MULTITIME PROPAGATORS AND THE CONSISTENCY CONDITION
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For a nonrelativistic quantum system of \( N \) particles, the wave function is a function of \( 3N \) spatial coordinates and one temporal coordinate. The relativistic generalization of this wave function is a function of \( N \) time variables known as the multitime wave function, and its evolution is described by \( N \) Schrödinger equations, one for each time variable. To guarantee the existence of a nontrivial common solution of these \( N \) equations, the \( N \) Hamiltonians must satisfy a compatibility condition known as the integrability condition. In this work, the integrability condition is expressed in terms of Lagrangians. The time evolution of a wave function with \( N \) time variables is derived in Feynman’s picture of quantum mechanics. However, these evolutions are compatible if and only if the \( N \) Lagrangians satisfy a certain relation called the consistency condition, which can be expressed in terms of Wilson line. As a consequence of this consistency condition, the evolution of the wave function gives rise to a key feature called the “path-independence” property on the space of time variables. This suggests that one must consider all possible paths not only on the space of dependent variables (spatial variables) but also on the space of independent variables (temporal variables). Geometrically, this consistency condition can be regarded as a zero-curvature condition and the multitime evolutions can be treated as compatible parallel transport processes on the flat space of time variables.

Keywords: multitime, propagator, quantum

DOI: 10.1134/S0040577922020040

1. Introduction

In nonrelativistic quantum mechanics, the wave function for \( N \) particles can be written in the form

\[ \Psi(q_1, q_2, \ldots, q_N, t), \]

where \( q_k \in \mathbb{R}^d \), \( k = 1, 2, \ldots, N \). If we ask for the relativistic counterpart of this wave function, we encounter a difficulty: because there is only one time variable in the wave function, it is not clear how Lorentz transformations should be performed. The argument of \( \Psi \) can be treated as a collection of \( N \) simultaneous spacetime points \((t, q_1), \ldots, (t, q_N)\), which under a Lorentz transformation change to \((t', q'_1), \ldots, (t'_N, q'_N)\), where of course, \( t'_1 \neq t'_2 \neq \cdots \neq t'_N \) in general. It is then quite natural to introduce the multitime structure into the wave function \( \Phi(q_1, t_1, q_2, t_2, \ldots, q_N, t_N) \) to manifest the Lorentz transformation. This idea was first introduced by Dirac in 1932 [1]. Consequently, we have a set of partial differential equations

\[ i \frac{\partial}{\partial t_j} \Phi(q_1, t_1, q_2, t_2, \ldots, q_N, t_N) = H_j \Phi(q_1, t_1, q_2, t_2, \ldots, q_N, t_N), \quad j = 1, 2, \ldots, N, \]

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S. Sungted is supported by the Development and Promotion of Science and Technology Talents Project (DPST).

1We set \( \hbar = 1 \) throughout the text.
where $H_j$ is a self-adjoint Hamiltonian for the $j$th particle. These multitime systems are compatible (equivalently, a common nontrivial solution $Φ$ exists) if and only if the relations

$$\frac{\partial H_j}{\partial t_k} - \frac{\partial H_k}{\partial t_j} - i[H_j, H_k] = 0 \quad \forall j \neq k$$

hold. This is known as the consistency or integrability condition [2]–[6].

The idea of the multitime wave function formalism could possibly be useful in many aspects. For example, Petrat and Tumulka [7] demonstrated that the relevant interacting quantum field theories can be reformulated in terms of multitime wave functions, and hence the multitime wave function, the Tomonaga–Schwinger, and the Heisenberg approaches are equivalent. The consistency condition of the multitime formulation explains why the process of a fermion decay into two fermions cannot happen in nature [8]. Lienert, Petrat, and Tumulka [9] pointed out that the multitime wave function can be considered within the discrete action principle and can be applied to study cellular automata.

Here comes the main question of this work. What is the Lagrangian analogue of the consistency condition? This question is natural to be asked because normally in physics we can choose to work with Hamiltonian or Lagrangian descriptions. In this paper, the variational principle plays a central role in obtaining the consistency (integrability) condition and the quantum multitime evolution is studied in the framework of Feynman’s path integral expressed in terms of a Wilson line.

The remainder of this paper is organized as follows. In Sec. 2, a brief review of the Hamiltonian approach at both classical and quantum levels is given. In Sec. 3, the derivation of the integrability condition via a variational principle is explained in the classical case. After that, the multitime propagators are constructed and quantum multitime evolution is studied. The conclusions are given in Sec. 4.

2. Hamiltonian approach

2.1. Classical case. In this section, we give a short review of the multitime structure in the context of classical mechanics, together with the derivation of the consistency criterion known as the Hamiltonian commutativity condition [6], [10].

We suppose there is a set of Hamiltonians $\{H_1, H_2, \ldots, H_N\}$ and a multitime Hamilton–Jacobi function $Φ$ associated with a set of time variables $t = (t_1, t_2, \ldots, t_N)$, where $t_j \in \mathbb{R}$. We then seek solutions of a set of the first-order differential equations

$$\frac{\partial}{\partial t_j} Φ(q, t) + H_j(q, t, \frac{\partial}{\partial q} Φ(q, t)) = 0, \quad q \in \mathbb{R}^d, \quad j = 1, 2, \ldots, N. \quad (2.1)$$

It is well known that these are the multitime Hamilton–Jacobi equations, and the system is overdetermined. Then, to obtain a nontrivial common solution, we may need all Hamiltonians to commute in an appropriate way, known as the Hamiltonian commuting flows. To obtain that particular consistency condition, we consider the compatibility of flows with respect to $t_i$ and $t_j$:

$$\frac{\partial^2}{\partial t_j \partial t_i} Φ = \frac{-\partial H_j}{\partial t_j} + \frac{\partial H_i}{\partial (\partial Φ/\partial q_k)} \left( \frac{\partial H_j}{\partial q_k} + \frac{\partial H_j}{\partial (\partial Φ/\partial q_l)} \cdot \frac{\partial^2}{\partial q_k \partial q_l} Φ \right),$$

$$\frac{\partial^2}{\partial t_i \partial t_j} Φ = \frac{-\partial H_i}{\partial t_i} + \frac{\partial H_j}{\partial (\partial Φ/\partial q_k)} \left( \frac{\partial H_i}{\partial q_k} + \frac{\partial H_i}{\partial (\partial Φ/\partial q_l)} \cdot \frac{\partial^2}{\partial q_k \partial q_l} Φ \right). \quad (2.2)$$

The compatibility requires

$$\left( \frac{\partial^2}{\partial t_j \partial t_i} - \frac{\partial^2}{\partial t_i \partial t_j} \right) Φ = 0, \quad (2.3)$$

which leads to the condition

$$\frac{\partial H_i}{\partial t_j} + \frac{\partial H_i}{\partial t_i} - \{H_i, H_j\} = 0, \quad (2.4)$$

where $\{·, ·\}$ is the standard Poisson bracket.
2.2. Quantum case.

Single-time case. A natural way to pass from the classical level to the quantum level in terms of the Hamiltonian function is the Schrödinger approach. Given a state $\Psi(q, t)$ defined in the Hilbert space $\mathcal{H}$, the Schrödinger equation for a particle of mass $m$ trapped in a potential $V$ is

$$i \frac{\partial \Psi}{\partial t} = H \Psi,$$  \hspace{1cm} (2.5)

where $H$ is the Hamiltonian operator.

The time evolution of the wave function can be considered in terms of the unitary operator $U(t', t)$, where $t' > t$, such that

$$\Psi(q', t') = U(t', t)\Psi(q, t),$$  \hspace{1cm} (2.6)

where

$$U(t', t) = \exp \left( -i \int_{t}^{t'} H(\tau) d\tau \right).$$  \hspace{1cm} (2.7)

It might happen that the Hamiltonian operators evaluated at different moments of time do not commute, i.e., $[H(s), H(s')] \neq 0$. In that case, the time evolution operator becomes

$$U(t, s) = \mathcal{T} \exp \left( -i \int_{t}^{s} H(T, s) dT \right) =$$

$$= I + \sum_{n=1}^{\infty} (-i)^n \int_{s}^{T_1} dT_1 \int_{s}^{T_2} dT_2 \ldots \int_{s}^{T_{n-1}} (H(T_1, s)H(T_2, s) \ldots H(T_n, s)) dT_n,$$  \hspace{1cm} (2.8)

where $\mathcal{T}$ is the time ordering operator and this expansion is known as the Dyson series [11]. However, in this study, we restrict ourself to the case of time-independent Hamiltonians.

Multi-time case. In this case, we suppose that there are $N$ particles in the system and $(q_1, q_2, \ldots, q_N)$ is a set of coordinates. The single-time wave function is given by $\Psi(q_1, q_2, \ldots, q_N, t)$ and the relativistic version is $\Phi(q_1, t_1, q_2, t_2, \ldots, q_N, t_N)$, which satisfies $N$ separable time-dependent Schrödinger equations [1], [12]

$$\left( H_j + \frac{1}{i} \frac{\partial}{\partial t_j} \right) \Phi(q_1, t_1, q_2, t_2, \ldots, q_N, t_N) = 0, \hspace{1cm} j = 1, 2, \ldots, N,$$  \hspace{1cm} (2.9)

where $H_j$ are the free Schrödinger Hamiltonians (or free Dirac Hamiltonians). The ordinary probability amplitude is retrieved by setting all time coordinates equal:

$$\Phi(q_1, t, q_2, t, \ldots, q_N, t) = \Psi(q_1, q_2, \ldots, q_N, t).$$  \hspace{1cm} (2.10)

Here, the single-time wave function $\Psi$ satisfies the standard Schrödinger equation (2.5) and $H = \sum_{j=1}^{N} H_j$. Equations (2.10) and (2.5) suggest that the multitime wave function coincides with the single-time wave function with respect to the Lorentz frame on configurations of $N$ spacetime points [9].

Here comes an interesting feature of the system of equations (2.9). The multitime evolution must satisfy a certain condition. We suppose that the multitime wave function evolves from the initial point $(0, 0)$ to the final point $(t_1, t_2)$.$^2$ In the case of time-independent Hamiltonians, we define $U_1(t_1) = e^{-iH_1t_1}$ as the unitary time operator in the $t_1$ direction and $U_2(t_2) = e^{-iH_2t_2}$ as the unitary time operator in the $t_2$ direction. There are two different ways for the system to evolve:

$$\Phi(t_1, t_2) = e^{-iH_2t_2} \Phi(t_1, 0) = e^{-iH_2t_2} e^{-iH_1t_1} \Phi(0, 0) = U_2 U_1 \Phi(0, 0),$$

$$\Phi(t_1, t_2) = e^{-iH_1t_1} \Phi(0, t_2) = e^{-iH_1t_1} e^{-iH_2t_2} \Phi(0, 0) = U_1 U_2 \Phi(0, 0).$$  \hspace{1cm} (2.11)

It follows from these equations that the evolution is compatible if and only if $[H_1, H_2] = 0$, which is called the consistency condition or the integrability criterion for the multitime evolution, see Fig. 1.

$^2$For simplicity, we consider only two time variables.
In the case of time-dependent Hamiltonians, we can obtain the consistency condition in the form [5]

\[ \frac{\partial H_j}{\partial t_k} - \frac{\partial H_k}{\partial t_j} - i[H_j, H_k] = 0 \quad \forall j \neq k. \]  

(2.12)

This equation can be regarded as a quantum analogue of Eq. (2.4).

**Remark.** The wave function \( \Phi \) is defined only on space-like configurations. For a fixed number of particles, the system of multitime equations with interaction potentials automatically violates consistency condition [5]. Nevertheless, there is a special initial datum, with all time variables set equal to zero. In spite of the inconsistency, the system of \( N \) Schrödinger equations can be solved simultaneously [13].

Condition (2.12) implies the path-independence feature of the time evolution in the context of multitime quantum theory. This can be seen by the following construction. If we consider the path that is parameterized by \( \gamma \) (see Fig. 2a), where \( \gamma \) goes from the initial point \( \gamma(0) = \vec{t} = (t_1^i, t_2^i, \ldots, t_N^i) \) to the final point \( \gamma(1) = \vec{t} = (t_1^f, t_2^f, \ldots, t_N^f) \), then the time evolution operator along this particular path is given by

\[ U_\gamma = \text{Exp} \left( -i \int_{\gamma} \sum_j H_j \, dt_j \right). \]  

(2.13)

Another path is parameterized by \( \gamma' \) (see Fig. 2a), where \( \gamma' \) goes from the initial point \( \gamma'(0) = \vec{t}' = (t_1^i, t_2^i, \ldots, t_N^i) \) to the final point \( \gamma'(1) = \vec{t}' = (t_1^f, t_2^f, \ldots, t_N^f) \); the time evolution operator along this path is given by

\[ U_{\gamma'} = \text{Exp} \left( -i \int_{\gamma'} \sum_j H_j \, dt_j \right). \]  

(2.14)

The path-independence feature requires the condition \( U_\gamma = U_{\gamma'} \).
In the language of geometry, we can represent the path-independence feature as parallel transport. To see this, we define the covariant derivative $\nabla_j = \partial_j - iA_j$, where $\partial_j = \partial/\partial t_j$ and the connection coefficient is $A_j = -H_j$. Then $U_\gamma$ can be treated as the parallel transport operator along the path $\gamma$, known as a path-ordered integral or a Wilson line. For an arbitrary loop $\gamma$ (see Fig. 2b), we can express the transport operator in the form

$$U_\gamma = \exp\left( -i \oint_\gamma \sum_j H_j dt_j \right),$$

(2.15)

which is known as the Wilson loop. Then the path-independence property is nothing but the requirement that all closed paths $\gamma$ have trivial holonomy, i.e., $U_\gamma = I$. Consequently, a gauge connection has trivial holonomies if and only if its curvature $F$, defined as

$$F_{jk} \equiv -\frac{\partial H_k}{\partial t_j} + \frac{\partial H_j}{\partial t_k} - i[H_j, H_k],$$

(2.16)

vanishes [5]:

$$F_{jk} = 0 \quad \forall j \neq k.$$  

(2.17)

With the definition of the curvature, we can rewrite the exponent in the expression for the transport operator as

$$-i \oint_{\partial \Sigma} \sum_j H_j dt_j = -i \int \int_{\Sigma} \sum_{ij} F_{ij} dt_i \land dt_j,$$

(2.18)

where $\Sigma$ is a 2-dimensional surface whose boundary is $\partial \Sigma$.

Obviously, condition (2.17) is identical to (2.12), and we can therefore consider the consistency condition in terms of curvature. We know that curvature is the tool to test the difference between a vector and its parallel transport along a closed path. If the directions of the initial and the final vectors coincide, there is no curvature of the surface, $F_{jk} = 0$, which means a flat surface. Therefore, consistency condition (2.12) for the multitime wave function can be treated as a zero-curvature condition.

### 3. Lagrangian approach

In the preceding section, we described the consistency conditions for multitime evolution at both classical and quantum levels in terms of the Hamiltonian picture. In this section, we express the consistency condition, in both classical and quantum cases, in terms of a Lagrangian.

#### 3.1. Classical case.

We start with giving the action functional along a path $\Gamma$ on the space of time variables, as shown in Fig. 3 in the case of two time variables,

$$S_\Gamma[t] = \int_\Gamma \sum_{i=1}^N L_i dt_i,$$

(3.1)

where $L_i = L_i(dq_i/dt_i, q_i; t)$ is the Lagrangian for the $i$th particle. We introduce a new variable $\sigma_0 \leq \sigma \leq \sigma_1$ such that $(t_1(\sigma), t_2(\sigma), \ldots, t_N(\sigma))$. Action (3.1) then becomes

$$S_\Gamma[t(\sigma)] = \int_{\sigma_0}^{\sigma_1} L d\sigma, \quad \text{where} \quad L = \sum_{i=1}^N \frac{dt_i}{d\sigma}.$$  

(3.2)
Fig. 3. Variation of a path on the space of two time variables.

To express the consistency condition for multitime evolution, we consider the time variation $t_i \rightarrow t_i + \delta t_i$ resulting in a new path $\Gamma'$ with the action

$$S_{\Gamma'}[t(\sigma) + \delta t(\sigma)] = \int_{\sigma_0}^{\sigma_1} \left( \sum_{i=1}^{N} L_i(t + \delta t) \frac{d(t_i + \delta t_i)}{d\sigma} \right) d\sigma. \tag{3.3}$$

Using the Taylor series expansion and ignoring higher-order terms, we express each Lagrangian as

$$L_i(t + \delta t) = L_i(t) + \sum_{j=1}^{N} \delta t_j \frac{\partial L_i}{\partial t_j} + \cdots, \quad i = 1, 2, \ldots, N. \tag{3.4}$$

The variation of the action is given by

$$S_{\Gamma'}[t(\sigma) + \delta t(\sigma)] - S_{\Gamma}[t] \equiv \delta S \approx \int_{\sigma_0}^{\sigma_1} \left\{ \sum_{i=1}^{N} \left( \sum_{j=1}^{N} \delta t_j \frac{\partial L_i}{\partial t_j} \right) \frac{dt_i}{d\sigma} + \sum_{i=1}^{N} L_i \frac{d\delta t_i}{d\sigma} \right\} d\sigma. \tag{3.5}$$

After integration by parts, (3.5) becomes

$$\delta S = \int_{\sigma_0}^{\sigma_1} \left\{ \sum_{i, j=1}^{N} \delta t_{ij} \left( \frac{\partial L_j}{\partial t_i} - \frac{\partial L_i}{\partial t_j} \right) \frac{dt_i}{d\sigma} \right\} d\sigma \quad \forall i \neq j. \tag{3.6}$$

Imposing the condition $\delta S = 0$, we obtain

$$\frac{\partial L_j}{\partial t_i} = \frac{\partial L_i}{\partial t_j} \quad \forall i \neq j. \tag{3.7}$$

This equation is nothing but the consistency condition for the multitime evolution in Lagrangian terms.\(^3\) Consequently, under condition (3.7), the action remains the same under path variations on the space of time variables. This is nothing but the path-independence feature of the evolution on the space of time variables.

\(^3\)This equation was first derived in a different context, that of an integrable 1-dimensional many-body system [14], where it also expresses a consistency condition.
Remark. We note that we can vary the action with respect to the coordinate variables, resulting in a set of Euler–Lagrange equations together with constraints [14].

Equation (3.7) can also be obtained from the geometric standpoint. We suppose that \( \alpha \) is a differential \((k-1)\)-form. The generalized Stokes theorem states that the integral of its exterior derivative over a smooth oriented \( k \)-dimensional manifold \( \Sigma \) is equal to its integral along the boundary \( \partial \Sigma \) of the manifold \( \Sigma \) [15]:

\[
\int_{\partial \Sigma} \alpha = \int \int_{\Sigma} d\alpha. \tag{3.8}
\]

We now introduce an object \( dS \) given by

\[
dS = \sum_{i=1}^{N} L_i \, dt_i, \tag{3.9}
\]

as a 1-form on the \( N \)-dimensional space of independent variables, and the action (3.1) then becomes \( S = \int_{\Gamma} dS \). With the the exterior derivative applied to the smooth function coefficients that are the Lagrangian in this case, Eq. (3.8) becomes

\[
\int_{\partial \Sigma} \sum_{i=1}^{N} L_i \, dt_i = \int \int_{\Sigma} \sum_{1 \leq i < j \leq N} \left( \frac{\partial L_i}{\partial t_i} - \frac{\partial L_i}{\partial t_j} \right) dt_i \wedge dt_j. \tag{3.10}
\]

The left-hand side here is equivalent to \( \int_{\Gamma} dS - \int_{\Gamma'} dS \). The right-hand side of (3.10) vanishes because the exterior derivative operating on a closed form gives zero. Therefore, we obtain

\[
\frac{\partial L_i}{\partial t_i} - \frac{\partial L_i}{\partial t_j} = 0, \quad i, j = 1, 2, \ldots, N, \quad i \neq j. \tag{3.11}
\]

These are the consistency conditions for the system that evolves in the \( N \)-dimensional space of independent variables. The main point is that Eq. (3.10) is the Lagrangian version of parallel transport (see Eq. (2.18)) if we define

\[
F_{ij} = \frac{\partial L_i}{\partial t_i} - \frac{\partial L_i}{\partial t_j}, \quad i, j = 1, 2, \ldots, N, \quad i \neq j. \tag{3.12}
\]

Consequently, consistency condition (3.11) of multitime evolution can be treated as a zero-curvature condition in terms of the Lagrangians.

We find that condition (3.7) is violated if there is interaction. To see this, we give a simple example as follows. Given

\[
L_1 = \frac{m \dot{q}_1^2}{2} + k q_1 q_2, \quad L_2 = \frac{m \dot{q}_2^2}{2}, \tag{3.13}
\]

where \( q_1 = q_1(t_1), \) \( q_2 = q_2(t_2), \) and \( k \) is a constant, we have

\[
\frac{\partial L_1}{\partial t_2} = k q_1 \frac{\partial q_2}{\partial t_2}, \quad \frac{\partial L_2}{\partial t_1} = 0. \tag{3.14}
\]

Thus the interaction leads to inconsistency. We see later in the quantum case that the interaction also gives incompatible quantum evolution in terms of propagators.
3.2. Quantum case. To express the quantum version of the consistency condition in Lagrangian terms, the appropriate approach is the Feynman path integration method. We first briefly recall some basic ingredients.

**Single-time case.** The main mathematical object in this section is the propagator

\[ K(q^f, t^f; q^i, t^i) = \langle q^f | U(t^f - t^i) | q^i \rangle. \]  

(3.15)

The propagator provides the probability amplitude for a particle to travel from the initial point \((q^i, t^i)\) to the final point \((q^f, t^f)\). If we introduce the time \(t_1\) such that \(t^f > t_1 > t^i\), the propagator can be factored as

\[ K(q^f, t^f; q^i, t^i) = \langle q^f | U(t^f - t_1 + t_1 - t^i) | q^i \rangle = \langle q^f | U(t^f - t_1) \int |q_1\rangle \langle q_1 | U(t_1 - t^i) | q^i \rangle \ dq_1 = \int K(q^f, t^f; q_1, t_1) K(q_1, t_1; q^i, t^i) \ dq_1. \]  

(3.16)

This equation suggests that the transition amplitude of the quantum particle from the initial point to the final point must be taken into account for all possible points \(q_1\) at the time \(t_1\). We can divide the time interval into \(n\) parts such that \(t^f \equiv t_n > t_{n-1} > t_{n-2} > \cdots > t_2 > t_1 > t_0 \equiv t^i\), resulting in

\[ K(q_n, t_n; q_0, t_0) = \left( \prod_{k=1}^{n-1} \int dq_k \right) \prod_{k=0}^{n-1} K(q_{k+1}, t_{k+1}; q_k, t_k). \]  

(3.17)

The discrete propagator is given by [16]

\[ K(q_{k+1}, t_{k+1}; q_k, t_k) = \sqrt{\frac{m}{2\pi i (t_{k+1} - t_k)}} e^{i(t_{k+1} - t_k) L(q_k, q_{k+1})} \]  

(3.18)

where

\[ L(q_k, q_{k+1}) = \frac{m}{2} \left( \frac{q_{k+1} - q_k}{t_{k+1} - t_k} \right)^2 - V(q_k). \]

Taking \(t_{k+1} - t_k \equiv \Delta t \to 0\) and \(n \to \infty\), propagator (3.17) can be written as

\[ K(q^f, t^f; q^i, t^i) = \int_{q^i}^{q^f} \mathcal{D}[q(t)] e^{iS[q(t)]}, \]  

(3.19)

where

\[ \int_{q^i}^{q^f} \mathcal{D}[q(t)] \equiv \lim_{n \to \infty} \left( \frac{m}{2\pi i \Delta t} \right)^{n/2} \left( \prod_{k=1}^{n-1} \int dq_k \right), \quad S[q(t)] = \int_t^{t_f} L(q, q; t) \ dt; \]

Here, \(L(q, q; t) = T(q) - V(q)\) is the standard single-time Lagrangian.

**Remark.** The explicit form of the propagator can be obtained in the case of a quadratic Lagrangian,

\[ K(q^f, t^f; q^i, t^i) = F(t^f - t^i) e^{iS_c}, \quad F(t^f - t^i) = \sqrt{\frac{1}{2\pi i} \left| \frac{\partial^2 S_c}{\partial q^f \partial q^{f'}} \right|}. \]  

(3.20)

where \(S_c\) is the classical action [17].

**Multi-time case.** Next, we discuss the consistency condition for the multitime evolution in the Feynman picture.
from the initial point (\(q_1, t_1, t_2\)) to the final point (\(t_1', t_2'\)).

**Compatible evolution.** For simplicity, we consider the evolution of the multitime wave function from the initial point \((t_1, t_2)\) to the final point \((t_1', t_2')\) (see Fig. 4), along two different paths in the context of Feynman path integration on the space of time variables.

Along the first path (solid line in Fig. 4), the multitime wave function evolves from the point \((t_1, t_2)\) to the point \((t_1', t_2')\) first under the unitary operator \(U_1\) from \(t_1\) to \(t_1'\) and then under the unitary operator \(U_2\) from \(t_2\) to \(t_2'\). The lower-half path \(\gamma\) in Fig. 4 corresponds to the propagator

\[
\langle q'_1, q'_2 | \Phi_{\gamma}(t_1', t_2') \rangle = \langle q'_1, q'_2 | U_2 U_1 | \Phi(t_1, t_2) \rangle = \\
= \iint \langle q'_1, q'_2 | U_2 | q_2 \rangle \langle q_2, q_1 | \Phi(t_1, t_2) \rangle \, dq_1 \, dq_2 \\
= \iint \langle q'_2 | U_2 | q_2 \rangle \langle q_2, q_1 | \Phi(t_1, t_2) \rangle \, dq_1 \, dq_2 \\
= \iint \langle q'_2 | U_2 | q_2 \rangle \delta(q'_2 - q_1) \delta(q_2 - q_2) \Phi(q_1, q_2, t_1, t_2) \, dq_1 \, dq_2 = \\
= \int \langle q'_2 | U_2 | q_2 \rangle \Phi(q_1, q_2, t_1, t_2) \, dq_2,
\]

and

\[
\Phi_{\gamma}(q'_1, q'_2, t_1', t_2') = \int K_2(q'_2, t_2'; q_2, t_2) K_1(q'_1, t_1'; q_1, t_1) \Phi(q_1, q_2, t_1, t_2) \, dq_1 \, dq_2.
\]

Along the second path (dashed line in Fig. 4), the multitime wave function evolves from \((t_1, t_2)\) to \((t_1', t_2')\) as

\[
\langle q'_1, q'_2 | \Phi_{\gamma}(t_1', t_2') \rangle = \langle q'_1, q'_2 | U_2 U_1 | \Phi(t_1, t_2) \rangle = \\
= \iint \langle q'_1, q'_2 | U_1 | q_1 \rangle \langle q_1, q_2 | \Phi(t_1, t_2) \rangle \, dq_1 \, dq_2 \\
= \iint \langle q'_1 | U_1 | q_1 \rangle \langle q_1, q_2 | \Phi(t_1, t_2) \rangle \, dq_1 \, dq_2 \\
= \iint \langle q'_1 | U_1 | q_1 \rangle \delta(q'_1 - q_1) \delta(q_1 - q_1) \Phi(q_1, q_2, t_1, t_2) \, dq_1 \, dq_2 = \\
= \int \langle q'_1 | U_1 | q_1 \rangle \Phi(q_1, q_2, t_1, t_2) \, dq_1 \, dq_2,
\]

and

\[
\Phi_{\gamma}(q'_1, q'_2, t_1', t_2') = \int K_1(q'_1, t_1'; q_1, t_1) K_2(q'_2, t_2'; q_2, t_2) \Phi(q_1, q_2, t_1, t_2) \, dq_1 \, dq_2.
\]

To make both transitions compatible, we require that \(\Phi_{\gamma}(q'_1, q'_2, t_1', t_2') = \Phi_{\gamma}(q'_1, q'_2, t_1', t_2')\), resulting in

\[
\iint \{K_2 K_1 - K_1 K_2\} \Phi(q_1, q_2, t_1, t_2) \, dq_1 \, dq_2 = 0. \tag{3.21}
\]
We now define

\[ K_\gamma(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2) = K_\gamma(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2) = K_\gamma(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2). \]

(3.22)

to be the respective lower-half and upper-half propagators. Because \( \Phi(q_1, q_2, t_1, t_2) \) cannot be zero, it follows from (3.21) that

\[ K_\gamma(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2) = K_\gamma(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2). \]

Here, we obtain the consistency condition for multitime evolution in terms of propagators. This equation is nothing but the propagator commutativity property \([K_1, K_2] = 0\), reflecting the path-independence property of the propagator on the space of time variables.

We can treat the commuting propagators in (3.22) as the parallel transport operation in Lagrangian terms. We can write the propagators in terms of the Wilson lines associated with the paths \( \gamma \) and \( \gamma' \):

\[ K_\gamma(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2) = \int_{q_2}^{q_2'} \mathcal{D}[\tilde{q}_2(\tilde{t}_2)] \int_{q_1}^{q_1'} \mathcal{D}[\tilde{q}_1(\tilde{t}_1)] e^{i \int_{\tilde{t}_1}^{\tilde{t}_2} (L_1(\tilde{q}_1, \partial_{\tilde{t}_1} \tilde{q}_1) d\tilde{t}_1 + L_2(\tilde{q}_2, \partial_{\tilde{t}_2} \tilde{q}_2) d\tilde{t}_2)}, \]

\[ K_{\gamma'}(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2) = \int_{q_1}^{q_1'} \mathcal{D}[\tilde{q}_1(\tilde{t}_1)] \int_{q_2}^{q_2'} \mathcal{D}[\tilde{q}_2(\tilde{t}_2)] e^{i \int_{\tilde{t}_1}^{\tilde{t}_2} (L_1(\tilde{q}_1, \partial_{\tilde{t}_1} \tilde{q}_1) d\tilde{t}_1 + L_2(\tilde{q}_2, \partial_{\tilde{t}_2} \tilde{q}_2) d\tilde{t}_2)}. \]

(3.23)

Equation (3.22) gives the invariance property of the propagators sharing the endpoints \( K_\gamma = K_{\gamma'} \). The result in (3.23) can be easily extended to the case of \( N \) time variables and the propagator written in terms of the Wilson line \( \gamma \),

\[ K_\gamma(q_1', t_1', q_2', t_2'; q_1, t_1, q_2, t_2; \ldots; q_N', t_N'; q_1, t_1, q_2, t_2; \ldots; q_N, t_N) = \mathcal{P} \prod_{i=1}^{N} \int_{q_i}^{q_i'} \mathcal{D}[\tilde{q}_i(\tilde{t}_i)] e^{i \int_{\tilde{t}_1}^{\tilde{t}_N} (L_1(\tilde{q}_i, \partial_{\tilde{t}_1} \tilde{q}_i) d\tilde{t}_i)}, \]

(3.24)

where \( \mathcal{P} \) stands for permutation. Therefore, propagator (3.24) is invariant under the permutation.

**Time loops.** We now consider another type of evolution, called the loop transition. Before proceeding with the calculation, we need to establish some useful relations. We start with discussing the transition of the wave function from \((q, t)\) to \((q', t')\):

\[ \Phi(q', t') = \int K(q', t'; q, t) \Phi(q, t) dq. \]

(3.25)

Next, we consider the transition from \((q', t')\) to \((\tilde{q}, \tilde{t})\):

\[ \Phi(\tilde{q}, \tilde{t}) = \int K(\tilde{q}, \tilde{t}; q', t') \Phi(q', t') dq'. \]

(3.26)

Combining (3.26) with (3.25), we obtain

\[ \Phi(\tilde{q}, \tilde{t}) = \int \int K(\tilde{q}, \tilde{t}; q', t') K(q', t'; q, t) \Phi(q, t) dq' dq = \int \delta(q - \tilde{q}) \Phi(q, \tilde{t}) dq = \Phi(\tilde{q}, \tilde{t}), \]

(3.27)

To change transition (3.27) to a loop transition, we impose the condition

\[ \Phi(\tilde{q}, \tilde{t}) = \int \int K(\tilde{q}, \tilde{t}; q', t') K(q', t'; q, t) \Phi(q, t) dq' dq = \int \delta(q - \tilde{q}) \Phi(q, \tilde{t}) dq = \Phi(\tilde{q}, \tilde{t}), \]

(3.28)
which requires that
\[ \delta(\hat{q} - q) = \int K(\hat{q}, \hat{t}; q', t') K(q', t'; q, t) \, dq' = K(\hat{q}, t; q, t), \] (3.28)
where \( \hat{t} - t = \delta t \to 0 \). Equivalently, (3.28) can be expressed in terms of Lagrangians as
\[ \delta(\hat{q} - q) = \lim_{\delta t \to 0} \int dq' \left[ \int_q^{q'} D[\hat{q}(\hat{t})] \, e^{i \int_{\hat{t}}^{\hat{t}'} L(\hat{q}, \partial_{\hat{t}} \hat{q}) \, d\hat{t}} \right] = \lim_{\delta t \to 0} \int dq' \int_q^{q'} D[\hat{q}(\hat{t})] \, e^{i \int_{\hat{t}}^{\hat{t}'} L(\hat{q}, \partial_{\hat{t}} \hat{q}) \, d\hat{t}} + f_{\hat{t}}' L(q, \partial q) \, dt = \lim_{\delta t \to 0} \int_q^{q'} D[\hat{q}(\hat{t})] \, e^{i \int_{\hat{t}}^{\hat{t}'} L(q, \partial q) \, d\hat{t}}. \] (3.29)

We now consider loop evolution. We define \( U_1(t'_1 - t_1) \) as the time evolution operator from \( t_1 \) to \( t'_1 \), \( U_2(t'_2 - t_2) \) as the time evolution operator from \( t_2 \) to \( t'_2 \), \( U_1[\hat{t}_1 - t'_1] \) as the time evolution operator from \( t'_1 \) to \( \hat{t}_1 \), and \( U_2[\hat{t}_2 - t'_2] \) as the time evolution operator from \( t'_2 \) to \( \hat{t}_2 \).

**Fig. 5.** (a) Evolution from the initial point \((q_1, q_2, t_1, t_2)\) to the final point \((\hat{q}_1, \hat{q}_2, \hat{t}_1, \hat{t}_2)\) and (b) loop evolution \( \gamma \) can be obtained by imposing \( \hat{q}_1 = q_1 \) and \( \hat{t}_i = t_i \), where \( i = 1, 2 \).

The transition map shown in Fig. 5a can be expressed as
\[ \langle \hat{q}_1, \hat{q}_2 \rangle = \langle q_1, q_2 \rangle U_2 U_1 U_2 U_1 |\Phi(t_1, t_2)\rangle, \]
and
\[ \Phi(\hat{q}_1, \hat{q}_2, \hat{t}_1, \hat{t}_2) = \int \int \langle \hat{q}_1, \hat{q}_2 | U_2 U_1 | q_1, q_2 \rangle \langle q_2, q_1 | \Phi(t_1, t_2) \rangle \, dq_1 \, dq_2 = \int \int \int \langle \hat{q}_2 | U_2(t'_2 - t_2) | \hat{q}_2 \rangle \langle q_2 | U_2(t'_2 - t_2) | q_2 \rangle \, dq_2 \times \int \langle \hat{q}_1 | U_1(t'_1 - t_1) | \hat{q}_1 \rangle \langle q_1 | U_1(t'_1 - t_1) | q_1 \rangle \, dq_1 \, dq_2 = \int \int K_2(\hat{q}_2, \hat{t}_2; t'_2, q_2) K_2(q_2, t_2) \, dq_2 \times \int K_1(\hat{q}_1, \hat{t}_1; q_1, t_1) K_1(\hat{q}_1, t_1; q_1, t_1) \, dq_1 \, dq_2. \] (3.30)

The full derivation of (3.30) is given in the Appendix. Using condition (3.28) with \( \hat{t}_1 - t_1 = \delta t_1 \to 0 \) and \( \hat{t}_2 - t_2 = \delta t_2 \to 0 \), we obtain
\[ \delta(\hat{q}_2 - q_2) = \int K_2(\hat{q}_2, \hat{t}_2; t'_2, q_2) K_2(q_2, t_2) \, dq_2 = K_2(\hat{q}_2, t'_2; q_2, t_2), \] (3.31)
\[ \delta(\hat{q}_1 - q_1) = \int K_1(\hat{q}_1, \hat{t}_1; q_1, t_1) K_1(q_1, t_1) \, dq_1 = K_1(\hat{q}_1, t_1; q_1, t_1). \] (3.32)
Substituting these equations in (3.30), we arrive at

$$
\Phi(\hat{q}_1, \hat{q}_2, t_1, t_2) = \int \delta(\hat{q}_2 - q_2)\delta(\hat{q}_1 - q_1)\Phi(q_1, q_2, t_1, t_2) \, dq_1 \, dq_2 = \Phi(\hat{q}_1, \hat{q}_2, t_1, t_2);
$$

which gives the loop evolution shown in Fig. 5b.

Next, the condition for the propagator in (3.30) can be expressed in terms of the Lagrangians as

$$
\delta(\hat{q}_2 - q_2)\delta(\hat{q}_1 - q_1) = \int K_2(\hat{q}_2, \hat{t}_2; \hat{q}_1, \hat{t}_1)K_2(\hat{q}_2, \hat{t}_1'; \hat{q}_1', \hat{t}_1') K_1(\hat{q}_1, t_1; \hat{q}_1', t_1) \, dq_1
$$

$$
= \int \left[ \int_{\hat{q}_2}^{\hat{q}_1} \mathcal{D}[\hat{q}_2(\hat{t}_2)] \, e^{i \int_{\hat{t}_2}^{\hat{t}_1} L_2(\hat{q}_2, \partial_{\hat{t}_2} \hat{q}_2) \, d\hat{t}_2} \right] \left[ \int_{\hat{q}_2}^{\hat{q}_1} \mathcal{D}[\hat{q}_2(\hat{t}_2)] \, e^{i \int_{\hat{t}_2}^{\hat{t}_1} L_2(\hat{q}_2, \partial_{\hat{t}_2} \hat{q}_2) \, d\hat{t}_2} \right] \, d\hat{q}_2 \times
$$

$$
\times \int \left[ \int_{\hat{q}_1}^{\hat{q}_1} \mathcal{D}[\hat{q}_1(\hat{t}_1)] \, e^{i \int_{\hat{t}_1}^{\hat{q}_1} L_1(\hat{q}_1, \partial_{\hat{t}_1} \hat{q}_1) \, d\hat{t}_1} \right] \left[ \int_{\hat{q}_1}^{\hat{q}_1} \mathcal{D}[\hat{q}_1(\hat{t}_1)] \, e^{i \int_{\hat{t}_1}^{\hat{q}_1} L_1(\hat{q}_1, \partial_{\hat{t}_1} \hat{q}_1) \, d\hat{t}_1} \right] \, dq_1 = \left[ \int_{\hat{q}_2}^{\hat{q}_1} \mathcal{D}[\hat{q}_2(\hat{t}_2)] \, e^{i \int_{\hat{t}_2}^{\hat{t}_1} L_2(\hat{q}_2, \partial_{\hat{t}_2} \hat{q}_2) \, d\hat{t}_2} \right] \left[ \int_{\hat{q}_1}^{\hat{q}_1} \mathcal{D}[\hat{q}_1(\hat{t}_1)] \, e^{i \int_{\hat{t}_1}^{\hat{q}_1} L_1(\hat{q}_1, \partial_{\hat{t}_1} \hat{q}_1) \, d\hat{t}_1} \right]. (3.33)
$$

Taking $\delta t_1 \to 0$ and $\delta t_2 \to 0$, we obtain

$$
\delta(\hat{q}_2 - q_2)\delta(\hat{q}_1 - q_1) = \lim_{\delta t_2 \to 0} \int_{\hat{q}_2}^{\hat{q}_1} \mathcal{D}[\hat{q}_2] \, e^{i \int_{\hat{t}_2}^{\hat{t}_1} L_2(\hat{q}_2, \partial_{\hat{t}_2} \hat{q}_2) \, d\hat{t}_2} \times
$$

$$
\times \lim_{\delta t_1 \to 0} \int_{\hat{q}_1}^{\hat{q}_1} \mathcal{D}[\hat{q}_1] \, e^{i \int_{\hat{t}_1}^{\hat{q}_1} L_1(\hat{q}_1, \partial_{\hat{t}_1} \hat{q}_1) \, d\hat{t}_1} = \int_{\hat{q}_2}^{\hat{q}_1} \mathcal{D}[\hat{q}_2] \, e^{i \int_{\hat{t}_2}^{\hat{t}_1} L_2(\hat{q}_2, \partial_{\hat{t}_2} \hat{q}_2) \, d\hat{t}_2} \int_{\hat{q}_1}^{\hat{q}_1} \mathcal{D}[\hat{q}_1] \, e^{i \int_{\hat{t}_1}^{\hat{q}_1} L_1(\hat{q}_1, \partial_{\hat{t}_1} \hat{q}_1) \, d\hat{t}_1}. (3.34)
$$

This expression can be immediately extended to the case of $N$ time variables as

$$
\prod_{k=1}^{N} \int_{\hat{q}_k}^{\hat{q}_k} \mathcal{D}[\hat{q}_k] \, e^{i \int_{\hat{t}_k}^{\hat{t}_k} L_k(\hat{q}_k, \partial_{\hat{t}_k} \hat{q}_k) \, d\hat{t}_k} = \prod_{k=1}^{N} \delta(\hat{q}_k - q_k). (3.35)
$$

In the language of Wilson lines, we have the propagator for the loop $\gamma$ in the form

$$
K_\gamma(\hat{q}_1, t_1, \hat{q}_2, t_2, \ldots, \hat{q}_N, t_N; \hat{q}_1, t_1, \hat{q}_2, t_2, \ldots, \hat{q}_N, t_N) = \prod_{k=1}^{N} \mathcal{D}[\hat{q}_k] \, e^{i \int_{\hat{t}_k}^{\hat{t}_k} L_k(\hat{q}_k, \partial_{\hat{t}_k} \hat{q}_k) \, d\hat{t}_k} = I. (3.36)
$$

From (3.36), we deduce the following. The quantum transition between two endpoints acquires no contribution from loops. In other words, loops can be excluded from the entire evolution, as shown in Fig. 6.
We now compute the propagator along the time variables $t_i$, where $i = 1, 2$:

$$K_i(\hat{q}_i, \hat{t}_i; \hat{q}_i, t_i) = \int K_i(\hat{q}_i, \hat{t}_i; \hat{q}_i, t_i)K_i(\hat{q}_i, t_i; q_i, t_i) d\hat{q}_i =$$

$$= \int \sqrt{\frac{m}{2\pi i(\hat{t}_i - t_i)}} \sqrt{\frac{m}{2\pi i(t_i' - t_i)}} e^{\frac{-im}{2i}((\hat{q}_i - \hat{q}_i)^2)} e^{\frac{-im}{2i}((\hat{q}_i - q_i)^2)} d\hat{q}_i =$$

$$= \int \sqrt{\frac{m}{2\pi i(\hat{t}_i - t_i')}} \sqrt{\frac{m}{2\pi i(t_i' - t_i)}} e^{\frac{-im}{2i}(\hat{q}_i^2 - 2\hat{q}_i\hat{q}_i + \hat{q}_i^2)} e^{\frac{-im}{2i}(\hat{q}_i^2 - 2\hat{q}_i q_i + q_i^2)} d\hat{q}_i =$$

$$= \int \frac{m}{2\pi i} \sqrt{\frac{1}{(\hat{t}_i - t_i')(t_i' - t_i)}} e^{\frac{-im}{2i}(\hat{q}_i^2 - 2\hat{q}_i q_i + q_i^2)} \times$$

$$\times e^{\frac{-imq_i}{2i(\hat{t}_i - t_i')}} \sqrt{\frac{m}{2\pi i(\hat{t}_i - t_i)}} e^{\frac{-im}{2i}(\hat{q}_i^2 - 2\hat{q}_i q_i + q_i^2)} d\hat{q}_i =$$

$$= \frac{m}{2\pi i} \sqrt{\frac{2\pi}{(-i\hat{m})(\hat{t}_i - t_i)}} e^{\frac{-imq_i}{2i(\hat{t}_i - t_i')}} \sqrt{\frac{m}{2\pi i(\hat{t}_i - t_i)}} e^{\frac{-im}{2i}(\hat{q}_i^2 - 2\hat{q}_i q_i + q_i^2)} \times$$

$$\times e^{\frac{-imq_i}{2i(\hat{t}_i - t_i')}} e^{\frac{-imq_i}{2i(\hat{t}_i - t_i')}} =$$

$$= \sqrt{\frac{m}{2\pi i(t_i - t_i)}} e^{\frac{-im}{2i(\hat{t}_i - t_i')}} (1 - \frac{q_i^2 - 2\hat{q}_i q_i + q_i^2}{(\hat{t}_i - t_i')}} \sqrt{\frac{m}{2\pi i(t_i - t_i')}} e^{\frac{-im}{2i(\hat{t}_i - t_i')}} =$$

$$= \sqrt{\frac{m}{2\pi i(t_i - t_i)}} e^{\frac{-im}{2i(\hat{t}_i - t_i')}} (\hat{q}_i^2 - 2\hat{q}_i q_i + q_i^2) = \sqrt{\frac{m}{2\pi i(t_i - t_i')}} e^{\frac{-im}{2i(\hat{t}_i - t_i')}} (\hat{q}_i - q_i)^2. \quad (3.38)$$

Imposing $\hat{t}_i - t_i = \delta t_i$ and taking $\delta t_i \to 0$, we obtain [17]

$$K_i(\hat{q}_i, t_i; q_i, t_i) = \lim_{\delta t_i \to 0} \sqrt{\frac{m}{2\pi i\delta t_i}} e^{\frac{-im}{2i(\hat{q}_i - q_i)^2}} = \delta(\hat{q}_i - q_i), \quad (3.39)$$

which does reproduce (3.31) for $i = 2$ and (3.32) for $i = 1$.

**Including interaction.** We finally consider a system with interaction. For simplicity, we work with the Hamiltonian for a two-particle system

$$H = H_1 + H_2 + V_{12}, \quad (3.40)$$
where $V_{12}$ is the potential representing interaction between the particles and $H_i$ is the free Hamiltonian for the $i$th particle. We do the calculations for the paths shown in Fig. 4.

We first define the unitary operators

$$U_1(t'_1, t_1) = e^{-i(H_1 + V_{12})(t'_1 - t_1)}, \quad U_2(t'_2, t_2) = e^{-iH_2(t'_2 - t_2)}.$$ 

Then the propagator for the lower-corner path is given by

$$K_\downarrow(q'_1, t'_1; q'_2, t'_2; q_1, t_1, q_2, t_2) = \langle q'_1, q'_2 | U_2 U_1 | q_1, q_2 \rangle =$$

$$= \int \int \langle q'_1, q'_2 | U_2 | \tilde{q}_2, \tilde{q}_1 \rangle \langle \tilde{q}_2, \tilde{q}_1 | U_1 | q_1, q_2 \rangle d\tilde{q}_1 d\tilde{q}_2 =$$

$$= \int \int \langle q'_1 | U_1 | \tilde{q}_1 \rangle \langle q'_2 | U_2 | \tilde{q}_2 \rangle \langle \tilde{q}_2, \tilde{q}_1 | U_1 | q_1, q_2 \rangle d\tilde{q}_1 d\tilde{q}_2 =$$

$$= \int \int \delta(q'_1 - \tilde{q}_1) K_2(q'_2, t'_2; \tilde{q}_2, t_2) \langle \tilde{q}_2, \tilde{q}_1 | U_1 | q_1, q_2 \rangle d\tilde{q}_1 d\tilde{q}_2 =$$

$$= K_2(q'_2, t'_2; \tilde{q}_2, t_2) \langle \tilde{q}_2 | (q'_1 | U_1 | q_1) | q_2 \rangle d\tilde{q}_2 =$$

$$= K_2(q'_2, t'_2; \tilde{q}_2, t_2) G(q'_1, t'_1; q_1, t_1; \tilde{q}_2, t_2; q_2, t_2) d\tilde{q}_2 =$$

$$= K_2(q'_2, t'_2; \tilde{q}_2, t_2) \delta(\tilde{q}_2 - q_2) G(q'_1, t'_1; q_1, t_1; \tilde{q}_2, t_2; q_2, t_2) d\tilde{q}_2 =$$

$$= K_2(q'_2, t'_2; q_2, t_2) G(q'_1, t'_1; q_1, t_1; q_2, t_2; q_2, t_2). \quad (3.41)$$

and the propagator for the upper-corner path is

$$K_\uparrow(q'_1, t'_1; q'_2, t'_2; q_1, t_1, q_2, t_2) = \langle q'_1, q'_2 | U_1 U_2 | q_1, q_2 \rangle =$$

$$= \int \int \langle q'_1, q'_2 | U_1 | \tilde{q}_1, \tilde{q}_2 \rangle \langle \tilde{q}_1, \tilde{q}_2 | U_2 | q_1, q_2 \rangle d\tilde{q}_1 d\tilde{q}_2 =$$

$$= \int \int \langle q'_1, q'_2 | U_1 | \tilde{q}_1, \tilde{q}_2 \rangle \langle \tilde{q}_1 | U_2 | q_1, q_2 \rangle d\tilde{q}_1 d\tilde{q}_2 =$$

$$= \int \int \langle q'_1 | U_1 | \tilde{q}_1 \rangle \langle q'_2 | U_2 | \tilde{q}_2 \rangle \delta(\tilde{q}_1 - q_1) K_2(\tilde{q}_2, t'_2; q_2, t_2) d\tilde{q}_1 d\tilde{q}_2 =$$

$$= \int \langle q'_2 | (q'_1 | U_1 | q_1) | \tilde{q}_2 \rangle K_2(\tilde{q}_2, t'_2; q_2, t_2) d\tilde{q}_2 =$$

$$= \int \langle q'_2 | G'(q'_1, t'_1; q_1, t_1; \tilde{q}_2) \rangle K_2(\tilde{q}_2, t'_2; q_2, t_2) d\tilde{q}_2 =$$

$$= \int \delta(\tilde{q}_2 - q_2) G'(q'_1, t'_1; q_1, t_1; q'_2, t'_2; \tilde{q}_2, t'_2) K_2(\tilde{q}_2, t'_2; q_2, t_2) d\tilde{q}_2 =$$

$$= G'(q'_1, t'_1; q_1, t_1; q'_2, t'_2; t'_2) K_2(q'_2, t'_2; q_2, t_2). \quad (3.42)$$

We find that the propagators for the upper and lower paths are not the same. This implies that the quantum evolution of the system with interaction is path-dependent. Of course, this path-dependence feature is a direct consequence of the violation of consistency condition (3.7).
Example 2. We give an explicit example choosing the Lagrangians

\[ L_1 = \frac{m(\dot{q}_1)^2}{2} + kq_2 \]

\[ L_2 = \frac{m(\dot{q}_2)^2}{2} + Fq_1, \]

where \( F = kq_2 \). The propagator for a free particle under the action of a constant force \( F \), described by the Lagrangian \( L = m\dot{q}^2/2 + Fq \), is given by [16]

\[ K^F(q', t'; q, t) = \sqrt{\frac{m}{2\pi i(t'-t)}} e^{i\left\{ \frac{1}{2} \frac{(q' - q)^2}{t'-t} + \frac{2}{2} (q' + q)(t'-t) - \frac{k^2}{2m} (t'-t)^3 \right\}}, \]

We consider the transition in Fig. 4. The propagator of the lower-corner path can be written as

\[ K_L(q'_1, t'_1; q_1, t_1) = \left\langle q'_1, q'_2 | U_2 U_1 | q_1, q_2 \right\rangle = \frac{m}{2\pi i(t'_2 - t_2)} e^{i\frac{(q'_2 - q_2)^2}{t'_2 - t_2}} \times \int \langle q_2 \rangle \sqrt{\frac{m}{2\pi i(t'_1 - t_1)}} e^{i\left\{ \frac{1}{2} \frac{(q_1 - q_1)^2}{t'_1 - t_1} + \frac{2}{2} (q_1 + q)(t'_1 - t_1) - \frac{k^2}{2m} (t'_1 - t_1)^3 \right\}} | q_2 \rangle d\bar{q}_2 = \]

\[ K_2(q'_2, t'_2; q_2, t_2) K_1(q'_1, t'_1; q_1, t_1) \int \langle q_2 | q_1 \rangle e^{i\left\{ \frac{kq_2}{2} (q_1 + q)(t'_1 - t_1) - \frac{k^2q_2^2}{2m} (t'_1 - t_1)^3 \right\}} d\bar{q}_2 = \]

\[ K_2(q'_2, t'_2; q_2, t_2) K_1(q'_1, t'_1; q_1, t_1) \int \delta(q_2 - q_2) e^{i\left\{ \frac{kq_2}{2} (q_1 + q)(t'_1 - t_1) - \frac{k^2q_2^2}{2m} (t'_1 - t_1)^3 \right\}} d\bar{q}_2 = \]

\[ K_2(q'_2, t'_2; q_2, t_2) K_1(q'_1, t'_1; q_1, t_1) e^{i\left\{ \frac{kq_2}{2} (q_1 + q)(t'_1 - t_1) - \frac{k^2q_2^2}{2m} (t'_1 - t_1)^3 \right\}}. \]

We do the same computation for the upper-corner path, with the result

\[ K_U(q'_1, t'_1; q_2, t_2) = \left\langle q'_1, q'_2 | U_2 U_1 | q_1, q_2 \right\rangle = \frac{m}{2\pi i(t'_2 - t_2)} e^{i\frac{(q'_2 - q_2)^2}{t'_2 - t_2}} \times \int \langle q_2 \rangle \sqrt{\frac{m}{2\pi i(t'_1 - t_1)}} e^{i\left\{ \frac{1}{2} \frac{(q_1 - q_1)^2}{t'_1 - t_1} + \frac{2}{2} (q_1 + q)(t'_1 - t_1) - \frac{k^2}{2m} (t'_1 - t_1)^3 \right\}} | q_2 \rangle d\bar{q}_2 = \]

\[ K_1(q'_1, t'_1; q_1, t_1) K_2(q'_2, t'_2; q_2, t_2) \int \langle q_2 | q_1 \rangle e^{i\left\{ \frac{kq_2}{2} (q_1 + q)(t'_1 - t_1) - \frac{k^2q_2^2}{2m} (t'_1 - t_1)^3 \right\}} d\bar{q}_2 = \]

\[ K_1(q'_1, t'_1; q_1, t_1) K_2(q'_2, t'_2; q_2, t_2) \int \delta(q_2 - q_2) e^{i\left\{ \frac{kq_2}{2} (q_1 + q)(t'_1 - t_1) - \frac{k^2q_2^2}{2m} (t'_1 - t_1)^3 \right\}} d\bar{q}_2 = \]

\[ K_1(q'_1, t'_1; q_1, t_1) K_2(q'_2, t'_2; q_2, t_2) e^{i\left\{ \frac{kq_2}{2} (q_1 + q)(t'_1 - t_1) - \frac{k^2q_2^2}{2m} (t'_1 - t_1)^3 \right\}}. \]
This simple calculation shows that interaction violates relation (3.7) and hence the commutativity of the propagators. Of course, path independence is no longer applicable. From the standpoint of geometry, the presence of interaction can be viewed as a curvature of the temporal space and therefore the parallel transport along different paths gives different results.

4. Conclusion

We expressed the consistency condition for multitime evolution in terms of Lagrangians as a consequence of the variation of the action on the space of time variables. This consistency condition implies that the action is invariant under local deformations, with fixed endpoints, of a path on the space of time variables. Actually, if we consider a continuous path to be made of tiny discrete elements, then the path-independence property in the continuous-time case is a direct consequence of the path independence in the discrete-time case. Furthermore, with this property, there is a family of paths (homotopy), sharing the endpoints, that can be continuously transformed to each other in the N-dimensional space of time variables.

We derived the consistency condition for multitime quantum evolution in terms of Feynman’s path integrals. The important point is the path-independence feature of the multitime propagator, which can be summarized as follows. In general, there are infinitely many paths from the initial point to the final point on the space of time variables (see Fig. 7 in the case of two time variables). With a set of Lagrangians \( \{L_1, L_2, \ldots, L_N\} \) satisfying the consistency condition, the propagator remains unchanged under variations of the path on the space of time variables, which is of course nothing but the path-independence feature of the multitime propagator. This suggests that, apart from taking all possible paths in the configuration space, as we normally do in the standard single-time path integration, we may need to also take all possible paths in the space of time variables in the case of multitime path integration.\(^4\) In geometric terms, the path-independence feature can be expressed in terms of parallel transport on the flat space of time variables because the curvature vanishes. Then the consistency condition for a set of Lagrangians can be viewed as a zero-curvature condition.

\[ (\hat{q}_1, \hat{q}_2) |\Phi(\hat{t}_1, \hat{t}_2)\rangle = (\hat{q}_1, \hat{q}_2) |U_2'U_1'U_2U_1|\Phi(t_1, t_2)\rangle, \]

\(^4\)This terminology also arises in the context of integrable systems [18].
and

\[ \Phi(q_1, \dot{q}_2, \dot{t}_1, \dot{t}_2) = \iint \langle \dot{q}_1, \dot{q}_2 | U_2' U_1 | q_1, q_2 \rangle \langle q_2, q_1 | \Phi(t_1, t_2) \rangle \mathrm{d}q_1 \mathrm{d}q_2 = \]

\[ = \iint \langle \dot{q}_1, \dot{q}_2 | U_2' U_1 | q_1, q_2 \rangle \Phi(q_1, q_2, t_1, t_2) \mathrm{d}q_1 \mathrm{d}q_2 = \]

\[ = \iint \iint \iint \iint \iint \iint \iint \langle \dot{q}_1, \dot{q}_2 | U_2' U_1 | q_1, q_2 \rangle \langle \dot{q}_1, \dot{q}_2 | q_2, q_1 \rangle \Phi(q_1, q_2, t_1, t_2) \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_1 \mathrm{d}q_2 \]

\[ = \iint \iint \iint \iint \iint \iint \iint \langle \dot{q}_1, \dot{q}_2 | U_2' U_1 | q_1, q_2 \rangle \langle \dot{q}_1, \dot{q}_2 | q_2, q_1 \rangle \Phi(q_1, q_2, t_1, t_2) \times \]

\[ \times \delta(\dot{q}_2 - q_2) \Phi(q_1, q_2, t_1, t_2) \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_1 \mathrm{d}q_2 = \]

\[ = \iint \iint \iint \iint \iint \iint \iint \langle \dot{q}_1, \dot{q}_2 | U_2' U_1 | q_1, q_2 \rangle \langle \dot{q}_1, \dot{q}_2 | q_2, q_1 \rangle \Phi(q_1, q_2, t_1, t_2) \times \]

\[ \times \delta(\dot{q}_2 - q_2) \Phi(q_1, q_2, t_1, t_2) \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_1 \mathrm{d}q_2 = \]

\[ = \iint \iint \iint \iint \iint \iint \iint \langle \dot{q}_1, \dot{q}_2 | U_2' U_1 | q_1, q_2 \rangle \langle \dot{q}_1, \dot{q}_2 | q_2, q_1 \rangle \Phi(q_1, q_2, t_1, t_2) \times \]

\[ \times \delta(\dot{q}_2 - q_2) \Phi(q_1, q_2, t_1, t_2) \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_1 \mathrm{d}q_2 = \]

\[ = \iint \iint \iint \iint \iint \iint \iint \langle \dot{q}_1, \dot{q}_2 | U_2' U_1 | q_1, q_2 \rangle \langle \dot{q}_1, \dot{q}_2 | q_2, q_1 \rangle \Phi(q_1, q_2, t_1, t_2) \times \]

\[ \times \delta(\dot{q}_2 - q_2) \Phi(q_1, q_2, t_1, t_2) \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_1 \mathrm{d}q_2 = \]

This proves expression (3.30) for the counter-clockwise transition in a two-time system.
Acknowledgments. We thank Pichet Vanichchapongjaroen for the valuable discussion.

Conflicts of interest. The authors declare no conflicts of interest.

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