The Spectrum of Period-Doubling Hamiltonian

Qinghui Liu¹, Yanhui Qu², Xiao Yao³

1 Department of Computer Science, Beijing Institute of Technology, Beijing 100081, People’s Republic of China. E-mail: qhliu@bit.edu.cn
2 Department of Mathematical Science, Tsinghua University, Beijing 100084, People’s Republic of China. E-mail: yhqu@tsinghua.edu.cn
3 School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, People’s Republic of China. E-mail: yaoxiao@nankai.edu.cn

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Abstract: In this paper, we show the following: the Hausdorff dimension of the spectrum of period-doubling Hamiltonian is bigger than \( \log \alpha / \log 4 \), where \( \alpha \) is the Golden number; there exists a dense uncountable subset of the spectrum such that for each energy in this set, the related trace orbit is unbounded, which is in contrast with a recent result of Carvalho (Nonlinearity 33, 2020); we give a complete characterization for the structure of gaps and the gap labelling of the spectrum. All of these results are consequences of an intrinsic coding of the spectrum we construct in this paper.

1. Introduction

1.1. Background and motivation. Since the pioneer works [19, 26], the discrete Schrödinger operators with substitutional potentials have been extensively studied. In these two works, a crucial concept–trace map, were introduced, which became a standard and powerful tool for studying the spectral properties of this class of operators. Using trace map technique, Bovier and Ghez [7] showed that if the potential is generated by primitive substitution, then, under certain technical condition, the spectrum of the related operator is of zero Lebesgue measure. Liu et. al. [24] removed this technical condition. Lenz [21] also proved the same result by using cocycle dynamical technique, which is quite different from the previous two papers. Damanik and Lenz [17] showed that if the potential is generated by a minimal subshift satisfying the so-called Boshernitzan condition, then the result still holds. We note that the subshift related to a primitive substitution is minimal and satisfies the Boshernitzan condition.

Three operators are heavily studied: the Fibonacci, Thue-Morse and period-doubling Hamiltonians. The potentials of these Hamiltonians are generated by Fibonacci, Thue-Morse and period-doubling substitutions, respectively. We note that Fibonacci substitution is invertible, while the Thue-Morse and period-doubling substitutions are not (see Appendix B for a brief introduction on substitution). The differences in substitution rules for these models have great impacts on the corresponding dynamical systems induced
by the trace polynomials. For the Fibonacci case, the induced dynamical system is a
diffeomorphism on a cubic surface, which has a close relation with Hénon map on $\mathbb{C}^2$
There has been much progress for understanding the dynamics of Hénon map in the past
three decades. As a resonance, many powerful tools have been developed in the study of
the spectrum of the Fibonacci Hamiltonian, see [8,13–16]. While, for the Thue-Morse
and period-doubling models, the corresponding two dimensional dynamical systems are
completely different: they are not diffeomorphisms at all, see for example [3,6,23] and
the present work. The dynamics of two dimensional polynomial endomorphisms (which
are not automorphisms) are also quite developed, but usually the questions that people
study involve either generic properties, or a very specific map somehow important due to
a specific application. The latter are the cases of the trace maps for period-doubling and
Thue-Morse Hamiltonians –They are highly degenerate polynomial endomorphisms,
and their dynamical properties have to be studied “by bare hands”, since the degeneracy
makes most of the general theory not applicable.
Fibonacci Hamiltonian was introduced in [19,26], which soon became the most
popular model for quasicrystal. Let $\{\hat{h}_n : n \geq 1\}$ be the trace polynomials related to
Fibonacci Hamiltonian, Casdagli [10] defined the pseudospectrum of the operator:
$$\hat{B}_{\infty} := \left\{ E \in \mathbb{R} : \{\hat{h}_n(E) : n \geq 1\} \text{is bounded} \right\}$$
and showed that $\hat{B}_{\infty}$ is of zero Lebesgue measure. Sütő [31] showed that $\hat{B}_{\infty}$ coincides
with the spectrum $\hat{\sigma}_{\lambda}$ of Fibonacci Hamiltonian. Dynamically, $\hat{B}_{\infty}$ can be redefined as
follows (see for example [10]). Let
$$\hat{f}(x, y, z) := (y, z, yz - x)$$
be the trace map of Fibonacci Hamiltonian, then $\hat{f} : S_{\lambda} \to S_{\lambda}$ is a diffeomorphism,
where $\lambda > 0$ is the coupling constant and
$$S_{\lambda} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - xyz = 4 + \lambda^2\}.$$ 
Moreover, for any $n \geq 0$ and $E \in \mathbb{R}$, one has
$$\hat{f}(\hat{h}_n(E), \hat{h}_{n+1}(E), \hat{h}_{n+2}(E)) = (\hat{h}_{n+1}(E), \hat{h}_{n+2}(E), \hat{h}_{n+3}(E)).$$
For any $E$, write the initial condition as $\hat{\ell}(E) := (2, E, E - \lambda)$. Then
$$\hat{B}_{\infty} = \left\{ E \in \mathbb{R} : \{\hat{f}^n(\hat{\ell}(E)) : n \geq 1\} \text{is bounded} \right\}.$$ 
These two papers ([10,31]) suggest a way of characterizing the spectrum via bound-
edness of trace orbits, compare (1). Along this direction, Damanik [11] showed that if
the potential is generated by an invertible primitive substitution over two letters, then the
spectrum of the related operator still coincides with $\hat{B}_{\infty}$. In [10], Casdagli essentially
coded the spectrum by a kind of subshift of finite type when the coupling $\lambda > 16$. Ray-
mond [29] developed this approach to give coding for Sturmian Hamiltonian, which has
Sturmian sequence as its potential. We note that Sturmian sequence is not necessarily
generated by single substitution. Through this coding, he could compute the integrated
density of states(IDS) of all the energies in an explicit way when $\lambda > 4$. His approach
also gave a proof for gap opening and gap labelling of Sturmian Hamiltonian. The Hausdorff dimension of $\hat{\sigma}_\lambda$ has been well understood, see [8, 12–14, 16, 20, 29]. In particular, the following property is shown in [12]:

$$\lim_{\lambda \to \infty} (\dim_H \hat{\sigma}_\lambda) \log \lambda = \log(1 + \sqrt{2}).$$

This implies that $\dim_H \hat{\sigma}_\lambda \to 0$ with the speed $1/\log \lambda$ when $\lambda \to \infty$. In the remarkable paper [16], almost all interesting spectral properties of Fibonacci Hamiltonian are established for all the coupling $\lambda > 0$.

Next we discuss Thue-Morse Hamiltonian (TMH). This operator has potential generated by Thue-Morse substitution, which is primitive but non invertible, as we have mentioned above. TMH was studied in many works at 1980’s via trace maps, see for example [1–4, 25]. In particular, the gap labelling and gap opening properties of the operator were studied in [2, 4, 25]. However, more detailed structure of the spectrum (such as a coding) is not known compare with the Fibonacci case. This is reflected by the fact that the computation of the dimension of the spectrum is very difficult. Recently it was shown in [22] that for any $\lambda > 0$, the spectrum $\tilde{\sigma}_\lambda$ of TMH satisfies

$$\dim_H \tilde{\sigma}_\lambda \geq \frac{\log 2}{140 \log 2.1} = 0.00667 \ldots$$

Also, a numerical estimation for the box dimension of the spectrum was provided in [27]. Indicated by the results of [10, 31], one may expect that the spectrum also coincides with $\tilde{B}_\infty$, where

$$\tilde{B}_\infty := \left\{ E \in \mathbb{R} : \{ \tilde{h}_n(E) : n \geq 1 \} \text{ is bounded} \right\},$$

where $\{ \tilde{h}_n : n \geq 0 \}$ are the trace polynomials related to TMH. This problem was studied in [23]. There, a Thue-Morse trace map $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$\tilde{f}(x, y) := (x^2(y - 2) + 2, x^2y^2(y - 2) + 2).$$

Note that $\tilde{f}$ is not a diffeomorphism. Indeed it is not even surjective and is generically 4-to-1. For any $E \in \mathbb{R}$ and any $n \geq 1$,

$$\tilde{f}(\tilde{h}_n(E), \tilde{h}_{n+1}(E)) = (\tilde{h}_{n+2}(E), \tilde{h}_{n+3}(E)).$$

Write the initial condition as $\tilde{\ell}(E) := (E^2 - \lambda^2 - 2, (E^2 - \lambda^2)^2 - 4E^2 + 2)$, then

$$\tilde{B}_\infty = \left\{ E \in \mathbb{R} : \{ \tilde{f}^n(\tilde{\ell}(E)) : n \geq 1 \} \text{ is bounded} \right\}.$$
that of Fibonacci Hamiltonian, then use this coding to study the dimension, the IDS and gap labeling of the spectrum in a concrete manner, as Raymond did for Sturmian Hamiltonian ([29]).

Most recently, Carvalho [9] showed that for PDH, the spectrum also coincides with the set of energies related to bounded trace orbits. However, our experience in [23] seems suggest that the opposite result should hold, just like the Thue-Morse case. Our second motivation is to understand the problem of existence of unbounded trace orbit completely.

1.2. Main results. Now we set up the setting and state the main results of the paper.

1.2.1. Basic definitions. Let \( \eta \) be the period-doubling substitution: \( \eta(a) = ab, \eta(b) = aa \). It is seen that \( \eta^{2n}(a) \) is both a prefix and suffix of \( \eta^{2(n+1)}(a) \). Define a two-sided sequence \( \xi \) as

\[
\xi := \lim_{n \to \infty} \eta^{2n}(a) | \eta^{2n}(a) = \cdots \xi(-2) \xi(-1) | \xi(0) \xi(1) \cdots .
\]

Define the period-doubling potential \( V_\xi = (V_\xi(n))_{n \in \mathbb{Z}} \) by, \( V_\xi(n) = 1 \) if \( \xi(n) = a \) and \( V_\xi(n) = -1 \) if \( \xi(n) = b \) for \( n \in \mathbb{Z} \). Take \( \lambda \in \mathbb{R} \) and \( \lambda \neq 0 \). Let \( H_\lambda V_\xi \) be the discrete Schrödinger operator acting on \( \ell^2(\mathbb{Z}) \) with potential \( \lambda V_\xi \), i.e., for any \( n \in \mathbb{Z} \),

\[
(H_\lambda V_\xi \psi)_n = \psi_{n+1} + \psi_{n-1} + \lambda V_\xi(n) \psi_n, \quad \forall \psi \in \ell^2(\mathbb{Z}).
\]

We call \( H_\lambda V_\xi \) the period-doubling Hamiltonian (PDH). Denote by \( \sigma(H_\lambda V_\xi) \) the spectrum of \( H_\lambda V_\xi \). From now on, we only consider PDH, so we will write

\[
H_\lambda := H_\lambda V_\xi \quad \text{and} \quad \sigma_\lambda := \sigma(H_\lambda V_\xi). \quad (2)
\]

It is a general fact that \( \sigma_{-\lambda} = -\sigma_\lambda \). So in the following, without loss of generality, we always assume \( \lambda > 0 \).

Given \( E \in \mathbb{R} \), define an anti-homomorphism \( \tau : \text{FG}(a, b) \to \text{SL}(2, \mathbb{R}) \) as

\[
\tau(a) := \begin{bmatrix} E - \lambda & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \tau(b) := \begin{bmatrix} E + \lambda & -1 \\ 1 & 0 \end{bmatrix}
\]

and \( \tau(a_1 \cdots a_n) := \tau(a_n) \cdots \tau(a_1) \), where \( \text{FG}(a, b) \) is the free group generated by \( a, b \).

Define

\[
h_n(E) := \text{tr}(\tau(\eta^n(a))). \quad (3)
\]

The function \( h_n \) is called the \( n \)-th trace polynomial related to PDH. Since the length of the word \( \eta^n(a) \) is \( 2^n \), the leading term of \( h_n(E) \) is \( E^{2^n} \). Hence, we have \( \deg(h_n) = 2^n \).

For each \( E \in \mathbb{R} \), we define

\[
\mathcal{O}(E) := \{h_n(E) : n \geq 0\} \quad (4)
\]

and call \( \mathcal{O}(E) \) the trace orbit of \( E \). Define

\[
B_\infty := \{E \in \mathbb{R} : \mathcal{O}(E) \text{ is bounded}\}.
\]

We say \( E \in \sigma_\lambda \) is of \( \infty \)-type if \( \mathcal{O}(E) \) is unbounded. Define

\[
\delta_\infty := \{E \in \sigma_\lambda : E \text{ is of } \infty \text{-type}\}. \quad (5)
\]
1.2.2. Lower bound for the Hausdorff dimension of the spectrum. At first we have the following estimate for the dimension of the spectrum:

**Theorem 1.1.** Let $\alpha = (1 + \sqrt{5})/2$. Then for any $\lambda > 0$, 
\[
\dim_H \sigma_\lambda \geq \frac{\log \alpha}{\log 4} = 0.34712 \cdots
\]

**Remark 1.2.**
1) In [27], a numerical estimation for $\dim_H \sigma_\lambda$ is given. The numerical lower bound of $\dim_H \sigma_\lambda$ is around 0.75. It is natural to guess that $\dim_H \sigma_\lambda \to 1$ when $\lambda \to 0$. However the method of this paper seems hard to be applied to deal with this problem.

2) Indeed, $\log \alpha$ is the entropy of certain subshift of finite type and $\log 4$ can be viewed as an upper bound of the Lyapunov exponents, see Remark 5.7 for explanation. So the lower bound is kind of Young’s dimension formula ([33]).

3) As we have mentioned, in [22] we obtained a lower bound 0.00667 · · · for the Hausdorff dimension of the spectrum of TMH. Let us make a comparison on the related methods. In [22], we showed that, after a suitable renormalization, the trace polynomials $\{h_n : n \geq 1\}$ are exponentially close to the model family $\{2 \cos 2^n E : n \geq 1\}$. Using this closeness, we can construct a nested covering structure such that the ratios of lengths between the son intervals and the father intervals are bounded from below. By this, we can show that the dimension of the limit set has a universal lower bound (which is 0.00667 · · ·, very small and far from optimal). Also by the construction, the limit set is a subset of the spectrum, so we obtain the dimension lower bound for the spectrum. It is desirable to adapt the method to the PDH case. It is true that one can still find a model family $\{2 \cos 2^n \sqrt{E} : n \geq 1\}$ such that after a suitable renormalization, the trace polynomials $\{h_n : n \geq 1\}$ of PDH are close to them. However it is much harder to show the exponential convergence. It is even harder to construct a good Cantor subset of the spectrum if it was possible. Here we take a totally different approach. Due to the explicit coding of the spectrum (see Theorem 1.5), we can construct a nested covering structure with nice separation property. This structure determines a limit set, which is a subset of the spectrum. Every energy in this subset has a bounded trace orbit and the bound is 2. This bound together with the recurrence relation of the trace polynomials imply that the lengths of the bands in level $n$ are bigger than $4^{-n}$. Now it is not hard to show that the dimension of the limit set has a good lower bound.

4) We also note that it is doable to apply the method in the present paper to the TMH to give a much better lower bound for the Hausdorff dimension of the spectrum.

1.2.3. Existence of unbounded trace orbits. Define the set of zeros of trace polynomials as
\[
Z^o := \bigcup_{n \text{ odd}} \{h_n = 0\}; \quad Z^e := \bigcup_{n \text{ even}} \{h_n = 0\}; \quad Z := Z^o \cup Z^e.
\]  

**Theorem 1.3.** We have $Z \subset B_\infty \subset \sigma_\lambda$ and $\sigma_\infty = \sigma_\lambda \setminus B_\infty$. Both $B_\infty$ and $\sigma_\infty$ are dense in $\sigma_\lambda$ and uncountable.

**Remark 1.4.**
1) As we have mentioned, Carvalho [9] claimed that $\sigma_\infty$ is empty. Here we show the contrary. See Appendix A for some remarks related to this conflict. We will see soon that $\infty$-type energies are closely related to the gap edges of the spectrum.
2) In [23], for TMH, we showed that \( \tilde{E}_\infty \) is dense in \( \tilde{\sigma}_\lambda \) and uncountable. But we did not show that \( \tilde{B}_\infty \) is a subset of \( \tilde{\sigma}_\lambda \). Here, we show that \( \{ B_\infty, \tilde{E}_\infty \} \) form a partition of \( \sigma_\lambda \).

3) Similar to the TMH case, we will construct a period-doubling trace map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) (see (72)) such that for any \( E \in \mathbb{R} \) and any \( n \geq 0 \),

\[
f(h_n(E), h_{n+1}(E)) = (h_{n+2}(E), h_{n+3}(E)).
\]

If we write the initial condition as \( \ell(E) := (E - \lambda, E^2 - \lambda^2 - 2) \), then

\[
B_\infty = \{ E \in \mathbb{R} : \{ f^n(\ell(E)) : n \geq 1 \} \text{ is bounded} \}.
\]

We will see that the dynamical properties of \( f \) play an essential role for showing the injectivity of the coding map.

Both results are consequences of an intrinsic coding of the spectrum. So we turn to this coding map.

1.2.4. Coding of the spectrum. We will code the spectrum \( \sigma_\lambda \) via a subshift of finite type. Very roughly, we will show that each band in the \( n \)-th periodic approximation is of certain type and this band contains several bands in the \((n+1)\)-th periodic approximation with a fixed configuration of types. We will define an alphabet to record the types and design several admissible rules to record the type evolution. More precise explanation will be given in Sect. 1.3.3.

Now we define such a subshift. Define an alphabet \( A \) as

\[
A := \{0_e, 0_o, 1_e, 1_o, 2_e, 2_o, 3_{el}, 3_{er}, 3_{ol}, 3_{or}\}.
\]

Define the admissible rules as

\[
\begin{align*}
0_e & \rightarrow \{3_{ol}, 1_o, 3_{or}\}; & 0_o & \rightarrow \{3_{el}, 1_e, 3_{er}\}; & 1_e & \rightarrow \{3_{ol}, 2_o\}; & 1_o & \rightarrow \{2_e, 3_{er}\}; \\
2_e & \rightarrow \{1_o, 3_{or}, 2_o\}; & 2_o & \rightarrow \{2_e, 3_{el}, 1_e\}; & 3_{el}, 3_{er} & \rightarrow 0_o; & 3_{ol}, 3_{or} & \rightarrow 0_e.
\end{align*}
\]

See Fig. 1 for the related directed graph \( G \) (we will explain the extra information such as the colors and the labels of the edges, the dashed red square etc. later). It is seen that the graph \( G \) is irreducible. Consequently, \( G \) is connected. Define the adjacency matrix \( A = [a_{\alpha\beta}] \) of \( G \) as

\[
\begin{cases}
a_{\alpha\beta} = 1, & \text{if } \alpha \to \beta; \\a_{\alpha\beta} = 0, & \text{otherwise.}
\end{cases}
\]

Let \( \Omega_A \) be the related subshift of finite type:

\[
\Omega_A := \{ \omega \in A^\mathbb{N} : a_{\omega_j\omega_{j+1}} = 1, j \geq 0 \}.
\]

Define \( d : \Omega_A \times \Omega_A \to [0, \infty) \) as

\[
d(\omega, \hat{\omega}) := \begin{cases}2^{-|\omega \wedge \hat{\omega}|}, & \text{if } \omega \neq \hat{\omega} \\0, & \text{otherwise.}\end{cases}
\]

It is standard to check that \( d \) is a metric and \( (\Omega_A, d) \) is compact. Define the symbolic space

\[
\Omega_\infty := \{ \omega \in \Omega_A : \omega_0 \in \{3_{el}, 0_e\}\}.
\]
Fig. 1. The directed graph $G$ related to the subshift

\[ \Omega_\infty \text{ is a union of two cylinders: } \Omega_\infty = [3_{el}] \cup [0_e], \text{ so it is compact.} \]

Next we define a total order \( \preceq \) on \( \Omega_\infty \), which is essentially induced by the standard order \( \leq \) on \( \mathbb{R} \). (We will explain this in Sects. 1.3.5 and 4.2).

Define \( \preceq \) to be the smallest partial order on \( A \) such that the following holds:

\[
3_{el} \prec 1_o \prec 3_{or} \prec 2_o; \quad 2_e \prec 3_{el} \prec 1_e \prec 3_{er}; \quad 3_{el} \prec 0_e.
\] (10)

Define a relation \( \preceq \) on \( \Omega_\infty \) as follows. For any \( \omega \in \Omega_\infty \), let \( \omega \preceq \omega \).

We will show that \( \preceq \) is a total order on \( \Omega_\infty \), see Lemma 4.5.

**Theorem 1.5.** There exists an order-preserving homeomorphism \( \pi : (\Omega_\infty, \preceq) \to (\sigma_\lambda, \leq) \), where \( \leq \) is the standard order on \( \mathbb{R} \).

We call \( \pi \) the coding map of \( \sigma_\lambda \).

**Remark 1.6.** In some sense, we construct a “Markov partition” of the spectrum \( \sigma_\lambda \) through the coding given in Theorem 1.5. The idea of constructing combinatoric models (Markov partition, Young tower, Yoccoz puzzle, inducing scheme, etc.) plays a central role for understanding the hyperbolicity of the dynamical system, which can date back to Smale ([30]). For the spectrum of discrete Schrödinger operator, the idea of constructing certain Markov partition can date back to Casdagli for Fibonacci Hamiltonian ([10]) and Raymond for Sturmian Hamiltonian ([29]). Some recent progress have been made by Cantat ([8]) and Damanik, Gorodetski, Yessen ([13–16]), which heavily rely on the dynamics of \( \hat{f} \) on the surface \( S_\lambda \).

To some extent, all the constructions of the combinatoric models relies on the bounded distortion (complex bound) arguments. For the PDH model, the existence of the energies with unbounded trace orbits is the main obstruction for the bounded distortion phenomenon, which does not happen in the Fibonacci model. This makes the construction of combinatoric model extremely hard, we need to be very careful in discussion of various combinatorics of the zeros, bands, ends of the bands and related issues.

**Remark 1.7.** Let us make a comparison with [29] on the methods of constructing codings. For both classes of Hamiltonians, let \( \sigma_n \) be the \( n \)-th periodic approximation of the spectrum \( \sigma (H_\lambda) \), one can show that \( \{ \sigma_n \cup \sigma_{n+1} : n \geq 1 \} \) is a nested sequence of coverings of the spectrum. By choosing an optimal covering in each level, it is not hard to construct a symbolic space \( \Omega \) and a surjective coding \( \pi : \Omega \to \sigma (H_\lambda) \). Usually it is much harder to get a bijective coding. To achieve this, one need some disjoint condition for bands.
in \( \sigma_n \) and \( \sigma_{n+1} \). That is why in [29], the technical assumption \( \lambda > 4 \) is needed. We remark that when \( \lambda \downarrow 0 \), the overlapping of \( \sigma_n \) and \( \sigma_{n+1} \) is unavoidable. This presents an obstruction for a bijective coding. In our case, many bands in the \( n \)-th optimal covering do overlap with each other (at least for small \( \lambda \)). One can see Fig. 3 (b) for some intuition of this situation. A big challenge is to show that as \( n \) grows, this kind of overlapping will disappear. By a detailed analysis on the band configurations of \( \sigma_n \cup \sigma_{n+1} \cup \sigma_{n+2} \), finally we succeed to show that all the overlapping disappear and gaps of \( \sigma(H_z) \) show up. Surprisingly, we find that the edges of this kind of gaps are related to unbounded trace orbits. Indeed, the bijectivity of \( \pi \) follows from the study of the unbounded trace orbits.

1.2.5. Gaps, zeros of trace polynomials and \( \infty \)-type energies. With this coding map in hand, we can give a complete description for the gaps of the spectrum. We will see that the gap edges, zeros of trace polynomials and \( \infty \)-type energies are related in a delicate but beautiful way.

We will see that, if an energy is a gap edge, then its coding is eventually 2-periodic. Figure 1 suggests that there are 7 types of eventually 2-periodic codings:

\[
\begin{align*}
\mathcal{E}_1^0 &:= \{ \omega \in \Omega_{\infty} : \omega = w(2_01_2)_{\infty} \text{ for some } w \in \Omega_s \} \\
\mathcal{E}_0^r &:= \{ \omega \in \Omega_{\infty} : \omega = w(0_33_1)_{\infty} \text{ for some } w \in \Omega_s \} \\
\mathcal{E}_1^e &:= \{ \omega \in \Omega_{\infty} : \omega = w(2_10_2)_{\infty} \text{ for some } w \in \Omega_s \} \\
\mathcal{E}_e^r &:= \{ \omega \in \Omega_{\infty} : \omega = w(2_20_2)_{\infty} \text{ for some } w \in \Omega_s \} \\
\mathcal{E}_l &:= \{ \omega \in \Omega_{\infty} : \omega = w(0_33_2)_{\infty} \text{ for some } w \in \Omega_s \} \\
\mathcal{E}_r &:= \{ \omega \in \Omega_{\infty} : \omega = w(2_13_2)_{\infty} \text{ for some } w \in \Omega_s \} \\
\mathcal{F} &:= \{ \omega \in \Omega_{\infty} : \omega = w(2_20_2)_{\infty} \text{ for some } w \in \Omega_s \}
\end{align*}
\]

where \( \Omega_s \) is the set of finite admissible words. Define two special codings as follows:

\[
\omega_* := 3_{el}(0_33_1)_{\infty}; \quad \omega^* := (0_33_2)_{\infty}.
\]

Denote the set of gaps of \( \sigma_{\lambda} \) by \( \mathcal{G} \). We have the following complete description on the structure of \( \mathcal{G} \).

**Theorem 1.8.** With the notations above, we have:

i) \( \min \sigma_{\lambda} = \pi(\omega_*) \) and \( \max \sigma_{\lambda} = \pi(\omega^*) \).

ii) (Type-I gaps): There exists a bijection \( \iota^0 : \pi(\mathcal{E}_i^0) \to \pi(\mathcal{E}_i^0 \setminus \{\omega_*\}) \) such that \( E < \iota^0(E) \) for any \( E \in \pi(\mathcal{E}_i^0) \) and

\[
\mathcal{G}_i^0 := \{(E, \iota^0(E)) : E \in \pi(\mathcal{E}_i^0)\} \subset \mathcal{G}.
\]

There exists a bijection \( \iota^e : \pi(\mathcal{E}_i^e) \to \pi(\mathcal{E}_i^e \setminus \{\omega^*\}) \) such that \( \iota^e(E) < E \) for any \( E \in \pi(\mathcal{E}_i^e) \) and

\[
\mathcal{G}_i^e := \{(\iota^e(E), E) : E \in \pi(\mathcal{E}_i^e)\} \subset \mathcal{G}.
\]

iii) (Type-II gaps): There exists a bijection \( \iota : \pi(\mathcal{E}_l) \to \pi(\mathcal{E}_r) \) such that \( E < \iota(E) \) for any \( E \in \pi(\mathcal{E}_l) \) and

\[
\mathcal{G}_{11} := \{(E, \iota(E)) : E \in \pi(\mathcal{E}_l)\} \subset \mathcal{G}.
\]

iv) We have \( \mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_{11} \), where \( \mathcal{G}_1 := \mathcal{G}_i^0 \cup \mathcal{G}_i^e \).
v) (Coding of zeros): \( \pi(\mathcal{E}_o) = \mathcal{Z}_o \) and \( \pi(\mathcal{E}_e) = \mathcal{Z}_e \).

vi) (\( \infty \)-type energies): For any \( E \in \{ \mathcal{E}_e, \mathcal{E}_o, \mathcal{E}_l, \mathcal{E}_r \} \), we have

\[
\pi(\mathcal{E}) \subset \mathcal{E}_\infty \quad \text{and} \quad \pi(\mathcal{E}) \text{ is dense in } \sigma_\lambda.
\]

We say that the energy \( E \in \sigma_\lambda \) is of type \( \infty \) (type \( \infty \) II) if \( E \in \pi(\mathcal{E}_e \cup \mathcal{E}_o) \) (\( \pi(\mathcal{E}_l \cup \mathcal{E}_r) \)).

Remark 1.9.

1) For a gap in \( \mathcal{G}_I \), one of its end point is a zero of trace polynomials, another endpoint if of type \( \infty I \). For a gap in \( \mathcal{G}_{II} \), both endpoints of it are of type \( \infty II \). So the codings of all the gap edges are eventually 2-periodic. However, by ii)-iv), \( E \in \pi(\mathcal{F}) \) is not an edge of any gap. See Theorem 1.11 and Remark 1.12 for the special role played by \( \mathcal{F} \).

2) The sets \( \mathcal{G}_e I, \mathcal{G}_o I, \mathcal{G}_{II} \) are corresponding to the green, blue and pink edges of \( \mathcal{G} \), respectively, see Fig. 1. We also remark that \( \mathcal{G}_I \) are exactly the gaps in [6] Theorem 4 which open linearly, while \( \mathcal{G}_{II} \) are exactly the gaps which open exponentially.

1.2.6. Integrated density of states and gap labelling of the spectrum. Now we compute the IDS for every energy in the spectrum. The labels on the edges of the graph \( \mathcal{G} \) play an essential role in this computation.

Define the infinite binary tree \( \Sigma_\infty \) and the evaluation map \( \varepsilon : \Sigma_\infty \to [0, 1] \) as

\[
\Sigma_\infty := \{0, 1\}^\infty; \quad \varepsilon(\sigma) = \sum_{n \geq 1} \frac{\sigma_n}{2^n}.
\]

Define a map \( \Pi : \Omega_\infty \to \Sigma_\infty \) as follows: given \( \omega \in \Omega_\infty \), follow the infinite path \( \omega = \omega_0\omega_1\omega_2 \cdots \) in \( \mathcal{G} \), record the label \( w_i \) on the edge \( \omega_{i-1}\omega_i \) and concatenate them to get

\[
\Pi(\omega) := i(\omega)w_1w_2w_3 \cdots \in \Sigma_\infty,
\]

where \( i(\omega) = 0 \) if \( \omega_0 = 0_e \) and \( i(\omega) = 0 \) if \( \omega_0 = 3_{el} \). Here we use the convention that \( \emptyset w = w\emptyset = w \) for a finite word \( w \).

Denote by \( N(E) \) the IDS of \( E \in \sigma_\lambda \). We have the following:

Theorem 1.10. The map \( \Pi : \Omega_\infty \to \Sigma_\infty \) is surjective and for any \( \omega \in \Omega_\infty \),

\[
N(\pi(\omega)) = \varepsilon(\Pi(\omega)).
\]

In other words, we have the following commutative diagram:

\[
\begin{array}{ccc}
\Omega_\infty & \xrightarrow{\pi} & \sigma_\lambda \\
\downarrow{\Pi} & & \downarrow{N} \\
\Sigma_\infty & \xrightarrow{\varepsilon} & [0, 1]
\end{array}
\]

Later, we will show that \( \Pi \) is not injective. Indeed, we have \( \Pi(\tilde{\mathcal{E}}_l) = \Pi(\tilde{\mathcal{E}}_r) \), and \( \Pi : \Omega_\infty \setminus \tilde{\mathcal{E}}_r \to \Sigma_\infty \) is bijective, see Proposition 4.8 for detail. We also point out that \( \Pi \) play an essential role in the study of the symbolic space \( \Omega_\infty \).

It was conjectured in [25] and proved in [6] that all the possible gaps of \( \sigma_\lambda \) are open and labeled by dyadic numbers or dyadic numbers divided by 3 in \( (0, 1) \). However, to the best of our knowledge, the complete characterization of the labels of these gaps...
seems unknown in the literature (compare [6] Theorem 4 (iv) and [5] Sect. 5.4). With the coding map in hand, we can determine the exact set of numbers which label the gaps. It is known that \( N \) is constant on a gap \( G \) of the spectrum and we denote this number by \( N(G) \).

**Theorem 1.11.** We have the gap labelling:

\[
\{N(G) : G \in \mathcal{G}_I\} = \emptyset \cap (0, 1) \quad \text{and} \quad \{N(G) : G \in \mathcal{G}_{II}\} = \mathcal{E},
\]

where \( \mathcal{G} \) is the set of positive dyadic numbers and

\[
\mathcal{E} = \mathcal{E}(\Pi(\tilde{E})) = \left((\mathcal{D}/3 \setminus \mathcal{D}) \cap (0, 1)\right) \setminus \mathcal{E}(\Pi(F)). \tag{15}
\]

**Remark 1.12.** Assume \( s \in (0, 1) \) is a dyadic number divided by 3. (15) tells us that \( s \) is a label of some gap of \( \sigma_\lambda \) if and only if \( s \notin \mathcal{E}(\Pi(F)) \).

1.3. Sketch of the main ideas. Very roughly, our idea is the following: using periodic approximations to construct a family of nested optimal coverings; find out all the possible band configurations between consecutive levels; define the types of bands and figure out the evolution laws of types; finally construct the symbolic space. We explain more details in the following.

1.3.1. Nested structure. The family of coverings is a nested structure which we define now.

For any \( n \geq 0 \), let \( \mathcal{I}_n = \{I_{n,1}, \ldots, I_{n,k_n}\} \) be a family of compact, non-degenerate intervals. \( \mathcal{I} := \{\mathcal{I}_n : n \geq 0\} \) is called a nested structure (NS), if

i) \( \mathcal{I} \) is optimal: for any \( n \geq 0 \), \( i \neq j \), we have \( I_{n,i} \not\subset I_{n,j} \);

ii) \( \mathcal{I} \) is nested: for any \( I \in \mathcal{I}_{n+1} \), there is a unique \( J \in \mathcal{I}_n \) such that \( I \subset J \);

iii) \( \mathcal{I} \) is minimal: for any \( J \in \mathcal{I}_n \), there exists \( I \in \mathcal{I}_{n+1} \) such that \( I \subset J \).

Assume \( \mathcal{I} := \{\mathcal{I}_n : n \geq 0\} \) is a NS. By ii), the sequence of nonempty compact sets \( \bigcup_{i=1}^{k_n} I_{n,i} : n \geq 0 \) is decreasing. Define

\[
A(\mathcal{I}) := \bigcap_{n \geq 0} \bigcup_{i=1}^{k_n} I_{n,i}.
\]

We call \( A(\mathcal{I}) \) the limit set of \( \mathcal{I} \).

Assume \( \mathcal{I} = \{\mathcal{I}_n : n \geq 0\} \) is a NS. \( \mathcal{I} \) is called a separating nested structure (SNS), if moreover,

iv) \( \mathcal{I} \) is separating: for each \( n \), \( \mathcal{I}_n \) is a disjoint family.

Assume \( \mathcal{I} = \{\mathcal{I}_n : n \geq 0\} \) is a NS. Fix some \( I \in \mathcal{I}_n \). An interval \( J \in \mathcal{I}_{n+1} \) is called a son interval of \( I \) if \( J \subset I \).

We will construct a NS such that its limit set is \( \sigma_\lambda \). Then we choose a sub-NS from it such that this sub-NS is a SNS and for which we can estimate the dimension.
1.3.2. NS via Periodic approximations. For all \( n \geq 0 \), define the \( n \)-th periodic approximation of \( \sigma_\lambda \) as
\[
\sigma_{\lambda,n} := \{ E \in \mathbb{R} : |h_n(E)| \leq 2 \}.
\] (16)

By Floquet theory, \( \sigma_{\lambda,n} \) is made of \( 2^n \) non-overlapping intervals. Following the convention, we call these intervals bands. We denote this family of bands by \( \mathcal{B}_n \) and code it by \( \Sigma_n = \{0, 1\}^n \) as
\[
\mathcal{B}_n := \{ B_\sigma : \sigma \in \Sigma_n \}.
\] (17)

See Sect. 2.1.2 for detail. It is well-known that \( \sigma_{\lambda,n} \to \sigma_\lambda \) in Hausdorff distance. In general, \( \mathcal{B}_n \) is not a covering of \( \sigma_\lambda \). However we will show that \( \{ \mathcal{B}_n \cup \mathcal{B}_{n+1} : n \geq 0 \} \) is a family of nested coverings. We will see that \( \mathcal{B}_n \cup \mathcal{B}_{n+1} \) is always not optimal. To get an optimal covering, we simply delete those \( B \in \mathcal{B}_{n+1} \) which is contained in some \( \hat{B} \in \mathcal{B}_n \). Denote by \( \mathcal{B}_n \) the resulting optimal covering. Formally
\[
\mathcal{B}_n := \mathcal{B}_n \cup \{ B \in \mathcal{B}_{n+1} : B \not\subset \hat{B} \text{ for any } \hat{B} \in \mathcal{B}_n \}.
\] (18)

Now \( \mathcal{B} := \{ \mathcal{B}_n : n \geq 0 \} \) become a NS and its limit set coincides with \( \sigma_\lambda \), see Corollary 3.10. In principle, one can construct a symbolic space and the related coding map of the spectrum by following the sequence \( \{ \mathcal{B}_n : n \geq 0 \} \). However, we warn that when \( \lambda > 0 \) is small, \( \mathcal{B} \) is never a SNS. As a consequence, it is very difficult to show the injectivity of the coding map.

To decide which bands in \( \mathcal{B}_{n+1} \) should be deleted, we need to study the configurations of bands in \( \mathcal{B}_n \cup \mathcal{B}_{n+1} \). This is the content of Lemma 2.3—the first key lemma of the paper.

1.3.3. Types, graph-directed structure and coding. The next step is to study how the bands in \( \mathcal{B}_{n+1} \) are situated in their father bands in \( \mathcal{B}_n \). The crucial concept here is the type of the band.

By performing a renormalization process, we observe that every band in \( \mathcal{B}_n \) has certain type. The type of \( B \in \mathcal{B}_n \) determines its son intervals \( \hat{B} \in \mathcal{B}_{n+1} \) and their types, see Fig. 2 for some intuition. By detailed analysis, we obtain an alphabet \( \mathcal{A} \) for the types and a directed graph \( \mathcal{G} \) for the evolution of types.

Recall that \( \mathcal{B}_n \) is the union of \( \mathcal{B}_n \) and a subset of \( \mathcal{B}_{n+1} \). Roughly speaking, there are four types for \( B \in \mathcal{B}_n \): \( 0, 1, 2, 3 \). For \( k = 0, 1, 2 \), \( B \) has type \( k \) if \( B \in \mathcal{B}_n \) and \( \# \partial B \cap \mathcal{R}_{n-1} = k \) where \( \mathcal{R}_{n-1} \) is the set of zeros of trace polynomials \( h_0, \cdots, h_{n-1} \) (see (26)). \( B \) has type 3 if \( B \not\in \mathcal{B}_n \). Since we aim to give an order-preserving coding for the spectrum, the order of the son intervals become important. With this in mind, we observe that the bands with same types but with different parities of levels have different evolution laws: they are reflectional symmetric. This observation suggests that we need to distinguish \( 0^0, 0^-_e \) etc. This explains the definition (7) except for type 3, for which we need to distinguish left and right further. The evolution laws (8) of types are hidden in Lemma 3.1—the second key lemma of this paper. Now we can build a related directed graph \( \mathcal{G} \).

Once \( \mathcal{G} \) is constructed, we can first code the covering \( \mathcal{B}_n \) by \( \Omega_n \) (the set of admissible words of level \( n \), see (40)) as \( \mathcal{B}_n := \{ I_w : w \in \Omega_n \} \) and then code the spectrum \( \sigma_\lambda = A(\mathcal{B}) \) by the symbolic space \( \Omega_\infty \) as \( \pi(\omega) := \bigcap_n I_{\omega|_n} \). As we have mentioned, it is easy to show that \( \pi \) is surjective and continuous. However to show that \( \pi \) is injective, we need further information on \( \infty \)-type energies, see Sect. 1.3.5.
1.3.4. Sub-SNS and the lower bound of the dimension. Now consider the sub-alphabet \( \tilde{A} = \{1_e, 1_o, 2_e, 2_o\} \) and the sub directed graph \( \tilde{G} \) (enclosed by the red square in Fig. 1) with restricted admissible rules:

\[
1_e \rightarrow 2_o; \quad 1_o \rightarrow 2_e; \quad 2_e \rightarrow 1_o, 2_o; \quad 2_o \rightarrow 1_e, 2_e.
\]

Consider the sub-NS \( \tilde{B} := \{\tilde{B}_n: n \geq 1\} \) defined by

\[
\tilde{B}_n := \{I_{0_e,1_o,w}: w = w_1 \cdots w_n, w_i \in \tilde{A}\}.
\]

Obviously its limit set \( \tilde{A}(\tilde{B}) \) is a subset of the spectrum. Indeed, each energy in \( \tilde{A}(\tilde{B}) \) has a bounded trace orbit of bound 2. In particular, \( \tilde{A}(\tilde{B}) \subset B_\infty \). Dynamically, if \( E \in \tilde{A}(\tilde{B}) \), then for any \( n \geq 0 \),

\[
f^n(\ell(E)) \in [-2, 2] \times [-2, 2].
\]

In some sense, the limit set \( \tilde{A}(\tilde{B}) \) is the portion of \( \sigma_\lambda \) on which the trace polynomials behave in a uniformly hyperbolic way. We can show that \( \tilde{B} \) is a SNS and there is a uniform estimation for the length of the bands in \( \tilde{B}_n \), see Proposition 5.6. Now by a general result for SNS (see Proposition 5.5), we obtain a lower bound for the Hausdorff dimension of the spectrum.

1.3.5. Orders, gaps and \( \infty \)-type energies. Now we explain the orders on \( \Omega_\infty \). There are two orders on \( \Omega_\infty \), both are induced from orders on \( A \). So we start from the orders on \( A \).

We have two goals for designing an order on \( A \): (1) to remember the order of the son bands inside a father band; (2) to remember whether two neighbour son bands overlap. As we have mentioned several times, if \( \lambda \) is small, the overlapping of some son bands are unavoidable. To obtain a nice coding, it is essential to remember whether two bands overlap. For this purpose, we define the following two “orders” to compare two bands.

Assume \( I = [a, b], J = [c, d] \). We define

\[
\begin{cases}
I < J, & \text{if } a < c; b < d, \\
I \prec J, & \text{if } b < c.
\end{cases}
\tag{19}
\]

Relation \( I < J \) means that \( I \) and \( J \) are disjoint and \( J \) is on the right of \( I \). While \( I \prec J \) means that \( I \not\subset J, J \not\subset I \) and \( J \) is on the right of \( I \), however \( I \) and \( J \) may overlap. It is obvious that \( I < J \) implies \( I \prec J \). Lemmas 3.6–3.8 summarize the order configurations of son bands for all types of bands. These motivate the definitions of two orders \( \leq \) and \( \preceq \) on \( A \) (see Sects. 1.2.4 and 4.2.1).

Now we can extend these orders to \( \Omega_\infty \) in an obvious way to get the two orders \( \leq \) and \( \preceq \) on \( \Omega_\infty \). We will show that \( \preceq \) is a total order on \( \Omega_\infty \), which is the order appearing in Theorem 1.5. The order \( \leq \) is stronger than \( \preceq \) in the following sense: if \( \omega < \omega' \), then \( \pi(\omega) < \pi(\omega') \); however, if \( \omega < \omega' \) but \( \omega \not\prec \omega' \), we only have \( \pi(\omega) \leq \pi(\omega') \). It is easy to show the former statement, which follows from the disjointness of the bands. It is quite subtle to show the latter statement because the condition means that the bands containing \( \pi(\omega) \) and \( \pi(\omega') \) overlap with each other. We obtain the inequality by a detailed analysis on the combinatorics of \( \mathcal{Z} \). Note that our ultimate goal is to show the strict inequality, which is even harder. We need to study the dynamics of the trace map \( f \).

Next we study the gap of \( \sigma_\lambda \). Since \( (\Omega_\infty, \leq) \) is totally ordered, the notion of gap for \( \Omega_\infty \) is available. We will at first study the gap structure on \( \Omega_\infty \) (see Theorem 4.6), and then descend it to \( \sigma_\lambda \) by the coding map \( \pi \).
Now we discuss $\infty$-type energy. Recall that $B_n \cup B_{n+1}$ is a covering of the spectrum, this implies that for any $E \in \sigma_\lambda$ and any $n \geq 0$, we have $|h_n(E)| \leq 2$ or $|h_{n+1}(E)| \leq 2$. Inspired by the Thue-Morse case [23], to produce a unbounded trace orbit, we can ask for the following mechanism: there exists $N$ such that
\[
|h_{N+2k}(E)| \leq 2; \quad |h_{N+2k+1}(E)| > 2, \quad \text{for any } k.
\]
Dynamically, this means that for $n$ large enough (if $N$ is even),
\[
f^n(\ell(E)) \in [-2, 2] \times [-2, 2].
\]
If $E$ has coding $\omega$, (20) implies that necessarily $\omega$ is eventually $0_\alpha 3_\alpha 0_\alpha 3_\alpha \cdots$. With this in mind, we finally succeed to show that all the energies with coding eventually $(0_\alpha 3_\beta)^\infty$ are of $\infty$-type.

As one will see in Sect. 6, it is relatively easy to show that $E$ is of $\infty$-type if the coding of $E$ is eventually $(0_\alpha 3_\alpha)^\infty$ or $(0_\alpha 3_\alpha)^\infty$; however it is much harder to show that $E$ is of $\infty$-type if the coding of $E$ is eventually $(0_\alpha 3_\alpha)^\infty$ or $(0_\alpha 3_\alpha)^\infty$. For the latter case, we need to study the local dynamics of $f$ around the fixed point $(-1, -1)$. The injectivity of $\pi$ is also a consequence of this study.

1.3.6. Labelled directed graph and integrated density of states. At last, we explain the labels on the edges of $G$. Assume $B_\sigma \in B_n$ has type $\alpha$, $B_\tau \in B_{n+1}$ has type $\beta$ and $B_\tau \subset B_\sigma$. Then we will show that $\alpha \to \beta$ and $\sigma$ is a prefix of $\tau$. Write $\tau = \sigma \sigma'$, then the label of the edge $\alpha \beta$ is $\sigma'$. Now by Lemmas 3.5–3.8, we can label all the edges in $G$.

Next we compute the IDS for $E \in Z$. To do this we need to study the structure of $Z$, see Proposition 2.6. Now, since $Z$ is dense in $\sigma_\lambda$, by the continuity of $N$, we can compute the IDS for all $E \in \sigma_\lambda$.

Now we come to the last point–gap labelling. By Theorem 1.8, we know the coding of the gap edges. To compute the IDS of gaps, we just follow the path
\[
E \to \pi^{-1}(E) \to \Pi \circ \pi^{-1}(E) \to \varepsilon \circ \Pi \circ \pi^{-1}(E).
\]

Final remark about the proof: As we will see in all the properties which need to distinguish the parities of the level $n$ that the statements for the odd case are completely symmetric with that for the even case, so are the proofs. So in the rest of the paper, usually we only give the proofs for the odd case and leave the routine checking for the even case to the reader.

1.4. Structure of the paper. The rest of this paper is organized as follows. In Sect. 2, we study the band configurations of $B_n \cup B_{n+1}$ and give a characterization for $B_n$, we also study the properties of $Z$. In Sect. 3, we introduce the concept of type, derive the evolution laws and orders of the types, and determine the labels of the directed edges. In Sect. 4, we study the symbolic space $\Omega_\infty$. In Sect. 5, we prove a weak version of Theorem 1.5, and apply it to prove Theorem 1.1. In Sect. 6, by a detailed study of $\infty$-type energies, we prove Theorems 1.3, 1.5 and 1.8. In Sect. 7, we prove Theorems 1.10 and 1.11. In Sect. 8, we prove the technical lemmas and the related results. We also include four appendices in this paper. In Appendix A, we give some remarks on the paper [9]. In Appendix B, we give a brief introduction on substitution. In Appendix C, we give a characterization for the gaps of $\Sigma_\infty$. In Appendix D, we afford two tables of indexes.
2. Optimal Covering $\mathcal{B}_n$ and the Property of $Z$

At first, we study the band configurations of $\mathcal{B}_n \cup \mathcal{B}_{n+1}$ and give a characterization for $\mathcal{B}_n$. Then we study the property of $Z$.

2.1. The optimal covering $\mathcal{B}_n$.

2.1.1. Coverings of $\sigma_\lambda$ via periodic approximations. It is known that ([6,25]) the sequence of trace polynomials of PDH $\{h_n : n \geq 0\}$ defined by (3) satisfy the following initial conditions and recurrence relation:

\[
\begin{cases}
    h_0(E) = E - \lambda, & h_1(E) = E^2 - \lambda^2 - 2; \\
    h_{n+1}(E) = h_n(E)(h_{n-1}(E) - 2) - 2, & (\forall n \geq 1).
\end{cases}
\]  

We have defined $\sigma_{\lambda,n}$ in (16) and $\mathcal{B}_n$ in (17). By Floquet theory (see for example [32, Chapter 4]), $\sigma_{\lambda,n}$ is the spectrum of $H_{V(n)}$, where $V^{(n)}$ is $2^n$-periodic and

\[V^{(n)}(j) = \lambda V_\xi(j), \quad j = 0, \ldots, 2^n - 1.\]

The set $\sigma_{\lambda,n}$ is called the $n$-th periodic approximation of $\sigma_\lambda$. We know that $\mathcal{B}_n$ is a non-overlapping family. Indeed, we can say more:

**Proposition 2.1.** For any $n \geq 1$, all the possible gaps of $\sigma_{\lambda,n}$ are open, i.e., $\mathcal{B}_n$ is a disjoint family.

We will prove this proposition in Sect. 8.1.3. In general, $\mathcal{B}_n$ is not a covering of $\sigma_\lambda$. However, we have

**Lemma 2.2.** i) For any $n \geq 0$,

\[\sigma_{\lambda,n+2} \subset \sigma_{\lambda,n} \cup \sigma_{\lambda,n+1}.\]

Thus $\{\sigma_{\lambda,n} \cup \sigma_{\lambda,n+1} : n \geq 0\}$ is a decreasing sequence. Moreover,

\[\sigma_\lambda = \bigcap_{n \geq 0} (\sigma_{\lambda,n} \cup \sigma_{\lambda,n+1}).\]  

ii) We have

\[\max_{B \in \mathcal{B}_n} |B| \to 0, \quad n \to \infty.\]

See [22] Lemma A.2. for a proof (there all the results are proven for TMH, but the proof for PDH is the same).

As a consequence, $\mathcal{B}_n \cup \mathcal{B}_{n+1}$ is a covering of $\sigma_\lambda$. 

2.1.2. The coding of \( B_n \). We mentioned that \( B_n \cup B_{n+1} \) is not an optimal covering of \( \sigma \).

To construct an optimal covering, we need to throw some bands in \( B_{n+1} \). To proceed, we give a coding for \( B_n \).

Assume \( n \geq 1 \). Let \( \Sigma_n := \{0, 1\}^n \) and \( M_n := \{0, 1, \ldots, 2^n - 1\} \). By convention, \( \Sigma_0 := \{\emptyset\} \) and \( M_0 = \{0\} \). We endow \( \Sigma_n \) with the lexicographical order and denote this order by \( \preceq \). If \( \sigma \neq 1^n(\sigma \neq 0^n) \), denote the successor (predecessor) of \( \sigma \) by \( \sigma^+(\sigma^-) \).

For \( \sigma = \sigma_1 \cdots \sigma_n \in \Sigma_n \) and \( k \leq n \), we define the \( k \)-th prefix of \( \sigma \) as

\[
\sigma|_k := \sigma_1 \cdots \sigma_k. 
\]

Assume \( n, k \geq 0 \) and \( \sigma \in \Sigma_n, \tau \in \Sigma_k \). Then \( \sigma \wedge \tau \) is the maximal common prefix of \( \sigma \) and \( \tau; \sigma \tau \in \Sigma_{n+k} \) is the concatenation of \( \sigma \) and \( \tau \). If \( \sigma \) is a prefix of \( \tau \), we write \( \sigma \prec \tau \).

We denote the length of \( \sigma \) by \( |\sigma| \). By convention, \( |\emptyset| = 0 \).

When \( n = 0 \), define \( \varpi_0 : \Sigma_0 \to M_0 \) as \( \varpi_0(\emptyset) := 0 \). When \( n \geq 1 \), define \( \varpi_n : \Sigma_n \to M_n \) as

\[
\varpi_n(\sigma) := \sum_{j=1}^{n} \sigma_j 2^{n-j} = 2^n \sum_{j=1}^{n} \frac{\sigma_j}{2^j}, \quad \text{where} \quad \sigma = \sigma_1 \cdots \sigma_n. \tag{24}
\]

It is direct to check that \( \varpi_n : \Sigma_n \to M_n \) is bijective and order-preserving.

Now we list the \( 2^n \) bands in \( B_n \) from left to right and numbering \( 0, \ldots, 2^n - 1 \). Then we denote the \( j \)-th bands by \( B_{\varpi_n^{-1}(j)} \). In this way, we give a coding for \( B_n \).

For any \( n \geq 0 \), combining with Proposition 2.1, we have

\[
B_n = \{ B_\sigma : \sigma \in \Sigma_n \} \quad \text{and} \quad B_\sigma < B_{\sigma'} \text{ if } \sigma < \sigma'.
\]

Now we have

\[
\sigma_{\lambda,n} = \bigcup_{B \in B_n} B = \bigcup_{\sigma \in \Sigma_n} B_\sigma. \tag{25}
\]

We say that all the bands in \( B_n \) are of level-\( n \).

For any \( \sigma \in \Sigma_n \), write

\[
[a_\sigma, b_\sigma] := B_\sigma.
\]

For any \( n \geq 0 \), define

\[
\mathcal{Z}_n := \{ E \in \mathbb{R} : h_n(E) = 0 \} \quad \text{and} \quad \mathcal{R}_n := \bigcup_{j=0}^{n} \mathcal{Z}_j. \tag{26}
\]

By Floquet theory, in each band \( B_\sigma \), there is exactly one zero of \( h_n \), which we denote by \( z_\sigma \). So we have

\[
a_\sigma < z_\sigma < b_\sigma ; \quad \mathcal{Z}_n = \{ z_\sigma : \sigma \in \Sigma_n \} ; \quad z_\sigma < z_\tau \text{ if } \sigma < \tau. \tag{27}
\]

As an example, we compute \( B_0 \) and \( B_1 \). By (21), we have

\[
\begin{cases} 
B_0 = \{0\}, & B_1 = \{0, 1\}, \\
B_0 = \{\lambda - 2, \lambda + 2\}, & B_1 = \{-\sqrt{\lambda^2 + 4}, -\lambda\}, \\
B_0 = \{\lambda - 2, \lambda + 2\}.
\end{cases} \tag{28}
\]
2.1.3. The band configurations of $B_n \cup B_{n+1} \cup B_{n+2}$

Recall that we have defined two “orders” for bands in (19). The band configurations of $B_n \cup B_{n+1}$ is summarized in the following technical lemma:

**Lemma 2.3.** Fix $n \geq 0$ and $\sigma \in \Sigma_n$.

(i) Assume $n$ is odd. Then $b_{\sigma 0} = z_{\sigma}$, $B_{\sigma 0} \subset B_{\sigma}$ and $a_{\sigma 1} \notin \mathcal{R}_n$. Moreover,

\[
\begin{align*}
B_{\sigma 0} &= [a_{\sigma}, z_{\sigma}], \quad \text{if } a_{\sigma} \in \mathcal{R}_{n-1}, \\
B_{\sigma 0} &\subset (a_{\sigma}, z_{\sigma}], \quad \text{if } a_{\sigma} \notin \mathcal{R}_{n-1},
\end{align*}
\]

\[
\begin{align*}
B_{\sigma 1} &= (z_{\sigma}, b_{\sigma}], \quad \text{if } b_{\sigma} \in \mathcal{R}_{n-1}, \\
B_{\sigma 1} &\subset (z_{\sigma}, b_{\sigma}), \quad \text{if } b_{\sigma} \notin \mathcal{R}_{n-1}.
\end{align*}
\]

(ii) Assume $n$ is even. Then $a_{\sigma 1} = z_{\sigma}$, $B_{\sigma 1} \subset B_{\sigma}$ and $b_{\sigma 0} \notin \mathcal{R}_n$. Moreover,

\[
\begin{align*}
B_{\sigma 0} &= [a_{\sigma}, z_{\sigma}], \quad \text{if } a_{\sigma} \in \mathcal{R}_{n-1}, \\
B_{\sigma 1} &\subset [a_{\sigma}, z_{\sigma}], \quad \text{if } a_{\sigma} \notin \mathcal{R}_{n-1},
\end{align*}
\]

\[
\begin{align*}
B_{\sigma 0} &= (z_{\sigma}, b_{\sigma}], \quad \text{if } b_{\sigma} \in \mathcal{R}_{n-1}, \\
B_{\sigma 0} &\subset (z_{\sigma}, b_{\sigma}), \quad \text{if } b_{\sigma} \notin \mathcal{R}_{n-1}.
\end{align*}
\]

(iii) $B_{\sigma 0} < B_{\sigma 1}$. If $\sigma > 0^n$, then $B_{\sigma 0} < B_{\sigma 0}^-$; if $\sigma < 1^n$, then $B_{\sigma 1} < B_{\sigma 0}^+$.

Here we use the convention that $\mathcal{R}_{-1} = \emptyset$.

One can see Fig. 2 for some concrete examples.

We will prove this lemma in Sect. 8.2. For the moment, we will use it to characterize the bands in $\mathcal{B}_n$.

Recall that $\mathcal{B}_n$ is defined by (18). By Proposition 2.1 and Lemma 2.2 i), $\mathcal{B}_n$ is an optimal covering of $\sigma_\lambda$. By the definition, we always have $B_{n} \subset \mathcal{B}_n$. Now we give a characterization for a band $B \in B_{n+1}$ in $\mathcal{B}_n$.

**Proposition 2.4.** Assume $\sigma \in \Sigma_{n+1}$. Then $B_\sigma \in \mathcal{B}_n$ if and only if $B_\sigma \not\subset B_{\sigma 1}^\sigma$.
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Proof. Write \( \tau := \sigma |_n \). If \( B_{\sigma} \subset B_{\tau} \in B_n \), by (18), \( B_{\sigma} \notin \mathcal{B}_n \).

Now assume \( B_{\sigma} \not\subset B_{\tau} \). By Lemma 2.3 iii),

\[
B_{\tau^{-}} < B_{\sigma}, \quad \text{if } \tau > 0^n; \quad B_{\sigma} < B_{\tau^{+}}, \quad \text{if } \tau < 1^n.
\]

By the coding and the order of the bands in \( B_n \), we conclude that \( B_{\sigma} \not\subset B \) for any \( B \in B_n \). So by (18), \( B_{\sigma} \in \mathcal{B}_n \). \( \square \)

As an example, by (28), we have

\[
B_1 \subset B_{\emptyset} \quad \text{and} \quad B_0 < B_{\emptyset}.
\] (29)

Consequently we have (see Fig. 2a)

\[
\mathcal{B}_0 = \{ B_0, B_{\emptyset} \}.
\]

By direct computation, we have (see also Fig. 2a)

\[
\mathcal{B}_1 = \{ B_0, B_1, B_{01}, B_{11} \}.
\]

2.2. The property of \( Z \). At first, we have the following simple but useful observation:

Lemma 2.5. i) For \( n \geq 0 \), if \( E \in Z_n \), then

\[
h_{n+1}(E) = -2 \quad \text{and} \quad h_k(E) = 2 \quad \text{for any } k \geq n + 2.
\] (30)

Consequently, \( Z_n \cap Z_m = \emptyset \) if \( n \neq m \).

ii) For \( n \geq 1 \) and \( \sigma \in \Sigma_n \),

\[
\text{int}(B_{\sigma}) \cap R_{n-1} = \emptyset \quad \text{and} \quad \text{int}(B_{\sigma}) \cap R_n = \{ z_{\sigma} \}.
\]

iii) We have \( Z \subset B_{\infty} \cap \sigma_{\lambda} \) and \( Z \) is dense in \( \sigma_{\lambda} \).

Proof. i) By (21), one check directly that the (30) holds. WLOG assume \( n < m \). By (30), \( h_{n+1}(Z_n) = \{-2\} \) and \( h_k(Z_n) = \{2\} \) for \( k \geq n + 2 \). Hence \( Z_n \cap Z_m = \emptyset \).

ii) If \( E \in R_{n-1} \), then by i), \( h_n(E) = \pm 2 \). By Floquet theory, \( h_n(\text{int}(B_{\sigma})) = (-2, 2) \). So \( \text{int}(B_{\sigma}) \cap R_{n-1} = \emptyset \). Still by Floquet theory, \( \text{int}(B_{\sigma}) \cap Z_n = \{ z_{\sigma} \} \). Since \( R_n = R_{n-1} \cup Z_n \), we have \( \text{int}(B_{\sigma}) \cap R_n = \{ z_{\sigma} \} \).

iii) By i), \( Z \subset B_{\infty} \). By i) and (22), \( Z \subset \sigma_{\lambda} \). Fix any \( E \in \sigma_{\lambda} \). For any \( n \geq 0 \), by (22), there exists \( \sigma^{(n)} \in \Sigma_n \cup \Sigma_{n+1} \) such that \( E \in B_{\sigma^{(n)}} \). Since \( z_{\sigma^{(n)}} \in B_{\sigma^{(n)}} \), by (23), we have \( |z_{\sigma^{(n)}} - E| \to 0 \). So \( Z \) is dense in \( \sigma_{\lambda} \). \( \square \)
2.2.1. The order-preserving coding of \( Z \). Recall that \( Z \) is defined by (6). Define

\[
\Sigma_* := \bigcup_{n=0}^{\infty} \Sigma_n.
\]  

(31)

By (27) and Lemma 2.5 i), the map \( \sigma \rightarrow z_{\sigma} \) is a bijection between \( \Sigma_* \) and \( Z \). Thus we obtain a coding of \( Z \) as

\[
Z = \{z_{\sigma} : \sigma \in \Sigma_*\}.
\]

Define \( d : (\Sigma_* \cup \Sigma_{\infty}) \times (\Sigma_* \cup \Sigma_{\infty}) \rightarrow [0, \infty) \) as

\[
d(\sigma, \hat{\sigma}) := \begin{cases} 
2^{-|\sigma \wedge \hat{\sigma}|}, & \text{if } \sigma \neq \hat{\sigma} \\
0, & \text{otherwise}
\end{cases}
\]

It is standard to check that \( d \) is a metric on \( \Sigma_* \cup \Sigma_{\infty} \).

In the following, we define an order on \( \Sigma_* \) such that the coding map \( \sigma \rightarrow z_{\sigma} \) preserves the orders.

At first we define an order \( \preceq \) on \( \{0, \emptyset, 1\} \) as

\[
0 \prec \emptyset \prec 1.
\]

For each \( n \geq 1 \), let \( \preceq \) be the lexicographical order on \( \{0, \emptyset, 1\}^n \).

Now we define an order \( \preceq \) on \( \Sigma_* \) as follows. Given \( \sigma, \tau \in \Sigma_n \) and \( \tilde{\sigma}, \tilde{\tau} \in \Sigma_m \). Assume \( n \leq m \). Define \( \tilde{\sigma} := \sigma \emptyset^{m-n} \). Then define

\[
\begin{cases} 
\sigma \prec \tau & \text{if } \tilde{\sigma} \prec \tilde{\tau} \\
\sigma = \tau & \text{if } \tilde{\sigma} = \tilde{\tau} \\
\tau < \sigma & \text{if } \tau < \tilde{\sigma}
\end{cases}
\]

(32)

One check directly that \( \preceq \) is a total order on \( \Sigma_* \).

**Proposition 2.6.** Assume \( \sigma, \tau \in \Sigma_* \). Then

\[
z_{\sigma} < z_{\tau} \text{ if and only if } \sigma \prec \tau.
\]

It has the following consequence:

**Corollary 2.7.** If \( n \geq 1 \) and \( \sigma \in \Sigma_n \), then for any \( E \in \text{int}(B_{\sigma}) \),

\[
\prod_{j=0}^{n-1} h_j(E) \begin{cases} < 0, & \text{if } \sigma_n = 0 \\
> 0, & \text{if } \sigma_n = 1.
\end{cases}
\]

We will prove Proposition 2.6 and Corollary 2.7 in Sect. 8.1.1.

2.2.2. The set \( Z \) and the band edges. Recall that \( \mathcal{R}_{-1} = \emptyset \).

**Lemma 2.8.** Assume \( n, t \geq 0 \) and \( \sigma \in \Sigma_n \). Then

i) \( b_{\sigma 0^t} < a_{\sigma 10^t} \) and \( z_{\sigma} = \begin{cases} b_{\sigma 0^t}, & \text{if } n \text{ is odd; } \\
a_{\sigma 10^t}, & \text{if } n \text{ is even.}
\end{cases} \)

ii) If \( \sigma \in \mathcal{R}_{n-1} \), then \( a_{\sigma 0^t} = a_{\sigma} \); if \( \beta_{\sigma} \in \mathcal{R}_{n-1} \), then \( b_{\sigma 1^0} = b_{\sigma} \).

iii) If \( \sigma \in \mathcal{R}_{n-1} \), then \( a_{\sigma 0^t} = a_{\sigma} \); if \( \beta_{\sigma} \in \mathcal{R}_{n-1} \), then \( b_{\sigma 1^0} = b_{\sigma} \).

iv) If \( \sigma < 1^n \), then \#(\{b_{\sigma}, a_{\sigma}^+\} \cap \mathcal{R}_{n-1}) = 1.

We will prove this lemma in Sect. 8.1.3.
2.3. Two useful basic facts. We record two basic facts, which are frequently used in the whole paper.

The following lemma is fundamental for us.

**Lemma 2.9.** For any \( n \geq 0 \) and \( E \in \mathbb{R} \),

\[
h_{n+1}(E) - (h_n^2(E) - 2) = (-1)^n 2\lambda \prod_{c \in \mathcal{R}_n} (E - c) = (-1)^n 2\lambda \prod_{j=0}^{n} h_j(E). \tag{33}
\]

**Proof.** We prove it by induction. For \( n = 0 \), we have

\[
h_1(E) - (h_0^2(E) - 2) = 2\lambda(E - \lambda) = 2\lambda h_0(E).
\]

Then (33) holds for \( n = 0 \).

Assume (33) holds for \( n = k \geq 0 \). Now consider \( n = k + 1 \). By (21) and induction hypothesis, we have

\[
h_{k+1}(E) - (h_k^2(E) - 2) = h_k(E)(h_{k-1}^2(E) - 2) - (h_k^2(E) - 2) = -h_k(E) \left( h_k(E) - (h_{k-1}^2(E) - 2) \right) = (-1)^k 2\lambda \prod_{j=0}^{k} h_j(E) = (-1)^k 2\lambda \prod_{c \in \mathcal{R}_k} (E - c).
\]

Hence, the result holds for \( n = k + 1 \). By induction, the result follows. \( \Box \)

The following is standard Floquet theory:

**Proposition 2.10.**

i) For each \( \sigma \in \Sigma_n \), \( h_n : B_\sigma \to [-2, 2] \) is a homeomorphism. If \( \sigma_n = 0 \) (\( \sigma_n = 1 \)), then \( h_n'(E) < 0 \) (\( h_n'(E) > 0 \)) for \( E \in \text{int}(B_\sigma) \). Consequently

\[
h_n(a_\sigma) = 2, \quad h_n(b_\sigma) = -2 \quad (h_n(a_\sigma) = -2, \quad h_n(b_\sigma) = 2).
\]

ii) If \( \sigma \neq 1^n \), then the gap \((b_\sigma, a_\sigma^+)\) is open if and only if \( h_n'(b_\sigma) \neq 0 \) or \( h_n'(a_\sigma^-) \neq 0 \).

iii) A point \( E \) is an endpoint of some band of level \( n \) if \( h_n(E) = \pm 2 \). If \( h_n(E) = -2 \) and \( h_n'(E) < 0(h_n'(E) > 0) \), then \( E = b_\sigma(a_\sigma) \) for some \( \sigma \in \Sigma_n \). If \( h_n(E) = 2 \) and \( h_n'(E) < 0(h_n'(E) > 0) \), then \( E = a_\sigma(b_\sigma) \) for some \( \sigma \in \Sigma_n \).

3. Types, Evolution Laws, Orders and Labels

In this section, at first we study the band configurations of \( B_n \cup B_{n+1} \cup B_{n+2} \). Then we introduce the concept of type, which is fundamental for this paper. Next we derive the evolution laws and orders of the types, and determine the labels of the directed edges. At last, we show that \( \{ B_n : n \geq 0 \} \) is a NS with limit set \( \sigma_\lambda \).
3.1. Band configurations of $B_n \cup B_{n+1} \cup B_{n+2}$. Now we present the second technical lemma of our paper, which summarizes all the possible band configurations of $B_n \cup B_{n+1} \cup B_{n+2}$:

**Lemma 3.1.** Fix $n \geq 0$ and $\sigma \in \Sigma_n$.

i) Assume $n$ is odd. Then $a_{\sigma 10}, b_{\sigma 10} \notin \mathcal{R}_{n+1}$ and

$$B_{\sigma 00} \subset B_{\sigma}, \ B_{\sigma 10} \subset \text{int}(B_{\sigma}); \ B_{\sigma 01} \subset B_{\sigma 0}; \ B_{\sigma 0} < B_{\sigma 10} < B_{\sigma 1}.$$ Moreover we have

$$\begin{cases} B_{\sigma 00} \subset B_{\sigma 0}, & \text{if } a_{\sigma} \in \mathcal{R}_{n-1} \\ B_{\sigma 00} < B_{\sigma 0}, & \text{if } a_{\sigma} \notin \mathcal{R}_{n-1} \end{cases} \quad \begin{cases} B_{\sigma 11} \subset B_{\sigma 1} \subset B_{\sigma}, & \text{if } b_{\sigma} \in \mathcal{R}_{n-1} \\ B_{\sigma} < B_{\sigma 11}, & \text{if } b_{\sigma} \notin \mathcal{R}_{n-1} \end{cases}$$

If $a_{\sigma} \notin \mathcal{R}_{n-1}$, then $a_{\sigma 00}, b_{\sigma 00} \notin \mathcal{R}_{n+1}$ and $B_{\sigma 00} \subset \text{int}(B_{\sigma})$.

ii) Assume $n$ is even. Then $a_{\sigma 01}, b_{\sigma 01} \notin \mathcal{R}_{n+1}$ and

$$B_{\sigma 11} \subset B_{\sigma}, \ B_{\sigma 01} \subset \text{int}(B_{\sigma}); \ B_{\sigma 10} \subset B_{\sigma 1}; \ B_{\sigma 0} < B_{\sigma 01} < B_{\sigma 1}.$$ Moreover we have

$$\begin{cases} B_{\sigma 11} \subset B_{\sigma 1}, & \text{if } b_{\sigma} \in \mathcal{R}_{n-1} \\ B_{\sigma 1} < B_{\sigma 11}, & \text{if } b_{\sigma} \notin \mathcal{R}_{n-1} \end{cases} \quad \begin{cases} B_{\sigma 00} \subset B_{\sigma 0} \subset B_{\sigma}, & \text{if } a_{\sigma} \in \mathcal{R}_{n-1} \\ B_{\sigma 00} < B_{\sigma}, & \text{if } a_{\sigma} \notin \mathcal{R}_{n-1} \end{cases}$$

If $b_{\sigma} \notin \mathcal{R}_{n-1}$, then $a_{\sigma 11}, b_{\sigma 11} \notin \mathcal{R}_{n+1}$ and $B_{\sigma 11} \subset \text{int}(B_{\sigma})$.

Combine Lemmas 2.3 and 3.1, we have

**Corollary 3.2.** i) Assume $n \geq 1$ and $\sigma \in \Sigma_n$. Then

$$B_{\sigma} \subset B_{\sigma|_{n-1}} \iff B_{\sigma} \cap \mathcal{R}_{n-1} \neq \emptyset \iff \partial B_{\sigma} \cap \mathcal{R}_{n-1} \neq \emptyset.$$ ii) Assume $n \geq 1$ and $\sigma \in \Sigma_n$. Then

$$B_{\sigma} \subset B_{\sigma|_{n-1}} \text{ or } B_{\sigma} \subset B_{\sigma|_{n-2}}.$$ (For $n = 1$, $B_{\sigma|_0} := B_{\emptyset}$ and $B_{\sigma|_{-1}} := \mathbb{R}$).

iii) Assume $n \geq 0$ and $\sigma \in \Sigma_n$, $\tau \in \Sigma_n \cup \Sigma_{n+1} \cup \Sigma_{n+2}$. If $B_{\tau} \subset B_{\sigma}$, then $\sigma \prec \tau$.

We will prove Lemma 3.1 and Corollary 3.2 in Sect. 8.3. Now we apply them to derive the concept of type and figure out the evolution laws.

3.2. The types of the bands. Lemmas 2.3 and 3.1 suggest that for a band $B_{\sigma}$ of level $n$, once we know that whether $a_{\sigma}, b_{\sigma} \in \mathcal{R}_{n-1}$, the local configurations of $B_{\sigma}$ and $B_{\sigma \tau}$ with $|\tau| \leq 2$ are completely determined. This observation motivates the following definition for types of bands.
Definition 3.3. Assume $n \geq 0$, $0 \leq \kappa \leq 2$ and $B \in \mathcal{B}_n$.

At first we assume $B \in \mathcal{B}_n$. We say $B$ has type $\kappa_0(\kappa_e)$, if $n$ is odd (even), and $\# \partial B \cap \mathcal{R}_{n-1} = \kappa$.

(Recall that $\mathcal{R}_{n-1} = \emptyset$.)

Next we assume $B \notin \mathcal{B}_n$. Then $B \in \mathcal{B}_{n+1}$. There exists a unique $\sigma \in \Sigma_{n+1}$ such that $B = B_\sigma$. Since $B \in \mathcal{B}_n$, by Proposition 2.4, $B \notin B_{\sigma|n}$. By Corollary 3.2 ii), $B \subset B_{\sigma|n-1}$.

If $B_{\sigma|n} \subset B_{\sigma|n-1}$.

We say $B$ has type $3_{ol}(3_{el})$ if $n$ is odd (even) and $B < B_{\sigma|n}$.

We say $B$ has type $3_{or}(3_{er})$ if $n$ is odd (even) and $B_{\sigma|n} < B$.

If $B_{\sigma|n} \nsubseteq B_{\sigma|n-1}$.

We say $B$ has type $3_{ol}(3_{el})$ if $n$ is odd (even) and $B_{\sigma|n} < B_{\sigma|n-1}$.

We say $B$ has type $3_{or}(3_{er})$ if $n$ is odd (even) and $B_{\sigma|n-1} < B_{\sigma|n}$.

By the definition above, each band in $\mathcal{B}_n$ has one and only one type.

Example 3.4. Recall that $\mathcal{B}_0 = \{B_0, B_\emptyset\}$. By the definition, $B_\emptyset$ has type $0_e$, $B_0$ has type $3_{el}$, see Fig. 2 (a). $\mathcal{B}_1 = \{B_0, B_1, B_{01}, B_{11}\}$. $B_0$ has type $0_o$, $B_1$ has type $1_o$, $B_{01}$ has type $3_{ol}$ and $B_{11}$ have type $3_{or}$, see Fig. 2a.

3.3. The evolution laws and orders of the types, the labels of the directed edges. At first, we study type $3_o$ bands.

Lemma 3.5. Assume $B_\sigma \in \mathcal{B}_n$ has type $3_{ol}$ or $3_{or}$ ($3_{el}$ or $3_{er}$). Then $B_\sigma$ contains exactly one band in $\mathcal{B}_{n+1}$: $B_\sigma$ itself with type $0_o$ ($0_o$).

Proof. We only show the odd case. Assume $B_\sigma \in \mathcal{B}_n$ has type $3_{ol}$ or $3_{or}$, then $n$ is odd and $\sigma \in \Sigma_{n+1}$. By Proposition 2.4, $B_\sigma \nsubseteq B_{\sigma|n}$. By Corollary 3.2 i), $B_\sigma \cap \mathcal{R}_n = \emptyset$. Thus $B_\sigma \in \mathcal{B}_{n+1} \subset \mathcal{B}_{n+1}$ has type $0_e$. By the definition of $\mathcal{B}_{n+1}$, $B_\sigma$ is the only band in $\mathcal{B}_{n+1}$ which is contained in $B_\sigma$. \hfill \Box

Before continuing, we do the following observation. Take $B_\sigma \in \mathcal{B}_n$ and $B_\tau \in \mathcal{B}_{n+1}$. Assume $B_\tau \subset B_\sigma$. Then $\sigma \in \Sigma_n$, $\tau \in \Sigma_{n+1}$ or $\Sigma_{n+2}$. By Corollary 3.2 iii), $\sigma \prec \tau$. So

$$\tau \in \{\sigma 0, \sigma 1, \sigma 00, \sigma 01, \sigma 10, \sigma 11\}. \quad (34)$$

Next, we study type $0_o$ bands.

Lemma 3.6. Assume $B_\sigma \in \mathcal{B}_n$.

i) If $B_\sigma$ has type $0_o$, then $B_\sigma$ contains exactly three bands in $\mathcal{B}_{n+1}$ with the order:

$$B_{\sigma 00} < B_{\sigma 0} < B_{\sigma 10}.$$ and with the types $3_{el}$, $1_e$, $3_{er}$, respectively. Moreover,

$$b_{\sigma 0} = z_\sigma; \quad B_{\sigma 00}, B_{\sigma 10} \subset \text{int}(B_\sigma).$$

ii) If $B_\sigma$ has type $0_e$, then $B_\sigma$ contains exactly three bands in $\mathcal{B}_{n+1}$ with the order:

$$B_{\sigma 01} < B_{\sigma 1} < B_{\sigma 11},$$ and with the types $3_{ol}$, $1_o$, $3_{or}$, respectively. Moreover,

$$a_{\sigma 1} = z_\sigma; \quad B_{\sigma 01}, B_{\sigma 11} \subset \text{int}(B_\sigma).$$
Proof. We only show i), so we assume \( n \) is odd. By the assumption, \( \sigma \in \Sigma_n \) and \( a_{\sigma}, b_{\sigma} \notin R_{n-1} \). Assume \( B_{\tau} \in B_{n+1} \) and \( B_{\tau} \subset B_{\sigma} \), then \( \tau \) satisfies (34).

By Lemma 2.3 i) and Lemma 2.8 iii),
\[
B_{\sigma 0} \subset B_{\sigma}; \ B_{\sigma} < B_{\sigma 1}; \ a_{\sigma 0} \notin R_n; \ b_{\sigma 0} = z_{\sigma} \in R_n.
\]
So \( B_{\sigma 0} \in B_{n+1} \) has type \( 1_e \) and \( B_{\sigma 1} \not\subset B_{\sigma} \).

By Lemma 3.1 i),
\[
\begin{cases}
B_{\sigma 00}, B_{\sigma 10} \subset \text{int}(B_{\sigma}); & B_{\sigma 00} < B_{\sigma 0} < B_{\sigma 10} < B_{\sigma 1}, \\
B_{\sigma 01} \subset B_{\sigma 0}; & B_{\sigma} < B_{\sigma 11}.
\end{cases}
\]
So \( B_{\sigma 11} \not\subset B_{\sigma} \) and \( B_{\sigma 01} \notin B_{n+1}; B_{\sigma 00} \notin B_{\sigma 0}, B_{\sigma 10} \not\subset B_{\sigma 1} \). By Proposition 2.4, \( B_{\sigma 00}, B_{\sigma 10} \in B_{n+1} \). Since \( B_{\sigma 0} \subset B_{\sigma} \) and \( B_{\sigma 00} < B_{\sigma 0} \), by the definition, \( B_{\sigma 00} \) has type \( 3_{el} \). Since \( B_{\sigma 1} \not\subset B_{\sigma} \) and \( B_{\sigma} < B_{\sigma 1} \), by the definition, \( B_{\sigma 10} \) has type \( 3_{er} \).

As two examples, \( B_{\emptyset} \) has type \( 0_e \) and \( B_0 \) has type \( 0_o \), see Fig. 2a, b for the order configurations of their son intervals.

Now, we study type \( 1_* \) bands.

Lemma 3.7. Assume \( B_{\sigma} \in B_n \).

i) If \( B_{\sigma} \) has type \( 1_o \), then \( B_{\sigma} \) contains exactly two bands in \( B_{n+1} \) with the order:
\[
B_{\sigma 0} < B_{\sigma 10},
\]
and with the type \( 2_e, 3_{er} \), respectively. Moreover \( B_{\sigma 0} = [a_{\sigma}, z_{\sigma}] \) and \( B_{\sigma 10} \subset \text{int}(B_{\sigma}) \).

ii) If \( B_{\sigma} \) has type \( 1_e \), then \( B_{\sigma} \) contains exactly two bands in \( B_{n+1} \) with the order:
\[
B_{\sigma 01} < B_{\sigma 1},
\]
and with the type \( 3_{ol}, 2_o \), respectively. Moreover \( B_{\sigma 1} = [z_{\sigma}, b_{\sigma}] \) and \( B_{\sigma 01} \subset \text{int}(B_{\sigma}) \).

Proof. We only show i), so we assume \( n \) is odd. By the assumption, \( \sigma \in \Sigma_n \) and \( \#\{a_{\sigma}, b_{\sigma}\} \cap R_{n-1} = 1 \). Assume \( B_{\tau} \in B_{n+1} \) and \( B_{\tau} \subset B_{\sigma} \), then \( \tau \) satisfies (34).

At first, we claim that \( a_{\sigma} \in R_{n-1} \) and \( b_{\sigma} \notin R_{n-1} \). Indeed, since \( \partial B_{\sigma} \cap R_{n-1} \neq \emptyset \), by Corollary 3.2 i), \( B_{\sigma} \subset B_{\sigma|_{n-1}} \). Write \( \hat{\sigma} = \sigma|_{n-1} \), notice that \( n-1 \) is even. By applying Lemma 2.3 ii) to \( \hat{\sigma} \), we have \( b_{\hat{\sigma} 0} \notin R_{n-1} \) and \( a_{\hat{\sigma} 1} = z_{\hat{\sigma}} \in R_{n-1} \). Since \( \#\{a_{\sigma}, b_{\sigma}\} \cap R_{n-1} = 1 \), we have \( a_{\sigma} \in R_{n-1} \) and \( b_{\sigma} \notin R_{n-1} \) in either case of \( \sigma = \hat{\sigma} 0 \) or \( \hat{\sigma} = \hat{\sigma} 1 \). So the claim holds.

By Lemma 2.3 i), \( B_{\sigma 0} = [a_{\sigma}, z_{\sigma}] \subset B_{\sigma} \) and \( B_{\sigma} < B_{\sigma 1} \). Since \( a_{\sigma} \in R_{n-1} \) and \( z_{\sigma} \in R_n \), \( B_{\sigma 0} \in B_{n+1} \) has type \( 2_e \). \( B_{\sigma 1} \not\subset B_{\sigma} \).

By Lemma 3.1 i),
\[
B_{\sigma 00}, B_{\sigma 01} \subset B_{\sigma 0}; \ B_{\sigma 10} \subset \text{int}(B_{\sigma}); \ B_{\sigma 10} < B_{\sigma 1}; \ B_{\sigma} < B_{\sigma 11}; \ B_{\sigma 0} < B_{\sigma 10}.
\]
So \( B_{\sigma 11} \not\subset B_{\sigma} \) and \( B_{\sigma 00}, B_{\sigma 01} \notin B_{n+1} \). Since \( B_{\sigma 10} \not\subset B_{\sigma 1} \), by Proposition 2.4, \( B_{\sigma 10} \in B_{n+1} \). Since \( B_{\sigma} < B_{\sigma 1}, B_{\sigma 10} \) has type \( 3_{er} \). So the result holds.

As two examples, \( B_{\emptyset} \) has type \( 1_e \) and \( B_1 \) has type \( 1_o \), see Fig. 2c, d for the order configurations of their son intervals.

At last, we study type \( 2_* \) bands.
Lemma 3.8. Assume $B_\sigma \in \mathcal{B}_n$.

i) If $B_\sigma$ has type $2_\sigma$, then $B_\sigma$ contains exactly three bands in $\mathcal{B}_{n+1}$ with the order:

$$B_{\sigma 0} < B_{\sigma 10} < B_{\sigma 1},$$

and with the type $2_e, 3_{el}, 1_e$, respectively.

ii) If $B_\sigma$ has type $2_e$, then $B_\sigma$ contains exactly three bands in $\mathcal{B}_{n+1}$ with the order:

$$B_{\sigma 0} < B_{\sigma 01} < B_{\sigma 1},$$

and with the type $1_e, 3_{or}, 2_{or}$, respectively.

Proof. We only show i), so we assume $n$ is odd. By the assumption, $\sigma \in \Sigma_n$ and $a_\sigma, b_\sigma \in \mathcal{R}_{n-1}$. By Lemma 2.8 ii), $b_\sigma = b_\sigma$. Assume $B_\tau \in \mathcal{B}_{n+1}$ and $B_\tau \subset B_\sigma$, then $\tau$ satisfies (34).

By Lemma 2.3 i), $B_{\sigma 0} = \{a_\sigma, z_\sigma\}, B_{\sigma 1} \subset B_\sigma, a_\sigma \not\in \mathcal{R}_n$ and $b_\sigma = b_\sigma \in \mathcal{R}_n$. So $B_{\sigma 1} \in \mathcal{B}_{n+1}$ has type $1_e$. Since $a_\sigma \in \mathcal{R}_{n-1}$ and $z_\sigma \in \mathcal{R}_n$, $B_{\sigma 0} \in \mathcal{B}_{n+1}$ has type $2_e$.

By Lemma 3.1 i),

$$B_{\sigma 00}, B_{\sigma 01} \subset B_{\sigma 0}; \quad B_{\sigma 10} \subset B_{\sigma}; \quad B_{\sigma 11} \subset B_{\sigma 1}; \quad B_{\sigma 0} < B_{\sigma 10} < B_{\sigma 1}.$$  

So $B_{\sigma 00}, B_{\sigma 01}, B_{\sigma 11} \not\in \mathcal{B}_{n+1}$. Since $B_{\sigma 10} \not\in \mathcal{B}_{\sigma 1}$, by Proposition 2.4, $B_{\sigma 10} \in \mathcal{B}_{n+1}$. Since $B_{\sigma 1} \subset B_{\sigma}$ and $B_{\sigma 10} < B_{\sigma 1}$, by the definition, $B_{\sigma 10}$ has type $3_{el}$. ☐

As two examples, $B_{10}$ has type $2_e$ and $B_{101}$ has type $2_\sigma$, see Fig. 2e, f for the order configurations of their son intervals.

We summarize Lemmas 3.5–3.8 in a symbolic way as follows:

$$3_{ol}, 3_{or} \not\rightarrow 0_e; \quad 3_{el}, 3_{er} \not\rightarrow 0_o$$

$$\left\{ \begin{align*}
0_o &\rightarrow 00; \quad 0_o \rightarrow 01; \quad 0_o \rightarrow 10; \quad 0_o \rightarrow 11; \\
0_e &\rightarrow 00; \quad 0_e \rightarrow 01; \quad 0_e \rightarrow 10; \quad 0_e \rightarrow 11;
\end{align*} \right.$$  

$$\left\{ \begin{align*}
1_o &\rightarrow 00; \quad 1_o \rightarrow 01; \quad 1_o \rightarrow 10; \quad 1_o \rightarrow 11; \\
1_e &\rightarrow 00; \quad 1_e \rightarrow 01; \quad 1_e \rightarrow 10; \quad 1_e \rightarrow 11;
\end{align*} \right.$$  

$$\left\{ \begin{align*}
2_o &\rightarrow 00; \quad 2_o \rightarrow 01; \quad 2_o \rightarrow 10; \quad 2_o \rightarrow 11; \\
2_e &\rightarrow 00; \quad 2_e \rightarrow 01; \quad 2_e \rightarrow 10; \quad 2_e \rightarrow 11;
\end{align*} \right.$$  

Here $\alpha \xrightarrow{\tau} \beta$ means type $\alpha$ can evolve to type $\beta$ and the directed edge $\alpha \beta$ is labeled by $\tau$.

Finally, consider $\mathcal{B}_0 = \{B_0, B_\emptyset\}$. By (29), $B_0 < B_\emptyset$. By Example 3.4, $B_0$ has type $3_{el}$ and $B_\emptyset$ has type $0_e$. So we complete the order of the types by

$$3_{el} < 0_e.$$  

(39)
3.4. \( \{ \mathcal{B}_n : n \geq 0 \} \) as a NS. At first we have

**Proposition 3.9.** For each \( \tilde{B} \in \mathcal{B}_{n+1} \), there is a unique band \( B \in \mathcal{B}_n \) such that \( \tilde{B} \subset B \).

**Proof.** Let \( \sigma \in \Sigma_{n+1} \cup \Sigma_{n+2} \) be the unique word such that \( \tilde{B} = B_\sigma \). If \( B \in \mathcal{B}_n \) is such that \( \tilde{B} \subset B \), then by Corollary 3.2 iii), the only two possibilities are \( B = B_\sigma\mid_n \) or \( B_\sigma\mid_{n+1} \).

If \( \tilde{B} \) has type \( 3_* \), then \( \sigma \in \Sigma_{n+2} \). By Proposition 2.4, \( B_\sigma \not\subset B_\sigma\mid_n \). By Corollary 3.2 ii), \( B_\sigma \subset B_\sigma\mid_n \in \mathcal{B}_n \subset \mathcal{B}_n \).

If \( \tilde{B} \) has type \( 1_* \) or \( 2_* \), then \( \sigma \in \Sigma_{n+1} \) and \( \partial B_\sigma \cap \mathcal{R}_n \neq \emptyset \). By Corollary 3.2 i), \( B_\sigma \not\subset B_\sigma\mid_n \). By Proposition 2.4, \( B_\sigma \in \mathcal{B}_n \) and \( \tilde{B} = B_\sigma \).

In all the cases, there is a unique \( B \in \mathcal{B}_n \) such that \( \tilde{B} \subset B \). \( \square \)

See Sect. 1.3.1 for the definition of NS. Now we can show that

**Corollary 3.10.** \( \mathcal{B} = \{ \mathcal{B}_n : n \geq 0 \} \) is a NS and \( A(\mathcal{B}) = \sigma_\lambda \).

**Proof.** By Proposition 2.1, \( \mathcal{B}_n \) is a disjoint family. Combine with the definition (18) of \( \mathcal{B}_n \), \( \mathcal{B} \) is optimal. By Proposition 3.9, \( \mathcal{B} \) is nested. By Lemmas 3.5–3.8, each band in \( \mathcal{B}_n \) contains at least one band in \( \mathcal{B}_{n+1} \), so \( \mathcal{B} \) is minimal. By the definition of NS (see Sect. 1.3.1), \( \mathcal{B} \) is a NS. By (25) and the definition of \( \mathcal{B}_n \),

\[
\bigcup_{B \in \mathcal{B}_n} B = \sigma_{\lambda, n} \cup \sigma_{\lambda, n+1}.
\]

By (22), \( \sigma_\lambda \) is the limit set of \( \mathcal{B} \). \( \square \)

As a final complement, we show the following

**Lemma 3.11.** If \( B \in \mathcal{B}_n \) has type \( 0_\epsilon(0_\sigma) \), then \( h_n \) is increasing (decreasing) on \( B \).

**Proof.** We only show the case that \( B \) has type \( 0_\sigma \). The proof of the other case is the same.

If \( n = 0 \), then \( B = B_\emptyset \) and \( h_0(E) = E - \lambda \), so the result holds.

Now assume \( n \geq 1 \) and \( n \) is even. Assume \( B = B_\sigma \) for some \( \sigma \in \Sigma_n \). We claim that \( \sigma = 1 \). If otherwise \( \sigma = \tau 0 \) for some \( \tau \in \Sigma_{n-1} \). Then by Lemma 2.3 i), we have \( b_{\sigma} = z_\tau \in \mathcal{R}_{n-1} \), which contradicts with the fact that \( B \) has type \( 0_\sigma \). Now by Proposition 2.10 i), \( h_n \) is increasing on \( B \). \( \square \)

4. Symbolic Space \( \Omega_{\infty} \)

We have defined the symbolic space \( \Omega_{\infty} \) together with an order \( \preceq \) on it in Sect. 1.2.4. Through Sect. 3, we understand that \( \mathcal{A} \), defined by (7), is just the set of all the possible types; the admissible rules in (8) are just summaries of the evolution laws appeared in (35)-(38). These information tell us how to construct the directed graph \( \mathcal{G} \). The labels of edges of \( \mathcal{G} \) can also be read off from (35)-(38). Half of the order relations appeared in (36)-(39) also motivate the definition of \( \preceq \) on \( \Omega_{\infty} \). Another half of these relations suggest a new partial order \( \preceq \) on \( \Omega_{\infty} \), which will be defined and studied here.

In this section, our main goal is to give a characterization for the gaps of \( \Omega_{\infty} \), see Sect. 4.3. The main tool is the map \( \Pi : \Omega_{\infty} \to \Sigma_{\infty} \) introduced in (14), which will be studied in detail in Sect. 4.3.2. All the properties of \( \Pi \) can be proven in a purely combinatorial way, except one – the surjectivity of \( \Pi \). To prove the surjectivity of \( \Pi \), we need to use the “geometric realization” of \( \Omega_n \) as a bridge. This is the content of Sect. 4.1. In Sect. 4.2, we study the two orders in \( \Omega_{\infty} \).
4.1. Coding of $\mathcal{B}_n$ and surjectivity of $\Pi$.

Define the set of admissible words as

$$\Omega_n := \{ w \in \mathcal{A}^{n+1} : w_0 \in \{3_{el}, 0_e\}, a_{w_j}w_{j+1} = 1 \}, \quad n \geq 0 \quad \text{and} \quad \Omega_* := \bigcup_{n \geq 0} \Omega_n. \quad (40)$$

For $\omega = \omega_0\omega_1 \cdots \in \Omega_* \cup \Omega_\infty$, write $\omega|_n := \omega_0 \cdots \omega_n$. For $\omega, \omega' \in \Omega_* \cup \Omega_\infty$, write $\omega \wedge \omega'$ for their maximal common prefix. For $w \in \Omega_n$, define the cylinder as

$$[w] := \{ \omega \in \Omega_\infty : \omega|_n = w \}.$$  

Define the label-assigning map $\mathcal{L} : \{ \alpha \beta : \alpha \rightarrow \beta \} \rightarrow \{ \emptyset, 0, 1, 00, 01, 10, 11 \}$ as follows: $\mathcal{L}(\alpha \beta)$ is the label of the edge $\alpha \beta$.

By (36)-(39), the following holds:

$$\alpha \rightarrow \beta, \beta' \quad \text{and} \quad \beta < \beta' \Rightarrow \mathcal{L}(\alpha \beta) = 0*; \quad \mathcal{L}(\alpha \beta') = 1* \quad . \quad (41)$$

Define a map $\Pi_* : \Omega_* \rightarrow \Sigma_*$ as

$$\Pi_*(w) := i(w)\mathcal{L}(w_0w_1)\mathcal{L}(w_1w_2) \cdots \mathcal{L}(w_{n-1}w_n), \quad w \in \Omega_n, \quad (42)$$

where $i(w) = \emptyset$ if $w_0 = 0_e$ and $i(w) = 0$ if $w_0 = 3_{el}$.

By (35)-(38) and (14), it is direct to check that, for any $\omega \in \Omega_\infty$,

$$\Pi_*(\omega|_n) \sqsubset \Pi(\omega); \quad \Pi(\omega) = \lim_{n \rightarrow \infty} \Pi_*(\omega|_n). \quad (43)$$

4.1.1. Coding $\mathcal{B}_n$ by $\Omega_n$. We summarize Lemmas 3.5–3.8 in the following Corollary, the proof of which is by direct checking.

**Corollary 4.1.** Assume $B \in \mathcal{B}_n$ has type $\alpha$. Then $B$ contains a band $\tilde{B} \in \mathcal{B}_{n+1}$ of type $\beta$ if and only if $\alpha \rightarrow \beta$. Moreover, if $\alpha \rightarrow \beta$, then $B$ contains exactly one band $\tilde{B} \in \mathcal{B}_{n+1}$ of type $\beta$.

Now we can code $\mathcal{B}_n$ by $\Omega_n$ inductively as follows.

Recall that $\mathcal{B}_0 = \{ B_0, B_\emptyset \}; B_0$ has type $3_{el}$ and $B_\emptyset$ has type $0_e$ and by (29), $B_0 \not< B_\emptyset$. We code $\mathcal{B}_0$ by $\Omega_0 = \{ 3_{el}, 0_e \}$ as

$$I_{3_{el}} := B_0; \quad I_{0_e} := B_\emptyset.$$  

Assume we have coded $\mathcal{B}_n$ by $\Omega_n$:

$$\mathcal{B}_n = \{ I_w : w \in \Omega_n \}, \quad I_w \text{ has type } w_n \quad \text{and} \quad I_v \neq I_w \text{ if } v \neq w.$$  

Take any $u = u_0 \cdots u_{n+1} \in \Omega_{n+1}$, then $u|_n \in \Omega_n$ and $u_n \rightarrow u_{n+1}$. By Corollary 4.1, there exists a unique $\hat{B} \in \mathcal{B}_{n+1}$ with type $u_{n+1}$ such that $\hat{B} \subset I_{u|_n}$. Define $I_u := \hat{B}$. Then

$$\{ I_w : w \in \Omega_{n+1} \} \subset \mathcal{B}_{n+1}.$$  

Now take any $\hat{B} \in \mathcal{B}_{n+1}$, assume it has type $\beta$. By Proposition 3.9, there exists a unique $B \in \mathcal{B}_n$ such that $\hat{B} \subset B$. By induction hypothesis, there is a unique $w \in \Omega_n$ such that
by Corollary 4.1, \( w_n \to \beta \) and \( B = I_w \beta \) and no other \( u \in \Sigma_{n+1} \) satisfies \( B = I_u \). Thus

\[
\mathcal{B}_{n+1} = \{ I_w : w \in \Omega_{n+1} \},
\]

\( I_w \) has type \( w_{n+1} \) and \( I_u \equiv I_w \) if \( v \neq w \).

By induction, we complete the coding process.

Combine with Lemmas 3.5–3.8, the above coding process also implies that for \( w \in \Omega_* \),

\[
I_w = B_{\Pi(w)}.
\]

Since \( \omega|_n \in \Omega_n \) for any \( \omega \in \Omega_\infty \), we have \( B_{\Pi(\omega|_n)} = I_{\omega|_n} \in \mathcal{B}_n \). As a consequence,

\[
|\Pi(\omega|_n)| = n + 1.
\]

By Corollary 4.1, we also have that: if \( v \in \Omega_n \) and \( w \in \Omega_{n-1} \), then

\[
I_v \subset I_w \iff w < v.
\]

4.1.2. Surjectivity of \( \Pi \). As an application of the coding of \( \mathcal{B}_n \), we can show that \( \Pi \) is surjective.

**Proposition 4.2.** The map \( \Pi : \Omega_\infty \to \Sigma_\infty \) is surjective.

**Proof.** Given \( \sigma \in \Sigma_\infty \). For each \( n \geq 0 \), we always have \( B_{\sigma|_n} \in \mathcal{B}_n \). By the coding of \( \mathcal{B}_n \), there exists a unique \( w(n) \in \Omega_n \) such that \( I_{w(n)} = B_{\sigma|_n} \).

**Claim:** For each \( n \geq 2 \), \( w(n-1) \leq w(n) \) or \( w(n-2) < w(n) \).

By Corollary 3.2 iii), \( \tilde{B} = B_{\sigma|_{n-1}} \).

That is, \( I_{w(n)} \subset I_{w(n-1)} \). By (46), \( w(n-1) < w(n) \).

If \( B_{\sigma|_n} \in \mathcal{B}_{n-1} \), then \( B_{\sigma|_n} \subset \tilde{B} \) for some \( \tilde{B} \in \mathcal{B}_{n-1} \). Assume \( u(n-1) \in \Omega_{n-1} \) is such that \( B_{\sigma|_n} = I_{\hat{u}(n-1)} \). By (46), we have \( w(n-2) < \hat{w}(n-1) \). We also have \( I_{w(n)} = I_{\hat{u}(n-1)} \), again by (46), we have \( \hat{w}(n-1) < w(n) \). So, \( w(n-2) < w(n) \). □

By the claim, we have that

\[
\text{for any } n > k \geq 0, \text{ either } w(k) < w(n) \text{ or } w(k+1) < w(n).
\]

Define an integer sequence \( (m_k)_{k \geq 1} \) and a decreasing sequence \( (\Xi_k)_{k \geq 1} \) with \( \Xi_k \subset \mathbb{N} \) and \( \#\Xi_k = \infty \) inductively as follows. Write \( \Xi_0 := \mathbb{N} \). By (47), at least one of \( \{ n \in \Xi_0 : w(0) < w(n) \} \) and \( \{ n \in \Xi_0 : w(1) < w(n) \} \) is infinite. If \( \{ n \in \Xi_0 : w(0) < w(n) \} \) is infinite, define \( m_1 := 0 \). If otherwise, then \( \{ n \in \Xi_0 : w(1) < w(n) \} \) is infinite, define \( m_1 := 1 \). Define \( \Xi_1 := \{ n \in \Xi_0 : w(m_1) < w(n) \} \). Then \( \#\Xi_1 = \infty \).

Assume \( m_k \) and \( \Xi_k \) has been defined with the desired property. By (47), at least one of \( \{ n \in \Xi_k : w(m_k + 1) < w(n) \} \) and \( \{ n \in \Xi_k : w(m_k + 2) < w(n) \} \) is infinite. If \( \{ n \in \Xi_k : w(m_k + 1) < w(n) \} \) is infinite, define \( m_{k+1} := m_k + 1 \). Otherwise, the set \( \{ n \in \Xi_k : w(m_k + 2) < w(n) \} \) is infinite, define \( m_{k+1} := m_k + 2 \). Define \( \Xi_{k+1} := \{ n \in \Xi_k : w(m_{k+1}) < w(n) \} \). Hence \( \Xi_{k+1} \subset \Xi_k \) and is infinite.

By induction, we finish the definition.

Now we claim that for any \( k, w(m_k) < w(m_{k+1}) \). Indeed, by the defining process, we have \( w(m_k) < w(n) \) for any \( n \in \Xi_k \). Choose some \( n \in \Xi_{k+1} \), then \( n \in \Xi_k \) since \( \Xi_{k+1} \subset \Xi_k \). Hence we have \( w(m_k) < w(m_{k+1}) < w(n) \). Since \( |w(m_k)| < |w(m_{k+1})| \), we conclude that \( w(m_k) < w_m \).

Let \( \omega = \lim_{k \to \infty} w(m_k) \). Then we have \( \Pi(\omega) = \sigma \). □

It is an interesting question to find a purely combinatorial proof for the surjectivity of \( \Pi \).
4.2. Two orders on $\Omega_\infty$.

4.2.1. Two orders on $A$. Recall that $\preceq$ is the smallest partial order on $A$ such that the following holds (see (10)):

$$3_{ol} < 1_o < 3_{or} < 2_o; \quad 2_e < 3_{el} < 1_e < 3_{er}; \quad 3_{el} < 0_e. \tag{48}$$

Define $\preceq$ to be the smallest partial order on $A$ such that the following holds:

$$3_{ol} < 1_o < 2_o; \quad 3_{ol} < 3_{or} < 2_o; \quad 2_e < 3_{el} < 3_{er}; \quad 2_e < 1_e < 3_{er}. \tag{49}$$

(Here we view partial order as a subset of $A \times A$.)

By (48) and (49), it is direct to check that if $\alpha < \beta$, then $\alpha < \beta$.

Remark 4.3. The idea for defining (48) and (49) is as follows. At first, we impose

$$3_{ol} < 1_o < 3_{or} < 2_o; \quad 2_e < 3_{el} < 1_e < 3_{er}; \quad 3_{el} < 0_e,$$

which is a summary of the order relations appeared in (36)-(39). Since $<$ is stronger than $\preceq$, by transitivity, we get (48) and (49).

With this coding, we can explain two orders (48) and (49) as follows:

Corollary 4.4. Assume $w \in \Omega_n$ and $w_n \rightarrow \alpha, \beta$. Then

$$I_{w\alpha} < I_{w\beta} \iff \alpha < \beta; \quad I_{w\alpha} < I_{w\beta} \iff \alpha < \beta.$$

The proof is again by applying Lemmas 3.6–3.8 and checking directly.

4.2.2. Induced orders on $\Omega_\infty$. We have defined $\preceq$ in Sect. 1.2.4. Now we define a new relation $\preceq$ on $\Omega_\infty$ as follows. For any $\omega \in \Omega_\infty$, let $\omega \preceq \omega$. Given $\omega, \hat{\omega} \in \Omega_\infty$ and $\omega \neq \hat{\omega}$. Assume $n$ is such that $\omega|_{n-1} = \hat{\omega}|_{n-1}$ and $\omega_n \neq \hat{\omega}_n$. Let $\omega < \hat{\omega}$ if $\omega_n < \hat{\omega}_n$.

Lemma 4.5. Both $\leq$ and $\preceq$ are partial orders on $\Omega_\infty$. Moreover, $\leq$ is a total order. If $\omega < \hat{\omega}$, then $\omega < \hat{\omega}$.

Proof. It is routine to check that both $\preceq$ and $\leq$ are partial orders.

Now we show that $\preceq$ is a total order. Assume $\omega, \hat{\omega} \in \Omega_\infty$ and $\omega \neq \hat{\omega}$. Let $n \geq 0$ be the minimal integer such that $\omega_n \neq \hat{\omega}_n$. If $n = 0$, then $\omega_0 \neq \hat{\omega}_0$. By (9), $\{\omega_0, \hat{\omega}_0\} = \{3_{el}, 0_e\}$. By (48), we can compare $\omega_0$ and $\hat{\omega}_0$. Then by the definition, we can compare $\omega$ and $\hat{\omega}$. If $n \geq 1$, then $\omega \land \hat{\omega} = \omega|_{n-1} = \hat{\omega}|_{n-1}$ and by (8), $\omega_{n-1} = \hat{\omega}_{n-1} \in \{0_e, 0_o, 1_e, 1_o, 2_e, 2_o\}$. Now by (8) and (48), we can compare $\omega_n$ and $\hat{\omega}_n$. So, we can compare $\omega$ and $\hat{\omega}$.

Since in $A$, $\alpha < \beta$ implies $\alpha < \beta$, by the definitions, the last statement holds. □

Take $\omega, \hat{\omega} \in \Omega_\infty$. Assume $\omega < \hat{\omega}$. Define the open interval $(\omega, \hat{\omega})$ as

$$(\omega, \hat{\omega}) := [\omega \in \Omega_\infty : \omega < \hat{\omega} < \hat{\omega}].$$

We call $(\omega, \hat{\omega})$ a gap of $\Omega_\infty$ if $(\omega, \hat{\omega}) = \emptyset$. We also say that $\omega$ ( $\hat{\omega}$) is the left (right) edge of the gap.
4.3. Characterization of the gaps of $\Omega_\infty$. We denote the set of the gaps of $\Omega_\infty$ by $\mathcal{G}$. The following result is a symbolic version of Theorem 1.8:

**Theorem 4.6.** For $\Omega_\infty$, we have

i) $\omega_n \leq \omega \leq \omega^*$ for any $\omega \in \Omega_\infty$.

ii) (Type-I gaps): There exists a bijection $\ell^o : \mathcal{E}^o_i \rightarrow \mathcal{E}^e_r \setminus \{\omega^*\}$ such that $\omega < \ell^o(\omega)$ for any $\omega \in \mathcal{E}^o_i$ and

$$G^o_I := \{(\omega^*, \ell^o(\omega)) : \omega \in \mathcal{E}^o_i\} \subset \mathcal{G}. \tag{50}$$

There exists a bijection $\ell^e : \mathcal{E}^e_r \rightarrow \mathcal{E}^e_r \setminus \{\omega^*\}$ such that $\ell^e(\omega) < \omega$ for any $\omega \in \mathcal{E}^e_r$ and

$$G^e_I := \{((\ell^e(\omega), \omega) : \omega \in \mathcal{E}^e_r\} \subset \mathcal{G}. \tag{51}$$

iii) (Type-II gaps): There exists a bijection $\ell : \tilde{\mathcal{E}}^o_I \rightarrow \tilde{\mathcal{E}}^e_r$ such that $\omega < \ell(\omega)$ for any $\omega \in \tilde{\mathcal{E}}^o_I$ and

$$G_{II} := \{(\omega, \ell(\omega)) : \omega \in \tilde{\mathcal{E}}^o_I\} \subset \mathcal{G}. \tag{52}$$

iv) $\mathcal{G} = G_I \cup G_{II}$, where $G_I := G^o_I \cup G^e_I$.

We will prove Theorem 4.6 by using the map $\Pi$. But at first, we need to know how to define $\ell$, $\ell^o$, and $\ell^e$.

4.3.1. Definitions and bijectivities of $\ell$, $\ell^o$, $\ell^e$. At first, we define $\ell : \tilde{\mathcal{E}}^o_I \rightarrow \tilde{\mathcal{E}}^e_r$. Assume $\omega \in \tilde{\mathcal{E}}^o_I$. Then there exists $n$ such that $\omega = \omega|_{n-1}3_{el}0_3er(0_3er)\infty$ with $\omega_n = 1_0$; or $\omega = \omega|_{n}0_3er(0_03er)\infty$ with $\omega_n = 3_{el}$. By tracing the graph $G$ backward, in the former case, $\omega_n = 1_0$ and hence $n \geq 1$, $\omega_{n-1} = 0_e$ or $2_e$; in the latter case, $\omega_n = 3_{el}$ and hence $n \geq 2$, $\omega_{n-1} = 0_o$ or $2_o$, or $n = 0$. Now define

$$\ell(\omega) := \begin{cases} \omega|_{n-3}3_{el}(0_3er)\infty, & \text{if } \omega = \omega|_{n-1}3_{el}0_3er(0_03er)\infty, n \geq 1 \\ \omega|_{n-1}3_{el}(0_3er)\infty, & \text{if } \omega = \omega|_{n-1}3_{el}0_3er(0_03er)\infty, n \geq 2 \\ (0_3er)\infty, & \text{if } \omega = 3_{el}(0_03er)\infty. \end{cases} \tag{53}$$
See Fig. 3b for an illustration of the definition of \( \ell \) and for some intuition of a gap of \( \infty_{I} \)-type.

Next, we define \( \ell^{o} : \mathcal{E}_{I}^{o} \to \mathcal{E}_{r}^{o} \setminus \{ \omega_{n} \} \). Assume \( \omega \in \mathcal{E}_{I}^{o} \). Then there exists \( n \) such that either \( \omega = \omega_{|n}(2_{o}1_{e})^{\infty} \) with \( \omega_{n} \neq 1_{e} \); or \( \omega = \omega_{|n}(2_{o}1_{e})^{\infty} \) with \( \omega_{n} \neq 2_{o} \). By tracing the graph \( \mathbb{G} \) backward, in the former case, \( \omega_{n} = 2_{e} \) and hence \( n \geq 2 \), \( \omega_{n-1} = 1_{o} \) or \( 2_{o} \); in the latter case, \( \omega_{n} = 0_{o} \) and hence \( n \geq 1 \). Now define

\[
\ell^{o}(\omega) := \begin{cases} 
\omega_{n-1}0_{o}3_{er}(03_{o}3_{e})^{\infty}, & \text{if } \omega = \omega_{n-1}0_{o}2_{o}(2_{o}1_{e})^{\infty}, n \geq 2 \\
\omega_{n-2}0_{o}3_{er}(03_{o}3_{e})^{\infty}, & \text{if } \omega = \omega_{n-2}0_{o}2_{o}(2_{o}1_{e})^{\infty}, n \geq 2 \\
\omega_{n-1}0_{o}3_{er}(03_{o}3_{e})^{\infty}, & \text{if } \omega = \omega_{n-1}0_{o}1_{e}(2_{o}1_{e})^{\infty}, n \geq 1
\end{cases}
\]  

(54)

See Fig. 3a for an illustration of the definition of \( \ell^{o} \) and for some intuition of a gap of \( \infty_{I} \)-type.

Now, we define \( \ell^{e} : \mathcal{E}_{I}^{e} \to \mathcal{E}_{r}^{e} \setminus \{ \omega^{*} \} \). Assume \( \omega \in \mathcal{E}_{I}^{e} \). Then there exists \( n \) such that either \( \omega = \omega_{|n}(2_{e}1_{o})^{\infty} \) with \( \omega_{n} \neq 1_{o} \); or \( \omega = \omega_{|n}(2_{e}1_{o})^{\infty} \) with \( \omega_{n} \neq 2_{e} \). By tracing the graph \( \mathbb{G} \) backward, in the former case, \( \omega_{n} = 2_{o} \) and hence \( n \geq 3 \), \( \omega_{n-1} = 1_{o} \) or \( 2_{e} \); in the latter case, \( \omega_{n} = 0_{e} \) and hence \( n \geq 0 \). Now define

\[
\ell^{e}(\omega) := \begin{cases} 
\omega_{n-2}2_{e}3_{er}(03_{e}3_{o})^{\infty}, & \text{if } \omega = \omega_{n-2}2_{e}1_{o}(2_{o}1_{e})^{\infty}, n \geq 3 \\
\omega_{n-2}2_{e}3_{er}(03_{e}3_{o})^{\infty}, & \text{if } \omega = \omega_{n-2}2_{e}2_{o}(2_{o}1_{e})^{\infty}, n \geq 3 \\
\omega_{n-1}0_{e}3_{er}(03_{e}3_{o})^{\infty}, & \text{if } \omega = \omega_{n-1}0_{e}1_{o}(2_{o}1_{e})^{\infty}, n \geq 0
\end{cases}
\]  

(55)

Lemma 4.7. i) The map \( \tilde{\ell} : \tilde{\mathcal{E}}_{I} \to \tilde{\mathcal{E}}_{r} \) is a bijection and \( \omega < \ell(\omega) \) for any \( \omega \in \tilde{\mathcal{E}}_{I} \).

ii) The map \( \ell^{o} : \mathcal{E}_{I}^{o} \to \mathcal{E}_{r}^{o} \setminus \{ \omega_{n} \} \) is a bijection and \( \omega < \ell^{o}(\omega) \) for any \( \omega \in \mathcal{E}_{I}^{o} \).

iii) The map \( \ell^{e} : \mathcal{E}_{r}^{e} \to \mathcal{E}_{I}^{e} \setminus \{ \omega^{*} \} \) is a bijection and \( \ell^{e}(\omega) < \omega \) for any \( \omega \in \mathcal{E}_{r}^{e} \).

Proof. i) By (53) and (48), we have \( \omega < \ell(\omega) \).

To show the bijection of \( \ell \), we simply reverse the definition of \( \ell \) and construct another map \( \eta : \mathcal{E}_{r} \to \mathcal{E}_{I} \), then check \( \eta \) is the inverse of \( \ell \). Assume \( \omega \in \mathcal{E}_{I} \). Then either there exists \( n \geq 1 \) such that \( \omega = \omega_{|n}(03_{o}3_{e})^{\infty} \) with \( \omega_{n} \neq 3_{o} \), or there exists \( n \geq 0 \) such that \( \omega = \omega_{|n}(0e3_{o})^{\infty} \) with \( \omega_{n} \neq 0_{e} \), or \( \omega = (0e3_{o})^{\infty} \). By tracing the graph \( \mathbb{G} \) backward, in the first case, \( \omega_{n} = 3_{o} \) and hence \( \omega_{n-1} = 0_{e} \) or \( 2_{e} \). In the second case, \( \omega_{n} = 1_{o} \) and hence \( n \geq 2 \), \( \omega_{n-1} = 0_{o} \) or \( 2_{o} \). Now define

\[
\eta(\omega) := \begin{cases} 
\omega_{|n-1}3_{er}(03_{o}3_{e})^{\infty}, & \text{if } \omega = \omega_{|n-1}3_{er}(03_{o}3_{e})^{\infty}, n \geq 1 \\
\omega_{|n}3_{er}(03_{o}3_{e})^{\infty}, & \text{if } \omega = \omega_{|n}3_{er}(03_{o}3_{e})^{\infty}, n \geq 2 \\
3_{er}(03_{o}3_{e})^{\infty}, & \text{if } \omega = (0e3_{o})^{\infty}
\end{cases}
\]

One check directly that \( \eta \circ \ell = \text{Id}_{\tilde{\mathcal{E}}_{r}} \) and \( \ell \circ \eta = \text{Id}_{\tilde{\mathcal{E}}_{I}} \). So \( \ell \) is a bijection.

ii) and iii) can be proven similarly, we omit the proof. \( \Box \)

4.3.2. The property of the map \( \Pi : \Omega_{\infty} \to \Sigma_{\infty} \). We endow \( \Sigma_{\infty} \) with the lexicographical order \( \leq \). Define

\[
\Sigma^{e} := \bigcup_{n \geq 0} \Sigma_{2n+1}; \quad \Sigma^{o} := \bigcup_{n \geq 0} \Sigma_{2n}; \quad \Sigma_{\infty}^{(2)} := \{ \sigma(01)^{\infty} : \sigma \in \Sigma_{e} \}.
\]  

(56)

The basic property of \( \Pi \) is summarized in the following proposition:
Proposition 4.8. i) For any $\omega \in \tilde{\mathcal{E}}_i$, we have $\Pi(\omega) = \Pi(\ell(\omega))$. Consequently
\[ \Pi(\tilde{\mathcal{E}}_i) = \Pi(\tilde{\mathcal{E}}_r) \subset \Sigma^{(2)} \].  
(57)

ii) Assume $\sigma \in \Sigma_{\infty}$. If $\#\Pi^{-1}(\{\sigma\}) \geq 2$, then there exists $\omega \in \tilde{\mathcal{E}}_i$ such that
\[ \Pi^{-1}(\{\sigma\}) = \{\omega, \ell(\omega)\}. 
Consequently, $\#\Pi^{-1}(\{\sigma\}) \geq 2$ if and only if $\sigma \in \Pi(\tilde{\mathcal{E}}_i)$.

iii) $\Pi$ is surjective and order-preserving.
\[ \Pi(\omega^*) = 0^\infty; \quad \Pi(\omega^*_*) = 1^\infty. \]
(58)

Moreover, $\Pi : \Omega_{\infty} \setminus \tilde{\mathcal{E}}_r \to \Sigma_{\infty}$ is bijective.
iv) If $\omega \in \mathcal{E}_i^0$, then there exists a unique $\sigma \in \Sigma^0_{\infty}$ such that
\[ \Pi(\omega) = \sigma 01^\infty; \quad \Pi(\ell^0(\omega)) = \sigma 10^\infty. \]
(59)

If $\omega \in \mathcal{E}_r^e$, then there exists a unique $\sigma \in \Sigma^e_{\infty}$ such that
\[ \Pi(\omega) = \sigma 10^\infty; \quad \Pi(\ell^e(\omega)) = \sigma 01^\infty. \]
(60)

The following restrictions of $\Pi$ are bijections:
\[
\begin{align*}
\Pi_{ol} : \mathcal{E}_i^o &\to \{\sigma 01^\infty : \sigma \in \Sigma^o_{\infty}\}; \\
\Pi_{or} : \mathcal{E}_r^o \setminus \{\omega_*\} &\to \{\sigma 10^\infty : \sigma \in \Sigma^o_{\infty}\}; \\
\Pi_{el} : \mathcal{E}_i^e \setminus \{\omega_*\} &\to \{\sigma 01^\infty : \sigma \in \Sigma^e_{\infty}\}; \\
\Pi_{er} : \mathcal{E}_r^e &\to \{\sigma 10^\infty : \sigma \in \Sigma^e_{\infty}\}.
\end{align*}
\]
(61)

v) $\Pi : \tilde{\mathcal{E}}_i \cup \tilde{\mathcal{E}}_r \cup \mathcal{F} \to \Sigma^{(2)}_{\infty}$ is bijective. Consequently
\[ \tilde{\mathcal{E}}_i \cup \tilde{\mathcal{E}}_r \cup \mathcal{F} = \Pi^{-1}(\Sigma^{(2)}_{\infty}). \]
(62)

Proof. i) Assume $\omega \in \tilde{\mathcal{E}}_i$. By (53), there are three cases.

If $\omega = \omega|_{n-1} 0 3_{er} (0 3_{er})^\infty$, then $\omega_{n-1} = 0e$ or $2e$ and $\ell(\omega) = \omega|_{n-1} 3_{or} (0 3_{ol})^\infty$.

By tracing on the graph $\mathcal{G}$, for $m \geq n$,
\[
\begin{align*}
\Pi_*(\omega|_m) &= \Pi_*(\omega|_{n-1}) \delta(10)^{\lfloor(m+1-n)/2\rfloor}, \\
\Pi_*(\ell(\omega)|_m) &= \Pi_*(\omega|_{n-1}) \delta(01)^{\lfloor(m-n)/2\rfloor}.
\end{align*}
\]
(63)

Here $\delta = 1$ if $\omega_{n-1} = 0e$ and $\delta = 0$ if $\omega_{n-1} = 2e$.

If $\omega = \omega|_{n-1} 0 3_{er} (0 3_{er})^\infty$, then $\omega_{n-1} = 0o$ or $2o$ and $\ell(\omega) = \omega|_{n-1} 1 3_{el} (0 3_{ol})^\infty$.

By tracing on the graph $\mathcal{G}$, for $m \geq n$,
\[
\begin{align*}
\Pi_*(\omega|_m) &= \Pi_*(\omega|_{n-1}) \delta(01)^{\lfloor(m-n)/2\rfloor}, \\
\Pi_*(\ell(\omega)|_m) &= \Pi_*(\omega|_{n-1}) \delta(01)^{\lfloor(m+1-n)/2\rfloor}.
\end{align*}
\]
(64)

Here $\delta = 0$ if $\omega_{n-1} = 0o$ and $\delta = 1$ if $\omega_{n-1} = 2o$.

If $\omega = 3_{el} (0 3_{er})^\infty$, then $\ell(\omega) = (0 3_{ol})^\infty$. Notice that, in this case $i(\omega) = 0$ and $i(\ell(\omega)) = \emptyset$. By tracing on the graph $\mathcal{G}$, for $m \geq 0$,
\[
\begin{align*}
\Pi_*(\omega|_m) &= 0(10)^{\lfloor m/2\rfloor}, \\
\Pi_*(\ell(\omega)|_m) &= (01)^{\lfloor(m+1)/2\rfloor}.
\end{align*}
\]
(65)
In all cases, by (43) we have
\[
\Pi(\omega) = \lim_{m \to \infty} \Pi_*(\omega|m) = \lim_{m \to \infty} \Pi_*(\ell(\omega)|m) = \Pi(\ell(\omega)).
\]
Combine with Lemma 4.7 i), (57) holds.

ii) Assume \#\Pi^{-1}(\{\sigma\}) \geq 2. Take any \omega, \hat{\omega} \in \Pi^{-1}(\{\sigma\}). WLOG, we assume \omega < \hat{\omega}. Let \upsilon := \omega \land \hat{\omega}, and assume \upsilon \in \Omega_{n-1}. Then \omega_n < \hat{\omega}_n. By (41), \omega_n \not< \hat{\omega}_n. So either \n \geq 1 and
\[
(w_{n-1}, \omega_n, \hat{\omega}_n) \in \{(0_3, 3_\text{el}, 1_\text{e}), (2_\text{o}, 3_\text{el}, 1_\text{e}), (0_3, 1_\text{o}, 3_\text{or}), (2_\text{e}, 1_\text{o}, 3_\text{or})\};
\]
or \n = 0, \upsilon = \emptyset and \omega_0 = 3_\text{el}, \hat{\omega}_0 = 0_\text{e}.
At first assume \n \geq 1 and \(w_{n-1}, \omega_n, \hat{\omega}_n) = (0_3, 3_\text{el}, 1_\text{e})\). Then
\[
\Pi_*(\omega|n) = \Pi_*(w3_\text{el}) = 001; \quad \Pi_*(\hat{\omega}|n) = \Pi_*(w1_\text{e}) = 001, \quad \text{where} \quad \tau = \Pi_*(w).
\]
Since \(\Pi(\omega) = \sigma = \Pi(\hat{\omega})\), we have \(\mathcal{L}(\omega_{n+1}) = \mathcal{L}(1\text{e}\hat{\omega}_{n+1}) = 0\). By checking \(G\), the only possibility is \(\hat{\omega}_{n+1} = 3_\text{or}\). So \(\mathcal{L}(\omega_{n+1}) = 01\) and \(\Pi_*(\hat{\omega}|n+1) = \tau0011\). Again by \(\Pi(\omega) = \Pi(\hat{\omega})\), we have
\[
\mathcal{L}(\omega_{n+1}) = \mathcal{L}(1\text{e}0_30_3) = \mathcal{L}(0_30_3) = \mathcal{L}(0_30_30_3) = 1 * .
\]
By checking \(G\), the only possibility is \(\omega_{n+1} = 3_{er}\). As a consequence \(\Pi_*(\omega|n+2) = \tau0010\). This process can continue infinitely. By induction, we get
\[
\omega = w3_\text{el}(0_30_3) = \in \tilde{E}_{\ell}; \quad \hat{\omega} = w1_\text{e}0_30_3 = \in \tilde{E}_r .
\]
By (53), \(\hat{\omega} = \ell(\omega)\).
For the other four cases, the same proof shows that
\[
\omega \in \tilde{E}_{\ell}; \quad \hat{\omega} \in \tilde{E}_r \quad \text{and} \quad \hat{\omega} = \ell(\omega).
\]
This implies that
\[
\Pi^{-1}(\{\sigma\}) \subset \tilde{E}_{\ell} \cup \tilde{E}_r, \quad \#\tilde{E}_{\ell} \cap \Pi^{-1}(\{\sigma\}), \#\tilde{E}_r \cap \Pi^{-1}(\{\sigma\}) \leq 1.
\]
So the first statement follows.
If \#\Pi^{-1}(\{\sigma\}) \geq 2, then \Pi^{-1}(\{\sigma\}) = \{\omega, \ell(\omega)\} for some \omega \in \tilde{E}_{\ell}, hence \sigma \in \Pi(\tilde{E}_{\ell}).
If \sigma \in \Pi(\tilde{E}_{\ell}), then by i), \sigma \in \Pi(\tilde{E}_r), so \#\Pi^{-1}(\{\sigma\}) \geq 2. Thus the second statement holds.

iii) By Proposition 4.2, \Pi is surjective. Now we show that \Pi preserves the order. Given \omega \not< \hat{\omega}. Then there exists \n \geq 0 such that \omega|\n-1 = \hat{\omega}|\n-1 =: \upsilon \in \omega_n \land \hat{\omega}_n. If \omega_n \not< \hat{\omega}_n, then by (41),
\[
\mathcal{L}(w_{\n-1}\omega_n) = 0*; \quad \mathcal{L}(w_{\n-1}\hat{\omega}_n) = 1 * .
\]
Hence \(\Pi(\omega) < \Pi(\hat{\omega})\).
Now assume \(\omega|\n \not< \hat{\omega}|\n\). Then
\[
(\omega_n, \hat{\omega}_n) \in \{(3_\text{el}, 1_\text{e}), (1_\text{o}, 3_\text{or}), (3_\text{el}, 0_\text{e})\}.
\]
At first assume \(\omega_n, \hat{\omega}_n) = (3_\text{el}, 1_\text{e})\). So either \omega = \omega|\n-1_{3_\text{el}}(0_30_3) := \omega or there exists \m \geq 0 such that \(\omega = \omega|\m-1_{3_\text{el}}(0_30_3)\alpha \cdots\) with \alpha = 1_\text{e} or 3_\text{el}\). Since
\[
\mathcal{L}(0_31_\text{e}) = 0; \quad \mathcal{L}(0_33_\text{el}) = 00; \quad \mathcal{L}(0_33_\text{er}) = 10.
\]
We conclude that $\Pi(\omega) \leq \Pi(\hat{\omega})$.

The same argument shows that $\Pi(\ell(\hat{\omega})) \leq \Pi(\hat{\omega})$. So by i) we have

$$\Pi(\omega) \leq \Pi(\hat{\omega}) = \Pi(\ell(\hat{\omega})) \leq \Pi(\hat{\omega}).$$

For the other two cases, the same argument shows that $\Pi(\omega) \leq \Pi(\hat{\omega})$. Hence $\Pi$ is order preserving.

By (12) and the definition of $\Pi$, one get (58) by tracing the graph $G$.

By ii) $\Pi^{-1}(\Pi(\tilde{E}_l)) = \tilde{E}_l \cup \tilde{E}_r$ and $\Pi : \tilde{E}_l \to \Pi(\tilde{E}_l)$ is bijective. Since $\Pi$ is surjective, the restriction $\Pi : \omega_\infty \setminus (\tilde{E}_l \cup \tilde{E}_r) \to \Sigma_\infty \setminus \Pi(\tilde{E}_l)$ is also surjective. Again by ii), this restriction is injective. So it is bijective. Combine these two bijections, we conclude that $\Pi : \Omega_\infty \setminus \tilde{E}_r \to \Sigma_\infty$ is a bijection.

iv) (59) and (60) follow from (54) and (55) by using the definition of $\Pi$ and tracing on $G$. Now we show that all the restriction maps in (61) are bijections. By iii), they are all injections. Now we show that they are surjections. We take the first one as an example. Since $\Pi$ is surjective, for each $e \in \Sigma_n^\circ$, there exists some $\omega \in \Omega_\infty$ such that $\Pi(\omega) = \sigma 0 1^\infty$. We only need to show that $\omega \in \tilde{E}_l^\circ$. We delete all the edges of $G$ with label containing 0. Then only two connected subgraphs are left:

$$0_e \xrightarrow{11} 3_{or} \xrightarrow{\emptyset} 0_e; \quad 1_e \xrightarrow{1} 2_o \xrightarrow{1} 1_e.$$  

Each of these subgraphs can generates $1^\infty$ and $1^\infty$ can only be generated by these two subgraphs. Thus $\omega \in \tilde{E}_l^\circ$ or $\tilde{E}_r^\circ$. However, by (58) and (60), if $\omega \in \tilde{E}_r^\circ$, then $\Pi(\omega) = 1^\infty$ or $\sigma 0 1^\infty$ with $\sigma \in \Sigma_n^\circ$. So we conclude that $\omega \in \tilde{E}_l^\circ$. That is, $\Pi_{or}$ is surjective. The other three cases can be proven by exactly the same way.

v) By (11) and tracing on $G$, it is seen that $\Pi(\mathcal{F}) \subset \Sigma_\infty^{(2)}$. Combine with (57), we have $\Pi(\tilde{E}_l \cup \mathcal{F}) \subset \Sigma_\infty^{(2)}$. By iii), The restriction of $\Pi$ on $\tilde{E}_l \cup \mathcal{F}$ is injective. Thus we only need to show that $\Sigma_\infty^{(2)} \subset \Pi(\tilde{E}_l \cup \mathcal{F})$.

Fix any $\sigma \in \Sigma_\infty^{(2)}$. Since $\Pi$ is surjective, there exists $\omega \in \Omega_\infty$ such that $\Pi(\omega) = \sigma$.

At first, we claim that $1_e$ does not appear in $\omega$ infinitely often (i.o.). Indeed if otherwise, by tracing backward on $G$, one of the following words must appear in $\omega$ i.o.:

$$1_o 3_{or} 0_o 1_e; \quad 0_o 3_{or} 0_o 1_e; \quad 2_o 3_{el} 0_o 1_e; \quad 0_o 3_{el} 0_o 1_e; \quad 2_e 2_o 1_e; \quad 1_e 2_o 1_e.$$  

By applying $\mathcal{L}$ to these words, we see that for the first four cases, 00 will appear and for the last two cases, 11 will appear. So 00 or 11 will appears in $\Pi(\omega)$ i.o., a contradiction.

The same proof shows that $1_o$ also only appear in $\omega$ finite times. Now we go back to the graph $\tilde{G}$ and delete $1_o$ and $1_e$ and all the edges connected to $1_o$ and $1_e$. We also delete $0_e 3_{or}$ and $0_o 3_{el}$ since they have labels 11 and 00, respectively. Now the remaining part of the graph has only three closed paths:

$$0_e \xrightarrow{01} 3_{ol} \xrightarrow{\emptyset} 0_e; \quad 0_o \xrightarrow{10} 3_{er} \xrightarrow{\emptyset} 0_o; \quad 2_o \xrightarrow{0} 2_e \xrightarrow{1} 2_o,$$

and (01) can be realized by them and only by them. So $\omega$ can only be eventually $(ab)^\infty$, with

$$ab \in \{0_o 3_{er}, 0_e 3_{ol}, 2_e 2_o\}.$$  

That is, $\omega \in \tilde{E}_l \cup \tilde{E}_r \cup \mathcal{F}$.

Now combine with i), $\Pi(\omega) \in \Pi(\tilde{E}_l \cup \mathcal{F})$. Hence, $\Sigma_\infty^{(2)} \subset \Pi(\tilde{E}_l \cup \mathcal{F})$. Then the restriction of $\Pi$ on $\tilde{E}_l \cup \mathcal{F}$ is bijective.

Now by i), (62) follows. \qed
Remark 4.9. Recall that we have defined an order $\preceq$ on $\Sigma_*$ by (32). If $(\omega, \hat{\omega}) \in G_{II}$, by (63)-(65), we conclude that for any $m, m' > |\omega \wedge \hat{\omega}|$, 

$$\Pi_*(\omega|_m) < \Pi_*(\hat{\omega}|_{m'}).$$

4.3.3. Proof of Theorem 4.6. Before giving the formal proof, let us explain the mechanism to form a gap in $\Omega_\infty$. At first, if $(\sigma, \hat{\sigma})$ is gap of $\Sigma_\infty$, then it can be lifted to a gap of $\Omega_\infty$ via $\Pi$. By Proposition C.1, the gap of $\Sigma_\infty$ has the form $(\tau01^{\infty}, \tau10^{\infty})$. Secondly, if $\Pi^{-1}(\sigma)$ has more than one elements, then by Proposition 4.8 ii), it has exactly two elements, and the elements form a gap. These are the only two ways to form a gap.

Proof of Theorem 4.6. i) Fix any $\omega \neq \omega_*$. By (58), $\Pi(\omega_*) = 0^{\infty} \notin \Sigma_\infty$. By Proposition 4.8 ii), $\Pi(\omega) \neq 0^{\infty}$. We claim that $\omega_* < \omega$. Indeed, if otherwise, $\omega \preceq \omega_*$, then $\Pi(\omega) \leq \Pi(\omega_*) = 0^{\infty}$. Since $0^{\infty}$ is the minimum of $\Sigma_\infty$, we have $\Pi(\omega) = 0^{\infty}$, a contradiction. So $\omega_* < \omega$.

The same argument shows that if $\omega \neq \omega^*$, then $\omega < \omega^*$. So the result follows.

ii) Combine with Lemma 4.7, we only need to show (50) and (51). Fix $\omega \in \mathcal{E}_{\ell}^0$. By (59), there exists $\sigma \in \Sigma_*$ such that $\Pi(\omega) = \sigma01^{\infty} =: \tau$ and $\Pi(\ell^0(\omega)) = \sigma10^{\infty} =: \hat{\tau}$. By Proposition C.1, $(\tau, \hat{\tau})$ is a gap of $\Sigma_\infty$. We claim that $(\omega, \ell^0(\omega))$ is a gap of $\Omega_\infty$. Indeed if otherwise, there exists $\hat{\omega}$ such that $\omega < \hat{\omega} < \ell^0(\omega)$, so

$$\tau \leq \Pi(\hat{\omega}) \leq \hat{\tau}.$$

So either $\Pi(\hat{\omega}) = \tau$ or $\hat{\tau}$. In the former case, $\Pi^{-1}(\{\tau\}) \supset \{\omega, \hat{\omega}\}$. So by Proposition 4.8 ii), $\tau \in \Sigma_\infty$, which is a contradiction. In the latter case, we get a contradiction by the same reasoning. So $(\omega, \ell^0(\omega))$ is a gap of $\Omega_\infty$ and (50) holds.

The same proof shows that (51) holds.

iii) Combine with Lemma 4.7, we only need to show (52). Take $\omega \in \tilde{\mathcal{E}}_I$, we claim that $(\omega, \ell(\omega))$ is a gap. Indeed if otherwise, there exists $\tilde{\omega}$ such that $\omega < \tilde{\omega} < \ell(\omega)$, so

$$\Pi(\omega) \leq \Pi(\tilde{\omega}) \leq \Pi(\ell(\omega)).$$

By Proposition 4.8 i), $\Pi(\omega) = \Pi(\ell(\omega)) =: \sigma$. So $\Pi^{-1}(\{\sigma\}) \supset \{\omega, \tilde{\omega}, \ell(\omega)\}$, which contradicts with Proposition 4.8 ii). So (52) holds.

iv) Now assume $(\omega, \hat{\omega}) \in \mathcal{G}$. Then either $\Pi(\omega) = \Pi(\hat{\omega}) =: \sigma$, or $\Pi(\omega) < \Pi(\hat{\omega})$.

In the former case, by Proposition 4.8 ii), $\omega \in \mathcal{E}_I$ and $\hat{\omega} = \ell(\omega)$. So $(\omega, \hat{\omega}) \in \mathcal{G}_{II}$.

In the latter case, we claim that $(\Pi(\omega), \Pi(\hat{\omega}))$ is a gap of $\Sigma_\infty$. Indeed, if otherwise, there exists $\sigma \in \Sigma_\infty$ such that $\Pi(\omega) < \sigma < \Pi(\hat{\omega})$. Assume $\Pi(\hat{\omega}) = \sigma$, then $\hat{\omega} \neq \omega$, $\omega$. Since $\Pi$ preserve the order, we must have $\omega < \hat{\omega} < \hat{\omega}$, a contradiction.

So there exists $\sigma \in \Sigma_*$ such that $\Pi(\omega) = \sigma01^{\infty}$ and $\Pi(\hat{\omega}) = \sigma10^{\infty}$. By Proposition 4.8 iv) and ii), either $\omega \in \mathcal{E}_{I}^\sigma$, $\hat{\omega} = \ell^0(\omega)$, or $\hat{\omega} \in \mathcal{E}_{I}^\sigma$, $\omega = \ell^\sigma(\hat{\omega})$. So $(\omega, \hat{\omega}) \in \mathcal{G}_{I}$.

As a result, $\mathcal{G} \subset \mathcal{G}_I \cup \mathcal{G}_{II}$. Combine with ii) and iii), we get the equality. \hfill \Box

We end with the following lemma, which will be used to show that $\pi$ is order-preserving.

Lemma 4.10. Assume $\omega < \hat{\omega}$ but $\omega \neq \hat{\omega}$. Then there exists $(\tau, \hat{\tau}) \in \mathcal{G}_{II}$ such that $\omega \leq \tau$ and $\hat{\tau} \leq \hat{\omega}$. Moreover if $\omega \neq \tau(\hat{\tau} \neq \hat{\omega})$, then $\omega < \tau(\hat{\tau} < \hat{\omega})$. 

Proof. Assume \( n \geq 0 \) is such that \( \omega|_{n-1} = \hat{\omega}|_{n-1} =: w \) and \( \omega|_{n} < \hat{\omega}|_{n} \). Since \( \omega \neq \hat{\omega} \),
\[
(\omega_{n}, \hat{\omega}_{n}) \in \{(3_{el}, 1_{e}), (1_{o}, 3_{or}), (3_{el}, 0_{e})\}.
\]
At first assume \((\omega_{n}, \hat{\omega}_{n}) = (3_{el}, 1_{e})\). So either \( \omega = \omega|_{n-1}3_{el}(0_{0}3_{er})^{\infty} =: \tau \in \hat{\mathcal{C}}_{l} \) or there exists \( m \geq 0 \) such that \( \omega = \omega|_{n-1}3_{el}(0_{0}3_{er})^{m}0_{0}\alpha \cdots \) with \( \alpha = 1_{e} \) or \( 3_{el} \). In the latter case, by (49), \( \omega < \tau \). The same argument shows that \( \ell(\tau) \leq \hat{\omega} \) and if \( \omega \neq \ell(\tau) \), then \( \ell(\tau) < \hat{\omega} \).

For the case that \((\omega_{n}, \hat{\omega}_{n}) = (1_{o}, 3_{or}) \) or \((3_{el}, 0_{e})\), the proof is the same. \( \Box \)

5. Coding and the Hausdorff Dimension of the Spectrum

In this section, we prove a weak version of Theorem 1.5, which is enough for estimating the dimension of the spectrum. Then we prove Theorem 1.1.

5.1. Coding of the spectrum \( \sigma_{\lambda} \). For any \( \omega \in \Omega_{\infty} \), we have \( \omega|_{n} \in \Omega_{n} \). Hence \( I_{\omega|_{n}} \in \mathcal{B}_{n} \).

By the construction, we have \( I_{\omega|_{n+1}} \subset I_{\omega|_{n}} \). By (23), \( |I_{\omega|_{n}}| \rightarrow 0 \). Hence \( \bigcap_{n \geq 0} I_{\omega|_{n}} \) is a singleton. By Corollary 3.10, this point is in \( \sigma_{\lambda} \).

We define the coding map \( \pi : \Omega_{\infty} \rightarrow \sigma_{\lambda} \) as
\[
\pi(\omega) := \bigcap_{n \geq 0} I_{\omega|_{n}}.
\]

We have the following weak version of Theorem 1.5:

**Proposition 5.1.** \( \pi : (\Omega_{\infty}, \preceq) \rightarrow (\sigma_{\lambda}, \preceq) \) is continuous, surjective and preserves the orders.

To prove it, we need two lemmas.

**Lemma 5.2.** Given \( \omega, \hat{\omega} \in \Omega_{\infty} \). If \( \omega < \hat{\omega} \), then \( \pi(\omega) \not< \pi(\hat{\omega}) \).

**Proof.** Assume \( n \) is such that \( \omega|_{n-1} = \hat{\omega}|_{n-1} \) and \( \omega|_{n} < \hat{\omega}|_{n} \). By Corollary 4.4, \( I_{\omega|_{n}} \not< I_{\hat{\omega}|_{n}} \). Since \( \pi(\omega) \in I_{\omega|_{n}} \) and \( \pi(\hat{\omega}) \in I_{\hat{\omega}|_{n}} \), we conclude that \( \pi(\omega) \not< \pi(\hat{\omega}) \). \( \Box \)

**Lemma 5.3.** If \((\omega, \hat{\omega}) \in \mathcal{G}_{11} \), then \( \pi(\omega) \leq \pi(\hat{\omega}) \).

**Proof.** Write
\[
\sigma^{(n)} = \Pi_{s}(\omega|_{n}); \quad \hat{\sigma}^{(n)} = \Pi_{s}(\hat{\omega}|_{n}).
\]
By Remark 4.9, if \( n > |\omega \wedge \hat{\omega}| \), then \( \sigma^{(n)} \not< \hat{\sigma}^{(n)} \). By Proposition 2.6, \( z_{\sigma^{(n)}} \not< z_{\hat{\sigma}^{(n)}} \).

On the other hand, by (44),
\[
\pi(\omega) \in I_{\omega|_{n}} = B_{\sigma^{(n)}}; \quad \pi(\hat{\omega}) \in I_{\hat{\omega}|_{n}} = B_{\hat{\sigma}^{(n)}}.
\]
By (23), the length of the bands \( B_{\sigma^{(n)}} \) and \( B_{\hat{\sigma}^{(n)}} \) tends to zero as \( n \) tends to infinity. Since \( z_{\sigma^{(n)}} \in B_{\sigma^{(n)}} \) and \( z_{\hat{\sigma}^{(n)}} \in B_{\hat{\sigma}^{(n)}} \), we have
\[
\pi(\omega) = \lim_{n} z_{\sigma^{(n)}} \leq \lim_{n} z_{\hat{\sigma}^{(n)}} = \pi(\hat{\omega})
\]
So the result follows. \( \Box \)
Remark 5.4. In Lemma 6.8, we will show that the strict inequality holds. Here we just mention that the proof is highly nontrivial. We need to study the $\infty II$ energies to finally reach the strict inequality.

Proof of Proposition 5.1. By Corollary 3.10, $\pi$ is surjective. Assume $|\omega \land \hat{\omega}| = n$, then $d(\omega, \hat{\omega}) = 2^{-n}$. We have

$$|\pi(\omega) - \pi(\hat{\omega})| \leq |I_{\omega \land \hat{\omega}}|.$$ 

Combine with (23), $\pi$ is continuous.

Assume $\omega \prec \hat{\omega}$. If $\omega < \hat{\omega}$, by Lemma 5.2, $\pi(\omega) < \pi(\hat{\omega})$. If $\omega \neq \hat{\omega}$, by Lemma 4.10, there exists $(\tau, \hat{\tau}) \in G_{II}$ such that $\omega \preceq \tau$ and $\hat{\tau} \preceq \hat{\omega}$, moreover if $\omega \neq \tau (\hat{\tau} \neq \hat{\omega})$, then $\omega < \tau (\hat{\tau} < \hat{\omega})$. By Lemmas 5.2 and 5.3,

$$\pi(\omega) \leq \pi(\tau) \leq \pi(\hat{\tau}) \leq \pi(\hat{\omega}).$$

So $\pi$ is order-preserving. \hfill \Box

5.2. Lower bound for the Hausdorff dimension of $\sigma_\lambda$. In this subsection, at first we present a sufficient condition for estimating from below the Hausdorff dimension of the limit set of a SNS. Then we apply it to a sub-SNS of $\{\mathcal{B}_n : n \geq 0\}$ to obtain a lower bound for the Hausdorff dimension of $\sigma_\lambda$.

5.2.1. Dimension estimation for limit set of SNS. We need the following general result on the dimension of limit set of SNS, which is also useful in its own right.

Proposition 5.5. Let $\mathcal{I} = \{\mathcal{I}_n : n \geq 0\}$ be a SNS. Assume $\mathcal{I}$ satisfies:

i) There exist $\lambda \in (0, 1)$ and $C > 0$ such that

$$|I| \geq C \lambda^n, \ \forall n \geq 0, \forall I \in \mathcal{I}_n.$$ 

ii) There exists $C' > 0$ such that for any $n, k \geq 0$, and $I, I' \in \mathcal{I}_n$,

$$\frac{\#\{J \in \mathcal{I}_{n+k} : J \subset I\}}{\#\{J \in \mathcal{I}_{n+k} : J \subset I'\}} \leq C'.$$

Then the limit set satisfies

$$\dim_H A(\mathcal{I}) \geq \liminf_{n \to \infty} \frac{\log \#\mathcal{I}_n}{-n \log \lambda}.$$ 

Proof. For any $n \geq 0$, let

$$\kappa_n = \#\mathcal{I}_n, \ \alpha = \liminf_{n \to \infty} \frac{n}{\sqrt[n]{\kappa_n}}, \ t = -\log \alpha / \log \lambda.$$ 

If $\alpha = 1$, we are done. Suppose $\alpha > 1$. Take any $0 < \delta < t$. There exists $0 < \varepsilon < \alpha$ such that, for any large enough $n$,

$$t - \delta < \frac{-n \log(\alpha - \varepsilon)}{(n+1) \log \lambda + \log C}.$$ 

Note that, for any large enough $n$, $\kappa_n > (\alpha - \varepsilon)^n$. 
For any $n \geq 0$, define a probability measure $\mu_n$ on $\mathbb{R}$ as follows: $\mu_n$ is supported on $\bigcup_{B \in \mathcal{I}_n} B$. For any $B \in \mathcal{I}_n$,

$$\mu_n(B) = \kappa_n^{-1},$$

and $\mu_n$ is uniform on the interval $B$.

Take any $k, n > 0$. Let $B \in \mathcal{I}_n$. Suppose there are $m$ intervals in $\mathcal{I}_{n+k}$ that is contained in $B$. Then $\mu_{n+k}(B) = m/\kappa_{n+k}$. By ii),

$$m/C' \leq \frac{\kappa_{n+k}}{\kappa_n} \leq C'm.$$

So we have

$$\kappa_n^{-1}/C' \leq \mu_{n+k}(B) \leq C'\kappa_n^{-1}.
\tag{66}$$

Take a weak-star limit of $\{\mu_n\}_{n \geq 1}$, say $\mu$. Since $\mathcal{I}$ is a SNS, $\mu$ is a probability measure supported on $A(\mathcal{I})$. Moreover, by (66), for any $n > 0$ and $B \in \mathcal{I}_n$,

$$\kappa_n^{-1}/C' \leq \mu(B) \leq C'\kappa_n^{-1}.$$

For any open interval $U$ with $|U|$ small enough, there exists $k \in \mathbb{N}$ such that

$$C\lambda^{k+1} < |U| \leq C^{k+1}.$$

At most two intervals in $\mathcal{I}_k$ intersect $U$. Then

$$\mu(U) \leq 2C'\kappa_k^{-1} \leq 2C'(\alpha - \varepsilon)^{-k} \leq 2C'|U|^{\frac{-k\log(\alpha - \varepsilon)}{\lambda^{k+1} \log C}} \leq 2C'|U|^t - \delta.$$

By mass distribution principle,

$$\dim_H A(\mathcal{I}) \geq t - \delta.$$

Since $\delta > 0$ can be arbitrarily small, the result follows.

5.2.2. **Lower bound for** $\dim_H \sigma_\lambda$. Now we construct a sub-NS of $\{\mathcal{B}_n : n \geq 0\}$ such that it is a SNS and satisfies the conditions in Proposition 5.5.

Define a sub-alphabet $\tilde{A} = \{1_e, 1_o, 2_e, 2_o\}$ of $A$ and a sub directed graph $\tilde{G}$ with restricted admissible relation:

$$1_e \to 2_o; \quad 1_o \to 2_e; \quad 2_e \to 1_o, 2_o; \quad 2_o \to 1_e, 2_e.$$

Consider the sub-NS $\tilde{\mathcal{B}} = \{\tilde{\mathcal{B}}_n : n \geq 1\}$ defined by

$$\tilde{\mathcal{B}}_n := \{I_{0,1,0, w} \in \mathcal{B}_{n+1} : w = w_1 \cdots w_n, w_i \in \tilde{A}\}.$$

**Proposition 5.6.** $\tilde{\mathcal{B}}$ is a SNS and its limit set $A(\tilde{\mathcal{B}}) \subset \sigma_\lambda \cap B_\infty$. Moreover There exists constant $C > 0$ such that for all $n, k \geq 0$ and $I, I' \in \tilde{\mathcal{B}}_n$,

$$|I| \geq C 4^{-n} \quad \text{and} \quad \frac{\#\{J \in \tilde{\mathcal{B}}_{n+k} : J \subset I\}}{\#\{J \in \tilde{\mathcal{B}}_{n+k} : J \subset I'\}} \leq 2.$$

Proof. One check directly that \( \tilde{\mathcal{B}} \) is a sub-NS of \( \mathcal{B} \), so \( A(\tilde{\mathcal{B}}) \subset A(\mathcal{B}) = \sigma_1 \). Moreover, since \( 2e < 1e \) and \( 1o < 2o \), combine with Corollary 4.4, one conclude that \( \tilde{\mathcal{B}} \) is a SNS.

Take any \( n > 0 \) and any \( I \in \mathcal{B}_n \). We claim that for any \( 1 \leq k \leq n + 1 \) and any \( x \in I \),

\[
|h_k(x)| \leq 2.
\]

Indeed, assume \( I = I_{0,1,0}w \) with \( w \in \tilde{\mathcal{A}}^n \), then by (44) and tracing on \( \mathcal{G} \), for any \( k = 0, \ldots, n \) we have

\[
I^{(k)} := I_{0,1,0}w_1 \cdots w_k = B_1 \sigma_1 \cdots \sigma_k, \text{ where } \sigma_i \in \{0, 1\}.
\]

Thus \( I^{(k)} \in \mathcal{B}_{k+1} \) and \( I = I^{(n)} \subset I^{(n-1)} \subset \cdots \subset I^{(0)} = \mathcal{B}_1 \). Since \( |h_k| \leq 2 \) on any \( B \in \mathcal{B}_k \), the claim follows.

The claim implies immediately that \( A(\tilde{\mathcal{B}}) \subset B_\infty \).

Recall that

\[
h_{k+1}(x) = h_k(x)(h_{k-1}^2(x) - 2) - 2.
\]

we have

\[
h_{k+1}'(x) = (h_{k-1}^2(x) - 2)h_k'(x) + 2h_{k-1}(x)h_k(x)h_{k-1}'(x).
\]

Then, for \( x \in I \) and \( 2 \leq k \leq n \),

\[
|h_{k+1}'(x)| \leq 2|h_k'(x)| + 8|h_{k-1}'(x)|.
\]

This implies there exists \( c > 0 \) depending only on \( B_1 \) such that, for \( x \in I \) and \( 1 \leq k \leq n + 1 \),

\[
|h_k'(x)| \leq c4^k.
\]

By Floquet theory, \( h_{n+1} \) is monotone on \( I \in \mathcal{B}_{n+1} \) and \( h_{n+1}(I) = [-2, 2] \). Writing \( I = [x_0, x_1] \), we have

\[
4 = \int_{x_0}^{x_1} |h_{n+1}'(x)|dx \leq c(x_1 - x_0)4^{n+1}.
\]

Thus \( |I| = x_1 - x_0 \geq c^{-1}4^{-n} \).

It is seen that \( \tilde{\mathcal{B}} \) is corresponding to a subshift of finite type with alphabet \( \{1_o, 2_o, 1_e, 2_e\} \) and incidence matrix

\[
\tilde{\mathcal{A}} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}.
\]

By this, one can show easily that

\[
\frac{\#\{J \in \tilde{\mathcal{B}}_{n+k} : J \subset I\}}{\#\{J \in \tilde{\mathcal{B}}_{n+k} : J \subset I'\}} \leq \frac{F_k}{F_{k-1}},
\]

where \( \{F_k : k \geq 0\} \) is the Fibonacci sequence defined by \( F_0 = 1 \), \( F_1 = 2 \) and \( F_{n+1} = F_n + F_{n-1} \). Then the result follows easily. \( \square \)
Proof of Theorem 1.1.} By the definition of \( \tilde{A} \), it is seen that \( \# \tilde{A}_n = F_n \). It is well known that \( F_n \sim \alpha^n \). By Propositions 5.6 and 5.5,
\[
\dim_H \sigma_\lambda \geq \dim_H A(\tilde{A}) \geq \liminf_{n \to \infty} \frac{\log \# \tilde{A}_n}{n \log 4} = \frac{\log \alpha}{\log 4}.
\]

\( \square \)

Remark 5.7. 1) Indeed, we can start from any band \( B \) with type in \( \tilde{A} \) and do the same construction by following the subgraph \( \tilde{G} \). Then we construct a Cantor subset of \( \sigma_\lambda \cap B \) with the same dimension estimation. This means that the “local” dimension of \( \sigma_\lambda \) is also uniformly bounded from below.

2) If we view the SNS \( \tilde{B} \) as a dynamical system coded by the subshift \( \Omega_{\tilde{A}} \), then \( \log \alpha \) is just the topological entropy of the system and \( \log 4 \) can be viewed as an upper bound of the Lyapunov exponents.

6. \( \infty \)-type Energies, Gaps of the Spectrum and Zeros of Trace Polynomials

In this section, at first, we establish the existence of \( \infty \)-type energies; then we prove Theorems 1.3, 1.5 and 1.8.

6.1. \( \infty_I \)-type energies. We will show that \( E \in \pi(\mathcal{E}_c^r \cup \mathcal{E}_g^r) \) is an \( \infty \)-type energy. Recall that in Sect. 1.2.5 we call it \( \infty_I \)-type energy. Indeed, we can be more precise:

Proposition 6.1. i) The set \( \pi(\mathcal{E}_c^r) \) is dense in \( \sigma_\lambda \). If \( E \in \pi(\mathcal{E}_c^r) \), then \( h_{2n}(E) > \sqrt{2} \) and
\[
\lim_{n \to \infty} h_{2n+1}(E) \to \infty \quad \text{and} \quad \lim_{n \to \infty} h_{2n}(E) \to \sqrt{2}.
\]

ii) If \( \mathcal{E}_g^r \) is dense in \( \sigma_\lambda \). If \( E \in \pi(\mathcal{E}_g^r) \), then \( h_{2n+1}(E) > \sqrt{2} \) and
\[
\lim_{n \to \infty} h_{2n}(E) \to \infty \quad \text{and} \quad \lim_{n \to \infty} h_{2n+1}(E) \to \sqrt{2}.
\]

We only prove i), since the proof of ii) is the same.

Lemma 6.2. Suppose \( \sigma \in \Sigma_n \) and \( B_\sigma \) has type 0. Then
\[
b_{\sigma_1} < b_{\sigma_{111}} < b_{\sigma_{11}} < b_\sigma,
\]
and
\[
h_{r+1}(E) \geq 2 \quad \text{for} \quad E \in [b_{\sigma_1}, b_\sigma],
\]
\[
h_r(1) \geq \sqrt{2} \quad \text{for} \quad E \in [a_{\sigma_{11}}, b_\sigma].
\]

Proof. Since \( B_\sigma \) has type 0, \( n \) is even and \( b_\sigma \notin \mathcal{R}_{n-1} \). By Lemma 2.8 i), \( a_{\sigma_1} = z_\sigma \). By Lemma 2.8 iii), \( b_{\sigma_{1k}} \notin \mathcal{R}_{n+k-1} \) for any \( k \geq 1 \). Applying Lemma 3.1 i) to \( B_{\sigma_1} \), we have \( B_{\sigma_1} < B_{\sigma_{111}} \). So \( b_{\sigma_1} < b_{\sigma_{111}} \). Applying Lemma 2.3 ii) to \( B_{\sigma_{111}} \), we have \( B_{\sigma_{111}} \subset \text{int}(B_{\sigma_{11}}) \). So \( b_{\sigma_{111}} < b_{\sigma_{11}} \). By Lemma 3.1 ii), \( a_{\sigma_{11}} \notin \mathcal{R}_{n+1} \) and \( B_{\sigma_1} < B_{\sigma_{11}} \subset \text{int}(B_\sigma) \). So \( z_\sigma = a_{\sigma_{11}} < a_{\sigma_{111}} \). Thus (69) follows.

If \( \sigma = 1^n \), then \( h_{1n+1}(E) = 1 \). Hence \( h_{n+1}(E) \geq 2 \) for any \( E \in [b_{\sigma_1}, \infty) \). Now assume \( \sigma < 1^n \). Since \( b_\sigma \notin \mathcal{R}_{n-1} \), by Lemma 2.8 iv),
Proof of Proposition 6.1. We only show i). At first we show that $\pi(\mathcal{E}_f^\sigma)$ is dense in $\sigma$. Since $\{I_w : w \in \Omega_n\}$ is a covering of the spectrum and $\max \{|I_w| : w \in \Omega_n\} \to 0$ as $n \to \infty$, we only need to show that $I_w \cap \pi(\mathcal{E}_f^\sigma) \neq \emptyset$ for any $w \in \Omega_n$. Since $G$ is connected, for any $\alpha \in A$, there exists an admissible path $a_\alpha := a_{\alpha_1} \cdots a_{\alpha_k}$. Assume $w \in \Omega_n$, define $v := w|_{m-1} u_{w,m}$, then $I_v$ has type $0_e$, $I_v \subset I_w$ and $\pi(w3or(0_e3or)^\infty) \in I_v \cap \pi(\mathcal{E}_f^\sigma)$. Hence $I_w \cap \pi(\mathcal{E}_f^\sigma) \neq \emptyset$.

Assume $E \in \pi(\mathcal{E}_f^\sigma)$, then there exist some even $n$ and $w \in \Omega_n$ with $w_n = 0_e$ such that $E = \pi(w(3or0_e)^\infty)$. Write $\sigma = \Pi_\sigma(w)$, then $\sigma \in \Sigma_n$. For any $k \geq 0$, by (42) and tracing on $G$, we have $\Pi_\sigma(w(3or0_e)^k) = \sigma 1^{2k}$. By (44),

$$B_{\sigma 1^{2k}} = I_{w(3or0_e)^k}.$$

Write $\sigma^{(k)} := \sigma 1^{2k}$. Then $B_{\sigma^{(k)}}$ has type $0_e$. By the definition of $\pi$, we have $E \in B_{\sigma^{(k)}}$. We claim that $E \geq b_{\sigma 1^{2k+1}}$. Indeed, by applying (69) to every $\sigma^{(k)}$, we have

$$b_{\sigma 1} < b_{\sigma 1^3} < \cdots < b_{\sigma 1^{2k+1}} < \cdots < b_{\sigma 1^{2k}} < b_{\sigma 1^2} < b_{\sigma}.$$

Since $E \in B_{\sigma^{(k)}}$ and the length of $B_{\sigma^{(k)}}$ tend to 0, we conclude that

$$E = \inf_{k \geq 1} b_{\sigma 1^{2k}} \geq \sup_{k \geq 1} b_{\sigma 1^{2k+1}}.$$

So $E \in [b_{\sigma^{(k)}}, b_{\sigma^{(k)}}]$. By (70),

$$h_{n+2k+1}(E) \geq 2.$$

Since $E \in B_{\sigma^{(k+1)}} = [a_{\sigma^{(k)}}, b_{\sigma^{(k)}}] \subset B_{\sigma^{(k)}} = [a_{\sigma^{(k)}}, b_{\sigma^{(k)}}]$, by (71),

$$h_{n+2k}(E) \geq \sqrt{2}.$$

By Lemma 3.6 ii), $E \in B_{\sigma 1^4} \subset \text{int}(B_{\sigma 1^2})$. By Lemma 2.5 ii), $E \notin \mathcal{R}_{n+1}$. Now for any $N \geq 1$, since $E \in B_{\sigma 1^{2N}}$, we have $|h_{n+2N}(E)| \leq 2$. Then by Lemma 2.9,

$$|h_{n+2N+1}(E)| \geq 2\lambda \prod_{j=0}^{n+2N} |h_j(E)| - |h_{n+2N}^2(E) - 2| \geq 2^{3N/2+1} \lambda \prod_{j=0}^{n} |h_j(E)| - 6.$$
Hence $|h_{n+2N+1}(E)| \to \infty$. Now by (21),

$$|h^2_{n+2N}(E) - 2| = \frac{|h_{n+2N-2}(E) + 2|}{|h_{n+2N-1}(E)|} \leq \frac{4}{|h_{n+2N-1}(E)|} \to 0.$$ 

So $|h_{n+2N}(E)| \to \sqrt{2}$.

Now by (71) and (70), the equation (68) holds. 

6.2. $\infty_{11}$-type energies. We will show that $E \in \pi(\tilde{E}_l \cup \tilde{E}_r)$ is also an $\infty$-type energy. Recall that in Sect. 1.2.5 we call it $\infty_{11}$-type energy.

**Proposition 6.3.** i) Both $\pi(\tilde{E}_l)$ and $\pi(\tilde{E}_r)$ are dense in $\sigma_\lambda$.

ii) Assume $(\omega, \hat{\omega}) \in \mathcal{G}_{11}$, then

$$\lim_{n \to \infty} |h_{2n}(\pi(\omega))| = \lim_{n \to \infty} |h_{2n+1}(\pi(\hat{\omega}))| = \infty.$$ 

Although this proposition looks similar with Proposition 6.1, the proof of which is much more involved. We need to study a dynamical system induced by the recurrence relation of trace polynomials, which is inspired by our previous work [23].

Define the period-doubling trace map $f : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$f(x, y) = \left( y(x^2 - 2) - 2, [y(x^2 - 2) - 2](y^2 - 2) - 2 \right).$$

(72)

By (21), for any $n \geq 0$ and $E \in \mathbb{R}$, we have

$$(h_{n+2}(E), h_{n+3}(E)) = f(h_n(E), h_{n+1}(E)).$$

We have

$$Df(x, y) = \begin{bmatrix} 2xy & x^2 - 2 \\ 2xy(y^2 - 2)(x^2 - 2)(3y^2 - 2) - 4y & 2xy \end{bmatrix}.$$ 

So we have

$$\det Df(x, y) = 4xy^2(y(x^2 - 2) - 2).$$

Define a simply connected domain $U$ (enclosed by the red curve in Fig. 4a) as

$$U := \{(x, y) : x < 0, y(x^2 - 2) - 2 < 0\}.$$
Define a compact set $D$ (enclosed by the blue curve in Fig. 4a) as

$$D := \{(x, y) : -\sqrt{2} \leq x, y \leq 0; y \leq x^2 - 2\}.$$

Notice that $(-1, -1) \in \partial D$. Write

$$A := (-\sqrt{2}, -\sqrt{2}), \quad B := (-\sqrt{2}, 0), \quad C := (-2 - \sqrt{2}, -\sqrt{2}), \quad F := (-1, -1).$$

Then $A, B, C$ are the vertices of the “triangle” $D$. Define

$$\Gamma := \{(-\sqrt{2}, y) : -\sqrt{2} \leq y \leq 0\},$$

$$\Upsilon := \{(x, -\sqrt{2}) : -\sqrt{2} \leq x \leq -2 - \sqrt{2}\},$$

$$\Lambda := \{(x, x^2 - 2) : -\sqrt{2} \leq x \leq -2 - \sqrt{2}\}.$$

Then $B, C, F \in \Lambda$ and $\partial D = \Gamma \cup \Upsilon \cup \Lambda$, see Fig. 4b.

6.2.1. Local dynamical property of $f$. We summarize the needed properties of $f$ in the following proposition.

**Proposition 6.4.** i) $f : U \to f(U)$ is a diffeomorphism and $D \subset f(U)$. $f$ has a unique fixed point $F$ in $U$.

ii) Let $g : f(U) \to U$ be the inverse of $f$, then $g$ has a unique fixed point $F$ in $f(U)$.

Moreover, $g(D) \subset D \cap U$.

iii) $g(\Lambda) \subset \Lambda$ and $g^n(\Lambda) \downarrow \{F\}$.

iv) Write $g = (g_1, g_2)$. For any $(x, y) \in D$ and $i = 1, 2$,

$$\frac{\partial g_i}{\partial x}(x, y), \quad \frac{\partial g_i}{\partial y}(x, y) > 0. \tag{73}$$

**Proof.** i) Write $(x_1, y_1) := f(x, y)$, we have

$$y_1 = x_1 \frac{(x_1 + 2)^2}{(x^2 - 2)^2} - 2x_1 - 2.$$

Since $x_1 < 0$ for $(x, y) \in U$, we have $y_1 < -2x_1 - 2$. Moreover, for $-2 \leq x < 0$, $y_1 < x_1^2/4 + x_1^2 - x_1 - 2$. Now by computing the image of the fiber $U_a := U \cap \{y = a\}$ under $f$ for all $a < 0$, one can verify that

$$f(U) = \{(x, y) : x < 0; \quad y < -2x - 2; \quad y < \frac{x^3}{4} + x^2 - x - 2 \text{ if } -2 \leq x < 0\}.$$

On check directly that $D \subset f(U)$. Define a function $g$ as

$$g(x, y) := \left( -\sqrt{2 - \frac{2 + x}{\sqrt{2 + \frac{2 + y}{x}}}}, -\sqrt{2 + \frac{2 + y}{x}} \right).$$

One check directly that $g$ can be defined on $f(U)$ and is smooth, and

$$g \circ f \mid_U = \text{Id}_U; \quad f \circ g \mid_{f(U)} = \text{Id}_{f(U)}.$$

So $f : U \to f(U)$ is a diffeomorphism and $g$ is the inverse of $f$. 

By direct computation, the fixed points of $f$ are
\[ (-1, -1), (2, 2), (-\alpha, \alpha - 1), (\alpha^{-1}, -(1 + \alpha^{-1})), \]
where $\alpha = (\sqrt{5} + 1)/2$. So $F$ is the only fixed point of $f$ in $U$.

ii) The first assertion is a restatement of i). Fix any $(x, y) \in D$. By direct computation,
\[-\sqrt{2} < -1.3 \leq g_1(x, y) \leq -1.2 < -\sqrt{2} - \sqrt{2}; \quad -\sqrt{2} < -1.3 \leq g_2(x, y) \leq -0.7.\]
Moreover,
\[ g_2(x, y) - g_1^2(x, y) + 2 = \frac{y - x^2 + 2}{xg_2(x, y)} \leq 0. \]
By the definitions of $D$ and $U$, we conclude that $g(D) \subset D \cap U$.

iii) Define $\phi : \mathbb{R} \to \mathbb{R}$ as $\phi(x) = x^2 - 2$. Write $a := -\sqrt{2}, b := -\sqrt{2 - \sqrt{2}}$. Then
\[ \Lambda = \{(x, \phi(x)) : x \in [a, b]\}. \]
By the definition of $f$, for any $x \in \mathbb{R}$,
\[ f(x, \phi(x)) = (\phi^2(x), \phi^3(x)), \]
where $\phi^n$ is the $n$-time iteration of $\phi$. By solving $\phi^2(x) = a$ and $b$, there exist $a_1, b_1$ such that
\[ a < a_1 = -1.11 \cdots < -1 < b_1 < b = -0.76 \cdots; \quad \phi^2(a_1) = a; \quad \phi^2(b_1) = b. \]
Since $[\phi^2(x)]' = 4x(x^2 - 2)$, for any $x \in [a_1, b_1]$,
\[ [\phi^2(x)]' \geq 3. \tag{74} \]
Define $\Lambda_1 := \{(x, \phi(x)) : x \in [a_1, b_1]\}$, then $\Lambda_1 \subset \Lambda$ and $f(\Lambda_1) = \Lambda$. Since $\Lambda \subset D \subset f(U)$ and $g : f(U) \to U$ is a diffeomorphism, we conclude that
\[ g(\Lambda) = \Lambda_1 \subset \Lambda. \]
Notice that $\phi^2(-1) = -1$. By (74) and contraction principle, for any $n \geq 2$ one can find $a_n, b_n$ such that $a_n > a_1, b_n < b_1$ and $a_n \uparrow -1, b_n \downarrow -1$ and $\phi^{2n}(a_n) = a, \phi^{2n}(b_n) = b$. By repeating the above proof, we get
\[ g^n(\Lambda) = \Lambda_n := \{(x, \phi(x)) : x \in [a_n, b_n]\} \downarrow \{F\}. \]
iv) By ii), we have $-1.3 \leq g_1(x, y), g_2(x, y) \leq -0.7$. Thus
\[ \frac{\partial g_1}{\partial x} = \frac{1}{2g_1g_2} + \frac{(2 - g_1^2)(2 - g_2^2)}{4xg_1g_2^2} > 0, \quad \frac{\partial g_1}{\partial y} = \frac{2 - g_1^2}{4xg_1g_2^2} > 0; \]
\[ \frac{\partial g_2}{\partial x} = \frac{2 - g_2^2}{2xg_2} > 0, \quad \frac{\partial g_2}{\partial y} = \frac{1}{2xg_2} > 0. \]
So, (73) holds. \qed
Define a partial order on \( \mathbb{R}^2 \) as follows. Assume \( p = (x, y) \) and \( q = (\tilde{x}, \tilde{y}) \), we say \( p \leq q \) if \( x \leq \tilde{x}, \ y \leq \tilde{y} \). Assume \( C \) is a continuous curve in \( \mathbb{R}^2 \) with a parametrization \( \gamma : [a, b] \to C \) such that \( \gamma(t) \leq \gamma(s) \) whenever \( t < s \), then we call \( C \) an increasing curve in \( \mathbb{R}^2 \).

**Lemma 6.5.** If \( p, q \in D \) and \( p \leq q \), then \( g(p) \leq g(q) \).

**Proof.** Notice that, if \( p, q \in D \) and \( p \leq q \), then for any \( \tilde{p} \in \mathbb{R}^2 \) with \( p \leq \tilde{p} \leq q \), we have \( \tilde{p} \in D \). By (73), \( g_1(x, \cdot), g_1(\cdot, y), g_2(x, \cdot), g_2(\cdot, y) \) are all strictly increasing. Now assume \( p = (x, y), q = (\tilde{x}, \tilde{y}) \) and \( p \leq q \), then  

\[
\begin{align*}
g_1(p) &= g_1(x, y) \leq g_1(\tilde{x}, y) \leq g_1(\tilde{x}, \tilde{y}) = g_1(q), \\
g_2(p) &= g_2(x, y) \leq g_2(\tilde{x}, y) \leq g_2(\tilde{x}, \tilde{y}) = g_2(q).
\end{align*}
\]

Hence \( g(p) \leq g(q) \). \( \square \)

Now we have the following consequence.

**Proposition 6.6.** Assume \( p \in D \). Then \( f^n(p) \in D \) for any \( n \in \mathbb{N} \) if and only if \( p = F \).

**Proof.** Since \( f(F) = F \) and \( F \in D \), we have \( f^n(F) \in D \) for any \( n \). So the if part holds.

To prove the only if part, we need the following claim:

**Claim:** We have  

\[
g^n(D) \downarrow \{(-1, -1)\}. \tag{75}
\]

\(<\) since \( g(F) = F \), we have \( F \in g^n(D) \) for any \( n \geq 0 \). By Proposition 6.4 ii), \( g(D) \subseteq D \). So \( \{g^n(D) : n \geq 0\} \) is a decreasing sequence of compact sets. To prove (75), we only need to show that the diameter of \( g^n(D) \) tends to 0.

Recall that, \( A, B, C \) are the vertices of the “triangle” \( D \). Define

\[
A_n := g^n(A), \quad B_n := g^n(B), \quad C_n := g^n(C).
\]

Since \( B, C \in \Lambda \), By Proposition 6.4 iii), \( B_n, C_n \to F \). We also have

\[
A \leq A_1 = \left( -\sqrt{2 - \frac{2 - \sqrt{2}}{\sqrt{3 - \sqrt{2}}}}, -\sqrt{3 - \sqrt{2}} \right), \quad A, A_1 \in D.
\]

We have \( A_n \in D \) since \( g(D) \subseteq D \). By Lemma 6.5,

\[
A \leq A_1 \leq A_2 \leq \cdots
\]

Hence \( A_\infty := \lim_n A_n \) exists and \( A_\infty \in D \), since \( g \) is continuous on \( D \), \( g(A_\infty) = A_\infty \). By Proposition 6.4 ii), \( g \) has a unique fixed point \( F \) in \( D \), so \( A_\infty = F \).

Recall that \( \Gamma \cup \Upsilon \cup \Lambda \) is the boundary of \( D \). By Proposition 6.4 i) and ii), \( g : f(U) \to U \) is a diffeomorphism and \( D \subseteq f(U) \). Since \( g^n(D) \subseteq D \), we conclude that \( g^n(\Gamma) \cup g^n(\Upsilon) \cup g^n(\Lambda) \) is the boundary of \( g^n(D) \). Notice that \( \Gamma \) is the vertical interval \( AB \) which is an increasing interval. By Lemma 6.5, \( g^n(\Gamma) \) is also increasing and has endpoints \( A_n, B_n \). Similarly, \( g^n(\Upsilon) \) is increasing and has endpoints \( A_n, C_n \). By Proposition 6.4 iii), \( g^n(\Lambda) \subseteq \Lambda \) is the segment of \( \Lambda \) with endpoints \( B_n, C_n \). So we conclude that

\[
g^n(D) \subset [x_{A_n}, x_{C_n}] \times [y_{A_n}, y_{B_n}] \].
Since \( A_n, B_n, C_n \to F \), we conclude that \( \text{diam}(g^n(D)) \to 0 \).

By the claim, we can write \( D \setminus \{F\} \) as the following disjoint union:

\[
D \setminus \{F\} = \bigcup_{n \geq 1} (g^{n-1}(D) \setminus g^n(D)).
\]

Now if \( p \in D \setminus \{F\} \), then there exists \( m \geq 1 \) such that \( p \in g^{m-1}(D) \setminus g^m(D) \). So we have

\[
f^m(p) \notin f(f^{m-1}(g^{m-1}(D) \setminus g^m(D))) = f(D \setminus g(D)).
\]

By Proposition 6.4 ii), \( g(D) \subset D \cap U \). Notice that \( B, C \not\in U \) and \( D \setminus \{B, C\} \subset U \), so \( g(D) \subset D \setminus \{B, C\} \subset U \). Since \( f \) is injective on \( U \), we conclude that

\[
f(D \setminus \{B, C\}) \setminus g(D) = f(D \setminus \{B, C\}) \setminus D.
\]

Hence \( f((D \setminus \{B, C\}) \setminus g(D)) \cap D = \emptyset \). On the other hand, by direct computation,

\[
f(B) = (-2, 2), \quad f(C) = (0, -2) \notin D.
\]

So we have

\[
D \cap f(D \setminus g(D)) = D \cap (f((D \setminus \{B, C\}) \setminus g(D)) \cup \{f(B), f(C)\}) = \emptyset.
\]

That is, \( f^m(p) \notin D \). So the only if part follows. \( \square \)

6.2.2. Proof of Proposition 6.3. Assume \((\omega, \hat{\omega}) \in \mathcal{G}_{11}\). Write \( E_1 = \pi(\omega) \) and \( E_2 = \pi(\hat{\omega}) \). By Lemma 5.3, \( E_1 \leq E_2 \).

**Lemma 6.7.** If \( E_1 = E_2 =: E \), then there exists \( m \in \mathbb{N} \) such that for any \( k \geq m \),

\[
(h_k(E), h_{k+1}(E)) \in \text{int}(D).
\]

If \( E_1 < E_2 \), then as \( k \to \infty \),

\[
|h_{2k}(E_1)|, |h_{2k+1}(E_2)| \to \infty.
\]

**Proof.** By (53), \( \omega \) takes one of the following two forms:

\[
w_{103_{e3}}(0_{3e3})^\infty; \quad w_{3e3}(0_{3e3})^\infty.
\]

We prove the lemma for \( \omega = w_{103_{e3}}(0_{3e3})^\infty \), and leave the other case to the reader. Assume \( w \in \Omega_n \). In this case, \( n \) is even, \( w_n = 0_e \) or \( 2_e \) and \( \hat{\omega} = \ell(\omega) = w_{3_{03}}(0_{3_{03}})^\infty \). For \( k \geq 1 \), by (63),

\[
\begin{align*}
\Pi^*(\omega|_{n+2k+1}) &= \Pi^*(w)\delta(10)^k =: \sigma^{(k)}, \\
\Pi^*(\hat{\omega}|_{n+2k+2}) &= \Pi^*(w)\delta(01)^k =: \hat{\sigma}^{(k)}.
\end{align*}
\]

Here \( \delta = 1 \) if \( w_n = 0_e \) and \( \delta = 0 \) if \( w_n = 2_e \). Notice that \( \hat{\sigma}^{(k)} = \sigma^{(k)}1 \). By (44),

\[
I_{\omega|_{n+2k+1}} = I_{w_{103_{e3}}(0_e)^k} = B_{\sigma^{(k)}}, \quad I_{\hat{\omega}|_{n+2k+2}} = I_{w_{3_{03}}(0_e)^k} = B_{\hat{\sigma}^{(k)}}.
\]
We have $E_1 \in B_{\sigma(k)}$ and $E_2 \in B_{\hat{\sigma}(k)}$. Notice that $B_{\sigma(k)}$ has type $0_o$ and $B_{\hat{\sigma}(k)}$ has type $0_e$. By applying Lemma 3.6 i) to $B_{\sigma(k)}$, we have

$$z_{\sigma(k)} = b_{\sigma(k)0} < a_{\sigma(k+1)} < E_1 < b_{\sigma(k+1)} < b_{\sigma(k)}.$$  

(76)

In particular, $E_1 \in \text{int}(B_{\sigma(k)})$. By Proposition 2.10 i), $h_{n+2(k+1)+1}$ is decreasing on $B_{\sigma(k)}$, so $h_{n+2(k+1)+1}(b_{\sigma(k+1)}) = -2$. Hence by (21),

$$-2 = h_{n+2(k+1)}(b_{\sigma(k+1)}) (h_{n+2(k+1)}(b_{\sigma(k+1)}) - 2) - 2.$$  

Since $B_{\sigma(k+1)}$ has type $0_o$, $b_{\sigma(k+1)} \notin \mathcal{R}_{n+2(k+1)}$. Thus $h_{n+2k+1}(b_{\sigma(k+1)}) = \pm \sqrt{2}$. Since $h_{n+2k+1}$ is decreasing on $B_{\sigma(k)}$, by (80),

$$-\sqrt{2} = h_{n+2k+1}(b_{\sigma(k+1)}) < h_{n+2k+1}(E_1) < h_{n+2k+1}(z_{\sigma(k)}) = 0.$$  

(77)

On the other hand, since $E_1 \in \text{int}(B_{\sigma(k)})$ with $|\sigma(k)| = n + 2k + 1$ and $\sigma(k)$ ends with 0, by Lemma 2.9 and Corollary 2.7,

$$h_{n+2k+1}(E_1) - (h_{n+2k}^2(E_1) - 2) = (-1)^{n+2k} 2\sqrt{\lambda} \prod_{j=0}^{n+2k} h_j(E_1) < 0.$$  

(78)

By a symmetric argument, one can show that

$$-\sqrt{2} < h_{n+2k+2}(E_2) < 0 \quad \text{and} \quad h_{n+2k+2}(E_2) - (h_{n+2k+1}^2(E_2) - 2) < 0.$$  

(79)

Now assume $E_1 = E_2 =: E$. By (77), (78) and (79), for any $k \geq n + 3 =: m$,

$$-\sqrt{2} < h_k(E) < 0; \quad h_{k+1}(E) < h_k^2(E) - 2.$$  

By the definition of $D$, we have $(h_k(E), h_{k+1}(E)) \in \text{int}(D)$.

Next assume $E_1 < E_2$. By (80), we have $z_{\sigma(k)} < E_1$. By a symmetric argument, we have $E_2 < z_{\hat{\sigma}(k)}$. So

$$z_{\sigma(k)} < E_1 < E_2 < z_{\hat{\sigma}(k)}.$$  

Since the length of $B_{\sigma(k)}$ and $B_{\hat{\sigma}(k)}$ tend to 0, There exists $K \geq n + 3$ such that for any $k \geq K$, $B_{\sigma(k)} < B_{\hat{\sigma}(k)}$. Recall that $\hat{\sigma}(k) = \sigma(k+1)$. So $b_{\sigma(k)} < a_{\hat{\sigma}(k)} = a_{\sigma(k+1)}$. By Proposition 2.10 i), $h_{n+2k+2}$ is decreasing on $B_{\sigma(k)}0$ and increasing on $B_{\sigma(k+1)}$. By Floquet theory, for any $E \in (b_{\sigma(k)0}, a_{\sigma(k+1)})$, we have $h_{n+2k+2}(E) < -2$. By (80), $b_{\sigma(k)0} < E_1 < b_{\sigma(k)} < a_{\sigma(k+1)}$, so

$$h_{n+2k+2}(E_1) < -2.$$  

(80)

We claim that

$$h_{n+2k+1}(E_1) \leq -1.$$  

(81)

If otherwise, by (77), we have $-1 < h_{n+2k+1}(E_1) < 0$. Then

$$h_{n+2k+3}(E_1) = h_{n+2k+2}(E_1) (h_{n+2k+1}^2(E_1) - 2) - 2 \geq 0,$$

which contradicts with (77).
By (78), \( E_1 \notin \mathcal{Z} \). Since \( E_1 \in B_{\sigma^*(\bar{t})} \), we have \(|h_{n+2k+1}(E_1)| \leq 2\). Now for any \( N > K \), by Lemma 2.9 and (80), (81),

\[
|h_{n+2N+1}(E_1)| \geq 2\lambda \prod_{j=0}^{n+2N+1} |h_j(E_1)| - |h_{n+2N+1}^2(E_1) - 2|
\]

\[
\geq \lambda 2^{(N-K)/2} \prod_{j=0}^{n+2K} |h_j(E_1)| - 2.
\]

Hence \(|h_{n+2N+2}(E_1)| \to \infty \) as \( N \to \infty \).

By exactly the same proof, we can show that \(|h_{n+2N+1}(E_2)| \to \infty \) as \( N \to \infty \). So the proof is finished. \( \square \)

As an application, we can improve Lemma 5.3 as follows:

**Lemma 6.8.** If \((\omega, \hat{\omega}) \in \mathcal{G}_{11} \), then \( \pi(\omega) < \pi(\hat{\omega}) \).

**Proof.** We prove it by contradiction. Assume \( \pi(\omega) = \pi(\hat{\omega}) =: E \). By Lemma 6.7, there exists \( m \in \mathbb{N} \) such that for any \( k \geq m \), \((h_k(E), h_{k+1}(E)) \in \text{int}(D) \). Recall that \( f(h_k(E), h_{k+1}(E)) = (h_{k+2}(E), h_{k+3}(E)) \), thus for any \( k \in \mathbb{N} \), we have

\[
f^k(h_m(E), h_{m+1}(E)) = (h_{m+2k}(E), h_{m+2k+1}(E)) \in \text{int}(D).
\]

By Proposition 6.6, \((h_m(E), h_{m+1}(E)) = F \). However, \( F \in \partial D \), a contradiction. So \( \pi(\omega) < \pi(\hat{\omega}) \). \( \square \)

**Proof of Proposition 6.3.** i) The proof is the same as Proposition 6.1.

ii) It is a direct consequence of Lemma 6.8 and Lemma 6.7. \( \square \)

### 6.3. Proof of Theorems 1.3, 1.5 and 1.8.

**Proof of Theorem 1.3.** At first we show \( B_{\infty} \subset \sigma_{\lambda} \). Fix \( E \in B_{\infty} \), we claim that for any \( n \geq 0 \), \(|h_n(E)| \leq 2 \) or \(|h_{n+1}(E)| \leq 2 \). If otherwise, there exists \( n_0 \geq 0 \) and \( \delta > 0 \) such that \(|h_{n_0}(E)|, |h_{n_0+1}(E)| \geq 2 + \delta \). By (21),

\[
|h_{n_0+2}(E)| \geq |h_{n_0+1}(E)|(|h_{n_0}^2(E) - 2) - 2 \geq 2 + 10\delta.
\]

Using (21) again, we have \(|h_{n_0+3}(E)| \geq 2 + 10\delta \). Now by induction, it is seen that

\[
|h_{n_0+k}(E)| \geq 2 + 10^{[k/2]}\delta.
\]

Thus \(|h_n(E)| \to \infty \), which contradicts with \( E \in B_{\infty} \). So the claim holds. Now by (22), we conclude that \( E \in \sigma_{\lambda} \). Thus \( B_{\infty} \subset \sigma_{\lambda} \).

By Lemma 2.5 iii), \( \mathcal{Z} \subset B_{\infty} \) and \( B_{\infty} \) is dense in \( \sigma_{\lambda} \). By Proposition 5.6 and (67),

\[
\dim H B_{\infty} \geq \dim H A(\tilde{\mathcal{B}}) \geq \log \alpha / \log 4 > 0,
\]

so \( B_{\infty} \) is uncountable.

By the definition of \( \mathcal{E}_{\infty} \), \( \mathcal{E}_{\infty} = \sigma_{\lambda} \setminus B_{\infty} \) always holds. Now we show that \( \mathcal{E}_{\infty} \) is dense in \( \sigma_{\lambda} \) and uncountable.

By Proposition 6.1, \( \pi(\mathcal{E}_{\infty}^f) \subset \mathcal{E}_{\infty} \) and is dense in \( \sigma_{\lambda} \). So \( \mathcal{E}_{\infty} \) is dense in \( \sigma_{\lambda} \).
Next we show that $E_\infty$ is uncountable. We prove it by contradiction. Otherwise, $E_\infty$ is countable.

Let $E_\infty = \{\gamma_i\}_{i=1}^\infty$. Since $\sigma_\lambda$ has no isolated point, we conclude that $\{\gamma_i\}$ is nowhere dense in $\sigma_\lambda$ for each $i$. Denote

$$T := \sigma_\lambda \setminus E_\infty = \{E \in \sigma_\lambda : O(E) \text{ is bounded}\}.$$ 

We can write $T$ as a countable union of closed subsets,

$$T = \bigcup_{k=1}^\infty A_k,$$

where

$$A_k = \bigcap_{n=0}^\infty \{E \in \sigma_\lambda : |h_n(E)| \leq k\}.$$

Since $E_\infty$ is dense in $\sigma_\lambda$ and $E_\infty \cap A_k = \emptyset$, $A_k$ is nowhere dense for each $k \in \mathbb{N}$. Thus

$$\sigma_\lambda = \bigcup_{i=1}^\infty \{\gamma_i\} \cup \bigcup_{k=1}^\infty A_k,$$

i.e., $\sigma_\lambda$ is a countable union of nowhere dense closed sets. This contradicts with Baire’s category theorem.

We remark that, once we know that $E_\infty$ is dense in $\sigma_\lambda$, the rest proof of the theorem is purely topological.

**Proof of Theorem 1.5.** By Proposition 5.1, we only need to show that $\pi$ is a homeomorphism. Since $\Omega_\infty$ is compact and $\sigma_\lambda$ is Hausdorff, it suffice to show that $\pi$ is injective.

Assume $\omega \neq \hat{\omega}$. WLOG, we assume $\omega < \hat{\omega}$. If $\omega < \hat{\omega}$, then by Lemma 5.2, $\pi(\omega) < \pi(\hat{\omega})$. If $\omega \neq \hat{\omega}$, by Lemma 4.10, there exists $(\tau, \hat{\tau}) \in G_{II}$ such that $\omega \preceq \tau$ and $\hat{\tau} \preceq \hat{\omega}$. By Proposition 5.1 and Lemma 6.8,

$$\pi(\omega) \leq \pi(\tau) < \pi(\hat{\tau}) \leq \pi(\hat{\omega}).$$

Hence $\pi$ is injective. \hfill \Box

**Proof of Theorem 1.8.** i)-iv) By Theorem 1.5, $(a, b)$ is a gap of $\sigma_\lambda$ if and only if $(\pi^{-1}(a), \pi^{-1}(b))$ is a gap of $\Omega_\infty$. So i)-iv) follows directly from i)-iv) of Theorem 4.6 and Theorem 1.5.

v) At first, we prove $\pi(E^o_\lambda) = Z^o$. Recall that $Z^o = \{z_\sigma : \sigma \in \Sigma^o\}$. Fix any $\sigma \in \Sigma^o$, by (61), there exists a unique $\omega_\sigma \in E^o_\lambda$ such that $\Pi(\omega) = \sigma 01^\infty$. Moreover

$$\{\omega_\sigma : \sigma \in \Sigma^o\} = E^o_\lambda.$$

We claim that $\pi(\omega_\sigma) = z_\sigma$. Once we show the claim, we immediately get $\pi(E^o_\lambda) = Z^o$. Now we show the claim. At first, by Lemma 2.8 i), we have $z_\sigma = b_{\sigma 01^k}$ for any $k \geq 0$. On the other hand, by the definition of $\Pi$, there exists $N \in \mathbb{N}$ such that when $m \geq N$, $\Pi_\lambda(\omega_\sigma |_m) = \sigma 01^m$. Note the $\{I_{\omega_\sigma |_m} : m \geq 1\}$ is a decreasing set sequence, so we have

$$\pi(\omega_\sigma) = \bigcap_{m \geq 0} I_{\omega_\sigma |_m} = \bigcap_{m \geq N} I_{\omega_\sigma |_m} = \bigcap_{m \geq N} B_{\sigma 01^m}.$$

Consequently $z_\sigma \in \bigcap_{m \geq N} B_{\sigma 01^m}$. Thus $\pi(\omega_\sigma) = z_\sigma$.

The proof of $\pi(E^e_\lambda) = Z^e$ is the same.

vi) It is a direct consequence of Propositions 6.1 and 6.3. \hfill \Box
7. IDS and Gap Labelling

In this section, we prove Theorems 1.10 and 1.11.

7.1. Computation of IDS.

7.1.1. The IDS for $\mathcal{Z}$. Recall that $\mathcal{Z} = \{z_\sigma : \sigma \in \Sigma_*\}; \sigma_n$ is defined by (24); an order $\preceq$ on $\Sigma_*$ is defined by (32). We have

**Lemma 7.1.** For $\sigma \in \Sigma_n$, we have

$$N(z_\sigma) = \sum_{i=1}^{n} \frac{\sigma_i}{2^i} + \frac{1}{2^{n+1}}.$$  \hfill (82)

Consequently,

$$N(\mathcal{Z}) = \{N(z_\sigma) : \sigma \in \Sigma_*\} = \left\{\frac{j}{2^n} : n \geq 1; 1 \leq j < 2^n\right\} = \varnothing \cap (0, 1).$$ \hfill (83)

**Proof.** By the property of IDS and Proposition 2.6, we have

$$N(z_\sigma) = \lim_{m \to \infty} \frac{\#\{z \in \mathcal{Z}_m : z \leq z_\sigma\}}{2^m} = \lim_{m \to \infty} \frac{\#\{\alpha \in \Sigma_m : \alpha \preceq \sigma\}}{2^m} = \lim_{m \to \infty} \frac{\sigma_m(\sigma 01^{m-n-1} + 1)}{2^m} = \lim_{m \to \infty} \left(\sum_{j=1}^{n} \frac{\sigma_j}{2^j} + \sum_{j=n+2}^{m} 2^{-j} + 2^{-m}\right) = \sum_{j=1}^{n} \frac{\sigma_j}{2^j} + \frac{1}{2^{n+1}}.$$

Hence we have

$$\{N(z_\sigma) : \sigma \in \Sigma_0\} = \{1/2\}; \{N(z_\sigma) : \sigma \in \Sigma_1\} = \{1/4, 3/4\}; \cdots$$

Then (83) holds. \hfill \Box

7.1.2. The IDS for $\sigma_\lambda$. Now we compute the IDS for any energy in the spectrum.

**Proof of Theorem 1.10.** By Proposition 4.8 iii), $\Pi$ is surjective.

Now fix any $\omega \in \Omega_{\infty}$. Write $E = \pi(\omega)$ and $\sigma = \Pi(\omega)$. For any $n$, by (43), we have $\Pi_*(\omega|_n) \preceq \sigma$ and by (45), we have $k_n := |\Pi_*(\omega|_n)| = n$ or $n + 1$. So

$$E = \pi(\omega) = \bigcap_{n \geq 0} I_{\omega|_n} = \bigcap_{n \geq 0} B_{\sigma|_{k_n}}.$$

By (23), $|B_{\sigma|_{k_n}}| \to 0$ as $n \to \infty$. Since $z_{\sigma|_{k_n}} \in B_{\sigma|_{k_n}}$, we have $z_{\sigma|_{k_n}} \to E$. Since $N$ is continuous, by (82), we have

$$N(E) = \lim_{n \to \infty} N(z_{\sigma|_{k_n}}) = \lim_{n \to \infty} \sum_{i=1}^{k_n} \frac{\sigma_i}{2^i} + \frac{1}{2^{k_n+1}} = \sum_{i=1}^{\infty} \frac{\sigma_i}{2^i} = \varepsilon(\sigma).$$

That is, $N(\pi(\omega)) = \varepsilon(\Pi(\omega))$. \hfill \Box
7.2. Gap labelling of $\sigma_\lambda$.

Proof of Theorem 1.11. By Theorem 1.8 ii), v) and (83), we have

$$\{N(G) : G \in \mathcal{G}_2\} = \{N(\pi(\omega)) : \omega \in \mathcal{E}_1^0 \cup \mathcal{E}_1^c\} = \{N(z) : z \in \mathcal{Z}\} = \mathcal{D} \cap (0, 1).$$

It is well known that the restriction of $\varepsilon$ on $\Sigma_1^{(2)}$ is injective and

$$\mathcal{D}/3 \setminus \mathcal{D}) \cap (0, 1) = \varepsilon(\Sigma_1^{(2)}). \quad (84)$$

By Theorem 1.8 iii), Theorem 1.10, Proposition 4.8 v), injectivity of $\varepsilon$ on $\Sigma_1^{(2)}$ and (84), we have

$$\{N(G) : G \in \mathcal{G}_II\} = \{N(\pi(\omega)) : \omega \in \tilde{\mathcal{E}}_I\} = \varepsilon(\Pi_1(\omega)) : \omega \in \tilde{\mathcal{E}}_I\} = \varepsilon(\Pi_1(\tilde{\mathcal{E}}_I)) = \varepsilon(\mathcal{D}/\mathcal{D}) \cap (0, 1) \setminus \varepsilon(\Pi(\mathcal{F})).$$

So the result follows. $\square$

8. Proof of Technical Lemmas and Related Results

In this section, we give the proofs of the technical lemmas and related results in Sects. 2 and 3.

8.1. The properties of $\mathcal{Z}$.

8.1.1. The order of $\mathcal{Z}$. The following lemma tells us how the zeros in $\mathcal{R}_n$ are ordered.

Lemma 8.1. $\# \mathcal{R}_n = 2^{n+1} - 1$. Moreover $\mathcal{Z}_n$ interlaces $\mathcal{R}_{n-1}$. That is, if we list the elements of $\mathcal{R}_{n-1}$ as

$$r_1^{(n-1)} < r_2^{(n-1)} < \cdots < r_{2^{n-1}}^{(n-1)},$$

then $\mathcal{R}_n$ is ordered as

$$z_{\sigma_n^{-1}(0)}^{(n-1)} < r_1^{(n-1)} < z_{\sigma_n^{-1}(1)}^{(n-1)} < r_2^{(n-1)} < \cdots < z_{\sigma_n^{-1}(2^n-2)}^{(n-1)} < r_{2^n-1}^{(n-1)} < z_{\sigma_n^{-1}(2^n-1)}^{(n-1)}.$$ \quad (85)

In particular, for any $\sigma \in \Sigma_n$ we have

$$B_\sigma \subset \left[r_{\sigma_n^{-1}(0)}^{(n-1)}, r_{\sigma_n^{-1}(n)}^{(n-1)}\right]. \quad (86)$$

Here we use the convention: $r_0^{(n-1)} = -\infty$ and $r_{2^n}^{(n-1)} = \infty$. 
Proof. Notice that \( \#Z_n = 2^n \). By Lemma 2.5 i), \( Z_n \cap Z_m = \emptyset \) for \( n \neq m \). Hence,

\[
\#R_n = \sum_{j=0}^{n} \#Z_j = 2^{n+1} - 1.
\]

We prove (85) by induction. By (21),

\[
z_0 = -\sqrt{2 + \lambda^2}; \quad z_1 = \sqrt{2 + \lambda^2}; \quad z_0 = \sqrt{2 + \lambda^2}.
\]

So \( R_0 = Z_0 = \{z_0\} \) and \( Z_1 = \{z_0, z_1\} \). Since \( z_0 < z_0 < z_1 \), (85) holds for \( n = 1 \).

Now assume the result holds for \( n < k \). We consider \( R_k \). By Lemma 2.5 i) and induction hypothesis,

\[
\begin{align*}
\{r_2^{(k-1)}, r_4^{(k-1)}, \ldots, r_{2^{k-2}}^{(k-1)}\} &= R_{k-2} \subset \{h = 2\}; \\
\{r_1^{(k-1)}, r_3^{(k-1)}, \ldots, r_{2^{k-1}}^{(k-1)}\} &= Z_{k-1} \subset \{h = -2\}.
\end{align*}
\]

Thus in each interval \( (r_i^{(k-1)}, r_{i+1}^{(k-1)}) \) there is an element of \( Z_k \). Moreover since \( h_k(x) \to +\infty \) when \(|x| \to \infty \), there is one element of \( Z_k \) in \( (r_0^{(k-1)}, r_1^{(k-1)}) \) and one element of \( Z_k \) in \( (r_{2^{k-1}}, r_{2^k}) \). Since \( \#Z_k = 2^k \), we get all elements of \( Z_k \). Thus the result holds for \( n = k \).

By induction, (85) holds.

By (85), \( z_\sigma \in [r_{\sigma n(\sigma)}, r_{\sigma n(\sigma)+1}] \). If \( \sigma \neq 0^n \) or \( 1^n \), by (87), \( h_n(r_{\sigma n(\sigma)}) = \pm 2 \) and \( r_{\sigma n(\sigma)+1} = \pm 2 \). If \( \sigma = 0^n \), \( h_n(r_0^{(n-1)}) = \infty \) and \( h_n(r_1^{(n-1)}) = -2 \). If \( \sigma = 1^n \), \( h_n(r_{2^n-1}) = \infty \) and \( h_n(r_{2^n}) = -2 \). By Proposition 2.10 i), (86) holds. \(\square\)

**Corollary 8.2.** For any \( n, t \geq 0 \) and \( \sigma \in \Sigma_n \), we have

\[
z_{\sigma 0^t} < z_\sigma < z_{\sigma 1^t}.
\]

**Proof.** Fix any \( \sigma \in \Sigma_n \).

Claim: \( z_{\sigma 0} < z_\sigma < z_{\sigma 1} \) are three consecutive numbers in \( R_{n+1} \).

\(\triangleq\) By (85), \( z_\sigma \) is the \( (2 \sigma_n(\sigma) + 1) \)-th number in \( R_n \). So

\[
z_\sigma = r_{2 \sigma_n(\sigma) + 1}^{(n)} = r_{\sigma n(\sigma)+1}^{(n)} = r_{\sigma n+1(\sigma)}^{(n)}.
\]

By applying (85) for \( R_{n+1} \), we conclude that

\[
z_{\sigma 0} < r_{\sigma n+1(\sigma)}^{(n)} = z_\sigma < z_{\sigma 1}
\]

are three consecutive numbers in \( R_{n+1} \).

Applying the claim to \( 0 \), we know that \( z_{\sigma 0} < z_{\sigma 01} \) are two consecutive numbers in \( R_{n+2} \). Since \( z_\sigma \in R_{n+2} \) and \( z_\sigma < z_\sigma \), we have \( z_{\sigma 0} < z_{\sigma 01} < z_\sigma \). A symmetric argument shows that \( z_\sigma < z_{\sigma 10} \). We can continue this process. By an inductive argument, we get (88). \(\square\)
Proof of Proposition 2.6. Since the map \( \sigma \to z_\sigma \) is a bijection between \( \Sigma_\ast \) and \( \mathcal{Z} \), we only need to show that if \( \sigma < \tau \), then \( z_\sigma < z_\tau \).

Assume \( \sigma < \tau \) and \( |\sigma| = m, |\tau| = n \). Write \( \theta = \sigma \wedge \tau \).

If \( |\theta| < m, n \), then \( \sigma = \theta 0^* \leq \theta 0^{1,1-|\theta|} \in \Sigma_m \) and \( \tau = \theta 1^* \geq \theta 10^{n-1-|\theta|} \in \Sigma_n \).

By (27) and Corollary 8.2,

\[
|z_\sigma| \leq |z_{\theta 0^{1,1-|\theta|}}| < |z_\theta| < |z_{\theta 10^{n-1-|\theta|}}| \leq z_\tau.
\]

If \( |\theta| = m \), then \( \sigma < \tau \). Since \( \sigma < \tau \), we have \( \tau = \sigma 1^* \geq \sigma 10^{n-1-m} \). Again by (27) and Corollary 8.2,

\[
z_\sigma < z_{\sigma 10^{n-1-m}} \leq z_\tau.
\]

If \( |\theta| = n \), the same proof shows that \( z_\sigma < z_\tau \).

\( \Box \)

Proof of Corollary 2.7. By (86),

\[
\text{int}(B_\sigma) \subset (r_{\sigma n(\sigma)}^{(n-1)}, r_{\sigma n(\sigma)+1}^{(n-1)}) =: J_\sigma.
\]

So we only need to show the following claim.

Claim: If \( \sigma_n = 0(= 1) \), then \( \prod_{j=0}^{n-1} h_j(E) < 0(> 0) \) for any \( E \in J_\sigma \).

\( \triangleright \) We show it by induction. If \( n = 1 \), then \( J_0 = (-\infty, \lambda) \) and \( J_1 = (\lambda, \infty) \). Since \( h_0(E) = E - \lambda \), \( h_0(E) < 0 \) for \( E \in J_0 \) and \( h_0(E) > 0 \) for \( E \in J_1 \).

Assume the result holds for \( n = k \). Now fix any \( \sigma \in \Sigma_k \). By (85), for any \( j \) we have

\[
r_j^{(k-1)} = r_j^{(k)}.
\]

Together with (89), we have

\[
J_\sigma 0 = (r_{\sigma k+1(\sigma 0)}^{(k)}, r_{\sigma k+1(\sigma 0)+1}^{(k)}) = (r_{\sigma k(\sigma)}^{(k-1)}, z_\sigma),
J_\sigma 1 = (r_{\sigma k+1(\sigma 1)}^{(k)}, r_{\sigma k+1(\sigma 1)+1}^{(k)}) = (z_\sigma, r_{\sigma k(\sigma)+1}^{(k-1)}).
\]

Take \( E \in \text{int}(J_\sigma 0) \). If \( \sigma_k = 0 \), then by the induction hypothesis, \( \prod_{j=0}^{k-1} h_j(E) < 0 \). By Proposition 2.10 i), \( h_k \) is decreasing in a neighbourhood of \( z_\sigma \). Since \( z_\sigma - \prod_{j=0}^{k-1} h_j(E) > 0 \), we conclude that \( h_k(E) < 0 \). If \( \sigma_k = 1 \), then by the induction hypothesis, \( \prod_{j=0}^{k-1} h_j(E) > 0 \). By a symmetric argument, \( h_k(E) < 0 \). So we always have \( \prod_{j=0}^{k} h_j(E) < 0 \).

Take \( E \in \text{int}(J_\sigma 1) \). By the same argument, we can show that \( \prod_{j=0}^{k} h_j(E) > 0 \).

By induction, the claim holds.

\( \Box \)

8.1.2. Local behaviors of trace polynomials at zeros. Next we study the monotone properties of the trace polynomials around a zero.

Lemma 8.3. Assume \( n \geq 0 \) and \( \sigma \in \Sigma_n \). Then

\[
\text{sgn}(h_{n+1}'(z_\sigma)) = (-1)^n; \quad \text{sgn}(h_{n+k}'(z_\sigma)) = (-1)^{n+1}, \quad (k \geq 2).
\]
Proof. If \( n = 0 \), then \( z_\sigma = z_\emptyset = \lambda \) and \( h'_{n+1}(z_\sigma) = h'_0(\lambda) = 2\lambda > 0 \).
For \( n \geq 1 \), by the recurrence relation (21), we have,
\[
h'_{n+1}(E) = h'_n(E)(h^2_{n-1}(E) - 2) + 2h_n(E)h_{n-1}(E)h'_{n-1}(E).
\] (91)
Take \( E = z_\sigma \). Since \( h_n(z_\sigma) = 0 \), we get
\[
h'_{n+1}(z_\sigma) = h'_n(z_\sigma)(h^2_{n-1}(z_\sigma) - 2).
\] (92)
By Lemma 2.9, we have
\[
-(h^2_{n-1}(z_\sigma) - 2) = h_n(z_\sigma) - (h^2_{n-1}(z_\sigma) - 2) = (-1)^{n-1}2\lambda \prod_{c \in \mathcal{R}_{n-1}} (z_\sigma - c).
\] (93)
By Proposition 2.10 i),
\[
\text{sgn}(h'_n(z_\sigma)) = (-1)^{\sigma_n+1}.
\]
By (85) and (24),
\[
\text{sgn} \left( \prod_{c \in \mathcal{R}_{n-1}} (z_\sigma - c) \right) = (-1)^{\#\mathcal{R}_{n-1} - \sigma_n(\sigma)} = (-1)^{\sigma_n+1}.
\]
Combining (92) and (93), the sign of \( h'_{n+1}(z_\sigma) \) is \((-1)^n\). So the first equation of (90) holds.
By Lemma 2.5 i), \( h_{n+1}(z_\sigma) = -2 \), \( h_{n+k}(z_\sigma) = 2 \) for \( k \geq 2 \). Then by (91), for any \( n \geq 0 \),
\[
\begin{cases}
h'_{n+2}(z_\sigma) = -2h'_{n+1}(z_\sigma); \\
h'_{n+3}(z_\sigma) = 2h'_{n+2}(z_\sigma) - 8h'_{n+1}(z_\sigma); \\
h'_{n+k}(z_\sigma) = 2h'_{n+k-1}(z_\sigma) + 8h'_{n+k-2}(z_\sigma), \quad (k \geq 4).
\end{cases}
\]
From this, we conclude that, there exists an increasing positive integer sequence \( \{\tau_k : k \geq 2\} \) such that \( \tau_2 = 2 \), \( \tau_3 = 12 \), \( \tau_{k+2} = 2\tau_{k+1} + 8\tau_k \), \( (k \geq 2) \) and
\[
h'_{n+k}(z_\sigma) = -\tau_k h'_{n+1}(z_\sigma), \quad k \geq 2.
\]
So the second equation of (90) holds. \( \square \)

8.1.3. The relation between \( Z \) and band edges.

Proof of Lemma 2.8. i) By Lemma 2.5 i),
\[
h_{n+1}(z_\sigma) = -2; \quad h_{n+1+t}(z_\sigma) = 2, \quad \forall t \geq 1.
\] (94)
By Proposition 2.10 iii), \( z_\sigma \) must be an endpoint of some band in \( B_{n+1+t} \). By Corollary 8.2, \( z_{\sigma 01} < z_\sigma < z_{\sigma 10} \). Since \( (\sigma 01)^+ = \sigma 10^t \), \( B_{\sigma 01} \) and \( B_{\sigma 10} \) are two consecutive bands in \( B_{n+1+t} \). Hence \( z_\sigma \) can only be \( b_{\sigma 01} \) or \( a_{\sigma 10} \).
If \( n \) is odd, by Lemma 8.3,
\[
h'_{n+1}(z_\sigma) < 0 \quad \text{and} \quad h'_{n+1+t}(z_\sigma) > 0, \quad (t \geq 1).
\]
By (94) and Proposition 2.10 iii), \( z_\sigma = b_{\sigma 01} \).
If $n$ is even, by Lemma 8.3,
\[ h'_{n+1}(z) > 0 \quad \text{and} \quad h'_{n+1}(z) < 0 (t \geq 1). \]

By (94) and Proposition 2.10 iii), $z = a_{10'}$. In both cases, by Proposition 2.10 ii), $b_{01'} < a_{10'} = a_{10'}$.

ii) If $a_{0} < R_{n-1}$, then by i), $\sigma = \hat{\sigma}^{10'}$ for some $\hat{\sigma} \in \Sigma_{s}, s \geq 0$ and $z_{\hat{\sigma}} = a_{\hat{\sigma}10'}$ for any $t \geq 0$. This implies $a_{\sigma} = a_{\sigma10'}$ for any $t > 0$.

By Lemma 2.8 iv), $\sigma 0$ for some $\hat{\sigma} \in \Sigma_{s}, s \geq 0$ and $z_{\hat{\sigma}} = b_{\hat{\sigma}01'}$ for any $t \geq 0$. This implies $b_{\sigma} = b_{\sigma01'}$ for any $t > 0$.

iii) If $a_{0} < R_{n-1}$, then by ii), $a_{0} = a_{\sigma} \in R_{n-1} \subset R_{n}$. If $a_{0} \in R_{n}$, then by i), $\sigma 0 = \hat{\sigma}^{10'}$ for some $\hat{\sigma} \in \Sigma_{s}, s \geq 0$ and $z_{\hat{\sigma}} = a_{\hat{\sigma}10'}$ for any $t \geq 0$. In this case, we must have $s \geq 1$. So we have $a_{\sigma} = a_{\hat{\sigma}10'} = z_{\hat{\sigma}} \in R_{n-1}$. Thus the first assertion holds. The same proof shows that the second assertion holds.

iv) By Lemma 8.1, $(z_{\sigma}, z_{\sigma}+) \cap R_{n-1} = 1$. By Lemma 2.5 ii),
\[ ((z_{\sigma}, b_{\sigma}) (a_{\sigma}+, z_{\sigma}+)) \cap R_{n-1} = \emptyset. \]

By Floquet theory and Lemma 2.5 i),
\[ h_{n}(b_{\sigma}, a_{\sigma}+) \subset (-\infty, -2) \cup (2, \infty) \quad \text{and} \quad h_{n}(R_{n-1}) \subset \{\pm 2\}. \]

So $(b_{\sigma}, a_{\sigma}+) \cap R_{n-1} = \emptyset$. Combine these facts, the result follows.

\[ \Box \]

Proof of Proposition 2.1. The possible gap of $\sigma_{n}$ has the form: $(b_{\sigma}, a_{\sigma}+)$ with $\sigma \neq 1^{n}$. By Lemma 2.8 iv), $(b_{\sigma}, a_{\sigma}+) \cap R_{n-1} = \{z_{n}\}$. By Lemma 8.3, $h'_{n}(z_{n}) \neq 0$. By Proposition 2.10 ii), the gap $(b_{w}, a_{w+})$ is open.

8.2. Proof of Lemma 2.3. At first we need the following lemma:

Lemma 8.4. Assume $n \geq 1$ and $\sigma \in \Sigma_{s}$.

i) If $n$ is odd, then
\[ \begin{cases} h_{n+1}(a_{\sigma}) > 2, & \text{if } a_{\sigma} \notin R_{n-1}; \\ h_{n+1}(b_{\sigma}) < 2, & \text{if } b_{\sigma} \notin R_{n-1}. \end{cases} \]

ii) If $n$ is even, then
\[ \begin{cases} h_{n+1}(b_{\sigma}) > 2, & \text{if } b_{\sigma} \notin R_{n-1}; \\ h_{n+1}(a_{\sigma}) < 2, & \text{if } a_{\sigma} \notin R_{n-1}. \end{cases} \]

Proof. By (85),
\[ \#(R_{n} \cap (-\infty, z_{n})) = 2\sigma_{n}(\sigma). \]

i) At first assume $a_{\sigma} \notin R_{n-1}$. Since $h_{n}(a_{\sigma}) = \pm 2$, and $R_{n} = R_{n-1} \cup \mathcal{Z}_{n}, a_{\sigma} \notin R_{n}$.

By Lemma 2.9, we have
\[ h_{n+1}(a_{\sigma}) = h^{2}_{n}(a_{\sigma}) - 2 + (-1)^{n}2\lambda \prod_{c \in R_{n}}(a_{\sigma} - c) = 2 - 2\lambda \prod_{c \in R_{n}}(a_{\sigma} - c). \]
By Lemma 2.5 ii), \((a_\sigma, z_\sigma) \cap \mathcal{R}_n = \emptyset\). By Lemma 8.1, we have

\[
\#(\mathcal{R}_n \cap (a_\sigma, +\infty)) = 2^{n+1} - 1 - \#(\mathcal{R}_n \cap (-\infty, a_\sigma)) = 2^{n+1} - 2 - 2\sigma_n(\sigma).
\]

Thus \(\prod_{c \in \mathcal{R}_n} (a_\sigma - c) < 0\), and consequently \(h_{n+1}(a_\sigma) > 2\).

Now assume \(b_\sigma \notin \mathcal{R}_{n-1}\). Then by the same argument, \(b_\sigma \notin \mathcal{R}_n\). Still by Lemma 2.9,

\[
h_{n+1}(b_\sigma) = h_n^2(b_\sigma) - 2 + (-1)^n2\lambda \prod_{c \in \mathcal{R}_n} (b_\sigma - c) = 2 - 2\lambda \prod_{c \in \mathcal{R}_n} (b_\sigma - c).
\]

By Lemma 2.5 ii), \([z_\sigma, b_\sigma] \cap \mathcal{R}_n = [z_\sigma, b_\sigma) \cap \mathcal{R}_n = [z_\sigma, b_\sigma)\). By Lemma 8.1, we have

\[
\#(\mathcal{R}_n \cap (b_\sigma, +\infty)) = 2^{n+1} - 1 - \#(\mathcal{R}_n \cap (-\infty, b_\sigma)) = 2^{n+1} - 2 - 2\sigma_n(\sigma).
\]

Thus \(\prod_{c \in \mathcal{R}_n} (b_\sigma - c) > 0\), and consequently \(h_{n+1}(b_\sigma) < 2\).

\(\Box\)

**Proof of Lemma 2.3.** i) By Lemma 2.8 i), \(b_{\sigma_0} = z_\sigma \in \mathcal{R}_n\). By Lemma 2.8 iv),

\[a_{\sigma_1} = a_{(\sigma_0)^+} \notin \mathcal{R}_n.\]

If \(a_\sigma \in \mathcal{R}_{n-1}\), by Lemma 2.8 ii), \(a_\sigma = a_{\sigma_0}\). Hence \(B_{\sigma_0} = [a_{\sigma_0}, b_{\sigma_0}] = [a_\sigma, z_\sigma]\).

If \(a_\sigma \notin \mathcal{R}_{n-1}\), by Lemma 8.4 i), \(h_{n+1}(a_\sigma) > 2\). By Proposition 2.10 i), \(h_{n+1}(B_{\sigma_0}) = [-2, 2]\). Since \(a_\sigma < z_\sigma = b_{\sigma_0}\), we conclude that \(a_\sigma < a_{\sigma_0}\). Hence \(B_{\sigma_0} \subset (a_\sigma, z_\sigma]\).

In both cases, we have \(B_{\sigma_0} \subset B_{\sigma_1}\).

By Proposition 2.1, \(b_{\sigma_0} < a_{(\sigma_0)^+} = a_{\sigma_1}\). So \(z_\sigma = b_{\sigma_0} < a_{\sigma_1}\).

If \(b_\sigma \in \mathcal{R}_{n-1}\), by Lemma 2.8 ii), \(b_\sigma = b_{\sigma_0}\). So we have \(B_{\sigma_1} \subset (z_\sigma, b_\sigma]\).

If \(b_\sigma \notin \mathcal{R}_{n-1}\), by Lemma 8.4 i), \(h_{n+1}(b_\sigma) < 2\). If \(\sigma = 1^n\), then \(b_{\sigma_1} = b_{1^{n+1}}\) is the largest root of \(h_{n+1}(x) = 2\). Since \(h_{n+1}(b_\sigma) < 2\) and \(h_{n+1}(x) \to \infty\) when \(x \to \infty\), we conclude that \(b_\sigma < b_{\sigma_1}\). Now assume \(\sigma < 1^n\). By Lemma 2.8 iv), \(a_{\sigma_0} \in \mathcal{R}_{n-1}\). By Lemma 2.8 ii), \(a_{\sigma_0} = a_{\sigma_0}^+\).

By Proposition 2.1,

\[b_{\sigma} < a_{\sigma^+}; \quad b_{\sigma_1} < a_{\sigma_0} = a_{\sigma^+}^+; \quad b_{\sigma_1} < a_{\sigma_0}^+= a_{\sigma^+}^+; \quad b_{\sigma_0} < a_{\sigma}^+\]

By Proposition 2.10 i), \(h_{n+1}\) is increasing on \(B_{\sigma_1}\) and decreasing on \(B_{\sigma_0}\), so

\[h_{n+1}([b_{\sigma_1}, a_{\sigma^+}]) = h_{n+1}([b_{\sigma_1}, a_{\sigma_0}]) \subset [2, \infty).\]

Since \(h_{n+1}(b_\sigma) < 2\), we conclude that

\[b_{\sigma} < b_{\sigma_1} < a_{\sigma^+}\quad (95)\]

Together with \(z_\sigma < a_{\sigma_1}\), we always have \([z_\sigma, b_{\sigma}] < B_{\sigma_1}\).

ii) The proof is the same as i). In particular, we have that if \(\sigma > 0^n\), then

\[b_{\sigma} < a_{\sigma_0} < a_{\sigma^+}.\quad (96)\]

iii) By Proposition 2.1, \(B_{\sigma_0} < B_{\sigma_1}\). If \(n\) is odd and \(\sigma > 0^n\), then \(B_{\sigma} < B_{\sigma_0}\) and \(B_{\sigma_0} \subset B_{\sigma}\). Hence \(B_{\sigma} < B_{\sigma_0}\). If \(n\) is even and \(\sigma > 0^n\), then by (96), \(B_{\sigma} < B_{\sigma_0}\). If \(n\) is odd and \(\sigma < 1^n\), then by (95), \(B_{\sigma_1} < B_{\sigma^+}\). If \(n\) is even and \(\sigma < 1^n\), then \(B_{\sigma} < B_{\sigma^+}\) and \(B_{\sigma_1} \subset B_{\sigma}\). Hence \(B_{\sigma_1} < B_{\sigma^+}\). \(\Box\)
8.3. Proof of Lemma 3.1 and Corollary 3.2.

Proof of Lemma 3.1. i) Applying Lemma 2.3 i) to \( \sigma \) and ii) to \( \hat{\sigma} = \sigma 0 \), we have
\[
a_{\sigma 1} \notin \mathcal{R}_n, \ B_{\sigma 0} \subset B_{\sigma} \ 	ext{and} \ B_{\sigma 01} \subset B_{\sigma 0}.
\]
Applying Lemma 2.8 iii) and Lemma 2.3 ii), iii) to \( \hat{\sigma} = \sigma 1 \), we have
\[
a_{\sigma 10}, b_{\sigma 10} \notin \mathcal{R}_{n+1}, \ B_{\sigma 0} = B_{(\sigma 1)^-} < B_{\sigma 10} < B_{\sigma 1}.
\] (97)
If \( a_{\sigma} \in \mathcal{R}_{n-1} \), by Lemma 2.8 ii), \( a_{\sigma 0} = a_{\sigma} \in \mathcal{R}_{n-1} \subset \mathcal{R}_n \). Applying Lemma 2.3 ii) to \( \hat{\sigma} = \sigma 0 \), we have
\[
B_{\sigma 00} \subset B_{\sigma 0} \subset B_{\sigma}.
\] (98)
If \( a_{\sigma} \notin \mathcal{R}_{n-1} \), by Lemma 2.8 iii), \( a_{\sigma 0} \notin \mathcal{R}_n \) and hence \( a_{\sigma 00} \notin \mathcal{R}_{n+1} \). Applying Lemma 2.3 ii) to \( \hat{\sigma} = \sigma 0 \), we have
\[
b_{\sigma 00} \notin \mathcal{R}_{n+1}, \ B_{\sigma 00} < B_{\sigma 0}.
\]
Now we show \( a_{\sigma} < a_{\sigma 00} \). By Lemma 8.4 i), \( h_{n+1}(a_{\sigma}) > 2 \). Hence
\[
h_{n+2}(a_{\sigma}) = h_{n+1}(a_{\sigma})(h_n^2(a_{\sigma}) - 2) - 2 > 2.
\]
Since \( a_{\sigma 01} \in B_{\sigma 01} \subset B_{\sigma 0} \subset B_{\sigma} \), we have \( a_{\sigma} \leq a_{\sigma 01} \). By Proposition 2.10 i),
\[
h_{n+2}(a_{\sigma 01}) = -2.
\]
Since \( a_{\sigma 00} \) is the maximal solution of \( h_{n+2}(E) = 2 \) which are less than \( a_{\sigma 01} \), we conclude that \( a_{\sigma} < a_{\sigma 00} \). Since \( b_{\sigma 00} < b_{\sigma 01} \) and by Lemma 2.8 i), \( b_{\sigma 01} = b_{\sigma 0} = z_{\sigma} \), we have
\[
B_{\sigma 00} = [a_{\sigma 00}, b_{\sigma 00}] \subset (a_{\sigma}, z_{\sigma}) \subset \text{int}(B_{\sigma}).
\] (99)
Applying Lemma 2.3 ii) to \( \hat{\sigma} = \sigma 1 \), we always have
\[
B_{\sigma 11} \subset B_{\sigma 1}.
\]
If \( b_{\sigma} \in \mathcal{R}_{n-1} \), by Lemma 2.3 i), \( B_{\sigma 1} \subset B_{\sigma} \). Hence
\[
B_{\sigma 11} \subset B_{\sigma 1} \subset B_{\sigma}.
\]
By (97), \( B_{\sigma 0} < B_{\sigma 10} < B_{\sigma 1} \). Since \( B_{\sigma 0}, B_{\sigma 1} \subset B_{\sigma} \), we have
\[
B_{\sigma 10} = [a_{\sigma 10}, b_{\sigma 10}] \subset (a_{\sigma 0}, b_{\sigma 1}) \subset \text{int}(B_{\sigma}).
\] (100)
If \( b_{\sigma} \notin \mathcal{R}_{n-1} \), we claim that
\[
b_{\sigma 10} < b_{\sigma} < b_{\sigma 11}.
\]
In fact, by Lemma 2.8 iv), either \( \sigma = 1^n \), or \( a_{\sigma^+} \in \mathcal{R}_{n-1} \). By Lemma 8.4 i), \( h_{n+1}(b_{\sigma}) < 2 \). Hence
\[
h_{n+2}(b_{\sigma}) = h_{n+1}(b_{\sigma})(h_n^2(b_{\sigma}) - 2) - 2 < 2.
\]
If \( \sigma = 1^n \), then since \( h_{n+2}(E) \to \infty \) as \( E \to \infty \), there is a root of \( h_{n+2}(E) = 2 \) in \( (b_{\sigma}, \infty) \). Since \( b_{\sigma 11} = b_{1^{n+2}} \) is the largest root of \( h_{n+2}(E) = 2 \), we conclude that \( b_{\sigma} < b_{\sigma 11} \). Now assume \( \sigma \neq 1^n \), then \( a_{\sigma^+} \in \mathcal{R}_{n-1} \). By Lemma 2.8 ii), \( a_{\sigma^+} = a_{\sigma^+} \).
By Proposition 2.10 i), $h_{n+2}$ is increasing on $B_{\sigma_{11}}$ and decreasing on $B_{(\sigma_{11})^+} = B_{\sigma^{+00}}$. Hence by Floquet theory,

$$h_{n+2}([b_{\sigma_{11}}, a_{\sigma^+}]) = h_{n+2}([b_{\sigma_{11}}, a_{\sigma^{+00}}]) \subset [2, \infty).$$

Since $b_\sigma < a_{\sigma^+}$ and $h_{n+2}(b_\sigma) < 2$, we conclude that $b_\sigma < b_{\sigma_{11}} < a_{\sigma^+}$. As a result, we always have $b_\sigma < b_{\sigma_{11}}$.

On the other hand, we have

$$-2 = h_{n+2}(b_{\sigma_{10}}) = h_{n+1}(b_{\sigma_{10}})(h_{n}^2(b_{\sigma_{10}}) - 2) - 2.$$

So $h_{n+1}(b_{\sigma_{10}}) = 0$ or $|h_{n}(b_{\sigma_{10}})| = \sqrt{2}$. By (97), $b_{\sigma_{10}} \notin \mathcal{R}_{n+1}$. So $|h_{n}(b_{\sigma_{10}})| = \sqrt{2}$. If $\sigma = 1^n$, then since $h_n(E) \geq 2$ for any $E \geq b_\sigma$, we have $b_{\sigma_{10}} < b_\sigma$. Now assume $\sigma \neq 1^n$.

By Floquet theory, $h_n([b_{\sigma}, a_{\sigma^+}]) \cap (-2, 2) = \emptyset$. Since $b_{\sigma_{10}} < b_{\sigma_{11}} < a_{\sigma^{+00}} = a_{\sigma^+}$ and $h_n(b_{\sigma_{10}}) \in (-2, 2)$, we conclude that $b_{\sigma_{10}} < b_\sigma$. As a result, we always have $b_{\sigma_{10}} < b_\sigma$. Hence the claim holds.

By (97), $B_{\sigma_0} < B_{\sigma_{10}}$, so we have $a_\sigma < z_\sigma = b_{\sigma_0} < a_{\sigma_{10}} < a_{\sigma_{11}}$. Together with the claim above, we have

$$B_\sigma < B_{\sigma_{11}}.$$

By $B_{\sigma_0} < B_{\sigma_{10}}$ and the claim above, we also conclude that

$$B_{\sigma_{10}} = [a_{\sigma_{10}}, b_{\sigma_{10}}] \subset (b_{\sigma_0}, b_\sigma) = (z_\sigma, b_\sigma) \subset \text{int}(B_\sigma). \quad (101)$$

By (98) and (99), $B_{\sigma_{00}} \subset B_\sigma$ always holds.

By (100) and (101), $B_{\sigma_{10}} \subset \text{int}(B_\sigma)$ always holds.

ii) The proof is the same as i). \qed

Proof of Corollary 3.2. i) By Lemma 2.5 ii), $\text{int}(B_\sigma) \cap \mathcal{R}_{n-1} = \emptyset$. So $B_\sigma \cap \mathcal{R}_{n-1} \neq \emptyset$ if and only if $\partial B_\sigma \cap \mathcal{R}_{n-1} \neq \emptyset$. Write $\hat{\sigma} := \sigma|_{n-1}$.

At first assume $\partial B_\sigma \cap \mathcal{R}_{n-1} \neq \emptyset$. If $a_\sigma \in \mathcal{R}_{n-1}$, then by Lemma 2.8 i), there exist $m \leq n$ even and $\tau \in \Sigma_m$ such that $\sigma = \tau 1^n$ and $z_\tau = a_{\tau 1^n} = a_\sigma$. If $m = n - 1$, applying Lemma 2.3 ii) to $\tau$, we have $B_\sigma = B_{\tau 1} \subset B_{\tau} = B_{\hat{\sigma}}$. If $m \leq n - 2$, then $a_{\hat{\sigma}} = z_\tau \in \mathcal{R}_{n-2}$. By Lemma 2.3, we always have $B_\sigma = B_{\sigma_0} \subset B_{\hat{\sigma}}$. If $b_\sigma \in \mathcal{R}_{n-1}$, the same proof shows that $B_\sigma \subset B_{\hat{\sigma}}$.

Next assume $B_\sigma \subset B_{\hat{\sigma}}$. If $n$ is even, by applying Lemma 2.3 i) to $\hat{\sigma}$, either $\sigma = \hat{\sigma} 0$, hence $b_\sigma = z_{\hat{\sigma}} \in \mathcal{R}_{n-1}$; or $\sigma = \hat{\sigma} 1$, hence $b_{\hat{\sigma}} \in \mathcal{R}_{n-2}$. In the latter case, by Lemma 2.8 ii), $b_\sigma = b_{\hat{\sigma}} = b_{\hat{\sigma}} \in \mathcal{R}_{n-1}$. Thus we always have $\partial B_\sigma \cap \mathcal{R}_{n-1} \neq \emptyset$. If $n$ is odd, the same proof shows that $\partial B_\sigma \cap \mathcal{R}_{n-1} \neq \emptyset$.

ii) Write $\tilde{\sigma} := \sigma|_{n-1}$ and $\tilde{\delta} := \sigma|_{n-2}$. Assume $B_\sigma \not\subset B_{\tilde{\delta}}$. If $n = 1$, by the convention, $B_{\delta} = B_\delta$ and $B_{\tilde{\delta}} = \mathbb{R}$. So $B_{\delta} \subset B_{\tilde{\delta}}$. Now assume $n \geq 2$. At first, assume $n$ is odd. By applying Lemma 2.3 ii) to $\tilde{\delta}$, the only possibility is that $a_{\tilde{\delta}} \notin \mathcal{R}_{n-2}$ and $\sigma = \tilde{\sigma} \sigma_{n-1} 0$. Now by applying Lemma 3.1 i) to $\tilde{\delta}$, we conclude that $B_\sigma = B_{\tilde{\sigma} \sigma_{n-1}} \subset B_{\tilde{\delta}}$. If $n$ is even, the same proof show that $B_\sigma \subset B_{\tilde{\delta}}$. So ii) follows.

iii) At first assume $\tau \in \Sigma_n$. By Proposition 2.1, we must have $\sigma = \tau$. Next assume $\tau \in \Sigma_{n+1}$. By Lemma 2.3 iii), if $\sigma \neq \tau|_n$, then $B_{\tau|_n} \cap B_\sigma \neq \emptyset$. Consequently $B_{\tau|_n} \cap B_\sigma \neq \emptyset$. So we must have $\sigma = \tau|_n$. If $B_\tau \subset B_{\tau|_n}$, by Proposition 2.1, we must have $\sigma = \tau|_n$.

As a result, $\sigma = \tau|_n < \tau$. \qed
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A. Some Remarks on the Paper [9]

In this appendix, following essentially one of the referees’ comments, we point out several gaps appearing in Lemmas 2.25, 2.26 and Proposition 2.28 of [9]. We note that the main result Theorem 1.1 of [9] is a direct consequence of Proposition 2.28, and Lemmas 2.25 and 2.26 are the key inductive steps towards the proof of Proposition 2.28. In [9], the major part of the study of the trace map dynamics is done in variables denoted by \((s, t) \in \mathbb{R}^2\), after a conjugation of the well-known two dimensional map associated with the period-doubling substitution \(H(x, y) = (xy - 2, x^2 - 2)\). As explained in [9], the map \(H\) is conjugated to \(F(s, t) = (2 + st - s^2, st)\).

Define the basin of unstable points of \(F\) as

\[
U_0 = \{(s, t) | s < -2, t > 0\}, \quad V_0 = \{(s, t) | \sqrt{2} < s < t\}.
\]

Denote the set of points whose orbits are disjoint with \(U_0 \cup V_0\) as

\[
\mathcal{S} := \left( \bigcup_{n \geq 0} F^{-n}(U_0 \cup V_0) \right)^c.
\]

In Proposition 2.28 of [9], the author claimed that:

*If \((s_0, t_0) \in \mathcal{S}, then \((F^n(s_0, t_0))_{n \geq 1}\) is bounded.*

Fix \((s_0, t_0) \in \mathbb{R}^2\). For any \(n \geq 0\), define

\[
(s_n, t_n) := F^n(s_0, t_0), \quad c_n := s_n(s_n - t_n).
\]

It is equivalent to start from \((s_0, c_0)\), and for \(n \geq 0\), iterate by,

\[
s_{n+1} = 2 - c_n, \quad c_{n+1} = (2 - c_n)(2 - s_n^2).
\]

In [9], the following two-step iteration of \(c_n\) is studied:

\[
c_{n+2} = (2 + (2 - c_n)(s_n^2 - 2))(2 - (2 - c_n)^2). \quad (102)
\]

In Lemma 2.26 (a), the author claimed that:

*There exists \(u \in (2 + \sqrt{2}, 3.56)\) such that if

\[
|s_0| \geq 3, \quad 2 + \sqrt{2} < c_0 \leq u,
\]

*then \(c_2 \leq 2 - \sqrt{2} and F^5(s_0, t_0) \in V_0.\)
This is not the case. The reason is as follows. Fix any \( \varepsilon \in (0, u - 2 - \sqrt{2}) \) and let \( c_0(\varepsilon) := 2 + \sqrt{2} + \varepsilon \). By (102), we have
\[
c_2(\varepsilon) = \varepsilon(2\sqrt{2} + \varepsilon)[(\sqrt{2} + \varepsilon)(s_0^2 - 2) - 2].
\]

If \( s_0 \) is large enough, we could have \( c_2(\varepsilon) > 2 - \sqrt{2} \).

Given \( |s_n| \geq 3 \), one can view (102) as an expanding map in three relevant intervals for \( c_n : n \geq 0 \) and \( c_n : n \geq 0 \) such that \( s_{2n} \to -\infty \) and \( c_{2n} \in (2 + \sqrt{2}, 3.56) \) for all \( n \geq 0 \). Indeed this is compatible with Proposition 6.1 of the present paper. For any \( E \in \pi(\mathcal{E}_f^e) \), by Proposition 6.1, we have \( h_{2m}(E) \geq \sqrt{2} \) and
\[
\lim_{m \to \infty} h_{2m+1}(E) = \infty, \quad \lim_{m \to \infty} h_{2m}(E) = \sqrt{2}.
\]

We write \( h_m = h_m(E) \) for simplicity. If we define
\[
s_0 := -h_{2m+1}; \quad c_0 = 2 + h_{2m+2}.
\]

Then one can check that
\[
s_n = -h_{2m+n+1}; \quad t_n = s_n^2 + s_{n-1}^2 - 2; \quad c_n = 2 + h_{2m+n+2}.
\]

Consequently, we have \( c_{2n} \geq 2 + \sqrt{2} \) and
\[
s_{2n} \to -\infty, \quad c_{2n} \to 2 + \sqrt{2}.
\]

If \( m \) is large enough, we can check directly that \( (s_0, t_0) \in \mathcal{G} \). However since \( s_{2n} \to -\infty \), \( (F^n (s_0, t_0))_{n \geq 1} \) is unbounded. Thus Proposition 2.28 is false.

There is a similar gap in Lemma 2.25 (a), which we will not explain in detail. We just remark that exactly the same argument can be used to show the gap. Moreover, by using some results in the proof of Proposition 6.3, one can construct another point \((\hat{s}_0, \hat{t}_0) \in \mathcal{G}\) such that the related sequences \( \{\hat{s}_n : n \geq 1\}, \{\hat{c}_n : n \geq 1\} \) satisfy
\[
\hat{s}_{2n} \to -\infty, \quad \hat{c}_{2n} \to 2 - \sqrt{2}.
\]

Again, we obtain a counter example of Proposition 2.28.

**B. Substitutions–Basic Definitions and Examples**

In this appendix, we give a brief introduction on substitutions. For general theory of substitutions, in particular, the connection between substitutions and dynamical systems, see [18,28].
Assume \( \kappa \geq 2 \) and \( \mathbb{A} = \{a_1, \ldots, a_\kappa\} \). We call \( \mathbb{A} \) an alphabet and call each \( a_i \) a letter. We define the set of finite words \( \mathbb{A}_+ \) as

\[
\mathbb{A}_+ := \bigcup_{n \geq 1} \mathbb{A}^n.
\]

Define the concatenation operation \( * : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}_+ \) as

\[
(u_1 \cdots u_n) * (v_1 \cdots v_m) := u_1 \cdots u_n v_1 \cdots v_m.
\]

Then \( (\mathbb{A}_+, *) \) become a semigroup.

A substitution \( \zeta \) over \( \mathbb{A} \) is a map \( \zeta : \mathbb{A} \to \mathbb{A}_+ \). Given a substitution \( \zeta \) over \( \mathbb{A} \), we can extend it to a morphism of \( \mathbb{A}_+ \) or \( \mathbb{A}^N \) by concatenation as

\[
\zeta(u_1 u_2 \cdots) := \zeta(u_1) \ast \zeta(u_2) \ast \cdots
\]

If \( \zeta, \tilde{\zeta} \) are substitutions over \( \mathbb{A} \), so is \( \zeta \circ \tilde{\zeta} \). In particular, the \( k \)-th iteration \( \zeta^k \) of \( \zeta \) is also a substitution.

Assume \( \zeta \) is a substitution over \( \mathbb{A} \). \( u \in \mathbb{A}^N \) is called a fixed point of \( \zeta \) if \( \zeta(u) = u \). \( u \in \mathbb{A}^N \) is called a periodic point of \( \zeta \) if \( \zeta^k(u) = u \) for some \( k \geq 1 \).

A substitution \( \zeta \) over \( \mathbb{A} \) is primitive if there exists \( k \geq 1 \) such that, for any \( a, b \in \mathbb{A} \), the letter \( a \) occurs in \( \sigma^k(b) \).

Let \( \mathrm{FG}(\mathbb{A}) \) be the free group with generators \( \mathbb{A} \). Assume \( \zeta \) is a substitution over \( \mathbb{A} \). \( \zeta \) can be extended to a group homomorphism on \( \mathrm{FG}(\mathbb{A}) \) by defining

\[
\zeta(a^{-1}) := (\zeta(a))^{-1} \quad \text{and} \quad \zeta(\emptyset) = \emptyset,
\]

where \( \emptyset \) is the identity of \( \mathrm{FG}(\mathbb{A}) \). \( \emptyset \) can be viewed as the empty word. If \( \zeta \) is invertible as a group homomorphism on \( \mathrm{FG}(\mathbb{A}) \), then we call \( \zeta \) an invertible substitution.

Now we give three famous examples, which appear in this paper.

**Example B.1.** Let \( \mathbb{A} = \{a, b\} \). The Fibonacci substitution \( \zeta_1 \) is defined as

\[
\zeta_1(a) := ab; \quad \zeta_1(b) := a.
\]

\( \zeta_1 \) is primitive and invertible. Indeed, the inverse of \( \zeta_1 \) is the group homomorphism determined by the “substitution” \( \eta \):

\[
\eta(a) := b; \quad \eta(b) := b^{-1}a.
\]

The Fibonacci sequence is the unique fixed point \( \zeta_1^\infty(a) \) of \( \zeta_1 \), where

\[
\zeta_1^\infty(a) := \lim_{n \to \infty} \zeta_1^n(a) = abaababa \cdots
\]

**Example B.2.** Let \( \mathbb{A} = \{a, b\} \). The Thue-Morse substitution \( \zeta_2 \) is defined as

\[
\zeta_2(a) := ab; \quad \zeta_2(b) := ba.
\]

\( \zeta_2 \) is primitive but not invertible. It has two fixed points: \( \zeta_2^\infty(a) \) and \( \zeta_2^\infty(b) \), where

\[
\zeta_2^\infty(a) := \lim_{n \to \infty} \zeta_2^n(a) = abbabaab \cdots\]

Either \( \zeta_2^\infty(a) \) or \( \zeta_2^\infty(b) \) is called the Thue-Morse sequence.
Example B.3. Let $A = \{a, b\}$. The period-doubling substitution $\zeta_3$ is defined as

$$\zeta_3(a) := ab; \quad \zeta_3(b) := aa.$$ 

$\zeta_3$ is also primitive but not invertible. It has one fixed point:

$$\zeta_3^\infty(a) := \lim_{n \to \infty} \zeta_3^n(a) = abaaabab \cdots$$ 

$\zeta_3^\infty(a)$ is called the period-doubling sequence.

C. The Characterization of the Gaps of $\Sigma_\infty$

In this appendix, we show the following:

**Proposition C.1.** The set of gaps of $(\Sigma_\infty, \leq)$ is

$$\{(\sigma 01^\infty, \sigma 10^\infty) : \sigma \in \Sigma_+\}.$$

**Proof.** Given $\sigma \in \Sigma_+$, we claim that $(\sigma 01^\infty, \sigma 10^\infty) = \emptyset$. If otherwise, there exists $\tau \in \Sigma_+$ such that $\sigma 01^\infty < \tau < \sigma 10^\infty$. If there exists $j \leq |\sigma|$ such that $\sigma_j \neq \tau_j$, then either $\sigma 01^\infty, \sigma 10^\infty < \tau$ holds or $\tau < \sigma 01^\infty, \sigma 10^\infty$ holds, which is a contradiction. So $\sigma < \tau$. If $\tau_{|\sigma|+1} = 0(1)$, then

$$\tau = \sigma 0 \cdots \leq \sigma 01^\infty, \quad (\sigma 10^\infty \leq \sigma 1 \cdots = \tau)$$

which contradicts with the fact that $\sigma 01^\infty < \tau (\tau < \sigma 10^\infty)$. So the claim holds. By the definition of gap, $(\sigma 01^\infty, \sigma 10^\infty)$ is a gap of $\Sigma_\infty$.

Now assume $(\tau, \hat{\tau})$ is a gap of $\Sigma_\infty$. Assume $\tau \land \hat{\tau} = \sigma$ and $|\sigma| = n$. Then

$$\tau_{n+1} = 0, \quad \hat{\tau}_{n+1} = 1.$$ 

We claim that $\tau_j = 1$ for any $j \geq n + 2$. If otherwise, there exists $j_0 \geq n + 2$ such that

$$\tau_j = 1, \quad n + 2 \leq j < j_0, \quad \tau_{j_0} = 0.$$ 

Then we have

$$\tau = \sigma 01^{j_0-(n+2)} \cdots < \sigma 01^\infty < \hat{\tau},$$

which contradicts with the fact that $(\tau, \hat{\tau})$ is a gap. So we must have $\tau = \sigma 01^\infty$. The same argument shows that $\hat{\tau} = \sigma 10^\infty$. That is, $(\tau, \hat{\tau}) = (\sigma 01^\infty, \sigma 10^\infty)$. \qed

D. Two Tables

For the reader’s convenience, we include two tables of indexes in this appendix. One is for the notations used in this paper, and the other one is for the various orders used in this paper.

In this paper, two types of orders are defined in various spaces: one is $\leq$, another is $\preceq$. When $\leq$ is used, it means standard or strong, depending on the context. When $\preceq$ is used, it means non standard or weak.
The Spectrum of Period-Doubling Hamiltonian

Table 1. Index of notations

| Symbol | Description |
|--------|-------------|
| $\xi, V_\xi$ | P-D sequence and potential, see Sect. 1.2.1 |
| $H_j, \sigma_j$ | PDH, the spectrum, see (2) |
| $h_n(E)$ | Trace polynomial, see (3) |
| $\theta(E)$ | Trace orbit of $E$, see (4) |
| $\mathcal{E}_{\infty}$ | The set of $\infty$-type energies, see (5) |
| $\mathcal{A}, \mathcal{G}, A = [a_{ij}]$ | Alphabet, graph, adjacency matrix, see Sect. 1.2.4 |
| $\Omega_\mathcal{A}, \Omega_\mathcal{G}$ | Symbolic spaces, see Sect. 1.2.4 |
| $\mathcal{E}^0, \mathcal{E}^e, \mathcal{Z}, \mathcal{Z}_n, \mathcal{R}_n$ | The set of admissible words, see (40) |
| $\mathcal{E}^0, \mathcal{E}^e, \mathcal{E}^f, \mathcal{E}_f, \mathcal{E}_r, \mathcal{F}$ | Zeros of $|h_n| : n \geq 0$, see (6), (26) |
| $\omega_n, \omega^0$ | Eventually 2-periodic codings, see (11) |
| $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_II, \mathcal{G}_e$ | Minimum and maximum of $\Omega_\mathcal{G}$, see (12) |
| $\Sigma_n, \Sigma^0, \Sigma^e, \Sigma_\mathcal{G}, \Sigma_\mathcal{G}^{(2)}$ | The set of gaps of $\sigma_j$, see Theorem 1.8 |
| $I, I(E)$ | The set of gaps of $\Omega_\mathcal{G}$, see Theorem 4.6 |
| $\sigma_\alpha, n$ | The set of admissible words, see (48), (49) |
| $\mathcal{B}_n$ | Family of bands of level $n$, see (17) |
| $\mathcal{B}_\sigma$ | Optimal covering of level $n$, see (18) |
| $\mathcal{A}, \mathcal{C}, \mathcal{B}$ | Binary trees, see (13), (31), (56), Sect. 2.1.2 |
| $\mathcal{M}_n, \mathcal{M}_0$ | NS(SNS), limit set, see Sect. 1.3.1 |
| $\sigma^*, \sigma^+, \sigma|k| |\sigma|$ | Sub-alphabet, graph, NS, see Sect. 1.3.4 and 5.2.2 |
| $\sigma \wedge \tau, \sigma \prec \tau$ | The set $\{0, \ldots, 2^n-1\}$, its coding, see Sect. 2.1.2, (24) |
| $B_\sigma, a_\sigma, b_{\sigma, 0}$ | Successor, predecessor, prefix, length, see Sect. 2.1.2 |
| $I_w$ | Common prefix, prefix, see Sect. 2.1.2 |
| $\pi : (\Omega_{\infty}, \leq) \rightarrow (\sigma_\lambda, \leq)$ | Band, left and right endpoint, zero, see Sect. 2.1.2 |
| $\Pi : (\Omega_{\infty}, \leq) \rightarrow (\Sigma_{\infty}, \leq)$ | Rename of band, see Sect. 4.1.1 |
| $\varepsilon : \Sigma_{\infty} \rightarrow [0, 1]$ | Coding map, see Sect. 1.2.4 and 5.1 |
| $N : \sigma_\lambda \rightarrow [0, 1], N(E), N(G)$ | Another coding map, see Sect. 1.2.6 and 4.3.2 |
| $\xi, \Pi_\lambda$ | Evaluation map, see Sect. 1.2.6 |
| $\ell, \ell^\varepsilon$ | IDS, see Sect. 1.2.6 |
| $\xi, \ell^\varepsilon$ | Label-assigning map, its extension on $\Omega_\mathcal{G}$, see Sect. 4.1 |
| $\xi, \ell^\varepsilon$ | Bijections between gap edges, see Sect. 4.3 |

Table 2. Index for various orders

| $I < J, I < J$ | Two orders for bands, see (19) |
| $\sigma \leq \tau$ | Lexicographical order on $\Sigma_\mathcal{G}$ and $\Sigma_\mathcal{G}^{\infty}$, see Sects. 2.1.2, 4.3.2 |
| $\sigma \preceq \tau$ | Total order on $\Lambda_{\mathcal{G}}$, see (32) |
| $\alpha < \beta, \alpha < \beta$ | Two partial orders on $\mathcal{A}$, see (48), (49) |
| $\omega \leq \omega^0, \omega \leq \omega^I$ | Two partial orders on $\Omega_{\infty}$, see Sect. 4.2.2 |
| $(x, y) \leq (x', y')$ | Partial order on $\mathbb{R}^2$, see Sect. 6.2.1 |

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