Phase space interference and the WKB approximation for squeezed number states

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Abstract

Squeezed number states for a single mode Hamiltonian are investigated from two complementary points of view. Firstly the more relevant features of their photon distribution are discussed using the WKB wave functions. In particular the oscillations of the distribution and the parity behavior are derived and compared with the exact results. The accuracy is verified and it is shown that for high photon number it fails to reproduce the true distribution. This is contrasted with the fact that for moderate squeezing the WKB approximation gives the analytical justification to the interpretation of the oscillations as the result of the interference of areas with definite phases in phase space. It is shown with a computation at high squeezing using a modified prescription for the phase space representation which is based on Wigner-Cohen distributions that the failure of the WKB approximation does not invalidate the area overlap picture.
1 Introduction

The investigation in phase space of the properties of quantum systems contributes with a remarkable picture of those systems in which the states are described by areas in such a way that the emergence of diverse quantum phenomena \[1, 2, 3\] appears as the result of the interference of the portions of these areas which overlap. The area which represent each state may be fixed from considerations in the context of the old Bohr-Sommerfeld approach \[1, 2\], with the aid of phase-space distributions \[3\] or using the WKB approximate wave functions \[4\]. This last approach has the advantage of being more systematic but is constrained by the limitations of the approximation. In this paper we illustrate both terms in this sentence discussing the photon distribution of squeezed number states\[5\]. The photon distribution of squeezed number states is parity sensible and present characteristic oscillations with increasing photon number until it decay and vanish for sufficiently large photon number. These oscillations which may be computed exactly \[3, 6, 7\] are similar to the ones which appear for squeezed states which been discussed thoroughly using the phase space description \[2\]. It is then interesting to investigate also how this approach may also be applied to the case of the squeezed number states. A first step in this direction was taken in Ref. \[6\] where it was shown that the main characteristics of the distribution are consistent with interference in phase space picture. Here we develop with detail the WKB approach of the phase space picture to investigate how the parity restrictions may be enforced and under which conditions the approximation breaks down. The outcome is that the WKB approximation is not able to represent the last maximum of the distribution in the high photon number region. This failure may be traced to to the limitations of the WKB approximation and not to the phase space picture which as we show below is still realized (at least for high squeezing) using the more direct scheme to assign the areas to the states which relies on the Wigner-Cohen distributions.

2 The WKB approximation

Let us begin discussing the WKB approximation for squeezed number states and its relation with the phase space picture. We work with the normalized one mode Hamiltonian

\[
H = \frac{1}{2} p^2 + \frac{1}{2} x^2 = a^\dagger a + \frac{1}{2} \tag{1}
\]
and denote by $|n>\rangle$ the eigenstates of $H$ and of the number operator $N = a^\dagger a$ with eigenvalues $n + 1/2$ and $n$.

The squeezed number states \[5\] are defined by

$$|n, r>\rangle = \mathcal{S}(r) |n>\rangle$$

(2)

with the squeeze operator $S(r)$ given by

$$S(r) = \exp \left( \frac{1}{2} r (a^\dagger)^2 - \frac{1}{2} r a^2 \right).$$

(3)

(We are taking the phase parameter of the squeezing equal to zero since the photon distribution does not depend on it). These states are eigenstates with eigenvalues $(n+1/2)$ of the transformed Hamiltonian $H' = S(r)HS(r)^{-1}$ which is shown to be equal to \[5\],

$$H' = \frac{1}{2} p^2 e^{-2r} + \frac{1}{2} x^2 e^{2r}.$$

(4)

Alternatively the squeezed number states may be interpreted as the Fock states of a related system with mass $e^{2r}$ and the same frequency. For a more detailed description of these states see \[5\] \[6\] \[7\]. The WKB wave functions for these states are given by \[8\] \[4\],

$$\Phi_n^{(r)}(x) = \frac{2}{C_n} \left( T_n^{(r)} p_n^{(r)}(x) \right)^{-1/2} \cos \left( S_n^{(r)}(x) - \pi/4 \right),$$

(5)

where $C_n$ is a normalization constant. The value of $p_n^{(r)}$ is obtained from the equation

$$n + 1/2 = \frac{1}{2} (p_n^{(r)}(x))^2 e^{-2r} + \frac{1}{2} x^2 e^{2r}$$

(6)

and is given in terms of the classical turning point value $\varepsilon_n^{(r)} = e^{-r} \sqrt{2m + 1}$ by

$$p_n^{(r)}(x) = e^{2r}(\varepsilon_n^{(r)})^2 - x^2)^{1/2}$$

$$= e^r p_n^{(0)}(xe^{-r})$$

(7)

The phase function is given by,

$$S_n^{(r)}(x) = \int_x^{\varepsilon_n^{(r)}} p_n^{(r)}(x')dx'$$

$$= S_n^{(0)}(xe^{-r}),$$

(8)
and the normalization constant has been written in terms of the period of the related classical motion,

\[ T_n^{(r)} = \int_{\varepsilon_n^{(r)}}^{\varepsilon_n^{(r)}} \frac{dx'}{p_n^{(r)}(x')} = 2\pi e^{-2r} \quad (9) \]

For \( r = 0 \) we recover the description of the number states. The WKB wave functions are compressed by the squeezing in the same form as the exact states,

\[ \Phi_n^{(r)}(x) = e^{r/2}\Phi_n^{(0)}(e^r x) \quad (10) \]

We note that \( p_n^{(r)} \) is a symmetric function and that

\[ \int_{\varepsilon_n^{(r)}}^{\varepsilon_n^{(r)}} p_n^{(r)}(x')dx' = \pi/2 + n\pi \quad (11) \]

Hence the phase function \( S_n(x) \) satisfies the relation

\[ S_n^{(r)}(-x) - \pi/4 = -(S_n(x) - \pi/4) + n\pi \quad (12) \]

and one can show that the WKB wave functions satisfy the parity conditions of the exact states.

\[ \Phi_n^{(r)}(-x) = (-1)^n\Phi_n^{(r)}(x) \quad (13) \]

Again for \( r = 0 \) we are dealing with the number states.

The WKB wave functions are meaningless near the classical return points where they diverge. Also we note that the approximation holds better for higher values of \( n \). The normalization of the wave functions is such that

\[ |C_n|^2 = 1 + \frac{1}{\pi} \int_{\varepsilon_n^{(r)}}^{\varepsilon_n^{(r)}} \frac{\sin(2S_m^{(r)}(x))}{p_m^{(r)}(x)} \, dx \quad (14) \]

with the integral in the right term vanishing for large \( n \).

### 3 Photon statistics in the WKB approximation

Now we turn to the computation of the photon distribution of the squeezed number states within this approximation. It is given by

\[ P_{mn} = |W_{mn}|^2 \quad (15) \]
Figure 1: The four points in phase space which contribute to the photon distribution amplitude

with

\[ W_{mn} = \langle m|n, r \rangle = \int_{-\infty}^{\infty} \Phi_m^{(0)}(x) \Phi_n^{(r)}(x) dx. \]  

(16)

Using the WKB wave functions is shown that, as was suggested in References [4, 6] the probability amplitude \( W_{mn} \) may be expressed in terms of a phase \( \varphi_{mn} \) and an overlapping area \( A_{mn} \) in the form,

\[
W_{mn} = (A_{mn})^{1/2} e^{i\varphi_{mn}} + (A_{mn})^{1/2} e^{-i\varphi_{mn}} \\
+ (A_{mn})^{1/2} e^{i\varphi'_{mn}} + (A_{mn})^{1/2} e^{-i\varphi'_{mn}}
\]

(17)

In this expression \( A_{mn} \) is given by

\[
A_{mn} = 2\pi \left( T_m T_n^{(r)} p_m^2(X_c) \right)^{-1} \left| \frac{d}{dx} p_m(X_c) - \frac{d}{dx} p_n^{(r)}(X_c) \right|^{-1}
\]

(18)

and is shown to be the overlapping area between a circular ring of interior radius \( \sqrt{2m} \) and exterior radius \( \sqrt{2m + 1} \) of total area \( 2\pi \) representing the \( m \) Fock state in phase space and a deformed elliptical ring of equal area which corresponds to the \( n \) squeezed number state. The phases are given by

\[
\varphi_{mn} = S_m^{(0)}(X_c) - S_n^{(r)}(X_c) - \pi/4, \\
\varphi'_{mn} = S_m^{(0)}(-X_c) - S_n^{(r)}(-X_c) + \pi/4
\]

(19)

where \( X_c \) is the point where,

\[
p_m^{(0)}(X_c) = p_n^{(r)}(X_c)
\]

(20)
Figure 2: Geometrical interpretation of $S_m(X_c)$

as can be shown in Figure (3). We have then,

$$X_c = \left( \frac{e^{2r}(2n+1) - (2m+1)}{e^{4r} - 1} \right)^{1/2}.$$ \hspace{1cm} (21)

From the properties of $S_m^{(r)}$ (Ec. 12) we obtain:

$$\varphi_{mn}' = -\varphi_{mn} + (m-n)\pi$$ \hspace{1cm} (22)

Using (22) and (17) the probability amplitude is given by

$$W_{mn} = 2(A_{mn})^{1/2} \cos(\varphi_{mn}) (1 + \cos((m-n)\pi)).$$ \hspace{1cm} (23)

This result is in agreement with the form of the amplitude proposed in [6] and in particular (22) gives the correct parity behavior of the distribution.

The connection with the geometrical construction is obtained noting that $S_m^{(0)}(X_c)$ is given by the shadowed area in Fig.(3). Then,

$$S_m^{(0)}(X_c) = \frac{1}{2} \theta(\varepsilon_m^{(0)})^2 - \frac{1}{2} X_c \left( (\varepsilon_m^{(0)})^2 - X_c^2 \right)^{1/2}$$ \hspace{1cm} (24)

where

$$\theta = \arccos \left\{ \frac{X_c}{\varepsilon_m^{(0)}} \right\}$$ \hspace{1cm} (25)

In the same way we have,

$$S_n^{(r)}(X_c) = \frac{1}{2} \theta(\varepsilon_n^{(r)})^2 - \frac{1}{2} X_c \left( (\varepsilon_n^{(r)})^2 - X_c^2 \right)^{1/2}$$ \hspace{1cm} (26)
Figure 3: Photon distribution $P_{mn}$ for a squeezed number state with $n = 5$ and $r = 2$. Left: WKB approximation. Right: Exact computation

where

$$\theta = \arccos \left( \frac{X_c}{\varepsilon_{(r)}} \right)$$

(27)

In Fig (3) we show the WKB approximation for the photon distribution of a squeezed number state with $n = 5$ and $r = 2$ compared with the exact computation. The approximation is faithful for photon numbers below the last maximum.

4 Wigner-Cohen distributions and interference in phase space

The failure of the approximation presented in the last section for large photon numbers is related to the breakdown of the validity of the WKB approximation. Nevertheless as we show below it does not exclude the description of this part of the distribution using the area overlapping in phase space. To see this let us introduce $F_n^{(r)}(x, p)$ to denote any of the phase space Wigner-Cohen distributions [9] associated to the squeezed number states which satisfies,
\[
\int F_n^{(r)}(x, p) dp = |\psi_n^{(r)}(p)|^2 \tag{28}
\]
and
\[
\int F_n^{(r)}(x, p) dx = |\bar{\psi}_n^{(r)}(p)|^2 \tag{29}
\]
where \(\psi_n^{(r)}(x)\) and \(\bar{\psi}_n^{(r)}(p)\) are the wave functions of the squeezed number states in the position and momentum representations respectively.

The photon number distribution may be obtained by integrating \(F_n^{(r)}(x, p)\) over the \(m\)-th ring \(\Omega_m\) associated to the number state \(|m\rangle\) which is the area between to circles of radii \(\sqrt{2m}\) and \(\sqrt{2m+2}\) respectively,

\[
A_{mn} = \frac{1}{2} \int_{\Omega_m} dp dx F_n^{(r)}(x, p) \tag{30}
\]

For a highly squeezed number state the integral in the configuration space variable \(q\) can approximately taken over the whole line

\[
A_{mn} \approx \int_{\sqrt{2m}}^{\sqrt{2m+2}} |\bar{\psi}_n^{(r)}(p)|^2 dp \tag{31}
\]
and by supposing a smooth behavior in the ring we get,

\[
A_{mn} \approx (\sqrt{2m+1})^2 (\sqrt{2m+2} - \sqrt{2m}) \tag{32}
\]
Taking into account the parity of the functions we finally have,

\[
P_{mn} \approx |1 + \cos((m-n)\pi)|^2 A_{mn} \tag{33}
\]
Substituting the values of the waves functions we obtain,

\[
P_{mn} \approx (1 + \cos((m-n)\pi))^2 \left( \sqrt{2m+2} - \sqrt{2m} \right) e^{-r} e^{-(2m+1)e^{-2r}} (H_n(\sqrt{2m+1}e^{-r}))^2 \tag{34}
\]
which is in an excellent agreement with the exact value as shown in Fig. [4] in the particular case of \(n = 5\) and \(r = 2\). The agreement stills holds for photon numbers beyond the last oscillation where the WKB approximation breaks down. This expression also shows that for large squeezing the photon number oscillation are driven by the oscillations of the wave function in momentum space.
Figure 4: Photon distribution $P_{mn}$ for a squeezed number state $n = 5$ and $r = 2$. Left: Phase space computation. Right: Exact result

5 Conclusion

We have computed the oscillations photon number distribution of squeezed number states within the WKB approximation making connection with the phase space description given in Ref. [4]. In this case there are four sectors where the areas overlap and give a contribution to the probability amplitude [6]. We show that for high photon number the WKB approximation fails to describe the last oscillation. Finally we note that a description in terms of interference of the overlapping areas in phase space may be recovered, but not based in the WKB approximation but in the Wigner-Cohen distributions. For large squeezing this prescription reproduces the photon number distribution for the whole range.

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