Quasi-arithmetic means of covariance functions with potential applications to space-time data

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Abstract

The theory of quasi-arithmetic means is a powerful tool in the study of covariance functions across space-time. In the present study we use quasi-arithmetic functionals to make inferences about the permissibility of averages of functions that are not, in general, permissible covariance functions. This is the case, e.g., of the geometric and harmonic averages, for which we obtain permissibility criteria. Also, some important inequalities involving covariance functions and preference relations as well as algebraic properties can be derived by means of the proposed approach. In particular, we show that quasi-arithmetic covariances allow for ordering and preference relations, for a Jensen-type inequality and for a minimal and maximal element of their class. The general results shown in this paper are then applied to study of spatial and spatiotemporal random fields. In particular, we discuss the representation and smoothness properties of a weakly stationary random field with a quasi-arithmetic covariance function. Also, we show that the generator of the quasi-arithmetic means can be used as a link function in order to build a space-time nonseparable structure starting from the spatial and temporal margins, a procedure that is technically sound for those working with copulas. Several examples of new families of stationary covariances obtainable with this procedure are shown. Finally, we use quasi-arithmetic functionals to generalise existing results concerning the construction of nonstationary spatial covariances and discuss the applicability and limits of this generalisation.

KeyWords: Completely monotone functions, Jensen’s type inequalities, Nonseparability, Nonstationarity, Quasi-arithmetic functionals, Random Fields, Smoothness, Space-time covariances.

Subj-class: Probability; Statistics.

MSC class: 60G60; 86A32.
1 Introduction

The importance of quasi-arithmetic means has been well understood at least since the 1930s, and a number of writers have since contributed to their characterisation and to the study of their properties. In particular, Kolmogorov (1930) and Nagumo (1930) derived, independently of each other, necessary and sufficient conditions for the quasi-arithmeticity of the mean, that is, for the existence of a continuous strictly monotonic function $f$ such that, for $x_1, \ldots, x_n$ in some real interval, the function $(x_1, \ldots, x_n) \mapsto M_n(x_1, \ldots, x_n) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right)$ is a mean.

Using this result, they partially modified the classical Cauchy (1821) internality and Chisini’s (1929) invariance properties. As pointed out by Marichal (2000), the Kolmogorov reflexive property is equivalent to the Cauchy internality, and both are accepted by statisticians as requisites for means.

Early works on the concept of mean include Bonferroni (1926), de Finetti (1931), Gini (1938), Dodd (1940) and Aczél (1948). More recent contributions are the works of Wimp (1986), Hutník (2006), Matkowski (1999;2002) Jarczyk and Matkowski (2000), J. Marichal (2000), Daróczy and Hajdu (2005), and Abrahamovic et al. (2006). Quasi-arithmetic means, in particular, have been applied in several disciplines. Their functional form has been used in the theory of copulas under the name of Archimedean copulas (Genest and MacKay, 1986) and a rich literature can be found under this name. In the theory of aggregation operators and fuzzy measures, a growing literature related to the use of quasi-arithmetics includes the works of Frank (1979), Hajek (1998), Kolesarova (2001), Klement et al. (1999), Grabish (1995) and Calvo and Mesiar (2001).

Despite the extensive quasi-arithmetic means literature, to the best of our knowledge, there is no published work relating quasi-arithmetic means with covariance functions, whose properties have been extensively studied both in mathematical analysis and statistical fields. In particular, the study of covariance functions is intimately related to that of positive definite functions, the latter being the subject of a considerable literature in a variety of fields, such as mathematical analysis, abelian semigroup theory, spatial statistics and geostatistics. For basic facts about positive definite functions, we refer the interested reader to Berg and Forst (1975) and to Berg et al. (1984). The importance of positive definite functions in the determination of permissible covariance functions (ordinary and generalised) in spatial statistics was studied in detail by Christakos (1984). Subsequent considerations, in a spatial and a spatiotemporal context, include the works of Christakos
Fundamental properties of covariance functions may be inferred by studying collections of them considered as convex cones closed in the topology of point-wise convergence. In this paper, quasi-arithmetic averages and positive definite functions are combined to gain valuable insight concerning certain covariance properties. Also, we apply the general results obtained by our analysis based on the concept of quasi-arithmeticity to build new classes of stationary and nonstationary space-time covariance functions. In such a context, we seek answers to the following questions:

(1) Consider an arbitrary number \( n \in \mathbb{N} \) of covariance functions, not necessarily defined in the same space. Their arithmetic average and product (i.e., the geometric product raised to the \( n \)-th power) are valid covariance functions, and so is the \( k \)-th power average of covariance functions \((k \text{ is a natural positive number})\). But, what about other types of averages? Since quasi-arithmetic means constitute a general group that includes the arithmetic, geometric, power and logarithmic means as special cases, it seems natural to use quasi-arithmetic representations in order to derive positive definiteness conditions for quasi-arithmetic averages of covariance functions.

(2) Can we use the properties of quasi-arithmetic means to establish important inequalities, ordering and preference relations and minimal and maximal elements within the class of covariance functions?

(3) Is it possible to find a class of link functions that, when applied to the \( k \) covariances, can generate valid nonseparable covariances? If this is the case, other potentially desirable properties should be examined, e.g., this approach should be as general as possible and include famous constructions and separability as special cases; and it should preserve the margins.

In view of the above considerations, the paper is organised as follows: Section 2 discusses the background, notation and proposed methodology; it also provides a very brief introduction to positive definite functions. In Section 3, the main theoretical results are presented. In particular, we propose permissibility criteria for the quasi-arithmetic mean of covariance functions. We derive important covariance inequalities of Jensen’s type as well as ordering and preference relations between covariance functions. Finally, minimal and maximal elements of the quasi-arithmetic covariance class are identified, and an associativity property of this class is provided. In Section 4,
we apply our results to the construction of new families of space-time non-separable stationary covariance functions. Also, we extend Stein (2005b) result to a more general class of spatial nonstationary covariance functions. Several examples of stationary and nonstationary covariances are proposed and their mathematical properties discussed. Finally, we study the properties of quasi-arithmetic random fields (defined later in the paper) in terms of mean square differentiability and variance. Section 5 concludes with a critical discussion of the preceding analysis. All the proofs of the original results derived in this paper can be found in the Appendix.

2 Background and Methodology

2.1 Covariance functions: characterisation and basic properties

In this section, we shall assume the real mapping $C : \mathbb{X} \times \mathbb{X} \subseteq \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ to be continuous and Lebesgue measurable on the domain $\mathbb{X} \times \mathbb{X}$, where $\mathbb{X}$ can be either a compact space or the entire $d$-dimensional Euclidean space, $d \in \mathbb{N}$. $C$ is a covariance function if and only if it is positive definite, that is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j C(x_i, x_j) \geq 0 \quad (1)$$

for any finite set of real coefficients $\{c_i\}_{i=1}^{n}$, and for $x_1, \ldots, x_n \in \mathbb{X}$. Christakos (1984) calls the covariance condition (1) permissibility, and throughout the paper we shall use both this term and that of positive definiteness to characterise valid covariance functions. A subclass of positive definite functions, called stationary, is obtained if

$$C(x_i, x_j) := \tilde{C}_0(x),$$

with $x = x_i - x_j$ and $\tilde{C}_0 : \mathbb{X} \to \mathbb{R}$ positive definite and such that $\tilde{C}_0(0) < \infty$.

As shown by Bochner (1933), condition (1) is then equivalent to the requirement that $\tilde{C}_0$ is the Fourier transform $\mathcal{F}$ of a positive bounded measure $\tilde{C}_0$ with support in $\mathbb{R}^d$, that is

$$\tilde{C}_0(x) := \mathcal{F}[\tilde{C}_0](x) = \int_{\mathbb{R}^d} e^{i \omega \cdot x} d\tilde{C}_0(\omega).$$

Additionally, if $\tilde{C}_0$ is absolutely continuous with respect to the Lebesgue measure (ensured if $\tilde{C}_0 \in L_1(\mathbb{X})$), then the expression above can be written
as a function of \( dC_0(\omega) = \hat{c}_0(\omega) d\omega \), where \( \hat{c}_0 \) is called the spectral density of \( \tilde{C}_0 \). For a detailed mathematical discussion of the Fourier representation in a spatial-temporal statistics context, we refer the interested reader to volumes by Yaglom (1987) and Christakos (1992).

In what follows, we shall consider some interesting restrictions on the general class of stationary covariance functions. But first, we need to introduce some standard notation for arbitrary partitions and operations between vectors. In order to manipulate arbitrary decompositions of nonnegative integer numbers, let us consider the set \( \exp(\mathbb{Z}_+) = \emptyset \cup \mathbb{Z}_+ \cup \mathbb{Z}_+^2 \cup \mathbb{Z}_+^3 \cup \ldots \) (disjoint union). An element \( d \) of \( \exp(\mathbb{Z}_+) \) can be expressed either as \( d = \emptyset \) or as \( d = (d_1, d_2, \ldots, d_n) \) if \( d \in \mathbb{Z}_+^n \) with \( n \geq 1 \). In the latter case we denote by \( n(d) = n \) the dimension of \( d \) and \( |d| = \sum_{i=1}^n d_i \) the length of \( d \). Both values are taken to be 0 whenever \( d = \emptyset \). For \( d, d' \in \exp(\mathbb{Z}_+) \) we say that \( d \leq d' \) if and only if \( n(d) = n(d') \) and \( d_i \leq d'_i \) for all \( i = 1, 2, \ldots, n(d) \). Usual vector operations are possible only between elements of the same dimension. Vectors with all components equal are denoted in bold symbols, such as \( \mathbf{0}, \mathbf{1} \).

Now, the restrictions considered in this paper are of the following type:

1. The isotropic case:

   \[
   \tilde{C}_0(\mathbf{x}) := \tilde{C}_1(||\mathbf{x}||), \quad \mathbf{x} \in \mathbb{R},
   \]

   that is, \( C \) is said to be represented by the positive definite function \( \tilde{C}_1 : \mathbb{R}_+ \rightarrow \mathbb{R} \), that is rotation-translation invariant (or radially symmetric), and where \( ||.|| \) denotes the Euclidean norm. This is the most popular case in spatial and spatiotemporal statistics (Mátern, 1960; Yaglom, 1987, Christakos and Papanicolaou, 2000).

2. The component-wise isotropic case: Let us consider the \( d \)-dimensional space \( \mathbb{R}^d \), and let \( d \) be an element of \( \exp(\mathbb{Z}_+) \) such that \( |d| = d \) and \( 1 \leq d \). Thus, one can create opportune partitions of the spatial lag vector \( \mathbf{x} \in \mathbb{R}^d \) in the following way. If \( d = (d_1, d_2, \ldots, d_n) \) and \( \mathbf{x} \in \mathbb{R}^d \) we can always write

   \[
   \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \ldots \times \mathbb{R}^{d_n}
   \]

   so that:

   (i)- \( \tilde{C}_0(\mathbf{x}) = \tilde{C}_0(\mathbf{k}) \) for any \( \mathbf{x}, \mathbf{k} \in \mathbb{R}^d \) if and only if \( ||\mathbf{x}_i|| = ||\mathbf{k}_i|| \) for all \( i = 1, 2, \ldots, n \).
(ii)- The resulting covariance admits the representation

\[
\tilde{C}_0(x) := \tilde{C}_1(\|x_1\|, \ldots, \|x_n\|) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \prod_{i=1}^{n} \Omega_d(\|x_i\| r_i) dF(r_1, \ldots, r_n)
\]

with \(\Omega_d(t) = \Gamma(d/2) \left( \frac{t}{2} \right)^{\frac{d}{2} - 1} J_{d-2}(t) \), \(J_d(.)\) denoting the Bessel function of the first kind of order \(d\) (Abrahamowitz and Stegun, 1965), \(F\) a \(n\)-variate distribution function and \(\tilde{C}_1 : \mathbb{R}^n \to \mathbb{R}\) positive definite. Thus, equation (2) is a special case of (3) \(|\mathbf{d}| = d\) and \(\mathbf{d} := d\), a scalar) and its corresponding integral representation can be readily obtained. The special case \(\mathbf{d} = 1\) and \(|\mathbf{d}| = d\) is particularly interesting in the subsequent sections of this paper, as the function \((x_1, \ldots, x_n) \mapsto \tilde{C}_1(|x_1|, \ldots, |x_n|)\), with \(\tilde{C}_1\) positive definite, does not depend on the Euclidean norm, but on the Manhattan or city block distance, with important implications in spatial and spatiotemporal statistics as pointed out by Christakos (2000) and Banerjee (2004).

It is worth noticing that covariance functions of the type (3) have a property called reflection symmetry (Lu and Zimmerman, 2005), or full symmetry (Christakos and Hristopulos, 1998). This means that \(C(x_1, \ldots, x_i, \ldots, x_n) = C(x_1, \ldots, -x_i, \ldots, x_n) = \ldots = C(-x_1, \ldots, -x_i, \ldots, -x_n)\). Whenever no confusion arises, in the remainder of the paper we shall drop the under- and super-script denoting a stationary or stationary and isotropic covariance function, respectively.

As far as the basic properties of covariance functions are concerned, assume that \(C_i : \mathbb{R}^{d_i} \to \mathbb{R}_{+}\) \((i = 1, \ldots, n)\) are positive, continuous and integrable stationary covariance functions with \(d_i \in \mathbb{Z}_{+}\) and let \(\mathbf{d} = (d_1, \ldots, d_n)\) such that \(|\mathbf{d}| = d\). It is well known that some mean operators preserve the permissibility of the resulting structure. In particular, if we assume, without loss of generality, that the \(\theta_i\) are nonnegative weights summing up to one, the following are then permissible on \(\mathbb{R}^{d}\):

1 The arithmetic average

\[
C_A(x) = \sum_{i=1}^{n} \theta_i C_i(x_i),
\]

2 the non weighted geometric average up to a power \(n\),

\[
C_G(x) = \prod_{i=1}^{n} C_i(x_i),
\]

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the $k$-power average ($k \in \mathbb{Z}_+$), up to a power $k$,

$$C_{\mu^k}(x) = \sum_{i=1}^{n} \theta_i C_i^k(x_i).$$

4 Scale and power mixtures of covariance functions (Christakos, 1992),

$$C(x) = \int_{\Theta} C(x; \theta) d\mu(\theta),$$

for $\mu$ a positive measure and $\theta \in \Theta \subseteq \mathbb{R}^p$, $p \in \mathbb{N}$.

2.2 The methodology. Quasi-arithmetic multivariate com-

positions

Quasi-arithmetic averages have been extensively treated by Hardy et al. (1934). Our methodology generalises the concept of quasi-arithmetic averages and introduces new formalisms and notation, since our aim is to find a class of compositions of covariance functions that satisfies desirable proper-

eties.

Let $\Phi$ be the class of real-valued functions $\varphi$ defined in some domain $D(\varphi) \subset \mathbb{R}$, admitting a proper inverse $\varphi^{-1}$ defined in $D(\varphi^{-1}) \subset \mathbb{R}$ and such that $\varphi(\varphi^{-1}(t)) = t$ for all $t \in D(\varphi^{-1})$. Also, let $\Phi_c$ and $\Phi_{cm}$ be the subclasses of $\Phi$ obtained by restricting $\varphi$ to be, respectively, convex or completely monotone on the positive real line. Let us call a quasi-arithmetic class of functionals the class

$$\Omega := \left\{ \psi : D(\varphi^{-1}) \times \cdots \times D(\varphi^{-1}) \rightarrow \mathbb{R} : \psi(u) = \varphi\left(\sum_{i=1}^{n} \theta_i \varphi^{-1}(u_i)\right), \quad \varphi \in \Phi \right\},$$

(4)

where $\theta_i$ are nonnegative weights and $u = (u_1, \ldots, u_n)'$, for $n \geq 2$ positive integer. Also, we shall call $\Omega_c$ and $\Omega_{cm}$ the subclasses of $\Omega$ when restricting $\varphi$ to belong, respectively, to $\Phi_c$ and $\Phi_{cm}$.

If $\psi \in \Omega$, then we should write $\varphi_{\psi}$ as the function such that: for any nonnegative vector $u$, $\psi(u) = \varphi_{\psi}\left(\sum_{i=1}^{n} \theta_i \varphi_{\psi}^{-1}(u_i)\right)$. For ease of notation, we simply write $\varphi$ instead of $\varphi_{\psi}$, whenever no confusion arises.

Next, we introduce a new class of functionals that will be used extensively throughout the paper.

Definition 1. Quasi-arithmetic compositions If $f_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}_+$ such that $\cup_i^n f_i(\mathbb{R}^{d_i}) \subset D(\varphi^{-1})$ for some $\varphi \in \Phi$, the quasi-arithmetic composition
Table 1: Examples of quasi-arithmetic compositions for some possible choices of the generating function $\varphi \in \Phi$.

| $\varphi(t)$ | $\varphi^{-1}(t)$ | $Q_\psi(f_1, f_2)(\omega)$ | Remarks |
|--------------|-------------------|-----------------------------|---------|
| $\exp(-t)$   | $-\ln t$          | $\prod_{i=1}^{n} f_i(x_i)^{\theta_i}$ | $f_i : \mathbb{R}^{d_i} \to [0, \infty)$  
$\ln 0 = -\infty$  
$\exp(-\infty) = 0$  
$\sum \theta_i = 1$ |
| $1/t$        | $1/t$             | $\frac{\sum_{i=1}^{n} \theta_i \exp(-\theta_i f_i(x_i))}{\sum_{i=1}^{n} \theta_i f_i(x_i)}$ | $f_i : \mathbb{R}^{d_i} \to [0, \infty)$  
$1/0 = \infty$, $1/\infty = 0$  
$0/0 = 0$  
$\sum \theta_i = 1$ |
| $M(1 - t/M)^+$ | $M(1 - t/M)^+$ | $\sum_{i=1}^{n} \theta_i f_i(x_i)$ | $f_i : \mathbb{R}^{d_i} \to [0, M)$  
for some $M > 0$  
$(u)^+ = \max(u, 0)$  
$\sum \theta_i = 1$ |
| $-\ln t$     | $\exp(-t)$       | $-\ln \left( \sum_{i=1}^{n} \exp(-\theta_i f_i(x_i)) \right)$ | $f_1, f_2 : \mathbb{R}^{d} \to \mathbb{R}$ |

The quasi-arithmetic composition $Q_\psi(f_1, \ldots, f_n)(x)$ of $f_1, f_2, \ldots, f_n$ with generating function $\psi \in \mathcal{Q}$ is defined as the functional

$$Q_\psi(f_1, \ldots, f_n)(x) = \psi\left(f_1(x_1), \ldots, f_n(x_n)\right)$$

for $x = (x_1, \ldots, x_n)^\prime$, $x_i \in \mathbb{R}^{d_i}$, $d = (d_1, \ldots, d_n)^\prime$ and $|d| = d$.

Throughout the paper, we refer to $\psi \in \mathcal{Q}$ or the corresponding $\varphi \in \Phi$ as the generating functions of $Q_\psi$. Note that $Q_\psi(f, \ldots, f) = f$ for any function $f$ and generating function $\psi$.

**Example 1.** In Table 1 we present four basic examples of the quasi-arithmetic composition considered above. Some conventions are needed in order to deal with possibly ill-defined values.

Ordering relations as well as a minimal element can be found among the set of quasi-arithmetic compositions of $n \in \mathbb{N}$ fixed functions indexed by convex generating functions (a maximal element can also be found only when both fixed functions are upper bounded). From now on, we shall write

$$Q_G(f_1, \ldots, f_n) = \prod_{i=1}^{n} f_i^{\theta_i},$$
which is the quasi-arithmetic composition associated with \( \varphi(t) = \exp(-t) \) that generates a geometric average (constraints on the weights are specified in Table I). Also, let

\[
\mathcal{Q}_A(f_1, \ldots, f_n) = \sum_{i=1}^{n} \theta_i f_i,
\]

for \( f_1, \ldots, f_n : \mathbb{R}^d \to [0, M] \), be the quasi-arithmetic composition associated with \( \varphi(t) = M(1 - t/M)_+ \) that generates the arithmetic average. Finally, we shall call

\[
\mathcal{Q}_H(f_1, \ldots, f_n) = \frac{\sum_{i=1}^{n} \theta_i}{\sum_{i=1}^{n} \frac{\theta_i}{f_i(x_i)}},
\]

the quasi-arithmetic composition associated with \( \varphi(t) = 1/t \) and generating the harmonic average.

We shall write \( g_1 \leq g_2 \) whenever \( g_1(x_1, \ldots, x_n) \leq g_2(x_1, \ldots, x_n) \) for all \( x = (x_1, \ldots, x_n)' \in \mathbb{R}^d \). Finally, recall that a function \( g \) is subadditive whenever \( g(a + b) \leq g(a) + g(b) \) for all \( a, b \) in its domain. Thus, we have the following result:

**Proposition 1.** Given any set of functions \( f_1, \ldots, f_n \) and generating functions \( \varphi, \varphi_1, \varphi_2 \in \Phi_{cm} \) (which \( \psi, \psi_1, \psi_2 \in \Omega_{cm} \) are associated with, respectively), and given the same set of weights \( \{\theta_i\}_{i=1}^{n} \) for any pairwise comparison, we have the following point-wise order relations,

- \( \mathcal{Q}_G(f_1, \ldots, f_n) \leq \mathcal{Q}_\psi(f_1, \ldots, f_n) \leq \sum_{i=1}^{n} \theta_i f_i; \)
- \( \mathcal{Q}_\psi(f_1, \ldots, f_n) \leq \mathcal{Q}_A(f_1, \ldots, f_n) \) whenever \( f_1, \ldots, f_n : \mathbb{R}^d \to [0, M] \);
- \( \mathcal{Q}_{\psi_1}(f_1, \ldots, f_n) \leq \mathcal{Q}_{\psi_2}(f_1, \ldots, f_n) \) if and only if the function \( \varphi_1^{-1} \circ \varphi_2 \) is subadditive.

This result extends to the functional case those reported by Nelsen (1999). Its proof can be found in the Appendix.

### 2.3 Other useful notions and notation

A real mapping \( \gamma : \mathbb{X} \subseteq \mathbb{R}^d \to \mathbb{R} \) is called an intrinsically stationary variogram (Matheron, 1965) if it is conditionally negative definite, that is,

\[
\sum_{i=1}^{n} \sum_{i=j}^{n} a_i a_j \gamma(x_i - x_j) \leq 0
\]
for all finite collections of real weights $a_i$ summing up to zero and all points $\mathbf{x}_i \in \mathbb{X}$. The restriction to the isotropic case is analogical to that of covariance functions, that is $\gamma(\mathbf{x}) := \tilde{\gamma}(\|\mathbf{x}\|)$, $\mathbf{x} \in \mathbb{X}$ and $\tilde{\gamma} : \mathbb{R}_+ \to \mathbb{R}$ conditionally negative definite.

A completely monotone function $\varphi$ is a positive function defined on the positive real line and satisfying

$(-1)^n \varphi^{(n)}(t) \geq 0, \quad t > 0,$

for all $n \in \mathbb{N}$. Completely monotone functions are characterised in Bernstein’s theorem (see Feller, 1966, p.439) as the Laplace transforms of positive and bounded measures. By a theorem of Schoenberg (1938), a function $C$ is radially symmetric and positive definite on any $d$-dimensional Euclidean space $\mathbb{R}^d$ if and only if $C(\mathbf{x}) := \varphi(\|\mathbf{x}\|^2)$, $\mathbf{x} \in \mathbb{R}^d$, with $\varphi$ completely monotonic on the positive real line.

Bernstein functions are positive functions defined on the positive real line, whose first derivative is completely monotonic. Once again, an intimate connection with (negative) definiteness arises, as $\gamma$ is a radially symmetric and conditionally definite function on any $d$-dimensional Euclidean space $\mathbb{R}^d$ if and only if $\gamma(\mathbf{x}) := \mathcal{B}(\|\mathbf{x}\|^2)$, with $\mathcal{B}$ a Bernstein function.

Sufficient conditions for positive definiteness are stated in Pólya’s criteria (Berg and Forst, 1975) in $\mathbb{R}^1$; and in Christakos (1984, 1992) and Gneiting (2001), who extend criteria of the Pólya type to $\mathbb{R}^d$.

### 3 Theoretical Results

In this section we present theoretical results in a general setting, e.g. working with arbitrary partitions of $d$-dimensional spaces as explained previously. In particular, we shall obtain permissibility criteria for quasi-arithmetic averages of covariance functions on $\mathbb{R}^d$. This will be done for (a) a general case in which the respective arguments of the covariance functions used for the quasi-arithmetic average have no restrictions; (b) the restriction to the component-wise isotropic case; and (c) the further restriction to covariances that are isotropic and defined on the real line. Also, we shall show some properties of this construction. In particular, we refer to the associativity of quasi-arithmetic functionals and to the extension of ordering relations in Proposition 1 to the case of compositions of covariance functions. The proof of these new results can be found in the Appendix.

**Proposition 2.** (a) **General case** Let $\varphi \in \Phi_{cm}$ and $C_i : \mathbb{R}^d_i \to \mathbb{R}$ ($i = 1, \ldots, n$) be continuous stationary covariance functions such that $\bigcup_i C_i(\mathbb{R}^d_i) \subset$
D(φ⁻¹) and \( d = (d_1, \ldots, d_n)' \), \(|d| = d\). If the functions \( x_i \mapsto φ^{-1} \circ C_i(x_i), \ i = 1, \ldots, n \), are intrinsically stationary variograms on \( \mathbb{R}^{d_i} \), then

\[
Q(\psi) \left( C_1, \ldots, C_n \right) (x_1, \ldots, x_n)
\]

is a stationary covariance function on \( \mathbb{R}^d \). 

(b) Component-wise isotropy Let \( φ \in \Phi_{cm} \) and \( C_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R} (i = 1, \ldots, n) \) be continuous stationary and isotropic covariance functions such that \( \bigcup_i^n C_i(\mathbb{R}^{d_i}) \subset D(φ^{-1}) \) and \( d = (d_1, \ldots, d_n)' \), \(|d| = d\). If the functions \( x \mapsto φ^{-1} \circ C_i(x) \) are Bernstein functions on the positive real line, then

\[
Q(\psi) \left( C_1, \ldots, C_n \right) (|x_1|, \ldots, |x_n|)
\]

is a stationary and fully symmetric covariance function on \( \mathbb{R}^d \).

(c) Univariate covariances Let \( φ \in \Phi_{cm} \) and \( C_i : \mathbb{R} \rightarrow \mathbb{R} (i = 1, \ldots, n) \) be continuous stationary and isotropic covariance functions defined on the real line such that \( \bigcup_i^n C_i(\mathbb{R}) \subset D(φ^{-1}) \) and \(|d| = n\). If the functions \( x \mapsto φ^{-1} \circ C_i(x) \) are continuous, increasing and concave on the positive real line, then

\[
Q(\psi) \left( C_1, \ldots, C_n \right) (|x_1|, \ldots, |x_n|)
\]

is a stationary and fully symmetric covariance function on \( \mathbb{R}^n \).

It is worth noticing that case (3) represents a covariance permissibility condition that does not depend on the Euclidean metric, as it is function of the Manhattan or city block distance. For a detailed discussion of the limitations of the Euclidean norm-dependent covariances, see Banerjee (2004).

The previous result is of particular importance, since it has implications both in terms of averages of covariance functions and mixtures thereof. In the following we derive sufficient permissibility conditions for geometric and harmonic averages of covariance functions.

**Corollary 1. (Geometric average)** Let \( C_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R} (i = 1, \ldots, n) \) be continuous permissible covariance functions. Let \( \theta_i (i = 1 \ldots, n) \) be non-negative weights summing up to one. If the functions \( x \mapsto -\ln (C_i(x)) \), \( x > 0 \) satisfy any of the relevant conditions described in (a),(b) and (c) of Proposition 2 above, then

\[
Q(\psi)(C_1, \ldots, C_n) = \prod_{i=1}^n C_i^{\theta_i}
\]

is a covariance function.
An example of this setting can be found by using the function \( x \rightarrow (1 + x^\delta)^{-\varepsilon} \), \( x \) positive argument, \( \delta \in (0, 2] \) and \( \varepsilon \) positive, also known as generalised Cauchy class (Gneiting and Schlather, 2004). One can verify that the composition of this function with the natural logarithm is continuous, increasing and concave on the positive real line for \( \delta \in (0, 1] \). Another function satisfying these requirements is the function \( x \rightarrow \exp(-x^\delta) \), which is completely monotonic for \( \delta \in (0, 1] \). It should be stressed that these permissibility criteria do not apply to compactly supported covariance functions, such as the spherical model (Christakos, 1992).

**Corollary 2. (Harmonic average)** Let \( C_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R} \) \( (i = 1, \ldots, n) \) be continuous permissible covariance functions. If the functions \( x \rightarrow C_i(x)^{-1}, \) \( x > 0 \) satisfy any of the relevant conditions described in (a), (b) and (c) of Proposition 2 above, then for \( \theta_i \geq 0 \) such that \( \sum_i \theta_i = 1 \), the

\[
Q_H(C_1, \ldots, C_n) = \frac{\sum_i \theta_i}{\sum_i \frac{\theta_i}{C_i}}
\]

is a covariance function.

In order to complete the picture about quasi-arithmetic covariance functions, it would be desirable to establish at least some of their algebraic properties. The following results show some important features of the theoretical construction obtained from Proposition 2.

**Proposition 3. (Associativity).** Consider the same arrangements as in Proposition 2 and set \( \theta_i = 1/n \) \( (i = 1, \ldots, n) \). Let \( \varphi_1 \in \Phi_{cm} \) and \( \varphi_2 \in \Phi \). If the functions \( x \rightarrow \varphi_1^{-1} \circ \varphi_2(x), \) \( x \rightarrow \varphi_2^{-1} \circ C_i(x) \) \( (i = 1, \ldots, k) \) and \( x \rightarrow \varphi_1^{-1} \circ C_j(x) \) \( (j = k + 1, \ldots, n; t > 0 \) and \( k < n) \) satisfy any of the relevant conditions described in (a), (b) and (c) of Proposition 2 above, then

\[
Q_{\psi_1} (Q_{\psi_2}(C_1, \ldots, C_k), C_{k+1}, \ldots, C_n) (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n), \quad (9)
\]

\[
Q_{\psi_1} (Q_{\psi_2}(C_1, \ldots, C_k), C_{k+1}, \ldots, C_n) (\|x_1\|, \ldots, \|x_k\|, \|x_{k+1}\|, \ldots, \|x_n\|) \quad (10)
\]

and

\[
Q_{\psi_1} (Q_{\psi_2}(C_1, \ldots, C_k), C_{k+1}, \ldots, C_n) (|x_1|, \ldots, |x_k|, |x_{k+1}|, \ldots, |x_n|) \quad (11)
\]

and accordingly any coherent permutation of \( \varphi_1, \varphi_2 \) with \( C_i \) \( (i = 1, \ldots, n) \) are covariance functions.
One may notice that we deviate from the associativity condition defined in Bemporad (1926) and (in a more general form called decomposability) in Marichal (2000). Nevertheless, it can be readily verified that the associativity condition of Bemporad is always satisfied for our construction, whereas the strong and weak decomposability of Marichal (2000) is certainly satisfied by construction (5), but not, in general, by constructions (6) and (7).

**Proposition 4. (Ordering and preference relations)** For any set of covariance functions $C_1, \ldots, C_n$ and any arbitrary generating functions $\varphi, \varphi_1, \varphi_2 \in \Phi_{cm}$ with associated $\psi, \psi_1, \psi_2 \in \Omega_{cm}$, if (i) the constraints in either (a), (b), or (c) of Proposition 2 are satisfied and (ii) the same set of weights $\{\theta_i\}_{i=1}^k$ is used for any pairwise comparison, then we have the point-wise order relations,

1. $Q_G(C_1, \ldots, C_n) \leq Q_\psi(C_1, \ldots, C_n) \leq \sum_{i=1}^n \theta_i C_i$;
2. $Q_H(C_1, \ldots, C_n) \leq Q_A(C_1, \ldots, C_n)$ whenever $C_1, \ldots, C_n : \mathbb{R}^{d_i} \to [0, M]$;
3. $Q_{\psi_1}(C_1, \ldots, C_n) \leq Q_{\psi_2}(C_1, \ldots, C_n)$ if and only if the function $\varphi_1^{-1} \circ \varphi_2$ is subadditive.

The proof is omitted as it follows the same arguments as that of Proposition 1. It should be stressed that the first inequality shows that quasi-arithmetic covariances satisfy Jensen’s inequality. This fact has some implications on the variance of the stationary random field generated by a quasi-arithmetic operator. This will be discussed in the subsequent sections. Also, it is important to specify that other results, in particular inequalities involving quasi-arithmetic operators, can be readily extended to the case of quasi-arithmetic compositions of covariance functions. This is the case, for instance, of Theorem 1 in Abramovich et al. (2004).

4 Using quasi-arithmetic functionals in the construction of nonseparable space-time covariance functions

4.1 Review of space-time covariance functions

Natural (physical, health, cultural etc.) systems involve various attributes, such as atmospheric pollutant concentrations, precipitation fields, income distributions, and mortality fields. These attributes are characterised by spatial-temporal variability and uncertainty that may be due to epistemic
and ontologic factors. In view of the prohibiting costs of spatially dense monitoring networks, one often aims to develop a mathematical model of the natural system in a continuous space-time domain, based on sequential observations at a limited number of monitoring stations. This kind of problem has been a motivation for the development of the spatio-temporal random field (S/TRF) theory; see Christakos (1990, 1991, 1992) and Christakos and Hristopulos (1998) for a detailed discussion of the ordinary and generalised S/TRF theory and its various applications. In the following, we slightly change notation in order to be consistent with classical nomenclature in the Geostatistical literature (Cressie and Huang, 1999; Gneiting, 2002). Let \( \{ Z(s,t), (s,t) \in \mathbb{R}^d \times \mathbb{R} \} \) be a real-valued S/TRF, where \( s, t \) denote respectively the spatial and temporal position. Then, the function 
\[
C_{s,t}(s_1,t_1,s_2,t_2) = \text{cov}(Z(s_1,t_1), Z(s_2,t_2)),
\]
defined on the product space \( \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \), is the covariance function of the associated S/TRF \( Z \) if and only if it is permissible, i.e. satisfies (1). When referring to the spatial index, the term homogeneity instead of weak stationarity is equivalently adopted (Christakos, 1990, 1992). Thus, under the assumption of spatial homogeneity and temporal stationarity (sometimes, simply called spatio-temporal stationarity in the weak sense), the underlying S/TRF, ergo denoted by \( H/S \), has finite and constant mean and the covariance function, defined on the product space \( \mathbb{R}^d \times \mathbb{R} \), is such that 
\[
\text{cov}(Z(s_1,t_1), Z(s_2,t_2)) = C_{s,t}(h,u),
\]
with \( (h,u) = (s_i - s_j, t_i - t_j) \in \mathbb{R}^d \times \mathbb{R} \) denoting the spatio-temporal separation vector, and the \( h \) and \( u \) denoting the spatial and the temporal lags, respectively. The special case of isotropy in the spatial component and symmetry in the the temporal one is denoted as 
\[
C_{s,t}(h,u) := \tilde{C}_{s,t}(\|h\|, |u|),
\]
where the \( \|\cdot\| \) denotes the Euclidean norm. Obviously, (12) is fully symmetric.

Another popular assumptions concerning the S/TRF model is that of separability, that is (Christakos and Hristopulos, 1998; Mitchell et al., 2004)
\[
C_{s,t}(h,u) = C_{s,t}(h,0)C_{s,t}(0,u).
\]
In other words, separability means that the spatio-temporal covariance structure factors into a purely spatial and a purely temporal component, which allows for computationally efficient estimation and inference. Consequently, separable covariance models have been popular even in situations in which they are not physically justifiable. Another interesting aspect is that separable covariances are also fully symmetric, whereas the converse is not necessarily true.
It has been argued in the relevant literature that separable models allow for ease of computation and dimensionality reduction, as the space-time covariance matrix is obtained through the Kronecker product of the marginal spatial and temporal ones. However, separability is an unrealistic assumption for many applications, since it implies a considerable loss of information about important interactions between the spatial and temporal variations. Therefore, various techniques have been introduced for generating different classes of nonseparable spatio-temporal covariance models. Most of these techniques have been developed in the context of applied stochastics analysis and include H/S as well as non-H/S covariances (e.g., Yaglom, 1948; Gandin and Boltenkov, 1967; Monin and Yaglom, 1965, 67; Christakos, 1990, 1991, 1992, 2000; Christakos and Hristopulos, 1998; Christakos et al., 2002, 2005; Kolovos et al., 2004). Also, covariance models have been developed in the context of spatio-temporal statistics (e.g., Jones and Zhang, 1997; Cressie and Huang, 1999; De Cesare et al., 2001; Fuentes and Smith, 2001; Gneiting, 2002; Ma, 2003; Fernández-Casal et al., 2003; Stein, 2005a; Porcu et al., 2006a).

Being the Laplace transform of positive bounded measures, completely monotone functions are particularly appealing for the construction of stationary and nonstationary space-time covariances. In particular, they are intimately connected with the concept of mixture-based covariance functions, that has been repeatedly used by several authors. In the stationary case, see Gneiting (2002), Ma (2002;2003), Ehm et al. (2003), Fernández-Casal et al. (2003), Porcu et al. (2006a). In the nonstationary case, Paciorek and Schervish (2004;2006), Stein (2005b), and Porcu et al. (2006b) have made use of this technique.

Also, mixture-based covariances have been developed with less sophisticated instruments than completely monotone functions. This is the case of the so-called product sum model (De Cesare, 2001) and their extensions (De Iaco et al., 2002; Porcu et al., 2006c). This group of authors builds nonseparable space-time covariances through simple application of the basic properties of covariances seen as a convex cone. The mixture-based procedures and the basic properties of covariance functions are properly combined in the present paper.

In the following, a stationary RF with a quasi-arithmetic covariance function will be called quasi-arithmetic random field and denoted with the acronym QARF.
4.2 On the representation and smoothness properties of QARF

In this section we focus on the representation of a QARF and discuss the smoothness properties in terms of (mean square) partial differentiability. These properties are intimately related to those of the associated covariance function.

Let \( Z_i(s) \) be univariate mutually independent continuous weakly stationary Gaussian random processes defined on the real line \((i = 1, \ldots, (d + 1)\); \( s \in \mathbb{R}; \) and \( d \in \mathbb{Z}_+\)). In particular, let the process \( Z_{d+1} \) be continuously indexed by time \( t \). Consider also a \((d + 1)\)-dimensional nonnegative random vector \( R = (R_1, \ldots, R_{d+1})' \) with \( R_i \) independent of \( Z_i \). Let the univariate covariances \( C_{s_i} \) and the temporal covariance \( C_t \) be respectively associated with \( Z_i, i = 1, \ldots, d \) and \( Z_{d+1} \). In the following we shall assume these covariances to be stationary, symmetric, and of the type \( C_{s_i}(h_i) = \exp(-\nu_i(|h_i|)), i = 1, \ldots, d, \) and \( C_{t}(u) = \exp(-\nu_t(|u|)), \) with \( \nu_i = \phi^{-1} \circ C_{s_i}, \nu_t = \phi^{-1} \circ C_{t}, \) where the \( \phi \in \Phi_{cm} \) and \( C_i \) are positive definite and such that the compositions \( \nu_i \) are continuous, increasing and concave on the positive real line. Positive definiteness of this construction is guaranteed by direct application of the theorem of Schoenberg (1938) and according to a Pólya type criterion (see Berg and Forst (1975), Proposition 10.6).

We are interested in inspecting the properties of the following stationary spatio-temporal scale mixture-based random field, defined on \( \mathbb{R}^d \times \mathbb{R}, \)

\[
Z(s, t) = Z_{d+1}(tR_{d+1}) \prod_{i=1}^{d} Z_i(s_iR_i),
\]

where \( s = (s_1, \ldots, s_d)' \in \mathbb{R}^d \) and \( t \in \mathbb{R}. \) It can be easily seen that the covariance structure associated to this random field is nonseparable, as

\[
C_{s,t}(h, u) = \int_{\mathbb{R}_{d+1}^+} \exp \left( - \sum_{i=1}^{d} \nu_i(|h_i|)r_i - \nu_t(|u|)r_{d+1} \right) dF(r),
\]

with \( h = (h_1, \ldots, h_d)' \in \mathbb{R}^d, \) \( u \in \mathbb{R} \) and \( r = (r_1, \ldots, r_{d+1})' \in \mathbb{R}^{d+1}, \) and where \( F \) is the \((d + 1)\)- variate distribution function associated to the random vector \( R. \) If \( F \) is absolutely continuous with respect to the Lebesgue measure, then previous representation can be reformulated with respect to the \((d + 1)\)- variate density, say \( f, \) that is

\[
C_{s,t}(h, u) = \int_{\mathbb{R}_{d+1}^+} \exp \left( - \sum_{i=1}^{d} \nu_i(|h_i|)r_i - \nu_t(|u|)r_{d+1} \right) f(r)dr.
\]
It can be seen easily that this construction allows for the case of separability if and only if the integrating \((d+1)\)-dimensional measure \(F\) (or equivalently its associated density \(f\)) factorises into the product of \((d+1)\) marginal ones, \(i.e.\) if the nonnegative random vector \(R\) has mutually independent components.

Now, if we suppose that the measure \(F\) is concentrated on the line \(r_1 = \ldots = r_{d+1} = r\) and that \(\varphi^{-1}\) is such that \(\varphi := \mathcal{L}[F]\), \(i.e.\) the Laplace transform of the positive univariate measure \(F\), then we obtain that the QARF is a special case of \((15)\). For this random field, further inferences may be made about its mean square partial differentiability if, additionally, we assume the function \(t \mapsto \exp(-\nu_i(t))\) to be absolutely integrable on the positive real line \((i = 1, \ldots, (d+1))\). In this case, one can show (technicalities can be found in the appendix) that the \(k\)-th order mean square partial derivative with respect to the \(i\)-th coordinate exists and is finite whenever the function \(\chi(r) = \int_{[0,\infty)} \omega_i^2 \tilde{c}_i(\omega_i; r)d\omega_i\), with \(\tilde{c}_i := \mathcal{F}^{-1}[\exp(-r\nu_i)] = \int_{\mathbb{R}} \exp(-i\omega_i h_i - r\nu_i(h_i))dh_i\) is measurable with respect to \(F\) \((i = 1, \ldots, d)\).

It should be stressed that Proposition \(\text{IV}\) gives some more information about the characteristics of the underlying QARF in terms of variance, as QARF can be ordered with respect to their minimum or maximum variance. Thus, the QARF generated by \(\psi := A\) has the largest variance among all the other QARF, for any choice of \(\psi\).

Finally, observe that the trivial quasi-arithmetic composition, obtained by setting \(C_i := \varphi \in \Phi_{cm} \ (i = 1, \ldots, d)\) preserves permissibility and results in a model of the type

\[
C_s(h) = \varphi(\theta'h)
\]

where \(\theta = (\theta_1, \ldots, \theta_d)' \in \mathbb{R}^d\) and \(h = (h_1, \ldots, h_d)'\) is the arbitrary partition of the spatial lag vector. Thus, the trivial case results in a composition of a completely monotonic function with an affine function, and can be used for modelling geometric anisotropies.

4.3 Applications of quasi-arithmeticity in the construction of stationary nonseparable space-time covariances

The results presented in the previous section can be useful in the construction of space-time covariance functions. For this purpose, some considerations are in order. The functional in equation \((14)\) should be adapted in the spatio-temporal case. It does not make sense to consider a weighted average of a spatial covariance with a temporal one. Also, the use of weights forbids, in this case, the construction of nonseparable model admitting separability.
as a special case. For this reason, we suggest to reduce the class in equation (4) to the case of trivial weights, e.g. \( \theta_i = 1, \forall i \). Then, it is more appropriate to call the class \((4)\) Archimedean, in analogy with the class built in Genest and MacKay (1986). It should be mentioned that one can easily prove that the restriction to trivial weights does not affect the permissibility of the resulting covariance function, provided that either one of the constraints imposed in cases \((a), (b)\) or \((c)\) of Proposition 2 are fulfilled. By analogy with the construction of copulas, and following Genest and MacKay (1986), the definition of generator for \( \varphi \in \Phi_{cm} \) is more meaningful.

Working this way, one can obtain some new families of covariance functions whose analytical expressions are familiar for those interested in probabilistic modelling through copulas. It should be noticed that all the families we propose in this section include separable covariance models as special cases, depending on the parameter values of the generators.

**Example 1. The Clayton family**

Consider the completely monotone function

\[
\varphi(x) = (1 + x)^{-1/\lambda_1}, \quad x > 0, \tag{16}
\]

where \( \lambda_1 \) is a nonnegative parameter with inverse \( \varphi^{-1}(y) = y^{-\lambda_1} - 1 \). Observe that (16) is the generator of the Clayton family of copulas; refer to Genest and MacKay (1986) for mathematical details about this class.

Also, consider the covariance functions \( C_s(h) = (1 + \|h\|)^{-1/\lambda_2} \) and \( C_t(u) = (1 + |u|)^{-1/\lambda_3} \), for \( \lambda_2, \lambda_3 \) positive parameters. It is easy to verify that \( \varphi^{-1}(C_i(y)) = (1 + y)^{\lambda_1/\lambda_i} \ (i = 2, 3) \) is a positive function whose first derivative is completely monotone if and only if \( \lambda_1 < \lambda_i \). Under this constraint, and applying case \((b)\) of Proposition 2 we find that

\[
C_{s,t}(h, u) = Q_{\psi}(C_s, C_t)(h, u) = \sigma^2 \left[(1 + \|h\|)^{\rho_1} + (1 + |u|)^{\rho_2} - 1\right]^{-1/\lambda_1} \tag{17}
\]

is a valid nonseparable stationary fully symmetric spatio-temporal covariance function, with \( \rho_i = \lambda_1/\lambda_i, \ \lambda_i > 0 \ (i = 2, 3) \) and \( \sigma^2 \) a nonnegative parameter denoting the variance of the underlying S/T process. It is interesting that both margins (the spatial and the temporal one) belong to the generalised Cauchy class. This is desirable for those interested in the local and global behaviour of the underlying random field.

Another covariance function that preserves Cauchy margins can be obtained through the following *iter*. Consider the function \( x \mapsto \varphi(x) = x^{-\alpha}, \ t > 0, \) that belongs to \( \Phi_{cm} \) for any positive \( \alpha \), being the Laplace transform
of the function $x \mapsto x^{\alpha-1}/\Gamma(\alpha)$, with $\alpha$ positive parameter and $\Gamma$ the Euler Gamma function. Also, for the spatial margin consider $C_s(h) = (1+\|h\|^{\delta})^{-\varepsilon}$ that belongs to $\Phi_{cm}$ if and only if $\delta \in (0,2]$ and $\varepsilon$ is strictly positive. Finally, let the temporal margin be of the type $C_t(u) = |u|^{-\rho}$, which is not a stationary covariance function but respects the composition criteria, as $\varphi^{-1} \circ C_t$ is continuous, increasing and concave on the positive real line if and only if $\alpha < \rho$. Then, it is easy to prove that $Q_\psi(C_s,C_t)(h,u)$ is a valid space-time function if and only if $\alpha < \varepsilon$ and $\alpha < \rho$, and that this covariance function has margins of the Cauchy type.

**Example 2. The Gumbel-Hougard family**

Consider the completely monotone function, the so-called positive stable Laplace transform

$$\varphi(x) = e^{-x^{1/\lambda_1}}, \quad x > 0,$$

(18)

where $\lambda_1 \geq 1$. Equation (18) admits the inverse $\varphi^{-1}(y) = (-\ln(y))^{\lambda_1}$. This is the generator of the Gumbel-Hougard family of copulas, whose mathematical construction and details are exhaustively described in Nelsen (1999). Consider two respectively spatial and temporal covariance functions admitting the same analytical form, i.e. $C_s(h) = \exp(-\|h\|^{1/\lambda_2})$ and $C_t(u) = \exp(-|u|^{1/\lambda_3})$. Then, it can be easily verified that $\varphi^{-1}(C_s(y)) = y^{\lambda_1/\lambda_2}$, and $\varphi^{-1}(C_t(y)) = y^{\lambda_1/\lambda_3}$ are always positive for $y > 0$ and possess completely monotone first derivatives if and only if $\lambda_1 < \lambda_i$, $i = 2,3$. So we get that

$$C_{s,t}(h,u) = Q_\psi(C_s,C_t)(h,u) = \sigma^2 \exp\left(-\|h\|^{\rho_1} + |u|^{\rho_2})^{1/\lambda_1}\right),$$

(19)

is a permissible nonseparable stationary fully symmetric spatio-temporal covariance function, with $\rho_i = \lambda_1/\lambda_i$, $i = 2,3$ and $\sigma^2$ as before.

**Example 3. The power series family**

The so-called power series

$$\varphi(x) = 1 - (1 - \exp(-x))^{1/\lambda_1}, \quad x > 0,$$

(20)

with $\lambda_1 \geq 1$, admits the inverse $\varphi^{-1}(y) = -\ln(1 - (1 - y)^{\lambda_1})$. Suppose a spatial and a temporal covariance function of the same analytical form as in Example 2. The composition $\varphi^{-1}(C_i(y)) = -\ln(1 - (1 - \exp(-y))^{\lambda_1/\lambda_i})$, 20
i = 2, 3, is always positive for \( y > 0 \) and admits a completely monotone first derivative if and only if \( \lambda_1 < \lambda_i \). So we get that

\[
Q_\psi(C_S, C_T)(h, u) = 1 - (1 - \exp(-\|h\|))^{\rho_1} - (1 - \exp(-|u|))^{\rho_2}
+ (1 - \exp(-\|h\|))^{\rho_1} (1 - \exp(-|u|))^{\rho_2}
\]  

(21)

is a permissible nonseparable stationary fully symmetric spatio-temporal covariance function, with \( \rho_i = \lambda_1/\lambda_i \) (\( i = 2, 3 \)).

**Example 4. The semiparametric Frank family**

Here we show that a nonstationary covariance function can be obtained, starting from the proposed setting, even if the arguments of the quasi-arithmetic functional are not covariance functions.

The Frank family of copulas (Genest, 1987) is generated by the function

\[
\varphi(x) = \frac{1}{\lambda} \ln \left( 1 - (1 - e^{-\lambda}) e^{-x} \right)
\]

with inverse \( \varphi^{-1}(y) = -\ln \left( (1 - e^{-\lambda y}) / (1 - e^{-\lambda}) \right) \). Nelsen (1999) shows that, for \( \lambda \) positive, \( \varphi \) is the composition of an absolutely monotonic function with a completely monotonic one, i.e. a completely monotonic function. As far as the inverse is concerned, it is easy to see that \( \varphi^{-1} \circ \gamma \) (\( \gamma \) an intrinsically stationary variogram) is negative definite. Thus, the

\[
C_{s,t}(h, u) = -\frac{1}{\lambda} \ln \left( 1 + \frac{(1 - e^{-\lambda \gamma_S(h)})(1 - e^{-\lambda \gamma_T(u)})}{1 - e^{-\lambda}} \right),
\]

for \( \gamma_S, \gamma_T \) intrinsically stationary variograms defined on \( \mathbb{R}^d \) and \( \mathbb{R} \), respectively, is a stationary space-time covariance function.

**4.4 Quasi-arithmeticity and nonstationarity in space**

The direct construction of spatial covariances that are nonstationary is anything but a trivial fact. Only few contributions refer to this kind of construction: see, among them, Christakos (1990, 1991), Christakos and Hristopoulos (1998) and Kolovos et al. (2004). More recent contributions can be found in Stein (2005b) and Paciorek and Schervish (2006). It seems that something more could be done concerning the construction of nonstationary covariances, knowing that stationarity is very often an unrealistic assumption for several physical and natural processes.

In this section, we show that both approaches discussed in Stein (2005b) and Paciorek and Schervish (2006), admit a natural extension through the
use of quasi-arithmetic functionals. We need to introduce some more notation concerning the restriction of the class $\Phi_{cm}$. For abuse of notation, a completely monotonic function is the Laplace transform of some positive and bounded measure, so that $\varphi \in \Phi_{cm}$ if and only if $\varphi := L[F]$.

**Theorem 1.** Let $\Sigma$ be a mapping from $\mathbb{R}^p \times \mathbb{R}^p$ to positive definite $p \times p$ matrices, $F$ a nonnegative measure on $\mathbb{R}_+$, $\varphi_1, \varphi_2 \in \Phi_{cm}$ and $g$ a nonnegative function such that, for any fixed $s \in \mathbb{R}^p$, $h_s = \varphi_2^{-1} \circ g(\cdot;s) \in L^1(F)$. Define $\Sigma(s_1,s_2) = 1/2(\Sigma(s_1) + \Sigma(s_2))$ and $Q(s_1,s_2) = (s_1 - s_2)^\top \Sigma(s_1,s_2)^{-1}(s_1 - s_2)$. Then,

$$
C_s(s_1,s_2) = \frac{|\Sigma(s_1)|^{1/4}|\Sigma(s_2)|^{1/4}}{|\Sigma(s_1,s_2)|^{1/2}} \int_0^\infty \varphi_1(Q(s_1,s_2)\tau) \varphi_2(g_{s_1}, g_{s_2})(\tau) dF(\tau)
$$

(22)

with $\theta_i = 1$, $i = 1,2$, is a nonstationary covariance function on $\mathbb{R}^d \times \mathbb{R}^d$.

Some comments are in order. One can see that Stein’s (2005b) result is a special case of (22), under the choice $\varphi_1(t) = \exp(-t)$, $t$ positive, and $\psi_2 := G$. So is the result in Paciorek and Schervish (2006). Nevertheless, the form we propose has some drawbacks that need to be noticed explicitly. The first problem is that it is very difficult to obtain a closed form for expression (22), unless one chooses $\psi_2 := G$. Secondly, if one’s purpose is to generalise the Matérn covariance function (Matérn, 1960) to the nonstationary case, then the problem has already been solved by Stein (2005b) and his approach is in our opinion the most suitable, as he finds a Matérn type covariance that allows for a spatially varying smoothness parameter and for local geometric anisotropy. On the one hand, the Matérn covariance possesses some desirable features (i.e., it allows for arbitrary levels of smoothness of the associated random field). On the other hand, there are other covariance functions that are of considerable interest to the statistical and scientific communities, such as the nonstationary ones. In particular, we refer to the Cauchy class, whose properties (in terms of decoupling of the global and local behaviour of the associated random field) have been discussed in Gneiting and Schlather (2004). After several trials, we did not succeed in obtaining a Cauchy type nonstationary covariance. Several examples of covariances can be derived that belong to the class (22) and can be numerically integrated. Here, we propose some closed form that is obtained by letting $\psi_2 := G$. As a first example, take $dF(\tau) = d\tau$, $\varphi_1(\tau) = \tau^{\lambda-1}$, that is completely monotonic for $\lambda \in (0,1)$, $g(\tau;\alpha_i,\nu_i) = (1 + \alpha(s_i)\tau)^{-\nu(s_i)}$, where $\alpha$ and $\nu$ are supposed to be strictly positive functions of $s_i$ ($i = 1,2$) and additionally $0 < \alpha(s_i), \nu(s_i) < \pi$. One can readily verify that all integrability conditions
in Theorem 1 are satisfied. Letting $k = \frac{\left| \sum(s_1) \right|^{1/4} \left| \sum(s_2) \right|^{1/4}}{\left| \sum(s_1, s_2) \right|^{1/2}} Q(s_1, s_2)^{\lambda - 1}$ and using [3.259.3] in Gradshteyn and Ryzhik (1980), one obtains the following covariance function

$$C_s(s_1, s_2) = k\alpha(s_1)^{-\lambda} B(\lambda, \nu(s_1) + \nu(s_2) - \lambda) 2F_1 \left( \nu(s_2), \lambda; \nu(s_1) + \nu(s_2); 1 - \frac{\alpha(s_2)}{\alpha(s_1)} \right),$$

where $B(.,.)$ is the Beta function, and $2F_1(.,.,.,.)$ is the Gauss hypergeometric function.

Another example is obtained by considering $F(d\tau) = \exp(-\tau)d\tau$, $\varphi_1(\tau) = \tau^{\nu-1}$, with $\nu \in (0, 1)$, $g(\tau; s_i) = \exp(-\alpha(s_i)\tau)$, with $\alpha(s_i)$ strictly positive ($i = 1, 2$) and using [3.478.4] of Gradshteyn and Ryzhik (1980), we find that

$$C_s(s_1, s_2) = 2k \left( \frac{\alpha(s_1) + \alpha(s_2)}{2} \right)^{-\nu/2} K_{\nu} \left( 2 \left( \frac{\alpha(s_1) + \alpha(s_2)}{2} \right)^{1/2} \right)$$

is a nonstationary spatial covariance that allows for local geometric anisotropy, but has a fixed smoothing parameter.

It should be stressed that numerical integration under the setting (22) could outperform previously proposed models if the objective is to find a different type of interaction between the local parameters characterising the integrating function $g$. In all the examples proposed by Paciorek and Schervish (2006) and Stein (2005b) the varying-smoothing, spatially adaptive parameter is obtained as the semi-sum of a parameter acting on the location $s_1$ with another one depending on the location $s_2$. This is a serious limitation of the method, as only one type of interaction can be achieved. Starting from a different setting, Pintore and Holmes (2004) obtain the same type of spatially adaptive smoothing parameters. Thus, quasi-arithmetic functionals could be of help, at least through numerical integration. Finding a closed form for a functional different than $Q_G$ remains an open problem to which we did not find any solution for the meantime.

5 Conclusions and discussion

In this work, novel results are presented concerning permissible spatio-temporal covariance functions in terms of the theory of quasi arithmetic means, and valuable insight is gained about their space-time structure. The theory of quasi arithmetic means is used, together with the relevant permissibility criteria, to derive new classes of nonseparable space-time covariances and to investigate their properties in considerable detail.
There are several possible avenues for research based on the results of the present paper. From the spatial and spatiotemporal statistics perspectives, the QARF representation considered in this paper seems very promising. E.g., the QARF may provide a naive separability testing procedure, as follows. Consider the set $\Psi_{cm}$ of all possible generators of the QARF class and let $\Theta$ be the set of parameter vectors indexing the generators, i.e. $\Theta : \{ \theta \in \mathbb{R}^p : \varphi_{\theta} \in \Phi_{cm} \}$. Thus, testing for separability of the covariance function associated to the QARF is equivalent to testing for the null hypothesis $\varphi_0 := \varphi_0(x; \theta_0) = \exp(-\theta_0 x)$, $x > 0$. This is a topic worth of further investigation.

Another interesting topic is that of generator estimation. A tempting choice would be to consider the approach proposed by Genest and Rivest (1993), and that in Ferguson et al. (2000) who find a serial version of the Kendall’s tau and a relationship between this concordance index and the generating function $\varphi$. In this case, one would extend the result of these authors at least to the lattice $\mathbb{Z}^2$.

Future research effort would also focus on a deeper study of the properties of quasi-arithmetic covariances along several directions. For instance, it would be interesting to find limit properties of the covariance functions specified through this construction.

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6 Appendix

6.1 Proof of Proposition 1

Let us begin by showing the final part, i.e. that

$$Q_{\psi_1}(f_1, \ldots, f_n) \leq A_{\psi_2}(f_1, \ldots, f_n), \quad (23)$$

for $\psi_1, \psi_2 \in \Omega_{cm}$, if and only if $\varphi_1^{-1} \circ \varphi_2$ is subadditive.

Let $g = \varphi_1^{-1} \circ \varphi_2$ and denote $x_i = \varphi_2^{-1} \circ f_i(x_i)$.

Thus, (23) is equivalent to

$$\varphi_1 \left( \sum_{i=1}^n \theta_i g(x_i) \right) \leq \varphi_2 \left( \sum_{i=1}^n \theta_i x_i \right). \quad (24)$$
For the necessity, assume $A_{\psi_1} \leq A_{\psi_2}$. Applying $\varphi_1^{-1}$ in both sides of (24) we obtain the result.

For the if part, assume (24) holds. Thus, applying $\varphi_1$ to both sides of the same inequality the result holds.

Let us now prove that $A_{\Pi}(f_1, \ldots, f_n) \leq A_{\psi}(f_1, \ldots, f_n) \leq \sum \theta_i f_i$.

The lower bound in the inequality is direct consequence of Lemma 4.4.3, Corollary 4.6.3 in Nelsen (1999, p. 110) and references therein, while the upper bound is a direct property of convex functions.

### 6.2 Proof of Proposition 2

Recall that, for Bernstein’s theorem (Feller, 1966, p.439), $\varphi \in \Phi_{cm}$ if and only if

$$\varphi(t) = \int_0^{\infty} e^{-rt} dF(r),$$

with $F$ a positive and bounded measure. Now, for the proof of (a), notice that, if $\varphi \in \Phi_{cm}$, equation (25) can be written as

$$\int_0^{\infty} \exp \left(-r \sum_{i=1}^{n} \theta_i \varphi^{-1}(C_i(h_i)) \right) dF(r).$$

Now, observe that if, for every $i$, $\varphi^{-1} \circ C_i$ is a variogram, then so is the sum $\sum_{i=1}^{n} \theta_i \varphi^{-1} \circ C_i$. Then, by Schoenberg theorem (1939), we have that for every positive $r$, the integrand in the formula above is a covariance function. So is the positive scale mixture of covariances. This completes the proof.

For (b), it is sufficient to notice that, by Schoenberg theorem (1938), the mapping $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is the variogram associated to an intrinsically stationary and isotropic random field if and only if $\gamma(h) = \psi(||h||^2)$, for $\psi$ a Bernstein function. The rest of the proof comes from the same arguments of point (a).

For (c), notice that, being continuous, increasing and concave on $[0, \infty)$, each of the functions $\varphi^{-1}(C_i)$, is negative definite on $\mathbb{R}$, according to a Pólya type criterion (see Berg and Forst (1975), Proposition 10.6). So is their sum. Since $\varphi \in \Phi_{cm}$, by Schoenberg’s theorem (cf. Berg and Forst, 1975), we get once again the result.

### 6.3 Proof of Proposition 3

We shall only prove the result in (a), as (10) and (11) follow the same argument. Recall that here we impose $\theta_i = 1/n \forall i$. Call $h^{(i)} = (||h_1||, \ldots, ||h_k||)^t$
and $h(2) = (||h_{k+1}||, \ldots, ||h_n||)'$, $k < n$. Also, let $f_1 = 1/n \sum_{j=k+1}^{n} \varphi_1^{-1} \circ C_j$, $f_2 = \varphi_1^{-1} \circ \varphi_2$ and $f_3 = 1/n \sum_{i=1}^{k} \varphi_2^{-1} \circ C_i$. Thus, equation (9) can be written, using Bernstein’s theorem, as

$$\int_0^\infty e^{-rf_1(h(2))-rf_2 \circ f_3(h(1))} dF_1(r)$$

where $F_1$ is the distribution associated to its Laplace transform $\varphi_1$. Thus, the proof follows straight by arguments of the previous proofs.

6.4 Proof of Theorem 1

The result is a direct consequence of the work of Stein (2005b). First, observe that $C(s_1, s_1) = \int_0^\infty g(\tau; s_1) dF(s_1) < \infty$. Now, we need to show that $\sum_{i=1}^{n} a_i a_j C(s_i, s_j) \geq 0$, for every finite system of arbitrary real constants $a_i$, $i = 1, \ldots, n$. Write $K_{\tau, \tau', r, r'}$ for the normal density centered at $s_i$ and with covariance matrix $(\tau \tau')^{-1}$. Also, by abuse of notation, $\varphi_1 := L[F_1]$ and $\varphi_2 := L[F_2]$. Then, by a convolution argument in Paciorek (2003, p.27), by Bernstein’s theorem, and using Fubini’s theorem, we get

$$\sum_{i,j=1}^{n} a_i a_j C(s_i, s_j) = \pi^{p/2} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left( \int_{\mathbb{R}^p} \hat{K}_{\tau, \tau', r, r'}(u) a_i a_j \hat{C}(\omega) d\omega \right)^2 dF(r) dF_1(r) dF_2(r')$$

Thus, the proof is completed.

6.5 M.S. differentiability of the QARF

Recall that the existence of the $k$-th order $i$-th mean square partial derivative of $Z$ is related to the existence of the $2k$-th order $i$-th mean square partial derivative of the covariance function. This is in turn related to the spectral moments through the following formula (Adler, 1981, p.31):

$$(-1)^k \frac{\partial^{2k} C(h)}{\partial h_i^{2k}}|_{h=0} = \int_{\mathbb{R}^d} \omega_i^{2k} d\hat{C}(\omega) < \infty.$$
\[ \int_{\mathbb{R}^d} \omega_i^{2k} \hat{c}(\omega) d\omega \propto \int_{\mathbb{R}^d} \omega_i^{2k} \int_{\mathbb{R}^d} e^{-i\omega' \hat{h} C(h)} dh d\omega \\
= \int_{\mathbb{R}^d} \omega_i^{2k} \int_{\mathbb{R}^d} e^{-i\omega' \hat{h}} \int_0^\infty e^{-r \sum_i \theta_i \nu_i(|h_i|)} dF(r) dh d\omega \\
= \int_{\mathbb{R}^d} \int_0^\infty \omega_i^{2k} \left( \int_{\mathbb{R}^{d-1}} e^{-i\tilde{\omega}' \tilde{h}} e^{-r \sum_{j \neq i} \theta_j \nu_j(|h_j|)} d\tilde{h} \right) \left( \int_{\mathbb{R}} e^{-r \theta_i \nu_i(|h_i|)} dh_i \right) dF(r) d\omega \\
\]

where \( \tilde{h} = (h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_d)' \in \mathbb{R}^{d-1} \) and where we repeatedly make use of Fubini's theorem for standard integrability criteria. Now, observe that, for the assumption of absolute integrability of \( \exp(-r \theta_i \nu_i(x)) \) for any positive \( r \), the last equality can be written as

\[ \int_0^\infty \int_{\mathbb{R}^d} \omega_i^{2k} \hat{c}(\omega; r) \hat{c}(\omega_i; r) d\omega dF(r) \]

where \( \hat{c}(\omega; r) := F^{-1}[C_r](\omega) \), with \( C_r(h) = \exp(-r \sum_{j \neq i} \theta_j \nu_j(|h_j|)) \) and where \( \hat{c}(\omega_i; r) = F^{-1}[C_r](\omega_i) \), with \( C_r(h_i) = \exp(-r \theta_i \nu_i(|h_i|)) \). Then, by noticing that \( \int_{\mathbb{R}^{d-1}} \hat{c}(\omega; r) d\omega = C_r(0) < \infty \), one obtains that the integral above is finite if and only if the function \( r \mapsto \int_{\mathbb{R}} \omega_i^{2k} \hat{c}(\omega_i; r) d\omega_i \) is \( F \)-measurable. Thus, the proof is completed.

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