Some remarks on varieties of pairs of commuting upper triangular matrices and an interpretation of commuting varieties

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Abstract

It is known that the variety of pairs of $n \times n$ commuting upper triangular matrices isn’t a complete intersection for infinitely many values of $n$; we show that there exists $m$ such that this happens if and only if $n > m$. We also show that $m < 18$ and that it could be found by determining the dimension of the variety of pairs of commuting strictly upper triangular matrices. Then we define a natural map from the variety of pairs of commuting $n \times n$ matrices onto a subvariety defined by linear equations of the grassmannian of subspaces of $K^{n^2}$ of codimension 2.

1 Introduction

Let $T_n$ be the set of all $n \times n$ upper triangular matrices over an algebraically closed field $K$; let $T_n$ be the subset of $T_n$ of all the invertible matrices. Let

$$CT_n = \{(X,Y) \in T_n \times T_n : [X,Y] = 0\}.$$  

Let $U_n$ be the subset of $T_n$ of all the strictly upper triangular matrices and let

$$NT_n = CT_n \cap (U_n \times U_n).$$

It is known that there exist infinitely many values of $n$ such that $CT_n$ and $NT_n$ are not irreducible and are not complete intersections. The determination of the smallest $n$ such that these properties occur is an open problem which
has recently interested several mathematicians.
The action of $T_n$ on $U_n$ hasn’t finitely many orbits; a classification of them can be found in [1]. Hence many of the arguments which are used in the study of commuting varieties cannot be applied here.

In section 2 we show that $CT_n$ is a complete intersection if and only if its irreducible components have the same dimension, and that there exist natural numbers $m$, $m'$ such that $CT_n$ isn’t a complete intersection if and only if $n > m$ and is reducible if and only if $n > m'$. Similar results hold for $NT_n$; moreover we prove that $m$ and $m'$ have the previous property according to the dimension of $NT_m$, $NT_{m'}$. Then we give examples which prove that $m < 18$ and $m' < 17$.

Many results of section 2 were independently obtained by Allan Keeton, as results of his Ph.D. thesis. More precisely, Keeton communicated to the author that he had obtained the following results:

a) $CT_n$ is an irreducible, normal complete intersection if $n \leq 8$;
b) $CT_n$ is not normal if $n \geq 16$;
c) $CT_n$ is reducible if $n \geq 17$;
d) $CT_n$ is not a complete intersection and not of pure dimension if $n \geq 18$.

In section 3 we consider the variety $\mathcal{C}(n, K)$ of all the pairs of commuting $n \times n$ matrices, which we regard as a subvariety of $\mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1}$. We denote by $\mathcal{C}_0(n, K)$ the subvariety of $\mathcal{C}(n, K)$ of pairs of equal elements, then we define a map $\gamma_n$ from $\mathcal{C}(n, K) \setminus \mathcal{C}_0(n, K)$ into the grassmannian $G(2, K^{n^2})$ of all the subspaces of $K^{n^2}$ of codimension 2. The fibers of this map are the orbits of $\mathcal{C}(n, K) \setminus \mathcal{C}_0(n, K)$ under the natural action of $\text{GL}(2, K)$ on $\mathcal{C}(n, K) \setminus \mathcal{C}_0(n, K)$.

We get that the image of $\gamma_2$ is a linear complete intersection subvariety of the projective space of dimension 5 in which $G(2, K^4)$ is defined.

2 Some remarks on varieties of pairs of commuting upper triangular matrices

We will denote by $(X, Y)$ a generic element of $T_n \times T_n$. The entries of $[X, Y]$ give $\frac{n(n-1)}{2}$ equations for $CT_n$ and $\frac{(n-1)(n-2)}{2}$ equations for $NT_n$.

Let $CT_n^0$ be the Zariski closure of the subset of $CT_n$ of all the pairs $(X, Y)$
such that $X$ and $Y$ are regular (that is have minimum polynomial of degree $n$); let $NT_n^0$ be the same Zariski closure in $NT_n$.

** Proposition 1** We have:

i) $CT_n^0$ is irreducible of dimension $\frac{n(n + 3)}{2}$; this is the minimum dimension of the irreducible components of $CT_n$.

\[ n(n + 1) - 1; \] this is the minimum dimension of the irreducible components of $NT_n$.

\[ CT_n^0 \]

\[ \frac{n(n + 1)}{2} \]

\[ + 1 \]

\[ n(n + 3) \]

\[ = \frac{n(n + 3)}{2} \]

\[ + n \]

Moreover \[
\frac{n(n + 3)}{2} = \dim (T_n \times T_n) - \frac{n(n - 1)}{2},
\]

which shows i). The same argument can be used for ii).

\[ \frac{n(n + 3)}{2} \]

By the irreducibility of the centralizer in $T_n$ and in $U_n$ we get the following result.

** Proposition 2** If $(X, Y) \in CT_n (NT_n)$ and $X$ or $Y$ commutes with regular matrices of $T_n (U_n)$ then $(X, Y) \in CT_n^0 (NT_n^0)$.

We observe that any irreducible component of $CT_n (NT_n)$ is stable under the action of $T_n$. Moreover any irreducible component of $CT_n$ is stable with respect to the action of $K^2$ defined by

\[ (x, y) \cdot (X, Y) = (X + xI_n, Y + yI_n); \]

hence the subset of the pairs of nonsingular matrices is dense in any irreducible component of $CT_n$.

We denote by \{ $e_1, \ldots, e_n$ \} the canonical basis of $K^n$ and by $M(p, q)$ the set of all $p \times q$ matrices over $K$.

** Proposition 3** If $CT_{n-1} (NT_{n-1})$ isn't irreducible or isn't a complete intersection, the same holds for $CT_n (NT_n)$. 
Proof. We first prove the claim for $CT_n$.
Let $CT^1_{n-1}$ be an irreducible subvariety of $CT_{n-1}$ different from $CT^0_{n-1}$ and let
\[ \dim CT^1_{n-1} = \frac{(n-1)(n+2)}{2} + k, \quad k \geq 0. \]
Let $T'_n$ be the subspace of $T_n$ of all the endomorphisms which stabilize $\langle e_1 \rangle$ and $\langle e_2, \ldots, e_n \rangle$. Let $T'_n = T_n \cap T'_n$. Let
\[ \Gamma = \{(X, Y) \in CT_n : X = \begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & Y' \end{pmatrix}, \quad (X', Y') \in CT^1_{n-1} \}. \]
Let $\Gamma'$ be the orbit of $\Gamma$ under the action of $T_n$. If $(X', Y') \in CT^1_{n-1}$, rank $X'$, rank $Y' = n-1$ and $G \in T_n$ then we have that $G \cdot (X, Y) \in \Gamma$ iff $G \in T'_n$.
Hence
\[ \dim \Gamma' = \dim T_n - \dim T'_n + \dim CT^1_{n-1} = \frac{n(n+1)}{2} - \left( \frac{n(n-1)}{2} + 1 \right) + \frac{(n-1)(n+2)}{2} + k = \frac{n(n+3)}{2} + k - 2. \]
Let $CT^1_n$ be the orbit of $\Gamma'$ under the action of $K^2$; we have
\[ \dim CT^1_n = \frac{n(n+3)}{2} + k. \]
Since $X'$ and $Y'$ aren’t regular, $CT^1_n \neq CT^0_n$, which shows the claim.
We now prove the claim for $NT_n$. Let $NT^1_{n-1}$ be a subvariety of $NT_{n-1}$ and let
\[ NT^1_n = \{ (X, Y) \in NT_n : X = \begin{pmatrix} 0 & \widetilde{X} \\ 0 & \widetilde{X}' \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & \widetilde{Y} \\ 0 & Y' \end{pmatrix}, \quad (X', Y') \in NT^1_{n-1}, \widetilde{X}, \widetilde{Y} \in M(1, n-1) \}. \]
The equations for $NT^1_n$ as subvariety of
\[ NT^1_{n-1} \times M(1, n-1) \times M(1, n-1) \]
are given by $\widetilde{X}Y' - \widetilde{Y}X' = 0$, hence
\[ \dim NT^1_n \geq \dim NT^1_{n-1} + n. \]
If \( \dim NT_{n-1}^1 = \frac{n(n-1)}{2} - 1 + k, \ k \geq 0 \) then \( \dim NT_n \geq \frac{n(n+1)}{2} - 1 + k \), which proves the claim.

For \( X \in T_n \) let \( f(t) = (t - \lambda_1)^{m_1} \cdots (t - \lambda_r)^{m_r} \) be the minimum polynomial of \( X \). For \( i = 1, \ldots, r \) let \( f_i(t) = f(t) : (t - \lambda_i)^{m_i} \) and let \( g_1(t), \ldots, g_r(t) \) be such that \( \sum_{i=1}^{r} g_i(t)f_i(t) = 1 \); then \( \sum_{i=1}^{r} g_i(X)f_i(X) = I_n \). Hence the matrices \( I_i = g_i(X)f_i(X), \ i = 1, \ldots, r, \) are orthogonal projections of \( K^n \) on \( K^n \) and the image of \( I_i \) is \( \ker (X - \lambda_i I_n)^{m_i} \). Then we get the following result.

**Lemma 4** Let \( (X, Y) \in CT_n \). There exist \( G \in T_n \) and a partition \( \{E_1, \ldots, E_r\} \) of \( \{Ge_1, \ldots, Ge_n\} \) such that \( \ker (X - \lambda_i I_n)^{m_i} = \langle E_i \rangle \) and \( \langle E_i \rangle \) is stable with respect to \( X \) and \( Y \) for \( i = 1, \ldots, r \).

**Proof.** The matrices \( I_i \) for \( i = 1, \ldots, r \) are upper triangular and commute with the matrices of the centralizer of \( X \). If \( j \in \{1, \ldots, n\} \) there exists a unique \( i \in \{1, \ldots, r\} \) such that the entry of \( I_i \) of indices \( (j, j) \) is 1. Let \( G \in T_n \) be such that \( Ge_j = I_ie_j \); this gives a partition with the required property.

We set \( n_i = |E_i| \) for \( i = 1, \ldots, r \). By Lemma 4 we get the following results.

**Proposition 5** Let \( CT_{n-1} \) be a complete intersection and let \( (X, Y) \in CT_n \) be such that \( X \) or \( Y \) has at least two eigenvalues. Then \( (X, Y) \) doesn't belong to any irreducible component of dimension greater than \( \frac{n(n+3)}{2} \).

**Proof.** If \( (X, Y) \) belongs to an irreducible component then the subset of it of all the pairs such that at least one of the matrices has more than one eigenvalue is dense. Let \( E = \{E_1, \ldots, E_r\} \) be a partition of \( \{e_1, \ldots, e_n\} \) such that \( r \geq 2 \). Let \( T_E \) be the subset of \( T_n \) of all the endomorphisms which stabilize \( \langle E_i \rangle \) for \( i = 1, \ldots, r \). Let \( T_E = T_n \cap T_E \) and let \( CT_E = CT_n \cap (T_E \times T_E) \). By Proposition 3 we have \( \dim CT_E = \sum_{i=1}^{r} \frac{n_i(n_i + 3)}{2} \), hence the dimension of the orbit of \( CT_E \) under the action of \( T_n \) is less or equal than

\[
\dim T_n - \dim T_E + \dim CT_E = \frac{n(n+1)}{2} - \sum_{i=1}^{r} \frac{n_i(n_i + 1)}{2} + \sum_{i=1}^{r} \frac{n_i(n_i + 3)}{2} = \frac{n(n+3)}{2}.
\]
Hence the claim follows by Lemma 4.

**Proposition 6** Let $CT_{n-1}$ be irreducible and let $(X, Y) \in CT_n$ be such that $X$ or $Y$ has at least two eigenvalues. Then $(X, Y) \in CT_n^0$.

**Proof.** Let us assume that $X$ has $r \geq 2$ eigenvalues and let $\{E_1, \ldots, E_r\}$ be as in Lemma 4. Let $CT_n^E$ be the subvariety of $CT_n$ of all the pairs of matrices which stabilize $\langle E_i \rangle$ for $i = 1, \ldots, r$. Then we have

$$CT_n^E \cong CT_{n_1} \times \cdots \times CT_{n_r}.$$ 

By Proposition 3, $CT_{n_i}$ is irreducible for $i = 1, \ldots, r$. Hence $CT_n^E$ is irreducible and the subset of $CT_E$ of all the pairs of regular matrices is dense, which shows the claim.

By Propositions 5 and 6 we get that, in order to determine the values of $n$ such that $CT_n$ isn’t irreducible or isn’t a complete intersection, we can look for irreducible components which have as elements only pairs of matrices with only one eigenvalue. Hence it is enough to determine the dimension of $NT_n$.

Let $X = (x_{i,j}), Y = (y_{i,j}), X_{i,j} = \begin{pmatrix} x_{i,j} \\ y_{i,j} \end{pmatrix}$. The condition $[X, Y] = 0$ gives the following equations for $NT_n$:

$$\sum_{k=i+1}^{j-1} \text{det} (X_{i,k} \quad X_{k,j}) = 0 \quad i = 1, \ldots, n - 2, \quad j = i + 2, \ldots, n. \quad (1)$$

We observe that this system of equations is invariant under the involution of $U_n$ defined by $z_{i,j} \mapsto z_{n+1-j, n+1-i}$.

Any irreducible component of $NT_n$ different from $NT_n^0$ is contained in a subvariety of $NT_n$ defined by some equations of the form $X_{i,j} = 0, i < j$. Moreover the following result holds.

**Lemma 7** If $X \in U_n$ and there exist $h, k \in \{1, \ldots, n\}$ such that $x_{h,k}$ vanishes on the orbit of $X$ under the action of $T_n$ then also $x_{i,j}$ vanishes on that orbit for $i, j = h, \ldots, k$.

**Proof.** The claim follows from the fact that in the orbit under $T_m$ of any nonzero matrix of $U_m$ there exists a matrix such that its entry of indices $(1, m)$ isn’t 0.
Corollary 8 Let $NT^*_n$ be an irreducible component of $NT_n$ different from $NT^0_n$. There exists $s \in \{1, \ldots, n-1\}$ and subsets $J_1, \ldots, J_s$ of $\{1, \ldots, n\}$, such that:

i) if $h \in \{1, \ldots, s\}$, $i, j \in J_h$, $l \in \{1, \ldots, n\}$ and $i < l < j$ then $l \in J_h$;

ii) $J_1 \cup \cdots \cup J_s = \{1, \ldots, n\}$;

iii) $X_{i,j}$ is 0 on $NT^*_n$ iff there exists $h \in \{1, \ldots, s\}$ such that $i, j \in J_h$.

Let $\Upsilon_n$ be the set of all the partitions $J = \{J_1, \ldots, J_s\}$ of $\{1, \ldots, n\}$ such that $s \in \{1, \ldots, n-1\}$ and $J_1, \ldots, J_s$ have the property i) of Corollary 8.

We assume that if $h, k \in \{1, \ldots, s\}$ and $h < k$ the elements of $J_h$ are smaller than those of $J_k$. If $J \in \Upsilon_n$ we denote by $U^J_n$ the subvariety of $U_n$ defined by the equations

$$x_{i,j} = 0, \quad i, j \in J_h, \quad h = 1, \ldots, s$$

and we set $NT^J_n = NT_n \cap (U^J_n \times U^J_n)$.

If $NT^*_n$ is an irreducible component of $NT_n$ different from $NT^0_n$ there exists $J \in \Upsilon_n$ such that $NT^*_n \subseteq NT^J_n$.

Example A The variety $NT_4$ is defined by the equations:

$$\det (X_{1,2} \quad X_{2,3}) = 0, \quad \det (X_{2,3} \quad X_{3,4}) = 0,$$

$$\det (X_{1,2} \quad X_{2,4}) + \det (X_{1,3} \quad X_{3,4}) = 0.$$

Let $NT^1_4$ be the subvariety of $NT_4$ defined by the equations $X_{2,3} = 0$ and let $NT^0_4$ be the subvariety of $NT_4$ defined by the equations

$$\rank (X_{1,2} \quad X_{2,3} \quad X_{3,4}) \leq 1.$$
which shows the claim. Let \( X_{1,2} = \gamma X_{2,3}, \gamma \neq 0 \), and let \( X_{3,4} = 0 \). Let us consider the subvariety of \( NT_n^0 \), parametrized by \( \delta \), defined by \( X_{i,j} = X_{i,j} \) for \((i,j) = (2,3), (1,2), (1,3), X_{3,4} = \delta X_{2,3}, X_{2,4} = X_{2,4} + \delta X_{1,3} \); for \( \delta \neq 0 \) we get pairs of regular elements. We could use a similar argument if \( X_{1,2} = 0 \) and \( X_{3,4} \neq 0 \), which shows the claim.

Let \( J \in \Upsilon_n \) and let \( Z_{h,k} = (x_{i,j}) \), \( W_{h,k} = (y_{i,j}) \), \( i \in J_h, j \in J_k \). We can write the equations of \( NT_J^* \) in \( U_J^* \times U_J^* \) as follows:

\[
\sum_{i=h+1}^{k-1} Z_{h,i} W_{i,k} - W_{h,i} Z_{i,k} = 0 \quad h = 1, \ldots, s - 2, \quad k = h + 2, \ldots, s .
\]

This can be also written in the following way:

\[
\begin{pmatrix}
Z_{h,h+1} & -W_{h,h+1} & \cdots & Z_{h,k-1} & -W_{h,k-1}
\end{pmatrix}
\begin{pmatrix}
W_{h+1,k} \\
Z_{h+1,k} \\
\vdots \\
W_{k-1,k} \\
Z_{k-1,k}
\end{pmatrix} = 0
\]

\( h = 1, \ldots, s - 2, \quad k = h + 2, \ldots, s . \)

Let \( V_{m,p,q} = \{(A, B) \in M(m,p) \times M(p,q) : AB = 0\} \). We can determine a lower bound of the dimension of \( NT_J^* \) by the following elementary result.

**Lemma 9** The irreducible components of \( V_{m,p,q} \) are the subvarieties

\[ V_{m,p,q}^{a,b} = \{(A, B) \in V_{m,p,q} : \text{rank } A \leq a, \ \text{rank } B \leq b\} \]

where \((a,b)\) is maximal such that \( b \leq \min \{p, q\}, \ a \leq \min \{p - b, m\} \). We have:

1) \( \dim V_{m,p,q}^{a,b} = a(p + m - a) + b(p + q - b) - ab \);

2) \( V_{m,p,q} \) is a complete intersection iff \( p \geq m + q - 1 \).

By the following example W.V. Vasconcelos observed that \( CT_n \) isn’t a complete intersection for infinitely many values of \( n \).

**Example B** [4]. Let \( n = 3m \) and let \( J \in \Upsilon_{3m} \) be defined by \( J_1 = \{1, \ldots, m\} \),
$J_2 = \{m + 1, \ldots, 2m\}$, $J_3 = \{2m + 1, \ldots, 3m\}$. If $(X, Y)$ belongs to the subvariety of $U_n \times U_n$ defined by

$$X_{i,j} = 0, \quad i, j \in J_h, \quad h = 1, 2, 3$$

then for $i \notin J_3$ or $j \notin J_3$ the entry of $[X, Y]$ of indices $(i, j)$ is 0. Hence $\dim NT_3^m \geq 3m^2 + 3m^2 - m^2 = 5m^2$. We have $\dim CT_3^0 = \frac{9m(m+1)}{2}$, which for $m \geq 10$ is smaller than $5m^2$, hence $CT_3^0$ isn’t a complete intersection for $m \geq 10$.

**Example C** Let $n = 18$ and let $J \in \Upsilon_{18}$ be such that $|J_1| = 1$, $|J_2| = 5$, $|J_3| = 6$, $|J_4| = 5$, $|J_5| = 1$. The condition $[X, Y] = 0$ gives 48 equations for $NT_1^J$ as subvariety of $U_1^J \times U_1^J$. Hence

$$\dim NT_1^J \geq \dim (U_1^J \times U_1^J) - 48 = 188.$$ 

Then the dimension of the orbit of $NT_1^J$ under the action of $K^2$ is greater or equal than 190. Since $\dim CT_1^0 = 189$, $CT_1$ isn’t a complete intersection.

**Example D** Let $n = 17$ and let $J \in \Upsilon_1$ be such that $|J_1| = 2$, $|J_2| = 4$, $|J_3| = 5$, $|J_4| = 4$, $|J_5| = 2$. The condition $[X, Y] = 0$ gives 56 equations for $NT_1^J$ as subvariety of $U_1^J \times U_1^J$. Hence

$$\dim NT_1^J \geq \dim (U_1^J \times U_1^J) - 56 = 168.$$ 

Then the dimension of the orbit of $NT_1^J$ under the action of $K^2$ is greater or equal than 170. Since $\dim CT_1^0 = 170$, $CT_1$ is reducible.

### 3 An interpretation of commuting varieties

Let $M(n, K)$ be the set of $n \times n$ matrices over $K$, which we regard as a projective space of dimension $n^2 - 1$. Let

$$\mathcal{C}(n, K) = \{(X, Y) \in M(n, K) \times M(n, K) : [X, Y] = 0\}.$$ 

For $X, Y \in M(n, K)$ we set $X = (x_{i,j}), Y = (y_{i,j})$, where $i, j \in \{1, \ldots, n\}$, and $X_{i,j} = \begin{pmatrix} x_{i,j} \\ y_{i,j} \end{pmatrix}$. As a generalization of equations 1, the equations for $\mathcal{C}(n, K)$ given by the condition $[X, Y] = 0$ can be written as follows:

$$\sum_{k=1}^{n} \det \begin{pmatrix} X_{i,k} & X_{k,j} \end{pmatrix} = 0 \quad i, j = 1, \ldots, n.$$
Let
\[ C_0(n, K) = \{ (X, Y) \in C(n, K) : \det \begin{pmatrix} X_{i,j} & X_{h,k} \end{pmatrix} = 0 \text{ for any } (i, j), (h, k) \in \{1, \ldots, n\} \times \{1, \ldots, n\} \}. \]

For \((i, j), (h, k) \in \{1, \ldots, n\} \times \{1, \ldots, n\}\) we denote by \(p_{(i,j)(h,k)}\) the Plücker coordinates of subspaces of codimension 2 of \(K^{n^2}\); we denote by \(G(2, K^{n^2})\) the grassmannian of those subspaces.

There is a natural map \(\gamma_n\) from \(C(n, K) \setminus C_0(n, K)\) into \(G(2, K^{n^2})\), defined by associating to \((X, Y)\) the subspace having the following Plücker coordinates:
\[ p_{(i,j)(h,k)} = \det \begin{pmatrix} X_{i,j} & X_{h,k} \end{pmatrix} \]
for \((i, j), (h, k) \in \{1, \ldots, n\} \times \{1, \ldots, n\}\). The image of \(\gamma_n\) is the subvariety of \(G(2, K^{n^2})\) defined by the following linear equations:
\[ \sum_{k=1}^{n} p_{(i,k)(k,j)} = 0, \quad i, j = 1, \ldots, n. \]

The group \(GL(2, K)\) acts on \(C(n, K) \setminus C_0(n, K)\) by the following rule:
\[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot (X, Y) = (aX + bY, cX + dY) \]
for \(\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL(2, K)\) and \((X, Y) \in C(n, K) \setminus C_0(n, K)\). The fibers of \(\gamma_n\) are the orbits of \(C(n, K) \setminus C_0(n, K)\) under the action of \(GL(2, K)\).

We can define a similar map for \(CT_h\) and \(NT_n\), as a restriction of \(\gamma_n\). As an example we illustrate this geometrical interpretation for \(NT_4\).

**Example A** We regard \(U_4\) as a projective variety of dimension 5 (whose coordinates have indices \((1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\)). We consider the elements of \(U_4\) as hyperplanes of \(\mathbb{P}^5\). The map \(\gamma_4\) associates to any pair of different hyperplanes of \(NT_4\) the subspace of \(\mathbb{P}^5\) given by their intersection.

The image of \(NT_4\) by \(\gamma_4\) is defined by the equations:
\[ p_{(1,2)(2,3)} = 0, \, p_{(2,3)(3,4)} = 0, \, p_{(1,2)(2,4)} + p_{(1,3)(3,4)} = 0. \]

The inverse image under \(\gamma_4\) of the subset of all subspaces of \(\mathbb{P}^5\) of codimension 2 such that \(p_{(1,2)(3,4)} = 0\) is the irreducible component \(NT_4^0\) of \(NT_4\). The set
of pairs of hyperplanes such that the coordinate of indices (2, 3) is 0 (that is, "parallel" to the (2, 3)-axis) is the irreducible component $N_{T}^1$ of $NT_4$.

**Example E**  The image of $C(2, K) \setminus C_0(2, K)$ under the map $\gamma_2$ is a subvariety of $G(2, K^4)$ defined by the following equations:

$$p(1,2)(2,1) = 0 ,$$

$$p(1,1)(1,2) + p(1,2)(2,2) = 0 ,$$

$$p(2,1)(1,1) + p(2,2)(2,1) = 0 .$$

The variety $G(2, K^4)$ is a subvariety of a projective space of dimension 5 defined by the equation

$$p(1,1)(2,2)p(1,2)(2,1) - p(1,1)(1,2)p(2,2)(2,1) + p(1,1)(1,2)p(2,2)(1,2) = 0 .$$

If we consider subvarieties of that projective space, we have the following results. The subvariety defined by equations 3, 4 and 5 has two irreducible components; one of them is $\gamma_2(C(2, K) \setminus C_0(2, K))$ (see [3]). The subvariety defined by the equations 2, 3 and 4 is contained in $G(2, K^4)$, hence it is $\gamma_2(C(2, K) \setminus C_0(2, K))$. Then $\gamma_2(C(2, K) \setminus C_0(2, K))$ is linear complete intersection as subvariety of that projective space.

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**References**

[1] W.H. Hesselink, A classification of the nilpotent upper triangular matrices, Compositio Math. 55 (1985). 89-133

[2] A. Keeton, Ph.D. thesis (University of California).

[3] A. Knutson, Some schemes related to the commuting variety, J. Algebr. Geom. 14, No. 2, 283-294 (2005).

[4] W.V. Vasconcelos, Arithmetic of Blowup Algebras, London Mathematical Society, Lecture Note Series 195, Cambridge University Press (1994).