Massive gravity from Dirichlet boundary conditions

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Abstract

We propose an explicit non-linear realization of massive gravity, which relies on the introduction of a spurious compact extra dimension, on which we impose half-Newmann and half-Dirichlet boundary conditions. At the linearized level, we recover the expected gravitational exchange amplitude between two sources mediated by a massive Fierz-Pauli spin-2 field, while cubic interactions in the additional helicity-0 mode give rise to the expected Vainshtein mechanism. We also show that this framework can accommodate for a flat four-dimensional geometry in the presence of a cosmological constant, putting this framework on a good footing for the study of degravitation.

Keywords: massive gravity, degravitation, extra dimensions

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1. Introduction

While laboratory experiments, solar systems tests and cosmological observations have all been in complete agreement with General Relativity for now almost a century, these bounds do not eliminate the possibility for the graviton to bear a small hard mass $m \lesssim 6 \times 10^{-32}$eV, [1]. Conversely, the main obstacle in giving the graviton a mass lies in the theoretical constraints rather than the observational ones, as explicit non-linear realizations of massive gravity are hard to construct. The Dvali-Gabadadze-Porrati (DGP) model is the first realization of soft massive gravity, where the graviton can be thought of as a resonance, or a superposition of massive modes [2]. This model was then extended to higher dimensions, [3, 4], where gravity becomes even weaker at large distances, and could exhibit a "degravitation" mechanism, by which the cosmological constant could be large but gravitate weakly on the geometry [5]. Such a degravitation mechanism is also "expected" to be present if the graviton bears a hard mass. An explicit realization of a theory of a hard mass gravity was proposed in [6], which appeared while this work was in progress, and relies on the same mechanism.

This framework is based on the presence of a "spurious" compact extra dimension on which we impose Dirichlet boundary condition on one end and Neumann (Israel) on the other, where our 4d world stands. The techniques used throughout this study, in particular the introduction of a St"uckelberg field to restore 4d gauge invariance, are in no way original to this work, however the introduction on the spurious extra dimension provides a geometrical interpretation of massive gravity, for which non-linearities can be tracked down explicitly. Furthermore, this model is of high interest for the study of degravitation, providing a framework where explicit solutions with a cosmological constant can be understood and more general cosmological solution can be studied numerically.

We also show that when diffeomorphism is broken along the extra dimension, one recovers an effective 4d theory of gravity where the graviton has a constant mass. Moreover, this class of model can also accommodate a fully 5d diffeomorphism invariant theory for which the 4d effective graviton has a soft mass and is free of any ghost-like instability at the non-linear level.

We proceed as follows: We first show in section 2 how our mechanism works for a scalar field toy-model before presenting the full spin-2 analogue in section 3. We then recover the expected gravitational exchange amplitude between two conserved sources for a theory of massive gravity in section 4 and derive the decoupling limit for a specific class of models where higher extrinsic curvature terms are present in section 5, while the decoupling limit in the more general case is deferred for later studies. We also discuss on the number of physical degrees of freedom and comment on the stability (presence of ghosts) in this class of model. We then present in section 6 solutions capable of "hiding" a 4d cosmological constant by curving the extra dimension and keeping the 3-brane flat, which is of high importance for the degravitation mechanism. Finally, we discuss soft massive gravity in the appendix 7, which is obtained when restoring 5d gauge invariance along the extra dimension.

2. Scalar Field Toy model

Before diving into the technical subtleties of the gravitational case, we focus to start with on the idea using a scalar field toy-model. Let $\varphi(x^\mu, \omega)$ be a massless scalar field living in a 5d space-time $(x^\mu, \omega)$ where the coordinates $x^\mu$, $\mu = 0, \cdots, 3$
describe our four transverse dimensions, while the fifth coordinate $\omega$ is compact, $0 \leq \omega \leq \tilde{\omega}$, and we choose the $\omega$ coordinate to be dimensionless. We explicitly break the 5d Lorentz invariance by omitting the kinetic term along the transverse direction in the bulk
\[ S = \int_0^\infty d\omega \, d^4x \left( \frac{M_4^2}{2} (\partial_\omega \phi)^2 + \delta(\tilde{\omega} - \omega) L_4 \right), \]
while these kinetic terms are present on the brane:
\[ L_4 = -\frac{M_4^2}{2} \phi \Box \phi + \phi J(x), \] (2)
where $\Box$ is the 4d d'Alembertian and $J$ the source localized on the 3-brane. A shift in the brane position $\tilde{\omega}$ is equivalent to rescaling the 5d scale $M_5$ and without loss of generality, we set $\tilde{\omega} = 1$ and $M_4^2 = M_4^2 m^2$, where $M_4$ is the 4d Planck scale and $m$ is a mass parameter. The boundary condition on the brane at $\omega = 1$ is set using the standard Neumann or Israeli Matching Conditions, while at $\omega = 0$, we choose to impose the Dirichlet boundary condition:
\[ \phi(x, \omega)|_{\omega=0} = 0 \]
\[ -M_4^2 m^2 \partial_\omega \phi|_{\omega=1} = -M_4^2 \Box \phi + J. \] (4)
Solving the bulk equation of motion with the previous boundary condition, the field profile is therefore $\phi(x, \omega) = \tilde{\phi}(x)\omega$, where $\tilde{\phi}$ is the induced value of the field on the brane, satisfying the 4d effective equation of motion on the brane,
\[ M_4^2 (\Box - m^2) \tilde{\phi} = J(x) \] (5)
and hence behaving as a massive scalar field from a 4d point of view. Needless to say that this is a very convoluted way to obtain a massive scalar field theory, but for gravity, it would be extremely difficult to do so otherwise.

3. Massive Gravity

The extension of this model to a spin-2 field is straightforward. We consider a 4d metric $q_{\mu\nu}(x^\xi, \omega)$ living in the previous 5d space-time. 5d diffeomorphism is here again explicitly broken, but 4d gauge invariance is preserved using the standard trick of introducing a St"uckelberg field $M^{\mu}(x^\xi, \omega)$ which shifts under a 4d gauge transformation $x^\xi \rightarrow \tilde{x}^\xi(x^\xi, \omega)$ as:
\[ q_{\mu\nu} \rightarrow \tilde{q}_{\mu\nu} = q_{\mu\nu} + \frac{\partial x^\xi}{\partial \xi^\nu} \frac{\partial x^\eta}{\partial \xi^\mu}, \] (6)
\[ N^{\mu} \rightarrow \tilde{N}^{\mu} = N^{\mu} + \delta^{\mu}_{\nu} \frac{\partial x^\eta}{\partial \xi^\nu} + \partial_\omega \tilde{x}^\nu \] (7)
so that the "extrinsic curvature"
\[ K_{\mu\nu} = \frac{1}{2} L_\omega q_{\mu\nu} = \frac{1}{2} \left( \partial_\omega q_{\mu\nu} - D_\nu N_\mu \right) \] (8)
transform as 4d tensor. Hereafter, the 4d metric $q_{\mu\nu}$ is used to express the covariant derivative $D_\nu$ as well as to raise and lower the indices.

Similarly as for the scalar field toy-model, we then construct the 5d bulk action by considering the equivalent of the "5d curvature" $R_5[\phi, M] = R_4[\phi] + K^{\mu\nu} K_{\mu\nu}$ but omitting the contribution from the 4d kinetic term $R_4$
\[ S_K = \frac{M_4^2 \eta^2}{2} \int_0^1 d\omega \, d^4x \sqrt{-q} \left( K^{\mu\nu} K_{\mu\nu} \right). \] (9)
Notice that the specific combination $K^{\mu\nu} K_{\mu\nu}$ that appears when expressing the 5d curvature in terms of the 4d one, is precisely what will give to the specific Fierz-Pauli combination, which is the only ghost-free linear realization of massive gravity that respects 4d diffeomorphism invariance. The 4d curvature is yet present on the brane at $\omega = 1$ which holds the action
\[ S_4 = \int d^4x \sqrt{-g} \left( \frac{M_4^2}{2} R_4 - L_4 \right), \] (10)
where $L_4$ is the Lagrangian for matter fields confined to the 3-brane. Working in terms of the two dynamical variables $q_{\mu\nu}$ and $M^\mu$, the Israeli matching conditions are used to determine the boundary condition on the brane at $\omega = 1$, while we impose Dirichlet boundary condition at $\omega = 0$: $q_{\mu\nu}(x^\xi, \omega)|_{\omega=0} = \eta_{\mu\nu}$ and $M^\mu|_{\omega=0} = 0$. (11)
Notice that if we had restricted ourselves to theories that only have the restricted gauge symmetry $\xi^\nu \rightarrow \tilde{\xi}^\nu(x^\xi, \omega)$, the action (9) would be gauge invariant without the need of the St"uckelberg field, but the Dirichlet boundary condition would break 4d gauge invariance. The extended symmetry $\xi^\nu \rightarrow \tilde{\xi}^\nu(x^\xi, \omega)$ and the St"uckelberg field therefore play a crucial role.

Differentiating the bulk action with respect to the St"uckelberg field yields the "Codacci" equation
\[ D_\nu K_{\nu\rho} - \partial_\rho K = 0, \] (12)
while differentiating the action with respect to the metric leads to the modified "Gauss" equation:
\[ M_4^2 \eta^2 \left( L_\omega (K^{\mu\nu} - K \delta^{\mu\nu}) + K K^{\mu\nu} - \frac{1}{2} (K^2 + K^\mu_\rho K^{\nu}_\rho) \eta^{\mu\nu} \right) \]
\[ = \delta(\omega - 1) \left( T^{\mu\nu}_\omega - M_4^2 G^{(4)d}) \right) \] (13)
where the Lie derivative of a (1, 1)-tensor is
\[ L_\omega F^{\mu\nu}_\omega = (\partial_\omega - \eta^{\rho}_{\mu} \partial_{\rho}) F^{\mu\nu}_\omega + F^{\rho}_\omega \partial_{\rho} N^{\mu} - F^{\mu}_\rho N^{\rho} \] (14)
In the absence of any gravitational source $L_\omega$, the field $M^\rho$ vanishes and the 4d metric is flat $q_{\mu\nu} = \eta_{\mu\nu}$ as in standard general relativity. In what follows, we show that this theory behaves as a theory of massive gravity at the linear level.

4. Effective Boundary Action

We derive in this section the effective action on the 3-brane for small perturbations around the vacuum solution, $q_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x, \omega)$, sourced by a 4d stress-energy tensor $T_{\mu\nu}$, localized on the brane at $\omega = 1$. We follow the same approach as that used in [7]. In terms of the variable $H_{\mu\nu}$,
\[ H_{\mu\nu} = h_{\mu\nu} - \partial_\mu M_{\nu} = h_{\mu\nu} - (\partial_\mu M_{\nu} + \partial_\nu M_{\mu}), \] (15)
the bulk action is then of the form

$$\mathcal{L}_k = -\frac{M_4^2m^2}{8} \partial_\mu H^{\mu\nu} \partial_\nu (H_{\mu\nu} - H_{\eta\mu\nu}),$$

(16)

where $H_4 = H^{\mu\nu}$. The field $H_{\mu\nu}$ is hence linear in the fifth variable $\omega$, and the Dirichlet boundary condition at $\omega = 0$ sets

$$H_{\mu\nu}(x^\mu, \omega) = \bar{H}_{\mu\nu}(x^\mu),$$

(17)

where hereafter bar quantities represent the induced value of the fields on the brane. Using this expression in $\mathcal{L}_k$, leads after integration by part to the 4d boundary term at $\omega = 1$:

$$\mathcal{L}^{\text{bdy}}_{k} = -\frac{M_4^2m^2}{8} \bar{H}^{\mu\nu} (H_{\mu\nu} - \bar{H}_{\eta\mu\nu}),$$

(18)

which is precisely the mass term of a standard Fierz-Pauli massive theory of gravity at the linearized level. To this induced boundary action, we add the brane Einstein-Hilbert term

$$\mathcal{L}^{\text{bdy}}_{\text{eff}} = \frac{M_4^2}{4} \left( \partial_\mu \bar{h}^{\mu\nu}_3 - \frac{1}{2} \partial_\mu \bar{h}_4 - m^2 \bar{M}_4 \right)^2,$$

(19)

The resulting boundary action is then

$$\mathcal{L}^{\text{bdy}}_{\text{eff}} = \frac{M_4^2}{8} \left[ F^{\mu\nu}_4 \right] + \frac{1}{2} \partial_\mu \bar{h}_4 - m^2 \bar{M}_4 \right],$$

(20)

with $F^{\mu\nu}_4 = \partial_\mu \bar{h}_4 - \partial_\nu \bar{h}_4$, and the second line corresponds to the action of a Proca field coupled to $\bar{h}_4$. Notice that in the absence of this coupling, the Proca or Stückelberg field would be irrelevant. When coupling these fields to conserved matter, only the scalar mode in the Stückelberg field is excited, and the resulting gravitational exchange amplitude between two sources is then

$$\mathcal{A} \sim \frac{2}{M_4^2} \int d^4 x T^{\mu\nu}_4 \left[ T^{\mu\nu}_4 - \frac{1}{3} T_{\eta\mu\nu} \right],$$

(21)

corresponding to the expected gravitational exchange amplitude due to a massive graviton. In particular, we notice the standard factor $1/3$ $T$ instead of $1/2$ $T$ which appears in massive gravity and signals the presence of an extra helicity-0 mode hidden in the Stückelberg field. As observed by van Dam-Veltman and Zakharov (vDVZ), this factor remains $1/3$ even in the massless limit and is at the origin of the well-known vDVZ discontinuity, [8]. The resolution to this puzzle lies in the observation that close enough to any source, the extra scalar mode is strongly coupled. [9]. Non-linearities dominate over the linear term and effectively freeze the field. This is most easily understood by studying the decoupling limit.

5. Decoupling limit

Following the same prescription as in [7, 10], we work from now on in the high energy limit $\Box \gg m^2$, and focus on the scalar mode, $\bar{M}_4 = -\partial_\mu \pi$. The helicity-0 mode then decouples when changing variable to $\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu} + m^2 \eta_{\mu\nu}$, and the effective boundary action simplifies to

$$\mathcal{L}^{\text{bdy}} \approx \frac{M_4^2}{4} \bar{h}^{\mu\nu} \left( \bar{h}^{\nu}_{\mu} \right) - \frac{3}{2} m^2 \pi \Box \pi.$$  

(22)

The small kinetic term of $\pi$ is precisely what resolves the vDVZ discontinuity, similarly as in DGP, [7]. In the small mass limit, higher order interactions in $\pi$ dominate over the quadratic term and effectively freeze the extra excitations out. To see the strong coupling at work, let us find out the most important interaction present beyond this quadratic action. We work for that in terms of the canonically normalized variables $\bar{h}_{\mu\nu} = M_4 \bar{h}_{\mu\nu}$ and $\bar{\pi} = m^2 M_4 \phi \pi$. A general bulk interaction between $\pi$ and $h^2_{\mu\nu}$ will give rise to a boundary term of the form

$$\mathcal{L}^{\text{bdy}}(\pi) \sim \frac{M_4^2}{4} \bar{h}^{\mu\nu} \left( \bar{h}^{\nu}_{\mu} \right) - \frac{3}{2} m^2 \pi \Box \pi.$$  

(23)

We immediately see that interactions with the helicity-2 mode $h^2_{\mu\nu}$ bear an important coupling scale and will hence be suppressed. Setting $p = 0$, the strong coupling scale for this kind of interaction is

$$\Lambda_q \sim \frac{M_4}{m^2}.$$  

(24)

The lowest interaction scale therefore occurs for cubic interactions $q = 3$, as expected from [10], giving rise to the strong coupling scale

$$\Lambda_3 \sim \left( \frac{m^2}{M_4} \right)^{1/3}.$$  

(25)

We can quickly convince ourselves that such cubic interactions generically exist in a theory of massive gravity, although they are absent in the specific theory at hand as the Stückelberg field $M^\mu = -\partial_\mu \pi/m^2 M_4$ only comes to quadratic order in the action. For the cubic interactions with scale $\Lambda_5$ to be present, the action should include cubic terms in the Stückelberg field such as $(\partial_\mu M_4)^3$ not present in the model considered thus far. However such terms will typically be present if higher order terms in the extrinsic curvature are present.

5.1. In the presence of $K^3$ terms

Generically we expect to generate higher order in the extrinsic curvature by quantum interactions, without modifying the linearized arguments provided so far. Such interactions are typically of the form

$$\mathcal{L}_{K^3} = \frac{M_4^2 m^2}{2} \left( \alpha K^3 - (\alpha + \beta)K^2_{\mu\nu} + K_{\mu\nu}^3 \right).$$  

(26)

with $\alpha$ and $\beta$ arbitrary dimensionless parameters. We focus on the scalar mode $h^2_{\mu\nu} = m^2 \Pi_{\mu\nu}$, and $M_4 = -\partial_\mu \Pi$, which extends in the bulk as $\Pi = \pi(x) \omega$. At high energy, these terms
contribute to the boundary action with the following cubic interactions
\[ \mathcal{L}_{K}^{(3)} = \frac{1}{2 \Lambda^3} (\alpha \Box \hat{\nabla})^3 - (\alpha + \beta)(\Box \hat{\nabla})(\partial_{\mu} \partial_{\nu} \hat{\nabla})^2 + \beta (\partial_{\mu} \partial_{\nu} \hat{\nabla})^3 \]  
(27)

which dominate over the quadratic term at the energy scale \( \Lambda_5 \). These cubic interactions are precisely the ones expected for a typical theory of massive gravity in \([10, 5]\), and are the ones responsible for the Vainshtein mechanism. \([9]\). At least in the decoupling limit, this theory exhibits the degravitation behavior [5] and therefore represents a powerful tool to study this mechanism further in a fully non-linear scenario.

5.2. General \( K^n \) terms

In more generality, one may expect the extrinsic curvature interactions to come in at the order \( n \geq 2 \). They will then generate interactions of the form
\[ \mathcal{L}_{K^n} \sim m^2 \left( \frac{\Box \hat{\nabla}}{M_4^2 m^2} \right)^n \sim \frac{1}{\Lambda^{n-3} \left( \partial \hat{\nabla} \right)^n}, \]
(28)

with the strong coupling scale
\[ \Lambda_\ast = \left( m^{2n-1} M_4 \right)^{\frac{n}{2}}, \]
(29)
in particular we recover the strong coupling scale \( \Lambda_\ast = \Lambda_5 = \left( m^4 M_4 \right)^{1/3} \), when extrinsic curvature interactions are included already at cubic order (\( n = 3 \)), whereas if the theory is free of such interactions or \( n \to \infty \), the strong coupling scale is \( \Lambda_\ast = \Lambda_3 = \left( m^2 M_4 \right)^{1/3} \).

5.3. Ghosts and physical degrees of freedom

As soon as higher extrinsic curvature terms are present, they result in interactions that are relevant at the scale \( \Lambda_\ast \). In that case, the theory inexorably manifests a ghost at the non-linear order. This can be seen by the presence of the higher order derivative operators of the form \( \Box \hat{\nabla} \) that appears in the equation of motion of (27) or (28). Such a ghost is expected in a theory of hard mass gravity, and is usually refereed to as the Boulware-Deser ghost, \([11, 12]\). This theory has 10 degrees in the metric and 4 in the Stückelberg field, but the gauge invariance makes only 6 of them physical, like in a usual theory of massive gravity around a general background. The Stückelberg field contributes 4 additional degrees of freedom, compared to the only 2 present in a theory of massless gravity.

However, when perturbing to first order around flat space-time, only 5 degrees of freedom are excited, \( (M_4) \) plays the role of a Proca field, with only 3 degrees of freedom, one of them being the helicity-0 mode \( \pi \), while the two helicity-1 modes decouple when considering conserved sources), as expected from a usual Fierz-Pauli massive theory of gravity. At the non-linear level, the 6th mode is typically excited and propagates a ghost, at least when higher extrinsic curvature terms are present, or in other words when the strong coupling scale is below \( \Lambda_3 \).

5.4. In the absence of higher extrinsic curvature terms

We emphasize however that when the theory is exempted of any higher extrinsic curvature term \( K^n \) (with \( n > 2 \)), all interactions with coupling scale \( \Lambda_\ast \) with \( 1/5 \leq \nu < 1/3 \) disappear. Indeed, in that case interactions of the form (23) are only possible with \( 0 \leq \nu \leq 2 \), since the Stückelberg field only comes in at quadratic order in the action. In that case the associated strong coupling scale is then \( \Lambda_3 = (m^2 M_4)^{1/3} \), and interactions becoming important at that scale can be of any order. The situation is then far more subtle. In particular, it has been shown in \([6]\) that the Hamiltonian density remains positive definite for appropriate choice of boundary conditions when these \( K^n \) terms are absent. Furthermore, the strong coupling scale in this case is the same as in the DGP model \([2]\) (or its extension in the appendix), for which no ghost-like instability is manifest non-linearly. Understanding whether the theory (9) has an underlying symmetry that keeps only 5 physical degrees of freedom non-linearly, or in other words whether or not the Boulware-Deser ghost manifests itself in that case and if so at which scale therefore deserves more attention and will presented in some later work, \([13]\). Before concluding we show that this model of massive gravity can be of great interest for cosmology as it can accommodate for flat solutions in the presence of a cosmological constant on the brane.

6. Flat Solutions with Tension

We show here that such models present solutions which are very similar to the codimension-2 “deficit-angle” configuration, that carry a tension but keep the 4d geometry flat. Indeed, including a cosmological constant \( \lambda_4 \) on the brane, gives rise to the following metric profile: \( q_{\mu \nu} = a^2(\omega)\eta_{\mu \nu} \) with
\[ a^2(\omega) = \frac{\omega + \omega_0}{\omega_0}, \]
(30)

where \( \omega_0 \) is a positive constant, related to \( \lambda_4 \) via the Israel Matching condition:
\[ \lambda_4 = \frac{3 M_s^2}{2(1 + \omega_0)}. \]
(31)
The 4d geometry on the brane is flat, and the cosmological constant on the brane is carried by the bulk profile of the metric. Notice however, that similarly to the deficit-angle case for codimension-2 branes, such solutions only exist when the tension is smaller than a maximal value \( \lambda_{\text{max}} = \frac{3}{2} M_s^2 \). When higher order terms in the extrinsic curvature are included, this bound can be increased slightly but not by a significant order of magnitude.

7. Discussion

In this paper, we constructed a class of models, first presented in \([6]\), giving rise to theories of massive gravity, with hard or soft masses depending on the details of the setup. This model relies on the presence of a spurious compactified extra dimension on which we impose half-Neumann, half-Dirichlet boundary conditions. For definiteness, we have focused most of this
paper on a theory of massive gravity with a constant mass, for which 5d diffeomorphism is broken, and refer to the appendix for other kinds of solutions. In the case of a hard mass, we recover the usual decoupling limit with strong coupling scale \((m^4M_5)^{1/5} \leq \Lambda_5 < (m^2M_5)^{1/3}\) and show the presence of the Boulware-Deser ghost when higher order terms in the extrinsic curvature are considered. When such terms are absent, all the interactions with coupling scale \(\Lambda_5 < (m^2M_5)^{1/3}\) disappear and the decoupling limit is more subtle. In particular, using a Hamiltonian approach, it has been shown in [6], that the energy is positive for appropriate choices of boundary conditions.

In parallel, this model allows us to understand strong gravitation behavior for large \(\bar{y}\), while the opposite limit gives rise to a constant mass, similar to Cascading Gravity, [4].

\[ \Lambda \sim -\frac{2}{M_5^2} \int d^4x \ T^{\mu\nu} \frac{1}{\Box - m^4} \left( T_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} \phi \right), \]  

(34)

where the graviton mass is

\[ m^2(\Box) = m_5^2 \coth(\bar{y} \Box), \]  

(35)

with \(m_5 = M_5^2/M_3^2\). In particular, we recover the standard DGP behavior for large \(\bar{y}\), while the opposite limit gives rise to a constant mass, similar to Cascading Gravity, [4].

Notice that the Dirichlet boundary condition at \(y = 0\) has projected out the zero mode, and we do not recover 4d gravity in the infrared limit, despite having a compactified extra dimension. Had we impose the Neumann boundary conditions \(\partial_i h_{AB}(y) = 0\) or periodic boundary conditions, \(h_{AB}(0) = h_{AB}(\bar{y})\), the zero mode would survive and would be the dominant one in the infrared.

In this case, the decoupling limit arises precisely in the same way as in DGP, [7]. The main difference with the model presented in (9) is the presence of the lapse, which plays a crucial role. The \(\pi\) mode decouples at the strong scale \(\Lambda_3 = (m_3^2M_3)^{1/3}\) and its equation of motion is then

\[ 3\Box \hat{\chi} + \frac{1}{\Lambda_3^2} \left( \Box \hat{\chi} \right)^2 - (\partial_i \partial_j \hat{\chi})^2 = -\frac{T}{M_3}. \]  

(37)

As already hinted in this limit, where the 5d diffeomorphism is restored, the theory is free of any ghost-like instability, when working around the standard branch. However, similarly as the DGP model, this will not provide a satisfactory framework for gravitation, since it cannot accommodate for stable static solutions in the presence of a tension. We can check this statement explicitly, by deriving the effective Friedmann equation on the brane. For that, we consider the bulk metric

\[ ds^2 = dy^2 - \frac{1}{1 + ky^2} \left( dx^2 + (1 + ky) \delta_{ij} dx^i dx^j \right), \]  

(38)

where \(k\) is a free parameter, analogue to the spatial curvature which can be scaled to 1. If the brane is located at \(y = \bar{y}(t)\), the induced extrinsic curvature on the brane is then

\[ K_{ij} = -\frac{k}{2 \sqrt{1 - a^2(t) \bar{y}^2}} \delta_{ij}, \]  

(39)

where \(a^2(t) = (1 + ky(t))\), and the resulting Friedmann equation in the presence of a fluid with energy density \(\rho\) is

\[ M_3^2 \left( 3H^2 + \frac{m_5^2}{2} \sqrt{4H^2 + \frac{k}{a^2}} \right) = \rho. \]  

(40)

When \(k/a^2 \ll H^2\), we recover the intermediary regime analogue to DGP,

\[ M_3^2(3H^2 + m_5 H) = \rho, \]  

(41)

while in the opposite limit, the corrections just play the role of a spatial curvature term.

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Appendix: Soft Massive gravity

To finish, we show in this appendix that when 5d Lorentz invariance is restored, the graviton acquires a soft mass, similarly as in DGP or Cascading Gravity, and is free of Boulware-Deser ghost instabilities. Indeed, when considering the 5d diffeomorphism invariant action

\[ S_s = \frac{M_5^2}{2} \int_0^\infty dy^4 \sqrt{-g_s} R_s, \]  

(32)

working now in terms of the dimensionful direction \(y\) which remains compactified. Imposing the Dirichlet boundary condition at \(y = 0\) and the Neumann one at \(y = \bar{y}\), the metric perturbations satisfy the following bulk profile in 5d de Donder gauge,

\[ h_{AB}(x, y) = \frac{\sinh(\sqrt{\bar{y}} \nabla)}{\sinh(\sqrt{y} \nabla)} h_{AB}(x) \]  

(33)

where \(\nabla = \sqrt{-\bar{g}}\). In terms of the graviton mass \(m(\Box)\), the gravitational exchange amplitude between two sources at \(\bar{y}\) is

\[ A = \frac{2}{M_5^2} \int d^4x \ T^{\mu\nu} \frac{1}{\Box - m^2(\Box)} T_{\mu\nu} \]  

(42)

where the graviton mass is

\[ m^2(\Box) = m_5^2 \coth(\bar{y} \Box), \]  

(43)

with \(m_5 = M_5^2/M_3^2\). In particular, we recover the standard DGP behavior for large \(\bar{y}\), while the opposite limit gives rise to a constant mass, similar to Cascading Gravity, [4].

\[ \bar{y} \Box \gg 1, \ m^2 \to m_5 \]  

(44)

\[ \bar{y} \Box \ll 1, \ m^2 \to \frac{m_5}{\bar{y}}. \]  

(45)
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