New Fractional Modelling, Analysis and Control of the Three Coupled Multiscale Non-Linear Buffering System

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Abstract
This study aims to investigate the complicated dynamical $HCO_3^-/CO_2$ buffering system using fractional operators which is not been investigated yet. We consider a new fractional mathematical model in the frame of fractional-order differential equations. In the proposed fractional-order model, we apply the Caputo-Fabrizio fractional operator with an exponential kernel. Then to solve the derived system of fractional equations, we suggest a quadratic numerical technique and prove its stability and convergence. Also, accurate control for the proposed system is considered. Behaviors of the approximate solutions for the considered model are provided by choosing different values of fractional orders along with integer order. Each figure manifests and compares the numerical solutions under selected orders. Figures, show how the results can be affected by changing the fractional orders.

Keywords Mathematical modelling · Caputo-Fabrizio derivative · Robust control · Numerical method · Fractional buffering model

Introduction and Motivation

In recent years, researchers have paid a considerable concentration to the non-integer differential and integral operators because these types of operators are effective instruments for the modelling of real-world phenoms with complicated dynamics [1–6]. Studies on fractional-order problems show that these types of operators can manifest intricate dynamical behaviours more perfectly than integer order systems. Oscillatory and complex behaviour of Caputo-Fabrizio fractional order HIV-1 infection model was reported in [7]. Effect of vaccination to control COVID-19 with fractal fractional operator can be read in [8]. Also, modeling and analysis of fractional order Ebola virus model with Mittag-Leffler kernel can be seen in [9]. This is because fractional-order operators can identify the qualities of memory effects as a crucial perspective of numerous real-world phenoms. Modelling and simulations of the
SEIR and Blood Coagulation systems using Atangana-Baleanu-Caputo derivative is done in [10]. Also, complex dynamics of multi strain TB model under nonlocal and nonsingular fractal fractional operator was studied in [11]. In [12] modeling the dynamics of novel coronavirus (COVID-19) via stochastic epidemic model can be read. Modeling and numerical investigation of fractional-order bovine babesiosis disease can be found in [13]. A Fractal Fractional Model for Cervical Cancer due to Human Papillomavirus Infection has been done in [14]. Recently, FC has been identified as an attractive area for modelling and investigating dynamical systems by disordered characteristics [15–17]. Using these types of operators we can choose any wanted order. In fact, we have much more selection of orders to see the behaviour of the considered problem. In this regard, various definitions have been developed for fractional operators. Indeed, Caputo fractional derivative is one of the leading concepts in this regard. Many analytical and numerical investigations have been done containing this sense. Numerical solutions of the fractional-order Korteweg-de Vries problem has been investigated using a combination of two techniques in [18]. Indeed, approximate solutions of two-dimensional time-fractional Burgers equation using the discontinuous Galerkin method and a finite difference method can be read in [19]. Another study regarding Caputo sense is about solving the fractional-order Burger-Huxley problem. In fact, this system has been solved numerically, by applying a geometric numerical scheme. Full detail of the work can be seen in [20]. In another fractional work, numerical and exact solutions for the time-fractional Burgers’ equation have been obtained [21]. In fact, the expansion method and the Cole-Hopf transformation have been applied to do this goal. To see more studies on the Caputo frame, refer to [22–25].

But, despite the beneficial features of this fractional sense, the principal limitation of the operator is about the kernel singularity of the Caputo derivative. The non-local frame of the fractional operator has presented finite options to make effective codes, for it needs all past information to be taken into consideration while simulating. This is what we say the tendance of ongoing memory, that creates computing more dear and dull. In this respect, there are many short memory principles that are employed to decrease the computing cost and the efficacy of rounding-off error accumulation while employing numerical algorithms, thereby making the short memory principles quite useful in solving non-integer order problems. Fortunately, in order to defeat this concern, new definitions of fractional derivatives have been developed such as the Caputo-Fabrizio (CF) [26] and Atangana-Baleanu derivatives [27]. The mentioned fractional operators exhibit various asymptotic actions and are more reliable than the classic ones [28, 29]. Some other interesting works can be read in [30–38]. We should examine the characteristics of such operators, and effectual approaches should be evolved to create such operators more useful in real-world systems. Prompted by the preceding argumentation, the chief purpose of this study is to utilize the CF fractional operator for modelling and investigating the non-linear dynamical $HCO_3^-/CO_2$ buffering system for the first time. Notable differences between the integer-order system and non-integer-order one can be realized by comparing the model performance in two integer and non-integer-order frames. Furthermore, some unknown traits of the studied model are picked out by the suggested non-integer system, whereas such properties cannot be recognized using ODEs. Additionally, we develop an effective quadratic approach is developed to get the approximate solutions of the considered model. To model fractionally the considered system some basic definitions are provided in the following lines.

Now, we shortly present preliminaries regarding the Caputo-Fabrizio fractional derivative features.
Definition 1.1 [39] For each \( f \in \mathbb{H} \) and \( 0 < \alpha < 1 \) the Caputo-Fabrizio (CF) fractional derivative and its related fractional integral operator are, respectively, presented by
\[
\text{CF}D_t^\alpha (u(t)) = \frac{1}{(1-\alpha)} \int_0^t \text{exp}(-\alpha/(1-\alpha) (t-z)) u'(z)dz, \quad 0 < \alpha \leq 1, \quad (1)
\]
\[
\text{CF}I_t^\alpha (u(t)) = (1-\alpha)u(t) + \int_0^t u(z)dz, \quad 0 < \alpha \leq 1, \quad (2)
\]
By taking \( \alpha = 1 \), Eq. (1) gives the integer-order integral yields to \( u(t) \) for \( \alpha = 0 \). Now, we demonstrate some helpful relations of the CF definition [40, 41] - for \( f_1 \) and \( f_2 \in \mathbb{H}^1(0, T) \) and \( c_1, c_2 \in \mathbb{R} \), the CF derivative and integral are linear operators:
\[
\text{CF}D_t^\alpha (c_1 u_1(t) + c_2 u_2(t)) = c_1 \text{CF}D_t^\alpha (u_1(t)) + c_2 \text{CF}D_t^\alpha (u_2(t)), \quad (3)
\]
- If we consider \( u(t) = u_c \) as a constant function, the CF operator is zero, as
\[
\text{CF}D_t^\alpha (u_c) = 0, \quad (4)
\]
- Because the CF operator (1) is the convolution integral of \( \frac{du(t)}{dt} \) and \( \text{exp}(\frac{-\alpha}{1-\alpha} t) \), using the convolution theory for the Laplace transform, we get
\[
\mathcal{L}[\text{CF}D_t^\alpha (u(t))] = \frac{s F(s) - f(0)}{s + \alpha(1-s)}, \quad F(s) = \mathcal{L}[f(t)], \quad (5)
\]
This numerical study is prepared as follows. A new mathematical using fractional Caputo-Fabrizio operator is developed in Section 2. Also, in Sect. 3, a quadratic scheme is introduced and stability and convergent of this method are shown, successfully. Robust control for the non-linear dynamical buffering systems can be read in Sect. 4. Numerical experiments for the proposed model using different values of fractional orders and initial conditions can be observed in Sect. 5 and numerical experiments to show the effectiveness of the proposed approach for solving the considered fractional model are provided in Sect. 6. Finally, we provide the conclusion of this numerical investigation on the nonlinear dynamical \( HCO_3^- / CO_2 \) buffering system in Sect. 7.

**Novel Model**

Now we consider the integer order model of acid-base homeostatic \( HCO_3^- / CO_2 \) process reported in [42] as
\[
\frac{d}{dt} A(t) = l_1 - l_2 A(t) - l_3 A(t)B(t) + l_4 C(t),
\]
\[
\frac{d}{dt} B(t) = l_5 + l_6 C(t) - l_7 B(t) - l_3 A(t)B(t) + l_4 C(t),
\]
\[
\frac{d}{dt} C(t) = l_8 - l_9 l_{10} C(t) + l_3 A(t)B(t) - l_4 C(t).
\]
In the above system \( l_1 \) depicts the cellular reproduction of \( H^+ \) and \( l_2 \) shows the \( H^+ \) lack. Moreover, we indicate the hydration and dehydration response measures by \( l_3 \) and \( l_4 \), respectively, where the amounts have been arranged to reveal the carbonic anhydrase behaviour. Indeed, \( l_5 \) indicates the \( HCO_3^- \) treatment and/or completion. Also, acid secretion degree is represented by \( l_6 \). The measure related to renal filtration for \( HCO_3^- \) is denoted by \( l_7 \) and

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We depict the cellular reproduction of \( CO_2 \) with \( l_8 \). Also, we illustrate the ventilation measure by \( l_9 \) and \( l_{10} \) is minute amount ventilation. Indeed, \( A(t) \) indicates the concentration of the free hydrogen ions, \( B(t) \) denotes the condensation of bicarbonate and \( C(t) \) shows the concentration of carbon dioxide.

In order to make the fractional model of the above system, we replace the classic derivative with the fractional Caputo-Fabrizio derivative. So, we have the following one

\[
\begin{align*}
\mathcal{D}_t^\alpha A(t) &= l_1 - l_2 A(t) - l_3 A(t) B(t) + l_4 C(t), \\
\mathcal{D}_t^\alpha B(t) &= l_5 + l_6 C(t) - l_7 B(t) - l_8 A(t) B(t) + l_4 C(t), \\
\mathcal{D}_t^\alpha C(t) &= l_8 - l_9 l_{10} C(t) + l_3 A(t) B(t) - l_4 C(t), \\
\end{align*}
\]

Subjected to the initial conditions \( A(0) = A_0, B(0) = B_0 \) and \( C(0) = C_0 \).

**Quadratic Numerical Technique**

In order to get the approximate solutions of the proposed model (7), we suggest a quadratic numerical strategy. First, we convert the system (7) into its corresponded integral equation. After that, a numerical scheme based on the trapezoidal method is presented to obtain the numerical solutions of the derived integral equation. To use the offered method in the framework of CF, we write Eq. (7) in the next form:

\[
\begin{cases}
\mathcal{D}_t^\alpha \Xi(t) = \Phi(\Xi(t)), \quad 0 \leq t \leq T_f < \infty, \\
\Xi(0) = \Xi_0,
\end{cases}
\]

where \( \Xi(t) = (A(t), B(t), C(t)) \), \( \Phi \) is the vector function

\[
\Phi(\Xi(t)) = \begin{bmatrix}
l_1 - l_2 A(t) - l_3 A(t) B(t) + l_4 C(t), \\
l_5 + l_6 C(t) - l_7 B(t) - l_8 A(t) B(t) + l_4 C(t), \\
l_8 - l_9 l_{10} C(t) + l_3 A(t) B(t) - l_4 C(t),
\end{bmatrix},
\]

meeting Lipschitz condition

\[
\|\Phi(\Xi_1(t)) - \Phi(\Xi_2(t))\| \leq L\|\Xi_1(t) - \Xi_2(t)\|, \quad L > 0,
\]

and \( \Xi_0 = (A(0), B(0), C(0)) \) involves initial conditions. By implementing relation (1) on Eq. (7), we get

\[
\Xi(t) = \Xi_0 + \mathcal{D}_t^\alpha \mathcal{E}_t^\alpha(\Xi(t)), \quad 0 \leq t \leq T < \infty,
\]

We set \( 0 = t_0 < t_1 < \cdots < t_S = T_f \) on \([0, T_f]\) via step \( h = \frac{T_f}{S} = t_{s+1} + t_s \), where \( S > 0 \) is an integer, and \( t_S = s h, s = 0, 1, 2, \cdots, S - 1 \). Also, we present the approximation of \( \Xi(t_s) \) via \( \Xi_s \). After that, \( \Phi(\Xi(\tau)) \) on subinterval \([t_k, t_{k+1}]\) is approximated by the piecewise linear interpolation as:

\[
\Phi(\Xi(\tau))|_{[t_k, t_{k+1}]} \approx \frac{t_{k+1} - \tau}{t_{k+1} - t_k} \Phi(\Xi_k) + \frac{\tau - t_k}{t_{k+1} - t_k} \Phi(\Xi_{k+1}), \quad 0 \leq k \leq s.
\]

To possess a proper approximation solution of Eq. (11), the next discretization for (1) can be a proper option:

\[
\mathcal{D}_t^\alpha \mathcal{T}(\Xi(t_{s+1})) = (1 - \alpha) \Phi(\Xi(t_{s+1})) + \alpha \int_0^{t_{s+1}} \Phi(\Xi(\tau))d\tau,
\]
Now, by replacing relation (12) into (13), we get

\[ \Xi_{s+1} = \Xi_0 + (1 - \alpha)\Phi(\Xi_{s+1}) + \alpha h \sum_{k=0}^{s+1} a_{s+1,k} \Phi(\Xi_k), \quad s = 0, 1, \ldots, S - 1, \quad (14) \]

where the coefficient \(a_{s+1,k}\) is in the following form:

\[
\begin{aligned}
    a_{s+1,0} &= \frac{1}{2}, \\
    a_{s+1,k} &= 1, \quad k = 1, 2, \ldots, s, \\
    a_{s+1,s+1} &= \frac{1}{2},
\end{aligned}
\quad (15)

We examine the convergence as well as stability of the scheme (14) using following theories.

**Theorem 3.1** Numerical scheme (14) is conditionally stable.

**Proof** We denote the perturbation of \(\Xi_0\) and \(\Xi_s\) \((s = 0, 1, \ldots, S - 1)\) via \(\tilde{\Xi}_0\) and \(\tilde{\Xi}_s\), respectively. Thus, regarding the numerical technique (14), it results

\[
\Xi_{s+1} + \tilde{\Xi}_{s+1} = \Xi_0 + \tilde{\Xi}_0 + (1 - \alpha)\Phi(\Xi_{s+1} + \tilde{\Xi}_{s+1}) + \alpha h \sum_{k=0}^{s+1} a_{s+1,k} \Phi(\Xi_k + \tilde{\Xi}_k), \quad (16)
\]

Now, we replace Eq. (14) into Eq. (16), so we get

\[
\tilde{\Xi}_{s+1} = \tilde{\Xi}_0 + (1 - \alpha)(\Phi(\Xi_{s+1} + \tilde{\Xi}_{s+1} - \Phi(\Xi_{s+1})) + \alpha h \sum_{k=0}^{s+1} a_{s+1,k} (\Phi(\Xi_k + \tilde{\Xi}_k) - \Phi(\Xi_k)), \quad (17)
\]

Applying triangle inequality as well as Lipschitz condition, by Eq.(17), yields

\[
\|\tilde{\Xi}_{s+1}\| = \|\tilde{\Xi}_0 + (1 - \alpha)(\Phi(\Xi_{s+1} + \tilde{\Xi}_{s+1} - \Phi(\Xi_{s+1})) + \alpha h \sum_{k=0}^{s+1} a_{s+1,k} (\Phi(\Xi_k + \tilde{\Xi}_k) - \Phi(\Xi_k))\|
\]

\[
\leq \|\tilde{\Xi}_0\| + (1 - q)\|\Phi(\Xi_{s+1} + \tilde{\Xi}_{s+1}) - \Phi(\Xi_{s+1})\|
\]

\[
+ \alpha h \sum_{k=0}^{s+1} a_{s+1,k} \|\Phi(\Xi_k + \tilde{\Xi}_k) - \Phi(\Xi_k)\|
\]

\[
\leq \|\tilde{\Xi}_0\| + (1 - q)L\|\tilde{\Xi}_{s+1}\| + \alpha h L \sum_{k=0}^{s+1} a_{s+1,k} \|\tilde{\Xi}_k\|
\]

\[
\leq \|\tilde{\Xi}_0\| + \alpha h L \sum_{k=0}^{s+1} \|\tilde{\Xi}_k\| + L(1 - \alpha) + \alpha h a_{s+1,s+1} \|\tilde{\Xi}_{s+1}\|, \quad (18)
\]

Using Eq. (15), we obtain

\[
\|\tilde{\Xi}_{s+1}\| \leq \|\tilde{\Xi}_0\| + \alpha h L \sum_{k=0}^{s} \|\tilde{\Xi}_k\| + L \left(1 - \alpha + \frac{\alpha h}{2}\right) \|\tilde{\Xi}_{s+1}\|, \quad (19)
\]
For $L, h, \alpha$ such that $L \left(1 - \alpha + \frac{\alpha h}{2}\right) < 1$, we gain

$$
\| \tilde{\mathcal{E}}_{s+1} \| \leq \phi(\alpha, h) \| \tilde{\mathcal{E}}_0 \| + \phi(\alpha, h) ahL \sum_{k=0}^s \| \tilde{\mathcal{E}}_k \|, 
$$

(20)

where

$$
\phi(\alpha, h) = \frac{1}{1 - L(1 - \alpha + \frac{h\alpha}{2})},
$$

(21)

Moreover, for a constant $C_\phi$ such that for a adequately small $h$, we own

$$
1 < \phi(\alpha, h) < C_\phi,
$$

(22)

Thus

$$
\| \tilde{\mathcal{E}}_{s+1} \| \leq C_\phi \| \tilde{\mathcal{E}}_0 \| + C_\phi ahL \sum_{k=0}^s \| \tilde{\mathcal{E}}_k \|. 
$$

(23)

Finally, using Lemma 3.3 in [39] and using the Grönwall inequality, $\| \tilde{\mathcal{E}}_{s+1} \| \leq C_1 \| \tilde{\mathcal{E}}_0 \|$ can be obtained, where in $C_1$ is a constant.

\[\square\]

**Theorem 3.2** Numerical method (20) is conditionally convergent.

**Proof** We compute the difference between the exact solution $\Xi(t_{m+1})$ and the approximate solution $F_{m+1}$ (Eq. (14)) via

$$
\Xi(t_{s+1}) - \Xi_{s+1} = (1 - \alpha)(\phi(\Xi(t_{s+1})) - \Phi(\Xi_{s+1}))
$$

$$
+ \alpha \left[ \int_0^{t_{s+1}} \Phi(\Xi(\tau))d\tau - h \sum_{k=0}^{s+1} a_{s+1,k}\Phi(k) \right]
$$

$$
= (1 - \alpha)(\Phi(\Xi(t_{s+1})) - \Phi(\Xi_{s+1}))
$$

$$
+ \alpha \left[ \int_0^{t_{s+1}} \Phi(\Xi(\tau))d\tau - h \sum_{k=0}^{s+1} a_{s+1,k}\Phi(k) \right]
$$

$$
= \alpha h \sum_{k=0}^{s+1} a_{s+1,k}(\Phi(\Xi(t_k)) - \Phi(\Xi(k))), 
$$

(24)

Now, we apply the triangle inequality and Lipschitz condition, along with Lemma 3.1 in [39]

$$
\| \Xi(t_{s+1}) - \Xi_{s+1} \| \leq L(1 - \alpha)\| \Xi(t_{s+1}) - \Xi_{s+1} \| + \frac{1}{2} \alpha Ch^2 + ahL \sum_{k=0}^{s+1} a_{s+1,k} \| \Xi(t_k) 
$$

$$
- \Xi_k \| \leq L \left(1 - \alpha + \frac{\alpha h}{2}\right) \| \Xi(t_{s+1}) - \Xi_{s+1} \| + \frac{1}{2} \alpha Ch^2 
$$

$$
+ \alpha hL \sum_{s=0}^{s+1} a_{s+1,k} \| \Xi(t_k) - \Xi_k \|.
$$

(25)
where $C$ is a general constant. Using Eq. (15) and the parameters $L$, $h$, $\alpha$ meeting the inequality $L(1 - \alpha + \frac{qh}{2}) < 1$, we own

$$\| \Xi(t_{s+1}) - \Xi_{s+1} \| \leq \phi(\alpha, h) \frac{1}{2} \alpha C T h^2 + \phi(\alpha, h) \frac{h}{L} \sum_{k=0}^{s} \| \Xi(t_k) - \Xi_k \|, \quad (26)$$

where $\phi(\alpha, h)$ is taken into account from Eq. (21) and satisfies inequality (22). At the end, using Lemma 3.3 in [39] and applying Grönwall inequality, yields

$$\| \Xi(t_{s+1}) - \Xi_{s+1} \| \leq C_1 h^2. \quad (27)$$

where $C_1$ is a generic constant.

\[\Box\]

**Robust Control for Non-Linear Dynamical Buffering System**

We consider the following system as

$$\begin{align*}
\mathcal{C}_0 \mathcal{D}_t^\alpha & \mathcal{A}(t) = l_1 - l_2 A(t) - l_3 A(t) B(t) + l_4 C(t), \\
\mathcal{C}_0 \mathcal{D}_t^\alpha & \mathcal{B}(t) = l_5 + l_6 C(t) - l_7 B(t) - l_3 A(t) B(t) + l_4 C(t), \\
\mathcal{C}_0 \mathcal{D}_t^\alpha & \mathcal{C}(t) = l_8 - l_9 l_{10} C(t) + l_3 A(t) B(t) - l_4 C(t),
\end{align*} \quad (28)$$

Now we impose $u_A$, $u_B$ and $u_C$ on the above model. So, we have

$$\begin{align*}
\mathcal{C}_0 \mathcal{D}_t^\alpha & \mathcal{A}(t) = l_1 - l_2 A(t) - l_3 A(t) B(t) + l_4 C(t) + u_A, \\
\mathcal{C}_0 \mathcal{D}_t^\alpha & \mathcal{B}(t) = l_5 + l_6 C(t) - l_7 B(t) - l_3 A(t) B(t) + l_4 C(t) + u_B, \\
\mathcal{C}_0 \mathcal{D}_t^\alpha & \mathcal{C}(t) = l_8 - l_9 l_{10} C(t) + l_3 A(t) B(t) - l_4 C(t) + u_C, \quad (29)
\end{align*}$$

The purpose of the control is to overcome the chaotic performance of the model. We set $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ as equilibrium points of the model.

$$\begin{align*}
\mathcal{C}_0 \mathcal{D}_t^\alpha & \tilde{A}(t) = l_1 - l_2 \tilde{A}(t) - l_3 \tilde{A}(t) \tilde{B}(t) + l_4 \tilde{C}(t), \\
\mathcal{C}_0 \mathcal{D}_t^\alpha & \tilde{B}(t) = l_5 + l_6 \tilde{C}(t) - l_7 \tilde{B}(t) - l_3 \tilde{A}(t) \tilde{B}(t) + l_4 \tilde{C}(t), \\
\mathcal{C}_0 \mathcal{D}_t^\alpha & \tilde{C}(t) = l_8 - l_9 l_{10} \tilde{C}(t) + l_3 \tilde{A}(t) \tilde{B}(t) - l_4 \tilde{C}(t),
\end{align*} \quad (30)$$

Then by defining a control error we have the following relations

$$\begin{align*}
e_A(t) & = A(t) - \tilde{A}(t), \\
e_B(t) & = B(t) - \tilde{B}(t), \\
e_C(t) & = C(t) - \tilde{C}(t),
\end{align*} \quad (31)$$

Thus

$$\begin{align*}
\mathcal{C}_0 \mathcal{D}_t^\alpha & e_A(t) = l_1 - l_2 e_A - l_3 e_A(t) e_B + l_4 (e_C + \tilde{C}) + u_A, \\
\mathcal{C}_0 \mathcal{D}_t^\alpha & e_B(t) = l_5 + l_6 (e_C + \tilde{C}) - l_7 e_B - l_3 e_A e_B + l_4 (e_C + \tilde{C}) + u_B, \\
\mathcal{C}_0 \mathcal{D}_t^\alpha & e_C(t) = l_8 - l_9 l_{10} e_C + l_3 (e_A + \tilde{A}) (e_B + \tilde{B}) - l_4 e_C + u_C, \quad (32)
\end{align*}$$

As stated, the performance of chaos would be overcome in the model, thus points of equilibrium will be set at zero i.e $\tilde{A} = \tilde{B} = \tilde{C} = 0$, so we have

$$\mathcal{C}_0 \mathcal{D}_t^\alpha e_A(t) = l_1 - l_2 e_A - l_3 e_A e_B + l_4 e_C + u_A,$$
From the above system, the control law is represented by the following relations:

\[
\begin{align*}
  u_A &= -l_1 + l_2 e_A + l_3 e_A e_B - l_4 e_C - \Phi_A e_A, \\
  u_B &= -l_5 - l_6 e_C - l_7 e_B + l_3 e_A e_B - l_4 e_C - \Phi_B e_B, \\
  u_C &= -l_8 + l_9 l_{10} e_C - l_3 e_A e_B - l_4 e_C - \Phi_C e_C.
\end{align*}
\]

(34)

**Theorem 4.1** System (28) is stable in asymptotic manner for the control derived as Eq. (34).
Fig. 2 Approximate solutions for different values of $\alpha$ and ICs $A(0) = 0.05$, $B(0) = 100$ and $C(0) = 1$

**Proof** We use the Lyapunov function to show the stability of the considered controller (34). The proposed functions is as follows

$$\Upsilon(t) = \frac{e_A^2 + e_B^2 + e_C^2}{2},$$

(35)

Thus, for the above function we have the following derivative

$$\frac{C\Phi F}{0} D_t^\alpha \Upsilon(t) = e_A \frac{C\Phi F}{0} D_t^\alpha e_A(t) + e_B \frac{C\Phi F}{0} D_t^\alpha e_B(t) + e_C \frac{C\Phi F}{0} D_t^\alpha e_C(t),$$

(36)

Now, using Eqs. (33) and (34), the next relations can be gained

$$\frac{C\Phi F}{0} D_t^\alpha e_A(t) = l_1 - l_2 e_A - l_3 e_A e_B + l_4 e_C - l_5 + l_2 e_A + l_3 e_A e_B - l_4 e_C - \Phi_A e_A,$$

$$\frac{C\Phi F}{0} D_t^\alpha e_B(t) = l_5 + l_6 e_C - l_7 e_B - l_3 e_A e_B + l_4 e_C - l_5 - l_6 e_C + l_7 e_B + l_3 e_A e_B - l_4 e_C - \Phi_B e_B,$$

$$\frac{C\Phi F}{0} D_t^\alpha e_C(t) = l_8 - l_9 l_{10} e_C + l_3 e_A e_B - l_4 e_C - l_8 - l_9 l_{10} e_C - l_3 e_A e_B + l_4 e_C - \Phi_C e_C,$$

(37)
Fig. 3  Chaotic behaviour of solutions for different values of $\alpha$ and ICs
$A(0) = 0.05$, $B(0) = 100$ and $C(0) = 1$

\[
\begin{align*}
CF_{\alpha}^{D_{\alpha}^t}e_A(t) &= -\Phi_Ae_A, \\
CF_{\alpha}^{D_{\alpha}^t}e_B(t) &= -\Phi_Be_B, \\
CF_{\alpha}^{D_{\alpha}^t}e_C(t) &= -\Phi_ce_C, \\
\end{align*}
\]

(38)

Now, by substituting Eq. (38) into Eq. (36), we obtain

\[
CF_{\alpha}^{D_{\alpha}^t}\gamma(t) = -\Phi_Ae_A^2 - \Phi_Be_B^2 - \Phi_ce_C^2 \leq 0.
\]

(39)

So regarding $\Phi_A, \Phi_B, \Phi_C > 0$ indicates that the Eq.(39) is negative or zero, therefore we prove that the considered controller is stable.

\square

Numerical Experiments

To see the performance of the offered method on the proposed fractional model of buffering system, we select different values of fractional orders and initial conditions (ICs) to show the accuracy and capability of the proposed method for solving fractional dynamical systems under different conditions. Figures of approximate solutions are provided for each case. For the first case, we take ICs as $A(0) = 0.05$, $B(0) = 100$ and $C(0) = 1$ along with various amounts of fractional orders $\alpha = 0.95, 0.97, 0.99$ and integer order $\alpha = 1$ and

\[
\begin{align*}
l_1 &= 1.2 \times 10^{-6}, l_2 = 30.151, l_3 = 3.437 \times 10^{10}, l_4 = 2.736 \times 10^4, l_5 = 0, l_6 = 7.09 \times 10^{-3}l_7 = 3.5461 \times 10^{-4}, l_8 = 3 \times 10^{-6}, l_9 = 2.5 \times 10^{-2}, l_{10} = 0.1.\end{align*}
\]

Figures 3, 4, and 5 are responsible to show the behaviours of approximate solutions for the first case. To be exact, Fig. 1 shows the approximate solutions of each state variable containing considerable fluctuations. Strong fluctuations of the solutions can be seen for the considered time period. Fig. 2 show the 2D behaviour of solutions and Fig. 3 is showing the chaotic behaviour of the numerical solutions, successfully. For the second case, we consider ICs as $A(0) = 0.11$, $B(0) = 190$ and $C(0) = 1.05$ and the fractional orders are the same of the first case. For this case, Figs. 4, 5 and 6 are dedicated to manifest the numerical solutions of the proposed model for the chosen ICs and fractional orders. Fig. 4 depicts the numerical solutions of each state variable
Fig. 4 Approximate solutions for different values of $\alpha$ and ICs $A(0) = 0.11$, $B(0) = 190$ and $C(0) = 1.05$ in a considerable time and Figs. 5 and 6 are illustrating 2D and 3D behaviours of the solutions, respectively.

**Conclusion**

In the current study, the complex behaviour of the non-linear buffering system was investigated in the frame of fractional concept for the first time. Firstly, fractional operator in the sense of CF was considered for modelling the non-linear buffering system. After that, a quadratic numerical technique is considered to reveal the behaviour of the approximate solutions. For different values of fractional orders and ICs, relevant figures are reported to manifest how fractional and integer orders affect the approximate solutions of the recommended fractional model. Interesting chaotic patterns were obtained through changing orders. Each figure reveals and compares the approximate solutions on both integer and non-integer
orders. Also, accurate control is provided for the studied model. All in all, in this numerical investigation, accurate, flexible and successful performance of fractional Caputo-Fabrizio operator for the non-linear buffering system was done.
Fig. 6 Chaotic behaviour of solutions for different values of $\alpha$ and ICs

$A(0) = 0.11$, $B(0) = 190$ and $C(0) = 1.05$

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Declarations

Conflict of Interest  The authors declared that there is no conflict of interest whatsoever in this manuscript.

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