1. It is known that in the four-dimensional Riemannian space the complex bispinor generates a number of tensors: scalar, pseudo-scalar, vector, pseudo-vector, antisymmetric tensor. This paper solves the inverse problem: the above tensors are arbitrarily given, it is necessary to find a bispinor (bispinors) reproducing the tensors. The algorithm for this mapping constitutes construction of Hermitean matrix $M$ from the tensors and finding its eigenvalue spectrum. A solution to the inverse problem exists only when $M$ is nonnegatively definite. Under this condition a matrix $Z$ satisfying equation $M = ZZ^+$ can be found. One and the same system of tensor values can be used to construct the matrix $Z$ accurate to an arbitrary factor on the left-hand side, viz. unitary matrix $U$ in polar expansion $Z = H \cdot U$. The matrix $Z$ is shown to be expandable to a set of bispinors, for which the unitary matrix $U$ is responsible for the internal (gauge) degrees of freedom. Thus, a group of gauge transformations depends only on the Riemannian space dimension, signature, and the number field used. The constructed algorithm for mapping tensors to bispinors admits extension to Riemannian spaces of a higher dimension.

2. Bispinor matrix $Z$

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¹The term “paper” means a reference to two papers by M.V.Gorbatenko and A.V.Pushkin “On correspondence between tensors and bispinors” [1], [2].
At a given field of metric tensor, \( g_{\alpha\beta}(x) \), the field of Dirac matrices (DM), \( \gamma_{\alpha}(x) \), is determined by relation

\[
[\gamma_{\alpha}(x), \gamma_{\beta}(x)]_{+} = 2g_{\alpha\beta}(x). \tag{1}
\]

Hereafter we call DM \( \gamma_{\alpha}(x) \) as world matrices.

Suppose, the DM are implemented over the real number field. In this case, if relation (1) is satisfied by DM \( \gamma_{\alpha}(x) \), it is satisfied by three more DM systems, namely: \(-\gamma_{\alpha}(x)\); \(\gamma^{T}_{\alpha}(x)\); \(-\gamma^{T}_{\alpha}(x)\). Here \(\gamma^{T}_{\alpha}(x)\) are DM produced from DM \(\gamma_{\alpha}(x)\) through transposition.

The existing discrete transposition operation allows construction of non-trivial fields of matrix scalars, vectors, and other tensors from DM. A simplest example of the matrix scalar is \((\gamma^{\nu}(x)\gamma^{T}_{\nu}(x))\). It is easy to notice that at DM transformations by rule

\[
\gamma_{\alpha}(x) \rightarrow \gamma'_{\alpha}(x) = T(x)\gamma_{\alpha}(x)T^{-1}(x), \tag{2}
\]

DM \(-\gamma_{\alpha}(x)\); \(\gamma^{T}_{\alpha}(x)\); \(-\gamma^{T}_{\alpha}(x)\) are transformed similarly, provided the non-singular matrices \(T(x)\) are orthogonal, that is

\[
T^{T}(x) = T^{-1}(x). \tag{3}
\]

Hence, any relations involving matrix tensors, which have been constructed involving \(\gamma_{\alpha}(x)\); \(-\gamma_{\alpha}(x)\); \(\gamma^{T}_{\alpha}(x)\); \(-\gamma^{T}_{\alpha}(x)\), retain their form at transformations (3) with orthogonal matrices \(T(x)\). We treat this feature of matrix relations as \(T(x)\) invariance of the DM apparatus. At each Riemannian space point a set of orthogonal matrices form group \(O(4,R)\). \(T(x)\) invariance of the DM apparatus can be interpreted as an invariance with respect to a manner of matrix row and column numbering.

Introduce a field of frame-of-reference vectors \(H_{k}^{\alpha}(x)\); these satisfy relations

\[
g_{\alpha\beta}(x) = H_{m}^{\alpha}(x)H_{n}^{\beta}(x)g_{mn} \tag{4},
\]

where \(g_{mn}\) is a metric tensor in the Minkowski tangent space at the points having coordinates \((x)\). Assume that Galilean coordinates are used in the

\(^{2}\)Greek and Latin letters take on the same values, 0, 1, 2, 3. The difference in the letters is that the Greek letters mean that the value responds to transformations of the world coordinates in the Riemannian space, while the Roman ones imply responding to coordinate transformations in the local tangent space. The Roman subscripts from the alphabet beginning also denote space subscripts 1, 2, 3 hereinafter.
tangent spaces, so that tensor $g_{mn}$ has diagonal form, with numbers $(-1, 1, 1, 1)$ appearing along the diagonal. At the world coordinate transformation $x \to x'(x)$ $H^k_\alpha(x)$ are transformed by the law of ordinary vectors,

$$H^k_\alpha(x) \to H'^k_\alpha(x') = \left( \partial x^\mu / \partial x'^\alpha \right) \cdot H^k_\mu(x). \quad (5)$$

$H^k_\alpha(x)$ values also respond to transformations of coordinates in the tangent spaces. Under our assumptions on the choice of the Galilean coordinates in the tangent spaces, the tangent space coordinate transformations are Lorentz transformations,

$$H^k_\alpha(x) \to H'^k_\alpha(x) = w^k_p(x) \cdot H^p_\alpha(x). \quad (6)$$

$w^k_p(x)$ satisfy the relations

$$w^m_p(x) w^n_q(x) \cdot g_{mn} = g_{pq}. \quad (7)$$

Relations (6) mean in essence that the metric $g_{mn}$ remains unchained at the Lorentz transformations.

The availability of frame-of-reference vectors $H^k_\alpha(x)$ at each Riemannian space point allows introduction, alongside the world Dirac matrices $\gamma_\alpha(x)$, one more DM type, i.e. local DM $\gamma_k(x)$,

$$\gamma_k(x) = H^k_\alpha(x) \gamma_\alpha(x). \quad (8)$$

The local DM satisfy the relation

$$[\gamma_m(x), \gamma_n(x)]_+ = 2g_{mn}. \quad (9)$$

A feature of relation (9) is that its right-hand side is independent on coordinates, while $\gamma_k(x)$ appearing in the left-hand side are, generally speaking, dependent on them. At DM world coordinate transformations $\gamma_k(x)$ behave like scalars, while at Lorentz transformations $\gamma_k(x)$ behave similar to frame-of-reference vectors. As a result, relation (8) retains its form at either transformation type. At $T(x)$ transformations of (8) we have:

$$\gamma_k(x) \to \gamma'_k(x) = T(x) \gamma_k(x) T^{-1}(x), \quad (10)$$

so that (8) form is unchanged.
At each Riemannian space point, alongside the world DM, $\gamma_\alpha (x)$, and the local DM, $\gamma_k (x)$, one more DM system can be introduced without loss of generality, which we call as "doubly local DM system" and denote as $\gamma_k^0$.

Formally, DM $\gamma_k^0$, like $\gamma_k (x)$, are introduced using a relation similar to (9), that is relation

$$\begin{array}{c}
[\gamma_m^0, \gamma_n^0]_+ = 2g_{mn}.
\end{array} \tag{11}$$

However $\gamma_k^0$ differ from $\gamma_k (x)$ in two features. First, these are independent on coordinates. Second, these do not change at Lorentz transformations in tangent spaces. The second difference can be valid only under the condition that the matrix subscripts in $\gamma_k^0$ are of another nature than those in $\gamma_k (x)$. The difference in the matrix subscripts manifests itself at Lorentz transformations:

$$\begin{array}{c}
\gamma_k (x) \rightarrow \gamma'_k (x) = w_k^p (x) \gamma_p (x), \\
\gamma_k^0 \rightarrow \gamma_k^{0'} = w_k^p (x) \cdot L (x) \gamma_p^0 L^{-1} (x) = \gamma_k^0.
\end{array} \tag{12}$$

The matrix subscripts in $\gamma_k^0$ respond to the Lorentz transformations, however, taking into account that the vector frame-of-reference subscript also responds to the Lorentz transformations, it turns out that the resultant effect of the Lorentz transformations on $\gamma_k^0$ is zero.

As $\gamma_k (x)$ and $\gamma_k^0$ satisfy relations (9), (11) with one and the same metric tensor $g_{mn}$ in the right-hand side, then, by the Pauli theorem, these values should be related as

$$\gamma_k (x) = R (x) \gamma_k^0 R^{-1} (x), \tag{13}$$

Here $R (x)$ is the field of the nonsingular matrix, which we refer to as frame-of-reference matrix (the meaning of the name will become clear right now). From (13) and (8) it follows that

$$\gamma_\alpha (x) = H_\alpha^k (x) \cdot R (x) \gamma_k^0 R^{-1} (x). \tag{14}$$

The frame-of-reference matrix has a specific feature: its matrix subscripts behave differently at invariant $T (x)$ transformations and at $L (x)$ transformations. From (12), (13) it follows that at combination of the transformations

$$R (x) \rightarrow R^' (x) = T (x) R (x) L^{-1} (x). \tag{15}$$
If $T(x)$ transformations are considered as the world transformations and $L(x)$ transformations as the local, then it can be stated that in matrix $R(x)$ one subscript is of world nature and the other of the local. The situation is similar to the one which takes place in frame-of-reference vectors $H^k_\alpha(x)$, in which one subscript is associated with the world coordinates and the second with the local as well. It is by virtue of this analogy that matrix $R(x)$ is called frame-of-reference matrix. By the way, in $0\gamma_k$ both vector and matrix subscripts are local, that is why $0\gamma_k$ are called doubly local.

Above we noted that availability of the discrete transposition operation allowed construct nontrivial fields of matrix scalars, vectors, and other tensors from DM. However, there is a class of DM, for which all nontrivial scalar, vector and other fields can be made trivial, that is can be converted to coordinate independent constants. A DM system of this type is the ordinary Majorana system which will be used later on.

Along with the world invariant $T(x)$ transformations, local transformations with similar features can be introduced. Denote the local orthogonal transformations as $O(x)$ transformations.

Hereafter for product $R^{-1}(x) \cdot \Phi(\gamma)$, where $\Phi(\gamma)$ is some scalar function of DM $\gamma_\alpha(x)$; $-\gamma_\alpha(x)$; $\gamma_\alpha^T(x)$; $-\gamma_\alpha^T(x)$, we use notation $Z(x)$,

$$Z(x) = R^{-1}(x) \cdot \Phi(\gamma). \quad (16)$$

Object $Z(x)$ is called as bispinor matrix. In contrast to $R^{-1}$, matrix $Z$ can have zero determinant$^5$. At invariant $T(x)$ transformations and at the $L(x)$ transformations the bispinor matrix $Z(x)$ is transformed, as it follows from (16), in the same manner as matrix $R^{-1}$, that is

$$Z(x) \rightarrow Z(x) R(x) = L(x) Z(x) T^{-1}(x). \quad (17)$$

From any DM system it is possible to construct a complete system of matrices $4 \times 4$, composed of 16 matrices.

Hereafter of concern to us is implementation of the complete matrix systems using the doubly local DM systems. The systems convenient for our purposes are those composed of 10 symmetric and 6 antisymmetric matrices; these systems appear in Table 1.

---

$^3$The condition of matrix $Z$ equality to the nonsingular frame-of-reference matrix discussed in ref. [3] is, strictly speaking, an additional hypothesis. Our following consideration is not related to the hypothesis and includes the case, where the rank of $Z$ is less than 4.
### Table 1. Complete matrix systems

| System | Symmetric matrices | Antisymmetric matrices |
|--------|--------------------|------------------------|
| 1      | $\gamma_k D^{-1}; S_{mn} D^{-1}$ | $\gamma_5 \gamma_k D^{-1}; \gamma_5 D^{-1}$ |
| 2      | $\gamma_5 \gamma_k C^{-1}; S_{mn} C^{-1}$ | $\gamma_5 C^{-1}; \gamma_k C^{-1}; \gamma_5 C^{-1}$ |

If the field of vector $j^\alpha(x)$ and the field of antisymmetric tensor $H^{\alpha\beta}(x)$ are given, then system 1 can be used to construct the scalar symmetric matrix,

$$Y(x) = \left( \frac{1}{4} j^k(x) \right) \cdot 0_k 0 D^{-1} + \left( -\frac{1}{8} H^{mn}(x) \right) \cdot S_{mn} 0 D^{-1}. \quad (18)$$

at each point. Here

$$j^k(x) = H^k\alpha(x) j^\alpha(x), \quad H^{mn}(x) = H^m\alpha(x) H^n\beta(x) H^{\alpha\beta}(x). \quad (19)$$

Matrix fields of type (17) are characterized with a certain type of their subscripts. Thus, in case (17) field $Y(x)$ has two local subscripts. This allows, if necessary, changing components $j^k(x), H^{mn}(x)$ in type (19) expansions through Lorentz rotations of local frame-of-reference.

When using $Y(x)$ field types, the invariance condition should be ensured: the subscript types in the left-hand and right-hand sides of the relations should coincide. If this condition is met, covariance and invariance with respect to $w^m_n(x)$ and $T(x)$ and $O(x)$ transformations is preserved in (17) type expressions, despite using the doubly local DM and Roman tensor subscripts in these expressions.

Matrix $Z$ can be represented as a direct sum of 4 bispinors using projectors $P_{\eta\lambda} (\eta, \lambda = \pm)$ constructed from doubly local DM and satisfying the conditions of completeness and orthonormality:

$$P_{++} + P_{+-} + P_{-+} + P_{--} = E, \quad P_{\eta\lambda} \cdot P_{\eta'\lambda'} = \delta_{\eta\eta'} \delta_{\lambda\lambda'} P_{\eta\lambda}. \quad (20)$$

The representation of matrix $Z$ is:

$$Z = ZP_{++} + ZP_{+-} + ZP_{-+} + ZP_{--}. \quad (21)$$

Separate addends $\Psi_{\eta\lambda} \equiv ZP_{\eta\lambda}$ in the right-hand side of (21) have 4 parameters and are transformed at the Lorentz transformations of local frame-of-reference according to law
\[ \Psi'_{\eta\lambda} (x) = L (x) \Psi_{\eta\lambda} (x) . \]  

(22)

\( \Psi_{\eta\lambda} \) is equivalent to ordinary 4-component column bispinor. Write the bispinor matrix as a so-called polar expansion

\[ Z (x) = H (x) \cdot U^{-1} (x) , \]  

(23)

Here \( H (x) \) is a symmetric and nonnegatively definite matrix and \( U^{-1} (x) \) is an orthogonal one. The possibility that any square real matrix can be represented as (23) follows from the classic theory of matrices. Here multiplier \( H (x) \) is uniquely defined in the polar expansion. We call matrix \( H (x) \) as amplitude and \( U^{-1} (x) \) as phase.

Consider behavior of each multiplier in (23) at the Lorentz transformations of local frame-of-reference. At such transformations, according to Table 1, the bispinor matrix is transformed by law

\[ Z (x) \rightarrow Z' (x) = L (x) Z (x) . \]  

(24)

Upon the Lorentz transformation of the local frame-of-reference the bispinor matrix can again be represented in a form similar to (23),

\[ Z' (x) = H' (x) \cdot U'^{-1} (x) . \]  

(25)

\( H' (x) \) and \( U'^{-1} (x) \), as it follows from (23), (24), (25), should satisfy relation:

\[ H' (x) \cdot U'^{-1} (x) = L (x) \cdot H (x) \cdot U^{-1} (x) . \]  

(26)

However, relation (26) by no means dictates any definite transformation rules for each of the multipliers in the polar expansion. The rules can be specified only at special form of matrices \( L (x) \): when the matrices describe spatial rotations and, hence, are orthogonal, that is when

\[ L (x) = R (x) , \quad R^T (x) = R^{-1} (x) . \]  

(27)

In the case of (27) the multipliers are transformed as follows:

\[
\begin{align*}
H (x) & \rightarrow H' (x) = R (x) H (x) R^T (x) , \\
U^{-1} (x) & \rightarrow U'^{-1} (x) = R (x) U^{-1} (x) .
\end{align*}
\]  

(28)
The fact that in the general case there is no definite law of transformation for each of the multipliers in the polar expansion leads to an important conclusion, which is formulated below. Keeping in mind that the amplitude is a symmetric matrix, we can expand it by a complete system of symmetric matrices, for example, by system 1 from Table 1.

\[
H(x) = -v_0 \cdot \gamma_0 \bar{D}^{-1} + v_b \cdot \gamma_b \bar{D}^{-1} + \omega_{ab} \cdot S_{ab} \bar{D}^{-1} - 2\omega_{0b} \cdot S_{0b} \bar{D}^{-1}. \tag{29}
\]

The coefficients in the expansion are found in a standard manner. However, expansion coefficients \((v_0, v_b), (\omega_{ab}, \omega_{0b})\) are not components of the local vector and the antisymmetric tensor, respectively. If they had been such, expression (24) would have been invariant and transformed by rule \(H(x) \rightarrow H'(x) = L(x) H(x) L^T(x)\) at the Lorentz transformations of the local frame-of-reference. But the rule, together with the rule of transformation of bispinor matrix (24), lead to the fact that the phase multiplier has to be transformed by rule \(U^{-1}(x) \rightarrow U'^{-1}(x) = L^{T^{-1}}(x) U^{-1}(x)\). The last rule leads to a contradiction: on the Lorentz transformation the phase multiplier is no longer orthogonal. It is this contradiction that the above statement follows from: coefficients \((v_0, v_b), (\omega_{ab}, \omega_{0b})\) do not generate the local vector and the antisymmetric tensor, respectively.

Note that the situation in expansion (29) of amplitude matrix \(H(x)\) basically differs from expansion (17) for scalar symmetric matrix field \(Y(x)\). One difference is that in the case of \(Y(x)\) the vector field and the antisymmetric tensor field were given a priori, while in the case of \(H(x)\) there was no similar a priori requirement. The other difference is that in the case of \(H(x)\) it is additionally required that product \(H(x) \cdot U^{-1}(x)\) have the same subscript type as the bispinor matrix.

3. Formulation of the principal proposition

Assume that five real tensor values presented in Table 2 are given at some Riemann space point.

Table 2. List of tensors

| Tensor value                  | Notation |
|-------------------------------|----------|
| Scalar                        | \(m\)    |
| Vector                        | \(j^\alpha\) |
| Pseudo-vector                 | \(s_\alpha\) |
| Anti-symmetric tensor         | \(H_{\alpha\beta}\) |
| Pseudo-scalar                 | \(n\)    |
Assume that an arbitrary, but fixed implementation of the Dirac symbols as complex matrices $4 \times 4$ is taken. Let $D$ be a nonsingular matrix connecting two local Dirac matrix systems, $\gamma_k$ and $-\gamma_k^+$,

$$D \gamma_k D^{-1} = -\gamma_k^+.$$  \hspace{1cm} (30)

Matrix $D$ is determined by relation (1) with an accuracy of multiplication by an arbitrary complex number. Using this freedom, it is always possible to make that matrices $D$, $D^{-1}$ be anti-Hermitean. $D^{-1}$ and Dirac matrices can be used to construct the following complete system of matrices $4 \times 4$ composed solely of Hermitean matrices:

$$-i D^{-1}, \gamma_\alpha D^{-1}, -i \gamma_5 \gamma_\alpha D^{-1}, -S_{\alpha\beta} D^{-1}, i \gamma_5 D^{-1}$$  \hspace{1cm} (31)

Construct matrix $M$ with the following algorithm using the given tensors and Hermitean matrix system (31):

$$M = \frac{1}{4} \left( -i D^{-1} m + \gamma_\alpha D^{-1} j^\alpha - i \gamma_5 \gamma_\alpha D^{-1} s^\alpha - S_{\alpha\beta} D^{-1} \left( \frac{1}{2} H^{\alpha\beta} \right) + i \gamma_5 D^{-1} n \right)$$  \hspace{1cm} (32)

Assume that the matrix $M$ is non-negatively definite, that is all of its four eigenvalues ($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) are nonnegative. Denote the rank of the matrix $M$ as $r$; clear that $r$ can take values from 0 to 4, with the rank of the matrix $M$ being able to become zero only when $M = 0$.

If the nonnegativity condition is met, then valid are

**Propositions**

1. Let $H$ be Hermitean matrix in binary expansion $M = H \overline{H}$, there are no more than $2^r$ unitarily nonequivalent matrices $H$, each of which is correspondent with one and the same set of tensors listed in Table 2. By the unitary nonequivalence is meant that different Hermitean multipliers $H_1, H_2$ in the binary expansion can not be related as $H_1 = U_{12} H_2 U_{12}^+$, where $U_{12}$ is unitary.

2. Matrix $Z$ coincides with arithmetic root of matrix $M$ with an accuracy of the unitary multiplier $U$ on the right, that is $Z = h U$, where $h$ is the matrix among matrices $H$, which is nonnegative.

3. The tensors listed in Table 2 relate to each of matrices $Z$ as

$$m \equiv i S p (Z^+ D Z), \quad j^\alpha \equiv S p (Z^+ D \gamma^\alpha Z), \quad s_\alpha \equiv i S p (Z^+ D \gamma_5 \gamma_\alpha Z), \quad H_{\alpha\beta} \equiv S p (Z^+ D S_{\alpha\beta} Z), \quad n \equiv i S p (Z^+ D \gamma_5 Z)$$  \hspace{1cm} (33)
4. Solution of the problem of finding matrix \( Z \) with using Dirac matrices for basis

Write the expansion for \( Z \) through Dirac matrices:

\[
Z = a \cdot E + i A_0 \cdot \gamma_0 + A_k \cdot \gamma_k + i B_0 \cdot \gamma_5 \gamma_0 + B_k \cdot \gamma_5 \gamma_k + i b \cdot \gamma_5 + C_k \cdot \gamma_0 \gamma_k + i h_k \cdot \gamma_5 \gamma_0 \gamma_k .
\] (34)

As we assume that the matrix \( Z \) is Hermitean, it follows that in expansion (34) the coefficients of the Hermitean matrices are real and those of the anti-Hermitean are imaginary.

Multiply matrix \( Z \) by matrix \( Z^+ \).

\[
ZZ^+ = E \cdot \{ a^2 + A^2 + B^2 + C^2 + b^2 + A_0^2 + B_0^2 + h^2 \} + 2 \gamma_k \cdot \{ a A_k + \varepsilon_{kab} B_a C_b + B_0 h_k \} + 2 \gamma_5 \gamma_k \cdot \{ a B_k - \varepsilon_{kab} A_a C_b - A_0 h_k \} + 2 i \gamma_0 \gamma_k \cdot \{ a C_k + \varepsilon_{kab} A_a B_b + b h_k \} + 2 i \gamma_5 \gamma_0 \gamma_k \cdot \{ a h_k + A_k B_0 - A_0 B_k + b C_k \}
\] (35)

Set obtained expression (34) equal to expression (31).

\[
\begin{align*}
a^2 + A^2 + B^2 + C^2 + b^2 + A_0^2 + B_0^2 + h^2 &= \frac{1}{4} j^0 \\
a A_k + \varepsilon_{kab} B_a C_b + B_0 h_k &= - \frac{1}{2} H_{0k} \\
2 \gamma_0 \gamma_k \cdot \{ a B_k - \varepsilon_{kab} A_a C_b - A_0 h_k \} &= \frac{1}{16} \varepsilon_{kpq} H_{pq} \\
2 i \gamma_0 \gamma_k \cdot \{ a C_k + \varepsilon_{kab} A_a B_b + b h_k \} &= \frac{1}{8} j_k; \quad a b + (Ch) = \frac{1}{8} s_0; \quad a A_0 - (Bh) = \frac{1}{8} m \\
2 i \gamma_5 \gamma_k \cdot \{ a h_k + A_k B_0 - A_0 B_k + b C_k \} &= - \frac{1}{8} s_k
\end{align*}
\] (36)

If it is possible to solve system (36), then the answer to the question formulated in the title of this section will be given. The solution procedure consists in expression of the values appearing in the left-hand sides of the equations, through those appearing in the right-hand sides of the equations,

\[
a, \quad b, \quad A_0, \quad A_k, \quad B_0, \quad B_k, \quad C_k, \quad h_k,
\] (37)

through those appearing in the right-hand sides of the equations,

\[
j_0, \quad H_{0k}, \quad H_{pq}, \quad j_k, \quad s_0, \quad m, \quad n, \quad s_k.
\] (38)

System (36) becomes somewhat reduced in the number of unknowns, if we are manipulating over the real number field. In doing so only the following remain from among values (37), (38):

\[
a, \quad A_k, \quad B_k, \quad C_k; \quad j_0, \quad F_{0k}, \quad F_{pq}, \quad j_k.
\] (39)
If the general normalization of desired values to \( \sqrt{\frac{1}{4}j^0} \) and given values to \( \sqrt{\frac{1}{4}j^0} \) is introduced, then, taking
\[
\vec{x} \equiv \frac{\vec{A}}{\sqrt{\frac{1}{4}j^0}}; \quad \vec{y} \equiv \frac{\vec{B}}{\sqrt{\frac{1}{4}j^0}}; \quad \vec{z} \equiv \frac{\vec{C}}{\sqrt{\frac{1}{4}j^0}}; \quad a_k \equiv -\frac{1}{2} H_{0k} j^0; \quad b_k \equiv \frac{1}{4} \varepsilon_{kpq} \frac{H_{pq}}{j^0}; \quad c_k \equiv \frac{1}{2} j^0
\]
system (36) becomes:
\[
a_2 + \vec{x}_2 + \vec{y}_2 + \vec{z}_2 = 1; \quad a \cdot \vec{x} + [\vec{y}; \vec{z}] = \vec{a}; \quad a \cdot \vec{y} + [\vec{z}; \vec{x}] = \vec{b}; \quad a \cdot \vec{z} + [\vec{x}; \vec{y}] = \vec{c}.
\]

(40)

Now it is possible to construct solutions to the last nine equations of system (40) explicitly.

A solution to the characteristic equation for matrix \( M \), defined by formula (32) is
\[
\lambda_1 = j + \sqrt{u^2 + v^2 + 2w}; \quad \lambda_2 = j + \sqrt{u^2 + v^2 - 2w}; \quad \lambda_3 = j - \sqrt{u^2 + v^2 - 2w}; \quad \lambda_4 = j - \sqrt{u^2 + v^2 + 2w}
\]

(41)

Here the following notations are used:
\[
\begin{align*}
\vec{j} & \equiv \sqrt{(-g_{\mu\nu} j^{\mu} j^{\nu})} \geq 0, \quad e^\alpha \equiv j^\alpha / \sqrt{(-g_{\mu\nu} j^{\mu} j^{\nu})}, \\
u_\alpha & \equiv H_{\alpha\nu} e^\nu, \quad u^2 \equiv g^{\alpha\beta} u_\alpha u_\beta, \\
v_\alpha & \equiv \frac{1}{2} E_{\alpha\mu\nu\lambda} H^{\mu\nu} e^\lambda, \quad v^2 \equiv g^{\alpha\beta} v_\alpha v_\beta, \\
w_\alpha & \equiv \left[ \delta_\beta^\alpha + e_\alpha e_\beta \right] H_{\beta\mu} H^{\mu\nu} e_\nu, \quad w \equiv \sqrt{g^{\alpha\beta} w_\alpha w_\beta}.
\end{align*}
\]

(42)

From formulas (42) it follows, that vectors \( u^\alpha, v^\alpha, w^\alpha \) are orthogonal to vector \( j^\alpha \), in addition, vectors \( u^\alpha, v^\alpha \) are orthogonal to vector \( w^\alpha \). The least eigenvalue is \( \lambda_4 \). For the matrix \( M \) to be non-negative, the following condition has to be met:
\[
j \geq \sqrt{u^2 + v^2 + 2w}.
\]

(43)

Inequality (43) is the only condition for solvability of our problem.

Then matrix \( Z \) was constructed explicitly with the diagonalizing matrix \( V \). The result was validated with a computer program of symbol computations.

The mapping of the world tensors on the amplitude part in the polar expansion of the bispinor matrix found at a choice of local frame-of-reference
The mapping between the Riemannian variety tensor items and matrix $Z$ composed of four bispinors constructed in this paper solves the problem. The uniqueness in the proposed construction is achieved thanks to the fact that at $Z$ matrix generation by the tensor invariants the entire arbitrariness is localized in the gauge transformation unitary matrix responsible for the internal degrees of freedom of half-integer spin particles. Hence we arrive at a very uncommon physical corollary: the existence of the spin structure of particles as of a purely geometric item of the Riemannian variety is only possible, given other internal geometric degrees of freedom in the particles, with the
gauge symmetry group describing these degrees of freedom being uniquely determined by the Riemannian variety dimension “n” in accordance with the chain \( n \leftrightarrow N \times N \) - dimension \( N \) (=dimensions of matrices \( Z \)) of the Dirac matrices in space \( \leftrightarrow \) dimension of the unitary group of matrices \( U \) acting on the matrix \( Z \) on the right \( \leftrightarrow \) gauge group.

The case of four-dimensional matrices \( Z_{4R} \) over the real number field considered in the paper corresponds to nonzero spin and zero electric charge particles. In particular, neutron and neutrino are such particles. Our conclusion agrees with presently known experimental data: all spin-containing neutral particles are members of multiplets in internal symmetry group representations.

Note that the gauge group appearing in the case of \( Z_{4R} \) is group \( O(4) \) which is direct product \( SO(3) \times SU(2) \) (+ all non-intrinsic automorphisms in this product). Physically, this is a direct hint about possible correspondence with the gauge group of electroweak interactions.

The second most important corollary of the spinor structure geometrization method proposed in the paper is that the \( Z \) matrix field dynamics completely depends on tensor fields, which eventually reduces to dynamic equations for the Riemannian variety. Naturally, the constructive derivation of the general dynamic equations for \( Z \) matrix from equations of the general relativity theory (GRT) will require solution of several very difficult problems pertaining to the GRT equations themselves. These are primarily: the problem of representation of Riemannian variety global properties which are solutions to dynamic equations. The second problem group pertains to development of algebraic methods for integration of partial differential equations which allow expression of solutions in terms of Riemannian variety tensor algebraic invariants. Finally, the third problem is construction of the energy-momentum tensor, i.e. the right-hand side in the GRT equations, at which the solutions to the equations provide the tensor invariants defined on the Riemannian variety which lead to non-negatively definite matrix \( M \) (see formula (32)). It turns out that our spinor geometrization method automatically leads to formulation of conditions for the Riemannian variety global structure and then, eventually, for the energy-momentum tensor property. These conditions result from studying the covariant expressions for the tensor invariants comprising the condition of \( M \) matrix non-negativity (see formulas (12) and (13)), namely: a) existence of globally-definite time-like vector \( j^\alpha \), i.e. 3+1 structure of the Riemannian variety; b) existence of global partition
2+1 on 3D hyper-surface (i.e. two vectors, $u_\alpha$ and $v_\alpha$, generate a 2D surface orthogonal to vector $w_\alpha$, see formula (42)).

Summing up items a) and b), the result can be expressed as a single condition: for the geometrized spin structure to exist, the Riemannian variety has to admit the 2+2 global structure (naturally, we can not argue here that it is this that the whole set of sufficient conditions for $M$ matrix non-negativity reduces to). It should be noted that this conclusion completely agrees with J. Wheeler’s guess about potential properties of the Riemannian varieties necessary for adequate description of $1/2$-spin particles.

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