SPECTRUM OF THE FRACTIONAL $p-$LAPLACIAN IN $\mathbb{R}^N$ AND DECAY ESTIMATE FOR POSITIVE SOLUTIONS OF A SCHRÖDINGER EQUATION

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Abstract. In this paper, we prove the existence of unbounded sequence of eigenvalues for the fractional $p-$Laplacean with weight in $\mathbb{R}^N$. We also show a nonexistence result when the weight has positive integral. In addition, we show some qualitative properties of the first eigenfunction including a sharp decay estimate. Finally, we extend the decay result to the positive solutions of a Schrödinger type equation.

1. Introduction and main results

In this paper, we study the following eigenvalue problem

\[(−\Delta_p)^s u(x) = \lambda g(x)|u(x)|^{p−2}u(x) \quad \text{in } \mathbb{R}^N,\]

where $0 < s < 1$, $p > 1$, and $g$ is a weight function satisfying some conditions to be specified later. Here $(-\Delta_p)^s$ denotes the fractional $p-$Laplacean operator, that is

\[(-\Delta_p)^s u(x) := 2 \text{ P.V.} \int_{\mathbb{R}^N} \frac{|u(x)−u(y)|^{p−2}(u(x)−u(y))}{|x−y|^{N+sp}} dy,\]

where P.V. is a commonly used abbreviation for in the principal value sense.

Before we describe our principal results, we will give some motivations. Nonlocal equation of $p-$Laplace type where introduced in [6, 7, 15, 28]. Fractional Sobolev spaces semi-norms (see for an introduction to the topic in [19] and below) are natural in the weak form and functional associated with the operator $(-\Delta_p)^s$, therefore eigenvalues can be studied in bounded domains using variational methods see [31, 16, 24, 10, 11].

All spectrum in bounded domain for the fractional Laplacian ($p=2$) is studied in [34], see also [33]. The variational unbounded sequence of eigenvalues of the fractional $p-$Laplacian is studied in [11].

In unbounded domain, in particular $\mathbb{R}^N$, some weight function with some condition needs to be introduce, moreover non-existence may also appear. Spectrum in $\mathbb{R}^N$ for local problems with weights are studied in [13, 2, 26, 21, 3]. As far as we know, there isn’t an extension of these type of results for the nonlocal setting even when $p=2$. Therefore, one of the main purpose of this work is study the spectrum in $\mathbb{R}^N$ in a nonlocal setting.

Finally, let observe that the eigenvalues are a starting point in study some type bifurcation results in $\mathbb{R}^N$, see for example [22] and [14] in the local case. Other type of bifurcation results in $\mathbb{R}^N$ for the fractional Laplacian can be found in [20]. While in [16], the authors show a bifurcation results in bounded domain for the fractional $p-$Laplacian.

Now we will describe our results. As in the local case, we will split the discussion in to two cases $sp < N$ and $sp \geq N$, where different approaches are needed. But first we need to introduce the theoretical framework for them.
The fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$ is defined to be the set of functions $u \in L^p(\mathbb{R}^N)$ such that

$$\|u\|_{s,p}^p := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy < \infty.$$  

While, the closure of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{s,p}$ is denoted by $W^{s,p} (\mathbb{R}^N)$. For more details about the spaces, see Section 2.

**Definition 1.1.** Let $g \in L^\infty (\mathbb{R}^N)$ be such that $g \not\equiv 0$.

(i) If $sp < N$ and $g \in L^{N/sp}(\mathbb{R}^N)$ we say that a pair $(u, \lambda) \in W^{s,p}(\mathbb{R}^N) \times \mathbb{R}$ is a weak solution of (1) if

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}} \, dx \, dy = \lambda \int_{\mathbb{R}^N} g(x)|u(x)|^{p-2}u(x)\varphi(x) \, dx,$$

for all $\varphi \in C^\infty_0(\mathbb{R}^N)$.

(ii) If $sp \geq N$, we say that a pair $(u, \lambda) \in W^{s,p}(\mathbb{R}^N) \times \mathbb{R}$ is a weak solution of (1) if (2) holds for all $\varphi \in C^\infty_0(\mathbb{R}^N)$.

(iii) In both cases, a pair $(u, \lambda)$ is called an eigenpair, in which case, $\lambda$ is called an eigenvalue and $u$ a corresponding eigenfunction.

Lastly, $g_+(x) := \max\{g(x), 0\}$ and $g_-(x) := \max\{-g(x), 0\}$.

Our first aim is extended the results for the local case given by [3, 26] to the nonlocal case. Let start with the case $sp < N$.

**Theorem 1.1.** Assume that $sp < N$ and $g \in L^\infty(\mathbb{R}^N) \cap L^{N/sp}(\mathbb{R}^N)$.

(i) If $g_+ \not\equiv 0$ then there exists a sequence of eigenpairs $\{(u_n, \lambda_n(g))\}_{n \in \mathbb{N}}$ such that

$$\int_{\mathbb{R}^N} g(x)|u_n(x)|^p \, dx = 1 \quad \forall n \in \mathbb{N},$$

and

$$0 < \lambda_1(g) < \lambda_2(g) \leq \cdots \leq \lambda_n(g) \to \infty.$$  

Moreover

$$\lambda_1(g) = \min \left\{ \|u\|_{s,p}^p : u \in W^{s,p}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} g(x)|u(x)|^p \, dx = 1 \right\},$$

is a simple eigenvalue with constant sign eigenfunction.

(ii) If $g_+ \not\equiv 0$ then there exist two sequence of eigenpairs $\{(u_n, \lambda_n^+(g))\}_{n \in \mathbb{N}}$ and $\{(v_n, \lambda_n^-(g))\}_{n \in \mathbb{N}}$ such that

$$\int_{\mathbb{R}^N} g(x)|u_n(x)|^p \, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} g(x)|v_n(x)|^p \, dx = -1$$

and

$$0 < \lambda_1^+(g) < \lambda_2^+(g) \leq \cdots \leq \lambda_n^+(g) \to \infty \quad \text{and} \quad 0 > \lambda_1^-(g) > \lambda_2^-(g) \geq \cdots \geq \lambda_n^-(g) \to -\infty.$$  

Moreover

$$\lambda_1^+(g) = \min \left\{ \|u\|_{s,p}^p : u \in W^{s,p}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} g(x)|u(x)|^p \, dx = 1 \right\},$$

$$-\lambda_1^-(g) = \min \left\{ \|u\|_{s,p}^p : u \in W^{s,p}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} g(x)|u(x)|^p \, dx = -1 \right\},$$

are simple eigenvalues with constant sign eigenfunctions.
Let now discuss the case $sp \geq N$ and give the following result.

**Theorem 1.2.** Assume $sp \geq N$ and $g = g_1 - g_2$ satisfies

- $g_1(x) \geq 0$ a.e. in $\mathbb{R}^N$ and $g_1 \in L^\infty(\mathbb{R}^N) \cap L^{\infty/N}(\mathbb{R}^N) \setminus \{0\}$, with $N_0 \in \mathbb{N}$ such that $N_0 > sp$;
- $g_2(x) \geq \varepsilon > 0$ a.e. in $\mathbb{R}^N$.

Then there exists a sequence of eigenpairs $\{(u_n, \lambda_n)\}_{n \in \mathbb{N}}$ such that

$$
\int_{\mathbb{R}^N} g(x)|u_n(x)|^p \, dx = 1 \quad \forall n \in \mathbb{N},
$$

and

$$
0 < \lambda_1(g) < \lambda_2(g) \leq \cdots \leq \lambda_n(g) \to \infty.
$$

Moreover

$$
\lambda_1(g) = \min \left\{ \|u\|_{L^p}^p : u \in W^{s,p}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} g(x)|u(x)|^p \, dx = 1 \right\},
$$

is a simple eigenvalue with constant sign eigenfunction.

We also give the following non-existence results.

**Theorem 1.3.** If $sp > N$, $g \in L^\infty(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} g(x)dx > 0$ then there is not a positive principle eigenvalue.

The existence part, as in the local case, is based on Lusternik–Schnirelman principle, for details see [12, 35, 36].

On the other hand, if we assume $g \in L^\infty(\mathbb{R}^N)$ then the solution found in the above theorems are Hölder continuous (see Section 5). Therefore we get

$$
\lim_{|x| \to +\infty} u(x) = 0
$$

for any eigenfunction. Motivated by this result, we study the asymptotic behaviour of positive eigenfunctions. More precisely, our fourth result is a sharp decay estimate for the positive eigenfunction $u$ associated with $\lambda_1$ given by Theorem 1.2.

**Theorem 1.4.** Assume $sp \geq N$ and $g = g_1 - g_2$ satisfies

- $g_1(x) \geq 0$ a.e. in $\mathbb{R}^N$ and $g_1 \in L^\infty(\mathbb{R}^N) \cap L^{\infty/N}(\mathbb{R}^N) \setminus \{0\}$, with $N_0 \in \mathbb{N}$ such that $N_0 > sp$;
- $g_2(x) \geq \varepsilon > 0$ a.e. in $\mathbb{R}^N$ and $g_2 \in L^\infty(\mathbb{R}^N)$;
- $g(x) < -\delta < 0$ for $|x|$ large enough.

Let $u \in W^{s,p}(\mathbb{R}^N)$ be a positive eigenfunction associated to $\lambda_1(g)$. Then there exists $k > 1$ such that

$$
C_1|x|^{-\frac{sp}{s}} \leq u(x) \leq C_2|x|^{-\frac{sp}{s}}
$$

for any $|x| > k$ and some positive constants $C_1$ and $C_2$.

The base in established this sharp decay estimate is Lemma 7.1, that is a nonlinear version of [8, Lemma 2.1](case $p = 2$). This lemma is the computation of the fractional $p$–Laplacean for a power like function at infinity that give good sub and super-solutions. Moreover, these sub and super-solutions can also be used to prove decay estimate Schrödinger type equations, such result, in the case $p = 2$, can be found in [23]. We remark that our nonlinear version of [8, Lemma 2.1], can be useful for other proposes like for example in the study of parabolic problems as in [8].
We would also like to remark that the above theorem shows a difference between the local and nonlocal cases since in the local case the eigenfunctions decay exponentially at infinity, see \[26\].

Finally, we are concerned with the decay rate at infinity of all positive ground state solutions of the next autonomous Schrödinger equations
\[
\begin{aligned}
\left\{ \begin{array}{ll}
(-\Delta_p)^s u(x) + \mu |u|^{p-2}u & = f(u) \quad \text{in } \mathbb{R}^N, \\
u & \in W^{s,p}(\mathbb{R}^N) \\
u(x) & > 0 
\end{array} \right.
\end{aligned}
\]

for all \(x \in \mathbb{R}^N, \mu > 0\).

The existence of at least one positive ground state solution of (4) was recently proved in \[4\] under standard assumptions on \(f\) including subcritical growth, for details see \((f_1)-(f_5)\) in Section 7 and \[4\] where also other references for existence results can be found.

In our last main results, we prove that the positive ground state solutions of (4) also satisfies (3) for large \(|x|\) large.

**Theorem 1.5.** Let \(sp < N\) and suppose that \(f\) verifies \((f_1)-(f_5)\). If \(v\) is a positive ground state solutions of (4) there is \(k >> 1\) such that
\[
C_1 |x|^{-\frac{N+sp}{p-1}} \leq v(x) \leq C_2 |x|^{-\frac{N+sp}{p-1}} \quad \forall |x| > k
\]
for some positive constant \(C_1\) and \(C_2\).

To end this introduction, we want to mention that, as far as we know, the main results of this work are new also in the linear case \(p = 2\) that corresponds to the fractional Laplacian.

The rest of this paper is organized as follows: in Section 2, we introduce the the notation a preview some preliminaries. In Section 3 (resp, Section 4), we study the case \(sp < N\) (resp. \(sp \geq N\)). In Section 5, we obtained the Hölder regularity. In Section 6, we study the principal eigenvalue. Finally, in Section 7, we prove the decay estimates.

2. Notations and preliminaries

For the benefit of the reader, we start by including the basic tools that will be needed in subsequent sections. The known results are generally stated without proofs, but we provide references where they can be found. In additional, we take this opportunity to introduce some of our notational conventions.

2.1. Sobolev spaces. Let \(\Omega\) be an open subset of \(N\)-dimensional euclidean space \(\mathbb{R}^N\). Let \(C^\infty(\Omega)\) denote the space of infinitely differentiable functions on \(\Omega\); by \(C_0^\infty(\mathbb{R}^N)\) we denote the space of functions in \(C^\infty(\mathbb{R}^N)\) with compact support on \(\Omega\).

Let \(1 \leq p \leq \infty\) and \(L^p(\Omega)\) be the space of Lebesgue measurable functions \(u\) on \(\Omega\), such that
\[
\|u\|_{L^p(\Omega)} := \begin{cases}
\left( \int_\Omega |u(x)|^p \, dx \right)^{1/p} & \text{if } 1 \leq p < \infty, \\
\inf \{ M : |u(x)| \leq M \text{ for almost every } x \} & \text{if } p = \infty,
\end{cases}
\]
is finite. If \(\Omega = \mathbb{R}^N\), we simply use the notation \(|u|_p\) instead of \(|u|_{L^p(\mathbb{R}^N)}\).
Let $0 < s < 1$ and $1 < p < \infty$. The fractional Sobolev spaces $W^{s,p}(\Omega)$ is defined to be the set of functions $u \in L^p(\Omega)$ such that
\[
|u|_{W^{s,p}(\Omega)}^p := \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty.
\]
The fractional Sobolev spaces admit the following norm
\[
\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + |u|_{W^{s,p}(\Omega)}^p\right)^{\frac{1}{p}}.
\]
The space $W^{s,p}(\Omega)$ endowed with the norm $\|\cdot\|_{W^{s,p}(\Omega)}$ is a reflexive Banach space. We denote by $\tilde{W}^{s,p}(\Omega)$ the space of all $u \in W^{s,p}(\Omega)$ such that $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$, where $\tilde{u}$ is the extension by zero of $u$.

The closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm
\[
[u]_{p,s}^p := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy
\]
is denoted by $W^{s,p}(\mathbb{R}^N)$.

In the case $sp < N$, we consider the spaces
\[
\mathcal{X}_0^{a,p}(\Omega) := \left\{ u \in L^{p^*_s}(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega, \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty \right\},
\]
and
\[
\mathcal{X}^{a,p}(\mathbb{R}^N) := \left\{ u \in L^{p^*_s}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty \right\},
\]
where $p^*_s = \frac{pN}{N - sp}$.

The proof of the following results can be found in [32, page 521].

**Theorem 2.1.** If $sp < N$, then for an arbitrary function $u \in \mathcal{W}^{s,p}(\mathbb{R}^N)$ there holds
\[
\|u\|_{p^*_s}^p \leq C(N, p) \frac{s(1 - s)}{(N - sp)^{sp}} [u]_{p,s}^p
\]
where $C(N, p)$ is a function of $N$ and $p$.

In other words, if $sp < N$ then $\mathcal{W}^{s,p}(\mathbb{R}^N) \subseteq \mathcal{X}^{a,p}(\mathbb{R}^N)$. In fact, adapting ideas of the proofs of Proposition 4.27 in [18] and Proposition 2.4 in [30] we can proof the following result.

**Theorem 2.2.** If $sp < N$ then $\mathcal{W}^{s,p}(\mathbb{R}^N) = \mathcal{X}^{a,p}(\mathbb{R}^N)$.

**Proof.** By Theorem 2.1, we only need to show that $\mathcal{X}^{a,p}(\mathbb{R}^N) \subseteq \mathcal{W}^{s,p}(\mathbb{R}^N)$.

Let $u \in \mathcal{X}^{a,p}(\mathbb{R}^N)$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that
\[
0 \leq \varphi \leq 1 \text{ in } \mathbb{R}^N, \varphi(x) = 1 \text{ if } |x| \leq 1 \text{ and } \varphi(x) = 0 \text{ if } |x| \geq 2.
\]
We define $\varphi_n(x) := \varphi(x/n)$ and $u_n(x) = \varphi_n(x)u(x)$.

**Step 1.** We claim that $u_n \in \mathcal{W}^{s,p}(\mathbb{R}^N)$.

Since $\varphi$ has compact support and $u \in L^{p^*_s}(\mathbb{R}^N)$, we have that $u_n \in L^{p}(\mathbb{R}^N)$. Therefore we only need to show that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty.
\]
Observe that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \\
\leq C \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N + sp}} \, dx \, dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi_n(x) - \varphi_n(y)|^p|u(x)|^p}{|x - y|^{N + sp}} \, dx \, dy \right).
\]

Then, to prove (6) it suffices to show that the second term on the right-hand sides of the above inequality is finite.

Let \( B_n = \{ x \in \mathbb{R}^N : |x| < n \} \) and \( B_n^c = \mathbb{R}^N \setminus B_n \). Then
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi_n(x) - \varphi_n(y)|^p|u(x)|^p}{|x - y|^{N + sp}} \, dx \, dy = \int_{B_n} \int_{B_n^c} \frac{\varphi_n(x) - \varphi_n(x)|^p|u(x)|^p}{|x - y|^{N + sp}} \, dx \, dy + \int_{B_n} \int_{B_n^c} \frac{\varphi_n(x) - \varphi_n(y)|^p|u(x)|^p}{|x - y|^{N + sp}} \, dx \, dy
\]
\[
= I_1 + I_2.
\]

By (5) and Hölder’s inequality, we get
\[
I_1 \leq C \left\{ \int_{B_n^c \cap B_n} \frac{|\varphi_n(x) - 1|^p|u(x)|^p}{(|x| - n)^{sp}} \, dx + \| u \|_{p^*_s} \int_{B_n} \left( \int_{B_n^c} \frac{dx}{|x - y|^{N + Ns/2}} \right)^{s/p} \, dy \right\}
\]
\[
\leq C\| u \|_{p^*_s} \left\{ \left( \int_{B_n^c \cap B_n} \frac{|\varphi_n(x) - \varphi_n(x)|^p(\delta/|x|)^{N_s}}{(|x| - n)^{N}} \, dx \right)^{s/p} + 1 \right\}
\]
\[
\leq C\| u \|_{p^*_s} \left\{ \left( \int_{B_n^c \cap B_n} \|\nabla \varphi_n\|_{\infty}^{N_s - N} (|x| - n)^{N_s - N} \, dx \right)^{s/p} + 1 \right\}
\]
\[
< \infty.
\]

On the other hand, again by (5) and Hölder’s inequality,
\[
I_2 = \int_{B_n^c \cap B_n} \int_{B_n^c \cap B_n} \frac{|\varphi_n(x) - \varphi_n(y)|^p|u(x)|^p}{|x - y|^{N + sp}} \, dx \, dy + \int_{B_n^c \cap B_n} \int_{B_n^c \cap B_n} \frac{\varphi_n(x) - \varphi_n(y)|^p|u(x)|^p}{|x - y|^{N + sp}} \, dx \, dy
\]
\[
\leq \int_{B_n} \int_{B_n} \frac{\varphi_n(x) - \varphi_n(y)|^p|u(x)|^p}{|x - y|^{N + sp}} \, dx \, dy + \int_{B_n^c \cap B_n} \int_{B_n^c \cap B_n} \frac{\varphi_n(x) - \varphi_n(y)|^p}{|x - y|^{N + sp}} \, dx \, dy
\]
\[
\leq C \left\{ \int_{B_n} \int_{B_n} \frac{|x - y|^{p(1-s) - N}|u(x)|^p \, dx \, dy + \int_{B_n} \frac{|\varphi_n(x)|^p|u(x)|^p}{(2n - |x|)^{s/p}} \, dx \right\}
\]
\[
+ \| u \|_{p^*_s} \int_{B_n} \frac{|\varphi_n(x)|^p}{(2n - |x|)^{s/p}} \, dx \, dy
\]
\[
\leq C \left\{ \| u \|_{p^*_s} + \int_{B_n} \frac{(2n - |x|)^{p(1-s)}}{|x|} \, dx + \int_{B_n} \frac{|\varphi_n(y)|^p}{(2n - |y|)^{N}} \, dy \right\}
\]
\[
\leq C \left\{ \| u \|_{p^*_s} + \int_{B_n} (2n - |y|)^{p-N} \, dy \right\}
\]
\[
< \infty.
\]

Therefore \( I_1 + I_2 < \infty \) then (6) holds.
Step 2. We now claim that \([u_n - u]_{s,p} \to 0\).

It is clear that \(u_n \to u\) strongly in \(L^{p^*}(\mathbb{R}^N)\). Our first step is to prove that \([u_n - u]_{s,p} \to 0\) as \(n \to \infty\). Observe that

\[
[u_n - u]_{s,p}^p = 2 \int_{B_1^n \cap B_2^n} \frac{v_n(x,y)}{|x - y|^{N + sp}} dy dx + \int_{B_2^n} \frac{v_n(x,y)}{|x - y|^{N + sp}} dy dx \leq 2 J_n^1 + J_n^2,
\]

where \(v_n(x,y) = |(u_n(x) - u(x)) - (u_n(y) - u(y))|^p\). Then, we only need to show that \(J_n^1\) and \(J_n^2\) converge to 0.

Let’s prove that \(J_n^1 \to 0\) as \(n \to \infty\). By (5) and Hölder’s inequality,

\[
J_n^1 \leq \int_{B_1^n \cap B_2^n} \frac{|1 - \varphi_n(y)| |u(y)|^p}{|y - n|^{sp}} \frac{dxdy}{|x - y|^{N + sp}} + \int_{B_2^n} \frac{|u(y)|^p}{|x - y|^{N + sp}} dy dx \\
\leq C(N, s, p) \int_{B_1^n \cap B_2^n} \frac{|\varphi_n(y)| |u(y)|^p}{|y - n|^{sp}} \frac{dxdy}{|x - y|^{N + sp}} \\
+ \int_{B_2^n} \left( \int_{B_2^n} \frac{|u(y)|^p}{|x - y|^{N + sp}} \right)^{sp/N} dx \left( \int_{B_2^n} |u(y)|^{p^*} dy \right)^{N - sp/N} \\
\leq \frac{C(N, s, p, \varphi)}{n^p} \left( \int_{B_2^n} \frac{(|y - n|)^{sp}}{|x - y|^{N + sp}} \right)^{sp/N} \left( \int_{B_2^n} |u(y)|^{p^*} dy \right)^{N - sp/N} \\
+ C(N, s, p) \left( \int_{B_2^n} |u(y)|^{p^*} dy \right)^{N - sp/N},
\]

by a simple change of variable

\[
J_n^1 \leq C(N, s, p, \varphi) \left( \int_{B_2^n} \frac{(|y - n|)^{sp}}{|x - y|^{N + sp}} \right)^{sp/N} \left( \int_{B_2^n} |u(y)|^{p^*} dy \right)^{N - sp/N} \\
+ C(N, s, p) \left( \int_{B_2^n} |u(y)|^{p^*} dy \right)^{N - sp/N} \\
\leq C(N, s, p, \varphi) \left( \int_{B_2^n} |u(y)|^{p^*} dy \right)^{N - sp/N} \to 0 \text{ since } u \in L^{p^*}(\mathbb{R}^N).
\]

Our next aim is to show that \(J_n^2 \to 0\) as \(n \to \infty\). Observe that for any \(x, y \in B_1^n\) we have

\[
u_n(x) - u_n(y) = \begin{cases} 
\varphi_n(x)u_n(x) & \text{if } x \in B_1^n \cap B_2^n, y \in B_2^n, \\
-\varphi_n(y)u_n(y) & \text{if } y \in B_1^n \cap B_2^n, x \in B_2^n, \\
\varphi_n(x)u_n(x) - \varphi_n(x)u_n(y) & \text{if } x, y \in B_1^n \cap B_2^n, \\
0 & \text{if } x, y \in B_2^n.
\end{cases}
\]
Then
\[
J_n^2 \leq C(p) \left\{ \int_{B_n^c \cap B_n} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy + 2 \int_{B_n^c \cap B_n} \int_{B_n^c} \frac{|\varphi_n(y)u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy 
\right. 
\left. + \int_{B_n^c \cap B_n} \int_{B_n^c} \frac{|\varphi_n(x)|^p |u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy 
\right. 
\right. 
\left. + \int_{B_n^c \cap B_n} \int_{B_n^c} \frac{|\varphi_n(x) - \varphi_n(y)|^p |u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \right\} 
\leq C(p) \left\{ 2 \int_{B_n^c} \int_{B_n^c} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy + 2 \int_{B_n^c \cap B_n} \int_{B_n^c} \frac{|\varphi_n(y)u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy 
\right. 
\right. 
\left. + \int_{B_n^c \cap B_n} \int_{B_n^c} \frac{|\varphi_n(x) - \varphi_n(y)|^p |u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \right\}.
\]

Since \( u \in X^{s,p}(\mathbb{R}^N) \), by dominated convergence theorem, we get
\[
\int_{B_n^c} \int_{B_n^c} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \to 0 \quad \text{as} \quad n \to \infty.
\]

On the other hand, by Hölder’s inequality
\[
\int_{B_n^c \cap B_n} \int_{B_n^c} \frac{|\varphi_n(y)u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy = C(N, s, p) \int_{B_n^c \cap B_n} \frac{|\varphi_n(y)u(y)|^p}{|2n - |y||^{sp}} \, dy 
\leq C(N, s, p, \varphi) \int_{B_n^c \cap B_n} |2n - |y||^{(1-s)p} |u(y)|^p \, dy 
\leq C(N, s, p, \varphi) \left( \int_{B_n^c} |u(y)|^{p*r} \, dy \right)^{\frac{N - sp}{N}} \to 0 \quad \text{since} \quad u \in L^{p*}(\mathbb{R}^N).
\]

Finally
\[
\int_{B_n^c \cap B_n} \int_{B_n^c \cap B_n} \frac{|\varphi_n(x) - \varphi_n(y)|^p |u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy 
\leq C(\varphi) \int_{B_n^c \cap B_n} \int_{B_n^c \cap B_n} |x - y|^{p(1-s) - N} |u(y)|^p \, dx \, dy 
\leq C(N, s, p, \varphi) \int_{B_n^c} |u(y)|^{p} \, dy 
\leq C(N, s, p, \varphi) \left( \int_{B_n^c} |u(y)|^{p*r} \, dy \right)^{\frac{N - sp}{N}} \to 0 \quad \text{since} \quad u \in L^{p*}(\mathbb{R}^N).
\]

Hence, \( J_n^2 \to 0 \) as \( n \to \infty \).

**Step 3.** Finally, we show that \( u \in W^{s,p}(\mathbb{R}^N) \).

By step 1, we have that \( \{u_n\}_{n \in \mathbb{N}} \subset W^{s,p}(\mathbb{R}^N) \). Then for all \( n \in \mathbb{N} \) there is \( \phi_n \in C_0^\infty(\mathbb{R}^N) \) such that \( \|u_n - \phi_n\|_{s,p} \leq 1/n \). Therefore
\[
[u - \phi_n]_{s,p} \leq \|u - u_n\|_{s,p} + \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty \quad \text{(step 2)}.
\]

Thus \( u \in W^{s,p}(\mathbb{R}^N) \). \( \Box \)

It is easy to see that \( W^{s,p}(\mathbb{R}^N) \) is a reflexive Banach space. Now the proof of the following results is standard.
Corollary 1. Let $sp < N$, and $g \in L^{N/p}(\mathbb{R}^N)$. Then there is a positive constant $C$ such that
\[
\int_{\mathbb{R}^N} g(x)|u(x)|^p \, dx \leq C\|u\|_{s,p}^p
\]
for all $u \in W^{s,p}(\mathbb{R}^N)$.

Corollary 2. Let $sp < N$, and $g \in L^{N/p}(\mathbb{R}^N)$. If \{u_n\}$_{n\in\mathbb{N}}$ is a sequence of \(W^{s,p}(\mathbb{R}^N)\) such that $u_n \rightharpoonup u$ weakly in $W^{s,p}(\mathbb{R}^N)$, then there is a subsequence \{u_{n_k}\}$_{k\in\mathbb{N}}$ such that
\[
\int_{\mathbb{R}^N} g(x)|u_{n_k}(x)|^p \, dx \to \int_{\mathbb{R}^N} g(x)|u(x)|^p \, dx,
\]
\[
\int_{\mathbb{R}^N} g(x)|u_{n_k}(x)|^{p-2}u_{n_k}(x)u(x) \, dx \to \int_{\mathbb{R}^N} g(x)|u(x)|^{p-2}u(x)u(x) \, dx.
\]

For more details about these spaces and their use, we refer the reader to [1, 18, 19, 25, 27, 32].

2.2. The principal eigenvalue in bounded domain. We start by introducing the definition of eigenpair in bounded domain.

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, $s \in (0, 1)$, $p \in (1, \infty)$ and $g \in \mathcal{A}(\Omega) := \{f \in L^\infty(\Omega) : |\{x \in \Omega : f(x) > 0\}| > 0\}$. We say that a pair $(u, \mu) \in W^{s,p}(\Omega) \times \mathbb{R}$ is a weak solution of
\[
\begin{cases}
(-\Delta_p)^s u(x) = \mu g(x)|u(x)|^{p-2}u(x) & \text{in } \Omega, \\
u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
if
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, dx \, dy = \mu \int_{\mathbb{R}^N} g(x)|u(x)|^{p-2}u(x)\varphi(x) \, dx
\]
for all $\varphi \in C^\infty_0(\Omega)$. A pair $(u, \mu)$ is called a Dirichlet eigenpair if $u$ is nontrivial and $(u, \mu)$ is a weak solution of (7). In which case, $\mu$ is called an Dirichlet eigenvalue and $u$ a corresponding eigenfunction.

Given $g \in \mathcal{A}(\Omega)$, the first Dirichlet eigenvalue is
\[
\mu_1(\Omega, g) := \min \left\{ \|u\|_{s,p}^p : u \in \overline{W}^{s,p}(\Omega) \text{ and } \int_{\mathbb{R}^N} g(x)|u(x)|^p \, dx = 1 \right\}.
\]

Moreover, we have the following result.

Theorem 2.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, $s \in (0, 1)$, $p \in (1, \infty)$, and $g \in \mathcal{A}$. There exists a positive function $u \in \overline{W}^{s,p}(\Omega) \cap L^\infty(\mathbb{R}^N)$, such that
\begin{itemize}
  \item $u$ is a minimizer of (8);
  \item $(u, \mu_1(\Omega, g))$ is a weak solution of (7).
\end{itemize}
Furthermore $\mu_1(\Omega, g)$ is simple.

Proof. See [16, Section 4].

Finally by a scaling argument, we have the following result.

Lemma 2.1. Let $B_R$ be the ball of center 0 and radius $R$ and $\mu_R = \mu_1(B_R, 1)$. Then
\[
\mu_1(B_R, 1) \leq \frac{1}{R^p}\mu_1(B_1, 1).
\]
3. Case $sp < N$

As mentioned in the introduction we shall establish the existence of a sequence of eigenvalues using the Lusternik–Schnirelman principle, see \cite{12,35,36}.

Let us consider
\[ \mathcal{M} := \left\{ u \in W^{s,p}(\mathbb{R}^N) : p \Psi(u) := \int_{\mathbb{R}^N} g(x)|u(x)|^p \, dx = 1 \right\}, \]
and $\Phi : W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$
\[ \Phi(u) := \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy. \]
It is known that $\Phi$ is weakly lower semicontinuous and that $\Phi$ and $\Psi$ are of class $C^1$.

Observe that, $(u, \lambda) \in \mathcal{M} \times \mathbb{R}$ is an eigenpair if only if $u$ is a critical point of $\Phi$ restricted to the manifold $\mathcal{M}$. Then, we are looking for the critical points of $\Phi$ restricted to the manifold $\mathcal{M}$. To find them, we will use the Lusternik–Schnirelman principle. For this reason we need to show Palais-Smale condition for the functional $\Phi$ on $\mathcal{M}$.

**Lemma 3.1.** Assume that $sp < N$, $g \in L^\infty(\mathbb{R}^N) \cap L^{N/(sp)}(\mathbb{R}^N)$ and $g \not\equiv 0$. Then the functional $\Phi$ satisfies the Palais–Smale condition on $\mathcal{M}$.

**Proof.** Given a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ such that $\{\Phi(u_n)\}_{n \in \mathbb{N}}$ is bounded and
\[ \Phi'(u_n) - \left[ [u_n]_{s,p} \right]^p \Phi'(u_n) \to 0 \quad \text{as} \quad n \to \infty, \] we want to show that there exist a function $u \in W^{s,p}(\mathbb{R}^N)$ and a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ of $\{u_n\}_{n \in \mathbb{N}}$ such that
\[ u_{n_k} \to u \quad \text{strongly in} \quad W^{s,p}(\mathbb{R}^N). \]

Since $W^{s,p}(\mathbb{R}^N) \subset W^{s,p}(\Omega)$ for any bounded smooth domain $\Omega$ and $\{\Phi(u_n)\}_{n \in \mathbb{N}}$ is bounded, there exist a function $u \in W^{s,p}(\mathbb{R}^N)$ and a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ of $\{u_n\}_{n \in \mathbb{N}}$ such that
\[ u_{n_k} \rightharpoonup u \quad \text{weakly in} \quad W^{s,p}(\mathbb{R}^N), \]
\[ u_{n_k} \to u \quad \text{strongly in} \quad L^p(\Omega), \]
for any bounded smooth domain $\Omega$. Furthermore, since $g \in L^{N/(sp)}(\mathbb{R}^N)$, by Corollary 2 we have that
\[ \int_{\mathbb{R}^N} g(x)|u_{n_k}(x)|^p \, dx \to \int_{\mathbb{R}^N} g(x)|u(x)|^p \, dx, \]
\[ \int_{\mathbb{R}^N} g(x)|u_{n_k}(x)|^{p-2} u_{n_k}(x) u(x) \, dx \to \int_{\mathbb{R}^N} g(x)|u(x)|^{p-2} u_{n_k}(x) u(x) \, dx. \]
Therefore $u \in \mathcal{M}$.

On the other hand, by (9) we get
\[ [u_{n_k}]_{s,p} p \left[ u \right]_{s,p} \geq \]
\[ \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{n_k}(x) - u_{n_k}(y)|^{p-2}(u_{n_k}(x) - u_{n_k}(y))(u(x) - u(y))}{|x-y|^{N+ps}} \, dx \, dy \]
\[ = [u_{n_k}]_{s,p} \int_{\mathbb{R}^N} g(x)|u_{n_k}(x)|^{p-2} u_{n_k}(x) u(x) \, dx + o(1). \]
Then, by (10) and (11), we have
\[ [u]_{s,p} \geq \liminf_{k \to \infty} [u_{n_k}]_{s,p} \geq [u]_{s,p}, \]
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that is

$$\liminf_{k \to \infty} \|u_{n_k}\|_{s,p} = \|u\|_{s,p}.$$  

Hence, taking a subsequence if necessary, $u_{n_k} \to u$ strongly in $W^{s,p}(\mathbb{R}^N)$.

Thus, by the Lusternik–Schnirelman principle, we get (i) of Theorem 1.1. For (ii) we consider additionally

$$\mathcal{M}^- \coloneqq \left\{ u \in W^{s,p}(\mathbb{R}^N) : p\Psi(u) \coloneqq \int_{\mathbb{R}^N} g(x)|u(x)|^p dx = -1 \right\},$$

that is not empty by assumption and so again by the Lusternik-Schnirelman principle we get the second sequence. The principal eigenvalue results of Theorem 1.1 are discussed in Section 6.

4. CASES $sp \geq N$

4.1. Nonexistence result. Our next aim is to show nonexistent result Theorem 1.3. The next result will be one of the keys to prove our nonexistence result.

Lemma 4.1. If $sp > N$, $g \in L^\infty(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} g(x) dx > 0$ then

$$\mu_1(B_R, g) \to 0 \text{ as } R \to \infty.$$  

Here $B_R$ denotes the ball of center 0 and radius $R$.

Proof. Let $\varphi \in C_0^\infty(B_1)$ be such that $\varphi(0) = 1$. For all $R > 1$ we define

$$\varphi_R(x) \coloneqq \varphi \left( \frac{x}{R} \right) \in C_0^\infty(B_R).$$

If $\int_{\mathbb{R}^N} g(x) dx < \infty$, by dominated convergence theorem

$$\int_{\mathbb{R}^N} g(x) |\varphi_R(x)|^p dx \to \int_{\mathbb{R}^N} g(x) dx$$

as $R \to \infty$.

On the other hand, if $\int_{\mathbb{R}^N} g(x) dx = \infty$, by dominated convergence theorem

$$\int_{\mathbb{R}^N} g_-(x) |\varphi_R(x)|^p dx \to \int_{\mathbb{R}^N} g_-(x) dx$$

as $R \to \infty$, and by Fatou’s lemma

$$\liminf_{R \to \infty} \int_{\mathbb{R}^N} g_+(x) |\varphi_R(x)|^p dx \geq \int_{\mathbb{R}^N} g_+(x) dx = \infty.$$  

Therefore, in both cases,

$$\int_{\mathbb{R}^N} g(x) |\varphi_R(x)|^p dx \to \int_{\mathbb{R}^N} g(x) dx$$

as $R \to \infty$.

Since $\int_{\mathbb{R}^N} g(x) dx > 0$, there exists $R_0 > 1$ such that

$$g \in \mathcal{A}(B_R) \quad \text{and} \quad \int_{\mathbb{R}^N} g(x) |\varphi_R(x)|^p dx > 0 \quad \forall R > R_0.$$
Then,

\[
0 < \mu_1(B_R, g) \leq \frac{\|\varphi_{R}^p\|_{s,p}}{\int_{\mathbb{R}^N} g(x)|\varphi_R(x)|^p dx} = \frac{1}{R^{sp-N}} \int_{\mathbb{R}^N} g(x)|\varphi_R(x)|^p dx \to 0 \text{ as } R \to \infty
\]

because \(sp > N\).

We now prove our non-existence result.

**Proof of Theorem 1.3.** Assume by contradiction the existence of a weak solution \((\lambda, u)\) of (1) such that \(\lambda > 0\) and \(u > 0\).

Since \(\int_{\mathbb{R}^N} g(x)dx > 0\) there exists \(R_0 > 1\) such that \(g \in A(B_R)\) for all \(R > R_0\).

Then, by Theorem 2.3, for any \(R > R_0\) there exists a positive function \(v_R \in \tilde{W}^{s,p}(B_R) \cap L^\infty(\mathbb{R}^N)\) such that \((v_R, \mu_1(B_R, g))\) is a weak solution of (7) with \(\Omega = B_R\) and

\[
\int_{\Omega} g(x)v_R(x)^p dx = 1.
\]

Let \(n \in \mathbb{N}\) and \(u_n(x) := u(x) + 1/n\). Since \(v_R, u \in W^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\) and \(v_R = 0\) in \(\mathbb{R}^N \setminus B_R\), we have that

\[
w_m(x) = \frac{v_R(x)^p}{u_m(x)^{p-1}} \in \tilde{W}^{s,p}(B_R).
\]

For further details we refer the reader to [16, Proof of Theorem 4.8].

By the discrete version of Picone’s identity (see [5]), we have

\[
0 \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_R(y) - v_R(x)|^p}{|x - y|^{n+sp}} dxdy - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|x - y|^{n+sp}} \left(\frac{v_R(y)^p}{u_m(y)^{p-1}} - \frac{v_R(x)^p}{u_m(x)^{p-1}}\right) dxdy \\
\leq \mu_1(B_R, g) - \lambda \int_{B_R} g(x)u(x)^{p-1}\frac{v_R(x)^p}{u_m(x)^{p-1}} dx.
\]

Then, by dominated convergence theorem, we have

\[
0 \leq \mu_1(B_R, g) - \lambda
\]

that is

\[
\lambda \leq \mu_1(B_R, g) \quad \forall R > R_0.
\]

Therefore \(\lambda = 0\) since \(\mu_1(B_R, g) \to 0\) as \(R \to \infty\) by Lemma 4.1. This contradiction establishes the result.

\(\square\)

### 4.2. Existence result

It follows from the proof of Theorem 1.1 that to show the existence of a sequence of eigenvalues we need to prove the following: if \(\{u_n\}_{n \in \mathbb{N}}\) is contained in

\[
\mathcal{N} := \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} g(x)|u|^p dx = 1 \right\}
\]

and \(\Phi(u_n)\) is bounded then \(\{u_n\}_{n \in \mathbb{N}}\) is bounded in \(W^{s,p}(\mathbb{R}^N)\). To prove this we will adapt ideas of [3] for the nonlocal case.

We now want to prove existence result Theorem 1.2. The key in the proof of Theorem 1.2 is the following result.
Then there is a constant \( C = C(N, M, s, p) \) such that
\[
\int_{\mathbb{R}^N} |f(x)||\varphi(x)|^p \, dx \leq CR^{\nu/pN_0} \|f\|_{s,p} (\|\varphi\|_{s,p}^p + \mu_1(B_R^M, 1))\|\varphi\|_p^p
\]
for in \( \varphi \in C_0^\infty(\mathbb{R}^N) \).

**Proof.** Let \((v, \mu_1(B_R^M, 1))\) be a Dirichlet eigenpair such that \( v > 0 \) in \( B_R^M \) and
\[
\int_{\mathbb{R}^M} |v(x)|^p \, dx = 1.
\]
Given \( \varphi \in C_0^\infty(\mathbb{R}^N) \), as in the proof of [3, Lemma 3], we define
\[
h, \tau : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}
\]
\[
h(x, y) := f(x)\chi_{B_R^M}(y) \quad \text{and} \quad \tau(x, y) := \varphi(x)v(y), \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^M
\]
where \( \chi_{B_R^M}(y) \) is the characteristic function \( B_R^M \).

Observe that
\[
\int_{\mathbb{R}^N} |f(x)||\varphi(x)|^p \, dx = \int_{\mathbb{R}^N \times \mathbb{R}^M} |h(x, y)||\tau(x, y)|^p \, dx \, dy.
\]

**Claim.** \( \tau \in W^{s,p}(\mathbb{R}^N \times \mathbb{R}^M). \) Moreover
\[
\int_{\mathbb{R}^N \times \mathbb{R}^M} \int_{\mathbb{R}^N \times \mathbb{R}^M} |\tau(x, y) - \tau(w, z)|^p \, dx \, dy \, dz \leq C \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} \left| \frac{\varphi(x) - \varphi(w)}{|x - w|^{N + s}} \right|^p \, dx \, dw + \mu_1(B_R^M, 1) \int_{\mathbb{R}^N} |\varphi(w)|^p \, dw \right),
\]
with the constant \( C \) depending only on \( N, M, s \) and \( p \).

Note that
\[
\int_{\mathbb{R}^N \times \mathbb{R}^M} |\tau(x, y)|^p \, dx \, dy = \left( \int_{\mathbb{R}^N} |\varphi(x)|^p \, dx \right) \left( \int_{\mathbb{R}^M} |v(y)|^p \, dy \right).
\]
Therefore \( \tau \in L^p(\mathbb{R}^N \times \mathbb{R}^M) \).

On the other hand
\[
\int_{\mathbb{R}^N \times \mathbb{R}^M} \int_{\mathbb{R}^N \times \mathbb{R}^M} |\tau(x, y) - \tau(w, z)|^p \, dx \, dy \, dz \leq
\]
\[
C(p) \left( \int_{\mathbb{R}^N \times \mathbb{R}^M} \int_{\mathbb{R}^N \times \mathbb{R}^M} \frac{|\tau(x, y) - \tau(w, z)|^p}{|x - w|^{N + 2s} + |y - z|^{N + 2s}} \, dx \, dy \, dz \right)
\]
\[
+ \int_{\mathbb{R}^N \times \mathbb{R}^M} \int_{\mathbb{R}^N \times \mathbb{R}^M} |\tau(w, y) - \tau(w, z)|^p \, dx \, dy \, dz \right).
\]

For the first term on the right hand side of the previous inequality we have
\[
\int_{\mathbb{R}^N \times \mathbb{R}^M} \int_{\mathbb{R}^N \times \mathbb{R}^M} \frac{|\tau(x, y) - \tau(w, z)|^p}{|x - w|^{N + 2s} + |y - z|^{N + 2s}} \, dx \, dy \, dz =
\]
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(w)|^p \int_{\mathbb{R}^M} |v(y)|^p \int_{\mathbb{R}^M} \frac{dz \, dy \, dx}{|x - w|^{N + s} + |y - z|^{N + s}}
\]
\[
= C(N, M) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(w)|^p \, dx \, dw \int_0^\infty \int_0^{sM-1} \frac{r^{N-1}}{(1 + r^2)^{N+s/2}} \, dr \, dw
\]
\[
= C(N, M, s, p) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(w)|^p \, dx \, dw.
\]
Similarly
\[
\iint_{\mathbb{R}^N \times \mathbb{R}^M} \iint_{\mathbb{R}^N \times \mathbb{R}^M} \frac{|\tau(w,y) - \tau(w,z)|^p}{(|x-w|^2 + |y-z|^2)^{\frac{N}{2} - sp}} \, dx \, dy \, dw \, dz = \\
= C(N, M, s, p) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(y) - v(z)|^p \, dy \, dz \int_{\mathbb{R}^M} |\varphi(w)|^p \, dw \\
= C(N, M, s, p) \mu_1(B^M_R, 1) \int_{\mathbb{R}^M} |\varphi(w)|^p \, dw.
\]

Therefore, by (14), (15) and (16), we get (13).

Since $N_0 = N + M > sp$ and $\tau \in W^{s,p}(\mathbb{R}^N \times \mathbb{R}^M)$, we have that $|\tau|^p \in L^{N_0/(N_0 - sp)}(\mathbb{R}^N \times \mathbb{R}^M)$. Then by (12), Hölder’s inequality, Theorem 2.1 and (13), we get
\[
\int_{\mathbb{R}^N} |f(x)||\varphi(x)|^p \, dx \leq CR^{M+p}|N_0|\|f\|_{N_0/sp} \left(\|\varphi\|_{s,p} + \mu_1(B^M_R, 1)|\varphi|_{p}^p\right)
\]
with the constant $C$ depending only on $N, M, s$ and $p$. □

**Corollary 3.** Assume $sp \geq N$ and $g = g_1 - g_2$ satisfies
\begin{itemize}
  \item $g_1(x) \geq 0$ a.e. in $\mathbb{R}^N$ and $g_1 \in L^\infty(\mathbb{R}^N) \cap L^{N_0/sp}(\mathbb{R}^N) \setminus \{0\}$, with $N_0 \in \mathbb{N}$ such that $N_0 > sp$;
  \item $g_2(x) \geq \varepsilon > 0$ a.e. in $\mathbb{R}^N$.
\end{itemize}
Then there is a constant $C$ such that
\[
\|u\|^p_p \leq C(1 + \|u\|^p_{s,p}),
\]
for all $u \in \mathcal{N}$.

**Proof.** Let $u \in \mathcal{N}$. Then by Theorem 4.1 and Lemma 2.1 we get
\[
\varepsilon \int_{\mathbb{R}^N} |u(x)|^p \, dx \leq \int_{\mathbb{R}^N} g_2(x)|u(x)|^p \, dx = 1 + \int_{\mathbb{R}^N} g_1(x)|u(x)|^p \, dx \\
\leq 1 + CR^{M+p}|N_0|\|g_1\|_{N_0/sp} \left(\|u\|^p_{s,p} + \mu_1(B^M_R, 1)|u|_{p}^p\right) \\
\leq 1 + CR^{N_0/(N_0 - sp)}\|g_1\|_{N_0/sp} \left(\|u\|^p_{s,p} + \frac{1}{R^sp}\mu_1(B_1, 1)|u|_{p}^p\right)
\]
where $C$ is a constant independent of $u$ and $R$. Then for $R$ large enough, we have that there is a constant $C$ independent of $u$ such that
\[
\|u\|^p_p \leq C(1 + \|u\|^p_{s,p}).
\]

□

Using Corollary 3 and proceeding as in the proof of Lemma 3.1 we can prove that the functional $\Phi$ satisfies the Palis–Smale condition on $\mathcal{N}$.

**Lemma 4.2.** Let $s \in (0, 1)$ and $p \in (1, \infty)$ be such that $sp \geq N$. Assume $g = g_1 - g_2$ satisfies
\begin{itemize}
  \item $g_1(x) \geq 0$ a.e. in $\mathbb{R}^N$ and $g_1 \in L^\infty(\mathbb{R}^N) \cap L^{N_0/sp}(\mathbb{R}^N) \setminus \{0\}$, with $N_0 \in \mathbb{N}$ such that $N_0 > sp$;
  \item $g_2(x) \geq \varepsilon > 0$ a.e. in $\mathbb{R}^N$.
\end{itemize}
Then the functional $\Phi$ satisfies the Palais–Smale condition on $\mathcal{N}$.

Finally, by the Lusternik–Schnirelman principle, Theorem 1.2 follows.
5. Hölder regularity

In this section, we will show that if $(u, \lambda)$ is an eigenvalue then $u \in L^\infty(\mathbb{R}^N) \cap C^r(\mathbb{R}^N)$ for some $\gamma \in (0, 1)$.

By the fractional Sobolev embedding theorem, we know that if $sp > N$ then $W^{s,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N) \cap C^{s-N/p}(\mathbb{R}^N)$. Then, we have the following result.

Lemma 5.1. If $sp > N$ and $(u, \lambda)$ is an eigenpair then $u \in L^\infty(\mathbb{R}^N) \cap C^{s-N/p}(\mathbb{R}^N)$.

In the case $sp \leq N$, we need assume that $g \in L^\infty(\mathbb{R}^N)$ to prove the result.

Lemma 5.2. Assume $sp \leq N$ and $g \in L^\infty(\mathbb{R}^N)$. If $(u, \lambda)$ is an eigenpair then $u \in L^\infty(\mathbb{R}^N)$.

Proof. We split the proof in two cases.

Case $sp < N$. We begin by observing that $(-u, \lambda)$ is an eigenpair since $(u, \lambda)$ is. Therefore, it is enough to prove that $u_+ \in L^\infty(\mathbb{R}^N)$.

Observe that $u_+ \in W^{s,p}(\mathbb{R}^N)$ because

$$|u_+(x) - u_+(y)| \leq |u(x) - u(y)| \quad \forall x, y \in \mathbb{R}^N.$$ 

We now intend to prove by induction on $n$ that

$$u_+ \in L^{p_{\gamma_n}}(\mathbb{R}^N), \quad \gamma_n = \left(\frac{N}{N - sp}\right)^n$$

for all $n \in \mathbb{N}$, with the estimate

$$\|u_+\|_{p_{\gamma_{n+1}}} \leq K^{1\gamma_n} u_+ \|u_+\|_{p_n},$$

for some positive constant $K = K(N, s, p, \lambda, \|g\|_{L^\infty(\mathbb{R}^N)})$.

Since $u_+ \in W^{s,p}(\mathbb{R}^N)$, we have that $u \in L^{p_{\gamma_n}}(\mathbb{R}^N)$. Assume now $u_+ \in L^{p_{\gamma_n}}(\mathbb{R}^N)$ for some positive integer $n$. We want to show that $u_+ \in L^{p_{\gamma_{n+1}}}(\mathbb{R}^N)$.

Let us define, for any positive integer $k$

$$v_k(x) = \min\{k, u_+(x)\}.$$

Since $u_+ \in W^{s,p}(\mathbb{R}^N)$ and for any $\alpha > 1$

$$|(v_k(x))^\alpha - (v_k(y))^\alpha| \leq \alpha(v_k(x) + v_k(y))^{\alpha-1}|v_k(x) - v_k(y)|$$

$$\leq \alpha(2k)^{\alpha-1}|u_+(x) - u_+(y)| \quad \forall x, y \in \mathbb{R}^N.$$ 

we get $v_k \in L^{p_{\gamma_n}}(\mathbb{R}^N) \cap W^{s,p}(\mathbb{R}^N)$ for any $\alpha > 1$.

On the other hand, if $u_+(x) \geq u_+(y)$ then $v_k(x) \geq v_k(y)$ and there is $\theta \in (v_k(y), v_k(x))$ such that

$$|(v_k(x))^{\gamma_n} - (v_k(y))^{\gamma_n}|^p \leq \gamma_n p \theta^{(\gamma_n-1)p} |v_k(x) - v_k(y)|^p$$

$$\leq \gamma_n p \theta^{(\gamma_n-1)p} |u_+(x) - u_+(y)|^{p-2}(u_+(x) - u_+(y))v_k(x)$$

$$\leq \gamma_n p \theta^{(\gamma_n-1)p} |u_+(x) - u_+(y)|^{p-2}(u_+(x) - u_+(y))v_k(x)$$

$$\leq \gamma_n p |u_+(x) - u_+(y)|^{p-2}(u_+(x) - u_+(y))v_k(x)^{(\gamma_n-1)p+1}$$

$$- \gamma_n p |u_+(x) - u_+(y)|^{p-2}(u_+(x) - u_+(y))v_k(x)^{(\gamma_n-1)p+1}.$$ 

Then, taking $\alpha_n = (\gamma_n - 1)p + 1$, we get

$$|(v_k(x))^{\gamma_n} - (v_k(y))^{\gamma_n}|^p \leq$$

$$\leq |u_+(x) - u_+(y)|^{p-2}(u_+(x) - u_+(y))[(v_k(x))^{\alpha_n} - (v_k(y))^{\alpha_n}]$$

$$\leq |u(x) - u(y)|^{p-2}(u(x) - u(y))[(v_k(x))^{\alpha_n} - (v_k(y))^{\alpha_n}].$$
By symmetry we also get the last inequality when $u_+(y) \geq u_+(x)$. Thus, by Theorem 2.1, there is a positive constant $C = C(N, s, p)$ such that
\[
C^p \|v_k^n\|_{p_2^{p^*}}^p \leq \|v_k^n\|_{s, p} \leq \gamma_n^p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v_k^n(x))^\alpha_n - (v_k^n(y))^\alpha_n}{|x - y|^{N+sp}} \, dx \, dy.
\]
Therefore, since $(u, \lambda)$ is an eigenpair, we get
\[
C^p \|v_k^n\|_{\gamma_n+1}^p = C^p \|v_k^n\|_{p_2^{p^*}}^p \leq \lambda \gamma_n^p \|g\|_{L^\infty(\mathbb{R}^N)} \|u_+\|_{\gamma_n+1}^p.
\]
Then there is a positive constant $K = K(N, s, p, \lambda, \|g\|_{L^\infty(\mathbb{R}^N)})$ such that
\[
\|v_k^n\|_{\gamma_n+1} \leq K^{1/\gamma_1} \gamma_n^{1/\gamma_1} \|u_+\|_{\gamma_n+1}^p.
\]
Letting $k \to \infty$, by Fatou’s Lemma
\[
\|u_+\|_{\gamma_n+1} \leq K^{1/\gamma_1} \gamma_n^{1/\gamma_1} \|u_+\|_{\gamma_n+1}^p.
\]
Hence $u_+ \in L^{\gamma_n+1}(\mathbb{R}^N)$ and
\[
\|u_+\|_{\gamma_n+1} \leq \prod_{i=1}^n K^{1/\gamma_i} \|u_+\|_{\gamma_i+1} = K^{\sum_{i=1}^n 1/\gamma_i} \sum_{i=1}^n \frac{1/\gamma_1}{1/\gamma_1} \|u_+\|_{\gamma_i+1}^p.
\]
Note that
\[
\lim_{n \to \infty} K^{\gamma_n^{p-1}/\gamma_1} = K^{1/(\gamma_1-1)} \text{ and } \lim_{n \to \infty} \gamma_n^{(\gamma_1-1)/(\gamma_1-1)} = \gamma_1^{1/(\gamma_1-1)}.
\]
Now, it is easy to check that $u_+ \in L^\infty(\mathbb{R}^N)$.

**Case sp = N.** Let $\tilde{s} \in (0, s)$ be such that $\tilde{s}p < N$. By [16, Lemma 2.3] there is a constant $K = K(N, \tilde{s}, s, p)$ such that
\[
\|v\|_{s,p}^p \leq \|v\|_{\tilde{s},p}^p + K \|v\|_p^p \quad \forall v \in W^{s,p}(\mathbb{R}^N).
\]
Thus, by Theorem 2.1, we get that there is a constant $C = C(N, \tilde{s}, p)$ such that
\[
C^p \|v\|_{p_2^p}^p \leq \|v\|_{\tilde{s},p}^p \leq \|v\|_{s,p}^p + K \|v\|_p^p \quad \forall v \in W^{s,p}(\mathbb{R}^N)
\]
that is
\[
C^p \|v\|_{p_2^p}^p \leq \|v\|_{s,p}^p + K \|v\|_p^p \quad \forall v \in W^{s,p}(\mathbb{R}^N).
\]
Now, taking $\gamma_n = \left(\frac{N}{N-s,p}\right)^n$ and proceeding as in the previous case, we can conclude that $u \in L^\infty(\mathbb{R}^N)$. \qed

Then, by the previous lemma and [27, Corollary 5.5] we have the following result.

**Lemma 5.3.** Assume $sp \leq N$ and $g \in L^\infty(\mathbb{R}^N)$. If $(u, \lambda)$ is an eigenpair then $u \in L^\infty(\mathbb{R}^N) \cap C^\gamma(\mathbb{R}^N)$ for some $\gamma \in (0, 1)$.
6. Principal eigenvalues

Now we will collect some relevant properties of the principal eigenvalues and their eigenfunctions. To simplify matters, in the remainder of this section we write \( \lambda_1 = \lambda_1(g) \) \((\lambda_1^\pm = \lambda_1^\mp(g))\).

Our next lemma follows from the following inequality
\[
\|u\|_{s,p} < [u]_{sp}
\]
for any \( u \) such that \( u_{\pm} \neq 0 \).

**Lemma 6.1.** If \( u \) is an eigenfunction associated to \( \lambda_1(\lambda_1^\pm) \) then either \( u_+ \equiv 0 \) or \( u_- \equiv 0 \).

Moreover, by [17, Theorems 1.2 and 1.4], we get the following results.

**Lemma 6.2.** Assume that \( sp < N \) and \( g \in L^\infty(\mathbb{R}^N) \cap L^{\infty/sp}(\mathbb{R}^N) \). If \( g_+ \neq 0 \) \((g_{\pm} \neq 0)\) and \( u \) is an eigenfunction associated to \( \lambda_1(\lambda_1^\pm) \) then either \( u > 0 \) or \( u < 0 \) in whole \( \mathbb{R}^N \).

**Lemma 6.3.** Let \( s \in (0,1) \) and \( p \in (1,\infty) \) be such that \( sp \geq N \). Assume \( g = g_1 - g_2 \) satisfies
- \( g_1(x) \geq 0 \ a.e. \ in \ \mathbb{R}^N \) and \( g_1 \in L^\infty(\mathbb{R}^N) \cap L^{\infty/sp}(\mathbb{R}^N) \setminus \{0\}, \) with \( N_0 \in \mathbb{N} \) such that \( N_0 > sp \);
- \( g_2(x) \geq \varepsilon > 0 \ a.e. \ in \ \mathbb{R}^N \).

If \( u \) is an eigenfunction associated to \( \lambda_1 \) then either \( u > 0 \) or \( u < 0 \) a.e. in \( \mathbb{R}^N \).

If in addition \( g_2 \in C(\mathbb{R}^N) \) or \( g_2 \in L^\infty(\mathbb{R}^N) \) then either \( u > 0 \) or \( u < 0 \) in whole \( \mathbb{R}^N \).

The proof of the result given below follows from a careful reading of [16, proof of Theorem 4.8]

**Lemma 6.4.** If \( u \) is an eigenfunction associated to \( \lambda_1(\lambda_1^\pm) \) such that \( u > 0 \ a.e. \ in \ \mathbb{R}^N \) and \( \lambda \geq \lambda_1(\lambda \geq \pm \lambda_1^\pm) \) is such that there exists a nonnegative eigenfunction \( v \) associated to \( \lambda(\pm \lambda) \) then \( \lambda = \lambda_1(\lambda = \pm \lambda_1^\pm) \) and there is \( k \in \mathbb{R} \) such that \( v = ku_1(v = ku_1^\pm) \) in \( \mathbb{R}^N \).

Then, by Lemmas 6.1, 6.2, 6.3 and 6.4, we have the next two theorems that give the last part of our main theorems.

**Theorem 6.1.** If \( sp < N \), \( g \in L^\infty(\mathbb{R}^N) \cap L^{\infty/sp}(\mathbb{R}^N) \), \( g_+ \neq 0 \) \((g_{\pm} \neq 0)\) then \( \lambda_1(\lambda_1^\pm) \) is simple and all its eigenfunctions are of constant sign.

**Theorem 6.2.** If \( sp \geq N \) and \( g = g_1 - g_2 \) satisfies
- \( g_1(x) \geq 0 \ a.e. \ in \ \mathbb{R}^N \) and \( g_1 \in L^\infty(\mathbb{R}^N) \cap L^{\infty/sp}(\mathbb{R}^N) \setminus \{0\}, \) with \( N_0 \in \mathbb{N} \) such that \( N_0 > sp \);
- \( g_2(x) \geq \varepsilon > 0 \ a.e. \ in \ \mathbb{R}^N \),

then \( \lambda_1 \) is simple and all its eigenfunctions are of constant sign.

7. Decay estimates

Finally, we study the decay rate at infinity of
- all positive eigenfunctions associated to \( \lambda_1(g) \) in the case \( sp \geq N \);  
- all positive ground state solutions of the autonomous Schrödinger equations in the case \( sp < N \).

For this reason, we give an nonlinear version of [8, Lemma 2.1].
Lemma 7.1. Let $\Upsilon \in C^2(\mathbb{R}^N)$ be a positive function such that $\Upsilon$ is radially symmetric and decreasing. Assume also that

$$\Upsilon(x) \leq \frac{C_1}{|x|^p}, \quad 0 \neq |\nabla \Upsilon(x)| \leq \frac{C_2}{|x|^\alpha}, \quad |D^2 \varphi(x)| \leq \frac{C_3}{|x|^{\alpha+2}}$$

for some $\alpha > 0$ and for $|x|$ large enough. Then there is $k >> 1$ such that

$$\left| (-\Delta_p)^s \Upsilon(x) \right| \leq \begin{cases} \frac{c_1}{|x|^{\alpha(p-1)+ps}} & \text{if } \alpha(p-1) < N, \\ \frac{c_2 \log(|x|)}{|x|^{N+ps}} & \text{if } \alpha(p-1) = N, \\ \frac{c_3}{|x|^{N+ps}} & \text{if } \alpha(p-1) > N, \end{cases}$$

for all $|x| \geq k$. Here $c_1, c_2, c_3$ are positive constants that depend only on $\alpha, s, p, N$ and $\|\Upsilon\|_{C^2(\mathbb{R}^N)}$.

Moreover, if $\alpha(p-1) > N$ and $|x| > k$, we have

$$(-\Delta_p)^s \Upsilon(x) \leq \frac{c_4}{|x|^{N+sp}},$$

for some positive constant $c_4$.

Proof. By [29, Lemma 3.6], we have that $(-\Delta_p)^s \Upsilon(x)$ is finite for all $|x|$ large enough.

Now we proceed as in the proof of Lemma 2.1 in [8]. From now on $|x|$ is large enough.

$$(-\Delta_p)^s \Upsilon(x) = \int_{\mathbb{R}^N} \frac{|\Upsilon(x) - \Upsilon(y)|^{p-2}(\Upsilon(x) - \Upsilon(y))}{|x - y|^{N+ps}} dy$$

$$= \int_{|y| > \frac{3}{2}|x|} \frac{|\Upsilon(x) - \Upsilon(y)|^{p-2}(\Upsilon(x) - \Upsilon(y))}{|x - y|^{N+ps}} dy$$

$$+ \int_{\left\{ \frac{|y|}{2} \leq |y| \leq \frac{3}{2}|x| \right\} \setminus B_{\frac{|x|}{2}}(x)} \frac{|\Upsilon(x) - \Upsilon(y)|^{p-2}(\Upsilon(x) - \Upsilon(y))}{|x - y|^{N+ps}} dy$$

$$+ \int_{B_{\frac{|x|}{2}}(x)} \frac{|\Upsilon(x) - \Upsilon(y)|^{p-2}(\Upsilon(x) - \Upsilon(y))}{|x - y|^{N+ps}} dy$$

$$+ \int_{B_{\frac{|x|}{2}}(0)} \frac{|\Upsilon(x) - \Upsilon(y)|^{p-2}(\Upsilon(x) - \Upsilon(y))}{|x - y|^{N+ps}} dy$$

$$= I_1 + I_2 + I_3 + I_4.$$ 

If $|y| > \frac{3}{2}|x|$ then $\Upsilon(y) < \Upsilon(x)$ and therefore

$$0 \leq I_1 \leq |\Upsilon(x)|^{p-1} \int_{|y| > \frac{3}{2}|x|} \frac{dy}{|x - y|^{N+sp}}$$

$$= C(N, s, p) |\Upsilon(x)|^{p-1} \frac{1}{|x|^{\alpha(p-1)+ps}}.$$ 

In the same way

$$|I_2| \leq \left| \Upsilon \left( \frac{x}{2} \right) \right|^{p-1} \int_{\left\{ \frac{|y|}{2} \leq |y| \leq \frac{3}{2}|x| \right\} \setminus B_{\frac{|x|}{2}}(x)} \frac{1}{|x - y|^{N+ps}} dy \leq \frac{C(N, s, p, C_1, \alpha)}{|x|^{\alpha(p-1)+ps}}.$$
On the other hand, if \( |y| \leq \frac{|x|}{2} \) then \( Y(x) \leq Y(y) \) and \( |x - y| > \frac{|x|}{2} \). Therefore

\[
|I_4| \leq \frac{2N+ps}{|x|^{N+ps}} \int_{B_{\frac{|x|}{2}}(0)} |Y(y)|^{p-1} dy
\leq \frac{C(N, s, p)}{|x|^{N+ps}} \int_{B_{\frac{|x|}{2}}(0)} |Y(y)|^{p-1} dy = C(N, s, p) I_4'.
\]

As in [8], we have the following estimate for \( |x| \) large enough

\( \bullet \) If \( \alpha(p - 1) > N \) then \( I_4' < \infty \) and \( |I_4| \leq \frac{\Gamma_1}{|x|^{N+sp}} \);

\( \cdot \) If \( \alpha(p - 1) < N \) then \( I_4' \) grows like \( |x|^{N-\alpha(p-1)} \) and \( |I_4| \leq \frac{\Gamma_2}{|x|^{\alpha(p-1)+sp}} \);

\( \bullet \) Finally, if \( \alpha(p - 1) = N \) we get \( |I_4| \leq \frac{\Gamma_3 \log(|x|)}{|x|^{N+ps}} \).

Here \( \Gamma_1, \Gamma_2, \Gamma_3 \) are positive constant that depend only on \( N, s, p, C_1, \alpha \) and \( \|Y\|_{L^\infty(\mathbb{R}^N)} \).

The real difference with the linear case is observed in the estimate of \( |I_3| \). It follows from the proof of Lemma 3.6 in [29] that there is a positive constant \( c \) depending on \( N \) and \( p \) such that

\[
|I_3| \leq \begin{cases} 
\frac{\tau \omega^{p-2} \left( \frac{|x|}{2} \right)^{p(1-s)} + \tau^{p-1} \left( \frac{|x|}{2} \right)^{p(1-s)}}{c} & \text{if } p \geq 2, \\
\tau^{p-1} \left( \frac{|x|}{2} \right)^{p-2+p(1-s)} & \text{if } \frac{2}{2-s} < p < 2, \\
\tau \omega^{p-2} \left( \frac{|x|}{2} \right)^{p(1-s)} & \text{if } 1 < p \leq \frac{2}{2-s},
\end{cases}
\]

where \( \tau = \sup \left\{ |D^2 Y(y)| : y \in B_{\frac{|x|}{2}}(x) \right\} \) and \( \omega = \sup \left\{ |\nabla Y(y)| : y \in B_{\frac{|x|}{2}}(x) \right\} \).

Since \( \frac{|x|}{2} \leq |y| \) for any \( y \in B_{\frac{|x|}{2}}(x) \), by (17) and (23), we have

\[
|I_3| \leq \frac{C}{|x|^{\alpha(p-1)+N}}
\]

where \( C \) is a positive constant depending on \( N, s, p, C_2, C_3 \) and \( \alpha \).

By (20), (21), (22) and (24) we get (18).

To end the proof we show (19) For \( x \) large enough, by (20), (21), (22) and (24) there is a positive constant \( C \) such that

\[
(-\Delta_p)^s Y(x) = I_1 + I_2 + I_3 + I_4 \leq I_4 + \frac{C}{|x|^{\alpha(p-1)+ps}}.
\]

On the other hand, since \( |y| \leq \frac{|x|}{2} \) implies that \( Y(x) \leq Y(y) \) and \( |x - y| < \frac{3}{2}|x| \), we get

\[
I_4 = -\int_{B_{\frac{|x|}{2}}(0)} \frac{(Y(y) - Y(x))^{p-1}}{|x - y|^{N+ps}} dy
\leq -\left( \frac{2}{3} \right)^{N+ps} \frac{1}{|x|^{N+ps}} \int_{B_{\frac{|x|}{2}}(0)} (Y(y) - Y(x))^{p-1} dy.
\]
Now, if $|x| > 2$ we have that
\[
I_4 \leq -\left(\frac{2}{3}\right)^{N+ps} \int_{B_1(0)} \frac{1}{|x|^{N+ps}} (\Upsilon(y) - \Upsilon(x))^{p-1} dy
\]
\[
\leq -\left(\frac{2}{3}\right)^{N+ps} \int_{B_1(0)} \frac{1}{|x|^{N+ps}} (\Upsilon(y) - \Upsilon(x))^{p-1} dy
\]
\[
\leq -\left(\frac{2}{3}\right)^{N+ps} \int_{B_1(0)} \frac{1}{|x|^{N+ps}} (\Upsilon(y) - \Upsilon(z))^{p-1} dy
\]
for any $z \in \partial B_1(0)$. That is, there is a positive constant $C$ such that
\[
(26) \quad I_4 \leq -\frac{C}{|x|^{N+ps}} \quad \forall |x| > 2.
\]
Then by (25) and (26), there is a positive constant $C$ such that
\[
(-\Delta_p)^s \Upsilon(x) \leq -\frac{C}{|x|^{N+ps}}
\]
for all $x$ large enough and $\alpha$ such that $\alpha(p - 1) > N$.

The other result that play an important role in the proof of decay estimates is the next comparison principle.

**Theorem 7.1** (Comparison principle). Let $\Omega \subset \mathbb{R}^N$ be an open set, $V \in L^\infty(\mathbb{R}^N)$ $V \geq 0$ in $\Omega$ and $u, v \in W^{s,p}(\mathbb{R}^N)$ satisfy $u \leq v$ in $\mathbb{R}^N \setminus \Omega$ and
\[
(-\Delta_p)^s u + V(x)|u(x)|^{p-2}u(x) \leq (-\Delta_p)^s v + V(x)|v(x)|^{p-2}v(x) \quad \text{in } \Omega,
\]
that is
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)\varphi(x) \, dx
\]
\[
\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^N} V(x)|v(x)|^{p-2}v(x)\varphi(x) \, dx
\]
whenever $\varphi \in \tilde{W}^{s,p}(\Omega)$, $\varphi \geq 0$. Then $u \leq v$ in $\Omega$.

**Proof.** Let’s start by observing that $\varphi = (u - v)_+ \in \tilde{W}^{s,p}(\Omega)$, since $u, v \in W^{s,p}(\mathbb{R}^N)$, and for any $x, y \in \mathbb{R}^N$ we have
\[
|\varphi(x) - \varphi(y)| \leq |u(x) - v(x) - (u(y) - v(y))| \leq |u(x) - u(y)| + |v(x) - v(y)|.
\]
The proof follows by the argument of [17, Proposition 2.5]. See also [31, Lemma 9] and [9, Theorem 2.6].

First we show the decay rate at infinity of all positive eigenfunctions associated to $\lambda_1(g)$ in the case $sp \geq N$. 

Proof of Theorem 1.4. We give only the proof of the left hand side of the inequality, the proof of the right hand side is similar.

Let us observe that, by assumptions, we have
\[
0 \leq (-\Delta_p)^s u(x) + \lambda_1(g)\|g_2\|_{\infty}|u(x)|^{p-1}
\]
for $|x|$ large enough.

On the other hand, taking $\alpha = \frac{N + ps}{p - 1}$ and $\Upsilon \in C^2(\mathbb{R}^N)$ a positive function such that $\Upsilon$ is radially symmetric, decreasing and
\[
\Upsilon(x) = \frac{1}{|x|^{\alpha}} \quad \forall|x| > 1,
\]
by Lemma 7.1, we have that there exists $k \gg 1$ such that for any $|x| > k$
\[(27) \quad (-\Delta_p)^s \Upsilon(x) + c_1 |\Upsilon(x)|^{p-2} \Upsilon(x) \leq 0 \]
for some positive constants $c_1$ and $c_2$.

We next set
\[
\phi(x) = K \Upsilon(Rx)
\]
where $K$ and $R$ are positive constant that will be selected below. Then, by (27),
\[
(-\Delta_p)^s \phi(x) = \frac{K}{R^{sp}} (-\Delta_p)^s \Upsilon(Rx) \leq -\frac{c_1}{R^{sp}} |\phi(x)|^{p-2} \phi(x) \quad \forall|x| > \frac{k}{R}
\]
Taking
\[
R = \left(\frac{c_1}{\lambda_1(g)\|g_2\|_{\infty}}\right)^{\frac{1}{sp}} \quad \text{and} \quad k_1 = \frac{k}{R},
\]
we have
\[
(-\Delta_p)^s \phi(x) + \lambda_1(g)\|g_2\|_{\infty}|\phi(x)|^{p-2} \phi(x) \leq 0 \quad \forall|x| > k_1.
\]
Notice that $\phi(x)$ is classical solution and then a strong solution and weak solution, for details see [27].

On the other hand, by Lemmas 6.3, 5.1 and 5.3, we can choose $K > 0$ so that $u(x) \geq \phi(x)$ in $|x| < k_1$.

Finally, by Theorem 7.1, we get $u(x) \geq \phi(x)$ in $\mathbb{R}^N$. Therefore
\[
\frac{K}{R^{\frac{1}{sp} + \frac{1}{p}}} |x|^{-\frac{1}{sp} - \frac{1}{p}} \leq u(x)
\]
for all $|x| > k_1$. \[\square\]

Notice that in the case $sp < N$ one side bound can be obtained but the other is not possible since the assumption $g(x) < -\delta < 0$ for $|x|$ large enough is not compatible with the assumptions of the existence results.

Lastly, we study the decay rate at infinity of all positive ground state solutions of the next autonomous Schrödinger equations
\[
(-\Delta_p)^s u(x) + \mu |u|^{p-2} u = f(u) \quad \text{in } \mathbb{R}^N,
\]
\[(28) \quad u \in W^{s,p}(\mathbb{R}^N), \quad u(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \mu > 0.
\]

The existence of at least one positive ground state solution of (28) was recently proved in [4] under the following assumptions: $sp < N$ and the nonlinearity $f: \mathbb{R} \to \mathbb{R}$ satisfies the next conditions
\[
(f_1) \quad f \in C(\mathbb{R}) \text{ and } f(t) = 0 \text{ for all } t < 0;
\]
\[
(f_2) \quad \lim_{|t| \to 0} \frac{|f(t)|}{|t|^{p-1}} = 0;
\]
\[
(f_3) \quad \lim_{|t| \to \infty} \frac{|f(t)|}{|t|^{p-1}} = 0.
\]
There is \( q \in (p, p^*_s) \) such that
\[
\lim_{|t| \to \infty} \frac{|f(t)|}{|t|^{\vartheta - 1}} = 0;
\]

there is \( \vartheta > p \) such that
\[
0 < \vartheta \int_0^t f(\tau) d\tau \leq tf(t) \quad \text{for all } t > 0;
\]

the map \( t \to f(t) \) is increasing in \((0, +\infty)\).

In addition, in [4, Remark 3.2], the authors observe that if \( v \) is a positive ground state solutions of (28) then \( v \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N) \). In fact, by [27, Corollary 5.5], we can conclude that \( v \in C^\gamma(\mathbb{R}^N) \) for some \( \gamma \in (0, 1) \). Therefore
\[
v(x) \to 0 \quad \text{as } |x| \to 0
\]

since \( v \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap C^\gamma(\mathbb{R}^N) \).

Our last result shows the rate decay of \( v \) at infinity.

**Proof of Theorem 1.5.** Observe that, by (29) and \((f_2)\) there is a \( k > 1 \) such that
\[
f(v(x)) < \frac{\mu}{2} |v(x)|^{p-2} v(x) \quad \forall |x| > k.
\]

Then
\[
(-\Delta)^s v(x) + \frac{\mu}{2} |v(x)|^{p-2} v(x) \leq 0 \leq (-\Delta)^s v(x) + \mu |v(x)|^{p-2} v(x)
\]

for \( |x| \) large enough.

The remain of the proof is entirely analogous to the proof of Theorem 1.4. \( \square \)

**Acknowledgements.** L. D. P. was partially supported by PICT2012 0153 from ANPCyT (Argentina). A. Q. was partially supported by Fondecyt Grant No. 1151180 and Programa Basal, CMM. U. de Chile

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