Dynamical Topology Change in String Theory

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ABSTRACT

Exact string solutions are presented, providing backgrounds where a dynamical change of topology is occurring. This is induced by the time variation of a modulus field. Some lessons are drawn concerning the region of validity of effective theories and how they can be glued together, using stringy information in the region where the topology changes.

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1 Introduction, Results and Conclusions

In this work we shall present some exact solutions to string theory in 3+1 dimensions where *topology changes dynamically*. The number of exact solutions to string theory where the background fields indicate some interesting behaviour is not large, but we can however try to get to some conclusions from studying the exact solutions we posses. This is the first time we have a model where we can study dynamical topology change in a region where the curvature is strong (of the order of the Planck scale) and where the $a'$ expansion (or organization) of the effective field theory breaks down. However we can extend at the string level our description past the strong curvature region (where the topology change occurs) towards another asymptotic region where we have a different low energy field theory. The topology change is described by a modulus field that varies with time.

Studies on string theory have so far given hints for the presence of several interesting phenomena, like the existence of a minimal distance [1], finiteness at short distances [2], smooth topology change [3, 4], spacetime duality symmetries [5], spacetime dimensionality of spacetime [11], existence of maximal (Hagedorn) temperature and subsequent phase transitions [12] etc. The lessons we learn from the exact solutions we are going to describe essentially corroborate some items on the list above and we would like to present them in a somewhat general context.

We will start from the simple case of a compactification on a flat torus to indicate the idea and we will eventually translate for non-flat backgrounds. The spectrum of string physical states in a given compactification can be generically separated in three kinds of sub-spectra.

i) The first one consists of Kaluza-Klein-like effective field theory modes (or momentum modes). The masses of such modes are always proportional to the typical compactification scale $M_c = 1/\sqrt{a' R}$ where $R$ is a typical radius.

ii) Another set contains the winding modes which exist here because of two reasons. The first is that the string is an extended object. The second is that the target space has a non-trivial $\pi_1$ so the string can wind around in a topologically non-contractible way. In a small volume compact space, these modes are always super-heavy, since their mass is inversely proportional to the compactification volume $M_c$, (more precisely, it is proportional to $1/\sqrt{a' M_c}$).

iii) The third class of states are purely stringy states constructed from the string oscillator operators. Their masses are proportional to the string scale $M_{str} = 1/\sqrt{a'}$.

This separation of the spectrum is strictly correct in toroidal backgrounds. However, further analysis indicates that one can extend the notions of Kaluza-Klein (KK) type modes and winding modes to at least non-flat backgrounds with some Killing sym-

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*Ideas of a similar form have already been presented in [13].
†When there are more than one compact dimensions one can still define the concept of a scale [14].
metries. This generalization comes with the help of the duality symmetries present in such background fields \[\mathcal{B}\]. However, the “winding” states are not associated with non-contractible circles of the manifold. They appear as winding configurations in a (usually) contractible circle associated with a Killing coordinate \[\mathcal{B}\]. An example would be given by a winding in the Cartan torus of a group, which is however contractible inside the full group manifold. Because of the above, such modes are special combinations of zero modes as well as oscillator excitations. There may be dual versions of the model where such states are in fact associated with non-contractible circles of the dual manifold. We will see such a case in section 2. Although the above seems to apply to backgrounds with Killing symmetries, we feel that it might be more general. There are indications \[\mathcal{B}\] that one can have duality symmetries without the presence of isometries.

We will illustrate better the issues above in two cases. The first corresponds to Cartan deformations in a WZW model. There, although the Cartan torus is contractible, the role of windings and momenta are played by the eigenvalues of the left and right Cartan generators. The duality map here involves also the oscillators (that is, the currents). We will analyse an example of this sort in the rest of the paper. The second kind of behaviour concerns models of the \[SL(2, R)/U(1)\] type. There, one has zero modes and oscillator excitations (here we mean the non-compact parafermions), however, unlike the standard case, the energies associated to such oscillators are \(k\)-dependent. Duality interchanges zero modes with energies \(1/k\) with high lying modes with energies \(\sim k\).

Once we have the picture above concerning different types of string excitations we can state that the notion of a String Effective field theory makes sense when the “winding” states as well as the oscillator states are much heavier than the field theory-like KK states. Here, we will assume the presence of a single scale (except \(\alpha'\)). If there are more such scales then one has to investigate the different regimes. In any such regime, our discussion below is applicable.

Using the \(\alpha'\)-expansion one finds the effective theory which is applicable to low energy processes \((E_t < M_{st})\) among the massless states, as well as the lowest lying KK states, provided, that the typical mass scale \(M_c\), (which could be a compactification scale, gravitational curvature scale etc ) is much below the string scale \(M_{st}\).

In the last few years, a lot of activity was devoted in understanding the effective theories of strings at genus zero, \[\mathcal{B}\] and in some cases the genus one corrections were included \[\mathcal{B}\]. The output of this study confirms that the winding and the \(\mathcal{O}(M_{st})\) string superheavy modes can be integrated out and one can define consistently the string low energy theory in terms of the massless and lowest massive KK states. Thus, the perturbative string solutions are well described by classical gravity coupled to some gauge and matter fields with unified gauge, Yukawa and self interactions. As long as we stay in the regime where \(E_t, \tilde{M}_c < M_{st}\), our description of the physics in terms the the effective theory is good with well defined and calculable \(\mathcal{O}(E_t/M_{st}), \mathcal{O}(M_c/M_{st})\) corrections.

The situation above changes drastically once the mass of ”winding” type states becomes smaller than the string scale and when (usually at the same time) the KK modes
have masses above the string scale. This is the case when the typical $M_c$ scale is larger than $M_{st}$. When this occurs, the relevant modes are not any more the KK ones but rather the winding ones. Thanks to the well known by now generalized string duality [3]-[11],[4] (e.g. the generalization of the well known R to 1/R torroidal duality) it is possible to find an alternative effective field theory description by means of $\mathcal{O}(E_{\gamma}\frac{M_c}{M_{st}})$ dual expansion, where one uses the dual background which is characterized by $\tilde{M}_c$ instead of the initial one characterized by a high mass scale. The dual mass scale $\tilde{M}_c = \frac{M_{st}^2}{M_c}$ is small when the $M_c$ becomes big and vice-versa. We observe that in both extreme cases, either (i) $M_c < M_{st} < \tilde{M}_c$ or (ii) $M_c > M_{st} > \tilde{M}_c$, a field theory description exists in terms of the original curved background metric in the first case or in terms of its dual in the second case. This observation is of main importance since it extends the notion of the effective field theories in backgrounds with associated high mass scales (due to size or curvature).

Strictly speaking, there can be many dual backgrounds which correspond to the same string solution and it is an open problem if it is always possible to map all regions of spacetime with high associated scales to ones with small such scales. In fact, this is not always necessary, since even for regions which are strongly curved or have small volume, we can have a well defined effective description. Examples could be the region close to special symmetry points in torroidal compactifications, where although the torus has volume of order one, we can easily handle the low energy spontaneously broken gauge theory. Then, a general string solution would give rise to a set of effective field theories defined in restricted regions of space-time $(x_\mu)_I$, $I=1,2,3...$ with $M_I$ smaller or of the same order as $M_{st}$; If $T_{I,J}$ is the boundary region among $(x_\mu)_I$ and $(x_\mu)_J$, then on $T_{I,J}$ we have almost degenerate effective characteristic scales $M_I \sim M_{st} \sim M_J$. In such regions, the effective field theory description of regions $I, J$ break down (individually), and the full string theory is needed in order to have a smooth transition between the two. A goal in that direction would be to establish some simple rules that would provide the extra (stringy) information that would patch the field theoretic regions together. This effectively amounts to a reorganization of the $\alpha'$ expansion. The models we are proposing in this paper could very well serve as a laboratory towards answering this question. This is certainly important, since many interesting phenomena happen precisely at such regions in string theory. We can mention, the patching of dual solutions in cosmological contexts, [18] and global effective theories for large regions of internal moduli spaces.

A related issue here is, that with each region, one has an associated geometry and spatial topology, as dictated by the effective field theory. It turns out that moving from one region to another not only the geometry can change but also the topology. Examples in the context of Calabi-Yau compactifications [3] and more simple models, [4] have been given. There is another important point about topology change that we would like to stress here. Where topology change happens, depends crucially on the values of some of the parameters in the background. One such parameter is always $\alpha'$, but usually, in string

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[‡] Here by stringy we mean exact to all orders in $\alpha'$, or equivalently the full CFT description.

[§] In simple backgrounds, like torroidal ones this patching can be effectively done [26] by constructing an effective action, containing an infinite number of fields.
backgrounds, there are others, like various different levels for non-simple WZW and their descendant CFTs, various radii or related moduli etc. The absolute judge concerning topology change is the effective field theory.

As we will see in later sections, the solutions we will describe can be viewed as a time dependent background, where a modulus of space (or internal space) changes with time. At $t = 0$ we have a manifold with the topology of a disk times a line which evolves at $t = \infty$ to a manifold with the topology of a cylinder times a line. The topology changes in the intermediate (stringy) region. Of course, the backgrounds we are describing are time-dependent and thus, strictly speaking, we lose the meaning of energy (or mass in the compactified case). However, we will show that there are regimes where the time dependence is adiabatic, and where it make sense to use an approximate concept of energy or mass. Solutions with time dependent radii have been found in the past, solving the string equations to leading order in $\alpha'$. The advantage of the solutions we present is that they are exact string solutions (to all orders in $\alpha'$).

In this context, we will be able to demonstrate our general picture described above. One of the models considered, is $SL(2,R)_k$ or $SU(2)_k$ with its Cartan deformed. This deformation is parametrized by a new continuous parameter, $R$ such that at $R = 1$ we have the SU(2) model. This family of models was considered in [4] as a simple example of topology change. The topology change happens only at the boundary $R = 0, \infty$. Before we consider any time variation of $R$, let us illustrate a few features of the static model. The spectrum has the form [20]:

$$L_0 + \bar{L}_0 = 2j(j + 1) - \frac{m^2 + \bar{m}^2}{k} + \frac{1}{2k} \left[ (m - \bar{m} + kM)^2 R^2 + \frac{(m + \bar{m} + kN)^2}{R^2} \right] + N' + \bar{N}'$$

where $m, \bar{m} \in [0, 2k], M, N, N', \bar{N}' \in Z$. For $R \sim 1$ the low-lying spectrum is close to that of $S^3$. The effective field theory contains only the low-lying spectrum and the geometry is certainly that of $S^3$. However, for $R >> 1$, the low lying spectrum is dominated from the Cartan contributions and the geometry, from the effective field theory point of view is that of $S^1$. The leftover piece, which is that of $SU(2)_k/U(1)$ is strongly curved (for $k \sim \mathcal{O}(1)$) and thus not visible at low energy. Thus, we see that the change of effective field theory happens in the $R >> 1$ or $R << 1$ regions and is certainly not describable in the language field theory. In the half line $0 < R < \infty$, we need one effective field theory to describe the region for $R << 1$ and $R >> 1$ and another in-between.

The modulus $R$ can be made to be time dependent without destroying the exactness of the solution. Moreover as we show in the main text there are regions where this dependence is adiabatic. Thus, in the (euclidean) $SL(2,R)_k$ case, we can make the topology change picture described above, dynamical.

Another application of such solutions could be in considering strings at finite temperature. They would describe a string ensemble with temperature that varies (adiabatically)

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\*In [4] a variation was given where the topology change happens in the interior of the $\sigma$-model moduli space.
in space. It might be interesting to entertain such an idea in more detail, in order to
investigate temperature gradients in string theory. We will comment on this possibility
in the last section.

2 The time-independent model

We would like to construct exact classical solutions to string theory where some moduli
vary with time and their variation can induce topology change, in the sense presented in
the introduction. In order to do that, we will start from a solution where the modulus
is time independent, but arbitrary (alternatively speaking, there is no potential for it, in
the effective action). This will be the subject of this section where we will use the results
of [1]. In the next section we will proceed to include time dependent moduli.

The models we will analyse are related with the Cartan deformations of SU(2)_k or
SL(2)_k and its euclidean continuation

H^3. We will briefly describe the σ-model action for this radius deformation. For more
details we refer the reader to [1].

Although the deformation of SU(2)_k was originally found using O(2,2,R) deforma-
tions, [2] we will present a different approach [1], which has the advantage of showing
that the σ-model action and dilaton we will thus obtain, are exact to all orders in the
α′ expansion (in some scheme).

Conformal perturbation theory indicates that there is a line of theories obtained by
perturbing around the SU(2) or SL(2) WZW model by ∫ J bar J^3. The theories along the
line have a U(1)_L × U(1)_R chiral symmetry. Thus the σ-model action of these theories
must satisfy at least the following three properties:

1) It should have U(1)_L × U(1)_R chiral symmetry along the line.

2) It should have the group property: δS ∼ ∫ J bar J^3 at any point of the line.

3) At a specific point it should reduce to the known action of the SU(2) or SL(2)
WZW model.

4) The σ-model should be conformally invariant.

In [1] it was shown that properties (1-3) above imply that the σ-model action is:

\[ S(R) = \frac{k}{2\pi} \int d^2 z \left( \frac{(1 - \Sigma)\partial\theta\bar{\partial}\theta + R^2(1 + \Sigma)\partial\bar{\theta}\bar{\partial}\bar{\theta} + (1 + \Sigma)(\partial\bar{\theta}\bar{\partial}\theta - \partial\theta\bar{\partial}\bar{\theta})}{1 + \Sigma + R^2(1 - \Sigma)} + \partial x\bar{\partial}x \right) \]

(2.1)

The fact that O(2,2,R) transformations can produce marginal current-current perturbations infinitesimally was observed in [3].
where the angles $\theta, \tilde{\theta}$ take values in $[0, 2\pi]$. For $SU(2)$, $\Sigma = \cos 2x$ with $x \in [0, \pi/2] \cup [\pi, 3\pi/2]$ whereas for $SL(2, R)^{\ast \ast} \Sigma = \cosh 2x$ with $x$ real. $R$ is the parameter which varies along the line.

So far we have not imposed conformal invariance. At one-loop the $\beta$-function equations determine the dilaton, which so far was not fixed:

$$\phi(x, R) = \log\left[1 + \frac{1 - R^2}{1 + R^2} \Sigma(x)\right] + f(R). \quad (2.2)$$

where $f(R)$ is an arbitrary $R$-dependent constant. The only way (2.1) can change consistent with our requirements (1-3) is by a redefinition of $R$, which implies that there is a scheme in $\sigma$-model perturbation theory where the metric and the antisymmetric tensor receive no higher order corrections in $\alpha'$. In such a case also the dilaton receives no higher order corrections.††

The $R$-dependent constant in (2.2) can be fixed by the requirements that $\sqrt{G(R)}e^{\phi(R)}$ (which represents the physical string coupling) is invariant along the line. This implies that $f(R) = \log(R + 1/R)$ and that

$$\phi(x, R) = \log[(1 - \Sigma(x))R + (1 + \Sigma(x))/R] + \phi_0. \quad (2.3)$$

The constant $\phi_0$ in (2.3) is $R$-independent. When $R \to 0, \infty$, the dilaton has the appropriate asymptotic behaviour. This is an independent way of fixing the constant $R$-dependent part of the dilaton and the result coincides with the previous method.

Several observations are in order here, [4].

• For the line of theories above one has $R \to 1/R$ duality. If we parametrize $R$ around the WZW point $R = 1$ as $R = 1 + \epsilon + O(\epsilon^2)$ then the duality transformation becomes $\epsilon \to -\epsilon + O(\epsilon^2)$. This is equivalent to the Weyl invariance ($J^3, \bar{J}^3 \to -J^3, \bar{J}^3$) of the theory at $R = 1$. The fact that the remnants of the Weyl invariance can be retained far away is guaranteed by the fact the perturbation theory converges and it can also organized in order to preserve duality order by order. Non-perturbative effects (in $\alpha'$) exist only for $SU(2)$ and there a glimpse at the exact partition function [20] indicate that they don’t spoil the symmetry.

• At $R = 0$ the model becomes a direct product of a non-compact boson and the vector coset $SU(2)_k/U(1)_V$ (or $SL(2, R)^{\ast \ast}_k/U(1)_V$). Strictly speaking, the radius of $\tilde{\theta}$ becomes zero, but this is equivalent to a non-compact boson (as can be verified from the exact partition function in the $SU(2)$ case). There is also a constant antisymmetric tensor piece, but in the limit it can be safely dropped since it couples to a non-compact coordinate.

**By $SL(2, R)$ in this work we mean its Euclidean continuation, see for example [2].

††The dilaton $\beta$-function (central charge) does get corrections. This is what is happening also in the WZW model. However, like in that case, one can replace $k$ with $k + 2$ in front of the action. Then the central charge is given by the classical and 1-loop piece only, without spoiling the vanishing of the other $\beta$-functions.
• At $R = \infty$ the model becomes a direct product of a non-compact boson ($\theta$, this time) and the axial coset $SU(2)_k/U(1)_A$ (or $SL(2, R)_k/U(1)_A$).

• The three observations above imply that the axial and the vector cosets are equivalent CFTs.

We will look now at the geometry along the line, $[4]$. The $\sigma$-model metric in the case of deformed $SU(2)$ (in the coordinates $\theta, \tilde{\theta}, x$) is given by

$$G \sim k \begin{pmatrix}
\sin^2 x & \cos^2 x & R^2 \cos x & 0 \\
\cos^2 x + R^2 \sin^2 x & 0 & 0 & 0 \\
0 & R^2 \cos^2 x & \cos^2 x + R^2 \sin^2 x & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  \hspace{1cm} (2.4)

The scalar curvature $\hat{R}$ is

$$\hat{R} = \frac{2 - 5R^2 + 2(R^4 - 1)\sin^2 x}{k (1 + (R^2 - 1)\sin^2 x)^2}.$$ \hspace{1cm} (2.5)

The manifold is regular except at the end-points where

$$\hat{R}(R = 0) = -\frac{4}{k \cos^2 x}, \quad \hat{R}(R = \infty) = -\frac{4}{k \sin^2 x}.$$ \hspace{1cm} (2.6)

At $R = 1$ we get the constant curvature of $S^3$, $\hat{R} = 6/k$.

It should be noted that the geometric data (metric, curvature, etc.) are invariant under $R \rightarrow 1/R$ and $x \rightarrow \pi/2 - x$. Another interesting object is the volume of the manifold as a function of $R$ that can be computed to be

$$V(R) \sim \frac{R \log R}{R^2 - 1}$$ \hspace{1cm} (2.7)

satisfying $V(R) = V(1/R)$. The volume becomes singular only at the boundaries of moduli space, $R = 0, \infty$. To be more precise, it vanishes there, reflecting the vanishing of the radius of one of the angles (either $\theta$ or $\tilde{\theta}$). It is interesting to note though that at the limits, the model factorizes to a theory that has zero volume (corresponding to one of the angles) and another with infinite volume (the coset, this is due to the semiclassical singularity). The zero volume space dominates the infinite volume one. The “string” volume, $\int e^\phi \sqrt{G}$ is constant along the line.

The topology of the manifold is $S^3$ except at the end-points. Since one of the circles there shrinks to a point we have really a collapse of the manifold.

For $SL(2, R)$, the trigonometric functions in (2.3) are replaced by the corresponding hyperbolic functions. Here the manifold has a curvature singularity for $0 \leq R < 1$. Similar remarks apply to the Euclidean version, the 3-d hyperboloid. In this case the manifold has always infinite volume.

The $R$ marginal deformation generates a continuous family of CFTs that interpolate between two manifolds with different topology: the “cigar” shape ($R = \infty$) with the topology of the disk and the “trumpet” shape ($R = 0$) with the topology of a cylinder (this happens...
in the $H^3_+$ case). In the $SU(2)$ case, the $S^3$ group manifold is deformed to the direct product of a two disc and a large circle.

In [4] we have given also a continuous family of theories, where the topology change happens in the interior of moduli space. There, for example, an $S^3$ would evolve into a disk times a finite radius circle. We will not pursue further these backgrounds here.

### 3 Dynamical Topology Change

We would like to make the static picture described in the previous section dynamical, i.e. evolving in real time. In order to do that, we would need to add time into the $\sigma$-model (2.1). Thus, we will consider the following 4-d $\sigma$-model action:

$$S_{3+1} = S(R(t)) - \frac{1}{2\pi} \int d^2 z \partial t \bar{\partial} t - \frac{1}{8\pi} \int d^2 z \frac{R(2)\Phi(x,t)}{1 + \Sigma + R^2(t)(1 - \Sigma)}$$

$$= \frac{k}{2\pi} \int d^2 z \left( \frac{(1 - \Sigma)\partial \bar{\theta} \partial \tilde{\theta} + R^2(t)(1 + \Sigma)(\partial \bar{\theta} \partial \tilde{\theta} - \partial \tilde{\theta} \partial \bar{\theta})}{1 + \Sigma + R^2(t)(1 - \Sigma)} + \partial x \bar{\partial} x \right) - \frac{1}{2\pi} \int d^2 z \partial t \bar{\partial} t - \frac{1}{8\pi} \int d^2 z R(2)\Phi(x,t)$$

where $S_{1+2}(R(t))$ is the action in (2.1) but where the radius has (an unspecified for the moment) time dependence. We would like to find exact string solutions of the form (3.1). We would start by imposing conformal invariance at one-loop level. Later we will show that our one-loop solutions are in fact exact to all orders in $\alpha'$ (in a certain scheme).

The one-loop $\beta$-function equation imply

$$\frac{R'''}{R''} = \frac{R''}{R'} + \frac{R'}{R}$$

for $R(t)$

$$\Phi(x,t) = \log \left[ \frac{1 + \Sigma(x) + R^2(t)(1 - \Sigma(x))}{R'(t)} \right] + \text{constant}$$

for the dilaton and

$$\delta^{(1)}_{C} = -\frac{6}{k} + \frac{3}{2} \frac{R''}{R'^2} \left( \frac{2}{R' - \frac{R''}{R'}} \right)$$

for the one-loop correction to the central charge. It should be noted that (3.2) is invariant under $R(t) \rightarrow 1/R(t)$.

Equation (3.2) has three classes of solutions corresponding to flat space, $SU(2)/U(1)$ or $SL(2,R)/U(1)$ coset models and their duals. In more detail, (3.2) can be integrated to

$$R' = C_1 R^2 + C_2$$

$1/R(t)$ is a solution of the same equation with $(C_1, C_2) \rightarrow (-C_2, -C_1)$. The solutions are (up to shifts in $t$):
\[(i)\] \(C_1 = 0:\)
\[R^2(t) = C_2^2 t^2\]  \hspace{1cm} \text{(3.6)}

For \(t \in [0, \infty), \ R^2 \in [0, \infty).\)

\[(ib)\] \(C_2 = 0:\)
\[R^2(t) = \frac{1}{C_1^2 t^2}\]  \hspace{1cm} \text{(3.7)}

For \(t \in [0, \infty), \ R^2 \in [0, \infty).\)

\[(ii)\] \(C_1 C_2 > 0:\)
\[R(t) = \sqrt{\frac{C_2}{C_1}} \tan\left(\sqrt{C_1 C_2} t\right)\]  \hspace{1cm} \text{(3.8)}

For \(t \in [0, \pi/2\sqrt{C_1 C_2}], \ R^2 \in [0, \infty).\)

\[(iii)\] \(C_1 C_2 < 0:\)
\[R(t) = \sqrt{-\frac{C_2}{C_1}} \tanh\left(\sqrt{|C_1 C_2|} t\right)\]  \hspace{1cm} \text{and}  
\[R(t) = \sqrt{-\frac{C_2}{C_1}} \coth\left(\sqrt{|C_1 C_2|} t\right)\]  \hspace{1cm} \text{(3.9)}

Here, for the \textit{tanh} solution, \(R^2 \in [0, 1]\) whereas for the \textit{coth} solution \(R^2 \in [1, \infty).\)

For all of the above \(\delta (1)c = \pm \frac{6}{k} + 4C_1 C_2,\) where the \(-\) and \(+\) corresponds to SU(2) and SL(2,R).

We can also discuss here the adiabaticity conditions for the solutions above. The backgrounds we describe change with time so, strictly speaking there is no conserved energy. However if there is an adiabatic region where masses change slowly then one could still use the concept of energy to a good approximation. The adiabaticity condition, as we will show later on is \(|R'/R| \ll 1\) which using (3.5) becomes \(|C_1 R + C_2/R| \ll 1.\)

For (i) the adiabatic region is at \(t >> 1.\)

In case (ii) which corresponds to \(SU(2)/U(1)\) in the dual version, we obtain that \(|R'/R| \geq 2\sqrt{C_1 C_2}.\) Thus, there is always a lower bound in adiabaticity which tends to zero only if \(k \to \infty,\) where \(k\) is the central element of the \(SU(2)_k/U(1)\) theory.

Finally in case (iii) the adiabatic region is again \(t >> 1.\) In cases (ii) and (iii) the “size” of the adiabatic region is governed by \(|C_1 C_2|,\) in the sense that the adiabaticity condition implies \(\sqrt{C_1 C_2} t >> 1.\) We can blow-up that region by making \(|C_1 C_2|\) arbitrarily small. In fact, we can even arrange for the total system to have \(c = 4\) and still be able to blow up the adiabatic region.

As we will now show, the presence of the solutions (i-iii) is not accidental. If we do a duality transformation in (3.1) with respect to the \(\theta\) angle we obtain \(\tilde{S} = \tilde{S}_1 + \tilde{S}_2\) with

\[
\tilde{S}_1 = \frac{1}{2\pi} \int d^2 z \left[ -\partial t \bar{\partial} t + \frac{R^2(t)}{k} \partial \bar{\theta} \partial \theta \right] + \frac{1}{8\pi} \int d^2 z \ R^{(2)} \log[R'(t)]
\]  \hspace{1cm} \text{(3.10)}

\[
\tilde{S}_2 = \frac{k}{2\pi} \int d^2 z \left[ \partial x \bar{\partial} x + \frac{1 + \Sigma(x)}{1 - \Sigma(x)} \partial \phi \bar{\partial} \phi \right] - \frac{1}{8\pi} \int d^2 z \ R^{(2)} \log[1 - \Sigma(x)]
\]  \hspace{1cm} \text{(3.11)}
with $\phi = \tilde{\theta} + \theta/k$. Using this redefinition $\phi$ is an independent angle, so one might think that the two models are decoupled. This is not the case though due to the presence of $1/k$ in the redefinition. We will discuss this coupling in more detail below. $\tilde{S}_2$ describes the $SU(2)/U(1)_V$ or $SL(2,R)/U(1)_V$ coset.

Of course, in this “dual” formulation, we know the exact conformal field theory associated to $\tilde{S}_1$ corresponding to the three types of solutions found above. Case (i) is flat 2-d space (we could also have a linear dilaton but we will not entertain this further). Case (ii) corresponds to the Minkowski version of the $SU(2)_{k'}/U(1)$ coset model, (with $k' \sim 1/C_1C_2$). Finally case (iii) corresponds to the 2-d black-hole [23].

Another way to state what was said above is that there is an $O(2,2)$ transformation [25, 26] which maps the model (3.1) to a product of two 2-d target spaces.

Start from the action (3.1) and perform the following $O(2,2)$ transformation: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \quad c = \frac{1}{k} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad d = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.12)$$

We thus obtain (3.10,3.11) with an extra duality transformation on $\tilde{\theta}$ which gives instead of $\tilde{S}_1$:

$$\tilde{S}_1' = \frac{1}{2\pi} \int d^2z \left[ -\partial t \bar{\partial} t + \frac{k}{R^2(t)} \partial \bar{\partial} \tilde{\theta} \right] - \frac{1}{8\pi} \int d^2z R^{(2)} \log[R^2/R'] \quad (3.13)$$

The coupling between the original models is hidden here in the fact that the $O(2,2)$ matrix has fractional elements.

To summarize: in the dual formulation, the original theory corresponds to the product of two conformal fields theories, one being $SU(2)_{k}/U(1)$ and the other is one of the $SU(2)_{k}/U(1)$, $SL(2,R)_{k}/U(1)$, flat 2-d space or its dual. Thus, this proves that the original theory is an exact CFT.

As it is obvious from (3.11), the two CFTs corresponding to $\tilde{S}_1$ and $\tilde{S}_2$ are coupled. The type of coupling that exists is precisely the one which transforms a parafermionic theory times a free boson to an $SU(2)_{k}$ theory. This is a direct coupling of the parafermionic $Z_k$ charge to the translational $Z_k$ charge of the compact boson. In terms of the $\sigma$-model angles, a translation of $\tilde{\theta}$ by $2\pi$ induces a translation of $\phi$ by $2\pi/k$. This implies that the operators in the spectrum of the theory must have the mod-k residues of their angular momenta (corresponding to $\tilde{\theta}$ and $\phi$) the same. For scattering amplitudes, this just affects only the way one couples the operators of the two theories.

We can also write now the exact central charge of the model:

$$c = 2\frac{k + 1}{k - 2} + \frac{3 + 2C_1C_2}{3 - 4C_1C_2} \quad (3.14)$$

*In fact the solutions we describe are special lines in the general O(2,2) moduli space described in [26].
Observe that we can choose $C_1 C_2 = \frac{3}{2} (4 - k)$ so that the total central charge is exactly 4. This is important because, if the central charge differs substantially from four, we lose the clear four-dimensional interpretation, whereas if it close to four then the extra compactified theory will have many low lying states which again will interfere with those coming from the four dimensional part. When $c = 4$, the background admits N=4 superconformal symmetry \cite{24}. Observe also that we can have this extra symmetry, and at the same time, by choosing $k \rightarrow \infty$, be in the semiclassical regime, where time evolution is adiabatic for a long time.

There are also some global possibilities to be addressed. Let us first start from the Euclidean case ($t \rightarrow it$). There, we have two options: $SU(2)$ or $SL(2, R)$ (the Euclidean version, $H_3^+$). We will focus on $SU(2)_k$ in the sequel although using also $SL(2, R)$ might give some interesting cosmological models with a non-compact spatial slice. Concerning the time dependence of the radius $R(t)$, we have the three different cases, (i,ii,iii).

Once we go to the Minkowski case, there are several possibilities. We will consider first the case where $t$ is the time. If we look at the dual action (3.10) then it is obvious that there are two distinct possibilities for the coordinate $\tilde{\theta}$: that it has a finite radius, or that it is a non-compact coordinate. The former case is a suitable orbifold of the latter with respect to a discrete infinite abelian group.

There is another possibility in which $x$ is taken as time, by doing the $x \rightarrow ix$ continuation in the Euclidean case. For the possibilities (ii,iii) this model has a similar interpretation as before with $t \leftrightarrow x$.

We will study here the semiclassical picture of these solutions. If we consider the time-independent case, for the deformed $SU(2)_k$ theory at radius $R$, it is not difficult to show that the eigenfunctions of the Laplacian (specified by the metric (2.4) and the dilaton (2.3) as $\Box_3 \equiv \frac{\epsilon_{\mu\nu} \partial_{\mu} e^\Phi G_{\mu\nu}}{e^\Phi}$) are the same as those for SU(2), namely the standard D-functions, $D_{j,m,\tilde{m}}(x, \theta, \tilde{\theta}) = e^{i(m-\tilde{m})\theta/2 + i(m+\tilde{m})\tilde{\theta}/2} d_{m,\tilde{m}}(x)$. The energy ($L_0 + \bar{L}_0$) eigenvalues change though:

$$E_3(R) = \frac{1}{k} \left[ 2j(j + 1) - m^2 - \tilde{m}^2 + \frac{1}{2} (m - \tilde{m})^2 R^2 + \frac{1}{2} \frac{(m + \tilde{m})^2}{R^2} \right] + \mathcal{O}(1/k^2) \quad (3.15)$$

in agreement with the exact formula, (1.1) with $M, N, N', \tilde{N}' = 0$. If we now consider the four-dimensional case then we obtain that the 4-d stringy Laplacian is given as

$$\Box_4 = \frac{\partial^2}{\partial t^2} + \left( R'/R - R''/R' \right) \frac{\partial}{\partial t} + \Box_3 \quad (3.16)$$

Changing variables from $t \rightarrow R(t)$, and separating variables we obtain that the wavefunctions can be written in the form $F_{j,m,\tilde{m}}(R)D_{j,m,\tilde{m}}(x, \theta, \tilde{\theta})$ where $F(R)$ satisfies the following second order differential equation:

$$F'' + \frac{1}{R} F' + \frac{E_4 - E_3(R)}{(C_1 R^2 + C_2)^2} F = 0 \quad (3.17)$$

where $E_3(R)$ is given in (3.15) and we used (3.3). The behaviour of solutions varies for cases (i-iii). In case (i) it is easy to see from (3.17) that the spectrum has a continuum.
part (when either \( m \pm \bar{m} = 0 \) is valid) whose wavefunctions are Bessel functions as well as a discrete part whose wavefunctions are localized at \( t = 0 \).

For cases (ii,iii) the wavefunctions are hypergeometric function in the variable \( x = -C_1 R^2 / C_2, \ x \geq 0 \) corresponds to case (iii) while \( x \leq 0 \) to case (ii). The spectrum in both cases has continuous and discrete parts. These wavefunctions correspond to specific oscillator states in the dual coset model (3.10).

The semiclassical wavefunctions described above give a picture of the zero mode geometry which however does not reflect the behaviour of the whole spectrum. This can be seen by looking at the zero mode spectrum in the dual formulation (3.11). The Laplacian here factorizes into a sum of

\[
\frac{\partial^2}{\partial t^2} + \left( \frac{R'}{R} + \frac{R''}{R'} \right) \frac{\partial}{\partial t} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2}
\]

and the laplacian of the \( SU(2)/U(1) \) coset. Their wavefunctions though, are coupled as was mentioned earlier. Observe that although (3.16) is invariant under \( R(t) \rightarrow 1/R(t) \), (3.18) is certainly not.

In case (i), for example, we have almost the same wavefuctions on the spatial slice (although the \( d \)-functions are vector \( m = \bar{m} \)) but the time dependent part of the wavefunction has continuous spectrum only and behaves as a plane wave for large \( t \). This case has been analysed in detail in [9]. For case (ii) we obtain similar results as in the original version while in case (iii) again we have different zero mode spectrum.

For fixed time, the transverse spectrum of energies (or masses) behaves as \( E \sim m^2 / R^2 + n^2 R^2 \). The adiabaticity condition is essentially \( |\dot{E}/E| << 1 \) which translates to \( |\dot{R}(m^2 / R^2 - n^2 R^2)/R(m^2 / R^2 + n^2 R^2)| << 1 \) and eventually to \( |\dot{R}/R| << 1 \) as advocated earlier.

4 The Physical Interpretation

In this section we will discuss in more detail the physical interpretation of some of the models presented in the previous section.

A) Case (iii) time dependence which gives a model dual to \( SL(2, R)_{k'}/O(1, 1) \otimes SU(2)_{k}/U(1) \).

We have chosen \( C_1 = -C_2 = \sqrt{3/2k'} \) in order to have the standard \( SL(2, R) \) model.\[1\] Here in terms of the radial time dependence we have two branches (see (3.9)). However both of them participate in this Minkowski signature solution. The best way to see that is to write the dual model in light-cone like coordinates. The action (3.10) is precisely that of the 2-d black hole [23], with metric \( ds^2 = k'dudv/(1-u^2) \) and the two branches in (iii) are realized in regions I and V while there is case (ii), that is the Minkowski version of \( SU(2)/U(1) \) that appears in between, [27]. Thus the evolution of the radius has three

\[1\] Since \( C_2/C_1 \) determines the radius of \( \hat{\theta} \) we can go to other rational values by orbifolding.
patches. In the first (corresponding to region I), \( R(t) \sim \tanh t \). At \( t \to \infty \), \( R(t) \equiv 1 \). The effective string coupling is given by

\[
\begin{align*}
g_{\text{eff}}^2 &= e^{-\phi} = \frac{e^{-\phi_0} R'(t)}{1 + \Sigma(x) + R^2(t)(1 - \Sigma(x))} \tag{4.1}
\end{align*}
\]

where we used (3.3) and where \( \phi_0 \) is the arbitrary constant part of the dilaton. Thus, at \( t \to \infty \), \( g_{\text{eff}} \to 0 \) exponentially fast. At \( t \to 0 \), \( R(t) \sim t \) and the string coupling is finite and given by the (anisotropic) \( SU(2)/U(1) \) dilaton. At this point the time dependence is matched to case (ii) with \( t = 0 \) and continues till \( t \to \pi \sqrt{k'/6} \) where \( R(t) \sim \infty \) and \( g_{\text{eff}} \) is finite. The adiabatic period in this model is around the "horizon" in dual model. Finally, from that point on, the \( t \)-dependence is matched into the \( \coth \) branch at \( t = 0 \) with \( R(t) \sim 1/t \) and evolves till \( t \to \infty \), where \( R(t) \sim 1 \) and the coupling is again exponentially small.

As argued in the previous section, we will adjust \( k + k' = 4 \) so that the central charge is \( c = 4 \). If \( k \gg 1 \) then \( S^3 \) KK states are always low lying although the lowest part of the spectrum is dominated by the Cartan if \( R \gg 1 \) or \( R \ll 1 \). Thus, as we argued in the introduction, in the periods of time where the radius is either large or small

the low energy spectrum is dominated by the Cartan, whereas in between there is a stringy region. There is no topology change here.

In order to give a physical interpretation for these solutions we have to use the \( \sigma \)-model picture very carefully. Consider a period in time where \( R(t) \to 0, \infty \). As we have seen in section 2, one of the radii in the \( \sigma \)-model goes to zero and the spatial volume also goes to zero (qf. (2.7)). One would say that during the period the “universe” heads for a “big crunch”. However, from the low energy point of view things seem different. the inhabitants of this world see a large circular direction which expands (and becomes non-compact in the limit) but the excitations (momenta) of the rest of the spatial dimensions are large so that they effectively disappear at low energy. Thus it is a ”big squeeze” but for reason completely different from those coming naively from the \( \sigma \)-model (see also [28]). That is, directions we thought were tiny, are not, and those we thought are finite, are not visible. The effective string coupling in such periods becomes exponentially small.

Once now we are in the regions where \( R(t) \sim O(1) \) there is a unique low energy scale set by \( \alpha' \) (or equivalently \( k \)) and the low energy effective spectrum is that of a (deformed) \( S^3 \). The string coupling here is finite, space dependent and can be made almost everywhere arbitrarily small by adjusting \( \phi_0 \).

Thus we see that although the volume and \( \sigma \)-model geometry of the “universe” implies that we have a expanding or contracting universe, the low energy effective theory has a very different picture of it. These issues are very important when we try to construct stringy realistic cosmological models.

Instead, we could have the same dependence as above for \( R(t) \) but deforming \( H^3_+ \) now. Here we have an example of dynamical topology change. The change happens in the region around \( R \sim 1 \) as discussed in section 2. Similar remarks apply here as for the
case discussed above.

B) Case (ii) time dependence which gives a model dual to $SU(2)_{k}/U(1) \otimes SU(2)_{k′}/U(1)$ and its Minkowski continuation.

In this model the radius goes from $R = 0$ at $t = 0$ to $R = \infty$ at $t = \pi \sqrt{k′/6}$. Similar remarks like in case A apply here. The only difference is that here we have an oscillating “universe”. At $t << \sqrt{k′}$, $R(t) \sim t$, while around $t = \pi \sqrt{k′/6}$, $R(t) \sim 1/(t - \pi \sqrt{k′/6})$. Like (A), at the boundaries $R = 0, \infty$ the effective string coupling is finite and given by the dilaton of the SU(2)/U(1) model. In between the model has an effective $S^3$ topology, but unlike model (A) the effective string coupling does not vanish at $R = 1$ but is finite and constant there.

Although this model appears (from the $\sigma$-model geometry point of view) to correspond to an eternally expanding and recontracting universe, the real story is different, as in (A). The model starts out as a large $S^1$ while the rest of space is curled up and turns into a continuously deforming squashed $S^3$ until it comes back to its original squeezed state.

C) Case (ia,ib) time dependence which gives a model dual to $R^2 \otimes SU(2)_{k}/U(1)$. This case is qualitatively similar to (B) in the sense that the radius evolves from 0 to $\infty$. There is no periodicity in time however.

Another interpretation of the models above can be given. If one considers the $SU(2)_{k}$ as a part of the compactified dimensions of string theory, the model then provides a time-variation of the masses (and other data like couplings) of the particle spectrum. This would describe a higher dimensional universe at $t = 0$, where some of its dimensions compactify as time go by, until their scale becomes of the order of the Planck scale (this would correspond to either the $tanh$ or the $coth$ solution (case (iii)). This is a very interesting cosmological solution which in our opinion deserves further study.

The stringy cosmological models presented above seem to have some features which are not compatible with standard cosmological scenarios (due to anisotropy etc). It may be though, that isotropy is something that is achieved later in the history of the universe. In particular, consider the product of flat 2-d space times the $SL(2, R)_{k}/O(1, 1)$ coset. In the first of ref. [29] it was shown that for large times the spacetime was isotropic. The dual version of this model would give a universe where the Cartan radius $R$ of $SL(2, R)$ changes with $t$ as $R \sim t$. This would mean that there are particles in this theory which see asymptotically an isotropic space.

The solutions we presented have some extra interesting features, apart from the fact that being exact string solutions they can give us a chance to understand stringy phenomena in the context of cosmology. In particular, although anisotropic inflation has not been analysed so far we can remark that what plays here the role of the scale factor is $R(t)$ and there are regimes where it satisfies the usual conditions for inflation, $R′, R′′ > 0$, namely, cases (ib,ii) and the $tanh$ branch of (iii). It remains however to be seen if such

\footnote{It is very similar to the model presented in [29], and is in fact in the same $O(2, 2, R)$-moduli space of the $\sigma$-model [24].}
conditions remain valid in the presence of anisotropy. Many interesting isotropic $\sigma$-model solutions exist, however we don’t know at present if they can be made exact solutions. This is a serious drawback since some of the interesting physics is supposed to happen at strong curvatures where our confidence on the solution breaks down, [18].

As was mentioned in the introduction, another interesting application of the Euclidean version of (3.10,3.11), is the identification of the angle $\theta$ with Euclidean time. In this case $R(t)$§ plays the role of the inverse temperature (Matsubara interpretation) which varies spatially. In this context, we have a background describing a thermal ensemble of strings with a spatially non-uniform temperature. More precisely there is a temperature gradient in the $t$ direction. Studying further this kind of solutions might give us useful information about Hagedorn-like instabilities and phase transitions in some regions of space.

Finally, some generalizations of the solutions above can be envisaged using the same ideas as those discussed in this paper. One would be higher dimensional generalizations of the above providing interesting cosmological solutions where some non-compact dimensions compactify. The other, would be to find all paths inside $O(d,d)$ which provide exact string solutions. Looking at the boundaries of $O(d,d)$ moduli space may be the hint for finding such solutions.

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§Remember that now $t$ is a spatial coordinate.
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