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THE BRAUER GROUP OF A LOCALLY COMPACT GROUPOID

By Alexander Kumjian, Paul S. Muhly, Jean N. Renault, and Dana P. Williams

Abstract. We define the Brauer group Br(G) of a locally compact groupoid G to be the set of Morita equivalence classes of pairs (A, α) consisting of an elementary C*-bundle A over G(0) satisfying Fell's condition and an action α of G on A by *-isomorphisms. When G is the transformation groupoid X × H, then Br(G) is the equivariant Brauer group Br_H(X). In addition to proving that Br(G) is a group, we prove three isomorphism results. First we show that if G and H are equivalent groupoids, then Br(G) and Br(H) are isomorphic. This generalizes the result that if G and H are groups acting freely and properly on a space X, say G on the left and H on the right, then Br_G(X/H) and Br_H(G\X) are isomorphic. Secondly we show that the subgroup Br_1(G) of Br(G) consisting of classes [A, α] with A having trivial Dixmier-Douady invariant is isomorphic to a quotient E(G) of the collection Tw(G) of twists over G. Finally we prove that Br(G) is isomorphic to the inductive limit Ext(G, T) of the groups E(GX) where X varies over all principal G spaces X and GX is the imprimitivity groupoid associated to X.

1. Introduction. This paper is about two groups that are naturally associated to a locally compact groupoid (with Haar system) and an isomorphism between them. The first group, denoted Br(G) and called the Brauer group of G, where G is the groupoid in question, is a collection of equivalence classes of actions of G on certain bundles of C*-algebras. The second group, Ext(G, T), is a group that organizes all the extensions by T of all the groupoids that are equivalent to G in the sense of [35]. The isomorphism between these two groups or, more accurately, our analysis of it, may be viewed as a simultaneous generalization of Mackey’s analysis of projective representations of locally compact groups [30], on the one hand, and the Dixmier-Douady analysis of continuous trace C*-algebras ([10] and [8]), on the other. Moreover, our work provides a global perspective from which to view a number of recent studies in the theory of C*-dynamical systems. See, in particular, [5], [27], [32], [39], [43], [44], [45], and [46]. Further, it suggests a way to generalize to groupoids Moore’s Borel cohomology theory for locally compact groups without becoming entangled in measure-theoretic difficulties.

To expand upon these remarks, and to help motivate further the need for our theory, we shall begin with Mackey’s analysis [30], but first, we wish to emphasize the usual separability assumptions that will be made throughout this paper. All groups, groupoids, and spaces will be locally compact, Hausdorff, and
second countable. Thus, in particular, they are paracompact. All Hilbert spaces will be separable. They may be finite dimensional as well as infinite dimensional. Likewise, all $C^*$-algebras under discussion will be assumed to be separable. Since we are interested mainly in groupoids with Haar systems, we always assume our groupoids have open range and source maps.

Mackey was inspired, in part, by earlier investigations of Wigner [57] and Bargmann [2] concerned with the mathematical foundations of quantum mechanics and, in particular, with the problem of classifying the so-called automorphic representations of locally compact groups on (infinite dimensional) Hilbert space. The problem (formulated a bit differently than in [57], [2], and [30]) is to classify the continuous representations $\alpha$ of a locally compact group $G$ as automorphisms of the $C^*$-algebra $K = K(H)$ of all compact operators on a separable infinite dimensional Hilbert space $H$. The meaning of “continuity” is that for each compact operator $k \in K$, the function $g \mapsto \alpha_g(k)$ is continuous from $G$ to $K$ with the norm topology. Two such representations $\alpha_1$ and $\alpha_2$ are considered equivalent if there is a $*$-automorphism $\varphi : K \to K$ such that $\varphi \circ \alpha_1 = \alpha_2 \circ \varphi$. The collection of equivalence classes of such representations will be denoted $\mathcal{R}(G)$. With respect to the process of forming tensor products, it is not difficult to see that $\mathcal{R}(G)$ becomes a semigroup with identity in a natural way. The problem is to describe this semigroup.

Owing to Wigner’s theorem [57] that all $*$-automorphisms of $K(H)$ are implemented by unitary operators on the underlying Hilbert space $H$, and the fact that two unitaries on $H$ implement the same automorphism if and only if one is a unimodular scalar multiple of the other, one may summarize part of the analysis as follows: Every $\alpha$ is determined by a unimodular Borel function $\sigma$ on $G \times G$ satisfying identities making it a 2-cocycle in a certain cohomology theory and a unitary $\sigma$-representation $U$ of $G$ with values in the unitary group of $H$. (These are also called projective representations of $G$.) The relation between $U$ and $\sigma$ is found in the equation $U_{gh} = \sigma(g, h)U_gU_h$, for $g, h \in G$. The formula for $\alpha$, then, is $\alpha_g(k) = U_gkU_g^{-1}$, $k \in K$, for $g \in G$. Two $\alpha$’s are equivalent if and only if they are spatially equivalent, and in terms of the $U$’s and $\sigma$’s, this happens if and only if $(U_1, \sigma_1)$ is unitarily equivalent (in the obvious sense) to $(U_2, \sigma_2)$ where $\sigma_2$ is cohomologous to $\sigma_2$, meaning that there is a unimodular Borel function $b$ on $G$ with the property that $\sigma'_2(g, h) = b(gh)b(g)b(h) \cdot \sigma_2(g, h)$. The upshot is that to understand $\mathcal{R}(G)$, one must identify $H^2(G, \mathbb{T})$, and for each $[\sigma] \in H^2(G, \mathbb{T})$, one must identify the unitary equivalence classes of $\sigma$-unitary representations of $G$.

As we are emphasizing, the group $H^2(G, \mathbb{T})$ is a measure-theoretic object that cannot be avoided, if one wants to think in terms of projective representations of $G$. As Mackey noted in [30], however, one may avoid measure theoretic difficulties (or at least push them into the background) if one thinks in terms of extensions and an Ext-group, instead of $H^2(G, \mathbb{T})$. The point is, thanks to Mackey [29, Theorem 7.1] and Weil [56], the correspondence between extensions and 2-cohomology that one learns about in finite group theory extends to the
theory of locally compact groups as follows. With each two-cocycle $\sigma$ one can form a certain \textit{locally compact topological} group $G^\sigma$. Set-theoretically $G^\sigma$ is the product of the circle $\mathbb{T}$ with $G$, $\mathbb{T} \times G$, but with the multiplication $(t_1, g_1)(t_2, g_2) = (t_1 t_2 \sigma(g_1, g_2), g_1 g_2)$. In general, the topology on $G^\sigma$ is \textit{not} the product topology. One focuses on the Borel structure of $G^\sigma$, i.e., the product Borel structure, and notes that Haar measure on $\mathbb{T}$ times Haar measure on $G$ endows $G^\sigma$ with a left invariant Borel measure. The theorems of Mackey and Weil, then, force the existence of the topology. One can prove without difficulty that two cocycles give rise to topologically isomorphic extensions if and only if they are cohomologous. Thus, one obtains, as anticipated, a natural isomorphism between $\text{Ext}(G, \mathbb{T})$ and $H^2(G, \mathbb{T})$.

The measure theoretic technology is pushed further into the background when one recognizes how to realize directly an extension, given an automorphic representation $\alpha$ of $G$. Indeed, if $H$ is the underlying Hilbert space and if $\mathcal{U}(H)$ denotes the set of unitary operators on $H$ endowed with the $\ast$-strong operator topology, simply form $E(\alpha) = \{(g, U) \in G \times \mathcal{U}(H) \mid \text{Ad}(U) = \alpha_g\}$. It is not difficult to see that with the relative topology, $E(\alpha)$ is a locally compact group and an extension of $G$ by $\mathbb{T}$. Further, the relation between $E(\alpha)$ and $G^\sigma$ is given, via the $\sigma$-unitary representation, $U^\sigma$, of $G$ determined by $\alpha$, by the formula

$$(t, g) \rightarrow (g, tU^\sigma_g), \quad (t, g) \in G^\sigma.$$ 

It is not difficult to see that two elements $\alpha, \beta \in \mathcal{R}(G)$ give isomorphic extensions $E(\alpha)$ and $E(\beta)$ if and only if $\alpha$ and $\beta$ are \textit{exterior equivalent} in the sense that there is a unitary valued function $u$ on $G$ such that $\beta_g = \text{Ad}(u_g) \circ \alpha_g$ for all $g \in G$ and such that $u_{gh} = u_g \alpha_g(u_h)$ for all $g, h \in G$. Further, it is not difficult to see that $\alpha$ is exterior equivalent to the identity (i.e., the trivial action) if and only if $\alpha$ is implemented by a \textit{unitary representation} of $G$. Thus, the map $\alpha \rightarrow E(\alpha)$ sets up an isomorphism between $\mathcal{R}(G)/\mathcal{S}$, where $\mathcal{S}$ is the subsemigroup of spatially implemented automorphic representations, and $\text{Ext}(G, \mathbb{T})$. This shows, in particular, that $\mathcal{R}(G)/\mathcal{S}$ has the structure of a group. Further, the class of $E(\alpha)$ in $\text{Ext}(G, \mathbb{T})$ (and also the class of the cocycle $\sigma$ corresponding to $\alpha$) has come to be known as the \textit{Mackey obstruction} to implementing $\alpha$ by a unitary representation of $G$. For reasons that will be further clarified in a moment, we shall write $\text{Br}(G)$ for $\mathcal{R}(G)/\mathcal{S}$ and call this group the \textit{Brauer group} of $G$. We thus have the isomorphisms

$$\text{Br}(G) \cong H^2(G, \mathbb{T}) \cong \text{Ext}(G, \mathbb{T}).$$

We turn now to the second source of motivation for this paper, the Dixmier-Douady theory of continuous trace $C^*$-algebras ([10] and [8]; these are summarized in Chapter 10 of [9]). However, we shall adopt the somewhat more contemporary perspective first articulated by Green in [16] and by Taylor [55].
(What follows is amplified in [5], which is based in turn on [46].) The quickest way to say that a $C^*$-algebra $A$, say, is continuous trace is to say that it has Hausdorff spectrum $T$ and that there is a bundle $A$ of $C^*$-algebras over $T$, each fibre $A(t)$ of which is isomorphic to the algebra of compact operators on some Hilbert space (such bundles are called elementary $C^*$-bundles), and satisfying Fell’s condition (see Definition 2.11, below), such that $A = C_0(T; A)$, the $C^*$-algebra of continuous sections of $A$ that vanish at infinity on $T$. If $A$ is unital, then $T$ is compact and all the fibres of $A$ are unital (and therefore finite dimensional). Evidently, in this case, the center of $A$ is $C(T)$. Further in this case, as Grothendieck observed [18], $A$ may be viewed as a separable algebra over $C(T)$ in the sense of pure algebra. One is then led to think about classifying algebras like $A$ up to Morita equivalence over $C(T)$. When one does this, the result has a group structure, denoted $\text{Br} (T)$, that generalizes the Brauer group of central simple algebras over a field. Green recognized in [15] that this idea works in the nonunital setting as well, provided one uses Rieffel’s notion of strong Morita equivalence [52].

In a bit more detail, let us consider continuous trace $C^*$-algebras $A$ with a prescribed spectrum $T$. Then we shall say that two such algebras $A_1$ and $A_2$ are strongly Morita equivalent (in the sense of Rieffel [52]) via an equivalence $\mathcal{X}$ that respects $T$, or more simply that $\mathcal{X}$ is a strong Morita equivalence over $T$, if $\mathcal{X}$ implements a strong Morita equivalence between $A_1$ and $A_2$ in such a way that the actions of $C_0(T)$ on $\mathcal{X}$ are the same, where $C_0(T)$ is viewed as the center of each of $A_1$ and $A_2$. If $A_i$ is realized as $C_0(T; A_i)$, $i = 1, 2$, then such an $\mathcal{X}$ exists if and only if there is a bundle $\mathcal{X}$ over $T$ such that for each $t \in T$, $\mathcal{X}(t)$ implements a strong Morita equivalence between $A_1(t)$ and $A_2(t)$ and such that $\mathcal{X} = C_0(T; \mathcal{X})$. Such a bundle always exists locally in the sense that each point $t \in T$ has a neighborhood $U$ such that the ideal $C_0(U; A_1|_U)$ in $A_1$ is strongly Morita equivalent to the ideal $C_0(U; A_2|_U)$ in $A_2$ via an equivalence that respects $U$; i.e., there is a bundle $\mathcal{X}_U$ over $U$ so that $\mathcal{X}(t)$ implements a strong Morita equivalence between $A_1(t)$ and $A_2(t)$ for all $t \in U$. The problem is to glue the $\mathcal{X}_U$ together. The obstruction to doing this is an element in the sheaf cohomology group $H^2(T, \mathcal{S})$, where $\mathcal{S}$ is the sheaf of all continuous $\mathbb{T}$-valued functions on $T$. Indeed, given the continuous trace $C^*$-algebra $A = C_0(T; A)$ one can find an open cover $U = \{U_i\}$ of $T$ such that each ideal $C_0(U_i; A|_{U_i})$ is strongly Morita equivalent to $C_0(U_i)$ over $U_i$. The bundle $\mathcal{X}_{U_i}$ associated with this equivalence is, in fact, a Hilbert bundle, i.e., the fibres $\mathcal{X}(t)$ are Hilbert spaces, and one has that $A|_{U_i}$ is isomorphic to $\mathcal{K}(\mathcal{X}_{U_i})$. On overlaps, $U_i \cap U_j$, the bundles $\mathcal{K}(\mathcal{X}_{U_i}|_{U_i \cap U_j})$ and $\mathcal{K}(\mathcal{X}_{U_j}|_{U_i \cap U_j})$, being strongly Morita equivalent (and separable), are stably isomorphic. Assuming, without loss of generality, that the bundles are stable, we may take the fibres $\mathcal{X}(t)$ to be infinite dimensional. We may then infer that there are unitary valued functions $u_{ij}$ on $U_i \cap U_j$ such that $u_{ij}(t): \mathcal{X}(t) \to \mathcal{X}(t)$ and such that $\text{Ad} (u_{ij})$ implements a bundle isomorphism from $\mathcal{K}(\mathcal{X}_{U_i}|_{U_i \cap U_j})$ to $\mathcal{K}(\mathcal{X}_{U_j}|_{U_i \cap U_j})$. If one compares $u_{ik}$ with $u_{ij}u_{jk}$ on a triple overlap, $U_i \cap U_j \cap U_k$, one sees that there is a unimodular function
Given continuous trace $C^*$-algebras $A_1$ and $A_2$ with spectra $T$, one can form $A_1 \otimes_{C_0(T)} A_2$. In terms of bundles, the bundle for this tensor product is a pointwise tensor product of the bundles $A_1$ and $A_2$ representing $A_1$ and $A_2$, respectively. Notice that if, for $i = 1, 2$, $A'_i$ is strongly Morita equivalent to $A_i$ over $T$, then $A_1 \otimes_{C_0(T)} A_2$ is also strongly Morita equivalent to $A'_1 \otimes_{C_0(T)} A'_2$ over $T$. Thus if, as above, we let $\text{Br}(T)$ be the collection of equivalence classes of continuous trace $C^*$-algebras with spectrum $T$, where the equivalence relation is “strong Morita equivalence over $T$,” then $\text{Br}(T)$ becomes a semigroup under the operation of tensoring over $C_0(T)$. In fact, $\text{Br}(T)$ is a group, the Brauer group of $T$: the identity is represented by $C_0(T)$ and the inverse of $[A]$ is represented by the conjugate of $A$, $\overline{A}$. That is, $\overline{A}$ is the same set as $A$, with the same product and addition, but if $b: A \to \overline{A}$ is the identity map, then $\lambda \cdot b(a) = b(\overline{\lambda} \cdot a)$, $\lambda \in \mathbb{C}$, $a \in A$. Alternatively, one may think of $\overline{A}$ as the opposite algebra of $A$. The main result of [8], then, is that the map $[A] \to \delta(A)$ is an group isomorphism of $\text{Br}(T)$ with $H^2(T, \mathcal{S})$.

Now the surjectivity of $\delta$ was proved in [10] using the fact that the unitary group of an infinite dimensional Hilbert space is contractible. Subsequently, in independent investigations, the third author [49] and Raeburn and Taylor [44] showed that $\delta$ is onto using groupoids. The idea behind both approaches, and one that is important for this paper, may be traced back to Mackey’s pioneering article [31, p. 1190]. In this paper, Mackey showed how to describe the topological cohomology of a space in terms of the groupoid cohomology of certain groupoids constructed from covers of the space. It is important for our purposes to have some detail at our disposal about how this is done.

So fix the locally compact Hausdorff space $T$ and a locally finite cover $U = \{U_i\}_{i \in I}$ of $T$. Let $X = \bigsqcup_{i \in I} U_i = \{(i, t): t \in U_i\}$ and define $\psi: X \to T$ by $\psi(i, t) = t$. Then, of course, $\psi$ is a local homeomorphism and the set $G^\psi := \bigsqcup_{i, j} U_i \cap U_j = \{(i, t, j): t \in U_i \cap U_j\} = X *^\psi X$ has a naturally defined structure of a locally compact groupoid that is $r$-discrete in the sense of [48]. (Since we always assume the the range map is open, a groupoid is $r$-discrete here exactly when the range map is a local homeomorphism [48, Proposition I.2.8].) The unit space of $G^\psi$ is $X$ and two triples $(i_1, t_1, j_1)$ and $(i_2, t_2, j_2)$ are composable (or multipliable) if and only if $j_1 = i_2$ and $t_1 = t_2$, in which case $(i_1, t_1, j_1) \cdot (j_1, t_1, j_2) = (i_1, t_1, j_2)$, and $(i_1, t_1, j_1)^{-1} = (j_1, t_1, i_1)$. If $\nu = \{\nu_{ijk}\}$ is a continuous 2-cocycle for $U$ with values in $\mathbb{T}$, then $\nu$ determines a continuous groupoid 2-cocycle $\hat{\nu}$ on $G^\psi$ with values in $\mathbb{T}$ by the formula

$$\hat{\nu}((i, t, j), (j, t, k)) = \nu_{ijk}(t).$$
And conversely, a continuous $\mathbb{T}$-valued, groupoid 2-cocycle on $G^\psi$ determines a 2-cocycle on $U$. Moreover, two cocycles, $\nu^1$ and $\nu^2$, are topologically cohomologous by a coboundary over $U$ if and only if the associated groupoid cocycles, $\tilde{\nu}^1$ and $\tilde{\nu}^2$, are cohomologous in the groupoid sense. Thus we find that the groupoid cohomology group $H^2(G, \mathbb{T})$ is isomorphic to the sheaf cohomology group $H^2(U, \mathcal{S})$ of the cover $U$.

Just as with groups, we prefer to think of extensions instead of groupoid cocycles. The groupoid 2-cocycle $\tilde{\nu}$ gives an extension of $G^\psi$ by the $(G^\psi)^{(0)} \times \mathbb{T}$, denoted $E_\nu$. Set-theoretically, $E_\nu = G^\psi \times \mathbb{T}$, and the product is given by the formula

$$( (i, t, j), z) \cdot ((j, t, k), w) = ((i, t, k), \nu_{ijk}(t)zw).$$

The topology on $E_\nu$ is the product topology because $\nu$ is continuous. The extension $E_\nu$ is called a twist over $G^\psi$ in [23]. In general, a twist $E$ over a locally compact groupoid $G$ is simply an extension $E$ of $G$ by the groupoid $G^{(0)} \times \mathbb{T}$. Such an $E$ has the natural structure of a principal $\mathbb{T}$-bundle over $G$ and two twists $E_1$ and $E_2$ are called equivalent if there is a groupoid isomorphism $\phi$: $E_1 \rightarrow E_2$ that is also a bundle map. With respect to the operation of forming fibred products, or Baer sums, the collection of twists over a locally compact groupoid $G$ modulo twist equivalence is a group, denoted Tw$(G)$. In the case that the groupoid is $G^\psi$, an equivalence $\phi$: $E_\nu \rightarrow E_\mu$ between two twists gives rise to continuous functions $\alpha_{ij}$: $U_i \cap U_j \rightarrow \mathbb{T}$ such that

$$\phi((i, t, j), z) = ((i, t, j), \alpha_{ij}(t)z),$$

because $\phi$ is a bundle map, and the $\alpha_{ij}$’s further satisfy the equation

$$\mu_{ijk}(t)\alpha_{ij}(t)\alpha_{jk}(t)\overline{\alpha_{ik}(t)} = \nu_{ijk}(t)$$

because $\phi$ is a homomorphism. Thus the cocycles $\nu$ and $\mu$ are cohomologous over $U$. Hence, we conclude that there are natural isomorphisms between the three groups $H^2(U, \mathcal{S})$, $H^2(G^\psi, \mathbb{T})$, and Tw$(G^\psi)$.

Returning to the groupoid approach to showing that the Dixmier-Douady map $\delta$ is surjective, discovered by the third author [49] and Raeburn and Taylor [44], let $[\nu]$ be a given element of $H^2(T, \mathcal{S})$ and choose a cover $U$ on which $[\nu]$ is represented by $\nu$. If $E_\nu$ is the associated twist, then the restricted groupoid $C^*$-algebra $C^*(G^\psi; E_\nu)$ is continuous trace with spectrum $T$ and $\delta(C^*(G^\psi; E_\nu)) = [\nu]$.

The relation between topological cohomology and groupoid cohomology is further strengthened through the notion of equivalence of groupoids [35] and leads to our definition of Ext$(G, \mathbb{T})$ for an arbitrary locally compact groupoid $G$. Indeed, if the space $T$ is viewed as a groupoid, the so-called cotrivial groupoid with unit space $T$, then $T$ is equivalent, in the sense of [35] to $G^\psi$. Moreover, if $V = \{V_j\}_{j \in J}$ is another cover of $T$, if $Y := \bigsqcup_{j \in J} V_j$, with local homeomorphism
\( \varphi: Y \to T \), given by \( \varphi(j,t) = t \), and if \( G^\varphi = Y \ast \varphi \) \( Y \) is the associated groupoid, then \( G^\psi \) and \( G^\varphi \) are equivalent. In fact, \( X \ast Y = \{(x,y) \in X \times Y: \psi(x) = \varphi(y)\} \) is a \( G^\psi, G^\varphi \)-equivalence in the sense of [35]. Notice, however, that \( X \ast Y \) comes from the cover \( U \cap V = \{U_i \cap V_j\}_{(i,j) \in I \times J} \) that refines \( U \) and \( V \). Furthermore, if \( G^\psi, \varphi = (X \ast Y) \ast (\psi, \varphi) (X \ast Y) \) is the associated groupoid, then \( G^\psi, \varphi \) is equivalent to both of \( G^\psi \) and \( G^\varphi \), and each of these groupoids is a homomorphic image of \( G^\psi, \varphi \). In this way, we find that the notion of “refinement” for covers is captured in the notion of ‘equivalence’ for groupoids. The collection of groupoids equivalent to \( T \) becomes directed through the process of taking homomorphisms. We thus arrive at the isomorphism

\[
H^2(T, S) \cong \lim_{\to} H^2(U, S) \cong \lim_{\to} \text{Tw} \left( G^\psi \right)
\]

that relates \( H^2(T, S) \) with groupoid cohomology and suggests how to proceed in general (see Sections 7 and 9).

For a fixed locally compact groupoid \( G \) with Haar system, we shall consider the collection \( \mathcal{P}(G) \) consisting of right principal \( G \) spaces (with equivariant \( s \)-systems in the sense of [50]). For each \( X \in \mathcal{P}(G) \), we let \( G^X \) denote the imprimitivity groupoid associated with \( X \). This is a canonical groupoid with Haar system equivalent to \( G \) (see [35]). We form the twist group \( \text{Tw}(G^X) \) defined above. Owing to the fact that \( X \) can carry nontrivial first cohomology, we actually take a quotient of \( \text{Tw}(G^X) \) reflecting this, getting a group that we call \( E(G^X) \). As we shall see, \( \mathcal{P}(G) \) forms a directed system and the family \( \{E(G^X)\}_{X \in \mathcal{P}(G)} \) is inductive. We define, then, \( \text{Ext}(G, \mathbb{T}) \) as the limit

\[
\lim_{\to} E(G^X).
\]

This is our way to generalize 2-cohomology of a space to groupoids. It looks like one could develop a whole cohomology theory along these lines, but at the moment, the technical difficulties seem formidable. In any case, this \( \text{Ext}(G, \mathbb{T}) \) suffices for our purposes here.

As we indicated above, \( \text{Br}(T) \) may be viewed in terms of strong Morita equivalence classes of elementary \( C^* \)-bundles over \( T \), each satisfying Fell’s condition. To generalize this notion to groupoids and to return to the Bargmann-Mackey-Wigner analysis, we proceed as follows. Fix a locally compact groupoid \( G \) with Haar system. We consider pairs \((A, \alpha)\) consisting of elementary \( C^* \)-bundles \( A \) over \( G^{(0)} \) satisfying Fell’s condition and continuous actions \( \alpha \) of \( G \) on \( A \). Of course, if \( G \) is a group, the bundles reduce to a single elementary \( C^* \)-algebra, i.e., the algebra of compact operators on a Hilbert space, and we are looking at its automorphic representations. We shall say that two pairs \((A, \alpha)\) and \((B, \beta)\) are \textit{Morita equivalent} if there is an \( A - B \)-imprimitivity bimodule bundle \( X \) over \( G^{(0)} \) that admits a continuous action \( V \) of \( G \) by isometric \( \mathbb{C} \)-linear isomorphisms such
that $V$ implements the actions $\alpha$ and $\beta$ in a natural way (see Definition 3.1). In the case when $G$ is a group and the bundles reduce to $K$, this notion reduces to \textit{exterior equivalence} described above. (See Section 8, below.) The Morita equivalence class of $(A, \alpha)$ will be denoted $[A, \alpha]$ and the collection of all such equivalence classes will be denoted by $\text{Br}(G)$. Under the operation of tensoring, as we shall see, $\text{Br}(G)$ becomes a group. The identity element is represented by the class of $(G(0) \times \mathbb{C}, \tau)$, where $\tau$ is equipped with the action coming from $G$, and the inverse of $[A, \alpha]$ is represented by $[\overline{A}, \overline{\alpha}]$, where $\overline{\alpha}$ is the bundle of conjugate or opposite $C^*$-algebras of $A$, and $\overline{\alpha}$ is $\alpha$. The details of the proof of this theorem, Theorem 3.7, take some time to develop (Sections 2 and 3), but the proof is conceptually straightforward.

It follows the lines of Theorem 3.6 in [5].

It should be emphasized that our $\text{Br}(G)$ coincides with $\text{Br}(T)$ when $G$ reduces to a space $T$ and that when $G$ is the transformation group groupoid, $X \times H$, determined by a locally compact group $H$ acting on locally compact space $X$, then our $\text{Br}(G)$ coincides with the \textit{equivariant Brauer group} $\text{Br}_H(X)$ analyzed in [5]. Of course, in the very special case when $X$ reduces to a point, so that $G$ may be viewed as the group $H$, we find that $\text{Br}(G) = \text{Br}_H(\{pt\}) \cong \mathcal{R}(G)/S$ described above.

Our ultimate goal is to prove Theorem 10.1, which asserts that $\text{Br}(G) \cong \text{Ext}(G, \mathbb{T})$. As we have said, this result contains, simultaneously, the isomorphism of $\text{Br}(G)$ ( = $\mathcal{R}(G)/S$) with the Borel cohomology group $H^2(G, \mathbb{T})$, when $G$ is a locally compact group, and the isomorphism $\text{Br}(T) \cong H^2(T, S)$ when $T$ is a space.

When the groupoid $G$ is the transformation group groupoid, $X \times H$, Crocker and Raeburn, together with the first and fourth authors, described $\text{Br}(G)$ ( = $\text{Br}_H(X)$) in terms of certain Moore cohomology groups ([33] and [34]) arising from various actions of $H$ on the objects that are involved in the analysis of $C^*$-dynamical systems coming from $H$ and continuous trace $C^*$-algebras with spectrum $X$ [5]. We shall not enter into the details here, except to say that the Moore cohomology groups are measure theoretic objects, like Mackey’s $H^2(G, \mathbb{T})$. Our approach gives a purely \textit{topological} description $\text{Br}(G)$. We concentrate on extensions, as emphasized above, and not on actions of $H$ on Polish groups, as in [5]. However, it remains to be seen what the precise relation is between our description of $\text{Br}(G)$ and that in [5]. In particular, it would be interesting to understand the “filtration” for $\text{Br}(G)$, discovered there, in terms of groupoid theoretic constructs. Also, as noted above, our analysis suggests that a cohomology theory based on $\mathcal{P}(G)$ might exist that generalizes the measure theoretic cohomology of Moore.

The first author [24] adapted the equivariant sheaf cohomology theory of Grothendieck [17] to cover an $\nu$-discrete groupoid $G$ acting on sheaves defined on the unit space $G(0)$, and he showed that $\text{Br}(G)$ is isomorphic to an equivariant sheaf cohomology group $H^2(G, S)$. Here $S$ is the sheaf of germs of continuous $\mathbb{T}$-valued functions on the unit space, $G(0)$, of $G$, with $G$ acting on $S$ in a natural
way. The equivariant sheaf cohomology was invented in part to obtain a deeper understanding of the group $\text{Tw}(G)$, which is an extension group. Our work gives a meaning to $H^1(G, S)$ and $H^2(G, S)$, at least, when $G$ is not $r$-discrete, but at the present time, we are not able to go beyond this. We believe that rationalizing and unifying the various cohomology theories that we have been discussing is one of the most important challenges for the theory in the future.

The paper is organized as follows. In Section 2, we present preliminaries on Banach bundles and related material. We define and discuss Morita equivalence of $C^*$-bundles, imprimitivity bimodule bundles, and actions of groupoids on bundles. We analyze in detail $\mathcal{B}r(G)$, the collection of all $C^*$-$G$-bundles $(A, \alpha)$, where $G$ is a groupoid and $A$ is an elementary $C^*$-bundle over $G^{(0)}$ that is acted upon by $G$ via $\alpha$. Most important, we present the fundamental construction that assigns to each principal $G$-space $X$ and each $C^*$-$G$-bundle, $(A, \alpha) \in \mathcal{B}r(G)$, an induced $C^*$-$G^X$-bundle $(A^X, \alpha^X) \in \mathcal{B}r(G^X)$, where $G^X$ is the imprimitivity groupoid of $G$ determined by $X$.

In the next section we define what it means for two systems in $\mathcal{B}r(G)$ to be Morita equivalent and we define $\text{Br}(G)$ to be $\mathcal{B}r(G)$ modulo this equivalence. We then show that $\text{Br}(G)$ is a group. As we have indicated, the details follow the lines found in [5].

The analysis of $\text{Br}(G)$ really is about three isomorphism theorems. The first is presented in Section 4, where we present an explicit isomorphism $\phi^X$ between $\text{Br}(G)$ and $\text{Br}(H)$ where $X$ is an equivalence between $G$ and $H$. This isomorphism theorem yields as an immediate corollary the main theorem in [27].

Section 5 is devoted to the relation between Haar systems on equivalent groupoids, on the one hand, and the notion of equivariant $s$-systems in the sense of [51], on the other. It turns out that the two notions are essentially the same and this fact plays an important rôle in our analysis.

Section 6 discusses the relation between the notions of groupoid equivalence and groupoid homomorphisms. In a sense it gives us a start to the generalization, to arbitrary locally compact groupoids, of the order structure on $\mathcal{P}(G)$ that is embodied in the concept of refinement for covers of a topological space. Most important for our purposes is the result, Corollary 6.6, that allows us to replace a system $(A, \alpha) \in \mathcal{B}r(G)$ by the system $(A^X, \alpha^X) \in \mathcal{B}r(G^X)$ for a suitable principal $G$-space $X$ such that the Dixmier-Douady class $\delta(A^X)$ vanishes.

In Section 7, we analyze twists over $G$, but we use ideas from [49] and [38] to replace the equivariant sheaf cohomology of [24] that was used for this purpose when $G$ is $r$-discrete. As we have implied, the sheaf cohomology theory does not seem to have a serviceable analogue in our more general setting. However, we still are able to show that the group of twists over $G$, $\text{Tw}(G)$, contains a subgroup of twists that is “parameterized” by principal circle bundles on $G^{(0)}$, i.e., by a certain $H^1$ group. The quotient of $\text{Tw}(G)$ by this subgroup is denoted $\mathcal{E}(G)$ and is called the generalized twist group.

With $\mathcal{E}(G)$ in hand, we are able to follow the first author’s lead in [24],
where $\text{Br} (G)$ is analyzed in terms of the subgroup $\text{Br}_0 (G)$ consisting of those equivalence classes of systems $(\mathcal{A}, \alpha)$ with trivial Dixmier-Douady invariant. For an equivalence class of $(\mathcal{A}, \alpha)$ in $\text{Br}_0 (G)$ one may define a natural twist $E(\alpha)$ over $G$. The definition is essentially Mackey’s definition of $E(\alpha)$ in the context of automorphic representations of locally compact groups described above. Section 8 is devoted to showing that every twist over $G$ is an $E(\alpha)$ for a suitable $\alpha$, Proposition 8.7, and to showing that $\text{Br}_0 (G)$ is naturally isomorphic to $E(G)$, Corollaries 8.6 and 8.9. This last result is our second isomorphism theorem.

Section 9 is devoted to the definition of $\text{Ext} (G, \mathcal{T})$ as an inductive limit of the generalized twist groups $\mathcal{E} (G^X)$, where $X$ runs over the collection of right principal $G$ spaces with equivariant $s$-systems, $\mathcal{P} (G)$. The problem here is to show that $\mathcal{P} (G)$ is directed. The solution is to rephrase the notion of “refinement” for covers in terms of equivalence of groupoids. The details require technology developed in Sections 6 and 7.

Finally, our third, and main, isomorphism theorem, Theorem 10.1, that asserts that $\text{Br} (G)$ is isomorphic to $\text{Ext} (G, \mathcal{T})$, is proved by using our first and second isomorphism theorems to realize embeddings of the $\mathcal{E} (G^X)$, $X \in \mathcal{P} (G)$, in a fashion that is compatible with the order on $\mathcal{P} (G)$. That is, we show that $\text{Br} (G)$ satisfies the universal properties of the inductive limit defining $\text{Ext} (G, \mathcal{T})$.

The last section is devoted to discussions of a number of examples that help to illustrate our theory and to establishing a connection between it and the third author’s work on dual groupoids.

2. Preliminaries. We will need to make considerable use of the notion of a Banach bundle. An excellent summary of the definitions and results needed can be found in §§13–14 of Chapter II of [13] (see also [12, §1]). We record some of the basic definitions and results here for the sake of completeness.

**Definition 2.1.** Suppose that $T$ is a locally compact space. A **Banach bundle** over $T$ is a topological space $\mathcal{A}$ together with a continuous, open surjection $p = p_A : \mathcal{A} \to T$ and Banach space structures on each fibre $p^{-1}\{\{t\}\} \ (t \in T)$ satisfying the following axioms.

- (a) The map $a \mapsto \|a\|$ is continuous from $\mathcal{A}$ to $\mathbb{R}^+$.
- (b) The map $(a, b) \mapsto a + b$ is continuous from $\mathcal{A} \times \mathcal{A}$.

\[ \mathcal{A} \ast \mathcal{A} := \{ (a, b) \in \mathcal{A} \times \mathcal{A} : p(a) = p(b) \} \]

- (c) For each $\lambda \in \mathbb{C}$, the map $b \mapsto \lambda b$ is continuous from $\mathcal{A}$ to $\mathcal{A}$.
- (d) If $\{a_t\}$ is a net in $\mathcal{A}$ such that $p(a_t) \to t$ in $T$ and such that $\|a_t\| \to 0$ in $\mathbb{R}$, then $a_t \to 0_t$ in $\mathcal{A}$. (Of course, $0_t$ denotes the zero element in the Banach space $p^{-1}\{\{t\}\}$.)}
A $C^*$-bundle over $T$ is a Banach bundle $p: \mathcal{A} \to T$ such that each fibre is a $C^*$-algebra satisfying, in addition to axioms (a)--(d) above, the following axioms.

(e) The map $(a, b) \mapsto ab$ is continuous from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$.
(f) The map $a \mapsto a^*$ is continuous from $\mathcal{A}$ to $\mathcal{A}$.

A continuous function $f: T \to \mathcal{A}$ such that $p \circ f = \text{id}_T$ is called a section of $\mathcal{A}$. The collection of sections $f$ for which $t \mapsto \|f(t)\|$ is in $C_0(T)$ is denoted by $\mathcal{A} := C_0(T; \mathcal{A})$.

Remark 2.2. If $\mathcal{A}$ is a Banach bundle (resp. $C^*$-bundle) over $T$, then $\mathcal{A} = C_0(T; \mathcal{A})$ is a Banach space (resp. $C^*$-algebra) with respect to the obvious pointwise operations and norm $\|f\| := \sup_{t \in T} \|f(t)\|$. We will use the notation $\mathcal{A}(t)$ for the fibre $p^{-1}\{t\}$. Furthermore, $C_0(T; \mathcal{A})$ is a central $C_b(T)$-bimodule.

Definition 2.3. A $C^*$-bundle is called elementary if every fibre is an elementary $C^*$-algebra; that is, every fibre is isomorphic to the compact operators on some complex Hilbert space.

Definition 2.4. If $p: \mathcal{A} \to T$ is a Banach bundle and if $q: X \to T$ is continuous, then the pull-back of $\mathcal{A}$ along $q$ is the Banach bundle $p': q^*\mathcal{A} \to X$, where

$$q^*\mathcal{A} := \{ (x, a) \in X \times \mathcal{A} : q(x) = p(a) \},$$

and $p'(x, a) := x$.

Remark 2.5. Fell refers to $q^*\mathcal{A}$ as the retraction of $\mathcal{A}$ by $q$. Note that the fibre of $q^*\mathcal{A}$ over $x$ can be identified with the fibre of $\mathcal{A}$ over $q(x)$. We will often make this identification without any fanfare.

If $p: \mathcal{A} \to T$ is a Banach bundle and $A = C_0(T; \mathcal{A})$, then we will often write $q^*(A)$ for $C_0(T; q^*\mathcal{A})$. If $\mathcal{A}$ is a $C^*$-bundle, then this is the same notation employed for the $C^*$-algebraic pull back

$$C_0(X) \otimes_{C(T)} A.$$ 

This should not be a source of confusion since the two are isomorphic by [45, Proposition 1.3].

Finally, if each fibre of $\mathcal{A}$ is a Hilbert space, then we refer to $\mathcal{A}$ as a Hilbert bundle. A straightforward polarization argument implies that $(a, b) \mapsto (a | b)$ is continuous from $\mathcal{A} \ast \mathcal{A}$ to $\mathbb{C}$. Therefore $\mathcal{A} = C_0(T; \mathcal{A})$ is a Hilbert $C_0(T)$-module [28]. This is in contrast to a Borel Hilbert bundle, where the (Borel) sections form a Hilbert space.

Definition 2.6. A Banach bundle $p: \mathcal{A} \to T$ has enough cross sections if given any $a \in \mathcal{A}$, there is a cross section $f \in C_0(T; \mathcal{A})$ such that $f(p(a)) = a$. 

Remark 2.7. Since we are assuming that our base spaces are locally compact, it follows from a result of Douady and dal Soglio-Hérault [11] that all our bundles have enough cross sections. A proof of their result appears in Appendix C of [13].

Remark 2.8. It follows from [9, Theorem 10.5.4] and Fell’s characterization of Banach bundles (e.g., [13, Theorem II.13.8]) that a CCR $C^*$-algebra has Hausdorff spectrum $T$ if and only if $A$ is isomorphic to the section algebra of an elementary $C^*$-bundle $\mathcal{A}$ over $T$.

Remark 2.9. Axiom (d) of Definition 2.1 ties the topology on a Banach bundle to its sections. Specifically, one can use [13, Proposition II.13.12] to characterize the topology as follows. If $p : \mathcal{A} \to T$ is a Banach bundle, and if $A = C_0(T; \mathcal{A})$, then a net $\{ a_i \}$ converges to $a$ in $\mathcal{A}$ if and only if (a) $\{ p(a_i) \} \to p(a)$, and (b) $\| a_i - f(p(a_i)) \| \to \| a - f(p(a)) \|$ for all $f \in A$. In fact, for the “if direction,” it suffices to take $f \in A$ with $f(p(a)) = a$.

Remark 2.10. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach bundles over a locally compact space $T$, and that $\phi : C_0(T; \mathcal{A}) \to C_0(T; \mathcal{B})$ is an isometric $C_0(T)$-isomorphism; i.e., $\phi$ respects the $C_0(T)$-module structures. Then we obtain a well-defined linear map $\Phi : \mathcal{A} \to \mathcal{B}$ by $\Phi(f(x)) = \phi(f)(x)$. This is a bundle isomorphism by [13, Proposition II.13.16].

Of course, the most studied sorts of bundles are those which are locally trivial. However, the bundles which arise from $C^*$-algebras (see Remark 2.8) are almost never naturally locally trivial, as the fibres are formally distinct spaces. Instead the proper notion is the following.

Definition 2.11. An elementary $C^*$-bundle over $T$ satisfies Fell’s condition if each $s \in T$ has a neighborhood $U$ such that there is a section $f$ with $f(t)$ a rank-one projection for all $t \in U$.

Remark 2.12. An elementary $C^*$-bundle satisfies Fell’s condition if and only if its section algebra is a continuous-trace $C^*$-algebra [9, Proposition 10.5.8].

We will need to use the notion of a groupoid action on a space. Let $G$ be a groupoid and let $X$ be a topological space together with a continuous open map $s$ from $X$ onto $G^{(0)}$. Form $X \ast G = \{ (x, \gamma) : s(x) = r(\gamma) \}$. (We have chosen not to write $s_X$, $s_G$, etc., as these notations eventually become too distracting.) To say that $G$ acts (continuously) on the right of $X$ means that there is a continuous map from $X \ast G$ to $X$, with the image of $(x, \gamma)$ denoted $x \cdot \gamma$, such that the following hold:

(a) $s(x \cdot \gamma) = s(\gamma),$

(b) $x \cdot (\alpha \beta) = (x \cdot \alpha) \cdot \beta$, for all $(\alpha, \beta) \in G^{(2)}$, and

(c) $x \cdot s(x) = x$ for all $x \in X$. 
We think of s as being a “generalized source” map for the action. Likewise a
(continuous) left action of G on X is determined by a continuous open surjection
r: X → G^{(0)}, and a continuous map from G × X to X satisfying the appropriate
analogues of (a), (b), and (c). In a like fashion, we think of r as a “generalized
range” map. When we speak of left or right actions in the sequel, the maps s and
r will be implicitly understood and will be referred to without additional
comment. Most of our definitions will be made in terms of right actions, but the
left-handed versions can be formulated with no difficulty.

As with group actions, if G acts on X (on the right, say), and if x ∈ X, the
orbit of x is simply \{ x · γ: γ ∈ G^{(x)} \}. The orbits partition X and we write X/G
for the quotient space with the quotient topology.

Consider a (right) action of G on X and let \( Ψ: X × G → X × X \) be defined by
the formula \( Ψ(x, γ) = (x, x · γ) \). Then we call the action free if \( Ψ \) is one-to-one. Alternatively, the action is free precisely when the equation \( x · γ = x \) implies that
γ = s(x). The action is called proper provided that \( Ψ \) is a proper map.

Now suppose that G is a locally compact groupoid and that p: \( A → G^{(0)} \)
is a Banach bundle (resp. \( C^* \)-bundle). A collection \( (A, α) \) of isometric isomorphisms (resp. *-isomorphisms) \( α_γ: A(s(γ)) → A(r(γ)) \) is called a G-action by isomorphisms or simply a G-action on \( A \) if \( γ · α := α_γ(α) \) makes \( (A, p) \) into a
(topological) left G-space as in [38, §2]. In this case, we say that \( A \) is a Banach
G-bundle (resp. \( C^* \)-G-bundle).

If \( α \) is a G-action on \( A \), then \( α \) induces a \( C_0(G^{(0)}) \)-linear isomorphism
\( α: s_G^*(A) → r_G^*(A) \) defined by \( α(f)(γ) := α_γ(f(γ)) \). It is not hard to see that
the converse holds.

**Lemma 2.13.** Let G be a locally compact groupoid. Suppose that p: \( A → G^{(0)} \)
is a Banach bundle, and that for each \( γ ∈ G, α_γ: A(s(γ)) → A(r(γ)) \) is an isometric
isomorphism such that \( α_γη = α_γ ∘ α_η \) if \( (γ, η) ∈ G^{(2)} \). Then \( (A, α) \) is a G-action by
isomorphisms if and only if \( α(f)(γ) := α_γ(f(γ)) \) defines an \( C_0(G^{(0)}) \)-isomorphism
of \( s_G^*(A) \) onto \( r_G^*(A) \).

**Proof.** We have already observed that if \( α \) is a G-action, then \( α: s_G^*(A) → r_G^*(A) \) is a \( C_0(G^{(0)}) \)-isomorphism.

To see the converse, the only issue is to observe that the action is continuous.
Suppose that \( (γ_i, a_i) → (γ, a) \) in \( s_G^*(A) \). Then there is a \( g ∈ s_G^*(A) \) such that
\( g(γ) = a \). Since \( α(g) ∈ r_G^*(A), α(g)(γ_i) → α(g)(γ) = γ · a \). On the other hand,
\[
\|α(g)(γ_i) - γ_i · a\| = \|α_γ_i(g(γ_i) - a_i)\| = \|g(γ_i) - a_i\|,
\]
which converges to zero. It now follows from Definition 2.1(d) that \( γ_i · a_i → γ · a \)
as required. \( \Box \)

**Definition 2.14.** Let G be a second countable locally compact groupoid. We
denote by \( \mathcal{B}(G) \) the collections of \( C^* \)-G-bundles \( (A, α) \) where \( A \) is an ele-
mentary $C^*$-bundle with separable fibres and which satisfies Fell’s condition. (Equivalently, $A = C_0(T; A)$ is a separable continuous-trace $C^*$-algebra.)

Now we turn to a construction which will be crucial in the sequel. Let $X$ be a $(H, G)$-equivalence—although, for the construction, it will suffice for $X$ to be a principal right $G$-space and a left $H$-space such that $(h \cdot x) \cdot \gamma = h \cdot (x \cdot \gamma)$ and such that $r_X: X \to H(0)$ induces a homeomorphism of $X/G$ with $H(0)$. Then $s_X^A = \{(x, a) \in X \times A : s_X(x) = p(a)\}$ is a (not necessarily locally compact) principal right $G$-space:

$$(x, a) \cdot \gamma := (x \cdot \gamma, a_{\gamma}^{-1}(a)).$$

Define $A^X$ to be the quotient space $s_X^A/G$. We will employ the notation $[x, a]$ to denote the orbit of $(x, a)$ in $A^X$. The map $(x, a) \mapsto r_X(x)$ defines a continuous surjection $p^X: A^X \to H(0)$. To see that $p^X$ is open, we use the criterion for a map to be open that convergent nets in the image have a subnet which can be lifted to the domain. An explicit statement and proof can be found, for example, in [13, Proposition II.13.2]. So, suppose that $r_X(x_1) \to r_X(x)$ in $H(0)$. Passing to a subnet and relabeling, we can assume that there are $\gamma_i$ in $G$ such that $x_i \cdot \gamma_i \to x$ in $X$. Since $p: A \to G(0)$ is open, we may as well assume that there are $a_i \in A$ such that $a_i \to a$ in $A$ and $p(a_i) = s(\gamma_i)$. Then $(x_i \cdot \gamma_i, a_i) \to (x, a)$ in $s_X^A$. The openness of $p^X$ follows from this. Finally, the map $a \mapsto [x, a]$ defines an isomorphism of $A(s_X(x))$ onto $A^X(r_X(x)) = (p^X)^{-1}(r_X(x))$. (Note that the operations and norm on $(p^X)^{-1}(r_X(x))$ are independent of our choice of representative for $r_X(x)$.)

If $A$ and $B$ are bundles over $T$, then a continuous map $\psi: A \to B$ is a bundle map if the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\psi} & B \\
p_A & \searrow & \swarrow p_B \\
& T & \\
\end{array}$$

commutes. We say $\psi$ is a bundle isomorphism if $\psi$ is a bijection, and $\psi$ is bicontinuous. If $A$ and $B$ are Banach bundles, then a bundle isomorphism $\psi$ is a Banach (resp. $C^*$-) bundle isomorphism if it is also an isomorphism (resp. $\ast$-isomorphism) of the fibres. Note that a continuous bundle map $\psi$ which is an isomorphism of each fibre is necessarily a Banach (resp. $C^*$-) bundle isomorphism [13, Proposition II.13.17].

**Proposition 2.15.** Let $G$ and $H$ be locally compact groupoids. Suppose that $X$ is a $(H, G)$-equivalence and that $A$ is a Banach $G$-bundle. Then $p^X: A^X \to H(0)$ is a Banach bundle. Furthermore, $\alpha^X_h[x, a] := [h \cdot x, a]$ defines a left $H$-action on $A^X$ making $(A^X, \alpha^X)$ an Banach $H$-bundle called the the induction of $A$ from $G$ to $H$ via $X$. In particular, if $(A, \alpha) \in \mathcal{B}(G)$, then $(A^X, \alpha^X) \in \mathcal{B}(H)$.
Proof. For the first assertion we must verify axioms (a)-(d) of Definition 2.1. For example, suppose that \([x_i, a_i] \to [x, a] \in \mathcal{A}^X\). Note that \(||(x_i, a_i)|| = ||a_i||\). Thus, if (a) fails, then we can pass to a subnet and relabel so that \(||a_i|| - ||a||\) \geq \epsilon > 0 for all \(i\). We can assume that there are \(\gamma_i \in G\) such that \((x_i \cdot \gamma_i, \alpha_{\gamma_i}^{-1}(a_i)) \to (x, a)\) in \(s_{\mathcal{A}}^X\). This leads to a contradiction since \(||a_i|| = ||\alpha_{\gamma_i}^{-1}(a_i)||\).

To establish (b), suppose that \([x_i, a_i] \to [x, a]\) and \([y_i, b_i] \to [y, b]\) in \(\mathcal{A}^X\) with \(r_X(x_i) = r_X(y_i)\) for all \(i\). We need to show that \([x_i, a_i] + [y_i, b_i] \to [x, a] + [y, b]\). But \(y_i = x_i \cdot \gamma_i\) for some \(\gamma_i \in G\) with \(s_G(\gamma_i) = p(b_i)\). Thus

\[
[x_i, a_i] + [y_i, b_i] = [x_i, a_i] + [x_i \cdot \gamma_i, b_i] = [x_i, a_i + \alpha_{\gamma_i}(b_i)].
\]

The result is that we can assume that \(y_i = x_i\) in the following. Therefore if (b) fails, we can pass to a subnet and relabel so that there is a neighborhood \(U\) of \([x, a+b]\) which contains no \([x_i, a_i+b_i]\). But we may as well also assume that there are \(\gamma_i, \eta_i \in G\) such that \((x_i \cdot \gamma_i, \alpha_{\gamma_i}^{-1}(a_i)) \to (x, a)\) and \((x_i \cdot \eta_i, \alpha_{\eta_i}^{-1}(b_i)) \to (x, b)\). Since \(X\) is a principal \(G\)-space, we may pass to yet another subnet and relabel so that we can assume \(\eta_i^{-1}\gamma_i \to s_X(x)\) in \(G\). Thus \((x_i \cdot \gamma_i, \alpha_{\gamma_i}^{-1}(b_i)) \to (x, b)\), and \((x_i \cdot \gamma_i, \alpha_{\gamma_i}^{-1}(a_i) + \alpha_{\gamma_i}^{-1}(b_i)) \to (x, a+b)\). This implies that \([x_i, a_i+b_i] \to [x, a+b]\), which contradicts our assumptions. Axioms (c) and (d) follow from similar (but easier) arguments.

Similarly, if \(\mathcal{A}\) is a \(C^*\)-bundle, then \(\mathcal{A}^X\) also satisfies axioms (e) and (f); that is, \(\mathcal{A}^X\) is a \(C^*\)-bundle which is clearly elementary if \(\mathcal{A}\) is.

It is straightforward to verify that \(\alpha^X\) is an action of \(H\) on \(\mathcal{A}^X\) by isomorphisms (or \(*\)-isomorphisms if \(\mathcal{A}\) is a \(C^*\)-bundle). It only remains to check that \(\mathcal{A}^X\) satisfies Fell’s condition if \(\mathcal{A}\) does. However, the map \((x, a) \mapsto (x, (x, a) \cdot G)\) is a Banach bundle isomorphism of \(s_{\mathcal{A}}^X\) onto \(r_X^* A^X\). Since \(\mathcal{A}\) is assumed to satisfy Fell’s condition, it is clear that \(s_{\mathcal{A}}^X\) does. Consequently, so does \(r_X^* A^X\). Therefore \(r_X^*(A^X) = C_0(X; r_X^* A^X)\) has continuous trace (Remark 2.12). Since \(r_X: X \to H^{(0)}\) is continuous, open, and surjective, [43, Lemma 1.2] implies that \(A^X\) has continuous trace. Therefore \(\mathcal{A}^X\) satisfies Fell’s condition as required.

If \(A\) and \(B\) are \(C^*\)-algebras and if \(X\) is an \(A-B\)-imprimitivity bimodule, then there is a homeomorphism \(h_X: \hat{A} \to \hat{B}\). If the spectrums of \(A\) and \(B\) have been identified with \(T\), there is no reason that \(h_X\) has to be the identity map. When \(A\) and \(B\) have spectrum (identified with) \(T\), then we call \(X\) a \(A-T\) \(B\)-imprimitivity bimodule when \(h_X = \text{id}_T\), and we say that \(A\) and \(B\) are Morita equivalent over \(T\) (cf., [46, p. 1035]). Note that if \(p: \mathcal{A} \to T\) is an elementary \(C^*\)-bundle, the spectrum of \(A = C_0(T; \mathcal{A})\) is naturally identified with \(T\).

Definition 2.16. Two \(C^*\)-bundles \(p_A: \mathcal{A} \to T\) and \(p_B: \mathcal{B} \to T\) with section algebras \(A\) and \(B\), respectively, are Morita equivalent if there exists an \(A-T\) \(B\)-imprimitivity bimodule \(X\).
It follows from [13, Theorem II.13.18] and [13, Corollary II.14.7] that $X = C_0(T; X)$ for a unique Banach bundle $q: X \to T$. Notice that the fibres $X(t)$ are exactly the quotient $A(t) - B(t)$-imprimitivity bimodules as described in [52, Corollary 3.2]. In particular if $f \in A$, $g \in B$, and $\xi, \eta \in X$, then we have the following:

$$
\begin{align*}
\langle \xi , \eta \rangle_A(t) &= \langle \xi(t) , \eta(t) \rangle_{A(t)} , \\
\langle \xi , \eta \rangle_B(t) &= \langle \xi(t) , \eta(t) \rangle_{B(t)} , \\
(f \cdot \xi)(t) &= f(t) \cdot \xi(t), \quad \text{and} \\
(\xi \cdot g)(t) &= \xi(t) \cdot g(t).
\end{align*}
$$

(2.1)

It follows that there are continuous maps $(x, y) \mapsto \langle \xi, \eta \rangle_A := \{(x, y) \in X \times X : q(x) = q(y)\}$ to $A$ and $(x, y) \mapsto \langle \xi, \eta \rangle_B$ from $X \times X \to B$ such that, for example, $\langle \xi(t) , \eta(t) \rangle_A := \langle \xi(t) , \eta(t) \rangle_{A(t)}$. Similarly, there are continuous maps $(a, x) \mapsto a \cdot x$ from $A \times X$ to $X$, and $(x, b) \mapsto x \cdot b$ from $X \times B$ to $X$.

**Definition 2.17.** Suppose that $p_A: A \to T$ and $p_B: B \to T$ are $C^*$-bundles with section algebras $A$ and $B$, respectively. Then a Banach bundle $q: X \to T$ is called an $A - B$-imprimitivity bimodule bundle if each fibre $X(t)$ is an $A(t) - B(t)$-imprimitivity bimodule such that the natural maps $(a, x) \mapsto a \cdot x$ from $A \times X$ to $X$ and $(x, b) \mapsto x \cdot b$ from $X \times B$ to $X$ are continuous.

As we will be working mostly with Banach bundles, it will be more natural here to work with imprimitivity bimodule bundles rather than imprimitivity bimodules. Fortunately our next result will make it easy to go from one to the other.

**Proposition 2.18.** Suppose that $p_A: A \to T$ and $p_B: B \to T$ are $C^*$-bundles with section algebras $A$ and $B$, respectively, and that $q: X \to T$ is an $A - B$-imprimitivity bimodule bundle. Then $X := C_0(T; X)$ is an $A - B$-imprimitivity bimodule bundle with actions and inner products defined as in (2.1). Conversely, if $X$ is an $A - T$ $B$-imprimitivity bimodule, then there is a unique $A - B$-imprimitivity bimodule bundle $X$ such that $X \cong C_0(T; X)$.

**Proof.** The last assertion follows from the preceding discussion. On the other hand, if $X = C_0(T; X)$, then it is immediate that $X$ is an $A - B$-bimodule. Next we observe that $(x, y) \mapsto \langle x , y \rangle_B := \langle x , y \rangle_{B(q(x))}$ is continuous from $X \times X$ to $B$; this follows from a polarization argument and the fact that $x \mapsto \|x\| = \langle x , x \rangle_B$ is continuous since $X$ is a Banach bundle. The Cauchy-Schwartz inequality for Hilbert modules implies that $\|\langle x , y \rangle_B\| \leq \|x\|\|y\|$. Consequently

$$
\langle \xi , \eta \rangle_B(t) := \langle \xi(t) , \eta(t) \rangle_B
$$

defines a $B$-valued inner product on $X$. Then ideal $\langle X , X \rangle_B$ is dense in $B$ as it
is a $C_0(T)$-submodule and $\langle X, X \rangle_B(t)$ is dense in $B(t)$ for each $t \in T$. Similar arguments apply to the $A$-valued inner product. The rest is straightforward.

3. The Brauer group. The notion of Morita equivalence for $C^*-G$-systems that we are about to define is crucial for all the work to follow. It is a natural generalization to groupoids of the idea of Morita equivalence for $C^*$-dynamical systems first investigated in [4] and [6].

**Definition 3.1.** Two $C^*-G$-systems $(A, \alpha)$ and $(B, \beta)$ are Morita equivalent if there is an $A-B$-imprimitivity bimodule bundle $\mathcal{X}$ which admits an action $V$ of $G$ by isomorphisms such that

$$\begin{align*}
\langle V_\gamma(x), V_\gamma(y) \rangle &= \alpha_\gamma (\langle x, y \rangle), \\
\langle V_\gamma(x), V_\gamma(y) \rangle &= \beta_\gamma (\langle x, y \rangle).
\end{align*}$$

In this case we will write $(A, \alpha) \sim_{(X, V)} (B, \beta)$.

**Lemma 3.2.** Morita equivalence of $C^*-G$-systems is an equivalence relation.

**Proof.** Since $(A, \alpha) \sim_{(A, \alpha)} (A, \alpha)$, “$\sim$” is reflexive. Recall that if $X$ is an $A-B$-imprimitivity bimodule, then one obtains a $B-A$-imprimitivity bimodule via the dual module $X^\sim$: $X^\sim$ coincides with $X$ as a set and, if $\iota: X \to X^\sim$ is the identity map, then the actions and inner products are given by

$$\begin{align*}
\iota(x) \cdot a &= \iota(a^* \cdot x) \\
\langle \iota(x), \iota(y) \rangle_A &= \langle x, y \rangle_B
\end{align*}$$

Thus if $\mathcal{X}$ is an $A-B$-imprimitivity bimodule bundle, then there is a $B-A$-imprimitivity bimodule bundle $\mathcal{X}^\sim$ which coincides with $\mathcal{X}$ as a set and is such that the identity map $\iota: \mathcal{X} \to \mathcal{X}^\sim$ maps the fibre $X(u)$ onto $X(u)^\sim$. Furthermore, $V_\gamma^\sim(\iota(a)) := \iota(V_\gamma(a))$ defines a $G$-action on $\mathcal{X}^\sim$ so that $(\mathcal{X}^\sim, V^\sim)$ implements an equivalence between $(B, \beta)$ and $(A, \alpha)$. Simply put: “$\sim$” is symmetric.

Now suppose

$$(A, \alpha) \sim_{(X, V)} (B, \beta) \sim_{(Y, W)} (C, \gamma).$$

Let $X = C_0(G^{(0)}; \mathcal{X})$ and $Y = C_0(G^{(0)}; \mathcal{Y})$. For each $u \in G^{(0)}$, let $Z(u) := X(u) \otimes_{B(u)} Y(u)$ be the usual $A(u) - C(u)$-imprimitivity bimodule (see, for example, [46, p. 1036]). If $\{\xi_1, \ldots, \xi_n\} \subseteq X$ and $\{\eta_1, \ldots, \eta_n\} \subseteq Y$, then

$$\left\| \sum_{i=1}^n \xi_i(u) \otimes \eta_i(u) \right\|^2 = \left\| \sum_{i,j=1}^n \langle \xi_i(u), \xi_j(u) \rangle_B \langle \eta_i(u), \eta_j(u) \rangle \right\|,$$
and the right-hand side is continuous in $u$. Moreover, $\{\xi(u) \otimes \eta(u) : \xi \in X$ and $\eta \in Y\}$ spans a dense subset of $Z(u)$ for each $u \in G^{(0)}$. Thus [13, Theorem II.13.18] implies that there is a Banach bundle $Z$ over $G^{(0)}$ having $Z(u)$ as fibres and such that $u \mapsto \xi(u) \otimes \eta(u)$ belongs to $Z = C_0(G^{(0)}; Z)$ for all $\eta \in Y$ and $\xi \in X$. Thus it remains only to see that the $A$- and $B$-actions are continuous. We consider only the $A$-action as the argument for the $B$-action is similar.

Suppose that $(a_i, z_i) \to (a, z)$ in $A \star Z$. We aim to show that $a_i \cdot z_i \to a \cdot z$. For convenience, let $u = p_Z(z)$ and $u_i = p_Z(z_i)$. Fix $\epsilon > 0$ and choose $z_0 := \sum z_i$ such that each $\xi_j \in X$, $\eta_j \in Y$, and $\|a \cdot z - a \cdot z_0\| < \epsilon$. Therefore we eventually have

$$\left\|a_i \cdot \left(\sum_{j=1}^n \xi_j(u_i) \otimes \eta_j(u_i)\right) - a_i \cdot z_i\right\| < \epsilon,$$

while

$$a_i \cdot \left(\sum_{j=1}^n \xi_j(u_i) \otimes \eta_j(u_i)\right) \to a \cdot z_0.$$

It follows from [13, Proposition II.13.12] that $a_i \cdot z_i \to a \cdot z$. This completes the proof that “$\sim$” is transitive, and hence that Morita equivalence of systems is an equivalence relation.

**Definition 3.3.** The collection $\text{Br}(G)$ of Morita equivalence classes of systems in $\mathcal{B}r(G)$ is called the Brauer group of $G$.

Of course, the terminology anticipates that fact (Theorem 3.7) that $\text{Br}(G)$ carries a natural group structure that we shall define in a moment.

**Remark 3.4.** Suppose that $(X, H)$ is a second countable locally compact transformation group, and that $G = X \times H$ is the corresponding transformation group groupoid. Then our $\text{Br}(G)$ is the “equivariant Brauer group” $\text{Br}_H(X)$ of $[5, 27, 39]$.

If $A$ and $B$ are elementary $C^*$-bundles over $T$, then there is a unique elementary $C^*$-bundle $A \otimes B$ over $T \times T$ with fibre $A(t) \otimes B(s)$ over $(t, s)$ and such that $(t, s) \mapsto f(t) \otimes g(s)$ is a section for all $f \in A = C_0(T; A)$ and $g \in B = C_0(T; B)$. This bundle clearly satisfies Fell’s condition if $A$ and $B$ do, as does its restriction $A \otimes_T B$ to $\Delta = \{ (u, u) \in T \times T \}$. By identifying $\Delta$ with $T$, we will view $A \otimes_T B$ as an elementary $C^*$-bundle over $T$. Notice that $C_0(T; A \otimes_T B)$ is the balanced tensor product $A \otimes_{C(T)} B$. Consequently it should not prove confusing to write $A \otimes_T B$ for $C_0(T; A \otimes_T B)$.

Now assume $(A, \alpha), (B, \beta) \in \mathcal{B}r(G)$. We want to equip $A \otimes_{G^{(0)}} B$ with a $G$-action $\alpha \otimes \beta$ such that $(A \otimes_{G^{(0)}} B, \alpha \otimes \beta) \in \mathcal{B}r(G)$. Of course, $\alpha_\gamma \otimes \beta_\gamma$ is a
*-isomorphism of \( A(s_G(\gamma)) \otimes B(s_G(\gamma)) \) onto \( A(r_G(\gamma)) \otimes B(r_G(\gamma)) \); thus, \( A \otimes_{G(0)} B \) admits a \( G \)-action provided we show that this action is continuous. There is a map \((a, b) \mapsto a \otimes b\) from \( A \ast B = \{ (a, b) \in A \times B : p_A(a) = p_B(b) \} \) into \( A \otimes_{G(0)} B \), and it is clearly continuous. Suppose that \((\gamma_i, c_i) \mapsto (\gamma, c)\) in \((A \otimes_{G(0)} B) \ast G\). Let \( u = s_G(\gamma) \) and \( u_i = s_G(\gamma_i) \). Fix \( \epsilon > 0 \). Choose \( z = \sum_{j=1}^n a_i^j \otimes b_i^j \in A \otimes_{G(0)} B(u) \) such that \( \| z - c \| < \epsilon \). Choosing sections in \( A \) and \( B \) through \( a_i^j \) and \( b_i^j \), respectively, we obtain elements \((a_i^j, b_i^j) \mapsto (a_i^j, b_i^j)\) in \( A \ast B \) with \( p_A(a_i^j) = u_i = p_B(b_i^j) \). Then
\[
z_i := \sum_{j=1}^n a_i^j \otimes b_i^j \rightarrow z.
\]

On the other hand, \( \gamma_i \cdot z_i \rightarrow \gamma \cdot z \). Moreover, \( \| \gamma \cdot c - \gamma \cdot z \| = \| c - z \| < \epsilon \), and \( \| \gamma_i \cdot z_i - \gamma_i \cdot c_i \| = \| z_i - c_i \| \). Therefore we eventually have \( \| \gamma_i \cdot z_i - \gamma_i \cdot c_i \| < \epsilon \).

It follows that \( \gamma_i \cdot c_i \rightarrow \gamma \cdot c \) as required.

Now suppose that
\[
(A, \alpha) \sim_{(x, V)} (C, \gamma), \quad \text{and} \quad \quad (B, \beta) \sim_{(y, W)} (D, \delta).
\]

For each \( u \in G(0) \), the external tensor product \( Z(u) := X(u) \otimes Y(u) \) is an \( A(u) \otimes B(u) - C(u) \otimes D(u) \)-imprimitivity bimodule (see [21, §1.2] or [5, §2]). If \( \xi_1, \ldots, \xi_n \in X \) and \( \eta_1, \ldots, \eta_n \in Y \), then
\[
\left\| \sum_{i=1}^n \xi_i(u) \otimes \eta_i(u) \right\|^2 = \left\| \sum_{i=1}^n \langle \xi_i(u), \xi_j(u) \rangle_{C} \otimes \langle \eta_i(u), \eta_j(u) \rangle_{D} \right\|,
\]

and the right-hand side is continuous in \( u \). Since \( \{ f(u) \otimes g(u) : f \in X \) and \( g \in Y \} \) spans a dense subspace of \( Z(u) \), there is a Banach bundle \( Z := X \otimes Y \), having \( Z(u) \) as fibre over \( u \), such that \( u \mapsto f(u) \otimes g(u) \) is a section for all \( f \in X \) and \( g \in Y \). Note that \( Z \) is the bundle associated to the global external tensor product \( X \otimes Y \). There are isometries \( V_\gamma \otimes W_\gamma : X(s_G(\gamma)) \otimes Y(s_G(\gamma)) \rightarrow X(r_G(\gamma)) \otimes Y(r_G(\gamma)) \) which define a left \( G \)-action by isomorphisms on \( Z \), and it is not hard to see that this action is continuous. Thus \((X \otimes Y, V \otimes W)\) implements an equivalence between \((A \otimes_{G(0)} B, \alpha \otimes \beta)\) and \((C \otimes_{G(0)} D, \gamma \otimes \delta)\).

**Definition 3.5.** Let \( T^{(0)} = T = G(0) \times \mathbb{C} \) be the trivial line bundle over \( G^{(0)} \), and let \( \tau \) be the action of \( G \) on \( T \) given by \( \gamma \cdot (s_G(\gamma), \lambda) := (r_G(\gamma), \lambda) \).

**Proposition 3.6.** The binary operation
\[
(3.1) \quad [A, \alpha][B, \beta] := [A \otimes_{G(0)} B, \alpha \otimes \beta]
\]
is well defined on $\text{Br}(G)$. With respect to this operation, $\text{Br}(G)$ is an abelian semi-
group with identity equal to the class $[\mathcal{T}, \tau]$. 

Proof. The preceding discussion shows that the operation is well defined. Since equivariant Banach bundle isomorphisms certainly provide Morita equivalences of systems, it is not hard to check that the operation is commutative and

associative.

The isomorphism $\phi \otimes \varphi \mapsto \phi \cdot \varphi$ of $C_0(G^{(0)}) \otimes G^{(0)}$ onto $A$ induces a Banach bundle isomorphism of $\mathcal{T} \otimes G^{(0)} \mathcal{A}$ onto $\mathcal{A}$ taking $\lambda \otimes a$ to $\lambda a$. Since this map is equivariant, it follows that $[A, \alpha][\mathcal{T}, \tau] = [A, \alpha]$; that is, $[\mathcal{T}, \tau]$ is an identity as claimed. 

In order to see that $\text{Br}(G)$ is a group, we will need to introduce the conjugate bundle. If $p : \mathcal{A} \to G^{(0)}$ is a Banach bundle, then let $\mathcal{A}$ be the topological space $\mathcal{A}$ and let $\iota : \mathcal{A} \to \mathcal{A}$ be the identity map. Then $\overline{p} : \mathcal{A} \to G^{(0)}$ defined by $\overline{p}(\iota(a)) = \iota(p(a))$ is a Banach bundle over $G^{(0)}$ with fibre $\tilde{\mathcal{A}}(t)$ identified with the conjugate

Banach space $\overline{\mathcal{A}}(t)$ (see the discussion preceding [5, Remark 3.5]). Furthermore $C_0(G^{(0)}; \mathcal{A}) = C_0(G^{(0)}; \overline{\mathcal{A}})$, so we may write $\overline{\mathcal{A}}$ for the section algebra of $\mathcal{A}$. If $(A, \alpha) \in \mathbb{Br}(G)$, then, $\overline{\alpha} = \iota(\alpha \gamma)$ is a $C^\ast-G$-action and $(\mathcal{A}, \overline{\alpha}) \in \mathbb{Br}(G)$.

Theorem 3.7. If $G$ is a locally compact groupoid with paracompact unit space $G^{(0)}$, then $\text{Br}(G)$ is an abelian group with addition defined by (3.1). The inverse of $[A, \alpha]$ is $[\mathcal{A}, \overline{\alpha}]$.

Proof. At this point, it will suffice to show that for any $(A, \alpha) \in \mathbb{Br}(G)$,

$(A \otimes G^{(0)}; \mathcal{A}, \alpha \otimes \overline{\alpha}) \sim (\mathcal{T}, \tau)$. Our proof will follow the lines of [5, Theorem 3.6], and we will use some of the same notation. In particular, since $A = C_0(G^{(0)}; \mathcal{A})$

has continuous trace, it follows from [46, Lemmas 6.1 and 6.2] that there are compact sets $F_i \subseteq G^{(0)}$ whose interiors form a cover $U = \{ \bigcup (F_i) : i \in I \}$ of $G^{(0)}$ such that

(a) for each $i \in I$, there is an $A^{F_i} - F_i C(F_i)$-imprimitivity bimodule $X_i$, and

(b) for each $i, j \in I$, there is an imprimitivity bimodule isomorphism $g_{ij} : X_{ij} \to X_{ij}$.

Consequently, there is an imprimitivity bimodule bundle $\mathcal{X}_i$ with $C_0(G^{(0)}; \mathcal{X}_i) = X_i$, and the fibres $X_i(u)$ are Hilbert spaces. If $u \in G^{(0)}$, then $g_{ij}$ determines a Hilbert space isomorphism $g_{ij}(t) : X_j(t) \to X_i(t)$. The Dixmier-Douady class of $\mathcal{A}$ is determined by the cocycle $\nu = \{ \nu_{ijk} \} \in H^2(U, \mathcal{S})$, where for each $t \in F_{ijk}$

$$g_{ij}(t) \circ g_{jk}(t) = \nu_{ijk}(t) g_{ik}(t).$$

We can then define $g_{ij} : X_{ij} \to X_{ij}$ by $g_{ij}(\iota(x)) := \iota(g_{ij}(x))$, and $h_{ij} := g_{ij} \otimes g_{ij} : X_{ij} \otimes C(F_i) X_{ij} \to X_{ij} \otimes C(F_i) X_{ij}$ as in [5]. We obtain an $A \otimes G^{(0)} \mathcal{A} - G^{(0)} C_0(G^{(0)})$-
imprimitivity bimodule by forming 
\[ \mathcal{Y}' := \left\{ (y_i) \in \prod X_i \otimes_{C(F_i)} X_i : h_{ij}(t)(y_j(t)) = y_i(t) \right\}, \]
and noticing that if \( t \in F_{ij} \) and \((x_k), (y_k) \in \mathcal{Y}'\), then
\[ \langle x_i, y_i \rangle_{C(F_i)}(t) = \langle x_j, y_j \rangle_{C(F_j)}(t). \]
Then we can define \( \langle (x_k), (y_k) \rangle_{C_0(G^{(0)})}(t) = \langle x_i, y_i \rangle_{C(F_i)}(t) \) when \( t \in F_i \), and
\[ \mathcal{Y} = \left\{ y \in \mathcal{Y}' : t \mapsto \langle y, y \rangle_{C_0(G^{(0)})}(t) \text{ vanishes at } \infty \right\}. \]
It is shown in [5, 46] that \( \mathcal{Y} \) is an \( A \otimes_{G^{(0)}} \overline{A} - C_0(G^{(0)}) \)-imprimitivity bimodule, and that \( \mathcal{Y} \) is isomorphic as a (full) Hilbert \( C_0(G^{(0)}) \)-module to
\[ N = \left\{ a \in A : t \mapsto \text{tr}(a^*a(t)) \text{ is in } C_0(G^{(0)}) \right\}. \]
The isomorphism \( \Phi : \mathcal{Y} \rightarrow N \) is given by \( \Phi((y_k))(t) = \Phi_i(y_i)(t) \) if \( t \in F_i \), where \( \Phi_i : X_i \otimes_{C(F_i)} X_i \rightarrow N^{F_i} \) is given by \( \Phi_i(x \otimes t(y)) \)(t) = \( A_{F_i}\langle x, y \rangle(t) \). Now \( N = C_0(G^{(0)}; N) \) for some Banach bundle \( N \). It also follows from the uniqueness condition in [13, Theorem II.13.18] that we can identify \( N \) with
\[ \{ a \in A : \text{tr}(a^*a) < \infty \} \]
with the relative topology. (The fibres of \( N \) are the ideals of Hilbert-Schmidt operators in the appropriate fibre of \( A \).) Since each \( \alpha_g \) is an isomorphism of elementary \( C^* \)-algebras, and therefore trace preserving, \( \alpha_g(N) \subseteq N \). It follows that we can define an continuous action \( V \) of \( G \) on \( N \) by isomorphisms via \( V_\gamma(a) := \alpha_{\gamma}(a) \).

**Lemma 3.8.** Let \( (\mathcal{B}, \tau) \in \text{Br}(G) \). Suppose that \( q : X \rightarrow T \) is an \( A - \mathcal{B} \)-imprimitivity bimodule bundle, and that \( V \) is a continuous action of \( G \) on \( X \) by isomorphisms such that \( \tau_\gamma(\langle x, y \rangle_\mathcal{B} \cdot b) = \langle V_\gamma(x), V_\gamma(y) \rangle_\mathcal{B} \cdot \tau_\gamma(b) \). Then \( \alpha_\gamma(a) := V_\gamma a V_\gamma^{-1} \) defines an isomorphism of \( A(s_G(\gamma)) \) onto \( A(r_G(\gamma)) \), such that \( (A, \alpha) \in \text{Br}(G) \), and \( (A, \alpha) \sim_{(X,V)} (\mathcal{B}, \tau) \).

**Proof.** Each \( a \in A(u) \) defines a linear operator \( x \mapsto a \cdot x \in \mathcal{L}(X(u)) \), and we can identify \( A(u) \) with \( K(X(u)) \). If \( \gamma \in G \) and \( a \in A(s_G(\gamma)) \), then \( x \mapsto V_\gamma a V_\gamma^{-1} \) is clearly in \( \mathcal{L}(X(r_G(\gamma))) \). If \( z \in X(r_G(\gamma)) \), then
\[
V_{\gamma, a}(x, y) V_{\gamma}^{-1}(z) = V_\gamma(x \cdot (y, V_{\gamma}^{-1}(z))_\mathcal{B}) = V_\gamma(x) \cdot \tau_\gamma(\langle y, V_{\gamma}^{-1}(z) \rangle_\mathcal{B})
\]
\[ = V_\gamma(x) \cdot \langle V_\gamma(x), z \rangle_{\mathcal{A}} \]
\[ = (V_\gamma(x), V_g(y))_{\mathcal{A}} \cdot z. \]

It follows that \( V_\gamma \{ x, y \} V_\gamma^{-1} \in K(\mathcal{X}(r_G(\gamma))) \cong A(r_G(\gamma)). \) Thus we can view \( V_\gamma aV_\gamma^{-1} \) as an element of \( A(r_G(\gamma)) \) for each \( a \in A(s_G(\gamma)) \). The continuity of \( V \) ensures that \( \alpha \) is continuous. The rest is routine. \( \square \)

**Completion of the proof of Theorem 3.7.** Note that \( K(\mathcal{N}) \) is Morita equivalent to \( C_0(G^{(0)}) \), and, since \( Y \) and \( N \) are isomorphic, there is an isomorphism \( Q: A \otimes_{G^{(0)}} \mathcal{A} \to K(\mathcal{N}) \) defined by \( Q(a \otimes i(b))(n) = abc^* \) [5, Corollary 3.12]. It is clear from the uniqueness of the trace on an elementary \( C^* \)-algebra that

\[ \langle V_\gamma(n), V_\gamma(m) \rangle_{c_0(G^{(0)})}(r_G(\gamma)) = \langle n, m \rangle_{c_0(G^{(0)})}(s_G(\gamma)). \]

Therefore it follows from Lemma 3.8 and the fact that \( K(\mathcal{N}) \) is Morita equivalent to \( C_0(G^{(0)}) \), that there is a \( C^* \)-\( G \)-bundle action \( \beta \) on \( A \otimes_{G^{(0)}} \mathcal{A} \) such that

\[ (A \otimes_{G^{(0)}} \mathcal{A}, \beta) \sim (\mathcal{G}, \tau). \]

In particular

\[ Q \circ \beta \cdot \langle f(\gamma) \otimes i(g(\gamma)) \rangle \cdot c \]
\[ = V_\gamma(Q(f(\gamma) \otimes i(g(\gamma)))(c) \]
\[ = \alpha_\gamma(f(\gamma)a_{\gamma}^{-1}(c)g(\gamma))^* \]
\[ = Q(\alpha_\gamma(f(\gamma)) \otimes i(\alpha_\gamma(g(\gamma))))(c). \]

Therefore \( \beta = \alpha \otimes \tilde{\alpha} \). This completes the proof that \( Br(G) \) is a group. \( \square \)

**4. The first isomorphism result.** The main objective of this section is to prove that equivalent groupoids have isomorphic Brauer groups. Specifically:

**Theorem 4.1.** Suppose that \( Z \) is a \((H,G)\)-equivalence. Then \( \phi^Z([A, \alpha]) := [A^Z, \alpha^Z] \) defines a group isomorphism of \( Br(G) \) onto \( Br(H) \).

This is the natural groupoid analogue of the main result in [27] (see Proposition 11.5).

The main tool here will be the fundamental construction of Proposition 2.15 (which applies to any Banach bundle). Our first result is that this construction respects Morita equivalence so that \( \phi^X \) is well defined.

**Lemma 4.2.** Suppose that \( Z \) is a \((H,G)\)-equivalence, and that \((A, \alpha) \sim_{(X,H)} (\mathcal{B}, \beta) \) in \( \mathcal{B}r(G) \). Then \((A^Z, \alpha^Z) \sim_{(X^Z,H)} (\mathcal{B}^Z, \beta^Z) \) in \( \mathcal{B}r(H) \).

**Proof.** As usual, we will denote the section algebras of \( A^Z, \mathcal{B}^Z, \) and \( X^Z \) by \( A^Z, \mathcal{B}^Z, \) and \( X^Z \), respectively. We obtain a well-defined continuous map from
\[ A^\mathbb{Z} \ast \mathbb{X}^\mathbb{Z} \] to \[ \mathbb{X}^\mathbb{Z} \] by

\[ (z \cdot \gamma, a) \cdot (z \cdot \eta, \xi) := (z, \alpha_{z}(a) \cdot V_{\eta}(\xi)); \]

similarly, there is a continuous map from \[ \mathbb{X}^\mathbb{Z} \ast \mathbb{B}^\mathbb{Z} \] to \[ \mathbb{X}^\mathbb{Z} \] given by

\[ (z \cdot \eta, \xi) \cdot (z \cdot \gamma, b) = (z, V_{\eta}(\xi) \cdot \beta_{z}(b)). \]

Furthermore, there are well-defined maps \( \langle \cdot , \cdot \rangle_{gZ} \) and \( \langle \cdot , \cdot \rangle_{gZ} \) from \( \mathbb{X}^\mathbb{Z} \ast \mathbb{X}^\mathbb{Z} \) to \( \mathbb{B}^\mathbb{Z} \) and \( \mathbb{A}^\mathbb{Z} \), respectively, given by

\[ \langle [z \cdot \gamma, \xi], [z \cdot \eta, \xi'] \rangle_{gZ} = [z, \langle \xi, \xi' \rangle_{gZ}], \text{ and} \]

\[ \langle \langle [z \cdot \gamma, \xi], [z \cdot \eta, \xi'] \rangle \rangle_{gZ} = [z, \langle \xi, \xi' \rangle_{gZ}]. \]

It is not difficult to see that these formulae equip \( \mathbb{X}^2(\mathbb{V}) \) with an \( \mathbb{A}^2(\mathbb{V}) - \mathbb{B}^2(\mathbb{V}) \)-imprimitivity bimodule structure. Since the norm on \( \mathbb{X}^2(\mathbb{V}) \) induced by the inner products coincides with the Banach bundle norm, it follows that \( \mathbb{X}^\mathbb{Z} \) is an \( \mathbb{A}^\mathbb{Z} - \mathbb{B}^\mathbb{Z} \)-imprimitivity bimodule bundle. Since

\[ \langle V^Z_{hZ}(\langle [z \cdot \gamma, \xi] \rangle), V^Z_{hZ}(\langle [z \cdot \eta, \xi'] \rangle) \rangle_{gZ} = \langle [h \cdot z \cdot \gamma, \xi], [h \cdot z \cdot \eta, \xi'] \rangle_{gZ} \]

\[ = [h \cdot z, \langle V_{\gamma}(\xi), V_{\eta}(\xi') \rangle_{gZ}] \]

\[ = \beta^{n}_{Z}(\langle [z \cdot \gamma, \xi], [z \cdot \eta, \xi'] \rangle_{gZ}), \]

and similarly for \( \alpha^{Z} \), we see that \( \langle \mathbb{X}^Z, V^Z \rangle \) implements the desired equivalence. \( \square \)

The next lemma will be exactly what we need to prove that \( \phi^{X} \) is a homomorphism.

**Lemma 4.3.** Suppose that \( p_{A} : A \rightarrow T \) and \( p_{B} : B \rightarrow T \) are elementary \( C^{*} \)-bundles, and that \( q : X \rightarrow T \) is continuous. Then the map \( (x, \sum_{i=1}^{n} a_{i} \otimes b_{i}) \mapsto \sum_{i=1}^{n} (x, a_{i}) \otimes (x, b_{i}) \) extends to a Banach bundle isomorphism of \( q^{*}(A \otimes_{T} B) \) onto \( q^{*}A \otimes_{X} q^{*}B \).

**Proof.** The given map is isometric on fibres and therefore clearly extends to a map \( \Phi \) from \( q^{*}(A \otimes_{T} B) \) onto \( q^{*}A \otimes_{X} q^{*}B \) which preserves fibres and is an isometry on each fibre. Thus it will follow that \( \Phi \) is a Banach bundle isomorphism once we see that it is continuous. But if \( f \) and \( g \) are sections of \( A \) and \( B \), respectively, then \( q^{*}(f \otimes g) \) given by \( x \mapsto (x, f(q(x)) \otimes g(q(x))) \) is a section of \( q^{*}(A \otimes_{T} B) \) and \( \Phi \circ q^{*}(f \otimes g) = q^{*}f \otimes q^{*}g \) is a section of \( q^{*}A \otimes_{X} q^{*}B \) (here \( q^{*}f(x) := (x, f(q(x))) \) and \( q^{*}g(x) := (x, g(q(x))) \)). Now the result follows from \([13, \text{Proposition II.13.16}]\). \( \square \)
Corollary 4.4. Suppose that \((A, \alpha), (\mathcal{B}, \beta) \in \mathcal{B}r(G)\) and that \(X\) is a \((H, G)\)-equivalence. Then \((A \otimes_{G(0)} \mathcal{B})^X, (\alpha \otimes \beta)^X\) is equivalent to \((A^X \otimes_{H(0)} \mathcal{B}^X, \alpha^X \otimes \beta^X)\) in \(\mathcal{B}r(H)\).

Proof. The result follows immediately from the lemma and the observation that \([ (a, x) \otimes (b, x) ] \mapsto [x, a] \otimes [x, b]\) is an \(H\)-invariant Banach bundle isomorphism of \((s^X \mathcal{A} \otimes_X s^Y \mathcal{B}) / G\) onto \(A^X \otimes_{X/G} \mathcal{B}^X\).

It follows from Lemma 4.2 and Corollary 4.4 that if \(X\) is a \((H, G)\)-equivalence, then \(\phi^X\) is a well-defined homomorphism from \(\mathcal{B}r(G)\) into \(\mathcal{B}r(H)\). We can finish the proof of Theorem 4.1 by exhibiting an inverse for \(\phi^X\). Our next result will provide the necessary “calculus” of these maps; this will allow us not only to finish the proof of Theorem 4.1, but will be useful in Section 8 as well.

First recall that groupoid equivalence is indeed an equivalence relation. Reflexivity follows from noting that \(G\) is a \((G, G)\)-equivalence. If \(X\) is a \((H, G)\)-equivalence, then the opposite space \(X^{op}\) is a \((G, H)\)-equivalence: if \(\iota: X \to X^{op}\) is the identity map then the left \(G\)-action on \(X^{op}\) is given by \(\gamma \cdot \iota(x) := \iota(x, \gamma^{-1})\) and the right \(H\)-action is given by \(\iota(x) \cdot h := \iota(h^{-1} \cdot x)\). If \(X\) is a \((H, G)\)-equivalence and \(Y\) is a \((G, K)\)-equivalence, then we obtain an \((H, K)\)-equivalence by forming \(X \ast G Y\) which is the orbit space of \(X \ast Y = \{(x, y) \in X \times Y: s_X(x) = r_Y(y)\}\) by the diagonal \(G\)-action: \((x, y) \cdot \gamma := (x, \gamma, \gamma^{-1} \cdot y)\). Furthermore two \((G_1, G_2)\)-equivalences \(Z_1\) and \(Z_2\) are isomorphic if there is a homeomorphism \(\phi: Z_1 \to Z_2\) such that \(\phi(\gamma_1 \cdot z \cdot \gamma_2) = \gamma_1 \cdot \phi(z) \cdot \gamma_2\) for all \((\gamma_1, z, \gamma_2) \in G_1 \ast Z \ast G_2\).

More generally, if \(X\) is a principal right \(G\)-space, then we will also write \(X^{op}\) for the corresponding principal left \(G\)-space. Then we can form \(G^X := X^X \otimes^G X^{op}\) as above. Then \(G^X\) has a natural groupoid structure (cf., e.g., [38, pp. 119+]). In particular, the unit space of \(G^X\), \(\{[x, x]: x \in X\}\), can be identified with \(X/G\), and \(G^X\) acts on the left of \(X\) via \([y, x] \cdot x = y\). With this action \(X\) is a \((G^X, G)\)-equivalence, and we refer to \(G^X\) as the imprimitivity groupoid of \(X\) ([38, Theorem 3.5(1)]).

Lemma 4.5. Suppose that \(G, H, \text{ and } K\) are locally compact groupoids.

(a) If \(X\) is a \((H, G)\)-equivalence and \(Y\) is a \((G, K)\)-equivalence, then

\[
\phi^{X \ast G Y} = \phi^X \circ \phi^Y.
\]

(b) If \(X\) and \(Y\) are isomorphic as \((H, G)\)-equivalences, then \(\phi^X = \phi^Y\).

(c) Viewing \(G\) is a \((G, G)\)-equivalence, \(\phi^G = \text{id}_G\).

(d) If \(X\) is a \((H, G)\)-equivalence, then

\[
\phi^{X \otimes^H X} = \text{id}_{\mathcal{B}r(G)} \quad \text{and} \quad \phi^{X \otimes^G G^X \otimes^H X} = \text{id}_{\mathcal{B}r(H)}.
\]
(e) If $X$ is a $(H, G)$-equivalence and $Y$ is a $(K, G)$-equivalence, then

$$\phi^X = \phi^Y \circ \phi^{X \ast g Y} \circ \phi^{Y \ast g X}.$$  

**Proof.** Suppose that $(A, \alpha) \in \text{Br}(K)$. We need to prove that $(A^X \ast g Y, \alpha^X \ast g Y) \simeq ((A^Y)^X, (\alpha^Y)^X)$. But $[x, [y, a]] = [x_1, [y_1, a_1]]$ in

$$(A^Y)^X = \{ [x, [y, a]] : s_X(x) = r_Y(y) \text{ and } s_Y(y) = p_{\lambda}(a) \}$$

if and only if there are $\gamma, \eta \in G$ such that $x_1 = x \cdot \gamma$, $y_1 = \gamma \cdot y \cdot \eta$, and $a_1 = \alpha^{-1}(a)$. That is, if and only if $[[x, y], a] = [[x_1, y_1], a_1]$ in

$$A^X \ast g Y = \{ [[x, y], a] : s_X(x) = r_Y(y) \text{ and } s_Y(y) = p_{\lambda}(a) \}.$$  

Thus we have a well-defined bijection $[x, [y, a]] \mapsto [[x, y], a]$ from $(A^Y)^X$ onto $A^X \ast g Y$. It is not hard to see that this is an equivariant Banach bundle map. Part (a) follows.

Parts (b) and (c) are straightforward. Part (d) follows from (b) and (c). (It should be noted that $H$ is only isomorphic to $X \ast g X^{op}$ and one may wonder if this isomorphism intervenes in (d). The point of (b) is that it doesn’t.) Finally, (e) follows from (c).

**Proof of Theorem 4.1.** We have already observed that $\phi^X$ is a homomorphism. It follows from Lemma 4.5 that $\phi^{X^{op}}$ is an inverse for $\phi^X$.

5. Haar systems for imprimitivity groupoids. Suppose that $X$ and $Y$ are locally compact (Hausdorff) spaces and that $\pi : X \to Y$ is a continuous, open, surjection. Then $C_c(X)$ is a $C_c(Y)$-module. A $C_c(Y)$-module map $\beta : C_c(X) \to C_c(Y)$ with the property that $f \geq 0$ implies $\beta(f) \geq 0$ is called a $\pi$-system. Notice that for each $y \in Y$ there is a Radon measure $\beta_y$ with $\text{supp}(\beta_y) \subseteq \pi^{-1}(y)$ such that

$$\beta(f)(y) = \int_X f(x) \, d\beta_y(x).$$

We say that $\beta$ is full if $\text{supp}(\beta_y) = \pi^{-1}(y)$ for all $y \in Y$. If in addition, $X$ and $Y$ are (right) $G$-spaces, then we say that $\beta$ is equivariant if

$$\int_X f(x \cdot \gamma) \, d\beta_y(x) = \int_X f(x) \, d\beta_{y \cdot \gamma}(x) \quad \text{for all } (y, \gamma) \in Y \ast G.$$  

We will make use of the following result from [50] where it was proved for left actions. We restate it for convenience.
LEMMA 5.1. [50, Lemme 1.3] Suppose that $X$ and $Y$ are principal right $G$-spaces, and that $\pi: X \to Y$ is a continuous, open, equivariant surjection. Let $\pi: X/G \to Y/G$ be the induced map.

(a) If $\beta$ is an equivariant $\pi$-system, then

$$\hat{\beta}(f)(y \cdot G) := \int_{X/G} f(x \cdot G) d\beta(x)$$

is a well-defined $\pi$-system.

(b) If $\tau$ is a $\pi$-system, then there is a unique equivariant $\pi$-system $\beta$ such that $\hat{\beta} = \tau$.

Our interest in Lemma 5.1 here is that it provides a characterization of when imprimitivity groupoids carry a Haar system.

PROPOSITION 5.2. Suppose that $G$ and $H$ are locally compact groupoids and that $X$ is a $(H, G)$-equivalence. Then $H$ has a Haar system if and only if $X$ has a full equivariant $s$-system where $s: X \to G^{(0)}$ is the source map for the principal $G$-action on $X$.

Proof. We can identify $H$ with $G^X = X_G^* X_{op}$. Suppose that $\alpha$ is a full equivariant $s$-system on $X$. Let $\pi: X_* X_{op} \to X$ be the projection map: $\pi(x, t(y)) = x$. Let $\beta_x := \delta_\{x\} \times \alpha_{s(x)}$. Then it is not difficult to verify that

$$\beta(f)(x) = \int_{X_* X_{op}} f(x, t(y)) d\alpha_{s(x)}(y)$$

is a full $\pi_\ell$-system. Furthermore since $\alpha$ is equivariant,

$$\int_{X_* X_{op}} f(x, t(y)) d\beta_{x, \gamma}(x, y) = \int_{X_* X_{op}} f(x \cdot \gamma, t(y)) d\alpha_{s(x, \gamma)}(y)$$

$$= \int_{X_* X_{op}} f(x \cdot \gamma, t(y \cdot \gamma)) d\alpha_{s(x)}(y)$$

$$= \int_{X_* X_{op}} f(x \cdot \gamma, \gamma^{-1} \cdot t(y)) d\beta_x(x, y);$$

it follows that $\beta$ is equivariant as well. Using the first part of Lemma 5.1, we obtain a $\pi_\ell$-system $\beta: C_c(G^X) \to C_c(X/G)$. It is not difficult to verify that $\lambda^\nu G := \beta_{sG}$ is a Haar system for $G^X$. For example, invariance follows from the computation:

$$\int_{G^X} f([x, y][u, v]) d\lambda^{\nu G}([u, v]) = \int_{G^X} f([x, v]) d\beta_{sG} = \int_{G^X} f([x, v]) d\alpha_{s(y)}(v)$$

$$= \int_{G^X} f([x, v]) d\alpha_{s(x)}(v), \text{ since } s(x) = s(y),$$

$$= \int_{G^X} f([x, v]) d\lambda^{\nu G}([x, v]).$$
For the converse, notice that a Haar system $\lambda$ for $G^X$ defines a $\pi_L$-system. Using the second part of Lemma 5.1, there is a unique $\pi_L$-system $\beta$ such that $\beta$ coincides with $\lambda$. But $\beta$ must be of the form $b_x = \delta_{\{x\}} \times \alpha_x$ with supp $\alpha_x = S^{-1}(x)$. Furthermore, $\alpha_x$ depends only on $s(x)$ and $\alpha = \{ \alpha_u \}_{u \in G(0)}$ is a full equivariant $s$-system as required.

In the sequel, we write $\mathcal{P}(G)$ for the collection of second countable principal right $G$-spaces which have full equivariant $s$-systems. By Proposition 5.2, these are precisely the principal $G$-spaces whose imprimitivity groupoids have Haar systems. We shall see in Sections 6 and 9 that $\mathcal{P}(G)$ may be viewed as a generalization of the collection of all open covers of $G(0)$, directed by refinement.

### 6. The graph of a homomorphism

This section contains two key results for our analysis. The first, Lemma 6.2, gives a serviceable sufficient condition for a homomorphism between two groupoids to implement an equivalence. The second, Corollary 6.6, shows that if $[A, \alpha] \in \text{Br}(G)$, then there is a space $X \in \mathcal{P}(G)$ such that $\delta(A^X) = 0$. Our approach to understanding $\text{Br}(G)$ is to analyze it “through the eyes” of $A^X$.

If $\phi : H \to G$ is any continuous groupoid homomorphism, then we can define a principal right $G$-space called the graph of $\phi$ as follows:

$$\text{Gr}(\phi) := \{(u, \gamma) \in H(0) \times G : \phi(u) = r(\gamma)\}.$$  

If, for example, $\phi|_{H(0)}$ is an open surjection onto $G(0)$, then $s(u, \gamma) := s(\gamma)$ is clearly an open surjection of $\text{Gr}(\phi)$ onto $G(0)$, and $\text{Gr}(\phi)$ is a principal right $G$-space: $(u, \gamma) \cdot \gamma' = (u, \gamma \gamma')$. Moreover, we claim that $r(u, \gamma) := r(\gamma)$ is open. (We have opted not to decorate every range and source map in the sequel. In particular, writing $r_{\text{Gr}(\phi)}(u, \gamma) = r_G(\gamma)$ in place of $r(u, \gamma) := r(\gamma)$ does not seem to make these expressions easier to sort out. Instead, we hope our meaning will be clear from context.) If $v_i \to v$ in $G(0)$, then we can pass to a subnet and relabel, and assume that there are $\gamma_i \to \gamma$ in $G$ such that $r(\gamma_i) = v_i$. Since $\phi|_{H(0)}$ is open and surjective, we can pass to a subnet and relabel and assume that there are $u_i \to u$ in $H(0)$ and $\phi(u_i) = v_i$. Thus $(u_i, \gamma_i)$ converges to $(u, \gamma)$ in $\text{Gr}(\phi)$; it follows that $s$ is open. Therefore $\text{Gr}(\phi)$ is also a left $H$-space: $h \cdot (s(h), \gamma) = (r(h), \phi(h) \gamma)$. We are most interested in situations where $\text{Gr}(\phi)$ is actually a $(H, G)$-equivalence; in this case we say that $\phi$ induces an equivalence between $H$ and $G$.

**Example 6.1.** Suppose that $\phi$ is a groupoid automorphism of $G$. Then $\text{Gr}(\phi)$ is easily seen to be a $(G, G)$-equivalence, and $\phi$ induces an equivalence between $G$ and $G$.

Our next lemma gives some simple criteria for a homomorphism to induce an equivalence; we have made no attempt to find the most general such criteria—only ones that are easy to check in our applications.
Lemma 6.2. Suppose that $\phi : H \to G$ is a continuous groupoid homomorphism that satisfies the following.

(a) $\phi$ is surjective.

(b) $\phi|_{H^{(0)}}$ is open.

(c) If $\{h_i\}$ is a net in $H$ such that $\{r(h_i)\}$ and $\{s(h_i)\}$ converge in $H^{(0)}$, while $\{\phi(h_i)\}$ converges in $G$, then $\{h_i\}$ has a convergent subnet.

(d) The restriction of $\phi$ to the isotropy subgroupoid $A = \{h \in H : r(h) = s(h)\}$ has trivial kernel; that is, $\ker(\phi) \cap A = H^{(0)}$, where $\ker(\phi) = \phi^{-1}(G^{(0)})$.

(e) If $u, v \in H^{(0)}$ and if $\phi(u) = \phi(v)$, then there exists $h \in \ker(\phi)$ such that $h \cdot v = u$.

Then $\phi$ induces an equivalence between $H$ and $G$. If only assertions (a) and (b) hold, then $\Gr(\phi)$ is a principal right $G$-space and a $H$-space such that $h \cdot (x \cdot \gamma) = (h \cdot x) \cdot \gamma$ and $\Gr(\phi)/G \cong H^{(0)}$.

Proof. The $G$-action on $\Gr(\phi)$ is always free and proper (provided that $\phi(H^{(0)}) = G^{(0)}$ and $\phi|_{H^{(0)}}$ is open). If $\phi|_{H^{(0)}}$ is open, then, as pointed out above, $\Gr(\phi)$ is a right $H$-space, and this action will be free and proper in view of (d) and (e), respectively. It remains to show only that $s : \Gr(\phi) \to G^{(0)}$ and $r : \Gr(\phi) \to H^{(0)}$ induce homeomorphisms of $G^{(0)}$ with $H \setminus \Gr(\phi)$ and $H^{(0)}$ with $\Gr(\phi)/G$, respectively.

Assume that $(u, \gamma)$ and $(v, \eta)$ are in $\Gr(\phi)$ and that $s(u, \gamma) = s(\gamma) = s(\eta) = s(v, \eta)$. By (a), there is an $h \in H$ such that $\phi(h) = \eta \gamma^{-1}$. Then $\phi(s(h)) = r(\gamma) = \phi(u)$ and $\phi(r(h)) = r(\eta) = \phi(v)$. It follows from (e), that there are $h', h'' \in \ker(\phi)$ such that $h' \cdot r(h) = v$ and $h' \cdot u = s(h)$. Then $\phi(h' hh'') = \eta \gamma^{-1}$ and $h' hh'' \cdot (u, \gamma) = (v, \eta)$. Thus $s$ induces a bijection of $H \setminus \Gr(\phi)$ onto $G^{(0)}$. Now suppose that $w_i \to w$ in $G^{(0)}$ and that $(u_i, \gamma_i) \in \Gr(\phi)$ satisfies $s(u_i, \gamma_i) = s(\gamma_i) = w_i$. Passing to a subnet and relabeling we can assume that there are $\gamma_i \in G$ such that $\gamma_i \to \gamma$ and $s(\gamma_i) = w_i$. In view of (b), we can even assume that there are $u_i \to u$ with $\phi(u_i) = r(\gamma_i)$. Then $(u_i, \gamma_i) \to (u, \gamma)$ in $\Gr(\phi)$. It follows that $s$ is open and induces a homeomorphism of $H \setminus \Gr(\phi)$ with $G^{(0)}$ as claimed.

The case for $r : \Gr(\phi) \to H^{(0)}$ is more straightforward. It is immediate that $r$ induces a continuous bijection of $\Gr(\phi)/G$ onto $H^{(0)}$. Furthermore if $(u, \gamma) \in \Gr(\phi)$ and if $u_i \to u$, then we can pass to a subnet and relabel and assume there are $\gamma_i \to \gamma$ in $G$ with $r(\gamma_i) = \phi(u_i)$. Then $(u_i, \gamma_i) \in \Gr(\phi)$, and $(u_i, \gamma_i) \to (u, \gamma)$. Therefore $r$ is open and always induces a homeomorphism of $\Gr(\phi)/G$ onto $H^{(0)}$.

Example 6.3. (Local Trivialization) Let $U = \{U_i\}$ be a locally finite open cover of $G^{(0)}$. Let $X := \coprod U_i$ and $\psi : X \to G^{(0)}$ the obvious map—note that $\psi$ is a continuous, open surjection. Then we can build a groupoid $G^{\psi} := \coprod U_i G^{U_i}$ as follows. We write elements of $G^{\psi}$ as triples $(j, \gamma, i)$ with $r(\gamma) \in U_j$ and $s(\gamma) \in U_i$. Each $(i, u) \in X$ is identified with $(i, u, i) \in G^{\psi}$. Thus we can define...
Let \( r(j, \gamma, i) = (j, r(\gamma)) \) and \( s(j, \gamma, i) = (i, s(\gamma)) \). Then we can identify the unit space of \( G^\psi \) with \( X \) where the groupoid operations on \( G^\psi \) are

\[
(i, \gamma, j)(j, \eta, k) = (i, \gamma \eta, k) \quad \text{and} \quad (i, \gamma, j)^{-1} = (j, \gamma^{-1}, i).
\]

The map \( \phi: G^\psi \to G \) given by \( \phi(i, \gamma, j) = \gamma \) is a homomorphism extending \( \psi \). It is easy to see that \( \phi \) satisfies conditions (a)–(e) of Lemma 6.2 so that \( \phi \) induces an equivalence of \( G^\psi \) and \( G \).

**Example 6.4.** (Pull-Back Actions) Suppose that \( \phi: H \to G \) is a groupoid homomorphism, and that \( \psi := \phi|_{H(0)} \) is surjective. Then if \( (A, \alpha) \in \mathcal{Br}(G) \), \( \psi^*A \) is an elementary \( C^* \)-bundle over \( H(0) \) satisfying Fell’s condition. Furthermore, we obtain an action of \( H \) on \( \psi^*A \) as follows: \( h \cdot (s(h), a) = (r(h), \alpha_{\phi(h)}(a)) \). We denote this action by \( \phi^* \alpha \). Observe that \( (\psi^*A, \phi^* \alpha) \in \mathcal{Br}(H) \).

The next result shows how the previous two examples are related.

**Lemma 6.5.** Suppose that \( \psi: H(0) \to G(0) \) is a continuous open surjection, and that \( \phi: H \to G \) is a homomorphism from \( H \) to \( G \) and which extends \( \psi \). If \( (A, \alpha) \in \mathcal{Br}(G) \), then

\[
(A^{\text{Gr}(\phi)}, \alpha^{\text{Gr}(\phi)}) \sim (\psi^*A, \phi^* \alpha) \quad \text{in} \ \mathcal{Br}(H).
\]

**Proof.** The map \([u, \gamma, a] \mapsto (u, \alpha_u(a))\) is a \( H \)-equivariant Banach bundle map.

**Corollary 6.6.** Suppose that \( (A, \alpha) \in \mathcal{Br}(G) \) and that \( G \) has a Haar system \( \{ \lambda^u \}_{u \in G(0)} \). Then there is a groupoid \( H \) with a Haar system and an \((H, G)\)-equivalence \( Z \) such that the Dixmier-Douady class \( \delta(A^Z) \) is trivial in \( H^3(H(0); \mathbb{Z}) \).

**Proof.** Since \( A \) satisfies Fell’s condition, we can find a locally finite open cover \( U = \{ U_i \} \) of \( G(0) \) such that \( \delta(A|_{U_i}) = 0 \) for all \( i \). If \( X := \bigsqcup U_i \) and \( \psi: X \to G(0) \) is the usual map, then \( \psi^*A \cong \bigoplus A|_{U_i} \). Consequently, \( \delta(\psi^*A) = 0 \). On the other hand, we can let \( H := G^\psi \) as in Example 6.3 so that \( Z := \text{Gr}(\phi) \) is a \((H, G)\)-equivalence, and Lemma 6.5 implies that \( A^{\text{Gr}(\phi)} \) is Morita equivalent over \( H(0) \) to \( \psi^*(A) \) (isomorphic, in fact). Since the Dixmier-Douady class is invariant under Morita equivalence (e.g., [46, Theorem 3.5]), \( \delta(A^{\text{Gr}(\phi)}) = 0 \) as desired.

To show that \( H \) has a Haar system, it will suffice to show that \( Z \) has a full equivariant \( s \)-system (Proposition 5.2). But \( Z = \bigsqcup_i G^{U_i} = \{(i, \gamma): r(\gamma) \in U_i\} \). Then

\[
\alpha(f)(u) := \sum_i \int_{G} f(i, \gamma) \, d\lambda_u(\gamma)
\]

does the trick.
7. Generalized twists over $H$: $E(H)$. Suppose that $H$ is a locally compact groupoid. We need to recall the definition of a twist over $H$ or a $\mathbb{T}$-groupoid as defined, for example, in [23] or [38, §3]. Simply put, a twist over $H$ is a principal circle bundle $j: E \to H$ equipped with a groupoid structure such that $E$ is a groupoid extension of $H$ by $H^{(0)} \times \mathbb{T}$:

$$H^{(0)} \xrightarrow{i} E \xrightarrow{j} H.$$  

The collection of equivalence classes of twists is a group $\text{Tw}(H)$ with respect to Baer sum. Thus if $E_1$ and $E_2$ are twists over $H$, then the sum of their classes is the class of the quotient $E_1 + E_2 := E_1 \# \mathbb{T} E_2$ of $E_1 \ast E_2 := \{(e,f) \in E_1 \times E_2: j_1(e) = j_2(f)\}$ by the diagonal $\mathbb{T}$-action $t \cdot (e,f) := (t \cdot e, t^{-1} \cdot f)$.

As developed in [23, 49, 38], a twist over $G$ should be regarded as a replacement for a Borel 2-cocycle. The collection of twists forms a substitute for the groupoid cohomology group $H^2(G, \mathbb{T})$ defined in [48]. However for our purposes, a weaker notion of equivalence of extensions is appropriate. Recall that strict equivalence means there is a commutative diagram

$$
\begin{array}{ccccccc}
H^{(0)} & \longrightarrow & H^{(0)} \times \mathbb{T} & \xrightarrow{i_1} & E_1 & \xrightarrow{j_1} & H \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{(0)} & \longrightarrow & H^{(0)} \times \mathbb{T} & \xrightarrow{i_2} & E_2 & \xrightarrow{j_2} & H
\end{array}
$$

(7.1)

where $\phi$ is a groupoid isomorphism. That is, $E_1$ and $E_2$ are isomorphic over $H$.

We want to weaken this notion in two steps. In rough terms, the first is to replace the isomorphism $\phi$ in (7.1) with an equivalence of $\mathbb{T}$-groupoids while maintaining equality in the other vertical positions. The second step will be to allow the groupoid $H$ to vary over equivalent groupoids. To make these statements precise will take a bit of work. First recall that two twists $E$ and $F$ over $H$ are equivalent as $\mathbb{T}$-groupoids if there is an $(E,F)$-$\mathbb{T}$-equivalence as defined in [38, Definition 3.1]; that is, if there is an ordinary $(E,F)$-equivalence $Z$ for which the $\mathbb{T}$-actions on $Z$ induced by $E$ and $F$ coincide. It is straightforward to check that the quotient $Z/\mathbb{T}$ is naturally an $(H,H)$-equivalence. We define two twists $E$ and $F$ to be equivalent over $H$ if there is a $(E,F)$-$\mathbb{T}$-equivalence $Z$ such that $Z/\mathbb{T}$ is isomorphic to $H$ (as $(H,H)$-equivalences). In this case we write $E \sim_H F$, and say that $E$ and $F$ are equivalent over $H$. 

If $[E] = [F]$ in $\text{Tw}(H)$, then there is a twist isomorphism

$$
\begin{array}{c}
E \\
\downarrow \phi \\
F
\end{array}
$$

which is a bundle isomorphism as well as a groupoid homomorphism. In particular, $\text{Gr}(\phi)$ is an $(E, F)$-T-equivalence, and $\text{Gr}(\phi)/\mathbb{T}$ satisfies $j_1(e)[s(e), f] = [r(e), \phi(e)f]$. Since $j_1(e) = j_2(\phi(e))$, the map $[u, f] \mapsto (u, j_2(f))$ is an isomorphism of $\text{Gr}(\phi)/\mathbb{T}$ onto $\text{Gr}(\text{id}_H)$. Thus $E \sim_H F$ in this case. The converse is not valid (see Remark 7.4). To examine this, consider the group homomorphism $\epsilon: H^1(H^{(0)}, S) \to \text{Tw}(H)$ defined on page 124 of [38]. Recall that $\epsilon$ is defined as follows. We identify $H^1(H^{(0)}, S)$ with isomorphism classes of principal $\mathbb{T}$-bundles over $H^{(0)}$. If $p: \Lambda \to H^{(0)}$ is such a bundle, we let $\tilde{\Lambda}$ be the conjugate bundle. Note that viewing $\Lambda$ as a right $\mathbb{T}$-space, $\tilde{\Lambda}$ is the opposite left $\mathbb{T}$-space: $t \cdot \iota(\xi) = \iota(\xi \cdot t^{-1})$. Then let $\Lambda^{* \mathbb{T}}_{\mathbb{T}} \tilde{\Lambda}$ be the $\mathbb{T}$-bundle over $H^{(0)} \times H^{(0)}$ obtained as the quotient of $\Lambda \times \tilde{\Lambda}$ by the diagonal $\mathbb{T}$-action: $t \cdot (\xi, \iota(\eta)) = (\xi \cdot t, t^{-1} \cdot \iota(\eta))$. Then $\epsilon([\Lambda])$ is the class of the pull-back $\pi^*(\Lambda^{* \mathbb{T}}_{\mathbb{T}} \tilde{\Lambda})$ where $\pi: H \to H^{(0)} \times H^{(0)}$ is the map $\pi(h) = (r(h), s(h))$.

If $E$ is a principal $\mathbb{T}$-bundle over $H$ and $\Lambda$ is a principal $\mathbb{T}$-bundle over $H^{(0)}$, then the quotient $\Lambda^{* \mathbb{T}}_{\mathbb{T}} E$ of $\Lambda^{*} E := \{ (\xi, e) \in \Lambda \times E : p(\xi) = r(e) \}$ by the $\mathbb{T}$-action $t \cdot (\xi, e) := (\xi \cdot t, t^{-1} e)$ is also a principal $\mathbb{T}$-bundle over $H$. There is a similar quotient $\Lambda^{* \mathbb{T}}_{\mathbb{T}} \tilde{\Lambda}$ of $\Lambda^{*} \tilde{\Lambda} := \{ (e, \eta) \in E \times \tilde{\Lambda} : s(e) = p(\eta) \}$. Our interest in these constructions is that if $E$ is also a twist over $H$, then the same is true of

$$
(7.2) \quad \Lambda^{* \mathbb{T}}_{\mathbb{T}} E^{* \mathbb{T}} \tilde{\Lambda} := (\Lambda^{* \mathbb{T}}_{\mathbb{T}} E)^{* \mathbb{T}} \tilde{\Lambda} \cong \Lambda^{* \mathbb{T}}_{\mathbb{T}} (E^{* \mathbb{T}} \tilde{\Lambda});
$$

we have $r([\xi, e, \iota(\eta)]) = p(\xi)$, $s([\xi, e, \iota(\eta)]) = p(\eta)$, and $[\xi, e, \iota(\eta)][\eta \cdot t, f, \iota(\zeta)] = [\xi, tef, \iota(\zeta)]$.

**Lemma 7.1.** Suppose that $E$ is a twist over $H$ and that $\Lambda$ is a principal $\mathbb{T}$-bundle over $H^{(0)}$. Then

$$
[E] + \epsilon([\Lambda]) = [\Lambda^{* \mathbb{T}}_{\mathbb{T}} E^{* \mathbb{T}} \tilde{\Lambda}].
$$

**Proof.** It is not difficult to see that $[E] + \epsilon([\Lambda])$ is represented by the quotient of $\{ (e, h, \xi, \iota(\eta)) \in E \times H \times \Lambda \times \tilde{\Lambda} : j(e) = h, r(h) = p(\xi), s(h) = p(\eta) \}$ by the $\mathbb{T}^2$-action $(t, s) \cdot (e, h, \xi, \iota(\eta)) = (e \cdot t^{-1}, h, \xi \cdot ts, s^{-1} \iota(\eta))$. Then $(e, h, \xi, \iota(\eta)) \mapsto (\xi, e, \iota(\eta))$ induces a twist isomorphism. $\square$

**Lemma 7.2.** Suppose that $E$ and $F$ are $\mathbb{T}$-groupoids over $H$. Then $E \sim_H F$ if and only if there is a principal $\mathbb{T}$-bundle over $H^{(0)}$ such that

$$
[F] = [\Lambda^{* \mathbb{T}}_{\mathbb{T}} E^{* \mathbb{T}} \tilde{\Lambda}].
$$
Proof. Suppose that $E \sim_H F$ and that $Z$ is an $(E,F)$-$\mathbb{T}$-equivalence such that $Z/\mathbb{T}$ is isomorphic to $H$ as an $(H,H)$-equivalence. Composing with the orbit map then gives $Z$ the structure of a $\mathbb{T}$-bundle over $H$, $p : Z \to H$ and the restriction to $H(0)$ is therefore a principal $\mathbb{T}$-bundle $\Lambda$ over $H(0)$. Since $Z/\mathbb{T} \cong H$, we have $s(p(z)) = s(z)$ and $r(p(z)) = r(z)$. Therefore we obtain a map from $\Lambda \ast E = \{ (\xi,e) \in \Lambda \times E : p(\xi) = r(e) \}$ to $E$ by restricting the action map from $Z \ast E \to Z$. The restriction is invariant with respect to the given $\mathbb{T}$-action on $\Lambda \ast E$ and defines a continuous map $\phi$ from $\Lambda \ast E$ to $Z$. It is not difficult to see that $\phi$ is a bundle map, and hence a bundle isomorphism (cf., e.g., [19, Theorem 3.2]). Therefore $F$ is groupoid isomorphic to the imprimitivity groupoid $E_{\Lambda^*}^\Lambda = (\Lambda^* E)_E^* (\Lambda^* E)_{op}$. The latter is clearly the quotient of $\{ (\xi,e,\eta,f) \in \Lambda \times E \times \Lambda \times E : p(\xi) = r(e), p(\eta) = r(f), \text{and } s(e) = s(f) \}$ by the action of $T^2 \times E : (t,s,g) \cdot (\xi,e,\eta,f) = (\xi \cdot t, r^{-1} \cdot eg, \eta \cdot s, s^{-1} \cdot fg)$. The map $[\xi,e,\eta,f] \mapsto [\eta,ef^{-1},t(\eta)]$ is a groupoid isomorphism onto $\Lambda^* E^\Lambda \Lambda$. On the other hand, given $f \in F$ and $\xi \in \Lambda$ with $r(\xi) = s(f)$, the isomorphism of $F$ with $Z^* E^\Lambda$ takes $f$ to $[f \cdot \xi, \xi]$. Choose $\eta \in \Lambda$ and $e_f \in E$ such that

$$\eta \cdot e_f = f \cdot \xi. \tag{7.3}$$

Then the induced isomorphism of $F$ with $\Lambda^* E^\Lambda \Lambda$ takes $f$ to $[\eta, e_f, \xi]$. But (7.3) and the fact that $Z/\mathbb{T} \cong H$ implies that

$$j_F(f) = j_F(f) \cdot (\xi) = p(\eta) \cdot j_E(e_f) = j_E(e_f).$$

It follows that the isomorphism of $F$ with $\Lambda^* E^\Lambda \Lambda$ is also a bundle isomorphism. This proves “only if” implication.

On the other hand, if $[F] = [\Lambda^* E^\Lambda \Lambda]$ for some $\mathbb{T}$-bundle over $H(0)$, then $F$ is the imprimitivity groupoid for $\Lambda^* E$. Then $(\Lambda^* E)/\mathbb{T} \cong \text{Gr(id}_H)$. This completes the proof. \qed

We will denote the trivial twist over $H$ by $H \times \mathbb{T}$.

**Corollary 7.3.** The collection of classes $[E]$ in $\text{Tw}(H)$ such that $E \sim_H H \times \mathbb{T}$ is a subgroup $W$ of $\text{Tw}(H)$. In particular, $\sim_H$ is an equivalence relation on $\text{Tw}(H)$, and $\mathcal{E}(H) := \text{Tw}(H)/\sim_H$ can be identified with the quotient group $\text{Tw}(H)/W$. Addition in $\mathcal{E}(H)$ is defined by

$$[[E]] + [[F]] = [[E \ast F]].$$

**Proof.** By the previous lemma, $W = \epsilon(H^1(H(0),S))$, and $\sim_H$ equivalence classes coincide with $W$-cosets. \qed

**Remark 7.4.** The subgroup $W$ in Corollary 7.3 can be nontrivial. For example, if $\psi : X \to T$ is a local homeomorphism and if $H$ is the corresponding equivalence
relation $R(\psi)$ in $X \times X$, then the six term exact sequence in either [22] or [38] implies that $W$ is the cokernel of the pull-back $\psi^* : H^1(T, \mathcal{S}) \rightarrow H^1(X, \mathcal{S})$. This can certainly be nontrivial; for example, let $X = T = \mathbb{T}^2$ and $\psi$ a double cover.

8. The second isomorphism result. The goal of this section is Theorem 8.3, which identifies $\mathcal{E}(G)$ with the special subgroup of $\text{Br}(G)$ consisting of those elementary $G$-bundles with vanishing Dixmier-Douady invariant.

It is well known that Morita equivalence classes of continuous-trace $C^*$-algebras with spectrum $T$ are parameterized by classes in $H^3(T; \mathbb{Z})$ (cf., e.g., [46, Theorem 3.5]). If $A = C_0(T; \mathcal{A})$ is a continuous-trace $C^*$-algebra with spectrum $T$, then the corresponding class in $H^3(T; \mathbb{Z})$ is denoted by $\delta(A)$ (or $\delta(\mathcal{A})$) and is called the Dixmier-Douady class of $A$ (or $\mathcal{A}$). In [46], $\delta(A)$ is constructed as the obstruction to $A$ being Morita equivalent to $C_0(T)$. The latter is equivalent to the existence of a Hilbert $C_0(T)$-module $\mathcal{H}$ such that $A \cong K(\mathcal{H})$. The classical interpretation of $\delta(\mathcal{A})$ is as the obstruction to the bundle $A$ arising as the bundle $K(\mathcal{H})$ associated to a Hilbert bundle $\mathcal{H}$ ([9, Theorem 10.7.15]). Of course, if $\mathcal{H} = C_0(T; \mathcal{H})$ and $A = C_0(T; \mathcal{A})$, then $K(\mathcal{H}) = C_0(T; K(\mathcal{H}))$. Despite the fourth author’s fondness for the methods of [46], the bundle interpretation is the more natural one here. However, as the Dixmier-Douady class is invariant under Morita equivalence (over the spectrum), we note that if $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent $C^*$-bundles satisfying Fell’s condition, then $\delta(A) = \delta(B)$. This allows us to make the following definition.

**Definition 8.1.** If $G$ is a second countable locally compact groupoid, then

$$\text{Br}_0(G) := \{ (A, \alpha) \in \text{Br}(G) : \delta(A) = 0 \},$$

and

$$\text{Br}_0(G) := \{ [A, \alpha] \in \text{Br}(G) : (A, \alpha) \in \text{Br}_0(G) \}.$$
in \(\mathcal{Br}_0(G)\), where
\[
\pi_{\gamma}(s(\gamma), T) := (r(\gamma), \pi_{\gamma}(T)).
\]
Clearly, if \((G^{(0)} \times K, \alpha) \in \mathcal{Br}_0(G)\), then \(\alpha = \tilde{\rho}\) for some \(\rho \in \mathcal{R}(G)\). It is a pleasant surprise that all elements of \(\mathcal{Br}_0(G)\) have a representative of the form of (8.1).

**Lemma 8.2.** If \(G\) is a second countable locally compact groupoid and if \((A, \alpha) \in \mathcal{Br}_0(G)\), then there is a \(\pi \in \mathcal{R}(G)\) such that
\[
(A, \alpha) \sim (G^{(0)} \times K, \tilde{\pi}).
\]

*Proof.* Since \(\delta(A) = 0\), there is a Hilbert bundle \(\mathcal{H}\) such that \(A = \mathcal{K}(\mathcal{H})\) ([9, Theorem 10.7.15]). Let \(H = C_0(G^{(0)}; \mathcal{H})\). Since \(A = C_0(G^{(0)}; A)\) is separable, \(H\) is countably generated and [28, Proposition 7.4(ii)] implies that the Hilbert \(C_0(G^{(0)})\)-modules \(H \otimes \ell^2\) and \(H_{C_0(G^{(0)})} := C_0(G^{(0)}) \otimes \ell^2\) are isomorphic. It follows from Remark 2.10 that
\[
(8.2) \quad \mathcal{K}(\mathcal{H}) \otimes (G^{(0)} \times \ell^2) \cong G^{(0)} \times \ell^2
\]
as Hilbert bundles over \(G^{(0)}\). Since \(\tau_{G^{(0)}} \sim (G^{(0)} \times K, \tau \otimes 1)\) acts as the identity in \(\mathcal{Br}(G)\) and since
\[
\mathcal{K}(\mathcal{H}) \otimes (G^{(0)} \times K) \cong \mathcal{K}(\mathcal{H} \otimes (G^{(0)} \times \ell^2)),
\]
it follows that
\[
(A, \alpha) = (\mathcal{K}(\mathcal{H}), \alpha) \sim \mathcal{K}(\mathcal{H}) \otimes (G^{(0)} \times K), \alpha \otimes (\tau \otimes 1)
\]
\[
\sim (\mathcal{K}(\mathcal{H} \otimes (G^{(0)} \times \ell^2)), \alpha \otimes (\tau \otimes 1)),
\]
which is covariantly isomorphic to \((\mathcal{K}(G^{(0)} \times \ell^2), \tilde{\alpha})\) for some action \(\tilde{\alpha}\). As remarked above, \(\tilde{\alpha}\) must be of the form \(\tilde{\pi}\) for some \(\pi \in \mathcal{R}(G)\). \(\square\)

If \(\pi \in \mathcal{R}(G)\), then there is an associated twist over \(G\) defined by
\[
E(\pi) := \{ (\gamma, U) \in G \times U(\ell^2); \, \pi_{\gamma} = \text{Ad}(U) \}\.
\]
The main result of this section is the following.

**Theorem 8.3.** Suppose that \(G\) is a second countable locally compact groupoid. Then
\[
[G^{(0)} \times K, \tilde{\pi}] \mapsto [E(\pi)]
\]
defines an isomorphism \(\Theta_G\) of \(\mathcal{Br}_0(G)\) onto \(\mathcal{E}(G)\).
The proof is fairly involved and is broken into a number of intermediate results. Naturally, the main tool is the relationship between Morita equivalence of systems and the elements of $\mathcal{R}(G)$. For this we need a couple of definitions.

We say that $\pi$ and $\rho$ in $\mathcal{R}(G)$ are exterior equivalent if there is a continuous map $u: G \to U(\ell^2)$ such that

(a) $\rho_\gamma = \text{Ad}(u_\gamma) \circ \pi_\gamma$ for all $\gamma \in G$, and

(b) $u_{\gamma\eta} = u_\gamma \pi_{\gamma}(u_\eta)$ for all $(\gamma, \eta) \in G^{(2)}$.

In this case, we write $\pi \sim_{\text{ext}} \rho$ and call $u$ a 1-cocycle implementing this equivalence.

It is not hard to see that $E(\pi)$ and $E(\eta)$ are isomorphic elements of $\text{Tw}(G)$ if and only if $\pi \sim_{\text{ext}} \rho$. To see this, suppose that $u$ is a 1-cocycle as above. Then $(\gamma, V) \mapsto (\gamma, u_\gamma V)$ is a continuous twist isomorphism of $E(\pi)$ onto $E(\rho)$. Conversely, if $\phi: E(\pi) \to E(\rho)$ is a twist isomorphism, then since $U(\ell^2)$ is a group,

$$\phi(\gamma, V) = (\gamma, u_\gamma V)$$

for some continuous function $u: G \to U(\ell^2)$. Since $\phi$ is also a groupoid homomorphism, it is immediate that $u$ is a 1-cocycle implementing an exterior equivalence between $\pi$ and $\rho$.

We say that $\pi$ and $\eta$ are cohomologous (written $\pi \sim_{\text{coh}} \rho$) if there is a continuous function $\phi: G^{(0)} \to \text{Aut}(K)$ such that

$$\pi_\gamma \circ \phi_{\delta(\gamma)} = \phi_{\tau(\gamma)} \circ \rho_\gamma.$$ 

It is not hard to see that $(G^{(0)} \times K, \pi)$ and $(G^{(0)} \times K, \bar{\rho})$ are covariantly isomorphic if and only if $\pi \sim_{\text{coh}} \rho$.

The main tool in the proof of Theorem 8.3 is the following.

**Proposition 8.4.** [32], [24, Proposition 14] Suppose that $G$ is a second countable locally compact groupoid, and that $\pi, \rho \in \mathcal{R}(G)$. Then

$$(G^{(0)} \times K, \pi) \sim (G^{(0)} \times K, \bar{\rho})$$

if and only if there is a $\sigma \in \mathcal{R}(G)$ such that

$$\pi \sim_{\text{ext}} \sigma \quad \text{and} \quad \sigma \sim_{\text{coh}} \rho.$$ 

**Proof.** The “if” direction is clear. For the “only if” direction, let $(X, V)$ implement an equivalence between $(G^{(0)} \times K, \pi)$ and $(G^{(0)} \times K, \bar{\rho})$, and let $X = C_0(G^{(0)}; X)$, $A = G^{(0)} \times K$, and $\bar{A} = C_0(G^{(0)}; \bar{A}) = C_0(G^{(0)}, K)$. Then $X$ is an $A_{\text{G0}}$-imprimitivity bimodule. By [3, Proposition 3.4], $X$ is isomorphic (as an $A - A$-imprimitivity bimodule) to $A_\kappa$ where $\kappa \in \text{Aut}(A)$. Since $X$ is an imprimitivity bimodule over $G^{(0)}$, the same must be true of $A_\kappa$ (using [42, Proposition 1.11],
for example). Then $\kappa \in \text{Aut}_{C_0(G^{(0)})}(A)$ and must be given by a continuous function $\phi_\gamma : G^{(0)} \to \text{Aut}(K)$. Therefore we may as well assume that $X = G^{(0)} \times K$ and that

$$
\langle (u, T), (u, S) \rangle_A = (u, \phi_u(T^*S)) \quad \langle (u, T), (u, S) \rangle_A = (u, TS^*)
$$

$$
\langle V_\gamma(s(\gamma), T), V_\gamma(s(\gamma), S) \rangle_A = \tilde{\rho}_\gamma((r(\gamma), T), (r(\gamma), S)) = (r(\gamma), \rho_\gamma(\phi(r(\gamma)(T^*S)))
$$

$$
\langle V_\gamma(s(\gamma), T), V_\gamma(s(\gamma), S) \rangle_A = \tilde{\pi}_\gamma((r(\gamma), T), (r(\gamma), S)) = (r(\gamma), \pi_\gamma(TS^*))
$$

with appropriate left and right actions.

Recall that we have an isomorphism $V : s^*A \to r^*A$ defined by $V(f)(\gamma) := V_\gamma(f(\gamma))$. Then it is straightforward to compute that $V(fg) = \tilde{\pi}(f)V(g)$, where $\tilde{\pi}(f)(\gamma) = \tilde{\pi}_\gamma(f(\gamma))$. Thus we can define $L : r^*A \to r^*A$ by $L(f) := V(\tilde{\pi}^{-1}(f))$. Note that by construction, $L(fg) = fL(g)$. Of course, $r^*A$ is isomorphic to $C_0(G, K)$ and is a left Hilbert $C_0(G, K)$-module. Furthermore, $L$ is an adjointable operator on $r^*A$:

$$
L^*(f)(\gamma) := \tilde{\pi}_\gamma^{-1}(V_{\gamma^{-1}}(f(\gamma))).
$$

However $L(r^*A)$ is isomorphic to $M(K(r^*A)) \cong M(C_0(G, K))$ by [16, Lemma 16]. But $M(C_0(G, K)) \cong C_b(G, B(\ell^2))$, where $B(\ell^2)$ denotes the bounded operators on $\ell^2$ with the $*$-strong topology [1, Corollary 3.4]. (Formally, we should have the strict topology on $B(\ell^2)$, but these topologies coincide on bounded subsets.) Therefore $L(f) = f \cdot u^*$ for some $u \in C_b(G, B(\ell^2))$, where $f \cdot u^*(\gamma) = (r(\gamma), Tu^*_\gamma)$ if $f(\gamma) = (r(\gamma), T)$. Alternatively, we can define $\bar{U}^* \in r^*A$ by $\bar{U}_\gamma = (r(\gamma), u^*_\gamma)$ and $L(f) = f\bar{U}^*$. In any case, we can compute that for all $f \in r^*A$

$$
\langle r^*_A(L(f), L(f)) \rangle_{r^*_A} = r^*_A(f^*, f) = r^*_A(L^*(f), L^*(f)),
$$

while on the other hand

$$
\langle r^*_A(L(f), L(f)) \rangle_{r^*_A}(\gamma) = f(\gamma)\bar{U}_\gamma \bar{U}^*_\gamma f(\gamma)^*.
$$

Therefore $u_\gamma u^*_\gamma = I$. Similarly, $u^*_\gamma u_\gamma = I$ and $u$ is a continuous function from $G$ into $U(\ell^2)$. Since $V(f) = \tilde{\pi}(f)\bar{U}^*$ and since $V$ is an action, it follows that $u_\gamma u = u_\gamma \pi_\gamma(u_\gamma)$. Thus if we define $\sigma \in \mathcal{R}(G)$ by

$$
\sigma_\gamma := \text{Ad}(u_\gamma) \circ \pi_\gamma.
$$

then $\pi \sim_{\text{coh}} \sigma$ by construction.

If $f, g \in X = C_0(G^{(0)}; X)$, then

$$
\langle V_\gamma(f(\gamma)), V_\gamma(g(\gamma)) \rangle = \tilde{\phi}_\gamma(V_\gamma(f(s(\gamma)))^*V_\gamma(g(s(\gamma)))
$$

$$
= \tilde{\phi}_\gamma(\bar{U}_\gamma \tilde{\pi}_\gamma(f(s(\gamma))^*g(s(\gamma)))\bar{U}^*_\gamma)
$$

$$
= \tilde{\phi}_\gamma(\bar{\pi}_\gamma(f(s(\gamma))^*g(s(\gamma)))).
$$
Since the left-hand side is also equal to \( \bar{\rho}_\gamma \circ \bar{\phi}_{s(\gamma)}(f(s(\gamma))^* g(s(\gamma))) \), it follows that \( \phi_{r(\gamma)} \circ \sigma_\gamma = \rho_\gamma \circ \phi_{s(\gamma)} \). That is, \( \sigma \sim_{\text{coh}} \rho \); this completes the proof. \( \square \)

**Corollary 8.5.** If \( G \) is a second countable locally compact groupoid, if \( \pi, \rho \in \mathcal{R}(G) \), and if

\[
(G^{(0)} \times K, \pi) \sim (G^{(0)} \times K, \rho),
\]

then \( \|E(\pi)\| = \|E(\rho)\| \) in \( \mathcal{E}(G) \).

**Proof.** Using the proposition, there is a \( \sigma \in \mathcal{R}(G) \) such that \( \pi \sim_{\text{ext}} \sigma \) and \( \sigma \sim_{\text{coh}} \rho \). Since exterior equivalence implies that \( E(\pi) \) and \( E(\sigma) \) are isomorphic as twists, it will suffice to show that \( E(\sigma) \sim_H E(\rho) \). Let \( \phi \colon G^{(0)} \to \text{Aut}(K) \) be such that

\[
\sigma_\gamma \circ \phi_{s(\gamma)} = \phi_{r(\gamma)} \circ \rho_\gamma.
\]

Let

\[
\Lambda := \{ (u, W) \in G^{(0)} \times U(\ell^2) : \phi_u = \text{Ad}(W) \}.
\]

Then \( \Lambda \) is a principal circle bundle over \( G^{(0)} \), and \( \epsilon(\Lambda) \) is the class of

\[
\{ [U, \gamma, V] \in U(\ell^2) \times G \times U(\ell^2)/\mathbb{T} : \text{Ad}(U) = \phi_{r(\gamma)} \text{ and Ad}(V) = \phi_{s(\gamma)} \},
\]

where \( t \cdot (U, \gamma, V) = (tU, \gamma, tV) \). Then

\[
[[U, \gamma, V], (\gamma, W)] \mapsto (\gamma, UWV^*)
\]

is a twist isomorphism of \( \epsilon(\Lambda) + E(\rho) \) onto \( E(\sigma) \). This completes the proof (Corollary 7.3). \( \square \)

**Corollary 8.6.** The map \( \Theta_G([G^{(0)} \times K, \bar{\pi}]) = [E(\pi)] \) is a well-defined injective homomorphism from \( \text{Br}_0(G) \) to \( \mathcal{E}(G) \).

**Proof.** Lemma 8.2 implies every class in \( \text{Br}_0(G) \) has the form \( [G^{(0)} \times K, \pi] \), and \( \Theta_G \) is well-defined by Proposition 8.4. To see that \( \Theta_G \) is a homomorphism, we need only consider

\[
(G^{(0)} \times K, \pi) \otimes (G^{(0)} \times K, \rho) = (G^{(0)} \times K \otimes K, \bar{\pi} \otimes \bar{\rho}).
\]

Fix a Hilbert space isomorphism \( W \colon \ell^2 \otimes \ell^2 \to \ell^2 \), and let \( \phi := \text{Ad}(W) \colon K \otimes K \to K \) be the corresponding isomorphism. Then (8.3) is covariantly isomorphic to \( (G^{(0)} \times K, \bar{\sigma}) \) where \( \sigma_\gamma = \phi \circ (\pi_\gamma \otimes \rho_\gamma) \circ \phi^{-1} \). Then

\[
E(\pi) + E(\rho) = \{ [U, \gamma, V] \in U(\ell^2) \times G \times U(\ell^2)/\mathbb{T} : \text{Ad}(U) = \pi_\gamma \text{ and Ad}(V) = \rho_\gamma \},
\]
where \( t \cdot (U, \gamma, V) = (tU, \gamma, t^{-1}V) \). Therefore

\[
[U, \gamma, V] \mapsto (\gamma, W(U \otimes B)W^*)
\]
is a twist isomorphism of \( E(\pi) + E(\rho) \) onto \( E(\sigma) \). It follows that \( \Theta_G \) is an homomorphism.

Thus to see that \( \Theta_G \) is injective, it suffices to show it has trivial kernel. However if \( \|[E(\pi)]\| = 0 \), then \( E(\pi) = \epsilon(c) \) for some \( c \in H^1(G^{(0)}, \mathcal{S}) \). But by [40, Theorem 2.1], there is a continuous function \( \sigma: G^{(0)} \to \text{Aut}(K) \) such that \( \alpha \in \text{Aut}_{C_0(G^{(0)})}(C_0(G^{(0)}), K) \) defined by \( \alpha(f)(u) = \sigma_u(f(u)) \) has Phillips-Raeburn obstruction \( \zeta(\alpha) = c \). Then \( c = [\Lambda] \) where

\[
\Lambda = \{ (u, W) \in G^{(0)} \times U(\ell^2): \text{Ad}(U) = \sigma_u \}.
\]

Then

\[
\epsilon(\Lambda) = \{ [R, \gamma, S] \in U(\ell^2) \times G \times U(\ell^2) / \mathbb{T}: \text{Ad}(R) = \sigma_{r(\gamma)} \text{ and } \text{Ad}(S) = \sigma_{s(\gamma)} \},
\]

where \( t \cdot (R, \gamma, S) = (tR, \gamma, tS) \). If we define \( \rho \in \mathcal{R}(G) \) by \( \rho_\gamma = \sigma_{r(\gamma)}\sigma_{s(\gamma)}^{-1} \), then

\[
[R, \gamma, S] \mapsto (\gamma, RS^*)
\]
is a twist isomorphism of \( E(\pi) = \epsilon(\Lambda) \) onto \( E(\phi) \). Since \( \rho \) is cohomologous to the trivial homomorphism, the result follows from (the easy half) of Proposition 8.4. \( \square \)

This leaves only surjectivity. At this point, our methods require the existence of a Haar system.

**Proposition 8.7.** Suppose that \( G \) is a second countable locally compact groupoid with a Haar system \( \{ \lambda^u \}_{u \in G^{(0)}} \). Then if \( E \in \text{Tw}(G) \), there is a \( \pi \in \mathcal{R}(G) \) such that \( E \cong E(\pi) \).

Recall from [24] that a *twist representation* of \( E \in \text{Tw}(G) \) is a pair \((\mathcal{K}, U)\) consisting of a Hilbert bundle \( \mathcal{K} \) and an action \( U \) of \( E \) on \( \mathcal{K} \) such that

\[
U_{t e} = t U_e \quad \text{for all } t \in \mathbb{T}.
\]

Then \( \alpha_{j(e)} := \text{Ad}(U_e) \) is a well-defined action of \( G \) on \( \mathcal{K}(\mathcal{K}) \), and \( (\mathcal{K}(\mathcal{K}), \alpha) \in \mathcal{B}_0(G) \). Naturally we say that \( (\mathcal{K}, U) \) is a *separable* twist representation if \( H = C_0(G^{(0)}); \mathcal{K} \) is separable.

**Lemma 8.8.** Suppose that \( G \) is a second countable locally compact groupoid and that \( E \in \text{Tw}(G) \). If \((\mathcal{K}, U)\) is a separable twist representation of \( E \) and if
is the associated element of $\mathfrak{B}r_0(G)$, then

$$\Theta(G([\mathcal{K}(\mathcal{H}), \alpha]) = \|E\|.$$  

**Proof.** It follows from (8.2) that there is a bundle isomorphism $V = \{V_u\}$ of $\mathcal{H} \otimes (G^{(0)} \times \ell^2)$ onto $G^{(0)} \times \ell^2$ which intertwines $U \otimes (\tau \otimes 1)$ with an action $\breve{U}$ on the trivial bundle. Note that

$$\breve{U}_e = V_{r(e)}(U_e \otimes (\tau_e \otimes 1))V_{s(e)}^*.$$  

Thus $\breve{U}$ is continuous and there exists a continuous homomorphism $W: E \to U(\ell^2)$ such that

$$\breve{U}_e(s(e), \xi) = (r(e), W_e(\xi)).$$  

This forces $W_{r(e)} = tW_e$. Then $\pi_{j(e)} := \text{Ad}(W_e)$ is an element of $\mathcal{R}(G)$ and

$$(\mathcal{K}(\mathcal{H}), \alpha) \sim (G^{(0)} \times K, \pi).$$

It is immediate that $e \mapsto (j(e), W_e)$ is a twist isomorphism of $E$ and $E(\pi)$.  

**Proof of Proposition 8.7.** At this point, all we need to do is to produce a twist representation of $E \in \text{Tw}(G)$. As in [37], we consider

$$C_c(G; E) := \{f \in C_c(E): f(t \cdot e) = tf(e)\}.$$  

Notice that $C_c(G; E)$ is a pre-Hilbert $C_0(G^{(0)})$-module:

$$\langle f, g \rangle_{C_0(G^{(0)})}(u) := \int_G \overline{f(e)}g(e) d\lambda^u(j(e)).$$

The completion is a Hilbert $C_0(G^{(0)})$-module $\mathcal{H} = C_0(G^{(0)}; \mathcal{H})$, where $\mathcal{H}$ is a Hilbert bundle with fibres $H_u$ equal to the completion of $C_c(r^{-1}(u))$ with respect to the restriction of (8.4). If $e \in E$, then define $U_e : C_c(r^{-1}(s(e))) \to C_c(r^{-1}(r(e)))$ by $U_e(f)(e') = f(e^{-1}e')$. Then $U_e$ is easily seen to extend to a unitary isomorphism of $H_{s(e)}$ onto $H_{r(e)}$ such that $U_{e'e} = U_e \circ U_{e'}$. To see that $U$ is a continuous action of $E$, we want to apply Lemma 2.13 to $U: s^*\mathcal{H} = C_0(G; s^*\mathcal{H}) \to r^*\mathcal{H} = C_0(G; r^*\mathcal{H})$. Note that $\Gamma := C_c(G; E)$ can be viewed as a dense subspace of sections in $\mathcal{H}$. If $f \in C_c(G; E)$ and $\phi \in C_c(E)$, then

$$\phi \otimes s^*f(e) := (e, \phi(e)f(s(e)))$$

is in $s^*\mathcal{H}$ and the collection $s^*\Gamma$ of such sections is dense in $s^*\mathcal{H}$ by [13, Proposition II.14.6].
Let \( C_c(G; E^{(2)}) \) be the pre-Hilbert \( C_0(E) \)-module consisting of continuous compactly supported functions on \( E^{(2)} = \{(e, e') \in E \times E: s(e) = r(e')\} \) such that \( f(e, t \cdot e') = tf(e, e') \) with

\[
\langle f, g \rangle_{C_0(E)}(e) := \int_G f(e, e') d\lambda \cdot s(e)(e').
\]

Each element of \( s^* \Gamma \) defines an element of \( C_c(G; E^{(2)}): \phi \otimes s^* f(e, e') = \phi(e)f(e') \). The image of \( s^* \Gamma \) is therefore dense in \( C_c(G; E^{(2)}) \) by [13, Proposition II.14.6]. This allows us to identify \( s^* H \) with the completion of \( C_c(G; E^{(2)}) \). Similarly, we can identify \( r^* H \) with the completion of \( C_c(G; E_s^* E) \) where \( E_s^* E := \{(e, e') \in E \times E: s(e) = s(e')\} \). Therefore \( L(f)(e, e') := f(e, e^{-1}e') \) extends to an isomorphism of \( s^* H \) onto \( r^* H \). It follows that \( U \) is a continuous action as claimed. It only remains to observe that \( U_{r,e} = tU_e; \) thus, \((\mathcal{H}, U)\) is a twist representation. \( \square \)

**Proof of Theorem 8.3.** \( \Theta_G \) is an injective homomorphism by Corollary 8.6 and surjective by Proposition 8.7. \( \square \)

In Section 10, we will show that \( Br(G) \) can be computed in terms of the \( \mathcal{E}(H) \) for groupoids \( H \) equivalent to \( G \). However if \( G^{(0)} \) is a reasonable space, it is easy to recover \( Br(G) \) from the generalized twists over a single equivalent groupoid.

**Corollary 8.9.** Suppose that \( G \) is a second countable locally compact groupoid with a Haar system, and that \( G^{(0)} \) is locally contractible. Then there is an equivalent groupoid \( H \) such that \( Br(G) \) is isomorphic to \( \mathcal{E}(H) \).

**Proof.** Let \( U = \{U_i\}_{i \in I} \) be a locally finite cover of \( G^{(0)} \) by contractible open sets. In fact, any cover for which \( H^3(U_i; \mathbb{Z}) = \{0\} \) for all \( i \) will do. Anyway, if \( (A, \alpha) \in \mathfrak{C}(G) \), then \( \delta(A|_{U_i}) = 0 \) for all \( i \). Then if \( X := \coprod U_i, \psi: X \to G^{(0)}, \) and \( \phi: G^\psi \to G \) are as in Corollary 6.6, then \( Z := \text{Gr}(\phi) \) is a \((G^\psi, G)\)-equivalence and \( \phi^Z([A, \alpha]) \in Br_0(G^\psi) \). Since \( \phi^Z \) is an isomorphism, \( Br_0(G^\psi) = Br(G^\psi) \). Now the result follows from Theorem 8.3. \( \square \)

**9. Ext(\( G, \mathbb{T} \)).** As we saw in Section 7, \( \sim_H \) is indeed a weaker notion of equivalence of extensions and is the “first step” alluded to earlier. The next step is to allow the groupoid \( H \) to vary over equivalent groupoids. The idea is that extensions of \( H' \) and \( H'' \) should be compared in \( \mathcal{E}(H) \) where \( H \) is a “larger” groupoid equivalent to \( H' \) and \( H'' \). This process is formalized by taking an inductive limit. It turns out to be appropriate to work with principal \( G \)-spaces rather than equivalent groupoids.

Let \( \mathcal{P}(G) \) be the collection of second countable principal right \( G \)-spaces which have a full equivariant \( s \)-system. By Proposition 5.2, these are precisely the principal \( G \)-spaces whose imprimitivity groupoid has a Haar system. (We will eventually need separability, and therefore second countability, for some of our
results. At the moment, some restriction on the “size” of the elements of $\mathcal{P}(G)$ is necessary in order that it be a set.)

Motivated by Example 6.1, if $X, Y \in \mathcal{P}(G)$, then we will say that $X$ is a \textit{refinement of} $Y$ if there is a continuous, open, $G$-equivariant surjection $\phi: X \to Y$. In this event we will write $Y \preceq X$, or $Y \preceq_{\phi} X$ if we want to keep the map $\phi$ in evidence. Note that $\mathcal{P}(G)$ is a directed set with respect to $\preceq$: given $X, Y \in \mathcal{P}(G)$, let $Z := X \ast Y = \{(x, y) \in X \times Y : s(x) = s(y)\}$. Then $Z$ is a principal $G$-space with respect to the diagonal action. If $\alpha$ and $\beta$ are full equivariant $s$-systems on $X$ and $Y$, respectively, then

$$\pi(f)(u) := \int_X \int_Y f(x, y) \, d\beta_u(y) \, d\alpha_u(x)$$

is a full equivariant $s$-system for $Z$; that is, $Z \in \mathcal{P}(G)$. Using the projection maps, we see that that $X \preceq Z$ and $Y \preceq Z$.

If $Y \preceq_{\phi} X$, then there is an associated groupoid homomorphism $\tilde{\phi}: G^X \to G^Y$ given by $\tilde{\phi}[x, \iota(z)] = [\phi(x), \iota(\phi(z))]$. Using Lemma 6.2, it is not difficult to see that $\tilde{\phi}$ induces an equivalence between $G^X$ and $G^Y$. (In fact, the equivalence is isomorphic to $X^*_G Y^{\text{op}}$ which is independent of $\phi$ (Lemma 10.2).) More generally, if $E$ is a twist over $G^Y$, then

$$E^\phi := \tilde{\phi}^*(E) = \{ (e, [x, \iota(z)]) \in E \times G^X : j(e) = [\phi(x), \iota(\phi(z))] \}$$

is a $\mathbb{T}$-groupoid over $G^X$ and we claim that the bundle map $\tilde{\phi}: E^\phi \to E$ given by $\tilde{\phi}(e, [x, \iota(z)]) = e$ also induces an equivalence between $E^\phi$ and $E$. To verify this, we check conditions (a)–(e) of Lemma 6.2. Conditions (a) and (b) are straightforward. For (c), suppose that $\{(e_i, [x_i, \iota(w_i)])\}$ satisfies $r(e_i, [x_i, \iota(w_i)]) = [x_i, \iota(x_i)] \to [x, \iota(x)]$ and $s(e_i, [x_i, \iota(w_i)]) = [w_i, \iota(w_i)] \to [w, \iota(w)]$ in $G^X$, while $e_i \to e$ in $E$. It follows that $[\tilde{\phi}(x_i), \iota(\phi(w_i))] \to [\phi(x), \iota(\phi(w))]$ in $G^Y$. By passing to a subnet and relabelling, we may assume that there are $\gamma_i \in G$ such that $[\phi(x_i), \iota(\phi(w_i))] \to (\phi(x), \iota(\phi(w)))$ in $Y \ast Y$. Similarly, we may assume that there are $\eta_i \in G$ such that $x_i \cdot \eta_i \to x$ in $X$. Then $\phi(x_i) \cdot \eta_i \to \phi(x)$. Since $Y$ is a principal $G$-space, we can assume that $\eta_i^{-1} \gamma_i \to s(x)$. Thus $x_i \cdot \gamma_i \to x$; similarly, $w_i \gamma_i \to w$ and $[x_i, \iota(w_i)] \to [x, \iota(w)]$ in $G^X$; this establishes (c).

To verify (d), suppose that $[x, \iota(x)] = [w, \iota(w)]$. Then $w = x \cdot \gamma$ for some $\gamma \in G$. But if $\tilde{\phi}(e, [x, \iota(x \cdot \gamma)]) \in E(0)$, then $e \in E(0)$ and $[\tilde{\phi}(x), \iota(\phi(x \cdot \gamma))] = [\phi(x), \iota(\phi(x))]$. Thus $\gamma = s(x)$ and (d) holds.

Finally if $u = \phi(x) = \phi(w)$, then $(i([u, \iota(u)], 1), [x, \iota(x)]) \in \ker(\phi)$ and maps $(i([u, \iota(u)], 1), [w, \iota(w)])$ to $(i([u, u], 1), [x, \iota(x)])$. Thus (e) is valid and we have proved the claim.

Now suppose that $Y \preceq_{\phi} X$ and that $E = \pi^*_G(X^\phi \Lambda)$ for some principal $\mathbb{T}$-bundle over $Y/G$. Then routine computations reveal that $E^\phi \cong \pi^*_G(X^\phi \Lambda^\phi \Lambda)$ where $\Lambda^\phi$ is the pull-back of $\Lambda$ with respect to the induced map of $X/G$ onto $Y/G$. Since
induces a homomorphism of $\text{Tw}(G^Y)$ to $\text{Tw}(G^X)$, we obtain a homomorphism $\Phi$ from $\mathcal{E}(G^Y)$ to $\mathcal{E}(G^X)$. We want to show that if $Y \preceq X$ and if $\Psi$ is the induced homomorphism from $\mathcal{E}(G^Y)$ to $\mathcal{E}(G^X)$, then $\Psi = \Phi$. Then we will have a canonical homomorphism $\Phi_{Y,X}: \mathcal{E}(G^Y) \rightarrow \mathcal{E}(G^X)$ whenever $Y \preceq X$. To show that $\Psi = \Phi$, it will suffice to see that $E^\phi \sim_{G^X} E^\psi$. However, we already know, for example, that $\text{Gr}(\tilde{\phi})$ is a $(E^\phi, E^\psi)$-$\mathbb{T}$-equivalence. Thus $Z := \text{Gr}(\tilde{\phi})^\phi \text{Gr}(\tilde{\psi})^\psi$ is a $(E^\phi, E^\psi)$-$\mathbb{T}$-equivalence, and we need to see that $Z/\mathbb{T} \simeq H$. However it is straightforward to check that

$$Z \cong \{([x, \iota(x)], e, [w, \iota(w)]) \in X/G \times E \times X/G:\quad r(e) = [\phi(x), \iota(\phi(w))] \text{ and } s(e) = [\phi(w), \iota(\phi(w))]\},$$

where

$$(f, [z, \iota(x)]) \cdot ([x, \iota(x)], e, [w, \iota(w)]) = ([z, \iota(z)], fe, [w, \iota(w)])$$

and

$$([x, \iota(x)], e, [w, \iota(w)]) \cdot (f, [w, \iota(z)]) = ([x, \iota(x)], ef, [z, \iota(z)]).$$

In particular,

$$Z/\mathbb{T} \cong \{([x, \iota(x)], h, [w, \iota(w)]) \in X/G \times G^Y \times X/G:\quad r(h) = [\phi(x), \iota(\phi(w))] \text{ and } s(h) = [\phi(w), \iota(\phi(w))]\}.$$

Then $[x, \iota(w)] \mapsto ([x, \iota(x)], [\phi(x), \iota(\phi(w))], [w, \iota(w)])$ is the required isomorphism of $G^X$ and $Z/\mathbb{T}$.

To summarize, we have proved most of the following.

**Theorem 9.1.** Suppose that $G$ is a second countable locally compact groupoid. If $Y \preceq X$ in $\mathcal{P}(G)$, then $\tilde{\phi}( [x, \iota(w)]) := [\phi(x), \iota(\phi(w))]$ defines a groupoid homomorphism of $G^X$ onto $G^Y$. If $E$ is a twist over $G^Y$, then let $E^\phi$ be the pull-back by $\tilde{\phi}$. Then $[E] \mapsto [E^\phi]$ is a well-defined homomorphism $\Phi_{Y,X}$ of $\mathcal{E}(G^Y)$ to $\mathcal{E}(G^X)$ which depends only on $X$ and $Y$ and not on the choice of $\phi$. In particular, $\{G^X, \Phi_{Y,X}\}$ is a directed system of groups.

**Proof.** It only remains to show that if $X \preceq Y \preceq Z$, then $\Phi_{X,Z} = \Phi_{Y,Z} \circ \Phi_{X,Y}$. However this follows immediately from the fact that $(E^\phi)^\psi \cong E^{\psi \circ \phi}$. ∎

As described in the introduction, the two dimensional cohomology of $G$ should be related to twists over $G$—or at least to generalized twists over imprimitivity groupoids $G^X$ for $X \in \mathcal{P}(G)$. Naturally, twists relative to different refinements should be compared in a common refinement. Just as in ordinary co-
homology theories, this process is formalized by taking an inductive limit. This is possible in view of the previous theorem.

Definition 9.2. If $G$ is a second countable locally compact groupoid, then

$$\text{Ext}(G, \mathbb{T}) := \lim_{\longrightarrow} \mathcal{E}(G^X).$$

10. Third isomorphism result. Using our first two isomorphism results (Theorems 4.1 and 8.3), we see that for each $X \in \mathcal{P}(G)$, there is an injective homomorphism $\tau_X: \mathcal{E}(G^X) \rightarrow \text{Br}(G)$ given by

$$\tau_X := \phi^{X^\text{op}} \circ \Theta^{-1}_{G^X}.$$

We will write $h_X$ for the canonical homomorphisms of $\mathcal{E}(G^X)$ into $\text{Ext}(G, \mathbb{T})$.

Theorem 10.1. The monomorphisms $\tau_X: \mathcal{E}(G^X) \rightarrow \text{Br}(G)$ are compatible with the directing homomorphisms $\Phi_{X,Y}: \mathcal{E}(G^X) \rightarrow \mathcal{E}(G^Y)$. In particular, there is an isomorphism $\tau$ of $\text{Ext}(G, \mathbb{T})$ onto $\text{Br}(G)$ defined by

$$\tau(h_X([E])) := \tau_X([E]).$$

Furthermore, the canonical homomorphisms $h_X: \mathcal{E}(G^X) \rightarrow \text{Ext}(G, \mathbb{T})$ are injective for all $X \in \mathcal{P}(G)$.

Lemma 10.2. If $X \preceq \phi Y$ and if $Z := \text{Gr}(\tilde{\phi})$, then $Z$ is isomorphic to $Y^*_G X^{\text{op}}$ as $(G^Y, G^X)$-equivalences.

Proof. Each element in $\text{Gr}(\tilde{\phi})$ has a representative of the form $([y, \iota(y)], [\phi(y), \iota(x)])$. Furthermore the map $([y, \iota(y)], [\phi(y), \iota(x)]) \mapsto [y, \iota(x)]$ is a well-defined homeomorphism of $\text{Gr}(\tilde{\phi})$ onto $Y^*_G X^{\text{op}}$. It is not difficult to see that this map intertwines the $G^Y$- and $G^X$-actions and is therefore an isomorphism of equivalences.

Lemma 10.3. If $X \preceq \phi Y$ and if $Z := \text{Gr}(\tilde{\phi})$, then for all $[A, \alpha] \in \text{Br}_0(G)$,

$$\Phi_{X,Y}(\Theta_{G^X}([A, \alpha])) = \Theta_{G^Y}(\phi^{Z}([A, \alpha])).$$

Proof. By Lemma 8.2 there is a $\pi \in \mathcal{R}(G)$ such that $(A, \alpha) \sim (X/G \times K, \tilde{\pi})$; therefore $\Theta_{G^X}([A, \alpha]) = [[E(\pi)]]$ where $E(\pi) = \{ ([x, \iota(w)], U) \in G^X \times U(\ell^2): \text{Ad}(U) = \pi_{[x, \iota(w)]} \}$. Therefore since $\phi^{Z}$ is well defined,

$$\phi^{Z}([A, \alpha]) = [A^{Z}, \alpha^{Z}] = [(X/G \times K)^Z, \tilde{\pi}^Z].$$
which by Lemma 6.5
\[ ([\bar{\phi}]_{Y/G})^*(X/G \times K), \tilde{\delta}^* \alpha] = [Y/G \times K, \bar{\rho}], \]
where \( \rho_{[y,z]} = \pi_{[\phi(y), \phi(z)]} \). Thus,
\[ \Theta_{G Y} \circ \phi^Z([A, \alpha]) = \|E(\rho)\| = \|E(\pi)\| \]
\[ := \Phi_{X,Y}(\|E(\pi)\|) = \Phi_{X,Y}(\Theta_{G X}([A, \alpha])). \quad \square \]

**Proof of Theorem 10.1.** Since the \( \tau_X \) are monomorphisms, in order to show that \( \tau \) is a well-defined monomorphism and that the \( h_X \) are injective, it will suffice to see that
\[ \mathcal{E}(G^X) \xrightarrow{\Phi_{X,Y}} \mathcal{E}(G^Y) \]
\[ \tau_X \quad \tau_Y \]
\[ \text{Br}(G) \]
(10.1)
commutes for all \( X \preceq Y \). But
\[ \tau_Y \circ \Phi_{X,Y} = \phi^{Y \text{op}} \circ \Theta^{-1}_{G^Y} \circ \Phi_{X,Y}; \]
which, by Lemmas 10.2, 10.3, and 4.5(b), is equal to
\[ \phi^{Y \text{op}} \circ \phi^X \circ \Theta^{-1}_{G^X}; \]
which, by Lemma 4.5(e), is equal to
\[ \phi^{X \text{op}} \circ \Theta^{-1}_{G^X} = \tau_X. \]
In other words, (10.1) commutes.

It only remains to verify that \( \tau \) is surjective. But if \( [A, \alpha] \in \text{Br}(G) \), then there is an \( X \in \mathcal{P}(G) \) such that \( \phi^X([A, \alpha]) = [A^X, \alpha^X] \in \text{Br}_0(G) \) (Corollary 6.6). But then Lemma 4.5(a)&(d) imply that
\[ \tau_X(\Theta_{G^X}([A^X, \alpha^X])) = [A, \alpha]. \]
This suffices as the image of \( \tau \) is the union of the images of the \( \tau_X (X \in \mathcal{P}(G)) \).
\[ \square \]

**11. Examples and additional remarks.** In this section, we collect together several examples that are of interest and that help to illustrate the general theory.
we have developed. We also outline an alternate perspective on our analysis that is revealed by the third author’s analysis of the dual groupoid of a C*-algebra.

We begin by reviewing the examples discussed in the introduction. So, let $G$ be a locally compact group. Since $G$ has but one unit, the collection of C*-G-bundles (Definition 2.14) coincides with the collection of automorphic representations of $G$ described at the very beginning. Further, since the one point space consisting of the unit of $G$ carries no cohomology, $\text{Br}(G) = \text{Br}_0(G)$, and Proposition 8.4 is the assertion that $\text{Br}(G)$ is really the collection of exterior equivalence classes of automorphic actions of $G$. (One does not have to worry about cohomology equivalence of actions, since for groups this amounts to unitary equivalence.) Thus there is complete consistency between the definition of $\text{Br}(G)$ that we gave in the introduction and the definition we gave of $\text{Br}(G)$ for general locally compact groupoids $G$ in the body of this paper. Theorem 8.3, coupled with the fact that $\text{Br}(G) = \text{Br}_0(G)$, guarantees that $\text{Br}(G)/\mathbb{Z} = E(G)$. On the other hand, because $G(0)$ reduces to a point, $E(G)$ reduces to $\text{Tw}(G)$ which, in turn, is isomorphic to the group of all group extensions of $T$ by $G$. Granted that this is isomorphic to $H^2(G, \mathbb{T})$, our analysis really does recapture the essence of Mackey’s (et al.) analysis of automorphic representations of locally compact groups.

However, it does more. The collection of groupoids that are equivalent to a locally compact group is precisely the collection of all transitive groupoids. This yields

**Proposition 11.1.** If $G$ is a transitive groupoid, then

$$\text{Br}(G) \cong \text{Br}(G_u) \cong H^2(G_u, \mathbb{T}),$$

where $G_u$ is the isotropy group of any (and hence every) unit.

**Proof.** This, of course, is a consequence of our Theorem 4.1 and Theorem 2.2B in [35]. (See also Theorem 3.5 in [7].) However, it should be emphasized that our standing assumption about groupoids, that they are locally compact, Hausdorff, second countable, and have Haar systems, plays an important rôle here. Theorem 2.2B of [35] says that a transitive groupoid $G$ satisfying these assumptions is equivalent to the isotropy group $G_u$ of any point $u \in G(0)$. Of course $G_u$ is a locally compact (second countable) group. So, Theorem 4.1 and the fact that $\text{Br}(G_u) \cong H^2(G_u, \mathbb{T})$ complete the proof.

Turning now to the other “extreme,” the case of spaces, observe that a groupoid $G$ is equivalent, in the sense of [35], to a space precisely when it is principal and acts freely and properly on its unit space. (Such a groupoid is called a proper principal groupoid.) Indeed, if $G$ is equivalent to a space $T$, say, then the fact that $G$ has these properties is straightforward to verify from the definition of the imprimitivity groupoid of an equivalence connecting $G$ and $T$. The
converse assertion is Proposition 2.2 of [36]. In fact, in this case, $G$ is equivalent to $G^{(0)}/G$. Now in the introduction, we already have noted that the Brauer group of a space $T$, $\text{Br}(T)$, is isomorphic to $H^2(T, S)$ via the Dixmier-Douady map $\delta$. (Here, and throughout this section we use $S$, as before, to denote the sheaf of germs of continuous $\mathbb{T}$-valued functions on whatever space is appropriate for the discussion.) This is Green’s formulation of Dixmier and Douady’s theory in [15]. This fact, Proposition 2.2 of [36], and Theorem 4.1 combine to prove

**Proposition 11.2.** If $G$ is a proper principal groupoid, then $\text{Br}(G) \cong H^2(G^{(0)}/G, S)$.

Several other applications of Theorem 4.1 are worth pointing out. Recall that a groupoid $\Gamma$ is called $r$-discrete if its range map $r$ is a local homeomorphism. This definition is at slight odds with [48], but now has been universally adopted. The reason is that $r$ is a local homeomorphism if and only if $\Gamma$ is $r$-discrete in the sense of [48] and counting measure is a Haar system for $\Gamma$. In [24], $r$-discrete groupoids in the sense we are using the term are called sheaf groupoids. In that paper, the first author developed a cohomology theory for $r$-discrete groupoids acting on sheaves of abelian groups for the purpose of identifying Brauer groups. Fix an $r$-discrete groupoid $\Gamma$, let $S$ denote the sheaf of germs of continuous $\mathbb{T}$-valued functions on $\Gamma^{(0)}$, and let $\Gamma$ act on $S$ in the usual way. Then the first author shows how to build a cohomology theory $H^*(\Gamma, S)$, which reduces to Grothendieck’s equivariant cohomology theory [17], $H^*_G(X, S)$, when $\Gamma$ is the transformation group groupoid $X \times G$ determined by the action of a discrete group $G$ on a locally compact space $X$. It may also be worthwhile to point out that certain equivariant sheaf cohomology groups have been “computed” in [17] (using spectral sequences) and in [24] (using an analog of Milnor’s $\text{lim}^1$ sequence). Theorem 4.19 of [24] may be phrased as

**Proposition 11.3.** If $\Gamma$ is an $r$-discrete groupoid, then $\text{Br}(\Gamma) \cong H^2(\Gamma, S)$.

Evidently this result contains the case when $\Gamma$ is a space, and so recovers Green’s perspective on the Dixmier-Douady theory. Also, this result together with our Theorem 10.1 suggests that for a general locally compact groupoid $G$ of the kind we have been discussing, there ought to be a cohomology theory $H^*(G, S)$ that generalizes $H^*(\Gamma, S)$. In any event, what we can assert is

**Proposition 11.4.** If $G$ is a locally compact groupoid of the kind we have been discussing, and if $G$ is equivalent, in the sense of [35] to an $r$-discrete groupoid $\Gamma$, then $\text{Br}(G) \cong H^2(\Gamma, S)$.

Of course, the proof is simply to cite Theorem 4.1. However, the range of applicability of this theorem is quite large. Many groupoids of interest are known to be equivalent to $r$-discrete groupoids. In particular, the holonomy groupoid of a foliation has this property. See [35] for a discussion of this. Also, the groupoids arising in the theory of hyperbolic dynamical systems and Smale spaces have this
property (see [41] and [53]). The exact range of applicability of this proposition
is not known. There are groupoids that are not equivalent to *r*-discrete groupoids.
In fact, a locally compact group is equivalent to an *r*-discrete groupoid if and
only if the group is itself discrete. (To see this, simply note, as we have noted
earlier, that a groupoid Π is equivalent to a group G iff Π is transitive. Further, Π
is equivalent to any isotropy group Π|_u. So, if Π is *r*-discrete, then G is equivalent
to the discrete group Π|_u. However, it is easy to see that two groups are equivalent
as groupoids if and only if they are isomorphic as topological groups.) As far as
we know, the situation for principal groupoids is unknown. There are, however,
shreds of evidence that make the problem quite piquant. For instance, Ramsay
[47] shows that if G is a locally compact principal groupoid, then there is a Borel
transversal. This means that there is a Borel set T ⊆ G(0) such that, among other
things, the Borel groupoid G|_T has countable orbits and is (Borel) equivalent
to G (via the Borel equivalence s^{-1}(T)). On the other hand, Schwartzman [54]
gives a necessary and sufficient condition for a flow to be isomorphic to a flow
built under a function. (See the discussion on page 282 and note that the horo-
cyclic flow on the cosphere bundle of a compact Riemann surface satisfies the
condition described there [14].) Thus there are flows whose associated transforma-
tion group groupoids are not equivalent to ℤ-transformation group groupoids
via topological transversals; i.e., a continuous flow need not be isomorphic to a flow built under a function (cf., Corollary 11.6). Whether a transformation
group groupoid determined by a flow is equivalent to an *r*-discrete groupoid in
some more complicated fashion remains to be determined.

Theorem 4.1 also yields the main result of [27, Theorem 1]. Recall, as we
remarked in Remark 3.4, if the groupoid under consideration is a transformation
group groupoid, T × G, say, then Br (T × G) is the equivariant Brauer group
Br_Γ (T) studied in [5], [27], and [39].

**Proposition 11.5.** [27, Theorem 1] Suppose that X is a second countable loc-
cally compact Hausdorff space, and that G and H are second countable, locally
compact groups. Suppose that G acts on the left of X, while H acts on the right, so
the actions commute, and suppose that both actions are free and proper. Then the
equivariant Brauer groups, Br_G (X/H) and Br_H (G\X), are isomorphic.

**Proof.** The point is that the transformation group groupoids, G × X/H and
G\X × H, are equivalent by Example 2.4 of [35]. Hence, Theorem 4.1 yields the
result.

This result and Corollary 6.1 of [5] yield

**Corollary 11.6.** Suppose that X is a second countable locally compact space
with countable (integral) cohomology groups, H^0(X; ℤ), H^1(X; ℤ), and H^2(X, ℤ).
If G is the transformation group groupoid associated with an action of the integers
ℤ on X, then Br (G) is isomorphic to H^2(Y, S) where Y is the space of the flow
obtained by suspending the homeomorphism giving the action of ℤ on X.
Remark 11.7. The hypotheses on $X$ are not very restrictive. For example, $H^n(X;\mathbb{Z})$ is countable for all $n$ if $X$ has the homotopy type of a compact metric space [43, Lemma 0.3].

Proof. Recall how the suspension is obtained. Let $\tau$ be the homeomorphism that defines the action of $\mathbb{Z}$ on $X$, let $\hat{Y} = X \times \mathbb{R}$, and define $\bar{\tau}$ on $\hat{Y}$ via the formula $\bar{\tau}(x, t) = (\tau x, t + 1)$. Then the $\mathbb{Z}$ action determined by $\bar{\tau}$ is free and proper and it commutes with the $\mathbb{R}$ action on $\hat{Y}$ given by translation in the second variable. The space $Y$ is the quotient $\hat{Y}/\mathbb{Z}$. Evidently, the action of $\mathbb{R}$ on $\hat{Y}$ is also free and proper. It is clear that $\mathbb{R}\setminus \hat{Y} \times \mathbb{Z}$ is $\times \mathbb{Z}$. The suspension of $\tau$ is the transformation groupoid $\mathbb{R} \times Y$, that is $\mathbb{R} \times \hat{Y}/\mathbb{Z}$. By Proposition 11.5, $\text{Br}(G) \cong \text{Br}_\mathbb{R}(Y)$. We claim that our hypotheses on the cohomology of $X$ guarantee that $H^1(Y;\mathbb{Z})$ and $H^2(Y;\mathbb{Z})$ are countable. This will suffice as we can then apply Corollary 6.1 of [5] to conclude that $\text{Br}_\mathbb{R}(Y)$ is isomorphic to $H^2(Y,S)$.

To prove the claim, let $q: \mathbb{R} \times \mathbb{R} \to Y$ be the quotient map. Define $Y_1 := q(X \times (0, \frac{1}{2}))$ and $Y_2 := q(X \times (\frac{1}{2}, 1))$. Then $Y_1 \cup Y_2 = Y$. Furthermore, $Y_1$ and $Y_2$ are each homeomorphic to $X \times (0, 1)$. Consequently, $H^n(Y_i;\mathbb{Z}) \cong H^n(X;\mathbb{Z})$ for $i = 1, 2$; thus the former is countable provided $n = 1, 2$. Similarly, $Y_1 \cap Y_2$ is homeomorphic to a disjoint union of two copies of $X \times (0, 1)$, and $H^n(Y_1 \cap Y_2;\mathbb{Z})$ is countable for $n = 1, 2$. The usual Mayer-Vietoris sequence [20, II.5.10] gives an exact sequence

$$
\cdots \to H^{n-1}(Y_1 \cap Y_2;\mathbb{Z}) \to H^1(Y_1 \cup Y_2;\mathbb{Z}) \to H^1(Y_1;\mathbb{Z}) \oplus H^1(Y_2;\mathbb{Z}) \to \cdots.
$$

It follows that $H^n(Y_1 \cup Y_2;\mathbb{Z}) = H^n(Y;\mathbb{Z})$ is countable for $n = 1, 2$ as claimed. \square

Recently there has been considerable interest in the groupoid $G_E$ and associated groupoid $C^*$-algebra $C^*(G_E)$ built from the path space corresponding to a directed graph $E$ [25, 26]. The $C^*$-algebras $C^*(G_E)$ can be quite interesting. For example, if $A$ is the connectivity matrix of a finite directed graph $E$, then one obtains the Cuntz-Krieger algebras $O_A$.

We show that the Brauer group of a groupoid associated to a directed graph is trivial. Let $E$ be a (nontrivial) countable directed graph. We use the notation of [25]: the set of vertices is denoted $E^0$, the set of edges is denoted $E^1$ and the structure maps are written $r, s: E^1 \to E^0$. We will assume that $E$ is locally finite (i.e., $r$ and $s$ are finite to one) and that it has no sinks (i.e., $s: E^1 \to E^0$ is surjective). One forms the associated groupoid $G_E$ as follows: the unit space $G_E(0)$ is identified with the infinite path space $E^\infty = \{(e_i)_{i=1}^\infty: e_i \in E^1 \text{ and } r(e_i) = s(e_{i+1}) \}$; the topology of $E^\infty$ is generated by a countable basis of compact open sets, the cylinder sets associated to finite paths. The graph groupoid consists of the collection of triples

$$
G_E = \{(e, n, f): e, f \in E^\infty, n \in \mathbb{Z}, \text{ and } e_i = f_{i-n} \text{ for } i \text{ sufficiently large} \};
$$
the range and source maps are given by \( r(e, n, f) = e \) and \( s(e, n, f) = f \), respectively. Then \((e, n, f)(f, m, g) = (e, n + m, g)\) and \((e, n, f)^{-1} = (f, -n, e)\). The topology of \( G_E \) is also generated by a countable basis of compact open sets, the cylinder sets associated to pairs of finite paths ending in a common vertex.

It will be of some use to have available the following factorization of elements of the groupoid which are not units. For each edge \( e_0 \in E^1 \) set \( V(e_0) = \{(e, 1, f) : e, f \in E^\infty, e_1 = e_0, \text{ and } e_i = f_{i-1} \text{ for } i > 1\} \subset G_E\). Given \((e, n, f) \notin G_E^{(0)}\), there are \(k, l \geq 0\) (at least one of which is nonzero), with \(n = k - l\) so that \(e_{k+i} = f_{l+i}\) for \(i \geq 1\) and elements \(\xi_i \in V(e_i)\) and \(\eta_j \in V(f_j)\) for \(1 \leq i \leq k\) and \(1 \leq j \leq l\) such that \((e, n, f) = \xi_1 \ldots \xi_k \eta_l^{-1} \ldots \eta_1^{-1}\). There is a unique factorization of this form of minimal length.

**Proposition 11.8.** Suppose that \( E \) is a locally finite directed graph with no sinks, and that \( G_E \) is the associated locally compact groupoid constructed above. Then

\[
\text{Br}(G_E) = \{0\}.
\]

**Proof.** Since the unit space \( E^\infty \) of \( G_E \) is zero-dimensional, \( H^2(E^\infty, S) = \{0\} \) and so \( \text{Br}(G_E) = \text{Br}_0(G_E) \). By the second isomorphism result (Theorem 8.3), \( \text{Br}_0(G_E) \) is a quotient of \( \text{Tw}(G_E) \); thus, it suffices to show that \( \text{Tw}(G_E) = \{0\} \). Suppose that \( F \) is a twist over \( G_E \) with twist map \( j : F \to G_E \); we will show that there is a trivializing section. That is, a continuous section of twist map which is also a groupoid homomorphism. Let \( \iota : E^\infty \to F \) be the embedding which identifies the unit spaces of \( G_E \) and \( F \). Since the twist viewed as a principal \( \mathbb{T} \)-bundle is trivial (\( G_E \) has a basis of compact open sets), there is continuous section \( \sigma : \bigcup_{x \in E^1} V(x) \to F \) of the bundle restricted to \( \bigcup_{x \in E^1} V(x) \). We extend \( \sigma \) as follows: for \( \gamma \in E^\infty = G_E^{(0)} \), set \( \sigma(\gamma) = \iota(\gamma) \), for \( \gamma = (e, n, f) \notin G_E^{(0)} \), using the above factorization \((e, n, f) = \xi_1 \ldots \xi_k \eta_l^{-1} \ldots \eta_1^{-1}\), set

\[
\sigma(e, n, f) = \sigma(\xi_1) \ldots \sigma(\xi_k) \sigma(\eta_l)^{-1} \ldots \sigma(\eta_1)^{-1}.
\]

One checks that \( \sigma \) is well defined, and that it is a morphism of groupoids as required. \( \square \)

We turn now to an alternate viewpoint on the material we have presented that is based on the concept of the dual groupoid of a \( C^* \)-algebra. So fix a \( C^* \)-algebra \( A \). As a set, the dual groupoid of \( A \), denoted \( G(A) \), consists of the extreme points \( \omega \) of the unit ball of its dual \( A^* \). By taking the left and right polar decompositions of the functional \( \omega \) we may write \( \omega = xu = uy \), where \( x \) and \( y \) are pure states and \( u \) is a partial isometry that is uniquely determined. We may therefore write the elements \( \omega \in G(A) \) as triples \( \omega = (x, u, y) \). Since the partial isometry \( u \) in such a triple is unique, the circle \( \mathbb{T} \) acts freely on \( G(A) \) via the formula \( t(x, u, y) := (x, tu, y) \). This makes \( G(A) \) a principal \( \mathbb{T} \)-space with quotient
$R(A)$, the graph of the unitary equivalence relation on the pure state space of $A$, $P(A)$. The groupoid structure on $G(A)$ is given by $(x,u,y)(y,v,z) = (x,u,v,y)$ and $(x,u,y)^{-1} = (y,u^*,x)$. The presentation of $G(A)$ as triples in this way gives the canonical extension

$$P(A) \times \mathbb{T} \looparrowright G(A) \looparrowright R(A).$$

In the case when $A$ is the $C^*$-algebra of compact operators, $K(H)$, on the Hilbert space $H$, its pure state space may be viewed as the projective space $P(H) := S(H)/\mathbb{T}$, where $S(H)$ is the unit sphere of $H$ endowed with the weak topology. The dual groupoid, $G(H) := G(A)$, then is the quotient of the trivial groupoid $S(H) \times S(H)$ by the diagonal action of $\mathbb{T}$. It is evident in this case that the natural topology on $G(H)$ gives $G(H)$ the structure of a Polish groupoid. The canonical extension in this case is

$$P(H) \times \mathbb{T} \looparrowright G(H) \looparrowright P(H) \times P(H).$$

We may view the dual groupoid $G(H)$ as the groupoid of minimal partial isometries $u$ with $r(u) = uu^*$ and $s(u) = u^*u$. That is, the map $(x,u,y) \mapsto u$ is an isomorphism of $G(H)$ with the collection of all minimal partial isometries on $H$.

Let $a \mapsto \gamma(a)$ be an automorphism of a $C^*$-algebra $A$. Then $\gamma$ acts on $A^*$ by transposition: For $\omega \in A^*$, $\omega \gamma$ is defined by the formula $\langle \omega \gamma, a \rangle = \langle \omega, \gamma(a) \rangle$. With this notation, we have $(x,u,y)\gamma = (x\gamma, \gamma^{-1}(u), y\gamma)$, for $(x,u,y) \in G(A)$. If $A = K(H)$ and $\alpha$ is a group homomorphism of the locally compact group $G$ into the projective unitary group $PU(H) \cong \text{Aut}(K(H))$, then as we have seen earlier, the Mackey obstruction $E(\alpha)$ of this projective representation is the pull back by $\alpha$ of the universal extension

$$1 \longrightarrow \mathbb{T} \longrightarrow U(H) \longrightarrow PU(H) \longrightarrow 1$$

given by $\text{Ad}$. Explicitly,

$$E(\alpha) = \{ (U, \gamma) \in U(H) \times G: \text{Ad}(U) = \alpha(\gamma) \}. $$

The multiplication law is just $(U, \gamma)(V, \gamma') = (UV, \gamma \gamma')$. As we mentioned in the introduction, this extension splits if and only if the action is implemented. Indeed, just define the isomorphism $\mathbb{T} \times G \rightarrow E$ by $(z, \gamma) \mapsto (zU(\gamma), \gamma)$ and conversely.

The dual groupoid provides an alternative but equivalent definition of the Mackey obstruction. We view $\alpha$ as an action of $G$ on the $C^*$-algebra $K(H)$. Therefore $G$ acts continuously on $G(H)$ on the right as above. The semi-direct product $G(H) \rtimes G$ has its multiplication law given by $(u, \gamma)(v, \gamma') = (uv, \gamma \gamma')$. One can check that $Z = S(H) \times G$ is a $\mathbb{T}$-equivalence between $E$ and $G(H) \rtimes G$ in the sense of Section 7. Here $E$ acts on the left of $Z$ according to the formula
\[(U, \gamma)(\xi, \gamma') = (U\xi, \gamma'\gamma)\] and \(G(H) \ast G\) acts on the right by \((\xi, \gamma)(\gamma' - 1)(\xi\langle \xi' \rangle), \gamma') = (\xi', \gamma'\gamma').\) (Of course, \(\langle \xi \rangle \langle \xi' \rangle\) denotes the rank one operator that maps \(\xi\) to \(\xi'.\)) Note that this alternative definition leads us to consider groupoids and principal \(G\)-spaces that are no longer locally compact.

More generally, let \(G\) be a locally compact groupoid. Let \((A, \alpha)\) be a representative element of \(\text{Br} (G)\) and write \(A = C_0(G^{(0)}; A)\). Then the dual groupoid \(G(A)\) is a groupoid bundle over \(G^{(0)}\) whose fiber above \(x\) is the dual groupoid \(G(A(x))\).

The topology on \(G(A)\) is defined naturally in terms of local fields of minimal partial isometries (which exist by virtue of Fell’s condition). For every \(\gamma \in G\), the isomorphism \(\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))\) induces the isomorphism \(\omega \mapsto \omega \gamma\) from \(G(A(r(\gamma)))\) onto \(G(A(s(\gamma)))\) such that \(\langle \omega \gamma, a \rangle = \langle \omega, \gamma(a) \rangle\). If we write as before an element \(\omega\) of \(G(A(x))\) as a minimal partial isometry \(u \in A(x)\), then \(\omega \gamma\) is given by \(\gamma^{-1}(u)\). We define the semi-direct product groupoid \(G(A) \ast G\) as the set of pairs \((u, \gamma)\) such that \(p(u) = r(\gamma)\). The topology is the relative topology.

The multiplication law is given as above:

\[ (u, \gamma)(\gamma^{-1}(v), \gamma') = (uv, \gamma\gamma') \]

and the inverse is given by

\[ (u, \gamma)^* = (\gamma^{-1}(u), \gamma^{-1}) \].

It is a Polish groupoid extension:

\[ P(A) \times \mathbb{T} \longrightarrow G(A) \ast G \longrightarrow R(A) \ast G. \]

Note, in this case, that \(R(A)\) is a Polish equivalence relation and that \(R(A) \ast G\) is a Polish groupoid equivalent to \(G\) via the equivalence \(P(A) \ast G\).

In our definition of \(\text{Ext} (G, \mathbb{T})\), we consider only principal \(G\)-spaces that are \(locally\ compact\) and that possess an equivariant \(s\)-system. Therefore, this extension \(G(A) \ast G\) does not quite define an element of \(\text{Ext} (G, \mathbb{T})\). However, we have the following result:

**Proposition 11.9.** Let \(G\) be a locally compact groupoid with Haar system and let \((A, \alpha)\) be a \(C^*\)-\(G\)-system defining an element of \(\text{Br} (G)\). Then, the extension \(G(A) \ast G\) defined by the dual groupoid as above is \(\mathbb{T}\)-equivalent, in the sense of Section 7, to any of the extensions provided by \((A, \alpha)\) via the isomorphism of Theorem 10.1.

**Proof.** Let us first assume that \((A, \alpha)\) determines an element in \(\text{Br}_0 (G)\). Thus, there exists a continuous field \(\mathcal{H}\) of Hilbert spaces \(x \mapsto H(x)\) such that \(A(x) = K(H(x))\). The dual groupoid \(G(\mathcal{H}) := G(A)\) is then a bundle over \(G^{(0)}\) with fiber \(G(H(x))\). Just as in the case of a group discussed above, one can check that \(Z := S(\mathcal{H}) \ast G\), where \(S(\mathcal{H})\) is the sphere bundle of \(\mathcal{H}\), is a \(\mathbb{T}\)-equivalence between
$G(3\xi) \star G$ and the extension

$$E = \{(U, \gamma) \in U(3\xi) \star G: \text{Ad}(U) = \alpha(\gamma)\}$$

in the sense defined in Section 7. In the general case, we may apply Corollary 6.6 to find a groupoid $\mathcal{G}$ and a $(\mathcal{G}, G)$-equivalence $Z$ such that $(A^Z, \alpha^Z) \in \text{Br}_0(\mathcal{G})$. One can check that $G(A) \star Z$ is the desired $\mathbb{T}$-equivalence between $G(A) \star \mathcal{G}$ and $G(A^Z) \star G$.

Thus, if one is willing to widen the class of “equivalent extensions” to include groupoids that are not necessarily locally compact, one finds that the association $(A, \alpha) \mapsto G(A) \star G$ gives a canonical representative for an element in $\text{Ext}(G, \mathbb{T})$ directly in terms of the elementary $C^*-G$-bundle $(A, \alpha)$. This perspective was emphasized by the third author in [49], where, in the special case that the groupoid is a space, $T$, he showed how to view the equivalence, $G(A) \star Z$, in the preceding proof as the Dixmier-Douady invariant for $A = C_0(T; A)$.

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