An Algebraic Bootstrap for Dimensionally Reduced Quantum Gravity

M. Niedermaier

Department of Physics
100 Allen Hall, University of Pittsburgh
Pittsburgh, PA 15260, USA

H. Samtleben

Laboratoire de Physique Théorique
de l’Ecole Normale Supérieure
24 Rue Lhomond, 75231 Paris Cédex 05, France

Abstract

Cylindrical gravitational waves of Einstein gravity are described by an integrable system (Ernst system) whose quantization is a long standing problem. We propose to bootstrap the quantum theory along the following lines: The quantum theory is described in terms of matrix elements e.g. of the metric operator between spectral-transformed multi-vielbein configurations. These matrix elements are computed exactly as solutions of a recursive system of functional equations, which in turn is derived from an underlying quadratic algebra. The Poisson algebra emerging in its classical limit links the spectral-transformed vielbein and the non-local conserved charges and can be derived from first principles within the Ernst system.

Among the noteworthy features of the quantum theory are: (i) The issue of (non-)renormalizability is sidestepped and (ii) there is an apparently unavoidable “spontaneous” breakdown of the $SL(2,\mathbb{R})$ symmetry that is a remnant of the 4D diffeomorphism invariance in the compactified dimensions.

*E-mail: nie@prospero.phyast.pitt.edu
§E-mail: henning@lpt.ens.fr
¶UMR 8548: Unité Mixte de Recherche du Centre National de la Recherche Scientifique et de l’Ecole Normale Supérieure.
1. Introduction

Attempts to construct a quantum theory of gravity based on a functional integral formulation have so far been unsuccessful. Initially this was thought to be a breakdown only of its perturbative expansion. Meanwhile various reasonably looking discretized versions of the functional integral have (in all likelihood) failed to produce a continuum limit of the desired form. This may indicate the necessity to incorporate specific forms of matter, it may indicate a failure of the functional integral approach, or the analogy to the quantization of conventional field theories may just be physically misleading altogether. In any case it seems desirable to locate the source of the problem more clearly by studying model situations sufficiently complex to make (non-)renormalizability an issue, but tractable enough to be mathematically controllable.

An intriguing such system is the Ernst system, an infinite dimensional subsector (the two Killing vector field reduction) of the full phase space of general relativity; see [1]–[6] and references therein. Together with its abelian truncation it has become a prominent testing ground to explore certain quantum issues of gravity; see e.g. [7]–[13]. The reduced phase space is equivalent to that of a two-dimensional diffeomorphism invariant field theory. The latter couples 2D gravity via a dilaton field $\rho$ to a 2D matter system equivalent to the noncompact $O(1,2)$ nonlinear sigma-model. This means, the matter degrees of freedom are mappings $n: \Sigma \to H_2$ from the 2D spacetime manifold $\Sigma$ to the hyperboloid $H_2 = \{ n = (n^0, n^1, n^2) \in \mathbb{R}^{1,2} | n \cdot n = (n^0)^2 - (n^1)^2 - (n^2)^2 = 1, n^0 > 0 \}$. If $h_{\mu\nu}$ denotes the (Lorentzian) metric on $\Sigma$ and $R^{(2)}(h)$ its scalar curvature, the action of the two-dimensional system is given by

$$S = \int_{\Sigma} d^2x \sqrt{-h} \left\{ -\rho R^{(2)}(h) + \frac{1}{2} \rho h^{\mu\nu} \partial_\mu n \cdot \partial_\nu n \right\}, \quad (1.1)$$

where ‘ $\cdot$ ’ is the bilinear form on $\mathbb{R}^{1,2}$. The relation to the coset action principles usually employed in the literature and that to the original Ernst variables is described in appendix A.

Depending on the signature of $\Sigma$, this sector physically describes either stationary axisymmetric solutions or gravitational waves with additional symmetries. The latter case comprises – depending on the norm of $\partial_\mu \rho$ – the Gowdy universes, colliding plane wave solutions, and (the case considered here) cylindrical gravitational waves. The original Einstein-Rosen waves form a collinearly polarized subsector, they have $n^1 \equiv 0$, and are thus described by the abelian $O(1,1)$ subtheory of (1.1).

At first sight, the action (1.1) seems perfectly amenable to conventional quantum field theoretical techniques. Upon closer inspection however, one is quickly led to address the following questions:

(i) Can one expect 2D conformal invariance to be unbroken?
(ii) What is the physics of the flat space sigma-model with target space $H_2$?

(iii) Is the theory (1.1) renormalizable in perturbation theory?

Let us briefly comment on these issues:

(i) Flat space non-linear sigma-models are known to exhibit dynamical mass generation, destroying the 2D conformal invariance of the classical theory. In the gauge $h_{\mu\nu} = e^{2\sigma} \eta_{\mu\nu}$ (where $\eta_{\mu\nu}$ is the flat 2D Minkowski metric) the classical system (1.1) likewise exhibits a conformal symmetry, and off-hand there is no reason why it should not again be broken in the quantum theory. Rather, there are strong indications for a dynamical breaking of the conformal symmetry from other 2D quantum gravity models [14, 15]. In any case, it seems advisable not to employ 2D conformal invariance as a guiding principle to construct the quantum theory.

(ii) Although sigma-models with a non-compact target space have been studied for some time [16]–[21], even basic qualitative features are unknown. Perturbative renormalizability in the presence of an infrared cutoff should largely parallel that of the compact case [22]. However the free field Fock space carrying a nonlinear realization of the $O(1, N)$ group action has indefinite metric [21] and the projection onto a physical state space is not fully understood. Likewise the extent to which the results [23, 24] on the infrared finiteness carry over has to be examined. One might hope a non-perturbative construction to be feasible via the lattice approach. Specifically, the results of a lowest order large $N$ analysis of the $O(1, N)$ models [16, 17, 18] suggest that a non-trivial continuum limit theory might emerge when using spacelike hyperbolic variables $n \cdot n = -1/g_0$, $g_0 > 0$, and sending the bare coupling constant $g_0$ to zero. (Classically this is the region where the Hamiltonian density fails to be positive semi-definite.) Relying on these indications one would further guess that the resulting QFT is massive, has unbroken $O(1, N)$ symmetry, a positive definite physical Hilbert space, and a unitary S-matrix. However this scenario has not been corroborated so far.

(iii) Here, renormalizability should mean in particular that the target space geometry $H_2$ is left intact by the renormalization process. In conformal gauge one may take $\rho$ as a loop counting parameter, in which case the model is 1-loop renormalizable in the background field expansion [25]. For higher loops however one cannot expect off-hand that the results for the generalized Riemannian sigma-models [26, 27] used in the context of string theory, employing a much weaker notion of renormalizability, will carry over. Let us emphasize that the answer is not automatic even if one takes the (ultraviolet and infrared) renormalizability of the non-compact sigma-models in flat space for granted. One way to see this is to fix the 2D diffeomorphism invariance in (1.1) completely and to identify $\rho$ with one of the coordinates on $\Sigma$ (this corresponds to the 4D Weyl coordinates). Then (1.1) becomes a flat space action, though with an explicit coordinate dependence which in particular destroys Poincaré
invariance. Clearly the presumed renormalizability of the Poincaré invariant flat space theory is not very indicative for the behavior of the other system.

In summary none of the conventional quantum field theoretical techniques presents itself to construct a quantum theory for (1.1). We propose therefore to bootstrap the quantum theory from structures linked to its classical integrability. In upshot the quantum theory is described in terms of matrix elements e.g. of the metric operator between certain spectral-transformed multi-vielbein configurations. These matrix elements are described exactly as solutions of a recursive system of functional equations, which in turn are derived from an underlying dynamical algebra. The Poisson algebra emerging in its classical limit links the spectral-transformed vielbein and the non-local conserved charges and can be derived from first principles within the Ernst system. Schematically one can summarize the approach as follows:

\[
\begin{array}{ccc}
\text{Dynamical algebra} & \rightarrow & \text{Functional equations} \\
& & \text{for sequences of meromorphic functions} \\
& & \rightarrow \text{Exact matrix elements in (renormalized) quantum theory}
\end{array}
\]

We first briefly describe the ingredients of the above scheme and then comment on why we expect it to yield a viable quantum theory for the Ernst system.

The data for the dynamical algebra \( \mathcal{D} \) are: A solution \( R \) of the Yang-Baxter equation, a real parameter \( \beta \), and a choice of \(*\)-operation. There are two sets of generators \( T^\pm(\theta)^b_a \) and \( W_a(\theta) \), where \( \theta \in \mathbb{C} \), and the indices \( a,b \) refer to a basis in a finite dimensional vector space. The algebra \( \mathcal{D} = \mathcal{D}(R, \beta, *) \) associated with the data is basically the most general simple associative algebra, where “simple” means that all ideals have been divided out. In the case at hand the data are as follows: \( R \) is the rational \( sl_2 \) \( R \)-matrix multiplied with a scalar function that ensures a suitable unitarity and crossing condition. The parameter \( \beta \) vanishes, which corresponds to the case where the algebra has an enlarged center. The latter roughly speaking ensures that the quantum theory has the same number of dynamical degrees of freedom as the classical theory. For any other value of \( \beta \) this correspondence would be violated and degrees of freedom transversal to the reduced phase space would become dynamical (without describing a consistent bigger portion of the full phase space). The possible \(*\)-structures turn out to fall into several equivalence classes; the proper one is selected by matching it against that of the classical Poisson algebra.

Once the correct dynamical algebra \( \mathcal{D} \) has been identified one considers linear functionals \( \mathcal{D} \ni X \rightarrow \langle \Theta | X | \Omega \rangle \in \mathbb{C} \) over \( \mathcal{D} \), where the vectors \( \langle \Theta | \) and \( | \Omega \rangle \) are characterized by the conditions: \( T^+(\theta)^b_a | \Omega \rangle = \gamma^+_b a | \Omega \rangle \) and \( \langle \Theta | T^-(\theta)^b_a = \gamma^-_a b | \Theta \rangle \). The \( \gamma^\pm \) are numerical matrices subject to certain consistency conditions, and in general \( \langle \Theta | \Omega \rangle = 0 \). Given such a functional one can introduce the sequence of functions

\[
f_{a_0 \ldots a_n}(\theta_0, \ldots, \theta_1) = \langle \Theta | W_{a_0}(\theta_0) \ldots W_{a_n}(\theta_1) | \Omega \rangle , \quad n \geq 1 . 
\]  

(1.2)

The relations of the dynamical algebra then imply that this sequence satisfies a recursive system of functional equations, where the consistency of the underlying algebra ensures the
consistency of the functional equations. Conversely the original functional over $\mathcal{D}$ provides an abstract solution to the functional equations. The functional equations can be grouped into two sets (I) and (II). The set (I) characterizes the functions (1.2) for fixed $n$, the second set (II) prescribes how the solutions of (I) are arranged into sequences. Essentially (II) stipulates that the functions (1.2) have simple poles whenever two $\theta$-variables differ by a fixed purely imaginary number (which in physical units equals $il_{pl}^2/l_z$, $l_{pl}$ being the Planck length and $l_z$ the unit length along the symmetry axis) and that the residues at these poles are linked to a function (1.2) in $n-2$ variables.

Having arrived at the functional equations one can in principle forget about their derivation and take them as the starting point. The aim then is to construct explicit solutions in the form of sequences of meromorphic functions $f^{(n)} = f_{a_1\ldots a_n}(\theta_1, \ldots, \theta_n)$. To understand their physical interpretation the analogy to the form factor approach [28] to relativistic integrable QFTs is useful. For these systems a similar interplay between a dynamical algebra and a system of functional equations (the so-called form factor equations) exists [29, 30]. The solutions $f^{(n)}$ in that case describe the form factors of the QFT, i.e. matrix elements of some local operator between the physical vacuum and an asymptotic multi-particle state. In the case at hand of course a conventional quantum field theoretical framework is not available; in particular scattering states in the usual sense are unlikely to exist. Nevertheless the functions $f^{(n)}$ can still be interpreted as the matrix elements of some operator between a ground state and a “multi-$W_a(\theta)$” configuration. From the analysis of the semi-classical limit one finds that the $W_a(\theta)$ are the quantum counterparts of a spectral-transformed vielbein variable. Moreover the analogy to the $\mathbb{CP}_1$ system (cf. appendix A) suggests to view them as confined degrees of freedom rather than generators of scattering states. Similarly as in the form factor approach the identification of the operator whose matrix elements are obtained in that way requires external input. Of course “identification” here to some extent just means making contact to a conventional quantum field theoretical formulation where the operator in question is constructed as a composite operator from a set of fundamental field operators. As explained before such a more conventional formulation is presently not available for the Ernst system, so that the identification of the operator underlying a sequence $f^{(n)}$ in this sense has to be left for future work.

Note that at no stage any renormalization procedure entered. This is a known, yet striking, feature of the form factor approach. Mathematically it can be understood in terms of the extreme rigidity of the underlying dynamical algebra, which simply does not allow for interesting continuous automorphisms that could account for a renormalization process. The same is true in the present setting and suggests that the functions $f^{(n)}$ indeed are exact matrix elements that do not require renormalization.

Physically our most important finding is an apparently unavoidable “spontaneous” breakdown of the global $SL(2, \mathbb{R})$ invariance that is a remnant of the original 4D diffeomorphism invariance in the Killing coordinates. Technically this emerges because the consistency conditions on the matrices $\gamma^\pm$ mentioned before eq. (1.2) do not admit a $SL(2, \mathbb{R})$ invariant solution. As a consequence the functions $f^{(n)}$ are invariant only under a maximal
compact $SO(2)$ subgroup. Despite the trivial technical origin the symmetry breaking is a genuine dynamical feature intimately linked to the structure of the dynamical algebra; in particular it disappears in the semi-classical limit.

As explained before we also regard it as more likely than not, that the 2D conformal invariance of the classical theory (1.1) will be broken in the quantum theory. A-fortiori then also the diffeomorphism invariance in the two non-Killing coordinates of the Ernst system will be lost. Together both remnants of the original 4D diffeomorphism invariance (i.e. that it the Killing and in the non-Killing coordinates) appear to be broken in the quantum theory for dynamical reasons. Being a field theoretical phenomenon that does not have a counterpart in systems with finitely many degrees of freedom, the result may well have significance beyond the symmetry-reduced theory.

The article is organized as follows. In the next section, the dynamical algebra $\mathcal{D}$ is introduced and the functional equations (I), (II) for the matrix elements (1.2) are derived. In section 3 we discuss the semi-classical limit of the construction and explain the relation to the phase space of the classical Ernst system. Also various aspects of the symmetry breaking are detailed. A compilation of useful action principles for the matter sigma-model is deferred to appendix A. In section 4 a solution technique for the functional equations (I), (II) is described. We adapt techniques from the algebraic Bethe ansatz which are summarized in appendix B. In particular, the apparently new concept of “sequential” Bethe roots and Bethe vectors is introduced. Finally a list of explicit solutions for the functions (1.2) with $n \leq 4$ is collected in appendix C.

2. Dynamical algebra and functional equations

Here we describe the dynamical algebra and derive the functional equations (I), (II) for the objects (1.2). The discussion is naturally organized into two steps. First an algebra $\mathcal{D}_I$ is introduced giving rise to the functional equations (I). Then $\mathcal{D}_I$ is shown to still contain two-sided ideals; the factor algebra obtained by dividing out these ideals is the full dynamical algebra $\mathcal{D}$ and gives rise to the additional functional equations (II). We begin by describing $\mathcal{D}_I$ and initially keep the data $(R, \beta, \ast)$ generic.

2.1 $W$-extended Yangian doubles

The algebras $\mathcal{D}_I$ are centrally extended Yangian doubles $DY_\beta(R)$ with generators $T^\pm(\theta)^b_a$ supplemented by generators $W^a(\theta)$. We write $WY(R, \beta, \ast)$, where $R$ is a is a solution of the Yang-Baxter equation satisfying unitarity and crossing. Further $\beta$ is a real parameter and $\ast$ refers to a choice of $\ast$-operation. Lower indices refer to a basis in a finite dimensional vector space $V$; upper indices refer to the dual basis, where indices are raised and lowered
by means of the constant charge conjugation matrix $C_{ab}$ and its inverse $C^{ba}$, associated with the given $R$-matrix. For the moment we only need the data $R$ and $\beta$; the possible $*$-operations will be discussed below. To any $R$-matrix and parameter $\beta$ one can assign an associative algebra $WY$ with unity by postulating the following exchange relations among its generators:

\begin{align}
(T1) \quad R^{cd}_{mn}(\theta_{12}) T^\pm(\theta_1)_a^n T^\pm(\theta_2)_b^m &= T^\pm(\theta_2)_b^c T^\pm(\theta_1)_a^d R^{mn}_{ab}(\theta_{12}), \\
& \quad R^{cd}_{mn}(\theta_{12}) T^+(\theta_1)_a^n T^-(\theta_2)_b^m = T^-(\theta_2)_b^c T^+(\theta_1)_a^d R^{mn}_{ab}(\theta_{12} + 2i\hbar - i\hbar\beta/\pi),
\end{align}

\begin{align}
(T2) \quad C_{mn} T^\pm(\theta)_a^m T^\pm(\theta - i\hbar)_b^n &= C_{ab}, \\
& \quad C^{mn} T^\pm(\theta)_a^m T^\pm(\theta + i\hbar)_b^n = C^{ab}.
\end{align}

\begin{align}
(TW) \quad T^-(\theta_1)_a^c W_b(\theta_2) &= R^{dc}_{ab}(\theta_{12}) W_c(\theta_2) T^-(\theta_1)_d^c, \\
& \quad T^+(\theta_1)_a^c W_b(\theta_2) = R^{dc}_{ab}(\theta_{12} + i2\hbar - i\hbar\beta/\pi) W_c(\theta_2) T^+(\theta_1)_d^c.
\end{align}

\begin{align}
(WW) \quad W_a(\theta_1) W_b(\theta_2) &= R^{dc}_{ab}(\theta_{12}) W_c(\theta_2) W_d(\theta_1), \quad \text{Re} \theta_{12} \neq 0,
\end{align}

with $\theta_{12} := \theta_1 - \theta_2$. The parameter $\hbar$ is included for later convenience; it can be given any non-zero value by a rescaling of the $\theta$ variables. For convenience we also assume that real boosts in the $\theta$ variables are unitarily implemented, i.e. $e^{i\lambda K} X(\theta) e^{-i\lambda K} = X(\theta + \lambda)$, with $\lambda \in \mathbb{R}$, for any generator $X(\theta)$ of the algebra. For completeness we also note the precise form of the Yang Baxter equation

\begin{align}
R^{mn}_{ab}(\theta_{12}) R^{kp}_{nc}(\theta_{13}) R^{ij}_{mp}(\theta_{23}) &= R^{nm}_{bc}(\theta_{23}) R^{pi}_{am}(\theta_{13}) R^{kj}_{pn}(\theta_{12}), \quad (2.1)
\end{align}

and the conditions of unitarity and real analyticity

\begin{align}
R^{mn}_{ab}(\theta) R^{cd}_{nm}(-\theta) = \delta_a^d \delta_b^c, \quad [R^{cd}_{ab}(\theta)]^* = R^{cd}_{ab}(-\theta^*). \quad (2.2)
\end{align}

We add a few remarks. The algebra $DY_\beta(R)$ is a well-known structure. For $\beta = 2\pi$ it can be viewed as a presentation of the quantum double of some underlying infinite dimensional Hopf algebra \cite{B1}. The (TW) and (WW) relations are then characteristic for the intertwining operators between quantum double modules \cite{K1, K2}. Particular cases are the Yangian double or the quantum double of $U_q(\hat{g})$, in which case the parameter $\beta$ can be related to the central extension via $c = 2i(1 - \beta/2\pi)$. Here we do not make use of the co-algebra structure and always treat $\beta$ as a real numerical parameter. The case of the critical level with enlarged center \cite{C1, C2} in our conventions corresponds to $\beta = 0$; for the $sl_2$ Yangian case it will be studied in detail below.
In preparation, we introduce the following quadratic element which turns to play a decisive role in the WY algebras

\[ D_{ab}(\theta) := C_{cd} T^-(\theta + i\hbar) a T^+(\theta) b, \quad (2.3) \]

It enjoys the following exchange relations

\[ R_{ab}^{mn}(\theta_{21} + i\hbar \beta / \pi - 2i\hbar) D_{cn}(\theta_1) W_m(\theta_2) = R_{ac}^{mn}(\theta_{12} + i\hbar) W_m(\theta_2) D_{nb}(\theta_1), \quad (2.4a) \]

\[ R_{ab}^{mn}(\theta_{21}) D_{cn}(\theta_1) T^+(\theta_2) d = R_{ac}^{mn}(\theta_{12} - i\hbar + i\hbar \beta / \pi) T^+(\theta_2) m D_{nb}(\theta_1), \quad (2.4b) \]

\[ R_{ab}^{mn}(\theta_{21} + i\hbar \beta / \pi - 2i\hbar) D_{cn}(\theta_1) T^- (\theta_2) d = R_{ac}^{mn}(\theta_{12} + i\hbar) T^- (\theta_2) m D_{nb}(\theta_1), \quad (2.4c) \]

\[ R_{ab}^{mn}(\theta_{21}) R_{mc}^{kl}(\theta_{21} + i\hbar \beta / \pi - i\hbar) D_{nk}(\theta_1) D_{ld}(\theta_2) = R_{cd}^{mn}(\theta_{12}) R_{bn}^{kl}(\theta_{12} + i\hbar \beta / \pi - i\hbar) D_{al}(\theta_2) D_{km}(\theta_1). \quad (2.4d) \]

In particular, it follows from these relations that the (generalized) quantum current \( D(\theta) = C_{a b} D_{a b}(\theta) \) lies in the center of the algebra \( W Y(R, 0, \ast) \), for any \( R \)-matrix obeying a standard crossing relation with a symmetric charge conjugation matrix \( C_{ab} \). Much of the construction described in the rest of the paper could therefore be transferred to a fairly general class of infinite dimensional quantum algebras.

With the application to the quantum Ernst system in mind however we specialize already at this point to the \( s l_2 \) Yangian \( R \)-matrix \([36, 37, 38]\). Its charge conjugation matrix is anti-symmetric which enforces a slight modification of the above scheme. The relevant \( R \)-matrix is then given by

\[ R_{ab}^{cd}(\theta) = r(\theta) \left[ -\frac{\theta}{i\hbar - \theta} \delta_a^c \delta_b^d + \frac{i\hbar}{i\hbar - \theta} \delta_a^d \delta_b^c \right], \quad a, b, \ldots = 1, 2, \quad (2.5) \]

where

\[ r(\theta) = \frac{\operatorname{ch} \frac{\pi \theta}{2\hbar} + i \operatorname{sh} \frac{\pi \theta}{2\hbar}}{\operatorname{ch} \frac{\pi \theta}{2\hbar} - i \operatorname{sh} \frac{\pi \theta}{2\hbar}} \frac{\Gamma \left( \frac{1}{2} + \frac{\theta}{2i\hbar} \right) \Gamma \left( -\frac{\theta}{2i\hbar} \right)}{\Gamma \left( \frac{1}{2} - \frac{\theta}{2i\hbar} \right) \Gamma \left( \frac{\theta}{2i\hbar} \right)} , \]

satisfies

\[ r(\theta) r(\theta - i\hbar) = 1 - \frac{i\hbar}{\theta}, \quad r(\theta) = 1 - \frac{i\hbar}{2\theta} + \frac{1}{8} \left( \frac{i\hbar}{\theta} \right)^2 + O \left( (i\hbar / \theta)^3 \right) . \]

Further \( r(0) = -1 \) and \( r(i\hbar + \delta) = -\delta / i\hbar, r(-i\hbar + \delta) = i\hbar / \delta, \) for \( \delta \to 0 \). For later reference we list the main properties of the \( R \)-matrix \((2.5)\). In addition to the Yang-Baxter equation \((2.1)\) and \((2.2)\) one has a sign modified crossing invariance

\[ R_{ab}^{dc}(\theta) = -C_{aa'} C_{dd'} R_{d'b}^{a'c}(i\hbar - \theta) \quad \text{with} \quad C_{ab} = i\varepsilon_{ab} \cdot \quad (2.6) \]
The R-matrix (2.5) has no poles in the strip $0 \leq \text{Im } \theta \leq i\hbar$ but a simple zero at $\theta = i\hbar/2$. At $\theta = 0, i\hbar$ it becomes a projector

$$R^{cd}_{ab}(0) = -\delta^{d}_{a} \delta^{c}_{b},$$
$$R^{cd}_{ab}(i\hbar) = -\delta^{d}_{a} \delta^{c}_{b} + \delta^{d}_{c} \delta^{a}_{b} = C_{ab} C^{cd}. \quad (2.7)$$

The semi-classical expansion is

$$R^{cd}_{ab}(\theta) = \delta^{c}_{a} \delta^{d}_{b} - \frac{i\hbar}{\theta} \Omega^{cd}_{ab} - \left(\frac{i\hbar}{\theta}\right)^{2} \left(\delta^{d}_{a} \delta^{c}_{b} - \frac{5}{8} \delta^{c}_{a} \delta^{d}_{b}\right) + O\left((i\hbar/\theta)^{3}\right), \quad (2.8)$$

with

$$\Omega^{cd}_{ab} = -\frac{1}{2} \delta^{c}_{a} \delta^{d}_{b} - \frac{1}{2} \delta^{d}_{c} \delta^{a}_{b}. \quad (2.10)$$

Next we determine the possible *-operations of the WY-algebras with the R-matrix (2.5). Starting with a general linear ansatz one finds

$$\sigma W_{a}(\theta) = F^{a'}_{a} W_{a'}(\theta^{*} + i\hbar)$$
$$\sigma T^{+}(\theta)_{b}^{a} = F^{a'}_{a} E_{b'}^{a} T^{-}(\theta^{*} + i\hbar/\pi - i\hbar)_{a'}^{b'}$$
$$\sigma T^{-}(\theta)_{b}^{a} = (E^{-1})_{b'}^{b} (F^{-1})_{a'}^{a} T^{+}(\theta^{*} + i\hbar/\pi - i\hbar)_{a'}^{b'}, \quad (2.9)$$

where $E, F$ are $GL(2, \mathbb{C})$ matrices satisfying

$$EE^{*} = \mathbb{1}, \quad \epsilon \in \{\pm 1\}, \quad FF^{*} = \mathbb{1}, \quad \det E \cdot \det F = 1. \quad (2.10)$$

Any solution of (2.10) yields a consistent *-operation (2.9), for any value of $\beta$. We omitted a trivial overall shift by a purely imaginary number in the arguments on the right hand side of (2.9). We also omitted scalar prefactors on the right hand side which can be removed by a rescaling of the generators. The operator $D_{ab}(\theta)$ is basically hermitian with respect to any of the *-operations (2.9), (2.10)

$$\sigma D_{ab}(\theta) = \epsilon D_{ba}(\theta^{*} + i\hbar/\pi - 2i\hbar), \quad \sigma D(\theta) = \epsilon D(\theta^{*} + i\hbar/\pi - 2i\hbar), \quad (2.11)$$

where $D(\theta) := C^{ab} D_{ab}(\theta)$.

Clearly, most of these *-operations will be equivalent in being related by an automorphism of the WY-algebra. Such automorphisms are provided by $SL(2, \mathbb{C})$ basis transformations $W_{a}(\theta) \rightarrow f^{a'}_{a} W_{a'}(\theta), \ T^{\pm}(\theta)_{a}^{b} \rightarrow f^{a'}_{a} g^{b'}_{b} T^{\pm}(\theta)_{a'}^{b'}, \ f, g \in SL(2, \mathbb{C})$, under which the *-structure (2.9) transforms covariantly as

$$E \mapsto g^{-1} E g^{*}, \quad \epsilon \mapsto \epsilon, \quad F \mapsto f^{*} F f^{-1}. \quad (2.12)$$

It is not hard to classify the inequivalent *-structures in the general case. With regard to the Ernst system however we restrict attention to real linear transformations and thus
require the matrices $E, F$ to be real. This leaves four cases for the possible $*$-operations, corresponding to the sign choices $\text{sign}(\det F) = \text{sign}(\det E)$ and $\epsilon \in \{\pm 1\}$. Consider first $F$: If $\det F = -1$ then $F = A \sigma^1$, where $A \in SO(2)$ and $\sigma^j$, $j = 1, 2, 3$, are the Pauli matrices. If $\det F = 1$ then $F = \pm \mathbb{I}$. In the former case one can achieve $F = \sigma^3$ by a similarity transformation; in the latter case one can take $F = \mathbb{I}$, because the sign can be absorbed either into $E$ or into a rescaling of the generators. It turns out that $\det F = 1$ is the case relevant for the Ernst system, so for brevity we consider the possible $E$’s only for $\det E = 1$. The general solution of $E^2 = \epsilon \mathbb{I}$, $E \in SL(2, \mathbb{R})$, then is readily worked out. For $\epsilon = 1$ it leaves only $E = \pm \mathbb{1}$, for $\epsilon = -1$ one finds a two-parameter family of $E$’s; by a similarity transformation each of its members can be mapped onto $E = i\sigma^2$. In summary, we always take $F = \mathbb{1}$ in (2.9), which leaves only two inequivalent $*$-structures implemented by real $E$ matrices, namely

$$
\epsilon = 1 : \quad E = \mathbb{1} \quad \text{and} \quad \epsilon = -1 : \quad E = i\sigma^2. \quad (2.13)
$$

Note that with the second choice the $SL(2, \mathbb{R})$ basis transformations acting on the upper index are restricted to the $SO(2) \subset SL(2, \mathbb{R})$ subgroup leaving $E$ fixed.

From now on $\mathcal{D}_1$ will denote the algebra $WY(R, 0, \ast)$ with the $R$-matrix (2.5), the parameter $\beta = 0$, and the $*$-operation (2.9) with $F = \mathbb{1}$ and $E$ given by one of the matrices in (2.13). The case with $\epsilon = -1$ will turn out to be the one relevant for the Ernst system. Often we shall treat the $\epsilon = 1$ case as well in order to emphasize the crucial differences entailed by the seemingly minor flip. For convenient reference let us note explicitly

$$
\begin{align*}
\sigma W_a(\theta) &= W_a(\theta^* + i\hbar) \\
\sigma T^+ (\theta)^b_a &= E^b \epsilon (\theta^* - i\hbar)^b_a \\
\sigma T^- (\theta)^b_a &= \epsilon^b \epsilon (\theta^* - i\hbar)^b_a,
\end{align*}
$$

with $E$ as above, as the $*$-operation of $\mathcal{D}_1$. $SL(2, \mathbb{R})$ transformations acting on the lower index are $*$-automorphisms of $\mathcal{D}_1$, and similarly $SO(2)$ rotations acting on the upper index. Generic $SL(2, \mathbb{R})$ transformations acting on the upper index in contrast are automorphisms but do not preserve the $*$-structure. Rather the matrix $E$ transforms covariantly as $E \rightarrow g^{-1} E g$, $g \in SL(2, \mathbb{R})$.

In addition $\mathcal{D}_1$ admits some simple $*$-automorphisms given by $\theta$-dependent rescalings of the generators. Explicitly

$$
W_a(\theta) \rightarrow \omega(\theta) W_a(\theta), \quad T^\pm (\theta)^b_a \rightarrow \kappa^\pm(\theta) T^\pm (\theta)^b_a, \quad (2.15)
$$

are $*$-automorphisms of $\mathcal{D}_1$ provided the scalar functions $\omega(\theta), \kappa^\pm(\theta)$ obey

$$
\begin{align*}
\omega(\theta)^* &= \omega(\theta^* + i\hbar), \\
\kappa^\pm(\theta)^* &= \kappa^\mp(\theta^* + i\hbar), \quad \kappa^\pm(\theta) \kappa^\pm(\theta \pm i\hbar) = 1. \quad (2.16)
\end{align*}
$$
The last equation in particular entails that $\kappa^\pm(\theta)$ are $2i\hbar$ periodic functions.

2.2 Diagonalizing the center at the critical level $\beta = 0$

For $\beta = 0$ the quantum current is central. Explicitly

$$[D(\theta_1), W_a(\theta_2)] = 0 = [D(\theta_1), T^\mp(\theta_2)_a^b], \quad \text{Re}\, \theta_{21} \neq 0.$$  \hspace{1cm} (2.17)

The second equation is well known \cite{34}. The first one follows similarly from (2.4), which also explains the origin of the CDD-like sinh-prefactor in (2.5).

Since $D(\theta)$ is central it is natural to search for representations of $D_I$ on which $D(\theta)$ acts like a multiple of the unit operator. The Fock space representations of the Yangian double at the critical level inherited from a free field realization \cite{39, 35} do not have this property. Experience from other contexts suggests to search for appropriate representations in terms of functionals over the algebra $D_I$.

Specifically we consider vector functionals (called “T-invariant")

$$D_I \ni X \rightarrow \langle X \rangle = \langle \Theta | X | \Omega \rangle,$$  \hspace{1cm} (2.18)

built from a pair of vectors $| \Omega \rangle$ and $\langle \Theta |$ satisfying

$$T^+(\theta^b_a | \Omega \rangle = \gamma^+(\theta)_{b}^{a} | \Omega \rangle, \quad \langle \Theta | T^-(\theta)_{a}^{b} = \gamma^-(\theta)_{a}^{b} | \Theta \rangle,$$  \hspace{1cm} (2.19)

where $\gamma^\pm(\theta)$ are numerical matrices which according to (T1), (T2) carry one-dimensional representations of the Yangian algebra $Y(sl_2)$ respectively. This implies $\gamma^\pm(\theta) = \kappa^\pm(\theta)\gamma^\pm$ with $2i\hbar$-periodic scalar functions $\kappa^\pm(\theta)$ and constant matrices $\gamma^\pm$, satisfying the relations $\kappa^\pm(\theta)\kappa^\pm(\theta \pm i\hbar) = 1$ and $C_{cd}\gamma^\pm_{a}^{c} \gamma^\pm_{b}^{d} = C_{ab}$. It is natural to supplement the conditions on $\kappa^\pm(\theta)$ by the first condition in (2.16). A rescaling (2.13) of the $T^\pm$ generators then allows one to dispense of the $\theta$-dependence of the $\gamma^\pm$ matrices in (2.19). Henceforth we shall use constant $\gamma^\pm$ matrices. Hermiticity $\langle \sigma(X) \rangle = \langle X \rangle^*$ then imposes the conditions

$$[\gamma^+_{a}^{b} |_{a}^{*} = \gamma^{-}_{b}^{a} E_{b}^{a}, \quad [\gamma^{-}_{a}^{b} |_{a}^{*} = \epsilon \gamma^{+}_{a}^{b} E_{b}^{a}.$$  \hspace{1cm} (2.20)

\footnote{However, the relations (2.17) do not imply that the antisymmetric part of $D_{ab}(\theta)$ decouples algebraically. Defining

$$M_{ab}(\theta) := \frac{i}{2}(D_{ab}(\theta) + D_{ba}(\theta)),$$

the relations (2.4) (at $\beta = 0$) do not hold with $D_{ab}(\theta)$ replaced by $M_{ab}(\theta)$. We shall see in section 3 how to separate the symmetric part of $D_{ab}(\theta)$ in the classical limit.}
We shall mainly need the combination
\[
\Gamma^b_a := -C_{mn} \gamma^{-m}_{\alpha} C^{\alpha b}_k , \quad \text{satisfying}
\]
\[
(\Gamma^{-1})^b_a = -C^{\alpha b} C_{mn} \gamma^{-m}_{\alpha} \gamma^+ n , \quad [\Gamma^b_a]^* = \epsilon (\Gamma^{-1})^b_a ,
\]
\[
\Gamma^b_a + \epsilon [\Gamma^b_a]^* = \Gamma^a_a \mathbb{I} , \quad \det \Gamma = 1 . \tag{2.21}
\]

Observe that the value (2.18) of the central element \( D(\theta) \) is given by the trace \( \Gamma^a_a \) and reinforces the distinct features of the \( \epsilon = 1 \) and \( \epsilon = -1 \) involutions in (2.9):
\[
\langle \Theta | D(\theta) | \Omega \rangle = \Gamma^a_a \langle \Theta | \Omega \rangle , \\
\Gamma^a_a = \begin{cases} 
-2 , & \text{if } \epsilon = 1 \text{ and } \gamma^\pm \text{ real} \\
0 , & \text{if } \epsilon = -1 \text{ and } \gamma^\pm \text{ real}. \tag{2.22}
\end{cases}
\]

Natural choices are: \( \gamma^\pm = \mathbb{I} \) for \( \epsilon = 1 \), and \( \gamma^- = \mathbb{I} \), \( \gamma^+ = E \) for \( \epsilon = -1 \), in which case
\[
\Gamma^b_a = -\delta_a^b \text{ and } \Gamma^a_a = -E_a^a , \text{ respectively.}
\]

Clearly any \( T \)-invariant functional (2.18) is uniquely determined by its values on strings of \( W \)-generators, for which we introduce some extra notation
\[
f_A(\theta) := f_{a_n, a_{n-1}, \ldots, a_1}(\theta_N, \ldots, \theta_1) := \langle W_{a_n}(\theta_N) \ldots W_{a_1}(\theta_1) \rangle , \tag{2.23}
\]
where \( \text{Re} \theta_{ij} \neq 0, \ i \neq j, \ \theta = (\theta_N, \ldots, \theta_1), \ A = (a_N, \ldots, a_1) \). Sometimes also the shorthand \( f^{(v)} \) for the value of \( \langle \cdot \cdot \cdot \rangle \) on a string of \( n \ W \)-generators will be used. With these definitions one computes
\[
\langle W_{a_n}(\theta_N) \ldots W_{a_{k+1}}(\theta_{k+1}) D(\theta_0) W_{a_k}(\theta_k) \ldots W_{a_1}(\theta_1) \rangle = T(\theta_0|\theta)^B_A \langle W_{b_n}(\theta_N) \ldots W_{b_1}(\theta_1) \rangle , \tag{2.24}
\]
where \( T(\theta_0|\theta), \ \theta = (\theta_N, \ldots, \theta_1) \), is basically the familiar transfer matrix
\[
T(\theta_0|\theta)^B_A = \Gamma^a_a T^b_a(\theta_0 + i\hbar|\theta)^B_A , \\
T^b_a(\theta_0|\theta_N, \ldots, \theta_1)^{b_n \ldots b_1}_{a_n \ldots a_1} := R^{b_n}_{c_n a_n}(\theta_{N,0}) \ldots R^{b_1}_{c_1 a_1}(\theta_{1,0}) . \tag{2.25}
\]

Implicit in (2.23) are two important features reflecting the fact that \( D(\theta_0) \) is central: The right hand side of (2.23) is \( k \) independent and for fixed \( \theta \in \mathbb{C}^n \) and varying \( \theta_0 \in \mathbb{C} \) the \( T(\theta_0|\theta) \) form a one-parameter family of commuting matrices. Hence they can be simultaneously diagonalized and on the eigenvectors \( D(\theta_0) \) will act like a multiple of the unit operator. We are thus lead to restrict attention to those functionals (2.18) (or later a subset thereof) for which the functions (2.23) obey
\[
T(\theta_0|\theta)^B_A f_B(\theta) = \tau(\theta_0|\theta) f_A(\theta) . \tag{2.26}
\]
We note the following hermiticity properties of the $\mathcal{T}$-matrices and their eigenvalues:

\[
\left[\mathcal{T}(\theta_0|\theta)B_A\right]^* = \epsilon \mathcal{T}(\theta_0^* - 2i\hbar|\theta^*T + i\hbar)B_A^T, \\
\tau(\theta_0|\theta)^* = \epsilon \tau(\theta_0^* - 2i\hbar|\theta^*T + i\hbar), \tag{2.27}
\]

which follow from (2.11), (2.24) and the general hermiticity condition $\langle \sigma(X) \rangle = \langle X \rangle^*$. The notation is $\theta^T = (\theta_1, \ldots, \theta_n), A^T = (a_1, \ldots, a_n), \text{etc.}$ A further important property of the eigenvalues $\tau(\theta_0|\theta)$ of (2.26) is

\[
\tau(\theta_k - i\hbar|\theta) \tau(\theta_k - 2i\hbar|\theta) = 1, \quad k = 1, \ldots, N. \tag{2.28}
\]

To derive this, consider the matrix $\mathcal{T}(\theta_0|\theta)\mathcal{T}(\theta_0 - i\hbar|\theta)$, which describes the action of $D(\theta_0)D(\theta_0 - i\hbar)$ within the matrix elements (2.23). Then

\[
\left[\mathcal{T}(\theta_0|\theta)\mathcal{T}(\theta_0 - i\hbar|\theta)B_A\right]_{\theta_0 = \theta_k - i\hbar} = \delta_A^B, \quad k = 1, \ldots, N, \tag{2.29}
\]

as may be verified by direct computation from (2.25).

The matrix $\Gamma^b_a$ in (2.25) can be thought of as describing the deviation from the $SL(2, \mathbb{R})$ symmetry. In particular $\mathcal{T}(\theta_0|\theta)$ is invariant only under the subgroup of $SL(2, \mathbb{R})$ matrices obeying

\[
\Gamma^c_a \Lambda^b_c = \Gamma^b_a \Lambda^c_c. \tag{2.30}
\]

For $\epsilon = 1$ one may take $\Gamma = -\mathbb{I}$ and the condition is empty. For $\epsilon = -1$ the solutions of (2.30) can be seen to generate a maximal compact subgroup $SO(2) \subset SL(2, \mathbb{R})$. This holds irrespective of any further constraints on the matrix $\Gamma$, but for the reasons outlined in section 2.3 we take $\Gamma$ to be real. Of course $\Gamma$ still has to obey the constraints in (2.21) and the most general real solution to them may be parameterized as

\[
\Gamma(\varphi, \nu) := \begin{pmatrix} \text{sh}\varphi & \nu \text{ch}\varphi \\ -\frac{1}{\nu} \nu \text{ch}\varphi & -\text{sh}\varphi \end{pmatrix}, \quad 0 \neq \nu \in \mathbb{R}, \ \varphi \in \mathbb{R}. \tag{2.31}
\]

An $SL(2, \mathbb{R})$ basis transformation in (2.24) maps $\Gamma$ in (2.25) onto $g\Gamma g^{-1}, g \in SL(2, \mathbb{R})$. The transformed $\Gamma$ matrix still obeys the constraints in (2.21) and hence can be parameterized as in (2.31), however with different values for $\varphi$ and $\nu$. The $\Gamma$ matrices therefore define a conjugacy class in $SL(2, \mathbb{R})$. The transfer matrix $\mathcal{T}$ and its eigenvalues depend on the representative $g\Gamma g^{-1}$, while the eigenvalues are class-functions, i.e. depend only on the conjugacy class. In particular the eigenvalues of $\Gamma$ itself are $\pm i$, independent of $\varphi$ and $\nu$. For any fixed matrix $\Gamma$ the solutions of (2.30) then generate the missing compact conjugacy class of $SL(2, \mathbb{R})$. Explicitly the solutions of (2.30) are given by

\[
\Lambda^b_a(\phi) = \delta^b_a \cos \phi + \Gamma(\varphi, \nu)^b_a \sin \phi \quad \text{for} \quad \epsilon = -1, \quad 0 \leq \phi < 2\pi. \tag{2.32}
\]
As anticipated, they generate an \( SO(2) \) subgroup of \( SL(2, \mathbb{R}) \). Furthermore, they satisfy \( C^\alpha \Lambda^\alpha_\beta_\gamma (\phi) C^\gamma_\alpha (\phi) = \Lambda^\beta_\alpha (\phi) \). The invariance of \( \mathcal{T}(\theta_0|\theta) \) is expressed by

\[
\mathcal{T}(\theta_0|\theta)_A^C \Lambda^\beta_\alpha (\phi) \ldots \Lambda^\gamma_\alpha (\phi) = \Lambda^\beta_\alpha (\phi) \ldots \Lambda^\gamma_\alpha (\phi) \mathcal{T}(\theta_0|\theta)_C^B .
\] (2.33)

In particular, (2.33) allows one to break up the eigenvalue problem (2.26) into subsectors of fixed \( SO(2) \) charge; cf. appendix B.

The diagonalization (2.26) of \( \mathcal{T} \) of course is the object of the Bethe Ansatz. We shall be interested in solutions which are in addition equivariant with respect to the usual representation of the permutation group \( S_N \) on the space of tensor-valued functions. Whence we require

\[
f_A(\theta) = L_s(\theta)^B_A f_B(s^{-1}\theta) , \quad \forall s \in S_N .
\] (2.34)

It suffices to specify the action of the generators \( s_1, \ldots, s_{N-1} \) of \( S_N \):

\[
s_j(\theta_1, \ldots, \theta_N) = (\theta_1, \ldots, \theta_j, \theta_{j+1}, \ldots, \theta_N) , \quad 1 \leq j \leq N - 1 ,
\]

\[
L_{s_j}(\theta)_A^B = \delta^b_0 \ldots \delta^b_{j+2} R^b_{a_j+1} (\theta_{j+1,j}) \delta^b_{j+1} \ldots \delta^b_1 ,
\] (2.35)

and the product

\[
L_{s s'}(\theta) = L_s(\theta)L_{s'}(s^{-1}\theta) , \quad \forall s, s' \in S_N .
\] (2.36)

Not surprisingly, the system (2.26) is compatible with (2.34) provided the eigenvalues \( \tau(\theta_0|\theta) \) are completely symmetric in \( \theta = (\theta_0, \ldots, \theta_1) \), which we henceforth assume to be the case. It suffices to verify the asserted compatibility for the generators of \( S_N \); this in turn follows from the identities

\[
L_{s_j}(s_f(\theta)_A^B \mathcal{T}(\theta_0|\theta)_A^C L_{s_j}(\theta)_B^C = \mathcal{T}(\theta_0|s_f(\theta)_A^B , \quad 1 \leq j \leq N - 1 .
\] (2.37)

### 2.3 Further properties of the eigenvectors

The joint solutions (2.26) and (2.34) enjoy a number of other remarkable properties. First they are also solutions to an asymptotic form of the deformed Knizhnik-Zamolodchikov equations (KZE) \( ^0 \), where in the present conventions \( 2(1 - \beta/2\pi) \) parameterizes the level. The critical level \( \beta = 0 \) corresponds to the limiting case where these equations degenerate into an eigenvalue problem for mutually commuting matrices \( Q_{k} \), namely (see e.g. \( ^1 \))

\[
Q_k(\theta)^B_A f_B(\theta) = q_k(\theta) f_A(\theta) ,
\] (2.38)

with

\[
Q_k(\theta)^B_A := -\Gamma_c^d T^c_{a_k}(\theta_k|\theta_N, \ldots, \theta_{k+1})_{a_k \ldots a_{k+1}} T^b_{k}(\theta_k|\theta_{k-1}, \ldots, \theta_{1})_{a_{k-1} \ldots a_{1}} .
\] (2.39)
To see the relation to (2.26) note
\[
\mathcal{T}(\theta_0)\bigg|_{\theta_0 = \theta_k - i\beta} = Q_k(\theta)^B_A.
\] (2.40)

Hence (2.38) is a consequence of (2.26) with \( q_k(\theta) = \tau(\theta_k - i\beta) \). The converse is also true as we show in appendix B2. The \( \mathcal{T} \) eigenvalue problem (2.26) and the seemingly weaker \( Q_k \) eigenvalue problem (2.38) therefore are equivalent for \( \beta = 0 \).

For \( \beta \neq 0 \) the deformed KZE has been found to have an algebraic counterpart [30]:

\[
W_a(\theta + i\beta/\pi) = C_{mn}T^{-}(\theta + i\beta/\pi)^m W_k(\theta) T^{+}(\theta + i\beta/\pi - i\beta)^n C^{kl},
\] (2.41)

the matrix elements (2.23) automatically solve the deformed KZE with level \( i(2 - \beta/\pi) \), or equivalently a cyclic equation with shift parameter \( \beta \). Moreover the relation (2.41) endows the algebra with a “modular structure” characteristic for a quantum system at finite temperature \( 1/\beta \). Here we show that much of this structure survives in the limit \( \beta \rightarrow 0 \).

We begin by showing that the joint solutions of (2.26) and (2.34) also enjoy the following cyclic property
\[
f_A(\theta) = \Omega(\theta)^B_A f_B(\Omega^{-1}\theta).
\] (2.42)

Here \( \Omega \) is the Coxeter element \( \Omega = s_{N-1} \cdots s_1 \in S_N \), acting by cyclic permutation \( \Omega(\theta_N, \ldots, \theta_1) = (\theta_1, \theta_N, \ldots, \theta_2) \) on elements of \( \mathcal{C}^N \) and
\[
\Omega(\theta)^B_A := -\tau(\theta_N - 2i\beta) \Gamma_{a_1}^{b_1} \delta_b^{b_1} \cdots \delta_b^{b_2}.
\] (2.43)

Of course one could also have used (2.34) to obtain the relation \( f_A(\theta) = L_{\Omega}(\theta)^B_A f_A(\Omega^{-1}\theta) \), where \( L_{\Omega}(\theta) \) is the representation matrix of \( \Omega \in S_N \) defined by (2.35), (2.36). Consistency is ensured by the fact that
\[
L_{\Omega}(\theta)^B_A f_B(\Omega^{-1}\theta) = \Omega(\theta)^B_A f_B(\Omega^{-1}\theta),
\] (2.44)

on the joint solutions of (2.26) and (2.34). In other words, although \( \Omega(\theta) \) and \( L_{\Omega}(\theta) \) are distinct as matrices, they act in the same way on the solutions of (2.26) and (2.34). This follows from
\[
f_A(\theta) \equiv (2.28) \quad \tau(\theta_N - 2i\beta) \mathcal{T}(\theta_N - i\beta)^B_A f_B(\theta)
\]
\[
\equiv (2.23) \quad -\tau(\theta_N - 2i\beta) \Gamma_{a_1}^{d} T^{b_c}_{d} \tau(\theta_N - i\beta)_{\theta_{N-1}, \ldots, \theta_1}^{b_c \cdots b_1} f_B(\theta)
\]
\[
\equiv (2.35),(2.36) \quad -\tau(\theta_N - 2i\beta) \Gamma_{a_1}^{d} L_{\Omega^{-1}}(\Omega^{-1}\theta)^B_A f_B(\theta)
\]
\[
\equiv (2.43) \quad \Omega(\theta)^B_A L_{\Omega^{-1}}(\Omega^{-1}\theta)^C f_C(\theta),
\]
and together with (2.34) we obtain (2.42) and (2.44). One may check that consistently the matrix $T$ has the cyclic property
\[
T(\theta_0|\theta)_{AB} = T(\theta_0|\Omega^{-1}\theta)_{a_{n-1}...a_1|c} \Gamma_{a_k} (\Gamma^{-1})_{d}^{b_n} = \Omega(\theta)_{A'}^A T(\theta_0|\Omega^{-1}\theta)_{A'}^{B'} \Omega^{-1}(\theta)_{B'}^B .
\] (2.45)

From here one can show that the cyclic equation (2.42) is in fact equivalent to the $Q_k$ eigenvalue problem: Specializing (2.45) to $\theta_0 = \theta_k - i\hbar$ yields cyclicity relations for the $Q_k$ matrices and their eigenvalues
\[
Q_{k-1}(\Omega\theta)_{AB} = Q_k(\theta)_{a_{n-1}...a_1|c} \Gamma_{a_k} (\Gamma^{-1})_{d}^{b_n} ,
q_{k-1}(\Omega\theta) = \tau(\theta_k - i\hbar|\theta) = q_k(\theta) .
\] (2.46)

In particular modulo the exchange relations (2.34) the eigenvalue equation (2.38) for $k = n$, say, entails all others. On the other hand from the computation before (2.45) one also sees that the cyclic equation (2.42) and the $Q_N$ eigenvalue equation are equivalent.

A further property of the $Q_k$ matrices and their eigenvalues is unravelled by iterating the cyclic equation (2.42)
\[
\prod_{k=1}^N q_k(\theta) f_A(\theta) = (-)^N \Gamma_{a_n}^{b_n} ... \Gamma_{a_1}^{b_1} f_B(\theta) ,
[Q_1(\theta) ... Q_N(\theta)]_{AB} = (-)^N \prod_{j=1}^N \Gamma_{a_j}^{b_j} ,
\] (2.47)

where the second equation follows from the first one together with the fact that for generic $\theta$’s the matrices $Q_k$ have maximal rank. From (2.28) one finds $q_k(\theta)q_k(\theta - i\hbar) = \epsilon$. Finally the $q_k(\theta)$ have two properties that are readily seen only in the Bethe ansatz construction relegated to appendix B: For real $\theta$ they are pure phases and the phase is a gradient; see also [41, 42]. Explicitly, $q_k(\theta)^* = q_k(\theta^*)^{-1}$ and
\[
q_k(\theta) = e^{i\partial_k \Delta(\theta)} , \quad \text{with} \quad \partial_k \Delta(\theta) = \delta(\Omega^{N-k}\theta) , \quad k = 1, \ldots, N ,
\] (2.48)

where $\delta(\theta) = \delta(\theta^*) := -i \ln \tau(\theta_N - i\hbar|\theta)$. The fact that the $q_k(\theta)$ are pure phases is linked to having $\Gamma$ in (2.21) chosen to be real.

Real $\Gamma$-matrices are also natural because they allow one to introduce a quadratic form on the space of solutions to (2.26), (2.34). Consider the following quadratic form on the space of $V^{\otimes N}$-valued functions
\[
\langle f, g \rangle = \int d^N \theta f_A(\theta)^* C^{AB} g_B(\theta) .
\] (2.49)
It is manifestly hermitian \(\langle f, g \rangle^* = \langle g, f \rangle\) and \(SL(2, \mathbb{R})\) invariant. Equations (2.34), (2.38), and (2.42) give rise to symmetry operations which are unitary with respect to \(\langle \cdot, \cdot \rangle\) in the sense that

\[
\langle f, g \rangle = \langle f', g' \rangle \quad \text{for}
\]

\[
f_A' (\theta) = L_s (\theta)^B f_B (s^{-1} \theta) , \quad s \in S_N , \tag{2.50a}
\]

\[
f_A' (\theta) = q_k (\theta)^{-1} Q_k^B f_B (\theta) , \tag{2.50b}
\]

\[
f_A' (\theta) = \Omega (\theta)^B f_B (\Omega^{-1} \theta) . \tag{2.50c}
\]

Hence, for real \(\Gamma^b\), \(\langle \cdot, \cdot \rangle\) induces a quadratic form \(\langle \cdot, \cdot \rangle_{sol}\) on the space of functions \(f' = f\), which are the joint solutions of (2.34), (2.26) \(\iff\) (2.34), (2.38) \(\iff\) (2.34), (2.42).

Now let us return to the algebraic description. Since for \(\beta = 0\) (and only then) the KZE (2.38) and the cyclic equation (2.42) are consequences of the \(T\)-eigenvalue problem (2.26) and (2.34) one expects that in the algebra \(D_1\) extra algebraic relations hold which imply (2.38), (2.42) for the matrix elements (2.23). This is indeed the case and the relevant relations are

\[
C_{mn} T^{-}(\theta)^n_a W_k (\theta) T^+(\theta - i \hbar)_l^m C^{kl} = -W_a (\theta) D (\theta - i \hbar) , \tag{2.51a}
\]

\[
D (\theta - 2i \hbar) D (\theta - i \hbar) W_a (\theta) = W_a (\theta) . \tag{2.51b}
\]

The first relation can be verified simply by pushing \(T^{-}\) to the right using (TW). Equation (2.51b) then is required for compatibility with the \(\ast\)-operations (2.9). Indeed, applying \(\sigma\) to (2.51a) yields

\[
C_{mn} T^{-}(\theta - i \hbar)^n_k W_l (\theta) T^+(\theta - 2i \hbar)_l^m C^{kl} = -D (\theta - 2i \hbar) W_a (\theta) . \tag{2.52}
\]

Employing associativity and (T2) one recovers (2.51a) iff (2.51b) holds. Inserting now (2.51a) into the \(k\)-th position of a matrix element (2.23) precisely produces the \(k\)-th KZE-equation (2.38). Using the (WW) relations any of them can be seen to be equivalent to the cyclic equation (2.42). Finally the algebraic consistency relation (2.51b) amounts to (2.29) when used within the matrix elements (2.23).

Let us summarize the construction until here. For \(\beta = 0\), the quantum current lies in the center of the algebra \(D_1\). Searching for \(T\)-invariant functionals (2.18), (2.19) on which it acts like a multiple of the unit operator, leads to the eigenvalue problem (2.26). We were interested in those solutions which also enjoy the \(R\)-matrix exchange relations (2.34) and found that they have a number of remarkable bonus properties. Most notably, they satisfy the asymptotic form (2.38) of the deformed KZE, which we showed to be equivalent to the cyclic equation (2.42). Moreover on the space of permutation equivariant functions
the three requirements: \( \mathcal{T} \) eigenvalue problem (2.26), \( Q_k \) eigenvalue problem, and the cyclic condition (2.42) are all equivalent. The actual solution e.g. of the \( \mathcal{T} \) eigenvalue problem (2.26), (2.34) is relegated to appendix B. All this was for a fixed number \( n \) of variables \( \theta_j \). Next we show that the eigenvectors and eigenvalues for different \( n \) can naturally be arranged into sequences.

### 2.4 Sequences of eigenvectors

Apart from the non-trivial center, the algebra \( \mathcal{D}_1 = WY(R, 0, \ast) \) contains further two-sided ideals which should be divided out. The following relations (R) can be checked to arise in this way and we suspect them to be the only ones

\[
W_a(\theta + i\hbar) \cdot W_b(\theta) = \lambda(\theta) C_{ab} D(\theta - i\hbar),
\]

\[
C^{ab} W_a(\theta - i\hbar) \cdot W_b(\theta) = \lambda(\theta - i\hbar) D(\theta - 2i\hbar),
\]

where the function \( \lambda(\theta) \) satisfies \( \lambda(\theta)^\ast = \epsilon \lambda(\theta^\ast) \). Under the action of the \( \ast \)-automorphism (2.15) \( \lambda(\theta) \) changes according to

\[
\lambda(\theta) \longrightarrow \lambda(\theta) \frac{\omega(\theta) \omega(\theta + i\hbar)}{\kappa^{-}(\theta) \kappa^{+}(\theta + i\hbar)}.
\]  

(2.53)

We will later comment on specific choices for \( \lambda(\theta) \). The algebra \( \mathcal{D}_1 \) where in addition the relations (R) are imposed, is our complete dynamical algebra \( \mathcal{D} \).

The operator product \( W_a(\theta_1)W_b(\theta_2) \) turns out to be singular as \( \theta_{21} \to \pm i\hbar \), with a first order pole. The ‘·’ in (R) indicates a normal product defined roughly by taking the residue at the pole. A more precise definition will be given below in terms of its action within the \( T \)-invariant functionals. For the moment we are only interested in the algebraic properties of the relations (R).

A stronger, uncontracted version of the second relation is

\[
W_a(\theta - i\hbar) \cdot W_b(\theta) = \lambda(\theta - i\hbar) D_{ab}(\theta - 2i\hbar).
\]  

(2.54)

Let us momentarily denote these relations by (R1), (R2) and (R3), respectively. Recall that the exchange relations (TW) are valid also for \( \text{Re} \theta_{12} = 0 \), while for the (WW) relations these points are a-priori excluded. Consider the following formal extension of (WW) to \( \theta_{12} = i\hbar \),

\[
W_a(\theta + i\hbar) \cdot W_b(\theta) = -R_{ab}^{cl}(i\hbar) W_d(\theta) \cdot W_c(\theta + i\hbar),
\]  

(2.55)

\footnote{It appears to be a general rule that reducibility of a tensor product of fundamental representations is always caused by a pole in the \( R \)-matrix \[43, 44\]. According to (TW) the \( W_a \) play the role of intertwiners between two such representations.}
The extra minus sign of (2.55) with respect to (WW) is in accordance with the residue interpretation of the ‘·’-product. Then:

\[(R2) \implies (R3) \text{ by means of (2.51), (TW)},\]
\[(R1) \iff (R2), (R3) \text{ by means of (2.55).} \quad (2.56)\]

The first implication can be seen by starting from \(W_a(\theta - ih) \cdot W_b(\theta)\), replacing \(W_a(\theta - ih)\) in favor of \(-D(\theta - 2ih)^{-1}C_{mn}T^{-1}(\theta - ih)_a^mW_k(\theta - ih)T^+(\theta - 2ih)_a^nC^{kl}\), then using (TW) to move \(T^+\) to the right and finally applying (R2). Thus (R2) and (R3) are equivalent in \(\mathcal{D}\) and both are readily seen to be formally equivalent to (R1) once one is allowed to use (2.55). In order to avoid potential troubles with the formal identity (2.55), however, we postulate both equations (R) independently, keeping in mind that they are formally related by (2.55).

We can now restore topological concepts by calling a T-invariant functional (2.18) analytic if:

(i) The dependence of the values \(f^{(n)}\) on the parameters \(\theta_N, \ldots, \theta_1\) is locally analytic, possibly with branch points but without cuts.

(ii) For \(X, Y \in \mathcal{D}_1\) the expectation value \(\langle X W_a(\theta_1) W_b(\theta_2) Y \rangle\) has a simple pole at \(\theta_{12} = ih\) with residue \(\lambda(\theta_2) \langle X C_{ab} D(\theta_2 - ih) Y \rangle\); similarly \(\langle X C^{ab} W_a(\theta_1) W_b(\theta_2) Y \rangle\) has a simple pole at \(\theta_{12} = -ih\) with residue \(\lambda(\theta_2 - ih) \langle X D(\theta_2 - 2ih) Y \rangle\). In particular this defines the ‘·’ product in (R).

From now on we assume all functionals (2.18) to be analytic in this sense. The relations (R) then imply the \(N \rightarrow N-2\) recursive relations (II) given below for the functions (2.23). They link the eigenfunctions and eigenvalues of the eigenvalue problem (2.26), (2.34) in \(N\) and in \(N-2\) variables.

In summary we arrive at the following system of functional equations:

(I) \[\mathcal{T}(\theta_0|\theta)^B_A f_B(\theta) = \tau(\theta_0|\theta) f_A(\theta), \quad (a)\]
\[f_A(\theta) = R_{a_{k+1}a_k}(\theta_{k+1,k}) f_{a_{a_k}c_{a_k} \ldots c_{a_1}}(\theta_N, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_1), \quad (b)\]

where \(N\) is fixed, \(k = 1, \ldots, N-1\), and \(\mathcal{T}\) is the transfer matrix (2.25). The solutions of the functional equations (I) for \(N\) and \(N-2\) are linked by

(II) \[\text{Res}_{\theta_{k+1} = \theta_k + ih} f_A(\theta) = \lambda(\theta_k) \tau(\theta_k - ih|p_k \theta) C_{a_{k+1}a_k} f_{p_k A}(p_k \theta), \quad (a)\]
\[\text{Res}_{\theta_{k+1} = \theta_k - ih} C^{a_{k+1}a_k} f_A(\theta) = \lambda(\theta_k - ih) \tau(\theta_k - 2ih|p_k \theta) f_{p_k A}(p_k \theta), \quad (b)\]
\[\tau(\theta_0|\theta)\bigg|_{\theta_{k+1} = \theta_k \pm ih} = \tau(\theta_0|p_k \theta). \quad (c)\]

18
Here \( k = 1, \ldots, n - 1 \), and we adopted the notation
\[ p_k \theta = (\theta_n, \ldots, \theta_{k+2}, \theta_{k-1}, \ldots, \theta_1), \]
\[ p_k A = (a_n, \ldots, a_{k+2}, a_{k-1}, \ldots, a_1). \]
Equation (IIc) arises as consistency condition for (IIa,b) whenever their right hand sides are non-vanishing, using the properties
\[
C_{b_k+1b_k} T(\theta_0|\theta)^B_A|_{\theta_k+1=\theta_k} = C_{a_k+1a_k} T(\theta_0|p_k\theta)^{p_kB}_{p_kA},
\]
\[
C_{a_k+1a_k} T(\theta_0|\theta)^B_A|_{\theta_k+1=\theta_k} = C_{b_k+1b_k} T(\theta_0|p_k\theta)^{p_kB}_{p_kA},
\]
(2.59)
of the transfer matrix. Observe that specializing here \( \theta_0 \) to \( \theta_k - i\hbar \) yields relations reciprocal to (2.40). Similarly on the level of the eigenvalues (IIc) for \( \theta_0 = \theta_k - i\hbar \) gives
\[
q_k(\theta) |_{\theta_k+1=\theta_k} = \tau(\theta_k - i\hbar|p_k\theta),
\]
\[
q_{k+1}(\theta) |_{\theta_k+1=\theta_k} = \tau(\theta_k - 2i\hbar|p_k\theta).
\]
The restrictions of \( \tau(\theta_0|\theta) \) to \( \theta_0 = \theta_k - i\hbar \) and to \( \theta_{k+1} = \theta_k \pm i\hbar \) commute. For completeness let us also note the recursive equation implied by (2.54)
\[
\text{Res}_{\theta_k+1=\theta_k-i\hbar} f_A(\theta) = -\frac{1}{2}(\theta_k - i\hbar) C_{b_k+1b_k} Q_{k+1}(\theta)^B_A|_{\theta_k+1=\theta_k} f_{p_kB}(p_k\theta).
\]
(2.62)
Clearly, viewed as a projection, \( n \to n - 2 \), the recursive operation defined through (II) must have a large kernel. Simply counting the dimensions one expects \( 2^n \) eigenvectors on the left hand side of (II) to be mapped onto only \( 2^{n-2} \) on the right hand side. Conversely given a solution of (I) in \( n - 2 \) variables it is non-trivial that the equations (II) can be ‘integrated’ to a function of \( n \) variables within the class of solutions of (I). The explanation why this is possible stems from the underlying algebraic framework.

Recall the notation \( D \) for the algebra \( D_1 = WY(R, 0, \ast) \) supplemented by the relations (R). As in section 2.2 we can consider “T-invariant” functionals, now over the algebra \( D \), i.e.
\[
\langle \; \rangle : D \longrightarrow \mathbb{C}, \quad X \longrightarrow \langle X \rangle,
\]
\[
\langle T^-(\theta_0)^b_a X \rangle = \gamma^{-b}_a \langle X \rangle, \quad \langle X T^+(\theta_0)^b_a \rangle = \gamma^{+b}_a \langle X \rangle,
\]
(2.63)
keeping all other notations. Again, such a functional will be completely determined by its values on strings of \( W \)-generators. By construction, it gives rise to a solution of the system of functional equations (I), (II) with the identification \( f_A(\theta) := \langle W_A(\theta) \rangle \), and vice versa. Symbolically we arrive at a one-to-one correspondence:

| T-invariant Functional over \( D \) | \( \longleftrightarrow \) | Sequence of Solutions of (I), (II) |

19
3. Semi-classical limit and phase space of the Ernst system

Here we study the semi-classical limit of the regular part of the dynamical algebra and show that essentially the same Poisson algebra describes the classical phase space of the Ernst system. In particular this lends a physical interpretation to the variables $T^{\pm}(\theta)^b_a$ and $W_a(\theta)$.

3.1 Semi-classical limit of the dynamical algebra

We begin with the classical limit of the algebra $D_I$ and assume that semi-classically the operator products in $D_I$ behave like a Moyal product, i.e.

$$XY = X^{cl} Y^{cl} - \frac{i\hbar}{2} \{X^{cl}, Y^{cl}\} + O(\hbar^2),$$

where $X^{cl}, Y^{cl}$ are the corresponding functions on phase space. Usually we shall drop the superscript “cl” when no confusion is possible. Upon expansion in $\hbar$ the exchange relations (T1), (TW), (WW) then provide the symplectic structure of a classical Poisson algebra:

$$\{T^{\pm}(\theta_1)^d_a, T^{\pm}(\theta_2)^c_b\} = \frac{1}{\theta_{12}} \left( T^{\pm}(\theta_1)^d_e T^{\pm}(\theta_2)^c_j \Omega_{ae}^{cf} - \Omega_{ef}^{dc} T^{\pm}(\theta_1)_a^{c} T^{\pm}(\theta_2)_b^{f} \right), \quad (3.1a)$$

$$\{T^{+}(\theta_1)^d_a, T^{-}(\theta_2)^c_b\} = \frac{1}{\theta_{12}} \left( T^{+}(\theta_1)^d_e T^{-}(\theta_2)^c_j \Omega_{ae}^{cf} - \Omega_{ef}^{dc} T^{+}(\theta_1)_a^{c} T^{-}(\theta_2)_b^{f} \right), \quad (3.1b)$$

$$\{T^{\pm}(\theta_1)^f_a, W_b(\theta_2)\} = \frac{1}{\theta_{12}} T^{\pm}(\theta_1)^f_c W_d(\theta_2) \Omega_{ab}^{cd}, \quad (3.1c)$$

$$\{W_a(\theta_1), W_b(\theta_2)\} = \frac{1}{\theta_{12}} W_c(\theta_1) W_d(\theta_2) \Omega_{ab}^{cd}, \quad (3.1d)$$

with $\Omega_{ab}^{cd}$ from (2.8). In particular the operators $T^{\pm}(\theta)^b_a$ have turned into classical $2 \times 2$ matrices which due to (T2) have unit determinant

$$C_{cd} T^{\pm}(\theta)^c_a T^{\pm}(\theta)^d_b = C_{ab}. \quad (3.2)$$

The $\ast$-structure (2.14) translates into the classical hermiticity relations

$$[T^{+}(\theta)^b_a]^\ast = E^b_a T^{-(\theta^*\!)}_a, \quad [T^{-}(\theta)^b_a]^\ast = \epsilon E^b_a T^{+(\theta^*\!)}_a, \quad [W_a(\theta)]^\ast = W_a(\theta^*). \quad (3.3)$$

The classical analogue of the matrix $D_{ab}(\theta)$ from (2.3) is given by

$$D_{ab}(\theta) = C_{cd} T^{-(\theta)^c_a} T^{+(\theta)^d_b}, \quad \text{with} \quad [D_{ab}(\theta)]^\ast = \epsilon D_{ba}(\theta^*). \quad (3.4)$$
Separating the symmetric hermitian and the anti-symmetric part for $\epsilon = -1$ yields

$$D_{ab}(\theta) = -i\mathcal{M}_{ab}(\theta) - \frac{1}{2}C_{ab}D(\theta), \quad D(\theta) = C^{ab}D_{ab}(\theta),$$

$$\det \mathcal{M}_{ab}(\theta) + \frac{1}{4}D(\theta)^2 = 1 = -\det D_{ab}(\theta).$$

(3.5)

Since for $\epsilon = -1$ $D(\theta)$ is purely imaginary, one has in particular $\det \mathcal{M}(\theta) \geq 1$.

We proceed by showing that the antisymmetric part of $D_{ab}(\theta)$ (and hence also $\det \mathcal{M}_{ab}$) is not a dynamical degree of freedom of the Poisson algebra (3.1)–(3.3). One indication is the fact that it Poisson-commutes with the generators $T^\pm(\theta)_a^b$ and $W_a(\theta)$. More specifically, there exists an automorphism of the Poisson algebra such that $D(\theta)$ is mapped onto a prescribed numerical constant, e.g. $D(\theta) \equiv 0$ for $\epsilon = -1$, yielding $\det \mathcal{M}(\theta) = 1$. To find the automorphism consider first the following rotation:

$$T^+(\theta)_a^b \longrightarrow y_+ T^+(\theta)_a^b + y_- T^-(\theta)_a^b,$$

$$T^-(\theta)_a^b \longrightarrow y_+ T^+(\theta)_a^b + y_- T^-(\theta)_a^b,$$

$$W_a(\theta) \longrightarrow W_a(\theta),$$

(3.6)

which obviously is a homomorphism of the Poisson algebra (3.1). Note, that the parameters $y_\pm$ may depend on $\theta$ here. If we further specify them to be of the form

$$y_{\pm} = \cos \alpha_{\pm} \sqrt{1 - \frac{1}{2}D(\theta)\sin(2\alpha_{\pm})}, \quad y_{\pm\mp} = \frac{\sin \alpha_{\pm}}{\sqrt{1 - \frac{1}{2}D(\theta)\sin(2\alpha_{\pm})}},$$

with $\alpha_{\pm} = \alpha_{\pm}(\theta)$, the map (3.6) extends to an automorphism of the full structure (3.1), (3.2). The condition

$$\alpha_{+}(\theta) = \epsilon \alpha_{-}(\theta)^*,$$

(3.7)

finally ensures compatibility with hermiticity (3.3). For $\epsilon = 1$ thus $\alpha_{+} = \alpha_{-} =: \alpha$ must be real-valued, as is $D(\theta)$. For $\epsilon = -1$ both $D(\theta)$ and $\alpha_{+} = \alpha_{-} =: i\alpha$, $\alpha \in \mathbb{R}$ are purely imaginary. Under this automorphism the antisymmetric part of the matrix $D_{ab}(\theta)$ transforms as

$$D(\theta) \longrightarrow D(\theta)_\alpha := \frac{D(\theta) - 2\sin(2\alpha)}{1 - \frac{1}{2}D(\theta)\sin(2\alpha)} \quad \text{for } \epsilon = 1$$

$$D(\theta) \longrightarrow D(\theta)_\alpha := \frac{D(\theta) - 2i\sinh(2\alpha)}{1 - \frac{1}{2}D(\theta)\sinh(2\alpha)} \quad \text{for } \epsilon = -1.$$

(3.8)

For $\epsilon = 1$ there are two disjoint orbits of $D(\theta) \in \mathbb{R}$ under the Poisson automorphism (3.6), the interval $[-2, 2]$ and its complement. In particular, the fixpoints of (3.8) at $D = \pm 2$ are fake and correspond to a noninvertible map (3.6). For $\epsilon = -1$, on the other hand, the Poisson automorphism (3.6) acts transitively on $D(\theta) \in i\mathbb{R}$. This means starting from any non-zero value of $D(\theta)$, this automorphism can be used to define new generators
of the Poisson algebra (3.1)–(3.3) for which the new \( D(\theta) \equiv D(\theta) a \) vanishes. This fact ensures that one can always work with symmetric \( SL(2, \mathbb{R}) \) matrices

\[
\mathcal{M}_{ab}(\theta) := iD_{ab}(\theta) \equiv -\frac{1}{2} \arcsinh\left( \frac{1}{2} iD(\theta) \right) = \mathcal{M}_{ba}(\theta) .
\]  

(3.9)

Further one can assume \( \mathcal{M}(\theta) \) to have positive trace. This is because a vanishing trace would contradict the positive determinant, so that \( \text{Tr} \mathcal{M}(\theta) \) must be either positive or negative. Since \( T^{\pm}(\theta)_a^b \rightarrow \pm T^{\pm}(\theta)_a^b, W_a(\theta) \rightarrow W_a(\theta) \) is a *-automorphism of the Poisson algebra (3.1), (3.2), (3.3) one can take \( \text{Tr} \mathcal{M}(\theta) > 0 \). The Poisson brackets of \( \mathcal{M}(\theta) \) follow from the classical limit of (2.4d)

\[
\{ \mathcal{M}_{ab}(\theta_1), \mathcal{M}_{cd}(\theta_2) \} = \frac{1}{\vartheta_{12}} \left( \Omega_{ac}^{mn} \mathcal{M}_{mb}(\theta_1)\mathcal{M}_{nd}(\theta_2) + \mathcal{M}_{am}(\theta_1)\mathcal{M}_{cn}(\theta_2)\Omega_{bd}^{mn} \right) \quad (3.10)
\]

\[
+ \frac{1}{\vartheta_{12}} \left( \mathcal{M}_{am}(\theta_1)\Omega_{bc}^{mn} \mathcal{M}_{nd}(\theta_2) + \mathcal{M}_{mb}(\theta_1)\Omega_{ad}^{mn} \mathcal{M}_{cn}(\theta_2) \right) .
\]

The Poisson algebra (3.10) turns out to provide a direct link to the phase space of the Ernst system which will be detailed in section 3.2.

We focused on the regular part \( \mathcal{D}_1 \) of the dynamical algebra here because the various operations invoked: “Taking the semi-classical limit”, “Taking the residue at \( \vartheta_{12} = \pm i\hbar \)” and “Applying the automorphism (3.6)” are mutually non-commuting. In particular taking the semi-classical limit of the relation (R) would require further specifications. However since we introduced topological concepts only on the level of the matrix elements, not for the algebra, it is convenient to discuss the semi-classical limit of the recursive structure directly on the level of the functional equations (I), (II) and their solutions; cf. section 4.

### 3.2 Phase space of the classical Ernst system

The classical phase space of the Ernst system can be described in various ways. A non-redundant parameterization is in terms of gauge invariant symmetric \( SL(2, \mathbb{R}) \) matrices \( \mathcal{M}_{ab}(\theta) \), which can be viewed as the “scattering data” from each of which a classical solution can be reconstructed. These matrices can be shown, starting from the canonical Poisson brackets associated with the action (1.1), to carry the Poisson structure (3.10) [43].

Hence in this non-redundant parameterization there is a direct correspondence between the phase space of the Ernst system and the subsector (3.3), (3.10) of the Poisson manifold emerging in the classical limit of the dynamical algebra \( \mathcal{D} \).

In the quantum algebra we saw in section 2.2 that the antisymmetric part of \( D_{ab}(\theta) \) does not decouple algebraically. This enforced to work with the bigger algebra generated by \( T^{\pm}(\theta)_a^b, W_a(\theta) \), and to implement the decoupling in terms of an eigenvalue problem. We now show that a Poisson algebra essentially equivalent to (3.1) also naturally emerges in
the Ernst system. Moreover, this lends a physical interpretation to the variables \( T^\pm(\theta)_a \) and \( W_a(\theta) \) in (3.1).

We begin by recalling that the scalar sector of the Ernst system is given by an \( SL(2, \mathbb{R}) \) valued matrix \( V^m_a(x) \) which essentially contains the vierbein components of the compactified dimensions. The model is invariant under global \( SL(2, \mathbb{R}) \) and local \( SO(2) \) transformations

\[
V^m_a(x) \mapsto g^b_a V^m_b(x) h^m_n(x), \quad \text{with} \quad g^b_a \in SL(2, \mathbb{R}), \quad h^m_n(x) \in SO(2).
\]  

(3.11)

This invariance has its roots in the four-dimensional theory, \( SL(2, \mathbb{R}) \) descending from linear diffeomorphisms in the “compactified” coordinates, \( SO(2) \) being a remnant of the corresponding part of the local Lorentz group. The bilinear combination

\[
M_{ab}(x) = V^m_a(x) V^m_b(x) \delta_{mn},
\]

(3.12)

is invariant under local \( SO(2) \) transformations and corresponds to the metric components in the ‘compactified’ dimensions. Note that the symmetric \( \delta_{mn} \) symbol appearing here is invariant only under the \( SO(2) \) subgroup of \( SL(2) \).

The dynamics of the Ernst system is captured by a Lax pair \[3, 4\] whose spectral parameter – in contrast to the flat space integrable systems – depends explicitly on the space-time coordinates, see \[3\] for a review. For definiteness, we focus on the case of cylindrical gravitational waves. In particular, the worldsheet then has Lorentzian signature and can be covered by coordinates \( x = (t, r) \), \( t \in \mathbb{R} \) and \( r > 0 \). For the description of the linear system, light-cone coordinates \( x^\pm = t \pm r \) are most convenient. It is then given by

\[
\partial_\pm \hat{V}(x; \gamma) = \hat{V}(x; \gamma)L_\pm(x; \gamma),
\]

(3.13)

with

\[
L_\pm(x; \gamma) = Q_\pm(x) + \frac{1 \mp \gamma}{1 \pm \gamma} P_\pm(x).
\]

Here, \( Q_\pm \) and \( P_\pm \) are the compact and non-compact components of the \( sl(2, \mathbb{R}) \)-valued current \( \mathcal{V}^{-1} \partial_\pm \mathcal{V} \) lying in \( so(2) \) and its orthogonal complement, respectively. The spectral parameter \( \gamma \) is given by the following explicit function

\[
\gamma(x; \theta) = \frac{1}{r} \left( t - \theta - \sqrt{(\theta - t)^2 - r^2} \right),
\]

(3.14)

of the 2D coordinates \( x = (t, r) \) and a constant \( \theta \) which may be understood as the underlying constant spectral parameter of (3.13).

---

\[3\] We use abstract index notation in this section to indicate the transformation behavior of the objects: Indices \( a, b, \ldots \) from the beginning of the alphabet refer to covariance under \( SL(2, \mathbb{R}) \), whereas indices \( k, l, \ldots \) from the middle of the alphabet refer to covariance under local \( SO(2) \) rotations.
The associated monodromy matrices $U(r, r', t | \theta)$ are obtained in the usual way as path ordered exponentials of the Lax connection

$$U(r, r', t | \theta) := \tilde{V}^{-1}(t, r; \gamma(t, r; \theta)) \tilde{V}(t, r'; \gamma(t, r'; \theta))$$

$$= \mathcal{P} \exp \int_r^{r'} dz \, L_1(t, z; \gamma(t, z; \theta)),$$

which are unique functionals of the connection $L$ ordered exponentials of the Lax connection $\theta$ lives on the twofold covering of the complex $U$ plane. The parameter $\theta$ spatial infinity – corresponding to the classical sector of gravitational waves with regularity of the currents at the spatial boundaries and time-independence at the notation. Restoring it momentarily of (3.14) varying on the real $\theta$-axis while $z$ runs from $r$ to $r'$. The monodromy matrix $U(r, r', t | \theta)$ hence is well defined for $\theta \notin \mathbb{R}$. Under (3.11) it transforms as

$$U(r, r', t | \theta)_m^a \mapsto h^{-1}(t, r)^k_m U(r, r', t | \theta)^k_i h(t, r')^i_m,$$

in particular, it is invariant under the global $SL(2, \mathbb{R})$ transformations.

Assuming regularity of the currents at the spatial boundaries and time-independence at spatial infinity – corresponding to the classical sector of gravitational waves with regular Ernst potential on the symmetry axis – we define the following objects for real values of the parameter $\theta$:

$$T^+(\theta)^b_a := i \lim_{\delta \to 0} V_0(t)^n_a U^k_n(0, \infty, t | \theta + i \delta) (V_\infty)_c^l \delta_{kl} C^{cb},$$

$$T^-(\theta)^b_a := \lim_{\delta \to 0} V_0(t)^n_a U^k_n(0, \infty, t | \theta - i \delta) (V_\infty)_c^l \delta_{kl} C^{cb},$$

$$W^m_a(\theta) := V_0(t)^n_a U^m_n(0, |\theta-t|, t | \theta),$$

with

$$V_0(t) := V(0, t), \quad V_\infty := V(\infty, t) = V(\infty).$$

The $T^\pm(\theta)$ are conserved and the $W(\theta)$ are conserved up to a local gauge transformation

$$\partial_t T^\pm(\theta)^b_a = 0$$

$$\partial_t W^m_a(\theta) = \begin{cases} W^m_a(\theta) Q^m_{\pm}(t, \theta-t)^m_n & \text{for } \theta > t \\ W^m_a(\theta) Q^m_{\pm}(t, \theta-t)^m_n & \text{for } \theta < t \end{cases}.$$ 

The fact that the last equation is not 2D covariant is due to the explicit appearance of the time $t$ in the integration boundary of $W^m_a(\theta)$. Observe however that according to equation (3.18) the time derivative of $W^m_a(\theta)$ is continuous in $\theta$ since $Q^m_{1}(t, r=0) = 0$, as follows from the field equations derived from (2.13) and regularity of the vielbein $V^m_a$ on the symmetry axis. As indicated we shall usually suppress the time-dependence of $W^m_a(\theta)$ in the notation. Restoring it momentarily $W^m_a(\theta) \sim W_a(t, \theta)$ one finds for the time evolution (e.g. for $\theta > t$)

$$W(t, \theta) = W(0, \theta) e^{i \phi(t, \theta) \sigma^2}, \quad \phi(t, \theta) = -i \int_0^t ds \, Tr[Q_+(s, \theta-s) \sigma^2],$$

$$W(t, \theta) = W(0, \theta) e^{i \phi(t, \theta) \sigma^2}, \quad \phi(t, \theta) = -i \int_0^t ds \, Tr[Q_+(s, \theta-s) \sigma^2],$$

(3.19)
where $\sigma^2$ is the Pauli matrix and $\phi(t, \theta)$ is real for real $\theta$. Note, that in contrast to $W(t, \theta)$, the function $\phi(t, \theta)$ is defined by integrating over a null line in spacetime, and may hence not be considered as canonical object on a fixed time slice. In particular, this makes it difficult to compute its Poisson brackets.

The matrices $T^{\pm}(\theta)$ satisfy (3.2) whereas the $W(\theta)$ obey

$$C_{mn} W^m_a(\theta) W^n_b(\theta) = C_{ab}, \quad C^{ab} W^m_a(\theta) W^n_b(\theta) = C^{mn}. \quad (3.20)$$

Under the symmetry transformations (3.11), the matrices (3.17) behave as

$$T^{\pm}(\theta)^b_a \mapsto g^b_d T^{\pm}(\theta)^d_c (g^{-1})^c_b , \quad (3.21)$$

$$W^m_a(\theta) \mapsto g^d_a W^m_d(\theta) h^m_n(t, |\theta - t|).$$

Clearly any gauge invariant quantity build from these monodromy matrices will be time independent. A gauge invariant object of particular interest is the bilinear combination

$$\mathcal{M}_{ab}(\theta) = i T^{-} (\theta)^c_a T^{+} (\theta)^d_b C_{dc} = W^m_a(\theta) W^n_b(\theta) \delta_{mn}. \quad (3.22)$$

The second equality follows from the definition (3.17) in the limit $t \to \infty$ and shows that the matrix $\mathcal{M}_{ab}$ is symmetric in the indices $a, b$, and has positive trace. In particular, for the $T^{\pm}(\theta)^b_a$ defined by (3.17), the combination $D(\theta) = C^{ab} C_{cd} T^{-} (\theta)^c_a T^{+} (\theta)^d_b$ vanishes automatically. The decomposition of $\mathcal{M}(\theta)$ into the product of $T^{\pm}$ corresponds to the Riemann-Hilbert decomposition; the matrices $T^{\pm}(\theta)$ are holomorphic in the upper resp. lower half of the complex $\theta$-plane. It may further be shown that

$$\mathcal{M}_{ab}(\theta \in \mathbb{R}) = M_{ab}(t = \theta, r = 0), \quad (3.23)$$

i.e. this matrix coincides with the physical scalar fields on the axis $r = 0$ \[.45\]. From the viewpoint of the inverse scattering transform, equation (3.23) is a striking result. Usually, the scattering data associated with a given solution “live at” timelike infinity and have no direct relation to the original field variables. In contrast, for the Ernst system, equation (3.23) means that the scattering data live on the symmetry axis $r = 0$ and are directly related to the original vielbein variables. The second Gauss-like decomposition of $\mathcal{M}$ into the bilinear product of $W$’s in (3.22) therefore corresponds to the decomposition of the metric into a spectral-transformed vielbein. As anticipated by the notation, the matrices (3.22) can be identified with (3.9) and the decomposition in (3.22) may be viewed as the classical analogue of (2.54).

The fact that the scattering data have an interpretation in terms of the original field variables imposes an interesting causality constraint, which also clarifies the structure of the monodromy matrices $W^m_a(\theta)$: Monodromy matrices of the form (3.17b) (path ordered integrals over finite space intervals without any specification of conditions on
the physical fields at the boundary of the interval) do not arise in the usual flat space integrable systems. The raison d’être in the Ernst system is the space-time dependence of the spectral parameter (3.14). The Lax connection (3.13) degenerates at certain points in space-time though the physical currents remain regular. This happens at the spatial boundaries \( r = 0, \infty \), but also, curiously, at \( r = |\theta - t| \), – a point by no means distinguished in the physical space-time. However this point does have special significance for the causal past of the point \((\theta, 0)\) on the symmetry axis. For given \( t_0 \) and \( \theta \) (with \( t_0 < \theta \), say) consider the intersection of the causal past of \((\theta, 0)\) with the \( t = t_0 \) surface, i.e. the interval \([0, \theta - t_0]\).

According to causality in the \((t, r)\) Lorentzian space, the vielbein on the symmetry axis at time \( t = \theta \) should be a functional of the initial data on the interval \([0, \theta - t_0]\) only, which due to the range of integration in (3.17b) it indeed is; cf. Fig. 1.

![Figure 1: The spectral transformed vielbein \( W^m_a(\theta) \)](image)

We proceed by studying the hermitian structure and the Poisson brackets of the monodromy matrices (3.17). The hermiticity \( U(r, r' | \theta)^* = U(r, r' | \theta^*) \) implies

\[
[T^\pm(\theta)^b_a]^* = \mp i T^\mp(\theta^*)^c_a M^\infty_{cd} C^{db} ,
\]

\[
[W^m_a(\theta)]^* = W^m_a(\theta^*) ,
\]

where the constant matrix \( M^\infty_{cd} \equiv (V^m_{\infty})^m_a (V^m_{\infty})^n_b \delta_{mn} \) defines a positive definite bilinear form on \( SL(2, \mathbb{R}) \). Due to the explicit appearance of \( M^\infty_{cd} \) in (3.24), this hermitian structure is not invariant but transforms covariantly under (3.21). With

\[
E^b_a = - i M^\infty_{ac} C^{cb} ,
\]

we recover (3.3), i.e. the classical limit of (2.9). Fixing the \( SL(2, \mathbb{R}) \) symmetry (3.21) by setting \((V^m_{\infty})^m_a = \delta^m_a \) corresponds to the choice \( E = i\sigma^2 \) in (2.13). Having done so, one is still left with \( SL(2, \mathbb{R}) \) basis transformations acting on the lower index

\[
T^\pm(\theta)^b_a \mapsto g^d_a T^\pm(\theta)^b_d , \quad W^m_a(\theta) \mapsto g^d_a W^m_d(\theta) .
\]

(3.25)
The linear transformations (3.27) provide a *-automorphism of the Poisson algebra even after (3.21) has been “eaten up” by fixing the *-structure (e.g. $\mathcal{M}_a^c = \delta_{ac}$) and even when working with SO(2) gauge fixed $W^m_a(\theta)$ matrices. We shall return to the distinction between (3.21) and (3.26) in section 3.3.

The Poisson brackets of the monodromy matrices $T^\pm$ coincide with (3.1a,b), where again the coordinate dependence of the spectral parameter $\gamma$ plays a crucial role in the computation [45]. Evaluating the general formula from [45], one similarly obtains for the matrices $W^m_a(\theta)$

\[
\{T^\pm(\theta_1)_a^e, W^m_b(\theta_2)\} = \frac{1}{\theta_{12}} T^\pm(\theta_1)_c^e W^m_d(\theta_2) \Omega_{cd}^{ab} \\
- \frac{1}{2\theta_{12}} W^l_b(\theta_2) T^\pm(\theta_1)_c^e U^k_a(\theta_2 | \theta_1) (U^{-1})^c_l(\theta_2 | \theta_1) E^m_k E^n_l,
\]

\[
\{W^m_a(\theta_1), W^m_b(\theta_2)\} = \frac{1}{\theta_{12}} W^c_m(\theta_1) W^d_m(\theta_2) \Omega_{cd}^{ab}
- \frac{\chi(\theta_1)}{2\theta_{12}} W^l_b(\theta_2) W^m_c(\theta_1) U^k_a(\theta_2 | \theta_1) (U^{-1})^c_j(\theta_2 | \theta_1) E^m_k E^n_l
- \frac{\chi(\theta_2)}{2\theta_{12}} W^k_a(\theta_1) W^m_c(\theta_2) U^l_b(\theta_1 | \theta_2) (U^{-1})^c_j(\theta_1 | \theta_2) E^m_k E^n_l,
\]

with the step function $\chi(\theta) = \text{sign}(\theta)$, and the matrix $E^m_n = \delta_{nk} C^{kn} = (i\sigma^2)^m_n$ here playing the role of an SO(2) invariant tensor. Further

\[
U^k_a(\theta_2 | \theta_1) := V_0(t)^m_a U^k_m(0, |\theta_2-t|, t | \theta_1), \\
(U^{-1})^k_a(\theta_2 | \theta_1) := U^m_k(|\theta_2-t|, 0, t | \theta_1) V_0(t)^m_a C_{mn} C^{ba}.
\]

Evidently the Poisson structure is not closed but contains the transition matrices $U(\theta_2 | \theta_1)$, etc. on the right hand side. Nevertheless the Jacobi identities can be verified, and (3.26) defines a consistent Poisson structure on the phase space of the Ernst system. As remarked earlier, the combination $D(\theta) = C^{ab} C_{cd} T^-(\theta)_a^e T^+(\theta)_b^c$ here vanishes automatically.

Consistency thus requires that $D(\theta)$ also Poisson commutes with $W^m_a(\theta)$ (with respect to the brackets (3.26)), – which indeed can be verified to be the case. Moreover it can be shown that (3.26) induces the Poisson structure (3.10) for the gauge invariant phase space functions $\mathcal{M}_{ab}(\theta)$, i.e. the same as the somewhat simpler Poisson structure (3.1) obtained from the classical limit of our dynamical algebra. Thus, up to redundancies, (3.26) can be regarded as equivalent to (3.1c,d) in the sense that both induce the same structure on the space of objects invariant under the SO(2) gauge transformations in (3.11).4

\[4\] A similar point of view is e.g. usually adopted to study the symplectic structures on the moduli space of flat connections on Riemann surfaces [10].
U-terms in (3.26) can be viewed as being a remnant of the gauge dynamics (3.19). The problem is that (3.26) can not readily be rewritten as a Poisson bracket structure on the initial data \( W^m_a(0, \theta) \). This is because although the time evolution (3.19) can be viewed as a gauge transformation, the transformation is a nonlocal functional of the dynamical variable \( Q(x) \), so that at some point non-equal-time Poisson brackets would have to be evaluated. In principle, however, we view the \( W_a(\theta) \) in (3.1) as being gauge fixed or gauge invariant and time-independent versions of a linear combination of \( W^1_a(\theta) \) and \( W^2_a(\theta) \). We have not been able so far to properly map the Poisson brackets (3.26) onto (3.1) for such a combination. Our main argument that it should be possible is, that on gauge invariant objects like \( M(\theta) \) both induce the same Poisson structure.

One can also check that the counting of degrees of freedom works out: There are two types of redundancies in (3.26). First (3.26) is invariant under the symmetry transformations (3.21). In particular the local gauge transformations \( W^m_a(\theta) \rightarrow W^n_a(\theta) h^m_n(t, |\theta - t|) \) effectively remove one degree of freedom. In addition (3.26) has a one-dimensional Poisson center generated by the determinant \( \det W^m_a(\theta) \). Thus (3.26) contains only two physical degrees of freedom for the \( W^m_a(\theta) \) fields, just as (3.1c,d).

Taking the equivalence of (3.26) and (3.1c,d) for granted, the recursive relations (R) can be viewed as a quantum implementation of the identity (3.22) and the determinant condition (3.20). To see this set

\[
W^\pm_a(\theta) = W^m_a(\theta) (\Upsilon^{-1})^\pm_m, \quad (3.27)
\]

where \( \text{Ad} \Upsilon : SL(2, \mathbb{R}) \rightarrow SU(1, 1) \) is the isomorphism (A.11) or (B.2). Then the \( W^\pm_a(\theta) \) are complex fields transforming as \( W^\pm_a(\theta) \rightarrow W^\pm_a(\theta) e^{\pm i\phi(t, |\theta - t|)} \) under \( SO(2) \) gauge transformations with \( h = \cos \phi \mathbb{1} + \sin \phi \mathbb{E} \). Further, they obey

\[
W^\pm_a(\theta) W^\mp_b(\theta) = \frac{1}{2} (\mathcal{M}_{ab}(\theta) \mp C_{ab}), \quad (3.28)
\]

using the definition (3.22) and the determinant condition (3.20). In the quantum theory (3.28) will turn into a singular operator product. Parallel to (R) one can stipulate

\[
W^\pm_a(\theta - i\hbar) \cdot W^\mp_b(\theta) = \frac{i}{2} D_{ab}(\theta - 2i\hbar), \quad (3.29)
\]

while all ‘\cdot’ products of \( W^+ \) with itself and of \( W^- \) with itself are supposed to vanish. For simplicity we set \( \lambda = i \) here and only noted the counterpart of (R2), i.e. (2.54); the interplay with the other versions is analogous to (R1) – (R3). The obvious \( * \)-operation is \( \sigma W^\pm_a(\theta) = W^\pm_a(\theta^* + i\hbar) \). Finally, consider the linear combinations

\[
V_a(\theta) = e^{i\phi(\theta)} W^+_a(\theta) + e^{-i\phi(\theta)} W^-_a(\theta), \quad (3.30)
\]

with parameter \( \phi(\theta) = \phi(t, \theta) \), obeying \( \phi(\theta)^* = \phi(\theta^* + i\hbar) \). If in addition we take \( \phi(\theta) \) to be \( i\hbar \)-periodic, the ‘\cdot’ products (3.29) etc. for \( W^\pm_a(\theta) \) imply (R) for \( V_a(\theta) \) with \( \lambda = i \). If
the $W^\pm$ in (3.30) are rescaled by $\omega(\theta)$, obeying $\omega(\theta)^* = \omega(\theta^* \pm i\hbar)$, the same holds with $\lambda(\theta) = i\omega(\theta)\omega(\theta + i\hbar)$, a form we shall use later. In principle we could have developed the entire formalism of section 2 for an enlarged quantum algebra with a pair of $W_a^\pm(\theta)$ generators and exchange relations like those in $D_1$ for both of them. The advantage of working with the linear combinations (3.30) is that the recursive relations are simpler because the same rule applies to each pair of $V$-generators. We do not expect substantial difficulties in deriving a system of functional equations analogous to (I), (II) for matrix elements of mixed strings of $W^\pm$ generators. Observe however that according to (3.29) the recursive structure in any case determines only the ‘propagation’ of the gauge invariant parts. For example, matrix elements with only $W^+$ would only be constrained by the analogue of the functional equations (I).

As remarked before we prefer to analyze the semi-classical limit directly on the level of the functional equations (I), (II) and their solutions. Observe however that comparing the formal $\hbar \to 0$ limit of (3.29) with (3.28) and (3.3) one can match the expressions by taking $D(\theta) = \pm 2i$. Clearly the Poisson algebra (3.26) admits an automorphism analogous to (3.6). Starting from $D = 0$ one can achieve $D_\alpha = \pm 2i$ by taking $\text{sh}2\alpha = \pm 1$. The conjecture that the Poisson structures (3.1) and (3.26) are fully equivalent, translates into one concerning the status of the field $\phi(\theta)$. As long as it is regarded as an independent parameter, the linear combination (3.30) will obey Poisson brackets of the form (3.1), modified by the extra $U$-terms. By allowing $\phi(\theta)$ to become dynamical (as in (3.13)) one may hope to render the $V_a(\theta)$ time independent and either gauge fixed or gauge invariant. At the same time the $U$-terms should disappear, making the correspondence $V_a(\theta) \sim W_a(\theta)$ precise.

Summarizing, in the classical limit the dynamical algebra $D_1$ gives rise to a phase space parametrized by symmetric $SL(2,\mathbb{R})$ matrices $M$ equipped with the Poisson structure (3.10) and the two alternative Riemann-Hilbert and Gauss decompositions into matrices $T^\pm$ and $W$, respectively. The resulting Poisson algebra (3.1)–(3.3) essentially coincides with that deduced from the fundamental Poisson brackets in the classical Ernst system; admittedly yet with some loose ends.

### 3.3 Symmetry breaking

In retrospect we can now also highlight some aspects of our quantum formulation. As noted in section 2.1, the antisymmetric part of $D_{ab}(\theta)$ does not decouple algebraically within $D_1$. Thus in order to avoid that the quantum theory has an extra dynamical operator field $D(\theta)$, one is forced to go to the critical level $\beta = 0$, where $D(\theta)$ becomes central. However an automorphism of the form (3.6), (3.8) no longer exists in the quantum theory; the spectrum of $D(\theta)$ on the state space described by the matrix elements (2.23) is a characteristic feature of the system. Remarkably the state space exhibits a (“spontaneous”) breakdown of the classical $SL(2,\mathbb{R})$ invariance that is a is a remnant of the original four dimensional diffeomorphism invariance in the Killing coordinates: As we have seen before
for $\epsilon = -1$ the eigenvalue problem (2.26) is invariant only under the SO(2) subgroup, while the algebra itself and the spectrum of $D(\theta)$ still are fully $SL(2, \mathbb{R})$ invariant. This suggests that, in contrast to the prevalent assumptions in many approaches to quantum gravity, diffeomorphism invariance is not sacrosanct; it might be broken for dynamical reasons.

In view of the possible implications it seems worthwhile to critically reexamine the line of argument and to see whether, in the context of the present framework, the conclusion can be avoided. To address the issue, it is convenient to introduce shorthands for the different $SL(2, \mathbb{R})$ actions involved. Let $SL(2, \mathbb{R})_D$ be the $SL(2, \mathbb{R})$ action (3.21) inherited via (3.17) from the linear diffeomorphisms in the Killing coordinates. Let $SL(2, \mathbb{R})_L$ be the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.21) inherited via (3.11) from the linear diffeomorphisms in the Killing coordinates. Let $SL(2, \mathbb{R})_L$ be the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index. Finally denote by $SL(2, \mathbb{R})_U$ the $SL(2, \mathbb{R})$ action (3.25) acting by basis transformations on the lower index.
of the *-structure (3.3) or (2.9) with respect to the upper index. Since the definition of \( D(\theta) \) involves a contraction over an upper index pair, one might argue, that the violation of \( SL(2, \mathbb{R}) \) invariance already enters at this point. However, this is not the case. Though not invariant, the *-structure (3.3) is \( SL(2, \mathbb{R}) \) covariant. An \( SL_2 U \) rotation of the form
\[
W_a(\theta) \rightarrow W_a(\theta), \quad T^\pm(\theta)_b^a \rightarrow g_b^b T^\pm(\theta)_b^a
\]
provides an automorphism of the algebra \( D \) under which the *-structure (2.9) transforms covariantly as in (2.12). But in the hermiticity equations (2.11) the matrix \( E \) drops out; both the operator \( D(\theta) \) and its hermiticity condition (2.11) are invariant under both the \( SL_2 U \) and the \( SL_2 L \) actions of the \( SL(2, \mathbb{R}) \) algebra. However its matrix elements (2.23) are not! It is clear that the phenomenon disappears in the classical limit because then \( D(\theta) \) can be set to zero. In summary we conclude that the classical \( SL_2 L \) symmetry, and by (3.31) therefore the \( SL_2 D \) invariance, being a remnant of the 4D diffeomorphisms in the Killing coordinates, is “spontaneously” broken in the quantum theory.

Because of the coupling to gravity, no conflict with the Coleman-Mermin-Wagner theorem \[47, 48\] arises. In lattice formulations usually also the compactness of the global symmetry group is assumed in order to ensure the existence of the regularized functional integral. If, as in Coleman’s version \[48\], the existence of the quantum field theory and the Noether current is postulated, the result should hold also for non-compact groups. However the Poincaré invariance and the cluster property are essential for the argument. For the Ernst system the former is manifestly absent and the latter would at least require a re-interpretation. In the statistical mechanics context, one way to look at the Mermin-Wagner theorem is as a tug of war between entropy and energy. For a flat space sigma-model in 2D the entropy always wins, forcing the system to remain in a disordered state even at low temperatures. From this perspective the above result indicates that the coupling to gravity changes this entropy–energy balance such that a breaking of the \( SL(2, \mathbb{R}) \) symmetry becomes possible.

Concerning the worldsheet diffeomorphism invariance, it is clear that if 2D conformal invariance is broken, so is invariance under diffeomorphisms. Classically the constraints induced by fixing the conformal gauge \( h_{\mu \nu} = e^{2\sigma} \eta_{\mu \nu} \) in the action (1.1) are
\[
T_{\pm \pm} = 2 \partial_+ \rho \partial_- \sigma - \rho \text{Tr}[P_\pm P_\pm] .
\]

(3.32)

Here \( \sigma = -\frac{1}{2} \ln |\partial_+ \rho \partial_- \rho| \) is a conformal scalar and \( P_\mu \) is the coset part of the current appearing in (3.13). Their Poisson brackets form two commuting copies of the Virasoro-Witt algebra. Further (3.32) can be checked to generate infinitesimal conformal transformations on gauge invariant objects, otherwise an additional \( SO(2) \) gauge transformation is induced. In particular, for the conserved charges (3.14) and Weyl coordinates \( \rho = r \) one has \( \{ T_{\pm \pm}(t, r) , T^\pm(\theta)_b^a \} = 0 \), and modulo some technical subtleties a similar equation holds for \( W_{a^m}(\theta) \). Classically the (weakly) vanishing of the constraints is compatible with the equation of motion for \( \sigma \). The latter can be integrated to express \( \sigma \) in terms of \( \rho \) and \( P_\mu \). Off-shell the essential dynamical features of \( \sigma \) should be captured by the quantum counterparts of its Poisson brackets with \( T^\pm(\theta)_b^a \) and \( W_{a^m}(\theta) \). The latter turn out to be
remarkably simple

\[
\{\sigma(t, 0), T^\pm(\theta)_a^b\} = \partial_\theta T^\pm(\theta), \quad \{\sigma(t, 0), W^m_a(\theta)\} = \partial_\theta W^m_a(\theta), \quad (3.33)
\]

assuming that \(\sigma(t, r)\) vanishes for \(r \to \infty\). The obvious quantum counterparts are

\[
e^{i\lambda K}T^\pm(\theta)_a^b e^{-iK\lambda} = T^\pm(\theta + \lambda)_a^b, \quad e^{i\lambda K}W_a(\theta)e^{-iK\lambda} = W_a(\theta + \lambda). \quad (3.34)
\]

That is, translations in \(\theta\) are unitarily implemented, and the generator \(K\) is the quantum counterpart of the conformal factor in the 2D metric.

4. Exact matrix elements

We return now to the functional equations (I), (II) of section 2. As outlined, its solutions are conjectured to describe exact matrix elements in the quantum theory, without the need for any renormalization. We begin by describing a solution algorithm for (I), (II).

4.1 Solution algorithm

Before turning to the solution procedure, let us note some simple structural features of (I) and (II). Clearly a solution of (I) is determined only up to multiplication by a scalar function completely symmetric in \(\theta_n, \ldots, \theta_1\). The recursive equations (II) cut down this ambiguity to scalar, symmetric functions solving

\[
P(\theta)|_{\theta_{k+1}=\theta_k \pm i\hbar} = P(p_k\theta). \quad (4.1)
\]

In other words, if \((f^{(n)})_{n \geq N_0}\) (with \(n\) even or odd depending on the starting member) is a sequence solving (I), (II) and \((P^{(n)})_{n \geq N_0}\) is a sequence solving (4.1), then the sequence obtained by pointwise multiplication, \((P^{(n)}f^{(n)})_{n \geq N_0}\), again is a solution of (I), (II). According to (IIc), the eigenvalues sequences are an example of a sequence (4.1) but there are many others, e.g. power sums \(\sum_{j=1}^N e^{s\pi \theta_j/\hbar}\), with \(s\) odd, or any smooth scalar function of \(\theta_N + \ldots + \theta_1 - 2(t_1 + \ldots + t_\Lambda)\), where \(t_\alpha\) are the Bethe roots of appendix B. Usually one will be interested in solutions of (I), (II) which are ‘minimal’ in the sense that one cannot ‘naturally’ split off a solution of (4.1). Apart from these obvious ambiguities we expect that associated with each starting member \(f^{(N_0)}\) there is basically only a single sequence solving (I), (II).
To actually find solutions of (I), (II) we make an ansatz of the form

\[ f_A(\theta) = \tilde{f}_A(\theta) c^{(n)}(\theta) \prod_{k>l} \psi(\theta_{kl}) \prod_{k>l} \psi(\theta_{kl} - i\hbar), \quad (4.2) \]

where \( c^{(n)}(\theta) \) absorbs the \( \lambda(\theta) \) dependence and \( \psi(\theta) \) satisfies

\[ \psi(\theta) = r(\theta) \psi(-\theta), \]
\[ \psi(-\theta) \psi(\theta - i\hbar) = -1. \quad (4.3) \]

The rationale for (4.3) is that the first relation effectively replaces (Ib) by exchange relations with a rational \( R \)-matrix, the second one turns out to achieve the same for the recursive relations (IIa,b). The solution of (4.3) analytic in the strip \(-\hbar/2 < \text{Im} \theta < \hbar/2\) is given by

\[ \psi(\theta) = \tanh \frac{\pi \theta}{2\hbar} \exp \left\{ i \int_0^\infty \frac{dt}{t} \frac{h(t)}{\cosh^2 t} \sin \frac{t}{2\hbar} (i\hbar + 2\theta) \right\}, \]

where \( h(t) = 2 e^{-t/2} + e^{-t} \). \quad (4.4)

The functional equations (4.3) are readily verified by means of the integral representation

\[ r(\theta) = -\exp \left\{ i \int_0^\infty \frac{dt}{t} h(t) \sin \frac{\theta}{\hbar} \right\}. \quad (4.5) \]

In addition \( \psi(\theta) \) has the following properties: It has a simple pole at \( \theta = -i\hbar \) with residue \( \hbar \psi_0 \), where \( \psi_0 := i \psi(i\hbar) = \lim_{\delta \to 0} \delta / \psi(h\delta / \pi) \approx 1.54678 \). The only real zero is at \( \theta = 0 \). Further it obeys \( \psi(\theta)^* = -\psi(\theta^*) \) and \( \psi(\theta) \to i \) for \( \theta \to \pm \infty \), cf. Fig. 3.

It remains to specify the prefactors \( c^{(n)}(\theta) \) in (4.2), supposed to absorb the \( \lambda(\theta) \) dependence and overall constants. For definiteness we assume \( \lambda(\theta) \) to be of the form

\[ \lambda(\theta) = i \omega(\theta) \omega(\theta + i\hbar), \quad (4.6) \]

where \( \omega(\theta) \) satisfies \( \omega(\theta)^* = \omega(\theta^* + i\hbar) \). The overall \( i \) in (4.6) takes care of the \( \epsilon = -1 \) hermiticity property. Comparing with (2.53) one sees that (4.6) amounts to fixing the normalization of the \( T^\pm \) (i.e. \( \kappa^\pm(\theta) \equiv 1 \)) while still allowing for automorphisms (2.13) with a nontrivial \( \omega(\theta) \). The restriction to \( \kappa^\pm(\theta) = 1 \) is not indispensible; however since their \( \hbar \to 0 \) limits can only be \( \pm 1 \) anyhow, not much generality is lost. Conversely any prescribed \( \lambda(\theta) \) enjoying suitable analyticity and fall-off properties can be written in the factorized form (4.6). With \( \lambda(\theta) \) of the form (4.6) we then take

\[ c^{(n)}(\theta) = c^{(N_0)}(\theta_0) \ldots \omega(\theta_{N_0+1}) \left( \frac{-1}{\psi_0} \right)^{(N-N_0)/2}, \quad N \geq N_0. \quad (4.7) \]
Finally we enter with the ansatz (4.2) into the functional equations (I) and (II). The equations (I) translate into

\[ \mathcal{T}(\theta_0|\theta) f_A(\theta) = \tau(\theta_0|\theta) f_A(\theta), \]  

\[ T(\theta_0|\theta) = R_{a_{k+1}a_k} c d \mathcal{T}_{a_{k+1}a_1}(s_k \theta), \]

where

\[ R(\theta)_{ab} := -\frac{1}{\theta + i\hbar} [\theta \delta_a^c \delta_b^d - i\hbar \delta_a^d \delta_b^c]. \]

\[ \mathcal{T}(\theta_0|\theta) := \prod_{j=1}^N (\theta_j - 2i\hbar) r(\theta - \theta_j) \mathcal{T}(\theta_0|\theta). \]  

The redefinition of \( \mathcal{T} \) renders the components of \( \mathcal{T}(\theta_0|\theta) \) polynomial in the \( \theta_j - i\hbar \). The associated eigenvalues \( \tau(\theta_0|\theta) \) turn out to be rational functions of \( \theta_j, \theta_0 \) and the Bethe roots, cf. (B.18). The solutions \( \tilde{f} \) in \( N \) and \( N-2 \) variables are now linked by

\[ \tilde{f}_A(\theta) \bigg|_{\theta_{k+1} = \theta_k + i\hbar} = \mathcal{T}(\theta_k - i\hbar|p_k \theta) \prod_{j \neq k+1,k} (\theta_{jk} + i\hbar) C_{a_{k+1}a_k} \tilde{f}_{p_k A}(p_k \theta), \]  

\[ C^{a_{k+1}a_k} \tilde{f}_A(\theta) \bigg|_{\theta_{k+1} = \theta_k - i\hbar} = -2\mathcal{T}(\theta_k - 2i\hbar|p_k \theta) \prod_{j \neq k+1,k} (\theta_{jk} + 2i\hbar) \tilde{f}_{p_k A}(p_k \theta), \]  

\[ \mathcal{T}(\theta_0|\theta) \bigg|_{\theta_{k+1} = \theta_k + i\hbar} = \theta_{k0}(\theta_{k0} - 2i\hbar) \mathcal{T}(\theta_0|p_k \theta), \]  

\[ \mathcal{T}(\theta_0|\theta) \bigg|_{\theta_{k+1} = \theta_k - i\hbar} = (\theta_{k0} - i\hbar)(\theta_{k0} - 3i\hbar) \mathcal{T}(\theta_0|p_k \theta), \]  

\[ (4.10c) \]
where again (4.10c) applies only when the right hand sides of (4.10a,b) are nonvanishing. Clearly any one of the eqs (4.10c) implies the other, we noted them both for symmetry. The system of eqs (4.8), (4.10) for \( f \) is now ‘quasirational’ in the sense that all ingredients are rational functions in \( \theta_j, \theta_0 \) and the Bethe roots, while the Bethe roots themselves are algebraic functions of the \( \theta_j \).

The proper hermiticity requirements are

\[
\begin{align*}
[\mathcal{F}_A(\theta)]^* &= \mathcal{F}_{AT}(\theta^{*T} + i\hbar), \\
[\mathcal{F}_{e;\sigma_n...\sigma_1}(\theta)]^* &= \mathcal{F}_{e;\sigma_1...\sigma_n}(\theta^{*T} + i\hbar).
\end{align*}
\] (4.11)

The first one is just the general hermiticity requirement expressed in terms of \( \mathcal{F} \). The second one is the transcription into the “charged basis” introduced in appendix B1. In brief one can switch to a basis in \( V^{\otimes N} \) on which the matrices \( \Gamma \) and \( \Lambda \) in (2.31), (2.32) are diagonalized. The components of \( f^{(N)} \) in the new basis are denoted by \( f_{e;\sigma_n...\sigma_1}(\theta) \), where \( \sigma_j \in \{ \pm 1 \} \) and \( e = \sigma_N + \ldots + \sigma_1 \). Under a \( U(1) \approx SO(2) \) rotation (2.32) they transform as \( f_e(\theta) \rightarrow e^{ie\phi}f_e(\theta) \), i.e. with charge \( e \). It is easy to see that the functional equations (I), (II), or (4.8), (4.10), then split up into decoupled sectors of fixed charge \( e \in \{ N, N-2, \ldots, -N \} \), making the charged basis particularly useful for their analysis. In group theoretical terms the basis transformation is related to the isomorphism \( SL(2, \mathbb{R}) \rightarrow SU(1,1) \).

Let us now address the problem of solving the functional equations (4.8), (4.10), in the charged basis. As indicated eq. (4.8a) can be solved by means of the Bethe ansatz; some details are provided in appendix B. In upshot an eigenvector \( w_{e;\sigma_n...\sigma_1}(\theta) \) is constructed by introducing a set of auxiliary variables \( t_1, \ldots, t_{\Lambda}, \Lambda = (N - e)/2 \), such that a candidate eigenvector \( w_{e;\sigma_n...\sigma_1}(t(\theta)|\theta) = w_{e;\sigma_n...\sigma_1}(t_1, \ldots, t_{\Lambda}|\theta_N, \ldots, \theta_1) \) turns into a proper eigenvector, provided the parameters are turned into judiciously chosen functions of the \( \theta_j \)'s, i.e.

\[
w_{e;\sigma_n...\sigma_1}(\theta) = w_{e;\sigma_n...\sigma_1}(t(\theta)|\theta), \quad t(\theta): \text{solution of BAE (B.19)}.
\] (4.12)

Here BAE are the Bethe ansatz equations whose solutions are (complicated) algebraic functions \( t_\alpha(\theta), \alpha = 1, \ldots, \Lambda \), completely symmetric in the \( \theta_j \)'s. The candidate eigenvector can be chosen to be polynomial in the \( \theta_j \) and the auxiliary parameters \( t_\alpha \). A Bethe eigenvector \( w_e(\theta) \) solving (4.8a) will not necessarily obey (4.8b). By (B.31) we know however that

\[
\mathcal{F}_{e;\sigma_n...\sigma_1}(\theta) = \phi_e(\theta) \prod_{k>1} i \theta_{kl} + i\hbar w_{e;\sigma_n...\sigma_1}(\theta),
\] (4.13)

solves both (4.8a) and (4.8b). Here \( \phi_e(\theta) \) is a completely symmetric function in \( \theta_j \) not constrained by (4.8a,b) in any way.

In general a solution of (4.8a,b) will not satisfy (4.10). The generic expression (4.13) however contains two pieces of information left unspecified so far. First the choice of a specific Bethe eigenvector and second the choice of a specific symmetric function \( \phi_e(\theta) \). The obvious way to proceed is to try adjusting these two ingredients such that also the recursive
equations (4.10) are satisfied. As explained in appendix B the choice of the proper Bethe eigenvector amounts to a choice of the proper Bethe root. For the charge \( e \) sector one typically expects \( \binom{n}{\Lambda} \) independent eigenvectors with distinct eigenvalues (though there may be some degeneracies). We expect that typically one and only one of them in addition satisfies the recursive equations (4.10a,b) in which case the associated eigenvalue satisfies (4.10c). Moreover the sequential eigenvector and eigenvalue can already be identified on the level of the Bethe roots. We call a solution of the BAE (B.19) in \( n \) variables \( \theta_j \) a “sequential” \( \Lambda \)-tuple of Bethe roots, if it is real for real \( \theta_j \) and satisfies

\[
\left. t_\alpha(\theta) \right|_{\theta_{k+1}=\theta_k+i\hbar} = t_\alpha(p_k \theta), \quad \alpha = 1, \ldots, \Lambda - 1, \\
\left. t_\Lambda(\theta) \right|_{\theta_{k+1}=\theta_k\pm i\hbar} = \theta_k \pm \frac{i\hbar}{2}, \quad k = 1, \ldots, n-1,
\]

(4.14)

where on the right hand side of the first equation a sequential \( (\Lambda-1) \)-tuple in \( n-2 \) variables occurs. Observe that the combination \( e = N-2\Lambda \) is invariant under \( N \mapsto N-2, \; \Lambda \mapsto \Lambda-1; \) as indicated it can be identified with the conserved \( U(1) \simeq SO(2) \) charge carried by \( f_e^{(n)} \). In other words the “sequential” solutions to the BAE can naturally be arranged into sequences of fixed charge \( e \) where consecutive members are linked by (4.14). In appendix B we show that such sequential Bethe roots exist and that the eigenvalues \( \tau(\theta_0|\theta) \) associated with a sequence satisfy the recursive equations (IIc). Upon restriction to \( \theta_{k+1} = \theta_k \pm i\hbar \) the corresponding eigenvectors should then always become proportional to the right hand side of (4.10a,b).

Having fixed the Bethe eigenvector in (4.13) to be one associated with a sequential Bethe root, the only freedom left resides in the symmetric function \( \phi_e(\theta) \). The aim now is to adjust it such that eqs (4.10a,b) hold identically. Since in the \( n \rightarrow n-2 \) recursion step the Bethe roots \( t_\alpha, \alpha < \Lambda, \) enter via \( \tau(\theta_0|p_k \theta) \), it is natural to require that \( \phi_e(\theta) \) is a rational function in \( \theta_n, \ldots, \theta_1 \) and \( t_1, \ldots, t_{\Lambda-1} \), invariant under shifts \( \theta_j \rightarrow \theta_j + \text{const} \). Schematically,

\[
\phi_e(\theta) = \phi_e(\theta^*)^* : \quad \text{symmetric, shift invariant,}
\]

\[
\text{rational in } \theta_n, \ldots, \theta_1 \text{ and } t_1, \ldots, t_{\Lambda-1}.
\]

(4.15)

We have verified these features for \( n \leq 4 \) by explicit computation and are confident that they are generic.

In summary we arrive at the following solution procedure for the recursive functional equations (I), (II) or their ‘quasirational’ form (4.8), (4.10):

(a) For given \( n \) and given \( SO(2) \) charge \( e = N-2, N-4, \ldots, \) \( -N+2 \) compute the trial eigenvector \( w_e(t|\theta) \) via (B.17). It contains \( \Lambda = \frac{1}{2}(N-e) \) Bethe parameters and has \( \binom{n}{\Lambda} \) independent components.
(b) Verify that for any Bethe root satisfying (4.14) the restrictions of $f_e(\theta)$ in (4.13) to $\theta_{k+1} = \theta_k \pm i\hbar$ are proportional to the right hand sides of (4.10a,b). Multiply by a scalar function $\phi_e(\theta)$ of the type (4.15) such that (4.10a,b) holds.

(c) When feasible compute the sequential $\Lambda$-tuple of Bethe roots explicitly. Repeat the procedure for $n \mapsto n+2$.

The explicit form of the sequential Bethe roots will usually be available only for small $n$; its existence is ensured by the results in appendix B. Together (a) – (c) produces a solution to the ‘quasirational’ system (4.8), (4.10). Inserting into (4.2) then yields a solution of the original system of functional equations (I), (II). For clarity’s sake let us emphasize that we have not proven step (b) to work always. However we verified it on sufficiently non-trivial examples to conjecture with some confidence that it does. A plausibility argument of course stems from the very algebraic construction used to derive (I), (II). A $T$-invariant functional is an “$N$-independent” object, once it exists at all it will automatically produce a non-terminating sequence of functions $f^{(n)}$ solving (I), (II). The case $e = \pm n$ is excluded in (a) because those solutions of (4.8) necessarily vanish upon restriction to $\theta_{k+1} = \theta_k \pm i\hbar$. However they will naturally serve as a starting member $f^{(n_0)}$ to a sequence. The above procedure then yields semi-infinite sequences $(f^{(n)})_{n \geq |e|}$, which we expect to be basically uniquely determined by their starting member $f^{(|e|)}$. To illustrate the scheme, we have collected the first few members of the charge $e = 0, \pm 1, \pm 2$ sequences in appendix C.

### 4.2 Semi-classical limit

Let us now consider the classical limit of these solutions. Recalling from (4.2), (4.13)

$$f_{e;\sigma_n...\sigma_1}(\theta) = c^{(n)}(\theta) \phi_e(\theta) w_{e;\sigma_n...\sigma_1}(\theta) \prod_{k>l} i \psi(\theta_{kl}) \frac{i \psi(\theta_{kl})}{\theta_{kl}^2 + \hbar^2}, \quad (4.16)$$

this amounts to studying the $\hbar \to 0$ limit of the various ingredients. For the transcendental function $\psi(\theta)$ in (4.4) one simply has (cf. Fig. 1)

$$\psi(\theta) = i + O(\hbar). \quad (4.17)$$

For the scalar prefactor $\phi_e(\theta)$ the explicit results of appendix C suggest that

$$\phi_e(\theta) = \phi_e^{cl}(\theta) + O(\hbar), \quad (4.18)$$

where $\phi_e^{cl}(\theta)$ is a ratio of homogeneous polynomials in $\theta_n, \ldots, \theta_1$. However provided $\phi_e(\theta)$ depends on the Bethe roots (which will generically be the case) the corresponding factor will no longer be symmetric in $\theta_n, \ldots, \theta_1$. This feature is related to the curious behavior
of the Bethe roots in the $\hbar \to 0$ limit discussed in appendix B4. Recall for real $\theta_N, \ldots, \theta_1$ the sequential Bethe roots are real-valued and completely symmetric in all variables. For $\hbar \to 0$ however one has

$$t_\alpha(\theta) = \theta_{j(\alpha)} + O(\hbar^2), \quad \alpha = 1, \ldots, \Lambda,$$

(4.19)

where the index $j(\alpha) \in \{1, \ldots, n\}$ of the preferred variable $\theta_{j(\alpha)}$ is determined by the choice of branch in the Bethe root and the relative size of the $\theta$ variables. Further $j(\alpha) \neq j(\beta)$ for $\alpha \neq \beta$. The Bethe vector (B.17) contains an explicit power of $\hbar^\Lambda = \hbar^{(N-e)/2}$ (since the matrix operator $B(t|\theta)$ is $O(\hbar)$). Taking out this explicit power we define

$$w_{e;\sigma_N\ldots\sigma_1}^{cl}(\theta) = \lim_{\hbar \to 0} \hbar^{-\Lambda} w_{e;\sigma_N\ldots\sigma_1}(\theta).$$

(4.20)

As shown in appendix B4, $w_{e;\sigma_N\ldots\sigma_1}^{cl}(\theta)$ has only one non-vanishing component

$$w_{e;\sigma_N\ldots\sigma_1}^{cl}(\theta) = 0 \quad \text{unless} \quad (\sigma_N, \ldots, \sigma_1) = (\epsilon_N, \ldots, \epsilon_1),$$

(4.21)

where $(\epsilon_N, \ldots, \epsilon_1) \in \{\pm\}^N$ is a particular sign configuration of charge $e$. The sign pattern $(\epsilon_N, \ldots, \epsilon_1)$ of the non-vanishing component is determined by the asymptotics (4.19) of the Bethe roots $t_1, \ldots, t_\Lambda$ as follows: Let $I_\Lambda = \{j(1), \ldots, j(\Lambda)\} \subset \{1, \ldots, N\}$ be the subset of indices appearing on the right hand side of (4.19). Then

$$\epsilon_j = \begin{cases} 1 & \text{if} \quad j \notin I_\Lambda \\ -1 & \text{if} \quad j \in I_\Lambda. \end{cases}$$

(4.22)

Viewed as a scalar function of the $\theta$’s the component $w_{e;\epsilon_N\ldots\epsilon_1}(\theta)$ is a homogeneous polynomial of degree $(N-1)\Lambda$, however again no longer a symmetric one.

Combining (4.16) – (4.22) we conclude that for $N > N_0$ the leading term $f_{e;\sigma_N\ldots\sigma_1}^{cl}(\theta)$ in the $\hbar$-expansion of $f_{e;\sigma_N\ldots\sigma_1}(\theta)$ is given by

$$f_{e;\sigma_N\ldots\sigma_1}^{cl}(\theta) = d^{(N)}_e(\theta) \phi^{cl}_e(\theta) w_{e;\sigma_N\ldots\sigma_1}^{cl}(\theta) \prod_{k>l}^{1} \frac{1}{\theta_{kl}^2},$$

where

$$f_{e;\sigma_N\ldots\sigma_1}(\theta) = \hbar^\Lambda f_{e;\sigma_N\ldots\sigma_1}^{cl}(\theta) + O(\hbar^{\Lambda+1}),$$

(4.23)

where $\Lambda = (N - e)/2$ as before and

$$d^{(N)}_e(\theta) = (-)^{N(N-1)/2} c^{(N)}(\theta)^{cl}.$$

(4.24)

Here we assumed that $\omega(\theta)$ in (4.6) has a regular $\hbar \to 0$ limit $\omega^{cl}(\theta)$ in terms of which the limit $c^{(N)}(\theta)^{cl} = \lim_{\hbar \to 0} c^{(N)}(\theta)$ is defined. As remarked before a fixed relative size of the
variables $\theta_N, \ldots, \theta_1$ is implicit in (4.23), say $\theta_N > \ldots > \theta_1$. The results for other orderings then are compatible with the classical limit of the exchange relations in (I), i.e.

$$f_{e;\sigma_N,\ldots,\sigma_1}^{cl}(\theta_N, \ldots, \theta_1) = f_{e;\sigma_N,\ldots,\sigma_k,\sigma_{k+1},\ldots,\sigma_1}^{cl}(\theta_N, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_1).$$

(4.25)

In particular $f_{e;\sigma_N,\ldots,\sigma_1}^{cl}(\theta)$ is separately symmetric in all $\theta_j$ with $\sigma_j = 1$ and all $\theta_j$ with $\sigma_j = -1$. As described before they are also eigenvectors of the semi-classical transfer matrix (B.39) with eigenvalues (B.40).

It is then natural to ask whether the $f^{cl}$ also obey a classical counterpart of the recursive relations (II). This is not automatic since one cannot a-priori expect the operation of taking the residue at $\theta_{k+1} = \theta_k \pm i\hbar$ to commute with the limit $\hbar \to 0$. Examples where they don’t commute are the function $\psi(\theta_{k+1},k)$, the eigenvalues of the transfer matrix, or the Bethe roots. Nevertheless for the final solutions it turns out that both operations do commute, – up to a numerical factor $Z$:

$$\lim_{\hbar \to 0} \left[ \hbar^{-\Lambda} \text{Res}_{\theta_{k+1}=\theta_k+i\hbar} f_{e;\sigma_N,\ldots,\sigma_1}^{cl}(\theta) \right] = Z \text{Res}_{\theta_{k+1}=\theta_k} f_{e;\sigma_N,\ldots,\sigma_1}^{cl}(\theta),$$

(4.26)

and similarly for the appropriate residue at $\theta_{k+1} = \theta_k - i\hbar$. The derivation of (4.26) is deferred to Appendix B. In principle the proportionality constant could depend on the solution considered. The explicit checks of (4.26) on the $N \leq 4$ solutions however suggest that it is universal and given by: $Z = (-)^N \psi_0 / 2, \psi_0 \approx 1.54678$. Using (4.26) in (IIa,b) one finds that both reduce to a single recurrent relation for $f^{cl}$

$$Z \text{Res}_{\theta_{k+1}=\theta_k} f_{e;\sigma_N,\ldots,\epsilon_1}^{cl}(\theta) = \chi^{cl}(\theta_k) C_{\epsilon_{k+1}+\epsilon_k} f_{e;\epsilon_N,\ldots,\epsilon_k+2,\epsilon_{k+1},\ldots,\epsilon_1}^{cl}(p_k \theta) - \sum_{j \neq k+1,k} \frac{\epsilon_j}{\theta_{kj}}.\quad(4.27)$$

The last factor on the right hand side equals the leading term, restricted to $\theta_{k+1} = \theta_k$ and $\epsilon_{k+1} + \epsilon_k = 0$, in the $\hbar$ expansion (B.40) of the transfer matrix eigenvalues. This is the consistency condition on (4.27) analogous to (IIc). ((IIc) itself cannot naively be expanded in powers of $\hbar$, due to the non-commutativity of the two operations mentioned earlier.)

In summary, the solutions of the functional equations (I), (II) admit a consistent semi-classical expansion. The leading term (4.23) of this expansion has, for a given ordering of $\theta_N, \ldots, \theta_1$, only one non-vanishing component; different orderings being related by (4.25). Further these leading terms are themselves linked by the recurrence relation (4.27). Of course it is tempting to ask whether the leading terms have a direct interpretation in the classical theory. This is beyond the scope of the present paper; however the vertex operator formalism of [49, 50] should be the appropriate framework to address the issue.
5. Conclusions

Motivated by the fact that none of the conventional field theoretical techniques presents itself to develop a quantum theory for the Ernst system, we proposed to bootstrap the quantum theory from structures linked to its classical integrability. Starting from a few reasonable assumptions on the nature of the quantum counterparts of these integrable structures, a very rigid computational framework emerged. The eventual outcome are meromorphic functions \( f^{(N)}(\theta) \), conjectured to describe exact matrix elements in the quantum theory, without the need for any renormalization.

On a technical level, our main result is the system of functional equations (I), (II) for the functions \( f^{(N)}(\theta) \). One of the equations is a standard eigenvalue problem for the transfer matrix, which is why the functions \( f^{(N)}(\theta) \) can be viewed as Bethe eigenvectors, however very special ones. The special feature is that they are members of semi-infinite sequences in a similar way the form factors of an integrable QFT are; consecutive members being linked by recursive relations. The solution procedure of the functional equations for ordinary form factors [28] \( (\beta = 2\pi) \) and for replica deformed ones [30] \( (\beta \text{ generic}) \) can be viewed as selecting those special solutions of the deformed KZ equation enjoying such extra recursive relations. Employing the integral representation for the latter [40, 41, 42], the integrands can be seen to obey recursive relations similar to our \( \beta = 0 \) ones [51]. This might indicate that the \( \beta = 0 \) system of functional equations found here can serve as a master system, from whose solutions the solutions of the \( \beta \neq 0 \) systems can be obtained by a universal integral transformation.

Physically, the most interesting finding is the “spontaneous” breakdown of the SL(2, R) symmetry that is a remnant of the 4D diffeomorphism invariance in the ‘compactified’ dimensions. The matrix elements \( f^{(N)}(\theta) \) and hence the state space generated by the physical operators supposed to underly them are covariant only under the SO(2) subgroup, despite the fact that the algebraic framework is fully SL(2, R) covariant. Clearly in a next step one should try to gain a better understanding of the field theoretical meaning of the matrix elements \( f^{(N)}(\theta) \) and how physically interesting quantities can be computed in terms of them. To this end it would be important to develop, at least to some extent, a more conventional field theoretical formulation. A combination of perturbative and semi-classical techniques should be adequate for this purpose.

Non-perturbatively one might aim at developing a dynamical triangulations approach, using the exact results proposed here, or quantities computed thereof, as a guideline. In particular, it would be important to understand the statistical mechanics origin of the dynamical breaking of the SL(2, R) symmetry, e.g. as a tug of war between energy and entropy as in conventional spin models. The mechanism may well contain clues for the breaking of diffeomorphism invariance beyond the symmetry reduced theory.

**Acknowledgements:** We wish to thank H. DeVega and D. Korotkin for valuable discussions. The work of M. N. was supported by NSF grant 97-22097; that of H. S. by EU contract ERBFMRX-CT96-0012.
Appendix A: Action principles for coset sigma models

As outlined in the introduction the 2D matter sector of the Ernst system is a nonlinear sigma-model whose target space is a (noncompact) homogeneous space. Such sigma-models are known as coset sigma models, they are classically integrable and several action principles and parameterizations turn out to be useful. Although in the bulk of the paper no direct use of these actions is made, both the parameterizations employed and the origin of the (gauge) symmetries is best understood in the terms of these action principles. For simplicity we omit the coupling to 2D gravity here, each of the coset sigma-models can be coupled to gravity as in (1.1).

Coset sigma models describe the dynamics of generalized harmonic maps from a 2-dim. spacetime $\Sigma$ to a homogeneous space of the form $G/H$, where $G$ is a (simple) matrix Lie group and $H$ a maximal subgroup of $G$. For the purposes of this appendix, we take $\Sigma$ to be 2-dim. Minkowski space with signature $(+, -)$. There are two useful action principles for these coset sigma models. A gauge theoretical one

$$S[V,Q] = \frac{1}{2} \int d^2x \text{Tr}[D_\mu V V^{-1} D^\mu V V^{-1}] , \quad (A.1)$$

where $V$ is a group-valued field transforming as $V \rightarrow V h$ under an $H$-valued gauge transformation, and $Q_\mu$ is the associated connection ensuring that $D_\mu V = \partial_\mu V - V Q_\mu$ transforms covariantly. Clearly the gauge symmetry removes $\text{dim } H$ degrees of freedom leaving $\text{dim } G/H$ physical ones. Alternatively, one can choose a non-redundant parameterization of the coset space by matrices $M \in G$ obeying a suitable quadratic constraint $M \tau_0(M) = \pm \mathbb{I}$, where $\tau_0$ is an involutive outer automorphism of $G$. The subgroup $H$ can then be characterized as being fixed by a related involutive automorphism $\tau$ of $G$, given by $\tau(g) = g_0^{-1} \tau_0(g) g_0$, for some fixed $g_0 \in G$ which likewise satisfies $g_0 \tau_0(g_0) = \pm \mathbb{I}$. Explicitly, the matrices $M$ can be constructed as $M = V g_0^{-1} \tau_0(V^{-1})$; they are gauge invariant and parameterize the coset space as $V$ runs through $G$. Using $\partial_\mu M M^{-1} = 2 D_\mu V V^{-1}$, the action (A.1) becomes

$$S[M] = \frac{1}{8} \int d^2x \text{Tr}[\partial_\mu M M^{-1} \partial^\mu M M^{-1}] , \quad M \tau_0(M) = \pm \mathbb{I} . \quad (A.2)$$

Specifically we are interested in the case $G/H = SL(2, \mathbb{R})/SO(2) \simeq SU(1,1)/U(1)$. The theory with a compact target space $SU(2)/U(1)$ can be seen to have two useful reformulations. One as a $U(1)$ gauge theory on the projective space $\mathbb{R}P^2 =: \mathbb{C}P_1$, known as the $\mathbb{C}P_1$ model. The second one parameterizes the coset matrices $M$ in (A.2) by real 3-dim. unit vectors and yields the $S_2$ Heisenberg spin-model. (The latter is also known as the $O(3)$ nonlinear sigma-model, though $O(3)/O(2)$ would be the proper coset notation.) The aim in the following is to derive similar reformulations for the noncompact $SU(1,1)/U(1)$ model.
We begin with the counterpart of the $\mathbb{C}P_1$ formulation and choose an explicit parameterization of the $SU(1, 1)$ matrix fields. Since $\mathcal{V} \in SU(1, 1)$ iff $g\sigma^3 g^\dagger = \sigma^3$ one has
\[
\mathcal{V} = \begin{pmatrix} z_1^* & z_2 \\ z_2^* & z_1 \end{pmatrix}, \quad |z_1|^2 - |z_2|^2 = 1, \\
\quad h = \begin{pmatrix} e^{-i\gamma} & 0 \\ 0 & e^{i\gamma} \end{pmatrix}, \quad \gamma \in \mathbb{R}.
\]

Here and below $\sigma^j, j = 1, 2, 3$, are the Pauli matrices. The automorphism $\tau$ in the case at hand is given by $\tau(g) = \sigma^3 g \sigma^3$, $g \in SU(1, 1)$, i.e. $g_0 = -i\sigma^3$ and $\tau_0 = id$ in the general framework outlined before. On the component fields $\tau$ acts as $\tau(z_1) = z_1, \tau(z_2) = -z_2$. The condition $h \in H \simeq U(1)$ iff $\tau(h) = h$ then yields the above parameterization of $h$. The gauge transformations $\mathcal{V} \to \mathcal{V}h$ simply amount to $z_1 \to z_1 e^{i\gamma}, z_2 \to z_2 e^{i\gamma}$ and $A_\mu \to A_\mu + \partial_\mu \gamma$, where $Q_\mu = -iA_\mu \sigma^3$. Inserting this parameterization of $\mathcal{V}$ and $Q_\mu$ into the action (A.1) one finds
\[
S[z, A] = \int d^2 x (\partial^\mu z^\dagger + iA_\mu z^\dagger) \cdot (\partial_\mu z - iA_\mu z), \quad z^\dagger \cdot z = 1.
\]

where $z^\dagger = (z_1^*, z_2^*)$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $w^\dagger = w_1^* z_1 - w_2^* z_2$.

This is a $U(1)$ gauge theory on the ‘Lorentzian' projective space $\mathbb{CP}^{1,1}$, and is a non-compact analogue of the $\mathbb{C}P_1$ model. The $SU(1, 1)/U(1)$ coset theory can therefore be viewed as a matrix version of (A.2).

The second reformulation of the $SU(1, 1)/U(1)$ theory starts from the coset action (A.2) and yields a hyperbolic counterpart of the $S_2$ spin-model. In upshot it simply amounts to parameterizing the matrices $M$ by elements of the 2-dim. hyperboloid $H_2$, replacing the sphere $S_2$ in the compact case. Following the general construction we first compute the gauge invariant matrix
\[
M = \mathcal{V}i\sigma^3 \mathcal{V}^{-1} = \begin{pmatrix} i(|z_1|^2 + |z_2|^2) \\ 2iz_2^* z_1 \\ -2iz_2 z_1^* \\ -i(|z_1|^2 + |z_2|^2) \end{pmatrix}, \quad M^2 = -\mathbb{I}.
\]

Elements $Y$ of the Lie algebra $su(1, 1)$ are characterized by $-\sigma^3 Y = Y^\dagger \sigma^3$. A convenient basis is $\tau^0 = i\sigma^3, \tau^1 = \sigma^1, \tau^2 = \sigma^2$. For any real triplet $n = (n^0, n^1, n^2)$ the matrix $\sum_j n^j \tau^j$ defines an $U(1, 1)$ matrix which squares to the multiple $-(n^0)^2 + (n^1)^2 + (n^2)^2$ of the unit matrix. The relations
\[
M = \sum_j n^j \tau^j \iff in^j = z^\dagger \cdot \tau^j z, \quad j = 0, 1, 2,
\]

(where the '$\cdot$' is that of (A.4)) then provide an isomorphism
\[
SU(1, 1)/U(1) \to H_2, \\
H_2 = \{ n \in \mathbb{R}^{1,2} \mid n \cdot n = (n^0)^2 - (n^1)^2 - (n^2)^2 = 1, \ n^0 > 0 \}.
\]
where \( R^{1,2} \) is the ambient Lorentzian vector space of signature \((+,\,-\,-\,-)\). Substituting (A.6) into the coset action (A.2) one obtains

\[
S[n] = -\frac{1}{4} \int d^2 x \, \partial_\mu n \cdot \partial^\mu n \, , \quad n \cdot n = 1 ,
\]  

(A.8)

where the bilinear form \( \cdot \cdot \) is that of \( R^{1,2} \). An alternative way of arriving at (A.8) would have been to vary (A.4) with respect to \( A_\mu \) and substitute the (algebraic) equation of motion back into the action. We preferred the above route in order to have the link (A.6) to the coset matrices \( M \) at our disposal. The action (A.8) is also convenient to verify that the energy density of the system is positive definite

\[
\mathcal{H} = -\frac{1}{2} [\partial_0 n \cdot \partial_0 n + \partial_1 n \cdot \partial_1 n] > 0 .
\]

(A.9)

To see (A.9) one differentiates the constraint and uses Schwarz inequality

\[
(\partial_\mu n^0)^2 = \frac{1}{(n^0)^2} \left( \sum_{j=1}^2 n^j \partial_\mu n^j \right) < (\partial_\mu n^1)^2 + (\partial_\mu n^2)^2 , \quad \mu = 0, 1 .
\]

(A.10)

It is essential here that the hyperboloid is timelike, for a spacelike hyperboloid the energy density would be indefinite.

In the above we started with the complex \( SU(1,1)/U(1) \) coset space because it is the description most convenient in the bulk of the paper. The dimensional reduction of the 4D Einstein-Hilbert action however initially yields a \( SL(2, \mathbb{R})/SO(2) \) matter sigma-model [2]. The relation is given by the isomorphism

\[
\text{Ad} \Upsilon : SL(2, \mathbb{R}) \rightarrow SU(1,1) , \quad \Upsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - i & \cr \cr 1 & i \end{pmatrix} , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} z_1^* & z_2 \\ z_2^* & z_1 \end{pmatrix} , \quad z_1 = \frac{1}{2}[(a + d) - i(b - c)] , \quad z_2 = \frac{1}{2}[(a - d) - i(b + c)] .
\]

(A.11)

For the \( H_2 \) variables one gets

\[
n^0 = \frac{1}{2}(a^2 + b^2 + c^2 + d^2) , \quad n^1 = -(ac + bd) , \quad n^2 = \frac{1}{2}(a^2 + b^2 - c^2 - d^2) .
\]

(A.12)

The automorphism \( \tau \) for \( SL(2, \mathbb{R}) \) is inner and is given by \( \tau(g) = (g^T)^{-1} = (i\sigma^2)^{-1}g(i\sigma^2) \), i.e. \( \tau(a) = d , \tau(b) = -c \) on the above components. It evidently selects the proper \( SO(2) \) subgroup which is mapped onto the diagonal \( SU(1,1) \) matrices in (A.3). On the gauge invariant \( H_2 \) variables \( \tau \) (in either the \( SL(2, \mathbb{R}) \) or the \( SU(1,1) \) version) acts as
\( \tau(n^0) = n^0, \ \tau(n^1) = -n^1, \ \tau(n^2) = -n^2, \) which maps \( H_2 \) onto itself and has only the trivial fixed point \( n = (1, 0, 0) \). In the coset space \( SL(2, \mathbb{R})/SO(2) \) one can pick representatives given by upper triangular \( SL(2, \mathbb{R}) \) matrices. A conventional parameterization is

\[
\mathcal{V} = \begin{pmatrix}
\Delta^{1/2} & B\Delta^{-1/2} \\
0 & \Delta^{-1/2}
\end{pmatrix}, \quad \Delta^{1/2} > 0, \ B \in \mathbb{R}.
\]  

(A.13)

The combination \( \mathcal{E} = \Delta + iB \) then is the “Ernst potential” used in the general relativity literature. In this parameterization the ‘hyperbolic spins’ read

\[
n^0 = \frac{\Delta^2 + B^2 + 1}{2\Delta}, \quad n^1 = -\frac{B}{\Delta}, \quad n^2 = \frac{\Delta^2 + B^2 - 1}{2\Delta}.
\]  

(A.14)

In particular the Einstein-Rosen waves have \( B \equiv 0 \), i.e. \( n^1 \equiv 0 \), and are thus described by a \( O(1, 1) \) matter sigma-model. For the gauge invariant \( M \) matrix in the \( SL(2, \mathbb{R}) \) description one obtains

\[
M = \mathcal{V}i\sigma^2\mathcal{V}^{-1} = \mathcal{V}V^T i\sigma^2 = \begin{pmatrix}
n^1 & n^2 + n^0 \\
n^2 - n^0 & -n^1
\end{pmatrix},
\]  

(A.15)

related to the \( SU(1, 1) \) version \((A.5), (A.6)\) by \( \text{Ad}Y \), as it should. Contact to the parameterization used e.g. in \((3.23)\) is made by lowering one index, \( M_{ab} = iM_b^cC_{ycb} \), etc.

In the case of the compact coset space \( SU(2)/U(1) \) the counterpart of the action \((A.4)\) yields the classical \( CP_1 \) model, the counterpart of \((A.8)\) yields the \( O(3) \) nonlinear sigma-model. Heuristically one can then think of the \( n \)-fields as “mesonic bound states” of the “quark-doublets” \( z \), via \( n^j = z^\dagger \sigma^j z \). The analogy is correct in the sense that in the quantum field theory only the \( n \)-field generates scattering states.

**Appendix B: Diagonalization of \( \mathcal{T} \)**

Here we describe the solution of the eigenvalue problem \((2.26), (2.34)\) for case \( \epsilon = -1 \), where only the \( SO(2) \subset SL(2, \mathbb{R}) \) invariance remains. The remaining abelian symmetry is nevertheless useful in that it still allows one to decompose the full problem into pieces of lower dimensionality.
B.1 Decomposition into charge \( e \) sectors

To this end we switch to a basis in \( V^\otimes n \) diagonalizing \( \Gamma \). With the choice \( \Gamma = \Gamma(\varphi, \lambda) \) in (2.31) the eigenvalues are \( \pm i \) and we simply label the components with respect to the new basis by the sign of the corresponding eigenvalue, i.e.

\[
w_{\sigma_n \ldots \sigma_1}(\theta) = \Upsilon_{\sigma_n}^{a_n} \ldots \Upsilon_{\sigma_1}^{a_1} w_{a_n \ldots a_1}(\theta) , \quad \sigma_j \in \{ \pm \}. \quad (B.1)
\]

For the inverse matrix we use the index staggering \( (\Upsilon^{-1})_a^\sigma \), such that in \( \Gamma^\tau_\sigma = \Upsilon_a^\sigma \Upsilon_b^\tau (\Upsilon^{-1})_b^\sigma \) the nonvanishing components of \( \Gamma^\tau_\sigma \) are \( \Gamma^+_- = i = -\Gamma^-_+ \). Explicitly, one finds for \( \nu > 0 \)

\[
\Upsilon \Gamma \Upsilon^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \text{with} \quad \Upsilon = \Upsilon_a^\sigma = \frac{1}{\sqrt{2\nu \cosh \varphi}} \begin{pmatrix} \cosh \varphi & \nu(-i + \sinh \varphi) \\ \cosh \varphi & \nu(i + \sinh \varphi) \end{pmatrix}. \quad (B.2)
\]

It obeys \( \det \Upsilon = i \) and \( (\Upsilon_a^\sigma)^* = \Upsilon_a^{-\sigma} \). Clearly, the same transformation also diagonalizes the matrix \( \Lambda(\phi) \) in (2.32), i.e. \( \Lambda^\tau_\sigma(\phi) = \delta^\tau_\sigma e^{i(\sigma + \tau)\phi} \). The charge conjugation matrix becomes \( C_{\sigma\tau} = -\epsilon_{\sigma\tau} \). For simplicity we shall refer to the basis (B.2) as the “charged basis” (regarding the phase \( e^{\pm i\varphi} \) as associated with a \( U(1) \) charge) and to the original basis as the “real” basis. We write \( e \) for the \( U(1) \) charge of \( w_{\sigma_n \ldots \sigma_1}(\theta) \), i.e. \( e = \sigma_n + \ldots + \sigma_1 \) is the number of + minus the number of – in a multi-index.

In group theoretical terms \( \text{Ad} \Upsilon \) yields a two-parameter family of automorphisms \( SL(2, \mathbb{R}) \rightarrow SU(1, 1) \) generalizing (A.11). It suffices to verify this on the level of the Lie algebras. Let \( \tau^0 = i\sigma^3, \quad \tau^1 = \sigma^1, \quad \tau^2 = \sigma^2 \) denote the basis of \( su(1, 1) \) used before. The matrices \( \Upsilon^{-1}\tau^j\Upsilon, \quad j = 0, 1, 2 \), then are readily checked to be real and trace-free, and hence can serve as a (non-standard) basis of \( sl(2, \mathbb{R}) \).

Returning to the eigenvalue problem (2.26) it follows from (2.33) that

\[
\mathcal{T}(\theta_0|\theta)\tau_{\sigma_n \ldots \sigma_1} = 0 , \quad \text{unless} \quad \sigma_N + \ldots + \sigma_1 = \tau_N + \ldots + \tau_1 . \quad (B.3)
\]

In the charged basis (2.26) therefore decomposes into decoupled sectors of dimension

\[
m_{e}(N) = \begin{pmatrix} N \\ \Lambda \end{pmatrix} = \begin{pmatrix} N \\ N - \Lambda \end{pmatrix} = m_{-e}(N) , \quad \Lambda := \frac{1}{2}(N - e) . \quad (B.4)
\]

Explicitly we write

\[
\mathcal{T}_e(\theta_0|\theta)w_e(\theta) = \tau_e(\theta_0|\theta)w_e(\theta) , \quad e = N, N - 2, \ldots, -N + 2, -N , \quad (B.5)
\]

where \( w_e(\theta) \) is a column vector of length \( m_e(N) \). For convenience we split off here a scalar factor

\[
\mathcal{T}(\theta_0|\theta) = \prod_{j=1}^{N} \frac{r(\theta_{j0} - i\hbar)}{\theta_{j0} - 2i\hbar} \mathcal{T}(\theta_0|\theta) , \quad (B.6)
\]

45
which renders the components of $\mathcal{T}(\theta_0|\theta)$ polynomial in the $\theta_{j0} - i\hbar$. Similarly the KZE eigenvalue problem (2.38) splits up into decoupled sectors with fixed charge. The eigenvalues $q_{e;k}(\theta)$ in the charge $e$ sector are phases and in addition obey $\prod_{k=1}^{N} q_{e}(Q^{k-N}\theta) = (-)^N i^e$, which follows from (2.47). Again it is convenient to split off a scalar function from the $Q_k(\theta)$ matrices and write

$$Q_k(\theta) = -i \prod_{j \neq k} r(\theta_{jk}) Q_k(\theta) , \quad \mathcal{T}(\theta_k - i\hbar|\theta) = \hbar \prod_{j \neq k} (\theta_{jk} - i\hbar) Q_k(\theta) . \quad \text{(B.7)}$$

The reduced eigenvalues in the charge $e$ sector defined by

$$\overline{Q}_{e;k}(\theta) w_e(\theta) = \overline{q}_{e;k}(\theta) w_e(\theta) , \quad \text{i.e.} \quad \overline{q}_{e;k}(\theta) = \frac{1}{i} \prod_{j \neq k} \frac{1}{\theta_{jk} - i\hbar} \tau_e(\theta_k - i\hbar|\theta) , \quad \text{(B.8)}$$

will then satisfy

$$\prod_{k=1}^{N} \overline{q}_{e;k}(\theta) = i^{e-N} . \quad \text{(B.9)}$$

The $\mathcal{T}$ eigenvalue problem (B.3) is in fact equivalent to the seemingly weaker $\overline{Q}_{e;k}$ eigenvalue problem (B.8). To see this note that $\mathcal{T}(\theta_0|\theta)$ is a polynomial in $\theta_0$ of degree $N-1$, due to the tracelessness of $\Gamma$. The second relation in (B.7) therefore amounts to $n$ linear equations for the $n$ matrix-valued coefficients of this polynomial. Provided all variables $\theta_j$ are distinct these equations turn out to be independent, so that the $\mathcal{T}(\theta_0|\theta)$ matrix is uniquely determined by the $Q_k(\theta)$ matrices. By expanding the $\mathcal{T}$ eigenvalue problem into powers of $\theta_0$ one sees that also the eigenvalues $\tau_e(\theta_0|\theta)$ must be polynomials of degree $N-1$ in $\theta_0$. By the same token they will be uniquely determined by their values at $n$ points, i.e. by the eigenvalues $\overline{q}_{e;k}(\theta)$. Explicitly one finds

$$\tau_e(\theta_0|\theta) = \hbar \sum_{j=1}^{N} \overline{q}_{e;j}(\theta) \prod_{k \neq j} \frac{(\theta_{0k} + i\hbar)(\theta_{kj} - i\hbar)}{\theta_{jk}} . \quad \text{(B.10)}$$

Finally let us note a number of useful involution properties linking the spectrum in the charge $e$ and the charge $-e$ sector. Since

$$\mathcal{T}(\theta_0|\theta)^{\tau_{e_1}...\tau_{e_1}} = \mathcal{T}(\theta_0|\theta)^{-\tau_{e_1}...-\tau_{e_1}} , \quad \text{(B.11)}$$

one infers:

If $\tau_e(\theta_0|\theta) \in \text{Spec} \mathcal{T}_e(\theta_0|\theta)$ then $-\tau_e(\theta_0|\theta) \in \text{Spec} \mathcal{T}_{-e}(\theta_0|\theta)$ ,
If $q_{e;k}(\theta) \in \text{Spec} Q_{e;k}(\theta)$ then $-q_{e;k}(\theta) \in \text{Spec} Q_{-e;k}(\theta)$ . \quad \text{(B.12)}
In particular for charge $e = 0$ this means all eigenvalues come in pairs differing only by a sign. For $e \neq 0$ the sign flip $e \mapsto -e$ amounts to $\Lambda \mapsto N - \Lambda$ so that by (B.4) one expects, at least for generic rapidities, the eigenvalues to be in 1-1 correspondence. Under complex conjugation one has \( [w_{\sigma_n...\sigma_1}(\theta)]^* = w_{-\sigma_1...-\sigma_n}(\theta^* + i\hbar) \) and a counterpart of the first equation in (2.27). Combined with (B.11) this implies

\[
\text{If } \tau_e(\theta_0|\theta) \in \text{Spec} \mathcal{T}_e(\theta_0|\theta) \text{ then } \tau_e(\theta_0^* - 2i\hbar|\theta^* + i\hbar)^* \in \text{Spec} \mathcal{T}_e(\theta_0|\theta). \quad (B.13)
\]

For the corresponding eigenvectors one can choose normalizations such that

\[
[w_{\sigma_n...\sigma_1}(\theta)]^* = w_{\sigma_1...\sigma_n}(\theta^* + i\hbar), \quad (B.14)
\]

which also matches the properties of the Bethe Ansatz vectors (B.17) below.

### B.2 Bethe ansatz equations

The aim in the following is to compute the eigenvalues and eigenvectors in (B.5) explicitly. For small $n$ this can be done by brute force but for generic $n$ it is useful to parameterize the solutions in terms of the roots of the Bethe equations. The literature on the Bethe Ansatz is enormous, some guidance can be obtained from the book [52] and e.g. the following papers [53, 54, 55, 56]. Transferred to the present context the construction can be outlined as follows: Denote by

\[
\Omega_A := (\mathcal{Y}^{-1})_{a_n}^+ (\mathcal{Y}^{-1})_{a_{n-1}}^+ \cdots (\mathcal{Y}^{-1})_{a_1}^+, \quad (B.15)
\]

a cyclic vector on which the operator $\Gamma_T$ from (2.25) in the charged basis acts as follows

\[
\mathcal{Y}_a^b \Gamma_a (\mathcal{Y}^{-1})_c^T \mathcal{T}_b(\theta_0 + i\hbar|\theta)^B_A \Omega_B = \begin{pmatrix} i \prod_j (\theta_{0j} + 2i\hbar) \Omega_A^* \\ 0 \\ -i \prod_j (\theta_{0j} + i\hbar) \Omega_A \end{pmatrix}. \quad (B.16)
\]

Here, $\mathcal{T}_b(\theta_0|\theta)^B_A$ denotes the monodromy matrix (2.23) with the same prefactor taken out as in (B.6). Following the Bethe Ansatz procedure we generate candidate eigenstates from $\Omega$ by the repeated action of $B(t|\theta) := \mathcal{Y}_a^b \Gamma_a (\mathcal{Y}^{-1})_c^T \mathcal{T}_b(t + i\hbar|\theta)$. The matrix operators $B(t|\theta)$ are commuting for different values of $t$ and each $B(t|\theta)$ lowers the $SO(2)$ charge of a candidate eigenstate of $\mathcal{T}$ by the two units. The candidate eigenstates can be made proper eigenstates by turning the parameters $t_\alpha$ into judiciously chosen functions of the \( \theta_j \). In upshot one obtains eigenvectors

\[
w_e(\theta) = \prod_{\alpha=1}^{\Lambda} B(t_\alpha|\theta) \Omega, \quad \Lambda := \frac{1}{2}(N-e), \quad (B.17)
\]
with eigenvalues

\[
\tau_e(\theta_0|\theta) = \frac{i \prod_j (\theta_j - 2i\bar{h}) \prod_\alpha (\theta_0 - t_\alpha + i\bar{h}/2)}{\prod_\alpha (\theta_0 - t_\alpha + 3i\bar{h}/2)} - \frac{i \prod_j (\theta_j - i\bar{h}) \prod_\alpha (\theta_0 - t_\alpha + 5i\bar{h}/2)}{\prod_\alpha (\theta_0 - t_\alpha + 3i\bar{h}/2)}, \tag{B.18}
\]

where the Bethe roots \( t_\alpha \) are solutions of the following Bethe Ansatz equations (BAE)

\[
\prod_{j=1}^n \frac{\theta_j - t_\alpha - i\bar{h}/2}{\theta_j - t_\alpha + i\bar{h}/2} = -\prod_{\beta \neq \alpha} \frac{t_\beta - t_\alpha - i\bar{h}}{t_\beta - t_\alpha + i\bar{h}}, \quad \alpha = 1, \ldots, \Lambda. \tag{B.19}
\]

The only modification of the BAE as compared to the standard case \( \epsilon = 1 \) is the sign on the r.h.s. which comes from the ratio of the eigenvalues of \( \Gamma \). These equations ensure that \( \tau_e(\theta_0|\theta) \) is indeed a polynomial in \( \theta_0 \) of degree \( n - 1 \), as anticipated in section B1:

\[
\tau_e(\theta_0|\theta) = (-1)^{n-1} \bar{h} \theta_0^{n-1} (N-2\Lambda)
- (-1)^{n-1} \bar{h} \theta_0^{n-2} \left[ (N-2\Lambda-1) \sum_j (\theta_j - 3i\bar{h}/2) + 2 \sum_\alpha (t_\alpha - 3i\bar{h}/2) \right]
+ \ldots
= : \sum_{p=0}^{N-1} \theta_0^p \tau_{e;p}(\theta), \tag{B.20}
\]

where the coefficients obey \( \tau_{e;p}(\theta)^* = \tau_{e;p}(\theta^* + 3i\bar{h}) \). For \( e = N \) no Bethe roots are required and (B.17), (B.18) should be interpreted as \( w_{N;A}(\theta) = \Omega_A \) and

\[
\tau_N(\theta_0|\theta) = i \prod_j (\theta_j - 2i\bar{h}) - i \prod_j (\theta_j - i\bar{h}), \tag{B.22}
\]

from which one computes \( \tau_{N;k} = 1, k = 1, \ldots, N \). Since for \( e = N \) the eigenvalue problem (B.5) is one-dimensional it follows from (B.12) that \( \tau_{-N}(\theta_0|\theta) = -\tau_{N}(\theta_0|\theta) \) and \( \tau_{-N;k} = -1 \). In the expressions (B.18) for the eigenvalues coming out of the Bethe Ansatz however this is not obvious as now \( N \) Bethe roots are required, rather than none as in the charge \( N \) sector. More generally the charge \( e \) and charge \(-e\) sector enter asymmetrically in the Bethe Ansatz construction, though by (B.12) they are practically identical.

The eigenvalues \( \tau_{e;k}(\theta) \) of the asymptotic qKZE equations are accordingly given by

\[
\tau_{\pm N;k} = \pm 1, \quad \tau_{e;k}(\theta) = \prod_\alpha \frac{\theta_k - t_\alpha - i\bar{h}/2}{\theta_k - t_\alpha + i\bar{h}/2}, \quad |e| \leq N - 2. \tag{B.23}
\]

The explicit form (B.23) allows to directly check many of the properties which we have derived in the main text. Inspection of the BAE (B.19) shows that for real \( \theta_j \) the Bethe
roots appear in (possibly degenerate) complex conjugate pairs \((t_\alpha, t_\alpha^*)\). The \(\overline{\eta}_{c;k}(\theta)\) therefore indeed are pure phases, consistent with (2.48). Assuming the Bethe roots to be completely symmetric functions of the \(\theta_j\) eqs (B.23) also makes manifest the cyclic property (2.46) of the \(\overline{\eta}_{c;k}(\theta)\). For the product of the eigenvalues \(\overline{\eta}_{c;k}(\theta)\) we obtain with (B.19) and (B.23):

\[
\prod_k \overline{\eta}_{c;k}(\theta) = (-)^{\Lambda} \prod_{\beta \neq \alpha} \frac{t_\beta - t_\alpha - i\hbar}{t_\beta - t_\alpha + i\hbar} = i^{N-e} \tag{B.24}
\]

confirming (B.9) and thus (2.47).

Finally the logarithmic derivative of the BAE yields

\[
\frac{i\hbar}{(t_k - t_\alpha - i\hbar/2)(t_k - t_\alpha + i\hbar/2)} = \sum_{\beta \neq \alpha} \frac{2i\hbar \partial_k (t_\beta - t_\alpha)}{(t_\beta - t_\alpha)^2 - (i\hbar)^2} + \sum_j \frac{i\hbar \partial_k t_\alpha}{(t_j - t_\alpha - i\hbar/2)(t_j - t_\alpha + i\hbar/2)} , \tag{B.25}
\]

which may be used to prove that

\[
\partial_k \ln \overline{\eta}_{c;l}(\theta) = \partial_l \ln \overline{\eta}_{c;k}(\theta) , \tag{B.26}
\]

as claimed in (2.48) and in accordance with [41, 42].

Let us also briefly comment on the solutions to the BAE. In analogy to the homogeneous case (all \(\theta_j\) equal [56]) one expects that only the solutions with \(t_\alpha \neq t_\beta, \alpha \neq \beta\), are relevant. Assuming that \(t_\alpha - t_\beta \neq 0, \pm i\hbar\) the eqs (B.19) can be rewritten in polynomial form. Specifically they constitute a system of \(\Lambda\) polynomial equations of degree \(\Lambda + N - 1\) for the unknowns \(t_\alpha - \frac{1}{N}(\theta_N + \ldots + \theta_1)\), \(\alpha = 1, \ldots, \Lambda\), whose coefficients are symmetric polynomials in \(\tilde{\theta}_j = \theta_j - \frac{1}{N}(\theta_N + \ldots + \theta_1)\), \(j = 1, \ldots, N\). The point of adding and subtracting the ‘center of mass term’ is that the \(\tilde{\theta}_j\)’s are boost invariant. In particular it follows that

\[
t_\alpha - \frac{1}{N}(\theta_N + \ldots + \theta_1) \text{ are boost invariant} , \tag{B.27}
\]

i.e. are (completely symmetric) functions of the differences \(\theta_{jk}\) only.

It may be instructive to exemplify the construction for the simplest case \(N = 2\):

\[e = 2: \text{ Bethe roots: } \emptyset \]

\[
\overline{\eta}_2(\theta_0|\theta_2, \theta_1) = \hbar (\theta_{20} + \theta_{10} - 3i\hbar) , \quad \overline{\eta}_{2,2} = \overline{\eta}_{2,1} = 1 ,
\]

\[e = 0: \text{ Bethe roots: } t = \frac{1}{2} \left[ \theta_1 + \theta_2 \pm \sqrt{\hbar^2 + \theta_{12}^2} \right] ,
\]

\[
\overline{\eta}_0(\theta_0|\theta_2, \theta_1) = \pm i\hbar \sqrt{\hbar^2 + \theta_{21}^2} , \quad \overline{\eta}_{0,2}(\theta_2, \theta_1) = \pm i\sqrt{\frac{i\hbar + \theta_{12}}{i\hbar - \theta_{12}}} = \overline{\eta}_{0,1}(\theta_1, \theta_2) ,
\]

49
Let us now return to the eigenvectors (B.17). Clearly any eigenvector is only determined up to multiplication by an arbitrary scalar function, or the corresponding linear combinations in the case with degeneracies. The Bethe eigenvectors (B.17) will in general not obey the exchange relations

\[ f_{e;\tau_n...\tau_1}(\theta) = R(\theta) \prod_{k>l} \frac{i}{\theta_{kl} + i \hbar} w_{e;\sigma_n...\sigma_1}(\theta) , \]

(B.31)

solves both (B.3) and (B.29). It is manifestly polynomial in the Bethe roots and rational in the \( \theta_j \)'s.
B.3 Sequential Bethe roots

Let us examine the behavior of the Bethe ansatz equations and their solutions under pinching $\theta_{k+1} \to \theta_k \pm i\hbar$ of the insertions $\theta_N, \ldots, \theta_1$. The relations (II) imply that the $SO(2)$ charge of the eigenvectors is conserved under $\theta_{k+1} \to \theta_k \pm i\hbar$, i.e.

$$n \to n-2, \quad \epsilon \to \epsilon, \quad \Lambda \to \Lambda-1. \quad (B.32)$$

This suggests that the Bethe roots describing these (special) eigenvectors might likewise be related. Indeed the BAE (B.19) are consistent with the following $n \to n-2$ reduction of its solutions

$$t_\Lambda(\theta) \bigg|_{\theta_{k+1}=\theta_k \pm i\hbar} = \theta_k \pm i\hbar \frac{\hbar}{2}$$
$$t_\alpha(\theta) \bigg|_{\theta_{k+1}=\theta_k \pm i\hbar} = t_\alpha(p_k \theta), \quad \text{for } \alpha < \Lambda, \quad (B.33)$$

Since the Bethe roots are symmetric in all $\theta_j$ it suffices to verify (B.33) for the $\theta_{k+1}=\theta_k+i\hbar$ case. The other then formally follows from applying the $\theta_{k+1}=\theta_k-i\hbar$ reduction to $\theta'= (\theta_N, \ldots, \theta_{k+1}, \theta_k-i\hbar, \theta_{k-1}, \ldots, \theta_1)$. It is easy to verify that with (B.33) the BAE (B.19) for $\alpha < \Lambda$ reduce to the BAE with $n-2$ insertions for the $t_\alpha(p_k \theta)$. The equation for $\alpha = \Lambda$ is a bit more subtle and requires to specify the limit in which the pinched configuration is approached. Entering with the ansatz

$$t_\Lambda(\theta) = \theta_k + i\hbar \frac{\hbar}{2} + \delta/Z(\theta) + o(\delta), \quad \text{for } \theta_{k+1} = \theta_k + i\hbar + \delta, \quad (B.34)$$

into the $\alpha = \Lambda$ BAE one obtains for $n > 2$

$$\prod_{\beta<\Lambda} \frac{t_\beta-\theta_k-3i\hbar/2}{t_\beta-\theta_k+i\hbar/2} = -\left(\frac{-i\hbar}{-\delta/Z}\right) \prod_{j \neq k,k+1} \frac{\theta_j-\theta_k-i\hbar}{\theta_j-\theta_k} \quad (B.35)$$

$$= (1-Z) \prod_{j \neq k,k+1} \frac{\theta_j-\theta_k-i\hbar}{\theta_j-\theta_k}. \quad (B.35)$$

This can be taken to define $Z = Z(\theta)$ in (B.34), showing the consistency of the reduction rule for $t_\Lambda(\theta)$ as $\delta \to 0$.

Of course, not every solution of the BAE will satisfy (B.33), in fact the vast majority will not. The argument shows however that under the same 'genericity assumptions' under which solutions exist at all, there also exists at each recursion step $n-2 \mapsto n$ at least one $\Lambda$-tuple of Bethe roots enjoying the property (B.33). In addition (B.33) is compatible with the following reality condition

$$t_\alpha(\theta)^* = t_\alpha(\theta^*), \quad \alpha = 1, \ldots, \Lambda. \quad (B.36)$$
Here we refer to the observation after (B.22) that the solutions of the BAE come in pairs \((t_a(\theta)^*, t_a(\theta^*))\), where in general \(t_a(\theta)^* \neq t_a(\theta^*)\). We call a solution of the BAE a “sequential” tuple of Bethe roots, if all roots are distinct, real in the sense of (B.36), and satisfy (B.33).\footnote{The concept appears to be new. The only article we are aware of where a pinching of inhomogeneities is considered, is \cite{[53].}}

To justify the terminology let us consider the behavior of the eigenvalues \(\tau(\theta_0|\theta)\) under \(\theta_{k+1} \to \theta_k + i\hbar\). From (B.33) one finds

\[
\tau_e(\theta_0|\theta) \xrightarrow{\theta_{k+1} \to \theta_k + i\hbar} i\theta_{k0}(\theta_{k0} - 2i\hbar) \prod_{\alpha < \Lambda} (\theta_0 - t_\alpha + 3i\hbar/2)^{-1} \times \\
\times \left[ \prod_{j \neq k, k+1} (\theta_{j0} - 2i\hbar) \prod_{\alpha < \Lambda} (\theta_0 - t_\alpha + i\hbar/2) \right] \\
= \theta_{k0}(\theta_{k0} - 2i\hbar) \tau_e(\theta_0|\theta_k),
\]

and similarly for \(\theta_{k+1} = \theta_k - i\hbar\). Hence for any \(\Lambda\)-tuple of Bethe roots satisfying (B.33) the associated eigenvalues satisfy the recursive relation (IIc) from section 2.4

\[
\tau_e(\theta_0|\theta) \bigg|_{\theta_{k+1} = \theta_k \pm i\hbar} = \tau_e(\theta_0|\theta_k).
\]

One can also easily verify (2.61) i.e. the fact that the limits \(\theta_0 \to \theta_k - i\hbar\) and \(\theta_{k+1} \to \theta_k + i\hbar\) of \(\tau(\theta_0|\theta)\) commute.

### B.4 Semi-classical limit

The semiclassical limit of the transfer matrix \(T\) (2.23) follows directly from (2.8):

\[
T(\theta_0|\theta)_A = i\hbar \sum_k \frac{\Gamma_k}{\theta_{0k}} + (i\hbar)^2 \left( -\frac{1}{2} \sum_k \frac{\Gamma_k^2}{\theta_{0k}^2} + \sum_k H_k \right) + O(\hbar^3),
\]

with \((\Gamma_k)_A = \delta_{a_n}^{b_n} \cdots \delta_{a_k}^{b_k} \cdots \delta_{a_1}^{b_1}\)

\[
H_k = \sum_{l \neq k} \frac{\Omega_{kl} (\Gamma_k + \Gamma_l)}{\theta_{kl}}, \quad (\Omega_{kl})_A = \delta_{a_n}^{b_n} \cdots \Omega_{a_k a_l}^{b_k b_l} \cdots \delta_{a_1}^{b_1},
\]

and \(\Omega_{ab}^{cd}\) from (2.8). This expansion is valid either in the sense of a formal power series in \(\hbar\) or, with a numerical \(\hbar\), in the region \(\text{Im} \theta_{0k} \gg \hbar, \text{Im} \theta_{lk} \gg \hbar, l \neq k\), in order to prevent a mixing of different powers of \(\hbar\). The absence of a term of order \(\hbar^0\) in (B.39) is due to the tracelessness (2.22) of \(\Gamma\) and distinguishes this case from the usual situation \(\epsilon = 1\) (see e.g. \cite{[54, 57]}). The same fact implies \(\Gamma_k^2 = -1\) and furthermore that the operators \(\Gamma_k\) and
Hamiltonians $H_k$ form a family of mutually commuting operators. (In fact the $H_k$ can be viewed as the Hamiltonians of an abelian $SO(2)$ Knizhnik-Zamolodchikov system). Simultaneous diagonalization of the $\Gamma_k$ yields eigenvectors with only one nonvanishing component $w_{\varepsilon_\ldots\varepsilon_1}$, $(\varepsilon_N, \ldots, \varepsilon_1) \in \{\pm\}^N$, in the charged basis, $\sum_j \epsilon_j = e$. On these eigenvectors the $H_k$ already act diagonally. Thus the first terms in the semiclassical expansion of the eigenvalues $\tau$ are given by:

$$\tau(\theta_0|\theta) = -\hbar \sum_k \frac{\epsilon_k}{\theta_{0k}} + i\hbar^2 \left( \frac{1}{2} \sum_k \frac{\epsilon_k}{\theta_{0k}^2} - \sum_{k \neq l, \epsilon_k = \epsilon_l} \frac{\epsilon_k}{\theta_{0k} \theta_{kl}} \right) + O(\hbar^3) \quad (B.40)$$

This phenomenon can also be understood in terms of the Bethe ansatz. Examination of the explicit solutions of the Bethe roots for $n = 2, 3$ indicates that the symmetry in $\theta_N, \ldots, \theta_1$ gets lost in the limit $\hbar \to 0$ and that they typically behave like

$$t_\alpha(\theta) = \theta_{j(\alpha)} + (i\hbar)^2 s_\alpha(\theta) + O(\hbar^3), \quad (B.41)$$

for some $j(\alpha) \in \{1, \ldots, n\}$ with $j(\alpha) \neq j(\beta)$ for $\alpha \neq \beta$. This curious behavior is directly linked to the seemingly innocent sign flip in the Bethe ansatz equations (B.19). Indeed, entering with (B.41) into the BAE and matching coefficients in powers of $\hbar$ one finds at $O(\hbar)$

$$s_\alpha(\theta) = \frac{1}{4} \sum_{j \neq j(\alpha)} \frac{1}{\theta_j - \theta_{j(\alpha)}} - \frac{1}{2} \sum_{\beta \neq \alpha} \frac{1}{\theta_{j(\beta)} - \theta_{j(\alpha)}}. \quad (B.42)$$

Generally one can show that the Bethe roots admit a power series expansion in $\hbar$ (in the region $\text{Im} \theta_{kl} \gg \hbar, k \neq l$) whose coefficients are uniquely determined by the assignment $\alpha \to j(\alpha)$ in (B.41).

Expanding the BA expression for the eigenvalues (B.18) one obtains

$$\tau_e(\theta_0|\theta) = \hbar \left( \sum_j \frac{1}{\theta_{j0}} - 2 \sum_\alpha \frac{1}{t_\alpha - \theta_0} \right) + O(\hbar^2),$$

$$= \hbar \left( \sum_{j \notin I_\Lambda} \frac{1}{\theta_{j0}} - \sum_{j \in I_\Lambda} \frac{1}{\theta_{j0}} \right) + O(\hbar^2). \quad (B.43)$$

In the second line we inserted (B.41) and denoted by $I_\Lambda = \{j(1), \ldots, j(\Lambda)\} \subset \{1, \ldots, n\}$ the subset of $j$’s appearing on the right hand side of (B.41). Comparing now with the result (B.40) we conclude that $\epsilon_j = 1$ if $j \notin I_\Lambda$ and $\epsilon_j = -1$ if $j \in I_\Lambda$. A similar computation then yields

$$\bar{q}_{e,k}(\theta) = \epsilon_k + O(\hbar), \quad (B.44)$$

which one can also check to be consistent with (B.10).
Appendix C: Explicit solutions for $N \leq 4$

Here we illustrate the solution procedure for the functional equations (I), (II) outlined in section 4 and list the first few members of the charge $e = 0, \pm 1, \pm 2$ sequences. The eigenvectors will be given in the charged basis (B.1); we denote by $\mathcal{F}_{e_0, \ldots, e_1}(\theta)$, $e_j \in \{ \pm \}$ the $N$-th member of the charge $e$ sequence in this basis. Taking advantage of the duality described in appendix B one can restrict attention to positive charges.

Let us begin with the charge $e = 1$ sector. For $N = 1$ one will naturally take $f_{1;\sigma} = \mathcal{F}_{1;\sigma}$ be prescribed non-zero constants which serve as the starting member of the sequence. For later convenience we take $f_{1;\sigma} = \delta_{\sigma,1}$ and $c^{(1)} = 1$. To determine the $n = 3$ member we follow the procedure (a) – (c) described in section 4. The components of the Bethe trial vector (B.17) are

$$w_{1;++}(\theta) = \bar{h}u_3 u_2, \quad w_{1;+--}(\theta) = \bar{h}u_3(u_1 - i\bar{h}), \quad w_{1;--+}(\theta) = \bar{h}(u_2 - i\bar{h})(u_1 - i\bar{h}),$$

(C.1)

where $u_j = \theta_j - t + i\bar{h}/2$ and $t$ is the Bethe parameter. The ansatz (4.13) reads

$$\mathcal{F}_{1;\sigma_3\sigma_2\sigma_1}(\theta) = c_{\sigma_3\sigma_2}(\theta)w_{1;\sigma_3\sigma_2\sigma_1}(\theta),$$

(C.2)

For step (b) one first verifies that the consistency conditions

$$\mathcal{F}_{1;\sigma_3\sigma_2\sigma_1}(\theta) \bigg|_{\theta_3 = \theta_2 + i\bar{h}, t = \theta_2 + i\bar{h}/2} \sim C_{\sigma_3\sigma_2}\mathcal{F}_{1;\sigma_1},$$

$$C_{\sigma_3\sigma_2}\mathcal{F}_{1;\sigma_3\sigma_2\sigma_1}(\theta) \bigg|_{\theta_3 = \theta_2 - i\bar{h}, t = \theta_2 - i\bar{h}/2} \sim \mathcal{F}_{1;\sigma_1},$$

(C.3)

(and a similar pair for $k = 1$) are obeyed. Following (4.13) the symmetric function $\phi_1(\theta)$ searched for to turn (4.10) into identities should be a rational, boost invariant function of the $\theta_j$. Naturally one will select the one with the smallest possible numerator and denominator degrees. Since $\tau(\theta_0|\theta_1) = \bar{h}$ this fixes

$$\mathcal{F}_{1;\sigma_3\sigma_2\sigma_1}(\theta) = 2i\frac{\theta_3^2 + \theta_2^2 - \theta_3\theta_2 - \theta_3\theta_1 - \theta_2\theta_1 + 3\bar{h}^2}{(\theta_21 + i\bar{h})(\theta_31 + i\bar{h})(\theta_32 + i\bar{h})}w_{1;\sigma_3\sigma_2\sigma_1}(\theta).$$

(C.4)

So far only the existence of the sequential Bethe root, i.e. its defining properties (4.14) have been used. For $N = 3$ one can still find it explicitly, a presentation valid for $\theta \in \mathbb{R}^3$. 
is
\[
t(\theta) = \frac{1}{3}(\theta_1 + \theta_2 + \theta_3) - \frac{1}{6}i s^{1/3} + \frac{i}{6}s^{-1/3}[9h^2 + 4(\theta_{12}\theta_{13} + \theta_{23}\theta_{21} + \theta_{31}\theta_{32})]
\]
\[
s := 4i(\theta_{13} + \theta_{23})(\theta_{12} + \theta_{32})(\theta_{21} + \theta_{31})
\]
\[
+ [(9h^2 + 4(\theta_{12}\theta_{13} + \theta_{23}\theta_{21} + \theta_{31}\theta_{32}))^3 - 16(\theta_{13} + \theta_{23})^2(\theta_{12} + \theta_{32})^2(\theta_{21} + \theta_{31})^2]^{1/2},
\]
where the expression under the square root is positive for all \( \theta \in \mathbb{R}^3 \). Further \( s^{1/3} \) is defined to be the cube root of \( s \) that is real for \( \theta_3 = \frac{1}{2}(\theta_1 + \theta_2) \) (and cyclic) and equals the positive square root of \([9h^2 + 4(\theta_{12}\theta_{13} + \theta_{23}\theta_{21} + \theta_{31}\theta_{32})]\). With this choice one has
\[
[s^{1/3}]^* = [9h^2 + 4(\theta_{12}\theta_{13} + \theta_{23}\theta_{21} + \theta_{31}\theta_{32})] s^{-1/3}, \quad \text{for } \theta \in \mathbb{R}^3,
\]
and \( t(\theta) \) is indeed real for \( \theta \in \mathbb{R}^3 \). It is also instructive to study the branch points of this Bethe root. They are located at the zeros of the square root in \( (C.3) \) and have no intersection with the strip \( |\text{Im} \theta| < h \). E.g., as a function of \( \theta_3 \) the Bethe root has 4 branch points of order 2 such that moving \( \theta_3 \) around two of them interchanges the two non-sequential Bethe roots whereas the other two separate the sequential Bethe root from the non-sequential ones. Under pinching \( \theta_2 \rightarrow \theta_1 + ih \) the latter two vanish at complex infinity which is just in agreement with the desired behavior \( \{1, 1, 1\} \).

Having illustrated the procedure for the charge \( e = 1 \) case we now just present the results for the \( e = 2 \) and \( e = 0 \) series. For \( e = 2 \) we take \( \mathcal{T}^{++}_+(\theta) = -i(\theta_{21} - ih) \) with \( c^{(2)} = 1 \) as the starting member. The \( n = 4 \) member is conveniently expressed in terms of the Bethe trial vectors, which for \( n = 4, \ e = 2 \) read
\[
w_{2,+++}(\theta) = hu_4 u_3 u_2,
\]
\[
w_{2,++-}(\theta) = hu_4 u_3 (u_1 - ih),
\]
\[
w_{2,+-+}(\theta) = hu_4 (u_2 - ih)(u_1 - ih),
\]
\[
w_{2,-++}(\theta) = h(u_3 - ih)(u_2 - ih)(u_1 - ih),
\]
with \( u_j = \theta_j - t + ih/2 \). The symmetric rational function \( \phi_2(\theta) \) is conveniently described in terms of a basis of boost invariant symmetric polynomials
\[
\tau_2 = \hat{\theta}_4 \hat{\theta}_3 + \hat{\theta}_4 \hat{\theta}_2 + \hat{\theta}_4 \hat{\theta}_1 + \hat{\theta}_3 \hat{\theta}_2 + \hat{\theta}_3 \hat{\theta}_1 + \hat{\theta}_2 \hat{\theta}_1,
\]
\[
\tau_3 = \hat{\theta}_4 \hat{\theta}_3 \hat{\theta}_2 + \hat{\theta}_4 \hat{\theta}_3 \hat{\theta}_1 + \hat{\theta}_4 \hat{\theta}_2 \hat{\theta}_1 + \hat{\theta}_3 \hat{\theta}_2 \hat{\theta}_1,
\]
\[
\tau_4 = \hat{\theta}_4 \hat{\theta}_3 \hat{\theta}_2 \hat{\theta}_1,
\]
where \( \hat{\theta}_j = \theta_j - \frac{1}{4}\sum_k \theta_k \). Explicitly it is given by
\[
\phi_2(\theta) = -16\tau_3[12\tau_4 + \tau_2^2 - 8h^2\tau_2 + 7h^4],
\]
\[55\]
and satisfies

\[ \phi_2(\theta) \bigg|_{\theta_4=\theta_3+i\hbar} = 2(\theta_{31} + \theta_{32} + i\hbar)(\theta_{21}^2 + \hbar^2)(\theta_{32} + 2i\hbar)(\theta_{31} + 2i\hbar)(\theta_{32} - i\hbar)(\theta_{31} - i\hbar). \]  

(C.9)

The final result is

\[ \mathcal{F}_{2;\sigma_4\sigma_3\sigma_2\sigma_1}(\theta) = \phi_2(\theta) \prod_{k>l} \frac{i}{\theta_{kl} + i\hbar} w_{2;\sigma_4\sigma_3\sigma_2\sigma_1}(\theta). \]  

(C.10)

For the $e = 0$ sequence one has two options, it can start at $N_0 = 0$ or at $N_0 = 2$. Of course already the $N = 2$ members will be different and accordingly two distinct sequences will emerge. For the $N_0 = 0$ series one naturally takes $f = \mathcal{F} = \hbar$ with $c^{(0)} = 1$ as the starting member. The next member of the series is then given by

\[ \mathcal{F}_{0;+-}(\theta) = \frac{2i}{\theta_{21} + i\hbar} hu_2, \]
\[ \mathcal{F}_{0;--}(\theta) = \frac{2i}{\theta_{21} + i\hbar} h(u_1 - i\hbar). \]  

(C.11)

Alternatively one can consider an $e = 0$ series starting at $N_0 = 2$. An appropriate starting member then is

\[ \mathcal{F}_{0;+-}(\theta) = i(\theta_{21} - i\hbar) hu_2, \]
\[ \mathcal{F}_{0;--}(\theta) = i(\theta_{21} - i\hbar) h(u_1 - i\hbar), \]  

(C.12)

and we take $c^{(2)} = 1$. Equivalently this amounts to having $\phi_0(\theta_2, \theta_1) = 2$ for the $N_0 = 0$ series and $\phi_0(\theta_2, \theta_1) = \theta_{21}^2 + \hbar^2$ for the $N_0 = 2$ series.

To describe the $N = 4$ members of both sequences we again first note the the Bethe trial vectors. For $N = 4$, $e = 0$ there are two Bethe parameters $t_1, t_2$. We set $u_j := \theta_j - t_1 + i\hbar/2$, $v_j := \theta_j - t_2 + i\hbar/2$, in terms of which the Bethe trial vectors come out to be

\[ w_{0;++--}(\theta) = \hbar^2 v_4 v_3 u_4 u_3 \left( u_2 v_1 + u_1 v_2 - i\hbar(u_1 + v_2) - \hbar^2 \right) \]
\[ w_{0;+-++}(\theta) = \hbar^2 v_4 u_4 \left( u_3 u_2 (v_2 - i\hbar)v_1 + (u_2 - i\hbar)(u_1 - i\hbar)(v_3 - i\hbar)v_2 \right) \]
\[ - \hbar^4 v_4 u_4 u_3 (u_1 - i\hbar) \]
\[ w_{0;+--+}(\theta) = \hbar^2 v_4 (v_1 - i\hbar) u_4 (u_1 - i\hbar) \left( u_3 v_2 + u_2 v_3 - i\hbar(u_2 + v_3) - \hbar^2 \right) \]
\[ w_{0;--++}(\theta) = \hbar^2 (v_3 - i\hbar)(v_2 - i\hbar)v_1 u_4 u_3 u_2 + \hbar^2 (v_4 - i\hbar)v_3 v_2 (u_3 - i\hbar)(u_2 - i\hbar)(u_1 - i\hbar) \]
\[ - \hbar^4 (v_3 - i\hbar) u_4 u_3 (u_1 - i\hbar) - \hbar^4 v_4 u_4 (u_2 - i\hbar)(u_1 - i\hbar) \]
\[
\begin{align*}
    w_{0,-++}(\theta) &= \hbar^2 (v_1 - i\hbar)(u_1 - i\hbar) \left( (v_3 - i\hbar)v_2 u_4 u_3 + (v_4 - i\hbar)v_3(u_3 - i\hbar)(u_2 - i\hbar) \right) \\
    &\quad - \hbar^4 (v_1 - i\hbar)(u_1 - i\hbar)u_4(u_2 - i\hbar) \\
    w_{0,-++}(\theta) &= \hbar^2 (v_2 - i\hbar)(v_1 - i\hbar)(u_2 - i\hbar)(u_1 - i\hbar) \left( u_4 v_3 + u_3 v_4 - i\hbar(u_3 + v_4) - \hbar^2 \right)
\end{align*}
\]

As required they are invariant under \(u_j \leftrightarrow v_j\) and enjoy the property (B.14).

The symmetric multiplier functions \(\phi_0(\theta)\) are now given by the product of a symmetric polynomial \(v_0(\theta)\) and a factor \(u_0(\theta)\) depending on the first Bethe root. Explicitly

\[
\begin{align*}
    N_0 &= 0 \text{ series:} \quad \phi_0(\theta) = u_0(\theta)[12\tau_4 + \tau_2^2 - 8\hbar^2\tau_2 + 7\hbar^4] \tag{C.13} \\
    N_0 &= 2 \text{ series:} \quad \phi_0(\theta) = u_0(\theta)[16\tau_4\tau_2 - 18\tau_3^2 - 4\tau_2^3 + 18\hbar^2\tau_2^2 - 40\hbar^2\tau_4 - 24\hbar^4\tau_2 + 10\hbar^6],
\end{align*}
\]

where

\[
u_0(\theta) = \frac{2}{\hbar} \frac{U_+ + U_-}{U_+ U_-} \left[ i(U_+ - U_-) + \frac{1}{2\hbar} (U_+ + U_-) (4t_1 - (\theta_4 + \theta_3 + \theta_2 + \theta_1)) \right],
\]

with \(U_{\pm} = \prod_{j=1}^{4} (\theta_j - t_1 \pm i\hbar/2) \tag{C.14}\).

\(u_0(\theta)\) is completely symmetric, boost invariant and real for real \(\theta\)'s. Using

\[
2i\hbar \frac{U_+ - U_-}{U_+ + U_-} \bigg|_{\theta_4 = \theta_3 + i\hbar} = 3\theta_3 + i\hbar - \theta_1 - \theta_2 \mp \sqrt{\theta_3^2 + \hbar^2},
\]

one verifies

\[
u_0(\theta) \bigg|_{\theta_4 = \theta_3 + i\hbar} = \frac{\pm 16\sqrt{\theta_3^2 + \hbar^2}}{(3i\hbar + \theta_32 + \theta_31 \mp \sqrt{\theta_3^2 + \hbar^2})(-i\hbar + \theta_32 + \theta_31 \mp \sqrt{\theta_3^2 + \hbar^2})},
\]

and further (4.10). Finally the \(N = 4\) member of the two \(e = 0\) series is given by

\[
\mathcal{T}_{0,\sigma_4\sigma_3\sigma_2}(\theta) = \phi_0(\theta) \prod_{k>l} \frac{i}{\theta_{kl} + i\hbar} w_{0,\sigma_4\sigma_3\sigma_2}(\theta), \quad \text{(C.15)}
\]

with \(\phi_0(\theta)\) given in (C.13).

The semi-classical limit of these \(N \leq 4\) solutions is readily taken, and one can verify the general pattern described in section 4.2. Specifically let us verify the semi-classical residue equation (4.27), and along the way determine the constant \(\mathcal{Z}\) in (4.26). It is convenient to work with the reduced functions \(\mathcal{F}\). So, in a first step we note the counterparts of equations (1.23) - (4.27) in terms of \(\mathcal{F}\). One finds

\[
\mathcal{F}^{cl}_{e;\sigma_4...\sigma_1}(\theta) = \phi_e^{cl}(\theta) w_{e;\sigma_4...\sigma_1}^{cl}(\theta) \prod_{k>l} \frac{i}{\theta_{kl}}, \quad \text{where}
\]

\[
\mathcal{F}^{cl}_{e;\sigma_4...\sigma_1}(\theta) = \hbar^A \mathcal{F}^{cl}_{e;\sigma_4...\sigma_1}(\theta) + O(\hbar^{A+1}). \quad \text{(C.16)}
\]
If we assume analogously to (4.26)
\[
\lim_{\hbar \to 0} \left[ \hbar^{-1} \mathcal{F}_{e;\sigma_n;\ldots;\sigma_1}(\theta) \right]_{\theta_{k+1} = \theta_k + i\hbar} = \mathcal{Z} \left[ \mathcal{F}_{e;\sigma_n;\ldots;\sigma_1}(\theta) \right]_{\theta_{k+1} = \theta_k + i\hbar}.
\] (C.17)
and similarly for \( \theta_{k+1} = \theta_k - i\hbar \), the recursive equations (4.10) turn into
\[
\mathcal{Z} \mathcal{F}_{e;\sigma_n;\ldots;\sigma_1}(\theta) \bigg|_{\theta_{k+1} = \theta_k} = C_{\epsilon_k+1,\epsilon_k} \prod_{l \neq k+1,k} \theta_{kl}^2 \left( - \sum_{j \neq k+1, k} \frac{\epsilon_j}{\theta_{kj}} \right). \] (C.18)
On the other hand \( f \) and \( \tilde{f} \) are related by (4.12), while \( f^{cl} \) and \( \tilde{f}^1 \) are related by
\[
f^{cl}_{e;\sigma_n;\ldots;\sigma_1}(\theta) = \mathcal{F}_{e;\sigma_n;\ldots;\sigma_1}(\theta) d^{(s)}(\theta) \prod_{k \neq 1} \frac{-i}{\theta_{kl}}. \] (C.19)
Matching (4.26) against (C.17) one finds
\[
Z = (-)^{n-1} \psi_0 \mathcal{Z}, \quad \psi_0 \approx 1.54678. \] (C.20)
It remains to verify (C.18) and to determine \( \mathcal{Z} \). To this end we first note
\[
\lim_{\hbar \to 0} \left[ \phi_{e}(\theta) \right]_{\theta_{k+1} = \theta_k \pm i\hbar} = \left[ \lim_{\hbar \to 0} \phi_{e}(\theta) \right]_{\theta_{k+1} = \theta_k}, \] (C.21)
which for the \( \phi_{e}(\theta) \) involving Bethe roots is not quite automatic. It follows however from the observation that the classical limit of the \( t_\alpha \) (B.41) and the pinching operation (B.33) commute in the relevant situations: (C.21) is only relevant when the right hand side of (C.18) is non-vanishing; that is when \( \epsilon_{k+1} \neq \epsilon_k \), and when a branch of the Bethe roots is selected by having all but \( \theta_{k+1} \) real, say. With these specifications one can choose a labeling of the Bethe roots such that \( j(\alpha) \neq k, k+1 \) for all \( \alpha < \Lambda \). Indeed, since either \( k \in I_\Lambda \) or \( k+1 \in I_\Lambda \), only one of the corresponding \( \theta_j \)'s appears on the right hand side of (B.41), which one can label to be \( \theta_{j(\Lambda)} \). This ensures the asserted commutativity with the pinching operation (B.33).
A similar argument can then be applied to the remainder \( \mathcal{F}_{e}(\theta)/\phi_{e}(\theta) \). The components of the Bethe vectors are symmetric polynomials in the Bethe roots, and after canceling common terms against the \( \prod_{k \neq 1} 1/\theta_{kl} \) numerator, the operation to be performed on the left hand side of (C.17) is known to have a regular limit. The result must thus be proportional to the right hand side. A proportionality constant different from 1 can arise as a remnant of the before-mentioned cancellations. In principle, the constant \( \mathcal{Z} \) could depend on the solution considered. However the explicit evaluation for the \( n \leq 4 \) solutions suggests the universal value \( \mathcal{Z} = -1/2 \).
To see this, note e.g.

\[
\phi_1^{cl}(\theta)\big|_{\theta_3=\theta_2} = -2(\theta_{21})^2,
\]

\[
\phi_2^{cl}(\theta)\big|_{\theta_4=\theta_3} = -2(\theta_{13} + \theta_{23})(\theta_{32}\theta_{31}\theta_{21})^2,
\]

\[
\phi_0^{cl}(\theta)\big|_{\theta_4=\theta_3} = -4(\theta_{21})^3(\theta_{32})^2, \quad n_0 = 2 \text{ series},
\]

\[
\phi_0^{cl}(\theta)\big|_{\theta_4=\theta_3} = -4\theta_{21}(\theta_{32})^2, \quad n_0 = 0 \text{ series}.
\]  (C.22)

From here one can readily verify (C.18) with \(Z = -1/2\).

References

[1] F. Ernst, New formulation of the axially symmetric gravitational field problem, Phys. Rev. 167 (1968) 1175.

[2] R. Geroch, A method for generating solutions of Einstein’s equations, J. Math. Phys. 12 (1971) 918.

[3] V. Belinskii and V. Zakharov, Integration of the Einstein equations by means of the inverse scattering problem technique and construction of exact soliton solutions, Sov. Phys. JETP 48 (1978) 985.

[4] D. Maison, Are the stationary, axially symmetric Einstein equations completely integrable?, Phys. Rev. Lett. 41 (1978) 521.

[5] P. Breitenlohner and D. Maison, On the Geroch group, Ann. Inst. H. Poincaré. Phys. Théor. 46 (1987) 215.

[6] H. Nicolai, Two-dimensional gravities and supergravities as integrable systems, in: Recent Aspects of Quantum Fields, eds. H. Mitter and H. Gausterer, Berlin (1991) . Springer-Verlag.

[7] K. Kuchař, Canonical quantization of cylindrical gravitational waves, Phys. Rev. D4 (1971) 955.

[8] A. Ashtekar and M. Pierri, Probing quantum gravity through exactly soluble midisuperspaces. 1, J. Math. Phys. 37 (1996) 6250.

[9] A. Ashtekar, Large quantum gravity effects: Unforeseen limitations of the classical theory, Phys. Rev. Lett. 77 (1996) 4864.

[10] D. Korotkin and H. Nicolai, Isomonodromic quantization of dimensionally reduced gravity, Nucl. Phys. B475 (1996) 397.
[11] G. A. Mena Marugán, Canonical quantization of the Gowdy model, Phys. Rev. D56 (1997) 908.

[12] J. Cruz, A. Mikovic and J. Navarro-Salas, Free field realization of cylindrically symmetric Einstein gravity, Phys. Lett. B437 (1998) 273.

[13] D. Korotkin and H. Samtleben, Canonical quantization of cylindrical gravitational waves with two polarizations, Phys. Rev. Lett. 80 (1998) 14.

[14] I. Klebanov, I. Kogan and A. Polyakov, Gravitational dressing of renormalization group, Phys. Rev. Lett. 71 (1993) 3243.

[15] J. Ambjørn and K. Ghoroku, 2d quantum gravity coupled to renormalizable matter fields, Int. J. Mod. Phys. A9 (1994) 5689.

[16] M. Gomes and Y. K. Ha, Noncompact sigma model and dynamical mass generation, Phys. Lett. 145B (1984) 235.

[17] Y. K. Ha, Noncompact symmetries in field theories with indefinite metric, Nucl. Phys. B256 (1985) 687.

[18] T. Morozumi and S. Nojiri, An analysis of noncompact nonlinear sigma models, Prog. Theor. Phys. 75 (1986) 677.

[19] M. Gomes and Y. K. Ha, Dynamical gauge boson in $SU(N,1)$-type $\sigma$ models, Phys. Rev. Lett. 58 (1987) 2390.

[20] S. A. Brunini, M. Gomes and A. J. da Silva, Remarks on noncompact sigma models, Phys. Rev. D38 (1988) 706.

[21] J. W. van Holten, Quantum noncompact sigma models, J. Math. Phys. 28 (1987) 1420.

[22] J. Zinn-Justin, Quantum field theory and critical phenomena. Clarendon, Oxford, UK (1989).

[23] F. David, Cancellations of infrared divergences in the two- dimensional nonlinear sigma models, Commun. Math. Phys. 81 (1981) 149.

[24] S. Elitzur, The applicability of perturbation expansion to two-dimensional Goldstone systems, Nucl. Phys. B212 (1983) 501.

[25] B. de Wit, M. T. Grisaru, H. Nicolai and E. Rabinovici, Two loop finiteness of $d = 2$ supergravity, Phys. Lett. B286 (1992) 78.

[26] D. H. Friedan, Nonlinear models in $2 + \epsilon$ dimensions, Ann. Phys. 163 (1985) 318.

[27] C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, Strings in background fields, Nucl. Phys. B262 (1985) 593.

[28] F. A. Smirnov, Form factors in completely integrable models of quantum field theory, no. 14 in Advanced series in mathematical physics. World Scientific, Singapore (1992).
[29] M. R. Niedermaier, An algebraic approach to form-factors, Nucl. Phys. B440 (1995) 603.

[30] M. R. Niedermaier, Form-factors, thermal states and modular structures, Nucl. Phys. B519 (1998) 517.

[31] V. Drinfel’d, Quantum groups, in: Proc. Int. Congress Math., Berkeley 1986, pp. 798–820. AMS, Providence RI (1986).

[32] A. LeClair and F. A. Smirnov, Infinite quantum group symmetry of fields in massive 2-D quantum field theory, Int. J. Mod. Phys. A7 (1992) 2997.

[33] D. Bernard and A. LeClair, The quantum double in integrable quantum field theory, Nucl. Phys. B399 (1993) 709.

[34] N. Reshetikhin and M. Semenov-Tian-Shansky, Central extensions of quantum current groups, Lett. Math. Phys. 19 (1990) 133.

[35] E. Frenkel and N. Reshetikhin, Quantum affine algebras and deformation of Virasoro and W-algebras, Commun. Math. Phys. 178 (1996) 237.

[36] E. Sklyanin, On the complete integrability of the Landau-Lifshitz equation, Preprint LOMI E-3-79, Leningrad (1979).

[37] L. Faddeev, E. Sklyanin and L. Takhtajan, Quantum inverse scattering method, Theoret. Math. Phys. 40 (1979) 194.

[38] V. Drinfel’d, Hopf algebras and the quantum Yang-Baxter equation, Soviet Math. Dokl. 32 (1985) 254.

[39] H. Konno, Free field representation of level $\kappa$ Yangian double $DY(sl(2))(\kappa)$ and deformation of Wakimoto modules, Lett. Math. Phys. 40 (1997) 321.

[40] I. Frenkel and N. Reshetikhin, Quantum affine algebras and holonomic difference equations, Commun. Math. Phys. 146 (1992) 1.

[41] V. Tarasov and A. Varchenko, Asymptotic solutions to the quantized Knizhnik-Zamolodchikov equation and Bethe vectors, in: Mathematics in St. Petersburg, eds. A. B. et al., pp. 235–273. Am. Math. Soc. (1996).

[42] V. Tarasov and A. Varchenko, Geometry of $q$-hypergeometric functions as a bridge between Yangians and quantum affine algebras, Inventiones Mathematicae 128 (1997) 501.

[43] V. Chari and A. Pressley, A Guide to Quantum Groups. Cambridge University Press, Cambridge (1994).

[44] E. Frenkel and E. Mukhin, Combinatorics of $q$-characters of finite-dimensional representations of quantum affine algebras, Preprint math.QA/9911112.

[45] D. Korotkin and H. Samtleben, Yangian symmetry in integrable quantum gravity, Nucl. Phys. B527 (1998) 657.
[46] V. Fock and A. Rosly, Poisson structures on moduli of flat connections on Riemann surfaces and \( r \)-matrices, Preprint ITEP 72-92, Moscow (1992).

[47] N. D. Mermin and H. Wagner, Absence of ferromagnetism or antiferromagnetism in one- dimensional or two-dimensional isotropic Heisenberg models, Phys. Rev. Lett. 17 (1966) 1133.

[48] S. Coleman, There are no Goldstone bosons in two dimensions, Commun. Math. Phys. 31 (1973) 259.

[49] D. Bernard and B. Julia, Twisted self-duality of dimensionally reduced gravity and vertex operators, Nucl. Phys. B547 (1999) 427.

[50] D. Bernard and N. Regnault, Vertex operator solutions of 2d dimensionally reduced gravity, Preprint SPhT-99-017, solv-int/9902017.

[51] M. Pillin, Replica deformation of the \( su(2) \) invariant Thirring model via solutions of the qKZ equation, Preprint KCL-MTH-99-28, hep-th/9907147.

[52] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, Quantum inverse scattering method and correlation functions. Cambridge University Press, Cambridge (1993).

[53] A. N. Kirillov and N. Y. Reshetikhin, The Yangians, Bethe ansatz and combinatorics, Lett. Math. Phys. 12 (1986) 199.

[54] H. Babujian, Off-shell Bethe ansatz equations and \( N \)-point correlators in the \( SU(2) \) WZNW theory, J. Phys. A26 (1993) 6981.

[55] J. M. Maillet and J. S. de Santos, Drinfel’d twists and algebraic Bethe ansatz, Preprint ENSLAPP 601-96, q-alg/9612012.

[56] L. D. Faddeev, Algebraic aspects of the Bethe ansatz, Int. J. Mod. Phys. 10 (1995) 1845.

[57] B. Feigin, E. Frenkel and N. Reshetikhin, Gaudin model, Bethe ansatz and critical level, Commun. Math. Phys. 166 (1994) 27.