A NOTE ON SEPARATING FUNCTION SETS

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Abstract. We study separating function sets. We find some necessary and sufficient conditions for $C_p(X)$ or $C^2_p(X)$ to have a point-separating subspace that is a metric space with certain nice properties. One of the corollaries to our discussion is that for a zero-dimensional $X$, $C_p(X)$ has a discrete point-separating space if and only if $C^2_p(X)$ does.

1. Introduction

To start our discussion, recall, that $F \subset C^n_p(X)$ is point-separating if for any distinct $x, y \in X$ there exists $\langle f_1, ..., f_n \rangle \in F$ such that $f_i(x) \neq f_i(y)$ for some $i \leq n$. In this paper we are concerned with the following general problem.

Problem. Let $P$ be a nice property. Describe "$C_p(X)$ (or $C^m_p(X)$) having a point-separating subspace with $P$" in terms of the topology of $X$, $X^n$, or $X^\omega$.

In this study, $P$ is the property of being a discrete space, a countable union of discrete subspaces, a metric compactum, or a discrete group. We obtain two characterizations of spaces $X$ for which $C^2_p(X)$ has a discrete point-separation subspace (Theorems 2.9 and 2.10, and 2.17). One of the characterizations is consistent and may have a chance for a ZFC proof. We also characterize zero-dimensional $Z$ with point-separating discrete subspaces in $C_p(X)$ (Theorems 2.13 and 2.14, and 2.18). Questions of similar nature are quite popular among topologists interested in $C_p$-theory and have been considered in many papers.

In notation and terminology we follow [2]. All spaces under consideration are assumed Tychonoff and infinite. By $s(X)$ we denote the supremum of cardinalities of discrete subspaces of $X$. By $iw(X)$ we denote the smallest weight of a Tychonoff subtopology of $X$. When we say that $D$ is a discrete subspace of $X$, $D$ need not be closed in $X$. By $\sigma_X(x^*)$ we denote the subspace of $X^\omega$ that consists of all points that differ from $x^*$ by finitely many coordinates. Since $\sigma_X(x)$ and $\sigma_X(y)$ are obviously homeomorphic we may simply write $\sigma_X$ and, as usual, refer to it as $\sigma$-product of $X^\omega$. A standard neighborhood of $f$ in $C_p(X)$

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is in form \( U(x_1, ..., x_n; B_1, ..., B_n) = \{ g \in C_p(X) : g(x_i) \in B_i \} \), where \( B_i \) is open interval with rational endpoints for each \( i \). Zero-dimensionality is understood in the sense of \( \dim \).

2. DISCRETE POINT-SEPARATING SUBSPACES

Our first goal is to find a characterization of those infinite \( X \) for which \( C_p(X) \) or \( C^2_p(X) \) has a point-separating discrete subspace. We achieve the goal within a wide class of spaces. We start with a few helping lemmas.

The following Lemma is almost identical to Proposition II.5.5 in [1] but due to cofinality restrictions we have to prove it using a similar argument.

**Lemma 2.1.** (version of [1, II.5.5]) Assume that \( C^2_p(X) \) has a discrete subspace of size \( \tau \). Then the following hold:

1. If \( cf(\tau) > \omega \), then \( s(X^n) \geq \tau \) for some \( n \in \omega \).
2. If \( cf(\tau) = \omega \), then \( s(\sigma_X) \geq \tau \).

**Proof.** Since part (2) is an obvious consequence of part (1), we will prove part (1) only. Let \( D \subseteq C^2_p(X) \) be a \( \tau \)-sized discrete subspace. For each \( \langle f, g \rangle \in D \) fix standard neighborhoods \( U_f = U_f(x_1^f, ..., x_{n_f}^f, I_1^f, ..., I_{m_f}^f) \) and \( V_g = V_g(y_1^g, ..., y_{m_g}^g, J_1^g, ..., J_{m_g}^g) \) such that \( U_f \times V_g \) contains \( \langle f, g \rangle \) and misses \( D \setminus \{ \langle f, g \rangle \} \). Since \( cf(\tau) > \omega \) we can find \( n^*, m^* \in \omega \), \( \langle I_i : i \leq n^* \rangle \), \( \langle J_i : i \leq m^* \rangle \), and a \( \tau \)-sized \( D' \subseteq D \) such that \( n_f = n^* \), \( m_g = m^* \), \( \langle I_i^f : i \leq n^* \rangle = \langle I_i : i \leq n^* \rangle \), and \( \langle J_i^g : i \leq m^* \rangle = \langle J_i : i \leq m^* \rangle \) for each \( \langle f, g \rangle \in D' \). We can now conclude that for any distinct \( \langle f, g \rangle, \langle f', g' \rangle \in D' \), either \( \langle x_i^f : i \leq n^* \rangle \neq \langle x_i^{f'} : i \leq n^* \rangle \) or \( \langle y_i^g : i \leq m^* \rangle \neq \langle y_i^{g'} : i \leq m^* \rangle \). Therefore, the set \( S = \{ x_1^f, ..., x_{n^*}^f, y_1^g, ..., y_{m^*}^g : \langle f, g \rangle \in D' \} \) is \( \tau \)-sized. To show that \( S \) is a discrete subspace of \( X^{n^*+m^*} \), for each \( \langle f, g \rangle \in D' \), put \( U_f = f^{-1}(I_1) \times ... \times f^{-1}(I_{n^*}) \) and \( V_g = g^{-1}(J_1) \times ... \times g^{-1}(J_{m^*}) \). Clearly \( U_f \times V_g \) is a neighborhood of \( \langle x_1^f, ..., x_{n^*}^f, y_1^g, ..., y_{m^*}^g \rangle \) in \( X^{n^*+m^*} \). Next, fix \( \langle f', g' \rangle \in D' \setminus \{ \langle f, g \rangle \} \). By the choice of our neighborhoods, we may assume that \( f \notin U_f \). Therefore, there exists \( i \leq n^* \) such that \( f(x_i^f) \notin I_i \). Therefore, \( x_i^f \notin f^{-1}(I_i) \), which implies that \( \langle x_i^f, ..., x_{n^*}^f, y_i^g, ..., y_{m^*}^g \rangle \notin U_f \times V_g \). \( \square \)

Note that if \( C_p(X) \) or \( C^2_p(X) \) has a discrete point-separating subspace of an infinite size \( \tau \), then \( \tau \geq i \omega(X) \). If in addition \( cf(\tau) > \omega \), then, by Lemma 2.1, \( s(X^n) \geq \tau \geq i \omega(X) \) for some \( n \). Thus, the following statement holds.
Theorem 2.2. Assume that $C_p(X)$ or $C^2_p(X)$ has a discrete point-separating subspace of size $\tau$ with $cf(\tau) > \omega$. Then $s(X^n) \geq \tau \geq iw(X)$ for some natural number $n$.

We are now ready to formulate and prove two necessary conditions for $C_p(X)$ and $C^2_p(X)$ to have a point-separating discrete subspace.

Theorem 2.3. If $C^2_p(X)$ has a discrete point-separating subspace, then $s(\sigma_X) \geq iw(X)$.

Proof. Put $\tau = iw(X)$. If $\tau$ is countable, then $X$ has a countable network. Since $X$ is infinite, it contains an infinite countable subspace. Hence, $s(\sigma_X) \geq iw(X)$.

We now assume that $\tau$ is uncountable. By Theorem 2.2 we may assume that $cf(\tau) = \omega$. Fix a strictly increasing sequence of cardinals $\tau_n$ of uncountable cofinalities so that $\tau = \sum_n \tau_n$. Since any point-separating subset of $C^2_p(X)$ must have size at least $\tau$, there exists a discrete subset of cardinality $\tau_n$ in $C^2_p(X)$ for each $n$. By Lemma 2.1, there exists a discrete subset $D_n$ in some finite power of $X$ for each $n$. Therefore, $s(\sigma_X) \geq \tau$. $\square$

In all future arguments, the cases when a discrete point-separating subspace is finite can be handled as in Theorem 2.3 and will therefore not be considered.

For our next observation we need Zenor’s theorem [7] stating that if $s(X \times Y) \leq \tau \geq \omega$ then either $hl(X) \leq \tau$ or $hd(Y) \leq \tau$.

Theorem 2.4. Assume Generalized Continuum Hypothesis. If $C^2_p(X)$ has a discrete point-separating subspace, then $s(X^n) \geq iw(X)$ for some $n \in \omega$.

Proof. Put $\tau = iw(X)$. By Theorem 2.2 we may assume that $\tau$ is an infinite cardinal of countable cofinality. Assume the contrary. Then $s(X^4) = \lambda < \tau$. By Zenor’s theorem, $hl(X^2) \leq \lambda$ or $d(X^2) \leq \lambda$. If the former is the case, then the off-diagonal part of $X^2$ can be covered by $\lambda$-many functionally closed boxes, which implies that $iw(X) < \tau$. If $d(X^2) \leq \lambda$, then by Generalized Continuum Hypothesis, $w(X^2)$ is at most $2^\lambda < \tau$. Since both cases lead to contradictions, the statement is proved. $\square$

The assumptions in Theorem 2.4 prompts the following questions.

Question 2.5. Does Theorem 2.4 hold in ZFC?
Note that if one can construct a space $X$ such that $s(X^n) = \omega_n$ for all natural numbers $n$ and $iw(X) = \omega_\omega$, then the answer to Question 2.5 is a “no”.

At this point one may wonder if our study is justified. In other words, are we studying a non-empty class? Let $X$ be an non-metrizable compact space such that $X^n$ is hereditary separable for each $n$. Such a space exists. A consistent example of such a space is Ivanov’s modification [5] of Fedorchuk’s example [3]. Since $X^n$ is hereditarily separable, by Lemma 2.1, no discrete subspace of $C^2_p(X)$ or $C_p(X)$ is uncountable. Since $X$ is not submetrizable, we conclude that no countable subspace of $C^2_p(X)$ or $C_p(X)$ is point-separating. Let us summarize this observation as follows.

**Example 2.6.** There exists a consistent example of a compactum $X$ such that neither $C_p(X)$ nor $C^2_p(X)$ has a discrete separating subspace.

The authors believe that in some models of ZFC, no such example may exist, meaning that any space may have a discrete in itself point-separating function set.

**Question 2.7.** Is there a ZFC example of a space $X$ such that no discrete subspace of $C_p(X)$ ($C^n_p(X)$) is point-separating?

We will next reverse the statement of Theorem 2.2, which will bring us to the promised characterizations.

**Theorem 2.8.** If $X^n$ has a discrete subspace of size $iw(X)$ for some natural number $n$, then $C^2_p(X)$ has a point-separating discrete subspace.

**Proof.** Let $n$ be the smallest that satisfies the hypothesis of the lemma and put $\tau = iw(X)$. By the choice of $n$ there exists a $\tau$-sized discrete subspace $D$ of $X^n$ with the following property:

**Property:** $|\{x(i) : i \leq n\}| = n$ for each $x \in D$.

Let $\mathcal{T}$ be a Tychonoff subtopology of the topology of $X$ of weight $\tau$. Fix a $\tau$-sized network $\mathcal{N}$ for $\langle X, \mathcal{T} \rangle$ that consists of functionally closed subsets. Let $\mathcal{P}$ be the set of all pairs $\langle A, B \rangle$ of disjoint elements of $\mathcal{N}$. Enumerate $\mathcal{P}$ and $D$ as $\{\langle A_\alpha, B_\alpha \rangle : \alpha < \tau\}$ and $\{d_\alpha : \alpha < \tau\}$. Since $D$ is a discrete subspace, for each $\alpha < \tau$ we can fix a functionally closed set $B_\alpha^i \times \ldots \times B_\alpha^n$ that contains $d_\alpha$ in its interior and misses $D \setminus \{d_\alpha\}$. By Property, we may assume that $B_i^\alpha \cap B_j^\alpha = \emptyset$ for distinct $i$ and $j$.

We will next construct our desired subspace $\{(f_\alpha, g_\alpha) : \alpha < \tau\}$ of $C_p(X)$.
Definition of \( f_\alpha \), where \( \alpha < \tau \): Let \( S_\alpha \) be a functionally closed subset of \( X \) such that \( X \setminus S_\alpha \) can be written as a union of \( L_\alpha \) and \( R_\alpha \) so that the following hold.

1. \( cl_X(L_\alpha) \cap cl_X(R_\alpha) \subseteq S_\alpha \);
2. \( A_\alpha \subseteq L_\alpha \) and \( B_\alpha \subseteq R_\alpha \);
3. \( d_\alpha(i) \in L_\alpha \) if \( d_\alpha(i) \notin B_\alpha \), and \( d_\alpha(i) \in R_\alpha \) if \( d_\alpha(i) \in B_\alpha \).

Such an \( S_\alpha \) exists since \( A_\alpha \) and \( B_\alpha \) are functionally separable sets and the coordinate set of \( d_\alpha \) is finite. Let \( f_{\alpha,i} : L_\alpha \cup S_\alpha \to [-1,0] \) be any continuous function that has the following properties:

\[
\begin{align*}
L1: & \quad f_{\alpha,i}^{-1}(\{0\}) = S_\alpha; \\
L2: & \quad (\{d_\alpha(i) : i \leq n\} \cap L_\alpha) \subset f_{\alpha,i}^{-1}([-1,-1/3)) \subset \bigcup \{B_\alpha^\alpha : d_\alpha(i) \in L_\alpha\}.
\end{align*}
\]

This can be done since \( B_\alpha^\alpha \)'s form a disjoint finite collection of functionally closed sets.

To show that \( F = \{(f_\alpha,g_\alpha) : \alpha < \tau\} \) is a point-separating discrete subspace. To show that \( F \) is point-separating, fix distinct \( a,b \) in \( X \). Since \( \mathcal{N} \) is a network, there exist disjoint \( A,B \in \mathcal{N} \) that contain \( a \) and \( b \), respectively. Then \( \langle A,B \rangle = \langle A_\alpha,B_\alpha \rangle \in \mathcal{P} \). By the definition of \( f_\alpha \), \( f_\alpha(a) = f_\alpha(i) < 0 \) and \( f_\alpha(b) = f_\alpha(i,b) > 0 \).

It remains to show that \( F \) is discrete in itself, fix \( \alpha \). Put

\[
U_\alpha = \{f : f(d_\alpha(i)) < -1/3 \text{ if } d_\alpha(i) \in L_\alpha, f(d_\alpha(i)) > 1/3 \text{ if } d_\alpha(i) \in R_\alpha\}
\]

\[
V_\alpha = \{g : g(d_\alpha(i)) \in (i-1/3,i+1/3)\}
\]

Clearly, \( U_\alpha \times V_\alpha \) is a neighborhood of \( (f_\alpha,g_\alpha) \). To show that this neighborhood misses the rest of \( F \), fix \( \beta \neq \alpha \). There exists \( i \leq n \) such that \( d_\alpha(i) \notin B_\beta^\beta \). We have two possible cases.

Case 1: This case’s assumption is that \( d_\alpha(i) \notin \bigcup_{j \leq n} B_j^\beta \). By L2 and R2 of the definition of \( f_\beta \), we have \( f_\beta(d_\alpha(i)) \in (-1/3,1/3) \). Hence \( f_\beta \notin U_\alpha \).

Therefore, \( \langle f_\beta,g_\beta \rangle \notin U_\alpha \times V_\alpha \).

Case 2: Assume Case 1 does not take place. Then there exists \( j \leq n \) such that \( d_\alpha(i) \in B_j^\beta \). By the choice of \( i \), we have \( i \neq j \). Therefore,
$g_\beta(d_\alpha(i)) \notin (i - 1/3, i + 1/3)$. Hence $g_\beta \notin V_\alpha$. Therefore, $\langle f_\beta, g_\beta \rangle \notin U_\alpha \times V_\alpha$.

Statements 2.2, 2.8, and 2.4 result in the following criteria.

**Theorem 2.9.** Let a space $X$ have $iw(X)$ of uncountable cofinality. Then $C_p^2(X)$ has a point-separating discrete subspace if and only if $s(X^n) \geq iw(X)$ for some $n$.

**Theorem 2.10.** Assume Generalized Continuum Hypothesis. Then $C_p^2(X)$ has a point-separating discrete subspace if and only if $s(X^n) \geq iw(X)$ for some $n$.

Note that criteria 2.9 and 2.10 would hold for $C_p(X)$ if we could prove Theorem 2.8 for $C_p(X)$.

**Question 2.11.** Assume that $X^n$ has a discrete subspace of size $iw(X)$ for some natural number $n$. Is it true that $C_p(X)$ has a discrete point-separating set?

Using an argument somewhat similar to that of Theorem 2.8 we will next show that Question 2.11 has an affirmative answer if we assume that $C$ is zero-dimensional.

**Theorem 2.12.** Assume that $X$ is zero-dimensional. If $X^n$ has a discrete subspace of size $iw(X)$, then $C_p(X)$ has a point-separating discrete subspace.

**Proof.** Let $n$ be the smallest that satisfies the hypothesis of the lemma and put $\tau = iw(X)$. By the choice of $n$ there exists a $\tau$-sized discrete subspace $D$ of $X^n$ with the following property:

*Property:* $|\{x(i) : i \leq n\}| = n$ for each $x \in D$.

Let $T$ be a Tychonoff subtopology of the topology of $X$ of weight $\tau$. Due to zero-dimensionality of $X$ and the factorization theorem of Mardesic [6], we may assume that $T$ is zero-dimensional too. Fix a $\tau$-sized network $N$ for $(X,T)$ that consists of clopen subsets. Let $P$ be the set of all pairs $\langle A, B \rangle$ of disjoint elements of $N$. Enumerate $P$ and $D$ as $\{\langle A_\alpha, B_\alpha \rangle : \alpha < \tau \}$ and $\{d_\alpha : \alpha < \tau \}$. We will next construct our desired subspace in $C_p(X)$.

*Definition of $f_\alpha$, where $\alpha < \tau$* Since $D$ is a discrete subspace, we can fix a clopen box $U_1^\alpha \times ... \times U_n^\alpha$ that contains $d_\alpha$ and misses $D \setminus \{d_\alpha\}$. By *Property*, we may assume that $U_i^\alpha \cap U_j^\alpha = \emptyset$ if $i \neq j$. Since $A_\alpha$ and $B_\alpha$ are disjoint,
we may assume that each $U_i$ meets at most one of the sets $A_\alpha$ and $B_\alpha$. Define $f_\alpha : X \to \{0, 1, 2, ..., n + 1\}$ by letting $f_\alpha(U_i) = \{i\}$, $f_\alpha(A_\alpha \setminus \bigcup_{i \leq n} U_i) = \{0\}$, and $f_\alpha(X \setminus (A_\alpha \cup U_1 \cup ... \cup U_n)) = \{n + 1\}$.

It remains to show that $F = \{f_\alpha : \alpha < \tau\}$ is a point-separating discrete subspace. To show that $F$ is point-separating, fix distinct $a, b$ in $X$. Since $\mathcal{N}$ is a network, there exist disjoint $A, B \in \mathcal{N}$ that contain $a$ and $b$, respectively. Then $\langle A, B \rangle = \langle A_\alpha, B_\alpha \rangle \in \mathcal{P}$. Since no $U_i^a$ meets both $A_\alpha$ and $B_\alpha$ at the same time, $f_\alpha(A_\alpha)$ misses $f_\alpha(B_\alpha)$.

To show that $F$ is discrete in itself, fix $f_\alpha$ and put $V_\alpha = \{f : f(d_\alpha(i)) \in (i - 1/3, i + 1/3), i \leq n\}$. Next fix any $\beta \neq \alpha$. Then there exists $i \leq n$ such that $d_\alpha(i) \notin U_i^\beta$. Therefore, $f_\beta(d_\alpha(i)) \notin (i - 1/3, i + 1/3)$. Hence, $f_\beta \notin U_\alpha$. □

Note that Theorems 2.9 and 2.10 are now true for $C_p(X)$ provided $X$ is zero-dimensional. Let us state the new versions for reference.

**Theorem 2.13.** Let a zero-dimensional space $X$ have $iw(X)$ of uncountable cofinality. Then $C_p(X)$ has a point-separating discrete subspace if and only if $s(X^n) \geq iw(X)$ for some natural number $n$.

**Theorem 2.14.** Assume Generalized Continuum Hypothesis. Let $X$ be zero-dimensional. Then $C_p(X)$ has a point-separating discrete subspace if and only if $s(X^n) \geq iw(X)$ for some $n$.

For our final characterization discussion we would like to extract a technical statement from the proof of Theorem 2.8 and prove one helpful proposition.

**Lemma 2.15.** Assume that a finite power of $X$ has a discrete subspace of size $\lambda$. Let $\{\langle A_\alpha, B_\alpha \rangle : \alpha < \lambda\}$ be a family of pairs of functionally closed disjoint subsets of $X$. Then there exists a discrete subspace $F$ in $C^2_p(X)$ with the following property:

(*) If $a \in A_\alpha$ and $b \in B_\alpha$ for some $\alpha < \lambda$, then $f(a) \neq f(b)$ for some $f \in F$.

**Proposition 2.16.** Let $C^m_p(X)$ contain a point-separating subspace which is a countable union of discrete subspaces. Then $C^m_p(X)$ has a discrete point-separating subspace.
Proof. We will prove the statement for $m = 2$. Let $D = \cup_n D_n$ be a point-separating set of $C_p^2(X)$, where each $D_n$ is a discrete subspace. For each $n$, fix a homeomorphism $h_n : \mathbb{R} \to (n, n + 1)$. Put $E_n = \{ (h_n \circ f, h_n \circ g) : (f, g) \in D_n \}$. Clearly, $E_n$ separates $x$ and $y$ if and only if $D_n$ does. Also, $E_n$ is a discrete subspace of $C_p^2(X)$. Since all functions in $(\cup_k E_k) \setminus E_n$ target $\mathbb{R} \setminus (n, n + 1)$, we conclude that the closure of $(\cup_k E_k) \setminus E_n$ misses $E_n$. Therefore, $\cup_n E_n$ is a point-separating discrete subspace of $C_p^2(X)$. □

Theorem 2.17. $C_p^2(X)$ has a discrete point-separating subspace if and only if $s(\sigma_X) \geq iw(X)$.

Proof. Necessity is done in Theorem 2.3. To prove sufficiency, put $\tau = iw(X)$. Let $\mathcal{N}$ be a $\tau$-sized family of functionally closed subsets of $X$ that is a network for some Tychonoff subtopology of $X$. Let $\mathcal{P}$ consist of all pairs of disjoint elements of $\mathcal{N}$. For each $n$ we can find a discrete subset $D_n$ of $\sigma_X$ that lives in a copy of some finite power of $X$ so that $\tau = \sum_n |D_n|$. Next represent $\mathcal{P}$ as $\bigcup \mathcal{P}_n$, where $|\mathcal{P}_n| = |D_n|$. Applying Lemma 2.15 to $\mathcal{P}_n$ and $D_n$ for each $n$, we find a point-separating subspace in $C_p^2(X)$ that is a countable union of discrete subspaces. By Proposition 2.16, $C_p(X)$ contains a discrete point-separating subspace. □

An argument identical to that of Theorem 2.17 leads to the following statement for $C_p(X)$.

Theorem 2.18. Assume that $X$ is zero-dimensional. Then $C_p(X)$ has a discrete point-separating subspace if and only if $s(\sigma_X) \geq iw(X)$.

Theorems 2.17 and 2.18 imply the following.

Corollary 2.19. Let $X$ be a zero-dimensional space. Then $C_p(X)$ has a point-separating discrete subspaces if and only if $C_p^2(X)$ does.

Note that the image of a point-separating family under a homeomorphism need not be point-separating. Indeed, $\{id_{[0,1]}\}$ is a point-separating subspace of $C_p([0, 1])$. However, one can construct an automorphism on $C_p([0, 1])$ that carries $\{id_{[0,1]}\}$ to $\{\vec{0}\}$ which is not point-separating. In connection with this observation, it would be interesting to know if having a discrete point-separating subspace is preserved by homeomorphisms among function spaces. The answer is affirmative and to show it we will use the fact [1, 1, 1, 6] that if $C_p(X)$ and $C_p(Y)$ are homeomorphic then $iw(X) = iw(Y)$. 


Theorem 2.20. Let $X$ and $Y$ be $t$-equivalent. If $C^2_p(X)$ has a discrete point-separating subspace, then so does $C^2_p(Y)$.

Proof. Fix a homeomorphism $\phi : C^2_p(X) \to C^2_p(Y)$ and a discrete point-separating subspace $D$ of $C^2_p(X)$.

Assume, first, that $|D|$ is finite. Then $iw(X) = \omega$. Hence $iw(Y) = \omega$. Since $Y$ is infinite, it contains a an infinite countable subspace. By Theorem 2.17, $C^2_p(Y)$ contains a discrete point-separating subspace.

We now assume that $|D|$ is infinite. Then $|D| \geq iw(X)$. Therefore, $|\phi(D)| \geq iw(Y)$. By Lemma 2.1, $s(\sigma_Y) \geq |\phi(D)| \geq iw(Y)$. By Theorem 2.17, $C^2_p(Y)$ contains a discrete point-separating subspace. \hfill $\Box$

Repeating the argument of Theorem 2.20, we obtain the following.

Theorem 2.21. Let $X$ and $Y$ be zero-dimensional and $t$-equivalent. If $C_p(X)$ has a discrete point-separating subspace, then so does $C_p(Y)$.

While being a discrete subspace is already a nice property, it would be interesting to know when $C_p(X)$ or its finite power has a discrete point-separating subspace which is in addition a subgroup. Note that any discrete subgroup is closed. In addition, $C_p(X)$ can be covered by countably many shifts of any neighborhood of zero-function. Therefore, any discrete subgroup of $C_p(X)$ is countable. These observations lead to the following question.

Question 2.22. Let $X$ be a separable metric space. Is it true that $C_p(X)$ has a discrete point-separating subgroup?

It is worth noting that separable metric spaces have many pretty point-separating subspaces as backed up by the next two statement.

Theorem 2.23. $C_p(X)$ has a point-separating subset homeomorphic to $[0,1]$ if and only $X$ admits a continuous injection into $\mathbb{R}^\omega$.

Proof. To prove necessity, let $F \subset C_p(X)$ a point-separating family homeomorphic to $[0,1]$. Then any dense subset of $F$ is point-separating too. Therefore, $C_p(X)$ has a countable point-separating family. Therefore, $X$ continuously injects into $\mathbb{R}^\omega$.

To prove sufficiency we need the following claim.

Claim. $\mathbb{R}^\omega$ embeds into $C_p([0,1])$. 

To prove the claim, note that $C_p(\omega) = \mathbb{R}^\omega$ embeds into $C_p(\mathbb{R})$ since $\omega$ is closed in $\mathbb{R}$. By Gulko-Hmyleva theorem [4] that $\mathbb{R}$ and $[0, 1]$ are $t$-equivalent, we conclude that, $\mathbb{R}^\omega$ embeds into $C_p([0, 1])$. The claim is proved.

By Claim X injects into $C_p([0, 1])$. Let $F$ be the image of such an injection. Due to homogeneity we may assume that the identity function is in $F$. Therefore, $F$ generates the topology of $[0, 1]$. Consider the evaluation map the evaluation function $\Psi_F : [0, 1] \rightarrow C_p(F)$. Since $F$ generates the topology of $[0, 1]$ , we conclude that $\Psi_F([0, 1])$ generates the topology of $F$. If $h : X \rightarrow F$ is a continuous bijection then the map $H : C_p(F) \rightarrow C_p(X)$ is a continuous injection, where $H(f) = hf$. Clearly, $H(\Psi_F([0, 1])) = [0, 1]$ is point separating. □

**Theorem 2.24.** Let $X$ be a separable metric space. Then $C_p(X)$ has a topology-generating subspace homeomorphic to $[0, 1]$.

**Proof.** Embed $X$ into $C_p([0, 1])$ so that the image $F$ contains the identity map. The evaluation function $\Psi_F : [0, 1] \rightarrow C_p(F)$. Since $F$ generates the topology of $[0, 1]$ and therefore $\Psi_F([0, 1])$ generates the topology of $F$. Since $F = X$, we conclude that $[0, 1] = \Psi_F([0, 1])$ generates the topology of $F = X$. □

Note that Theorem 2.24 cannot be reversed. Indeed, $[0, 1]$ generates the topology of $C_p([0, 1])$ but the latter is not metrizable.

We would like to finish with two problems that are naturally prompted by our study.

**Question 2.25.** Characterize spaces $X$ for which $C_p(X)$ has a closed discrete point-separating subset.

**Question 2.26.** Characterize spaces $X$ for which $C_p(X)$ has a (closed) discrete topology-generating subset.

At last, the unattained goal of the paper is left as the following question.

**Question 2.27.** Assume that $C_p(X)$ has a discrete subspace of size $\text{iw}(X)$. Is it true that $C_p(X)$ has a discrete point-separating set?
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