Collapse of the Hierarchy of Constant-Depth Exact Quantum Circuits

Yasuhiro Takahashi and Seiichiro Tani
NTT Communication Science Laboratories, NTT Corporation
{takahashi.yasuhiro,tani.seiichiro}@lab.ntt.co.jp

Abstract

We study the quantum complexity class $\text{QNC}_0^f$ of quantum operations implementable exactly by constant-depth polynomial-size quantum circuits with unbounded fan-out gates (called $\text{QNC}_0^f$ circuits). Our main result is that the quantum OR operation is in $\text{QNC}_0^f$, which is an affirmative answer to the question of Høyer and Špalek. In sharp contrast to the strict hierarchy of the classical complexity classes: $\text{NC}_0^f \subsetneq \text{AC}_0^f \subsetneq \text{TC}_0^f$, our result with Høyer and Špalek’s one implies the collapse of the hierarchy of the corresponding quantum ones: $\text{QNC}_0^f = \text{QAC}_0^f = \text{QTC}_0^f$. Then, we show that there exists a constant-depth subquadratic-size quantum circuit for the quantum threshold operation. Lastly, we show that, if the quantum Fourier transform modulo a prime is in $\text{QNC}_0^f$, there exists a polynomial-time exact classical algorithm for a discrete logarithm problem using a $\text{QNC}_0^f$ oracle. This implies that, under a plausible assumption, there exists a classically hard problem that is solvable exactly by a $\text{QNC}_0^f$ circuit with gates for the quantum Fourier transform.

1 Introduction and Summary of Results

Quantum computers are expected to solve some problems much faster than classical computers (e.g. Shor’s factoring algorithm [21]). It is, however, still difficult to realize a quantum computer that can perform quantum algorithms for a reasonably large input size. A major obstacle to realizing a quantum computer is that, even if we can prepare many qubits, we can use them only for a short time due to the coherence time. In order to use such fragile qubits effectively, it is important to understand the possibilities and limitations of using them. This motivates us to study the computational power of quantum circuits with a small amount of computation time [16, 12, 10, 15, 9, 2, 3].

In this paper, we focus on the theoretical analysis of the computational power of constant-depth polynomial-size quantum circuits, which allows us to analyze that of polylogarithmic-depth ones. The elementary gates are one-qubit, CNOT, and unbounded fan-out gates. The unbounded fan-out gate is an analog of the classical one normally assumed to be an elementary gate for the theoretical study of classical circuits [24]. The gate on $n+1$ qubits makes $n$ copies of a classical source bit in a superposition and, in particular, the gate on two qubits is a CNOT gate. It is theoretically interesting to deal with the gate as an elementary gate since the use of the gate clarifies many differences between quantum and classical circuits [12, 15] and connects the quantum circuit model with the one-way model [6].

There are three important settings for studying constant-depth classical circuits. All the settings allow the use of (classical) unbounded fan-out gates. The first setting deals with constant-depth polynomial-size classical circuits consisting of NOT gates and OR and AND gates with bounded fan-in. The classical complexity class $\text{NC}_0^f$ is the class of problems solvable by (uniform families of) the classical circuits in the setting. The second setting is the first one augmented with OR and AND gates with unbounded fan-in, which defines the class $\text{AC}_0^f$. The third setting is the second one augmented with threshold gates with unbounded fan-in, which defines the class $\text{TC}_0^f$. The threshold gate implements the threshold function that outputs the bit representing whether the Hamming weight of the input is less than a pre-determined threshold. These classes form a strict hierarchy: $\text{NC}_0^f \subsetneq \text{AC}_0^f \subsetneq \text{TC}_0^f$ [11, 24].

Some authors consider the quantum counterparts of the above settings [16, 12, 15]. Although it is difficult to determine what the correct counterparts are, we regard the following settings as the counterparts [12], where all the settings allow the use of unbounded fan-out gates. The first setting
The quantum complexity class $QNC_0$, which corresponds to $NC^0$, is the class of quantum operations implementable exactly by (uniform families of) the quantum circuits in the setting (called $QNC_0$ circuits). The second setting is the first one augmented with a quantum version of OR gates with unbounded fan-in, which defines the class $QAC_0$, corresponding to $AC^0$. The third setting is the second one augmented with a quantum version of threshold gates with unbounded fan-in, which defines the class $QTC_0$, corresponding to $TC^0$. It holds that $QNC_0^f \subseteq QAC_0^f = QTC_0^f$ [15].

First, in order to study the relationship between $QNC_0^f$ and $QAC_0^f$, we consider the question posed by Høyer and Špalek [15] as to whether an $O(1)$-depth poly$(n)$-size quantum circuit can be constructed for the quantum operation $OR_n$, which computes the OR function on $n$ bits. They showed that there exists an $O(\log^* n)$-depth $O(n \log n)$-size quantum circuit. It is a repetition of the OR reduction, which is represented as an $O(1)$-depth circuit that exactly reduces the computation of the OR function on $n$ bits to that on $O(\log n)$ bits. Based on their work, we give an affirmative answer to the question:

**Theorem 1** There exists an $O(1)$-depth $O(n \log n)$-size quantum circuit for $OR_n$.

Theorem 1 immediately implies that $OR_n$ is in $QNC_0^f$ and thus $QNC_0^f = QAC_0^f$. Since $QAC_0^f = QTC_0^f$ as described above, the hierarchy of $QNC_0^f$, $QAC_0^f$, and $QTC_0^f$ collapses, i.e., $QNC_0^f = QAC_0^f = QTC_0^f$. This is a sharp contrast to the strict hierarchy of the corresponding classical classes: $NC^0 \subsetneq AC^0 \subsetneq TC^0$. More generally, Theorem 1 with Høyer and Špalek’s result immediately implies that the hierarchy of polylogarithmic-depth exact quantum circuits collapses, i.e., $QNC_0^k = QAC_0^k = QTC_0^k$ for any integer $k \geq 0$, where $QNC_0^k$, $QAC_0^k$, and $QTC_0^k$ are defined similarly to $QNC_0^f$, $QAC_0^f$, and $QTC_0^f$, respectively, except that they deal with $O(\log^k n)$-depth circuits in place of $O(1)$-depth ones.

Our idea for constructing the circuit is that, after we apply Høyer and Špalek’s OR reduction, we compute the OR function on $O$ bits to that on $O(\log n)$ bits. Based on their work, we give an affirmative answer to the question:

**Theorem 2** There exist the following $O(1)$-depth quantum circuits for $TH_n^t$:

- An $O(n \log n)$-size circuit for any $1 \leq t \leq \log n$ or $n - \log n \leq t \leq n$.
- An $O(\sqrt{n \log n})$-size circuit for any $\log n \leq t \leq \lfloor n/2 \rfloor$.
- An $O(\sqrt{n-t} \log n)$-size circuit for any $\lceil n/2 \rceil \leq t \leq n - \log n$.

Theorem 2 implies the size difference between the $QNC_0^f$ and $QTC_0^f$ circuits for implementing the same quantum operation. Let $U_n$ be a quantum operation on $n$ qubits. Let us assume that we have an optimal-size $QTC_0^f$ circuit for $U_n$ and its size is represented by some polynomial $s(n)$. Similarly, let $t(n)$ (at least $s(n)$) be the optimal $QNC_0^f$ circuit size. The definition of $QNC_0^f$ only implies that $t(n)$ is bounded above by poly$(n)$. Theorem 2 tells us more about this: $t(n)$ is $O(s(n) \sqrt{n \log n})$ for any $n \geq \log n$. This is because we can obtain an $O(s(n) \sqrt{n \log n})$-size $QNC_0^f$ circuit for $U_n$, by transforming every threshold gate in the optimal-size $QTC_0^f$ circuit into the $QNC_0^f$ circuit by Theorem 2.

A key ingredient of the circuits in Theorem 2 is an $O(1)$-depth $O(n^2)$-size quantum circuit for the quantum counting operation, which computes the counting function on $n$ bits that outputs the
binary representation of the Hamming weight of the input. Our idea for constructing the circuit is that, after we apply Høyer and Špalek’s OR reduction, we implement a particular type of the quantum Fourier transform (QFT) on \(O(\log n)\) qubits in depth \(O(1)\) and with size exponential in \(\log n\). The QFT part performs many projective measurements in parallel and applies the circuit in Theorem 1 to the classical outcomes of the measurements to estimate the phase of a Fourier state. It is similar to the \(O(\log n)\)-depth \(O(n \log n)\)-size quantum circuit for approximating the QFT on \(n\) qubits \[5\]. The main difference is that the QFT part requires exponentially more gates than those in \[8\] to construct an \(O(1)\)-depth exact circuit. Nevertheless, the size is still \(\text{poly}(n)\) since the input size is \(O(\log n)\).

Lastly, we apply Theorem 1 to studying the relationship between \(\text{QNC}_f^0\) and efficient classical computation. More concretely, based on Theorem 1, we study the existence of a classically hard problem that is solvable exactly by a \(\text{QNC}_f^0\) circuit, where a problem is said to be classically hard if it cannot be solved by a polynomial-time bounded-error classical algorithm. To do this, we consider the question of whether a polynomial-time exact classical algorithm using a \(\text{QNC}_f^0\) oracle can be constructed for a discrete logarithm problem (DLP) that seems classically hard. Here, the \(\text{QNC}_f^0\) oracle solves, in classical constant time, a problem that is solvable exactly by a \(\text{QNC}_f^0\) circuit. Such an algorithm for the DLP implies the existence of the desired problem under the plausible assumption that the DLP is classically hard. This is because the algorithm with a polynomial-time bounded-error classical simulation of the \(\text{QNC}_f^0\) oracle would imply that the DLP is not classically hard.

Based on Shor’s bounded-error quantum algorithm for the general DLP \[21\], Høyer and Špalek showed that there exists a polynomial-time bounded-error classical algorithm using a bounded-error version of the \(\text{QNC}_f^0\) oracle \[15\]. It is, however, difficult to directly transform the algorithm into an exact one. Based on van Dam’s exact quantum algorithm for the general DLP \[23\], which is simpler than Mosca and Zalka’s \[17\], we show that, using Theorem 1, under an assumption about the QFT, there exists the desired algorithm for a particular type of the DLP that seems classically hard:

**Theorem 3** Let \(q\) be a safe prime, i.e., a prime of the form \(2p + 1\) for some prime \(p\), and \(n = \lfloor \log q \rfloor\). If the QFT modulo \(p\) is in \(\text{QNC}_f^0\), there exists a \(\text{poly}(n)\)-time exact classical algorithm for the DLP over the multiplicative group of integers modulo \(q\) using the \(\text{QNC}_f^0\) oracle.

We note that, as in the cryptographic literature, we assume that there exist infinitely many safe primes. Since we require the assumption about the QFT, Theorem 3 does not imply the existence of the above-mentioned problem (under a plausible assumption). It, however, allows us to deepen our understanding of the relationship among \(\text{QNC}_f^0\), the QFT, and efficient classical computation. In fact, it implies that, under the plausible assumption that the DLP in Theorem 3 is classically hard, there exists a classically hard problem that is solvable exactly by a \(\text{QNC}_f^0\) circuit with gates for the QFT modulo \(p\).

Theorem 3 suggests the following key problem for further understanding the relationship between \(\text{QNC}_f^0\) and efficient classical or quantum computation: Is the QFT modulo \(p\) in \(\text{QNC}_f^0\)? If this is the case, Theorem 3 implies the existence of a classically hard problem that is solvable exactly by a \(\text{QNC}_f^0\) circuit (under a plausible assumption). If not, \(\text{QNC}_f^0\) is strictly weaker than efficient quantum computation, more precisely, it is strictly contained in the class of quantum operations implementable approximately (or even exactly) by polynomial-size quantum circuits. This is because the QFT modulo \(p\) is in the latter class \[17\] \[13\] \[15\]. We leave the problem about the QFT modulo \(p\) as an open problem.

The main components of (a slightly modified version of) van Dam’s algorithm for the DLP are the QFT modulo \(p\), arithmetic operations such as modular exponentiation, and an amplitude amplification procedure \[5\]. Our rigorous analysis of the algorithm shows that these components excluding the QFT can be implemented by using the OR functions and iterated multiplications with values pre-computed by polynomial-time exact classical algorithms. This analysis with Theorem 1 implies Theorem 3.

The remainder of this paper is organized as follows. In Section 2, we give some definitions and the idea of the OR reduction to describe our results precisely. In Sections 3 and 4, we describe the circuits in Theorems 1 and 2, respectively. In Section 5, we describe the algorithm in Theorem 3. In Section 6, we give some open problems. Most of the proofs are given in Appendix A.

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1 We deal with not only a decision problem, but also a relation problem, where a relation problem can have many valid (polynomial-length) outputs for an input. An algorithm for solving such a problem outputs any one of them \[1\].
2 Preliminaries

2.1 Quantum Circuits and Complexity Classes

We use the standard notation for quantum states and the standard diagrams for quantum circuits [18]. A quantum circuit consists of elementary gates, where the elementary gates are one-qubit, CNOT, and unbounded fan-out gates (unless otherwise stated). An unbounded fan-out gate on \( k + 1 \) qubits implements the quantum operation defined as

\[
\left( \begin{array}{c}
|y\rangle \\
\vdots \\
|y\rangle
\end{array} \right) \mapsto \left( \begin{array}{c}
|y\rangle \\
\vdots \\
|y\rangle \\
|y\rangle \\
\vdots \\
|y\rangle \\
|y\rangle \\
\vdots \\
|y\rangle
\end{array} \right),
\]

where \( y, x_j \in \{0, 1\}, k \geq 1 \), and \( \oplus \) denotes addition modulo 2. The first input qubit, i.e., the qubit in state \( |y\rangle \), is called the control qubit. When \( k = 1 \), the gate is a CNOT gate. Since an unbounded fan-out gate makes copies of a classical source bit, we may say “copy” when we apply this gate. The complexity measures of a quantum circuit are its size and depth. The size of a quantum circuit is defined as the total size of all elementary gates in it, where the size of an elementary gate is defined as the number of qubits affected by the gate. The depth of a quantum circuit is defined as follows. Input qubits are considered to have depth 0. For each gate \( G \), the depth of \( G \) is equal to 1 plus the maximal depth of a gate on which \( G \) depends. The depth of a quantum circuit is defined as the maximal depth of a gate in it. Intuitively, the depth is the number of layers in the circuit, where a layer consists of gates that can be applied in parallel. A quantum circuit can use ancillary qubits initialized to \( |0\rangle \).

For any \( a = a_0 \cdots a_{n-1} \in \{0, 1\}^n \setminus \{0^n\} \), the parity function with value \( a \) on \( n \) bits, denoted as \( \text{PA}_n^a \), is defined as \( \text{PA}_n^a(x) = \bigoplus_{j=0}^{n-1} a_j x_j \), where \( x = x_0 \cdots x_{n-1} \in \{0, 1\}^n \). We denote \( \text{PA}_n^n \) as \( \text{PA}_n \). For example, \( \text{PA}_2^0(x) = x_0 \), \( \text{PA}_2^1(x) = x_1 \), and \( \text{PA}_2^2(x) = x_0 \oplus x_1 \). For any integer \( 1 \leq t \leq n \), the threshold function with a threshold \( t \) on \( n \) bits, denoted as \( \text{TH}_n^t \), is defined as \( \text{TH}_n^t(x) = 1 \) if \( |x| \geq t \) and 0 otherwise, where \( x = x_0 \cdots x_{n-1} \in \{0, 1\}^n \) and \( |x| = \sum_{j=0}^{n-1} x_j \), the Hamming weight of \( x \). The OR function on \( n \) bits, denoted as \( \text{OR}_n \), is defined as \( \text{TH}_n^1 \). The AND function on \( n \) bits, denoted as \( \text{AND}_n \), is defined as \( \text{TH}_n^n \). For any integer \( 1 \leq t \leq n \), the exact function with value \( t \) on \( n \) bits, denoted as \( \text{EX}_n^t \), is defined similarly to \( \text{TH}_n^t \) except that \( |x| \geq t \) in the definition of \( \text{TH}_n^t \) is replaced with \( |x| = t \). The function \( \text{EX}_n^0 \) is defined as the negation of \( \text{OR}_n \). The quantum operation for computing \( \text{PA}_n^a \) is defined as

\[
\left( \begin{array}{c}
|z\rangle \\
\vdots \\
|z\rangle
\end{array} \right) \mapsto \left( \begin{array}{c}
|z\rangle \\
\vdots \\
|z\rangle \\
|z\rangle \\
\vdots \\
|z\rangle \\
|z\rangle \\
\vdots \\
|z\rangle
\end{array} \right) \bigoplus \text{PA}_n^a(x),
\]

where \( x_j, z \in \{0, 1\} \) and \( x = x_0 \cdots x_{n-1} \). For simplicity, this operation is also denoted as \( \text{PA}_n^a \). The quantum operations \( \text{TH}_n^t, \text{OR}_n, \text{AND}_n, \) and \( \text{EX}_n^t \) are defined similarly. For any integer \( m > 0 \), the quantum Fourier transform modulo \( m \), denoted as \( F_m \), is the quantum operation on \( \log m \) qubits defined as \( |x\rangle \mapsto \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{2\pi i xy/m} |y\rangle \), where \( 0 \leq x \leq m - 1 \) and \( \omega_m = e^{2\pi i/m} \).

The quantum complexity class \( \text{QNC}_0^t \) is the class of quantum operations implementable exactly by (uniform families of) constant-depth polynomial-size quantum circuits consisting of the elementary gates described above. The definition of \( \text{QAC}_0^t \) is the same as that of \( \text{QNC}_0^t \) except that quantum circuits can use a gate for \( \text{OR}_n \) as an elementary gate for any \( k \) bounded above by an arbitrary \( \text{poly}(n) \) for input length \( n \). The definition of \( \text{QTC}_0^t \) is the same as that of \( \text{QAC}_0^t \) except that quantum circuits can use a gate for \( \text{TH}_n^t \) as an elementary gate for any \( k \) bounded above by an arbitrary \( \text{poly}(n) \) and \( 1 \leq t \leq k \). Although some authors assume that quantum circuits can use only a bounded number of distinct one-qubit gates [15], we do not assume this since we consider the exact setting. Thus, the complexity classes in this paper are equal to or larger than those in the papers that considered only a bounded number of distinct one-qubit gates. We note, however, that one-qubit gates used in our circuits are only Hadamard gates \( H \) and \( Z(\pm \pi/2^k) \) gates for any integer \( k \geq 0 \), where, for any \( \theta \in \mathbb{R} \),

\[
H = \frac{1}{\sqrt{2}} \left( \begin{array}{cc}
1 & 1 \\
1 & -1
\end{array} \right), \quad Z(\theta) = \left( \begin{array}{cc}
1 & 0 \\
0 & e^{i\theta}
\end{array} \right).
\]
of the circuit, which will be used in our circuits. We want to compute OR
an
the circuit exactly reduces the problem of computing OR
m
by using unbounded fan-out gates, all the states
be an input state, where
the circuit outputs the m-qubit state \( \bigotimes_{k=0}^{m-1} H |\varphi_k\rangle \), where
for any \( 0 \leq k \leq m-1 \). If \( |x| = |x_0 \cdots x_{n-1}| = 0 \), \( H |\varphi_k\rangle = |0\rangle \) for any \( 0 \leq k \leq m-1 \) and thus the output state is \( |0\rangle^{\otimes m} \). If \( |x| \geq 1 \), there exist \( 0 \leq a \leq m-1 \) and \( b \geq 0 \) such that \( |x| = 2^a (2^b + 1) \). A direct calculation shows that \( H |\varphi_a\rangle = |1\rangle \) and thus the output state is orthogonal to \( |0\rangle^{\otimes m} \). Therefore, the circuit exactly reduces the problem of computing OR
m
to that of computing OR
n
. For any \( 0 \leq k \leq m-1 \), \( |\varphi_k\rangle \) can be prepared by an \( O(1) \)-depth \( O(n) \)-size quantum circuit as depicted in Fig. 21. By using unbounded fan-out gates, all the states \( |\varphi_k\rangle \) can be prepared in parallel and thus the depth and size of the circuit for the OR reduction are \( O(1) \) and \( O(nm) = O(n \log n) \), respectively.

2.2 Høyer and Špalek’s OR Reduction

The OR reduction is described as an \( O(1) \)-depth \( O(n \log n) \)-size quantum circuit for exactly reducing the problem of computing OR
n
, where \( m = \lceil \log(n+1) \rceil \). We explain the idea of the circuit, which will be used in our circuits. We want to compute OR
n
and let \( |x| = |x_0 \cdots x_{n-1}| \) be an input state, where \( x_j \in \{0, 1\} \). The circuit outputs the m-qubit state \( \bigotimes_{k=0}^{m-1} H |\varphi_k\rangle \), where

\[
|\varphi_k\rangle = \frac{|0\rangle + e^{i\frac{\pi}{2^k}} |1\rangle}{\sqrt{2}}
\]

for any \( 0 \leq k \leq m-1 \). This is shown by using the Fourier expansion of \( f_n \), more precisely, by replacing the Fourier basis in the Fourier expansion of \( f_n \) with a basis consisting of the parity functions \( \text{PA}_a^n \). In particular, the following representation of OR
n
can be obtained by using the Fourier expansion of OR
n
. The proof is given in Appendix A.1.

Lemma 1 For any \( x \in \{0, 1\}^n \), \( \text{OR}_n(x) = \frac{1}{2^{n-1}} \sum_{a \in \{0, 1\}^n \setminus \{0^n\}} \text{PA}_a^n(x) \).

The representation of OR
n
implies an \( O(1) \)-depth \( O(n2^n) \)-size quantum circuit for OR
n
. The idea is that, when the input \( x \) is given, we compute \( \text{PA}_a^n(x) \) for every \( a \) in parallel and prepare the state \( \langle 0 \rangle^{\otimes (2^n-1)} + (-1)^{\text{OR}_n(x)} |1\rangle^{\otimes (2^n-1)} \) based on the representation. Applying an unbounded fan-out gate and a Hadamard gate to the state gives the desired state \( |\text{OR}_n(x)\rangle \). The point is that there exists an \( O(1) \)-depth \( O(|a|) \)-size quantum circuit for \( \text{PA}_a^n \), consisting of Hadamard gates and an unbounded fan-out gate as depicted in Fig. 2 [12].

3 Circuit for the OR Function

3.1 Exponential-Size Circuit

For any Boolean function \( f_n : \{0, 1\}^n \rightarrow \{0, 1\} \) satisfying \( f_n(0^n) = 0 \), there exists a set of real numbers \( \{r_a\}_{a \in \{0, 1\}^n \setminus \{0^n\}} \) such that

\[
f_n(x) = \sum_{a \in \{0, 1\}^n \setminus \{0^n\}} r_a \text{PA}_a^n(x)
\]

for any \( x \in \{0, 1\}^n \). This is shown by using the Fourier expansion of \( f_n \) [19], more precisely, by replacing the Fourier basis in the Fourier expansion of \( f_n \) with a basis consisting of the parity functions \( \text{PA}_a^n \). In particular, the following representation of OR
n
can be obtained by using the Fourier expansion of OR
n
. The proof is given in Appendix A.1.
To describe the circuit for OR$_n$ more precisely, let $|x\rangle = |x_0\rangle \cdots |x_{n-1}\rangle$ be an input state. The circuit is described as follows:

1. Copy the input state $|x\rangle$ and apply the circuit for PA$_n^a$ to each copy for every $a \in \{0,1\}^n \setminus \{0^n\}$ in parallel to prepare the state $\bigotimes_{a \in \{0,1\}^n \setminus \{0^n\}} |PA_n^a(x)\rangle$.

2. Apply a Hadamard gate and an unbounded fan-out gate to ancillary qubits (initialized to $|0\rangle$) to prepare the $(2^n - 1)$-qubit state $(|0\rangle^{\otimes(2^n-1)} + |1\rangle^{\otimes(2^n-1)})/\sqrt{2}$.

3. Apply controlled-$Z(\pi/2^{n-1})$ gates in parallel to the states in Steps 1 and 2 to prepare the state

$$\frac{|0\rangle^{\otimes(2^n-1)} + e^{i\pi/2^{n-1}} \sum_{a \in \{0,1\}^n \setminus \{0^n\}} PA_n^a(x) |1\rangle^{\otimes(2^n-1)}}{\sqrt{2}} = |0\rangle^{\otimes(2^n-1)} + (-1)^{OR_n(x)} |1\rangle^{\otimes(2^n-1)},$$

where Lemma 1 implies the equation.

4. Apply an unbounded fan-out gate and a Hadamard gate to the state in Step 3 to prepare the desired state $|OR_n(x)\rangle$.

For any $0 \leq j \leq n - 1$, let $e(j) = e_0 \cdots e_{n-1} \in \{0,1\}^n$ such that $e_k = 1$ if $k = j$ and 0 otherwise. In Step 1, since the input state $|x_j\rangle = |PA_n^{e(j)}(x)\rangle$, it suffices to prepare the state $|PA_n^a(x)\rangle$ for every $a \in \{0,1\}^n$ such that $|a| \geq 2$. To prepare the states in parallel, we require the state $|x_j\rangle^{\otimes(2^n-1)}$ for any $0 \leq j \leq n - 1$. Thus, before applying the circuit for PA$_n^a$, we apply an unbounded fan-out gate to the input qubit in state $|x_j\rangle$ and $2^n-1$ ancillary qubits for every $0 \leq j \leq n - 1$ in parallel. In Step 2, we apply an unbounded fan-out gate to the ancillary qubits in state $(H|0\rangle) |0\rangle^{\otimes(2^n-2)}$. In Step 3, we use the qubit in state $|PA_n^a(x)\rangle$ as the control qubit of the controlled-$Z(\pi/2^{n-1})$ gate. In Step 4, we first apply an unbounded fan-out gate to the state in Step 3 to disentangle the last $2^n - 2$ qubits and obtain the state $(|0\rangle + (-1)^{OR_n(x)} |1\rangle)/\sqrt{2}$. Thus, the Hadamard gate outputs the desired state. By the construction, the depth of the whole circuit does not depend on $n$. Since Step 1 is the dominant part and uses $n$ unbounded fan-out gates on $2^{n-1}$ qubits, the size of the whole circuit is $O(n2^n)$. This implies the following lemma. The details of the proof are given in Appendix A.2.

**Lemma 2** There exists an $O(1)$-depth $O(n2^n)$-size quantum circuit for OR$_n$.

**Remark:** Hoban et al. considered a restricted model of measurement-based quantum computation, where the adaptivity of measurements is removed [14]. They showed that, if we are allowed to use the $(2^n - 1)$-qubit state in Step 2, any Boolean function $f_n$ can be computed exactly in the model by the procedure based on the above-mentioned representation of $f_n$. The circuit in Lemma 2 can be considered as a simulation of the procedure for computing OR$_n$ in the model. The unbounded fan-out gates are mainly used for preparing the $(2^n - 1)$-qubit state and for computing PA$_n^a$.

### 3.2 Proof of Theorem 1

We show Theorem 1 using Høyer and Špalek’s OR reduction and Lemma 2. Let $|x\rangle = |x_0\rangle \cdots |x_{n-1}\rangle$ be an input state. The circuit is described as follows:
1. Apply Høyer and Špalek’s OR reduction to the input state $|x\rangle$ to prepare the $m$-qubit state $\bigotimes_{k=0}^{m-1} H|\varphi_k\rangle$, where $m = \lceil \log(n+1) \rceil$.

2. Apply the circuit in Lemma 2 to the state in Step 1 to prepare the desired state $|\text{OR}_n(x)\rangle$.

Since Step 1 exactly reduces the problem of computing $\text{OR}_n$ to that of computing $\text{OR}_m$ in depth $O(1)$ and with size $O(n \log n)$, Step 2 outputs the desired state. Since the input size to Step 2 is $m$, the depth and size of the circuit in Step 2 are $O(1)$ and $O(m^2n^2) = O(n \log n)$, respectively. Thus, the depth and size of the whole circuit are $O(1)$ and $O(n \log n)$, respectively. This completes the proof.

Theorem 1 immediately implies that $\text{OR}_n$ is in $\text{QNC}_1^0$ and thus the following relationship holds:

Corollary 1 $\text{QNC}_1^0 = \text{QAC}_1^0$.

Since $\text{QAC}_1^0 = \text{QTC}_1^0$ [15], it holds that $\text{QNC}_1^0 = \text{QAC}_1^0 = \text{QTC}_1^0$. Corollary 1 and the relationship $\text{QAC}_1^0 = \text{QTC}_1^0$ immediately imply that $\text{QNC}_1^0 = \text{QAC}_1^0$ and $\text{QAC}_1^0 = \text{QTC}_1^0$, respectively, for any integer $k \geq 0$, where $\text{QNC}_1^k$, $\text{QAC}_1^k$, and $\text{QTC}_1^k$ are defined similarly to $\text{QNC}_0^m$, $\text{QAC}_0^m$, and $\text{QTC}_0^m$, respectively, except that they deal with $O(\log k n)$-depth circuits in place of $O(1)$-depth ones. Therefore, more generally, it holds that $\text{QNC}_1^k = \text{QAC}_1^k = \text{QTC}_1^k$ for any integer $k \geq 0$.

For any integer constant $c \geq 1$, the size of the circuit in Theorem 1 can be decreased to $O(n \log^c n)$ without increasing the depth asymptotically, where $\log^c(n)$ is the $c$-times iterated logarithm $\log \cdots \log n$. To show this, we divide the $n$ input qubits into $n/\log n$ blocks of $\log n$ qubits. For each block, we apply the circuit in Theorem 1 to compute $\text{OR}_{\log n}$. We obtain $n/\log n$ output qubits and apply the circuit again to the output qubits to compute $\text{OR}_{\log n}$, which yields the desired output. The depth and size of the whole circuit are $O(1)$ and $O(n \log^2 n)$, respectively. Using the resulting circuit, we repeat this size-reduction procedure. After $c - 1$ times repetition, we obtain an $O(n \log^c n)$-size circuit.

The circuit for $\text{OR}_n$ yields a circuit for $\text{EX}_n^t$ [15]. To construct the circuit, it suffices to prepare $Z(-t\pi/2^k)|\varphi_k\rangle$ in place of $|\varphi_k\rangle$ in Høyer and Špalek’s OR reduction and to negate the final output of the circuit in Theorem 1. This is done by only adding a $Z(-t\pi/2^k)$ gate for every $0 \leq k \leq m - 1$ and a NOT gate. Thus, the depth and size of the resulting circuit are asymptotically the same as those in Theorem 1. This yields an $O(1)$-depth $O(n \log n)$-size quantum circuit for $\text{EX}_n^t$ for any $0 \leq t \leq n$.

4 Circuit for the Threshold Function

First, we describe a constant-depth circuit for $\text{TH}_n^t$ based on the constant-depth circuits for $\text{EX}_n^k$ described above. Then, we describe another constant-depth circuit for $\text{TH}_n^t$ based on a circuit for the counting function. Next, we combine these two circuits to show Theorem 2.

4.1 Exact-Function-Based and Counting-Function-Based Circuits

We first consider a constant-depth circuit for $\text{TH}_n^k$ based on the circuits for $\text{EX}_n^k$ when $1 \leq t \leq \lceil n/2 \rceil$. Let $|x\rangle = |x_0\rangle \cdots |x_{n-1}\rangle$ be an input state. The circuit is described as follows:

1. Copy the input state $|x\rangle$ and apply the circuit for $\text{EX}_n^k$ to each copy for every $0 \leq k \leq t - 1$ in parallel to prepare the state $\bigotimes_{k=0}^{t-1} |\text{EX}_n^k(x)\rangle$.

2. Apply the circuit for $\text{PA}_t$ and a NOT gate to the state in Step 1 to prepare the state $|\bigoplus_{k=t}^{n-1} \text{EX}_n^k(x) \oplus 1\rangle$.

If $|x| \geq t$, $\text{EX}_n^k(x) = 0$ for every $0 \leq k \leq t - 1$. If $|x| < t$, there exists exactly one $0 \leq k \leq t - 1$ such that $\text{EX}_n^k(x) = 1$. Thus, the state in Step 2 is equal to the desired state $|\text{TH}_n^t(x)\rangle$. The depth and size of the circuit in Step 1 are $O(1)$ and $O(t n \log n)$, respectively. As depicted in Fig. 2 the depth and size of the circuit for $\text{PA}_t$ are $O(1)$ and $O(t)$, respectively. Thus, the depth and size of the whole circuit are $O(1)$ and $O(t n \log n)$, respectively. When $\lceil n/2 \rceil \leq t \leq n$, we modify the circuit in such a way that it prepares the state $|\bigoplus_{k=t}^{n-1} \text{EX}_n^k(x)\rangle$ in Step 2. This implies the following lemma:
Lemma 3 There exist the following $O(1)$-depth quantum circuits for TH$^k_n$:

- An $O(tn \log n)$-size circuit for any $1 \leq t \leq \lceil n/2 \rceil$.
- An $O((n - t + 1)n \log n)$-size circuit for any $\lceil n/2 \rceil \leq t \leq n$.

When $t$ is an integer constant, the size is $O(n \log n)$. On the other hand, when $t = \lceil n/2 \rceil$, in other words, for the majority function, the size is $O(n^2 \log n)$.

We define the counting function on $n$ bits, denoted as $CO_n$, as $CO_n(x) = s_0 \cdots s_{m-1}$, where $x \in \{0, 1\}^n$, $s_j \in \{0, 1\}$, $m = \lceil \log (n+1) \rceil$, and $|x| = \sum_{j=0}^{m-1} s_j 2^j$. It computes the binary representation of the Hamming weight of the input. The quantum operation for computing $CO_n$ is defined as

\[
\left( \bigotimes_{j=0}^{n-1} |x_j\rangle \right) \left( \bigotimes_{j=0}^{m-1} |z_j\rangle \right) \mapsto \left( \bigotimes_{j=0}^{n-1} |x_j\rangle \right) \left( \bigotimes_{j=0}^{m-1} |z_j \oplus s_j\rangle \right),
\]

where $x_j, z_j \in \{0, 1\}$. This operation is also denoted as $CO_n$.

We construct a constant-depth circuit for $CO_n$. Let $|x\rangle$ be an input state. Since $|x\rangle = \sum_{j=0}^{m-1} s_j 2^j |\varphi_k\rangle$ in H"{o}yer and Špalek’s OR reduction is $(|0\rangle + e^{i \pi \sum_{j=0}^{n} 2^j |1\rangle})/\sqrt{2}$. This implies that $|\varphi_0\rangle \cdots |\varphi_{m-1}\rangle = F_{2^n} |s_0\rangle \cdots |s_{m-1}\rangle$. Thus, to obtain the desired state $|s_0\rangle \cdots |s_{m-1}\rangle$, it suffices to implement the following type of the inverse of the QFT: $|x\rangle (F_{2^n} |s_0\rangle \cdots |s_{m-1}\rangle) \mapsto |x\rangle |s_0\rangle \cdots |s_{m-1}\rangle$.

Our idea for implementing this operation is to perform $A(\theta)$-measurements on many $|\varphi_k\rangle$’s in parallel for appropriate $\theta$’s, where, for any $\theta \in \mathbb{R}$, an $A(\theta)$-measurement is the one-qubit projective measurement in the basis $(|0\rangle + e^{i \theta} |1\rangle)/\sqrt{2}, (|0\rangle - e^{i \theta} |1\rangle)/\sqrt{2}$, which correspond to the classical outcomes 0 and 1, respectively. The classical outcomes imply each $s_k$ exactly.

For example, when $m = 3$, we first prepare the state $|\varphi_0\rangle |\varphi_1\rangle^{\otimes 2} |\varphi_2\rangle^{\otimes 4}$ with a slightly modified version of H"{o}yer and Špalek’s OR reduction, where

\[
|\varphi_0\rangle = \frac{|0\rangle + e^{i \pi s_0} |1\rangle}{\sqrt{2}}, \quad |\varphi_1\rangle = \frac{|0\rangle + e^{i \pi (s_1 + s_2 2^0)} |1\rangle}{\sqrt{2}}, \quad |\varphi_2\rangle = \frac{|0\rangle + e^{i \pi (s_2 2^0 + s_3 2^1 + s_4 2^2)} |1\rangle}{\sqrt{2}}.
\]

We can easily obtain $s_0$ since it is equal to the classical outcome $s_0^0$ of an $A(0)$-measurement on $|\varphi_0\rangle$. The value $s_1$ is determined depending on $s_0$. When $s_0 = 0$, $s_1$ is equal to the classical outcome $s_1^0$ of an $A(0)$-measurement on $|\varphi_1\rangle$. When $s_0 = 1$, $s_1$ is equal to the classical outcome $s_1^1$ of an $A(\pi/2)$-measurement on $|\varphi_1\rangle$. In other words, $s_1 = s_0^{s_1}$. Similarly, we perform $A(0)$-, $A(\pi/4)$-, $A(\pi/2)$-, and $A(3\pi/4)$-measurements on $|\varphi_2\rangle^{\otimes 4}$ and let $s_2^{00}, s_2^{01}, s_2^{01}, s_2^{11}$ be the classical outcomes, respectively. By the definition of the measurements, $s_2 = s_2^{s_1 s_3}$. These relationships imply

\[
s_1 = [s_1^0 (1 \oplus s_0^0) \oplus s_1^1 (1 \oplus 1 \oplus s_0^0)],
\]
\[
s_2 = [s_2^{00} (1 \oplus 0 \oplus s_0^0) (1 \oplus 0 \oplus s_1^0)] \oplus [s_2^{01} (1 \oplus 1 \oplus s_0^0) (1 \oplus 0 \oplus s_1^0)] \oplus [s_2^{01} (1 \oplus 0 \oplus s_0^0) (1 \oplus 1 \oplus s_1^0)] \oplus [s_2^{11} (1 \oplus 1 \oplus s_0^0) (1 \oplus 1 \oplus s_1^0)].
\]

Thus, if we have sufficiently many copies of the classical outcomes, we can compute $s_k$ for every $1 \leq k \leq m - 1$ in parallel using the circuits for AND$_{k+1}$ and PA$_{2k}$. We note that we can perform all the above measurements in parallel. We define the function $t_k(y)$ on $k$ bits as $t_k(y) = y^k \wedge \sum_{j=0}^{k-1} (1 \oplus y_j \oplus s_j^{y_0 \cdots y_{j-1}})$ for any $y = y_0 \cdots y_{k-1} \in \{0, 1\}^k$, where the value $s_j^{y_0 \cdots y_{j-1}}$ is regarded as $s_j^y$ when $j = 0$. It holds that $s_1 = t_1(0) \oplus t_1(1)$ and $s_2 = t_2(00) \oplus t_2(10) \oplus t_2(01) \oplus t_2(11)$.

To describe the circuit for $CO_n$ more precisely and generally, let $|x\rangle = |x_0\rangle \cdots |x_{n-1}\rangle$ be an input state. The circuit is described as follows:

1. Apply a slightly modified version of H"{o}yer and Špalek’s OR reduction to the input state $|x\rangle$ to prepare the state $\bigotimes_{k=0}^{m-1} |\varphi_k\rangle^{\otimes 2^k}$.
2. Perform $A(0)$- and $A(\pi \sum_{j=0}^{k-1} 2^j |y_j\rangle)$-measurements for every $1 \leq k \leq m - 1$ and $y = y_0 \cdots y_{k-1} \in \{0, 1\}^k$ in parallel on the state in Step 1 to obtain the values $s_0^x, s_1^y, s_2^z \in \{0, 1\}$ such that $s_0 = s_0^x$ and $s_k = s_k^{s_0 \cdots s_{k-1}}$. 

3. Prepare $2^m - 1$ copies of the state $|s_0^x\rangle$ and $2^{m-k} - 1$ copies of the state $|s_k^y\rangle$ and apply the circuit for AND$_{k+1}^x$ (constructed by the circuit for OR$_{k+1}$ in Section 3) to the states for every $1 \leq k \leq m - 1$ and $y \in \{0,1\}^k$ in parallel to prepare the state $\bigotimes_{1 \leq k \leq m-1, y \in \{0,1\}^k} |t_k(y)\rangle$.

4. Apply the circuit for PA$_{2k}$ for every $1 \leq k \leq m - 1$ in parallel to the state in Step 3 to prepare the state $|s_0^x\rangle \bigotimes_{1 \leq k \leq m-1} |\bigoplus_{y \in \{0,1\}^k} t_k(y)\rangle$.

Since $t_k(y) = s_k$ if $y = s_0 \cdots s_{k-1}$ and 0 otherwise for any $1 \leq k \leq m - 1$, $\bigoplus_{y \in \{0,1\}^k} t_k(y) = s_k$.

Thus, Step 4 outputs the desired state. By the construction, the depth of the whole circuit does not depend on $n$. Since Step 1 is the dominant part and the state in Step 1 can be prepared with a circuit of size $O(n \sum_{k=0}^{m-1} 2^k) = O(n^2)$, the size of the whole circuit is $O(n^2)$. This implies the following lemma. The details of the proof are given in Appendix A.3.

**Lemma 4** There exists an $O(1)$-depth $O(n^2)$-size quantum circuit for CO$_n$.

Lemma 4 yields an $O(1)$-depth $O(n^2)$-size quantum circuit for TH$_n^k$. To construct the circuit, it suffices to add a circuit for comparing $t$ with the output of the circuit for CO$_n$. We can construct an $O(1)$-depth poly$(m)$-size quantum circuit for the comparison using the circuit for addition in [7].

4.2 Combination of the Two Circuits

A careful combination of the circuits in Lemmas 3 and 4 yields a smaller circuit for TH$_n^k$. We explain the idea in the case when $1 \leq t \leq \lceil n/2 \rceil$. When the input $x$ is given, before using the first circuit in Lemma 3, we compute some low-order bits (not all the bits!) of the binary representation of $|x\rangle$ by the circuit in Lemma 4. Since we know the low-order bits, it is not necessary to check whether EX$_n^k(x) = 1$ for every $0 \leq k \leq t - 1$ as in Lemma 3. It suffices to consider $0 \leq k \leq t - 1$ such that the low-order bits of the binary representation of $k$ are equal to those computed by the circuit in Lemma 4. The number of $k$'s we need to consider is decreased and thus the size of the whole circuit can be decreased.

More precisely, the circuit is described as follows:

1. Apply the circuit in Lemma 4 to the input state $|x\rangle = |x_0\rangle \cdots |x_{n-1}\rangle$ to prepare the state $|s_0\rangle \cdots |s_{l-1}\rangle$, where $s_0 \cdots s_{l-1}$ are the $l$ low-order bits of the binary representation of $|x\rangle$ and $l$ is an integer satisfying $0 \leq l < \lceil \log(t+1) \rceil$.

2. Apply the first circuit in Lemma 3 to the input state $|x\rangle$ to prepare the state $|\bigoplus_{k} \text{EX}_n^k(x) \oplus 1\rangle$, where we consider only $0 \leq k \leq t - 1$ such that the $l$ low-order bits of the binary representation of $k$ are equal to $s_0 \cdots s_{l-1}$.

Step 2 outputs the desired state as in Lemma 3. It is obvious that the depth does not depend on $n$. The size of the circuit in Step 1 is $O(2^l n)$ and that in Step 2 is $O(2^{-l} t n \log n)$ since there are at most $2^{-l} t$ $k$'s we need to consider. The same idea with the second circuit in Lemma 3 works when $\lceil n/2 \rceil \leq t \leq n$. This implies the following lemma. The details of the proof are given in Appendix A.4.

**Lemma 5** There exist the following $O(1)$-depth quantum circuits for TH$_n^k$:

- An $O(2^l n + 2^{-l} tn \log n)$-size circuit for any $1 \leq t \leq \lceil n/2 \rceil$ and $0 \leq l < \lceil \log(t+1) \rceil$.

- An $O(2^l n + 2^{-l}(n - t + 1) n \log n + n \log n)$-size circuit for any $\lceil n/2 \rceil \leq t \leq n$ and $0 \leq l < \lceil \log(t+1) \rceil$.

By setting $l$ appropriately depending on $t$, Lemma 5 implies Theorem 2. The proof is given in Appendix A.5. The size of the circuit for TH$_n^{\lceil n/2 \rceil}$ in Lemma 3 is $O(n^2 \log n)$ and it can be decreased to $O(n^2)$ by Lemma 4. Theorem 2 with $t = \lceil n/2 \rceil$ yields an even smaller circuit.

**Corollary 2** There exists an $O(1)$-depth $O(n\sqrt{n \log n})$-size quantum circuit for TH$_n^{\lceil n/2 \rceil}$.
5 Discrete Logarithm Algorithm Using a QNC\textsubscript{f} Oracle

Let \( q > 5 \) be a safe prime, i.e., a prime of the form \( q = 2p + 1 \) for some prime \( p > 2 \). In the following, as in the cryptographic literature, we assume that there exist infinitely many safe primes. Let \( G_q = (\mathbb{Z}/q\mathbb{Z})^*, \) the multiplicative group of integers modulo \( q \). It is known that there exists a generator \( 1 < g_q \leq q-1 \) of \( G_q \) and thus \( G_q = \{ g_q^0 = 1, g_q^1, \ldots, g_q^{q-2} \} \) and \( g_q^{q-1} \equiv 1 \mod q \). The discrete logarithm problem (DLP) over \( G_q \) (with respect to given \( q \) and \( g_q \)) is to find \( 0 \leq l_q \leq 2 \) such that \( g_q^{l_q} \equiv x_q \mod q \) for an input \( x_q \in G_q \), where the problem size is \( n = \lceil \log q \rceil \) and the order of \( G_q \), i.e., \( q-1 \) and its decomposition \( 2p \) are known. Since it seems difficult to reduce the DLP over \( G_q \) to DLP’s over groups of sufficiently small orders, it is plausible that it cannot be solved by a polynomial-time bounded-error classical algorithm, in other words, that the DLP over \( G_q \) is classically hard.

Although we can directly consider the DLP over \( G_q \), for simplicity, we consider simpler DLP’s obtained by the reduction method in \cite{3}. Since the order of \( G_q \) is \( 2p \) and \( \gcd(2, p) = 1 \), the DLP over \( G_q \) with an input \( x_q \) can be reduced to the following two DLP’s by a polynomial-time exact classical algorithm. One is the DLP over the group generated by \( g_q^p \) with the input \( x_q^2 \), which is solvable by a polynomial-time exact classical algorithm since the order of \( g_q^p \) is \( 2 \). The other is the DLP over the group \( G \) generated by \( g = g_q^2 \) with the input \( x = x_q^2 \). Thus, to show Theorem 3, it suffices to show that, if \( F_p \) is in QNC\textsubscript{f}, there exists a polynomial-time exact classical algorithm for the DLP over \( G \) using the QNC\textsubscript{f} oracle, which solves, in classical constant time, a problem that is solvable exactly by a QNC\textsubscript{f} circuit.

We analyze (a slightly modified version of) van Dam’s exact algorithm for the DLP \cite{23}, which consists of two parts. The first part is independent of the input \( x \in G \) and transforms the state \( |0\rangle^{\otimes (m+n)} \) into the state \( \frac{1}{\sqrt{p}} \sum_{s=1}^{p-1} |s\rangle |\chi^s\rangle \) as follows, where \( m = \lceil \log p \rceil \), the \( n \)-qubit state \( |\chi^s\rangle = \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \omega_p^{sr} |g^r \mod q\rangle \) for \( 0 \leq s \leq p-1 \), and \( \omega_p = e^{2\pi i/p} \):

1. Apply \( F_p \) to the first \( m \) qubits of the state \( |0\rangle^{\otimes (m+n)} \) to prepare the state \( \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} |r\rangle |0\rangle^{\otimes n} \).
2. Apply the modular exponentiation operation \( |r\rangle |0\rangle \rightarrow |r\rangle |g^r \mod q\rangle \) to the state in Step 1 to prepare the state \( \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} |r\rangle |g^r \mod q\rangle \).
3. Apply \( F_p \) to the first \( m \) qubits of the state in Step 2 to prepare the state \( \frac{1}{\sqrt{p}} \sum_{s=0}^{p-1} |s\rangle |\chi^s\rangle \).
4. Apply the amplitude amplification procedure to prepare the state \( \frac{1}{\sqrt{p}} \sum_{s=1}^{p-1} |s\rangle |\chi^s\rangle \).

Steps 1 and 3 are in QNC\textsubscript{f} by our assumption. Since \( g^r \equiv \prod_{j=0}^{m-1} g_2^{2^{r_j}} \mod q \) and \( r = \sum_{j=0}^{m-1} 2^{r_j} \) and \( r_j \in \{0, 1\} \), the modular exponentiation operation in Step 2 can be implemented by using the iterated multiplication operation with the values \( g_2^{2^j} \) mod \( q \) that can be precomputed by a polynomial-time exact classical algorithm \cite{8}. It holds that QNC\textsubscript{f} = QTC\textsubscript{f} as shown in Section 3 and QTC\textsubscript{f} includes arithmetic operations\textsuperscript{2} such as the iterated multiplication operation \cite{22}. Thus, Step 2 is in QNC\textsubscript{f}.

The procedure in Step 4 is similar to the one in \cite{4}. We define the algorithm \( \mathcal{A}' \) as Steps 1, 2, and 3, and the good state \( |\mathcal{A}'\rangle = \frac{1}{\sqrt{p}} \sum_{s=1}^{p-1} |s\rangle |\chi^s\rangle \). Since \( \langle \mathcal{A}' | \mathcal{A}' \rangle = 1 - 1/p \), it is easy to transform \( \mathcal{A}' \) into a new algorithm \( \mathcal{A} \) with success probability \( 1/2 \) using one ancillary qubit. Thus, we require only one application of a Grover iteration with \( \mathcal{A} \). The Grover iteration includes operations that change the phases of the states \( |0\rangle^{\otimes k} \) with some \( k \leq m + n + 1 \). These operations can be implemented by using OR\textsubscript{n}, which is in QNC\textsubscript{f} as shown in Section 3. Thus, the whole procedure in Step 4 is in QNC\textsubscript{f}.

For the input \( x = x_q^2 \equiv g^l \in G \) \( (0 \leq l \leq p-1) \), the second part of van Dam’s exact algorithm transforms the state \( \frac{1}{\sqrt{p}} \sum_{s=1}^{p-1} |s\rangle |\chi^s\rangle |0\rangle^{\otimes m} \) into the state \( \frac{1}{\sqrt{p}} \sum_{s=1}^{p-1} |s\rangle |\chi^s\rangle (s \mod p) \) as follows:

5. Apply \( F_p \) to the last \( m \) qubits to prepare the state \( \frac{1}{\sqrt{p}} \sum_{s=1}^{p-1} |s\rangle |\chi^s\rangle \left( \frac{1}{\sqrt{p}} \sum_{\alpha=0}^{p-1} |\alpha\rangle \right) \).

\textsuperscript{2}To show this, we need to show that the “weighted” threshold gates are in QTC\textsubscript{f}. We can simply show this as in \cite{15}.
6. Apply $D_x : |y\rangle|\alpha\rangle \mapsto |y \cdot x^{-\alpha} \mod q\rangle|\alpha\rangle$ (0 ≤ $y \leq q - 1$, 0 ≤ $\alpha \leq p - 1$) to the last $n + m$ qubits of the state in Step 5 to prepare the state \( \frac{1}{\sqrt{p}} \sum_{s=1}^{p-1} |s\rangle|\chi^s\rangle \left( \frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} \omega_p^s|n\rangle|\alpha\rangle \right) \). Note that $D_x|\chi^s\rangle|\alpha\rangle = \omega_p^s|\chi^s\rangle|\alpha\rangle$.

7. Apply $F_p^{-1}$ (as in Step 5) to prepare the state \( \frac{1}{\sqrt{p}} \sum_{s=1}^{p-1} |s\rangle|\chi^s\rangle|sl \mod p\rangle \).

One-qubit projective measurements in the basis $|0\rangle$, $|1\rangle$ on the state in Step 7 yield the classical outcomes $s$ and $sl \mod p$ for some $1 \leq s \leq p - 1$. Since $(s, p) = 1$, we can compute $sl \cdot s^{-1} \mod p = l$, which is the desired result, by a poly$(n)$-time exact classical algorithm. Steps 5 and 7 are in $QNC^0$ since, as in Step 2, $D_x$ can be implemented by using arithmetic operations with the pre-computed values $x^j \mod q$ and $(x^{-1})^j \mod q$. This analysis implies Theorem 3. The details of the proof are given in Appendix A.6.

As described above, the (relation) problem of finding $s$ and $sl \mod p$ for some $1 \leq s \leq p - 1$ for the input $x \equiv g^j \in G$ with the pre-computed values can be solved exactly by the $QNC^0$ circuit with gates for $F_p$. On the other hand, the problem is classically hard under the plausible assumption that the DLP over $G_q$ is classically hard, since otherwise we can easily show that the plausible assumption does not hold. Thus, under the plausible assumption, there exists a classically hard problem that is solvable exactly by a $QNC^0$ circuit with gates for $F_p$.

6 Open Problems

Interesting challenges would be to find ways of improving our quantum circuits and to further study the relationships between the complexity classes. We give some examples of such problems:

- Does there exist an $O(1)$-depth $O(n)$-size exact or approximate quantum circuit for OR$_n$?
- Does there exist an $O(1)$-depth $O(n \log n)$-size exact quantum circuit for TH$_n^t$ for any $1 \leq t \leq n$?
- Does it hold that $F_p$ is in $QNC^0_f$?
- The classes QAC$^0$ and QTC$^0$ are defined similarly to $QAC^0_f$ and $QTC^0_f$, respectively, except that unbounded fan-out gates are not allowed. Does it hold that $QAC^0 \subseteq QAC^0_f$ or $QTC^0 \subseteq QTC^0_f$?
- Does there exist a fundamental gate that is as powerful as an unbounded fan-out gate?

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A Proofs

A.1 Proof of Lemma 1

We show this lemma by induction on \( n \) (without using the Fourier expansion of OR\(_n\) explicitly). It is obvious that the lemma holds when \( n = 1 \). We assume that it holds when \( n = k \). For any \( x = x_0 \cdots x_k \in \{0,1\}^{k+1} \),

\[
\frac{1}{2^k} \sum_{a \in \{0,1\}^{k+1} \setminus \{0^{k+1}\}} \text{PA}^a_{k+1}(x) = \frac{1}{2^k} \sum_{a \in \{0,1\}^k \setminus \{0^k\}} \text{PA}^a_k(x_0 \cdots x_{k-1})
\]

\[
+ \frac{x_k}{2^k} + \frac{1}{2^k} \sum_{a \in \{0,1\}^k \setminus \{0^k\}} (\text{PA}^a_k(x_0 \cdots x_{k-1}) \oplus x_k)
\]

\[
= \begin{cases} \text{OR}_k(x_0 \cdots x_{k-1}), & \text{if } x_k = 0, \\ 1, & \text{otherwise}, \end{cases}
\]

where the induction hypothesis implies the second equation. The value is equal to OR\(_{k+1}(x_0 \cdots x_k)\). Thus, when \( n = k + 1 \), the lemma holds as desired.

A.2 Proof of Lemma 2

Let \(|x\rangle = |x_0\rangle \cdots |x_{n-1}\rangle\) be an input state. As described in Section 3.1, we prepare the states \(|x_j\rangle^{\otimes (2^n-1)}\) for any \( 0 \leq j \leq n-1 \), \(|\text{PA}^a_n(x)\rangle\) for any \( a \in \{0,1\}^n \) such that \(|a| \geq 2\), and the \((2^n-1)\)-qubit state

\[
\frac{|0\rangle^{\otimes (2^n-1)} + |1\rangle^{\otimes (2^n-1)}}{\sqrt{2}}.
\]

Thus, we prepare the registers \( R_j \) for storing the state \(|x_j\rangle^{\otimes (2^n-1)}\) for any \( 0 \leq j \leq n-1 \), \( S \) for storing all the states \(|\text{PA}^a_n(x)\rangle\), and \( T \) for storing the \((2^n-1)\)-qubit state. All the registers consist of qubits initialized to \(|0\rangle\). The numbers of qubits in \( R_j \), \( S \), and \( T \) are \( 2^n-1 \), \( 2^n-n-1 \), and \( 2^n-1 \), respectively. The circuit is described as follows:

1. Copy the input state \(|x\rangle\) and apply the circuit for \( \text{PA}^a_n \) to each copy for every \( a \in \{0,1\}^n \setminus \{0^n\} \) in parallel.
   (a) For each \( 0 \leq j \leq n-1 \):
       Apply an unbounded fan-out gate to the input qubit in state \(|x_j\rangle\) and all the qubits in \( R_j \), where the input qubit is used as the control qubit.
   (b) For each \( 0 \leq j \leq n-1 \):
       Apply Hadamard gates to all the qubits in \( R_j \).
   (c) Apply Hadamard gates to all the qubits in \( S \).
   (d) For each \( a = a_0 \cdots a_{n-1} \in \{0,1\}^n \) such that \(|a| \geq 2\):
       Apply an unbounded fan-out gate to a qubit in \( R_{j_0}, \ldots, \) a qubit in \( R_{j_{|a|-1}} \), and a qubit in \( S \), where the qubit in \( S \) is used as the control qubit and \( j_0, \ldots, j_{|a|-1} \) is a unique sequence of the non-negative integers satisfying \( a_{j_0} = \cdots = a_{j_{|a|-1}} = 1 \) and \( j_0 < \cdots < j_{|a|-1} \). All the gates and the qubits are arranged so that all the gates can be applied in parallel.
   (e) This step is the same as Step 1-(b).
   (f) This step is the same as Step 1-(c).

2. Apply a Hadamard gate and an unbounded fan-out gate to ancillary qubits.
   (a) Apply a Hadamard gate to a qubit in \( T \).
   (b) Apply an unbounded fan-out gate to all the qubits in \( T \), where the qubit to which a Hadamard gate is applied in Step 2-(a) is used as the control qubit.
3. Apply controlled-Z($\pi/2^{n-1}$) gates in parallel to the states in Steps 1 and 2.

   (a) For each $0 \leq j \leq n - 1$:
   
   Apply a controlled-Z($\pi/2^{n-1}$) gate to the input qubit in state $|x_j\rangle$ and a qubit in $T$.

   (b) For each qubit in $S$:
   
   Apply a controlled-Z($\pi/2^{n-1}$) gate to the qubit in $S$ and a qubit in $T$.

All the gates and the qubits are arranged so that all the gates can be applied in parallel.

4. Apply an unbounded fan-out gate and a Hadamard gate to the state in Step 3.

   (a) This step is the same as Step 2-(b).

   (b) This step is the same as Step 2-(a).

The circuit for $n = 3$ is depicted in Fig. 3.

The correctness of the circuit is described as follows. Step 1-(a) transforms the state of $R_j$ into the state $|x_j\rangle \otimes (2^{n-1} - 1)$. Since $PA_n^a(x)$ can be computed by a combination of Hadamard gates and an unbounded fan-out gate as depicted in Fig. 2, Step 1-(f) stores the state $|PA_n^a(x)\rangle$ in $S$ for any $a \in \{0,1\}^n$ such that $|a| \geq 2$. Step 2-(a) prepares the state $(H|0\rangle)|0\rangle \otimes (2^{n-2})$ and thus Step 2-(b) transforms the state of $T$ into the $(2^n - 1)$-qubit state

$$|0\rangle \otimes (2^n - 1) + |1\rangle \otimes (2^n - 1).$$

Step 3 transforms the $(2^n - 1)$-qubit state into

$$|0\rangle \otimes (2^n - 1) + e^{i \frac{\pi}{2^{n-1}}} \sum_{a \in \{0,1\}^n \setminus \{0^n\}} PA_n^a(x)|1\rangle \otimes (2^n - 1),$$

which is equal to

$$|0\rangle \otimes (2^n - 1) + (-1)^{OR_n(x)}|1\rangle \otimes (2^n - 1)$$

by Lemma 1. Since Step 4-(a) yields the state

$$|0\rangle + (-1)^{OR_n(x)}|1\rangle,$$
Step 4-(b) outputs the desired state $|\text{OR}_n(x)\rangle$.

By the construction, the depth of the whole circuit does not depend on $n$. Since Step 1-(a) is the dominant part and uses $n$ unbounded fan-out gates on $2^{n-1}$ qubits, the size of the whole circuit is $O(n2^n)$. Thus, the depth and size of the whole circuit are $O(1)$ and $O(n2^n)$, respectively.

### A.3 Proof of Lemma 4

Let $|x\rangle = |x_0\rangle \cdots |x_{n-1}\rangle$ be an input state. As described in Section 4.1, we prepare the $(2^m - 1)$-qubit state $\bigotimes_{k=0}^{m-1} |\varphi_k\rangle^{\otimes k}$, $2^m - 1$ copies of the state $|s_0^y\rangle$ and $2^{m-k} - 1$ copies of the state $|s_k^y\rangle$, and the state $|t_k(y)\rangle$ for any $1 \leq k \leq m - 1$ and $y \in \{0, 1\}^k$. Thus, we prepare the registers $R$ for storing the $(2^m - 1)$-qubit state, $S_0^\varepsilon$ for storing the copies of the state $|s_0^y\rangle$, $S_k^\varepsilon$ for storing the copies of the state $|s_k^y\rangle$ for any $1 \leq k \leq m - 1$ and $y \in \{0, 1\}^k$, $T_0$ for storing the state $|s_0^y\rangle$, and $T_k$ for storing all the states $|t_k(y)\rangle$ for any $1 \leq k \leq m - 1$. All the registers consist of qubits initialized to $|0\rangle$. The numbers of qubits in $R$, $S_0^\varepsilon$, $S_k^\varepsilon$, and $T_k$ are $2^m - 1$, $2^m - 1$, $2^{m-k} - 1$, and $2^k$, respectively. The circuit is described as follows:

1. Apply a slightly modified version of Høyer and Špalek’s OR reduction to the input state $|x\rangle$, where the output is stored in $R$.

2. Perform $A(0)$- and $A(\pi \sum_{j=0}^{k-1} \frac{y_j}{2^j})$-measurements for every $1 \leq k \leq m - 1$ and $y = y_0 \cdots y_{k-1} \in \{0, 1\}^k$ in parallel on the state of $R$.

   (a) Perform an $A(0)$-measurement on the state $|\varphi_0\rangle$ of $R$ and let $s_0^\varepsilon$ be the classical outcome of the measurement.

   (b) For each $1 \leq k \leq m - 1$ and $y = y_0 \cdots y_{k-1} \in \{0, 1\}^k$:

      Perform an $A(\pi \sum_{j=0}^{k-1} \frac{y_j}{2^j})$-measurement on the state $|\varphi_k\rangle$ of $R$ and let $s_k^\varepsilon$ be the classical outcome of the measurement.

3. Prepare $2^m - 1$ copies of the state $|s_0^\varepsilon\rangle$ and $2^{m-k} - 1$ copies of the state $|s_k^y\rangle$ and apply the circuit for AND$_{k+1}$ (constructed by the circuit for OR$_{k+1}$ in Section 3) to the states for every $1 \leq k \leq m - 1$ and $y \in \{0, 1\}^k$ in parallel.

   (a) Apply NOT gates to all the qubits in $S_0^\varepsilon$ if $s_0^\varepsilon = 1$.

   (b) For each $1 \leq k \leq m - 1$ and $y = y_0 \cdots y_{k-1} \in \{0, 1\}^k$:

      Apply NOT gates to all the qubits in $S_k^\varepsilon$ if $s_k^\varepsilon = 1$.

   (c) Apply a CNOT gate to a qubit in $S_0^\varepsilon$ and the qubit in $T_0$, where the qubit in $S_0^\varepsilon$ is used as the control qubit.

   (d) For each $1 \leq k \leq m - 1$ and $y = y_0 \cdots y_{k-1} \in \{0, 1\}^k$:

      - Apply a NOT gate to a qubit (not used in Step 3-(c)) in $S_0^\varepsilon$ if $y_0 = 0$, a NOT gate to a qubit in $S_1^\varepsilon$ if $y_1 = 0$, . . . , and a NOT gate to a qubit in $S_{k-1}^\varepsilon$ if $y_{k-1} = 0$. All the gates and the qubits are arranged so that all the gates can be applied in parallel.

      - Apply a gate for AND$_{k+1}$ to the qubit in $S_0^\varepsilon$, the qubit in $S_1^\varepsilon$, . . . , the qubit in $S_{k-1}^\varepsilon$, a qubit in $S_k^\varepsilon$, and a qubit in $T_k$, where the output is stored in $T_k$. All the gates and the qubits are arranged so that all the gates can be applied in parallel.

4. Apply the circuit for PA$_{2k}$ for every $1 \leq k \leq m - 1$ in parallel to the state in Step 3.

   (a) For each $1 \leq k \leq m - 1$:

      Apply Hadamard gates to all the qubits in $T_k$.

   (b) For each $1 \leq k \leq m - 1$:

      Apply an unbounded fan-out gate to all the qubits in $T_k$.

   (c) This step is the same as Step 4-(a).
The circuit for Steps 3-(c) and 3-(d) for $m = 3$ is depicted in Fig. 4. The first half of the whole circuit contains many one-qubit projective measurements and unitary operations depending on the classical outcomes of the measurements. We can replace them with unitary operations including controlled operations and with measurements in the computational basis only at the end of the circuit by using the well-known method of coherently implementing measurements [18].

The correctness of the circuit is described as follows. Step 1 transforms the state of $R$ into the state $\bigotimes_{k=0}^{m-1} |\varphi_k\rangle \otimes |2^k\rangle$. Step 2 yields the values $s^y_0, s^y_k \in \{0, 1\}$. By the definition of the measurements, it holds that $s_0 = s^y_0$ and $s_k = s^y_0 \cdots s^y_k$ for any $1 \leq k \leq m - 1$. Steps 3-(a) and 3-(b) transform the state of $S^y_0$ into $|s^y_0\rangle \otimes |2^{m-1}\rangle$ and the state of $S^y_k$ into $|s^y_k\rangle \otimes |2^{m-k-1}\rangle$ for any $1 \leq k \leq m - 1$. Steps 3-(c) and 3-(d) transform the state of $T_0$ into $|s^0_0\rangle$ and the state of $T_k$ into $\bigotimes_{y \in \{0, 1\}^k} |t_k(y)\rangle$. Step 4 transforms the state of a qubit in $T_k$ into $|\bigoplus_{y \in \{0, 1\}^k} t_k(y)\rangle$ for any $1 \leq k \leq m - 1$. For any $1 \leq k \leq m - 1$ and $y \in \{0, 1\}^k$,

$$t_k(y) = s^y_k \bigwedge_{j=0}^{k-1} (1 + y_j \oplus s^y_{j+1}) = s^y_k \bigwedge_{j=0}^{k-1} (1 + y_j \oplus s^0_{j+1}) = s^y_k \bigwedge_{j=0}^{k-1} (1 + y_j \oplus s_j)$$

and thus

$$t_k(y) = \begin{cases} s_k, & \text{if } y = s_0 \cdots s_{k-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\bigoplus_{y \in \{0, 1\}^k} t_k(y) = s_k$ for any $1 \leq k \leq m - 1$. Thus, Step 4 outputs the desired state $|s_k\rangle$ for any $1 \leq k \leq m - 1$.

By the construction, the depth of the whole circuit does not depend on $n$. Since Step 1 is the dominant part and the state $\bigotimes_{k=0}^{m-1} |\varphi_k\rangle \otimes |2^k\rangle$ in Step 1 can be prepared with a circuit of size $O(n \sum_{k=0}^{m-1} 2^k) = O(n^2)$ as in Høyer and Spalek’s OR reduction, the size of the whole circuit is $O(n^2)$. Therefore, the depth and size of the whole circuit are $O(1)$ and $O(n^2)$, respectively.

### A.4 Proof of Lemma 5

Let $t$ be an integer satisfying $1 \leq t \leq \lceil n/2 \rceil$ and $|x\rangle = |x_0\rangle \cdots |x_{n-1}\rangle$ be an input state. Let $l$ be an integer satisfying $0 \leq l < \lceil \log(t + 1) \rceil$. This means that $l$ is less than the length of the binary representation of $t$. Let $t_0 \cdots t_{l-1}$ be the $l$ low-order bits of the binary representation of $t$, where $t_0$ is
the lowest-order bit. Note that the value \( t - \sum_{j=0}^{l-1} t_j 2^j \) is positive and is a multiple of \( 2^l \). The first circuit is described as follows:

1. Apply the circuit in Lemma 4 to the input state \( |x\rangle \), where we regard \( m \) in the proof of Lemma 4 as \( l \). Let \( |s_0 \cdots s_{l-1}\rangle \) be the output. In other words, \( s_0 \cdots s_{l-1} \) are the \( l \) low-order bits of the binary representation of \( |x\rangle \), where \( s_0 \) is the lowest-order bit.

2. Apply the first circuit in Lemma 3 to the input state \( |x\rangle \), where we consider only \( 0 \leq k \leq t - 1 \) such that the \( l \) low-order bits of the binary representation of \( k \) are equal to \( s_0 \cdots s_{l-1} \). More concretely,

\[
k = M 2^l + \sum_{j=0}^{l-1} s_j 2^j
\]

for any integer \( M \) satisfying

\[
0 \leq M \leq \frac{t - \sum_{j=0}^{l-1} t_j 2^j}{2^l - 1}
\]

if \( \sum_{j=0}^{l-1} t_j 2^j \geq \sum_{j=0}^{l-1} s_j 2^j \) and

\[
0 \leq M \leq \frac{t - \sum_{j=0}^{l-1} t_j 2^j}{2^l} - 1
\]

otherwise.

We note that, before Step 2, we prepare all the binary representations of \( k \) satisfying the above conditions by applying unbounded fan-out gates and NOT gates to ancillary qubits (initialized to \( |0\rangle \)).

As in the proof of Lemma 3, the circuit outputs the desired state \( |\Theta_l^k(x)\rangle \) and the depth of the whole circuit does not depend on \( n \). The sizes of the circuits in Steps 1 and 2 are \( O(2^l n) \) and \( O(2^{-l} n \log(2^l n)) \), respectively, since \( M \leq t/2^l \). Thus, the depth and size of the whole circuit are \( O(1) \) and \( O(2^l n + 2^{-l} n \log(n)) \), respectively. To construct the second circuit, we use the second circuit in Lemma 3, where we consider only \( t \leq k \leq n \) such that the \( l \) low-order bits of the binary representation of \( k \) are \( s_0 \cdots s_{l-1} \). The number of \( k \)'s we need to consider is bounded above by \( (n - t + 1)/2^l + 2 \) and thus the depth and size of the resulting circuit are \( O(1) \) and \( O(2^l n + 2^{-l} (n - t + 1) n \log(n) + n \log(n)) \), respectively.

### A.5 Proof of Theorem 2

For any \( 1 \leq t \leq \log n \), it holds that \( 0 \leq \lceil \log(t+1) \rceil - 1 < \lceil \log(t+1) \rceil \) and thus we set \( l = \lceil \log(t+1) \rceil - 1 \) in the first circuit in Lemma 5. This yields an \( O(n \log n) \)-size circuit. For any \( \log n \leq t \leq \lceil n/2 \rceil \), it holds that \( 0 \leq \lceil \log \sqrt{t} \log n \rceil - 1 < \lceil \log(t+1) \rceil \) and thus we set \( l = \lceil \log \sqrt{t} \log n \rceil - 1 \) in the first circuit in Lemma 5. This yields an \( O(n \sqrt{t} \log n) \)-size circuit. For any \( \lceil n/2 \rceil \leq t \leq n - \log n \), it holds that \( 0 \leq \lceil \log \sqrt{(n-t+1) \log n} \rceil - 1 < \lceil \log(t+1) \rceil \) and thus we set \( l = \lceil \log \sqrt{(n-t+1) \log n} \rceil - 1 \) in the second circuit in Lemma 5. This yields an \( O(n \sqrt{(n-t) \log n}) \)-size circuit. For any \( n - \log n \leq t \leq n \), it holds that \( 0 \leq \lceil \log(n-t+2) \rceil - 1 < \lceil \log(t+1) \rceil \) and thus we set \( l = \lceil \log(n-t+2) \rceil - 1 \) in the second circuit in Lemma 5. This yields an \( O(n \log n) \)-size circuit.

### A.6 Proof of Theorem 3

As described in Section 5, it suffices to show that, if \( F_p \) is in \( \text{QNC}_l^p \), there exists a poly\((n)\)-time exact classical algorithm for the DLP over \( G \) using the \( \text{QNC}_l^p \) oracle. We consider a slightly modified version of van Dam’s exact algorithm for the DLP. The main difference is that the slightly modified version does not include intermediate measurements. This allows us to consider an exact algorithm with a simple structure: a poly\((n)\)-time classical pre-processing, a query to the \( \text{QNC}_l^p \) oracle, and a poly\((n)\)-time classical post-processing.

Let \( x \equiv g^t \in G \) \((0 \leq l \leq p - 1)\) be an input. In the classical pre-processing step, we compute the values \( g^{2^j} \mod q \), \( x^{2^j} \mod q \), and \( (x^{-1})^{2^j} \mod q \) \((0 \leq j \leq m - 1)\) by a poly\((n)\)-time exact classical
algorithm. By a query to the QNC\textsuperscript{T} oracle, we solve the problem of finding \( s \) and \( sl \mod p \) for some \( 1 \leq s \leq p - 1 \) using the pre-computed values. In the classical post-processing step, using the values \( s \) and \( sl \mod p \) obtained from the QNC\textsuperscript{T} oracle, we compute \( sl \cdot s^{-1} \mod p = l \), which is the desired output, by a \( \text{poly}(n) \)-time exact classical algorithm. This can always be done since \( \gcd(s, p) = 1 \) for any \( 1 \leq s \leq p - 1 \). Thus, the only problem is to show that the QNC\textsuperscript{T} oracle can solve the problem, in other words, to show that the problem can be solved exactly by a QNC\textsuperscript{T} circuit (if \( F_p \) is in QNC\textsuperscript{T}).

The quantum algorithm for solving the problem consists of two parts \( Q_1 \) and \( Q_2 \). We note that we can use the pre-computed values described above in the quantum algorithm. The first part \( Q_1 \) transforms the state \(|0\rangle^{\otimes(m+n+1)}\) into the state

\[
\frac{1}{\sqrt{p-1}} \sum_{s=0}^{p-1} |s\rangle |x^s\rangle |1\rangle,
\]

which is independent of the input \( x \). To define \( Q_1 \), we define the following algorithm as \( A \), where the input state is \(|0\rangle^{\otimes(m+n+1)}\):

1. Apply \( F_p \) to the first \( m \) qubits of the input state.
2. Apply the modular exponentiation operation \(|r\rangle|0\rangle \rightarrow |r\rangle|g^r \mod q\rangle\) to the state in Step 1.
3. Apply \( F_p \) to the first \( m \) qubits of the state in Step 2.
4. Apply the one-qubit unitary operation defined by

\[
\frac{1}{\sqrt{2(p-1)}} \begin{pmatrix} \sqrt{p-2} & -\sqrt{p} \\ \sqrt{p} & \sqrt{p-2} \end{pmatrix}
\]


to the last one qubit of the state in Step 3.

A direct calculation shows that \( A \) transforms the input state into

\[
\frac{1}{\sqrt{p}} \sum_{s=0}^{p-1} |s\rangle |x^s\rangle \left( \sqrt{\frac{p-2}{2(p-1)}} |0\rangle + \sqrt{\frac{p}{2(p-1)}} |1\rangle \right)
\]

\[
= \frac{1}{\sqrt{2(p-1)}} \sum_{s=1}^{p-1} |s\rangle |x^s\rangle |1\rangle + \frac{1}{\sqrt{2(p-1)}} |0\rangle |x^0\rangle |1\rangle + \sqrt{\frac{p-2}{2p(p-1)}} \sum_{s=0}^{p-1} |s\rangle |x^s\rangle |0\rangle.
\]

We define

\[
|A\rangle = \frac{1}{\sqrt{2(p-1)}} \sum_{s=1}^{p-1} |s\rangle |x^s\rangle |1\rangle.
\]

It holds that \( \langle A|A \rangle = 1/2 \). Let \( S_{|0\rangle} \) be the quantum operation that changes the phase of a state by \( i \) if and only if the state is \(|0\rangle^{\otimes(m+n+1)}\). Similarly, let \( S_A \) be the quantum operation that changes the phase of a state by \( i \) if and only if the state of the first \( m \) qubits is not \(|0\rangle^m\) and the state of the last one qubit is \(|1\rangle\). We define the Grover iteration \( G = AS_{|0\rangle}A^{-1}S_A \) and the first part \( Q_1 = GA \).

The correctness of \( Q_1 \) follows from the direct calculation as in the amplitude amplification procedure in [4, 5].

The argument in Section 5 implies that \( A \) is in QNC\textsuperscript{T}. Moreover, by Theorem 1, \( S_{|0\rangle} \) and \( S_A \) are in QNC\textsuperscript{T}. Thus, \( Q_1 \) is in QNC\textsuperscript{T}. We note that the last qubit in state \(|1\rangle\) is not important for the second part \( Q_2 \) describe below (and thus can be ignored below) and that the pre-computed values used in Step 2 on ancillary qubits have no effect on the amplitude amplification procedure.

Recall that the quantum operation \( D_x \) is defined as

\[
|y\rangle|\alpha\rangle \mapsto |y \cdot x^{-\alpha} \mod q\rangle|\alpha\rangle,
\]

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where $0 \leq y \leq q - 1$ and $0 \leq \alpha \leq p - 1$. Before considering the second part $Q_2$, we show that the relationship

$$D_x|\chi^s\rangle|\alpha\rangle = \omega_p^{sla}|\chi^s\rangle|\alpha\rangle$$

holds for any $0 \leq s \leq p - 1$ and $0 \leq \alpha \leq p - 1$ by the following direct calculation:

$$D_x|\chi^s\rangle|\alpha\rangle = \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \omega_p^{sr} D_x|g^r \bmod q\rangle|\alpha\rangle = \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \omega_p^{sr} |g^r \cdot x^{-\alpha} \bmod q\rangle|\alpha\rangle$$

$$= \omega_p^{sla} \frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} \omega_p^{s(r-\alpha)} |g^{r-\alpha} \bmod q\rangle|\alpha\rangle = \omega_p^{sla}|\chi^s\rangle|\alpha\rangle.$$

We consider the second part $Q_2$ that transforms the input state

$$\frac{1}{\sqrt{p-1}} \sum_{s=1}^{p-1} |s\rangle|\chi^s\rangle|0\rangle \otimes m,$$

which is obtained by $Q_1$ with $m$ qubits initialized to $|0\rangle$, into the state

$$\frac{1}{\sqrt{p-1}} \sum_{s=1}^{p-1} |s\rangle|\chi^s\rangle|sl \bmod p\rangle.$$

We define the following algorithm as $Q_2$:

5. Apply $F_p$ to the last $m$ qubits of the input state.

6. Apply $D_x$ to the last $n + m$ qubits of the state in Step 5.

7. Apply $F_p^{-1}$ to the last $m$ qubits of the state in Step 6.

The correctness of $Q_2$ is described as follows. Step 5 transforms the input state into the state

$$\frac{1}{\sqrt{p-1}} \sum_{s=1}^{p-1} |s\rangle|\chi^s\rangle \left( \frac{1}{\sqrt{p}} \sum_{\alpha=0}^{p-1} |\alpha\rangle \right).$$

By the relationship shown above, Step 6 transforms the state in Step 5 into the state

$$\frac{1}{\sqrt{p-1}} \sum_{s=1}^{p-1} |s\rangle|\chi^s\rangle \left( \frac{1}{\sqrt{p}} \sum_{\alpha=0}^{p-1} \omega_p^{sla} |\alpha\rangle \right).$$

Step 7 transforms the state in Step 6 into the desired state

$$\frac{1}{\sqrt{p-1}} \sum_{s=1}^{p-1} |s\rangle|\chi^s\rangle|sl \bmod p\rangle.$$

We perform one-qubit projective measurements in the basis $|0\rangle, |1\rangle$ on the first $m$ qubits and the last $m$ qubits of the state in Step 7. This yields the classical outcomes $s$ and $sl \bmod p$ for some $1 \leq s \leq p - 1$.

Steps 5 and 7 are in $\text{QNC}^0_l$ by our assumption. In Step 6, as in Step 2 of $\mathcal{A}$, $D_x$ is implemented by using the iterated multiplication operation with the pre-computed values $(x^2 \bmod q$ and $(x^{-1})^2 \bmod q)$ and the modular multiplication operation as follows:

$$|y\rangle|\alpha\rangle|0\rangle \otimes 2n \mapsto |y\rangle|\alpha\rangle|x^{-\alpha} \bmod q\rangle|0\rangle \otimes n$$

$$\mapsto |y\rangle|\alpha\rangle|x^{-\alpha} \bmod q\rangle|y \cdot x^{-\alpha} \bmod q\rangle$$

$$\mapsto |y\rangle|\alpha\rangle|0\rangle \otimes n |y \cdot x^{-\alpha} \bmod q\rangle$$

$$\mapsto |y\rangle|\alpha\rangle|x^\alpha \bmod q\rangle|y \cdot x^{-\alpha} \bmod q\rangle$$

$$\mapsto |0\rangle \otimes n |\alpha\rangle|x^\alpha \bmod q\rangle|y \cdot x^{-\alpha} \bmod q\rangle$$

$$\mapsto |0\rangle \otimes n |\alpha\rangle|0\rangle \otimes n |y \cdot x^{-\alpha} \bmod q\rangle.$$

Since $\text{QNC}^0_l = \text{QTC}^0_l$ as shown in Section 3 and $\text{QTC}^0_l$ includes the iterated multiplication operation and the modular multiplication operation [22], Step 6 is in $\text{QNC}^0_l$. Therefore, $Q_2$ is in $\text{QNC}^0_l$. 

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