Color degeneracy of topological defects in quadratic band touching systems

Bitan Roy

1Department of Physics, Lehigh University, Bethlehem, Pennsylvania, 18015, USA

(Dated: April 29, 2020)

We study two-dimensional fermionic quadratic band touching (QBT) systems in the presence of vortex and skyrmion of insulating and superconducting masses. A prototypical example of such systems is the Bernal bilayer graphene that supports eight zero-energy modes in the presence of a mass vortex with the requisite U(1) symmetry. Inside the vortex core, additional ten masses that close an SO(5) algebra can develop local expectation values by splitting the zero modes in five and ten different ways by lifting its SO(4) and SU(2) chiral symmetries, respectively. In particular, each SU(2) chiral symmetry can be broken by three distinct copies of chiral-triplet mass orders, giving rise to the notion of the color or flavor degeneracy among the competing orders. By contrast, a skyrmion of three anticommuting masses supports additional six masses in its core, and possesses an SU(2) isospin quantum number, besides the usual generalized U(1) charge. Consequently, charge $4e$ Kekule pair-density-waves can develop in the skyrmion core of Néel layer antiferromagnet, while a skyrmion of quantum spin Hall insulator in addition supports a mundane $s$-wave pairing. We also analyze the internal algebra of competing orders in the core of these defects on checkerboard or Kagome lattice that supports only a single copy of QBT.

I. INTRODUCTION

The transition between two distinct broken symmetry phases, even though commonly believed to be first-order, can be continuous when two order-parameters are related via a chiral rotation (see below). Such unconventional continuous phase transition possibly takes place through proliferation of real space singularities, known as topological defects, when one order resides inside the defect core of the other, giving rise to the notion of dual orders and deconfined criticality [1, 2]. One well studied example of such dual or competing orders is the Néel antiferromagnet and valence bond solid in two-dimensional frustrated spin models of insulating systems [3–7].

The notion of competing orders becomes more transparent, when they can be described as composite objects of underlying fermionic degrees of freedom. In this respect, massless Dirac fermions, realized in monolayer graphene (MLG), $d$-wave superconductor, honeycomb Kondo-Heisenberg model, constitute an ideal platform to capture the competing orders [8–20]. Namely, in a multicomponent spinor basis (arising from the sublattice or orbital, valley, and spin degrees of freedom) ordered phases are represented by Dirac bilinears. Two competing orders are then described by mutually anticommuting Dirac matrices, which when in addition anticommute with the Dirac Hamiltonian, are named masses. Naturally, the generators of the chiral rotation between any two competing masses commute with the Dirac Hamiltonian, manifesting its chiral symmetry [21, 22].

However, representation of ordered phases in terms of Dirac matrices is not limited to the Dirac materials, rather quite natural for any multiband systems. And here we address competing orders that can be found in the core of topological defects, such as vortex and skyrmion, in planar fermionic systems, where the valence and conduction bands in the normal state display biquadratic touching, also known as Luttinger materials. The Bernal stacked bilayer graphene (BLG) is an ideal place to realize such unusual gapless fermionic excitations [23]. Theoretical studies have shown that quadratic band touching (QBT) in Bernal BLG can be unstable toward the formation of various broken symmetry phases, the exact nature of which depends on a number of microscopic details, such as the relative strength of various finite range components of the Coulomb interactions [24–39], even if that may require a finite interaction couplings [40–42]. A number of ordered phases has also been observed in experiments in the presence or absence of external magnetic and electric fields [43–50]. Therefore, understanding the role of topological defects and competing orders in QBT systems is a timely topic of pressing importance.

Various theoretical works in the recent past have discussed the possibilities of topological defects and dual orders in BLG in the absence [51–53] as well as in the presence of magnetic fields [54, 55], and also predicted a charge $4e$ $s$-wave superconductor induced by skyrmion of topological quantum spin Hall insulator (QSHI) [51, 52]. Despite these commendable efforts, the internal algebra of competing orders in QBT systems still remains unexplored, and constitutes the central theme of the present work. Here we use the real Clifford algebra of anticommuting matrices to address this question [56].

The most tantalizing outcomes are possibly the following. We find that skyrmions of both Néel antiferromagnet and QSHI in BLG accommodate charge $4e$ spin-singlet pair-density-waves, assuming two distinct Kekule patterns on honeycomb lattice, whereas the later one in addition sustains a mundane $s$-wave pairing. On the other hand, all three singlet pairings support topological QSHI in the mixed phase, while the Néel order can only be realized in the vortex phase of spin-singlet Kekule superconductors. Thus, in BLG the competing orders are not unique, giving rise to the notion of a flavor or color degeneracy (defined precisely below) among them. Now we present an extended summary of our main findings.
A. Extended summary of results

The differences in the internal algebra of competing orders in MLG and BLG root into the dispersion of non-interacting fermions, which respectively scales linearly and quadratically with the momentum in these two systems. Consequently, the number of mass matrices that can develop a uniform spectral gap at the band touching points via spontaneous lifting of discrete and/or continuous symmetries are $36$ [12] and $28$ [57] in MLG and BLG, respectively, despite both of them possessing the same symmetry [23], see Table I. 1 Also in stark contrast to Dirac systems, we show that QBT does not necessarily encounter the fermion doubling, and one can realize a two-component QBT for spinless fermions on two-dimensional, such as checkerboard and Kagome [58], lattices. A simple algebraic proof of this statement is offed in Appendix A. Here, such a realization is named 'single-flavored QBT', while the QBTs in Bernal BLG is coined 'valley-degenerate QBT'.

(1) In a single-flavored QBT system, a vortex of any two mutually anticommuting masses [see Table I] hosts two states at precise zero-energy and each of them are two-fold degenerate, yielding total four zero-energy states. 2 But, three competing mass matrices can split the zero-energy manifold by developing finite expectation values, and they close an SU(2) algebra, see Fig. 1. For example, the zero-energy modes bound to the vortex of an s-wave superconductor supports all three components of the QSHI. On the other hand, a skyrmion of QSHI accommodates the s-wave pairing.

(2) A real space vortex of two anticommuting masses with the requisite U(1) symmetry accommodates doubly-degenerate four, thus total eight states at zero energy in a valley-degenerate QBT system. The sub-space of zero-energy states altogether supports ten masses. For example, a vortex of translational symmetry breaking Kekule current orders sustains the layer polarized state (1), Néel layer antiferromagnet (3), the real (3) and imaginary (3) components of the spin-triplet $f$-wave pairing. Quantities in the parentheses indicate the number of matrices required to describe a particular order, see Table I. Other examples are discussed in Sec. IV B 1. Irrespective of these details, the ten masses close an SO(5) algebra. 3

(3) Any set of ten masses can be organized into five sets of four mutually anticommutating masses, closing an SO(4) algebra [see Fig. 2 and Appendix B]. Therefore, if the system chooses to split the zero-energy manifold by lifting its SO(4) chiral symmetry, there are five such choices. On the other hand, an SO(5) group has ten SO(3) or SU(2) subgroups. Thus, zero-energy subspace can also be split by breaking its SU(2) chiral symmetry in ten different ways. But, each set of SU(2) generators rotate between three distinct set of three mutually anticommuting masses, see Fig. 3. Hence, each SU(2) chiral symmetry of zero modes can be lifted in three different patterns, leading the notion of the flavor or color degeneracy among the competing orders inside the vortex core. For example, either the Néel layer antiferromagnet or the real and imaginary components of triplet $f$-wave pairing can split the zero modes bound to the vortex of singlet Kekule current orders by spontaneously lifting the SU(2) spin rotational symmetry.

(4) In the presence of an underlying skyrmion of three mutually anticommutating masses, there is no bound state at zero energy. But, the bound states at finite energies possess an SU(2)$\otimes$U(1) chiral symmetry. While the generator of U(1) rotation captures the generalized charge of the skyrmion, the SU(2) generators correspond to its isospin, see Fig. 4. Altogether a skyrmion core supports six induced masses. The U(1) charge causes rotation among three distinct copies of induces masses, while each SU(2) generator rotates between two distinct flavors of masses. Thus by developing finite expectation value of its charge or isospin quantum number, a skyrmion core can support degenerate flavors of competing induced masses, giving rise the color degeneracy among competing orders in its core. Consequently, one can construct multiple copies of five mutually anticommutating masses [see Sec. IV B 2], the right number to sustain a Wess-Zumino-Witten (WZW) term in $d = 2$ [59, 60], after integrating out the fermions [58, 61]. However, due to the color degeneracy one can construct charge-WZW and isospin-WZW terms (defined more precisely in Sec. IV B 2), which can be responsible for continuous and possibly deconfined phase transitions between competing phases that can also be tested in quantum Monte Carlo simulations [62, 64].

| Mass order | Matrix | $I_{uu}$ | $\hat{S}$ | $I_T$ |
|------------|--------|---------|--------|------|
| QAHI       | $\tau_0 \otimes \sigma_0 \otimes \alpha_2$ | $-$ | $\checkmark$ | $-$ |
| QSHI       | $\tau_1 \otimes \sigma \otimes \alpha_2$ | $-$ | $\times$ | $+$ |
| s-wave pairing | $\tau_1 \otimes \sigma \otimes \alpha_0$ | + | $\checkmark$ | $(-, +)$ |

TABLE I: All masses in a single-flavored QBT system in checkerboard lattice [55], and their transformation under the exchanges of two sublattices ($I_{uu}$), rotation of spin quantization axes ($\hat{S}$), and reversal of time ($I_T$). Here, $+(-)$ corresponds to even(odd), and $\checkmark$ and $\times$ reflect weather a mass operator preserves a particular symmetry or not, respectively. Three sets of Pauli matrices ($\tau_i$), ($\sigma_\mu$) and ($\alpha_\mu$) operate on the Nambu, spin and sublattice indices, respectively, with $\mu = 0, \ldots, 3$. The real and imaginary components of the s-wave pairing appear with $\tau_1$ and $\tau_2$, respectively.

1 The maximal number of mutually anti-commuting masses is five in MLG [12, 16], while that is six in BLG [57].

2 Such two-fold degeneracy of each zero mode is protected by a pseudo time-reversal symmetry [51], discussed in Sec. III A. In a Dirac material, such as MLG, the six masses bound to the vortex zero-modes close an SU(2)$\otimes$SU(2)$\otimes$SO(4) algebra [10].

3 In a Dirac material, such as MLG, the six masses bound to the vortex zero-modes close an SU(2)$\otimes$SU(2)$\otimes$SO(4) algebra [10].
TABLE II: All masses in Bernal BLG (supporting valley-degenerate QBTs) that anticommute with $\hat{H}_{0}^{\text{BLC}}$, see Eq. (1) [57]. First eight candidates represent insulating, and last four to fully gapped superconducting states. Among the insulating masses, first four are spin-singlet, while the remaining ones are spin-triplet. From the third to seventh column we display the transformations of these masses under the exchanges of the layers ($I_{uv}$), valleys ($I_{\eta}$), rotation of the spin quantization axis ($I_{\sigma}$), and reversal of time ($I_{T}$). The Pauli matrices $\{\tau_{\mu}\}$, $\{\eta_{\mu}\}$ and $\{\sigma_{\mu}\}$ operate on Nambu or particle-hole, spin, valley and layer indices, respectively, where $\mu = 0, \cdots, 3$. Rest of the notations are the same as in Table I.

### B. Organization

The rest of the paper is organized in the following way. In the next section, we discuss the microscopic models leading to both single-flavored and valley-degenerate QBTs, and all possible mass orders therein, see Tables I and II. Topological defects, such as vortex and skyrmions, and the bound states in their cores are discussed in Sec. III. Sec. IV is devoted to the derivation of the internal algebra among competing orders in the defect cores using the real representation of the Clifford algebra. We support these findings through some concrete examples in Sec. V and summarize the results in Sec. VI. Additional discussions are relegated to the appendices.

## II. MASS IN QBT SYSTEMS

We begin the discussion by considering microscopic models for QBTs. Unlike the situation in two-dimensional Dirac materials, displaying linear touching of the valence and conduction bands, for which the minimal representation must be four component (for spinless fermions), a two-component QBT can be realized in two-dimensional lattices with finite-range hopping. Such realizations are compatible with the requirement of the time-reversal symmetry, see Appendix A. Nevertheless, it is also conceivable to realize four-component QBTs in two-dimensional lattices, such as in Bernal stacked BLG in the presence of intralayer nearest-neighbor and interlayer dimer hopping elements. In this system two copies of two-component QBT are realized near two inequivalent corners, also known as the valleys, of the hexagonal Brillouin zone [23]. Below we write down the low-energy models of these systems and tabulate all possible mass orders therein, see Table I and II.

### A. Single-flavored QBT

The simplest microscopic model, supporting a single copy of QBT can be realized on a checkerboard lattice. To accommodate all possible masses in such a system, we introduce an eight-component Nambu spinor $\Psi = (\Psi_{p}, \Psi_{h})^{\top}$, where $\Psi_{p}$ and $\Psi_{h}$ are two four-component spinors, with $\Psi_{p}^{\top} = (\Psi_{p,\uparrow}, \Psi_{p,\downarrow})$ and $\Psi_{h}^{\top} = (\Psi_{h,\uparrow}, -\Psi_{h,\downarrow})$. The two-component spinors are

$$\Psi_{p,\sigma} = [u_{\sigma}, v_{\sigma}] (k), \quad \Psi_{h,\sigma} = [u_{\sigma}^\dagger, v_{\sigma}^\dagger] (-k).$$

(1)

Here $u_{\sigma}(k)$ and $v_{\sigma}(k)$ correspond to fermion annihilation operators on two sublattices of the checkerboard lattice with momentum $k$, measured from the band touching $\Gamma = (0, 0)$ point, and spin projection $\sigma = \uparrow, \downarrow$. In this basis, the low-energy Hamiltonian near the $\Gamma$ point is

$$\hat{H}_{0}^{\text{SF}} = \tau_{3} \otimes \sigma_{0} \otimes [\alpha_{1}d_{2}(k) + \alpha_{3}d_{1}(k)],$$

(2)

where

$$d_{1}(k) = \frac{k^{2} - k_{y}^{2}}{2m_{3}}, \quad d_{2}(k) = \frac{2k_{x}k_{y}}{2m_{3}},$$

(3)

and $m_{\ast}$ has the dimension of mass. Three sets of Pauli matrices $\{\alpha_{\mu}\}$, $\{\sigma_{\mu}\}$ and $\{\tau_{\mu}\}$ operate on the sublattice, spin and Nambu indices, respectively, where $\mu = 0, 1, 2, 3$, and `$\otimes$' represents a direct or tensor product. Throughout we neglect the particle-hole ansitropy.
The above Hamiltonian ($\hat{H}_0^{\text{SF}}$) is invariant under the (1) exchange to two sublattices ($u \leftrightarrow v$), generated by $I_{uv} = \tau_0 \otimes \sigma_0 \otimes \alpha_1$, under which $(k_x, k_y) \rightarrow (k_y, k_x)$, (2) reversal of time, generated by the antiunitary operator $I_T = (\tau_0 \otimes \sigma_2 \otimes \alpha_0) K$, where $K$ is the complex conjugation, and (3) rotation of the spin quantization axis, generated by $\tilde{S} = \tau_0 \otimes \sigma \otimes \alpha_0$.

Various mass orders in this system that uniformly gap the QBT point and their transformation under various discrete ($I_{uv}, I_T$) and continuous ($\tilde{S}$) symmetries of $\hat{H}_0^{\text{SF}}$ are shown in Table IV. Altogether, a single-flavored QBT supports three physical masses, namely the quantum anomalous Hall insulator (QAHI), QSHI, and spin-singlet s-wave pairing. But, it requires six matrices to describe them \[55\]. Notice that QAHI only anticommutates with $\hat{H}_0^{\text{SF}}$, but commutes with remaining two masses. Hence, for the following discussion it does not play any role.

### B. Valley-degenerate QBTs: Bernal BLG

Next we focus on the QBTs in Bernal BLG. Unlike the previous example, BLG accommodates two copies of QBT, yielding the valley degeneracy. The corresponding sixteen dimensional low-energy Hamiltonian reads

$$\hat{H}_0^{\text{BLG}} = \eta_3 \otimes \sigma_0 \otimes [[\eta_0 \otimes \alpha_1] d_1(\mathbf{k}) + (\eta_3 \otimes \alpha_2) d_2(\mathbf{k})],$$

where the newly introduced set of Pauli matrices $\{\eta_{\mu}\}$ operate on the valley index. The sixteen-component Nambu spinor basis is $\Psi = (\Psi_p, \Psi_y)^\dagger$, where $\Psi_p^\dagger = (\Psi_{p,+}, \Psi_{p,-})$ and $\Psi_y^\dagger = (\Psi_{y,+}, -\Psi_{y,-})$ are two eight-component spinors. The four-component spinors are

$$\Psi_{p,\sigma} = [u_{+,\sigma}, v_{+,\sigma}, u_{-,\sigma}, v_{-,\sigma}] (\mathbf{k}),$$

$$\Psi_{y,\sigma} = [v_{+,\sigma}, u_{+,\sigma}, v_{-,\sigma}, u_{-,\sigma}] (-\mathbf{k}),$$

where $u_{\pm,\sigma}(\mathbf{k})$ and $v_{\pm,\sigma}(\mathbf{k})$ are the fermionic annihilation operators on two complimentary layers, with Fourier component localized around the nonequivalent valleys at $\pm \mathbf{K}$, spin projection $\sigma = \uparrow, \downarrow$, and momentum $\mathbf{k}$, measured from the corresponding valley.

The non-interacting Hamiltonian ($\hat{H}_0^{\text{BLG}}$) remains invariant under the following symmetries.

1. Exchange of two layers: $I_{uv} = \tau_0 \otimes \sigma_0 \otimes \eta_0 \otimes \alpha_1$.
2. Exchange of two valleys: $I_K = \tau_0 \otimes \sigma_0 \otimes \eta_1 \otimes \alpha_0$.
3. Reversal of time: $I_T = (\tau_0 \otimes \sigma_2 \otimes \eta_1 \otimes \alpha_0) K$.
4. Rotation of the spin quantization axis: $\tilde{S} = \tau_0 \otimes \sigma \otimes \eta_0 \otimes \alpha_0$.
5. $U(1)$ translational symmetry: $I_{tr} = \tau_3 \otimes \sigma_0 \otimes \eta_3 \otimes \alpha_0$.

It should be noted that the exchange of two layers and valleys are accompanied by the inversions of momentum axes $k_y \rightarrow -k_y$ and $k_z \rightarrow -k_z$, respectively. Therefore, all mass orders can be classified according to their transformation under these symmetries, see Table IV.

Altogether Bernal BLG supports twelve different symmetry breaking mass orders, among which eight (four) are insulators (superconductors). But, one requires 28 matrices to describe them \[57\]. Note that in BLG Kekule valence bond solids \[69\] (both spin-singlet and spin-triplet) no longer represent masses. They are replaced by Kekule current orders ($K_E, K_O, \tilde{K_E}$ and $\tilde{K_O}$). Furthermore, two Kekule spin-triplet mass superconductors in the pairing channels in MLG \[70\] are replaced by spin-singlet Kekule pairings in BLG ($sK_1, sK_2, pK_1$, and $pK_2$), reducing the number of mass matrices in BLG to 28 from 36 in MLG \[12\]. These differences will play important roles in the internal algebra of competing orders inside the core of topological defects, which we discuss next.

### III. TOPOLOGICAL DEFECTS

In this section, we introduce topological defects inside various mass ordered phases. Specifically, we consider vortex and skyrmion, and highlight the structure of the bound states in their cores. This will allow us to construct the internal algebra of competing orders in the core of these defects, discussed in Secs. IV and V.

#### A. Vortex

The effective single-particle Hamiltonian for a vortex-like point defect involving two anticommuting masses in QBT systems assume the following universal form

$$H_{\text{vor}} = \gamma_1 \frac{\partial^2_y}{2m_s} + \gamma_2 \frac{\partial^2_y}{2m_s} + |m(r)| \left( \gamma_3 C_{n\phi} + \gamma_5 S_{n\phi} \right),$$

where $C_{n\phi} = \cos(n\phi)$, $S_{n\phi} = \sin(n\phi)$, with $\phi$ as the polar angle and $r$ as the radial coordinate in the $xy$ plane. The radial profile of $m(r)$ is $m(r \rightarrow 0) = 0$ and $m(r \rightarrow \infty) = m_0$, otherwise arbitrary, where $m_0$ is a constant. For concreteness, we consider vortex of unit vorticity ($n = 1$), as it is the most stable and energetically favored topological defect. Here, $\gamma_3$'s are Hermitian matrices satisfying the anticommuting Clifford algebra $\{\gamma_j, \gamma_k\} = 2\delta_{jk}$. Therefore, $\gamma_3$ and $\gamma_5$ are the mass matrices. Since $H_{\text{vor}}$ involves four mutually anticommuting $\gamma$ matrices, their minimal dimensionality is four.

It was shown by Herbut and Lu \[51\] that due to the QBT in the normal phase, the above Hamiltonian describing a unit vortex supports two modes at precise zero energy. Such two-fold degeneracy of the zero-energy manifold and rest of the spectrum is assured by an antitunitary operator $J_K = U K$, where $U$ is a unitary operator, such that $[H_{\text{vor}}, J_K] = 0$ and $J_K^2 = -1$. Therefore, $J_K$...
the vortex Hamiltonian can always be cast as a orthog-

tion involves only four mutually anticommuting matrices,

single-particle Hamiltonian describing a vortex configura-

dimensional Hermitian matrices. Since the effective

will find out that competing orders can split them

system supports their competing mass

the vortex, constituted by the

FIG. 1: Triangle of three mutually anticommuting masses,

M3, M4 and M5, that also anticommute with the eight-
dimensional vortex Hamiltonian \( H^{\text{Num}}_{\text{vor}} \) [see Eq. (14)] in a

single-flavored QBT system. Three arms represent SU(2) ro-

motions, generated by \( E_{jk} = iM_jM_k \).

plays the role of a pseudo time-reversal operator. Exist-

tence of such antiunitary operator does not depend on

the choice of representation of the \( \gamma \) matrices. With-

out any loss of generality, we choose \( \gamma_1 \) and \( \gamma_3 \) to be

purely imaginary, and \( \gamma_1 \) and \( \gamma_2 \) to be purely real. Then,

\( U = \gamma_1\gamma_3 \). While \( J_K \) endows each energy eigenvalue

a two-fold degeneracy, existence of the midgap states is

guaranteed by the spectral symmetry, generated by an

unitary operator \( \gamma_0 \), such that \( \{ H_{\text{vor}}, \gamma_0 \} = 0 \). In par-

icular, \( \gamma_0 = \gamma_1\gamma_2\gamma_3\gamma_5 \) is the fifth anticommuting four-
dimensional Hermitian \( \gamma \) matrix [7].

From the above discussion, we can also infer the

competing order in core of the mass vortex. Since

\( \{ H_{\text{vor}}, \gamma_0 \} = 0 \), the two zero-energy modes are the eigen-

states of \( \gamma_0 \) with eigenvalues +1 or −1. Therefore, filled or

empty zero modes yields a finite expectation value of

the mass operator \( \gamma_0 \), i.e. \( \langle \gamma_0 \rangle \neq 0 \). Then in the core of

the vortex, constituted by the \( \gamma_3 \) and \( \gamma_5 \) masses, the system

supports their competing mass \( \gamma_0 \), since \( \{ \gamma_0, \gamma_j \} = 0 \)

for \( j = 1 \) and \( 2 \) (thus qualifying as a mass), as well as

\( j = 3 \) and \( 5 \) (hence, a competing order). For the minimal

model in Eq. (6), a finite expectation value of \( \gamma_0 \) mass

places the zero modes at a finite energy. However, for

single-flavored and valley-degenerate QBT systems, soon

we will find out that competing orders can split them

symmetrically about the zero energy.

A single-flavored QBT system is described by eight-
dimensional Hermitian matrices. Since the effective

single-particle Hamiltonian describing a vortex configura-
tion involves only four mutually anticommuting matrices,

and their irreducible representation is four-dimensional,

the vortex Hamiltonian can always be cast as an orthog-
onal sum of two copies of \( H_{\text{vor}} \). As a result, the core of

a mass vortex hosts four zero energy modes. Following

the same line of arguments, one can convince herself that

there exists eight zero energy modes in the core of a mass

vortex in valley-degenerate QBT systems. In the follow-
ings sections, we will discuss the possible competing or-
ders and their internal algebra in such higher-dimensional

zero-energy manifolds.

B. Skyrmion

Next we consider a skyrmion of mass orders. It involves

three mutually anticommuting mass matrices. The cor-

responding effective single-particle Hamiltonian is

\[
H_{\text{skyr}} = \gamma_1 \frac{\partial^2_x - \partial^2_y}{2m_x} + \gamma_2 \frac{2\partial_x \partial_y}{2m_x} + m_1(r)\gamma_3 + m_2(r)\gamma_3 + m_3(r)\gamma_5. \tag{7}
\]

For an underlying skyrmion of unit skyrmion number

\[
m(r) = m_0 \left( \frac{-2\lambda}{r^2 + \lambda^2} C_\gamma, \frac{2r\lambda}{r^2 + \lambda^2} S_\gamma, \frac{r^2 - \lambda^2}{r^2 + \lambda^2} \right). \tag{8}
\]

where the parameter \( \lambda \) determines its core size. Note

that \( H_{\text{skyr}} \) exhausts all five mutually anticommuting four-
dimensional \( \gamma \) matrices. Therefore, we cannot find any

unitary (or antiunitary) matrix that fully anticommutes with

\( H_{\text{skyr}} \) and all states (including the bound ones) re-

side at finite energies. Nonetheless, they continue to en-

joy the two-fold degeneracy, as \( [H_{\text{skyr}}, J_K] = 0 \).

One can render the loss of the spectral symmetry in the

following way. Say, we begin with two zero-energy modes

(these eigenstates of \( \gamma_0 \) with eigenvalues +1 or −1)

bound to the core of a vortex, and subsequently intro-

duce the third component of the mass \( \gamma_0 \) such that it

changes sign as we approach the boundary of the system

\( r \to \infty \) from its origin \( r = 0 \). Therefore, addition of the \( \gamma_0 \) mass besides constituting a skyrmion texture,

pushes the zero-modes bound to a vortex to finite ener-
gies. As a direct consequence of the spectral asymmetry,

the core of the skyrmion becomes electrically charged of

charge +e or −e (depending on the sign of \( m_0 \)). The

corresponding operator is \( Q_{\text{elec}} = I_4 \), where \( I_n \) is an n-
dimensional identity matrix, which is the product of five

mutually anticommuting matrices appearing in \( H_{\text{skyr}} \)

\[
Q_{\text{skyr}} = Q_{\text{elec}} = \gamma_1\gamma_2\gamma_3\gamma_5\gamma_0 = I_4. \tag{9}
\]

where \( Q_{\text{skyr}} \) is the generalized charge of a skyrmion [17].

For single-flavored and valley-degenerate QBT sys-
tems, the Hamiltonian operator in the presence of a back-
ground skyrmion of three mutually anticommuting mass

orders can be cast as direct or orthogonal sum of two

and four copies of \( H_{\text{skyr}} \), respectively. Such decomposition

allows skyrmion to acquire chiral charges, while being
electrically neutral. Note that any Hermitian op-
operator that commutes with the noninteracting Hamilto-
nian (\( H_0^{\text{SF}} \) and \( H_0^{\text{BFG}} \)), such as \( \hat{S} \), generates the chiral
symmetry of the system, and qualifies as a chiral charge of the skyrmion. On the other hand, any mass operator that anticommutes with $H_{\text{sky}}$ can develop a finite expectation value in the core of a skyrmion.

IV. REAL CLIFFORD ALGEBRA AND COMPETING ORDERS

In this section, we derive the internal algebra of competing orders in the core of topological defects using the real representation of Clifford algebra. In order to describe various insulating and superconducting mass gaps within a unified representation, it is useful to double the number of fermionic components (Nambu doubling), and include both particle and hole in the spinor representation. The resulting massive Nambu Hamiltonian is

$$H^\text{Nam}_m (k) = H^\text{Nam}_0 (k) + mM. \tag{10}$$

The kinetic energy part of $H^\text{Nam}_m (k)$ is given by

$$H^\text{Nam}_0 (k) = H_0 (k) \oplus H_0^T (-k) \equiv \sum_{j=1,2} \Gamma_j d_j (k), \tag{11}$$

where $\Gamma_8, M$ are eight and sixteen dimensional Hermitian matrices for single-flavored and valley-degenerate QBT systems, respectively. Here

$$H_0 (k) = \beta_1 d_1 (k) + \beta_2 d_2 (k), \tag{12}$$

and $\beta_j$ are mutually anticommuting four and eight dimensional matrices for these two systems. The Hermitian matrix $M$ represents a mass order, when it satisfies the anticommutation relation $\{ \Gamma_j, M \} = 0$. The fully gapped spectra of $H^\text{Nam}_m (k)$, namely $\pm \sqrt{\left( k^2 / (2m) \right)^2 + m^2}$, then extend over positive and negative energies.

By construction the Nambu Hamiltonian $H^\text{Nam}_m (k)$ preserves the particle-hole symmetry, generated by the antiunitary operator $I_{ph} = (\sigma_1 \otimes I_n) K$, and $\{ H^\text{Nam}_m (k), I_{ph} \} = 0$, with $n = 4$ and 8 for single-flavored and valley-degenerate QBT systems, respectively, and $\sigma_1$ is the real off-diagonal Pauli matrix. Since, $I_{ph}^2 = +1$, it is always possible to find a representation, known as ‘Majorana representation’, in which $I_{ph} = K$, and $H^\text{Nam}_m (k)$ is purely imaginary. In real space representation, the operators $d_j (k \to -i \nabla)$ is real. Thus two matrices appearing in the kinetic energy $\{ \Gamma_1 \text{ and } \Gamma_2 \}$, as well as any mass matrix $M$ are imaginary. This is strikingly different from the Dirac system, where due to the linear dependence of $d_j (k) \sim k_j$ on spatial components of momentum, $\Gamma_j$’s are real.

A. Single-flavored QBT

For single-flavored QBT the Nambu Hamiltonian in Eq. (11) is eight dimensional, and $i\Gamma_j$ are purely real. Since $iM$ is also real, we first seek to answer the following questions. What is the maximal number of $q$, so that for $p \geq 0$ the dimensionality of the real representation is eight, and mutually anticommuting $p + q$ matrices satisfy the Clifford algebra $C(p, q)$? The answer is seven. They constitute $C(0, 7)$ Clifford algebra. Two of them, namely $\Gamma_1$ and $\Gamma_2$, be two imaginary kinetic energy matrices, and $M_j$ are five mutually anticommuting mass matrices, with $j = 1, \cdots, 5$. There exists another imaginary Hermitian matrix $i\Gamma_1 \Gamma_2$ that anticommutes with the kinetic energy and satisfies the requisite criteria of a mass matrix. But, $i\Gamma_1 \Gamma_2$ commutes with five other mass matrices. Therefore, single-flavored QBT system altogether supports six mass matrices, which we show explicitly in Table I. The $i\Gamma_1 \Gamma_2$ mass can be identified as the QAHI.

Next we consider topological defects in such a system.

1. Vortex

If we construct a vortex out of two mutually anticommuting masses, say $M_1$ and $M_2$, according to

$$M_{\text{vor}} (x) = |m(r)| [M_1 C_\phi + M_2 S_\phi]. \tag{13}$$

Then the vortex Hamiltonian, defined as

$$H^\text{Nam}_{\text{vor}} = H^\text{Nam}_0 (k \to -i \nabla) + M_{\text{vor}} (x), \tag{14}$$

supports four zero-energy modes. Note that four mutually anticommuting matrices in $H^\text{Nam}_{\text{vor}}$ close a $C(4, 0)$ algebra. Thus the eight-dimensional matrix $H^\text{Nam}_{\text{vor}}$ can be decomposed as orthogonal sum of two identical copies of the four-dimensional Hamiltonian $H_{\text{vor}}$, shown in Eq. (6). Each of them hosts two zero energy states [51]. Consequently, the eight dimensional vortex Hamiltonian $H^\text{Nam}_{\text{vor}}$ supports $2 \times 2 = 4$ states at precise zero energy.

Any mass matrix that anticommutes with $H^\text{Nam}_{\text{vor}}$ can acquire finite expectation value inside the vortex core by splitting the zero-energy manifold. The number of such matrices is only three, and they are $M_3, M_4, M_5$, which together close an SU(2) algebra. They can be placed at three vertices of a triangle, see Fig. 1. Three generators of the SU(2) rotations are $\{ E_{34}, E_{45}, E_{53} \}$, where $E_{jk} = iM_j M_k$. Also note that $E_{jk}$ commute with $H^\text{Nam}_{\text{vor}}$, thus generating its chiral symmetry.

2. Skyrmion

Next we proceed to construct a skyrmion out of three mutually anticommuting masses, say $M_1, M_2$ and $M_3$,

4 The Clifford algebra $C(p, q)$ defines a set of $p + q$ mutually anticommuting matrices, where $p(q)$ of them squares to $+1(-1)$. 

imaginary mass, namely \(i\Gamma_1\Gamma_2\), which anticommutes with the noninteracting Hamiltonian, but commutes with rest of the masses. It is identified as the QAII, and does not play any role in the forthcoming discussion.

### 1. Vortex

First we focus on vortex constituted by two mutually anticommuting mass matrices \(M_1\) and \(M_2\), following the protocol in Eqs. (13) and (14). But, now all matrices \((\Gamma_1, \Gamma_2, M_1, M_2)\) are sixteen-dimensional. Since these four matrices satisfy \(C(4,0)\) algebra, \(H_{\text{vor}}^{\text{Nam}}\) can be cast as orthogonal sum of four copies of \(H_{\text{vor}}\), see Eq. (6). Consequently, \(H_{\text{vor}}^{\text{Nam}}\) supports eight zero-energy modes.

Any operator, say \(X\), that anti-commutes with \(H_{\text{vor}}^{\text{Nam}}\) can acquire a finite expectation value by splitting the subspace of the zero-energy states. To establish the internal structure of such competing orders we need to search for all imaginary matrices \(X\) that satisfy the anticommutation relations \(\{X, \Gamma_j\} = \{X, M_j\} = 0\), for \(j = 1, 2\). One can immediately find at least four candidates for \(X\): \(M_3, M_4, M_5, \) and \(M_6\). However, they do not exhaust all possibilities. In terms of four imaginary matrices appearing in \(H_{\text{vor}}^{\text{Nam}}\), we can define another Hermitian matrix

\[
E = \Gamma_1\Gamma_2 M_1 M_2. \tag{17}
\]

Even though \(\langle H_{\text{vor}}^{\text{Nam}}, E \rangle = 0\), \(E\) by construction is a real. So, \(E\) is not a mass. Nevertheless, we can define the following six imaginary Hermitian matrices

\[
M_{j k} = i E M_j M_k, \tag{18}
\]

where \(3 \leq j, k \leq 6\), but with \(j \neq k\) and \(j > k\), which anticommute with \(H_{\text{vor}}^{\text{Nam}}\). Hence, altogether there are ten masses, anicommuting with \(H_{\text{vor}}^{\text{Nam}}\). Any one of them can acquire finite expectation value by splitting the eight-dimensional subspace of zero energy states.

In order to demonstrate the two-fold degeneracy of the zero-energy manifold, we search for all possible candidates for the sixteen-dimensional unitary operator \(U\),

---

5 This is so because seven mutually anticommuting matrices satisfy the constraint \(i\Gamma_1\Gamma_2 M_1 M_2 M_3 M_4 M_5 \propto I_8\).
such that we can define the pseudo time-reversal operator $J_K = U K$, satisfying $J_K^2 = -1$ and $[H_V, J_K] = 0$. Since $\Gamma_1, \Gamma_2, M_1, M_2$ are imaginary and $J_K^2 = -1$, the imaginary unitary operator $U$ must satisfy $[H_V, U] = 0$. Due to the enlarged dimensionality of $H_{\text{Non}}$, in fact there are ten possible choices of $U$, given by

$$U \in \{ M_3, M_4, M_5, M_6, M_{34}, M_{35}, M_{36}, M_{45}, M_{46}, M_{56} \}.$$ 

Therefore, any one of the ten masses that anticommutes with $H_{\text{Non}}$ can be a candidate for $U$. Since all mass matrices are Hermitian and imaginary $J_K^2 = -1$. If one of them, say $M_3$, acquires local expectation value $(m_3)$ near the vortex core, there are still six candidates for $U$, namely $M_j$ and $M_{3j}$ with $j = 4, 5, 6$, such that $[J_K, H_{\text{Non}}] + m_3 M_3] = 0$. Therefore, split zero energy modes continue to enjoy the two-fold degeneracy.

A question arises quite naturally. What is the internal algebra among these 10 competing masses? Notice each member of the set of 10 masses matrices, say $M_3$, anticommutes with 6 other masses (namely, $M_4, M_5, M_6, M_{34}, M_{35}, M_{36}$), and commutes with 3 other masses (namely, $M_{45}, M_{46}, M_{56}$). Such an algebra is the defining property of an SO(5) group, constituted by product matrices. Therefore, 10 masses that can develop finite expectation value within the zero-energy subspace close an SO(5) algebra. By contrast, in a Dirac system (such as MLG) six mass orders in the vortex core satisfy SU(2)$\otimes$SU(2) algebra [10], which is isomorphic to SO(4). The 10 generators of SO(5) rotations (each of them causing U(1) rotation between two specific mutually anticommuting masses) are given by

$$G \in \{ EM_3, EM_4, EM_5, EM_6, E_{34}, E_{35}, E_{36}, E_{45}, E_{46}, E_{56} \},$$

where $E_{jk} = i M_j M_k$. Each generator anticommutes (commutes) with 6 (3) other generators, and they close an SO(5) algebra. See also Appendix [3].

An SO(5) group has five SO(4) subgroups. Between any two of them there exists three common generators, precisely the number of common planes between two four-dimensional subspaces of a five-dimensional sphere. The generators of each SO(4) subgroup are shown in blue in Fig. 2 and in Appendix [3] we explicitly show that each of them satisfies SO(4)$\cong$SU(2)$\otimes$SU(2) algebra. In addition, one can construct the following five ‘four-tuplets’ of four mutually anticommuting masses belonging to the SO(4) subgroups

$$\begin{align*}
(a) & \equiv \{ M_3, M_4, M_5, M_6 \}, \\
(b) & \equiv \{ M_3, M_{34}, M_{35}, M_{36} \}, \\
(c) & \equiv \{ M_4, M_{34}, M_{45}, M_{46} \}, \\
(d) & \equiv \{ M_5, M_{35}, M_{45}, M_{56} \}, \\
(e) & \equiv \{ M_6, M_{36}, M_{46}, M_{56} \}.
\end{align*}$$

In Fig. 2 masses are shown in red, and four masses from each SO(4) subgroup reside at the vertices of a square.
Four masses belonging to any SO(4) subgroup are mutually anticommuting. This reconciles with the fact that the maximal number of mutually anticommuting mass matrices is six in BLG. Therefore, if the system chooses to split the zero-energy manifold by breaking SO(4) chiral symmetry of $H_{\text{vor}}^{\text{Nam}}$, it can be accomplished in five different patterns.

On the other hand, an SO(5) group has ten SO(3) or SU(2) subgroups.\(^6\) Their generators are the following

\[
(I) \equiv (E_{34}, E_{35}, E_{45}), (II) \equiv (E_{35}, E_{36}, E_{45}), (III) \equiv (E_{45}, E_{46}, E_{56}), (IV) \equiv (E_{46}, E_{34}, E_{36}), (V) \equiv (E_{45}, E_{55}, E_{45}), (VI) \equiv (E_{55}, E_{56}, E_{36}), (VII) \equiv (E_{45}, E_{56}, E_{46}), (VIII) \equiv (E_{35}, E_{55}, E_{35}), (IX) \equiv (E_{35}, E_{65}, E_{36}), (X) \equiv (E_{35}, E_{45}, E_{34}).
\]

For any $j = 1, \cdots, X$, three SU(2) generators ($A_a$, $\mathbb{I}_a$) satisfy the group algebra $[A_a, \mathbb{I}_b] = i\epsilon_{a,b,\delta} A_\delta$, where $\epsilon_{a,b,\delta}$ is the fully antisymmetric Levi-Civita symbol. As shown in Fig. 3, each set of SU(2) generators can rotate between three distinct flavors of three mutually anticommuting masses, occupying the vertices of three triangles $j\alpha$, where $\alpha = A, B$ and $C$. Therefore, if the system chooses to split the zero-energy manifold by breaking its SU(2) chiral symmetry, there are ten choices ($j = 1, \cdots, X$). And each SU(2) chiral symmetry can be broken by three flavors ($jA, jB$ and $jC$) of chiral-triplet masses. Such extra three-fold degeneracy among triplet mass orders is termed the flavor or color degeneracy. In Sec. V B 1, we show its explicit examples.

2. Skyrmion

Now we focus on skyrmion [see Eq. (15)], with $\Gamma_1$, $\Gamma_2$, $M_1$, $M_2$ and $M_3$ as sixteen-dimensional imaginary Hermitian matrices. In the presence of an underlying skyrmion there is no bound state at zero energy. But, the ones at finite energies still possess two-fold degeneracy, guaranteed by the pseudo time-reversal operator $J_K$, with $U \in \{M_4, M_5, M_6, M_{34}, M_{35}, M_{36}\}$. There are two sets of three mutually anticommuting masses that close SU(2) algebra and also anticommute with $H_{\text{skyrm}}^{\text{Nam}}$. They are

\[
\{M_4, M_5, M_6\} \text{ and } \{M_{34}, M_{35}, M_{36}\}.
\]

and placed at three vertices of two triangles, see Fig. 4. The generators of SU(2) rotations ($E_{45}, E_{56}, E_{64}$) are, however, identical for two SU(2) triangles. In addition to the intra-triangle SU(2) symmetry, there exist an inter-triangle U(1) symmetry, generated by

\[
Y = EM_3 = \Gamma_1 \Gamma_2 M_1 M_2 M_3, \quad (22)
\]

that rotates between two masses residing at identical vertices of two triangles. The generator of the U(1) symmetry is the product of five mutually anticommuting matrices appearing in $H_{\text{skyrm}}^{\text{Nam}}$. As $[H_{\text{skyrm}}^{\text{Nam}}, E_{ij}] = [H_{\text{skyrm}}^{\text{Nam}}, Y] = 0$, and $[E_{ij}, Y] = 0$, the bound states in the core of the skyrmion possesses SU(2) $\otimes$ U(1) chiral symmetry, which gets broken by the induced masses. While $Y$ determines the generalized charge of the skyrmion ($Q_{\text{skyrm}}$), three generators of the SU(2) rotations correspond to its isospin. The notion of the generalized charge is also germane in Dirac [17] and single-flavored QBT systems [see Sec. IV A 2]. But, ‘isospin’ of the skyrmion is unique to valley-degenerate QBT systems.

Notice that the U(1) charge of the skyrmion rotates between three pairs of distinct induced masses, residing at any two equivalent vertices of two triangles, while each generator of isospin SU(2) symmetry rotates between two copies distinct induced masses, residing at the end of identical arms of two triangles, see Fig. 4. Therefore, the induced U(1) or SU(2) quantum number of skyrmion gives rise to the color degeneracy among the competing orders in its core, which we exemplify in Sec. V B 2.

In terms of the masses related by the U(1) rotation, generated by $Q_{\text{skyrm}} = Y$, we can define three copies of five-tuplet of mutually anticommuting masses

\[
\mathcal{T}^{\text{charge}}_1 = \{M_1, M_2, M_3, M_4, M_{34}\}, \\
\mathcal{T}^{\text{charge}}_2 = \{M_1, M_2, M_3, M_5, M_{35}\}, \\
\mathcal{T}^{\text{charge}}_3 = \{M_1, M_2, M_3, M_6, M_{36}\}. \quad (23)
\]

The existence of five mutually anticommuting masses gives the right number to support WZW term in $d = 2$ [59, 60]. Here such a topological term is named charge-
Due to the flavor degeneracy, the same charge-WZW term can arise for three copies of induced masses. One can also construct isospin-WZW term from the following two five tuplets of anticommuting masses

\[ \mathcal{T}_{\text{isospin},1} = \{ M_1, M_2, M_3, M_4, M_5 \} , \]
\[ \mathcal{T}_{\text{isospin},2} = \{ M_1, M_2, M_3, M_34, M_{35} \} , \]

(24)

where the U(1) rotation between the induced masses, namely \((M_4, M_5)\) and \((M_{34}, M_{35})\), is generated by one of the generators of isospin SU(2) symmetry, namely \(E_{45}\). Therefore, same isospin-WZW term can arise for two copies of five tuplets of masses. Isospin-WZW terms can also be derived for the induced masses, related via U(1) rotation generated by \(E_{46}\) and \(E_{56}\). The WZW term in believed to be responsible for continuous (and possibly deconfined) quantum phase transition. Therefore, in valley-flavored QBT systems such an unconventional quantum phase transition can take the system to a variety of competing broken symmetry phases, due to their flavor degeneracy in the skyrmion core, a detailed analysis of which is left for a future investigation.

V. EXAMPLES

Upon establishing the internal algebra of competing orders in the core of topological defects of mass orders, we now discuss some physically pertinent examples for both single-flavored and valley-degenerate QBT systems.

A. single-flavored QBT

To this end we refer to Table I for all masses in this systems and Sec. II A for the corresponding definition of eight-component spinor. First, we consider a vortex of easy-plane components of QSHI. Therefore, in Eq. (13)

(M_1, M_2) = \tau_3 \otimes (\sigma_1, \sigma_2) \otimes \sigma_0 ,

and three mutually anticommuting masses are the easy-axis QSHI, and real and imaginary components of the singlet s-wave pairing. Any one of these three masses can split the zero-energy manifold of \(H_{\text{sym}}^{\text{spin}}\) [see Eq. (14)], and acquire a finite expectation value. On the other hand, a vortex inside the s-wave paired state, i.e. when

(M_1, M_2) = (\tau_1, \tau_2) \otimes \sigma_0 \sigma_0 ,

supports all three components of the QSHI inside its core.

Next we consider a skyrmion of QSHI. The generalized charge of such a skyrmion

\[ Q_{\text{skyr}} = \tau_3 \otimes \sigma_0 \otimes \sigma_0 , \]

(25)

is the standard electric charge \(Q_{\text{elec}}\) in the Nambu basis. Note that \(Q_{\text{skyr}}\) generates a U(1) rotation between the real and imaginary components of s-wave pairing. Therefore, core of the skyrmion of QSHI supports s-wave pairing, and a vortex of s-wave superconductor allows the local formation of QSHI.

B. Valley-degenerate QBT

Next we discuss the competing phases in the core of vortex and skyrmion in Bernal BLG. Readers should consult Table I for sixteen-dimensional representation of all masses, and Sec. II B for the definition of the spinor basis.

1. Vortex

Due to a large number of masses, one can construct a myriad of vortices out of any two mutually anticommuting masses shown in Table I. However, we restrict the discussion to some physically pertinent situations.

(1) Vortex of spin-singlet Kekule currents \((M_1 = K_E, M_2 = K_O)\): Ten masses that anticommute with \(K_E\) and \(K_O\) are the layer polarized (LP) and layer antiferromagnet (\(\hat{N}\)) states, and spin-triplet f-wave superconductor \((\bar{F}_1, \bar{F}_2)\). Four-tuplets of masses forming the SO(4) subgroups of mutually anticommuting masses are

\[ \{ \text{LP}, F_1^1, F_2^1, F_1^3 \} , \{ \text{LP}, F_1^2, F_2^2, F_1^3 \} , \{ N^1, N^2, F_1^1, F_2^1 \} , \{ N^2, N^3, F_1^1, F_2^1 \} . \]

If the zero-energy manifold gets split by spontaneously breaking the SU(2) spin rotational symmetry \((\hat{S})\), it can be accomplished by nucleating either the Néel layer antiferromagnet \((\hat{N})\), real \((\bar{F}_1)\) or imaginary \((\bar{F}_2)\) components of the spin-triplet f-wave pairing, representing the color degeneracy of competing orders in the vortex core.

(2) Vortex of easy-plane layer anti-ferromagnet \((M_1 = N^1, M_2 = N^2)\) supports singlet Kekule currents \((K_E\) and \(K_O)\), Kekule pair-density-waves \((sK_1, sK_2, pK_1\) and \(pK_2)\), and the easy-axis component of QSHI \((SH)\), layer anti-ferromagnet \((N^3)\) and f-wave pairing \((F_1^1\) and \(F_2^2)\). Five SO(4) subgroups of competing masses are

\[ \{ N^3, K_E, sK_1, sK_2 \} , \{ N^3, K_O, pK_1, pK_2 \} , \{ F_1^3, F_2^3, K_O, K_E \} \}
\[ \{ SH^3, F_2^2, pK_2, sK_1 \} , \{ SH^3, F_2^2, pK_1, sK_2 \} . \]

In contrast to a Dirac system (such as MLG), the vortex core of easy-plane Néel layer anti-ferromagnet supports spin-singlet pair-density-waves in BLG.

(3) A vortex in the easy-plane of QSHI \((M_1 = SH^3, M_2 = SH^3)\) supports easy-axis layer anti-ferromagnet \((N^3)\), QSHI \((SH^3)\), and Kekule pair-currents \((K_E^3\) and \(K_O^3)\), s-wave pairing \((S_1)\) and \((S_2)\), and Kekule pair-density-waves \((sK_1, sK_2, pK_1\) and \(pK_2)\). The associated four-tuplets of mutually anticommuting masses are

\[ \{ SH^3, S_1, pK_1, sK_2 \} , \{ SH^3, S_2, pK_2, S_1 \} , \{ N^3, K_E^3 \} . \]
In contrast to a similar situation in MLG, where the vortex zero modes only support the s-wave pairing, in Bernal BLG they can also accommodate translational symmetry breaking Kekule pairings. So far, we discussed vortex in various insulating phases of BLG, discerning sufficient differences with their counterparts in MLG. Next we discuss vortex of superconducting masses in this system.

(4) First we consider a vortex of s-wave pairing, with \( M_1 = S_1 \) and \( M_2 = S_2 \). It supports layer polarized state (LP), QSHI (SH), and Kekule spin-currents \((\vec{K}_E \text{ and } K_O)\). The four-tuplet of masses are

\[
\{ \text{SH}^1, \text{SH}^2, K_O^0, K_E^0 \}, \{ \text{SH}^2, \text{SH}^3, K_O^1, K_E^1 \}, \{ \text{SH}^3, \text{SH}^1, K_O^0, K_E^0 \}, \{ \text{LP}, K_O^0, K_O^1, K_O^2 \}, \{ \text{LP}, K_E^0, K_E^1, K_E^2 \}.
\]

If the zero-energy manifold gets split by lifting the SU(2) spin rotational symmetry, it can be accompanied by QSHI (SH) or two spin-triplet Kekule currents \((\vec{K}_E \text{ and } K_O)\), manifesting the flavor degeneracy.

(5) In the vortex core of spin-singlet s-Kekule pairing \((M_1 = sK_1, M_2 = sK_2)\), one can find layer antiferromagnet (~\(\vec{N}\)), QSHI (SH), and specific components of spin-singlet \((\vec{K}_E)\) and spin-triplet \((K_O)\) Kekule currents. The corresponding four-tuplet of masses are

\[
\{ \text{SH}^1, \text{SH}^2, N^3, K_O^1 \}, \{ \text{SH}^2, \text{SH}^3, N^1, K_O^1 \}, \{ \text{SH}^1, \text{SH}^3, N^2, K_O^2 \}, \{ \text{N}^1, N^2, N^3, K_E \}, \{ K_O^0, K_O^1, K_O^2, K_E \}.
\]

On the other hand, the SU(2) spin rotational symmetry of the zero modes can be lifted by layer antiferromagnet (~\(\vec{N}\)), QSHI (SH) or spin Kekule current \((K_O)\), manifesting the color degeneracy among the competing orders. A similar algebra among ten masses in the vortex of p-Kekule superconductor can be constructed after taking \(K_O \to K_E\) and \(K_O \to K_E\). Therefore, vortex core of all spin-singlet superconductors (the s-wave and two Kekule ones) supports topological QSHI.

(6) Finally, we focus on the vortex phase of the spin-triplet f-wave pairing. For concreteness, we orient the superconducting order parameter along the z-direction (easy-axis), i.e., \( M_1 = F_3^1 \), \( M_2 = F_3^2 \). Inside the vortex core, one then finds layer polarized state (LP), easy-plane components of layer anti-ferromagnet \((N^1, N^2)\) and spin-triplet Kekule currents \((K_E^1, K_E^2, K_O^1, K_O^2)\), easy-axis QSHI (SH), and singlet Kekule currents \((K_E \text{ and } K_O)\). The five sets of four mutually anticommuting masses are

\[
\{ N^1, \text{SH}^2, K_E^1, K_O^1 \}, \{ N^2, \text{SH}^3, K_E^2, K_O^2 \}, \{ K_E, K_O, N^1, N^2 \}, \{ \text{LP}, K_E, K_O, K_O^2 \}, \{ \text{LP}, K_O, K_E, K_E^2 \}.
\]

Therefore, all four gapped superconductors support QSHI and some translational symmetry breaking masses in the vortex core. It is also interesting to notice that the vortex phase of pair-density waves additionally supports the Néel layer antiferromagnet.

### VI. SUMMARY AND DISCUSSION

To summarize, here we unveil the internal algebra of competing orders inside the core of the topological defects, such as vortex and skyrmion, of various ordered phases in two-dimensional fermionic systems that in the normal phase are described by biquadratic touching of the valence and conduction bands. We consider two realizations of such systems, describing single-flavored and valley-degenerate QBTs, respectively realized on the checkerboard or Kagome lattice [50] and Bernal BLG [24]. In the former system, four zero-energy

\[
Q_{\text{Néel} \text{ skyr}} = \tau_3 \otimes \sigma_0 \otimes \eta_3 \otimes \alpha_0.
\]

is the chiral or valley charge \((Q_{ch})\), which changes sign between two valley. The valley charge rotates between the following three pairs of masses, (1) \( K_E \) and \( K_O \), (2) \( sK_1 \) and \( pK_2 \), and (3) \( sK_2 \) and \( pK_1 \), manifesting the flavor degeneracy of competing order. One generator of the SU(2) isospin is the electric charge \( Q_{\text{elec}} = \tau_3 \otimes \sigma_0 \otimes \eta_0 \otimes \alpha_0 \), which rotates between the real and imaginary components of the s-Kekule \((sK_1 \text{ and } sK_2)\) and p-Kekule \((pK_1 \text{ and } pK_2)\) pair-density-waves. Therefore, skyrmion core of Néel order can become charged by nucleating a specific Kekule superconductor.

(2) QSHI, with \( M_j = SH^j \) for \( j = 1, 2, 3 \). Six competing masses are the real and imaginary components of s-wave \((s_1, s_2)\), s-Kekule \((sK_1, sK_2)\) and p-Kekule \((pK_1, pK_2)\) pairings. The U(1) charge of this skyrmion

\[
Q_{\text{skyr}} = \tau_3 \otimes \sigma_0 \otimes \eta_0 \otimes \alpha_0.
\]

is the regular electric charge \((Q_{\text{elec}})\), which rotates between the real and imaginary components of three singlet pairings. One generator of SU(2) isospin is the chiral charge \( Q_{\text{ch}} \). Hence, the core of a skyrmion of QSHI can host three different types of spin-singlet superconductors, leading to the notion of the color degeneracy among competing orders. In contrast, only the s-wave pairing can be realized in the skyrmion core of QSHI in MLG [11].

7 In Dirac system, such as MLG, a skyrmion of the Néel order does not permit any superconducting mass in its core [12].
modes bound to the vortex can be split by three competing masses, while the core of a skyrmion possesses a unique charge and supports a doublet of masses. For example, zero modes bound to the vortex of the s-wave superconductor get split by the QSHI, while a skyrmion of QSHI becomes electrically charged and sustains s-wave pairing in its core.

The internal algebra of competing orders in valley-degenerate QBT systems is much richer. For example, eight zero-energy vortex modes can support ten masses that close an SO(5) algebra. While there are five possible patterns for splitting the zero modes by lifting its SO(4) chiral symmetry, they can also be split by spontaneously breaking the SU(2) chiral symmetry in ten different ways. Most interestingly, each SU(2) symmetry can be broken by three distinct set of chiral triplet masses, giving rise to the notion of color or flavor degeneracy of competing orders inside the vortex core. As a concrete example of such flavor degeneracy, we note that zero modes bound to the vortex of Kekule current orders can be split by spontaneously breaking the SU(2) spin rotational symmetry by either Néel layer antiferromagnet or the real and imaginary components of the spin-triplet f-wave pairing.

On the other hand, a skyrmion composed of three mutually anticommuting masses possesses an SU(2)⊗U(1) chiral symmetry, and therefore supports a generalized U(1) charge and SU(2) isospin. While the U(1) charge rotates between three distinct pairs of masses, each generator of SU(2) isospin symmetry rotates between two distinct pairs of masses, once again yielding the color or flavor degeneracy among competing orders within the skyrmion core. As a concrete outcome of such a rich algebraic structure, we note that skyrmions of QSHI and Néel antiferromagnet supports singlet Kekule pairings, while the vortex phase of spin-singlet pair-density-waves (s-Kekule and p-Kekule) supports both insulating masses.

A question of practical importance arise quite naturally. How to stabilize a real space vortex in an ordered phase? Notice that an easy-plane configuration of Néel layer antiferromagnet or topological QSHI can be realized in the presence of an in-plane external magnetic field \[\text{[36]},\] which only couples to the spin of electrons (Zeeman coupling) without causing the Landau quantization \[\text{[73]},\] restricting these two orderparameters within the easy-plane, thereby providing the requisite U(1) symmetry to support a vortex. Superconducting vortex can be realized in BLG by bringing a type-II superconductor, such as Nb, to close proximity and applying a magnetic field such that \(H_{c1} < H \ll H_{c2}.\) On the other hand, a two-component mass order, such as the spin-singlet Kekule current, is expected to support vortex defect deep inside the ordered phase.

Deep inside a triplet ordered phase, such as layer antiferromagent and QSHI, skyrmions are expected to appear naturally. Recently it has been shown that singlet s-wave pairing can be nucleated through the condensation of skyrmions of QSHI in MLG \[\text{[62]},\] Furthermore, continuous quantum phase transition between two distinct broken symmetry phases in the presence of topological WZW terms can now be demonstrated in quantum Monte Carlo simulations within the half-filled zeroth Landau level of MLG, without encountering the infamous sign problem \[\text{[63, 64]},\] These recent developments are encouraging, and should be applicable for Bernal BLG, where continuous phase transition driven by charge- and isospins-WZW terms can be tested numerically.

**Acknowledgments**

This work was supported by the Startup grant of Bitan Roy from Lehigh University. Author thanks Igor F. Herbut for useful discussions and correspondences, and Max Planck Institute for the Physics of Complex Systems, Dresden, Germany for hospitality.

**Appendix A: No doubling for QBT**

Low energy excitations around a QBT point in a 2D Brillouin zone is described by the effective Hamiltonian

\[H_{QBT}(k) = \alpha_1 d_1(k) + \alpha_2 d_2(k),\]  

where \(d_j(k)\) are defined in Eq. \[\text{[3]},\] and \(\alpha_1\) and \(\alpha_2\) are mutually anti-commuting Hermitian matrices. But their dimensionality remains unspecified for now. If there exists another Hermitian matrix, say \(\beta\), which anti-commutes with both \(\alpha_1\) and \(\alpha_2\), spectral symmetry of the energy eigenvalues is guaranteed. Next we ask the following question. What is the minimum dimensionality of \(\alpha_i\), so that \(H_{QBT}(k)\) is time-reversal invariant?

Let us assume \(\alpha_i\)s are two-dimensional matrices. The maximal number of mutually anti-commuting two-dimensional Hermitian matrices is three, and they close a \(C(3, 0)\) algebra. Two of them are purely real, while the remaining one is purely imaginary. One can immediately identify them as the Pauli matrices. Without any loose of generality, we can choose \(\alpha_1\) and \(\alpha_2\) to be purely real.

The time reversal symmetry is represented by an anti-unitary operator \(I_t = AK\), where \(A\) is a unitary matrix. As we focus on the time-reversal symmetric system,

\[I_t H_{QBT}(k) I_t^{-1} = H'_{QBT}(-k),\]  

since it describes the motion of spinless free fermions on real space. Moreover, for spinless fermions one must have \(I_t^2 = +1\ \text{[66]},\) Note that \(d_1(k), d_2(k)\) do not change sign under the reversal of time. Since we have taken \(\alpha_1\) and \(\alpha_2\) to be real, Eq. \[\text{(A2)},\] is satisfied when

\([A, \alpha_1] = [A, \alpha_2] = 0.\]  

For two-dimensional matrices, there exist only one matrix which commutes with all the three mutually anticommuting Pauli matrices, the identity matrix \((\sigma_0)\) with trace
2. The time-reversal operator is, therefore, \( I_t = K \), and \( I_t^2 = +1 \), which is basis independent. Therefore, when valence and conduction band display quadratic touching, the minimal representation of such a system can be two-component, and therefore the system does not necessarily encounter the fermion doubling. On the other hand, when \( d_1(k) = v k_x \), and \( d_2(k) = v k_y \), where \( v \) is the Fermi velocity, the minimal representation of \( \alpha_1 \) and \( \alpha_2 \) is four-dimensional for spinless fermions in two dimensions \( \Delta \), which leads to the notion of fermion doubling for chiral Dirac fermions, such as in MLG, according to the Nielsen-Ninomiya theorem \( [67] \).

Appendix B: Generators of SO(5), SO(4)

Ten generator of an SO(5) group can be labeled as \( J_{\alpha \beta} \), where \( \alpha, \beta = 2, \cdots, 6 \), which satisfy \( J_{\alpha \beta} = -J_{\beta \alpha} \). In addition, they satisfy the following commutation relation

\[
[J_{\alpha \beta}, J_{\mu \nu}] = i \left[ \delta_{\beta \mu} J_{\alpha \nu} + \delta_{\alpha \nu} J_{\beta \mu} - \delta_{\beta \nu} J_{\alpha \mu} - \delta_{\alpha \mu} J_{\beta \nu} \right].
\]

(B1)

In order to show that ten generator from Eq. (19), close an SO(5) algebra we write the first four entries of \( \mathcal{G} \) as

\[
EM_j = \Gamma_1 \Gamma_2 M_1 M_2 M_j \equiv 2 J_{2j},
\]

(B2)

such that \( J_{2j} = -J_{2j} \) for \( j = 3, 4, 5, 6 \). The rest of the six entries from \( \mathcal{G} \) can be expressed as \( E j_k = 2 J_{j_k} \) for \( j = 3, 4, 5, 6 \), and they also satisfy the antisymmetry property. Now it is straightforward to show that ten generators appearing in \( \mathcal{G} \), expressed as \( J_{\alpha \beta} \), where \( \alpha, \beta = 2, \cdots, 6 \), satisfy the commutation relation in Eq. (B1).

Next we show that five sets of six generators appearing in Fig. 2(a)-(c) (in blue) close SO(4) algebra. For concreteness, we focus on the six generators appearing in Fig. 2(a). The following the same steps, one can show that other four sets of six generators also close SO(4) algebra. An SO(4) group has six generators, namely

\[
A = (A_1, A_2, A_3), \quad B = (B_1, B_2, B_3),
\]

(B3)

satisfying the commutation relations

\[
[A_j, A_k] = i \epsilon_{jkl} A_l, [B_j, B_k] = i \epsilon_{jkl} B_l, [A_j, B_k] = i \epsilon_{jkl} B_l,
\]

(B4)

for \( j, k, l = 1, 2, 3 \). From the six generators appearing in Fig. 2(a), we choose

\[
A_1 = - E_{34} 2, \quad A_2 = - E_{45} 2, \quad A_3 = - E_{53} 2,
\]

\[
B_1 = E_{56} 2, \quad B_2 = B_3 = E_{36} 2.
\]

(B5)

It is now straightforward to show that for these choices of \( A \) and \( B \), the commutation relations from Eq. (B4) are satisfies. To show that SO(4) group is isomorphic to SU(2)⊗SU(2), we construct six new generators

\[
X_j = \frac{1}{2} (A_j + B_j), \quad Y_j = \frac{1}{2} (A_j - B_j),
\]

(B6)

for \( j = 1, 2, 3 \). It is now straightforward to show that individually \( X \) and \( Y \) close SU(2) algebra, but these two sets of three generators commute with each other, i.e.,

\[
[X_j, X_k] = i \epsilon_{jkl} X_l, [Y_j, Y_k] = i \epsilon_{jkl} Y_l, [X_j, Y_k] = 0.
\]

(B7)

for \( j, k, l = 1, 2, 3 \). Therefore, \( X \) and \( Y \) are the generators of two decoupled SU(2), and SO(4)⊗SU(2)⊗SU(2).

[1] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev and M. P. A. Fisher, Science 303, 1490 (2004).
[2] T. Senthil, L. Balents, A. Vishwanath, and M. P. A. Fisher, Phys. Rev. B 70, 144407 (2004).
[3] A. W. Sandvik, Phys. Rev. Lett. 98, 227202 (2007).
[4] R. K. Kaul and A. W. Sandvik, Phys. Rev. Lett. 108, 137201 (2012).
[5] P. Nahum, P. Serna, J. T. Chalker, M. Ortiz, and A. M. Somoza, Phys. Rev. Lett. 115, 267203 (2015).
[6] A. Nahum, J. T. Chalker, P. Serna, M. Ortiz, and A. M. Somoza, Phys. Rev. X 5, 041048 (2015).
[7] A. F. Albuquerque, D. Schwandt, B. Hétényi, S. Capponi, M. Mambrini, and A. M. Läuchli, Phys. Rev. B 84, 024406 (2011).
[8] A. G. Abanov and P. Weigman, Nucl. Phys. B 570, 685 (2000).
[9] A. Tanaka and X. Hu, Phys. Rev. Lett. 95, 036402 (2005).
[10] T. Senthil and M. P. A. Fisher, Phys. Rev. B 74, 064405 (2006).
[11] T. Grover and T. Senthil, Phys. Rev. Lett. 100, 156804 (2008).
[12] S. Ryu, C. Mudry, C-Y. Hou, C. Chamon, Phys. Rev. B 80, 205319 (2009).
[13] B. Seradjeh, J. E. Moore, and M. Franz, Phys. Rev. Lett. 103, 066402 (2009).
[14] P. Ghaemi, S. Ryu and D-H. Lee, Phys. Rev. B 81, 081403(R) (2010).
[15] L. Fu, S. Sachdev, C. Xu, Phys. Rev. B 83, 165123 (2011).
[16] I. F. Herbut, Phys. Rev. B 85, 085304 (2012).
[17] I. F. Herbut, C-K. Lu, and B. Roy, Phys. Rev. B 86, 075101 (2012).
[18] C-H. Hsu, and S. Chakravarty, Phys. Rev. B 87, 085114 (2013).
[19] P. Goswami and Q. Si, Phys. Rev. B 89, 045124 (2014).
[20] C-C. Liu, P. Goswami, and Q. Si, Phys. Rev. B 96, 125101 (2017).
[21] I. F. Herbut, V. Juricic, B. Roy, Phys. Rev. B 79, 085116 (2009).
[22] B. Roy, P. Goswami, and V. Juricic, Phys. Rev. B 97, 205117 (2018).
[23] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. 81, 109
