Chiral properties of the fixed point action of the Schwinger model*

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Abstract

We study the spectrum properties for a recently constructed fixed point lattice Dirac operator. We also consider the problem of the extraction of the fermion condensate, both by direct computation, and through the Banks-Casher formula by analyzing the density of eigenvalues of a redefined antihermitean lattice Dirac operator.

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1 Introduction

One of the most annoying drawbacks of the lattice discretization of a fermionic theory is the impossibility\(^1\) to avoid the explicit breaking of the chiral symmetry. In the case of a simple discretization, the Wilson action, the breaking of the symmetry is so bad that no trace of the chiral properties of the continuum theory is kept in the lattice theory. This is of particular relevance for attacking QCD by lattice techniques. At the classical level, the chiral properties of the zero-modes of the Dirac operator are lost, and the Atiyah-Singer theorem of the continuum theory\(^2\) has no strict correspondence on the lattice. At the quantum level, the explicit breaking of the symmetry induces an additive renormalization of the quark mass, the chiral limit being attained through a fine tuning of the bare parameters; moreover, chiral currents undergo a finite renormalization, mixing among operators with different (nominal) chirality occurs, and no order parameter for the spontaneous breaking of the chiral symmetry can be defined in a natural way. ‘Accidental’ zero-modes (related to the so-called ‘exceptional configurations’) are causing troubles in Monte Carlo simulations.

In an early paper\(^3\) of lattice quantum field theory, Ginsparg and Wilson already gave the key to make definite what is meant by chiral limit in the framework of a lattice theory breaking explicitly chiral symmetry, providing a general condition – we will refer to it as to the Ginsparg-Wilson Condition (GWC) – for the fermion matrix of the lattice theory. They showed in particular that any action fulfilling the GWC reproduces the correct triangular anomaly. The condition was found by requiring that the lattice action is obtained as the fixed point (FP) action of a block-spin transformation (BST), in the universality class of a continuum chiral-symmetric action. In this way, the breaking of the symmetry is introduced in the lattice action by the BST itself and not ‘by hand’. Recently it has been shown\(^4\) that the GWC can be solved in a rather independent approach, namely in the so-called overlap formalism, and the corresponding symmetry has been identified\(^5\).

Ginsparg and Wilson’s observation had no practical applications until the technology for the construction of FP actions of lattice gauge theory was improved following Hasenfratz and Niedermayer’s ideas\(^6\)\(^\text{-}\)\(^8\). In a recent series of papers it has been shown that the GWC is a sufficient con-

\(^1\)With the precondition that very general properties of the continuum theory, like locality, are preserved.
dition for the restoration of the main features of the continuum (symmetric) theory; in particular, at the classical level, the Atiyah-Singer theorem finds correspondence on the lattice [9, 10], excluding the possibility of ‘accidental’ zero-modes; at the quantum level, no fine tuning, mixing and current renormalization occur, and a natural definition for an order parameter of the spontaneous breaking of the chiral symmetry is possible [11].

Monte Carlo calculations require actions with finite number of couplings, while the FP action is extended over all distances (even with exponential damping of the couplings). It is therefore clear that only a parametrized form of the FP action is of practical relevance. The point to be checked is to what extent such an approximation of the FP action is able to reproduce the nice properties of the ‘ideal’ FP action. The construction of the FP action of QCD is - because of technical problems - yet still far away (for some pioneering attempts, see [12]), and meanwhile we try to practice with toy models.

Recently, a parameterization of the FP action of the Schwinger model for the non-overlapping BST (noBST) (we refer to this parameterization as to $pA_{FP}$ subsequently, while to the ‘ideal’ FP action simply as to $A_{FP}$) was found [13] in a Monte Carlo approach. The fermionic action is parametrized in terms of bilinear expressions in the fermion fields, connected by paths made up of compact gauge link variables, whereas the gauge action was the non-compact simple fixed point action. Here we give a brief account of a study of the spectral properties of this Dirac operator, concerning the numerical verification of some of the chiral properties of the FP action in the case of the $pA_{FP}$. A more detailed report will be given elsewhere. We focus our attention on the spectral properties of the FP Dirac operator, verifying the lattice Atiyah-Singer theorem. A first verification of this theorem was accomplished in a different approach in [14]. Moreover, we address the problem of the extraction of the fermion condensate, both from the direct computation and through the Banks-Casher formula [15].

2 Chiral Properties of the FP Action

The GWC for the fermion matrix $h_{x,x'}$ for massless fermions reads

$$\frac{1}{2} \{ h_{x,x'}, \gamma^5 \} = (h \gamma^5 R h)_{x,x'},$$

(1)
where $R_{x,x'}$ is a local matrix in coordinate space, i.e. whose matrix elements vanish exponentially with the distance. Any FP action of a BST satisfies the GWC: $R_{x,x'}$ is fixed by the average matrix of the fermion BST, and it can be in general assumed to be trivial in Dirac space.

In the case of the non-overlapping BST (noBST) considered in [13], one has $R_{x,x'} = \frac{1}{2} \delta_{x,x'}$, and the GWC assumes the elegant form

$$h + h^\dagger = h^\dagger h = hh^\dagger .$$

(2)

where also the hermiticity property $h^\dagger = \gamma^5 h \gamma^5$ (fulfilled by the FP action in this study as well as by other actions) has been taken into account.

2.1 The spectrum of the Dirac operator

Eq. (2) implies non trivial properties of the spectrum of the Dirac operator, namely the fermion matrix $h$:

i. $[h, h^\dagger] = 0$, i.e. $h$ is a normal operator; as a consequence, its eigenvectors form a complete orthonormal set.

ii. The spectrum lies on a unitary circle in the complex plane centered at $(1, 0)$.

iii. The property (i), together with the hermiticity property of the fermion matrix, implies (denoting with $v_\lambda$ an eigenvector of $h$ with eigenvalue $\lambda$):

$$\gamma^5 v_\lambda = v_\lambda^* , \quad \text{if } \lambda \neq \lambda^*$$

$$\gamma^5 v_\lambda = \pm v_\lambda , \quad \text{if } \lambda = \lambda^* \in \mathbb{R} .$$

(3)

So, just as in the continuum the eigenvectors of complex-conjugated eigenvalues form chirality doublets; moreover all real-modes have definite chirality. (For a general FP action this property holds only for the zero-modes.)

It was shown in [10] that (1) ensures the existence of a lattice version of the Atiyah-Singer theorem, which can be stated in the form:

$$Q_{FP} = - \sum_{\{v_0\}} (v_0, \gamma^5 v_0) ,$$

(4)
where \( Q_{FP} \) is the fixed point topological charge \(^{[16, 17]}\) of the background gauge configuration. \( Q_{FP} \) provides a lattice definition of the topological charge of a gauge configuration which depends on the RG blocking procedure for the gauge sector.

### 2.2 The fermion condensate

In \(^{[9]}\), a subtraction procedure for the fermion condensate – inspired by \(^{[11]}\) – was proposed:

\[
\langle \bar{\psi} \psi \rangle_{\text{sub}} = -\frac{1}{V} \langle \text{tr}(h^{-1} - R) \rangle_{\text{gauge}},
\]

where \( V \) is the (finite) space-time volume. Because of \(^{[11]}\) the quantity in brackets of the r.h.s of the above equation vanishes, except when a zero-mode of \( h \) occurs, in which case a regulator-mass \( \mu \) must be introduced, \( h \rightarrow h_{\mu} = h + \mu \). For the number of flavors \( n_f > 1 \), the contribution of the zero-modes to the gluon average vanishes when \( \mu \rightarrow 0 \) because of the damping effect of the fermion determinant, and we conclude that the finite-volume subtracted fermion condensate vanishes in the chiral limit \(^{[9]}\). The case \( n_f = 1 \) is peculiar since \( \langle \bar{\psi} \psi \rangle_{\text{sub}} \neq 0 \) even in a finite volume. The configurations responsible in this case for the non-zero fermion condensate in a finite volume are those from the \( |Q| = 1 \) sector; indeed, if \( h \) has just one zero-mode – which is possible because of the (lattice) Atiyah-Singer theorem only for \( |Q| = 1 \) – the quantity \( \text{tr} h^{-1}(\det h) \) has a non-zero limit when \( \mu \rightarrow 0 \).

The effect of the subtraction is in general (i.e. for any \( n_f \)) just to remove the spurious contribution of the \( Q = 0 \) sector introduced by the explicit breaking of the chiral symmetry of the FP action.

The situation in the infinite volume, i.e. when the fermion condensate is obtained through the sequence of limits \( \lim_{\mu \rightarrow 0} \lim_{V \rightarrow \infty} \), is somehow different: in this case the role of the zero-modes is irrelevant,\(^2\) the quasi-zero-modes being responsible for the non-zero fermion condensate as the Banks-Casher formula shows.

We observe that \(^{[5]}\) may be rewritten in the form

\[
\langle \bar{\psi} \psi \rangle_{\text{sub}} = -\frac{1}{V} \langle \text{tr}(\tilde{h}^{-1}) \rangle_{\text{gauge}}, \quad \tilde{h} = h (1 - R h)^{-1}.
\]

The redefined fermion matrix \( \tilde{h} \) has the non trivial properties:

\[
\tilde{h}^\dagger = -\tilde{h}, \quad \{\tilde{h}, \gamma^5\} = 0,
\]

\(^2\)We are grateful to P. Hasenfratz for having driven our attention on this point.
from which it follows that \( \tilde{h} \) has a purely imaginary spectrum,
\[
\tilde{h} v = -i \tilde{\lambda} v , \quad \tilde{\lambda} \in \mathbb{R} ;
\] (8)
of course, the zero-modes of \( h \) coincide with those of \( \tilde{h} \).

The replacement \( \lambda \to \tilde{\lambda} \) allows in a sense to reconstruct the ‘chirally invariant information’ of the spectrum of \( h \). One can show that the spectral density of \( \tilde{h} \), \( \rho(\tilde{\lambda}) \) complies (in the limit: \( \lim_{a \to 0} \lim_{V \to \infty} \)) with the the Banks-Casher formula for the subtracted fermion condensate:
\[
\langle \bar{\psi} \psi \rangle_{\text{sub}} = -\pi \rho(0) .
\] (9)
In the case of the noBST, one has
\[
\tilde{\lambda} = \lambda \left(1 - \frac{\lambda}{2}\right)^{-1} \]
and the spectrum of \( \tilde{h} \) is obtained by mapping the spectrum of \( h \) (which, we recall, lies on a unitary circle in the complex plane) onto the imaginary axis by the stereographic projection.

3 Numerical Results: the Schwinger Model

The parametrized FP action \( pA_{FP} \) of the Schwinger model in [13] was obtained in the non-compact formulation for the pure-gauge sector. A first difficulty arises in this respect, since it is not possible to give a lattice definition of the topological charge in terms of the non-compact gauge variables. The most natural definition, i.e. \( 1/2\pi \sum_x F_{12}(x) \) vanishes identically on the torus. A definition in terms of compact variables is instead available, the so-called ‘geometrical’ charge:
\[
Q_G = \frac{1}{2\pi} \sum_x \text{Im} \ln(U_{12}(x)) .
\] (11)

Measuring the topological charge after the compactification of the gauge variables according to (11) reveals however an unnatural suppression of non-zero values even at moderate \( \beta \)'s, due to the non-compact nature of the gauge action.
Ideally, one should work with the FP action of the compact theory, the corresponding FP topological charge operator coinciding in this case with the expression \[11\] \[14\]. Since a parameterization of a compact FP action is not yet available, here, as an approximation, we take the parametrized FP fermion matrix of \[13\] (which is compact by construction) and simply replace the original action of the pure-gauge sector with the compact Wilson action. Of course this approximation is expected to introduce additional deviations from the behaviour of the ideal FP action \(A_{FP}\) when topological fluctuations are important in the statistical ensemble.

The fermion part of \(pA_{FP}\) has the form

\[
\bar{\psi} h_p(U) \psi = \sum_{i=0}^{3} \sum_{x,f} \rho_i(f) \bar{\psi}(x) \sigma_i U(x,f) \psi(x+\delta f).
\] (12)

Here \(h_p(U)\) is the parametrized lattice Dirac operator, \(f\) denotes a closed loop through \(x\) or a path from the lattice site \(x\) to \(x+\delta f\) (distance vector \(\delta f\)) and \(U(x,f)\) is the parallel transporter along this path. The \(\sigma_i\)-matrices denote the Pauli matrices for \(i = 1, 2, 3\) and the unit matrix for \(i = 0\). We make the identification: \(\gamma^0 = \sigma_1\), \(\gamma^1 = \sigma_2\), \(\gamma^5 = \sigma_3\). The action obeys the usual symmetries as discussed in \[13\]; altogether it has 429 terms per site.

We constructed gauge field configurations according to (a) the compact Wilson action \(S_W\) and (b) the non-compact action \(S_{NC} = (\beta/2) \sum_x F_{12}^2(x)\). In the case (a) the configurations were generated with a Metropolis Monte Carlo update, separating configurations by a number of updates of twice the size of the integrated autocorrelation length for the (geometric) topological charge, which scales with \(\beta\) exponentially (e.g. approximately \(\tau_Q \approx \exp(1.67 \beta - 3)\) for \(\beta > 2\) on \(16^2\) lattices; for comparable observations see \[18\]). The measured distributions for the topological charge agree with the observations in \[19\]. In situation (b) the configurations were generated completely independently according to the Gaussian measure (respecting the gauge d.f.).

For each of those gauge field configurations we analyzed the Dirac operator eigenvalues (and sometimes eigenvectors) and the determinant. So most of the results presented may be considered quenched results, although some of the observables have been weighted with the determinant. The latter mentioned numbers are for the full, dynamical system, although maybe plagued

\[3\] Except for a set of measure zero, when \(U_{12}(x) = -1\) for some \(x\), in which case the prescription \[11\] is not well-defined.
Figure 1: Eigenvalues of the lattice Dirac operator (12) at the values of $\beta = 2, 4, 6$ (from left to right), on $16^2$-lattices and sampled over 25 gauge configurations each, according to the compact (upper row) or non-compact (lower row) gauge action.

by the expectation value of the determinant in the denominator – as we will discuss below.

### 3.1 The spectrum

Fig. 1 shows the spectrum of the studied lattice Dirac operator (12) collectively for 25 gauge configurations (uncorrelated to the amount discussed earlier; we collected data for several hundred such configurations, but do not plot them in order to prevent the figure files becoming too large. The overall behaviour is well represented in the figures). At larger values of $\beta$ we notice excellent agreement with the expected circular shape for $A_{FP}$. For smaller $\beta$ the spectrum roughens, although the circular shape generally is kept.

One has to keep in mind that the numerically determined parametrized Dirac operator (12) is truncated and optimized on a finite sample of (non-
compact) gauge field configurations. Thus it is not surprising, that the spectral shape is closer to the optimum for the configurations sampled according the non-compact gauge action in Fig. 1.

Subsequently we discuss (except when explicitly stated differently) our results for the compact gauge action, for $n_f = 1$ and lattice volume $16^2$

We note, that there are configurations with real eigenvalues. We checked the eigenvectors for those and confirm that these modes have definite chirality, indeed, as expected from (3). Also, we can clearly distinguish the real values around zero from those around 2 (right-hand part of the spectrum). In fact each real quasi-zero-mode has three real ‘doubler’-partners at the other end of the spectrum. The chiralities of the partners add up to the opposite value of the chirality of the quasi-zero-mode(s).

We may identify these real eigenvalues (around zero) with zero-modes and relate their number $n_0$ with the geometrically (i.e. from the gauge field configuration) defined topological charge $Q_G$. We find agreement in the following sense: The ratio of the number of configurations, where these numbers coincide over those, where they do not, approaches unity in the limit $\beta \to \infty$. In fact, the approach is faster than that observed in the situation of the Wilson Dirac-operator [19]. We find that ratio to be 0.887, 0.999 and 1.000 at $\beta = 2, 4$ and 6, respectively.

Whereas $A_{FP}$ has no real modes at negative values, here we do observe those, in particular towards smaller $\beta$. This is expected due to the general roughening (cf. Fig. 1) and is equivalent to the (albeit more frequent) occurrence of ‘exceptional’ configurations for the Wilson fermion action already below $\kappa_c$.

In order to estimate the scaling behaviour of the deviations of the spectrum from the ideal circular shape, we defined a mean deviation $|\lambda - 1|$ from the unit circle in the region close to $\lambda = 0$, in an angular window of $|\arg(1-\lambda)| < \pi/4$. Fig. 2 shows the behaviour of the average width (standard deviation) $\sigma$ of that distribution with regard to $\beta$. The log-log plot demonstrates a behaviour of $\sigma \propto 1/\beta^{2.41} \approx a^5$. The parametrized action $pA_{FP}$ is truncated in a finite range (7x7 in our case). Heuristically this implies an error for the eigenvalues, considered as dimension-one gauge-invariant operators of the gauge field, in the form of some operator of higher dimension $k$.

From the observed deviation we estimate an effective value $k \approx 5$. Since similar behaviour (differing just in an overall multiplicative factor) is observed for both types of gauge action, we think that we may justify the observed deviation by the truncation.
Figure 2: Scaling behaviour of the distribution width $\sigma$ for the spectra observed in Fig. 1 a–c (full circles, compact gauge action) and Fig. 1 d–f (diamonds, non-compact gauge action) as discussed in the text. The lines are the result of a linear fit.

### 3.2 The fermion condensate

**Spectral density:** The spectral density for the projected eigenvalues $\tilde{\lambda}$ (see (10)), $\tilde{\rho}(\tilde{\lambda})$, has been calculated both in the quenched and unquenched situation. In the unquenched case, the individual configurations are weighted by the value of the determinant, which amounts to sampling in the presence of dynamical fermions. The fluctuations are in this case higher, since one has to divide by the sum over determinant values, and as is well known this involves cancellation of large terms and therefore substantial statistical errors.

The results are displayed in Fig. 3 for a sample of 2000 configurations. In the quenched case we observe in the central part of the spectrum a peak (smoothed by the binning procedure) produced by the (quasi-) zero-modes. It is removed when they are discarded from the sample (cf. Fig. 3). This peak is absent in the unquenched situation as expected: The zero-modes are suppressed due to the determinant. Apart from this, the unquenching does not change the main features of the distribution, only introducing higher fluctuations. For both, the so ‘cleaned’ distribution as well as for the full, unquenched case, we observe good agreement of $\rho(0)$ with the theoretically expected value for the condensate at infinite volume $\langle \bar{\psi}\psi \rangle = -e^7/(2\pi \sqrt{\beta \pi})$.
through the Banks-Casher relation \([10]\). The theoretical value in the finite volume case (torus) is also available \([20]\); for the physical volume here considered the correction is less than 4%.

Our conclusions from these observations are:

- The partial unquenching (by omitting the zero-modes) gives similar results as the complete unquenching (by including the determinant).

- The zero-modes produce a substantial finite-volume effect; indeed their elimination allows us to approximately recover the Casher-Banks relation even in the quenched case.

**Direct computation:** Alternatively we may attempt to derive \(\langle \bar{\psi} \psi \rangle\) directly from \(\text{tr}(\tilde{h}^{-1})\) according to \([6]\).

The standard procedure to get the infinite volume fermion condensate is to take the succession of limits \(\lim_{\eta \to 0} \lim_{V \to \infty}\). We argue that we have indeed accomplished something equivalent to this in the previous calculation with the spectral density, via the Banks-Casher formula: For that quantity, the volume effects are small (after discarding quasi zero-modes), so that the
Figure 4: We show $-\langle \bar{\psi} \psi \rangle_{\text{sub},\mu}$ determined from the same sample of configurations as discussed in Fig. 3.

$V \to \infty$ limit is effectively attained, and the limit $\tilde{\lambda} \to 0$ replaces the limit $\mu \to 0$. Note, that these two limits are complementary, since $\mu$ represents the real part of the eigenvalues of $\tilde{h}$, while $\tilde{\lambda}$ the imaginary part. One can show [21] that the two approaches, the direct computation and the one via the Banks-Casher formula, are consistent even in a finite volume, if $\mu \gg 1/V \langle \bar{\psi} \psi \rangle$.

First of all, we have checked that, as expected from the discussion of the previous section, the subtracted fermion condensate (6) is zero (within error bars) in the trivial topological sector. We then studied the full condensate, i.e. including also (quasi-) zero-modes. Fig. 4 gives $-\langle \bar{\psi} \psi \rangle_{\text{sub},\mu}$ for the discussed sample of 2000 configuration ($\beta = 4$, volume $16^2$) determined for several values of the regulator mass $\mu$. The values are of course strongly correlated since each point contains the same set of eigenvalues of $h$. The comparison with the (infinite volume) continuum value shows good agreement near $\mu = 0$. In fact, it turns out, that the regulator here is not really important. Essentially all quasi-zero-modes are not exact zero modes and so the regularization of the determinant (vanishing for an exact zero mode) and the inverse eigenvalue (diverging) is not necessary.

Obviously a better study of the finite volume dependence for the presented quantities is in order. However, already from the results presented here we conclude, that – at least for the $n_f = 1$ Schwinger model – even approximate fixed point actions provide excellent possibilities to study chiral properties.
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References

[1] H. Nielsen and M. Ninomiya, Nucl. Phys. B 185 (1981) 20; ibid. 193 (1981) 173.

[2] M. Atiyah and I. M. Singer, Ann. Math. 93 (1971) 139.

[3] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D 25 (1982) 2649.

[4] H. Neuberger, RU-98-03, hep-lat/9801031, 1998; R. Narayanan and H. Neuberger, Phys. Rev. Lett. 71 (1993) 3251; Nucl. Phys. B 433 (1993) 305.

[5] M. Lüscher, hep-lat/9802011, 1998.

[6] P. Hasenfratz and F. Niedermayer, Nucl. Phys. B 414 (1994) 785.

[7] T. DeGrand, A. Hasenfratz, P. Hasenfratz, and F. Niedermayer, Nucl. Phys. B 454 (1995) 587, 615.

[8] W. Bietenholz and U.-J. Wiese, Nucl. Phys. B 464 (1996) 319.

[9] P. Hasenfratz, Nucl. Phys.B (Proc. Suppl.) 63A-C (1997) 53.

[10] P. Hasenfratz, V. Laliena, and F. Niedermayer, BUTP-98/1, hep-lat/9801021, 1998.

[11] P. Hasenfratz, BUTP-98/4, hep-lat/9802007, 1998.

[12] T. DeGrand, Nucl. Phys. B (Proc. Suppl.) 63A-C (1998) 913.

[13] C. B. Lang and T. K. Pany, Nucl. Phys. B (Proc. Suppl.) 63A-C (1998) 898; Nucl. Phys. B 513 (1998) 645.

[14] F. Farchioni and V. Laliena, hep-lat/9802009, 1998.

[15] T. Banks and A. Casher, Nucl. Phys. 169 (1980) 103.
[16] M. Blatter, R. Burkhalter, P. Hasenfratz, and F. Niedermayer, Phys. Rev. D 53 (1996) 923. M. D’Elia, F. Farchioni, and A. Papa, Phys. Rev. D 55 (1997) 2274.

[17] T. DeGrand, A. Hasenfratz, and D. Zhu, Nucl. Phys. B 475 (1996) 321; ibid. B 478 (1996) 349.

[18] S. Elser and B. Bunk, Nucl. Phys. B (Proc. Suppl.) 63A-C (1998) 940. P. de Forcrand, J. E. Hetrick, T. Takaishi, and A. J. van der Sijs, ibid. 63A-C (1998) 679.

[19] C. R. Gattringer, I. Hip, and C. B. Lang, Phys. Lett. B 409 (1997) 371; Nucl. Phys. B 508 (1997) 329.

[20] I. Sachs and A. Wipf, Helv. Phys. Acta 65 (1992) 653.

[21] H. Leutwyler and A. Smilga, Phys. Rev. D 46 (1992) 5607.