METRIC DIOPHANTINE APPROXIMATION OVER A LOCAL FIELD OF POSITIVE CHARACTERISTIC

ANISH GHOSH

ABSTRACT. We establish the conjectures of Sprindžuk over a local field of positive characteristic. The method of Kleinbock-Margulis for the characteristic zero case is adapted.

1. Introduction

In this paper, we present a proof of the strong extremality of non-degenerate manifolds over a local field of positive characteristic.

1.1. Preliminary Notation. Let $\mathbb{F}$ denote the finite field of $k = p^\nu$ elements. Let $\mathbb{K} = \mathbb{F}(X)$ be the ring of rational functions, $\mathcal{Z} = \mathbb{F}[X]$ the ring of polynomials, $\mathcal{O} = \mathbb{F}[[X^{-1}]]$ be the ring of formal series in $X^{-1}$ and $\mathcal{K} = \mathbb{F}((X^{-1}))$ denote the field of Laurent series. A typical element of $\mathcal{K}$ is of the form

$$a = \sum_{i=-n}^{\infty} a_i X^{-i}, a_i \in \mathbb{F}, a_{-n} \neq 0.$$ 

It is well known that one can define a non-Archimedean valuation on $\mathcal{K}$ (the “valuation at $\infty$”):

$$v(a) = \sup\{j \in \mathbb{Z}, a_i = 0 \ \forall \ i < j\}$$ 

The corresponding discrete valuation ring is $\mathcal{O}$ and $\mathcal{K}$ is its quotient field. The valuation above leads to an absolute value $|a| = k^{-v(a)}$ which in turn induces a metric $d(a, b) = |a - b|$ and $(\mathcal{K}, d)$ is a separable, complete, ultrametric, totally disconnected space. Moreover, any local field of positive characteristic is isomorphic to some $\mathcal{K}$ (cf. [34]). We will extend the norm to vectors by defining $|\mathbf{x}| = \max_i |x_i|$. Vectors will be denoted in boldface, and we will use the notation $|\mathbf{x}|$ for both vectors as well as elements of $\mathcal{K}$, relying on the context and typeface to make the distinction between the norms. The notation $|x|_+$ will stand for $\max(|x|, 1)$ and we will set $\Pi_+(\mathbf{x}) = \prod_{i=1}^{n} |x_i|_+$. The notation $[\ ]$ will be used to denote both the polynomial part of an element of $\mathcal{K}$ as well as the integer part of a real number. $B(\mathbf{x}, r)$ will denote the ball centered around $\mathbf{x}$ in $\mathcal{K}^n$ of radius $r$, and $B_r$ will denote $B(0, r)$. Haar

\footnote{Mathematics Subject Classification: Primary 11J83, Secondary 11K60.}
measure on $K^n$ will be referred to as $\lambda$, normalised so that the measure of $B_1$ is 1. For a map $f : U \subset K^r \to K^n$ and a ball $B \subset \mathcal{X}$, we will set $|f|_B = \sup_{x \in B} |f(x)|$.

1.2. Diophantine Approximation. Metric Diophantine approximation is primarily concerned with classifying points in a finite dimensional vector space over a field with regard to their approximation properties. The classification is done with respect to a measure, so a “typical” property is a property which holds or does not for almost every (hereafter abbreviated as a.e.) point with respect to the specified measure. For instance, one studies the set of $v$-approximable vectors,

**Definition 1.1.** $\mathcal{W}_v \overset{\text{def}}{=} \{ x \in K^n | |qx + p| < |q|^{-v}, \text{for infinitely many } q \in \mathbb{Z}^n \text{ and some } p \in \mathbb{Z} \}$.

And the set of badly approximable vectors,

**Definition 1.2.** $\mathcal{B} \overset{\text{def}}{=} \{ x \in K^n | \exists C > 0 \text{ such that } |p + q \cdot x| > \frac{C}{|q|^{n}} \text{ for every } q \in \mathbb{Z}^n \setminus \{0\}, p \in \mathbb{Z} \}$.

It has been shown by Kristensen ([22], [23]) that whenever $v > n$, $\mathcal{W}_v$ is a null set of Hausdorff dimension $n - 1 + \frac{n+1}{v+1}$, and that $\mathcal{B}$ is a null set of full Hausdorff dimension. A vector which is $v$-approximable for some $v > n$ is said to be very-well approximable (abbreviated as VWA). More generally one can define very well multiplicatively approximable (VWMA) vectors as follows:

**Definition 1.3.** A vector $x$ is VWMA if for some $\epsilon > 0$, there are infinitely many $q \in \mathbb{Z}^n$ such that

$$|p + q \cdot x| \leq \Pi_+ (q)^{-1 - \epsilon}$$

for some $p \in \mathbb{Z}$.

We now describe the set-up of Diophantine approximation with dependent quantities. A map $f = (f_1, \ldots, f_n) : K^r \to K^n$ will be called extremal (resp. strongly extremal) if for $\lambda$ a.e. $x$, $f(x)$ is not VWA (resp. VWMA). The theme of establishing extremality of maps began when Mahler ([24]) conjectured the extremality of $f : \mathbb{R} \to \mathbb{R}^n$ given by $f(x) = (x, x^2, \ldots, x^n)$.\footnote{The definitions of VWA and VWMA vectors over the field of real or $p$-adic numbers are analogous. The interested reader should consult one of the many references, for instance [7], [18], [20].} Mahler’s conjecture was proved by Sprindžuk (cf. [31]). Let $\mathcal{X}$ denote a metric space, $\mathcal{F}$ a valued field and $\mu$ a Borel measure on $\mathcal{X}$. We will call a map $f : \mathcal{X} \to \mathcal{F}^n$, non-planar at
$x_0 \in \mathcal{X}$ if for any neighborhood $B$ of $x_0$, the restrictions of $1, f_1, \ldots, f_n$ are linearly independent over $\mathcal{F}$. Let us now take $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{F} = \mathbb{R}$. The strong extremality of analytic non-planar $f$ in this case was conjectured by Sprindžuk (conjecture $H_2$, [32]). This conjecture was settled by D.Kleinbock and G.Margulis in [18], using newly developed tools from homogeneous dynamics. In fact, they relaxed the analyticity condition and replaced the non-planarity condition with an appropriate generalization called nondegeneracy (which we define precisely in section 3). See [15] for a nice survey of the problem. Sprindžhuk’s (and indeed Mahler’s) conjectures can be formulated over other local fields. In [31], Sprindžuk proved Mahler’s conjecture over the fields $\mathbb{Q}_p$ and $\mathbb{K}$. Following some partial results (see [20] for a brief historical survey), the methods of Kleinbock-Margulis were extended in [20] to settle the conjecture $H_2$ over $\mathbb{Q}_p$. In fact, the following more general theorem is obtained by the authors.

**Theorem 1.4.** [20] Let $S$ be a finite set of valuations of $\mathbb{Q}$, for any $v \in S$ take $k_v, d_v \in \mathbb{N}$ and an open subset $U_v \subseteq \mathbb{Q}_v^{d_v}$, and let $\lambda$ be the product of haar measures on $\mathbb{Q}_v^{d_v}$. Suppose that $f$ is of the form $(f^v)_{v \in S}$, where each $f^v$ is a $C^{k_v}$ map from $U_v$ into $\mathbb{Q}_v^{n_v}$ which is nondegenerate at $\lambda_v$-a.e. point of $U_v$. Then $f, \lambda$ is strongly extremal.

1.3. **Main Result and Structure of this paper.** In this paper, we establish the validity of Sprindžuk’s conjecture $H_2$ over a local field of characteristic $p > 0$. The structure of this paper is as follows. In section 2, we establish the the link between Diophantine approximation and flows on homogeneous spaces, record a proof of Mahler’s compactness criterion in characteristic $p$ and provide an application (after Dani) to bounded trajectories on the space of lattices. Section 3 is devoted to a discussion of non-degenerate and good maps, culminating in a theorem from [20] which relates these notions. Finally, in section 4 we use the results from prior sections, as well as a modified version of a measure estimate from [20] to prove the main theorem of this paper, a special case of which is as follows.

**Theorem 1.5.** Let $U \subset \mathcal{K}^d$ be an open set and $f = (f_1, \ldots, f_n) : U \to \mathcal{K}^n$ be a $C^l$ non-planar map. Then $f$ is strongly extremal.

**Acknowledgements.** The author would like to thank his advisor Prof.Dmitry Kleinbock for guidance and for providing the preprint [20] which served as inspiration. Thanks are also due to Prof.Barak Weiss for helpful discussions.
2. Reduction to a dynamical statement

2.1. Mahler’s compactness criterion. It is well known that $\text{SL}(n, \mathbb{Z})$ is a non-uniform lattice in $\text{SL}(n, \mathbb{K})$ (c.f. [30]), which means that the space $\Omega_n = G_n/\Gamma_n$ is a non-compact space of finite volume. $\text{SL}(n, \mathbb{K})$ acts transitively on the space of unimodular (i.e. covolume 1) lattices in $\mathbb{K}^n$, and the stabilizer of $\mathbb{Z}^n$ is $\text{SL}(n, \mathbb{Z})$. Hence $\Omega_n$ can be identified with the space of unimodular lattices in $\mathbb{K}^n$. Let $\Lambda$ be any (not-necessarily unimodular) lattice. Then $\det(\Lambda)$ will refer to $\det(g)$ where $g \in \text{GL}(n, \mathbb{K})$ and $\Lambda$ is of the form $g\mathbb{Z}^n$.

Following Mahler, we will call a real valued function $F$ on $\mathbb{K}^n$ a distance function if it satisfies the following three conditions.

1. $F(x) \geq 0 \forall x$.
2. $F(tx) = |t|F(x)$ for every $t \in \mathbb{K}, x \in \mathbb{K}^n$.
3. $F(x - y) \leq \max(F(x), F(y))$ for every $x, y \in \mathbb{K}$.

The function $F(x) = |x|$ is the prototype of a distance function. The structure of compact subsets of $\Omega_n$ is described by the Mahler Compactness Criterion which we will now state and prove. This is well known over the field of real numbers and a proof can be found for instance in [6]. We will need the following result from the geometry of numbers due to Mahler.

**Theorem 2.1.** [25] Let $F$ be a distance function on $\mathbb{K}^n$. There are $n$ independent lattice points $x_1, \ldots, x_n \in \mathbb{Z}^n$ with the following properties:

1. $F(x_1)$ is the minimum of $F(x)$ among all non-zero lattice points.
2. For $k \geq 2$, $F(x_k)$ is the minimum of $F(x)$ among all lattice points which are independent of $x_1, \ldots, x_{k-1}$.
3. The determinant of the points $x_1, \ldots, x_n$ is 1.
4. $0 < F(x_1) \leq \cdots \leq F(x_n)$ and

$$\prod_{i=1}^{n} F(x_i) = 1$$

For our purposes, a trivial modification of the above theorem will be required which extends it to all lattices. Notice that the above theorem is actually a statement about the successive minima of $B_1$ with respect to the standard lattice. To restate the theorem for an arbitrary lattice $\Lambda = g\mathbb{Z}^n$, $g \in \text{GL}(n, \mathbb{K})$ one needs to instead consider the successive
minima of the set $g^{-1}B_1$ with respect to the standard lattice. Thus we get the following corollary of theorem 2.1.

**Corollary 2.2.** Any $n-$dimensional lattice $\Lambda$ has a basis $x_1, \ldots, x_n$ such that

$$\prod_{i=1}^{n} |x_i| \leq |\text{det}(\Lambda)|$$

A subset $Q$ of $\Omega_n$ is said to be separated from 0, if there exists a non-empty neighborhood $B$ of 0 in $K^n$ such that $\Lambda \cap B = \{0\}$ for any lattice $\Lambda$ in $Q$. The following is the positive characteristic version of Mahler’s compactness criterion.

**Theorem 2.3.** A subset $Q$ of $\Omega_n$ is bounded if and only if it is separated from 0.

**Proof.** We omit the implication ($\Rightarrow$), as it is elementary and identical to the classical case. For the converse, notice that by corollary 2.2, we know that any lattice $\Lambda$ in $Q$ has a basis $a_1, \ldots, a_n$ such that

\begin{equation}
\prod_{i=1}^{n} |a_i| \leq 1
\end{equation}

Then, since the vectors $a_i$ are also bounded away from the origin by assumption, it follows that the norms of the vectors $a_i$ are uniformly bounded from above. We now apply the Bolzano-Weierstrass theorem to finish the proof. \qed

We get the following immediate:

**Corollary 2.4.** The set

$$Q_\epsilon \overset{def}{=} \{ \Lambda \in \Omega_n \mid |x| \geq \epsilon \ \forall \ x \in \Lambda \backslash \{0\} \}$$

is compact for every $\epsilon > 0$.

### 2.2. Dynamics and Diophantine Approximation.

In order to state Diophantine properties of vectors in dynamical language, we need some notation. Let $f$ be a map from an open subset of $K^d$ to $K^n$, and let $u_{f(x)}$ denote the matrix

\begin{equation}
u_{f(x)} \overset{def}{=} \begin{pmatrix}1 & f(x)^t \\ 0 & I_n \end{pmatrix}
\end{equation}

and let $\Lambda_{f(x)}$ denote the lattice $u_{f(x)} \mathbb{Z}^{n+1}$. In particular, if $f(x) = x$, we will denote the lattice by $\Lambda_x$. Let $t = (t_1, \ldots, t_n) \in \mathbb{Z}_+^n$ and set
\( t = \sum_{i=1}^{n} t_i \), we consider the action on \( \Lambda_{\varepsilon(x)} \) by semisimple elements of the form
\[
g_t = \text{diag}(X^t, X^{-t_1}, \ldots, X^{-t_n}).
\]
(2.3)

Define a function on the space \( \Omega_n \) in the following manner:
\[
\delta(\Lambda) \overset{\text{def}}{=} \inf_{v \in \Lambda \setminus \{0\}} |v|.
\]
(2.4)

The following theorem establishes a link between orbits on the space of lattices and Diophantine properties of vectors.

**Theorem 2.5.** Let \( \varepsilon > 0 \), \( x \in K^n \) and \( (p, q) \in \mathbb{Z}^{n+1} \) be such that (1.1) holds. Denote \( \Pi_+(q) \) by \( k^m \) and define
\[
r = k^{-\left\lceil \frac{m\varepsilon}{n+1} \right\rceil}.
\]
(2.5)

Choose \( t_i \in \mathbb{Z}_+ \) to satisfy \( |q_i|_+ = rk^{t_i} \). Then, \( \delta(g_t \Lambda x) \leq r \).

**Proof.** We need to prove the inequalities:
\[
k^t|p + q \cdot y| \leq r
\]
and
\[
k^{-t_i}|q_i| \leq r \ \forall \ i.
\]
(2.7)

The second follows immediately from the fact that \( |q_i| \leq |q| \) and the definition of \( t_i \). As for the first, assume that (1.3) holds. Then, we have
\[
|q \cdot y + p| \leq \Pi_+(q)^{-1-\varepsilon}.
\]

Since \( \Pi_+(q) = r^n k^t \), it follows that
\[
k^t|q + y \cdot p| \leq r^{-n} \Pi_+(q)^{-\varepsilon}
\]

Since \( k^{\frac{m\varepsilon}{n+1}} \geq k^{\frac{m\varepsilon}{n+1}} \), we see that \( k^{-m\varepsilon} \leq k^{-\left\lceil \frac{m\varepsilon}{n+1} \right\rceil(n+1)} \) which implies that \( \Pi_+(q)^{-\varepsilon} \leq r^{n+1} \). Thus,
\[
k^t|q + y \cdot p| \leq r^{-n} r^{n+1}.
\]

This completes the proof. \( \square \)

Writing \( r = k^{-\gamma} \) for a suitably chosen \( \gamma \) allows us to derive the following:

**Corollary 2.6.** Assume that \( x \in K^n \) is VWMA. Then there exists \( \gamma > 0 \) and infinitely many \( t \in \mathbb{Z}_+^n \) such that
\[
\delta(g_t \Lambda x) \leq k^{-\gamma t}.
\]

**Proof.** By theorem 2.5 and for \( \gamma \) as above, we can find an unbounded sequence \( t_k \in \mathbb{Z}_+^n \) such that \( \delta(g_{t_k} \Lambda x) \leq k^{-\gamma t_k} \). \( \square \)
Consequently, to show that a map $f : U \subset K^d \to K^n$ is strongly extremal, it is enough to show that any non-degenerate point has a neighborhood $B \subseteq U$ such that for a.e. point in the neighborhood and any $\gamma > 0$, there are at most finitely many $t \in \mathbb{Z}_+^n$ such that
\begin{equation}
\delta(g_t \Lambda f(x)) \leq k^{-\gamma t}
\end{equation}
For then, if we fix $t$ and define the set
\begin{equation}
E_t = \{x \in B \mid \delta(g_t \Lambda f(x)) \leq k^{-\gamma t}\}
\end{equation}
then theorem 1.3 will follow from an application of the Borel-Cantelli lemma if we are able to show that
\begin{equation}
\sum_{t \in \mathbb{Z}_+^n} \lambda(E_t) < \infty.
\end{equation}
Lemma 2.7 will be an easy consequence of the following theorem which will then complete the proof of theorem 1.5.

**Theorem 2.8.** Let $f$ be a $C^1$ map from an open subset $U \subset K^d$ to $K^n$, and assume that $f$ is nondegenerate at $x_0 \in U$. Then there exists a ball $B(x_0, r) \subset U$ and positive constants $C, \rho$ such that for any $t \in \mathbb{Z}_+^n$, any $s > 0$ and $0 < \epsilon \leq \rho$ one has
\begin{equation}
\lambda(\{x \in B \mid \delta(g_t \Lambda f(x)) < \epsilon\}) \leq (n + 1)C \left(\frac{\epsilon}{\rho}\right)^\alpha \lambda(B)
\end{equation}

2.3. Bounded trajectories. Let us digress a bit to provide an application of theorem 2.3. This result is originally due to Dani [8] who established it over the field of real numbers. For $t \in \mathbb{Z}$, let
\begin{equation}
g_t = \text{diag}(X^{nt}, X^{-t}, \ldots, X^{-t}).
\end{equation}

**Theorem 2.9.** The trajectory $\{g_t \Lambda x \mid t \in \mathbb{Z}_+\}$ is bounded if and only if $x$ is badly approximable.

**Proof.** Assume that $x$ is badly approximable and choose $\delta$ so that
\begin{equation}
C \frac{1}{\pi n t} > \delta > 0
\end{equation}
where $C$ is the constant in definition 1.2. Let $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$ and $\tilde{y} = (y_0, y) \in \mathbb{Z}^{n+1}$ be such that $g_{t'} u_x \tilde{y} \in B_\delta$ for some $t' \in \mathbb{Z}_+$. Keeping in mind that $|X| = k$, we have
\begin{equation}
k^{nt'} |\tilde{y} \cdot (1, x)| \leq \delta
\end{equation}
and
\begin{equation}
k^{-t'} |y| \leq \delta.
\end{equation}
From definition 1.2 and equations 2.12 and 2.13 it follows that
\[ \frac{C}{\delta^n k^{nt'}} \leq \frac{C}{|y|^n} < |\tilde{y} \cdot (1, x)| \leq \frac{\delta}{k^{nt'}} \]
which cannot happen in view of equation 2.11. Hence, \( g_t \Lambda \in \overline{B}_\delta = \{0\} \)
and by corollary 2.4 the trajectory is bounded.
For the converse, by theorem 2.3 there exists \( \delta > 0 \) such that \( |g_t u \tilde{y}| > \delta \) for every \( \tilde{y} \in \mathbb{Z}^{n+1} \). This implies that for every \( t \in \mathbb{Z} \),
\begin{align*}
(2.14) & \quad k^{nt'}|\tilde{y} \cdot (1, x)| > \delta \\
and \quad (2.15) & \quad k^{-t}|y| > \delta
\end{align*}
A choice of \( C = \delta^{n+1} \) can now be seen to ensure that \( x \) is badly approximable.

One can now decompose \( g \in \text{SL}(n+1, \mathcal{K}) \) into factors one of which is of the form 2.2 and then conclude (cf. proposition 2.12 in [8]) that

**Lemma 2.10.** The trajectory \( \{g_t g \mathbb{Z}^{n+1} \mid t \in \mathbb{Z}_+\} \) is bounded if and only if \( \{g_t \Lambda \mid t \in \mathbb{Z}_+\} \) is bounded.

As a corollary of theorem 2.9, lemma 2.10 and the main result in [23], it follows that

**Corollary 2.11.** The set
\[ Bdd_{n+1} \defeq \{ \Lambda \in \Omega_{n+1} \mid \{g_t \Lambda\} \text{ is a bounded trajectory} \} \]
has full Hausdorff dimension.

To put corollary 2.11 in context, we remark that in case \( G = \text{SL}(n+1, \mathbb{R}) \) and \( \Gamma = \text{SL}(n+1, \mathbb{Z}) \), the action of a one-parameter subgroup \( g_t \) not contained in a compact subgroup of \( G \), on \( G/\Gamma \) is ergodic (a special case of Moore’s ergodicity theorem cf. [35]). This implies that the set of bounded \( g_t \) orbits is a null set (with respect to the \( \text{SL}(n, \mathbb{R}) \)-invariant measure on \( G/\Gamma \)). The Kleinbock-Margulis bounded orbit theorem (cf. [19]) is a vast generalization of the “ampleness” of bounded trajectories as above, to semisimple flows on general homogeneous spaces of real Lie groups. Over \( \mathbb{Q}_p \), we know after Tamagawa that all lattices in \( \text{SL}(n, \mathbb{Q}_p) \) are cocompact (cf. [30]) and so all orbits are necessarily bounded. Over \( \mathcal{K} \), the ergodicity of semisimple flows has been established by G.Prasad (cf. [27]) and implies that for every \( n \in \mathbb{Z}_+ \), \( Bdd_n \) has measure 0 (with respect to the \( \text{SL}(n, \mathcal{K}) \)-invariant measure on \( \text{SL}(n, \mathcal{K})/\text{SL}(n, \mathbb{Z}) \)).
3. Ultrametric non-degenerate and good maps

We will first define single variable $C^n$ functions in the ultrametric case. Our definitions and treatment are from [29]. Let $U$ be a non-empty subset of $\mathcal{K}$ without isolated points. For $n \in \mathbb{N}$, define

**Definition 3.1.**

$$\nabla^n(U) = \{(x_1, \ldots, x_n) \in U, x_i \neq x_j \text{ for } i \neq j\}$$

The $n$-th order difference quotient of a function $f: U \rightarrow \mathcal{K}$ is the function $\Phi_n(f)$ defined inductively by $\Phi_0(f) = f$ and, for $n \in \mathbb{N}$, $(x_1, \ldots, x_{n+1}) \in \nabla^n(U)$ by

$$\Phi_n f(x_1, \ldots, x_{n+1}) = \frac{\Phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \Phi_{n-1} f(x_2, \ldots, x_{n+1})}{x_1 - x_2}$$

Note that the definition does not depend on the choice of variables, as all difference quotients are symmetric functions. A function $f$ on $\mathcal{K}$ is called a $C^n$ function if $\Phi_n f$ can be extended to a continuous function $\bar{\Phi}_n f : U^{n+1} \rightarrow \mathcal{K}$. We also define

$$D_n f(a) = \Phi_n f(a, \ldots, a), \ a \in U$$

We then have the following theorem (c.f. [29], Theorem 29.5)

**Theorem 3.2.** Let $f \in C^n(U \rightarrow \mathcal{K})$. Then, $f$ is $n$ times differentiable and

$$j! D_j f = f^j$$

for all $1 \leq j \leq n$.

An immediate corollary shows us why we must exercise a little caution in positive characteristic:

**Corollary 3.3.** Let $\text{char}(\mathcal{K}) = p$ and $f \in C^p(U \rightarrow \mathcal{K})$. Then $f^p = 0$.

To define $C^k$ functions in several variables, a generalization of the above notion is required. We will follow the notation set forth in [20]. Namely, we now consider a multiindex $\beta = (i_1, \ldots, i_d)$ and let

$$\Phi_\beta f = \Phi_{i_1} \circ \cdots \circ \Phi_{i_d} f$$

This difference order quotient is defined on the set $\nabla^{i_1} U_1 \times \cdots \times \nabla^{i_d} U_d$ and the $U_i$ are all non-empty subsets of $\mathcal{K}$ without isolated points. A function $f$ will then be said to belong to $C^k(U_1 \times \cdots \times U_d)$ if for any multiindex $\beta$ with $|\beta| = \sum_{j=1}^d i_j \leq k$, $\Phi_\beta f$ extends to a continuous function $\bar{\Phi}_\beta f : U_1^{i_1+1} \times \cdots \times U_d^{i_d+1}$. As in the one variable case, we have

$$(3.1) \quad \partial_\beta f(x_1, \ldots, x_d) = \beta! \bar{\Phi}_\beta(x_1, \ldots, x_1, \ldots, x_d, \ldots, x_d)$$
where $\beta! = \prod_{j=1}^{d} i_j!$.

We now wish to define non-degenerate functions in our situation. Over the field of real numbers, a function is said to be non-degenerate if the target space is spanned by the partial derivatives of the function. We will have to modify this slightly in view of corollary 3.3. Let $f = (f_1, \ldots, f_n)$ be a $C^m$ map from $U \subset \mathcal{K}^d$ to $\mathcal{K}^n$. For $l \leq m$, we will say that a point $y = f(x)$ is $l$ non-degenerate if the space $\mathcal{K}^n$ is spanned by the difference quotients $\Phi_\beta$ of $f$ at $x$ with $|\beta| \leq l$. For analytic functions, it follows that the linear independence of $1, f_1, \ldots, f_n$ is equivalent to all points of $f(x)$ being non-degenerate. We would also like to remark that for one variable, the definition of non-degeneracy does not correspond to the non-vanishing of the Wronskian. This is in contrast to the real variable case.

It follows easily that $f$ is $k$ non-degenerate at $x_0$ if and only if for any function $f$ of the form $f = c_0 + c \cdot f$, where $c_0 \in \mathcal{K}\{0\}$ and $c \in \mathcal{K}$ there exists a multiindex $\beta$ such that $|\beta| \leq k$ and $\Phi_\beta \neq 0$.

Before proceeding, we define an important class of functions. Let $\mathcal{X}$ denote a metric space, $\mu$ a locally finite Borel measure on $\mathcal{X}$ and $\mathcal{F}$ a locally compact field. For a ball $B \subset \mathcal{X}$, and a map $f : \mathcal{X} \rightarrow \mathcal{F}$ we set $|f|_{B, \mu} \overset{df}{=} |f|_{B \cap \text{supp } \mu}$.

**Definition 3.4.** Let $C$ and $\alpha$ be positive numbers and $V \subseteq \mathcal{X}$. A function $f : V \rightarrow \mathcal{F}$ is said to be $(C, \alpha)$-good on $V$ with respect to $\mu$ if for any open ball $B \subseteq V$, and for any $\epsilon > 0$, one has:

$$\mu\left(\left\{v \in B \mid |f(v)| < \epsilon \cdot |f(v)|_{B, \mu}\right\}\right) \leq C \epsilon^\alpha \mu(B).$$

We will be mostly concerned with the case when $\mathcal{X} = \mathcal{K}^d$ for some $d$. In this case, we will assume that $\mu$ is the normalized Haar measure $\lambda$ and simply refer to the map as $(C, \alpha)$-good. Some easy properties of $(C, \alpha)$-good functions are:

1. $f$ is $(C, \alpha)$-good on $V$ $\Rightarrow$ so is $cf \forall c \in \mathcal{K}$. (Here $\mathcal{F} = \mathcal{K}$).

2. $f_i$ $i \in I$ are $(C, \alpha)$-good $\Rightarrow$ so is $\sup_{i \in I} |f_i|$. (Here $\mathcal{F} = \mathbb{R}$).

Polynomials provide good examples of $(C, \alpha)$-good functions. In fact, we have the following lemma from [33].

**Lemma 3.5.** Let $\mathcal{F}$ be an ultrametric valued field. Then for any $k \in \mathbb{N}$, any polynomial $f \in \mathcal{F}[x]$ of degree not greater than $k$ is $(C, 1/k)$-good on $\mathcal{F}$, where $C$ is a constant depending on $k$ alone.
More generally, we will call a map \( f : U \subset K^d \to K^n \) good at \( x_0 \in U \) if there exists a neighborhood \( V \subset U \) of \( x_0 \) and positive \( C, \alpha \) such that any linear combination of \( 1, f_1, \ldots, f_n \) is \((C, \alpha)\) good on \( V \). We now state Proposition 4.2 from [20] which shows that non-degenerate functions are good.

**Theorem 3.6.** Let \( F \) be an ultrametric valued field and let \( f = (f_1, \ldots, f_n) \) be a \( C^1 \) map from an open subset \( U \subset F^d \) to \( F^n \) which is \( l \)-non-degenerate at \( x_0 \in U \). Then there is a neighborhood \( V \subset U \) of \( x_0 \) such that any linear combination of \( 1, f_1, \ldots, f_n \) is \((dl^{3-\frac{4}{d}}, \frac{1}{dl})\)-good on \( V \). In particular, the nondegeneracy of \( f \) at \( x_0 \) implies that \( f \) is good at \( x_0 \).

4. Quantitative non-divergence and applications

In this section, our aim is to establish theorem [16]. We first will need some notation. Let \( D \) be an integral domain, \( K \) its quotient field, and \( R \) denote a field containing \( K \) as a subfield. If \( \Delta \) is a \( D \)-submodule of \( R^m \), we will denote by \( R\Delta \) its \( R \)-linear span inside \( R^m \), and define the rank of \( \Delta \) to be

\[
\text{rk}(\Delta) = \dim_R(R\Delta)
\]

If \( \Delta \subset \Lambda \) and \( \Lambda \) is also a \( D \)-submodule, we will say that \( \Delta \) is primitive in \( \Lambda \) if any submodule of \( \Lambda \) of rank equal to \( \text{rk}(\Delta) \) which contains \( \Delta \) is equal to \( \Delta \), and we will call \( \Delta \) primitive if it is primitive in \( D^m \). It follows from Lemma 6.2 in [20] that \( \Delta \) is primitive if and only if

\[
\Delta = R\Delta \cap D^m.
\]

We also define

\[
\mathcal{B}(D, m) = \text{the set of nonzero primitive submodules of } D^m.
\]

and

\[
\mathcal{M}(R, D, m) = \{ g\Delta \mid g \in GL(m, R), \Delta \text{ is a submodule of } D^m. \}.
\]

Note that \( \mathcal{B}(D, m) \) is a poset ordered by inclusion of length \( m \). Moreover we have,

**Lemma 4.1.** Let \( \Gamma \) be a discrete \( \mathbb{Z} \)-submodule of \( K^m \). Then

\[
\Gamma = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_k
\]

where \( x_1, \ldots, x_k \) are linearly independent over \( K \). In particular, \( \Gamma \) is free and finitely generated.
Proof. Since $\Gamma \subset K^m$, we can take a maximal linearly independent (over $K$) set $\{v_1, \ldots, v_k\}$ of vectors. Let $\Gamma'$ denote the free $\mathbb{Z}$-module $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k$. Clearly, $\Gamma'$ is a $\mathbb{Z}$-submodule of $\Gamma$. Moreover, $\Gamma/\Gamma'$ is a discrete subset of the compact space $(Kv_1 + \cdots + Kv_k)/\Gamma'$, and is consequently finite. Thus $\Gamma'$ has finite index in $\Gamma$ and so $\Gamma$ is a free $\mathbb{Z}$-module of rank $k$. The existence and linear independence of the basis follows. □

Consequently, $\mathcal{M}(\mathcal{R}, \mathcal{D}, m)$ can be identified with the set of discrete $\mathbb{Z}$-submodules of $K^m$. We now wish to measure the size of such submodules. Let $\nu : \mathcal{M}(\mathcal{R}, \mathcal{D}, m) \to \mathbb{R}_+$ be a function. Following [20], we will call $\nu$ norm-like if the following three conditions are satisfied:

1. For any $\Delta, \Delta' \in \mathcal{M}(\mathcal{R}, \mathcal{D}, m)$, with $\Delta' \subset \Delta$ and $rk(\Delta) = rk(\Delta')$, one has $\nu(\Delta') \geq \nu(\Delta)$.
2. There exists $C_\nu > 0$ such that for any $\Delta \in \mathcal{M}(\mathcal{R}, \mathcal{D}, m)$ and any $\gamma \not\in \mathcal{R}\Delta$ one has $\nu(\Delta + \mathcal{D}\gamma) \leq C_\nu \nu(\Delta) \nu(\mathcal{D}\gamma)$.
3. For every submodule $\Delta$ of $\mathcal{D}^m$, the function $GL(m, \mathcal{R}) \to \mathbb{R}_+, g \to \nu(g\Delta)$ is continuous.

The following theorem is an ultrametric version of theorem 6.3 in [20]. The proof of the theorem is to a large extent identical to that in [20], or [18]. Rather than reproduce it, we point out the differences in the statement and provide the reader with justifications.

**Theorem 4.2.** Let $\mathcal{X}$ be a separable ultrametric space, $\mu$ denote a locally finite Borel measure on $\mathcal{X}$, and let $\mathcal{D} \subset K \subset \mathcal{R}$ be as above. For $m \in \mathbb{N}$, let a ball $B = B(x_0, r_0) \subset \mathcal{X}$ and a continuous map $h : B \to GL(m, \mathcal{R})$ be given. Let $\nu$ be a norm-like function on $\mathcal{M}(\mathcal{R}, \mathcal{D}, m)$. For any $\Delta \in \mathcal{B}(\mathcal{D}, m)$, denote by $\psi_\Delta$ the function $x \to \nu(h(x)\Delta)$ on $B$. Now suppose that for some $C, \alpha > 0$ and $0 < \rho < \frac{1}{C_\nu}$, the following three conditions are satisfied.

1. For every $\Delta \in \mathcal{B}(\mathcal{D}, m)$, the function $\psi_\Delta$ is $(C, \alpha)$-good on $B$.
2. For every $\Delta \in \mathcal{B}(\mathcal{D}, m)$, $|\psi_\Delta|_B, \mu \geq \rho$.
3. For every $x \in B \cap \text{supp } \mu$, $\# \{\Delta \in \mathcal{B}(\mathcal{D}, m) \mid \psi_\Delta(x) < \rho\} < \infty$.

Then for any positive $\epsilon \leq \rho$ one has

$$\mu \left( \left\{ x \in B \mid \nu(h(x)\gamma) < \epsilon \text{ for some } \gamma \in \mathcal{D}^m \setminus \{0\} \right\} \right) \leq mC \left( \frac{\epsilon}{\rho} \right)^\alpha \mu(B).$$
Theorem 6 in [20] differs from the above statement in two ways. Firstly, the domain of the map $h$ above is a dilate of $B$, namely it is $B(x_0, 3^m r_0)$. Secondly, it is proven for the class of Federer measures (see below), a restriction we no longer need. This rids the estimate of a constant. We elaborate on these below.

**Dilation of balls:** The proof of theorem 4.2 is based on a delicate induction argument. Essentially, a notion of “marked” points is introduced and it is established that the set of unmarked points has small measure. In the induction step, a collection of balls with centers inside $B$ is taken. However, these balls need not be contained in $B$, and therefore, one needs to dilate the ball $B$ and introduce a constraint on the measure $\mu$ so as to ensure that it behaves well with respect to dilations. This is the so-called Federer condition and it introduces an additional constant in the above estimate. However, in the case that $X$ is ultrametric, each of the above balls must be contained in $B$. Therefore we do not need to dilate the ball and restrict ourselves to Federer measures.

**Besicovitch constant:** The subsequent strategy is to cover the dilated ball $B$ and choose a countable sub-covering with some multiplicity (depending on $X$). The fact that this can be done is the content of the Besicovitch covering theorem (cf. [18] and the references therein). This introduces a constant (a power of the multiplicity) in the above estimate. For separable ultrametric spaces, as can be easily verified a subcovering with multiplicity one suffices.

To apply the above theorem, we take $\mathcal{D} = \mathcal{Z}$, and $\mathcal{R} = \mathcal{K}$. Let $e_0, e_1, \ldots, e_m$ denote the standard basis of $\mathcal{K}^m$. Let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_m}$ where $I = (i_1, \ldots, i_m)$. We extend this norm to the exterior algebra of $\mathcal{K}^m$. Namely, for $w = \sum_I w_I e_I$, we set $|w| = \max_I |w_I|$. Since $\Gamma$ is a finitely generated free $\mathcal{Z}$-module, we can choose a basis $v_1, v_2, \ldots, v_r$ (where $r$ is the rank of $\Gamma$ as a $\mathcal{Z}$-module) of $\Gamma$ and define

\begin{equation}
|\Gamma| = |v_1 \wedge v_2 \wedge \cdots \wedge v_r|
\end{equation}

Note that $\Gamma$ is a lattice in $\mathcal{K} \Gamma$ and that the vectors $v_i$ generate this space. Moreover, it turns out that

**Lemma 4.3.** The function $| |$ is norm like on $\mathfrak{M}(\mathcal{K}, \mathcal{Z}, m)$.

**Proof.** Property N3 is a consequence of the definition. To prove N2, we take $w$ representing $\Delta$, and $C_\nu = 1$. Then $w, \gamma$ is a basis for $\Delta + \mathcal{Z} \gamma$
and so it suffices to prove that $|w \wedge \gamma| \leq |w||\gamma|$. Let $w = \sum_{I} w_{I}e_{I}$ and $\gamma = \sum_{i=1}^{k} \gamma_{i}e_{i}$. Then

$$|w \wedge \gamma| \leq \max_{1 \leq i \leq k} \max_{I} |w_{I}\gamma_{i}| \leq \max_{I} |w_{I}| \max_{1 \leq i \leq k} |\gamma_{i}| = |w||\gamma|.$$ 

It is also straightforward to verify the veracity of N1. \hfill \Box

We thus have:

**Theorem 4.4.** Let $m, d \in \mathbb{N}$, $C, \alpha > 0$ and $0 < \rho < 1$ be given. Let a ball $B = B(x_{0}, r_{0}) \subset K^{d}$ and a continuous map $h : B \to GL(m, K)$ be given. For any $\Delta \in \mathcal{B}(Z, m)$, let $\psi_{\Delta}(x) = |h(x)\Delta|$, $x \in B$. Assume that

1. For every $\Delta \in \mathcal{B}(Z, m)$, the function $\psi_{\Delta}$ is $(C, \alpha)$-good on $B$.
2. For every $\Delta \in \mathcal{B}(Z, m)$, $|\psi_{\Delta}|_{B} \geq \rho$.
3. For every $x \in B$, $\sharp\{\Delta \in \mathcal{B}(Z, m) \mid \psi_{\Delta}(x) < \rho\} < \infty$.

Then for any positive $\epsilon \leq \rho$ one has

$$\lambda\left(\{x \in B \mid \delta(h(x)Z^{m}) < \epsilon\}\right) \leq mC\left(\frac{\epsilon}{\rho}\right)^{\alpha}\lambda(B).$$

**Proof.** We apply theorem 4.2. Lemma 4.3 guarantees the norm-like behavior of $| |$ whereas condition (3) follows from the discreteness of $\wedge^{r}(Z^{m})$ in $\wedge^{r}(K^{m})$. Further, if $\delta(h(x)Z^{m}) < \epsilon$ then there exists a non-zero vector $w \in Z^{m}$ such that $|h(x)w| < \epsilon$. \hfill \Box

We now complete the proof of theorem 2.8 using:

**Theorem 4.5.** Let $f = (f_{1}, \ldots, f_{n})$ be a $C^{d}$ map from a ball $B \subset K^{d}$ to $K^{n}$ which satisfies the following two conditions:

1. For any $c = (c_{0}, \ldots, c_{n}) \in K^{n+1}$, $c_{0} + \sum_{i=1}^{n} c_{i}f_{i}$ is $(C, \alpha)$-good on $B$.
2. For any $c \in K^{n}$ with $|c| \geq 1$,

$$|c_{0} + \sum_{i=1}^{n} c_{i}f_{i}|_{B} \geq \rho$$

Take any $\epsilon \leq \rho$ and set $h(x) = g_{t}u_{f(x)}$. Then,

$$\lambda(\{x \in B \mid \delta(h(x)Z^{n+1}) < \epsilon\}) \leq (n + 1)C\left(\frac{\epsilon}{\rho}\right)^{\alpha}\lambda(B).$$
Proof. Let us begin by describing the action of \( h(x) \) on \( \mathcal{B}(\mathcal{D}, m) \). To do this, we fix a basis (the standard one) \( e_0, e_1, \ldots, e_n \) of \( \mathcal{K}^{n+1} \). We now take a submodule \( \Gamma \in \mathcal{B}(\mathcal{D}, m) \), and an element \( w \in \Lambda^r(\mathcal{K}^{n+1}) \) of the form \( w = \sum_I w_I e_I \) representing \( \Gamma \). Then,

\[
u_{\Gamma(x)} w = \begin{cases} e_I \\ e_I + \sum_{i \in I} \pm f_i(x) e_{I \cup \{0\} \setminus \{i\}} & 0 \in I \\ \text{else.} & \end{cases}
\]

and so we have

\[
u_{\Gamma(x)} w = \sum_{0 \notin I} w_I e_I + \sum_{0 \in I} \left( w_I + \sum_{i \not\in I} \pm w_{I \cup \{i\} \setminus \{0\}} f_i(x) \right) e_I
\]

If we now apply \( g_{\Gamma} \) to both sides of the above equation, we get

\[
u_{\Gamma(x)} w = \sum_I h_I(x) e_I
\]

where

\[
 h_I(x) = \begin{cases} (\prod_{i \in I} k^{-i}) w_I & 0 \notin I \\ (\prod_{i \not\in I} k^i)(w_I + \sum_{i \not\in I} \pm w_{I \cup \{i\} \setminus \{0\}} f_i(x)) & \text{else.}
\end{cases}
\]

Hence, all the coordinates of \( h_I(x) \) are of the form \( c_0 + \sum_{i=1}^n c_i f_i(x) \) for some \( c \in \mathcal{K}^{n+1} \). By assumption 1, any such combination is \( (C, \alpha) \)-good on \( \tilde{B} \). Then, by property 2 following definition 3.4, we have that \( \sup h_I \) is \( (C, \alpha) \)-good as well. Moreover, since \( w_I \in \mathcal{Z} \) for each \( I \), and at least one of them is non-zero, we can conclude that there exists \( I \) containing 0 such that \( h_I(x) = c_0 + \sum_{i=1}^n c_i f_i(x) \) and \( |c| \geq 1 \) which implies that \( |h_I|_B \geq \rho \). If we now define \( \psi_I(x) = |h(x)\Gamma| \), this means that \( |\psi|_B \geq \rho \). Now an application of theorem 4.3 completes the proof.

We now proceed to a proof of theorem 2.3 using theorem 4.3. Take \( U \subset \mathcal{K}^d \), \( f : U \to \mathcal{K}^n \), and \( x_0 \in U \). Using proposition 3.6, we can find a neighborhood \( V \subset U \) of \( x_0 \) such that any linear combination of 1, \( f_1, \ldots, f_n \) is \( (d^{\frac{1}{2}} + \frac{1}{m}) \)-good on \( V \). Choose a ball \( B = B(x_0, r) \subset V \). Then \( f \) and \( B \) will satisfy condition 1 of theorem 4.3. As for condition 2, it is an immediate consequence of the linear independence of 1, \( f_1, \ldots, f_n \) over \( \mathcal{K} \). Thus, an application of theorem 4.3 completes the proof.

Thus, it follows that for any \( t \in \mathbb{Z}_+^n \), \( \lambda(E_t) \leq d^{\frac{1}{2}} \left( \frac{k^{-d}}{\rho} \right)^\frac{1}{m} \) and so,

\[
\sum_{t \in \mathbb{Z}_+^n} \lambda(E_t) = \sum_{q=1}^\infty \sum_{t=t} q^{-d/q/d} \approx \sum_{q=1}^\infty q^n q^{-d/q/d} \text{ which converges.}
\]

This immediately implies lemma 2.7 thus completing the proof of Theorem 1.5.
5. **Dynamical Applications and concluding remarks**

5.1. **Dynamical Applications.** We now proceed to applications of a dynamical nature. Following work of G. Margulis [26], it has been known that orbits of unipotent flows on $\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$ are non-divergent. This was extended by S.G. Dani (cf. [9] and the references therein) in several important ways. Specifically, given a lattice $\Lambda$ in $\mathbb{R}^n$ and any unipotent flow $\{u_t\}_{t \in \mathbb{R}}$, it was shown that one can find a compact $K \subset \text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$ such that $u_t\Lambda$ spends most of its time in this compact set and a quantitative estimate on this time was obtained. Secondly, it was shown that under suitable conditions (i.e. unless the orbit of a lattice is contained in a proper closed subset), one could pick a compact set which works for any lattice, and these results were extended to general semi-simple Lie groups and their lattices. In [18], the authors obtain a quantitative improvement of Dani’s result (for the case $\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$) and in [20], these results were extended to the $S$-arithmetic case. The question of establishing unipotent non-divergence in characteristic $p$ was raised by S.G. Dani in [10]. Using theorem 4.4 and 2.3 it is possible to answer this question for $\text{SL}(n, \mathbb{K})/\text{SL}(n, \mathbb{Z})$. Specifically it can be shown that,

**Theorem 5.1.** Let $\Lambda \in \Omega_n$ be any lattice. Then there exist positive constants $C = C(n)$ and $\rho = \rho(\Lambda)$ such that for any one-parameter subgroup $\{u_t\}$ of $\text{SL}(n, \mathbb{K})$, for any ball $B \subset \mathbb{K}$ containing 0, and any $\epsilon \leq \rho$, we have

$$
(5.1) \quad \mu \left( \left\{ t \in B \mid \delta(u_t\Lambda) < \epsilon \right\} \right) \leq C \left( \frac{\epsilon}{\rho} \right)^{\frac{1}{n^2}} \mu(B).
$$

The proof will follow in a sequel [12] where we will also establish more general non-divergence results for $G/\Gamma$ where $G$ is the group of $\mathbb{K}$-points of a semi-simple algebraic group defined over $\mathbb{K}$ and $\Gamma$ is a lattice in $G$.

5.2. **More on Diophantine Approximation.** One can ask questions in a more general framework as introduced in [17] (see also [28]). Namely, one can study Diophantine properties of points with respect to measures, and show that a large class of measures (including measures supported on fractal subsets of $\mathbb{K}^\nu$) are strongly extremal. Definitions and details will appear in the author’s PhD. thesis. One can also seek to extend the results in this paper as well as [20] and obtain Khintchine-type theorems over ultrametric fields (cf. [5], [3], [1] for the real variable case, [2], [4], [21] for results over $\mathbb{Q}_p$ and [14], [11] for results over $\mathbb{K}$).
Finally, following [10] (see also [13]), it would be interesting to study Diophantine properties of affine subspaces over $\mathbb{Q}_p$ and $K$.

REFERENCES

[1] V.Beresnevich, A Groshev type theorem for convergence on manifolds, Acta Math Hungar. 94 (2002), No.1 – 2, 99 – 130.

[2] V.Beresnevich, V.Bernik and E.Kovalevskaya, On approximation of $p$-adic numbers by $p$-adic algebraic numbers, J.Number Theory, 111, (2005), No.1, 33 – 56.

[3] V.Beresnevich, V.Bernik, D.Kleinbock and G.A.Margulis, Metric Diophantine Approximation: the Khintchine-Groshev theorem for nondegenerate manifolds. Dedicated to Yuri.I.Manin on the occasion of his 65th birthday, Moscow Math. Journal2 (2002), No.2, 203 – 225.

[4] V.Beresnevich and E.Kovalevskaya, On Diophantine approximations of dependent quantities in the $p$-adic case, (Russian) Mat.Zametki, 73 (2003). No.1 – 2, 21 – 35.

[5] V.Bernik, D.Kleinbock and G.A.Margulis, Khintchine type theorems on manifolds : the convergence case for the standard and multiplicative versions, Internat.Math.Res.Notices (2001), No.9, 453 – 486.

[6] M. Bachir Bekka and Matthias Mayer, Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces, Cambridge University Press (2000).

[7] J.W.S. Cassels, An introduction to Diophantine Approximation, Cambridge tracts in Mathematics.vol 45.Cambridge Univ.Press, Cambridge, (1957).

[8] S.G.Dani, Divergent trajectories of flows on homogeneous spaces and Diophantine Approximation, J.Reine Angew.Math 360 (1985), 214.

[9] S.G.Dani, On orbits of unipotent flows on homogeneous spaces,2. Ergodic Theory Dynamical systems 6 (1986), 167 – 182.

[10] S.G.Dani, Continuous equivariant images of lattice-actions on boundaries, Ann. Math., 119 (1984), 111 – 119.

[11] M. M. Dodson, S. Kristensen and J. Levesley, A quantitative Khintchine-Groshev type theorem over a field of formal series, to appear in Indag. Math. (N.S.).

[12] Anish Ghosh, Orbits of unipotent flows in finite characteristic, In preparation.
[13] Anish Ghosh, *A Khintchine-type theorem for hyperplanes*, J.London Math.Soc. To appear.

[14] Kae Inoue and Hitoshi Nakada, *On metric Diophantine approximation in positive characteristic*, Acta Arith. 110 (2003), no.3, 205 – 218.

[15] D.Kleinbock, *Some applications of homogeneous dynamics to number theory*, Smooth ergodic theory and its applications, Proc.Sympos.Pure.Math.69, AMS, Providence RI, (2001), 639 – 660.

[16] D.Kleinbock, *Extremal subspaces and their submanifolds*, Geom.Funct.Anal. 13, (2003), no 2 437 – 466.

[17] D.Kleinbock, E.Lindenstrauss and B.Weiss, *On Fractal Measures and Diophantine Approximation*, Selecta Math. To appear.

[18] D.Kleinbock and G.A.Margulis, *Flows on homogeneous spaces and Diophantine Approximation on Manifolds*, Ann. Math., 148 (1998), 339 – 360.

[19] D.Kleinbock and G.A.Margulis, *Bounded orbits of nonquasiunipotent flows on homogeneous spaces*, Amer.Math.Soc.Transl. 171 (1996), 141 – 172.

[20] D.Kleinbock and G.Tomanov, *Flows on S-arithmetic homogeneous spaces and applications to metric Diophantine Approximation*, Preprint, http://front.math.ucdavis.edu/math.NT/0506510.

[21] E.Kovalevskaya, *$p$-adic variant of the convergence Khintchine theorem for curves over $\mathbb{Z}_p$*, Acta Math. Inform. Univ. Ostraviensis 10 (2002), No. 1, 71 – 78.

[22] Simon Kristensen, *On well approximable matrices over a field of formal series*, Math.Proc.Camb.Phil.Soc. (2003), 135(2), 255 – 268.

[23] Simon Kristensen, *Badly Approximable systems of linear forms over a field of formal series*, Preprint.

[24] K.Mahler, *Über das Mass der Menge aller S-Zahlen*, Math Ann. 106, 1932, 131 – 139.

[25] Kurt Mahler, *An analogue to Minkowski’s Geometry of numbers in a field of series*, Ann.Math., 2nd Ser.1941, Vol.42, No.2. 488 – 522.

[26] G.A.Margulis, *On the action of unipotent group in the space of lattices*, Proceedings of the Summer School on group representations, (Budapest 1971), Académiai Kiado, Budapest, (1975), 365 – 370.

[27] G.Prasad, *Strong approximation theorem for semi-simple groups over function fields*, Ann. Math., 105 (1977), 553 – 572.
[28] A. Pollington and S. Velani, *Metric Diophantine Approximation and ‘absolutely friendly’ measures*, Preprint.

[29] W. H. Schikhof, *Ultrametric Calculus, an introduction to p-adic analysis*, Cambridge studies in advanced mathematics, 4. Cambridge University Press, (1984).

[30] Jean-Pierre Serre, *Arbres, Amalgames, SL₂*. Asterisque no.46, Soc.Math.France, (1977).

[31] V. G. Sprindžuk, *Mahler’s Problem in Metric Number theory* (Translated from the Russian by B. Volkmann), Translations of Mathematical Monographs, AMS, Vol25, (1969).

[32] V. G. Sprindžuk, *Achievements and problems in Diophantine Approximation theory*, Russian Math. Surveys 35 (1980), 1 – 80.

[33] G. Tomanov, *Orbits on homogeneous spaces of arithmetic origin and approximations*, Advanced Studies in Pure Mathematics, 26 (2000), 265 – 297.

[34] A. Weil, *Basic Number Theory*, Springer-Verlag, New York, (1995).

[35] R. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston, (1984).

Anish Ghosh
MS 050, Brandeis University
415 South Street
Waltham, MA-02454
U.S.A.
ghosh@brandeis.edu