Unconventional superconductivity as a synchronization problem in nuclear oscillator networks

V. Velasco¹ and M. B. Silva Neto¹

¹Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, Rio de Janeiro, Brazil

We formulate the problem of unconventional $d$–wave superconductivity, with phase fluctuations, pseudogap phenomenon, and local Cooper pairs, in terms of a synchronization problem in random, quantum dissipative, elasto-nuclear oscillator networks. The nodes of the network correspond to localized, collective quadrupolar vibrations of nuclei-like, elastic inhomogeneities embedded in a dissipative medium. Electrons interacting with such vibrations form local Cooper pairs, with a superfluid $d$–wave pseudogap $\Delta_{PG}$, due to an effective, short range attractive interaction of $d_{x^2−y^2}$ character. Phase coherent, bulk superconductivity, with a $d$–wave gap $\Delta$, is stabilized when the oscillator network is asymptotically entangled in a nearly decoherence-free environment. Phase coherence will in turn be destroyed, at $T_c$, when the thermal noise becomes comparable to the coupling between oscillators, the superfluid density $\kappa$. The $2\Delta/k_BT_c$ is a function of Kuramoto’s order parameter, $r = \sqrt{1 - K_0/K}$, for the loss of synchronization at $K_0$, and is much larger than the nonuniversal $2\Delta_{PG}/k_BT^*$ ratio, where $T^*$ is the temperature at which $\Delta_{PG}$ is completely destroyed by thermal fluctuations. We discuss our findings in connection to the available data for various unconventionally high-temperature superconductors.

I. INTRODUCTION

Quadrupolar vibrations are ubiquitous in nature and often constitute the fundamental normal modes of vibration in a plethora of different physical systems. They arise as the lowest frequency modes of vibration in wineglasses [11], and also as the most relevant tectonic field for density perturbations induced by earthquake ruptures [2]. Accelerating masses moving through spacetime produce ripples that propagate as transverse, quadrupolar gravitational waves [3], and quadrupolar vibrations of the inner crust in a neutron star controls its transient cooling, when coupled to a dissipative, outer thermal bath [4]. At smaller scales, quadrupolar surface vibrations in finite nuclei are known to contribute to the giant quadrupolar resonance [5]. Most importantly, these very same quadrupolar surface vibrations contribute also to a remarkable emergent phenomenon in nuclear matter: superfluidity [6]. For finite nuclei, this is the mechanism behind the opening of superfluid gaps, $\Delta$, in nuclear spectra, for nucleons that minimize their energy, in the presence of a short range attractive nuclear potential, by moving in Cooper paired, time-reversed orbits [7].

The spontaneous emergence of collective behaviour in large oscillator networks is also a phenomenon that has applications in many branches of science. These include the description of the synchronous flash of fireflies [3] and the generation of alpha rhythms in the brain [9]. Huygen’s pendulum clocks, weakly coupled through a wooden beam [10], or metronomes sharing a common base [11] are examples of anti-phase and in-phase classical synchronization, respectively. The dynamics of fast spins coupled to slow exchange interactions in XY spin glasses [12] and the frequency locking in superconducting Josephson junction arrays [13] are, on the other hand, examples of emergent collective behaviour in quantum oscillator networks. In all those cases, the model that has become the simplest paradigm for the synchronization phenomenon is the Kuramoto model [14]. It relies basically on two properties: i) the couplings between the node oscillators in the network: a superfluid density, $\kappa$; and ii) the presence of white noise, quenched, $\delta$, or thermal, $k_BT$, provided by an environment. Then, partial or full synchronization is achieved when the couplings, $K$, outgrow the noise, $\delta, k_BT$, while phase coherence is entirely lost otherwise. The synchronization phase transition in Kuramoto’s model is described by a complex order parameter with a real part $r = 0$, for no–, $0 < r < 1$, for partial–, and $r = 1$, for full–synchronization [15].

In this work we formulate the problem of unconventional $d$–wave superconductivity in terms of a synchronization problem in random, quantum dissipative, nuclear oscillator networks. Our basic requirement is the existence of elastic inhomogeneities as the ones shown in Fig. 1 [16]. We use elasticity theory to calculate the collective, normal modes of vibration at these elastic insertions and show that electrons couple to their low-lying quadrupolar mode. An effective, short range, $d_{x^2−y^2}$–wave attractive, two-particle interaction, leads to the formation of time-reversed, local Cooper pairs, in close analogy to nuclear superfluidity. We calculate the quantum wavefunctions for localized, collective quadrupolar vibrations forming a connected, nuclear oscillator network through their overlap, $K$, and show that, for $K$ stronger than the decoherence noise of the environment, $\delta, k_BT$, phase locking occurs and bulk superconductivity emerges. Finally, we calculate the bulk gap, $\Delta$, the transition temperature, $T_c$, and compare their ratio, $2\Delta/k_BT_c$, given as a function of Kuramoto’s order parameter, $r$, to the available data for various compounds.

Figure 1. Adapted from ref. [16]. Left: gap inhomogeneities deduced from STM spectra at: (a) $T = 60K$ (below), (b) $T = 64K$ (around), and (c) $T = 70K$ (above), the critical $T_c = 65K$ in overdoped $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$. Right: topographic STM image of the positions where the gaps were measured. The STM images clearly show that local Cooper pairs form at random elastic inhomogeneities.
II. QUADRUPOLAR VIBRATIONS

We begin by showing that, already at the classical level, the lowest energetic mode of vibration for an elastic sphere embedded in an infinite homogeneous medium, with different elasto-mechanical properties such as the Lamé parameters, \( \lambda \) and \( \mu \), and the density, \( \rho \), is the quadrupole mode. For that we need to solve Navier’s equation for the displacement

\[
\mathbf{u}(r, \theta, \phi, t) = \mathbf{u}(r, \theta, \phi) \exp(i\Omega t),
\]

which, for the simplest case of a uniform, elastic, isotropic, ideal medium, reads [17]

\[
\frac{\partial^2 \mathbf{u}}{\partial t^2} + \mu \nabla \times (\nabla \times \mathbf{u}) - (\lambda + 2\mu)\nabla \cdot \mathbf{u} = \mathbf{F}. \tag{2}
\]

In eq. (2) \( \mathbf{F} \) represents all volume and surface equilibrium external forces that are provided by the crystalline host enclosing the spherical elastic insertion. The stationary, homogeneous, \( \mathbf{F} = 0 \), solutions to (2) are of the form

\[
\mathbf{u}(r, \theta, \phi) = u_r \mathbf{\hat{r}} + u_\theta \mathbf{\hat{\theta}} + u_\phi \mathbf{\hat{\phi}},
\]

and two types of vibrations arise from the continuity conditions: torsional and spheroidal. Torsional vibrations are characterized by the vanishing of the radial displacement, \( u_r = 0 \), and by the vanishing of divergent displacements, \( \nabla \cdot \mathbf{u} = 0 \). Spheroidal vibrations, on the other hand, are characterized by the vanishing of the radial part of circulation displacements, \( \nabla \times \mathbf{u} \). In what follows we shall be interested in considering radial spheroidal solutions to eq. (2) of the kind [18]

\[
\begin{align*}
    u_r &= \sum_{\ell, m} \left[ A_{\ell, m} d_1(k_p r) + B_{\ell, m} k_s r \right] Y_{\ell, m}, \\
    u_\theta &= \sum_{\ell, m} \left[ A_{\ell, m} b_1(k_p r) + B_{\ell, m} k_s r \right] \frac{\partial Y_{\ell, m}}{\partial \theta}, \\
    u_\phi &= \sum_{\ell, m} \left[ A_{\ell, m} b_1(k_p r) + B_{\ell, m} k_s r \right] \frac{1}{\sin(\theta)} \frac{\partial Y_{\ell, m}}{\partial \phi},
\end{align*}
\]

where \( k_p^2 = \Omega^2 c_s^2 \) and \( k_s^2 = \Omega^2 c_p^2 \), with \( c_s \) and \( c_p \) being the velocities of the transverse and longitudinal elastic waves, respectively [18]. The coefficients \( A_{\ell, m} \) and \( B_{\ell, m} \) are fixed by the boundary conditions of continuity for the displacement, \( \mathbf{u} \), and by the radial, \( \sigma_{rr} \), and shearing, \( \tau_{r\theta} \) and \( \tau_{r\phi} \), stresses, at the radius of the inhomogeneous insertion, \( r = R_0 \), relevant when \( \mathbf{F} \neq 0 \). As usual, \( Y_{\ell, m}(\theta, \phi) \) are the spherical harmonics, and we have defined the function \( b_1(z) = h_1(z) \), for vibrations of the crystalline host, outside the sphere, and the function \( b_1(z) = j_1(z) \) for vibrations of the embedded elastic inhomogeneity, with \( j_1(z) \) and \( h_1(z) \) being the spherical Bessel and Hankel functions, respectively. The use of spherical Bessel and Hankel functions ensures that the displacement, \( \mathbf{u} \), vanishes both at the origin and at infinity, which are natural boundary conditions for any finite elastic deformation. Finally, we have also defined \( d_1(z) = b_1(z) - zh_1(z) \), and \( d_2(z) = (\ell + 1)b_1(z) - 2h_1(z) \), for compactness.

The normal frequencies, \( \Omega \), are found, as usual, from the zeroes of the determinant for the system of coupled equations defined by eq. (2), and their real or complex characters are determined by the ratio between the shear moduli of the enclosing medium and the inclusion, \( \rho = \beta^2 \eta / \rho_i \). Here \( \rho = \rho_e / \rho_i \) is the ratio between the external (e) and internal (i) densities, and \( \beta = c_{se} / c_{si} \) is the ratio between the transverse elastic velocities outside and inside the sphere. For the case of freely vibrating elastic objects (or for a very soft enclosing medium) then \( p \rightarrow 0 \) and all normal frequencies are real, \( \Omega \in \mathbb{R} \); once excited the insertion is set into permanent, stationary vibration. For radial-spheroidal elastic vibrations embedded in an equally elastic, homogeneous crystalline host, however, then \( p \approx 1 \) and we end up with the transcendental equation [18]

\[
\ell(\eta - 1) + g_e(\epsilon, \chi) - \eta g_i(\alpha, \chi) = 0,
\]

where \( \chi = \Omega R_0 / c_{se} \) is a dimensionless frequency written in terms of the transverse elastic velocity in the exterior, \( c_{se} \), while \( \alpha = c_{pe} / c_{pi} \) and \( \zeta = c_{se} / c_{pe} \) are ratios between the transverse and longitudinal elastic velocities outside and inside the sphere. The functions \( g_e(z) = j_{\ell + 1}(z) / j_\ell(z) \), and \( g_i(z) = z h_{\ell + 1}(z) / h_\ell(z) \), are also defined inside and outside the sphere, respectively [18]. In this case, eq. (3) produces, instead, complex solutions, \( \Omega \in \mathbb{C} \), with a real part, \( Re[\Omega] \neq 0 \), that sets the natural frequency of vibration and an imaginary part, \( Im[\Omega] = \gamma \neq 0 \), that provides damping, which can be naturally understood as the decay of the localized, normal vibrations into divergent spherical elastic waves of the enclosing host [18]. Finally, the lowest \( Re[\Omega] \neq 0 \) mode of vibration corresponds to \( \ell = 2 \), localized, collective quadrupolar vibrations, see bottom of Fig. 2 just like for the vibrations of wineglasses [1], or at the core of neutron stars [4], or at the surface of nuclei [2]. The monopole, \( \ell = 0 \), is a breathing mode associated to volume changes at the insertion, see bottom of Fig. 2 while dipole, \( \ell = 1 \), corresponds to translations of the center of mass of the insertion, see bottom of Fig. 2, and both \( \ell = 0, 1 \) solutions are thus too energetically costly.
III. THE $d_{x^2-y^2}$ PARTICLE VIBRATION COUPLING

Once we established that the lowest mode of vibration is the $\ell = 2$ quadrupole mode, let us now proceed and narrow the true vibrational ground state down to the $d_{x^2-y^2}$ mode, which will be always valid for layered, anisotropic systems such as the high temperature cuprates. For that, we recall that quadrupolar deformations from a spherical equilibrium can be parametrized in terms of collective coordinates $\alpha_{2,m}$ \[ R = R_0 \left( 1 + \sum_{m=-2}^{m=+2} \alpha_{2,m} Y^m_2 \right). \] (4)

The equilibrium configuration requires that $\alpha_{2,m} = \alpha_{2,-m}$ and, since the radius of a sphere is always a real quantity, $R \in \mathcal{R}$, one must also impose that $\alpha_{2,0} = \alpha_{2,-0} = 0$, eliminating deformations associated to combinations of $Y^m_2$ and $Y^{-m}_2$. Furthermore, the $R \in \mathcal{R}$ constraint also leads to $\alpha_{2,0} \in \mathcal{R}$ and $\alpha_{2,2} = \alpha_{2,-2} \in \mathcal{R}$, eliminating also deformations associated to combinations of $Y^2_2$ and $Y^{-2}_2$ containing the imaginary unit, $i$. Altogether, these constraints exclude completely any deformations associated to the three $l_{2g}$ orbitals

\[
Y_{2,1c} = \frac{1}{\sqrt{2}} (Y^1_2 - Y^{-1}_2) = \sqrt{\frac{15}{4\pi}} \left( \frac{xz}{r^2} \right),
\]

\[
Y_{2,1s} = \frac{i}{\sqrt{2}} (Y^1_2 + Y^{-1}_2) = \sqrt{\frac{15}{4\pi}} \left( \frac{yz}{r^2} \right),
\]

\[
Y_{2,2s} = \frac{i}{\sqrt{2}} (Y^{-2}_2 - Y^2_2) = \sqrt{\frac{15}{4\pi}} \left( \frac{xy}{r^2} \right).
\]

The $\alpha_{2,0} \in \mathcal{R}$ and $\alpha_{2,2} = \alpha_{2,-2} \in \mathcal{R}$ constraints leave us then with only two symmetry allowed, real, $e_g$ orbitals

\[
Y_{2,0} = Y_2^0 = \sqrt{\frac{5}{16\pi}} (3 \frac{z^2}{r^2} - 1),
\]

\[
Y_{2,2c} = \frac{1}{\sqrt{2}} (Y^{-2}_2 + Y^2_2) = \sqrt{\frac{15}{16\pi}} \left( \frac{x^2 - y^2}{r^2} \right),
\]

where $s, c$ stand for sine and cosine, respectively.

For isotropic or weakly anisotropic systems the two $e_g$ deformations in eq. (5) are degenerate (or nearly) and the lowest elastic quadrupolar vibrations should contain admixtures between the two $Y_{2,0}$ and $Y_{2,2c}$ orbitals. This might, perhaps, be relevant to the description of the two-gap structure observed by ARPES in some cuprates \[20\]. For strongly anisotropic, weakly-coupled, layered structured systems, however, such as the majority of the high-temperature cuprates, the degeneracy between the two $e_g$ orbitals is lifted because of the large energy cost for planar stretches required by the $Y_{2,0}$ oblate and prolate deformations (see Fig. 3 left). In this case, the true, lowest order, radial-spheroidal vibration mode is the quadrupolar $d_{x^2-y^2}$ mode, associated to the $Y_{2,2c}$ orbital (see Fig. 3 right). It is worth point out that such elastic $d_{x^2-y^2}$ mode breaks the $C_4$ rotational symmetry of the lattice and may be related to the nematicity recently observed in several layered cuprates, in the pseudogap phase \[21,22\].

Figure 3. Equatorial cuts for the $Y_{2,0}$ (left) and $Y_{2,2c}$ (right) radial-like spheroidal displacements of eq. (5). For layered, anisotropic systems, such as the cuprates, the configuration which minimizes the elastic energy is $Y_{2,2c}$ (right), and the true ground state for elastically deformable vibrations corresponds to the $d_{x^2-y^2}$ symmetry.

Before we are able to write down the particle vibration coupling (PVC) Hamiltonian we must first quantize quadrupolar vibrations. A quantum mechanical Hamiltonian for deformations $Y_{2,0}$ and $Y_{2,2c}$ can be written in terms of the Bohr shape variables, $\alpha_{2,0} = \beta \cos \gamma$ and $\alpha_{2,2} = \alpha_{2,-2} = (\beta/\sqrt{2}) \sin \gamma$ \[19\], such that $\beta^2 = \sum_{m=-2}^{m=+2} \alpha_{2,m}^2$ and thus

\[
H = -\frac{\hbar^2}{2B} \sum_m \frac{\partial^2}{\partial \alpha_{2,m} \partial \alpha_{2,m}^*} + C \beta^2,
\]

where $B$ is some inertial parameter for the quadrupolar vibration of normal frequency $\Omega = \sqrt{C/B}$. This is a five dimensional problem in which the separable eigenfunctions of $H$, $\Psi(\beta, \gamma, \theta) = f(\beta) \Phi(\gamma, \theta)$, are written not only in terms of the two Bohr shape variables, $\beta$ and $\gamma$, but also in terms of three Euler angles, $\theta_{1,2,3}$. In what follows we fix the crystal-to-laboratory reference frames, through a convenient choice for the Euler angle, $\theta$, and we find for the radial part

\[
f(\beta) = F_{n,\tau} \beta^\tau e^{-s^2/2} L_{n + 3/2}^\tau (s^2 \beta^2),
\]

with spectrum $E_{n,\tau} = \hbar \Omega (N + 3/2)$, and $N = 2n + \tau$ \[24\]. Here $F_{n,\tau}$ is a normalization constant, $n$ is the index of the radial solution, $s = (BC/\hbar^2)^{1/4}$ is the oscillator stiffness, $L_{n + 3/2}^\tau$ is a Laguerre polynomial, and $\tau$ is the seniority \[24\].

Particles of effective mass, $m^*$, moving in the vicinity of an elastic inhomogeneity are subject to a local potential, $U(r)$, entering the single-particle Schrödinger’s equation as

\[
\left[ -\frac{\hbar^2}{2m^*} \nabla^2 + U(r) \right] \varphi_k(r) = \epsilon(k) \varphi_k(r),
\]

from which one obtains the single-particle wave functions, $\varphi_k(r)$, and single-particle energies, $\epsilon(k)$. The linear deformation potential for small displacements, $\alpha^2 \ll \alpha$, reads \[6\]

\[
\delta U(r) = -R_0 \frac{\partial U(r)}{\partial r} \sum_m \alpha_{2,m} Y_{2,m}.
\]
and the matrix element for a process where an electron, at initial state $k$, scatters off an elastic insertion, setting it into vibration of the $Y_{2,2c}$ type, and goes into a final state $k'$ is \[ \Gamma_{k,k'} = -\Gamma_0 \int d^3r \varphi^*_k(r) \left[ R_0 \frac{\partial U(r)}{\partial r} Y_{2,2c}(r) \right] \varphi_k(r), \] \[ (9) \]

where $\varphi_k(r)$, $\varphi^*_k(r)$ are the single-particle wave functions solutions to eq. (6). Here

$$
\Gamma_0 = \sqrt{\frac{\hbar}{2\beta\Omega}} = (2,2c|\hat{\alpha}_{2,2c}|0,0),
$$

is the reduced matrix element for the quadrupolar deformations, $\hat{\alpha}_{2,2c}$, which, in second quantized form and in the Heisenberg picture, are given as

$$
\hat{\alpha}_{2,2c}(t) = \sqrt{\frac{\hbar}{2\beta\Omega}} \left( b_{2,2c}^\dagger e^{-i\Omega t} + b_{2,2c} e^{i\Omega t} \right), \]

(10)

where $b_{2,2c}^\dagger$ and $b_{2,2c}$ are bosonic, phonon creation and annihilation operators for the $Y_{2,2c}$ quadrupolar vibrational modes with frequencies $\Omega = \sqrt{C/B}$. We recognize eq. (10) as the second quantized, quantum mechanical version of eq. (1).

Within the adiabatic, Born-Oppenheimer approximation the motion and quadrupolar vibrations occur at very different time scales and to all purposes the phases $e^{\pm i\Omega t}$ appearing in eq. (10) can be omitted. We are then ready to write down the local PVC Hamiltonian between conduction electrons and the $d_{x^2-y^2}$ localized, collective quadrupolar vibrations in Born-Oppenheimer approximation as

$$
H_{\text{pvc}} = \sum_{k,k',\sigma} \Gamma_{k,k'} c_{k',\sigma}^\dagger c_{k,\sigma} (b_{2,2c}^\dagger + b_{2,2c}), \]

(11)

where $c_{k,\sigma}, c_{k,\sigma}^\dagger$ are the usual fermionic, creation and annihilation operators for electrons with wavevector $k$ and spin $\sigma$, associated to the single-particle wavefunctions $\varphi_k(r)$, and with dispersion given by the single-particle energies $\epsilon_k$.

\section*{IV. LOCAL QUADRUPOLAR SUPERCONDUCTIVITY}

Consider now the problem of a band of conduction electrons interacting with an isolated, localized, collective quadrupolar vibration, as described by the total Hamiltonian

$$
H = \sum_{k,\sigma} \xi_k c_{k,\sigma}^\dagger c_{k,\sigma} + \hbar \Omega \left( b_{2,2c}^\dagger b_{2,2c} + \frac{5}{2} \right) + H_{\text{pvc}}, \]

(12)

where $\langle b_{2,2c}^\dagger b_{2,2c} \rangle = N$ and $\xi_k = \epsilon_k - \mu$, with dispersion relation $\epsilon_k$ relative to the chemical potential $\mu$. The particle-vibration-coupling in (11) produces, in second order perturbation theory, an effective two-particle interaction

$$
V_{k,k'} = \left| \Gamma_{k,k'} \right|^2 \frac{\hbar \Omega}{(\xi_k - \xi_{k'})^2 - (\hbar \Omega)^2},
$$

\[ (13) \]

that is attractive for electrons close to the Fermi surface, $|\xi_k|, |\xi_{k'}| \ll h \Omega$. The explicit form of such two-particle interaction is obtained through a canonical transformation to eliminate the phonons producing a BCS Hamiltonian \[ (23) \]

$$
H = \sum_{k,\sigma} \xi_k c_{k,\sigma}^\dagger c_{k,\sigma} + \sum_{k,k'} V_{k,k'} c_{k,\sigma}^\dagger c_{k',\sigma}, \]

showing that electrons minimize their energy by moving in Cooper paired, time-reversed orbits. From this BCS Hamiltonian one makes the usual mean-field decoupling of the quartic interaction leading to the self-consistent gap equation

\begin{equation}
\Delta_k = -\frac{\sum_{k'} V_{k,k'} \Delta_{k'}}{2 \sqrt{\xi_k^2 + (\Delta_k)^2} \tanh \left( \frac{\sqrt{\xi_k^2 + (\Delta_k)^2}}{2k_B T} \right)}, \tag{12}
\end{equation}

Since by construction $V_{k,k'}$ is separable we may write \[ (26,28) \]

$$
V_{k,k'} = -V_0 \eta(k) \eta(k') w(k) w(k') \Theta(|\xi_k| - h \Omega) \Theta(|\xi_{k'}| - h \Omega),
$$

\[ (13) \]

is a momentum dependent form factor \[ (23,29) \], obtained from the Fourier transform of a Gogny-type short range interaction \[ (20) \], in terms of a material dependent parameter $b \sim 1/R_0$. At zero temperature, $T = 0$, the pseudogap equation reads

$$
1 = V_0 \sum_{k'} \frac{\eta^2(k') w^2(k') \Theta(|\xi_k| - h \Omega)}{2 \sqrt{\xi_k^2 + \eta^2(k') w^2(k') \Delta_{PG}^2}}, \]

\[ (14) \]

As usual we perform the average $\eta^2(k') \rightarrow \langle \eta^2 \rangle_F S$, where $\langle \cdots \rangle_F S$ stands for angular average over the Fermi surface, and since $h \Omega \ll \epsilon_F$ we can set $w^2(k) \rightarrow w^2(k_F)$ to arrive at the local (pseudo) gap at $T = 0$

$$
\Delta_{PG} = \frac{2h \Omega}{\sqrt{\langle \eta^2 \rangle_F S w(k_F)}} \exp \left\{ -\frac{1}{\lambda \langle \eta^2 \rangle_F S w^2(k_F)} \right\},
$$

\[ (15) \]

where $\lambda = N(\epsilon_F)/V_0$ is proportional to the density of states at the Fermi level. At the temperature $T^*$ the local pseudogap, $\Delta_{PG}$, is completely destroyed by thermal fluctuations. In order to calculate $T^*$ one needs to solve the gap equation (12) for $\Delta_{PG}(T^*) = 0$

$$
1 = V_0 \sum_{k'} \frac{\eta^2(k') w^2(k') \tanh (|\xi_{k'}|/2k_B T^*) \Theta(|\xi_{k'}| - h \Omega)}{2|\xi_{k'}|},
$$

\[ (16) \]

whose solution is, after performing the same averaging procedures as done for $\Delta_{PG}$, given by

$$
k_B T^* = 1.13 h \Omega \exp \left\{ -\frac{1}{\lambda \langle \eta^2 \rangle_F S w^2(k_F)} \right\}, \tag{14}
$$

\[ (17) \]

We are now ready to calculate the pseudo-gap-to-$T^*$ ratio

$$
\frac{2\Delta_{PG}}{k_B T^*} = \frac{3.53}{\sqrt{\langle \eta^2 \rangle_F S w(k_F)}}, \tag{15}
$$

\[ (18) \]

which is a nonuniversal ratio, due to the factor $w(k_F)$, larger than BCS’s 3.53, and can, in fact, be as large as 8 \[ (16) \].
associated, via fluctuation-dissipation theorem, to the damping, $\text{Im} [\Omega_i] \equiv \gamma \neq 0$, that controls the decay of the localized quadrupolar vibrations into thermal modes of the bath $[31]$. For $N$ independent oscillators, $K_{ij} = 0, \forall i,j$, the solution to Kuramoto’s model in eq. (19) is simply

$$\theta_i(t) = \Omega_i t + \phi_i,$$

where $\phi_i$ is an arbitrary phase constant. In this case, the second-quantized displacements at the independent quadrupolar oscillators are written in Heisenberg’s representation as

$$\hat{\alpha}_2,2c,i(t) = \sqrt{\frac{\hbar}{2B\Omega_i}} (b_{2,2c,i}^e - i\theta_i(t) + b_{2,2c,i}^e i\theta_i(t)).$$  (20)

Second order perturbation theory then gives rise to attractive, two-particle interactions at each independent oscillator, and to local superconductivity, as described earlier, being, however, incoherent and unable to produce bulk superconductivity.

On the other hand, when $K_{ij} \neq 0$, Kuramoto’s order parameter, defined as $[15]$

$$r(t) = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)},$$

quantifies the overall entanglement of the oscillator network, in terms of the relative strengths between fluctuations and hybridization. If the spread of $g(w)$ is larger than $K_{ij}$, which is the case of dilute, distant elastic insertions, $s^2 |r_i - r_j|^2 \gg 1$, then all oscillators perform individual cycles, all phases $\theta_i(t)$ are uniformly distributed over a circle $[0, 2\pi]$, and $r = 0$: synchronization is not possible. In Fig. 4 (left) we show a dilute, random collection of quadrupolar oscillators whose overlap is not enough to overcome the spread of their distributed, individual natural frequencies. As a result, although at each elastic inhomogeneity attractive two-particle interactions emerge, their local Cooper pairs do not exhibit phase coherence and bulk superconductivity is not possible. For stronger overlaps, $K_{ij}$, however, when elastic insertions are more abundant and closer together, $s^2 |r_i - r_j|^2 \sim 1$, larger number of oscillators start to group together into clusters of a definite phase, $\Theta$, and then $r \neq 0$: partial or full synchronization is achieved. In Fig. 4 (right) we show a larger, random collection of quadrupolar oscillators whose overlap is now enough to overcome the spread of the distributed, individual natural frequencies. In this case the attractive two-particle interactions at all elastic inhomogeneities give rise to coherent Cooper pairing and to bulk superconductivity. The surge of bulk superconductivity in random oscillator networks, as $N \to \infty$, can thus be described by Kuramoto’s order parameter, $r$, which, by using a mean field approximation, $K_{ij} = K, \forall i,j$, is found as $[31]$

$$r(\delta, T) = \sqrt{1 - \frac{K_{c}(\delta, T)}{K}},$$

$$\frac{2}{K_{c}(\delta, T)} = \int_{-\infty}^{\infty} dw \frac{g(D(w) + \Omega)}{w^2 + 1},$$

with $D(w) = \gamma k_B T w$. $[31]$ Furthermore, the equivalence between Kuramoto’s and the XY model $[31] [32]$ allows us to identify $K$ as the superfluid density which controls a second order, synchronization phase transition for $K \geq K_c(\delta, T)$.
where \( b^\dagger_\alpha, b_\sigma \) are bosonic, global phonon creation and annihilation operators for the \( \Theta \)—phase-locked, globally entangled oscillator network, associated to the asymptotic, lowest frequency, \( \Omega_\sigma \), corresponding to the \( \sigma \)—bonded eigenvector \( \hat{\alpha}'_\sigma \).

The globally entangled form for the PVC produces, in second order perturbation theory and within the spirit of the adiabatic, Born-Oppenheimer approximation, an effective, real, attractive two-particle interaction which for electrons with \( |\xi_k|, |\xi_k'| \ll \hbar\Omega_\sigma \) is separable and can be written as

\[
\Gamma_{k,k'} = -r^2 \nabla_0 \eta(k) \eta(k') \Theta(|\xi_k| - \hbar\Omega_\sigma) \Theta(|\xi_{k'}| - \hbar\Omega_\sigma),
\]

\[
\Delta_k = \Delta\eta(k) r e^{i\Theta},
\]

where \( \nabla_0 = |\Gamma_{k,k}'|^2 / \hbar\Omega_\sigma \), corresponds to the convolution of all short range interactions at the local elastic insertions. From the self-consistent gap equation in eq. (12) and the BCS interaction in eq. (24), the \( T = 0 \) bulk gap thus becomes

\[
\Delta = \frac{2\hbar\Omega_\sigma}{r(\delta,0)\sqrt{\eta^2}_{FS}} \exp\left\{-\frac{1}{r^2(\delta,0)\bar{X}(\eta^2)_{FS}}\right\},
\]

where \( \bar{X} = N(\epsilon_F)\nabla_0 \). The bulk gap, \( \Delta \), is then a function of the superfluid density, \( K \), through Kuramoto’s order parameter, \( r(\delta,0) = \sqrt{1 - K_c(\delta,0)/K} \), and it’s evolution across the synchronization phase transition is shown in Fig. 5 for: (A) no inhomogeneities and \( r = 0 \); (B)-(C) few inhomogeneities and \( 0 < r < 1 \); and (D) several inhomogeneities and \( r \rightarrow 1 \).

The critical temperature, in turn, is obtained as solution to the gap equation (12) and the BCS interaction (24) at \( \Delta = 0 \)

\[
k_{B}T_c = 1.13\hbar\Omega_\sigma \exp\left\{-\frac{1}{r^2(\delta,0)\bar{X}(\eta^2)_{FS}}\right\},
\]

which is a transcendental equation for \( T_c \) given in terms of \( r(\delta, T_c) = \sqrt{1 - K_c(\delta, T_c)/K} \). Nevertheless, the ratio between \( \Delta \) and \( T_c \) can be calculated for \( \gamma k_{B}T_c \ll r^2(\delta,0)K \)

\[
\frac{2\Delta}{k_{B}T_c} = \frac{3.53/\sqrt{\eta^2_{FS}}}{(1 - K_c/K)} \exp\left\{ \frac{2\gamma k_{B}T_c / K}{(1 - K_c/K)^2\bar{X}(\eta^2)_{FS}} \right\},
\]

and is exponentially larger than \( 2\Delta_{PG}/k_{B}T^* \) because of dissipation, \( \gamma \neq 0 \), associated to the decay of the collective quadrupolar vibrations into thermal modes of the bath

\[2\Delta/k_{B}T_c \gg 2\Delta_{PG}(k_F)/k_{B}T^* \text{ for } r^2(\delta, T_c) \ll r^2(\delta,0).\]

VII. COMPARISON TO EXPERIMENTS

In figure 6 we show the behaviour of (13) and (27) as functions of the superfluid density \( K \). The \( \Delta_{PG}/T^* \) ratio does not depend on \( K \) and, apart from a nonuniversal factor, \( w(k_F) \), it is given by \( 3.53/\sqrt{\eta^2_{FS}} \approx 3.53\sqrt{2} = 4.99 \). The \( \Delta/T_c \) ratio, in turn, depends exponentially on \( K \).

Even if in cuprates the Uemura plot suggests that the ratio \( k_{B}T_c / K \approx \text{const} \) [36], a \( K \) dependence in (27) is provided by \( r(\delta,0) = \sqrt{1 - K_c/K} \) appearing both as prefactor and as
Localised vibrations in cuprates have been observed with Raman spectroscopy [38], inelastic neutron scattering [39], and ultrafast optical coherent spectroscopy [40], but their role to the mechanism of superconductivity remains controversial. STM spectroscopy, on the other hand, show unambiguously that local Cooper pairs form at elastic inhomogeneities seen in STM topographic images, as shown in Fig. 1 [16]. In this work, we showed that collective quadrupolar vibrations localized at random, elastic inhomogeneities, are able to capture several key features of unconventional superconductivity. Using concepts from elasticity theory, nuclear superfluidity, and the emergence of collective behaviour in oscillator networks, we have shown that the phenomena of phase fluctuations, pseudogap formation, local $d_{x^2-y^2}- wave Cooper pairing, and phase coherent, bulk high-temperature superconductivity arise naturally from a synchronization problem in nuclear oscillator networks. Furthermore, we speculate that the quadrupolar nature of the node oscillators may be associated to the nematicity observed in the pseudogap phase [21–23], while its double degeneracy, for weakly anisotropic systems, may be relevant to the understanding of the two-gap structure also observed in the pseudogap phase [20]. We hope our formulation of unconventional superconductivity, may help to shed some light into the remarkable emergent phenomenon of unconventional high-temperature superconductivity.

Table I. $\eta$ values extracted from [37]. Rough estimates for $T_c$ from [26], assuming full synchronization, $r \to 1$, nanometer-size insertions, and the weak-coupling parameter, $\lambda = 0.8$.

| Comp./Param. | $\eta$ | $\zeta$ | $R_0$(nm) | $\Omega_0$(eV) | $T_c$(K) |
|-------------|--------|--------|-----------|--------------|----------|
| YBa$_2$Cu$_3$O$_{7-x}$ | 0.50 | 0.50 | 0.624 | 0.4 | 0.056 | 86 |
| Bi$_2$Sr$_2$CaCu$_2$O$_{8+x}$ | 0.54 | 0.54 | 0.609 | 0.3 | 0.049 | 74 |

VIII. CONCLUSION

MBSN acknowledges Roberta Zambrini for useful comments. This work was supported by CAPES.
[17] L. D. Landau, E. M. Lifshitz, A. M. Kosevich, and L. P. Pitaevskii, in *Theory of Elasticity*, vol. 7, Elsevier, 3rd ed. (1986).

[18] V. A. Dubrovskiy, V. S. Morochniki. *Izvezeiya, Earth Physics*, Vol 17, 494 (1981).

[19] A. Bohr and B. R. Mottelson, in *Nuclear Structure*, World Scientific Publishing Co. Pte. Ltd. (1975).

[20] T. Yoshida, M. Hashimoto, S. Ideta, A. Fujimori, K. Tanaka, N. Mannella, Z. Hussain, Z.-X. Shen, M. Kubota, K. Ono, S. Komita, Y. Ando, H. Eisaki, and S. Uchida, Phys. Rev. Lett. **103**, 037004 (2009).

[21] R. Daou et al., *Nature* **463**, 519 (2010).

[22] E. Fradkin, S. A. Kivelson, M. J. Lawler, J. P. Eisenstein and A. P. Mackenzie, Annu. Rev. Cond. Matt. **1**, 153 (2010).

[23] Y. Sato et al., *Nat. Phys*. **13**, 1074 (2017).

[24] T. M. Corrigan, F. J. Margetan, and S. A. Williams, Phys. Rev. C **14**, 2279 (1976).

[25] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[26] D. Gogny, Nucl. Phys. A **237**, 399 (1975).

[27] R. Kennedy, L. Wilets, and E. M. Henley, Phys. Rev. **133**, B1131 (1964).

[28] A. Sedrakian, T. T. S. Kuo, H. Müther, and P. Schuck, Phys. Lett. B **576**, 68 (2003).

[29] F. Chappert, N. Pillet, M. Girod, and J.-F. Berger, Phys. Rev. C**91**, 034312 (2015).

[30] G. Manzano, F. Galve, G. L. Giorgi, E. Hernández-García, and R. Zambrini, *Scientific Reports* **3:**1439, 1 (2013).

[31] H. Sakaguchi, Prog. Theo. Phys. **79**, 39 (1988).

[32] T. Uezu, T. Kimoto, S. Kiyokawa, and M. Okada, J. Phys. Soc. Jpn. **84**, 033001 (2015).

[33] T. Hanaguri, K. Iwaya, Y. Kohsaka, T. Machida, T. Watachige, S. Kasahara, T. Shibauuchi, and Y. Matsuda, Sci. Adv. **4**, 1 (2018).

[34] I. H. de Mendoza, L. A. Pachón, J. Gómez-Gardythes, and D. Zueco, Phys. Rev. E**90**, 052904 (2014).

[35] T. Nakano, N. Momono, M. Oda, and M. Ido, J. Phys. Soc. Jpn. **67**, 2622 (1998).

[36] Y. J. Uemura, *Solid State Communications* **126**, 425 (2002).

[37] S. Lin, Y. He, C. Wei, and Z. Shen, *Supercon. Sci. Technol.* **2**, 145 (1989).

[38] P. A. Litvinchuk, C. Thomsen, and M. Cardona, in *Physical Properties of High Temperature Superconductors*, ed. D. M. Ginsberg (Singapore: World Scientific) 1, 375 (1992).

[39] T. Egami, *Physica C: Superconductivity and its Applications*, **460**, 267 (2007).

[40] F. Novelli, G. Giovannetti, A. Avella, F. Cilento, L. Patthey, M. Radovic, M. Capone, F. Parmigiani, and D. Fausti, Phys. Rev. B**95**, 174524 (2017).