The space of generalized $G_2$-theta functions

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Abstract. Let $G_2$ be the exceptional Lie group of automorphisms of the complex Cayley algebra and $C$ be a smooth, connected, projective curve of genus at least 2. Using the map obtained from extension of structure groups, we prove explicit links between the space of generalized $G_2$-theta functions over $C$ and spaces of generalized theta functions associated to the classical Lie groups $SL_2$ and $SL_3$.

Key words. Generalized theta functions, moduli stacks, principal bundles.

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1. Introduction

Throughout this paper we fix a smooth, connected, projective curve $C$ of genus at least 2. For a complex Lie group $G$ we denote by $\mathcal{M}_C(G)$ the moduli stack of principal $G$-bundles and by $\mathcal{L}$ the ample line bundle that generates the Picard group $\text{Pic}(\mathcal{M}_C(G))$. The spaces $H^0(\mathcal{M}_C(G), \mathcal{L}^l)$ of generalized $G$-theta functions of level $l$ are well-known for classical Lie groups but less understood for exceptional Lie groups. Let $G_2$ be the smallest exceptional Lie group: the group of automorphisms of the complex Cayley algebra. Our aim is to relate the space of generalized $G_2$-theta functions $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ of level one to other spaces of generalized theta functions associated to classical Lie groups.

Using the Verlinde formula, which gives the dimension of the space of generalized $G$-theta functions for any simple and simply-connected Lie group $G$, we observe a numerical coincidence:

$$\dim H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) = 2^g \dim H^0(\mathcal{M}_C(G_2), \mathcal{L}).$$

In addition, we link together $\dim H^0(\mathcal{M}_C(G_2), \mathcal{L})$ and $\dim H^0(\mathcal{M}_C(SL_3), \mathcal{L})$. Our aim is to give a geometric interpretation of these dimension equalities.

According to the Borel-De Siebenthal classification [BDS49], the groups $SL_3$ and $SO_4$ appear as the two maximal subgroups of $G_2$ among the connected subgroups of $G_2$ of maximal rank. We define two linear maps by pull-back of the corresponding extension maps: on the one hand

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \to H^0(\mathcal{M}_C(SL_3), \mathcal{L})$$

and on the other hand using the isogeny $SL_2 \times SL_2 \to SO_4$:

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \to H^0(\mathcal{M}_C(SL_2), \mathcal{L}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3).$$
These maps take values in the invariant part by the duality involution for the first map and by the action of 2-torsion elements of the Jacobian for the second one. We denote these invariant spaces by $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$ and $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$ respectively.

Using the natural isomorphism proved in [BNR89]:

$$H^0(\mathcal{M}_C(SL_3), \mathcal{L})^* \simeq H^0(\text{Pic}^{g-1}(C), 3\Theta)$$

where $\Theta = \{ L \in \text{Pic}^{g-1}(C) | h^0(C, L) > 0 \}$, we prove the following theorem:

Theorem A.

The linear map

$$\Phi : H^0(\mathcal{M}_C(G_2), \mathcal{L}) \to H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$$

obtained by pull-back of the extension map $\mathcal{M}_C(SL_3) \to \mathcal{M}_C(G_2)$ is surjective for a general curve and is an isomorphism when the genus of the curve equals 2.

A curve is said satisfying the cubic normality when the multiplication map $\text{Sym}^3 H^0(\mathcal{M}_C(SL_2), \mathcal{L}) \to H^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)$ is surjective. Using an explicit basis of $H^0(\mathcal{M}_C(SL_2), \mathcal{L}^2)$ described in [Bea91], we prove the theorem:

Theorem B.

The linear map

$$\Psi : H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \to [H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$$

obtained by pull-back of the extension map $\mathcal{M}_C(SO_4) \to \mathcal{M}_C(G_2)$ is an isomorphism for a general curve satisfying the cubic normality.

Notation. We use the following notations:

- $C$ a smooth, connected, projective curve of genus $g$ at least 2,
- $G$ a connected and simply-connected simple complex Lie group,
- $\mathcal{M}_C(G)$ the moduli stack of principal $G$-bundles on $C$,
- $K_C$ the canonical bundle on $C$,
- $\text{Pic}^{g-1}(C)$ the Picard group parametrizing line bundles on $C$ of degree $g-1$,
- $h^0(X, \mathcal{L}) = \dim H^0(X, \mathcal{L})$.

2. Principal $G_2$-bundles arising from vector bundles of rank two and three

2.1. The octonions algebra. Let $\mathbb{O}$ be the complex algebra of the octonions, $\text{Im}(\mathbb{O})$ the 7-dimensional subalgebra of the imaginary part of $\mathbb{O}$ and $B_0 = (e_1, \ldots, e_7)$ the canonical basis of $\text{Im}(\mathbb{O})$ (see [Bae02]). The exceptional Lie group $G_2$ is the group of automorphisms of the octonions.

In Appendix A we give the multiplication table in $\text{Im}(\mathbb{O})$ by the Fano diagram and we introduce another basis $B_1 = (y_1, \ldots, y_7)$ of $\text{Im}(\mathbb{O})$ so
that \((y_1, y_2, y_3)\) and \((y_4, y_5, y_6)\) are isotropic and orthogonal and \(y_7\) is orthogonal to both of these subspaces. This basis is defined in Appendix A, as well as two other basis obtained by permutation of elements of \(B_1: B_2 = (y_2, y_3, y_4, y_5, y_6, y_1, y_7)\) and \(B_3 = (y_1, y_2, y_4, y_5, y_3, y_6, y_7)\).

In the following paragraphs, we use local sections of rank-7 vector bundles satisfying multiplication rules of \(B_2\) or \(B_3\).

2.2. Principal \(G_2\)-bundles admitting a reduction. We introduce the notion of non-degenerated trilinear form on \(\text{Im}(\mathcal{O})\) as Engel did it in [Enga] and [Engb]. A trilinear form \(\omega\) on \(\text{Im}(\mathcal{O})\) is said non-degenerated if the associated bilinear symmetric form \(B_\omega\) is non-degenerated where \(B_\omega(x, y) = \omega(x, \cdot, \cdot) \land \omega(\cdot, y, \cdot) \land \omega(\cdot, \cdot, \cdot) \ \forall x, y \in \text{Im}(\mathcal{O})\).

**Lemma 2.1.**

Giving a principal \(G_2\)-bundle is equivalent to giving a rank-7 vector bundle with a non-degenerated trilinear form.

**Proof.** Let \(P\) be a principal \(G_2\)-bundle on \(C\) and \(V\) the associated rank-7 vector bundle and let \(\omega\) be any non-degenerated trilinear form on \(\text{Im}(\mathcal{O})\). By construction, \(V\) has a reduction to \(G_2\), i.e. it exists a section \(\sigma: C \to GL_7/G_2\). The Lie group \(G_2\) is the stabilizer \(\text{Stab}_{GL_7}(\omega)\) under the action of \(SL_7\). Besides, under the action of \(GL_7\), \(\text{Stab}_{GL_7}(\omega_0) \simeq G_2 \times \mathbb{Z}/3\mathbb{Z}\). Then

\[
\sigma: C \xrightarrow{\simeq} GL_7/G_2 \to GL_7/(G_2 \times \mathbb{Z}/3\mathbb{Z}) \simeq GL_7/\text{Stab}_{GL_7}(\omega) \simeq \text{Orb}_{GL_7}(\omega).
\]

In addition, the orbit \(\text{Orb}_{GL_7}(\omega)\) is the set of all the non-degenerated trilinear form on \(\text{Im}(\mathcal{O})\). So, \(V\) is fitted with a non-degenerated trilinear form. Reciprocally any rank-7 vector bundle fitted with a non-degenerated trilinear form defines a \(G_2\)-vector bundle.

For a principal \(G_2\)-bundle, we use the non-degenerate trilinear form \(\omega\) on \(\mathcal{V}\), locally defined by \(\omega(x, y, z) = -\text{Re}[(xy)z]\).

According to the Borel-De Siebenthal classification (see [BDS49]), \(SL_3\) and \(SO_4\) are, up to conjugation, the two maximal subgroups of \(G_2\) among the connected subgroups of \(G_2\) of maximal rank. Using the inclusion \(G_2 \subset SO_7 = SO(\text{Im}(\mathcal{O}))\) both of the following lemma describe the rank-7 vector bundle (and the non-degenerate trilinear form) associated to a principal \(G_2\)-bundle which admits either a \(SL_3\)-reduction or a \(SO_4\)-reduction.

Note that \(M_C(SO_4)\) has two connected components distinguished by the second Stiefel Whitney class. We only make here explicit computations with regards to the connected component \(M^+_C(SO_4)\) of \(M_C(SO_4)\) containing the trivial bundle.

2.2.1. Principal \(G_2\)-bundles arising from rank-3 vector bundles.

**Lemma 2.2.**

Let \(E\) be a rank-3 vector bundle with trivial determinant and let \(E(G_2)\) be his associated principal \(G_2\)-bundle and \(\mathcal{V}\) be his associated rank-7 vector bundle.
Then, \( \mathcal{V} \) has the following decomposition and the local sections basis \( B_2 \) is adapted to this decomposition:

\[
\mathcal{V} = E \oplus E^* \oplus \mathcal{O}_C.
\]

The non-degenerate trilinear form \( \omega \) is defined by the following local conditions:

1. \( \Lambda^3 E \cong \Lambda^3 E^* \cong \mathbb{C} \) and \( \omega(y_2, y_3, y_4) = \omega(y_5, y_6, y_7) = -\sqrt{2} \).
2. On \( E \times E^* \times \mathcal{O}_C \):
   \[
   \omega(y_2, y_5, y_7) = \omega(y_3, y_6, y_7) = \omega(y_4, y_1, y_7) = i,
   \]
3. All other computation, not obtainable by permutation of the previous triplets, equals zero.

**Proof.** Under the action of \( SL_3 \), \( \text{Im}(\mathcal{O}) \) decomposes into \( SL_3 \)-modules:

\[
\text{Im}(\mathcal{O}) = \langle y_2, y_3, y_4 \rangle \oplus \langle y_5, y_6, y_7 \rangle
= \mathbb{C}^3 \oplus (\mathbb{C}^3)^* \oplus \mathbb{C}
\]

where \( \{y_2, y_3, y_4, y_5, y_6, y_7\} \) is the basis \( B_2 \) defined in Appendix A; \( \langle y_2, y_3, y_4 \rangle \) is an isotropic subspace of dimension 3, dual of \( \langle y_5, y_6, y_7 \rangle \).

So, the 7-rank vector bundle \( \mathcal{V} \) associated to a rank-3 vector bundle \( E \) is:

\[
\mathcal{V} = E \times_{SL_3} \left( \mathbb{C}^3 \oplus (\mathbb{C}^3)^* \oplus \mathbb{C} \right),
\]

\[
\mathcal{V} = E \oplus E^* \oplus \mathcal{O}_C.
\]

Evaluations given for \( \omega \) are deduced from Table 5 of Appendix A. \( \square \)

2.2.2. **Principal \( G_2 \)-bundles arising from two rank-2 vector bundles.** The following lemma makes explicit the vector bundle associated to a principal \( G_2 \)-bundle extension of an element of \( M_C^+ (SO_4) \), using the surjective map \( M_C (SL_2) \times M_C (SL_2) \to M_C^+ (SO_4) \).

**Lemma 2.3.**
Let \( E, F \) be two rank-2 vector bundles of trivial determinant. Denote by \( (E, F) \) the associated principal \( SO_4 \)-bundle, \( P \) the associated principal \( G_2 \)-bundle and \( \mathcal{V} \) the associated rank-7 vector bundle. Then, \( \mathcal{V} \) has the following decomposition and the local sections basis \( B_3 \) is adapted to this decomposition:

\[
\mathcal{V} = E^* \otimes F \oplus \text{End}_0(F).
\]

The non-degenerate trilinear form \( \omega \) on \( \mathcal{V} \) is defined by the following local conditions:

1. On \( (E^* \otimes F)^3 \) and on \( (E^* \otimes F) \times (\text{End}_0(F))^2 \), \( \omega \) is identically zero,
2. On \( (E^* \otimes F)^2 \times \text{End}_0(F) \):
   \[
   \omega(y_2, y_3, y_5) = \omega(y_5, y_1, y_6) = \sqrt{2},
   \omega(y_4, y_1, y_7) = \omega(y_2, y_5, y_7) = i,
   \]
3. \( \Lambda^3 \text{End}_0(F) \cong \mathbb{C} \) and \( \omega(y_3, y_6, y_7) = i \),
4. All other computation, not obtainable by permutation of the previous triplets, equals zero.
Proof. The groups $SO_4$ and $SL_2 \times SL_2$ are isogenous. Under the action of $SO_4$, $\text{Im}(O)$ has the following decomposition:

$$\text{Im}(O) = \langle y_1, y_2, y_4, y_5 \rangle \oplus \langle y_3, y_6, y_7 \rangle,$$

$$\simeq M^* \otimes N \oplus \text{End}_0(N)$$

where $M, N$ are 2-dimensional; an element $(A, B)$ of $SO_4$ ($A, B \in SL_2$) acts on $M^* \otimes N$ by $A \otimes B$ and by conjugation $\text{End}_0(N)$.

So, the rank-7 vector bundle $\mathcal{V}$ associated to $(E, F)(G_2)$, when $E, F$ are two rank-7 vector bundle of trivial determinant, is:

$$\mathcal{V} = E^* \otimes F \oplus \text{End}_0(F).$$

Evaluations given for $\omega$ are deduced from Table 6 of Appendix A. □

3. Equalities between dimensions of spaces of generalized theta functions

Here are some dimension counts, using the Verlinde Formula, to calculate $h^0(\mathcal{M}_C(G_2), \mathcal{L})$ and $h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3)$.

**Proposition 3.1.**

Dimensional equalities between the following spaces of generalized theta functions occur:

$$h^0(\mathcal{M}_C(G_2), \mathcal{L}) = \left( \frac{5 + \sqrt{5}}{2} \right)^{g-1} + \left( \frac{5 - \sqrt{5}}{2} \right)^{g-1},$$

$$h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) = 2^g \left[ \left( \frac{5 + \sqrt{5}}{2} \right)^{g-1} + \left( \frac{5 - \sqrt{5}}{2} \right)^{g-1} \right],$$

so,

$$h^0(\mathcal{M}_C(SL_2), \mathcal{L}^3) = 2^g h^0(\mathcal{M}_C(G_2), \mathcal{L}).$$

**Proof.** See Appendix B and C □

4. Surjectivities and isomorphisms between $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ and $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$

To avoid confusion, we sometimes specify the group $G$ in the notation of the generator of the Picard group $\mathcal{L}$ writing $\mathcal{L}_G$.

Let $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$ be the invariant part of $H^0(\mathcal{M}_C(SL_3), \mathcal{L})$ by the duality involution: the eigenspace of $H^0(\mathcal{M}_C(SL_3), \mathcal{L})$ associated to the eigenvalue 1 under the natural involution $E \mapsto \sigma(E) = E^*$. We consider the extension map $i : \mathcal{M}_C(SL_3) \rightarrow \mathcal{M}_C(G_2)$ which associates to a rank-3 vector bundle of trivial determinant the associated principal $G_2$-bundle. The pull-back $i^*(\mathcal{L}_{G_2})$ equals $\mathcal{L}_{SL_3}$. 
Theorem 4.1.
The extension map $i : \mathcal{M}_C(SL_3) \to \mathcal{M}_C(G_2)$ induces by pull-back a linear map between the following spaces of generalized theta functions:

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \to H^0(\mathcal{M}_C(SL_3), i^*(\mathcal{L}_{G_2})).$$

This map takes values in $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$. We denote by $\Phi$ this map:

$$\Phi : H^0(\mathcal{M}_C(G_2), \mathcal{L}) \to H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+.$$  

Proof.

(1) The pull-back $i^*(\mathcal{L}_{G_2})$ equals $\mathcal{L}_{SL_3}$ Indeed, let $E$ be a rank-3 vector bundle. By Lemma 2.2, the rank-7 vector bundle associated to $E$ is $E \oplus E^* \oplus \mathcal{O}_C$. We study the commutative diagram:

$$\begin{array}{ccc}
\mathcal{M}_C(G_2) & \xrightarrow{\rho_1} & \mathcal{M}_C(SL_3) \\
i & & \downarrow{\rho_2} \\
\mathcal{M}_C(SL_3) & \xrightarrow{\rho_3} & \mathcal{M}_C(SL_3) \times \mathcal{M}_C(SL_3)
\end{array}$$

where $i$ and $\rho_1$ are maps of extension of group of structure and $\forall E \in \mathcal{M}_C(SL_3)$, $\rho_3(E) = (E, E^*)$ and $\forall (E, F) \in \mathcal{M}_C(SL_3) \times \mathcal{M}_C(SL_3)$, $\rho_2(E, F) = E \oplus F \oplus \mathcal{O}_C$. By Proposition 2.6 of [LS97], applied with $SL_7$, $G_2$ and the irreducible representation $\rho_1$ of highest weight $\varpi_1$, the Dynkin index of $d(\rho_1)$ equals 2. Therefore, $\rho_1^*(D_{SL_7}) = \mathcal{L}_{G_2}$ where $D_{SL_7}$ the determinant bundle generator of $\text{Pic}(\mathcal{M}_C(SL_7))$ (see [KRN94] and [LS97]). In addition, by the same proposition, $\rho_3^*(D_{SL_7}) = \mathcal{L}_{SL_3} \cong \mathcal{L}_{SL_3}$ and $\rho_1^*(\rho_2^*(D_{SL_7})) = \mathcal{L}_{SL_3}^2$. So, $\Phi^*(\mathcal{L}_{G_2}^2 = \mathcal{L}_{SL_3}$ so that $i^*(\mathcal{L}_{G_2}) = \mathcal{L}_{SL_3}$ since the Picard group $\text{Pic}(\mathcal{M}_C(SL_3))$ is isomorphic to $\mathbb{Z}$.

(2) We show that the image of the linear map $\Phi$ is contained in $H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+$. The morphism $i$ is $\sigma$-invariant: for all $E \in \mathcal{M}_C(SL_3)$, the $G_2$-principal bundles $E(G_2)$ and $E^*(G_2)$ are isomorphic. Indeed, the Weyl group $W(SL_3)$ is contained in $W(G_2)$ since so are there normalizer; and $W(SL_3)$ is a subgroup of $W(G_2)$ of index 2. We consider $\mathcal{L}$ in the Weyl group $W(G_2) \backslash W(SL_3)$ and $g \in G_2$ a representative of the equivalence class of $\mathcal{L}$. Then, $g \not\in SL_3$. Let $C_g$ be the inner automorphism of $G_2$ induced by $g$. As the subalgebra $\mathfrak{sl}_3$ of $\mathfrak{g}_2$ corresponds to the long roots and as each element of the Weyl group $W(G_2)$ respects the Killing form on $\mathfrak{g}_2$, $C_g(SL_3)$ is contained in $SL_3$. The restriction of $C_g$ to $SL_3$ is then an exterior automorphism of $SL_3$ which we call $\alpha : SL_3 \to SL_3$. This automorphism exchanges the two fundamental representations of $SL_3$. So, $\alpha$ induces an automorphism $\alpha$ on $\mathcal{M}_C(SL_3)$ such that, $\forall E \in \mathcal{M}_C(SL_3)$, $\alpha(E) = E^*$. Consider the following commutative diagram, where $\tilde{C}_g$ is the inner
An automorphism given by \( g \):
\[
\begin{array}{ccc}
\mathcal{M}_C(SL_3) & \hookrightarrow & \mathcal{M}_C(G_2) \\
\tilde{\alpha} & | & \tilde{C}_g \\
\mathcal{M}_C(SL_3) & \hookrightarrow & \mathcal{M}_C(G_2).
\end{array}
\]

Then, \( \forall E \in \mathcal{M}_C(SL_3) \),
\[
E^*(G_2) = \tilde{\alpha}(E)(G_2) = \tilde{C}_g(E(G_2)) \simeq E(G_2)
\]
since \( \tilde{C}_g \) is an inner automorphism. Thus, \( i(E) \) and \( i(\sigma(E)) \) are isomorphic.

The \( \sigma \)-invariance of \( i \) implies that the image of \( \Phi \) is contained in one of the two eigenspaces of \( H^0(\mathcal{M}_C(SL_3), \mathcal{L}) \). As \( \sigma^*(\mathcal{L}_{SL_3}) \simeq \mathcal{L}_{SL_3} \), which is the isomorphism which implies identity over the trivial bundle, we get \( \sigma(i^*(\mathcal{L}_{G_2})) = i^*(\sigma(\mathcal{L}_{SL_3})) = \sigma(\mathcal{L}_{SL_3}) = \sigma^*(\mathcal{L}_{SL_3}) \). Thus, the image of \( \Phi \) is contained in \( H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+ \), the eigenspace relative to the eigenvalue 1.

\[ \square \]

We remind two points of vocabulary: an even theta-characteristic \( \kappa \) on a curve \( C \) is an element \( \kappa \) of \( \text{Pic}^{g-1}(C) \) such that \( \kappa \otimes \kappa = K_C \) and \( h^0(C, \kappa) \) is even; a curve \( C \) is said without effective theta-constant if \( h^0(C, \kappa) = 0 \) for all even theta-characteristic \( \kappa \). The set of all even theta-characteristics is named \( \Theta^{\text{even}}(C) \).

**Theorem 4.2.**

The linear map \( \Phi : H^0(\mathcal{M}_C(G_2), \mathcal{L}) \to H^0(\mathcal{M}_C(SL_3), \mathcal{L})_+ \)

1. is surjective when the curve \( C \) is without effective theta-constant.
2. is an isomorphism if the genus of \( C \) equals 2.

**Proof.**

(1) Let \( C \) be a curve without effective theta-constant and consider the following diagram:

\[
\begin{array}{ccc}
\mathcal{M}_C(G_2) & \xrightarrow{\rho} & \mathcal{M}_C(SL_7) \\
\downarrow{i} & & \downarrow{\varphi} \\
\mathcal{M}_C(SL_3) & & \\
\end{array}
\]

We introduce the element \( \Delta_\kappa \) defined for each even theta-characteristic \( \kappa \):
\[
\Delta_\kappa = \{ P \in \mathcal{M}_C(SL_7) \mid h^0(C, P(C^7) \otimes \kappa) > 0 \}.\]
These $\Delta_\kappa$ are Cartier divisors, so they define, up to a scalar, an element of $H^0(MC(SL_7), \mathcal{L})$. The image $\Phi(\rho_1^*(\Delta_\kappa)) = \rho^*(\Delta_\kappa)$ is

$$
\rho^*(\Delta_\kappa) = \begin{cases} 
E \in \mathcal{M}_C(SL_3) | h^0(C, E \otimes E^* \otimes O_C) \otimes \kappa > 0, \\
E \in \mathcal{M}_C(SL_3) | h^0(C, E \otimes \kappa) + h^0(C, E^* \otimes \kappa) + h^0(C, \kappa) > 0, \\
E \in \mathcal{M}_C(SL_3) | h^0(C, E \otimes \kappa) + h^0(C, E^* \otimes \kappa) > 0, \\
because C \text{ is without effective theta-constant,} \\
\{E \in \mathcal{M}_C(SL_3) | 2h^0(C, E \otimes \kappa) > 0\}, \text{ by Serre duality.}
\end{cases}
$$

Thus, $\rho^*(\Delta_\kappa) = 2H_\kappa$ where $H_\kappa := \{E \in \mathcal{M}_C(SL_3) | h^0(C, E \otimes \kappa) > 0\}$. Therefore, to show the surjectivity of $\Phi$, it suffices to show that $\{H_\kappa | \kappa \in \Theta^{even}(C)\}$ generates $H^0(MC(SL_3), \mathcal{L})_+$. We consider

$$
\Theta = \{L \in \text{Pic}^{g-1}(C) | h^0(C, L) > 0\}
$$

and the natural map between the spaces $H^0(MC(SL_3), \mathcal{L})_+$ and $H^0(\text{Pic}^{g-1}(C), 3\Theta)$. By Theorem 3 of [BNR89], this map is an isomorphism and, besides, it is equivariant for the two involutions on $H^0(MC(SL_3), \mathcal{L})_+$ and $H^0(\text{Pic}^{g-1}(C), 3\Theta)$ (respectively $E \mapsto E^*$ and $L \mapsto K_C \otimes L^{-1}$). So, the components (+) and (−) of each part are in correspondence: $H^0(MC(SL_3), \mathcal{L})_+$ is isomorphic to $H^0(\text{Pic}^{g-1}(C), 3\Theta)_+$. Denote by $\varphi$ this isomorphism $\varphi : \mathbb{P}H^0(MC(SL_3), \mathcal{L})_+ \sim \mathbb{P}H^0(\text{Pic}^{g-1}(C), 3\Theta)_+$. For all even theta-characteristic $\kappa$, the image $\varphi(H_\kappa)$ is $\varphi_{3\Theta}(\kappa)$ where $\varphi_{3\Theta}$ is the following map:

$$
\varphi_{3\Theta} : \text{Pic}^{g-1}(C) \rightarrow \mathbb{P}H^0(\text{Pic}^{g-1}(C), 3\Theta)^*_+ = 3\Theta^*_+.
$$

The set $\{\varphi_{3\Theta}(\kappa) | \kappa \in \Theta^{even}(C)\}$ generates $3\Theta^*_+$. Indeed, in the following commutative diagram

$$
\begin{CD}
|4\Theta|^*_+ @>>> |3\Theta|^*_+ \\
@VVV @VVV \\
\text{Pic}^{g-1}(C) @<2g-1<< |3\Theta|^*_+,
\end{CD}
$$

the map $|4\Theta|^*_+ \rightarrow |3\Theta|^*_+$ is surjective because it is induced by the inclusion $D \in H^0(C, 3\Theta)_+ \rightarrow D + \Theta \in H^0(C, 4\Theta)_+$. In addition, by [KPS09], when $C$ is without effective theta-constant, $\{\varphi_{3\Theta}(\kappa) | \kappa \in \Theta^{even}(C)\}$ is a base of $|4\Theta|^*_+$ (the number of even theta-characteristics equals $2^g - (2^g + 1)$ which equals the linear dimension of $|4\Theta|^*_+$). Thus, $\{\varphi_{3\Theta}(\kappa) | \kappa \in \Theta^{even}(C)\}$ generates $3\Theta^*_+$ and $\{H_\kappa | \kappa \in \Theta^{even}(C)\}$ generates the space $H^0(MC(SL_3), \mathcal{L})_+$. As we have shown that $H_\kappa$ equals $\Phi(\rho_1(\Delta_\kappa))$ for all $\kappa$ even theta-characteristic, the map $\Phi$ is surjective.

(2) By [BNR89], the dimension of $H^0(MC(SL_3), \mathcal{L})_+$ equals the dimension of $H^0(\text{Pic}^{g-1}(C), 3\Theta)_+$, that is $\frac{3g+1}{2}$. When the genus of $C$ is 2,
the dimension of $H^0(\text{Pic}^g-1(C), 3\Theta)_+$ equals 5 which is also the dimension of $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ by Proposition 3.1.(1). A curve of genus 2 is without effective theta-constant. So, by this dimension equality and by the point (1) of the theorem, $\Phi$ is an isomorphism when the genus of $C$ equals 2.

\[ \square \]

5. Isomorphisms between spaces of generalized $G_2$-theta functions and generalized $SL_2$-theta functions

Let $JC[2]$ be the group of 2-torsion elements of the Jacobian: $JC[2] = \{ \alpha \in \text{Pic}^0(C) \mid \alpha \otimes \alpha = \mathcal{O}_C \}$. This group acts on $\mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2}$: for $\alpha \in JC[2]$ and $(E, F) \in \mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2}$, we associate $(E \otimes \alpha, F \otimes \alpha)$. Let $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$ be the invariant part of $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]$ under the action of the element of the Jacobian group $JC[2]$.

We study the subgroup $SO_4$ of $G_2$, which is isogenous to $SL_2 \times SL_2$, and the linear map induced by pull-back by the extension map $j : \mathcal{M}_C(SO_4) \to \mathcal{M}_C(G_2)$ which associates to two rank-2 vector bundle of trivial determinant the associated principal $G_2$-bundle. The pull-back $j^*(\mathcal{L}_{G_2})$ equals $\mathcal{L}_{SL_2} \boxtimes \mathcal{L}_{SL_2}^3$.

**Theorem 5.1.**
The extension map $j : \mathcal{M}_C(SO_4) \to \mathcal{M}_C(G_2)$ induces by pull-back a linear map between the following spaces of generalized theta functions:

$$H^0(\mathcal{M}_C(G_2), \mathcal{L}) \to H^0(\mathcal{M}_C(SO_4), j^*(\mathcal{L}_{G_2})).$$

This map takes values in $[H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$.

We denote by $\Psi$ this map:

$$\Psi : H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \to [H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_C(SL_2), \mathcal{L}_{SL_2}^3)]_0$$

**Proof.** (1) Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{M}_C(G_2) & \xrightarrow{\rho_1} & \mathcal{M}_C(SL_2) \\
\downarrow{j} & & \downarrow{\rho_4} \\
\mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) & \xrightarrow{(f_1, f_2)} & \mathcal{M}_C(SL_4) \times \mathcal{M}_C(SL_3)
\end{array}$$

where $j$ and $\rho_1$ are the extension maps, $f_1(M, N) = M^* \otimes N$, $f_2(M, N) = \text{End}_0(N)$ and $\rho_4(A, B) = A \oplus B$.

As in the previous section, we calculate explicitly $j^*(\mathcal{L}_{G_2})$.

Let $\mathcal{D}_{SL_2}$ be the determinant bundle of $\mathcal{M}_C(SL_2)$ and $\text{pr}_1$ and $\text{pr}_2$ the canonical projections of $SL_2 \times SL_2$. We get

$$f_1^*(\mathcal{L}_{SL_4}) = \text{pr}_1^*(\mathcal{L}_{SL_2})^2 \otimes \text{pr}_2^*(\mathcal{L}_{SL_2})^2 = \mathcal{L}_{SL_2}^2 \boxtimes \mathcal{L}_{SL_2}^2.$$
and according to Table B of [Sor00]:

\[ f_2^*(\mathcal{L}_{SL_3}) = \rho_2(L_{SL_2})^4, \]

since \( f_2 \) is associated to the adjoint representation of \( SL_2 \), which has Dynkin index 4.

so

\[
\rho_1^*(\mathcal{D}) = \mathcal{L}_{SL_4} \boxtimes \mathcal{L}_{SL_3},
\]

so

\[
j^*(\mathcal{L}_{G_2}^4) = (\rho_1, \rho_2)^*(\mathcal{L}_{SL_4} \boxtimes \mathcal{L}_{SL_3}),
\]

\[
j^*(\mathcal{L}_{G_2}^4) = f_1^*(\mathcal{L}_{SL_4}) \otimes f_2^*(\mathcal{L}_{SL_3}),
\]

\[
j^*(\mathcal{L}_{G_2}^4) = [\rho_1^*(\mathcal{L}_{SL_2})^2 \otimes \rho_2^*(\mathcal{L}_{SL_2})^2] \otimes \rho_2^*(\mathcal{L}_{SL_2})^4,
\]

so

\[
j^*(\mathcal{L}_{G_2}^4) = \rho_1^*(\mathcal{L}_{SL_2})^2 \otimes \rho_2^*(\mathcal{L}_{SL_2})^6.
\]

We get

\[
j^*(\mathcal{L}_{G_2}) = \mathcal{L}_{SL_2} \boxtimes \mathcal{L}_{SL_2}^3.
\]

(2) The morphism \( j : \mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2} \to \mathcal{M}_{G_2} \) is invariant under the action of \( JC[2] \): \((E,F) \in \mathcal{M}_{SL_2} \times \mathcal{M}_{SL_2} \) and \( \alpha \in JC[2] \) then the rank-7 vector bundle associated by \( j \) to \((E \otimes \alpha, F \otimes \alpha)\) is

\[
(E \otimes \alpha)^* \otimes (F \otimes \alpha) \oplus \text{End}_0(F \otimes \alpha)
\]

\[
= E^* \otimes F \otimes \alpha^* \otimes \alpha \oplus \text{End}_0(F \otimes \alpha),
\]

\[
= E \otimes F \oplus \text{End}_0(F) = j(E,F).
\]

Therefore, the image of \( \Psi \) is contained in the expected vector space \([H^0(\mathcal{M}_{C(SL_2)}, \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_{C(SL_2)}, \mathcal{L}_{SL_2}^3)])_0\).

\[
\square
\]

Before going further in the study of the morphism \( j \), we compare the dimensions of the involved sets.

**Lemma 5.2.**

The dimension of the space \([H^0(\mathcal{M}_{C(SL_2)}, \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_{C(SL_2)}, \mathcal{L}_{SL_2}^3)])_0\) equals the dimension of \( H^0(\mathcal{M}_{C(G_2)}, \mathcal{L}_{G_2}) \).

**Proof.** By Proposition 3.1,(3), we notice the remarkable following relation:

\[
2^g h^0(\mathcal{M}_{C(G_2)}, \mathcal{L}) = h^0(\mathcal{M}_{C(SL_2)}, \mathcal{L}^3). \tag{4}
\]

So, as \( h^0(\mathcal{M}_{C(SL_2), \mathcal{L}}) = 2^g \) (see [Bea88]), we get

\[
dim \left( [H^0(\mathcal{M}_{C(SL_2)}, \mathcal{L}_{SL_2}) \otimes H^0(\mathcal{M}_{C(SL_2)}, \mathcal{L}_{SL_2}^3)])_0 \right) = \frac{1}{2^g} \cdot h^0(\mathcal{M}_{C(SL_2), \mathcal{L}_{SL_2}}) \cdot h^0(\mathcal{M}_{C(SL_2), \mathcal{L}_{SL_2}^3}) = \frac{1}{2^g} \times 2^g \times 2^g h^0(\mathcal{M}_{C(SL_2), \mathcal{L}_{SL_2}}) \times h^0(\mathcal{M}_{C(SL_2), \mathcal{L}_{SL_2}^3}) \]

because \( |JC[2]| = 2^2g \), by (4).

\[
= h^0(\mathcal{M}_{C(G_2)}, \mathcal{L}_{G_2}).
\]

\[
\square
\]

All the following results are based on the *cubic normality conjecture*. Its statement is:
Conjecture 5.3.
For a general curve $C$, the multiplication map
\[ \eta : \text{Sym}^3 H^0(\mathcal{M}_C(\text{SL}_2), \mathcal{L}_{\text{SL}_2}) \to H^0(\mathcal{M}_C(\text{SL}_2), \mathcal{L}_{\text{SL}_2}^3) \]
is surjective.

When the previous map $\eta$ is surjective, we say that the curve $C$ satisfies cubic normality.

Proposition 5.4.
Cubic normality holds for all curve of genus 2, all non hyper-elliptic curve of genus 3 and all curve of genus 4 without effective theta-constant.

Proof. For a curve of genus 2, $\mathcal{M}_C(\text{SL}_2)$ is isomorphic to $\mathbb{P}^3$ and $\mathcal{L}_{\text{SL}_2}$ to $\mathcal{O}(1)$ (see [NR75]). A non-hyper-elliptic curve of genus 3 is a Coble quartic (see [NR87]). The cubic normality is true in both of these cases. For a general curve of genus 4 without effective theta-constant, cubic normality is proved in Theorem 4.1 of [OP99].

When this conjecture is true, we get this theorem:

Theorem 5.5.
Let $C$ be a curve of genus at least 2 without effective theta-constant and satisfying the cubic normality and let $\Psi$ be the map defined in Theorem 5.1:
\[ \Psi : H^0(\mathcal{M}_C(G_2), \mathcal{L}_{G_2}) \to \left[ H^0(\mathcal{M}_C(\text{SL}_2), \mathcal{L}_{\text{SL}_2}) \otimes H^0(\mathcal{M}_C(\text{SL}_2), \mathcal{L}_{\text{SL}_2}^3) \right]_0. \]

1. The map $\Psi$ is an isomorphism,
2. The space of generalized $G_2$-theta functions $H^0(\mathcal{M}_C(G_2), \mathcal{L})$ is linearly generated by the divisors $\rho^*_0(\Delta_\kappa)$ for $\kappa$ even theta-characteristic, where $\rho_0$ is the extension morphism $\rho_0 : \mathcal{M}_C(G_2) \to \mathcal{M}_C(\text{SO}_7)$.

Proof. (1) According to the dimension equality proved in Lemma 5.2, it suffices to prove the surjectivity of $\Psi$.

Denote by $V$ the vector space $H^0(\mathcal{M}_C(\text{SL}_2), \mathcal{L})$.

Using the notations of [Bea91], we associate to each even theta-characteristic $\kappa$ an element $d_\kappa$ of $H^0(\mathcal{M}_C(\text{SL}_2), \mathcal{L}^{\otimes 2})$ and an element $\xi_\kappa$ of $V \otimes V$. For each even theta-characteristic $\kappa$, $d_\kappa$ is the section of $H^0(\mathcal{M}_C(\text{SL}_2), \mathcal{L}^{\otimes 2})$ such that $D_\kappa$ is the divisor of the zeros of $d_\kappa$, where $D_\kappa = \{ S \in \mathcal{M}_C(\text{SL}_2) \mid h^0(C, \text{End}_0(S) \otimes \kappa) > 0 \}$. Consider the following maps:
\[ \rho^*_0 : H^0(\mathcal{M}_C^+(\text{SO}_7), \mathcal{L}) \to H^0(\mathcal{M}_C(G_2), \mathcal{L}) \]
and $\beta : [V \otimes V \otimes H^0(\mathcal{M}_C(\text{SL}_2), \mathcal{L}^2)]_0 \to [V \otimes H^0(\mathcal{M}_C(\text{SL}_2), \mathcal{L}^3)]_0$.
where $\beta(A, B, D) = (A, BD)$. For any even theta-characteristic $\kappa$, the image $\Psi(\rho_0^*(\Delta_\kappa))$ equals $\beta(\xi_\kappa \otimes d_\kappa)$. Indeed, $\Psi$ is induced by:

$$
j : \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) \rightarrow \mathcal{M}_C(G_2)
(E, F) \mapsto \text{Hom}(E, F) \oplus \text{End}_0(F);
$$

the pull-back $\Psi(\rho_0^*(\Delta_\kappa))$ is the sum of two divisors:

$$
\begin{align*}
\Delta_1 &= \{(E, F) \in \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) \mid h^0(C, \text{End}_0(F) \otimes \kappa) > 0\}, \\
\Delta_2 &= \{(E, F) \in \mathcal{M}_C(SL_2) \times \mathcal{M}_C(SL_2) \mid h^0(C, \text{Hom}(E, F) \otimes \kappa) > 0\}.
\end{align*}
$$

In addition, $O(\Delta_1) = O_C \boxtimes L^2$ and $O(\Delta_2) = L \boxtimes L$ (see [Bea91]) and more precisely:

$$
\Delta_1 = \text{Zeros}(d_\kappa) \text{ et } \Delta_2 = \text{Zeros}(\xi_\kappa).
$$

When the curve $C$ is of genus at least 2 without effective theta-constant, it is proved in [BNR89] that the map

$$
\varphi_0^* : \text{Sym}^2 V \rightarrow H^0(M_C(SL_2), L^2)
\xi_\kappa \mapsto d_\kappa
$$

is an isomorphism. We identify $\text{Sym}^2 V$ with the invariant space of $V \otimes V$ under the involution $a \otimes b \mapsto b \otimes a$. By Theorem 1.2 and Proposition A.5 of [Bea91], the set $\{d_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$ is a basis of $H^0(\mathcal{M}_C(SL_2), L^2)$ and $\{\xi_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$ is a basis of $\text{Sym}^2 V$. Then, the vector space $[V \otimes V \otimes h^0(\mathcal{M}_C(SL_2), L^2)]_0$ is generated by $\{\xi_\kappa \otimes d_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$. Thus, to prove the surjectivity of the map $\Psi$, it is sufficient to show the surjectivity of the map $\beta$. Consider the following diagram:

$$
\begin{array}{ccc}
V \otimes H^0(\mathcal{M}_C(SL_2), L^2) & \rightarrow & H^0(\mathcal{M}_C(SL_2), L^3) \\
\uparrow & & \uparrow \eta \\
V \otimes V \otimes V & \rightarrow & \text{Sym}^3 V.
\end{array}
$$

With the hypothesis of cubic normality, the map $\eta$ is surjective. Therefore the map $V \otimes H^0(\mathcal{M}_C(SL_2), L^2) \rightarrow H^0(\mathcal{M}_C(SL_2), L^3)$ is also surjective. By restriction to invariant sections under the action of $JC[2]$, $\beta$ is surjective.

The map $\Psi$ is thus an isomorphism.

(2) The point (2) is a consequence of the previous facts: for each element $\kappa \in \Theta^{\text{even}}(C)$, the image of $\rho_0^*(\Delta_\kappa)$ by $\Psi$ is $\xi_\kappa \otimes d_\kappa$. As $\{\xi_\kappa \otimes d_\kappa \mid \kappa \in \Theta^{\text{even}}(C)\}$ generates $[V \otimes V \otimes h^0(\mathcal{M}_C(SL_2), L^2)]$, the set $\{\rho_0^*(\Delta_\kappa) \mid \kappa \in \Theta^{\text{even}}(C)\}$ generates $H^0(\mathcal{M}_C(G_2), L)$.
Remark 5.6. By Proposition 5.4 the linear map $\Psi$ is an isomorphism for each curve of genus 2, each non hyperelliptic curve of genus 3 and each curve of genus 4 without effective theta-constant.

Appendix A. The octonion algebra

Let $B_0 = \{e_1, \ldots, e_7\}$ be the canonical basis of the subalgebra of the imaginary part of octonions. The multiplication rules are:

- $\forall i \in \{1, \ldots, 7\}$, $e_i^2 = -1$,
- $e_i e_j = -e_j e_i = e_k$ when $(e_i, e_j, e_k)$ are three points on the same edge of the oriented Fano diagram.

We introduced the basis $B_1 = \{y_1, \ldots, y_7\}$ obtained by basis change by the change of basis matrix $P = \frac{\sqrt{2}}{2} \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & -i & 0 & 0 & i & 0 \\
-i & 0 & 0 & i & 0 & 0 \\
0 & 0 & -i & 0 & 0 & i \\
0 & 0 & 0 & 0 & 0 & \sqrt{2}
\end{pmatrix}$. The
canonical quadratic form on $\text{Im}O$ expressed in the basis $B_1$ is $Q = \begin{pmatrix} 0 & I_3 & 0 \\ I_3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

In the basis $B_2 = \{y_2, y_3, y_4, y_5, y_6, y_1, y_7\}$, the multiplication table is

\[
\begin{array}{cccccccc}
\otimes & y_2 & y_3 & y_4 & y_5 & y_6 & y_1 & y_7 \\
y_2 & 0 & -\sqrt{2}y_1 & \sqrt{2}y_6 & -1 + iy_7 & 0 & 0 & -iy_2 \\
y_3 & \sqrt{2}y_1 & 0 & -\sqrt{2}y_5 & 0 & -1 + iy_7 & 0 & -iy_3 \\
y_4 & -\sqrt{2}y_6 & \sqrt{2}y_5 & 0 & 0 & 0 & -1 + iy_7 & -iy_4 \\
y_5 & -1 - iy_7 & 0 & 0 & 0 & -\sqrt{2}y_4 & \sqrt{2}y_3 & iy_5 \\
y_6 & 0 & -1 - iy_7 & 0 & \sqrt{2}y_4 & 0 & -\sqrt{2}y_2 & iy_6 \\
y_1 & 0 & 0 & -1 - iy_7 & -\sqrt{2}y_3 & \sqrt{2}y_2 & 0 & iy_1 \\
y_7 & iy_2 & iy_3 & iy_4 & -iy_5 & -iy_6 & -iy_1 & -1 \\
\end{array}
\]

(5)

In the basis $B_3 = \{y_1, y_2, y_4, y_5, y_3, y_6, y_7\}$, the multiplication table is

\[
\begin{array}{cccccccc}
\otimes & y_1 & y_2 & y_4 & y_5 & y_3 & y_6 & y_7 \\
y_1 & 0 & 0 & -1 - iy_7 & -\sqrt{2}y_3 & 0 & \sqrt{2}y_2 & iy_1 \\
y_2 & 0 & 0 & \sqrt{2}y_6 & -1 + iy_7 & -\sqrt{2}y_1 & 0 & -iy_2 \\
y_4 & -1 + iy_7 & -\sqrt{2}y_6 & 0 & 0 & \sqrt{2}y_5 & 0 & -iy_4 \\
y_5 & \sqrt{2}y_3 & -1 - iy_7 & 0 & 0 & 0 & -\sqrt{2}y_4 & iy_5 \\
y_3 & 0 & \sqrt{2}y_1 & -\sqrt{2}y_5 & 0 & 0 & 0 & -1 + iy_7 & -iy_3 \\
y_6 & -\sqrt{2}y_2 & 0 & 0 & \sqrt{2}y_4 & -1 - iy_7 & 0 & iy_6 \\
y_7 & -iy_1 & iy_2 & iy_4 & -iy_5 & iy_3 & -iy_6 & -1 \\
\end{array}
\]

(6)

**Appendix B. Computation of $h^0(\mathcal{M}_C(G_2), L)$**

First, we recall the Verlinde Formula.

**Proposition B.1 (Verlinde Formula).**

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ of classical type or type $\mathfrak{g}_2$, $\mathcal{L}$ be the ample canonical line bundle on $\mathcal{M}_C(G)$ and $i$ be a positive integer. The integer $h^0(\mathcal{M}_C(G), \mathcal{L}^i)$ is given by the following relation:

$$h^0(\mathcal{M}_C(G), \mathcal{L}^i) = (\# T_i)^{g-1} \sum_{\mu \in P_+ \cap \Delta_+} \prod_{\alpha \in \Delta_+} \left[ 2 \sin \left( \frac{\pi (\alpha, \mu + \rho)}{i + g^*} \right) \right]^{2-2g}.$$
where
\[
\#T_i = (i + g^*)^{\text{rk}(g)} \#(P/Q)\#(Q/Q_g), \quad \langle \cdot, \cdot \rangle \text{ is the Killing form},
\]
where \( \text{rk}(g) \) is the rank of \( g \), \( P \) is the weight lattice, \( Q \) is the root lattice, \( Q_g \) is the long root lattice, \( \rho = \frac{1}{2} \sum_{\alpha_j \in \Delta_+} \alpha_j \).

\[
P_1 = \{ \text{dominant weights } \mu \mid \langle \mu, \theta \rangle \leq i \},
\]
\( \theta \) is the maximal positive root,
\[
\Delta_+ = \{ \text{positive roots of } g \},
\]
g* is the dual Coxeter number of the group \( G \).

**Proof.** See the survey [Sor96].

The root system of \( G_2 \) has six positive roots \( \alpha_1, \ldots, \alpha_6 \), with two simple roots, called \( \alpha_1 \) and \( \alpha_2 \).

For \( h^0(M_{C(G_2), L}) \), the data used in the Verlinde Formula are \( g^* = 4 \), \( \text{rk}(g_2) = 2 \), \( \#T_1 = (1 + 4)^2 \times 1 \times 3 = 75 \), \( \rho = 5\alpha_1 + 3\alpha_2 \), \( \theta = \alpha_6 = 3\alpha_1 + 2\alpha_2 \) and the fundamental weights are \( \varpi_1 = \alpha_5 \) and \( \varpi_2 = \alpha_6 \).

To describe \( P_1 \), we describe the Killing form. The angle formed between \( \alpha_1 \) and \( \alpha_2 \) is \( 5\pi/6 \) and the ratio \( \|\alpha_2\|/\|\alpha_1\| = \sqrt{3} \). In order to normalize the Killing form, we impose the norm of maximal positive root \( \|\theta\| \) equals 2.

As \( \theta = \alpha_6 \), \( \|\alpha_2\| = \|\theta\| = 2 \) and then \( \|\alpha_1\| = 2/3 \) and \( \langle \alpha_1, \alpha_2 \rangle = -1 \). The evaluations of the Killing form on \( \varpi_1 \) and \( \theta \) are \( \langle \varpi_1, \theta \rangle = 1 \) and \( \langle \varpi_2, \theta \rangle = 2 \); so \( P_1 = \{0, \varpi_1\} \).

The evaluation of the Killing form on each positive root and \( \rho \) added to each value of \( P_1 \) are the following:

\[
\begin{align*}
(\alpha_1, \rho) &= 1/3 \quad \text{and} \quad (\alpha_1, \varpi_1 + \rho) = 2/3, \\
(\alpha_2, \rho) &= 1, \quad (\alpha_2, \varpi_1 + \rho) = 1, \\
(\alpha_3, \rho) &= 4/3, \quad (\alpha_3, \varpi_1 + \rho) = 5/3, \\
(\alpha_4, \rho) &= 5/3, \quad (\alpha_4, \varpi_1 + \rho) = 7/3, \\
(\alpha_5, \rho) &= 2, \quad (\alpha_5, \varpi_1 + \rho) = 3, \\
(\alpha_6, \rho) &= 3, \quad (\alpha_6, \varpi_1 + \rho) = 4.
\end{align*}
\]

So, by the Verlinde formula, the dimension \( h^0(M_{C(G_2), L}) \) is

\[
h^0(M_{C(G_2), L}) = \left( \frac{2 \alpha^0}{2\pi} \right)^{-9} \left[ \sin^2 \left( \frac{\pi}{15} \right) \sin^2 \left( \frac{4\pi}{15} \right) \sin^2 \left( \frac{\pi}{6} \right) \sin^4 \left( \frac{2\pi}{5} \right) \right]^{1-g} + \left[ \sin^2 \left( \frac{3\pi}{10} \right) \sin^2 \left( \frac{4\pi}{15} \right) \sin^2 \left( \frac{2\pi}{5} \right) \sin^4 \left( \frac{\pi}{6} \right) \right]^{1-g}.
\]

To obtain a compact formula, we express these trigonometric products in \( \mathbb{Q}(\sqrt{5}) \):

\[
\begin{align*}
\sin^2 \left( \frac{\pi}{6} \right) &= \frac{1}{2} (5 - \sqrt{5}), \quad \sin^2 \left( \frac{3\pi}{15} \right) \sin^2 \left( \frac{4\pi}{15} \right) = \frac{1}{2} (3 - \sqrt{5}), \\
\sin^2 \left( \frac{2\pi}{5} \right) &= \frac{1}{8} (5 + \sqrt{5}), \quad \sin^2 \left( \frac{3\pi}{10} \right) \sin^2 \left( \frac{12\pi}{15} \right) = \frac{1}{8} (3 + \sqrt{5}), \\
\sin^4 \left( \frac{\pi}{5} \right) &= \frac{5}{2} (3 - \sqrt{5}), \quad \sin^4 \left( \frac{2\pi}{5} \right) = \frac{5}{2} (3 + \sqrt{5}).
\end{align*}
\]
So,
\[ h^0(M_C(G_2), \mathcal{L}) = \left( \frac{20}{25} \right)^{1-g} \left[ \left( \frac{1}{25} \right) \left( \frac{1}{25} \right) \left( \frac{25}{25} \right) \left( 3 - \sqrt{5} \right) \left( 5 - \sqrt{5} \right) \left( 3 + \sqrt{5} \right) \right]^{1-g} + \left[ \left( \frac{1}{25} \right) \left( \frac{1}{25} \right) \left( \frac{25}{25} \right) \left( 3 + \sqrt{5} \right) \left( 5 + \sqrt{5} \right) \left( 3 - \sqrt{5} \right) \right]^{1-g} , \]
\[ = \left( \frac{5 + \sqrt{5}}{10} \right)^{1-g} + \left( \frac{5 + \sqrt{5}}{10} \right)^{1-g} = \left( \frac{5 + \sqrt{5}}{10} \right)^{g-1} + \left( \frac{5 - \sqrt{5}}{10} \right)^{g-1} . \]

APPENDIX C. COMPUTATION OF \( h^0(M_C(SL_2), \mathcal{L}^3) \)

Using the previous notations, the evaluation on \( \alpha \) and each element of \( \mu + \rho \), where \( \mu \in \mathcal{P}_3 \), are the following:
\[ \langle \alpha, \rho \rangle = 1, \text{ and } \langle \alpha, 2\varpi_1 + \rho \rangle = 3, \]
\[ \langle \alpha, \varpi_1 + \rho \rangle = 2, \quad \langle \alpha, 3\varpi_1 + \rho \rangle = 4. \]

By the Verlinde Formula,
\[ h^0(M_C(SL_2), \mathcal{L}^3) = (10)^{g-1} \left( 2 \right)^{2-2g} \left[ \sin \left( \frac{\pi}{5} \right) \right]^{2-2g} + \left[ \sin \left( \frac{2\pi}{5} \right) \right]^{2-2g} + \left[ \sin \left( \frac{3\pi}{5} \right) \right]^{2-2g} + \left[ \sin \left( \frac{4\pi}{5} \right) \right]^{2-2g} , \]
\[ = \left( \frac{2}{5} \right)^{g-1} \left[ \sin \left( \frac{\pi}{5} \right) \right]^{1-g} + \left[ \sin \left( \frac{2\pi}{5} \right) \right]^{1-g} , \]
\[ = 2 \left( \frac{2}{5} \right)^{g-1} \left( \frac{\sqrt{5}}{5} \right)^{g-1} + \left( \frac{5 + \sqrt{5}}{10} \right)^{g-1} , \]
\[ = 2^g \left( \frac{5 + \sqrt{5}}{10} \right)^{g-1} + \left( \frac{5 - \sqrt{5}}{10} \right)^{g-1} . \]

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