The Number of Two-Term Tilting Complexes over Symmetric Algebras with Radical Cube Zero

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Abstract. In this paper, we compute the number of two-term tilting complexes for an arbitrary symmetric algebra with radical cube zero over an algebraically closed field. First, we give a complete list of symmetric algebras with radical cube zero having only finitely many isomorphism classes of two-term tilting complexes in terms of their associated graphs. Secondly, we enumerate the number of two-term tilting complexes for each case in the list.

1. Introduction

Tilting theory plays an important role in the study of many areas of mathematics. A central notion of tilting theory is a tilting complex which is a generalization of a progenerator in Morita theory. Indeed, its endomorphism algebra is derived equivalent to the original algebra [16]. Hence, it is a natural problem to give a classification of tilting complexes for a given algebra.

In this paper, we study the classification of two-term tilting complexes for an arbitrary symmetric algebra with radical cube zero over an algebraically closed field $\k$. Symmetric algebras with radical cube zero have been studied by Okuyama [15], Benson [7], and Erdmann–Solberg [11], and also appear in several areas such as [8,13,17]. Recently, Green–Schroll [12] showed that this class is precisely the Brauer configuration algebras with radical cube zero.

The study of symmetric algebras $A$ with radical cube zero can be reduced to that of algebras with radical square zero. For example, as an application of $\tau$-tilting theory [3], we find in Proposition 3.1 that the functor $- \otimes_A (A/socA)$ gives a bijection

$$2\text{-tilt } A \longrightarrow 2\text{-silt } (A/socA).$$
Here, we denote by $2$-tilt $A$ (respectively, $2$-silt $A$) the set of isomorphism classes of basic two-term tilting (respectively two-term silting) complexes for $A$. Notice that tilting complexes coincide with silting complexes for a symmetric algebra $A$ [4, Example 2.8].

In [2, 5, 19], they study two-term silting theory (or equivalently $\tau$-tilting theory) for algebras with radical square zero. The first author [2] gives a characterization of algebras with radical square zero which are $\tau$-tilting finite (i.e., having only finitely many isomorphism classes of basic two-term silting complexes) by using the notion of single-quivers; see Proposition 2.3(2). Using this result, we give a complete list of $\tau$-tilting finite symmetric algebras with radical cube zero as follows. Let $A$ be a basic connected finite-dimensional symmetric $k$-algebra with radical cube zero. Let $Q$ be the Gabriel quiver of $A$ and $Q^\circ$ be the quiver obtained from $Q$ by deleting all loops. We show in Definition-Proposition 3.3 that $Q^\circ$ is the double quiver $Q_G$ (see Definition 3.2) of a finite connected (undirected) graph $G$ with no loops, i.e., $Q^\circ = Q_G$. We call $G$ the graph of $A$.

**Theorem 1.1.** Let $A$ be a basic connected finite-dimensional symmetric $k$-algebra with radical cube zero. Then, the following conditions are equivalent.

1. $A$ is $\tau$-tilting finite (or equivalently, $2$-tilt $A$ is finite).
2. The graph of $A$ is one of graphs in the following list:

\[
\begin{align*}
(\tilde{A}_n) & \quad 1 \rightarrow 2 \rightarrow \cdots \rightarrow n \\
(\mathbb{D}_n) & \quad 4 \leq n \\
(\mathbb{E}_7) & \quad 4 \\
(\mathbb{E}_8) & \quad 4 \\
(\tilde{\Lambda}_{n-1}) & \quad n: \text{odd} \\
(I_n) & \quad 4 \leq n \\
(II_n) & \quad 5 \leq n \leq 8 \\
(III) & \\
(IV) & \\
(V) & \\
\end{align*}
\]

The second author [5] classifies two-term silting complexes for an arbitrary algebra with radical square zero by using tilting modules over a path algebra (see Proposition 2.3(1)). Since the cardinality of the set of isomorphism classes of tilting modules over a path algebra is well known, this provides us an explicit way to compute the number of them. We use this result to determine the number $\#2$-tilt $A$ for each graph $G$ in the list of Theorem 1.1.
Theorem 1.2. In Theorem 1.1, the number $\# 2$-tilt $A$ depends only on the graph $G$ of $A$ and is given as follows:

| $G$   | $A_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $E_{n-1}$ | $I_n$ | $II_5$ | $II_6$ | $II_7$ | $II_8$ | $III$ | $IV$ | $V$ |
|-------|-------|-------|-------|-------|-------|-----------|-------|--------|--------|--------|--------|------|-----|-----|
| $\# 2$-tilt $A$ | $(\frac{2n}{n})$ | $a_n$ | 1700 | 8872 | 54066 | $2^{2(n-1)}$ | $b_n$ | 632 | 2936 | 14024 | 75240 | 3288 | 4056 | 17328 |

Here, for any $n \geq 4$, let $a_n := 6 \cdot 4^{n-2} - 2 \binom{2(n-2)}{n-2}$ and $b_n := 6 \cdot 4^{n-2} + 2 \binom{2n}{n} - 4 \binom{2(n-1)}{n-2} - 4 \binom{2(n-2)}{n-2}$.

We remark that the numbers for Dynkin graphs of type $A$, $D$ and $E$ in the list are precisely the biCatalan numbers introduced by [6] in the context of Coxeter–Catalan combinatorics. Our results for Dynkin graphs are independently obtained by [9] in the study of biCambrian lattices for preprojective algebras.

We also remark that we can generalize our results for Brauer configuration algebras in terms of multiplicities. A Brauer configuration algebra is defined by a configuration and a multiplicity function. The configuration of a Brauer configuration algebra with radical cube zero corresponds to a graph [12]. By [10], one can show that the number of two-term tilting complexes over Brauer configuration algebras is independent of the multiplicity. Therefore, we can also apply our results to any Brauer configuration algebra obtained by replacing the multiplicity of a Brauer configuration associated with a graph in the list of Theorem 1.1.

This paper is organized as follows. In Sect. 2, we recall the definition of algebras with radical square zero and their two-term silting theory which are needed in this paper. In Sect. 3, we study symmetric algebras with radical cube zero together with the correspondence algebra with radical square zero. Our main results are Theorem 3.4 and Corollary 3.5, which provide us an explicit way to compute the number of two-term tilting complexes for a given symmetric algebra with radical cube zero. In Sect. 4, we prove Theorems 1.1 and 1.2 by using results shown in the previous section.

2. Preliminaries

Throughout this paper, $k$ is an algebraically closed field. We recall that any basic connected finite-dimensional $k$-algebra $A$ is isomorphic to a bound quiver algebra $kQ/I$, where $Q$ is a finite connected quiver and $I$ is an admissible ideal in the path algebra $kQ$ of the quiver $Q$. We call $Q_A := Q$ the Gabriel quiver of $A$. Remark that all graphs appeared in this paper may have multiple edges.

2.1. Silting Complexes

Let $A$ be a basic (not necessary connected) finite-dimensional $k$-algebra. We denote by $\text{mod} A$ the category of finitely generated right $A$-modules and by $\text{proj} A$ the category of finitely generated projective right $A$-modules. Let $\mathcal{K}^b(\text{proj} A)$ be the homotopy category of bounded complexes of objects of $\text{proj} A$. 
For a complex $X \in K^b(\text{proj } A)$, we say that $X$ is basic if it is a direct sum of pairwise non-isomorphic indecomposable objects.

**Definition 2.1.** A complex $T$ in $K^b(\text{proj } A)$ is said to be presilting if it satisfies $$\text{Hom}_{K^b(\text{proj } A)}(T, T[i]) = 0$$ for all positive integers $i$. A presilting complex $T$ is called a silting complex if it satisfies $$\text{thick } T = K^b(\text{proj } A),$$ where $\text{thick } T$ is the smallest triangulated full subcategory which contains $T$ and is closed under taking direct summands. In addition, a silting complex $T$ is called a tilting complex if $$\text{Hom}_{K^b(\text{proj } A)}(T, T[i]) = 0$$ for all non-zero integers $i$.

We restrict our interest to the set of two-term silting complexes. Here, a complex $T = (T^i, d^i)$ in $K^b(\text{proj } A)$ is said to be two-term if it is isomorphic to a complex concentrated only in degree 0 and $-1$, that is, $$(T^{-1} \xrightarrow{d^{-1}} T^0) = \cdots \to 0 \to T^{-1} \xrightarrow{d^{-1}} T^0 \to 0 \to \cdots.$$ We denote by $2\text{-silt } A$ (respectively, $2\text{-tilt } A$) the set of isomorphism classes of basic two-term silting (respectively, two-term tilting) complexes for $A$.

Now, we call $M \in \text{mod } A$ a tilting module if all the following conditions are satisfied: (i) the projective dimension of $M$ is at most 1, (ii) $\text{Ext}^1_A(M, M) = 0$, and (iii) $|M| = |A|$, where $|M|$ denotes the number of pairwise non-isomorphic indecomposable direct summands of $M$. We denote by $\text{tilt } A$ the set of isomorphism classes of basic tilting $A$-modules. By definition, we can naturally regard a tilting $A$-module $M$ as a tilting complex. More precisely, by taking a minimal projective presentation $P_1 \xrightarrow{f} P_0 \to M \to 0$ of $M$ in $\text{mod } A$, the two-term complex $(P_1 \xrightarrow{f} P_0)$ provides a tilting complex in $K^b(\text{proj } A)$.

The number of tilting modules over a path algebra of a Dynkin quiver is well known.

**Proposition 2.2.** (See [14] for example). Let $Q$ be a quiver whose underlying graph $\Delta$ is one of Dynkin graphs of type $A$, $D$, and $E$. Then, the number $\# \text{tilt } kQ$ is given by the following table and does not depend on the orientation of $Q$:

| $\Delta$ | $A_n$ $(n \geq 1)$ | $D_n$ $(n \geq 4)$ | $E_6$ | $E_7$ | $E_8$ |
|----------|---------------------|---------------------|-------|-------|-------|
| $\# \text{tilt } kQ$ | $\frac{1}{n+1} \binom{2n}{n}$ | $\frac{3n-4}{2n} \binom{2(n-1)}{n-1}$ | 418   | 2431  | 17342 |

More generally, if $Q$ is a disjoint union of Dynkin quivers $Q_\lambda$ ($\lambda \in \Lambda$), then we have

$$\# \text{tilt } kQ = \prod_{\lambda \in \Lambda} \# \text{tilt } kQ_\lambda,$$

and this number is completely determined by a collection of the underlying graphs $\Delta_\lambda$ of $Q_\lambda$ for all $\lambda \in \Lambda$ as in Proposition 2.2.
2.2. Algebras with Radical Square Zero

Let $A$ be a basic connected finite-dimensional $k$-algebra. We say that $A$ is an algebra with radical square zero (respectively, radical cube zero) if $J^2 = 0$ but $J \neq 0$ (respectively, $J^3 = 0$ but $J^2 \neq 0$), where $J$ is the Jacobson radical of $A$. For simplicity, we abbreviate an algebra with radical square zero (respectively, radical cube zero) by an RSZ (respectively, RCZ) algebra.

We first recall that any basic connected finite-dimensional RSZ $k$-algebra $A$ is isomorphic to a bound quiver algebra $kQ/I$, where $Q := Q_A$ is the Gabriel quiver of $A$ and $I$ is the two-sided ideal in $kQ$ generated by all paths of length 2.

Next, let $Q = (Q_0, Q_1)$ be a finite connected quiver, where $Q_0$ is the vertex set and $Q_1$ is the arrow set. We denote by $Q^{\text{op}}$ the opposite quiver of $Q$. For a map $\epsilon: Q_0 \to \{\pm 1\}$, we define a quiver $Q_\epsilon$, called a single-quiver of $Q$, as follows:

- The set of vertices is $Q_0$.
- We draw an arrow $a: i \to j$ in $Q_\epsilon$ whenever there exists an arrow $a: i \to j$ with $\epsilon(i) = +1$ and $\epsilon(j) = -1$.

Note that $Q_\epsilon$ is bipartite, that is, each vertex is a sink or a source (an isolated vertex is a sink and source), but not connected in general. Since it has no loops by definition, we have $Q_\epsilon = (Q^\circ)_\epsilon$, where $Q^\circ$ denotes the quiver obtained from $Q$ by deleting all loops.

We give a connection between two-term silting complexes for an RSZ algebra and tilting modules over path algebras.

**Proposition 2.3.** Let $A$ be a basic connected finite-dimensional RSZ $k$-algebra and $Q_A$ the Gabriel quiver of $A$. Let $Q := (Q_A)^\circ$ be the quiver obtained from $Q_A$ by deleting all loops. Then, the following statements hold.

1. [5, Theorem 1.1] There is a bijection

   $$2\text{-silt } A \to \bigsqcup_{\epsilon: Q_0 \to \{\pm 1\}} \text{tilt } k(Q_\epsilon)^{\text{op}}.$$

2. [2,5] The following conditions are equivalent.
   (a) $2\text{-silt } A$ is finite.
   (b) For every map $\epsilon: Q_0 \to \{\pm 1\}$, the underlying graph of the single-quiver $Q_\epsilon$ is a disjoint union of Dynkin graphs of type $A$, $D$ and $E$.

3. If one of equivalent conditions of (2) holds, we have

   $$\# 2\text{-silt } A = \sum_{\epsilon: Q_0 \to \{\pm 1\}} \# \text{tilt } k(Q_\epsilon)^{\text{op}}.$$

We remark that we can replace the quiver $Q$ with the Gabriel quiver $Q_A$ of $A$ in Proposition 2.3, since we have $(Q_A)_\epsilon = Q_\epsilon$ for any map $\epsilon: Q_0 \to \{\pm 1\}$. 
3. Two-Term Tilting Complexes over Symmetric RCZ Algebras

Let $A$ be a basic connected finite-dimensional symmetric RCZ $k$-algebra. By [12, Lemma 5.2], every indecomposable projective $A$-module has Loewy length 3 and its simple socle is contained in the square of the Jacobson radical of $A$. Then, $\overline{A} := A/\operatorname{soc}A$ is an RSZ algebra and the Gabriel quiver of $\overline{A}$ coincides with the Gabriel quiver of $A$.

The following is basic. Here, we remember that silting complexes coincide with tilting complexes for a symmetric algebra $A$ [4, Example 2.8]. In particular, $2\text{-tilt } A = 2\text{-silt } A$.

**Proposition 3.1.** [1, Theorem 3.3]. Let $A$ be a basic connected finite-dimensional symmetric RCZ $k$-algebra and $\overline{A} := A/\operatorname{soc}A$. Then, the functor $- \otimes_A \overline{A}$ gives a bijection 

$$2\text{-tilt } A \longrightarrow 2\text{-silt } \overline{A}.$$ 

Next, the following observations provide us a combinatorial framework of studying two-term tilting complexes over symmetric RCZ algebras.

**Definition 3.2.** For a finite connected graph $G$ with no loops, we define a quiver $Q_G$ as follows.

- The set of vertices of $Q_G$ is the set of vertices of $G$.
- We draw two arrows $a^*: i \to j$ and $a^{**}: j \to i$ whenever there exists an edge $a$ of $G$ connecting $i$ and $j$.

We call $Q_G$ the double quiver of $G$. Notice that $Q_G$ has no loops since so does $G$.

**Definition-Proposition 3.3.** Let $A$ be a basic connected finite-dimensional symmetric RCZ $k$-algebra. Let $Q_A$ be the Gabriel quiver of $A$ and $Q := (Q_A)^\circ$ the quiver obtained from $Q_A$ by deleting all loops. Then, $Q$ is the double quiver $Q_G$ of a finite connected (undirected) graph $G$ with no loops. We call $G$ the graph of $A$.

**Proof.** For the Gabriel quiver $Q_A$ of $A$, let $\pi: kQ_A \to A$ be a canonical surjection. For any vertex $i$ of $Q_A$, let $P_i$ be the indecomposable projective $A$-module corresponding to $i$. By definition, $P_i$ has Loewy length 3 and its simple socle is isomorphic to the simple top $S_i := P_i/P_iJ$.

We recall from [12, Theorem 5.6] that our algebra $A$ is special multiserial (we refer to [12, Definition 2.2] for the definition of special multiserial algebras). Then, each arrow $a: i \to j$ of $Q_A$ determines the unique arrow $\sigma(a)$, such that $\pi(\sigma(a)) \neq 0$, and the correspondence $\sigma$ gives a permutation of the set of arrows of $Q_A$, see [12, Definition 4.8]. Here, we remark that there is an arrow $a$, such that $a = \sigma(a)$ in general (in this case, $a$ must be a loop). Since $P_i$ has Loewy length 3 and $A$ is symmetric special multiserial, we have $\sigma^2 = \operatorname{id}$ by [12, Lemma 4.9(3)]. In particular, for an arrow $a: i \to j$, we have $0 \neq \pi(\sigma(a)) \in \operatorname{soc}P_i$ and $0 \neq \pi(\sigma(\sigma(a))) \in \operatorname{soc}P_j$.

Now, we can restrict the permutation $\sigma$ to the subset consisting of all arrows which are not loops. Then, we define a finite undirected graph $G$ as
follows: The set of vertices of $G$ bijectively corresponds to the set of vertices of $Q_A$, and the set of edges of $G$ is naturally given by the set of unordered pairs \{a, \sigma(a)\} for all arrows $a$ of $Q_A$ which are not loops. Then, $G$ is the desired one as $(Q_A)\circ = Q_G$ from our construction. □

As we mentioned before, the algebras $A$ and $\overline{A} := A/\text{soc}A$ have the same Gabriel quiver $Q_A = Q_{\overline{A}}$. Therefore, $(Q_A)\circ = (Q_{\overline{A}})\circ$ is the double quiver $Q_G$ of a common finite connected graph $G$ with no loops by Definition-Proposition 3.3.

**Theorem 3.4.** Let $A$ be a basic connected finite-dimensional symmetric RCZ $k$-algebra and $\overline{A} := A/\text{soc}A$. Let $Q_A$ be the Gabriel quiver of $A$ and $Q := (Q_A)\circ$ the quiver obtained from $Q_A$ by deleting all loops.

1. The following conditions are equivalent.
   (a) 2-tilt $A$ is finite.
   (b) 2-silt $\overline{A}$ is finite.
   (c) For every map $\epsilon: Q_0 \to \{\pm 1\}$, the underlying graph of the single-quiver $Q_\epsilon$ is a disjoint union of Dynkin graphs of type $A, D$ and $E$.

2. Fix any vertex $v \in Q_0$. If one of the equivalent conditions in (1) is satisfied, then the following equalities hold:

$$\# \text{ 2-tilt } A = \# \text{ 2-silt } \overline{A} = 2 \cdot \sum_{\epsilon: Q_0 \to \{\pm 1\}} \# \text{ tilt } kQ_\epsilon.$$

**Proof.** (1) It follows from Propositions 2.3(2) and 3.1.

(2) By Proposition 3.1, we have $\# \text{ 2-tilt } A = \# \text{ 2-silt } \overline{A}$. We show the second equality. Let $v$ be a vertex in $Q$. By Proposition 2.3(1), we have

$$\# \text{ 2-silt } \overline{A} = \sum_{\epsilon: Q_0 \to \{\pm 1\}} \# \text{ tilt } (Q_\epsilon)^{\text{op}}$$

$$= \sum_{\epsilon: Q_0 \to \{\pm 1\}} \# \text{ tilt } (Q_\epsilon)^{\text{op}} + \sum_{\epsilon: Q_0 \to \{\pm 1\}} \# \text{ tilt } (Q_{-\epsilon})^{\text{op}}.$$

For a map $\epsilon: Q_0 \to \{\pm 1\}$, we define a map $-\epsilon: Q_0 \to \{\pm 1\}$ by $(-\epsilon)(i) := -\epsilon(i)$ for all $i \in Q_0$. Since $Q$ is the double quiver of the graph $G$ of $A$, we have $Q_{-\epsilon} = (Q_\epsilon)^{\text{op}}$. This implies that $Q_\epsilon$ and $Q_{-\epsilon}$ have the same underlying graph $\Delta$. By our assumption, $\Delta$ is a disjoint union of Dynkin graphs. Thus, we obtain $\# \text{ tilt } kQ_\epsilon = \# \text{ tilt } kQ_{-\epsilon}$, because the number of non-isomorphic tilting modules over a path algebra of Dynkin type does not depend on orientation; see Proposition 2.2. Hence, we have

$$\sum_{\epsilon: Q_0 \to \{\pm 1\}} \# \text{ tilt } kQ_\epsilon = \sum_{\epsilon: Q_0 \to \{\pm 1\}} \# \text{ tilt } (Q_\epsilon)^{\text{op}} = \sum_{\epsilon: Q_0 \to \{\pm 1\}} \# \text{ tilt } (Q_{-\epsilon})^{\text{op}}.$$

This finishes the proof. □
For our convenience, we restate Theorem 3.4 in terms of undirected graphs. Let $G = (G_0, G_1)$ be a finite connected graph with no loops, where $G_0$ is the set of vertices and $G_1$ is the set of edges. For each map $\epsilon: G_0 \to \{\pm 1\}$, let $G_\epsilon$ be the graph obtained from $G$ by removing all edges between vertices $i, j$ with $\epsilon(i) = \epsilon(j)$. From our construction, $G_\epsilon$ is precisely the underlying graph of the quiver $Q_\epsilon$, where $Q := QC$ is the double quiver of $G$ with vertex set $Q_0 = G_0$. In particular, $Q_\epsilon$ is a disjoint union of Dynkin quivers if and only if $G_\epsilon$ is a disjoint union of Dynkin graphs.

Now, we recall that, for a quiver $Q$ whose underlying graph $\Delta$ is a disjoint union of Dynkin graphs, the number $\#\text{tilt}^2 Q$ does not depend on orientation of $Q$ and given by (2.1). Then, we set $|\Delta| := \#\text{tilt}^2 Q$.

**Corollary 3.5.** Let $A$ be a basic connected finite-dimensional symmetric RCZ $k$-algebra and $G$ the graph of $A$.

(1) The following conditions are equivalent.

(a) $2\text{-tilt} A$ is finite.

(b) For every map $\epsilon: G_0 \to \{\pm 1\}$, the graph $G_\epsilon$ is a disjoint union of Dynkin graphs of type $A$, $D$ and $E$.

(2) Assume that, for any $\epsilon: G_0 \to \{\pm 1\}$, the graph $G_\epsilon$ is a disjoint union of Dynkin graphs $\Delta_{\epsilon, \lambda}$ ($\lambda \in \Lambda_\epsilon$). Then, for a fixed vertex $v$ of $G$, the number $\#2\text{-tilt} A$ is equal to

$$2 \cdot \sum_{\epsilon: G_0 \to \{\pm 1\}} |G_\epsilon| = 2 \cdot \sum_{\epsilon: G_0 \to \{\pm 1\}} \prod_{\lambda \in \Lambda_\epsilon} |\Delta_{\epsilon, \lambda}|.$$  \hfill (3.1)

**Proof.** Let $Q := (Q_A)^\circ$, where $Q_A$ is the Gabriel quiver of $A$. Then, $Q = Q_G$ holds by Definition-Proposition 3.3. Thus, the assertion follows from Theorem 3.4, since $G_\epsilon$ coincides with the underlying graph of $Q_\epsilon$ for any map $\epsilon: G_0 \to \{\pm 1\}$. \hfill $\Box$

**Definition 3.6.** Keeping the notations in Corollary 3.5(2), we write $\|G\|$ for the number given by the left-hand side of (3.1).

**Example 3.7.** (1) Let $Q$ be a quiver whose underlying graph $\Delta$ is one of Dynkin graphs of type $A$, $D$ and $E$. Let $A$ be the trivial extension of the path algebra $kQ$ of $Q$ by a minimal co-generator. It is easy to see that $A$ is a symmetric RCZ algebra if $Q$ is bipartite. In this case, the Gabriel quiver of $A$ is precisely the double quiver $Q_\Delta$ of $\Delta$; in other words, the graph of $A$ is $\Delta$. On the other hand, $Q^\text{op}$ also determines the symmetric RCZ algebra, which is naturally isomorphic to $A$.

(2) Let $\Delta = E_6$ and let $A$ be the symmetric RCZ algebra obtained as in (1). In Fig. 1, we describe single-quivers of $Q := Q_{E_6}$ associated with maps $\epsilon$ with $\epsilon(6) = +1$. Here, the notation $i^\sigma$ denotes the vertex $i$ with $\epsilon(i) = \sigma \in \{\pm 1\}$. Using the Corollary 3.5, we find that there are 1700 isomorphism classes of basic two-term tilting complexes over $A$ as in the list of Theorem 1.2.
Figure 1. A half of single-quivers of the double quiver of $E_6$

4. Proof of Main Theorem

In this section, we prove Theorems 1.1 and 1.2. Throughout this section, $G$ is a finite connected graph with no loops.

4.1. Proof of Theorem 1.1

By Corollary 3.5(1), the proof is completed with the following proposition.

Proposition 4.1. Let $G$ be a connected finite graph with no loops. Then, the graph $G_{\epsilon}$ is a disjoint union of Dynkin graphs of type $A$, $D$ and $E$ for every map $\epsilon: G_0 \rightarrow \{\pm 1\}$ if and only if $G$ is one of the list in Theorem 1.1.

In the following, we give a proof of Proposition 4.1 by removing extended Dynkin graphs from the collection $G_{\epsilon}$ of subgraphs of $G$. We start with removing extended Dynkin graphs of type $\tilde{A}$. A graph is called an $n$-cycle if it is a
cycle with exactly \( n \) vertices. In particular, it is called an odd-cycle if \( n \) is odd, and an even-cycle if \( n \) even.

**Lemma 4.2.** The following statements are equivalent:

1. There exists a map \( \epsilon: G_0 \to \{ \pm 1 \} \) such that \( G_\epsilon \) contains an extended Dynkin graph of type \( \tilde{A} \) as a subgraph.
2. \( G \) contains an even-cycle as a subgraph.

**Proof.** (2) \( \Rightarrow \) (1): Let \( G' \) be a subgraph of \( G \) which is an even-cycle. Since an even-cycle is a bipartite graph, there exists a map \( \epsilon: G_0 \to \{ \pm 1 \} \), such that the underlying graph of \( G_\epsilon \) contains \( G' \) as a subgraph. Hence, the assertion follows.

(1) \( \Rightarrow \) (2): Assume that for some map \( \epsilon: G_0 \to \{ \pm 1 \} \), the graph \( G_\epsilon \) contains an extended Dynkin graph \( G' \) of type \( \tilde{A} \). Since \( G_\epsilon \) is bipartite, so is \( G' \). Hence, \( G' \) is an even-cycle and a subgraph of \( G \). This finishes the proof. \( \square \)

By Lemma 4.2, we may assume that \( G \) contains no even-cycle as a subgraph. In particular, \( G \) has no multiple edges. We give a connection between our graphs \( G_\epsilon \) and subtrees of \( G \). Recall that a subtree of \( G \) is a connected subgraph of \( G \) without cycles.

**Proposition 4.3.** Assume that \( G \) contains no even-cycle as a subgraph. Let \( G' \) be a connected graph. Then, the following statements are equivalent.

1. There exists a map \( \epsilon: G_0 \to \{ \pm 1 \} \), such that \( G_\epsilon \) contains \( G' \) as a subgraph.
2. \( G' \) is a subtree of \( G \).

**Proof.** (2) \( \Rightarrow \) (1) is clear. We show (1) \( \Rightarrow \) (2). Since \( G \) has no even-cycle as a subgraph, the graph \( G_\epsilon \) is a tree by Lemma 4.2. Thus, the subgraph \( G' \) is also a tree. This implies that \( G' \) is a subtree of \( G \). \( \square \)

For a tree, we have the following result.

**Corollary 4.4.** Assume \( G \) is a tree. Then, the graph \( G_\epsilon \) is a disjoint union of Dynkin graphs of type \( A, D \) and \( E \) for each map \( \epsilon: G_0 \to \{ \pm 1 \} \) if and only if \( G \) is a Dynkin graph of type \( A, D \) and \( E \).

**Proof.** It is well known that \( G \) is a Dynkin graph if and only if all subtrees of \( G \) are Dynkin graphs. The assertion follows from Proposition 4.3. \( \square \)

We remove extended Dynkin graphs of type \( \tilde{D} \). Assume that \( G \) contains at least two odd-cycles. If two odd-cycles have at least two common vertices, then \( G \) has an even-cycle. Since \( G \) contains no even-cycles as a subgraph, two odd-cycles share at most one vertex. If they share exactly one vertex (respectively, no vertices), then \( G \) contains \( \tilde{D}_4 \) (respectively, \( \tilde{D}_{n \geq 5} \)) as a subgraph.
Therefore, there exists a subtree $G'$ of $G$, such that $G'$ is an extended Dynkin graph of type $\tilde{D}$. Moreover, by Proposition 4.3, there exists a map $\epsilon: G_0 \to \{\pm 1\}$, such that $G_\epsilon$ contains an extended Dynkin graph of type $\tilde{D}$ as a subgraph. Hence, we may assume that $G$ contains at most one odd-cycle. By Corollary 4.4, it is enough to consider the case where $G$ contains exactly one odd-cycle. Namely, $G$ consists of an odd-cycle, such that each vertex $v$ in the odd-cycle is attached to a tree $T_v$.

\[ \begin{array}{c}
bullet & \bullet & v_1 & \bullet & \bullet \\
v_2 & & & & v_3 \\
\end{array} \]

**Lemma 4.5.** Fix an integer $k \geq 1$ and $n := 2k + 1$. Assume that $G$ consists of an $n$-cycle, such that each vertex $v$ in the $n$-cycle is attached to a tree $T_v$. Then, the following statements are equivalent:

1. There exists a map $\epsilon: G_0 \to \{\pm 1\}$, such that $G_\epsilon$ contains an extended Dynkin graph of type $\tilde{D}$ as a subgraph.
2. $G$ contains an extended Dynkin graph of type $\tilde{D}$ as a subgraph.
3. $G$ satisfies one of the following conditions.
   - There is a vertex $v$ in the $n$-cycle, such that the degree is at least four.
   - There is a vertex $v$ in the $n$-cycle, such that the degree is exactly three and $T_v$ is not a Dynkin graph of type $A$.
   - $k \geq 2$ and there are at least two vertices in the $n$-cycle, such that the degrees are at least three.

**Proof.** (1)$\Leftrightarrow$(2) follows from Proposition 4.3. Moreover, we can easily check (2)$\Leftrightarrow$(3), because $\tilde{D}_4$ has exactly one vertex whose degree is exactly four and $\tilde{D}_l$ ($l \geq 5$) has exactly two vertices whose degree are exactly three. \[ \square \]

Fix an integer $k \geq 1$ and $n := 2k + 1$. By Lemma 4.5, we may assume that $G$ is one of the following graphs:
Without loss of generality, we can assume $l_1 \leq l_2 \leq l_3$ for the case $k = 1$.

Finally, we remove extended Dynkin graphs of type $\tilde{E}$.

**Lemma 4.6.** Fix an integer $k \geq 1$ and $n := 2k + 1$.

(1) Assume that $k = 1$. The following graphs (i), (ii), and (iii) are the minimal graphs in the forms of $G$ containing an extended Dynkin graph of type $\tilde{E}$.
(2) Assume that $k \geq 2$. The following graphs (iv) and (v) are the minimal graphs in the forms of $G$ containing an extended Dynkin graph of type $\widetilde{E}$.

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {$1$};
\node (2) at (-1,-1) {$2$};
\node (3) at (1,-1) {$3$};
\node (4) at (2,0) {$4$};
\node (5) at (0,-2) {$5$};
\draw (1) -- (2);
\draw (1) -- (3);
\draw (1) -- (4);
\draw (1) -- (5);
\node (11) at (6,0) {$1$};
\node (22) at (5,-1) {$2$};
\node (33) at (7,-1) {$3$};
\node (44) at (8,0) {$n$};
\node (55) at (7,-2) {$n-1$};
\node (66) at (8,-2) {$n-2$};
\draw (11) -- (22);
\draw (11) -- (33);
\draw (11) -- (44);
\draw (11) -- (55);
\end{tikzpicture}
\end{center}

(iv) $k = 2$  
(v) $k \geq 3$

Proof. (1) Deleting the edge between 1 and 2 in the graphs (i), (ii), and (iii), we can find $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$, respectively. Moreover, if we delete a vertex $i_j$ in the graphs (i), (ii), and (iii), then the new graphs contain no extended Dynkin graphs of type $\widetilde{E}$ as a subgraph. Hence, the graphs (i), (ii), and (iii) are minimal graphs containing $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$, respectively.

(2) Deleting the edge between 3 and 4 in (iv) (respectively, 4 and 5 in (v)), we can find $\widetilde{E}_6$ (respectively, $\widetilde{E}_7$). Moreover, we can easily check that the graphs (iv) and (v) are minimal graphs containing an extended Dynkin graph of type $\widetilde{E}$.

Now, we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. If $G$ is a tree, then the assertion follows from Corollary 4.4. We assume that $G$ is not a tree. By Lemma 4.2, we may assume that $G$ does not contain even-cycles as subgraphs. Then, $G$ does not contain extended Dynkin graphs as subgraphs if and only if $G$ is one of the following classes:

- $(I_n)_{n \geq 4}$ in Theorem 1.1 (2),
- proper connected non-tree subgraphs appearing in Lemma 4.6 (i)-(v).

The second class coincides with the graphs $(\widetilde{A}_{n-1})_{n: \text{odd}}$, $(I_n)_{4 \leq n \leq 8}$, $(II_n)_{5 \leq n \leq 8}$, (III), (IV), and (V) in Theorem 1.1 (2). Hence, the assertion follows from Proposition 4.3.

We finish this subsection with proving Theorem 1.1.

Proof of Theorem 1.1. The result follows from Corollary 3.5 (1) and Proposition 4.1.

4.2. Proof of Theorem 1.2
We just compute the number of two-term tilting complexes for each graph in the list of Theorem 1.1. Our calculation is based on Theorem 3.4 and Corollary 3.5. For our purpose, we assume that $G$ is a graph appearing in the list.
of Theorem 1.1 and let \( A \) be a basic connected finite-dimensional symmetric RCZ algebra whose graph is \( G \).

Keeping above notations, we determine the number \( \# 2\text{-}\text{tilt} A \), or equivalently, \( \| G \| \) in Definition 3.6. First, for types \( A \) and \( \tilde{A} \), the number is already computed by [5]:

**Proposition 4.7.** [5, Theorem 1.2] The following equality holds:

\[
\# 2\text{-}\text{tilt} A = \begin{cases} 
\binom{2n}{n} & \text{if } G = \tilde{A}_n, \\
2^{2n-1} & \text{if } G = \tilde{A}_{n-1} \text{ for odd } n.
\end{cases}
\]

Secondly, we consider the case where \( G \) is a Dynkin graph of type \( D \). For simplicity, let \( c_0 = 1, c_l := \binom{2l}{l} \) for each \( l \geq 1 \). Then, we have \( \| A_l \| = c_l \) for all \( l \geq 1 \) by Proposition 4.7. In addition, let \( \| A_0 \| := 2 \).

**Proposition 4.8.** Let \( n \geq 4 \) and \( G = D_n \). Then, we have

\[
\# 2\text{-}\text{tilt} A = 6 \cdot 4^{n-2} - 2c_{n-2}.
\]

**Proof.** Let \( G \) be a graph as follows:

\[
\begin{array}{c}
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \rightarrow 4 \rightarrow \cdots \rightarrow n.
\end{array}
\]

By Corollary 3.5, we have

\[
\# 2\text{-}\text{tilt} A = 2 \cdot \sum_{\epsilon \colon Q_0 \rightarrow \{ \pm 1 \} \atop \epsilon(3) = +1} | G_\epsilon |.
\]  

We study the right-hand side of (4.1). Let \( M \) be the set of maps \( \epsilon \colon G_0 \rightarrow \{ \pm 1 \} \), such that \( \epsilon(3) = +1 \). Clearly, \( M \) is a disjoint union of the following subsets:

- \( M_1 := \{ \epsilon \in M \mid \epsilon(1) = \epsilon(2) = \epsilon(3) \} \).
- \( M_2 := \{ \epsilon \in M \mid \epsilon(1) = -\epsilon(2) = \epsilon(3) \} \).
- \( M_3 := \{ \epsilon \in M \mid -\epsilon(1) = \epsilon(2) = \epsilon(3) \} \).
- \( M_4 := \{ \epsilon \in M \mid -\epsilon(1) = -\epsilon(2) = \epsilon(3) = \epsilon(4) \} \).
- \( M_5 := \{ \epsilon \in M \mid -\epsilon(1) = \epsilon(2) = \epsilon(3) = -\epsilon(4) \} = \bigcup_{t=4}^n M_5(t) \), where

\[
M_5(t) := \left\{ \epsilon \in M_5 \big| t = \min \{ 4 \leq j \leq n \mid \epsilon(j) = \epsilon(j+1) \} \right\}.
\]

From now, we compute \( n(i) := \sum_{\epsilon \in M_i} | G_\epsilon | \) for each \( i \in \{ 1, \ldots, 5 \} \). In the following, the notation \( i \sim j \) is replaced by an edge connecting \( i \) and \( j \) if \( \epsilon(i) \neq \epsilon(j) \), otherwise nothing between them.

(i) Let \( \epsilon \in M_1 \). Then, \( G_\epsilon \) is given by

\[
1 \sim 3 \sim 4 \sim \cdots \sim n-1 \sim n.
\]
Let $G'$ be the subgraph of $G$ obtained by removing the vertices $\{1, 2\}$. Then, we have $|G_\epsilon| = |G'_\epsilon|_{\{3, \ldots, n\}}$. Since $G'$ is a Dynkin graph of type $\tilde{A}_{n-2}$, we obtain

$$2n(1) = 2 \cdot \sum_{\epsilon: G_\epsilon \rightarrow \{\pm 1\} \epsilon(3) = +1} |G'_\epsilon| = \|\tilde{A}_{n-2}\| = c_{n-2},$$

where the last equality follows from Proposition 4.7.

By an argument similar to (1), we can calculate other cases.

(ii) For each $\epsilon \in M_2$, the graph $G_\epsilon$ is given by

$$1 \rightarrow 3 \sim 4 \sim \cdots \sim n-1 \sim \sim n.$$  

Then, we can check $2n(2) = \|\tilde{A}_{n-1}\| - \|\tilde{A}_{n-2}\| = c_{n-1} - c_{n-2}$.

(iii) By the symmetry of $G$, we have $n(3) = n(2)$.

(iv) Let $\epsilon \in M_4$. Then, $G_\epsilon$ is described as

$$1 \rightarrow 3 \rightarrow 4 \sim \cdots \sim n-1 \sim n.$$  

Thus, we find that $2n(4) = \|\tilde{A}_3\| \cdot \|\tilde{A}_{n-3}\| = 5c_{n-3}$.

(v) For $\epsilon \in M_5(t)$, the graph $G_\epsilon$ is given by

$$1 \rightarrow 3 \rightarrow 4 \rightarrow \cdots \rightarrow t \sim t+1 \sim \cdots \sim n.$$  

Then, we obtain

$$2n(5) = \sum_{t=4}^{n} |D_t| \cdot \|\tilde{A}_{n-t}\| = \frac{3n-4}{2n} c_{n-1} \cdot 2c_0 + \sum_{t=4}^{n-1} \frac{3t-4}{2t} c_{t-1} c_{n-t}$$  

$$= \frac{3n-4}{2n} c_{n-1} + \sum_{t=4}^{n} \frac{3t-4}{2t} c_{t-1} c_{n-t}.$$  

To finish the proof, we need the following lemma.

**Lemma 4.9.** For any positive integer $n$, the following equalities hold:

1. $\sum_{t=1}^{n} c_{t-1} c_{n-t} = 4^{n-1}$.
2. $\sum_{t=1}^{n} \frac{1}{t} c_{t-1} c_{n-t} = \frac{1}{2} c_n$.

**Proof.** Let $C_n := \frac{1}{n+1} c_n$ be the $n$-th Catalan number. Then we have

$$C_n = \sum_{t=1}^{n} C_{t-1} C_{n-t}, \quad \quad (4.2)$$

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}, \quad \quad (4.3)$$
By Lemma 4.9, we obtain the equality
\[ \sum_{t=1}^{n} \frac{3t - 4}{2t} c_{t-1}c_{n-t} = \frac{3}{2} \sum_{t=4}^{n} c_{t-1}c_{n-t} - 2 \sum_{t=4}^{n} \frac{1}{t} c_{t-1}c_{n-t} = \frac{3}{2} (4^{n-1} - c_{n-1} - 2c_{n-2} - 6c_{n-3}) - 2(\frac{1}{2}c_{n} - c_{n-1} - c_{n-2} - 2c_{n-3}) = 6 \cdot 4^{n-2} + \frac{1}{2}c_{n} - c_{n-1} - c_{n-2} - 5c_{n-3}. \]

By (i)–(v), we have
\[ \# 2\text{-tilt } A = c_{n-2} + 2(c_{n-1} - c_{n-2}) + 5c_{n-3} + 6 \cdot 4^{n-2} - c_{n} + \frac{2n - 2}{n} c_{n-1} - c_{n-2} - 5c_{n-3} = 6 \cdot 4^{n-2} - c_{n} + 4n \cdot 2c_{n-1} - 2c_{n-2} = 6 \cdot 4^{n-2} - 2c_{n-2}, \]
where the last equality follows from \( c_{n} = \frac{2(2n-1)}{n} c_{n-1}. \) \( \Box \)

Thirdly, we give an enumeration for type (I). The number is obtained by using the result on type \( \mathbb{D}. \)

**Proposition 4.10.** If \( G = I_n, \) then we have
\[ \# 2\text{-tilt } A = 6 \cdot 4^{n-2} + 2c_{n} - 4c_{n-1} - 4c_{n-2}. \]

**Proof.** Let \( G \) be a graph as follows:
\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\end{array}
\begin{array}{c}
\text{3} \\
\text{4} \\
\cdots \\
\text{n} \\
\end{array}
\]

Using a similar method of the proof of Proposition 4.8, we calculate the right-hand side of
\[ \# 2\text{-tilt } A = 2 \sum_{\epsilon: G_0 \rightarrow \{\pm 1\}} |G_{\epsilon}|. \]
Let $M$ and $M_i$ ($1 \leq i \leq 5$) be sets of maps given in the proof of Proposition 4.8 and $m(i) := \sum_{\epsilon \in M_i} |G_{\epsilon}|$. For each map $\epsilon \in M_1 \cup M_4 \cup M_5$, we have $G_{\epsilon} = (\mathbb{D}_n)_\epsilon$. Hence, for each $i \in \{1, 4, 5\}$, we have

$$m(i) = \sum_{\epsilon \in M_i} |G_{\epsilon}| = \sum_{\epsilon \in M_i} |(\mathbb{D}_n)_\epsilon| = n(i).$$

Since $m(2) = m(3)$ holds by the symmetry of $G$, we have only to calculate $m(2)$. For each map $\epsilon \in M_2 \cup M_4 \cup M_5$, we have

$$G_{\epsilon} = (D_n)_{\epsilon}.$$

Hence, for each $i \in \{1, 4, 5\}$, we have

$$m(i) = \sum_{\epsilon \in M_i} |G_{\epsilon}| = \sum_{\epsilon \in M_i} |(D_n)_{\epsilon}| = n(i).$$

Since $m(2) = m(3)$ holds by the symmetry of $G$, we have only to calculate $m(2)$. For each map $\epsilon \in M_2$, the graph $G_{\epsilon}$ is given by

$$1 \sim \sim \sim 4 \sim \sim \sim \cdots \sim \sim n - 1 \sim \sim n.$$

Then, the calculation of $m(2)$ is reduced to that of Dynkin graphs of type $A$. In fact, let $G'$ be the Dynkin graph $A_n$. Then, we have

$$m(2) = \sum_{\epsilon : G'_0 \to \{\pm 1\}} |G'_\epsilon| - \sum_{\epsilon : G'_0 \to \{\pm 1\}} |G'_\epsilon| - \sum_{\epsilon : G'_0 \to \{\pm 1\}} |G'_\epsilon|$$

$$= \frac{1}{2} \|A_n\| - \frac{1}{4} \|A_2\| \cdot \|A_{n-2}\| - n(2).$$

Therefore, we obtain

$$\# \text{2-tilt } A = 2(m(1) + m(2) + m(3) + m(4) + m(5))$$

$$= 2(n(1) + 2n(2) + n(4) + n(5)) - 4n(2) + 4m(2)$$

$$= \|D_n\| - 4n(2) + 4m(2)$$

$$= 6 \cdot 4^{n-2} - 2c_{n-2} - 4n(2) + 2c_n - 6c_{n-2} - 4n(2)$$

$$= 6 \cdot 4^{n-2} + 2c_n - 4c_{n-1} - 4c_{n-2}.$$

This finishes the proof. \hfill \Box

For the remaining finite series $E$, (II), (III), (IV), and (V), we just compute the number by using the formula (3.1) in Corollary 3.5(2).

**Proposition 4.11.** For each case $E$, (II), (III), (IV) and (V), the number of isomorphism classes of basic two-term tilting complexes is given by the table of Theorem 1.2.

**Proof.** The number for $E_6$ is shown in Example 3.7(2) and the others are similar. The detail is left to the reader. \hfill \Box

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