Can Quantum Wormholes really help set $\Lambda \to 0$?

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Abstract

We find the quantum analogues of wormholes obtained by Carlini and Mijić (CM), who analytically continued closed universe models. The CM requirement that the strong energy condition ($\gamma > 2/3$) be satisfied is shown to be consistent with the Hawking-Page conjecture for quantum wormholes as solutions of the Wheeler-DeWitt equation. The presence of a cosmological constant $\Lambda$ violates such a condition and so prevents wormholes occurring. It is therefore inconsistent to invoke these wormholes to make $\Lambda$ a dynamical variable: used in arguments which suggest $\Lambda \to 0$.

We analyse instead a simple model with just $\Lambda$ present. Differing results are obtained depending on the boundary conditions applied. In the Euclidean regime only the Hartle-Hawking boundary condition gives the factor $\exp(1/\Lambda)$ but is badly behaved for negative $\Lambda$. Tunneling boundary conditions suggest an initially large value for $\Lambda$. Whereas in the Lorentzian region all boundary conditions suggest an initially large value of $\Lambda$ for spatial curvature $k = 1$. This differs from the previously obtained result of Strominger[28] for such models.
1 Introduction

One possible solution to the cosmological constant \( \Lambda \) problem that has attracted a lot of interest is due to the idea that wormhole solutions can lead \( \Lambda \) to become a dynamical variable with a distribution function \( P(\Lambda) \) \[1\] - for a review of this proposal see eg. ref.[2]. It is suggested that this function is peaked, due to De Sitter instantons, with the Baum-Hawking factor \( P(\Lambda) \sim exp(1/\Lambda) \) \[3,4\] so predicting \( \Lambda \to 0 \) \[1\]. Wormholes are used in two distinct ways in such arguments, firstly they are used to justify why \( \Lambda \) should be treated as a dynamical quantum variable instead of a usual classical variable. This is the most important aspect, as it allows one to make predictions of the possible values of \( \Lambda \). Wormholes have further been used in connecting many universes together which produces a further exponentiation \[1\] i.e.

\[
P(\Lambda) \sim exp\left(exp\left(1/\Lambda\right)\right)
\] (1)

This is only useful if the first factor \( \sim 1/\Lambda \) is correct. In other words, this aspect of wormholes only exaggerates any underlying behaviour.

Carlini and Mijić \[5\] have greatly expanded the number of possible wormhole solutions by considering an analytic continuation of closed Friedman-Robertson-Walker (FRW) universes. For a perfect fluid equation of state, closed universes require that the strong energy condition be satisfied i.e. \( \gamma > 2/3 \). By using an adaption of an approach used by Ellis and Madsen \[6\] the value of \( \gamma \) can be fixed. This in turn can be formulated in terms of a scalar field model with a rather complicated potential. This enables you to see the significance of the strong energy condition in setting up closed universe models. It is the continuation of this theory to the Euclidean domain which then gives wormhole solutions. One drawback of this approach is that the analytic continuation is rather ad hoc with the matter and gravitational parts being treated differently \[5\].

In a different vein, because the number of known wormhole solutions had appeared so limited, Hawking and Page (HP) considered that solutions of the Wheeler-DeWitt (WDW) equation could more generally represent wormholes \[7\]. For such wormholes they suggested that the quantum mechanical wavefunction \( \Psi \) decay exponentially for large scale factor \( a \) so as to represent Euclidean space, and that \( \Psi \) be well behaved as \( a \to 0 \): so that no singularities are present.
By starting with a scalar field with a potential that fixes a value of $\gamma$ we intend deriving the corresponding WDW equation. Because the WDW equation is independent of the Lapse chosen, the Euclidean regime is already included in the formalism. There is thus no need in making any arbitrary continuation and the quantum versions of the Carlini-Mijić wormholes can be found. Such solutions are found to obey the Hawking-Page behaviour when the matter obeys the strong energy condition ($\gamma > 2/3$). This is the same condition that CM required for a closed universe; that could then be analytically continued to a wormhole [5].

Another point raised by the CM wormholes is whether spatial curvature $k = 1$ is really necessary for a wormhole. Although all previously considered wormholes have this feature it is not strictly necessary for a closed universe. Certain unstable fields are known to cause recollapse [8,9] but for simplicity the case of a -ve cosmological constant can be considered.

We first review the classical closed universe models from which we are going to derive their corresponding WDW equations. First we take a bulk matter source with a perfect fluid equation of state $p = (\gamma - 1)\rho$, ($p$ and $\rho$ are the pressure and energy density respectively).

Working in a Lorentzian metric

\[
ds^2 = -a^{4-3\gamma}dt^2 + a^2(t)d\Omega_3^2
\]  

where we are using the same lapse to aid calculations as in Refs.[5], the scale factor $a$ is given by

\[
a(t) = \left[a_o^{3\gamma - 2} - (1 - 3\gamma/2)a^2t^2\right]^{1/(3\gamma - 2)}
\]

With $a_o$ an arbitrary constant which is the maximum size of the Robertson-Walker universe. Using the same approach as in Ellis-Madsen [6] we can convert to a scalar field $\phi$ whose trajectory is given by

\[
\phi = \frac{2}{3\gamma - 2} \gamma^{1/2} \tanh^{-1} \left[ \frac{3\gamma - 2}{2a_o^{1/2(3\gamma - 2)/2}}t \right]
\]

The solution have a potential of the form

\[
V(\phi) = \frac{2 - \gamma}{2a_o^{1/2(3\gamma - 2)}} \cosh^{\frac{6\gamma}{(3\gamma - 2)}} \left[ \frac{(3\gamma - 2)\phi}{2\sqrt{\gamma}} \right]
\]
Notice the difference in the preceding expressions with their Euclidean counterparts in refs.[5]. These expressions are valid for a closed universe analogous expressions could be obtained for \( k = -1 \). By using eq.(3) and (4) the potential can be rewritten as a function of the scale factor.

\[
V(\phi) \equiv V(a) = \frac{V_m}{a^{3\gamma}}
\]

where the constant \( V_m = (1 - \gamma/2)a_o^{3\gamma - 2} \).

2 Quantum Wormholes

We consider solutions of the Wheeler-DeWitt equation, see e.g. [10-12]

\[
\left( \frac{\partial}{\partial a^2} + \frac{p}{a} \frac{\partial}{\partial a} - \frac{1}{a^2} \frac{\partial}{\partial \phi^2} - ka^2 + a^4 V(\phi) \right) \Psi(a, \phi) = 0
\]

(7)

\( p \) is a factor ordering correction and \( k \) the spatial curvature \( \pm 1,0 \).

By using the potential \( V(\phi) \) which corresponds to a fixed value of \( \gamma \) allows the WDW equation to be separable.

The WDW equation simplifies to:

\[
\left( a^2 \frac{d^2}{da^2} + pa \frac{da}{da} + q^2 + V_m a^{6-3\gamma} - ka^4 \right) \Psi(a) = 0
\]

(8)

\[
\left( \frac{d^2}{d\phi^2} + q^2 \right) \Psi(\phi) = 0
\]

(9)

with \( q \) the separation constant.

We can get some idea as to when a Euclidean domain occurs at large \( a \) by considering the sign of the potential \( U \) in the analogous equation [10]

\[
\left( \frac{d^2}{da^2} + U \right) \Psi(a) = 0
\]

(10)

When \( U > 0 \) oscillating solutions occur which represent Lorentzian metrics.

So in order to obtain a wormhole (an asymptotically Euclidean regime for large scale factor) we require \( U < 0 \). Returning to our eq.(8), and setting the unimportant in this regard factor \( p = 0 \), this occurs for

\[
V_m a^{4-3\gamma} - ka^2 < 0
\]

(11)
Therefore for the usual case of positive potential \( V_m > 0 \) we require \( 2 > 4 - 3\gamma \) i.e. \( \gamma > 2/3 \) and \( k = 1 \) for such behaviour. If we have a negative potential \( (V_m < 0) \) then this is reversed ; \( \gamma < 2/3 \) and \( k = \pm 1 \) can give such a solution. This is an example that shows that \( k = 1 \) is not strictly necessary to obtain wormhole solutions - at least in the CM and HP sense.

We have ignored the term involving the separation constant \( q/a^2 \) which would only be important at small \( a \). This strong energy condition is the same as that obtained by CM for the occurrence of wormhole solutions. Recently Kim and Page [13] seem to find also that quantum wormholes are incompatible with a cosmological constant. However we find here a stronger condition : matter sources violating the strong energy condition are incompatible with wormholes obeying the HP conditions. The presence of any matter source with \( \gamma < 2/3 \) will eventually dominate for large \( a \) and prevent the Euclidean wormhole.

A quantum wormhole also requires suitable behaviour for small \( a \). As \( a \to 0 \) we can ignore the \( ka^4 \) term since \( 4 > 6 - 3\gamma \) when \( \gamma > 2/3 \). In this case the WDW equation simplifies to a Bessel equation with solution

\[
\Psi(a) \sim J_{iq}(a^{3-3\gamma/2} \sqrt{V_m}) + Y_{iq}(a^{3-3\gamma/2} \sqrt{V_m})
\] (12)

Using the asymptote \( J_\nu(z) \sim z^\nu \) as \( z \to 0 \) enables the solution to be expressed as

\[
\Psi(a) \sim exp\{iq(3-3\gamma/2)lna\}
\] (13)

The other Bessel function would simply have a (-) sign in the exponent of eq.(13) since \( Y_\nu(z) \sim -z^{-\nu} \). The foregoing argument would proceed in the same fashion. We have also taken \( p = 1 \) although the analysis could easily be extended to include other cases.

The problem now is that as \( a \to 0 \) the \( lna \to -\infty \) causes infinite oscillations to occur, the wavefunction cannot be regarded as a wormhole in its present form as the oscillations represent a singularity [7]. The full solution is:

\[
\Psi(a, \phi) \sim exp\{iq[(3-3\gamma/2)lna + \phi]\}
\] (14)

By integrating over the separation constant we can eliminate this singularity at the origin. This is the same as performing a Fourier Transform [14,15] or like forming a wave-packet solution [16]. Now the integral

\[
\Psi \sim \int exp\{iq[(3-3\gamma/2)lna + \phi]\} dq
\] (15)
Is of the form $\int \exp(ixt)dt$ which by means of the Riemann-Lebesgue Lemma (see eg. ref.[17]) $\to 0$ as $x \to \infty$. The wavefunction is now, in effect damped as $a \to 0$ or $\phi \to \infty$. Such solutions now obey the HP regularity condition and can be taken to be wormholes.

We now return to the full WDW equation; although it has been simplified to an ordinary differential equation it is still not straightforward to obtain analytic solutions for all $\gamma$. You could proceed by finding approximate WKB solutions; but instead we consider only certain values of $\gamma$ that are exactly soluble - this still enables us to emphasize important properties that any solution in the range $2 \geq \gamma > 2/3$ will have. We should point out that throughout this section we are only interested in the existence of wormhole solutions, and set arbitrary coefficients accordingly. In theory such coefficients should be determined by the boundary conditions applied. The usually applied boundary conditions of Hartle-Hawking [18] or the Tunneling one [19], would in general contradict such solutions - they also generally require $q = 0$ - see ref.[20]. We have already seen that the wave-packet type solution is more in keeping with the HP conditions for a quantum wormhole.

a) $\gamma = 2$ or $V(\phi) = 0$.

This is the minimally coupled massless scalar field. The solution of equations (8) and (9) is cf.ref.[7]

$$\Psi \sim e^{iq\phi} \left\{ J_{iq/2}(ia^2/2) + Y_{iq/2}(ia^2/2) \right\}$$

(16)

This has an oscillation phase for $a < q$ and an exponential fall off for $a > q$ : cf. Fig.(1). Let us call this type(I) behaviour of the wavefunction. Classically this is a closed universe with a forbidden region when $a > q$, can this type(I) wavefunction behaviour be considered a wormhole as claimed by Hawking and Page [7]? A classical wormhole is known to occur for a totally imaginary scalar field. This corresponds to a change of sign in the $\partial^2/\partial \phi^2$ term . In order to keep plane wave solutions in $\phi$ the separation constant would change sign $q^2 \to -q^2$ and the solution is

$$\Psi \sim e^{iq\phi} \left\{ K_q(a^2/2) + I_q(a^2/2) \right\}$$

(17)

This no longer has an oscillating domain and is instead Euclidean for all $a$ cf. Fig.(2). Although if the Modified Bessel function $K$ is chosen the wave function is divergent for $a \to 0$. We can call such a wavefunction type(II)
behaviour and is the sort of wavefunction that corresponds to situations that classically would allow wormholes.

b) $\gamma = 4/3$: Radiation or a conformally coupled scalar field. A second example is that of radiation $\gamma = 4/3$ which allows eq.(8) to be written as cf.ref.[21]

$$\left(\frac{d^2}{da^2} + V_m - a^2\right)\Psi(a) = 0$$

(18)

This is in the form of a Parabolic cylinder equation with solutions in terms of confluent hypergeometric functions -see eg.[22]

$$\Psi(a) \simeq \exp(-a^2/2) \, _1F_1\left(\frac{1}{4}(1 - V_m); 1/2; a^2\right)$$

$$+ \exp(-a^2/2) \, _1F_1\left(\frac{1}{4}(3 - V_m); 3/2; a^2\right)$$

(19)

For $V_m > 1$ we get an oscillation for small $a$ with the Euclidean regime again for large $a$ see Fig.(1)- this is again a type(I) wavefunction. As $V_m$ is lowered the Euclidean regime gets closer to the origin. With $V_m \leq 0$ it is totally Euclidean and becomes type (II) see Fig.(2). This is now a negative energy density which should classically be able to support a wormhole solution.

Whether type(I) or the more restrictive type (II) behaviour is the correct description for the wavefunction to describe wormholes there is something of a problem here. We would expect a Euclidean regime to occur at small size and not to correspond to a region beyond the size of a Lorentzian one. The type(II) description has a Euclidean region at small size, but the presence of any matter source violating the strong energy condition will make it have Lorentzian behaviour for larger $a$. It would no longer obey the HP description and requires matter sources which anyway have classical wormholes solutions. If the type (I) wavefunction correctly describes a wormholes then the hope of Hawking and Page, that arbitrary matter sources could have wormhole solutions would appear not be be true.

Can we not simply include a $\Lambda$ term together with a matter source with $\gamma > 2/3$ ? After all the $\Lambda$ term only dominates at large size and many classical wormholes can be constructed also with $\Lambda$ present: this results in the wormhole being attached to Euclidean De Sitter space, see e.g. [23,24]. In the WDW equation however, the De Sitter regime is necessarily Lorentzian
at large size. The wormhole behaviour at small size is also Lorentzian and
the presence of $\Lambda$ can only potentially result in a barrier between these two
regions. The solution would instead represent a tunnelling event between
these two regions. In order to get the important $\exp(1/\Lambda)$ factors we need
rather to attach the wormhole to Euclidean De Sitter spaces. In this sense the
solutions of the WDW equation are less general than required in Coleman’s
approach [1] since they cannot incorporate such geometries.

The main conclusion of this section is that quantum wormholes are pre-
vented by the presence of matter sources violating $\gamma < 2/3$ (which includes
a $\Lambda$ term). With this contradiction it is therefore uncertain what role such
solutions of the WDW equation can have in the setting to zero of the cosmo-
logical constant. One approach out of this impasse, the 3rd quantization,
is to simply piece together many separate wavefunctions each representing
a separate universe [25]. Instead we stick with the usual (2nd quantized)
WDW equation and next consider the case, without wormholes, of just a
cosmological constant $\Lambda$ present.

3 Cosmological constant model

When only a cosmological constant is present the WDW equation takes the
simplified form.

$$\left( \frac{d}{da^2} - U \right) \Psi(a) = 0$$

(20)

Where the WDW potential $U$ for a closed $k = 1$ universe is given by

$$U = a^2 - \Lambda a^4$$

(21)

The potential is sketched in Fig.(2). This has been studied by many authors
especially as the case of quantum tunnelling to a Lorentzian universe [26]. We
follow particularly the analysis of Lavrelashrili et.al. [27] and Strominger[28].

1 There is an interesting analogy with the problem found by Verbin and Davidson [24]
in the case of the conformally coupled wormhole. When a potential $\lambda \phi^4$ was introduced
the wormhole cannot exist (its size $<<$ Planck size) if $\Lambda \neq 0$. 
The WKB solutions have the form cf. [27]

\[ \Psi = \frac{1}{\sqrt{|U(a)|}} \exp \left( \pm \int \sqrt{U(a)} da \right) \]  

(22)

where the ‘action’ \( S = - \int \sqrt{U} da \) is given by

\[ - \int a(1 - \Lambda a^2)^{1/2} da = \frac{(1 - \Lambda a^2)^{3/2}}{3\Lambda} \]  

(23)

Taking the limits between \( a = \Lambda^{-1/2} \) and \( a = 0 \) gives the solutions

\[ \Psi_{\pm} \sim \exp(\pm 1/\Lambda). \]  

(24)

The (+) sign corresponds to the Hartle-Hawking (HH)[18] boundary condition \( \exp(-S) \) and the (-) sign to the tunnelling one \( \exp(-|S|) \) - see for example [11,12].

If we assume that the probability of having a specific \( \Lambda \) is \( P(\Lambda) \sim \Psi^2 \sim \exp(\pm 2/\Lambda) \) then the two approaches predict \( \Lambda \to 0 \) and \( \Lambda \to \infty \) respectively. For the HH case we appear to get the suppression of \( \Lambda \) but the opposite for the tunnelling case. However the tunnelling occurs through the barrier to \( U = 0 \) where \( a^2 \sim 1/\Lambda \). When the barrier is small i.e. when \( \Lambda \) is large tunnelling is enhanced: this is somewhat analogous to the application of an electric potential to an atom which allows electron to tunnel away. In this case a large value of \( \Lambda \) is enhancing the possibility of the universe tunnelling into existence.

According to Strominger [28] because the scale factor \( a \) today is very large the value of \( \Lambda \sim a^{-2} \) is very small as required to fit observation. This does not however agree with the usual interpretation of quantum cosmology which is that of predicting initial conditions. As quantum tunneling is expected to occur to an initial size of roughly Planck dimensions the initial value of \( \Lambda \) is correspondingly big \( \sim 1 \) in Planck units. If instead the initial scale factor was large (and so the initial value of \( \Lambda \sim \text{small} \) it would mean that the Euclidean domain would extend to large sizes. It would then be inconsistent with the present structure of space-time which appears Lorentzian down to at least sizes of \( \sim 10^{-20} \) meters.

We consider next a problem that has arisen for the case of a \(-ve\) cosmological constant. The equivalent expression to eq. (23) is

\[ S = - \frac{(1 + |\Lambda a^2|^{3/2}}{3|\Lambda|} \]  

(25)
It is no longer clear what integration limits have to be placed on $a$. Choosing them from $a$ to $a = 0$ gives the solutions

$$\Psi \sim \exp \pm \left[ \frac{1 + |\Lambda|a^2}{3|\Lambda|} - \frac{1}{3|\Lambda|} \right]$$

(26)

If we keep track of the signs then the $(+)$ one corresponds to the HH case and will be dominated by large

$$\sim \left[ \frac{1 + |\Lambda|a^2}{3|\Lambda|} - \frac{1}{3|\Lambda|} \right]$$

(27)

i.e. by $\sqrt{|\Lambda|a^3}$ large. This is the problem found by Lavrelashrili et. al.[27]

It is however uncertain that this makes any sense and is rather an artifact of the HH wavefunction been peaked around the exponentially increasing solution. cf. Fig.(2) in Ref.[26].

There is another reason to discount this solution. If the cosmological constant was absent the wave function would be

$$\Psi \sim \exp \left( \pm a^2/2 \right)$$

(28)

If we choose the $(+)$ sign there is a contradiction with our notions of classical behaviour since the universe would apparently prefer to have large size. Rather the other sign is more correctly peaked around $a = 0$.

The other $(-)$ sign solution in eq.(26) formally appears to predict $\Lambda \to 0$ if $a \neq 0$. But since there is no barrier to tunnel through the tunneling condition will simply imply that the universe stays at the origin $a = 0$ and $\Lambda$ is left undefined. One can seemingly obtain large or small $|\Lambda|$ depending on how one applies the boundary condition (the limits of integration in eq.(22) ). When considering a -ve $\Lambda$ it seems that we should conclude that the universe will wish to stay at the origin and no predictions about $\Lambda$ should be drawn from the factor $\exp \sqrt{|\Lambda|a^3}$. We limit our discussion to the case of positive values of $\Lambda$ again from now.

In this Euclidean region we have found that the possible values of $\Lambda$ depend

\footnote{If we had not subtracted the part corresponding to $a = 0$ we would also find a divergence when $|\Lambda| \to 0$}

\footnote{It is not necessary, as done in ref.[27] to include matter fields or to consider a third quantized theory to obtain this dominant $\sqrt{|\Lambda|a^3}$ factor.}
upon the choice of boundary conditions. This point has also been mentioned by Kiefer [29] in the context of wave packet solutions to the WDW equation. Because the choice of boundary conditions is not known a priori, it seems that to simply choose the boundary condition that solves the cosmological constant problem is merely to pass the problem down the line. What is required is a measure of solutions to the WDW equation which give either large or small \( \Lambda \). Any possible wave function

\[
\Psi \sim \alpha \exp(1/\Lambda) + \beta \exp(-1/\Lambda)
\]  

(\( \alpha, \beta \) arbitrary complex coefficients) will have the critical behaviour at \( \Lambda = 0 \) for \( \alpha \) small. However, it would make \( D >> 1 \) (defined in ref.[30]) but according to the measure given in ref.[30] a typical wavefunction has \( D < 1 \): it is therefore more like a tunneling one i.e. of the form \( \exp(-|S|) \). If this measure is correct it would suggest that a large value of \( \Lambda \) should be expected. Until a choice of boundary conditions can be justified, or the measure of solutions is known, we cannot know if such Baum-Hawking factors are important for determining the likely cosmological constant.

We consider next what happens when the universe starts in a Lorentzian region where the WKB wave functions have the oscillating behaviour \( \Psi \sim \exp(\pm iS) \). The exponents in the terms \( \exp(\pm iS) \) no longer have any critical influence, but instead the pre-factor contains any dominant behaviour. We do however have to exclude the Euclidean regime from the expression for the action cf. eq.(23) i.e. the lower limit in the integral is taken to be \( a = \Lambda^{-1/2} \). Otherwise we would simply introduce the factors \( \exp(\pm 1/\Lambda) \) again and reach similar conclusions.\footnote{For the same reason these factors appear when you wish to normalize the wavefunction as \( a \to 0 \) -see e.g.[31].}

Typically the wavefunction has the form

\[
\Psi \sim \frac{1}{a\sqrt{a^2\Lambda - 1}} \left( e^{iS} + e^{-iS} \right)
\]

(30)

There is a similar peak around \( a^2\Lambda \sim 1 \) as the WDW potential is zero. For \( a \) fixed and \( a^2\Lambda >> 1 \) then \( \Psi^2 \sim 1/\Lambda \) so that larger values of \( \Lambda \) are suppressed inversely. These are again initial conditions to be followed by
classical evolution, and it would appear correct to assume the quantum behavior made predictions for an initially small universe. The initial value of $\Lambda$ would therefore appear large which would produce an inflationary phase. This prediction occurs for both HH and tunneling boundary conditions since they only determine which combination of $\exp(iS)$ and $\exp(-iS)$ to take. There is a heuristic reason to see this: in the Lorentzian region the HH and tunneling solutions look almost alike (damped oscillations) and so should not differ much in their predictions. Contrast this with their behaviour in the Euclidean regime - see eg. Fig.(11.2) in ref. [12].

Similar predictions could be made if we considered a spatially open $k = -1$ model together with a -ve $\Lambda$. This has Lorentzian behaviour for small $a$ beyond which is a Euclidean regime. The peak would be around $a^2|\Lambda| \sim -k$. Notice how the spatial curvature is crucial for any predictions about $\Lambda$. If we set $k = 0$ then $P(\Lambda) \sim 1/(a^4\Lambda)$ and we would obtain the prediction that $\Lambda \to 0$, although without the exponential peak.

It might appear that this property $a^2\Lambda \sim 1$ is an artifact of using WKB solutions which are simply blowing up at the turning point $U \equiv a^2 - \Lambda a^4 = 0$. However exact solutions of the WDW equation can be found and this behaviour remains. For example the equation [33]

$$\left(\frac{d^2}{da^2} + \frac{p}{a} \frac{da}{d a} - U\right) \Psi(a) = 0$$ (31)

has solutions

$$\Psi \sim a^{1 - p/2} \left\{ J_{(1-p)/4}(\sqrt{-U}a) + Y_{(1-p)/4}(\sqrt{-U}a) \right\}$$ (32)

Since both $J_\nu(x)$ and $Y_\nu(x)$ both decrease for increasing $x$ they both take their maximum value when $x = 0$ and as we require $a > 0$ this occurs for $U = 0$, so again when $\Lambda \sim 1/a^2$. For $p \neq 1$ this is slightly modified $\sqrt{-U}a \sim small$.

The addition of additional matter fields is likely to round off this spike at $\Lambda \sim 1/a^2$ cf. ref.[34]. We see that the initial value of $\Lambda$ is expected to be large in the Lorentzian regime provided $a$ is not large in Planck units. If the

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5 We should perhaps be careful and say the prediction is not strongly dependent on boundary conditions since there might be more unusual ways of imposing them cf. Ref.[32]. Note the slight discrepancy with Cline[32] who using a Lorentzian path integral approach concluded that $P(\Lambda) \sim 1$ so that any $\Lambda$ is equally likely. In the Euclidean region he found that only special boundary conditions gave the $\exp(1/\Lambda)$ factor - agreeing with us.
initial size of the universe is ‘big’ ~ 1 cm the probable value of $\Lambda$ is smaller but still huge compared to its present value cf. Ref.[28].

Let us finally try to understand the Euclidean regime results (when $U \geq 0$) in terms of the solutions eq.(31). For factor ordering $p = 1$ the solutions simplify to

$$\Psi \sim K_0(aU^{1/2}) + I_0(aU^{1/2})$$

(33)

Since $K_0(x) \to \infty$ as $x \to 0$ it picks out the $U = 0$ or $\Lambda \sim 1/a^2$ case. The other Bessel function $I$ increases with increasing $aU^{1/2} \equiv a^2(1 - \Lambda a^2)^{1/2}$ so is maximized for $\Lambda = 0$, or negative $\Lambda$ if we allow it. This is as expected since the tunneling boundary condition is the decaying solution $K$ and the HH one a mixture of both $I$ and $K$ - see eg.ref.[35].

4 Conclusions

Using the Carlini-Mijić approach of fixing $\gamma$ we have found that quantum wormholes occur only for matter sources that obey the strong energy condition (the opposite to what is required for inflationary behaviour). The solutions obtained also obey the HP description for $\Psi$ to describe a wormhole: they are asymptotically Euclidean and well behaved (if we take a Fourier transform) as $a \to 0$. It might be argued that this has little to do with actual matter sources. For example a massive scalar field $V(\phi) = m^2\phi^2$ has a range $0 \leq \gamma \leq 2$ and so does not necessarily violate the condition $\gamma > 2/3$. This is why a quantum wormhole could still be obtained for the massive scalar field [7]. However if wormholes only occur for matter sources which obey the strong energy condition it seems a contradiction to invoke them to a situation which violates such conditions i.e. a $\Lambda$ term.

This requirement that $\gamma > 2/3$ is easier to satisfy than the corresponding condition for classical wormholes (that the Ricci tensor have negative eigenvalues [36]). But perhaps a still more general form of quantum wormhole is required which might exist even when the strong energy condition is violated. Otherwise the (3rd quantized) many universe approach, which is even more speculative could be considered.

Even the present HP definition of a quantum wormhole seems suspect since they can appear ‘more quantum’ Euclidean on large scales. We would instead expect Euclidean regimes only to occur on small ~ Planck scales when quantum gravity is expected to be important.
Instead of considering wormhole solutions of the WDW equation we took a simple model with only a cosmological constant present. In the Euclidean regime the initial value of $\Lambda$ is expected large if tunneling boundary conditions or tunneling like behaviour is correct. This would not be suitable for setting $\Lambda$ small but would allow for an Inflation regime to proceed. If we consider HH boundary conditions then you can get the factor $exp(1/\Lambda)$ (or if $\Lambda$ is -ve the anomalous $exp(\sqrt{|\Lambda|}a^3$ factor). The two behaviours are complementary and the occurrence of one would prevent the other: we could not have Inflation together with $\Lambda \to 0$ unless some other dynamical mechanism could give the two mechanisms differing time scales cf. Ref.[37].

We then considered the purely Lorentzian regime and found that the predictions in this case are not dependent on the boundary conditions. We found an initial value of $\Lambda \sim k/a^2$ and since we expect the universe to start small, due to quantum gravity processes, the corresponding value of $\Lambda$ is large. We are still left with the problem of why $\Lambda$ should be a dynamical variable with a distribution function.

If wormholes are not allowed when $\Lambda$ is present, it would be a contradiction to then invoke them to make $\Lambda$ a dynamical variable. The 3-form (axion) field might still work in this regard even if its wormhole solution cannot be invoked cf.ref.[2].

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5 References

1. S. Coleman, Nucl. Phys. B 310 (1988) p.643.

2. S. Weinberg, Rev. of Mod. Phys. 61 (1989) p.1.

3. E. Baum, Phys. Lett. 133 B (1983) p.185.

4. S.W. Hawking, Phys. Lett. 134 B (1983) p.403.

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6In this regard we essentially agree with the analysis of Strominger [28], except for his conclusion that this implies $\Lambda \to \sim 0$. 

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5. A. Carlini and M. Mijić, “Spacetime wormholes as analytic continuation of closed expanding universes” SISSA preprint 91A (1991).
   A. Carlini, preprint SISSA 65/92/A (1992).
   A. Carlini and M. Martellini, Class. Quan. Grav. 9 (1992) p.629.
6. G.F.R. Ellis and M.S. Madsen, Class. Quan. Grav. 8 (1991) p.667.
7. S.W. Hawking and D.N. Page, Phys. Rev. D 42 (1990) p.2655.
8. L.H. Ford, Phys. Lett. 110 A (1985) p.21.
9. A. Vilenkin, Phys. Rev. D 33 (1986) p.3560.
10. J.J. Halliwell, in Quantum Cosmology and Baby Universes eds. S. Coleman et al. (World Scientific, Singapore) 1991.
11. A.D. Linde, Particle Physics and Inflationary Cosmology (Harwood, Switzerland 1990).
12. E.W. Kolb and M.S. Turner, The Early Universe, (Addison- Wesley, USA 1990).
13. S.P Kim and D. N. Page, Phys. Rev. D 45 (1992) p.R3296.
14. A. Zhuk, Phys. Rev. D 45 (1992) p.1192.
15. L.J. Garay, Phys. Rev. D 44 (1991) p.1059.
16. C. Kiefer, Phys. Rev. D 38 (1988) p.1761.
17. C.M. Bender and S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, (McGraw-Hill, Singapore 1984).
18. J.B. Hartle and S.W. Hawking, Phys. Rev. D 28 (1983) p.2960.
19. A. Vilenkin, Phys. Rev. D 30 (1984) p.549
   A.D. Linde, Sov. Phys. JEPT 60 (1984) p.211
   V.A. Rubakov, Phys. Lett. 148 B (1984) p.280.
   Y.B. Zeldovich and A.A. Starobinski, Sov. Astron. Lett. (1984) p.135.
20. M.B. Mijić, M.S. Morris and W. Suen, Phys. Rev. D 39 (1989) p.1496
21. P.F. Gonzalez-Diaz, Mod. Phys. Lett. A 5 (1990) p.1307.
22. E. Merzbacher, Quantum Mechanics (Wiley, New York) 1961.
23. J.J. Halliwell and R. Laflamme, Class. Quan. Grav. 6 (1989) p.1839.
24. Y. Verbin and A. Davidson, Nucl. Phys. B 339 (1990) p.545.
25. V.A. Rubakov, Phys. Lett. 214 B (1988) p.503.
   A. Hosoya and M. Morikawa, Phys. Rev. D 39 (1989) p.1123.
26. A. Vilenkin, Phys. Rev. D 37 (1988) p.888.
27. G. Lavrelashrili, V.A. Rubakov and P.G. Tinyakov, in Gravitation and Quantum Cosmology eds. A. Zichichi et al. (Plenum Press, New York, 1991) p.87.
28. A. Strominger, Nucl. Phys. B 319 (1989) p.722.
29. C. Kiefer, Nucl. Phys. B 341 (1990) p. 273.
30. G.W. Gibbons and L.P. Grishchuk, Nucl. Phys. B 313 (1989) p.736.
   L.P. Grishchuk and Y.V. Sidorov, Sov. Phys. JEPT 67 (1988) p.1533.
31. E. Fahri, Phys. Lett. B 219 (1989) p.403.
32. J. M. Cline, Phys. Lett. 224 B (1989) p.53.
33. D.H. Coule, Class. Quan. Grav. 9 (1992) p.2353.
34. T. Banks, Nucl. Phys. B 309 (1988) p.493.
35. D.N. Page, in Proceedings of Banff summer research institute on Gravitation eds. R. Mann and P. Wesson (World Scientific, Singapore) 1991.
36. S.B. Giddings and A. Strominger, Nucl. Phys. B 306 (1988) p.890.
37. T. Fukuyama and M. Morikawa, Time dependence of Coleman-Hawking Mechanism Kyoto preprint (1988).
6 Figures

Fig. 1
An example from eq.(19) of a quantum wormhole solution. 
\( \Psi(a) \sim \exp(-a^2/2) \, _1F_1(-6; 1/2; a^2) \). This oscillation region with an exponential decay beyond is what we term type (I) behaviour. It corresponds to a matter source which do not have classical wormholes but satisfy the HP conditions. All Figures obtained using Mathematica.

Fig. 2
As the potential is reduced the Euclidean regime reaches the origin. We plot the modified Bessel function \( \Psi(a) \sim K_{1/2}(a^2) \). This is now the more restrictive type (II) behaviour which have classical wormholes.

Fig. 3
The Wheeler-DeWitt potential \( U \). The Euclidean regime has \( U \geq 0 \) beyond which it is Lorentzian. The tunneling boundary condition describes the decay from the origin to \( a = \Lambda^{-1/2} \).