DERIVED ℓ-ADIC ZETA FUNCTIONS

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ABSTRACT. We lift the classical Hasse–Weil zeta function of varieties over a finite field to a map of spectra with domain the Grothendieck spectrum of varieties constructed by Campbell and Zakharevich. We use this map to prove that the Grothendieck spectrum of varieties contains nontrivial geometric information in its higher homotopy groups by showing that the map $S \rightarrow K(V_k)$ induced by the inclusion of 0-dimensional varieties is not surjective on $\pi_1$ for a wide range of fields $k$. The methods used in this paper should generalize to lifting other motivic measures to maps of $K$-theory spectra.

1. INTRODUCTION

Let $k$ be a field. The Grothendieck ring of varieties over $k$, denoted $K_0(V_k)$, is the abelian group generated by isomorphism classes $[X]$ of $k$-varieties, with the relation $[X] = [Z] + [X - Z]$ for $Z \hookrightarrow X$ a closed inclusion; the ring structure is given by the Cartesian product. Many important invariants of varieties induce “motivic measures”, i.e. ring homomorphisms $K_0(V_k) \rightarrow A$; for example point counts (for $k$ finite) or Hodge numbers (for $k \subset \mathbb{C}$) produce such homomorphisms.

The first and third authors (see [Cam, Zak17, CWZ]) have constructed a higher Grothendieck ring of varieties, namely, a spectrum $K(V_k)$ such that $\pi_0 K(V_k) \cong K_0(V_k)$. Two natural questions arise: 1) Do classical motivic measures lift to maps of spectra? 2) What arithmetic or geometric information do the higher homotopy groups $K_i(V_k)$ encode?

One of the most important invariants of a variety is its Hasse–Weil zeta function

$$X \mapsto \zeta_X(s) := \prod_{x \in X_{cl}} (1 - |\kappa(x)|^{-s})^{-1},$$

where $X_{cl}$ denotes the set of closed points in $X$ and $\kappa(x)$ denotes the residue field at $x$. As beautifully explained in [Ram15], the Hasse–Weil zeta function of a variety over a finite field descends to a motivic measure

$$\zeta_-(s): K_0(V_{F_q}) \longrightarrow W(\mathbb{Z})$$

from the Grothendieck ring of varieties to the (big) Witt vectors of the integers. The main result of this paper is a lift of this homomorphism to a map of $K$-theory spectra.

Let $k$ be a field, $k^s$ a separable closure, and $\ell \neq \text{char}(k)$ a prime. Denote by $\text{Rep}_{\text{cts}}(\text{Gal}(k^s/k) ; \mathbb{Z}_\ell)$ the category of finitely generated continuous representations of $\text{Gal}(k^s/k)$ over $\mathbb{Z}_\ell$.

**Theorem 1.1.** The function from $k$-varieties to continuous $\text{Gal}(k^s/k)$ representations

$$X \mapsto H^*_{et,c}(X \times_k k^s ; \mathbb{Z}_\ell)$$

lifts to a map of $K$-theory spectra

$$\zeta: K(V_k) \longrightarrow K(\text{Rep}_{\text{cts}}(\text{Gal}(k^s/k) ; \mathbb{Z}_\ell)).$$

We refer to this map as the derived $\ell$-adic zeta function; it is a homotopical enrichment of the standard zeta function. When $k$ is a finite field, this map is exactly the lift of the Hasse–Weil zeta function.
Corollary 1.2. Let $k = \mathbb{F}_q$ be a finite field and $\ell \nmid q$ a prime. The Hasse–Weil zeta function of $\mathbb{F}_q$-varieties lifts to a map of spectra

$$\zeta: K(V_{\mathbb{F}_q}) \longrightarrow K(\text{Rep}_{cts}(\text{Gal}(\mathbb{F}_q/\mathbb{F}_q); \mathbb{Z}_\ell)),$$

which fits into a commuting square

$$K_0(V_{\mathbb{F}_q}) \xrightarrow{\pi_0 \zeta} K_0(\text{Rep}_{cts}(\text{Gal}(\mathbb{F}_q/\mathbb{F}_q); \mathbb{Z}_\ell)) \xrightarrow{\zeta-(s)} W(\mathbb{Z}) \xrightarrow{\det(1-\text{Frob} \zeta)} W(\mathbb{Z}_\ell)$$

after applying $\pi_0$.

For the second question, one might begin by asking: are there nontrivial elements in $K_i(V_k)$ for $i > 0$? The inclusion of the category of zero-dimensional $k$-varieties defines a map $\mathbb{S} \longrightarrow K(V_k)$. For a finite field $k = \mathbb{F}_q$, the map $X \mapsto X(\mathbb{F}_q)$ defines a map $K(V_{\mathbb{F}_q}) \longrightarrow \mathbb{S}$ which splits the above map, yielding $K(V_{\mathbb{F}_q}) \cong \mathbb{S} \oplus \tilde{K}(V_{\mathbb{F}_q})$. For general $k$, one can define $\tilde{K}(V_k)$ as the homotopy cofiber of the map $\mathbb{S} \longrightarrow K(V_k)$. While $\pi_i(\mathbb{S})$ is a rich and interesting object, the classes corresponding to this summand are somewhat uninteresting from the point of algebraic geometry. One could then ask:

Do there exist nontrivial elements in $\tilde{K}_i(V_k)$ for $i > 0$?

We use the derived zeta function to answer this question affirmatively.

Theorem 1.3. The group $\tilde{K}_1(V_k)$ is nontrivial whenever $k$ is a subfield of $\mathbb{R}$, a finite field with $|k| \equiv 3 \pmod{4}$, or a global or local field with a place of cardinality $3 \pmod{4}$.

To prove this theorem we use the derived 2-adic zeta function. On $\pi_1$, the derived 2-adic zeta function takes the form $K_1(V_k) \longrightarrow K_1(\text{Aut}(\mathbb{Q}_2))$. The group $K_1(\text{Aut}(\mathbb{Q}_2))$ is relatively well-understood. In [Gra79], Grayson constructs a homomorphism $\sigma_2: K_1(\text{Aut}(\mathbb{Q}_2)) \longrightarrow K_2(\mathbb{Q}_2)$. By Moore’s Theorem (see e.g. [Mil71 Appendix]), the 2-adic Hilbert symbol induces a (split) surjection $(-,-): K_2(\mathbb{Q}_2) \rightarrow \mathbb{Z}/2\mathbb{Z}$, and by composing these maps, we produce a map

$$h_2: K_1(V_k) \longrightarrow K_1(\text{Aut}(\mathbb{Q}_2)) \longrightarrow K_2(\mathbb{Q}_2) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

We then show that this map is surjective and also that this map is trivial on the image of $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2\mathbb{Z}$ in $K_1(V_k)$.

In order to construct the derived zeta function, we use a $K$-theory machinery first created by the first author in [Cam]. The usual categories one wants to work with as inputs for a $K$-theory machine are Waldhausen categories [Wal85]. Unfortunately, these do not work to produce $K(V_k)$, which is why the first and third authors introduced their formalisms. In [Cam], the difficulty is circumvented by defining a modification of Waldhausen categories called $SW$-categories (the $S$ is for scissors) where one can define algebraic $K$-theory for $V_k$ in much the same way one does for Waldhausen categories. However, in order to get maps $K(C) \longrightarrow K(W)$ where $C$ is an $SW$-category and $W$ is a Waldhausen category, one needs the notion of a “$W$-exact functor” introduced in [Cam]. It needs to satisfy certain variance conditions reminiscent of push-pull formulæ (see Section 2 for details). To construct the derived $\ell$-adic zeta function, we take the $SW$-category $V_k$ and the Waldhausen category $\text{Ch}^b(\text{Rep}_{cts}(\text{Gal}(k^s/k); \mathbb{Z}_\ell))$. To from one to the other we need to use compactly-supported étale cohomology. Unfortunately, this is not sufficiently functorial; to resolve this, we construct a helper category which decorates each variety with a compactification. Once the compactification is chosen, compactly-supported étale cohomology is functorial and satisfies the axioms of a $W$-exact functor. Applying $K$-theory, we obtain the derived $\ell$-adic zeta function.
We view this work as part of a larger program to lift motivic measures to the spectral/homotopical level. For example, the outline we follow should adapt to give lifts for other cohomologically defined motivic measures, e.g. $p$-adic zeta functions, Serre polynomials, and the Gillet–Soulé measure \cite{GS96}. One might similarly ask for lifts of Kapranov’s motivic zeta function, or of the motivic measure used by Larsen and Lunts \cite{LL03} to show that motivic zeta function is not rational as a map out of $K_0(V_C)$. See Section 6 for a more detailed discussion.

This paper is organized as follows. In Section 2 we quickly review Waldhausen $K$-theory and introduce SW-categories. In Section 3 we review the background and necessary results from $\ell$-adic cohomology and Galois representations. In Section 4 we prove Theorem 1.1 and Corollary 1.2. In Section 5 we use the derived zeta function to do calculations and prove Theorem 1.3. We close, in Section 6, by discussing questions for future work.

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Notation 1.4. Throughout, when dealing with schemes or varieties, we will let $Z \hookrightarrow Y$ denote a closed inclusion and $X \overset{\circ}{\to} Y$ denote an open inclusion.

2. SW-categories and K-Theory

In \cite{Zak17}, the third author defines a spectrum $K(V_k)$ whose zeroth homotopy group is the Grothendieck ring of varieties over $k$. In \cite{Cam}, the first author gives an alternate construction of this spectrum. In this paper, we use the latter construction to produce maps out of $K(V_k)$, so we will review the structure necessary to produce this spectrum.

Most definitions of $K$-theory work with categories where a suitable notion of quotient exists, for example Quillen’s exact categories \cite{Qui73} or Waldhausen’s categories \cite{Wal85}. These notions of quotient are then used to define the exact sequences that $K$-theory is defined to “split.” When dealing with the category of varieties, we have no such quotients. Instead, our “exact sequences” are sequences of the form $Y \hookrightarrow X \overset{\sim}{\to} (X - Y)$ where the first map is a closed inclusion and the second is an open inclusion. The notion of an SW-category is meant to modify Waldhausen’s definition of categories with cofibrations and weak equivalences to allow the use of such “exact sequences.” For ease of reading, we review Waldhausen’s construction before recalling the first author’s construction.

Definition 2.1 (Waldhausen category, \cite[Section 1.2]{Wal85}). A Waldhausen category is a category $C$ equipped with two distinguished subcategories: cofibrations and weak equivalences, denoted $\text{co}(C)$ and $\text{w}(C)$. The arrows in $\text{co}(C)$ will be denoted by hooked arrows $\hookrightarrow$. Arrows representing weak equivalences will be decorated with $\sim$. These categories satisfy the following axioms:

1. $C$ has a zero object $\emptyset$.
2. All isomorphisms are contained in $\text{co}(C)$ and $\text{w}(C)$.
3. For all objects $A$ of $C$, the morphism $\emptyset \to A$ is a cofibration.
4. (pushouts) For any diagram

$$
\begin{array}{ccc}
C & \to & A \\
\downarrow & & \downarrow \\
& & \downarrow \\
& & B
\end{array}
$$

where $A \to B$ is a cofibration, the pushout exists and the morphism $C \to B \cup_A C$ is a cofibration.

\footnote{Referred to as a “category with cofibrations and weak equivalences” by Waldhausen.}
(5) **(gluing)** For any diagram

\[
\begin{array}{ccc}
C & \xleftarrow{A} & B \\
\downarrow & & \downarrow \\
C' & \xleftarrow{A'} & B'
\end{array}
\]

where the vertical morphisms are weak equivalences the induced morphism

\[
B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'
\]

is also a weak equivalence.

Before we define Waldhausen’s $K$-theory we will need two extra definitions.

**Definition 2.2.** Let $[n]$ be the ordered set $\{0 < \ldots < n\}$ considered as a category. We define $\text{Ar}[n]$ to be the full subcategory of $[n] \times [n]$ consisting of pairs $(i, j)$ with $i \leq j$.

**Definition 2.3 (S•-construction, see [Rog92, Definition 1.3]).** Let $\mathcal{C}$ be a Waldhausen category. We define $S_n\mathcal{C}$ to be the category of functors

\[X: \text{Ar}[n] \rightarrow \mathcal{C}\]

with morphisms natural transformations, subject to the conditions

- $X_{i,i} = \emptyset$, the initial object
- When $j < k$, $X_{i,j} \rightarrow X_{i,k}$ is a cofibration.
- For any $i < j < k$ the square

\[
\begin{array}{ccc}
X_{i,j} & \rightarrow & X_{i,k} \\
\downarrow & & \downarrow \\
\emptyset & \rightarrow & X_{j,k}
\end{array}
\]

is a pushout for all $i < j < k$.

**Remark 2.4.** The $S_n\mathcal{C}$ assemble to form a simplicial category (i.e. a simplicial object in the category of small categories).

We now define the algebraic $K$-theory spectrum of a Waldhausen category.

**Definition 2.5.** Let $\mathcal{C}$ be a Waldhausen category. Let $wS_n\mathcal{C}$ denote the subcategory of weak equivalences of $S_n\mathcal{C}$ and let $N_* wS_n\mathcal{C}$ denote the nerve of that category. The topological space $K^f(\mathcal{C})$ is defined by

\[K^f(\mathcal{C}) = \Omega |N_* wS_n\mathcal{C}|\]

where $|-|$ denotes the geometric realization of a bisimplicial set. The spectrum $K(\mathcal{C})$ is defined by taking a (functorial) fibrant-cofibrant replacement in the category of spectra of the spectrum whose $m$-th space is

\[|N_* wS_n \cdots S_n\mathcal{C}|,\]

m times

The most important example of a Waldhausen category for the purposes of this paper is the following:

**Example 2.6.** Let $\mathcal{E}$ be any exact category. If we define the admissible monomorphisms to be the cofibrations and the isomorphisms to be the weak equivalences then the Waldhausen $K$-theory of $\mathcal{E}$ and Quillen’s $K$-theory of $\mathcal{E}$ are equivalent.
Now let \( \text{Ch}^b(\mathcal{E}) \) be the category of homologically bounded chain complexes in \( \mathcal{E} \). We define the cofibrations to be the levelwise cofibrations, and the weak equivalences to be the quasi-isomorphisms. Then there is a functor \( \cdot[0]: \mathcal{E} \rightarrow \text{Ch}^b(\mathcal{E}) \) which includes \( \mathcal{E} \) as the chain complexes concentrated at level 0. After applying \( K \), this inclusion becomes a weak equivalence (see \cite[Theorem 1.11.7]{TT90}). On \( K_0 \) the inverse to the inclusion map is exactly the Euler characteristic.

We now turn to defining \( SW \)-categories. As much of the intuition necessary for working with these comes from Waldhausen’s \( S_* \)-construction we describe the structures in parallel language.

**Definition 2.7** (\( SW \)-category \cite[Definition 3.23]{Cam}). An \( SW \)-category is a category equipped with three distinguished subcategories: cofibrations, fibrations, and weak equivalences, denoted \( \text{co}(\mathcal{C}) \), \( \text{fib}(\mathcal{C}) \) and \( \text{w}(\mathcal{C}) \). The arrows in \( \text{co}(\mathcal{C}) \) will be denoted by hooked arrows \( \hookrightarrow \) and the arrows in \( \text{fib}(\mathcal{C}) \) will be denoted by \( \circlearrowleft \). Arrows representing weak equivalences will be decorated with \( \sim \). We require the following.

1. \( \mathcal{C} \) has an initial object, denoted \( \emptyset \).
2. All isomorphisms are contained in \( \text{fib}(\mathcal{C}) \), \( \text{co}(\mathcal{C}) \), and \( \text{w}(\mathcal{C}) \).
3. (pullbacks) Pullbacks along fibrations and cofibrations exist and satisfy base change.
4. (subtraction) Given a cofibration \( Z \hookrightarrow X \), there is a notion of “subtracting” \( Z \) from \( X \). That is, there is a family of subtraction sequences. Subtraction sequences are diagrams \( Z \hookrightarrow X \circlearrowleft Y \), and they are required to satisfy the following properties
   a. Every cofibration \( Z \hookrightarrow X \) participates in a subtraction sequence \( Z \hookrightarrow X \circlearrowleft Y \) such that \( Y \) is unique up to unique isomorphism. We informally write \( Y = X - Z \) in this case.
   b. Subtraction sequences are respected by base change.
5. (pushouts) Given a diagram

\[
\begin{array}{ccc}
W & \xrightarrow{f} & X \\
g \downarrow & & \downarrow \\
Y & \xrightarrow{} & Z
\end{array}
\]

with both \( f \) and \( g \) cofibrations, a pushout \( Z \) exists and both dotted arrows are cofibrations. Furthermore, every diagram of this form is required to be a pullback.
6. (pushout products) Given a pullback diagram

\[
\begin{array}{ccc}
W & \xrightarrow{} & X \\
Y & \xrightarrow{} & Z
\end{array}
\]

where all arrows are cofibrations, the map \( X \amalg_W Y \rightarrow Z \) is a cofibration.
7. (subtraction and pushouts) Given a diagram

\[
\begin{array}{ccc}
X' & \xleftarrow{W'\leftarrow} & Y' \\
X & \xleftarrow{W} & Y \\
X'' & \xleftarrow{W''} & Y''
\end{array}
\]

where the columns are subtraction sequences and upper left and upper right squares are pullback squares, then the pushouts along the rows form a subtraction sequence:

\[ X' \amalg_W Y \leftarrow X \amalg_W Y \leftarrow X'' \amalg_W Y'' \]
(8) **(gluing)** Given the diagram

\[
\begin{array}{ccccccccc}
Y & \xrightarrow{\sim} & X & \xrightarrow{\sim} & Z \\
\downarrow & & \downarrow & & \downarrow \\
Y' & \xleftarrow{\sim} & X' & \xleftarrow{\sim} & Z'
\end{array}
\]

where the vertical arrows are weak equivalences, there is an induced weak equivalence

\[Y \amalg_X Z \xrightarrow{\sim} Y' \amalg_{X'} Z'.\]

(9) **(subtraction and weak equivalences)** Given a commuting square

\[
\begin{array}{cccc}
X & \xrightarrow{\sim} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{\sim} & Y'
\end{array}
\]

there is a weak equivalence

\[Y - X \xrightarrow{\sim} Y' - X'.\]

This may look like quite a bit of data. The first four axioms are meant to codify the notion of “subtraction” in an arbitrary category with cofibrations and fibrations standing in for open and closed embeddings in topological spaces. Note that the term “fibration” is used here to invoke the idea of being “dual” to cofibration, rather than of being analogous to a fibration in a model category. Axioms 5-7 are essentially technical in nature, providing the exact conditions under which \(K\)-theory may be defined.

There are many examples of these kinds of categories. The following is the motivating example.

**Example 2.8.** \(\mathcal{V}_k\), the category of varieties over a field \(k\), is an \(SW\)-category, where cofibrations are closed immersions, fibrations are open immersions, and the weak equivalences are isomorphisms. (This is proven in detail in [Cam, Prop. 3.26]) The subtraction sequences are defined as follows. Given a closed inclusion \(i: Z \to X\), \(i\) determines a homeomorphism of \(Z\) onto a closed set \(i(Z)\). We consider the open set \(X - i(Z)\) and give it a scheme structure by restricting the structure sheaf on \(X\). Thus

\[X - Z = (X - i(Z), \mathcal{O}_X|_{X-Z}).\]

The difficult part of showing that \(\mathcal{V}_k\) is an \(SW\)-category is showing that pushouts along closed inclusions exist. This is proven in [Sch05].

The definition of an \(SW\)-category is designed to provide exactly the structure needed to carry out a Waldhausen-style \(S_*\)-construction when we have subtraction instead of quotients. However, we need one auxiliary definition.

**Definition 2.9.** We define \(\tilde{\text{Ar}}[n]\) to be the full subcategory of \([n]^{\text{op}} \times [n]\) consisting of pairs \((i, j)\) with \(i \leq j\).

**Definition 2.10 (\(\tilde{S}_*\)-construction).** Let \(\mathcal{C}\) be an \(SW\)-category. We define \(\tilde{S}_i\mathcal{C}\) to be the category with objects functors

\[X: \tilde{\text{Ar}}[n] \to \mathcal{C}\]

with morphisms natural transformations, subject to the conditions

- \(X_{i,i} = \emptyset\), the initial object.
- When \(j < k\), \(X_{i,j} \to X_{i,k}\) is a cofibration.
- The subdiagram

\[X_{i,j} \to X_{i,k} \leftarrow X_{j,k}\]

is a subtraction sequence for all \(i < j < k\).
Remark 2.11. The $\tilde{S}_nC$ assemble to form a simplicial category (i.e. a simplicial object in the category of small categories).

We may finally define the algebraic $K$-theory spectrum of an $SW$-category.

Definition 2.12. Let $C$ be an $SW$-category. Let $w\tilde{S}_nC$ denote the subcategory of weak equivalence of $\tilde{S}_nC$ and let $N_*w\tilde{S}_nC$ denote the nerve of that category. The topological space $K^t(C)$ is defined by

$$K^t(C) = \Omega|N_*w\tilde{S}_nC|$$

where $|−|$ denotes the geometric realization of a bisimplicial set. The spectrum $K(C)$ is defined by taking a (functorial) fibrant-cofibrant replacement in the category of spectra of the spectrum whose $m$-th space is

$$|N_*w\tilde{S}_n\cdots\tilde{S}_nC|,$$

There is a notion of exact functors for $SW$-categories:

Definition 2.13. Let $C, D$ be $SW$-categories. A functor $F: C \rightarrow D$ will be called exact if

1. $F$ preserves the initial object: $F(\emptyset) = \emptyset$.
2. $F$ preserves subtraction sequences
3. $F$ preserves pushout diagrams.

Proposition 2.14. Let $C$ and $D$ be $SW$-categories and let $F: C \rightarrow D$ be an exact functor. Then $F$ descends to a map of spectra $K(C) \rightarrow K(D)$.

Proof. This follows directly from the definition of exact functor. \qed

Most of the maps in which we are interested do not have $SW$-categories as codomains; instead, we wish to be able to construct a functor from an $SW$-category to a Waldhausen category. This requires we a different definition in order to define the map of $K$-theories, since we cannot just hit the source and target with the $\tilde{S}_n$ construction or $S_n$-construction. In fact, because of the change in variance, the proper notion is not a functor at all — instead it will be a pair of functors, one covariant and one contravariant. One should keep in mind here compactly supported cohomology, which is covariant on open inclusions and contravariant on closed.

Definition 2.15. Let $C$ be an $SW$-category and $D$ a Waldhausen category. A $W$-exact functor from $C$ to $D$ is a pair of functors $(F!, F^!)$ such that

1. $F!$ is a covariant functor $F!: \text{fib}(C) \rightarrow D$ from the category of fibrations in $C$ to $D$. For a map $i$ we abbreviate $F!(i)$ to $i!$.
2. $F^!$ a contravariant functor $F^!: \text{co}(C)^{op} \rightarrow D$, from the category of cofibrations in $C$ to $D$. For a map $j$ we abbreviate $F^!(j)$ to $j^!$.
3. For objects $X \in C$, $F^!(X) = F!(X)$ and we denote both by $F(X)$.
4. (base change) For every cartesian diagram in $C$

$$\begin{array}{ccc}
X & \xrightarrow{\gamma} & Z \\
\downarrow{i} & & \downarrow{j^!} \\
Y & \xrightarrow{j} & W
\end{array}$$
where the horizontal maps are cofibrations and the vertical maps are open maps, we obtain a commuting diagram

\[
\begin{array}{ccc}
F(X) & \xleftarrow{j^!} & F(Z) \\
\downarrow{i_!} & & \downarrow{(j')_!} \\
F(Y) & \xleftarrow{(i')_!} & F(W)
\end{array}
\]

i.e.

\[i_! \circ j^! = (j')_! \circ (i')_!\]

(5) For a subtraction sequence in \(\mathcal{C}\)

\[
X \xleftarrow{i} Y \xrightarrow{j} Y - X
\]

the sequence

\[
F(X) \xrightarrow{i_!} F(Y) \xrightarrow{j^!} F(X - Y)
\]

must be a cofiber sequence in \(\mathcal{D}\).

**Remark 2.16.** For notational ease, we denote \(W\)-exact functors by \((F_!, F^!): \mathcal{C} \rightarrow \mathcal{D}\) or even \(F: \mathcal{C} \rightarrow \mathcal{D}\), when no confusion can arise.

**Remark 2.17.** This definition is dual to the one in [Cam]. In [Cam], the goal of the construction of a \(W\)-exact functor was to have a functor which is covariant on cofibrations and contravariant on fibrations. However, in our case the opposite is required. The two definitions are functionally equivalent, however, since in our case the codomain of the desired \(W\)-exact functors have biWaldhausen structures (see [TT90, Definition 1.2.4]).

Having defined this, one can prove the following.

**Proposition 2.18.** [Cam Prop. 5.3] Let \(\mathcal{C}\) be an \(\mathcal{SW}\)-category and \(\mathcal{D}\) a Waldhausen category and let \((F_!, F^!): \mathcal{C} \rightarrow \mathcal{D}\) be a \(W\)-exact functor. Then there is a spectrum map

\[K(\mathcal{C}) \rightarrow K(\mathcal{D}).\]

As a consequence of the definition of \(K\)-theory, we obtain the following result, which we will need for the proof of Theorem 5.2. The proof is just as for Waldhausen \(K\)-theory as in [Wal85 §1.5].

**Proposition 2.19.** Let \(X\) be an object in an \(\mathcal{SW}\)-category (resp. Waldhausen category) \(\mathcal{C}\). There is a homomorphism \(\xi_X \: \text{Aut}(X) \rightarrow K_1(\mathcal{C})\), which is natural in \(\mathcal{C}\) in the sense that for any exact (resp. \(W\)-exact) functor \(F: \mathcal{C} \rightarrow \mathcal{D}\), \(\xi_{F(X)} = \pi_1 F \circ \xi_X\).

### 3. Preliminaries on \(\ell\)-adic Cohomology

In this section we recall standard facts we will need about \(\ell\)-adic cohomology with its continuous Galois action. We take [Del77] and [FK80] as standard references.
3.1. Continuous Galois Representations. Let $k$ be a field, let $k^s$ be a separable closure, and let $\ell \neq \text{char}(k)$ be a prime. The separable Galois group $\text{Gal}(k^s/k)$ is a profinite group, and canonically carries the profinite topology

$$\text{Gal}(k^s/k) = \varprojlim_{L/k \text{ separable}} \text{Gal}(L/k)$$

where the finite groups $\text{Gal}(L/k)$ are discrete, and the limit is in the category of topological groups. Let $R$ be a ring. Recall that for a (discrete) $R$-module $A$, a continuous representation $\text{Gal}(k^s/k) \to \text{Aut}_R(A)$ is one which factors through a finite subgroup $\text{Gal}(k^s/k) \to \text{Gal}(L/k) \to \text{Aut}_R(A)$ for some separable $L/k$. Denote by $\text{Rep}^{\text{cts}}(\text{Gal}(k^s/k); R)$ the category of finitely generated continuous representations of $\text{Gal}(k^s/k)$ over $R$. Recall the following (cf. [Del77, Arcata II.4.4]).

**Proposition 3.1.** Let $k$ be a field, and $k^s$ a separable closure. Let $R$ be a ring. Denote by $\text{Sh}(\text{Spec}(k); R)$ the category of étale sheaves of (discrete) finitely generated $R$-modules on $\text{Spec}(k)$. The functor

$$\text{Sh}(\text{Spec}(k); R) \to \text{Rep}^{\text{cts}}(\text{Gal}(k^s/k); R)$$

$$F \mapsto F|_{\text{Spec}(k^s)}$$

is an equivalence of categories.

We are especially interested in continuous $\ell$-adic representations. Recall the following reformulation of the category of finitely generated $\mathbb{Z}_\ell$-modules (we follow the presentation of [FK80, Ch I.12]). We consider projective diagrams

$$F: \mathbb{Z}_\geq \to \text{Mod}(\mathbb{Z}_\ell),$$

where $\mathbb{Z}_\geq$ is the category whose objects are integers and where there is a unique morphism $m \to n$ whenever $m \geq n$. The category of diagrams is an abelian category, and in particular has images, kernels, cokernels, etc. defined pointwise. For $r \in \mathbb{Z}$, denote by $F[r]$ the shifted diagram, i.e. with $F[r]_m := F_{r+m}$.

**Definition 3.2.** A projective diagram $F: \mathbb{Z}_\geq \to \text{Mod}(\mathbb{Z}_\ell)$ satisfies:

1. the Mittag–Leffler (ML) condition if for every $n$, there exists $t \geq n$ such that for all $m \geq t$,

$$\text{Image}(F_m \to F_n) = \text{Image}(F_t \to F_n),$$

2. the Mittag–Leffler–Artin–Rees (MLAR) condition if there exists some $t \geq 0$ such that for all $r \geq t$,

$$\text{Image}(F[r] \to F) = \text{Image}(F[t] \to F).$$

**Definition 3.3.** Define $\text{Pro}_{\text{MLAR}}(\mathbb{Z}_\ell)$ to be the procategory of MLAR projective systems in which each $F_n$ is torsion. Concretely, for two such systems $F$ and $G$,

$$\text{hom}_{\text{Pro}_{\text{MLAR}}(\mathbb{Z}_\ell)}(F, G) := \varprojlim_{r \geq 0} \text{hom}(F[r], G).$$

The key purpose of Mittag–Leffler systems is that on such systems, the inverse limit is an exact functor. The ML property is frequently satisfied, for instance, if all the modules $F_n$ are finite length, then $F$ is an ML system.

**Definition 3.4.** An $\ell$-adic system is a projective diagram $F$ such that $F_n = 0$ for $n < 0$ and for all $n$

1. $F_n$ is a module of finite length,
2. $\ell^{n+1}F_n = 0$, and
3. the map $F_{n+1} \to F_n$ induces an isomorphism

$$F_{n+1}/\ell^{n+1}F_{n+1} \cong F_n.$$
An $A-R$ $\ell$-adic system is any object of $\text{Pro}_{\text{MLAR}}(\mathbb{Z}_\ell)$ which is isomorphic to an $\ell$-adic system. Denote by $AR(\ell) \subset \text{Pro}_{\text{MLAR}}(\mathbb{Z}_\ell)$ the full sub-category of $A-R$-$\ell$-adic systems.

We can now give the promised reformulation of the category $\text{Mod}_f(\mathbb{Z}_\ell)$ of finitely generated $\mathbb{Z}_\ell$-modules (cf. e.g. [FK80, Proposition I.12.4])

**Proposition 3.5.** The inverse limit

$$\lim_{\leftarrow} : AR(\ell) \longrightarrow \text{Mod}_f(\mathbb{Z}_\ell)$$

is an equivalence of exact categories.

We can use the proposition to give a similar reformulation of the category $\text{Rep}_{\text{cts}}(\text{Gal}(k^s/k); \mathbb{Z}_\ell)$.

Recall that $\mathbb{Z}_\ell$ is a profinite ring, with the profinite (equivalently “adic”) topology. Similarly, for any finitely generated $\mathbb{Z}_\ell$-module $A$, the group $\text{Aut}_{\mathbb{Z}_\ell}(A)$ is canonically a topological group, with the profinite topology. Mutatis mutandis, we obtain from Definitions 3.3 and 3.4 a notion of $\ell$-adic systems of continuous $\text{Gal}(k^s/k)$ representations, and a category $\text{Rep}_{\text{cts}}^{AR}(\text{Gal}(k^s/k); \ell)$ of such. Concretely, objects are given by projective systems

$$\cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots$$

where for each $n$, $F_n$ is a continuous representation of $\text{Gal}(k^s/k)$ in finitely generated $\mathbb{Z}/\ell^n\mathbb{Z}$-modules and the analogous conditions to those of Definition 3.4 hold. Analogously to Proposition 3.5 we have the following.

**Proposition 3.6.** The inverse limit

$$\lim_{\leftarrow} : \text{Rep}_{\text{cts}}^{AR}(\text{Gal}(k^s/k); \ell) \longrightarrow \text{Rep}_{\text{cts}}(\text{Gal}(k^s/k); \mathbb{Z}_\ell)$$

is an equivalence of exact categories.

### 3.2. $\ell$-adic Sheaves

We now extend the above to sheaves. We consider schemes $X$ for which $\ell \in O(X)$ is invertible. Mutatis mutandis, we obtain from Definition 3.3 a definition of MLAR projective systems of $\ell$-torsion étale sheaves, and the category $\text{Pro}_{\text{MLAR}}(\text{Sh}(X), \mathbb{Z}_\ell)$ of such.

**Definition 3.7.** Let $X$ be a scheme and $\ell$ a prime invertible on $X$. An $\ell$-adic sheaf on $X$ is projective system $F : \mathbb{Z}_\geq \longrightarrow \text{Sh}(X; \mathbb{Z})$ of étale sheaves of abelian groups on $X$ such that

1. The sheaves $F_n$ are constructible for all $n$,
2. $F(n) = 0$ for $n < 0$,
3. The map $F_{n+1} \longrightarrow F_n$ induces isomorphisms

$$F_{n+1} \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \cong F_n.$$

An $A-R$-$\ell$-adic sheaf is any object of $\text{Pro}_{\text{MLAR}}(\text{Sh}(X), \mathbb{Z}_\ell)$ which is isomorphic to an $\ell$-adic sheaf. We denote the category of $A-R$-$\ell$-adic sheaves by $\text{Sh}(X; \mathbb{Z}_\ell)$.

The following is the key theorem we use for $\ell$-adic sheaves (cf. [FK80, Theorem I.12.15]).

**Theorem 3.8** (Finiteness Theorem for $\ell$-adic Sheaves). Let $f : X \longrightarrow S$ be a compactifiable mapping, $\ell$ a prime invertible on $X$, and $n \mapsto F_n$ an $A-R$-$\ell$-adic sheaf on $X$. Then for all $\nu \geq 0$, the system $n \mapsto R^\nu f_!(F_n)$ is an $A-R$-$\ell$-adic sheaf on $S$.

Using the smooth base change theorem (cf. e.g. [FK80, Theorem I.7.3]) and Propositions 3.1 and 3.5 above, we immediately deduce the following.

**Corollary 3.9.** Let $k$ be a field, $k^s$ a separable closure, and let $X$ be a variety of finite type over $k$. Then for any $A-R$-$\ell$-adic sheaf $F$ on $X$, a choice of compactification $j : X \hookrightarrow \bar{X}$ and a choice of functorial flasque resolution determine an extension of the assignment $F \mapsto H^*_\ell(X/k^s; F)$
to a functor

$$\text{Sh}(X; \mathbb{Z}_\ell) \longrightarrow \text{Ch}^b(\text{Rep}_{cts}(\text{Gal}(k^s/k); \mathbb{Z}_\ell))$$

$$F \mapsto R\Gamma(j_!F)|_{k^s}$$

4. The Derived Zeta Function

Our goal in this section is to prove Theorem 1.1 and Corollary 1.2. For convenience we restate Theorem 1.1 here.

**Theorem 1.1.** Let $k$ be a field and let $k^s$ be a separable closure of $k$. Let $\ell \neq \text{char}(k)$ be a prime, and denote by $\text{Rep}_{cts}(\text{Gal}(k^s/k); \mathbb{Z}_\ell)$ the category of finitely generated continuous representations of $\text{Gal}(k^s/k)$ over $\mathbb{Z}_\ell$. For varieties $X$ over $k$, the assignment

$$X \mapsto \text{Gal}(k^s/k) \otimes H^*_\text{et,c}(X \times_k k^s; \mathbb{Z}_\ell)$$

lifts to a map of $K$-theory spectra

$$\zeta: K(V_k) \longrightarrow K(\text{Rep}_{cts}(\text{Gal}(k^s/k); \mathbb{Z}_\ell)).$$

We refer to $\zeta$ as the derived $\ell$-adic zeta function. The results of Almkvist [Alm78] and Grayson [Gra79] imply the following corollary.

**Corollary 4.2.** Let $k$ be a field, $k^s$ a separable closure, $\ell \neq \text{char}(k)$ a prime, and $g \in \text{Gal}(k^s/k)$ any element. Then the assignment

$$X \mapsto g \otimes H^*_\text{et,c}(X \times_k k^s; \mathbb{Z}_\ell)$$

lifts to a map of $K$-theory spectra

$$g_\ast \circ \zeta: K(V_k) \longrightarrow K(\text{Aut}(\mathbb{Z}_\ell)).$$

Passing to $K_0$ and taking the characteristic polynomial, we recover the “$g$-Zeta-function”

$$\zeta_g: K_0(V_k) \longrightarrow W(\mathbb{Z}_\ell)$$

$$[X] \mapsto \det(1 - g^*t; H^*_\text{et,c}(X \times_k k^s; \mathbb{Z}_\ell)).$$

Taking $k = \mathbb{F}_q$ and $g$ to be Frobenius, the “$g$-Zeta-function” is precisely the Hasse–Weil zeta function (cf. e.g. the discussion on p. 171-174 of [FK80]); in this special case Corollary 1.2 is exactly Corollary 4.2.

The proof of Theorem 1.1 occupies the rest of this section. The main steps of the proof are as follows:

**Step 1:** We define an SW-category $V_k^{cptd}$ of varieties with a specified compactification,

**Step 2:** We show that the forgetful map $U: V_k^{cptd} \longrightarrow V_k$ induces an equivalence on $K$-theory, and

**Step 3:** We show that the assignment (4.1) can be lifted to a $W$-exact functor

$$F: V_k^{cptd} \longrightarrow \text{Ch}^b(\text{Rep}_{cts}(\text{Gal}(k^s/k); \mathbb{Z}_\ell)).$$

By applying $K$-theory to the two functors from Steps 2 and 3 as well as the 0-level inclusion $[0]: \text{Rep}_{cts}(\text{Gal}(k^s/k); \mathbb{Z}_\ell) \longrightarrow \text{Ch}^b(\text{Rep}_{cts}(\text{Gal}(k^s/k); \mathbb{Z}_\ell))$ (see Example 2.6) we get a diagram of
spectra

\[
\begin{array}{ccc}
K(\text{Rep}_{\text{cts}}(\text{Gal}(k^a/k); \mathbb{Z}_\ell))) & \xrightarrow{\cdot [0]} & \chi \\
K(V^\text{cptd}_k) & \xrightarrow{K(F)} & K(\text{Ch}^b(\text{Rep}_{\text{cts}}(\text{Gal}(k^a/k); \mathbb{Z}_\ell))) \\
K(U) & \xrightarrow{\cdot h} & K(V^\text{cptd}_k)
\end{array}
\]

Here, \( \chi \) is the inverse equivalence to \( \cdot [0] \) corresponding to the Euler characteristic and \( h \) is the inverse equivalence to \( K(U) \). Both of these exist and are well-defined and unique up to a contractible space of choices since all of the spectra are fibrant and cofibrant. We define

\[ \zeta := \chi \circ K(F) \circ h. \]

We now proceed to the proofs of the individual steps.

**Step 1: The SW-Category \( V^\text{cptd}_k \).** Throughout this section, let \( k \) be an arbitrary field. In order to define the compactly-supported \( \ell \)-adic cohomology of a variety \( X \), one must first choose a compactification \( X \to \overline{X} \). To manage these choices, we construct an SW-category of varieties together with a choice of compactification, and we show that forgetting the choices induces an equivalence on \( K \)-theory.

**Definition 4.3.** Let \( k \) be a field. We define the SW-category \( V^\text{cptd}_k \) as follows. Objects of \( V^\text{cptd}_k \) are open embeddings \( X \to \overline{X} \) where \( X \) is a variety over \( k \) and \( \overline{X} \) is a proper \( k \)-variety. The morphisms are commuting squares

\[
\begin{array}{ccc}
X \xrightarrow{\circ} & \overline{X} \\
\downarrow f & & \downarrow \overline{f} \\
Y & \xrightarrow{\circ} & \overline{Y}
\end{array}
\]

A morphism \( (X \xrightarrow{\circ} \overline{X}) \to (Y \xrightarrow{\circ} \overline{Y}) \) is a

- **cofibration:** if both \( f \) and \( \overline{f} \) are closed embeddings,
- **fibration:** if both \( f \) and \( \overline{f} \) are open embeddings, and
- **weak equivalence:** if \( f \) is an isomorphism.

A sequence \( (X \xrightarrow{\circ} \overline{X}) \to (Y \xrightarrow{\circ} \overline{Y}) \leftarrow (Z \xrightarrow{\circ} \overline{Z}) \) is a subtraction sequence if \( \overline{Z} = \overline{Y} \) and \( X \to Y \leftarrow Z \) is a subtraction sequence in \( V_k \).

**Lemma 4.4.** The category \( V^\text{cptd}_k \) with cofibrations, fibrations, weak equivalences, and subtraction defined as above satisfies the axioms of an SW-category.

**Proof.** We verify the axioms in turn. Observe the following:

1. The object \((\emptyset, \emptyset)\) is initial.
2. By definition, isomorphisms are cofibrations, fibrations and weak equivalences.
3. Pullbacks exist and preserve cofibrations and fibrations because all constructions are defined objectwise.
4. Subtraction satisfies this axiom because \( V_k \) is an SW-category.
5. Pushouts along diagrams where both legs are cofibrations exist; they are defined objectwise. If \( \overline{X}, \overline{Y}, \overline{Z} \) are compactifications of \( X, Y, Z \) respectively, then \( \overline{Y} \amalg \overline{X} \amalg \overline{Z} \) is a compactification of \( Y \amalg X \amalg Z \), by the cancellative property of proper morphisms. Pushout diagrams of this form are pullbacks, as this is the case in \( V_k \).
6. The pushout product axiom holds because all constructions are defined objectwise.
(7) We next verify that subtraction behaves appropriately, i.e., given a diagram

\[
\begin{array}{ccc}
(X', \overline{X'}) & \leftarrow & (W', \overline{W'}) \rightarrow (Y', \overline{Y'}) \\
\downarrow & & \downarrow \\
(X, \overline{X}) & \leftarrow & (W, \overline{W}) \rightarrow (Y, \overline{Y}) \\
\circ & & \circ \\
(X'', \overline{X}) & \leftarrow & (W'', \overline{W}) \rightarrow (Y'', \overline{Y})
\end{array}
\]

where the top two squares are pullback squares and the rows are subtraction sequences, we would like the pushouts of the columns to form a subtraction sequence. But this follows from the definition of subtraction sequence in \(V_{k}^{cptd}\) and the corresponding statement for \(V_k\).

(8) We now verify that gluing holds. Indeed, given a diagram

\[
\begin{array}{ccc}
(Y, \overline{Y}) & \rightarrow & (X, \overline{X}) \rightarrow (Z, \overline{Z}), \\
\sim & & \sim \\
(Y, \overline{Y}) & \rightarrow & (X, \overline{X}) \rightarrow (Z, \overline{Z})
\end{array}
\]

the pushout along the top row is weakly equivalent to the pushout along the bottom row, by the corresponding gluing statement for \(V_k\) and the definition of weak equivalence.

(9) Finally, that subtraction is respected is an immediate consequence of the definition of subtraction and weak equivalence in \(V_{k}^{cptd}\).

\[\square\]

**Step 2: A \(K\)-theory Equivalence.**

**Lemma 4.5.** The forgetful map \(V_{k}^{cptd} \rightarrow V_k\) induces an equivalence on \(K\)-theory.

**Proof.** It suffices to prove that, for \(n \geq 0\), the map \(w\overline{S}_n(V_{k}^{cptd}) \rightarrow w\overline{S}_n(V_k)\) induces a weak equivalence on geometric realizations. By Quillen’s Theorem A \([\text{Qui}73, \text{Theorem A}]\), it suffices to show that for any \(\alpha \in w\overline{S}_n(V_k)\), the undercategory \(\alpha/w\overline{S}_n(V_{k}^{cptd})\) is cofiltering.

For this, we first observe that the category \(\alpha/w\overline{S}_n(V_{k}^{cptd})\) is non-empty. Indeed, by Nagata \([\text{Nag}62, \text{Theorem 4.3}]\)\(^2\), every \(k\)-variety \(X\) admits an open embedding \(X \xrightarrow{\alpha} \overline{X}\) into a proper \(k\)-variety \(\overline{X}\). More generally, if \(\alpha\) is a sequence of closed immersions

\[
X_0 \hookrightarrow \cdots \hookrightarrow X_n,
\]

then any compactification \(X_n \xrightarrow{\alpha} \overline{X}_n\) determines an object \((\alpha, \overline{\alpha})\) of \(w\overline{S}_n(V_{k}^{cptd})\)

\[
(X_0 \xrightarrow{\alpha} \overline{X}_0) \hookrightarrow \cdots \hookrightarrow (X_n \xrightarrow{\alpha} \overline{X}_n),
\]

where, for \(i < n\), \(\overline{X}_i\) denotes the closure of \(X_i\) in \(\overline{X}_n\). Then \(((\alpha, \overline{\alpha}), 1_\alpha)\) is an object of \(\alpha/w\overline{S}_n(V_{k}^{cptd})\), so this is non-empty.

We now show that \(\alpha/w\overline{S}_n(V_{k}^{cptd})\) is a preorder: that there is at most one arrow between any two objects. Consider two parallel morphisms

\[
\sigma, \tau: ((\beta, \overline{\beta}), f: \alpha \rightarrow \beta) \rightarrow ((\gamma, \overline{\gamma}), g: \alpha \rightarrow \gamma).
\]

\[\square\]

\(^2\)For a modern treatment of Nagata’s Theorem, see \([\text{Con}07]\), esp. Theorem 4.1.
The data of \( \sigma \) and \( \tau \) consist of morphisms \( (\beta, \beta) \to (\gamma, \tilde{\gamma}) \) in \( w\hat{S}_k(\mathcal{V}_k^{\text{cptd}}) \) which are compatible with the morphisms from \( \alpha \). Since these are weak equivalences, they are isomorphisms \( \beta \to \gamma \); thus \( \sigma \) and \( \tau \) agree on the dense open subsets given by \( \beta \) and \( \gamma \), and thus they must be equal.

It remains to show that for every pair of objects in \( \alpha/w\hat{S}_n(\mathcal{V}_k^{\text{cptd}}) \), there exists a third object which maps into each of them. This follows from the fact that given any two compactifications \( X \to \bar{X} \) and \( X \to \bar{X}' \), there exists a third compactification \( X \to \bar{X}'' \) dominating both (e.g. take the closure of \( X \) inside \( \bar{X} \times \bar{X}' \), as in [Del77 §IV-10.5]).

**Step 3: The \( W \)-Exact Functor.** By Proposition 5.6, it suffices to construct a \( W \)-exact functor

\begin{equation}
F: \mathcal{V}_k^{\text{cptd}} \longrightarrow \text{Ch}^b(\text{Rep}_{cts}\text{AR}(\text{Gal}(k^s/k); \ell))
\end{equation}

\( \text{Ch}^b(\text{Rep}_{cts}(\text{Gal}(k^s/k); \mathbb{Z}/\ell^n\mathbb{Z})) \). We then show these fit together to define an \( \text{A} - \text{R-\ell} \)-adic system of continuous Galois representations.

We start by defining the functor

\[ F^\text{a}_n: \text{fib}(\mathcal{V}_k^{\text{cptd}}) \longrightarrow \text{Ch}^b(\text{Rep}_{cts}(\text{Gal}(k^s/k); \mathbb{Z}/\ell^n\mathbb{Z})). \]

On objects, it is given by

\begin{equation}
F^\text{a}_n(X) := (\text{Gal}(k^s/k) \circ R\Gamma\gamma_X(\mathbb{Z}/\ell^n\mathbb{Z})_{X|k^s}).
\end{equation}

Note that the (strict) continuous action of \( \text{Gal}(k^s/k) \) follows from the functoriality of \( \gamma_X(\mathbb{Z}/\ell^n\mathbb{Z}) \) and our choice of a functorial flasque resolution.

We now define \( F^\text{a}_n \) on morphisms. By definition, a fibration \( (j, \gamma) \) in \( \mathcal{V}_k^{\text{cptd}} \) consists of a commuting square

\[
\begin{array}{ccc}
U & \xrightarrow{j} & X \\
\gamma_U & \downarrow & \gamma_X \\
X & \xrightarrow{\gamma} & \bar{X}
\end{array}
\]

where all maps are open embeddings and \( \bar{X} \) is proper. Given such, we obtain a map

\[ F^\text{a}_n(j, \gamma): R\Gamma\gamma_U(\mathbb{Z}/\ell^n\mathbb{Z})_{U|k^s} \longrightarrow R\Gamma\gamma_{X}(\mathbb{Z}/\ell^n\mathbb{Z})_{X|k^s} \]

by applying \( R\Gamma \) to the composition of morphisms in \( \text{Sh}((\bar{X} \times_k k^s)) \)

\[
\gamma_U(\mathbb{Z}/\ell^n\mathbb{Z})_{U|k^s} \cong \gamma_X(\mathbb{Z}/\ell^n\mathbb{Z})_{X|k^s} \cong \gamma_X(\mathbb{Z}/\ell^n\mathbb{Z})_{X|k^s} \cong \gamma_X(\mathbb{Z}/\ell^n\mathbb{Z})_{X|k^s} \]

where \( \epsilon \) is the counit of the adjunction \( j^! \dashv j^* \) (which exists because \( j \) is étale), and where the first two isomorphisms come from the canonical identifications \( \gamma_U = \gamma_X \) and \( (\mathbb{Z}/\ell^n\mathbb{Z})_U \cong j^*(\mathbb{Z}/\ell^n\mathbb{Z})_X \).

**Lemma 4.8.** The assignment \( (j, \gamma) \mapsto F^\text{a}_n(j, \gamma) \) is functorial on \( \text{fib}(\mathcal{V}_k^{\text{cptd}}) \).
Proof. It suffices to show that the assignment is functorial before applying \( R\Gamma \). The definition immediately implies that identities are mapped to identities, so we only need to check that composition is respected. For this, note that a composable pair of fibrations in \( \mathcal{V}^{\text{cptd}}_k \) consists of a commuting diagram

\[
\begin{array}{ccc}
U & \xrightarrow{j_1} & X \\
\gamma_U & \downarrow & \gamma_X \\
Y & \xrightarrow{\gamma_Y} & Y
\end{array}
\]

Given such, we obtain a string of morphisms (in \( \text{Sh}(\bar{Y} \times_k k^s) \))

\[
\gamma_{U!}(\mathbb{Z}/\ell^n\mathbb{Z})|_{k^s} \cong \gamma_X!(\mathbb{Z}/\ell^n\mathbb{Z})|_{k^s} \cong \gamma_Y!(\mathbb{Z}/\ell^n\mathbb{Z})|_{k^s}
\]

where \( \epsilon_i \) denotes the counit of the adjunction \( j_i! \vdash j_i^* \) for \( i = 1, 2 \). Moreover, under the canonical isomorphisms used to define \( F^n_i(j_i, \gamma) \) this string becomes isomorphic to \( \gamma_Y! \) applied to the string of morphisms (in \( \text{Sh}(Y \times_k k^s) \))

\[
\bar{j}_2!j_1!j_2^* \cong \bar{j}_2!j_1!j_2^* \cong \bar{j}_2!j_1!j_2^*
\]

which is equal to the counit of the adjunction \( j_2! \vdash j_1^* \). We thus conclude that \( F^n_i(j_1, \gamma) \circ F^n_i(j_2, \gamma) = F^n_i(j_2j_1, \gamma) \).

We now define the functor

\[
F^n_i : \mathbf{co}(\mathcal{V}^{\text{cptd}}_k) \rightarrow \text{Ch}^b(\text{Rep}_{\text{cts}}(\text{Gal}(k^s/k); \mathbb{Z}/\ell^n\mathbb{Z})).
\]

On objects, it is equal to \( F^m_i \) (see [17]). By definition, a cofibration \((i, \gamma)\) in \( \mathcal{V}^{\text{cptd}}_k \) consists of a commuting square

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\gamma_Z & \downarrow & \gamma_X \\
\bar{Z} & \xrightarrow{i} & \bar{X}
\end{array}
\]

where the horizontal maps are closed embeddings, the vertical maps are open embeddings, and \( Z \) and \( X \) are proper. Given such, we obtain a map

\[
F^n_i(i, \gamma) : R\Gamma \gamma_X!(\mathbb{Z}/\ell^n\mathbb{Z})|_{k^s} \rightarrow R\Gamma \gamma_Z!(\mathbb{Z}/\ell^n\mathbb{Z})|_{k^s}
\]

by applying \( R\Gamma \) to the composition of morphisms (in \( \text{Sh}(\bar{X} \times_k k^s) \))

\[
\gamma_X!(\mathbb{Z}/\ell^n\mathbb{Z})|_{k^s} \cong \gamma_X!i_!i^*(\mathbb{Z}/\ell^n\mathbb{Z})|_{k^s} \cong \gamma_X!i!i^*(\mathbb{Z}/\ell^n\mathbb{Z})|_{k^s} \cong \bar{i}_!\gamma_Z!(\mathbb{Z}/\ell^n\mathbb{Z})|_{k^s}
\]

where \( \eta \) is the unit of the adjunction \( i^* \vdash i_* \), and where the last two isomorphisms come from the canonical identifications \( i_! = i_* \) (because \( i \) is proper), \( \gamma_X!i_! = \bar{i}_!\gamma_Z! \) (by functoriality of \( ! \)) and \( (\mathbb{Z}/\ell^n\mathbb{Z})_X \cong i^*(\mathbb{Z}/\ell^n\mathbb{Z})_X \). Finally, we note that we are implicitly identifying \( R\Gamma\mathcal{Z} \) with \( R\Gamma\mathcal{X}\mathcal{Z} \) using that \( i_* \cong R\Gamma\mathcal{Z} \) (because \( \bar{i} \) is a closed embedding and we are using the Godement resolution!)

Lemma 4.9. The functor \( F^n_i \) is well-defined.
Lemma 4.10. The collection of functors \( F \) of sheaves on \( F \to \) functors \( X \) which is equal to the unit of the adjunction \( \gamma \) morphisms (in Sh(\( \mathcal{V}_{k}^{\text{ptd}} \)))

Proof. As above, it suffices to show that the assignment is functorial before applying \( R\Gamma \). The \( \eta \) immediately implies that identities are mapped to identities, so we only need to check \( \gamma \) composition is respected. A composable pair of cofibrations in \( \mathcal{V}_{k}^{\text{ptd}} \) consists of a commuting diagram

\[
\begin{array}{c}
\begin{array}{ccc}
W & \xrightarrow{i_2} & Z & \xrightarrow{i_1} & X \\
\gamma_W & & \gamma_Z & & \gamma_X \\
W & \xrightarrow{i_2} & Z & \xrightarrow{i_1} & \bar{X}
\end{array}
\end{array}
\]

Given such, we obtain a string of morphisms (in Sh(\( \bar{X} \times_k k^s \)))

\[
\gamma_X! (\mathbb{Z}/\ell^n \mathbb{Z})_{X|k^s} \overset{\gamma_X! (\mathbb{Z}/\ell^n \mathbb{Z})_{X|k^s}}{\longrightarrow} \gamma_X! i_1^* i_1^* (\mathbb{Z}/\ell^n \mathbb{Z})_{X|k^s} \simeq \gamma_X! i_1^* (\mathbb{Z}/\ell^n \mathbb{Z})_{X|k^s} \overset{\gamma_X! i_2^* (\eta_1)}{\longrightarrow} \gamma_X! i_2^* i_2^* (\mathbb{Z}/\ell^n \mathbb{Z})_{W|k^s} \simeq \gamma_X! i_2^* (\mathbb{Z}/\ell^n \mathbb{Z})_{W|k^s} \simeq \gamma_X! i_1^* i_2^* (\mathbb{Z}/\ell^n \mathbb{Z})_{W|k^s} \simeq \gamma_X! i_1^* (\mathbb{Z}/\ell^n \mathbb{Z})_{W|k^s},
\]

where \( \epsilon_a \) denotes the unit of the adjunction \( i_a^* \dashv i_a^* \) for \( a = 1, 2 \). Moreover, under the isomorphisms used to define \( (i_a, \gamma)^a \), this string becomes isomorphic to \( \gamma_X! \) applied to the string of morphisms (in Sh(\( X \)))

\[
(\mathbb{Z}/\ell^n \mathbb{Z})_X \overset{\eta_1}{\longrightarrow} i_1^* i_2^* (\mathbb{Z}/\ell^n \mathbb{Z})_X \overset{i_1^* (\eta_2 \circ \eta_1^* i_2^*)}{\longrightarrow} i_2^* i_1^* i_2^* (\mathbb{Z}/\ell^n \mathbb{Z})_X
\]

which is equal to the unit of the adjunction \( i_2^* \dashv i_1^* \). We thus conclude that \( F_i^j (i_2, \gamma) \circ F_i^j (1, \gamma) = F_i^j (i_1 i_2, \gamma) \).

Lemma 4.10. The collection of functors \( \{ F_i^j \} \) define a W-exact functor

\[
F : \mathcal{V}_{k}^{\text{ptd}} \longrightarrow \text{Ch}^b (\text{Rep}_{\text{cts}}^{AR} (\Gamma_{\text{Gal}(k^s/k); \ell})).
\]

Proof. The maps \( F_{i+1}^i \to F_{i+1}^i \otimes \mathbb{Z} \to F_i^i \) and similarly for \( F^n \) endow the collections \( \{ F_i^i \} \) and \( \{ F^n \} \) with the structure of projective systems. From the construction, for each fixed \( X, \{ F_i^i (X) \} \) and \( \{ F^n (X) \} \) are obtained by applying \( R\Gamma \) to A–R-\( \ell \)-adic systems of sheaves on \( X \times_k k^s \). Therefore, by [FK80] Theorem 1.12.15, \( \{ F_i^i (X) \} \) and \( \{ F^n (X) \} \) are A–R-\( \ell \)-adic complexes of sheaves on \( k^s \), i.e. A–R-\( \ell \)-adic complexes of continuous \( \text{Gal}(k^s/k) \)-modules. So, we indeed have functors

\[
F_i^i : \text{co} (\mathcal{V}_{k}^{\text{ptd}}) \longrightarrow \text{Ch}^b (\text{Rep}_{\text{cts}}^{AR} (\text{Gal}(k^s/k); \ell))
\]

\[
F_i^i : \text{fib} (\mathcal{V}_{k}^{\text{ptd}}) \longrightarrow \text{Ch}^b (\text{Rep}_{\text{cts}}^{AR} (\text{Gal}(k^s/k); \ell)).
\]

It remains to verify that these are W-exact.

That \( F \) takes subtraction sequences to cofiber sequences is classical: given a subtraction sequence in \( \mathcal{V}_{k}^{\text{ptd}} \)

\[
\begin{array}{c}
\begin{array}{ccc}
Z & \xrightarrow{i} & X & \xrightarrow{j} & U \\
\gamma_Z & & \gamma_X & & \gamma_U \\
Z & \xrightarrow{i} & \bar{X} & \xrightarrow{1} & \bar{X}
\end{array}
\end{array}
\]
we obtain a sequence (in Sh(\(\mathcal{X} \times_k k^s\))

\[
0 \to \gamma_{U!*}(\mathbb{Z}/\ell^n\mathbb{Z})_{U|k^s} \xrightarrow{j} \gamma_{X!*}(\mathbb{Z}/\ell^n\mathbb{Z})_{X|k^s} \xrightarrow{i^*} i_*\gamma_{Z!*}(\mathbb{Z}/\ell^n\mathbb{Z})_{Z|k^s} \to 0
\]

and we see that this is exact by direct inspection (i.e. by verifying exactness on stalks). Applying \(R\Gamma\), we obtain an exact sequence in \(\text{Ch}^b(\text{Rep}_{cts}^{\text{AR}}(\text{Gal}(k^s/k); \ell))\) (since exactness in \(\text{Ch}^b(\text{Rep}_{cts}^{\text{AR}}(\text{Gal}(k^s/k); \ell))\) only depends on the exactness of the underlying complex in \(\text{Ch}^b(\text{AR}(\ell))\).

It remains to verify the base change axiom. It suffices to prove that it holds for \(F_n := (F^n_{1,n}, F^n_{1})\).

Given a commuting square in \(V^{\text{profd}}_k\) with horizontal arrows fibrations and vertical arrows cofibrations

\[
\begin{array}{cccc}
\gamma & \downarrow & \gamma
\
W & \rightarrow & Z
\
i_W & \downarrow & i_Z
\
U & \rightarrow & X
\end{array}
\]

we obtain a diagram in Sh(\(\mathcal{X} \times_k k^s\))

\[
\begin{array}{cccc}
\gamma_{U!*}(\mathbb{Z}/\ell^n\mathbb{Z})_{U|k^s} & \xrightarrow{\gamma_{U!*}(\eta_W)} & \gamma_{U!*}i_W!i_W^*(\mathbb{Z}/\ell^n\mathbb{Z})_{U|k^s}
\
\gamma_{X!*}j_U!(\mathbb{Z}/\ell^n\mathbb{Z})_{X|k^s} & \xrightarrow{\gamma_{X!*}(\eta_W \circ j_U)} & \gamma_{X!*}i_W!i_W^*j_U!(\mathbb{Z}/\ell^n\mathbb{Z})_{X|k^s}
\
\gamma_{X!*}(\mathbb{Z}/\ell^n\mathbb{Z})_{X|k^s} & \xrightarrow{\gamma_{X!*}(\eta_Z)} & \gamma_{X!*}i_Z!i_Z^*(\mathbb{Z}/\ell^n\mathbb{Z})_{X|k^s}
\end{array}
\]

To verify the base change axiom for \(F_n\), it suffices to prove that the above diagram commutes. The upper two squares and the lower right square commute by inspection. Note also that in the lower middle vertical arrow, we are implicitly using the identification

\[
\gamma_{X!*}j_U!(\mathbb{Z}/\ell^n\mathbb{Z})_X = \gamma_{X!*}i_Z!i_Z^*j_U!(\mathbb{Z}/\ell^n\mathbb{Z})_X
\]

It remains to verify that the lower left square commutes. For this, it suffices to show that the square (in Sh(\(\mathcal{X}\)))

\[
\begin{array}{cccc}
j_U!j_U^!(\mathbb{Z}/\ell^n\mathbb{Z})_X & \xrightarrow{j_U!*}(\eta_W \circ j_U^!)} & j_U!i_W!i_W^*j_U!(\mathbb{Z}/\ell^n\mathbb{Z})_X
\
\epsilon_U & \downarrow & i_Z!*i_Z^!(\mathbb{Z}/\ell^n\mathbb{Z})_X
\
(\mathbb{Z}/\ell^n\mathbb{Z})_X & \rightarrow & i_Z!*i_Z^!(\mathbb{Z}/\ell^n\mathbb{Z})_X
\end{array}
\]

commutes. Unpacking the definitions of these maps on sections, we see that, for \(V \to X\) étale,

\[
\eta_Z \circ \epsilon_U : j_U!j_U^!(\mathbb{Z}/\ell^n\mathbb{Z})_X \to i_Z!*i_Z^!(\mathbb{Z}/\ell^n\mathbb{Z})_X
\]
is equal to the inclusion of $0$ if $V \longrightarrow X$ does not factor through the inclusion of $U$, and if $V \longrightarrow X$ does factor through $U$, then $\eta_Z \circ \epsilon_U$ is given by the canonical map

$$j_U^*(\mathbb{Z}/\ell^n\mathbb{Z})_X(V) \longrightarrow \lim_{Z \times_X V \longrightarrow V'} (\mathbb{Z}/\ell^n\mathbb{Z})_X(V') = i_{Z,*}i_Z^*(\mathbb{Z}/\ell^n\mathbb{Z})_X$$

where the limit is over all factorizations of the map $Z \times_X V \longrightarrow Z \longrightarrow X$ through étale $V' \longrightarrow X$. On the other hand, the definitions give that

$$i_{Z,*}(\epsilon_W \circ i_Z^*) : j_U^!(\eta_W \circ j_U^*): j_U^!(\mathbb{Z}/\ell^n\mathbb{Z})_X \longrightarrow i_{Z,*}i_Z^*(\mathbb{Z}/\ell^n\mathbb{Z})_X$$

is equal to the inclusion of $0$ either if $V \longrightarrow X$ does not factor through the inclusion of $U$ in $X$ or if $Z \times_X V \longrightarrow Z$ does not factor through the inclusion of $W$ in $Z$. If $V \longrightarrow X$ and $Z \times_X V \longrightarrow Z$ do factor through $U$ and $W$ then the map is given by the map (4.13).

We conclude that the square (4.12) commutes if and only if: for any $V \longrightarrow X$ étale, $Z \times_X V \longrightarrow Z$ factors through the inclusion of $W$ in $Z$ if $V \longrightarrow X$ factors through the inclusion of $U$ in $X$. A necessary and sufficient condition for this is that $W = Z \times_X U$. But this is precisely the assumption of the base change axiom. We thus conclude the proof for $F_n$ for each $n$ and thus that the base change axiom holds for $F$.

5. Nontrivial Classes in the Higher K-Theory of Varieties

In this section we use the derived zeta function to detect nontrivial classes in the higher $K$-theory of $V_k$. Before we begin, note that whenever $k$ is finite the functors

$$\text{FinSet} \longrightarrow V_k \quad \text{and} \quad V_k \longrightarrow \text{FinSet}$$

$$A \mapsto \coprod_A \text{Spec}(k) \quad \text{and} \quad X \mapsto X(k)$$

give a splitting of the $K$-theory spectrum $K(V_k)$ as $K(V_k) \simeq \mathbb{S} \oplus \widetilde{K}(V_k)$. For infinite fields $k$, this splitting does not exist, but one can still define $\widetilde{K}(V_k)$ by using the homotopy cofiber of the map $\mathbb{S} \longrightarrow K(V_k)$,

$$\widetilde{K}(V_k) = \text{hocofib} (\mathbb{S} \longrightarrow K(V_k)).$$

The sphere $\mathbb{S}$ of course has extremely rich higher homotopy groups, but these are somewhat uninteresting from the perspective of algebraic varieties. A priori it may be the case that all higher homotopy groups of $K(V_k)$ are in the image of the homotopy groups of $\mathbb{S}$.

In this section, we show that this is not the case by using $\zeta$ to identify a nonzero element in $\widetilde{K}_1(V_k)$ for $k = \mathbb{F}_q$ (with $q \equiv 3 \mod 4$), $k$ a global or local field with a place of cardinality equivalent to $3$ mod $4$, or $k \subset \mathbb{R}$. For such $k$, we construct a surjective homomorphism $h_2 : K_1(V_k) \longrightarrow \mathbb{Z}/2$ such that the composition $\pi_1(\mathbb{S}) \longrightarrow K_1(V_k) \xrightarrow{h_2} \mathbb{Z}/2$ is trivial. The map $h_2$ is defined as follows. Let $\star$ be the operation defined in [Mil71, §8]. Grayson [Gra79] has shown that given a pair of commuting automorphisms $f, g$ on a $\mathbb{Q}_\ell$-vector space $V$, the map

$$(f, g) \mapsto f^{-1} \star g,$$

induces a homomorphism $\sigma_\ell : K_1(\text{Aut}(\mathbb{Q}_\ell)) \longrightarrow K_2(\mathbb{Q}_\ell)$. Moore’s Theorem (see either the appendix to [Mil71], or Theorem 57 of [Wei05] and its proof) shows that the mod-$\ell$ Hilbert symbol gives a split surjection $(-, -)_\ell : K_2(\mathbb{Q}_\ell) \longrightarrow \mu(\mathbb{Q}_\ell)$ onto the roots of unity in $\mathbb{Q}_\ell$ with kernel an uncountable uniquely divisible abelian group $U_2(\mathbb{Q}_\ell)$. For $q$ odd, we then define $h_2$ to be the composition

$$K_1(V_k) \xrightarrow{\pi_1(Frob_{\ell} \circ \zeta)} K_1(\text{Aut}(\mathbb{Q}_2)) \xrightarrow{\sigma_2} K_2(\mathbb{Q}_2) \xrightarrow{(-, -)_2} \mathbb{Z}/2,$$

where $Frob_{\ell}$ is the map defined in Corollary 4.2 for $g = Frob$ and $(-, -)_2$ is the 2-adic Hilbert symbol. For $k$ a global or local field with a place of cardinality equivalent to $3$ mod $4$, we pick a
The map \( h \) to \(-1\), and thus gives the desired element in \( \tilde{\mathbb{Q}} \) direct sum this place, the above computation similarly shows that the map \( \sigma \) to \(-1\) for any Frobenius element \( h \) in \( \mathbb{Q}_p \) with a place of cardinality \( 3 \mod 4 \), then, for any Frobenius element \( \phi \) for this place, and define \( h_2 \) to be the composition

\[
K_1(V_k) \xrightarrow{\pi_1(\phi \circ \mathcal{Q})} K_1(\text{Aut}(\mathbb{Q}_2)) \xrightarrow{\sigma_2} K_2(\mathbb{Q}_2) \xrightarrow{(-, -)_2} \mathbb{Z}/2,
\]

Similarly for \( k \in \mathbb{R} \), we define \( h_2 \) to be the composition

\[
K_1(V_k) \xrightarrow{\pi_1(\tau \circ \mathcal{Q})} K_1(\text{Aut}(\mathbb{Q}_2)) \xrightarrow{\sigma_2} K_2(\mathbb{Q}_2) \xrightarrow{(-, -)_2} \mathbb{Z}/2,
\]

where \( \tau \) denotes complex conjugation and \( \tau_\ast \) is the map defined in Corollary 4.2 for \( g = \tau \).

Fix a variety \( X \), and consider the diagram

\[
\begin{array}{ccc}
\xi_{(-,-)} & \xrightarrow{h_2} & \mathbb{Z}/2 \\
\downarrow \quad \quad & & \\
\text{Aut}(X) & \xrightarrow{\xi_-} & K_1(V_k)
\end{array}
\]

where \( \xi_- \) takes \( \alpha \) to \( \xi_\alpha \) and \( \xi_{(-,-)} \) takes \( \alpha \) to \( \xi_{(\alpha, \alpha)} \), as defined in Proposition 2.19. If \( X \) is proper then \( \xi_{(-,-)} \) exists, and the diagram commutes with the dotted arrow added.

**Theorem 5.2.** For \( k = \mathbb{F}_q \) for \( q \equiv 3 \mod 4 \), for \( k \) a global or local field with a place of cardinality equivalent to \( 3 \mod 4 \), or for \( k \subset \mathbb{R} \), the map \( h_2 \) defined above detects a nontrivial class in \( K_1(V_k) \). In particular, the spectrum \( \tilde{K}(V_k) \) is not an Eilenberg–MacLane spectrum.

Theorem 1.3 is a direct consequence of this theorem.

**Remark 5.3.** We expect that \( \tilde{K}(V_k) \) is not an Eilenberg–MacLane spectrum in general, however the particular class that we use \( h_2 \) to detect requires the assumptions on \( k \). Using the methods of this paper, it should be possible to find a different example that gives a nontrivial class for all odd \( q \), all global and local fields with a place of odd cardinality, and all non-algebraically closed subfields of \( \mathbb{C} \). For even \( q \), one would need the \( p \)-adic (rather than the \( \ell \)-adic analogue of the derived zeta function to employ the present approach.

**Proof.** Let \( \eta \in \pi_1(\mathbb{S}) \) be the nonzero element. We will show that \( h_2 \) is nonzero but contains \( \eta \) in its kernel. This shows that \( \tilde{K}_1(V_k) \neq 0 \) and therefore that \( \tilde{K}(V_k) \) is not an Eilenberg–MacLane spectrum.

First, let \( X \) be two points, and let \( \tau \) be the transposition of these two points. Then \( \xi_{\tau} = \eta \) inside \( K_1(V_k) \), by definition. Tracing through the definition, \( h(\xi_{\tau}) = (-1, 1)_2 = 1 \in \mathbb{Z}/2 \).

We now exhibit an \( X \) and an \( \alpha \) such that its image in \( \mathbb{Z}/2 \) is nontrivial. Let \( X = \mathbb{P}^1 \coprod \mathbb{P}^1 \) and let \( \alpha \) be the transposition of the two lines. For \( k = \mathbb{F}_q \), we can write \( \alpha \circ (H^2_{et,c}(X|_{\mathbb{F}_q}; \mathbb{Q}_2), \text{Frob}_q^X) \) as a direct sum

\[
(1 \circ \mathbb{Q}_2(0)) \oplus (-1 \circ \mathbb{Q}_2(0)) \oplus (1 \circ \mathbb{Q}_2(-1)) \oplus (-1 \circ \mathbb{Q}_2(-1)).
\]

The map \( \sigma_2 \) sends everything with the identity acting on it to the unit, so the image of this under \( h_2 \) is \((-1, q)_2 \), which, when \( q \equiv 3 \mod 4 \), is \(-1 \). This gives the desired element in \( \tilde{K}_1 \). If \( k \) is a global or local field with a place of cardinality \( 3 \mod 4 \), then, for any Frobenius element \( \phi \) for this place, the above computation similarly shows that the map \( h_2 \) takes the class \( \tau \circ \mathbb{P}^1 \coprod \mathbb{P}^1 \) to \(-1\), and thus gives the desired element in \( \tilde{K}_1(V_k) \).

For \( k \subset \mathbb{R} \), we can consider the same \( X \) and \( \alpha \). We can write \( \alpha \circ (H^2_{et,c}(X|_\mathbb{C}; \mathbb{Q}_2), \tau) \) as a direct sum

\[
(1 \circ (\mathbb{Q}_2, 1)) \oplus (-1 \circ (\mathbb{Q}_2, 1)) \oplus (1 \circ (\mathbb{Q}_2, -1)) \oplus (-1 \circ (\mathbb{Q}_2, -1)).
\]

The map \( \sigma_2 \) sends everything with the identity acting on it to the unit, so the image of this under \( h_2 \) is \((-1, -1)_2 = -1 \). This gives the desired element in \( \tilde{K}_1 \). \( \square \)
6. Questions for Future Work

**Indecomposable Elements in** $K(V_k)$. Theorem 5.2 establishes that there are non-trivial classes in the higher $K$-theory of varieties that do not come from the sphere spectrum. However, one could ask a more refined question: since $K(V_k)$ is an $E_\infty$-ring spectrum, its homotopy groups $K_*(V_k)$ form a ring. We therefore have a ready supply of elements of $K_*(V_k)$: those in the image of the multiplication
\[ \beta: K_0(V_k) \otimes \pi_*(S) \longrightarrow K_0(V_k) \otimes K_*(V_k) \longrightarrow K_*(V_k). \]
We call such elements **decomposable**. A priori, it may be the case that this map is surjective, and that therefore all higher homotopy groups of $K(V_k)$ are decomposable. The example constructed in Section 5 is decomposable, since this can just be written as $\eta \cdot [P_1]$.

**Question 6.1. Indecomposable elements.** Do there exist indecomposable elements in $K_*(V_k)$?

As we explain in Remark 6.5 below, we do not expect the map $h_2$ to be able to distinguish decomposable from non-decomposable elements. Instead, we hope that by expanding the collection of derived motivic measures, a suitable invariant could be found.

**Other Derived Motivic Measures.** We expect the derived $\ell$-adic zeta function constructed in this paper to be just one of many derived motivic measures, i.e. maps of spectra
\[ K(V_k) \longrightarrow A \]
for some spectrum $A$.

The recipe followed in this paper should adapt to construct derived motivic measures for other cohomological invariants. We took $\ell$-adic cohomology as the basis for our derived zeta function. One would like analogous maps for the other Weil cohomology theories. We hope that by expanding the collection of derived motivic measures, a suitable invariant could be found.

**Problem 6.2. Derived $p$-adic zeta functions.** Let $k$ be a perfect field of characteristic $p$ with Witt vectors $W(k)$. Construct a map of $K$-theory spectra
\[ K(V_k) \longrightarrow K(\text{Aut}(W(k))) \]
which lifts the function sending a variety $X$ to $H^*_{\text{rig},c}(X/W(k))$ to its compactly supported rigid cohomology (with constant coefficients) acted on by the Frobenius automorphism.

We expect that the construction should parallel that in Section 4, with the category $\mathcal{V}_{\text{ctd}}$ replaced by a category of varieties $X$ equipped with a choice of compactification $X \hookrightarrow \bar{X}$, and a choice of map of admissible triples $(X, Y, \mathcal{Y}) \longrightarrow (\bar{X}, \bar{Y}, \mathcal{Y})$ extending $X \hookrightarrow \bar{X}$ as in [Ber86, Section 3], and with rigid cohomology replacing the $\ell$-adic constructions. Note that, Tsuzuki’s finiteness theorem [Tsu03, Theorem 5.1.1] will play an essential role in defining the $W$-exact functor.

**Problem 6.3. Derived Serre Polynomial.** Let $k$ be a field of characteristic 0. Construct a map of $K$-theory spectra taking values in the $K$-theory of integral mixed Hodge structures
\[ K(V_k) \longrightarrow K(MHS_\mathbb{Z}) \]
which lifts the function sending a variety $X$ to $H^*_c(X(\mathbb{C}); \mathbb{Z})$ with its canonical mixed Hodge structure.

We expect that the construction should parallel that in Section 4, with the category $\mathcal{V}_{\text{ctd}}$ replaced by a category of varieties $X$ equipped with a choice of compactification $X \hookrightarrow \bar{X}$, and a choice of cubical hyperresolution $\tilde{X}_\bullet \longrightarrow \bar{X}$ of the pair $(\bar{X}, \bar{X} - X)$ (see e.g. [PS08, Chapter 5]), and with logarithmic differential forms in lieu of the $\ell$-adic constructions.

The framework of motives suggests that the derived $\ell$-adic zeta function, along with the two maps described above, should factor through a derived motivic measure built from motivic cohomology.
Problem 6.4. Derived Gillet–Soulé. Let $k$ be a field admitting resolution of singularities. Construct a map of $K$-theory spectra taking values in the $K$-theory of integral Chow motives over $k$

$$K(V_k) \longrightarrow K(M_k)$$

which lifts the motivic measure of Gillet–Soulé [GS96] sending a $k$-variety $X$ to its compactly supported integral Chow motive. Prove that the derived $\ell$-adic zeta function and the derived Serre polynomial factor through this map.

We expect the construction should be parallel to those above. In particular, the replacement of $V_{cptd}$ should be the same as for mixed Hodge structures. For general fields $k$, one might expect to have a motivic measure based on integral Voevodsky motives through which all of the above maps factor.

Remark 6.5. That the derived $\ell$-adic zeta function should factor through integral compactly supported motivic cohomology suggests that the invariant $h_2: K_1(V_k) \longrightarrow K_2(\mathbb{Q})$ in Section 5 should factor through the composition

$$K_1(\text{Aut}(\mathbb{Z})) \longrightarrow K_2(\mathbb{Z}) \longrightarrow K_2(\mathbb{Q}) \longrightarrow K_2(\mathbb{Q}_2).$$

We highlight three implications of this expected factoring:

1. It underscores the importance of the 2-adic Hilbert symbol, as opposed to the $\ell$-adic Hilbert symbol for $\ell \neq 2$. Indeed, by Tate’s computation of $K_2(\mathbb{Q})$ (see e.g. [Mil71, Theorem 11.6]), the map $K_2(\mathbb{Z}) \longrightarrow K_2(\mathbb{Q})$ is split injective, with the splitting given by the 2-adic Hilbert symbol. In particular, the Hilbert symbols $(-,-)_\ell$ for $\ell \neq 2$ identically vanish on $K_2(\mathbb{Z})$. Further, no classes outside the summand $\mu(\mathbb{Q}_2) \subset K_2(\mathbb{Q}_2)$ are in the image of $K_2(\mathbb{Z})$.

2. It suggests that we should not expect the map $h_2$ to be able to distinguish indecomposable elements in $K_1(V_k)$. Indeed, Milnor’s computation [Mil71 Corollary 10.2] shows that the map $K_1(\text{Aut}(\mathbb{Z})) \longrightarrow K_2(\mathbb{Z})$ is surjective and the nontrivial class in $K_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ is mapped onto by decomposable classes.

3. It suggests that the higher invariants of the derived zeta functions should be in some sense independent of $\ell$. It would be fruitful to understand this more precisely!

Moving further away from cohomological invariants, one of the richest motivic measures is Kapranov’s motivic zeta function

$$K_0(V_k) \longrightarrow W(K_0(V_k))$$

$$[X] \mapsto \sum_{i=0}^{\infty} [\text{Sym}^i(X)] t^i.$$

Question 6.6. Derived motivic zeta function. Does Kapranov’s motivic zeta function lift to a map of $K$-theory spectra?

For motivation, recall that Weil’s realization that the Hasse-Weil zeta function of a variety over a finite field can be obtained cohomologically provided a robust strategy for proving that the zeta function of such varieties is rational. Similarly, a lift of Kapranov’s motivic zeta function to a map of $K$-theory spectra might be expected to go a long way toward proving that the motivic zeta function is rational, in an appropriate sense.

Recall, however, that purely as a map out of $K_0(V_C)$, Kapranov’s motivic zeta function is not rational. This was proven by Larsen and Lunts [LL03], and the key tool in their proof was a motivic measure

$$\mu_{LL}: K_0(V_C) \longrightarrow \mathbb{Z}[SB_C]$$
taking values in the free abelian group on stable birational equivalences classes of complex varieties. Note that, since $\mathbb{P}^1 \sim_{SB} \ast$, Larsen and Lunts’ measure takes the class of the affine line to 0. In particular, it still may be the case that Kapranov’s motivic zeta is rational after inverting the affine line, or performing some other modification of $K_0(V_k)$ (cf. [LL04]). This underpins the “in the appropriate sense” above.

**Question 6.7.** Derived Larsen–Lunts. Does the Larsen–Lunts measure lift to a map of $K$-theory spectra?

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