Abstract.

This article will introduce a new iteration method called the zenali iteration method for the approximation of fixed points. We show that our iteration process is faster than the current leading iterations like Mann, Ishikawa, Noor, D- iterations, and \( \mathcal{K}^* \)- iteration for new contraction mappings called \( \delta ZA \) -- quasi contraction mappings. And we proved that all these iterations (Mann, Ishikawa, Noor, D- iterations and \( \mathcal{K}^* \)- iteration) equivalent to approximate fixed points of \( \delta ZA \) -- quasi contraction. We support our analytic proof by a numerical example, data dependence result for contraction mappings type \( \delta ZA \) by employing zenali iteration also discussed.

Keywords: Mann and \( \mathcal{K}^* \)- iteration, \( \delta ZA \) -- quasi contraction mappings.

1. Introduction

The fixed point theory is one of the most important theories that play an important and fundamental role to solve many problems in various fields of science and knowledge such as Geometry, game theory, chemistry, etc. numerical calculation of fixed points for nonlinear operators is also an active research problem at present for nonlinear analysis due to its applications in balance problems, variable inequality, image coding, computer simulation and more. For that, many authors have created a large number of algorithms to approximate the fixed point for different types of applications for example see [1-9]. The well-known Banach contraction theorem uses the Picard iteration mechanism for fixed point approximation. This paper consists of three sections section one converges the zenali iteration with all these iterations. In section two rate of converge, section three equivalent, section four numerical example with real datasets. Many of the other well-known iterative methods are those of Mann [10], Ishikawa [11], D- iteration [12], Picard S iteration [13] , \( \mathcal{K}^* \)-iteration [14] , Noor iteration [15].
Let $\mathcal{M}$ be a uniformly convex Banach space, $\emptyset \neq C$ be a closed-convex subset of $\mathcal{M}$. We recall some definitions of those iterations as:

1- Let $\langle s_n \rangle, \langle t_n \rangle$ and $\langle u_n \rangle$ are sequences lies in $(0,1)$. The following iteration $\langle d_n \rangle$ is called $D$-iteration and defined as follows:

$$d_0 \in C, \quad s_n = (1 - u_n)d_n + u_n Td_n,$$

$$t_n = (1 - t_n)Td_n + t_n Ts_n, \quad d_{n+1} = (1 - s_n)Ts_n + s_n Tt_n.$$

2- Let $z_n \in C$. The following iteration $\langle z_n \rangle$ is called Picard iteration and defined as follows:

$$z_{n+1} = Tz_n, \quad n \in N.$$

3- Let $\langle s_n \rangle$ be a sequence in $(0,1)$. The following iteration is called Mann iteration and defined as follows:

$$r_0 \in C, \quad r_{n+1} = (1 - s_n)r_n + s_n Tr_n, \quad n \in N.$$

4- Let $\langle s_n \rangle, \langle t_n \rangle$ and $\langle u_n \rangle$ be real sequences in $(0,1)$. The following iteration is called Ishikawa iteration and defined as follows

$$w_0 \in C, \quad w_{n+1} = (1 - s_n)w_n + s_n Tw_n,$$

$$d_n = (1 - t_n)w_n + t_n Tw_n.$$

5- Let $\langle s_n \rangle$ and $\langle t_n \rangle$ are sequences lies in $(0,1)$. The following iteration is called Picard $\delta$-iteration and defined as follows:

$$n \in N, \quad h_{n+1} = Tl_n,$$

$$l_n = (1 - s_n)Tl_n + s_n Te_n, \quad e_n = (1 - t_n)e_n + t_n Th_n.$$

6- Let $\langle s_n \rangle, \langle t_n \rangle$ and $\langle u_n \rangle$ are sequences in $(0,1)$. The following iteration $\langle q_n \rangle$ is called $\mathcal{K}$-iteration, and defined as follows:

$$q_{n+1} = Tp_n, \quad p_n = T((1 - s_n)o_n + s_n To_n) \quad \text{and} \quad o_n = (1 - t_n)o_n + t_n Tq_n.$$

7- Let $\langle s_n \rangle, \langle t_n \rangle$ and $\langle u_n \rangle$ are sequences in $(0,1)$. The following iteration $\langle y_n \rangle$ is called Noor iteration and defined as follows:

$$y_0 \in C, \quad y_{n+1} = (1 - s_n)y_n + s_n Ty_n,$$

$$q_n = (1 - s_n)y_n + s_n Tp_n,$$

$$p_n = (1 - u_n)y_n + u_n Ty_n, \quad n \in N.$$

Definition 1.1 [16]: Let $\langle v_n \rangle, \langle u_n \rangle$ are sequences lies in $\mathbb{R}$ converge to $V$ and $\langle u_n \rangle$ converge to $U$, and Let $\langle v_n \rangle$ such that $Z = \lim_{n \to \infty} \frac{|v_n - V|}{|u_n - U|}$

1. If $Z = 0$. Then The sequence $\langle v_n \rangle$ is converge to $V$ faster then $\langle u_n \rangle$ converge to $U$.

2. If $0 < Z < \infty$ → $\langle x_n \rangle$ and $\langle u_n \rangle$ have the same rate of convergence.
Lemma 1.2 [17]: Let $\mathcal{M}$ be a uniformly convex Banach space and $(\mathcal{J}_n)_{n=0}^{\infty}$ be any sequence such that $0 < p \leq J_n \leq r < 1$, for some $p, r \in R$ and for all $n \geq 1$. Let $(\mathcal{V}_n)_{n=0}^{\infty}$ and $(\mathcal{U}_n)_{n=0}^{\infty}$ be a nonnegative real sequences of $\mathcal{M}$ such that $\limsup_{n \to \infty} ||\mathcal{V}_n|| \leq r, \limsup_{n \to \infty} ||\mathcal{U}_n|| \leq r$ and

$$\lim_{n \to \infty} \left( J_n \mathcal{V}_n - (1 - J_n) \mathcal{U}_n \right) = r \text{ for some } r \geq 0. \text{ Then, } \lim_{n \to \infty} ||\mathcal{V}_n - \mathcal{U}_n|| = 0.$$

Lemma 1.3 [18]: Let $(\mathcal{V}_n)_{n=0}^{\infty}$ and $(\mathcal{U}_n)_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following condition:

$$\mathcal{V}_{n+1} \leq (1 - \mu_n)\mathcal{V}_n + \mathcal{U}_n, \text{ where } \mu_n \in (0,1), \text{ for all } n \geq n_0, \sum_{n=1}^{\infty} \mu_n = \infty$$

and $\frac{\mu_n}{\mu_{n+1}} \to 0 \text{ as } n \to \infty$. Then $\lim_{n \to \infty} \mathcal{V}_n = 0$.

2. Main Results

In this section, we introduced a new iteration process known as Zenali Iteration

and new contraction mappings called a $\delta\mathcal{Z}\mathcal{A}$ – quasi contraction mappings.

Definition 2.1: Let $\langle s_n \rangle, \langle t_n \rangle$ and $\langle u_n \rangle$ are sequences in $(0,1)$ and $\mathcal{T}: \mathcal{C} \to \mathcal{C}$. The following iteration is called Zenali iteration and defined as follows

$$x_0 \in \mathcal{C}, \quad x_{n+1} = \mathcal{T}y_n$$

$$y_n = \mathcal{T}((1 - s_n)x_n + s_n\mathcal{T}x_n),$$

$$z_n = \mathcal{T}((1 - t_n)x_n + t_n\mathcal{T}x_n).$$

Definition 2.2: Let $\mathcal{T}$ be a self mapping on $\mathcal{C}$, then $\mathcal{T}$ called a $\delta\mathcal{Z}\mathcal{A}$ – quasi contraction for all $x, y \in \mathcal{C}$, if $||\mathcal{T}x - \mathcal{T}y|| \leq \delta ||x - y|| + \mathcal{A}(nx, my)$ where

$$\mathcal{A}(nx, my) = \min\{n||x - \mathcal{T}x||, m||y - \mathcal{T}y||, n,m||x - \mathcal{T}y||, n,m||y - \mathcal{T}x||\}$$

for some $0 < \delta \leq 1$, $\mathcal{Z} \geq 0$ and $n, m \geq 0$.

Lemma 2.3: Let $\mathcal{C}$ be a nonempty convex and closed subset of a Banach space $\mathcal{M}$ and let $\mathcal{T}: \mathcal{C} \to \mathcal{C}$ a $\delta\mathcal{Z}\mathcal{A}$ – quasi contraction mapping. Suppose that $(x_n)$ the Zenali iteration in $\mathcal{C}$.

If $\mathcal{F}(\mathcal{T}) \neq \emptyset$, then $1- \|z_n - p\| \leq \|x_n - p\|$ and $\|y_n - p\| \leq \|x_n - p\|$.

$2- \lim_{n \to \infty} \|x_n - p\|$ exists, for all $n \in \mathbb{N}$.

Proof. Let $p$ be a fixed point of $\mathcal{T}$. Then the following inequalities hold

$$\|z_n - p\| = \|\mathcal{T}((1 - t_n)x_n - t_n\mathcal{T}x_n) - p\|$$

$$\leq \delta \|((1 - t_n)x_n - t_n\mathcal{T}x_n) - p\| + \mathcal{A}(n((1 - t_n)x_n - t_n\mathcal{T}x_n), mp).$$

Since $\|\mathcal{T}p - p\| \to 0$ as $n \to \infty$ then, $\mathcal{Z}\mathcal{A}(n((1 - t_n)x_n - t_n\mathcal{T}x_n), mp) = 0$

$$\|z_n - p\| \leq (1 - t_n)\|x_n - p\| + \delta t_n\|x_n - p\| + t_n\mathcal{Z}\mathcal{A}(nx_n, mp).$$

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\[\|x_n - p\| + Z \min\{n \|x_n - T x_n\|, m \|p - T p\|, n m \|x_n - T p\|, m n \|p - T x_n\|}\]

\[\|z_n - p\| \leq \|x_n - p\| \quad (2.1)\]

And,

\[\|y_n - p\| = \|T[(1 - s_n)z_n + s_n T z_n] - p\|
\leq \delta \|z_n - p\| + \delta s_n \|z_n - p\| + s_n Z A (n\|z_n - p\|)\]

Since \[\|p - T p\| \to 0 \text{ as } n \to \infty.\] Then,

\[Z A(1 - s_n) z_n + s_n T z_n, m p) = 0\]

\[\|y_n - p\| \leq (1 - s_n) \|z_n - p\| + \delta s_n \|z_n - p\| + s_n Z A (n z_n, m p)\]

\[\leq (1 - s_n (1 - \delta)) \|z_n - p\| + \delta m \|z_n - T z_n\|\]

\[, m \|p - T p\|, n m \|z_n - T p\|, m n \|p - T z_n\|\]

\[\|y_n - p\| \leq \|z_n - p\|. \quad (2.2)\]

Using inequality (2.1) in (2.2), it follows that:

\[\|y_n - p\| \leq \|x_n - p\| \quad (2.3)\]

And, \[\|x_{n+1} - p\| = \|T y_n - p\| \leq \delta \|y_n - p\| + \delta m \|z_n - p\|\]

\[\leq \|y_n - p\| \quad (2.4)\]

Using inequality (2.3), inequality (2.4) becomes

\[\|x_{n+1} - p\| \leq \|x_n - p\| \text{ for all } n \in N. \quad (2.5)\]

So, \[\{\|x_n - p\|\}\] is decreasing, for each \(p \in F(T)\), this implies that the sequence

\[\lim_{n \to \infty} \|x_n - p\| \text{ exists.}\]

**Theorem 2.4:** Let \(\mathcal{M}\) be a uniformly convex Banach space, \(\mathcal{C}\) be nonempty convex and closed subset of \(\mathcal{M}\) and let \(T; \mathcal{C} \to \mathcal{C}\) a \(\delta Z A -\) quasi contraction mapping. Suppose that \(\{x_n\}\) the Zenali iteration in \(\mathcal{C}\). Then \(F(T) \neq \emptyset \) if and only if \(\{x_n\}\) is bounded and \(\lim_{n \to \infty} \|x_n - T x_n\| = 0\).

**Proof.** Since \(p \in F(T)\), from (2.5) we get

\[\|x_{n+1} - p\| \leq \|x_n - p\| \leq \cdots \leq \|x_0 - p\| \text{ for all } n \in N.\]

Thus, \(\{x_n\}\) is bounded set in \(\mathcal{C}\).

Put,

\[r = \lim_{n \to \infty} \|x_n - p\| \quad (2.6)\]

And, \(\lim_{n \to \infty} \|T x_n - p\| \leq \lim_{n \to \infty} (\delta \|x_n - p\| + Z A (n x_n, m p))\)

\[\leq \lim_{n \to \infty} (\|x_n - p\| + Z \min\{n \|x_n - T x_n\|\})\]

\[, m \|p - T p\|, n m \|x_n - T p\|, m n \|p - T x_n\|\)]
\[
\lim_{n \to \infty} \|T x_n - p\| \leq \lim_{n \to \infty} \|x_n - p\| \tag{2.7}
\]

Using inequality (2.6) in (2.7) becomes, \(\lim_{n \to \infty} \|T x_n - p\| \leq r\) \hspace{1cm} (2.8)

From Equations (2.3) and (2.6), we have
\[
\liminf_{n \to \infty} \|y_n - p\| \leq r. \tag{2.9}
\]

Similarly by using (2.1) and (2.6), we have
\[
\limsup_{n \to \infty} \|z_n - p\| \leq r \tag{2.10}
\]

Now,
\[
r = \liminf_{n \to \infty} \|x_{n+1} - p\| = \liminf_{n \to \infty} \|T y_n - p\|
\leq \liminf_{n \to \infty} (\|y_n - p\| + Z\mathcal{A}(ny_n, mp))
\leq \liminf_{n \to \infty} (\|y_n - p\| + Z\min{n\|y_n - Tw_n\|, mn\|p - Tw_n\|})
\]

Then, we get \(r \leq \liminf_{n \to \infty} \|y_n - p\|\). \hspace{1cm} (2.11)

Having in mind (2.2), inequality (2.11) becomes, \(r \leq \liminf_{n \to \infty} \|z_n - p\|\) \hspace{1cm} (2.12)

From Eqs (2.10) and (2.12), we obtain
\[
r = \lim_{n \to \infty} \|z_n - p\| = \lim_{n \to \infty} (\|T((1 - t_n)x_n - t_nTx_n) - p\|)
\leq \delta \lim_{n \to \infty} ((1 - t_n)(x_n - p)) - t_n(Tx_n - p)
\]
\[+Z\mathcal{A}(n((1 - t_n)x_n - t_nTx_n), mp).\]

Since \(\|p - p\| \to 0\) as \(n \to \infty\) then, \(Z\mathcal{A}(n((1 - t_n)x_n - t_nTx_n), mp) = 0\)
\[
\leq \lim_{n \to \infty} \delta(1 - t_n(1 - \delta)) \|x_n - p\| + Z\min{n\|x_n - Tw_n\|, mn\|p - Tw_n\|}
\leq \lim_{n \to \infty} \|x_n - p\| = r
\]
So,
\[
r \leq \|(1 - t_n)(x_n - p) - t_n(Tx_n - p)\| \leq r
\]
Then \(\|(1 - t_n)(x_n - p) - t_n(Tx_n - p)\| = r\) \hspace{1cm} (2.13)

Thus From Eqs (2.6), (2.8), (2.13) and lemma (1.2) we obtain, \(\lim_{n \to \infty} \|x_n - Tw_n\| = 0\).

Now, we prove that \(F(T) \neq \emptyset\)

Let \(p \in A(C, (x_n)) \Rightarrow r(C, (x_n)) = r(p, (x_n))\)

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Theorem 2.6: Let $\mathcal{C}$ be a nonempty convex and closed subset of a Banach space $\mathcal{M}$ and let $T: \mathcal{C} \to \mathcal{C}$ be a $\delta$-quasi contraction mapping. Suppose that $\langle r_n \rangle$ the Mann iteration in $\mathcal{C}$, if $F(T) \neq \emptyset$, then $\lim_{n \to \infty} \| r_n - p \|$ exists.

Proof: Let $p$ be a fixed point of $T$. The following inequalities hold:

$$\| r_{n+1} - p \| = \| (1 - a_n) r_n + a_n T r_n - p \|$$

$$\leq (1 - a_n) \| r_n - p \| + a_n \| T r_n - p \|$$

$$\leq (1 - a_n) \| r_n - p \| + \delta a_n \| r_n - p \| + a_n \mathcal{Z} \mathcal{A} (nr_n, mp)$$

$$\leq (1 - a_n (1 - \delta)) \| r_n - p \|$$

$$\| r_{n+1} - p \| \leq \| r_n - p \|$$

(2.14)

So, $\{ \| r_n - p \| \}$ is decreasing, for each $p \in F(T)$, this implies that the sequence $\lim_{n \to \infty} \| r_n - p \|$ exists.

(2.15)

Theorem 2.6: Let $\mathcal{M}$ be a uniformly convex Banach space, $\mathcal{C}$ be nonempty convex and closed subset of $\mathcal{M}$ and let $T: \mathcal{C} \to \mathcal{C}$ be a $\delta$-quasi contraction mapping. Suppose that $\langle r_n \rangle$ the Mann iteration in $\mathcal{C}$. Then $F(T) \neq \emptyset$ if and only if $\langle r_n \rangle$ is bounded and $\lim_{n \to \infty} \| r_n - T r_n \| = 0$.

Proof. Since $p \in F(T)$, from (2.14) we get:

$$\| r_{n+1} - p \| \leq \| r_n - p \| \leq ... \leq \| r_0 - p \|$$

for all $n \in \mathbb{N}$.

Thus, $\langle r_n \rangle$ is bounded set in $\mathcal{C}$.

Put,

$$r = \lim_{n \to \infty} \| r_n - p \|$$

(2.15)

$$\limsup_{n \to \infty} \| T r_n - p \| \leq \limsup_{n \to \infty} (\| r_n - p \| + \mathcal{Z} \mathcal{A} (nr_n, mp))$$

$$\leq \limsup_{n \to \infty} \| r_n - p \|$$

(2.16)

Having in mind (2.15), inequality (2.16) becomes:

$$\limsup_{n \to \infty} \| T r_n - p \| \leq r$$

(2.17)

Now, $r = \lim_{n \to \infty} \| r_{n+1} - p \| = \lim_{n \to \infty} (1 - s_n) r_n + s_n T r_n - p$
\[ \lim_{{n \to \infty}} \left\| (1 - s_n) (r_n - \mu) + s_n (T r_n - \mu) \right\| = 0 \]  

(2.18)

Thus From Eqs (2.15), (2.17), (2.18) and lemma(1.2) we obtain, \( \lim_{{n \to \infty}} \left\| r_n - T r_n \right\| = 0 \).

Now, we prove that \( F(T) \neq \emptyset \). By the same proof way of the previous theorem.

Now, we will study the equivalent between many of iterations by using a \( \delta \mathcal{A} \) - quasi contraction mappings.

Theorem 2.7: Let \( C \) closed a nonempty convex and subset of a Banach space \( X \), \( T \) be a \( \delta \mathcal{A} \) - quasi contraction mapping on \( C \) and has a unique fixed point \( \mu \). Consider the Zenali iteration and Mann iteration with real sequences. Then the following a assertions are equivalent:

The Mann iteration converges to \( \mu \).

The Zenali iteration converges to \( \mu \).

Proof. We show that (i) \( \implies \) (ii) that is, if the Mann iteration converges, then the Zenali iteration does too. Since, the Mann iteration converges to \( \mu \)

\[ \Rightarrow \left\| r_n - \mu \right\| \to 0 \text{ as } n \to \infty. \]

Now, consider Mann and the Zenali iterations, we have:

\[ \left\| r_{n+1} - x_{n+1} \right\| = \left\| (1 - s_n) r_n + s_n T r_n - T y_n \right\| \]

\[ \leq (1 - s_n) \left\| r_n - T y_n \right\| + \delta s_n \left\| r_n - y_n \right\| + s_n \mathcal{A} (nr_n, my_n) \]

\[ \leq (1 - s_n) \left\| r_n - T r_n \right\| + \delta (1 - s_n) \left\| r_n - y_n \right\| \]

\[ + (1 - s_n) L m (r_n, y_n) + \delta s_n \left\| r_n - y_n \right\| + s_n \mathcal{A} (nr_n, my_n) \]

\[ = (1 - s_n) \left\| r_n - T r_n \right\| + \delta \left\| r_n - y_n \right\| + \mathcal{A} (nr_n, my_n) \]  

(2.19)

\[ \left\| r_n - y_n \right\| = \left\| r_n - T(1 - s_n) z_n - s_n T z_n \right\| \]

\[ \leq \left\| r_n - T r_n \right\| + \left\| T r_n - T(1 - s_n) z_n - s_n T z_n \right\| \]

\[ \leq \left\| r_n - T r_n \right\| + \delta \left\| r_n - (1 - s_n) z_n - s_n T z_n \right\| \]

\[ + \mathcal{A} (nr_n, m [(1 - s_n) z_n + s_n T z_n]) \]

\[ \leq \left\| r_n - T r_n \right\| + \delta (1 - s_n) \left\| r_n - z_n \right\| + \delta s_n \left\| r_n - T r_n \right\| \]

\[ + \delta^2 s_n \left\| r_n - z_n \right\| + \delta s_n \mathcal{A} (nr_n, mz_n) + \mathcal{A} (nr_n, m [(1 - s_n) z_n + s_n T z_n]) \]

\[ = (1 + \delta s_n) \left\| r_n - T r_n \right\| + (\delta (1 - s_n) + \delta^2 s_n) \left\| r_n - z_n \right\| \]

\[ + \delta s_n \mathcal{A} (nr_n, mz_n) + \mathcal{A} (nr_n, m [(1 - s_n) z_n + s_n T z_n]) \]  

(2.20)

\[ \left\| r_n - z_n \right\| = \left\| r_n - T(1 - t_n) x_n - t_n T x_n \right\| \]

\[ \leq \left\| r_n - T r_n \right\| + \left\| T r_n - T(1 - t_n) x_n - t_n T x_n \right\| \]

\[ \leq \left\| r_n - T r_n \right\| + \left\| T r_n - T(1 - t_n) x_n - t_n T x_n \right\| \]
\begin{equation}
\begin{aligned}
\| r_n - \mathcal{T} r_n \| &+ \delta \| r_n - (1 - t_n) x_n - t_n \mathcal{T} x_n \| + Z \mathcal{A} (nr_n, mx_n) \\
\leq \| r_n - \mathcal{T} r_n \| + \delta (1 - t_n) \| r_n - x_n \| + \delta t_n \| r_n - \mathcal{T} x_n \| + Z \mathcal{A} (nr_n, mx_n) \\
\leq \| r_n - \mathcal{T} r_n \| + \delta (1 - t_n) \| r_n - x_n \| + \delta t_n \| r_n - \mathcal{T} r_n \| + \delta^2 \| r_n - x_n \| + \delta t_n Z \mathcal{A} (nr_n, mx_n) + Z \mathcal{A} (nr_n, mx_n) \\
= (1 + \delta t_n) \| r_n - \mathcal{T} r_n \| + (\delta (1 - t_n) + \delta^2 t_n) \| r_n - x_n \| + (1 + \delta t_n) Z \mathcal{A} (nr_n, mx_n)
\end{aligned}
\end{equation}

Substituting (2.21) in (2.20), we obtain
\begin{equation}
\begin{aligned}
\| r_n - y_n \| &\leq (1 + \delta s_n) \| r_n - \mathcal{T} r_n \| + (\delta (1 - s_n) + \delta^2 s_n) (1 + \delta t_n) \| r_n - \mathcal{T} r_n \| \\
&\quad + (\delta (1 - s_n) + \delta^2 a_n) (\delta (1 - t_n) + \delta^2 t_n) \| r_n - x_n \| \\
&\quad + (\delta (1 - s_n) + \delta^2 s_n) (1 + \delta t_n) Z \mathcal{A} (nr_n, mx_n) \\
&\quad + (1 + \delta s_n) Z \mathcal{A} (nr_n, mz_n)
\end{aligned}
\end{equation}

Substituting (2.22) in (2.19), we obtain
\begin{equation}
\begin{aligned}
\| r_{n+1} - x_{n+1} \| &\leq (1 - s_n) \| r_n - \mathcal{T} r_n \| + \delta (1 + s_n) \| r_n - \mathcal{T} r_n \|
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\quad + \delta (\delta (1 - s_n) + \delta^2 s_n) (1 + \delta t_n) \| r_n - \mathcal{T} r_n \| \\
&\quad + \delta^2 (1 - s_n (1 - \delta)) (1 - t_n (1 - \delta)) \| r_n - x_n \| \\
&\quad + \delta (\delta (1 - s_n) + \delta^2 s_n) Z \mathcal{A} (nr_n, mx_n) \\
&\quad + \delta (1 + \delta s_n) Z \mathcal{A} (nr_n, mz_n) + Z \mathcal{A} (nr_n, my_n)
\end{aligned}
\end{equation}

Let \( \mu_n = s_n (1 - \delta) \in (0, 1) \),
\( \nu_n = \| r_n - x_n \| \)
\begin{equation}
\begin{aligned}
U_n = [(1 - s_n) + \delta (1 + s_n) + \delta (\delta (1 - s_n) + \delta^2 s_n)(1 + \delta t_n)] \| r_n - \mathcal{T} r_n \|
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&\quad + \delta (\delta (1 - s_n) + \delta^2 s_n)(1 + \delta t_n) Z \min \{ n \| r_n - \mathcal{T} r_n \|, m \| x_n - \mathcal{T} x_n \|, n m \| r_n - \mathcal{T} x_n \|, m n \| x_n - \mathcal{T} x_n \|, m n \| z_n - \mathcal{T} z_n \|^2, mn \| y_n - \mathcal{T} y_n \|^2, \mu_n \| y_n - \mathcal{T} y_n \|, \mu_n \| z_n - \mathcal{T} z_n \|, \mu_n \| z_n - \mathcal{T} x_n \|, mn \| y_n - \mathcal{T} y_n \|, mn \| z_n - \mathcal{T} z_n \|, mn \| z_n - \mathcal{T} x_n \| \}
\end{aligned}
\end{equation}
Furthermore, using $T \rho = \rho$ and $\|r_n - \rho\| \to 0$, we have

\[
\|r_n - T r_n\| = \|r_n - \rho + T \rho - T r_n\| \\
\leq \|r_n - \rho\| + \delta \|r_n - \rho\| + \mathcal{A}(nr_n, mp) \\
= (1 + \delta) \|r_n - \rho\| + \mathcal{A}(nr_n, mp)
\]

Then, $\|r_n - T r_n\| \to 0$.

Now, because of these results, we get $U_n \to 0$.

By applying lemma (1.3), we obtain $V_n = \|r_n - x_n\| \to 0$ as $n \to 0$.

Consequently, $\|r_{n+1} - x_{n+1}\| \to 0$ as $n \to 0$. Therefore,

\[
\|x_n - \rho\| \leq \|r_n - x_n\| + \|r_n - \rho\| \to 0 \quad \text{as } n \to \infty
\]

Now, we show that, (ii) $\Rightarrow$ (i).

Since, the Zenali iteration converges to $\rho \Rightarrow \|x_n - \rho\| \to 0$ as $n \to \infty$.

Now, consider the following

\[
\|x_{n+1} - r_{n+1}\| \leq \|x_{n+1} - \rho\| + \|r_{n+1} - \rho\| \\
= \|T y_n - \rho\| + \|(1 - s_n) r_n + s_n T r_n - \rho\| \\
\leq \delta \|T y_n - \rho\| + s_n \mathcal{A}(ny_n, mp) + (1 - s_n) \|r_n - \rho\| \\
+ \delta s_n \|r_n - \rho\| + s_n \mathcal{A}(nr_n, mp) \\
= \delta \|y_n - \rho\| + s_n \mathcal{A}(ny_n, mp) + (1 - s_n(1 - \delta)) \|r_n - \rho\| \\
+ s_n L m (r_n, \rho)
\]

(2.23)

\[
\|y_n - \rho\| = \|T[(1 - s_n) z_n + s_n T z_n] - \rho\| \\
\leq \delta \|(1 - s_n) z_n + s_n T z_n - \rho\| + s_n \mathcal{A}(nz_n, mp) \\
\leq \delta (1 - s_n) \|z_n - \rho\| + \delta s_n \|z_n - \rho\| + s_n \mathcal{A}(nz_n, mp) + \mathcal{A}(nz_n, mp) \\
\leq \delta (1 - s_n(1 - \delta)) \|z_n - \rho\| + (\delta s_n + 1) s_n \mathcal{A}(nz_n, mp)
\]

(2.24)

\[
\|z_n - \rho\| = \|T[(1 - t_n) x_n + t_n T x_n] - \rho\| \\
\leq \delta \|(1 - t_n) x_n + t_n T x_n - \rho\| + \mathcal{A}(n((1 - t_n) x_n + t_n T x_n), mp) \\
\leq \delta (1 - t_n) \|x_n - \rho\| + \delta^2 t_n \|x_n - \rho\| + \delta t_n \mathcal{A}(nx_n, mp) \\
\leq \delta (1 - t_n(1 - \delta)) \|x_n - \rho\| + (\delta t_n + 1) \delta t_n \mathcal{A}(nx_n, mp)
\]

(2.25)

Substituting (2.25) in (2.24), we obtain
\[ \| y_n - p \| \leq \delta^2 \left( 1 - s_n(1 - \delta) \right) \left( 1 - t_n(1 - \delta) \right) \| x_n - p \| + \delta \left( 1 - s_n(1 - \delta) \right) \]

\[ (\delta t_n + 1)Z\mathcal{A} (nx_n, mp) + (\delta s_n + 1)Z\mathcal{A} (nz_n, mp) \quad (2.26) \]

Substituting (2.26) in (2.23), we obtain

\[ \| x_{n+1} - r_{n+1} \| \leq \delta^3 \left( 1 - s_n(1 - \delta) \right) \left( 1 - t_n(1 - \delta) \right) \| x_n - p \| + (1 - s_n(1 - \delta)) \]

\[ \| r_n - x_n + x_n - p \| + \delta^2 \left( 1 - s_n(1 - \delta) \right) (\delta t_n + 1)Z\mathcal{A} (nx_n, mp) \]

\[ + Z\mathcal{A} (nz_n, mp) + \delta (\delta s_n + 1)Z\mathcal{A} (nz_n, mp) + s_nZ\mathcal{A} (nr_n, mp) \]

\[ \leq \left[ (1 - s_n(1 - \delta)) + (1 - s_n(1 - \delta)) \right] \| x_n - p \| \]

\[ + (1 - s_n(1 - \delta)) \| x_n - r_n \| + \delta^2 \left( 1 - s_n(1 - \delta) \right) (\delta t_n + 1) \]

\[ Z\min \{ n \| x_n - T x_n \|, m \| p - T p \|, n m \| y_n - T y_n \|, mn \| p - T y_n \| \} \]

\[ + Z\min \{ n \| y_n - T y_n \|, m \| p - T p \|, n m \| x_n - T x_n \|, mn \| p - T x_n \| \} \]

\[ + \delta (\delta s_n + 1)Z\min \{ n \| z_n - T z_n \|, m \| p - T p \|, n m \| z_n - T p \|, mn \| p - T z_n \| \} \]

\[ , mn \| r_n - T r_n \| + s_nZ\min \{ n \| r_n - T r_n \|, m \| p - T p \| \]

\[ , mn \| r_n - T r_n \|, mn \| p - T r_n \| \} \]

Let \( \mu_n = s_n(1 - \delta) \in (0, 1) \), \( V_n = \| x_n - r_n \| \)

\[ U_n = \left[ (1 - s_n(1 - \delta)) + (1 - s_n(1 - \delta)) \right] \| x_n - p \| + \delta^2 \left( 1 - s_n(1 - \delta) \right) (\delta t_n + 1) \]

\[ Z\min \{ n \| x_n - T x_n \|, m \| p - T p \|, n m \| x_n - T p \|, mn \| p - T x_n \| \} \]

\[ + Z\min \{ n \| y_n - T y_n \|, m \| p - T p \|, n m \| y_n - T p \|, mn \| p - T y_n \| \} \]

\[ + \delta (\delta s_n + 1)Z\min \{ n \| z_n - T z_n \|, m \| p - T p \|, n m \| z_n - T p \|, mn \| p - T z_n \| \} \]

\[ + s_nZ\min \{ n \| r_n - T r_n \|, m \| p - T p \|, n m \| r_n - T p \|, mn \| p - T r_n \| \} \]

Since, \( \| x_n - p \| \rightarrow 0 \) as \( n \rightarrow \infty \). So, we get \( U_n \rightarrow 0 \), thus from lemma (1.3), we get

\[ V_n = \| x_n - r_n \| \rightarrow 0 \text{ as } n \rightarrow 0 \]. Consequently, \( \| x_{n+1} - r_{n+1} \| \rightarrow 0 \text{ as } n \rightarrow 0 \)

Therefore, \( \| r_n - p \| \leq \| r_n - x_n \| + \| x_n - p \| \rightarrow 0 \text{ as } n \rightarrow \infty \). \:

Now, we will prove that our new iteration is faster than many know iterations. By using new contraction mappings.

Theorem 2.8: Let \( T \) be a \( \delta Z\mathcal{A} \) - quasi contraction mapping on \( C \). Suppose that the iterations Zenali iteration, Ishikawa iteration and Mann iteration converge to \( p \in \mathcal{F}(T) \) where 0 < \( \sigma \leq u_n \), \( s_n, t_n < 1, \forall n \in \mathbb{N} \). Then the Zenali iteration converges faster than of Mann iteration and Ishikawa iteration.

Proof. Consider Zenali iteration, we obtain

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\[ \|x_{n+1} - p\| = \|Ty_n - p\| \leq \delta \|y_n - p\| + ZA(ny_n, mp) \]
\[ = \delta \|T((1 - s_n)z_n + s_nTz_n) - p\| \]
\[ \leq \delta^2 \|((1 - s_n)z_n + s_nTz_n) - p\| + ZA(n(1 - s_n)z_n + s_nTz_n), mp) \]

Since \( \|T - p\| \to 0 \) as \( n \to \infty \), \( ZA(n(1 - s_n)z_n + s_nTz_n), m, p) = 0 \)
\[ = \delta^2 \|((1 - s_n)(z_n - p) + s_n(Tz_n - p)) \]
\[ \leq \delta^2 [(1 - s_n)\|z_n - p\| + \delta s_n\|z_n - p\| + ZA(nz_n, mp)] \]
\[ = \delta^2 [((1 - s_n) + \delta s_n)\|z_n - p\|] \]
\[ = \delta^2 [(1 - s_n(1 - \delta))] T((1 - t_n)x_n + t_nTz_n) - p \]
\[ \leq \delta^2 [(1 - s_n(1 - \delta)) \|x_n - p\| + ZA(nx_n, mp)] \]
\[ = \delta^2 [(1 - s_n(1 - \delta)) \|z_n - p\|] \]
\[ \leq \delta^2 [(1 - \nu(1 - \delta))^2 \|x_n - p\|] \]
\[ \leq (\delta^3 [1 - \nu(1 - \delta)]^2)^n \|x_0 - p\| \]

Suppose that \( ZA_n = (\delta^3 [1 - \nu(1 - \delta)]^2)^n \|x_0 - p\| \)

Consider the Mann iteration, we have
\[ \|r_{n+1} - p\| = \|(1 - s_n)r_n + s_nTr_n - p\| \]
\[ = \|(1 - s_n)(r_n - p) + s_n(Tr_n - p)\| \]
\[ \leq (1 - s_n)\|r_n - p\| + s_n\|Tr_n - p\| \]
\[ \leq (1 - s_n)\|r_n - p\| + \delta s_n\|r_n - p\| + ZA(n r_n, mp) \]
\[ \leq \|1 - \nu(1 - \delta)\|\|r_n - p\| + Z\min\{n \|r_n - Tr_n\|, m \|p - Tp\| \}
\[ , \min\{n \|r_n - Tp\|, m \|p - Tr_n\| \} \]
\[ \leq [1 - \nu(1 - \delta)]^n \|r_0 - p\| \]

Suppose that \( S_n = [1 - \nu(1 - \delta)]^n \|r_0 - p\| \)

Here, after simple compute, we have
Then, the Zenali iteration converges to \( p \) faster than Ishikawa iteration and Mann iteration.

**Theorem 2.9**: Let \( T \) be a \( \delta \mathcal{ZA} \) - quasi contraction self-mapping on \( C \). Suppose that the Zenali iteration and D - iteration converge to the same fixed point \( p \) of \( T \) where \( 0 < \nu \leq u_n, s_n, t_n < 1, \forall n \in N \). Then, the Zenali iteration converges faster than D - iteration.

**Proof.** Form D - iteration, we obtain

\[
\| d_{n+1} - p \| = \| (1 - s_n) Ts_n + s_n Tt_n - p \| \\
\leq \delta (1 - s_n) \| s_n - p \| + (1 - s_n) Z\mathcal{A} (ns_n, mp) \\
+ \delta s_n \| t_n - p \| + s_n Z\mathcal{A} (nt_n, mp) \\
= \delta [(1 - s_n) \| s_n - p \| + s_n \| (1 - t_n) Ts_n - p \|] \\
\leq \delta [(1 - s_n) \| s_n - p \| + \delta s_n (1 - t_n) \| d_n - p \| \\
+ \delta s_n (1 - t_n) \| s_n - p \|] + Z\mathcal{A} (ns_n, mp) + Z\mathcal{A} (nt_n, mp) \\
\leq \delta [(1 - s_n) + \delta s_n (1 - t_n) \| s_n - p \| + \delta s_n (1 - t_n) \| d_n - p \|] \\
eq \delta [(1 - s_n (1 - \delta t_n)) (1 - u_n) d_n + u_n Td_n - p] \\
+ \delta s_n (1 - t_n) \| d_n - p \| \\
\leq \delta [(1 - s_n (1 - \delta t_n)) (1 - u_n) \| d_n - p \| + \delta u_n \| d_n - p \|] \\
+ \delta s_n (1 - t_n) \| d_n - p \| + Z\mathcal{A} (nt_n, mp) \\
\leq \delta [(1 - s_n (1 - \delta t_n)) (1 - u_n) \| d_n - p \| \\
+ \delta s_n (1 - t_n) \| d_n - p \|] \\
\leq \delta [(1 - s_n (1 - \delta t_n)) \| d_n - p \| + \delta s_n (1 - t_n) \| d_n - p \|] \\
eq \delta [(1 - \nu (1 - \delta)) \| d_n - p \|] \\
\vdots \\
\leq [\delta ((1 - \nu (1 - \delta))]^n \| d_0 - p \| \\
Let D_n = [\delta ((1 - \nu (1 - \delta))]^n \| d_0 - p \|

Form Zenali-Iteration, we have, \( ZA_n = (\delta^3 \ [1 - \nu (1 - \delta) \]^2) \| x_0 - p \| \)

\[
\frac{ZA_n}{d_n} = \frac{(\delta^3 \ [1 - \nu (1 - \delta) \]^2) \| x_0 - p \|}{\| d_0 - p \|} 
\rightarrow 0 \text{ as } n \rightarrow \infty. \text{ Thus } \langle x_n \rangle \text{ converges to } p \text{ faster than } \langle d_n \rangle.

So, the Zenali-Iteration converges faster than D - iteration.
We proof other iterations by the same proof way of the previous Theorem.

Example 2.10: Let $\mathcal{N} = \mathbb{R}$ and $\mathcal{C} = [0,100]$. and $\mathcal{T}$ be a mapping on $\mathcal{C}$ defined by $\tilde{x} = \sqrt{\tilde{x}^2 - 9\tilde{x} + 54}$, for all $x \in \mathcal{C}$, such that $\mathcal{T}$ is a $\delta \mathcal{A} - \text{quasi contraction mapping}$ and unique fixed point say $\mathcal{p} = 6$.

Take $< s_n > = < t_n > = < u_n > = \frac{3}{4}, v = \frac{1}{2}$ with initial value 30.

| n  | Zenali | $K^*$ | D- | Ishikawa | Mann |
|----|-------|------|----|----------|------|
| 1  | 30    | 30   | 30 | 30       | 30   |
| 2  | 13.9156 | 17.1404 | 21.1334 | 25.0120 | 27.1150 |
| 3  | 6.1717 | 7.9203 | 13.2989 | 20.2548 | 24.2908 |
| 4  | 6.0006 | 6.0388 | 7.8776 | 15.8509 | 21.5421 |
| 5  | 6.0000 | 6.0004 | 6.1725 | 12.0133 | 18.8893 |
| 6  | 6.0000 | 6.0007 | 6.0087 | 9.0688 | 16.3607 |
| 7  | 6.0007 | 7.2820 | 13.9954 |
| 8  | 6.0000 | 6.4668 | 11.8476 |
| 9  | 6.1601 | 9.8476 |
| 10 | 6.0537 | 8.4901 |
| 11 | 6.0179 | 7.4083 |
| 12 | 6.0060 | 6.7247 |
| 13 | 6.0020 | 6.3468 |
| 14 | 6.0007 | 6.1587 |
| 15 | 6.0002 | 6.0709 |
| 16 | 6.0001 | 6.0313 |
| 17 | 6.0000 | 6.0137 |
| 18 |       |       |     | 6.0011 |
| 19 |       |       |     | 6.0005 |
| 20 |       |       |     | 6.0001 |
| 21 |       |       |     | 6.0000 |

3. Conclusion
In this section, a new iteration method for approximation of fixed points and a new contraction mappings called $\delta Z\mathcal{A} -$ quasi contraction mappings are introduced. Also, we proved that our iteration process is faster than the existing leading iterations like Mann, Ishikawa, Noor, D-iterations and $\mathcal{K}^*$- iteration and proved that all these iterations are equivalent to approximate fixed points of $\delta Z\mathcal{A} -$ quasi contraction.

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