Transformation of the spinless Salpeter equation

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I. INTRODUCTION

Relativistic few-body problem has received a great attention in many fields of researches including particle physics. Solution of the problem in a way fully consistent with all requirements imposed by special relativity and within the framework of Quantum Field Theory (QFT) is one of the great challenges in theoretical elementary particle physics. The most complete results here exist for the case of two particles. However, even the simplest relativistic two-body (R2B) bound-state problem has several principal difficulties.

There are two main aspects of the problem: 1) the scattering of two particles and 2) bound state of the particles. These are two main R2B problems in quantum physics. The scattering and bound-state problems in quantum physics are related to solution of a single covariant equation, but for different boundary conditions. The R2B bound-state problem has several principal difficulties. Main of them concern $\alpha$) the equation of motion and $\beta$) the interaction potential. Judging from the large variety of approaches attempted even in recent years, this problem has no generally agreed-upon solution [1, 2].

Description of the R2B bound systems has a long history. There have been proposed several wave equations for the description of bound states within relativistic quantum theory. Solution of the R2B problem in a way fully consistent with all requirements imposed by special relativity and within the framework of QFT is founded on the four-dimensional (4D) covariant Bethe-Salpeter equation (BS) [3, 4]. The better-known work is an integral BS equation in momentum space that is obtained directly from QFT.

The 4D covariant BS equation governs all the bound states and the scattering in R2B problem; it is appropriate framework for the description of the R2B bound-state problem within QFT [2]. However, the BS equation cannot be solved in general. Many attempts to apply the BS formalism to the R2B bound-state problem give series of difficulties, such as the relative time problem, the impossibility to determine the BS interaction kernel beyond the tight limits of perturbation theory and many others [5].

There exist various reductions of the BS equation [2, 6–10]. For a variety of reasons most of the attempts are not appropriate for the treatment of highly relativistic effects like those necessary for the calculation of bound states. Many authors have developed noncovariant instantaneous truncations of the BS equation [2, 8]. A better known is the Salpeter work [6]. These and many other difficulties are the sources of the numerous attempts to reformulate
the R2B problem [2, 7, 8, 11, 12]. This rather long list of authors and papers include different formulations of the R2B problem and related applications to QED and QCD. Most of the references are related to constraint dynamics [12].

There are a number of strategies in computational treatments of QCD that emerge in the study of meson spectroscopy. One is to set up a discrete lattice analog of the full QFT. Another is to first make analytic approximations which replace the QFT problem by a classical variational problem involving an effective Lagrange function and action. The latter approach has been exploited and gave a detailed account of applications of the R2B Dirac equations of constraint dynamics to the meson quark-antiquark bound states [13]. Applications of R2B Dirac Equations to the meson spectrum with three versus two covariant interactions, SU(3) mixing, and comparison to a quasipotential approach were considered in [14]. The R2B Dirac equations of Constraint Dynamics have dual origins. On the one hand they arise as one of the many quasipotential reductions of the BS equation. On the other they arise independently from the development of a consistent covariant approach to the R2B problem in relativistic classical mechanics independent of QFT [15].

In this work we consider these two aspects and then go on to discuss applications to the Hydrogen atom and hadron spectroscopy. We start with R2B problem in relativistic classical mechanics Using relativistic kinematics and the correspondence principle, we deduce a two-particle wave equation. The interaction of particles (quarks) is described by the modified funnel-type Lorentz-scalar Cornell potential. We obtain two exact asymptotic solutions of the equation which are used to write the complex-mass formula for the bound system. The last part of the work explains the importance we put on numerical tests of the model and some speculative theoretical results concerning the Hydrogen atom.

II. QUASIPOTENTIAL REDUCTION OF THE BS EQUATION

The homogeneous BS equation governs all the bound states. However, numerous attempts to apply the BS formalism to relativistic bound-state problems give series of difficulties. Its inherent complexity usually prevents to find the exact solutions or results in the appearance of excitations in the relative time variable of the bound-state constituents (abnormal solutions), which are difficult to interpret in the framework of quantum physics [5]. Usually, calculations are carried out with the help of phenomenological and relativistic mod-
The BS equation [3, 4], which is the basic bound state equation in QFT, has been revealed inadequate for quantitative calculation. In practice, the BS equation has been used in QED in the Coulomb gauge, which is a noncovariant gauge. Because of the instantaneous nature of the dominant part of the photon propagator, one is able to transform the original 4D equation into a 3D one and to avoid the previous difficulties [18, 19]. However, the latter gauge has its own limitations. It necessitates a different treatment of exchanged photons and of photons entering in radiative corrections. Additional complications arise when QED is mixed with other interactions, where already covariant propagators are present. In this respect, the wave equations obtained in the framework of constraint theory [9, 11, 12, 20] have been shown to provide a satisfactory answer to the requirement of a covariant treatment of perturbation theory in the bound state problem [11].

More valuable are methods which provide either exact or approximate analytic solutions for various forms of differential equations. They may be remedied in three-dimensional reductions of the BS equation. In most cases the analytic solution can be found if original equation is reduced to the Schrödinger-type wave equation. The most well-known of the resulting bound-state equations is the one proposed by Salpeter [6]. There exist many other approaches to bound-state problem.

Two body BS equation [2–4] for spin-zero bound states is

$$G_0^{-1} \Psi \equiv (p_1^2 + m_1^2)(p_2^2 + m_2^2) \Psi = K \Psi, \quad (1)$$

where $G_0 = G_{0,1}G_{0,2}$ is free propagator of particles. The irreducible BS kernel $K$ would in general contain charge renormalization, vacuum polarization graphs and could contain self-energy terms transferred from the inverse propagators. The kernel $K$ is obtained from the off-mass-shell scattering amplitude,

$$T = K + KG_0 T. \quad (2)$$

Recent work with static models has indicated, that abnormal solutions disappear if one includes all ladder and cross ladder diagrams [11]. This supports Wick’s conjecture on defects of ladder approximations. In the mean time numerous 3D quasipotential reductions of the BS equation had been proposed.

Reductions of the BS equation can be obtained from iterating this equation around a 3D Lorentz invariant hypersurface in relative momentum ($p$) space. This leads to invariant
3D wave equations for relative motion. The resultant 3D wave equation is not unique, but depends on the nature of the 3D hypersurface. One can choose Todorov’s quasipotential equation [9] which has this Schrödinger-like form

\[ [p^2 + \Phi(x_1 - x_2)]\psi = \kappa^2(w)\psi, \tag{3} \]

where the quasipotential \( \Phi \) is related to the scattering amplitude, 3D hyperfine restriction on the relative momentum \( p \) is defined by \( p \cdot P\psi = 0, P = p_1 + p_2 \). The effective eigenvalue in (3) is

\[ \kappa^2(w) = \frac{1}{4w^2}[w^2 - (m_1 - m_2)^2][w^2 - (m_1 + m_2)^2], \tag{4} \]

with \( w = \sqrt{P^2} \) the c.m. invariant energy.

If one uses a scheme that adapts Eikonal approximation for ladder, cross ladder, and constraint diagrams to bound states applied through all orders, it gives for scalar exchange the quasipotential

\[ \Phi = 2m_w S + S^2, \tag{5} \]

while for vector exchange

\[ \Phi = 2\epsilon_w A - A^2. \tag{6} \]

The kinematical variables

\[ m_w = \frac{m_1m_2}{w}, \tag{7} \]

\[ \epsilon_w = \frac{w^2 - m_1^2 - m_2^2}{2w}, \tag{8} \]

satisfy the Einstein relation

\[ \kappa^2(w) = \epsilon_w^2 - m_w^2, \tag{9} \]

and corresponds to the energy and reduced mass for the fictitious particle of relative motion. The effects of ladder and cross ladder diagrams thus embedded in their c.m. energy dependencies.

These forces \( \Phi \) to depend on \( x_1 - x_2 \) only through the transverse component, \( x_1^\mu \). Thus, in the c.m. frame, the hypersurface restriction \( p \cdot P\psi = 0 \) not only eliminates the relative energy \([p\psi = (0, \mathbf{p})\psi = 0]\) but implies that the relative time does not appear \([x_1^\mu = (0, \mathbf{r})]\).
III. THE INTERACTION POTENTIAL

The nonrelativistic (NR) quantum mechanics shows very good results in describing bound states; this is partly because the potential is NR concept. In relativistic mechanics one faces with different kind of speculations around the potential, because of absence of a strict definition of the potential in this theory. In NR formulation, the $H$ atom, for example, is described by the Schrödinger equation and is usually considered as an electron moving in the external field generated by the proton static electric field given by the Coulomb potential. In relativistic case, the binding energy of an electron in a static Coulomb field (the external electric field of a point nucleus of charge $Ze$ with infinite mass) is determined predominantly by the Dirac eigenvalue \[21\]. The spectroscopic data are usually analyzed with the use of the Sommerfeld’s fine-structure formula \[22\],

One should note that, in these calculations the $S$ states start to be destroyed above $Z = 137$, and that the $P$ states being destroyed above $Z = 274$. Similar situation we observe from the result of the Klein-Gordon wave equation, which predicts $S$ states being destroyed above $Z = 68$ and $P$ states destroyed above $Z = 82$. Besides, the radial $S$-wave function $R(r)$ diverges as $r \to 0$. These problems are general for all Lorentz-vector potentials which have been used in these calculations \[23\]. In general, there are two different relativistic versions: the potential is considered either as the zero component of a four-vector, a Lorentz-scalar or their mixture \[24\]; its nature is a serious problem of relativistic potential models \[25\].

This problem is very important in hadron physics where, for the vector-like confining potential, there are no normalizable solutions \[25, 26\]. There are normalizable solutions for scalar-like potentials, but not for vector-like. This issue was investigated in \[23, 27\]; it was shown that the effective interaction has to be Lorentz-scalar in order to confine quarks and gluons. The relativistic correction for the case of the Lorentz-vector potential is different from that for the case of the Lorentz-scalar potential \[28\].

Quarkonia as quark-antiquark bound states are simplest among mesons. The quarkonium universal mass formula and “saturating” Regge trajectories were derived in \[27\] and in \[29, 30\] applied for gluonia (glueballs). The mass formula was obtained by interpolating between NR heavy $Q\bar{Q}$ quark system and ultra-relativistic limiting case of light $q\bar{q}$ mesons for the
The Cornell potential [31, 32],

\[ V(r) = V_S(r) + V_L(r) \equiv -\frac{4}{3} \frac{\alpha_S}{r} + \sigma r. \]  

(10)

The short-range Coulomb-type term \( V_S(r) \), originating from one-gluon exchange, dominates for heavy mesons and the linear one \( V_L(r) \), which models the string tension, dominates for light mesons. Parameters \( \alpha_S \) and \( \sigma \) are directly related to basic physical quantities of mesons.

The Cornell potential (10) is fixed by the two free parameters, \( \alpha_S \) and \( \sigma \). However, the strong coupling \( \alpha_S \) in QCD is a function \( \alpha_S(Q^2) \) of virtuality \( Q^2 \) or \( \alpha_S(r) \) in configuration space. The potential can be modified by introducing the \( \alpha_S(r) \)-dependence, which is unknown. A possible modification of \( \alpha_S(r) \) was introduced in [29],

\[ V_{QCD}(r) = -\frac{4}{3} \frac{\alpha_S(r)}{r} + \sigma r, \quad \alpha_S(r) = \frac{1}{b_0 \ln\left[1/(\Lambda r)^2 + (2\mu_g/\Lambda)^2\right]}, \]  

(11)

where \( b_0 = (33 - 2n_f)/12\pi \), \( n_f \) is number of flavors, \( \mu_g = \mu(Q^2) \) — gluon mass at \( Q^2 = 0 \), \( \Lambda \) is the QCD scale parameter. The running coupling \( \alpha_S(r) \) in (11) is frozen at \( r \to \infty \), \( \alpha_\infty = \frac{1}{2} [b_0 \ln(2\mu_g/\Lambda)]^{-1} \), and is in agreement with the asymptotic freedom properties, i.e., \( \alpha_S(r \to 0) \to 0 \).

In this work we consider and analyze general coordinate-space relativistic spinless Salpeter (SS) equation for two-body system [6]. In the c.m. frame, the SS equation has the form \((\hbar = c = 1)\)

\[ \left[ \sqrt{(-i\nabla)^2 + m_1^2} + \sqrt{(-i\nabla)^2 + m_2^2} + V(r) \right] \psi(\vec{r}) = E \psi(\vec{r}) = 0, \]  

(12)

where \( V(r) \) is the potential (for simplicity we consider separable spherically symmetric potential). It is a problem to find the analytic solution of this equation; it can not be reduced to the second-order differential equation of the Shrödinger type. The problem originates from two square root operators which cause a serious difficulties.

**IV. TRANSFORMATION OF THE SS EQUATION**

Standard relativistic approaches for R2B systems run into serious difficulties in solving known relativistic wave equations. Consider the problem in Relativistic Quantum Mechanics (RQM). The formulation of RQM differs from NR QM by the replacement of invariance under Galilean transformations with invariance under Poincaré transformations. The RQM
is also known in the literature as relativistic Hamiltonian dynamics or Poincaré-invariant QM with direct interaction [33]. There are three equivalent forms in the RQM called “instant”, “point”, and “light-front” forms.

The dynamics of many-particle system in the RQM is specified by expressing ten generators of the Poincaré group, $\hat{M}_{\mu\nu}$ and $\hat{W}_\mu$, in terms of dynamical variables. In the constructing generators for interacting systems it is customary to start with the generators of the corresponding non-interacting system; the interaction is added in the way that is consistent with Poincaré algebra. In the relativistic case it is necessary to add an interaction $V$ to more than one generator in order to satisfy the commutation relations of the Poincaré algebra.

The interaction of a relativistic particle with the 4-momentum $p_\mu$ moving in the external field $A_\mu(x)$ is introduced in QED according to the gauge invariance principle, $p_\mu \rightarrow P_\mu = p_\mu - eA_\mu$. The description in the “point” form of RQM implies that the mass operators $\hat{M}^\mu_{\mu\nu}$ are the same as for non-interacting particles, i.e., $\hat{M}^\mu_{\mu\nu} = M^\mu_{\mu\nu}$, and these interaction terms can be presented only in the form of the 4-momentum operators $\hat{W}^\mu$ [34].

Consider the R2B problem in classic relativistic theory. Two particles with 4-momenta $p_1^\mu$, $p_2^\mu$ and the interaction field $W^\mu(q_1, q_2)$ together compose a closed conservative system, which can be characterised by the 4-vector $P^\mu$,

$$P^\mu = p_1^\mu + p_2^\mu + W^\mu(q_1, q_2),$$

where the space-time coordinates $q_1^\mu$, $q_2^\mu$ and 4-momenta $p_1^\mu$, $p_2^\mu$ are conjugate variables, $P_\mu P^\mu = M^2$; here $M$ is the system’s invariant mass. Underline, that no external field and each particle of the system can be considered as moving source of the interaction field; the interacting particles and the potential are a unified system. There are the following consequences of (13) and they are key in our approach.

The 4-vector (13) describes free motion of the bound system and can be presented as,

$$E = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} + W_0(q_1, q_2) = \text{const},$$

$$P = p_1 + p_2 + W(q_1, q_2) = \text{const},$$

describing the energy and momentum conservation laws. The energy (14) and total momentum (15) of the system are the constants of motion. By definition, for conservative systems, the intergals (14) and (15) can not depend on time explicitly. This means the interaction $W(q_1, q_2)$ should not depend on time, i.e., $W(q_1, q_2) \Rightarrow V(r_1, r_2)$. 

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It is well known that the potential as a function in 3D-space is defined by the propagator $D(q^2)$ (Green function) of the virtual particle as a carrier of interaction, where $q = p_1 - p_2$ is the transferred momentum. In case of the Coulomb potential the propagator is $D(q^2) = -1/q^2$; the Fourier transform of $4\pi\alpha D(q^2)$ gives the Coulomb potential, $V(r) = -\alpha/r$. The relative momentum $q$ is conjugate to the relative vector $r = r_1 - r_2$, therefore, one can accept that $V(r_1, r_2) = V(r)$ [5]. If the potential is spherically symmetric, one can write $V(r) => V(r)$, where $r = |r|$. Thus, the system’s relative time $\tau = t_1 - t_2 = 0$ (instantaneous interaction).

Equations (14) and (15) in the c.m. frame are

$$M = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} + V(r),$$

$$P = p_1 + p_2 + W(r_1, r_2) = 0,$$

where $p = p_1 = -p_2$ that follows from the equality $p_1 + p_2 = 0$; this means that $W(r_1, r_2) = 0$. The system’s mass (16) in the c.m. frame is Lorentz-scalar. In case of free particles ($V = 0$) the invariant mass $M = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2}$ can be transformed for $p^2$ as

$$p^2 = \frac{1}{4s}(s - m_-^2)(s - m_+^2) \equiv k^2,$$

which is relativistic invariant, $s = M^2$ is the Mandelstam’s invariant, $m_- = m_1 - m_2$, $m_+ = m_1 + m_2$.

Equation (14) is the zeroth component of the 4-vector (13) and the potential $W_0$ is Lorentz-vector. But, in the c.m. frame the mass (16) is Lorentz-scalar; and what about the potential $V$? Is it still Lorentz-vector? To show that the potential is Lorentz-scalar, let us reconsider (16) as follows. The relativistic total energy $\epsilon_i(p)$ ($i = 1, 2$) of particles in (16) given by $\epsilon_i^2(p) = p_i^2 + m_i^2$ can be represented as sum of the kinetic energy $\tau_i(p)$ and the particle rest mass $m_i$, i.e., $\epsilon_i(p) = \tau_i(p) + m_i$. Then the system’s total energy (invariant mass) (16) can be written in the form $M = \sqrt{p_1^2 + m_1^2(r)} + \sqrt{p_2^2 + m_2^2(r)}$, where $m_{1,2}(r) = m_{1,2} + \frac{1}{2}V(r)$ are the distance-dependent particle masses [35] and (18) with the use of $m_1(r)$ and $m_2(r)$ takes the form,

$$p^2 = K(s) \left[s - (m_+ + V)^2\right] \equiv k^2 - U(s, r),$$

where $K(s) = (s - m_-^2)/4s$, $k^2$ is squared invariant momentum given by (18) and $U(s, r) = K(s) [2m_+ V + V^2]$ is the potential function. The equation (19) is the relativistic analogy of the NR expression $p^2 = 2\mu [E - V(r)] \equiv k^2 - U(E, r)$. 

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The equality (19) with the help of the fundamental correspondence principle gives the two-particle spinless wave equation,

$$\left[ (-i\vec{\nabla})^2 + k^2 - U(s, r) \right] \psi(r) = 0.$$  \hspace{1cm} (20)

The equation (20) can not be solved by known methods for the potential (11). Here we use the quasiclassical (QC) method and solve another wave equation [28, 36]. Compare (20) with the one (3). Is there any difference between them?

V. SOLUTION OF THE QC WAVE EQUATION

Solution of the Shrödinger-type’s wave equation (20) can be found by the QC method developed in [36]. In our method one solves the QC wave equation derivation of which is reduced to replacement of the operator $\vec{\nabla}^2$ in (20) by the canonical operator $\Delta^c$ without the first derivatives, acting onto the state function $\Psi(\vec{r}) = \sqrt{\det g_{ij}} \psi(\vec{r})$, where $g_{ij}$ is the metric tensor. Thus, instead of (20) one solves the QC equation, for the potential (11),

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + s - m^2 \right\} \left[ s - \left( m_+ - \frac{4}{3} \frac{\alpha_s(r)}{r} + \sigma r \right) \right]^2 \Psi(r) = 0. \hspace{1cm} (21)

This equation is separated. Solution of the angular equation was obtained in [36] by the QC method in the complex plane, that gives $M_l = (l + 1/2)\hbar$, for the angular momentum eigenvalues. These angular eigenmomenta are universal for all spherically symmetric potentials in relativistic and NR cases.

The radial problem has four turning points and cannot be solved by standard methods. We consider the problem separately by the QC method for the short-range Coulomb term (heavy mesons) and the long-range linear term (light mesons). The QC method reproduces the exact energy eigenvalues for all known solvable problems in quantum mechanics [28, 36]. The radial QC wave equation of (21) for the Coulomb term has two turning points and the phase-space integral is found in the complex plane with the use of the residue theory and method of stereographic projection [36, 37] that gives

$$M_N^2 = \left( \sqrt{\epsilon_N^2} \pm \sqrt{(\epsilon_N^2)^*} \right)^2 \equiv 4 \left[ \text{Re}\{\epsilon_N^2\} \pm \text{Im}\{\epsilon_N^2\} \right],$$  \hspace{1cm} (22)

where $\epsilon_N^2 = \frac{1}{4} m_+^2 (1 - v_N^2) + \frac{1}{2} m_+ m_- v_N$, $v_N = \frac{3}{2} \alpha_{\infty}/N$, $N = k + l + 1$.

Large distances in hadron physics are related to the problem of confinement. The radial problem of (21) for the linear term has four turning points, i.e., two cuts between these
points. The phase-space integral in this case is found by the same method of stereographic projection as above that results in the cubic equation [35]:

\[ s^3 + a_1 s^2 + a_2 s + a_3 = 0, \]

where

\[ a_1 = 16\tilde{\alpha}_\infty \sigma - m^2, \quad a_2 = 64\sigma^2 \left(\tilde{\alpha}_\infty^2 - \tilde{N}^2 - \tilde{\alpha}_\infty m^2 / 4\sigma\right), \quad a_3 = -(8\tilde{\alpha}_\infty \sigma m_-)^2, \quad \tilde{N} = N + k + \frac{1}{2}, \]

\[ \tilde{\alpha}_\infty = \frac{3}{4}\alpha_\infty. \]

The first root \( s_1(N) \) of this equation gives the physical solution (complex eigenmasses), \( M_1^2(N) = s_1(N) \), for the squared invariant mass.

Two exact asymptotic solutions obtained such a way are used to derive the interpolating mass formula. The interpolation procedure for these two solutions [27] is used to derive the resonance’s mass formula:

\[
M_N^2 = (m_1 + m_2)^2 \left(1 - v_N^2\right) \pm 2i(m_1^2 - m_2^2)v_N + \text{Re}\{M_1^2(N)\}. \tag{23}
\]

The real part of the square root of (23) defines the centered masses and its imaginary part defines the total widths, \( \Gamma_N^{\text{TOT}} = -2\text{Im}\{M_N\} \), of resonances [37, 38]. The real-part mass in (23) exactly coincides with the universal mass formula obtained independently by another method with the use of the two-point Padé approximant [27] and is very transparent physically, as well as the Coulomb potential.

The free fit to the data shows a good agreement for the light and heavy \( Q\bar{q} \) meson resonances. Note, that the gluon mass in the independent fitting is the same, \( m_g = 416 \text{ MeV} \). Besides, it is the same for glueballs [29]. the \( d \) quark effective mass is also practically the same, i.e., \( m_d \approx 273 \text{ MeV} \), for the light and heavy resonances.

It describes equally well the mass spectra of all \( q\bar{q} \) and \( Q\bar{Q} \) mesons ranging from the \( u\bar{d} \) (\( d\bar{d}, u\bar{u}, s\bar{s} \)) states up to the heaviest known \( b\bar{b} \) systems [27] and glueballs [29, 30] as well. Besides, it allows one to get the Regge trajectories as analytic functions in the whole region from solution of the cubic equation for the angular momentum \( J(M^2) \) [27]; the Regge trajectories including the Pomeron [29, 30] are “saturating” and appears to be successful in many applications [39–41].

In our QC method not only the total energy, but also momentum of a particle-wave in bound state is the constant of motion. Solution of the QC wave equation in the whole region is written in elementary functions as [28, 36, 42],

\[
\bar{R}_n(r) = C_n \left\{ \begin{array}{ll}
\frac{1}{\sqrt{2}} \exp(|p_n|r - \phi_1), & r < r_1, \\
\cos(|p_n|r - \phi_1 - \frac{\pi}{4}), & r_1 \leq r \leq r_2, \\
\frac{(-1)^n}{\sqrt{2}} \exp(-|p_n|r + \phi_2), & r > r_2,
\end{array} \right. \tag{24}
\]
TABLE I. The masses of the $\rho^{\pm}(ud)$-meson resonances

| Meson       | $J^P$ | $E_n^{ex}$ | $E_n^{th}$ | Parameters in (23) |
|-------------|-------|------------|------------|--------------------|
| $\rho(1S)$  | 1−−   | 776        | 776        | $\Lambda = 500$ MeV |
| $a_2(1P)$   | 2++   | 1318       | 1314       | $\mu_g = 416$ MeV  |
| $\rho_3(1D)$| 3−−   | 1689       | 1689       | $\sigma = 0.139$ GeV$^2$ |
| $a_4(1F)$   | 4++   | 1996       | 1993       | $m_d = 276$ MeV     |
| $\rho(1G)$  | 5−−   | 2255       |            | $m_u = 129$ MeV     |
| $\rho(2S)$  | 1−−   | 1717       | 1682       |                    |
| $\rho(2P)$  | 2++   | 1990       |            |                    |
| $\rho(2D)$  | 3−−   | 2254       |            |                    |

TABLE II. The masses of the $D^{*\pm}(cd)$-meson resonances

| Meson       | $J^P$ | $E_n^{ex}$ | $E_n^{th}$ | Parameters in (23) |
|-------------|-------|------------|------------|--------------------|
| $D^*(1S)$   | 1−−   | 2010       | 2010       | $\Lambda = 446$ MeV |
| $D^*_2(1P)$ | 2++   | 2460       | 2464       | $m_g = 416$ MeV     |
| $D^*_3(1D)$ | 3−−   | 2845       |            | $\sigma = 0.249$ GeV$^2$ |
| $D^*_4(1F)$ | 4++   | 3178       |            | $m_c = 1163$ MeV    |
| $D^*_5(1G)$ | 5−−   | 3478       |            | $m_d = 271$ MeV     |
| $D^*(2S)$   | 1−−   | 1820       | 2821       |                    |
| $D^*(2P)$   | 2++   | 2011       | 3166       |                    |
| $D^*(2D)$   | 3−−   | 3471       |            |                    |

where $C_n = \sqrt{2|p_n|/\pi(n + \frac{1}{2}) + 1}$ is the normalization coefficient, $p_n$ is the corresponding eigenmomentum, $\phi_1 = -\pi(n + \frac{1}{2})/2$ and $\phi_2 = \pi(n + \frac{1}{2})/2$ are the values of the phase-space integral at the turning points $x_1$ and $x_2$, respectively. In the classically allowed region $[x_1, x_2]$, the solution is

$$\bar{R}_n(r) = C_n \cos \left( |p_n| r + \frac{\pi}{2} n \right),$$

i.e., has the form of a standing wave. This solution is appropriate for two-turning-point problems both in non-relativistic and relativistic cases with the corresponding eigenmomenta.
We use this fact in the present work.

To demonstrate its efficiency we calculate the leading-state masses of the $\rho$ and $D^{*}$ meson resonances (see tables, where masses are in MeV).

\section*{CONCLUSION}

The constituent quark picture could be questioned since potential models have serious difficulties because the potential is non-relativistic concept. However, in spite of non-relativistic phenomenological nature, the potential approach is used with success to describe mesons as bound states of quarks.

We have modeled meson resonances to be the quasi-stationary states of two quarks interacting by the QCD-inspired funnel-type potential with the coordinate dependent strong coupling, $\alpha_{S}(r)$. Using the complex analysis, we have derived the meson complex-mass formula (23), in which the real and imaginary parts are exact expressions. This approach allows to simultaneously describe in the unified way the centered masses and total widths of resonances. We have shown here the results only for unflavored and charmed meson resonances, however, we have obtained a good description for strange and beauty mesons as well [43].

\begin{thebibliography}{9}

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