The Gauss map of a complete minimal surface with finite total curvature

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Abstract

In [15] Robert Osserman proved that the image of the Gauss map of a complete, non flat minimal surface in \( \mathbb{R}^3 \) with finite total curvature miss at most 3 points. In this paper we prove that the Gauss map of such a minimal immersions omit at most 2 points. This is a sharp result since the Gauss map of the catenoid omits exactly two points. In fact we prove this result for a wider class of isometric immersions, that share the basic differential topological properties of the complete minimal surfaces of finite total curvature.

1 Introduction

Let \( M \) be a complete minimal surface in \( \mathbb{R}^3 \) with Gauss map \( G: M \rightarrow S^2 \). It is well known that \( G \) is a holomorphic map from \( M \), viewed as a Riemann surface, to the Riemann sphere \( S^2 \). A classical problem in minimal surface theory was to know which results from the classical complex function theory remain true for the Gauss map. For instance, is there a Picard theorem for the Gauss map? L. Nirenberg conjectured that the Gauss map of a complete minimal surface \( M \) of \( \mathbb{R}^3 \) can not omit a neighbourhood of a point unless \( M \) is a plane. Nirenberg’s conjecture, a generalization of Bernstein’s theorem, also viewed as a weak form of Picard’s theorem, was settled in the affirmative by R. Osserman [14], [15]. Let \( Y = S^2 \setminus G(M) \) be the complement of the image of the Gauss map and let \( \sharp Y \) be its cardinality. Osserman showed that

i. If \( M \) has infinite total curvature then every point of \( S^2 \setminus Y \) is covered infinitely many times and \( Y \) has capacity zero, [15, Cor. p. 356].

ii. If \( M \) has finite total curvature then it is conformally equivalent to a compact Riemann surface \( \overline{M} \) of genus \( g \) minus a finite number of points \( p_1, \ldots, p_m \) and the Gauss map \( G \) extends to a branched covering map \( G: \overline{M} \rightarrow S^2 \), [15, Thm. 3.1].

iii. If \( M \) has finite total curvature then \( \sharp Y \leq 3 \) unless \( M \) is a plane and if \( \sharp Y = 3 \) then \( M \) has genus \( g \geq 1 \) and the total curvature

\[
\iint_M K \, d\mu \leq -12\pi
\]

see [15, Thm. 3.3 & 3.3A].
Osserman’s result was greatly improved by F. Xavier [18] who showed that, in general, \( \sharp Y \leq 6 \). Then H. Fujimoto [5] proved the sharp upper bound \( \sharp Y \leq 4 \) answering fully Nirenberg’s conjecture.

Regarding the finite total curvature case, item ii., it seems that most likely the Gauss map omits only two points. This would be a sharp bound since the Gauss map of the catenoid omits exactly two points. More evidences for this claim can be found in [19], where A. Weitsman and F. Xavier showed that if \( \sharp Y = 3 \) then

\[
\iint_M Kd\mu < -16\pi,
\]

followed by Y. Fang [4], who proved that the Gauss map of a non-flat minimal surface with finite total curvature omits at most two points unless

\[
\iint_M Kd\mu < -20\pi.
\]

It should be noted that L. Barbosa, R. Fukuoka and F. Mercuri [1] considered a class of surfaces that share the basic differential topological properties with minimal surfaces of finite total curvature and they proved, under some non-degeneracy conditions, that the Gauss map of those surfaces omits at most two points.

In this paper we will settle Osserman’s problem, as a corollary of our main result, Theorem 2. We will prove the following

**Theorem 1.** Let \( M \subset \mathbb{R}^3 \) be an immersed complete minimal surface with finite total curvature. Then its Gauss map omits at most two points unless \( M \) is a plane.

It should be remarked that R. Miyaoka and K. Sato [11] constructed examples of complete minimal surfaces in \( \mathbb{R}^3 \) with finite total curvature and Gauss map missing two non-antipodal points.

In order to describe better the settings we will be considering in the rest of this paper, let us recall some basic facts about finite total curvature minimal surfaces. If \( M \) is an oriented complete non-flat minimal surface of \( \mathbb{R}^3 \) with finite total curvature, then

a. \( M \) is conformally equivalent to \( \overline{M} \setminus E \) where \( \overline{M} \) is a compact Riemann surface and \( E = \{ w_1, \ldots, w_m \} \subset \overline{M} \) is a finite set of points.

b. The Gauss map \( G: M \to S^2 \) extends to a smooth map, denoted by the same symbol, \( G: \overline{M} \to S^2 \).

c. The map \( G: \overline{M} \to S^2 \) is a branched covering map.

A point \( w \in E \) or, sometimes, a punctured neighbourhood of \( w \), is called an end. Since the Gauss map is defined at such a point \( w \in E \), we have a tangent plane at \( w \), namely \( G(w)^{\perp} \). It follows from the above properties that the image of the immersion of a small punctured neighbourhood of \( w \) projects (orthogonally) onto the complement of a disk in \( G(w)^{\perp} \) as a finite covering map of order \( I(w) \). The number \( I(w) \) is called the geometric index of \( w \), see

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1The Gauss map of Scherk’s surface omits four points.
Since the branching points of $G$, i.e. the points of zero Gaussian curvature and, possibly, the ends, are isolated, a punctured neighbourhood of such a point $v$ is mapped onto its image as a covering map of order $\nu(v)$. The number $\beta(v) = \nu(v) - 1$ is called the branching number at $v$. Observe that if $v$ is not a branching point then $\beta(v) = 0$. The following topological relations are well known

\[(\text{RH}) \quad \text{Riemann-Hurwitz:} \quad -2\deg(G) = \chi(\overline{M}) + \sum_{w \in \overline{M}} \beta(w),\]

\[(\text{TC}) \quad \text{Total curvature:} \quad -2\deg(G) = -\chi(\overline{M}) + \sharp E + \sum_{i=1}^{m} I(w_i),\]

where $\chi(\overline{M})$ is the Euler characteristic of $\overline{M}$, $I(w_i)$ is the geometric index of $w_i \in E$ and $\deg(G)$ is the degree of the Gauss map $G$.

By a suitable combination of these two relations, Osserman [15] proved that $\sharp Y \leq 3$.

**Remark 1.1.** The Riemann-Hurwitz relation, also known as Hurwitz’s theorem, is a general fact in covering space theory. The total curvature relation was first obtained by Osserman [15] as an inequality, using complex analysis and the Weierstrass representation formula for minimal surfaces. The precise relation (equality) and the properness property of the immersion was established by L. Jorge and W. Meeks [8] for submanifolds of $\mathbb{R}^n$. Thus the proof of Osserman’s upper bound $\sharp Y \leq 3$ is of a topological nature and depends only on the properties a., b., c. of the Gauss map and not on the minimality of the immersion.

The two relations above seem to be not enough to find the best number of possible missing points. We will consider the immersion, in some elected position, and project it orthogonally over a fixed horizontal plane. Then new topological relations come up involving the Euler characteristic and the number of cusp points. All these relations together imply that for a wide class of immersions, including the minimal immersions with finite total curvature, it is not possible to have a Gauss map missing more than two points, unless the surface is a plane.

The structure of this paper is as follows. First we set the basic facts about singularities of smooth maps. We follow H. Whitney [20] who introduced the class of excellent maps $F: M^2 \to \mathbb{R}^2$ (see Section 2). We start with a subset of excellent maps, those where $M^2$ is complete analytic isometrically immersed into $\mathbb{R}^3$ with Gaussian curvature $K \leq 0$ and finite total curvature and the Gauss map extending smoothly to a map from $\overline{M}$ to the sphere $S^2$. The admissible maps $F: M \to \mathbb{R}^2$ are these orthogonal projections over the horizontal plane having the singular set without ends or zero Gaussian curvature (see Section 3). We call the class of such immersions $\mathcal{AE}$. The class $\mathcal{AE}^*$ is the closure of this family in the $C^3$ compact-open topology (see Section 3). It turns out that the Riemann-Hurwitz relation and the total curvature formula hold for all elements of $\mathcal{AE}^*$. When we project one of those immersions over the horizontal plane and count the number of cusp points, (see Section 4), some new relations appear. In
fact one of them involving the Euler characteristic and cusp points generalizes those of R. Thom [16, Thm. 9], A. Haefleger [6], H. I. Levine [9] and many others. We should point out that to count the cusp points the work of Whitney [21] plays an important role (see Section 4.3). A suitable combination of all of those relations yields that $\sharp Y \leq 2$ (see Section 5). The Theorem 1 follows from the following general result.

**Theorem 2.** For all $f \in \mathcal{AE}^*$ the Gauss map $G$ omits at most two points.

Finally, in Section 6, we will comment on some problems that seem to be important to the theory.

## 2 Singularities of smooth maps

In this section we will recall some basic facts about singularities of smooth maps between surfaces following closely H. Whitney [20]. Let $F: M_1 \to M_2$ be a smooth map between two Riemannian surfaces and let $\omega_i$ be the volume form of $M_i$. The Jacobian of $F$ is the map $J = J(F): M_1 \to \mathbb{R}$, characterized by

$$F^*\omega_2 = J\omega_1$$

where $F^*\omega_2$ is the pull back of the volume form $\omega_2$.

**Definition 2.1** (Whitney, [20]). Let $f: M_1 \to M_2$ be a differentiable map between surfaces.

- A singular point of $F$ is a point $p \in M_1$ such that $J(p) = 0$. The set of singular points of $F$ will be denoted by $\Sigma$.
- A point $p \in M_1$ is a good point if $p \notin \Sigma$ or $p \in \Sigma$ and $\nabla J(p) \neq 0$.
- $F$ is a good map if all points in $M_1$ are good points.

Clearly $dF(p)$ has rank 2 if $p \notin \Sigma$.

**Lemma 2.2** (Whitney, [20] Lemma 4a & 4b). If $F$ is a good map then $\Sigma$ is a regular curve and $\text{rank}(dF) \geq 1$.

**Example 2.3.** Let $f: M \to \mathbb{R}^3$ be an isometric immersion, $P: \mathbb{R}^3 \to \mathbb{R}^2$ be the orthogonal projection $P(x) = x - \langle x, e_3 \rangle e_3$ and let $F: M \to \mathbb{R}^2$ be given by $F = P \circ f$. Let $G: M \to S^2$ be the Gauss map of $M$ and let $A: TM \to TM$ be the shape operator. Take isothermal parameters $f(u,v)$ in $M$. Then we have

$$F^*(dx_1 \wedge dx_2) = df_1 \wedge df_2 = (f_{1u}f_{2v} - f_{1v}f_{2u})du \wedge dv = G_3\omega.$$ (1)

Hence

$$J = G_3 = \langle G, e_3 \rangle$$

Now $\nabla J = -Ae_3^\top$ where $e_3^\top$ is the orthogonal projection of $e_3$ over the tangent space. Since $e_3^\top = e_3$ over $\Gamma$ we get

$$|\nabla J| = \frac{1}{\lambda} \sqrt{|G_{3u}^2 + G_{3v}^2|} = |A(e_3)|$$ (2)

Then if the Gaussian curvature $K$ of $M$ satisfies

$$K(p) \neq 0, \quad \text{for all } p \in \Sigma$$ (3)

the map $F$ is a good map.
Definition 2.4 (Whitney,[20]). Let $F : M_1 \to M_2$ be a good map and let $t \mapsto \Gamma(t)$ be a local regular parametrization of $\Sigma$.

- A point $p = \Gamma(t)$ is a folder point if $\Gamma'(t) \notin \ker(dF)$.
- A point $p = \Gamma(t)$ is a singularity of second order if $\Gamma'(t) \in \ker(dF)$.
- A point $p = \Gamma(t)$ is a cusp point if it is a singularity of second order and, in addition,

$$\nabla_{\Gamma'} \nabla_{\Gamma'} F \circ \Gamma(t) \neq 0.$$

Lemma 2.5 (Whitney,[20]). Let $F : M_1 \to M_2$ be a good map $p \in \Sigma$. Then, in suitable coordinates of neigbourhoods of $p$ and $F(p)$, we have

$$F(u,v) = (u^2,v) \text{ if } p \text{ is a folder, } F(u,v) = (uv - u^3,v) \text{ if } p \text{ is a cusp.}$$

In particular cusp points are isolated.

Definition 2.6. Let $F : M_1 \to M_2$ be a smooth map. We say that

- $F$ is excellent if all singularities of second order are cusps,
- $F$ is almost excellent if the set $C$ of singularities of second order does not have accumulation points.

Remark 2.7. Since cusp points are isolated, an excellent map is almost excellent.

Let $C$ be the set of all singularity of second order of an orthogonal projection $F : M \to \mathbb{R}^2$ of some isometric immersion $f : M \to \mathbb{R}^3$.

Lemma 2.8. Assume that $f$ is analytic and $F = P \circ f$ is good. Let $\Gamma$ be a connected component of the singular set of $F$. If $C \cap \Gamma$ contains accumulation points, then $F \circ \Gamma$ is constant and $\Gamma$ is a straight line. In addition if $\Sigma$ is compact then all singularity of second order are isolated.

Example 2.9. Consider an analytic isometric immersions $f : M \to \mathbb{R}^3$ where $M$ is conformally equivalent to a compact Riemann surface $\overline{M}$ minus a finite set $E$ of ends with the Gauss map $G : M \to S^2$ extending continuously to $\overline{M}$. Set $F = P \circ f$ for the orthogonal projection of $f$ over the horizontal plane and denote the shape operator by $A : TM \to TM$ and the Gaussian curvature by $K(p) = \det(A_p)$. The singular set of $F$ is given by

$$\Sigma = G^{-1}(S^1).$$

It follows from [8] that $f$ is proper and the shape operator $AX = -D_X G$ is bounded, that is,

$$\|A\|_{\infty} = \sup_{p \in \overline{M}}\|A(p)\| < \infty.$$ 

If $|\epsilon| < \epsilon_o$ where

$$\epsilon_o = \|A\|_{\infty}^{-1},$$

then $M_\epsilon : f_\epsilon = f + \epsilon G$ is an immersion with the same Gauss map $G_\epsilon = G$. Hence the singular set $\Sigma_\epsilon$ is the same or more precisely $\Sigma_\epsilon = f_\epsilon(\Sigma)$. The number of connected components in both cases are equal. We also have that $M_\epsilon \to M$ by
parts in any $C^k$ topology for $k \geq 3$ and the geometric index of one end in both cases are equal (see [8]). If $G(E) \cap S^1 = \emptyset$, outside a solid cylinder $B^2_R \times \mathbb{R}$ each end of the hole family is a multi graph over $\{x \in \mathbb{R}^2 \mid |x| \geq R\}$ with fibre the geometric index of the end. There are no singularities of the hole family in this region. Complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature fits this conditions.

In addition to the above conditions we assume that

(i) $K(p) \neq 0$, for all $p \in \Sigma$,

(ii) $G(E) \cap S^1 = \emptyset$.

From example(2.3) we have that $F = P \circ f$ is a good map. Let $p \in \Sigma$ be a singularity of second order and $\Gamma(t)$ one arc length parametrization with $\Gamma(0) = p$. We have

$$\langle G(t), e_3 \rangle = 0, \quad G(t) = G \circ \Gamma(t), \quad \Gamma \in \Sigma$$

and $e_3 \in T_pM$ for all $p \in \Sigma$. This implies

$$\langle A\Gamma', e_3 \rangle = 0$$ (4)

The singularity of second order are the points $\Gamma(t)$ where

$$0 = \nabla_{\Gamma'} F = \Gamma'(t) - \langle \Gamma'(t), e_3 \rangle e_3$$

or

$$\Gamma'(t) = \langle \Gamma'(t), e_3 \rangle e_3.$$

Then $\Gamma'(t)$ is an asymptotic direction. If the singularity of second order is degenerated then

$$0 = \nabla_{\Gamma'} \nabla_{\Gamma'} F = \Gamma'' - \langle \Gamma'', e_3 \rangle e_3,$$

Since $\Gamma'(0) = \pm e_3$ it follows that a second order singularity is degenerated if and only if

$$\Gamma''(0) = 0$$ (5)

Follows from (i) and (ii) and lemma(2.8) that all singularity of second order of $F$ are isolated. If $\Gamma(0)$ is a degenerated second order singularity of $F$ then $\Gamma''(0) = 0$ and

$$\Gamma''(0) = \Gamma''(0) + \epsilon G''(0) = -\epsilon (\nabla_{\Gamma'} A\Gamma' + |A(e_3)|G) \neq 0, \quad \text{for all } 0 < |\epsilon| \leq \epsilon_0$$

Since $|A(e_3)| \neq 0$ it follows that the parametrization by arc length of $\Gamma_\epsilon(t)$ has no zero curvature at the origin. Then $F_\epsilon = P \circ f_\epsilon$ are excellent maps for $0 < |\epsilon| < \epsilon_0$.

**Remark 2.10.** Rotations of the catenoid give examples of good maps whose orthogonal projection is not excellent beside orthogonal projection of parallel surfaces are excellent. Consider the parametrization of the catenoid given by

$$f(t, u) = (\cos(t) \cosh(u), \sin(t) \cosh(u), u) \quad 0 \leq t \leq 2\pi, \quad u \in \mathbb{R}.$$
We rotate $f$ in the plane $[e_2,e_3]$ from $e_3$ to $e_2$ of an angle $\delta$, $0 \leq \delta \leq \pi/2$ to get a new immersion $f_\delta$ given by

$$f_\delta(t,u) = \left( \cos(t) \cosh(u), \cos(\delta) \sin(t) \cosh(u) + u \sin(\delta), -\sin(\delta) \sin(t) \cosh(u) + u \cos(\delta) \right).$$

Let $\delta_o$ be the angle where one of the lines tangent to both catenaria of the plane $[e_2,e_3]$ becomes vertical, that is, $\delta_o$ is solution of the equation

$$\delta = \frac{\cosh(\delta)}{\sinh(\delta)}. \quad (7)$$

Then we have the following situations:
1. $F_{\delta_o} = P \circ f_{\delta_o}$ are excellent for $0 < |\epsilon| < \epsilon_o$,
2. $F_{\delta_o}$ is not excellent and $F_{\delta_o}(\Sigma(f_{\delta_o}))$ is a segment of line,
3. $F(\Sigma_{\delta_o})$ are convex curves of the plane for $0 \leq \delta < \delta_o$ or $\delta = \delta_o$ and $0 < \epsilon < \epsilon_o$,
4. $F(\Sigma_{\delta_o})$ has exactly 4 cusp points for $\delta_o < \delta < \pi/2$ or $\delta = \delta_o$ and $-\epsilon_o < \epsilon < 0$.

## 3 The class $\mathcal{AE}$, $\mathcal{AE'}$ and $\mathcal{AE}^*$

### 3.1 The class $\mathcal{AE}$ and $\mathcal{AE'}$

Let $f: M \to \mathbb{R}^3$ be an isometric immersion of a Riemannian surface with Gauss map $G: M \to S^2$ and let $A: TM \to TM$ be the shape operator, given by $AX = -D_XG$. The Gaussian curvature at $p$ is then given by $K(p) = \det(A_p)$. Let $F: M \to \mathbb{R}^2$ be $F = P \circ f$ where $P: \mathbb{R}^3 \to \mathbb{R}^2$ is the orthogonal projection. The map $F$ has singular set $\Sigma = G^{-1}(S^1)$ where

$$S^1 = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1\}.$$

We will be concerned in triples $(M,f,F)$ of isometric immersions $f: M \to \mathbb{R}^3$ with orthogonal projection $F = P \circ f$ that satisfy the following conditions:

- **(AE.1)** $M$ is complete, analytic and $f: M^2 \to \mathbb{R}^3$ is analytic.
- **(AE.2)** $M^2$ is conformally equivalent to a compact surface $\overline{M}$ minus a finite set $E = \{w_1, \ldots, w_m\}$, the Gaussian curvature $K(p) \leq 0$ and $K(p) = 0$ at most in a finite set of $M$.
- **(AE.3)** The Gauss map of $M$ extends to a smooth branched recovering map $G: \overline{M} \to S^2$.
- **(AE.4)** The Gaussian curvature $K(x) \neq 0$ for all $x \in \Sigma \cup G^{-1}(e_3)$.
- **(AE.5)** $G(w) \notin S^1 \cup \{e_3\}$ for all $w \in E$. 


**Definition 3.1.** We will call the class of triples with the above properties $\mathcal{AE}$.

We will see that if $(M, f, F) \in \mathcal{AE}$ then $F$ is almost excellent. Will be important to consider the following class

**Definition 3.2.** The set of triples $(M, f, F) \in \mathcal{AE}$ such that $F$ is excellent will be denoted by $\mathcal{AE}'$.

The conditions (AE.1), (AE.2), (AE.3) imply that $f$ is a proper map, has finite total curvature and for any sequence of positive one loop curves going to one infinite point $w \in E$ the integral of the geodesic curvature converges to a number $2\pi I(w)$ where the positive integer $I(w)$ is geometric index at $w$ (see [8]). In particular the total curvature formula holds for all $f \in \mathcal{AE}$ where the degree of $G: \overline{M} \to \mathbb{S}^2$ is given by

$$
\text{deg}(G) = \frac{1}{4\pi} \int_M K dM
$$

where $K$ is the Gaussian curvature of $M$. In general if one split $M$ as $M_\pm = G^{-1}(S^2_\pm)$ where $S^2_\pm$ are the points $x \in \mathbb{S}^2$ such that the third coordinate $x_3 \geq 0$ or $x_3 \leq 0$ respectively then the degree is the total curvature divided by $2\pi$ and the total curvature formula holds. Following [8] we get the next result.

**Proposition 3.3.** If $(M, f) \in \mathcal{AE}$ and $\Omega$ is a connected component of $\overline{M}_\pm$ then

$$
-\text{deg}(G|\Omega) - \frac{1}{2\pi} \int_{\partial \Omega} k_g = -\chi(\overline{\Omega}) + \sharp(E \cap \overline{\Omega}) + \sum_{E \cap \overline{\Omega}} I(w) \quad (9)
$$

In particular if $n = \text{deg}(G)$ the total curvature formula holds:

$$
\begin{align*}
\frac{1}{2\pi} \int_{\partial M_+} k_g &= -\chi(\overline{M}_\pm) + \sharp(E_\pm) + \sum_{w \in E_\pm} I(w) \quad (10) \\
2n &= -\chi(\overline{M}) + \sharp(E) + \sum_{w \in E} I(w) \quad (11)
\end{align*}
$$

where $E_\pm = E \cap \overline{M}_\pm$.

The condition (AE.4) implies that $F$ is good. In fact, taking a point $p \in \Sigma$ as origin and writing down a neighborhood of $p$ at $M$ as a graph of $\varphi(u, v)$ with $G(p) = e_1$ the Jacobian of $F$ is given by the partial derivative $\varphi_\nu$ and the Gaussian curvature by $K(p) = \det H_\varphi(p)$, where $H_\varphi$ is the Hessian of $\varphi$. One have

$$
\nabla J = (\varphi_\nu, \varphi_\nu)
$$

and $K(p) \neq 0$ implies $\nabla J(p) \neq 0$.

The last condition (AE.5) implies that the singular set is a finite union of compact Jordan curves. Comes up that each connected component of $\Sigma$ has at most a finite number of singularity of second order since $M$ is analytic. All of them are isolated singularity of second order. Thus all $f \in \mathcal{AE}$ has the orthogonal projection $F$ as almost excellent map.

Take $f \in \mathcal{AE}$ and define $h: \overline{M} \to \mathbb{R}$ by $h(x) = \langle e_3, G(x) \rangle$, $x \in \overline{M}$. The gradient of $h$ is given by $\nabla h(x) = -Ae_3^\perp$, $x \in \overline{M}$. Then a critical point of $h$ is given by $G(x) = e_3$ or $K(x) = 0$. 

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Definition 3.4. The critical points of $\nabla h(x) = 0$ with $K(x) = 0$ are the branching points of $G: \overline{M} \to \mathbb{S}^2$ and the order $\beta(x)$ is the index of $\nabla h$ at $x \in \overline{M}$.

If we calculate the index of $\nabla h$ at some no critical point $x$ we get $\beta(x) = 0$. Then we extend $\beta: \overline{M} \to \mathbb{Z}$ to preserve the classical notation.

By the Sard’s theorem one can move $M$ if necessary to get $h$ a Morse function such that there is no end or branching point $x$ with $G(x) = e_3$. Take a connected component $\Omega$ of $\overline{M}$. Then $h|\partial \Omega \equiv 0$ and we can complete $\Omega$ to a compact topological surface adding $k$ disks where $k$ is the number of connected component of $\partial \Omega$. Identifying each disk as a point we get a local maximum or minimum of $h$ according $\nabla h$ points outside or inside $\Omega$. Observe that each critical point $x \in E$ is a branching point. Isolating each branch point of $G$ by small disks and counting the index of the singularity we get the Euler characteristic of $\Omega$. The Euler characteristic is the same for small motion of $M$. We conclude

Proposition 3.5. Let $(M, f) \in \mathcal{AE}$ and $\Omega$ a connected component of $\overline{M}$. Then

$$\chi(\Omega) = n - \sum_{w \in \Omega} \beta(w)$$

(12)

In particular we have

$$n = \chi(M) + \sum_{z \in M} \beta(z)$$

(13)

$$2n = \chi(M) + \sum_{z \in M} \beta(z)$$

(14)

3.2 The topology in the space of functions: The class $\mathcal{AE}^*$

Let us introduce the space of functions where our results work. Fix a compact Riemann surface $\overline{M}$, a finite set of ends $E \subset \overline{M}$ and $M = \overline{M} \setminus E$. Let $C^*(\overline{M}, E)$ be the space of immersions $f: M \to \mathbb{R}^3$ that are complete in the induced metric and the Gauss map $G: M \to \mathbb{S}^2$ extends to a continuous map from $\overline{M}$ to $\mathbb{S}^2$. The topology of $C^*(\overline{M}, E)$ is given by the family of semi metrics

$$d_K(f_1, f_2) = \sup_{x \in \overline{M}} d_{\mathbb{S}^2}(G_1(x), G_2(x)) + \|f_1 - f_2\|_{C^3, K}$$

where $G_j$ is the Gauss map of $f_j$, $K$ is a compact set of $M$ and $\|\cdot\|_{C^3, K}$ is the usual $C^3$ metric take over $K$. Let $C^*$ be the class of all $C^*(\overline{M}, E)$ for all $\overline{M}$ and $E$ possible with the topology given by the family of semi metrics

$$d_K(f_1, f_2) = \begin{cases} 
1, & \text{if } \overline{M}_1 \neq \overline{M}_2 \text{ or } E_1 \neq E_2, \\
\sup_{x \in \overline{M}} d_{\mathbb{S}^2}(G_1(x), G_2(x)) + \|f_1 - f_2\|_{C^3, K}, & \text{if } \overline{M}_1 = \overline{M}_2 \text{ and } E_1 = E_2.
\end{cases}$$

where $K$ is a compact set of one of surfaces $M_j \setminus E_j$, $j = 1, 2$.

Definition 3.6. The topological space $\mathcal{AE}^*$ is the closure of $\mathcal{AE}$ in $C^*$ with the induced topology.
Since all maps in $\mathcal{AE}$ are good maps (see, example(2.3)) and have bounded second fundamental form the example(2.9) implies the following result.

**Lemma 3.7.** The class $\mathcal{AE}'$ is dense in $\mathcal{AE}^*$ in the $C^*$ topology. In particular $\mathcal{AE}'$ is open and dense into $\mathcal{AE}$.

Let $f \in \mathcal{AE}^*$ with Gauss map $G$ and set of ends $E = \{w_1, \cdots, w_k\}$. We know from [8] that there is $R_0 > 0$ such that

(i) $f^{-1}(\mathbb{R}^3 \setminus (B_R^3(0) \times \mathbb{R})) = \cup V_j R$, where $V_j R$ is a neighbourhood of one end $w_j$ in $\overline{M}$, for all $R \geq R_0$ and all $j$,

(ii) $F = P \circ f \colon (V_j R \setminus \{w_j\}) \to \mathbb{R}^2 \setminus B_R^3(0)$ is a covering with fibre the geometric index $I(w_j)$, for all $R \geq R_0$ and for all $j$.

Choose $\delta_0 > 0$ such that

$$B_{\delta_0}^3(y_j) \subset G(V_j R_0), \quad y_j = G(w_j), \quad 1 \leq j \leq k,$$

and all balls disjoints. There is $R > R_0$ such that

$$G(V_j R) \subset B_{\delta_0}^3(y_j), \quad 1 \leq j \leq k. \quad (15)$$

Fix some $y_j \in G(E)$ and take $Z = \cup V_j R$ where $\{w_j\} = V_j R \cap E$ satisfies $G(w_j) = y_j$. Now we are in position to bend the set $Z$ to get a new $C^3$ immersion $\tilde{f}$ that just move $y_j$ to a point $y$ close to $y_j$. Choose some $\delta_1 > 0$ such that

$$\sin(\delta_1) < \text{dist}_{\delta_2}(y_j, f(\partial Z))/4$$

and $y \in B_{\delta_1}^3(y_j)$. Take a $C^\infty$ function $\psi \colon \mathbb{R}^3 \to \mathbb{R}$, $0 \leq \psi \leq 1$, with $\psi \equiv 0$ for $|x| \leq 2R$ and $\psi \equiv 1$ for $|x| \geq 3R$. Define $h \colon \mathbb{R}^3 \to \mathbb{R}^3$ by $h(x) = (1 - \psi(x))x + \psi(x)A_\delta x$ where $\delta$ is the angle between $y_j$ and $y$ and $A_\delta$ is the rotation of angle $\delta > 0$ moving $y_j$ to $y$. Let $\tilde{f} \colon M \to \mathbb{R}^3$ be defined by $\tilde{f}(x) = f(x)$ if $x \in M \setminus Z$ and $\tilde{f}(x) = h(f(x))$ for $x \in Z$. Since $f$ is proper the set $K = f^{-1}(B_R^3(0))$ is compact in $M$. Then there is a open set $\mathcal{U}$ is the space $C^3(K, \mathbb{R}^3)$ where maps in $\mathcal{U}$ are immersions with orthogonal projection excellent maps (see [20]). Then we can make $\delta_1$ small to have $\tilde{f}|K \in \mathcal{U}$. Hence we have the following result.

**Proposition 3.8.** Let $f \in \mathcal{AE}^*$ with $F = P \circ f$ excellent map and $\tilde{f}$ be the bent constructed above. Let $G$ and $\tilde{G}$ be the respective Gauss maps. Then

1. $\tilde{f} \in \mathcal{AE}^*$ and $\tilde{F} = P \circ \tilde{f}$ is excellent map,

2. $G(x) = \tilde{G}(x)$ for all $x \notin Z$ and $\tilde{G}(Z) \subset B_{\delta_1}^3(y_j)$,

3. $\tilde{G}(w_j) = y$ for all $w_j \in G^{-1}(y_j)$ for any chose of $y \in B_{\delta_1}^3(y_j)$,

4. If $G$ miss $y_j$ then $\tilde{G}$ miss $y$,

5. $G(w) = \tilde{G}(w)$ for all $w \in E \setminus G^{-1}(y_j)$,

6. The number of missing points of $G$ and $\tilde{G}$ are the same,
7. Both singular set of $F$ and $\tilde{F}$ are the same and the geometric index are equal in each end,

8. If the Gauss map of $f$ has all missing points in $S^1$ there are bent immersions $\tilde{f}$ whose Gauss map has all missing points in $S^2$,

9. $\mathcal{AE}^*$ is closed for bent operation,

10. All true formulas type total curvature, Riemann-Hurwitz or Euler characteristic in the class $\mathcal{AE}'$ are also true in the class $\mathcal{AE}^*$.

4 Counting cusps

We will count the cusps points in three different ways. First we will calculate the total geodesic curvature of each singular connected component $\Gamma$. It is zero in the absence of cusps points but counts the cusp points adding $0$ or $\pm 1$ according the geometry of the surface.

When we project $\Gamma$ over the horizontal plane we find another way to count the cusp points. In this case we get the degree of the Gauss map given by the number of cusp points.

At the end we find new relations involving cusps, geometric number of ends, the degree of Gauss map and Euler characteristic of the surface.

Without exception all immersions considered here are in the class $\mathcal{AE}'$.

4.1 First counting of cusps

Let $\Gamma$ be a connected component of the singular set of some $F = P \circ f$, with $f \in \mathcal{AE}'$. Denote by $c(\Gamma)$ the cardinality of $C \cap \Gamma$ and assume that $c(\Gamma) \neq 0$. We order the set of cusp points in $\Gamma$ by

$$p_1, \cdots, p_k, p_{k+1}, \quad p_1 = p_{k+1}. \quad (16)$$

Let $\gamma$ be the orthogonal projection of $\Gamma$ into $\mathbb{R}^2$ and $q_1, \cdots, q_k$, the corresponding projections of the singularities. The singular points divide $\gamma$ and $\Gamma$ into a finite collection of regular arts $\gamma_j$ and $\Gamma_j$, $1 \leq j \leq k$. Set $t_0 = 0$ and $t_{j+1} - t_j$ equal to the length of $\gamma_j$ and parametrize each $\gamma_j$ by arc length in the interval $[t_j, t_{j+1}]$. In this way we get a general parametrization analytic by parts of $\gamma$ and

$$\Gamma(t) = \gamma(t) + y(t)e_3 \quad (17)$$

is analytic.

Before we go further we should observe an elementary fact about the orientation of boundary curves of $M_\pm$. Let $\Gamma$ be a connected component of $\Sigma$ with orthogonal projection $\gamma = P(\Gamma)$ and endowed with the orientation of $M_+$. We know that $\gamma$ is regular by parts with regular arcs locally convex and the Gauss map $G|\Gamma$ is horizontal and changes sides in adjoins arcs. Let $\{E_1, E_2\}$ be a positive frame of $M_+$ adapted to $\Gamma$. Then $E_1 = \Gamma/|\Gamma|$ and $E_2$ point inside $M_+$. In a convex arc of $\gamma$ with $G$ pointing to the convex side we have $(E_2, e_3) > 0$ and $\Gamma/|\Gamma| = E_2 \wedge G$. Then $\gamma$ is clockwise oriented. For $\Gamma$ endowed with $M_-$ orientation one gets $\gamma$ counterclockwise oriented. The singular set $\Sigma$ is the boundary of $M_+$ and $M_-$ and $P(\Sigma)$ is clockwise or counterclockwise oriented respectively.
Lemma 4.1. If we endowed a connected component $\Gamma$ with the $M_+$ orientation we get $\gamma$ clockwise oriented and counterclockwise oriented endowed with the $M_-$ orientation.

We consider each connected component $\Gamma$ of $\Sigma$ with the orientation induced from $M_+$. This implies that $\gamma = P(\Gamma)$ has the clockwise orientation of the plane. Let's take over $\Gamma$ the parametrization (17) with clockwise orientation coming from lemma (4.1). Let $\{E_1, E_2\}$ be a positive orthogonal frame in the boundary of $M_+$ given by

$$
\begin{aligned}
E_1 &= \frac{1}{W} \Gamma' \\
E_2 &= \frac{1}{W} G \wedge \Gamma'
\end{aligned}
$$

(18)

Let $\{\gamma', \eta_j\}$ be the Frenet frame of $\gamma$ according the positive orientation of $\mathbb{R}^2$. Let $k_g$ be the geodesic curvature of $\Gamma$. Then

$$
k_g = \frac{1}{W^3} \langle \Gamma' \wedge \Gamma'', G \rangle
$$

(19)

$$
= \frac{1}{W^3} (\kappa e_3 - y'' \eta - y' \kappa \gamma', G),
$$

(20)

$$
= -\mu(\Gamma_j) \frac{y''}{W^3}, \quad \mu(\Gamma_j) = (G(t), \eta_j(t)),
$$

(21)

over the arc $\gamma_j(t)$, $t_i \leq t \leq t_{i+1}$. The number $\mu(\Gamma_j) = \pm 1$ is the sign of $\langle G, \eta_j \rangle$ and changes at adjacent arcs, that is, $\mu(\Gamma_{j+1}) = -\mu(\Gamma_j)$. Observe that $y'(t) = \tan(\alpha_j(t))$ where $\alpha_j(t)$ is the angle between $\Gamma'(t)$ and $\gamma'_j(t)$. We have

$$
\alpha_j(t_j) = \begin{cases} 
\pi/2 & \text{if } \Gamma'(t_j) = We_3 \\
-\pi/2 & \text{if } \Gamma'(t_j) = -We_3
\end{cases}
$$

(22)

and the same happens at $t_{j+1}$. The first condition $\Gamma'(t_j) = We_3$ means $y$ is increasing close to $t_j$ and decreasing in the second case. Then

$$
\int_{P_{j+1}} k_g dS = -\mu(\Gamma_j) \int_{t_j}^{t_{j+1}} \frac{y''}{1+y^2} dt
$$

(23)

$$
= -\mu(\Gamma_j) (\alpha(t_{j+1}) - \alpha(t_j))
$$

(24)

$$
= -\pi \mu(\Gamma_j) \begin{cases} 
-1 & \text{if } \Gamma'(t_i) = We_3 = -\Gamma'(t_{i+1}) \\
1 & \text{if } \Gamma'(t_i) = -We_3 = -\Gamma'(t_{i+1}) \\
0 & \text{if } \Gamma'(t_i) = \Gamma'(t_{i+1})
\end{cases}
$$

(25)

Setting $\rho(\Gamma_j) = [\alpha_j(t_{j+1}) - \alpha_j(t_j)]/\pi$ we get

$$
\int_{\Gamma} k_g dS = -\pi \sum_{j=0}^{k-1} \mu(\Gamma_j) \rho(\Gamma_j)
$$

(26)

We use the notation

$$
\omega_f(\Gamma) = \sum_{j=0}^{k-1} \mu(\Gamma_j) \rho(\Gamma_j) \quad \text{and} \quad \omega_f(\Sigma) = \sum_{\Gamma \in \Sigma} \omega_f(\Gamma)
$$

(27)
Lemma 4.2. Let \( f \in \mathcal{AE}' \) and \( \Gamma \) a connected component of \( \Sigma \) endowed with the orientation of \( M_+ \). Then

\[
\int_{\Gamma} k_g dS = 0, \quad c(\Gamma) = 0, \tag{28}
\]

\[
\frac{1}{2\pi} \int_{\Gamma} k_g = -\frac{1}{2} \omega_f(\Gamma), \quad \frac{1}{2\pi} \int_{\partial M_+} k_g = -\frac{1}{2} \omega_f(\Sigma). \tag{29}
\]

If

\[
\int_{p_i}^{p_{i+1}} k_g \neq 0, \quad \int_{p_{i+j}}^{p_{i+j+1}} k_g = 0, \quad 1 \leq j < k - 1, \quad \int_{p_{i+k}}^{p_{i+k+1}} k_g \neq 0 \tag{30}
\]

then

\[
\int_{p_i}^{p_{i+1}} k_g = (-1)^k \int_{p_{i+k}}^{p_{i+k+1}} k_g. \tag{31}
\]

Corollary 4.3. If \( f \in \mathcal{AE}' \) and \( \Omega \) is a connected component of \( \overline{M}_\pm \) then

\[
-\deg(G|\Omega) \pm \frac{1}{2} \omega_f(\partial \Omega) = -\chi(\overline{\Omega}) + \sharp(\mathcal{E} \cap \overline{\Omega}) + \sum_{w \in \mathcal{E} \cap \overline{\Omega}} I(w) \tag{32}
\]

In particular if \(-n = \deg(G)\) the total curvature formula holds:

\[
n \pm \frac{1}{2} \omega_f(\partial M_\pm) = -\chi(\overline{M}_\pm) + \sharp(E_\pm) + \sum_{w \in E_\pm} I(w) \tag{33}
\]

where \( E_\pm = E \cap \overline{M}_\pm \).

Remark 4.4. The Enneper surfaces oriented with Gauss map being the north pole at the end has only one singular curve \( \Gamma \) with 4 singularity of second order and total geodesic curvature \( 4\pi \). The catenoid with Gauss map pointing inside and rotated from the vertical position of an angle bigger than angle \( \delta_0 \) defined in (7) has also only one singular curve \( \Gamma \) with 4 singularity of second order but with zero total geodesic curvature. Two opposite arcs have winding number zero and the other two have the same absolute value and different sign.

4.2 Whitney’s rotational number

Hassler Whitney in [21] study the rotational number of a \( C^1 \) immersed curve \( \gamma : S^1 \to \mathbb{R}^2 \). The rotational number \( \varrho(\gamma) \) is the total angle through which the tangent turns while traversing the curve (see [21]). Whitney’s first results is the following

Theorem 4.5 (Withney, Theorem 1 [21]). Two regular closed curves \( \gamma_0 \) and \( \gamma_1 \) are regular homotopics \(^2\) if and only if \( \varrho(\gamma_0) = \varrho(\gamma_1) \).

In [21] a point \( q \in \gamma \) is called a simple crossing point if there are \( t_1 < t_2 \) such that \( \gamma(t_j) = q, \ j = 1, 2 \), and the vectors \( \gamma'(t_1) \) and \( \gamma'(t_2) \) are linearly independent. Whitney define \( \gamma \) to be normal if all singularities are simple crossing

\(^2\)i.e. homotopic through regular closed curves
points and he shows that the set of normal curves is open and dense in the $C^1$ topology (see [21]). He associated to each crossing point $q$ a number $\Theta(q)$ in the following way: $\Theta(q) = 1$ if $\{\gamma'(t_1), \gamma'(t_2)\}$ is a negative base of $R^2$, and $\Theta(q) = -1$ if this base is positive. Set

$$\Theta_+(\gamma) = \sum_{\Theta(q) > 0} 1, \quad \text{and} \quad \Theta_-(\gamma) = \sum_{\Theta(q) < 0} 1,$$

where $q$ varies over the set of simple cross points of $\gamma$. An outside starting point is a global support point of $\gamma$. He proves the following:

**Theorem 4.6** (Whitney, Theorem 2 [21]). If $\gamma$ is normal with a outside starting point then

$$\varrho(\gamma) = 2\pi(\mu + \Theta_+(\gamma) - \Theta_-(\gamma)), \quad \mu = \pm 1.$$  \hspace{1cm} (34)

Moving the outside starting point $\gamma(0)$ to the origin with with the second coordinate $y \geq 0$ we have $\mu = 1$ or $-1$ according $\gamma'(0) = e_1$ or $-e_1$ respectively.

**Remark 4.7.** In fact if two simple crossing points $q_1$ and $q_2$ are such that $\gamma(t_1) = \gamma(t'_1) = q_1$ and $\gamma(t_2) = \gamma(t'_2) = q_2$ where $t_1 < t_2 < t'_1 < t'_2$, and $\gamma((t_1, t_2))$ and $\gamma((t'_1, t'_2))$ do not intersect then $\Theta(q_1) + \Theta(q_2) = 0$ and $\gamma$ is homotopic to another curve $\tilde{\gamma}$ that do not have this kind of intersection. The only crossing point that can not be removed is of type $\gamma(t_1) = \gamma(t_2)$ with $t_1 < t_2$. The proof is standard. Then we can assume that $\Theta_\pm(q), q \in \gamma$, counts only the true simple cross points of sign positive or negative, that is the true loops of $\gamma$. Denote by $\Theta^t_\pm(\gamma)$ the counting of true simple cross points $q$ of $\gamma$ corresponding to $\Theta_\pm(q) > 0$. The Whitney result became

$$\varrho(\gamma) = 2\pi(\mu + \Theta^t_+(\gamma) - \Theta^t_-(\gamma)), \quad \mu = \pm 1.$$  \hspace{1cm} (35)

**Corollary 4.8.** Let $\gamma: S^1 \to R^2$ be a $C^1$ immersion with $\Theta_-(\gamma) = \Theta^t_+(\gamma) = 0$. Then $\varrho(\gamma) = 2\pi\mu$, and $\gamma$ is homotopic to $\pm S^1$ according with the sign of $\mu$.

Since $f \in \mathcal{AE}'$ the orthogonal projection $F: M \to R^2$ of $f$ is an excellent map. Let $\Omega$ be a connected component of $M_+$ or $M_-$. We induce the flat metric of $R^2$ in $\Omega$ making the pull back by $F$. We want to calculate the total curvature of $\Omega$ with this flat metric to get new geometric relations involving the cusp points and Euler characteristic.

Let $\Gamma \subset \partial \Omega$ be a connected component. If $c(\Gamma) \neq 0$ then $\gamma = F(\Gamma)$ is not a regular curve. Let take regular approximation of $\gamma$ in the following way. First choose $\delta > 0$ if $\Omega \subset M_+$ or $\delta < 0$ if $\Omega \subset M_-$. Let $I_\delta$ the interval with extreme points $\delta$ and 0 in way that

$$U_\delta = J^{-1}(I_\delta) \subset \Omega$$

is a finite union of regular annulus with regular boundary and

$$F(\partial U_\delta) = \gamma_\delta \cup \gamma.$$  \hspace{1cm} (37)

If $0 < |\delta' | < |\delta|$ then applying the Gauss-Bonnet theorem to the annulus $U_\delta \setminus U_{\delta'}$, we get

$$\int_{\gamma_\delta} k_g = \int_{\gamma_{\delta'}} k_g$$

we get
where \( k_g \) is the geodesic curvature. Then the limit

\[
\int_{\gamma_\delta} k_g = \lim_{\delta' \to 0} \int_{\gamma_{\delta'}} k_g
\]

exists. For the moment let us consider only \( \gamma_\delta \) and \( \gamma \) as one curve. If the regular arcs of \( \gamma \) has a intersection with the same tangent line one can move the position of \( f(M) \) a little bit to get that all intersections are simple cross points and there are no crossings in a neighborhood of each cusp point. Since \( \gamma_\delta \) is close to \( \gamma \) we can assume that \( \gamma_\delta \) are normal for all small \( \delta \). By the Whitney’s normal form in neighborhood of a cusp point (see lemma(2.2)) the curve \( \gamma_\delta \) gives a loop around the cusp point or is an embedded convex arc. Then if \( p \in \Gamma \) is a cusp point of \( F \) and \( q = P(p) \), we can cut off a small arc \( \sigma_{q\delta} \) of \( \gamma_\delta \) such that either

(cp1) \( \sigma_{q\delta} \) is a loop with total curvature converging to \(-\pi\) when \( \delta \to 0 \).

or

(cp2) \( \sigma_{q\delta} \) is a convex arc with total curvature converging to \(+\pi\) when \( \delta \to 0 \),

after doing the blow up of the singularity. We associated to the curves \( \sigma_{q\delta} \) the following counting of cusps

**Definition 4.9.** Let \( \sigma_{q\delta} \) the approximation of cusp points given by lemma(2.2). We define

1. \( a_\Omega(\gamma) \) is the number of cusps \( p \) of \( F \) where \( \sigma_{q\delta} \) is an embedded convex arc.
2. \( b_\Omega(\gamma) \) is the number of cusps \( p \) of \( F \) where \( \sigma_{q\delta} \) is a loop.

Observe that \( \gamma_\delta \setminus \cup \sigma_{q\delta} \) converges to \( \gamma \) in the \( C^3 \) topology. Taking the limit as \( \delta \to 0 \), we get

\[
\frac{1}{2\pi} \int_{\gamma_\delta} k_g = \lim_{\delta' \to 0} \int_{\gamma_{\delta'}} k_g = \frac{1}{2\pi} \int_{\gamma} \kappa + \frac{1}{2} a_\Omega(\gamma) - \frac{1}{2} b_\Omega(\gamma) \quad (39)
\]

Since \( M \) has negative Gaussian curvature in a neighborhood of \( \Gamma \) we have that \( \gamma_\delta \setminus \cup \sigma_{q\delta} \) and the loops \( \sigma_{q\delta} \) are clockwise oriented and the convex arcs \( \sigma_{q\delta} \) counterclockwise oriented. The true simple cross points \( q' \in \gamma_\delta \) that are not loops of type \( \sigma_{q\delta} \) converges to true simple cross points of regular arcs of \( \gamma \). For that true simple crossing point \( q' \) we have \( \Theta(q') = -1 \). Then

\[
\Theta^+_{\gamma}(\gamma_\delta \setminus \cup \sigma_{q\delta}) = 0 \quad (40)
\]

If \( q' \) is the corner of one loop \( \sigma_{q\delta} \) we have \( \Theta^+_{\gamma}(q') = 0 \) and \( \Theta^+_{\gamma}(q') = 1 \). Then

\[
\Theta^+_{\gamma}(\gamma_\delta) = 0, \quad \text{and} \quad \Theta^+_{\gamma}(\gamma_\delta) = \Theta(\gamma) + b_\Omega(\gamma) \quad (41)
\]

where \( \Theta(\gamma) \) is the number of true simple crossing points of regular arcs of \( \gamma \). We summarize all these results in the next lemma. A first touching point of a curve \( \gamma \) is an absolute maximum or minimum of some linear functional of \( \mathbb{R}^2 \).

**Lemma 4.10.** Let \( q_\delta \in \gamma_\delta \) a first touching point and let \( \lim_{\delta \to 0} q_\delta = q \in \gamma \). We have

15
(1) If \( q \) is a folder of \( F \) (regular point of \( \gamma \)) or is a cusp point with \( \sigma_q \delta \) a loop then \( \mu = -1 \).

(2) If \( q \) is a cusp point of \( \gamma \) with \( \sigma_q \delta \) an embedded convex arc then \( \mu = 1 \),

(3) The Whitney’s rotation number becomes

\[
\Theta_\gamma = 0, \quad \frac{1}{2\pi} \int_{\gamma} k_\gamma = \mu - \Theta_\gamma \quad (42)
\]

If \( \kappa \) is the geodesic curvature of \( \gamma \) as boundary of \( F(\Omega) \) then

(4) The total geodesic curvature of \( \gamma \) is given by

\[
\frac{1}{2\pi} \int_{\gamma} \kappa = \mu - \Theta_\gamma + \frac{1}{2} b_U(\gamma) - \frac{1}{2} a_U(\gamma) \quad (43)
\]

\[
= \mu - \Theta(\gamma) - \frac{1}{2} c(\Gamma) \quad (44)
\]

If \( \Gamma \) has no cusp point for \( F \) then \( \gamma'' : \gamma \to S^1 \) is regular and \( \deg(G|\Gamma) = \deg(\gamma'') \) Setting \( \nu(\Gamma) = |\deg(G|\Gamma)| \) we get

\[
\deg(G|\Gamma) = -\nu(\Gamma) \quad (45)
\]

If \( \Gamma \) has cusp points for \( F \) then by Whitney’s theorem (see lemma(2.2)) the map \( \eta : \Gamma \to \mathbb{P}^1, \eta = [(P \circ \Gamma)'] \) is continuous and \( \deg(\eta) = 2\deg(G|\Gamma) \). The degree of \( \eta \) is given by

\[
\frac{1}{\pi} \int_{\gamma} \kappa.
\]

Then

\[
\deg(G|\Gamma) = \frac{1}{2\pi} \int_{\gamma} \kappa \quad (46)
\]

\[
= \mu - \Theta(\gamma) - \frac{1}{2} c(\gamma) \quad (47)
\]

Lemma 4.11. Set \( n = -\deg(G) \), \( n_\Omega = -\deg(G|\Omega) \) and \( \Omega \) a connected component of \( \mathbb{M}_\pm \). We have

\[
\deg(G|\Gamma) = \frac{1}{2\pi} \int_{\gamma} \kappa \quad (48)
\]

and

\[
\deg(G|\Gamma) = \begin{cases} 
\nu(\Gamma), & \text{if } c(\Gamma) = 0 \\
-\mu + \Theta(\gamma) + \frac{1}{2} c(\gamma), & \text{if } c(\Gamma) > 0 
\end{cases} \quad (49)
\]

If \( \Gamma \) varies in \( \partial \Omega \) then

\[
n_\Omega = -\sum c(\Gamma) \nu(\Gamma) + \Theta(\partial \Omega) + \frac{1}{2} c(\partial \Omega) \quad (50)
\]

If \( \Gamma \) varies in \( \Sigma \) then

\[
c(\Sigma) = 2 \left( n + \sum c(\Gamma) = 0 \nu(\Gamma) - \Theta(\Sigma) \right) \quad (51)
\]
4.3 Second counting of cusps

Now we are in position to calculate the Euler characteristic of a connected component of \(M\) using the number of true simple cross points and cusp points of \(F = P \circ f, f \in \mathcal{AE}'\). Let started with compact connected component.

**Lemma 4.12.** Let \(\Omega\) be a connected compact component of \(M_+\) or \(M_-\) and \(\Gamma\) be a connected component of the boundary. Then

1. \(\mu(\gamma_{\delta}) = 1\), if \(\gamma = P \circ \Gamma\) contains a first touching point,
2. \(c(\partial \Omega) \geq a_\Omega(\partial \Omega) \geq 3\)
3. \(\partial P(\Omega) \subset P(\partial \Omega)\)

**Proof.** The item 3 is obviously since for \(q \in \partial P(\Omega)\) we have \(P^{-1}(q)\) inside the singular set, that is, \(P^{-1}(q) \subset \partial \Omega\).

Moving a line \(L\) as we did in the prove of lemma(4.10) we get the first touching point \(q \in P(\partial \Omega)\) and \(P(\Omega)\) is one side of \(L\). Let \(\gamma = P(\Gamma)\) be a connected component containing some \(p \in \partial \Omega\) with \(P(p) = q\). Let \(\gamma_{\delta} \subset \Omega\) the regular approximation of \(\gamma = P(\Gamma)\). Remember that \(\gamma\) is locally strictly convex at a regular point \(q\) and \(\Omega\) is locally a graphic over the concave side of \(\gamma\). Since \(P(\Omega)\) is one side of \(L\) the point \(q\) can not be regular. By the same reason the arc \(\sigma_{q\delta}\) can not be a loop. Then \(\sigma_{q\delta}\) must be an embedded convex arc positive oriented and by lemma(4.10)(3) we have \(\mu = 1\).

If we move \(L \ni q\) keeping \(q\) fixed we get at least two more distinct points \(q_1\) and \(q_2\) with \(\sigma_{q_i\delta}\) a convex embedded arc what conclude the proof.

**Lemma 4.13** (Main lemma). If \(\Omega\) is a topological disk with \(\partial \Omega = \Gamma\) then

1. \(\mu = 1\)
2. \(b_\Omega(\Gamma) = \Theta_+^L(\gamma) = 0\)
3. \(|\text{deg}(G|\Omega)| = 1 + \sum_\Omega \beta(z)\)
4. \(c(\Gamma) = a_\Omega(\Gamma) = 2 + 2|\text{deg}(G|\Omega)|\)

**Proof.** We know from lemma(4.12)(1) that \(\mu = 1\). Making the pull back of the flat metric over \(\Omega\), applying the Gauss-Bonnet theorem to \(\Omega \setminus U_\delta\) and using lemma(4.10)(3) we get

\[1 = \chi(\Omega) = \frac{1}{2\pi} \int_{\gamma_{\delta}} k_g = 1 - \Theta_+^L(\gamma)\]

Then \(\Theta_+^L(\gamma) = 0\) implying \(b_\Omega(\gamma) = 0\). his proves item (2) and that \(c(\Gamma) = a_\Omega(\Gamma)\). The item (3) is just the Rimann-Hurwirtz relation for \(\Omega\). The item (4) follows from (4.11).

Let \(\Omega\) be a non compact connected component of \(\overline{M}_\pm\) with boundary \(\Gamma = \partial \Omega\) and \(U_\delta\) the neighbourhoods of the boundary of \(\Omega\) given in (36). We cut off neighbourhoods of the ends of \(\Omega\) and of the boundary to get a compact domain given by

\[\Omega_\delta = (B_2^{R^2} \times 0) \cap (\Omega \setminus U_\epsilon)\]
If we take \( \delta \) small enough we have that each neighbourhood of one end \( w \) of \( \Omega \) is a recovering of the unbounded annulus \( \mathbb{R}^2 \setminus B_{|\theta|}^\mathbb{R}^2(0) \) with fiber the geometric index \( I(w) \). This gives that if we induce the flat metric on \( \Omega \) then total geodesic curvature of the neighbourhoods of infinity is just \( 2\pi \sum I(w) \) with \( w \) varying in the set of ends \( \hat{E}_\Omega = E \cap \Omega \). One also has \( \chi(\Omega_\delta) = \chi(\Omega) - \frac{1}{2}E_\Omega \). Then the Gauss-Bonnet theorem gives

\[
\chi(\Omega) = \frac{1}{2\pi} \int_{\gamma_\delta} k_g + \frac{1}{2}E_\Omega + \sum_{E_\Omega} I(w).
\]

Using (39), (32), (33) and (48) we get the following results

**Proposition 4.14.** If \( \Omega \) is a connected component of \( \overline{\mathbb{M}}_{\pm} \) with boundary \( \Gamma \) then

\[
\chi(\Omega) = \deg(G|\Omega) + \frac{1}{2}a_\Omega(\Gamma) - \frac{1}{2}b_\Omega(\Gamma) + \frac{1}{2}E_\Omega + \sum_{E_\Omega} I(w)
\]

(54)

\[
\omega(\Gamma) = b_\Omega(\Gamma) - a_\Omega(\Gamma)
\]

(55)

In particular we have

\[
\chi(\overline{\mathbb{M}}_{\pm}) = -n \pm \frac{1}{2}a_{M_{\pm}}(\Sigma) \mp \frac{1}{2}b_{M_{\pm}}(\Sigma) + \frac{1}{2}E_{\pm} + \sum_{E_{\pm}} I(w)
\]

(56)

\[
\omega(\Sigma) = b_{M_{\pm}}(\Sigma) - a_{M_{\pm}}(\Sigma)
\]

(57)

**Remark 4.15.** If a single curve \( \Gamma \) is in the boundary of \( \Omega_1 \subset M_+ \) and \( \Omega_2 \subset M_- \) then \( a_{\Omega_1}(\gamma) = b_{\Omega_2}(\gamma) \) and \( b_{\Omega_1}(\gamma) = a_{\Omega_2}(\gamma) \). We also have \( c(\gamma) = a(\gamma) + b(\gamma) \).

### 5 The proof of Theorem 2

Take \( f \in \mathcal{AE}^* \) defined in \( M = \overline{\mathbb{M}} \setminus E \) with Gauss map \( G \) extending to \( \overline{\mathbb{M}} \) and and set \( Y = S^2 \setminus G(M) \). Choose \( \delta > 0 \) such that the balls \( B_{\delta}^S(z) \), \( z \in G(E) \), isolated the points of \( G(E) \) and the image by \( f \) of each connected component of \( G^{-1}(B_{\delta}^S(z)) \) is a multi graphic over some neighborhood of the infinite of the horizontal plane. Take the compact set \( K \subset M \) defined by \( K := \overline{\mathbb{M}} \setminus G^{-1}(\cup B_{\delta}^S(z)) \), \( z \in G(E) \). Let \( f_j \in \mathcal{AE}' \) be a sequence converging to \( f \) with \( d_K(f, f_j) < 1 \) for all \( j \) and set of missing points of the Gauss map \( Y_j \). Then all \( f_j \) are defined in \( \overline{\mathbb{M}} \setminus E \) and each Gauss map \( G_j \) of \( f_j \) extends to a continuous map from \( \overline{\mathbb{M}} \) to \( S^2 \). If we take \( \delta \) small enough we can assume that the set of ends that are covered by \( G(K) \) are also covered by \( G_j(K) \) for all \( j \). We also have \( G^{-1}(Y) = G_j^{-1}(Y_j) \) for all \( j \) and the sets \( Y \) and \( Y_j \) are \( \delta \)-closed in the sphere \( S^2 \). We have that all formulas deduced here are true for \( f \) and that the cardinality of \( Y \) and \( Y_j \) are the same for all \( j \).
We use the convention “$X_\pm = X \cap M_\pm$” to describe restrictions of facts to $M_\pm$. Observe that $G^{-1}(Y) \subset E$. Denote $l = \sharp(Y), \ l_\pm = \sharp(Y_\pm)$, and

\[ I = \sum_{E} I(w), \quad I_\pm = \sum_{E_\pm} I(w), \quad I = I_+ + I_- \]

Also

\[ \mathcal{B} = \sum_{M} \beta(w), \quad \mathcal{B}_\pm^\infty = \sum_{G^{-1}(Y_\pm)} \beta(w), \quad \mathcal{B}'_\pm = \mathcal{B} - \mathcal{B}_\pm^\infty \]

\[ \mathcal{E} = \sharp(E), \quad \mathcal{E}_\pm^\infty = \sharp(G^{-1}(Y) \cap S^2_\pm), \quad \mathcal{E}'_\pm = \mathcal{E} - \mathcal{E}_\pm^\infty \]

Since the Gauss map $G_\cdot : \overline{M_\pm} \to S^2_\pm$ is a branched recovering with $n$ points in the fibres over the regular values, one gets

\[ l \ n = \sharp(G^{-1}(Y)) + \mathcal{B}, \quad l_\pm \ n = \sharp(G^{-1}(Y_\pm)) + \mathcal{B}_\pm \]

It follows from corollary(33) equation(13) that

\[ (2 - l_\pm)n = \frac{\omega(M_\pm)}{2} + \mathcal{B}'_\pm + \mathcal{E}'_\pm + I_\pm \]  

(58)

Adding these two equations we get

\[ (4 - l)n = \mathcal{B}' + \mathcal{E}' + I > 0, \]  

(59)

where $\mathcal{B}' = \mathcal{B}'_+ + \mathcal{B}'_- \text{ and } \mathcal{E}' = \mathcal{E}'_+ + \mathcal{E}'_-$. Then $0 \leq l \leq 3$. Suppose that $\sharp Y = 3$ what implies $\sharp Y_j = 3$ for all $j$. Let $Y_j$ and $Y$ be the planes defined by $Y_j$ and $Y$ respectively. If $Y$ and all $Y_j$ pass by the origin one can band any $f_j$ to get a new immersion $\tilde{f}_j$ with the corresponding set of missing points $\tilde{Y}_j$ of the Gauss map $\tilde{G}_j$ defining a plane $\tilde{Y}_j$ not passing by the origin. Therefor we can assume that $f \in \mathcal{A}\mathcal{E}'$ and that $l_+ = l = 3$. The equations (58) and (33) give

\[ -n + \frac{\omega(M_+)}{2} = \mathcal{B}'_+ + \mathcal{E}'_+ + I_+ \]  

(60)

\[ n - \frac{\omega(M_+)}{2} = -\chi(M_-) + +\mathcal{E}_- + I_- \]  

(61)

Hence

\[ \chi(M_-) = \mathcal{B}'_+ + \mathcal{E}' + I > 0. \]

(62)

Let $\{D_1, \ldots, D_s\}$ be the collection of topological disks of $M_-$ and write

\[ M_- = \Omega \cup \bigcup_{j+1}^{s} D_j. \]

Set $\Gamma = \partial \Omega, \ -m = \deg(G|\Omega), \ -m_j = \deg(G|D_j)$ and $\beta$ and $\beta_j$ for the total order of branch points of the Gauss map in $\Omega$ and $D_j$ respectively. By
Lemma (4.13) we have
\[
\begin{align*}
n &= m + \sum m_j \quad (63) \\
\mathcal{B}_- &= \beta + \sum \beta_j \quad (64) \\
\omega(M_+) &= \omega_+(\Gamma) - 2s - 2 \sum m_j = \omega(\Gamma_+) - 2 \sum \beta_j - 4s \quad (65) \\
\frac{\omega_+(\Gamma)}{2} &= n + s + \sum m_j + \mathcal{B}_+ + \mathcal{E}_+ + \mathcal{I}_+ \\
&= n + 2s + \sum \beta_j + \mathcal{B}_+ + \mathcal{E}_+ + \mathcal{I}_+ \quad (66)
\end{align*}
\]

Using Riemann-Hurwitz and the total curvature formula for \(\Omega\) and the fact \(n = \mathcal{B}^\prime + \mathcal{E}^\prime + \mathcal{I}\) that comes from \(l = 3\) into (59) one gets
\[
\begin{align*}
m - \beta &= \chi(\Omega) \\
&= -m + \frac{\omega_+(\Gamma)}{2} + \mathcal{E}_- + \mathcal{I}_- \\
&= 2s + \sum m_j + \sum \beta_j + \mathcal{B}_+ + \mathcal{E}_+ + \mathcal{I} \\
m &= 2s + \sum m_j + \mathcal{B} + \mathcal{E}_+ + \mathcal{I} \\
0 &= 2s + 2 \sum m_j
\end{align*}
\]

what is impossible. Then we can not have \(l_+ = 3\) or \(l_- = 3\) what prove the theorem 2.

6 Questions

Let \(M\) be in the class \(\mathcal{AE}^*\) with the Gauss map missing two points. One can rotate \(M\) applying a linear orthogonal operator \(A\) on the image of \(M\) to get the surface in a new position \(AM\). For an open and dense set of the orthogonal group we have \(AM \in \mathcal{AE}^*\). If for those \(A\) we have \(\omega(AM_+) = 0\) then the Gauss map miss exactly one point in each \(AM_+\) and \(AM_-\). This implies that the two missing points are antipodal.

If we bent a catenoid \(M\) in the vertical position like we did in section (3.2) we get a surface \(M_\epsilon\) whose Gauss map miss 2 points in the same hemisphere when \(\epsilon > 0\) without cusp points. Then \(\omega(M_+) = 0\) but in some positions the singular set of \(M_\epsilon\) is one single Jordan curve bounding a disk with \(\omega_+(\Sigma) \neq 0\). The regular catenoid for any position lying in the class \(\mathcal{AE}^*\) has \(\omega_+(\Sigma) = 0\).

In [11] R. Miyaoka and K. Sato constructed examples of minimal spheres (or tori) punctured tree times whose Gauss map miss two points that are not antipodes (antipodes for tori, see Proposition (3.1) and (3.2)). Then the two missed points of the Gauss map are not obligated to be antipodal even in the minimal case.

**Question 1.** Let \(M\) be a complete minimal surface with finite total curvature and Gauss map missing 2 points. When are they antipodal? Is it equivalent to have \(\omega_A(M_+) = 0\) for \(A\) varying in the orthogonal group of \(\mathbb{R}^3\)? Perhaps the answer depends only on the fact that the Gauss map miss 2 points. In general it will be interesting to classify all minimal surfaces whose Gauss map miss exactly two points, one point or \(\omega(\Sigma) = 0\).
In the minimal case we have that $M$ is a catenoid or Enneper surface if the total curvature is $-4\pi$. If we consider the total curvature equal to $-8\pi$ and $g = 1$ we get one of the examples of [3], according with F. J. Lopes[10].

**Question 2.** Are there examples of complete minimal surfaces of genus $g$, one end of Enneper type ($I(w) = 3$) or two ends of catenoid type and total curvature $-4\pi(g + 1)$? In the Enneper case do we have that $\overline{M}_{-}$ is a disk? Are they unique?

In the class $\mathcal{AE}$ the domains are not necessarily planar domains. For example, in one of the examples of [3] the unbounded connected component is a torus minus a disk. In the bent catenoid in horizontal position we get for the unbounded connected component of $M_{\pm}$ a disk punctured twice. In the theory of complete minimal surfaces with finite total curvature it is important to know how the unbounded connected component is determinate by the genus at least in the embedded case.

**Question 3.** Find relationship between the genus and the number of ends in an unbounded connected component of $M_{\pm}$. If not in general at least for $M$ minimal and embedded.

It will be very important to know all possible compact domain that appear in $M_{\pm}$. There are positions of the bent catenoid where the singular set is boundary of a disk. We also have a disk in one of the examples of [3]. In lemma (4.12) we saw that it is impossible to have $\Omega$ compact in $M$ and the orthogonal projection of each curve in $\partial\Omega$ regular.

**Question 4.** What kind of compact connected component $\Omega \subset M_{\pm}$ could exist? Are they always disks?

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