AN IMPROVED BOUND ON DISTILLABLE ENTANGLEMENT

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October 30, 2000

Abstract. The best bound known on 2-locally distillable entanglement is that of Vedral and Plenio, involving a certain measure of entanglement based on relative entropy. It turns out that a related argument can be used to give an even stronger bound; we give this bound, and examine some of its properties. In particular, and in contrast to the earlier bounds, the new bound is not additive in general. We give an example of a state for which the bound fails to be additive, as well as a number of states for which the bound is additive.

One of the central problems in quantum information theory is that of entanglement distillation [1], the production of maximally entangled states from partially entangled states. The purpose of the present note is to give a new upper bound on the rate at which entanglement can be distilled.

In general, if $C$ is a given class of physical operations, we define $C$-distillable entanglement as follows:

Definition. The $C$-distillable entanglement of a state $\rho$ on a state space $V_A \otimes V_B$ is the maximum number $D_C(\rho)$ such that there exists a sequence

$$T_i : (V_A \otimes V_B)^{\otimes n_i} \rightarrow V_i \otimes V_i$$

of operations from $C$, with $n_i \rightarrow \infty$,

$$\frac{1}{n_i} \log_2 \text{dim } V_i \rightarrow D_C(\rho),$$

and

$$F(T_i(\rho^{\otimes n})) \rightarrow 1.$$
curious phenomenon. The first bound, entanglement of formation, is most easily proved for 2-local operations. However, the second, stronger, bound of Vedral and Plenio ([3], see also [4]) applies to a larger class of operations, namely that of separable operations. The bound of the present note carries this even further, both strengthening the bound and enlarging the set of allowed operations. This phenomenon is rather counter-intuitive, since enlarging the class of allowed operations would be expected to increase the distillable entanglement.

The new bound

As in [4], the key to the new bound is the observation that if we are given a process that distills entanglement from \( \rho \) at a given rate and high fidelity, and apply the process to a different state \( \sigma \), then there is a limit to how much the fidelity can be reduced by doing so. Given an upper bound on the fidelity any process of that rate can obtain from \( \sigma \), we may be able to deduce that no process can obtain high fidelity from \( \rho \).

We will state this in some generality, to support possible future applications.

**Theorem 1.** Let \( C \) be some class of operations. Suppose we are given a state \( \sigma \) on \( V_A \otimes V_B \) and an increasing, left continuous, function \( \alpha : \mathbb{R}^+ \to \mathbb{R}^+ \) such that any operation

\[
\mathcal{T} : (V_A \otimes V_B)^{\otimes n} \to V \otimes V
\]

from \( C \) satisfies

\[
\frac{1}{n} \log_2 F_{\mathcal{T}}(\sigma^{\otimes n}) \leq -\alpha \left( \frac{1}{n} \log_2 \dim V \right).
\]

Then for any other state \( \rho \),

\[
\alpha(D_C(\rho)) \leq S(\rho|\sigma),
\]

where

\[
S(\rho|\sigma) = \text{Tr}(\rho(\log_2(\rho) - \log_2(\sigma)))
\]

is the relative entropy of \( \rho \) and \( \sigma \).

**Proof.** The crucial observation is that for any operator \( \mathcal{T} \) the function \( F_{\mathcal{T}} \) is linear on density operators, and is bounded between 0 and 1. It follows that we can write it in the form

\[
F_{\mathcal{T}}(\omega) = \text{Tr}(F_{\mathcal{T}} \omega)
\]

for some (uniquely determined) operator \( F_{\mathcal{T}} \) such that \( F_{\mathcal{T}} \) and \( 1 - F_{\mathcal{T}} \) are both positive.

For \( \epsilon > 0 \), let \( n, V, \mathcal{T} \) give a process from \( C \) with

\[
\frac{1}{n} \log_2 \dim V \geq D_C(\rho)(1 - \epsilon),
\]

and

\[
\text{Tr}(F_{\mathcal{T}} \rho^{\otimes n}) \geq 1 - \epsilon.
\]

On the other hand, by assumption,

\[
\text{Tr}(F_{\mathcal{T}} \sigma^{\otimes n}) \leq 2^{-n \alpha \left( \frac{1}{n} \log_2 \dim V \right)}.
\]
Lemma 1 below then tells us that
\[
S(\rho||\sigma) \geq \limsup_{\epsilon \to 0} \alpha\left(\frac{1}{n} \log_2 \dim V\right)
\geq \limsup_{\epsilon \to 0} \alpha((1 - \epsilon)D_C(\rho))
= \alpha(D_C(\rho)).
\]

\[\Box\]

Remark. The above argument is a hybrid of the arguments of [3] and [4]. In particular, it should be noted that the above argument does not require any assumption that the resulting bound be additive.

**Lemma 1.** Let \(\rho\) and \(\sigma\) be states on a common Hilbert space \(V\), such that \(S(\rho||\sigma)\) is finite. For \(n \in \mathbb{Z}^+\) and \(0 < \epsilon < 1\), define
\[
R(n, \epsilon) = \inf_{\pi} \\{ \log_2 \text{Tr}(\sigma^{\otimes n} \pi) \},
\]
where \(\pi\) ranges over positive operators on \(V^{\otimes n}\) with both \(\pi\) and \(1 - \pi\) positive, and such that
\[
\text{Tr}(\rho^{\otimes n} \pi) \geq 1 - \epsilon.
\]
Then
\[
\liminf_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} R(n, \epsilon) \geq -S(\rho||\sigma).
\]

**Proof.** From the condition on \(\pi\), it follows that \(\{\pi, 1 - \pi\}\) is a POVM. Consequently, Uhlmann’s monotonicity theorem ([5], theorem 1.5) tells us that, writing
\[
p_\rho = \text{Tr}(\rho^{\otimes n} \pi),
p_\sigma = \text{Tr}(\sigma^{\otimes n} \pi),
\]
we have
\[
nS(\rho||\sigma) \geq p_\rho((\log_2(p_\rho) - \log_2(p_\sigma)) + (1 - p_\rho)((\log_2(1 - p_\rho) - \log_2(1 - p_\sigma))).
\]
Now, this in turn is bounded below by
\[-1 - p_\rho \log_2(p_\sigma) - (1 - p_\rho) \log_2(1 - p_\sigma).
\]
If we divide by \(n\), the first and third terms will be negligible unless
\[
p_\sigma = \text{Tr}(\sigma^{\otimes n} \pi) > \frac{1}{2},
\]
say. But such a \(\pi\) could not possibly provide a counterexample to the lemma. It remains to consider the second term. But
\[-\frac{1}{n} p_\rho \log_2(p_\sigma) \geq -(1 - \epsilon)\frac{1}{n} \log_2(p_\sigma).\]
It follows that
\[-(1 - \epsilon) \lim_{n \to \infty} \frac{1}{n} \log_2(p_n) \leq S(\rho||\sigma).\]
The lemma follows by taking the limit as $\epsilon \to 0$. □

Remark. Indeed, it is the case that
\[\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} R(n, \epsilon) = -S(\rho||\sigma)\]
([5], equation 1.31), in which the operator $\pi$ may be assumed to be a projection. In particular the conclusion of theorem 1 is the strongest that can be made from its hypotheses.

To obtain a bound, then, we need to find states $\sigma$ for which we can bound the fidelity. Let $\Gamma$ denote the partial transpose operator of [6]. Let $C_\Gamma$ be the set of positive-partial-transpose (p.p.t.) superoperators, that is completely positive, trace-preserving superoperators $S$ such that the superoperator

\[S^\Gamma : \rho \mapsto (S(\rho^\Gamma))^\Gamma\]
is also completely positive. It is not too difficult to see that any separable superoperator is also p.p.t., and thus any 2-local superoperator is p.p.t. Similarly, we will say a state $\rho$ is p.p.t. if $\rho^\Gamma$ is positive semi-definite. Note that the creation of a p.p.t. state is a p.p.t. operation. We will write “$\Gamma$-distillable entanglement” for “$C_\Gamma$-distillable entanglement”.

The key observation [7] is that p.p.t. states must have a $\Gamma$-distillable entanglement of 0, because the fidelity of a p.p.t. state can be bounded away from 1. Indeed:

**Lemma 2.** Let $\sigma$ be a p.p.t. state on a space $V \otimes V$. Then

\[F(\sigma) \leq \frac{1}{\dim V}.\]

**Proof.** We have

\[
F(\sigma) = \text{Tr}(\Phi^+(V)\Phi^+(V)^\dagger_\sigma) = \text{Tr}((\Phi^+(V)\Phi^+(V)^\dagger)^\Gamma_\sigma^\Gamma).
\]

But

\[
(\Phi^+(V)\Phi^+(V)^\dagger)^\Gamma = \frac{1}{\dim V} \sum_{i,j} |ij\rangle\langle ji|,
\]
and thus has eigenvalues of absolute value at most $1/\dim V$. So

\[
\text{Tr}((\Phi^+(V)\Phi^+(V)^\dagger)^\Gamma_\sigma^\Gamma) \leq \frac{1}{\dim V} \text{Tr}(\sigma^\Gamma) = \frac{1}{\dim V}.
\]

□

Remark. The new bound is essentially an update of the Vedral-Plenio bound to take this observation into account.
**Theorem 2.** For any state $\rho$ and any p.p.t. state $\sigma$ on the same bipartite Hilbert space,

$$D_\Gamma(\rho) \leq S(\rho||\sigma).$$

**Proof.** It suffices to show that the function $\alpha$ of theorem 1 can be taken to be 1. In other words, we must show that for any p.p.t. superoperator

$$T : (V_A \otimes V_B)^{\otimes n} \rightarrow V \otimes V,$$

we have

$$F_T(\sigma^{\otimes n}) \leq \frac{1}{\dim V}.$$  

But the image of a p.p.t. operator under a p.p.t. superoperator is p.p.t., since

$$T(\omega)^\Gamma = T^\Gamma(\omega^\Gamma),$$

so the lemma applies. □

The statement that

$$F_T(\sigma^{\otimes n}) \leq \frac{1}{\dim V}$$

for a p.p.t. superoperator $T$ and a p.p.t. state $\sigma$ is a special case of the following:

**Theorem 3.** Let $T$ be a p.p.t. superoperator with output dimension $K$ and associated fidelity operator $F_T$. Then

$$-\frac{1}{K} \leq F_T \leq \frac{1}{K},$$

where for Hermitian operators $A$ and $B$, $A \leq B$ means that $B - A$ is positive semi-definite.

**Proof.** It suffices to show that for any density operator $\rho$,

$$-\frac{1}{K} \leq \text{Tr}(F_T^\Gamma \rho) \leq \frac{1}{K}.$$

But

$$\text{Tr}(F_T^\Gamma \rho) = \text{Tr}(F_T \rho^\Gamma)$$

$$= \text{Tr}(\Phi^+(K)\Phi^+(K)^\dagger T(\rho^\Gamma))$$

$$= \text{Tr}(\Phi^+(K)\Phi^+(K)^\dagger (T(\rho))^\Gamma)$$

$$= \text{Tr}((\Phi^+(K)\Phi^+(K)^\dagger)^\Gamma T^\Gamma(\rho)).$$

Since $-(1/K) \leq (\Phi^+(K)\Phi^+(K)^\dagger)^\Gamma \leq (1/K)$, and $T^\Gamma(\rho)$ is a density operator, the result follows. □

Remarks. (1) It is an open question whether this inequality, together with the inequality

$$0 \leq F_T \leq 1$$

which holds for all superoperators, can be used to give a stronger bound than that of Theorem 2, which essentially only uses the inequalities one at a time. (2) This statement is essentially a generalization of equation (16) of [4].
OPTIMIZING $\sigma$

To obtain the full strength of the bound of theorem 2, it is necessary to optimize the choice of $\sigma$. In the sequel, we will say that a p.p.t. $\sigma$ is optimal for $\rho$ if

$$ S(\rho||\sigma) = \min_{\sigma'} S(\rho||\sigma') \overset{\text{def}}{=} B_{\Gamma}(\rho) $$

where $\sigma'$ ranges over all p.p.t. states.

**Theorem 4.** Suppose $\rho$ is a positive definite state. Then the p.p.t. state $\sigma$ is optimal for $\rho$ if and only if, setting

$$ K = 1 - D_{\sigma} \text{Tr}(\rho \log(\sigma)),$$

where $D_{\sigma}$ is the matrix derivative, we have

$$ \sigma^{\Gamma} K^{\Gamma} = 0 \quad \text{and} \quad K^{\Gamma} > 0. $$

**Proof.** We note first that $\sigma$ must also be positive definite. Otherwise, $S(\rho||\sigma)$ would be infinite, but this is impossible, since $S(\rho||\frac{1}{\text{dim } \rho}) < \infty$. Thus we must solve the optimization problem:

Minimize $S(\rho||\sigma)$ subject to the constraints $\text{Tr } \sigma = 1$, $\sigma > 0$ and $\sigma^{\Gamma} \geq 0$.

Now, $S(\rho||\sigma)$ is a convex function ([5], theorem 1.4), that is:

$$ S(\rho||a\sigma_1 + (1 - a)\sigma_2) \leq aS(\rho||\sigma_1) + (1 - a)S(\rho||\sigma_2). $$

Moreover, the set of p.p.t. density operators is convex. Thus $\sigma$ is optimal for $\rho$ if and only if it satisfies the Karush-Kuhn-Tucker conditions ([8], Theorem 2.1.4).

For the set of p.p.t. density operators, this becomes:

$$ D_{\sigma} \text{Tr}(\rho \log(\sigma)) + \lambda + K = 0, $$

for some number $\lambda$ and Hermitian matrix $K$, where $K^{\Gamma}$ is positive and supported on the kernel of $\sigma^{\Gamma}$. Multiplying on the left by $\sigma$ and taking a trace, we find

$$ -\lambda = \text{Tr}(\sigma D_{\sigma} \text{Tr}(\rho \log(\sigma))) $$
$$ = \frac{d}{dt} \text{Tr}(\rho \log(\sigma + t\sigma)) $$
$$ = \frac{d}{dt} \text{Tr}(\rho \log(1 + t)) $$
$$ = 1. $$

□

If $\rho$ is only semi-definite, then $\sigma$ can be semi-definite, and the condition is somewhat more complicated:

$$ 1 - D_{\sigma} \text{Tr}(\rho \log \sigma) = K + L, $$

where $\sigma L = 0$, $\sigma^{\Gamma} K^{\Gamma} = 0$, and both $L$ and $K^{\Gamma}$ are positive semi-definite. For simplicity, we will assume that $\rho$ is definite in the proofs below, but in each case, the proof can be adapted to this more complicated case.
Remark. In the published version of this paper, the positivity condition on $K$ was overlooked. The resulting condition is still valid (for $\rho$ definite and not p.p.t.) when $\sigma$ is a smooth point on the boundary of the set of p.p.t. operators; the fact that the optimal $\sigma$ must be on the boundary forces $K$ to have at least one positive eigenvalue. Unfortunately, a tensor product of two boundary points is never a smooth point on the larger cone, and thus most of the additivity results of the published version are invalid.

One consequence of theorem 4 is that for a specific $\sigma$, it is reasonably straightforward to determine the set of $\rho$ for which it is optimal. For instance, if we consider the p.p.t. state

$$\sigma = \begin{pmatrix}
\frac{1}{6} & 0 & 0 & 0 \\
0 & \frac{55}{144} & -\frac{5}{144} & 0 \\
0 & -\frac{1}{6} & \frac{11}{144} & 0 \\
0 & 0 & 0 & \frac{1}{6}
\end{pmatrix},$$

we find that $\sigma$ is optimal for (among others) the state

$$\rho = \begin{pmatrix}
\frac{1}{12} & 0 & 0 & 0 & 0 & \frac{1}{12} \\
0 & \frac{45007}{90000} & -\frac{7}{15}x & -\frac{1201}{3750} & -\frac{49}{750} & 0 \\
0 & -\frac{1201}{3750} & \frac{1497}{50000} & -\frac{39093}{15000} & -\frac{7}{15}x & 0 \\
0 & 0 & 0 & \frac{1}{12} & 0 & -\frac{1201}{3750} \\
0 & 0 & 0 & 0 & \frac{1}{12} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{12}
\end{pmatrix},$$

where $x = 1/\ln(73/23)$. However, $\sigma \otimes \sigma$ is not optimal for $\rho \otimes \rho$. It follows immediately that

$$B_{\Gamma}(r \otimes \rho) < 2B_{\Gamma}(\rho).$$

Thus the new bound is not additive.

Since $B_{\Gamma}$ is certainly subadditive, we can regularize to a stronger bound:

$$\tilde{B}_{\Gamma}(\rho) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n}B_{\Gamma}(\rho^\otimes n).$$

It is not clear how to compute this bound, however. It turns out, however, that there are a number of states for which the bound is additive, and in particular $B_{\Gamma} = \tilde{B}_{\Gamma}$. The simplest sufficient condition seems to be

**Theorem 5.** Let $\rho$ be a definite state such that there exists $\sigma$ optimal for $\rho$ that commutes with $\rho$. Suppose further that

$$D_\sigma(\text{Tr}(\rho \log(\sigma)))^\Gamma \geq 0.$$

Then for any other state $\rho'$,

$$B_{\Gamma}(\rho \otimes \rho') = B_{\Gamma}(\rho) + B_{\Gamma}(\rho').$$

Similarly, if

$$D_\sigma(\text{Tr}(\rho \log(\sigma)))^\Gamma \geq -1,$$

then

$$B_{\Gamma}(\rho^\otimes n) = nB_{\Gamma}(\rho).$$
for all \( n \geq 1 \).

**Proof.** Let \( \sigma' \) be a state optimal for \( \rho' \); assume \( \rho' \) is definite (otherwise write it as a limit of definite states). We need only show that \( \sigma \otimes \sigma' \) is optimal for \( \rho \otimes \rho' \). We can thus apply theorem 4. The crucial observation is that

\[
D_{\sigma \otimes \sigma'} \operatorname{Tr}(\rho \otimes \rho' \log(\sigma \otimes \sigma')) = D_{\sigma} (\operatorname{Tr}(\rho) \log(\sigma)) \otimes D_{\sigma'} (\operatorname{Tr}(\rho') \log(\sigma'))
\]

whenever \([\rho, \sigma] = 0\). We may simultaneously diagonalize \( \rho \) and \( \sigma \). Then by basic properties of the matrix derivative, we have

\[
D_{\sigma \otimes \sigma'} \operatorname{Tr}(\rho \otimes \rho' \log(\sigma \otimes \sigma')) = \frac{d}{dt} \log(\sigma \otimes \sigma' + t\rho \otimes \rho')
\]

But the argument to the logarithm is block-diagonal; we can therefore consider each block independently. In other words, it suffices to consider the case in which \( \sigma \) and \( \rho \) are scalars. But then

\[
\frac{d}{dt} \log(\sigma \sigma' + t\rho \rho') = \rho/\sigma \frac{d}{dt} \log(\sigma' + t\rho'),
\]

as desired.

Then the first condition of theorem 4 requires

\[
(\sigma \otimes \sigma')^T (D_{\sigma} (\operatorname{Tr}(\rho) \log(\sigma)) \otimes D_{\sigma'} (\operatorname{Tr}(\rho') \log(\sigma')) - 1)^T = 0,
\]

which is straightforward to verify. The other condition is that

\[
(D_{\sigma} (\operatorname{Tr}(\rho) \log(\sigma)) \otimes D_{\sigma'} (\operatorname{Tr}(\rho') \log(\sigma')))^T < 1.
\]

But this follows since

\[
D_{\sigma} (\operatorname{Tr}(\rho) \log(\sigma))^T \geq 0 \\
D_{\sigma'} (\operatorname{Tr}(\rho') \log(\sigma'))^T < 1.
\]

The self-additivity claim follows similarly. \( \square \)

**Remark.** Note that when \([\rho, \sigma] = 0\), \( D_{\sigma} (\operatorname{Tr}(\rho \log(\sigma))) = \rho \sigma^{-1} \).

**Remark.** This, again, is weaker than the published version; we thank R. F. Werner and K. G. H. Vollbrecht for telling us of the counterexample they found [9].

In particular, this applies when \( \rho = \sigma \), so

**Corollary 1.** If \( \rho \) is p.p.t., then for any other state \( \rho' \),

\[
B_\Gamma(\rho \otimes \rho') = B_\Gamma(\rho').
\]

**Exploiting symmetries**

The most powerful tool for computing an optimal \( \sigma \) seems to be the following result:
Theorem 6. Let $\rho$ be an arbitrary state on $V_A \otimes V_B$. Let $G$ be a subgroup of $U(V_A) \otimes U(V_B)$ consisting of matrices $U_A \otimes U_B$ such that

$$(U_A \otimes U_B)\rho(U_A \otimes U_B) = \rho.$$ 

Then there exists some $\sigma$ optimal for $\rho$ with

$$(U_A \otimes U_B)\sigma(U_A \otimes U_B) = \sigma$$

for all $U_A \otimes U_B \in G$.

Proof. There certainly exists some $\sigma$ optimal for $\rho$. The point, then, is that for any $U_A, U_B$,

$$(U_A \otimes U_B)\sigma(U_A \otimes U_B) = \sigma$$

is p.p.t., and optimal for

$$(U_A \otimes U_B)\rho(U_A \otimes U_B).$$

In particular, this is true for $U_A \otimes U_B \in G$. But $G$ is a closed subgroup of a compact Lie group, so there is a unique invariant probability measure on $G$. Define

$$\sigma' = E_{U_A \otimes U_B \in G}(U_A \otimes U_B)\sigma(U_A \otimes U_B).$$

By convexity, $\sigma'$ is p.p.t., and

$$S(\rho||\sigma') \leq S(\rho||\sigma).$$

Since $\sigma$ is optimal for $\rho$, so is $\sigma'$. Since $\sigma'$ is clearly preserved by $G$, we are done. □

Example 1. Let $\rho$ be a state of the form

$$a \Phi^+(V) \Phi^+(V) + b,$$

with $\rho^+ > 0$ (an isotropic state). Then we may take $G$ to be the group

$$\{U \otimes \overline{U} : U \in U(V)\}.$$ 

But, in fact, $G$ forces $\rho$, and thus $\sigma$ to have that form. So there are numbers $c$ and $d$ with

$$\sigma = c \Phi^+(V) \Phi^+(V) + d.$$ 

Then $[\rho, \sigma] = 0$, and it is straightforward to solve for $c$ and $d$:

$$\sigma = \frac{1}{\dim V + 1} \Phi^+(V) \Phi^+(V) + \frac{1}{\dim V(\dim V + 1)}.$$

In other words, $\sigma$ is the isotropic state of fidelity $1/\dim V$. Computing $S(\rho||\sigma)$, we find:
Theorem 7. Let $\rho$ be the isotropic state of fidelity $F \geq \frac{1}{K}$ and dimension $K$. Then

$$B_T(\rho) = \log_2 K + F \log_2 F + (1 - F) \log_2((1 - F)/(K - 1)).$$

Moreover, for all $n \geq 1$,

$$B_T(\rho^\otimes n) = nB_T(\rho).$$

The second statement follows from theorem 5; we readily verify

$$(D_\sigma \Tr(\rho \log \sigma))^T \geq -1.$$ 

Remark. (1) If $\rho$ is maximally entangled, then, while there exists an optimal $\sigma$ with full symmetry, there also exist optimal $\sigma$ with much smaller symmetry groups.

(2) Since $B_T$ is a lower bound on entanglement of formation (by the same proof used in [3] to show that the Vedral-Plenio bound is less than entanglement of formation), this theorem gives a lower bound on the entanglement of formation of an isotropic state.

Example 2. In Example 1, the symmetries of $\rho$ were enough to force $[\rho, \sigma] = 0$, and thus to force the bound to be additive with respect to $\rho$ (modulo the derivative bound condition). Here we use that idea to give a large family of states for which the bound is additive.

Let $G$ be a finite abelian group. For an element $g \in G$ and a character $\chi : G \to \mathbb{C}$, define the “generalized Bell state” associated to $g$ and $\chi$ to be

$$v_{g,\chi} = \frac{1}{\sqrt{|G|}} \sum_{h \in G} \chi(h)|h, h - g\rangle.$$ 

If $G = \mathbb{Z}_2$, we recover the usual Bell states. In general, the generalized Bell states for a given group $G$ form an orthonormal basis of maximally entangled states.

Theorem 8. Suppose $\rho$ is a mixture of generalized Bell states for the group $G$. Then there exists $\sigma$ optimal for $\rho$ which is also a mixture of generalized Bell states for $G$, and thus $[\rho, \sigma] = 0$.

Proof. Let $V_G$ be the Hilbert space with basis $|h\rangle$ for $h \in G$. Define operators $X(g)$ and $Z(\chi)$ by

$$X(g) |h\rangle = |h + g\rangle,$$
$$Z(\chi) |h\rangle = \chi(h) |h\rangle.$$ 

Let $G$ be the group generated by $X(g) \otimes X(g)$ for all $g$ and $Z(\chi) \otimes Z(\chi)$ for all $\chi$. Then each generalized Bell state is a common eigenvector for $G$, since we compute

$$X(g') \otimes X(g') v_{g,\chi} = \chi(g') v_{g,\chi},$$
$$Z(\chi') \otimes Z(\chi') v_{g,\chi} = \chi'(g) v_{g,\chi}.$$ 

Moreover, for any two generalized Bell states, there exists an element of $G$ that distinguishes them. From this, it follows that a state is invariant under $G$ if and only if it is a mixture of generalized Bell states. Since any two such mixtures commute, it follows that $[\rho, \sigma] = 0$ for some $\sigma$ optimal for $\rho$. □
For $G = \mathbb{Z}_2$, we can explicitly solve for $\sigma$; if

$$\rho = av_{00}v_{00}^\dagger + bv_{01}v_{01}^\dagger + cv_{10}v_{10}^\dagger + dv_{11}v_{11}^\dagger,$$

with $a \geq \frac{1}{2} \geq b \geq c \geq d$, then

$$\sigma = \frac{1}{2}v_{00}v_{00}^\dagger + \frac{b}{2(1-a)}v_{01}v_{01}^\dagger + \frac{c}{2(1-a)}v_{10}v_{10}^\dagger + \frac{d}{2(1-a)}v_{11}v_{11}^\dagger,$$

and

$$B_\Gamma(\rho) = 1 + a \log_2 a + (1 - a) \log_2(1 - a),$$

agreeing with the bound of [4]. In general, it appears to be rather more difficult to compute the optimal $\sigma$ analytically. Of particular interest would be the case $\mathbb{Z}_2^n$, corresponding to mixtures of tensor products of the usual Bell states.

**Maximally correlated states**

We close with consideration of a class of states for which it is a reasonable conjecture that the p.p.t. bound is not only tight, but is in fact equal to the 1-locally distillable entanglement. This also gives examples of states for which the bound is additive, even though theorem 5 does not apply.

Say that a state $\rho$ on $V \otimes V$ is maximally correlated if for any classical measurement on $V$, Alice and Bob will always obtain the same result. In other words, $\rho$ is of the form

$$\rho = \sum_{i,j} \alpha_{ij} |ii\rangle\langle jj|$$

for some Hermitian, trace 1 operator $\alpha$ on $V$.

**Theorem 9.** If $\rho$ is maximally correlated, then

$$B_\Gamma(\rho) = S(\text{Tr}_A(\rho)) - S(\rho).$$

For any other maximally correlated state $\rho'$,

$$B_\Gamma(\rho \otimes \rho') = B_\Gamma(\rho) + B_\Gamma(\rho').$$

**Proof.** We first show that the state

$$\sigma = \sum_i \alpha_i |ii\rangle\langle ii|$$

is optimal for $\rho$. Certainly, $\sigma$ is p.p.t. (indeed, it is manifestly separable), so it remains to apply the Karush-Kuhn-Tucker condition. Of course, $\sigma$ is only semi-definite, so we must use the more complicated condition following Theorem 4.

We have

$$D_\sigma \text{Tr}(\rho \log(\sigma)) = \sum_{i,j} \alpha_{ij} f(\alpha_{ii}, \alpha_{jj}) |ii\rangle\langle jj|,$$

where

$$f(\alpha, \beta) = \frac{\log \alpha - \log \beta}{\alpha - \beta}.$$
except that
\[ f(\alpha, \alpha) = \frac{1}{\alpha}. \]

We thus choose
\[ L = \sum_{i \neq j} \lambda_{ij} |ij \rangle \langle ij | \]
\[ K = -\sum_{i \neq j} \alpha_{ij} f(\alpha_{ii}, \alpha_{jj}) |ii \rangle \langle jj | + \sum_{i \neq j} (1 - \lambda_{ij}) |ij \rangle \langle ij |, \]
for suitable numbers \( \lambda_{ij} = \lambda_{ji} \). We observe that
\[ \sigma L = \sigma^T K^T = 0, \]
so it remains to verify that \( L, K^T \geq 0 \). Clearly \( L \geq 0 \) if and only if each \( \lambda_{ij} \geq 0 \), while \( K^T \) is essentially a block matrix with \( 2 \times 2 \) blocks
\[ \begin{pmatrix} 1 - \lambda_{ij} & -\alpha_{ij} f(\alpha_{ii}, \alpha_{jj}) \\ -\alpha_{ji} f(\alpha_{ii}, \alpha_{jj}) & 1 - \lambda_{ij} \end{pmatrix}. \]

We thus obtain the conditions
\[ 0 \leq \lambda_{ij} \leq 1 \]
\[ 1 - \lambda_{ij} \geq |\alpha_{ij}| f(\alpha_{ii}, \alpha_{jj}). \]

We can choose \( \lambda_{ij} \) satisfying these conditions if and only if
\[ |\alpha_{ij}| f(\alpha_{ii}, \alpha_{jj}) \leq 1. \]

But, since \( \alpha \) is positive,
\[ (|\alpha_{ij}| f(\alpha_{ii}, \alpha_{jj}))^2 \leq \alpha_{ii} \alpha_{jj} f(\alpha_{ii}, \alpha_{jj})^2 = \frac{\beta \log(\beta)}{(1 - \beta)^2} \leq 1, \]
where \( \beta = \alpha_{jj}/\alpha_{ii} \).

Additivity follows from the fact that the tensor product of maximally correlated states is maximally correlated. \( \square \)

In particular,

**Corollary 2.** If \( \rho \) is a pure state, then
\[ B_\Gamma(\rho) = E_f(\rho). \]

**Proof.** Increasing the dimension of \( V_A \) or \( V_B \) as necessary to make the dimensions equal, we find that \( \rho \) is locally equivalent to a maximally correlated state. The result then follows immediately from theorem 9. \( \square \)

Here, in fact, \( D_1(\rho) = B_\Gamma(\rho) \), by the fact [10] that pure states can be distilled at a rate equal to their entanglement of formation. We also have
Corollary 3. Let $\rho$ be a maximally correlated state on a $2 \times 2$ dimensional Hilbert space. Suppose $\text{Tr}_A(\rho) = 1/2$. Then

$$D_1(\rho) = B_1(\rho) = 1 - S(\rho).$$

Proof. This is equivalent to a mixture of two Bell states. As remarked in [4], we can distill this using classical error correcting codes, and find

$$D_1(\rho) = 1 - S(\rho).$$

Since these two examples are at opposite extremes in a certain sense, the following conjecture seems reasonable:

Conjecture. For any maximally correlated state $\rho$,

$$D_1(\rho) = B_1(\rho).$$

Acknowledgements

The author would like to thank J. Smolin and especially P. Shor for helpful conversations.

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