WIENER ESTIMATES ON MODULATION SPACES

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Abstract. We characterise modulation spaces by suitable Wiener estimates on the short-time Fourier transforms of the involved functions and distributions. We use the results to refine some formulae on periodic distributions with Lebesgue estimates on their coefficients.

0. Introduction

In the paper we characterise Gelfand-Shilov spaces of functions and distributions, modulation spaces and Gevrey classes in background of various kinds of Wiener estimates. We apply the results to deduce some refined formulae on periodic functions and distributions, given in [24].

Essential motivations arised in [24] on characterizations of certain spaces of periodic functions and distributions. In fact, it follows from [24] that if \( q \in (0, \infty] \) and \( f \) is a \( 2\pi \)-periodic Gelfand-Shilov distribution on \( \mathbb{R}^d \) with Fourier coefficients \( c(f, \alpha), \alpha \in \mathbb{Z}^d \), then

\[
\{c(f, \alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell^q \iff f \in M^{\infty,q}.
\] (0.1)

Here \( M^{\infty,q} \) is the (unweighted) modulation spaces with Lebesgue parameters \( \infty \) and \( q \). (See Section 1 or [24] for notations.) We note that a proof of (0.1) in the case \( q \in [1, \infty] \) can be found in e. g. [21], and with some extensions in [19].

An alternative formulation of (0.1) is

\[
\{c(f, \alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell^q \iff \xi \mapsto \|V_\phi f(\cdot, \xi)\|_{L^q(\mathbb{R}^d)} \in L^q.
\] (0.1)′

By observing that periodicity of \( f \) induce the same periodicity for \( x \mapsto |V_\phi f(x, \xi)| \), it follows that (0.1)′ is the same as

\[
\{c(f, \alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell^q \iff \xi \mapsto \|V_\phi f(\cdot, \xi)\|_{L^q([0,2\pi]^d)} \in L^q.
\] (0.1)″

In Section 2 we show that the latter equivalence hold true with \( L^r([0,2\pi]^d) \) norm in place of \( L^\infty([0,2\pi]^d) \) norm for every \( r \in (0, \infty] \). That is, we improve (0.1)″ into

\[
\{c(f, \alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell^q \iff \xi \mapsto \|V_\phi f(\cdot, \xi)\|_{L^r([0,2\pi]^d)} \in L^q.
\] (0.1)‴

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In particular, if $q < \infty$ and choosing $q = r$, then we obtain

$$\sum_{\alpha \in \mathbb{Z}^d} |c(f, \alpha)|^q < \infty \Leftrightarrow \int_{[0,2\pi]^d \times \mathbb{R}^d} |V_\phi f(x, \xi)|^q dx d\xi < \infty. \tag{0.2}$$

More generally, we deduce weighted versions of these identities. Since our weights include general moderate weights which are allowed possess exponential types growth and decays, we formulate our results in the framework of Gelfand-Shilov spaces of functions and distributions.

The improved equivalence (0.1)'' can in the case $q, r \in [1, \infty]$ be obtained from (0.1)'' by a suitable combination of Hölder’s and Young’s inequalities and the inequality

$$F(X) \lesssim \int_{\Omega} \Phi(X - Y) F(Y) dY, \quad X \in \Omega = [0, 2\pi]^d \times \mathbb{R}^d, \tag{0.3}$$

where

$$\Phi(x, \xi) = \sum_{k \in \mathbb{Z}^d} |V_\phi(x - 2\pi k, \xi)| \quad \text{and} \quad F(X) = |V_\phi f(X)|,$$

which follows from Lemma 1.3.3 in [12] for $2\pi$-periodic distributions $f$. It follows that this case can be handled by straight-forward modifications of the methods that are used when establishing basic results for classical modulation spaces in [3] and in Chapter 11 in [12].

In our situation, the parameters $q$ and $r$ are, more generally, allowed to belong to the full interval $(0, \infty]$ instead of $[1, \infty]$. The classical approaches in [3, 5, 6, 12] are then insufficient because they require convex structures in the topology of the involved vector spaces. This convexity is absent when $q < 1$ or $r < 1$.

We manage our more general situation by using techniques based on ideas in [9, 17, 18, 22] and which can handle Lebesgue and Wiener spaces which are quasi-Banach spaces but may fail to be Banach spaces. Especially we shall follow a main idea in [9, 22] and replace the usual convolution, used in [3, 5, 6, 12], by a semi-continuous version which is less sensitive when convexity is lacking in the topological structures.

For the semi-continuous convolution we deduce in Section 2 the needed Lebesgue and Wiener estimates. In the end we achieve in Section 2 various types of characterizations of modulation spaces in terms of Wiener norm estimates on the short-time Fourier transforms of the functions and (ultra-)distributions under considerations. For example, as special case of Propositions 1.15 after Proposition 2.4 we have for $p, q, r \in (0, \infty]$ that

$$\|f\|_{M_{p,q}} \asymp \|a\|_{\ell_{p,q}} \quad \text{when} \quad a(j) = \|V_\phi f\|_{L^r(j+[0,1]^d)} \tag{0.4}$$

Similar facts hold true for those Wiener amalgam spaces which are Fourier images of modulation spaces of the form $M_{p,q}$. In particular our results can be used to deduce certain invarians properties concerning
the choice of local component in the Wiener amalgam quasi-norm. (See also Proposition 2.6.) Here we remark that for Wiener amalgam spaces which at the same time are Banach spaces, the approaches are often less complicated and there are several examples on other Banach spaces (e.g. suitable modulation spaces) to furnish the local component in the Wiener amalgam norms. (See e.g. [7, 8] and the references therein.)

We also present some applications on periodic elements which gives (0.1)" and (0.2) as special cases. (See Propositions 2.7 and 1.18.)

The Wiener spaces under considerations can also be described in terms of coorbit spaces, whose general theory was founded by Feichtinger and Gröchenig in [5, 6] and further developed in different ways, e.g. by Rauhut in [17, 18]. Since our investigations in Section 2 concern quasi-Banach spaces which may fail to be Banach spaces, our investigations are especially linked to Rauhut’s analysis in [17, 18]. In this context, a part of our analysis on modulation spaces can be formulated as coorbit norm estimates of short-time Fourier transforms with local component in $L^r$-spaces with $r \in (0, \infty]$ and global component in other Lebesgue spaces. Proposition 1.15 in Section 2 then shows that different choices of $r$ give rise to equivalent norm estimates on short-time Fourier transforms. Again we remark that if $r$ belongs to the subset $[1, \infty]$ of $(0, \infty]$ and that all involved spaces are Banach spaces, then our results can be obtained in other less complicated ways, given in e.g. Chapters 11 and 12 in [12].

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1. Preliminaries

In this section we recall some basic facts. We start by discussing Gelfand-Shilov spaces and their properties. Thereafter we recall some properties of modulation spaces and discuss different aspects of periodic distributions.

1.1. Gelfand-Shilov spaces and Gevrey classes. Let $0 < s, \sigma \in \mathbb{R}$ be fixed. Then the Gelfand-Shilov space $S^s_\sigma(\mathbb{R}^d)$ ($\Sigma^s_\sigma(\mathbb{R}^d)$) of Roumieu type (Beurling type) with parameters $s$ and $\sigma$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$
\|f\|_{S^s_\sigma} \equiv \sup_{h \in \mathbb{N}^d} \frac{|x^\alpha \partial^\beta f(x)|}{h! |\alpha|! |\beta|! \sigma}
$$

(1.1)
is finite for some $h > 0$ (for every $h > 0$). Here the supremum should be taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. We equip $S^s_\sigma(\mathbb{R}^d)$ ($\Sigma^s_\sigma(\mathbb{R}^d)$) by
the canonical inductive limit topology (projective limit topology) with respect to \( h > 0 \), induced by the semi-norms in (1.1).

The Gelfand-Shilov distribution spaces \((\mathcal{S}_a^\sigma)'(\mathbb{R}^d)\) and \((\Sigma_a^\sigma)'(\mathbb{R}^d)\) are the dual spaces of \(\mathcal{S}_a^\sigma(\mathbb{R}^d)\) and \(\Sigma_a^\sigma(\mathbb{R}^d)\), respectively. As for the Gelfand-Shilov spaces there is a canonical projective limit topology (projective limit topology) for \((\mathcal{S}_a^\sigma)'(\mathbb{R}^d)\) \((\Sigma_a^\sigma)'(\mathbb{R}^d)\). (Cf. [10, 13, 16] For convenience we set

\[
\mathcal{S}_s = \mathcal{S}_s^s, \quad \mathcal{S}_s' = (\mathcal{S}_s^s)', \quad \Sigma_s = \Sigma_s^s \quad \text{and} \quad \Sigma_s' = (\Sigma_s^s)'.
\]

From now on we let \( \mathcal{F} \) be the Fourier transform which takes the form

\[
(\mathcal{F} f)(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)e^{-i(x,\xi)} \, dx
\]

when \( f \in L^1(\mathbb{R}^d) \). Here \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product on \( \mathbb{R}^d \). The map \( \mathcal{F} \) extends uniquely to homeomorphisms on \( \mathcal{S}(\mathbb{R}^d) \), from \((\mathcal{S}_s^\sigma)'(\mathbb{R}^d)\) to \(\mathcal{S}_{\sigma}(\mathbb{R}^d)\) and from \((\Sigma_a^\sigma)'(\mathbb{R}^d)\) to \(\Sigma_{\sigma}(\mathbb{R}^d)\). Furthermore, \( \mathcal{F} \) restricts to homeomorphisms on \( \mathcal{S}_a(\mathbb{R}^d) \), from \(\mathcal{S}_a^\sigma(\mathbb{R}^d)\) to \(\mathcal{S}_{\sigma}(\mathbb{R}^d)\) and from \(\Sigma_a^\sigma(\mathbb{R}^d)\) to \(\Sigma_{\sigma}(\mathbb{R}^d)\), and to a unitary operator on \( L^2(\mathbb{R}^d) \).

Next we consider Gevrey classes on \( \mathbb{R}^d \). Let \( \sigma \geq 0 \). For any compact set \( K \subseteq \mathbb{R}^d \), \( h > 0 \) and \( f \in C^\infty(K) \) let

\[
\| f \|_{K, h, \sigma} = \sup_{\alpha \in \mathbb{N}^d} \left( \frac{\| \partial^\alpha f \|_{L^\infty(K)}}{h^{\alpha_1+1} \alpha!} \right). \quad (1.2)
\]

The Gevrey class \( \mathcal{E}_\sigma(K) \) \((\mathcal{E}_{0,\sigma}(K))\) of order \( \sigma \) and of Roumieu type (of Beurling type) is the set of all \( f \in C^\infty(K) \) such that (1.2) is finite for some (for every) \( h > 0 \). We equipp \( \mathcal{E}_\sigma(K) \) \((\mathcal{E}_{0,\sigma}(K))\) by the inductive (projective) limit topology with respect to \( h > 0 \), supplied by the seminorms in (1.2). Finally if \( \{K_j\}_{j \geq 1} \) is an exhausted sets of compact subsets of \( \mathbb{R}^d \), then let

\[
\mathcal{E}_\sigma(\mathbb{R}^d) = \operatorname{proj \, lim} \mathcal{E}_\sigma(K_j) \quad \text{and} \quad \mathcal{E}_{0,\sigma}(\mathbb{R}^d) = \operatorname{proj \, lim} \mathcal{E}_{0,\sigma}(K_j).
\]

In particular,

\[
\mathcal{E}_\sigma(\mathbb{R}^d) = \bigcap_{j \geq 1} \mathcal{E}_\sigma(K_j) \quad \text{and} \quad \mathcal{E}_{0,\sigma}(\mathbb{R}^d) = \bigcap_{j \geq 1} \mathcal{E}_{0,\sigma}(K_j).
\]

It is clear that \( \mathcal{E}_{0,0}(\mathbb{R}^d) \) contains all constant functions on \( \mathbb{R}^d \), and that \( \mathcal{E}_0(\mathbb{R}^d) \setminus \mathcal{E}_{0,0}(\mathbb{R}^d) \) contains all non-constant trigonometric polynomials.

1.2. Ordered, dual and phase split bases. Our discussions involving Zak transforms, periodicity, modulation spaces and Wiener spaces are done in terms of suitable bases.

**Definition 1.1.** Let \( E = \{e_1, \ldots, e_d\} \) be an ordered basis of \( \mathbb{R}^d \). Then \( E' \) denotes the basis of \( e'_1, \ldots, e'_d \) in \( \mathbb{R}^d \) which satisfies

\[
\langle e_j, e'_k \rangle = 2\pi \delta_{jk} \quad \text{for every} \quad j, k = 1, \ldots, d.
\]
The corresponding lattices are given by
\[ \Lambda_E = \{ n_1 e_1 + \cdots + n_d e_d; (n_1, \ldots, n_d) \in \mathbb{Z}^d \}, \]
and
\[ \Lambda'_E = \Lambda'_E = \{ \nu_1 e'_1 + \cdots + \nu_d e'_d; (\nu_1, \ldots, \nu_d) \in \mathbb{Z}^d \}. \]
The sets \( E' \) and \( \Lambda'_E \) are called the dual basis and dual lattice of \( E \) and \( \Lambda_E \), respectively. If \( E_1, E_2 \) are ordered bases of \( \mathbb{R}^d \) such that a permutation of \( E_2 \) is the dual basis for \( E_1 \), then the pair \( (E_1, E_2) \) are called permuted dual bases (to each others on \( \mathbb{R}^d \)).

**Remark 1.2.** Evidently, if \( E \) is the same as in Definition 1.1, then there is a matrix \( T_E \) with \( E \) as the image of the standard basis in \( \mathbb{R}^d \). Then \( E' \) is the image of the standard basis under the map \( T_E' = 2\pi(T_E^{-1})^t \).

Two ordered bases on \( \mathbb{R}^d \) can be used to construct a uniquely defined ordered basis for \( \mathbb{R}^{2d} \) as in the following definition.

**Definition 1.3.** Let \( E_1, E_2 \) be ordered bases of \( \mathbb{R}^d \),
\[ V_1 = \{ (x, 0) \in \mathbb{R}^{2d}; x \in \mathbb{R}^d \}, \quad V_2 = \{ (0, \xi) \in \mathbb{R}^{2d}; \xi \in \mathbb{R}^d \} \]
and let \( \pi_j \) from \( \mathbb{R}^{2d} \) to \( \mathbb{R}^d \), \( j = 1, 2 \), be the projections
\[ \pi_1(x, \xi) = x \quad \text{and} \quad \pi_2(x, \xi) = \xi. \]
Then \( E_1 \times E_2 \) is the ordered basis \( \{e_1, \ldots, e_{2d}\} \) of \( \mathbb{R}^{2d} \) such that
\[ \{e_1, \ldots, e_d\} \subseteq V_1, \quad E_1 = \{\pi_1(e_1), \ldots, \pi_1(e_d)\}, \]
\[ \{e_{d+1}, \ldots, e_{2d}\} \subseteq V_2 \quad \text{and} \quad E_2 = \{\pi_2(e_{d+1}), \ldots, \pi_2(e_{2d})\}. \]
In the phase space it is convenient to consider phase split bases, which are defined as follows.

**Definition 1.4.** Let \( V_1, V_2, \pi_1 \) and \( \pi_2 \) be as in Definition 1.3 \( E \) be an ordered basis of the phase space \( \mathbb{R}^{2d} \) and let \( E_0 \subseteq E \). Then \( E \) is called phase split (with respect to \( E_0 \)), if the following is true:

1. the span of \( E_0 \) and \( E \setminus E_0 \) equal \( V_1 \) and \( V_2 \), respectively;
2. let \( E_1 = \pi_1(E_0) \) and \( E_2 = \pi_2(E \setminus E_0) \) be the bases in \( \mathbb{R}^d \) which preserves the orders from \( E_0 \) and \( E \setminus E_0 \). Then \( (E_1, E_2) \) are permuted dual bases.

If \( E \) is a phase split basis with respect to \( E_0 \) and that \( E_0 \) consists of the first \( d \) vectors in \( E \), then \( E \) is called strongly phase split (with respect to \( E_0 \)).

In Definition 1.4 it is understood that the orderings of \( E_0 \) and \( E \setminus E_0 \) are inherited from the ordering in \( E \).
Remark 1.5. Let $E$ and $E_j$, $j = 0, 1, 2$ be the same as in Definition 1.4. It is evident that $E_0$ and $E \setminus E_0$ consist of $d$ elements, and that $E_1$ and $E_2$ are uniquely defined. The pair $(E_1, E_2)$ is called the pair of permuted dual bases, induced by $E$ and $E_0$.

On the other hand, suppose that $(E_1, E_2)$ is a pair of permuted dual bases to each others on $\mathbb{R}^d$. Then it is clear that for $E_1 \times E_2 = \{e_1, \ldots, e_{2d}\}$ in Definition 1.3 and $E_0 = \{e_1, \ldots, e_d\}$, we have that $E_0$ and $E$ fulfills all properties in Definition 1.3. In this case, $E_1 \times E_2$ is called the phase split basis (of $E$).

1.3. Invariant quasi-Banach spaces and spaces of mixed quasi-normed spaces of Lebesgue types. We recall that a quasi-norm $\| \cdot \|_\mathcal{B}$ of order $r \in (0, 1]$ on the vector-space $\mathcal{B}$ over $\mathbf{C}$ is a nonnegative functional on $\mathcal{B}$ which satisfies

$$
\| f + g \|_\mathcal{B} \leq 2^{\frac{1}{r}-1}(\| f \|_\mathcal{B} + \| g \|_\mathcal{B}), \quad f, g \in \mathcal{B},
$$

(1.3)

$$
\| \alpha \cdot f \|_\mathcal{B} = |\alpha| \cdot \| f \|_\mathcal{B},
$$

(1.4)

and

$$
\| f \|_\mathcal{B} = 0 \iff f = 0.
$$

The space $\mathcal{B}$ is then called a quasi-norm space. A complete quasi-norm space is called a quasi-Banach space. If $\mathcal{B}$ is a quasi-Banach space with quasi-norm satisfying (1.3) then by [1, 20] there is an equivalent quasi-norm to $\| \cdot \|_\mathcal{B}$ which additionally satisfies

$$
\| f + g \|_\mathcal{B} \leq 2 \| f \|_\mathcal{B} + \| g \|_\mathcal{B}, \quad f, g \in \mathcal{B}.
$$

(1.4)

From now on we always assume that the quasi-norm of the quasi-Banach space $\mathcal{B}$ is chosen in such way that both (1.3) and (1.4) hold.

Before giving the definition of $v$-invariant spaces, we recall some facts on weight functions.

A weight or weight function on $\mathbb{R}^d$ is a positive function $\omega \in L_{1,\infty}^\infty(\mathbb{R}^d)$ such that $1/\omega \in L_{1,\infty}^\infty(\mathbb{R}^d)$. The weight $\omega$ is called moderate, if there is a positive weight $v$ on $\mathbb{R}^d$ such that

$$
\omega(x+y) \lesssim \omega(x)v(y), \quad x, y \in \mathbb{R}^d.
$$

(1.5)

If $\omega$ and $v$ are weights on $\mathbb{R}^d$ such that (1.5) holds, then $\omega$ is also called $v$-moderate. We note that (1.5) implies that $\omega$ fulfills the estimates

$$
v(-x)^{-1} \lesssim \omega(x) \lesssim v(x), \quad x \in \mathbb{R}^d.
$$

(1.6)

We let $\mathcal{P}_E(\mathbb{R}^d)$ be the set of all moderate weights on $\mathbb{R}^d$.

It can be proved that if $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, then $\omega$ is $v$-moderate for some $v(x) = e^{rx}$, provided the positive constant $r$ is large enough (cf. [13]).
In particular, (1.6) shows that for any \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \), there is a constant \( r > 0 \) such that
\[
e^{-r|x|} \lesssim \omega(x) \lesssim e^{r|x|}, \quad x \in \mathbb{R}^d.
\]

We say that \( v \) is submultiplicative if \( v \) is even and (1.5) holds with \( \omega = v \). In the sequel, \( v \) and \( v_j \) for \( j \geq 0 \), always stand for submultiplicative weights if nothing else is stated. The next definition is similar to [5, Section 3] in the Banach space case.

**Definition 1.6.** Let \( r \in (0,1] \), \( v \in \mathcal{P}_E(\mathbb{R}^d) \) and let \( \mathcal{B} = \mathcal{B}(\mathbb{R}^d) \subseteq L^r_{\text{loc}}(\mathbb{R}^d) \) be a quasi-Banach space such that \( \Sigma_1(\mathbb{R}^d) \subseteq \mathcal{B}(\mathbb{R}^d) \). Then \( \mathcal{B} \) is called \( v \)-invariant on \( \mathbb{R}^d \) if the following is true:

1. \( x \mapsto f(x+y) \) belongs to \( \mathcal{B} \) for every \( f \in \mathcal{B} \) and \( y \in \mathbb{R}^d \).
2. There is a constant \( C > 0 \) such that \( \|f_1\|_{\mathcal{B}} \leq C\|f_2\|_{\mathcal{B}} \) when \( f_1, f_2 \in \mathcal{B} \) are such that \( |f_1| \leq |f_2| \). Moreover,
\[
\|f(\cdot + y)\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{B}} v(y), \quad f \in \mathcal{B}, \; y \in \mathbb{R}^d.
\]

Let \( \mathcal{B} \) be as in Definition 1.6. \( E \) be a basis for \( \mathbb{R}^d \) and let \( \kappa(E) \) be the closed parallelepiped spanned by \( E \). The discrete version, \( \ell_{\mathcal{B},E} = \ell_{\mathcal{B},E}(\Lambda_E) \), of \( \mathcal{B} \) with respect to \( E \) is the set of all \( a \in \ell_0(\Lambda_E) \) such that
\[
\|a\|_{\ell_{\mathcal{B},E}} = \left\| \sum_{j \in \Lambda_E} a(j) \chi_{j+\kappa(E)} \right\|_{\mathcal{B}}
\]
is finite.

An important example on \( v \)-invariant spaces concerns mixed quasi-norm spaces of Lebesgue type, given in the following definition.

**Definition 1.7.** Let \( E = \{e_1, \ldots, e_d\} \) be an ordered basis of \( \mathbb{R}^d \), \( \kappa(E) \) be the parallelepiped spanned by \( E \), \( \omega \in \mathcal{P}_E(\mathbb{R}^d) \) \( q = (q_1, \ldots, q_d) \in (0, \infty]^d \) and \( r = \min(1, q) \). If \( f \in L^r_{\text{loc}}(\mathbb{R}^d) \), then
\[
\|f\|_{L^q_{E,\omega}(\mathbb{R}^d)} \equiv \|g_{d-1}\|_{L^q(\mathbb{R}^d)}
\]
where \( g_k : \mathbb{R}^{d-k} \to \mathbb{R}, \; k = 0, \ldots, d-1 \), are inductively defined as
\[
g_0(x_1, \ldots, x_{d}) \equiv |f(x_1 e_1 + \cdots + x_d e_d)\omega(x_1 e_1 + \cdots + x_d e_d)|,
\]
and
\[
g_k(z_k) \equiv \|g_{k-1}(\cdot, z_k)\|_{L^q(\mathbb{R}^d)}, \quad z_k \in \mathbb{R}^{d-k}, \; k = 1, \ldots, d-1.
\]
If \( \Omega \subseteq \mathbb{R}^d \) is measurable, then \( L^q_{E,\omega}(\mathbb{R}^d) \) consists of all \( f \in L^r_{\text{loc}}(\Omega) \) with finite quasi-norm
\[
\|f\|_{L^q_{E,\omega}(\mathbb{R}^d)\Omega} \equiv \|f\|_{L^q_{E,\omega}(\mathbb{R}^d)}, \quad f_\Omega(x) \equiv \begin{cases} f(x), & \text{when } x \in \Omega \\ 0, & \text{when } x \notin \Omega. \end{cases}
\]
The space \( L^q_{E,\omega}(\mathbb{R}^d) \) is called \( E \)-split Lebesgue space (with respect to \( \omega \), \( q \) and \( \Omega \).
We let $\ell^p_{E,\omega}(\Lambda_E)$ be the discrete version of $\mathcal{B} = L^p_{E,\omega}(\mathbb{R}^d)$ when $p \in (0, \infty]^d$.

Suppose that $E$ and $\Lambda$ are the same as in Definition 1.7. Then we let $(\ell^0_E)'(\Lambda)$ be the set of all formal sequences $\{a(j)\}_{j \in \Lambda}$, and we let $\ell^0_E(\Lambda)$ be the set of all such sequences such that at most finite numbers of $a(j)$ are non-zero.

Remark 1.8. Evidently, $L^q_{E,\omega}(\Omega)$ and $\ell^q_{E,\omega}(\Lambda)$ in Definition 1.7 are quasi-Banach spaces of order $\min(p,1)$. We set

$$L^q_E = L^q_{E,\omega} \quad \text{and} \quad \ell^q_E = \ell^q_{E,\omega}$$

when $\omega = 1$. For convenience we identify $q = (q_1, \ldots, q_d) \in (0, \infty]^d$ with $q \in (0, \infty]$ when considering spaces involving Lebesgue exponents. In particular,

$$L^q_{E,\omega} = L^q_{E,\omega}, \quad L^q_E = L^q_E, \quad \ell^q_{E,\omega} = \ell^q_{E,\omega} \quad \text{and} \quad \ell^q_E = \ell^q_E$$

for such $q$, and notice that these spaces agree with

$$L^q(\omega), \quad L^q, \quad \ell^q(\omega) \quad \text{and} \quad \ell^q,$$

respectively, with equivalent quasi-norms.

1.4. Modulation and Wiener spaces. We consider a general class of modulation spaces given in the following definition (cf. [4]).

Definition 1.9. Let $\omega, \upsilon \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that $\omega$ is $\upsilon$-moderate, $\mathcal{B}$ be a $\upsilon$-invariant quasi-Banach space on $\mathbb{R}^{2d}$, and let $\phi \in \mathcal{S}_{1/2}(\mathbb{R}^d) \setminus 0$. Then the modulation space $M(\omega, \mathcal{B})$ consists of all $f \in \mathcal{S}'_{1/2}(\mathbb{R}^d)$ such that

$$\|f\|_{M(\omega, \mathcal{B})} \equiv \|V_\phi f \cdot \omega\|_\mathcal{B}$$

is finite.

An important family of modulation spaces which contains the classical modulation spaces, introduced by Feichtinger in [3], is given next.

Definition 1.10. Let $p, q \in (0, \infty]^d$, $E_1$ and $E_2$ be ordered bases of $\mathbb{R}^d$, $E = E_1 \times E_2$, $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$ and let $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$. For any $f \in \Sigma_1(\mathbb{R}^d)$ set

$$\|f\|_{M^{p,q}_{E,\omega}(\omega)} \equiv \|H_{1,f,E_1,p,\omega}\|_{L^p_{E_2}},$$

where $H_{1,f,E_1,p,\omega}(\xi) \equiv \|V_\phi f(\cdot, \xi)\omega(\cdot, \xi)\|_{L^p_{E_1}}$

and

$$\|f\|_{W^{p,q}_{E,\omega}(\omega)} \equiv \|H_{2,f,E_2,q,\omega}\|_{L^p_{E_1}},$$

where $H_{2,f,E_2,q,\omega}(x) \equiv \|V_\phi f(x, \cdot)\omega(x, \cdot)\|_{L^q_{E_2}}$.
The modulation space $M_{E,\omega}^{p,q}(\mathbb{R}^d)$ consists of all $f \in \Sigma_1(\mathbb{R}^d)$ such that $\|f\|_{M_{E,\omega}^{p,q}} < \infty$.

The theory of modulation spaces has developed in different ways since they were introduced in [3] by Feichtinger. (Cf. e.g. [4, 9, 12, 22].) For example, let $p, q, E, \omega$ and $v$ be the same as in Definition 1.9 and 1.10 and let $B = L_{E}^{p,q}(\mathbb{R}^{d})$ and $r = \min(1, p, q)$. Then $M(\omega, B) = M_{E,\omega}^{p,q}(\mathbb{R}^d)$ is a quasi-Banach space. Moreover, $f \in M_{E,\omega}^{p,q}(\mathbb{R}^d)$ if and only if $V_{\omega} f \in L_{E}^{p,q}(\mathbb{R}^{d})$, and different choices of $\omega$ give rise to equivalent quasi-norms in Definition 1.10. We also note that for any such $B$, then

$$\Sigma_1(\mathbb{R}^d) \subseteq M_{E,\omega}^{p,q}(\mathbb{R}^d) \subseteq \Sigma'_1(\mathbb{R}^d).$$

Similar facts hold for the space $W_{E,\omega}^{p,q}(\mathbb{R}^d)$. (Cf. [9, 22].)

We shall consider various kinds of Wiener spaces involved later on when finding different characterizations of modulation spaces. The following type of Wiener spaces can essentially be found in e.g. [5, 9, 12], and is related to coorbit spaces of Lebesgue spaces.

**Definition 1.11.** Let $r \in (0, \infty]^d$, $\omega_0 \in \mathcal{P}_E(\mathbb{R}^d)$, $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$, $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$, $E \subseteq \mathbb{R}^d$ be an ordered basis, and let $\kappa(\phi)$ be the closed parallelepiped spanned by $E$. Also let $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_0 = \mathcal{B}_0(\mathbb{R}^d)$ be invariant QB-space on $\mathbb{R}^d$, $f$ and $F$ be measurable on $\mathbb{R}^d$ respective $\mathbb{R}^{2d}$, $F_\omega = F \cdot \omega$, and let $\ell_{\mathcal{B}, E}(\Lambda_E)$ be the discrete version of $\mathcal{B}$ with respect to $E$.

1. Then $\|f\|_{W_k^r(\omega, E)}$ is given by

$$\|f\|_{W_k^r(\omega, E)} = \|h_{E, \omega_0, \omega, f}\|_{\ell_{\mathcal{B}, E}(\Lambda_E)},$$

where

$$h_{E, \omega_0, \omega, f}(j) = \|f\|_{L_{E}^{r}(j + \kappa(\phi))}\omega_0(j), \quad j \in \Lambda_E.$$  

The set $W_k^r(\omega, E)$ consists of all measurable $f$ on $\mathbb{R}^d$ such that $\|f\|_{W_k^r(\omega, E)} < \infty$.

2. Then $\|F\|_{W_k^r(\omega, E)}$ consists of all measurable $F$ on $\mathbb{R}^{2d}$ such that $\|F\|_{W_k^r(\omega, E)} < \infty$, $k = 1, 2$.

The space $W_k^r(\omega_0, E)$ in Definition 1.11 is essentially a Wiener amalgam space with $L_{E}^{r}$ as local (quasi-)norm and $\mathcal{B}$ or $\ell_{\mathcal{B}, E}(\Lambda_E)$ as global component. They are also related to coorbit spaces. (See [2, 5, 7, 17, 18].)
In fact, $W^\infty(\omega_0, \ell^p)$ in Definition 1.11 (i.e. the case $r = (\infty, \ldots, \infty)$ and $E$ is the standard basis) is the coorbit space of $L^p(\mathbb{R}^d)$ with weight $\omega_0$, and is sometimes denoted by

$$\text{Co}(L^p(\omega_0)(\mathbb{R}^d)) \text{ or } W(L^p(\omega_0)) = W(L^p(\omega)(\mathbb{R}^d)),$$

in the literature (cf. [12, 17, 18]).

**Remark 1.12.** Let $p$, $\omega_0$, $\omega$, $E$, $B$, $B_0$, $f$ and $F$ be the same as in Definition 1.11. Evidently, by using the fact that $\omega_0$ is $\nu_0$-moderate for some $\nu_0$, it follows that

$$\|f \cdot \omega_0\|_{W^q_2(1, \ell_{B,E})} \lesssim \|f\|_{W^q_2(\omega_0, \ell_{B,E})}$$

and

$$\|F \cdot \omega\|_{W^q_2(1, \ell_{B,E}; B_0)} = \|F\|_{W^q_2(\omega, \ell_{B,E}; B_0)}$$

for $k = 1, 2$. Furthermore,

$$W^q_{1,E}(\omega, \ell_{B,E}; B_0) = \omega^{-1} \cdot W^q_E(1, \ell_{B,E}; B_0)$$

and

$$W^q_{2,E}(\omega, \ell_{B,E}; B_0) = \omega^{-1} \cdot B_0(\mathbb{R}^d; W^q_E(1, \ell_{B,E})).$$

Here and in what follows, $\mathcal{B}(\mathbb{R}^d; B_0) = \mathcal{B}(\mathbb{R}^d; B_0(\mathbb{R}^{d_0}))$ is the set of all functions $g$ in $\mathcal{B}$ with values in $B_0$, which are equipped with the quasi-norm

$$\|g\|_{\mathcal{B}(\mathbb{R}^d; B_0)} = \|g_0\|_B, \quad g_0(x) = \|g(x)\|_{B_0},$$

when $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_0(\mathbb{R}^{d_0})$ are invariant QBF-spaces.

Later on we discuss periodicity in the framework of certain modulation spaces which are related to spaces which are defined by imposing $L^\infty$-conditions on the configuration variable of corresponding short-time Fourier transforms.

**Definition 1.13.** Let $E$, $r$, $B_0$ and $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$ be the same as in Definition 1.11 and let $\psi \in \Sigma_1(\mathbb{R}^d) \setminus 0$. Then $M^r_E(\omega, \mathcal{B}_0)$ and $W^r_E(\omega, \mathcal{B}_0)$ are the sets of all $f \in \Sigma_1(\mathbb{R}^d)$ such that

$$\|f\|_{M^r_E(\omega, \mathcal{B}_0)} \equiv \|V_{\phi f}\|_{w^r_{1,E}(\omega, \ell_{B,E}; \mathcal{B}_0)}$$

respectively

$$\|f\|_{W^r_E(\omega, \mathcal{B}_0)} \equiv \|V_{\phi f}\|_{w^r_{2,E}(\omega, \ell_{B,E}; \mathcal{B}_0)}$$

are finite.

**Remark 1.14.** For the spaces in Definition 1.11 we set $W^{q_0, r_0} = W^r$, when $r_0 = (r_1, \ldots, r_d) \in (0, \infty]^d$, and $r = (q_0, \ldots, q_0, r_1, \ldots, r_d) \in (0, \infty)^{2d}$,
and similarly for other types of exponents and for the spaces in Definitions 1.10 and 1.13. (See also Remark 1.8.) We also set
\[ M_{E_1,\omega}^{\infty,q} = M_{E_2,\omega}^{\infty,q} \quad \text{and} \quad W_{E_1,\omega}^{\infty,q} = W_{E_2,\omega}^{\infty,q}, \]
when \( E_1, E_2 \) are ordered bases of \( \mathbb{R}^d \) and \( E = E_1 \times E_2 \), for spaces in Definition 1.10 since these spaces are independent of \( E_1 \).

In Section 2 we prove that if \( \mathcal{B}_0 \) is an \( E \)-split Lebesgue space on \( \mathbb{R}^d \) and \( \omega(x, \xi) \in \mathcal{P}_{E}(\mathbb{R}^{2d}) \) which is constant with respect to the \( x \) variable, then \( M_{E}(\omega, \mathcal{B}_0) \) and \( W_{E}(\omega, \mathcal{B}_0) \) are independent of \( r \) and agree with modulation spaces of the form in Definition 1.9 (cf. Proposition 2.6). The next result is a reformulation of [22, Proposition 3.4], and indicates how Wiener spaces are connected to modulation spaces. The proof is therefore omitted. Here, let
\[(\Theta_{\rho})v(x, \xi) = v(x, \xi)(x, \xi)^\rho, \quad \text{where} \quad \rho \geq 2d \left( \frac{1}{r} - 1 \right), \tag{1.8}\]
for any submultiplicative \( v \in \mathcal{P}_{E}(\mathbb{R}^{2d}) \) and \( r \in (0, 1] \). It follows that \( L_{E(\Theta_{\rho})}(\mathbb{R}^{2d}) \) is continuously embedded in \( L_{E}(\mathbb{R}^{2d}) \), giving that \( M_{(\Theta_{\rho})}(\mathbb{R}^{2d}) \subseteq M_{E}(\mathbb{R}^{2d}) \). Hence if \( \phi \in M_{(\Theta_{\rho})}(\mathbb{R}^{2d}) \setminus \{0\} \), \( \varepsilon_0 \) is chosen such that \( S_{\phi, \rho}^{\Lambda} \) is invertible on \( M_{(\Theta_{\rho})}(\mathbb{R}^{2d}) \) for every \( \Lambda =\varepsilon\Lambda_{E} \), \( \varepsilon \in (0, \varepsilon_0] \), it follows that both \( \phi \) and its canonical dual with respect to \( \Lambda \) belong to \( M_{(\Theta_{\rho})}(\mathbb{R}^{2d}) \). Notice that such \( \varepsilon_0 > 0 \) exists in view of [11, Theorem 5].

**Proposition 1.15.** Let \( E \) be a phase split basis for \( \mathbb{R}^{2d} \), \( p \in (0, \infty)^{2d} \), \( r = \min(1, p) \), \( \omega, v \in \mathcal{P}_{E}(\mathbb{R}^{2d}) \) be such that \( \omega \) is \( v \)-moderate, \( \rho \) and \( \Theta_{\rho}v \) be as in (1.8) with strict inequality when \( r < 1 \), and let \( \phi_1, \phi_2 \in M_{(\Theta_{\rho})}(\mathbb{R}^{d}) \setminus \{0\} \). Then
\[ \|f\|_{M_{E_1,\omega}^{p}} \lesssim \|V_{\phi_1}f\|_{L_{E_1,\omega}^{p}} \lesssim \|V_{\phi_2}f\|_{W_{E_1,\omega}^{p}}, \quad f \in S_{1/2}^{1/2}(\mathbb{R}^{d}). \]

In particular, if \( f \in S_{1/2}^{1/2}(\mathbb{R}^{d}) \), then
\[ f \in M_{E_1,\omega}^{p}(\mathbb{R}^{2d}) \iff V_{\phi_1}f \in L_{E_1,\omega}^{p}(\mathbb{R}^{2d}) \iff V_{\phi_2}f \in W_{E_1,\omega}^{p}(\mathbb{R}^{d}). \]

In Section 2 we extend this result in such way that we may replace \( W_{E}^{p}(\omega, \ell_{E}^{p}) \) by \( W_{E}(\omega, \ell_{E}^{p}) \) for any \( r > 0 \).

1.5. **Classes of periodic elements.** We consider spaces of periodic Gevrey functions and their duals.

Let \( s, \sigma \in \mathbb{R}_{+} \) be such that \( s + t \geq 1 \), \( f \in (\mathcal{S}_{\sigma})'(\mathbb{R}^{d}) \), \( E \) be a basis of \( \mathbb{R}^{d} \) and let \( E_{0} \subseteq E \). Then \( f \) is called \( E_{0} \)-periodic if \( f(x + y) = f(x) \) for every \( x \in \mathbb{R}^{d} \) and \( y \in E_{0} \).

We note that for any \( \Lambda_{E} \)-periodic function \( f \in C_{\infty}(\mathbb{R}^{d}) \), we have
\[ f = \sum_{\alpha \in \Lambda_{E}} c(f, \alpha)e^{i(\cdot, \alpha)}, \tag{1.9}\]
where \( c(f, \alpha) \) are the Fourier coefficients given by
\[
c(f, \alpha) = |\kappa(E)|^{-1}(f, e^{it \cdot \alpha})_{L^2(E)}.
\]

For any \( s \geq 0 \) and basis \( E \subseteq \mathbb{R}^d \) we let \( \mathcal{E}^E_{0,\sigma}(\mathbb{R}^d) \) and \( \mathcal{E}^E_{\sigma}(\mathbb{R}^d) \) be the sets of all \( E \)-periodic elements in \( \mathcal{E}_{0,\sigma}(\mathbb{R}^d) \) and in \( \mathcal{E}_{\sigma}(\mathbb{R}^d) \), respectively. Evidently,
\[
\mathcal{E}^E_{\sigma}(\mathbb{R}^d) \cong \mathcal{E}_{\sigma}(\mathbb{R}^d / \Lambda_E) \quad \text{and} \quad \mathcal{E}^E_{0,\sigma}(\mathbb{R}^d) \cong \mathcal{E}_{0,\sigma}(\mathbb{R}^d / \Lambda_E),
\]
which is a common approach in the literature.

**Remark 1.16.** Let \( E \) be an ordered basis on \( \mathbb{R}^d \) and \( V \) be a topological space of functions or (ultra-)distributions on \( \mathbb{R}^d \). Then we use the convention that \( V_E \) (\( E \) as upper case index) denotes the \( E \) periodic elements in \( V \), while \( V_E \) (\( E \) as lower case index) is the space analogous to \( V \) when \( E \) is used as basis.

Let \( s, s_0, \sigma, \sigma_0 > 0 \) be such that \( s + \sigma \geq 1 \), \( s_0 + \sigma_0 \geq 1 \) and \( (s_0, \sigma_0) \neq \left( \frac{1}{2}, \frac{1}{2} \right) \). Then we recall that the duals \( (\mathcal{E}^E_{\sigma})'(\mathbb{R}^d) \) and \( (\mathcal{E}^E_{0,\sigma})'(\mathbb{R}^d) \) of \( \mathcal{E}^E_{\sigma}(\mathbb{R}^d) \) and \( \mathcal{E}^E_{0,\sigma}(\mathbb{R}^d) \), respectively, can be identified with the \( E \)-periodic elements in \( (\mathcal{S}^s)'(\mathbb{R}^d) \) and \( (\Sigma_{s_0}^\sigma)'(\mathbb{R}^d) \) respectively via unique extension of the form
\[
(f, \phi)_E = \sum_{\alpha \in \Lambda_E} c(f, \alpha) \overline{c(\phi, \alpha)}
\]
on \( \mathcal{E}^E_{0,\sigma_0}(\mathbb{R}^d) \times \mathcal{E}^E_{0,\sigma}(\mathbb{R}^d) \). We also let \( (\mathcal{E}^E_{\sigma})'(\mathbb{R}^d) \) be the set of all formal expansions in (1.9) and \( \mathcal{E}^E_{0,\sigma}(\mathbb{R}^d) \) be the set of all formal expansions in (1.9) such that at most finite numbers of \( c(f, \alpha) \) are non-zero (cf. 24). We refer to [15, 24] for more characterizations of \( \mathcal{E}^E_{\sigma}, \mathcal{E}^E_{0,\sigma} \) and their duals.

The following definition takes care of spaces of formal expansions (1.9) with coefficients obeying specific quasi-norm estimates.

**Definition 1.17.** Let \( E \) be a basis of \( \mathbb{R}^d \), \( \mathcal{B} \) be a quasi-Banach space continuously embedded in \( \mathcal{L}^r_0(\Lambda'_E) \) and let \( \omega_0 \) be a weight on \( \mathbb{R}^d \). Then \( \mathcal{L}^E(\omega_0, \mathcal{B}) \) consists of all \( f \in (\mathcal{E}^E)'(\mathbb{R}^d) \) such that
\[
\|f\|_{\mathcal{L}^E(\omega_0, \mathcal{B})} = \|\{c(f, \alpha)\omega_0(\alpha)\}_{\alpha \in \Lambda'_E}\|_\mathcal{B}
\]
is finite.

If \( \omega_0 \in \mathcal{P}_E(\mathbb{R}^d) \) and \( \omega(x, \xi) = \omega_0(\xi) \), then
\[
\|f\|_{\mathcal{L}^E_0(\omega, \mathcal{B})} = \|g\omega_0\|_{\mathcal{B}},
\]
when \( g(\xi) = \|V_\phi f(\cdot, \xi)\|_{L^r_0(\kappa(E))}, \quad f \in (\mathcal{E}^E)'(\mathbb{R}^d), \) (1.10)
and
\[
\|f\|_{\mathcal{L}^E_{0,\sigma}(\omega, \mathcal{B})} = \|h\|_{L^r_{\kappa}(\kappa(E))},
\]
when \( h(x) = \|V_\phi f(x, \cdot)\omega_0\|_{\mathcal{B}}, \quad f \in (\mathcal{E}^E)'(\mathbb{R}^d), \) (1.11)
because the \(E\)-periodicity of \(x \mapsto |V_\phi f(x, \xi)|\) when \(f\) is \(E\) periodic gives
\[
g(\xi) = \|V_\phi f(\cdot, \xi)\|_{L_{E}(\kappa(E))} = \|V_\phi f(\cdot, \xi)\|_{L_{E}(x, \kappa(E))},
\]
\[
\|h\|_{L_{E}(\kappa(E))} = \|h\|_{L_{E}(x, \kappa(E))}, \quad x \in \mathbb{R}^d.
\]

**Proposition 1.18.** Let \(E\) be a basis of \(\mathbb{R}^d\), \(r \in (0, 1]\), \(\mathcal{B} \subseteq L_{loc}^1(\mathbb{R}^d)\) be an \(E^\prime\)-split Lebesgue space, \(\ell_{\mathcal{B}, E}(\Lambda_E)\) be its discrete version, \(\omega_0 \in \mathcal{P}_E(\mathbb{R}^d)\) and let \(\omega(x, \xi) = \omega_0(\xi)\) when \(x, \xi \in \mathbb{R}^d\). Then
\[
\mathcal{L}_E^E(\omega_0, \ell_{\mathcal{B}, E}) = \mathcal{M}_E^\infty(\omega, \mathcal{B}) \cap (\mathcal{E}_0^E)'(\mathbb{R}^d) = \mathcal{W}_E^\infty(\omega, \mathcal{B}) \cap (\mathcal{E}_0^E)'(\mathbb{R}^d).
\]

When proving that \(\mathcal{W}_E^r(\omega, \ell_{\mathcal{B}, E}^p)\) is independent of \(r \in (0, \infty]^d\) in Section 2, as announced earlier, it will at the same time follow that if \(\mathcal{B}\) is a suitable quasi-norm space of Lebesgue type, then
\[
\mathcal{M}_E^r(\omega, \mathcal{B}) = \mathcal{W}_E^r(\omega, \mathcal{B}) \quad \text{when} \quad \omega \in \mathcal{P}_E(\mathbb{R}^d)
\]
for every \(r_1, r_2 \in (0, \infty]^d\).

**Remark 1.19.** The link between periodic Gelfand-Shilov distributions and formal Fourier series expansions is given by the formula
\[
\langle f, \phi \rangle = (2\pi)^d \sum_{\alpha \in \Lambda_E} c(f, \alpha)\hat{\phi}(-\alpha).
\]

## 2. Estimates on Wiener spaces and periodic elements in modulation spaces

In this section we deduce equivalences between various Wiener (quasi-)norm estimates on short-time Fourier transforms. Especially we prove that (1.13) holds for every \(r_1, r_2 \in (0, \infty]^d\).

### 2.1. Estimates of Wiener spaces.

We begin with the following inclusion between the different Wiener spaces in the previous section.

**Proposition 2.1.** Let \((E_1, E_2)\) be permuted dual bases of \(\mathbb{R}^d\), \(E = E_1 \times E_2\), \(p, q, r \in (0, \infty]^d\) \(r_1 \in (0, \min(p, q, r)]\), \(r_2 \in (0, \min(q)]\), and let \(\omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^d)\) be such that
\[
\omega_1(x, \xi) = \omega_2(\xi, x), \quad x, \xi \in \mathbb{R}^d.
\]
Then
\[
\mathcal{W}_{E}^{r_1}(\omega, \ell_{E_1}^{p,q}(\Lambda_{E})) \hookrightarrow \mathcal{W}_{1,E_1}(\omega, \ell_{E_1}^{p}(\Lambda_{E_1}), L_{E_2}^{q}(\mathbb{R}^d))
\]
\[
\hookrightarrow \mathcal{W}_{E}^{r_1}(\omega, \ell_{E}^{p,q}(\Lambda_{E})) \quad (2.1)
\]
and
\[
\mathcal{W}_{E'}^{r_1}(\omega, \ell_{E'}^{q,p}(\Lambda_{E'})) \hookrightarrow \mathcal{W}_{2,E_2}(\omega, \ell_{E_2}^{p}(\Lambda_{E'}), L_{E_1}^{q}(\mathbb{R}^d))
\]
\[
\hookrightarrow \mathcal{W}_{E'}^{r_1}(\omega, \ell_{E'}^{q,p}(\Lambda_{E})). \quad (2.2)
\]
Remark 2.2. For the involved spaces in Proposition 2.1 it follows from Hölder’s inequality that
\[ W_{1,E_1}(\omega, \ell^{p_1}_E(\Lambda_{E_1})), \ W_{2,E_2}^{q_2}(\omega, \ell^{p_2}_{E_2}(\Lambda_{E_2})), \ W_{2,E_2}^{r_2}(\Lambda_{E_2}), L^{q_1}_{E_1}(R^d)) \]
and
\[ W_{E}^{p}(\omega, \ell^{p}_{E}(\Lambda_{E})) \]
increase with respect to \( p \) and decrease with respect to \( r \).

We need the following lemma for the proof of Proposition 2.1.

Lemma 2.3. Let \( \omega \in \mathcal{P}_E(R^d) \), \( E \) be an ordered basis of \( R^d \), \( \kappa(E) \) the parallelepiped spanned by \( E \), \( p \in (0, \infty]^d \), \( r \in (0, \min(p)] \) and let \( f \) be measurable on \( R^d \). Then
\[
\|a\|_{\ell^p_E(\Lambda_E)} \lesssim \|f\|_{L^p_E(\omega, \ell^p_E(\Lambda_E))}, \quad a(j) = \|f\|_{L^r_E(j + \kappa(E))\omega(j)}. \tag{2.3}
\]

Proof. Let \( f \) be measurable on \( R^d \), \( g_k \) be the same as in Definition 1.7, \( T_E \) be the linear map which maps the standard basis into \( E \), \( Q_k = [0, 1]^k \), and let \( p_k = (p_{k+1}, \ldots, p_d) \), when \( k \geq 1 \). Then
\[
\|f\|_{L^r_E(T_E(j) + \kappa(E))\omega(j)} \asymp \|f \cdot \omega\|_{L^r_E(T_E(j) + \kappa(E))} \asymp \|g_0\|_{L^r_E(j + Q_d)}, \quad j \in Z^d.
\]
This reduce the situation to the case that \( E \) is the standard basis, \( \kappa(E) = Q_d \) and \( \omega = 1 \). Moreover, by replacing \(|f|^r \) with \( f \) and \( p_j \) by \( p_j \), \( j = 1, \ldots, d \), we may assume that \( r = 1 \) (and that each \( p_j \geq 1 \)).

By induction it suffices to prove that if
\[
a_k(l) = \|g_k\|_{L^1(j + Q_{d-k})}, \quad l \in Z^{d-k},
\]
then
\[
\|a_k\|_{\ell^{p_k}(Z^{d-k})} \lesssim \|a_{k+1}\|_{\ell^{p_{k+1}}(Z^{d-k-1})}, \quad k = 0, \ldots, d - 1, \tag{2.4}
\]
since \( \|a_0\|_{\ell^{p_0}(Z^d)} \) is equal to the left-hand side of (2.3), and \( a_d = \|a_d\|_{\ell^{\infty}(Z^d)} \) is equal to the right-hand side of (2.3).

Let \( m \in Z^{d-k-1} \) be fixed. We only prove (2.4) in the case \( p_{k+1} < \infty \). The case \( p_{k+1} = \infty \) will follow by similar arguments and is left for the reader. By first using Minkowski’s inequality and then Hölder’s
inequality we get
\[
\|a_k(\cdot, m)\|_{\ell^{p_{k+1}}(\mathbb{Z})} = \left(\sum_{l_1 \in \mathbb{Z}} \|g_k\|_{L^1([l_1, m] + Q_{d-k})}^{p_{k+1}}\right)^{\frac{1}{p_{k+1}}}
\]

\[
= \left(\sum_{l_1 \in \mathbb{Z}} \left(\int_{m+Q_{d-k-1}} \left(\int_{l_1+Q_1} g_k(t, y) \, dy\right) \, dt\right)^{p_{k+1}}\right)^{\frac{1}{p_{k+1}}}
\]

\[
\leq \int_{m+Q_{d-k-1}} \left(\sum_{l_1 \in \mathbb{Z}} \left(\int_{l_1+Q_1} g_k(t, y)^{p_{k+1}} \, dy\right)\right)^{\frac{1}{p_{k+1}}}
\]

\[
= \int_{m+Q_{d-k-1}} \left(\int_{\mathbb{R}} g_k(t, y)^{p_{k+1}} \, dt\right)^{\frac{1}{p_{k+1}}} \, dy = \int_{m+Q_{d-k-1}} g_{k+1}(y) \, dy.
\]

Hence,
\[
\|a_k(\cdot, m)\|_{\ell^{p_{k+1}}(\mathbb{Z})} \lesssim a_{k+1}(m), \quad m \in \mathbb{Z}^{d-k-1}.
\]

By applying the $\ell^{p_{k+1}}(\mathbb{Z}^{d-k-1})$-norm on (2.5) we get (2.4), and thereby (2.3).

**Proof of Proposition 2.1** Since the map $F \mapsto F \cdot \omega$ is homeomorphic between the involved spaces and their corresponding non-weighted versions, we may assume that $\omega_1 = \omega_2 = 1$. Furthermore, by a linear change of variables, we may assume that $E_1$ is the standard basis and $E_2 = 2\pi E_1$. Then $\kappa(E_1) = Q_d$, $E_1' = E_2$ and $E_2' = E_1$.

Let $F$ be measurable on $\mathbb{R}^{2d}$,

\[
f_{1,r}(\xi, j) = \|F(\cdot, \xi)\|_{L^r(j + \kappa(E_1))}, \quad g_1(\xi) = \|f_{1,r}(\xi, \cdot)\|_p
\]

and

\[
G_1(j, t) = \|F\|_{L^{r, \infty}_{(j + \kappa(E_1) \times E_2)}}.
\]

Then
\[
g_1 \leq g \equiv \sum_{t + \Lambda E_2} \left(\|g\|_{L^\infty(\cdot + 2\pi Q)}\right) \cdot \chi_{\cdot + 2\pi Q},
\]

and
\[
\|F\|_{W^r_{E_1}(1, \ell^p, L^q)} = \|g_1\|_{L^p} \leq \|g\|_{L^p} = \|G_1\|_{\ell^p, q} \approx \|F\|_{W^{r, \infty}_{E_1}(1, \ell^p, L^q)}.
\]

This implies that $W^{r, \infty}_{E_1}(1, \ell^p, L^q) \hookrightarrow W^r_{E_1}(1, \ell^p, L^q)$, and the first inclusion in (2.1) follows.
In order to prove the second inclusion in (2.1), we may assume that \( r_0 < \infty \), since otherwise the result is trivial. Let 

\[
\psi(\xi) = \|f_{1,r_0}(\xi, \cdot)\|_{L^p}, \quad a(\iota) = \|\psi\|_{L^{q}(i+\kappa(E_2))}
\]

and 

\[
H_1(j, \iota) = \|f_{1,r_0}(\cdot, j)\|_{L^{r_0}(i+\kappa(E_2))}.
\]

Then 

\[
\|\psi\|_{L^q(\mathbb{R}^d)} = \|F\|_{W^{r_0}_{1,E_1}(1,\ell^p,L^q)} \quad \text{and} \quad H_1 \|_{\ell^p,q} = \|F\|_{W^{r_0}_{E_1}(1,\ell^p,q)}.
\]

By Minkowski’s inequality and the fact that \( \min(p) \geq r_0 \) we get 

\[
\|H_1(\cdot, \iota)\|_{\ell^p} = \left\| \left( \int_{i+\kappa(E_2)} f_{1,r_0}(\xi, \cdot)^{r_0} d\xi \right)^{\frac{1}{r_0}} \right\|_{\ell^p} 
\]

\[
\leq \left( \int_{i+\kappa(E_2)} \|f_{1,r_0}(\xi, \cdot)^{r_0}\|_{\ell^{p/r_0}} d\xi \right)^{\frac{1}{r_0}} = a(\iota).
\]

Hence \( H_1 \|_{\ell^p,q} \leq \|a\|_{\ell^q} \). By Lemma 2.3 it follows that \( \|a\|_{\ell^q} \leq \|\psi\|_{L^q} \), and the second inclusion of (2.1) follows by combining these relations.

It remains to prove (2.2). Again we may assume that \( r_2 < \infty \), since otherwise the result is trivial. Let 

\[
f_{2,q}(x, \iota) = \|F(x, \cdot)\|_{L^q(i+\kappa(E_2))}, \quad f_3(x) = \|F(x, \cdot)\|_{L^q(\mathbb{R}^d)},
\]

\[
H_{2,q_1,q_2}(\iota, j) = \|f_{2,q_1}(\cdot, \iota)\|_{L^{q_2}(j+\kappa(E_1))}, \quad \text{and} \quad H_{2,q} = H_{2,q,q}
\]

when \( q, q_1, q_2 \in (0, \infty] \). Then the fact that \( r_2 \leq \min(q) \), Minkowski’s inequality and Lemma 2.3 give 

\[
\|H_{2,r_2}(\cdot, j)\|_{\ell^q} \leq \left( \int_{j+\kappa(E_1)} \|f_{2,r_2}(x, \cdot)\|_{\ell^{q_2}}^{r_2} dx \right)^{\frac{1}{r_2}} 
\]

\[
\leq \left( \int_{j+\kappa(E_1)} \|F(x, \cdot)\|_{\ell^{q_2}}^{r_2} dx \right)^{\frac{1}{r_2}}.
\]

By applying the \( \ell^p \) norm on the latter inequality we get 

\[
\|F\|_{W^{r_2}_{E_1}(1,\ell^p,q)} \leq \|F\|_{W^{r_2}_{E_2}(1,\ell^p,L^q)},
\]

and the second relation in (2.2) follows.
On the other hand, we have
\[
\left( \int_{j+\kappa(E_1)} \| F(x, \cdot) \|_{L^q(R^d)}^{r_2} \, dx \right)^{\frac{1}{r_2}} \lesssim \left( \int_{j+\kappa(E_1)} \| f_{2,\infty}(x, \cdot) \|_{L^q(R^d)}^{r_2} \, dx \right)^{\frac{1}{r_2}} \leq H_{2,\infty}(\cdot, j) \| \ell_q \|
\]
Again, by applying the $\ell^p$ norm with respect to the $j$ variable, we get
\[
\| F \|_{W^r_{x,\ell^p}(1, \ell^q)} \leq \| F \|_{W^r_{x,\ell^p}(1, \ell^q)},
\]
and the first relation in (2.2) follows.

2.2. Wiener estimates on short-time Fourier transforms, and modulation spaces. Essential parts of our analysis are based on Lebesgue estimates of the semi-discrete convolution with respect to (the ordered) basis $E$ in $\mathbb{R}^d$, given by
\[
(a *_{[E]} f)(x) = \sum_{j \in \Lambda_E} a(j) f(x - j), \quad (2.6)
\]
when $f \in \mathcal{S}'(\mathbb{R}^d)$ and $a \in \ell_0(\Lambda_E)$.

The next result is an extension of [22, Proposition 2.1] and [9, Lemma 2.6], but a special case of [23, Theorem 2.1]. The proof is therefore omitted. Here the domain of integration is of the form
\[
I = \{ x_1 e_1 + \cdots + x_d e_d ; x_k \in J_k \}, \quad J_k = \begin{cases} [0,1], & e_k \in E_0 \\ \mathbb{R}, & e_k \notin E_0 \end{cases} \quad (2.7)
\]

**Proposition 2.4.** Let $E$ be an ordered basis of $\mathbb{R}^d$, $E_0 \subseteq E$, $I$ be given by (2.7), $\omega, v \in \mathcal{P}_E(\mathbb{R}^d)$ be such that $\omega$ is $v$-moderate, and let $p, r \in (0, \infty]^d$ be such that
\[
r_k \leq \min_{m \leq k}(1, p_m).
\]
Also let $f$ be measurable on $\mathbb{R}^d$ such that $|f|$ is $E_0$-periodic and $f \in L^p_{E,\omega}(I)$. Then the map $a \mapsto a *_{[E]} f$ from $\ell_0(\Lambda_E)$ to $L^p_{E,\omega}(I)$ extends uniquely to a linear and continuous map from $\ell^r_{E,\omega}(\Lambda_E)$ to $L^p_{E,\omega}(I)$, and
\[
\| a *_{[E]} f \|_{L^p_{E,\omega}(I)} \leq C \| a \|_{\ell^r_{E,\omega}(\Lambda_E)} \| f \|_{L^p_{E,\omega}(I)}, \quad (2.8)
\]
for some constant $C > 0$ which is independent of $a \in \ell^r_{E,\omega}(\Lambda_E)$ and measurable $f$ on $\mathbb{R}^d$ such that $|f|$ is $E_0$-periodic.

We have now the following result, which agrees with Proposition 1.15 when $r = (\infty, \ldots, \infty)$.

**Proposition 1.15.** Let $E$ be a phase split basis for $\mathbb{R}^{2d}$, $p, r \in (0, \infty]^d$, $r \in (0, \min(1, p)]$, $\omega, v \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that $\omega$ is $v$-moderate, $\rho
and $\Theta, \rho : \rho$ as in \(1.18\) with strict inequality when $r < 1$, and let $\phi_1, \phi_2 \in M^1(\Theta, \rho)(\mathbb{R}^d) \setminus 0$. Then

$$
\|f\|_{M^p_{E, \omega}(\mathbb{R}^d)} \prec \|V_{\phi_1}f\|_{L^p_{E, \omega}(\mathbb{R}^d)} \prec \|V_{\phi_2}f\|_{W^p_{E}(\omega, \xi^p_{E}(\Lambda_E))}, \quad f \in S'_{1/2}(\mathbb{R}^d).
$$

In particular, if $f \in S'_{1/2}(\mathbb{R}^d)$, then

$$
f \in M^p_{E, \omega}(\mathbb{R}^{2d}) \iff V_{\phi_1}f \in L^p_{E, \omega}(\mathbb{R}^{2d}) \iff V_{\phi_2}f \in W^p_{E}(\omega, \xi^p_{E}(\Lambda_E)).
$$

We need the following lemma for the proof.

**Lemma 2.5.** Let $p \in (0, \infty]$, $r > 0$, $(x_0, \xi_0) \in \mathbb{R}^{2d}$ be fixed, and let $\phi \in S_{1/2}(\mathbb{R}^d)$ be a Gaussian. Then

$$
|V_{\phi}f(x_0, \xi_0)| \leq C\|V_{\phi}f\|_{L^p(B_r(x_0, \xi_0))}, \quad f \in S'_{1/2}(\mathbb{R}^d),
$$

where the constant $C$ is independent of $(x_0, \xi_0)$ and $f$.

When proving Lemma \[2.5\] we may first reduce ourself to the case that the Gaussian $\phi$ should be centered at origin, by straight-forward arguments involving pullbacks with translations. The result then follows by using the same arguments as in \[9, Lemma 2.3\] and its proof, based on the fact that

$$
z \mapsto F_{\phi}(z) \equiv e^{c_3|z|^2 + c_2(z, w) + c_1|w|^3}V_{\phi}f(x, \xi), \quad z = x + i\xi
$$

is an entire function for some choice of the constant $c_1$ (depending on $\phi$).

**Proof of Proposition \[1.15\].** Let $F = V_{\phi}f$, $F_0 = V_{\phi_0}f$, $\kappa(E)$ be the (closed) parallelepiped which is spanned by $E = \{e_1, \ldots, e_{2d}\}$, and let

$$
\kappa_M(E) = \{x_1e_1 + \cdots + x_{2d}e_{2d} : |x_k| \leq 2, \; k = 1, \ldots, 2d\}.
$$

Also choose $r_0 > 0$ small enough such that

$$
\kappa(E) + B_{r_0}(0, 0) \subseteq \kappa_M(E)
$$

The result holds when $r = (\infty, \ldots, \infty)$, in view of Proposition \[1.15\]. By Hölder’s inequality we also have

$$
\|V_{\phi}f\|_{W^p_{E}(\omega, \xi^p_{E})} \lesssim \|V_{\phi}f\|_{W^p_{E}(\omega, \xi^p_{E})}.
$$

We need to prove the reversed inequality

$$
\|V_{\phi}f\|_{W^p_{E}(\omega, \xi^p_{E})} \lesssim \|V_{\phi}f\|_{W^p_{E}(\omega, \xi^p_{E})},
$$

and it suffices to prove this for $r = (r, \ldots, r)$ for some $r \in (0, 1]$ in view of Hölder’s inequality.

First we consider the case when $\phi = \phi_0$. If $r > 0$ is small enough and $j \in \Lambda_E$, then Lemma \[2.5\] gives for some $(x_j, \xi_j) \in j + \kappa(E)$ that

$$
\|V_{\phi_0}f\|_{L^\infty(j + \kappa(E))} = |V_{\phi_0}f(x_j, \xi_j)| \lesssim \|V_{\phi_0}f\|_{L^r(B_r(x_j, \xi_j))} \leq \|V_{\phi_0}f\|_{L^r(j + \kappa_M(E))}
$$
Hence,
\[
\|V_{\phi_0}f\|_{W_{E}^{\omega}(\omega, \ell_{p}E)} = \|\{\|V_{\phi_0}f\|_{L^{\infty}(j+\kappa(E)\omega(j))}\}_{j\in\Lambda_E}\|_{\ell_{p}E(\Lambda_{E})}
\lesssim \|\{\|V_{\phi_0}f\|_{L^{r}(j+\kappa(E)\omega(j))}\}_{j\in\Lambda_E}\|_{\ell_{p}E(\Lambda_{E})} = \|V_{\phi_0}f\|_{W_{E}^{r}(\omega, \ell_{p}E)}.
\]
and (2.10) holds for \(\phi = \phi_{0}\).

Next suppose that \(\phi\) is arbitrary and let \(n \geq 1\) be a large enough integer such that if \(E_{n} = 1_{n} \cdot E \equiv \{e_{1}/n, \ldots, e_{2d}/n\}\) and \(\Lambda = 1_{n}\Lambda_{E} = \Lambda_{E_{n}}\), then
\[
\{\phi(\cdot - k)e^{i\langle \cdot, \kappa \rangle}\}_{(k, \kappa)\in\Lambda}
\]
is a frame. Since \(\phi \in M_{1}^{1}(\Theta_{\rho}, v)\), it follows that its canonical dual \(\psi\) also belongs to \(M_{1}^{1}(\Theta_{\rho}, v)\) (cf. [11, Theorem S]). Consequently, any \(f\) possess the expansions
\[
f = \sum_{(k, \kappa)\in\Lambda} V_{\phi}(k, \kappa)f(\cdot - k)e^{i\langle \cdot, \kappa \rangle}
= \sum_{(k, \kappa)\in\Lambda} V_{\psi}(k, \kappa)\phi(\cdot - k)e^{i\langle \cdot, \kappa \rangle}
\]
with suitable interpretation of convergences.

Let
\[
F_{0} = |V_{\phi_0}f| \cdot \omega, \quad F = |V_{\phi}f| \cdot \omega, \quad \text{and} \quad a(k) = |V_{\phi}(0, -k)|.
\]
As in the proofs of [9, Theorem 3.1] and [22, Proposition 3.1] we use the fact that
\[
|V_{\phi_0}f| \leq (2\pi)^{-\frac{d}{2}}a_{[E_{n}]}|V_{\phi}f|,
\]
which follows from
\[
|V_{\phi_0}f(x, \xi)| = (2\pi)^{-\frac{d}{2}}|(f, e^{i\langle \cdot, \xi \rangle}\phi_{0}(\cdot - x))| 
\leq (2\pi)^{-\frac{d}{2}} \sum_{(k, \kappa)\in\Lambda} |(V_{\phi}(0))(k, \kappa)||f, e^{i\langle \cdot, \xi + \kappa \rangle}\phi(\cdot - x - k)| 
= (2\pi)^{-\frac{d}{2}} \sum_{(k, \kappa)\in\Lambda} |(V_{\phi}(0))(k, \kappa)||V_{\phi}f(x + k, \xi + \kappa)| 
= (a_{[E_{n}]}|V_{\phi}f|)(x, \xi).
\]
Here we have used (2.11) with \(\phi_{0}\) in place of \(f\), in the inequality. By using that
\[
\omega(x, \xi) \lesssim v(k, \kappa)\omega(x + k, \xi + \kappa),
\]
(2.12) gives
\[
F_{0} \lesssim (a \cdot v)_{[E_{n}]} F.
\]
If we set
\[ b_0(j) = \int_{j+\kappa(E)} |F_0(X)|^r dX \quad \text{and} \quad b(j) = \int_{j+\kappa(E)} |F(X)|^r dX, \quad j \in \Lambda, \]
integrate (2.13) and use the fact that \( r \leq 1 \), we get for \( j \in \Lambda \) that
\[
b_0(j) \lesssim \int_{j+\kappa(E)} \left( \sum_{k \in \Lambda} (a(k)v(k)|F(X-k)| \right)^r dX \lesssim \sum_{k \in \Lambda} (a(k)v(k))^r \int_{j+\kappa(E)} |F(X-k)|^r dX = ((a \cdot v)^* b)(j),
\]
where \( * \) is the discrete convolution with respect to the lattice \( \Lambda \).

Let \( q = p/r \). Then \( \min(q) \geq 1 \), and Young’s inequality applied on the last inequality gives
\[
\|F_0\|_{W^p(1,p_E^\Lambda)} = \|b_0^\frac{1}{r}\|_{p_E^\Lambda}\|b\|_{p_E^\Lambda} \lesssim \|\|a \cdot v\|^r \|b\|_{p_E^\Lambda}\|b\|_{p_E^\Lambda} \leq \|\|a\|_{\ell_p(\Lambda)}\|b\|_{\ell_p(\Lambda)} \leq \|a\|_{\ell_{p,q}(\Lambda)}\|b\|_{\ell_{p,q}(\Lambda)} \lesssim \|\phi\|_{M_{(p,q)}^1}\|b\|_{\ell_p(\Lambda)}. \tag{2.14}
\]
In the last steps we have used Hölder’s inequality and
\[
\|V_{\phi_0} \phi\|_{L^1_{(p,q)}(\mathbb{R}^d)} \asymp \|\|V_{\phi_0} \phi\|_{L^p_{(j+\kappa(E))}(\Theta_{\rho})(j)}\|_{\ell_p(\Lambda)} \asymp \|\phi\|_{M_{(p,q)}^1}.
\]

We have
\[
\|b_0^\frac{1}{r}\|_{p_E^\Lambda} \asymp \|\|F\|_{L^p_{(j+\kappa(E))}}\|_{\ell^2(\Lambda)} \}
\]
where \( \Lambda \) and \( \Lambda \) are lattices such that \( \Lambda \) contains \( \Lambda_E \), and \( \Lambda \) is \( n \) times as dense as \( \Lambda_E \). From these facts it follows by straight-forward computations that
\[
\|\|F\|_{L^p_{(j+\kappa(E))}}\|_{\ell^2(\Lambda)} \asymp \|\|F\|_{L^p_{(j+\kappa(E))}}\|_{\ell^2(\Lambda)} \}
\]
Here the second relation follows from the fact that \( \omega(x) \asymp \omega(j) \) when \( j \in \Lambda_E \) and \( x \in j+\kappa(E) \), which follows from (1.5). By combining these relations with (2.14) we get
\[
\|F_0\|_{W^p(1,p_E^\Lambda)} \lesssim \|F\|_{W^p(1,p_E^\Lambda)}. \]
Hence, Proposition 1.15 and the fact that we have already proved (2.10) when \( \phi \) equals \( \phi_0 \) gives

\[
\|V_\phi f\|_{W^r_E(\omega, \ell_{q}^p)} \asymp \|V_{\phi_0} f\|_{W^r_E(\omega, \ell_{q}^p)} \asymp \|F_0\|_{W^r_E(1, \ell_{q}^p)} \lesssim \|F\|_{W^r_E(1, \ell_{q}^p)} \asymp \|V_\phi f\|_{W^r_E(\omega, \ell_{q}^p)}.
\]

By combining Proposition 1.15 with Proposition 2.1 and Remark 2.2 we get the following.

**Proposition 2.6.** Let \( E_0 \) be a basis for \( R^d \), \( E'_0 \) be its dual basis, \( E = E_0 \times E'_0 \), \( q, r \in (0, \infty]^d \), \( \omega_0, v_0 \in \mathcal{P}_E(R^d) \) be such that \( \omega_0 \) is \( v_0 \)-moderate, \( \omega(x, \xi) = \omega_0(\xi), v(x, \xi) = v_0(\xi) \), \( \Theta_{\rho v} \) be as in (1.8) with strict inequality when \( r < 1 \), and let \( \phi \in M^1(\Theta_{\rho v})(R^d) \setminus 0 \). Then

\[
M^\infty_{E_0}(\omega) = M^\infty_{E_0}(\omega, L^q_{E_0}(R^d)), \quad W^\infty_{E_0}(\omega) = W^r_{E_0}(\omega, L^q_{E_0}(R^d)),
\]

and

\[
\|f\|_{M^\infty_{E_0}(\omega)} \asymp \|V_\phi f\|_{W^r_{E_0}(\omega, \ell_{q}^p, \ell_{q}^p)}; \quad \|f\|_{W^\infty_{E_0}(\omega)} \asymp \|V_\phi f\|_{W^r_{E_0}(\omega, \ell_{q}^p, \ell_{q}^p)}.
\]

2.3. **Periodic elements in modulation spaces.** By a straightforward combination of Propositions 1.18 and 2.6 we get the following. The details are left for the reader.

**Proposition 2.7.** Let \( E_0 \) be a basis for \( R^d \), \( E'_0 \) be its dual basis, \( E = E_0 \times E'_0 \), \( q, r \in (0, \infty]^d \), \( \omega_0, v_0 \in \mathcal{P}_E(R^d) \) be such that \( \omega_0 \) is \( v_0 \)-moderate, \( \omega(x, \xi) = \omega_0(\xi), v(x, \xi) = v_0(\xi) \), \( \Theta_{\rho v} \) be as in (1.8) with strict inequality when \( r < 1 \), and let \( \phi \in M^1(\Theta_{\rho v})(R^d) \setminus 0 \). Then

\[
\|f\|_{M^\infty_{E_0}(\omega)} \asymp \|f\|_{W^\infty_{E_0}(\omega)} \asymp \|f\|_{M^\infty_{E_0}(\omega, L^q_{E_0})}
\]

\[
\asymp \|f\|_{W^\infty_{E_0}(\omega, L^q_{E_0})} \asymp \|f\|_{L^p_{E_0}(\omega, L^q_{E_0}(N_{E_0}))}, \quad f \in (E^0_{E_0})'(R^d). \tag{2.15}
\]

As an immediate consequence of the previous result we get the following extension of Proposition 1.18. The details are left for the reader.

**Proposition 1.18.** Let \( E \) be a basis of the \( R^d \), \( r \in (0, \infty]^d \), \( r \in (0, 1] \), \( B \subseteq L^r_{\text{loc}}(R^d) \) be an \( E' \)-split Lebesgue space, \( \ell_{\rho v}(E) \) its discrete version, and let \( \omega = \rho \in \mathcal{P}_E(R^d) \). Then

\[
\mathcal{E}_E(\omega, \ell_{\rho v}) = M^\infty(\omega, B) \cap \mathcal{E}_E(\omega) = W^r_E(\omega, B) \cap (E^0_E)'(R^d).
\]

**Remark 2.8.** Let \( E_0 = \{e_1, \ldots, e_d\}, \quad E'_0 = \{\varepsilon_1, \ldots, \varepsilon_d\}, \quad q = (q_1, \ldots, q_d), \quad r = (r_1, \ldots, r_d), \omega, v \) and \( \phi \) be the same as in Proposition 2.7, and let \( r_0 \leq \min(r) \) and \( f \in f \in (E^0_{E_0})'(R^d) \) with Fourier series expansion \( 1.9 \). Then (1.10), (1.12) and (2.15) imply that

\[
\|V_\phi f \cdot \omega\|_{L^{q,r}_{E_0}(\omega, \ell_{q,r}(R^d \times \kappa(E_0)))} \asymp \|c(f, \cdot)\|_{L^{q,r}_{E_0}(\omega_0)}.
\]

(2.16)
Let \( \| \cdot \| \) be the quasi-norm on the left-hand side of (2.16), after the orders of the involved \( L_{q_k}^{\kappa_k}(\kappa(e_k)) \) quasi-norms have been permuted in such way that the internal order of the hitting \( L_{q_k}^{\kappa_k}(\kappa(E_0)) \) quasi-norms remains the same. Then

\[
\| F \|_{L_{E_0}^{q_0}(\kappa(E_0) \times \mathbb{R}^d)} \lesssim \| F \|_{L_{E_0}^{\infty,q}(\kappa(E_0) \times \mathbb{R}^d)};
\]

by repeated application of Hölder’s inequality. A combination of (2.16) and (2.17) give

\[
\| V \varphi f \cdot \omega \| \approx \| c(f, \cdot) \|_{\ell_{q_E}^{\kappa(E_0)\times\mathbb{R}^d}}.
\]

In particular, if \( e_j \) are the same as in Remark 1.5, \( E_* \) is the ordered basis \( \{ e_1, e_{d+1}, \ldots, e_d, e_{2d} \} \) of \( \mathbb{R}^{2d} \),

\[
\Omega = \left\{ y_1 e_1 + \cdots + y_{2d} e_{2d} : 0 \leq y_j \leq 1 \text{ and } y_{d+j} \in \mathbb{R}, j = 1, \ldots, d \right\}
\]

and \( q_0 = (q_1, q_1, q_2, \ldots, q_d, q_d) \in (0, \infty)^{2d} \), then

\[
\| V \varphi f \cdot \omega \|_{L_{E'}^{q_0}(\Omega)} \approx \| c(f, \cdot) \|_{\ell_{E_0'}^{q_0}(\omega_0)}.
\]

Remark 2.9. With the same notation as in the previous remark, we note that if \( E'_0 \) is the standard basis of \( \mathbb{R}^d \), \( X_j = (x_j, \xi_j) \), \( j = 1, \ldots, d \), \( I = \mathbb{R} \times [0, 2\pi] \) and \( \max(q) < \infty \), then (2.19) is the same as

\[
\left( \int_I \left( \cdots \left( \int_I | V \varphi f(x, \xi) \omega_0(\xi)|^q \right)^{\frac{1}{q_1}} \cdots \right)^{\frac{1}{q_d}} \cdot \right)^{\frac{1}{p_d}} \cdot \left( \cdots \left( \sum_{\alpha_1 \in \mathbb{Z}} | c(f, \alpha) \omega_0(\alpha)|^{\frac{2}{\alpha_1^q}} \right)^{\frac{1}{q_1}} \cdots \right)^{\frac{1}{q_d}}
\]

\[
\left( \cdots \left( \sum_{\alpha \in \mathbb{Z}} | c(f, \alpha) \omega_0(\alpha)|^{\frac{2}{\alpha^q}} \right)^{\frac{1}{q_1}} \cdots \right)^{\frac{1}{q_d}}
\]

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