Stable self-spinning mode of a waved-dressed particle

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Abstract

Self-propelled droplets are composed of droplets driven by the waves they emit when bouncing on a vertically vibrated bath. Their dynamics is based on an interplay between the waves and their source. The existence of self-spinning modes is still controversial. Here, we show experimentally that these modes are stable for a class of droplets and emerge spontaneously from noise fluctuations. We perform a discrete stability analysis to confirm experimental observations. In addition, we show that these self-spinning modes provide a unique opportunity for a direct experimental measurement of parameters used in the wave-driven droplet models found in the literature to enable comparison and calibration.
Since their discovery in 2005, self-propelled bouncing droplets, called walkers, have drawn much interest because they exhibit an original non-quantum macroscopic wave-particle duality. When placed on a vertically vibrated bath, droplets bounce indefinitely and become self-propelled, driven by the waves they emit. The wave field that pilots the droplet is a superposition of the elementary waves produced at each bounce and sustained for a tunable time which depends on the bath acceleration. This memory time is associated with the Faraday instability for which the liquid surface destabilizes above a threshold. This memory based duality has been investigated in numerous experiments.

The memory endows this waved-dressed particle with a non-markovian dynamics. The wave field which drives the droplet holds the signature of the positions of the previous impacts visited by the droplet along its trajectory. This so-called path-memory dynamics produces many original phenomena among which the existence of several stable states. The quest for a stable self-rotating mode which would possess an intrinsic non-zero angular momentum and would find some analogy with a spin is still controversial.

The possibility of such modes was investigated theoretically by Oza et al. driven by experiments of walkers placed in a rotating frame. Using a continuous approximation model, they found that spinning could be maintained even for vanishing Coriolis force but for parameters non accessible to experimental conditions. The same conclusion was drawn by a stability analysis performed with a discrete-time model based on first principles in a specific parameter regime. These results seems to contradict experimental observations performed with walkers released from a harmonic potential. However, the observed states lasted only for a few memory times arguably due to experimental noise. Here, we show experimentally that self-spinning orbits are stable with several hundred rotations in a new regime of small and slow droplets. We develop a discrete stability analysis of these orbits confirming the experimental findings. We also show that self-orbiting modes analysis enables an experimental calibration of the parameters used in the many existing models.
Figure 1: (a) Schematics of the experimental setup. A color Schlieren technique encodes the slope direction of the wave field. (b) Typical snapshot of the wave field and its associated droplet superimposed with a top view graphic of the container. Walkers going in a straight line.
(c) large fast walker \((v \approx 10\ \text{mm.s}^{-1}, D_w = 0.57\ \text{mm})\) and (d) small slow walker \((v \approx 2\ \text{mm.s}^{-1}, D_w = 0.43\ \text{mm})\), \(Me \approx 40\) and \(v_{\text{exc}} = 72\ \text{Hz}\).

Figure 1a shows a schematic of the experimental setup. The container is fixed on a shaker (LDS V455) vibrating vertically at frequency \(v_{\text{exc}}\) between 60 Hz and 80 Hz. A linear air bearing is used to avoid transverse vibrations\textsuperscript{28}. The whole experiment is confined in a Plexiglass box to limit air flows. The 3D printed square container consists of a 60 mm wide, 8.5 mm deep square cavity surrounded by a 55 mm wide shallow bank. This profile is optimized to attenuate the waves emitted by the walls using a 0.5 mm deep shallow liquid section and minimize wave reflection from the cavity using a 30 mm wide, 2 mm high chamfer. The cavity is filled with silicon oil of viscosity \(\mu = 20 \cdot 10^{-3}\ \text{Pa.s}\), density \(\rho = 950\ \text{kg.m}^{-3}\) and surface tension \(\gamma = 0.0206\ \text{N.m}^{-1}\). The droplets are produced by a piezoelectric generator similar to the one in ref. \textsuperscript{29}. The droplets have a tunable diameter from 0.4 mm to 1 mm with a 10 \(\mu\)m precision.

The illumination is performed with a radial colored Schlieren mask projected on the liquid surface using a Fresnel lens which enables color-coding the direction of the surface slope\textsuperscript{30}. The camera placed above the container is synchronized with the Faraday frequency \(v_F = 1/T_F = v_{\text{exc}}/2\) with a tunable relative phase to tune the recorded amplitude of the wave field. The bouncing droplets which have undergone period doubling are strobed. Their trajectories are retrieved from image analysis. Figure 1b shows a typical snapshot with the bouncing droplet with its wave field superimposed on a schematic top view of the cavity. The wave field surrounding the droplet extends over 10 Faraday wavelengths \(\lambda_F\), well beyond the deep region of the cavity with no detectable field reflection nor field discontinuity in the shallow region. This ensures that the influence of the size and boundaries of the cavity on the walker dynamics is limited. The memory \(Me\) is retrieved for a given acceleration amplitude by measuring the characteristic decay time \(\tau_{Me} = T_F Me\) for an instantaneous point perturbation of the liquid surface.

Small and slow droplets should be good candidates for self-spinning modes since the centrifugal force is proportional to their mass and to the square of their speed. The minimal size is set by the vertical dynamics of the droplet\textsuperscript{32-34}. While most studies use a typical droplet diameter of \(D_w \approx 0.7\ \text{mm}\), we experimentally found the period doubled bouncing regime could be maintained down to droplet diameter of \(D_w \approx 0.43\ \text{mm}\) for \(v_{\text{exc}} = 70\ \text{Hz}\). Small droplets are much slower than large ones\textsuperscript{32}, typically \(v \approx 2\ \text{mm.s}^{-1}\) versus \(v \approx 10\ \text{mm.s}^{-1}\). Figure 1c and 1d shows the associated wave field for a straight trajectory with \(\lambda_F = 5.2\ \text{mm}\) at \(Me \approx\)
40. Fast droplets create typical interference patterns resulting from a memory length \( l_{Me} = v \tau_{Me} \) persistent over several \( \lambda_F \). For slow droplets, \( l_{Me} \approx \lambda_F \) resulting in a nearly circular wave field. Although their characteristics increase the possibility of an equilibrium, these slow walkers have not been investigated in previous studies on self-rotation.
Figure 2: (a) Trajectory of a walker in a clockwise self-orbiting mode with color coded speed ($Me \approx 70$, $\nu_{exc} = 70$ Hz, $D_w = 0.46$ mm). The bold final segment represents the memory length $l_{Me} = \nu \tau_{Me}$. Trajectory of the center of curvature (red line). (b) Snapshot of the associated wave field and the superimposed trajectory (black line) with
transparency proportional to attenuation (see supplementary Movie)\textsuperscript{31}. (c) Time evolution of the instantaneous orbital radius \(R/\lambda_F\) and speed \(v/v_F\) and their associated distributions.

Figure 2a shows a typical self-orbiting trajectory with 76 revolutions for a droplet with \(D_w = 0.46\) mm and \(Me \approx 70\) (see Supplementary Movie)\textsuperscript{31}. Typically, droplets enter self-spinning trajectories either spontaneously or after a small perturbation. Undetectable noise fluctuations seem to be sufficient to trigger the self-spinning mode in contrasts with previous studies for which the droplet needed to be set into the spinning orbit\textsuperscript{27}. Self-spinning can occur anywhere in the cavity before slowly drifting in a random direction (red line) in agreement with the negligible influence of the boundaries. The orbiting period is approximately twice the memory time \(\tau_{Me}\) (\(\tau_{Me}\) is in bold line in Fig. 2a). Stable self-orbiting trajectories of several hundred rotations are observed for forcing frequencies \(70\) Hz \(\leq v_{exc} \leq 76\) Hz, wave field memories \(70 \leq Me \leq 100\) and droplet diameters \(0.43\) mm \(\leq D_w \leq 0.56\) mm.

Figure 2b is a typical snapshot of a self-orbiting walker. The transparency of the trajectory line indicates the vanishing contribution of successive impacts to the wave field. This wave field has a spatio-temporal helical symmetry and can be decomposed primarily into a zero and a first order Bessel functions centered on the orbit\textsuperscript{27}. The former creates a local axial slope \(\nabla h^A(r)\) at the droplet position \(r\), which gives a centripetal force balancing the centrifugal inertial effects. The latter rotates with the droplet creating a local tangential slope \(\nabla h^T(r)\) for self-propulsion. Figure 2c shows the time evolution of the instantaneous orbital radius \(R/\lambda_F\) and the speed \(v/v_F\) of the walker normalized by the Faraday wavelength \(\lambda_F\) and the wave phase velocity \(v_F = \lambda_F/T_F\) respectively. Both exhibit small oscillations at the orbital period around \(R \approx 0.4\lambda_F\) and \(v \approx 0.014v_F\) respectively. Their distributions have similar Gaussian profiles with a narrow standard deviation (\(\sigma/R \approx 4.3\%\) and \(\sigma/v \approx 3.6\%\) respectively). Their fluctuations are correlated (with a linear correlation coefficient of 0.75) indicating a coupling between the droplet propulsion and the radial force provided by the wave field.

The walker dynamics is described by a simple discrete path-memory model which takes the essence of the dynamics\textsuperscript{2,3,27}. The vertical and horizontal motion of the walkers are decoupled. Walkers undergo parabolic jumps between inelastic bounces. Following Durey \textit{et al.}\textsuperscript{6,26}, we consider the bouncing time as instantaneous and the horizontal velocity \(v_{n+1}\) after the \(n^{th}\) bounce constant. The position of the \((n+1)^{th}\) bounce \(r_{n+1}\) is given by iteration: \(r_{n+1} = r_n + v_{n+1}T_F\). The impact induces a change in the horizontal velocity satisfying
\[
\frac{m}{T_F} (\mathbf{v}_{n+1} - \mathbf{v}_n) = -D \left( \frac{\mathbf{v}_{n+1} + \mathbf{v}_n}{2} \right) - C \nabla h(\mathbf{r}_n)
\]

with \( m \) being the droplet mass, \( \nabla h(\mathbf{r}_n) \) the surface gradient at \( \mathbf{r}_n \), \( C \) and \( D \) the wave coupling and the friction constant respectively. The change in velocity is driven by a friction-like force originating mainly in the shearing of the air layer between the drop and the bath, and a wave or memory force, proportional to the local surface slope. Neglecting viscosity and wave propagation, the wave field \( h(\mathbf{r}) \) can be describe as a superposition of \( J_0 \) Bessel functions with initial amplitude \( h_0 \) centered at the impact positions along the trajectory and decaying exponentially with a memory time \( \tau_F \). At the time of the \( n^{th} \) impact, it satisfies
\[
h(\mathbf{r}) = h_0 \sum_{q=1}^{\infty} J_0(k_F|\mathbf{r} - \mathbf{r}_{n-q}|) \exp\left(-\frac{q}{M_e}\right)
\]

The dynamics of the walkers is thus fully characterized by the two parameters \( C h_0 \) and \( D \). For a walker of mass \( m \) and speed \( v \) in an orbit of radius \( R \), eq. (1) projected along the radial and tangential directions gives respectively
\[
F_\perp(R, v) = \frac{m v^2}{R} - C \nabla h^\perp(\mathbf{r}_n)
\]
\[
F_\parallel(R, v) = -D v - C \nabla h^\parallel(\mathbf{r}_n)
\]

For the spinning mode values \((R, v)\), the force balance writes \( F_\perp(R, v) = 0 \) and \( F_\parallel(R, v) = 0 \). The centripetal wave force equilibrates the inertial force involving the parameter \( C h_0 \) only (eq. 3). It equals to \( 1.56 \cdot 10^{-10} \) N in the present case, a value approximately 25 times smaller than that of “standard” walkers. The tangential component yields the parameter \( D \) (eq. 4).

Considering the walker in a given circular trajectory, the bouncing positions are on the vertices of a regular polygon from which \( \nabla h^\perp(\mathbf{r}_n) \) and \( \nabla h^\parallel(\mathbf{r}_n) \) can be computed. From the experimental observations, we obtain \( C h_0 = 2.22 \cdot 10^{-13} \) N.m and \( D = 1.03 \cdot 10^{-6} \) kg.s\(^{-1}\) which are values in the range of ones calculated from the hydrodynamic model.

Decoupling the parameters in eq (3) and (4) allows experimental calibration using measurements on self-orbiting modes. All the proposed models ultimately determine two parameters associated with the wave propulsion and the damping force. However, the diversity of hypothesis and formalisms make it difficult to find an exact correspondence between these parameters. An experimental calibration would enable comparison. In addition, when these parameters are rooted in the hydrodynamic of the bouncing, this calibration would provide an additional validation of the hydrodynamic hypothesis.

We use the calibrated values to run simulations based on eq. (1) and (2) and observe the outcome on the walker dynamics with the first \( N \) impacts placed on a self-orbiting mode as
initial conditions. For $Me = 70$ and $N > 75$, the walker enters a stable self-orbiting mode confirming the experimental findings and validating the calibrations a posteriori.

Figure 3: Radial force equilibrium $F_\perp(R, v) = 0$ (blue line) and tangential one $F_\parallel(R, v) = 0$ (black line) in the $(R, v)$ plane for $Me = 70$ (a) and $Me = 90$ (b). Intersection points (crosses) associated to
possible self-spinning modes labeled A, B and C, D, E, F respectively. (c) Eigenvalues (black dots) for the discrete linear stability matrix around equilibrium point A (inset: close up). Two eigenvalues associated with the translational invariance (blue circles), the rotational one (cyan circle) and the least-stable eigenvalue (green points). (d) Modulus of the most-unstable eigenvalue (color coded) of the self-orbiting mode in the \((R/\lambda_F, Me)\) plane. \(J_0(k_F R) = 0\) (dashed red lines) and \(J_1(k_F R) = 0\) (dashed black lines).

Using values from the calibration, we explore the existence and stability of the self-orbiting modes. These modes characterized by \((R, v)\) must simultaneously satisfy \(F_\perp(R, v) = 0\) and \(F_\parallel(R, v) = 0\). Figure 3a and 3b show the two curves (blue and black lines respectively) in the \((R, v)\) plane for \(Me = 70\) and \(Me = 90\) respectively. The radial wave force is mainly associated to a centered Bessel mode \(J_0(k_F r)\) with an excitation amplitude proportional to \(Me J_0(k_F R)\) at high memory.\(^{27}\) Hence, the radial wave force exerted on an orbit of radius \(R\) is proportional to \(Me J_0(k_F R) J_1(k_F R)\). As \(Me\) increases, the condition \(F_\perp(R, v) = 0\) is asymptotically fulfilled by \(J_0(k_F R) \approx 0\) or \(J_1(k_F R) \approx 0\). The tangential wave force is mainly associated with a centered \(J_1\) Bessel function rotating with the droplet. For \(R \gtrsim 0.25\lambda_F\), the driving force equilibrates the shearing force resulting in a nearly constant droplet speed of \(v/v_F \approx 0.014\). Two possible self-spinning modes can be found for \(Me = 70\) (labeled A and B in Fig. 3a). Couples of new possible modes appear as \(Me\) increases, e.g. four modes appear for \(Me = 90\) (see Fig. 3b).

We use the discrete iterative path-memory model to assess the stability of the modes.\(^{6,26}\) We reformulate the walker dynamics as a non-linear iterative map on a \(p\) dimensional complex space. The walker dynamics is fully determined at the time of the \(n^\text{th}\) impact by the previous ones. The walker state is given by a vector \(Z_n\) containing the positions of the previous impacts written in the complex plane. The \(q^\text{th}\) component of the vector \(z_n^q\) contains the \((q-1)^\text{th}\) impact, \(z_n^q = r_{n-q+1} \in \mathbb{C}\) with \(q \in \mathbb{N}^+, z_1^n\) being the position of the \(n^\text{th}\) impact. The walker dynamics is calculated iteratively using a function \(f\) such that \(Z_{n+1} = f(Z_n)\) (see supplementary materials).\(^{36}\) In practice, the exponential wave field decreases with a characteristic time \(\tau_{Me}\), \(Z_n\) can be truncated to the \(p\) previous impacts satisfying \(e^{-p/\tau_{Me}} \approx 0.1\%\).

A walker with a state vector \(Z_n\) in a self-spinning mode \((R, v)\) is rotated at each step. Hence, \(f(Z_n) = rot_\theta(Z_n)\), \(rot_\theta\) being a rotation in \(\mathbb{C}^p\) by an angle \(\theta = 2 \arcsin(v T_F/2 R) \approx v T_F/R\). \(Z_n\) is thus a fixed point of the nonlinear operator \(g = rot_{-\theta} \circ f\). The linear stability analysis of the self-spinning mode can be performed using \(g\) at point \(Z_n\). The mode is stable if the eigenvalues of the jacobian matrix of \(g\) evaluated at \(Z_n\) are within the unit circle. The
eigenvalues outside are linearly unstable if real and oscillatory unstable otherwise. Figure 3c shows the eigenvalues found for $Me = 70$ associated with point A (Fig. 3a) with $Ch_0$ and $D$ calibrated experimentally. The mode is stable with all the eigenvalues inside the unit circle. The eigenvalues associated with translational (blue circles) and rotational invariance (cyan circle) are discarded for the stability analysis. Figure 3d shows the stability map of the possible self-orbiting modes of radius $R/\lambda_F$ versus $Me$ given by the highest eigenvalue norm. Modes labeled in Fig. 3a and 3b are plotted. The orbits associated with the centered $J_0$ Bessel function are stable for $Me$ spanning from 51 to 93 (modes $A$ and $C$). All the other modes are found unstable, meaning that only the smallest orbits are stable in agreement with the experimental findings (see fig. 2). The instability associated with the upper (resp. lower) branches of the curves is linear (resp. oscillatory). At high memory ($Me > 100$), the orbits appear experimentally unstable possibly due to the limited cavity size.

It is interesting to compare the present stability results based on a discrete model with the ones based on the continuous approximation using integro-differential equations. Using the experimental calibration, the self-orbiting mode is found unstable ($\kappa_0 = 0.29$) but very close to stability ($\kappa_0 \lesssim 0.2$, with $C = 0.33$, $\sin \Phi = 0.16$). Small droplets could slightly change the hydrodynamics parameters to make reach stability.

The discrete stability analysis can also be applied to “standard” walkers in previous experiments for which orbits could be maintained only for a limited times. The parameter values are retrieved from the calibration method ($Ch_0 = 8.8 \times 10^{-12}$ N.m and $D = 7.2 \times 10^{-6}$ kg.s$^{-1}$). The discrete stability analysis finds stability for a range of intermediate memories. For high memories, simulations show that the self-orbiting is wobbling but maintained while it destabilizes at even higher memories. The model of Durey et al. based on first principles with an instantaneous contact approximation could possibly find self-spinning stability with this new set of parameters. The finding of the new stable self-orbiting walkers together with the proposed calibration method should definitely help compare and rationalize the many walking models with their quite diverse hypotheses from continuous to instantaneous approximations.

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A discrete path-memory model for the walking droplet is detailed, where its dynamics is reformulated as a non-linear iterative map on a \( p \)-dimensional complex space. The linear stability analysis of the self-spinning mode is performed by computing the eigenvalues of the jacobian matrix of the suitable operator.

### 1 Non-linear Discrete \( p \)-dimensional model

For a system with Memory \( M_e \), only a certain number of impacts along the droplet trajectory have a significant influence on the field. Indeed, the local Faraday mode excited at each impact decays exponentially and its influence can be considered to vanish after a certain time. We consider that the wave field created by a single impact is vanishing if its amplitude is less than 0.1 percent of the maximum amplitude.

The walking droplet motion is modeled by the dynamics of the discrete set of the last \( p \) impacts of the droplets. \( p \) increases with the memory of the system. The principle of this model is described in (2)

At iteration \( n \), the positions of the previous \( p^{th} \) impacts are:

\[
\vec{z}_n^{p} = \begin{bmatrix} x_n^p \\ y_n^p \end{bmatrix}
\]

With \( \vec{z}_n^1 \) the position of the droplet at iteration \( n \).

For clarity, we will consider \( \vec{z}_n^p = x_n^p + iy_n^p \in \mathbb{C} \) and explicit the dynamics as the evolution of a finite set of points in the complex plane.

\( \vec{z}_n^1 \) is the position of the droplet just before the impact at iteration \( n \), and its speed just before it gets an impulse from the field is:

\[
\vec{v}_n^1 = \frac{\vec{z}_n^1 - \vec{z}_n^2}{T_F}
\]

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Figure 1: Notations for the liquid and the equivalent disk with notations from the pendulum
Where $T_F$ is the Faraday period. The position of the next impact is:

$$\vec{z}_1^{n+1} = \vec{z}_n + T_F \vec{v}_1^{n+1}$$  \hspace{1cm} (3)

where the speed $\vec{v}_1^{n+1}$ just after the current $n^{th}$ impact is calculated following a discrete impulse from the field:

$$m \frac{(\vec{v}_1^{n+1} - \vec{v}_1^n)}{T_F} = -D \left( \frac{\vec{v}_1^{n+1} + \vec{v}_1^n}{2} \right) - C \vec{\nabla} h(\vec{z}_1^n)$$  \hspace{1cm} (4)

The discrete impulse at impact $n$ has 2 components.

- The viscous dissipation during contact time $-D(\frac{\vec{v}_1^{n+1} + \vec{v}_1^n}{2})$ is equally divided before and after the field impulse.

- The field contribution $-C \vec{\nabla} h(\vec{z}_1^n)$ is its slope at position of impact $n$, computed from the previous impacts, according to:

$$\vec{\nabla} h(\vec{z}_1^n) = -h_0 k_F \sum_{k=2}^p \frac{J_1(k_F \| \vec{z}_1^n - \vec{z}_k^n \|)}{\| \vec{z}_1^n - \vec{z}_k^n \|} (\vec{z}_1^n - \vec{z}_k^n) e^{-(k-1)/Me}$$  \hspace{1cm} (5)

where $h_0$ is the maximum amplitude of the wave field generated by one impact.

This yields the following recurrence formula for the position of the droplet:

$$\vec{z}_1^{n+1} = I \vec{z}_n + J \vec{z}_1^n + K \sum_{k=2}^p \frac{J_1(k_F \| \vec{z}_1^n - \vec{z}_k^n \|)}{\| \vec{z}_1^n - \vec{z}_k^n \|} (\vec{z}_1^n - \vec{z}_k^n) e^{-(k-1)/Me}$$  \hspace{1cm} (6)

with $I = 1 + \frac{(1 - DT_F)}{DT_F}$, $J = 1 - I$, and $K = \frac{Ch_0 T_F^2 k_F}{m(1 - DT_F)}$.

The position of all other previous impacts is updated according to a shift:

$$\vec{z}_j^{n+1} = \vec{z}_j^n - 1 \text{ for } j \geq 2$$  \hspace{1cm} (7)

We now write the state vector as a $p$ dimensional vector in $\mathbb{C}^p$:

$$\vec{Z}_n = \begin{bmatrix} \vec{z}_1^n \\ \vdots \\ \vec{z}_p^n \end{bmatrix}$$  \hspace{1cm} (8)

The evolution of the state vector is given by $f : \mathbb{C}^p \rightarrow \mathbb{C}^p$. 

16
\[ \vec{Z}_{n+1} = f(\vec{Z}_n) = f(\vec{z}^1_n, \ldots, \vec{z}^p_n) \]  

(9)

that is:

\[
\begin{bmatrix}
\vec{z}^1_{n+1} \\
\vdots \\
\vec{z}^j_{n+1} \\
\vdots \\
\vec{z}^p_{n+1}
\end{bmatrix}
= f(\begin{bmatrix}
\vec{z}^1_n \\
\vdots \\
\vec{z}^j_n \\
\vdots \\
\vec{z}^p_n
\end{bmatrix}) =
\begin{bmatrix}
f_1(\vec{z}^1_n, \ldots, \vec{z}^p_n) \\
\vdots \\
f_j(\vec{z}^1_n, \ldots, \vec{z}^p_n) \\
\vdots \\
f_p(\vec{z}^1_n, \ldots, \vec{z}^p_n)
\end{bmatrix}
\]  

(10)

From equations (6) we get:

\[ f_1(\vec{z}^1_n, \ldots, \vec{z}^p_n) = I\vec{z}^1_n + J\vec{z}^2_n + K \sum_{k=2}^p J_k(k_F||\vec{z}^k_n - \vec{z}^k_n||) ||\vec{z}^k_n - \vec{z}^k_n|| (\vec{z}^k_n - \vec{z}^k_n)e^{-(k-1)/M} \]  

(11)

For \(2 \leq j \leq p\), we get from (11):

\[ f_j(\vec{z}^1_n, \ldots, \vec{z}^p_n) = \vec{z}^j_{n-1} \]  

(12)

This completes the description of the discrete dynamical system as a non-linear operator \(f\) acting on \(C^p\).

2 Circular orbit and rotation operator

Let \((s^*_n)\) be a circular solution \((s^*_{n+1} = f(s^*_n))\) associated with radius \(R\) and pulsation \(\Omega\). At each iteration, each point of the state vector rotates by a constant angle \(\theta = \Omega T_F\):

\[
\begin{bmatrix}
R e^{i(n\Omega T_F)} \\
\vdots \\
R e^{i((n-p+1)\Omega T_F)}
\end{bmatrix}
= 
\begin{bmatrix}
R e^{i\theta} \\
\vdots \\
R e^{i((n-p+1)\theta)}
\end{bmatrix}
\]  

(13)

We now define a rotation operator \(rot_\theta\) acting on the state vector, which rotates each component \(\vec{z}^j_n\) (\(j = 1 \ldots p\)) by an angle \(\theta\):

\[
rot_\theta(\begin{bmatrix}
\vec{z}^1_n \\
\vdots \\
\vec{z}^p_n
\end{bmatrix}) = 
\begin{bmatrix}
e^{i\theta} \vec{z}^1_n \\
\vdots \\
e^{i\theta} \vec{z}^p_n
\end{bmatrix}
\]  

(14)

That is: \(rot_\theta(\vec{Z}_n) = e^{i\theta} \vec{Z}_n\)
We notice that $s_{n+1} = rot_{\theta}(s_n)$, hence $s_n$ is a fixed point of the non linear operator $g$ defined by $g = rot_{-\theta} \circ f$ for any $n$.

We also note from (11) and (12) that:

\[ rot_{-\theta} \circ f = f \circ rot_{-\theta} \]  

(15)

and since $f$ and $rot_{-\theta}$ commute, so do $g$ and $rot_{\theta}$.

This is the so called equivariance condition (19). This commutativity condition will allow us to deduce the stability criteria of the circular orbits from the stability analysis of the fixed point $\tilde{s}$ of operator $g$.

3 Stability analysis

We now consider the state vector as part of $\mathbb{R}^{2p}$ instead of $\mathbb{C}^p$, and modify the operators $f$, $g$ and $rot_{\theta}$ accordingly. This will allow us to compute Jacobian matrices.

Let’s call $\tilde{s}$ a fixed point of operator $g$, corresponding to a state vector taken at any iteration of a circular solution of dynamical system $f$. For a small perturbation $\epsilon$ around $s$, we have at first order:

\[ g(\tilde{s} + \epsilon) \approx g(\tilde{s}) + J_g(\tilde{s}).\epsilon = \tilde{s} + J_g(\tilde{s}).\epsilon \]  

(16)

where $J_g(\tilde{s})$ is the Jacobian of operator $g$ at $\tilde{s}$. Hence:

\[ g^{(n)}(\tilde{s} + \epsilon) \approx \tilde{s} + J_g^{(n)}(\tilde{s}).\epsilon \]  

(17)

If the eigenvalues of $J_g(\tilde{s})$ all have a modulus strictly inferior to 1, $s$ is a stable stationary point for $g$. Now, let’s look at the relative evolution of a slightly perturbed circular solution under $f$

\[
\begin{align*}
& f^{(n)}(\tilde{s} + \epsilon) - f^{(n)}(\tilde{s}) \\
& = (rot_{\theta} \circ g(\tilde{s} + \epsilon))^{(n)} - (rot_{\theta} \circ g(\tilde{s}))^{(n)} \\
& = rot_{\theta}^{(n)} \circ g^{(n)}(\tilde{s} + \epsilon) - rot_{\theta}^{(n)} \circ g^{(n)}(\tilde{s}) \quad \text{(because $g$ and $rot_{\theta}$ commute)} \\
& = rot_{n\theta} \circ (\tilde{s} + J_g^{(n)}(\tilde{s}).\epsilon) - rot_{n\theta}(\tilde{s}) \quad \text{(from (17))} \\
& = rot_{n\theta}(J_g^{(n)}(\tilde{s}).\epsilon)
\end{align*}
\]  

(18)

The norm of the difference between a state vector corresponding to the stable circular trajectory and one corresponding to the perturbed trajectory evolves according to $J_g(\tilde{s})$.

In other words, the stability of a circular trajectory under $f$ is given by the eigenvalues of the Jacobian of $g = rot_{-\theta} \circ f$ taken at any state vector in the stable circular trajectory.
\[ J_g(\vec{s}) = J_{\text{rot} \circ f}(\vec{s}) = J_{\text{rot} \circ (f(\vec{s}))} J_f(\vec{s}) = \text{rot} \circ J_f(\vec{s}) \] (19)

To compute the Jacobian \( J_f(\vec{s}) \), we now need to write explicitly the state vector as a 2p dimensional vector:

\[
\vec{r}_n = \begin{bmatrix}
x_1^n \\
y_1^n \\
... \\
x_i^n \\
y_i^n \\
... \\
x_p^n \\
y_p^n
\end{bmatrix}
\] (20)

The dynamics is now an evolution given by \( f : \mathbb{R}^{2p} \rightarrow \mathbb{R}^{2p} \)

\[
\vec{r}_{n+1} = f(\vec{r}_n) = f(x_1^n, y_1^n, \ldots, x_i^n, y_i^n, \ldots, x_p^n, y_p^n)
\] (21)

that is:

\[
\begin{bmatrix}
x_{n+1}^1 \\
y_{n+1}^1 \\
... \\
x_{n+1}^i \\
y_{n+1}^i \\
... \\
x_{n+1}^p \\
y_{n+1}^p
\end{bmatrix} = f(\begin{bmatrix}
x_1^n \\
y_1^n \\
... \\
x_i^n \\
y_i^n \\
... \\
x_p^n \\
y_p^n
\end{bmatrix}) = \begin{bmatrix}
f_1(x_1^n, y_1^n, \ldots, x_i^n, y_i^n, \ldots, x_p^n, y_p^n) \\
f_2(x_1^n, y_1^n, \ldots, x_i^n, y_i^n, \ldots, x_p^n, y_p^n) \\
... \\
f_{2i-1}(x_1^n, y_1^n, \ldots, x_i^n, y_i^n, \ldots, x_p^n, y_p^n) \\
f_{2i}(x_1^n, y_1^n, \ldots, x_i^n, y_i^n, \ldots, x_p^n, y_p^n) \\
... \\
f_{2p-1}(x_1^n, y_1^n, \ldots, x_i^n, y_i^n, \ldots, x_p^n, y_p^n) \\
f_{2p}(x_1^n, y_1^n, \ldots, x_i^n, y_i^n, \ldots, x_p^n, y_p^n)
\end{bmatrix}
\] (22)

From equations (6) we get:

\[
f_1(x_1^n, y_1^n, \ldots, x_i^n, y_i^n, \ldots, x_p^n, y_p^n)
\]

\[ = I_a x_n^1 + J x_n^2 + K \sum_{k=2}^p \frac{J_1(kF \sqrt{(x_n^k - x_n^k)^2} + (y_n^k - y_n^k)^2)}{(x_n^k - x_n^k)^2 + (y_n^k - y_n^k)^2} (x_1^n - x_k^n)e^{-(k-1)/Me}
\] (23)

\[
f_2(x_1^n, y_1^n, \ldots, x_i^n, y_i^n, \ldots, x_p^n, y_p^n)
\]

\[ = I_b y_n^1 + J y_n^2 + K \sum_{k=2}^p \frac{J_1(kF \sqrt{(x_n^k - x_n^k)^2} + (y_n^k - y_n^k)^2)}{(x_n^k - x_n^k)^2 + (y_n^k - y_n^k)^2} (y_1^n - y_k^n)e^{-(k-1)/Me}
\] (24)
with

$$I = 1 + \left(1 - \frac{DT_{m}}{2m}\right), J = 1 - I,$$

and

$$K = \frac{C h_{0} T_{F}^{2} k_{F}}{m(1 - \frac{DT_{m}}{2m})}.$$

For $2 \leq i \leq p$, we get from (7):

$$f_{2i-1}(x_{n}, y_{n}, \ldots, x_{i}, y_{i}, \ldots, x_{p}, y_{p}) = x_{i-1}$$

(25)

$$f_{2i}(x_{n}, y_{n}, \ldots, x_{i}, y_{i}, \ldots, x_{p}, y_{p}) = y_{i-1}$$

(26)

The $\text{rot}_{\theta}$ operator is a block diagonal square $2p$ matrix

$$\text{rot}_{\theta} = \begin{bmatrix}
\cos \theta - \sin \theta & 0 & 0 & \cdots & 0 & 0 \\
\sin \theta, \cos \theta & \cos \theta - \sin \theta & 0 & \cdots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \cos \theta - \sin \theta & 0 & \cdots & 0 \\
& & \sin \theta, \cos \theta & \cos \theta - \sin \theta & \cdots & 0 \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \cos \theta - \sin \theta & 0 \\
& & & & & \sin \theta, \cos \theta & \cos \theta - \sin \theta \\
\end{bmatrix}$$

(27)

Eventually, a state vector taken at iteration 0 of a circular solution of dynamical system $f$ is written from [13] as:

$$\vec{s} = \begin{bmatrix}
R \\
0 \\
\vdots \\
R \cos (1 - i)\theta \\
R \sin (1 - i)\theta \\
\vdots \\
R \cos (1 - p)\theta \\
R \sin (1 - p)\theta
\end{bmatrix}$$

(28)

Equations 23, 24, 25, 26, 27, 28 allow to compute:

$$J_{g}(\vec{s}) = \text{rot}_{-\theta} . J_{f}(\vec{s})$$

(29)

with numerical tools. (Matlab symbolic package toolbox)

The spectral radius of $J_{g}(\vec{s})$ determines the stability of the self-orbit, after the eigenvalues 1 and $e^{\pm i\theta}$ corresponding respectively to the rotational and translational invariance of the dynamics are removed.