REDUCTION METHOD FOR REPRESENTATIONS OF QUEER LIE SUPERALGEBRAS

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Abstract. We develop a reduction procedure which provides an equivalence from an arbitrary block of the BGG category for the queer Lie superalgebra $q(n)$ to a $\mathbb{Z} \pm s$-weights ($s \in \mathbb{C}$) block of a BGG category for finite direct sum of queer Lie superalgebras. We give descriptions of blocks. We also establish equivalences between certain maximal parabolic subcategories for $q(n)$ and blocks of atypicality-one of the category of finite-dimensional modules for $\mathfrak{gl}(\ell|n-\ell)$.

1. Introduction

1.1. The character problem for finite-dimensional irreducible modules over queer Lie superalgebras $q(n)$ was first solved by Penkov and Serganova [PS1, PS2]. They provided an algorithm for computing the coefficient $a_{\lambda\mu}$ of the character of the irreducible $q(n)$-module $L(\mu)$ in the expansion of the character of the associated Euler characteristic $E(\lambda)$ for given dominant weights $\lambda, \mu$.

In [Br2] Brundan developed a different approach to computing the coefficient $a_{\lambda\mu}$ for integer dominant weights $\lambda, \mu$. Let $F^n$ be the $n$th exterior power of the natural representation of type B quantum group $U_q(b_{\infty})$ with infinite rank (cf. [JMO]). It was proved that the transition matrix $(a_{\lambda\mu})$ is given by the transpose of the transition matrix between canonical basis and the natural monomial basis of $F^n$ at $q = 1$. This gives all irreducible characters of finite-dimensional integer weight modules in terms of a combinatorial algorithm for computing canonical bases. A new interpretation of the irreducible characters of finite-dimensional half-integer weight modules was given in [CK] and [CKW] as well.

The celebrated Brundan’s Kazhdan-Lusztig conjecture [Br1] for the BGG category of integer weight $\mathfrak{gl}(m|n)$-modules has been proved by Cheng, Lam and Wang in [CLW] (also see [BLW]). Furthermore, in [CMW], by using twisting functors and parabolic induction functors Cheng, Mazorchuk and Wang reduced the irreducible character problem of an arbitrary weight to the problem of integer weight.

In the present paper, we study the problem analogous to [CMW] for the queer Lie superalgebra. One of the main goals is to study the (indecomposable) blocks of the
BGG category for queer Lie superalgebra. In particular, we will prove equivalence of categories between certain blocks for \( q(n) \) and \( \mathfrak{g}l(n - \ell) \).

Throughout the paper we denote by \( \mathfrak{g} \) the queer Lie superalgebra \( q(n) \) with the standard Cartan subalgebra \( \mathfrak{h} \) for a fixed integer \( n \geq 1 \). Let \( \mathcal{O}^\theta \) denote the BGG category (see, e.g., [BT Section 3]) of finitely generated \( \mathfrak{g} \)-modules which are locally finite \( \mathfrak{b} \)-modules and semisimple \( \mathfrak{h}_0 \)-modules. Note that morphisms in \( \mathcal{O}^\theta \) are even. For a finite direct sum of queer Lie superalgebras and reductive Lie algebras, we have analogous notation of its BGG category. Let \( m \in \mathbb{Z}_+ \), \( 0 \leq \ell \leq m \) and \( s \in \mathbb{C} \). If \( m \geq 1 \), let \( \Lambda_{s^t}(m) \subset \mathbb{C}^m 

\begin{align*}
(1.1) \quad \Lambda_{s^t}(m) := \left\{ \lambda = (\lambda_1, \ldots, \lambda_m) \mid \begin{array}{ll}
(1) & \lambda_i \equiv s \mod \mathbb{Z} \text{ for } 1 \leq i \leq \ell, \\
(2) & \lambda_i \equiv -s \mod \mathbb{Z} \text{ for } \ell + 1 \leq i \leq n.
\end{array} \right\}.
\end{align*}

We define \( q(0) \) and \( \Lambda_{s^t}(0) \) to be 0 and the empty set, respectively. For each \( \lambda \in \mathfrak{h}_0^* \), we shall assign a specific irreducible module \( L(\lambda) \) of highest weight \( \lambda \) and then define the corresponding block \( \mathcal{O}_\lambda^\theta \), see the definitions in Section 2.2. The following theorem is the first main result of this paper.

**Theorem 1.1.** Let \( \lambda \in \mathfrak{h}_0^* \). Then \( \mathcal{O}_\lambda^\theta \) is equivalent to a block \( \mathcal{O}_\mu^l \) of a Levi subalgebra \( I = q(n_1) \times q(n_2) \times \cdots \times q(n_k) \subseteq \mathfrak{g} \) with \( \sum_{i=1}^k n_i = n \) and the weight \( \mu \) of the form

\begin{equation}
(1.2) \quad \mu \in \Lambda_{s^t_1}(n_1) \times \Lambda_{s^t_2}(n_2) \times \cdots \times \Lambda_{s^t_k}(n_k),
\end{equation}

such that \( s_i \neq \pm s_j \mod \mathbb{Z} \) for all \( i \neq j \).

Accordingly, the study of blocks of \( \mathcal{O}^\theta \) is reduced to the study of blocks of the following three types: (i) \( s = 0 \) a BGG category \( \mathcal{O}_{n,\mathbb{Z}} \) of the \( q(n) \)-modules of integer weights, see, e.g., [Br2]. (ii) \( s \in \mathbb{Z} + \frac{1}{2} \) a BGG category \( \mathcal{O}_{n,\frac{1}{2}+\mathbb{Z}} \) of the \( q(n) \)-modules of half-integer weights, see, e.g., [CK, CKW]. (iii) \( s \notin \mathbb{Z}/2 \) a BGG category \( \mathcal{O}_{n,s^t} \) of the \( q(n) \)-modules of \( \pm s \)-weights”, see the definition in Section 4.2.

1.2. Let \( \mathfrak{g}l(\ell|n - \ell) \) be the general linear Lie superalgebra with the standard Cartan subalgebra \( \mathfrak{h}_{\ell|n-\ell} \) for \( 1 \leq \ell \leq n \). Another main result of the present paper is to establish an equivalence between a block \( \mathcal{F}_\lambda \) of certain maximal parabolic categories \( \mathcal{F} \) for \( q(n) \) and certain block of atypicality-one of the finite-dimensional module category \( \mathcal{F}_{\ell|n-\ell} \) for \( \mathfrak{g}l(\ell|n - \ell) \), see the definitions in Sections 1.1 and 4.2. Their identical linkage principle (see Lemma 4.1) is the first piece of evidence to support such an equivalence.

For a weight \( \lambda \in \mathfrak{h}_0^* \), or \( \lambda \in \mathfrak{h}_0^{\ell|n-\ell} \), we denote by \( \sharp \lambda \) the atypicality degree of \( \lambda \) (see, e.g., [CW Definitions 2.29, 2.49]). According to [Ser98 Theorem 2.6] and [BS12, Theorem 1.1] the blocks \( (\mathcal{F}_{\ell|n-\ell})_\lambda \) (see Section 4.1) for all \( \ell, n - \ell \) with the same \( \sharp \lambda \) are equivalent. More precisely, the endomorphism ring of projective generator of \( (\mathcal{F}_{\ell|n-\ell})_\lambda \) is isomorphic to the opposite ring of the diagram algebra \( K_1^\infty \) (see, e.g., [BS12 Introduction]). In particular, \( K_1^\infty \) is the path algebra of the infinite quiver.
Theorem 1.2. Let $L(\lambda) \in \mathcal{F}$ with $\sharp \lambda = 1$. Then the endomorphism ring of the projective generator of $\mathcal{F}_\lambda$ is isomorphic to $(K^{\infty}_1)^{\text{op}}$.

1.3. The paper is organized as follows. In Section 2, we recall definitions of queer Lie superalgebras, general linear Lie superalgebras and their categories of modules.

In Section 3, an approach of reduction similar to [CMW] is established for queer Lie superalgebras. Equivalences of blocks via twisting functors and parabolic induction functors are established. In addition, a description of decomposition of blocks of $\mathcal{O}$ is given in Theorem 3.8.

In Section 4, we recall the category of finite-dimensional modules for $\text{gl}((\ell | n - \ell))$ and introduce certain maximal parabolic category for $q(n)$. A correspondence preserving linkage principles between their irreducibles is established. Finally, we compute the endomorphism ring of projective generator to obtain Theorem 1.2.

Acknowledgments. The author is very grateful to Shun-Jen Cheng for numerous helpful comments and suggestions.

2. Preliminaries

2.1. Lie superalgebras $\mathfrak{gl}$ and $q$. For positive integers $m, n \geq 1$, let $\mathbb{C}^{m|n}$ be the complex superspace of dimension $(m|n)$. Let $\{v_1, \ldots, v_\bar{m}\}$ be an ordered basis for the even subspace $\mathbb{C}^{m|0}$ and $\{v_1, \ldots, v_n\}$ be an ordered basis for the odd subspace $\mathbb{C}^{0|n}$ so that the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ may be realized as $(m + n) \times (m + n)$ complex matrices indexed by $I(m|n) := \{1 < \cdots < \bar{m} < 1 < \cdots < n\}$:

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
$$

where $A, B, C$ and $D$ are respectively $m \times m$, $m \times n$, $n \times m$, $n \times n$ matrices. For $m = n$, the subspace

$$
\mathfrak{g} := q(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \middle| A, B : n \times n \text{ matrices} \right\}
$$

forms a subalgebra of $\mathfrak{gl}(n|n)$ called the queer Lie superalgebra.

Let $E_{ab}$ be the elementary matrix in $\mathfrak{gl}(m|n)$ with $(a, b)$-entry 1 and other entries 0, for $a, b \in I(m|n)$. Then $\{e_{ij}, \tilde{e}_{ij} | 1 \leq i, j \leq n\}$ is a linear basis for $\mathfrak{g}$, where $e_{ij} = E_{ij} + E_{ij}$ and $\tilde{e}_{ij} = E_{ij} + E_{ij}$. Note that the even subalgebra $\mathfrak{g}_0$ is spanned by $\{e_{ij} | 1 \leq i, j \leq n\}$, which is isomorphic to the general linear Lie algebra $\mathfrak{gl}(n)$.
Let \( \mathfrak{h}_{m|n} \) and \( \mathfrak{h}_{m|n}^* \) be respectively the standard Cartan subalgebra of \( \mathfrak{gl}(m|n) \) and its dual space, with linear bases \( \{ E_{ji} | i \in I(m|n) \} \) and \( \{ \delta_i | i \in I(m|n) \} \) such that \( \delta_i(E_{jj}) = \delta_{ij} \).

Let \( \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \) be the standard Cartan subalgebra of \( \mathfrak{g} \), with linear bases \( \{ h_i := e_{ii} | 1 \leq i \leq n \} \) and \( \{ \bar{h}_i := \bar{e}_{ii} | 1 \leq i \leq n \} \) of \( \mathfrak{h}_0 \) and \( \mathfrak{h}_1 \), respectively. Let \( \{ \varepsilon_i | 1 \leq i \leq n \} \) be the basis of \( \mathfrak{h}_0^* \) dual to \( \{ h_i | 1 \leq i \leq n \} \). We define a symmetric bilinear form \( (,\cdot) \) on \( \mathfrak{h}_0^* \) by \( (\varepsilon_i,\varepsilon_j) = \delta_{ij} \), for \( 1 \leq i, j \leq n \).

We denote by \( \Phi, \Phi_0, \Phi_1 \) the sets of roots, even roots and odd roots of \( \mathfrak{g} \), respectively. Let \( \Phi^+ = \Phi_0^+ \cup \Phi_1^+ \) be the set of positive roots with respect to its standard Borel subalgebra \( \mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1 \), which consists of matrices of the form (2.2) with \( A \) and \( B \) upper triangular. Denote the set of negative roots by \( \Phi^- := \Phi \setminus \Phi^+ \). Ignoring the parity we have \( \Phi_0 = \Phi_1 = \{ \varepsilon_i - \varepsilon_j | 1 \leq i, j \leq n \} \) and \( \Phi^+ = \{ \varepsilon_i - \varepsilon_j | 1 \leq i < j \leq n \} \). We denote by \( \leq \) the partial order on \( \mathfrak{h}_0^* \) defined by using \( \Phi^+ \). The Weyl group \( W \) of \( \mathfrak{g} \) is defined to be the Weyl group of the reductive Lie algebra \( \mathfrak{g}_0 \) and hence acts naturally on \( \mathfrak{h}_0^* \) by permutation. We also denote by \( s_\alpha \), the reflection associated to a root \( \alpha \in \Phi^+ \).

For a given root \( \alpha = \varepsilon_i - \varepsilon_j \in \Phi \), let \( \alpha_i := \varepsilon_i + \varepsilon_j \). For each \( \lambda \in \mathfrak{h}_0^* \), we have the integral root system \( \Phi_\lambda := \{ \alpha \in \Phi | \lambda, \alpha \in \mathbb{Z} \} \) and the integral Weyl group \( W_\lambda \) defined to be the subgroup of \( W \) generated by all reflections \( s_\alpha \), \( \alpha \in \Phi_\lambda \).

### 2.2. Categories of modules

Let \( V = V^+_\lambda \oplus V^-_\lambda \) be a superspace. For a given homogeneous element \( v \in V_i \) \( (i \in \mathbb{Z}_2) \), we let \( \bar{v} = i \) denote its parity. Let \( \Pi \) denote the parity change functor on the category of superspaces. Let \( \Pi^0 \) be the identity functor. For a \( \mathfrak{g} \)-module \( M \) and \( \mu \in \mathfrak{h}_0^* \), let \( M_\mu := \{ m \in M | h \cdot m = \mu(h)m, \text{ for } h \in \mathfrak{h}_0 \} \) denote its \( \mu \)-weight space. If \( M \) has a weight space decomposition \( M = \bigoplus_{\mu \in \mathfrak{h}_0^*} M_\mu \), its character is given as usual by \( \text{ch}M = \sum_{\mu \in \mathfrak{h}_0^*} \dim M_\mu e^\mu \), where \( e \) is an indeterminate. In particular, we have the root space decomposition \( \mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha) \) with respect to the adjoint representation of \( \mathfrak{g} \).

Let \( \lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in \mathfrak{h}_0^* \), and consider the symmetric bilinear form on \( \mathfrak{h}_1^* \) defined by \( (\cdot,\cdot) : = \lambda(\cdot,\cdot) \). Let \( \ell(\lambda) \) be the number of \( i \)'s with \( \lambda_i \neq 0 \) and \( \delta(\lambda) = 0 \) (resp. \( \delta(\lambda) = 1 \)) if \( \ell(\lambda) \) is even (resp. odd). Let \( 1 \leq i_1 < i_2 < \cdots < i_{\ell(\lambda)} \leq n \) such that \( \lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{\ell(\lambda)}} \) are non-zero. Denote by \( [\cdot] \) the ceiling function. Then the space

\[
\mathfrak{h}_1' := \left( \bigoplus_{j \neq i_1, \ldots, i_{\ell(\lambda)}} \mathbb{C} h_j \right) \oplus \left( \bigoplus_{k=1}^{\ell(\lambda) - \lceil \ell(\lambda)/2 \rceil} \mathbb{C} (\sqrt{\lambda_{i_{2k-1}}}, \sqrt{\lambda_{i_{2k}}}) \right),
\]

is a maximal isotropic subspace of \( \mathfrak{h}_1 \) associated to \( \langle \cdot, \cdot \rangle_\lambda \). Put \( h' = \mathfrak{h}_0 \oplus \mathfrak{h}_1' \). Let \( \mathbb{C} v_\lambda \) be the one-dimensional \( h' \)-module with \( \text{ch} = 0 \), \( h \cdot v_\lambda = \lambda(h)v_\lambda \) and \( h' \cdot v_\lambda = 0 \) for \( h \in \mathfrak{h}_0 \), \( h' \in \mathfrak{h}_1' \). Then \( I_\lambda := \text{Ind}_{h'}^{h} \mathbb{C} v_\lambda \) is an irreducible \( h \)-module of dimension \( 2^{\ell(\lambda)}/2 \) (see, e.g., [CW, Section 1.5.4]). We let \( M(\lambda) := \text{Ind}_{h'}^{h} I_\lambda \) be the Verma module, where \( I_\lambda \) is extended to a \( h \)-module in a trivial way, and define \( L(\lambda) \) to be the unique irreducible quotient of \( M(\lambda) \).
Let $\mathcal{O}^\vartheta$ denote the BGG category (see, e.g., [Fr Section 3]) of finitely generated $\mathfrak{g}$-modules which are locally finite over $\mathfrak{b}$ and semisimple over $\mathfrak{h}_0$. Note that the morphisms in $\mathcal{O}^\vartheta$ are even. It is known (see, e.g., [CW Section 1.5.4]) that $L(\lambda) \cong \Pi LL(\lambda)$ if and only if $\delta(\lambda) = 1$. Therefore we have the following.

**Lemma 2.1.** \{\(L(\lambda)\), with $\delta(\lambda) = 1\} \cup \{L(\lambda), \Pi LL(\lambda), \lambda \in h_0^\ast \text{ with } \delta(\lambda) = 0\}$ is a complete set of irreducible $\mathfrak{g}$-modules in $\mathcal{O}^\vartheta$ up to isomorphism.

We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. As in the case of Lie algebras, the BGG category $\mathcal{O}^\vartheta$ of $\mathfrak{g}$ has a decomposition into subcategories corresponding to central characters $\chi_{\lambda} : Z(\mathfrak{g}) \to \mathbb{C}$ for $\lambda \in h_0^\ast$. We have a refined decomposition by the linkage principle (see, e.g., [CW Section 2.3])

\[
\mathcal{O}^\vartheta = \bigoplus_{\lambda \in h_0^\ast / \sim} \mathcal{O}^\vartheta_\lambda,
\]

where the equivalence relation $\sim$ on $h_0^\ast$ is defined by

\[
\lambda \sim \mu \text{ if and only if } \chi_{\lambda} = \chi_{\mu} \text{ and } \mu \in \lambda + Z\Phi,
\]

and $\mathcal{O}^\vartheta_\lambda$ is the Serre subcategory of $\mathcal{O}^\vartheta$ generated by simple objects with highest weight $\mu$ such that $\lambda \sim \mu$. The subcategories $\mathcal{O}^\vartheta_\lambda$ are decomposable in general.

For a finite direct sum of queer Lie superalgebras and reductive Lie algebras, we have analogous notation and decomposition of its BGG category. When there is no confusion, we denote $\mathcal{O}^\vartheta$ by $\mathcal{O}$. For $\lambda \in h_0^\ast$, denote the block of $\mathcal{O}$ containing $L(\lambda)$ by $\mathcal{O}_\lambda$. Namely, it is the Serre subcategory generated by the set of vertices in the connected component of the Ext-quiver for $\mathcal{O}$ containing $L(\lambda)$.

3. Equivalences and Reductions for Blocks of Queer Lie Superalgebra

3.1. **Equivalence using twisting functors.** For a simple root $\alpha \in \Phi^+$, we can define the *twisting functor* $T_\alpha$ associated to $\alpha$. The twisting functor was originally defined by Arkhipov in [Ar] and further investigated in more detail in [AS], [KM], [CMW], [AL], [MS], [GGT3], [KM]. Recall the precise definition of $T_\alpha$ as follows. First, fix a non-zero root vector $X \in (\mathfrak{g}_0)^{-\alpha}$. Since the adjoint action of $X$ on $\mathfrak{g}$ is nilpotent, by using a standard argument (see e.g., [MO00, Lemma 4.2]) we can form the Ore localization $U'_\alpha$ of $U(\mathfrak{g})$ with respect to the set of powers of $X$. Since $X$ is not a zero divisor in $U(\mathfrak{g})$, $U(\mathfrak{g})$ can be viewed as an associative subalgebra of $U'_\alpha$. The quotient $U'_\alpha/U(\mathfrak{g})$ has the induced structure of a $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule. Let $\varphi = \varphi_\alpha$ be an automorphism of $\mathfrak{g}$ that maps $(\mathfrak{g}_i)_\beta$ to $(\mathfrak{g}_i)_{\alpha(\beta)}$ for all simple root $\beta$ and $i \in \{0, 1\}$. Finally, consider the bimodule $\varphi U_\alpha$, which is obtained from $U_\alpha$ by twisting the left action of $U(\mathfrak{g})$ by $\varphi$. We also have an analogous construction with respect to the subalgebra $\mathfrak{g}_0$ to obtain the $U(\mathfrak{g}_0)$-$U(\mathfrak{g}_0)$-bimodule $\varphi U^0_\alpha$.

Now we are in a position to define twisting functors:

\[
T_\alpha(-) := \varphi U_\alpha \otimes - : \mathcal{O}^\vartheta \to \mathcal{O}^\vartheta \text{ and } T_\alpha^0(-) := \varphi U^0_\alpha \otimes - : \mathcal{O}^{\mathfrak{g}_0} \to \mathcal{O}^{\mathfrak{g}_0}.
\]
Then $T_\alpha$ and $T^0_\alpha$ have right adjoints $K_\alpha$ and $K^0_\alpha$, respectively (see, e.g., [AS]). Let $\mathcal{D}^b(\mathcal{O})$ and $\mathcal{D}^b(\mathcal{O}^{(0)})$ be the bounded derived categories of $\mathcal{O}$ and $\mathcal{O}^{(0)}$, respectively. It is not hard to prove that $T_\alpha$ and $T^0_\alpha$ are right exact functors. Let $L_iT_\alpha$, $L_iT^0_\alpha$ the $i$-th left derived functors of $T_\alpha$, $T^0_\alpha$, respectively. It was proved in [AS] that $L_iT^0_\alpha = 0$ for $i > 1$ and $L_1T^0_\alpha$ is isomorphic to the functor of taking the maximal submodule on which the action of $\mathfrak{g}_{-\lambda}$ is locally nilpotent. Similarly, we have analogous definition for right derived endofunctors $\mathcal{R}^iK_\alpha$ and $\mathcal{R}^iK^0_\alpha$ of $K_\alpha$ and $K^0_\alpha$, respectively. Furthermore, $\mathcal{R}^iK^0_\alpha = 0$ for $i > 1$ and $\mathcal{R}^1K^0_\alpha$ is isomorphic to the functor of taking the maximal subquotient on which the action of $\mathfrak{g}_{-\lambda}$ is locally nilpotent.

The star action $* \circ s_\alpha$ on weights had been introduced in [GG13 Introduction] and [CM]: $s_\alpha * \lambda := s_\alpha \lambda$ if $(\lambda, \alpha) \neq 0$ and $s_\alpha * \lambda := s_\alpha \lambda - \alpha$ if $(\lambda, \alpha) = 0$. We call the former an $\alpha$-typical weight and the later an $\alpha$-atypical weight (also see [GG13 Section 1.2.3]). The following theorem is inspired by [CM] Proposition 8.6.

**Theorem 3.1.** Let $\lambda \in h^*_0$ and $\alpha \in \Phi^+$ be a simple root such that $(\lambda, \alpha) \notin \mathbb{Z}$. Then $\Pi^i \circ T_\alpha : \mathcal{O}_\lambda \to \mathcal{O}_{s_\alpha \lambda}$ is an equivalence with inverse $\Pi^j \circ K_\alpha : \mathcal{O}_{s_\alpha \lambda} \to \mathcal{O}_\lambda$ for some $i, j \in \{0, 1\}$.

**Proof.** We claim that $T_\alpha$ and $K_\alpha$ are exact functors on $\mathcal{O}_\lambda$ and $\mathcal{O}_{s_\alpha \lambda}$, respectively. To see this, we first note that $L_1T^0_\alpha$ and $\mathcal{R}^1K^0_\alpha$ vanish at each simple $\mathfrak{g}_{00}$-module of highest weight $\mu$ with $(\mu, \alpha) \notin \mathbb{Z}$ (e.g., [Mar Chapter 3]). Next we recall that $\text{Res}_{\mathfrak{g}_{00}}^{\mathfrak{g}_0} \circ L_iT_\alpha = L_iT^0_\alpha \circ \text{Res}_{\mathfrak{g}_{00}}^{\mathfrak{g}_0}$ and $\text{Res}_{\mathfrak{g}_{00}}^{\mathfrak{g}_0} \circ \mathcal{R}^iK_\alpha = \mathcal{R}^iK^0_\alpha \circ \text{Res}_{\mathfrak{g}_{00}}^{\mathfrak{g}_0}$ (e.g. [CM Lemma 5.1]) for all $i \geq 0$. This means that $L_iT_\alpha M = \mathcal{R}^iK_\alpha M' = 0$ for all $M \in \mathcal{O}_\lambda$, $M' \in \mathcal{O}_{s_\alpha \lambda}$ and $i \geq 1$. As a conclusion, $T_\alpha$ and $K_\alpha$ are exact functors on $\mathcal{O}_\lambda$ and $\mathcal{O}_{s_\alpha \lambda}$, respectively. For $\mu \in h^*_0$ with $(\mu, \alpha) \notin \mathbb{Z}$, it is proved in [CM Lemma 5.8] that $T_\alpha L(\mu)$ is simple with $T^2_\alpha L(\mu) \in \{L(\mu), \Pi L(\mu)\}$. By a similar argument we can show that $K_\alpha L(\mu)$ is also simple with $K^2_\alpha L(\mu) \in \{L(\mu), \Pi L(\mu)\}$. That is, that $T_\alpha$ and $K_\alpha$ preserve simple objects of $\mathcal{O}_\lambda$ and $\mathcal{O}_{s_\alpha \lambda}$, respectively. Finally recall that $\text{ch} T_\alpha M(\mu) = \text{ch} M(s_\alpha \mu)$ [CM Lemma 5.5] for all $\mu \in h^*_0$. From this together with the fact that $\text{Hom}_{\mathcal{O}}(T_\alpha L, L') = \text{Hom}_{\mathcal{O}}(L, K_\alpha L')$ for all simple objects $L, L' \in \mathcal{O}$, we conclude that $T_\alpha$ sends objects of $\mathcal{O}_\lambda$ to objects of $\mathcal{O}_{s_\alpha \lambda}$ and $K_\alpha$ sends objects of $\mathcal{O}_{s_\alpha \lambda}$ to objects of $\Pi^j \mathcal{O}_\lambda$, for some $i = 0, 1$. Consequently, the restrictions of $T_\alpha$ and $K_\alpha$ make $T_\alpha : \mathcal{O}_\lambda \to \Pi^i \mathcal{O}_{s_\alpha \lambda}$ an equivalence with inverse $K_\alpha : \mathcal{O}_{s_\alpha \lambda} \to \Pi^j \mathcal{O}_\lambda$, for some $i, j \in \{0, 1\}$. \hfill $\Box$

**Remark 3.2.** Let $\lambda, \alpha$ be as in Theorem 3.1. It is worth pointing out that $T_\alpha L(\lambda)$ only depends on whether $\lambda$ is $\alpha$-typical or $\alpha$-atypical. That is, it was determined in [CM Corollary 8.15]: By the classification of simple $\mathfrak{g}(2)$-highest weight modules in [Mar10] we have $[T_\alpha L(\lambda) : \Pi^i L(s_\alpha * \lambda)] \neq 0$, for some $i = 0, 1$. This is also proved in [GG13 Proposition 4.7.1]. As a consequence, we have $T_\alpha L(\lambda) = \Pi^1 L(s_\alpha * \lambda)$ for some $i = 0, 1$.

**Example 3.3.** Let $n = 3$ and $\lambda := (-\pi, \pi, -\pi)$. Then by Theorem 3.1 and Remark 3.2 $T_{\lambda - \epsilon_2} : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$ is an equivalence sending $L(\lambda)$ to $\Pi^1 L(\lambda - (\epsilon_1 - \epsilon_2))$ for some $i = 0, 1$, where $\lambda = (\pi, -\pi, -\pi)$. 

3.2. Equivalence using parabolic induction functor. The goal of this section is to show that the parabolic induction functors give equivalences of blocks under some suitable condition. For given integers \( \ell, m \) with \( 1 \leq \ell \leq m \) and \( s \in \mathbb{C} \), recall the set \( \Lambda_{\ell}(m) \). In this section, we consider blocks \( \mathcal{O}_\lambda \) with the weight \( \lambda \in \mathfrak{h}_0^* \) of the following form

\[
(3.2) \quad \lambda \in \Lambda_{s_1}(n_1) \times \cdots \times \Lambda_{s_k}(n_k) \ 	ext{such that} \ s_1 = 0, \ s_2 = \frac{1}{2} \text{ and } s_i \neq \pm s_j \mod \mathbb{Z},
\]

for all \( i \neq j \). We define \( \mathfrak{g}_\lambda := \{ \alpha \in \Phi \mid (\lambda, \alpha) \in \mathbb{Z} \} \cup \Phi_\lambda \) and \( \mathfrak{l}_\lambda := \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \mathfrak{g}_\lambda} \mathfrak{g}_\alpha \right) \) to be the Levi subalgebra associated to \( \lambda \). In this case, we denote by \( u_\lambda \) the corresponding nilradical. Furthermore, we have isomorphisms

\[
(3.3) \quad W_\lambda \cong \mathfrak{g}_{n_1} \times \mathfrak{g}_{n_2} \times (\mathfrak{g}_{\ell_3} \times \mathfrak{g}_{n_3 - \ell_3}) \times \cdots \times (\mathfrak{g}_{\ell_k} \times \mathfrak{g}_{n_k - \ell_k}),
\]

and \( \mathfrak{l}_\lambda \cong \mathfrak{q}(n_1) \times \mathfrak{q}(n_2) \times \cdots \times \mathfrak{q}(n_k) \). In order to prove that the parabolic induction functors are equivalences in this setting, we first recall the following well-known characterization of central characters (see, e.g., [CW, Theorem 2.48]).

**Lemma 3.4.** For \( \lambda, \mu \in \mathfrak{h}_0^* \), \( \chi_\lambda = \chi_\mu \) if and only if there exist \( w \in W_\lambda \), \( \{ k_j \}_j \subset \mathbb{C} \), and a subset of mutually orthogonal roots \( \{ \alpha_j \}_j \) such that \( \mu = w(\lambda - \sum_j k_j \alpha_j) \) and \( (\lambda, \alpha_j) = 0 \) for all \( j \).

Define a relation \( \sim \) on \( \mathfrak{h}_0^* \) as follows. For \( \lambda, \mu \in \mathfrak{h}_0^* \) we let \( \lambda \sim \mu \) if there exist \( w \in W_\lambda \), \( \{ k_j \}_j \subset \mathbb{Z} \), and a subset of mutually orthogonal roots \( \{ \alpha_j \}_j \) such that \( \mu = w(\lambda - \sum_j k_j \alpha_j) \) and \( (\lambda, \alpha_j) = 0 \) for all \( j \). The following lemma shows that \( \sim \) and \( \approx \) coincide in our setting.

**Lemma 3.5.** Let \( \lambda \in \mathfrak{h}_0^* \) be of the form \( (3.2) \). Then \( \mu \sim \lambda \) if and only if \( \mu \approx \lambda \). In particular, if \( \Pi^iL(\mu) \in \mathcal{O}_\lambda \), for some \( i = 1, 2 \), then \( \mu \sim \lambda \).

**Proof.** Since \( \chi_\lambda = \chi_\mu \) we have \( \mu = w(\lambda - \sum_j k_j \alpha_j) \) for some \( w \in W_\lambda \), \( \{ k_j \}_j \subset \mathbb{C} \) and \( \{ \alpha_j \}_j \subset \Phi \) such that \( (\lambda, \alpha_j) = 0 \) for all \( j \) by Lemma 3.4. Furthermore, we have \( \lambda \in \mu + \mathbb{Z}\Phi \). It follows that \( w \in W_\lambda \) and \( k_j \in \mathbb{Z} \) for all \( j \). This completes the proof. \( \square \)

The following theorem is inspired by [CMW, Proposition 3.6].

**Theorem 3.6.** Let \( \lambda \in \mathfrak{h}_0^* \) be of the form \( (3.2) \). Let \( l := l_{\lambda, u} := u_\lambda \). Then there are \( i, j \in \{ 0, 1 \} \) such that the parabolic induction functor \( \Pi^i \circ \text{Ind}^0_{L^0} : \mathcal{O}^0_\lambda \rightarrow \mathcal{O}^0_\lambda \) is an equivalence, with inverse equivalence \( \Pi^j \circ \text{Res}^0_\lambda : \mathcal{O}_\lambda \rightarrow \mathcal{O}^0_\lambda \) defined by \( M \mapsto M^u \), where \( M^u \) is the maximal trivial \( \mathfrak{u} \)-submodule of \( M \).

**Proof.** As in the proof of [CMW, Proposition 3.6], it suffices to show that \( \text{Ind}^0_{L^0} \) is irreducible for each irreducible \( \mathfrak{u} \)-module \( L^0_\mu \) of highest weight \( \mu \). We first assume that \( \zeta \in \mathfrak{h}_0^* \) is a weight of a non-zero singular vector in \( \text{Ind}^0_{L^0} \). Then by Lemma 3.5 there exist \( w \in W_\mu, \{ k_j \}_j \subset \mathbb{Z} \), and a subset of mutually orthogonal roots \( \{ \alpha_j \}_j \) such that \( \zeta = w(\mu - \sum_j k_j \alpha_j) \) and \( (\mu, \alpha_j) = 0 \) for all \( j \) (note that \( \Phi_\lambda = \Phi_\mu \)). On the
other hand, by consideration of the weights of $\text{Ind}_{\mu+u}^\mu L_\mu^0$, we have $\zeta \in \mu - \sum_{\alpha \in \Phi_+} Z_{\geq 0} \alpha$. Hence $\zeta \in \mu - \sum_{\alpha \in \Phi_+} Z_{\geq 0} \alpha$ by (3.3). This means that every subquotient of $\text{Ind}_{\mu+u}^\mu L_\mu^0$ intersects $L_\mu^0$ and so $\text{Ind}_{\mu+u}^\mu L_\mu^0$ is irreducible. This completes the proof. \hfill \square

**Proof of Theorem 3.7.** Let $\lambda \in h_0^\ast$. We can first apply a sequence of suitable twisting functors (see Theorem 3.1) to $\mathcal{O}_\lambda$ and obtain an equivalent block $\mathcal{O}_\lambda$ such that $\lambda \in \Lambda_{s_1} \times \Lambda_{s_2} \times \cdots \times \Lambda_{s_k}$ and $s_i \neq \pm s_j \mod Z$ for all $i \neq j$. Next we can apply the parabolic induction functor (see Theorem 3.6) to obtain an equivalent block of the desired Levi subalgebra. This completes the proof. \hfill \square

**Example 3.7.** Let $\lambda := (1, 1, -\pi, \frac{3}{2}, \pi, -\frac{3}{2}, -\pi)$. Then by applying a sequence of twisting functors $\Pi^i \circ T_\alpha$ with some $i_\alpha \in \{0, 1\}$ in Theorem 3.1 related to $\alpha$-typical weights, we may transform $\lambda$ to the weight $\tilde{\lambda} = (1, \frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, -\pi, -\pi, -\pi)$, which gives an equivalence from $\mathcal{O}_\lambda$ to $\mathcal{O}_{\tilde{\lambda}}$ sending $L(\lambda)$ to $L(\tilde{\lambda})$. Then we apply the twisting functor $\Pi^i \circ T_{\varepsilon_5 - \varepsilon_6}$ with some $i = 0, 1$ to obtain the weight $\tilde{\lambda} = (1, \frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, -\pi, -\pi, -\pi)$ and an equivalence $\mathcal{O}_\lambda$ to $\mathcal{O}_{\tilde{\lambda}}$ which sends $L(\lambda)$ to $L(\tilde{\lambda} - (\varepsilon_5 - \varepsilon_6))$. Next we use the parabolic induction functors. Define $\alpha := (\varepsilon_5 - \varepsilon_6)$. Note that $\tilde{\lambda}, \tilde{\lambda} - \alpha \in \Lambda_{\varepsilon_5}(1) \times \Lambda_{\varepsilon_6}(2) \times \Lambda_{\varepsilon_6}(1) \times \Lambda_{\varepsilon_6}(3)$ and $I_{\tilde{\lambda} - \alpha} = I_{\tilde{\lambda}} \approx q(1) \times q(2) \times q(1) \times q(3)$. By Theorem 3.6 there is an equivalence from $\mathcal{O}_\lambda$ to $\mathcal{O}_{\tilde{\lambda}}$ sending $L(\lambda)$ to the irreducible $\mathfrak{l}$-module with highest weight $\tilde{\lambda} - \alpha$.

3.3. **Description of blocks.**

**Theorem 3.8.** Let $\lambda, \mu \in h_0^\ast$. Then $\Pi^i L(\mu) \in \mathcal{O}_\lambda$ for some $i = 0, 1$ if and only if $\mu \approx \lambda$.

**Proof.** First assume that $\lambda \in h_0^\ast$ is of the form (3.2). Thanks to Lemma 3.5 it remains to show that $\mu \approx \lambda$ implies $\Pi^i L(\mu) \in \mathcal{O}_\lambda$ for some $i = 0, 1$. Recall the fundamental lemma in [PS2, Proposition 2.1] by Penkov and Serganova. It follows from $\text{Hom}_0(M(\lambda - \alpha), \Pi^i M(\lambda)) \neq 0$ for some $j = 0, 1$, for all $\alpha \in \Phi_+$ with $(\lambda, \alpha) = 0$ that $\Pi^i M(\lambda - \alpha) \in \mathcal{O}_\lambda$ for some $i = 0, 1$. Therefore we may assume that $\mu$ is of the form $s(\lambda)$, for some reflection $s \in W_\lambda$ corresponding to a simple root $\varepsilon_i - \varepsilon_{i+1}$. In this case, we have $\lambda_i - \lambda_{i+1} = k \in \mathbb{Z}$. Without loss of generality, assume that $k > 0$. Let $v_{\lambda} \in M(\lambda)$ be a highest weight vector, it is not hard to compute that $\mathcal{O}_{s(\lambda)} \rightarrow \mathcal{O}_\lambda$ an equivalence constructed by using a sequence of twisting functors in Theorem 3.1. For $\zeta, \zeta' \in h_0^\ast$ and simple reflection $s \in W$, note that $s \ast \zeta \approx s \ast \zeta'$ if and only if $\zeta \approx \zeta'$. The theorem now follows by Remark 3.2. \hfill \square

**Remark 3.9.** If $\ell(\lambda)$ is odd, then $\mathcal{O}_\lambda$ is the Serre subcategory generated by $\{L(\mu) | \mu \approx \lambda\}$.
4. Equivalences of Certain Maximal Parabolic Subcategory

In this section, we fix non-negative integers \( n, \ell \) with \( n \geq \ell \) and \( s \not\in \mathbb{Z}/2 \). The goal of this section is to establish an equivalence between certain block of atypicality-one of finite-dimensional category for \( \mathfrak{gl}(\ell|n-\ell) \) and some block of certain maximal parabolic subcategory for \( \mathfrak{q}(n) \).

4.1. Finite-dimensional representations of \( \mathfrak{gl}(\ell|n-\ell) \). We denote by \( \tilde{\mathcal{F}}_{\ell|n-\ell} \) the category of integral weight, finite-dimensional \( \mathfrak{gl}(\ell|n-\ell) \)-modules with even morphisms. Let \( \Lambda^a := \oplus_{i=1}^{n} \mathbb{Z} \delta_i \) be the weight lattice. Recall that the set of all irreducible objects (up to parity) of \( \tilde{\mathcal{F}}_{\ell|n-\ell} \) are parametrized by its highest weight \( \lambda \) in \( \Lambda^{a,+} := \{ \lambda \in \Lambda^a | \lambda_i \geq \lambda_{i+1}, \text{ for } 1 \leq i < \ell \text{ and } \ell \leq i < n \}. \) We define \( |\lambda| := (\lambda, \sum_{i=\ell+1}^{n} \delta_i) \) (mod 2). Recall that for a given \( M \in \tilde{\mathcal{F}}_{\ell|n-\ell} \), there is a decomposition \( M = M_+ \oplus M_- \) of \( \mathfrak{gl}(\ell|n-\ell) \)-modules, where \( M_+ := \oplus_{\mu \in \Lambda^a} (M_\mu |_{\mu}) \) and \( M_- := \oplus_{\mu \in \Lambda^a} (M_\mu |_{\mu+1}) \). This induces a decomposition \( \mathcal{F}_{\ell|n-\ell} = \tilde{\mathcal{F}}_{\ell|n-\ell} \oplus \Pi \mathcal{F}_{\ell|n-\ell} \) (see, e.g., [Br1 Section 4-e]), where \( \mathcal{F}_{\ell|n-\ell} \) is the full subcategory consisting of all \( M \in \tilde{\mathcal{F}}_{\ell|n-\ell} \) such that \( M = M_+ \) (resp. \( M = M_- \)).

As we mentioned in Section 4, the diagram algebra \( K_{\ell}^{\infty} \) is the path algebra of a certain infinite quiver. Therefore we can identify \( (K_{\ell}^{\infty})^{op} \) as the associative algebra generated by elements \( \{ z_i, x_j, y_{kl} \}_{i,j,k} \) and relations

\[
\begin{align*}
z_ic &= cz_i = y_iy_j = x_jx_i = 0, \\
x_iz_i &= z_{i+1}, \quad y_ix_i = z_i,
\end{align*}
\]

for all \( i, j \in \mathbb{Z}, c \in \{ x, y \}_{s,t} \in \mathbb{Z} \).

4.2. Parabolic categories of \( \mathfrak{q}(n) \) and Equivalences. We define a bijection \( \alpha : \Lambda_{\mathfrak{q}^s}(n) \to \Lambda^a \) by

\[
(4.1) \quad \lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i \in \Lambda_{\mathfrak{q}^s}(n) \quad \mapsto \lambda^a := \sum_{i=1}^{\ell} (\lambda_i - s) \delta_i + \sum_{i=\ell+1}^{n} (\lambda_i + s) \delta_i - \rho \in \Lambda^a,
\]

where \( \rho := \sum_{i=1}^{\ell} -(\ell - i + 1) \delta_i + \sum_{i=\ell+1}^{n} (i - \ell) \delta_i \).

Let \( \chi_{\lambda}^a \) be the central character of \( \mathfrak{gl}(\ell|n-\ell) \) corresponding to \( \lambda \in \mathfrak{h}_{\ell|n-\ell}^* \). We first consider the linkage principles under this bijection.

Denote by \( \Psi := \{ \delta_i - \delta_j | 1 \leq i \neq j \leq n \} \) the root system of \( \mathfrak{gl}(\ell|n-\ell) \). Recall that the linkage principle in (2.5) for \( \mathfrak{q}(n) \) defines an equivalence relation \( \sim \). The following lemma follows from Lemma [3.5] and the proof of [CMW Proposition 3.3].

**Lemma 4.1.** Let \( \lambda, \mu \in \Lambda_{\mathfrak{q}^s}(n) \). Then \( \lambda \sim \mu \) if and only if \( \chi_{\lambda}^a = \chi_{\mu}^a \) and \( \mu^a \in \Lambda^a + \mathbb{Z}\Psi \).
We define the set \( \Lambda^+_s(n) := \{ \lambda \in \Lambda_s(n) \mid \lambda_i > \lambda_{i+1}, \text{ for } 1 \leq i < \ell \text{ and } \ell \leq i < n \} \). Note that we have \( \Lambda^a := (\Lambda^+_s(n))^a \). For arbitrary \( \lambda \in \Lambda^+_s(n) \) we have a Levi subalgebra \( \mathfrak{l} := \mathfrak{h} \oplus (\oplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha) \cong q(\ell) \times q(n-\ell) \) and the maximal parabolic subalgebras \( \mathfrak{p} := \mathfrak{h} \oplus (\oplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha) \). Let \( \mathfrak{u} \) be the corresponding nilradical of \( \mathfrak{p} \). We denote by \( \mathcal{O} \) the maximal parabolic subcategory of \( \mathfrak{g} \)-modules. We define \( \mathcal{O}_{n,s} \) to be the full subcategory of \( \mathfrak{g} \)-modules with weights in \( \Lambda^+_s(n) \) and \( \mathcal{F} := \mathcal{O}_p \cap \mathcal{O}_{n,s} \) its maximal parabolic subcategory. For each \( M \in \mathcal{F} \), note that \( M \) is also \( l \)-semisimple since all weights of \( M \) are \( l \)-typical. As a conclusion, if \( \Pi^i L(\mu) \in \mathcal{F} \) for some \( i = 0, 1 \) then we have \( \mu \in \Lambda^+_s(n) \).

Let \( \lambda \in \Lambda^+_s(n) \). Note that every irreducible \( l \)-module of highest weight \( \lambda \) can be extended to a \( \mathfrak{p} \)-module by letting \( \mathfrak{u} \) act trivially. We define \( L^0(\lambda) \) to be the finite-dimensional irreducible \( l \)-module with highest weight space \( I_\lambda \). Therefore the corresponding parabolic Verma module \( K(\lambda) := \text{Ind}^p_{\mathfrak{l}} L^0(\lambda) \) has the irreducible quotient \( L(\lambda) \). Furthermore, we note that \( K(\lambda) \) is \( \mathfrak{p} \)-locally finite and all the \( l \)-weights of \( K(\lambda) \) are \( l \)-typical. Therefore we have \( K(\lambda), L(\lambda) \in \mathcal{F} \). Consequently, \( \mathcal{F} \) is the Serre subcategory of \( \mathcal{O} \) generated by \( \{ \Pi^i L(\lambda) | \lambda \in \Lambda^+_s(n), \ i \in \{0, 1\} \} \).

For \( \lambda \in \Lambda^+_s(n) \). We also denote by \( P(\lambda) \) and \( U(\lambda) \) the projective cover of \( L(\lambda) \) and the tilting module corresponding to \( \lambda \) in \( \mathcal{O} \), respectively. For their definitions and existences, we refer to \cite[Proposition 1.7]{Mar14} and \cite[Theorem 2]{Mar14}. Note that all weights of \( P(\lambda), U(\lambda) \) are in \( \Lambda^+_s(n) \) since they are indecomposable (by definition). That is, \( P(\lambda), U(\lambda) \in \mathcal{F} \).

Let \( P \) be the free abelian group on basis \( \{ \epsilon_a \}_{a \in \mathbb{Z}} \). Let \( \text{wt}(\cdot) : \Lambda^+_s(n) \to P \) be the weight function defined by (c.f. \cite[Section 2-c]{Br2})

\[
\text{wt}(\lambda) := \sum_{i=1}^{\ell} \epsilon_{\lambda_i-s} + \sum_{i=\ell+1}^{n} (-\epsilon_{-(\lambda_i+s)}),
\]

By Lemma 3.3, we have \( \chi_\lambda = \chi_\mu \) if and only if \( \text{wt}(\lambda) = \text{wt}(\mu) \). By (2.4), we have decomposition \( \mathcal{F} = \bigoplus_{\lambda \in \mathcal{O}_{n,s}} \mathcal{F}_\lambda = \bigoplus_{\gamma \in \mathcal{P}\mathcal{F}_\gamma} \mathcal{F}_\gamma \) according to central characters \( \chi_\lambda \) with \( \text{wt}(\lambda) = \gamma \).

Let \( \mathbb{C}^{n|n} \) and \( (\mathbb{C}^{n|n})^* \) be the standard representation and its dual, respectively. Denote the projection functor from \( \mathcal{F} \) to \( \mathcal{F}_\gamma \) by \( \text{pr}_\gamma \). We define the translation functors \( E_a, F_a : \mathcal{F} \to \mathcal{F} \) as follows

\[
E_a(M) := \text{pr}_\gamma(\epsilon_a-\epsilon_{a+1})(M \otimes (\mathbb{C}^{n|n})^*), \quad F_a(M) := \text{pr}_\gamma(\epsilon_a-\epsilon_{a+1})(M \otimes \mathbb{C}^{n|n}),
\]

for all \( M \in \mathcal{F}_\gamma, \gamma \in P, a \in \mathbb{Z} \). For each \( a \in \mathbb{Z} \), it is not hard to see that both \( E_a \) and \( F_a \) are exact and bi-adjoint to each other. We write \( \lambda \to \gamma \mu \) if \( \lambda, \mu \in \Lambda^+_s(n) \) and there exists \( 1 \leq i \leq \ell \) such that \( \lambda_i = \mu_i - 1 = a + s \) or there exists \( \ell + 1 \leq i \leq n \) such that \( \lambda_i = \mu_i - 1 = -a - 1 - s \), and in addition, \( \lambda_j = \mu_j \) for all \( i \neq j \). We have the following lemma.
Lemma 4.2. Let $\lambda \in \Lambda^+_q(n)$. Then both $E_aK(\lambda)$ and $F_aK(\lambda)$ have flags of parabolic Verma modules and we have the following formula:

\begin{align}
(4.4) \quad & \text{ch}E_aK(\lambda) = 2 \sum_{\mu \rightarrow a\lambda} \text{ch}K(\mu), \\
(4.5) \quad & \text{ch}F_aK(\lambda) = 2 \sum_{\lambda \rightarrow a\mu} \text{ch}K(\mu).
\end{align}

Proof. We first note that we have

\begin{equation}
K(\lambda) \otimes \mathbb{C}^{n|n} \cong U(\mathfrak{g}) \otimes_p (L^0(\lambda) \otimes \mathbb{C}^{n|n}).
\end{equation}

This implies that $K(\lambda) \otimes \mathbb{C}^{n|n}$ (and so the summand of $F_aK(\lambda)$) has a filtration whose non-zero subquotients are parabolic Verma modules. Note the $l_0 = \mathfrak{gl}(\ell) \times \mathfrak{gl}(n-\ell)$ with the Cartan subalgebra $\mathfrak{h}_0 = \bigoplus_{i=1}^{n} \mathfrak{c}e_i$ and its dual $\mathfrak{h}_0^* = \bigoplus_{i=1}^{n} \mathfrak{c}e_i^*$. Let $(1) \zeta := \sum_{i=1}^{n} \lambda_i e_i$ and $\zeta^{(1)} := \sum_{i=\ell+1}^{n} \lambda_i e_i$, and $\rho_0^\ell, p_0^{n-\ell}$ be Weyl vectors for the general linear Lie algebras $\mathfrak{gl}(\ell)$ and $\mathfrak{gl}(n-\ell)$, respectively. Then the module in (4.6) has character

\begin{equation}
\text{ch} \left( K(\lambda) \otimes \mathbb{C}^{n|n} \right) = 2 \cdot D \cdot \sum_{i=1}^{n} \left( e^{\varepsilon_i} \cdot s_{(1)\lambda - \rho_0^\ell} \cdot s_{(1)\lambda - p_0^{n-\ell}} \right),
\end{equation}

where $s_{(1)\lambda - \rho_0^\ell}$ and $s_{(1)\lambda - p_0^{n-\ell}}$ are respectively Schur functions corresponding to $(1)\lambda - \rho_0^\ell$ and $(1)\lambda - p_0^{n-\ell}$, respectively, and

\begin{align}
(4.8) \quad & D = 2^{[n/2]} \prod_{1 \leq i < j \leq \ell} \frac{1 + e^{-\varepsilon_i + \varepsilon_j}}{1 - e^{-\varepsilon_i + \varepsilon_j}} \cdot \prod_{\ell+1 \leq i < j \leq n} \frac{1 + e^{-\varepsilon_i + \varepsilon_j}}{1 - e^{-\varepsilon_i + \varepsilon_j}}, \\
(4.9) \quad & D \cdot s_{(1)\zeta - \rho_0^\ell} \cdot s_{(1)\zeta - p_0^{n-\ell}} = \text{ch}K(\zeta),
\end{align}

for all $\zeta \in \Lambda^+_q(n)$ (c.f. [Pe4 Theorem 2] and [CW Section 3.1.3]). By the Pieri formula (see, e.g., [Mac95 Lemma 5.16]), we may conclude that

\begin{equation}
\text{ch} \left( K(\lambda) \otimes \mathbb{C}^{n|n} \right) = 2 \sum_{\mu} \text{ch}K(\mu),
\end{equation}

where the summation is over $\mu$ such that $\mu = \lambda + \varepsilon_i$, for some $1 \leq i \leq n$, and $\mu_1 > \cdots > \mu_\ell, \mu_{\ell+1} > \cdots > \mu_n$. From the definition of $F_a$, we obtain (4.5).

Formula (4.4) can be obtained by a similar argument. \hfill \square

Let $\lambda \in \Lambda^+_q(n)$. It is not hard to prove that both $E_aU(\lambda)$ and $F_aU(\lambda)$ are direct sums of tilting modules (see, e.g., [Br1 Corollary 4.27]). Furthermore, we have the following lemma.

Lemma 4.3. Let $\lambda \in \Lambda^+_q(n)$. Then the multiplicity of each non-zero tilting summand of $E_aU(\lambda)$ and $F_aU(\lambda)$ is even.

Proof. We first let $U(\nu_1), \ldots, U(\nu_p)$ be the (not necessarily distinct) direct summands of $F_aU(\lambda)$. Suppose on the contrary that $[F_aU(\lambda) : U(\nu_i)]$ is odd for some $1 \leq i \leq p$. Without loss of generality, we let $1 \leq p' \leq p$ such that $[F_aU(\lambda) : U(\nu_i)]$ are odd for all $i \leq p'$, and $[F_aU(\lambda) : U(\nu_{i'})]$ are even for all $i' > p'$, and in addition, $\nu_1$ is a maximal
element in \(\{\nu_i | 1 \leq i \leq p'\}\). Since the coefficients of \(\{\operatorname{ch}K(\mu) | \mu \in \mathfrak{h}_0^+\}\) in \(\operatorname{ch}F_a U(\lambda)\) are even by Lemma 4.2, we have

\[
\sum_{i=1}^{p'} \operatorname{ch}U(\nu_i) = \operatorname{ch}F_a U(\lambda) - \sum_{i=p'+1}^{p} \operatorname{ch}U(\nu_i) = 2 \sum_{i=1}^{q} \operatorname{ch}K(\mu_i),
\]

(4.10) for some (not necessarily distinct) \(\mu_1, \mu_2, \ldots, \mu_q \in \mathfrak{h}_0^+\). For a given \(\nu \in \mathfrak{h}_0^+\), note that \(\operatorname{ch}U(\nu) \in \operatorname{ch}K(\nu) + \bigoplus_{\zeta < \nu} \mathbb{Z}_{\geq 0} \operatorname{ch}K(\zeta)\). This means that the coefficient of \(\operatorname{ch}K(\nu_i)\) in each \(\operatorname{ch}U(\nu_j)\)'s is zero for all \(1 \leq j \leq p'\) with \(\nu_j \neq \nu_i\). But \([F_a U(\lambda) : U(\nu_1)] = |\{j | 1 \leq j \leq p', \nu_j = \nu_1\}|\) is the coefficient of \(\operatorname{ch}K(\nu_1)\) in \(\sum_{i=1}^{p'} \operatorname{ch}U(\nu_i)\) contradicting (4.10).

The result for \(E_a U(\lambda)\) can be obtained by a similar argument. This completes the proof. \(\square\)

By Lemma 4.3 and algorithm in \cite[Procedure 3.20]{Brundan}, we have the following lemma:

**Lemma 4.4.** Let \(\lambda \in \Lambda^+_{\tilde{\varphi}}(n)\) with \(\# \lambda = 1\). There is an operation \(Z_{\lambda}\) consisting of first applying the operators \(\tilde{X}_a\) 's coming from Brundan’s procedure \cite[Procedure 3.20]{Brundan} and then taking a direct summand of a direct sum of two isomorphic copies such that

\[
\operatorname{ch}Z_{\lambda} U(t_\lambda) = \operatorname{ch}K(\lambda) + \operatorname{ch}K(\lambda^-),
\]

(4.11) where \(t_\lambda \in \Lambda^+_{\tilde{\varphi}}(n)\) is typical, and \(\lambda^- = w(\lambda - k\alpha)\), where \(\alpha \in \Phi^+\) with \((\lambda, \overline{\tau}) = 0\) and \(k \in \mathbb{N}\) is the smallest positive integer such that \(\lambda - k\alpha\) is \(W\)-conjugate to an element in \(\Lambda^+_{\tilde{\varphi}}(n)\), and \(w \in W\) is such that \(w(\lambda - k\alpha) \in \Lambda^+_{\tilde{\varphi}}(n)\).

**Remark 4.5.** Note that the map \(\lambda \in \Lambda^+_{\tilde{\varphi}}(n) \mapsto \lambda^- \in \Lambda^+_{\tilde{\varphi}}(n)\) define a bijection on \(\Lambda^+_{\tilde{\varphi}}(n)\). We denote by \(\lambda^+\) the unique element in \(\Lambda^+_{\tilde{\varphi}}(n)\) such that \((\lambda^+)^- = \lambda\).

**Remark 4.6.** Since the translation functors are exact and both bi-adjoint to each other, they preserve projective and injective modules. In particular, \(Z_{\lambda} U(t_\lambda)\) is both a projective cover and an injective hull.

Let \(\tau : \mathfrak{g} \to \mathfrak{g}\) be the anti-automorphism on \(\mathfrak{g}\) defined by \(\tau(e_{ij}) = e_{ji}\) and \(\tau(c_{ij}) = c_{ji}\) (see, e.g., \cite[Example 7.10]{Brundan}). Then \(\tau\) induces a contravariant auto-equivalence on \(\mathcal{O}\) and \(\mathcal{F}\) (see, e.g., \cite[Section 3.2]{HumR} and \cite[Section 2.1]{Ger}). For a given \(M \in \mathcal{F}\), let \(M^\tau\) be the image of \(\tau\). Since \(\operatorname{ch}L(\lambda)^\tau = \operatorname{ch}L(\lambda)\), we have \(L(\lambda)^\tau \in \{L(\lambda), \Pi \lambda L(\lambda)\}\). Note that \(\ell(\lambda) = n\) for each \(\lambda \in \Lambda_{\tilde{\varphi}}(n)\), we thus have the following lemma:

**Lemma 4.7.** (\cite[Lemma 7]{Li}) Let \(n\) be even and \(\lambda \in \Lambda_{\tilde{\varphi}}(n)\). Then we have

\[
L(\lambda)^\tau \cong \begin{cases} \Pi \lambda L(\lambda) & \text{if } n \equiv 2 \mod 4, \\ L(\lambda) & \text{if } n \equiv 0 \mod 4. \end{cases}
\]

(4.12)

The following corollary is an immediate consequence by Lemma 4.7.

**Corollary 4.8.** Let \(M \in \mathcal{F}\). If \(n \equiv 0 \mod 4\) then \(M^\tau\) and \(M\) have identical set of composition factors. If \(n \equiv 2 \mod 4\) then \(M^\tau\) and \(\Pi M\) have identical set of composition factors.
For a given $M \in \mathcal{O}$, denote by $\text{rad} M$ and $\text{soc} M$ the radical and socle of $M$, respectively. We now can prove the following BGG reciprocity.

**Lemma 4.9.** *(BGG reciprocity)* Let $\lambda, \mu \in \Lambda^+_s(n)$. Then we have

$$(4.13) \quad (P(\lambda) : \Pi^i K(\mu)) = [\Pi^i K(\mu) : L(\lambda)].$$

for $i = 0, 1$.

**Proof.** We first assume $n$ is even. Let $\lambda', \mu' \in \Lambda^+_s(n)$ be arbitrary. Recall that $L(\lambda') \not\cong \Pi L(\lambda')$ in this case. By a similar arguments of consideration on (highest) weights as in [Hum08 Theorem 3.3(c)], we may conclude that

$$(4.14) \quad \text{Ext}_F(\Pi^i K(\lambda'), K(\mu')^\tau) = 0,$$

$$(4.15) \quad \dim \text{Hom}_F(\Pi^i K(\lambda'), K(\mu')^\tau) \neq 0 \text{ implies that } \lambda' = \mu'.$$

Furthermore, by Lemma 4.7 we have

$$(4.16) \quad \text{soc}(\mu')^\tau = L(\mu')^\tau \cong \begin{cases} \Pi L(\mu') & \text{if } n \equiv 2 \text{ mod } 4, \\ L(\mu') & \text{if } n \equiv 0 \text{ mod } 4. \end{cases}$$

By a similar proof as in [Hum08 Theorem 3.3(c)], we may conclude that

$$(4.17) \quad \dim \text{Hom}_F(\Pi^i K(\lambda'), (K(\mu')^\tau) = \begin{cases} 0 & \text{if } \lambda' \neq \mu', \\ 1 & \text{if } \lambda' = \mu' \text{ and } n \equiv 2 \text{ mod } 4, \\ 0 & \text{if } \lambda' = \mu' \text{ and } n \equiv 0 \text{ mod } 4. \end{cases}$$

$$(4.18) \quad \dim \text{Hom}_F(\Pi^i K(\lambda'), (\Pi K(\mu'))^\tau) = \begin{cases} 0 & \text{if } \lambda' \neq \mu', \\ 1 & \text{if } \lambda' = \mu' \text{ and } n \equiv 0 \text{ mod } 4, \\ 0 & \text{if } \lambda' = \mu' \text{ and } n \equiv 2 \text{ mod } 4. \end{cases}$$

Since $P(\lambda)$ has a flag of parabolic Verma modules, as a conclusion, we have

$$(4.19) \quad (\Pi^i P(\lambda) : K(\mu)) = \begin{cases} \dim \text{Hom}_F(\Pi^i P(\lambda), (\Pi K(\mu))^\tau) & \text{if } n \equiv 2 - 2i \text{ mod } 4, \\ \dim \text{Hom}_F(\Pi^i P(\lambda), K(\mu)^\tau) & \text{if } n \equiv 2i \text{ mod } 4. \end{cases}$$

Recall that $\dim \text{Hom}_F(\Pi^i P(\lambda), M) = [M : \Pi^i L(\lambda)]$ for all $M \in \mathcal{F}$ and $\lambda \in \Lambda^+_s(n)$ (see, e.g., [Hum08 Section 3.9]). By Corollary 4.8 we have $[(\Pi K(\mu))^\tau : \Pi^i L(\lambda)] = [K(\mu) : \Pi^i L(\lambda)]$ for $n \equiv 2 \text{ mod } 4$ (resp. $[K(\mu)^\tau : \Pi^i L(\lambda)] = [K(\mu) : \Pi^i L(\lambda)]$ for $n \equiv 0 \text{ mod } 4$).

The proof of this lemma follows provided that $n$ is even.

Recall that $L(\lambda') \cong \Pi L(\lambda')$ if $n$ is odd. By a similar argument, the proof for odd $n$ can be obtained. This completes the proof. \qed

Let $\lambda \in \Lambda^+_s(n)$ with $\sharp \lambda = 1$. Define $\Lambda_{\lambda} := \{\lambda^i | i \in \mathbb{Z}\}$ by the recursive relation $\lambda^{i+1} = (\lambda^i)^+$ and $\lambda^0 = \lambda$. We define irreducible modules $L_{\lambda^i}$ and parabolic Verma modules $K_{\lambda^i}$ for $i \in \mathbb{Z}$ as follows. First, $L_{\lambda^0} := L(\lambda)$ and $K_{\lambda^0} := K(\lambda)$. Then $L_{\lambda^i}$ for $i \neq 0$ is defined by the recursive relation coming from the short exact sequences:

$$(4.20) \quad 0 \to L_{\lambda^{i-1}} \to K_{\lambda^i} \to L_{\lambda^i} \to 0,$$
for \(i \in \mathbb{Z}\). Let \(P_{\lambda^i}\) be the projective cover of \(L_{\lambda^i}\). Then by Lemma 4.9, we have short exact sequence
\[
(4.21) \quad 0 \to K_{\lambda^{i+1}} \to P_{\lambda^i} \to K_{\lambda^i} \to 0,
\]
for \(i \in \mathbb{Z}\). Let \(F_\lambda\) be the block of \(F\) containing \(L(\lambda)\). Namely, it is the Serre subcategory generated by the set of vertices in the connected component of the Ext-quiver for \(F\) containing \(L(\lambda)\).

**Corollary 4.10.** Let \(\lambda \in \Lambda^+_s(n)\) with \(\sharp \lambda = 1\). Then \(L(\lambda)\) and \(\Pi L(\lambda)\) are in different blocks if and only if \(n\) is even. Furthermore, \(F_\lambda\) is the Serre subcategory of \(F\) generated by \(\{L_\mu| \mu \in \Lambda_\lambda\}\).

**Proof.** By Lemma 4.4 and Lemma 4.9, the set \(\{L_{\lambda^i}, L_{\lambda^i+1}, L_{\lambda^i-1}\}\) is the set of all composition factors of \(P_{\lambda^i}\). Recall that
\[
(4.22) \quad \text{Ext}_F(L_{\lambda^i}, \Pi^j L(\mu)) \cong \text{Hom}_F(\text{rad} P_{\lambda^i}/\text{rad}^2 P_{\lambda^i}, \Pi^j L(\mu)),
\]
for all \(i \in \mathbb{Z}, j = 0,1\) and \(\mu \in \Lambda^+_s(n)\). This means that \(\{L_{\lambda^i}| i \in \mathbb{Z}\}\) is the complete set of irreducible objects of \(F_\lambda\) by (4.20). Since for each \(i \in \mathbb{Z}\) the object \(L_{\lambda^i}\) is of highest weight \(\lambda\) if and only if \(i = 0\) (i.e. \(\lambda^i = \lambda\)), we may conclude that \(\Pi L(\lambda) \not\in F_\lambda\) if and only if \(n\) is even by Lemma 2.4. \(\square\)

**Example 4.11.** (The Ext-quiver for \(q(2)\)) Let \(n = 2\). Let \(s \notin \mathbb{Z}/2, \alpha := \varepsilon_1 - \varepsilon_2, \bar{s} \in s + \mathbb{Z}\) and \(\lambda := \bar{s} \alpha \in \Lambda^+_s(2)\). In this case, we have \(K(\lambda) = M(\lambda)\) and \(I_\lambda = \text{Ind}_V^H \mathbb{C}v_\lambda\) is a two dimensional irreducible \(h\)-module with \(h' = h_0 \oplus \mathbb{C}(h_1 + h_2)\), where \(h'\) acts on \(\mathbb{C}v_\lambda\) trivially. For a given irreducible \(h\)-module \(V\) of \(h_0\)-weight \(\lambda - \alpha\), we have that \(V \cong I_{\lambda-\alpha}\) (resp. \(V \cong \Pi I_{\lambda-\alpha}\) ) if and only if \((\bar{h}_1 + \bar{h}_2)V_0 = 0\) (resp. \((\bar{h}_1 - \bar{h}_2)V_0 = 0\)). It is not hard to compute that \(u := e_{21}h_1 v_\lambda - \bar{h}_2 v_\lambda\) is a singular vector in \(M(\lambda)\) with \(\bar{h}_1 u = -\bar{h}_2 u\) (see, e.g., [CW] Lemma 2.44(2)). Note that \(\bar{\sigma} = \bar{1}\), therefore we have short exact sequence
\[
(4.23) \quad 0 \to \Pi L(\lambda - \alpha) \to K(\lambda) \to L(\lambda) \to 0.
\]
Thus, \(\text{Ext}_F(L(\lambda), \Pi L(\lambda - \alpha)) \neq 0\). By Lemma 4.9 we may conclude that \(P(\lambda)\) has a Verma flag
\[
(4.24) \quad 0 \to \Pi K(\lambda + \alpha) \to P(\lambda) \to K(\lambda) \to 0,
\]
and so \(\{L(\lambda), L(\lambda), \Pi L(\lambda + \alpha), \Pi L(\lambda - \alpha)\}\) is the complete set of composition factors of \(P(\lambda)\). Therefore the set of objects of \(F_\lambda\) is \(\{\Pi^k L(\lambda + k\alpha)|k \in \mathbb{Z}\}\). Furthermore, if we apply \(\Pi \circ \tau\) to the short exact sequence (4.23), then it follows that \(\text{Ext}_F(\Pi L(\lambda - \alpha), L(\lambda)) \neq 0\). Replace \(\lambda\) by arbitrary \(\lambda + k\alpha\), we may conclude that the Ext-quivers of \(F_\lambda\) and that of the principal block \((F_{\bar{1}1})_0\) for \(gl(1|1)\) are the same.

By a similar argument, we may generalize results in Example 4.11. More precisely, let \(\bar{\tau} := \Pi \circ \tau\) (resp. \(\bar{\tau} := \tau\)) if \(n \equiv 2\) mod 4 (resp. \(n \equiv 0\) mod 4). Let \(\lambda \in \Lambda^+_s(n)\) with \(\sharp \lambda = 1\). By applying \(\bar{\tau}\) to the non-trivial short exact sequence (4.20), it follows from
that \( \text{rad}(\lambda)/\text{rad}^2(\lambda) \cong L_{\lambda^+} \oplus L_{\lambda^-} \). As a consequence, we obtain the following lemma.

**Lemma 4.12.** Let \( \lambda \in \Lambda^+_{s\ell}(n) \) with \( \sharp \lambda = 1 \). Then there are exactly four distinct proper submodules of \( P(\lambda) \):

\[
A_\lambda \cong K_{\lambda^+}, \quad B_\lambda, \quad \text{rad}(\lambda) = A_\lambda + B_\lambda, \quad \text{rad}^2(\lambda) = \text{soc}(\lambda) \cong L_\lambda.
\]

Furthermore, we have the following short exact sequences:

\[
0 \to A_\lambda \to \text{rad}(\lambda) \to L_{\lambda^-} \to 0,
\]

\[
0 \to B_\lambda \to \text{rad}(\lambda) \to L_{\lambda^+} \to 0,
\]

\[
0 \to L_\lambda \to A_\lambda \to L_{\lambda^+} \to 0,
\]

\[
0 \to L_\lambda \to B_\lambda \to L_{\lambda^-} \to 0.
\]

We are now in a position to state the main theorem in this section.

**Theorem 4.13.** Let \( \lambda \in \Lambda^+_{s\ell}(n) \) with \( \sharp \lambda = 1 \). Then \( \mathcal{F}_\lambda \) is equivalent to \( (\mathcal{F}_{(n-1)} \Lambda\mathcal{F}) \).

**Proof.** Our goal is to prove that the endomorphism algebra \( \text{End}_{\mathcal{F}_\lambda}(\bigoplus_{\mu \in \Lambda_\lambda} P_\mu) \) of projective generator \( \bigoplus_{\mu \in \Lambda_\lambda} P_\mu \) for \( \mathcal{F}_\lambda \) is isomorphic to \( (K_1^\infty)^{op} \). We fix \( \mu = \lambda^i \in \Lambda_\lambda \). It follows from Lemma 4.12 that \( \text{End}_{\mathcal{F}_\lambda}(P_\mu, P_{\mu'}) = 0 \) if \( \mu' \notin \{\mu^+, \mu^-, \mu, \mu'\} \). Since \( \dim_{\text{End}_{\mathcal{F}_\lambda}}(L_\mu, L_{\mu'}) = \delta_{\mu, \mu'} \) for all \( \mu, \mu' \in \Lambda_\lambda \), it is not hard to prove that (see, e.g., [Hum08, Section 3.9])

\[
\dim_{\text{End}_{\mathcal{F}_\lambda}}(P_\mu, P_{\mu'}) = [P_{\mu'} : L_\mu] = \begin{cases} 1 & \text{in case } \mu' \in \{\mu^+, \mu^-, \mu\}, \\ 2 & \text{in case } \mu' = \mu. \end{cases}
\]

We now construct bases for \( \text{Hom}_{\mathcal{F}_\lambda}(P_\mu, P_{\mu'}) \), for \( \mu' \in \{\mu, \mu^+, \mu^-, \mu'\} \):

1. A Basis for \( \text{Hom}_{\mathcal{F}_\lambda}(P_\mu, P_{\mu'}) \): First note that the canonical epimorphism \( P_\mu \to L_\mu \) gives an endomorphism of \( \tilde{z}_i \in \text{End}_{\mathcal{F}_\lambda} P_\mu \) by mapping \( P_\mu \) onto \( \text{soc}(P_\mu) \subset P_\mu \). Let \( 1_i \in \text{End}_{\mathcal{F}_\lambda} P_\mu \) be the identity map, then we may conclude that \( \text{End}_{\mathcal{F}_\lambda} P_\mu \) is generated by \( 1_i, \tilde{z}_i \) by (4.30).

2. A Basis for \( \text{Hom}_{\mathcal{F}_\lambda}(P_\mu, P_{\mu'}) \): Next note the composition of homomorphisms

\[
\tilde{y}_{i-1} : P_\mu \to P_\mu/A_\mu \cong A_{\mu^-} \hookrightarrow P_{\mu^-},
\]

gives a non-zero element \( \tilde{y}_{i-1} \in \text{Hom}_{\mathcal{F}_\lambda}(P_\mu, P_{\mu^-}) \), and so \( \mathcal{C}\tilde{y}_{i-1} = \text{Hom}_{\mathcal{F}_\lambda}(P_\mu, P_{\mu^-}) \) by (4.30) again.

3. A Basis for \( \text{Hom}_{\mathcal{F}_\lambda}(P_\mu, P_{\mu^+}) \): Note that \( \dim_{\text{End}_{\mathcal{F}_\lambda}}(P_\mu, P_{\mu^+}) = 1 \) by (4.30). Let \( \bar{x}_i \in \text{Hom}_{\mathcal{F}_\lambda}(P_\mu, P_{\mu^+}) \) be a non-zero element. Note that each composition factor of \( \bar{x}_i(P_\mu) \) lies in \( \{L_{\mu^+}, L_{\mu}, L_{\mu^+}\} \) because \( \bar{x}_i(P_\mu) \subset \text{rad}(P_{\mu^+}) \). Since every non-trivial quotient of \( P_\mu \) has composition factor \( L_\mu \), we may conclude that \( \bar{x}_i(P_\mu) = B_\mu \). Consequently, \( \bar{x}_i \) can be expressed as the following composition of homomorphisms

\[
\bar{x}_i : P_\mu \to P_\mu/B_\mu \cong B_\mu \to P_{\mu^+}.
\]
It is not hard to compute the relations $\widetilde{x}_j\widetilde{x}_{j+1} = \widetilde{y}_j\widetilde{y}_{j+1} = 0$ for all $j \in \mathbb{Z}$ by (4.31) and (4.32). Therefore we may conclude that $\widetilde{z}_i c = c\widetilde{z}_i = \widetilde{y}_i\widetilde{y}_j = \widetilde{x}_j\widetilde{x}_j = 0$ for all $i, j \in \mathbb{Z}, c \in \{\widetilde{x}_i, \widetilde{y}_i\}_{i \in \mathbb{Z}}$. Finally, we note that $\widetilde{y}_i\widetilde{x}_i$ and $\widetilde{x}_i\widetilde{y}_i$ can be expressed as the following composition of homomorphisms

\begin{equation}
\widetilde{y}_i\widetilde{x}_i : P_\mu \rightarrow P_\mu / B_\mu \cong B_\mu^+ \rightarrow \frac{A_\mu^+ + B_\mu^+}{A_\mu^+} \cong \text{soc}P_\mu \subset P_\mu,
\end{equation}

\begin{equation}
\widetilde{x}_i\widetilde{y}_i : P_\mu^+ \rightarrow P_\mu^+ / A_\mu^+ \cong A_\mu \rightarrow \frac{A_\mu + B_\mu}{B_\mu} \cong \text{soc}P_\mu^+ \subset P_\mu^+.
\end{equation}

It is not hard to see that $\widetilde{y}_i\widetilde{x}_i = \widetilde{z}_i$ and $\widetilde{x}_i\widetilde{y}_i = \widetilde{z}_{i+1}$ for all $i \in \mathbb{Z}$. Therefore, we have an isomorphism from $\text{End}_{\mathcal{F}_\lambda}(\bigoplus_{\lambda \in \Lambda}P_\mu)$ to $(K^\infty)^{\text{op}}$ sending $\widetilde{x}_i, \widetilde{y}_i, \widetilde{z}_i$ to $x_i, y_i, z_i$, respectively. This completes the proof. \hfill \Box

References

[AL] H.H. Andersen, N. Lauritzen; Twisted Verma modules. Studies in memory of Issai Schur, Progr. Math., 210, Birkhauser Boston, Boston, MA, 2003, 1-26

[AS] H. H. Andersen and C. Stroppel; Twisting functors on $\mathcal{O}$. Represent. Theory 7 (2003), 681-699.

[Ar] S. Arkhipov; Semi-infinite cohomology of associative algebras and bar duality. Internat. Math. Res. Notices 1997, 833-863.

[Br1] J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{gl}(m|n)$, J. Amer. Math. Soc. 16 (2003), 185–231.

[Br2] J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{q}(n)$, Adv. Math. 182 (2004), 28–77.

[Br3] J. Brundan, Tilting modules for Lie superalgebras, Comm. Algebra 32 (2004), 2251–2268.

[BLW] J. Brundan, I. Losev, B. Webster, Tensor product categorifications and the super Kazhdan-Lusztig conjecture, preprint, arXiv:1310.0349.

[BS12] J. Brundan, C. Stroppel, Highest weight categories arising from Khovanov’s diagram algebra IV: the general linear supergroup, J. Eur. Math. Soc. 14 (2012), 373–419.

[CK] S.-J. Cheng and J.-H. Kwon, Finite-dimensional half-integer weight modules over queer Lie superalgebras, Commun. Math. Phys., to appear.

[CKW] S.-J. Cheng, J.-H. Kwon and W. Wang, Character formulae for queer Lie superalgebras and canonical bases of type C, preprint (2015).

[CLW] S.-J. Cheng, N. Lam, and W. Wang, Brundan-Kazhdan-Lusztig conjecture for general linear Lie superalgebras, Duke Math. J. 110 (2015), 617–695.

[CM] K. Coulembier and V. Mazorchuk, Primitive ideals, twisting functors and star actions for classical Lie superalgebras. Accepted in J. Reine Ang. Math. doi: 10.1515/crelle-2014-0079.

[CMW] S.-J. Cheng, V. Mazorchuk, and W. Wang, Equivalence of blocks for the general linear Lie superalgebra, Lett. Math. Phys. 103 (2013), 1313–1327.

[CW] S.-J. Cheng and W. Wang, Dualities and Representations of Lie Superalgebras. Graduate Studies in Mathematics 144. American Mathematical Society, Providence, RI, 2012.

[FM] A. Frisk and V. Mazorchuk, Regular Strongly Typical Blocks of $\mathcal{O}^\circ$, Commun. Math. Phys. 291 (2009), 533542.

[Fr] A. Frisk, Typical blocks of the category $\mathcal{O}$ for the queer Lie superalgebra, J. Algebra Appl. 6 (2007), no. 5, 731–778.
[Go] M. Gorelik, *On the ghost centre of Lie superalgebras*. Ann. Inst. Fourier (Grenoble) **50** (2000), no. 6, 1745–1764 (2001).

[GG13] M. Gorelik, D. Grantcharov, *Bounded highest weight modules over q(n)*. Int. Math. Res. Not. (2013) doi:10.1093/imrn/rnt14

[Ger98] J. Germoni, *Indecomposable representations of special linear Lie superalgebras*, J. Algebra **209** (1998) 367–401.

[Hum08] J. Humphreys, Representations of semisimple Lie algebras in the BGG category O, Graduate Studies in Mathematics, vol. **94**, American Mathematical Society, Providence, RI, 2008.

[JMO] N. Jing, K. Misra, and M. Okado, *q-Wedge Modules for Quantized Enveloping Algebra of Classical Type*, J. Algebra **230** (2000), 518–539.

[KM] O. Khomenko and V. Mazorchuk; *On Arkhipov’s and Enright’s functors*. Math. Z. **249** (2005), 357–386.

[Mac95] I. G. Macdonald, “Symmetric functions and Hall polynomials”, Second Edition, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[Mar10] V. Mazorchuk, *Classification of simple q_2-supermodules*. Tohoku Math. J. (2) **62** (2010), no. 3, 401-426.

[Mar14] V. Mazorchuk, *Paraholic category O for classical Lie superalgebras*. In: Gorelik, M.; Papi, P. (Eds.) Springer INdAM Series 7. Springer International Publishing, Cham (2014).

[Mar] V. Mazorchuk; *Lectures on sl_2(C)-modules*. Imperial College Press, London, 2010.

[MS] V. Mazorchuk, C. Stroppel, *On functors associated to a simple root*. J. Algebra **314** (2007), no. 1, 97-128.

[MO00] O. Mathieu, *Classification of irreducible weight modules*. Annales de l’institut Fourier **50**.2 (2000) 537–592.

[Pe] I. Penkov, *Characters of typical irreducible finite-dimensional q(n)-modules*, Funct. Anal. App. **20** (1986), 30–37.

[PS1] I. Penkov and V. Serganova, *Characters of Finite-Dimensional Irreducible q(n)-Modules*, Lett. Math. Phys. **40** (1997), 147–158.

[PS2] I. Penkov and V. Serganova, *Characters of irreducible G-modules and cohomology of G/P for the supergroup G = Q(N)*, J. Math. Sci., **84** (1997), 1382–1412.

[Ser96] V. Serganova, *Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra gl(m|n)*, Selecta Math. (N.S.) **2** (1996), 607–651.

[Ser98] V. Serganova, *Characters of irreducible representations of simple Lie superalgebras*, Doc. Math., Extra Volume ICM II (1998), 583–593.