KÄHLER METRICS ON SINGULAR TORIC VARIETIES

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ABSTRACT. We extend Guillemin’s formula for Kähler potentials on toric manifolds to singular quotients of $\mathbb{C}^N$ and $\mathbb{C}P^N$.

1. INTRODUCTION

Let $G$ be a torus with Lie algebra $\mathfrak{g}$ and integral lattice $\mathbb{Z}_G \subset \mathfrak{g}$. Let $u_1, \ldots, u_N \in \mathbb{Z}_G$ be a set of primitive vectors which span $\mathfrak{g}$ over $\mathbb{R}$. Let $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ and let

$$P = P_{u,\lambda} := \{\eta \in \mathfrak{g}^* \mid \langle \eta, u_j \rangle - \lambda_j \geq 0, \quad 1 \leq j \leq N\}$$

be the corresponding polyhedral set. We assume that $P$ has a non-empty interior and that the collection of inequalities defining $P$ is minimal: if we drop the condition that $\langle \eta, u_j \rangle - \lambda_j \geq 0$ for some index $j$ then the resulting set is strictly bigger than $P$.

A well-known construction of Delzant, suitably tweaked, produces a symplectic stratified space $M_P$ with an effective Hamiltonian action of the torus $G$ and associated moment map $\phi = \phi_P : M_P \to \mathfrak{g}^*$ such that $\phi(M_P) = P$. We will review the construction below. The space $M_P$ is a symplectic quotient of $\mathbb{C}^N$ by a compact abelian subgroup $K$ of the standard torus $T^N$. Therefore, by a theorem of Heinzner and Loose $M_P$ is a complex analytic space [HL]. Moreover $M_P$ is a Kähler space [HL (3.5)] and [HHuL]. Even though in general the space $M_P$ is singular, the preimages of open faces of $P$ under the moment map $\phi_P$ are smooth Kähler manifolds. The main result of the paper are formulas for the Kähler forms on these manifolds. In particular we will show that the Kähler form $\omega$ on the preimage $\phi_P^{-1}(\hat{P})$ of the interior $\hat{P}$ of the polyhedral set $P$ is given by the formula (1.1) below:

$$\omega = \sqrt{-1} \partial \bar{\partial} \phi^* \left( \sum_{j=1}^{N} \lambda_j \log(u_j - \lambda_j) + u_j \right),$$

where we think of $u_j \in \mathbb{Z}_G$ as a function on $\mathfrak{g}^*$.

Formula (1.1) was originally proved by Guillemin in the case where $M_P$ is a compact manifold (and thus $P$ is a simple unimodular polytope, also known as a Delzant polytope). It was extended to the case of compact orbifolds by Abreu [A]. Calderbank, David and Gauduchon gave two new proofs of Guillemin’s formula (for orbifolds) in [CDG]. One of their proofs was simplified further in [BG].

As we just mentioned, for generic values of $\lambda$ the polyhedral set $P$ is simple and consequently $M_P$ is at worst an orbifold. But for arbitrary values of $\lambda$ it may have more serious singularities. Of particular interest is the singular case where $P$ is a cone on a simple polytope. Then there is only one singular point, and the link of the singularity is a Sasakian

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orbifold. Such orbifolds, especially the ones with Sasaki-Einstein metrics, have attracted some attention in string theory. They play a role in the AdS/CFT correspondence [MS].

If the polyhedral set $P$ is a polytope, i.e., if $P$ is compact, then as a symplectic space $M_P$ may also be obtained as a symplectic quotient of $\mathbb{C}P^N$. In this case the Fubini-Study form on $\mathbb{C}P^N$ will induce a Kähler structure on $M_P$, which is different from the one induced by the flat metric on $\mathbb{C}^N$ even in the case where $M_P$ is smooth. We will give a formula for this Kähler structure as well.

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2. The “Delzant” Construction: Toric Varieties as Kähler Quotients

It will be convenient for us to fix the following notation. As in the introduction, let $G$ be a torus with Lie algebra $\mathfrak{g}$ and integral lattice $\mathbb{Z}_G \subset \mathfrak{g}$. Let $u_1, \ldots, u_N \in \mathbb{Z}_G$ be a set of primitive vectors which span $\mathfrak{g}$ over $\mathbb{R}$. Let $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ and let

$$P = P_{u, \lambda} := \{ \eta \in \mathfrak{g}^* | \langle \eta, u_j \rangle - \lambda_j \geq 0, \ 1 \leq j \leq N \}$$

be the corresponding polyhedral set. As above we assume that $P$ has the non-empty interior and that the collection of inequalities defining $P$ is minimal. Let $A : \mathbb{Z}^N \to \mathbb{Z}_G$ be the $\mathbb{Z}$-linear map given by

$$A(x_1, \ldots, x_N) = \sum x_i u_i.$$ 

That is, $A$ is defined by sending the standard basis vector $e_i$ of $\mathbb{Z}^N$ to $u_i$. Let $A$ also denote the $\mathbb{R}$-linear extension $\mathbb{R}^N \to \mathfrak{g}$. Let $\mathfrak{k} = \ker A$ and let $\mathfrak{b} : \mathfrak{k} \to \mathbb{R}^N$ denote the inclusion. The map $A$ induces a surjective map of Lie groups $\overline{A} : T^N = \mathbb{R}^N/\mathbb{Z}^N \to \mathfrak{g}/\mathbb{Z}_G = G$.

Let $K = \ker \overline{A}$ and let $\mathfrak{b} : K \to T^N$ denote the corresponding inclusion. The group $K$ is a compact abelian group which need not be connected. It’s easy to see that the Lie algebra of $K$ is $\mathfrak{k}$.

We have a short exact sequence of abelian Lie algebras:

$$0 \to \mathfrak{k} \overset{B}{\to} \mathbb{R}^N \overset{A}{\to} \mathfrak{g} \to 0.$$ 

Let

$$0 \to \mathfrak{g}^* \overset{A^*}{\to} (\mathbb{R}^N)^* \overset{B^*}{\to} \mathfrak{k}^* \to 0$$

be the dual sequence. Note that $\ker B^* = A^*(\mathfrak{g}^*) = \mathfrak{t}^0$ where $\mathfrak{t}^0$ denotes the annihilator of $\mathfrak{t}$ in $(\mathbb{R}^N)^*$. Let $\{e_i^*\}$ denote the dual basis of $(\mathbb{R}^N)^*$ and let $\lambda = \sum \lambda_i e_i^*$. We note that

$$(B^*)^{-1}(B^*(-\lambda)) = -\lambda + \mathfrak{t}^0 = -\lambda + A^*(\mathfrak{g}^*).$$

In particular $(B^*)^{-1}(B^*(-\lambda))$ is the image of the affine embedding

$$t_\lambda : \mathfrak{g}^* \hookrightarrow (\mathbb{R}^N)^*, \quad t_\lambda(\ell) = -\lambda + A^*(\ell).$$

Lemma 2.1. Let $P$ be the polyhedral set defined by (2.1) above.

1. There exists a Kähler space $M_P$ with an effective holomorphic Hamiltonian action of the torus $G$ so that the image of the associated moment map $\phi_P : M_P \to \mathfrak{g}^*$ is $P.$
(2) For every open face \( \breve{F} \), the preimage \( \phi_{\breve{F}}^{-1}(\breve{F}) \) is the Kähler quotient of a complex torus \( (\mathbb{C}^\times)^{N_F} \) by a compact subgroup \( K_F \) of the compact torus \( T^{N_F} \subset (\mathbb{C}^\times)^{N_F} \). Here the number \( N_F \) and the group \( K_F \) depend on the face \( F \).

(3) If the set \( P \) is bounded, then \( M_P \) can also be constructed as a Kähler quotient of \( \mathbb{CP}^N \).

Proof. For every index \( i \) and any \( \eta \in g^* \)

\[
(2.3) \quad \langle \eta, A\eta \rangle - \lambda_i = \langle A^*\eta, e_i \rangle - \sum \lambda_j \eta_j = \langle A^*\eta - \lambda, e_i \rangle = \langle \iota_\lambda(\eta), e_i \rangle.
\]

Therefore

\[
\iota_\lambda(P) = \{ \ell \in (\mathbb{R}^N)^* | \langle \ell, e_i \rangle \geq 0, 1 \leq i \leq N \} \cap \iota_\lambda(g^*).
\]

More generally, if \( \breve{F} \subset P \) is an open face, there is a unique subset \( I_F = I \subset \{1, \ldots, N\} \) so that

\[
(2.4) \quad \breve{F} = \bigcap_{j \notin I} \{ \eta \in g^* | \langle \eta, u_j \rangle - \lambda_j > 0 \} \cap \bigcap_{j \in I} \{ \eta \in g^* | \langle \eta, u_j \rangle - \lambda_j = 0 \}.
\]

Therefore

\[
(2.5) \quad \iota_\lambda(\breve{F}) = \iota_\lambda(g^*) \cap \bigcap_{j \notin I} \{ \ell \in (\mathbb{R}^N)^* | \langle \ell, e_i \rangle > 0 \} \cap \bigcap_{j \in I} \{ \ell \in (\mathbb{R}^N)^* | \langle \ell, e_i \rangle = 0 \}.
\]

The moment map \( \phi \) for the action of \( T^N \) on \( (\mathbb{C}^N, \sqrt{-1} \sum dz_j \wedge d\bar{z}_j) \) is given by

\[
\phi(z) = \sum |z_j|^2 e_j^*.
\]

Hence

\[
\phi(\mathbb{C}^N) = \{ \ell \in (\mathbb{R}^N)^* | \langle \ell, e_i \rangle \geq 0, \quad 1 \leq i \leq N \}.
\]

The moment map \( \phi_K \) for the action of \( K \) on \( \mathbb{C}^N \) is the composition

\[
\phi_K = B^* \circ \phi.
\]

Let \( \nu = B^*(-\lambda) \). We argue that

\[
\phi(\phi_K^{-1}(\nu)) = \iota_\lambda(P).
\]

Indeed,

\[
\phi_K^{-1}(\nu) = \phi^{-1}(B^* \circ \phi^{-1}(\nu))
= \phi^{-1}(B^* \circ \phi^{-1}(B^*(-\lambda)))
= \phi^{-1}(\iota_\lambda(g^*))
= \phi^{-1}(\iota_\lambda(P))
\]

Therefore

\[
\phi(\phi_K^{-1}(\nu)) = \iota_\lambda(P).
\]

The restriction \( \phi|_{\phi_K^{-1}(\nu)} \) descends to a map \( \breve{\phi} : M_P \equiv \phi_K^{-1}(\nu)/K \rightarrow \iota_\lambda(g^*) \). It is not hard to see that the composition \( \phi_P \) of \( \breve{\phi} \) with the isomorphism \( \iota_\lambda(g^*) \xrightarrow{\sim} g^* \) is a moment map for the action of \( G \) on the symplectic quotient (symplectic stratified space) \( M_P \). Since the isomorphism \( \iota_\lambda(g^*) \rightarrow g^* \) obviously maps \( \iota_\lambda(P) \) to \( P \), we conclude that the image of \( \phi_P : M_P \rightarrow g^* \) is exactly \( P \). This proves (1).
To prove (2) we define a bit more notation. For a subset \( I \subset \{1, \ldots, N\} \) we define the corresponding coordinate subspace
\[
V_I := \{ z \in \mathbb{C}^N \mid j \in I \Rightarrow z_j = 0 \}.
\]
Its “interior” \( \hat{V}_I \) is defined by
\[
\hat{V}_I := \{ z \in \mathbb{C}^N \mid j \in I \Rightarrow z_j = 0 \}.
\]
Also, let
\[
T^N_1 := \{ a \in T^N \mid j \notin I \Rightarrow a_j = 1 \}.
\]
The sets \( V_I, \hat{V}_I \) are Kähler submanifolds of \( \mathbb{C}^N \) preserved by the action of \( T^N \). They are both fixed by \( T^N_1 \) with \( \hat{V}_I \) being precisely the set of points of orbit type \( T^N_1 \).

The restriction \( \phi_K|_{\hat{V}_I} \) is a moment map for the action of \( K \) on \( \hat{V}_I \). Moreover, for any \( \nu \in \mathfrak{k}^* \)
\[
\phi_K^{-1}(\nu) \cap \hat{V}_I = (\phi_K|_{\hat{V}_I})^{-1}(\nu).
\]
Hence
\[
(\phi_K^{-1}(\nu) \cap \hat{V}_I)/K = (\phi_K|_{\hat{V}_I})^{-1}(\nu)/K.
\]
While the action of \( K \) on \( \hat{V}_I \) need not be free, the action of
\[
K_I := K/(K \cap T^N_1)
\]
on \( \hat{V}_I \) is free. Therefore, the quotient \( (\phi_K^{-1}(\nu) \cap \hat{V}_I)/K \) may be interpreted as a regular Kähler quotient of \( \hat{V}_I \) by the Hamiltonian action of \( K_I \):
\[
(2.7) \quad (\phi_K^{-1}(\nu) \cap \hat{V}_I)/K = \hat{V}_I/_{\nu} K_I
\]
for an appropriate value \( \nu_1 \in \mathfrak{k}_I^* \) of the \( K_I \) moment map.

Given a face \( F \), let \( I = I_F \) be the corresponding subset of \( \{1, \ldots, N\} \). Then, by (2.5),
\[
(2.8) \quad \{ z \in \mathbb{C}^N \mid \phi(z) \in \iota_{\lambda}(\hat{F}) \} =
\{ z \in \mathbb{C}^N \mid \phi(z) = \iota_{\lambda}(g^*), \langle \phi(z), e_j \rangle > 0 \text{ for } j \notin I, \langle \phi(z), e_j \rangle = 0 \text{ for } j \in I \}
= \phi_K^{-1}(\nu) \cap \hat{V}_I.
\]
Therefore,
\[
\phi_K^{-1}(\nu) \cap \hat{V}_I = \phi^{-1}(\iota_{\lambda}(\hat{F})).
\]
It follows from the definition of \( \phi_P \) that
\[
(\phi_K^{-1}(\nu) \cap \hat{V}_I)/K = \phi_P^{-1}(\hat{F}).
\]
By (2.7) we conclude that
\[
(2.9) \quad \phi_P^{-1}(\hat{F}) = \hat{V}_I/_{\nu} K_I.
\]
This proves (2).

If \( P \) is compact, then \( \iota_{\lambda}(P) \subset (\mathbb{R}^N)^* \) is bounded. Hence \( \iota_{\lambda}(P) \) is contained in a sufficiently large multiple of the standard simplex. Any such simplex is the image of \( \mathbb{C}P^N \) under the moment map for the standard action of \( T^N \) with the Kähler form on \( \mathbb{C}P^N \) being the appropriate multiple of the standard Fubini-Study form. This proves (3). \( \square \)
Remark 2.2. It follows from the results of Heinzner and his collaborators [H], in particular from [HHu], that the action of \( G \) on \( M_P \) extends to an action the complexified group \( G^\mathbb{C} \). This action of \( G^\mathbb{C} \) has a dense open orbit. In other words \( M_P \) is a toric \( \mathbb{C} \)hähler space.

3. Kähler potentials, Legendre transforms and symplectic quotients

We start this section by recalling a result of Guillemin (Theorem 4.2 and Theorem 4.3 in [G]):

\textbf{Lemma 3.1.} Suppose the action of \( T^N \) on \( (\mathbb{C}^\times)^N = \mathbb{R}^N \times \sqrt{-1}T^N \) preserves a Kähler form \( \omega \) and is Hamiltonian. Then there exists a \( T^N \)-invariant function \( f \) on \( (\mathbb{C}^\times)^N \) such that \( \omega = i\partial\bar{\partial}f \). Additionally

\[ L_f \circ \pi : (\mathbb{C}^\times)^N \rightarrow (\mathbb{R}^N)^* \]

is a moment map for the action of \( T^N \) on \((\mathbb{C}^\times)^N, \omega \). Here \( \pi : \mathbb{R}^N \times \sqrt{-1}T^N \rightarrow \mathbb{R}^N \) is the projection and \( L_f : \mathbb{R}^N \rightarrow (\mathbb{R}^N)^* \) is the Legendre transform of \( f \), where we have identified \( f \in C^\infty((\mathbb{C}^\times)^N)^*, \mathbb{R}^N \) with a function on \( \mathbb{R}^N \).

The same result holds with \( (\mathbb{C}^\times)^N \) replaced by \( U \times \sqrt{-1}T^N \) for any contractible open set \( U \subset \mathbb{R}^N \).

\textbf{Lemma 3.2.} Let \( f : V \rightarrow \mathbb{R} \) be a (strictly) convex function on a finite dimensional vector space \( V \), let \( A : W \rightarrow V \) be an injective linear map, \( x \in V \) be a point and

\[ j : W \rightarrow V, \quad j(w) = Aw + x \]

an affine map. Then \( f \circ j : W \rightarrow \mathbb{R} \) is (strictly) convex and the associated Legendre transform \( L_{f\circ j} : W \rightarrow W^* \) is given by

\[ L_{f\circ j} = A^* \circ L_f \circ j, \]

where \( A^* : V^* \rightarrow W^* \) is the dual map.

\textbf{Proof.} By the chain rule and the definition of the Legendre transform, \( L_{f\circ j}(w) = d(f\circ j)_w = df_{j(w)} \circ dj_w = L_f(j(w)) \circ A = A^* \circ L_f \circ j(w) \) for any \( w \in W \).

\textbf{Lemma 3.3.} Let \( f \in C^\infty(\mathbb{R}^N) \) be a strictly convex function and \( \omega = \sqrt{-1}\partial\bar{\partial}\pi_N^* f \) the corresponding \( T^N \)-invariant Kähler form on \( (\mathbb{C}^\times)^N = \mathbb{R}^N \times \sqrt{-1}T^N \) (here \( \pi_N : (\mathbb{C}^\times)^N \rightarrow \mathbb{R}^N \) is the projection). Let \( \Phi = L_f \circ \pi_N : (\mathbb{C}^\times)^N \rightarrow (\mathbb{R}^N)^* \) denote the associated moment map.

Let \( K \subset T^N \) be a closed subgroup and let \( G = T^N/K \). For any \( \nu \in (\mathbb{R}^N)^* \) the symplectic quotient \( (\mathbb{C}^\times)^N//\sqrt{K} \) is biholomorphic to \( U \times \sqrt{-1}T^G \subset g \times \sqrt{-1}T^G = G^\mathbb{C} \) where \( U \subset g \) is an open contractible set. Hence the reduced Kähler form \( \omega_\nu \) has a potential \( f_\nu \).

Moreover, the Legendre-Fenchel dual \( f_\nu^* \) of the Kähler potential \( f_\nu \) is given by

\[ (3.1) \quad f_\nu^* = f^* \circ \iota_\lambda \]

where \( \iota_\lambda : g^* \rightarrow (\mathbb{R}^N)^* \) is the affine embedding \( (2.2) \) and \( -\lambda \) is a point in \( (B^*)^{-1}(\nu) \).

\textbf{Proof.} It is no loss of generality to assume that the group \( K \) is connected. Then \( T^N \simeq K \times G \). Consequently \( \mathbb{R}^N \simeq \mathfrak{k} \times g \) and the short exact sequence

\[ 0 \rightarrow \mathfrak{k} \xrightarrow{B} \mathbb{R}^N \xrightarrow{A} g \rightarrow 0. \]

splits. Let

\[ \pi_K : \mathbb{R}^N \rightarrow \mathfrak{k} \quad \text{and} \quad \iota_\theta : g \rightarrow \mathbb{R}^N \]

denote the maps defined by the splitting. The moment map \( \phi_K : (\mathbb{C}^*)^N \to \mathfrak{k}^* \) for the action of \( K \) on \(((\mathbb{C}^*)^N, \omega)\) is the composition

\[
\phi_K = B^* \circ \phi = B^* \circ L_f \circ \pi_N.
\]

Let

\[
\Delta = (B^*)^{-1}(\nu) \cap \phi((\mathbb{C}^*)^N) = (B^*)^{-1}(\nu) \cap \mathcal{L}_f(\mathbb{R}^N).
\]

Then \( \Delta \) is the intersection of an affine hyperplane with a convex set, hence is contractible.

Since the action of \( K \) on \( \phi_K^{-1}(\nu) \) is free, \( K \cdot \phi_K^{-1}(\nu) \) is an open subset of \((\mathbb{C}^*)^N\) and \( K^C \) acts freely on it. Moreover, for each \( x \in \phi_K^{-1}(\nu) \) the orbit \( K^C \cdot x \) intersects the level set \( \phi_K^{-1}(\nu) \) transversely and

\[
K^C \cdot x \cap \phi_K^{-1}(\nu) = K \cdot x
\]

(See [GS, pp. 526–527]). It follows that the restriction

\[
\pi_K|_{\mathcal{L}_f^{-1}(\Delta)} : \mathcal{L}_f^{-1}(\Delta) \to \mathfrak{k}
\]

is 1-1 and a local diffeomorphism. Hence

\[
U = \pi_K(\mathcal{L}_f^{-1}(\Delta))
\]

is a contractible open set.

On the other hand, the restriction \( \omega|_{\phi_K^{-1}(\nu)} \) descends to a Kähler form \( \omega_\nu \) on the symplectic quotient

\[
(\mathbb{C}^*)^N//\nu K := \phi_K^{-1}(\nu)/K.
\]

Moreover, since \( \omega \) is \( T^N \) invariant, \( \omega_\nu \) is \( G \)-invariant. Note that

\[
(\mathbb{C}^*)^N//\nu K \cong U \times \sqrt{-1}G \subset G^C.
\]

By Lemma 3.1 there exists \( f_\nu \in C^\infty(U) \) such that

\[
\omega_\nu = \sqrt{-1}\partial\bar{\partial}f_\nu.
\]

The potential \( f_\nu \) defines a moment map

\[
\phi_G : U \times \sqrt{-1}G \to \mathfrak{g}^*
\]

with

\[
\phi_G = \mathcal{L}_{f_\nu} \circ \pi_G
\]

where \( \pi_G : U \times \sqrt{-1}G \to U \) is the projection. Moreover, by adjusting \( f_\nu \) [BG] we may arrange for the diagram

\[
\begin{array}{ccc}
\phi_K^{-1}(\nu) & \xrightarrow{\phi} & \Delta \subset (\mathbb{R}^N)^* \\
\downarrow \quad /K & & \uparrow \text{id} \\
U \times \sqrt{-1}G & \xrightarrow{\phi_G} & \mathfrak{g}^*
\end{array}
\]

(3.2)

to commute. That is, the moment map \( \phi_G \) is defined up to a constant and the potential \( f_\nu \) is defined up to a pluri-harmonic \( G \)-invariant function. By adding an appropriate pluri-harmonic function to \( f_\nu \) we can change \( \phi_G \) by any constant we want. Since \( \phi = \mathcal{L}_f \circ \pi_N \)
and since $\phi_G = L_{f_v} \circ \pi_G$, it follows from (3.2) that the diagram below commutes as well:

$$
\begin{array}{ccc}
L_f^{-1}(\Delta) & \xrightarrow{L_f} & \Delta \subset (\mathbb{R}^N)^* \\
\pi_K & & \uparrow \iota_L \\
U & \xrightarrow{L_{f_v}} & \mathfrak{g}^*
\end{array}
$$

(3.3)

Since $(L_{f_v})^{-1} = L_{f_v}^*$, where $f_v^*$ is the Legendre-Fenchel dual of $f_v$,

$$
L_{f_v}^* = \pi_K \circ (L_f)^{-1} \circ \iota_L = \pi_K \circ (L_{f_v}) \circ \iota_L.
$$

By Lemma 3.2

$$
L_{f_v}^* = L_{f_v}^* \circ \iota_L.
$$

Therefore, up to a constant,

$$
f_v^* = f^* \circ \iota_L.
$$

□

4. FROM POTENTIALS TO DUAL POTENTIALS AND BACK AGAIN

We start by making two observations. Let $V$ be a real finite dimensional vector space, $V^*$ its dual, $\mathcal{O} \subset V$ an open set, $\varphi \in C^\infty(\mathcal{O})$ a strictly convex function, $L_\varphi : \mathcal{O} \to V^*$ the Legendre transform (which we assume to be invertible), $\mathcal{O}^* = L_\varphi(\mathcal{O})$ and $\varphi^* \in C^\infty(\mathcal{O}^*)$ the Fenchel dual of $\varphi$.

**Lemma 4.1.** Under the above assumptions, $\varphi = (L_\varphi)^* h$ where $h : \mathcal{O}^* \to \mathbb{R}$ is given by

$$
h(\eta) = \langle \eta, (d\varphi^*)_\eta \rangle - \varphi^*(\eta)
$$

where we think of $(d\varphi^*)_\eta \in T_\eta^* \mathcal{O}^*$ as an element of $(V^*)^* = V$.

**Proof.** By definition of the Fenchel dual

$$
\varphi(s) + \varphi^*(\eta) = \langle \eta, s \rangle
$$

for $\eta = L_\varphi(s)$. Hence

$$
\varphi(s) = \langle \eta, s \rangle - \varphi^*(\eta) = \langle \eta, (L_\varphi)^{-1}(\eta) \rangle - \varphi^*(\eta) = \langle \eta, L_{\varphi^*}(\eta) \rangle - \varphi^*(\eta)
$$

and the result follows since $L_{\varphi^*}(\eta) = (d\varphi^*)_\eta$. □

**Lemma 4.2.** We keep the above notation. Suppose additionally that the dual potential $\varphi^*$ has the following special form:

$$
\varphi^*(\eta) = \sum_{i=1}^N f_i(u_i(\eta) - \lambda_i),
$$

where $u_1, \ldots, u_N$ are vectors in $V$ (thought of as linear functionals $u_i : V^* \to \mathbb{R}$), $\lambda_i \in \mathbb{R}$ are constants and $f_i$'s are functions of one variable. Then

$$
h(\eta) = \sum_{i=1}^N \left( f_i'(u_i(\eta) - \lambda_i) u_i(\eta) - f_i(u_i(\eta) - \lambda_i) \right).
$$
Proof. Observe that \(d(f_i \circ (u_i - \lambda_i))_\eta = f'_i(u_i(\eta) - \lambda_i)d(u_i - \lambda_i)_\eta = f'_i(u_i(\eta) - \lambda_i) u_i\) since \(u_i\) is linear. Hence

\[
\langle \eta, (d\varphi^*)_\eta \rangle = \langle \eta, \sum f'_i(u_i(\eta) - \lambda_i)u_i \rangle = \sum f'_i(u_i(\eta) - \lambda_i) u_i(\eta)
\]

and (4.1) follows from Lemma 4.1 above.

Example 4.3. We use the lemma above to argue that for the standard action of \(\mathbb{T}^N\) on \((\mathbb{C}^N, \sqrt{-1} \partial \bar{\partial} \|z\|^2)\), the dual potential \(\varphi^*\) is given by

\[
\varphi^* = \sum_{i=1}^N e_i \log e_i,
\]

where \(e_1, \ldots, e_N\) is the standard basis of \(\mathbb{R}^N = \text{Lie}(\mathbb{T}^N)\).

Indeed, the homogeneous moment map \(\Phi : \mathbb{C}^N \to (\mathbb{R}^N)^*\) for the standard action of \(\mathbb{T}^N\) is given by

\[
\Phi(z) = \sum |z_j|^2 e^*_j,
\]

where \(\{e^*_j\}\) is the basis dual to \(\{e_j\}\). Hence

\[
\|z\|^2 = \Phi^*(\sum e_j).
\]

On the other hand, if \(\varphi^* = \sum e_j \log e_j\), then

\[
\varphi^* = \sum f \circ e_j
\]

where \(f(x) = x \log x\). Since \(f'(x) = \log x + 1\), equation (4.1) becomes

\[
h = \sum (\log e_j + 1) e_j - \sum e_j \log e_j = \sum e_j.
\]

Therefore, \(\varphi^* = \sum e_j \log e_j\) is, indeed, the dual potential.

We are now in position to prove (1.1).

Theorem 4.4. Let \(G\) be a torus, \(P \subset \mathfrak{g}^*\) the polyhedral set defined by (2.1), \(M_P = \mathbb{C}^N//\sqrt{\mathfrak{k}}\) the Kähler \(G\)-space with moment map \(\varphi_P : M_P \to \mathfrak{g}^*\) constructed in Lemma 2.1(1). Then the Kähler form \(\omega_P\) on \(\hat{M}_P := \varphi_P^{-1}(P)\) is given by:

\[
\omega_P = \sqrt{-1} \partial \bar{\partial} \varphi_P^*\left(\sum_{j=1}^N \lambda_j \log(u_j - \lambda_j) + u_j\right).
\]

Proof. By Lemma 2.1, \(\hat{M}_P = \mathbb{C}^N//\sqrt{\mathfrak{k}}\) where \(\mathfrak{k} \subset \mathbb{T}^N\) is a closed subgroup. By Lemma 3.3, the dual potential \(\varphi_P^*\) on \(\hat{P}\) is given by

\[
\varphi_P^* = \varphi^* \circ \iota_\lambda
\]

where \(\varphi^*\) is the potential on the open orthant in \((\mathbb{R}^N)^*\) dual to the flat metric potential \(\varphi(z) = \|z\|^2\) on \((\mathbb{C}^\times)^N\). By Example 4.3 \(\varphi^* = \sum e_j \log e_j\). Since \(\iota^*_\lambda e_j = u_j - \lambda_j\),

\[
(4.2) \quad \varphi_P^* = \sum (u_j - \lambda_j) \log(u_j - \lambda_j).
\]

By Lemmas 4.1 and 4.2, the potential \(\varphi_P\) is given by

\[
\varphi_P = \varphi_P^* h
\]

where \(h\) is the dual potential on the open orthant in \((\mathbb{R}^N)^*\).
where

\[ h = \sum (\log(u_j - \lambda_j) + 1)u_j - \sum (u_j - \lambda_j) \log(u_j - \lambda_j), \]

(c.f. (4.1)). Therefore

\[ \varphi_P = \phi_P^* \left( \sum_{i=1}^{N} (\lambda_j \log(u_j - \lambda_j) + u_j) \right) \]

and we are done. \(\square\)

5. Kähler potentials on the preimages of faces

Once again let \( P \subset g^* \) be a polyhedral set given by (2.1). Recall that in section 2 we canonically associated to this set a Kähler quotient \( M_P \) of \( \mathbb{C}^N \) which carries an effective holomorphic and Hamiltonian action of the torus \( G \) with a moment map \( \varphi_P : M_P \to g^* \). Let \( F \subset P \) be a face. Its interior \( \tilde{F} \) is given by:

\[ \tilde{F} = \bigcap_{j \not\in I} \{ \eta \in g^* \mid \langle \eta, u_j \rangle - \lambda_j > 0 \} \cap \bigcap_{j \in I} \{ \eta \in g^* \mid \langle \eta, u_j \rangle - \lambda_j = 0 \} \]

for some nonempty subset \( I \) of \( \{1, \ldots, N\} \). We have seen in the proof of Lemma 2.1 that the preimage

\[ M_{\tilde{F}} := \varphi_P^{-1}(\tilde{F}) \]

is the Kähler quotient of \( \tilde{V}_I \) by a compact abelian group \( K_I \). Therefore there is a potential \( \varphi_I^* \in C^\infty(\tilde{F}) \) dual to the Kähler potential \( \varphi_I \) on \( M_{\tilde{F}} \). The goal of this section is to compute the dual potential \( \varphi_I^* \) “explicitly.” Lemmas 4.1 and 4.2 will then give us an analogue of (1.1) for the Kähler metric on \( M_{\tilde{F}} \).

The Kähler potential \( \varphi_I \) on \( \tilde{V}_I \) for the flat metric induced from \( \mathbb{C}^N \) is given by

\[ \varphi_I(z) = \sum_{j \not\in I} |z_j|^2. \]

The restriction of the moment map \( \varphi : \mathbb{C}^N \to (\mathbb{R}^N)^* \) to \( \tilde{V}_I \) is a moment map for the action of the torus

\[ H_I := \mathbb{T}^N / \mathbb{T}_I^N. \]

Note that

\[ \varphi(\tilde{V}_I) = \{ \sum a_i e_i^* \mid a_i > 0 \}. \]

This set is an open subset in \( \text{span}_{i \not\in I} \{ e_i^* \} \simeq \mathfrak{h}_I^* \). From now on we identify \( \mathfrak{h}_I^* \) with \( \text{span}_{i \not\in I} \{ e_i^* \} \). The dual potential \( \varphi_I^* \in C^\infty(\varphi(\tilde{V}_I)) \) is easily seen to be

\[ \varphi_I^* = \sum_{j \not\in I} e_j \log e_j. \]

The manifold \( M_{\tilde{F}} \) is a Hamiltonian \( G \) space, but the group \( G \) doesn’t act effectively. So we cannot yet apply Lemma 3.3 as we would like. Let \( G_I \) denote the quotient of \( G \) that does act effectively on \( M_{\tilde{F}} \). It is isomorphic to the quotient \( H_I / K_I \). The dual of its Lie algebra \( g_I^* \) is naturally embedded in \( g^* \):

\[ g_I^* = \{ \eta \in g^* \mid \langle \eta, u_i \rangle = 0 \text{ for all } i \not\in I \}. \]
Note also that the affine span \( \text{affspan} \tilde{F} \subset g^* \) is the translation of \( g_i^* \) by an element \( \eta_0 \in \tilde{F} \), as it should be. Let \( \gamma_1 : g_i^* \to \text{affspan} \tilde{F} \subset g^* \) denote the affine embedding. Then there exists an affine embedding \( \iota : g_i^* \hookrightarrow h_i^* \) so that the diagram

\[
\begin{array}{ccc}
\tilde{h}_i^* & \longrightarrow & (\mathbb{R}^N)^* \\
\uvec & \uparrow & \upvec \\
\gamma_1 \quad & \quad & \gamma_1 \\
g_i^* & \longrightarrow & g^*
\end{array}
\]

(5.1)

commutes. Here the top arrow simply identifies \( \tilde{h}_i^* \) with \( \text{span}_{i \in I}(e_i^*) \). Since \( \gamma_1 \) is an embedding, we may think of \( \varphi_F^* \) as living on \( \tilde{F} \subset \gamma_1(g_i^*) \). Therefore, by Lemma 3.3

(5.2)

\[
\varphi_F^* = (\varphi_i^* \circ \iota_\lambda)|_{\tilde{F}}.
\]

Let

\[
v_j = u_j|_{\tilde{F}}.
\]

These functions are affine, but not necessarily linear. Then

\[
(e_j \circ \iota_\lambda)|_{\tilde{F}} = (u_j - \lambda_j)|_{\tilde{F}} = v_j - \lambda_j.
\]

Therefore

\[
\varphi_F^* = (\varphi_i^* \circ \iota_\lambda)|_{\tilde{F}} = \sum_{j \not \in I} (v_j - \lambda_j) \log(v_j - \lambda_j).
\]

To get a nicer formula for the potential on \( M_{\tilde{F}} \) we now make a simplifying assumption, namely, that \( 0 \in \tilde{F} \). Then \( v_i = u_i|_{\tilde{g}_i} \) and, in particular, it is linear for all \( i \). Hence Lemmas 4.1 and 4.2 apply, and we obtain:

**Theorem 5.1.** Under the simplifying assumption above, the Kähler form \( \omega_F \) on \( M_{\tilde{F}} \) is given by

\[
\omega_F = \sqrt{-1} \partial \bar{\partial} (\varphi_P|_{M_{\tilde{F}}})^* \left( \sum_{j \not \in I} \lambda_j \log(v_j - \lambda_j) + v_j \right)
\]

Alternatively we may take the isomorphism \( \gamma_1 : g_i^* \to \text{affspan} \tilde{F} \) explicitly into account and think of \( \varphi_F^* \) as living on an open subset of \( g_i^* \). Then, by Lemma 3.3

\[
\varphi_F^* = \varphi_1^* \circ \iota_\lambda \circ \gamma_1.
\]

Since

\[
e_i \circ \iota_\lambda \circ \gamma_1 = u_i|_{\tilde{g}_i} + u_i(\eta_0) - \lambda_i,
\]

we get

\[
\varphi_F^* = \sum_{i \not \in I} (u_j|_{\tilde{g}_i} + u_j(\eta_0) - \lambda_j) \log(u_j|_{\tilde{g}_i} + u_j(\eta_0) - \lambda_j).
\]

We conclude:

**Theorem 5.2.** The Kähler form \( \omega_F \) on \( M_{\tilde{F}} \) is given by

\[
\omega_F = \sqrt{-1} \partial \bar{\partial} \left( (\varphi_P|_{M_{\tilde{F}}})^* \sum_{j \not \in I} ((\lambda_j - u_j(\eta_0)) \log(u_j|_{\tilde{g}_i} + u_j(\eta_0) - \lambda_j) + u_j|_{\tilde{g}_i}) \right)
\]
5.1. Variations on the theme. The same technique allows us to prove a variant of \( \text{(1.1)} \). We keep the notation above. Suppose that the polyhedral set \( P \) is compact. That is, suppose that \( P \) is actually a polytope. Then

\[
\iota^\lambda(P) \subset \{ \ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_j \rangle \geq 0 \text{ for all } j \}
\]

is bounded. Hence there is \( R > 0 \) such that \( \iota^\lambda(P) \) is contained in a scaled copy \( \Delta_R \) of the standard simplex:

\[
\Delta_R = \{ \ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_j \rangle \geq 0 \text{ for all } j \text{ and } \sum \langle \ell, e_j \rangle \leq R \}.
\]

Since \( \Delta_1 \) is the moment map image of \( \mathbb{CP}^N \) under the standard action of \( T_N \), it follows that \( M_P \) is also a symplectic quotient of \( (\mathbb{CP}^N, R\omega_{FS}) \) by the action of the compact abelian Lie group \( K \) defined earlier (\( \omega_{FS} \) denotes the Fubini-Study form) (c.f. Lemma 2.1 (3)). Since

\[
\Delta_R = \{ \ell \in (\mathbb{R}^N)^* \mid \langle \ell, e_j \rangle \geq 0, 1 \leq j \leq N, \langle \ell, -\sum e_j \rangle + R \geq 0 \},
\]

it follows from (3.1) that the potential \( f^* \) dual to the potential for \( R\omega_{FS} \) on \( \Delta_R \) is given by

\[
f^* = \sum e_j \log e_j + (R - \sum e_j) \log (R - \sum e_j).
\]

Consequently the potential \( f^*_y \) dual to the potential on the quotient \( (\mathbb{CP}^N//_y K, \omega_P) \) is

\[
f^*_y = \sum (u_j - \lambda_j) \log (u_j - \lambda_j) + (R - \sum (u_j - \lambda_j)) \log (R - \sum (u_j - \lambda_j)).
\]

By Lemma 4.1 the reduced Kähler form \( \omega_P \) is

\[
\omega_P = \sqrt{-1} \partial \bar{\partial} f^* h
\]

where

\[
h(\eta) = \langle \eta, (df_y)_{\eta} \rangle - f_y(\eta).
\]

A computation similar to the ones in the previous sections gives

\[
(5.3) \quad \sqrt{-1} \partial \bar{\partial} f^* h = \sum \lambda_j \log (u_j - \lambda_j) - (R + \sum \lambda_j) \log (R - \sum (u_j - \lambda_j)).
\]

We have proved:

**Theorem 5.3.** Let \( G \) be a torus, \( P \subset \mathfrak{g}^* \) the polyhedral set defined by (2.1) which happens to be compact, \( M_P = (\mathbb{CP}^N, R\omega_{FS})//_y K \) the Kähler \( G \)-space with moment map \( \phi_P : M_P \to \mathfrak{g}^* \) constructed in Lemma 2.1 (3). Then the Kähler form \( \omega_P \) on \( \hat{M}_P := \phi_P^{-1}(\hat{P}) \) is given by:

\[
\omega_P = \sqrt{-1} \partial \bar{\partial} f^* \left( \sum \lambda_j \log (u_j - \lambda_j) - (R + \sum \lambda_j) \log (R - \sum (u_j - \lambda_j)) \right).
\]

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