Robustness of negativity of the Wigner function to dissipation

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Non-Gaussian quantum states, described by negative valued Wigner functions, are important both for fundamental tests of quantum physics and for emerging quantum information technologies. However, they are vulnerable to dissipation. It is known, that the Wigner functions negativity could exist only if the overall quantum efficiency $\eta$ of the setup is higher than $1/2$. Here we prove that this condition is not only necessary but also a sufficient one: the negativity always persists while this condition is fulfilled. At the same time, in the case of bright (multi-photon) non-Gaussian quantum states, the negativity dependence on $\eta$ is highly non-linear. With the loss of several photons, it drops by orders of magnitude, hampering its experimental detection.

\begin{equation}
\eta > \frac{1}{2}
\end{equation}

where $\eta$ is the overall quantum efficiency of the setup (including, e.g., the detection quantum efficiency) is the necessary condition for the Wigner function’s negativity. But the natural question arises, is this condition also a sufficient one? It is reasonable to expect, that some pure non-Gaussian states, in particular bright (multi-photon) ones, are more fragile than other ones and lose their negativity at bigger values of $\eta$ than $1/2$. The broadly accepted rule is that the negativity vanishes with the loss of a few photons.

In this paper we show that the conditions (1) is not only necessary but also a sufficient one: while it is fulfilled, the negativity of pure non-Gaussian states persists. At the same time, we show also that in the case of bright quantum states, the negativity dependence on $\eta$ is highly non-linear. After the loss of several photons, it drops by orders of magnitude, which could make its experimental detection problematic.

In this section, we review the main features of the QPDs, known in literature, and introduce the main notations, used in this paper.

Let $\hat{a}$, $\hat{a}^\dagger$ be the annihilation and creation operator a harmonic oscillator (e.g., a mode of electromagnetic field), satisfying the standard commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. Introduce the $s$-parametrized set of characteristic functions as follows [1, 2]:

\begin{equation}
C(z, s) = \text{Tr} \left[ \hat{b} e^{i(z' \hat{a} + \hat{a}^\dagger z)} \right] e^{s|z|^2/2},
\end{equation}

where $\rho$ is the density matrix, $s$ is a real parameter with $|s| \leq 1$, and $z$ a complex number. The inverse Fourier transform of $C(z, s)$ produces the corresponding s-parametrized set of the QPDs:

\begin{equation}
W(\alpha, s) = \int_{-\infty}^{\infty} C(z, s) e^{-i(z' \alpha + \alpha^* z)} \frac{dz^2}{\pi^2},
\end{equation}

where $\alpha = \frac{a+ip}{\sqrt{2}}$. The important special cases of $W(\alpha, s)$ are the Husimi Q-function, the Wigner function $W$, and the Glauber’s P-distribution:

\begin{equation}
W(\alpha, -1) = Q(\alpha), \quad W(\alpha, 0) = W(\alpha), \quad W(\alpha, 1) = P(\alpha).
\end{equation}

They can be expressed through each other using the following relation:

\begin{equation}
C(z, s') = C(z, s) e^{(s'-s)|z|^2/2}.
\end{equation}

\* The first two authors contributed equally to this work.
Consider now the dissipation. Following approach of Ref. [17], we model it by means of an imaginary beamsplitter with the power transmissivity $\eta$, which couples the considered mode with the effective heat bath mode, which we assume to be in the ground state $|0_B\rangle$, see Fig. 1. The density operator of the resulting quantum state is equal to

$$\hat{\rho}(\eta) = \text{Tr}_B (\hat{U} \hat{\rho} \otimes |0_B\rangle \langle 0_B| \hat{U}^\dagger), \quad (6)$$

where $\text{Tr}_B$ is the partial trace taken over the heat bath subspace and the unitary operator $\hat{U}$ describes the beamsplitter action:

$$\hat{U}^\dagger \hat{a} \hat{U} = \sqrt{\eta} \hat{a} + \sqrt{1-\eta} \hat{a}_B,$$

where $\hat{a}_B$ is the annihilation operator of the heat bath mode.

It follows from Eqs. (2, 6, 7), that the resulting characteristic function is equal to

$$C(z, s, \eta) = \text{Tr} \left[ \hat{\rho}(\eta) e^{i(z' \hat{a} + z \hat{a}^\dagger)} \right] e^{s|z|^2/2}$$

$$= C(\sqrt{\eta}z, s) \exp \left[ -\frac{(1-s)(1-\eta)}{2} |z|^2 \right], \quad (8)$$

or, alternatively,

$$C(z, s, \eta) = C(z', s'). \quad (9)$$

Here

$$z' = \sqrt{\eta}z, \quad s' = \frac{s+\eta-1}{\eta} \quad (10)$$

are, respectively, the reduced by dissipation of value $z$ and the new effective value of the parameter $s$.

Inverse Fourier transform of Eq. (8) gives the corresponding QPD after the losses:

$$W(\alpha, s, \eta) = \int_{-\infty}^{\infty} C(z, s, \eta) e^{-i(z' a + z a^\dagger)} \frac{d^2z}{\pi^2}$$

$$= \int_{-\infty}^{\infty} W(\beta, s) B(\alpha - \sqrt{\eta} \beta, s, \eta) d^2\beta, \quad (11)$$

where

$$B(\alpha, s, \eta) = \frac{2}{\pi(1-s)(1-\eta)} \exp \left[ -\frac{2|\alpha|^2}{(1-s)(1-\eta)} \right] \quad (12)$$

is the Gaussian blurring kernel.

c. Sufficient condition for the Wigner function negativity. It follows from Eqs. (9, 10) that if $s < 1$, then the losses transform the initial QPD into the more smooth one with $s' < s$. In particular, is $s = 0$ (the Wigner function) and $\eta = 1/2$, then $s' = -1$, which corresponds to the non-negative everywhere Husimi function. So indeed (1) is the necessary condition for the Wigner function’s negativity [17].

Now we prove the sufficiency of this condition. Let $W(\phi)(\alpha, s)$ be the QPD of some pure non-Gaussian quantum state $|\phi\rangle$ and $s_0$ be the maximal value of $s$ for which $W(\phi)(\alpha, s)$ is non-negative everywhere. Evidently, $s_0 \geq -1$ because the Husimi function $W(\alpha, -1)$ of any quantum state is non-negative everywhere, and $s_0 < 0$ because our quantum state is a non-Gaussian one and therefore its Wigner function $W(\phi)(\alpha, 0)$ takes negative values. Let also $W(\beta)(\alpha, s)$ be the QPD of a coherent state $|\beta\rangle$.

We use the proof by contradiction. Suppose that $s_0 > -1$ and consider the scalar product

$$\langle W(\phi), W(\beta) \rangle = \int_{-\infty}^{\infty} W(\phi)(\alpha, s_0) W(\beta)(\alpha, -s_0) d^2\alpha. \quad (13)$$

Note that $W(\phi)(\alpha, s_0) \geq 0$ due to the definition of $s_0$ and $W(\beta)(\alpha, -s_0) > 0$ because it is a Gaussian function. Therefore,

$$\langle W(\phi), W(\beta) \rangle > 0. \quad (14)$$

At the same time, it is easy to show that (13) is invariant with respect to $s_0$. Therefore,

$$\langle W(\phi), W(\beta) \rangle = \int_{-\infty}^{\infty} W(\phi)(\alpha, 0) W(\beta)(\alpha, 0) d^2\alpha > 0 \quad (15)$$

for all coherent states $|\beta\rangle$. According to the reasoning of Hudson [8], this means that $W(\phi)(\alpha, 0)$ is a Gaussian function. So, we came to the contradiction.

However, if $s_0 = -1$, then $W(\beta)(\alpha, -s_0) = \delta^2(\alpha - \beta)$, where $\delta^2$ is the two-dimensional $\delta$-function, and the strict inequality (14) relaxes to $\langle W(\phi), W(\beta) \rangle \geq 0$. In this cases, no contradiction arises. Therefore, for all pure non-Gaussian states, $s_0 = -1$, which means that the condition (1) is the sufficient one for preserving the Wigner function’s negativity.

It have to be emphasized, that this is not the case for mixed quantum states. For example, in the case of the mix $\hat{\rho} = \frac{1}{2} \left[ |0\rangle \langle 0| + |1\rangle \langle 1| + \frac{1}{2} (|0\rangle \langle 1| + |1\rangle \langle 0|) \right]$, negativity exists only if $\eta > 7/8$.

d. Rate of the negativity degradation. Here we explore dependence of the Wigner function negativity on the factor $\eta$. Several quantitative measure of the negativity are known. In this short paper, we use the minimal value of the Wigner function as such a measure, because it is simple, while providing closed analytical equations. At the same time, in the case of the Wigner functions with several local minima, this measure gives only semi-qualitative estimates of the negativity. The detailed analysis based on the volume of the negative part of the Wigner function [18] will be done in our next paper [19].

We assume here that the Wigner function minimum corresponds to $a_m = 0$. If this is not the case, then the quantum state could be adjusted by applying the displacement operator.
\( \hat{D}(\alpha_m) = e^{\frac{1}{2}(\alpha_m \hat{a} - \alpha_m \hat{a}^\dagger)} \), which also takes into account the drift of the minimum due to dissipation.

It is shown in App. A, that the value of a lossy Wigner function at \( \alpha = 0 \) can be presented as follows:

\[
W_0(\eta) \equiv W(0, 0, \eta) = \frac{2}{\pi} \langle 1 - 2\eta \rangle^{\frac{1}{2}}
\]

\[
= \frac{2}{\pi} \sum_{n=0}^{\infty} (1 - 2\eta)^n \rho_{nn},
\]

where \( \rho_{nn} = \langle n|\hat{\rho}|n \rangle \) are the diagonal values of the corresponding density matrix in the number of quanta \( \hat{n} = \hat{a}^\dagger \hat{a} \) representation. Evidently, to obtain the minimal possible value of \( W_0 \), all even coefficients \( \rho_{nn} \) should be equal to zero:

\[
\forall n = 2k, \ k \geq 0 : \rho_{nn} = 0.
\]

We assume that the quantum states which we consider here satisfy this condition. In this case,

\[
W_0(\eta) = -\frac{2}{\pi} \langle (2\eta - 1)^{\frac{1}{2}} \rangle.
\]

In the case of small losses, \( 1 - \eta \ll 1 \), \( W_0 \) can be approximated as follows:

\[
W_0(\eta) \approx -\frac{2}{\pi} \langle e^{-2(1-\eta)\hat{n}} \rangle = -\frac{2}{\pi} \sum_{n=0}^{\infty} e^{-2(1-\eta)n} \rho_{nn}.
\]

This form suggests that the negativity actually depends on the product \( (1 - \eta)n_c \), where \( n_c \) is some characteristic number of quanta depending on the probability distribution \( \{\rho_{nn}\} \). In particular, if this distribution is well concentrated around its mean value \( \hat{n} \), with the variance \( \delta n \ll \hat{n} \), then evidently \( n_c = \hat{n} \) and

\[
W_0(\eta) \approx -\frac{2}{\pi} e^{-\varepsilon},
\]

where

\[
\varepsilon = (1 - \eta)\hat{n}
\]

is the mean number of the lost quanta.

Consider three characteristic examples, satisfying the condition (17) and being important from the practical point of view.

(i) Fock state \( |n\rangle \) with the odd number of quanta \( n = 2k + 1 \).

In this case,

\[
W_0(\eta) = -\frac{2}{\pi} \langle 2(\eta - 1)^{\frac{1}{2}} \rangle = -\frac{2}{\pi} \left(1 - \frac{2e}{n}\right)^n,
\]

(ii) “Odd” Schrodinger cat state \( |20 \rangle \)

\[
|\tilde{\alpha}_0\rangle = \frac{|\alpha_0\rangle - -|\alpha_0\rangle}{\sqrt{2(1 - e^{-2|\alpha_0|^2})}}
\]

(without loss of generality, we assume that \( 3\alpha = 0 \)). It follows from Eqs. (8, 11, 23) that

\[
W_0(\eta) = -\frac{2}{\pi} e^{-2(1-\eta)|\alpha|^2} - e^{-2\eta|\alpha|^2}
\]

and the mean number of quanta is equal to

\[
\langle n \rangle = \alpha^2 \frac{1 + e^{-\alpha^2}}{1 - e^{-\alpha^2}}.
\]

(iii) Squeezed single quantum state \( \hat{S}|1\rangle \), where \( \hat{S} = e^{\frac{1}{2}(\hat{a}^\dagger - \hat{a})} \) is the squeeze operator and \( r \) is the squeeze factor [21]. In this case (see e.g., Ref. [22]),

\[
W_0(\eta) = \frac{2}{\pi} \frac{1 - 2\eta}{(1 + 4\eta(1 - \eta) \sinh^2 r)^{3/2}}.
\]
and 
\[
\langle n \rangle = 1 + 3 \sinh^2 r . 
\]  \hspace{1cm} (27)

In Fig. 2, the absolute values of the Wigner functions (22, 24, 26) are plotted as functions of \( \varepsilon \). In the first two cases, the asymptotics (20) is also plotted. It can be seen from these plots that for the first two examples, \( W_0 \) converges quickly to the asymptotic value (20). This is not the case for the third one. The reason for this is evident. The number of quanta distribution of the squeezed single photons state is a very broad one, with \( \delta n \sim \hat{n} \) and with a large contribution from the Fock states with small values of \( n \). Therefore, the asymptotics (18) cannot be used for this quantum state.

e. Conclusion. We showed that the condition (1) is not only the necessary [17], but also the sufficient one for the negativity of Wigner function of any pure non-Gaussian state. This means, that in the strict mathematical sense, the negativity always persists for all values of the quantum efficiency \( \eta \) down to 1/2. We explored also the dependence of the Wigner function negativity on \( \eta \), using the minimal value of the Wigner function as a simple semi-qualitative measure of negativity. We showed, that in the case of bright (multi-photon) quantum states, that dependence is very nonlinear. After the loss of several photons, which corresponds to \( 1 - \eta \simeq 1/(n! \), it could drop by orders of magnitude, which could make its experimental detection problematic.

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Appendix A: Proof of Eq. (16)

For the case of \( s = 0 \), Eq. (8) can be presented as follows:

\[
C(z, s, \eta) = \text{Tr} (\hat{\mathcal{K}}) .
\]  \hspace{1cm} (A1)

where

\[
\hat{\mathcal{K}}(z) = e^{i\sqrt{\eta}z} e^{i\sqrt{\eta} z^*} e^{-|z|^2/2}
= \sum_{k,l=1}^{\infty} \frac{(i\sqrt{\eta})^k (i\sqrt{\eta})^{l^*}}{k! l!} \hat{a}^k \hat{a}^l e^{-|z|^2/2} . \]  \hspace{1cm} (A2)

Inverse Fourier transform of this operator:

\[
\hat{\mathcal{G}}(\alpha) = \int_{-\infty}^\infty \hat{\mathcal{K}}(z) e^{-i(z^* \alpha + z \alpha^*)} d^2 z = \frac{2}{\pi} \sum_{k,l=1}^{\infty} \frac{(-2\eta)\hat{\alpha}^k \hat{\alpha}^l}{k!} \frac{1}{\delta \alpha^k \delta \alpha^{k+l}} .
\]  \hspace{1cm} (A3)

Its value at \( \alpha = 0 \):

\[
\hat{\mathcal{G}}(0) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-2\eta)\hat{\alpha}^k \hat{\alpha}^l}{k!} .
\]  \hspace{1cm} (A4)

Note that \([\hat{\mathcal{G}}(0), \hat{a}] = 0\). Therefore,

\[
W(0,0,\eta) = \text{Tr} [\hat{\rho} \hat{\mathcal{G}}(0)] = \frac{2}{\pi} \sum_{k,n=0}^{\infty} \frac{(-2\eta)^k}{k!} \rho_{nn} \langle n | \hat{a}^k \hat{a}^l | n \rangle
= \frac{2}{\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-2\eta)^k = \frac{2}{\pi} \sum_{n=0}^{\infty} \rho_{nn} (1 - 2\eta)^n .
\]  \hspace{1cm} (A5)

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