A REMARK ON THE CONVERGENCE OF THE DOUGLAS-RACHFORD ITERATION IN A NON-CONVEX SETTING

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Dedicated to the memory of Jonathan M. Borwein

Abstract. Using the construction of a Lyapunov function from [Ben15], it is shown that the Douglas-Rachford iteration with respect to a sphere and a line in $\mathbb{R}^d$ is robustly $KL$-stable. This implies a convergence which is stronger than uniform convergence on compact sets.

1. Introduction

Given a set $A \subseteq \mathbb{R}^d$, define the projection operator $P_A : \mathbb{R}^d \to \mathbb{R}^d$,

$$P_A(x) = \left\{ y \in A \mid \|x - y\| = \inf_{z \in A} \|x - z\| \right\}.$$ 

In general $P_A$ can be multi-valued. Here and in what follows, $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^d$. Define also the reflection operator by $R_A = 2P_A - I$, $I$ being the identity operator on $\mathbb{R}^d$. Given two sets $A, B \subseteq \mathbb{R}^d$ define the Douglas-Rachford operator,

$$T_{A,B} = \frac{I + R_B R_A}{2}. \quad (1.1)$$

Given $x \in \mathbb{R}^d$, define the sequence $\{x_n\}_{n=0}^{\infty} \subseteq \mathbb{R}^d$ by the recursive condition

$$x_{n+1} = T_{A,B} x_n = T_{A,B}^n x_0, \quad x_0 = x. \quad (1.2)$$

The sequence defined in (1.2) is known as the Douglas-Rachford iteration of $x$. A well known question concerns the asymptotic behaviour of this sequence. This question has application in the case where both $A$ and $B$ are convex, as well as in the case where one of the sets is not convex. See for example [BCL02, LM79] for the convex case and [ERT07, GE08] for the non-convex case.

In the case $A$ is convex, it is known that the projection map $P_A$ is firmly non-expansive, that is, for all $x, y \in \mathbb{R}^d$,

$$\|P_A x - P_A y\|^2 + \|(I - P_A)x - (I - P_A)y\|^2 \leq \|x - y\|^2.$$ 

See for example [GK90, Thm. 12.2]. It then follows that if both $A$ and $B$ are convex, $T_{A,B}$ is also firmly non-expansive. See for example [GK90, Thm. 12.1]. From the non-expansiveness of $T_{A,B}$ it follows that $\{x_n\}_{n=0}^{\infty}$ is norm convergent, with norm convergence replaced by weak convergence in the case of an infinite dimensional space.

While the convex case is well understood, much less is known in the non-convex setting, when either $A$ or $B$ is not convex. One of the simplest non-convex cases is the case of a sphere and a line. This case is of particular interest also because the sphere is a model of many reconstruction problems in which only the magnitude of a phase is measured. This case was studied in [AAB13, Ben15, BS11]. Other non-convex cases were considered in [AABT16, BG16, HL13, Pha16].

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Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors in $\mathbb{R}^d$. Given $\lambda \in \mathbb{R}$, define the following sets,

$$L_\lambda = \{t \mathbf{e}_1 + \lambda \mathbf{e}_2 \mid t \in \mathbb{R}\}, \quad S = \{x \in \mathbb{R}^d \mid \|x\| = 1\},$$

that is, a line and the unit Euclidean sphere in $\mathbb{R}^d$. In this case, the Douglas-Rachford operator is given explicitly by the following formula,

$$T_{S,L_\lambda} x = \frac{\langle x, \mathbf{e}_1 \rangle}{\|x\|} \mathbf{e}_1 + \left(\left(1 - \frac{1}{\|x\|}\right) \langle x, \mathbf{e}_2 \rangle + \lambda\right) \mathbf{e}_2 + \left(1 - \frac{1}{\|x\|}\right) \sum_{j=3}^{d} \langle x, \mathbf{e}_j \rangle \mathbf{e}_j, \quad (1.3)$$

where here and in what follows, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^d$. Define also the following sets

$$H_0 = \{x \in \mathbb{R}^d \mid \langle x, \mathbf{e}_1 \rangle = 0\} \setminus \{0\},$$

$$H_+ = \{x \in \mathbb{R}^d \mid \langle x, \mathbf{e}_1 \rangle > 0\},$$

$$H_- = \{x \in \mathbb{R}^d \mid \langle x, \mathbf{e}_1 \rangle < 0\}. \quad (1.4)$$

By (1.3), all three sets are invariant under $T_{S,L_\lambda}$. Assuming that $\lambda \in [0, 1]$ (the case $\lambda \in [-1, 0]$ is completely analogous), the only fixed points of $T_{S,L_\lambda}$ are the intersection points of $L_\lambda$ and $S$,

$$x^* = \sqrt{1 - \lambda^2} \mathbf{e}_1 + \lambda \mathbf{e}_2 \quad \text{and} \quad x_* = -\sqrt{1 - \lambda^2} \mathbf{e}_1 + \lambda \mathbf{e}_2. \quad (1.5)$$

In [BS11] it was shown that when $\lambda \in (0, 1)$, the Douglas-Rachford iteration is locally norm convergent around the intersection points, and later in [AAB13] an explicit domain of convergence was given. In the case $\lambda = 0$, it was shown in [BS11] that in fact we have global convergence on $H_+ \cup H_+$. In the case $\lambda = 1$, it was also shown in [BS11] that if $x_0 \in H_+ \cup H_+$, then the Douglas-Rachford iteration converges to a point of the form $\hat{y} \mathbf{e}_2$, where $\hat{y} \in (1, \infty)$. Finally, it was shown in [BS11], that when $\lambda > 1$ and $x_0 \in H_- \cup H_+$ or when $x_0 \in H_0$, the Douglas-Rachford iteration is divergent. In [Ben15] it was shown that for every $\lambda \in (0, 1)$, the Douglas-Rachford iteration converges globally in norm whenever $x_0 \in H_+ \cup H_+$. From (1.3), it follows that the behaviour of $T_{S,L_\lambda}$ on $H_+$ and $H_-$ are similar. Thus, it is enough to consider the case where the initial point is in $H_+$. In this case, the only intersection point we need to consider is $x^*$. Define the following domain $\Delta \subseteq \mathbb{R}^d$,

$$\Delta = \{x \in \mathbb{R}^d \mid \langle x, \mathbf{e}_1 \rangle \in (0, 1]\}. \quad (1.6)$$

It follows from (1.3) that

$$T_{S,L_\lambda}(\Delta) \subseteq T_{S,L_\lambda}(H_+) \subseteq \Delta. \quad (1.7)$$

In [Ben15], an ingenious construction of a Lyapunov function for $T_{S,L_\lambda}$ was presented, which implied global convergence of the iteration (1.2). Given $x \in \mathbb{R}^d$ and $t > 0$, let $B(x, t)$ denote the open Euclidean ball centred at $x$ with radius $t$. The following is the main result of [Ben15].

**Theorem 1.1** ([Ben15]). Assume that $\lambda \in [0, 1)$. Define the function $F : \Delta \to \mathbb{R}$ as follows,

$$F(x) = \frac{1}{2}\|x - \lambda \mathbf{e}_2\|^2 - \lambda \log \left(1 + \sqrt{1 - \langle x, \mathbf{e}_1 \rangle^2}\right) + \lambda \sqrt{1 - \langle x, \mathbf{e}_1 \rangle^2} + (\lambda - 1) \log \langle x, \mathbf{e}_1 \rangle. \quad (1.8)$$

Then $F$ satisfies $F(x^*) < F(x)$ for all $x \in \Delta \setminus \{x^*\}$ and $F(T_{S,L_\lambda} x) \leq F(x)$ for all $x \in \Delta$. Moreover, for every $t > 0$,

$$\sup_{x \in \Delta \setminus B(x^*, t)} \left[F(T_{S,L_\lambda} x) - F(x)\right] < 0.$$

In particular, $F(T_{S,L_\lambda} x) = F(x)$ if and only if $x = x^*$. 


Note that in [Ben15], the main result is stated for \( \lambda \in (0,1) \), but it is in fact true for the case \( \lambda = 0 \) as well.

If \( K \subseteq H_+ \) and for \( n = 1,2,\ldots \) we define \( f_n : K \to \mathbb{R} \) by \( f_n(x) = F(T^n_{S,L_\lambda} x) - F(x) \), then by Theorem 1.1 \( \{f_n\}_{n=1}^\infty \) is a decreasing sequence of continuous functions which converges point-wise to 0. Therefore, if \( K \) is compact, then by Dini’s convergence theorem, \( \{f_n\}_{n=1}^\infty \) converges uniformly to 0 on \( K \). This in particular implies that \( T^n_{S,L_\lambda} x \xrightarrow{n \to \infty} x^* \) uniformly for \( x \in K \). Here and in what follows any convergence of vectors means convergence in the Euclidean norm on \( \mathbb{R}^d \). However, using Theorem 2.3 below, it is shown that on compact sets in \( H_+ \) we obtain a type of convergence which is stronger than uniform convergence. See Section 2.2 below for the exact formulation.

The rest of the note is organised as follows. In Section 2.1, we recall some preliminaries and notations from the theory of discrete time dynamical systems. Then, in Section 2.2, we state the main results in the note, which are then proved in Section 3.

2. Preliminaries and statement of the main results

2.1. Stability of discrete time dynamical systems. Assume that \( D \) is a set in \( \mathbb{R}^d \), and let \( T : D \rightrightarrows D \) be a set-valued map from \( D \) to subsets of \( D \). For \( n \in \mathbb{Z}_+ = \{0,1,2,\ldots\} \), consider the difference inclusion with initial condition

\[
x_{n+1} \in T x_n, \quad x_0 \in D.
\]

Let \( S(x,T) \) be the set of solutions to (2.1) with \( x_0 = x \), and let \( \phi(x,n) \in S(x,T) \) denote a solution to (2.1), that is, \( \phi : D \times \mathbb{Z}_+ \to D \) is such that \( \phi(x,0) = x \) and \( \phi(x,n+1) \in T(\phi(x,n)) \) for all \( n \in \mathbb{Z}_+ \). For \( K \subseteq D \) let

\[
S(K,T) = \bigcup_{x \in K} S(x,T).
\]

Next, recall some definitions regarding the stability of the system (2.1). Let \( \mathbb{R}_+ = [0,\infty) \). A function \( \beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) is said to belong to class \( K \mathcal{L} \) if for every \( t \geq 0 \), \( \beta(\cdot,t) \) is continuous, strictly increasing, and \( \beta(0,t) = 0 \), and also for every \( s \geq 0 \), \( \beta(s,\cdot) \) is non-increasing, and satisfies \( \beta(s,t) \xrightarrow{t \to \infty} 0 \). We have the following definition.

**Definition 2.1.** Assume that \( D \subseteq \mathbb{R}^d \) and \( \omega_1,\omega_2 : D \to \mathbb{R}_+ \) are continuous functions. The difference inclusion (2.1) is said to be \( K \mathcal{L} \)-stable with respect to \( (\omega_1,\omega_2) \) if there exists \( \beta \in K \mathcal{L} \) such that for every \( x \in D \), every \( \phi \in S(x) \), and every \( n \in \mathbb{Z}_+ \),

\[
\omega_1(\phi(x,n)) \leq \beta(\omega_2(x),n).
\]

Let \( \sigma : D \to \mathbb{R}_+ \) be such that \( B[x,\sigma(x)] \subseteq D \) for all \( x \in D \), where here and in what follows \( B[x,r] \) denotes the closed ball around \( x \) with radius \( r \) with respect to the Euclidean norm. Given a set \( K \subseteq D \), define

\[
K_\sigma = \bigcup_{x \in K} B[x,\sigma(x)].
\]

Note that \( K_\sigma \subseteq D \). Given an operator \( T : D \rightrightarrows D \), define also \( T_\sigma \), the \( \sigma \)-perturbation of \( T \),

\[
T_\sigma x = \bigcup_{y \in T B[x,\sigma(x)]} B[y,\sigma(y)].
\]

Note that if \( K \subseteq D \), then \( T_\sigma(K) = (T(K_\sigma))_\sigma \). Denote by \( S_\sigma(x,T) \) the set of solutions to the perturbed difference inclusion \( x_{n+1} \in T_\sigma x_n \) with initial condition \( x_0 = x \). Note that in particular,
we have \( S_0(x, T) = S(x, T) \), where here 0 denotes the constant zero function. As before, for \( K \subseteq D \), let

\[
S_\sigma(K, T) = \bigcup_{x \in K} S_\sigma(x, T).
\]

Given a continuous function \( \omega_1 : D \to \mathbb{R}_+ \) define the following set

\[
A_\sigma = \left\{ x \in D \mid \sup_{n \in \mathbb{Z}_+} \sup_{\phi \in S_\sigma(x)} \omega_1(\phi(x, n)) = 0 \right\},
\]

and let

\[
A = \left\{ x \in D \mid \sup_{n \in \mathbb{Z}_+} \sup_{\phi \in S(x)} \omega_1(\phi(x, n)) = 0 \right\}. \tag{2.2}
\]

Next, recall the notion of robust \( KL \)-stability.

**Definition 2.2.** Assume that \( D \subseteq \mathbb{R}^d \) and \( \omega_1, \omega_2 : D \to \mathbb{R}_+ \) are continuous functions. The difference inclusion (2.1) is said to be robustly \( KL \)-stable with respect to \( (\omega_1, \omega_2) \) on \( D \) if there exists a continuous function \( \sigma : D \to \mathbb{R}_+ \) such that

1. For all \( x \in D \), \( B[x, \sigma(x)] \subseteq D \);
2. For all \( x \in D \setminus A, \sigma(x) > 0 \);
3. \( A_\sigma = A \);
4. The difference inclusion \( x_{n+1} \in T_\sigma x_n \) is \( KL \)-stable with respect to \( (\omega_1, \omega_2) \) on \( D \).

**Remark 2.1.** Note that if \( x_{n+1} \in T_\sigma x_n \) is \( KL \)-stable with respect to \( (\omega_1, \omega_2) \) and \( \tau : D \to \mathbb{R}_+ \) is such that \( \tau \leq \sigma \) on \( D \), then \( x_{n+1} \in T_\tau x_n \) is \( KL \)-stable with respect to \( (\omega_1, \omega_2) \) on \( D \) as well, since \( S_\tau(K, T) \subseteq S_\sigma(K, T) \).\( \diamond \)

A function \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) is said to belong to class \( \mathcal{K} \) if it is continuous, strictly increasing, and \( \alpha(0) = 0 \). A function \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) is said to belong to class \( \mathcal{K}_\infty \) if \( \alpha \in \mathcal{K} \) and in addition \( \lim_{t \to \infty} \alpha(t) = \infty \). A function \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) is said to be positive definite if \( \alpha(t) = 0 \) if and only if \( t = 0 \).

Recall next the following notion of a Lyapunov function.

**Definition 2.3.** Assume that \( D \subseteq \mathbb{R}^d \) and \( \omega_1, \omega_2 : D \to \mathbb{R}_+ \) are continuous functions. A function

\[
V : D \to \mathbb{R}_+
\]

is said to be a Lyapunov function with respect to \( (\omega_1, \omega_2) \) on \( D \) for the difference inclusion (2.1) if there exist \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \) and a continuous positive definite function \( \alpha \) such that for all \( x \in D \),

\[
\alpha_1(\omega_1(x)) \leq V(x) \leq \alpha_2(\omega_2(x)), \tag{2.3}
\]

\[
\sup_{y \in T_\tau x} V(y) \leq V(x) - \alpha(V(x)), \tag{2.4}
\]

\[
V(x) = 0 \iff x \in A. \tag{2.5}
\]

It is known that there is a close connection between the stability properties of a dynamical system and the existence and properties of a Lyapunov function. Originally this was known for continuous time dynamical systems, but results for the discrete time case have also been obtained. See for example the survey [Kel15]. In particular, the following result is Theorem 2.8 in [KT05].

**Theorem 2.1** ([KT05]). Assume that \( D \subseteq \mathbb{R}^d \) is open, and \( T : D \rightrightarrows D \) is such that \( Tx \) is compact and non-empty for all \( x \in D \). Assume also that there exists a continuous Lyapunov function on \( D \) with respect to two continuous functions \( \omega_1, \omega_2 : D \to \mathbb{R}_+ \). Then the difference inclusion (2.1) is robustly \( KL \)-stable with respect to \( (\omega_1, \omega_2) \).
2.2. Statement of the main results. Using the explicit construction from Theorem 1.1 in Theorem 2.1, we obtain the following result.

**Theorem 2.2.** Assume that \( \lambda \in [0,1) \) and let \( H_+ \) be the domain defined in (1.4). Then there exists a continuous Lyapunov function \( V : H_+ \to \mathbb{R}_+ \) in the sense of Definition 2.3 such that the Douglas-Rachford iteration \( x_{n+1} = T_{S,L_\lambda}x_n, x_0 \in H_+ \), is robustly KL-stable with respect to \( \omega_1 = \omega_2 = V \).

Theorem 2.2 implies the following convergence result for the Douglas-Rachford iteration.

**Theorem 2.3.** Assume that \( \lambda \in [0,1) \) and let \( H_+ \) be the domain defined in (1.4). Then there exists a function \( \tau : H_+ \to \mathbb{R}_+ \) which satisfies \( \tau(x) > 0 \) for all \( x \in H_+ \setminus \{x^*\} \), and such that for every compact set \( K \subseteq H_+, \phi \in S_\tau(K,T_{S,L_\lambda}) \) converges uniformly to \( x^* \), that is,

\[
\lim_{n \to \infty} \sup_{\phi \in S_\tau(K,T_{S,L_\lambda})} \|\phi(x,n) - x^*\| = 0.
\]

Theorem 2.3 says that if we consider paths which are ‘close enough’ to the paths resulting from the Douglas-Rachford iteration, we still have uniform convergence.

**Remark 2.2.** Regarding the ‘boundary’ case \( \lambda = 1 \), it was shown in [BS11] that in this case we have global convergence on \( H_+ \), even though it need not converge to the intersection point. See Theorem 6.12 in [BS11]. It would be interesting to see whether a result similar to Theorem 2.2 or Theorem 2.3 can be obtained in this case.

3. Proof of Theorem 2.2 and Theorem 2.3

For the domain \( H_+ \) as defined in (1.4), define \( V : H_+ \to \mathbb{R}_+ \),

\[
V(x) = F(T_{S,L_\lambda}x) - F(x^*).
\]

By (1.7), \( T_{S,L_\lambda}x \in \Delta \) whenever \( x \in H_+ \), where \( \Delta \) is defined as in (1.6). Thus, \( V \) is well defined, and by Theorem 1.1, \( V(x) \geq 0 \). We would like to show that \( V \) is a continuous Lyapunov function that satisfies the conditions of Definition 2.3. The following proposition shows that condition (2.4) holds for this choice of \( V \).

**Proposition 3.1.** Let \( \lambda \in [0,1) \), and let \( V : H_+ \to \mathbb{R}_+ \) be defined as in (3.1). Then there exists a continuous, positive definite function \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) such that for every \( x \in H_+ \),

\[
V(T_{S,L_\lambda}x) \leq V(x) - \alpha(V(x)).
\]

**Proof.** Let \( \Delta \) be the domain defined in (1.6), and define \( U,W : \Delta \to \mathbb{R}_+ \),

\[
U(x) = F(x) - F(x^*),
\]

and

\[
W(x) = U(x) - U(T_{S,L_\lambda}x) = F(x) - F(T_{S,L_\lambda}x).
\]

By Theorem 1.1, both \( U \) and \( W \) are continuous and positive on \( \Delta \), and are equal to 0 if and only if \( x = x^* \). Define also \( g : \mathbb{R}_+ \to \mathbb{R}_+ \),

\[
g(t) = \inf_{x \in \Delta \setminus B(x^*,t)} W(x).
\]

Clearly \( g \) is non-decreasing. Also, by Theorem 1.1, \( g \) is non-negative and \( g(t) = 0 \) if and only if \( t = 0 \), and so \( g \) is positive definite. Note that by (3.3), combined with (1.3) and (1.8),

\[
W(x) = \psi(x) + (\lambda - 1) \left( \log \langle x, e_1 \rangle - \log \left( \frac{\langle x, e_1 \rangle}{\|x\|} \right) \right) = \psi(x) + (\lambda - 1) \log \|x\|,
\]

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where \( \psi \) is a continuous function on \( \{ x \in \mathbb{R}^d \mid -1 \leq \langle x, \mathbf{e}_1 \rangle \leq 1 \} \). Also, \( \log \| x \| \) is continuous on \( \mathbb{R}^d \setminus \{ 0 \} \) which contains \( \Delta \). Since \( g \) is non-decreasing, in order to show that it is continuous it is enough to show that for every \( \varepsilon > 0 \) and \( t \geq 0 \), there exists \( \delta > 0 \) such that \( g(t + \delta) \leq g(t) + \varepsilon \). Indeed, let \( t \geq 0 \) and \( \varepsilon > 0 \). Assume that \( x \in \Delta \setminus B(x^*, t) \) is such that

\[
W(x) \leq g(t) + \varepsilon.
\]

Since by (3.4) \( W \) is continuous on \( \Delta \) and \( W(x) \xrightarrow{\varepsilon \to 0} \infty \), it follows that we may choose \( x \in \text{int}(\Delta \setminus B(x^*, t)) \), the interior of \( \Delta \setminus B(x^*, t) \). Given \( \delta > 0 \), define \( x_\delta \) as follows,

\[
x_\delta = x^* + \left( 1 + \frac{\delta}{\| x - x^* \|} \right) (x - x^*).
\]

Then \( \| x_\delta - x \| = \delta \) and \( \| x_\delta - x^* \| = \| x - x^* \| + \delta \). Since \( x \in \Delta \setminus B(x^*, t) \), we have \( \| x - x^* \| \geq t \) and so \( \| x_\delta - x^* \| \geq t + \delta \). Since \( x \in \text{int}(\Delta \setminus B(x^*, t)) \), if \( \delta \) is sufficiently small then we may assume that \( x_\delta \in \Delta \). Altogether, we have that \( x_\delta \in \Delta \setminus B(x^*, t + \delta) \). Since \( W \) is continuous on \( \Delta \), if \( \delta \) is sufficiently small, we have \( W(x_\delta) \leq W(x) + \varepsilon \). Therefore,

\[
g(t + \delta) = \inf_{x \in \Delta \setminus B(x^*, t + \delta)} W(x) \leq W(x_\delta) \leq W(x) + \varepsilon \leq g(t) + 2\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, this shows that \( g \) is continuous.

Next, define \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \),

\[
\alpha(t) = \inf \{ g(\| y - x^* \|) \mid y \in \Delta, \ U(y) \geq t \}.
\]

Then \( \alpha(0) = 0 \), and \( \alpha \) is non-negative and non-decreasing. We would also like to show that \( \alpha \) is positive definite and continuous. Indeed, assume that \( \alpha(0) = 0 \). Then for every \( \varepsilon > 0 \), there exists \( y_\varepsilon \in \Delta \) such that \( g(\| y_\varepsilon - x^* \|) \leq \varepsilon \). Since \( g \) is positive definite and continuous, it follows that \( y_\varepsilon \xrightarrow{\varepsilon \to 0} x^* \). Thus, we must have \( t = 0 \). This shows that \( \alpha \) is positive definite. In order to prove that \( \alpha \) is continuous, let \( \varepsilon > 0 \) and \( t \geq 0 \). Let \( y \in \Delta \) with \( U(y) \geq t \) be such that \( g(\| y - x^* \|) \leq \alpha(t) + \varepsilon \). Let \( \delta > 0 \), and choose \( y_\delta \in \Delta \) such that \( U(y_\delta) \geq t + \delta \). Since \( U \) is continuous on \( \Delta \), it follows that we can choose \( y_\delta \) such that \( y_\delta \xrightarrow{\delta \to 0} y \). Since \( g \) is continuous, if \( \delta \) is sufficiently small, then

\[
g(\| y_\delta - x^* \|) \leq g(\| y - x^* \|) + \varepsilon \leq \alpha(t) + 2\varepsilon,
\]

and taking the infimum over the left hand side gives

\[
\alpha(t + \delta) \leq \alpha(t) + 2\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary and since \( \alpha \) is non-decreasing, it follows that \( \alpha \) is continuous.

To conclude the proof, note that

\[
U(x) - U(T_{S,L_\lambda}x) \geq \inf_{y \in \Delta \setminus B(x^*, \| x - x^* \|)} [U(y) - U(T_{S,L_\lambda}y)] = g(\| x - x^* \|) \geq \alpha(U(x)).
\]

By (1.7), \( T_{S,L_\lambda}x \in \Delta \) whenever \( x \in H_+ \). Thus,

\[
V(x) \stackrel{(3.1)}{=} F(T_{S,L_\lambda}x) - F(x^*) \stackrel{(3.2)}{=} U(T_{S,L_\lambda}x).
\]

Therefore,

\[
V(x) - V(T_{S,L_\lambda}x) = U(T_{S,L_\lambda}x) - U(T^2_{S,L_\lambda}x) \geq \alpha(U(T_{S,L_\lambda}x)) = \alpha(V(x)),
\]

and the proof is complete.

We are now in a position to prove Theorem 2.2 and Theorem 2.3.
Proof of Theorem 2.2. By the definition of $V$ (3.1), together with (1.7) and (1.8), it follows that $V$ is continuous on $H_+$, which is an open set. Therefore, since we choose $\omega_1 = \omega_2 = V$, all functions are continuous. Clearly by the choice of $\omega_1$, $\omega_2$, it follows that (2.3) holds with $f_1(t) = f_2(t) = t$, which is of class $\mathcal{K}_\infty$. Condition (2.4) holds by Proposition 3.1. Finally, it follows directly from (1.3) that if $x \in H_+$ is such that $T_{S,L_x}x = x^*$, then $x = x^*$. Thus, by Theorem 1.1 and (3.1), we have that $V(x) = 0 \iff x = x^*$. On the other hand, by the definition of $A$ (2.2) and the choice of $\omega_1$, 

$$x \in A \iff \sup_{n \in \mathbb{Z}_+} \sup_{\phi \in S(x)} V(\phi(n,x)) = 0 \iff \phi(n,x) = x^*, \forall n \in \mathbb{Z}_+, \forall \phi \in S(x).$$

Since the only fixed point of $T_{S,L_x}$ in $H_+$ is $x^*$, it follows that $x \in A \iff x = x^*$. Hence, (2.5) holds as well, and so $V$ is a continuous Lyapunov function on $H_+$ in the sense of Definition 2.3 with respect to $(\omega_1,\omega_2)$. Therefore, by Theorem 2.1, the Douglas-Rachford iteration (1.2) is robustly $\mathcal{K}\mathcal{L}$-stable on $H_+$, and this completes the proof. \hfill \Box

Proof of Theorem 2.3. By Theorem 2.2, the Douglas-Rachford iteration (1.2) is robustly $\mathcal{K}\mathcal{L}$-stable on $H_+$. Therefore, there exists $\sigma : H_+ \to \mathbb{R}_+$ which satisfies $\sigma(x) = 0 \iff x = x^*$ and such that the $\sigma$-perturbation of $T_{S,L_x}$ is $\mathcal{K}\mathcal{L}$-stable on $H_+$. Define $\tau : H_+ \to \mathbb{R}_+$,

$$\tau(x) = \min \left\{ \sigma(x), \frac{1}{2} \langle x, e_1 \rangle \right\}. \hspace{1cm} (3.6)$$

Note that $\tau \leq \sigma$ and also $\tau(x) = 0 \iff x = x^*$. Note also that by Remark 2.1, the $\tau$-perturbation of $T_{S,L_x}$ is $\mathcal{K}\mathcal{L}$-stable. Thus, by Theorem 2.2, there exists a function $\beta$ of class $\mathcal{K}\mathcal{L}$, such that for all $x \in K$, all $\phi \in S_\tau(x, T_{S,L_x})$, and all $n \in \mathbb{Z}_+$,

$$V(\phi(x,n)) \leq \beta(V(x,n)). \hspace{1cm} (3.7)$$

If $K \subseteq H_+$ is compact, it is bounded and there exists $b \in (0,\infty)$ such that $\inf_{x \in K} \langle x, e_1 \rangle \geq b$. Therefore, by the definition of $\tau$, it follows that $K_\tau$ is bounded and $\inf_{x \in K_\tau} \langle x, e_1 \rangle \geq \frac{1}{2}b > 0$. This means that $K_\tau$, the closure of $K_\tau$, is compact and satisfies $K_\tau \subseteq H_+$. Since $V$ is continuous on $H_+$, there exists $M \in (0,\infty)$ such that $\sup_{x \in K_\tau} V(x) = M$. Therefore, given $n \in \mathbb{Z}_+$,

$$\sup_{\phi \in S_\tau(K,T_{S,L_x})} V(\phi(x,n)) \leq \sup_{x \in K_\tau} \beta(V(x,n)) \leq \beta \left( \sup_{x \in K_\tau} V(x,n) \right) = \beta(M,n), \hspace{1cm} (3.8)$$

where in (*) we used the fact that $\beta(\cdot, n)$ is increasing for all $n \in \mathbb{Z}_+$. Thus, since $\beta(M,n) \xrightarrow{n \to \infty} 0$ for all $M \in \mathbb{R}_+$, (3.8) gives

$$\lim_{n \to \infty} \sup_{\phi \in S_\tau(K,T_{S,L_x})} V(\phi(x,n)) = 0. \hspace{1cm} (3.9)$$

Assume that there exist $\{x_n\}_{n=0}^\infty \subseteq K$, $\{\phi_n\}_{n=0}^\infty \subseteq S_\tau(K,T_{S,L_x})$, and $\varepsilon > 0$ such that

$$\inf_{n \in \mathbb{Z}_+} \|\phi_n(x_n,n) - x^*\| \geq \varepsilon.$$

Then since $T_{S,L_x}$ is continuous on $H_+$ and since $T_{S,L_x}x = x \iff x = x^*$, there exists $\varepsilon' > 0$ such that

$$\inf_{n \in \mathbb{Z}_+} \|T_{S,L_x}(\phi_n(x_n,n)) - x^*\| \geq \varepsilon',$$

and so by Theorem 1.1, there exists $\varepsilon'' > 0$ such that

$$\inf_{n \in \mathbb{Z}_+} V(\phi_n(x_n,n)) = \inf_{n \in \mathbb{Z}_+} \left[ F(T_{S,L_x}(\phi_n(x_n,n))) - F(x^*) \right] \geq \varepsilon''.$$
But this is a contradiction to (3.9). Therefore,
\[
\lim_{n \to \infty} \sup_{\phi \in \mathcal{S}(K,T,S,L)} \|\phi(x,n) - x^*\| = 0,
\]
and this completes the proof. \qed

**Remark 3.1.** The choice of 1/2 in the definition of \( \tau \) (3.6) is not of any significance. One can choose any number in (0, 1).

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