Contribution of the hybrid inflation waterfall to the primordial curvature perturbation

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I. INTRODUCTION

The primordial curvature perturbation ζ is one of the most important features of the early universe. On each scale it is relevant until the era of horizon entry at the epoch \( k = aH \), and provides the principle (perhaps the only) initial condition for the subsequent evolution of all other perturbations. At the beginning of the known history of the universe, when the temperature is around 1 MeV, observation has established the existence of ζ at wavenumbers between \( k \sim a_0 H_0 \) (corresponding to the size of the observable universe) and \( k \sim e^{15} a_0 H_0 \). On these ‘cosmological scales’ its spectrum \( P_\zeta(k) \) is almost scale-invariant with value \( P_\zeta = (5 \times 10^{-5})^2 \).

The primordial curvature perturbation presumably exists also on much shorter scales and at much earlier times. Although not directly observable, this can have a significant effect on the evolution of the early universe. The most dramatic possibility is black hole formation; requiring wavenumbers between \( \sim k \) and \( \sim k \sigma \), where \( \sigma \) is the heavy field and \( \chi \) is the heavy field and \( \phi \) is an oscillating field that is initially homogeneous. It begins with an era, during which \( \phi \) remains homogeneous so that the field equation for \( \chi_k \) is

\[
\ddot{\chi}_k = -\left( m^2(t) + k^2 \right) \chi_k, \tag{1}
\]

A contribution ζ to the curvature perturbation will be generated during the waterfall that ends hybrid inflation, that may be significant on small scales. In particular, it may lead to excessive black hole formation. We here consider standard hybrid inflation, where the tachyonic mass of the waterfall field is much bigger than the Hubble parameter. We calculate ζ in the simplest case, and see why earlier calculations of ζ are incorrect.

\#1 We use the standard cosmology notation. The comoving wavenumber \( k \) of a Fourier component defines a comoving scale \( 1/k \) so that smaller scales have bigger \( k \), but \( k \) itself is loosely referred to as the scale. The physical wavenumber is \( k/a(t) \) and the Hubble parameter is \( H \equiv \dot{a}/a \), while \( a_0 \) and \( H_0 \) are evaluated at the present epoch. During inflation scales ‘leave the horizon’ when \( aH \) exceeds \( k \) and after inflation they enter it when \( aH \) falls below \( k \).

\#2 A scalar field \( \sigma \) with canonically kinetic term is said to be light during inflation if \( |m_\sigma^2| \ll H^2 \), where \( m_\sigma^2 \equiv \partial^2 V/\partial \sigma^2 \) and \( V \) is the scalar field potential, and heavy if \( |m_\sigma^2| \gg H^2 \). A vector field perturbation might also contribute to ζ.
with \( m^2(t) = m^2 + g^2 \phi^2(t) \) and \( m^2 \) the mass-squared of \( \chi \). If \( m^2(t) \) is dominated by the second term, the solution can grow exponentially, converting the vacuum fluctuation of \( \chi_k \) to a classical perturbation. The contribution to the curvature perturbation generated during this era is calculated in [6] assuming the inflation potential \( V \propto \phi^2 \), and in [7] for the smooth/mutated hybrid inflation [8] potential.

In this paper we consider instead tachyonic preheating, that ends hybrid inflation. Here the interaction is the same as before, but the waterfall field \( \chi \) has a negative (tachyonic) mass-squared. If \( \phi \) falls below a critical value during inflation, \( m^2(\phi) \) becomes negative and \( \chi_k \) can grow exponentially, again converting its vacuum fluctuation to a classical perturbation. To facilitate the calculation we make some fairly restrictive assumptions about the waterfall, which will be relaxed in future papers.

The layout of the paper is as follows. In Section II we recall the definition and properties of \( \zeta \). In Section III we recall the basics of hybrid inflation. In Section IV we lay down our assumptions and describe the evolution of waterfall field. In Section V we define the region of parameter space in which our assumptions are consistent. In Section VI we calculate the pressure and energy density of the waterfall field. In Section VII we justify our assumptions. In Section VIII we calculate \( \zeta_{\chi} \), the contribution of the waterfall field perturbation to the curvature perturbation, by integrating the perturbation theory expression for \( \dot{\zeta} \). In Section IX we review the \( \delta N \) approach and show that it reproduces the previously-calculated result for \( \zeta_{\chi} \). In Section X we see that previous calculations of \( \zeta_{\chi} \) are incorrect. In Section XI we conclude, pointing to the need for a better understanding of the ultra-violet cutoff in the context of cosmological scalar field calculations.

II. PRIMORDIAL CURVATURE PERTURBATION \( \zeta \)

A. Cosmological perturbations

Let us recall some basic concepts, described for instance in [9]. Given a function \( f(x, t) \) in our Universe one can define the perturbation \( \delta f \):

\[
\delta f(x, t) = f(t) + \delta f(x, t),
\]

where \( f(t) \) refers to some Robertson-Walker (unperturbed or background) universe whose line element is

\[
ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j = a^2(t) \left( -d\eta^2 + \delta_{ij} dx^i dx^j \right).
\]

In the background universe the energy density \( \rho \) and pressure \( p \) are related by the continuity equation

\[
\dot{\rho} = -3H(\rho + p),
\]

where \( H \equiv \dot{a}/a \). Also, Einstein gravity is assumed corresponding to the Friedmann equation \( \rho = 3M_P^2 H^2 \) with \( M_P = (8\pi G)^{-1/2} = 2 \times 10^{18} \text{GeV} \).

The coordinates \( (x, t) \) label points in the background universe, and also points in our Universe. The latter labeling (gauge) defines slices (fixed \( t \)) and threads (fixed \( x \)) in our Universe. Let us denote \( \delta f(x, t) \) in some chosen gauge by \( g(x, t) \). We will need the first-order gauge transformation for the case that \( f \) is specified by a single number. Going to a new coordinate system with time coordinate \( \tilde{t}(t, x) \), it is

\[
\tilde{g} - g = -\tilde{f} (\tilde{t} - t)
= \tilde{f} \delta t,
\]

where \( \delta t \) is the time shift going from a slice of uniform \( t \), to one of uniform \( \tilde{t} \) with the same numerical value.

We will also need to describe the statistical properties of \( g \) at fixed \( t \), which are simplest in Fourier space:

\[
g_k(t) = \int d^3x e^{-ik \cdot x} g(x, t).
\]
When considering the statistical properties one ignores the spatial average (zero mode), which can be absorbed into the background unless \( f \) corresponds to anisotropy.

Considering an ensemble of universes, we assume that the statistical properties are invariant under translations (statistical homogeneity) and rotations (statistical isotropy). By virtue of the former, ensemble averages can be replaced by spatial averages within the particular realization of the ensemble that corresponds to our Universe (ergodic theorem) \( \textbf{Q} \).

Statistical homogeneity implies \( \langle g_k \rangle = 0 \). The two-point correlator defines the spectrum:

\[
\langle g_k g_p \rangle = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{p})(2\pi^2/k^3)P_g(k). \tag{8}
\]

An equivalent quantity, also called the spectrum, is \( P_g \equiv (2\pi^2/k^3)P_g \).

A gaussian perturbation is defined as one with no correlation between the \( \chi_n \), except that implied by the reality condition \( \chi_{-\mathbf{k}} = \chi^*_\mathbf{k} \). Its only correlators are the two-point correlator, and the disconnected \( 2^n \) point correlators starting with

\[
\langle g_{k_1} g_{k_2} g_{k_3} g_{k_4} \rangle = \langle g_{k_1} g_{k_2} \rangle \langle g_{k_3} g_{k_4} \rangle + \text{permutations}. \tag{9}
\]

Non-gaussianity of \( g \) is specified by its bispectrum \( B_g \), trispectrum \( T_g \) etc., defined by

\[
\langle g_{k_1} g_{k_2} \rangle = (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2)B_g, \tag{10}
\]

\[
\langle g_{k_1} g_{k_2} g_{k_3} g_{k_4} \rangle = (2\pi)^3 \delta^3(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)T_g, \tag{11}
\]

etc., where the subscript c denotes the connected part of the correlator that has to be added to the disconnected part.

The mean-square perturbation \( \langle g^2 \rangle \) can be taken as the spatial average of \( g^2(x) \), and is given by

\[
\langle g^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k P_g(k) = \int_0^\infty \frac{dk}{k} P_g(k). \tag{12}
\]

After smoothing \( g \) on a scale \( L \) this becomes

\[
\langle g^2 \rangle = \int_0^{L^{-1}} \frac{dk}{k} P_g(k). \tag{13}
\]

For a gaussian perturbation, the probability distribution of \( g(x) \) is gaussian so that \( \langle g^n \rangle \) vanishes for odd \( n \) and is equal to \((n-1)!!(g^2)^{n/2}\) for even \( n \). For a non-gaussian perturbation

\[
\langle g^3 \rangle = (2\pi)^{-6} \int d^3k_1 d^3k_2 B_g(k_1, k_2),
\]

\[
\langle g^4 \rangle_c \equiv \langle g^4 \rangle - 3\langle g^2 \rangle^2 = (2\pi)^{-9} \int d^3k_1 d^3k_2 d^3k_3 T_g(k_1, k_2, k_3), \tag{14}
\]

etc.. The skew, kurtosis etc. of the probability distribution are \( S_g \equiv \langle g^3 \rangle / \langle g^2 \rangle^{3/2} \), \( K_g \equiv \langle g^4 \rangle_c / \langle g^2 \rangle^2 \) etc.. If they are all \( \ll 1 \) \( g \) can be regarded as almost gaussian.

We will also be interested in the spectrum of \( g^2 \). Using Eq. \( \textbf{10} \) with the convolution theorem

\[
\langle g^2 \rangle_k = \frac{1}{(2\pi)^3} \int d^3k' g_{k'} g_{-k'}, \tag{15}
\]

one finds \( \textbf{10} \)

\[
P_{g^2}(k) = \frac{2}{(2\pi)^3} \int d^3k' P_g(k') P_g(|k - k'|). \tag{16}
\]

\( ^3 \) A function is said to be smooth on the scale \( L \) if its Fourier components are negligible in the regime \( k > L^{-1} \). Smoothing means that we remove Fourier components in this regime.
B. Primordial curvature perturbation $\zeta$

In the gauge with the comoving threading and the slicing of uniform energy density $\rho$, the spatial metric defines $\zeta$ non-perturbatively through
\[
g_{ij}(x, t) \equiv a^2(x, t) \left( e^{2h(x, t)} \right)_{ij}, \quad \zeta \equiv \ln \left( \frac{a(x, t)}{a(t)} \right) \equiv \delta \ln a. \tag{17} \tag{18}
\]
In this expression $\text{Tr} \, h = 0$, so that the last factor has unit determinant and $a(x, t)$ is the local scale factor such that a comoving volume element is proportional to $a^3$. For this definition to be useful, we need to smooth the metric and the stress-energy tensor on some fixed super-horizon scale $L_\zeta$ (physical scale $L_\zeta a(t)$). The smoothing scale must be below the shortest scale of interest, and to encompass the biggest possible range of scales it should not be much bigger than the Hubble scale at the end of inflation. Then we assume, by virtue of the smoothing, that spatial gradients have a negligible effect on the evolution of the stress-energy tensor. This is the separate universe assumption, that is useful also for the evolution of other quantities (we will invoke it later for scalar fields). The separate universe assumption will be valid if relevant spatial gradients are negligible, which will be the case if $L_\zeta$ is sufficiently far outside the horizon. One generally assumes that the separate universes are isotropic (which in our case means that the energy-momentum tensor and the expansion are isotropic) so that the separate universes are Robertson-Walker as opposed to more general Bianchi universes. That will be the case if scalar field inflation sets the initial condition for the Universe, as we are assuming in this paper. The following treatment of $\zeta$ goes through however even if the separate universes are anisotropic as might be the case if vector fields are involved [3].

By virtue of the separate universe assumption, the energy continuity equation (4) applies at each location. Since we have chosen the slicing of uniform $\rho$, the local continuity equation reads
\[
\dot{\rho}(t) = -3 \frac{\dot{a}(x, t)}{a(x, t)} [\rho(t) + p(x, t)] \tag{19}
\]
\[
= -3 \left[ H(t) + \dot{\zeta}(x, t) \right] [\rho(t) + p(t) + \delta p_{\text{nad}}(x, t)], \tag{20}
\]
where $\delta p_{\text{nad}}$ is the pressure perturbation on the slicing of uniform density (non-adiabatic pressure perturbation). Subtracting the original continuity equation (4) gives
\[
\dot{\zeta}(x, t) = -H(t) \frac{\delta p_{\text{nad}}(x, t)}{\rho(t) + p(t) + \delta p_{\text{nad}}(x, t)}. \tag{21}
\]
During any era when $p(\rho)$ is a unique function (the same at each location), $\delta p_{\text{nad}}$ vanishes and $\zeta$ is time-independent.

To first order in $\delta p_{\text{nad}}$ we have,
\[
\dot{\zeta}(x, t) = - \frac{H(t)}{\rho(t) + p(t)} \delta p_{\text{nad}}(x, t), \tag{22}
\]
with
\[
\delta p_{\text{nad}}(x, t) = \delta p(x, t) - \frac{\dot{p}(t)}{\rho(t)} \delta \rho(x, t), \tag{23}
\]
which is valid in any gauge. Also, using Eqs. (6) and (18),
\[
\zeta(x, t) = H \delta t = -H \delta \rho/\dot{\rho} = \frac{1}{3} \frac{\delta \rho}{\rho + p}, \tag{24}
\]
where $\delta \rho$ is the perturbation on the slicing of uniform $a(x, t)$ (flat slicing).

A second-order calculation of $\zeta$ is needed only to treat very small non-gaussianity corresponding to reduced bispectrum $|f_{NL}| \ll 1$. On cosmological scales, such non-gaussianity
will eventually be measurable (and is expected if $\zeta$ comes from a curvaton-type mechanism [9, 11, 12]). But there is no hope of detecting such non-gaussianity on much smaller scales.

Before continuing we emphasize the following point. A realistic Robertson-Walker universe will have inhomogeneities, and one has to take their spatial average in order to determine the evolution of the spatially-homogeneous scale factor. This is true in particular for each of the separate universes. Various kinds of small-scale inhomogeneity could contribute to the spatially-averaged energy density and pressure. The generation of the classical perturbation of light fields from the vacuum continues after the smoothing scale leaves the horizon until at least the end of inflation, on successively smaller scales. The effect of this perturbation on the energy density and pressure during inflation is removed by the smoothing (because it is linear to high accuracy in the field perturbation) but the light field perturbations existing at the end of inflation might have an effect on the subsequent evolution, for instance on massless preheating [13]. There could be cosmic strings or other localized energy density configurations, whose spacing must be much less than the smoothing scale for the separate universe assumption to be valid. Finally, through preheating, both heavy and light fields can acquire classical perturbations from the vacuum fluctuation on sub-horizon scales. It is this effect that will be important for us, as it was for the ordinary preheating calculations [6, 7]. To calculate $\zeta$, one must first calculate $\rho$ and $p$ including the sub-horizon scale field perturbations, and then smooth $\rho$ and $p$ on a super-horizon scale.

III. HYBRID INFLATION

In this paper we consider scalar fields with canonical kinetic terms. Their contributions to the energy density and pressure are

$$\rho = V + \frac{1}{2} \sum \dot{\phi}_i^2 + \frac{1}{2} \sum |\nabla \phi_i|^2$$

(25)

$$p = -V + \frac{1}{2} \sum \dot{\phi}_i^2 + \frac{1}{6} \sum |\nabla \phi_i|^2,$$

(26)

where $V$ is the scalar field potential. Ignoring the metric perturbation (back-reaction) their field equations are

$$\ddot{\phi}_i + 3H \dot{\phi}_i + \frac{\partial V}{\partial \phi_i} - \nabla^2 \phi_i = 0.$$  

(27)

A. Slow-roll inflation

We consider single-field slow-roll inflation, in which the inflaton field $\phi$ has negligible interaction with any other field, and is the only one with time-dependence. Consider first the unperturbed (spatially homogeneous) universe. The energy density and pressure are

$$\rho_\phi = V(\phi) + \frac{1}{2} \dot{\phi}^2$$

(28)

$$p_\phi = -V(\phi) + \frac{1}{2} \dot{\phi}^2,$$

(29)

and the field equation is

$$\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0.$$  

(30)

Slow-roll inflation corresponds to almost exponential expansion:

$$|\dot{H}| \ll H^2 \quad \Leftrightarrow \quad \dot{\phi}^2 \ll M_p^2 H^2 \quad (\text{slow roll}),$$

(31)

#4 In that case one may have to worry about which realization of the ensemble of fluctuations is the observable universe [13].
which leads to the approximations
\[ 3M_P^2 H^2 = V(\phi), \quad 3H\dot{\phi} = -V'(\phi). \] (32)

(An over-dot means differentiation with respect to \( t \) while a prime means differentiation with respect to the displayed argument.) The first derivative of Eq. (31) is also supposed to be valid, leading to
\[ |\ddot{\phi}| \ll H|\dot{\phi}|, \quad |V''| \ll H^2. \] (33)

For some purposes one or more higher derivatives of Eq. (31) are also supposed to be valid leading to further relations, but we shall not need them.

Ignoring back-reaction, the first-order perturbation \( \delta \phi \) satisfies
\[ \ddot{\delta \phi}_k + 3H\dot{\delta \phi}_k + (k/a)^2 \delta \phi_k = -V''(\phi(t))\delta \phi_k. \] (34)

Back-reaction vanishes in the slow-roll limit \( \dot{\phi}(t) \to 0 \), in any gauge whose slicing is non-singular in that limit \[9\]. The slicing of the widely used longitudinal gauge is non-singular, and so is the flat slicing \[9\]. Assuming Einstein gravity, back-reaction is a small effect in these gauges. The slicing with uniform energy density is singular in the limit, but \( \delta \phi \) then vanishes.

As each scale leaves the horizon during slow-roll inflation, the vacuum fluctuation generates a nearly gaussian classical perturbation \( \delta \phi_k \) with spectrum \((H/2\pi)^2\). We denote the contribution of this perturbation to the curvature perturbation by \( \zeta_\phi \) (to first order in the field perturbations, it is the only contribution during single-field inflation). After smoothing \( \phi \) on a super-horizon scale, Eq. (30) is valid at each location, with the overdot meaning \( d/dt_{\text{pr}} \) and \( H \equiv \left[ da(x, t_{\text{pr}})/dt_{\text{pr}} \right]/a(x, t_{\text{pr}}) \), where \( t_{\text{pr}} \) is the proper time.\[5\]

By virtue of the slow-roll approximation \( \dot{\phi} \) is determined by \( \phi \), which means \( \zeta_\phi \) is conserved. It also means that the slicing of uniform \( \rho \) is the same as the slicing of uniform \( \phi \), so that to first order in \( \delta \phi \)
\[ \zeta_\phi = -H\delta \phi/\dot{\phi}. \] (35)

This gives
\[ P_{\zeta_\phi}(k) = \left( H^2/2\pi\dot{\phi} \right)^2, \] (36)

where the right hand side is evaluated at horizon exit.

### B. Hybrid inflation

Hybrid inflation was proposed and named in \[15\] as a way of getting the low inflation energy scale that might be required by the axion isocurvature perturbation \[10, 16, 17\]. (See \[18\] for earlier realizations of hybrid inflation.) It was subsequently found \[19–22\] to be a powerful tool for model building especially in the context of supersymmetry.\[6\]

Until hybrid inflation nears its end a ‘waterfall field’ \( \chi \), which has nonzero vev, is fixed at the origin by its interaction with \( \phi \), up to a vacuum fluctuation which is ignored. This displacement of \( \chi \) from its vev gives a constant contribution to \( V \) which dominates the total, leading to single-field slow-roll inflation. After \( \phi \) falls below some critical value \( \phi_c \), the waterfall field develops a nonzero value \( \chi(x, t) \) which eventually ends inflation. That process is called the

\[5\] To first order in \( \delta \phi \), the effect of back-reaction on Eq. (34) takes care of the difference between \( H(t) \) and \( H(x, t) \) and between \( t_{\text{pr}} \) and coordinate time (see for instance Eq. (5) of \[14\]).

\[6\] More recently, there is interest in hybrid inflation with a non-canonical kinetic term, in particular DBI inflation \[23\]. The only essential feature, needed to make sense of the hybrid inflation paradigm, is that the potential dominates the energy density.
waterfall; in other words, the waterfall begins when \( \phi \) falls through its critical value, and ends when inflation ends.

We will adopt the potential

\[
V(\phi, \chi) = V_0 + V(\phi) + \frac{1}{2}m^2(\phi)\chi^2 + \frac{1}{4}\lambda\chi^4
\]

\[
m^2(\phi) \equiv g^2\phi^2 - m^2 \equiv g^2(\phi^2 - \phi^2_c),
\]

with \( 0 < \lambda \ll 1 \) and \( 0 < g \ll 1 \). The potential is invoked for \( \phi \lesssim \phi_{\text{obs}} \), the value when the observable universe leaves the horizon. The effective mass-squared \( m^2(\phi) \) goes negative when \( \phi \) falls below \( \phi_c \equiv m/g \), the waterfall then commencing.\(^7\)

The vev of \( \phi \) vanishes and we take \( V(\phi) \) to vanish at the vev. The inflationary potential is supposed to be dominated by \( V_0 \) which means \( V(\phi) \ll V_0 \). The requirements that \( V \) and \( \partial V/\partial \chi \) vanish in the vacuum give the vev \( \chi_0 \) and the inflation scale \( V_0 \simeq 3M_P^2H^2 \):

\[
\chi_0^2 = \frac{m^2}{\lambda}, \quad V_0 = \frac{m^4}{4\lambda} \simeq 3M_P^2H^2.
\]

The following relations are useful:

\[
\frac{12H^2}{m^2} = \frac{\chi_0^2}{M_P^2}, \quad \frac{12H^2}{M_P^2} = \frac{\lambda}{M_P^4}.
\]

To justify the omission of higher powers of \( \chi \) we will assume \( \chi_0 \ll M_P \). This implies \( m/H \gg 1 \), which is in any case the standard assumption for hybrid inflation.\(^8\) Also, to justify the omission of high powers of \( \phi \) in \( V(\phi) \), we will assume \( \phi_{\text{obs}} \ll M_P \). That requires

\[
\phi_c \equiv m/g \ll M_P \quad \text{(small } \phi \text{)}.
\]

It also requires a tensor perturbation well below the present observational limit, implying

\[
H/M_P \lesssim 10^{-5} \quad \text{(tensor bound)}.
\]

Successful BBN requires an inflation scale \( \sqrt{M_PH} \gtrsim \text{MeV} \) corresponding to

\[
H/M_P > 10^{-42} \quad \text{(BBN),}
\]

but viable models of the early universe generally require a far higher value.

The potential \( \text{[37]} \) was proposed in \( 13 \), with \( V(\phi) = m_0^2\phi^2/2 \). If one demands \( \zeta \simeq \zeta_{\phi} \), it gives spectral index \( n > 1 \) in contradiction with observation. Many forms of \( V(\phi) \) have been proposed that are consistent with \( \zeta \simeq \zeta_{\phi} \), \( 9, 25 \), and we will not assume any particular form.

Minor variants of Eq. \( \text{[37]} \) would make little difference to our analysis. The interaction \( g^2\phi^2\chi^2 \) might be replaced by \( \phi^2\chi^{2+n}/\Lambda^n \) where \( \Lambda \) is a uv cutoff, or the term \( \lambda\chi^4 \) might be replaced by \( \chi^{1+n}/\Lambda^n \). For our purpose, these variants are equivalent to allowing (respectively) \( g \) and \( \lambda \) to be many orders of magnitude below unity. Also, \( \phi \) might have two or more components that vary during inflation. Most of our treatment of the waterfall will apply to that case, if \( \phi \) is the field pointing along the inflationary trajectory when the waterfall commences.

More drastic modifications are also possible, including inverted hybrid inflation \( 26 \) where \( \phi \) is increasing during inflation, and mutated/smooth hybrid inflation \( 8 \) where the waterfall field varies during inflation. Also, the waterfall potential might have a local minimum at

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\(^7\) Taking account of the inhomogeneity in \( \phi \), the waterfall begins at each location when \( \phi(x, t) = \phi_c = m/g \).

Since \( m \gg H \), the perturbation \( \delta \phi \approx H / m \) is small compared with \( \phi_c \), which means that this is a small effect.

\(^8\) At least for the usually-considered potentials, this ensures that the waterfall takes at most a few Hubble times. In the opposite case the the waterfall can end with an era of two-field slow-roll inflation, involving \( \chi \) as well as \( \phi \).
the origin so that the waterfall proceeds by bubble formation \[19, 27\]. Our analysis does not apply to those cases.

In this account of hybrid inflation we have taken \(\phi\) and \(\chi\) to be real fields. More generally they may correspond to directions in a field space that provides a representation of some non-Abelian symmetry group (the GUT symmetry, for the waterfall field of GUT inflation). That introduces some numerical factors without changing the structure of the equations. For the waterfall field it also avoids the formation of domain walls at locations where \(\chi = 0\), which would be fatal to the cosmology. There may instead be cosmic strings, which are harmless if the inflation scale is not too high. For clarity we pretend that \(\phi\) and \(\chi\) are real fields.

### IV. WATERFALL FIELD \(\chi\)

#### A. Linear era

Most papers on the waterfall make two basic assumptions. First, back-reaction is ignored so that

\[
\ddot{\phi} + 3H\dot{\phi} - \nabla^2 \phi = -V'(\phi) - g^2 \chi^2 \phi, \\
\ddot{\chi} + 3H\dot{\chi} - \nabla^2 \chi = -m^2(\phi)\chi - \lambda \chi^3. \tag{44}
\]

Second, the waterfall is assumed to begin with an era when the last term of each equation is negligible; during that era, the evolution of \(\chi\) is linear and we call it the linear era. Since \(\phi\) is smooth on the horizon scale, its negligible evolution at each location is that of an unperturbed universe (i.e. its spatial gradient is negligible).\(^{9}\) Choosing a gauge whose slicing corresponds to uniform \(\phi\),

\[
\ddot{\chi}_k + 3H\dot{\chi}_k + [(k/a)^2 + m^2(\phi(t))] \chi_k = 0. \tag{46}
\]

Even if slow-roll fails during the waterfall, the slow-roll initial condition ensures that \(\dot{\phi}\) continues to be determined by \(\phi\), which means that the slicing is also one of uniform \(\rho_\phi\) and \(p_\phi\).

The energy density and pressure of \(\chi\) are

\[
\rho_\chi = m^2(\phi)\chi^2 + \frac{1}{2} \chi^2 + \frac{1}{2} |\nabla \chi|^2, \tag{47}
\]

\[
p_\chi = -m^2(\phi)\chi^2 + \frac{1}{2} \chi^2 + \frac{1}{6} |\nabla \chi|^2. \tag{48}
\]

#### B. Regime of parameter space for our calculation

In this paper we focus on the simplest regime of parameter space, which will be identified in Section \[\text{V}\]. In that regime the following conditions are satisfied: (i) slow-roll inflation continues until the end of the linear era, (ii) that era takes much less than a Hubble time (iii) the change in \(\phi\) during that era is negligible. The first two conditions imply that the change in \(\dot{\phi}\) is also negligible. Under these conditions, the assumptions of the previous subsection will be justified in Section \[\text{VII}\].

Since the changes in \(\phi\) and \(\dot{\phi}\) are negligible, \(m^2(\phi(t))\) decreases linearly with time. Setting \(t = 0\) at the beginning of the waterfall and using the dimensionless time \(\tau \equiv \mu t\) we write

\[
m^2(\phi(t)) = -\mu^3 t, \quad \mu^3 \equiv -2g^2\phi\dot{\phi} = -2gm^2\dot{\phi}. \tag{49}
\]

\(^9\) If the waterfall takes several Hubble times, the scales on which \(\phi\) is inhomogeneous will start to come inside the horizon but that effect will not be very significant.
Our assumptions mean that we can ignore the changes in $H$ and $a$ during the linear era, and we set $a = 1$. The assumption that the linear era takes much less than a Hubble time is

$$(H/\mu)\tau_{nl} \ll 1 \quad (Ht_{nl} \ll 1).$$

The assumption $\phi \simeq \phi_c$ corresponds to

$$(\mu/m)^2 \ll \tau_{nl}^{-1} \quad (\phi_{nl} \simeq \phi_c).$$

We are implicitly assuming that $\chi_k$, generated from the vacuum fluctuation, can be treated as a classical field, and we shall see that the condition for this is $\tau_{nl} \gg 1$. As a result, Eqs. (50) and (51) require the hierarchy $H \ll \mu \ll m$.

To see when the linear era ends, let us include the small second terms on the right hand sides of Eqs. (44) and (45). Since the change in $\phi$ is negligible and the perturbation $\delta \phi$ is small, the unperturbed $\phi(t)$ (the spatial average) satisfies

$$\ddot{\phi} + 3H\dot{\phi} \simeq 3H\dot{\phi} = -V'\phi_c - g^2\phi_c\langle \chi^2 \rangle,$$

where $\langle \chi^2 \rangle$ is the spatial average of $\chi^2$. Subtracting this from the full equation for $\phi$ we find

$$\ddot{\delta \phi} + 3H\dot{\delta \phi} - \nabla^2 \delta \phi = -V''\phi_c\delta \phi - g^2\chi^2\delta \phi - g^2\phi_c\delta \chi^2,$$

where

$$\delta \chi^2 \equiv \chi^2 - \langle \chi^2 \rangle.$$

Finally, Eq. (45) becomes

$$\ddot{\chi} + 3H\dot{\chi} - \nabla^2 \chi = -m^2\phi(t)\chi - \lambda\chi^3 - \delta(m^2)\chi$$

where $\delta(m^2) \equiv m^2(\phi(x, t)) - m^2(\phi(t))$. The linear era ends when the first term on the right hand side ceases to dominate, for one of these equations. For Eq. (52) this happens at roughly the epoch when $\langle \chi^2 \rangle$ has the value

$$\chi_{nl}^2 \equiv 3H|\dot{\phi}|/gm.$$

The constraints (31) and (41) imply $\chi_{nl} \ll \chi_0$, which means that the second term on the right hand side of Eq. (55) is smaller than the first one at this epoch. We will show in Section VII that the last terms of Eqs. (55) and (53) can also be ignored, which means that $\chi_{nl}$ marks the end of the linear era. Afterward, the last term of Eq. (52) will cause $\phi(x, t)$ to quickly decrease, and $m^2(\phi)$ will quickly move to its final value ending inflation and (by definition) the waterfall.

C. Solution of the waterfall field equation

By virtue of our condition (ii) we can ignore the expansion of the universe, setting $H = 0$ in Eq. (44) to get

$$\frac{d^2\chi_k(\tau)}{d\tau^2} = x(\tau, k)\chi_k(\tau), \quad x \equiv \tau - k^2/\mu^2 = -(m^2(\tau) + k^2)/\mu^2.$$

The solutions are the Airy functions Ai(x) and Bi(x).

In the quantum theory, $\chi_k$ becomes an operator $\hat{\chi}_k$. Working in the Heisenberg picture we write

$$\hat{\chi}_k(\tau) = \chi_k(\tau)\hat{a}_k + \chi_k^*(\tau)\hat{a}_{-k},$$

$$[\hat{a}_k, \hat{a}_p] = (2\pi)^3\delta^3(k - p).$$

The mode function $\chi_k$ satisfies Eq. (57) and its Wronskian is normalized to $-1$. We choose

$$\chi_k = \sqrt{\pi/2\mu} [\text{Bi}(x) + i\text{Ai}(x)].$$
Then, in the early-time regime \( x \ll -1 \), \( E_\phi^2 \equiv m_\phi^2(\tau) + k^2 \) is slowly varying (\( |\dot{E}_k| \ll E_\phi^2 \)) and the theory describes particles with mass-squared \( m_\phi^2(t) \). Indeed, the solution in this regime is
\[
\chi_k(\tau) = (2\mu)^{-1/2} |x|^{-1/4} e^{-i\pi/4} e^{-\frac{2}{\mu t}|x|^{3/2}},
\]
which can be written
\[
\chi_k(t) = \frac{1}{\sqrt{2E_k}} e^{-i \int^t dt E_k}.
\]
We choose the state vector as the vacuum, such that \( \hat{a}_k |0\rangle = 0 \). A significant occupation number is excluded since the resulting positive pressure would spoil inflation \cite{28}.

**D. Classical waterfall field \( \chi(x,t) \)**

We will assume that \( \tau \) becomes much bigger than 1 during the linear era. Then there exists a regime \( x \equiv \tau - (k/\mu)^2 \gg 1 \) in which
\[
\chi_k(\tau) \simeq (2\mu)^{-1/2} x^{-1/4} e^{\frac{2}{\mu t} x^{3/2}}, \quad \hat{\chi}_k \simeq \mu \sqrt{x} \chi_k,
\]
the errors vanishing in the limit \( x \to \infty \). In this regime \( \hat{\chi}_k(\tau) = \chi_k(\tau)(\hat{a}_k + \hat{a}_{-k}) \) to high accuracy. As a result, \( \hat{\chi}_k \) is a practically constant operator times a c-number, which means that \( \chi_k \) is a classical quantity in the WKB sense. By this, we mean that a suitable measurement of \( \chi_k \) at a given time will give a state that corresponds to a practically definite value \( \chi_k \) far into future \cite{9}. (We have nothing to say about the cosmic Schrödinger’s Cat problem that now presents itself.) After the measurement, \( \chi_k(\tau) \) is classical and
\[
\chi_k(\tau) \propto \chi_k(\tau).
\]

Keeping only the classical modes \( \chi_k \), we generate a classical field \( \chi(x,t) \). Its spatial average vanishes and we can treat it as a gaussian perturbation. Its spectrum is defined by Eq. \( \star \), with the ensemble corresponding to the different outcomes of the measurement that is supposed to have been made to produce \( \chi_k(t) \). The expectation value in Eq. \( \star \) can therefore be identified with the vacuum expectation value \( \langle \chi_k \hat{\chi}_p \rangle \), which gives \( P_{\chi}(k,\tau) = \chi^2_k(\tau) \). The classical mode function \( \chi_k(\tau) \) is nonzero only in the regime \( x \equiv \tau - k^2/\mu^2 \gg 1 \), where it is given by Eq. \( \star \). At \( k = 0 \) we have
\[
P_{\chi}(0,\tau) = \chi^2_{k=0}(\tau) = (2\mu \tau^{1/2})^{-1} e^{\frac{4}{\tau} \tau^{3/2}}.
\]
At fixed \( \tau \), \( P_{\chi} \) is a decreasing function of \( k \). At \( k^2 \ll \mu^2 \tau \),
\[
P_{\chi}(k,\tau) = \chi^2_k(\tau) \simeq \chi^2_{k=0}(\tau) e^{-2\sqrt{\tau k^2/\mu^2}}
\]
\[
= \chi^2_{k=0}(\tau) e^{-k^2/k^2_{\ast}(\tau)}, \quad k^2_{\ast}(\tau) \equiv \mu^2/2\sqrt{\tau},
\]
which means that \( P_{\chi}(k) \) decreases exponentially in the regime \( k^2_{\ast}(\tau) \ll k^2 \ll \mu^2 \tau \). Modes with \( k \gg k_{\ast}(\tau) \) can therefore be ignored. The dominant modes (corresponding to the peak of \( P_{\chi} \) at fixed \( \tau \)) have \( k \sim k_{\ast}(\tau) \). From Eq. \( \star \) we have \( k_{\ast}(\tau) \gg H \) which means that the dominant modes are sub-horizon.

From Eq. \( \star \) we see that \#10
\[
\langle \chi^2(\tau) \rangle \sim P_{\chi}(0,\tau) k^2_{\ast}(\tau).
\]

\#10 The argument in \( \langle \chi^2(\tau) \rangle \) is inserted to remind us that the expectation value of \( \chi^2 \) (defined as the ensemble average or equivalently the spatial average for a given ensemble) depends only on time and not on position.
Doing the integral we find
\[ \langle \chi^2(\tau) \rangle = \frac{4\pi}{(2\pi)^3} P_\chi(0, \tau) \int_0^\infty dk d^2k e^{-(k^2/k^2_x(\tau))} = (2\pi)^{-3/2} P_\chi(0, \tau) k_x^3(\tau). \] (69)

From Eqs. (65) and (69), \( \tau_{nl} \sim (\ln(\chi_{nl}/\mu))^{2/3} \), with \( \chi_{nl} \) given by Eq. (56). To have a classical era \( \tau \gg 1 \) we require \( \tau_{nl} \gg 1 \). To get an upper bound on \( \tau_{nl} \) we use \( \chi_{nl} \ll \chi_0 \ll M_P \) with \( \mu \gg H \), to find \( \tau_{nl} \lesssim (\ln(M_P/H))^{2/3} \). According to Eqs. (12) and (13), this upper bound on \( \tau_{nl} \) is of order 5 to 20 with the lower end of the range far more likely.

Since the dominant modes have \( k^2 \sim k^2_x(\tau) \ll \mu^2 \tau \), we can set \( x \simeq \tau \) to get \( \tilde{\chi}_k \simeq \mu \sqrt{x} \tilde{x} \).

At a typical location we have to a good approximation
\[ \tilde{\chi}(x, \tau) = \mu \sqrt{\tau} \chi(x, \tau). \] (70)

We also have
\[ \langle |\nabla \chi|^2 \rangle \sim \int d^3k k^2 |\chi_k|^2 \sim k_x^2(\tau) \langle \chi^2 \rangle \ll \langle \chi^2 \rangle = \mu^2 \tau \chi^2. \] (71)

At a typical location, \( |\nabla \chi| \) is of order \( k \chi \), which is negligible compared with the typical time-derivative \( \mu \sqrt{\tau} \chi \). Integrating Eq. (70) over an extended time interval gives
\[ \chi(x, \tau) \propto e^{(2/3)\tau^{3/2}}. \] (72)

According to Eqs. (65), (70), and (71), this approximation ignores the time-dependence of \( k \chi \), (and less seriously, the prefactor in Eq. (65)). That approximation is good over a time interval \( \Delta \tau \ll \tau \), and we will use it over such an interval ending at \( \tau \simeq \tau_{nl} \).

These results may not hold near places where \( \chi(x, \tau) \) vanishes, because there is then a cancellation between different Fourier components and the relation \( \chi_k \simeq \mu \sqrt{\tau} \chi_k \) cannot be expected to give Eq. (70). The places where \( \chi \) vanishes are those in which cosmologically disastrous domain walls would form if \( \chi \) were really a single field and where cosmic strings might form in a more realistic case. At a typical location \( \nabla \chi/\chi \) is of order \( k \chi \), therefore the typical spacing between these places is of order \( k^{-1} \). But at a distance \( L \) from one of the places, \( \chi \) is typically of order \( L k \chi \sim L k \chi \langle \chi^2 \rangle^{1/2} \), which is well below its typical value only within a small region \( L \ll k^{-1} \). Outside this region, there is no strong cancellation between Fourier components, and we may expect the approximations to be valid.

V. CONSTRAINTS ON THE PARAMETER SPACE

In this section we identify the part of parameter space in which our assumptions are consistent. There are four independent parameters, which we will take to be the dimensionless quantities\(^\#1\)

\[ g \ll 1, \quad H_P \equiv H/M_P \ll 1, \quad H_m \equiv H/m \ll 1, \] (73)

and

\[ f \equiv (5 \times 10^{-5})^{-1} H^2/2\pi \phi = (5 \times 10^{-5})^{-1} \mathcal{P}_{\zeta}\left(k_{\text{end}}\right). \] (74)

Inflation models are usually constructed so that \( \mathcal{P}_{\zeta(\phi)} \) accounts for the observed \( \mathcal{P}_{\zeta} \) on cosmological scales. Then, if \( \mathcal{P}_{\zeta(\phi)} \) is nearly scale-independent we will have \( f \sim 1 \). More generally there is an upper bound
\[ f \lesssim 2 \times 10^3 \] (75)

\(^\#1\) Strictly speaking, we cannot consider \( \zeta_{\text{end}}(k_{\text{end}}) \), because \( \zeta \) is usefully defined only after smoothing on a super-horizon scale. But the analogue of \( \zeta \) defined on the slicing orthogonal to comoving worldlines (usually denoted by \( \mathcal{R} \)) remains useful with a smaller smoothing scale and Eq. (64) as well as the black hole bound are valid with \( \mathcal{P}_\zeta \) replaced by \( \mathcal{P}_R \).
corresponding to the bound \( P_{\zeta} \lesssim 10^{-2} \) that is required to avoid excessive black hole production from \( \zeta \).\footnote{Since \( \phi \) and \( \chi \) have independent vacuum fluctuations, \( P_{\chi} \) is the sum of \( P_{\zeta\phi} \) and \( P_{\zeta\chi} \), and the black hole bound applies to each of them.}

In terms of these four parameters, useful relations are

\[
\tau_{nl} \sim \left[ \ln(H_m g^{-1/5} f^{-1/5}) \right]^{2/3}, \tag{76}
\]

\[
\mu^3 \sim 10^4 g H^3 H_m^{-1} f^{-1}, \tag{77}
\]

and the various constraints become

\[
10^{-42} < H_P \lesssim 10^{-5} \quad \text{(BBN \& tensor)} \tag{78}
\]

\[
g H_P \ll 10^{-3} f \quad \text{(slow roll)} \tag{79}
\]

\[
g H_m \ll H_m \quad \text{(small \( \phi \))} \tag{80}
\]

\[
H_P \ll H_m^2 \quad (\lambda \ll 1) \tag{81}
\]

\[
10^{3/2} g H_m^2 \ll f \quad (\phi_{nl} \simeq \phi_c) \tag{82}
\]

\[
g f^{1/5} \ll H_m \quad (\tau_{nl} \gg 1) \tag{83}
\]

\[
10^{-42} < H_P \lesssim 10^{-5} \quad \text{(BBN \& tensor)} \tag{84}
\]

The constraint \( \tau_{nl} \gg 1 \) corresponds to the existence of a classical regime, which fails if \( g f^{1/5} \gtrsim H_m \). As seen in Section IV C, there is an absolute bound \( \tau_{nl} \lesssim 20 \) with \( \tau_{nl} \lesssim 10 \) more likely. Eqs. (82) and (84) require \( f \ll (10/\tau_{nl})^3 \) which is stronger than Eq. (78). If this bound on \( f \) is saturated, Eqs. (82) and (84) become \( g H_m^{-1} \sim 1 \) and then Eq. (83) becomes \( g \ll (10 \tau_{nl}^{1/2})^{-1} \). For smaller \( f \) the range for \( g H_m^{-1} \) becomes wider and Eq. (83) bounds the product \( g H_m^2 \).

**VI. PRESSURE AND ENERGY DENSITY OF \( \chi \)**

Taking the spatial gradient to be negligible (ie. excluding places where \( \chi \) is close to zero), Eqs. (47), (48), and (70) give

\[
\rho_\chi(x, \tau) \simeq -\frac{1}{2} \mu^2 \tau \chi^2(x, \tau) + \frac{1}{2} \chi^2(x, \tau) \simeq 0 \tag{86}
\]

\[
p_\chi(x, \tau) \simeq \frac{1}{2} \mu^2 \tau \chi^2(x, \tau) + \frac{1}{2} \chi^2(x, \tau) \simeq 2 \tau \chi^2 \simeq m^2(\phi)^2 \chi^2. \tag{87}
\]

The cancellation of the terms in the first expression occurs because we have ignored the expansion of the universe when calculating \( \chi \), so that energy conservation prevents \( \rho_\chi \) moving away from its initially zero value. Instead of taking the expansion into account directly, we invoke the local energy continuity equation for \( \rho_\chi \), which will be valid to a good approximation because the gradient of \( \chi \) is negligible. This gives \( \dot{\rho}_\chi(x, t) = -3 H p_\chi(x, t) \).

Using Eq. (72) to integrate \( \dot{\rho} \) gives\footnote{Eq. (72) will be a good approximation because the integral is dominated by values of \( t \) close to \( t_{nl} \).}

\[
\rho_\chi(x, t) = -\frac{3 H}{2 \mu \sqrt{\tau}} p_\chi(x, t) = -\frac{3 H}{2 \tau^{3/2}} p_\chi(x, t). \tag{88}
\]

Since \( H t_{nl} \ll 1 \) and \( \tau \gg 1 \), this gives \( |\rho_\chi| \ll |p_\chi| \) as is required for consistency.

To a good approximation the spatially averaged pressure is

\[
\langle p_\chi(\tau) \rangle = \mu^2 \tau \langle \chi^2(\tau) \rangle = \langle \chi^2 \rangle \tag{89}
\]
and the pressure perturbation is

\[ \delta p_{\chi}(x, \tau) = \mu^2 \tau \delta \chi^2(x, t) = \delta \chi^2, \tag{90} \]

where

\[ \delta \chi^2(x, \tau) \equiv \chi^2(x, \tau) - \langle \chi^2(\tau) \rangle, \quad \chi^2(x, \tau) \equiv \chi^2(x, \tau) - \langle \chi^2(\tau) \rangle. \tag{91} \]

Multiplying Eqs. (89) and (90) by \(-1/2\mu \sqrt{\tau}\) we arrive at the corresponding expressions for \(\rho_{\chi}\).

Our goal is to evaluate the contribution of \(\chi\) to the curvature perturbation \(\zeta\), which depends linearly on \(\delta p_{\chi}\) and \(\delta \chi^2\), and hence on \(\delta \chi^2\). Since \(\zeta\) is by definition smooth on a super-horizon scale we are interested only in \(\delta \chi^2\) smoothed on such a scale. Let us determine its statistical properties, starting with its spectrum.

Super-horizon scales satisfy a fortiori \(k < k_s(\tau)\), which means that we can set \(k = 0\) within the integral (10). This gives roughly

\[ P_{\delta \chi^2}(\tau, k) \sim P_{\chi}^2(\tau, 0) k_s^3(\tau). \tag{92} \]

Doing the integral and using Eq. (89), and going to \(P_{\delta \chi^2}\), we find

\[ P_{\delta \chi^2}(\tau, k) = \frac{1}{\sqrt{\pi}} \langle \chi^2(\tau) \rangle^2 [k/k_s(\tau)]^3. \tag{93} \]

Taking the smoothing scale to be the horizon scale, Eq. (13) gives

\[ \frac{\langle (\delta \chi^2)^2 \rangle}{\langle \chi^2 \rangle^2} = \frac{1}{3} \frac{P_{\delta \chi^2}(H)}{\langle \chi^2 \rangle^2} \sim (k_s(\tau)/H)^{-3} \ll 1, \tag{94} \]

and for a bigger smoothing scale \(L\) we should replace \(H\) by \(L^{-1}\), making the bound tighter. We see that after smoothing on a super-horizon scale, \(\chi^2\) is almost homogeneous.

Using the convolution theorem one finds at \(k \ll k_s(\tau)\) the scale-independent expressions

\[ B_{\delta \chi^2}(\tau) \sim P_{\chi}^3(\tau, 0) k_s^3(\tau), \quad T_{\delta \chi^2}(\tau) \sim P_{\chi}^4(\tau, 0) k_s^3(\tau), \tag{95} \]

e tc. After smoothing \(\delta \chi^2\) on the horizon scale,

\[ \langle (\delta \chi^2)^3 \rangle \sim P_{\chi}^3 k_s^3(\tau) H^6, \quad \langle (\delta \chi^2)^4 \rangle \sim P_{\chi}^4 k_s^3(\tau) H^6, \tag{96} \]

e tc. From these follow the skew, kurtosis etc. of the probability distribution of \(\chi^2\), which are seen to be small:

\[ S_{\delta \chi^2} \sim (k_s(\tau)/H)^{-3/2} \ll 1, \quad K_{\delta \chi^2} \sim (k_s(\tau)/H)^{-3} \ll 1. \tag{97} \]

We conclude that after smoothing on a super-horizon scale, \(\delta \chi^2\) is almost gaussian.

In all of this, we have taken the spatial average of \(\chi\) to vanish. That is true within an indefinitely large region because \(\chi\) is supposed to be generated entirely from the vacuum fluctuation. But to make contact with observation we should consider a finite box, that is not many orders of magnitude bigger than the region in which the observations are made (30). Denote the average within the box by \(\bar{\chi}\) we have

\[ \chi(x) = \bar{\chi} + \chi_{\text{calc}}(x) \tag{98} \]

where the second term is the one we calculated. This gives

\[ \delta \chi^2 = \delta \chi^2_{\text{calc}} + 2 \bar{\chi} \chi_{\text{calc}}. \tag{99} \]

For our results to be valid we need \((\delta \chi^2_{\text{calc}})^2\) to be much bigger than \(\bar{\chi}^2(\chi^2_{\text{calc}})\), when \(\delta \chi^2_{\text{calc}}\) and \(\chi_{\text{calc}}\) are smoothed on any scale \(k^{-1}\) much smaller than the box size \(L\). The expected value of \(\bar{\chi}^2\) for a random location of the box is \(\langle \chi^2_{\text{calc}} \rangle\) where now \(\chi^2_{\text{calc}}\) is smoothed on the scale \(L\). We therefore expect

\[ \frac{\langle (\delta \chi^2_{\text{calc}})^2 \rangle}{\chi^2(\chi_{\text{calc}})} \sim \frac{P_{\delta \chi^2}(k)}{P_{\chi}(L^{-1}) P_{\chi}(k)} \sim (kL)^3 \gg 1. \tag{100} \]

We conclude that the spatial average \(\bar{\chi}\) is negligible, unless we live at a a very untypical location.
VII. JUSTIFYING OUR ASSUMPTIONS

A. Neglect of back-reaction

According to the calculation of $\chi$ in Section IV, the expansion of the universe is negligible and the dominant modes of $\chi$ have $k \sim k_* \gg H$. Therefore, the calculation can be regarded as a flat spacetime calculation, formulated at each epoch within a locally inertial frame corresponding to a box whose size $L$ satisfies $H^{-1} \ll L \ll k^{-1}$, that is at rest with respect to the cosmic fluid. Since spatial gradients are negligible, each element of the cosmic fluid is free-falling.

Since the calculation within each box takes place in practically flat spacetime, back-reaction will have practically no effect on it. It follows that our calculation is valid globally, in the gauge with free-falling threads, and with the time coordinate corresponding to proper time along each thread (a synchronous gauge).

B. Neglected terms in Eqs. (53) and (55)

Let us restore the neglected terms in Eqs. (53) and (55). We write $\delta \phi = \delta \phi_1 + \delta \phi_2$ and $\chi = \chi_1 + \chi_2$, where $\delta \phi_1$ and $\chi_1$ are the solutions of Eqs. (34) and (46), that we considered earlier. Then Eqs. (53) and (55) give

$$\ddot{\delta \phi}_2 + 3H \dot{\delta \phi}_2 - \nabla^2 \delta \phi_2 = - \left[ V''(\phi(t)) + g^2 \chi_1^2(t) \right] \delta \phi_2 - g^2 \phi_c \delta \chi^2, \quad (101)$$

$$\ddot{\chi}_2 + 3H \dot{\chi}_2 - \nabla^2 \chi_2 = 2g^2 \phi_c \chi_1 \delta \phi_2. \quad (102)$$

where $\delta \chi^2 = \chi_2^2 - \langle \chi_2^2 \rangle$ as before. In writing these equations, we assumed $|\delta \phi_2| \ll \phi_c$ and $|\chi_2| \ll |\chi_1|$, and we will verify the self-consistency of these assumptions.

Recall that $\langle \delta \chi_2^2 \rangle = 2\mu \sqrt{\tau} \delta \chi_1^2$ and $\langle \delta \chi_1^2 \rangle = 4 \mu^2 \tau \delta \chi_1^2$. Also, $|V''| \ll H^2$, $H \ll \mu$ and $g^2 \chi_{nl}^2 \sim g^2 H \mu^3 / m^2 \ll \mu^2$. It follows that the first term on the right hand side of Eq. (101) can be ignored, and that its solution is

$$\delta \phi_2 = - \frac{g^2 \phi_c}{4 \tau \mu^2} \delta \chi_1^2. \quad (103)$$

Putting this into Eq. (102) and remembering that $\chi_1 \propto \exp[(2/3) \tau^{3/2}]$ we get

$$\chi_2 = 2 \frac{g^2}{\tau} \phi_c \chi_1 \delta \phi_2 / \mu^2. \quad (104)$$

Using $|\delta \chi_2^2| \sim |\chi_1^2| < \chi_{nl}^2$ one easily verifies that $|\delta \phi_2| \ll \phi_c$ and $|\chi_2| \ll |\chi_1|$. The neglect of back-reaction in Eqs. (101) and (102) is justified because the dominant mode of $\delta \phi_2$ and $\chi_2$ is sub-horizon, being the dominant mode $k_*$ of $\chi_1$.

Since $\chi_2$ is much less than $\chi_1$, it has a negligible effect on $\zeta_\chi$. The effect of $\delta \phi_2$ is also negligible, because its contribution $\delta p_2 = - V'' \delta \phi_2$ to $\delta p$ is small compared with that of $\delta \chi$:

$$\left| \frac{\delta p_2}{\delta p_\chi} \right| = \frac{3}{8 \pi^2 \mu} \frac{H}{\mu} \ll 1. \quad (105)$$

C. Justifying the assumption of a linear era

The above discussion shows that the assumption of a linear waterfall era is self-consistent, and that such an era will end only at the epoch $\langle \chi_2^2 \rangle = \chi_{nl}^2$ when the last term of Eq. (53)
becomes significant. What about the possibility that, although self-consistent, the assumption is actually wrong? For the assumption to hold, it is essential that $V'$ dominates the evolution during some initial era of the waterfall. In the opposite case where $V'$ is completely negligible, both $\phi$ and $\chi$ will be generated entirely by the potential $\frac{1}{2}g^2(\phi^2 - \phi_c^2)\chi^2$. This case is discussed in [31]. Noting that the most rapid growth will be in the direction of steepest descent of the potential, they find $\sqrt{2}(\phi_c - \phi) \simeq \chi$ and

$$V = V_0 - \frac{gm}{\sqrt{2}}\chi^3. \quad (106)$$

This potential leads to a non-linear equation for $\chi(x, t)$, which has been considered in [32]. To handle it, they give $\langle \chi^2 \rangle$ an initial value coming from its vacuum fluctuation. Of course that requires a prescription for handling the uv divergence of $\langle \chi^2 \rangle$ that is different from ours which is to simply drop modes that are in the quantum regime. (We comment on this issue at the end of Section XI)

The authors of [32] find that $\chi$ grows like $\exp(k_{\text{quant}}t)$ with $k_{\text{quant}} \sim gm$. This growth rate may be compared with the growth rate $\chi \propto \exp[\frac{\tilde{A}}{2}(\tau)^{3/2}]$ that we found by assuming the existence of an initial linear era, where $\tau \equiv \mu t$ with $\tau \gg 1$. The latter growth rate will be bigger than the former if $gm \ll \mu \tau^{1/2}$, and [31] suggest that this will be the criterion for the linear era to exist. That criterion is satisfied with our assumptions, because Eqs. (77), (75), and (84) imply $gm \ll \mu$. We conclude that the assumption of an initial linear era has some justification, though there is as yet no conclusive proof.

**VIII. THE WATERFALL CONTRIBUTION TO $\zeta$**

**A. Calculating the waterfall contribution**

The contribution to $\zeta$ generated at a given epoch during the waterfall is

$$\zeta_x(x, \tau) = \zeta(x, \tau) - \zeta(x, \tau_1), \quad (107)$$

where $\tau_1 \sim 1$ is the epoch when $\chi$ becomes classical. We are interested in $\zeta_x(x, \tau_{nl})$.

Following the procedure of [6] used for ordinary preheating, let us calculate $\zeta_x$ by integrating $\zeta$. Since $\zeta_0$ is constant, $\delta \rho_{\text{had}}$ receives no contribution from $\delta \rho_{\phi}$. Using Eqs. (28), (29), (87) and (88) and remembering that $|\rho_{\chi}| \ll |\rho_{\chi}|$, we find

$$\delta \rho_{\text{had}} = \delta \rho_{\phi} - \frac{\delta \rho_{\phi}}{\rho} \delta \rho_{\chi} \quad (108)$$

Using this result and invoking Eq. (72), this gives

$$\zeta_x(x, \tau_{nl}) = -\frac{H}{\mu} \int_{\tau_1}^{\tau_{nl}} \frac{\delta \rho_{\text{had}}(x, \tau)}{\langle \chi^2(\tau) \rangle + \phi^2} d\tau = \frac{1}{3} \frac{\delta \rho_{\chi}(x, \tau_{nl})}{\langle \chi^2(\tau_{nl}) \rangle + \phi^2} - \frac{1}{3} \frac{\delta \rho_{\chi}(x, \tau_1)}{\langle \chi^2(\tau_1) \rangle + \phi^2}, \quad (110)$$

where

$$\frac{1}{3} \frac{\delta \rho_{\chi}(x, \tau)}{\langle \chi^2(\tau) \rangle + \phi^2} = -\frac{H}{2\mu\sqrt{\tau}} \frac{\langle \chi^2(\tau) \rangle}{\langle \chi^2(\tau) \rangle + \phi^2}. \quad (111)$$

From Eqs. (56) and (89) we find

$$\langle \chi^2(\tau_{nl}) \rangle / \phi^2 = 6H\tau_{nl}. \quad (112)$$

We are assuming $H\tau_{nl} \ll 1$, which suggests $6H\tau_{nl} \ll 1$ hence $\langle \chi^2(\tau_{nl}) \rangle \ll \phi^2$. Even if $6H\tau_{nl}$ is somewhat bigger than 1, $\langle \chi^2 \rangle \ll \phi^2$ until $\tau$ is close to $\tau_{nl}$; indeed, from Eq. (72) we have $\langle \chi^2 \rangle \sim \phi^2$ at the epoch $\tau_1$ given by

$$\tau_1 \sim \ln[(\chi^2_{\text{nl}})/\phi^2]/2_{\text{nl}}^{1/2} \ll \tau_{nl}. \quad (113)$$
It follows that the integral (110) is dominated by the range \( \tau_{nl} - \tau \ll \tau_{nl} \), which means that Eq. (72) will be a good approximation. In either case, the integral of Eq. (110) is dominated by its upper limit so that to a good approximation

\[
\zeta(x, \tau_{nl}) = \frac{1}{3} \frac{\delta \rho(x, \tau_{nl})}{\langle \dot{\chi}^2(\tau_{nl}) \rangle + \phi^2}.
\]

In the regime \( 6Ht_{nl} \gg 1 \) we have the following interesting result. Here, Eq. (108) becomes

\[
\delta p_{nad} = \delta p_{\chi} - \frac{\dot{p}_{\chi}}{p_{\chi}} \delta p_{\chi},
\]

with \( |\delta p_{nad}| \ll |\delta p_{\chi}| \). Therefore \( p_{\chi} \) is (almost) uniform on the slice of uniform \( \rho \) (practically coinciding with the slice of uniform \( \rho_{\chi} \)). That in turn implies that \( \chi^2(t, x) \) is almost uniform on the slice of uniform \( \rho \).

The spectrum of \( \zeta_{\chi}(x, \tau_{nl}) \) can be written

\[
P_{\zeta_{\chi}}(k) = \frac{36}{\sqrt{2\pi}} \frac{\tau_{nl}^{-21/4} (Ht_{nl})^7}{(1 + 6Ht_{nl})^2} \left( \frac{k}{H} \right)^3.
\]

This is the main result of our paper. It holds only in the regime of parameter space for which \( Ht_{nl} \ll 1 \), and only for \( \tau_{nl} \gg 1 \) (so that \( \chi \) is classical). Also, since \( \zeta \) is defined only after smoothing on a super-horizon scale, it holds only for \( H \gg k \). Requiring say \( (Ht_{nl})^{-1} \), \( \tau_{nl} \) and \( H/k \) all bigger than 2 we see that \( P_{\zeta_{\chi}} \) is far below the black hole bound \( P_{\zeta} \sim 10^{-2} \).

Taken to apply at say \( (H/k)^3 = 10^2 \), the black hole bound implies on a scale \( k = H \) leaving the horizon \( N \) Hubble times before the end of inflation \( P_{\zeta_{\chi}} \ll e^{-3N} \). Considering the shortest cosmological scale, the observed value \( P_{\zeta}(k) \sim 10^{-9} \) requires \( N < 7 \) which is unlikely with any reasonable post-inflationary cosmology even with a low inflation scale. We conclude that the black hole constraint on the scale \( k = H \) almost certainly makes \( P_{\zeta_{\chi}} \) negligible on cosmological scales.

IX. THE \( \delta N \) APPROACH

The calculation of \( \zeta_{\chi} \) in the previous section uses cosmological perturbation theory. For the contributions of light field perturbations to \( \zeta \), the \( \delta N \) approach provides a powerful alternative making no reference to cosmological perturbation theory. In this section, we recall the \( \delta N \) formula and its application to light fields. Then see how it works for the waterfall field.

A. Initial flat slice

One may extend Eq. (17) to describe a generic sequence of fixed-\( t \) slices, subject only to the condition that the initial slice has unit determinant for \( g_{ij} \) (called a flat slice) while the final slice still has uniform density. Since the local scale factor is uniform on the initial slice, \( \zeta \) on the final slice is given by

\[
\zeta(x, t) = \delta N(x, t) \equiv N(x, t) - N(t),
\]

where \( N(x, t) \) is the number of \( e \)-folds of expansion at a given location,

\[
N(x, t) \equiv \int_{t_1}^{t(\rho)} dt' \dot{a}(x, t')/a(x, t'),
\]

\#15 That regime can hardly exist with our restriction \( Ht_{nl} \ll 1 \), but it turns out that the result still holds in a regime \( Ht_{nl} \gtrsim 1 \).
and \(N(t)\) is the expansion in the unperturbed universe with scale factor \(a(t)\). The expansion is from the initial flat slice at time \(t_1\), to the final uniform-\(\rho\) slice at time \(t\), and we suppress the argument \(t_1\) of \(N\) because \(\delta N\) is independent of \(t_1\). This formula was given to first order in [34] and non-linearly in [35].

Going to the proper time \(t_{\text{pr}}\) at each location, Eq. (118) becomes

\[
N(x, t_{\text{pr}}) \equiv \int_{t_{\text{pr}}(x)}^{t_{\text{pr}}(\rho, x)} H(x, t') dt'.
\]  

(119)

(Without loss of generality one usually takes \(t_{1\text{pr}}\) to be independent of \(x\)). For each of the separate universes, \(t_{\text{pr}}\) is the cosmic time appearing the Robertson-Walker line element, and \(H\) is the Hubble parameter given by \(3M_p^2H^2 = \rho\). Since the final slice has homogeneous \(\rho\), the final value of \(H\) is also homogeneous.

**B. Initial uniform-density slice**

The contribution to \(\zeta\) that is generated between times \(t_1\) and \(t\) is

\[
\zeta(x, t) - \zeta(x, t_1) = \delta N(x, t, t_1) = N(x, t, t_1) - N(t, t_1)
\]

(120)

\[
N(x, t, t_1) = \int_{t_{\text{pr}}(\rho_1)}^{t_{\text{pr}}(\rho, x)} dt' \dot{a}(x, t')/a(x, t'),
\]

(121)

where both the initial and final slices have uniform \(\rho\).

Going to Proper time,

\[
N(x, t_{\text{pr}}, t_{\text{pr}1}) = \int_{t_{\text{pr}}(\rho_1, x)}^{t_{\text{pr}}(\rho, x)} H(x, t'_{\text{pr}}) dt'_{\text{pr}},
\]

(122)

with both the initial and final values of \(H\) determined by the relation \(3M_p^2H^2 = \rho\).

**C. \(\delta N\) for the contribution of light fields**

1. Initial flat slice

Let us now choose the initial epoch to be a very few Hubble times after the smoothing scale \(L_\zeta\) of \(\zeta\) leaves the horizon during inflation, and denote it by \(t_\ast\). At this stage the light fields are smooth on the scale \(L_\zeta\), because their classical perturbation exists only on scales much bigger than the horizon \((k \ll aH)\) while their quantum fluctuation has been dropped.

One assumes that the values of one or more of the light fields, at the epoch \(t_\ast\), provide at each location the initial condition for the evolution of the separate universe. Then, with \(N\) defined by Eq. (118), we have in terms of these values

\[
\zeta(x, t) = \delta N = N(t, \phi_1^*(x), \phi_2^*(x), \cdots) - \langle N(t, \phi_1^*, \phi_2^*, \cdots) \rangle,
\]

(123)

where \(\phi_i^*\) without an argument denote the unperturbed values, and the perturbations \(\delta \phi_i^* \equiv \phi_i^*(x) - \phi_i^*\) are defined on a flat slice.

In practice, \(\zeta\) is well-approximated by a few terms of a power-series in the field perturbations:

\[
\zeta(x, t) = \sum N_i(t) \delta \phi_i^*(x) + \frac{1}{2} \sum N_{ij}(t) \delta \phi_i^*(x) \delta \phi_j^*(x) + \cdots.
\]

(124)

#16 Considering the time-dependence of \(\zeta\) during inflation, which was also the focus in [34], the first-order formula is also given in [39] and its non-linear generalization is given in [37, 38].
In this formula, the partial derivatives \( N_i \) etc. are evaluated at the unperturbed point in field space.

Taking \( t \) to be any time after \( \zeta \) has stopped varying, we arrive at the quantity constrained by observation denoted simply by \( \zeta(x) \). Since the observed \( \zeta \) is gaussian to high accuracy, \( (124) \) should be dominated on cosmological scales by the first term. This term was given in \( 34 \). The power series was given in \( 39 \), where it was used to calculate the bispectrum etc. that correspond to non-gaussianity in a couple models.\(^{17}\)

If there is single-field slow-roll inflation and the only contribution to \( \zeta \) comes from the inflaton \( \phi \), the perturbation \( \delta \phi_s \) just represents a shift along the inflaton trajectory which does not affect the subsequent relationship \( p(\rho) \). Then \( \zeta \) is time-independent, and the first term of Eq. \( (124) \) dominates. To first order in \( \delta \rho_s \), the time shift is \( \delta t = \delta \phi_s / \dot{\phi}_s \), which with \( \delta N = -H\delta t \) gives Eq. \( (35) \). If there is multi-field slow-roll inflation with two or more fields \( \phi_i \), we can put these fields into Eq. \( (124) \) to arrive at \( \zeta(x, t) \) during slow-roll inflation. For each of the separate universes, the final epoch should be one of fixed \( V \simeq \rho \), which in general will mean different final values for the fields.

\[ \text{Initial uniform-} \rho \text{ slice} \]

So far we have taken \( N \) to be defined from an initial flat slice, leading to Eq. \( (124) \) with \( t_s \) an epoch soon after the smoothing scale \( L_s \) leaves the horizon. This is definitely the formula to use if all of the light field contributions become significant during inflation (so that we deal with multi-field inflation). There is another scenario though, where a different approach works better. This is the curvaton-type scenario, in which the contribution of some light field \( \sigma \) starts to become significant only at some epoch \( t_1 \) after inflation is over. For this to make sense, the smoothing scale \( L_s \) chosen for \( \zeta \) has to be outside the horizon at the epoch \( t_1 \). In principle we should keep the effect on \( \rho \) and \( p \) of the modes of \( \sigma \) that have entered the horizon during the era \( t_s < t < t_1 \), but such modes are likely to have redshifted away (i.e. \( |\partial^2 V/\partial \sigma^2| \gtrsim H^2 \) is likely during this era) and they are ignored. Also, the coupling of \( \sigma \) to other fields is supposed to be negligible during this era. The evolution of \( \sigma \) at each location is then given by the unperturbed field equation, leading to a relation \( \chi_1(\chi_s) \) and

\[ \delta \chi(t_1, x) = \frac{d\chi_1}{d\chi_s} \delta \chi_s + \frac{1}{2} \frac{d^2\chi_1}{d\chi_s^2} (\delta \chi_s)^2 + \cdots. \quad (125) \]

The field \( \sigma(x, t_1) \) is supposed to set the initial condition at \( t_1 \) so that

\[ \zeta_\chi(x, t) = N_\chi(t, t_1) \delta \chi(t_1, x) + \frac{1}{2} N_{\chi\chi}(t, t_1) [\delta \chi(t_1, x)]^2 + \cdots. \quad (126) \]

where a subscript denotes a derivative, \( N_\chi = \partial N / \partial \chi \) etc.. Eqs. \( (125) \) and \( (126) \) give \( \zeta_\chi \) in terms of the nearly-gaussian perturbation \( \delta \chi_s \) which has the spectrum \( (H/2\pi)^2 \).

Following \( 33 \), this version of the \( \delta N \) formula is generally used for the curvaton-type contribution, usually with the assumption that \( \zeta_\phi \) is negligible so that \( \zeta_\chi \) is the observed quantity. With that assumption, there is actually no need to assume slow-roll inflation; all one needs is the assumption of inflation with a specified Hubble parameter \( H(t) \) \( 11 \).

\[ \text{D. } \delta N \text{ for the waterfall contribution} \]

Now we show that the \( \delta N \) formula \( (122) \) provides a direct derivation of Eq. \( (114) \) that we used to calculate the waterfall field contribution \( \zeta_\chi \). Our calculation sets \( H \) equal to a constant. Also, it begins with a slice of uniform \( \phi \), and subsequent slices are separated by

\[ \text{\footnotesize \#17 Considering the time-dependence of } \zeta \text{ during inflation, which is also the focus in } 34, \text{ the first-order formula was also given in } 36 \text{ and its non-linear generalization was considered (without a specific application) in } 37, 38. \]
uniform amounts of proper time, so that \( \phi \) remains uniform on these slices and \( \delta \rho = \delta \rho_\chi \).

Using these results, Eq. (122) indeed gives to first order in \( \delta \rho_\chi \):

\[
\zeta_\chi (x, t_{nl}) = H [\delta t(x, t_{nl}) - \delta t(x, t_1)] \\
\approx - \frac{H}{\rho(t_{nl})} \delta \rho_\chi (x, t_{nl}) = \frac{1}{3} \frac{\delta \rho_\chi (x, t_{nl})}{\rho(t_{nl})},
\]

(128)

where \( \delta t \) is the time interval from the uniform proper time slicing to the uniform \( \rho \) slicing. In the second line we dropped \( \delta t(x, t_1) \) which as seen in Section VII is negligible.

One might be concerned that setting \( H \) equal to a constant is incompatible with \( 3M_\ast^2 H^2 = \rho \). But the non-uniformity of \( H \) implied by this relation gives a negligible contribution to \( N \):

\[
\int_{t_1}^{t_{nl}} \delta H dt = \frac{1}{2} \int_{t_1}^{t_{nl}} H \frac{\delta \rho_\chi}{\rho} dt < \frac{1}{2} H \frac{\delta \rho_\chi (x, t_{nl})}{\rho(t_{nl})},
\]

(129)

which is much less than Eq. (128), justify the assumption that \( H \) is practically uniform.

This use of the \( \delta N \) formula looks very different from the one for light fields, but it can be made to resemble the latter using the local evolution equation Eq. (72) for \( \chi \). (Recall that this equation is needed if we are to use Eq. (128) within the framework of the present paper, because it is invoked for the calculation of \( \rho_\chi \).) With this equation, the time \( t_{nl}(x) \) required to give the final energy density a uniform value \( \rho(t_{nl}) \) is given by

\[
\rho(t_{nl}) = \rho_\phi(t_{nl}(x)) = \frac{3H\mu\sqrt{\tau_1}}{2} x^2(x, \tau_1) e^{\frac{3}{2}x_{nl}(x)}.
\]

(130)

This gives \( N \equiv (H/\mu)\tau_{nl}(x) \) as a function of \( x^2(x, \tau_1) \), and taking its first order perturbation we get

\[
\zeta_\chi (x, \rho(t)) = \delta N \approx \left. \frac{dN(x^2, \rho)}{d(x^2)} \right|_{x^2=(x^2)} \delta x^2(x, t_1).
\]

(131)

Since its input is the same, this will reproduce Eq. (122) (hence Eq. (114)) as one can check. For instance, in the simplest case that \( \rho_\phi(x, \tau_{nl}) \) has a negligible perturbation, corresponding to \( \langle x^2(\tau_{nl}) \rangle \gg \delta^2 \), it gives

\[
\langle x^2(\tau_{nl}) \rangle = \frac{4}{3} (\mu N/H)^{3/2} = \text{const} - \ln \left[ \chi^2(x, \tau_1) \right],
\]

(132)

so that Eq. (131) gives

\[
\zeta_\chi = \frac{H}{2\mu\sqrt{\tau_{nl}}} \frac{\delta x^2(x, \tau_1)}{\langle x^2(\tau_{nl}) \rangle} = \frac{H}{2\mu\sqrt{\tau_{nl}}} \frac{\delta x^2(x, \tau_{nl})}{\langle x^2(\tau_{nl}) \rangle},
\]

(133)

which indeed coincides with Eq. (114) when \( \langle x^2(\tau_{nl}) \rangle \gg \delta^2 \).

We emphasize that Eq. (131), which invokes \( \delta x^2(x, t_1) \) smoothed on a super-horizon scale, is likely to hold only during the linear era. There is no reason to think that the smoothed \( \delta x^2 \) will determine the evolution of \( \zeta \) after that era. Rather, in order to calculate that evolution, one should calculate \( \rho \) and \( p \) keeping all modes of \( \chi \) and \( \phi \) and then smooth them on a super-horizon scale.

X. EARLIER CALCULATIONS

Several papers [31, 32, 40–44] consider the evolution of the waterfall field, without considering its effect on the curvature perturbation. Among them, [41, 43] invoke the assumptions laid down in Section IV B. We will not comment on these papers, but turn instead to those that consider the curvature perturbation. In doing so, we exclude the non-standard scenario \( |m_\chi| \ll H \) considered in [45] and part of [46, 47].
Several papers [48, 51] treat the waterfall as two-field inflation with nonzero $\langle \chi \rangle_{nl}$. For that to be valid, we would need to be in the extreme $|m_\chi| \ll H$ scenario (considered in parts of [48, 47]) where $m_\chi^2(t) \ll H^2$ already when the observable universe leaves the horizon. The calculation in these papers is therefore incorrect, and all of them except [51] reach the incorrect conclusion that $\zeta_\chi$ depends linearly on $\chi$, with $\mathcal{P}_{\zeta_\chi}$ nearly scale invariant. (The calculation of [51], setting without justification $\langle \chi \rangle = \sqrt{\langle \chi^2 \rangle}$ reaches a basically correct result even though the reasoning is incorrect; it makes $\zeta_\chi$ quadratic in $\delta \chi$, with $\mathcal{P}_{\zeta_\chi} \propto k^3$ and with very very roughly $\mathcal{P}_{\zeta_\chi} \sim 1$ at $k = H$.)

Two papers [52, 53] invoke the function $N$ considered at the end of the previous section. They consider hybrid inflation with the potential $V(\phi) = \frac{1}{2} m_\phi^2 \phi^2$. The parameters are adjusted to give $f = 1$, and are chosen so that the waterfall takes a small number of Hubble times, not much less than a Hubble time as we are assuming. For the present purpose that difference can be ignored, because the qualitative evolution of $\chi$ and $\zeta$ is similar in the two cases [53]. With these parameters, $\langle \chi^2(t_{nl}) \rangle \gg \dot{\phi}^2$, which as we saw at the end of Section VIII means that the slicing of uniform $\rho$ is also one of uniform $\chi$. The latter is assumed by the authors, without proof. In [52] $N$ is calculated numerically, while in [53] it is calculated analytically.

Both papers invoke a quantity, denoted by $\delta \chi_L$, which is $\chi$ smoothed on the scale of the horizon at the beginning of the waterfall. In [52], $N$ is regarded as a function of $\chi$ rather than $\chi^2$, and is inserted into an expression

$$\zeta_\chi(x) = \frac{d^2(N)_L}{d\chi^2} \bigg|_{\chi=0} (\delta \chi_L(x,t_1))^2.$$  \hspace{1cm} (134)

The subscript attached to $(N)_L$ indicates that a smoothing procedure has been applied, which is necessary because $N$ itself diverges at $\chi_L = 0$. This divergence occurs because, with their choice of parameters, the energy density at the epoch $t_{nl}$ is less than $V_0$. Such an energy density cannot be achieved if $\chi$ is fixed at 0.

This expression for $\zeta_\chi$ differs from (131) in several ways. First, the expansion is about $\chi = 0$ whereas it should be about $\chi^2 = \langle \chi^2 \rangle$. Second $(\delta \chi_L)^2$ (which is smoothed and then squared) should be replaced by $\delta \chi^2$ (which is squared and then smoothed). Third, $(N)_L$ should be replaced by the unsmoothed quantity $N$. As it happens though, none of these errors will affect the rough order of magnitude, because with their parameter choice the following hold: (i) the dominant mode of $\chi$ is of order $H$ so that the smoothed quantity $\delta \chi_L$ will not be very different from $\chi_L$ itself, (ii) ignoring the smoothing, the Fourier components of $\delta \chi_L^2$ are the same as those of $\delta \chi^2$ (iii) $N(\chi_L^2)$ is approximately linear and close to $(N(\chi_L^2))_L$ away from the origin.

In [52], the correct Eq. (131) is used, except that $\delta \chi^2$ is replaced by $\delta \chi_L^2$. Because of this replacement, $P_{\delta \chi^2}(k) \sim P_{\chi^2}(0)k^4$ is replaced by $P_{\delta \chi_L^2} \sim P_{\chi_L^2}(0)H^3$ (since $\delta \chi_L$ is smooth on the Hubble scale) which means that $\mathcal{P}_{\zeta_\chi}$ is underestimated by a factor $(H/k_s)^3$, which with their parameters is roughly $(1/3)^3$ #18. In a footnote of [53], an argument is given which is intended to justify the dropping of short-scale modes when evaluating the perturbation in $\chi^2$. They write $\chi = \chi_L + \chi_S$, and to define $\chi_S$ the universe is divided into boxes with size $\sim H^{-1}$. Within each box $\chi_S$ is defined as the usual Fourier Series valid within that box, and all Fourier coefficients are generated as usual from the vacuum fluctuation. This proposal is not viable, because for a typical realization of the ensemble (presumed to correspond to the observed universe) $\chi$ defined in this way has a large discontinuity at the boundary of each box in violation of the field equation.

In the Appendix of [53], $\zeta_\chi$ is calculated by integrating $\dot{\zeta}$, but they get a result that is bigger than the one from $\delta N$, by a factor $\ln[\langle \chi^2(t_{nl}) \rangle/\dot{\phi}^2(t_{nl})]$. The discrepancy arises because they identify $\delta p_{nad}$ with with $\delta p_\chi$, which makes $\zeta$ constant in the regime $\langle \chi^2(t) \rangle \gg \dot{\phi}^2$. As we see from Eq. (109), $\delta p_{nad}$ in that regime actually decays exponentially.

#18 We are denoting the dominant scale of their calculated $\chi$ by $k_s$, the same symbol that we used for its dominant scale in our calculation.
The papers we considered so far contain errors of principle. We turn now to papers [47, 54] where the method is in principle correct, but leads to excessively complicated equations. Presumably because of this complication, the expressions for \( \zeta \chi \) in [47, 54] are not correct. The method is to consider equations that include, in addition to the perturbations included in our equations, the second order metric perturbation as well as the product \( \chi \delta \phi \). We demonstrated in Section [VI] that the latter has a negligible effect and we pointed out in Section [II B] that the former will also have a negligible effect. The motivation for including these small extra terms seems to be the following. First, field perturbations are defined as perturbations away from zero, so that \( \chi \) is regarded as a perturbation. Second, ‘first order’ equations are by definition linear in the field perturbations, so that in this order \( \chi \) gives no contribution to \( \rho \) or \( p \). Finally, the next (‘second’) order is defined to involve equations containing terms of the form \( x^2, xy \) and \( y^2 \), where \( x \) is a first-order field perturbation and \( y \) is a first-order metric perturbation. It is the inclusion of the \( y^2 \) terms that causes the main complication in the equations.

As has been noticed elsewhere [39, 47, 55], the expression for \( \zeta \chi \) in [54] is manifestly incorrect because it is not a local function of \( \chi \). The expression in [47] (Eq. (57)) is local, but still not correct. In particular, it should in the regime \( 6Ht_{nl} \ll 1 \) coincide (to first order in \( \delta p \chi \)) with Eq. (114), whereas it differs by having \( -m^2 \) instead of our \( -\mu^2 \tau \equiv m^2(\phi) \) (and by a factor 2). Further calculations using the same expression were done in [57].

XI. CONCLUSION

In this paper we have considered \( \zeta \chi \), the amount by which \( \zeta \) changes during the linear evolution of the waterfall field. We have calculated its spectrum in the regime of parameter space in which slow-roll inflation continues during the linear era, with the waterfall mass-squared decreasing linearly and the waterfall takes much less \( Ht_{nl} \ll 1 \) Hubble times. The spectrum, given by Eq. (116), is proportional to \( k^3 \) and amply satisfies the black hole bound \( P_\zeta \lesssim 10^{-2} \) on those super-horizon scales \( k \ll H \) for which \( \zeta \) is defined. The full parameter space for standard hybrid inflation will be explored elsewhere [33], using instead of \( \zeta \) the quantity \( R \) mentioned before Eq. (73).

We made the fundamental assumption, shared by all previous authors except those of [31], that the waterfall does indeed begin with a the linear era. We have justified that assumption using the work of [31], who invoke in turn the calculation of [32]. But the justification is not quite complete, because the latter calculation invokes a nonzero value of \( \langle \chi^2 \rangle \) at the start of the waterfall, coming from the vacuum fluctuation, whereas we have set that value to zero.

This brings us to an issue of principle, namely the regularization of uv divergences. We adopted the simplest procedure, of dropping at a given epoch all Fourier modes (of the waterfall and inflaton fields) that are in the quantum as opposed to the classical regime. That procedure will violate at some level the energy continuity equation, which follows from the complete field equations including all modes. With the assumptions laid down in Section [IV B] the violation will be negligible because the dominant classical modes (with \( k \sim k_* \)) appear as soon as \( \tau \) becomes significantly bigger than 1; as a result, the subsequent creation of classical modes, which violates the energy continuity equation, can be ignored. But there seems to be no guarantee that the violation will be negligible in all cases where a classical field is created from the vacuum.

More seriously, the procedure of dropping the quantum modes will not make sense at all for the region of parameter space in which the waterfall field fails to enter the classical regime. (We identified one such a region in Section [V] working with our particular assumptions.) In that region, the waterfall becomes an essentially quantum phenomenon requiring a formalism quite different from the one that we presented.

The procedure of dropping the quantum regime is crude, but something certainly has to

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#19 The paper [47] builds on [46] and corrects some errors in the latter.  
#20 Since the equations in [54] are erroneous, we will not consider [56] which uses them to consider a metric perturbation related to \( \zeta \).
be done because the energy density, pressure and mean-square fields diverge if the entire quantum regime is kept. Indeed, an ultra-violet cutoff at some scale $k/a = \Lambda$ much bigger than the particle masses gives a constant energy density and pressure $\rho = 3p = \Lambda^4/16\pi^2$ which violates the energy continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$.

Several alternatives to simply dropping the quantum regime have been proposed that can apply to hybrid inflation (for example \cite{41, 43, 60} in addition to the procedure of \cite{32} that we mentioned before) and it is not clear how to choose between them if they give different results.

XII. ACKNOWLEDGMENTS

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