PRIMARY DECOMPOSITION OVER PARTIALLY ORDERED GROUPS

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Abstract. Over any partially ordered abelian group whose positive cone is closed in an appropriate sense and has finitely many faces, modules that satisfy a weak finiteness condition admit finite primary decompositions. This conclusion rests on the introduction of basic notions in the relevant generality, such as closedness of partially ordered abelian groups, faces and their coprimary modules, and finiteness conditions as well local and global support functors for modules over partially ordered groups.

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1. Introduction

Primary decomposition yields concrete answers in combinatorial commutative algebra, particularly in monomial [MS05, Example 7.13] and binomial [ES96, KM14, KMO16] contexts. These answers come most naturally in the presence of a multigrading that is positive [MS05, §8.1]. When the grading is by a torsion-free abelian group, positivity is equivalent to the group being partially ordered: one element precedes another if their difference lies in the positive cone of elements greater than 0.

What happens when the grading set isn’t necessarily discrete? Substantial parts of commutative algebra—especially homological algebra including the syzygy theorem [Mil20a]—generalize to modules over arbitrary posets and have no need to rest on an underlying ring. Alas, the part of the theory relating to primary decomposition is not amenable to arbitrary posets, because of a lack of natural prime ideals and inability to localize. Rather, the natural setting to carry out primary decomposition is, in the

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best tradition of classical mathematics [Bir42, Cli40, Rie40], over partially ordered abelian groups (Definition 2.2). Those provide an optimally general context in which posets have a notion of “face” along which to localize—and to do so without altering the ambient poset. That is, a partially ordered group \( Q \) has an origin, namely its identity \( 0 \), and hence a positive cone \( Q^+ \) of elements it precedes. A face of \( Q \) is a submonoid of \( Q^+ \) that is also a downset therein (Definition 2.8). And as everything takes place inside of the ambient group \( Q \), every localization of a \( Q \)-module along a face (Definition 4.1) remains a \( Q \)-module.

Primary decomposition (Theorem 5.8) expresses a given module \( M \) as a submodule of a direct sum of coprimary modules (Definition 4.9 and Theorem 4.13), each with an essential submodule consisting of coprimary elements (Definition 4.11). Isolating all coprimary elements functorially requires localization, after which local support functors (Definition 4.8) do the job, as in ordinary commutative algebra and algebraic geometry.

**Example 1.1.** The downset \( D \) in \( \mathbb{R}^2 \) consisting of all points beneath the upper branch of the hyperbola \( xy = 1 \) canonically decomposes (Theorem 3.10) as the union of its subsets of coprimary elements of various types: every red point in the
- leftmost subset on the right dies when pushed over to the right or up far enough;
- middle subset dies in the localization of \( D \) along the \( x \)-axis (Definition 3.2 or Definition 4.1) when pushed up far enough; and
- rightmost subset dies locally along the \( y \)-axis when pushed over far enough.

**Example 1.2.** The union in Example 1.1 results in a canonical primary decomposition

\[
\begin{bmatrix}
\kappa & \kappa & \kappa \\
\kappa & \kappa & \kappa
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\kappa & \kappa & \kappa \\
\kappa & \kappa & \kappa
\end{bmatrix}
\oplus
\begin{bmatrix}
\kappa & \kappa & \kappa \\
\kappa & \kappa & \kappa
\end{bmatrix}
\oplus
\begin{bmatrix}
\kappa & \kappa & \kappa \\
\kappa & \kappa & \kappa
\end{bmatrix}
\]

of the downset module \( \kappa[D] \) over \( \mathbb{R}^2 \) (Corollary 3.11). Elements in the lower-left quadrant locally die any type of death.

It bears emphasizing that primary decomposition of downset modules, or equivalently, expressions of downsets as unions of coprimary downsets (cogenerated by the \( \tau \)-coprimary elements for some face \( \tau \); see Definition 3.9), is canonical by Theorem 3.10 and Corollary 3.11, generalizing the canonical primary decomposition of monomial
ideals in ordinary polynomial rings. However, notably lacking from primary decomposition theory over arbitrary polyhedral partially ordered abelian groups is a notion of minimality—alas, a lack that is intrinsic.

**Example 1.3.** Although three death types occur in \( D \) in Example 1.1, and hence in the union there, the final two summands in the primary decomposition of \( \mathbb{k}[D] \) in Example 1.2 are redundant. One can, of course, simply omit the redundant summands, but for arbitrary polyhedral partially ordered groups no criterion is known for detecting a priori which summands should be omitted.

The failure of minimality here stems from geometry that can only occur in partially ordered groups more general than finitely generated free ones. More specifically, although \( D \) contains, for instance, elements that die deaths of type “\( x \)-axis”, the boundary of \( D \) fails to contain an actual translate of the face of \( \mathbb{R}^2_+ \) that is the positive \( x \)-axis. This can be seen as a certain failure of localization to commute with taking homomorphisms into \( \mathbb{k}[D] \) (Remark 4.7); it is the source of much of the subtlety in the theory developed in the sequel [Mil20c] to this paper, whose purpose is partly to rectify, for \( \mathbb{R}^n \)-modules (equivalently, \( \mathbb{R}^n \)-graded modules over the real-exponent polynomial ring \( \mathbb{k}[\mathbb{R}_n^+] \)), the failure of minimality in Example 1.2.

In general, the innovation in this paper is getting the hypotheses right and selecting the appropriate one of the usually many equivalent formulations for any given claim or proof. The failure of localization to commute with taking homomorphisms is a quintessential feature of the more general theory that guides the sometimes nonobvious choices required.

Substantial impetus for this work comes from applied topology, where the focus is on modules over posets whose underlying partially ordered groups are real vector spaces [Les15, Mil15, KS18, KS19, Mil20b]; see [Mil17] for context (and an early draft of this paper in §3 there). The view toward algorithmic computation draws the focus to the case where \( Q \) is polyhedral, meaning that it has only finitely many faces (Definition 2.8). This notion is apparently new for arbitrary partially ordered abelian groups. Its role here is to guarantee finiteness of primary decomposition of downset-finite modules (Theorem 5.8). To illuminate the meaning of this main result in the context of persistent homology, multiparameter features can die in many ways, persisting indefinitely as some of the parameters increase without limit but dying when any of the others increase sufficiently. In persistence language, a single element in a module over a partially ordered group can a priori be mortal or immortal in more than one way. But some elements die “pure deaths” of only a single type \( \tau \). These are the \( \tau \)-coprimary elements for a face \( \tau \). In the concrete setting of a partially ordered real vector space with closed positive cone, as in more general settings, a coprimary element is characterized (Example 2.10, Definition 4.11, and Theorem 4.13) as

1. \( \tau \)-**persistent**: it lives when pushed up arbitrarily along the face \( \tau \); and
2. \( \overline{\tau} \)-**transient**: it eventually dies when pushed up in any direction outside of \( \tau \).
Primary decomposition tells the fortune of every element: its death types are teased apart as the “pure death types” of the coprimary summands where the element lands with nonzero image. In the ordinary situation of one parameter, the only distinction being made here is that a feature can be mortal or immortal. Beyond the intrinsic mathematical value, decomposing a module according to these distinctions has concrete benefits for statistical analysis using multipersistence [MT20].

2. Polyhedral partially ordered groups

Definition 2.1. Let $Q$ be a partially ordered set (poset) and $\leq$ its partial order. A module over $Q$ (or a $Q$-module) is

• a $Q$-graded vector space $M = \bigoplus_{q \in Q} M_q$ with
  • a homomorphism $M_q \to M_{q'}$ whenever $q \leq q'$ in $Q$ such that
  • $M_q \to M_{q''}$ equals the composite $M_q \to M_{q'} \to M_{q''}$ whenever $q \leq q' \leq q''$.

A homomorphism $M \to N$ of $Q$-modules is a degree-preserving linear map, or equivalently a collection of vector space homomorphisms $M_q \to N_q$, that commute with the structure homomorphisms $M_q \to M_{q'}$ and $N_q \to N_{q'}$.

Example 2.3. The finitely generated free abelian group $Q = \mathbb{Z}^n$ can be partially ordered with any positive cone $Q_+$, polyhedral or otherwise, resulting in a discrete partially ordered group. The free commutative monoid $Q_+ = \mathbb{N}^n$ of integer vectors with nonnegative coordinates is the most common instance and serves as a well behaved, well known foundational case. For notational clarity, $\mathbb{Z}_n^+$ always means the nonnegative orthant in $\mathbb{Z}^n$, which induces the standard componentwise partial order on $\mathbb{Z}^n$. Other partial orders can be specified using notation $Q \cong \mathbb{Z}^n$ with arbitrary positive cone $Q_+$.

Example 2.4. The group $Q = \mathbb{R}^n$ can be partially ordered with any positive cone $Q_+$, polyhedral or otherwise, closed, open (away from the origin 0) or anywhere in between, resulting in a real partially ordered group. The orthant $Q_+ = \mathbb{R}_+^n$ of vectors with nonnegative coordinates is most useful for purposes such as multipersistence. For notational clarity, $\mathbb{R}_+^n$ always means the nonnegative orthant in $\mathbb{R}^n$, which induces the standard componentwise partial order on $\mathbb{R}^n$. Other partial orders can be specified using notation $Q \cong \mathbb{R}^n$ with arbitrary positive cone $Q_+$.

Example 2.5. Definition 2.2 allows the group to have torsion. Thus the submonoid of $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $[1]$ and $[1]$ is a positive cone in the group. There is a continuous version in which the resulting partial order is easier to see geometrically:
Q = \mathbb{R} \times \mathbb{R}/\mathbb{Z} with \(Q_+\) generated by \([1,0]\) and \([1,1]\). In the figure, the blue center line is the first factor \(\mathbb{R}\), with origin \(0\) at the fat blue dot. The positive cone \(Q_+\) is shaded.

The following allows the free use of the language of either \(Q\)-modules or \(Q\)-graded \(k[Q_+]\)-modules, as appropriate to the context.

**Lemma 2.6.** A module over a partially ordered abelian group \(Q\) is the same thing as a \(Q\)-graded module over the monoid algebra \(k[Q_+]\) of the positive cone. \(\square\)

**Example 2.7.** When \(Q = \mathbb{Z}^n\) and \(Q_+ = \mathbb{N}^n\), the relevant monoid algebra is the polynomial ring \(k[\mathbb{N}^n] = k[x]\), where \(x = x_1, \ldots, x_n\) is a sequence of \(n\) commuting variables. This is the classical case; see [MS05, §8.1], for example.

Primary decomposition of \(Q\)-modules depends on certain finiteness conditions. In ordinary commutative algebra, where \(Q = \mathbb{Z}^n\), the finiteness comes from \(Q_+\), which is assumed to be finitely generated (so it is an affine semigroup). This condition implies that finitely generated \(Q\)-modules are noetherian: every increasing chain of submodules stabilizes. Primary decomposition is then usually derived as a special case of the theory for finitely generated modules over noetherian rings. But the noetherian condition is stronger than necessary: it suffices for the positive cone to have finitely many faces, in the following sense, along with the (much weaker) “downset-finite” replacement for the noetherian condition in Section 5. To the author’s knowledge, the notion of polyhedral partially ordered group is new, and there is no existing literature on primary decomposition in this setting.

**Definition 2.8.** A face of the positive cone \(Q_+\) of a partially ordered group \(Q\) is a submonoid \(\sigma \subseteq Q_+\) such that \(Q_+ \setminus \sigma\) is an ideal of the monoid \(Q_+\). Sometimes it is simpler to say that \(\sigma\) is a face of \(Q\). Call \(Q\) polyhedral if it has only finitely many faces.

Polyhedrality suffices to prove existence of (finite) primary decomposition (Theorem 5.8) in the presence of weak finiteness condition on the module being decomposed. However, many of the ingredients, such as localization along or taking support on a face (Definition 3.2), make sense also under a different sort of hypothesis.

**Definition 2.9.** Let \(Q\) be a partially ordered group.

1. A ray of the positive cone \(Q_+\) is a face that is totally ordered as a partially ordered submonoid of \(Q\).

2. The partially ordered group \(Q\) is closed if the complement \(Q_+ \setminus \tau\) of each face \(\tau\) is generated as an upset (i.e., as an ideal) of \(Q_+\) by \(\rho \setminus \{0\}\) for the rays \(\rho \not\subseteq \tau\).
Example 2.10. Any real partially ordered group (Example 2.4) $Q$ whose positive cone $Q_+$ is closed in the usual topology on $Q$ is a closed partially ordered group by the Krein–Milman theorem: $Q_+$ is the set of nonnegative real linear combinations of vectors on extreme rays of $Q_+$. For instance, a non-polyhedral closed partial order on $Q = \mathbb{R}^3$ results by taking $Q_+$ to be a cone over a disk, such as either half of the cone $x^2 + y^2 \leq z^2$. In contrast, if $Q_+$ is an intersection of finitely many closed half-spaces, then there are only finitely many extreme rays. (This case is crucial in applications—see [KS19, Mil20c], for instance.) Even in the polyhedral case the cone need not be rational; that is, the vectors that generate it—or the linear functions defining the closed half-spaces whose intersection is $Q_+$—need not have rational entries.

Example 2.11. Any discrete partially ordered group (Example 2.3) whose positive cone is a finitely generated submonoid is automatically both polyhedral and closed [MS05, Lemma 7.12]. A discrete partially ordered group can also have a positive cone that is not a finitely generated submonoid, such as $Q = \mathbb{Z}^2$ with $Q_+ = C \cap \mathbb{Z}^2$ for the cone $C \subseteq \mathbb{R}^2$ generated by $[\frac{1}{2}]$ and $[\frac{1}{\pi}]$. This particular irrational cone yields a partially ordered group that is polyhedral but not closed. Indeed, there are fewer than the expected faces, because only some of the faces of $C$ result in faces of $Q_+$ itself. The image of $Q$ is not discrete in the quotient of $Q \otimes \mathbb{R}$ modulo the subgroup spanned by the irrational real face, which can have unexpected consequences for the algebra of poset modules under localization along such a face.

Example 2.12. The cylindrical group $Q$ in Example 2.5 has two faces: the origin $0$ (the fat blue dot) and $\mathbb{R}_+$ (the rightmost half of the horizontal blue center line).

3. Primary decomposition of downsets

Definition 3.1. Fix a poset $Q$. The vector space $\mathbb{k}[Q] = \bigoplus_{q \in Q} \mathbb{k}$ that assigns $\mathbb{k}$ to every point of $Q$ is a $Q$-module with identity maps on $\mathbb{k}$. More generally,

1. an upset (also called a dual order ideal) $U \subseteq Q$, meaning a subset closed under going upward in $Q$ (so $U + Q_+ = U$, when $Q$ is a partially ordered group) determines an indicator submodule or upset module $\mathbb{k}[U] \subseteq \mathbb{k}[Q]$; and

2. dually, a downset (also called an order ideal) $D \subseteq Q$, meaning a subset closed under going downward in $Q$ (so $D - Q_+ = D$, when $Q$ is a partially ordered group) determines an indicator quotient module or downset module $\mathbb{k}[Q] \twoheadrightarrow \mathbb{k}[D]$.

Definition 3.2. Fix a face $\tau$ of the positive cone $Q_+$ in a polyhedral or closed partially ordered group $Q$ and a downset $D \subseteq Q$. Write $\mathbb{Z}\tau$ for the subgroup of $Q$ generated by $\tau$.

1. The localization of $D$ along $\tau$ is the subset

$$D_\tau = \{ q \in D \mid q + \tau \subseteq D \}.$$

2. An element $q \in D$ is globally supported on $\tau$ if $q \notin D_{\tau'}$ whenever $\tau' \not\subseteq \tau$. 
3. The part of $D$ globally supported on $\tau$ is
\[ \Gamma_\tau D = \{ q \in D \mid q \text{ is globally supported on } \tau \} . \]

4. An element $q \in D$ is locally supported on $\tau$ if $q$ is globally supported on $\tau$ in $D_\tau$.

5. The local $\tau$-support of $D$ is the subset $\Gamma_\tau(D_\tau) \subseteq D$ consisting of elements globally supported on $\tau$ in the localization $D_\tau$.

6. The $\tau$-primary component of $D$ is the downset
\[ P_\tau(D) = \Gamma_\tau(D_\tau) - Q_+ \]
cogenerated by the local $\tau$-support of $D$.

**Example 3.3.** The local $\tau$-supports of the under-hyperbola downset in Example 1.1 are the subsets depicted on the right-hand side there, for the faces $\tau = 0$, $x$-axis, and $y$-axis, respectively. The corresponding primary components are depicted in Example 1.2. In contrast, the global support on (say) the $y$-axis consists of the part of the local support that sits strictly above the $x$-axis, and the global support at $0$ is the part of $D$ strictly in the positive quadrant.

This example demonstrates that the $\tau$-primary component of $D$ in Definition 3.2 need not be supported on $\tau$. Indeed, $D = P_0(D)$ here, and points outside of $Q_+$ are not supported at the origin, being instead locally supported at either the $x$-axis (if the point is below the $x$-axis) or the $y$-axis (if the point is behind the $y$-axis).

**Remark 3.4.** Definition 3.2 makes formal sense in any partially ordered group, but extreme caution is recommended without the closed or polyhedral assumptions. Indeed, without such assumptions, faces can be virtually present, such as the irrational face in Example 2.11 or a missing face in a real polyhedron that is not closed. In such cases, aspects of Definition 3.2 might produce unintended output. The natural generality for the concepts in Definition 3.2 is unclear.

**Example 3.5.** The coprincipal downset $a + \tau - Q_+$ inside of $Q = \mathbb{Z}^n$ cogenerated by $a$ along $\tau$ is globally supported along $\tau$. It also equals its own localization along $\tau$, so it equals its local $\tau$-support and is its own $\tau$-primary component. Note that when $Q_+ = \mathbb{N}^n$, faces of $Q_+$ correspond to subsets of $[n] = \{1, \ldots, n\}$, the correspondence being $\tau \leftrightarrow \chi(\tau)$, where $\chi(\tau) = \{ i \in [n] \mid e_i \in \tau \}$ is the characteristic subset of $\tau$ in $[n]$.

(The vector $e_i$ is the standard basis vector whose only nonzero entry is 1 in slot $i$.)

**Remark 3.6.** The localization of $D$ along $\tau$ is acted on freely by $\tau$. Indeed, $D_\tau$ is the union of those cosets of $\mathbb{Z} \tau$ each of which is already contained in $D$. The minor point being made here is that the coset $q + \mathbb{Z} \tau$ is entirely contained in $D$ as soon as $q + \tau \subseteq D$ because $D$ is a downset: $q + \mathbb{Z} \tau = q + \tau - \tau \subseteq q + \tau - Q_+ \subseteq D$ if $q + \tau \subseteq D$.

**Remark 3.7.** The localization of $D$ is defined to reflect localization at the level of $Q$-modules: enforcing invertibility of structure homomorphisms $k[D]_q \rightarrow k[D]_{q+f}$ for $f \in \tau$ results in a localized indicator module $k[D][\mathbb{Z} \tau] = k[D_\tau]$; see Definition 4.1.
Example 3.8. Fix a downset $D$ in a partially ordered group $Q$ that is closed (Definition 2.9 and subsequent examples). An element $q \in D$ is globally supported on $\tau$ if and only if it lands outside of $D$ when pushed far enough up in any direction outside of $\tau$—that is, every $f \in Q_+ \setminus \tau$ has a nonnegative integer multiple $\lambda f$ with $\lambda f + q \not\in D$.

One implication is easy: if every $f \in Q_+ \setminus \tau$ has $\lambda f + q \not\in D$ for some $\lambda \in \mathbb{N}$, then any element $f' \in \tau' \setminus \tau$ has a multiple $\lambda f' \in \tau'$ such that $\lambda f' + q \not\in D$, so $q \not\in D_{\tau'}$. For the other direction, use Definition 2.9: $q \in \Gamma_\tau D \Rightarrow q \not\in D_\rho$ for all rays $\rho$ of $Q_+$ that are not contained in $\tau$, so along each such ray $\rho$ there is a group element $v_\rho$ with $v_\rho + q \not\in D$. Given $f \in Q_+ \setminus \tau$, choose $\lambda \in \mathbb{N}$ big enough so that $\lambda f \succeq v_\rho$ for some $\rho$. This argument is the purpose of the closed hypothesis; see the proof of Theorem 4.13.

Definition 3.9. Fix a downset $D$ in a polyhedral partially ordered group $Q$.

1. The downset $D$ is coprimary if $D = P_\tau (D)$ for some face $\tau$ of the positive cone $Q_+$. If $\tau$ needs to specified then $D$ is called $\tau$-coprimary.

2. A primary decomposition of $D$ is an expression $D = \bigcup_{i=1}^n D_i$ of coprimary downsets $D_i$, called components of the decomposition.

Theorem 3.10. Every downset $D$ in a polyhedral partially ordered group $Q$ is the union $\bigcup_{\tau} \Gamma_\tau (D_\tau)$ of its local $\tau$-supports for all faces $\tau$ of the positive cone.

Proof. Given an element $q \in D$, finiteness of the number of faces implies the existence of a face $\tau$ that is maximal among those such that $q \in D_\tau$; note that $q \in D = D_0$ for the trivial face $0$ consisting of only the identity of $Q$. It follows immediately that $q$ is supported on $\tau$ in $D_\tau$. \hfill \Box

Corollary 3.11. Every downset $D$ in a polyhedral partially ordered group $Q$ has a canonical primary decomposition $D = \bigcup_{\tau} P_\tau (D)$, the union being over all faces $\tau$ of the positive cone with nonempty support $\Gamma_\tau (D_\tau)$.

Remark 3.12. The union in Theorem 3.10 is not necessarily disjoint. Nor, consequently, is the union in Corollary 3.11. There is a related union, however, that is disjoint: the sets $(\Gamma_\tau D) \cap D_\tau$ do not overlap. But their union need not be all of $D$; try Example 3.3, where the negative quadrant intersects none of the sets $(\Gamma_\tau D) \cap D_\tau$.

Algebraically, $(\Gamma_\tau D) \cap D_\tau$ should be interpreted as taking the elements of $D$ globally supported on $\tau$ and then taking their images in the localization along $\tau$, which deletes the elements that aren’t locally supported on $\tau$. That is, $(\Gamma_\tau D) \cap D_\tau$ is the set of graded degrees from $Q$ where the image of $\Gamma_\tau \mathbb{k}[D] \to \mathbb{k}[D]_\tau$ is nonzero.

Example 3.13. The decomposition in Theorem 3.10—and hence Corollary 3.11—is not necessarily minimal: it might be that some of the canonically defined components can be omitted. This occurs, for instance, in Example 1.2. The general phenomenon, as in this hyperbola example, stems from geometry of the elements in $D_\tau$ supported on $\tau$, which need not be bounded in any sense, even in the quotient $Q/\mathbb{Z}_\tau$. In contrast, for (say) quotients by monomial ideals in the polynomial ring $\mathbb{k}[\mathbb{N}^n]$, only finitely
many elements have support at the origin, and the downset they cogen-erate is consequently artinian.

4. Localization and support

Definition 4.1. Fix a face $\tau$ of a partially ordered group $Q$. The localization of a $Q$-module $M$ along $\tau$ is the tensor product $M_\tau = M \otimes_{k[D]} k[Q_+ + \mathbb{Z}\tau]$, viewing $M$ as a $Q$-graded $k[Q_+]$-module. The submodule of $M$ globally supported on $\tau$ is $\Gamma_\tau M = \bigcap_{\tau' \not\subseteq \tau} \ker(M \to M_{\tau'}) = \ker(M \to \prod_{\tau' \not\subseteq \tau} M_{\tau'})$.

Example 4.2. Definition 3.2 says that $1_q \in k[D]_q = k$ lies in $\Gamma_\tau k[D]$ if and only if $q \notin D_\tau$, because $q \notin D_\tau'$ if and only if $1_q \mapsto 0$ under localization of $k[D]$ along $\tau'$.

Example 4.3. The global supports of the indicator subquotient for the interval

in $\mathbb{R}^2$ on the left-hand side of this display are the indicator subquotients for the intervals on the right-hand side, each labeled by the relevant face $\tau$. Caution: this example is not to be confused with Examples 1.1, 1.2, 3.3, and 3.13, where the curve is a hyperbola whose asymptotes are the two axes. In contrast, here the upper boundary of the interval has the vertical axis as an asymptote, whereas the horizontal axis is exactly parallel to the positive end of the upper boundary.

Lemma 4.4. The kernel of any natural transformation between two exact covariant functors is left-exact. In more detail, if $\alpha$ and $\beta$ are two exact covariant functors $A \to B$ for abelian categories $A$ and $B$, and $\gamma_X : \alpha(X) \to \beta(X)$ naturally for all objects $X$ of $A$, then the association $X \mapsto \ker \gamma_X$ is a left-exact covariant functor $A \to B$.

Proof. This can be checked by diagram chase or spectral sequence.

Proposition 4.5. The global support functor $\Gamma_\tau$ is left-exact.

Proof. Use Lemma 4.4: global support is the kernel of the natural transformation from the identity to a direct product of localizations.
Proposition 4.6. For modules over a polyhedral partially ordered group, localization commutes with taking support: \((\Gamma_{\tau'} M)_\tau = \Gamma_{\tau'} (M_\tau)\), and both sides are 0 unless \(\tau' \supseteq \tau\).

Proof. Localization along \(\tau\) is exact, so 
\[
\ker(M \to M_{\tau''})_\tau = \ker(M_\tau \to (M_{\tau''})_\tau) = \ker(M_\tau \to (M_{\tau''})_\tau).
\]
Since localization along \(\tau\) commutes with finite intersections of submodules, \((\Gamma_{\tau'} M)_\tau\) is the intersection of the leftmost of these modules over the faces \(\tau'' \nsubseteq \tau',\) of which there are only finitely many by the polyhedral hypothesis. But \(\Gamma_{\tau'} (M_\tau)\) equals the same intersection of the rightmost of these modules by definition. And if \(\tau' \nsubseteq \tau\) then one of these \(\tau''\) equals \(\tau\), so \(M_\tau \to (M_{\tau''})_\tau = M_\tau\) is the identity map, whose kernel is 0. \(\square\)

Remark 4.7. It is miraculous that localization commutes with taking support over general polyhedral partially ordered groups, because localization does not commute with taking relevant Hom functors in this setting. Indeed, this commutativity failure occurs even when the source module is a quotient \(k[D] = k[Q_+ \setminus U] = k[Q_+] / I\) modulo a graded ideal \(I = k[U]\) of the monoid algebra \(k[Q_+]\) of the positive cone; see [Mil20c, Remark 4.22] for an explanation with examples. The problem comes down to the homogeneous prime ideals of the monoid algebra \(k[Q_+]\) not being finitely generated, so the quotient \(k[\tau]\) fails to be finitely presented. However, taking the colimit of Hom functors of the form Hom\((k[D], M)\) for \(\tau\)-coprimary downsets \(D \subseteq Q_+\) eliminates the failure of commutativity.

Definition 4.8. Fix a \(Q\)-module \(M\) for a polyhedral partially ordered group \(Q\). The local \(\tau\)-support of \(M\) is the module \(\Gamma_\tau M_\tau\) of elements globally supported on \(\tau\) in the localization \(M_\tau\), or equivalently (by Proposition 4.6) the localization along \(\tau\) of the submodule of \(M\) globally supported on \(\tau\).

Definition 4.9. A module \(M\) over a polyhedral partially ordered group is coprimary if for some face \(\tau\), the localization map \(M \hookrightarrow M_\tau\) is injective and \(\Gamma_\tau M_\tau\) is an essential submodule of \(M_\tau\), meaning every nonzero submodule of \(M_\tau\) intersects \(\Gamma_\tau M_\tau\) nontrivially.

Remark 4.10. It is easy to check that over any polyhedral partially ordered group, if a module \(E\) is coprimary then it is \(\tau\)-coprimary for a unique face \(\tau\) of \(Q\).

The coprimary concept has an elementary, intuitive formulation in the language of persistence, when the ambient partially ordered group is polyhedral and closed.

Definition 4.11. Fix a face \(\tau\) of the positive cone \(Q_+\) in a partially ordered group \(Q\). A homogeneous element \(y \in M_q\) in a \(Q\)-module \(M\) is
1. \(\tau\)-persistent if it has nonzero image in \(M_{q'}\) for all \(q' \in q + \tau\);
2. \(\tau\)-transient if, for each \(f \in Q_+ \setminus \tau\), the image of \(y\) vanishes in \(M_{q'}\) whenever \(q' = q + \lambda f\) for \(\lambda \gg 0\);
3. \(\tau\)-coprimary if it is \(\tau\)-persistent and \(\overline{\tau}\)-transient.
Remark 4.12. It is an interesting exercise to check that every element of a coprimary module is coprimary when the polyhedral partially ordered group is discrete (Example 2.11) and closed (Definition 2.9).

Theorem 4.13. Fix a $Q$-module $M$ and a face $\tau$ of the positive cone $Q_+$ in a closed polyhedral partially ordered group $Q$. The module $M$ is $\tau$-coprimary if and only if every homogeneous element divides a $\tau$-coprimary element, where $y \in M_q$ divides $y' \in M_{q'}$ if $q \leq q'$ and $y$ has image $y'$ under the structure morphism $M_q \to M_{q'}$.

Proof. If $M$ is $\tau$-coprimary and $y \in M_q$ is a nonzero homogeneous element, then $y$ is $\tau$-persistent because $M$ is a submodule of $M_\tau$ on which $k[Z\tau]$ acts freely. On the other hand, $y$ divides a $\overline{\tau}$-transient element because $\Gamma_\tau M_\tau$ is an essential submodule of $M_\tau$: the submodule of $M_\tau$ generated by $y$ intersects $\Gamma_\tau M_\tau$ nontrivially. The closed hypothesis on $Q$ implies that an element supported on $\tau$ is $\overline{\tau}$-transient, as in Example 3.8.

The other direction does not require the closed hypothesis. Assume that every homogeneous element of $M$ divides a $\tau$-coprimary element. The graded component of the localization $M_\tau$ in degree $q \in Q$ is the direct limit of $M_{q'}'$ over $q' \in q + \tau$. If $y \in M_q$ lies in $\ker(M \to M_{\tau})$, then the image of $y$ must vanish in some $M_{q'}$, whence $y = 0$ to begin with, by $\tau$-persistence. On the other hand, that $\Gamma_\tau M_\tau$ is an essential submodule of $M_\tau$ follows because every $\overline{\tau}$-transient element is supported on $\tau$. \hfill \Box

5. Primary decomposition of modules

Definition 5.1. Fix a $Q$-module $M$ over a polyhedral partially ordered group $Q$. A primary decomposition of $M$ is an injection $M \hookrightarrow \bigoplus_{i=1}^{r} M/M_i$ into a direct sum of coprimary quotients $M/M_i$, called components of the decomposition.

Remark 5.2. Primary decomposition is usually phrased in terms of primary submodules $M_i \subseteq M$, which by definition have coprimary quotients $M/M_i$, satisfying $\bigcap_{i=1}^{r} M_i = 0$ in $M$. This is equivalent to Definition 5.1.

Example 5.3. By Theorem 4.13, a primary decomposition $D = \bigcup_{i=1}^{r} D_i$ of a downset $D$ yields a primary decomposition of the corresponding indicator quotient, namely $k[D] \hookrightarrow \bigoplus_{i=1}^{r} k[D_i]$ induced by the surjections $k[D] \twoheadrightarrow k[D_i]$. See, Example 1.2, for instance.

Example 5.4. The interval module in Example 4.3 has a primary decomposition

\[
\begin{bmatrix}
\text{k} & \text{k} & \text{k} & \text{k} \\
\text{k} & \text{k} & \text{k} & \text{k} \\
\text{k} & \text{k} & \text{k} & \text{k} \\
\text{k} & \text{k} & \text{k} & \text{k}
\end{bmatrix}
\]

in which the global support along each face is extended downward so as to become a quotient instead of a submodule of the original interval module.
Primary decomposition requires some notion of finiteness, both for the ambient context and for the module being decomposed. In usual commutative algebra, the noetherian condition serves both purposes. Here, the closed polyhedral condition provides ambient finiteness, and the following takes care of modules.

**Definition 5.5.** A downset hull of a module $M$ over an arbitrary poset is an injection $M \hookrightarrow \bigoplus_{j \in J} E_j$ with each $E_j$ being a downset module. The hull is finite if $J$ is finite. The module $M$ is downset-finite if it admits a finite downset hull.

**Remark 5.6.** The existence of primary decomposition in Theorem 5.8 is intended for modules that are tame [Mil20a, Definitions 2.6 and 2.11]. Roughly speaking, each such module is constant on finitely many regions that partition the poset. However, because primary decomposition deals only with essential submodules and nothing akin to generators—in the pictures, only phenomena near the upper boundary matter, not anything near the lower boundary—it only requires the downset half of the tame condition. In contrast, the tame condition is equivalent (by the syzygy theorem for modules over posets [Mil20a, Theorem 6.12]) to requiring a finite fringe presentation [Mil20a, Definition 3.16], which entails a finite upset cover in addition to a finite downset hull.

In general, if the monoid algebra $k[Q_+]$ is noetherian, then for $Q$-modules,

$$\text{noetherian} \Rightarrow \text{tame} \Rightarrow \text{downset-finite}.$$ 

**Example 5.7.** The implications in Remark 5.6 are strict even when $k[Q_+]$ is noetherian. The upset module $k\{x^ay^b \mid a+b \geq 0\}$ for the half-plane above the antidiagonal line is a tame but not noetherian $\mathbb{Z}^2$-module. For a downset-finite but not tame $\mathbb{Z}^2$-module, take the submodule of $k[\mathbb{Z}^2] \oplus k[\mathbb{Z}^2]$ that has degree $[\frac{a}{b}]$ component

$$\begin{cases} 
0 & \text{below the antidiagonal line } a+b=1, \\
\mathbb{k}^2 & \text{above the antidiagonal line } a+b=1, \\
\text{span}([\frac{a}{b}]) & \text{on the antidiagonal line } a+b=1.
\end{cases}$$

This module is visibly downset-finite—it is a submodule of a direct sum of two copies of the downset module $k[\mathbb{Z}^2]$—but it is not tame [Mil20a, Example 4.25].

**Theorem 5.8.** Every downset-finite module over a polyhedral partially ordered group admits a primary decomposition.

**Proof.** If $M \hookrightarrow \bigoplus_{j=1}^k E_j$ is a downset hull of the module $M$, and $E_j \hookrightarrow \bigoplus_{i=1}^\ell E_{ij}$ is a primary decomposition for each $j$ afforded by Corollary 3.11 and Example 5.3, then let $E^\tau$ be the direct sum of the downset modules $E_{ij}$ that are $\tau$-coprimary. Set $M^\tau = \ker(M \to E^\tau)$. Then $M/M^\tau$ is coprimary, being a submodule of a coprimary module. Moreover, $M \to \bigoplus_\tau M/M^\tau$ is injective because its kernel is the same as the kernel of $M \to \bigoplus_{ij} E_{ij}$, which is a composite of two injections and hence injective by construction. Therefore $M \to \bigoplus_\tau M/M^\tau$ is a primary decomposition. $\square$
Example 5.9. The finiteness of primary decomposition depends on the polyhedral condition that posits finiteness of the number of faces of the positive cone (Definition 2.8). When the positive cone has infinitely many faces, such as the positive half $Q_+$ of the right circular cone $x^2 + y^2 \leq z^2$ in $Q = \mathbb{R}^3$, the $Q$-module 
$$k[\partial Q_+] = k[Q_+]/k[Q^\circ_+]$$
does not admit a finite primary decomposition. The module $M = k[\partial Q_+]$ has a vector space of dimension 1 on the boundary of the positive cone and 0 elsewhere. Every face of the positive cone must get its own summand $M/M_i$ in Definition 5.1 for the homomorphism $M \to \bigoplus_{i=1}^r M/M_i$ there to be injective, and in that case the infinite number of faces would force the direct sum to become a direct product. This particular example, with the right circular cone, works as well in the discrete partially ordered group $\mathbb{Z}^3$ because the circle has infinitely many rational points.

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