi} = \frac{g}{a^2}, \tag{45}$$

with the two constants $h$ and $g$ satisfying

$$3h^2 + 3hg - \frac{1}{2}\omega g^2 = -\frac{1}{2} B^2, \tag{46}$$

$$(2\omega + 3)g(g + h) = B^2. \tag{47}$$

Eliminating the dark energy terms from the equations we find two possibilities $h = -(1/2)g$, or $3h = -(\omega + 3)g$. Only the first one gives rise to a solution when $B^2$ is positive, so we get

$$h = -\frac{1}{2}g = \pm \sqrt{\frac{2B^2}{2\omega + 3}}. \tag{48}$$

The two signs represent contracting and expanding phases. Solving for $a$ and $\phi$ reads

$$a^2(t) = 2ht, \tag{49}$$

$$\phi(t) = \frac{\phi_0}{t}, \tag{50}$$

where $\phi_0$ is the integration constant. This early universe behavior, $a(t) \sim t^{1/2}$, is the one corresponding to the normal (not dark) radiation source. This is a rather unexpected result due to the scalar field.

Secondly, in the large $a$ limit cosmological term dominates over the curvature and dark radiation terms. We get

$$3H^2 + 3H \frac{\dot{\phi}}{\phi} - \frac{1}{2}\omega \left(\frac{\dot{\phi}}{\phi}\right)^2 = \Lambda, \tag{51}$$

$$(2\omega + 3) \left(\frac{\ddot{\phi}}{\phi} + \left(\frac{\dot{\phi}}{\phi}\right)^2 + 3H \frac{\dot{\phi}}{\phi}\right) = 2\Lambda. \tag{52}$$

Similarly to the previous case, these two equations are solved by

$$H = h, \tag{53}$$

$$\frac{\dot{\phi}}{\phi} = g, \tag{54}$$

with $h$ and $g$ satisfying

$$3h^2 + 3hg - \frac{1}{2}\omega g^2 = \Lambda, \tag{55}$$

$$(2\omega + 3)g(g + h) = \Lambda. \tag{56}$$
In this case, we get either \( h = -\frac{1}{2}g \) or \( h = (\omega + 1)g \). Only the first is allowed for negative \( \Lambda \), and we find

\[ h = -\frac{g}{2} = \pm \sqrt{-\frac{4\Lambda}{2\omega + 3}}. \tag{57} \]

For the positive sign the solution represents the universe exponentially expanding. It is interesting to note that such a solution exists even for a negative cosmological constant. This is in sharp contrast to the Horava gravity case. Note that the effective cosmological constant \( 3h^2 \) is suppressed by the factor of \( \omega \).

To summarize, we have constructed a Brans-Dicke extension of the Horava-Lifshitz gravity with the detailed balance condition satisfied. We have investigated its IR limit and shown that the resulting IR theory is the Brans-Dicke theory with a negative cosmological constant and a dark radiation term. By studying its cosmological solutions we have shown that exponentially expanding solution at late time and power law expanding solution at early time can exist. This is in contrast with the pure gravity of Horava. Although we focused on the Brans-Dicke theory in this paper the analysis can be generalized to other non-minimally coupled scalar field gravity theory. It would be interesting to investigate further cosmological aspects of the resulting theories.

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We look for a Brans-Dicke type generalization of Horava-Lifshitz gravity. It is shown that such a generalization is possible within the detailed balance condition. Classically, the resulting theory reduces in the low energy limit to the usual Brans-Dicke theory with a negative cosmological constant for certain values of parameters. We then consider homogeneous, isotropic cosmology and study the effects of the new terms appearing in the model.

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I. INTRODUCTION

Recently, a new theory of gravity has been proposed by Horava[1–3]. This theory, being based on anisotropic scaling of space and time, breaks the spacetime symmetry. It has a much better UV behavior than the theories with the spacetime diffeomorphism symmetry, but expected to reduce to Einstein’s gravity in the infrared limit recovering the spacetime diffeomorphism symmetry\(^1\). Physical constants such as the speed of light, Newton’s constant, and cosmological constant all emerge from the relevant deformation of the non-relativistic theory at short distance. These interesting features as well as other related findings have received a great deal of attention[6, 7].

On the other hand, there are many alternative theories and extensions of the Einstein theory. In particular, various gravity models with scalar fields have been considered in the context of cosmology to explain the behaviors of the universe in the early stage as well as in the late stage[8, 9]. Therefore, it would be interesting to consider similar extensions in the context of Horava’s theory. In this regard, of particular interest is the one with a non-minimally coupled scalar field\(^2\), typical examples being the Brans-Dicke field\(^2\) and the dilaton field\(^2\).

In this paper, we take the Brans-Dicke field as a concrete example of the non-minimally coupled scalar field and consider its inclusion into the framework of Horava’s gravity. Originally, Horava introduced the concept of the detailed balance condition as a way of reducing the choice of the potential, motivated by analogous methods used in quantum critical systems. Recently, many serious problems were reported\(^1\) associated with strictly imposing this condition. However, some of the problems can be alleviated by softly breaking the condition. Although the fate of the detailed balance condition remains to be seen, it will be interesting to see if the detailed balance condition can be maintained when we try to non-minimally couple the scalar field to the Horava-Lifshitz gravity.

It turns out that such an extension is possible and it reduces to the four-dimensional Brans-Dicke theory with negative cosmological constant when only the lowest order derivative terms are kept and parameters of the theory are chosen to satisfy certain conditions.

We then study cosmological implication of the theory assuming homogeneity and isotropy, and including the curvature-squared terms. Because of the symmetries these higher order terms become a single term proportional to \(a^{-4}\) which can be regarded as the radiation with negative energy. However, it is not strictly so because the nor-
mal matter would couple with the inverse of the Brans-Dicke scalar field. We concentrate only on their effects on cosmology. We find several interesting features. We discuss them in Sec. 3.

II. CONSTRUCTION OF THE MODEL

Let us consider the four-dimensional Brans-Dicke theory\[11\], where the action is given by

$$S = \int d^4x \sqrt{-g} \left( \phi R - \omega \phi^{-1} g^{\mu
u} \partial_\mu \phi \partial_\nu \phi \right).$$

(1)

Decomposition of this action into \((3 + 1)\) form, including the speed of light, \(c\), yields (See Ref. [10], for instance.)

$$\sqrt{-g} \phi R \simeq N \sqrt{q} \phi \left( R + c^{-2} (K_{ab} K^{ab} - K^2) \right) - 2N \sqrt{q} c^{-2} K \pi - 2N \sqrt{q} D^2 \phi,$$

(2)

$$- \sqrt{-g} \omega \phi^{-1} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = N \sqrt{q} \omega \phi^{-1} c^{-2} \pi^2 - N \sqrt{q} \omega \phi^{-1} D^a \phi D_a \phi,$$

(3)

where the four metric \(g\) is decomposed into the lapse function \(N\), the shift vector \(N^a\) and the three metric \(g_{ab}\), and the corresponding three-dimensional covariant derivative and its scalar curvature are denoted respectively by \(D_a\), \(R\). The Brans-Dicke parameter is assumed positive, \(\omega > 0\). In the first equation irrelevant total divergence terms were dropped out. The time derivatives of the three-metric and the scalar field are encoded in the following quantities;

$$K_{ab} \equiv \frac{1}{2N} (g_{ab} - D_a N_b - D_b N_a),$$

(4)

$$\pi \equiv \frac{1}{N} \left( \frac{\dot{\phi}}{\chi} - N^a \partial_a \phi \right).$$

(5)

Using the above result the Brans-Dicke action can be split into the two parts \(S_{BD} = S^K_{BD} + S^\nu_{BD}\), where the kinetic and potential parts can be written after re-scaling of the scalar field \(\phi\) and the corresponding field \(\pi\) as

$$S^K_{BD} = \int d^3x N \sqrt{q} \left( \phi (K_{ab} K^{ab} - K^2) - 2K \pi + \omega \phi^{-1} \pi^2 \right),$$

(6)

$$S^\nu_{BD} = c^2 \int d^3x N \sqrt{q} \left( \phi R - 2D^2 \phi - \omega \phi^{-1} D^a \phi D_a \phi \right).$$

(7)

Note the factor of \(c^2\) in front of the potential term. For the later purpose regarding the detailed balance condition it is important to express the kinetic part in the following matrix form;

$$S^K_{BD} = \int d^3x N \sqrt{q} \left( K_{ac} \pi \right) \left( \phi G^{abcd} - q^{ab} \omega \phi^{-1} \right) \left( K_{ed} \pi \right) \left( K_{ed} \pi \right),$$

(8)

where

$$G^{abcd} = \frac{1}{2} \left( q^{ac} q^{bd} + q^{ad} q^{bc} - q^{ab} q^{cd} \right).$$

(9)

The matrix in the middle of the kinetic part of the action can be regarded as the supermetric on the space of \((g_{ab}, \phi)\), naturally extending the DeWitt metric on the space of three-metrics.

We intend to construct a Brans-Dicke type extension of Horava-Lifshitz gravity with the detailed balance condition. So, we choose the action of the form, \(S_{HLBD} = S^K_{HLBD} + S^\nu_{HLBD}\), where the kinetic part is

$$S^K_{HLBD} = \int d^3x N \sqrt{q} \left( K_{ac} \pi \right) \left( \phi G^{abcd}(\lambda) - q^{ab} \omega \phi^{-1} \right) \left( K_{ed} \pi \right) \left( K_{ed} \pi \right),$$

(10)

and the potential part is of the form

$$S^\nu_{HLBD} = - \int d^3x N \sqrt{q} \left( \frac{\delta W}{\delta q_{ab}} + \frac{\delta W}{\delta \phi} \right) \left( \phi G^{abcd}(\lambda) - q^{ab} \omega \phi^{-1} \right)^{-1} \left( \frac{\delta W}{\delta q_{ab}} + \frac{\delta W}{\delta \phi} \right),$$

(11)

for some suitable choice of function \(W(q, \phi)\). The supermetric \(G^{abcd}(\lambda)\) was slightly deformed compared to the Eq. \[9\] to include the parameter \(\lambda\) as usual,

$$G^{abcd}(\lambda) \equiv \frac{1}{2} \left( q^{ac} q^{bd} + q^{ad} q^{bc} - \lambda q^{ab} q^{cd} \right).$$

(12)
The factor of two was inserted in front of the variation of \( W \) with respect to \( \phi \) to compensate for different normalization in time derivatives in Eqs.(4) and (5). It is a straightforward matter to calculate the inverse supermetric. It comes out to be of form

\[
\left( \begin{array}{cc}
\phi^{-1} G_{abcd} - A q_{ab} & - A q_{cd} \\
-B \phi & B \phi
\end{array} \right),
\]

(13)

where

\[ G_{abcd} = \frac{1}{2} (q_{ac} q_{bd} + q_{ad} q_{bc}) - \bar{\lambda} q_{ab} q_{cd}, \]

(14)

with

\[ A = \frac{1}{\omega(3\lambda - 1) + 3}, \quad B = \frac{3\lambda - 1}{\omega(3\lambda - 1) + 3}, \quad \bar{\lambda} = \frac{1 + \omega \lambda}{\omega(3\lambda - 1) + 3}. \]

(15)

Note that this inverse supermetric is well-defined even for \( \lambda = 1/3 \) contrary to the pure gravity case and becomes singular instead when \( \lambda = (\omega - 3)/3\omega \), for instance when \( \lambda = 1 \) and \( \omega = -3/2 \) corresponding to the conformal scalar case (We assume \( \omega > 0 \) in this work.). If we take the limit of \( \omega \to \infty \), \( A \) and \( B \) vanish and \( \bar{\lambda} = \lambda/(3\lambda - 1) \), reproducing the pure gravity case.

We choose

\[ W = c_1 \int d^3 x \sqrt{q} \phi(R - 2\Lambda_b) - c_2 \int d^3 x \sqrt{q} \omega^{-1} D^a \phi D_a \phi. \]

(16)

In general all possible marginal and relevant terms can be included. The above choice of \( W \) corresponds to keeping only terms important in the infrared limit. Then, after a straightforward calculation Eq. (11) can be written as

\[
S_{HLBD}^V = \int dt d^3 x N \sqrt{q} \left\{ \alpha \phi + \beta (\phi R - \frac{c_2}{c_1} \omega^{-1} D^a \phi D_a \phi) + \gamma (-2 D^2 \phi) \right\}
- \int dt d^3 x N \sqrt{q} (Q^{ab} \phi^{-1} G_{abcd} Q^{cd} - 2 A Q^{ab} q_{ab} Q + B \phi Q^2),
\]

(17)

where

\[ \alpha = (c_1 \Lambda_b)^2 \frac{3\omega + 7 - 3\lambda}{\omega(3\lambda - 1) + 3} \]

(18)

\[ \beta = -(c_1)^2 \Lambda_b \frac{\omega + 5 - 3\lambda}{\omega(3\lambda - 1) + 3} \]

(19)

\[ \gamma = -(c_1)^2 \Lambda_b \frac{2(\omega + 1) - \frac{c_2}{c_1} \omega(4 - 3\lambda)}{\omega(3\lambda - 1) + 3} \]

(20)

and

\[ Q^{ab} \equiv c_1 \left( -\phi (R^{ab} - \frac{1}{2} R q^{ab}) + D^a D^b \phi - q^{ab} D^2 \phi \right), \]

(21)

\[ Q \equiv c_1 \frac{R}{2} - c_2 \left( -\phi^{-1} D^2 \phi + \frac{\omega}{2} \phi^{-1} D^a \phi D_a \phi \right). \]

(22)

The second line of Eq. (17) has quadratic terms only.

When \( c_1 = c_2 \) and \( \lambda = 1 \), the theory recovers four-dimensional diffeomorphism symmetry, as one can see from the fact that in the infrared limit the potential part of the action becomes

\[ S_{BDHLIR}^V = -(c_1)^2 \Lambda_b \frac{\omega + 2}{2\omega + 3} \int dt d^3 x N \sqrt{q} \left( \phi (R - 2\Lambda) - 2 D^2 \phi - \omega \phi^{-1} D^a \phi D_a \phi \right), \]

(23)

where

\[ \Lambda = \frac{3\omega + 4}{2(\omega + 2)} \Lambda_b. \]

(24)
This expression coincides with that of the Brans-Dicke theory except that the cosmological constant term is present. Comparison with the kinetic part yields the speed of light

$$c^2 = -\frac{(c_1)^2 \Lambda_b \omega + 2}{2\omega + 3}. \quad (25)$$

As in the case of the Horava gravity the constant $\Lambda_b$ must be negative, consequently allowing only negative cosmological constant $\Lambda$. The Newton constant is related to the scalar field $\phi$ as follows,

$$G_N = \frac{c^2}{16\pi \phi}. \quad (26)$$

### III. COSMOLOGICAL SOLUTIONS

Now, we consider the homogeneous, isotropic cosmology. We restrict ourself to the case of $\lambda = 1$, $c_1 = c_2$, and set the speed of light to unity, i.e., $c = 1$. We choose vanishing shift vector $N^a = 0$, and the three-metric to be the usual maximally symmetric ones with curvature constant $k = -1, 0, +1$,

$$ds^2 = a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right). \quad (1)$$

In this case the higher derivative terms become greatly simplified due to homogeneity and isotropy,

$$Q_{ab} = kc_1 \frac{\phi}{a^2} q_{ab}, \quad Q = 6kc_1 \frac{1}{a^2}. \quad (2)$$

The field equations become

$$3H^2 + 3H \frac{\dot{\phi}}{\phi} - \frac{1}{2} \omega (\frac{\dot{\phi}}{\phi})^2 = \frac{1}{2} \phi^{-1} \rho_m - \frac{3k}{a^2} + \Lambda - \frac{1}{2} \left(\frac{B^2}{a^4}\right),$$

$$-2H - 3H^2 - \frac{\ddot{\phi}}{\phi} - 2H \frac{\dot{\phi}}{\phi} - \frac{\omega}{2} (\frac{\dot{\phi}}{\phi})^2 = \frac{1}{2} \phi^{-1} \rho_m + \frac{k}{a^2} - \Lambda - \frac{1}{6} \left(\frac{B^2}{a^4}\right),$$

$$(2\omega + 3) \left(\frac{\ddot{\phi}}{\phi} + 3H \frac{\dot{\phi}}{\phi}\right) = \frac{1}{2} \phi^{-1} (\rho_m - 3p_m) + 2\Lambda + \frac{B^2}{a^4}, \quad (3)$$

together with the usual form of the continuity equation for the matter density $\rho_m$ for consistency, where $H \equiv (\dot{a}/a)$ is the Hubble constant and

$$B^2 = \frac{3\omega}{2\omega + 3} \left(\frac{c_1}{a^2}\right)^2 = \frac{3\omega (3\omega + 4)}{2(\omega + 2)} \frac{k^2}{(-\Lambda)}. \quad (4)$$

The first equation in Eq. 3 is the Friedmann equation of the Brans-Dicke theory with a negative cosmological term and the dark radiation term included. Again, we emphasize that they do not have $\phi^{-1}$ coupling in contrast to the normal matter. In the absence of those two terms the equations simply become those of the usual Brans-Dicke theory[17]. Therefore, we restrict our attention to the new effects resulting from those two terms.

For simplicity, assume that the matter is absent, i.e., $\rho_m = p_m = 0$. First, consider the case where the dark radiation like term dominates, so Eq. 3 reduces to

$$3H^2 + 3H \frac{\dot{\phi}}{\phi} - \frac{1}{2} \omega (\frac{\dot{\phi}}{\phi})^2 = -\frac{1}{2} \left(\frac{B^2}{a^4}\right),$$

$$-2H - 3H^2 - \frac{\ddot{\phi}}{\phi} - 2H \frac{\dot{\phi}}{\phi} - \frac{\omega}{2} (\frac{\dot{\phi}}{\phi})^2 = -\frac{1}{6} \left(\frac{B^2}{a^4}\right),$$

$$(2\omega + 3) \left(\frac{\ddot{\phi}}{\phi} + 3H \frac{\dot{\phi}}{\phi}\right) = \frac{B^2}{a^4}. \quad (5)$$

To further simplify these equation we set

$$X \equiv H + Y, \quad Y \equiv \frac{1}{2} \left(\frac{\dot{\phi}}{\phi}\right).$$
in terms of which, they can be written as

\[ 3X^2 - AY^2 = \frac{B^2}{2a^4}, \]

\[
\dot{X} = -\frac{3}{2}X^2 + XY - \frac{A}{2}Y^2 + \frac{1}{6} (\frac{B^2}{2a^4}),
\]

\[
A\dot{Y} = -3AXY + AY^2 + \frac{B^2}{2a^4},
\]

(7)

where \(A \equiv 2\omega + 3\). Setting

\[
\sqrt{3}X = \sqrt{\frac{B^2}{2a^4}} \sinh \theta,
\]

\[
\sqrt{AY} = \sqrt{-\frac{B^2}{2a^4}} \cosh \theta,
\]

(8)

Eq. (3) yields

\[
\dot{\theta} = -\sqrt{\frac{B^2}{2a^4}} \left( 1 + \frac{1}{\sqrt{3}} \cosh \theta + \frac{1}{\sqrt{A}} \sinh \theta \right).
\]

(9)

Note that \(\dot{\theta}\) is always negative for \(\omega > 0\), which means that \(\theta\) goes from negative infinity to positive infinity as time flows. Combine this result with

\[
\frac{\dot{a}}{a} = X - Y = \sqrt{\frac{B^2}{2a^4}} \left( -\frac{1}{\sqrt{A}} \cosh \theta + \frac{1}{\sqrt{3}} \sinh \theta \right)
\]

(10)

to get

\[
\frac{d\log a}{d\theta} = -\left( \frac{1}{\sqrt{3}} \sinh \theta - \frac{1}{\sqrt{A}} \cosh \theta \right).
\]

(11)

Although this equation can be integrated, the resulting expressions can be quite complicated, asymptotic behavior of \(a(t)\) at early and late times can be easily determined. A straightforward analysis shows that the scale factor vanishes at initial time and finite later time. The universe it describes expansion from a singularity and within a finite time it collapses.

When the cosmological term dominates, Eq. (3) becomes

\[ 3X^2 - AY^2 = \Lambda, \]

\[
\dot{X} = -\frac{3}{2}X^2 + XY - \frac{A}{2}Y^2 + \frac{\Lambda}{2},
\]

\[
A\dot{Y} = -3AXY + AY^2 + \Lambda.
\]

(12)

With

\[
\sqrt{3}X = \sqrt{-\Lambda} \sinh \theta,
\]

\[
\sqrt{AY} = \sqrt{-\Lambda} \cosh \theta,
\]

(13)

we find a slightly different equation for \(\theta\),

\[
\dot{\theta} = -\sqrt{-\Lambda} \left( \sqrt{3}\cosh \theta - \frac{1}{\sqrt{A}} \sinh \theta \right).
\]

(14)

General behavior of the solution is the same as the previous case.
IV. CONCLUSION AND DISCUSSION

To summarize, we constructed a Brans-Dicke type extension of the Horava-Lifshitz gravity maintaining the detailed balance condition. Although strict imposition of the detailed balance condition is known to have many problems, one can either break the condition or simply treat the resulting terms as an important contribution to the potential. We have not discussed the issue of projectability in this work. At this level, our model can be incorporated into any version.

Furthermore, we investigated its low energy limit and shown that the resulting theory is the Brans-Dicke theory with a negative cosmological constant. In Brans-Dicke theory one can incorporate cosmological constant term in two ways. One is treating it as a vacuum expectation value from the matter sector and the other is what we have done in this work.

We studied the curvature contribution up to quadratic order in the context of homogeneous and isotropic cosmology. The resulting theory is the Brans-Dicke theory with a negative cosmological constant and a radiation-like term with a negative energy density. They are somewhat different from usual matter in that they do not have $\phi^{-1}$ coupling. We analyzed their effects and showed that the resulting solution has the general behavior of big rip. This is in contrast with the pure gravity case of Horava.

Although we focused on the Brans-Dicke theory in this paper the analysis can be generalized to other non-minimally coupled scalar field gravity theory. It would be interesting to further investigate cosmological aspects of the resulting theories.

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