ON THE VOLUME AND THE NUMBER OF LATTICE POINTS OF SOME SEMIALGEBRAIC SETS

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Abstract. Let \( f = (f_1, \ldots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a polynomial map; \( G_f(r) = \{ x \in \mathbb{R}^n : |f_i(x)| \leq r, \ i = 1, \ldots, m \} \). We show that if \( f \) satisfies the Mikhailov - Gindikin condition then

(i) Volume \( |G_f(r)| \asymp r^\theta (\ln r)^k \)

(ii) Card \( (G_f(r) \cap \mathbb{Z}^n) \asymp r^{\theta'} (\ln r)^{k'} \), as \( r \rightarrow \infty \),

where the exponents \( \theta, k, \theta', k' \) are determined explicitly in terms of the Newton polyhedra of \( f \).

Moreover, the polynomial maps satisfy the Mikhailov - Gindikin condition form an open subset of the set of polynomial maps having the same Newton polyhedron.

Keywords and phrases: Newton polyhedron, the Mikhailov - Gindikin condition, sublevel set, lattice points.

1. Introduction

The study of the asymptotic behavior of the volume of sublevel sets and the number of lattice points has attracted a lot of researchers and has found many important applications.

In the middle 1970s, A.N. Varchenko and V.A. Vasiliev used Newton polyhedra to study the asymptotic behavior of the volume of sublevel sets and the integrals of real analytic functions in the degenerate situation ([21], [22], [23]). In particular, sharp estimates for the volume and the integrals were obtained in terms of Newton polyhedra for certain classes of the functions with an isolated minimum at zero.

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a polynomial map. For \( r > 0 \), put

\[
G_f(r) = \{ x \in \mathbb{R}^n : |f_i(x)| \leq r, \ i = 1, \ldots, m \}, \quad Z_f(r) = G_f(r) \cap \mathbb{Z}^n,
\]

where \( \mathbb{Z}^n = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : a_i \neq 0, \ i = 1, \ldots, n \} \). Let \( |G_f(r)| \) and Card \( Z_f(r) \) be correspondingly the volume of \( G_f(r) \) and the cardinal of \( Z_f(r) \).

In this paper, we are interested in computing explicitly the exponents arising in the asymptotic formulas for \( |G_f(r)| \) and Card \( Z_f(r) \), as \( r \rightarrow \infty \).

In the case of \( m = 1 \), the asymptotic behavior of \( |G_f(r)| \) plays an important role in many problems of the theory of pseudo-differential operators.

The asymptotic behavior of the volume of the set \( \{ x \in U : |f(x)| < r \} \), as \( r \rightarrow 0 \), where \( U \) is a small enough neighborhood of a singularity point, concerns the oscillatory integral operators and the scalar oscillatory integrals (see [5], [8], [9], [11], [12], [13], [16], [18], [19], [20]).

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Key words and phrases. Newton polyhedron, the Mikhailov - Gindikin condition, sublevel set, lattice points.
The asymptotic behavior of the volume of the set \( \{ x \in \mathbb{R}^n : |f(x)| < r \} \), as \( r \to \infty \), is used in [20] to estimate the number of eigenvalues of the Schrödinger operator in \( \mathbb{R}^n \).

In [4], the asymptotic behavior of \( \text{Card} Z^f(r) \), where \( f \) is a monomial map was computed and applied in the approximation theory.

For the set \( G^f(r) \), the following problems arise naturally.

(i) When are the qualities \( |G^f(r)| \) and \( \text{Card} Z^f(r) \) finite.

(ii) If they are finite, how to compute the exponents arising in the asymptotic formulas for these qualities?

If \( f \) is an arbitrary polynomial map then it is very difficult to provide satisfactory answers to these questions, even for the case \( n = 2 \) and \( m = 1 \). However, if the application \( f \) satisfies the so called Mikhailov - Gindikin condition, then we can give complete answers to these problems.

2. Statement of results

For a polynomial \( \varphi(x) = \sum a_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots, x_n] \), we call the support of \( \varphi \) the following set

\[
\text{supp}(\varphi) := \{ \alpha \in (\mathbb{N} \cup \{0\})^n : a_\alpha \neq 0 \}.
\]

Let \( f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m \). Put \( \Gamma(f) = \text{convex} \left( \bigcup_{i=1}^m \text{supp}(f_i) \right) \), the convex hull of the set \( \bigcup_{i=1}^m \text{supp}(f_i) \). We call \( \Gamma(f) \) the Newton polyhedron of \( f \).

Let \( f_i(x) = \sum a_i^\alpha x^\alpha \) and \( \Delta \) be a face of \( \Gamma(f) \). We put

\[
f_{i\Delta}(x) = \sum_{\alpha \in \Delta} a_i^\alpha x^\alpha.
\]

**Definition 2.1.** We say that \( f \) satisfies the Mikhailov-Gindikin condition if for any face \( \Delta \subset \Gamma(f) \), we have

\[
\max_{i=1}^m |f_i\Delta(x)| \neq 0, \quad i = 1, \ldots, m;
\]

in \( (\mathbb{R} \setminus \{0\})^n \).

We denote by \( \text{cone}\Gamma(f) \) the cone generated by \( \Gamma(f) \),

\[
\text{cone}\Gamma(f) = \{ y : y = \lambda x \text{ for } \lambda \geq 0 \text{ and } x \in \Gamma(f) \},
\]

and by \( \Delta^+(d) \) the diagonal of the positive orthant in \( \mathbb{R}^n \),

\[
\Delta^+(d) = \{(d_1, \ldots, d_n) \in \mathbb{R}^n^+ : d_1 = \ldots = d_n \}.
\]

Let \( D_{\infty}\Gamma(f) \) be the furthest point from the origin in the intersections of the diagonal \( \Delta^+(d) \) and \( \Gamma(f) \), and \( \Lambda_{\infty} \) be the face of smallest dimension of \( \Gamma(f) \), having \( D_{\infty}\Gamma(f) \) as its interior point.

We denote by \( k = \text{dim}\Lambda_{\infty}, D_{\infty}\Gamma(f) = (d_\infty, \ldots, d_\infty), \theta = \frac{1}{d_\infty} \) and \( v_n = (1, \ldots, 1) \).

The notation \( g \asymp h \) means that there exists positive constants \( K_1, K_2 \) such that

\[
K_1 h \leq g \leq K_2 h.
\]

**Theorem 2.2.** Let \( f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m \) be a polynomial map satisfying the Mikhailov-Gindikin condition. Then we have
(i) \( |G^f(r)| < \infty \), for any \( r > 0 \), if and only if the vector \((1, \ldots, 1)\) belongs to the interior of \(\text{cone} \Gamma(f)\).

(ii) If \( |G^f(r)| < \infty \), then we have

\[
|G^f(r)| \asymp r^\theta \ln^{n-k-1} r, \quad \text{as} \quad r \to \infty.
\]

Next, we construct the so called complete Newton polyhedron of \(f\).

For \(\alpha, \beta \in \mathbb{R}^n\), we shall write \(\alpha \preceq \beta\), if \(\alpha_j \leq \beta_j\) for all \(j = 1, \ldots, n\).

Definition 2.3. We call the complete Newton polyhedron of \(f\), the polyhedron \(\tilde{\Gamma}(f)\) obtained from \(\Gamma(f)\) by adding all the \(\alpha \in \mathbb{R}^n_+\) for which there exists \(\beta \in \Gamma(f)\), s.t. \(\alpha \preceq \beta\).

We denote by \(D_\infty \tilde{\Gamma}(f)\) the furthest point from the origin in the intersections of \(\Delta^+(d)\) and \(\tilde{\Gamma}(f)\). Put \(D_\infty \tilde{\Gamma}(f) = (\tilde{d}_\infty, \ldots, \tilde{d}_\infty)\) and \(\theta' = 1/\tilde{d}_\infty\). Let \(\Lambda_\infty\) be the face having smallest dimension of \(\tilde{\Gamma}(f)\) that contains the point \(D_\infty \tilde{\Gamma}(f)\) in its interior. Put \(k' = \dim \Lambda_\infty\).

Theorem 2.4. Let \(f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^m\) be a polynomial map satisfying the Mikhailov-Gindikin condition. Then we have

(i) \(\text{Card } Z^f(r) < \infty\) for all positive real numbers \(r\), if and only if \(\text{cone} \Gamma(f) \cap \mathbb{R}^n_+ = \emptyset\).

(ii) Moreover, if \(\text{Card } Z^f(r) < \infty\), we have

\[
\text{Card } Z^f(r) \asymp r^{\theta'} \ln^{n-k'-1} r, \quad \text{as} \quad r \to \infty.
\]

Remark 2.5.

(i) It follows from Theorems 2.2 and 2.4 that, under the Mikhailov - Gindikin condition, the equalities \(\theta = \theta', \ k = k'\) hold if and only if \(\Lambda = \tilde{\Lambda}\), i.e \(\Lambda\) is a common face of \(\Gamma(f)\) and \(\tilde{\Gamma}(f)\).

(ii) If \(f\) is a monomial map, the exponents in the asymptotic formulas for \(G^f(r)\) and \(\text{Card } Z^f(r)\) were computed by Dinh Dung in [4]. Note that this author has stated his result in terms of some linear programming problems and did not make use of Newton polyhedra.

Let \(\Gamma\) be a convex polytope in \(\mathbb{R}^n\). Assume that all the vertex of \(\Gamma\) belong to \((\mathbb{N} \cup \{0\})^n\).

We define

\[
\mathcal{M}_\Gamma := \left\{ f : \mathbb{R}^n \to \mathbb{R}^m : \bigcup_{i=1}^m \text{supp}(f_i) \subset \Gamma \right\}, \quad \mathcal{N}_\Gamma := \{ f : \mathbb{R}^n \to \mathbb{R}^m : \Gamma(f) = \Gamma \}, \quad \mathcal{D}_\Gamma := \{ f : \mathbb{R}^n \to \mathbb{R}^m : \Gamma(f) = \Gamma, \text{ and } f \text{ satisfies the Mikhailov-Gindikin condition} \}.
\]

By the lexicographic ordering in the set of monomials, we can identify \(\mathcal{M}_\Gamma\) with a finite dimension space over \(\mathbb{R}\), and \(\mathcal{N}_\Gamma\) and \(\mathcal{D}_\Gamma\) with subsets in this space.

Theorem 2.6. With the notations above, \(\mathcal{D}_\Gamma\) is an open subset in \(\mathcal{N}_\Gamma\), and, consequently, it is an open set in the space \(\mathcal{M}_\Gamma\).

3. PROOFS

Theorem 2.2 and Theorem 2.3 are direct consequences of two following facts

(i) Two-side estimation for polynomial functions satisfying the Mikhailov - Gindikin condition.
(ii) Asymptotic formulas for the volume and the number of lattice points in semi-algebraic sets, defined by monomial inequalities [4].

Let \( \varphi(x) = \sum a_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots, x_n] \), \( \Gamma(\varphi) \) be the Newton polyhedron of \( \varphi \) and \( V(\varphi) \) be the set of vertices of \( \Gamma(\varphi) \).

Put \( N(\varphi)(x) = \sum_{\alpha \in V(\varphi)} |x^\alpha| \).

**Theorem 3.1.** (see [6, p. 204]) Two conditions are equivalent

(i) There is \( c > 0 \) and \( \rho > 0 \) such that

\[
cN(\varphi)(x) \leq |\varphi(x)|, \quad x \in \mathbb{R}^n, \quad |x| > \rho.
\]

(ii) For any face \( \Delta \subset \Gamma(\varphi) \), and \( x \in (\mathbb{R} \setminus \{0\})^n, \quad |x| > \rho \), we have

\[
\varphi(x) \neq 0, \quad \text{and} \quad \varphi_\Delta(x) \neq 0.
\]

**Remark 3.2.** It follows from the theorem 3.1 and from ([6, Lemma 1.1]) that if \( \varphi \) satisfies condition (ii), then exist positive constants \( c_1, c_2 \) and \( \rho \) such that

\[
c_1N(\varphi)(x) \leq |\varphi(x)| \leq c_2N(\varphi)(x), \quad x \in \mathbb{R}^n, \quad |x| > \rho.
\]

Now, let us consider the system of monomials

\[\{x^{\alpha_1}, \ldots, x^{\alpha_s}\}, \quad \alpha^i \in (\mathbb{N} \cup \{0\})^n, \quad i = 1, \ldots, s.\]

For \( r > 0 \), put

\[G^\alpha(r) = \left\{ x \in \mathbb{R}^n : |x|^{\alpha^i} \leq r, \quad i = 1, \ldots, s \right\}.\]

In [4], Dinh Dung computed the first term in asymptotic formulas for volume of \( G^\alpha(r) \) and for the number of lattice points in \( Z^\alpha(r) = G^\alpha(r) \cap \mathbb{Z}^n \). We now recall his result.

Consider the following linear programming problem

\[
x_1 + \ldots + x_n \rightarrow \sup; \quad \left\{ \langle x, \alpha^i \rangle \leq 1, \quad i = 1, \ldots, s \right\} \quad x \in \mathbb{R}^n.
\]

Let \( \theta \) and \( M(\alpha) \) be correspondingly the optimal value and the solution set of this problem. Put \( p := \dim M(\alpha) \) and \( \Gamma(\alpha) := \text{conv } \{\alpha^1, \ldots, \alpha^s\} \), and let \( C(\alpha) \) be the cone generated by \( \Gamma(\alpha) \).

**Theorem 3.3.** ([4 Theorem 1]) The volume of \( G^\alpha(r) \) is finite for all \( r > 0 \) if and only if the vector \( (1, \ldots, 1) \) belongs to the interior of \( C(\alpha) \). Moreover, if volume of \( G^\alpha(r) \) is finite for all \( r > 0 \), then

\[\text{volume of } G^\alpha(r) \asymp r^\theta \ln^p r.\]

Next, consider the linear programming problem

\[
x_1 + \ldots + x_n \rightarrow \sup; \quad \left\{ \langle x, \alpha^i \rangle \leq 1, \quad i = 1, \ldots, s \right\} \quad x \in \mathbb{R}^n.\]
Theorem 3.4. ([4] Theorem 2) Card $Z^n(r)$ is finite for any $r > 0$ if and only if $C(\alpha) \cap \mathbb{R}^n \neq \emptyset$. Moreover, if this condition is satisfied, then

$$\text{Card } Z^n(r) \asymp r^{\theta'} \ln^{p'} r,$$

where $\theta'$ and $p'$ be correspondingly the optimal value and the dimension of the solution set of the linear programming problem (3.2).

Proof of Theorem 2.2

Assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies the Mikhailov-Gindikin condition. We put $N_f(x) := \sum_{\alpha \in V(f)} |x|^\alpha$, where $V(f)$ is the set of vertices of $\Gamma(f)$.

Lemma 3.5. If $f$ satisfies the Mikhailov-Gindikin condition then there exist positive constants $c_1$, $c_2$ and $\rho$ such that

$$c_1 N_f(x) \leq \max |f_i(x)| \leq c_2 N_f(x)$$

for all $x \in \mathbb{R}^n$, $|x| > \rho$.

Proof. Let $F = \sum_{i=1}^m f_i^2$ and $\Gamma(F)$ be the Newton polyhedron of $F$. It is not difficult to see that if $V(f) = \{\alpha_1, \ldots, \alpha_k\}$ is the set of vertices of $\Gamma(f)$ then the set $V(F) = \{2\alpha_1, \ldots, 2\alpha_k\}$ is that of $\Gamma(F)$. In consequence, for every face $\Delta'$ of $\Gamma(F)$, there exists a unique face $\Delta$ of $\Gamma(f)$ such that $\Delta' = 2\Delta$.

Claim 3.6. Let $\Delta'$ be a face of $\Gamma(F)$ and $\Delta$ be that of $\Gamma(f)$ such that $\Delta' = 2\Delta$. Then

$$F_{\Delta'}(x) = \sum_{i=1}^m f_i^2(x).$$

Proof.

We begin with a description of a face of a polyhedron in $\mathbb{R}^n$. Let $\Gamma$ be a polyhedron in $\mathbb{R}^n$, $\dim \Gamma = n$ and $\Delta$ be its face. Then there exists $q \in \mathbb{R}^n$ such that the restriction of $\langle x, q \rangle$ on $\Gamma$ attains its maximum value at $x$ if and only if $x \in \Delta$.

In fact, if $\dim \Delta = n - 1$ then $q$ is a normal vector of the hyperplane containing $\Delta$, and $q$ is determined uniquely within a positive factor. If $\dim \Delta < n - 1$, then $\Delta$ lies on the boundaries of some faces of dimension $n - 1$, say $\Delta_1, \ldots, \Delta_l$, where

$$\Delta_i = \{x \in \Gamma : \langle x, q_i \rangle = d(q_i)\}$$

and $d(q_i) = \sup_{x \in \Gamma} \langle x, q \rangle$. Then $\Delta$ can be represented by

$$\Delta = \{x \in \Gamma : \langle x, q \rangle = d(q)\}$$

with $q = \sum_{i=1}^l t_i q_i$, $\sum_{i=1}^l t_i = 1$ and $t_i > 0$, $i = 1, \ldots, l$.

Now, assume that

$$\Delta = \{x \in \Gamma(f) : \langle x, q \rangle = d(q)\} \quad \text{and} \quad \Delta' = \{x \in \Gamma(F) : \langle x, q \rangle = 2d(q)\}.$$

We write $f_i(x)$ in the form

$$f_i(x) = h_i(x) + g_i(x)$$

where $h_i(x) = f_{i\Delta}(x)$. In the sum

$$f_i^2(x) = h_i^2(x) + 2h_i(x)g_i(x) + g_i^2(x)$$

...
every monomial \(x^\alpha\) satisfying the condition \(\langle \alpha, q \rangle = 2d(q)\), can occur only in the first summand. Therefore
\[
F_{\Delta}(x) = \sum_{i=1}^{m} h_i^2(x) = \sum_{i=1}^{m} f_i^2(x).
\]
As consequence of this claim, since \(f\) satisfies the Mikhailov - Gindikin condition, \(F\) satisfies this condition too.

By Theorem 3.3 there exist \(c > 0\), \(c' > 0\), \(\rho > 0\) such that
\[
|x| > \rho \implies c \sum_{2\alpha' \in V(F)} |x^{2\alpha'}| \leq |F(x)| \leq c' \sum_{2\alpha' \in V(F)} |x^{2\alpha'}|
\]
for all \(x \in \mathbb{R}^n\), where \(V(F)\) is the set of vertices of \(\Gamma(F)\). And therefore, there exist positive constants \(c_1\), \(c_2\) and \(\rho_1\) such that
\[
c_1 \sum_{\alpha \in V(f)} |x^\alpha| \leq max_{i} |f_i(x)| \leq c_2 \sum_{\alpha \in V(f)} |x^\alpha|,
\]
for \(|x| \geq \rho_1\).

Put
\[
A(r) := \{x \in \mathbb{R}^n : |x|^\kappa \leq r, \quad \kappa \in V(f)\}, \quad B(r) := \{x \in \mathbb{R}^n : max_{i} |f_i(x)| \leq r\}.
\]
Now, it follows from Lemma 3.5 that there exist constants \(\rho_1\) and \(\rho_2\) such that
\[
||x|| \geq \rho \implies \rho_1 A(r) \leq |B(r)| \leq \rho_2 A(r).
\]
Since
\[
|B(r)| = |\{ |x| \leq \rho : max_{i} |f_i(x)| \leq r\}| \cup \{ |x| \geq \rho : max_{i} |f_i(x)| \leq r\}
\]
and
\[
|\{ |x| \leq \rho : max_{i} |f_i(x)| \leq r\}| \leq \{ |x| \in \mathbb{R}^n : ||x|| \leq \rho\}
\]
then
\[
|B(r)| \leq \{ |x| \geq \rho : max_{i} |f_i(x)| \leq r\}, \quad r \to \infty.
\]
Thus, by (3.3), the proof of Theorem 2.2 is reduced to the problem of computing the exponents in the asymptotic formula of \(|A(r)|\), as \(r \to \infty\). For this monomial case, the problem is solved already in [4].

Using Theorem 3.3 we have

(i) \(|A(r)| < \infty\) for any \(r > 0\) if and only if \(v_n \in int(coneV(f))\).

(ii) If \(|A(r)| < \infty\) then \(|A(r)| \approx r^{\tilde{\theta}} \ln^{\tilde{k}} r\), where \(\tilde{\theta}\) is the optimal value and \(\tilde{k}\) is the dimension of the solution set of the following linear programming problem
\[
x_1 + \ldots + x_n \to sup ;
\]
(3.4)
\[
\left\{ \begin{array}{l} \langle x, \alpha^i \rangle \leq 1, \quad \alpha^i \in V(f) = \{\alpha_1, \ldots, \alpha_s\} \\ x \in \mathbb{R}^n \end{array} \right\}
\]
To finish the proof of Theorem 2.2, it rest to prove that \(\tilde{\theta} = \theta\), and \(\tilde{k} = n - k - 1\), where the exponents \(\theta\) and \(k\) are determined in the statement of Theorem 2.2.
We write the linear programming problem above in the form

\[
\begin{align*}
\text{max} & \{ x_1 + \ldots + x_n \}, \\
& \alpha_1^1 x_1 + \ldots + \alpha_n^1 x_n \leq 1 \\
& \ldots \\
& \alpha_1^s x_1 + \ldots + \alpha_n^s x_n \leq 1 \\
(x_1, \ldots, x_n) \in \mathbb{R}^n,
\end{align*}
\]

where \( \alpha^i = (\alpha_1^i, \ldots, \alpha_n^i), \ i = 1, \ldots, s. \)

Let us consider the dual problem

\[
\begin{align*}
\text{min} & \{ u_1 + \ldots + u_s \}, \\
& \alpha_1^1 u_1 + \ldots + \alpha_n^1 u_s = 1 \\
& \ldots \\
& \alpha_1^s u_1 + \ldots + \alpha_n^s u_s = 1 \\
u_i \geq 0, \ i = 1, \ldots, s.
\end{align*}
\]

The system of linear equations in (3.6) can be rewritten

\[
(\sum_{i=1}^s u_i) \alpha^1 + \ldots + \left( \frac{u_s}{\sum_{i=1}^s u_i} \right) \alpha^s = \left( \frac{1}{\sum_{i=1}^s u_i}, \ldots, \frac{1}{\sum_{i=1}^s u_i} \right).
\]

The point in the left-hand side of (3.7) belongs to \( \text{conv}\{\alpha^1, \ldots, \alpha^s\} \), and the right-hand side is a point that belongs to \( \Delta^+(d) \). On the other hand, \( \sum_{i=1}^s u_i \) achieves the maximum value when \( \sum_{i=1}^s u_i \) reaches the minimum value. Thus \( \sum_{i=1}^s u_i \) achieves the minimum value at the point \( D_\infty \Gamma(f) = (d_\infty, \ldots, d_\infty) \) and \( \tilde{\theta} = \frac{1}{d_\infty} = \theta. \)

Put \( P := \{ x \in \mathbb{R}^n : \langle x, \alpha^i \rangle \leq 1, \ i = 1, \ldots, s \} \) and let \( P^* \) be the polar set of \( P \), i.e.

\[ P^* = \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \ \forall x \in P \}. \]

By (2), Theorem 9.1, p.57), we have

\[ P^* := \text{conv}\{O, \alpha^1, \ldots, \alpha^s\}. \]

According to The Bipolar Theorem (2), \( (P^*)^* = P. \)

Let \( \Lambda_{\max} \) be the solution set of the problem (3.4). Then \( \Lambda_{\max} \) is a face of \( P \). Put

\[ \Lambda_\infty = \{ y \in P^* : \langle x, y \rangle = 1, \ \forall x \in \Lambda_{\max} \}. \]

Then, \( \Lambda_\infty \) is a face of \( P^* \) and \( \Lambda_\infty^* \), the polar set of \( \Lambda_\infty \), is equal to \( \Lambda_{\max}. \) We see that if \( x \in \Lambda_{\max} \), then \( x_1 + \ldots + x_n = \theta. \) Therefore \( \langle D_\infty \Gamma(f), x \rangle = \frac{1}{\theta} (x_1 + \ldots + x_n) = 1 \), hence \( D_\infty \Gamma(f) \in \Lambda_\infty. \) Since \( \Lambda_\infty \) does not contain the origin, \( \Lambda_\infty \) is the face of \( \Gamma(f) \) containing the point \( D_\infty \Gamma(f). \)

Now, since \( \Lambda_\infty = \Lambda_{\max}^* \), we have

\[ \dim \Lambda_{\max} = k = n - \dim \Lambda_\infty - 1 = n - k - 1. \]
3.1. Proof of Theorem 2.4

As in the proof of Theorem 2.2, the proof of Theorem 2.4 is reduced to the problem of computing the exponents in the asymptotic formula $\text{card } Z^V(f)(r)$, as $r \to \infty$, where $\text{card } Z^V(f)(r) = A(r) \cap \mathbb{Z}^n$. Using Theorem 3.4 we have

(i) $\text{Card } Z^V(f)(r) < \infty$ for any $r > 0$ if and only if $\text{cone } V \cap \mathbb{R}^n \neq \emptyset$.

(ii) If $\text{Card } Z^V(f)(r) < \infty$ then $\text{Card } Z^V(f)(r) \simeq r^{\bar{\theta}'} \ln r'$, where $\bar{\theta}'$ is the optimal value and $k'$ is the dimension of the solution set of the following linear programming problem

\[
\begin{align*}
x_1 + \ldots + x_n & \to \sup; \\
\{ \langle x, \alpha^i \rangle \leq 1, \quad \alpha^i \in V(f) = \{ \alpha_1, \ldots, \alpha_s \} \\
x \in \mathbb{R}^n_+ .
\end{align*}
\]

(3.8)

We will show that $\bar{\theta}' = \theta'$, and $k' = n - k' - 1$, where the exponents $\theta'$ and $k'$ are determined in the statement of Theorem 2.4.

We write the linear programming problem 3.8 in the form

\[
\begin{align*}
\max \{ x_1 + \ldots + x_n \}, \\
\left\{ \begin{array}{l}
\alpha_1^1 x_1 + \ldots + \alpha_1^nx_n \leq 1 \\
\vdots \\
\alpha_s^1 x_1 + \ldots + \alpha_s^nx_n \leq 1 \\
(x_1, \ldots, x_n) \in \mathbb{R}^n, \quad x_j \geq 0, \quad j = 1, \ldots, n,
\end{array} \right.
\end{align*}
\]

(3.9)

where $\alpha^i = (\alpha_1^i, \ldots, \alpha_s^i)$, $i = 1, \ldots, s$.

Let us consider the dual problem

\[
\begin{align*}
\min \{ u_1 + \ldots + u_s \}, \\
\left\{ \begin{array}{l}
\alpha_1^1 u_1 + \ldots + \alpha_1^nu_s \geq 1 \\
\vdots \\
\alpha_s^1 u_1 + \ldots + \alpha_s^nu_s \geq 1 \\
u_i \geq 0, \quad i = 1, \ldots, s .
\end{array} \right.
\end{align*}
\]

(3.10)

The system of linear inequations in (3.10) can be rewritten

\[
\left( \frac{u_1}{\sum_{i=1}^s u_i} \right) \alpha^1 + \ldots + \left( \frac{u_s}{\sum_{i=1}^s u_i} \right) \alpha^s \geq \left( \frac{1}{\sum_{i=1}^s u_i}, \ldots, \frac{1}{\sum_{i=1}^s u_i} \right).
\]

(3.11)

Put $\gamma^f = \left( \frac{u_1}{\sum_{i=1}^s u_i} \right) \alpha^1 + \ldots + \left( \frac{u_s}{\sum_{i=1}^s u_i} \right) \alpha^s$ and $\gamma^r = \left( \frac{1}{\sum_{i=1}^s u_i}, \ldots, \frac{1}{\sum_{i=1}^s u_i} \right)$. Since $\gamma^f \in \Gamma(f)$ and $\gamma^r \leq \gamma^f$, we have $\gamma^r \in \Gamma(f)$, the complete Newton polyhedron of $f$.

On the other hand, $\frac{1}{\sum_{i=1}^s u_i}$ achieves the maximum value when $\sum_{i=1}^s u_i$ reaches the minimum value. It follows that $\sum_{i=1}^s u_i$ reaches the minimum value at the point $D_{\infty} \tilde{\Gamma}(f) = (\tilde{d}_\infty, \ldots, \tilde{d}_\infty)$ and $\bar{\theta}' = \frac{1}{\tilde{d}_\infty} = \theta'$.

Put $\tilde{P} = \{ x \in \mathbb{R}^n_+ : \langle x, \alpha^i \rangle \leq 1, \quad i = 1, \ldots, s \}$. Then, $\tilde{P}$ is a bounded convex polyhedron having faces which intersect the axes $Ox_j$ at the points $A_j$, $j = 1, \ldots, n$, and containing the
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origin $O$. Hence,
\[ \tilde{P} = \text{conv}\{O, A_1, \ldots, A_n, \alpha_1, \ldots, \alpha_s\}. \]

From the properties of polar sets (see [10, 24]) we have, \( \tilde{P}^* = \bigcap_{\beta \in \text{V}(\tilde{P})} K(\beta, 1) \), where \( K(\beta, 1) = \{ x \in \mathbb{R}^n : \langle \beta, x \rangle \leq 1 \} \) and \( V(\tilde{P}) \) is the set of the vertices of \( \tilde{P} \). Hence,
\[ \tilde{P}^* \cap \mathbb{R}^n_+ = \tilde{\Gamma}(f). \]

Let \( \tilde{\Lambda}_{\text{max}} \) be the solution set of the problems (3.3). Put
\[ \tilde{\Lambda}_\infty := \{ y \in \tilde{P}^* : \langle x, y \rangle = 1, \text{ for all } x \in \tilde{P} \}. \]

Then, \( \tilde{\Lambda}_\infty \) is a face of \( \tilde{P}^* \) and \( (\tilde{\Lambda}_{\text{max}})^* = \tilde{\Lambda}_\infty \). We see that if \( x \in \tilde{\Lambda}_{\text{max}} \), then \( x_1 + \ldots + x_n = \theta' \).
Therefore \( \langle D_{\text{max}}\tilde{\Gamma}(f), x \rangle = \frac{1}{\theta'}(x_1 + \ldots + x_n) = 1 \). Hence, \( D_{\text{max}}\tilde{\Gamma}(f) \in \tilde{\Lambda}_\infty \). Since \( \tilde{\Lambda}_\infty \) does not contain the origin, \( \tilde{\Lambda}_\infty \) is the face of \( \tilde{\Gamma}(f) \), which contains the point \( D_{\text{max}}\tilde{\Gamma}(f) \) and
\[ \dim \tilde{\Lambda}_{\text{max}} = k' = n - \dim \tilde{\Lambda}_\infty - 1 = n - k' - 1. \]

3.2. Proof of Theorem 2.6

Put \( \Omega := 2\Gamma \) and \( N_\Omega := \{ h \in \mathbb{R}[x_1, \ldots, x_n] : \Gamma(h) = \Omega \} \). We consider the map
\[ F : N_\Gamma \rightarrow N_\Omega, \quad g = (g_1, \ldots, g_m) \mapsto F_g = \sum_{i=1}^m g_i^2, \]
where \( F_g(x) = \sum_{i=1}^m g_i^2(x) \). It is obvious that \( F \) is a continuous mapping. Put
\[ A_\Omega := \{ f \in N_\Omega : \text{there exist } r > 0, c > 0 \text{ such that } \| x \| \geq r \Rightarrow f(x) \geq c \sum_{\alpha \in \text{V}(\Omega)} x^\alpha \}. \]

Claim 3.7. \( g = (g_1, \ldots, g_m) \in D_\Gamma \) if and only if \( F_g \in A_\Omega \).

Proof. Let \( g = (g_1, \ldots, g_m) \in D_\Gamma \). Then \( \Gamma(g) = \Gamma \), and for any face \( \Delta \) of \( \Gamma(g) \), we have
\[ \max |g_i, \Delta(x)| \neq 0 \text{ for all } x \in (\mathbb{R}^n \setminus \{0\})^n. \]
Let \( \Delta' \) be a face of \( \Omega \), \( \Delta' = 2\Delta \). By Claim 3.6 we have
\[ F_g(x) = \sum_{i=1}^m g_i^2(x) \neq 0, \text{ and } (F_g)_{\Delta'}(x) = \sum_{i=1}^m g_i^2, \Delta(x) \neq 0. \]
Therefore \( F_g \) satisfies the Mikhailov - Gindikin. By Theorem 3.1, there exists \( c > 0 \) and \( \rho > 0 \) such that
\[ |F_g(x)| \geq c \sum_{\alpha \in \text{V}_\Omega} |x^\alpha| = c \sum_{\alpha \in \text{V}_\Omega} x^\alpha, \text{ for all } x \in \mathbb{R}^n \text{ satisfying } |x| > \rho. \]

Since all the point \( \alpha \in \text{V}_\Omega \) have even coordinates.

We now prove the converse. Let \( F_g \in A_\Omega \). Then \( \Gamma(F_g) = 2\Gamma(g) \) and there exist the numbers \( r > 0, c > 0 \) such that
\[ \| x \| \geq r \Rightarrow F_g(x) = \sum_{i=1}^m g_i^2(x) \geq c \sum_{\alpha \in \text{V}_\Omega} x^\alpha. \]
Let \( \Delta \) be a face of \( \Gamma(g) \), and \( \Delta' = 2\Delta \) be the corresponding face of \( \Omega \). Let \( q = (\rho_1, \ldots, \rho_n) \) be an interior point of the normal cone of \( \Delta \). Then
\[
\Delta = \{ x \in \Gamma : \langle x, q \rangle = d(q) \},
\]
and \( \langle x, q \rangle < d(q) \) if \( x \in \Gamma \setminus \Delta \).
Take a point \( x_0 \in (\mathbb{R} \setminus \{0\})^n \), we see that
\[
F_g(t^{\rho_1}x_1^0, \ldots, t^{\rho_n}x_n^0) = (F_g)_\Delta'(t^{\rho_1}x_1^0, \ldots, t^{\rho_n}x_n^0) + \text{lower order terms in } t
\]
Hence
\[
t^{2d(q)}(F_g)_\Delta'(x_0) + o(t^{2d(q)}) \geq c \sum_{\alpha \in V_\Omega} x^\alpha,
\]
for \( t \) sufficiently large.
By Claim 3.6,
\[
(F_g)_\Delta'(x_0) = \sum_{i=1}^m g_i^2 \Delta(x_0) > 0.
\]
It follows that,
\[
\max |g_i(\Delta(x_0))| \neq 0.
\]

**Claim 3.8.** \( A_\Omega \) is an open subset of \( N_\Omega \).

**Proof.** Assume \( f(x) = \sum c_\alpha x^\alpha \in A_\Omega \). Therefore, there exist \( r > 0, c > 0 \) such that
\[
||x|| \geq r \implies f(x) \geq c \sum_{\alpha \in V_\Omega} x^\alpha.
\]
We shall show that there exists a number \( \epsilon > 0 \) such that, if \( |\delta_\alpha| < \epsilon \), for any \( \alpha \in \Omega \), then
\[
\tilde{f}(x) := \sum_{\alpha \in \Omega} (c_\alpha + \delta_\alpha)x^\alpha \in A_\Omega.
\]
In fact, if \( ||x|| \geq r \) then
\[
(3.12) \quad \tilde{f}(x) \geq \sum_{\alpha \in \Omega} c_\alpha x^\alpha - \sum_{\alpha \in \Omega} |\delta_\alpha||x|^\alpha.
\]
By [7] (Lemma 1, p. 160), if \( \alpha \in \Omega \cap \mathbb{N}^n \) then
\[
(3.13) \quad |x|^\alpha \leq \sum_{\alpha \in V_\Omega} x^\alpha.
\]
Thus
\[
(3.14) \quad \sum_{\alpha \in \Omega \cap \mathbb{N}^n} |\delta_\alpha||x|^\alpha \leq \sum_{\alpha \in \Omega \cap \mathbb{N}^n} \epsilon |x|^\alpha \leq \epsilon \eta \sum_{\alpha \in V_\Omega} x^\alpha,
\]
where \( \eta \) is the number of integer points in \( \Omega \).
Combining the inequalities (3.12), (3.13), and (3.14) we get the following inequality
\[
\tilde{f}(x) \geq (c - \epsilon \eta) \sum_{\alpha \in V_\Omega} x^\alpha,
\]
for all \( x \in \mathbb{R}^n \) satisfying \( ||x|| \geq r \).
Thus, if $\epsilon = \frac{c}{2\eta}$ then $\tilde{f}(x) \geq \frac{c}{2} \sum_{a \in V_{\Omega}} x^a$. Therefore $\tilde{f}(x) \in A_{\Omega}$ and the claim 3.8 is proved.

We continue the proof of Theorem 2.6.

Assume that $g_0 \in D_{\Gamma}$. We will show that there exists an open neighborhood $U(g_0)$, s.t. $U(g_0) \subset D_{\Gamma}$. By the claim 3.7 since $g_0 \in D_{\Gamma}$ we have

$$F(g_0) \in A_{\Omega}.$$  

By Claim 3.8 there exist an open set $V \subset A_{\Omega}$, containing $F(g_0)$. The mapping $F : N_{\Gamma} \rightarrow N_{\Omega}$ is continuous, then there exists an open neighborhood $U(g_0)$ of $g_0$, such that

$$F(U(g_0)) \subset V.$$  

Takes any element $g \in U(g_0)$, we have $F(g) \in V$. Hence $F(g) \in A_{\Omega}$. Now, it follows from Claim 3.7 $g \in D_{\Gamma}$. Thus the open set $U(g_0)$ is contained $D_{\Gamma}$.

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