On moments and strong local Hölder regularity of solutions of stochastic differential equations and of their spatial derivative processes

Anselm Hudde\textsuperscript{1}, Martin Hutzenthaler\textsuperscript{1}, and Sara Mazzonetto\textsuperscript{2}

\textsuperscript{1}Faculty of Mathematics, University of Duisburg-Essen, Germany
\textsuperscript{2}Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France

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Abstract

Spatial differentiability of solutions of stochastic differential equations (SDEs) is a classical question in stochastic analysis. The case of coefficients with globally Lipschitz continuous derivatives is well understood in the literature. Counterexamples with smooth and bounded coefficients demonstrate that the non-globally Lipschitz case is more subtle. In this article we establish conditions, including a suitable local monotonicity property, which provide existence of continuously differentiable solutions of SDEs, moment estimates and strong local Hölder regularity.

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1 Introduction

For a number of tools such as Taylor expansions, the Itô-Aleksiev-Gröbner formula in [9], or the backward Itô-Ventzell formula in [4] it is convenient to have a solution of a stochastic differentiable equation (SDE) which is continuously differentiable in the starting point.

In the literature it is well-known that spatially differentiable solutions of SDEs exist if the first derivatives of the coefficient functions exist and are globally Hölder continuous; see, e.g., [13, Theorem 4.6.5]. The classical approach is to prove with a Kolmogorov-Chentsov continuity argument that difference quotients have continuous versions and to infer from this the existence of differentiable solutions. We note that the proof of [13, Theorem 4.6.5] leaves the gap that [13, Theorem 4.6.5] does not directly guarantee existence of a continuous extension of the difference quotient at  through the domain (Proposition 2.2 below closes this gap) and the proof of [13, Theorem 4.6.5] does not explain how to get a differentiable solution from a continuous version of difference quotients (Lemma 2.1 below closes this gap). Moreover, we note that the conditions of [13, Theorem 1.4.1] are rarely satisfied by SDEs from applications; cf. examples in [2, Chapter 4].

In the non-globally Lipschitz case one might also hope for differentiable solutions if the coefficient functions are smooth. However, this is not the case. Example 2.6 in [15] shows that there exist SDEs with smooth and bounded coefficients for which there exists no solution which is continuous in the starting point. In particular, SDEs with smooth coefficients do not necessarily have differentiable solutions.

The main contribution of this article is to derive conditions which allow at the same time to ensure existence of continuously differentiable solutions of order 0, 1, or 2, to bound moments and to establish strong local Hölder regularity. Our central assumption is the local monotonicity-type assumption 1 below which seems to be new. Roughly speaking, this condition requires $\mu'$ and $\|\sigma'\|$ to be bounded from above in a suitable sense by a Lyapunov-type function where $\mu$ is the drift coefficient and $\sigma$ is the diffusion coefficient of the SDE. The following theorem, Theorem 1.1, illustrates our main results and is a special case of Corollary 5.5 below which in turn is derived from Theorem 5.5 below. The proof of Theorem 1.1 is therefore omitted. We note that Theorem 1.1 and Theorem 3.4 below in particular imply strong completeness of the SDE.

Theorem 1.1 (Existence of a $C^2$-solution). Let $d \in \mathbb{N}$, $T, c, \alpha \in (0, \infty)$, $p \in (6(d+3)(1+1/c)^3, \infty)$, let $\|\cdot\|$, $\langle \cdot, \cdot \rangle$ denote the standard norm and the standard scalar product on $\mathbb{R}^d$, let $\|\cdot\|_F$ denote the Frobenius norm on $\mathbb{R}^{d \times d}$, let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]})$ be a filtered probability space satisfying the usual conditions, let $W: [0,T] \times \Omega \to \mathbb{R}^d$ be a standard $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process, let $\mu \in C^3(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in C^3(\mathbb{R}^d, \mathbb{R}^{d \times d})$, $V \in C^2(\mathbb{R}^d, [0, \infty))$, $\tilde{V} \in C(\mathbb{R}^d, [0, \infty))$, assume that $\mu''$ and $\sigma''$ grow at most polynomially at infinity, assume that $\exists \gamma \in (0, \infty)$: $\sup_{x \in \mathbb{R}^d} \|x\|^{\gamma}/(1+V(x)) < \infty$, and assume for all $x, y, v \in \mathbb{R}^d$ that

$$
\left\langle v, \mu^\prime(\lambda(x-y)+y)\right\rangle + \frac{p-1}{2} \|\int_0^1 \sigma^\prime(\lambda(x-y)+y)\,d\lambda\right\|_F^2 \leq \|v\|^2 \cdot \left( c + \frac{\tilde{V}(x)+V(y)}{4pT^{\alpha/2}} + \frac{\tilde{V}(x)+V(y)}{4pT^{\alpha/2}} \right)
$$

$$
\left\langle \mu(x), (\nabla V)(x) \right\rangle + \frac{1}{2} \text{trace}\left( \sigma(x)[\sigma(x)^* \text{Hess} V](x) \right) + \frac{i}{2} \|\sigma(x)^*(\nabla V)(x)\|^2 + \tilde{V}(x) \leq \alpha V(x) + c.
$$

(1)

Then there exists a measurable function $X: \{ (s,t) \in [0,T]: s \leq t \} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ such that

(i) for every $\omega \in \Omega$ it holds that $X(\omega) \in C^{0,2}[\{ (s,t) \in [0,T]: s \leq t \} \times \mathbb{R}^d, \mathbb{R}^d]$ and

(ii) for all $x \in \mathbb{R}^d$, $s \in [0,T]$, $t \in [s,T]$ it holds a.s. that

$$
X_{s,t}^x = x + \int_s^t \mu(X_{s,r}^x)\,dr + \int_s^t \sigma(X_{s,r}^x)\,dW_r.
$$

1'usual conditions' means that for all $t \in [0,T]$ it holds that $\{A \in \mathcal{F}: \mathbb{P}(A) = 0\} \subseteq \mathcal{F}_t = \cap_{s \in (t,T] \cup \{T\}} \mathcal{F}_s$. 

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The paper is organized as follows. In Section 2 we provide preliminary results used for the proofs of the main results. In Sections 3, 4, and 5 we establish existence of $C^0$, $C^1$, and $C^2$ solutions respectively of SDEs using suitable strong local Hölder estimates obtained in Subsections 3.1, 4.1, and 5.1.

2 Preliminary results

In Subsection 2.1 we prove that if there exists a continuous version of difference quotients, then there exists a differentiable version. Moreover, we recall for the convenience of the reader results from the literature which are used in our proofs of existence of solutions of SDEs which are differentiable in the initial value. More precisely, we will use in our proofs a local Kolmogorov-Chentsov continuity theorem (see Subsection 2.2 below), a stochastic Gronwall lemma (see Subsection 2.3 below), and exponential moment estimates (see Subsection 2.4 below).

2.1 Inferring a differentiable version from continuity of difference quotients

The following lemma shows, informally speaking, that if the difference quotient of a continuous random field has a continuous version, then the random field is differentiable almost surely.

Lemma 2.1 (Continuity of the difference quotient implies differentiability). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(H, \langle \cdot, \cdot \rangle_H, \| \|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \| \|_U)$ be separable $\mathbb{R}$-Hilbert spaces, assume that $H$ is finite-dimensional, let $\mathbb{H} \subseteq H$ be an orthonormal basis of $H$, let $O \subseteq H$ be an open subset, let $D \subseteq O$ be a countable dense subset, let $(T, T)$ be a topological space, let $\mathbb{T} \subseteq T$ be a countable dense subset, let $O \subseteq H \times \mathbb{R}$ be the set $O = \cap_{h \in \mathbb{H}} \{(x, p) \in O \times \mathbb{R} : x + hp \in O\}$, and let $X : T \times O \times \Omega \to U$ and $Z : T \times O \times H \times \Omega \to U$ be random fields satisfying that for all $(x, p) \in O \cap \{(D \times (\mathbb{Q} \setminus \{0\}))$, $h \in \mathbb{H}$, $t \in \mathbb{T}$ it holds a.s. that

$$Z_t(x, p, h) = \frac{X_t^{x+ph} - X_t^x}{p}$$

and satisfying that for all $h \in \mathbb{H}$ and almost all $\omega \in \Omega$ it holds that the functions $T \times O \ni (t, x) \mapsto X_t^x(\omega) \in U$ and $T \times O \ni (t, x, p) \mapsto Z_t(x, p, h, \omega) \in U$ are continuous. Then there exists a set $\Omega_0 \in \mathcal{F}$ such that

(i) $\mathbb{P}(\Omega_0) = 1$, and

(ii) it holds for all $\omega \in \Omega_0$ and all $t \in T$ that the mapping $O \ni x \mapsto X_t^x(\omega) \in U$ is continuously differentiable and it holds for all $\omega \in \Omega_0$, $t \in T$, $x \in O$, $v \in H$ that $\frac{\partial}{\partial x} X_t^x(\omega)v = \sum_{h \in \mathbb{H}} \langle v, h \rangle_H Z_t(x, 0, h, \omega)$.

Proof of Lemma 2.1. Without loss of generality we assume that $H \neq \{0\}$ and that $T \neq \emptyset$. Throughout this proof let $d \in \mathbb{N}$ denote the dimension of $H$ and let $h_1, \ldots, h_d \in H$ satisfy that $\{h_1, \ldots, h_d\} = \mathbb{H}$. By assumption there exists a set $\Omega_1 \in \mathcal{F}$ satisfying that $\mathbb{P}(\Omega_1) = 1$ and that for all $\omega \in \Omega_1$, $h \in \mathbb{H}$ it holds that the functions $T \times O \ni (t, x) \mapsto X_t^x(\omega) \in U$ and $T \times O \ni (t, x, p) \mapsto Z_t(x, p, h, \omega) \in U$ are continuous. Let $\Omega_0 \subseteq \Omega$ be the set satisfying that

$$\Omega_0 = \Omega_1 \cap \bigcap_{(x, p) \in O \cap (D \times \mathbb{Q})} \bigcap_{h \in \mathbb{H}} \bigcap_{t \in \mathbb{T}} \left\{ \omega \in \Omega : X_t^{x+ph}(\omega) - X_t^x(\omega) = p Z_t(x, p, h, \omega) \right\}.$$

The fact that $X$, $Z$ are random fields, the fact that for all $(x, p) \in O \cap (D \times \mathbb{Q})$, $h \in \mathbb{H}$, $t \in \mathbb{T}$ it holds a.s. that $p Z_t(x, p, h) = X_t^{x+ph} - X_t^x$, the fact that $D \times \mathbb{Q} \times \mathbb{H} \times \mathbb{T}$ is a countable set, the fact that $\Omega_1 \in \mathcal{F}$, and the fact that $\mathbb{P}(\Omega_1) = 1$ imply that $\Omega_0 \in \mathcal{F}$ and that $\mathbb{P}(\Omega_0) = 1$. This proves item (i).
Next, we prove item (iii). For this we first observe that density of $\mathbb{T}$ in $T$ and the fact that for all $h \in \mathbb{H}$, $\omega \in \Omega_1$ the functions $T \times O \ni (t, x) \mapsto X_t^x(\omega) \in U$ and $T \times O \ni (t, x, p) \mapsto Z_t(x, p, h, \omega) \in U$ are continuous imply that

$$\Omega_0 = \Omega_1 \cap \bigcap_{(x,p)\in O} \bigcap_{h \in \mathbb{H}} \bigcap_{t \in T} \left\{ \omega \in \Omega : X_t^{x,p,h}(\omega) - X_t^x(\omega) = pZ_t(x, p, h, \omega) \right\}. \quad (5)$$

For the rest of the proof, let $\omega \in \Omega_0$, $t \in T$, $x \in O$, and $r \in (0, \infty)$ with $\{ y \in H : \|y-x\| < 2r \} \subseteq O$ be fixed. Now for all $v \in H$, $i \in \{1, \ldots, d\}$ with $\|v\|_H < r$ it holds that

$$\|x + \sum_{j=1}^{i-1} \langle v, h_j \rangle_H h_j - x\|_H^2 + |\langle v, h_i \rangle_H|^2 = \sum_{j=1}^{i} (\langle v, h_j \rangle_H)^2 \leq \|v\|_H^2 < r^2. \quad (6)$$

This, a telescoping sum, the fact that $\|\cdot\|_H$ be a separable $\mathbb{H}$ for all $y, p \in \Omega$, and (iii) yield that for all $v \in H$ with $\|v\|_H < r$ it holds that

$$X_t^{x+v}(\omega) - X_t^x(\omega) - \sum_{h \in \mathbb{H}} \langle v, h \rangle_H Z_t(x, 0, h, \omega)
= \sum_{i=1}^{d} \left( X_t^{x+\sum_{j=1}^{i-1} \langle v, h_j \rangle_H h_j}(\omega) - X_t^{x+\sum_{j=1}^{i-1} \langle v, h_j \rangle_H h_j}(\omega) \right) - \sum_{i=1}^{d} \langle v, h_i \rangle_H Z_t(x, 0, h_i, \omega)
= \sum_{i=1}^{d} \langle v, h_i \rangle_H Z_t \left( x + \sum_{j=1}^{i-1} \langle v, h_j \rangle_H h_j \right) - \sum_{i=1}^{d} \langle v, h_i \rangle_H Z_t \left( x, 0, h_i, \omega \right). \quad (7)$$

This, the triangle inequality, and the Cauchy-Schwarz inequality show that for all $v \in H$ with $\|v\|_H \in (0, r)$ it holds that

$$\frac{\|X_t^{x+v}(\omega) - X_t^x(\omega) - \sum_{h \in \mathbb{H}} \langle v, h \rangle_H Z_t(x, 0, h, \omega)\|_U}{\|v\|_H}
= \frac{1}{\|v\|_H} \left| \left| \sum_{i=1}^{d} \langle v, h_i \rangle_H \left( Z_t \left( x + \sum_{j=1}^{i-1} \langle v, h_j \rangle_H h_j \right) - Z_t \left( x, 0, h_i, \omega \right) \right) \right|_U \right|
\leq \frac{\|v\|_H}{\|v\|_H} \left( \sum_{i=1}^{d} \left| \left| Z_t \left( x + \sum_{j=1}^{i-1} \langle v, h_j \rangle_H h_j \right) - Z_t \left( x, 0, h_i, \omega \right) \right|_U^2 \right)^{1/2} - r. \quad (8)$$

This, the fact that $H$ is finite-dimensional, and the fact that for all $h \in \mathbb{H}$ the function $O \ni (y, p) \mapsto Z(y, p, h, \omega) \in U$ is continuous in $(x, 0)$ yield that

$$\lim_{H \ni x \to 0} \frac{\|X_t^{x+v}(\omega) - X_t^x(\omega) - \sum_{h \in \mathbb{H}} \langle v, h \rangle_H Z_t(x, 0, h, \omega)\|_U}{\|v\|_H} = 0. \quad (9)$$

This together with continuity of the function $O \ni y \mapsto (H \ni v \mapsto \sum_{h \in \mathbb{H}} \langle v, h \rangle_H Z_t(y, 0, h, \omega) \in U) \in L(H, U)$ proves item (iii) and thus completes the proof of Lemma 2.1. \hfill \square

### 2.2 A local Komogorov-Chentsov continuity theorem

For the convenience of the reader the following proposition, Proposition 2.2, reformulates part of Corollary 3.12 in [2] which is a version of the Komogorov-Chentsov continuity theorem.

**Proposition 2.2** (A local Komogorov-Chentsov continuity theorem). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be a finite-dimensional $\mathbb{R}$-Hilbert space, let $D \subseteq H$ be a set, let $(E, \|\cdot\|_E)$ be a separable $\mathbb{R}$-Banach space, let $F \subseteq E$ be a closed subset, let $p \in (\dim(H), \infty)$,
\[ \alpha \in \left( \frac{\dim(H)}{p}, 1 \right], \text{ and let } X: D \times \Omega \to E \text{ be a random field which satisfies for all } n \in \mathbb{N}, \; z \in D \text{ that} \]
\[ \mathbb{P}(X(z) \in F) = 1, \; \mathbb{E}[\|X(z)\|_E^p] < \infty, \text{ and that} \]
\[ \sup \left\{ \left( \frac{\mathbb{E}[\|X(x) - X(y)\|_E^p]}{\|x-y\|_H^p} \right)^{\frac{1}{p}} : x, y \in D, \|x\|_H \leq n, \|y\|_H \leq n, x \neq y \right\} \cup \{0\} < \infty. \]

Then there exists a measurable function \( X: \Omega \times \Omega \to F \) which satisfies

(i) that for all \( \omega \in \Omega \) it holds that the function \( \mathcal{D} \ni x \mapsto X(x, \omega) \in F \) is continuous and

(ii) that for all \( x \in D \) it holds a.s. that \( X(x) = X(x) \).

### 2.3 A stochastic Gronwall inequality

For the convenience of the reader the following proposition, Proposition 2.3, reformulates part of Corollary 2.5 in [10]. It will be used to prove strong local Hölder estimates for difference quotients.

**Proposition 2.3 (A stochastic Gronwall inequality).** Let \( (H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H) \) and \( (U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U) \) be separable \( \mathbb{R} \)-Hilbert spaces, let \( p \in [2, \infty) \), \( T \in (0, \infty) \), let \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}) \) be a filtered probability space satisfying the usual conditions, let \( (W_t)_{t \in [0,T]} \) be an \( \mathbb{R}^d \)-cylindrical \( (\mathcal{F}_t)_{t \in [0,T]} \)-Wiener process, let \( X, a: [0, T] \times \Omega \to H, \; b: [0, T] \times \Omega \to HS(U,H), \; \alpha, \beta: [0, T] \times \Omega \to \mathbb{R} \) be measurable and adapted stochastic processes which satisfy that \( X \) has continuous sample paths, which satisfy that for all \( t \in [0, T] \) it holds a.s. that \( \int_0^T \|a_s\|_H + \|b_s\|_{HS(U,H)}^2 + |\alpha_s| ds < \infty \) and \( X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dW_s \), and which satisfy that it holds a.s. for Lebesgue-almost all \( t \in [0, T] \) that

\[ \left( X_t, q_t \right)_H + \left( \frac{1}{2} \|b_t\|_{HS(U,H)}^2 + \frac{p-2}{2} \frac{\|X_t \|_{\text{HS}(U,H)}^2}{\|X_t \|_H^2} \right) \leq \alpha_t \|X_t \|_H^2 + \frac{1}{2} \beta_t^2. \]

Then it holds for all \( t \in [0, T], \; q_1, q_2 \in (0, \infty) \) with \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{p} \) that

\[ \left\| X_t \right\|_{L^{q_1}(\mathbb{P}; H)} \leq \left\| \exp \left( \int_0^t \alpha_u \, du \right) \right\|_{L^{q_2}(\mathbb{P}; \mathbb{R})} \left( \left\| X_0 \right\|_{L^{q_1}(\mathbb{P}; H)}^2 + \int_0^t \beta_s^2 \, ds \right)^{\frac{1}{q_2}}. \]

### 2.4 Exponential moment estimates

In this subsection we collect two results from the literature which formalize a Lyapunov-method to derive (exponential) moment estimates. We will use these estimates to prove condition (10) for suitable difference quotients. In this subsection we frequently use the following setting.

**Setting 2.4.** Let \( (H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H) \) and \( (U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U) \) be separable \( \mathbb{R} \)-Hilbert spaces, let \( T \in [0, \infty) \), \( s \in [0, T] \), let \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}) \) be a filtered probability space satisfying the usual conditions, let \( W: [s, T] \times \Omega \to U \) be an adapted stochastic process such that \( (W_{s+t} - W_s)_{t \in [0,T-s]} \) is an \( \mathbb{R}^d \)-cylindrical \( (\mathcal{F}_{s+t})_{t \in [0,T-s]} \)-Wiener process, let \( O \subseteq H \) be an open set, let \( O \subseteq B(O) \), let \( \mu: O \to H \) and \( \sigma: O \to HS(U,H) \) be measurable functions, and let \( X: [s, T] \times \Omega \to \mathbb{R} \) be an adapted stochastic process with continuous sample paths which satisfies that for all \( t \in [s, T] \) it holds a.s. that \( \int_s^t \|\mu(X_r)\|_H + \|\sigma(X_r)\|_{HS(U,H)}^2 \, dr < \infty \) and \( X_t = X_s + \int_s^t \mu(X_r) \, dr + \int_s^t \sigma(X_r) \, dW_t \).

The next result, Lemma 2.5, is a slight generalization of Corollary 2.4 in [2] to arbitrary nonnegative starting times.

**Lemma 2.5 (Exponential moment estimates).** Assume Setting 2.4, let \( \alpha, \beta \in \mathbb{R}, \; V \in C^2(O, \mathbb{R}), \) and let \( V: [s, T] \times O \to \mathbb{R} \) be a measurable function which satisfies that it holds a.s. that
Then stochastic processes with continuous sample paths satisfying that for all
\[ \int_0^t \sigma(x) \, dW_s \leq \alpha V(x) + \beta, \]

(13)

Then
\[
\mathbb{E} \left[ \exp \left( \frac{V(X_t)}{e^{\alpha t}} + \int_0^t \frac{\beta}{e^{\alpha t}} \, ds \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{V(X_s)}{e^{\alpha s}} + \int_0^s \frac{\beta}{e^{\alpha s}} \, ds \right) \right] \in [0, \infty].
\]

(14)

Proof of Lemma 2.7. Corollary 3.3 in [10] (applied in the case \( T > s \) with \( T \cap T - s, \ (F_t)_{t \in [0,T-s]} \cap (F_{t+s})_{t \in [0,T-s]}, \ (W_t)_{t \in [0,T-s]} \cap (W_{t+s} - W_s)_{t \in [0,T-s]}, \ (X_t)_{t \in [0,T-s]} \cap (X_{t+s})_{t \in [0,T-s]}, \ U \cap \left( \left( [0,T-s] \times \Omega \right) \ni (t,x) \mapsto e^{-\alpha s} V(t+s,x) - e^{-\alpha s} \beta \in \mathbb{R} \right), \ U \cap (O \ni x \mapsto e^{-\alpha s} V(x) \in \mathbb{R}) \right) \) in the notation of Corollary 3.3 in [10] implies (14). This proves Lemma 2.7.

The next result, Lemma 2.6, is a consequence of Corollary 3.4 in [10].

Lemma 2.6 (Exponential moment condition implies moments). Assume Setting 2.4 let \( \alpha, \beta \in [0, \infty) \), \( V \in C^2(\Omega, [0, \infty)) \) satisfy for all \( x \in \Omega \) that
\[
\mathbb{E} \left[ \exp \left( \frac{V(X_t)}{e^{\alpha t}} + \int_0^t \frac{\beta}{e^{\alpha t}} \, ds \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{V(X_s)}{e^{\alpha s}} + \int_0^s \frac{\beta}{e^{\alpha s}} \, ds \right) \right] \in [0, \infty].
\]

(14)

Proof of Lemma 2.6. Corollary 3.4 in [10] (applied in the case \( T > s \) with \( T \cap T - s, \ (F_u)_{u \in [0,T-s]} \cap (F_{u+s})_{u \in [0,T-s]}, \ (W_u)_{u \in [0,T-s]} \cap (W_{u+s} - W_u)_{u \in [0,T-s]}, \ \mu \cap \left( \left( [0,T-s] \times \Omega \right) \ni (t,x) \mapsto \sigma(x) \in \text{HS}(U,H) \right), \ \tau \cap T - s, \ (X_u)_{u \in [0,T-s]} \cap (X_{u+s})_{u \in [0,T-s]}, \ U \cap \left( \left( [0,T-s] \times \Omega \right) \ni (t,x) \mapsto e^{-\alpha s} e^{-\alpha t} V(t,s) \in [0, \infty) \right), \ \beta \ni \beta e^{-\alpha s} \right. \) in the notation of Corollary 3.4 in [10] implies that
\[
\mathbb{E} \left[ \exp \left( \frac{V(X_t)}{e^{\alpha t}} + \int_0^t \frac{\beta}{e^{\alpha t}} \, ds \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{V(X_s)}{e^{\alpha s}} + \int_0^s \frac{\beta}{e^{\alpha s}} \, ds \right) \right] \in [0, \infty].
\]

(14)

This proves (14) and finishes the proof of Lemma 2.6.

The following result, Lemma 2.7, generalizes the \( k = 1 \) case of [2, Lemma 2.23] which is a special case of Lemma 2.7 with \( H = \mathbb{R}^d \), \( s = 0 \), \( V(t, \cdot) \equiv V(\cdot) \) for all \( t \in [0, T] \), \( X^{(1)} = X^x \), \( X^{(3)} = X^x \), \( X^{(2)} = X^y \), and \( X^{(4)} = X^y \) for \( d \in \mathbb{N}, x, y \in \Omega \).

Lemma 2.7. Assume Setting 2.4 let \( X(j) : [s,T] \times \Omega \rightarrow \mathbb{R}^d, j \in \{1, 2, 3, 4\} \) be \( (F_t)_{t \in [s,T]} \)-adapted stochastic processes with continuous sample paths satisfying that for all \( j \in \{1, 2, 3, 4\} \), \( t \in [s,T] \) it holds a.s. that
\[
\int_s^T \sigma(x) \, dW_r \leq \alpha V(x) + \beta, \]

(18)
let \( q, q_0, q_1 \in (0, \infty) \) satisfy \( \frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{q} \), and let \( \phi: [s, T] \to \mathbb{R} \) be a measurable function with \( \int_s^T |\phi(r)| \, dr < \infty \). Then it holds that

\[
\left\| \exp \left( \int_s^T \left( \phi(r) + \sum_{j=1}^d \left[ \frac{V_0(X_j(\cdot))}{4q_0(T-r)^{q_0}} + \frac{V(r,X_j(\cdot))}{4q_1 r^{q_1}} \right] \right) \, dr \right\|_{L^s(P; \mathbb{R})} \\
\leq \exp \left( \int_s^T \left( \phi(r) + \frac{\beta_0 (1 - \frac{2}{q_0})}{q_0 r^{q_0}} + \frac{\beta_0}{q_1 r^{q_1}} \right) \, dr \right) \prod_{i=0}^d \prod_{j=1}^d \left\| \exp \left( \frac{V_i(X_j(\cdot))}{4q_i r^{q_i}} \right) \right\|_{\mathcal{L}^4_q(P; \mathbb{R})}.
\]

\( (19) \)

Proof of Lemma 2.7. Without loss of generality we assume for the rest of the proof that \( \phi \equiv 0 \), otherwise divide by \( \exp(\int_s^T \phi(r) \, dr) \in (0, \infty) \). Hölder’s inequality together with \( \frac{1}{q} = 4 \frac{1}{4q_0} + 4 \frac{1}{4q_1} \), the fact that \( \int_s^T \frac{\beta_0 (1 - \frac{2}{q_0})}{q_0 r^{q_0}} \, dr = \int_s^T \frac{\beta_0}{q_0 r^{q_0}} \, du \), Jensen’s inequality, nonnegativity of \( V_1 \), and Lemma 2.5 (applied for every \( j \in \{1, 2, 3, 4\}, t \in [s, T] \) with \( T \wedge t, \alpha \wedge \alpha_0, \beta \wedge \beta_0, V \wedge V_0, V_0 \wedge V_0, V \wedge V_1, V_0 \wedge V_1 \) and \( X \wedge X^j \) in the notation of Lemma 2.5) show that

\[
\left\| \exp \left( \int_s^T \sum_{j=1}^d \left[ \frac{V_0(X_j(\cdot))}{4q_0(T-s)^{q_0}} + \frac{V(r,X_j(\cdot))}{4q_1 r^{q_1}} \right] \, dr \right) \right\|_{L^s(P; \mathbb{R})} \\
\leq \prod_{j=1}^d \left[ \left\| \exp \left( \int_s^T \left( \frac{1}{T-s} \int_s^T \frac{V_0(X_j(\cdot))}{4q_0} + \frac{V(r,X_j(\cdot))}{4q_1 r^{q_1}} \, du \right) \, dr \right) \right\|_{L^4_{q_0}(P; \mathbb{R})} \right]
\cdot \left[ \left\| \exp \left( \int_s^T \frac{V(r,X_j(\cdot))}{4q_1 r^{q_1}} \, dr \right) \right\|_{L^4_{q_1}(P; \mathbb{R})} \right]
\leq \prod_{j=1}^d \left[ \sup_{t \in [s, T]} \left( \mathbb{E} \left[ \exp \left( \int_s^T \frac{V_0(X_j(\cdot))}{4q_0} - \int_s^T \frac{\beta_0}{q_0 r^{q_0}} \, du \right) \right] \right)^{\frac{1}{4q_0}} \right]^{\frac{1}{4q_0}} \cdot \left[ \mathbb{E} \left[ \exp \left( \frac{V_1(X_j(\cdot))}{4q_1 r^{q_1}} \right) \right]^{\frac{1}{4q_1}} \right]^{\frac{1}{4q_1}}
\leq \prod_{j=1}^d \left[ \left( \mathbb{E} \left[ \exp \left( \frac{V_0(X_j(\cdot))}{4q_0} \right) \right] \right)^{\frac{1}{4q_0}} \left( \mathbb{E} \left[ \exp \left( \frac{V_1(X_j(\cdot))}{4q_1 r^{q_1}} \right) \right] \right)^{\frac{1}{4q_1}} \right] \prod_{i=0}^d \prod_{j=1}^d \left\| \exp \left( \frac{V_i(X_j(\cdot))}{4q_i r^{q_i}} \right) \right\|_{\mathcal{L}^4_q(P; \mathbb{R})}.
\]

\( (20) \)

This implies \( (19) \). The proof of Lemma 2.7 is thus completed. \( \square \)

3 Existence of a \( C^0 \)-solution

In this section we prove a strong local Hölder estimate for solutions of SDEs in Lemma 3.2 below, a moment estimate for the first derivative process in Lemma 3.3 below, and establish existence of a continuous solution under suitable assumptions in Theorem 3.4 below. First, we introduce the setting for these results.

Setting 3.1. Let \( (H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H) \) and \( (U, \langle \cdot, \cdot \rangle_U, \| \cdot \|_U) \) be separable \( \mathbb{R} \)-Hilbert spaces, let \( T \in (0, \infty) \), let \( O \subseteq H \) be an open set, let \( O \subseteq B(O) \), let \( \mu \in C(O, H), \sigma \in C(O, \operatorname{HS}(U, H)) \), let \( \alpha_0, \alpha_1, \beta_0, \beta_1 \in [0, \infty) \), \( V_0, V_1 \in C^2(O, [0, \infty)) \), let \( V: [0, T] \times O \to [0, \infty) \) be a measurable function, assume for all \( i \in \{0, 1\}, t \in [0, T], x \in O \) that

\[
\langle \mu(x), (\nabla V_i(x)) \rangle_H + \frac{1}{2} \text{tr} \left( \sigma(x)[\sigma(x)]^* (\text{Hess} V_i(x)) \right) \\
+ \frac{1}{2e^{\alpha r}} \| \sigma(x)[\sigma(x)]^* (\nabla V_i(x)) \|^2_H + 1(1) (i) \cdot V(t, x) \leq \alpha_i V_i(x) + \beta_i,
\]

(21)
let $\phi : [0, T] \to [0, \infty)$ be a measurable function satisfying that $\int_0^T \phi(t) \, dt < \infty$, let $p \in [2, \infty)$, $\theta \in [0, \infty)$, $q_0, q_1 \in (0, \infty)$ satisfy $\frac{\theta}{p(1+\theta)} = \frac{1}{q_0} + \frac{1}{q_1}$, assume for all $t \in [0, T]$, $x, y \in \mathcal{O}$ that

$$
\langle x - y, \mu(x) - \mu(y) \rangle_H + \| \sigma(x) - \sigma(y) \|_{HS(U,H)}^2 \leq \| x - y \|_H^2 \cdot \left( \phi(t) + \frac{V_0(x) + V_0(y)}{2q_0 t^{1-e^{\alpha t}}} + \frac{V_1(x) + V_1(y)}{2q_1 t^{1-e^{\beta t}}} \right),
$$

let $\gamma \leq [\frac{1}{p}, \infty)$, $c \in (0, \infty)$ satisfy for all $x \in \mathcal{O}$ that

$$\max \left\{ \| \mu(x) \|_H, \| \sigma(x) \|_{HS(U,H)} \right\} \leq c (1 + V_0(x))^\gamma,$$

let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$ be a filtered probability space satisfying the usual conditions, let $(W_t)_{t \in [0,T]}$ be an $\text{Id}_U$-cylindrical $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process, for all $s \in [0, T]$, $x \in \mathcal{O}$ let $X^x_{s,:} : [s, T] \times \Omega \to \mathcal{O}$ be an $(\mathcal{F}_t)_{t \in [s,T]}$-adapted stochastic process with continuous sample paths satisfying that for all $t \in [s, T]$ it holds a.s. that $\int_s^T (\tilde{V}(r, X^x_{s,r}) \, dr < \infty$ and

$$X^x_{s,t} = x + \int_s^t \mu(X^x_{s,r}) \, dr + \int_s^t \sigma(X^x_{s,r}) \, dW_r,$$

and let $\Delta_T = \{(s, t) \in [0, T]^2 : s \leq t\}$.

### 3.1 Strong local Hölder estimate

The following lemma proves strong local Hölder continuity of SDE solutions in the starting point, starting time, and terminal time. Lemma 3.2 improves existing results in [2, 3, 4, 8, 10].

**Lemma 3.2 (Strong local Hölder estimate).** Assume Setting 3.2 and let $s_1, s_2, t_1, t_2 \in [0, T]$, $x_1, x_2 \in \mathcal{O}$ satisfy that $s_1 \leq t_1$, $s_2 \leq t_2$ and $s_1 \leq s_2$. Then it holds that

$$
\| X^x_{s_1,t_1} - X^x_{s_2,t_2} \|_{L^p(\mathbb{P}; H)} \leq \sqrt{|t_1 - t_2|} \cdot \left( \frac{p \gamma}{e^{\theta \alpha T}} \right) p \gamma + e^{-\theta \alpha s_1} V_0(x_1) + \int_0^T \frac{V_0(x_1) + V_0(x_2)}{2q_0 t^{1-e^{\alpha T}}} \, du \right) \gamma \left( \sqrt{T} + p \right) + \| x_1 - x_2 \|_H \exp \left( \int_0^T \left( \phi(r) + \frac{\beta_0}{q_0 t^{1-e^{\alpha T}}} + \frac{\beta_1}{q_1 t^{1-e^{\beta T}}} \right) \, dr + \sum_{i=0}^{\infty} \frac{V_i(x_1) + V_i(x_2)}{2q_i t^{1-e^{\gamma T}}} \right)
$$

and $c \in (0, \infty)$ satisfy for all $x \in \mathcal{O}$ that

$$\max \left\{ \| \mu(x) \|_H, \| \sigma(x) \|_{HS(U,H)} \right\} \leq c (1 + V_0(x))^\gamma,$$

and $\Delta_T = \{(s, t) \in [0, T]^2 : s \leq t\}$.

### Proof of Lemma 3.2

**Proof of Lemma 3.2.** Lemma 3.8 in [10] (applied in the case $s_1 < T$ with $T \cap T - s_1 \cap (\mathcal{F}_t)_{t \in [0,T-s_1]} \cap (\mathcal{F}_{s_1+t})_{t \in [T-T-s_1]}$, $(W_t)_{t \in [0,T-s_1]} \cap (W_{s_1+t} - W_{s_1})_{t \in [0,T-s_1]}$, $\mu \cap ([0, T - s_1] \cdot \mathcal{O} \ni (r, x) \mapsto \mu(x) \in H)$, $\sigma \cap ([0, T - s_1] \cdot \mathcal{O} \ni (r, x) \mapsto \sigma(x) \in HS(U,H)$, $\tau \cap T - s_1, (X_t)_{t \in [0,T-t_1]} \cap (X_{s_1+t_1})_{t \in [0,T-t_1]}$, $(V_t)_{t \in [0,T-t_1]} \cap (V_{s_1+t_1})_{t \in [0,T-t_1]}$, $\beta_0 \cap e^{-\theta \alpha s_1}, \beta_1 = e^{-\theta \alpha s_1}, V_0 \cap (\mathcal{O} \ni x \mapsto e^{-\theta \alpha s_1} V_0(x) \in [0, \infty), V_1 \cap (\mathcal{O} \ni x \mapsto e^{-\theta \alpha s_1} V_1(x) \in [0, \infty), V \cap ([0, T - s_1] \cdot \mathcal{O} \ni (r, x) \mapsto e^{-\theta \alpha s_1} V(s_1+t, x) \in \mathbb{R}$, $\phi \cap (0, T - s_1) \ni t \mapsto \phi(s_1 + t) \in [0, \infty)$, $p \cap p(1 + \theta), q \cap p(1 + 1/\theta), q_0 \cap q_0, q_1 \cap q_1, t_1 \cap t_1 - s_1, t_2 \cap t_2 - s_1$ in the notation of Lemma 3.8 in [10] implies that

$$
\| X^x_{s_1,t_1} - X^x_{s_2,t_2} \|_{L^p(\mathbb{P}; H)} \leq \sqrt{|t_1 - t_2|} \cdot \left( \frac{p \gamma}{e^{\theta \alpha T}} \right) p \gamma + e^{-\theta \alpha s_1} V_0(x_1) + \int_0^T \frac{V_0(x_1) + V_0(x_2)}{2q_0 t^{1-e^{\alpha T}}} \, du \right) \gamma \left( \sqrt{T} + p \right) + \| x_1 - x_2 \|_H \exp \left( \int_0^T \left( \phi(r) + \frac{\beta_0}{q_0 t^{1-e^{\alpha T}}} + \frac{\beta_1}{q_1 t^{1-e^{\beta T}}} \right) \, dr + \sum_{i=0}^{\infty} \frac{V_i(x_1) + V_i(x_2)}{2q_i t^{1-e^{\gamma T}}} \right).
$$



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The fact that $\gamma \geq \frac{1}{p} \geq \frac{1}{p(1+\theta)}$ implies

\begin{equation}
\|X_{s_1,s_2}^{x_2} - x_2\|_{L^p(\mathbb{P}; H)} = \|X_{s_1,s_2}^{x_2} - X_{s_1,s_1}^{x_2}\|_{L^p(\mathbb{P}; H)}
\leq ce^{\alpha_0\gamma(s_2-s_1)}\left|p(1+\theta)\gamma + e^{-\alpha_0 s_1}V_0(x_2) + \int_{s_2-s_1}^{s_1} e^{-\alpha_0 s_1} du\right| (\sqrt{s_2 - s_1} + p(1+\theta))\sqrt{s_2 - s_1}.
\end{equation}

Moreover, the fact that for all $t \in [0, T - s_2]$ it holds a.s. that

\begin{equation}
X_{s_1,s_2+t}^{x_2} = X_{s_1,s_2}^{x_2} + \int_0^t \mu(X_{s_1,s_2+r}^{x_2}) dr + \int_0^t \sigma(X_{s_1,s_2+r}^{x_2}) d(W_{s_2+r} - W_{s_2})
\end{equation}

and that

\begin{equation}
X_{s_2,s_2+t}^{x_2} = x_2 + \int_0^t \mu(X_{s_2,s_2+r}^{x_2}) dr + \int_0^t \sigma(X_{s_2,s_2+r}^{x_2}) d(W_{s_2+r} - W_{s_2}),
\end{equation}

Lemma 3.8 in [10] (applied in the case $s_2 < T$ with $T \wedge T - s_2$, $(\mathbb{F}_t)_{t \in [0,T-s_2]} \wedge (\mathbb{F}_t)_{t \in [0,T-s_2]}$, $(\mathbb{W}_t)_{t \in [0,T-s_2]} \wedge (\mathbb{W}_t)_{t \in [0,T-s_2]}$, $(\mathbb{Y}_t)_{t \in [0,T-s_2]} \wedge (\mathbb{Y}_t)_{t \in [0,T-s_2]}$, $(\mathbb{V}_t)_{t \in [0,T-s_2]} \wedge (\mathbb{V}_t)_{t \in [0,T-s_2]}$, $(\mathbb{V}_t)_{t \in [0,T-s_2]} \wedge (\mathbb{V}_t)_{t \in [0,T-s_2]}$, $V_0 \wedge (O \supset x \mapsto e^{-\alpha_1 s_2}V_0(x) \in [0,\infty])$, $V_1 \wedge (O \supset x \mapsto e^{-\alpha_1 s_2}V_1(x) \in [0,\infty])$, $V \wedge (\{0, T - s_2\} \times O \supset (t, x) \mapsto e^{-\alpha_1 s_2}V(s_2 + t, x) \in \mathbb{R})$, $p \wedge p(1+\theta)$, $q \wedge q(1+1/\theta)$, $q_0 \wedge q_0$, $q_1 \wedge q_1$, $\phi \wedge (\{0, T - s_2\} \supset t \mapsto \phi(s_2 + t) \in [0,\infty])$, $t_1 \wedge t_2 - s_2$, $t_2 \wedge t_2 - s_2$ in the notation of Lemma 3.8 in [10], [27], nonnegativity of $V$, and Lemma 25 (applied for every $i \in \{0,1\}$ with $T \wedge s_2$, $s \wedge s_1$, $X \wedge (X_{s_i,t}^{x_2})_{t \in [s_2,s_1]}$, $\alpha \wedge \alpha_1$, $\beta \wedge \beta_1$, $V \wedge V_1$, $s \wedge 0$, $V \wedge 0$ in the notation of Lemma 25) imply that

\begin{equation}
\|X_{s_1,s_2}^{x_2} - X_{s_2,s_2}^{x_2}\|_{L^p(\mathbb{P}; H)}
\leq \left|p(1+\theta)\gamma + e^{-\alpha_0 s_1}V_0(x_2) + \int_{s_2-s_1}^{s_2} e^{-\alpha_0 s_1} du\right| (\sqrt{s_2 - s_1} + p(1+\theta))\sqrt{s_2 - s_1}
\end{equation}

\begin{align*}
&\cdot \left(1 + \prod_{i=0}^{T-s_2} \left(\exp\left(\frac{V_{(s_2,x_2)}}{2q_0 e^{\alpha_0 s_1}}\right) + \int_{s_2}^{s_1} e^{-\alpha_0 s_1} du\right)\right) \\
&\cdot \exp\left(\int_0^{T-s_2} \left(\phi(s_2 + r) + e^{-\alpha_0 s_1} \beta_0 + e^{-\alpha_1 s_2} \beta_1\right) dr + \int_{s_2}^{s_1} e^{-\alpha_0 s_1} du\right) \\
&\cdot \exp\left(\int_0^{T-s_2} \left(\phi(s_2 + r) + e^{-\alpha_0 s_1} \beta_0 + e^{-\alpha_1 s_2} \beta_1\right) dr + \int_{s_2}^{s_1} e^{-\alpha_0 s_1} du\right) \\
&\cdot \exp\left(\int_0^{T-s_2} \left(\phi(s_2 + r) + e^{-\alpha_0 s_1} \beta_0 + e^{-\alpha_1 s_2} \beta_1\right) dr + \int_{s_2}^{s_1} e^{-\alpha_0 s_1} du\right)
\end{align*}
Finally, the triangle inequality, \((20), (30)\), and nonnegativity of \(\phi, \alpha_0, \alpha_1, \beta_0, \beta_1, V_0, V_1\) yield that

\[
\|X_{s_1,t_1} - X_{s_2,t_2}\|_{L^p(\mathbb{P}; H)} \leq \|X_{s_1,t_1}^{x_1} - X_{s_2,t_2}^{x_2}\|_{L^p(\mathbb{P}; H)} + \|X_{s_1,t_2}^{x_2} - X_{s_2,t_2}\|_{L^p(\mathbb{P}; H)}
\]

\[
\leq \sqrt{|t_1 - t_2|} e^{\alpha_0 \gamma T} \|p\gamma + e^{-\alpha_0 s_1} V_0(x_1) + \int_{s_1}^{T} \|e^\gamma\| \gamma(\sqrt{T} + p)
\]

\[
+ \|x_1 - x_2\|_H \exp\left(\int_{s_1}^{T} \left(\phi(r) + \frac{\beta_0}{q e^{\alpha_0 r}} + \frac{\beta_1}{(1 + \theta) q e^{\alpha_0 r}}\right) dr + \sum_{i=0}^{1} \frac{V_i(x_2)}{2 q e^{\alpha_0 r_i}}\right)
\]

\[+ \exp\left(\int_{s_1}^{T} \left(\phi(r) + \frac{\beta_0}{q e^{\alpha_0 r}} + \frac{\beta_1}{(1 + \theta) q e^{\alpha_0 r}}\right) dr + \sum_{i=0}^{1} \frac{V_i(x_2)}{2 q e^{\alpha_0 r_i}}\right).
\]

This completes the proof of Lemma 3.2 \(\square\)

### 3.2 Moment estimates for the first derivative process

The following lemma, Lemma 3.3, provides a moment estimate for spatial derivatives of solutions of SDEs.

**Lemma 3.3** (Moment estimates for the first derivative process). Assume Setting 3.7, let \(D \subseteq \mathcal{O}\) be an open set, let \(s \in [0, T]\), \(t \in [s, T]\), let \(Y, Z : \Omega \rightarrow D\), \(Z : \Omega \rightarrow H\) be \(\mathbb{F}_s/\mathcal{B}(H)\)-measurable, assume that \(\sigma(\{X_{s,t}^x : x \in D\})\) and \(\mathbb{F}_s\) are independent, and assume for all \(\omega \in \Omega\) that \(D \ni x \mapsto X_{s,t}^x(\omega) \in C^1(D, H)\). Then it holds that

\[
\left\| \frac{\partial}{\partial x} X_{s,t}^x Z \right\|_{L^p(\mathbb{P}; H)} \leq \left\| Z \exp\left(\int_{s}^{t} \left(\phi(r) + \sum_{i=0}^{1} \frac{V_i(y)}{q e^{\alpha_0 r}}\right) dr + \sum_{i=0}^{1} \frac{V_i(y)}{q e^{\alpha_0 r}}\right) \right\|_{L^p(\mathbb{P}; H)}.
\]

**Proof of Lemma 3.3.** Independence of \(\sigma(\{X_{s,t}^x : x \in D\})\) and \(\mathbb{F}_s\), a disintegration formula (e.g. [12, Lemma 2.3]), the fact that for all \(\omega \in \Omega\) it holds that \((D \ni x \mapsto X_{s,t}^x(\omega) \in C^1(D, H)\), Fatou's lemma (e.g. Lemma 3.10 in [11]), and Lemma 3.2 (applied in the case \(t > 0\) for every \(y \in D\), \(z \in H\), \(h \in \{r \in \mathbb{R} \setminus \{0\} : y + z r \in D\}\) with \(T \cap t, s_1 \cap s, s_2 \cap s, t_1 \cap t, t_2 \cap t, x_1 \cap y + z h, x_2 \cap y\) in the notation of Lemma 3.2) yield that

\[
\left\| \frac{\partial}{\partial x} X_{s,t}^y Z \right\|_{L^p(\mathbb{P}; H)} = \int_{D \times H} \mathbb{E}\left[\left\| \frac{\partial}{\partial x} X_{s,t}^y z \right\|_{H}^{p}\right] p\left((Y, Z) \in d(y, z)\right)
\]

\[
= \int_{D \times H} \mathbb{E}\left[\liminf_{h \rightarrow 0}\|X_{s,t+h}^y - X_{s,t}^y\|_{H}^{P}\right] p\left((Y, Z) \in d(y, z)\right)
\]

\[
\leq \int_{D \times H} \mathbb{E}\left[\liminf_{h \rightarrow 0}\|X_{s,t+h}^y - X_{s,t}^y\|_{H}^{P}\right] p\left((Y, Z) \in d(y, z)\right)
\]

\[
\leq \int_{D \times H} \left\| \frac{y + z h - y}{h} \exp\left(\int_{s}^{t} \left(\phi(r) + \sum_{i=0}^{1} \frac{V_i(y)}{q e^{\alpha_0 r}}\right) dr + \sum_{i=0}^{1} \frac{V_i(y)}{q e^{\alpha_0 r}}\right)\right\|^P p\left((Y, Z) \in d(y, z)\right).
\]

This implies (32) and finishes the proof of Lemma 3.3 \(\square\)
The following theorem establishes existence of continuous solutions of SDEs. Theorem 3.4 improves existing results in \cite{2} and also existing results on strong completeness, e.g., in \cite{18, 11, 7, 6, 17, 16, 14}.

**Theorem 3.4** (Existence of a $C^0$-solution). Assume Setting 3.1 and assume that $V_0, V_1$ are bounded on every bounded subset of $O$, and assume that $\dim(H) < \infty$ and $p \in (2\dim(H) + 4, \infty)$. Then there exists a measurable function $X: \Delta_T \times O \times \Omega \rightarrow \overline{O}$ such that

(i) for all $x \in O$, $s \in [0, T]$ it holds a.s. that $(X^x_{s,t})_{t \in [s, T]} = (X^x_{s,t})_{t \in [s, T]}$ and

(ii) for every $\omega \in \Omega$ it holds that $X(\omega) \in C(\Delta_T \times \overline{O}, \overline{O})$.

**Proof of Theorem 3.4** Throughout this proof let $K_n \subseteq \mathbb{R} \times \mathbb{R} \times H$, $n \in \mathbb{N}$, be the sets which satisfy for all $n \in \mathbb{N}$ that $K_n = \{(s, t, x) \in \Delta_T \times O: s^2 + t^2 + \|x\|_H^2 \leq n^2\}$. Lemma 3.2 the fact that $\int_0^T \phi(r) \, dr < \infty$, and $\forall n \in \mathbb{N}$: sup $\left\{\{V_0(x) + V_1(\omega): (s, t, x) \in K_n\} \cup \{0\}\right\} < \infty$ yield for all $n \in \mathbb{N}$ that

$$\sup \left\{\left(\frac{\mathbb{E}[\|X^x_{s,t}\|_H]}{\|X^x_{s,t}\|_H}\right)^{\frac{1}{2}}: (s, t, x) \in K_n\} \cup \{0\}\right\} < \infty. \tag{34}$$

In particular this implies for all $n \in \mathbb{N}$ that

$$\sup \left\{\left(\frac{\mathbb{E}[\|X^x_{s,t}\|_H]}{\|X^x_{s,t}\|_H}\right)^{\frac{1}{2}}: (s, t, x) \in K_n\} \cup \{0\}\right\} \leq \sup \left\{\left(\frac{\mathbb{E}[\|X^x_{s,t} - X^x_{s,s}\|_H]}{\|X^x_{s,t} - X^x_{s,s}\|_H}\right)^{\frac{1}{2}}\sqrt{T} + \|x\|_H: (s, t, x) \in K_n\} \cup \{0\}\right\} < \infty. \tag{35}$$

This, \cite{17}, Proposition 2.2 (applied with $H \subseteq \mathbb{R} \times \mathbb{R} \times H$, $D \subseteq \Delta_T \times O$, $E \subseteq H$, $F \subseteq \overline{O}$, $Q \subseteq \frac{1}{2}$, $X \subseteq (\Delta_T \times O \subseteq (s, t, x) \mapsto X^x_{s,t} \subseteq \overline{O}$) in the notation of Proposition 2.2, and path continuity of $X^x_{s,t}$, $s \in [0, T]$, $x \in O$ establish the existence of a measurable function $\mathcal{X}: \Delta_T \times \overline{O} \times \Omega \rightarrow \overline{O}$ which satisfies that for all $\omega \in \Omega$ it holds that $\mathcal{X}(\omega) \in C(\Delta_T \times \overline{O}, \overline{O})$ and which satisfies that for all $x \in O$, $s \in [0, T]$ it holds a.s. that $(X^x_{s,t})_{t \in [s, T]} = (X^x_{s,t})_{t \in [s, T]}$. This completes the proof of Theorem 3.4 \hfill \Box

**4 Existence of a $C^1$-solution**

In this section we prove a strong local Hölder estimate for the first derivative process in Lemma 5.4 below, a moment estimate for the second derivative process in Lemma 4.3 below, and establish existence of a continuously differentiable solution under suitable assumptions in Theorem 4.6 below. First, we introduce the setting for these results.

**Setting 4.1.** Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space satisfying the usual conditions, let $(W_t)_{t \in [0, T]}$ be an $\mathbb{F}_t$-cylindrical $(\mathbb{F}_t)_{t \in [0, T]}$-Wiener process, let $\Delta_T = \{(s, t) \in [0, T]^2: s \leq t\}$, let $O \subseteq H$ be an open set, let $O \subseteq O$ be a convex set, let $\mu \in C^1(O, H)$, $\sigma \in C^1(O, HS(U, H))$, for all $s \in [0, T]$, $x \in O$ let $X^x_{s,t}: [s, T] \times O \rightarrow O$ be an $(\mathbb{F}_t)_{t \in [s, T]}$-adapted stochastic process with continuous sample paths which satisfies that for all $t \in [s, T]$ it holds a.s. that

$$X^x_{s,t} = x + \int_s^t \mu(X^x_{s,r}) \, dr + \int_s^t \sigma(X^x_{s,r}) \, dW_r, \tag{36}$$

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let $\alpha_0, \alpha_1, \beta_0, \beta_1, c \in [0, \infty)$, $V_0, V_1 \in C^2(O, [0, \infty))$, let $\bar{V} : [0, T] \times \Omega \to [0, \infty)$ be a measurable function, assume for all $i \in \{0, 1\}$, $t \in [0, T]$, $x \in \Omega$ that $\mathbb{P}\left( \int_t^T \bar{V}(r, X_{t,r}^x) \, dr < \infty \right) = 1$ and
\[
\left< \mu(x), (\nabla V_i)(x) \right>_H + \frac{1}{2} \text{trace} \left( \sigma(x)[\sigma(x)]^T (\text{Hess} V_i)(x) \right) + \frac{1}{2} \|\sigma(x)\|_2^2 \|\nabla V_i(x)\|_2^2 + 1(i) \cdot \bar{V}(t, x) \leq \alpha_i V_i(x) + \beta_i,
\]
\[(37)\]
let $\phi : [0, T] \to [0, \infty)$ be a measurable function which satisfies that $\int_0^T \phi(r) \, dr < \infty$, let $p \in [2, \infty)$, $\theta, \delta \in [0, \infty)$, $q_0, q_1 \in (0, \infty)$, $\gamma \in [\frac{1}{p}, \infty)$ satisfy that $\frac{\theta}{2p(1+\theta)^2(1+\delta)} = \frac{1}{q_0} + \frac{1}{q_1}$, assume that for all $t \in [0, T]$, $x, y, v, u, \bar{z}, \bar{y} \in \Omega, \bar{v} \in H \setminus \{0\}$ it holds that
\[
\left< v, \int_0^1 \mu'(\lambda x + (1 - \lambda) y) \, d\lambda \right>_H + \frac{1 + \delta}{2} \left< \frac{1}{0} \sigma'(\lambda x + (1 - \lambda) y) \, d\lambda \right>_H^2 \leq \|v\|_H^2 \cdot \left( \phi(t) + \frac{V_0(x) + V_0(y)}{2q_0 e^\alpha t} + \frac{\bar{V}(t, x) + \bar{V}(t, y)}{2q_1 e^{\alpha t}} \right),
\]
assume for all $x \in \Omega$ that
\[
\max \left\{ \|\mu(x)\|_H, \|\sigma(x)\|_H \right\} \leq c \left( 1 + V_0(x) \right) \gamma,
\]
assume for all $x, y, \omega \in \Omega$ that
\[
\max \left\{ \left< \frac{1}{0} \mu'(\lambda x + (1 - \lambda) y) \, d\lambda \right>_H, \left< \frac{1}{0} \sigma'(\lambda x + (1 - \lambda) y) \, d\lambda \right>_H \right\} \leq c \left( 2 + V_0(x) + V_0(y) \right) \gamma,
\]
assume for all $x_1, x_2, x_3, x_4 \in \Omega$ that
\[
\max \left\{ \left< \frac{1}{0} \mu'(\lambda x_1 + (1 - \lambda) x_2) - \mu'(\lambda x_3 + (1 - \lambda) x_4) \, d\lambda \right>_H, \left< \frac{1}{0} \sigma'(\lambda x_1 + (1 - \lambda) x_2) - \sigma'(\lambda x_3 + (1 - \lambda) x_4) \, d\lambda \right>_H \right\} \leq c \int \lambda \|x_1 - x_3\|_H + (1 - \lambda) \|x_2 - x_4\|_H \, d\lambda \left( 4 + \sum_{j=1}^4 V_0(x_i) \right) \gamma,
\]
and for all $(s, t) \in \Delta_T$, $x, v, \omega \in \Omega$ let $D_{s,t}^{x,h} : \Omega \to H$ be the function which satisfies that
\[
D_{s,t}^{x,h}(v) = \frac{X_{s,t}^{x+h} - X_{s,t}^x}{h}.
\]
\[(42)\]
**Lemma 4.2.** Assume Setting 4.1 and let $t \in [0, T]$, $x, y, \omega \in \Omega$. Then it holds that
\[
\left< x - y, \mu(x) - \mu(y) \right>_H + \frac{1}{2} \|\sigma(x) - \sigma(y)\|_H^2 \leq \left< x - y, \mu' \left( \lambda x + (1 - \lambda) y \right) d\lambda(x - y) \right>_H + \frac{1}{2} \left< \sigma' \left( \lambda x + (1 - \lambda) y \right) d\lambda(x - y) \right>_H^2 + \frac{2p(1+\theta)^2(1+\delta)-2}{2} \frac{\|x-y, \sigma(x) - \sigma(y)\|_H^2}{\|x-y\|_H^2}
\]
\[
\leq \|x - y\|_H^2 \cdot \left( \phi(t) + \frac{V_0(x) + V_0(y)}{2q_0 e^{\alpha t}} + \frac{\bar{V}(t, x) + \bar{V}(t, y)}{2q_1 e^{\alpha t}} \right).
\]
\[(43)\]
\[
\text{Proof of Lemma 4.2} \quad \text{Convexity of } \Omega, \text{ the fundamental theorem of calculus, and assumption (38) (applied in the case } x \neq y \text{ with } v \wedge x - y \text{ in the notation of assumption (38)) yield that}
\]
\[
\left< x - y, \mu(x) - \mu(y) \right>_H + \frac{1}{2} \|\sigma(x) - \sigma(y)\|_H^2 + \frac{2p(1+\theta)^2(1+\delta)-2}{2} \frac{\|x-y, \sigma(x) - \sigma(y)\|_H^2}{\|x-y\|_H^2}
\]
\[
= \left< x - y, \int_0^1 \mu'(\lambda x + (1 - \lambda) y) d\lambda(x - y) \right>_H + \frac{1}{2} \left< \sigma' \left( \lambda x + (1 - \lambda) y \right) d\lambda(x - y) \right>_H^2 + (p(1+\theta)^2(1+\delta)-1) \frac{\left< x-y, \sigma' \left( \lambda x + (1 - \lambda) y \right) d\lambda(x-y) \right>_H^2}{\|x-y\|_H^2}
\]
\[
\leq \|x - y\|_H^2 \cdot \left( \phi(t) + \frac{V_0(x) + V_0(y)}{2q_0 e^{\alpha t}} + \frac{\bar{V}(t, x) + \bar{V}(t, y)}{2q_1 e^{\alpha t}} \right).
\]
\[(44)\]
This completes the proof of Lemma 4.2. \qed
Throughout this proof let (Strong local H"older estimate for difference quotients)

The following lemma proves strong local H"older continuity of the difference quotients.

### 4.1 Strong local H"older estimate

The following lemma proves strong local H"older continuity of the difference quotients.

#### Lemma 4.4 (Strong local H"older estimate for difference quotients)

Assume Setting 4.1 and let $s_1, s_2, t_1, t_2 \in [0, T], x_1, x_2 \in \mathcal{O}, v_1, v_2 \in H, h_1, h_2 \in \mathbb{R} \setminus \{0\}$ satisfy that $s_1 \leq t_1, s_2 \leq t_2, x_1 + v_1 h_1, x_2 + v_2 h_2 \in \mathcal{O}$, and $s_1 \leq s_2$. Then it holds that

$$
\left\| D^{x_1,h_1}_{s_1,t_1}(v_1) - D^{x_2,h_2}_{s_2,t_2}(v_2) \right\|_{L^p(\mathbb{P},H)} \leq \left[ \left( \|x_1 - x_2\|_H + \|v_1 h_1 - v_2 h_2\|_H \right) \sqrt{t_2 - s_2} + \sqrt{|s_1 - s_2| + \sqrt{|t_1 - t_2|}} \right] \|v_1\|_H + \|v_1 - v_2\|_H
$$

$$
\cdot e^{2\alpha_{\gamma}T} \left( \frac{4p(1+\beta)^2(1+\gamma)(1+\gamma)}{\min(4,1)} + 2\sqrt{T} + \int_0^{2\beta_0 r} dr + 2 \max_{\tau \in \{0,1\}, j \in \{1,2\}} V_0(x_j + \omega_{j, h_j}) \right)^{2\gamma + 2}
$$

$$
\cdot (1 + c)^2 \max_{\tau \in \{0,1\}, j \in \{1,2\}} \exp \left( 3 \int_0^T (\phi(r) + \sum_{i=0}^1 \frac{b_i}{q_i e^{\gamma r}}) \, dr + 3 \sum_{i=0}^1 \frac{V_i(x_j + \omega_{j, h_j})}{q_i e^{\gamma r}} \right).
$$

#### Proof of Lemma 4.4

Throughout this proof let $Y, a, \zeta^\mu : [0, T - s_2] \times \Omega \rightarrow H, b, \zeta^\sigma : [0, T - s_2] \times \Omega \rightarrow \text{HS}(U, H)$, ad $\eta : [0, T - s_2] \times \Omega \rightarrow L(H, \text{HS}(U, H))$ be the functions which satisfy for all $r \in [0, T - s_2]$ that

$$
a_r = \int_0^1 \left( \mu \left( X^{x_1} \|x_1 - x_2\|_H + \|v_1 h_1 - v_2 h_2\|_H \right) \sqrt{t_2 - s_2} + \sqrt{|s_1 - s_2| + \sqrt{|t_1 - t_2|}} \right) \|v_1\|_H + \|v_1 - v_2\|_H
$$

$$
- \mu \left( X^{x_1} + \|x_1 - x_2\|_H + \|v_1 h_1 - v_2 h_2\|_H \right) \sqrt{t_2 - s_2} + \sqrt{|s_1 - s_2| + \sqrt{|t_1 - t_2|}} \right) \|v_1\|_H + \|v_1 - v_2\|_H
$$

$$
b_r = \int_0^1 \left( \sigma \left( X^{x_1} + \|x_1 - x_2\|_H + \|v_1 h_1 - v_2 h_2\|_H \right) \sqrt{t_2 - s_2} + \sqrt{|s_1 - s_2| + \sqrt{|t_1 - t_2|}} \right) \|v_1\|_H + \|v_1 - v_2\|_H
$$

$$
- \sigma \left( X^{x_1} + \|x_1 - x_2\|_H + \|v_1 h_1 - v_2 h_2\|_H \right) \sqrt{t_2 - s_2} + \sqrt{|s_1 - s_2| + \sqrt{|t_1 - t_2|}} \right) \|v_1\|_H + \|v_1 - v_2\|_H
$$

$$
Y_r = D^{x_1,h_1}_{s_1,s_2+r}(v_1) - D^{x_2,h_2}_{s_2,s_2+r}(v_2),
$$

(50)
\[ \eta_r = \int_0^1 \sigma'(X_{s_2,s_2+r}^x + \lambda(X_{s_2,s_2+r}^{x_1} + v h^2) - X_{s_2,s_2+r}^x) \, d\lambda, \]  
\[ \zeta_r^\mu = \int_0^1 \left( \mu' \left( \lambda X_{x_1,v}^{x_1} + (1 - \lambda) X_{x_1^{s_1},s_2^{s_1}+r}^{x_1} \right) - \mu'(\lambda X_{s_2,v}^{x_2} + (1 - \lambda) X_{s_2^{s_2},s_2^{s_2}+r}^{x_2}) \right) d\lambda D_t^{x_1,v}(v_1), \]  
\[ \zeta_r^\sigma = \int_0^1 \left( \sigma'(\lambda X_{x_1^{s_1},s_2^{s_1}+r}^{x_1} + (1 - \lambda) X_{x_1^{s_1},s_2^{s_1}+r}^{x_1}) - \sigma'(\lambda X_{s_2,v}^{x_2} + (1 - \lambda) X_{s_2^{s_2},s_2^{s_2}+r}^{x_2}) \right) d\lambda D_t^{x_1,v}(v_1). \]

Note that \( Y, a, b, \eta, \zeta^\mu, \zeta^\sigma \) are \((\mathbb{F}_{s_2+r})_{r \in [0,T-s_2]}\)-adapted stochastic processes with continuous sample paths and, therefore, measurable. Lemma 13 implies that for all \( t \in [0,T-s_2] \) it holds a.s. that

\[ Y_t = D_t^{x_1,v}(v_1) - D_t^{x_2,v}(v_2) = D_t^{x_1,v}(v_1) - v_2 + \int_0^{t} a_r \, dr + \int_0^{t} b_r \, dW_t \]

We consider the one-sided affine-linear growth condition for the Itô process \( Y \). Equations (18), (50), and (52) imply for all \( r \in [0,T-s_2] \) that

\[ a_r = \int_0^1 \mu'(X_{x_1,v}^{x_1} + \lambda(X_{x_1,v}^{x_1} - X_{x_1,v}^{x_1})) \, d\lambda D_t^{x_1,v}(v_1) \]

\[ \mu'(X_{x_2,v}^{x_2} + \lambda(X_{x_2,v}^{x_2} - X_{x_2,v}^{x_2})) \, d\lambda D_t^{x_1,v}(v_1) \]

\[ = \int_0^1 \mu'(X_{x_2,v}^{x_2} + \lambda(X_{x_2,v}^{x_2} - X_{x_2,v}^{x_2})) \, d\lambda Y_r \]

\[ + \int_0^1 \mu'(\lambda X_{x_1,v}^{x_1} + (1 - \lambda) X_{x_1,v}^{x_1}) - \mu'(\lambda X_{x_2,v}^{x_2} + (1 - \lambda) X_{x_2,v}^{x_2}) \, d\lambda Y_r + \zeta_r^\mu. \]

Analogously, equations (19), (51), and (53) imply for all \( r \in [0,T-s_2] \) that

\[ b_r = \int_0^1 \sigma'(X_{x_2,v}^{x_2} + \lambda(X_{x_2,v}^{x_2} - X_{x_2,v}^{x_2})) \, d\lambda Y_r + \zeta_r^\sigma = \eta_r Y_r + \zeta_r^\sigma. \]

Equation (55), the Cauchy-Schwarz inequality, and Young’s inequality yield for all \( r \in [0,T-s_2] \) that

\[ \langle Y_r, a_r \rangle_H \leq \left\langle Y_r, \int_0^1 \mu'(X_{s_2,v}^{x_2} + \lambda(X_{s_2,v}^{x_2} - X_{s_2,v}^{x_2})) \, d\lambda Y_r \right\rangle_H + \left\| Y_r \right\|_H \left\| \zeta_r^\mu \right\|_H \]

\[ \leq \left\langle Y_r, \int_0^1 \mu'(X_{s_2,v}^{x_2} + \lambda(X_{s_2,v}^{x_2} - X_{s_2,v}^{x_2})) \, d\lambda Y_r + \delta Y_r \right\rangle_H + \frac{1}{4\delta} \left\| \zeta_r^\mu \right\|_H^2. \]

Similarly, equation (56), the Cauchy-Schwarz inequality, and Young’s inequality imply for all
Next, (57), (58), (51), the fact that (57) imply for all \(r \in [0, T - s_2]\) that

\[
\frac{1}{2} \| b_r \|_{HS(U,H)}^2 + \frac{p(1+\theta)-2}{2} \| (Y_r, b_r) \|_{HS(U,H)}^2 + \frac{p(1+\theta)-2}{2} \| (Y_r, Y_r + \zeta^r) \|_{HS(U,H)}^2 \\
= \frac{1}{2} \| \eta_r \|_{HS(U,H)}^2 + \| (\eta_r, \zeta^r) \|_{HS(U,H)}^2 + \frac{p(1+\theta)-2}{2} \| (Y_r, Y_r, \zeta^r) \|_{HS(U,H)}^2 \\
+ \frac{p(1+\theta)-2}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 + \frac{p(1+\theta)-2}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 + \frac{p(1+\theta)-2}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 \\
\leq \frac{1}{2} \| \eta_r \|_{HS(U,H)}^2 + \| (\eta_r, \zeta^r) \|_{HS(U,H)}^2 + \frac{p(1+\theta)-2}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 + \frac{p(1+\theta)-2}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 \\
+ \frac{p(1+\theta)-2}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 + \frac{p(1+\theta)-2}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 \\
\leq \frac{1+\delta}{2} \| \eta_r \|_{HS(U,H)}^2 + \frac{(p(1+\theta)-2)(1+\delta)}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 + \frac{(p(1+\theta)-2)(1+\delta)}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 \\
= \frac{1+\delta}{2} \| \eta_r \|_{HS(U,H)}^2 + \frac{(p(1+\theta)-2)(1+\delta)}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 + \frac{(p(1+\theta)-2)(1+\delta)}{2} \| (Y_r, Y_r) \|_{HS(U,H)}^2 \\
(58)
\]

This, (51), nonnegativity of \(\phi, \psi, \bar{V}\), Proposition 2.3 (applied in the case \(s_2 < T\) for every \(z \in \mathbb{R}^d\) with \(p \leq p(1+\theta), \ T \leq T - s_2, \ P \subset P\cdot D_{s_2}^{x_1}(v_1) = z\), \(\mathbf{F}_t \cap [0, T - s_2] \subset \mathbf{F}_{s_2+t}\), \(W \cap (W_{s_2+t} - W_{s_2+r}) \in [0, T - s_2]\), \(X \cap Y\),

\[
\alpha \cap \left( \phi(s_2 + r) + \frac{V_0(X_{s_2+2r})}{2q1\epsilon_1^{x_1}(s_2+r)} \right) r \in [0, T - s_2] \quad (60)
\]

\[
\beta \cap \left( \left[ (p(1+\theta) - 1)(1+\delta^{-1}) \| \zeta^r \|_{HS(U,H)}^2 + \frac{1}{2} \| \zeta^r \|_{H}^2 \right]^{\frac{1}{2}} \quad r \in [0, T - s_2] \quad (61)
\]

\(q_1 \subset p, \ q_2 \subset p(1+\theta) / \theta \) in the notation of Proposition 2.3, the triangle inequality, the fact that \(\frac{\theta}{p(1+\theta)} \geq \frac{1}{q_0} + \frac{1}{q_1}\) and Lemma 2.7 (applied in the case \(t + s_2 > 0\) with \(T \cap t + s_2, s \cap s_2, X^1 \cap X_{s_2+2r}^{x_2+y_2}, X^2 \cap X_{s_2+2r}^{x_2+y_2}, X^3 \cap X_{s_2+2r}^{x_2+y_2}, X^4 \cap X_{s_2+2r}^{x_2+y_2}, q \cap 2p(1+\theta)(1+\delta) / \theta \) in the notation
of Lemma 2.7] that for all \( t \in [0, T - s_2] \) it holds a.s. that
\[
\left\| D_{s_1, s_2 + t}^{x_1, h_1} (v_1) - D_{s_2, s_2 + t}^{x_2, h_2} (v_2) \right\|_{L^p(P; \mathbb{F}_{s_2}; H)} = \left\| Y_t \right\|_{L^p(P; \mathbb{F}_{s_2}; H)} \leq \left( \left\| Y_0 \right\|_{L^p(\mathbb{F}_{s_2}; H)} + \int_0^t \left( \frac{(p(1+\theta) - 1)(1+\delta)}{\delta} \left\| \zeta^\theta \right\|_{\text{HS}(U, H)} + \frac{1}{2\delta} \left\| \zeta^\theta \right\|_{\text{HS}(U, H)}^2 \right)^2 \frac{ds}{L^p(1+\theta)/\delta(\mathbb{F}_{s_2}; R)} \right)^{\frac{1}{2}} \cdot \exp \left( \int_0^t \left( \phi(r) + \frac{2}{t \gamma_0 e^{\gamma_0 r}} + \frac{2}{t \gamma_1 e^{\gamma_1 r}} \right) dr + \sum_{i=0}^{\frac{1}{4}} V_i (x_2 + v_2 h_2) + (x_i) \right). \tag{62}
\]

This, the fact that \( p \geq 2 \), and the triangle inequality imply that
\[
\left\| D_{s_1, s_2 + t}^{x_1, h_1} (v_1) - D_{s_2, s_2 + t}^{x_2, h_2} (v_2) \right\|_{L^p(\mathbb{F}_{s_2}; H)} \leq \left\| D_{s_1, s_2 + t}^{x_1, h_1} (v_1) - D_{s_2, s_2 + t}^{x_2, h_2} (v_2) \right\|_{L^p(\mathbb{F}_{s_2}; H)} \left\| D_{s_1, s_2 + t}^{x_1, h_1} (v_1) \right\|_{L^p(\mathbb{F}_t; H)} \cdot \exp \left( \int_0^t \left( \phi(r) + \frac{2}{t \gamma_0 e^{\gamma_0 r}} + \frac{2}{t \gamma_1 e^{\gamma_1 r}} \right) dr + \sum_{i=0}^{\frac{1}{4}} V_i (x_2 + v_2 h_2) + (x_i) \right). \tag{63}
\]

Moreover, [52], the triangle inequality, and [11] yield for all \( r \in [0, T - s_2] \) that
\[
\left\| \zeta^\mu_r \right\|_{L^p(1+\theta)(\mathbb{F}_r; H)} \leq \left\| \mu \left( \lambda X_{s_1, s_2 + r}^{x_1, h_1} + (1 - \lambda) X_{s_1, s_2 + r}^{x_2, h_2} \right) \right\|_{L^p(H, H)} \left\| D_{s_1, s_2 + r}^{x_1, h_1} (v_1) \right\|_{H^p(\mathbb{F}_r; H)} \cdot \exp \left( \int_0^t \left( \phi(r) + \frac{2}{t \gamma_0 e^{\gamma_0 r}} + \frac{2}{t \gamma_1 e^{\gamma_1 r}} \right) dr + \sum_{i=0}^{\frac{1}{4}} V_i (x_2 + v_2 h_2) + (x_i) \right). \tag{64}
\]

This, Hölder’s inequality (applied with \( \frac{1}{p(1+\theta)} = \frac{1}{2p(1+\theta) + 1} + \frac{\delta}{p(1+\theta)(1+\delta)} \)), and the triangle inequality show for all \( r \in [0, T - s_2] \) that
\[
\left\| \zeta^\mu_r \right\|_{L^p(1+\theta)(\mathbb{F}_r; H)} \leq \left\| D_{s_1, s_2 + r}^{x_1, h_1} (v_1) \right\|_{H^p(\mathbb{F}_r; H)} \cdot \exp \left( \int_0^t \left( \phi(r) + \frac{2}{t \gamma_0 e^{\gamma_0 r}} + \frac{2}{t \gamma_1 e^{\gamma_1 r}} \right) dr + \sum_{i=0}^{\frac{1}{4}} V_i (x_2 + v_2 h_2) + (x_i) \right). \tag{65}
\]
Equation (42), Lemma 3.2, and Lemma 3.2 (applied for all $r \in [s_1, T]$ with $T \sim t_2$, $p \sim 2p(1 + \theta)(1 + \delta)$, $s_1 \sim s_1$, $s_2 \sim s_1$, $t_1 \sim s_2 + r$, $t_2 \sim s_2 + r$, $x_1 \sim x_1 + v_1 h_1$, $x_2 \sim x_1$ in the notation of Lemma 3.2) yield for all $r \in [0, t_2 - s_2]$ that

$$
\|D_{s_1, s_2}^{\beta_1, h_1}(v_1)\|_{L^{2p(1 + \theta)(1 + \delta)}(\mathbb{P}; H)} = \frac{1}{|t_1|} \|X_{s_1, s_2}^{x_1 + v_1 h_1} - X_{s_1, s_2 + r}^{x_1}\|_{L^{2p(1 + \theta)(1 + \delta)}(\mathbb{P}; H)} \leq \|v_1\|_H \exp \left( \int_{s_1}^{t_2} (\phi(t) + \sum_{i=0}^{2} \frac{\beta_i}{q_i e^{\gamma_i t}}) dt + \sum_{i=0}^{2} \frac{V_i(x_1 + v_1 h_1) + V_i(x_1)}{2q_i e^{\gamma_i t_1}} \right).$$

(66)

Moreover, (65), (66), (43), Lemma 3.2 (applied for all $r \in [0, t_2 - s_2]$), $\epsilon \in \{0, 1\}$ with $p \sim 2p(1 + \theta)(1 + \delta)$, $s_1 \sim s_1$, $s_2 \sim s_1$, $t_1 \sim s_2 + r$, $t_2 \sim s_2 + r$, $x_1 \sim x_1 + v_1 h_1$, $x_2 \sim x_2 + v_2 h_2$ in the notation of Lemma 3.2, and Lemma 2.6 yield for all $r \in [0, t_2 - s_2]$ that

$$
\|\mathcal{C}_r^\mu\|_{L^{p(1 + \theta)}(\mathbb{P}; H)} \leq \|v_1\|_H \exp \left( \int_{s_1}^{t_2} (\phi(t) + \sum_{i=0}^{2} \frac{\beta_i}{q_i e^{\gamma_i t}}) dt + \sum_{i=0}^{2} \frac{V_i(x_1 + v_1 h_1) + V_i(x_1)}{2q_i e^{\gamma_i t_1}} \right) \cdot c \max_{\epsilon \in \{0, 1\}} \exp \left( \int_{s_1}^{t_2} (\phi(t) + \sum_{i=0}^{2} \frac{\beta_i}{q_i e^{\gamma_i t}}) dt + \sum_{i=0}^{2} \frac{V_i(x_1 + v_1 h_1) + V_i(x_2 + v_2 h_2)}{2q_i e^{\gamma_i t_1}} \right) \left( \|x_1 + v_1 h_1 - x_2 - v_2 h_2\|_H + \sqrt{|s_1 - s_2|} \epsilon e^{\alpha_0 t} \right) \left( \|x_1 + v_1 h_1 - x_2 - v_2 h_2\|_H + \sqrt{|s_1 - s_2|} \right) \max_{z \in \{x_1, x_2, x_1 + v_1 h_1, x_2 + v_2 h_2\}} V_0(z) \gamma \right) \gamma
$$

(67)

and

$$
\|\mathcal{C}_r^\mu\|_{L^{p(1 + \theta)}(\mathbb{P}; H)} \leq \left( \|x_1 - x_2\|_H + \|v_1 h_1 - v_2 h_2\|_H + \sqrt{|s_1 - s_2|} \right) \cdot \|v_1\|_H c \max_{\epsilon \in \{0, 1\}, j \in \{1, 2\}} \exp \left( \int_{s_1}^{t_2} (\phi(t) + \sum_{i=0}^{2} \frac{\beta_i}{q_i e^{\gamma_i t}}) dt + \sum_{i=0}^{2} \frac{V_i(x_1 + v_1 h_1)}{q_i e^{\gamma_i t_1}} \right) \left( \|x_1 - x_2\|_H + \|v_1 h_1 - v_2 h_2\|_H + \sqrt{|s_1 - s_2|} \right) \max_{z \in \{x_1, x_2, x_1 + v_1 h_1, x_2 + v_2 h_2\}} V_0(x_j + v_j h_j) \right) \gamma \leq 1 + c).$$

(68)

An analogous argumentation shows for all $r \in [0, t_2 - s_2]$ that

$$
\|\mathcal{C}_r^\mu\|_{L^{p(1 + \theta)}(\mathbb{P}; H)} \leq \left( \|x_1 - x_2\|_H + \|v_1 h_1 - v_2 h_2\|_H + \sqrt{|s_1 - s_2|} \right) \cdot \|v_1\|_H c \max_{\epsilon \in \{0, 1\}, j \in \{1, 2\}} \exp \left( \int_{s_1}^{t_2} (\phi(t) + \sum_{i=0}^{2} \frac{\beta_i}{q_i e^{\gamma_i t}}) dt + \sum_{i=0}^{2} \frac{V_i(x_1 + v_1 h_1)}{q_i e^{\gamma_i t_1}} \right) \left( \|x_1 - x_2\|_H + \|v_1 h_1 - v_2 h_2\|_H + \sqrt{|s_1 - s_2|} \right) \max_{z \in \{x_1, x_2, x_1 + v_1 h_1, x_2 + v_2 h_2\}} V_0(x_j + v_j h_j) \right) \gamma \leq 1 + c).$$

(69)

Next, we derive a temporal regularity estimate. Lemma 4.3, the triangle inequality, the Burkholder-Davis-Gundy type inequality in [3, Lemma 7.7], Hölder’s inequality (applied with $\frac{1}{p} = \frac{1}{2p} + \frac{1}{2p}$),
and (60) prove for all $u_1 \in [s_1, T]$, $u_2 \in [u_1, T]$ that

$$
\|D_{s_1,u_2}^{x_1,h_1}(v_1) - D_{s_1,u_1}^{x_1,h_1}(v_1)\|_{L^p(P;H)} \\
\leq \int_{u_1}^{u_2} \left( \int_0^1 \mu'(\lambda X_{s_1,r}^{x_1,v_{11}} + (1 - \lambda)X_{s_1,r}^{x_1}) \, d\lambda \right) \|D_{s_1,r}^{x_1,h_1}(v_1)\|_H \, \|D_{s_1,r}^{x_1,h_1}(v_1)\|_{L^p(P;\mathbb{R})} \, dr \\
+ \left( \frac{p(p-1)}{2} \right) \int_{u_1}^{u_2} \left( \int_0^1 \sigma'(\lambda X_{s_1,r}^{x_1,v_{11}} + (1 - \lambda)X_{s_1,r}^{x_1}) \, d\lambda \right) \|D_{s_1,r}^{x_1,h_1}(v_1)\|_H \|D_{s_1,r}^{x_1,h_1}(v_1)\|_{L^p(P;\mathbb{R})} \, dr \right) ^{\frac{1}{2}} \\
\leq \int_{u_1}^{u_2} \|c(2 + V_0(X_{s_1,r}^{x_1,v_{11}}) + V_0(X_{s_1,r}^{x_1}))\|_{L^p(P;\mathbb{R})} \|D_{s_1,r}^{x_1,h_1}(v_1)\|_{L^p(P;H)} \, dr \\
+ \left( \frac{p(p-1)}{2} \right) \int_{u_1}^{u_2} \|c(2 + V_0(X_{s_1,r}^{x_1,v_{11}}) + V_0(X_{s_1,r}^{x_1}))\|_{L^p(P;\mathbb{R})} \|D_{s_1,r}^{x_1,h_1}(v_1)\|_{L^p(P;H)} \, dr \right) ^{\frac{1}{2}} .
$$

(70)

This, the triangle inequality, Lemma 2.6 (applied for all $r \in [s_1, T]$, $\iota \in \{0, 1\}$ with $T \land r$, $s \land s_1$, $P \land (P_i)_{i \in [s_1,r]}$, $W \land (W_i)_{i \in [s_1,r]}$, $X \land (X_{s_1,t}^{x_1,v_{11}})_{t \in [s_1,r]}$, $\alpha \land \alpha_0$, $\beta \land \beta_0$, $V \land V_0$, $t \land r$, $p \land 2p\gamma$ in the notation of Lemma 2.6), and (66) imply for all $u_1 \in [s_1, T]$, $u_2 \in [u_1, T]$ that

$$
\|D_{s_1,u_2}^{x_1,h_1}(v_1) - D_{s_1,u_1}^{x_1,h_1}(v_1)\|_{L^p(P;H)} \\
\leq (u_2 - u_1 + p\sqrt{u_2 - u_1}) \cdot c \cdot \sup_{r \in [u_1,u_2]} \left( \left( \sum_{i \in \{0,1\}} \|1 + V_0(X_{s_1,r}^{x_1,v_{11}})\|_{L^{p_\gamma}(P;\mathbb{R})} \right) \|D_{s_1,r}^{x_1,h_1}(v_1)\|_{L^p(P;H)} \right) \gamma \\
\leq \sqrt{u_2 - u_1}(\sqrt{T} + p) e^{\alpha_0\gamma u_2} \left( 4 \beta_0 + \int_0^{u_2} \frac{2\beta_0}{e^{\alpha_0}} \, dr + \sum_{i \in \{0,1\}} V_0(x_1 + v_1 h_1) \right)^\gamma \\
\cdot \|v_1\|_H \exp \left( \int_0^T \left( \phi(r) + \sum_{i=0}^1 \frac{1}{q_i e^\alpha r^{\alpha_{i+1}}} \right) \, dr + \sum_{i=0}^1 \frac{V_i(x_1 + v_1 h_1) + V_i(x_1)}{2q_i e^\alpha r^{\alpha_{i+1}}} \right)
$$

(71)

The triangle inequality and (63) yield that

$$
\|D_{s_1,t_1}^{x_1,h_1}(v_1) - D_{s_2,t_2}^{x_2,h_2}(v_2)\|_{L^p(P;H)} \\
\leq \|D_{s_1,t_1}^{x_1,h_1}(v_1) - D_{s_1,t_2}^{x_1,h_1}(v_1)\|_{L^p(P;H)} + \|D_{s_1,t_2}^{x_1,h_1}(v_1) - D_{s_2,t_2}^{x_2,h_2}(v_2)\|_{L^p(P;H)} \\
\leq \|D_{s_1,t_1}^{x_1,h_1}(v_1) - D_{s_1,t_2}^{x_1,h_1}(v_1)\|_{L^p(P;H)} \\
+ \|D_{s_1,t_2}^{x_1,h_1}(v_1) - v_1 + v_1 - v_2\|_{L^p(P;H)} \\
+ \sup_{r \in [0,t_2-s_2]} \max \left\{ \|\zeta_{\mu}\|_{L^{p(1+\theta)}(P;H)} \|\zeta_{\sigma}\|_{L^{p(1+\theta)}(P;HS(U,H))} \right\} \sqrt{(t_2 - s_2)^p (1 + \theta)(1 + \delta)} \\
\cdot \exp \left( \int_{s_2}^{t_2} \left( \phi(r) + \frac{\beta_0}{q_0 e^\alpha r^{\alpha_1}} + \frac{\beta_1}{q_1 e^\alpha r^{\alpha_1}} \right) \, dr + \sum_{i=0}^1 \frac{V_i(x_2 + v_2 h_2) + V_i(x_2)}{2q_i e^\alpha r^{\alpha_{i+1}}} \right).
$$

(72)
This, (71), (68), and (69) ensure that

$$\left\| D^{x_1,h_1} \left( v_1 \right) - D^{x_2,h_2} \left( v_2 \right) \right\|_{L^p(\mathbb{P},H)} \leq \sqrt{t_2 - t_1} \left( \sqrt{T} + p \right) e^{\alpha_0 \gamma T} \left( 4p \gamma + \int_0^T \frac{2\gamma_0}{e^{\alpha_0 r}} dr + \sum_{i \in \{0,1\}} V_0(x_1 + \nu v_1 h_1) \right)^\gamma$$

$$\cdot \left\| v_1 \right\|_H \exp \left( \int_0^{t_2} \left( \phi(r) + \sum_{i=0}^1 \frac{\beta_i}{q_i e^{\alpha_i r}} \right) dr + \sum_{i=0}^{t_2} \frac{V_0(x_1 + \nu v_1 h_1) + V_0(x_1)\left( \phi(r) - \frac{\beta_0}{q_0 e^{\alpha_0 r}} \right)}{2q_i e^{\alpha_i r_2}} \right)$$

$$+ \left[ \left\| v_1 - v_2 \right\|_H + \sqrt{s_2 - s_1} \left( \sqrt{T} + p \right) e^{\alpha_0 \gamma s_2} \left( 4p \gamma + \int_0^{t_2} \frac{2\gamma_0}{e^{\alpha_0 r}} dr + \sum_{i \in \{0,1\}} V_0(x_1 + \nu v_1 h_1) \right)^\gamma$$

$$\cdot \left\| v_1 \right\|_H \max_{i \in \{0,1\}, j \in \{1,2\}} \exp \left( 2 \int_0^{t_2} \left( \phi(r) + \sum_{i=0}^1 \frac{\beta_i}{q_i e^{\alpha_i r}} \right) dr + 2 \sum_{i=0}^1 \frac{V_0(x_1 + \nu v_1 h_1)}{q_i e^{\alpha_i r_2}} \right)$$

$$\cdot 4 \gamma^2 e^{2\alpha_0 \gamma T} \left( \frac{2(1+\gamma)p(1+\theta)^2(1+\delta)}{\min\{\delta,1\}} + \sqrt{T} + \int_0^T \frac{2\gamma_0}{e^{\alpha_0 r}} dr + \max_{i \in \{0,1\}, j \in \{1,2\}} V_0(x_j + \nu v_j h_j) \right)^{2\gamma+2}$$

$$\cdot \left( 1 + c \right) \sqrt{(t_2 - s_2)p(1+\theta)} \left( 1 + \delta^{-1} \right)$$

$$\cdot \exp \left( \int_0^{t_2} \left( \phi(r) - \frac{\beta_0}{q_0 e^{\alpha_0 r}} + \frac{\beta_1}{q_1 e^{\alpha_1 r}} \right) dr + \sum_{i=0}^1 \frac{V_0(x_1 + \nu v_1 h_1)}{q_i e^{\alpha_i r_2}} \right).$$

Consequently we obtain that

$$\left\| D^{x_1,h_1} \left( v_1 \right) - D^{x_2,h_2} \left( v_2 \right) \right\|_{L^p(\mathbb{P},H)} \leq \left[ \left( \left\| x_1 - x_2 \right\|_H + \left\| v_1 h_1 - v_2 h_2 \right\|_H \right) \sqrt{t_2 - s_2} + \sqrt{s_1 - s_2} + \sqrt{|t_1 - t_2|} \right] \left\| v_1 \right\|_H + \left\| v_1 - v_2 \right\|_H$$

$$\cdot e^{2\alpha_0 \gamma T} \left( \frac{4p(1+\theta)^2(1+\delta)^2}{\min\{\delta,1\}} + 2 \sqrt{T} + \int_0^T \frac{2\gamma_0}{e^{\alpha_0 r}} dr + 2 \max_{i \in \{0,1\}, j \in \{1,2\}} V_0(x_j + \nu v_j h_j) \right)^{2\gamma+2}$$

$$\cdot \left( 1 + c \right)^2 \max_{i \in \{0,1\}, j \in \{1,2\}} \exp \left( 3 \int_0^T \left( \phi(r) + \sum_{i=0}^1 \frac{\beta_i}{q_i e^{\alpha_i r}} \right) dr + 3 \sum_{i=0}^1 \frac{V_0(x_j + \nu v_j h_j)}{q_i e^{\alpha_i r_2}} \right).$$

This completes the proof of Lemma 4.4.

\[ \square \]

### 4.2 Moment estimates for the second-order derivative process

The following lemma, Lemma 4.5, provides a moment estimate for second-order spatial derivatives of solutions of SDEs.

**Lemma 4.5** (Moment estimates for the second-order derivative process). Assume Setting 4.4, let \( D \subseteq \mathcal{O} \) be an open set, let \( s \in [0,T] \), \( t \in [s,T] \), let \( Y: \Omega \to D \), \( Z_1, Z_2: \Omega \to H \) be \( \mathbb{F}_s/B(H) \)-measurable, assume that \( \sigma(\{X_{s,t}^x: x \in D\}) \) and \( \mathbb{F}_s \) are independent, and assume for all \( \omega \in \Omega \)
Lemma 4.4 (applied in the case \( \sigma \))

This, independence of \( \omega \)

Existence of a \( \mathbb{E} \) solution \( \mathbb{P}_t \)

Lemma 4.5 yields for all \( y \in D, z_1, z_2 \in H, h, \varepsilon \in (0, \infty) \) with \( y + h z_2 \in D \) that

This, independence of \( \sigma(\{X_{x,t}^\epsilon : x \in D\}) \) and \( \mathbb{P}_t \), a disintegration formula (e.g. [2, Lemma 2.3]), the fact that for all \( \omega \in \Omega \) it holds that \( (D \ni x \mapsto X_{x,t}^\epsilon(\omega) \in H) \in C^2(D, H) \), and Fatou’s lemma (e.g. Lemma 3.10 in [11]) yield that

This implies \((75)\) and finishes the proof of Lemma 4.5. \( \square \)

4.3 Existence of a \( C^1 \)-solution

The following theorem establishes existence of continuously differentiable solutions of SDEs.

Theorem 4.6 (Existence of a \( C^1 \)-solution). Assume Setting \( 4.2 \), assume that \( \dim(H) < \infty \), assume that \( O \subseteq \overline{O} \), and assume that \( p \in (2 \dim(H) + 6, \infty) \). Then there exists a measurable function \( \mathbb{X} : \Delta_T \times \overline{O} \times \Omega \to \overline{O} \) such that

\[
\mathbb{X}(r, s, t, x, y, z, v, w) = \begin{cases} 
X_{x,t}^\epsilon(y + h z_2 + j \varepsilon z_1) & \text{if } j \in \{0,1\}, h \geq 0, \varepsilon \geq 0, \\
X_{x,t}^\epsilon(y + h z_2) & \text{if } h \geq 0, j \geq 0, \varepsilon \geq 0.
\end{cases}
\]
(i) for all $x \in \mathcal{O}$, $s \in [0, T]$ it holds a.s. that $(\mathcal{X}^x_{s,t})_{t \in [s,T]} = (X^x_{s,t})_{t \in [s,T]}$, and

(ii) for every $\omega \in \Omega$ it holds that $\mathcal{X}(\omega) \in C^{0,1}(\Delta_T \times O, \overline{O})$.

Proof of Theorem 4.6. Without loss of generality we additionally assume throughout this proof that $\dim(H) \geq 1$ and that $O \neq \emptyset$. Throughout this proof let $o \in O$, $d \in \mathbb{N}$, $\mathbb{H} \subseteq H$ satisfy that $d = \dim(H)$ and that $\mathbb{H}$ is an orthonormal basis of $H$, let $\mathcal{O}^R$, $\mathcal{O}^E$ be the sets which satisfy that $\mathcal{O}^R = \cap_{n \in \mathbb{H}} \{ \langle x, h \rangle \in O \times \mathbb{R} : x + vh \in O \}$ and $\mathcal{O}^E = \cap_{n \in \mathbb{H}} \{ \langle x, h \rangle \in O \times (\mathbb{R} \setminus \{0\}) : x + vh \in O \}$ and let $K_n \subseteq \Delta_T \times H \times \mathbb{R}$, $n \in \mathbb{N}$, be the sets which satisfy for all $n \in \mathbb{N}$ that $K_n = \{ (s, t, x, h) \in \Delta_T \times O \times \mathbb{R} : s^2 + t^2 + \| x \|^2 + h^2 \leq n^2 \}$. Lemma 4.2 the fact that $p > 2d + 6 \geq 2d + 4$, and Theorem 3.4 (applied with $\rho \cap 2p(1 + \theta)(1 + \delta)$ in the notation of Theorem 3.4) yield that there exists a measurable function $\overline{X} : \Delta_T \times \mathcal{O} \times \Omega \to \overline{O}$ such that for all $x \in \mathcal{O}$, $s \in [0, T]$ it holds a.s. that $\overline{X}(\omega) \in C^0(\Delta_T \times O, \overline{O})$.

Next, Lemma 4.4 the fact that $\int_0^T \phi(r) dr < \infty$, and boundedness of the functions $V_0, V_1$ on each of the bounded subsets $\{ x \in \mathcal{O} : \exists s, t, h \in \mathbb{R} \text{ s.t. } (s, t, x, h) \in K_n \} \subseteq \mathcal{O}$, $n \in \mathbb{N}$, demonstrate for all $n \in \mathbb{N}$, $v \in \mathbb{H}$ that

$$\sup \left\{ \left( \mathbb{E} \left[ \| D^x_{s,t}(v) \|^2_H \right] \right)^{\frac{1}{2}} : (s, t, x, h) \in K_n \right\} < \infty.$$  \hspace{1cm} (78)

In particular this implies for all $n \in \mathbb{N}$, $v \in \mathbb{H}$ that

$$\sup \left\{ \left( \mathbb{E} \left[ \| D^x_{s,t}(v) \|^2_H \right] \right)^{\frac{1}{2}} : (s, t, x, h) \in K_n \right\} \leq \sup \left\{ \left( \mathbb{E} \left[ \| D^x_{s,t}(v) - D^x_{s,t}(v) \|^2_H \right] \right)^{\frac{1}{2}} : (s, t, x, h) \in K_n \right\} < \infty.$$  \hspace{1cm} (79)

This, (78), Proposition 2.2 (applied for every $v \in \mathbb{H}$ with $H \cap \mathbb{H} \times H \times \mathbb{R}, D \cap \Delta_T \times \mathcal{O}^E$, $E \cap H, F \cap H, p \cap p, \alpha \cap \frac{1}{2}, \mathcal{X} \cap \left( \left( \Delta_T \times \mathcal{O}^E \ni (s, t, x, h) \mapsto D^x_{s,t}(v) \in \Omega \right) \right)$ in the notation of Proposition 2.2), and path continuity of $D^x_{s,t}(v), (s, t, x, h) \in [0, T] \times \mathcal{O}^E \times \mathbb{H}$, establish for every $v \in \mathbb{H}$ the existence of a measurable function $D^x_{s,t}(v) : [0, T] \times \mathcal{O}^E \times \mathbb{H}$ which satisfies that for all $\omega \in \Omega$ it holds that $D^x_{s,t}(\omega) \in C(\Delta_T \times \mathcal{O}^E \times \mathcal{O}^E \times \mathbb{H})$ and which satisfies that for all $(s, t, x, h) \in \Delta_T \times \mathcal{O}^E$ it holds a.s. that $(D^x_{s,t}(v)(x, h))_{t \in [s,T]} = (D^x_{s,t}(v))_{t \in [s,T]}$. Note that $O \subseteq \mathcal{O}$ and convexity of $O$ imply that $\mathcal{O}^E \subseteq \mathcal{O}^E$. Let $\mathcal{D} : \Delta_T \times \mathcal{O}^E \times \mathcal{O}^E \times \mathcal{O} \to \mathbb{H}$ be the function which satisfies for all $(s, t, x, h) \in \Delta_T \times \mathcal{O}^E$, $v \in H$ that $\mathcal{D}_{s,t}(x, h, v) = \sum_{c \in \mathbb{H}} \langle v, e \rangle_H D^c_{s,t}(v, x, h)$. Next, we observe that for all $(s, t, x, h, v) \in [0, T] \times \mathcal{O} \times \mathbb{H}$ it holds a.s. for all $t \in [s, T]$ that

$$D^x_{s,t}(x, h, v) = D^x_{s,t}(x, h) = D^x_{s,t}(v) = \mathcal{X}^x_{s,t,h}(\omega) = \mathcal{X}^x_{s,t,h}(\omega).$$  \hspace{1cm} (80)

This, continuity of the random fields $\mathcal{X}$, $\mathcal{D}$, and Lemma 2.1 (applied with $U \cap H, T \cap \Delta_T, T \cap \Delta_T \cap \mathcal{O}^E, \mathcal{X} \cap \left( \left( \Delta_T \times \mathcal{O} \times \mathcal{O} \ni (s, t, x, \omega) \mapsto \mathcal{X}^x_{s,t} \in \mathcal{H} \right) \right)$, $\mathcal{Z} \cap \mathcal{D}$ in the notation of Lemma 2.1) prove that there exists $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and such that for all $\omega \in \Omega_0$, $(s, t) \in \Delta_T$ it holds that the mapping $O \ni x \mapsto \mathcal{X}^x_{s,t}(\omega) \in \mathcal{H}$ is continuously differentiable and it holds for all $x \in O$, $v \in H$ that

$$\mathcal{D}_{s,t}(x, h, v) = \mathcal{D}_{s,t}(x, h) = \mathcal{D}_{s,t}(v) = \mathcal{X}^x_{s,t,h}(\omega) = \mathcal{X}^x_{s,t,h}(\omega).$$  \hspace{1cm} (81)

This and continuity of $\mathcal{D}$ prove that for all $\omega \in \Omega_0$ it holds that $\mathcal{X}(\omega)|_{\Delta_T \times O} \in C^{0,1}(\Delta_T \times O, \overline{O})$. Let $\mathcal{X} : \Delta_T \times \mathcal{O} \times \mathcal{O} \to \overline{O}$ be the function which satisfies for all $(s, t, x, \omega) \in \Delta_T \times \mathcal{O} \times \mathcal{O}$ that $\mathcal{X}^x_{s,t}(\omega) = \mathcal{I}_{\Omega_0}(\omega) \mathcal{X}^x_{s,t}(\omega) + o_{\mathcal{P}(\Omega_0)}(\omega)$. Then it holds that $\mathcal{X}$ is measurable, that for all $x \in \mathcal{O}$, $s \in [0, T]$ it holds a.s. that $(\mathcal{X}_{s,t})_{t \in [s,T]} = (\mathcal{X}^x_{s,t})_{t \in [s,T]} = (X^x_{s,t})_{t \in [s,T]}$, and that for every $\omega \in \Omega$ it holds that $\mathcal{X}(\omega) \in C^{0,1}(\Delta_T \times O, \overline{O})$. This proves item (i) and item (ii) and finishes the proof of Theorem 1.6. \hfill \square
5 Existence of a $C^2$-solution

In this section we prove a strong local Hölder estimate in Lemma [5,4] below and establish existence of a twice continuously differentiable solution under suitable assumptions in Theorem [5,5] below. First, we introduce the setting for these results.

**Setting 5.1.** Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ be a filtered probability space satisfying the usual conditions, let $(W_t)_{t \in [0, T]}$ be an $\text{Id}_U$-cylindrical $(\mathcal{F}_t)_{t \in [0, T]}$-Wiener process, let $\Delta_T = \{(s, t) \in [0, T]^2 : s \leq t\}$, let $O \subseteq H$ be an open set, let $\mathcal{O} \subseteq O$ be a convex set, let $\mu \in C^2(O, H)$, $\sigma \in C^2(O, \mathbb{R}^2)$, let $X : \Omega \rightarrow C^0([\Delta_T \times O, \mathcal{O})$ be a function which satisfies for all $(s, t) \in \Delta_T$, $x \in \mathcal{O}$ that $X^x_{s,t} : [s, T] \times \Omega \rightarrow \mathcal{O}$ is an $(\mathcal{F}_t)_{t \in [s, T]}$-adapted stochastic process and that a.s. it holds that

\[ X^x_{s,t} = x + \int_s^t \mu(X^x_{s,r}) \, dr + \int_s^t \sigma(X^x_{s,r}) \, dW_r, \quad (s, t) \in \Delta_T \]  

(82)

let $\alpha_0, \alpha_1, \beta_0, \beta_1, c \in [0, \infty)$, $V_0, V_1 \in C^2(O, [0, \infty))$, let $\bar{V} : [0, T] \times \mathcal{O} \rightarrow [0, \infty)$ be a measurable function which satisfies for all $i \in \{0, 1\}, t \in [0, T], x \in \mathcal{O}$ that

\[ \left\langle \mu(x), (\nabla V_i)(x) \right\rangle_H + \frac{1}{2} \text{trace} \left( \sigma(x) \sigma(x)^* (\text{Hess} V_i)(x) \right) \]

\[ + \frac{1}{2c_1} \|\sigma(x)^*(\nabla V_i)(x)\|^2_U + 1(t)(i) \cdot \bar{V}(t, x) \leq \alpha_i V_i(x) + \beta_i, \]

let $\phi : [0, T] \rightarrow [0, \infty)$ be a measurable and integrable function, let $p \in [2, \infty)$, $\theta \in (0, \infty)$, $\delta \in (0, \infty)$, $q_0, q_1 \in (0, \infty)$ satisfy that

\[ \frac{\theta}{6p(1+\theta)^3(1+\delta)^2} = \frac{1}{q_0} + \frac{1}{q_1}, \]

assume for all $t \in [0, T], x, y \in \mathcal{O}, v \in H \setminus \{0\}$ that

\[ \left\langle v, \int_0^t \mu'(\lambda x + (1 - \lambda)y) + \delta \, d\lambda \right\rangle_H + \frac{1+\delta}{2} \left\| \int_0^t \sigma'(\lambda x + (1 - \lambda)y) \, d\lambda \right\|_{HS(U,H)}^2 \]

\[ + \frac{3p(1+\theta)^3(1+\delta)^2-1}{\|v\|_H^2} \left\| \left\langle v, \int_0^t \sigma'(\lambda x + (1 - \lambda)y) \, d\lambda \right\rangle_H \right\|_{HS(U,H)} \]

\[ \leq \|v\|_H^2 \left( \phi(t) + \frac{V_0(x) + V_0(y)}{2q_0 T e^{\theta t}} + \frac{\bar{V}(t,x) + \bar{V}(t,y)}{2q_1 e^{\theta t}} \right), \]

let $\gamma \in [\frac{1}{p}, \infty)$ satisfy for all $x \in \mathcal{O}$ that

\[ \max \left\{ \| \mu(x) \|_H, \| \sigma(x) \|_{HS(U,H)} \right\} \leq c(1 + V_0(x))^\gamma, \]

(85)

assume for all $x, y \in \mathcal{O}, i \in \{1, 2\}$ that

\[ \max \left\{ \left\| \int_0^t D^i \mu(\lambda x + (1 - \lambda)y) \, d\lambda \right\|_{L^1(H,H)}, \left\| \int_0^t D^i \sigma(\lambda x + (1 - \lambda)y) \, d\lambda \right\|_{L^1(H,HS(U,H))} \right\} \]

\[ \leq c(2 + V_0(x) + V_0(y))^\gamma, \]

(86)

assume for all $x_1, x_2, x_3, x_4 \in \mathcal{O}, i \in \{1, 2\}$ that

\[ \max \left\{ \left\| \int_0^t D^i \mu(\lambda x_1 + (1 - \lambda)x_2) - D^i \mu(\lambda x_3 + (1 - \lambda)x_4) \, d\lambda \right\|_{L^1(H,H)}, \right\}

\[ \left\| \int_0^t D^i \sigma(\lambda x_1 + (1 - \lambda)x_2) - D^i \sigma(\lambda x_3 + (1 - \lambda)x_4) \, d\lambda \right\|_{L^1(H,HS(U,H))} \}

\[ \leq c \int_0^t \lambda \|x_1 - x_2\|_H + (1 + \lambda)\|x_2 - x_4\|_H \, d\lambda \left( 4 + \sum_{j=1}^4 V_0(x_i) \right)^\gamma, \]

(87)

and for all $(s, t) \in \Delta_T$, $x \in \mathcal{O}, v, w \in H$, $h \in \mathbb{R} \setminus \{0\}$ with $x + hw \in \mathcal{O}$ let $D_{x,t}^h(w), D_{x,t}^h(v, w) : \Omega \rightarrow H$ be the functions which satisfy that

\[ D_{x,t}^h(w) = \frac{X_{s,t}^{x+hw} - X_{s,t}^x}{h}, \]

(88)
\[ D^{x,h}_{s,t}(v, w) = \frac{\partial}{\partial z} X^{x+h\omega}_{s,t}(v) - \frac{\partial}{\partial z} X^{x}_{s,t}(v) \]  

(89)

Lemma 5.2. Assume Setting 5.1 and let \( t \in [0, T] \), \( x, y \in \mathcal{O} \) satisfy \( x \neq y \). Then

\[
\langle x - y, \mu(x) - \mu(y) \rangle_H + \frac{1}{2} \| \sigma(x) - \sigma(y) \|^2_{HS(U,H)} \leq \| x - y \|^2_H \left( \phi(t) + \frac{V_0(x) + V_0(y)}{2q_U e^{\alpha t}} + \frac{V(t,x) + V(t,y)}{2q_U e^{\alpha t}} \right).
\]

(90)

Proof of Lemma 5.2 The fundamental theorem of calculus, convexity of \( \mathcal{O} \), and assumption (84) (applied with \( v \wedge x - y \) in the notation of assumption (84)) yield that

\[
\langle x - y, \mu(x) - \mu(y) \rangle_H + \frac{1}{2} \| \sigma(x) - \sigma(y) \|^2_{HS(U,H)} \leq \| x - y \|^2_H \left( \phi(t) + \frac{V_0(x) + V_0(y)}{2q_U e^{\alpha t}} + \frac{V(t,x) + V(t,y)}{2q_U e^{\alpha t}} \right).
\]

This completes the proof of Lemma 5.2  

Lemma 5.3 (Second order difference processes satisfy affine-linear SDEs). Assume Setting 5.1 and let \( (s, t) \in \Delta_T, \ x, v, w \in H, \ h \in \mathbb{R} \setminus \{0\} \) satisfy \( x + wh \in \mathcal{O} \). Then it holds a.s. that

\[
D^{x,h}_{s,t}(v, w) = \int_s^t \mu'(X^x_{s,r}) D^{x,h}_{s,r}(v, w) \, dr + \int_s^t \sigma'(X^x_{s,r}) D^{x,h}_{s,r}(v, w) \, dW_r
\]

\[ + \int_s^t \mu''(\lambda X^{x+wh}_{s,r} + (1 - \lambda) X^x_{s,r}) \, d\lambda \left( D^{x,h}_{s,r}(w), \frac{\partial}{\partial z} X^{x+wh}_{s,r}(v) \right) \, dr
\]

\[ + \int_s^t \sigma''(\lambda X^{x+wh}_{s,r} + (1 - \lambda) X^x_{s,r}) \, d\lambda \left( D^{x,h}_{s,r}(w), \frac{\partial}{\partial z} X^{x+wh}_{s,r}(v) \right) \, dW_r.
\]

(92)

Proof of Lemma 5.3 The chain rule, continuous differentiability of \( \mu, \sigma, \) and of \( X^x_{s,r}(\omega), \ r \in [s, t], \ \omega \in \Omega, \) and equation (82) imply that for all \( z \in O \) it holds a.s. that

\[
\frac{\partial}{\partial z} X^{x}_{s,t}(v) = v + \int_s^t \mu' X^{x}_{s,r} \frac{\partial}{\partial z} X^{z}_{s,r}(v) \, dr + \int_s^t \sigma' X^{x}_{s,r} \frac{\partial}{\partial z} X^{z}_{s,r}(v) \, dW_r.
\]

(93)

This and equation (82) yield that it holds a.s. that

\[
D^{x,h}_{s,t}(v, w) = \frac{v - w}{h} + \int_s^t \frac{\mu'(X^{x+wh}_{s,r})}{h} \frac{\partial}{\partial z} X^{x+wh}_{s,r}(v) - \frac{\mu'(X^x_{s,r})}{h} \frac{\partial}{\partial z} X^{x+wh}_{s,r}(v) \, dr
\]

\[ + \int_s^t \frac{\mu'(X^x_{s,r})}{h} \frac{\partial}{\partial z} X^{x}_{s,r}(v) - \frac{\mu'(X^x_{s,r})}{h} \frac{\partial}{\partial z} X^{x+wh}_{s,r}(v) \, dr
\]

\[ + \int_s^t \frac{\sigma'(X^{x+wh}_{s,r})}{h} \frac{\partial}{\partial z} X^{x+wh}_{s,r}(v) - \frac{\sigma'(X^x_{s,r})}{h} \frac{\partial}{\partial z} X^{x}_{s,r}(v) \, dW_r
\]

\[ + \int_s^t \frac{\sigma'(X^x_{s,r})}{h} \frac{\partial}{\partial z} X^{x+wh}_{s,r}(v) - \frac{\sigma'(X^x_{s,r})}{h} \frac{\partial}{\partial z} X^{x}_{s,r}(v) \, dW_r.
\]

(94)
This, \( \mu \in C^2(\Omega, H), \sigma \in C^2(\Omega, \text{HS}(U, H)), \) and the fundamental theorem of calculus yield that it holds a.s. that

\[
D_{s,t}^{x,h}(v, w) = \int_s^t \mu'(X_{s,r})D_{s,r}^{x,h}(v, w) \, dr + \int_s^t \sigma'(X_{s,r})D_{s,r}^{x,h}(v, w) \, dW_r
+ \int_s^t \int_0^1 \mu''(\lambda X_{s,r}^{x+wh} + (1 - \lambda)X_{s,r}^x) \, d\lambda D_{s,r}^{x,h}(w) \, \frac{d}{dr}X_{s,r}^{x+wh} \, dr
+ \int_s^t \int_0^1 \sigma''(\lambda X_{s,r}^{x+wh} + (1 - \lambda)X_{s,r}^x) \, d\lambda D_{s,r}^{x,h}(w) \, \frac{d}{dr}X_{s,r}^{x+wh} \, dW_r.
\]

(95)

The proof of Lemma [5,3] is thus completed.

\[\square\]

### 5.1 Strong local Hölder estimate

The following lemma proves strong local Hölder continuity of the second-order difference quotients.

**Lemma 5.4** (Strong local Hölder estimate for second order difference quotients). Assume Setting [7,1] and let \( s_1, s_2, t_1, t_2 \in [0, T], x_1, x_2 \in \Omega, v_1, v_2, w_1, w_2 \in H, h_1, h_2 \in \mathbb{R} \setminus \{0\} \) satisfy that \( s_1 \leq t_1, s_2 \leq t_2, x_1 + w_1 h_1, x_2 + w_2 h_2 \in O, \) and \( s_1 \leq s_2. \) Then it holds

\[
\begin{align*}
\left\| D_{s_1,t_1}^{x_1,h_1}(v_1, w_1) - D_{s_2,t_2}^{x_2,h_2}(v_2, w_2) \right\|_{L^p(P, H)} & \leq (1 + \max_{i \in \{1, 2\}} \|w_i\|H)^2 (1 + \|v_i\|H)^2 (1 + 4c)(p^2 + T) \\
& \cdot \left( \|x_1 - x_2\|_H + \|w_1 h_1 - w_2 h_2\|_H + \sqrt{|s_1 - s_2|} + \sqrt{|t_1 - t_2|} + \|w_1 - w_2\|_H + \|v_1 - v_2\|_H \right) \\
& \cdot \left[ e^{2\alpha_0 T} \left( 12 \frac{p(1+\theta)^3(1+\theta^2)^2}{\min\{\delta, 1\}} + 2\sqrt{T} + \int_0^T \frac{\theta_0}{e^{\alpha_0 t}} \, dt + 2 \max_{i \in \{0, 1\}, j \in \{1, 2\}} V_0(x_j + \iota w_j h_j) \right)^{2\gamma + 2} \\
& \cdot (1 + c)^2 \max_{i \in \{0, 1\}, j \in \{1, 2\}} \exp \left( 3 \int_0^T \left[ \phi(u) + \frac{1}{q_1 e^{\alpha_0 u}} \right] \, du + 3 \sum_{i=0}^{q_1 e^{\alpha_0 u}} V_0(x_i + \iota w_i h_i) \right)^5 \\
& \cdot \max_{i \in \{0, 1\}, i \in \{1, 2\}} 4^\gamma e^{\alpha_0 T} \left( \frac{p(1+\theta)(1+\delta)}{\delta} + \int_{s_1}^T \frac{\theta_0}{e^{\alpha_0 t}} \, dt + V_0(x_i + \iota w_i h_i) \right)^\gamma. 
\end{align*}
\]

Proof of Lemma [5,3] Without loss of generality we additionally assume throughout this proof that \( q_0 + q_1 < \infty \) (otherwise apply the result for each sufficiently large \( n \in \mathbb{N} \) with \( \theta_n, \delta_n, q_0, n, q_1, n \in (0, \infty) \) such that \( q_0, n = \min\{q_0, n\}, q_1, n = \min\{q_1, n\}, \) \( \delta_n(1+\delta_n)^3 = \theta_n e^{\alpha_0 n} \) and let \( n \to \infty). \) Throughout this proof let \( Y, a, \zeta^\nu : [0, T - s_2] \times \Omega \to \text{HS}(U, H), \) and \( \eta : [0, T - s_2] \times \Omega \to L(H, \text{HS}(U, H)) \) be the functions which satisfy for all \( r \in [0, T - s_2] \) that

\[
a_r = \mu'(X_{s_1, s_2+r}^{x_1})D_{s_1,s_2+r}^{x_1,h_1}(v_1, w_1) - \mu'(X_{s_2, s_2+r}^{x_2})D_{s_2,s_2+r}^{x_2,h_2}(v_2, w_2) \\
+ \int_0^1 \mu''(\lambda X_{s_1,s_2+r}^{x_1+w_1 h_1} + (1 - \lambda)X_{s_1,s_2+r}^{x_1}) \, d\lambda \left( D_{s_1,s_2+r}^{x_1,h_1}(w_1), \frac{\partial}{\partial x}X_{s_1,s_2+r}^{x_1+w_1 h_1}(v_1) \right),
\]

\[
b_r = \sigma'(X_{s_1, s_2+r}^{x_1})D_{s_1,s_2+r}^{x_1,h_1}(v_1, w_1) - \sigma'(X_{s_2, s_2+r}^{x_2})D_{s_2,s_2+r}^{x_2,h_2}(v_2, w_2) \\
+ \int_0^1 \sigma''(\lambda X_{s_1,s_2+r}^{x_1+w_1 h_1} + (1 - \lambda)X_{s_1,s_2+r}^{x_1}) \, d\lambda \left( D_{s_1,s_2+r}^{x_1,h_1}(w_1), \frac{\partial}{\partial x}X_{s_1,s_2+r}^{x_1+w_1 h_1}(v_1) \right),
\]

(97)

(98)
and

\[ Y_t = D^{x_{1}, h_1}_{s_1, s_2 + t}(v_1, w_1) - D^{x_{2}, h_2}_{s_2, s_2 + t}(v_2, w_2), \quad (99) \]

\[ \eta_t = \sigma'(X_{s_2, s_2 + r}^{x_2}), \quad (100) \]

\( \zeta_r^\mu = \left( \mu'(X_{s_1, s_2 + r}^{x_1}) - \mu'(X_{s_2, s_2 + r}^{x_2}) \right) D^{x_1, h_1}_{s_1, s_2 + r}(v_1, w_1) \]

\[ + \int_0^1 \mu'' \left( \lambda X_{s_1, s_2 + r}^{x_1 + w h_1} + (1 - \lambda) X_{s_1, s_2 + r}^{x_1} \right) d\lambda \left( D^{x_1, h_1}_{s_1, s_2 + r}(w_1), \frac{\partial}{\partial w} X_{s_1, s_2 + r}^{x_1 + w h_1}(v_1) \right) \]

\[ - \int_0^1 \mu'' \left( \lambda X_{s_2, s_2 + r}^{x_2 + w h_2} + (1 - \lambda) X_{s_2, s_2 + r}^{x_2} \right) d\lambda \left( D^{x_2, h_2}_{s_2, s_2 + r}(w_2), \frac{\partial}{\partial w} X_{s_2, s_2 + r}^{x_2 + w h_2}(v_2) \right), \quad (101) \]

\( \zeta_r^\sigma = \left( \sigma'(X_{s_1, s_2 + r}^{x_1}) - \sigma'(X_{s_2, s_2 + r}^{x_2}) \right) D^{x_1, h_1}_{s_1, s_2 + r}(v_1, w_1) \]

\[ + \int_0^1 \sigma'' \left( \lambda X_{s_1, s_2 + r}^{x_1 + w h_1} + (1 - \lambda) X_{s_1, s_2 + r}^{x_1} \right) d\lambda \left( D^{x_1, h_1}_{s_1, s_2 + r}(w_1), \frac{\partial}{\partial w} X_{s_1, s_2 + r}^{x_1 + w h_1}(v_1) \right) \]

\[ - \int_0^1 \sigma'' \left( \lambda X_{s_2, s_2 + r}^{x_2 + w h_2} + (1 - \lambda) X_{s_2, s_2 + r}^{x_2} \right) d\lambda \left( D^{x_2, h_2}_{s_2, s_2 + r}(w_2), \frac{\partial}{\partial w} X_{s_2, s_2 + r}^{x_2 + w h_2}(v_2) \right). \quad (102) \]

Note that \( Y, a, b, \eta, \zeta^\mu, \zeta^\sigma \) are \((\mathbb{F}_{s_2 + r})_{r \in [0, T - s_2]}\)-adapted stochastic processes with continuous sample paths and, therefore, measurable. Lemma 5.3 implies that for all \( t \in [0, T - s_2] \) it holds a.s. that

\[ Y_t = D^{x_1, h_1}_{s_1, s_2 + t}(v_1, w_1) - D^{x_2, h_2}_{s_2, s_2 + t}(v_2, w_2) = D^{x_1, h_1}_{s_1, s_2 + t}(v_1, w_1) + \int_{s_2}^{s_2 + t} a_r dW_r + \int_{s_2}^{s_2 + t} b_r dw \]

\[ = D^{x_1, h_1}_{s_1, s_2 + t}(v_1, w_1) + \int_{0}^{t} a_r dr + \int_{0}^{t} b_r d(W_{s_2 + r} - W_{s_2}). \quad (103) \]

We consider the one-sided affine-linear growth condition for the Itô process \( Y \). Equations \((97), (99), \) and \((101)\) imply for all \( r \in [0, T - s_2] \) that

\[ a_r = \mu'(X_{s_2, s_2 + r}^{x_2}) \left( D^{x_1, h_1}_{s_1, s_2 + r}(v_1, w_1) - D^{x_2, h_2}_{s_2, s_2 + r}(v_2, w_2) \right) \]

\[ + \left( \mu'(X_{s_1, s_2 + r}^{x_1}) - \mu'(X_{s_2, s_2 + r}^{x_2}) \right) D^{x_1, h_1}_{s_1, s_2 + r}(v_1, w_1) \]

\[ + \int_0^1 \mu'' \left( \lambda X_{s_1, s_2 + r}^{x_1 + w h_1} + (1 - \lambda) X_{s_1, s_2 + r}^{x_1} \right) d\lambda \left( D^{x_1, h_1}_{s_1, s_2 + r}(w_1), \frac{\partial}{\partial w} X_{s_1, s_2 + r}^{x_1 + w h_1}(v_1) \right) \]

\[ - \int_0^1 \mu'' \left( \lambda X_{s_2, s_2 + r}^{x_2 + w h_2} + (1 - \lambda) X_{s_2, s_2 + r}^{x_2} \right) d\lambda \left( D^{x_2, h_2}_{s_2, s_2 + r}(w_2), \frac{\partial}{\partial w} X_{s_2, s_2 + r}^{x_2 + w h_2}(v_2) \right), \quad (104) \]

Analogously, equations \((98), (99), (102), \) and \((100)\) imply for all \( r \in [0, T - s_2] \) that

\[ b_r = \sigma'(X_{s_2, s_2 + r}^{x_2}) Y_r + \zeta_r^\sigma = \eta_r Y_r + \zeta_r^\sigma. \quad (105) \]

Equation \((104)\), the Cauchy-Schwarz inequality, and Young’s inequality yield for all \( r \in [0, T - s_2] \) that

\[ \langle Y_r, a_r \rangle_H \leq \langle Y_r, \mu'(X_{s_2, s_2 + r}) Y_r \rangle_H + \| Y_r \|_H^2 \| \zeta_r^\mu \|_H \]

\[ \leq \langle Y_r, \left( \mu'(X_{s_2, s_2 + r}) + \delta \right) Y_r \rangle_H + \frac{1}{2\delta} \| \zeta_r^\sigma \|_H^2. \quad (106) \]

Similarly, equation \((105)\), the Cauchy-Schwarz inequality, and Young’s inequality imply for all
\[ r \in [0, T - s_2] \text{ that} \]
\[ \frac{1}{2} \left\| b_r \right\|^2_{H^2(U,H)} + \frac{p(1+\theta)-2}{2} \left\| (Y_r,Y_r)^{\#}_{H^2(U,H)} \right\|^2_{H^2(U,H)} = \frac{1}{2} \left\| \eta_r Y_r + \zeta^\sigma \right\|^2_{H^2(U,H)} + \frac{p(1+\theta)-2}{2} \left\| (Y_r,Y_r+\zeta^\sigma)^{\#}_{H^2(U,H)} \right\|^2_{H^2(U,H)} \]
\[ = \frac{1}{2} \left\| \eta_r Y_r \right\|^2_{H^2(U,H)} + \left\| \eta_r Y_r \right\|^2_{H^2(U,H)} + \frac{1}{2} \left\| \zeta^\sigma \right\|^2_{H^2(U,H)} + \frac{p(1+\theta)-2}{2} \left\| (Y_r,Y_r)^{\#}_{H^2(U,H)} \right\|^2_{H^2(U,H)} + \frac{p(1+\theta)-2}{2} \left\| (Y_r,Y_r)^{\#}_{H^2(U,H)} \right\|^2_{H^2(U,H)} \]
\[ \leq \frac{1}{2} \left\| \eta_r Y_r \right\|^2_{H^2(U,H)} + \left\| \eta_r Y_r \right\|^2_{H^2(U,H)} + \frac{1}{2} \left\| \zeta^\sigma \right\|^2_{H^2(U,H)} + \frac{p(1+\theta)-2}{2} \left\| (Y_r,Y_r)^{\#}_{H^2(U,H)} \right\|^2_{H^2(U,H)} + \frac{p(1+\theta)-2}{2} \left\| (Y_r,Y_r)^{\#}_{H^2(U,H)} \right\|^2_{H^2(U,H)} \]
\[ = \frac{1}{2} \left\| \eta_r Y_r \right\|^2_{H^2(U,H)} + \left( \frac{p(1+\theta)-2}{2} \left\| (Y_r,Y_r)^{\#}_{H^2(U,H)} \right\|^2_{H^2(U,H)} + \frac{1}{2} \left\| \zeta^\sigma \right\|^2_{H^2(U,H)} \right)^2 \]
This implies that
\[ \left\| D^{x_1,h_1}(v_1, w_1) - D^{x_2,h_2}(v_2, w_2) \right\|_{L^p(P;H)} \]
\[ = \left\| D^{x_1,h_1}(v_1, w_1) - D^{x_2,h_2}(v_2, w_2) \right\|_{L^p(P;H)} \]
\[ \leq \left( \left\| D^{x_1,h_1}(v_1, w_1) \right\|_{L^p(P;H)} + \sup_{r \in [0, T - s_2]} \max \left\{ \left\| \zeta^\mu_r \right\|_{L^p(1+\theta)(P;H)}, \left\| \zeta^\sigma_r \right\|_{L^p(1+\theta)(P;HS(U,H))} \right\} \cdot \sqrt{t_2 p(1+\theta)(1+\delta)} \cdot \exp \left( \int_{s_2}^T \left[ \phi(r) + \frac{\delta_0}{\theta_0 e^{\theta_0 r}} + \frac{\delta_1}{\theta_1 e^{\theta_1 r}} \right] \, dr + \sum_{i=0}^1 V_i(x_2) \right) \right) . \]

Moreover, \[ (101) \] the fact that for all \( a_1, a_2, a_3 \in \mathbb{R}, b_1, b_2, b_3 \in \mathbb{R} \) it holds that \( a_1 a_2 a_3 - b_1 b_2 b_3 = (a_1 - b_1)a_2 a_3 + b_1 (a_2 - b_2)a_3 + b_1 b_2 (a_3 - b_3) \), and the triangle inequality yield for all \( r \in [0, T - s_2] \) that
\[ \left\| \zeta^\mu_r \right\|_{L^p(1+\theta)(P;H)} \leq \left\| \mu'(X_{x_1,s_2+r}) - \mu'(X_{x_2,s_2+r}) \right\|_{L^p(H,H)} \left\| D^{x_1,h_1}(v_1, w_1) \right\|_{H} \left\| D^{x_2,h_2}(v_2, w_2) \right\|_{H} \]
\[ + \left\| \int_0^1 \mu''(\lambda X_{x_1,s_2+r} + (1 - \lambda) X_{x_2,s_2+r}) \left( \frac{\partial}{\partial x} X_{x_1,s_2+r}(v_1) \right) \right\|_{L^p(1+\theta)(P;H)} \]
\[ \cdot \left\| D^{x_1,h_1}(v_1, w_1) \right\|_{H} \left\| D^{x_2,h_2}(v_2, w_2) \right\|_{H} \]
\[ + \left\| \int_0^1 \mu''(\lambda X_{x_1,s_2+r} + (1 - \lambda) X_{x_2,s_2+r}) \left( \frac{\partial}{\partial x} X_{x_2,s_2+r}(v_2) \right) \right\|_{L^p(1+\theta)(P;H)} \]
\[ \cdot \left\| D^{x_2,h_2}(v_2, w_2) \right\|_{H} \left\| D^{x_2,h_2}(v_2, w_2) \right\|_{H} \].

Then \[ (56), (57), \] and Hölder’s inequality (applied with \( \frac{1}{p(1+\theta)} = \frac{3}{3p(1+\theta)(1+\delta)} + \frac{\delta}{p(1+\theta)(1+\delta)} \)) show for all \( r \in [0, T - s_2] \) that
\[ \left\| \zeta^\mu_r \right\|_{L^p(1+\theta)(P;H)} \leq \left( 1 + \max_{i \in \{1, 2\}} \left\| D^{x_1,h_1}(v_1, w_1) \right\|_{L^p(1+\theta)(H)} \left( 1 + \left\| \frac{\partial}{\partial x} X_{x_1,s_2+r}(v_1) \right\|_{L^p(1+\theta)(H)} \right) \right) \]
\[ \cdot \left( 2c \max_{i \in \{0, 1\}} \left\| X_{x_1,s_2+r} - X_{x_2,s_2+r} \right\|_{L^p(1+\theta)(H)} \right) \]
\[ + c \left\| D^{x_1,h_1}(v_1, w_1) - D^{x_2,h_2}(v_2, w_2) \right\|_{L^p(1+\theta)(H)} \]
\[ + c \left\| \frac{\partial}{\partial x} X_{x_1,s_2+r}(v_1) - \frac{\partial}{\partial x} X_{x_2,s_2+r}(v_2) \right\|_{L^p(1+\theta)(H)} \]
\[ \cdot \max_{i \in \{0, 1\}} \left( 4 + V_0(X_{x_1,s_2+r}) + V_0(X_{x_1,s_2+r}) + V_0(X_{x_2,s_2+r}) + V_0(X_{x_2,s_2+r}) \right) \frac{\delta}{p(1+\theta)(1+\delta)} . \]

(113)
This, Fatou’s lemma, and the triangle inequality yield for all $r \in [0, T - s_2]$ that

$$
\left\| \zeta_r \right\|_{L^{p(1+\theta)}(\mathbb{P};H)} \leq \left( 1 + \liminf_{t \to 0} \frac{D^{x_1+h_1\gamma_1,\varepsilon}_{1,2} (x_1) - D^{x_1+h_1\gamma_1}_{1,2} (x_1)}{h_1} \right) \cdot \left( 1 + \max_{i \in \{1, 2\}} \left\| \frac{X_{x_1+h_1\gamma_1,\varepsilon} - X_{x_1+h_1\gamma_1}}{h_1} \right\|_{L^{p(1+\theta)}(\mathbb{P};H)} \right)
$$

(114)

Lemma 5.2 and Lemma 3.2 (applied for all $t \in [0, T]$), $u_1 \in [0, t]$, $u_2 \in [u_1, t]$, $y_1, y_2 \in O$ with $O \cap O$, $p \sim 6p(1 + \theta)^2(1 + \delta)^2$, $s_1 \cap u_1$, $s_2 \cap u_2$, $t_1 \cap t$, $t_2 \cap t$, $x_1 \cap y_1$, $x_2 \cap y_2$, $v_1 \cap \tilde{v}_1$, $v_2 \cap \tilde{v}_2$, $h_1 \cap \tilde{h}_1$, $h_2 \cap \tilde{h}_2$ in the notation of Lemma 5.2 yield for all $t \in [0, T]$), $u_1, u_2 \in [0, t]$, $y_1, y_2 \in O$, $\tilde{v}_1, \tilde{v}_2 \in H$ with $u_1 \leq u_2$ that

$$
\left\| \frac{X_{u_1, t} - X_{u_2, t}}{L^{p(1+\theta)}(\mathbb{P};H)} \right\| \leq \left\| \frac{X_{u_1, t} - X_{u_2, t}}{L^{p(1+\theta)(1+\delta)}(\mathbb{P};H)} \right\|
$$

(115)

In addition, Lemma 5.3 (applied for all $t \in [0, T]$, $u_1 \in [0, t]$, $u_2 \in [u_1, t]$, $y_1, y_2 \in O$, $\tilde{h}_1, \tilde{h}_2 \in \mathbb{R} \setminus \{0\}$, $\tilde{v}_1, \tilde{v}_2 \in H$ satisfying that $y_1 + \tilde{h}_1 \tilde{v}_1, y_2 + \tilde{h}_2 \tilde{v}_2 \in O$ with $O \cap O$, $p \sim 3p(1 + \theta)(1 + \delta)$, $s_1 \cap u_1$, $s_2 \cap u_2$, $t_1 \cap t$, $t_2 \cap t$, $x_1 \cap y_1$, $x_2 \cap y_2$, $v_1 \cap \tilde{v}_1$, $v_2 \cap \tilde{v}_2$, $h_1 \cap \tilde{h}_1$, $h_2 \cap \tilde{h}_2$ in the notation of Lemma 5.3) yields for all $t \in [0, T]$, $u_1, u_2 \in [0, t]$, $y_1, y_2 \in O$, $\tilde{h}_1, \tilde{h}_2 \in \mathbb{R} \setminus \{0\}$, $\tilde{v}_1, \tilde{v}_2 \in H$ with $y_1 + \tilde{h}_1 \tilde{v}_1, y_2 + \tilde{h}_2 \tilde{v}_2 \in O$ and $u_1 \leq u_2$ that

$$
\left\| \frac{D^{y_1, h_1}(\tilde{v}_1) - D^{y_2, h_2}(\tilde{v}_2)}{L^{p(1+\theta)(1+\gamma)}(\mathbb{P};H)} \right\|
$$

(116)
Next, (114), (115) (applied for all \( r \in [0, T - s_2] \)), \( \varepsilon \in (0, 1) \) with \( t \bowtie s_2 + r \), \( u_1 \bowtie s_1 \), \( u_2 \bowtie s_1 \), \( y_1 \bowtie x_1 + h_1 w_1 \), \( y_2 \bowtie x_1 \), \( \tilde{h}_1 \bowtie \varepsilon \), \( \tilde{h}_2 = \varepsilon \), \( \tilde{v}_1 = v_1 \), \( \tilde{v}_2 = v_1 \) in the notation of (116), (115) (applied for all \( i \in \{ 1, 2 \} \)) with \( u_1 \bowtie s_1 \), \( u_2 \bowtie s_1 \), \( y_1 \bowtie x_1 + w_1 h_1 \), \( y_2 \bowtie x_1 \), \( \tilde{h}_1 \bowtie \varepsilon \), \( \tilde{h}_2 \bowtie \varepsilon \), \( \tilde{v}_1 \bowtie v_1 \), \( \tilde{v}_2 \bowtie v_2 \) in the notation of (115), (115) (applied for all \( i \in \{ 0, 1 \} \)) with \( u_1 \bowtie s_1 \), \( u_2 \bowtie s_1 \), \( y_1 \bowtie x_1 + w_1 h_1 \), \( y_2 \bowtie x_1 + w_2 h_2 \), \( \tilde{h}_1 \bowtie \varepsilon \), \( \tilde{h}_2 \bowtie \varepsilon \), \( \tilde{v}_1 \bowtie v_1 \), \( \tilde{v}_2 \bowtie v_2 \) in the notation of (116), (116) (applied for all \( \varepsilon \in (0, 1) \) with \( u_1 \bowtie s_1 \), \( u_2 \bowtie s_2 \), \( y_1 \bowtie x_1 \), \( y_2 \bowtie x_1 \), \( h_1 = 0 \), \( h_2 = 2 \), \( v_1 \bowtie w_1 \), \( v_2 \bowtie w_2 \), \( \gamma \bowtie \varepsilon \), \( \tilde{h}_1 \bowtie \varepsilon \), \( \tilde{h}_2 \bowtie \varepsilon \), \( \tilde{v}_1 \bowtie v_1 \), \( \tilde{v}_2 \bowtie v_2 \) in the notation of (116), (116) (applied for all \( i \in \{ 0, 1 \} \), \( i \in \{ 1, 2 \} \), \( r \in [0, T - s_2] \) with \( s \bowtie s_1 \), \( X \bowtie X_{s_1}^{x_1 + \varepsilon w_1} \), \( \alpha \bowtie \alpha \), \( \beta \bowtie \beta \), \( V \bowtie V \), \( t \bowtie s_2 + r \), \( p \bowtie \frac{p(1 + \sigma)(1 + p)}{2} \) in the notation of Lemma 2.6) yield for all \( r \in [0, T - s_2] \) that

\[
\left\| \phi^\mu \right\|_{L^p(P; H)} \leq (1 + \| w_1 \|_H \| v_1 \|_H ) (1 + \max_{i \in \{1, 2\}} \| w_i \|_H ) (1 + \| v_1 \|_H )
\]

\[
\cdot c \left( 4 \| x_1 - x_2 \|_H + 4 \| w_1 h_1 - w_2 h_2 \|_H + 4 \sqrt{\| s_1 - s_2 \|} + \| w_1 - w_2 \|_H + \| v_1 - v_2 \|_H \right) (1 + T)
\]

\[
\cdot e^{2 \alpha_0 T} \left( \frac{12 p(1 + \theta)^3(1 + \delta)^2(1 + \gamma)}{\min \{ 8, 1 \} } \right)^2 + 2 \sqrt{2 T} + \int_0^T \left( \frac{2 \alpha_0}{\epsilon w_1^2} \right) \max_{i \in \{0, 1\}, j \in \{1, 2\}} V_0(x_j + i w_j h_j)
\]

\[
\cdot (1 + c)^2 \max_{i \in \{0, 1\}, j \in \{1, 2\}} \exp \left[ \int_0^T [\phi(u) + \sum_{i=0}^1 \frac{\beta_i}{q_i e^{v_i u}}] du + 3 \sum_{i=0}^1 \frac{V_i(x_i + i w_i h_i)}{q_i} \right]
\]

\[
\cdot \max_{i \in \{0, 1\}, j \in \{1, 2\}} \gamma e^{\alpha_i(s_2 + r)} \gamma \left( \frac{p(1 + \theta)(1 + \delta)}{\delta} \right)^ \gamma + \int_{s_1}^{s_2 + r} \frac{\beta_i}{e^{v_i u}} du + V_0(x_i + i w_i h_i) \right)
\]

(117)

An analogous argumentation shows for all \( r \in [0, T - s_2] \) that

\[
\left\| \phi^\sigma \right\|_{L^p(P; HS(U,H), H)} \leq (1 + \max_{i \in \{1, 2\}} \| w_i \|_H )^2 (1 + \| v_1 \|_H )^2
\]

\[
\cdot 4 \| x_1 - x_2 \|_H + \| w_1 h_1 - w_2 h_2 \|_H + \sqrt{\| s_1 - s_2 \| + \| w_1 - w_2 \|_H + \| v_1 - v_2 \|_H )
\]

\[
\cdot e^{2 \alpha_0 T} \left( \frac{12 p(1 + \theta)^3(1 + \delta)^2(1 + \gamma)}{\min \{ 8, 1 \} } \right)^2 + 2 \sqrt{2 T} + \int_0^T \left( \frac{2 \alpha_0}{\epsilon w_1^2} \right) \max_{i \in \{0, 1\}, j \in \{1, 2\}} V_0(x_j + i w_j h_j)
\]

\[
\cdot (1 + c)^2 \max_{i \in \{0, 1\}, j \in \{1, 2\}} \exp \left[ \int_0^T [\phi(u) + \sum_{i=0}^1 \frac{\beta_i}{q_i e^{v_i u}}] du + 3 \sum_{i=0}^1 \frac{V_i(x_i + i w_i h_i)}{q_i} \right]
\]

\[
\cdot \max_{i \in \{0, 1\}, j \in \{1, 2\}} \gamma e^{\alpha_i(s_2 + r)} \gamma \left( \frac{p(1 + \theta)(1 + \delta)}{\delta} \right)^ \gamma + \int_{s_1}^{s_2 + r} \frac{\beta_i}{e^{v_i u}} du + V_0(x_i + i w_i h_i) \right)
\]

(118)

Next, Fatou’s lemma and (116) (applied for all \( t \in [s_1, T] \)), \( \varepsilon \in (0, \infty) \) which satisfy that \( x_1 + h_1 w_1 + e \varepsilon v_1, x_1 + e \varepsilon v_1 \in O \) with \( u_1 \bowtie s_1 \), \( u_2 \bowtie s_1 \), \( y_1 \bowtie x_1 + h_1 w_1 \), \( y_2 \bowtie x_1 \), \( \tilde{h}_1 \bowtie \varepsilon \), \( \tilde{h}_2 \bowtie \varepsilon \), \( \tilde{v}_1 \bowtie v_1 \), \( \tilde{v}_2 \bowtie v_2 \) in the notation of (116) prove for all \( t \in [s_1, T] \) that

\[
\left\| D^{s_1, t}_{s_1, t} (v_1, w_1) \right\|_{L^p(P; H)} \leq \left\| D^{s_1, t}_{s_1, t} (v_1, w_1) \right\|_{L^p(P; H)}
\]

\[
\leq \liminf_{\{ \varepsilon \in (0, \infty) : x_1 + h_1 w_1 + e \varepsilon v_1, x_1 + e \varepsilon v_1 \in O \} \varepsilon \to 0} \frac{\| D^{s_1, t}_{s_1, t} (v_1, w_1) - D^{s_1, t}_{s_1, t} (v_1) \|_{L^p(\partial_1)} + 1}{|h_1|}
\]

\[
\leq \| v_1 \|_H \sqrt{T - s_1}
\]

\[
\cdot e^{2 \alpha_0 T} \left( \frac{12 p(1 + \theta)^3(1 + \delta)^2(1 + \gamma)}{\min \{ 8, 1 \} } \right)^2 + 2 \sqrt{2 T} + \int_0^T \left( \frac{2 \alpha_0}{\epsilon w_1^2} \right) \max_{i \in \{0, 1\}} V_0(x_1 + i w_1 h_1)
\]

\[
\cdot \| v_1 \|_H (1 + c)^2 \max_{i \in \{0, 1\}} \exp \left[ \int_0^T [\phi(u) + \sum_{i=0}^1 \frac{\beta_i}{q_i e^{v_i u}}] du + 3 \sum_{i=0}^1 \frac{V_i(x_1 + i w_1 h_1)}{q_i} \right]
\]

(119)
This, \([112], [117],\) and \([118]\) show that
\[
\left\| D_{s_1,t_1}^{x_1,h_1} (v_1, w_1) - D_{s_2,t_2}^{x_2,h_2} (v_2, w_2) \right\|_{L^p(F;H)} \\
\leq (1 + \max_{i \in \{1,2\}} \|w_i\|_H)^2(1 + \|v_1\|_H)^2(1 + 4c) \\
\cdot \left( \|x_1 - x_2\|_H + \|w_1 h_1 - w_2 h_2\|_H + \sqrt{s_1 - s_2} + \|w_1 - w_2\|_H + \|v_1 - v_2\|_H \right)(1 + T) \\
\cdot e^{2\alpha_0 T} \left( \frac{12p(1+\theta)^2(1+\delta)^2(1+\gamma)}{\min\{\delta,1\}} + 2\sqrt{T} + \int_0^T \frac{2b_0}{\varepsilon^\alpha} du \right) \\
\cdot \left( 1 + c \right)^2 \max_{i \in \{0,1\}, j \in \{1,2\}} \frac{\|\phi(u)\|}{\|\phi^{\prime}(u)\|} \\
\cdot \left( \|v_i\|_H \|w_i\|_H \right) \left( \frac{p(1+\theta)(1+\gamma)}{\delta} \right) + \int_{s_i}^T \frac{\beta_i}{\varepsilon^\alpha} du + 2 \max_{i \in \{0,1\}, j \in \{1,2\}} \|V_0(x_j + \iota w_j h_j)\|^2 \\
\cdot \left( 1 + c \right)^2 \max_{i \in \{0,1\}, j \in \{1,2\}} \left( 3 \int_0^T \left( \frac{\beta_i}{\varepsilon^\alpha} \right) du + 3 \sum_{i=0}^1 \frac{V(x_i + \iota w_i h_i)}{q_i} \right) \\
\cdot (1 + c)^2 \max_{i \in \{0,1\}, j \in \{1,2\}} \exp \left( 3 \int_0^T \left[ \phi(u) + \sum_{i=0}^1 \frac{\beta_i}{\varepsilon^\alpha} \right] du + 3 \sum_{i=0}^1 \frac{V(x_i + \iota w_i h_i)}{q_i} \right) \\
\cdot \max_{i \in \{0,1\}, j \in \{1,2\}} \left( 4^\gamma e^{\alpha_0 T} \left( \frac{p(1+\theta)(1+\gamma)}{\delta} \right) + \int_{s_i}^T \frac{\beta_i}{\varepsilon^\alpha} du + V_0(x_i + \iota w_i h_i) \right)^\gamma. 
\]

Next, we derive a temporal regularity estimate. Lemma 5.3, the triangle inequality, the Burkholder-Davis-Gundy type inequality in \([2, \text{ Lemma 7.7}],\) Hölder’s inequality (applied with \(\frac{1}{p} = \frac{1}{3p} + \frac{1}{3p} + \frac{1}{3p}\)), and \([50]\) prove for all \(u_1 \in [s_1, T], u_2 \in [u_1, T]\) that
\[
\left\| D_{s_1,u_2}^{x_1,h_1} (v_1, w_1) - D_{s_1,u_1}^{x_1,h_1} (v_1, w_1) \right\|_{L^p(F;H)} \\
\leq \int_{u_1}^{u_2} \left\| \mu'(X_{s_1,r}) \right\|_{L^p(H;H)} \left\| D_{s_1,r}^{x_1,h_1} (v_1, w_1) \right\|_H \left\| \right\|_{L^p(F;\mathbb{R})} \left\| \mu'(X_{s_1,r}) \right\|_{L^p(H;H;L^p(U,H))} \\
\cdot \left\| D_{s_1,r}^{x_1,h_1} (v_1, w_1) \right\|_H \left\| \right\|_{L^p(F;\mathbb{R})} \\
\cdot \left( \|X_{s_1,r}^x + \iota w_1 h_1\|_H \right) \lambda \left( \frac{\|X_{s_1,r}^x + \iota w_1 h_1\|_H}{H} \right) \\
\cdot \lambda \left( \frac{\|X_{s_1,r}^x + \iota w_1 h_1\|_H}{H} \right) \\
\cdot \left( \frac{p(1+\theta)(1+\gamma)}{\delta} \right) + \int_{s_1}^{u_2} \frac{\beta_i}{\varepsilon^\alpha} du + V_0(x_1 + \iota w_1 h_1) \right)^\gamma. 
\]

This, the triangle inequality, Lemma 2.6, \([119],\) and \([115]\) yield that
\[
\left\| D_{s_1,t_1}^{x_1,h_1} (v_1, w_1) - D_{s_1,t_2}^{x_1,h_1} (v_1, w_1) \right\|_{L^p(F;H)} \\
\leq \sqrt{|t_1 - t_2|} \left( \sqrt{T} + p \right)e^{2\gamma e\alpha_0 T} \left( 6p \gamma + \int_{s_1}^T \frac{2b_0}{\varepsilon^\alpha} dr \right) \\
\cdot \left( \|v_1\|_H \|w_1\|_H \sqrt{T - s_1} + \|v_1\|_H \|w_1\|_H \right) \\
\cdot e^{2\alpha_0 T} \left( \frac{12p(1+\theta)^3(1+\delta)^2(1+\gamma)}{\min\{\delta,1\}} + 2\sqrt{T} + \int_0^T \frac{2b_0}{\varepsilon^\alpha} du \right) \\
\cdot \left( \|v_i\|_H \|w_i\|_H \right) \left( \frac{p(1+\theta)(1+\gamma)}{\delta} \right) + \int_{s_i}^T \frac{\beta_i}{\varepsilon^\alpha} du + 2 \max_{i \in \{0,1\}} \|V_0(x_i + \iota w_i h_i)\|^2 \\
\cdot \left( 1 + c \right)^2 \max_{i \in \{0,1\}, j \in \{1,2\}} \exp \left( 3 \int_0^T \left[ \phi(u) + \sum_{i=0}^1 \frac{\beta_i}{\varepsilon^\alpha} \right] du + 3 \sum_{i=0}^1 \frac{V(x_i + \iota w_i h_i)}{q_i} \right) \]
This, the triangle inequality, and \([120]\) yield that
\[
\left\| D_{s_1,t_1}^{x_1,h_1}(v_1, w_1) - D_{s_2,t_2}^{x_2,h_2}(v_2, w_2) \right\|_{L^p(\mathbb{P} ; H)} \\
\leq \left\| D_{s_1,t_1}^{x_1,h_1}(v_1, w_1) - D_{s_1,t_2}^{x_1,h_2}(v_1, w_1) \right\|_{L^p(\mathbb{P} ; H)} + \left\| D_{s_1,t_2}^{x_1,h_1}(v_1, w_1) - D_{s_2,t_2}^{x_2,h_2}(v_2, w_2) \right\|_{L^p(\mathbb{P} ; H)} \\
\leq (1 + \max_{i \in \{1, 2\}} \|w_i\|_H)^2 (1 + \|v_1\|_H)^2 (1 + 4c)(p^2 + T) \\
\cdot \left( \|x_1 - x_2\|_H + \|w_1 h_1 - w_2 h_2\|_H + \sqrt{|s_1 - s_2|} + \sqrt{|t_1 - t_2|} + \|w_1 - w_2\|_H + \|v_1 - v_2\|_H \right) \\
\cdot \left( e^{2\gamma_0 T} \left( \frac{12p(1+\theta)^3(1+\gamma)^2}{\min\{\delta, 1\}} + 2\sqrt{T} + \int_0^T \frac{2\delta_0}{e^{\gamma_0 u}} du + 2 \max_{i \in \{0, 1\}, j \in \{1, 2\}} V_0(x_j + i w_j h_j) \right)^{2\gamma+2} \\
\cdot \left( 1 + c \right)^2 \max_{i \in \{0, 1\}, j \in \{1, 2\}} \exp \left( 3 \int_0^T \left[ \phi(u) + \sum_{i=0}^{1} \frac{\beta_i}{q_i e^{\gamma_0 u}} \right] du + 3 \sum_{i=0}^{1} V_i(x_i + i w_i h_i) \right)^5 \\
\cdot \max_{i \in \{0, 1\}, j \in \{1, 2\}} 4^7 e^{\alpha_i T \gamma} \left( \frac{p(1+\theta)(1+\gamma)}{8} + \int_{s_i}^T \frac{\beta_i}{e^{\gamma_0 u}} du + V_0(x_i + i w_i h_i) \right)^{\gamma}. \tag{123} \right.
\]
This completes the proof of Lemma 5.4 \(\square\)

5.2 Existence of a \(C^2\)-solution

The following theorem establishes existence of twice continuously differentiable solutions of SDEs.

**Theorem 5.5 (Existence of a \(C^2\)-solution).** Let \((H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)\) and \((U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)\) be separable \(\mathbb{R}\)-Hilbert spaces, assume that \(\dim(H) < \infty\), let \(T \in (0, \infty)\), let \((\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})\) be a filtered probability space satisfying the usual conditions, let \((W_t)_{t \in [0,T]}\) be an \(\mathbb{R}^{d_U}\)-cylindrical \((\mathbb{F}_t)_{t \in [0,T]}\)-Wiener process, let \(\Delta_T = \{(s,t) \in [0,T]^2 : s \leq t\}\), let \(O \subseteq \mathbb{R}\) be an open and convex set, let \(\mu \in C^2(O, H), \sigma \in C^2(O, HS(U, H))\), for all \(s \in [0,T]\), \(x \in O\) let \(X^x_s : [s,T] \times O \to O\) be an \((\mathbb{F}_t)_{t \in [s,T]}\)-adapted stochastic process with continuous sample paths which satisfies that for all \(t \in [s,T]\) it holds a.s. that

\[
X^x_{s,t} = x + \int_s^t \mu(X^x_{s,r}) dr + \int_s^t \sigma(X^x_{s,r}) dW_r. \tag{124}
\]

let \(\alpha_0, \alpha_1, \beta_0, \beta_1, c \in [0, \infty)\), \(V_0, V_1 \in C^2(O, [0, \infty))\), let \(\tilde{V} : [0,T] \times O \to [0, \infty)\) be a measurable function which satisfies for all \(i \in \{0, 1\}, t \in [0,T]\), \(x \in O\) that \(\mathbb{P} \left( \int_0^T |\tilde{V}(r, X^x_{s,r})| dr < \infty \right) = 1\) and

\[
\left\langle \mu(x), \left( \nabla V_i(x) \right) \right\rangle_H + \frac{1}{2} \text{trace} \left( \sigma(x)^* \text{Hess} \quad V_i(x) \right) \tag{125}
\]

\[
+ \frac{1}{2e^{\gamma_0 u}} \|\sigma(x)^* \nabla V_i(x)\|_H^2 + 1_{\{1\}}(i) \cdot \tilde{V}(t, x) \leq \alpha_i V_i(x) + \beta_i,
\]

let \(\phi : [0,T] \to [0, \infty)\) be a measurable function which satisfies that \(\int_0^T \phi(r) dr < \infty\), let \(p \in (2 \dim(H) + 6, \infty)\), \(\theta \in [0, \infty)\), \(\delta \in (0, \infty)\), \(q_0, q_1 \in (0, \infty)\) satisfy that

\[
\frac{\theta}{6p(1+\theta)^3(1+\delta)^2} = \frac{1}{q_0} + \frac{1}{q_1},
\]

assume that for all \(t \in [0,T]\), \(x, y, v \in O, v, H \setminus \{0\}\) it holds that

\[
\left\langle v, \int_0^T \mu(\lambda x + (1-\lambda)y) + \delta d\lambda v \right\rangle_H + \frac{1+\delta}{2} \left\| \int_0^T \sigma(\lambda x + (1-\lambda)y) d\lambda v \right\|_{HS(U,H)}^2 \tag{126}
\]

\[
+ \left( 3p(1+\theta)^3(1+\delta)^2 - 1 \right) \left( \left\langle v, f \right\rangle_H \|\sigma(\lambda x + (1-\lambda)y) d\lambda v \right\|_{HS(U,H)}^2 \right) \leq \|v\|_H^2 \cdot \left( \phi(t) + \frac{V_0(x) + V_0(y)}{2q_0 T e^{\gamma_0 t}} + \frac{\tilde{V}(t,x) + \tilde{V}(t,y)}{2q_1 e^{\gamma_1 t}} \right),
\]

31
let $\gamma \in [\frac{1}{p}, \infty)$ satisfy that for all $x \in O$ it holds that

$$\max \left\{ \|\mu(x)\|_H, \|\sigma(x)\|_{HS(U,H)} \right\} \leq c(1 + V_0(x))^\gamma,$$

(127)
satisfy that for all $x, y \in O$, $i \in \{1, 2\}$ it holds that

$$\max \left\{ \left\| \int_0^1 D^i \mu(\lambda x + (1 - \lambda)y) \, d\lambda \right\|_{L^0(H,H)}, \left\| \int_0^1 D^i \sigma(\lambda x + (1 - \lambda)y) \, d\lambda \right\|_{L^0(H,HS(U,H))} \right\}$$

$$\leq c(2 + V_0(x) + V_0(y))^\gamma,$$

(128)
and satisfy that for all $x_1, x_2, x_3, x_4 \in O$, $i \in \{1, 2\}$ it holds that

$$\max \left\{ \left\| \int_0^1 D^i \mu(\lambda x_1 + (1 - \lambda)x_2) - D^i \mu(\lambda x_3 + (1 - \lambda)x_4) \, d\lambda \right\|_{L^0(H,H)}, \right\| \left\| \int_0^1 D^i \sigma(\lambda x_1 + (1 - \lambda)x_2) - D^i \sigma(\lambda x_3 + (1 - \lambda)x_4) \, d\lambda \right\|_{L^0(H,HS(U,H))} \right\}$$

$$\leq c \int_0^1 \lambda \|x_1 - x_3\|_H + (1 - \lambda)\|x_2 - x_4\|_H \, d\lambda \left( 4 + \sum_{j=1}^4 V_0(x_i) \right)^\gamma.$$

Then there exists a measurable function $\mathcal{X}': \Delta_T \times O \times \Omega \to \overline{O}$ such that

(i) for all $x \in O$, $s \in [0, T]$ it holds a.s. that $(\mathcal{X}'_{s,t})_{t \in [s,T]} = (X_{s,t})_{t \in [s,T]}$, and

(ii) for every $\omega \in \Omega$ it holds that $\mathcal{X}(\omega) \in C^{0,2}(\Delta_T \times O, \overline{O})$.

**Proof of Theorem 5.5.** Without loss of generality we additionally assume throughout this proof that $\dim(H) \geq 1$ and that $O \neq \emptyset$. Throughout this proof let $d \in \mathbb{N}$, $\mathbb{H} \subseteq H$ satisfy that $d = \dim(H)$, and that $\mathbb{H}$ is an orthonormal basis of $H$, let $O^R$, $O^R_0$ be the sets which satisfy that $O^R = \bigcap_{v \in \mathbb{H}} \{ (x, h) \in O \times \mathbb{R} : x + vh \in O \}$ and $O^R_0 = \{ (x, h) \in O^R : h \neq 0 \}$, and let $K_n \subseteq \Delta_T \times H \times \mathbb{R}$, $n \in \mathbb{N}$, be the sets which satisfy for all $n \in \mathbb{N}$ that $K_n = \{ (s, t, x, h) \in \Delta_T \times O^R_0 : s^2 + t^2 + \|x\|^2_H + h^2 \leq n^2, \inf(\{\|x - y\| : y \in O^\circ\} \cup \{2\}) \leq \frac{1}{n^2} \}$. The fact that $p > 2d + 6$ and Theorem 5.4 yield that there exists a measurable function $\mathcal{X}' : \Delta_T \times O \times \Omega \to \overline{O}$ such that for all $x \in O$, $s \in [0, T]$ it holds a.s. that $(\mathcal{X}'_{s,t})_{t \in [s,T]} = (X_{s,t})_{t \in [s,T]}$, and such that for every $\omega \in \Omega$ it holds that $\mathcal{X}(\omega) \in C^{0,1}(\Delta_T \times O, \overline{O})$. For the rest of this proof, for all $(s, t) \in \Delta_T$, $x \in O$, $v, w \in H$, $h \in \mathbb{R} \setminus \{0\}$ with $x + hw \in O$ let $D^r_{s,t}(v, w) : \Omega \to H$ be the function which satisfies that

$$D^r_{s,t}(v, w) = \frac{\partial}{\partial x} \mathcal{X}'_{s,t}^x + hw(v) - \frac{\partial}{\partial x} \mathcal{X}'_{s,t}^x(v).$$

(130)

Then Lemma 5.3, the fact that $\int_0^T \phi(r) \, dr < \infty$, and boundedness of the functions $V_0, V_1$ on each of the relatively compact subsets $\{ x \in O : \exists s, t, h \in \mathbb{R} \text{ s.t. } (s, t, x, h) \in K_n \} \subseteq O$, $n \in \mathbb{N}$, demonstrate for all $n \in \mathbb{N}$, $v, w \in \mathbb{H}$ that

$$\sup \left\{ \left\{ \mathbb{E} \left[ \left\| D^r_{s,t}(v, w) - D^r_{s,t}(v, w) \right\|_{H}^p \right] \right\}^{\frac{1}{p}} : (s, t, x, h) \in K_n \right\} < \infty.$$ 

(131)

In particular this implies for all $n \in \mathbb{N}$, $v, w \in \mathbb{H}$ that

$$\sup \left\{ \left\{ \mathbb{E} \left[ \left\| D^r_{s,t}(v, w) \right\|_{H}^p \right] \right\}^{\frac{1}{p}} : (s, t, x, h) \in K_n \right\} \cup \{0\} \leq \sup \left\{ \left\{ \mathbb{E} \left[ \left\| D^r_{s,t}(v, w) - D^r_{s,t}(v, w) \right\|_{H}^p \right] \right\}^{\frac{1}{p}} \sqrt{T} : (s, t, x, h) \in K_n, t \neq s \right\} \cup \{0\} < \infty.$$

(132)
This, (131), Proposition 2.2 (applied for every \( v, w \in \mathbb{H} \) with \( H \cap \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \), \( D \cap \Delta_T \times O_0^R \), \( E \cap H, F \cap H \), \( p \cap p, \alpha \cap 1/2 \), \( X \cap (\Delta_T \times O_0^R \ni (s, t, x, h) \mapsto D_{s,t}^{x,h}(v, w) \in H) \) in the notation of Proposition 2.2), and path continuity of \( D_{s,t}^{x,h}(v, w), (s, t, x, h, v, w) \in [0, T] \times O_0^R \times \mathbb{H} \times \mathbb{H} \), establish for all \( v, w \in \mathbb{H} \) the existence of a measurable function \( D^{v,w}: \Delta_T \times O_0^R \times \Omega \rightarrow H \) which satisfies that for all \( \omega \in \Omega \) it holds that \( D^{v,w}(\omega) \in C(\Delta_T \times O_0^R, H) \) and which satisfies that for all \( (s, t, x, h) \in \Delta_T \times O_0^R \) it holds a.s. that \( (D_{s,t}^{v,w}(x, h))_{t \in [s, T]} = (D_{s,t}^{x,h}(v, w))_{t \in [s, T]} \). Note that \( O_0^R \subseteq O_0^R \).

Let \( D: \Delta_T \times O_0^R \times H \times H \times \Omega \rightarrow H \) be the function which satisfies that for all \( (s, t, x, h) \in \Delta_T \times O_0^R \), \( v, w \in H \) it holds that \( D_{s,t}(x, h, v, w) = \sum_{e, \in H}(v, e)_{H} (w, e)_{H} D_{s,t}^{x,h}(x, h) \). Next, we observe that for all \( (s, t, x, h, v, w) \in [0, T] \times O_0^R \times \mathbb{H} \times \mathbb{H} \) it holds a.s. for all \( t \in [s, T] \) that

\[
D_{s,t}(x, h, v, w) = D_{s,t}^{v,w}(x, h) = D_{s,t}^{x,h}(v, w) = \frac{\partial}{\partial x} \tilde{X}_{s,t}^{x,h}(v) \bigg|_{h}.
\]

This, continuity of the random fields \( \frac{\partial}{\partial x} \tilde{X} \), and Lemma 2.1 (applied for all \( v \in \mathbb{H} \) with \( U \cap H, T \cap \Delta_T, T \cap \Delta_T \cap \mathbb{Q}^3, \mathcal{X} \cap (\Delta_T \times O \times \Omega \ni (s, t, x, \omega) \mapsto \frac{\partial}{\partial x} \tilde{X}_{s,t}^{x}(v, \omega) \in H), \mathcal{Z} \cap (\Delta_T \times O_0^R \times H \times \Omega \ni (s, t, x, h, w) \mapsto \tilde{D}_{s,t}(x, h, v, w, \omega) \in H) \) in the notation of Lemma 2.1) prove that there exists \( \Omega_0 \in \mathcal{F} \) such that \( \mathbb{P}(\Omega_0) = 1 \) and such that for all \( \omega \in \Omega_0 \), \( (s, t) \in \Delta_T \), \( v \in \mathbb{H} \) it holds that the mapping \( \Omega \ni x \mapsto \tilde{X}_{s,t}^{x}(v, \omega) \in H \) is continuously differentiable and it holds for all \( x \in O, v \in \mathbb{H}, w \in H \) that

\[
\frac{\partial}{\partial x} \frac{\partial}{\partial x} \tilde{X}_{s,t}^{x}(v, \omega)(w) = \sum_{h \in \mathbb{H}} \langle w, h \rangle_{H} D_{s,t}(x, 0, v, h, w) = D_{s,t}(x, 0, v, w, \omega).
\]

This and continuity of \( D \) prove that for all \( \omega \in \Omega_0 \) it holds that \( \tilde{X}(\omega) \in C^{0,2}(\Delta_T \times O_0^R, \Omega) \). Let \( \mathcal{X}: \Delta_T \times O \times \Omega \rightarrow \Omega \) be the function which satisfies for all \( (s, t, x, \omega) \in \Delta_T \times O \times \Omega \) that \( \mathcal{X}_{s,t}^{x}(\omega) = 1_{\Omega_0}(\omega) \tilde{X}_{s,t}^{x}(\omega) \). Then it holds that \( \mathcal{X} \) is measurable, that for all \( x \in O, s \in [0, T] \) it holds a.s. that \( (\mathcal{X}_{s,t}^{x})_{t \in [s, T]} = (\tilde{X}_{s,t}^{x})_{t \in [s, T]} = (\mathcal{X}_{s,t}^{x})_{t \in [s, T]} \), and that for every \( \omega \in \Omega \) it holds that \( \mathcal{X}(\omega) \in C^{0,2}(\Delta_T \times O, \Omega) \). This proves items (i) – (iii) and finishes the proof of Theorem 5.5.

The following corollary simplifies the assumptions of Theorems 5.5.

**Corollary 5.6 (Existence of a C^2-solution).** Let \( d, m \in \mathbb{N}, \alpha, \beta \in [0, \infty), T, c \in (0, \infty), p \in (6(d+3)(1+1/c)^3, \infty) \), let \( \| \cdot \|, \langle \cdot | \cdot \rangle \) denote the standard norm and the standard scalar product on \( \mathbb{R}^d \), let \( \| \cdot \|_m \) denote the standard norm on \( \mathbb{R}^m \), let \( \| \cdot \|_F \) denote the Frobenius norm on \( \mathbb{R}^{d \times m} \), let \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]} ) \) be a filtered probability space satisfying the usual conditions, let \( W: [0, T] \times \Omega \rightarrow \mathbb{R}^m \) be a standard \( (\mathcal{F}_t)_{t \in [0, T]} \)-Wiener process, let \( \mu \in C^3(\mathbb{R}^d, \mathbb{R}^d) \), \( \sigma \in C^3(\mathbb{R}^d, \mathbb{R}^{d \times m}) \), \( V \in C^2(\mathbb{R}^d, [0, \infty)) \), assume that \( \mu^a \) and \( \sigma^a \) grow at most polynomially at infinity, assume that \( \exists \gamma \in (0, \infty): \sup_{x \in \mathbb{R}^d} \| x \|^\gamma / (1 + V(x)) < \infty \), and assume for all \( x, y, v \in \mathbb{R}^d \) that

\[
\begin{align*}
\langle v, 1 \rangle^\gamma (\lambda(x-y) + y) d\lambda v + \frac{d-1}{2} \int_0^1 \sigma^T (\lambda(x-y) + y) d\lambda v^2 \| v \|_F^2 - \left( c + \frac{V(x) + V(y)}{4c^pT^{m+1}} \right) + \frac{V(x) + V(y)}{4c^pT^{m+1}},
\end{align*}
\]

\[
\begin{align*}
\mu(x), \langle \nabla V(x) \rangle + \frac{1}{2} \text{trace}(\sigma(x)\sigma(x)^* (\text{Hess} V)(x)) + \frac{1}{2} \| \sigma(x)^*(\nabla V)(x) \|_{2, \Omega}^2 + V(x) - \alpha V(x) + \beta.
\end{align*}
\]

(135)

Then there exists a measurable function \( X: \{(s, t) \in [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d \) such that

(i) for every \( \omega \in \Omega \) it holds that \( X(\omega) \in C^{0,2}(\{(s, t) \in [0, T] \times \mathbb{R}^d, \mathbb{R}^d \}) and

(ii) for all \( x \in \mathbb{R}^d, s \in [0, T], t \in [s, T] \) it holds a.s. that

\[
X_{s,t} = x + \int_s^t \mu(X_{s,r}) dr + \int_s^t \sigma(X_{s,r}) dW_r.
\]

(136)
Proof. The assumption that $\mu'''$ and $\sigma'''$ grow at most polynomially at infinity and the assumption 

$$\exists \gamma \in (0, \infty) : \sup_{x \in \mathbb{R}^d} \|x\|^\gamma/(1 + V(x)) < \infty$$

ensure that there exist $\gamma, \tilde{c} \in [1, \infty)$ such that for all $x \in \mathbb{R}^d$ it holds that

$$\max \{\|\mu(x)\|, \|\sigma(x)\|_F\} \leq \tilde{c}(1 + V(x))^{\gamma},$$

such that for all $x, y \in \mathbb{R}^d$, $i \in \{1, 2\}$ it holds that

$$\max \left\{\left\|\int_0^1 D^i \mu(\lambda x + (1 - \lambda)y) \, d\lambda\right\|_{L^1(\mathbb{R}^d, \mathbb{R}^d)}, \left\|\int_0^1 D^i \sigma(\lambda x + (1 - \lambda)y) \, d\lambda\right\|_{L^1(\mathbb{R}^d, \mathbb{R}^d)}\right\} \leq \tilde{c} (2 + V(x) + V(y))^{\gamma},$$

and such that for all $x_1, x_2, x_3, x_4 \in \mathbb{R}^d$, $i \in \{1, 2\}$ it holds that

$$\max \left\{\left\|\int_0^1 D^i \mu(\lambda x_1 + (1 - \lambda)x_2) - D^i \mu(\lambda x_3 + (1 - \lambda)x_4) \, d\lambda\right\|_{L^1(\mathbb{R}^d, \mathbb{R}^d)}, \left\|\int_0^1 D^i \sigma(\lambda x_1 + (1 - \lambda)x_2) - D^i \sigma(\lambda x_3 + (1 - \lambda)x_4) \, d\lambda\right\|_{L^1(\mathbb{R}^d, \mathbb{R}^d)}\right\} \leq \tilde{c} \int_0^1 \lambda \|x_1 - x_3\| + (1 - \lambda)\|x_2 - x_4\| \, d\lambda \left(4 + \sum_{j=1}^4 V(x_i)\right)^{\gamma}.$$

Moreover, let $\delta \in (0, p)$, $\tilde{p} \in (2d + 6, \infty)$ satisfy that $1 + \delta + 6\tilde{p}(1 + \frac{1}{c})(1 + \delta)^2 - 2 = p - 1$. In addition, local Lipschitz continuity of $\mu$, $\sigma$, and the Lyapunov condition in (135) yield existence of a global solution of the SDE with drift coefficient $\mu$ and diffusion coefficient $\sigma$; cf., e.g., [3]. The assertion follows then from Theorem 5.5 (applied with $H \sim \mathbb{R}^d$, $U \sim \mathbb{R}^m$, $O \sim \mathbb{R}^d$, $\alpha_0 \sim \alpha$, $\alpha_1 \sim \alpha$, $\beta_0 \sim \beta$, $\beta_1 \sim \beta$, $c \sim \tilde{c}$, $V_0 \sim V$, $V_1 \sim V$, $\phi \sim ([0, T] \ni t \mapsto c \in [0, \infty])$, $p \sim \tilde{p}$, $\theta \sim 1/c$, $q_0 \sim 2c(p - \delta)$, $q_1 \sim 2c(p - \delta)$ in the notation of Theorem 5.5). This completes the proof of Corollary 5.6. \qed

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References

[1] Stefano Attanasio. Stochastic flows of diffeomorphisms for one-dimensional SDE with discontinuous drift. Electronic Communications in Probability, 15:213–226, 2010.

[2] Sonja G. Cox, Martin Hutzenthaler, and Arnulf Jentzen. Local Lipschitz continuity in the initial value and strong completeness for nonlinear stochastic differential equations. Mem. Amer. Math. Soc. (accepted, arXiv:1309.5595v3), pages 1–94, 2013.

[3] Giuseppe Da Prato and Jerzy Zabczyk. Stochastic equations in infinite dimensions, volume 44 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1992. ISBN 0-521-38529-6.

[4] Pierre del Moral and Sumeetpal Sidhu Singh. Backward Itô-Ventzell and stochastic interpolation formulae. PhD thesis, INRIA, 2019.

[5] Shizan Fang and Tusheng Zhang. A study of a class of stochastic differential equations with non-Lipschitzian coefficients. Probab. Theory Related Fields, 132(3):356–390, 2005. ISSN 0178-8051. doi: 10.1007/s00440-004-0398-z. URL http://dx.doi.org/10.1007/s00440-004-0398-z
[6] Shizan Fang, Peter Imkeller, and Tusheng Zhang. Global flows for stochastic differential equations without global Lipschitz conditions. *Ann. Probab.*, 35(1):180–205, 2007. ISSN 0091-1798. doi: 10.1214/009117906000000412. URL http://dx.doi.org/10.1214/009117906000000412

[7] Franco Flandoli, Massimiliano Gubinelli, and Enrico Priola. Flow of diffeomorphisms for SDEs with unbounded Hölder continuous drift. *Bulletin des sciences mathematiques*, 134(4):405–422, 2010.

[8] István Gyöngy and Nicolai Krylov. Existence of strong solutions for Itô’s stochastic equations via approximations. *Probab. Theory Related Fields*, 105(2):143–158, 1996. ISSN 0178-8051. doi: 10.1007/BF01203833. URL http://dx.doi.org/10.1007/BF01203833

[9] Anselm Hudde, Martin Hutzenthaler, Arnulf Jentzen, and Sara Mazzonetto. On the Itô-Alekseev-Gröbner formula for stochastic differential equations. *arXiv preprint arXiv:1812.09857*, 2018.

[10] Anselm Hudde, Martin Hutzenthaler, and Sara Mazzonetto. A stochastic Gronwall inequality and applications to moments, strong completeness, strong local Lipschitz continuity, and perturbations. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 57(2):603–626, 2021. doi: 10.1214/20-AIHP1064. URL https://doi.org/10.1214/20-AIHP1064

[11] Martin Hutzenthaler and Arnulf Jentzen. Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients. *Mem. Amer. Math. Soc.*, 4:1–112, 2015.

[12] Martin Hutzenthaler, Arnulf Jentzen, Thomas Kruse, Tuan A. Nguyen, and Philippe von Wurstemberger. Overcoming the curse of dimensionality in the numerical approximation of semilinear parabolic partial differential equations. *arXiv:1807.01212*, 2018.

[13] Hiroshi Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. ISBN 0-521-35050-6.

[14] Xue-Mei Li. Strong $p$-completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds. *Probab. Theory Related Fields*, 100(4):485–511, 1994. ISSN 0178-8051. doi: 10.1007/BF01268991. URL http://dx.doi.org/10.1007/BF01268991.

[15] Xue-Mei Li and Michael Scheutzow. Lack of strong completeness for stochastic flows. *Ann. Probab.*, 39(4):1407–1421, 2011.

[16] Klaus Reiner Schenk-Hoppé. Deterministic and stochastic Duffing-van der Pol oscillators are non-explosive. *Z. Angew. Math. Phys.*, 47(5):740–759, 1996. ISSN 0044-2275. doi: 10.1007/BF00915273. URL http://dx.doi.org/10.1007/BF00915273

[17] Björn Schmalfuß. The random attractor of the stochastic Lorenz system. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 48:951–975, 1997.

[18] Xicheng Zhang. Stochastic flows and Bismut formulas for stochastic Hamiltonian systems. *Stochastic Process. Appl.*, 120(10):1929–1949, 2010. ISSN 0304-4149. doi: 10.1016/j.spa.2010.05.015. URL http://dx.doi.org/10.1016/j.spa.2010.05.015