Quantum Bayesian Statistical Inference

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Abstract

In this work a quantum analogue of Bayesian statistical inference is considered. Based on the notion of instrument, we propose a sequential measurement scheme from which observations needed for statistical inference are obtained. We further put forward a quantum analogue of Bayes rule, which states how a prior normal state of a quantum system updates under those observations. We then generalize the fundamental notions and results of Bayesian statistics according to the quantum Bayes rule. It is also noted that our theory retains the classical one as its special case. Finally, we investigate the limit of posterior normal state when the number of observations counts up to infinity.

Keywords: Instrument, Sequential Quantum Measurement, Bayesian Statistical Decision, Bayesian Statistical Inference, Large Sample Property

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1 Introduction

In the past century, quantum physics has flourished and merged with many fields. About fifty years ago, an interdisciplinary field called quantum statistics was born [1, 2]. In this field there are many studies in which Bayesian statistical methods are applied to quantum physics, as shown in [3–8]. What is at the heart of Bayesian statistics is the Bayes rule, which elaborates how a prior distribution updates under observations. The Bayes rule is undeniably classical - it is one of the significant results of conditional probability. Despite the impossibility to sufficiently describe quantum systems by classical probability, there are some similarities between the Bayes Rule and quantum
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state reduction [11]. Thus we wonder whether there is a quantum analogue of Bayesian statistics. To the best of our knowledge, although a few attempts have been made to quantize Bayes rule [9–15], there is still no study addressing a quantum analogue of Bayesian statistics. We then briefly review these articles.

In [9], the exact condition is explored w.r.t. the validity of the proposed quantum analogue of Bayes rule. The Bayes rule is generalized in [10] with the prior being a density matrix and the likelihood being a covariance matrix. A comparison is made between the Bayes rule and quantum state reduction in [11]. In order to update a prior density matrix, a novel Bayes rule is proposed in [12]. In particular, the principle of minimizing relative entropy is found to be the common guiding for both the classical Bayes rule and the novel Bayes rule.

In [13], a quantum analogue of Bayes rule is put forward based on the notion of operator valued measure and quantum random variable. The graphical framework for Bayesian statistical inference raised in [14] is sufficiently general to cover both the standard case and the proposals for quantum Bayesian inference in which degrees of belief are considered to be represented by density operators instead of probability distributions. In [15], the approach of maximizing quantum relative entropy is followed to study a quantum analogue of Bayes rule, resulting in some generalizations.

In this work we focus on the theoretical generalizations of Bayesian statistical decision and Bayesian statistical inference to the case where a quantum system is observed. Based on the notion of instrument, we propose a sequential measurement scheme (i.e. through successively performing a sequence of indirect measurements on the quantum system of interest), from which observations (data) needed for statistical inference are obtained. Furthermore, we put forward a quantum analogue of Bayes rule, which states how a prior normal state of a quantum system updates under those observations. In accordance with the quantum Bayes rule, we generalize the fundamental notions and results of Bayesian statistics. We also notice that when it comes to a given statistical decision problem, the Bayesian solution is probably no longer equivalent to the posterior solution in general. Basically, the posterior normal state no longer converges (except for special cases) when the sample size goes to infinity.

Our paper is organized as follows. Sect.2, Sect.3 and Sect.4 are devoted to presenting some general facts about operator valued measure, instrument and a family of posterior normal states respectively. Then in Sect.5 and Sect.6, we move on to our generalizations concerning the theory of Quantum Bayesian Statistical Decision and Quantum Bayesian Statistical Inference respectively. Finally in Sect.7, we turn our attention to the large sample properties of Quantum Bayesian Statistical Inference.

We now introduce some basic notations. Let \((X, \mathcal{A})\) be a measurable space and \(\mathbb{H}\) a complex Hilbert space. Denote by \(B(\mathbb{H})\) the set of bounded linear operators on \(\mathbb{H}\), by \(B_0(\mathbb{H})\) the set of compact operators on \(\mathbb{H}\), by \(B_{00}(\mathbb{H})\) the set
of finite rank operators on $H$ and by $\| \cdot \|_p$ the Schatten $p$-norm defined by
\[\|a\|_p = \left[\text{tr}(|a|^p)\right]^{1/p},\] (1)
where $|a| = (a^*a)^{1/2}$ and $a \in B(H)$. Denote by $B_p(H)$ the set of bounded linear operators with finite Schatten $p$-norm so that $B_1(H)$ denotes the set of trace class operators on $H$ and $B_2(H)$ denotes the set of Hilbert-Schmidt operators on $H$. Denote by $S(H)$ the set of states on $B(H)$ and $G(H)$ the closed convex set of density operators on $H$, i.e. $G(H) := \{ a \in B_1(H) : a \geq 0, \|a\|_1 = 1 \}$.

\section{Operator Valued Measure}

The notions and results here are generalizations of those in [16] and [17].

\textbf{Definition 2.1} A map $\nu : \mathcal{A} \rightarrow B(H)$ is called an operator valued measure (OVM) iff for any countable collection of sets $\{A_k\}_{k \in \mathbb{N}} \subseteq \mathcal{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ we have
\[\nu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \nu(A_k), \ \sigma\text{-weakly.} \] (2)

Moreover, $\nu$ is called positive iff $\nu(A) \geq 0$ for all $A \in \mathcal{A}$, normalized iff $\nu(X) = 1$ and projection valued iff $\nu^*(A) = \nu(A) = \nu^2(A)$ for all $A \in \mathcal{A}$. Normalized positive operator valued measures are called semispectral measures or observables while projection valued semispectral measures are called spectral measures or sharp observables.

The convex set of observables $\nu : \mathcal{A} \rightarrow B(H)$ is denoted by $O(\mathcal{A}, H)$. A convex combination of observables $\nu_1$ and $\nu_2$ in $O(\mathcal{A}, H)$ (i.e. $t\nu_1 + (1-t)\nu_2$, $0 < t < 1$) can be viewed as a randomization of measuring processes described by $\nu_1$ and $\nu_2$. Denote by $\text{Ext}O(\mathcal{A}, H)$ the set of extreme observables $\nu : \mathcal{A} \rightarrow B(H)$. An extreme observable $\nu \in \text{Ext}O(\mathcal{A}, H)$ cannot be obtained as a nontrivial convex combination of other observables. In other words, the measuring process described by $\nu$ involves no redundancy caused by probabilistically mixing statistically nonequivalent measuring processes.

A measure $\mu$ or an OVM $\nu$ is said to be $\sigma$-finite iff $X$ can be covered with countably many sets in $\mathcal{A}$ with finite measure. Let $\lambda_i : \mathcal{A} \rightarrow A_i$ be either measure or OVM for $i = 1, 2$. We say that $\lambda_1$ is absolutely continuous w.r.t. $\lambda_2$ (denoted $\lambda_1 \ll \lambda_2$) iff $\lambda_1(E) = 0$ for all $E \in \mathcal{A}$ satisfying $\lambda_2(E) = 0$.

Let $\nu : \mathcal{A} \rightarrow B(H)$ be an OVM. For any density operator $\rho \in G(H)$, the complex measure $\nu_{\rho}$ induced by $\nu$ is defined by
\[\nu_{\rho}(A) = \text{tr}[\rho \nu(A)], \ \forall A \in \mathcal{A}. \] (3)
Nonzero positive operators are mapped by $\text{tr}[\rho(\cdot)]$ to positive numbers in the case where $\rho$ is a full-rank density operator. Hence, if $\nu$ is a POVM then $\nu \ll \nu_\rho$ and $\nu_\rho \ll \nu$ for any full-rank density operator $\rho \in \mathcal{G}(\mathbb{H})$.

Assume that $\mathbb{H}$ is separable. Let $\{\psi_n\}$ be an orthonormal basis of $\mathbb{H}$ and denote by $\nu_{ij}$ the complex measure induced by the finite rank operator $|\psi_j\rangle\langle\psi_i|$, i.e.

$$\nu_{ij}(A) = \text{tr}[|\psi_j\rangle\langle\psi_i|\nu(A)], \forall A \in \mathcal{A}. \quad (4)$$

If $\nu : \mathcal{A} \to \mathcal{B}(\mathbb{H})$ is a POVM then the Radon-Nikodým derivative $d\nu/d\nu_\rho : X \to \mathcal{B}(\mathbb{H})$ of $\nu$ w.r.t. $\nu_\rho$ is defined implicitly by

$$\text{tr}[|\psi_j\rangle\langle\psi_i|\frac{d\nu}{d\nu_\rho}] = \frac{d\nu_{ij}}{d\nu_\rho}, \quad (5)$$

where $\rho \in \mathcal{G}(\mathbb{H})$ is a full-rank density operator. When $\mathbb{H}$ is of finite dimension $d\nu/d\nu_\rho$ always exists. However, $d\nu/d\nu_\rho$ may not exist when $\mathbb{H}$ is infinite dimensional. $d\nu/d\nu_\rho$ would exist for all full-rank density operators on $\mathbb{H}$ once it is found to exist for one of the full-rank density operators on $\mathbb{H}$. Furthermore, if $\nu$ is a POVM such that $d\nu/d\nu_\rho$ exists, then $d\nu/d\nu_\rho$ is positive and irrelevant of the choice of the orthonormal basis almost everywhere w.r.t. $\nu_\rho$.

We now introduce the notion of operator valued multimeasure. Let $(X_i, \mathcal{A}_i), i \in [n]$ and $(X, \mathcal{A})$ be measurable spaces. Let $\mathbb{H}$ denote a complex Hilbert space.

**Definition 2.2** A map $\kappa : \Pi_{i=1}^n \mathcal{A}_i \to \mathcal{B}(\mathbb{H})$ is called an operator valued multimeasure iff the maps $\kappa_i : \mathcal{A}_i \to \mathcal{B}(\mathbb{H}) := \kappa(A_1, \ldots, A_{i-1}, \cdot, A_{i+1} \ldots, A_n), i \in [n], \forall \Pi_{j \in [n]/\{i\}} A_j \in \Pi_{j \in [n]/\{i\}} \mathcal{A}_j$ are OVMs.

Moreover, $\kappa$ is called positive iff $\kappa \geq 0$ for all $\Pi_{i=1}^n A_i \in \Pi_{i=1}^n \mathcal{A}_i$ and normalized iff $\kappa(\Pi_{i=1}^n X_i) = 1$. Normalized positive operator valued multimeasures are called multiobservables.

We next give the definition of joint measurability for observables, which is of great importance in quantum measurement. Let $\nu_i \in \mathcal{O}(\mathcal{A}_i, \mathbb{H}), i \in [n]$, $\nu \in \mathcal{O}(\mathcal{A}, \mathbb{H})$ and $\tau \in \mathcal{O}(\Pi_{i=1}^n \mathcal{A}_i, \mathbb{H})$ be observables. Let $\kappa : \Pi_{i=1}^n \mathcal{A}_i \to \mathcal{B}(\mathbb{H})$ be a multiobservable.

**Definition 2.3**

(i) We say that $\nu_i, i \in [n]$ are functions of $\nu$ iff for any $i \in [n]$ there is a $\mathcal{A}_i$ measurable map $f_i : X \to X_i$ such that

$$\nu_i(A_i) = \nu[f_i^{-1}(A_i)], \forall A_i \in \mathcal{A}_i. \quad (6)$$

(ii) We say that $\nu_i, i \in [n]$ are smearings of $\nu$ iff for any $i \in [n]$ there is a Markov
kernel $K_i : X \times \mathcal{A}_i \to [0, 1]$ such that
\[
\nu_i(A_i) = \int_X K_i(x, A_i) \nu(dx), \quad \forall A_i \in \mathcal{A}_i. \tag{7}
\]

(iii) We say that $\nu_i, i \in [n]$ have $\kappa$ as a multiobservable iff for any $i \in [n]$ we have
\[
\nu_i(A_i) = \kappa(X_1, \cdots, X_{i-1}, A_i, X_{i+1}, \cdots, X_n), \quad \forall A_i \in \mathcal{A}_i. \tag{8}
\]

(iv) We say that $\nu_i, i \in [n]$ have $\tau$ as a joint observable iff for any $i \in [n]$ we have
\[
\nu_i(A_i) = \tau(X_1 \times \cdots \times X_{i-1} \times A_i \times X_{i+1} \times \cdots \times X_n), \quad \forall A_i \in \mathcal{A}_i. \tag{9}
\]

Let $(X_i, \mathcal{R}_i), i \in [n]$ be bigrams where $X_i$ is a Hausdorff topological space and $\mathcal{R}_i \subseteq 2^{X_i}$ a ring. Furthermore, $X_i$ is the union of countably many sets in $\mathcal{R}_i$ and $(X_i, \mathcal{R}_i)$ admits that all $\sigma$-additive functions $\mu : \mathcal{R}_i \to [0, +\infty)$ satisfy that there is a set $A' \in \mathcal{R}_i$ together with a compact set $K$ so that $A' \subseteq K \subseteq A$ and $\mu(A) - \mu(A') < \epsilon$ for any $A \in \mathcal{R}_i$ and $\epsilon > 0$. Let $(X_i, \mathcal{A}_i), i \in [n]$ be measurable spaces where $\mathcal{A}_i$ is the $\sigma$-algebra generated by $\mathcal{R}_i$ and $\nu_i \in \mathcal{O}(\mathcal{A}_i, \mathbb{H}), i \in [n]$ be observables. We call $(X_i, \mathcal{A}_i), i \in [n]$ type-A measurable spaces.

**Proposition 2.1** Let $\nu_i \in \mathcal{O}(\mathcal{A}_i, \mathbb{H}), i \in [n]$ where $(X_i, \mathcal{A}_i), i \in [n]$ are type-A measurable spaces. Then the following conditions are equivalent:

(i) $\nu_i, i \in [n]$ are functions of some observable;

(ii) $\nu_i, i \in [n]$ are smearings of some observable;

(iii) $\nu_i, i \in [n]$ have a multiobservable;

(iv) $\nu_i, i \in [n]$ have a joint observable.

Observables $\nu_i, i \in [n]$ are said to be jointly measurable or compatible iff they have one and thus all of the properties of Proposition 2.1.

### 3 Instrument

Here we introduce one of the important notions in quantum measurement called instrument, which will play an important role in the update of a prior normal state.

Let $\mathbb{H}_1, \mathbb{H}_2$ be complex Hilbert spaces and $\mathcal{M} \subseteq \mathcal{B}(\mathbb{H}_1), \mathcal{N} \subseteq \mathcal{B}(\mathbb{H}_2)$ von Neumann algebras. Denote by $\mathcal{M}_s$ (resp. $\mathcal{N}_s$) the predual of $\mathcal{M}$ (resp. $\mathcal{N}$), by $\mathcal{S}(\mathcal{M})$ (resp. $\mathcal{S}(\mathcal{N})$) the set of normal states on $\mathcal{M}$ (resp. $\mathcal{N}$), by $\langle \cdot, \cdot \rangle$ the duality pairing between $\mathcal{M}_s$ and $\mathcal{M}$ (resp. $\mathcal{N}_s$ and $\mathcal{N}$) and by $\mathcal{B}^+(\mathcal{M}_s, \mathcal{N}_s)$ the set of bounded positive linear maps from $\mathcal{M}_s$ to $\mathcal{N}_s$.

**Definition 3.1** A map $\Psi \in \mathcal{B}^+(\mathcal{M}_s, \mathcal{N}_s)$ is called a subtransition iff it satisfies $\langle \Psi \varphi, 1 \rangle \leq \langle \varphi, 1 \rangle$ for all $0 \leq \varphi \in \mathcal{M}_s$. Moreover, if $\Psi$ satisfies $\langle \Psi \varphi, 1 \rangle = \langle \varphi, 1 \rangle$ for all $\varphi \in \mathcal{M}_s$ then $\Psi$ is called a transition.
The dual of a subtransition (resp. transition) $\Psi$, denoted $\Phi$, is defined by $\langle \varphi, \Phi b \rangle = \langle \Psi \varphi, b \rangle$ for all $\varphi \in \mathcal{M}_*$ and $b \in \mathcal{N}$. It is a normal positive linear map from $\mathcal{N}$ to $\mathcal{M}$ such that $\Phi 1 \leq 1$ (resp. $\Phi 1 = 1$). If $\Psi$ is completely positive (CP), then $\Psi$ is called an operation (resp. channel).

**Definition 3.2** A map $\mathcal{J} : \mathcal{A} \to \mathcal{B}^+(\mathcal{M}_*, \mathcal{N}_*)$ is called an instrument (normalized subtransition valued measure) iff it satisfies

(i) $\mathcal{J}(X)$ is a transition;

(ii) If for any countable collection of sets $\{A_k\}_{k \in \mathbb{N}_+} \subseteq \mathcal{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ we have

\[
\mathcal{J} \left( \bigcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \mathcal{J}(A_k),
\]

(10)

where the convergence on the right side of the equation (10) is w.r.t. the strong operator topology of $\mathcal{B}^+(\mathcal{M}_*, \mathcal{N}_*)$ (endowed with the trace norm).

The dual of an instrument $\mathcal{J}$, denoted $\mathcal{J}^*$, is defined by $\langle \varphi, \mathcal{J}^*(A)b \rangle = \langle \mathcal{J}(A)\varphi, b \rangle$ for all $A \in \mathcal{A}$, $\varphi \in \mathcal{M}_*$ and $b \in \mathcal{N}$. It is a map from $\mathcal{A}$ to $\mathcal{L}^+(\mathcal{N}_*, \mathcal{M})$. The convex set of instruments $\mathcal{J} : \mathcal{A} \to \mathcal{B}^+(\mathcal{M}_*, \mathcal{N}_*)$ is denoted by $\text{Ins}(\mathcal{A}, \mathcal{M}_*, \mathcal{N}_*)$ and the set of extreme instruments $\mathcal{J} : \mathcal{A} \to \mathcal{B}^+(\mathcal{M}_*, \mathcal{N}_*)$ is denoted by $\text{ExtIns}(\mathcal{A}, \mathcal{M}_*, \mathcal{N}_*)$. The map $(\cdot)^* : \text{Ins}(\mathcal{A}, \mathcal{M}_*, \mathcal{N}_*) \to \text{Ins}^*(\mathcal{A}, \mathcal{M}_*, \mathcal{N}_*) := \mathcal{J} \mapsto \mathcal{J}^*$ is an affine bijection. If $\mathcal{J}(A)$ is an operation for all $A \in \mathcal{A}$, then $\mathcal{J}$ is called a CP instrument. The convex set of CP instruments $\mathcal{J} : \mathcal{A} \to \mathcal{B}^+(\mathcal{M}_*, \mathcal{N}_*)$ is denoted by $\text{CPIns}(\mathcal{A}, \mathcal{M}_*, \mathcal{N}_*)$ and its extreme elements by $\text{ExtCPIns}(\mathcal{A}, \mathcal{M}_*, \mathcal{N}_*)$.

Let $\mathcal{J} : \mathcal{A} \to \mathcal{B}^+(\mathcal{M}_*, \mathcal{N}_*)$ be an instrument. For any $b \in \mathcal{N}$, the OVM $\nu_b$ induced by $\mathcal{J}$ is defined by

\[
\nu_b(A) = \mathcal{J}^*(A)b, \quad \forall A \in \mathcal{A}.
\]

(11)

Then for any $\varphi \in \mathcal{M}_*$, the complex measure $\nu_{b,\varphi}$ induced by $\nu_b$ is defined by

\[
\nu_{b,\varphi}(A) = \langle \varphi, \nu_b(A) \rangle, \quad \forall A \in \mathcal{A}.
\]

(12)

We denote by $P_\varphi$ the probability measure induced by the observable $\nu_1$ induced by $\mathcal{J}$ for any $\varphi \in \mathcal{S}(\mathcal{M})$.

Let $\mathcal{J} : \mathcal{A} \to \mathcal{B}^+(\mathcal{M}_*)$ be an instrument. Then for any $A \in \mathcal{A}$, $P_\varphi(A)$ is interpreted as the probability of observing a measurement outcome $x \in A$ when the quantum system $S$ described by the von Neumann algebra $\mathcal{M}$ is in the normal state $\varphi \in \mathcal{S}(\mathcal{M})$ and a measurement of the observable $\nu_1 = \mathcal{J}^*1$ is performed. Furthermore, $\varphi_A := \mathcal{J}(A)/\langle \mathcal{J}(A)\varphi, 1 \rangle$ is the normal state the quantum system $S$ is in just after the measurement whose outcome lies in $A$ (provided that $\langle \mathcal{J}(A)\varphi, 1 \rangle > 0$).

We now introduce the notion of indirect measurement as a general class of models of quantum measurement for a quantum system described by a von
Neumann algebra. For any \( \phi \in \mathcal{N} \), let \( \Phi_{\phi} : \mathcal{M} \otimes \mathcal{N} \to \mathcal{M} \) be a unital normal completely positive linear map satisfying

\[
\langle \varphi, \Phi_{\phi} c \rangle = \langle \varphi \otimes \phi, c \rangle
\]

for all \( \varphi \in \mathcal{M}^* \) and \( c \in \mathcal{M} \otimes \mathcal{N} \).

**Definition 3.3** An indirect measurement for \((\mathcal{M}, \mathcal{A})\) is a quadruple \((H', \phi, U, \nu)\) where \(H'\) is a complex Hilbert space, \(\phi\) a normal state on \(B(H')\), \(U\) a unitary operator on \(\mathbb{H}_1 \otimes H'\) and \(\nu : \mathcal{A} \to B(H')\) a sharp observable, such that

\[
\Phi_{\phi} U^\dagger [a \otimes \nu(A)] U \in \mathcal{M}
\]

for all \(A \in \mathcal{A}\) and \(a \in \mathcal{M}\).

For any indirect measurement \((H', \phi, U, \nu)\) for \((\mathcal{M}, \mathcal{A})\), the predual of the map

\[
\mathcal{I}^* : \mathcal{A} \to \mathcal{L}^+(\mathcal{M}) := A \mapsto \Phi_{\phi} U^\dagger [\cdot \otimes \nu(A)] U
\]

is a CP instrument which describes the statistical properties of \((H', \phi, U, \nu)\). It is customary for us to call \(\mathcal{I}\) the corresponding CP instrument of \((H', \phi, U, \nu)\). Indirect measurements for \((\mathcal{M}, \mathcal{A})\), which share identical corresponding CP instruments, are said to be statistically equivalent. The results below show that under some mild conditions, the statistical properties of any indirect measurement can be captured by a CP instrument and any CP instrument can be realized by at least one indirect measurement.

**Proposition 3.1** [18] Assume that \(\mathcal{M}\) is \(\sigma\)-finite. Then a one-to-one correspondence is established by the map \(\mathcal{I}^*\) between the statistical equivalent classes of indirect measurements for \((\mathcal{M}, \mathcal{A})\) and \(\text{CPIns}(\mathcal{A}, \mathcal{M}_*)\) i.e. the set of CP instruments \(\mathcal{I} : \mathcal{A} \to \mathcal{B}^+(\mathcal{M}_*)\) that have the normal extension property.

**Proposition 3.2** [18] A one-to-one correspondence is established by the map \(\mathcal{I}^*\) between the statistical equivalent classes of indirect measurements for \((\mathcal{B}(\mathbb{H}_1), \mathcal{A})\) and \(\text{CPIns}(\mathcal{A}, \mathcal{B}(\mathbb{H}_1))\).

We next focus on the decomposition of instrument. Assume that \(\mathbb{H}_1, \mathbb{H}_2\) are separable. Let \(\{\psi_n\}\) be an orthonormal basis of \(\mathbb{H}_1\) and \(V_1 = \text{span}\{\psi_n : 1 \leq n < \dim(\mathbb{H}_1) + 1\}\) a dense subspace of \(\mathbb{H}_1\). Let \(V_1^\circ\) be the algebraic antidual of \(V_1\), i.e. \(V_1^\circ\) is the vector space of all antilinear functionals \(f : V_1 \to \mathbb{C}\). By denoting \(f_n = f(\psi_n)\) one sees that \(V_1^\circ\) can be identified with the vector space of formal series \(f = \sum_n f_n \psi_n\). Hence, \(V_1 \subseteq \mathbb{H}_1 \subseteq V_1^\circ\). Denote the dual pairing \(\langle \cdot | \cdot \rangle : V_1 \times V_1^\circ \to \mathbb{C}\) by \(\langle \eta, f \rangle \mapsto \sum_n \langle \eta | \psi_n \rangle f_n\) and \(\langle f | \eta \rangle := \overline{\langle \eta | f \rangle}\) for all \(\eta \in V_1\) and \(f \in V_1^\circ\). We say that a map \(\alpha : X \to V_1^\circ := x \mapsto \sum_n a_n(x) \psi_n\) is
weakly* \( \mathcal{A} \)-measurable iff for all \( n \geq 1 \) the map \( a_n : X \to \mathbb{C} := x \mapsto a_n(x) \) is \( \mathcal{A} \)-measurable. Note that if the map \( \alpha|_{\mathbb{H}_1} : X \to \mathbb{H}_1 \) is weakly* \( \mathcal{A} \)-measurable then the map \( \Gamma : X \times \mathbb{H}_1 \to \mathbb{C} := (x, \psi) \mapsto \langle \psi|\alpha|_{\mathbb{H}_1}(x) \rangle \) is \( \mathcal{A} \)-measurable for all \( \psi \in \mathbb{H}_1 \). For any linear operator \( A : V_1 \to \mathbb{H}_1 \), denote by \( A^\dagger \) its adjoint operator from \( \mathbb{H}_1 \) to \( V_1 \) defined by \( \langle \eta|A^\dagger|\psi \rangle = \langle A\eta|\psi \rangle \) for all \( \eta \in V_1 \) and \( \psi \in \mathbb{H}_1 \).

Let \( \mathcal{J} : \mathcal{A} \to B^+(B(\mathbb{H}_1)_*, B(\mathbb{H}_2)_*) \) be a CP instrument such that \( \nu_1 \ll \mu \) where \( \mu \) is a \( \sigma \)-finite measure. Then \( \nu_b \) is absolutely continuous w.r.t. \( \mu \) for all \( b \in \mathcal{N} \) since \( \nu_b(A) \leq \|b\|\nu_1(A) \) for all \( A \in \mathcal{A} \). Let \( \{\zeta_i\} \) be an orthonormal basis of \( \mathbb{H}_2 \).

**Lemma 3.3** [19]

(i) There are weakly* \( \mathcal{A} \)-measurable maps \( \Gamma_{lk} : X \to V_1^o \) such that the linear operators

\[
B_k(x) : V_1 \to \mathbb{H}_2 := \sum_l |\zeta_l\rangle\langle \Gamma_{lk}(x)|, \quad 1 \leq k < r(x) + 1
\]

are linearly independent for any \( x \in X \), and

\[
\langle \eta_1|\nu_b(A)\eta_2 \rangle = \sum_{l,m} \langle \zeta_l|b\zeta_m \rangle \int_A \sum_{k=1}^{r(x)} \langle \eta_1|\Gamma_{kl}(x)\rangle\langle \Gamma_{km}(x)|\eta_2 \rangle \mu(dx)
\]

\[
= \int_A \sum_{k=1}^{r(x)} \langle \eta_1|B_k(x)\rangle bB_k(x)\eta_2 \rangle \mu(dx), \quad \forall \eta_1, \eta_2 \in V_1
\]

for any \( A \in \mathcal{A} \).

(ii) For any \( A \in \mathcal{A} \),

\[
\nu_b(A) = \sum_{k=1}^{r(A)} C_k(A)^\dagger bC_k(A), \quad \sigma\text{-weakly},
\]

where \( r(A) \leq \dim \mathbb{H}_1 \times \dim \mathbb{H}_2 \) and the bounded linear operators \( C_k(A) : \mathbb{H}_1 \to \mathbb{H}_2, \quad 1 \leq k < r(A) + 1 \) are linearly independent.

The following result guarantees that under some mild conditions the sequential composite of instruments can be extended to an instrument uniquely. Let \( \mathbb{H} \) be a complex Hilbert space and \( (X_i, \mathcal{B}(X_i)), i \in [n] \) measurable spaces where \( X_i \) is a locally compact second countable Hausdorff topological space and \( \mathcal{B}(X_i) \) the Borel \( \sigma \)-algebra of \( X_i \). Let \( \mathcal{I}_i : \mathcal{B}(X_i) \to B^+(B(\mathbb{H})_*, i \in [n] \) be instruments.

**Proposition 3.4** There is a unique instrument \( \mathcal{I} \in \text{Ins}(\Pi_{i=1}^n \mathcal{B}(X_i), B(\mathbb{H})_*) \) such that

\[
\mathcal{I}(A_1 \times \cdots \times A_n) = \mathcal{I}_n(A_n) \circ \cdots \circ \mathcal{I}_1(A_1), \quad \forall A_i \in \mathcal{B}(X_i),
\]

and \( \mathcal{I} \) is called the composition of \( \mathcal{I}_i, i \in [n] \).

We next provide a result concerning commutativity of instruments and joint measurability of the observables induced by these instruments.
Proposition 3.5 Assume that \((X_i, \mathcal{B}(X_i)), i \in [n]\) are type-A measurable spaces and \(\mathcal{J}_i : \mathcal{B}(X_i) \to \mathcal{B}(\mathbb{H})^+, i \in [n]\) are CP instruments. If \(\mathcal{J}_i^* \) commutes with \(\mathcal{J}_i \circ \cdots \circ \mathcal{J}_{i+1} \) for any \(i \in [n]\) then the observables \(\nu_i, i \in [n]\) induced by \(\mathcal{J}_i, i \in [n]\) are jointly measurable and they have \(\mathcal{J}^* \) as a joint observable where \(\mathcal{J}\) is the composite of \(\mathcal{J}_i, i \in [n]\).

4 A Family of Posterior Normal States

Let \(\mathbb{H}\) be a complex Hilbert space and \(\mathcal{M} \subseteq \mathcal{B}(\mathbb{H})\) a von Neumann algebra. Denote by \(\mathcal{R}[\text{Ins}(\mathcal{A}, \mathcal{M}^*), \mathcal{S}(\mathcal{M}), X, \mathcal{S}(\mathcal{M})]\) the set of maps from \(\text{Ins}(\mathcal{A}, \mathcal{M}^*) \times \mathcal{S}(\mathcal{M}) \times X \to \mathcal{S}(\mathcal{M})\).

Definition 4.1 Let \((\mathcal{J}, \varphi) \in \text{Ins}(\mathcal{A}, \mathcal{M}^*) \times \mathcal{S}(\mathcal{M})\) and \(f \in \mathcal{R}\). Then the set \(\{f(\mathcal{J}, \varphi, x) : x \in X\}\) is called a disintegration w.r.t. \((\mathcal{J}, \varphi)\) iff

(i) \(\{f(\mathcal{J}, \varphi, x) : x \in X\}\) is weakly* \(P_{\varphi}\) measurable (i.e. the function \(g_a : X \to \mathbb{C} := x \mapsto f(\mathcal{J}, \varphi, x), a\) is \(P_{\varphi}\) measurable for any \(a \in \mathcal{M}\)).

(ii) For any \(a \in \mathcal{M}\) and \(A \in \mathcal{A}\),

\[
\int_A g_a(x)P_{\varphi}(dx) = \langle \mathcal{J}(A)\varphi, a \rangle. \tag{21}
\]

If \(\{f(\mathcal{J}, \varphi, x) : x \in X\}\) is a disintegration w.r.t. \((\mathcal{J}, \varphi)\), then it exists uniquely in the sense that if \(\{f'(\mathcal{J}, \varphi, x) : x \in X\}\) is another disintegration w.r.t. \((\mathcal{J}, \varphi)\) then for any \(a \in \mathcal{M}\), \(\langle f(\mathcal{J}, \varphi, x), a \rangle = \langle f'(\mathcal{J}, \varphi, x), a \rangle\) \(P_{\varphi}\)-a.s.

A disintegration \(\{f(\mathcal{J}, \varphi, x) : x \in X\}\) w.r.t. \((\mathcal{J}, \varphi)\) is called proper iff for any \(0 \leq a \in \mathcal{M}\), \(\langle \mathcal{J}(X)\varphi, a \rangle = 0\) implies \(g_a(x) = 0\) for all \(x \in X\). Although not obvious, for any \((\mathcal{J}, \varphi) \in \text{Ins}(\mathcal{A}, \mathcal{M}^*) \times \mathcal{S}(\mathcal{M})\) a proper disintegration w.r.t. \((\mathcal{J}, \varphi)\) exists.

A disintegration \(\{f(\mathcal{J}, \varphi, x) : x \in X\}\) w.r.t. \((\mathcal{J}, \varphi)\) is called a family of posterior normal states w.r.t. \((\mathcal{J}, \varphi)\) iff \(\{f(\mathcal{J}, \varphi, x) : x \in X\} \subseteq \mathcal{S}(\mathcal{M})\).

Proposition 4.1 [20] Let \(\mathcal{M}\) be the enveloping von Neumann algebra of a \(C^*\)-algebra \(\mathcal{C}\). If the sequential weak closure \(\sigma(\mathcal{C})\) of \(\mathcal{C}\) coincides with \(\mathcal{M}\), or if \(\mathcal{C}\) is an ideal of \(\mathcal{M}\), then for any \((\mathcal{J}, \varphi) \in \text{Ins}(\mathcal{A}, \mathcal{M}^*) \times \mathcal{S}(\mathcal{M})\) a proper family of posterior normal states w.r.t. \((\mathcal{J}, \varphi)\) exists.

Especially, for any \((\mathcal{J}, \varphi) \in \text{Ins}(\mathcal{A}, \mathcal{B}(\mathbb{H})^*) \times \mathcal{S}(\mathcal{B}(\mathbb{H}))\) a proper family of posterior normal states w.r.t. \((\mathcal{J}, \varphi)\) exists.

Proposition 4.2 Identify \(\mathcal{B}(\mathbb{H})^*\) with \(\mathcal{B}_1(\mathbb{H})\) and assume that \(\mathbb{H}\) is separable. Then for any \((\mathcal{J}, \rho) \in \text{CPIns}(\mathcal{A}, \mathcal{B}_1(\mathbb{H})) \times \mathcal{S}(\mathbb{H})\), a family of posterior normal states w.r.t. \((\mathcal{J}, \rho)\) can be chosen to be

\[
f(\mathcal{J}, \rho, x) = \frac{1}{s_\rho} \sum_{k=1}^{r(\mathcal{x})} B_k(x) \rho B_k(x)^\dagger, \quad \sigma\text{-weakly and } \mu\text{-a.e.} \tag{22}
\]

where \(s_\rho\) denotes the nonnegative Radon-Nikodým derivative of \(P_\rho\) w.r.t. \(\mu\).
Proof Let \( \{ \psi_n \} \) be an orthonormal basis of \( \mathbb{H} \) such that \( G(\mathbb{H}) \ni \rho = \sum_n \lambda_n | \psi_n \rangle \langle \psi_n | \) where \( \lambda_n \geq 0 \) and \( \sum_n \lambda_n = 1 \). By equation (21) and Lemma 3.3 we have for any \( A \in \mathcal{A} \) and \( 0 \leq a \in B(\mathbb{H}) \)

\[
\int_A \text{tr}[f(\mathcal{I}, \rho, x)a]P_\rho(dx) = \text{tr}[\rho \nu_a(A)]
\]

\[
= \sum_n \lambda_n \langle \psi_n | \nu_a(A) | \psi_n \rangle
\]

\[
= \sum_n \lambda_n \int_A \sum_{k=1}^{r(x)} \langle \psi_n | B_k(x) | aB_k(x) | \psi_n \rangle \mu(dx),
\]

which leads to

\[
\text{tr}[f(\mathcal{I}, \rho, x)a] = \frac{1}{s_\rho} \sum_n \lambda_n \sum_{k=1}^{r(x)} \langle \psi_n | B_k(x) \rangle \langle aB_k(x) | \psi_n \rangle, \quad \mu\text{-a.e.}
\]

by the monotone convergence theorem. Set \( a = 1 \) one gets

\[
s_\rho = \sum_n \lambda_n \sum_{k=1}^{r(x)} \langle \psi_n | B_k(x) \rangle \langle B_k(x) | \psi_n \rangle, \quad \mu\text{-a.e.}
\]

Since any \( a \in B(\mathbb{H}) \) can be written as a complex combination of positive operators on \( \mathbb{H} \), one sees that a family of posterior normal states w.r.t. \( (\mathcal{I}, \rho) \) can be chosen to be

\[
f(\mathcal{I}, \rho, x) = \frac{1}{s_\rho} \sum_{k=1}^{r(x)} B_k(x) \rho B_k(x)^\dagger, \quad \sigma\text{-weakly and } \mu\text{-a.e.}
\]

□

In particular, if \( \nu_1 \ll \nu_{1, \rho'} = P_\rho' \) where \( \rho' \in G(\mathbb{H}) \) is a full-rank density operator, then

\[
f(\mathcal{I}, \rho, x) = \frac{1}{s_\rho} \sum_{k=1}^{r(x)} B_k(x) \rho B_k(x)^\dagger, \quad \sigma\text{-weakly and } P_{\rho'}\text{-a.s.}
\]

Remark 4.1 The Bayes Rule.

Let \( (\Theta, \mathcal{E}, P), (X, \mathcal{A}, \mu) \) be two complete measure spaces where \( P \) is a probability measure and \( \mu \) a \( \sigma \)-finite measure. Let \( p(x|\theta) \) be a nonnegative \( \mathcal{E} \times \mathcal{A} \) measurable function satisfying

\[
\int_X p(x|\theta) \nu(dx) = 1, \quad \forall \theta \in \Theta,
\]

so that the function

\[
K : \Theta \times \mathcal{A} \to [0, 1] := (\theta, A) \mapsto \int_A p(x|\theta) \nu(dx)
\]

is a Markov kernel. Denote by \( L^\infty(\Theta, P) \) the abelian von Neumann algebra of essentially bounded functions from \( \Theta \) to \( \mathbb{C} \). Identify \( L^\infty(\Theta, P)_* \) with \( L^1(\Theta, P) \) and define an instrument \( \mathcal{I} \) by

\[
\mathcal{I}(A) \cdot = K(\theta, A) \cdot, \quad \forall A \in \mathcal{A}.
\]
Assume that a family of posterior normal states \( \{ \pi(\theta|x) : x \in X \} \) w.r.t. \( (\mathcal{A}, \pi(\theta)) \) exists and the map \( x \to \int_{\Theta} f(\theta)\pi(\theta|x)P(d\theta) \) is \( \mu \) measurable, where \( \pi(\theta) \in L^1_{+,1}(\Theta, P) \) is a normal state. Then by equation (21) we have
\[
\int_{A} \left[ \int_{\Theta} K(\theta, dx)\pi(\theta)P(d\theta) \right] \int_{\Theta} f(\theta)\pi(\theta|x)P(d\theta) = \int_{\Theta} f(\theta)K(\theta, A)\pi(\theta)P(d\theta)
\]
for all \( A \in \mathcal{A} \) and \( f(\theta) \in L^\infty_{+,1}(\Theta, P) \). By some technical tricks (Tonelli-Fubini Theorem etc.) we have
\[
\int_{A} \mu(dx) \int_{\Theta} p(x|\theta)\pi(\theta)P(d\theta) \int_{\Theta} f(\theta)\pi(\theta|x)P(d\theta) = \int_{A} \mu(dx) \int_{\Theta} f(\theta)p(x|\theta)\pi(\theta)P(d\theta)
\]
for all \( A \in \mathcal{A} \) and \( f(\theta) \in L^\infty_{+,1}(\Theta, P) \), which leads to
\[
\pi(\theta|x) = \int_{\Theta} p(x|\theta)\pi(\theta)P(d\theta) = p(x|\theta)\pi(\theta), \quad (P \times \mu)-a.e.
\]

A family of posterior normal states \( \{ f(\mathcal{A}, \varphi, x) : x \in X \} \) w.r.t. \( (\mathcal{A}, \varphi) \) is said to be strongly \( \mathcal{A} \) measurable iff there is a sequence \( \{ h_n \} \) of \( \mathcal{M}_* \)-valued simple functions on \( X \) such that \( \lim_{n \to \infty} \| f(\mathcal{A}, \varphi, x) - h_n(x) \| = 0 \) for all \( x \in X \) iff \( \{ f(\mathcal{A}, \varphi, x) : x \in X \} \) exists uniquely in the sense that if \( \{ f'(\mathcal{A}, \varphi, x) : x \in X \} \) is another family of posterior normal states w.r.t. \( (\mathcal{A}, \varphi) \) then \( f(\mathcal{A}, \varphi, x) = f'(\mathcal{A}, \varphi, x) \) \( P_{\varphi} \)-a.s. and the Bochner integral equation
\[
\mathcal{A}(\varphi) = \int_{A} f(\mathcal{A}, \varphi, x)P_{\varphi}(dx)
\]
holds for all \( A \in \mathcal{A} \).

**Proposition 4.3** [20] If for any decreasing sequence \( \{ p_n \} \) of projections in \( \mathcal{M} \) that satisfies \( \lim\inf_{n \to \infty} p_n = 0 \) we have
\[
\lim_{n \to \infty} \int_{X} q(x)\langle \mathcal{A}(dx)\varphi, p_n \rangle = 0
\]
uniformly for all \( q \in L^1(X, P_{\varphi}) \) that satisfies \( \| q \| \leq 1 \), where \( (\mathcal{A}, \varphi) \in \text{Ins}(\mathcal{A}, \mathcal{M}_*) \times \mathcal{G}(\mathcal{M}) \), then a proper strongly \( \mathcal{A} \) measurable family of posterior normal states w.r.t. \( (\mathcal{A}, \varphi) \) exists.

**Proposition 4.4** [18] Assume that \( \mathcal{M} \) is \( \sigma \)-finite. Then for any \( (\mathcal{A}, \varphi) \in \text{CPIns}(\mathcal{A}, \mathcal{M}_*) \times \mathcal{G}(\mathcal{M}) \), a strongly \( \mathcal{A} \) measurable family of posterior normal states w.r.t. \( (\mathcal{A}, \varphi) \) exists iff \( \mathcal{A} \) has the normal extension property.

The following result is more commonly used.

**Proposition 4.5** [20] For any \( (\mathcal{A}, \varphi) \in \text{CPIns}(\mathcal{A}, \mathcal{B}(\mathcal{H})) \times \mathcal{G}(\mathcal{B}(\mathcal{H})) \) a strongly \( \mathcal{A} \) measurable family of posterior normal states w.r.t. \( (\mathcal{A}, \varphi) \) exists.
5 Quantum Bayesian Statistical Decision

The notions here are parallel generalizations of the classical ones. Let $\mathbb{H}$ be a complex Hilbert space and $\mathcal{M} \subseteq \mathcal{B}(\mathbb{H})$ a von Neumann algebra.

**Definition 5.1** A quantum statistical model is a pair $(\nu, \Lambda)$ where $\nu : \mathcal{A} \to \mathcal{M}$ is an observable and $\Lambda \subseteq \mathcal{G}(\mathcal{M})$. Moreover, if $\Lambda = \{ \varphi(\theta) \in \mathcal{G}(\mathcal{M}) : \theta \in \Theta \}$, then $(\nu, \varphi(\theta))$ is called a parametric quantum statistical model.

Let $(Y, \mathcal{F}, \Pi)$ be a statistical model and $(\Pi, \mathcal{G})$ a measurable space. A map $\delta : X \to \Pi$ is called a randomized decision rule iff it is $\mathcal{A}$ measurable. Moreover, $\delta$ is called a nonrandomized decision rule iff all $\pi \in \Pi$ are degenerate. Denote by $\Delta$ (resp. $D$) a class of randomized decision rules (resp. nonrandomized decision rules) from $X$ to $\Pi$.

Let $\{ (X_\alpha, \mathcal{B}(X_\alpha)) : \alpha \in I \}$ be a family of measurable spaces where $X_\alpha$ is a locally compact second countable Hausdorff topological space and $\mathcal{B}(X_\alpha)$ the Borel $\sigma$-algebra of $X_\alpha$. Let $\{ J_\alpha \in \text{CPIns}(\mathcal{A}_\alpha, \mathcal{B}(\mathbb{H})_\alpha) : \alpha \in I \}$ be a family of CP instruments.

Assume that we are going to perform $n$ indirect measurements $\{ (H_i', \phi_i, U_i, \nu_i) \}_{i=1}^n$ of which the corresponding CP instruments are $\{ J_i : J_i \in \{ J_\alpha \} \}_{i=1}^n$ on the quantum system $S$ described by the von Neumann algebra $\mathcal{B}(\mathbb{H})$ sequentially. Denote by $X$ the Cartesian product $X_1 \times \cdots \times X_n$, by $\mathcal{J}$ the composite of $J_i, i \in [n]$ and by $\nu_1$ the observable induced by $J_1$.

**Remark 5.1**

(i) $J$ is a CP instrument.

(ii) $\nu_1$ is a joint observable of the observables $\nu'_i = J_1^*(X_1) \circ \cdots \circ J_{i-1}^*(X_{i-1}) \circ J_i^*(\cdot) \circ J_{i+1}^*(X_{i+1}) \circ \cdots \circ J_n^*(X_n), i \in [n]$. Since in general $\nu'_i \neq \nu_1$ for $i \in \{ 2, \ldots, n \}$, it is too naive to measure $\nu_i, i \in [n]$ jointly by measuring them sequentially.

Let $(\nu_1, \varphi(\theta))$ be a parametric quantum statistical model and $x = (x_1, \ldots, x_n) \in X$. Assume that for any given $\theta \in \Theta$ the loss function $L : \Theta \times Y \to \mathbb{R}_+$ is $\mathcal{F}$ measurable. Then the risk function

$$R(\theta, \delta) = E_{x|\theta}E_{y|x,\theta}[L(\theta, y)] = \int_X \langle J(dx)\varphi(\theta), 1 \rangle \int_Y L(\theta, y)\delta(x)(dy). \quad (38)$$

**Definition 5.2** If there is another randomized decision rule $\delta' \in \Delta$ such that

$$R(\theta, \delta') \leq R(\theta, \delta), \forall \theta \in \Theta \quad (39)$$

$$\exists \theta' \in \Theta \text{ s.t. } R(\theta', \delta') < R(\theta', \delta), \quad (40)$$

then $\delta$ is inadmissible.
Definition 5.3 If there is a nonrandomized decision rule \( \tilde{\delta} \in \mathcal{D} \) such that

\[
R(\theta, \tilde{\delta}) = \inf_{\delta \in \mathcal{D}} \sup_{\theta \in \Theta} R(\theta, \delta),
\]

then \( \tilde{\delta} \) is called minimax.

Let \( (\Theta, \mathcal{E}, P) \) be a probability space and \( \nu : \mathcal{E} \to \mathcal{B}(\mathbb{H}) \) an observable.

Definition 5.4 The function

\[
R : \Lambda \times \Delta \to \mathbb{R}_+ = E_{\theta}[R(\theta, \delta)]
\]

is called the quantum Bayes risk.

Moreover, if there is a nonrandomized decision rule \( \tilde{\delta} \in \mathcal{D} \) such that

\[
R(\varphi, \tilde{\delta}) = \inf_{\delta \in \mathcal{D}} R(\varphi, \delta),
\]

then \( \tilde{\delta} \) is called the quantum Bayes solution w.r.t. \((\varphi, \mathcal{D})\).

We next discuss the admissibility and minimax of quantum Bayesian solution.

Theorem 5.1 Assume that \( \Theta \) is a topological space and \( \mathcal{E} \) a \( \sigma \)-algebra containing all open subsets of \( \Theta \). Let \( \tilde{\delta} \) be the quantum Bayes solution w.r.t. \((\varphi, \mathcal{D})\). If

(i) \( P(E) > 0 \) for all open subsets \( E \) of \( \Theta \);
(ii) \( R(\varphi, \tilde{\delta}) < \infty \);
(iii) \( R(\theta, \delta) \) is continuous w.r.t. \( \theta \) for all \( \delta \in \mathcal{D} \),

then \( \tilde{\delta} \) is admissible.

Proof If \( \tilde{\delta} \) is inadmissible, then there is a nonrandomized decision rule \( \delta' \in \mathcal{D} \) such that

\[
R(\theta, \delta') \leq R(\theta, \tilde{\delta}), \quad \forall \theta \in \Theta
\]

\[
\exists \delta' \in \Theta \text{ s.t. } R(\theta, \delta') < R(\theta, \tilde{\delta}).
\]

By condition (iii), there is a positive number \( \epsilon > 0 \) together with an open neighbourhood \( U_{\epsilon}(\theta') \) such that

\[
R(\theta, \delta') \leq R(\theta, \tilde{\delta}) - \epsilon, \quad \forall \theta \in U_{\epsilon}(\theta').
\]

Then we have

\[
R(\varphi, \delta') = \int_{U_{\epsilon}(\theta')} R(\theta, \delta') P(d\theta) + \int_{\Theta \setminus U_{\epsilon}(\theta')} R(\theta, \delta') P(d\theta)
\]

\[
\leq \int_{U_{\epsilon}(\theta')} [R(\theta, \tilde{\delta}) - \epsilon] P(d\theta) + \int_{\Theta \setminus U_{\epsilon}(\theta')} R(\theta, \tilde{\delta}) P(d\theta)
\]

\[
= R(\varphi, \tilde{\delta}) - \epsilon P[U_{\epsilon}(\theta')]
\]

\[
< R(\varphi, \tilde{\delta}),
\]

which contradicts that \( \tilde{\delta} \) is the quantum Bayes solution w.r.t. \((\varphi, \mathcal{D})\). \( \square \)
Theorem 5.2 If $\hat{\delta}$ is the unique quantum Bayes solution w.r.t. $(\varphi, D)$, then $\hat{\delta}$ is admissible.

Proof If $\hat{\delta}$ is inadmissible, then there is a nonrandomized decision rule $\delta' \in D$ such that
\begin{align}
R(\theta, \delta') &\leq R(\theta, \hat{\delta}), \quad \forall \theta \in \Theta \quad (52) \\
\exists \theta' \in \Theta \text{ s.t. } R(\theta', \delta') &< R(\theta', \hat{\delta}). \quad (53)
\end{align}
Then we have
\begin{align}
R(\varphi, \delta') = \int_{\Theta} R(\theta, \delta')P(d\theta) \leq \int_{\Theta} R(\theta, \hat{\delta})P(d\theta) = R(\varphi, \hat{\delta}), \quad (54)
\end{align}
which shows that $\hat{\delta}$ is also a quantum Bayes solution w.r.t. $(\varphi, D)$ and this contradicts that $\hat{\delta}$ is the unique quantum Bayes solution w.r.t. $(\varphi, D)$. □

Theorem 5.3 Let $\tilde{\delta}$ be the quantum Bayes solution w.r.t. $(\varphi, D)$. If $R(\theta, \tilde{\delta}) = c$ for all $\theta \in \Theta$, then $\tilde{\delta}$ is minimax.

Proof On the one hand,
\begin{align}
c = R(\varphi, \tilde{\delta}) &= \inf_{\delta \in D} R(\varphi, \delta) \quad (55) \\
&\leq \sup_{\varphi' \in \Lambda} \inf_{\delta \in D} R(\varphi', \delta) \quad (56) \\
&\leq \inf_{\delta \in D} \sup_{\varphi' \in \Lambda} R(\varphi', \delta) \quad (57) \\
&\leq \inf_{\delta \in D} \sup_{\theta \in \Theta} R(\theta, \delta). \quad (58)
\end{align}
On the other hand,
\begin{align}
c = \sup_{\theta \in \Theta} R(\theta, \tilde{\delta}) &\geq \inf_{\delta \in D} \sup_{\theta \in \Theta} R(\theta, \delta). \quad (59)
\end{align}
□

Theorem 5.4 Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of normal states and $\tilde{\delta}_n$ the quantum Bayes solution w.r.t. $(\varphi_n, D)$. Let $\delta \in D$ be a nonrandomized decision rule. If
\begin{align}
\sup_{\theta \in \Theta} R(\theta, \delta) \leq \limsup_{n \to \infty} R(\varphi_n, \tilde{\delta}_n), \quad (60)
\end{align}
then $\delta$ is minimax.

Proof If $\delta$ is not minimax, then there is a nonrandomized decision rule $\delta' \in D$ such that
\begin{align}
\sup_{\theta \in \Theta} R(\theta, \delta') &< \sup_{\theta \in \Theta} R(\theta, \delta). \quad (61)
\end{align}
For any $n \geq 1$ we have
\begin{align}
R(\varphi_n, \tilde{\delta}_n) \leq R(\varphi_n, \delta') \quad (62)
\end{align}
Quantum Bayesian Statistical Inference

\[ \leq \sup_{\theta \in \Theta} R(\theta, \delta') \]
\[ < \sup_{\theta \in \Theta} R(\theta, \delta), \]
which contradicts the condition (60). □

**Theorem 5.5** Let \( \delta \in \mathcal{D} \) be a nonrandomized decision rule such that \( R(\theta, \delta) = c \) for all \( \theta \in \Theta \). If there is a sequence of normal states \( \{\varphi_n\}_{n=1}^\infty \) with \( \delta_n \) the quantum Bayes solution w.r.t. \( (\varphi_n, \mathcal{D}) \) such that
\[ \lim_{n \to \infty} R(\varphi_n, \delta_n) = c, \] (65)
then \( \delta \) is minimax.

**Proof** For any \( \theta \in \Theta \) we have
\[ \sup_{\theta \in \Theta} R(\theta, \delta) = c = \lim_{n \to \infty} R(\varphi_n, \delta_n). \] (66)
Thus by Theorem 5.4, \( \delta \) is minimax. □

We now introduce the quantum Bayes rule.

**Quantum Bayes Rule** If a quantum system \( S \) described by a von Neumann algebra \( \mathcal{M} \) is in the (prior) normal state \( \varphi \) before \( n \) indirect measurements \( \{(H_i, \phi_i, U_i, \nu_i)\}_{i=1}^n \) sequentially performed on \( \mathcal{M} \), then after these measurements \( S \) will be in the (posterior) normal state \( \varphi_x := f(\mathcal{J}_n, \cdot, x_n) \circ \cdots \circ f(\mathcal{J}_1, \varphi, x_1) \) if the measurement outcomes are \( x_1, \cdots, x_n \).

Denote by \( \{\varphi_x : x \in X\} \) the set \( \{f(\mathcal{J}_n, \cdot, x_n) \circ \cdots \circ f(\mathcal{J}_1, \varphi, x_1) : x \in X\} \) where \( \{f(\mathcal{J}_i, \cdot, x_i) : x_i \in X_i\} \) is a strongly \( \mathcal{A}_i \) measurable family of posterior normal states w.r.t. \( (\mathcal{J}_i, \cdot) \) for all \( i \in [n] \). Denote by \( \{\varphi_x : \varphi \in \Lambda\} \) the set \( \{f(\mathcal{J}_n, \cdot, x_n) \circ \cdots \circ f(\mathcal{J}_1, \varphi, x_1) : \varphi \in \Lambda\} \). Let \( \Lambda_x := \{\varphi_x : \varphi \in \Lambda\} \).

**Definition 5.5** The function
\[ R : \Lambda_x \times \Delta \to \mathbb{R}_+ := E\varphi_x E_y|\varphi_x L(\theta, y) \]
\[ = \int_{\Theta} \langle \varphi_x, \nu(d\theta) \rangle \int_Y L(\theta, y) \delta(x)(dy) \] (68)
is called the quantum posterior risk.

Moreover, if there is a nonrandomized decision rule \( \delta^* \in \mathcal{D} \) such that
\[ R(\varphi_x, \delta^*) = \inf_{\delta \in \mathcal{D}} R(\varphi_x, \delta), \] (69)
then \( \delta^* \) is called the quantum posterior solution.

**Remark 5.2** For a given quantum Bayesian statistical decision problem, the quantum Bayes solution \( \hat{\delta} \) and the quantum posterior solution \( \delta^* \) are probably not equivalent even if \( R(\varphi, \hat{\delta}) < \infty \) and since their definitions are quite different.
6 Quantum Bayesian Statistical Inference

In this section we focus on the (Bayesian) posterior statistical inference problem. Assume that \((X_\alpha, \mathcal{A}_\alpha) = (\Theta, \mathcal{B}(\mathbb{R})), \forall \alpha \in \mathcal{I}\). Denote by \(P_{\varphi_x}\) the probability measure induced by \(\nu\) for any \(\varphi_x \in \Lambda_x\), by \(F(\theta|x)\) the cumulative distribution function corresponding to \(P_{\varphi_x}\) and by \(p(\theta|x)\) the Radon-Nikodým derivative of \(P_{\varphi_x}\) w.r.t. its dominating measure.

6.1 Quantum Posterior Point Estimation

Assume that \((Y, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))\). Denote by \(\gamma_\delta : X \rightarrow Y := x \mapsto y \) s.t. \(\delta(x)(y) = 1\)

Theorem 6.1 Suppose the loss function \(L(\theta, y) = c(\theta)(\theta - y)^2\) where \(c(\theta) \geq 0\) for all \(\theta \in \Theta\), then the quantum posterior solution \(\delta^*\) is the nonrandomized decision rule such that

\[
\delta^*(x) \left\{ \frac{E_{\theta|x}[\theta c(\theta)]}{E_{\theta|x}[c(\theta)]} \right\} = 1. \tag{70}
\]

Specifically, if \(c(\theta) = 1\) for all \(\theta \in \Theta\), then the quantum posterior solution \(\delta^*\) is the nonrandomized decision rule such that

\[
\delta^*(x)[E_{\theta|x}(\theta)] = 1. \tag{71}
\]

Proof Straight calculation shows that the quantum posterior risk

\[
R(\varphi_x, \delta) = E_{\theta|x}E_{\theta|\theta,x}[c(\theta)(\theta - y)^2] = E_{\theta|x}\{c(\theta)[\theta - \gamma_\delta(x)]^2\} \tag{72}
\]

and apparently \(R(\varphi_x, \delta)\) takes its minimum when \(\gamma_\delta(x)\) satisfies

\[
\frac{d}{d\gamma_\delta}E_{\theta|x}\{c(\theta)[\theta - \gamma_\delta(x)]^2\} = 0, \tag{73}
\]

which is equivalent to \(\gamma_\delta(x) = E_{\theta|x}[\theta c(\theta)]/E_{\theta|x}[c(\theta)]\).

\[
\square
\]

Theorem 6.2 Suppose the loss function \(L(\theta, y) = \begin{cases} k_0(\theta - y) & \text{if } \theta \geq y \\ k_1(y - \theta) & \text{if } \theta < y \end{cases}\) where \(k_0, k_1 \geq 0\), then the quantum posterior solution \(\delta^*\) is the nonrandomized decision rule such that

\[
\delta^*(x)[\xi_{\theta|x}(\frac{k_1}{k_0 + k_1})] = 1. \tag{74}
\]

Specifically, if \(k_0 = k_1 = 1\), then the quantum posterior solution \(\delta^*\) is the nonrandomized decision rule such that

\[
\delta^*(x)[\xi_{\theta|x}(0.5)] = 1. \tag{75}
\]
The inequality above also holds when \( \delta_0 \) the nonrandomized decision rule such that \( \delta_0(x)[\xi_{\theta|x}(p)] = 1 \). Assume that \( y > \xi_{\theta|x}(p) \). A little calculation shows that

\[
L[\theta, \xi_{\theta|x}(p)] - L(\theta, y) = \begin{cases} 
  k_1[\xi_{\theta|x}(p) - y] & \theta \leq \xi_{\theta|x}(p) \\
  (k_0 + k_1)\theta - k_0\xi_{\theta|x}(p) - k_1y & \xi_{\theta|x}(p) < \theta < y \\
  k_0[y - \xi_{\theta|x}(p)] & \theta \geq y
\end{cases}
\]

and

\[
L[\theta, \xi_{\theta|x}(p)] - L(\theta, y) \leq k_0[\xi_{\theta|x}(p) - y]1_{(-\infty, \xi_{\theta|x}(p)]} + k_1[y - \xi_{\theta|x}(p)]1_{(\xi_{\theta|x}(p), +\infty)}
\]

(77)

since \((k_0 + k_1)\theta - k_0\xi_{\theta|x}(p) - k_1y < (k_0 + k_1)y - k_0\xi_{\theta|x}(p) - k_1y = k_0[y - \xi_{\theta|x}(p)]\).

Then for \( \gamma_\delta(x) > \xi_{\theta|x}(p) \) we have

\[
R(\varphi_x, \delta_0) - R(\varphi_x, \delta) = E_{\theta|x}\{L[\theta, \xi_{\theta|x}(p)] - L[\theta, \gamma_\delta(x)]\}
\]

(80)

\[
\leq k_0[\xi_{\theta|x}(p) - \gamma_\delta(x)]P_{\varphi_x}[\theta \leq \xi_{\theta|x}(p)] + k_1[\gamma_\delta(x) - \xi_{\theta|x}(p)]P_{\varphi_x}[\theta > \xi_{\theta|x}(p)]
\]

(81)

\[
\leq \frac{k_0k_1}{k_0 + k_1}[\xi_{\theta|x}(p) - \gamma_\delta(x)] + \frac{k_0k_1}{k_0 + k_1}[\gamma_\delta(x) - \xi_{\theta|x}(p)] = 0,
\]

(83)

which is equivalent to

\[
R(\varphi_x, \delta_0) \leq R(\varphi_x, \delta).
\]

(85)

The inequality above also holds when \( \gamma_\delta(x) < \xi_{\theta|x}(p) \) and thus the quantum posterior solution \( \delta^* = \delta_0 \).

**Theorem 6.3** Suppose the loss function \( L(\theta, y) = \begin{cases} 
  1 & \text{if } |\theta - y| > \epsilon, \\
  0 & \text{if } |\theta - y| \leq \epsilon
\end{cases}; \) \( p(\theta|x) \) is either continuous or discrete and has the unique maximum value, then as \( \epsilon \to 0 \) the quantum posterior solution is the nonrandomized decision rule such that

\[
\delta^*(x)[\max_\theta p(\theta|x)] = 1.
\]

(86)

**Proof** Straight calculation shows that

\[
R(\varphi_x, \delta) = E_{\theta|x}E_{y|x}|L(\theta, y)| = 1 - \int_{\gamma_\delta(x)+\epsilon}^{\gamma_\delta(x)+\epsilon} p(\theta|x)\mu(d\theta),
\]

(87)

where \( \mu \) is the Lebesgue measure or the counting measure. Apparently minimizing \( R(\varphi_x, \delta) \) is equivalent to maximizing \( \int_{\gamma_\delta(x)-\epsilon}^{\gamma_\delta(x)+\epsilon} p(\theta|x)\mu(d\theta) \) and thus \( \gamma_\delta(x) = \max_\theta p(\theta|x) \) as \( \epsilon \to 0 \).
Definition 6.1

\[ \hat{\theta}_E := E_{\theta|x}(\theta) \quad (89) \]

is called the quantum posterior mean estimator of \( \theta \);

\[ \hat{\theta}_Q := \xi_{\theta|x}(0.5) \quad (90) \]

is called the quantum posterior median estimator of \( \theta \);

\[ \hat{\theta}_M := \max_{\theta \in \Theta} p(\theta|x) \quad (91) \]

is called the quantum posterior mode estimator of \( \theta \).

### 6.2 Quantum Posterior Credible Interval

Assume that \( Y \) is the set of closed intervals on the real line and the loss function

\[ L(\theta, y) = k_0 l(y) + k_1 [1 - 1_y(\theta)] \]

where \( l(y) \) is the length of \( y \), \( 1_y(\theta) = \begin{cases} 1 & \text{if } \theta \in y \\ 0 & \text{if } \theta \notin y \end{cases} \) and \( k_0, k_1 \geq 0 \). Denote by \( \gamma_{\delta} : X \to Y := x \mapsto y \) s.t. \( \delta(x)(y) = 1 \).

Then the quantum posterior risk

\[ R(\varphi_x, \delta) = E_{\theta|x} E_{y|\theta,x} \{k_0 l(y) + k_1 [1 - 1_y(\theta)]\} \]

\[ = k_0 l[\gamma_{\delta}(x)] + k_1 E_{\theta|x} [1 - 1_{\gamma_{\delta}(x)}(\theta)] \quad (93) \]

\[ = k_0 l[\gamma_{\delta}(x)] + k_1 p_{\varphi_x}[\theta \notin \gamma_{\delta}(x)]. \]

In general, the quantum posterior solution \( \delta^* = \inf_{\delta \in D} k_0 l[\gamma_{\delta}(x)] + k_1 p_{\varphi_x}[\theta \notin \gamma_{\delta}(x)] \) is not easy to obtain. A common strategy is to keep \( l[\gamma_{\delta}(x)] \) as small as possible while controlling \( p_{\varphi_x}[\theta \notin y(x)] \) not to exceed a given small positive number.

**Definition 6.2** \( \gamma_{\delta}(x) \) is called a \( 1 - \alpha \) quantum posterior credible interval iff

\[ p_{\varphi_x}[\theta \in \gamma(x)] \geq 1 - \alpha, \]

where \( 0 < \alpha < 1 \).

For any \( x \in X \), let \( Y_x = \{ y \subseteq \mathbb{R} : y \text{ is } P_{\varphi_x} \text{ measurable} \} \). Denote by \( Y \) the set \( \bigcup_{x \in X} Y_x \) and by \( \gamma_{\delta} : X \to Y := x \mapsto y \) s.t. \( y \) is \( P_{\varphi_x} \) measurable and \( \delta(x)(y) = 1 \).

**Definition 6.3** \( \gamma_{\delta}(x) \) is called a \( 1 - \alpha \) quantum highest posterior density (QHPD) credible set iff

(i) \( P_{\varphi_x} \) is absolute continuous w.r.t. the Lebesgue measure;
(ii) \( P_{\varphi_x}[\theta \in \gamma_{\delta}(x)] \geq 1 - \alpha \) where \( 0 < \alpha < 1 \);
(iii) For any \( \theta_1 \in \gamma_{\delta}(x) \) and \( \theta_0 \notin \gamma_{\delta}(x) \) we have

\[ p(\theta_1|x) \geq p(\theta_0|x). \]

(96)
6.3 Quantum Posterior Hypothesis Testing

Assume that \((Y, \mathcal{F}) = ([n], 2^{[n]})\) and \(L(\theta, y) = \sum_{i=1}^{n} k_i \delta_{y_i} 1_{\Theta_i}(\theta)\) where \(k_i \geq 0\) for all \(i \in [n]\); \(\delta_{(\cdot)}\) is the Kronecker delta and \(\Theta_i, i \in [n]\) are pairwise disjoint \(P_{\varphi_x}\) measurable subsets of \(\Theta\). Denote by \(\gamma_{\delta} : X \rightarrow Y : x \mapsto y\) s.t. \(\delta(x)(y) = 1\).

**Theorem 6.4** The quantum posterior solution \(\delta^*\) is the nonrandomized decision rule such that

\[ \delta(x)[\arg \min_{i \in [n]} k_i P_{\varphi_x}(\theta \in \Theta_i)] = 1. \] (97)

**Proof** A little calculation shows that the quantum posterior risk

\[ R(\varphi_x, \delta) = E_{\theta|x} E_{y|\theta,x}[L(\theta, y)] \] (98)

\[ = \sum_{i=1}^{n} k_i \delta_{i\gamma_{\delta}(x)} P_{\varphi_x}(\theta \in \Theta_i) \] (99)

and apparently \(R(\varphi_x, \delta)\) takes its minimum when \(\gamma_{\delta}(x) = \arg \min_{i \in [n]} k_i P_{\varphi_x}(\theta \in \Theta_i)\).

**Definition 6.4** The quantum posterior solution \(\delta^*\) is called the quantum posterior testing rule of the multiple hypothesis testing \(H_i : \theta \in \Theta_i, i \in [n]\), (100)

where \(\{\Theta_i : i \in [n]\}\) is a \(P_{\varphi_x}\) measurable partition of \(\Theta\).

7 Large Sample Property

Generally the posterior normal state \(\varphi_x\) on \(\mathcal{M}\) does not converge when a sequential measurement scheme consists of countably infinite many CP instruments \(\{J_i\}_{i=1}^{n}\) is performed on \(\mathcal{M}\). For example, identify \(B(\mathbb{H})_s\) with \(B_1(\mathbb{H})\) and define \(J \in \text{CPIns}(\{\emptyset, x_0\}, B_1(\mathbb{H}))\) by

\[ J(x_0) \cdot u = u(\cdot)u^* \] (101)

where \(u \in B(\mathbb{H})\) is a unitary operator. Repeatedly apply \(n\) copies of \(J\) to the prior normal state \(\rho\) and one sees that the posterior normal state \(\rho_x\) does not converge as \(n \rightarrow \infty\) where \(x = (x_0, \cdots, x_0)\) is a tuple consists of \(n\) copies of \(x_0\). However, in some special cases the posterior normal state converges as the sample size goes to infinity.

Let \((X, \mathcal{A})\) be a measurable space and \(\mathbb{H}\) a separable complex Hilbert space. Define \(J \in \text{CPIns}(\mathcal{A}, B_1(\mathbb{H}))\) by

\[ J(A) \cdot = \sum_{i \in A} a_i(\cdot)a_i^*, \forall A \in \mathcal{A}, \] (102)
where \( a_i \in \mathcal{B}(\mathbb{H}) \) for all \( i \in \mathbb{N}_+ \) and \( \sum_i a_i^* a_i = 1 \) in the strong operator topology of \( \mathcal{B}(\mathbb{H}) \). Let \( \{ f_n : (\Pi_{i=1}^\infty \mathcal{G}_i, \Pi_{i=1}^\infty \mathcal{B}(\mathcal{G}_i)) \to (\mathcal{G}, \mathcal{B}(\mathcal{G})) : (\rho_1, \cdots, \rho_n, \cdots) \mapsto \rho_n \}_{n=1}^\infty \) be a sequence of \( \Pi_{i=1}^\infty \mathcal{B}(\mathcal{G}_i) \) measurable maps where \( \mathcal{G}_i = \mathcal{G}(\mathbb{H}), \mathcal{B}(\mathcal{G}_i) = \mathcal{B}(\mathcal{G}(\mathbb{H})) \) for all \( i \in \mathbb{N}_+ \). Then the function

\[
g : \mathcal{G}(\mathbb{H}) \times \mathcal{B}(\mathcal{G}(\mathbb{H})) \to [0, 1] := (\rho, A) \mapsto \sum_{i=1}^\infty \text{tr}(a_i \rho a_i^*) 1_{\rho_i}(A)
\]

is a Markov kernel where \( \rho_i = a_i \rho a_i^*/\text{tr}(a_i \rho a_i^*) \) so that \( \{ f_n \}_{n=1}^\infty \) is a homogeneous Markov chain. Denote by \( P_{\rho_0} \) the probability measure of \( \{ f_n \}_{n=1}^\infty \) initializing at \( \rho_0 \).

**Theorem 7.1** [21] Let \( \{ f_n \}_{n=1}^\infty \) be the homogeneous Markov chain started at \( \rho_0 \). Then there is a \( \{ \rho \in \mathcal{G}(\mathbb{H}) : \sum_i a_i \rho a_i^* = \rho \} \) valued \( \Pi_{i=1}^\infty \mathcal{B}(\mathcal{G}_i) \) measurable map \( f_0 \) such that

\[
\frac{1}{n} \sum_{i=1}^n f_n \to f_0, \text{ weakly}^* \text{ and } P_{\rho_0}\text{-a.s.}
\]

as \( n \to \infty \).

**Proposition 7.2** [21] For any \( m \in \mathbb{N}_+ \) we have

\[
\text{tr}(f_n^m) \to h^{(m)} \text{ \( P_{\rho_0}\text{-a.s.} \)}
\]

as \( n \to \infty \), where \( h^{(m)} \in L^1(\Pi_{i=1}^\infty \mathcal{G}_i, P_{\rho_0}) \).

Let \( \mathcal{C} \subseteq \mathcal{B}(\mathbb{H}) \) be a strongly compact algebra i.e. an algebra whose closed unit ball is relatively compact w.r.t. the strong operator topology of \( \mathcal{B}(\mathbb{H}) \). Let \( D = \{ \rho \in \mathcal{G}(\mathbb{H}) \cap \mathcal{B}_0(\mathbb{H}) : \rho^{1/2} \in \mathcal{C} \} \) and assume that \( a_i \in \mathcal{C} \) for all \( i \in \mathbb{N}_+ \). The following theorem shows that if \( f_n \) is not asymptotically extreme on \( D \) then every \( a_i, i \in \mathbb{N}_+ \) when restricted to a subspace of \( \mathbb{H} \) whose dimension is larger than 2, is proportional to an isometry.

**Proposition 7.3** [21] If there is a density operator \( \rho_0 \in D \) such that for a positive integer \( m_0 \in \mathbb{N}_+ \)

\[
P_{\rho_0}(h^{(m_0)} \neq 1) > 0,
\]

then there is a projection \( p \in \mathcal{B}(\mathbb{H}) \) with \( \text{rank}(p) \geq 2 \) such that

\[
p a_i^* a_i p = \lambda_i p, \quad \lambda_i \geq 0, \quad \forall i \in \mathbb{N}_+.
\]

In the case when \( \mathbb{H} \) is finite dimensional and all CP instruments \( \{ \mathcal{I} \}_{i=1}^\infty \) in the sequential measurement scheme are channels we have the following result. Assume that \( \mathbb{H} \simeq \mathbb{C}^k \) is of finite dimension. Let \( (X, \mathcal{A}, P) \) be a probability space and \( \mathfrak{T} : X \to X \) an invertible, measure preserving, ergodic map i.e. for all \( A \in \mathcal{A} \) satisfying \( \mathfrak{T}^{-1}(A) = A \) we have \( P(A) = 0 \) or 1. We say that...
a map $f \in B^+[M_k(\mathbb{C})]$ is strictly positive if $f(a) > 0$ for all $M_k^+(\mathbb{C})/\{0\}$. Let $f_0$ be a channel (on $M_k(\mathbb{C})$) valued $\mathcal{A}$ measurable map. Define $f_n(x) := f_0[\Sigma^n(x)], \forall x \in X, n \in \mathbb{Z}$.

**Theorem 7.4** Assume that there is a positive integer $N$ such that

\begin{align}
(1) \quad & \mathbb{P}(f_N \circ \cdots \circ f_0 \text{ is strictly positive}) > 0; \\
(2) \quad & \mathbb{P}[\ker(f_0^*) \cap M_k^+(\mathbb{C}) = \{0\}] = 1.
\end{align}

Then there is a positive number $\alpha > 0$, a $\mathcal{G}(\mathbb{C}^k)$ valued $\mathcal{A}$ measurable map $g_0$ and a sequence of maps $\{g_n := g_0[\Sigma^n(x)], \forall x \in X, n \in \mathbb{Z}\}$ such that for any $(m, z, n) \in \mathbb{Z}^3$ satisfying $m < n, m \leq z, z \leq n$ we have

$$\|f_n \circ \cdots \circ f_m(\rho) - g_n\|_1 \leq C_{\alpha, z} e^{-\alpha(n-m)}, \forall \rho \in \mathcal{G}(\mathbb{C}^k),$$

where $C_{\alpha, z}$ is finite $\mathbb{P}$-a.s.

**Proof** By Lemma 3.8, Lemma 3.13, Lemma 3.14 and Lemma 3.15 in [22]. □

Let $f$ be a channel on $M_k(\mathbb{C})$.

**Theorem 7.5** [23] Assume that the spectrum of $f$ has the unique eigenvalue $\{1\}$ on the unit circle and this eigenvalue is simple. Then there are positive numbers $C, \alpha > 0$ such that for any $n \in \mathbb{N}_+$ we have

$$\|f^n(\rho) - \rho_*\|_1 \leq C e^{-\alpha n}, \forall \rho \in \mathcal{G}(\mathbb{C}^k),$$

where $\rho_* \in \mathcal{G}(\mathbb{C}^k)$ satisfies $f(\rho_*) = \rho_*$. 

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