A Reverse Hölder Inequality for Extremal Sobolev Functions

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Received: 7 April 2014 / Accepted: 3 August 2014 / Published online: 15 August 2014
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Abstract Let $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^1$ boundary, and let $1 \leq p < \frac{2n}{n-2}$ (simply $p \geq 1$ if $n = 2$). The well-known Sobolev imbedding theorem and Rellich compactness implies that

$$C_p(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla f|^2 \, dm}{\left( \int_{\Omega} |f|^p \, dm \right)^{2/p}} : f \in W^{1,2}_0(\Omega), f \neq 0 \right\}$$

is a finite, positive number, and the infimum is achieved by a nontrivial extremal function $u$, which one can assume is positive inside $\Omega$. We prove that, for $1 \leq p \leq 2$ and for every $q > p$, there exists $K = K(n, p, q, C_p(\Omega)) > 0$ such that $\|u\|_{L^p(\Omega)} \geq K \|u\|_{L^q(\Omega)}$. This inequality, which reverses the classical Hölder inequality, mirrors results of G. Chiti for the first Dirichlet eigenfunction of the Laplacian and of M. van den Berg for the torsion function.

Keywords Best sobolev constant · Reverse Hölder inequality · Principal frequency · Torsional rigidity

Mathematics Subject Classification (2000) 35A23 · 35P15 · 35J05

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1 Introduction and Statement of Results

In 1972, Payne and Rayner [12] proved a reverse Hölder inequality for the eigenfunction $\phi$ of the Dirichlet Laplacian corresponding to the first eigenvalue $\lambda(D)$ of a bounded planar domain $D$. Specifically they proved that

$$\int_D \phi^2 \, dm \leq \frac{\lambda(D)}{4\pi} \left( \int_D \phi \, dm \right)^2,$$

with $dm$ being Lebesgue measure. The inequality is isoperimetric in the sense that equality holds if and only if $D$ is a disk.

It proved not to be entirely straightforward to extend this inequality to regions in higher dimensions. Payne and Rayner [13] obtained an isoperimetric extension to higher dimensions that they themselves described as not ‘entirely satisfactory’, since their inequality became trivial for regions of given volume but large eigenvalue. Kohler-Jobin [10] obtained an isoperimetric comparison between the $L^2$ and the $L^1$ norms of the eigenfunction that did not suffer from the defects of that of Payne and Rayner. Her inequality is

$$\int_{\Omega_1} \phi^2 \, dm \leq \frac{\lambda(\Omega_1)^{n/2}}{2^{n-1} \omega_n j_{n/2-1}^{n/2}} \left( \int_{\Omega_1} \phi \, dm \right)^2,$$

only leaving unanswered whether equality could hold for regions other than balls. Here $\Omega$ is a bounded domain in $\mathbb{R}^n$, the volume of the unit ball in $\mathbb{R}^n$ is denoted by $\omega_n$, and $j_m$ denotes the first positive zero of the Bessel function $J_m$. Kohler-Jobin [11] obtained further isoperimetric inequalities between the $L^1$ and $L^\infty$-norms and between the $L^p$ and $L^1$-norms of $\phi$ over the region $\Omega$. Subsequently, Chiti [6, 7] obtained, for $0 < r < s < \infty$, an inequality of the form

$$\left( \int_{\Omega_1} \phi^s \, dm \right)^{1/s} \leq K(n, r, s) \lambda(\Omega)^{\frac{r}{2}(1 - \frac{1}{r})} \left( \int_{\Omega_1} \phi^r \, dm \right)^{1/r} \quad (1)$$

where $\Omega$ is a bounded region in $\mathbb{R}^n$. The inequality is isoperimetric in that equality holds if and only if $\Omega$ is a ball.

The torsion function $u$ of an open set $\Omega$ in $\mathbb{R}^n$ is the solution of the problem $\Delta u = -2$ in $\Omega$ with zero Dirichlet boundary conditions. For $1 \leq r < s < \infty$, van den Berg [1, Theorem 1] obtains, as part of his work, an upper bound for the $L^s(\Omega)$-norm of the torsion function $u$ in terms of its $L^r(\Omega)$-norm, from which he concludes that if $u$ is in $L^r(\Omega)$ then it belongs to $L^s(\Omega)$ for all $s > r$. (A precursor of this result is Corollary 2 of [2].)

We aim herein to obtain a reverse Hölder inequality for the extremal Sobolev function that is similar to that of Chiti for the eigenfunction of the Laplacian. For a bounded domain $\Omega$ with $C^1$ boundary and for admissible values of $p$, we consider

$$\mathcal{C}_p(\Omega) = \inf \left\{ \Phi_p(f) : f \in W_0^{1,2}(\Omega), f \not\equiv 0 \right\} \quad \text{where} \quad \Phi_p(f) = \frac{\int_{\Omega} |\nabla f|^2 \, dm}{(\int_{\Omega} |f|^p \, dm)^{2/p}}. \quad (2)$$

Here the allowable range of exponents is $1 \leq p < \frac{2n}{n-2}$ if $n \geq 3$ and $p \geq 1$ if $n = 2$. We can see, using the change of variables $y = x/r$, that $\mathcal{C}_p$ obeys the scaling law

$$\mathcal{C}_p(r\Omega) = r^{n-2-\frac{2n}{p}} \mathcal{C}_p(\Omega) = r^{\alpha_{n,p}} \mathcal{C}_p(\Omega), \quad (3)$$

and observe that, within the allowable range of exponents, $\alpha_{n,p} < 0$. By Rellich compactness and the Sobolev embedding theorem, $\mathcal{C}_p(\Omega)$ is a finite, positive number, and is realized