The Minkowski norm and Hessian isometry induced by an isoparametric foliation on the unit sphere

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Abstract Let $M_t$ be an isoparametric foliation on the unit sphere $(S^{n-1}(1), g^{st})$ with $d$ principal curvatures. Using the spherical coordinates induced by $M_t$, we construct a Minkowski norm with the representation $F = r \sqrt{2f(t)}$, which generalizes the notions of $(\alpha, \beta)$-norm and $(\alpha_1, \alpha_2)$-norm. Using the technique of the spherical local frame, we give an exact and explicit answer to the question when $F = r \sqrt{2f(t)}$ really defines a Minkowski norm. Using the similar technique, we study the Hessian isometry $\Phi$ between two Minkowski norms induced by $M_t$, which preserves the orientation and fixes the spherical $\xi$-coordinates. There are two ways to describe this $\Phi$, either by a system of ODEs, or by its restriction to any normal plane for $M_t$, which is then reduced to a Hessian isometry between Minkowski norms on $\mathbb{R}^2$ satisfying certain symmetry and $(d)$-properties. When $d > 2$, we prove that this $\Phi$ can be obtained by gluing positive scalar multiplications and compositions of the Legendre transformation and positive scalar multiplications, so it must satisfy the $(d)$-property for any orthogonal decomposition $\mathbb{R}^n = V' + V''$, i.e., for any nonzero $x = x' + x''$ and $\Phi(x) = \xi = \xi' + \xi''$ with $x', \xi' \in V'$ and $x'', \xi'' \in V''$, we have $g^{F_1}(x'', x) = g^{F_2}(\xi'', \xi)$. As byproducts, we prove the following results. On the indicatrix $(S_F, g)$, where $F$ is a Minkowski norm induced by $M_t$ and $g$ is the Hessian metric, the foliation $N_t = S_F \cap \mathbb{R}^n > 0 M_0$ is isoparametric. Laugwitz Conjecture is valid for a Minkowski norm $F$ induced by $M_t$, i.e., if its Hessian metric $g$ is flat on $\mathbb{R}^n \{0\}$ with $n > 2$, then $F$ is Euclidean.

Keywords Minkowski norm, Hessian isometry, Hessian metric, isoparametric foliation, Laugwitz Conjecture, Legendre transformation

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1 Introduction

The classification of isoparametric foliations on the unit sphere $(S^{n-1}(1), g^{st})$ (if not otherwise specified, we will always assume $n > 2$) has been one of the most important geometric problems [60], with a history of eighty years since the time of Cartan [5,6]. There was much remarkable progress [8,11,12,16,27,34,47], and recently it was completely solved by Chi [13]. Meanwhile, researchers are eager to find applications and generalizations of this theory in geometry and topology. For example, its applications in Riemannian geometry and differential topology are concerned in [22,31,40,52,53]. Its generalization, the equifocal hypersurface, is studied in [21,51,55]. Its generalization to Finsler geometry is studied in [24–26,57,59]. More references can be found in the survey papers [20,39,54,56].
In this paper, we consider how to generalize and apply the isoparametric foliation on the unit spheres to Hessian geometry [45] for Minkowski norms. This work is inspired by the recent cowork [58] with Matveev, which implies the interesting connections with the study of Laugwitz Conjecture [29] in convex geometry [42] and Landsberg Unicorn Conjecture [33,44] in Finsler geometry [2].

In this paper, we only consider smooth and strongly convex Minkowski norms on finite-dimensional real vector spaces [2]. For example, a Minkowski norm on \( \mathbb{R}^n \) with \( n \geq 2 \) is a continuous function \( F : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) which is positive and smooth on \( \mathbb{R}^n \setminus \{0\} \), and satisfies the positive 1-homogeneity and the strong convexity (see [2] or Subsection 3.1). Then the Hessian of \( E = \frac{1}{2} F^2 \) is positive definite at each nonzero \( x \), which defines a Riemannian metric \( g = d^2 F \) on \( \mathbb{R}^n \setminus \{0\} \). For simplicity, we call it the Hessian metric of \( F \).

Since the Minkowski norm \( F \) is one-to-one determined by its indicatrix \( S_F = \{ x \in \mathbb{R}^n \mid F(x) = 1 \} \), the geometric properties of the Hessian metric \( g \) or its restriction to \( S_F \) help us understand the convexity of the domain enclosed by \( S_F \) (see [42,45] for more discussion on the relation between Hessian geometry and convex geometry).

Notice that this is only one important model in more general Hessian geometry. Hessian geometers have many other sources for the function \( E \) to construct the metric [23,29,30,46], toric Kähler geometry, the infinite-dimensional integrable system of hydrodynamic type, affine geometry of hypersurfaces, information geometry, etc. More involved discussion for Hessian geometry can be found in [45] and the references therein.

Now we come back to an isoparametric foliation \( M_t \) on the unit sphere \( (S^{n-1}(1), g^B) \subset \mathbb{R}^n \). Here, we parametrize \( M_t \) such that \( t = \text{dist}_{S^{n-1}(1)}(M_t, M_0) \in [0, \frac{\pi}{2}] \), where \( M_0 \) and \( M_{t/d} \) are the two focal submanifolds, and \( d \in \{1,2,3,4,6\} \) is the number of principal curvatures for each \( M_t \) with \( t \in (0, \frac{\pi}{2}) \) in \( (S^{n-1}(1), g^B) \) (see [35,36]). Associated with \( M_t \), we can define the (generalized) spherical coordinates \( (r,t,\xi) \in \mathbb{R}_{>0} \times (0, \frac{\pi}{2}) \times M_{t/d} \), i.e., \( x = (r, t, \xi) \) when \( |x| = r, x/|x| \in M_t \) and there exists a normal geodesic segment in \( (S^{n-1}(1), g^B) \) for this foliation, which connects \( x/|x| \) with \( \xi \) without passing the focal submanifolds. Furthermore, we introduce the spherical local frame induced by \( M_t \) (see Subsection 2.5), with which the standard flat metric \( g^B \) on \( \mathbb{R}^n \setminus \{0\} \) and its Levi-Civita connection can be explicitly calculated.

We can use the foliation \( M_t \) to define a Minkowski norm \( F \) on \( \mathbb{R}^n \) such that the restriction of \( F \) to each \( M_t \) is a constant function. We will simply call it a Minkowski norm induced by \( M_t \). When \( d = 1 \) or \( d = 2 \), the induced \( F \) admits a linear \( SO(n-1) \)- or \( O(k) \times O(n-k) \)-invariance, and is called an \( (\alpha,\beta) \)-norm or \( (\alpha_1,\alpha_2) \)-norm in some literature [10,15]. These norms have attracted much attention of Finsler geometers [28,32]. However, the induced Minkowski norms when \( d > 2 \) have been rarely studied.

By using the spherical \( r \)- and \( t \)-coordinates, the induced Minkowski norm \( F \) can be represented as \( F = r \sqrt{2f(t)} \). A natural and important question is the following.

**Question 1.1.** When does \( F = r \sqrt{2f(t)} \) define a Minkowski norm induced by \( M_t \)?

Notice that, besides the issue of strong convexity, the smoothness of \( F = r \sqrt{2f(t)} \) at \( \mathbb{R}_{>0} M_0 \) and \( \mathbb{R}_{>0} M_{t/d} \) is also subtle and crucial. We use the spherical local frame to calculate the Hessian of \( E = \frac{1}{2} f^2 = \int r^2 f(t) \) as in [58], and then completely answer Question 1.1 by the following theorem.

**Theorem A.** The spherical coordinate representation \( F = r \sqrt{2f(t)} \) defines a Minkowski norm induced by \( M_t \) if and only if \( f(t) \) can be extended to a positive smooth \( D_{2d} \)-invariant function on \( \mathbb{R}/(2\mathbb{Z} \pi) \) which satisfies

\[
2f(t) \frac{d^2}{dt^2} f(t) - \left( \frac{d}{dt} f(t) \right)^2 + 4f(t)^2 > 0
\]

everywhere, i.e., the polar coordinate representation \( \overline{F} = r \sqrt{2f(t)} \) defines a \( D_{2d} \)-invariant Minkowski norm on \( \mathbb{R}^2 \).

Here, \( \mathbb{R}^2 \) can be identified with any normal plane \( V \) for \( M_t \) (i.e., \( V \cap S^{n-1}(1) \) is a normal geodesic for \( M_t \)), and \( D_{2d} \) is the group \( \mathbb{Z}_2 \) when \( d = 1 \) and the dihedral group when \( d > 1 \), which can be interpreted as a Weyl group (see Subsection 2.4 for its explicit description and its action on \( \mathbb{R}^2 \) or \( \mathbb{R}/(2\mathbb{Z} \pi) \)).
Theorem A is a reformulation of Theorem 3.1. Its direct corollaries, Corollaries 3.2 and 3.3, where we take $d = 1$ and $d = 2$, respectively, reprove some known results for Minkowski norms of $(\alpha, \beta)$- and $(\alpha_1, \alpha_2)$-types [14, 15].

Let $F = r \sqrt{2f(t)}$ be a Minkowski norm induced by $M_t$. Then on its indicatrix $S_F$, there is a foliation $N_t = S_F \cap \mathbb{R}_{>0} M_t$ induced by $M_t$. Using the technique of the spherical local frame again, we prove the following theorem (see Theorem 3.7).

**Theorem B.** Let $F$ be a Minkowski norm induced by the isoparametric foliation $M_t$ on $(S^{n-1}(1), g^{st})$ and $g$ its Hessian metric. Then the foliation $N_t = S_F \cap \mathbb{R}_{>0} M_t$ on $(S_F, g)$ is isoparametric.

Theorem B provides more examples of isoparametric foliations. Indeed, when $M_t$ is homogeneous, i.e., it is induced by the isometric cohomogeneity one action of some compact connected Lie group $G$ (see [27, 48, 49] for its classification), the isometric $G$-action on $(S_F, g)$ is also of cohomogeneity one. So the $G$-orbits $N_t$ provide an isoparametric foliation on $(S_F, g)$. Though this shortcut to Theorem B is not valid for inhomogeneous $M_t$ of OT-FKM type (found by Ozeki, Takeuchi, Ferus, Karcher and Münzner) [18, 37, 38], it provides the most crucial hint, and it inspires us to more generally study the Hessian isometries between Minkowski norms. It is also remarkable that a similar correspondence has been found for isoparametric foliations on smooth homotopy spheres (see [19, Theorem 1.1]), where topology rather than geometry or Lie theory plays the main role.

Let $F_1$ and $F_2$ be two Minkowski norms on $\mathbb{R}^n$ with $n \geq 2$, and $g_1$ and $g_2$ be their Hessian metrics, respectively. Then a **Hessian isometry** $\Phi$ from $F_1$ to $F_2$ is a diffeomorphism on $\mathbb{R}^n \setminus \{0\}$ which is an isometry from $g_1$ to $g_2$ (see Subsection 4.1 for its basic properties and local version). A linear isomorphism $\Phi$ on $\mathbb{R}^n$ satisfying $F_1 = F_2 \circ \Phi$ naturally induces a Hessian isometry when restricted to $\mathbb{R}^n \setminus \{0\}$. We call it a **linear isometry** from $F_1$ to $F_2$.

As we have seen, the linear isometry provides us the hint and shortcut to Theorem B. Besides, it also helps us prove a special case of Laugwitz Conjecture [29], which improves [58, Corollary 1.7] (see Theorem 4.5).

**Theorem C.** Let $F$ be a Minkowski norm on $\mathbb{R}^n$ with $n > 2$ induced by the isoparametric foliation $M_t$ on $(S^{n-1}(1), g^{st})$. Suppose that its Hessian metric $g$ is flat on $\mathbb{R}^n \setminus \{0\}$. Then $F$ is Euclidean.

The (possibly) nonlinear Hessian isometry between two Minkowski norms induced by $M_t$ is more intriguing. Generally speaking, its complete classification is a hard problem which involves complicated case-by-case discussion. In this paper, we only concentrate on a subclass, i.e., we consider the triple $(F_1, F_2, \Phi)$, in which $F_1$ and $F_2$ are Minkowski norms induced by $M_t$, and the Hessian isometry $\Phi$ from $F_1$ to $F_2$ preserves the orientation and fixes the spherical $\xi$-coordinates. There are two ways to describe this triple.

We may start with the spherical coordinate representations for $(F_1, F_2, \Phi)$, i.e., $F_1 = r \sqrt{2f(t)}$, $F_2 = r \sqrt{2h(\theta)}$ (we use $\theta$ to define the spherical $t$-coordinate for $F_2$), and

$$\Phi : (r, t, \xi) \mapsto \left(\frac{r f(t)^{1/2}}{h(\theta(t))^{1/2}} \theta(t), \xi\right).$$

We find that $(f(t), h(\theta), \theta(t))$ must satisfy the $D_{2d}$-symmetry and the following ODE system:

$$\frac{1}{2f(t)} \frac{d^2}{dt^2} f(t) - \frac{1}{4f(t)^2} \left(\frac{d}{dt} f(t)\right)^2 + 1 = \left(\frac{d}{dt} \theta(t)\right)^2 \left(\frac{1}{2h(\theta(t))} \frac{d^2}{d\theta^2} h(\theta(t)) - \frac{1}{4h(\theta(t))^2} \left(\frac{d}{d\theta} h(\theta(t))\right)^2 + 1\right), \quad (1.1)$$

$$\sin^2 \left(t + \frac{k\pi}{d}\right) = \frac{\cos(t + \frac{k\pi}{d}) \sin(t + \frac{k\pi}{d})}{2f(t)} \frac{d}{dt} f(t)$$

$$= \sin^2 \left(\theta(t) + \frac{k\pi}{d}\right) + \frac{\cos(\theta(t) + \frac{k\pi}{d}) \sin(\theta(t) + \frac{k\pi}{d})}{2h(\theta(t))} \frac{d}{d\theta} h(\theta(t)) \quad (1.2)$$

for each $k \in \{0, \ldots, d - 1\}$. 


Alternatively, we may restrict \((F_1,F_2,\Phi)\) to any normal plane \(V\). With \(V\) identified with \(\mathbb{R}^2\) (see Subsection 2.4), we obtain a triple \((\mathcal{F}_1,\mathcal{F}_2,\Phi)\) with \(D_{2d}\)-symmetry, where both \(\mathcal{F}_1\) are Minkowski norms on \(\mathbb{R}^2\), and \(\Phi\) is a Hessian isometry between \(\mathcal{F}_1\). In particular, the ODE (1.2) can be interpreted as a \((d)\)-property, defined by the equality \(g^2_k(x^n,x) = g^2_k(x^\prime,x)\) for any nonzero \(x = x^\prime + x''\) and \(\Phi(x) = \pi = \pi' + \pi''\) with respect to a given orthogonal decomposition \(\mathbb{R}^n = V' + V''\) (see Definition 5.5 in Subsection 5.3 and its local version in Subsection 6.3).

Summarizing Theorems 5.3 and 5.6, we obtain the following complete description for Hessian isometries between two Minkowski norms induced by \(M_t\), which preserve the orientation and fix the \(\xi\)-coordinates.

**Theorem D.** Let \(M_t\) be any isoparametric foliation on \((S^{n-1},g^m)\) with \(d\) principal curvatures. Then there are one-to-one correspondences between any two of the following three sets:

1. The set of all the triples \((F_1,F_2,\Phi)\), in which both \(F_1\) and \(F_2\) are Minkowski norms induced by \(M_t\), and \(\Phi\) is a Hessian isometry from \(F_1\) to \(F_2\), which preserves the orientation and fixes the spherical \(\xi\)-coordinates.

2. The set of all the triples \((f(t),h(\theta),\theta(t))\) such that \(f(t)\) and \(h(\theta)\) are \(D_{2d}\)-invariant positive smooth functions on \(\mathbb{R}/(2\pi\mathbb{Z})\) satisfying the requirement in Theorem A, \(\theta(t)\) is a \(D_{2d}\)-equivariant orientation-preserving diffeomorphism on \(\mathbb{R}/(2\pi\mathbb{Z})\) fixing each point in \(\pi\mathbb{Z}\), and the triple is a solution of the ODE system for all \(t \in \mathbb{R}/(2\pi\mathbb{Z})\), which consists of (1.1) and (1.2) for all \(k \in \{0,\ldots, d-1\}\).

3. The set of all the triples \((\mathcal{F}_1,\mathcal{F}_2,\Phi)\), in which both \(\mathcal{F}_1\) are \(D_{2d}\)-invariant Minkowski norms on \(\mathbb{R}^2\), and \(\Phi\) is a \(D_{2d}\)-equivariant orientation-preserving Hessian isometry from \(\mathcal{F}_1\) to \(\mathcal{F}_2\) which satisfies the \((d)\)-property with respect to the decomposition

\[
\mathbb{R}^2 = V' + V'' = \mathbb{R}\left(\cos \left(-\frac{k\pi}{d}\right), \sin \left(-\frac{k\pi}{d}\right)\right) + \mathbb{R}\left(\cos \left(\frac{\pi}{2} - \frac{k\pi}{d}\right), \sin \left(\frac{\pi}{2} - \frac{k\pi}{d}\right)\right)
\]

for each \(k \in \{0,\ldots, d-1\}\).

The correspondences from (1) and (3) to (2) are provided by the spherical coordinate and polar coordinate representations, respectively. The correspondence between (1) and (3) is provided by the restriction to any normal plane \(V\) for \(M_t\) and an identification between \(V\) and \(\mathbb{R}^2\).

Finally, we consider the construction for the Hessian isometry \(\Phi\) in Theorem D.

The Legendre transformation (or its composition with a positive scalar multiplication) provides an important class of (possibly) nonlinear Hessian isometries [42]. Notice that in this paper we have used the standard inner product to identify \(\mathbb{R}^n\) with its dual. So for a Minkowski norm \(F\) induced by \(M_t\), its dual is also a Minkowski norm on \(\mathbb{R}^n\) induced by \(M_t\), and its Legendre transformation preserves the orientation and fixes the spherical \(\xi\)-coordinates. Theorem D (or Theorem 5.6) provides the one-to-one correspondence between Legendre transformations for Minkowski norms induced by \(M_t\) and Legendre transformations for \(D_{2d}\)-invariant Minkowski norms on \(\mathbb{R}^2\) (see Lemma 6.7 and Theorem 6.8 for the precise statements).

More examples for the Hessian isometry \(\Phi\) in Theorem D can be constructed by gluing positive scalar multiplications and the compositions of Legendre transformations and positive scalar multiplications (see Remark 7.3).

On the other hand, when we have \(d > 2\) for the foliation \(M_t\), this gluing construction can exhaust all the wanted \(\Phi\). The method for discussing the ODE system consisting of (1.1) and (1.2) with \(k = 0\) (which corresponds to the \((d)\)-property with \(k = 0\) in Theorem D(3)) has been given in [58], which enables us to locally determine the triple \((f(t),h(\theta),\theta(t))\) around a generic \(t_0 \in (0, \frac{\pi}{d})\). By the assumption \(d > 2\), Theorem D(3) requires essentially more \((d)\)-properties for the triple \((\mathcal{F}_1,\mathcal{F}_2,\Phi)\) (for example, the one with \(k = 1\)). Applying Lemma 6.6 accordingly, we prove the following theorem verifying our observation (see Theorem 7.2 for the more precise statement).

**Theorem E.** Any Hessian isometry between two Minkowski norms induced by an isoparametric foliation on \((S^{n-1},g^m)\) with \(d > 2\), which preserves the orientation and fixes the spherical \(\xi\)-coordinates can be constructed by gluing positive scalar multiplications and compositions of the Legendre transformation.
of $F_1$ and positive scalar multiplications. In particular, it satisfies the (d)-property for any orthogonal decomposition of $\mathbb{R}^n$.

When $d = 1$ or $d = 2$, [58, Theorems 1.4 and 1.5] provide a similar local description for $\Phi$.

The rest of this paper is organized as follows. In Section 2, we introduce the spherical coordinates and the spherical local frame induced by an isoparametric foliation $M_s$ on the unit sphere. In Section 3, we introduce the Minkowski norm induced by $M_s$ and prove Theorems A and B. In Section 4, we introduce the notion of the Hessian isometry and prove Theorem C. In Section 5, we study Hessian isometries between two Minkowski norms induced by $M_s$ and prove Theorem D for those which preserve the orientation and fix the spherical $\xi$-coordinates. In Section 6, we discuss the Legendre transformation and the (d)-property. In Section 7, we use the ODE method and the (d)-property to provide the local description for the Hessian isometry $\Phi$ in Theorem D when $d > 2$, and prove Theorem E.

2 Spherical coordinates and the spherical local frame induced by an isoparametric foliation on the unit sphere

2.1 The isoparametric function and the isoparametric foliation

An isoparametric function on a Riemannian manifold $(M, g)$ is a smooth function $p : M \to \mathbb{R}$ such that it is regular almost everywhere, and its gradient vector field $\text{grad} p$ and its Laplacian $\Delta p$ satisfy

$$g(\text{grad} p, \text{grad} p) = a \circ p \quad \text{and} \quad \Delta p = b \circ p$$

for some one-variable functions $a(s)$ and $b(s)$. For each regular value $s$ of $p$, its pre-image $M_s = p^{-1}(s)$ is called an isoparametric hypersurface [9]. We will also use $M_s$ to define the isoparametric foliation (i.e., the set of all the nonempty $M_s$'s). A geodesic is called normal for (the foliation) $M_s$, if it intersects each $M_s$ orthogonally.

The isoparametric foliation is called homogeneous if there exists a Lie group $G$ of isometric actions on $(M, g)$ such that each $M_s$ is a $G$-orbit, i.e., this isoparametric foliation is induced by the cohomogeneity-one isometric action of $G$. Indeed, any cohomogeneity-one isometric action can locally induce an isoparametric foliation.

2.2 The isoparametric foliation on a unit sphere

On a Euclidean space $\mathbb{R}^n$ with $n \geq 2$, we have the standard Euclidean inner product $(\cdot, \cdot)$, the standard Euclidean norm $| \cdot | = (\cdot)^{1/2}$ and the orthonormal linear coordinates $x = (x_1, \ldots, x_n)$. Meanwhile, we have the standard flat metric on $\mathbb{R}^n$, $g^st = dx_1^2 + \cdots + dx_n^2$. We will also use $g^st$ to define its restrictions to submanifolds.

Any isoparametric foliation on $(S^{n-1}(1), g^st)$ can be related to an isoparametric function $p : S^{n-1}(1) \to [-1, 1]$, which is the restriction of a homogeneous polynomial of degree $d \in \{1, 2, 3, 4, 6\}$ on $\mathbb{R}^n$, where $d$ is the number of principal curvatures. Furthermore, $\pm 1$ are the only critical values of $p(\cdot)$. In this foliation, each $M_s = p^{-1}(s)$ with $-1 < s < 1$ is a closed connected isoparametric hypersurface, and the two critical sets $M_{\pm 1}$ are the two focal submanifolds [35, 36].

There are only two subclasses of isoparametric foliations on the unit spheres [13]. One subclass is those homogeneous ones, which were classified in [27, 49]. The other subclass is of the OT-FKM type [18, 37, 38]. Notice that the OT-FKM type must have $d = 4$, and there is some overlap between the subclasses.

Consider any maximal extended normal geodesic $\gamma \subset (S^{n-1}(1), g^st)$ for $M_s$. It is a great circle, i.e., the intersection between a plane $V$ passing the origin and $S^{n-1}(1)$. We will simply call this $V$ a normal plane for (the foliation) $M_s$, because it coincides with the orthogonal normal complement of $T_x M_s$ in $\mathbb{R}^n = T_x(\mathbb{R}^n \setminus \{0\})$ for $x \in \gamma \cap M_s$. The intersection $\gamma \cap (M_{-1} \cup M_1)$ is the set of a pair of antipodal points when $d = 1$, or the vertex set of a regular $2d$-polygon when $d > 1$. The points in $\gamma \cap M_{-1}$ and in
2.4 The identification between a normal plane and a point.

Denote by dist$_{S^{n-1}(1)}(\cdot,\cdot)$ and dist$_{\gamma}(\cdot,\cdot)$ the distance functions on $(S^{n-1}(1), g^*)$ and $(\gamma, g^*)$, respectively. Then we have

$$\text{dist}_{S^{n-1}(1)}(M_{-1}, M_1) = \text{dist}_\gamma(\gamma \cap M_{-1}, \gamma \cap M_1) = \frac{\pi}{d}.$$  

For any $s \in (-1, 1)$, we have

$$c = \text{dist}_{S^{n-1}(1)}(M_{-1}, M_s) \in \left(0, \frac{\pi}{d}\right)$$

and

$$\gamma \cap M_s = \{ x \in \gamma \mid \text{dist}_\gamma(x, \gamma \cap M_{-1}) = c \}$$

contains $2d$ points. The principal curvatures of $M_s$ with respect to the normal direction represented by grad $p$ are exactly $\cot(c + \frac{k\pi}{d})$, $k = 0, \ldots, d - 1$. The multiplicities of these principal curvatures are crucial for the classification theory, which has been extensively studied (see [1, 17, 47, 50]).

2.3 Parametrization for an isoparametric foliation on the unit sphere

In the later discussion, we will always parametrize an isoparametric foliation on $(S^{n-1}(1), g^*)$ with $d$ principal curvatures as $M_t$ with $t \in [0, \frac{\pi}{d}]$ so that for any $x \in M_t$, we have

$$\text{dist}_{S^{n-1}(1)}(x, M_0) = \text{dist}_{S^{n-1}(1)}(M_t, M_0) = t.$$  

By this parametrization, $M_0$ and $M_{\pi/d}$ are the two focal submanifolds, and all the other $M_t$ are isoparametric hypersurfaces. Restricted to each normal geodesic segment realizing the distance from $M_0$ to $M_t$ with $0 < t \leq \frac{\pi}{d}$, $t$ is a $g^*$-arc length parameter.

Notice that the parameter $t$, when represented as $t(x) = \text{dist}_{S^{n-1}(1)}(x, M_0)$, is an isoparametric function on $(S^{n-1}(1) \setminus (M_0 \cup M_{\pi/d}), g^*)$ for the foliation $M_t$.

2.4 The identification between a normal plane and $\mathbb{R}^2$ with $D_{2d}$-action

Later we will frequently use the following identification between $\mathbb{R}^2$ and a normal plane for the isoparametric foliation $M_t$ on $(S^{n-1}(1), g^*)$.

Let $V \subset \mathbb{R}^n$ be a normal plane for $M_t$, i.e., $\gamma = V \cap S^{n-1}(1)$ is a normal geodesic for $M_t$ in $(S^{n-1}(1), g^*)$. We parametrize $\gamma$ as $\gamma(t)$ by its $g^*$-arc length, and require $\gamma(0) \in M_0$. Then $v_1 = \gamma(0)$ and $v_2 = \gamma(\frac{\pi}{d})$, when they are viewed as unit vectors, provide an orthonormal basis for $V$. We will identify $V$ with $\mathbb{R}^2$ such that $v_1$ and $v_2$ are identified with the standard orthonormal basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, respectively.

This identification depends on the choice of $v_1$ from $\gamma \cap M_0$ and the direction of curve $\gamma(t)$. Changing $v_1$ and changing the direction of $\gamma(t)$ result linear isometries of $\mathbb{R}^2$ which belong to the finite group $D_{2d}$. On the normal plane $V$ for $M_t$, $D_{2d}$ is the group of all the linear isometries which preserve $\gamma \cap M_0$. It is the dihedral group when $d > 1$, and $\mathbb{Z}_2$ when $d = 1$. For $\mathbb{R}^2$, $D_{2d}$ is generated by the right multiplications by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos \frac{2\pi}{d} & \sin \frac{2\pi}{d} \\ -\sin \frac{2\pi}{d} & \cos \frac{2\pi}{d} \end{pmatrix}$$

on row vectors. Alternatively, when we use the polar coordinates $(r, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}/(2\pi)$ for $x = (x_1, x_2)$ $= (r \cos t, r \sin t)$ on $\mathbb{R}^2$ or $x = x_1v_1 + x_2v_2 = r \cos tv_1 + r \sin tv_2$ on $V$, we have the corresponding $D_{2d}$-action on the space $\mathbb{R}/(2\pi)$ of all the polar $t$-coordinates, which is generated by the mappings $t \mapsto -t$ and $t \mapsto t + \frac{2\pi}{d}$.

Generally speaking, as long as the subjects we discuss later have the $D_{2d}$-symmetry (i.e., $D_{2d}$-invariancy or $D_{2d}$-equivariancy), then they do not depend on the identification when we translate them from $V$ to $\mathbb{R}^2$ or vice versa.
2.5 Spherical coordinates and the spherical local frame

Let $M_t$ be an isoparametric foliation on $(S^{n-1}, g^{st})$ with $d$ principal curvatures. For any point $x$ in the conic open subset

$$C(S^{n-1}(1)\setminus(M_0 \cup M_{\pi/d})) = \mathbb{R}_{>0}(S^{n-1}(1)\setminus(M_0 \cup M_{\pi/d})) = \mathbb{R}^n \setminus (\mathbb{R}_{>0}M_0 \cup \mathbb{R}_{>0}M_{\pi/d}),$$

its spherical coordinates induced by (the foliation) $M_t$, $(r, t, \xi) \in \mathbb{R}_{>0} \times (0, \frac{\pi}{d}) \times M_{\pi/2d}$, are determined by the following requirements: $r = |x| > 0$, $x/|x| \in M_t$, and there exists a normal geodesic segment in $(S^{n-1}(1), g^{st})$ for the given foliation, which connects $x$ with $\xi$ without passing the two focal submanifolds.

The mapping from $x$ to its spherical coordinates $(r, t, \xi)$ is a diffeomorphism between $C(S^{n-1}(1)\setminus(M_0 \times M_{\pi/d}))$ and $\mathbb{R}_{>0} \times (0, \frac{\pi}{d}) \times M_{\pi/2d}.

Then we construct the local frame in $C(S^{n-1}(1)\setminus(M_0 \cup M_{\pi/d}))$ with spherical $\xi$-coordinates contained in some sufficiently small open subset $U$ of $M_{\pi/2d}$. Here, the local frame means a set of smooth tangent vector fields defined on the same open subset, whose values at each point provide a basis of the tangent space.

For the spherical $r$- and $t$-coordinates induced by $M_t$, we have the tangent vector fields $\partial_r$ and $\partial_t$. That means that $\partial_r$ generates the rays setting off from the origin, and $\partial_t$ generates the normal geodesics for the foliation $M_{rt} = rM_t = \{rx \mid x \in M_t\}$ on $(S^{n-1}(r)\setminus(M_{0r} \cup M_{\pi/2d}, g^{st})$. Obviously, we have $|\partial_r|^2 = g^{st}(\partial_r, \partial_r) = 1$ and $|\partial_r, \partial_t| = 0$. Our convention for parametrizing $M_t$ implies $|\partial_t|^2 = r^2$.

The other local tangent vector fields, $X_1, \ldots, X_{n-2}$, are tangent to the foliation $M_{rt}$. Firstly, we construct them on $U \subset M_{\pi/2d}$, such that the following are satisfied:

(1) Each $X_i$ is a tangent vector field of constant length on $U$.

(2) For each $i$, there is a principal curvature value $\kappa_i = \cot t + \frac{k_i\pi}{d}$, $k_i \in \{0, \ldots, d-1\}$

for $M_t$ in $(S^{n+1}(1), g^{st})$ with respect to the normal direction $\partial_t$ such that the value of $X_i$ at each point of $U$ is an eigenvector for the eigenvalue $\kappa_i$ of the shape operator.

(3) At each point of $U$, the values of $X_i$ for all $1 \leq i \leq n-2$ provide a $g^{st}$-orthogonal basis for the tangent space of $M_{\pi/2d}$.

Notice that any $\xi \in M_{\pi/2d}$ has a sufficiently small neighborhood $U$ in $M_{\pi/2d}$ with no topological obstacle to the above construction of $X_1, \ldots, X_{n-2}$. If we ignore the multiplicities, then we have $\{k_1, \ldots, k_{n-2}\} = \{0, \ldots, d-1\}$.

Then we extend each $X_i$ such that

$$[\partial_r, X_i] = [\partial_t, X_i] = 0, \quad \forall 1 \leq i \leq n-2.$$

Indeed, if $X_i$ generates the local diffeomorphism $\rho_s$ on $M_{\pi/2d}$, then after the extension, it generates $\tilde{\rho}_s$ with the spherical coordinate representation $\tilde{\rho}_s(r, t, \xi) = (r, t, \rho_s(\xi))$. At each point, $X_1, \ldots, X_{n-2}$ linearly span the tangent space of $M_{rt}$. So their brackets $[X_i, X_j]$ are tangent to the foliation $M_{rt}$ as well. For simplicity, we set

$$[X_i, X_j] \equiv 0 \text{ (mod } X_1, \ldots, X_{n-2}).$$

Along any normal geodesic for $M_t$ in $(S^{n-1}, g^{st})$, each $X_i$ can be extended to a Jacobi field on the whole great circle, which vanishes at each $M_0$ and $M_{\pi/d}$. So $X_i/|X_i|$ are parallel along each $t$-curve (i.e., the spherical $r$- and $\xi$-coordinates are fixed) for $g^{st}$ on $\mathbb{R}^n \setminus \{0\}$. By the positive 1-homogeneity, we may set

$$|X_i|^2 = g^{st}(X_i, X_i) = r^2f_i(t) \quad \text{with } f_i(t) = a_i \sin^2 t + \frac{k_i\pi}{d}.$$

where $a_i$ is some positive constant and $k_i \in \{0, \ldots, d-1\}$ is the integer in (2.1).

Now we have constructed the $g^{st}$-orthogonal local frame $\{\partial_r, \partial_t, X_1, \ldots, X_{n-2}\}$. For simplicity, we will call it a spherical local frame induced by $M_t$. The above discussion can be summarized as the following lemma.
Lemma 2.1. Let \( \{\partial_r, \partial_t, X_1, \ldots, X_n\} \) be a spherical local frame induced by the isoparametric foliation \( M_t \) on \((S^{n-1}(1), g^n)\) with \( d \) principal curvatures. Then we have the following:

1. The brackets among tangent vector fields in this spherical local frame satisfy

\[
[\partial_r, \partial_t] = 0, \quad [\partial_r, X_i] = [\partial_t, X_i] = 0, \quad \forall i,
\]

\[
[X_i, X_j] \equiv 0 \pmod{X_1, \ldots, X_{n-2}}, \quad \forall i,j.
\]

2. The standard flat metric \( g^x \) on \( \mathbb{R}^n \setminus \{0\} \) can be represented as

\[
g^x = dr^2 + r^2 dt^2 + r^2 f_i(t)\theta_i^2 + \cdots + r^2 f_{n-2}(t)\theta_{n-2}^2,
\]

where \( \{dr, dt, \theta_1, \ldots, \theta_{n-2}\} \) is the dual frame for \( \{\partial_r, \partial_t, X_1, \ldots, X_{n-2}\} \), and \( f_i(t) \) is given in (2.2).

Using Lemma 2.1, we can further calculate the Levi-Civita connection of \((\mathbb{R}^n \setminus \{0\}, g^x)\).

Lemma 2.2. For the Levi-Civita connection \( \tilde{\nabla} \) of \((\mathbb{R}^n \setminus \{0\}, g^x)\), we have

\[
\tilde{\nabla}_{\partial_r} \partial_r = 0, \quad \tilde{\nabla}_{\partial_r} \partial_t = \tilde{\nabla}_{\partial_t} \partial_r = \frac{1}{r} \partial_t, \quad \tilde{\nabla}_{\partial_t} \partial_t = -r \partial_r,
\]

\[
\tilde{\nabla}_{\partial_r} X_i = \tilde{\nabla}_X \partial_r = \frac{1}{r} X_i, \quad \forall i, \quad \tilde{\nabla}_{\partial_t} X_i = \tilde{\nabla}_X \partial_t = \frac{1}{2f_i(t)} \frac{d}{dt} f_i(t) X_i, \quad \forall i,
\]

\[
\tilde{\nabla}_X X_i \equiv -rf_i(t) \partial_r - \frac{1}{2} \frac{d}{dt} f_i(t) \partial_t \pmod{X_1, \ldots, X_{n-2}}, \quad \forall i,
\]

\[
\tilde{\nabla}_X X_j \equiv 0 \pmod{X_1, \ldots, X_{n-2}}, \quad \forall i \neq j.
\]

3 The Minkowski norm induced by an isoparametric foliation

3.1 The Minkowski norm and the Hessian metric

A Minkowski norm \( F \) on \( \mathbb{R}^n \) with \( n \geq 2 \) is a continuous function satisfying the following conditions:

1. (Positiveness and smoothness) The restriction of \( F \) to \( \mathbb{R}^n \setminus \{0\} \) is positive and smooth.
2. (Positive 1-homogeneity) For any \( \lambda \geq 0 \) and any \( x \in \mathbb{R}^n \), \( F(\lambda x) = \lambda F(x) \).
3. (Strong convexity) For the linear coordinates \( x = (x_1, \ldots, x_n) \), the Hessian matrix \( \frac{\partial^2}{\partial x_i \partial x_j} F(x) \) is positive definite at any \( x \neq 0 \).

By its positive 1-homogeneity and strong convexity, the Minkowski norm \( F \) is totally determined by its indicatrix, \( S_F = \{ x \in \mathbb{R}^n \mid F(x) = 1 \} \), which is a smooth convex sphere surrounding the origin. On \( \mathbb{R}^n \setminus \{0\} \), the Minkowski norm \( F \) determines a Riemannian metric \( g = g(\cdot, \cdot) \) such that for any \( u, v \in \mathbb{R}^n = T_x(\mathbb{R}^n \setminus \{0\}) \), we have

\[
g(u, v) = \frac{\partial^2}{\partial s \partial t} \bigg|_{s=t=0} \left( \frac{1}{2} F(x + su + tv)^2 \right)
\]

at \( x \in \mathbb{R}^n \setminus \{0\} \). We call \( g \) the Hessian metric of \( F \) and use the same \( g \) to define its restriction to submanifolds of \( \mathbb{R}^n \setminus \{0\} \). Sometimes, when the norm \( F \) and the nonzero base vector \( x \) need to be specified, we define it as \( g^x_F(\cdot, \cdot) \).

For example, a Minkowski norm \( F \) is Euclidean if and only if its Hessian matrices are irrelevant to the choice of \( x \in \mathbb{R}^n \setminus \{0\} \), i.e., the Hessian metric \( g \) can be viewed as an inner product on \( \mathbb{R}^n \) which satisfies \( F(x) \equiv g(x, x)^{1/2} \) for every \( x \in \mathbb{R}^n \). An \((\alpha, \beta)\)-norm is of the form \( F = \alpha \varphi(\frac{x}{\beta}) \), in which \( \alpha \) is a Euclidean norm, \( \beta \) is a homogeneous linear function, and \( \varphi(s) \) is some positive one-variable function. An \((\alpha_1, \alpha_2)\)-norm is of the form \( F(x) = \alpha(x) \varphi\left(\frac{\alpha_1(x)}{\alpha_2(x)}\right) \), where \( \alpha \) is a Euclidean norm, \( \varphi(s) \) is some one-variable function, and \( x = x_1 + x_2 \) is with respect to a fixed \( \alpha \)-orthogonal decomposition \( \mathbb{R}^n = V_1 + V_2 \).

3.2 The induced Minkowski norm and a criterion theorem

Let \( M_t \) be an isoparametric foliation on \((S^{n-1}(1), g^n)\) with \( d \) principal curvatures. Then we can construct a Minkowski norm \( F \) on \( \mathbb{R}^n \), requiring it to be constant on each \( M_t \). For simplicity, we call it a Minkowski
norm induced by \( M_t \). Using the spherical coordinates, we can represent it as \( F = r \sqrt{2f(t)} \), where \( f(t) \) is some positive function on \([0, \frac{\pi}{2}]\).

Let us more closely observe the two features in the norm \( F = r \sqrt{2f(t)} \), the foliation \( M_t \) and the function \( f(t) \).

Firstly, when \( M_t \) satisfies \( d = 1 \) or \( d = 2 \), the induced Minkowski norm is an \((\alpha, \beta)\) or \((\alpha_1, \alpha_2)\)-norm. Indeed, by choosing suitable inner products, all \((\alpha, \beta)\) and \((\alpha_1, \alpha_2)\)-norms can be induced by an isoparametric foliation on the unit sphere with \( d = 1 \) or \( d = 2 \).

Secondly, the function \( f(t) \) for \( F = r \sqrt{2f(t)} \) can be determined by the restriction of \( F \) to any maximally extended normal geodesic for \( M_t \) in \((S^{n-1}(1), g^{n,1})\). So \( f(t) \) can be extended to a \( D_{2d}\)-invariant positive smooth function for \( t \in \mathbb{R}/(2\pi \mathbb{Z}) \), i.e., we always have \( f(t) = f(-t) \) and \( f(t) = f(t + \frac{2\pi}{2}) \). The function \( f(t) \) after the extension is used in the polar coordinate representation \( F = r \sqrt{2f(t)} \) for the restriction \( F = F |_V \) of \( F \) to any normal plane \( V \) for \( M_t \).

The following criterion theorem answers exactly and explicitly when the formal expression \( F = r \sqrt{2f(t)} \) really defines a Minkowski norm.

**Theorem 3.1.** Let \( M_t \) be an isoparametric foliation on \((S^{n-1}(1), g^{n,1})\) with \( d \) principal curvatures. Then the following are equivalent:

1. The function \( f(t) \) on \([0, \frac{\pi}{2}]\) defines a Minkowski norm \( F = r \sqrt{2f(t)} \) on \( \mathbb{R}^n \) induced by \( M_t \).
2. The function \( f(t) \) can be extended to a \( D_{2d}\)-invariant positive smooth function on \( \mathbb{R}/(2\mathbb{Z}\pi) \) such that
   \[
   2f(t) \frac{d^2}{dt^2} f(t) - \left( \frac{d}{dt} f(t) \right)^2 + 4f(t)^2 > 0 \tag{3.1}
   \]
   is satisfied everywhere.
3. The function \( f(t) \) after the extension (as indicated in (2)) defines a \( D_{2d}\)-invariant Minkowski norm on \( \mathbb{R}^2 \) with the polar coordinate representation \( F = r \sqrt{2f(t)} \).

Its proof is postponed to Subsection 3.4.

The special cases of Theorem 3.1 with \( d = 1 \) or \( d = 2 \) reprove the following known results for \((\alpha, \beta)\)-norms (see the discussion for [14, Proposition 5]) and \((\alpha_1, \alpha_2)\)-norms (see [15, Theorem 3.2]).

**Corollary 3.2.** Let \( \alpha, \beta \) and \( \varphi(s) \) be a Euclidean norm, a nonzero homogeneous linear function on \( \mathbb{R}^n \) with \( n \geq 2 \) and a positive one-variable function, respectively. Then \( F = \alpha \varphi(s) \) defines a Minkowski norm if and only if \( \varphi(s) \) is a positive smooth function on \([-b, b]\), where \( b \) is the \( \alpha \)-norm of \( \beta \), and \( \varphi(s) \) satisfies
   \[
   \varphi(s) - s \frac{d}{ds} \varphi(s) + (b^2 - |s|^2) \frac{d^2}{ds^2} \varphi(s) > 0 \tag{3.2}
   \]
on \([-b, b] \).

**Proof.** We first prove Corollary 3.2 when \( n \geq 2 \).

By using the suitable \( \alpha \)-orthonormal coordinates \((x_1, \ldots, x_n)\) and the corresponding spherical \( r \)- and \( t \)-coordinates, the expression \( F = \alpha \varphi(s) \) can be changed to \( F = r \varphi(b \cos t) = r \sqrt{2f(t)} \) with \( f(t) = \frac{1}{2} \varphi(b \cos t)^2 \) and \( t \in [0, \pi] \). Direct calculation shows
   \[
   2f(t) \frac{d^2}{dt^2} f(t) - \left( \frac{d}{dt} f(t) \right)^2 + 4f(t)^2 = \varphi(s)^3 \left( \varphi(s) - s \frac{d}{ds} \varphi(s) + (b^2 - |s|^2) \frac{d^2}{ds^2} \varphi(s) \right), \tag{3.3}
   \]
where \( s = b \cos t \).

Assume that \( F = \alpha \varphi(s) \) defines a Minkowski norm. Then by the equivalence between (1) and (2) in Theorem 3.1 for \( d = 1 \), \( f(t) \) is a smooth even function around \( t = 0 \). By L’Hospital rule and the implicit function theorem, we can find a function \( \psi(s) \) which is smooth at \( s = 0 \) such that \( f(t) = \psi(b^2 \sin^2 t) \) around \( t = 0 \). Then \( \varphi(s) = \psi(b^2 - s^2) \) is smooth at \( s = b \). The smoothness of \( \varphi(s) \) at \( s = -b \) can be similarly verified. Checking the other claims for \( \varphi(s) \) in Corollary 3.2 are easy routines.

Assume that \( \varphi(s) \) satisfies the requirements in Corollary 3.2. Then obviously \( f(t) = \frac{1}{2} \varphi(b \cos t)^2 \) is a positive smooth function on \( \mathbb{R} \) which satisfies \( f(t) = f(-t) \) and \( f(t) = f(t + 2\pi) \), i.e., \( f(t) \) is a function on \( \mathbb{R}/(2\pi) \) which is invariant with respect to the action of \( D_2 = \mathbb{Z}_2 \). The inequality (3.1) follows
immediately from (3.3) and (3.2). Finally, the equivalence between (1) and (2) in Theorem 3.1 for \( d = 1 \) tells us \( F = \alpha \varphi \left( \frac{t}{\alpha} \right) \) is a Minkowski norm.

To summarize, the above argument proves Corollary 3.2 when \( n > 2 \). When \( n = 2 \), we can use the equivalence between (2) and (3) in Theorem 3.1 and the similar argument to prove this corollary.

**Corollary 3.3.** Let \( \alpha \) be a Euclidean norm on \( \mathbb{R}^n \) with \( n \geq 2 \), \( \mathbb{R}^n = V' + V'' \) be an \( \alpha \)-orthogonal decomposition with \( 0 < \dim V' = m < n \), and \( \varphi(s) \) be some positive function on \( [0, 1] \). Then \( F(x) = \alpha \varphi \left( \frac{t}{\alpha} \right) \) defines a Minkowski norm if and only if both \( \varphi(s) \) and \( \psi(s) = \varphi(\sqrt{1 - s^2}) \) are smooth functions on \( [0, 1] \), and the inequality

\[
\varphi(s) - s \frac{d}{ds} \varphi(s) + (1 - |s|^2) \frac{d^2}{ds^2} \varphi(s) > 0
\]

is satisfied everywhere.

In the proof of Corollary 3.3, we need to apply Theorem 3.1 for \( d = 2 \) to discuss the case \( 2 \leq m \leq n - 2 \). As the argument for each case is very similar to that in the proof of Corollary 3.2, we skip the details.

### 3.3 Some calculation for the Hessian metric

The calculation for the Hessian metric \( g \) of the Minkowski norm \( F = r\sqrt{2f(t)} \) by a spherical local frame induced by \( M_t \) is the foundation for later discussion. It is a useful observation that \( g \) is in fact the second covariant derivative of \( E = \frac{1}{2}F^2 = r^2f(t) \) with respect to the Levi-Civita connection \( \tilde{\nabla} \) on \( (\mathbb{R}^n \setminus \{0\}, g^\alpha) \), so we have

\[
g(X, Y) = X \cdot (Y \cdot E) - (\tilde{\nabla}_X Y) \cdot E,
\]

in which \( \cdot \) denotes the directional derivative action of vector fields on differentiable functions and \( \tilde{\nabla} \) is the Levi-Civita connection on \( (\mathbb{R}^n \setminus \{0\}, g^\alpha) \). Using Lemma 2.2 and noticing \( X_i \cdot E = 0 \), we obtain the following lemma.

**Lemma 3.4.** Let \( \{\partial_t, \partial_1, X_1, \ldots, X_{n-2}\} \) be a spherical local frame, and \( F = r\sqrt{2f(t)} \) be a Minkowski norm, induced by the same isoparametric foliation \( M_t \) on \( (S^{n-1}, g^\alpha) \). Then we have

\[
g(\partial_t, \partial_t) = 2f(t), \quad g(\partial_t, \partial_i) = r^2 \frac{d^2}{dt^2} f(t) + 2r^2 f(t),
\]

\[
g(X_i, X_i) = r^2 \left( 2f_i(t) + \frac{1}{2} \frac{d}{dt} \left( f_i(t) \frac{d}{dt} f(t) \right) \right), \quad \forall i,
\]

\[
g(\partial_t, \partial_i) = r \frac{d}{dt} f(t), \quad g(\partial_t, X_i) = g(\partial_i, X_i) = 0, \quad \forall i, \quad g(X_i, X_j) = 0, \quad \forall i \neq j.
\]

We see from Lemma 3.4 that though the spherical local frame \( \{\partial_t, \partial_1, X_1, \ldots, X_{n-2}\} \) may not be \( g \)-orthogonal, it is close, i.e., by replacing \( \partial_t \) with \( T = \partial_t - \frac{r}{\sqrt{2f(t)}} \frac{d}{dt} f(t) \partial_t \), the frame \( \{\partial_t, T, X_1, \ldots, X_{n-2}\} \) is \( g \)-orthogonal. By using Lemma 3.4, the \( g \)-norm square of \( T \) can be easily calculated. To summarize, we have the following lemma.

**Lemma 3.5.** The local frame \( \{\partial_t, T, X_1, \ldots, X_{n-2}\} \) is \( g \)-orthogonal, in which \( T = \partial_t - \frac{r}{\sqrt{2f(t)}} \frac{d}{dt} f(t) \partial_t \), with

\[
g(T, T) = \frac{r^2}{2f(t)} \left( 2f(t) \left( \frac{d^2}{dt^2} f(t) - \left( \frac{d}{dt} f(t) \right)^2 + 4(f(t))^2 \right) \right)
\]

Let \( V \) be any normal plane for \( M_t \), which has a nonempty intersection with the defined domain for \( \{\partial_t, \partial_1, X_1, \ldots, X_{n-2}\} \). Then \( \partial_t \) can be smoothly extended to the one on \( V \setminus \{0\} \) corresponding to the polar \( t \)-coordinate. Since \( \partial_t \) can be globally defined on \( \mathbb{R}^n \setminus \{0\} \) (corresponding to the spherical \( r \)-coordinate), we see that \( T = \partial_t - \frac{r}{\sqrt{2f(t)}} \frac{d}{dt} f(t) \partial_t \) can be extended to a smooth tangent vector field on \( V \setminus \{0\} \) which is nonvanishing everywhere. By the positive 1-homogeneity, the smooth extension to \( V \setminus \{0\} \) for \( X_1, \ldots, X_{n-2} \) can also be observed. It is easy but useful to see that the nonzero values of \( X_1, \ldots, X_{n-2} \) provide a basis for \( T_x M_{r,t} \) when \( x \in V \cap M_{r,t} = V \cap rM_t \) with every \( r > 0 \) and \( t \in [0, \frac{\pi}{2}] \). We summarize these observations as the following lemma.
Lemma 3.6. Let $V$ be any normal plane for $M_t$, which has a nonempty intersection with the defined domain for $\{\partial_r, \partial_t, X_1, \ldots, X_{n-2}\}$. Then the restriction of $\{\partial_r, \partial_t, X_1, \ldots, X_{n-2}\}$ to this intersection can be canonically extended to $V \setminus \{0\}$ satisfying the following:

1. $\partial_r$ and $T = \partial_t - \frac{r}{2f(t)} \frac{df}{dt} \partial_r$ are nonvanishing everywhere on $V \setminus \{0\}$.

2. The nonzero values of $X_1, \ldots, X_{n-2}$ provide the basis of the tangent space for $M_{r,t}$ for every $r > 0$ and $t \in [0, \frac{\pi}{2}]$.

3.4 Proof of Theorem 3.1

We will mainly prove the equivalence between (1) and (2) in Theorem 3.1. The equivalence between (2) and (3) can be easily observed midway.

Firstly, we prove the claim in Theorem 3.1 from (2) to (1). We assume that $f(t)$ has been extended to a $D_{2d}$-invariant positive smooth function on $\mathbb{R}/(2\pi \mathbb{Z})$ as claimed in Theorem 3.1, i.e., we have

$$f(t) = f(-t), \quad f(t) = f\left(\frac{2\pi}{d} - t\right), \quad f(t) = f\left(t + \frac{2\pi}{d}\right),$$

(3.5)

$$2f(t) \frac{d^2}{dt^2} f(t) - \left(\frac{d}{dt} f(t)\right)^2 + 4f(t)^2 > 0$$

(3.6)

for every $t \in \mathbb{R}/(2\pi \mathbb{Z})$. Then we prove that $F = r \sqrt{2f(t)}$ is a Minkowski norm induced by $M_t$. Its positiveness, positive 1-homogeneity and smoothness on $C(S^{n-1}(1) \setminus (M_0 \cup M_{\pi/d}))$ are obvious.

To prove its smoothness at $\mathbb{R} > 0 M_0$, we can argue as follows. By the first equality in (3.5) and an exercise of L'Hopital rule, there exists a positive function $\psi(s)$ such that $f(t) = \psi(t^2)$ and $\psi(s)$ is smooth at $s = 0$. As a function on $(S^{n-1}(1) \setminus (M_0 \cup M_{\pi/d}), g^m)$, the spherical $t$-coordinate coincides with the distance $dist_{S^{n-1}(1)}(\cdot, M_0)$. Using the exponential map for the normal bundle of $M_0$ in $(S^{n-1}(1) \setminus (M_{\pi/d}, g^m))$, we see $t^2 = (dist_{S^{n-1}(1)}(\cdot, M_0))^2$ can be smoothly extended to a neighborhood of $M_0$ in $S^{n-1}(1)$. So $F = r \sqrt{2f(t)} = r \sqrt{2\psi(t^2)}$ is smooth at $\mathbb{R} > 0 M_0$.

Using the second equality in (3.5) and the similar argument, we can prove that $F = r \sqrt{2f(t)}$ is smooth at $\mathbb{R} > 0 M_{\pi/d}$. Then the smoothness is verified. Since we have already observed the positiveness, smoothness and positive 1-homogeneity for $F$, we see its indicatrix $S_F$ is a smooth sphere surrounding the origin.

Now we verify the strong convexity, which is the most essential part of the proof. Let $V$ be a normal plane for $M_t$. We consider any spherical local frame $\{\partial_r, \partial_t, X_1, \ldots, X_{n-2}\}$ induced by $M_t$, whose defined domain has a nonempty intersection with $V$. Define $\overline{F} = F|_V$. Then we have the polar coordinate representation $\overline{F} = r \sqrt{2f(t)}$.

By using $\{\partial_r, \partial_t, X_1, \ldots, X_{n-2}\}$ and $T = \partial_t - \frac{r}{2f(t)} \frac{df}{dt} \partial_r$, the Hessian $g(\cdot, \cdot)$ of $E = \frac{1}{2} F^2$ has the same representations as in Lemmas 3.4 and 3.5. For simplicity, we define $g(X, Y)$ for $X, Y \in \{\partial_r, T, X_1, \ldots, X_{n-2}\}$ as $g_{\alpha\beta}$, where $\alpha$ and $\beta$ are the indices in the ordered set $\{r, T, 1, \ldots, n-2\}$. For example, $g_{TT} = g(T, T)$ and $g_{rr} = g(\partial_r, \partial_r)$, etc. Then

$$(g_{\alpha\beta}) = \text{diag}(g_{rr}, g_{TT}, g_{11}, \ldots, g_{n-2,n-2})$$

is a diagonal matrix. By Lemma 3.6, when we restrict our discussion for $(g_{\alpha\beta})$ to $V$, the notion of $g_{\alpha\beta}$ can be smoothly extended everywhere on $V \setminus \{0\}$.

From Lemma 3.4, we know $g_{rr} = 2f(t) > 0$. By Lemma 3.5, the left-hand side of the inequality (3.6) coincides with $\frac{2f(t)}{r^2} g_{TT}$. So (3.6) implies that the upper left $2 \times 2$-block $\text{diag}(g_{rr}, g_{TT})$ in the Hessian matrix $(g_{\alpha\beta})$ is positive definite.

Before we go on to discuss the other diagonal entries $g_{ii}$ in $(g_{\alpha\beta})$, we digress to prove that $\overline{F} = F|_V$ is a Minkowski norm on $V$. Its strong convexity is the only nontrivial issue for us to consider. Since the polar coordinate representation $\overline{F} = r \sqrt{2f(t)}$ looks the same as the spherical coordinate representation for $F$, the Hessian matrix of $\overline{F}$ coincides with the upper left $2 \times 2$-block in $(g_{\alpha\beta})$. We have seen its positive definiteness in the previous discussion, so $\overline{F}$ is strongly convex. Indeed, this simple observation proves the equivalence between (2) and (3) in Theorem 3.1.
For the convenience when discussing $g_{ii}$, we denote by $N_t = S_F \cap \mathbb{R}_{>0} M_t$ the foliation on $S_F$ induced by $M_t$. We parametrize $S_F = \mathbf{V} \cap S_F$ as $c(t)$ with $t \in \mathbb{R}/(2\mathbb{Z} \pi)$, equivariantly with respect to the action of $D_{2d}$ such that $c(t) \in \mathbf{V} \cap N_t$ for all $t \in [0, \pi/2]$. Then $\gamma(t) = \frac{c(t)}{c(t)}$ is a maximal normal geodesic for $M_t$ on $(S^{n-1}(1), g^n)$ parametrized by its $g^n$-arc length with $\gamma(0) \in M_0$.

By Lemma 3.4, we have for each $1 \leq i \leq n - 2$,

$$
g_{ii} = r^2 \left(2f_i(t) f(t) + \frac{1}{2} \frac{d}{dt} f_i(t) \frac{d}{dt} f(t) \right)
= \frac{a_i}{(2f(t))^{1/2}} \sin \left( t + \frac{k_i \pi}{d} \right) \left( \sin \left( t + \frac{k_i \pi}{d} \right) (2f(t))^{1/2} + \cos(t + \frac{k_i \pi}{d}) \frac{d}{dt} f(t) \right)
$$

(3.7)

at $x = c(t)$ with $t \in [0, \pi/2]$. Here, the function $f_i(t) = a_i \sin^2(t + \frac{k_i \pi}{d})$ with positive constant $a_i$ and $k_i \in \{0, \ldots, n - 1\}$ is given in (2.2).

The factor $\sin(t + \frac{k_i \pi}{d})$ on the right-hand side of (3.7) is non-negative for $t \in [0, \pi/2]$ and $k_i \in \{0, \ldots, d-1\}$, and it vanishes if and only if

$$
either \quad (t, k_i) = (0, 0) \quad or \quad (t, k_i) = \left( \frac{\pi}{d}, d-1 \right).
$$

(3.8)

The factor

$$
\sin \left( t + \frac{k_i \pi}{d} \right) (2f(t))^{1/2} + \cos(t + \frac{k_i \pi}{d}) \frac{d}{dt} f(t)
$$

coincides with $g_{c(t)}(c(t), \gamma(\frac{\pi}{d} - \frac{k_i \pi}{d}))$, i.e., the derivative of

$$
F = \frac{1}{2} F^2 = r^2 f(t)
$$

in the direction of $\gamma(\frac{\pi}{d} - \frac{k_i \pi}{d})$ at $x = c(t)$ with $k_i \in \{0, \ldots, d-1\}$.

We define $s(t) \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ for $t \in [0, \pi/2]$ such that

$$
g_{c(t)}(c(t), \gamma(s(t) - \pi)) = g_{c(t)}(c(t), \gamma(s(t))) = 0
$$

and

$$
g_{c(t)}(c(t), \gamma(s)) > 0
$$

for all $s \in (s(t) - \pi, s(t))$. By the first two equalities in (3.5) from the $D_{2d}$-invariance, we see $s(0) = \frac{\pi}{2}$ and $s(\pi) = \frac{\pi}{2} + \frac{\pi}{2}$. When $t = 0$, the interval $(\frac{\pi}{2} - \frac{(d-1)\pi}{d}, \frac{\pi}{2})$ is contained in $(s(0) - \pi, s(0))$ with the same right end points. By the strong convexity of $F$, $s(t)$ is a strictly increasing continuous function on $[0, \frac{\pi}{2}]$. So $(\frac{\pi}{2} - \frac{(d-1)\pi}{d}, \frac{\pi}{2})$ stays in the moving interval $(s(t) - \pi, s(t))$ with $t$ increasing, until $t$ reaches $\frac{\pi}{d}$, and the two intervals have the same left end points.

This observation proves

$$
g_{c(t)}(c(t), \gamma(s)) \geq 0, \quad \forall t \in \left[0, \frac{\pi}{d}\right], \quad \forall s \in \left[\frac{\pi}{2} - \frac{(d-1)\pi}{d}, \frac{\pi}{2}\right],
$$

and the equality happens if and only if

$$
either \quad (t, s) = \left(0, \frac{\pi}{2}\right) \quad or \quad (t, s) = \left(\frac{\pi}{d}, \frac{\pi}{2} - \frac{\pi}{2}\right).
$$

So

$$
\sin \left( t + \frac{k_i \pi}{d} \right) (2f(t))^{1/2} + \cos(t + \frac{k_i \pi}{d}) \frac{d}{dt} f(t)
$$

is a non-negative factor in $g_{ii}$ as well, and it vanishes if and only if (3.8) happens.

Summarizing the above observations and using Lemmas 3.5 and 3.6 and the $D_{2d}$-symmetry, we see that $g_{ii}$ is strictly positive at $x \in S_F \setminus (N_0 \cup N_{\pi/d})$, and $g_{ii}$ vanishes at $x \in S_F \cap (N_0 \cup N_{\pi/d})$ if and only
if $X_t$ vanishes there. With the normal plane $V$ for $M_t$ changing arbitrarily, we see that $(g_{αβ})$ is positive definite on $S_F \backslash (N_0 \cup N_{π/d})$.

For $x \in N_0$, Lemmas 3.5 and 3.6 tell us that $T_x(\mathbb{R}_{>0}N_0) = T_x(\mathbb{R}_{>0}N_0) \subset \mathbb{R}^n$ has a $g$-orthogonal basis consisting of $\partial_r$ and all the nonzero values of $X_t$ at $x$. So the restriction of $g$ to $T_x(\mathbb{R}_{>0}N_0)$ is positive definite. On the other hand, for any nonzero vector $v$ in the $g$-orthogonal complement $V'$ of $T_x(\mathbb{R}_{>0}N_0)$ in $T_x(\mathbb{R}^n\{0\})$, $x$ and $v$ linearly span a normal plane for $M_t$, and up to a positive scalar change, $v$ coincides with the values at $x$ for the vector field $T$ on $V\{0\}$. By Lemma 3.5 and (3.6), we see $g(v,v) > 0$. So $g$ is positive definite on $V' \subset T_x(\mathbb{R}^n\{0\})$ as well.

This argument proves that $g$ is positive definite on $N_0$. The similar argument proves that $g$ is positive definite on $N_{π/d}$. By the positive 1-homogeneity, the proof for the strong convexity is done.

To summarize, the above argument proves the claim in Theorem 3.1 from (2) to (1).

Next, we prove the claim in Theorem 3.1 from (1) to (2), i.e., we assume that $F = r \sqrt{2f(t)}$ is a Minkowski norm and prove the properties of $f(t)$ claimed in Theorem 3.1. We just need to discuss the restriction of $F$ to any normal plane for $M_t$, and then we easily see that (1) implies (3). The equivalence between (2) and (3) has been observed midway, so the claim from (1) to (2) is proved.

### 3.5 The foliation on $S_F$ induced by $M_t$

In Subsection 3.4, we have mentioned the foliation $N_t = S_F \cap \mathbb{R}_{>0}M_t$ on $S_F$, which is induced by $M_t$ on $S^{n-1}(1)$. Now we prove the following theorem.

**Theorem 3.7.** Let $F = r \sqrt{2f(t)}$ be a Minkowski norm induced by the isoparametric foliation $M_t$ on the unit sphere $(S^{n-1}(1), g^{αβ})$ and $g$ be its Hessian metric. Then the foliation $N_t = S_F \cap \mathbb{R}_{>0}M_t$ on $(S_F, g)$ is isoparametric.

**Proof.** Denote by $d$ the number of principal curvatures for the foliation $M_t$ on $(S^{n-1}(1), g^{αβ})$. The spherical $t$-coordinate can be viewed as a regular smooth function on the conic open subset

$$C(S^{n-1}(1) \backslash (M_0 \cup M_{π/d})) = C(S_F \backslash (N_0 \cup N_{π/d})),$$

which is still denoted by $t$. Its level set provides the foliation $N_t$. We will first prove that the function $t |_{S_F \backslash (N_0 \cup N_{π/d})}$ is isoparametric on $(S_F \backslash (N_0 \cup N_{π/d}), g)$, where $g$ is the Hessian metric of $F = r \sqrt{2f(t)}$.

Let $\{\partial_1, \partial_2, X_1, \ldots, X_{n-2}\}$ be any spherical local frame induced by $M_t$ and let

$$T = \partial_t - \frac{r}{2f(t)} \frac{d}{dt} f(t) \partial_r.$$

Since $\partial_r \cdot t = 1$ and $\partial_1 \cdot t = X_t \cdot t = 0$, by Lemma 3.5, the gradient field $\text{grad}^E t$ on $C(S_F \backslash (N_0 \cup N_{π/d}))$ is

$$\text{grad}^E t = \left( \frac{2f(t)}{r^2(4f(t)^2 - (\frac{d}{dt} f(t))^2 + 2f(t) \frac{d^2}{dt^2} f(t))} \right) T,$$

and its pointwise $g$-norm square is

$$g(\text{grad}^E t, \text{grad}^E t) = \frac{2f(t)}{r^2(4f(t)^2 - (\frac{d}{dt} f(t))^2 + 2f(t) \frac{d^2}{dt^2} f(t))}.$$

If restricted to $S_F \backslash (N_0 \cup N_{π/d})$, where $r^2 = (2f(t))^{-1}$,

$$g(\text{grad}^E t, \text{grad}^E t) |_{S_F \backslash (N_0 \cup N_{π/d})} = \frac{4f(t)^2}{4f(t)^2 - (\frac{d}{dt} f(t))^2 + 2f(t) \frac{d^2}{dt^2} f(t)}$$

is a function of $t$.

Denote by $\text{Hess}(\cdot, \cdot)$ the Hessian with respect to $g$ on $\mathbb{R}^n \backslash \{0\}$. Then we have

$$\text{Hess}(X, X) t = X \cdot (X \cdot t) - (\nabla_X X) \cdot t = - (\nabla_X X) \cdot t$$
for each $X \in \{\partial_r, T, X_1, \ldots, X_{n-2}\}$. Here, $\nabla$ is the Levi-Civita connection of $(\mathbb{R}^n \setminus \{0\}, g)$. So the Laplacian $\Delta^E t$ on $C(S_F \setminus (N_0 \cup N_{e/d}), g)$ can be represented as

$$
\Delta^E t = \frac{\text{Hess}(\partial_r, \partial_r)t}{g(T, T)} + \frac{\text{Hess}(T, T)t}{g(T, T)} + \sum_{i=1}^{n-2} \frac{\text{Hess}(X_i, X_i)t}{g(X_i, X_i)}
$$

$$
= -\frac{\nabla_{\partial_r} \partial_r \cdot t}{g(\partial_r, \partial_r)} - \frac{\nabla_T T \cdot t}{g(T, T)} - \sum_{i=1}^{n-2} \frac{(\nabla_{X_i} X_i) \cdot t}{g(X_i, X_i)},
$$

(3.9)

where the spherical local frame is defined.

Using Lemma 3.4, we collect the following information for $\nabla$:

$$
\nabla_{\partial_r} \partial_r = 0,
$$

(3.10)

$$
\nabla_T T = \frac{\frac{d}{dt} f(t)^3 - 2 f(t) \frac{df}{dt} f(t)^2}{4 f(t)^3} \frac{df}{dt} f(t) + f(t)^2 \frac{d^2 f}{dt^2} f(t)
$$

$$
- f(t)^3 \frac{df}{dt} f(t) \frac{d^2 f}{dt^2} f(t) + 2 f(t)^2 \frac{d^2 f}{dt^2} f(t)
$$

(3.11)

$$
\nabla_{X_i} X_i = \frac{d}{dt} f(t) \frac{df}{dt} f(t)^2 - 4 f(t)^2 \frac{df}{dt} f(t) + f(t) \frac{d^2 f}{dt^2} f(t) - f(t)^2 \frac{d^2 f}{dt^2} f(t)
$$

$$
8 f(t)^2 - 2 f(t)^3 \frac{df}{dt} f(t) + 2 f(t)^2 \frac{d^2 f}{dt^2} f(t)
$$

(3.12)

Inputting the formula of $g(X_i, X_i)$ in Lemma 3.4, (3.4) in Lemma 3.5, and (3.10)–(3.12) into (3.9), we see that $\Delta^E t$ is the product of $r^{-2}$ and a function of $t$. In particular, its restriction to $S_F$ (where $r^{-2} = 2 f(t)$) only depends on the values of $t$.

Finally, we consider the gradient $\text{grad}^S(t \mid_{S^F \setminus (N_0 \cup N_{e/d})})$ and the Laplacian $\Delta^S(t \mid_{S^F \setminus (N_0 \cup N_{e/d})})$ on $(S^F \setminus (N_0 \cup N_{e/d}), g)$. Since the function $t$ is constant along each ray setting off from the origin, $\text{grad}^E t$ is tangent to $S_F$, so we have

$$
\text{grad}^S(t \mid_{S^F \setminus (N_0 \cup N_{e/d})}) = (\text{grad}^E t) \mid_{S^F \setminus (N_0 \cup N_{e/d})}.
$$

By [2, (14.3.10)], we also have

$$
\Delta^S(t \mid_{S^F \setminus (N_0 \cup N_{e/d})}) = (\Delta^E t) \mid_{S^F \setminus (N_0 \cup N_{e/d})}.
$$

So both $g(\text{grad}^S(t \mid_{S^F \setminus (N_0 \cup N_{e/d})}), \text{grad}^S(t \mid_{S^F \setminus (N_0 \cup N_{e/d})}))$ and $\Delta^S(t \mid_{S^F \setminus (N_0 \cup N_{e/d})})$ are functions on $S_F \setminus (N_0 \cup N_{e/d})$ which only depend on $t$.

To summarize, the above argument proves that $t$ is isoparametric on $(S_F \setminus (N_0 \cup N_{e/d}), g)$. Then we can choose $\psi(s)$ on $[0, \frac{\pi}{2}]$ which satisfies $\frac{d}{ds} \psi(s) > 0$ on $(0, \frac{\pi}{2})$ and can be extended to a smooth function on $\mathbb{R}$ satisfying $\psi(s) = \psi(-s)$ and $\psi(s) = \psi(\frac{\pi}{2} - s)$. Then the composition $\psi \circ t \mid_{S^F \setminus (N_0 \cup N_{e/d})}$ can be extended to a smooth function $p(\cdot)$ on $(S_F, g)$, whose level sets provide the foliation $N_t$. It is easy to verify the isoparametric property of $p(\cdot)$ from that of $t \mid_{S^F \setminus (N_0 \cup N_{e/d})}$. So the foliation $N_t$ on $(S_F, g \mid_{S_F})$ is isoparametric.

By Lemma 3.5 and Theorem 3.7, we see

$$
T = \partial_t - \frac{r}{2 f(t)} \frac{df(t)}{dt} \partial_r
$$

generates the normal geodesics for $N_t$ on $(S_F, g)$, which have the spherical coordinate representation $t \mapsto ((2 f(t))^{-1/2}, t, \xi)$ for any fixed $\xi \in M_0$. So we have the following corollary.

**Corollary 3.8.** Let $M_t$ be an isoparametric foliation on $(S^{n-1}(1), g^{st})$, for which we have induced the Minkowski norm $F = r \sqrt{2 f(t)}$ with the Hessian metric $g$, and the isoparametric foliation $N_t = S_T \cap \mathbb{R}_{>0} M_t$ on $(S_F, g)$. Then the intersection with $S_F$ provides a one-to-one correspondence between the set of all the normal planes for $M_t$ and the set of all the unparametrized maximally extended normal geodesics for $N_t$ in $(S_F, g)$.
4 The Hessian isometry and Laugwitz Conjecture

4.1 The Hessian isometry and the local Hessian isometry

Let $F_1$ and $F_2$ be two Minkowski norms on $\mathbb{R}^n$ with $n \geq 2$ and denote by $g_1 = g_1(\cdot, \cdot)$ and $g_2 = g_2(\cdot, \cdot)$ their Hessian metrics, respectively. A diffeomorphism $\Phi$ on $\mathbb{R}^n \setminus \{0\}$ is called a Hessian isometry from $F_1$ to $F_2$, if it is an isometry from $g_1$ to $g_2$.

Since the rays setting off from the origin provide the set of all the incomplete geodesics on $(\mathbb{R}^n \setminus \{0\}, g_1)$, and the Hessian metric $g_1$ on $\mathbb{R}^n \setminus \{0\}$ has the representation

$$g_1 = (dF_1)^2 + F_1^2(g_1|_{S_{F_1}}),$$

we have the following easy lemma.

**Lemma 4.1.** Any Hessian isometry $\Phi$ from $F_1$ to $F_2$ maps the indicatrix $S_{F_1}$ to the indicatrix $S_{F_2}$, and it is positively 1-homogeneous, i.e., $\Phi(\lambda x) = \lambda \Phi(x)$ for any $\lambda > 0$ and any $x \in \mathbb{R}^n \setminus \{0\}$. Conversely, any isometry from $(S_{F_1}, g_1)$ can be uniquely extended to a Hessian isometry from $F_1$ to $F_2$.

Besides the global Hessian isometry, a local Hessian isometry can be defined as an isometric diffeomorphism between two conic open subsets in $(\mathbb{R}^n \setminus \{0\}, g_1)$, respectively, satisfying the positive 1-homogeneity. An analog of Lemma 4.1 is valid for local Hessian isometries, i.e., the restriction to the indicatrix provides a one-to-one correspondence between local Hessian isometries from $F_1$ to $F_2$ and local isometries from $(S_{F_1}, g_1)$ to $(S_{F_2}, g_2)$.

4.2 The linear isometry and two applications

Any linear isomorphism $\Phi : (\mathbb{R}^n, F_1) \rightarrow (\mathbb{R}^n, F_2)$ which satisfies $F_1 = F_2 \circ \Phi$ naturally induces a Hessian isometry when it is restricted to $\mathbb{R}^n \setminus \{0\}$. For simplicity, we call it a linear isometry.

Here, we propose two applications of linear isometries for the Minkowski norm $g$.

Firstly, when the foliation $M_t$ is homogeneous, i.e., there exists a compact connected Lie subgroup $G$ of $SO(n)$ such that each $M_t$ is a $G$-orbit, then the following lemma shows us a shortcut to Theorem 3.7.

**Lemma 4.2.** Let $M_t$ be a homogeneous isoparametric foliation on $(S^{n-1}(1), g^n)$, and $F = r \sqrt{2f(t)}$ be a Minkowski norm induced by $M_t$. Then the foliation $N_t = S_F \cap \mathbb{R}_{>0} M_t$ on $(S_F, g)$ is also a homogeneous isoparametric foliation.

**Proof.** Let $G$ be the compact connected Lie subgroup of $SO(n)$ such that each $M_t$ is a $G$-orbit. Then the induced Minkowski norm $F = r \sqrt{2f(t)}$ is $G$-invariant. So the $G$-action on $(\mathbb{R}^n, F)$ is linearly isometric and it is of cohomogeneity one when restricted to $(S^F, g)$. Each $N_t = S_F \cap \mathbb{R}_{>0} M_t$ is $G$-orbit, and singular $G$-orbits only appear at the two ends, i.e., $N_0$ and $N_{\pi/d}$. By the theory of Riemannian manifolds of cohomogeneity one, the corresponding isoparametric function can be constructed.

Theorem 3.7 is a direct corollary of Lemma 4.2 when we have $d \in \{1, 2, 3, 6\}$ for the principal curvatures of $M_t$ and for some subcases with $d = 4$.

Secondly, we prove Laugwitz Conjecture in a special case. Laugwitz [29] conjectured that if the Hessian metric $g$ of a Minkowski norm $F$ is flat on $\mathbb{R}^n \setminus \{0\}$ with $n \geq 3$, then $F$ is Euclidean. When $F$ is reversible, i.e., $F(x) = F(-x)$, $\forall x \in \mathbb{R}^n$, or equivalently, when $F$ is absolutely 1-homogeneous, i.e., $F(\lambda x) = |\lambda| F(x)$, $\forall \lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, Brickell [4] proved this conjecture by the following theorem (see [41]).

**Theorem 4.3.** Let $F$ be a Minkowski norm on $\mathbb{R}^n$ with $n \geq 3$ satisfying the reversibility condition, i.e., $F(x) = F(-x)$, $\forall x \in \mathbb{R}^n$. If its Hessian metric $g$ is flat on $\mathbb{R}^n \setminus \{0\}$, then $F$ is Euclidean.

Recently, we proved the following theorem in [58].

**Theorem 4.4.** Laugwitz Conjecture is true for the class of Minkowski norms which are invariant with respect to the standard block diagonal $SO(n-1)$-action.

Now, we further strengthen it as follows.
Theorem 4.5. Laugwitz Conjecture is true for Minkowski norms induced by an isoparametric foliation on the unit sphere.

Proof. Let $M_1$ be an isoparametric foliation on $(S^{n-1}(1), g^M)$ with $n > 2$ and $d$ be the number of principal curvatures of $M_1$. Suppose that $F = r\sqrt{2f(t)}$ is a Minkowski norm induced by $M_1$ with a flat Hessian metric $g$ on $\mathbb{R}^n \setminus \{0\}$.

Case 1. We have $d \in \{2, 4, 6\}$. In this case, we only need to prove that $F$ is reversible, and then Theorem 4.5 follows from Theorem 4.3 immediately.

To prove our claim, we consider any $x \in S^{n-1}(1)$. Then there always exists a normal plane $V$ for $M_1$, which contains $x$. The antipodal map on $V$ is contained in $D_{2d}$ when $d$ is even. So the $D_{2d}$-invariance of $F = F|_V$ implies $F(x) = F(-x)$.

Case 2. We have $d = 1$. This case has already been proved by Theorem 4.4.

Case 3. We have $d = 3$. In this case, Cartan [6, 7] found the following explicit construction. We represent $\mathbb{R}^n$ with $n \in \{5, 8, 14, 26\}$ as

\[ \mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{F}^3 = \{(a, b, x, y, z) \mid \forall a, b \in \mathbb{R}, x, y, z \in \mathbb{F}\} \]

with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, respectively, such that the standard Euclidean norm is given by

\[ |(a, b, x, y, z)| = \sqrt{a^2 + b^2 + x^2 + y^2 + z^2}. \]

Then the isoparametric function for the foliation $M_1$ can be chosen as

\[ p = a^3 - 3ab^2 + \frac{3a}{2}(x^2 + y^2 - 2z^2) + \frac{3\sqrt{3}b}{2}(x^2 - y^2) + \frac{3\sqrt{3}}{2}((xy)z + (xy)z), \]

where $a^2 + b^2 + x^2 + y^2 + z^2 = 1$.

Both the standard Euclidean norm and the function $p(\cdot)$ are invariant for the action of

\[ \Phi(a, b, x, y, z) = (a, b, -x, -y, z). \]

So $\Phi$ is a linear isometry on $(\mathbb{R}^n, F)$. The fixed point set of $\Phi$ is the Euclidean subspace

\[ \mathbb{R}^{n'} = \{(a, b, 0, 0, z) \mid \forall a, b \in \mathbb{R}, z \in \mathbb{F}\} \]

with dimensions $n' = 3, 4, 6, 10$ when $n = 5, 8, 14, 26$, respectively. Restricted to the unit sphere

\[ S^{n'-1}(1) = \mathbb{R}^{n'} \cap S^{n-1}(1), \]

where we have $a^2 + b^2 + z^2 = 1$ and $x = y = 0$, the function $p(\cdot)$ is then given by

\[ p|_{S^{n'-1}(1)} = a^3 - 3ab^2 - 3az^2 = 4a^3 - 3a. \]

So the connected components of all $M'_1 = \mathbb{R}^{n'} \cap M_1$ provide a homogeneous isoparametric foliation on the unit sphere $(S^{n'-1}(1), g^{M'})$ induced by a standard block diagonal $SO(n' - 1)$-action.

Define the restriction $F' = F|_{\mathbb{R}^{n'}}$. Then $F'$ is invariant with respect to a standard block diagonal action of $SO(n' - 1)$. The Hessian metric $g'$ of $F'$ coincides with the restriction $g|_{\mathbb{R}^{n'} \setminus \{0\}}$. Since $\mathbb{R}^{n'} \setminus \{0\}$ is the fixed point set of the linear isometry $\Phi$ on $(\mathbb{R}^n \setminus \{0\}, g)$, it is totally geodesic. Because $g$ is flat on $\mathbb{R}^n \setminus \{0\}$, $g'$ is also flat on $\mathbb{R}^n \setminus \{0\}$. By Theorem 4.4, $F'$ is Euclidean.

If we use the spherical coordinates $(r, t, \xi) \in \mathbb{R}_{>0} \times (0, \pi) \times S^{n'-2}(1)$ on $\mathbb{R}^{n'}$ such that

\[ a = r \cos t \quad \text{and} \quad (b, z) = r \sin t \xi, \]

then $F'$ has the representation $F' = r\sqrt{2f(t)}$, where $f(t) = c_1 + c_2 \cos 2t$ for some constants $c_1$ and $c_2$ with $c_1 > |c_2|$. Meanwhile, $f(t)$ only depends on the values of

\[ p|_{S^{n'-1}(1)} = 4a^3 - 3a = 4\cos^3 t - 3\cos t = \cos 3t, \]

which has a zero derivative at $t = \frac{\pi}{3}$. So we have $\frac{d}{dt}f(t)|_{t=\pi/3} = 0$, i.e., $c_2 = 0$ and $f(t)$ is a positive constant function.

Since the same $f(t)$ is also used in the representation $F = r\sqrt{2f(t)}$, we see that $F$ must be Euclidean as well. The case $d = 3$ is proved. \qed
5 The Hessian isometry which preserves the orientation and fixes the spherical $\xi$-coordinates

5.1 Notations for the spherical coordinate representation

Let $M_t$ be an isoparametric foliation on $(S^{n-1}(1), g^d)$ with $d$ principal curvatures. When we mention the spherical coordinates, the spherical local frame and the induced Minkowski norm, etc., they are always referred to $M_t$. Let $F_1$ and $F_2$ be two induced Minkowski norms on $\mathbb{R}^n$. We denote their indicatrices and Hessian metrics by $S_{F_1}$ and $g_t$, respectively. On each $(S_{F_1}, g_t)$, we have the induced isoparametric foliation $N_{t,t} = S_{F_1} \cap \mathbb{R}_{>0} M_t$. To distinguish the two Minkowski norms, we use $\theta$ to define the spherical $t$-coordinate for $F_2$. So we have the spherical coordinate representations

$$F_1 = r\sqrt{2f(t)} \quad \text{and} \quad F_2 = r\sqrt{2h(\theta)}.$$  \hspace{1cm} (5.1)

Though they only use the values of $f(t)$ and $h(\theta)$ on $[0, \frac{\pi}{d}]$, as Theorem 3.1 indicates, both functions can and will be extended to $D_{2d}$-invariant positive smooth functions on $\mathbb{R}/(2\mathbb{Z} \pi)$, and they satisfy the inequality in Theorem 3.1.

Let $V$ be any normal plane for $M_t$. We parametrize $V \cap S^{n-1}(1)$ as $\gamma(t)$ with its $g^d$-arc length parameter $t \in \mathbb{R}/(2\mathbb{Z} \pi)$ and $\gamma(0) \in M_0$. The normal plane $V$ can be identified with $\mathbb{R}^2$ as indicated in Subsection 2.4, so that $v_1 = \gamma(0)$ and $v_2 = \gamma(\frac{\pi}{d})$ are mapped to $e_1 = (1, 0)$ and $e_2 = (0, 1)$, respectively. The polar coordinates $(r, t) \in \mathbb{R}_{>0} \times (\mathbb{R}/(2\pi))$ on $V$ is determined by

$$x = x_1 v_1 + x_2 v_2 = r \cos tv_1 + r \sin tv_2$$

for any $x \in V \setminus \{0\}$. Each interval $(\frac{k\pi}{d}, \frac{(k+1)\pi}{d})$ with $k \in \{0, \ldots, 2d-1\}$ for the polar $t$-coordinate determines a conic open subset of $V \setminus \{0\}$, which corresponds to a distinct spherical $\xi$-coordinate on $\mathbb{R}^n$.

The restrictions $F_1 = F_1|_V$ have the polar coordinate representations $F_1 = r\sqrt{2f(t)}$ and $F_2 = r\sqrt{2h(\theta)}$ (similarly we use $\theta$ to define the polar coordinate for $F_2$), where $f(t)$ and $h(\theta)$ are exactly those in (5.1) after the extension.

In this section, we discuss a Hessian isometry $\Phi$ from $F_1$ to $F_2$ which preserves the orientation and fixes the spherical $\xi$-coordinates, i.e., it satisfies the following conditions:

1. $\Phi$ is an orientation-preserving diffeomorphism on $\mathbb{R}^n \setminus \{0\}$.
2. $\Phi$ preserves the conic open subset

$$C(S^{n-1}(1) \setminus (M_0 \cup M_{\pi/d})) = \mathbb{R}^n \setminus (\mathbb{R}_{>0} M_0 \cup \mathbb{R}_{>0} M_{\pi/d}),$$

and for any $x \in C(S^{n-1}(1) \setminus (M_0 \cup M_{\pi/d}))$, $x$ and $\Phi(x)$ have the same $\xi$-coordinates.

By Lemma 4.1, we have $\Phi(S_{F_1}) = S_{F_2}$. Since the condition (2) requires that $\Phi$ fixes the spherical $\xi$-coordinates, our previous observation indicates that $\Phi$ preserves each arbitrarily chosen normal plane $V$, i.e., $\Phi|_V$ is a Hessian isometry from $F_1 = F_1|_V$ to $F_2 = F_2|_V$. Furthermore, $\Phi$ fixes each point in $V \cap (M_0 \cup M_{\pi/2d})$ and preserves each conic open subset in $V \setminus \{0\}$ with the polar $t$-coordinate in $(\frac{k\pi}{d}, \frac{(k+1)\pi}{d})$.

More discussion for the restriction to $V$ is postponed to Subsection 5.3. Here, we only need to apply Corollary 3.8 to notice that $\Phi$ maps each normal geodesic for the isoparametric foliation

$$N_{t,t} = S_{F_1} \cap \mathbb{R}_{>0} M_t$$

on $(S_{F_1}, g_1)$ to that for $N_{2,t} = S_{F_2} \cap \mathbb{R}_{>0} M_t$ on $(S_{F_2}, g_2)$. Meanwhile, $\Phi$ preserves the foliation $\mathbb{R}_{>0} M_t$ on $\mathbb{R}^n \setminus \{0\}$. So $\Phi$ maps each $N_{1,t} = S_{F_1} \cap \mathbb{R}_{>0} M_t$ to some $N_{2,\theta(t)} = S_{F_2} \cap \mathbb{R}_{>0} M_{\theta(t)}$. To summarize, $\Phi$ has the following spherical coordinate representation:

$$(r, t, \xi) \mapsto \left( \frac{r f(t)^{1/2}}{h(\theta(t))^{1/2}}, \theta(t), \xi \right).$$
Since \( \Phi \) is an orientation-preserving diffeomorphism by the condition (1), it can be observed immediately from the Jacobi matrix
\[
\begin{pmatrix}
\frac{f(t)^{1/2}}{h(\theta(t))^{1/2}} & \frac{h(\theta(t))-f(t)\frac{d}{dt}\theta(t)}{2f(t)(2h(\theta(t)))^{1/2}} & 0 \\
0 & \frac{d}{dt}\theta(t) & 0 \\
0 & 0 & \text{Id}
\end{pmatrix}
\]
for the tangent map \( \Phi \), that \( \theta(t) \) is a diffeomorphism on \( (0, \frac{\pi}{2}) \) with positive derivatives everywhere, and \( \theta(t) = t \) for \( t \in \{0, \frac{\pi}{2}\} \) by continuity. Indeed, \( \theta(t) \) can and will be extended to an orientation-preserving diffeomorphism on \( \mathbb{R}/(2\mathbb{Z}\pi) \) with a \( D_{2d} \)-equivariance, i.e.,
\[
\theta(-t) = -\theta(t)
\]
and
\[
\theta\left(t + \frac{2\pi}{d}\right) = \theta(t) + \frac{2\pi}{d},
\]
which will appear in the polar coordinate representation
\[
(r, t) \mapsto \left( \frac{rf(t)^{1/2}}{h(\theta(t))^{1/2}}, \theta(t) \right)
\]
for \( \overline{\Phi} = \Phi |_V \). After the extension, \( \theta(t) \) fixes each point in \( \frac{2\mathbb{Z}}{d} \subset \mathbb{R}/(2\mathbb{Z}\pi) \).

Summarizing the above argument, we obtain the following lemma for the spherical coordinate representation for \( \Phi \).

**Lemma 5.1.** Let \( F_1 = r\sqrt{2f(t)} \) and \( F_2 = r\sqrt{2h(\theta)} \) be two Minkowski norms induced by \( M_t \). Then any Hessian isometry \( \Phi \) from \( F_1 \) to \( F_2 \) which preserves the orientation and fixes the spherical \( \xi \)-coordinates has the spherical coordinate representation
\[
(r, t, \xi) \mapsto \left( \frac{rf(t)^{1/2}}{h(\theta(t))^{1/2}}, \theta(t), \xi \right),
\]
in which \( \theta(t) \) is a \( D_{2d} \)-equivariant orientation-preserving diffeomorphism on \( \mathbb{R}/(2\mathbb{Z}\pi) \) and fixes each point in \( \frac{2\mathbb{Z}}{d} \).

**5.2 The description by an ODE system**

Next, we consider a spherical local frame \( \{\partial_r, \partial_t, (\partial_\theta \text{ or } \partial_\phi \text{ for } F_2) \}, X_1, \ldots, X_{n-2} \), defined in the conic open subset of \( \mathbb{R}^n \setminus \{0\} \) which only requires the spherical \( \xi \)-coordinate to be contained in some open subset in \( M_{\pi/2d} \), i.e., its defined domain is preserved by \( \Phi \). We define
\[
T_1 = \partial_t - \frac{r}{2f(t)} \frac{d}{dt} f(t) \partial_r,
\]
and
\[
T_2 = \partial_\theta - \frac{r}{2h(\theta)} \frac{d}{d\theta} h(\theta) \partial_r,
\]
which are tangent to \( S_{F_1} \) and \( S_{F_2} \), respectively. Then we have the following lemma.

**Lemma 5.2.** The tangent map \( \Phi_* \) for \( \Phi \) satisfies
\[
\Phi_*(T_1) = \frac{d}{dt} \theta(t) T_2, \tag{5.3}
\]
\[
\Phi_*(X_i) = X_i, \quad \forall 1 \leq i \leq n - 2. \tag{5.4}
\]

**Proof.** We first prove (5.3). By the positive 1-homogeneity (i.e., Lemma 4.1), we may restrict our discussion to \( S_{F_1} \). On \( (S_{F_1}, g_1) \), \( T_1 \) generates a normal geodesic \( c_1(t) \) with \( t \in (0, \frac{\pi}{2}) \) for the isoparametric
foliation $N_{1,t}$. Its image $\Phi(c_t(1))$ coincides with a normal geodesic $c_2(\theta)$ with $\theta \in (0, \frac{\pi}{2})$ for the isoparametric foliation $N_{2,0}$ on $(S_{F_2}, g_2)$, i.e., an integral curve of $T_2$, up to a change of parameter $\theta = \theta(t)$. So we have $\Phi_t(T_1) = \frac{d}{dt}\theta(t) T_2$.

Then we prove (5.4). Let $s \mapsto ((2f(t))^{-1/2}, t, \xi(s))$ with any fixed $t \in (0, \frac{\pi}{2})$ be the spherical coordinate representation for an integral curve of $X_i$ on $S_{F_1}$. Then its $\Phi$-image has the spherical coordinate representation

$s \mapsto ((2h(\theta(t)))^{-1/2}, \theta(t), \xi(s))$,

which is still an integral curve of $X_i$. So we have $\Phi_t(X_i) = X_i, \forall 1 \leq i \leq n - 2$.

Since $\Phi$ is a Hessian isometry, (5.3) implies

$$g_1(T_1, T_1) = g_2(\Phi_t(T_1), \Phi_t(T_1)) = \left(\frac{d}{dt}\theta(t)\right)^2 g_2(T_2, T_2),$$

(5.5)

where the left-hand side is evaluated at $x = ((2f(t))^{-1/2}, t, \xi(s)) \in S_{F_1 \setminus (N_{1,0} \cup N_{1,\pi/d})}$ and the right-hand side is evaluated at $\Phi(x) = ((2h(\theta(t)))^{-1/2}, \theta(t), \xi(s)) \in S_{F_2 \setminus (N_{2,0} \cup N_{2,\pi/d})}$. Using (3.4) in Lemma 3.5, we obtain the following ODE:

$$\frac{1}{2f(t)} d^2 f(t) - \frac{1}{4f(t)^2} \left(\frac{d}{dt} f(t)\right)^2 + 1 = \left(\frac{d}{dt}\theta(t)\right)^2 \left(\frac{1}{2h(\theta(t))} \frac{d^2}{dt^2} h(\theta(t)) - \frac{1}{4h(\theta(t))^2} \left(\frac{d}{d\theta} h(\theta(t))\right)^2 + 1\right)$$

(5.6)

for $t \in (0, \frac{\pi}{2})$. By continuity and $D_{2d}$-symmetry, (5.6) for $t \in (0, \frac{\pi}{2})$ is equivalent to that for all $t \in \mathbb{R}/(2\mathbb{Z})$.

Similarly, (5.4) implies $g_1(X_i, X_i) = g_2(X_i, X_i)$ for each $i$. Using Lemma 3.4, we obtain the following ODEs:

$$f_i(t) + \frac{1}{4f(t)} \frac{d}{dt} f_i(t) \frac{d}{dt} f(t) = f_i(\theta(t)) + \frac{1}{4h(\theta(t))} f_i(\theta(t)) \frac{d}{d\theta} h(\theta(t))$$

(5.7)

for any $1 \leq i \leq n - 2$ and any $t \in (0, \frac{\pi}{2})$. Here, $f_i(t) = a_i \sin^2(t + \frac{2k\pi}{d})$ with constants $a_i > 0$ and $k_i \in \{0, \ldots, d - 1\}$. Because $\{k_1, \ldots, k_{n-2}\} = \{0, \ldots, d - 1\}$, we can reorganize (5.7) as

$$\sin^2\left(t + \frac{k\pi}{d}\right) + \frac{\cos(t + \frac{k\pi}{d}) \sin(t + \frac{k\pi}{d})}{2f(t)} \frac{d}{dt} f(t) = \sin^2\left(\theta(t) + \frac{k\pi}{d}\right) + \frac{\cos(\theta(t) + \frac{k\pi}{d}) \sin(\theta(t) + \frac{k\pi}{d})}{2h(\theta(t))} \frac{d}{d\theta} h(\theta(t))$$

(5.8)

for every $t \in (0, \frac{\pi}{2})$ and every $k \in \{0, \ldots, d - 1\}$.

Obviously, the ODEs in (5.8) are satisfied with all $k \in \mathbb{Z}$. Furthermore, they are satisfied for all $t \in \mathbb{R}/(2\mathbb{Z})$ as well. To prove this claim, we first observe that by continuity, (5.8) for each $k$ is valid at $t = 0$ and $t = \frac{\pi}{d}$. By using the properties $f(t) = f(-t)$, $h(\theta) = h(-\theta)$ and $\theta(-t) = -\theta(t)$, (5.8) with $k = k'$ for $t \in [-\frac{\pi}{d}, 0]$ can be deduced from (5.8) with $k = d - k'$ for $t \in [0, \frac{\pi}{d}]$. Then by using the symmetry with respect to $\mathbb{Z}_d \subset D_d$, i.e., the properties $f(t) = f(t + \frac{2\pi}{d})$, $h(\theta) = h(\theta + \frac{2\pi}{d})$ and $\theta(t + \frac{2\pi}{d}) = \theta(t) + \frac{2\pi}{d}$, (5.8) with $k = k'$ for $t \in \left[\frac{(2k')-1}{d}\pi, \frac{(2k'+1)}{d}\pi\right]$ can be deduced from (5.8) with $k = k' + 2k''$ for $t \in \left[-\frac{\pi}{d}, \frac{\pi}{d}\right]$.

The above argument tells us how to determine the triple $(f(t), h(\theta), \theta(t))$ from $(F_1, F_2, \Phi)$, and more importantly, list the properties that the triple $(f(t), h(\theta), \theta(t))$ must satisfy. Then we observe how to use the data $(f(t), h(\theta), \theta(t))$ with those properties to construct the wanted $(F_1, F_2, \Phi)$.

When the positive smooth $D_{2d}$-invariant functions $f(t)$ and $h(\theta)$ satisfy the inequality in Theorem 3.1, we can use them to construct the induced Minkowski norms $F_1 = r\sqrt{2f(t)}$ and $F_2 = r\sqrt{2h(\theta)}$.

When the $D_{2d}$-equivariant orientation-preserving diffeomorphism $\theta(t)$ on $\mathbb{R}/(2\mathbb{Z})$ fixes each point in $\frac{2\pi}{d}$, we can use it to construct a diffeomorphism $\Phi$ on $C(\mathbb{R}^{n-1}) \setminus (M_0 \cup M_{\pi/d})$ with the spherical coordinate representation (5.2), i.e., $(r, t, \xi) \mapsto (\frac{r(t)^{1/2}}{h(\theta(t))^{1/2}}, \theta(t), \xi)$. Then obviously $\Phi$ preserves the orientation.
Theorem 5.3. Let \( \theta \) be an isoparametric foliation on \((S^n-1, g^{st})\) with \(d\) principal curvatures. Then the spherical coordinate representations
\[
F_1 = r\sqrt{2f(t)}, \quad F_2 = r\sqrt{2h(\theta)}
\]
provide the one-to-one correspondence between the set of all the triples \((F_1, F_2, \Phi)\) satisfying the following:

1. \( F_1 \) and \( F_2 \) are Minkowski norms on \(\mathbb{R}^n\) induced by \(M_t\);
2. \( \Phi \) is a Hessian isometry from \( F_1 \) to \( F_2 \) which preserves the orientation and fixes the spherical \(\xi\)-coordinates

and the set of all the triples \((f(t), h(\theta), \theta(t))\) satisfying the following:

1. \( f(t) \) and \( h(\theta) \) are \(D_{2\theta}\)-invariant positive smooth functions on \(\mathbb{R}/(2\pi\mathbb{Z})\) satisfying the inequalities
\[
2f(t)\frac{d^2}{dt^2}f(t) - \left(\frac{d}{dt}f(t)\right)^2 + 4f(t)^2 > 0,
\]

\((5.9)\)
\[ 2h(\theta) \frac{d^2}{d\theta^2} h(\theta) - \left( \frac{d}{d\theta} h(\theta) \right)^2 + 4h(\theta)^2 > 0; \]  

(5.10)

(2) \( \theta(t) \) is a \( D_{2d} \)-equivariant orientation-preserving diffeomorphism on \( \mathbb{R}/(2\mathbb{Z}\pi) \) which fixes each point in \( \frac{2\pi}{d} \);

(3) the triple \( (f(t), h(\theta), \theta(t)) \) is a solution of the following ODE system for all \( t \in \mathbb{R}/(2\mathbb{Z}\pi) \):

\[
\frac{1}{2f(t)} \frac{d^2}{dt^2} f(t) - \frac{1}{4f(t)^2} \left( \frac{d}{dt} f(t) \right)^2 + 1
\]

\[
= \left( \frac{d}{dt} \theta(t) \right)^2 \left( \frac{1}{2h(\theta(t))} \frac{d^2}{d\theta^2} h(\theta(t)) - \frac{1}{4h(\theta(t))^2} \left( \frac{d}{d\theta} h(\theta(t)) \right)^2 + 1 \right),
\]

and

\[
\sin^2 \left( t + \frac{k\pi}{d} \right) + \frac{\cos(t + \frac{k\pi}{d}) \sin(t + \frac{k\pi}{d})}{2f(t)} \frac{d}{dt} f(t)
\]

\[
= \sin^2 \left( \theta(t) + \frac{k\pi}{d} \right) + \frac{\cos(\theta(t) + \frac{k\pi}{d}) \sin(\theta(t) + \frac{k\pi}{d})}{2h(\theta(t))} \frac{d}{d\theta} h(\theta(t))
\]

for each \( k \in \{0, \ldots, d-1\} \).

**Remark 5.4.** The way we present Theorem 5.3 is explicit and convenient. However, it contains some iteration. For example, when \( d \in \{1, 2, 3\} \), we only need to keep the ODE with \( k = 0 \) for (5.12). When \( d = 2 \), the two ODEs in (5.12) are equivalent, because their sum is an obvious equality. When \( d = 3 \), we can use (5.12) with \( k = 0 \) and the \( \mathbb{Z}_d \)-symmetry in \( D_{2d} \) to deduce the other ODEs in (5.12). Similarly, when \( d = 4 \) and \( d = 6 \), we only need the two ODEs with \( k = 0, 1 \) for (5.12).

### 5.3 The geometric description by the \( (d) \)-property

In this subsection, we study the correspondence between the triple \( (F_1, F_2, \Phi) \) in Theorem 5.3 and its restriction to a normal plane \( V \) for \( M_d \). Recall that \( F_1 = F_1|_V \) and \( F_2 = F_2|_V \) are two \( D_{2d} \)-invariant Minkowski norms on \( V \) with polar coordinate representations \( F_1 = r\sqrt{2f(t)} \) and \( F_2 = r\sqrt{2h(\theta)} \). In Subsection 5.2, we have seen that the restriction \( \overline{\Phi} = \Phi|_V \) is a Hessian isometry from \( F_1 \) and \( F_2 \) with the polar coordinate representation \( (r, t) \mapsto (r\sqrt{2f(t)}, \theta(t)) \). Here, \( (f(t), h(\theta), \theta(t)) \) is just the triple in Theorem 5.3 corresponding to \( (F_1, F_2, \Phi) \). The diffeomorphism \( \theta(t) \) on \( \mathbb{R}/(2\mathbb{Z}\pi) \) is \( D_{2d} \)-equivariant and preserves the orientation, so \( \overline{\Phi} \) is \( D_{2d} \)-equivariant and preserves the orientation as well. Furthermore, since \( \theta(t) \) fixes each point in \( \frac{2\pi}{d} \), \( \overline{\Phi} \) preserves each ray spanned by the points in \( V \cap (M_0 \cup M_{\pi/d}) \).

The restriction to \( V \) provides the correspondence from the triple \( (F_1, F_2, \Phi) \) to the triple \( (\overline{F}_1, \overline{F}_2, \overline{\Phi}) \), by which we can explain (5.9)–(5.12) in Theorem 5.3.

Theorem 5.1 indicates the two inequalities (5.9) and (5.10) just tell us \( \overline{F}_1 \) and \( \overline{F}_2 \) are Minkowski norms.

To explain (5.11), we denote by \( T_1 = \partial_t - \frac{\sqrt{2f(t)}}{f(t)} \frac{d}{dt} f(t) \partial_r \) and \( T_2 = \partial_{\theta} - \frac{\sqrt{2h(\theta)}}{h(\theta)} \frac{d}{d\theta} h(\theta) \partial_{\theta} \) the tangent vector fields on \( V \setminus \{0\} \) which generate \( S_{\overline{F}_1} \) and \( S_{\overline{F}_2} \), respectively. Here, \( \partial_r \) and \( \partial_{\theta} \) (or \( \partial_{\theta} \) for \( \overline{F}_2 \)) correspond to the polar \( r \)- and \( t \)-coordinates on \( V \). By Lemma 5.2 and the \( D_{2d} \)-symmetry, we have \( \overline{\Phi}_* (T_1) = \frac{d}{dt} \theta(t) T_2 \).

Then by (3.4) in Lemma 3.5, the ODE (5.11) just tells us that

\[
g_{\overline{F}_1}^{\overline{F}_1}(\overline{F}_1, T_1) = g_{\overline{F}_2}^{\overline{F}_2}(\overline{\Phi}_*(T_1), \overline{\Phi}_*(T_2)),
\]

i.e., \( \overline{\Phi} \) is a Hessian isometry between \( \overline{F}_1 \).

To explain (5.12), we recall that the orthonormal coordinates \((x_1, x_2)\) and polar coordinates \((r, t)\) (or \((r, \theta)\) where \( F_2 \) is concerned) of \( x \in V \setminus \{0\} \) are related by

\[
x = x_1 v_1 + x_2 v_2 = r \cos t v_1 + r \sin t v_2,
\]

in which \( v_1 \in V \cap M_0 \) and \( v_2 \) provide an orthonormal basis on \( V \).
Theorem 5.6. So at $v = x_1 v_1 + x_2 v_2$ in $S_{F_1}$, where $2r^2 f(t) = 1$ and $x_2 = r \sin t$, we can retrieve the triple $(\Phi)$. The right-hand side of (5.12) can be expressed similarly. So (5.12) tells us that with respect to the orthogonal decomposition

$$
V = V' + V'' = \mathbb{R} v_1 + \mathbb{R} v_2 \text{ (i.e., we have } V' = \mathbb{R} v_1 \text{ and } V'' = \mathbb{R} v_2)$$

for any $x = x' + x'' \in S_{F_1}$ and $\mathcal{F}(x) = \mathcal{F} + \mathcal{F}'$, using the identification in Subsection 2.4, which identifies each ray spanned by $\mathcal{F}$ and $\mathcal{F}'$, to the following equality is satisfied:

$$
g_{\mathcal{F}}(x'') = g_{\mathcal{F}'}(x'').$$

More generally, we define this property as follows.

Definition 5.5. A Hessian isometry $\Phi$ between two Minkowski norms $F_1$ and $F_2$ on $\mathbb{R}^n$ with $n \geq 2$ is said to satisfy the (d)-property with respect to the orthogonal decomposition $\mathbb{R}^n = V' + V''$, if for any nonzero $x = x' + x''$ and $\Phi(x) = \mathcal{F} + \mathcal{F}'$ with $x', \mathcal{F}' \in V'$ and $x'', \mathcal{F}'' \in V''$, we always have $g_{\mathcal{F}}(x'') = g_{\mathcal{F}'}(x'')$ (or equivalently, $g_{\mathcal{F}'}(x', x) = F_1(x)^2 - g_{\mathcal{F}}(x'' + x', x) = F_2(x)^2 - g_{\mathcal{F}'}(x'', x) = g_{\mathcal{F}''}(x', x)$).

So the ODE (5.11) with $k = 0$ can be interpreted as the (d)-property of $\Phi$ for $V = V' + V'' = \mathbb{R} v_1 + \mathbb{R} v_2$.

By a similar argument, (5.12) with $0 < k \leq d - 1$ can be interpreted as the (d)-property of $\mathcal{F}$ for the decomposition

$$
V = V' + V'' = \mathbb{R} (\cos \left( -\frac{k\pi}{d} \right) v_1 + \sin \left( -\frac{k\pi}{d} \right) v_2) + \mathbb{R} (\cos \left( \frac{\pi}{2} - \frac{k\pi}{d} \right) v_1 + \sin \left( \frac{\pi}{2} - \frac{k\pi}{d} \right) v_2).
$$

To summarize, we see that $(F_1, F_2, \Phi)$ in Theorem 5.3 determines $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F})$ which satisfies $D_{2d}$-symmetry and (d)-properties. Conversely, from any $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F})$ with $D_{2d}$-symmetry and (d)-properties, we can retrieve the triple $(f(t), h(\theta), \theta(t))$ in Theorem 5.3 and then use it to target $(F_1, F_2, \Phi)$.

Finally, we can transport $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F})$ to $\mathbb{R}^2$, using the identification in Subsection 2.4, which identifies $v_1$ and $v_2$ with $v_1 = (1, 0)$ and $v_2 = (0, 1)$ in $\mathbb{R}^2$, respectively. By the $D_{2d}$-symmetry, the triple after the translation is relevant to the choice of the identification between $V$ and $\mathbb{R}^2$. It does not depend on the choice of $V$ either. So we have the one-to-one correspondence in the following theorem.

Theorem 5.6. Let $M_t$ be an isoparametric foliation on $(S^{n-1}(1), g^\xi)$ with $d$ principal curvatures. Then the restriction to a normal plane $V$ for $M_t$, and the identification between $V$ and $\mathbb{R}^2$ provide a one-to-one correspondence between the set of all the triples $(F_1, F_2, \Phi)$ satisfying the following:

1. $F_1$ and $F_2$ are Minkowski norms on $\mathbb{R}^n$ induced by $M_t$;
2. $\Phi$ is a Hessian isometry from $F_1$ to $F_2$ which preserves the orientation and fixes the spherical $\xi$-coordinates

and the set of all the triples $(F_1, F_2, \Phi)$ satisfying the following:

1. $F_1$ and $F_2$ are two $D_{2d}$-invariant Minkowski norms on $\mathbb{R}^2$;
2. $\Phi$ is a $D_{2d}$-equivariant orientation-preserving Hessian isometry from $F_1$ to $F_2$, which preserves each ray spanned by $(\cos \left( \frac{k\pi}{d} \right), \sin \left( \frac{k\pi}{d} \right))$ for $k \in \{0, \ldots, 2d - 1\}$;
3. for each $k \in \{0, \ldots, d - 1\}$ with respect to the decomposition

$$
\mathbb{R}^2 = V' + V'' = \mathbb{R} \left( \cos \left( -\frac{k\pi}{d} \right), \sin \left( -\frac{k\pi}{d} \right) \right) + \mathbb{R} \left( \cos \left( \frac{\pi}{2} - \frac{k\pi}{d} \right), \sin \left( \frac{\pi}{2} - \frac{k\pi}{d} \right) \right),
$$

$\mathcal{F}$ satisfies the (d)-property, i.e., for any nonzero $x = x' + x''$ and $\mathcal{F}(x) = \mathcal{F} + \mathcal{F}'$ with $x', \mathcal{F}' \in V'$ and $x'', \mathcal{F}'' \in V''$, we always have $g_{\mathcal{F}}(x'') = g_{\mathcal{F}'}(x'')$.
Remark 5.7. When \( d = 1 \) or \( d = 2 \), Theorem 5.6 only requires \( \Phi \) to satisfy the (d)-property for the decomposition
\[
\mathbb{R}^2 = V' + V'' = \mathbb{R}e_1 + \mathbb{R}e_2,
\]
i.e., when \( d = 2 \), exchanging \( V' \) and \( V'' \) does not provide another restriction. On the other hand, when \( d > 2 \), essentially more (d)-properties for \( \Phi \) are required by Theorem 5.6. This phenomenon and its consequence will be discussed in the next two sections.

6 The Legendre transformation and the (d)-property

6.1 The Legendre transformation

In this paper, the Legendre transformation of a Minkowski norm \( F \) on \( \mathbb{R}^n \) with \( n \geq 2 \) is referred to the following. By the strong convexity of \( F \), we have the following orientation-preserving diffeomorphism:
\[
\Phi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}, \quad (x_1, \ldots, x_n) \mapsto \left( \frac{\partial \partial x_1 E, \ldots, \partial \partial x_n E}{\partial \partial x_1 E, \ldots, \partial \partial x_n E} \right), \quad (6.1)
\]
where \( E = \frac{1}{2}F^2. \) Obviously, \( \Phi \) is (locally) linear if and only if \( F \) is (locally) Euclidean. The image \( \Phi(S_F) \) is a strongly convex sphere surrounding the origin, so it determines a Minkowski norm \( \hat{F} \) of \( \mathbb{R}^n \). We denote by \( g \) and \( \hat{g} \) the Hessian metrics of \( F \) and \( \hat{F} \), respectively. It is crucial to know that with respect to the coordinates \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \backslash \{0\} \), the Hessian matrix of \( \hat{F} \) at \( \overline{x} = \Phi(x) \in \mathbb{R}^n \backslash \{0\} \) coincides with the inverse matrix \( (g^{ij}) \) for the Hessian matrix \( (g_{ij}) \) of \( F \) at \( x \) (see [2, Proposition 14.8.1] or [43, Lemma 3.1.2]). This observation together with the fact \( \Phi_* (\partial x_i) = \sum_j g_{ij} \partial x_j \) implies
\[
\hat{g}(\Phi_* (\partial x_i), \Phi_* (\partial x_j)) = \sum_{k,l} g_{kl} g^{jl} = g_{ij} = g_1 (\partial x_i, \partial x_j),
\]
i.e., \( \Phi \) is a Hessian isometry [42]. On the other hand, \( \Phi \) has the following involutive property. If we define \( \Phi(x) = \overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) = (\sum_i g_{1i} x_i, \ldots, \sum_i g_{ni} x_i) \) for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \backslash \{0\} \), then we see
\[
\Phi^{-1}(\overline{x}_1, \ldots, \overline{x}_n) = (x_1, \ldots, x_n) = \left( \sum_i g^{1i} \overline{x}_i, \ldots, \sum_i g^{ni} \overline{x}_i \right) = \left( \frac{\partial \partial E}{\partial \partial E}, \ldots, \frac{\partial \partial E}{\partial \partial E} \right),
\]
where \( \hat{E} = \frac{1}{2}\hat{F}^2 \), i.e., \( \Phi^{-1} \) is the Legendre transformation of \( \hat{F} \).

To summarize, we call \( \hat{F} \) the dual (Minkowski) norm of \( F \) and \( \Phi \) the Legendre transformation of \( F \).

Notice that our notion in (6.1) has implicitly used the standard Euclidean inner product to identify \( \mathbb{R}^n \) with its dual. However, it does not depend on the choice of orthonormal coordinates. So we have the following easy lemma.

Lemma 6.1. If a linear isometry on \((\mathbb{R}^n, F)\) preserves the standard inner product, then it commutes with the Legendre transformation of \( F \) and preserves the dual norm \( \hat{F} \).

6.2 Hessian isometries satisfying all the (d)-properties

From Theorem 5.6, we have seen the importance of the (d)-property. Indeed, the Legendre transformation is one of its origin.

Lemma 6.2. Let \( F_1 \) and \( F_2 \) be two Minkowski norms on \( \mathbb{R}^n \) with \( n \geq 2 \), and \( \Phi \) be a Hessian isometry from \( F_1 \) to \( F_2 \). If \( \Phi \) is a positive scalar multiplication or the composition of the Legendre transformation of \( F_1 \) and a positive scalar multiplication, then \( \Phi \) satisfies the (d)-property for every orthogonal decomposition.

Proof. Let \( \mathbb{R}^n = V' + V'' \) be any orthogonal decomposition. We prove that the Hessian isometry in the lemma satisfies the corresponding (d)-property.
Firstly, we prove the case where \( F_2(cx) = F_1(x) \) and \( \Phi(x) = cx \) for some constant \( c > 0 \). With respect to the above decomposition, if we have \( x = x' + x'' \in \mathbb{R}^n \setminus \{0\} \), then \( \pi = \Phi(x) = cx = \pi' + \pi'' \) satisfies \( \pi' = cx' \) and \( \pi'' = cx'' \). By the observation,

\[
\frac{d}{ds} (F_2(x + s \pi'''))^2 \bigg|_{s=0} = \frac{1}{2} \frac{d}{ds} (F_2(cx + cs \pi''))^2 \bigg|_{s=0} = \frac{1}{2} \frac{d}{ds} (F_1(x + s \pi''))^2 \bigg|_{s=0} = g^F_2(\pi', x).
\]

The (d)-property is proved.

Next, we prove the case where \( \Phi \) is the Legendre transformation of \( F_1 \). We may choose the orthonormal coordinates \( (x_1, \ldots, x_n) \) such that \( V' \) and \( V'' \) are given by \( x_{m+1} = \cdots = x_n = 0 \) and \( x_1 = \cdots = x_m = 0 \), respectively. Then for any nonzero \( x = (x_1, \ldots, x_n) = x' + x'' \) and \( \Phi(x) = \pi = (\pi_1, \ldots, \pi_n) = \pi' + \pi'' \) with \( x', \pi' \in V' \) and \( x'', \pi'' \in V'' \), we have

\[
g^F_2(\pi'', x) = \sum_{m+1 \leq i \leq n, 1 \leq j \leq n} \left( \sum_k x_k g_{ki} \right) \left( \sum_l x_l g_{jl} \right) = \sum_{m+1 \leq i \leq n, 1 \leq p \leq n} x_p g_{pi} x_i = g^F_1(x'', x),
\]

which proves the (d)-property of \( \Phi \).

Finally, we prove the case where \( \Phi \) is the composition of the Legendre transformation of \( F_1 \) and a positive scalar multiplication. The argument is a combination of the above two, or one may apply Lemma 6.5 below.

Besides the Legendre transformation, identity maps and their compositions with positive scalar multiplications, there exist many other Hessian isometries which satisfy the (d)-property for every orthogonal decomposition. Here, we propose a construction.

**Example 6.3.** Let \( C(U_1) \) and \( C(U_2) \) be two conic open subsets in \( \mathbb{R}^n \) with \( n \geq 2 \) such that their closures only intersect at the origin. We start with the standard Euclidean norm \( F_0 \) on \( \mathbb{R}^n \), and slightly deform it on \( C(U_1) \) and \( C(U_2) \) to obtain the new Minkowski norm \( F_1 \). The second new Minkowski norm \( F_2 \) is constructed by gluing \( F_1 \) on \( \mathbb{R}^n \setminus C(U_1) \) and the dual norm of \( F_1 \) on \( C(U_1) \). Then there is a Hessian isometry \( \Phi \) from \( F_1 \) to \( F_2 \) such that \( \Phi \) coincides with the Legendre transformation of \( F_1 \) on \( C(U_1) \) and the identity map elsewhere. This \( \Phi \) satisfies the (d)-property for every orthogonal decomposition. If \( F_1 \) is locally non-Euclidean on \( C(U_1) \) and \( C(U_2) \), then \( \Phi \) is not a positive scalar multiplication or the composition of a Legendre transformation and a positive scalar multiplication.

More examples can be constructed similarly, which may involve more (even infinitely many) conic open subsets \( C(U_i) \) and different scalar changes.

### 6.3 The local (d)-property

For the convenience of later discussion, we also introduce the local version for the (d)-property.

**Definition 6.4.** Let \( F_1 \) and \( F_2 \) be two Minkowski norms on \( \mathbb{R}^n \) with \( n \geq 2 \), \( C(U_1) \) be a connected conic open subset of \( \mathbb{R}^n \setminus \{0\} \), and \( \mathbb{R}^n = V' + V'' \) an orthogonal decomposition. Then a local Hessian isometry \( \Phi \) from \( F_1 \) to \( F_2 \) is said to satisfy the local (d)-property on \( C(U_1) \) for \( \mathbb{R}^n = V' + V'' \), if \( \Phi \) has the definition on \( C(U_1) \), and for any \( x = x' + x'' \in C(U_1) \) and \( \Phi(x) = \pi = \pi_1 + \pi_2 \) with \( x', \pi' \in V' \) and \( x'', \pi'' \in V'' \), we always have \( g^F_1(x', x) = g^F_2(\pi', \pi) \) (or equivalently, \( g^F_1(x', x) = F_1(x)^2 - g^F_2(\pi'', x) \)).

The following transitivity lemma for the local (d)-property is easy to see.

**Lemma 6.5.** Let \( F_i \) with \( 1 \leq i \leq 3 \) be three Minkowski norms on \( \mathbb{R}^n \) with \( n \geq 2 \), \( C(U_1) \) and \( C(U_2) \) be two connected conic open subsets of \( \mathbb{R}^n \setminus \{0\} \), \( \Phi_1 \) be a local Hessian isometry from \( F_1 \) to \( F_2 \) which maps \( C(U_1) \) into \( C(U_2) \), and \( \Phi_2 \) be a local Hessian isometry from \( F_2 \) to \( F_3 \) which has the definition
on $C(U_2)$. Then with respect to the same orthogonal decomposition $\mathbb{R}^n = V' + V''$, $\Phi = \Phi_2 \circ \Phi_1$ satisfies the (d)-property on $C(U_1)$ when each $\Phi_i$ satisfies the (d)-property on $C(U_i)$.

We will need the following lemma in Subsection 7.2.

**Lemma 6.6.** Let $F_1$ and $F_2$ be two Minkowski norms on $\mathbb{R}^2$, $C(U_1)$ be a connected conic open subset contained in the first quadrant $\{(x_1, x_2) \mid x_1 > 0, x_2 > 0\}$, and $\Phi$ be a local Hessian isometry from $F_1$ to $F_2$. Suppose that there exist positive constants $a$ and $b$ such that the local Hessian isometry $\Phi$ from $F_1$ to $F_2$ can be represented either as

$$\Phi(x_1, x_2) = (ax_1, bx_2), \quad \forall x = (x_1, x_2) \in C(U_1)$$

or as

$$\Phi(x_1, x_2) = \left( a \frac{\partial}{\partial x_1} E_1, b \frac{\partial}{\partial x_2} E_1 \right), \quad \forall x = (x_1, x_2) \in C(U_1),$$

where $E_1 = \frac{1}{2} F_1^2$. If $\Phi$ satisfies the local (d)-property on $C(U_1)$ for the decomposition $\mathbb{R}^2 = V' + V''$, in which $V'$ and $V''$ are spanned by $(\cos c, \sin c) = \cos ce_1 + \sin ce_2$ and $(-\sin c, \cos c) = -\sin ce_1 + \cos ce_2$, respectively, for some $c \notin \mathbb{Z} \pi/2$, then $\Phi_{|C(U_1)}$ is either a positive scalar multiplication or the composition of the Legendre transformation of $F_1$ and a positive scalar multiplication.

**Proof.** We first prove Lemma 6.6 when $\Phi(x_1, x_2) = (ax_1, bx_2)$, $\forall x = (x_1, x_2) \in C(U_1)$.

Notice that this $\Phi$ satisfies the local (d)-property on $C(U_1)$ for $\mathbb{R}^2 = V' + V'' = \mathbb{R} e_1 + \mathbb{R} e_2$. So we have

$$x_2 g_x^{F_1}(e_2, x) = x_2 g_x^{F_2}(e_2, \varpi) \quad \text{and} \quad x_1 g_x^{F_1}(e_1, x) = x_1 g_x^{F_2}(e_1, \varpi)$$

(6.2)

for any $x = (x_1, x_2) \in C(U_1)$ and $\Phi(x) = \varpi = (\varpi_1, \varpi_2)$.

On the other hand, the local (d)-property assumed in the lemma implies

$$(-x_1 \sin c + x_2 \cos c)g_x^{F_1}(-\sin ce_1 + \cos ce_2, x) = (-\varpi_1 \sin c + \varpi_2 \cos c)g_x^{F_2}(\sin ce_1 + \cos ce_2, \varpi)$$

(6.3)

for every

$$x = (x_1, x_2) = (x_1 \cos c + x_2 \sin c)(\cos c, \sin c) + (-x_1 \sin c + x_2 \cos c)(-\sin c, \cos c) \in C(U_1)$$

and

$$\varpi = \Phi(x) = (\varpi_1, \varpi_2) = (\varpi_1 \cos c + \varpi_2 \sin c)(\cos c, \sin c) + (-\varpi_1 \sin c + \varpi_2 \cos c)(-\sin c, \cos c).$$

Plugging (6.2), $\varpi_1 = ax_1$ and $\varpi_2 = bx_2$ into (6.3), and using the fact that $\sin c \cos c \neq 0$ because $c \notin \mathbb{Z} \pi/2$, we obtain

$$a(a-b)x_1 g_x^{F_1}(e_2, x) = b(a-b)x_2 g_x^{F_1}(e_1, x).$$

If $a = b$, there is nothing to prove. Otherwise, we obtain

$$ax_1 g_x^{F_1}(e_2, x) = bx_2 g_x^{F_1}(e_1, x)$$

(6.4)

for any $x = (x_1, x_2) \in C(U_1)$.

Now we switch to the polar coordinate representations $x = (x_1, x_2) = (r \cos t, r \sin t) \in C(U_1)$ and $F_1 = r \sqrt{2f(t)}$, and then (6.4) is translated to the ODE

$$(a \cos^2 t + b \sin^2 t) \frac{d}{dt} f(t) + (b-a) \sin 2tf(t) = 0,$$

which can be explicitly solved, i.e., for some positive constant $c'$, $f(t) = c' \frac{2a+b}{a-b} \cos 2t$. Then we see that $F_1_{|C(U_1)}$ coincides with the Euclidean norm

$$\sqrt{\frac{4c'}{|a-b|} (ax_1^2 + bx_2^2)}.$$
i.e., $\Phi|_{C(U_1)}$ coincides with the composition of the Legendre transformation of $F_1$ and the scalar multiplication by $\frac{a-b}{4m}$.

To summarize, we have proved Lemma 6.6 when $\Phi(x_1, x_2) = (ax_1, bx_2)$, $\forall x = (x_1, x_2) \in C(U_1)$. In particular, we see that $F_1|_{C(U_1)}$ is a positive constant multiple of $\sqrt{a_1^2 + b_1^2}$ when $a \neq b$.

Then we prove Lemma 6.6 when $\Phi(x_1, x_2) = (\frac{a}{\partial x_1} E_1, \frac{b}{\partial x_2} E_2)$ for any $x = (x_1, x_2) \in C(U_1)$.

Denote by $C(U_2) = \{(x_1, x_2) \mid (ax_1, bx_2) \in C(U_1)\}$ another connected conic open subset in the first quadrant, $F_3$ the Minkowski norm determined by $F_3(x, y) = F_1(ax, by)$, $\Phi_1(x, y) = (ax, by)$ the linear isometry from $F_3$ to $F_1$, and $\Phi_2$ the Legendre transformation of $F_2$. The composition $\Phi \circ \Phi_1$ coincides with the Legendre transformation of $F_3$ on $C(U_2)$. So by the involutive property of the Legendre transformation, $\Phi_2 \circ \Phi$ maps $(x_1, x_2)$ to $(\frac{x_1}{a}, \frac{x_2}{b})$ for any $x = (x_1, x_2) \in C(U_1)$. By Lemmas 6.2 and 6.5, $\Phi_2 \circ \Phi$ satisfies the local (d)-property in Lemma 6.6. As we have proved above, either $a = b$, i.e., $\Phi|_{C(U_1)}$ is the composition of the Legendre transformation of $F_1$ and a positive scalar multiplication, or $F_1|_{C(U_1)}$ is a positive multiple of $\sqrt{\frac{x_1^2}{a} + \frac{x_2^2}{b}}$.

6.4 The Legendre transformation of a Minkowski norm induced by $M_t$

Now we consider the Legendre transformation $\Phi$ of the Minkowski norm $F_1 = r\sqrt{2f(t)}$ induced by the isoparametric foliation $M_t$ on $(S^{n-1}(1), g^a)$. We prove the following lemma.

**Lemma 6.7.** Let $F_1 = r\sqrt{2f(t)}$ be a Minkowski norm induced by the isoparametric foliation $M_t$ on $(S^{n-1}(1), g^a)$ with $d$ principal curvatures. Then for its dual norm $F_2$ and the Legendre transformation $\Phi$ from $F_1$ to $F_2$, we have the following:

1. $F_2$ is also a Minkowski norm induced by $M_t$.
2. $\Phi$ preserves the orientation and fixes the spherical $\xi$-coordinates.
3. Let $V$ be any normal plane for $M_t$. Then the restriction $\overline{\Phi} = \Phi|_V$ coincides with the Legendre transformation of $\overline{F}_1 = F_1|_V$.

**Proof.** With the normal plane $V$ for $M_t$ arbitrarily chosen, we can find the orthonormal coordinates $(x_1, \ldots, x_n)$ for $\mathbb{R}^n$ such that $V$ is given by $x_3 = \cdots = x_n = 0$ and $(1, 0, \ldots, 0) \in M_0$. Around any point $x \in V \cap (\mathbb{R}_{>0} M_0 \cup \mathbb{R}_{\geq 0} M_{\pi/2/d})$, we can define a spherical local frame induced by $M_t$. The values of $X_1, \ldots, X_{n-2}$ span the same tangent subspace in $T_x(\mathbb{R}^n \setminus \{0\})$ as those of $\partial_{x_3}, \ldots, \partial_{x_n}$. By Lemma 3.4 and continuity, we have

$$\frac{\partial}{\partial x_i} E_1 \bigg|_{V \setminus \{0\}} = 0, \quad \forall 3 \leq i \leq n,$$

where $E_1 = 1/2 F_1^2$. So we have $\Phi(x_1, x_2, 0, \ldots, 0) = (\frac{\partial}{\partial x_1} E_1, \frac{\partial}{\partial x_2} E_1, 0, \ldots, 0)$, from which we see that $\Phi$ preserves $V \setminus \{0\}$ and $\overline{\Phi} = \Phi|_{V \setminus \{0\}}$ is the Legendre transformation for $\overline{F}_1 = F_1|_V$.

For different normal planes for $M_t$, $\overline{F}_2 = F_2|_{V}$, and its Legendre transformation $\overline{\Phi} = \Phi|_V$ are irrelevant to the choice of $V$. This observation implies that $F_2$ is a Minkowski norm induced by $M_t$, and $\Phi$ preserves the foliation.

Since $\overline{F}_1$ is $D_{2d}$-invariant, its Legendre transformation $\overline{\Phi}$ is $D_{2d}$-equivariant and orientation-preserving, and it preserves the ray spanned by each point in $V \cap (S^{n-1}(1) \setminus (M_0 \cup M_{\pi/2/d}))$. Let $(r, t) \mapsto \frac{r f(t)^{1/2}}{h(t)^{\pi/2}}, \theta(t)$ be the polar coordinate representation for $\overline{\Phi}$. Then $\theta$ fixes each point in $\mathbb{R}_{>0} \setminus \mathbb{R}_{\geq 0}$ and preserves the interval $(k\pi/2, (k+1)\pi/2)$ for each $k \in \{0, \ldots, 2d - 1\}$. Each interval $(k\pi/2, (k+1)\pi/2)$ for $t$ determines a conic open subset of $V \setminus \{0\}$ which corresponds to a distinct spherical $\xi$-coordinate in $\mathbb{R}^n$. With $V$ changing arbitrarily, we see from this observation that $\Phi$ fixes the spherical $\xi$-coordinates. Finally, from the spherical coordinate representation $(r, t, \xi) \mapsto \frac{r f(t)^{1/2}}{h(t)^{\pi/2}}, \theta(t), \xi$ for $\Phi$, which contains the same $\theta(t)$ but restricts $t$ to $(0, \frac{\pi}{2})$, we see that $\Phi$ preserves the orientation. \(\square\)
Lemma 6.7 indicates that the correspondence in Theorem 5.6 relates the Legendre transformations to Legendre transformations. On the other hand, for any Legendre transformation $\overline{\Phi}$ from the $D_{2d}$-invariant Minkowski norm $F_1$ on $\mathbb{R}^2$ to its dual $F_2$, the $(d)$-property requirement in Theorem 5.6 is met by Lemmas 6.1 and 6.2. So $\overline{\Phi}$ is related to a Hessian isometry $\Phi$ from $F_1$ to $F_2$. It must be the same one as in Lemma 6.7, i.e., the Legendre transformation of $F_1$, because the two coincide when restricted to each normal plane of $M_t$.

To summarize, we have the following one-to-one correspondence between Legendre transformations, which is not affected by the choice for the normal plane $V$ for $M_t$ or its identification with $\mathbb{R}^2$.

**Theorem 6.8.** Let $M_t$ be an isoparametric foliation on $(\mathbb{S}^n-1, g^2)$ with $d$ principal curvatures. Then the restriction to any normal plane $V$ for $M_t$ and the identification between $V$ and $\mathbb{R}^2$ provide a one-to-one correspondence between the set of Legendre transformations of Minkowski norms on $\mathbb{R}^n$ induced by $M_t$ and the set of Legendre transformations of $D_{2d}$-invariant Minkowski norms on $\mathbb{R}^2$.

**Remark 6.9.** The function $\theta(t)$ in the polar coordinate representation $(r, t) \mapsto (\frac{rf(t)^{1/2}}{h(\theta)^{1/2}}, \theta(t))$ for the Legendre transformation

$$\overline{\Phi}(x_1, x_2) = \left( \frac{\partial}{\partial x_1} E_1, \frac{\partial}{\partial x_2} E_1 \right)$$

for a $D_{2d}$-invariant $F_1 = r \sqrt{2f(t)}$ with $E_1 = \frac{1}{2} F_1^2 = r^2 f(t)$ has been calculated (with more generality) in [58], i.e.,

$$\theta(t) = \arccos \left( \frac{2 \cos tf(t) - \sin t f(t)}{(4f(t)^2 + (\frac{d}{dt} f(t))^2)^{1/2}} \right), \quad (6.5)$$

when $t \in (0, \pi)$. It is a solution of

$$\frac{d}{dt} \theta(t) = \frac{(2f(t) \frac{d^2}{d\theta^2} f(t) - \left( \frac{d}{dt} f(t) \right)^2 + 4f(t)^2 \sin \theta(t) \cos \theta(t)}{(\cos t \frac{d}{dt} f(t) + 2 \sin tf(t))(- \sin t \frac{d}{dt} f(t) + 2 \cos tf(t)^2)} \quad (6.6)$$

for $t \in (0, \frac{\pi}{2})$. The general solutions of (6.6) have also been given by [58], i.e.,

$$\theta(t) = \arccos \left( \frac{2 \cos tf(t) - \sin t f(t)}{[4(\cos^2 t + \frac{k \pi}{d} \sin^2 t) f(t)^2 + 4(\frac{k \pi}{d} - 1) \cos t \sin tf(t) \frac{d}{dt} f(t) + (\sin^2 t + \frac{k \pi}{d} \cos^2 t)(\frac{d}{dt} f(t)^2)]^{1/2}} \right), \quad (6.7)$$

which appears in the polar coordinate representation for the mapping

$$(x_1, x_2) \mapsto \left( a \frac{\partial}{\partial x_1} E_1, b \frac{\partial}{\partial x_2} E_1 \right)$$

with positive constants $a$ and $b$.

7 The ODE method and the local description

7.1 The ODE method

In this subsection, we analyze the ODE system for the triple $(f(t), h(\theta), \theta(t))$ consisting of (5.11) and (5.12) in Theorem 5.3, i.e.,

$$\frac{1}{2f(t)} \frac{d^2}{dt^2} f(t) - \frac{1}{4f(t)^2} \left( \frac{d}{dt} f(t) \right)^2 + 1 = \left( \frac{d}{dt} \theta(t) \right)^2 \left( \frac{1}{2h(\theta(t))} \frac{d^2}{d\theta^2} h(\theta(t)) \right) - \frac{1}{4h(\theta(t))^2} \left( \frac{d}{d\theta} f(t) \right)^2 + 1, \quad (7.1)$$

$$\sin^2 \left( t + \frac{k \pi}{d} \right) + \frac{\cos(t + \frac{k \pi}{d}) \sin(t + \frac{k \pi}{d})}{2f(t)} \frac{d}{dt} f(t)$$
We plug (7.3) and (7.4) into the right-hand side of (7.1) to erase technical complexity, we fix some $h(\theta(t))$ has been arbitrarily chosen and discuss how to locally determine $\theta(t)$ and $h(\theta(t))$ in a sufficiently small neighborhood of $t_0 \in (0, \frac{\pi}{2})$ as well, and discuss the ODE system for $t$ in a sufficiently small neighborhood of $t_0 \in (0, \frac{\pi}{2})$. We assume $d > 2$ because the cases where $d = 1$ and $d = 2$ have been discussed by [58, Theorems 1.4 and 1.5]. The ODE method in [58] for discussing (7.1) and (7.2) with $k = 0$ is still a main ingredient here. We sketch it as follows.

Rewriting (7.2) with $k = 0$ as

$$
\frac{1}{h(\theta(t))} \frac{d}{dt} h(\theta(t)) = \left(2 \sin^2 t + \frac{\cos t \sin t}{f(t)} \frac{d}{dt} f(t)\right) \csc \theta(t) \sec \theta(t) - 2 \tan \theta(t),
$$

and differentiating it with respect to $t$, we obtain

$$
\frac{d}{dt} \theta(t) \left(\frac{1}{h(\theta(t))} \frac{d^2}{dt^2} h(\theta(t)) - \frac{1}{h(\theta(t))^2} \left(\frac{d}{dt} h(\theta(t))\right)^2\right)
= \frac{d}{dt} \theta(t) \left(2 \sin^2 t + \frac{\cos t \sin t}{f(t)} \frac{d}{dt} f(t)\right) \left(\sec^2 \theta(t) - \csc^2 \theta(t)\right) - 2 \frac{d}{dt} \theta(t) \sec^2 \theta(t)
+ \left(4 \cos t \sin t + \frac{\cos^2 t - \sin^2 t}{f(t)} \frac{d}{dt} f(t) - \frac{\cos t \sin t}{f(t)} \left(\frac{d}{dt} f(t)\right)^2\right)
+ \frac{\cos t \sin t}{f(t)} \frac{d^2}{dt^2} f(t) \csc \theta(t) \sec \theta(t).
$$

We plug (7.3) and (7.4) into the right-hand side of (7.1) to erase $h(\theta(t))$ and its derivatives, and then we obtain a formal quadratic equation for $\frac{d}{dt} \theta(t)$,

$$
A \left(\frac{d}{dt} \theta(t)\right)^2 + B \left(\frac{d}{dt} \theta(t)\right) + C = 0,
$$

in which

$$
A = \frac{\cos t \sin t (\cos t \frac{d}{dt} f(t) + 2 \sin t f(t))(\sin t \frac{d}{dt} f(t) - 2 \cos t f(t))}{2 f(t)^2 \cos^2 \theta(t) \sin^2 \theta(t)},
$$

$$
B = \frac{\cos t \sin t \frac{d^2}{dt^2} f(t) - \cos t \sin t \frac{d}{dt} f(t)^2 + \cos^2 t - \sin^2 t \frac{d}{dt} f(t) + 4 \cos t \sin t}{\cos \theta(t) \sin \theta(t)},
$$

$$
C = -\frac{1}{f(t)} \frac{d^2}{dt^2} f(t) + \frac{1}{2 f(t)^2} \left(\frac{d}{dt} f(t)\right)^2 - 2.
$$

Since we have assumed $d > 2$, for $t \in (0, \frac{\pi}{2})$, we have $\theta(t) \in (0, \frac{\pi}{2}) \subset (0, \frac{\pi}{2})$. So from (6.6) in Remark 6.9, we see that

$$
\left(\cos t \frac{d}{dt} f(t) + 2 \sin t f(t)\right) \left(\sin t \frac{d}{dt} f(t) - 2 \cos t f(t)\right)
$$

is a positive factor in $A$ for $t \in (0, \frac{\pi}{2})$. To summarize, when $d > 2$, the $A$-coefficient in (7.5) is always nonzero for each $t \in (0, \frac{\pi}{2})$.

Direct calculation shows that for each $t$ close to $t_0$, the two solutions of (7.5) are

$$
\frac{\cos \theta(t) \sin \theta(t)}{\cos t \sin t} \quad \text{and} \quad \frac{-2 f(t) \frac{d^2}{dt^2} f(t) + (\frac{d}{dt} f(t))^2 - 4 f(t)^2}{(\cos t \frac{d}{dt} f(t) + 2 \sin t f(t))(\sin t \frac{d}{dt} f(t) - 2 \cos t f(t))} \cos \theta(t) \sin \theta(t).
$$
The discriminant of (7.5) is
\[ B^2 - 4AC = \left( \frac{\cos t \sin t \frac{d^2}{dt^2} f(t) + (\sin^2 t - \cos^2 t) \frac{d^2}{dt^2} f(t)}{\cos \theta(t) \sin \theta(t)} \right)^2, \]
which vanishes if and only if \( \cos t \sin t \frac{d^2}{dt^2} f(t) + (\sin^2 t - \cos^2 t) \frac{d^2}{dt^2} f(t) = 0. \)

### 7.2 The case-by-case discussion

There are two generic possibilities.

**Case 1.** We have \( \cos t_0 \sin t_0 \frac{d^2}{dt^2} f(t) + (\sin^2 t_0 - \cos^2 t_0) \frac{d^2}{dt^2} f(t) \neq 0. \) Then we have (see [58, Lemma 3.7]) the following lemma.

**Lemma 7.1.** If \( \cos t_0 \sin t_0 \frac{d^2}{dt^2} f(t) + (\sin^2 t_0 - \cos^2 t_0) \frac{d^2}{dt^2} f(t) \neq 0, \) then one of the following two cases must happen:

1. For all \( t \) sufficiently close to \( t_0, \) we have
   \[ \frac{d}{dt} \theta(t) = \frac{\cos \theta(t) \sin \theta(t)}{\cos t \sin t}. \]  
   (7.7)

2. For all \( t \) sufficiently close to \( t_0, \) we have
   \[ \frac{d}{dt} \theta(t) = \frac{(-2f(t) \frac{d^2}{dt^2} f(t) + (\frac{d}{dt} f(t))^2 - 4f(t)^2) \cos \theta(t) \sin \theta(t)}{(\cos t \frac{d}{dt} f(t) + 2 \sin tf(t))(\sin t \frac{d}{dt} f(t) - 2 \cos f(t))}. \]  
   (7.8)

Each of (7.7) and (7.8) has a unique solution for the initial value problem \( \theta_0 = \theta(t_0) \in (0, \frac{\pi}{2}) \). Then inputting this solution \( \theta(t) \) into (7.2) with \( k = 0, \) and using the initial value condition \( h(\theta_0) = h_0 > 0, \) we can locally determine \( h(\theta) \) around \( \theta_0, \) as well as the Hessian isometry \( \Phi : (r,t) \mapsto (\frac{r^2(\tan h(\theta))^2}{h_0(\theta)^2}, \theta(t)) \) from \( \mathcal{F}_1 = r\sqrt{2f(t)} \) to \( \mathcal{F}_2 = r\sqrt{2h(\theta)} \) on \( \mathbb{R}^2. \)

**Subcase 1.1.** (7.7) is satisfied for all \( t \) sufficiently close to \( t_0. \)

The ODE (7.7) has the solution
\[ \theta(t) = \arccos \left( \frac{a \cos t}{(a^2 \cos^2 t + b^2 \sin^2 t)^{1/2}} \right) \]
with suitable positive constants \( a \) and \( b \) to meet all the initial value requirements. In this subcase, \( \Phi \) coincides with the linear isomorphism \( (x_1, x_2) \mapsto (ax_1, bx_2) \) when the polar \( t \)-coordinate is close to \( t_0. \)

Here comes the speciality of \( d > 2. \) By Theorem 5.6, \( \Phi \) satisfies the \( (d) \)-property for the orthonormal decomposition
\[ \mathbb{R}^2 = V' + V'' = \mathbb{R} \left( \cos \left( -\frac{\pi}{d} \right), \sin \left( -\frac{\pi}{d} \right) \right) + \mathbb{R} \left( \cos \left( \frac{\pi}{2} - \frac{\pi}{d} \right), \sin \left( \frac{\pi}{2} - \frac{\pi}{d} \right) \right). \]

So by Lemma 6.6, \( \Phi \) locally coincides with a positive scalar multiplication or the composition of the Legendre transformation of \( \mathcal{F}_1 \) and a positive scalar multiplication.

**Subcase 1.2.** (7.8) is satisfied for all \( t \) sufficiently close to \( t_0. \)

The solution \( \theta(t) \) for (7.8) is provided in (6.7), in which the positive parameters \( a \) and \( b \) can be suitably chosen to meet all the initial value requirements. The corresponding \( \Phi \) is given by
\[ \Phi(x_1, x_2) = \left( a \frac{\partial}{\partial x_1} E_1, b \frac{\partial}{\partial x_2} E_1 \right) \]
with \( E_1 = \frac{1}{2} F_1^2 \) when the polar \( t \)-coordinate of \( x = (x_1, x_2) \) is sufficiently close to \( t_0. \) Using Theorem 5.6 and Lemma 6.6 for the speciality of \( d > 2 \) again, we see that \( \Phi \) locally coincides with a positive scalar multiplication or the composition of the Legendre transformation of \( \mathcal{F}_1 \) and a positive scalar multiplication.
Case 2. We have
\[ \cos t \sin t \frac{d^2}{dt^2} f(t) + (\sin^2 t - \cos^2 t) \frac{d^2}{dt^2} f(t) = 0 \]
for every \( t \) sufficiently close to \( t_0 \). In this case, the two ODEs in Lemma 7.1 are the same. So
\[ \frac{d}{dt} \theta(t) = \frac{\cos \theta(t) \sin \theta(t)}{\cos t \sin t} \]
is satisfied for all \( t \) sufficiently close to \( t_0 \). By the argument similar to that in Subcase 1.1, we see that \( \Phi \) locally coincides with a positive scalar multiplication or the composition of the Legendre transformation of \( F_1 \) and a positive scalar multiplication, when restricted to the conic open subset with polar \( t \)-coordinates sufficiently close to \( t_0 \).

### 7.3 Conclusion

We can translate the case-by-case discussion in Subsection 7.2 to the following theorem, which provides the local description for a Hessian isometry which preserves the orientation and fixes the spherical \( \xi \)-coordinates, between two Minkowski norms induced by the same isoparametric foliation \( M_t \) on \((S^{n-1}(1), g^t)\) with \( d > 2 \) principal curvatures.

**Theorem 7.2.** Let \( M_t \) be an isoparametric foliation on \((S^{n-1}(1), g^t)\) with \( d > 2 \) principal curvature values. Then for any Hessian isometry \( \Phi \) between two Minkowski norms \( F_1 \) and \( F_2 \) on \( \mathbb{R}^n \) induced by \( M_t \), there is a conic open dense subset \( C(U) \) in \( \mathbb{R}^n \) such that when \( U \) is the union of some hypersurfaces in the foliation \( M_t \) such that when restricted to each connected component of \( C(U) \), \( \Phi \) coincides either with a positive scalar multiplication or with the composition of the Legendre transformation of \( F_1 \) and a positive scalar multiplication. In particular, with respect to any orthogonal decomposition \( \mathbb{R}^n = V' + V'' \), \( \Phi \) satisfies the (d)-property, i.e., for any nonzero \( x = x' + x'' \) and \( \Phi(x) = \overrightarrow{x} = \overrightarrow{x'} + \overrightarrow{x''} \) with \( x', \overrightarrow{x'} \in V' \) and \( x'', \overrightarrow{x''} \in V'' \), we have \( g^F_1(x'', x) = g^F_2(\overrightarrow{x''}, \overrightarrow{x}) \).

**Proof.** Let \((\overrightarrow{F}_1, \overrightarrow{F}_2, \overrightarrow{F})\) be the triple corresponding to \((F_1, F_2, \Phi)\) given by Theorem 5.6. The above case-by-case discussion based on Theorem 5.3, Theorem 5.6 and Lemma 6.6 indicates the existence of an open dense subset \( \bigcap_{i=1}^{\infty} (c_i, d_i) \subset (0, \frac{\pi}{2}) \) such that when \( \overrightarrow{F} \) is restricted to each conic open subset \( C(U) \) determined by the polar coordinate condition \( t \in (c_i, d_i) \), it is either a positive scalar multiplication or the composition of a Legendre transformation and a positive scalar multiplication.

Let \( C(U_i) \) be the conic open subset in \( \mathbb{R}^n \) determined by the spherical coordinate condition \( t \in (a_i, b_i) \). Then \( C(U) = \bigcap_{i=1}^{\infty} C(U_i) \) is a conic dense open subset in \( \mathbb{R}^n \), and each \( C(U_i) \) is a connected component of \( C(U) \). When \( \overrightarrow{F}|_{C(U_i)} \) is a positive scalar multiplication, obviously so is \( \Phi|_{C(U_i)} \). When \( \overrightarrow{F}|_{C(U_i)} \) is the composition of a Legendre transformation and a positive scalar multiplication, so is \( \Phi|_{C(U_i)} \) by a local analog of Lemma 6.7.

By Lemma 6.2, we know that \( \Phi \) satisfies the local (d)-property on each \( C(U_i) \) for every orthogonal decomposition of \( \mathbb{R}^n \). Since \( C(U) = \bigcap_{i=1}^{\infty} C(U_i) \) is dense in \( \mathbb{R}^n \), by continuity, \( \Phi \) satisfies the (d)-property for every orthogonal decomposition of \( \mathbb{R}^n \).

**Remark 7.3.** The Hessian isometry \( \Phi \) in Theorem 5.6 between two Minkowski norms induced by \( M_t \) which preserves the orientation and fixes the spherical \( \xi \)-coordinates can be constructed as follows. Firstly, we use the similar technique to that in Example 6.3 to \( D_{2d} \)-equivariantly glue positive scalar multiplications and the compositions of Legendre transformations and positive scalar multiplications to construct the triple \((\overrightarrow{F}_1, \overrightarrow{F}_2, \overrightarrow{F})\), which meets all the requirements in Theorem 5.6. In particular, (d)-properties are satisfied by Lemma 6.2. Then Theorem 5.6 provides the corresponding \((F_1, F_2, \Phi)\), in which \( \Phi \) is the wanted Hessian isometry. Finally, the argument for Theorem 7.2 tells us that essentially this is the only construction when we have \( d > 2 \) for the isoparametric foliation \( M_t \) on \((S^{n-1}(1), g^t)\).

Finally, we remark that the local case-by-case discussion in Subsection 7.2 provides the following description for \( \theta(t) \) in Theorem 5.3 when \( d > 2 \).
Theorem 7.4. Let $M_t$ be an isoparametric foliation on $(S^{n-1}(1), g^t)$ with $d > 2$ principal curvatures. Then for any triple $(f(t), h(\theta), \theta(t))$ in the spherical coordinate representation in Theorem 5.3, there exists an open dense subset $\bigcap_{i=1}^{\infty} (c_i, d_i)$ of $(0, \frac{\pi}{2})$ such that when restricted to each $(c_i, d_i)$, we have

\[ \text{either } \theta(t) \equiv t \text{ or } \theta(t) \equiv \arccos \left( \frac{2 \cos tf(t) - \sin \frac{2t}{d} f(t)}{(4f(t)^2 + (\frac{2t}{d} f(t))^2)^{1/2}} \right). \]

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