Hilbert schemes of finite abelian group orbits and Gröbner fans

Tomohito Morita

Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152-8551
Japan

morita.t.ae@m.titech.ac.jp

Abstract

Let $G$ be a finite abelian subgroup of $PGL(r-1, K) = Aut(\mathbb{P}^{r-1})$. In this paper, we prove that the normalization of the $G$-orbit Hilbert scheme $\text{Hilb}^G(\mathbb{P}^{r-1})$ is described as a toric variety, which corresponds to the Gröbner fan for some homogeneous ideal $I$ of $K[x_1, \ldots, x_r]$.

Keywords: Gröbner fan, G-Hilbert schemes, toric singularity.
AMS classification: 13P10, 14L30, 14E15, 14M25

1 Introduction

Let $K$ be an algebraically closed field and $G$ a finite subgroup of $GL(n, K) \subset Aut(\mathbb{A}^n)$ of order prime to the characteristic of $K$. The $G$-orbit Hilbert scheme (or the Hilbert scheme of $G$-orbits) $\text{Hilb}^G(\mathbb{A}^n)$ is introduced by Ito and Nakamura [10]. The $G$-orbit Hilbert scheme is the scheme parameterizing all $G$-invariant smoothable zero-dimensional subschemes of $\mathbb{A}^n$ of length $m := |G|$. Here a smoothable zero-dimensional subscheme $Z$ of $\mathbb{A}^n$ of length $m$ is the subscheme of $\mathbb{A}^n$ for which $H^0(Z, \mathcal{O}_Z)$ is a regular representation of $G$.

Nakamura [11] proved that for a finite abelian group $G \subset GL(n, K)$, the normalization of $\text{Hilb}^G(\mathbb{A}^n)$ is the toric variety corresponding to the fan which is defined by the $G$-graph, which is defined in [11], and he also proved that $\text{Hilb}^G(\mathbb{A}^3)$ is a crepant smooth resolution of $\mathbb{A}^3/G$ if $G \subset SL(3, K)$. Furthermore, Ito [8] proved the following result. If $G$ is a finite cyclic group of $GL(2, \mathbb{C})$ and acts freely on $\mathbb{C}^2 \backslash \{0\}$, then $\text{Hilb}^G(\mathbb{C}^2)$ is described by the Gröbner fan for the ideal $I$ of $\mathbb{C}[x, y]$, corresponding to a subscheme which is a free $G$-orbit contained in $(\mathbb{C}^*)^2$.

In this paper, we consider the case when $G$ is a finite abelian subgroup of the diagonal subgroup of $PGL(r-1, K) = Aut(\mathbb{P}^{r-1})$. The $G$-orbit Hilbert scheme $\text{Hilb}^G(\mathbb{P}^{r-1})$ defined
as before, and the normalization of it is proved to be the toric variety corresponding to the Gröbner fan for a homogeneous ideal $I$ of $K[x_1, \ldots, x_r]$, corresponding to a subscheme which is a free $G$-orbit contained in $(K^*)^{r-1} \cong \{x_1 \cdots x_r \neq 0\} \subset \mathbb{P}^{r-1}$ (Theorem 4.2). The corresponding results on $\text{Hilb}^G(\mathbb{A}^{r-1})$ are easily deduced from it (Corollary 4.3). This gives an alternative proof and a generalization of Ito’s result.

The proof of our theorem consists of three steps. Note that this diagonal subgroup $T'$ of $\text{PGL}(r-1, K)$ is isomorphic to an algebraic torus $(K^*)^{r-1} \subset \mathbb{P}^{r-1}$ (Theorem 4.2). Let $\text{Hilb}^m(\mathbb{P}^{r-1})$ denote the Hilbert scheme of $m$ points in $\mathbb{P}^{r-1}$, where $m = |G|$. For any homogeneous ideal $I$ as in the theorem, $I$ defines a point of $\text{Hilb}^m(\mathbb{P}^{r-1})$, and we prove that the closure in $\text{Hilb}^G(\mathbb{P}^{r-1})$ of $T'$-orbit of $P$ is coincides with $\text{Hilb}^G(\mathbb{P}^{r-1})$. Next, we show that the normalization of $\text{Hilb}^G(\mathbb{P}^{r-1})$ is described by the state polytope of $I$, defined in (3.3) (Proposition 3.2). Finally, we show that the normal fan of the state polytope coincides with the Gröbner fan for $I$ by a theorem of Bayer and Morrison [1] (Theorem 3.4).

Gröbner fans are computable (for example by [5]). Hence we hope that the results in this paper are useful for the study on the $G$-orbit Hilbert schemes, especially in higher dimensional cases.

After we wrote up this paper, the author found the paper [3] by Craw, MacIagan, and Thomas, where they study the moduli space of McKay quiver representations. Our work is related to their theory in the case of the $G$-orbit Hilbert schemes, but the method is different. T. Yasuda also obtained related results in [15], but his method is different from ours. Y. Ito communicated to the author that Y. Sekiya independently obtained a similar result to ours.

Acknowledgments. I would like to thank Professor Takao Fujita for valuable advice and for pointing out some mistakes in English. I also would like to thank Professors Takeshi Kajiwara and Chikara Nakayama for helpful discussions and comments on earlier versions.

2 Gröbner basis and Gröbner fan

We recall the basic notations for Gröbner basis and Gröbner fan. These notations are based on the book by Sturmfels [13].

Let $K$ be any field and $K[X] = K[x_1, \ldots, x_n]$ the polynomial ring in $n$ indeterminates. Let $\mathbb{N}$ be the set of non-negative integers. For $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, let $X^a$ denote the monomial $x_1^{a_1} \cdots x_n^{a_n}$. By this correspondence $a \mapsto X^a$, the lattice $\mathbb{N}^n$ is embedded in $K[X]$ as a multiplicative semigroup and this image coincides with the set of monomials in $K[X]$.

2.1 Definition A total order $<$ on $\mathbb{N}^n$ is called a term order if $(0, \ldots, 0)$ is the unique minimal element, and $a < b$ implies $a + c < b + c$ for all $a, b, c \in \mathbb{N}^n$.

We denote by $\mathbb{R}_+$ the set of non-negative real numbers. Fix a weight vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ and a term order $<$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product of $\mathbb{R}^n$. 

2
2.2 Definition For a polynomial \( f = \sum_{c_i \neq 0} c_i X^{a_i} \), the initial term \( \text{in}_w(f) \) of \( f \) with respect to \( w \) is the sum of all terms \( c_i X^{a_i} \) such that the inner product \( \langle w, a_i \rangle \) is maximal. For an ideal \( I \) of \( K[X] \), the initial ideal \( \text{in}_w(I) \) is the ideal generated by the initial terms of all elements \( f \) in \( I \):

\[
\text{in}_w(I) = \langle \{ \text{in}_w(f) | f \in I \} \rangle .
\]

This ideal need not be a monomial ideal. However, \( \text{in}_w(I) \) is a monomial ideal if \( w \) is chosen sufficiently generic for a given \( I \).

2.3 Definition Let \( w \) be a weight vector in \( \mathbb{R}^n_+ \) and \( < \) the fixed term order. The weight term order \( <_w \) is the term order defined as follows:

\[
X^a <_w X^b \iff \langle a, w \rangle < \langle b, w \rangle , \text{ or } \langle a, w \rangle = \langle b, w \rangle \text{ and } a < b.
\]

2.4 Definition For a polynomial \( f \), the initial term \( \text{in}_{<_w}(f) \) of \( f \) with respect to \( w \) is the maximal term of \( f \) with respect to \( <_w \). For an ideal \( I \) of \( K[X] \), the initial ideal \( \text{in}_{<_w}(I) \) is the ideal generated by the initial terms of all elements \( f \) in \( I \):

\[
\text{in}_{<_w}(I) = \langle \{ \text{in}_{<_w}(f) | f \in I \} \rangle .
\]

2.5 Definition A finite subset \( \mathcal{G} \subset I \) is a Gröbner basis if \( \text{in}_{<_w}(I) \) is generated by \( \{ \text{in}_{<_w}(g) | g \in \mathcal{G} \} \).

It is known that \( I \) is generated by the Gröbner basis of \( I \).

2.6 Definition The Gröbner basis \( \mathcal{G} \) of \( I \) is reduced if for any two elements \( g, h \in \mathcal{G} \), no term of \( h \) is divisible by \( \text{in}_{<_w}(g) \).

It is known that the reduced Gröbner basis of \( I \) is unique.

2.7 Definition Two weight vectors \( w, w' \in \mathbb{R}^n \) are called \( I \)-equivalent (or simply equivalent) if \( \text{in}_w(I) = \text{in}_{w'}(I) \).

Then we can consider the equivalence classes of weight vectors.

2.8 Proposition ([13, Proposition 2.3]) For any ideal \( I \), and for any weight vector \( w \), the equivalence class \( c[w] \) of weight vectors is relatively open convex polyhedral cone. Moreover, if \( w \) is contained in \( \mathbb{R}^n_+ \) and chosen sufficiently generic, \( c[w] \) is given by the reduced Gröbner basis \( \mathcal{G} \) of \( I \) with respect to \( <_w \) as follows:

\[
c[w] = \{ w' \in \mathbb{R}^n | \text{in}_{w'}(g) = \text{in}_w(g) \text{ for any } g \in \mathcal{G} \}.
\]

2.9 Definition The Gröbner region \( GR(I) \) for \( I \) is the set of all \( w \in \mathbb{R}^n \) such that \( \text{in}_w(I) = \text{in}_{w'}(I) \) for some \( w' \in \mathbb{R}^n_+ \). Clearly, \( GR(I) \) contains \( \mathbb{R}^n_+ \).

2.10 Remark In general, the \( GR(I) \) does not coincide with \( \mathbb{R}^n \). However, when the ideal \( I \) is homogeneous, it is known that \( GR(I) \) coincides with \( \mathbb{R}^n \).
2.11 Definition The Gröbner fan $GF(I)$ for an ideal $I$ is the set consisting of the faces of the closed cones of the form $c[w]$ for some $w \in \mathbb{R}_+^n$. A closed cone $\sigma$ is called Gröbner cone if $\sigma \in GF(I)$.

2.12 Proposition ([13, Proposition 2.4]) The Gröbner fan for $I$ is a convex polyhedral fan.

2.13 Remark In general, the Gröbner fan for $I$ does not consist of strongly convex cones.

3 Weight polytope and state polytope

In the first half of this section, we explain that the normalization of the closure of the torus orbit of a point in a projective space is the toric variety corresponding to some polytope, which is called the weight polytope (Proposition 3.2).

In the latter half of this section, we consider the Hilbert scheme $\text{Hilb}_h(\mathbb{P}^{r-1})$ being embedded to the projective space $\mathbb{P}^n$ by the Plücker embedding. Then, the weight polytope corresponding to the normalization of the closure in $\text{Hilb}_h(\mathbb{P}^{r-1})$ of the torus orbit of a Hilbert point $I$ is called the state polytope of $I$. By a theorem of Bayer and Morrison [1], the normal fan of the state polytope coincides with the Gröbner fan for $I$ (Theorem 3.4). In conclusion, the normalization of the closure of the torus orbit of a Hilbert point $I$ is the toric variety corresponding to the Gröbner fan for $I$. We also prove that the normalization of the closure in $\text{Hilb}_h(\mathbb{A}^{r-1})$ of the torus orbit of $I$ in $\text{Hilb}_h(\mathbb{A}^{r-1})$ is the toric variety corresponding to the Gröbner fan for the dehomogenization of $I$ (Corollary 3.5).

Let $V$ be an $n$ dimensional vector space over an algebraically closed field $K$ and $T$ an algebraic torus of dimension $r$. Let $\rho : T \rightarrow GL(V)$ be a rational linear representation of $T$ such that $\rho(T)$ contains all scalar multiplications of $GL(V)$. Let the point of $\mathbb{P}(V)$ corresponding to $v \in V \setminus \{0\}$ will be denoted by $v$. Here $GL(V)$ (and hence $T$ via $\rho$) acts on $\mathbb{P}(V)$ in a natural way. For each $v \in \mathbb{P}(V)$, let $\overline{T \cdot v}$ be the closure in $\mathbb{P}(V)$ of the torus orbit of $v$. Then $\overline{T \cdot v}$ has the open dense orbit which is isomorphic to $T/\text{Stab}(v)$.

In general, $\overline{T \cdot v}$ is not a normal variety. But its normalization is the toric variety which contains $T/\text{Stab}(v)$ as an open dense orbit (cf. [12]). The corresponding fan of this toric variety is described by the weight polytope explained below.

Let $M$ be the character group of $T$ and $N$ the dual lattice of $M$. It is known that $V$ is the direct sum of its weight subspaces:

$$V = \bigoplus_{w \in M} V_w, \text{ where } V_w = \{v \in V | \rho(t) \cdot v = w(t) \cdot v \text{ for all } t \in T\}.$$

3.1 Definition Let $v = \sum_{w \in M} v_w$, where $v_w \in V_w$. The convex hull $\text{Wt}(v) \subset M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ of the set $\{w \in M | v_w \neq 0\}$ will be called the weight polytope of $v$.

Let $v$ be a point of $\mathbb{P}(V)$. We denote by $\text{Wt}(v)$ the weight polytope $\text{Wt}(v)$ of a lift $v$ of $v$. This definition is well-defined.
When $T \to T \cdot v$ is injective, the corresponding fan to the normalization of $\overline{T \cdot v}$ is nothing but the normal fan of $\text{Wt}(v)$ (see [12] Chapter 2.4).

In general, we take a sublattice of $M$ and construct the polytope corresponding to the toric variety as follows.

Let $M'$ be the character group of $T/\text{Stab}(v)$ and $N'$ the dual lattice of $M'$. $M'$ is identified with the sublattice of $M$ which is generated by $w_1 - w_2$ for any $w_1, w_2 \in \{w \in M | v_w \neq 0\}$, that is, $M' = \langle \{w_1 - w_2 \in M | v_{w_1}, v_{w_2} \neq 0\} \rangle$.

The next proposition follows from [12] Theorem 2.22.

**3.2 Proposition (cf. [12] Theorem 2.22)** The normalization of $\overline{T \cdot v}$ is isomorphic to the toric variety corresponding to the normal fan in $N'_{\mathbb{R}}$ of the polytope $\text{Wt}(v) - w \subset M'_{\mathbb{R}}$ for any $w \in \text{Wt}(v) \cap M$.

**Proof.** Let $X$ be the toric variety corresponding to the polytope $\text{Wt}(v) - w$ and let the set of weights $\{w_1, \ldots, w_s\} = \{w \in M | v_w \neq 0\}$. Take a basis of $V_w$ for each $w \in M$ which contains $v_w$ if $v_w \neq 0$. Gathering these bases, we get a base of $V$, and we may assume that the homogeneous coordinate of $v$ with respect to this base is given by $(1 : \ldots : 1 : 0 : \ldots : 0)$. We have the morphism $\phi : X \to \overline{T \cdot v} \subset \mathbb{P}(V)$ as follows:

$$\phi : X \to \overline{T \cdot v} \subset \mathbb{P}(V) ; x \mapsto (\chi_{w_1}(x) : \ldots : \chi_{w_s}(x) : 0 : \ldots : 0),$$

where $\chi_w$ is the character corresponding to $w$. Since $X$ is normal, the morphism $\phi$ factors through the normalization of $\overline{T \cdot v}$. The open torus orbit of $X$ is isomorphic to the open torus orbit of the normalization of $\overline{T \cdot v}$ by the morphism $\phi$. Moreover, the torus actions on $X$ and on the normalization of $\overline{T \cdot v}$ are compatible with the morphism $\phi$. Then the proposition follows from [12] Theorem 1.5. \qed

In the rest of this section, we consider the situation where the torus $T = (K^*)^r$ acts naturally on the vector space $V = N \otimes_{\mathbb{Z}} K \cong K^r$. Then we have a natural embedding $T \subset V$. Let $S$ be the symmetric algebra of the dual space of $V$, and put $\mathbb{P} = \text{Proj}(S)$. Then $T$ acts on $S = K[x_1, \ldots, x_r]$ in the following way:

$$T \times S \to S ; (t = (t_1, \ldots, t_r), f(X)) \mapsto t \cdot f(X) := f\left( \frac{x_1}{t_1}, \ldots, \frac{x_r}{t_r} \right).$$

Further $T$ acts on $\mathbb{P}$, too.

Let $h(x)$ be a polynomial and $\text{Hilb}_h(\mathbb{P})$ the Hilbert scheme corresponding to $h$. We identify $I \in \text{Hilb}_h(\mathbb{P})$ with the corresponding homogeneous ideal of $S$. Then $d$-graded part $I_d$ of $I$ is a subspace of $S_d$ of codimension $h(d)$ for any large integer $d \gg 0$. Let $G(l - h(d), S_d)$ be the Grassmannian variety of subspaces of $S_d$ of codimension $h(d)$, where $l$ is $\dim_K S_d = \binom{r + d - 1}{d}$. It is known that the mapping $\text{Hilb}_h(\mathbb{P}) \to G(l - h(d), S_d) ; I \mapsto I_d$ is a closed embedding for any large integer $d \gg 0$. By the Plücker embedding, we have $\text{Hilb}_h(\mathbb{P}) \subset G(l - h(d), S_d) \subset \mathbb{P}(\wedge^{l-h(d)} S_d) \cong \mathbb{P}^n$, where $n = \dim_K (\wedge^{l-h(d)} S_d) - 1$, and we
have a natural action of $T$ on $S_d$, and hence a natural action on $G(l - h(d), S_d)$, and also on $\mathbb{P}^n$.

Then we can apply the facts in the first half of this section to the torus orbit of a Hilbert point $I$.

3.3 Definition Let $I_d$ be the $d$-graded component of $I \in \text{Hilb}_h(\mathbb{P})$. The weight polytope of the Hilbert point $I_d \in \mathbb{P}^n$ is called the state polytope $St_d(I)$ of $I$ in degree $d$.

The following is deduced from a theorem of Bayer and Morrison [1] (cf. Sturmfels [14]). For reader’s convenience, we give here a slightly different proof.

3.4 Theorem ([14, Theorem 2.1]) Let $w \in St_d(I) \cap M$. Then the normal fan in $N'_R$ of the polytope $St_d(I) - w \subset M'_R$ coincides with the set of the images of the Gröbner cones for $I$ by the natural projection $N_R \rightarrow N'_R$.

Proof. It is enough to prove that the pull back of the normal fan of $St_d(I) - w$ coincides with the Gröbner fan for $I$. Therefore, we prove that the normal fan in $N_R$ of the state polytope coincides with the Gröbner fan for $I$. First note that each Gröbner cone contains the vector $(1, \ldots, 1)$, and the kernel of the projection $N_R \rightarrow N'_R$ also contains the vector $(1, \ldots, 1)$. Then we only have to prove that the intersection of the normal fan in $N_R$ of the state polytope and $\mathbb{R}_+^r \subset N_R$ coincides with the intersection of the Gröbner fan for $I$ and $\mathbb{R}_+^r$.

First, we prove that the latter is a subdivision of the former. We recall that the coordinate of $\mathbb{P}^n$ is given by the Plücker coordinate. Let $f_1, \ldots, f_k$ be a $K$-basis of $I_d \subset K[X]_d$, where $k = l - h(d)$. Then $f_1 \wedge \cdots \wedge f_k$ defines the Plücker coordinate of $I_d$. Let $S$ be the set of weights defined by its all non-zero components. Then, by definition, the state polytope of $I$ in degree $d$ is the convex hull of $S$. Let $\sigma$ be a maximal cone of the Gröbner fan for $I$ and $w \in \sigma \cap \mathbb{R}_+^r$ be a weight vector contained in the relative interior of $\sigma$. Let a set of monomials $X^{a_1}, \ldots, X^{a_k}$ be the basis of $\text{in}_w(I)_d$. Then the weight $v \in M_R$ corresponding to $X^{a_1} \wedge \cdots \wedge X^{a_k}$ is in $S$. For $v'(\neq v) \in S$, $v'$ is defined by wedge product of $X^{a_i}$, where $X^{a_i}$ is a term of $f_i$, and at least one term does not coincide with $X^{a_i}$. Hence we have $\langle w, v \rangle > \langle w, v' \rangle$ for any $v' \neq v$. This means that $v$ is a vertex of the state polytope and $w$ is contained in the normal cone of $v$. Hence $\sigma$ is contained in the normal cone of $v$. Therefore, the intersection of the Gröbner fan for $I$ and $\mathbb{R}_+^r$ is a subdivision of the intersection of the normal fan in $N_R$ of the state polytope and $\mathbb{R}_+^r$.

Let $\sigma' \neq \sigma$ be a maximal cone of the Gröbner fan for $I$ and $w' \in \sigma' \cap \mathbb{R}_+^r$ a weight vector contained in the relative interior of $\sigma'$. Let a set of monomials $X^{b_1}, \ldots, X^{b_k}$ be the basis of $\text{in}_w(I)_d$. Then we have $X^{a_1} \wedge \cdots \wedge X^{a_k} \neq X^{b_1} \wedge \cdots \wedge X^{b_k}$. Hence the intersection of the normal fan in $N_R$ of the state polytope and $\mathbb{R}_+^r$ coincides with the intersection of the Gröbner fan for $I$ and $\mathbb{R}_+^r$. \hfill \square

In the rest of this section, we consider the case where the dimension of $\text{Stab}(v)$ is one dimensional. Then $\text{Stab}(v)$ is generated by $(1, \ldots, 1)K^*$ and a finite abelian group. Therefore, $N_R'$ is identified with $N_R/(1, \ldots, 1)$.

3.5 Corollary Let $I'$ be an ideal of $K[\frac{x_1}{x_2}, \ldots, \frac{x_{r-1}}{x_r}]$ and $I \subset K[x_1, \ldots, x_r]$ the homogenization of $I'$. We assume that the stabilizer of $I$ is one dimensional. We take the image
of the part $e_1, \ldots, e_{r-1}$ of the standard basis of $N_\mathbb{R}$ as the basis of $N_\mathbb{P}$. Then the Gröbner fan for $I'$ consists of the images of the Gröbner cones for $I$ which contained in the Gröbner region $GR(I')$ for $I'$ in $N_\mathbb{P}$.

Proof. We only have to prove that the image $\sigma$ of a maximal Gröbner cone $\sigma$ for $I$ coincides with some maximal Gröbner cone for $I'$ if and only if the intersection of $\mathbb{R}^r_{+0} \cong \sum_{i=1}^{r-1} \mathbb{R}_+ e_i \subset N_\mathbb{P}$ and the relative interior of $\sigma$ is not empty.

The necessity is clear from the definition. We prove the sufficiency.

For a polynomial $f \in K[x_1, \ldots, x_r]$, we denote by $f'$ the dehomogenization of $f$. For a polynomial $f \in K[\frac{x_1}{r_1}, \ldots, \frac{x_{r-1}}{r_{r-1}}]$, we denote $f^h$ by the homogenization of $f$. Let $\sigma$ be a maximal Gröbner cone for $I$. In the following, we denote by $c^i$ the relative interior of a cone $c$. Let $w'_i$ be a weight vector in $\sigma \cap \mathbb{R}^r_{+0}$. Let $\tau$ be the Gröbner cone for $I'$ with respect to $w'_i$. Let $w'_{i_2}$ be a weight vector in $\tau^i$. Then we can take the reduced Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_k\}$ of $I'$ with respect to $<w'_i$ such that $\text{in}_{w'_i}(g_j) = \text{in}_{w'_i}(g_j)$ for any $g_j \in \mathcal{G}$.

Choose $w_1, w_2 \in N_\mathbb{R}$ lifts of $w'_i, w'_2$. Let $f$ be any homogeneous polynomial in $I$ and $f' \in I'$ dehomogenization of $f$. Let $\phi : N^r_\mathbb{R} \rightarrow N_\mathbb{R}$ be the lattice homomorphism as follows:

$$\phi : N^r_\mathbb{R} \rightarrow N_\mathbb{R} ; \overline{e_i} \rightarrow e_i.$$ Then we have

$$\text{in}_{w_{1}}(f) = \text{in}_{\phi(w'_i)}(f) = \text{in}_{w'_i}(f') \cdot x^t,$$

where $t = \text{deg} f - \text{deg} \text{in}_{w'_i}(f')$.

We have $\text{in}_{w'_i}(f') \in \langle \text{in}_{w'_i}(g_1), \ldots, \text{in}_{w'_i}(g_k) \rangle$, then we have $\text{in}_{w_{1}}(f) \in \langle \text{in}_{w_{1}}(g_1^h), \ldots, \text{in}_{w_{1}}(g_k^h) \rangle$. This implies $g_1^h, \ldots, g_k^h$ is the reduced Gröbner basis of $I$ with respect to $<_{\phi(w'_i)}$.

In the same way, we have $\text{in}_{w_{2}}(f) = \langle \text{in}_{w_{2}}(g_1^h), \ldots, \text{in}_{w_{2}}(g_k^h) \rangle$. Hence $w_1$ is $I$-equivalent to $w_2$. Thus we have $\tau \subset \sigma$.

On the other hand, let $w_3$ be any weight vector in $\sigma^i$. Then we have $\text{in}_{w_{3}}(I)$ is generated by $\text{in}_{\phi(w'_i)}(g_1), \ldots, \text{in}_{\phi(w'_i)}(g_k)$. Let $w'_3$ be the image of $w_3$. Let $f$ be a polynomial in $I'$ and $f^h$ the homogenization of $f$. We have $\text{in}_{w'_3}(f) = (\text{in}_{w'_3}(f^h))'$ and $\text{in}_{w'_3}(f^h)$ is contained in $\langle \text{in}_{w'_3}(g_1^h), \ldots, \text{in}_{w'_3}(g_k^h) \rangle$. Then we have

$$\text{in}_{w'_3}(f) \in \langle (\text{in}_{w'_3}(g_1^h))', \ldots, (\text{in}_{w'_3}(g_k^h))' \rangle = \langle (\text{in}_{\phi(w'_i)}(g_1^h))', \ldots, (\text{in}_{\phi(w'_i)}(g_k^h))' \rangle$$

$$= \langle \text{in}_{w'_i}(g_1), \ldots, \text{in}_{w'_i}(g_k) \rangle.$$

In particular, $\text{in}_{w'_3}(g_i)$ is contained in $\langle \text{in}_{w'_i}(g_1), \ldots, \text{in}_{w'_i}(g_k) \rangle$, and $\mathcal{G}$ is the reduced Gröbner basis of $I'$ with respect to $<_{w'_i}$. Then we have $\text{in}_{w'_3}(I') = \text{in}_{w'_i}(I')$. Hence $w'_3$ is $I'$-equivalent to $w'_i$. This implies $\sigma \subset \tau$. \hfill \Box

3.6 Corollary Let $I$ be a homogeneous ideal in $\text{Hilb}_h(\mathbb{P})$. We assume that the stabilizer $\text{Stab}(I)$ of $I$ is one dimensional and that the intersection of the algebraic torus of $\mathbb{P}$ and the closed set $\mathcal{M}$ defined by $I$ is an open dense subset of $\mathcal{M}$. Let $I_i \subset K[\frac{x_1}{r_1}, \ldots, \frac{x_{r-1}}{r_{r-1}}]$ be the dehomogenization of $I$ and $\Sigma_i$ the Gröbner fan for $I_i$ with respect to the basis $\{\overline{e_1}, \ldots, \overline{e_i}, \ldots, \overline{e_r}\}$ of $N^r_\mathbb{P}$. Then the set of the images of the Gröbner cones for $I$ coincides with the fan obtained as the union $\cup_i \Sigma_i$ of $\Sigma_i$. 

7
proof. This follows from Corollary 3.5 immediately. Note that the Gröbner region \( GR(I_i) \) for \( I_i \) contains \( \sum_{j \neq i} \mathbb{R}_+ e_j \cong \mathbb{R}_+^r \).

4 Main theorem and its applications

First, we recall the notation and the results in the previous section. Let \( K \) be an algebraically closed field and \( T \) the algebraic torus of dimension \( r \) which is identified with the diagonal subgroup of \( GL(r, K) \). Let \( \mathbb{P}^{r-1} \) be the \((r-1)\)-dimensional projective space over \( K \) and \( S \cong k[x_1, \ldots, x_r] \) the homogeneous coordinate ring of \( \mathbb{P}^{r-1} \), so \((x_1 : \ldots : x_r)\) is a homogeneous coordinate of \( \mathbb{P}^{r-1} \). Then \( T \) acts on \( \mathbb{P}^{r-1} \) defined by \((t_1, \ldots, t_r) \cdot (p_1 : \ldots : p_r) = (t_1 p_1 : \ldots : t_r p_r)\), and \( T \) acts on \( S \) defined by \((t_1, \ldots, t_r) \cdot f(x_1, \ldots, x_r) = f(t_1^{-1} x_1, \ldots, t_r^{-1} x_r)\). Then \( T \) acts on the Hilbert scheme \( \text{Hilb}^m(\mathbb{P}^{r-1}) \) of \( m \) points of \( \mathbb{P}^{r-1} \).

Let \( M \) be the character group of the torus \( T \) and \( N \) the dual lattice of \( M \). Since \( T \) acts on \( S \), the Gröbner fan for an ideal \( I \subset S \) is defined in \( N \). Let \( v \) be a point of \( \text{Hilb}^m(\mathbb{P}^{r-1}) \) and \( I \) the homogeneous ideal of \( S \) corresponding to \( v \). Then, by Proposition 3.2, the normalization of the closure in \( \text{Hilb}^m(\mathbb{P}^{r-1}) \) of the torus orbit of \( v \) is the toric variety corresponding to the state polytope defined in (3.3) for \( d \gg 0 \). Note that the state polytope is a translate of a subset of \( \mathbb{R}_+^r \) which consists of strongly convex cones.

Let \( N' \) be the dual lattice of \( M' \). We proved that the normal fan in \( N' \) of the state polytope coincides with the set of images of the Gröbner cones for \( I \) by the natural projection \( N_k \to N_k' \) (cf. Theorem 3.4). Then the set of images of the Gröbner cones for \( I \) becomes a fan which consists of strongly convex cones.

Next, following Nakamura [11], we introduce the \( G \)-orbit Hilbert schemes \( \text{Hilb}^G(\mathbb{P}^{r-1}) \). Let \( G \) be a finite abelian subgroup of \( GL(r, K) \) of order prime to the characteristic of \( K \). Let \( \phi : GL(r, K) \to PGL(r-1, K) \) be the natural projection. Let \( m \) be the order of \( G \). We assume that \( G \cong \phi(G) \) and that \( G \) is a subgroup of the diagonal subgroup of \( PGL(r-1, K) \). Then we can identify \( T/(1, \ldots, 1) K^* \) with an algebraic torus \( T' \) of \( \mathbb{P}^{r-1} \) and the actions of \( G \) and \( T \) are commutative. Since \( G \) and \( T \) also act on \( S \), the groups \( G \) and \( T \) act on the Hilbert scheme \( \text{Hilb}^m(\mathbb{P}^{r-1}) \).

Let \( \text{Ch}^m(\mathbb{P}^{r-1}) \) be the Chow variety of \( m \) points in \( \mathbb{P}^{r-1} \). We have a natural morphism \( \phi : \text{Hilb}^m(\mathbb{P}^{r-1}) \to \text{Ch}^m(\mathbb{P}^{r-1}) \), which is called the Hilbert-Chow morphism. Here \( G \) acts on \( \text{Hilb}^m(\mathbb{P}^{r-1}) \) and on \( \text{Ch}^m(\mathbb{P}^{r-1}) \), and \( \phi \) is \( G \)-equivariant. Therefore we have a natural morphism between their \( G \)-fixed point sets. It is known that the \( G \)-fixed point set of \( \text{Ch}^m(\mathbb{P}^{r-1}) \) contains \( U_r/G \) as a locally closed subset, where \( U_r = \{ x_r \neq 0 \} \subset \mathbb{P}^{r-1} \), and its closure is an irreducible component of the \( G \)-fixed point set of \( \text{Ch}^m(\mathbb{P}^{r-1}) \).

4.1 Definition The \( G \)-orbit Hilbert scheme \( \text{Hilb}^G(\mathbb{P}^{r-1}) \) is defined to be the unique irreducible component of the \( G \)-fixed point set of \( \text{Hilb}^m(\mathbb{P}^{r-1}) \) which dominates the closure of \( U_r/G \) by the map \( \phi \).

4.2 Theorem Let \( I \) be the \( G \)-invariant ideal of \( S \) whose zero set is contained in \( T' \subset \mathbb{P}^{r-1} \). Then the toric variety corresponding to the fan consisting of the images in \( N_k' \) of the Gröbner cones for \( I \) is isomorphic to the normalization of the \( G \)-orbit Hilbert scheme \( \text{Hilb}^G(\mathbb{P}^{r-1}) \).
Proof. By Proposition 3.2 and Theorem 3.4, it is enough to prove that Hilb^G(\mathbb{P}^{r-1}) is the closure of the torus orbit of I in \text{Hilb}^m(\mathbb{P}^{r-1}).

We can consider that the torus T' is the diagonal subgroup of PGL(r - 1, K). If there exist g, g' ∈ G and t ∈ T/(1, . . . , 1) ∼= T' such that gt = g't, then we have g = g' in T'. Here G acts freely on the open torus of \mathbb{P}^{r-1}.

We prove that the torus orbit of I coincides with the torus contained in Hilb^G(\mathbb{P}^n-1).

Let I' be a point of T · I, then there exist t ∈ T such that I' = t · I. Since G ⊂ T/(1, . . . , 1), I' is also a G-fixed point of \text{Hilb}^m(\mathbb{P}^{r-1}).

Conversely, let J be an ideal of S defined by a free G-orbit whose zero set is contained in T'. Then J determines distinct m points p_1, . . . , p_m of T' and I determines distinct m points q_1, . . . , q_m of T'. Take a t ∈ T satisfying t · p_1 = q_1. Then we have

\{q_1, . . . , q_m\} = \{g · q_1 | g ∈ G\} = \{g · t · p_1 | g ∈ G\} = \{t · g · p_1 | g ∈ G\} = t · \{p_1, . . . , p_m\}.

Therefore J is contained in T · I.

Since T · I is an open subset of Hilb^G(\mathbb{P}^{r-1}) and Hilb^G(\mathbb{P}^{r-1}) is irreducible, the closure of T · I coincides with Hilb^G(\mathbb{P}^{r-1}).

4.3 Corollary Let I be the G-invariant ideal of S whose zero set is contained in T' and I' ⊂ K[\frac{2}{3}, . . . , \frac{2}{3}] the dehomogenization of I. Then the toric variety corresponding to the Gröbner fan GF(I') for I' is isomorphic to the normalization of G-orbit Hilbert scheme Hilb^G(\mathbb{A}^{r-1}).

4.4 Remark In this case, the Gröbner fan for I' consists of strongly convex cones.

Proof. We denote by Hilb^G_{\text{norm}}(\mathbb{P}^{r-1}) (resp. Hilb^G_{\text{norm}}(\mathbb{A}^{r-1})) the normalization of Hilb^G(\mathbb{P}^{r-1}) (resp. Hilb^G(\mathbb{A}^{r-1})). A point p ∈ Hilb^G_{\text{norm}}(\mathbb{P}^{r-1}) is contained in Hilb^G_{\text{norm}}(\mathbb{A}^{r-1}) if and only if p is contained in the torus orbit corresponding to a Gröbner cone σ of GF(I'). Then this corollary follows from Corollary 3.5 and Theorem 4.2 immediately.

4.5 Corollary (Ito [8, Theorem 1.1]) Let G be a finite small cyclic group in GL(2, \mathbb{C}) and I the G-invariant ideal of \mathbb{C}[x, y] whose zero set is contained in (\mathbb{C}^*)^2. Then the toric variety corresponding to the Gröbner fan for I is isomorphic to the minimal resolution of \mathbb{C}^2/G.

Proof. Corollary 4.3 gives that the toric variety corresponding to the Gröbner fan for I is isomorphic to the normalization of G-orbit Hilbert scheme Hilb^G(\mathbb{A}^2). Ishii [7, theorem 3.1] shows that the G-orbit Hilbert scheme Hilb^G(\mathbb{C}^2) is the minimal resolution of \mathbb{C}^2/G when G is a finite small subgroup of GL(2, \mathbb{C}). Hence the proof completes.

5 Example

5.1 Example Let G be a cyclic group which is generated by the matrix

\[
\left(\begin{array}{ccc}
e & 0 & 0 \\
0 & e^3 & 0 \\
0 & 0 & 1
\end{array}\right),
\]

where e is a primitive fifth root of the unity. Let I be the ideal of K[x, y, z] which is
generated by \(x^3 - yz^2, x^2y - z^3\). The set of zeros of \(I\) is \(\{(1 : 1 : 1), (e : e^3 : 1), (e^2 : e^1 : 1), (e^3 : e^4 : 1), (e^4 : e^2 : 1)\}\). Then \(I\) is a \(G\)-invariant ideal whose zero set is contained in \((K^*)^2\).

We denote by \(S_w\) the reduced Gröbner basis with respect to \(w\) and \(c[w]\) the Gröbner cone with respect to \(w\). The Gröbner fan for \(I\) is defined by 11 maximal cones:

\[
\begin{align*}
S_{w_1} &= \{x^3 - yz^2, x^2y - z^3, xz^3 - y^2z^2, xy^3z^2 - z^6, y^5z^2 - z^7\} \\
S_{w_2} &= \{x^3 - yz^2, x^2y - z^3, y^2z^2 - xz^3\} \\
S_{w_3} &= \{yz^2 - x^3, x^2y - z^3, x^5 - z^5\} \\
S_{w_4} &= \{yz^2 - x^3, x^2y - z^3, z^5 - x^5\} \\
S_{w_5} &= \{yz^2 - x^3, z^3 - x^2y, x^2y^2 - x^3z\} \\
S_{w_6} &= \{yz^2 - x^3, z^3 - x^2y, x^3z - x^2y^2\} \\
S_{w_7} &= \{yz^2 - x^3, z^3 - x^2y, x^3z - x^2y^2, x^2y^3z - x^6, x^7 - x^2y^5\} \\
S_{w_8} &= \{yz^2 - x^3, z^3 - x^2y, x^3z - x^2y^2, x^6 - x^2y^3z\} \\
S_{w_9} &= \{x^3 - yz^2, z^3 - x^2y, y^2z^2 - xz^3\} \\
S_{w_{10}} &= \{x^3 - yz^2, x^2y - z^3, xz^3 - y^2z^2, z^6 - xy^3z^2\} \\
S_{w_{11}} &= \{x^3 - yz^2, x^2y - z^3, xz^3 - y^2z^2, xy^3z^2 - z^6, z^7 - y^4z^2\}
\end{align*}
\]

Figure 1: The Gröbner fan for \(I\)
5.2 Example  Let $G$ be a cyclic group which is generated by the matrix
\[
\begin{pmatrix}
0 & e^2 & 0 & 0 \\
0 & 0 & e^3 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
where $e$ is a primitive fifth root of the unity. Let $I$ be the ideal of $K[x, y, z, w]$ which is generated by $x^5 - w^5, x^2 - yw, x^3 - zw^2$. The set of zeros of $I$ is \{(1 : 1 : 1 : 1), (e : e^2 :...
\( e^3 : 1, (e^2 : e^4 : e : 1), (e^3 : e : e^4 : 1), (e^4 : e^3 : e^2 : 1) \). Then \( I \) is a \( G \)-invariant ideal
whose zero set is contained in \((K^*)^3\). Here \( G \) acts on \( \{w \neq 0\} \cong \mathbb{A}^3 \). Let \( I' \subset K[w, \frac{1}{w}, \frac{2}{w}] \)
be the dehomogenization of \( I \). The Gröbner fan for \( I' \) has 15 edges, 32 facets, and 18
maximal cones. 17 maximal cones are simplicial and nonsingular, but 1 maximal cone is
not simplicial. Then \( \text{Hilb}^G(\mathbb{A}^3) \) is singular.

![Gröbner fan for \( I' \)](image)

**Figure 3:** The Gröbner fan for \( I' \)

### References

[1] D. Bayer and I. Morrison, Gröbner bases and geometric invariant theory I. Initial
ideals and state polytopes, J. Symbolic Computation 6 (1988), 209–217.

[2] A. Craw, D. Maclagan, and R. R. Thomas, Moduli of McKay quiver representation
I: The coherent component, Proc. London Math. Soc. (2007), 95 (1):179–198.

[3] A. Craw, D. Maclagan, and R. R. Thomas, Moduli of McKay quiver representation
II: Gröbner basis techniques, to appear in J. Algebra.

[4] W. Fulton, Introduction to Toric Varieties, Princeton University Press.

[5] A. Jensen, Gfan - a software system for Gröbner fans. Available at
http://home.imf.au.dk/ajensen/software/gfan/gfan.html

[6] M. M. Kapranov, B. Sturmfels and A. V. Zelevinsky, Chow polytopes and general
resultants, Duke Math. J. Volume 67, Number 1 (1992), 189–218.

[7] A. Ishii, On the McKay correspondence for a finite small subgroup of \( GL(2, \mathbb{C}) \), J.
Reine Angew. Math. 549 (2002), 221–233.

[8] Y. Ito, Minimal resolution via Gröbner basis, Algebraic Geometry in East Asia,
(IIAS, 2001), World Scientific, (2003), 165–174.
[9] Y. Ito and I. Nakamura, Hilbert schemes and simple singularities, In New trends in algebraic geometry (Warwick, 1996), volume 264 of London Math. Soc. Lecture Note Ser., pages 151–233. Cambridge Univ. Press, Cambridge, 1999.

[10] Y. Ito and I. Nakamura, McKay correspondence and Hilbert schemes, Proc. Japan Acad. 72 (1996), 135–138.

[11] I. Nakamura, Hilbert schemes of abelian group orbits, J. Algebraic Geom., 10 (4) : 757–779, 2001.

[12] T. Oda, Convex Bodies and Algebraic Geometry, Ergeb. Math. Grenzeb. (3) 15, Springer-Verlag, Berlin, 1988.

[13] B. Sturmfels, Gröbner Bases and Convex Polytopes, Univ. Lect. Series, 8, AMS (1995).

[14] B. Sturmfels, Gröbner bases of toric varieties, Tohoku Math. J. 43 (1991), 249–261.

[15] T. Yasuda, Universal flattening of Frobenius, arXiv:0706.2700v3 [math.AG]

Tomohito Morita
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152-8551
Japan
morita.t.ae@m.titech.ac.jp