WALL-CROSSING IN GENUS ZERO LANDAU-GINZBURG THEORY

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Abstract. We study genus zero wall crossing for a family of moduli spaces introduced recently by Fan–Jarvis–Ruan. The family has a wall and chamber structure relative to a positive rational parameter $\epsilon$. For a Fermat quasi-homogeneous polynomial $W$ (not necessarily Calabi-Yau type), we study natural generating functions of invariants associated to these moduli spaces. Our wall-crossing formula relates the generating functions by showing that they all lie on the same Lagrangian cone associated to the Fan–Jarvis–Ruan–Witten theory of $W$. For arbitrarily small $\epsilon$, a specialization of our generating function is a hypergeometric series called the big $I$-function which determines the entire Lagrangian cone. As a special case of our wall-crossing, we obtain a new geometric interpretation of the Landau-Ginzburg mirror theorem.

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Introduction

Let $W(x_1,\ldots,x_N)$ be a quasi-homogeneous polynomial of weight $(w_1,\ldots,w_N)$ and degree $d$:

\begin{equation}
W(\lambda^{w_1} x_1, \ldots, \lambda^{w_N} x_N) = \lambda^d W(x_1,\ldots,x_N).
\end{equation}
We require that $\gcd(w_1, \ldots, w_N, d) = 1$ and we define the charges $q_j := \frac{w_j}{d}$ and set $q := \sum q_j$. We also require $W$ to be non-degenerate in the sense that the charges $q_j$ are uniquely determined from $W$ and the affine variety defined by $W$ is singular only at the origin. Let $X_W$ denote the hypersurface defined by the vanishing of $W$ in weighted projective space $\mathbb{P}(w_1, \ldots, w_N)$.

The Landau-Ginzburg/Calabi-Yau correspondence asserts the equivalence of two cohomological field theories (CohFTs):

1. Fan-Jarvis-Ruan-Witten (FJRW) theory of $W$ – defined via intersection numbers on moduli spaces of curves equipped with $W$-structures, and
2. Gromov-Witten (GW) theory of $X_W$ – defined via intersection numbers on moduli spaces of stable maps to $X_W$.

The correspondence was proved by Chiodo-Iritani-Ruan in genus zero for Calabi-Yau Fermat polynomials (i.e. $w_j \mid d$, $W = \sum x_j^{d/w_j}$, and $q = 1$) [CIR12]. Their proof developed a mirror theorem which relates the FJRW theory of $W$ to a hypergeometric series encoding the solutions of the Picard-Fuchs equations on the mirror family near the Gepner point. The correspondence with GW theory follows by analytically continuing the solutions from the Gepner point to the large complex structure limit where the analogous mirror theorem was proved by Givental and Lian–Liu–Yau [Giv 96, LLY97].

Rather than use mirror symmetry, Witten has suggested that for any such $W$ there exists a family of A-model CohFTs which interpolates directly between GW and FJRW theory, this is the theory of the gauged linear sigma model (GLSM) [Wit97]. One expects that the LG/CY correspondence can be realized by tracking the CohFTs across this family, leading to the idea of A-model wall-crossing. One motivation for the wall-crossing perspective is that it appears to give a better approach to the higher genus comparison.

In fact, Witten’s GLSM applies in much greater generality than hypersurfaces in weighted projective space and Fan–Jarvis–Ruan have recently provided a mathematically rigorous definition of the general GLSM [FJR]. In the case of hypersurfaces, the GLSM defines a family of CohFTs over the nonzero rational numbers which subdivides into a natural wall and chamber structure. Over the positive rationals lies the so-called geometric phase. This phase corresponds to a “quasi-map” version of stable maps with a field introduced by Chang–Li [CL11]. It is expected to be equivalent to the quasi-map theory defined by Ciocan-Fontanine–Kim–Maulik [CFK11].

Recently, Ciocan-Fontanine–Kim proved a wall-crossing formula which describes how the genus zero quasi-map invariants of $X_W$ change as the parameter varies [CFK13b, CFK13a]. Moreover, they proved that their wall-crossing formula recovers the mirror theorem for hypersurfaces by approaching the asymptotic chamber at $0^+$.

The purpose of this paper is to study the family of CohFTs which arise over the negative rationals corresponding to the Landau-Ginzburg phase. In particular, for a Fermat polynomial $W$ we prove a wall-crossing formula.
analogous to that of Ciocan-Fontanine–Kim. Moreover, we show that our wall-crossing formula recovers the LG mirror theorem by approaching the asymptotic chamber at 0−.

We now give a more in-depth overview of our results. We refer the reader to Section 1 for precise definitions.

Statement of Results. We study a family of moduli spaces \( R_{d,\vec{k},\vec{\epsilon}} \) where \( \epsilon \in \mathbb{Q} > 0 \) (for notational convenience, we renormalize to obtain a positive rational parameter). These moduli spaces parametrize \( \epsilon \)-stable pairs \((C,L)\) where \( C \) is a weighted stable orbifold curve with \( m \) marked orbifold points of weight one and \( n \) marked smooth points of weight \( \epsilon \), and \( L \) is an orbifold line bundle which is a \( d \)th root of an appropriate twist of the log canonical bundle:

\[
L^\otimes d \cong \omega_{\log}(-D_{\vec{l}})
\]

The vector \( \vec{k} \) records the multiplicities of the line bundles at the orbifold marked points.

For a Fermat polynomial, numerical invariants

\[
\langle \phi_{k_1} \psi^{j_1}, \ldots, \phi_{k_m} \psi^{j_m} \rvert \phi_{l_1}, \ldots, \phi_{l_n} \rangle_{m,n}^{W,\epsilon} \in \mathbb{Q}
\]

for narrow insertions \( \phi_i \in H'_W \) are obtained by capping tautological psi classes against a “virtual class” in \( R_{d,\vec{k},\vec{\epsilon}} \). We package these invariants in generating functions using the double bracket notation:

\[
\langle \langle \phi_{k_1} \psi^{j_1}, \ldots, \phi_{k_l} \psi^{j_l} \rangle \rangle_{W,\epsilon}^{W,\epsilon}(t,u) := \sum_{m,n} \frac{1}{m!n!} \langle \phi_{k_1} \psi^{j_1}, \ldots, \phi_{k_l} \psi^{j_l}, t(\psi)^m \rvert u^n \rangle_{W,\epsilon}^{W,\epsilon}
\]

where \( t(z) := \sum t^k \phi_k z^j \in H'_W[z] \), \( u := \sum u^k \phi_k \in H'_W \), and

\[
t(\psi)^m := t(\psi_1), \ldots, t(\psi_m).
\]

The sum is over all \( m, n \geq 0 \) for which the underlying moduli space exists (ie. we exclude terms where \( l + m + n < 3 \)). We remark that, in contrast to the double bracket notation typically used in GW theory, our notation encodes arbitrary descendant insertions.

The genus zero descendant potential is defined by

\[
F_W^\epsilon(t,u) := \langle \langle \rangle \rangle^{W,\epsilon}_0 = \sum_{m,n} \frac{1}{m!n!} \langle t(\psi)^m \rvert u^n \rangle_{m,n}^{W,\epsilon}.
\]

For \( \epsilon > 1 \) (denoted \( \epsilon = \infty \)), the invariants vanish for \( n > 0 \) and we recover the narrow FJRW descendant potential \( F^\infty(t) \).
For $\epsilon > 0$ we package the $t$ derivatives of $F^\epsilon$ in the large $J^\epsilon$-function:

$$J^\epsilon(t, u, z) := z \phi_0 \sum_{a_i \geq 0} \frac{1}{a_i!} \left( \frac{u^i \phi_i}{z} \right)^{a_i} \prod_{j=1}^{N} \prod_{0 \leq b < \sum_{a_i \langle iq_j \rangle}} b z$$

$$+ t(-z) + \sum_{k} \phi_k \left( \frac{\phi_k}{z - \psi_{m+1}} \right) \epsilon_1$$

where $\langle * \rangle$ denotes the fractional part of $*$. The initial sum in (3) corresponds to the unstable terms where the underlying moduli spaces in the double bracket do not exist. In Section 2.1 we give an equivariant interpretation of the terms in the $J$-functions, in particular we show how the unstable terms arise naturally in localization computations on graph spaces.

Following Givental, we encode the FJRW theory of $W$ in the graph of the differential $L := dF^\infty$. The tautological equations satisfied by $F^\infty(t)$ imply that $L$ has very special geometric properties. By definition, $J^\infty(t, -z)$ gives a generic point on $L$, our main result extends this property to all $\epsilon$.

**Theorem 1** (Theorem 1.11). Assume $W$ is Fermat. For all $\epsilon > 0$, $J^\epsilon(t, u, -z)$ is a $H[[u]]$-valued point on $L$.

**Theorem 2** (Theorem 3.1). For any $\epsilon_1, \epsilon_2 > 0$,

$$\frac{J^{\epsilon_1}(\tau^{\epsilon_1}(t, u), u, z)}{J_0^{\epsilon_1}(u)} = \frac{J^{\epsilon_2}(\tau^{\epsilon_2}(t, u), u, z)}{J_0^{\epsilon_2}(u)}$$

for some scalar functions $J_0^\epsilon(u) = 1 + O(u)$ and invertible change of variables $\tau^\epsilon(t, u)$. In particular, since $J$ encodes the $t$ derivatives of $F$, we conclude that any genus zero descendant invariant for $\epsilon_1$ which is not constant in $t$ can be recovered from descendant invariants for $\epsilon_2$.

The chamber $\epsilon = \infty$ is important as it recovers FJRW theory. Another important chamber occurs in the limit $\epsilon \to 0$ (denoted $\epsilon = 0$). Define the big $I$-function to be the formal series obtained by taking $\epsilon \to 0$ and $t = 0$ in (3):

$$I(u, z) = z \phi_0 \sum_{a_i \geq 0} \frac{1}{a_i!} \left( \frac{u^i \phi_i}{z} \right)^{a_i} \prod_{j=1}^{N} \prod_{0 \leq b < \sum_{a_i \langle iq_j \rangle}} b z.$$

The next result is obtained from Theorem 1 and the special geometry of $L$.

**Theorem 3** (Theorem 3.3). The big $I$-function $I(u, -z)$ is a $H[[u]]$-valued point of $L$. Moreover, $I(u, -z)$ determines the entire cone.
Theorem 3 can be viewed as a mirror theorem. It was proved previously by twisted theory formalism in the CY case ($q = 1$) by Chiodo–Iritani–Ruan [CIR12]. For $q \neq 1$, it is proved independently by Acosta [Aco]. If we specialize the big I-function by restricting to the degree $\leq 1$ part $u \in H^I_{\deg(\phi_i) \leq 1}$, then we obtain the small I-function:

$$I(u, z) := \left. I(u, z) \right|_{u^k = 0 \text{ if } \deg(\phi_k) > 1}.$$

As a special case of Theorem 1 we recover the usual LG mirror theorem which relates the small I-function to the FJRW $J$-function $J(t_0, z)$ defined by restricting $J^\infty$ to primary variables $t_0 = (t^k_0)$.

**Corollary 4 (Corollary 3.4).**

$$I(u, z) = I_0(u)z\phi_0 + I_1(u) + O\left(\frac{1}{z}\right)$$

and

$$\frac{I(u, z)}{I_0(u)} = J(\eta(u), z).$$

where $\eta(u) = \frac{I_1(u)}{I_0(u)}$ is the mirror map.

A new aspect of our proof of the mirror theorem is that we obtain an enumerative interpretation of the small I-function in terms of two point invariants at $\epsilon = 0$.

**Theorem 5 (Theorem 4.3).**

$$I(u, z) = zI_0(u)\phi_0 + z\sum_k \phi^k \left\langle \left\langle I_0(u)\phi_0, \frac{\phi_k}{z - \psi_2} \right\rangle \right\rangle^0_{2}(0, u)$$

Namely, after a correction of $\phi_0$ by $I_0(u)$ (to account for the failure of the string equation), the small I-function is equal to the small J-function without any change of variable.

The analog of Theorem 5 in the geometric phase was first proved by Cooper–Zinger [CZ12].

**Plan of the Paper.** In Section 1 we introduce weighted FJRW theory in the narrow, Fermat case, generalizing the original FJRW definitions. We define the moduli spaces, invariants, and generating functions which appear in Theorem 1. Our proof of the wall-crossing formula is a two step process contained in Section 2. In Section 2.1 we use torus actions on graph spaces to produce relations among the coefficients of the large $J$-functions. In Section 2.2 we give an explicit characterization of points on $L_W$ which reduces to the relations discussed in Section 2.1. In Section 3 we deduce Theorems 2 and 3 from Theorem 1. In Section 4 we exhibit a Birkhoff factorization of the large $J$-functions and we prove Theorem 5.
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1. Weighted Landau–Ginzburg Theory and Wall-Crossing

1.1. Moduli Spaces. The mathematical theory of GLSMs introduced recently by Fan-Jarvis-Ruan applies to a general situation of GIT-quotients. The construction greatly simplifies for Fermat hypersurfaces in weighted projective space. For the reader’s convenience, we present a self-contained description of the construction in this setting.

Definition 1.1. For $\epsilon \in \mathbb{Q}_{>0}$, a $(d, \epsilon)$-stable rational curve is a rational, connected, at worst nodal orbifold curve $C$ along with $m$ distinct orbifold marked points $x_1, \ldots, x_m$ and $n$ (not necessarily distinct) smooth marked points $y_1, \ldots, y_n$ satisfying

i. all nodes and orbifold marks $x_i$ have cyclic isotropy $\mu_d$ and the orbifold structure is trivial away from these points;

ii. $\text{mult}_z (\epsilon \sum [y_i]) \leq 1$ at all points $z \in C$;

iii. $\omega_{\log}(\epsilon \sum [y_i]) := \omega_C(\sum [x_i] + \epsilon \sum [y_i])$ is ample where $\omega_C$ is the orbifold dualizing sheaf and $[x_i]$ is the orbifold point $\mathcal{B}_{\mu_d}$.

We denote the moduli space of $(d, \epsilon)$-stable curves by $\overline{\mathcal{M}}_{0,m+\epsilon n}^d$.

By forgetting the orbifold structure, there is a morphism

$\overline{\mathcal{M}}_{0,m+\epsilon n}^d \to \overline{\mathcal{M}}_{0,m+\epsilon n}$

where the latter is the moduli space of Hassett weighted stable curves [Has03].

Definition 1.2. For $\vec{l} := (l_1, \ldots, l_n)$ with $0 \leq l_i \leq d - 1$, a $\vec{l}$-twisted $d$-spin structure on a $(d, \epsilon)$-stable curve consists of an orbifold line bundle $L$ and an isomorphism

$L^\otimes d \xrightarrow{\kappa} \omega_{\log} \otimes \mathcal{O}(- \sum l_i [y_i])$.

We denote the moduli space of $(d, \epsilon)$-stable curves with $\vec{l}$-twisted $d$-spin structures by $\mathcal{R}_{m,\epsilon,\vec{l}}^d$. 

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We recover the usual moduli spaces of \( d \)-spin curves by specializing \( n = 0 \). The spaces \( R^d_{m,\epsilon} \) consist of a disjoint union of components indexed by the multiplicity of the line bundle at the orbifold points \( x_i \).

**Definition 1.3.** For \( \tilde{k} = (k_1, \ldots, k_m) \) with \( 0 \leq k \leq d-1 \), let \( R^d_{\tilde{k},\epsilon} \) denote the component in \( R^d_{m,\epsilon} \) where \( \text{mult}_{x_i} L = k_i + 1 \).

**Remark 1.4.** The spaces \( R^d_{\tilde{k},\epsilon} \) considered here are slightly more general than the spaces which appear in the GLSM of \( W \). In particular, the GLSM is modeled on moduli spaces where \( \epsilon_i = 1 \) for all \( i \).

The isomorphism \( \kappa \) imposes a non-emptiness condition on the spaces \( R^d_{\tilde{k},\epsilon} \) by comparing degrees of line bundles. Explicitly, we compute

\[
\dim \left( R^d_{\tilde{k},\epsilon} \right) = \begin{cases} m + n - 3 + 2 \sum k_i + \sum \epsilon_i & \text{if } k \in d\mathbb{Z} \\ -1 & \text{else} \end{cases}
\]

**Graph spaces** \( R^G_{G,d,\tilde{k},\epsilon} \) are defined by enriching the objects in \( R^d_{\tilde{k},\epsilon} \) with a degree one map \( f : C \to P^1 \). The map to \( P^1 \) has the effect of parametrizing an irreducible component \( \hat{C} \subset C \) and we only require that \( \omega_{\log}(\epsilon \sum [y_i]) \) is ample on \( C \setminus \hat{C} \). The non-emptiness condition on graph spaces is the same as (4). When a graph space is nonempty, its dimension is three greater than the dimension of the corresponding non-graph space.

By forgetting the line bundle \( L \) and the orbifold structure on \( C \), the weighted spin spaces admit maps to Hassett weighted (graph) spaces:

\[
\theta : R^d_{k,\epsilon} \to \overline{M}_{0,m+n} \quad \text{and} \quad \theta : R^G_{k,\epsilon} \to \overline{M}_{0,m+n}(P^1,1).
\]

We define \( \psi \)-classes on \( R^G_{k,\epsilon} \) by \( \psi_i := c_1(\theta^*\mathbb{L}_i) \) where \( \mathbb{L}_i \) is the cotangent line bundle.

**1.2. Virtual Classes and Invariants.** We define the extended state space associated to \( W \) to be \( H_W := \mathbb{Q}^d \) with basis vectors denoted by \( \phi_k \) with \( 0 \leq k \leq d-1 \). In the definition of the unstable terms of \( J \), we use the multiplication on \( H_W \) defined by

\[
\phi_i \cdot \phi_j := \phi_{i+j} \mod d
\]

as a book keeping device. The reader should not confuse it with the quantum ring structure of the theory. The narrow sector in the state space \( H'_W = \bigoplus_{k \in \text{nar}} \mathbb{Q} \phi_k \) is obtained by restricting to the vectors indexed by the set

\[
\text{nar} := \{ k : \text{for all } j, \langle q_j (k + 1) \rangle \neq 0 \}.
\]

The complement of the narrow sector in the extended state space is referred to as the broad sector. There is a perfect pairing on \( H'_W \) defined by

\[
(\phi_i, \phi_j)_W := \delta_{i,d-2-j}.
\]

We set \( \phi^i := \phi_{d-2-i} \).
Remark 1.5. Narrow states have a geometric interpretation in terms of \( \mathcal{R}^d_{k_i l_i} \). Specifically, the condition \( k_i \in \text{nar} \) is equivalent to the condition that \( \text{mult}_x L_{w_j} \neq 0 \) for all \( j \). We similarly call an orbifold node \( z \) to be narrow if \( \text{mult}_z L_{w_j} \neq 0 \) for all \( j \) and we call it broad otherwise.

From this point on, assume \( W \) is Fermat so that \( w_j | d \) for all \( j \). With this assumption, the genus zero theory simplifies significantly due to the fact that we can write the virtual class as the Poincaré dual of the top Chern class of an appropriate vector bundle, which we call the obstruction bundle.

To define the obstruction bundle, we define integers \( s_{i,j} \) and \( l_{i,j} \) by

\[
l_i =: \frac{d}{w_j} + l_{i,j} \quad \text{for some} \quad 0 \leq l_{i,j} < \frac{d}{w_j}
\]

and we set

\[
L_j := L_{w_j} \otimes O \left( \sum_i s_{i,j} \left[ y_i \right] \right).
\]

We have the following vanishing result on cohomology.

Lemma 1.6 (Concavity). For \( \vec{k} = (k_1, \ldots, k_m) \) with \( k_i \in \text{nar} \) for all \( i \),

\[
(5) \quad H^0(C, L_j) = 0
\]

for all \( C \) points of \( \mathcal{R}^d_{k_i l_i} \).

Proof. Let \( Z \subset C \) be an irreducible component and denote the restriction of \( L_j \) to \( Z \) by \( L_j(Z) \). By definition,

\[
L_j(Z)^{\otimes \frac{d}{w_j}} \cong \left( \omega_{\log} \otimes O(- \sum_i l_{i,j} \left[ y_i \right]) \right) \big|_Z.
\]

In particular,

\[
\deg \left( L_j(Z)^{\otimes \frac{d}{w_j}} \right) \leq -2 + m + \frac{\# \{ Z \cap C \setminus Z \} }{d}.
\]

Pushing \( L_j(Z) \) forward to the coarse curve \( |Z| \), we obtain

\[
\deg \left( |L_j(Z)|^{\otimes \frac{d}{w_j}} \right) \leq -2 + m - \sum_{i=1}^m \frac{d}{w_j} \left( \frac{w_j(k_i+1)}{d} \right) + \frac{\# \{ Z \cap C \setminus Z \} }{d}.
\]

Since \( k_i \in \text{nar} \) and \( w_j | d \), \( \frac{d}{w_j} \left( \frac{w_j(k_i+1)}{d} \right) \geq 1 \) for all \( i \). Therefore,

\[
\deg \left( |L_j(Z)|^{\otimes \frac{d}{w_j}} \right) < \# \{ Z \cap C \setminus Z \} - 1.
\]

In particular, by an easy induction on the number of components of \( |C| \), this implies that \( H^0(|C|, |L_j|^{\otimes \frac{d}{w_j}}) = 0 \) and the lemma follows from the fact that

\[
H^0(C, L_j) = H^0(|C|, |L_j|) \hookrightarrow H^0(|C|, |L_j|^{\otimes \frac{d}{w_j}}).
\]

\( \square \)
Remark 1.7. We point out that the proof of Lemma 1.6 carries through unchanged even if one of the $k_i$ is broad.

Let $\pi : C \to R^d_{\tilde{k},l}$ be the universal curve. Due to concavity, $R^1\pi_*L_j$ is a vector bundle when $k_i \in \text{nar}$. We define the obstruction bundle as

$$
Ob := \bigoplus R^1\pi_*L_j
$$

and we define the virtual class in the narrow sector by

$$
\left[W_{\tilde{k},l}\right]^{\text{vir}} := c_{\text{top}}(Ob)^{\vee} \in H_*(R^d_{\tilde{k},l}; \mathbb{Q})
$$

where $*^{\vee}$ denotes the Poincaré dual of $*$. The (complex) homological degree of $\left[W_{\tilde{k},l}\right]^{\text{vir}}$ can be computed with orbifold Riemann-Roch:

$$
\deg_{C} \left(\left[W_{\tilde{k},l}\right]^{\text{vir}}\right) = N - 3 - 2q + m - \sum_i \deg(\phi_k) + n - \sum_{i,j} \langle q_j l_i \rangle
$$

where

$$
\deg(\phi_k) := -q + \sum_j \langle q_j (k + 1) \rangle = \sum_j \langle q_j k \rangle.
$$

The equality in $\deg(\phi_k)$ relies both on the fact that $k$ is narrow and that $W$ is Fermat.

Remark 1.8. We remark that the obstruction bundle defined here is exactly the obstruction bundle defined in FJRW theory of Fermat polynomials upon restricting $n = 0$. Moreover, it also agrees with the obstruction bundle in the GLSM of Fermat polynomials upon restricting $l_i = 1$ which forces $s_{i,j} = 0$ (cf. Remark 1.4).

We have the following useful vanishing property which plays a significant role in our localization computations.

Lemma 1.9 (Ramond Vanishing). If $D$ is a boundary divisor in $R^d_{\tilde{k},l}$ with a broad node,

$$
\left[W_{\tilde{k},l}\right]^{\text{vir}} \cap D = 0.
$$

Proof. Suppose that the broad node $z$ has orbifold structure $\frac{2}{d}$. Since it is broad, $\frac{w_j}{d}$ is an integer for some $w_j$. Letting $C_1$ and $C_2$ denote the components of $C$ separated by $z$, we have a short exact sequence on $D$:

$$
0 \to \pi_*L_j|_z \to R^1\pi_*L_j \to R^1\pi_*L_j|_{C_1} \oplus R^1\pi_*L_j|_{C_2} \to 0
$$

where the vanishing of the initial terms follows from concavity and Remark 1.7. Furthermore,

$$
\pi_* \left( L_j|_z \otimes \frac{d}{w_j} \right) \cong \pi_* (\omega C|_z).
$$
The latter is trivial. Therefore, \( c_1(\pi_* L_j | z) = 0 \) which implies
\[
e \left( \bigoplus_j R^1 \pi_* L_j \right) \cap D = 0.
\]
\( \square \)

**Remark 1.10.** Concavity fails in higher genus, even for \( W \) Fermat. However, an algebraic virtual class can still be defined in the narrow sector and Ramond vanishing continues to hold in higher genus if \( W \) is Fermat [FJR].

Numerical invariants are defined by integrating \( \psi \)-classes over the virtual class:
\[
\langle \phi_{k_1} \psi^{j_1}, \ldots, \phi_{k_m} \psi^{j_m} | \phi_{l_1}, \ldots, \phi_{l_n} \rangle_{W, \epsilon}^{G} := d \int \left[ W_{k, \epsilon} \right] \text{vir} \prod \psi^{j_i} \in \mathbb{Q}.
\]
For light points, we use \( \phi_i \) as a convenient book-keeping device. The readers should not confuse it with insertions at the orbifold points. The invariants are defined to vanish if any of the \( k_i, l_i \) are broad or if the underlying moduli space does not exist.

We define
\[
\langle \langle \phi_{k_1} \psi^{j_1}, \ldots, \phi_{k_m} \psi^{j_m} \rangle \rangle_{W, \epsilon}^{G} (t, u) := \sum_{m,n} \frac{1}{m! n!} \langle \phi_{k_1} \psi^{j_1}, \ldots, \phi_{k_m} \psi^{j_m}, t(\psi)^m | u^n \rangle_{W, \epsilon}^{G}.
\]
where \( t(z) := \sum t^k \phi_k z^j \in H^*_W[z], u := \sum u^k \phi_k \in H^*_W, \) and
\[
t(\psi)^m := t(\psi_1), \ldots, t(\psi_m).
\]

The genus zero descendant potential is
\[
\mathcal{F}_W (t, u) := \langle \langle \rangle \rangle_{W, \epsilon}^{G} = \sum_{m,n} \frac{1}{m! n!} \langle t(\psi)^m | u^n \rangle_{W, \epsilon}^{G}.
\]

Graph space invariants will be important in our proof of the wall-crossing formula. Concavity and Ramond vanishing continue to hold for graph spaces \( W_{k, \epsilon}^{G} \). We define a virtual class in graph spaces as in (5). It has homological degree
\[
\text{deg}_C \left( \left[ W_{k, \epsilon}^{G} \right] \text{vir} \right) = N - 2q + m - \sum_i \text{deg}(\phi_k) + n - \sum_{i,j} \langle a_j l_i \rangle.
\]
This allows us to define the graph invariants
\[
\langle \tau_{j_1}(\phi_{k_1}), \ldots, \tau_{j_m}(\phi_{k_m}) | \phi_{l_1}, \ldots, \phi_{l_n} \rangle_{m,n}^{G, W, \epsilon}
\]
as in (7).
1.3. Givental’s Symplectic Formalism. Following Givental, we define the infinite dimensional vector space \( \mathcal{H} := H'(z^{-1}) \) with symplectic form
\[
\Omega(f(z), g(z)) := \text{Res}_{z=0}(f(z), g(-z))W.
\]
\( \mathcal{H} \) has a natural polarization \( \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \) where
\[
\mathcal{H}^+ := H'[z] \quad \text{and} \quad \mathcal{H}^- := z^{-1}H'([z^{-1}])
\]
and we identify \( \mathcal{H} \) with the cotangent space \( T^*\mathcal{H}^+ \). There are Darboux coordinates \((q, p)\) on \( \mathcal{H} \) with \( q \in \mathcal{H}^+, p \in \mathcal{H}^- \). Via the dilaton shift
\[
q(z) := t(z) - \phi_0 z,
\]
we can think of \( \mathcal{F}^\infty(t) \) as a formal function on \( \mathcal{H}^+ \). We define \( L \) to be the graph of the differential of \( \mathcal{F}^\infty(t) \) in \( \mathcal{H} \):
\[
L := \{ (q, p) \in \mathcal{H} : p = \partial_q \mathcal{F}^\infty(t) \}.
\]
Givental showed that the tautological equations satisfied by \( \mathcal{F}^\infty \) are equivalent to the fact that \( L \) is the formal germ of a Lagrangian cone with vertex at the origin such that each tangent space \( T \) to the cone is tangent to the cone exactly along \( zT \) \cite{Givental04}.

1.4. Wall-Crossing Formula. Our wall crossing formula relates the large \( \mathcal{F}^\epsilon \)-functions which package the \( t \) derivatives of \( \mathcal{F}^\epsilon \):
\[
\mathcal{J}^\epsilon(t, u, z) := z \phi_0 \sum_{a_i \geq 0, \text{int} \atop \sum a_i \leq \frac{1}{\epsilon}} \frac{1}{a_i!} \left( \frac{u' \phi_i}{z} \right)^{a_i} \prod_{l=1}^N \prod_{0 < j < \sum a_i (\phi_i)} b z^j
\]
\[
+ t(-z) + \sum_k \phi_k \left\langle \left\langle \frac{\phi_k}{z - \psi_{m+1}} \right\rangle \right\rangle_1^\epsilon
\]
(8)
where \( \langle * \rangle \) denotes the fractional part of \( * \). When \( \epsilon > 1 \), we have
\[
\mathcal{J}^\infty(t, -z) = -z \phi_0 + t(z) + \sum_k \phi_k \left\langle \left\langle \frac{\phi_k}{-z - \psi_{m+1}} \right\rangle \right\rangle_1^\infty (t)
\]
which (by definition) is a generic point on \( \mathcal{L} \). More generally, by a \( \mathcal{H}[[u]] \)-valued point on \( \mathcal{L} \), we mean a formal series in \( H'[t, u](z^{-1}) \) which is of the form
\[
- z \phi_0 + \hat{t}(z) + \sum_k \phi_k \left\langle \left\langle \frac{\phi_k}{-z - \psi_{m+1}} \right\rangle \right\rangle_1^\infty (\hat{t})
\]
(9)
for some \( \hat{t}(z) = t(z) + O(u) \in \mathcal{H}^+[[u]] \). Our main result is the following.

**Theorem 1.11.** For all \( \epsilon > 0 \), \( \mathcal{J}^\epsilon(t, u, -z) \) is a \( \mathcal{H}[[u]] \)-valued point on \( \mathcal{L} \).
2. **Proof of Wall-Crossing**

2.1. **J-function Relations.** In this section we derive a set of universal relations satisfied by the coefficients of the \( J \)-functions. In the process, we give an equivariant description of the \( J \)-functions in terms of fixed-point contributions to certain auxiliary integrals on graph spaces.

To ease notation, we omit \( W \) when confusion does not arise.

**Lemma 2.1.** For every \( \epsilon > 0 \), the series

\[
(10) \quad (\partial_u^r J^\epsilon(t,u,z), \partial_t^s J^\epsilon(t,u,-z))
\]

is regular at \( z = 0 \) for all \( r,s \).

**Proof.** We prove the lemma via Atiyah-Bott localization on graph spaces, we proceed in several steps.

**Step 1: Equivariant Setup.** Let \( C^* \) act on \( \mathbb{P}^1 \) via \( \lambda z_0 : z_1 := [z_0 : \lambda z_1] \), so that the tangent bundle \( T\mathbb{P}^1 \) has weights 1 at 0 = [0 : 1] and \(-1 \) at \( \infty = [1 : 0] \). Define the equivariant class \([0] := c_1(\mathcal{O}(1)) \in H^{*\epsilon}(\mathbb{P}^1)\) where \( \mathcal{O}(1) \) is linearized with weights 1 at 0 and 0 at \( \infty \). Similarly define \([\infty] \) by linearizing \( \mathcal{O}(1) \) with weights 0 at 0 and \(-1 \) at \( \infty \).

The \( C^* \) action on \( \mathbb{P}^1 \) naturally induces an action on \( R_{\vec{k},\vec{l}}^{G,d} \) and it lifts canonically to an action on the obstruction bundle. There are (equivariant) evaluation maps

\[
ev_i, \tilde{ev}_j : R_{\vec{k},\vec{l}}^{G,d} \to \mathbb{P}^1
\]

which record the image of the \( x_i, y_j \), respectively, under the map \( f \).

Define invariants \( \langle \alpha | \beta | \gamma \rangle_{G,\epsilon}^{m,n} \) by cupping the integrand in the definition of \( \langle \alpha | \beta | \gamma \rangle_{G,\epsilon}^{m,n} \) with \( \gamma \). In the presence of a torus action, integration is the equivariant push-forward to a point.

**Step 2: Equivariant Interpretation of (10).** Consider the equivariant series

\[
(11) \quad \sum_{m,n \geq 0} \frac{1}{m!n!} (t(-\psi)^m, \phi_s | u^n, \phi_r | ev^*_{m+1}(\infty) \cup \tilde{ev}^*_{n+1}(0))_{G,\epsilon}^{m+1,n+1}.
\]

By definition, (11) is regular at \( z = 0 \). Therefore the lemma follows from the claim that (10) = (11). We prove this claim in Steps 3-5.

**Step 3: Localization Formula.** By the localization theorem, (11) can be computed as a sum of contributions from each \( C^* \) fixed locus. The fixed loci parametrize curves where all of the marked points and nodes are mapped to 0 and \( \infty \) via \( f \). We discard the loci where \( f(x_{m+1}) = 0 \) or \( f(y_{n+1}) = \infty \) because the integrand vanishes when restricted to these loci. For each \( \vec{k},\vec{l} \), we obtain a fixed locus by splitting the remaining \( m + n \) points over 0 and \( \infty \). Denote such a fixed locus by

\[
t : F_{\vec{k},\vec{l}}^{\vec{k}_0,\vec{l}_0} \to R_{\vec{k},\vec{l}}^{G,d} \ni (k,s), (l,r)
\]
where $\vec{k}_0, \vec{k}_\infty$ is a splitting of the vector $\vec{k}$ into subvectors of lengths $m_0, m_\infty$, similar for $\vec{l}$. By the Atiyah-Bott localization formula, the series (11) is equal to

$$\sum_F \frac{1}{m!n!} \int_F t^* \left( e(\text{Ob}) \cup t(-\psi)^{\vec{k}} \cup u^\vec{l} \cup ev_{m+1}^*([\infty]) \cup \tilde{ev}_{n+1}^*([0]) \right) \frac{e(N_F)}{e(\mathcal{N}_F)}$$

where the denominator is the equivariant Euler class of the normal bundle and

$$t(-\psi)^{\vec{k}} = t(-\psi_1)_{k_1}, \ldots, t(-\psi_m)_{k_m}$$

with $t(z)_k$ the coefficient of $\phi_k$ in $t(z)$, similar for $u^\vec{l}$.

**Step 4: Stable Terms.** Define a fixed locus to be stable if it has a node over 0 and $\infty$. For stable fixed loci, we have

$$F^{\vec{k}_0,\vec{l}_\infty}_{k_0,\ell_0} \cong \mathcal{R}^d_{(k_0,\vec{k}),t(l_0,\vec{l})} \times \mathcal{R}^d_{(\vec{k}_\infty,\vec{l}_\infty),s,d-2-k,t_{\vec{l}_\infty}}$$

where $k$ is uniquely determined from $\vec{k}_0, \vec{l}_0$, and the non-emptiness condition [4]. By Ramond vanishing, $t^* e(\text{Ob})$ vanishes whenever $k \notin \text{nar}$. On the other hand, when $k \in \text{nar}$ we compute

$$0 \to t^* \text{Ob} \to \text{Ob}_0 \oplus \text{Ob}_\infty \oplus \bigoplus_j H^1(\hat{C}, L_j|\hat{C}) \to 0$$

Since $\deg(L_j|\hat{C}) = 0$ in the stable case, the final term vanishes so

$$\bullet \ t^* (e(\text{Ob})) = \begin{cases} e(\text{Ob}_0)e(\text{Ob}_\infty) & s \in \text{nar} \\ 0 & s \notin \text{nar}. \end{cases}$$

We also compute

- $t^* \psi = \psi$,
- $t^* ev_{m+1}^*([\infty]) = -z$, and
- $t^* \tilde{ev}_{n+1}^*([0]) = z$.

On the other hand, the normal bundle in the denominator of (12) contributes factors

- $\frac{1}{d}(z - \psi_{m_0+1})$ and $\frac{1}{d}(-z - \psi_{m_\infty+1})$ from smoothing the nodes at 0 and $\infty$ on the orbifold curve, and
- $-z^2$ from deforming the map to $\mathbb{P}^1$.

Pulling together the bullet points above, we compute that for stable fixed loci the integral in (12) is equal to

$$\left( t(-\psi)^{\vec{k}_0}, \frac{\phi_k}{z - \psi_{m_0+1}} \right)_{m_0+1,n_0+1}^{\vec{l}_0,\vec{\phi}_r} \left( t(-\psi)^{\vec{k}_\infty}, \frac{\phi_{d-k}}{-z - \psi_{m_\infty+2}} \right)_{m_\infty+2,n_\infty}^{\vec{l}_\infty}$$

Notice that the left side of (15) captures the stable contributions to the coefficient of $\phi_k$ in $\partial_{\vec{u}^\vec{l}} \mathcal{J}^\epsilon(t, u, z)$ and the right side captures the stable contribution to the coefficient of $\phi_k$ in $\partial_{\vec{u}^\vec{l}} \mathcal{J}^\epsilon(t, u, -z)$.
Step 5: Unstable Terms. There are two cases, either \(m_\infty = n_\infty = 0\) or \(m_0 = 0\) and \(n_0 + 1 \leq \frac{1}{\epsilon}\).

In the first case, the fixed locus has a single \(\phi_s\) point over \(\infty\). The localization contribution in this case is obtained from (15) by setting \(k = s\) and replacing the right hand term with 1 which is the unstable contribution to the coefficient of \(\phi_k\),

\[
H^0(\infty, L_j|_\infty) \oplus H^1(\hat{C}, L_j|_{\hat{C}})
\]

In the second case, the fixed locus has \(n_0 + 1\) light points stacked up at \(0 \in \hat{C}\). \(L_j|_{\hat{C}}\) is now a negative bundle for all \(j\) so it contributes to the exact sequence (14). We compute

\[
e(\iota^* Ob) = e(Ob_\infty)e\left(\bigoplus_j H^0(\infty, L_j|_\infty) \oplus H^1(\hat{C}, L_j|_{\hat{C}})\right).
\]

If \(r + \sum (l_0)_i \notin \text{nar}\), then for some \(j\), \(H^0(\infty, L_j|_\infty)\) is a trivial bundle with trivial action and therefore \(e(\iota^* Ob) = 0\). If \(r + \sum (l_0)_i \in \text{nar}\), then \(H^0(\infty, L_j|_\infty) = 0\) for all \(j\) and

\[
e(\iota^* Ob) = e(Ob_\infty)e\left(\bigoplus_j H^1(\hat{C}, L_j|_{\hat{C}})\right)
\]

with

\[
L_j|\hat{C} \cong \mathcal{O} \left(-\left[\langle q_j r \rangle + \sum_i \langle q_j (l_0)_i \rangle \right] 0 - \left[\langle q_j r \rangle + \sum_i \langle q_j (l_0)_i \rangle \right] \infty \right).
\]

We compute

\[
e\left(\bigoplus_j H^1(\hat{C}, L_j|_{\hat{C}})\right) = \prod_{j=1}^N \prod_{0 \leq b < \langle q_j r \rangle + \sum_i \langle q_j (l_0)_i \rangle \atop \langle b \rangle = \langle q_j r \rangle + \sum_i \langle q_j (l_0)_i \rangle} \text{bz}.
\]

The normal bundle on this fixed locus contains a summand corresponding to deforming the points away from 0, this contributes an equivariant factor of \(z^{n_0+1}\). Pulling everything together, we see that the contributions from the second case of unstable loci is obtained from (15) by setting \(k = d-r-\sum (l_0)_i\) and replacing the left side with

\[
\frac{1}{z^{n_0}} \prod_{j=1}^N \prod_{0 \leq b < \langle q_j r \rangle + \sum_i \langle q_j (l_0)_i \rangle \atop \langle b \rangle = \langle q_j r \rangle + \sum_i \langle q_j (l_0)_i \rangle} \text{bz}
\]

which corresponds to the unstable contribution to the coefficient of \(\phi^k\) in \(\partial_u^c J^c(t, u, z)\).

Adding (15) over all stable and unstable loci proves the claim. \(\square\)
2.2. Cone Characterization. In this section we prove a characterization of the $H[[u]]$ valued points on $L$. This characterization along with Lemma 2.1 implies that the $J$-functions lie on $L$.

**Lemma 2.2.** Suppose $F(t,u,z) \in H'[t,u][[z^{-1}]]$ has the form

\[(16) \quad F(t,u,z) = z\phi_0 + t(-z) + f(u,-z) + F(t,u,z)\]

where

(i) $f(u,z) \in H'[[u]][z]$ with $f(0,z) = 0$,
(ii) $F(t,u,z) \in H'[t,u,z^{-1}]$ only has terms of degree $\geq 2$ in $t,u$, and
(iii) $F(t,u = 0, -z) \in L$.

Then $F(t,u,-z) \in L$ if and only if the series

\[(17) \quad (\partial_{u^r} F(t,u,z), \partial_{t^s} F(t,u,-z))\]

is regular at $z = 0$ for all $r,s$.

**Proof.** First suppose $F(t,u,z)$ satisfies (i)-(iii) and $F(t,u,-z)$ lies on $L$, we show that (17) is regular at $z = 0$. By definition, $F$ has the form

\[F(t,u,z) = z\phi_0 + \hat{t}(-z) + \sum_k \phi^k \left\langle \frac{\phi_k}{z - \psi_1} \right\rangle_1^\infty (\hat{t}).\]

where $\hat{t}(z) = t(z) + f(u,z)$. We have

\[\partial_{u^r} F(t,u,z) = \partial_{u^r} \hat{t}(-z) + \sum_k \phi^k \left\langle \partial_{u^r} t(\psi_1), \frac{\phi_k}{z - \psi_2} \right\rangle_2^\infty (\hat{t})\]

and

\[\partial_{t^s} F(t,u,-z) = \phi_s + \sum_k \phi^k \left\langle \phi_s, \frac{\phi_k}{-z - \psi_2} \right\rangle_2^\infty (\hat{t}).\]

Proceeding exactly as in the proof of Lemma 2.1 we see that the series (17) is equal to the equivariant series

\[(18) \quad \left\langle (\partial_{u^r} \hat{t}(\psi_1), \phi_s | ev_1^*([0]) \cup ev_1^*([\infty]) \right\rangle^G_2 \right\rangle^G_2 \]

which is regular at $z = 0$ by definition.

Now suppose $F$ satisfies (i)-(iii) and (17), we show that $F(t,u,-z)$ lies on $L$. To this end, we show that $F$ is uniquely determined from its regular part, its restriction to $u = 0$, and the recursions (17). To see this, write

\[F = \sum f_{\vec{m},\vec{n},j,s} t^{\vec{m}} u^{\vec{n}} \frac{\phi^s}{z^j}\]

where $t^{\vec{m}} = \sum (t_j^k)^{m_j}$, similar for $u^{\vec{n}}$. Then (17) determines the coefficients by lexicographic induction on $(|\vec{m}|, |\vec{n}|)$. Indeed, to compute the coefficient $f_{\vec{m},(\vec{n},r),j,s}$, we consider the relation

\[\left( \partial_{u^r} F(t,u,z), \partial_{t^s} F(t,u,-z) \right) \left[ \frac{t^{\vec{m}} u^{\vec{n}}}{z^j} \right] = 0.\]
There is an initial term equal to \((n^r + 1)f_{\vec{n},(\vec{n},r),i,s}\) and all other terms are determined by induction and \(f(u,z)\). This method recursively determines \(F(t,u,z)\) from \(F(t,u = 0,z)\).

Setting \(\hat{t}(z) = t(z) + f(u,z)\), we conclude that

\[
F(t,u,z) = z\phi_0 + \hat{t}(-z) + \sum_k \phi^k \left(1 - z - \psi_1\right) \langle \hat{t} \rangle
\]

because both sides agree along the restriction \(u = 0\), they have the same regular part, and they both satisfy the same recursion which determines them uniquely from these initial conditions. Therefore, \(F(t,u,-z) \in L\) \(\Box\)

Theorem 1.11 now follows immediately from Lemmas 2.1 and 2.2

3. Applications

We now extract several corollaries from Theorem 1.11. Define \(J_0^1(u) \in \mathbb{Q}[u]\) and \(J_1^1(u,z) \in \mathcal{H}_+[u]\) by

\[
\mathcal{J}^1(t,u,z) = J_0^1(u)\phi_0 z + t(z) + J_1^1(u,z) + O\left(\frac{1}{z}\right).
\]

Define the change of variable \(\tau^\ell(t,u)\) by

\[
\tau^\ell(t,u) := J_0^\ell(u) t(z) - J_1^\ell(u,z)
\]

and write \(\tau^\ell = (\tau^\ell(t,u))^\xi\). We have the following comparison of generating series.

**Theorem 3.1.**

\[
\frac{\mathcal{J}^{\ell_1}(\tau^\ell_1, u, z)}{J_0^{\ell_1}(u)} = \frac{\mathcal{J}^{\ell_2}(\tau^\ell_2, u, z)}{J_0^{\ell_2}(u)}
\]

**Proof.** Notice that \(J_0^0(u) = 1 + O(u)\), so both sides make sense as elements of \(\mathcal{H}[u]\). Since \(L\) is a cone, both sides of (19) lie on \(L\) by Theorem 1.11 (upon negating \(z\)). The change of variables is defined precisely so that the two sides of (19) agree in their regular parts. The result now follows from the simple observation that \(L\) is a graph over \(H^+\), meaning that points on \(L\) are uniquely determined by their regular part. \(\Box\)

We now turn our attention to the big \(I\) function

\[
\mathbb{I}(u,z) = z\phi_0 \sum_{a_i \geq 0} \prod_i \frac{1}{a_i!} \left(\frac{u^i \phi_i}{z}\right) \prod_{j=1}^N \prod_{0 \leq b < \sum_i a_i \langle i q_j \rangle} \prod_{(b) = \sum_i a_i \langle i q_j \rangle} b z.
\]

Since \(\mathbb{I}(u,z)\) has unbounded positive powers of \(z\), we need to work with a completion of our base ring \(\mathbb{Q}[u]\). In particular, we work over the \(u\)-adic completion of \(\mathbb{Q}[u]\) defined in terms of the total degree in the \(u^i\) where \(\deg(u^i) = i\). We take \(H\) to contain series \(\sum_{j \in \mathbb{Z}} h_j z^j\) which are possibly infinite in both directions, but we require that \(\lim_{j \to \infty} h_j \to 0\). With this convention, Theorem 1.11 implies that \(\mathbb{I}(u,-z)\) lies on \(L\).
One of the fundamental results in Givental’s approach to axiomatic GW theories [Giv04] is that the cone \(\mathcal{L}\) is swept by a finite-dimensional family of semi-infinite linear spaces. In particular, if \(T\) is a tangent space of \(\mathcal{L}\), then \(T\) is tangent to \(\mathcal{L}\) exactly along \(zT\), and

\[
\mathcal{L} = \bigcup_{t_0 \in H'} zT_{J(t_0, -z)} \mathcal{L}.
\]

where the big \(J\)-function \(J(t_0, z)\) is defined by restricting \(J^\infty(t, z)\) to primary variables \(t_0 = (t_0^k)\). In particular, \(I(u, -z) \in zT_{J(\sigma(u), -z)}\) where \(\sigma(u) \in H'\) is the unique point in the intersection

\[
zT_{I(u, -z)} \cap (-z + zH^-) \subset \mathcal{L} \cap (-z + zH^-).
\]

The map \(u \rightarrow \sigma(u)\) has the following important property.

**Lemma 3.2.** Let \(D\) be the dimension of \(H'\). Then \(\sigma\) defines an isomorphism \(\sigma : \mathbb{Q}^D \rightarrow H'\) when restricted to a formal neighborhood of the origin.

**Proof.** By definition,

\[
I(u, -z) = -z\phi_0 + u + O(u^2).
\]

Therefore, \(\sigma(u) = u + O(u^2)\) implying that \(\sigma(u)\) is invertible over \(\mathbb{Q}[[u]]\). \(\square\)

In particular, since \(J(t, -z)\) determines the entire cone, Lemma 3.2 implies the mirror theorem, proved previously by Chiodo–Iritani–Ruan for \(q = 1\) and independently by Acosta for \(q \neq 1\).

**Theorem 3.3** ([Aco] [CIR12]). \(I(u, -z)\) determines the entire cone \(\mathcal{L}\).

In fact, one can recursively invert \(\sigma(u)\) for low powers of \(u\) and thus determine the FJRW invariants purely in terms of \(I\). For examples of how this process is carried out explicitly, see [CCIT14].

A simple calculation shows that the power of \(z\) which appears in the \(\prod I(u^i)^{a_i}\) coefficient of the big \(I\)-function is at most

\[
1 + \sum_i a_i (\deg(\phi_i) - 1)
\]

Therefore, since the small \(I\)-function is defined by restricting to degree \(\leq 1\) insertions, it has the form

\[
I(u, z) = I_0(u) z\phi_0 + I_1(u) + O\left(\frac{1}{z}\right)
\]

for some \(I_0(u) = 1 + O(u) \in \mathbb{Q}[[u]]\) and \(I_1(z) \in H'[[u]]\). We similarly define the small FJRW \(J\)-function \(J(\xi, z)\) by restricting \(J^\infty\) to primary insertions of degree \(\leq 1\). Then we recover the usual LG mirror theorem which relates small \(I\) and \(J\) via a simple change of variables.

**Corollary 3.4.**

\[
\frac{I(u, z)}{I_0(u)} = J(q(u), z).
\]
\[ \eta(u) = \frac{I_1(u)}{I_0(u)}. \]

**Proof.** Both sides lie on \( \mathcal{L} \) (after negating \( z \)) and they agree in their regular part. \( \square \)

**Remark 3.5.** We note that the statement of Corollary 3.4 is most interesting for \( q \leq 1 \). For \( q > 1 \), the only state space element of degree \( \leq 1 \) is \( \phi_0 \) and the relation follows from the string equation in FJRW theory.

### 4. Birkhoff Factorizations

In this section we exhibit a Birkhoff factorization for the large \( J \)-functions. In particular, this specializes to an enumerative description of the small \( I \)-function and the mirror map in Corollary 3.4. The arguments in the section closely follow [CFK13b].

Define the large \( S \)-operator on \( H' \) by
\[
S^\epsilon_{t,u,z}(\phi_r) := \partial_{t_0}^1 J^\epsilon(t, u, z) = \phi_r + \sum_k \phi_k \left\langle \left\langle \phi_r, \frac{\phi_k}{z - \psi_2} \right\rangle_2 \right\rangle^\epsilon
\]
and its adjoint by
\[
(S^\epsilon)^*_{t,u,z}(\phi_r) := \phi_r + \sum_k \phi_k \left\langle \left\langle \phi_k, \frac{\phi_r}{z - \psi_2} \right\rangle_2 \right\rangle^\epsilon.
\]

**Lemma 4.1.** For all \( \epsilon > 0 \),
\[
(S^\epsilon)^* \circ S^\epsilon = Id.
\]

**Proof.** Observe that
\[
(S^\epsilon)^* (S^\epsilon(\phi_r)) = \sum_l \phi_l \left( \partial_{t_0}^1 J^\epsilon(t, u, z), \partial_{t_0}^{1-2-l} J^\epsilon(t, u, z) \right)
\]
which is regular in \( z \) by a proof identical to that of Lemma 2.1. The lemma then follows from the observation that
\[
(S^\epsilon)^* (S^\epsilon(\phi_r)) = \phi_r + O \left( \frac{1}{z} \right).
\]

Define the equivariant series
\[
P^\epsilon(t, u, z) = \sum_k \phi_k \left\langle \langle \phi_k | e_{u_1^*}([\infty]) \rangle \right\rangle_{1}^{G,\epsilon}
\]
which is regular in \( z \) by definition. We have the following Birkhoff factorization of the large \( J \)-functions.

**Lemma 4.2.** For all \( \epsilon > 0 \),
\[
z^{-1} J^\epsilon(t, u, z) = S^\epsilon (P^\epsilon(t, u, z)).
\]
Proof. Computing the $P$ series by localization exactly as in Lemma 2.1, we obtain
\[ P^\epsilon(t, u, z) = z^{-1} (S^\epsilon)^* (J^\epsilon(t, u, z)) . \]
The lemma then follows from Lemma 4.1. □

Consider the small $I$-function. We have
\[ P^0(0, u, z) = z^{-1} (S^\epsilon_0)^* (I(u, z)) = I_0(u)\phi_0 + O\left(\frac{1}{z}\right) . \]
Since $P$ is regular in $z$, we conclude that
\[ P^0(0, u, z) = I_0(u)\phi_0 . \]
Therefore, by (20) we obtain the following geometric interpretation of the small $I$-function and mirror map in terms of two-point $\epsilon = 0$ FJRW invariants.

**Theorem 4.3.**

\[ \frac{I(u, z)}{I_0(u)} = zS_{0, u, z}^0(\phi_0) = z\phi_0 + z \sum_k \phi^k \left\langle \frac{\phi_0}{z^2} \right\rangle^0_2 (0, u) \]

and
\[ \eta(u) = \sum_k \phi^k \left\langle \phi_0, \phi_k \right\rangle^0_2 (0, u) \]

For $\epsilon = 1$, it is not hard to see from the definitions and the string equation that $zS_{0, u, z}^1(\phi_0)$ is equal to the small FJRW $J$-function. For this reason, we define the **small $J^\epsilon$-function** for $0 < \epsilon \leq 1$ by
\[ J^\epsilon(u, z) := zS_{0, u, z}^\epsilon(\phi_0) \]

Theorem 4.3 asserts that the mirror theorem is obtained by tracking the two point series $J^\epsilon(u, z)$ as we vary $\epsilon$ from 1 to 0.

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