Description of $GL_3$-orbits on the quadruple projective varieties

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Abstract

This article gives a description of the diagonal $GL_3$-orbits on the quadruple projective variety $(\mathbb{P}^2)^4$. We give explicit representatives of orbits, and describe the closure relations of orbits. A distinguished feature of our setting is that it is the simplest case where $\text{diag}(GL_n)$ has infinitely many orbits but has an open orbit in the multiple projective space $(\mathbb{P}^{n-1})^m$.

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1 Introduction

The object of this article is an orbit decomposition of the flag variety $G/P$ under the action of a closed subgroup $H$ of a reductive group $G$ in the setting that $H$ has an open orbit in $G/P$ but $\#(H\backslash G/P) = \infty$.

The study of the double coset $H\backslash G/P$ is motivated by the representation theory. For example, for a real reductive algebraic group $G$, and for an algebraically defined closed subgroup $H$, Kobayashi-Oshima [7] proved that the finiteness (resp. boundedness) of the multiplicities of irreducible admissible representations $\pi$ of $G$ occurring in the induced representations $\text{Ind}_H^G(\tau)$ for finite-dimensional irreducible representations $\tau$ of $H$ is equivalent to the existence of open $H$-orbits (resp. $H_c$-orbits) on $G/P_G$ (resp. $G_c/B_G$), where $P_G$ is a minimal parabolic subgroup of $G$, and $B_G$ is a Borel subgroup of the complexification $G_c$ of $G$. A homogeneous space $G/H$ satisfying these equivalent conditions is called a real spherical variety (resp. spherical variety). In this case, the existence of an open $H$-orbit is known to be the
finiteness of $H$-orbits [1, 2, 14]. Needless to say, the latter implies the former, however, the converse may fail for a general parabolic subgroup $P$ of $G$. Even in such cases, the existence of open $H$-orbits or the finiteness of $H$-orbits on the flag variety $G/P$ gives useful information on the branching laws between representations of $H$ and representations of $G$ induced from characters of $P$ [4, 19].

Kobayashi-Oshima [7] also proved that the finiteness of the dimension of the space of symmetry breaking operators ($H$-intertwining operators from irreducible admissible representations of $G$ to those of $H$) is equivalent to the existence of open $P_H$-orbits on $G/P_G$ where $H$ is reductive and $P_H$ is its minimal parabolic subgroup. Furthermore, such pairs $(G, H)$ are classified by Kobayashi-Matsuki [6] under the assumption that $(G, H)$ are symmetric pairs. Moreover, an explicit description of the double coset $P_H \backslash G / P_G \cong \text{diag}(H) \backslash (G \times H) / (P_G \times P_H)$ parametrises the “families” of symmetry breaking operators as in the works [5, 8, 9] for the pair $(G, H) = (O(p + 1, q), O(p, q))$ which is in the classification given by [6]. Our research comes from these motivations.

So far, an explicit description of the double coset $H \backslash G / P$ has been studied only when $H$ has finitely many orbits in $G/P$. For instance, the descriptions of $H$-orbits on $G/P$ for symmetric pairs $(G, H)$ have been well studied by Matsuki [12, 13], generalising the Bruhat decomposition where $\text{diag}(G) \backslash (G \times G) / (P_1 \times P_2)$ may be identified with $P_1 \backslash G / P_2$. Matsuki [15, 16] gave also a description of diagonal action of orthogonal groups on multiple flag varieties $G^m / (P_1 \times P_2 \times \cdots \times P_n)$ under the assumption that the number of orbits is finite, referred to as “finite type”, which were also classified in that work. Note that the pair $(G, H)$ is no more a symmetric pair if $m \geq 3$, and multiple flag varieties of finite type were earlier classified by [10, 11] where $G$ is of type $A$ or $C$.

In this article, we are interested in the case where $\#(H \backslash G / P) = \infty$. We fix $\mathbb{K}$ to be an algebraically closed field with characteristic 0, and $Q_n$ to be the maximal parabolic subgroup of $GL_n$ such that $GL_n / Q_n \cong \mathbb{P}^{n-1}$ over $\mathbb{K}$. We at first prove the following.

**Theorem A** (see Theorems 2.1 and 2.2). Let $n, m$ be positive integers with $n \geq 2$, and $\mathbb{K}$ be an algebraically closed field with characteristic 0, then the followings hold:

1. the number of diag($GL_n$)-orbits on $(\mathbb{P}^{n-1})^m$ is finite if and only if $m \leq 3$;
2. there exists an open diag($GL_n$)-orbit on $(\mathbb{P}^{n-1})^m$ if and only if $m \leq n + 1$.

In particular, there exist infinitely many orbits and an open orbit simultaneously if and only if $4 \leq m \leq n + 1$. Hence, the case $(n, m) = (3, 4)$ can be regarded as the simplest case among them. In this article, we highlight this case, and give an explicit description of diag($GL_3$)-orbits on $(GL_3 / Q_3)^4 \cong (\mathbb{P}^2)^4$ and their closure relations. For this aim, we introduce the following diag($GL_3$)-invariant map with a finite image:

$$
\pi: \quad (\mathbb{P}^2)^4 \to \text{Map}(2^{\{1,2,3,4\}}, \mathbb{N}) \text{, where } \psi_{([v_i])_{i=1}^4} \mapsto (I \mapsto \dim \text{Span}\{v_i\}_{i \in I})
$$

(1.1)

Our strategy is to determine the image of $\pi$ in $\text{Map}(2^{\{1,2,3,4\}}, \mathbb{N})$, and to describe diag($GL_3$)-orbit decomposition of each fibre $\pi^{-1}(\varphi)$ for $\varphi \in \text{Image}(\pi)$. Here is a description of diag($GL_3$)-orbits on $(GL_3 / Q_3)^4 \cong (\mathbb{P}^2)^4$.
Theorem B (see Theorems 4.6, 4.8, and Propositions 5.2 and 5.3).  

(1) The map $\pi: (\mathbb{P}^2)^4 \to \text{Map}(2^{1,2,3,4}, \mathbb{N})$ is invariant under the diagonal action of $G = GL_3$.

(2) A map $\varphi \in \text{Map}(2^{1,2,3,4}, \mathbb{N})$ belongs to the image of $\pi$ if and only if $\varphi[*]$ appears in Figure 1 via the correspondence $\varphi \leftrightarrow \varphi[*]$ (see Definitions 4.3 and 4.7).

(3) For each $\varphi[*]$ listed in Figure 1, the fibre $\pi^{-1}(\varphi[*])$ is a single diag(G)-orbit unless $\varphi[*] = \varphi[6]$. The fibre $\pi^{-1}(\varphi[6])$ is decomposed into infinitely many diag(G)-orbits by means of 5-dimensional orbits $O(5;p)$ with parameters $p \in \mathbb{P}^1 \setminus \{0,1,\infty\}$, see Definition 4.7.

(4) If we fix $p \in \mathbb{P}^1 \setminus \{0,1,\infty\}$, then the closure relations among all fibres of $\varphi[*]$ and the orbit $O(5;p)$ are given by the following Hasse diagram in Figure 1.

![Figure 1: Hasse diagram](image1)

![Figure 2: mod $\mathcal{S}_4$](image2)

Remark 1.1.  

(1) Since $\text{diag}(G)\backslash (\mathbb{P}^2)^4$ admits a natural action of the symmetric group $\mathcal{S}_4$, one can consider the quotient $(\text{diag}(G)\backslash (\mathbb{P}^2)^4) / \mathcal{S}_4$ in Figure 2 where $O_p := \{p, \frac{1}{p}, 1 - p, 1 - \frac{1}{p}, 1 - \frac{1}{1-p} \}$.

(2) The symbol $\varphi[*] \in \text{Map}(2^{1,2,3,4}, \mathbb{N})$ has the following information: the dimension of the fibre $\pi^{-1}(\varphi[k;J]) \subset (\mathbb{P}^2)^4$ is $k$, where $J \subset \{1,2,3,4\}$ is another parameter. We observe that $\varphi[k;J]$ occurs in Figure 1 if and only if so is $\varphi[k;\sigma J]$ for $\sigma \in \mathcal{S}_4$.

Notation. We set $\mathbb{N} = \{0,1,2,3,\ldots\}$, and $[m]$ denotes the set $\{1,2,\ldots,m\} \subset \mathbb{N}$ for $m \in \mathbb{N}$.

We let $\mathbb{K}$ be an algebraically closed field with characteristic 0. For a vector $v \in \mathbb{K}^n \setminus \{0\}$, we write the $\mathbb{K}^*$-orbit through $v$ as $[v] \in \mathbb{P}^{n-1}$. Similarly for a matrix $(v_i)_{i=1}^m \in M(n,m)$ without any columns equal to $0$, the notation $[v_i]_{i=1}^m$ denotes the $m$-tuple of $\mathbb{P}^{n-1}$. For an $m$-tuple $[v_i]_{i=1}^m$ in $\mathbb{P}^{n-1}$, we write the subspace spanned by these $m$ vectors by $\langle v_i \rangle_{i=1}^m$. Furthermore, $\{e_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{K}^n$.
2 Existence of open orbits and finiteness of orbits

In this section, we prove Theorem A, which determines the existence of open orbits and finiteness of orbits under the diagonal action of $GL_n$ on the multiple projective space $(\mathbb{P}^{n-1})^m$.

Theorem 2.1. Let $n \geq 2$ and $m$ be positive integers. There exists an open $\text{diag}(GL_n)$-orbit on $(\mathbb{P}^{n-1})^m$ if and only if $n \geq m - 1$.

Proof. (1) If $n \geq m$, then we can take an element $[e_i]_{i=1}^m \in (\mathbb{P}^{n-1})^m$. Its stabiliser is
\[
\left\{ \begin{pmatrix} C \\ 0 \end{pmatrix} \right\} \quad C = \text{diag}(c_1, c_2, \ldots, c_m) \in GL_m \}
\]
and $\text{diag}(GL_n) \cdot [e_i]_{i=1}^m$ is the open orbit since $\dim(\text{diag}(GL_n) \cdot [e_i]_{i=1}^m) = \dim GL_n - (m + n(n - m)) = (n - 1)m = \dim(\mathbb{P}^{n-1})^m$.

(2) If $n + 1 = m$, then we can take a element $v = [v_i]_{i=1}^{m+1} \in (\mathbb{P}^{n-1})^{n+1}$ where $v_i := e_i$ for $1 \leq i \leq n$ and $v_{n+1} := \sum_{k=1}^n e_k$. If $g \in GL_n$ stabilises $[v_i] = [e_i]$ for $1 \leq i \leq n$, then it is a diagonal matrix, and if it stabilises $[v_{n+1}] = [\sum_{k=1}^n e_k] = [1]$, then it is actually a scalar matrix. Hence $\text{diag}(GL_n) \cdot v$ is the open orbit since $\dim(\text{diag}(GL_n) \cdot v) = \dim GL_n - 1 = (n - 1)(n + 1) = \dim(\mathbb{P}^{n-1})^{n+1}$.

(3) Let $n + 2 \leq m$. Remark that the centre of $GL_n$ acts trivially on $\mathbb{P}^{n-1}$, and the dimension of any $\text{diag}(GL_n)$-orbit in $(\mathbb{P}^{n-1})^m$ is less than $n^2$. Hence there is no open orbit since $\dim(\mathbb{P}^{n-1})^m = (n - 1)m > (n - 1)(n + 1) = n^2 - 1$.

\[\square\]

Theorem 2.2. Let $n \geq 2$ and $m$ be positive integers. There are only finitely many $\text{diag}(GL_n)$-orbits on $(\mathbb{P}^{n-1})^m$ if and only if $m \leq 3$.

Proof. (1) The action of $GL_n$ on $\mathbb{P}^{n-1}$ is transitive, hence there is only one orbit.

(2) From Bruhat decomposition, we have $\text{diag}(GL_n) \backslash (\mathbb{P}^{n-1})^2 = \text{diag}(GL_n) \backslash (GL_n/Q_n)^2 \cong Q_n \backslash GL_n/Q_n$ where $Q_n$ is the maximal parabolic subgroup of $GL_n$ with the Levi subgroup $GL_1 \times GL_{n-1}$. It has a one to one correspondence with $S_{n-1} \backslash S_n/S_{n-1}$, which is a two point set.

(3) If $m = 3$, we have
\[
\text{diag}(GL_n) \backslash (\mathbb{P}^{n-1})^3 = \text{diag}(GL_n) \backslash \left\{ \{v_1, v_2, v_3\} \in (\mathbb{P}^{n-1})^3 \mid [v_1] \neq [v_2] \neq [v_3] \neq [v_1]\right\}
\]
\[\Pi_{1 \leq i < j \leq 3} \text{diag}(GL_n) \backslash \left\{ [v_1, v_2, v_3] \in (\mathbb{P}^{n-1})^3 \mid [v_i] = [v_j]\right\}. \quad (2.1)\]

For $1 \leq i < j \leq 3$, we have
\[
\text{diag}(GL_n) \backslash \left\{ [v_1, v_2, v_3] \in (\mathbb{P}^{n-1})^3 \mid [v_i] = [v_j]\right\} \cong \text{diag}(GL_n) \backslash (\mathbb{P}^{n-1})^2, \quad (2.2)
\]
hence it is finite from the previous case. Furthermore, we have
\[
\left\{ [v_1, v_2, v_3] \in (\mathbb{P}^{n-1})^3 \mid [v_1] \neq [v_2] \neq [v_3] \neq [v_1]\right\}
\]
In Section 2, we observed the existence of open orbits and finiteness of orbits on \((\mathbb{P}^n)^m\). In this section, we discuss further general properties of \(GL_n\)-orbits on \((\mathbb{P}^n)^m\). Notably, we consider indecomposable splittings of configuration spaces.

**3 General structures of orbits on multiple projective spaces**

In Section 2, we observed the existence of open orbits and finiteness of orbits on \((\mathbb{P}^n)^m\) under the diagonal action of \(GL_n\). In this section, we consider further general properties of diagonal \(GL_n\)-orbits on \((\mathbb{P}^n)^m\).

### 3.1 Indecomposable splitting of configuration spaces

To simplify the description of \(GL_n\)-orbit decomposition of \((\mathbb{P}^n)^m\), we introduce some \(GL_n\)-invariant subsets of \((\mathbb{P}^n)^m\) and consider the decompositions into them.

For elements of \((\mathbb{P}^n)^m\), we can define the notion of indecomposability as follows:

**Definition 3.1.** For \(v = [v_i]_{i=1}^m \in (\mathbb{P}^n)^m\), we say \(v\) is decomposable if there exists a pair \(\emptyset \neq I, J \subset [m]\) such that \(I \sqcup J = [m]\) and \(\{v_i\}_{i \in I} \cap \{v_j\}_{j \in J} = \emptyset\). Conversely, if \(v\) does not have such a decomposition, we say \(v\) is indecomposable.

It is equivalent to the notion of indecomposability as objects in the flag category, and with this notation, we consider indecomposable splittings of elements in \((\mathbb{P}^n)^m\) as follows:

**Definition 3.2.** (1) Define a set \(\mathcal{P}_m\) as follows:

\[
\mathcal{P}_m := \left\{ \{(I_k, r_k)\}_{k=1}^l \left| \bigcap_{k=1}^l I_k = [m], \ I_k \neq \emptyset, \ r_k \in \mathbb{N} \right. \right\}.
\]

(2) Define the map \(\varpi : (\mathbb{P}^n)^m \to \mathcal{P}_m, \ v \mapsto \{(I_k, r_k)\}_{k=1}^l\) where

- i) \(\{v_i\}_{i=1}^m = \bigoplus_{k=1}^l \left\langle v_i \right\rangle_{i \in I_k}\) and \(r_k = \dim \left\langle v_i \right\rangle_{i \in I_k}\),
- ii) \(\{v_i\}_{i \in I_k} \in (\mathbb{P}^n)^m\) and \(I_k\) is indecomposable.

The map \(\varpi\) is well-defined since it is generally known that indecomposable splittings of flags are unique up to isomorphisms. However, we can also check its validity in this case. Let \(\{(I_k, r_k)\}_{k=1}^l\) and \(\{(J_k, s_k)\}_{k=1}^l\) \(\in \mathcal{P}_m\) satisfy the two conditions i) and ii) in Definition 3.2 for \([v_i]_{i=1}^m \in (\mathbb{P}^n)^m\). If \(I_k \cap J_{k^*} \neq \emptyset\) and \(J_k \setminus J_{k^*} \neq \emptyset\), then \(\{v_i\}_{i \in I_k \cap J_{k^*}} \cap \{v_i\}_{i \in I_k \setminus J_{k^*}} = \emptyset\) from i), and it contradicts to the indecomposability of \([v_i]_{i \in I_k}\) assumed in ii). Hence either \(I_k \cap J_{k^*} = \emptyset\) or \(I_k = J_{k^*}\) holds.
Example 3.3. For $(n, m) = (3, 4)$, we have
\[
\varpi([e_1, e_1, e_2, e_3]) = \{(\{1, 2\}, 1), (\{3\}, 1), (\{4\}, 1)\}, \\
\varpi([e_1, e_2, e_3, e_2 + e_3]) = \{(\{1\}, 1), (\{2, 3, 4\}, 2)\}.
\]

On the other hand, in this article, we observe mainly the following map:
\[
\pi: (\mathbb{P}^{n-1})^m \rightarrow \text{Map}(2^m, \mathbb{N}) \\
[v_i]_{i=1}^m \mapsto (I \mapsto \dim \langle v_i \rangle_{i \in I})
\]
(3.1)

We can characterise the correspondence between $\varpi$ and $\pi$ as bellow:

Lemma 3.4. For the maps $\varpi$ and $\pi$ defined in Definition 3.2 and (3.1), there exist following maps satisfying the diagram bellow.

\[
\begin{align*}
\text{Map}(2^m, \mathbb{N}) &\rightarrow \mathcal{P}_m \\
\pi &\mapsto \rho
\end{align*}
\]

Proof. Since the map $\pi$ is $GL_n$-invariant, we can define the map $\tilde{\pi}$ clearly.

Now for a map $\varphi: 2^m \rightarrow \mathbb{N}$, we define it to be decomposable if there exists a partition $I_1 \amalg I_2 = [m]$, $I_1, I_2 \neq \emptyset$ which satisfies $\varphi(I) = \varphi(\cap I) + \varphi(\cap I_2)$ for all $I \subseteq [m]$. Then we can define the map $\rho: \text{Map}(2^m, \mathbb{N}) \rightarrow \mathcal{P}_m$, $\varphi \mapsto \{(I_k, r_k)\}_{k=1}^m$ where

1. $\varphi(I) = \sum_{k=1}^m \varphi(I \cap I_k)$ for $I \subseteq [m]$, and $r_k = \varphi(I_k)$,
2. the restricted map $\varphi|_{I_k}$ is indecomposable for each $1 \leq k \leq l$.

The map $\rho$ is well-defined. Indeed, let $\{(I_k, r_k)\}_{k=1}^l$ and $\{(J_k, s_k)\}_{k=1}^{l'}$ satisfy the conditions (1) and (2) for $\varphi: 2^m \rightarrow \mathbb{N}$. If $I_k \cap J_k \neq \emptyset$ and $I_k \setminus J_k \neq \emptyset$, then for $I \subseteq I_k$,

\[
\begin{align*}
\varphi(I) &= \sum_{k''=1}^{l'} \varphi(J_{k''} \cap I) = \varphi(J_{k''} \cap I) + \sum_{k''=1}^{l'} \varphi(J_{k''} \cap I \setminus J_k) = \varphi(J_{k''} \cap I) + \varphi(I \setminus J_k) \\
&= \varphi(I_k \cap J_k \cap I) + \varphi(I_k \setminus J_k \cap I)
\end{align*}
\]
from (1), and it contradicts to the indecomposability of $\varphi|_{I_k}$ in (2). Hence either $I_k \cap J_k = \emptyset$ or $I_k = J_k$ holds, which concludes the well-definedness of $\rho$.

Now for $\varpi([v_i]_{i=1}^m) = \{(I_k, r_k)\}_{k=1}^m$, we have

1. From $\langle v_i \rangle_{i=1}^m = \bigoplus_{k=1}^l \langle v_i \rangle_{i \in I_k}$, we have $\dim \langle v_i \rangle_{i \in I} = \sum_{i=1}^l \langle v_i \rangle_{i \in I_k \cap I}$.
2. If $\emptyset \neq \cap J$, then from the indecomposability of $[v_i]_{i \in I_k}$, we have $\langle v_i \rangle_{i \in I_k \cap I} \cap \langle v_i \rangle_{i \in I_k \cap J'} \neq \emptyset$. Hence $\dim \langle v_i \rangle_{i \in I_k \cap I} < \dim \langle v_i \rangle_{i \in I_k \cap I} + \dim \langle v_i \rangle_{i \in I_k \cap J'}$.

Hence we have shown that $\rho \circ \pi = \varpi$. \qed
3.2 Some typical decompositions of multiple projective spaces

For the map \( \varpi \) defined in Definition 3.2, there are some remarkable properties.

**Lemma 3.5.** For the map \( \varpi: (\mathbb{P}^{n-1})^m \to \mathcal{P}_m \) defined in Definition 3.2, an element \( \{(I_k, r_k)\}_{k=1}^l \) of \( \mathcal{P}_m \) is contained in the image of \( \varpi \) if and only if the followings are satisfied:

1. \( r_k = 1 \), or \( 2 \leq r_k \leq \#I_k - 1 \) for each \( 1 \leq k \leq l \).
2. \( \sum_{k=1}^l r_k \leq n \).

**Proof.** If \( \varpi([v_i]_{i=1}^m) = \{(I_k, r_k)\}_{k=1}^l \), since \( \langle v_i \rangle_{i=1}^m = \bigoplus_{k=1}^l \langle v_i \rangle_{i \in I_k} \) and \( r_k = \dim \langle v_i \rangle_{i \in I_k} \), the second condition holds obviously. Furthermore, let us assume that the first condition fails, in other words, assume that \( 2, \#I_k \leq r_k \) is satisfied for some \( k \). Since \( \#I_k < r_k = \dim \langle v_i \rangle_{i \in I_k} \) cannot occur, the equality must hold. Hence \( \{v_i\}_{i \in I_k} \) is linearly independent, which contradicts to the indecomposability.

Conversely, if the two conditions are satisfied for \( \{(I_k, r_k)\}_{k=1}^l \in \mathcal{P}_m \), we define as follows:

1. \( R(k) := \sum_{k' < k} r_{k'} \) for \( 0 \leq k \leq l \) In particular, \( 0 = R(0) < R(1) < R(2) < \cdots R(l) \leq n \).
2. For \( I_k = \{i(1) < i(2) < \cdots < i(\#I_k)\} \), set
   \[
   v_{i(j)} := \begin{cases} 
   e_{R(k) + j} & 1 \leq j \leq r_k, \\
   \sum_{j' = 1}^{r_k} e_{R(k) + j'} & r_k + 1 \leq j \leq \#I_k.
   \end{cases}
   \]

Then we have \( \langle v_i \rangle_{i=1}^m = \langle e_j \rangle_{j=1}^{R(l)} = \bigoplus_{k=1}^l \langle e_{R(k) + j} \rangle_{j=1}^{r_k} = \bigoplus_{k=1}^l \langle v_i \rangle_{i \in I_k} \), and each \( \langle v_i \rangle_{i \in I_k} \) is indecomposable since the first \( r_k + 1 \) vectors are in a general position in the \( r_k \)-dimensional space. Hence \( \varpi([v_i]_{i=1}^m) = \{(I_k, r_k)\}_{k=1}^l \).

**Lemma 3.6.** If we define the subset of \( \mathcal{P}_m \) by

\[
\mathcal{P}_{n,m} := \{\{(I_k, r_k)\}_{k=1}^m \in \text{Image}(\varpi) \subset \mathcal{P}_m \mid r_k = 1, \text{ or } 2 \leq r_k \leq \#I_k - 1 (1 \leq k \leq l)\},
\]

then for each element \( \{(I_k, r_k)\}_{k=1}^l \in \mathcal{P}_{n,m} \), the fibre \( \varpi^{-1}(\{(I_k, r_k)\}_{k=1}^l) \subset (\mathbb{P}^{n-1})^m \) is a single \( GL_n \)-orbit. Furthermore, this orbit coincides with the fibre \( \pi^{-1}(\varphi) \) where

\[
\varphi: 2^m \to \mathbb{N}: I \mapsto \sum_{k=1}^l \min \{\#(I_k \cap I), r_k\}.
\]

In particular, we have the following bijections.

\[
\begin{array}{ccc}
GL_n(\mathbb{P}^{n-1})^m & \xrightarrow{\varpi} & \text{Image}(\varpi) \\
\xrightarrow{\pi} & \text{Image}(\pi) & \xrightarrow{\rho} \text{Image}(\varpi) \\
\end{array}
\]

\[
\begin{array}{ccc}
GL_n(\mathbb{P}^{n-1})^m & \xrightarrow{\varpi^{-1}} & \mathcal{P}_{n,m} \\
\xrightarrow{\rho^{-1}} & \mathcal{P}'_{n,m} & \xrightarrow{\sim} \mathcal{P}_{n,m}
\end{array}
\]

**Proof.** We define \( v_1, \ldots, v_m \) as in Lemma 3.5 for \( \{(I_k, r_k)\}_{k=1}^l \in \mathcal{P}_{n,m} \). Now let us consider an element \([w_i]_{i=1}^m \in \varpi^{-1}(\{(I_k, r_k)\}_{k=1}^l) \).
Definition 3.2 and Lemma 3.6. Then we can obtain a description of $G/Q$ (Definition 3.2) if and only if the tuple
\[
\{ w_i \}_{i \in I_k} \text{ for all } i \in I_k.
\]

For the case $2 \leq r_k = \# I_k - 1$, if $r_k$-tuple of $\{ w_i \}_{i \in I_k}$ is not linearly independent, then it spans strictly less than $r_k$-dimensional subspace of $\langle w_i \rangle_{i \in I_k}$, which does not contain the rest one vector. It contradicts to the indecomposability of $[w_i]_{i \in I_k}$. Hence all $r_k$-tuples of $\{ w_i \}_{i \in I_k}$ must be linearly independent.

From this observation, since $\{ w_{i(1)}, w_{i(2)}, \ldots, w_{i(r_k)} \}$ is a basis of the $r_k$-dimensional space $\langle w_i \rangle_{i \in I_k}$, there exists an linear combination $w_i(r_k+1) = \sum_{j=1}^{r_k} c_j w_{i(j)}$. If $c_1 = 0$, then $\{ w_{i(j)} \}_{j=2}^{r_k+1}$ is linearly dependent, which contradicts to the observation above. Hence $c_1 \neq 0$. Similarly we can prove $c_j \neq 0$ for all $1 \leq j \leq r_k$.

Now we can define a linear isomorphism $f_k$ between the $r_k$-dimensional spaces $\langle w_i \rangle_{i \in I_k} = \langle w_{i(j)} \rangle_{j=1}^{r_k}$ and $\langle v_i \rangle_{i \in I_k} = \langle e_{R(k-1)+j} \rangle_{j=1}^{r_k}$ by sending $c_j w_{i(j)}$ to $v_{i(j)} = e_{R(k-1)+j}$ for $1 \leq j \leq r_k$, which leads that
\[
f_k(w_i(r_k+1)) = f_k \left( \sum_{j=1}^{r_k} c_j w_{i(j)} \right) = \sum_{j=1}^{r_k} e_{R(k-1)+j} = v_i(r_k+1).
\]

Now, since $\langle w_i \rangle_{i \in I_k} = \bigoplus_{k=1}^{l} \langle w_i \rangle_{i \in I_k}$ and $\langle v_i \rangle_{i \in I_k} = \bigoplus_{k=1}^{l} \langle v_i \rangle_{i \in I_k}$, by taking the direct product of these linear isomorphisms $f_k: \langle w_i \rangle_{i \in I_k} \simeq \langle v_i \rangle_{i \in I_k}$ and some linear isomorphism between the complement space, we have obtained an isomorphism $g \in GL_n$ such that $g \cdot [w_i]_{i=1}^{l} = [v_i]_{i=1}^{l}$.

Now, since any $r_k$-tuple of $\{ w_i \}_{i \in I_k}$ are linearly independent, $\dim \langle w_i \rangle_{i \in I_k \cap I} = \min \{ \# (I_k \cap I), r_k \}$. Furthermore, since $\langle w_i \rangle_{i=1}^{l} = \bigoplus_{k=1}^{l} \langle w_i \rangle_{i \in I_k}$, we have
\[
\dim \langle w_i \rangle_{i \in I} = \sum_{k=1}^{l} \dim \langle w_i \rangle_{i \in I_k \cap I} = \sum_{k=1}^{l} \min \{ \# (I_k \cap I), r_k \}.
\]

Hence $\varpi^{-1}(\{ (I_k, r_k) \}_{k=1}^{l}) \subset \pi^{-1}(\varphi)$. On the hand, for an element $v \in \pi^{-1}(\rho)$, we have $\varpi(v) = \rho \circ \pi(v) = \rho(\varphi) = \{ (I_k, r_k) \}_{k=1}^{l}$, which concludes the converse inclusion. \hfill \Box

4 Description of orbits

From this section, we set $G$ to be the general linear group of the degree 3 over algebraically closed field $k$ with the characteristic 0, and $Q$ be the maximal parabolic subgroup of $G$ such that $G/Q \cong \mathbb{P}^2$.

In Section 4.1, we determine the image of the map $\varpi$ and its subset $\mathcal{P}_{3,4}$ defined in Definition 3.2 and Lemma 3.6. Then we can obtain a description of $G$-orbit decomposition of $\varpi^{-1}(S_{3,4}) \subset (\mathbb{P}^2)^4$ using $G$-invariant fibres of $\varpi$ and $\pi$ from Lemma 3.6 (see Theorem 4.6).

Then in Section 4.2, we describe the orbit decomposition of the $G$-invariant fibre of $\varpi$ on $\text{Image}(\varpi) \setminus \mathcal{P}_{3,4}$ to complete the description of all orbits (see Theorem 4.8).

4.1 Decomposition with the indecomposable splitting

First of all, we describe the image of the map $\varpi: (\mathbb{P}^2)^4 \rightarrow \mathcal{P}_4$ defined in Definition 3.2:

Lemma 4.1. For $\{ (I_k, r_k) \}_{k=1}^{l} \in \mathcal{P}_4$, it is contained in the image of $\varpi: (\mathbb{P}^2)^4 \rightarrow \mathcal{P}_4$ (see Definition 3.2) if and only if the tuple $\{ (\# I_k, r_k) \}_{k=1}^{l}$ is one of the followings:

$\{ (4, 1) \}, \{ (3, 1), (1, 1) \}, \{ (2, 1), (2, 1) \}, \{ (4, 2) \}, \{ (1, 1), (1, 1), (2, 1) \}, \{ (3, 2), (1, 1) \}, \{ (4, 3) \}.$
Proof. From Lemma 3.5, the tuple $\{(k, r_k)\}_{k=1}^l \in \mathcal{P}_4$ is in the image of $\varpi : (\mathbb{P}_2)^4 \rightarrow \text{Map}(2^4, \mathbb{N})$ if and only if $\sum_{k=1}^l r_k \leq 3$ and $r_k = 1$ or $2 \leq r_k \leq \#I_k - 1$ for each $1 \leq k \leq l$. Hence we can classify the image of $\varpi$ as follows:

1. If $l = 1$, then $\#I_1 = 4$ and the possibility of $r_1$ is equal to or less than $4 - 1 = 3$. Hence the possibilities of $\{(\#I_k, r_k)\}_{k=1}^l$ are $\{(4, 3), (4, 2), (4, 1)\}$.

2. If $l = 2$, the possibilities of partitions are $\{(\#I_1, \#I_2) = \{1, 3\}\}$ or $\{(\#I_1, \#I_2) = \{2, 2\}\}$. The first one corresponds to $\{(1, 1), (3, 2)\}$ or $\{(1, 1), (3, 1)\}$. The second one corresponds to $\{(2, 1), (2, 1)\}$.

3. If $l = 3$, the possibilities of partitions are $\{(\#I_1, \#I_2, \#I_3) = \{1, 1, 2\}\}$ hence it corresponds to $\{(1, 1), (1, 1), (2, 1)\}$.

4. If $l \geq 4$, then $\sum_{k=1}^l r_k \geq 4$ contradicts to the condition $\sum_{k=1}^l r_k \leq 3$.

\[\blacksquare\]

Remark 4.2. From the definition of $\mathcal{P}_{3,4}$ in Lemma 3.6, the tuple $\{(k, r_k)\}_{k=1}^l \in \text{Image}(\varpi)$ is in $\mathcal{P}_{3,4}$ if and only if $r_k = 1$ or $2 \leq r_k = \#I_k - 1$ for each $1 \leq k \leq l$. Hence from the classification in Lemma 4.1, we have $\text{Image}(\varpi) \setminus \mathcal{P}_{3,4} = \{\{[4], 2\}\}$.

Definition 4.3. We define maps $\varphi[*] \in \text{Map}(2^4, \mathbb{N})$ as in Table 1, according to the correspondence between $\{(k, r_k)\}_{k=1}^l \in \mathcal{P}_{3,4}$ classified in Lemma 4.1 and maps $2^4 \rightarrow \mathbb{N}$, $I \mapsto \sum_{k=1}^l \min\{\#I_k, r_k\}$.

| $\{(\#I_k, r_k)\}_{k=1}^l$ | $\{I_k\}_{k=1}^l$ | $\rho^{-1}((\#I_k, r_k)_{k=1}^l)$ | representative of the orbit $\pi^{-1}(\varphi[*])$ |
|---------------------------|------------------|-------------------------------|-------------------------------------|
| $\{(4, 1)\}$              | $\{(1, 2, 3, 4)\}$ | $\varphi[2]$                  | $[e_1, e_1, e_1, e_1]$ |
| $\{(1, 1), (3, 1)\}$      | $\{i\}, \{j, k, l\}$ | $\varphi[4; i]$               | $[e_1, e_2, e_2, e_2]$ if $i = 1$ |
| $\{(2, 1), (2, 1)\}$      | $\{i, j\}, \{k, l\}$ | $\varphi[4; i, j]$            | $[e_1, e_1, e_2, e_2]$ if $\{i, j\} = \{1, 2\}$ |
| $\{(1, 1), (1, 1), (2, 1)\}$ | $\{i\}, \{j\}, \{k, l\}$ | $\varphi[6; k, l]$ | $[e_1, e_2, e_3, e_3]$ if $\{k, l\} = \{3, 4\}$ |
| $\{(1, 1), (3, 2)\}$      | $\{i\}, \{j, k, l\}$ | $\varphi[7; i]$               | $[e_1, e_2, e_3, e_2 + e_3]$ if $i = 1$ |
| $\{(4, 3)\}$              | $\{(1, 2, 3, 4)\}$ | $\varphi[8]$                  | $[e_1, e_2, e_3, e_1 + e_2 + e_3]$ |

Table 1: Definition of maps $\varphi[*]$

Example 4.4. For instance, the map $\varphi[7; 4]$ corresponds to the partition $[4] = \{4\} \sqcup [3]$, and we have $\varphi[7; i](I) = \min\{\#(I \cap [4]), 1\} + \min\{\#(I \setminus [4]), 2\}$. In other words, $[v_i]_{i=1}^4$ is in the fibre $\pi^{-1}(\varphi[7; 1])$ if and only if $\{v_2, v_3\}$ is in a general position in a 2-dimensional space and $v_1$ is transversal to it.

Remark 4.5. We can see that $\varphi[4; i, j] = \varphi[4; j, i] = \varphi[4; k, l] = \varphi[4; l, k]$, $\varphi[5; i, j] = \varphi[5; j, i]$, $\varphi[6; i, j] = \varphi[6; j, i]$ where $\{1, 2, 3, 4\} = \{i, j, k, l\}$.

From Lemma 3.6, remark that for all $\rho^{-1}(\{(k, r_k)\}_{k=1}^l) = \varphi[*] \in \text{Map}(2^4, \mathbb{N})$ defined in Table 1, the fibre $\pi^{-1}(\varphi[*]) \subset (\mathbb{P}_2)^4$ is a single $G$-orbit through the element $v \in \varpi^{-1}(\{(k, r_k)\}_{k=1}^l)$ defined in Lemma 3.5 as listed in Table 1. Furthermore, the dimension of the orbit through it is $\dim G - \dim \text{Stab}_v = 3 \sum_{k=1}^l r_k - l$ since an isomorphism $g \in G$ stabilising $v$ has to be the scalar action on each $r_k$-dimensional space $\langle v_i \rangle_{i \in I_k}$. Hence, the dimension of each orbit $\pi^{-1}(\varphi[*])$ is denoted by the number before the semicolon.
Theorem 4.6. For the map $\varpi : (\mathbb{P}^2)^4 \to \mathcal{P}_4$ defined in Definition 3.2, each fibre of $\varpi$ at all elements in $\mathcal{P}_{3,4}' = \text{Image}(\varpi) \setminus \{(4,2)\}$ is a single $G$-orbit coincides with the fibre $\pi^{-1}(\varphi[*])$ listed in Table 1. Furthermore, representatives of these orbits are also listed in Table 1.

4.2 Description of infinitely many orbits

In Section 4.1, we described the decomposition of the multiple projective space $(G/Q)^4 \cong (\mathbb{P}^2)^4$ into a finite number of $G$-invariant fibres of $\varpi : (\mathbb{P}^2)^4 \to \mathcal{P}_4$ defined in Definition 3.2. Furthermore, for the subset $\mathcal{P}_{3,4}' \subset \text{Image}(\varpi)$ defined in Lemma 3.6, each fibre of $\varpi$ on it was single $G$-orbit (see Theorem 4.6). On the other hand, the fibre of $\{(4,2)\}$, which is the only element in $\text{Image}(\varpi) \setminus \mathcal{P}_{3,4}'$, is not. In this section, we describe the $G$-orbit decomposition of this extra fibre. For this aim, we define some notations as follows:

Definition 4.7. (1) The subset $(\mathbb{P}^1)'$ of $\mathbb{P}^1$ denotes $\mathbb{P}^1 \setminus \{[e_1], [e_2], [e_1 + e_2]\}$.

(2) Define the map $\varphi[6], \varphi[5; i, j] \in \text{Map}(2^{[4]}, \mathbb{N})$ for $1 \leq i < j \leq 4$ as follows:

$$\varphi[6] : I \mapsto \min\{\#I, 2\}, \quad \varphi[5; i, j] : I \mapsto \min\{\#(I/i \sim j), 2\}.$$

(3) For $p = [p_1 e_1 + p_2 e_2] \in \mathbb{P}^1$, we define a subset of $(\mathbb{P}^2)^4$ by

$$\mathcal{O}(5; p) : = \{ [v_i]_{i=1}^4 \in (\mathbb{P}^2)^4 \mid [v_i] \neq [v_2], \ v_3 = v_1 + v_2, \ v_4 = p_1 v_1 + p_2 v_2 \}.$$

Consider an element $v = [v_i]_{i=1}^4 \in (\mathbb{P}^2)^4$, then it lies in the fibre of $\{(4,2)\}$ if and only if $\{v_i\}_{i=1}^4$ spans 2-dimensional space and is indecomposable. If $\text{dim}(v_i)_{i=1}^4 = 2$, there exists a pair $\{v_i, v_j\}$ which forms a basis of this space. Then $\{v_i, v_j, v_k, v_l\}$ is indecomposable if and only if either $v_k$ or $v_l$ is contained in neither $\langle v_i \rangle$ nor $\langle v_j \rangle$ where $\{i, j, k, l\} = [4]$. Hence, $\varpi(v) = \{(4,2)\}$ if and only if its rank is 2 and there exists a fibre of $\{[v_i]\}_{i=1}^4$ which are distinct.

Let $\{[v_1], [v_2], [v_3]\}$ be a distinct triple, then there is an isomorphism $g \in GL_3$ sending it to $\{[e_1], [e_2], [e_1 + e_2]\}$. Then the line $[v_4]$ is sent to some $p = [p_1 e_1 + p_2 e_2] \in \mathbb{P}^1$. Remark that $p$ is independent of the choice of $G$-action since $G$-action stabilising $[e_1, e_2, e_1 + e_2]$ has to be a scalar action on this 2-dimensional space.

Now if $p \in (\mathbb{P}^1)'$, then we have $\pi(p) = \varphi[6]$. On the other hand, $\pi(p) = \varphi[5; 1, 4], \varphi[5; 2, 4]$, or $\varphi[5; 3, 4]$ if $p = [e_1], [e_2], [e_1 + e_2]$ with respectively. By considering other distinct triples, all $\varphi[5; *]$ can be covered, and we obtain the following:

Theorem 4.8. For the maps $\varpi : (\mathbb{P}^2)^4 \xrightarrow{\pi} \text{Map}(2^{[4]}, \mathbb{N}) \xrightarrow{\rho} \mathcal{P}_4$ defined in Definition 3.2, (3.1), and Lemma 3.4,

(1) for the only element $\{(4,2)\}$ in $\text{Image}(\varpi) \setminus \mathcal{P}_{3,4}'$, the fibre of $\rho|_{\text{Image}(\pi)}$ is fulfulled with the maps $\varphi[6]$ and $\varphi[5; i, j]$ (1 $\leq i < j \leq 4$) defined in Definition 4.7.

(2) Each fibre $\pi^{-1}(\varphi[5; i, j])$ is a single $G$-orbit through $\{e_1, e_1, e_2, e_1 + e_2\}$ (if $\{i, j\} = \{1, 2\}$), and the fibre $\pi^{-1}(\varphi[6])$ is decomposed into $G$-orbits as follows:

$$\pi^{-1}(\varphi[6]) = \coprod_{p \in (\mathbb{P}^1)'} \mathcal{O}(5; p) = \coprod_{p \in (\mathbb{P}^1)'} G \cdot [e_1, e_2, e_1 + e_2, p].$$
5 Closure relations among orbits

In this section, we determine the closure relations among $GL_3$-orbits on $(\mathbb{P}^2)^4$ and $GL_3$-invariant fibres described in Theorems 4.6 and 4.8.

For an integer $d$, consider the $\mathbb{K}$-submodule $\mathcal{H}^d_n$ of the $n$-variable polynomial ring consisting of all homogeneous polynomials of the degree $d$. Then the polynomial ring $Pol(M(n,m))$ over $n \times m$-matrices admits a $\mathbb{N}^m$-graded structure as $Pol(M(n,m)) = \bigoplus_{d \in \mathbb{N}^m} \mathcal{H}^d_{n,m}$ where $\mathcal{H}^{(d_1,d_2,\ldots,d_m)}_{n,m} := \mathcal{H}^{d_1}_n \otimes \mathcal{H}^{d_2}_n \otimes \cdots \otimes \mathcal{H}^{d_m}_n$. Then closed subsets in $(\mathbb{P}^{n-1})^m$ correspond to the zero-point sets $Z(I)$ of homogeneous ideals $I = \bigoplus I \cap \mathcal{H}^d_{n,m}$ of $Pol(M(n,m))$. In particular, the irreducibility of $Z(I)$ also corresponds to whether $I$ is primary or not.

**Definition 5.1.** For $\varphi, \psi \in \text{Map}(2^{[m]}, \mathbb{N})$, we say that $\varphi \leq \psi$ if $\varphi(I) \leq \psi(I)$ for all $I \subset [m]$.

With this notation, we state the result on the closure relations among $GL_3$-invariant fibres described in Theorems 4.6 and 4.8.

**Proposition 5.2.** For $\varphi \in \text{Image}(\pi) \subset \text{Map}(2^{[4]}, \mathbb{N})$, we have $\overline{\pi^{-1}(\varphi)} = \bigsqcup_{\psi \leq \varphi} \pi^{-1}(\psi)$.

**Proof.** For $\varphi \in \text{Map}(2^{[m]}, \mathbb{N})$, we have

$$\bigsqcup_{\psi \leq \varphi} \pi^{-1}(\psi) = \{ [v] \in (\mathbb{P}^{n-1})^m | \dim(v_i)_{i \in I} \leq \varphi(I) \text{ for all } I \subset [m] \} = \{ [v] \in (\mathbb{P}^{n-1})^m | \text{ all } (\varphi(I)+1)\text{-minors of } (v_i)_{i \in I} \text{ vanish for all } I \subset [m] \},$$

$$\pi^{-1}(\varphi) = \{ [v] \in (\mathbb{P}^{n-1})^m | \dim(v_i)_{i \in I} = \varphi(I) \text{ for all } I \subset [m] \} = \{ [v] \in (\mathbb{P}^{n-1})^m | \text{ there exists some non vanishing } (\varphi(I)-\text{minor of } (v_i)_{i \in I} \text{ for all } I \subset [m] \} \cap \bigsqcup_{\psi \leq \varphi} \pi^{-1}(\psi).$$

From this characterisation, $\bigsqcup_{\psi \leq \varphi} \pi^{-1}(\psi)$ is a closed subset defined by the homogeneous ideal generated by all $(\varphi(I)+1)$-minors consisting of columns contained in $I$, and $\pi^{-1}(\varphi)$ is open in it. Hence we have $\pi^{-1}(\varphi) \subset \bigsqcup_{\psi \leq \varphi} \pi^{-1}(\psi)$. Furthermore, since $\pi^{-1}(\varphi)$ is not empty for $\varphi \in \text{Image}(\pi)$, we can prove the converse inclusion by checking the irreducibility of $\bigsqcup_{\psi \leq \varphi} \pi^{-1}(\psi)$.

Now, let $X_{n,m}(r) \subset (\mathbb{P}^{n-1})^m$ denote the irreducible closed subset consisting of all matrices whose rank is less than or equal to $r$. Then for the maps $\varphi = \varphi[8], \varphi[7;i], \varphi[6;i,j], \varphi[6], \varphi[5;i,j], \varphi[4;*], \varphi[2]$ in $\text{Image}(\pi)$ introduced in Definitions 4.3 and 4.7, the closed subsets $\bigsqcup_{\psi \leq \varphi} \pi^{-1}(\psi)$ is naturally identified with $X_{3,4}(3), X_{3,3}(2) \times X_{3,1}(1), X_{3,3}(3), X_{3,4}(2), X_{3,3}(2), X_{3,2}(2),$ and $X_{3,1}(1)$ respectively. Hence they are all irreducible and the claim holds.

According to Theorems 4.6 and 4.8, the $GL_3$-invariant fibres $\pi^{-1}(\varphi)$ are all $GL_3$-orbits by themselves unless $\varphi = \varphi[6]$. On the other hand, $\pi^{-1}(\varphi[6])$ is decomposed in infinitely many orbits $\mathcal{O}(5;p)$ introduced in Definition 4.7. Hence in the next, we shall determine the closure of $\mathcal{O}(5;p)$, which completes the determination of all closure relations among orbits.

**Proposition 5.3.** For the orbits $\mathcal{O}(5;p) := GL_3 \cdot ([e_1],[e_2],[e_1+e_2],p)$, $\pi^{-1}(\varphi[4;i])$, and $\pi^{-1}(\varphi[2])$ where $p \in (\mathbb{P}^1)'$ and $1 \leq i \leq 4$ (see Definitions 4.3 and 4.7), we have

$$\overline{\mathcal{O}(5;p)} = \mathcal{O}(5;p) \bigsqcup \bigoplus_{i=1}^4 \pi^{-1}(\varphi[4;i]) \bigsqcup \pi^{-1}(\varphi[2]).$$
Lemma 5.4. For an linear inclusion \( \iota : \mathbb{K}^r \rightarrow \mathbb{K}^n \), the map below is a closed embedding.

\[
\tilde{\iota} : GL_r(\mathbb{P}^{r-1})^m \rightarrow GL_n(\mathbb{P}^{n-1})^m : [v_i]_{i=1}^m \mapsto [\iota(v_i)]_{i=1}^m.
\]

Proof. Since \( g \in GL_r \) defines a linear isomorphism in \( \text{Image}(\iota) \), there exists a linear isomorphism \( \tilde{g} \in GL_n \) such that \( \iota \circ g = \tilde{g} \circ \iota \) by taking the direct product with some linear isomorphism on the complement space. Hence the map \( \tilde{\iota} \) is well-defined. On the other hand, for a linear isomorphism \( \tilde{g} \in GL_n \) which satisfies \( \tilde{g} \cdot [\iota(v_i)]_{i=1}^m = [\tilde{g}(w_i)]_{i=1}^m \), it defines a linear isomorphism between \( \langle v_i \rangle_{i=1}^m \) and \( \langle w_i \rangle_{i=1}^m \) sending \( [v_i] \) to \([w_i]\) via \( \tilde{\iota} \). So there is an isomorphism \( g \in GL_r \) satisfying \( g[v_i] = [w_i] \). Hence the map \( \tilde{\iota} \) is injective.

Since the continuity is obvious, we shall only show that the map is closed. For a closed subset \( C \subseteq GL_r(\mathbb{P}^{r-1})^m \), there exists a \( GL_r \)-invariant zero point set \( Z(I) \subset (\mathbb{P}^{r-1})^m \) such that \( C = GL_r \setminus Z(I) \) where \( I \) is a homogeneous ideal in \( \text{Pol}(M(r, m)) \). Then we have \( \tilde{\iota}(C) = GL_n \setminus GL_n \cdot \iota(Z(I)) \).

Now we define a homogeneous ideal \( J \subset \text{Pol}(M(n, m)) \) generated by homogeneous polynomials \( f \circ \pi \) for all \( f \in I \) and linear maps \( p : \mathbb{K}^n \rightarrow \mathbb{K}^r \).

Let \( v = [v_i]_{i=1}^m \in Z(J) \cap X_{n,m}(r) \) where \( X_{n,m}(r) \) is a closed subset of \( (\mathbb{P}^{n-1})^m \) consisting of all elements with the rank less than \( r + 1 \). Then there exists a linear isomorphism \( g \in GL_n \) sending \( v \) into the \( r \)-dimensional space \( \text{Image}(\iota) \). Hence, taking some linear projection \( p : \mathbb{K}^n \rightarrow \mathbb{K}^r \) such that \( \iota p \) is the identity map on \( \text{Image}(\iota) \), we have \( v = g^{-1}gv = g^{-1}pGV \epsilon GL_n \cdot \iota(pgv), \; \text{and} \; f(pgv) = f \circ (pg)(v) = 0 \) for all \( f \in I \). Hence \( v \in GL_n \cdot \iota(Z(I)) \).

On the other hand, let \( v \in Z(I) \). Since \( Z(I) \) is \( GL_r \)-invariant, \( f(gv) = 0 \) for all \( g \in GL_r \). Hence \( f(xv) = 0 \) holds for all linear endomorphisms \( x \) on \( \mathbb{K}^r \), which leads that \( f \circ p(\pi(v)) = f(pGv) = 0 \) for all \( p : \mathbb{K}^n \rightarrow \mathbb{K}^r \). Hence \( \iota(Z(I)) \subset Z(J) \).

From these arguments, we have shown that \( GL_n \cdot \iota(Z(I)) = Z(J) \cap X_{n,m}(r) \), which leads that the map \( \tilde{\iota} \) is closed.

From this lemma, we only have to determine the closure of \( GL_2 \cdot [e_1, e_2, e_1 + e_2, p] = \tilde{\iota}^{-1}(O(5; p)) \subset (\mathbb{P}^1)^4 \). Remark that \( \tilde{\iota} \) intertwines the map \( \pi \). For this aim, we introduce the following notation: for a \( 2 \times 4 \)-matrix, \( [i, j] \) denotes the \( 2 \)-minor with the \( i \) and \( j \)-th columns. Then we define a homogeneous polynomial \( P_p \in \mathcal{H}_{2,4}^{(1,1,1,1)} \) for \( p = [p_1 e_1 + p_2 e_2] \in \mathbb{P}^1 \) by

\[
P_p(x) := p_1 |1, 2|4, 2| + p_1 |1, 4|2, 3| = -p_1 |1, 2|3, 4| + (p_1 - p_2) |1, 4|2, 3|.
\]

Remark that the equalities hold since \( |1, 2|3, 4| + |1, 3|4, 2| + |1, 4|2, 3| = 0 \) from the general property of determinants. Considering the \( S_4 \)-action on the columns, we have

\[
P_p((1, 2)x) = P_{(-p_2, -p_1)}(x), \; P_p((2, 3)x) = P_{(p_2 - p_1, p_2)}(x), \; P_p((3, 4)x) = P_{(-p_2, -p_1)}(x).
\]

Hence, for the group homomorphism \( \Psi : S_4 \rightarrow GL_2 \) generated by \( (1, 2), (3, 4) \mapsto (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}) \) and \( (2, 3) \mapsto (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}) \), we have

\[
P_p(\sigma^{-1}x) = P_{\Psi(\sigma)p}(x) \; \text{for} \; \sigma \in S_4.
\]

Remark that the kernel of \( \Psi \) is the Klein 4-group \( \{\text{id}, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\} \), and the image of \( \Psi \) acts on the 3 points \( \{0, 1, \infty\} \subset \mathbb{P}^1 \) effectively, which is isomorphic to \( S_3 \).
Lemma 5.5. The polynomial $P_p$ is irreducible if and only if $p \in (\mathbb{P}^1)'$ (see Definition 4.7).

Proof. Remark that $p$ is in $(\mathbb{P}^1)'$ if and only if $p_1 p_2 (p_1 - p_2) \neq 0$. Since the polynomial $P_p$ is of the homogeneous degree $(1, 1, 1, 1)$, we only have to check that it does not have any divisors with the homogeneous degree neither $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, nor $(1, 0, 0, 1)$.

By a direct computation, we have

$$P(x) = x_{11} x_{12} ((p_1 - p_2) x_{23} x_{24}) + x_{21} x_{22} ((p_1 - p_2) x_{13} x_{14})$$

$$+ x_{11} x_{12} (p_1 x_{13} x_{24} - p_2 x_{23} x_{14}) - x_{21} x_{12} (p_2 x_{13} x_{24} - p_1 x_{23} x_{14}).$$

Hence it is divided by a polynomial of the homogeneous degree $(1, 1, 0, 0)$ if and only if $p_1 = p_2$. Similarly, since $P_p((2, 3)|x) = P_{(p_2 - p_1, p_1)}(x)$ and $P_p((2, 4)|x) = P_{(p_1, p_1 - p_2)}(x)$, $P_p(x)$ is divided by a polynomial of the homogeneous degree $(1, 0, 1, 0)$ (resp. $(1, 0, 0, 1)$) if and only if $p_1 = 0$ (resp. $p_2 = 0$).

On the other hand, let $P_p(x)$ has a divisor $f(x_1)$ of the homogeneous degree $(1, 0, 0, 0)$. Since $P_p(x)$ is fixed under the action of $(1, 2)(3, 4)$, $(1, 3)(4, 2)$, and $(1, 4)(2, 3)$, it is factorised as $P_p(x) = c f(x_1) f(x_2) f(x_3) f(x_4)$, it can occur only if $p_1 p_2 (p_1 - p_2) = 0$ from the argument above.

Next, we observe the relationship between the polynomial $P_p$ introduced in (5.1) and the orbit $\text{GL}_2 \cdot [e_1, e_2, e_1 + e_2, p]$:

Lemma 5.6. For $v_p = [e_1, e_2, e_1 + e_2, p] \in (\mathbb{P}^1)^4$ and the homogeneous polynomial $P_p$ introduced in (5.1), we have the following:

$$\text{GL}_2 \cdot v_p = \left\{[v] \in (\mathbb{P}^1)^4 \mid |1, 2| |2, 3| |3, 1| \neq 0, P_p(v) = 0 \right\}. \quad (5.3)$$

Proof. First of all, $P_p(x)$ is relatively $\text{GL}_2$-invariant since all minors are relatively $\text{GL}_2$-invariant with the character $g \mapsto (\det g)^2$. Let $X_p \subset (\mathbb{P}^1)^4$ be the right-hand side of (5.3), then the orbit $\text{GL}_2 \cdot v_p$ is included in $X_p$ clearly from $v_p \in X_p$ and the $\text{GL}_2$-invariance of $P_p$.

Conversely, for $[v] = [v_1, v_2, v_3, v_4] \in X_p$, then $\{v_1, v_2\}$ is linearly independent and

$$|2, 3| v_1 + |3, 1| v_2 = -|1, 2| v_3$$

$$|1, 4| (p_1 |2, 3| v_1 + p_2 |3, 1| v_2) = p_1 |1, 4| |2, 3| v_1 + p_2 |1, 4| |3, 1| v_2$$

$$= -p_2 |1, 3| |4, 2| v_1 + p_2 |1, 4| |3, 1| v_2 = -p_2 |3, 1| |2, 4| v_1 + |4, 1| v_2$$

$$= p_2 |3, 1| |1, 2| v_4$$

Hence $v = (|2, 3| v_1 |3, 1| v_2) \cdot [e_1, e_2, e_1 + e_2, p] \in \text{GL}_2 \cdot v_p$. 

From Lemmas 5.5 and 5.6, the orbit $\text{GL}_2 \cdot v_p$ is open in the irreducible closed subset $Z(P_p)$ where $p \in (\mathbb{P}^1)'$. Hence we have $\text{GL}_2 \cdot v_p = Z(P_p)$, and shall only determine the set $Z(P_p)$.

Lemma 5.7. Let $p \in (\mathbb{P}^1)'$ and $v_p := [e_1, e_2, e_1 + e_2, p] \in \mathcal{O}_p$. Then the zero point set of the polynomial $P_p$ defined in (5.1) is given as follows:

$$Z(P_p) = \text{GL}_2 \cdot v_p \coprod_{i=1}^4 \pi^{-1}(\varphi[4; i]) \coprod \pi^{-1}(\varphi[2]). \quad (5.4)$$
Proof. We already have classified orbits in $GL_2((\mathbb{P}^1)^4 \hookrightarrow GL_3((\mathbb{P}^2)^4$ in Theorems 4.6 and 4.8. Hence, we shall only determine whether each of them is contained in $Z(P_p)$ or not.

If any 2-minors do not vanish, we only have to consider the orbit $GL_2 \cdot v_q = i^{-1}(\mathcal{O}(5; q))$ where $q \in (\mathbb{P}^1)^{\prime}$. Then since $P_p(v_q) = p_2q_1 - p_1q_2$, it is contained in $Z(P_p)$ if and only if $p = q$.

On the other hand, consider the case where exists some vanishing 2-minor. Since the polynomial $P_p$ is of the form $c_1[i, j][k, l] + c_2[i, k][l, j]$ for some $c_1c_2(c_1 - c_2) \neq 0$, if $P_p(v) = 0$, then there has to be a triple of columns of $v$ whose 2-minors all vanish. Conversely, if there exists a triple of columns whose 2-minors all vanish, then clearly $P_p(v) = 0$.

Combining these results, we completed the proof of the closure relations in Theorem B.

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