Consensus Beyond Thresholds:  
Generalized Byzantine Quorums Made Live

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Abstract

Existing Byzantine fault-tolerant (BFT) consensus protocols address only threshold failures, where the participating nodes fail independently of each other, each one fails equally likely, and the protocol’s guarantees follow from a simple bound on the number of faulty nodes. With the widespread deployment of Byzantine consensus in blockchains and distributed ledgers today, however, more sophisticated trust assumptions are needed. This paper presents the first implementation of BFT consensus with generalized quorums. It starts from a number of generalized trust structures motivated by practice and explores methods to specify and implement them efficiently. In particular, it expresses the trust assumption by a monotone Boolean formula (MBF) with threshold operators and by a monotone span program (MSP), a linear-algebraic model for computation. An implementation of HotStuff BFT consensus using these quorum systems is described as well and compared to the existing threshold model. Benchmarks with HotStuff running on up to 40 replicas demonstrate that the MBF specification incurs no significant slowdown, whereas the MSP expression affects latency and throughput noticeably due to the involved computations.

1 Introduction

Trust assumptions are a fundamental part of secure distributed computing protocols. On one hand, they capture the limits of a protocol’s safety properties, thus characterizing the domains in which it may be deployed safely. But on the other hand, they also impose limits on the potential of the protocol and, in some sense, the expressiveness and freedom of the parties, thus restricting the domains in which the protocol will be deployed. Byzantine quorum systems \cite{19} are the key abstraction for capturing the trust assumptions in distributed protocols where parties may behave maliciously. A Byzantine quorum system (BQS) is defined as a set of quorums, where a quorum is a set of parties that is sufficient to execute a particular task. A BQS is closely related with a fail-prone system, which contains the sets of parties that are tolerated to fail in an execution, through the following intersection property: any two quorums must intersect in a set of parties that is not expected to fail. Thus, BQS formalize the expected Byzantine failures and allow reasoning about the resilience of protocols using them.

We refer to a BQS that is allowed to contain arbitrary quorums as a generalized BQS, in contrast to a threshold BQS that defines quorums only by their cardinality. Generalized BQS have been intensely explored in the literature \cite{19}. For example, Malkhi \textit{et al.} \cite{20} study their load and availability, Hirt and Maurer \cite{13} use a very related notion for secure multiparty computation, Junqueira \textit{et al.} \cite{14} explore an equivalent formalization in terms of survivor sets, and Warns \textit{et al.} \cite{26} introduce a generalized model that unifies multiple such failure models.

Nevertheless, these works approach generalized BQS mainly from a theoretical perspective. When considering practical, state-of-the-art distributed protocols with Byzantine faults, especially state-machine

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replication (SMR) protocols in the blockchain space, one notices that threshold BQS are the only occurring trust structure. To name some examples, Aublin et al. [1] present an abstraction of an SMR protocol and build BFT algorithms as instances of it. Liu et al. [8] introduce cross fault-tolerance (XFT), a model that provides guarantees of crash fault-tolerance but tolerates a number of Byzantine faults. Buchman et al. [6] present Tendermint, a consensus protocol based on the classical PBFT [10] algorithm, making use of a novel gossip primitive. Finally, Yin et al. introduce HotStuff [27], a BFT SMR protocol with linear communication complexity. Threshold BQS have been researched and well understood in practice, but consensus using generalized BQS has been unexplored.

**Threshold is not enough.** However, faults and attacks on the nodes in a system often occur in a coordinated way and exhibit substantial dependencies in practice. Using Werner Vogels’ words [25]: “Many academics will confess to have made the assumption that failures of component are not correlated. This absolutely unrealistic assumption will come back to haunt you in real life, where failures frequently are correlated.”

In this sense, Cachin [7] gives an example of a distributed system where the parties are differentiated by location and operating system (OS). In this scenario, a vulnerability in an OS may result in all parties with that OS being exploited or a hostile action could corrupt all parties in a specific location. This example of a generalized BQS explicitly considers correlations and tolerates more failures than possible in the threshold model. It highlights the strictly richer trust assumptions we can specify and the resilience we can gain with protocols based on generalized BQS.

As another motivating example from the field of multiparty computation, Gennaro [12] studies verifiable secret sharing over non-threshold access structures, but using only formulas in disjunctive normal form to build the access structure. As a future direction he calls for a compact representation of an access structure, which would allow any BQS to be expressed, thus leading to more efficient and flexible MPC protocols. Moreover, Cramer et al. [11] propose MPC protocols over any general trust structure, assuming that the trust structure allows for MPC at all. They work with generalized linear secret sharing scheme, which is analogous to our focus on monotone span programs.

An important tool in encoding generalized BQS are monotone span programs [15, 5]. A monotone span program (MSP) is a linear algebraic model of computation, closely related to other models in the theory of computation, such as Boolean formulas and branching programs [15]. It is known that monotone span programs are more powerful than monotone circuits. Babai et al. [2] prove that there exist functions requiring an exponential-size formula that can be encoded by a linear-size MSP. Monotone span programs have also been proved equivalent to linear secret-sharing schemes [15, 5] and have been used to generalize existing threshold schemes in the fields of secret sharing and multiparty computation (MPC). Cramer et al. [11] provide constructions for general MPC protocols based on the MSP primitive.

Recent work on consensus protocols has started to consider trust models that generalize the traditional threshold assumption. Flexible Byzantine fault tolerance [18], for instance, considers diverse quorums where some nodes may choose a different threshold quorum. Asymmetric quorum systems [9] let each node specify its own quorum system.

**Contributions.** In this work we focus on generalized BQS and demonstrate the first BFT consensus protocol with generalized quorums. We describe all components necessary for generalized BQS-based protocols and investigate different ways to realize them. In particular, we address all the following topics:

**Encoding a generalized BQS.** We first consider a monotone Boolean formula (MBF) consisting of *and*, *or*, and *threshold* operators for specifying a BQS. Since monotone span programs are stronger than monotone Boolean formulas, as mentioned, we also investigate MSP for representing generalized BQS. We exhibit an algorithm for turning a BQS specification into an MSP. When the BQS is specified as a monotone formula, the size of the created MSP is linear in its inputs.

**Integrating generalized BQS with consensus.** For both representations (MBF and MSP), we show algorithms for checking quorum properties and for integrating them with distributed protocols. Comparing the implementations we observe that the MBF-based method generally performs better than
the MSP-based implementation because of the matrix manipulations required by the MSP. This provides the first unified treatment of the efficiencies of these methods and paves the way for their practical deployment.

**Generalized Byzantine quorum systems.** We apply our methods to generalized BQS as described in the literature. For the M-Grid BQS [20], which arranges $n$ nodes in a square and tolerates $O(\sqrt{n})$ Byzantine nodes, we construct the corresponding MSP and investigate its properties. We implement an attribute-defined BQS generalizing the OS and location-based example mentioned before and represent this as an MBF and as an MSP.

**HotStuff consensus with generalized BQS.** Last but not least, we address consensus, the central problem in distributed computing. Applying our approach, we realize consensus with generalized BQS by building on HotStuff [27], an efficient BFT consensus algorithm. This is the first BFT consensus implementation using a generalized trust assumption. In benchmarks with up to 40 replicas, we observe that the performance with the MBF representation is comparable to that of the threshold BQS. Using the same threshold trust structure, the MSP representation shows lower performance.

**Related work.** The exploration of generalized structures has a long background in the field of secret sharing. Benaloh and Leichter [4] present the first secret-sharing scheme for arbitrary monotone access structures. They use monotone Boolean formulas with and, or, and threshold operators to express the access structure and introduce a recursive secret-sharing construction. Their scheme is efficient for access structures that can be expressed with polynomially sized formulas. The MSP model was first used by Brickell [5] for secret sharing, although not explicitly identified as such. After Karchmer and Wigderson [15] formally defined MSP as a model for computation, it has been shown that linear secret-sharing schemes are equivalent to MSP [3].

Many constructions have been suggested for creating the MSP of a given access structure. Lewko and Waters [16], in a way similar to Benaloh and Leichter [4], suggested a general algorithm for converting any monotone Boolean formula to an MSP, that is however inefficient for access structures expressed with threshold operators. The notion of insertion was introduced in by Martin [21]. Nikov and Nikova [23] explored constructions for recursively building the MSP for an access structure from existing MSPs for smaller access structures and presented the definition of insertion used here.

**Organization.** The rest of the paper is organized as follows. Section 2 introduces the main concepts and important background. Section 3 presents our techniques for encoding a BQS. In Section 4 we describe HotStuff consensus algorithm with generalized BQS and prove its consistency and liveness properties. Section 5 subsequently evaluates an implementation of our generalized BQS methods using the HotStuff consensus protocol.

## 2 Preliminaries

**Parties and failures.** We denote as $\mathcal{P} = \{p_1, \ldots, p_n\}$ the set of all parties in a distributed protocol. Whenever describing properties of protocols, we consider Byzantine faults, meaning that faulty parties are allowed to take arbitrary steps, cooperate, and learn the internal state held by any of them. For a specific execution we denote as $B$ the set of the actually faulty parties.

**Definition 1 (Fail-prone system [13]).** A fail-prone system $\mathcal{F} \subseteq 2^\mathcal{P}$ is a set of subsets of $\mathcal{P}$ such that for every execution there is one fail-prone set $F \in \mathcal{F}$ with $B \subseteq F$. A fail-prone system is maximal, in the sense that no fail-prone set contains another one.

**Definition 2 (Byzantine quorum system [19]).** Let $\mathcal{F}$ be a fail-prone system. A Byzantine quorum system (BQS) $Q \subseteq 2^\mathcal{P}$ is a non-empty set of non-empty subsets of $\mathcal{P}$, such that no set is contained in another one, where each $Q \in Q$ is called a quorum, satisfying the following properties:
Consistency:
\[ \forall Q_1, Q_2 \in Q, \forall F \in F : Q_1 \cap Q_2 \not\subseteq F. \]

Availability:
\[ \forall F \in F : \exists Q \in Q : F \cap Q = \emptyset. \]

The definition actually corresponds to a Byzantine dissemination quorum system \([19]\). When a BQS is defined only by cardinality, i.e., it includes all the subsets of \(P\) of a given size, it is called a threshold BQS. When a BQS is allowed to contain arbitrary subsets of \(P\) it is called a generalized BQS.

**Definition 3 (\(Q^3\)-condition \([19, 13]\)).** Let \(F\) be a fail-prone system. We say that \(F\) satisfies the \(Q^3\)-condition whenever
\[ \forall F_1, F_2, F_3 \in F : P \not\subseteq F_1 \cup F_2 \cup F_3. \]

For threshold BQS, the \(Q^3\)-condition is equivalent to the requirement \(n > 3f\). Given a fail-prone system \(F\), a BQS for \(F\) exists if and only if \(F\) satisfies the \(Q^3\)-condition. In particular, if \(Q^3\) holds, then \(\overline{F} = \{F \mid F \in F\}\) is a BQS, called the canonical BQS of \(F\).

**Access structure \([4]\).** A BQS specifies the quorums that are self-sufficient for a particular task. In the literature on multiparty computation and secret sharing \([4, 13, 11]\), the term monotone access structure is used more often, whereby the basis of such an access structure is equivalent to our notion of a (minimal) quorum system, and every quorum is called a (minimal) authorized set. In the following, we will use BQS and access structure interchangeably and interpret both as the minimal collection of subsets of \(P\) with a certain property (i.e., redefining the access structure to its basis).

**Definition 4 (Insertion on access structures \([21, 23]\)).** Let \(A_1\) and \(A_2\) be two monotone access structures defined on two sets of parties \(P_1\) and \(P_2\), respectively, and let \(p_z \in P_1\) such that \(p_z \not\in P_2\). The insertion of \(A_2\) at \(p_z\), written as \(A_1(p_z \to A_2)\), is the minimal monotone access structure \(A_3\) defined on the set \(P_3 = (P_1 \setminus \{p_z\}) \cup P_2\) that satisfies the following: a set \(A \subseteq P_3\) is authorized in \(A_3\) if and only if the set \(A \cap P_1\) is authorized in \(A_1\) or the set \(A \cap P_1\) together with \(p_z\) is authorized in \(A_1\) and \(p_z\) is replaced in \(A\) by a set authorized in \(A_2\). Formally,
\[ A \in A_3 \Leftrightarrow A \cap P_1 \in A_1 \lor ((A \cap P_1) \cup \{p_z\} \in A_1 \land A \cap P_2 \in A_2) . \]

**Monotone span programs \([15]\).** Monotone span programs (MSP) have been introduced as a linear-algebraic model of computation. An MSP is a quadruple \((M, \rho, e_1, P)\), where \(M\) is an \(m \times d\) matrix over a finite field \(K\), \(\rho\) is a surjective function \(\{1, \ldots, m\} \to \{p_1, \ldots, p_n\}\) that labels each row of \(M\) with a party in \(P\), and \(e_1\) is the vector \((1, 0, \ldots, 0) \in K^d\), called the target vector. If \(r_i\) is a row of \(M\) and \(\rho(i) = p_j, p_j \in P\), we say that party \(p_j\) owns row \(r_i\). There is also a function \(\phi : P \to 2^{\{1, \ldots, m\}}\), such that \(\phi(p_j)\) is the set of rows owned by party \(p_j\). The size of the MSP is the number of its rows \(m\).

For any set \(A \subseteq P\) we define \(M_A\) to be the \(m_A \times d\) matrix obtained from \(M\) by keeping only the rows \(r_i\) with \(\rho(i) \in A\). Let \(M_A^\top\) denote the transpose of \(M_A\) and \(Im(M_A^\top)\) the span of the rows of \(M_A\). We say that the MSP accepts the set \(A\) if the rows of \(M_A\) span \(e_1\), i.e., \(e_1 \in Im(M_A^\top)\). Equivalently, there is a recombination vector \(\lambda_A\) such that \(\lambda_A M_A = e_1\). We say that the MSP rejects \(A\) otherwise. It follows that each MSP accepts exactly one monotone access structure and that each monotone access structure can be expressed in terms of an MSP \([15, 8]\).

One of the objectives of this work is to construct an MSP that encodes a given BQS, i.e., accepts exactly its quorums. Thus, when working with MSPs (in Section \([5, 2]\)), we start from a given BQS (and an implicit fail-prone system), such that consistency and availability of the BQS are satisfied. We usually express this in terms of the access structure equivalent to the BQS.
3 Techniques

3.1 Generalized Byzantine quorum systems encoded as formulas

It is crucial to internally encode the BQS using a data structure that is efficient, able to encode any possible BQS, and also offering an inexpensive method for checking whether a set is a quorum. In this section we show how the generalized trust assumptions of the system can be specified by the user in a structured way and encoded within the protocol as a Boolean formula.

We observe that it is enough to use only the threshold operator $\Theta_k^m(q_1,\ldots,q_m)$, which specifies that any subset of $\{q_1,\ldots,q_m\}$ with cardinality $k$ is a quorum. Each $q_i$ can be a literal, i.e. a party identifier, or a nested threshold operator. The threshold operator is the generalization of logical conjunction, that would require all $q_i$s to make a quorum, and logical disjunction, that would allow each of them alone to be a quorum – the first can be obtained for $k = m$ and the second for $k = 1$. Thus, the threshold operator is complete, in the sense that it can describe any possible BQS. What is more, the generalized trust assumptions can be specified in a structured, intuitive, and user-friendly manner by the protocol users in a standard format like JSON. This is aligned with the way users specify their quorum slices in Stellar Blockchain [22] using nested threshold operators. We call this nested threshold operator configuration given by the user a configuration description.

We use the notion of a monotone Boolean formula (MBF), a formula that consists of and, or, and threshold operators and literals that correspond to parties. An MBF $F$ describes a monotone function $2^P \to \{0,1\}$ in the following way; when $F$ consists only of a literal, then the value of $F$ on input $S \subseteq P$ is 1 if and only if $F \in S$; when $F$ is the threshold operator $\Theta_k^m(q_1,\ldots,q_m)$, then $F(S)$ is 1 if at least $k$ of the $q_1,\ldots,q_m$ are recursively evaluated to 1 on input $S$; and accordingly for the other operators. We say that an MBF $F$ implements a BQS $Q$ if it returns 1 on input a set $A \supseteq Q$, for $Q \in Q$, and 0 otherwise.

We use a tree data structure to store a BQS described through an MBF, where the internal nodes represent an operator, their children are the operands, and the leaves always represent a replica. Clearly, the size of the tree (defined as the number of nodes) is linear in the configuration description. We employ Algorithm 1 to evaluate whether a set is considered a quorum in the BQS implemented by a formula $F$. The runtime is linear in the size of $F$, given that the set membership operation returns in constant time.

Algorithm 1 Checking whether set $A$ is a quorum in the BQS implemented by formula $F$.

1: eval($F$, $A$)
2:     if $F$ is a literal then
3:         return ($F \in A$)
4:     else
5:         write $F = op(F_1,\ldots,F_m)$, where $op \in \{\land,\lor,\Theta\}$
6:         for each $F_i$ do
7:             $x_i \leftarrow$ eval($F_i$, $A$)
8:         return op($x_1,\ldots,x_m$)

A layered BQS. An example that highlights a more complex BQS that cannot be specified in the threshold model is a 2-layered-1-common BQS (2L1C). Let us consider two disjoint sets of parties, organized in two layers, with $k$ parties $A_0\ldots A_{k-1}$ on the first and $3k$ parties $B_0\ldots B_{3k-1}$ on the second. We may assume that the parties in the first layer are more trusted than those in the second layer. A quorum consists of a strict $2/3$ majority of the parties in the first layer plus, for each party $A_\ell$ of these, a 2 out-of 4 threshold from the set $\{B_{3\ell},\ldots,B_{3(\ell+1)}\}$, where indices are modulo $3k$. For $k \in \mathbb{N}$, the general formula of the BQS is

$$
\Theta_{k-\frac{1}{3}}^k \left( \{ A_\ell \land \Theta^2 \left( \{ B_m \} \right) \} \right), \text{ for } \ell \in \{0,\ldots,k-1\} \text{ and } m \in \{3\ell,\ldots,3(\ell+1) \mod 3k\}.
$$

(1)
3.2 Generalized Byzantine quorum systems as monotone span programs

Until now, we considered BQS that can be efficiently encoded using formulas. However, as already discussed, results in complexity theory suggest that MSPs can be superpolynomially stronger than monotone formulas. For this reason, we also investigate the capabilities of the MSP as the data structure that encodes a BQS. In this section we show how to instantiate an MSP from an MBF and how the MSP can be used to check for quorums. Later, we evaluate the MSP-based implementation and compare it with the one based on MBF. We remark, however, that constructing the MSP from an MBF is not the only option; in case a BQS is more efficiently described by an MSP than by a formula, we could plug the MSP directly in the protocol and use the same quorum-checking algorithms. Throughout this section, we formulate all our results in terms of the access structure implied by the given BQS, since we only focus on the quorums of the BQS and not its other properties.

In line with our previous terminology, we say that an MSP $\mathcal{M}$ implements an access structure $\mathcal{A}$ if it accepts exactly the sets in $\mathcal{A}$ and their supersets. In the following, let $\mathcal{M}^{(k)} = (M^{(k)}, \rho^{(k)}$, $e^{(k)}_1$, $\mathcal{P}^{(k)})$ be MSPs, where $M^{(k)}$ has dimensions $m_k \times d_k$, for $k \in \{1, 2, 3\}$. We denote the rows of each $M^{(k)}$ as $r_i^{(k)}$, for $1 \leq i \leq m_k$. We also denote the $j$th column in a row $r$ as $r[j]$, a range of columns $j_1$ to $j_2$ as $r[j_1 : j_2]$, a row with $\ell$ zero elements as $0^\ell$, and the concatenation of two rows $r$ and $r'$, that is an new vector of size $|r| + |r'|$, whose first elements are $r$ and the last are $r'$, as $r || r'$.

**Definition 5 (Insertion on MSPs [23])**. Let $r_z$ be a row of $M^{(1)}$ owned by $p_z \in \mathcal{P}^{(1)}$ – assuming without loss of generality it is unique. The insertion of $M^{(2)}$ in row $r_z$ of $M^{(1)}$, written as $M^{(1)}(r_z \rightarrow M^{(2)})$, is an MSP $M^{(3)}$, where $M^{(3)}$ has rows identical to $M^{(1)}$, except for $r_z$, which is repeated $m_2$ times in $M^{(3)}$, each time multiplied by the first column of $M^{(2)}$ and with the rest of the columns $2$ to $d_2$ of $M^{(2)}$ appended in the end. The function $\rho^{(3)}$ labels the rows of $M^{(3)}$ with the same owners as $\rho^{(1)}$, except for
Lemma 1. \[23\] If an MSP $V$ and an MSP $A$ exactly those sets $x$ the results of linear algebra, and because structure $\Theta e$ Proof.

$x$ with $\rho$ for each nested operator, so $\rho$ is used to get the row $r$ according to Definition 5. The function $\rho$ is a surjective function $\{1, \ldots, m_1 + m_2 - 1\} \to (\mathcal{P}^{(1)} \setminus \{p_z\}) \cup \mathcal{P}^{(2)}$ defined as

$$
\rho^{(3)}(i) = \begin{cases} 
\rho^{(1)}(i) & 1 \leq i < z - 1 \\
\rho^{(2)}(i - z + 1) & z - 1 \leq i < z + m_2 - 1 \\
\rho^{(1)}(i - m_2 + 1) & z + m_2 \leq i \leq m_1 + m_2 - 1
\end{cases}
$$

and $\rho^{(3)}$ is a surjective function $\{1, \ldots, m_1 + m_2 - 1\} \to (\mathcal{P}^{(1)} \setminus \{p_z\}) \cup \mathcal{P}^{(2)}$ defined as

Let Vandermonde-MSP $(n, t, \mathcal{P})$ be defined as the MSP $(V(n, t), \rho, e_1, \mathcal{P})$, with $\mathcal{P} = \{p_1, \ldots, p_n\}$, $V(n, t)$ the $n \times t$ Vandermonde matrix over a finite field $K$,

$$V(n, t) = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{t-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{t-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^{t-1} \end{pmatrix},$$

with $x_i \neq x_j \neq 0$, for $1 \leq i < j \leq n$, $\rho$ a function that maps row $r_i$ to party $p_i$, for $i \in \{1, \ldots, n\}$, and $e_1 = (1, 0, \ldots, 0) \in K^t$. Then, Vandermonde-MSP $(n, t, \mathcal{P})$ implements the $t$ out-of $n$ threshold access structure $\Theta^n_t(\mathcal{P})$.

Proof. Let $A \subseteq \mathcal{P}$ and $M_A$ the matrix consisting of the rows of $M$ owned by the members of $A$. From the results of linear algebra, and because $x_i$’s are pairwise different, we know that the rank of $M_A$ is maximal (that is, equal to $t$). Therefore, $M$ accepts exactly those sets $A$ with $|A| \geq t$.

Building the MSP that implements a generalized BQS. Based on the previous lemmata, we now present Algorithm[2] that gets as input an access structure, encoded as an MBF $F$, and outputs an MSP that implements it. The idea is to start with a Vandermonde matrix implementing the first in the hierarchy threshold operator and repeatedly perform insertions of the MSPs implementing the nested threshold operators.

Algorithm[2] works as follows. Let $F = \Theta^n_t(F_1, \ldots, F_m)$ be an MBF, where each $F_i$ can be a party or a nested threshold operator. The algorithm first creates the MSP for $F$ (lines 1–13) in the following way: it extracts the values $m, d$ and $F_1, \ldots, F_m$ from $F$ (line 2) and examines whether each $F_i$ is a party or a nested operator. In the second case, a fresh virtual party $v_i$ is created and associated with $F_i$ (the map $V_{\text{map}}$ is used to keep track of this association). A virtual party is treated exactly as an actual party, except it is used only during this construction. The MSP for $F$ is now created according to Lemma[2] and using both actual and virtual parties as the set $\mathcal{P}$. In the second part of the algorithm (lines 14–17) the MSPs for the nested operators (virtual parties $v_i$) are recursively created (line 13) and inserted in $M$, according to Definition[5]. The function $\phi$ related to the MSP $M$, that maps a party to the rows they own, is used to get the row $r_i$ of $M$ that was labeled with $v_i$. Notice that in line 10 a fresh variable is created for each nested operator, so $v_i$ owns a single row.

For the termination of the recursion, notice that, if $F$ does not contain any nested threshold operators, $V$ is the empty set when we reach line 14 and the algorithm returns. The next result therefore follows immediately from the definition of insertion and the fact that the algorithm starts with a $1 \times 1$ matrix.
Algorithm 2 Construction of an MSP from a monotone Boolean formula $F$.

1: buildMSP($F$)
2: let $\Theta_d^m(F_1, \ldots, F_m)$ be the formula $F$
3: $R \leftarrow \emptyset$
4: $V \leftarrow \emptyset$
5: $V_{\text{map}} \leftarrow \emptyset$
6: for each $F_i$ do
7:   if $F_i$ is a literal $p$ then
8:      $R \leftarrow R \cup \{p\}$
9:   else
10:      declare $v_i$ a new virtual party
11:      $V \leftarrow V \cup \{v_i\}$
12:      $V_{\text{map}} \leftarrow V_{\text{map}} \cup \{(v_i, F_i)\}$
13:      $M \leftarrow \text{Vandermonde-MSP}(m, d, R \cup V)$
14:   for each $v_i \in V$ do
15:      $M_2 \leftarrow \text{buildMSP}(V_{\text{map}}(v_i))$
16:      $r_i \leftarrow \phi(v_i)$
17:      $M \leftarrow M(r_i \rightarrow M_2)$
18: return $M$

Lemma 3. Let $F$ be an MBF that includes in total $c$ operators in the form $\Theta_d^m$. The matrix $M$ of the MSP constructed with Algorithm 2 has $m = \sum_i^k m_i - c + 1$ rows and $d = \sum_i^k d_i - c + 1$ columns.

Lemma 3 implies that the resulting matrix $M$ has size linear in the length of $F$. In the special case that each party appears only once in the access structure, $M$ has $n$ rows and at most $n$ columns, where $n = |P|$.

Checking for quorums. We now show how to determine whether a set constitutes a quorum using the MSP representation of the system and no other information about the BQS (e.g. whether it is a threshold or a generalized BQS, or whether it was specified using threshold or other operators).

We have seen that an MSP accepts a set $A$ if and only if the rows of $M_A$ span the vector $e$, or, equivalently, the linear system $M_A^T x = e$ has solutions for $x$. According to linear algebra, a necessary and sufficient condition for this is that the rank of $M_A^T$ is equal to the rank of the augmented matrix $M_A^T|e$. To check this condition, we perform Gaussian elimination on the augmented matrix $M_A^T|e$ and bring it in row echelon form. If it contains a row with only zeros in the coefficient part but a nonzero value in corresponding constant part, then the rank of $M_A^T|e$ is bigger that the rank of $M_A^T$ and $A$ is not an authorized group. Otherwise, $A$ is an authorized group.

Gaussian elimination has a cubic time complexity, so it is expensive to perform it every time we wish to check for a quorum. As an optimization we use the LUP-decomposition of matrix $M^T$, i.e. we calculate the $d \times d$ matrices $P$ and $L$, and the $d \times m$ matrix $U$, such that $PM^T = LU$. Then, for any set $A$ we get $PM_A^T = LU_A$, where $P$ and $L$ do not depend on $A$. In the initialization of the protocol we solve $Ly = Pe$ for $y$, where $y$ is a $d$-vector. Then, instead of the equation $M_A^T x = e$ we can work with the equation $U_A x = y$. In order to check whether a set $A$ is authorized, we now have to bring $U_A|y$ in row echelon form. Since $U_A$ is an upper triangular matrix, some computational steps are avoided.

Notice here that it might be the case that $A$ is a superset of an authorized group. These redundant parties can easily be identified from the echelon form, as they will correspond to the free variables of the system – variables whose corresponding column does not contain a pivot. Another situation worth to mention is that a party can own more than one rows of $M$. However, the algorithm described above also works in this case, since $M_A$ will contain all rows owned parties in $A$.  

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3.3 Concrete constructions of Byzantine quorum systems

We now consider two specific families of generalized BQS that have been studied in the literature and show how they can be encoded as MSPs.

**Attribute-based BQS.** Let \( \mathcal{X} = \{\chi_1, \ldots, \chi_r\} \) denote a set of attributes and \( \Psi \subseteq \mathcal{P} \times \mathcal{X} \) a relation between parties and attributes. We say that party \( p_j \) holds an attribute \( \chi \) whenever \( (p_j, \chi) \in \Psi \). An attribute-based MBF is a monotone Boolean formula \( F(\chi_1, \ldots, \chi_r) \) over the attributes \( \mathcal{X} \) and implements a BQS where a set \( A \subseteq \mathcal{P} \) is a quorum if the set \( \{\chi \in \mathcal{X} \mid \exists p \in A : (p, \chi) \in \Psi\} \) of the attributes collectively held by parties in \( A \) satisfy \( F \). By adding one more syntactic rule, we can also specify the requirement that an attribute is held by at least a number of parties. Let each \( \chi_i \in \mathcal{X} \) be related with \( L_i \) parties, i.e. \( p \in \mathcal{P} | (p, \chi_i) \in \Psi \) \( = \) \( L_i \), and let \( \ell_i \leq L_i \). Then, a formula \( F(\chi_1^{(\ell_1)}, \ldots, \chi_r^{(\ell_r)}) \) specifies that \( A \) is a quorum if, in addition to the aforementioned condition, \( |\{p \in A | (p, \chi_i) \in \Psi\}| \geq \ell_i \).

An MSP \( \mathcal{M} = (M, \rho, e_1, \mathcal{P}) \) that implements \( F(\chi_1^{(\ell_1)}, \ldots, \chi_r^{(\ell_r)}) \) can be constructed as follows. First, an MSP \( \mathcal{M}' = (M', \rho', e'_1, \mathcal{P}) \) is created for \( F(\chi_1, \ldots, \chi_r) \), using the methods presented in the previous sections. Then an insertion \( \mathcal{M}'(r_i \rightarrow \mathcal{M}_i) \) is performed for every \( \chi_i \), where \( \mathcal{M}_i \) is an MSP with \( M_i \) the \( L_i \times \ell_i \) Vandermonde matrix and \( \rho_i \) a function labelling the rows of \( M_i \) with the parties related with \( \chi_i \). Notice that the resulting MSP \( \mathcal{M} \) is defined on the set of parties \( \mathcal{P} \) and not the set of attributes \( \mathcal{X} \).

We now instantiate this using the attribute-based BQS mentioned earlier. Recall that there are two families of attributes, location and operating system. We use the attributes \( \{\chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}\} \) for the four different locations and the attributes \( \{\chi_{21}, \chi_{22}, \chi_{23}, \chi_{24}\} \) for the four different OS. The 16 parties are arranged in a four by four grid, so that each party is related with exactly one attribute from each family. The system tolerates the simultaneous failure of all parties in one location and all parties with a specific OS. Thus, a set is a quorum if it contains at least three parties with different OS for at least three different locations. This BQS is implemented by the attribute-based MBF

\[
\Theta_3^4(\chi_{11}^{(3)}, \chi_{12}^{(3)}, \chi_{13}^{(3)}, \chi_{14}^{(3)}) \land \Theta_3^4(\chi_{21}^{(3)}, \chi_{22}^{(3)}, \chi_{23}^{(3)}, \chi_{24}^{(3)})
\]

Following the method described above, a \( 4 \times 3 \) Vandermonde matrix will be inserted in every \( \chi_{ij}^{(3)} \) when creating the MSP, which, according to Lemma [3], will have dimensions \( 32 \times 22 \).

**The M-Grid BQS.** Malkhi et al. [20] proposed the M-Grid system, a family of BQS where \( n = k^2 \) parties are arranged in a \( k \times k \) grid and up to \( b \) parties are allowed to be Byzantine, with \( b \leq (\sqrt{n}+1)/2 \). A quorum consists of any \( \sqrt{b+1} \) rows and \( \sqrt{b+1} \) columns. Actually, the M-Grid was proposed as a Byzantine masking quorum system [19], a category of BQS that requires a stronger intersection property than the Byzantine dissemination quorum systems, but one can adapt the construction accordingly.

For a dissemination BQS, the requirement for \( b \) is \( b \leq \sqrt{n}-1 \) and a quorum consists of any \( \sqrt{b+2/1} \) rows and \( \sqrt{b+2/1} \) columns. To see this, notice that if two quorums \( Q_1 \) and \( Q_2 \) have a row or a column in common, then \( |Q_1 \cap Q_2| \geq \sqrt{n} \geq b+1 \). Otherwise, the intersection of \( Q_1 \)’s columns with \( Q_2 \)’s rows is disjoint from the intersection of \( Q_2 \)’s columns with \( Q_1 \)’s rows, so \( |Q_1 \cap Q_2| \geq 2\sqrt{b/2+1} \sqrt{b/2+1} > b+1 \). In both cases, the consistency property of a BQS is satisfied.

To encode the M-Grid BQS we define the attribute set \( \mathcal{X} = \{R_1, \ldots, R_k, C_1, \ldots, C_k\} \) and assign the party \( s_{ij} \) at row \( i \) and column \( j \) the attributes \( R_i \) and \( C_j \). The attribute-based MBF related to this BQS family is

\[
\Theta_{\sqrt{b+2/1}}^k(R_1^{(k)}, \ldots, R_k^{(k)}) \land \Theta_{\sqrt{b+2/1}}^k(C_1^{(k)}, \ldots, C_k^{(k)})
\]

The formula has \( 3 + 2k \) threshold operators, considering the and operator as a 2-out-of-2 threshold and recalling that our method inserts a \( k \times k \) MSP in the attributes \( R_i^{(k)} \) and \( C_j^{(k)} \). The resulting MSP that implements the M-Grid BQS has, \( 2n \) rows and \( 2n + 2(\sqrt{b/2+1} - k) < 2n \) columns according to Lemma [3].
4 Consensus using generalized quorums systems

HotStuff [27] is an efficient leader-driven Byzantine fault-tolerant state-machine replication (SMR) algorithm in the partially synchronous model. The nodes that take part in the protocol are separated into replicas, that actually run the protocol, and clients, that submit requests to the replicas and receive totally-ordered responses. The trust assumptions are specified by the number of replicas $n$ and the number of tolerated faults $f$. The replicas maintain a tree structure, whose nodes contain batches of clients’ commands and get committed in a monotonically increasing way. Two nodes conflict if none of them extends from the other.

HotStuff is presented in three versions, the so-called basic, chained, and implemented HotStuff. In the first, each view consists of four phases, called prepare, pre-commit, commit, and decide. In each phase, the leader waits for $n - f$ different vote messages from the replicas, constructs a quorum certificate (QC) upon receiving them, and starts the next phase by broadcasting this certificate to the replicas. The view changes in the end of the decide phase, or whenever the replicas time out waiting for a leader’s message. Each view has a deterministically determined leader. Out of these four phases, only the first makes progress, in the sense that the leader proposes a new node and the replicas evaluate the correctness of this proposal. The other three only exist to guard the safety of the protocol. Based on this remark, the authors pipeline the four phases into one and come up with the chained version. Now each view consists of a single generic phase, that serves as the prepare phase for the new node, as the pre-commit phase for the previous block and so on. Again, the leader awaits $n - f$ votes and constructs the certificate that starts the next view. Finally, the implemented version presents further optimizations, the most important being the extraction of the liveness mechanism into a module called the pacemaker. The actual prototype implementation of threshold HotStuff, which we also use for generalized HotStuff, is based on the implemented version.

The generalized HotStuff protocol is instantiated with a Byzantine quorum system, which specifies its trust assumptions. The leader now collects votes from a quorum of processes and constructs a QC by concatenating them. Upon receiving the QC, the replicas validate the signatures, as well as the fact that the voters indeed form a quorum. Each view can again have a deterministically determined leader, and a quorum of processes is required to trigger a view change against a faulty leader. The optimizations mentioned for threshold HotStuff also apply in generalized HotStuff.

The pseudocode of basic HotStuff with generalized BQS is presented in Algorithm 3. We give a brief description of the data structures used and refer to [27] for more details. A message consists of four fields, type, viewNumber, node, and justify. The type can be one of NEW-VIEW, PREPARE, PRE-COMMIT, COMMIT, DECIDE. The viewNumber is always populated with the current view number. The field node is used in the prepare phase by the leader to propose the new leaf node, as well as by replicas in vote messages. Finally, justify is always used by the leader to send a valid QC and by the replicas to send their prepareQC in a NEW-VIEW message. A vote message, sent by replicas, additionally contains a signature over the fields type, viewNumber, node. The QC data structure consists of four fields, type, viewNumber, node, and sig. The type can be one of PREPARE, PRE-COMMIT, COMMIT and is used to indicate the phase in which the votes used to construct the QC were cast. The fields viewNumber and node indicate the view in which the QC was created and the node it justifies, respectively. Finally, the field sig contains the signatures on the vote messages of the quorum that was used to construct the QC.

In the pseudocode we omit the details related to the signing and verification of the messages, the verification of a QC and the signing of the vote messages. We denote as $p_1$ the leader of a view. As in the original protocol, this could be any deterministic function from the view number to the replicas, as long as it eventually proposes a correct leader. If an interrupt happens when replicas are waiting for a message, line[43] is executed. The variables new-views, prepare-votes, precommit-votes, and commit-votes, used by the leader to store the votes until a quorum is received, are emptied in each view (not shown for brevity).

The safety of the HotStuff protocol as presented in [27] is based on the properties of threshold Byzantine quorum systems, namely the $n > 3f$ condition. In the generalized protocol the safety is reduced to the properties of the generalized BQS. Generalized and threshold HotStuff satisfy the same safety and liveness theorems, which we now present and prove for the generalized case.
Algorithm 3 Basic HotStuff, code for process $p_i$

| State | prepareQC $\leftarrow \bot$; lockedQC $\leftarrow \bot$; curView $\leftarrow 1$
| --- | --- |
| // PREPARE phase |  
1: **upon** receiving a message \([\text{NEW-VIEW, viewNumber, node, justify}]\) from $p_j$ // only leader $p_k$
2: \hspace{1cm} **such that** viewNumber $= \text{curView} - 1$ \do
3: \hspace{2cm} new-views[j] $\leftarrow$ justify
4: \hspace{2cm} \text{if exists} \(\{p_k \in \mathcal{P} \mid \text{new-views}[k] \neq \bot\} \in \mathcal{Q}\) then
5: \hspace{3cm} $V = \{\text{new-views}[k] \mid \text{new-views}[k] \neq \bot\}$; highQC $\leftarrow \arg\max_{v \in V}(v.\text{viewNumber})$
6: \hspace{3cm} curProposal $\leftarrow$ new node
7: \hspace{3cm} curProposal.parent $\leftarrow$ highQC.node; curProposal.cmd $\leftarrow$ client’s command
8: \hspace{3cm} send message \([\text{PREPARE, curView, curProposal, highQC}]\) to all $p_j \in \mathcal{P}$
9: **upon** receiving a message \([\text{PREPARE, viewNumber, node, justify}]\) from $p_k$ **such that** viewNumber $= \text{curView}$ \do
10: \hspace{1cm} if node extends from justify.node
11: \hspace{2cm} and (node extends from lockedQC.node
12: \hspace{3cm} or justify.viewNumber $>$ lockedQC.viewNumber) then
13: \hspace{4cm} send vote message \([\text{PREPARE, curView, node, }\bot]\) to $p_k$

| // PRE-COMMIT phase |  
14: **upon** receiving a vote message $v = \{\text{PREPARE, viewNumber, node, justify}]\) from $p_j$ // only leader $p_k$
15: \hspace{1cm} **such that** viewNumber $= \text{curView}$ \do
16: \hspace{2cm} prepare-votes[j] $\leftarrow v$
17: \hspace{2cm} \text{if exists} \(\{p_k \in \mathcal{P} \mid \text{prepare-votes}[k] \neq \bot\} \in \mathcal{Q}\) then
18: \hspace{3cm} $V = \{\text{prepare-votes}[k] \mid \text{prepare-votes}[k] \neq \bot\}$; prepareQC $\leftarrow \text{QC}(V)$
19: \hspace{3cm} send message \([\text{PRE-COMMIT, curView, }\bot, \text{prepareQC}]\) to all $p_j \in \mathcal{P}$

20: **upon** receiving a message \([\text{PRE-COMMIT, viewNumber, node, justify}]\) from $p_k$
21: \hspace{1cm} **such that** viewNumber $= \text{curView}$ and justify.type $= \text{PRE-COMMIT}$ \do
22: \hspace{2cm} prepareQC $\leftarrow$ justify
23: \hspace{3cm} send vote message \([\text{PRE-COMMIT, curView, justify.node, }\bot]\) to $p_k$

| // COMMIT phase |  
24: **upon** receiving a vote message $v = \{\text{PRE-COMMIT, viewNumber, node, justify}]\) from $p_j$ // only leader $p_k$
25: \hspace{1cm} **such that** viewNumber $= \text{curView}$ \do
26: \hspace{2cm} precommit-votes[j] $\leftarrow v$
27: \hspace{2cm} \text{if exists} \(\{p_k \in \mathcal{P} \mid \text{precommit-votes}[k] \neq \bot\} \in \mathcal{Q}\) then
28: \hspace{3cm} $V = \{\text{precommit-votes}[k] \mid \text{precommit-votes}[k] \neq \bot\}$; precommitQC $\leftarrow \text{QC}(V)$
29: \hspace{3cm} send message \([\text{COMMIT, curView, }\bot, \text{precommitQC}]\) to all $p_j \in \mathcal{P}$

30: **upon** receiving a message \([\text{COMMIT, viewNumber, node, justify}]\) from $p_k$
31: \hspace{1cm} **such that** viewNumber $= \text{curView}$ and justify.type $= \text{PRE-COMMIT}$ \do
32: \hspace{2cm} lockedQC $\leftarrow$ justify
33: \hspace{3cm} send vote message \([\text{COMMIT, curView, justify.node, }\bot]\) to $p_k$

| // DECIDE phase |  
34: **upon** receiving a vote message $v = \{\text{COMMIT, viewNumber, node, justify}]\) from $p_j$ // only leader $p_k$
35: \hspace{1cm} **such that** viewNumber $= \text{curView}$ \do
36: \hspace{2cm} commit-votes[j] $\leftarrow v$
37: \hspace{2cm} \text{if exists} \(\{p_k \in \mathcal{P} \mid \text{commit-votes}[k] \neq \bot\} \in \mathcal{Q}\) then
38: \hspace{3cm} $V = \{\text{commit-votes}[k] \mid \text{commit-votes}[k] \neq \bot\}$; commitQC $\leftarrow \text{QC}(V)$
39: \hspace{3cm} send message \([\text{DECIDE, curView, }\bot, \text{commitQC}]\) to all $p_j \in \mathcal{P}$

40: **upon** receiving a message \([\text{DECIDE, viewNumber, node, justify}]\) from $p_k$
41: \hspace{1cm} **such that** viewNumber $= \text{curView}$ and justify.type $= \text{COMMIT}$ \do
42: \hspace{2cm} output decide (justify.node)
43: \hspace{3cm} send message \([\text{NEW-VIEW, curView, }\bot, \text{prepareQC}]\) to $p_{k+1}$
Theorem 4. If \( w \) and \( b \) are conflicting nodes, they cannot be both decided, each by a correct replica.

Proof. Let \( qc_1 \) and \( qc_2 \) be the valid certificates, with \( qc_1 \) created with the votes of a quorum \( Q_1 \) and \( qc_2 \) with the votes of a quorum \( Q_2 \), that convinced the two replicas to decide, that is \( qc_1, type = COMMIT, qc_1, node = w \), \( qc_2, type = COMMIT, qc_2, node = b \). Also, let \( qc_1, viewNumber = v_1 \) and \( qc_2, viewNumber = v_2 \). First note that \( v_1 \) and \( v_2 \) cannot be the same. That would mean that the votes in \( Q_1 \) and \( Q_2 \) were cast in the same view, which would require the replicas in \( Q_1 \cap Q_2 \) to vote twice in that view. But this is impossible, since algorithm \( 3 \) allows replicas to vote only once in the \( commit \) phase.

For \( qc_2 \) to be created, according to algorithm \( 3 \), there must first have been a valid \( prepareQC \) for node \( b \). This could have been formed in view \( v_2 \) or in an earlier. Let \( v_s \) be the first view after \( v_1 \) in which a valid \( prepareQC \) \( qc_s \) was formed. So, \( qc_s, type = PREPARE, qc_s, node = b \) and \( qc_s, viewNumber = v_s \), and \( Q_s \) is a quorum of replicas, whose votes where used to create \( qc_s \).

Consider now a replica \( r \) that voted for \( qc_1 \) and \( qc_s \), i.e. \( r \in Q_1 \cap Q_s \). During view \( v_1 \), \( r \) must have received a valid \( precommitQC \) and set it to its \( lockedQC \), with \( lockedQC, node = w \), before casting its vote for the \( commitQC \) \( qc_1 \). Let us examine now the \( prepare \) phase of view \( v_s \), in which the leader proposed the new block \( b \), and specifically the conditions in lines \( 11 \) and \( 12 \). By the minimality of \( v_s \), \( r \) was still locked on \( lockedQC \) in that phase. By assumption \( b \) and \( w \) were conflicting nodes, so the condition in line \( 11 \) was \( FALSE \). Moreover, \( justify.viewNumber \) was not larger than \( lockedQC.viewNumber = u_1 \), again by the minimality of \( v_s \), because that would mean that a valid \( prepareQC \) was created in a view smaller than \( v_s \). So the condition in line \( 12 \) was also \( FALSE \). As a result, every replica in \( r \in Q_1 \cap Q_s \) must be faulty. But this contradicts the quorum intersection property, thus such \( qc_1 \) and \( qc_2 \) cannot exist.

Theorem 5. After GST, there exists a bounded time period \( T_f \) such that if all correct replicas remain in view \( v \) during \( T_f \) and the leader for view \( v \) is correct, then a decision is reached.

Proof. Assume a correct leader that collects \( new-view \) messages from a quorums \( Q_1 \) of replicas. Let \( qc_l \) be the highest \( lockedQC \) among all replicas. There must be at least a quorum \( Q_2 \) of replicas that have received (and voted for) a \( prepareQC \) \( qc_p \) that matches \( qc_l \). By the quorum intersection property, \( Q_1 \cap Q_2 \) contains a non-empty set of non-faulty replicas, through which the leader will learn \( qc_p \) and use it as its \( highQC \) in the \( prepare \) message. Since all the correct replicas remain in view \( v \), they will vote in all the phases and a decision will be reached.

In Appendix A, Algorithm \( 4 \), we also show the generalized implemented HotStuff, so as to document our changes with regard to [27].

5 Evaluation

We have implemented general BQS in HotStuff [27][1]. The new functionality has been added in the form of a C++ library into the existing code base. We use nholmann-json [17] to parse the user-defined quorum-specification file. We then generate the internal tree data structure and transform it also into a monotone span program. The extension also contains the functionality for determining whether an arbitrary set is a quorum, as described in the previous sections. We use Shoup’s NTL [24] for linear algebra over \( \mathbb{Z}_p \). As in the original version of HotStuff, our implementation uses secp256k1 for all signatures. The prototype code does not make use of threshold signatures, instead stores all the received votes for a block and verifies them independently. We keep the same logic for our generalized quorum votes.

[1] We used the prototype implementation available at https://github.com/hot-stuff/libhotstuff.
Table 1. The evaluated protocols.

| System         | BQS implementation in | Supported types of BQS |
|----------------|------------------------|-------------------------|
|                | System replicas        |                           |
| Counting-HotStuff | counting              |                           |
| Formula-HotStuff     | MBF                   | threshold & generalized |
| MSP-HotStuff      | MSP                   | threshold & generalized |
| MSP-Replicas     | MSP                   | threshold & generalized |
|                | System clients         |                           |
|                | counting              |                           |
|                | MBF                   | threshold & generalized |
|                | MSP                   | threshold & generalized |
|                | counting              | threshold & generalized |

Setup. In our evaluations, we report on benchmarks with four different versions of HotStuff that differ in the way how replicas and clients encode quorums. Their features are summarized in Table 1. In the original HotStuff algorithm (Counting-HotStuff), replicas and clients know the parameters $n$ and $f$, the number of total replicas and failures, respectively, and determine whether they have received messages from a quorum by counting. In Formula-HotStuff the replicas and the clients are given the Byzantine quorum system, which can be a threshold or a generalized BQS, encoded as a monotone Boolean formula. Here we use Algorithm 1 to check for quorums. For MSP-HotStuff, replicas and clients are given an MSP-encoded BQS, again threshold or generalized, and use the algorithm of Section 3.2 to decide whether a set of parties is a quorum. According to the standard practice, replicas use batching to amortize various expensive operations (signatures and potentially Gaussian elimination) over multiple requests. However, the clients collect responses individually for every single request. This incurs a large cost that is not part of the replication protocol per se but is due to the way how clients produce requests and check for quorums. For this reason, we experiment also with a fourth protocol, called MSP-Replicas, where only the replicas use an MSP. In this setting, the clients are mapped to replicas. Since the replicas receive and verify batches of requests at once, there is no further need to perform the quorum check on individual requests.

The evaluation in the original HotStuff paper [27] uses a batch size of 400 because the latency of batching becomes higher than the cost of cryptographic operations with larger batches. Hence, we run all our experiments with batch size 400. Finally, we work only with the three-phase HotStuff.

We use VMs on a leading cloud provider, with each replica or client running on a single VM with 16 vCPUs (Intel Xeon Broadwell, 2.6 GHz, or Intel Xeon Skylake, 2.7 GHz), 32 GB RAM, and SSD local storage. We use a varying number of VMs – up to 40 replicas and 32 clients. All experiments are done over the LAN inside one data center, with a RTT of less than 1 ms. As this setup eliminates most network delays, it exposes the overhead added by the generalized BQS code. For the same reason, we use only zero-sized request and response payloads. In realistic deployments (on a wide-area network and with significant payload data), the extra cost of generalized quorums would be less visible. All measurements are made on the client. Finally, the maximum available bandwidth among the VMs was measured by iperf as 1–2 Gbits per second.

Throughput vs. latency. We first measure throughput and latency in a small system with four replicas, with the goal of comparing the behavior of the four different quorum-system implementations. We use a threshold BQS because all four protocols can be instantiated with it, that is, in Counting-HotStuff, this is specified by two numbers, $n = 4$ and $f = 1$, in Formula-HotStuff by the $\Theta_3^4(P)$ MBF, and in the last two protocols by an MSP implementing the $\Theta_3^4(P)$ access structure. The reported values were produced by first fixing the request rate per client and increasing the number of clients from one to eight and then, with the number of clients fixed at eight, increasing the request rate even further for each of them, until the system saturates. The result is depicted in Figure 3.

All four protocols exhibit similar behavior. Counting-HotStuff saturates at 188.4K tx/sec, followed by Formula-HotStuff at 179.3K tx/sec, which is less than 5% lower. The peak throughput of MSP-based protocols are slightly lower. Specifically, MSP-Replicas delivers 175.5K tx/sec before saturation, which translates to an overhead of almost 7% compared to Counting-HotStuff, while MSP-HotStuff reaches roughly 167.8K tx/sec, for an overhead of 11%. The latency at the saturation point is about 11.5ms for
all protocols. We conclude that in a small system like this, with four parties, generalizing a protocol does not significantly impact its efficiency.

**Scalability.** In this evaluation we measure the throughput and latency in a system with a varying number of replicas. We use \( n = 3f + 1 \) replicas, for \( f \in \{1, \ldots, 10\} \), and a varying number of clients. The trust assumption is again a threshold quorum system with \( n \) replicas, of which up to \( f \) may fail, specified in the appropriate way for each system. For each \( n \) we increase the request rate per client and report the throughput and latency just before saturation. The question we want to answer is how the generalized protocols (Formula-HotStuff, MSP-HotStuff, MSP-Replicas) scale in comparison to Counting-HotStuff. The results are shown in Figure 4a (throughput) and Figure 4b (latency).

We notice that Counting-HotStuff and Formula-HotStuff scale up almost identically. In a system with 31 replicas they achieve a throughput of 80.5K and 78.7K tx/sec, respectively, with latencies of 29.6ms and 26ms. MSP-Replicas achieves throughput and latency very similar to Counting-HotStuff for low values of \( n \) and comparable to Counting-HotStuff for higher \( n \). At \( n = 13 \) the throughput of MSP-Replicas is 9% lower than that of Counting-HotStuff, while the latency is only 4% higher. With \( n = 31 \), throughput and latency of MSP-Replicas lie both approximately 35% behind the numbers for Counting-HotStuff. We conclude that the overhead added by the MSP-based quorum-checking code is relatively small for the replicas, considering all the other tasks they have to carry out, such as signature evaluation and message processing, especially when batching is used. However, the protocol where both the replicas and the clients use MSPs does not scale so well. This is because clients do not use batching but operate on the MSP matrix for every received response. Moreover, in the original HotStuff prototype implementation the clients do not verify the signatures on the response messages at all (!) and, therefore, this operation is very fast and lets the overhead of the MSP appear large. With signature verification enabled, as in a production system, additional cost incurred by the MSP representation would be much less visible.

**Scalability with generalized Byzantine quorum systems.** We now evaluate the protocols beyond threshold BQS. The question we want to answer with this benchmark is how they scale when instantiated with a generalized BQS, in comparison to when instantiated with a threshold BQS. We focus on Formula-HotStuff and MSP-Replicas, which perform best in the previous experiments, and run them on two different families of BQS. The first is the 2-layered-1-common generalized BQS presented in Section 3.1 and the second is a threshold BQS. For 2L1C we vary the parameter \( k \) from 4 to 10, resulting in a system with \( 4k \) parties, while the threshold BQS is specified by the MBF \( \Theta_{\left\lceil \frac{2n+1}{3} \right\rceil}^{\Theta} (p_1, \ldots, p_n) \), for
Figure 4. Scalability of the four protocols when instantiated with a threshold BQS.

Figure 5. Scalability of Formula-HotStuff and MSP-Replicas when running under two different trust assumptions, the generalized 2L1C for $k = 4, \ldots, 10$, resulting in $4k$ parties, and the 2/3 Byzantine threshold on a set of $4k$ parties.

$n = 4k$. We do not consider Counting-HotStuff in this benchmark because it cannot be instantiated with the generalized BQS.

In Figures 5a and 5b we report the throughput and latency, respectively. In this experiment we run two replica instances in every VM, so the values reported here are overall lower than in the previous benchmarks.

The performance of MSP-Replicas when running with the generalized and the threshold quorum specifications is similar. This is because in both cases the replicas have to perform Gaussian elimination on matrices of comparable dimensions. Formula-HotStuff also scales in a similar way for both families of trust assumptions, but this benchmark shows that its efficiency is slightly affected by the specified BQS. This is, first, because generalized BQS are implemented by longer monotone Boolean formulas, but also because generalized BQS have a (sometimes much) smaller number of quorums than threshold BQS, which might affect the leader when waiting for a quorum of votes. It is worth to mention that the MBF-based protocols perform better than the MSP-based ones also in this benchmark.
Discussion. Our benchmarks illustrate the added value of generalized BQS and demonstrate that they have small overhead. One can therefore specify complex, non-threshold trust assumptions in SMR protocols without significantly sacrificing efficiency. The MBF-based protocol performs consistently better than the MSP-based, which can be expected due to the higher implementation complexity. The performance of the MBF-based protocol was identical or comparable to the original threshold HotStuff, although it can be slightly affected by the complexity of the BQS, since more complex trust assumptions result in longer formulas. The protocol where both the replicas and the clients use the MSP does not scale well and can only be used in small systems. Nonetheless, in applications where all the nodes participate in the protocol, i.e., clients are not disjoint from servers, encoding the BQS as an MSP also results in high efficiency, as was shown by MSP-Replicas in the benchmarks. We anticipate that our work will pave the way for more protocols generalizing threshold trust assumptions. This can be combined with the novel ideas presented in the BFT literature, e.g. combination of crash and Byzantine faults [8] or peer-to-peer gossip [6].

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A Implemented HotStuff

The implemented version of HotStuff using generalized quorums is presented in Algorithm 4. The call isQuorum(votes[b]) checks whether the replicas in votes[b] constitute a quorum, using our algorithm described in 3.2. The function getLeader is not defined in HotStuff but is specified by the application. Procedure onBeat is also called by the leader in order to propose new clients’ commands at points specified by the application.
Algorithm 4 Implemented HotStuff, code for process $p_i$

**State**

$$\text{prepareQC} \leftarrow \bot; \text{lockedQC} \leftarrow \bot; \text{curView} \leftarrow 1$$

1: **procedure** createLeaf ($parent$, $cmd$, $qc$, $height$)
2: $b \leftarrow$ new node
3: $b.parent \leftarrow parent; b.cmd \leftarrow cmd$
4: $b.justify \leftarrow qc; b.height \leftarrow height$
5: **return*** $b$

6: **procedure** update ($b^*$)
7: $b'' \leftarrow b^*.justify.node; b' \leftarrow b''.justify.node; b \leftarrow b'.justify.node$
8: **updateQCHigh** ($b^*.justify$) //PRE-COMMIT phase on $b''$
9: if $b'.height > b_{lock}.height$ then //COMMIT phase on $b'$
10: $b_{lock} \leftarrow b'$
11: if $b''.parent = b'$ and $b'.parent = b$ then //DECIDE phase on $b$
12: onCommit($b$)
13: $b_{exec} \leftarrow b$

14: **procedure** onCommit ($b$)
15: if $b_{exec}.height < b.height$ then
16: onCommit($b.parent$)
17: execute($b.cmd$)

18: **procedure** onReceiveProposal ($m = \text{[GENERAL, $b_{new}$, $\bot$]}$)
19: if $b_{new}.height > vheight$ and ($b_{new} \text{ extends } b_{lock}$ or $b_{new}.justify.node.height > b_{lock}.height$) then
20: $vheight \leftarrow b_{new}.height$
21: send message $\text{[GENERAL-VOTE, $b_{new}$, $\bot$]}$ to getLeader()
22: **update**($b_{new}$)

23: **procedure** onReceiveVote ($m = \text{[GENERAL-VOTE, $b$, $\bot$]}$) from $p_j$
24: votes[$b$] $\leftarrow$ votes[$b$] $\cup \{ (j, m.sig) \}$
25: if $\text{isQuorum(votes[$b$])}$ then
26: $qc \leftarrow QC(\{v_j\}_{j=1}^k)$
27: **updateQCHigh**($qc$)

28: **function** onPropose ($b_{new}$, $cmd$, $qc_{high}$)
29: $b_{new} \leftarrow $ createLeaf ($b_{leaf}.cmd$, $qc_{high}$, $b_{leaf}.height + 1$)
30: send message $\text{[GENERAL, $b_{new}$, $\bot$]}$ to all $p_j \in \mathcal{P}$
31: **return*** $b_{new}$

32: **procedure** **updateQCHigh** ($qc'_{high}$)
33: if $qc'_{high}.node.height > qc_{high}.node.height$ then
34: $qc_{high} \leftarrow qc'_{high}$
35: $b_{leaf} \leftarrow qc_{high}.node$

36: **procedure** onBeat ($cmd$)
37: if $i = \text{getLeader()}$ then
38: $b_{leaf} \leftarrow $ onPropose ($b_{leaf}.cmd$, $qc_{high}$)

39: **procedure** onNextSyncView ($cmd$)
40: send message $\text{[NEW-VIEW, $\bot$, $qc_{high}$]}$ to getLeader()

41: **procedure** onReceiveNewView ($\text{[NEW-VIEW, $\bot$, $qc'_{high}$]}$)
42: **updateQCHigh** ($qc'_{high}$)