Projective MDS Codes Over $GF(27)$

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Abstract:
MDS code is a linear code that achieves equality in the Singleton bound, and projective MDS (PG-MDS) is MDS code with independents property of any two columns of its generator matrix. In this paper, elementary methods for modifying a PG-MDS code of dimensions 2, 3, as extending and lengthening, in order to find new incomplete PG-MDS codes have been used over $GF(27)$. Also, two complete PG-MDS codes over $GF(27)$ of length 16 and 28 have been found.

Key words: Conic, Finite field, Finite projective plane, Maximum distance separable codes.

Introduction:
Let $F_q = GF(q)$ denote the Galois field with $q$ elements and let $p$ denote the characteristic of $GF(q)$. An $[n, k]$-code $C$ over $F_q$ is a $k$-dimensional subspace of $F_q^n$. The parameter $n$ is called the length of $C$. The weight $wt(v)$ of a vector $v \in F_q^n$ is the number of non-zero coordinates of $v$. The minimum non-zero weight of all codewords in $C$ is called the minimum weight (Hamming distance) of $C$ and then an $[n,k]$-code with minimum weight $d$ is called an $[n,k,d]$-code. From Singleton Bound Theorem, the parameter $d$ has maximum value $n-k+1$, when $n$ and $k$ are given. The linear code that achieves equality in the Singleton bound is called, or MDS codes for short. The code which has minimum weight $d$ that correct $e$ errors that can be accrued is called $e$-error correcting code and $e = \left\lfloor \frac{d-1}{2} \right\rfloor$ (where $d$ is the smallest integer less than or equal to $x$, ex. $\left\lfloor \frac{5}{2} \right\rfloor = \left\lfloor \frac{4}{2} \right\rfloor = 2$).

To any an $[n,k]$-code $C$ over $F_q$, there is also another parameter $r = n-k$ called the redundancy, which represent the check digits added to the message to give protection against noise. The orthogonal complement of an $[n,k]$-code $C$ (the set of all vectors which are orthogonal to every vector in $C$), is called the dual code of $C$ with dimension $n-k$, and denoted by $C^\perp$. For more on linear codes, see (1).

Any linear code $C$ over $F_q$ with the three fundamental parameters, its length is $n$, its dimension is $k$, and its redundancy $r$ has a natural interpretation of each of these parameters. There are six basic modification techniques to linear codes depending on these three parameters. Each fixes one parameter and increases or decreases the other two parameters accordingly. These techniques are partitioned into three pairs and each member of a pair is the inverse process to the other as summarized below.

(i) Augmenting: Expurgating; Fix $n$; increase $k$; decrease $r$: decrease $k$; increase $r$.
(ii) Extending: Puncturing; Fix $k$; increase $n$; increase $r$: decrease $n$; decrease $r$.
(iii) Lengthening: Shortening; Fix $r$; increase $n$; increase $k$: decrease $n$; decrease $k$.

Since the redundancy of a code is its "dual dimension," each technique also has a natural dual technique.

The motivation of this research is based on two main ideas in the coding theory which can be summarized as follows:

1- It is well known algebraically; that two linear codes with same parameters will have the same efficiency if they are linearly isomorphism (equivalent).
2- The fundamental problem to find codes with two properties:
(i) reasonable information content (length is big enough),
(ii) reasonable error handling ability.

A number of researchers worked on these two ideas recently in the sense of modification, for instance, see (2). Grassl (3) presented a new table with bounds to good codes not MDS code (small length $n$ and large minimum Hamming distance $d$) for $2 \leq q \leq 9$. Emami and Pedram (4) use punctured and shortened methods to construct codes (optimal Linear codes) with minimum value of $n$ for certain dimension $k$ and minimum Hamming distance $d$. For further authors whose used shortening or puncturing structure of codes see; (5), (6) and the references therein. Some other researcher concentrate on the generator's entries of a linear code to achieve the singleton bound on the minimum distance; that is, code with maximum ability to correct errors (GM-MDS) (7), (8), (9), (10).

The main aim to this paper is to work with especial type of maximum distance separable (MDS) codes namely, projective MDS code (11) over $F_{27}$, since they provide the maximum protection against device failure for a given amount of redundancy; that is, the greatest error correcting capability (since error correcting capability is a function of minimum distance). To do that, an extending (dually, Lengthening) technique has been used.

The article is organized as follows. First section provided basic definitions and some properties of MDS and finite projective geometry. In second section, the inequivalent, incomplete projective MDS codes of dimension two have been constructed. Finally, in last section, the inequivalent, incomplete (complete) projective MDS codes of dimension three have been constructed, and three special complete MDS codes of lengths 16 and 28 have been founded using projective conic in the projective plane.

**Definitions and Basic Properties**

Any linear $[n,k]$-code can be defined by a $(k \times n)$ matrix $G$ or by a $(n-k) \times n$ matrix $H$ whose entries from $F_q$ as defined below.

**Definition 1:** (1) A generator matrix $G$ of an $[n,k]$-code $C$ is a $k \times n$ matrix whose rows form a basis for $C$. The standard form of a generator matrix $G$ is $[I_k A]$. A linear code for which any two columns of a generator matrix are linearly independent is called a projective code (PG-code). A linear code which cannot extend by adding columns to its generator matrix is called a complete code, otherwise it is called incomplete code.

**Definition 2:** (1) A parity check matrix $H$ of an $[n,k]$-code $C$ is a $(n-k) \times n$ matrix whose rows form basis for $C^\perp$. The standard form of a parity check matrix $H$ is $[A^T I_{(n-k)}]$.

**Theorem 1:** (1) An $[n,k]_q$-code $C$ is MDS if and only if the dual code $C^\perp$, $[n,n-k]$ is MDS.

Let $PG(2,q)$ denote the 2-dimensional projective space over $F_q$ (finite projective plane).

**Definition 3:** (12) A non-singular plane quadric (form of degree two) in $PG(2,q)$ is called a conic. A conic consists of $q + 1$ points no three of which are collinear.

During the paper, the notation $PG$-MDS will briefly refer to a projective MDS code.

**PG-MDS Code of Dimension 2 over $F_{27}$**

The technique used in this paper to check whether that two codes are projectively equivalent or not is as follows:

The $s \times r$ matrices $A$ is called projectively equivalent to $s \times r$ matrices $B$, and denoted by $A \cong_p B$ if there exist a non-singular $s \times s$ matrix $T$ such that matrix $TA$ transformed to $B$ by performing the following operations:

(i) make the last position of each row of $TA$, 0 or 1;
(ii) a permutation of the columns on $TA$.

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is only standard matrix generate the $PG$-MDS code $[2,2,1]$. This matrix can be extended to create a $PG$-MDS code $[3,2,2]$ by adding the column $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The $PG$-MDS code $[3,2,2]$ generated by the matrix $G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ in standard form is unique up to equivalence since, if $\tilde{G} = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$ is a generator matrix of another $PG$-MDS code $[3,2,2]$, $x,y,z \in F_q$, then there is a $2 \times 2$ non-singular matrix $T$ transform the matrix $G$ to $\tilde{G}$ as follows.

Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,q)$ such that $TG = \tilde{G}$. So, the following equations are deduced:

$a = \lambda x, \ b = \beta y, \ c = \lambda, \ d = \beta, \ \lambda x + \beta \beta = \gamma z, \ \lambda + \beta = \gamma$.
where $\lambda, \beta, z \in GF(q) \setminus \{0\}$. So, $T = \begin{bmatrix} Sx & Jy \end{bmatrix}$, where $S = \begin{bmatrix} 1 & \frac{y}{x-y} \\ \frac{1}{x-y} \end{bmatrix}$ and $J = \begin{bmatrix} 1 & \frac{z}{x-y} \end{bmatrix}$.

Let $G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = [ I^2 A_{2 \times 1} ]$. The matrix $G$ is of rank 2; that is, any two columns are linearly independent, so this gives the incomplete PG-MDS code with the parameters $[3,2,2]$. The $2 \times 3$ matrix $G$ can be extended in to $2 \times 4$ matrices, $G^*_{i}$ by adding appropriate 25 columns $\begin{bmatrix} \alpha^i \end{bmatrix}$ to $G$ from right side of $G$, where $\alpha^i \in F_{27}$, $i = 1,2,...,25$, such that the rows of the new matrix $G^*_{i}$ still linearly independent and any two columns is linearly independent. So, these 25 matrices, gives raise to 25 generated matrices $G^*_{i}$ of PG-MDS codes. Among these 25 matrices, only 5 of them are non equivalent as given the next theorem.

**Theorem 2**: Over $F_{27}$, there are only five inequivalent, incomplete PG-MDS codes with parameters $[4,2,3]$ and error correcting $e = 1$.

**Proof**: By searching for a non-singular $2 \times 2$ matrix, the equivalents generator matrices have been identified as shown below.

| $G^*_i \equiv_p G^*_j$ | row 1, row 2 of $T$ |
|-------------------------|------------------|
| $G^*_{1} \equiv_p G^*_{2}$ | $\alpha^{12}$, 1, 0, 1 |
| $G^*_{1} \equiv_p G^*_{10}$ | $\alpha^{23}$, $\alpha^{10}$, 0, 1 |
| $G^*_{1} \equiv_p G^*_{16}$ | $\alpha^{23}$, $\alpha^{10}$, 0, 1 |
| $G^*_{1} \equiv_p G^*_{24}$ | $\alpha^{10}$, $\alpha^{24}$, 0, 1 |
| $G^*_{1} \equiv_p G^*_{25}$ | $\alpha^{25}$, 0, 0, 1 |
| $G^*_{3} \equiv_p G^*_{4}$ | $\alpha^{17}$, $\alpha^{4}$, 0, 1 |
| $G^*_{3} \equiv_p G^*_{6}$ | $\alpha^{13}$, 1, 0, 1 |
| $G^*_{3} \equiv_p G^*_{20}$ | $\alpha^{4}$, $\alpha^{20}$, 0, 1 |
| $G^*_{3} \equiv_p G^*_{22}$ | $\alpha^{13}$, 1, 0, 1 |
| $G^*_{3} \equiv_p G^*_{23}$ | $\alpha^{23}$, 0, 0, 1 |
| $G^*_{5} \equiv_p G^*_{7}$ | $\alpha^{13}$, 1, 0, 1 |
| $G^*_{5} \equiv_p G^*_{11}$ | $\alpha^{19}$, $\alpha^{11}$, 0, 1 |
| $G^*_{5} \equiv_p G^*_{15}$ | $\alpha^{8}$, 1, 0, 1 |
| $G^*_{5} \equiv_p G^*_{19}$ | $\alpha^{6}$, $\alpha^{19}$, 0, 1 |
| $G^*_{5} \equiv_p G^*_{21}$ | $\alpha^{21}$, 1, 0, 1 |
| $G^*_{8} \equiv_p G^*_{9}$ | $\alpha^{14}$, $\alpha^{9}$, 0, 1 |
| $G^*_{8} \equiv_p G^*_{12}$ | $\alpha^{13}$, 1, 0, 1 |
| $G^*_{8} \equiv_p G^*_{14}$ | $\alpha^{14}$, 0, 1 |
| $G^*_{8} \equiv_p G^*_{17}$ | $\alpha^{5}$, 1, 0, 1 |
| $G^*_{8} \equiv_p G^*_{18}$ | $\alpha^{18}$, 1, 0, 1 |

Therefore, the only inequivalent ones are summarized below.

| $G_i$ | Row 1 | Row 2 |
|-------|-------|-------|
| $G_1$ | 1 0 1 | $\alpha^{13}$ | 0 1 1 1 |
| $G_2$ | 1 0 1 | $\alpha$ | 0 1 1 1 |
| $G_3$ | 1 0 1 | $\alpha^3$ | 0 1 1 1 |
| $G_4$ | 1 0 1 | $\alpha^5$ | 0 1 1 1 |
| $G_5$ | 1 0 1 | $\alpha^8$ | 0 1 1 1 |

To extend each matrix $G_i$, an appropriate column $\begin{bmatrix} \alpha^j \end{bmatrix}$ is added to $G_i$ for which $\alpha^j$ does not belong to the first row of $G_i$. So, there are $(q - 3)$ possibilities for $\begin{bmatrix} \alpha^j \end{bmatrix}$; that is, 24 possibility. Therefore, by this way, $5(q - 3) = 120$ codes can be constructed. This procedure will be used to extend the codes in this paper.

In the next theorems, only the inequivalent codes are presented.

**Theorem 3**: Over $F_{27}$, there are eight inequivalent PG-MDS codes with parameters $[5,2,4]$ and error correcting $e = 2$. These codes are given below.

| $M_i$ | Row 1 | Row 2 |
|-------|-------|-------|
| $M_1$ | 1 0 1 | $\alpha^{13}$ | $\alpha$ | 0 1 1 1 |
| $M_2$ | 1 0 1 | $\alpha$ | $\alpha^2$ | 0 1 1 1 |
| $M_3$ | 1 0 1 | $\alpha$ | $\alpha^3$ | 0 1 1 1 |
| $M_4$ | 1 0 1 | $\alpha$ | $\alpha^6$ | 0 1 1 1 |
| $M_5$ | 1 0 1 | $\alpha$ | $\alpha^7$ | 0 1 1 1 |
| $M_6$ | 1 0 1 | $\alpha$ | $\alpha^{12}$ | 0 1 1 1 |
| $M_7$ | 1 0 1 | $\alpha^3$ | $\alpha^7$ | 0 1 1 1 |
| $M_8$ | 1 0 1 | $\alpha^3$ | $\alpha^9$ | 0 1 1 1 |

From the eight $M_i$ PG-MDS codes, $8(q - 4) = 184$ projective codes can be constructed. These codes are given in the next theorem.

**Theorem 4**: Over $F_{27}$, there are 34 inequivalent, incomplete PG-MDS codes with parameters $[6,2,5]$ and error correcting $e = 2$ as given in Table 1.

| $N_i$ of length 6 | Row 1 | Row 2 |
|-------------------|-------|-------|
| $N_1$ | 1 0 1 | $\alpha^{13}$ | $\alpha$ | $\alpha^2$ | 0 1 1 1 1 1 |
| $N_2$ | 1 0 1 | $\alpha^{13}$ | $\alpha^3$ | 0 1 1 1 1 1 |
| $N_3$ | 1 0 1 | $\alpha^{13}$ | $\alpha^4$ | 0 1 1 1 1 1 |
| $N_4$ | 1 0 1 | $\alpha^{13}$ | $\alpha^5$ | 0 1 1 1 1 1 |
| $N_5$ | 1 0 1 | $\alpha^{13}$ | $\alpha^6$ | 0 1 1 1 1 1 |
| $N_6$ | 1 0 1 | $\alpha^{13}$ | $\alpha^7$ | 0 1 1 1 1 1 |
| $N_7$ | 1 0 1 | $\alpha^{13}$ | $\alpha^8$ | 0 1 1 1 1 1 |
| $N_8$ | 1 0 1 | $\alpha^{13}$ | $\alpha^{11}$ | 0 1 1 1 1 1 |
| $N_9$ | 1 0 1 | $\alpha^{13}$ | $\alpha^{12}$ | 0 1 1 1 1 1 |
| $N_{10}$ | 1 0 1 | $\alpha^{13}$ | $\alpha^{14}$ | 0 1 1 1 1 1 |
| $N_{11}$ | 1 0 1 | $\alpha^{13}$ | $\alpha^{15}$ | 0 1 1 1 1 1 |
| $N_{12}$ | 1 0 1 | $\alpha^{13}$ | $\alpha^{16}$ | 0 1 1 1 1 1 |
| $N_{13}$ | 1 0 1 | $\alpha^{13}$ | $\alpha^{19}$ | 0 1 1 1 1 1 |
New inequivalent, incomplete PG-MDS codes for fixed dimension \( k = 2 \) and length \( 7 \leq n \leq 14 \) over \( F_{27} \) can be constructed by means of a combinatorial computer search, and using the same technique in Theorem 2, 3 and 4. In the next theorem, the full details about these codes are given.

Let \( \ast_n \) denote the number of all codes with length \( n \) and \( \#_n \) denote the number of inequivalent ones.

**Theorem 5**: Over \( F_{27} \), for fixed \( k = 2 \) the inequivalent, incomplete PG-MDS codes with parameters given below exist.

\[
\begin{array}{cccccc}
\ast_n & \#_{n-1}(q - (n - 2)) & n & k & d & e \\
748 & 73 & 2 & 6 & 7 & 2 \\
1533 & 196 & 2 & 7 & 3 & 2 \\
3920 & 382 & 2 & 8 & 3 & 2 \\
7258 & 745 & 2 & 9 & 4 & 2 \\
13410 & 1142 & 2 & 10 & 4 & 2 \\
19414 & 1665 & 2 & 11 & 5 & 2 \\
26640 & 1976 & 2 & 12 & 5 & 2 \\
29640 & 2170 & 2 & 13 & 6 & 2 \\
\end{array}
\]

- **Example 1**: In this example, PG-MDS codes are given for each length \( n \), \( 8 \leq n \leq 14 \).

  (i) PG-MDS code with parameters \([7,2,6]\) and \(e = 2\).

  \[
  \mathcal{K}_7 = \begin{bmatrix}
  1 & 0 & 1 & a^{13} & a & a^2 & a^3 \\
  0 & 1 & 1 & 1 & 1 & 1 & 1
  \end{bmatrix}
  \]

  (ii) PG-MDS code with parameters \([8,2,7]\) and \(e = 3\).

  \[
  \mathcal{K}_8 = \begin{bmatrix}
  1 & 0 & 1 & a^{13} & a & a^2 & a^3 & a^6 \\
  0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
  \end{bmatrix}
  \]

  (iii) PG-MDS code with parameters \([9,2,8]\) and \(e = 4\).

  \[
  \mathcal{K}_9 = \begin{bmatrix}
  1 & 0 & 1 & a^{13} & a & a^2 & a^3 & a^6 & a^8 \\
  0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
  \end{bmatrix}
  \]

  (iv) PG-MDS code with parameters \([10,2,9]\) and \(e = 4\).

  \[
  \mathcal{K}_{10} = \begin{bmatrix}
  1 & 0 & 1 & a^{13} & a & a^2 & a^3 & a^6 & a^9 & a^{14} \\
  0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
  \end{bmatrix}
  \]

  (v) PG-MDS code with parameters \([11,2,10]\) and \(e = 5\).

  \[
  \mathcal{K}_{11} = \begin{bmatrix}
  1 & 0 & 1 & a^{13} & a & a^2 & a^3 & a^6 & a^8 & a^9 & a^{14} \\
  0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
  \end{bmatrix}
  \]

  (vi) PG-MDS code with parameters \([12,2,11]\) and \(e = 5\).

  \[
  \mathcal{K}_{12} = \begin{bmatrix}
  1 & 0 & 1 & a^{13} & a & a^2 & a^3 & a^6 & a^8 & a^9 & a^{14} & a^5 \\
  0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
  \end{bmatrix}
  \]

  (vii) PG-MDS code with parameters \([13,2,12]\) and \(e = 6\).

  \[
  \mathcal{K}_{13} = \begin{bmatrix}
  1 & 0 & 1 & a^{13} & a & a^2 & a^3 & a^6 & a^8 & a^9 & a^{14} & a^5 & a^7 \\
  0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
  \end{bmatrix}
  \]

  (viii) PG-MDS code with parameters \([14,2,13]\) and \(e = 6\).

  \[
  \mathcal{K}_{14} = \begin{bmatrix}
  1 & 0 & 1 & a^{13} & a & a^2 & a^3 & a^6 & a^8 & a^9 & a^{14} & a^5 & a^7 & a^{19} \\
  0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
  \end{bmatrix}
  \]

**Corollary 1**: Over \( F_{27} \), for fixed \( k = 2 \) the inequivalent, incomplete PG-MDS codes with length \( 15 \leq n \leq 26 \) exist.

**Proof**:

The first row of each generator matrix \( Q \) of a PG-MDS code \( C, [n',k] \) except the first element which is 1, in Theorems 2,3,4, and 5 has distinct elements from the base field \( F_{27} \). So, the matrix
\[ H = \begin{bmatrix} \alpha^i_1 & \alpha^i_2 & \ldots & \alpha^i_{(28-n')} \end{bmatrix}, \]

where \( \alpha^i_j \) belong to the complement of the first row of the generator matrix \( Q \), forming a generator matrix of a PG-MDS codes, \([28-n',2], 2 \leq n' \leq 14 \).

Therefore, the inequivalent, incomplete PG-MDS codes of the following parameters exist.

| \( n \) | \( k \) | \( d \) | \( e \) |
|---|---|---|---|
| 1976 | 15 | 2 | 14 | 6 |
| 1665 | 16 | 2 | 15 | 7 |
| 1142 | 17 | 2 | 16 | 7 |
| 745 | 18 | 2 | 17 | 8 |
| 382 | 19 | 2 | 18 | 8 |
| 196 | 20 | 2 | 19 | 9 |
| 73 | 21 | 2 | 20 | 9 |
| 34 | 22 | 2 | 21 | 10 |
| 8 | 23 | 2 | 22 | 10 |
| 5 | 24 | 2 | 23 | 11 |
| 1 | 25 | 2 | 24 | 11 |
| 1 | 26 | 2 | 25 | 12 |

**Corollary 2**: Over \( F_{27} \) for fixed distance \( d = 3 \) and \( e = 1 \), the inequivalent PG-MDS codes with parameters given below exist.

| \( n \) | \( k \) | \( d \) | \( e \) |
|---|---|---|---|
| 5 | 4 | 2 | 3 | 1 |
| 8 | 5 | 3 | 3 | 1 |
| 34 | 6 | 4 | 3 | 1 |
| 73 | 7 | 5 | 3 | 1 |
| 196 | 8 | 6 | 3 | 1 |
| 382 | 9 | 7 | 3 | 1 |
| 745 | 10 | 8 | 3 | 1 |
| 1142 | 11 | 9 | 3 | 1 |
| 1665 | 12 | 10 | 3 | 1 |
| 1976 | 13 | 11 | 3 | 1 |
| 2170 | 14 | 12 | 3 | 1 |
| 1976 | 15 | 13 | 3 | 1 |
| 1665 | 16 | 14 | 3 | 1 |
| 1142 | 17 | 15 | 3 | 1 |
| 745 | 18 | 16 | 3 | 1 |
| 382 | 19 | 17 | 3 | 1 |
| 196 | 20 | 18 | 3 | 1 |
| 73 | 21 | 19 | 3 | 1 |
| 34 | 22 | 20 | 3 | 1 |
| 8 | 23 | 21 | 3 | 1 |
| 5 | 24 | 22 | 3 | 1 |
| 1 | 25 | 23 | 3 | 1 |
| 1 | 26 | 24 | 3 | 1 |

**Proof:**

From Theorem 1, the dual code \( C^\perp \) of each code \( C \) in Theorem 5 is also inequivalent PG-MDS. Since each generator matrix \( Q \) of a PG-MDS code \( C, [n,k] \) in Theorems 2,3,4,5 is in standard form; that is, \( Q = [I_2 A_{2x(n-2)}] \), where \( A = \begin{bmatrix} 1 & \alpha^i_1 & \ldots & \alpha^i_{(n-3)} \end{bmatrix} \), so the parity-check matrix is \( H = [-A^T_{(n-2)x2}I_{(n-2)}] \). So, for any \( a,b \in GF(27)\setminus\{0\} \), if \( -a(1,a^i_1,\ldots,a^i_{(n-3)}) \pm b(1,1,\ldots,1) = 0 \), then \( a+b = 0 \) and \(-ad^j - b = 0, 1 \leq j \leq (n-3) \). Thus, \( b\alpha^i_j - b(\alpha^i_j - 1) = 0 \). But \( b \neq 0 \); that is, \( \alpha^i_j \neq 1 \) for all \( j \) which is contradicted with that each \( \alpha^i_j \) are distinct. Therefore, any two columns are linearly independent; that is, \( C^\perp \) is PG-code.

It clear that, each code, \([n,n-k,3] \) in Corollary 1 is a sub code of \([n+1, (n+1) - k,3] \), \( n > k \).

All the PG-codes in Theorems 2,3,4,5 are embedded in the complete PG-MDS code \([28,2,27] \) which is generated by the matrix

\[ D = \begin{bmatrix} 1 & \beta_{i_1} & \ldots & \beta_{i_{27}} \end{bmatrix}, \]

where all \( \beta_{i,j} \in F_{27} \) are distinct. Therefore, all PG-codes in Theorems 2,3,4,5 are incomplete codes.

**PG-MDS Code of Dimension 3 over \( F_{27} \)**

In this section, PG-MDS codes are constructed from these ones in section two by transferring the generator matrices into other matrices which each one formed a generator matrix to a PG-MDS code \([n,3,n-2] \), \( 3 \leq n \leq 28 \).

If \( \mathcal{W} = [c_1 \ldots c_n] = \begin{bmatrix} 1 & 0 & \ldots & \theta \end{bmatrix} \) is PG-MDS code \([n, 2, n-1] \), where \( c_i \) are the columns of \( \mathcal{W} \), then it could transfer to a PG-code \([n,3,n-2] \) with generator matrix \( \mathcal{W}^+ = [c_1^+ \ldots c_n^+] \) using the one to one map, \( T^+ \) as follows:

\[ T^+ = [x,y] \Rightarrow c_i^+ = [(x/y)^2 - a^{14}(x/y), a^{19}(1-a^{12}x/y), a^{16}x/y] \quad \text{if} \quad y \neq 0 \]

\[ c_i^+ = [1,0,0] \quad \text{if} \quad y = 0 \]

Therefore, the generator matrix \( G = \frac{\mathcal{W}^+}{[1,0,0]} \) of the unique PG-MDS code \([3,2,2] \) transfer to the matrix

\[ G^+ = \frac{[r_1^* r_3]}{[3]} = [c_1 c_2 c_3] = \begin{bmatrix} 1 & 0 & \alpha^{19} \\ 0 & 1 & \alpha^2 \\ 0 & 0 & 1 \end{bmatrix}. \]

It is not difficult to prove that the rows \( r_3, r_2 \), and \( r_3 \) are linearly independents and the columns \( c_i c_j \) are also linearly independents. Here, \( d = 1 \). Since, \( w(r_1) = w(r_2) = 1 \). So, \( G^+ \) formed generator matrix of the unique PG-MDS code \([3,3,1] \).

**Theorem 6**: Over \( F_{27} \), the inequivalent PG-MDS codes with parameters in Table 2 are exists.

**Table 2. PG-MDS Code of parameter \([n,3,n-2] \)**
Proof:

• $n = 5$: From Theorem 3, the eight generators $M_i$ are transformed by $T^*$ into the following matrices $M_i^+$ of rank 3 as shown below:

| $M_i^+$ | Row1 | Row2 | Row3 |
|---------|------|------|------|
| $M_1^+$ | $10\alpha^{19}\alpha^{19}\alpha^{24}$ | $01\alpha^2\alpha^5\alpha^6$ | 00111 |
| $M_2^+$ | $10\alpha^{19}\alpha^{19}\alpha^{26}$ | $01\alpha^2\alpha^6\alpha^6$ | 00111 |
| $M_3^+$ | $10\alpha^{19}\alpha^{24}\alpha^{26}$ | $01\alpha^2\alpha^6\alpha^{12}$ | 00111 |
| $M_4^+$ | $10\alpha^{19}\alpha^{24}\alpha^2$ | $01\alpha^2\alpha^6\alpha^5$ | 00111 |
| $M_5^+$ | $10\alpha^{19}\alpha^{24}\alpha^{24}$ | $01\alpha^2\alpha^6\alpha^{19}$ | 00111 |
| $M_6^+$ | $10\alpha^{19}\alpha^{24}\alpha^{4}$ | $01\alpha^2\alpha^6\alpha^{24}$ | 00111 |
| $M_7^+$ | $10\alpha^{19}\alpha^{6}\alpha^{22}$ | $01\alpha^2\alpha^6\alpha^{24}$ | 00111 |
| $M_8^+$ | $10\alpha^{19}\alpha^{6}\alpha^{6}$ | $01\alpha^2\alpha^{12}\alpha^1$ | 00111 |

These matrices, $M_i^+$ can be transformed into the matrices $L_i^+$ of rank 3 as in Table 3.

Table 3. Equivalent matrices

| $M_i^+$ | $L_i^+$ |
|---------|--------|
| $M_1^+$ | $\begin{pmatrix} G^+ \\ x \\ y \\ z \end{pmatrix}$ |
| $M_2^+$ | $\begin{pmatrix} G^+ \\ 0 \\ 0 \\ 1 \end{pmatrix}$ |
| $M_3^+$ | $\begin{pmatrix} G^+ \\ 0 \end{pmatrix}$ |
| $M_4^+$ | $\begin{pmatrix} G^+ \\ \alpha \end{pmatrix}$ |
| $M_5^+$ | $\begin{pmatrix} G^+ \\ \alpha \end{pmatrix}$ |
| $M_6^+$ | $\begin{pmatrix} G^+ \\ \alpha \end{pmatrix}$ |
| $M_7^+$ | $\begin{pmatrix} G^+ \\ \alpha \end{pmatrix}$ |
| $M_8^+$ | $\begin{pmatrix} G^+ \\ \alpha \end{pmatrix}$ |

Therefore, there are eight unique PG-MDS code with parameters [5,3,3] and generator matrix $L_i^+$.

By using the same technique the other results will deduce.

Theorem 7: There is a unique, complete PG-MDS code $C^*$ with parameters [28,3,26] over $F_{27}$.

Proof:

The $28 \times 3$ matrix $M_C$ with following rows has rank three and satisfies the projective conic $XY + \alpha^6XZ + \alpha^5YZ$.
\[ \alpha^{24} YZ; \] that is, the rows of \( M_{C^*} \) formed complete arc of degree 2 with 28 points which is Segre bound. Therefore, \( C^* \) has \( d = n - k + 1 = 26 \).

\[ M_{C^*} \text{ Matrix} \]

| \( C(1) \) | \( C(2) \) | \( C(3) \) | \( C(4) \) | \( C(5) \) | \( C(6) \) | \( C(7) \) |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | \( \alpha^4 \) | \( \alpha^3 \) | \( \alpha^8 \) | \( \alpha^9 \) |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 |

\[ C(8) \] \( \alpha^{10} \) \( \alpha^{12} \) \( \alpha^{21} \) \( \alpha^{24} \) \( \alpha^{18} \) \( \alpha^{20} \) \( \alpha \) \( \alpha^{21} \) \( \alpha \) \( \alpha \) \( \alpha \)

| \( C(15) \) | \( C(16) \) | \( C(17) \) | \( C(18) \) | \( C(19) \) | \( C(20) \) | \( C(21) \) |
|---|---|---|---|---|---|---|
| \( \alpha^7 \) | \( \alpha^{23} \) | \( \alpha^{19} \) | \( \alpha^5 \) | \( \alpha^2 \) | \( \alpha^{25} \) | \( \alpha \)
| \( \alpha^{25} \) | \( \alpha^{10} \) | \( \alpha^2 \) | \( \alpha^{12} \) | \( \alpha^5 \) | \( \alpha^{11} \) |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| \( C(22) \) | \( C(23) \) | \( C(24) \) | \( C(25) \) | \( C(26) \) | \( C(27) \) | \( C(28) \) |
|---|---|---|---|---|---|---|
| \( \alpha^{17} \) | \( \alpha^{15} \) | \( \alpha^9 \) | \( \alpha^{12} \) | \( \alpha^{16} \) | \( \alpha^4 \) | \( \alpha^4 \)
| \( \alpha^{15} \) | \( \alpha^{16} \) | \( \alpha^{20} \) | \( \alpha^{24} \) | \( \alpha^4 \) | \( \alpha^{14} \) |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Columns of \( \mathcal{U}_2 \):

| \( C(1) \) | \( C(2) \) | \( C(3) \) | \( C(4) \) | \( C(5) \) | \( C(6) \) | \( C(7) \) | \( C(8) \) |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | \( \alpha^{14} \) | \( \alpha^9 \) | \( \alpha^{10} \) | \( \alpha^{12} \) | \( \alpha^{14} \) |
| 0 | 0 | 1 | \( \alpha^{13} \) | \( \alpha^{23} \) | \( \alpha^3 \) | \( \alpha^{17} \) | \( \alpha^{18} \) |
| \( \alpha^{11} \) | \( \alpha^{20} \) | \( \alpha^{23} \) | \( \alpha^{19} \) | \( \alpha^5 \) | \( \alpha^2 \) | \( \alpha^{15} \) | \( \alpha^9 \) |
| \( \alpha^{19} \) | \( \alpha \) | \( \alpha^{10} \) | \( \alpha^2 \) | \( \alpha^{22} \) | \( \alpha^5 \) | \( \alpha^{16} \) | \( \alpha^{20} \) |

From Theorem 7, any two columns are linearly independents. So, it is enough to prove the completeness for \( \mathcal{U}_1(\mathcal{U}_2) \). To do this a computer program has been used to prove that each extra column, \( c = [a, b, 1] \) to \( \mathcal{U}_1(\mathcal{U}_2) \) will be linearly dependent with other two columns of \( \mathcal{U}_1(\mathcal{U}_2) \). So, the rank of \( \mathcal{U}_1(\mathcal{U}_2) \) will be less than 3.

**Conclusion:**

In this paper the extending and lengthening are used to conclude the existence of complete, projective MDS codes of dimension two and three over the finite field of order twenty-seven. Where if \( k = 2 \), codes of length \( n \), \( 4 \leq n \leq 26 \) and distance \( d, 3 \leq d \leq 25 \) are founded. Also, if \( k = 3 \), codes of length \( n, 4 \leq n \leq 26 \) and distance \( d, 2 \leq d \leq 24 \) are founded. Two complete, projective MDS have been computed of dimension three and length sixteen.

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- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
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**ترميزات الأسقاطية على GF(27)**

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الخلاصة:

الترميز MDS هو الترمز الخطي الذي يحقق المساواة في القيد المفرد، و MDS الأسقاطي (PG-MDS) هو ترميز مع خاصية الاستقلالية لأي عمودين من المصفوفة المولدة الخاصة به. في هذه البحث، تم استخدام الطرق الأولية لتعديل الترميز PG-MDS للأبعاد 2،3، مثل الامتداد والاطالة، من أجل إيجاد ترميزات PG-MDS جديدة غير مكتملة معرفة على الحقلGF(27). أيضاً، تم إيجاد اثنين من الترميزات كاملاً على الحقلGF(27) بطول 16 و 28.

الكلمات المفتاحية: المخروط، الحقل المنتهي، المستوى الأسقاطية المنتهي، ترميزات المسافة القصوى القابلة للفصل.