uτ-CONVERGENCE IN LOCALLY SOLID VECTOR LATTICES

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Abstract. Let \( x_\alpha \) be a net in a locally solid vector lattice \((X, \tau)\); we say that \( x_\alpha \) is unbounded \( \tau \)-convergent to a vector \( x \in X \) if \( |x_\alpha - x| \wedge w \overset{\tau}{\to} 0 \) for all \( w \in X_+ \). In this paper, we study general properties of unbounded \( \tau \)-convergence (shortly, uτ-convergence). uτ-Convergence generalizes unbounded norm convergence and unbounded absolute weak convergence in normed lattices that have been investigated recently. Besides, we introduce uτ-topology and study briefly metrizability and completeness of this topology.

1. Introduction and preliminaries

The subject of “unbounded convergence” has attracted many researchers\cite{25, 23, 11, 13, 9, 8, 27, 15, 5, 17, 16, 12, 22}. It is well-investigated in vector lattices and normed lattices\cite{11, 14, 13, 27}. In the present paper, we study unbounded convergence in locally solid vector lattices. Results in this article extend previous works\cite{8, 13, 15, 27}.

For a net \( x_\alpha \) in a vector lattice \( X \), we write \( x_\alpha \overset{o}{\to} x \), if \( x_\alpha \) converges to \( x \) in order. This means that there is a net \( y_\beta \), possibly over a different index set, such that \( y_\beta \downarrow 0 \) and, for every \( \beta \), there exists \( \alpha_\beta \) satisfying \( |x_\alpha - x| \leq y_\beta \) whenever \( \alpha \geq \alpha_\beta \). A net \( x_\alpha \) is unbounded order convergent to a vector \( x \in X \) if \( |x_\alpha - x| \wedge u \overset{o}{\to} 0 \) for every \( u \in X_+ \). We write \( x_\alpha \overset{uo}{\to} x \) and say that \( x_\alpha \) uo-converges to \( x \). Clearly, order convergence implies uo-convergence and they coincide for order bounded nets. For a measure space \((\Omega, \Sigma, \mu)\) and for a sequence \( f_n \) in \( L_p(\mu) \) (\( 0 \leq p \leq \infty \)), \( f_n \overset{uo}{\to} 0 \) iff \( f_n \to 0 \) almost everywhere (cf.\cite{13} Rem. 3.4). It is well known that almost everywhere convergence is not topological in general\cite{18}. Therefore, the uo-convergence might not be topological. Quite recently, it has been shown that order convergence is never topological in infinite dimensional vector lattices\cite{7}.

For a net \( x_\alpha \) in a normed lattice \((X, \|\cdot\|)\), we write \( x_\alpha \overset{\|\|}{\to} x \) if \( x_\alpha \) converges to \( x \) in norm. We say that \( x_\alpha \) unbounded norm converges to \( x \in X \) (or \( x_\alpha \)

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un-converges to \( x \) if \( |x_\alpha - x| \wedge u \rightarrow 0 \) for every \( u \in X_+ \). We write \( x_\alpha \rightarrow uaw x \).

Clearly, norm convergence implies un-convergence. The un-convergence is topological, and the corresponding topology (which is known as un-topology) was investigated in [15]. A net \( x_\alpha \) is unbounded absolute weak convergent to \( x \in X \) (or \( x_\alpha \) uaw-converges to \( x \)) if \( |x_\alpha - x| \wedge u \rightarrow 0 \) for all \( u \in X_+ \), where “\( w \)” refers the weak convergence. We write \( x_\alpha \rightarrow uaw x \). Absolute weak convergence implies uaw-convergence. The notions of uaw-convergence and uaw-topology were introduced in [27].

If \( X \) is a vector lattice, and \( \tau \) is a linear topology on \( X \) that has a base at zero consisting of solid sets, then the pair \((X, \tau)\) is called a locally solid vector lattice. It should be noted that all topologies considered throughout this article are assumed to be Hausdorff. It follows from [2] Thm. 2.28 that a linear topology \( \tau \) on a vector lattice \( X \) is locally solid iff it is generated by a family \( \{\rho_j\}_{j \in J} \) of Riesz pseudonorms. Moreover, if a family of Riesz pseudonorms generates a locally solid topology \( \tau \) on a vector lattice \( X \), then \( x_\alpha \overset{\tau}{\rightarrow} x \) in \( X \) iff \( \rho_j(x_\alpha - x) \rightarrow 0 \) in \( \mathbb{R} \) for each \( j \in J \). Since \( X \) is Hausdorff, then the family \( \{\rho_j\}_{j \in J} \) of Riesz pseudonorms is separating; i.e., if \( \rho_j(x) = 0 \) for all \( j \in J \), then \( x = 0 \). In this article, unless otherwise, the pair \((X, \tau)\) refers to as a locally solid vector lattice.

A subset \( A \) in a topological vector space \((X, \tau)\) is called topologically bounded (or simply \( \tau \)-bounded) if, for every \( \tau \)-neighborhood \( V \) of zero, there exists some \( \lambda > 0 \) such that \( A \subseteq \lambda V \). If \( \rho \) is a Riesz pseudonorm on a vector lattice \( X \) and \( x \in X \), then \( \frac{1}{n}\rho(x) \leq \rho(\frac{1}{n}x) \) for all \( n \in \mathbb{N} \). Indeed, if \( n \in \mathbb{N} \) then \( \rho(x) = \rho(n\frac{1}{n}x) \leq n\rho(\frac{1}{n}x) \). The following standard fact is included for the sake of completeness.

**Proposition 1.** Let \((X, \tau)\) be a locally solid vector lattice with a family of Riesz pseudonorms \( \{\rho_j\}_{j \in J} \) that generates the topology \( \tau \). If a subset \( A \) of \( X \) is \( \tau \)-bounded then \( \rho_j(A) \) is bounded in \( \mathbb{R} \) for any \( j \in J \).

**Proof.** Let \( A \subseteq X \) be \( \tau \)-bounded and \( j \in J \). Put \( V := \{x \in X : \rho_j(x) < 1\} \). Clearly, \( V \) is a neighborhood of zero in \( X \). Since \( A \) is \( \tau \)-bounded, there is \( \lambda > 0 \) satisfying \( A \subseteq \lambda V \). Thus \( \rho_j(\lambda a) \leq 1 \) for all \( a \in A \). There exists \( n \in \mathbb{N} \) with \( n > \lambda \). Now, \( \frac{1}{n}\rho_j(a) \leq \rho_j(\frac{1}{n}a) \leq \rho_j(\frac{1}{\lambda}a) \leq 1 \) for all \( a \in A \). Hence, \( \sup_{a \in A} \rho_j(a) \leq n < \infty \). \( \square \)

Next, we discuss the converse of the proposition above.

Let \( \{\rho_j\}_{j \in J} \) be a family of Riesz pseudonorms for a locally solid vector lattice \((X, \tau)\). For \( j \in J \), let \( \tilde{\rho}_j := \rho_j / \rho_0 \). Then \( \tilde{\rho}_j \) is a Riesz pseudonorm on \( X \). Moreover, the family \( \{\tilde{\rho}_j\}_{j \in J} \) generates the topology \( \tau \) on \( X \). Clearly, \( \tilde{\rho}_j(A) \leq 1 \) for any subset \( A \) of \( X \), but still we might have a subset that is not \( \tau \)-bounded.

Recall that a locally solid vector lattice \((X, \tau)\) is said to have the Lebesque property if \( x_\alpha \downarrow 0 \) in \( X \) implies \( x_\alpha \overset{\tau}{\rightarrow} 0 \); or equivalently \( x_\alpha \overset{\sigma}{\rightarrow} 0 \) implies
$x_\alpha \overset{\tau}{\rightarrow} 0$; and $(X, \tau)$ is said to have the $\sigma$-Lebesgue property if $x_n \downarrow 0$ in $X$ implies $x_n \overset{\tau}{\rightarrow} 0$. Finally, $(X, \tau)$ is said to have the Levi property if $0 \leq x_\alpha$ and the net $x_\alpha$ is $\tau$-bounded, then $x_\alpha$ has the supremum in $X$; and $(X, \tau)$ is said to have the $\sigma$-Levi property if $0 \leq x_\alpha$ and $x_n$ is $\tau$-bounded, then $x_n$ has supremum in $X$, see [2, Def. 3.16].

Let $X$ be a vector lattice, and take $0 \neq u \in X_+$. Then a net $x_\alpha$ in $X$ is said to be $u$-uniformly convergent to a vector $x \in X$ if, for each $\varepsilon > 0$, there exists some $\alpha_\varepsilon$ such that $|x_\alpha - x| \leq \varepsilon u$ holds for all $\alpha \geq \alpha_\varepsilon$; and $x_\alpha$ is said to be $u$-uniformly Cauchy if, for each $\varepsilon > 0$, there exists some $\alpha_\varepsilon$ such that, for all $\alpha, \alpha' \geq \alpha_\varepsilon$, we have $|x_\alpha - x_{\alpha'}| \leq \varepsilon u$. A vector lattice $X$ is said to be $u$-uniformly complete if every $u$-uniformly Cauchy sequence in $X$ is $u$-uniformly convergent; and $X$ is said to be uniformly complete if $X$ is $u$-uniformly complete for each $0 \neq u \in X_+$.

Let $X$ be a vector lattice. An element $0 \neq e \in X_+$ is called a strong unit if $I_e = X$ (equivalently, for every $x \geq 0$, there exists $n \in \mathbb{N}$ such that $x \leq ne$), and $0 \neq e \in X_+$ is called a weak unit if $B_e = X$ (equivalently, $x \wedge ne \uparrow x$ for every $x \in X_+$). Here $B_e$ denotes the band generated by $e$. If $(X, \tau)$ is a topological vector lattice, then $0 \neq e \in X_+$ is called a quasi-interior point, if the principal ideal $I_e$ is $\tau$-dense in $X$ [20, Def. II.6.1]. It is known that a vector lattice $X$ is said to be an $AM$-space if $\|x \vee y\| = \max\{|x|, |y|\}$ for all $x, y \in X$ with $x \wedge y = 0$.

Let $(X, \tau)$ be a sequentially complete locally solid vector lattice. Then it follows from the proof of [3] Cor. 2.59] that it is uniformly complete. So, for each $0 \neq u \in X_+$, let $I_u$ be the ideal generated by $u$ and $\|\cdot\|_u$ be the norm on $I_u$ given by

$$\|x\|_u = \inf\{r > 0 : |x| \leq ru\} \quad (x \in X).$$

Then, by [4] Thm. 2.58], the pair $(I_u, \|\cdot\|_u)$ is a Banach lattice. Now Theorem 3.4 in [1] implies that $(I_u, \|\cdot\|_u)$ is an $AM$-space with a strong unit $u$, and then, by [1] Thm. 3.6], it is lattice isometric (uniquely, up to a homeomorphism) to $C(K)$ for some compact Hausdorff space $K$ in such a way, that the strong unit $u$ is identified with the constant function $1$ on $K$.

For unexplained terminologies and notions we refer to [2, 3].

2. UNBOUNDED $\tau$-CONVERGENCE

Suppose $(X, \tau)$ is a locally solid vector lattice. Let $x_\alpha$ be a net in $X$. We say that $x_\alpha$ is unbounded $\tau$-convergent to $x \in X$ if, for any $w \in X_+$, we have $|x_\alpha - x| \wedge w \overset{\tau}{\rightarrow} 0$. In this case, we write $x_\alpha \overset{\tau}{\rightarrow} x$ and say that $x_\alpha \tau$-converges to $x$. Obviously, if $x_\alpha \overset{\tau}{\rightarrow} x$ then $x_\alpha \overset{\tau}{\rightarrow} x$. The converse holds if the net $x_\alpha$ is order bounded. Note also that $\tau$-convergence respects linear
and lattice operations. It is clear that $u\tau$-convergence is a generalization of un-convergence \cite{15} and, of $uw\tau$-convergence \cite{27}.

Let $\mathcal{N}_\tau$ be a neighborhood base at zero consisting of solid sets for $(X, \tau)$. For each $0 \neq w \in X_+$ and $V \in \mathcal{N}_\tau$, let

$$U_{V,w} := \{ x \in X : |x| \wedge w \in V \}.$$

It can be easily shown that the collection

$$\mathcal{N}_{u\tau} := \{ U_{V,w} : V \in \mathcal{N}_\tau, 0 \neq w \in X_+ \}$$

forms a neighborhood base at zero for a locally solid topology; we call it $u\tau$-topology, where $u$ refers to as unbounded. Moreover, $x_\alpha \xrightarrow{u\tau} 0$ iff $x_\alpha \to 0$ with respect to $u\tau$-topology. Indeed, suppose $x_\alpha \xrightarrow{u\tau} 0$. Given a neighborhood $U_{V,w} \in \mathcal{N}_{u\tau}$. Then there are $0 \neq w \in X_+$ and $V \in \mathcal{N}_\tau$ such that

$$U_{V,w} = \{ x \in X : |x| \wedge w \in V \}.$$

Now, $x_\alpha \xrightarrow{u\tau} 0$ implies $|x_\alpha| \wedge w \rightarrow 0$. So, there is $\alpha_0$ such that, for all $\alpha \geq \alpha_0$, we have $|x_\alpha| \wedge w \in V$. That is $x_\alpha \in U_{V,w}$ for all $\alpha \geq \alpha_0$. Thus, $x_\alpha \to 0$ in the $u\tau$-topology.

Conversely, assume $x_\alpha \to 0$ in the $u\tau$-topology. Given $0 \neq w \in X_+$ and $V \in \mathcal{N}_\tau$. Then, $U_{V,w}$ is a zero neighborhood in the $u\tau$-topology. So, there is $\alpha'$ such that $x_\alpha \in U_{V,w}$ for all $\alpha \geq \alpha'$. That is, $|x_\alpha| \wedge w \in V$ for all $\alpha \geq \alpha'$. Thus, $|x_\alpha| \wedge w \rightarrow 0$ or $x_\alpha \xrightarrow{u\tau} 0$. The locally solid $u\tau$-topology will be referred to as unbounded $\tau$-topology.

The neighborhood base at zero for the $u\tau$-topology on $X$ has an equivalent representation in terms of a family $(\rho_j)_{j \in J}$ of Riesz pseudonorms that generates the topology $\tau$. For $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$, let $V_{\varepsilon,w,j} := \{ x \in X : \rho_j(|x| \wedge w) < \varepsilon \}$. Clearly, the collection $\{ V_{\varepsilon,w,j} : \varepsilon > 0, 0 \neq w \in X_+, j \in J \}$ generates the $u\tau$-topology.

It is known that the topology of any linear topological space can be derived from a unique translation-invariant uniformity, i.e., any linear topological space is uniformisable (cf. \cite{21} Thm. 1.4]). It follows from \cite{10} Thm. 8.1.20 that any linear topological space is completely regular. In particular, the unbounded $\tau$-convergence is completely regular.

Since $x_\alpha \xrightarrow{\tau} 0$ implies $x_\alpha \xrightarrow{u\tau} 0$, then the $\tau$-topology in general is finer than $u\tau$-topology. The next result should be compared with \cite{15} Lm. 2.1.

**Lemma 1.** Let $(X, \tau)$ be a sequentially complete locally solid vector lattice, where $\tau$ is generated by a family $(\rho_j)_{j \in J}$ of Riesz pseudonorms. Let $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$. Then either $V_{\varepsilon,w,j}$ is contained in $[-w, w]$, or it contains a non-trivial ideal.

**Proof.** Suppose that $V_{\varepsilon,w,j}$ is not contained in $[-w, w]$. Then there exists $x \in V_{\varepsilon,w,j}$ such that $x \notin [-w, w]$. Replacing $x$ with $|x|$, we may assume $x > 0$. Since $x \notin [-w, w]$, then $y = (x - w)^+ > 0$. Now, letting $z = x \vee w$, we have that the ideal $I_z$ generated by $z$, is lattice and norm isomorphic to
$C(K)$ for some compact and Hausdorff space $K$, where $z$ corresponds to the constant function $1$. Also $x$, $y$, and $w$ in $I_2$ correspond to $x(t)$, $y(t)$, and $w(t)$ in $C(K)$ respectively.

Our aim is to show that for all $\alpha \geq 0$ and $t \in K$, we have

$$(\alpha y)(t) \wedge w(t) \leq x(t) \wedge w(t).$$

For this, note that $y(t) = (x - w)^+(t) = (x - w)(t) \vee 0$.

Let $t \in K$ be arbitrary.

- Case (1): If $(x - w)(t) > 0$, then $x(t) \wedge w(t) = w(t) \geq (\alpha y)(t) \wedge w(t)$ for all $\alpha \geq 0$, as desired.
- Case (2): If $(x - w)(t) < 0$, then $(\alpha y)(t) \wedge w(t) \leq (\alpha y)(t) = \alpha(x - w)(t) \vee 0 = 0 \leq x(t) \wedge w(t)$, as desired.

Hence, for all $\alpha \geq 0$ and $t \in K$, we have $(\alpha w)(t) \wedge w(t) \leq x(t) \wedge w(t)$ and so $(\alpha y) \wedge w \leq x \wedge w$ for all $\alpha \geq 0$. Note, that $\alpha y, w, x \in X_+$. Thus $\rho_j(\alpha y | \wedge w) \leq \rho_j(|x| \wedge w) < \varepsilon$, so $\alpha y \in V_{\varepsilon,w,j}$ and, since $V_{\varepsilon,w,j}$ is solid, then $I_2 \subseteq V_{\varepsilon,w,j}$.

Note that the sequential completeness in Lemma 1 can be removed, as we see in the following corollary.

**Theorem 1.** Let $(X, \tau)$ be a locally solid vector lattice, where $\tau$ is generated by a family $(\rho_j)_{j \in J}$ of Riesz pseudonorms. Let $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$. Then either $V_{\varepsilon,w,j}$ is contained in $[-w, w]$ or $V_{\varepsilon,w,j}$ contains a non-trivial ideal.

**Proof.** Given $\varepsilon > 0$, $j \in J$, and $0 \neq w \in X_+$. Let $(\hat{X}, \hat{\tau})$ be the topological completion of $(X, \tau)$. In particular, $(\hat{X}, \hat{\tau})$ is sequentially complete. Let $V_{\varepsilon,w,j} = \{ \hat{x} \in \hat{X} : \rho_j(\hat{x} \wedge w) < \varepsilon \}$. Then $V_{\varepsilon,w,j} = X \cap V_{\varepsilon,w,j}$. By Lemma 1 either $V_{\varepsilon,w,j}$ is a subset of $[-w, w]_\hat{X}$ in $\hat{X}$ or $V_{\varepsilon,w,j}$ contains a non-trivial ideal of $\hat{X}$. If $V_{\varepsilon,w,j} \subseteq [-w, w]_\hat{X}$, then

$$V_{\varepsilon,w,j} = X \cap V_{\varepsilon,w,j} \subseteq X \cap [-w, w]_\hat{X} = [-w, w]_\hat{X}.$$ 

If $V_{\varepsilon,w,j}$ contains a non-trivial ideal, then $V_{\varepsilon,w,j} \not\subseteq [-w, w]_\hat{X}$. So, there is $\hat{x} \in V_{\varepsilon,w,j}$ with $\hat{x} \notin [-w, w]_\hat{X}$. Since $[-w, w]_\hat{X}$ is $\hat{\tau}$-closed, then there is a solid neighborhood $N_{\hat{x}}$ of $\hat{x}$ in $\hat{X}$ such that $N_{\hat{x}} \cap [-w, w]_\hat{X} = \emptyset$. Hence, $N_{\hat{x}} \cap \hat{V}_{\varepsilon,w,j} \cap [-w, w]_\hat{X} = \emptyset$, and $N_{\hat{x}} \cap \hat{V}_{\varepsilon,w,j}$ is open in $\hat{X}$ with $\hat{x} \in N_{\hat{x}} \cap \hat{V}_{\varepsilon,w,j}$. By $\tau$-density of $X$ in $\hat{X}$, we may take $x \in X \cap N_{\hat{x}} \cap \hat{V}_{\varepsilon,w,j}$. Since $|x| \in X \cap N_{\hat{x}} \cap \hat{V}_{\varepsilon,w,j}$, we may also assume that $x \in X_+$.

Let $y := (x - w)^+$, then $y > 0$ and $y \in X_+$. By the same argument in Lemma 1, we get $(\alpha y) \wedge w \leq x \wedge w$ for all $\alpha \in \mathbb{R}_+$. Since $x \in \hat{V}_{\varepsilon,w,j}$, then $\alpha y \in \hat{V}_{\varepsilon,w,j}$ for all $\alpha \in \mathbb{R}_+$. But $\alpha y \in X_+$ for all $\alpha \in \mathbb{R}_+$ and, since $V_{\varepsilon,w,j} = X \cap \hat{V}_{\varepsilon,w,j}$, we get $\alpha y \in V_{\varepsilon,w,j}$ for all $\alpha \in \mathbb{R}_+$. Since $V_{\varepsilon,w,j}$ is solid, we conclude that the principal ideal $I_y$ taken in $X$ is a subset of $V_{\varepsilon,w,j}$. □
Lemma 2. Let \((X, \tau)\) be a locally solid vector lattice, where \(\tau\) is generated by a family \((\rho_j)_{j \in J}\) of Riesz pseudonorms. If \(V_{\varepsilon,w,j}\) is contained in \([-w,w]\), then \(w\) is a strong unit.

Proof. Suppose \(V_{\varepsilon,w,j} \subseteq [-w,w]\). Since \(V_{\varepsilon,w,j}\) is absorbing, for any \(x \in X_+\), there exist \(\alpha > 0\) such that \(\alpha x \in V_{\varepsilon,w,j}\), and so \(\alpha x \in [-w,w]\), or \(x \leq \frac{1}{\alpha} w\). Thus \(w\) is a strong unit, as desired. \(\square\)

Proposition 2. Let \(e \in X_+\). Then \(e\) is a quasi-interior point in \((X, \tau)\) if \(e\) is a quasi-interior point in the topological completion \((\hat{X}, \hat{\tau})\).

Proof. The backward implication is trivial.

For the forward implication let \(\hat{x} \in \hat{X}_+\). Our aim is to show that \(\hat{x} - \hat{x} \wedge ne \overset{\hat{\tau}}{\to} 0\) in \(\hat{X}\) as \(n \to \infty\). By \([2]\) Thm. 2.40, \(\hat{X}_+ = X^+_\perp\). So, there is a net \(x_\alpha\) in \(X_+\) such that \(x_\alpha \overset{\tau}{\to} \hat{x}\) in \(\hat{X}\). Let \(j \in J\) and \(\varepsilon > 0\). Since \(\hat{\rho}_j(x_\alpha - \hat{x}) \to 0\), then there is \(\alpha_\varepsilon\) satisfying

\[
(2.1) \quad \hat{\rho}_j(x_{\alpha_\varepsilon} - \hat{x}) < \varepsilon.
\]

Since \(e\) is a quasi-interior point in \(X\) and \(x_{\alpha_\varepsilon} \in X_+\), then \(x_{\alpha_\varepsilon} - x_{\alpha_\varepsilon} \wedge ne \overset{\tau}{\to} 0\) in \(X\) as \(n \to \infty\). Thus, there is \(n_\varepsilon \in \mathbb{N}\) such that

\[
(2.2) \quad \hat{\rho}_j(x_{\alpha_\varepsilon} - ne \wedge x_{\alpha_\varepsilon}) = \hat{\rho}_j(x_{\alpha_\varepsilon} - ne \wedge x_{\alpha_\varepsilon}) < \varepsilon \quad (\forall n \geq n_\varepsilon).
\]

Now, \(0 \leq \hat{x} - \hat{x} \wedge ne = \hat{x} - x_{\alpha_\varepsilon} + x_{\alpha_\varepsilon} - ne \wedge x_{\alpha_\varepsilon} + ne \wedge x_{\alpha_\varepsilon} - \hat{x} \wedge ne\). So \(\hat{\rho}_j(\hat{x} - \hat{x} \wedge ne) \leq \hat{\rho}_j(\hat{x} - x_{\alpha_\varepsilon}) + \hat{\rho}_j(x_{\alpha_\varepsilon} - ne \wedge x_{\alpha_\varepsilon}) + \hat{\rho}_j(ne \wedge x_{\alpha_\varepsilon} - \hat{x} \wedge ne)\).

For \(n \geq n_\varepsilon\), we have, by \((2.1)\), \((2.2)\), and \([3]\) Thm. 1.9(2), that

\[
\hat{\rho}_j(\hat{x} - \hat{x} \wedge ne) \leq \varepsilon + \varepsilon + \hat{\rho}_j(x_{\alpha_\varepsilon} - \hat{x}) \leq 3\varepsilon.
\]

Therefore, \(e\) is a quasi-interior point in \(\hat{X}\). \(\square\)

The technique used in the proof of \([15]\) Thm. 3.1] can be used in the following theorem as well, and so we omit its proof.

Theorem 2. Let \((X, \tau)\) be a sequentially complete locally solid vector lattice, where \(\tau\) is generated by a family \((\rho_j)_{j \in J}\) of Riesz pseudonorms. Let \(e \in X_+\). The following are equivalent:

1. \(e\) is a quasi-interior point;
2. for every net \(x_\alpha\) in \(X_+\), if \(x_\alpha \wedge e \overset{\tau}{\to} 0\) then \(x_\alpha \overset{u\tau}{\to} 0\);
3. for every sequence \(x_n\) in \(X_+\), if \(x_n \wedge e \overset{\tau}{\to} 0\) then \(x_n \overset{u\tau}{\to} 0\).

3. Unbounded \(\tau\)-convergence in sublattices

Let \(Y\) be a sublattice of a locally solid vector lattice \((X, \tau)\). If \(y_\alpha\) is a net in \(Y\) such that \(y_\alpha \overset{u\tau}{\to} 0\) in \(X\), then clearly, \(y_\alpha \overset{u\tau}{\to} 0\) in \(Y\). The converse does not hold in general. For example, the sequence \(e_n\) of standard unit vectors is \(un\)-null in \(e_0\), but not in \(\ell_\infty\). In this section, we study when the \(u\tau\)-convergence passes from a sublattice to the whole space.
Recall that a sublattice $Y$ of a vector lattice $X$ is majorizing if, for every $x \in X_+$, there exists $y \in Y_+$ with $x \leq y$. The following theorem extends [15, Thm. 4.3] to locally solid vector lattices.

**Theorem 3.** Let $(X, \tau)$ be a locally solid vector lattice and $Y$ be a sublattice of $X$. If $y_\alpha$ is a net in $Y$ and $y_\alpha \xrightarrow{\text{ut}} 0$ in $Y$, then $y_\alpha \xrightarrow{\text{ut}} 0$ in $X$ in each of the following cases:

1. $Y$ is majorizing in $X$;
2. $Y$ is $\tau$-dense in $X$;
3. $Y$ is a projection band in $X$.

**Proof.**

(1) Trivial.

(2) Let $u \in X_+$. Fix $\epsilon > 0$ and take $j \in J$. Since $Y$ is $\tau$-dense in $X$, then there is $v \in Y_+$ such that $\rho_j(u - v) < \epsilon$. But $y_\alpha \xrightarrow{\text{ut}} 0$ in $Y$ and so, in particular, $\rho_j(|y_\alpha| \wedge v) \rightarrow 0$. So there is $\alpha_0$ such that $\rho_j(|y_\alpha| \wedge v) < \epsilon$ for all $\alpha \geq \alpha_0$. It follows from $u \leq v + |u - v|$, that $|y_\alpha| \wedge u \leq |y_\alpha| \wedge v + |u - v|$, and so $\rho_j(|y_\alpha| \wedge u) \leq \rho_j(|y_\alpha| \wedge v) + \rho_j(u - v) < 2\epsilon$. Thus, $\rho_j(|y_\alpha| \wedge u) \rightarrow 0$ in $\mathbb{R}$. Since $j \in J$ was chosen arbitrary, we conclude that $y_\alpha \xrightarrow{\text{ut}} 0$ in $X$.

(3) Let $u \in X_+$. Then $u = v + w$, where $v \in Y_+$ and $w \in Y_+^d$. Now $|y_\alpha| \wedge u = |y_\alpha| \wedge v + |y_\alpha| \wedge w = |y_\alpha| \wedge v$, since $y_\alpha \in Y$. Then $|y_\alpha| \wedge u \xrightarrow{\text{ut}} 0$ in $X$.

**Corollary 1.** If $(X, \tau)$ is a locally solid vector lattice and $x_\alpha \xrightarrow{\text{ut}} 0$ in $X$, then $x_\alpha \xrightarrow{\text{ut}} 0$ in the Dedekind completion $X^\delta$ of $X$.

**Corollary 2.** If $(X, \tau)$ is a locally solid vector lattice and $x_\alpha \xrightarrow{\text{ut}} 0$ in $X$, then $x_\alpha \xrightarrow{\text{ut}} 0$ in the topological completion $\hat{X}$ of $X$.

The next result generalizes Corollary 4.6 in [15] and Proposition 16 in [27].

**Theorem 4.** Let $(X, \tau)$ be a topologically complete locally solid vector lattice that possesses the Lebesgue property, and $Y$ be a sublattice of $X$. If $y_\alpha \xrightarrow{\text{ut}} 0$ in $Y$, then $y_\alpha \xrightarrow{\text{ut}} 0$ in $X$.

**Proof.** Suppose $y_\alpha \xrightarrow{\text{ut}} 0$ in $Y$. By Theorem [31], $y_\alpha \xrightarrow{\text{ut}} 0$ in the ideal $I(Y)$ generated by $Y$ in $X$. By Theorem [32], $y_\alpha \xrightarrow{\text{ut}} 0$ in the closure $\overline{I(Y)}$ of $I(Y)$. It follows from [2] Thm. 3.7 that $\overline{I(Y)}$ is a band in $X$. Now, [2] Thm. 3.24 assures that $X$ is Dedekind complete, and so $\overline{I(Y)}$ is a projection band in $X$. Then $y_\alpha \xrightarrow{\text{ut}} 0$ in $X$, in view of Theorem [33].

Suppose that $(X, \tau)$ is a locally solid vector lattice possessing the Lebesgue property. Then, in view of [2] Thms. 3.23 and 3.26, its topological completion $(\hat{X}, \hat{\tau})$ possesses the Lebesgue property as well. Hence, by [2] Thm.
3.24], \( \hat{X} \) is Dedekind complete. Since \( X \subseteq \hat{X} \), there holds \( X^\delta \subseteq (\hat{X})^\delta = \hat{X} \).
So, \( X \subseteq X^\delta \subseteq \hat{X} \). Now, Theorem 4 assures that, given a net \( z_\alpha \) in \( X^\delta \), if \( z_\alpha \xrightarrow{ur} 0 \) in \( X^\delta \) then \( z_\alpha \xrightarrow{ur} 0 \) in \( \hat{X} \).

4. UNBOUNDED RELATIVELY UNFORMLY CONVERGENCE

In this section we discuss unbounded relatively uniformly convergence. Recall that a net \( x_\alpha \) in a vector lattice \( X \) is said to be relatively uniformly convergent to \( x \in X \) if, there is \( u \in X_+ \) such that for any \( n \in \mathbb{N} \), there exists \( \alpha_n \) satisfying \( |x_\alpha - x| \leq \frac{1}{n} u \) for \( \alpha \geq \alpha_n \). In this case we write \( x_\alpha \xrightarrow{ru} x \) and the vector \( u \in X_+ \) is called regulator, see [24 Def. III.11.1].

If \( x_\alpha \xrightarrow{ru} 0 \) in a locally solid vector lattice \( (X, \tau) \), then \( x_\alpha \xrightarrow{\tau} 0 \). Indeed, let \( V \) be a solid neighborhood at zero. Since \( x_\alpha \xrightarrow{ru} 0 \), then there is \( u \in X_+ \) such that, for a given \( \varepsilon > 0 \), there is \( \alpha_\varepsilon \) satisfying \( |x_\alpha| \leq \varepsilon u \) for all \( \alpha \geq \alpha_\varepsilon \). Since \( V \) is absorbing, there is \( c \geq 1 \) such that \( \frac{1}{c} u \in V \). There is some \( \alpha_0 \) such that \( |x_\alpha| \leq \frac{1}{c} u \) for all \( \alpha \geq \alpha_0 \). Since \( V \) is solid and \( |x_\alpha| \leq \frac{1}{c} u \) for all \( \alpha \geq \alpha_0 \), then \( x_\alpha \in V \) for all \( \alpha \geq \alpha_0 \). That is \( x_\alpha \xrightarrow{\tau} 0 \).

The following result might be considered as an \( ru \)-version of Theorem 1 in [7].

**Theorem 5.** Let \( X \) be a vector lattice. Then the following conditions are equivalent.

1. There exists a linear topology \( \tau \) on \( X \) such that, for any net \( x_\alpha \) in \( X \): \( x_\alpha \xrightarrow{ru} 0 \) iff \( x_\alpha \xrightarrow{\tau} 0 \).

2. There exists a norm \( \| \cdot \| \) on \( X \) such that, for any net \( x_\alpha \) in \( X \): \( x_\alpha \xrightarrow{ru} 0 \) iff \( \| x_\alpha \| \to 0 \).

3. \( X \) has a strong order unit.

**Proof.** (1) \( \Rightarrow \) (3) It follows from [7 Lem. 1].

(3) \( \Rightarrow \) (2) Let \( e \in X \) be a strong order unit. Then \( x_\alpha \xrightarrow{ru} 0 \) iff \( \| x_\alpha \|_e \to 0 \), where \( \| x \|_e := \inf \{ r : |x| \leq re \} \).

(2) \( \Rightarrow \) (1) It is trivial. \( \square \)

Let \( X \) be a vector lattice. A net \( x_\alpha \) in \( X \) is said to be unbounded relatively uniformly convergent to \( x \in X \) if \( |x_\alpha - x| \wedge w \xrightarrow{ru} 0 \) for all \( w \in X_+ \). In this case, we write \( x_\alpha \xrightarrow{uru} x \). Clearly, if \( x_\alpha \xrightarrow{uru} 0 \) in a locally solid vector lattice \( (X, \tau) \), then \( x_\alpha \xrightarrow{uru} 0 \).

In general, \( uru \)-convergence is also not topological. Indeed, consider the vector lattice \( L_1[0,1] \). It satisfies the diagonal property for order convergence by [19 Thm. 71.8]. Now, by combining Theorems 16.3, 16.9, and 68.8 in [19] we get that for any sequence \( f_n \) in \( L_1[0,1] \) \( f_n \xrightarrow{a.e.} 0 \) iff \( f_n \xrightarrow{ru} 0 \). In particular, \( f_n \xrightarrow{uo} 0 \) iff \( f_n \xrightarrow{uru} 0 \). But the \( wo \)-convergence in \( L_1[0,1] \) is equivalent to a.e.-convergence which is not topological, see [18].
However, in some vector lattices the uru-convergence could be topological. For example, if $X$ is a vector lattice with a strong unit $e$, it follows from Theorem 5 that ru-convergence is equivalent to the norm convergence $\|\cdot\|_e$, where $\|x\|_e := \inf\{\lambda > 0 : |x| \leq \lambda e\}, x \in X$. Thus uru-convergence in $X$ is topological.

Consider vector lattice $c_{00}$ of eventually zero sequences. It is well known that in $c_{00}$: $x_\alpha \xrightarrow{ru} 0$ iff $x_\alpha \xrightarrow{o} 0$. For the sake of completeness we include a proof of this fact. Clearly, $x_\alpha \xrightarrow{ru} 0 \Rightarrow x_\alpha \xrightarrow{o} 0$. For the converse, suppose $x_\alpha \xrightarrow{o} 0$ in $c_{00}$. Then there is a net $y_\beta \downarrow 0$ in $c_{00}$ such that, for any $\beta$, there is $\alpha_\beta$ satisfying $|x_\alpha| \leq y_\beta$ for all $\alpha \geq \alpha_\beta$. Let $e_n$ denote the sequence of standard unit vectors in $c_{00}$. Fix $\beta_0$. Then $y_{\beta_0} = c_1^\beta_0 e_{k_1} + \cdots + c_n^\beta_0 e_{k_n}$, $c_i^\beta_0 \in \mathbb{R}, i = 1, \ldots, n$. Since $y_\beta$ is decreasing, then $y_\beta \leq y_{\beta_0}$ for all $\beta \geq \beta_0$. So, $y_\beta = c_1^\beta e_{k_1} + \cdots + c_n^\beta e_{k_n}$ for all $\beta \geq \beta_0, c_i^\beta \in \mathbb{R}, i = 1, \ldots, n$. Since $y_\beta \downarrow 0$ then $\lim_{\beta \to 0} c_i^\beta = 0$ for all $i = 1, \ldots, n$. Let $u = e_{k_1} + \cdots + e_{k_n}$. Given $\varepsilon > 0$. Then, there is $\beta_\varepsilon \geq \beta_0$ such that $c_i^\beta < \varepsilon$ for all $\beta \geq \beta_\varepsilon$ for $i = 1, \ldots, n$. Consider $y_{\beta_\varepsilon}$, then there is $\alpha_\varepsilon$ such that $|x_\alpha| \leq y_{\beta_\varepsilon}$ for all $\alpha \geq \alpha_\varepsilon$. But $y_{\beta_\varepsilon} = c_1^\beta e_{k_1} + \cdots + c_n^\beta e_{k_n} \leq \varepsilon u$. So, $|x_\alpha| \leq \varepsilon u$ for all $\alpha \geq \alpha_\varepsilon$. That is $x_\alpha \xrightarrow{ru} 0$. Thus, the uru-convergence in $c_{00}$ coincides with the wo-convergence which is pointwise convergence and, therefore, is topological.

**Proposition 3.** Let $X$ be Lebesgue and complete metrizable locally solid vector lattice. then $x_\alpha \xrightarrow{ru} 0$ if $x_\alpha \xrightarrow{o} 0$.

**Proof.** The necessity is obvious. For the sufficiency assume that $x_\alpha \xrightarrow{o} 0$. Then there exists $y_\beta \downarrow 0$ such that for any $\beta$ there is $\alpha_\beta$ with $|x_\alpha| \leq y_\beta$ as $\alpha \geq \alpha_\beta$. Since $d(y_\beta, 0) \to 0$, there exists an increasing sequence $(\beta_k)_k$ of indeces with $d(ky_{\beta_k}, 0) \leq \frac{1}{2^k}$. Let $s_n = \sum_{k=1}^n k y_{\beta_k}$. We show the sequence $s_n$ is Cauchy. For $n > m$,

$$d(s_n, s_m) = d(s_n - s_m, 0) = d\left(\sum_{k=m+1}^n ky_{\beta_k}, 0\right) \leq \sum_{k=m+1}^n d(ky_{\beta_k}, 0) \leq \sum_{k=m+1}^n \frac{1}{2^k} \to 0, \text{ as } n, m \to \infty.$$ 

Since $X$ is complete, then the sequence $s_n$ converges to some $u \in X_+$. That is, $u := \sum_{k=1}^\infty ky_{\beta_k}$. Then

$$k|x_\alpha| \leq ky_{\beta_k} \leq u \quad (\forall \alpha \geq \alpha_{\beta_k})$$

which means that $x_\alpha \xrightarrow{ru} 0$. \square

Let $X = \mathbb{R}^\Omega$ be the vector lattice of all real-valued functions on a set $\Omega$. 

Proposition 4. In the vector lattice $X = \mathbb{R}^\Omega$, the following conditions are equivalent:

1. for any net $f_\alpha$ in $X$: $f_\alpha \xrightarrow{o} 0$ iff $f_\alpha \xrightarrow{ru} 0$;
2. $\Omega$ is countable.

Proof. (1) $\Rightarrow$ (2) Suppose $f_\alpha \xrightarrow{o} 0 \iff f_\alpha \xrightarrow{ru} 0$ for any sequence $f_\alpha$ in $X = \mathbb{R}^\Omega$. Our aim is to show that $\Omega$ is countable. Assume, in contrary, that $\Omega$ is uncountable. Let $F(\Omega)$ be the collection of all finite subsets of $\Omega$. For each $\alpha \in F(\Omega)$, put $f_\alpha = \chi_\alpha$. Clearly, $f_\alpha \uparrow 1$, where $1$ denotes the constant function one on $\Omega$. Then $1 - f_\alpha \downarrow 0$ or $1 - f_\alpha \xrightarrow{o} 0$ in $\mathbb{R}^\Omega$. So, there is $0 \leq g \in \mathbb{R}^\Omega$ such that, for any $\varepsilon > 0$, there exists $\alpha_\varepsilon$ satisfying $1 - f_\alpha \leq \varepsilon g$ for all $\alpha \geq \alpha_\varepsilon$. Let $n \in \mathbb{N}$. Then there is a finite set $\alpha_n \subseteq \Omega$ such that $1 - f_{\alpha_n} \leq \frac{1}{n} g$. Consequently, $g(x) \geq n$ for all $x \in \Omega \setminus \alpha_n$. Let $S = \cup_{n=1}^\infty \alpha_n$. Then $S$ is countable and $\Omega \setminus S \neq \emptyset$. Moreover, for each $x \in \Omega \setminus S$, we have $g(x) \geq n$ for all $n \in \mathbb{N}$, which is impossible.

(2) $\Rightarrow$ (1) Suppose that $\Omega$ is countable. So, we may assume that $X = s$, the space of all sequences. Since, from $x_\alpha \xrightarrow{ru} 0$ always follows that $x_\alpha \xrightarrow{o} 0$, it is enough to show that if $x_\alpha \xrightarrow{o} 0$ then $x_\alpha \xrightarrow{ru} 0$. To see this, let $(x_\alpha^n)_n = x_\alpha \xrightarrow{o} 0$. Then, the net $x_\alpha$ is eventually bounded, say $|x_\alpha| \leq u = (u_n)_n \in s$. Take $w := (nu_n)_n \in s$. We show that $x_\alpha \xrightarrow{ru} 0$ with the regulator $w$. Let $k \in \mathbb{N}$. Since $x_\alpha \xrightarrow{o} 0$, then for each $n \in \mathbb{N}$, $x_\alpha^n \xrightarrow{o} 0$ in $\mathbb{R}$. Hence, there is $\alpha_k$ such that $k|x_\alpha^n| < u_1$, $k|x_\alpha^2| < u_2$, $\cdots$, $k|x_\alpha^{k-1}| < u_{k-1}$ for all $\alpha \geq \alpha_k$. Note that for $n \geq k$, $k|x_\alpha^n| < u_n$. Therefore, $k|x_\alpha| < w$ for all $\alpha \geq \alpha_k$. \]

It follows from Proposition 4 that, for countable $\Omega$, the uru-convergence in $\mathbb{R}^\Omega$ coincides with the uo-convergence (which is pointwise) and therefore is topological. We do not know, whether or not the countability of $\Omega$ is necessary for the property that uru-convergence is topological in $\mathbb{R}^\Omega$.

5. Topological orthogonal systems and metrizability

A collection $\{e_\gamma\}_{\gamma \in \Gamma}$ of positive vectors in a vector lattice $X$ is called an orthogonal system if $e_\gamma \wedge e_{\gamma'} = 0$ for all $\gamma \neq \gamma'$. If, moreover, $x \wedge e_\gamma = 0$ for all $\gamma \in \Gamma$ implies $x = 0$, then $\{e_\gamma\}_{\gamma \in \Gamma}$ is called a maximal orthogonal system. It follows from Zorn’s Lemma that every vector lattice containing at least one non-zero element has a maximal orthogonal system. Motivated by Definition III.5.1 in [20], we introduce the following notion.

Definition 1. Let $(X, \tau)$ be a topological vector lattice. An orthogonal system $Q = \{e_\gamma\}_{\gamma \in \Gamma}$ of non-zero elements in $X_+$ is said to be a topological orthogonal system if the ideal $I_Q$ generated by $Q$ is $\tau$-dense in $X$.

Lemma 3. If $Q = \{e_\gamma\}_{\gamma \in \Gamma}$ is a topological orthogonal system in a topological vector lattice $(X, \tau)$, then $Q$ is a maximal orthogonal system in $X$. 
Proof. Assume $x \wedge e_\gamma = 0$ for all $\gamma \in \Gamma$. By the assumption, there is a net $x_\alpha$ in the ideal $I_Q$ such that $x_\alpha \xrightarrow{I} x$. Without lost of generality, we may assume $0 \leq x_\alpha \leq x$ for all $\alpha$. Since $x_\alpha \in I_Q$, then there are $0 < \mu_\alpha \in \mathbb{R}$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$, such that $0 \leq x_\alpha \leq \mu_\alpha(e_{\gamma_1} + e_{\gamma_2} + \cdots + e_{\gamma_n})$. So $0 \leq x_\alpha = x_\alpha \wedge x \leq \mu_\alpha(e_{\gamma_1} + e_{\gamma_2} + \cdots + e_{\gamma_n}) \wedge x = \mu_\alpha e_{\gamma_1} \wedge x + \cdots + \mu_\alpha e_{\gamma_n} \wedge x = 0$. Hence $x_\alpha = 0$ for all $\alpha$, and so $x = 0$. \hfill \Box

We recall the following construction from [20, p.169]. Let $X$ be a vector lattice and $Q = \{e_\gamma\}_{\gamma \in \Gamma}$ be a maximal orthogonal system of $X$. Let $\mathcal{F}(\Gamma)$ denote the collection of all finite subsets of $\Gamma$ ordered by inclusion. For each $(n, H) \in \mathbb{N} \times \mathcal{F}(\Gamma)$ and $x \in X_+$, define

$$x_{n,H} := \sum_{\gamma \in H} x \wedge ne_\gamma.$$ 

Clearly $\{x_{n,H} : (n, H) \in \mathbb{N} \times \mathcal{F}(\Gamma)\}$ is directed upward, and

$$x_{n,H} \leq x \quad \text{for all} \quad (n, H) \in \mathbb{N} \times \mathcal{F}(\Gamma).$$

Moreover, Proposition II.1.9 in [20] implies $x_{n,H} \uparrow x$.

**Theorem 6.** Let $Q = \{e_\gamma\}_{\gamma \in \Gamma}$ be an orthogonal system of a locally solid vector lattice $(X, \tau)$, then $Q$ is a topological orthogonal system iff we have $x_{n,H} \xrightarrow{I} x$ over $(n, H) \in \mathbb{N} \times \mathcal{F}(\Gamma)$ for each $x \in X_+$.

Proof. For the backward implication take $x \in X_+$. Since

$$x_{n,H} = \sum_{\gamma \in H} x \wedge ne_\gamma \leq n \sum_{\gamma \in H} e_\gamma,$$

then $x_{n,H} \in I_Q$ for each $(n, H) \in \mathbb{N} \times \mathcal{F}(\Gamma)$. Also, we have, by assumption, $x_{n,H} \xrightarrow{I} x$. Thus, $x \in T_Q$, i.e., $Q$ is a topological orthogonal system of $X$.

For the forward implication, note that $Q$ is a maximal orthogonal system, by Lemma 3. Let $x \in X_+$, and $j \in J$. Given $\varepsilon > 0$. Let $V_{\varepsilon,x,j} := \{z \in X : \rho_j(z - x) < \varepsilon\}$. Then $V_{\varepsilon,x,j}$ is a neighborhood of $x$ in the $\tau$-topology. Since $I_Q$ is dense in $X$ with respect to the $\tau$-topology, there is $x_{\varepsilon} \in I_Q$ with $0 \leq x_{\varepsilon} \leq x$ such that $\rho_j(x_{\varepsilon} - x) < \varepsilon$. Now, $x_{\varepsilon} \in I_Q$ implies that there are $H_{\varepsilon} \in \mathcal{F}(\Gamma)$ and $n_{\varepsilon} \in \mathbb{N}$ such that

$$x_{\varepsilon} \leq n_{\varepsilon} \sum_{\gamma \in H_{\varepsilon}} e_\gamma.$$ 

Let

$$w := x \wedge \sum_{\gamma \in H_{\varepsilon}} n_{\varepsilon}e_\gamma.$$ 

It follows from $0 \leq w \leq \sum_{\gamma \in H_{\varepsilon}} n_{\varepsilon}e_\gamma$ and the Riesz decomposition property, that, for each $\gamma \in H_{\varepsilon}$, there exists $y_\gamma$ with

$$0 \leq y_\gamma \leq n_{\varepsilon}e_\gamma.$$
such that

\[ w = \sum_{\gamma \in H_{\varepsilon}} y_{\gamma}. \]

From (5.3) and (5.5), we have

\[ y_{\gamma} \leq x \quad (\forall \gamma \in H_{\varepsilon}). \]

Also, (5.4) and (5.6) imply that \( y_{\gamma} \leq n_{\varepsilon} e_{\gamma} \wedge x \). Now

\[ w = \sum_{\gamma \in H_{\varepsilon}} y_{\gamma} \leq \sum_{\gamma \in H_{\varepsilon}} x \wedge n_{\varepsilon} e_{\gamma} = x_{n_{\varepsilon}, H_{\varepsilon}}. \]

But, from (5.2) and (5.3), we get

\[ 0 \leq x_{\varepsilon} \leq w. \]

Thus, it follows from (5.7), (5.8), and (5.1), that \( 0 \leq x_{\varepsilon} \leq x_{n_{\varepsilon}, H_{\varepsilon}} \leq x \).

Hence, \( 0 \leq x - x_{n_{\varepsilon}, H_{\varepsilon}} \leq x - x_{\varepsilon} \) and so \( \rho_{j}(x - x_{n_{\varepsilon}, H_{\varepsilon}}) \leq \rho_{j}(x - x_{n_{\varepsilon}, H_{\varepsilon}}) \)

for each \((n, H) \geq (n_{\varepsilon}, H_{\varepsilon})\). Therefore \( x_{n_{\varepsilon}, H_{\varepsilon}} \overset{\tau}{\to} x \). \( \square \)

The following corollary can be proven easily.

**Corollary 3.** Let \((X, \tau)\) be a locally solid vector lattice. The following statements are equivalent:

1. \( e \in X_{+} \) is a quasi-interior point;
2. for each \( x \in X_{+}, \quad x - x \wedge ne \overset{\tau}{\to} 0 \) as \( n \to \infty \).

**Corollary 4.** Let \((X, \tau)\) be a locally solid vector lattice possessing the \( \sigma \)-Lebesgue property. Then every weak unit in \( X \) is a quasi-interior point.

**Proof.** Let \( x \in X^{+} \), and let \( e \) be a weak unit. Then \( x \wedge ne \uparrow x \). So, by the \( \sigma \)-Lebesgue property, we get \( x - x \wedge ne \overset{\tau}{\to} 0 \) as \( n \to \infty \). \( \square \)

**Theorem 7.** Let \((X, \tau)\) be a locally solid vector lattice, and \( Q = \{ e_{\gamma} \}_{\gamma \in \Gamma} \) be a topological orthogonal system of \((X, \tau)\). Then \( x_{\alpha} \overset{\text{wrt}}{\to} 0 \) iff \( |x_{\alpha}| \wedge e_{\gamma} \overset{\tau}{\to} 0 \) for every \( \gamma \in \Gamma \).

**Proof.** The forward implication is trivial. For the backward implication, assume \( |x_{\alpha}| \wedge e_{\gamma} \overset{\tau}{\to} 0 \) for every \( \gamma \in \Gamma \). Let \( u \in X_{+}, \quad j \in J \). Fix \( \varepsilon > 0 \). We
have
\[
|x_\alpha| \land u = |x_\alpha| \land (u - u_{n,H} + u_{n,H}) \\
\leq |x_\alpha| \land (u - u_{n,H}) + |x_\alpha| \land u_{n,H} \\
\leq (u - u_{n,H}) + |x_\alpha| \land \sum_{\gamma \in H} u \land ne_\gamma \\
\leq (u - u_{n,H}) + |x_\alpha| \land \sum_{\gamma \in H} ne_\gamma \\
\leq (u - u_{n,H}) + n(|x_\alpha| \land \sum_{\gamma \in H} e_\gamma) \\
= (u - u_{n,H}) + n \sum_{\gamma \in H} |x_\alpha| \land e_\gamma.
\]

Now, Theorem 6 assures that \( u_{n,H} \overset{\tau}{\to} u \), and so, there exists \((n_{\varepsilon}, H_\varepsilon) \in \mathbb{N} \times \mathcal{F}(\Gamma)\) such that

\[
(5.9) \quad \rho_j(u - u_{n_{\varepsilon},H_\varepsilon}) < \varepsilon.
\]

Thus, \(|x_\alpha| \land u \leq u - u_{n_{\varepsilon},H_\varepsilon} + \sum_{\gamma \in H_\varepsilon} n_{\varepsilon}(e_\gamma \land |x_\alpha|)\). But, by the assumption, \(e_\gamma \land |x_\alpha| \overset{\tau}{\to} 0\) for all \(\gamma \in \Gamma\), and so \(n_{\varepsilon}(e_\gamma \land |x_\alpha|) \overset{\tau}{\to} 0\). Hence, there is \(\alpha_{\varepsilon,H_\varepsilon}\) such that

\[
(5.10) \quad \rho_j\left(n_{\varepsilon}(e_\gamma \land |x_\alpha|)\right) < \frac{\varepsilon}{|H_\varepsilon|} \quad (\forall \alpha \geq \alpha_{\varepsilon,H_\varepsilon}, \forall \gamma \in H_\varepsilon).
\]

Here \(|H_\varepsilon|\) denotes the cardinality of \(H_\varepsilon\). For \(\alpha \geq \alpha_{\varepsilon,H_\varepsilon}\), we have

\[
\rho_j(|x_\alpha| \land u) \leq \rho_j(u - u_{n_{\varepsilon},H_\varepsilon}) + \rho_j\left(n_{\varepsilon} \sum_{\gamma \in H_\varepsilon} |x_\alpha| \land e_\gamma\right) \\
\leq \varepsilon + \sum_{\gamma \in H_\varepsilon} \rho_j\left(n_{\varepsilon}(e_\gamma \land |x_\alpha|)\right) < \varepsilon + \sum_{\gamma \in H_\varepsilon} \frac{\varepsilon}{|H_\varepsilon|} = 2\varepsilon,
\]

where the second inequality follows from (5.9) and the third one from (5.10). Therefore, \(\rho_j(|x_\alpha| \land u) \to 0\), and so \(x_\alpha \overset{u{\tau}}{\to} 0\). \(\square\)

The following corollary is immediate.

**Corollary 5.** Let \((X, \tau)\) be a locally solid vector lattice, and \(e \in X_+\) be a quasi-interior point. Then \(x_\alpha \overset{u{\tau}}{\to} 0\) iff \(|x_\alpha| \land e \overset{\tau}{\to} 0\).

Recall that a topological vector space is metrizable iff it has a countable neighborhood base at zero, [2 Thm. 2.1]. In particular, a locally solid vector lattice \((X, \tau)\) is metrizable iff its topology \(\tau\) is generated by a countable family \((\rho_k)_{k \in \mathbb{N}}\) of Riesz pseudonorms. The following result gives a sufficient condition for the metrizability of \(u\tau\)-topology.
Let \((X, \tau)\) be a complete metrizable locally solid vector lattice. If \(X\) has a countable topological orthogonal system, then the \(u\tau\)-topology is metrizable.

**Proof.** First note that, since \((X, \tau)\) is metrizable, \(\tau\) is generated by a countable family \((\rho_k)_{k \in \mathbb{N}}\) of Riesz pseudonorms.

Now suppose \((\varepsilon_n)_{n \in \mathbb{N}}\) to be a topological orthogonal system. For each \(n \in \mathbb{N}\), put \(d_n(x, y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \rho_k(|x-y| \wedge \varepsilon_n)\). Note that each \(d_n\) is a semi-metric, and \(d_n(x, y) \leq 1\) for all \(x, y \in X\). If \(d_n(x, y) = 0\), then \(\rho_k(|x-y| \wedge \varepsilon_n) = 0\) for all \(k \in \mathbb{N}\), so \((|x-y| \wedge \varepsilon_n) = 0\). For \(x, y \in X\), let \(d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x, y)\). Clearly, \(d(x, y)\) is nonnegative and satisfies the triangle inequality, and \(d(x, y) = d(y, x)\) for all \(x, y \in X\). Now \(d(x, y) = 0\) iff \(d_n(x, y) = 0\) for all \(n \in \mathbb{N}\) iff \(\rho_k(|x-y| \wedge \varepsilon_n) = 0\) for all \(k \in \mathbb{N}\) iff \((|x-y| \wedge \varepsilon_n) = 0\) for all \(n \in \mathbb{N}\) iff \(|x-y| = 0\) iff \(x = y\). Thus \((X, d)\) is a metric space. Finally, it is easy to see from Theorem \(7\) that \(d\) generates the \(u\tau\)-topology. \(\square\)

Recall that a topological space \(X\) is called **submetrizable** if its topology is finer that some metric topology on \(X\).

**Proposition 6.** Let \((X, \tau)\) be a metrizable locally solid vector lattice. If \(X\) has a weak unit, then the \(u\tau\)-topology is submetrizable.

**Proof.** Note that, since \((X, \tau)\) is metrizable, then \(\tau\) is generated by a countable family \((\rho_k)_{k \in \mathbb{N}}\) of Riesz pseudonorms.

Suppose that \(e \in X_+\) is a weak unit. Put \(d(x, y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \rho_k(|x-y| \wedge e)\). Note that \(d(x, y) = 0\) iff \(\rho_k(|x-y| \wedge e) = 0\) for all \(k \in \mathbb{N}\). Since \(e\) is a weak unit, \(x = y\). It can easily be shown that \(d\) satisfies the triangle inequality. Assume \(x_\alpha \xrightarrow{u\tau} x\). Then, for all \(u \in X_+\), \(\rho_k(|x-y| \wedge u) \rightarrow 0\) for all \(k \in \mathbb{N}\). In particular, \(\rho_k(|x-y| \wedge e) \rightarrow 0\) for all \(k \in \mathbb{N}\). Then in a similar argument to \([24\text{ p.}200]\), it can be shown that \(x_\alpha \xrightarrow{d} x\). Therefore, the \(u\tau\)-topology is finer than the metric topology generated by \(d\), and hence \(u\tau\)-topology is submetrizable. \(\square\)

We do not know whether the converse of propositions \(5\) and \(6\) is true or not.

### 6. Unbounded \(\tau\)-Completeness

A subset \(A\) of a locally solid vector lattice \((X, \tau)\) is said to be **(sequentially) \(u\tau\)-complete** if, it is (sequentially) complete in the \(u\tau\)-topology. In this section, we relate sequential \(u\tau\)-completeness of subsets of \(X\) with the Lebesgue and Levi properties. First, we remind the following theorem.
Theorem 8. [26] Thm. 1' | If \((X, \tau)\) is a locally solid vector lattice, then the following statements are equivalent:

(1) \((X, \tau)\) has the Lebesgue and Levi properties;
(2) \(X\) is \(\tau\)-complete, and \(c_0\) is not lattice embeddable in \((X, \tau)\).

Recall that two locally solid vector lattices \((X_1, \tau_1)\) and \((X_2, \tau_2)\) are said to be isomorphic, if there exists a lattice isomorphism from \(X_1\) onto \(X_2\) that is also a homeomorphism; in other words, if there exists a mapping from \(X_1\) onto \(X_2\) that preserves the algebraic, the lattice, and the topological structures. A locally solid vector lattice \((X_1, \tau_1)\) is said to be lattice embeddable into another locally solid vector lattice \((X_2, \tau_2)\) if there exists a sublattice \(Y_2\) of \(X_2\) such that \((X_1, \tau_1)\) and \((Y_2, \tau_2)\) are isomorphic.

Note that \((X, \tau)\) can have the Lebesgue and Levi properties and simultaneously contains \(c_0\) as a sublattice, but not as a lattice embeddable copy. The following example illustrates this.

Example 1. Let \(s\) denote the vector lattice of all sequences in \(\mathbb{R}\) with coordinatewise ordering. Clearly, \(c_0\) is a sublattice of \(s\). Define the following separating family of Riesz pseudonorms

\[ \mathcal{R} := \{ \rho_j : \rho_j((x_n)_{n \in \mathbb{N}}) := |x_j| \} \]

for each \(j \in \mathbb{N}\) and \((x_n)_{n \in \mathbb{N}} \in s\). Then \(\mathcal{R}\) generates a locally solid topology \(\tau\) on \(s\). It can be easily shown that \((s, \tau)\) has the Lebesgue and Levi properties. Although \(c_0\) is a sublattice of \(s\), but \((c_0, \|\cdot\|_\infty)\) is not lattice embeddable in \((s, \tau)\). To see this, consider the sequence \(e_n\) of the standard unit vectors in \(c_0\). Then the sequence \(e_n\) is not norm null in \((c_0, \|\cdot\|_\infty)\), whereas \(e_n \overset{\tau}{\to} 0\) in \((s, \tau)\).

Proposition 7. Let \((X, \tau)\) be a complete locally solid vector lattice. If every \(\tau\)-bounded subset of \(X\) is sequentially \(\mu\tau\)-complete, then \(X\) has the Lebesgue and Levi properties.

Proof. Suppose \(X\) does not possess the Lebesgue or Levi properties. Then, by Theorem 8 \(c_0\) is lattice embeddable in \((X, \tau)\). Let \(s_n = \sum_{k=1}^{n} e_k\), where \(e_k\)’s denote the standard unit vectors in \(c_0\). Clearly, the sequence \(s_n\) is norm-bounded in \(c_0\) and so it is \(\tau\)-bounded in \((X, \tau)\). Note that \(\|e_k\|_\infty = 1 \Rightarrow 0\), and so \(e_k\) is not \(\tau\)-null. It follows from [15 Lm. 6.1] that \(s_n\) is \(\mu\tau\)-Cauchy in \(c_0\), but is not \(\mu\tau\)-convergent in \(c_0\). That is \(s_n\) is \(\mu\tau\)-Cauchy which is not \(\mu\tau\)-convergent, a contradiction. \(\square\)

Using the proof of the previous result and [26 Thm. 1’], one can easily prove the following result.

Proposition 8. Let \(X\) be a Dedekind complete vector lattice equipped with a sequentially complete topology \(\tau\). If every \(\tau\)-bounded subset of \(X\) is sequentially \(\mu\tau\)-complete, then \(X\) has the \(\sigma\)-Lebesgue and \(\sigma\)-Levi properties.
Clearly, every finite dimensional locally solid vector lattice \((X, \tau)\) is \(u\tau\)-complete. On the contrary of [15, Prop. 6.2], we provide an example of a \(\tau\)-complete locally solid vector lattice \((X, \tau)\) possessing the Lebesgue property such that it is \(u\tau\)-complete and \(\dim X = \infty\).

**Example 2.** Let \(X = s\) and \(R = (\rho_j)_{j \in \mathbb{N}}\) such that \(\rho_j((x_n)) := |x_j|\), where \((x_n)_{n \in \mathbb{N}} \in s\). It is easy to see that \((X, R)\) is \(\tau\)-complete and has the Lebesgue property. Now, we show that \((X, R)\) is \(u\tau\)-complete. Suppose \(x^\alpha\) is \(u\tau\)-Cauchy net. Then, for each \(u \in X_+\), we have \(|x^\alpha - x^\beta| \wedge u \xrightarrow{\tau} 0\). Now, \(u = u_n\) and, \(x^\alpha = x_n^\alpha\). Let \(j \in \mathbb{N}\), then \(\rho_j(|x^\alpha - x^\beta| \wedge u) \to 0\) in \(\mathbb{R}\) over \(\alpha, \beta\) iff \(|x_j^\alpha - x_j^\beta| \wedge u_j \to 0\) in \(\mathbb{R}\) iff \(|x_j^\alpha - x_j^\beta| \to 0\) in \(\mathbb{R}\) over \(\alpha, \beta\).

Thus, \((x_j^\alpha)\) is Cauchy in \(\mathbb{R}\) and so there is \(x_j \in \mathbb{R}\) such that \(x_j^\alpha \to x_j\) in \(\mathbb{R}\) over \(\alpha\). Let \(x = (x_j)_{j \in \mathbb{N}} \in s\), then, clearly, \(x^\alpha \xrightarrow{u\tau} x\).

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