Quasi-particles in the Chiral Potts model

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Abstract

We study the quasi-particle spectrum of the integrable three-state chiral Potts chain in the massive phase by combining a numerical study of the zeroes of associated transfer matrix eigenvalues with the exact results of the ferromagnetic three-state Potts chain and the three-state superintegrable chiral Potts model. We find that the spectrum is described in terms of quasi-particles with momenta restricted only to segments of the Brillouin zone $0 \leq P \leq 2\pi$ where the boundaries of the segments depend on the chiral angles of the model.

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1. Introduction

In 1985, von Gehlen and Rittenberg [1] made a profound discovery concerning the $Z_n$ symmetric chiral Potts spin chain [2] with the Hamiltonian

$$H_{CP} = A_0 + kA_1 , \quad (1.1)$$

where

$$A_0 = -\sum_{j=1}^{L} \sum_{n=1}^{N-1} \frac{e^{i(2n-N)}\phi/N}{\sin(\pi n/N)} (Z_j Z_{j+1})^n , \quad A_1 = -\sum_{j=1}^{L} \sum_{n=1}^{N-1} \frac{e^{i(2n-N)\bar{\phi}/N}}{\sin(\pi n/N)} X_j^n . \quad (1.2)$$

The matrices $X_j, Z_j$ are defined by

$$X_j = I_N \otimes \cdots \underbrace{X} \cdots \otimes I_N , \quad Z_j = I_N \otimes \cdots \underbrace{Z} \cdots \otimes I_N , \quad (1.3)$$

where $I_N$ is the $N \times N$ identity matrix, the elements of the $N \times N$ matrices $X$ and $Z$ are

$$X_{l,m} = \delta_{l,m+1} \pmod{N} , \quad Z_{l,m} = \delta_{l,m} \omega^{l-1} \quad (1.4)$$

and $\omega = e^{2\pi i/N}$. They demonstrated that in the special case,

$$\phi = \bar{\phi} = \pi/2 , \quad (1.5)$$

the following commutation relations hold:

$$[A_0, [A_0, [A_0, A_1]]] = \text{const} [A_0, A_1] , \quad [A_1, [A_1, [A_1, A_0]]] = \text{const} [A_1, A_0] , \quad (1.6)$$

and thus that $A_0$ and $A_1$ are embedded in the larger algebra

$$[A_l, A_m] = 4G_{l-m} , \quad [G_l, A_m] = 2A_{m+l} - 2A_{m-l} , \quad (1.7)$$

$$[G_l, G_m] = 0 .$$

This is exactly the same algebra that Onsager [3] discovered for the Ising model in 1944.

The special case (1.3) is now called the superintegrable case. The discovery that this case satisfies Onsager’s algebra was the first time since Onsager’s original paper that a new representation of this algebra had been discovered. This algebra is the source of many
remarkable simplifications, which make the study of the superintegrable chiral Potts model particularly interesting [8]–[11].

Several years later it was found [4]–[6] that if the chiral angles of (1.2) satisfy the restriction that
\[
\cos \phi = k \cos \bar{\phi}
\]
then the model (1.1) is integrable in the traditional sense, i.e. it can be derived from a family of commuting transfer matrices \(T(p, q)\):
\[
[T(p, q), T(p, q')] = 0,
\]
where \(T(p, q)\) is constructed from the Boltzmann weights
\[
\frac{W^v_{p,q}(n)}{W^v_{p,q}(0)} = \prod_{j=1}^{n} \left( \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j} \right) \quad \text{and} \quad \frac{W^h_{p,q}(n)}{W^h_{p,q}(0)} = \prod_{j=1}^{n} \left( \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j} \right)
\]
where \(a_p, b_p, c_p, d_p\) and \(a_q, b_q, c_q, d_q\) lie on the generalized elliptic curve
\[
a^N + kb^N = k'd^N, \quad ka^N + b^N = k'c^N, \quad k' = (1 - k^2)^{1/2}
\]
which for \(k \neq 0, 1, \infty\) has genus \(N^3 - 2N^2 + 1\). The transfer matrix is defined to be
\[
T_{l,l'}(p, q) = \prod_{j=1}^{L} W^v_{p,q}(l_j - l'_j)W^h_{p,q}(l_j - l'_j+1)
\]
with \(l = \{l_1, l_2, \ldots, l_L\}, l_j = 0, 1, \cdots N - 1\). The Hamiltonian (1.1) is obtained in the limit \(p \to q\) as
\[
T(p, q) = 1(1 + \text{const } u) + uH_{CP} + O(u^2)
\]
where \(u\) is a measure of the deviation of \(p\) from \(q\) and the chiral angles \(\phi\) and \(\bar{\phi}\) satisfy
\[
e^{2i\phi/N} = \omega^{1/2} \frac{a_p c_p}{b_p d_p}, \quad e^{2i\bar{\phi}/N} = \omega^{1/2} \frac{a_p d_p}{b_p c_p}.
\]

Unfortunately, since the integrability condition (1.9) is not as strong as the condition of Onsager’s algebra (1.7), the study of this case is in general substantially more difficult than the study of the superintegrable case (1.3). However there is one exception to this observation; namely the special case
\[
\phi = \bar{\phi} = 0.
\]
Here, from \((1.8)\), we have \(k = 1\) and from \((1.11)\) and \((1.14)\) we find
\[
a = \omega^{-1/2} b, \quad c = d. \tag{1.16}
\]
The Boltzmann weights \((1.11)\) reduce to the weights of the model of Fatteev and Zamolodchikov \([12]\)
\[
W^v(n) = \prod \left( \frac{\omega^{1/4} e^\lambda - \omega^{j-1/2}}{1 - \omega^{j-1/4} e^\lambda} \right), \quad W^h(n) = \prod_{j=1}^{n} \left( \frac{e^\lambda \omega^{j-1/4} - \omega^{-1/2}}{\omega^j - \omega^{1/4} e^\lambda} \right), \tag{1.17}
\]
where we have set \(\frac{a_{q/d_q}}{a_{p/d_p}} = \omega^{1/4} e^\lambda\). This model, which for the case \(N = 3\) is the three-state Potts model, is solvable by the methods of Bethe’s ansatz and conformal field theory.

One of the most interesting properties of the Hamiltonian \((1.1)\) with \((1.8)\) is the spectrum of excitations over the ground state. These excitations can be studied through the use of the functional equations satisfied \([9]\)–\([15]\) by the transfer matrix of the statistical system \((1.10)\)–\((1.12)\). In particular these equations have been used for the case \(N = 3\) to study both the superintegrable case \((1.9)\) \([11]\) and the scalar case \((1.15)\) (which is the critical three-state Potts model) \([16]\) \([17]\). In the first case, in the massive regime, the spectrum consists of two quasi-particles. In the second case, in the ferromagnetic regime, the spectrum has one genuine quasi-particle and two “ghost” particles which carry statistical information but no energy or momentum.

These two spectra are are not obviously of the same qualitative form. However, from the phase diagram of the general integrable chiral Potts model in \([18]\), it is clear that it is possible to smoothly connect these two points without passing through a massless, level crossing phase. Thus it should be possible to smoothly connect the spectra of these two special points together.

It is the purpose of this paper to study this connection for the case \(N = 3\). For convenience, we concentrate on the sector \(Q = 0\), where \(e^{2\pi i Q/3}\) are the eigenvalues of the spin rotation operator. In sec. 2, we summarize the exact results for the ferromagnetic three-state Potts and the superintegrable chiral Potts model. In sec. 3, we present the results of a numerical study of the zeroes of the transfer matrix in the general integrable case. We conclude in sec. 4 by using this study to connect the two exact spectra together, and we contrast our study with previous results \([19]\)–\([21]\). Our principle conclusions are that 1) there is a sense in which the superintegrable spectrum is more properly regarded as consisting of three rather than two quasi-particles 2) the momentum ranges for these three types of quasi-particle are different and 3) as we move smoothly from the superintegrable to the ferromagnetic three-state Potts point the allowed momentum range of two of the quasi-particles shrinks to zero while the range of the third expands to fill the entire Brillouin zone.
2. Spectrum of the superintegrable chiral and ferromagnetic three-state Potts models

The general quasi-particle form for an order one excitation spectrum over the ground state is

\[
\lim_{L \to \infty} (E_{ex} - E_{GS}) = \sum_{\alpha=1}^{N_S} \sum_{j_\alpha=1}^{m_\alpha} e_\alpha(P^\alpha_{j_\alpha})
\]  

(2.1)

and

\[
\lim_{L \to \infty} (P_{ex} - P_{GS}) = \sum_{\alpha=1}^{N_S} \sum_{j_\alpha=1}^{m_\alpha} P^\alpha_{j_\alpha}
\]

(2.2)

where \(N_S\) is the number of species of quasi-particles, \(m_\alpha\) is the number of quasi-particles of type \(\alpha\) in the states, \(e_\alpha(P)\) are the single particle energies, and the subscript “rules” indicates the restriction on the allowed choices for \(P^\alpha_{j_\alpha}\). If one of the rules is

\[
P^\alpha_{j_\alpha} \neq P^\alpha_{k_\alpha} \text{ for } j_\alpha \neq k_\alpha
\]

(2.3)

then the quasi-particles are said to be fermionic.

The spectrum of the ferromagnetic three-state Potts chain has been computed \[16\] \[17\] in terms of the zeroes \(\lambda_j\) of the eigenvalues \(\Lambda\) of the transfer matrix.

\[
\Lambda = \left( \frac{\sinh(\pi i/6) \sinh(\pi i/3)}{\sinh(\frac{\pi i}{2} - \lambda) \sinh(\frac{\pi i}{2} + \lambda)} \right)^L \prod_{k=1}^{2L} \frac{\sinh(\frac{1}{2}(\lambda - \lambda_k))}{\sinh(\frac{1}{2}(\frac{\pi i}{6} + \lambda_k))},
\]

(2.4)

as

\[
E = \sum_{k=1}^{2L} \cot \left( \frac{1}{2}(i\lambda_k + \frac{\pi}{6}) - \frac{2L}{3^{1/2}} \right),
\]

(2.5)

where \(\lambda_j\) satisfy the Bethe equation

\[
(-1)^{L+1} \left( \frac{\sinh(\frac{1}{2}(\lambda_j - 2iS\gamma))}{\sinh(\frac{1}{2}(\lambda_j + 2iS\gamma))} \right)^{2L} = \prod_{k=1}^{M} \frac{\sinh(\frac{1}{2}(\lambda_j - \lambda_k - 2i\gamma))}{\sinh(\frac{1}{2}(\lambda_j - \lambda_k + 2i\gamma))},
\]

(2.6)

with \(S = 1/4, \gamma = \pi/3\) and (for \(Q = 0\)) \(M = 2L\) From this it is found that as \(L \to \infty\) the allowed imaginary parts of \(\lambda_j\) are

\[
\text{Im } \lambda_j = 0, \pi, \pm\pi/3, \pm2\pi/3, \pm\pi/2
\]

(2.7)
which we refer to as $\pm, -2s, -2s$ and $ns$ respectively. We find that the spectrum is of the fermionic quasi-particle form, where there is only one type of single particle energy. In particular

$$\lim_{L \to \infty} (E(\{P_j\}) - E_{GS}) = \sum_{j=1}^{m_+} e_+(P_j^+) \quad (2.8)$$

where

$$e_+(P) = 6 \sin(P/2), \quad 0 \leq P \leq 2\pi, \quad (2.9)$$

and each energy occurs with a multiplicity

$$\left(\frac{1}{3}(m_+ - 2m_{ns} - m_{-2s})\right)\left(\frac{1}{3}(2m_+ - m_{ns} - 2m_{-2s})\right), \quad (2.10)$$

where the integers $m_+, m_{ns}$ and $m_{-2s}$ take on all nonnegative values subject only to the restriction (valid for the sector $Q = 0$ under consideration)

$$m_+ - 2m_{ns} - m_{-2s} \equiv 0 \pmod{3}. \quad (2.11)$$

This degeneracy factor can be interpreted as indicating that the $ns$ and $-2s$ excitations carry zero momentum and energy. We refer to these zero energy excitations as “ghost” excitations.

More precisely [22], the spectrum to order $1/L$ is given by

$$E(\{P\}) - E_{GS} = \sum_{\alpha = +, ns, -2s} \sum_{j_\alpha} e_\alpha(P_{j_\alpha}^\alpha), \quad (2.12)$$

where the energies of the “ghost” excitations are

$$e_{ns,-2s}(P_{j_\alpha}^\alpha) = 3P_{j_\alpha}^\alpha, \quad (2.13)$$

and the momenta $P_{j_\alpha}^\alpha$ obey the Fermi exclusion rule (2.3), and are chosen from sets with spacings $2\pi/L$ with the following limits

$$-\frac{\pi}{L} \left[\frac{1}{3}(m_+ - 2m_{ns} - m_{-2s}) - 1\right] \leq P_{j_\alpha}^+ \leq 2\pi + \frac{\pi}{L} \left[\frac{1}{3}(m_+ - 2m_{ns} - m_{-2s}) - 1\right], \quad (2.14)$$

$$-\frac{\pi}{L} \left[\frac{1}{3}(m_+ - 2m_{ns} - m_{-2s}) - 1\right] \leq P_{j_\alpha}^{-2s} \leq \frac{\pi}{L} \left[\frac{1}{3}(m_+ - 2m_{ns} - m_{-2s}) - 1\right],$$

$$-\frac{\pi}{L} \left[\frac{1}{3}(2m_+ - m_{ns} - 2m_{-2s}) - 1\right] \leq P_{j_\alpha}^{ns} \leq \frac{\pi}{L} \left[\frac{1}{3}(2m_+ - m_{ns} - 2m_{-2s}) - 1\right].$$
We refer to such momentum exclusion rules as generalized Fermi statistics. It is important to note that in (2.14) only the $+\text{ momentum}$ take on a macroscopic number of values (proportional to $L$). The $-s$ and the $n$s "ghost" momenta take on only a finite number of values (even as $L \to \infty$). We refer to such a momentum range as microscopic.

The energy spectrum of the superintegrable chiral Potts model has also been computed. At this point we find from (1.14) that

$$a_p = b_p, \quad c_p = d_p \quad (2.15)$$

and hence

$$a_p/d_p = \eta^{-1}, \quad \text{where} \quad \eta = [(1 + k)/(1 - k)]^{1/6}. \quad (2.16)$$

From the property of superintegrability it follows [10] that the spectrum of eigenvalues is decomposed into a number of sets of the form [7]

$$E = A + kB + 3 \sum_{j=1}^{m_E} \pm (1 + k^2 + a_j k)^{1/2} \quad (2.17)$$

where $A$, $B$, $a_j$ and the number of terms $m_E$ varies from set to set. More precisely we find [9] [11] that the eigenvalues of the transfer matrix are of the form

$$\Lambda_{si} = \frac{3^L (\eta a/d - 1)^L}{[(\eta a/d)^3 - 1]^L} \left( \frac{a}{d} \right)^{P_a} \left( \frac{b}{c} \right)^{P_b} \left( \frac{c}{d^3} \right)^{P_c} \prod_{l=1}^{m_P} \left( \frac{1 + \omega v_l \eta^2 ab/cd}{1 + \omega v_l} \right)^{m_E} \left( \frac{1 + k}{1 - k} \right)^{1/2} \left( \frac{a^3 + b^3}{2d^3} \pm w_l \frac{a^3 - b^3}{(1 + k)d^3} \right), \quad (2.18)$$

where we have used the sign convention for $v_l$ of [9]. These $v_l$ satisfy

$$\left( \frac{\omega^2 + v_k}{\omega + v_k} \right)^L = -\omega^{-(P_a + P_b)} \prod_{l=1}^{m_P} \left( \frac{v_k - \omega^2 v_l}{v_k - \omega v_l} \right), \quad (2.19)$$

The $w_l$ in (2.18) are

$$w_l^2 = \frac{1}{4}(1 - k)^2 + \frac{k}{1 - t_l^3} \quad (2.20)$$

where the $t_l$ are the roots of the polynomial

$$P(t) = t^{-(P_a + P_b)} \left[ (\omega^2 t - 1)^L (\omega t - 1)^L \omega^{P_a + P_b} \prod_{l=1}^{m_P} \left( \frac{1 + tv_l}{1 + t^3 v_l^3} \right) \right.$$

$$+ (t - 1)^L (\omega^2 t - 1)^L \prod_{l=1}^{m_P} \left( \frac{1 + \omega t v_l}{1 + t^3 v_l^3} \right) + (t - 1)^L (\omega t - 1)^L \omega^{-(P_a + P_b)} \prod_{l=1}^{m_P} \left( \frac{1 + \omega^2 t v_l}{1 + t^3 v_l^3} \right) \left. \right] \quad (2.21)$$
and $P_a$, $P_b$ and $P_c$ are integers. The eigenvalues of the superintegrable chain are thus

$$E_{SI} = A + Bk + 6 \sum_{l=1}^{m_E} \pm w_l. \quad (2.22)$$

In the small $k$ regime where there is a mass gap the ground state corresponds to the choice $m_P = 0$ in (2.21) and all minus signs in (2.22).

To obtain the excitation spectrum we need to know the allowed solutions $\lambda_l$ of the Bethe’s equation (2.19). This was studied in [11] where it was found that as $L \to \infty$ there are three allowed values for the imaginary parts of $\lambda_j$

$$\text{Im}\lambda_j = 0, \pi, \pm 2\pi/3, \quad (2.23)$$

which we call $+$, $-$, and $-2s$ respectively. These correspond to

$$v_l = v^+ > 0 , \quad v^- < 0 \quad \text{and} \quad v^{-2s}e^{\pm 2\pi/3}. \quad (2.24)$$

It was also shown that the eigenvalues obtained from the choices of the $\pm$ signs in (2.22) can be written in terms of the excitations $+$ and $-2s$ of the Bethe’s equation with a suitable alteration of the rules of combination. Thus it was shown that if we use the relation between momenta and $v_l^\alpha$ of

$$e^{-iP^r} = \left(\frac{1 + \omega^2 v^r}{1 + \omega v^r}\right) \quad \text{for} \quad r = \pm , \quad e^{-iP^{-2s}} = \left(\frac{1 + e^{-\pi i/3}v^{-2s}}{1 + e^{\pi i/3}v^{-2s}}\right), \quad (2.25)$$

then the order one excitations are of the fermionic quasi-particle form (2.1) where the single particle energies are

$$e_r(P^r) = 2|1 - k| + \frac{3}{\pi} \int_1^{\frac{1+k}{1-k}^{2/3}} dt \left(\frac{\omega v^r}{\omega tv^r + 1} + \frac{\omega^2 v^r}{\omega^2 tv^r + 1}\right) \left[\frac{4k}{t^3 - 1} - (1 - k)^2\right]^{1/2}, \quad (2.26)$$

where $r$ may be either $\pm$, with the range of $P^r$ restricted to

$$0 \leq P^+ \leq \frac{4\pi}{3} , \quad \frac{4\pi}{3} \leq P^- \leq 2\pi , \quad (2.27)$$

and

$$e_{-2s}(P^{-2s}) = 4|1 - k| + \frac{3}{\pi} \int_1^{\frac{1+k}{1-k}^{2/3}} dt \frac{v^{-2s}[4(v^{-2s}t)^2 + v^{-2st} + 1]}{(v^{-2s}t)^3 - 1} \left[\frac{4k}{t^3 - 1} - (1 - k)^2\right]^{1/2}, \quad (2.28)$$
with
\[ \frac{2\pi}{3} \leq P^{-2s} \leq 2\pi. \] (2.29)

We also have the restriction
\[ m_+ + m_- + 2m_{-2s} \equiv 0 \pmod{3}. \] (2.30)

Moreover this result may be extended to order $1/L$ by use of the counting rules of [11], where we note that the rewriting of the excitations associated with the $\pm$ signs in (2.22) in terms of $+$ and $-2s$ excitations is possible, because any set of roots $v_l = v, \omega v$ and $\omega^2 v$ satisfy (2.19) independently of the value of $v$. Thus the reduction in the number of states which results from choosing all the $\pm$ signs to be negative is exactly compensated for by elimination of the exclusion rule $-v^-_{k \neq v} \neq v^{-2s}$ of ref. [11]. Using this we find, to order $1/L$ (for $L \equiv 0 \pmod{3}$ where $P^a = P^b = 0$),
\[ E(\{P\}) - E_{GS} = \sum_{\alpha=+,\,-,\,-2s} \sum_{j_\alpha} e_\alpha(P^\alpha_{j_\alpha}), \] (2.31)

where the momenta $P^\alpha_{j_\alpha}$ obey the Fermi rule (2.3) and are chosen from sets with spacings $2\pi/L$ with the following limits
\[ -\frac{\pi}{L} \left[ \frac{1}{3}(m_+ - 2m_- - m_{-2s}) - 1 \right] \leq P^+_j \leq \frac{4\pi}{3} + \frac{\pi}{L} \left[ \frac{1}{3}(m_+ - 2m_- - m_{-2s}) - 1 \right], \]
\[ \frac{4\pi}{3} - \frac{\pi}{L} \left[ \frac{1}{3}(2m_+ - m_- - 2m_{-2s}) - 1 \right] \leq P^-_j \leq 2\pi + \frac{\pi}{L} \left[ \frac{1}{3}(2m_+ - m_- - 2m_{-2s}) - 1 \right], \]
\[ \frac{2\pi}{3} - \frac{\pi}{L} \left[ \frac{1}{3}(m_+ - 2m_- - m_{-2s}) - 1 \right] \leq P^{-2s}_j \leq 2\pi + \frac{\pi}{L} \left[ \frac{1}{3}(m_+ - 2m_- - m_{-2s}) - 1 \right]. \] (2.32)

Note that the $m_\alpha$ dependence of these restrictions is identical with that of the restrictions (2.14) of the ferromagnetic three-state Potts model if we make the identification:

\[ (+,\,-,\,-2s) \text{ superintegrable} = (+,ns,-2s) \text{ ferromagnetic three state Potts}. \] (2.33)

Moreover this identification makes the restrictions (2.11) and (2.30) identical. The only differences between the two cases are 1) the single particle energy functions are different, 2) the macroscopic momentum ranges of the $+$ excitation is $2\pi$ in (2.14) and is $4\pi/3$ in (2.32) and 3) the momentum ranges of the $-2s$ and the $ns$ excitations are microscopic in (2.14) whereas the ranges of $-$ and $-2s$ excitations in (2.32) are macroscopic. We conjecture that in the general case that the spectrum is of the form of the superintegrable spectrum where the single particle energies and the macroscopic momentum ranges depend on the chiral angles.
3. Spectrum in the general case

We now turn to an investigation of the excitation spectrum of the general integrable chiral Potts model and see what evidence can be obtained to support the conjecture of the previous section. An initial investigation of this spectrum was made in [18] by use of the functional equations of [14] and [17]. The single particle energy function was found to be

\[ e(v) = 2(1 - k)t_p^{3/2} \pm 3^{1/2}v t_p^{-1/2} \left\{ \frac{t_p^3 - 1 - (1 - k)^2}{(\omega t_p v + 1)(\omega^2 t_p v + 1)} \right\}^{1/2} \]

\[ + \frac{3v t_p^{3/2}}{\pi P} \int_1^{\frac{1 + k}{1 - k}} dt \left( \frac{\omega}{\omega t v + 1} + \frac{\omega^2}{\omega^2 t v + 1} \right) \left\{ \frac{(t^3 - 1)(1 + k)^2 - (1 - k)^2 t^3}{t^3 - t_p^3} \right\}^{1/2}, \]

where the ± sign is chosen to be + if \(0 \leq \phi \leq \pi/2\) and − if \(\pi/2 \leq \phi \leq \pi\), \(P\) indicates the principle value of the integral, \(t_p\) satisfies \(1 \leq t_p \leq (\frac{1 + k}{1 - k})^{2/3}\) and is defined to be

\[ t_p^3 = \frac{(1 + k)^2}{1 + k^2(1 - 2\cos^2 \phi) + 2k|\sin \phi|(1 - k^2\cos^2 \phi)^{1/2}}, \]

and the allowed values of \(v\) are equal to \(\omega^{1/2}\) times the location of the zeroes of the eigenvalues of the transfer matrix which satisfy an equation ((42) of [18]) which resembles a Bethe’s equation. However, this equation for the allowed \(v\) is in fact not a Bethe’s equation like (2.6) and we have not been able to relate the solutions \(v_j\) to the momentum of the state. Consequently we do not have an analytic verification of the conjecture.

To proceed further we have made a numerical study of the zeroes of the transfer matrix. For convenience we consider the case \(k = 1\). Here we define \(d^3 = [(1 - k)/(1 + k)]^{1/2} d^3\) and find that the curve (1.11) reduces to the Fermat curve

\[ a^3 + b^3 = 2d^3, \]

and that the superintegrable eigenvalue (2.18) reduces to

\[ \Lambda = \frac{2^L(a/d - 1)^L}{[(a/d)^3 - 1]^L} (\frac{a}{d})^P \prod_{l=1}^{m_P} \left( \frac{1 + \omega v_l a d}{a d} \right)^{m_E} \prod_{l=1}^{m_E} \left( 1 \pm w_l \frac{(a^3 - b^3)}{2d^3} \right). \]

For our purposes we rewrite this in terms of the variables

\[ \bar{t} = \omega^{-1/2} \frac{ab}{d^2}, \quad u = \frac{a^3 - b^3}{2d^3}, \]
as
\[ \Lambda = \frac{3^L (a/d - 1)^L}{[(a/d)^3 - 1]^L} \left( \frac{a}{d} \right)^{P_a} \left( \frac{b}{d} \right)^{P_b} \prod_{l=1}^{m_p} \left( \frac{1 - v_l \tilde{t}}{1 + \omega v_l} \right)^{m_p} \prod_{l=1}^{m_E} (1 - w_l u), \] (3.6)
where from (3.3) \( \tilde{t} \) and \( u \) satisfy
\[ u^2 = 1 + \tilde{t}^3. \] (3.7)
For arbitrary values of the chiral angle \( \phi \), the eigenvalues of the transfer matrix are still meromorphic functions on the Riemann surface (3.7) even though the form (3.6) is no longer valid.

In the superintegrable case we see from (3.6) that there are qualitatively two different types of zeros. The first type is specified by a value of \( \tilde{t}_l \), which satisfies
\[ \tilde{t}_l = v_l^{-1} \] (3.8)
and exists for both sheets (±) of \( u \). The second type exists only on one sheet of \( u \) (either + or −) and has three values of \( \tilde{t}_l \) related by
\[ \tilde{t}_l^{(2)} = \omega \tilde{t}_l^{(1)}, \quad \tilde{t}_l^{(3)} = \omega^2 \tilde{t}_l^{(1)} \] (3.9)
where \( \tilde{t}_l^{(1)} \) is real.

We have studied the zeroes of the transfer matrix for arbitrary values of the chiral angle \( \phi = \tilde{\phi} \) by use of the procedure previously used for three-state Potts case \([16]\). We will here illustrate the behavior of the zeroes by presenting the results for the nine eigenvalues in the \( Q = 0 \) sector for \( L = 3 \). This behavior is illustrated schematically in figs. 1–9. We summarize some of the features of these results in table 1.

There are many features of the motion of the zeroes which may be seen in these figures and we will explicitly comment on only a few of them. First, all zeroes move towards infinity as \( \phi \to 0 \). Second, we note that no zero ever moves from one sheet of \( u \) to the other, so the sign specifying the sheet in the superintegrable case acts like a good quantum number. Third, we see that as \( \phi \to 0 \) most of the zeroes stay close to the rays in the \( \tilde{t} \) plane \( \arg \tilde{t} = 0, \pm 2\pi/3 \). These correspond to the roots ± and \( \pm 2s \) of the three-state Potts model. However there are some roots, as seen in Figs. 2 and 6, which collide with the branch point at \( \tilde{t}^3 = -1 \) and move to infinity on somewhat less well defined rays. These are the \( ns \) roots of the three-state Potts model whose imaginary parts are subject to much more deviation from the asymptotic value than the other roots.
Table 1. A description of the zeroes of the transfer matrix studies in figures 1–9. The zeroes are indicated by their position in the $\bar{t}$ plane of (3.5) and the sign of $u$. We indicate in the columns $m_E$, $-2s$ and $E_{SI}$ the content and energy of the superintegrable case where $\phi = \pi/2$. The first (second) sign in the column $m_E$ refers to the root with the largest (smallest) value of $\bar{t}$. We indicate in the columns 3sP content and $E_{3sP}$ the content and energy of the three-state Potts case where $\phi = 0$.

| fig. | $P$ | $m_E$ | $+/-$ | $-2s$ | $E_{SI}$ | 3sP content | $E_{3sP}$ |
|------|-----|-------|-------|-------|---------|-------------|----------|
| 1    | 0   | 0     | 0     | 0     | -7.7459 | 3(2s)       | -8.87348 |
| 2    | 0   | +    | 0     | 0     | -3.4641 | (2s) (ns) 2(+) | -3.4641 |
| 3    | 0   | none | 3     | 0     | 0.0     | (2s) (-) 3(+) | 3.4641  |
| 4    | 0   | -    | 0     | 0     | 3.4641  | (2s) (-) 3(+) | 3.4641  |
| 5    | 0   | +    | 0     | 0     | 7.7454  | (-2s) 4(+)  | 5.40938 |
| 6    | $-2\pi/3$ | none | 1     | 0     | 0.0     | (2s) (ns) 2(+) | 0.0    |
| 7    | $-2\pi/3$ | none | 2     | 1     | 0.0     | (2s) (-) 3(+) | 0.0    |
| 8    | $2\pi/3$ | none | 1     | 0     | 0.0     | (2s) (-) 3(+) | 0.0    |
| 9    | $2\pi/3$ | none | 2     | 1     | 0.0     | (2s) (ns) 2(+) | 0.0    |

The most obvious property of the motion of these zeroes is that the zeroes change their character.

In Fig. 1 the positive real zeroes for $u$ negative move to the rays $\arg \bar{t} = \pm 2\pi/3$. This corresponds to the motion of the zeroes in the ground state of the massive phase.

In Fig. 2 the $u < 0$ roots on the rays $\arg \bar{t} = \pm 2\pi/3$ stay near the ray but the $u > 0$ roots that stay on $\arg \bar{t} = \pm 2\pi/3$ execute a motion that has one of them pairing with a $u < 0$ root to become an $ns$ pair as $\phi \to 0$. The remaining $u > 0$ root moves out along the positive real axis to become a $+$ root of the three-state Potts model.

Indeed in all the remaining figures there are several roots which undergo a qualitative change in going from the superintegrable case to the three-state Potts case. These qualitative changes must be accounted for in the solution of the pseudo-Bethe equation of [18].

4. Conclusions

There are several points to consider before the data presented in figures 1–9 can be used to discuss the conjecture presented at the end of sec. 2.
First, since the ground state of the superintegrable chiral Potts model at $k = 1$ is not the state with $m_+ = m_- = m_{-2s} = 0$, the identification of the ground state configuration of zeroes made in the previous section is not literally correct for the $L \to \infty$ limit. Indeed, the chiral Potts model is massless all along the line $k = 1$. However, for systems of such small size as $L = 3$ the level crossing phenomena which causes $m_+ = m_- = m_{-2s}$ not to be the ground state at $k = 1$ cannot be seen (in fact it is found that $L$ must be larger than 18 for level crossing to occur in the superintegrable case). Moreover we have made further numerical studies of the motion of the zeroes starting from a value of $k$ sufficiently small that the superintegrable case is massive and followed a path in $(\bar{\phi}, k)$ space that always lies in the massive phase. The only qualitative difference is that there is an additional square root branch point at $\bar{t}^3 = -\frac{(1 + k)}{(1 - k)}^2$ and this branch point does not cause a qualitative change in the motions of the zeroes in the figures 1-9.

Secondly, we must consider how much physical intuition should be invested in the zeroes of the transfer matrix. For the three-state Potts model and for the $r = 6$ RSOS model (which is related to the three-state Potts model by an orbifold construction [24] [26]) there is an alternative set of Bethe’s equations [27]–[29] which focus on the zeroes of an auxiliary objet called $Q$. This formalism is widely used to compute spectra of integrable models. For the general chiral Potts models there are many different functional equations [13] [14] and no one seems to be preferred for the purpose of giving physical interpretations. We use the $T$-matrix approach here for our convenience.

A major question of physical interest is the the uniqueness (or lack thereof) of the quasi-particle description of the spectrum. This problem is seen vividly if we compare our conjecture of sec. 2 with the numerical studies and interpretation of [19] [21]. In these studies the spectrum of the massless ferromagnetic three-state Potts model is interpreted as a quasi-particle spectrum made up of two excitations. This interpretation is completely consistent with the spectrum of the (massless) hard hexagon model ($r = 5$ RSOS at the I/II boundary) [31] [32] which is in the same universality class as the ferromagnetic three-state Potts model. Moreover, the hard hexagon model remains integrable in the massive (regime II) phase (although the three-state Potts model does not) and the two quasi-particle interpretation describes this massive spectrum [30]. In ref. [19], where the case $\phi = \bar{\phi}$ (which is integrable only for $k = 1$, $\phi = \bar{\phi} = 0$ and $\phi = \bar{\phi} = \pi/2$) is extensively studied, this two quasi-particle picture is used to interpret the numerical results.

It would appear that the two quasi-particle description of the spectrum in the case $\phi = \bar{\phi} = 0$, $k = 1$ is not the same as the description in terms of one quasi-particle
and two ghosts found in [16][17]. In order to justify our conjecture we must address this apparent contradiction. This question has been extensively studied at the level of the order $1/L$ excitations in [22]. The two quasi-particle form of the spectrum leads to the $q$-series character formula of Lepowsky and Primc [33] and the spectrum of sec. 2 leads to a different form which involves $q$-binomial coefficients [22]. However there is an identity on $q$-series [34] which makes these two different looking fermionic forms of the character equal. If the study of [19]-[21] is a correct interpretation of the numerical data then this identity of characters must have an extension to the order one spectrum. Let us assume such an identity and examine its consequences.

The existence of alternative forms of a massless order one spectrum was first seen in the Ising model where the spectrum of the transfer matrix at the critical temperature $T_c$ can be obtained either in terms of an odd number of quasi-particles if $T_c$ is approached from above, or an even number of quasi-particles if $T_c$ is approached from below. Spectra which can involve ghosts have been investigated in detail in the spin $1/2$ Heisenberg antiferromagnet by Faddeev and Takhtajan [35][36], who showed the equivalence of a two quasi-particle spectrum with the previously known spectrum [37][38] of one quasi-particle and an infinite number of ghosts. In the first case the two quasi-particles are interpreted [35][36] as spin $1/2$ spin waves. In the second case [37][38] the quasi-particles are the holes in the ground state distribution of roots of the Bethe’s equation and the ghosts (string solutions of the Bethe equations) which have zero energy and zero momentum are similar to the ghosts seen in the massive Thirring model [39]. The quasi-particles in [35]-[38] are the interacting counterparts of the free spinons of the $1/r^2$ Heisenberg spin chain [40]-[42]. The corresponding $1/L$ spectrum of the related conformal field theory has been investigated in [41] and [43]. The character identity between the two representations of the spectrum was proven in [44].

However it must be explicitly stated that the existence of an identity between different quasi-particle representations of the spectrum forces us to question what we mean by the “physical reality” of a quasi-particle description (at least for a massless spectrum). We do not seem to have the right to ascribe physical reality to something which is not unique. Indeed it is only the spectrum of the Hamiltonian which has a unique meaning. Any quasi-particle description of this spectrum imposes a basis on the Hilbert space and it is this basis to which we give the physical words of quasi-particle. But logically speaking a basis dependent statement can only be given physical reality if an additional condition has been given, which singles out the particular basis to be used. In the absence of such a condition
different bases cannot be distinguished and the non uniqueness of the description must be accepted. Moreover we point out the fact that many characters have been found to have more that one fermionic representation [45]-[46]. This seems to imply that at massless points of high symmetry there may be several different quasi-particle interpretations of exactly the same spectrum. This is the infinite dimensional analogy of finite dimensional degenerate perturbation theory, where different bases are used depending on the type of perturbation applied. The basis used here is appropriate to the chiral interaction which breaks the $Z_2$ charge conjugation symmetry of $S_3$ symmetric ferromagnetic Potts model. This basis also seems appropriate [22] for the integrable perturbation of the critical three-state Potts model which breaks $Z_3$ symmetry [47] [48]. The basis with two quasi-particles with equal energies for $Q = \pm 1$ seems to be appropriate for a (non-integrable) perturbation which preserves the full $S_3$ symmetry.

With this discussion of the symmetry of the ferromagnetic massless three-state Potts point we may now proceed to use the data of sec. 3 to discuss the conjecture of sec. 2. There are indeed two excitation energies: $e_{\pm}(P)$ has $Q = +1$ and $e_{-2s}(P)$ has $Q = -1$. The $-2s$ excitations can be thought of, in some sense, as bound states of the $\pm$ excitations and the motion of the zeroes with $u \geq 0$ from the rays $\arg \bar{t} = \pm 2\pi/3$ to the real axis can be thought of as an unbinding transition. As $\phi \to 0$ the number of these $-2s$ excitations decreases until at $\phi = \bar{\phi} = 0$ the number becomes microscopic (as is needed to reproduce the (2.14)). Moreover there is some difference between the $+$ and the $-$ excitations. Taken together they span the entire Brillouin zone $0 \leq P \leq 2\pi$ and as a function of the order one momentum the energy is continuous in this Brillouin zone. But nevertheless there is a break in the counting rules for the states, which separates the $+$ from the $-$ excitations. Some of the $-$ excitations move smoothly to the $ns$ excitations, which also must become microscopic in the limit $\phi = \bar{\phi} = 0$ in order to agree with (2.14). Of course with such small systems it is hardly possible to see these shifts from macroscopic to microscopic in any quantitative way. Nevertheless the existing data supports this picture in a qualitative fashion and it is in this sense that we say that the data of section 3 supports the conjecture of sec. 2.

It is sometimes said that the massless conformal field theories cannot be given a particle interpretation. This statement is based, in part, on the notion of quasi-particle as an isolated pole in a Greens function and, obviously, if massless particles are present there
are no isolated poles. However, for genuine physical applications this notion is not particularly useful, because massless excitations such as phonons and magnons are common in condensed matter physics and, moreover, all charged particles couple to massless photons. Indeed, we are used to the fact that different gauges may differ in their description of the “longitudinal photons.” In QED this non uniqueness is not considered to be a problem and it is often said that the Coulomb gauge is singled out as the one which should be considered as being physical. However, in the distinction between the various massive perturbations of the ferromagnetic Potts model we seem to be seeing cases where several different bases have physical relevance. It thus seems to us that it is more appropriate to say that conformal field theories not only do have a quasi-particle description but that they may in general have several different descriptions. The notion of perturbation of conformal field theory thus is related to the question of how many of such quasi-particle descriptions can be found.

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Figure captions

Figure 1 Schematic plot of the motion of the zeroes of the eigenvalue with $P = 0$, $Q = 0$ whose superintegrable content is $m_E = -, m_+ = m_- = m_{-2s} = 0$. The arrows indicate the motion of the zeroes as $\phi$ starts from $\pi/2$ and moves towards zero. The two zeroes which are initially on the real axis remain on the real axis until they collide at $\bar{t} = 0$ when $\phi = \pi/3$. We schematically represent this situation by lines which are off the real axis as a means of visualization. This convention for the motion of zeroes on the real axis will be used in all figures.

Figure 2 Schematic plot of the motion of the zeroes of the eigenvalue with $P = 0$, $Q = 0$ whose superintegrable content is $m_E = + -, m_+ = m_- = m_{-2s} = 0$.

Figure 3 Schematic plot of the motion of the zeroes of the eigenvalue with $P = 0$, $Q = 0$ whose superintegrable content is $m_+ = 3, m_- = m_{-2s} = 0$.

Figure 4 Schematic plot of the motion of the zeroes of the eigenvalue with $P = 0$, $Q = 0$ whose superintegrable content is $m_E = -, m_+ = m_- = m_{-2s} = 0$.

Figure 5 Schematic plot of the motion of the zeroes of the eigenvalue with $P = 0$, $Q = 0$ whose superintegrable content is $m_E = + +, m_+ = m_- = m_{-2s} = 0$.

Figure 6 Schematic plot of the motion of the zeroes of the eigenvalue with $P = -2\pi/3$, $Q = 0$ whose superintegrable content is $m_+ = m_{-2s} = 1$ and $m_- = 0$. 
Figure 7 Schematic plot of the motion of the zeroes of the eigenvalue with $P = -2\pi/3$, $Q = 0$ whose superintegrable content is $m_+ = 2$, $m_- = 1$, $m_{-2s} = 0$.

Figure 8 Schematic plot of the motion of the zeroes of the eigenvalue with $P = 2\pi/3$, $Q = 0$ whose superintegrable content is $m_+ = m_{-2s} = 1$ and $m_- = 0$.

Figure 9 Schematic plot of the motion of the zeroes of the eigenvalue with $P = 2\pi/3$, $Q = 0$ whose superintegrable content is $m_+ = 2$, $m_- = 1$, $m_{-2s} = 0$. 
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\[ Q = 0 \]
\[ P = 0 \]

**FIGURE NO. 2**
\( e^{\pi i/3} \)

\( e^{-\pi i/3} \)

\( Q = 0 \)

\( P = 0 \)
Q = 0
P = 0

FIGURE NO. 4
FIGURE NO. 5

$Q = 0$

$P = 0$

$1.898e^{2\pi i/3}$

$.5262e^{2\pi i/3}$

$e^{\pi i/3}$

$e^{-\pi i/3}$

$-1$
$Q = 0$

$P = -\frac{2\pi}{3}$

$1.595e^{\frac{2\pi}{3}} - 0.394^2$
$2 \pi^3 P = - \frac{2 \pi}{3}$

$Q = 0$

$e^{\pi i/3}$

$e^{-\pi i/3}$

FIGURE NO. 7
\[ Q = 0 \]
\[ P = \frac{2\pi}{3} \]
\[ Q = 0 \]
\[ P = \frac{2\pi}{3} \]