The solution of an open XXZ chain with arbitrary spin revisited

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Abstract. The Bethe ansatz solutions for an open XXZ spin chain with arbitrary spin with \( N \) sites and nondiagonal boundary terms are revisited. The anisotropy parameter, for cases considered here, has values \( \eta = i\pi \frac{r}{q} \), where \( r \) and \( q \) are positive integers with \( q \) restricted to odd integers. Numerical results are presented to support the solutions.

Keywords: integrable spin chains (vertex models), quantum integrability (Bethe ansatz)

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1. Introduction

Over the years, significant progress has been made on the solutions of the integrable open spin-1/2 XXZ quantum spin chain, in particular those with general or nondiagonal boundary terms. These works resulted in a number of approaches and methods which have been utilized in the solutions of the spectrum of the model. They include solutions obtained via certain functional relations obeyed by the transfer matrix of the model [1] for various special cases, such as that with boundary parameters obeying certain contraints [2–7] or that with root of unity values for the anisotropy parameter [8]. Works on the open XXZ spin chain such as that based on the deformed Onsager algebra [9–11] and solutions from functional relations based on Yang–Baxter algebra for the open XXZ model with general boundary terms [12] have also shed light on the nature of the solutions obtained for the open XXZ quantum spin chain. Further, an interesting method based on (generalized) coordinate Bethe ansatz has been used in solving the open XXZ spin chain (among others) with nondiagonal boundaries in [13,14]. More recent works based on the separation of variables method [15–17] and the inhomogeneous $T-Q$ equation approach [18] have served as important advancements to understand this crucial model.

Extension of the spin-1/2 solutions of the XXZ spin chain to include arbitrary spin $s$ ($s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$) have also received considerable interest. In [19], the Bethe ansatz solution for the open XXZ spin chain with alternating spins was constructed utilizing the
method of [3]. This solution relies on a certain constraint among the boundary parameters. In [20], a generalization of this constraint was found as well as a second set of Bethe ansatz equations necessary to obtain all the eigenvalues. One application of such a solution of the open XXZ chain with arbitrary spin is, the $s = 1$ case enables one to investigate the boundary version of the supersymmetric sine-Gordon model [21–23]. In particular, the Bethe ansatz solutions of the open spin-1 XXZ chain have been used to derive the nonlinear integral equations for the supersymmetric sine-Gordon model [24,25].

Motivated by these studies, we revisit solutions of an open spin-\(s\) XXZ spin chain studied in [26]. We extend the anisotropy parameter values to include $\eta = i\pi \frac{r}{q}$, where $r$ and $q$ are positive integers with $q$ assuming odd integer values. To avoid duplication, only the irreducible fractions for $\frac{r}{q}$ are considered. Part of the motivation to consider this problem is that to the extent that a number can be approximated by a rational number, this should in principle extend the solutions in [26] to include a larger class of imaginary values of $\eta$ than was presented earlier. In addition, to our knowledge, Bethe ansatz solution for open spin-\(s\) XXZ spin chain for the case considered here has not been given before. As in [26], we consider cases with at most two arbitrary boundary parameters. In the crucial work on open spin-\(s\) given in [20], Bethe ansatz solution presented there works when certain constraint are obeyed by the boundary parameters. Also, although in a recent important advance [18], the solution for arbitrary values of $\eta$ and the boundary parameters is given (with an unconventional term in the Bethe equations) for the open XXZ chain, such a solution has been written down only for $s = \frac{1}{2}$ case.

This paper is arranged as follows: in section 2, the transfer matrix of the open XXZ spin chain [27] is briefly reviewed. In addition, functional relations that the transfer matrices obey for $\eta = i\pi \frac{r}{q}$ are reviewed. This is followed by the derivation and presentation of the Bethe ansatz solutions for the open XXZ spin chain model with arbitrary spin for cases with two arbitrary boundary parameters in section 3. These solutions are restricted to roots of unity values of the anisotropy parameter, $\eta = i\pi \frac{r}{q}$, where $r$ and $q$ are positive integers (with $q$ assuming any odd positive integers). Numerical results are given in section 4, using $s = \frac{1}{2}$ and $s = 1$ as examples to support and to check for the completeness of the solutions given (presence of all $(2s+1)^N$ eigenvalues). We do this for selected values of number of sites $N$, $\eta$ and the boundary parameters. Some concluding remarks and potential future works follow in section 5.

2. Transfer matrices and functional relations

We present here a brief review on the commuting transfer matrices for \(N\)-site open XXZ quantum spin chain. The spin-1/2 transfer matrix $t^{(\frac{1}{2},\frac{1}{2})}(u)$, whose auxiliary space as well as each of its \(N\) quantum spaces are 2D is given by [27]

$$t^{(\frac{1}{2},\frac{1}{2})}(u) = \text{tr}_0 K_0^+(u) \ T_0(u) \ K_0^-(u) \ \hat{T}_0(u),$$

where the trace is taken over the ‘auxiliary space’ $0$. $T_0(u)$ and $\hat{T}_0(u)$ are the monodromy matrices given by

$$T_0(u) = R_{0N}(u) \cdots R_{01}(u), \quad \hat{T}_0(u) = R_{01}(u) \cdots R_{0N}(u),$$

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The $R$ matrix is given by

$$R(u) = \begin{pmatrix}
\sinh(u + \eta) & 0 & 0 & 0 \\
0 & \sinh u & \sinh \eta & 0 \\
0 & \sinh \eta & \sinh u & 0 \\
0 & 0 & 0 & \sinh(u + \eta)
\end{pmatrix},$$

(2.3)

where $\eta$ is the bulk anisotropy parameter; and $K^{\mp}(u)$ are $2 \times 2$ matrices whose components are given by $[28,29]$

$$K_{11}(u) = 2(\sinh \alpha_+ \cosh \beta_- \cosh u + \cosh \alpha_- \sinh \beta_- \sinh u)$$

$$K_{22}(u) = 2(\sinh \alpha_- \cosh \beta_+ \cosh u - \cosh \alpha_- \sinh \beta_+ \sinh u)$$

$$K_{12}(u) = e^{\theta_-} \sinh 2u, \quad K_{21}(u) = e^{-\theta_-} \sinh 2u,$$

(2.4)

and

$$K^{\pm}(u) = K^{-}(u - \eta)_{(\alpha_-, \beta_-, \theta_-, \theta_+)} \to (-\alpha_+, -\beta_+, \theta_+, \theta_-),$$

(2.5)

where $\alpha_\pm, \beta_\pm, \theta_\pm$ are the boundary parameters.

Using the fusion procedure $[30]$, one can similarly construct the open chain transfer matrix $t^{(s,j)}(u)$, whose auxiliary space is $(2j + 1)$-dimensional and each of its $N$ quantum spaces are $(2s + 1)$-dimensional $[31-33]$. The transfer matrix has the commutativity property for $j, j' \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and any $s \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}$,

$$[t^{(j,s)}(u), t^{(j',s)}(u')] = 0.$$

(2.6)

Furthermore, they also obey the fusion hierarchy $[31,32]$,

$$t^{(j-\frac{1}{2},s)}(u - j\eta) t^{(\frac{1}{2},s)}(u) = t^{(j,s)}(u - (j - \frac{1}{2})\eta) + \delta^{(s)}(u - \eta) t^{(j-1,s)}(u - (j + \frac{1}{2})\eta),$$

(2.7)

In (2.7), $j = \frac{3}{2}, \ldots$. In addition, $t^{(0,s)} = 1$, and $\delta^{(s)}(u)$ is given by

$$\delta^{(s)}(u) = \left[ \prod_{k=0}^{2s-1} \xi(u + (s - k + \frac{1}{2})\eta) \right]^{2N} \frac{\sinh(2u)\sinh(2u + 4\eta)}{\sinh(2u + \eta)\sinh(2u + 3\eta)} \times 2^s \sinh(u + \alpha_- + \eta) \sinh(u - \alpha_- + \eta) \cosh(u + \beta_- + \eta) \cosh(u - \beta_- + \eta) \times \sinh(u + \alpha_+ + \eta) \sinh(u - \alpha_+ + \eta) \cosh(u + \beta_+ + \eta) \cosh(u - \beta_+ + \eta).$$

(2.8)

where $\xi(u) = \sinh(u + \eta)\sinh(u - \eta)$. To avoid any unnecessary repetition, we urge the readers to refer to $[20]$ where the details on such a construction can be found.

Next, we review the $q$th order functional relations $[3,4]$, the ‘fundamental’ transfer matrix, $t^{(\frac{1}{2},s)}(u)$ (as well as each of the corresponding eigenvalues, $\Lambda^{(\frac{1}{2},s)}(u)$) obeys for bulk anisotropy values $\eta = i\pi\frac{r}{q}$, where $r$ and $q$ are positive integers. The functional relations take the following form,

$$t^{(\frac{1}{2},s)}(u)t^{(\frac{1}{2},s)}(u + \eta) \ldots t^{(\frac{1}{2},s)}(u + (q - 1)\eta)$$

$$-\delta^{(s)}(u - \eta) t^{(\frac{1}{2},s)}(u) t^{(\frac{1}{2},s)}(u + 2\eta) \ldots t^{(\frac{1}{2},s)}(u + (q - 2)\eta)$$

$$-\delta^{(s)}(u) t^{(\frac{1}{2},s)}(u + 2\eta) t^{(\frac{1}{2},s)}(u + 3\eta) \ldots t^{(\frac{1}{2},s)}(u + (q - 1)\eta)$$

$$-\delta^{(s)}(u + \eta) t^{(\frac{1}{2},s)}(u + 3\eta) t^{(\frac{1}{2},s)}(u + 4\eta) \ldots t^{(\frac{1}{2},s)}(u + (q - 1)\eta)$$

$$-\delta^{(s)}(u + 2\eta) t^{(\frac{1}{2},s)}(u + 4\eta) \ldots t^{(\frac{1}{2},s)}(u + (q - 1)\eta)$$

$$- \delta^{(s)}(u + (q - 2)\eta) t^{(\frac{1}{2},s)}(u + (q - 3)\eta)$$

$$+ \ldots \ = f(u).$$

(2.9)

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For example, for $q = 3$ and $q = 5$, the functional relations are

$$t^{(\frac{1}{2},-))(u) + \delta(s)(u - \eta)t^{(\frac{1}{2},-))(u + \eta) - \delta(s)(u + \eta)t^{(\frac{1}{2},-))(u + 2\eta)$$

$$- \delta(s)(u + \eta)t^{(\frac{1}{2},-))(u) = f(u).$$

(2.10)

and

$$t^{(\frac{1}{2},-))(u) + \delta(s)(u - \eta)t^{(\frac{1}{2},-))(u + \eta) + \delta(s)(u + \eta)t^{(\frac{1}{2},-))(u + 2\eta)$$

$$+ \delta(s)(u - 2\eta)t^{(\frac{1}{2},-))(u + 2\eta) - \delta(s)(u)t^{(\frac{1}{2},-))(u + 2\eta)$$

$$+ \delta(s)(u - \eta)t^{(\frac{1}{2},-))(u + \eta) - \delta(s)(u + \eta)t^{(\frac{1}{2},-))(u + 2\eta)$$

$$= f(u).$$

(2.11)

respectively.

We note here that the scalar function $f(u)$ that appears in the functional relations above, particularly for even $r$ values, differ slightly from that given in [26] where only the $r = 1$ case was considered. This minor difference in form however, is crucial in order for the functional relations to be obeyed by the ‘fundamental’ transfer matrix. We present below the scalar function $f(u)$ separately for odd $r$ and even $r$ cases respectively (when $q$ assumes any positive odd integer values):

Case 1. $r$ is a positive odd integer

$$f_0(u) = \begin{cases} (-1)^{N+1}2^{-4s(q-1)N}\text{sh}^{4sN}(qu), & s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \\ (-1)^{N+1}2^{-4s(q-1)N}\text{ch}^{4sN}(qu), & s = 1, 2, 3, \ldots \end{cases}$$

(2.12)

and

$$f_1(u) = (-1)^{N+1}2^{5-2s}\left(\text{sh}(q\alpha_-)\text{ch}(q\beta_-)\text{sh}(q\alpha_+)\text{ch}(q\beta_+) \text{ch}^2(qu)\right.$$

$$- \text{ch}(q\alpha_-)\text{sh}(q\beta_-)\text{ch}(q\alpha_+)\text{sh}(q\beta_+) \text{sh}^2(qu)$$

$$- (-1)^N \text{ch}(q(\theta_- - \theta_+)) \text{sh}^2(qu) \text{ch}^2(qu) \right),$$

(2.13)

for $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and

$$f_1(u) = (-1)^{N+1}2^{5-2s}\left(\text{sh}(q\alpha_-)\text{ch}(q\beta_-)\text{sh}(q\alpha_+)\text{ch}(q\beta_+) \text{ch}^2(qu)\right.$$

$$- \text{ch}(q\alpha_-)\text{sh}(q\beta_-)\text{ch}(q\alpha_+)\text{sh}(q\beta_+) \text{sh}^2(qu)$$

$$- \text{ch}(q(\theta_- - \theta_+)) \text{sh}^2(qu) \text{ch}^2(qu) \right),$$

(2.14)

for $s = 1, 2, 3, \ldots$.

Case 2. $r$ is a positive even integer

$$f_0(u) = (-1)^{N+2}2^{-4s(q-1)N}\text{sh}^{4sN}(qu),$$

(2.15)
and
\[
\begin{align*}
\frac{1}{u} &= (-1)^{N+1} 2^{5-2q} \left( \text{sh} (q\alpha_-) \text{ch} (q\beta_-) \text{sh} (q\alpha_+) \text{ch} (q\beta_+) \right. \\
&\quad \left. \text{h}^2 (qu) \right) \\
&\quad - \text{ch} (q\alpha_-) \text{sh} (q\beta_-) \text{ch} (q\alpha_+) \text{sh} (q\beta_+) \text{h} (qu) \\
&\quad + \text{ch} (q(\theta_- - \theta_+)) \text{sh}^2 (qu) \text{ch}^2 (qu),
\end{align*}
\]
for 
\[s = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\]

### 3. Bethe ansatz

Here we give the main result of this paper. Essentially, we revisit the Bethe ansatz solutions derived earlier in [26] for spin-\(s\) XXZ chain with nondiagonal boundary terms, using the method in [3,4], for cases with two arbitrary boundary parameters, namely any two from the \(\{\alpha_{\pm}, \beta_{\pm}\}\) set, e.g. \(\{\alpha_+, \beta_-\}\), \(\{\alpha_+, \alpha_-\}\), etc. We also set \(\theta_- = \theta_+ = \theta\), where \(\theta\) is arbitrary. The remaining boundary parameters are set to some fixed values. The reason behind these choices will be given below for all the cases treated here. Readers are also urged to refer to section 3.1 in [26], where such a discussion was also presented. In [26], we considered the case where \(\eta = \frac{i\pi}{q}\), where \(q\) assumed odd positive integer values. We note that solutions presented here, while bear resemblance to that given in [26], are worth reporting since they include a wider class of (imaginary) values of the anisotropy parameter \(\eta(= \frac{i\pi}{q})\) that we did not consider before, where \(\frac{p}{q}\) refers to an irreducible fraction.

#### 3.1. Case 1: one arbitrary \(\beta\) and one arbitrary \(\alpha\)

As the first case, we consider the solution for an open XXZ quantum spin chain with nondiagonal terms, where the arbitrary boundary parameters consist of one of the \(\beta\)'s and one of the \(\alpha\)'s from the \(\{\alpha_{\pm}, \beta_{\pm}\}\) set. The remaining ones are fixed, e.g. if \(\beta_-\), \(\alpha_+\) are arbitrary, then \(\beta_+ = \eta\), \(\alpha_+ = \frac{i\pi}{2}\) or other similar combinations. We set \(\theta_- = \theta_+ = \theta\), where \(\theta\) is arbitrary. The functional relation method used in this paper was proposed by Nepomechie in [3,4] to solve the spin-1/2 case, which in turn was used in [26] to study the spin-s case for \(\eta = \frac{i\pi}{q}\). When the functional relation (2.9) is expressed as the vanishing determinant of a certain matrix \(M\) (following [1]), one finds that (2.9) can be written as,

\[
\det M = 0,
\]

where \(M\) is given by the \(q \times q\) matrix

\[
M =
\begin{pmatrix}
\Lambda \left( \frac{1}{2}, \frac{1}{2} \right)(u) & -h(u) & 0 & \ldots & 0 & -h(-u + p\eta) \\
-h(-u) & \Lambda \left( \frac{1}{2}, \frac{1}{2} \right)(u + p\eta) & -h(u + p\eta) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-h(u + p^2\eta) & 0 & 0 & \ldots & -h(-u - p(p - 1)\eta) & \Lambda \left( \frac{1}{2}, \frac{1}{2} \right)(u + p^2\eta)
\end{pmatrix},
\]

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where \( p + 1 = q \). In the matrix above, successive rows are obtained by simultaneously shifting \( u \rightarrow u + p \eta \) and cyclically permuting the columns to the right provided that there exists a function \( h(u) \) with the following properties

\[
\begin{align*}
  h(u + 2i\pi) &= h(u + 2q\eta) = h(u), \\
  h(u + (q + 1)\eta) &= h(-u - (q + 1)\eta) = \delta(s)(u), \\
  \prod_{j=0}^{q-1} h(u + 2j\eta) + \prod_{j=0}^{q-1} h(-u - 2j\eta) &= f(u).
\end{align*}
\]

Equations (3.3)–(3.5) reduce the problem of finding \( h(u) \) to solving the following quadratic equation in \( z(u) \),

\[
(z(u))^2 - z(u)f(u) + \prod_{j=0}^{q-1} \delta(s)(u + (2j - 1)\eta) = 0,
\]

where

\[
z(u) = \prod_{j=0}^{q-1} h(u + 2j\eta).
\]

Our choice of boundary parameters as mentioned at the beginning of this section makes the discriminants of the corresponding quadratic equations to be perfect squares. Thus the factorizations such as (3.7) can be readily carried out. On the contrary, when all boundary parameters are arbitrary, the discriminant is no longer a perfect square, and factoring the result becomes a formidable challenge. Solving the quadratic equation (3.6) for \( z(u) \), after making use of (2.8) and (2.12)–(2.16), we obtain the following for \( h(u) \),

\[
h(u) = (-1)^{2sN} 4 \left[ \prod_{k=0}^{2s-1} \text{sh}(u + (s - k + 1/2)\eta) \right]^{2N} \frac{\text{sh}(2u + 2\eta)}{\text{sh}(2u + \eta)} \times \text{chu} \text{ch}(u - \eta)(\text{sh}u + (-1)^{2sN} \text{i} \chi \beta \text{sh}(u - \alpha) \frac{\text{ch}(\frac{1}{2}(u + \alpha + \eta))}{\text{ch}(\frac{1}{2}(u - \alpha - \eta))}
\]

for odd integer values of \( r \) and

\[
h(u) = -4 \left[ \prod_{k=0}^{2s-1} \text{sh}(u + (s - k + 1/2)\eta) \right]^{2N} \frac{\text{sh}(2u + 2\eta)}{\text{sh}(2u + \eta)} \times \text{chu} \text{ch}(u - \eta)(\text{sh}u - \text{i} \chi \beta \text{sh}(u - \alpha) \frac{\text{ch}(\frac{1}{2}(u + \alpha + \eta))}{\text{ch}(\frac{1}{2}(u - \alpha - \eta))}
\]

for even integer values of \( r \).

The structure of the matrix \( \mathcal{M} (3.2) \), suggests that its null eigenvector has the form \( (Q(u), Q(u + (q - 1)\eta), \ldots, Q(u + (q - 1)^2 \eta)) \), where \( Q(u) \) has the periodicity property

\[
Q(u + 2i\pi) = Q(u).
\]

The transfer matrix eigenvalues for the case considered here are therefore given by

\[
\Lambda^{(1/2\cdot s)}(u) = h(u) \frac{Q(u + (q - 1)\eta)}{Q(u)} + h(-u + (q - 1)\eta) \frac{Q(u - (q - 1)\eta)}{Q(u)}.
\]

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We recall that in [26], where \( \eta = i \pi / q \) for the same choice of boundary parameters, the form (3.11) was found to hold as is the case here when \( \eta = i \pi / 2 \). The \( h(u) \) function for any odd integer values of \( r \) given by (3.8), is the same as the one found in [26], where only \( r = 1 \) case was considered. However, from (3.9), we see that the \( h(u) \) function for even integer values of \( r \) differ a little from the one given by (3.8), with the former lacking the spin dependent term \( (2sN) \), while the latter is not. This is due to the fact that the \( f(u) \) function (see (2.12)–(2.16)) for these cases are indeed different as well.

The function \( Q(u) \) however has the same structure as in [26] and is given by,

\[
Q(u) = \prod_{j=1}^{M} \sinh \left( \frac{1}{2} (u - u_j) \right) \sinh \left( \frac{1}{2} (u + u_j - (q-1)\eta) \right),
\]

(3.12)

where

\[
M = 2sN + q - 1,
\]

(3.13)

It can be noted that \( Q(u) \) has the \( 2i\pi \)-periodicity and the crossing symmetry, \( Q(-u + (q-1)\eta) = Q(u) \). \( \{u_j\} \) are the Bethe roots which are also the zeros of \( Q(u) \). We remark that \( \alpha \) can be any one from \( \{\alpha_+\} \) and \( \beta \) can be any one from \( \{\beta_\pm\} \). For rescaling purpose, if one divides (3.11) by the factor \( g^{(1,s)}(u)^{2N} = \prod_{k=1}^{2s-1} \sinh(u + (s-k + 1/2)\eta) \), (3.11) assumes the following familiar form (see also [26]) in terms of the eigenvalues \( \tilde{\Lambda}^{(1,s)}(u) = \frac{\Lambda^{(1,s)}(u)}{g^{(1,s)}(u)^{2N}} \).

\[
\tilde{\Lambda}^{(1,s)}(u) = \tilde{h}(u) \frac{Q(u + (q-1)\eta)}{Q(u)} + \tilde{h}(-u + (q-1)\eta) \frac{Q(u - (q-1)\eta)}{Q(u)},
\]

(3.14)

where now

\[
\tilde{h}(u) = (-1)^{2sN} 4 \sinh^{2N}(u + (s + \frac{1}{2})\eta) \frac{\sinh(2u + 2\eta)}{\sinh(2u + \eta)}
\]

\[
\times \cosh(u - \eta)(\sinh(u + (s+1)\eta) \sinh(u - \alpha) \frac{\cosh(\frac{1}{2}(u + \alpha + \eta))}{\cosh(\frac{1}{2}(u - \alpha - \eta))}
\]

(3.15)

for odd integer values of \( r \) and

\[
\tilde{h}(u) = -4 \sinh^{2N}(u + (s + \frac{1}{2})\eta) \frac{\sinh(2u + 2\eta)}{\sinh(2u + \eta)}
\]

\[
\times \cosh(u - \eta)(\sinh(u + (s+1)\eta) \sinh(u - \alpha) \frac{\cosh(\frac{1}{2}(u + \alpha + \eta))}{\cosh(\frac{1}{2}(u - \alpha - \eta))}
\]

(3.16)

for even integer values of \( r \). The analyticity of \( \tilde{\Lambda}^{(1,s)}(u) \) gives the following Bethe ansatz equations,

\[
\frac{\tilde{h}(u_j)}{h(-u_j + (q-1)\eta)} = - \frac{Q(u_j - (q-1)\eta)}{Q(u_j + (q-1)\eta)}, \quad j = 1, \ldots, M.
\]

(3.17)

The above solution is confirmed numerically for small values of \( N \) and \( q \) for \( s = 1/2, 1 \) and \( 3/2 \). Finally, we remark that the \( h(u) \) given by (3.8) and (3.9) and therefore the \( \tilde{h}(u) \) given by (3.15) and (3.16) are found largely by trial and error and are not the only solutions. Other solutions for \( h(u) \) exist for which the number of Bethe roots \( M \) may also be different. The \( h(u) \) functions given here yield the smallest \( M \) value among other functions we tested. We find this beneficial in facilitating numerical works.
3.2. Case 2: arbitrary $\alpha_+$ and $\alpha_-$

In this section, we shall take $\alpha_+$ and $\alpha_-$ to be arbitrary while setting $\beta_+ = \eta, \beta_- = \theta_-$ = arbitrary. Our choice of boundary parameters for the present case, like for case 1, will make the discriminants of the corresponding quadratic equations (3.6) perfect squares so that the factorization involving the function $h(u)$ as in (3.7) can be readily accomplished. As in case 1, we rely on the functional relations (2.9) satisfied by $\Lambda$ squares so that the factorization involving the function $H(u)$ here in (3.18) and (3.21), despite only being slightly different to the one given in [26], is more suitable as far as each energy level.

In this section, we shall take $\alpha_+$ for even integer values of $r$ and $\alpha_-$ for odd integer values of $r$ and the structure of matrix $M$ suggests the same form for its null eigenvector as for the previous case. Consequently, the $Q(u)$ function is identical in form to (3.12) but with a different $M^1$, namely

$$M = 2sN + q + 1.$$  

(3.20)

The $T - Q$ equation and the Bethe ansatz equations are therefore given by (3.11) (and by (3.14) after the usual rescaling) and (3.17) respectively, where in (3.14) and (3.17), the function $\tilde{h}(u)$ is given by,

$$\tilde{h}(u) = (-1)^{2sN} 4 \left[ \prod_{k=0}^{2s-1} \text{sh}(u + (s - k + \frac{1}{2})\eta) \right]^{2N} \frac{\text{sh}(2u + 2\eta)}{\text{sh}(2u + \eta)}$$

$$\times \text{ch}(u + \eta) \text{ch}(u - \eta) \text{sh}(u + (1)^{2sN} \alpha_+) \text{sh}(u - \alpha_-)$$

$$\times \frac{\text{ch} \left( \frac{1}{2} (u + \alpha_- + \eta) \right) \text{ch} \left( \frac{1}{2} (u + \alpha_+ + \eta) \right)}{\text{ch} \left( \frac{1}{2} (u - \alpha_- - \eta) \right) \text{ch} \left( \frac{1}{2} (u - \alpha_+ - \eta) \right)}$$

(3.18)

for odd integer values of $r$.

(3.19)

for even integer values of $r$. The structure of matrix $M$ suggests the same form for its null eigenvector as for the previous case. Consequently, the $Q(u)$ function is identical in form to (3.12) but with a different $M^1$, namely

$$M = 2sN + q + 1.$$  

(3.20)

The $T - Q$ equation and the Bethe ansatz equations are therefore given by (3.11) (and by (3.14) after the usual rescaling) and (3.17) respectively, where in (3.14) and (3.17), the function $\tilde{h}(u)$ is given by,

$$\tilde{h}(u) = (-1)^{2sN} 4 \left[ \prod_{k=0}^{2s-1} \text{sh}(u + (s - k + \frac{1}{2})\eta) \right]^{2N} \frac{\text{sh}(2u + 2\eta)}{\text{sh}(2u + \eta)}$$

$$\times \text{ch}(u + \eta) \text{ch}(u - \eta) \text{sh}(u + (1)^{2sN} \alpha_+) \text{sh}(u - \alpha_-)$$

$$\times \frac{\text{ch} \left( \frac{1}{2} (u + \alpha_- + \eta) \right) \text{ch} \left( \frac{1}{2} (u + \alpha_+ + \eta) \right)}{\text{ch} \left( \frac{1}{2} (u - \alpha_- - \eta) \right) \text{ch} \left( \frac{1}{2} (u - \alpha_+ - \eta) \right)}$$

(3.21)

1 The $M$ here is different (smaller) than the corresponding $M$ given in [26]. This suggests that the $h(u)$ presented here in (3.18) and (3.21), despite only being slightly different to the one given in [26], is more suitable as far as numerical works are concerned, since the resulting Bethe ansatz equations give less number of Bethe roots for each energy level.

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for odd integer values of \( r \) and

\[
\tilde{h}(u) = -4\text{sh}^{2N}(u + (s + \frac{1}{2})\eta) \frac{\text{sh}(2u + 2\eta)}{\text{sh}(2u + \eta)} \\
\times \text{ch}(u + \eta) \text{ch}(u - \eta) \text{sh}(u - \alpha_+) \text{sh}(u - \alpha_-) \\
\times \frac{\text{ch} \left( \frac{1}{2}(u + \alpha_+ + \eta) \right)}{\text{ch} \left( \frac{1}{2}(u - \alpha_+ - \eta) \right)} \\
\times \frac{\text{ch} \left( \frac{1}{2}(u + \alpha_- + \eta) \right)}{\text{ch} \left( \frac{1}{2}(u - \alpha_- - \eta) \right) \text{ch} \left( \frac{1}{2}(u - \alpha_- - \eta) \right)}
\]

for even integer values of \( r \). As before, we see the presence of the \( 2sN \) term in \( h(u) \) and \( \tilde{h}(u) \) when \( r \) assumes odd integer values. The solution is also confirmed numerically for small values of \( N \) and \( q \) for \( s = 1/2, 1 \) and \( 3/2 \).

### 3.3. Case 3: arbitrary \( \beta_+ \) and \( \beta_- \)

As the final case, we consider the following combination of boundary parameters: \( \beta_\pm \) arbitrary, \( \alpha_\pm = \eta, \theta_+ = \theta_- = \) arbitrary. The corresponding \( \mathcal{M} \) matrix that gives the functional relation (2.9) when its determinant vanishes is,

\[
\mathcal{M} = \begin{pmatrix}
\Lambda(\frac{1}{2},s)(u) & -h(u) & 0 & \ldots & 0 & -h(-u - \eta) \\
-h(-u - (p + 1)\eta) & \Lambda(\frac{1}{2},s)(u + p\eta) & -h(u + p\eta) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-h(u + p^2\eta) & 0 & 0 & \ldots & -h(-u - (p^2 + 1)\eta) & \Lambda(\frac{1}{2},s)(u + p^2\eta)
\end{pmatrix}
\]

(3.22)

where \( p + 1 = q \). This is accomplished (for odd integer \( r \) values) if \( h(u) \) satisfies

\[
h(u + 2i\pi) = h(u + 2q\eta) = h(u), \tag{3.24}
\]

\[
h(u + (q + 1)\eta) h(-u - \eta) = \delta^{(s)}(u), \tag{3.25}
\]

\[
\prod_{j=0}^{q-1} h(u + 2j\eta) + \prod_{j=0}^{q-1} h(-u - (2j + 1)\eta) = f(u). \tag{3.26}
\]

The above conditions yield the following as a solution for \( h(u) \),

\[
h(u) = (-1)^{2sN} \left[ \prod_{k=0}^{2s-1} \text{sh}(u + (s - k + \frac{1}{2})\eta) \right]^{2N} \frac{\text{sh}(2u + 2\eta)}{\text{sh}(2u + \eta)} \times \text{sh}(u - \eta) \text{sh}(u + \eta)(\text{chu} - i\text{sh}\beta_\pm)(\text{chu} + (-1)^{2sN}i\text{sh}\beta_\pm)
\]

(3.27)

The \( T - Q \) equation for the transfer matrix eigenvalues now is given by

\[
\Lambda(\frac{1}{2},s)(u) = h(u) \frac{Q(u + (q - 1)\eta)}{Q(u)} + h(-u - \eta) \frac{Q(u - (q - 1)\eta)}{Q(u)}. \tag{3.28}
\]

Due to the common factor \( g^{(\frac{1}{2},s)}(u)^{2N} \), and using the crossing symmetry \( g^{(\frac{1}{2},s)}(u) = \pm g^{(\frac{1}{2},s)}(-u - \eta) \), the rescaled eigenvalues of \( \tilde{\Lambda}(\frac{1}{2},s)(u) \) are given by

\[
\tilde{\Lambda}(\frac{1}{2},s)(u) = \tilde{h}(u) \frac{Q(u + (q - 1)\eta)}{Q(u)} + \tilde{h}(-u - \eta) \frac{Q(u - (q - 1)\eta)}{Q(u)}, \tag{3.29}
\]

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where
\[ \tilde{h}(u) = (-1)^{2sN} 4 \text{sh}^{2N}(u + (s + \frac{1}{2}) \eta) \text{sh}(2u + 2\eta) \times \text{sh}(u - \eta) \text{sh}(u + \eta)(\text{chu} - i \text{sh} \beta_\text{-})(\text{chu} + (-1)^{2sN} i \text{sh} \beta_\text{+}) \] (3.30)

and as in the previous cases, the form of the null eigenvector of the matrix \( M \) gives the following for the \( Q(u) \) function,
\[ Q(u) = \prod_{j=1}^{M} \text{sh} \left( \frac{1}{2} (u - u_j) \right) \text{sh} \left( \frac{1}{2} (u + u_j + \eta) \right) , \] (3.31)

which satisfies \( Q(u + 2i\pi) = Q(u) \) and \( Q(-u - \eta) = Q(u) \), and
\[ M = 2sN + q - 1 . \] (3.32)

Finally, the analyticity of \( \tilde{A}^{(\frac{1}{2},s)}(u) \) yields the Bethe ansatz equations,
\[ \frac{\tilde{h}(u_j)}{h(-u_j - \eta)} = - \frac{Q(u_j - (q - 1)\eta)}{Q(u_j + (q - 1)\eta)} , \quad j = 1, \ldots, M . \] (3.33)

We stress that the results (3.27)–(3.33) work only for odd integer \( r \) values.

For even integer \( r \) values, the condition \( \det M = 0 \) with \( M \) given in (3.23), does not seem to produce the functional relation (2.9). The difficulty here is to find the function \( h(u) \) that satisfies the properties (3.25) and (3.26). Since our method of finding \( h(u) \) has been largely by trial and error, it is not clear whether an analogous \( h(u) \) can be obtained for this case using the matrix (3.23). Other available choices for \( M \) did not help either.

4. Hamiltonian, energy eigenvalues and numerical results for open spin-\( \frac{1}{2} \) and open spin-1 XXZ quantum spin chains

Here, we provide numerical support for the completeness of the Bethe ansatz solutions given in the previous section. We stress that these numerical studies are not a substitute for a proof for completeness of the solutions, but serve only as numerical verification. More specifically, we compute the energy eigenvalues of both open spin-\( \frac{1}{2} \) and open spin-1 XXZ spin chains. The complete energy levels and the Bethe roots used in the computations are tabulated in tables 1 and 2.

4.1. \( s = 1/2 \) case

The Hamiltonian for the open spin-1/2 XXZ quantum spin chain is given by [28, 29]
\[ H = \frac{1}{2} \sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \text{ch} \eta \ \sigma_n^z \sigma_{n+1}^z) \]
\[ + \frac{1}{2} \text{sh} \eta \left[ \text{cth} \alpha_- \text{th} \beta_- \sigma_1^z + \text{csch} \alpha_- \text{sech} \beta_- (\text{ch} \theta_- \sigma_1^z + i \text{sh} \theta_- \sigma_1^y) \right. \]
\[ - \left. \text{cth} \alpha_+ \text{th} \beta_+ \sigma_N^z + \text{csch} \alpha_+ \text{sech} \beta_+ (\text{ch} \theta_+ \sigma_N^z + i \text{sh} \theta_+ \sigma_N^y) \right] , \] (4.1)

where \( \sigma_x, \sigma_y, \sigma_z \) are the standard Pauli matrices, \( \eta \) is the bulk anisotropy parameter, \( \alpha_\pm, \beta_\pm, \theta_\pm \) are arbitrary boundary parameters, and \( N \) is the number of spins.

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Table 1. The complete set of $2^N$ energy levels and corresponding Bethe roots for $N = 4$, $s = 1/2$, $\eta = i7\pi/5$, $\alpha_- = 0.45i$, $\beta_- = \eta$, $\theta_- = 0.54$, $\alpha_+ = 0.87i$, $\beta_+ = \eta\theta_+$.  

| $E$         | Bethe roots, $\{u_k\}$                                                                 |
|-------------|---------------------------------------------------------------------------------------|
| $4.56711$   | $0.475167 + 0.000593i$, $0.475167 - 1.25723i$, $0.057772 + 1.88496i$, $0.057772 + i\eta$, $-2.19745i$, $-1.70664i$, $-2.68126i$, $0.314088i$, $-0.87i$, $1.57187i$ |
| $4.34568$   | $0.405517 + 0.666815i$, $0.405517 - 1.92345i$, $0.403252 - 0.628319i$, $0.0569468 - 2.70038i$, $0.0569468 + 1.44374i$, $2.36338i$, $-i\eta$, $-1.70664i$, $-2.82866i$, $-0.386637i$ |
| $3.05199$   | $0.693961 - 2.18827i$, $0.693961 + 0.931636i$, $-0.824282i$, $0.45i$, $-0.386637i$, $-1.96144i$, $-1.57051i$, $1.54118i$, $-2.8309i$, $2.80606i$ |
| $2.38474$   | $0.717734 + 0.933002i$, $0.717734 - 2.18964i$, $-0.323985i$, $-2.01785i$, $-1.56819i$, $-0.87i$, $-1.70664i$, $1.57883i$, $2.82555i$, $-2.70265i$ |
| $2.17816$   | $0.722701 - 2.18991i$, $0.722701 + 0.933271i$, $0.317914i$, $-0.949003i$, $-1.70664i$, $-2.81586i$, $2.19787i$, $0.767594i$, $1.44577i$, $-0.386637i$ |
| $0.994085$  | $0.590036 + 2.51327i$, $0.57252 - 0.628319i$, $-0.852939i$, $0.666397i$, $-1.70664i$, $0.312972i$, $1.57986i$, $2.81222i$, $1.5305i$ |
| $0.603975$  | $0.602144 + 2.51327i$, $0.585957 - 0.628319i$, $-1.96477i$, $-0.87i$, $0.45i$, $-0.335719i$, $-1.56682i$, $2.82363i$, $1.58342i$, $1.46666i$ |
| $0.243163$  | $0.609459 + 2.51327i$, $0.594107 - 0.628319i$, $0.322076i$, $-1.70664i$, $-0.952266i$, $-0.386637i$, $0.723371i$, $-2.80224i$, $2.19738i$, $1.46531i$ |
| $1.14152 - 0.195122i$ | $0.35887 - 2.71807i$, $0.330039 + 2.30814i$, $0.276567 + 0.656084i$, $0.087861 - 0.795393i$, $0.027171 - 0.308006i$, $0.015303 - 1.55285i$, $0.001193 + 2.82864i$, $0.000753 - 2.8471i$, $-0.386637i$, $0.45i$ |
| $1.14152 + 0.195122i$ | $0.35887 + 1.46144i$, $0.330039 + 2.71841i$, $0.276567 - 1.91272i$, $0.087861 - 0.461244i$, $0.027171 - 0.948631i$, $0.015303 + 0.296211i$, $0.001193 + 2.19791i$, $0.000753 + 1.59036i$, $0.45i$, $-0.386637i$ |
| $1.6454 - 0.036207i$ | $0.37612 - 2.71959i$, $0.347861 + 2.30928i$, $0.266598 + 0.632143i$, $0.126391 - 0.803703i$, $0.021899 + 1.52824i$, $0.013806 + 0.376725i$, $0.008631 - 0.942847i$, $0.000513 + 2.19931i$, $0.45i$, $-0.386637i$ |
| $1.6454 + 0.036207i$ | $0.37612 + 1.46295i$, $0.347861 + 2.71727i$, $0.266598 - 1.88878i$, $0.126391 - 0.452934i$, $0.021899 - 2.78488i$, $0.013806 - 1.63335i$, $0.008631 - 0.31379i$, $0.000513 + 2.82724i$, $-1.70664i$, $-0.87i$ |
| $1.87399 - 0.362703i$ | $0.382144 - 2.74196i$, $0.357472 + 2.28825i$, $0.228038 + 0.649592i$, $0.144987 - 0.887371i$, $0.120703 + 0.35646i$, $0.03521 - 2.7448i$, $0.000215 - 0.93752i$, $0.00002 + 2.19938i$, $0.45i$, $-0.386637i$ |
| $1.87399 + 0.362703i$ | $0.382144 + 1.48532i$, $0.357472 + 2.7383i$, $0.228038 - 1.90623i$, $0.144987 - 0.369266i$, $0.120703 - 1.61304i$, $0.03521 + 1.48816i$, $0.000215 - 0.319117i$, $0.000022 + 2.82716i$, $-1.70664i$, $-0.386637i$ |
| $3.41127$   | $0.426274 + 0.768867i$, $0.426274 - 2.0255i$, $0.380283 - 2.48686i$, $0.380283 + 1.23022i$, $0.264586 + 2.51327i$, $-0.46653i$, $-0.87i$, $-1.70664i$, $2.19908i$, $-0.942942i$ |
| $5.63582$   | $0.610828 + 2.51327i$, $0.585745 - 0.628319i$, $0.264927 - 2.30695i$, $0.264927 + 1.05031i$, $0.239528 + 2.51327i$, $-0.386637i$, $-0.786041i$, $0.45i$, $-0.313583i$, $2.19908i$ |
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Table 2. The complete set of $3^N$ energy levels and corresponding Bethe roots for $N = 2, s = 1, \eta = i4\pi/7, \alpha_- = i\pi/2, \beta_- = 0.651, \theta_- = 0.386, \alpha_+ = 0.734i, \beta_+ = \eta, \theta_+ = 0.386$.

| $E$                   | Bethe roots, $\{u_k\}$                                                                 |
|-----------------------|----------------------------------------------------------------------------------------|
| $-5.983890$           | $0.705 185 + 1.409 455 \ i, 0.705 185 + 3.078 533 \ i, 0.548 923 - 1.646 975 \ i, \quad$   |
|                       | $0.548 923 - 0.148 219 \ i, 0.210 780 + 3.018 164 \ i, 0.210 780 + 1.469 825 \ i, \quad$   |
|                       | $-0.367 144 \ i, 2.080 271 \ i, -2.080 446 \ i, -2.407 592 \ i \quad$                     |
| $-4.833822 - 0.089904$| $0.565 328 + 1.464 054 \ i, 0.560 227 + 3.079 482 \ i, 0.383 486 - 1.400 808 \ i, \quad$   |
|                       | $0.370 909 + 0.713 516 \ i, 0.359 969 - 2.475 472 \ i, 0.253 186 - 0.363 528 \ i, \quad$   |
|                       | $0.103 171 + 1.549 115 \ i, 0.000 260 + 2.081 055 \ i, 0.000 123 + 0.285 436 \ i, \quad$   |
|                       | $-2.407 592 \ i \quad$                                                                 |
| $-4.833822 + 0.089904$| $0.565 328 + 3.023 935 \ i, 0.560 227 + 1.408 507 \ i, 0.383 486 - 0.394 387 \ i, \quad$   |
|                       | $0.370 909 - 2.508 711 \ i, 0.359 969 + 0.680 276 \ i, 0.253 186 + 1.431 667 \ i, \quad$   |
|                       | $0.103 171 + 2.938 874 \ i, 0.000 260 + 2.406 933 \ i, 0.000 123 - 2.080 631 \ i, \quad$   |
|                       | $0.612 397 \ i \quad$                                                                   |
| $-2.835193 - 0.109209$| $0.577 091 + 1.584 580 \ i, 0.454 273 - 2.807 273 \ i, 0.444 406 + 0.861 744 \ i, \quad$   |
|                       | $0.443 204 - 0.977 039 \ i, 0.343 279 + 2.736 768 \ i, 0.251 255 - 1.788 773 \ i, \quad$   |
|                       | $0.016 001 + 0.153 832 \ i, 0.011 279 + 1.949 951 \ i, 0.001 045 + 0.732 900 \ i, \quad$   |
|                       | $-2.407 592 \ i \quad$                                                                   |
| $-2.835193 + 0.109209$| $0.577 091 + 2.903 409 \ i, 0.454 273 + 1.012 077 \ i, 0.444 406 - 2.656 940 \ i, \quad$   |
|                       | $0.443 204 - 0.818 156 \ i, 0.343 279 + 1.751 220 \ i, 0.251 255 - 0.006 423 \ i, \quad$   |
|                       | $0.016 001 - 1.949 027 \ i, 0.011 279 + 2.538 038 \ i, 0.001 045 - 2.528 096 \ i, \quad$   |
|                       | $0.612 397 \ i \quad$                                                                   |
| $-1.859189 - 0.040090$| $0.624 365 + 3.096 684 \ i, 0.613 412 + 1.442 585 \ i, 0.469 863 - 1.439 936 \ i, \quad$   |
|                       | $0.313 319 - 0.296 575 \ i, 0.171 303 + 1.588 756 \ i, 0.025 225 - 2.380 049 \ i, \quad$   |
|                       | $0.019 867 + 2.114 178 \ i, 0.019 825 + 0.318 497 \ i, 0.003 054 + 0.717 989 \ i, \quad$   |
|                       | $-2.407 592 \ i \quad$                                                                   |
| $-1.859189 + 0.040090$| $0.624 365 + 1.391 305 \ i, 0.613 412 + 3.045 404 \ i, 0.469 863 - 0.355 259 \ i, \quad$   |
|                       | $0.313 319 - 1.498 621 \ i, 0.171 303 + 2.899 233 \ i, 0.025 225 + 0.584 854 \ i, \quad$   |
|                       | $0.019 867 + 2.373 811 \ i, 0.019 825 - 2.113 692 \ i, 0.003 054 - 2.513 185 \ i, \quad$   |
|                       | $-2.407 592 \ i \quad$                                                                   |
| $-0.818351 - 0.180442$| $0.607 702 + 1.556 139 \ i, 0.466 814 - 3.064 030 \ i, 0.444 104 + 0.768 879 \ i, \quad$   |
|                       | $0.415 149 - 1.265 817 \ i, 0.343 162 - 2.450 324 \ i, 0.109 869 + 0.650 801 \ i, \quad$   |
|                       | $0.056 399 + 2.447 113 \ i, 0.055 560 - 2.041 603 \ i, 0.010 588 - 2.518 368 \ i, \quad$   |
|                       | $-2.407 592 \ i \quad$                                                                   |
| $-0.818351 + 0.180442$| $0.607 702 + 2.931 849 \ i, 0.466 814 + 1.268 834 \ i, 0.444 104 - 2.564 075 \ i, \quad$   |
|                       | $0.415 149 - 0.529 378 \ i, 0.343 162 + 0.655 129 \ i, 0.109 869 - 2.445 997 \ i, \quad$   |
|                       | $0.056 399 + 2.040 875 \ i, 0.055 561 + 0.246 407 \ i, 0.010 588 + 0.723 172 \ i, \quad$   |
|                       | $0.612 397 \ i \quad$                                                                   |

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For illustration purpose, we compute the energy eigenvalues of (4.1) for a particular case where the two arbitrary boundary parameters are $\alpha_-$ and $\alpha_+$, making use of the results found in section 3.2. The steps here can be repeated for any other desired combinations of boundary parameters. The Hamiltonian (4.1) is related to the first derivative of the transfer matrix, $\tilde{t}(\frac{j}{2}, \frac{j}{2})(u)$ [27],

$$\mathcal{H} = c_1^{(j)} \frac{d}{du} \tilde{t}(\frac{j}{2}, \frac{j}{2})(u) \bigg|_{u=0} + c_2^{(j)} \mathbb{I}, \quad (4.2)$$

where

$$c_1^{(j)} = -\frac{1}{16\alpha_- \alpha_+ \sinh \beta_- \sinh \beta_+ \sinh 2^{N-1} \eta \cosh \eta},$$
$$c_2^{(j)} = -\frac{\sinh^2 \eta + N \cosh^2 \eta}{2 \cosh \eta}, \quad (4.3)$$

and $\mathbb{I}$ is the identity matrix. For $s = 1/2$, $t(\frac{j}{2}, \frac{j}{2})(u) = \tilde{t}(\frac{j}{2}, \frac{j}{2})(u)$. Moreover, (4.2) implies that the energy eigenvalues are given by

$$E = c_1^{(j)} \frac{d}{du} \tilde{\Lambda}(\frac{j}{2}, \frac{j}{2})(u) \bigg|_{u=0} + c_2^{(j)}, \quad (4.4)$$

Hence, using the results (3.12)–(3.14) and (3.21)$^2$, one arrives at the following result for the energy eigenvalues in terms of Bethe roots $\{u_j\}$,

$$E = \frac{1}{2} \sinh \eta \cosh \eta \sum_{j=1}^{M} \frac{1}{\sinh (\frac{1}{2} u_j) \cosh (\frac{1}{2} (u_j + \eta))} + \frac{1}{2} N \cosh \eta - \frac{1}{2} \cosh \eta$$
$$+ \frac{1}{2} \sinh \left( - \coth \alpha_- + (-1)^N \coth \alpha_+ - (-1)^N \tanh \left( \frac{\alpha_+ - (-1)^N \eta}{2} \right) \right)$$
$$+ \tanh \left( \frac{\alpha_- + \eta}{2} \right)), \quad (4.5)$$

where $M = N + q + 1$. The energy eigenvalues computed from the Bethe roots using (4.5) for $N = 4$, which are tabulated in table 1, coincide with those obtained from direct diagonalization of (4.1).

### 4.2. $s = 1$ case

As for the spin-1/2 case, a brief review of the open spin-1 XXZ quantum spin chain is desirable at this point. The Hamiltonian is given by

$$\mathcal{H} = \sum_{n=1}^{N-1} H_{n,n+1} + H_b, \quad (4.6)$$

where the bulk term $H_{n,n+1}$ [35] and the boundary term $H_b$ [20, 34] are given by,

$$H_{n,n+1} = \sigma_n - (\sigma_n)^2 + 2 \sinh^2 \eta \left[ \sigma_n + (S_n^z)^2 + (S_{n+1}^z)^2 - (\sigma_n^z)^2 \right] - 4 \sinh^2 \left( \frac{\eta}{2} \right) \left( \sigma_n^+ \sigma_n^- + \sigma_n^z \sigma_n^z \right), \quad (4.7)$$

$^2$ The expression (4.5) is derived for odd integer values of $r$. Similar expression for even integer values of $r$ can also be derived.
and
\[ H_b = a_1(S^z_1)^2 + a_2S^z_1 + a_3(S^+_1)^2 + a_4(S^-_1)^2 + a_5S^+_1 S^-_1 + a_6S^z_1 S^-_1 \]
+ a_7S^z_1 S^+_1 + a_8S^-_1 S^+_1 + (a_j \leftrightarrow b_j \text{ and } 1 \leftrightarrow N), \tag{4.8} \]
respectively. In (4.7) and (4.8), the following definitions are used,
\[ \sigma_n = \vec{S}_n \cdot \vec{S}_{n+1}, \quad \sigma_n^+ = S^x_n S^y_{n+1} \pm S^y_n S^x_{n+1}, \quad \sigma_n^z = S^z_n S^z_{n+1}, \tag{4.9} \]
where \( \vec{S} \) are the \( su(2) \) spin-1 generators and \( S^\pm = S^x \pm iS^y \). The coefficients \( \{a_i\} \) in the boundary terms at site 1 are functions of the boundary parameters \((\alpha_-, \beta_-, \theta_-)\) and the bulk anisotropy parameter \( \eta \). They are given by,
\[
\begin{align*}
a_1 &= \frac{1}{4} a_0 \left( \text{ch} 2\alpha_- - \text{ch} 2\beta_- + \text{ch} \eta \right) \text{sh} 2\eta \text{sh} \eta, \\
a_2 &= \frac{1}{4} a_0 \text{sh} 2\alpha_- \text{sh} 2\beta_- \text{sh} 2\eta, \\
a_3 &= -\frac{1}{8} a_0 e^{2\eta} \text{sh} 2\eta \text{sh} \eta, \\
a_4 &= -\frac{1}{8} a_0 e^{-2\eta} \text{sh} 2\eta \text{sh} \eta, \\
a_5 &= a_0 e^{\eta} \left( \text{ch} \beta_- \text{sh} \alpha_- \text{ch} \eta \frac{\eta}{2} + \text{ch} \alpha_- \text{sh} \beta_- \text{sh} \eta \frac{\eta}{2} \right) \text{sh} \eta \text{ch} \frac{3}{2} \eta, \\
a_6 &= a_0 e^{-\eta} \left( \text{ch} \beta_- \text{sh} \alpha_- \text{ch} \eta \frac{\eta}{2} + \text{ch} \alpha_- \text{sh} \beta_- \text{sh} \eta \frac{\eta}{2} \right) \text{sh} \eta \text{ch} \frac{3}{2} \eta, \\
a_7 &= -a_0 e^{\eta} \left( \text{ch} \beta_- \text{sh} \alpha_- \text{ch} \eta \frac{\eta}{2} - \text{ch} \alpha_- \text{sh} \beta_- \text{sh} \eta \frac{\eta}{2} \right) \text{sh} \eta \text{ch} \frac{3}{2} \eta, \\
a_8 &= -a_0 e^{-\eta} \left( \text{ch} \beta_- \text{sh} \alpha_- \text{ch} \eta \frac{\eta}{2} - \text{ch} \alpha_- \text{sh} \beta_- \text{sh} \eta \frac{\eta}{2} \right) \text{sh} \eta \text{ch} \frac{3}{2} \eta, \tag{4.10} \end{align*} \]
where
\[
a_0 = \left[ \text{sh} \left( \alpha_- - \frac{\eta}{2} \right) \text{sh} \left( \alpha_- + \frac{\eta}{2} \right) \text{ch} \left( \beta_- - \frac{\eta}{2} \right) \text{ch} \left( \beta_- + \frac{\eta}{2} \right) \right]^{-1}. \tag{4.11} \]

Similarly, the coefficients \( \{b_i\} \) at site \( N \) are functions of the boundary parameters \((\alpha_+, \beta_+, \theta_+)\) and \( \eta \), are given by the following correspondence,
\[
b_i = a_i \bigg|_{\alpha_- \to \alpha_+, \beta_- \to \beta_+, \theta_- \to \theta_+}. \tag{4.12} \]

Next, we proceed to find an expression for the eigenvalues of the Hamiltonian (4.6) for the case considered in section 3.1, namely with two arbitrary boundary parameters, one from \( \{\alpha_\pm\} \) and the other from \( \{\beta_\pm\} \), e.g. \( \{\alpha_+, \beta_+\} \), etc. We set \( \theta_- = \theta_+ = \theta \), where \( \theta \) is arbitrary. The anisotropy parameter \( \eta \) is set to be \( \eta = \frac{\pi \xi}{q} \). The energy eigenvalues in terms of the rescaled transfer matrix eigenvalues \( \tilde{\Lambda}^{(1,1)}(u) \) is given by,
\[
E = c_1^{(1)} \frac{d}{du} \tilde{\Lambda}^{(1,1)}(u) \bigg|_{u=0} + c_2^{(1)}, \tag{4.13} \]
where
\[
c_1^{(1)} = \text{ch} \eta \left( 16[\text{sh} 2\eta \text{sh} \eta]^{2N} \text{sh} 3\eta \text{sh} \left( \alpha_- - \frac{\eta}{2} \right) \text{sh} \left( \alpha_- + \frac{\eta}{2} \right) \text{ch} \left( \beta_- - \frac{\eta}{2} \right) \text{ch} \left( \beta_- + \frac{\eta}{2} \right) \right. \]
\[
\times \left. \text{sh} \left( \alpha_+ - \frac{\eta}{2} \right) \text{sh} \left( \alpha_+ + \frac{\eta}{2} \right) \text{ch} \left( \beta_+ - \frac{\eta}{2} \right) \text{ch} \left( \beta_+ + \frac{\eta}{2} \right) \right) \bigg|^{-1}, \tag{4.14} \]
\[
doi:10.1088/1742-5468/2015/02/P02001 \]
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\[ c_2^{(1)} = -\frac{a_0}{4} b \chi_\eta - (N - 1)(4 + \chi_2 \eta) + 2N \chi_4 \eta \]

\[ -\frac{\sin \eta}{2d} \left\{ -2 \chi_2 \alpha_+ \left( \chi_\eta (3 + 7 \chi_2 \eta + \chi_4 \eta) + \chi_2 \beta_+ (4 + 5 \chi_2 \eta + 2 \chi_4 \eta) \right) \right. \]

\[ + 2 \chi_\eta \left( \chi_2 \beta_+ (3 + 7 \chi_2 \eta + \chi_4 \eta) + \chi_\eta (5 + 3 \chi_2 \eta + 3 \chi_4 \eta) \right) \}

\[ - \frac{\sin 2 \eta}{2d} \left\{ \chi_2 \beta_+ (2 + 4 \chi_3 \eta) + \chi_\eta (5 \chi_2 \eta + \chi_4 \eta) - 2 \chi_2 \alpha_+ (1 + \chi_2 \eta) \right. \]

\[ + \chi_2 \beta_+ (3 + \chi_2 \eta + 3 \chi_4 \eta) \} \cdot \] (4.15)

In (4.15), \( b \) and \( d \) are given by

\[ b = 2 \left( -\chi_2 \beta_- - \chi_3 \eta + \chi_2 \alpha_- (1 + \chi_2 \beta_- \chi_\eta) \right) \] (4.16)

and

\[ d = -4 \sin 3 \eta \sin \left( \alpha_+ + \frac{\eta}{2} \right) \sin \left( \alpha_+ - \frac{\eta}{2} \right) \sin \left( \beta_+ + \frac{\eta}{2} \right) \sin \left( \beta_+ - \frac{\eta}{2} \right) \] (4.17)

The rescaled spin-1 transfer matrix eigenvalues are given by\(^3\)

\[ \tilde{\Lambda}^{(1,1)}(u) = \gamma \Lambda^{(1,1)}(u) \] (4.18)

where \( \gamma = \frac{\sin (2u) \sin (2u + 2 \eta)}{\sin \sin (+ \eta) 2N} \) and \( \Lambda^{(1,1)}(u) \) is given by the result from fusion hierarchy (2.7). The analytic form of the energy eigenvalues in terms of Bethe roots \( \{ u_k \} \) then follows from (4.13),

\[ E = \frac{1}{2} \sin (2 \eta) \sin (\eta) \sum_{k=1}^{M} \sin \left( \frac{1}{2} (u_k + \frac{3 \eta}{2}) \right) \sin \left( \frac{1}{2} (u_k - \frac{\eta}{2}) \right) + c_1^{(1)}(A'(0) + B'(0) - C'(0)) + c_2^{(1)}, \] (4.19)

where

\[ A(u) = \tilde{z} \left( u + \frac{\eta}{2} \right) \tilde{z} \left( u - \frac{\eta}{2} \right) \]

\[ B(u) = \tilde{z} \left( -u + \left( q - \frac{1}{2} \right) \eta \right) \tilde{z} \left( u + \frac{\eta}{2} \right) \]

\[ C(u) = -\gamma \delta^{(1)} \left( u - \frac{\eta}{2} \right) \]

\[ \tilde{z}(u) = \sin (2u + \eta) \tilde{h}(u). \] (4.20)

We recall that \( M = 2N + q - 1 \). The expression (4.19) is derived here for even positive integer values of \( r \). Similar result can be derived when \( r \) assumes odd positive integers. This can then be used in the computation of complete energy levels from the Bethe roots given by (3.17). We tabulate the energies computed from the Bethe roots \( \{ u_k \} \), using (4.19) for some selected values of \( N, q, r \) (therefore \( \eta \)) and the boundary parameters \( \{ \alpha_\pm, \beta_\pm, \theta_\pm \} \) in table 2. These numerical results provide support to and illustrate the completeness of the Bethe ansatz equations, (3.17). We have verified that the energies given in table 2 coincide with those obtained from direct diagonalization of the open spin-1 XXZ chain Hamiltonian (4.6).

\(^3\) Following [20], the rescaled factor \( \gamma \) is introduced.
5. Conclusion

Bethe ansatz solutions of an open spin-$s$ XXZ quantum spin chain with nondiagonal boundary terms, derived from certain functional relations which the ‘fundamental’ transfer matrices, $t(\frac{1}{2}, s)(u)$ obey at roots of unity are revisited. The solutions given here include a wider class of anisotropic parameter $\eta$, namely $\eta = i\pi\frac{r}{q}$, where $r$ and $q$ are positive integers with $q$ assuming the odd integer values and $\frac{r}{q}$ corresponds to irreducible fractions. As far as we know, Bethe ansatz solution for such a case for open spin-$s$ XXZ chain has not been reported, except for the $s = \frac{1}{2}$ case in [18]. These considerations have motivated the present work. The solutions given here are for cases with any two arbitrary boundary parameters from the $\{\alpha_{\pm}, \beta_{\pm}\}$ set, while the remaining ones are fixed to some values. These solutions have been checked numerically for chains of length up to $N = 5$. Numerical support for the completeness of the Bethe ansatz solutions (using $s = 1/2$ and $s = 1$ as examples) are provided in tables 1 and 2, where all $(2s + 1)^N$ eigenvalues are given.

There remain problems that are worth investigating. It would be interesting to similarly investigate the solutions of the open quantum spin chain with alternating spins with nondiagonal boundary terms. One could also attempt to study the thermodynamics of such a model. Furthermore, Bethe ansatz solutions for open spin-$1$ XXZ quantum spin chain can be used to investigate the supersymmetric sine-Gordon model with boundaries via their nonlinear integral equations. It would be desirable to use the solution presented here to carry out such an analysis. We hope to address these questions in future.

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