Research Article

Frank Oertel*

An analysis of the Rüschendorf transform - with a view towards Sklar’s Theorem

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Abstract: We revisit Sklar’s Theorem and give another proof, primarily based on the use of right quantile functions. To this end we slightly generalise the distributional transform approach of Rüschendorf and facilitate some new results including a rigorous characterisation of an almost surely existing “left-invertibility” of distribution functions.

Keywords: Copulas, distributional transform, generalised inverse functions, Sklar’s Theorem

MSC: 26A27, 60E05, 60A99, 62H05

1. Introduction

The mathematical investigation of copulas started 1951, due to the following problem of M. Fréchet: suppose, one is given \( n \) random variables \( X_1, X_2, \ldots, X_n \), all defined on the same probability space \( (\Omega, \mathcal{F}, P) \), such that each random variable has a (non-necessarily continuous) distribution function \( F_i (i = 1, 2, \ldots, n) \). What can then be said about the set of all possible \( n \)-dimensional distribution functions of the random vector \( (X_1, X_2, \ldots, X_n) \) (cf. [7])? This question has an immediate answer if the random variables were assumed to be independent, since in this case there exists a unique \( n \)-dimensional distribution function of the random vector \( (X_1, X_2, \ldots, X_n) \), which is given by the product \( \Pi_{i=1}^{n} F_i \). However, if the random variables are not independent, there was no clear answer to M. Fréchet’s problem.

In [15], A. Sklar introduced the expression “copula” (referring to a grammatical term for a word that links a subject and predicate), and provided answers to some of the questions of M. Fréchet.

In the following couple of decades, copulas (which are precisely finite dimensional distribution functions with uniformly distributed marginals), were mainly used in the framework of probabilistic metric spaces (cf. e. g. [13, 14]). Later, probabilists and statisticians were interested in copulas, since copulas defined in a “natural way” nonparametric measures of dependence between random variables, allowing to include a mapping of tail dependencies. Since then, they began to play an important role in several areas of probability and statistics (including Markov processes and non-parametric statistics), in financial and actuarial mathematics (particularly with respect to the measurement of credit risk), and even in medicine and engineering.

One of the key results in the theory and applications of copulas, is Sklar’s Theorem (which actually was proven in [13] and not in [15]). It says:

Sklar’s Theorem. Let \( F \) be a \( n \)-dimensional distribution function with marginals \( F_1, \ldots, F_n \). Then there exists a copula \( C_F \), such that for all \( (x_1, \ldots, x_n) \in \mathbb{R}^n \) we have

\[
F(x_1, \ldots, x_n) = C_F(F_1(x_1), \ldots, F_n(x_n)).
\]
Furthermore, if $F$ is continuous, the copula $C_F$ is unique. Conversely, for any univariate distribution functions $H_1, \ldots, H_n$, and any copula $C$, the composition $C \circ (H_1, \ldots, H_n)$ defines a $n$-dimensional distribution function with marginals $H_1, \ldots, H_n$.

Since the original proof of (the general non-continuous case of) Sklar’s Theorem is rather complicated and technical, there have been several attempts to provide different and more lucidly appearing proofs, involving not only techniques from probability theory and statistics but also from topology and functional analysis (cf. [4]).

Among those different proofs of Sklar’s Theorem, there is an elegant, yet rather short proof, provided by L. Rüschendorf, originally published in [12]. He provided a very intuitive, and primarily probabilistic approach which allows to treat general distribution functions (including discrete parts and jumps) in a similar way as continuous distribution functions. To this end, he applied a generalised “distributional transform” which - according to [12] - has been used in statistics for a long time in relation to a construction of randomised tests. By making a consequent use of the properties of this generalised “distributional transform” together with Proposition 2.1 in [12], the proof of Sklar’s Theorem in fact follows immediately (cf. Theorem 2.2 in [12]). Irrespective of [12] the same idea was used in the (again rather short) proof of Lemma 3.2 in [11]. All key inputs for the proof of Sklar’s Theorem clearly are provided by Proposition 2.1 in [12]. However, the proof of the latter result is rather difficult to reconstruct. It says:

[12] - **Proposition 2.1.** Let $X, V$ be two random variables, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $V \sim U(0,1)$ and $V$ is independent of $X$. Let $F$ be the distribution function of the random variable $X$. Then $U := F_V(X) \sim U(0,1)$, and $X = F^{-}(U)$ $\mathbb{P}$-almost surely.

Here, $F^{-}(\alpha) = \inf \{ x \in \mathbb{R} : F(x) \geq \alpha \}$ denotes the (left-continuous) left $\alpha$-quantile of $F$ which in particular is the lowest generalised inverse of $F$ (cf. e.g. [14, Chapter 4.4], respectively [8, Definition 2]). In our paper we consistently adopt the very suitable symbolic notation of [14], respectively [8] to identify generalised inverse functions in general (cf. (2.2) and (2.3)).

While studying (and reconstructing) carefully the proof of Sklar’s Theorem built on Proposition 2.1 in [12], we recognise that it actually implements key mathematical objects which do not involve probability theory at all and play an important role beyond statistical applications.

The main contribution of our paper is to provide a thorough analysis of these mathematical building blocks by studying carefully properties of a real-valued (deterministic) function, used in the proof of Proposition 2.1 in [12]; the so-called “Rüschendorf transform”. We reveal some interesting structural properties of this function which to the best of our knowledge have not been published before, such as e. g. Theorem 2.12 which actually is a result on Lebesgue-Stieltjes measures, strongly built on the role of the right quantile function which seems to be not widely used in the literature (as opposed to the left quantile function).

Equipped with Theorem 2.12 we then revisit the proof of Proposition 2.1 in [12] (cf. also [10, Chapter 1.1.2]). However, in our approach Proposition 2.1 in [12] is an implication of Theorem 2.12 and Lemma 2.15. For sake of completeness we include a proof of Sklar’s Theorem again (cf. also [10, Chapter 1.1.2]) - yet as an implication of Theorem 2.12, finally leading to Remark 2.21.

Last but not least, by observing the significance of the jumps of the lowest generalised inverse, the proof of Theorem 2.12 indicates how to construct the $\mathbb{P}$-null set in Proposition 2.1 in [12] explicitly - leading to Theorem 2.18.

### 2. The Rüschendorf Transform

At the moment let us completely ignore randomness and probability theory. We “only” are working within a subclass of real-valued functions, all defined on the real line, and with suitable subsets of the real line.
Let \( F : \mathbb{R} \rightarrow \mathbb{R} \) be an arbitrary right-continuous and non-decreasing function. Let \( x \in \mathbb{R} \). Since \( F \) is non-decreasing, it is well-known that both, the left-hand limit
\[
F(x-) := \lim_{z \uparrow x} F(z) = \sup \{ F(z) : z \leq x \},
\]
and the right-hand limit
\[
F(x+) := \lim_{z \downarrow x} F(z) = \inf \{ F(z) : z \geq x \}
\]
are well-defined real numbers, satisfying \( F(x-) \leq F(x) \leq F(x+) \). Moreover, due to the assumed right-continuity of \( F \), it follows that \( F(x) = F(x+) \) for all \( x \in \mathbb{R} \). \( 0 \leq \Delta F(x) := F(x+) - F(x-) = F(x) - F(x-) \) denotes the (left-hand) “jump” of \( F \) at \( x \). We consider the following important transform of \( F \):

**Definition 2.1.** Let \( \lambda \in [0, 1] \) and \( x \in \mathbb{R} \). Put
\[
R_F(x, \lambda) := F_A(x) = F(x-) + \lambda \Delta F(x).
\]
We call the real-valued function \( R_F : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \) the Rüschendorf transform of \( F \). For given \( \lambda \in [0, 1] \), \( F_A : \mathbb{R} \rightarrow \mathbb{R} \) is called the Rüschendorf \( \lambda \)-transform of \( F \).

Clearly, we have the following equivalent representation of the Rüschendorf \( \lambda \)-transform \( F_A \):
\[
F_A(x) = (1 - \lambda)F(x-) + \lambda F(x) \text{ for all } x \in \mathbb{R}.
\]
In particular, for all \( (x, \lambda) \in \mathbb{R} \times [0, 1] \) the following inequality holds:
\[
F(x-) \leq F_A(x) \leq F(x). \quad (2.1)
\]
Moreover, \( F \) is continuous if and only if \( F(x-) = F_A(x) \) for all \( (x, \lambda) \in \mathbb{R} \times [0, 1] \), and for all \( (x, \lambda) \in \mathbb{R} \times [0, 1] \) we have \( F_0(x) = F(x-) = F_A(x-) \) and \( F_1(x) = F(x) = F_A(x+) \).

**Assumption 2.2.** In the following we assume throughout that \( F \) is bounded on \( \mathbb{R} \) (i.e., the range \( F(\mathbb{R}) \) is a bounded subset of \( \mathbb{R} \)), implying that \( F(\mathbb{R}) \subseteq [c_-, c^+] \) for some real numbers \( c_- < c^+ \). Moreover, let us assume that for any \( \alpha \in (c_-, c^+) \) the set \( \{ x \in \mathbb{R} : F(x) \geq \alpha \} \) is non-empty and bounded from below. WLOG, we may assume from now on that \( c_- = 0 \) and \( c^+ = 1 \) (else we would have to work with the function \( \frac{F - c_-}{c^+ - c_-} \)).

Although its proof (by contradiction) mostly is an easy calculus exercise with sequences, the following observation - which does not require a right-continuity assumption - should be explicitly noted (cf. also [5, 6, 13]):

**Remark 2.3.** Let \( G : \mathbb{R} \rightarrow [0, 1) \) an arbitrary non-decreasing function. Then the following statements are equivalent:

\( \text{(i)} \) \( \lim_{x \rightarrow -\infty} G(x) = 0 \) and \( \lim_{x \rightarrow +\infty} G(x) = 1 \);

\( \text{(ii)} \) For any \( \alpha \in (0, 1) \) the sets \( \{ x \in \mathbb{R} : G(x) < \alpha \} \) and \( \{ x \in \mathbb{R} : G(x) \geq \alpha \} \) both are non-empty;

\( \text{(iii)} \) For any \( \alpha \in (0, 1) \) the set \( \{ x \in \mathbb{R} : G(x) \geq \alpha \} \) is non-empty and bounded from below.

\( \text{(iv)} \) \( G^\wedge(\alpha) := \inf \{ x \in \mathbb{R} : G(x) \geq \alpha \} \) is a well-defined real number for any \( \alpha \in (0, 1) \).

Hence, given Assumption 2.2, the assumed right-continuity of \( F \) and Remark 2.3 imply that (possibly after shifting and stretching \( F \) adequately) \( F \) actually is a distribution function! Therefore, its generalised inverse function \( F^\wedge : (0, 1) \rightarrow \mathbb{R} \), given by
\[
F^\wedge(\alpha) := \inf \{ x \in \mathbb{R} : F(x) \geq \alpha \}, \quad (2.2)
\]
\* In particular, \( F \) cannot be a constant function on the whole real line.
is well-defined and satisfies
\[ -\infty < F^\wedge(a) \leq F^\wedge(a+) = \inf\{x \in \mathbb{R} : F(x) > a\} = \sup\{x \in \mathbb{R} : F(x) \leq a\} =: F^\vee(a) < \infty \quad (2.3) \]
for any \( a \in (0, 1) \) (cf. e.g. [9]). Actually, since \( F \) is assumed to be right-continuous, it follows that
\[ F^\wedge(a) = \min\{x \in \mathbb{R} : F(x) > a\} \]
for all \( a \in (0, 1) \) (cf. [5, Proposition 2.3(4)]). Moreover, the following important inequality is satisfied:
\[ F\left(F^\wedge(a) - \delta\right) < a \leq F\left(F^\wedge(a) + \varepsilon\right) \quad (2.4) \]
for all \( a \in (0, 1) \), \( \delta > 0 \), and for all \( \varepsilon > 0 \). Hence,
\[ F\left(F^\wedge(a) - \right) \leq a \leq F\left(F^\wedge(a) + \varepsilon\right) = F\left(F^\wedge(a)\right) \quad (2.5) \]
for all \( a \in (0, 1) \). Also recall from e.g. [14] that \( \{x \in \mathbb{R} : F(x) > a\} = [F^\wedge(a), \infty) \), respectively \( \{x \in \mathbb{R} : F(x) < a\} = (-\infty, F^\wedge(a)) \) for any \( a \in (0, 1) \).

Let us fix the distribution function \( F : \mathbb{R} \rightarrow [0, 1] \). Then by \( J_F := \{x \in \mathbb{R} : \Delta F(x) > 0\} \) we denote the set of all jumps of \( F \) which is well-known to be at most countable.

Throughout the remaining part of our paper, we follow the notation of [12] and put \( \xi := F^\wedge(a) \) for fixed \( 0 < a < 1 \). By taking a closer look at \( F^\wedge(F_A(x)) \), we firstly note the following observation.

**Remark 2.4.** Let \( \lambda \in [0, 1] \) and \( 0 < F_A(x) < 1 \). Then
\[ F^\wedge(F_A(x)) \leq x. \]

**Proof.** Fix \( \lambda \in [0, 1] \) and put \( \alpha := F_A(x) \), where \( x \in F_A^{-1}((0, 1)) \). Then \( F^\wedge(\alpha) \) is well-defined. Since \( F(x) \geq F_A(x) = \alpha \), the claim follows. \( \square \)

The next result shows an important part of the role of Rüschendorff transform which can be more easily understood if one sketches the graph of \( F \) including its jumps. Since \( J_F \) is at most countable, it follows that \( J_F = \{x_n : n \in M\} \), where either \( \text{card}(M) < \infty \) or \( M = \mathbb{N} \). By making use of this representation and the canonically defined function \( F : \mathbb{R} \rightarrow [0, 1], x \mapsto F(x^-) \) (cf. also [14, Chapter 4.4]) we arrive at the following

**Proposition 2.5.** Let \( x \in \mathbb{R} \). Then
\[ (F(x^-), F(x)) \subseteq \{ \alpha \in (0, 1) : x = F^\wedge(\alpha) \} \subseteq \left[ F(x^-), F(x) \right]. \]

In particular, if \( x_1 \neq x_2 \) then \( (F(x_1^-), F(x_1)) \cap (F(x_2^-), F(x_2)) = \emptyset \). Moreover,
\[ \bigcup_{x \in J_F} (F(x^-), F(x)) = \{ \alpha \in (0, 1) : \Delta F(F^\wedge(\alpha)) > 0 \text{ and } \alpha = F_A(F^\wedge(\alpha)) \text{ for some } 0 < \lambda < 1 \} \]
\[ = \{ F_A(x) : 0 < \lambda < 1 \text{ and } x \in J_F \} \]
\[ = R_F(J_F \times (0, 1)) \]
\[ = (0, 1) \setminus (F(\mathbb{R}) \cup \lim F^\wedge(\mathbb{R})), \]
implying that the mapping \( \Phi_F : (0, 1)^M \rightarrow \prod_{n=1}^{\text{card}(M)} (F(x_n^-), F(x_n)) \), \( (\lambda_n)_{n \in M} \mapsto (F_A(x_n))_{n \in M} \) is well-defined and bijective. Its inverse \( \Phi_F^{-1} : \prod_{n=1}^{\text{card}(M)} (F(x_n^-), F(x_n)) \rightarrow (0, 1)^M \) is given by
\[ \Phi_F^{-1}((\alpha_n)_{n \in M}) = \left( \frac{\alpha_n - F(F^\wedge(\alpha_n)\!)}{\Delta F^\wedge(\alpha_n)} \right)_{n \in M}. \]

**Proof.** To prove the first set inclusion, we may assume without loss of generality that \( F \) is not continuous in \( x \). So, let \( F(x^-) < \alpha < F(x) \). Then \( 0 < \alpha < 1 \) (else we would obtain the contradiction \( \alpha \leq 0 \leq F(x^-) \), respectively \( F(x) \leq 1 \leq \alpha \)) and \( F(x - \frac{1}{n}) < \alpha \leq F(x) \) for all \( n \in \mathbb{N} \). Hence, \( F^\wedge(\alpha) < x < F^\wedge(\alpha) + \frac{1}{n} \) for all \( n \in \mathbb{N} \) (cf. [5,
Thus, and first note that

\[ F(x-) \leq \alpha \leq F(x), \]

which gives the second set inclusion.

To verify the representation of the disjoint union \( \bigcup_{x \in F} (F(x-), F(x)) \) let \( \alpha \in (F(x-), F(x)) \) for some \( x \in J_{\xi} \).

Then \( x = F^\wedge(a) =: \xi \) and hence \( \Delta F(\xi) > 0 \) and \( \alpha \in (F(\xi-), F(\xi)) \). Put

\[ \lambda(a) := \frac{\alpha - F(\xi-) \Delta F(\xi)^{-1}.} {Then 0 < \lambda(a) < 1 and

\[ \alpha = F_{\lambda(a)}(\xi) = F_{\lambda(a)}(x). \]

Furthermore, a straightforward application of the inequality (2.4) (together with (2.5) and the monotonicity assumption on \( F \)) shows the graphically clear fact that there is no \( x \in J_{\xi} \) such that \( (F(x-), F(x)) \) contains elements of the form \( F(z) \), respectively \( F(w-) \) for some \( z, w \in \mathbb{R} \). Now, given the construction of \( \lambda(a) \) above and the listed properties of any of the sets \( (F(x_n-), F(x_n)) \), the assertion about the mapping \( \Phi_F \) follows immediately.

\[ \square \]

**Definition 2.6.** Let \( \alpha \in (0, 1) \) and \( \lambda \in [0, 1] \). Put:

\[ A_{\lambda,a} := \{ x \in \mathbb{R} : F_{\lambda}(x) \leq \alpha \}. \]

Firstly note that \( A_{\lambda,a} \) is non-empty. To see this, consider any \( x < \xi = F^\wedge(a) \). Then \( x < \xi - \delta \) for some \( \delta > 0 \).

Hence, \( F_{\lambda}(x) < F(x) < F_{\lambda,a}(x) \), \( (2.4) \). To motivate the following representation of the set \( A_{\lambda,a} \), let us assume for the moment that \( F \) is continuous at \( \xi \). Due to (2.5), it follows that \( F(\xi) = a \). Hence, in this case, \( F_{\lambda}(\xi) = F(\xi) = a \), implying that \( \xi = F^\wedge(a) \in A_{\lambda,a} \).

However, in the general (non-continuous) case, \( \xi = F^\wedge(a) \) need not be an element of the set \( A_{\lambda,a} \). Therefore (by fixing \( \alpha \in (0, 1) \) and \( \lambda \in [0, 1] \)), we are going to represent the set \( A_{\lambda,a} \) as a disjoint union of the following three subsets of the real line:

\[ A_{\lambda,a} := A_{\lambda,a} \cap \{ x \in \mathbb{R} : x > \xi \}, \]

\[ A_{\lambda,a} := A_{\lambda,a} \cap \{ x \in \mathbb{R} : x = \xi \}, \]

and

\[ A_{\lambda,a} := A_{\lambda,a} \cap \{ x \in \mathbb{R} : x < \xi \}. \]

Thus,

\[ A_{\lambda,a} = A_{\lambda,a}^+ \cup A_{\lambda,a}^- \cup A_{\lambda,a}^--. \]

Next, we are going to simplify the sets \( A_{\lambda,a}^+, A_{\lambda,a}^- \) and \( A_{\lambda,a}^- \) as far as possible. To this end, we have to analyse carefully the jump \( \Delta F^\wedge(a) \), implying that we have to check \( \xi = F^\wedge(a) = F^\wedge(a-) \) against the (finite) value of the largest generalised inverse of \( F \) (cf. [9] and [14, Chapter 4.4])

\[ \eta := F^\wedge(a) = \inf\{ x \in \mathbb{R} : F(x) > a \} = \sup\{ x \in \mathbb{R} : F(x) \leq a \} = F^\wedge(a+). \]

The inequality (2.5) is also satisfied for \( \eta \) (cf. [6, Lemma A.15]):

\[ F(\eta-) \leq \alpha \leq F(\eta). \]

The inequality (2.5) is also satisfied for \( \eta \) (cf. [6, Lemma A.15]):

\[ F(\eta-) \leq \alpha \leq F(\eta). \]

Note that since \( F \) is a distribution function, \( \eta \) (respectively \( \xi \)) is precisely the right (respectively left) \( \alpha \)-quantile of \( F \).

Clearly, \( \{ x \in \mathbb{R} : x > \xi \) and \( F(x) = a \} \subseteq A_{\lambda,a}^+ \) for every \( \lambda \in [0, 1] \). However, if \( 0 < \lambda \leq 1 \), we even obtain equality of both sets since:
Lemma 2.7. Let $0 < \lambda \leq 1$ and $\alpha \in (0, 1)$. Put $\xi := F^\wedge(a)$ and $\eta := F^\vee(a)$.

(i) If $\xi < \eta$, then $F(\xi) = \alpha = F(\eta) \not\in \bigcup_{x \in J_F} (F(x^-), F(x))$ and $\emptyset \neq \{ x \in \mathbb{R} : x > \xi$ and $F(x) = \alpha \}$. Moreover, the restricted function $F|_{A^+_{\lambda,a}} : A^+_{\lambda,a} \to \mathbb{R}$ is continuous, and

$$A^+_{\lambda,a} = \{ x \in \mathbb{R} : x > \xi$ and $F(x) = \alpha \} = \begin{cases} (\xi, \eta) & \text{if } F(\eta) > \alpha \\ (\xi, \eta) & \text{if } F(\eta) = \alpha \end{cases}$$ (2.7)

(ii) If $\xi = \eta$, then $A^+_{\lambda,a} = \emptyset$.

(iii) Furthermore,

$$J_F = \big\{ x \in \mathbb{R} : F^\wedge(u) = x = F^\vee(u) \text{ and } \Delta F(F^\wedge(u)) > 0 \text{ for some } u \in (0, 1) \big\} = \big\{ x \in \mathbb{R} : F^\wedge(u) = x = F^\vee(u) \text{ and } \Delta F(F^\wedge(u)) > 0 \text{ for some } u \in (0, 1) \big\}.$$

In particular, the following statements are equivalent:

(a) $0 < \Delta F^\wedge(a) = \eta - \xi$;

(b) $\{ x \in \mathbb{R} : x > \xi$ and $F(x) = \alpha \} \neq \emptyset$.

Proof. Put $B := \{ x \in \mathbb{R} : x > \xi$ and $F(x) = \alpha \}$. Clearly, we always have $B \subseteq A^+_{\lambda,a}$.

To verify (i), let $\xi < \eta$. Then $\xi < z_0 < \eta = \inf \{ x \in \mathbb{R} : F(x) > \alpha \}$ for some $z_0 \in \mathbb{R}$. Thus, $F(\xi) \leq F(z_0) \leq \alpha \leq F(\xi)$, implying that $z_0 \in B$ and $F(\xi) = \alpha$. Assume by contradiction that $F(\eta) < \alpha$. Then $F(\eta - \epsilon) < F(\xi)$ for all $\epsilon > 0$, implying the contradiction $\eta < \xi$. Hence, $F(\eta) = \alpha$. Proposition 2.5 therefore implies that $a = F(\xi) \not\in \bigcup_{x \in J_F} (F(x^-), F(x))$.

Let $x \in A^+_{\lambda,a} \supseteq B$. Assume by contradiction that $F|_{A^+_{\lambda,a}}$ is not continuous at $x$. Then $F(x^-) < F(x^-) + \lambda \Delta F(x) = F_1(x) < (a$ (since $\lambda > 0$). Since $x > \xi$, we have $\xi \leq x - \frac{1}{n}$ for some $n \in \mathbb{N}$. Thus,

$$a \leq F(\xi) \leq F \left( \xi + \frac{1}{2n} \right) \leq F \left( x - \frac{1}{2n} \right).$$

Hence, $a \leq F(x^-) < \alpha$, which is a contradiction. Thus, the restricted function $F|_{A^+_{\lambda,a}}$ is continuous on $A^+_{\lambda,a}$. Let $u \in A^+_{\lambda,a}$. Since $F$ is continuous at $u$, it follows that

$$a \leq F(\xi) \leq F(u) = F_1(u) \leq \alpha.$$

Thus, $\emptyset \neq A^+_{\lambda,a} = B$.

To prove (ii), suppose that $A^+_{\lambda,a}$ is non-empty. The previous calculations show that the existence of an element $u_0 \in A^+_{\lambda,a}$ already implies $F(\xi) = F(u_0) = \alpha$. Consequently, $\eta = F^\vee(a) = \sup \{ x \in \mathbb{R} : F(x) \leq \alpha \}$ cannot coincide with $\xi = F^\wedge(a)$ (since $\xi < \eta$), implying that $\xi < \eta$.

To finish the proof of (i), we have to verify (2.7). To this end, let $\xi < \eta$ and $x \in (\xi, \eta)$. Then there exists $\delta > 0$ such that $\xi < x - \delta < x < x + \delta < \eta = F^\vee(a) = \inf \{ u \in \mathbb{R} : F(u) > \alpha \}$. Consequently, $a \leq F(\xi) \leq F(x - \delta) \leq F(x) \leq F(x + \delta) \leq \alpha$. Thus,

$$(\xi, \eta) \subseteq \{ x \in \mathbb{R} : x > \xi$ and $F(x) = \alpha \} = B.$$

Moreover, [5, Proposition 2.3(6)] implies that

$$B = \{ x \in \mathbb{R} : x > \xi$ and $F(x) = \alpha \} \subseteq (\xi, \eta)'.$$

Hence,

$$(\xi, \eta) \subseteq B \subseteq (\xi, \eta).$$

If $F(\eta) > \alpha$, then $\eta \not\in B$ and hence $B = (\xi, \eta)$. If $F(\eta) = \alpha$, then $\xi < \eta \in B$ and hence $B = (\xi, \eta]$.

Statement (iii) is a direct implication of (i) and Proposition 2.5.

Regarding a visualisation of Lemma 2.7 consider the set $M_\alpha := \{ x \in \mathbb{R} : x \leq \xi$ and $F(x) = \alpha \} = \{ x \in \mathbb{R} : x = \xi$ and $F(x) = \alpha \} \in \{ \emptyset, \{ \xi \} \}$. Note that

$$\{ x \in \mathbb{R} : F(x) = \alpha \} = \{ x \in \mathbb{R} : x > \xi$ and $F(x) = \alpha \} \cup M_\alpha.$$
Thus, by joining Lemma 2.7 with Proposition 2.5 we immediately obtain the following tangible mathematical description of the (preimages of) “flat pieces” of $F$ (and hence allowing us to perfect related observations from e.g. [14, Chapter 4.4] and [5, Proposition 2.3, (6)], coherently):

**Corollary 2.8.** Let $0 < \lambda \leq 1$ and $\alpha \in (0, 1)$. Put $\xi := F^\lambda (a)$ and $\eta := F^\lambda (\alpha)$.

(i) If $\xi < \eta$, then

$$\emptyset \neq \{ x \in \mathbb{R} : F(x) = \alpha \} = \mathcal{A}_{\lambda, a}^+ \cup \{ \xi \} = \begin{cases} [\xi, \eta) & \text{if } F(\eta) > a \\ [\xi, \eta] & \text{if } F(\eta) = a \end{cases}$$

(ii) If $\xi = \eta$, then

$$\{ x \in \mathbb{R} : F(x) = \alpha \} = \begin{cases} \emptyset & \text{if } F(\eta) > a \\ \{ \xi \} & \text{if } F(\eta) = a \end{cases}$$

In particular, $F(\xi) = a$ if and only if $\{ x \in \mathbb{R} : F(x) = \alpha \} \neq \emptyset$, and $\eta - \xi = \Delta F^\lambda (a) = 0$ if and only if $\{ x \in \mathbb{R} : F(x) = \alpha \} \in \{ \emptyset, \{ \xi \} \}$, and if $\eta > \xi$, then $\Delta F(\eta) = 0$ if and only if $F(\eta) = a$.

**Remark 2.9.** Let $\alpha \in (0, 1)$. Then, according to [1, Corollary 1.1] for a large class of distribution functions $F$ any non-empty set $[\xi, \eta] = [F^\lambda (a), F^\lambda (\alpha)]$ even emerges as a set of optimal solutions of the so-called “single period newvendor problem” which asks for the minimisation of coherent risk measures, such as the conditional-value-at-risk (which coincides with Expected Shortfall), corresponding to a cost function, induced by random demand. Here, one should recall that recently the Basel Committee on Banking Supervision (BCBS) suggested in their updated consultative document “Fundamental review of the trading book” to implement Expected Shortfall at $\alpha = 97.5\%$ in a bank’s internal market risk model to calculate its minimum capital requirements with respect to market risk.

Let $\mathcal{B}(\mathbb{R})$ denote the set of all Borel subsets of $\mathbb{R}$. In the following, let $\mu_F : \mathcal{B}(\mathbb{R}) \longrightarrow [0, \infty]$ be the Lebesgue-Stieltjes measure of $F$. For a detailed description of the construction and properties of the Lebesgue-Stieltjes measure (including Lebesgue-Stieltjes integration), we refer the reader to e.g. [2] and [3]. For the convenience of the reader, we recall the following fundamental result (cf. [3, Theorem 12.4]):

**Theorem 2.10** (Lebesgue-Stieltjes measure). Let $G : \mathbb{R} \longrightarrow \mathbb{R}$ be an arbitrary non-decreasing and right-continuous function. Then there exists a unique Borel measure $\mu_G$ satisfying

$$\mu_G([x, y]) = G(y) - G(x)$$

for all $x, y \in \mathbb{R}$.

Clearly, this crucial result implies that $\mu_G([x, y]) = G(y) - G(x)$ and hence

$$\mu_G([y]) = \mu_G([x, y]) - \mu_G([x, y]) = G(y) - G(y) = \Delta G(y)$$

for all $y \in \mathbb{R}$. Moreover, $\mu_G(\mathbb{R}) = 0$ if and only if $G$ is a constant function on $\mathbb{R}$.

Returning to our distribution function $F$, a direct application of $\mu_F$ leads to another important implication of Lemma 2.7:

**Corollary 2.11.** Let $0 < \lambda \leq 1$ and $\alpha \in (0, 1)$. Then $\mathcal{A}_{\lambda, a}^+ \in \mathcal{B}(\mathbb{R})$, and

$$\mu_F(\mathcal{A}_{\lambda, a}^+) = 0.$$ 

In particular, if $\xi < \eta$, then

$$\mu_F(\{ x \in \mathbb{R} : F(x) = \alpha \}) = \Delta F(\xi) = \alpha - F(\xi-).$$

**Proof.** Nothing is to prove if $\mathcal{A}_{\lambda, a}^+ = \emptyset$. So, let $\mathcal{A}_{\lambda, a}^+ \neq \emptyset$. Then $\eta - \xi = \Delta F^\lambda (\alpha) > 0$. 


Suppose first that $F(\eta) > a$. Then
\[ A_{\lambda,a}^+ = (\xi, \eta) = \bigcup_{n=1}^{\infty} (\xi, \eta - \frac{1}{n}] . \]
Consequently, since in general $F(x) = \alpha = F(\xi)$ for all $x \in (\xi, \eta)$, it follows that
\[ \mu_F(A_{\lambda,a}^+) = \lim_{n \to \infty} \mu_F\left((\xi, \eta - \frac{1}{n}] \right) = \lim_{n \to \infty} (F(\eta - \frac{1}{n}) - F(\xi)) = \alpha - \alpha = 0 . \]
Now suppose that $F(\eta) = \alpha$. Then $\eta \in A_{\lambda,a}^+$, and it follows that $F$ is continuous at $\eta$. Thus, $\mu_F((\eta)) = \Delta F(\eta) = 0$. Since in this case
\[ A_{\lambda,a}^+ = (\xi, \eta) \cup \{ \eta \} , \]
it consequently follows that
\[ \mu_F(A_{\lambda,a}^+) = \lim_{n \to \infty} (F(\eta - \frac{1}{n}) - F(\xi)) + \mu_F((\eta)) = \alpha - \alpha + 0 = 0 . \]

Next, we are going to reveal in detail that the function $F$ is almost “left-invertible” at every $x \in \mathbb{R}$ which does not belong to the preimage $A_{\lambda,a}^+$ of a “flat piece” of $F$. More precisely:

**Theorem 2.12.** Let $0 < \lambda \leq 1$. Assume that $0 < F_\lambda < 1 \mu_F$-almost everywhere. Then
\[ \text{id}_\mathbb{R} = F^\land \circ F_\lambda \quad \mu_F\text{-almost everywhere} . \]
In particular, if $0 < F < 1 \mu_F$-almost everywhere, then
\[ \text{id}_\mathbb{R} = F^\land \circ F \quad \mu_F\text{-almost everywhere} . \]

**Proof.** Let $0 < \lambda \leq 1$. Consider the Borel set
\[ N_\lambda := \{ x \in \mathbb{R} : F_\lambda(x) = 0 \} \cup \{ x \in \mathbb{R} : F_\lambda(x) = 1 \} \cup \bigcup_{a \in J_{F^\land}} A_{\lambda,a}^+ , \]
where $J_{F^\land} := \{ a \in (0, 1) : \Delta F^\land(a) > 0 \}$ denotes the set of all jumps of the function $F^\land$. Since the (left-continuous) function $F^\land : (0, 1) \to \mathbb{R}$ is non-decreasing, $J_{F^\land}$ is at most countable. Hence, if $J_{F^\land} \neq \emptyset$, there exists a subset $M$ of $\mathbb{N}$, and a sequence $(a_n)_{n \in M}$, consisting of pairwise distinct elements $a_n \in J_{F^\land}$, such that $J_{F^\land} = \{ a_n : n \in \mathbb{N} \}$. Thus, $\bigcup_{a \in J_{F^\land}} A_{\lambda,a}^+ = \bigcup_{n \in M} A_{\lambda,a_n}^+$. Corollary 2.11 therefore implies that - in any case - $\mu_F(N_\lambda) = 0$ and hence $\mathbb{R} \setminus N_\lambda \neq \emptyset$ (since $F$ cannot be a constant function on the whole real line).

Let $x \in \mathbb{R} \setminus N_\lambda$. Put $a(x) := F_\lambda(x)$. Then $0 < a(x) < 1$, and $x \in \bigcap_{a \in J_{F^\land}} (\mathbb{R} \setminus A_{\lambda,a}^+)$. Thus, $\xi(x) := F^\land(a(x))$ is well-defined. Consider $\eta(x) := F^\land(a(x)) = F^\land(a(x)+)$.

First, let $J_{F^\land} = \emptyset$. Then $\xi(x) = \eta(x)$. Lemma 2.7 therefore implies that $A_{\lambda,a(x)}^+ = \emptyset$. In particular, $x \notin A_{\lambda,a(x)}^+$. Hence, since $F_\lambda(x) = a(x) \neq a(x)$, it consequently follows that
\[ x \leq \xi(x) = F^\land(a(x)) = F^\land(F_\lambda(x)) , \]
and hence $x = F^\land(F_\lambda(x))$.

Now let $J_{F^\land} \neq \emptyset$. If $a(x) \notin J_{F^\land}$, it follows again that $\xi(x) = \eta(x)$ and hence
\[ x \leq \xi(x) = F^\land(a(x)) = F^\land(F_\lambda(x)) \leq x , \]
as above. So, let $a(x) \in J_{F^\land}$. Then $a(x) = a_m$ for some $m \in \mathbb{M}$, and hence $A_{\lambda,a(x)}^+ = A_{\lambda,a_m}^+$. Since $x \in \mathbb{R} \setminus N_\lambda \subseteq \mathbb{R} \setminus A_{\lambda,a(x)}^+$, it follows once more again that $x \leq \xi(x) = F^\land(a(x))$, and hence
\[ x = \xi(x) = F^\land(a(x)) = F^\land(F_\lambda(x)) . \]

\[ \footnote{Note that by construction $N_\lambda = \{ x \in \mathbb{R} : F_\lambda(x) = 0 \} \cup \{ x \in \mathbb{R} : F_\lambda(x) = 1 \}$ if $J_{F^\land} = \emptyset$.} \]
Next, we consider the set $A_{\lambda,a}^-$. Again, in line with [12], we put $q := F(\xi-)$ and $\beta := \Delta F(\xi) \geq 0$. Then

$$q + \beta = F(\xi) \overset{(2.5)}{\geq} \alpha \geq q.$$ 

Obviously, we may write:

**Remark 2.13.** \(A_{\lambda,a}^- = \{ x \in \mathbb{R} : x = \xi \text{ and } \beta \lambda \leq a - q \} \).

Moreover, by using a similar argument like that one which has shown us that the set $A_{\lambda,a}$ is non-empty, we further obtain

**Remark 2.14.** \(A_{\lambda,a}^- = (-\infty, \xi) = \{ x \in \mathbb{R} : x < \xi \} \).

Observe that only the subset $A_{\lambda,a}^-$ of $A_{\lambda,a}$ does depend on the choice of $\lambda \in [0, 1]$.

### 2.1. The inclusion of randomness

In addition to our assumptions above, we now fix a given probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let $X : \Omega \rightarrow \mathbb{R}$ and $V : \Omega \rightarrow \mathbb{R}$ be two given random variables (on this probability space) such that $V \sim U(0, 1)$ is uniformly distributed over $(0, 1)$ and independent of $X$. In the following we consider the random variable $F_V(X)$, defined on \(\{ V \in (0, 1) \} \) as

$$F_V(X)(\omega) := F_{V(\omega)}(X(\omega)) = F(\xi),$$

where here $\omega \in \Omega$, $\lambda := V(\omega)$ and $x := X(\omega)$. Next, we have to evaluate $\mathbb{P}(F_V(X) \leq \alpha) = \mathbb{P}(F_V(X) \leq \alpha$ and $V \in (0, 1)$).

i.e., we wish to calculate

$$\mathbb{P}(F_V(X) \leq \alpha) = \mathbb{P}(\{ \omega \in \Omega : X(\omega) \in A_{V(\omega),a} \} \text{ and } V \in (0, 1)).$$

Due to our previous observations, we have

$$A_{V(\omega),a} = A_{V(\omega),a}^+ \cup A_{V(\omega),a}^- \cup A_{V(\omega),a}^-$$

for all $\omega \in \{ V \in (0, 1) \}$. Consequently, given the assumed independence of $V$ and $X$, Lemma 2.7 implies that:

$$\mathbb{P}(F_V(X) \leq \alpha) = \mathbb{P}(V(\omega) \in A_{V(\omega),a}^+ \text{ and } V \in (0, 1)) + \mathbb{P}(V(\omega) \in A_{V(\omega),a}^- \text{ and } V \in (0, 1))$$

+ \mathbb{P}(V(\omega) \in A_{V(\omega),a}^- \text{ and } V \in (0, 1))

= \mathbb{P}(X > \xi \text{ and } F(X) = \alpha) + \mathbb{P}(X = \xi) \mathbb{P}(\beta V \leq a - q \text{ and } V \in (0, 1)) + \mathbb{P}(X < \xi).$$

Apparently, to continue with the calculation of the respective probabilities, we have to consider the following two possible cases: $\beta = 0$ and $\beta > 0$:

(i) Let $\beta = 0$. Thus, since $\alpha - q \geq 0$, it follows that

$$\mathbb{P}(F_V(X) \leq \alpha) = \mathbb{P}(X > \xi \text{ and } F(X) = \alpha) + \mathbb{P}(X \leq \xi).$$

(ii) Let $\beta > 0$. Since $V \sim U(0, 1)$ is uniformly distributed over $(0, 1)$, we have

$$\mathbb{P}(\beta V \leq a - q \text{ and } V \in (0, 1)) = \mathbb{P} \left( V \leq \frac{a-q}{\beta} \right) = \frac{a-q}{\beta}. $$

Hence, since $\frac{a-q}{\beta} - 1 = \frac{a-F(\xi)}{\beta}$, it follows that

$$\mathbb{P}(F_V(X) \leq \alpha) = \mathbb{P}(X > \xi \text{ and } F(X) = \alpha) + \left( \frac{a-F(\xi)}{\beta} \right) \mathbb{P}(X = \xi) + \mathbb{P}(X \leq \xi).$$

---

\* Since $V \sim U(0, 1)$, we obviously have $\mathbb{P}(V \in (0, 1)) = 1$ and hence $\mathbb{P}(V \notin (0, 1)) = 0$.

\$ Here, $\{ X \in A_{V,a} \} := \{ \omega \in \Omega : X(\omega) \in A_{V(\omega),a} \}$ and $\{ X \in A_{V,a}^+ \} := \{ \omega \in \Omega : X(\omega) \in A_{V(\omega),a}^+ \}$, where $i \in \{ +, =, - \}$. 

---
Moreover, by taking into account that \( F(\xi) = \alpha \) in case (i) (since \( F \) is continuous at \( \xi \) if \( \beta = 0 \)), we have arrived at the following important

**Lemma 2.15.** Suppose that \( F : \mathbb{R} \rightarrow [0, 1] \) is an arbitrary distribution function. Let \( \alpha \in (0, 1) \). Put \( \xi := F^{-}(\alpha) \) and \( \beta := \Delta F(\xi) \). Let \( X, V \) be two random variables, both defined on the same probability space \((\Omega, \mathcal{F}, P)\), such that \( V \sim U(0, 1) \) and \( V \) is independent of \( X \). Then

\[
P(F(V) < \alpha) - \alpha = P(X > \xi \text{ and } F(X) = \alpha) + c_\beta (P(X = \xi) - \beta) + (P(X < \xi) - F(\xi)),
\]

where \( c_\beta := 0 \) if \( \beta = 0 \) and \( c_\beta := \frac{\alpha - F(\xi)}{\beta} \) if \( \beta \neq 0 \).

To conclude, let us slightly point towards the fact that Lemma 2.15 could also be viewed as a building block of a probabilistic limit theorem (whose detailed discussion would then exceed the main goal of this paper, though).

### 2.2. The role of the distribution function of \( X \)

From now on, \( F := F_X = P(X \leq \cdot) \) is given as the distribution function of a given random variable \( X \).

**Proposition 2.16.** Let \( X, V \) be two random variables, both defined on the same probability space \((\Omega, \mathcal{F}, P)\), such that \( V \sim U(0, 1) \) and \( V \) is independent of \( X \). Let \( F = F_X \) be the distribution function of \( X \). Then \( F_V(X) \sim U(0, 1) \) is a uniformly distributed random variable. Moreover,

\[
P(F(X) \leq \alpha) = \alpha = P(X \leq F^{-}(\alpha))
\]
on the set \( \{ \alpha \in (0, 1) : F^{-}(\alpha) < F^{+}(\alpha) \} \).

**Proof.** Let \( 0 < \alpha < 1 \). Lemma 2.15 · applied to \( F = F_X \cdot \) directly leads to

\[
P(F_V(X) \leq \alpha) - \alpha = P(X > \xi \text{ and } F(X) = \alpha).
\]

Corollary 2.11 further implies that for any \( 0 < \lambda < 1 \) we have

\[
P(X > \xi \text{ and } F(X) = \alpha) = P(X \in A_{\xi,\alpha}) = \mu_F(A_{\xi,\alpha}) = 0.
\]

Thus, we have

\[
P(F_V(X) \leq \alpha) = \alpha \text{ for any } 0 < \alpha < 1.
\]  

(2.8)

Consequently, \( \sigma \)-additivity of the probability measure \( P \) allows one to continuously extend (2.8) to the whole real line. Hence, \( F_V(X) \sim U(0, 1) \) is uniformly distributed.

Now let \( \alpha \in (0, 1) \) such that \( \xi := F^{-}(\alpha) < F^{+}(\alpha) =: \eta. \) Since \( F \) is the distribution function of \( X \), we have

\[
\mu_F = P(X < \cdot).
\]

Thus, Corollary 2.11 leads to

\[
P(F(X) = \alpha) = \mu_F(F = \alpha) = \alpha - F(\xi-) = P(X = \xi).
\]

Since always

\[
P(F(X) < \alpha) = P(X < \xi),
\]

it follows that

\[
P(F(X) \leq \alpha) = P(X \leq \xi) = F(\xi) = \alpha,
\]

and we are done.

**Remark 2.17.** It is well-known that in the case of a continuous distribution function, \( G = G_X \) say, \( G(X) \) is uniformly distributed over \((0, 1)\) (cf. e.g. [5, Proposition 3.1]). However, continuity of \( G \) is even a necessary condition for \( G(X) \) being uniformly distributed over \((0, 1)\). Else there were some \( x_0 \in \mathbb{R} \) such that

\[
0 < \Delta G(x_0) = G(x_0) - G(x_0-) = P(X = x_0) < P(G(X) \in G(x)) = 0.
\]
which would be a contradiction. Consequently, if \( F \) has non-zero jumps, \( F_Y(X) \) still would be uniformly distributed over \((0, 1)\) as opposed to \( F(X) \).

In order to complete the proof of statement of Proposition 2.1 in [12], let us recall that the assumed independence of the random variables \( X \) and \( V \) implies that the bivariate distribution function \( F_{(V,X)} \) of the random vector \((V, X)\) coincides with the product of the distribution functions \( F_V \) and \( F_X \). Moreover, since \( V \sim U(0, 1) \), it follows that on \( \mathcal{B}(\mathbb{R}^2) \) coincides with the Lebesgue measure \( m \). Hence, if \( \Phi : \mathbb{R}^2 \to \mathbb{R} \) denotes an arbitrary non-negative (or bounded) Borel function on \((\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mu_{F_V} \otimes \mu_{F_X})\), an immediate application of the Fubini-Tonelli Theorem leads to

\[
E[\Phi(V, X)] = E[\Phi(V, X)|_{V \in (0,1)}] = \int_0^1 E[\Phi(\lambda, X)] \, m(d\lambda) = \int_0^1 \left( \int \Phi(\lambda, x) \, \mu_F(dx) \right) m(d\lambda) \tag{2.9}
\]

**Theorem 2.18.** Let \( X, V \) be two random variables, defined on the same probability space \((\Omega, \mathcal{F}, P)\), such that \( V \sim U(0, 1) \) and \( V \) is independent of \( X \). Let \( F = F_X \) be the distribution function of the random variable \( X \). Then

\[
X = F^- (F_Y(X)) = F^- (F(X-) + V \Delta F(X)) \quad P\text{-almost surely.}
\]

If in addition \( P(0 < F(X) < 1) = 1 \) (for example, if \( F \) is continuous), then

\[
X = F^- (F(X)) \quad P\text{-almost surely}.
\]

**Proof.** Let \( B_\lambda := \{ x \in \mathbb{R} : F_A(x) = 0 \} \), where \( 0 < \lambda \leq 1 \). Then

\[
P(F_A(X) = 0) = E[\mathbb{I}_{B_\lambda}(X)]
\]

On the other hand, equality 2.9 clearly implies

\[
P(F_Y(X) = 0) = \int_0^1 E[\mathbb{I}_{B_\lambda}(X)] \, m(d\lambda).
\]

Hence, since \( F_Y(X) \sim U(0, 1) \), it follows that \( \int_0^1 E[\mathbb{I}_{B_\lambda}(X)] \, m(d\lambda) = 0 \), implying that for \( m \)-almost all \( \lambda \in (0, 1) \) we have

\[
\mu_F(F_A = 0) = P(F_A(X) = 0) = E[\mathbb{I}_{B_\lambda}(X)] = 0.
\]

Similarly, we obtain

\[
\mu_F(F_A = 1) = 0
\]

for \( m \)-almost all \( \lambda \in (0, 1) \). Hence, there exists an \( m \)-null set \( L \in \mathcal{B}((0, 1)) \) such that \( 0 < F_A < 1 \) \( \mu_F \)-almost everywhere for all \( \lambda \in (0, 1) \mid L := A \).

Thus, given the construction in the proof of Theorem 2.12, it follows that for all \( \lambda \in A \) there exists a \( \mu_F \)-Borel null set \( N_\lambda \), such that for any \( x \in \mathbb{R} \setminus N_\lambda \) the value \( F^-(F_A(x)) \) is well-defined and satisfies \( F^-(F_A(x)) = x \). Hence, since

\[
P(X \in N_V \text{ and } V \in A) \overset{(2.9)}{=} \int_A P(X \in N_\lambda) \, m(d\lambda) = \int_A \mu_F(N_\lambda) \, m(d\lambda) = 0,
\]

it consequently follows that \( X = F^- (F_Y(X)) = F^- (F(X-) + V \Delta F(X)) \) on the set \( \Omega \setminus N \subseteq \{ 0 < V \leq 1 \} \), where \( N := \{ V \notin A \} \cup \{ X \in N_V \text{ and } V \in A \} \).

If in addition \( P(0 < F(X) < 1) = 1 \), \( F^-(F(X)) \) is well-defined \( P\)-a.s. Consequently, since also \( F^-(F(X)) \) is non-decreasing, there exists a \( P \)-null set \( \tilde{N} \), satisfying \( \Omega \setminus \tilde{N} \subseteq \Omega \setminus N \subseteq \{ 0 < V \leq 1 \} \), such that

\[
X(\omega) = F^- (F(X(\omega)-) + V(\omega) \Delta F(X(\omega))) \leq F^- (F(X(\omega))) \leq X(\omega)
\]

for all \( \omega \in \Omega \setminus \tilde{N} \).
Remark 2.19. One might be easily lead to assume that already a direct application of Proposition 2.5 implies the first statement of Theorem 2.18. However, in the first instance Proposition 2.5 only implies that the equality $X = F^\wedge (F_Y (X)) = F^\vee (F(X) + V\Delta F(X))$ at least holds on the set $D := \{\omega \in \Omega : \Delta F(X(\omega)) > 0 \text{ and } 0 < V(\omega) < 1\}$. Now consider $N := \Omega \setminus D$. Then
\[ \mathbb{P}(N) = \mathbb{P}(\{\Delta F(X) = 0\}) = \mathbb{P}(\{\Delta F_Y (X) = 0\}) = \mathbb{P}(U = Y), \]
where $U := F_Y (X) \sim U(0, 1)$ and $Y := F_Y (X) = U - V\Delta F(X)$. However, in general we don’t know whether $U$ is independent of $Y$.

For the convenience of the reader, we conclude our paper with a full proof of the general version of Sklar’s Theorem, built on Theorem 2.18 (cf. also the proof of Theorem 1.2 in [10], respectively the short proof of Lemma 3.2 in [11]), complemented with another interesting and seemingly novel observation (Remark 2.21), induced by Lemma 2.7.

Corollary 2.20 (Sklar’s Theorem). Let $n \in \mathbb{N}$ and $F_{(X_1, \ldots, X_n)}$ be a joint $n$-variate distribution function of a random vector $(X_1, X_2, \ldots, X_n) : \Omega \rightarrow \mathbb{R}^n$ with marginals $F_i := F_{X_i}$ ($i = 1, 2, \ldots, n$). Then there exist a copula $C_F$ such that for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$
\[ F_{(X_1, \ldots, X_n)}(x_1, x_2, \ldots, x_n) = C_F(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)). \]
If all $F_i$ are continuous, then the copula $C_F$ is unique. Otherwise, $C$ is uniquely determined on $\prod_{i=1}^n F_i(\mathbb{R})$. Conversely, if $C$ is a copula and $H_1, H_2, \ldots, H_n$ are distribution functions, then the function $F$ defined by
\[ F(x_1, x_2, \ldots, x_n) := C(H_1(x_1), H_2(x_2), \ldots, H_n(x_n)) \]
is a joint distribution function with marginals $H_1, H_2, \ldots, H_n$.

Proof. Let $i \in \{1, 2, \ldots, n\}$ and $V_i \sim U(0, 1)$. On $0 < V_i \leq 1$ put $U_i := V_i\Delta F_i(X_i) + F_i(X_i-).$ According to Theorem 2.18 there exist null sets $M_1, M_2, \ldots, M_n \in \mathcal{F}$, such that on $\Omega \setminus M_i \subseteq \{0 < V \leq 1\} Z_i := F_i^\wedge (U_i)$ is well-defined and satisfies $X_i \equiv Z_i$ for every $i \in \{1, 2, \ldots, n\}$. Thus, $\mathbb{P}(M) = 0$, where $M := \bigcup_{i=1}^n M_i$.

Let $F_{(X_1, \ldots, X_n)}(x_1, x_2, \ldots, x_n) := \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)$ denote the $n$-variate distribution function of the random vector $(X_1, X_2, \ldots, X_n)$. Consider the copula
\[ C_F(y_1, y_2, \ldots, y_n) := \mathbb{P}(U_1 \leq y_1, U_1 \leq y_2, \ldots, U_n \leq y_n), \]
where $(y_1, y_2, \ldots, y_n) \in [0, 1]^n$. Since
\[ \{u \in (0, 1) : F_i^\wedge (u) \leq x_i\} = \{u \in (0, 1) : u \leq F_i(x_i)\} \]
for all $i \in \{1, 2, \ldots, n\}$ and $\mathbb{P}(M) = 0$, it consequently follows
\[ C_F(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) = \mathbb{P}(\{Z_1 \leq x_1, Z_2 \leq x_2, \ldots, Z_n \leq x_n\} \cap \mathbb{R} \setminus M) \]
\[ = \mathbb{P}(\{X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n\} \cap \mathbb{R} \setminus M) = F_{(X_1, \ldots, X_n)}(x_1, x_2, \ldots, x_n). \]

Combining Sklar’s Theorem with Lemma 2.7, we immediately obtain another interesting result:

Remark 2.21. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in (0, 1)^n$, satisfying $F_i^\wedge (\alpha_i) < F_i^\vee (\alpha_i)$ for all $i \in \{1, 2, \ldots, n\}$. Then
\[ C_F(\alpha_1, \alpha_2, \ldots, \alpha_n) = F_{(X_1, X_2, \ldots, X_n)}(F_1^\wedge (\alpha_1), F_2^\wedge (\alpha_2), \ldots, F_n^\wedge (\alpha_n)). \]

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