Does black hole radiance break supersymmetry?

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Arguments presented in an earlier paper, demonstrating the breakdown of global supersymmetry in Hawking radiation from a generic four-dimensional black hole with infalling massless scalar and spinor particles, are reexamined. Careful handling of the Grassmann-valued spinorial supersymmetry parameter is shown to lead to a situation wherein supersymmetry may not actually break. A comparative analysis in flat spacetime at finite temperature is also presented.

I. INTRODUCTION

It is now almost universally accepted that any generic black hole with infalling quantum matter fields radiates like a black body in equilibrium at a temperature $T_{bh} \sim \kappa/2\pi$, where $\kappa$ is the surface gravity at the horizon of the black hole \footnote{Supersymmetric black holes, i.e., those associated with a covariantly conserved Killing spinor, are usually extremal and do not emit Hawking radiation. Such special black holes are not considered here.}. Consequently, infalling bosonic fields are radiated out in a thermal Bose-Einstein (Planckian) distribution (modulo some 'grey-body' factors) while fermions are radiated out in a Fermi-Dirac distribution. On the basis of this alone, one might expect that the supersymmetry manifest at past null infinity in a system of bosons and fermions will not survive such radiation. This expectation seems to be borne out in detail in an investigation performed two years ago \footnote{email: partha@imsc.ernet.in}, in which the standard criterion of spontaneous supersymmetry breakdown is used, namely the non-vanishing (or otherwise) of the vacuum expectation value of the supersymmetry variation of a fermionic operator at future null infinity. The evaluation of the vacuum expectation value (vev) in ref. \footnote{email: partha@imsc.ernet.in} follows Hawking's original approach involving a zero temperature quantum field theory in the black hole background, and is generic in nature. However, in that derivation, as elsewhere, one has tended to ignore the fact that the spinorial supersymmetry parameter is actually an element of a Grassmann algebra (an $\alpha$-number) rather than a $\gamma$-number. In other words, unlike a $\gamma$-number parameter which commutes with all operators of the theory, the supersymmetry parameter anti-commutes with fermionic operators instead. Clearly, this may have serious implications for evaluation of Green's functions and the like involving strings of fermionic and bosonic operators, such as the issue of supersymmetry breaking entails. Our concern here is with possible ramifications for black hole radiance. Recall that the phenomenon of black hole radiance is based upon particle creation in the gravitational field of a black hole. Thus, evaluation of vevs of operators defined at future null infinity will involve matrix elements of such operators between states (at future null infinity) of non-zero fermion number. It is here that a naive handling of the supersymmetry parameter is most likely to differ from a careful one. If sharp disparities arise, the conventional wisdom that black hole radiance breaks supersymmetry, is bound to be challenged.

Section II of the paper is a brief recapitulation of the main tenets of the earlier work. In section III, we attempt a more careful evaluation of the relevant vacuum expectation value, to see if supersymmetry can indeed be preserved in black hole radiance. In section IV, a comparative analysis of the supersymmetric model in flat spacetime in presence of a heat bath at a finite temperature is presented. Our conclusions and outlook are presented in the final section.

II. THE EARLIER FORMULATION SURVEYED

We focus on a situation where, at past null infinity ($\mathcal{I}^-$), there exists a globally supersymmetric model of noninteracting massless complex scalar and chiral spinor fields. Now, any state on $\mathcal{I}^-$ will evolve into a state on the event horizon ($\mathcal{H}^+$), belonging to one of two mutually exclusive (in absence of backreaction) classes, viz., those which are purely outgoing, i.e., have zero Cauchy data on $\mathcal{H}^+$ and support on $\mathcal{I}^+$, and those which have zero Cauchy data on $\mathcal{I}^+$ and support on $\mathcal{H}^+$. As is well-known \footnote{email: partha@imsc.ernet.in}, an inherent ambiguity in the latter is chiefly responsible for the thermalization of the radiation received at $\mathcal{I}^+$. The scalar and spinor fields in our model have the following expansion (at $\mathcal{I}^-$),
\[
\phi = \sum_k \frac{1}{\sqrt{2\omega_k}} \left( a_k^B f_k^B + b_k^B \dagger f_k^B \right) \\
\psi_+ = \sum_k \frac{1}{\sqrt{2\omega_k}} \left( a_k^F f_k^F + b_k^F \dagger f_k^F \right) u_{k,+},
\]

where, the \{f_k\} are complete orthonormal sets of solutions of the respective field equations, with positive frequencies only at \(\mathcal{I}^-\), and \(u_{k,+}\) is a positively chiral spinor, reflecting the chirality of \(\psi_+\). The creation-annihilation operators obey the usual algebra, with \(B\) (\(F\) signifying Bose (Fermi)). The conserved Noether supersymmetry charge is given in terms of these creation-annihilation operators by (at \(\mathcal{I}^-\))

\[
Q_+(\mathcal{I}^-) = \sum_k \left( a_k^F b_k^B \dagger - b_k^F a_k^B \right) u_+(k),
\]

and annihilates the vacuum state \(|0_-\rangle\) defined by

\[
a_k^{B,F}|0_-\rangle = 0 = b_k^{B,F}|0_-\rangle.
\]

The existence of two disjoint classes of states at the horizon, as mentioned earlier, implies that the fields also admit the expansion

\[
\phi = \sum_k \frac{1}{\sqrt{2\omega_k}} \left( A_k^B p_k^B + B_k^B \dagger \bar{p}_k^B + A_k'^B q_k^B + B_k'^B \dagger \bar{q}_k^B \right) \\
\psi_+ = \sum_k \frac{1}{\sqrt{2\omega_k}} \left( A_k^F p_k^F + B_k^F \dagger \bar{p}_k^F + A_k'^F q_k^F + B_k'^F \dagger \bar{q}_k^F \right) u_+(k),
\]

where, \(\{p_k\}\) are purely outgoing orthonormal sets of solutions of the respective field equations with positive frequencies at \(\mathcal{I}^+\), while \(\{q_k\}\) are orthonormal sets of solutions with no outgoing component. The final vacuum state \(|0_+\rangle\), defined by the requirement

\[
A^{B,F}|0_+\rangle = 0 = B^{B,F}|0_+\rangle = A'^{B,F}|0_+\rangle = B'^{B,F}|0_+\rangle
\]

is not unique, because of the inherent ambiguity in defining positive frequency for the \(\{q_k\}\); in fact, one can write \(|0_+\rangle = |0_I\rangle|0_H\rangle\) with the unprimed (primed) operators acting on \(|0_I\rangle\) (\(|0_H\rangle\)). A supersymmetry charge \(Q(\mathcal{I}^+)\) may indeed be defined, analogously to eqn. \(\Box\), in terms of the unprimed operators, and that \(Q(\mathcal{I}^+) |0_+\rangle = 0\). Such a charge also satisfies the \(N = 1\) superalgebra at \(\mathcal{I}^+\).

The field operators \(a_k, b_k\) at \(\mathcal{I}^-\) are of course related to the \(A_k, B_k\) and \(A'^k, B'^k\) through the Bogoliubov transformations

\[
A_k^B = \sum_{k'} \left( \alpha_{kk'} a_{k'}^B + \beta_{kk'} b_{k'} \dagger \right) \\
B_k^B = \sum_{k'} \left( \alpha_{kk'} b_{k'}^B + \beta_{kk'} a_{k'} \dagger \right) \\
A_{k,+}^F = \sum_{k'} \left( \alpha_{kk'} a_{k',+}^F + \beta_{kk'} b_{k',-} \dagger \right) \\
B_{k,-}^F = \sum_{k'} \left( \alpha_{kk'} b_{k,-}^F + \beta_{kk'} a_{k',+} \right),
\]

and similarly for the primed operators. We notice in passing that

\[
Q(\mathcal{I}^-)|0_+\rangle \neq 0, \quad Q(\mathcal{I}^+)|0_-\rangle \neq 0, \quad \text{for } \beta^{B,F} \neq 0.
\]

The issue that we now wish to focus on is whether the radiated system of particles has \(N = 1\) spacetime supersymmetry. To address this question, recall that vacuum expectation values (vevs) of observables at future null infinity are defined by \(\Box\)

\[
\langle \mathcal{O} \rangle \equiv \langle 0_- | \mathcal{O} | 0_- \rangle = Tr \left( \rho \mathcal{O} \right)
\]
where, \( \rho \) is the density operator. The trace essentially averages over the (nonunique) states going through the horizon, thus rendering the vevs of observables (at \( I^+ \)) free of ambiguities. We also recall that a sufficient condition for spontaneous supersymmetry breaking is the existence of a fermionic operator \( \mathcal{O} \) which, upon a supertransformation, yields an operator with non-vanishing vev, i.e., \( \langle \delta_S \mathcal{O} \rangle \neq 0 \). Thus, if one is able to show that for all fermionic observables \( \mathcal{O}(I^+) \),

\[
\langle \delta_S \mathcal{O}(I^+) \rangle = 0 ,
\]

then we are guaranteed that the outgoing particles form a supermultiplet.

However, this is not the case, as is not difficult to see; for, consider the supercharge operator itself at \( I^+ \). Using the supersymmetry algebra, it can be shown that

\[
\langle \delta_S \bar{Q}_\alpha(I^+) \rangle = \epsilon^\beta \langle P_{\beta\dot{\alpha}}(I^+) \rangle ,
\]

where \( P_{\beta\dot{\alpha}} \) is the momentum operator of the theory and \( \epsilon^\beta \) the spinorial supersymmetry parameter. In our free field theory, the rhs of (10) is trivial to calculate, using eq. (6) above, so that we obtain, suppressing obvious indices,

\[
\langle \delta_S \bar{Q}(I^+) \rangle = \epsilon \sum_k \langle N_k^B + N_k^F \rangle = \epsilon \sum_{k,k'} \langle |\beta_k^B|^2 + |\beta_k^F| \rangle .
\]

Thus, supersymmetry is seemingly spontaneously broken in the sense described above, so long as the Bogoliubov coefficients \( \beta^{B,F} \) are non-vanishing In fact, we know from Hawking’s seminal work \( [1] \) that

\[
\langle N_k^B \rangle = \sum_{k'} |\beta_{kk'}^B|^2 = |t_k^B|^2 \left( e^{2\pi |k|/\kappa} - 1 \right)^{-1}
\]

\[
\langle N_k^F \rangle = \sum_{k'} |\beta_{kk'}^F|^2 = |t_k^F|^2 \left( e^{2\pi |k|/\kappa} + 1 \right)^{-1}.
\]

Here, \( \kappa \) is the surface gravity of the black hole, and \( |t_k^{B,F}|^2 \) the transmission coefficients through the potential barrier of the black hole for bosons, fermions respectively.

**III. A MORE CAREFUL EVALUATION**

Using the definition (8) above, the aim is to evaluate the trace \( Tr \left( \rho \left[ \epsilon \bar{Q} \right] \right) \). Treating \( \epsilon \bar{Q} \) as a bosonic operator and using the cyclicity of traces, it is easy to see that,

\[
\langle \delta_S \mathcal{O} \rangle = Tr \left( \left[ \bar{Q} \, , \, \rho \right] \epsilon \bar{Q} \right) .
\]

Thus, the issue of supersymmetry breakdown now depends crucially on whether the density operator \( \rho \) commutes with the supersymmetry generator \( \bar{Q} \) at \( I^+ \). As an operator relation this is not obvious since we do not know the density operator as a function of the basic field operators \( A^{B,F} \), \( B^{B,F} \). In Hawking’s approach, one can only unambiguously determine the diagonal elements of the density matrix. One would expect the determination of such elements to be enough to ascertain whether the vev \( \langle \delta_S \bar{Q} \rangle \) is non-vanishing.

The basic point of departure from earlier approaches (including ours) is the property that for any fermionic operator \( \mathcal{O} \), \( \epsilon \mathcal{O} = -\mathcal{O} \epsilon \). That is to say that \( \langle \epsilon , A^F \rangle = 0 \) and similarly for \( B^F \). It follows that, for normalized states (at \( I^+ \)) with \( n_k^F \) fermions with momentum \( k \), we must have

\[
\langle n_k^F \rangle | \epsilon | n_k^F \rangle = (-)^{n_k^F} \epsilon .
\]

In our earlier approach [3], the rhs of eq. (14) would not have contained the first factor. This does have an immediate import for our calculation of \( \langle \delta_S \mathcal{O} \rangle \) above in eq. (10). Rather than expanding the commutator in the variation \( \delta_S \bar{Q} \) as done above, we follow our earlier step eq. (8) of using the supersymmetry algebra and rewriting (8) as given in eq. (10),

\[
\langle \delta_S \bar{Q}_\alpha(I^+) \rangle = Tr \left( \epsilon^\beta \, P_{\beta\dot{\alpha}}(I^+) \rho \right) .
\]

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The Hilbert space of this non-interacting theory $\mathcal{H} \sim \mathcal{H}_B \otimes \mathcal{H}_F$ so that, changing to the occupation number basis of the infinite system of uncoupled boson and fermion oscillators, labelled by momentum $k$, the states are expressed as $|n^B_k, n^F_k\rangle$. Assuming without loss of generality, a discrete momentum spectrum, eq. (12) may be expressed as

$$\langle \delta \bar{Q}_\alpha (I^+) \rangle = e^\beta \sum_k k_{\beta \dot{\alpha}} \sum_{n^B_k,n^F_k} \rho_{n^B_k,n^F_k} (-)^n_f \left( n^B_k + n^F_k \right).$$ \hspace{1cm} (16)

Since the relevant density matrix elements are uniquely determined by states at future infinity (the horizon states being averaged over) where the system of particles are still non-interacting, the density operator can be factorized as $\rho = \rho^B \rho^F$ where $\rho^B(\rho^F)$ acts on bosonic (fermionic) states $|n^B_k\rangle (|n^F_k\rangle)$ alone respectively.

Recall now the elementary fact that for a given momentum $k$, $n^F_k = 0, 1$. Using eq. (14), (17) immediately yields

$$\langle \delta \bar{Q}_\alpha \rangle = e^\beta \sum_{k_{\beta \dot{\alpha}}} \sum_{n^B_k=0}^{\infty} n^B_k \left( \rho^B - \rho^B_{n^B_k-1,n^F_{k-1}} \right).$$ \hspace{1cm} (17)

Not surprisingly, bose-fermi pairing, characteristic of supersymmetric theories, seems to appear here as well. Thus, whether the $\text{lhs}$ factors)

$$\langle \delta \bar{Q}_\alpha \rangle \equiv e^\beta \sum_k k_{\beta \dot{\alpha}} \sum_{n^B_k=0}^{\infty} n^B_k \left( \rho^B - \rho^B_{n^B_k-1,n^F_{k-1}} \right).$$ \hspace{1cm} (18)

Thus, whether the $\text{rhs}$ of (18) vanishes or not, depends on the nature of the transmission coefficients (grey body factors) $|t^B_k|^2$, $|t^F_k|^2$. The calculation of these coefficients depends on explicit solutions of the matter field equations in specific black hole backgrounds - a task we do not attempt here. However, we note that in the limit $|t^B_k|^2 = |t^F_k|^2 = 1$, i.e., in the limit that the potential barrier of the black hole for outgoing particles is strictly reflectionless, $\langle \delta \bar{Q}\rangle = 0$. Alternatively, the $\text{lhs}$ of eq. (18) vanishes whenever,

$$|t^F_k|^2 = \frac{|t^B_k|^2 (e^{2\pi|k|/\kappa} + 1)}{e^{2\pi|k|/\kappa} - 1 + 2|t^B_k|^2}.$$ \hspace{1cm} (19)

Of course, both the above requirements are non-generic: the first, namely that the potential barrier is reflectionless, is extremely unrealistic. As for the second, namely eq. (19), this is something which only detailed calculation of grey body factors $|t^B_k|^2$ for specific black hole metrics can verify. Observe, using eq. (12) that eq. (19) can be rewritten as

$$\langle N^F_k \rangle = \frac{\langle N^B_k \rangle}{1 + 2\langle N^B_k \rangle}.$$ \hspace{1cm} (20)

IV. SUPERSYMMETRY IN MINKOWSKI SPACE AT FINITE TEMPERATURE

A comparison, with the behaviour of the system of massless supersymmetric bosons and fermions in flat spacetime at a finite temperature $\beta^{-1}$, is in order. In this case, we have full knowledge of the density operator as a function of the basic field operators $|3|$,

$$\rho = e^{-\beta H} / \text{Tr} e^{-\beta H},$$ \hspace{1cm} (21)

where, the Hamiltonian $H = H^B (A^B, B^B) + H^F (A^F, B^F)$. Using eq. (13), and recalling that there is no explicit supersymmetry breaking at $\mathcal{I}^+$ so that $[\bar{Q}, e^{-\beta H}] = 0$, we obtain,

$$\langle \delta \bar{Q}_\alpha \rangle = 0.$$ \hspace{1cm} (22)
More explicitly, taking recourse once again to (14), one can actually calculate the required vev: it is straightforward to see that the thermal average
\[ \langle \delta S \hat{Q} \rangle_\beta = \text{Tr} \{ \epsilon H e^{-\beta H} \}/\text{Tr} e^{-\beta H}. \]  
(23)

Now,
\[ \text{Tr} \{ \epsilon H e^{-\beta H} \} = -\frac{d}{d\beta} \text{Tr} e^{-\beta H}. \]  
(24)

Therefore, to obtain preservation of supersymmetry at a non-zero temperature, all we have to prove is that \( \text{Tr} \epsilon e^{\beta H} \) is independent of \( \beta \).

To see this, we use the fact that the Hamiltonian \( H \) is actually a sum of an infinite number of bosonic and fermionic (spin 1/2) harmonic oscillator Hamiltonians, each at a frequency \( \omega_k = |k| \). Thus,
\[ \text{Tr} e^{-\beta H} = \sum_k \sum_{n_k^B, n_k^F} \langle n_k^B, n_k^F | \epsilon e^{-\beta H} | n_k^B, n_k^F \rangle 
= \epsilon \sum_k \sum_{n_k^B, n_k^F} (-)^{n_k^F} \exp\{-\beta \omega_k \left( n_k^B + n_k^F \right) \} 
= \sum_k \sum_{n_k^B=0}^{\infty} \left( e^{-\beta \omega_k} - e^{-\beta (n_k^B + 1) \omega_k} \right). \]  
(25)

It is clear that there is a term by term cancellation for each value of \( n_k^B \) in the rhs of eq. (25), except for the first term for \( n_k^B = 0 \), which of course is independent of \( \beta \). Thus, just because bosons and fermions obey different statistics at a finite temperature, it is hasty to conclude that supersymmetry is broken. This fact was first pointed out by L. van Hove [4] and subsequently by other workers [5]. Our formulation here makes only implicit use of the Klein operator \((-)^{N_F}\) in contrast to its explicit use in those earlier papers.

V. CONCLUSIONS

The conditions which lead to \( \langle \delta S \hat{Q} \rangle = 0 \), namely that either the potential barrier of the black hole is reflectionless, or eq. (14) above holds, are both non-generic, requiring calculation of grey body factors for specific black holes. Unlike in section II where, by regarding the supersymmetry parameter as a c-number and blithely moving it outside of vevs, we were able to show generically that supersymmetry is broken, now it seems that the situation is actually more complicated. The comparison with the flat space finite temperature case in section IV underlines this feature fairly well: if none of the conditions for \( \langle \delta S \hat{Q} \rangle = 0 \) hold, then the results do not guarantee that a black body spectrum is all there is to black hole radiation. The disparity with the flat space behaviour needs to be probed more thoroughly. We hope to report on this in the near future.

It is important to point out that even if one succeeded in establishing \( \langle \delta S \hat{Q} \rangle = 0 \), in principle there could be other fermionic operators whose supersymmetry variations have nonvanishing vevs. However, the simplicity of the model under consideration makes it very unlikely that the foregoing will be challenged in any manner. It is of course quite another question if interacting fields are considered. The calculation of the vev will then have to be done perturbatively, in general, as one would analyze the flat space \( T \neq 0 \) situation. However, the latter situation has already been considered in [5]: appropriate use of the properties of the supersymmetry parameter seems to preserve all supersymmetry Ward identities. So our result should go through in that case as well.

I thank R. Kaul and A. Dasgupta for useful discussions.

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