Abstract

Filtering problems with general exponential quadratic criteria are investigated for Gauss-Markov processes. In this setting, the Linear Exponential Gaussian and Risk-Sensitive filtering problems are solved and it is shown that they may have different solutions.

Key words. Gauss-Markov process, optimal filtering, risk-sensitive filtering, exponential criteria, Riccati equation

AMS subject classifications. Primary 60G15. Secondary 60G44, 62M20.
1 Introduction

The so-called linear exponential Gaussian (LEG) and risk-sensitive (RS) filtering problems involve criteria which are exponentials of integral cost functionals. Before our paper [5], numerous results had been already reported in specific models, specially around Markov models, but without exhibiting the relationship between these two problems. See, e.g., Whittle [9], Speyer et al. [7], Elliott et al. [2], [3] and [4] for contributions. In our paper [5], we have solved the LEG and RS filtering problems for general Gaussian processes in the particular setting where the functional in the exponential is a singular quadratic functional. Moreover we have proved that actually in this case the solutions coincide. In the present paper the problems are revisited for Gauss-Markov processes but with a nonsingular quadratic functional in the exponential. In this setting the solutions are exhibited and we propose an example to show that they may be different.

It what follows all random variables and processes are defined on a given stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) satisfying the usual conditions and processes are \((\mathcal{F}_t)\)-adapted. We deal with a signal process \(X = (X_t, t \geq 0)\) in \(\mathbb{R}\) governed by the linear equation
\[
dX_t = a_t X_t \, dt + dB_t, \quad X_0 = 0,
\]
and an observation process \(Y = (Y_t, t \geq 0)\) in \(\mathbb{R}\) governed by the linear equation
\[
dY_t = A_t X_t \, dt + d\tilde{B}_t, \quad Y_0 = 0, \quad t \geq 0.
\]
Here \(a = (a_t, t \geq 0)\) and \(A = (A_t, t \geq 0)\) are continuous real-valued deterministic functions, \(B = (B_t, t \geq 0)\) and \(\tilde{B} = (\tilde{B}_t, t \geq 0)\) are independent standard one dimensional Brownian motions. Clearly the pair \((X, Y)\) is Gaussian.

For a given continuous deterministic function \(\Lambda = (\Lambda_s, 0 \leq s \leq T)\) with values in the set of nonnegative definite symmetric \(2 \times 2\) matrices
\[
\Lambda_s = \begin{pmatrix}
\Lambda_{11}(s) & \Lambda_{12}(s) \\
\Lambda_{12}(s) & \Lambda_{22}(s)
\end{pmatrix},
\]
such that \(\Lambda_{22}(s) \neq 0\), let us define by \(\bar{h} \in \mathcal{H}\) the solution of the LEG type filtering problem:
\[
\bar{h} = \arg\min_{h \in \mathcal{H}} \mathbb{E} \left[ \mu \exp \left\{ \frac{\mu}{2} \int_0^T (X_s h_s) \Lambda_s \left( \begin{array}{c} X_s \\ h_s \end{array} \right) \, ds \right\} \right].
\]
In this definition \( \mu \) is a real parameter and \( h = (h_s, 0 \leq s \leq T) \in \mathcal{H} \) means that \( h \) is a \((\mathcal{Y}_s)\)-adapted continuous process where \((\mathcal{Y}_s)\) is the natural filtration of \( Y \), i.e., \( \mathcal{Y}_s = \sigma(\{Y_u, 0 \leq u \leq s\}), 0 \leq s \leq T \).

We can also define \( \hat{h} \) as a solution of the following recursive equation, which is the basic definition of the \( RS \) type filtering problem:

\[
\hat{h}_t = \text{argmin}_{g \in \mathcal{Y}_t} \mathbb{E} \left[ \mu \exp \left\{ \frac{\mu}{2} (X_t, g) \Lambda_t \left( \frac{X_t}{g} \right) + \frac{\mu}{2} \int_0^t (X_s \hat{h}_s) \Lambda_s \left( \frac{X_s}{\hat{h}_s} \right) ds \right\} / \mathcal{Y}_t \right] ,
\]

where \( g \in \mathcal{Y}_t \) means that \( g \) is a \( \mathcal{Y}_t \)-measurable variable.

It is clear that \textit{risk-neutral} versions of these two problems (namely, dropping the exponentials in definitions (3)-(4), i.e., simply with quadratic criteria) are “equivalent”:

\[
\bar{h}_t = \hat{h}_t = -\frac{\Lambda_{12}(t)}{\Lambda_{22}(t)} \cdot \pi_t(X),
\]

where for any process \( \eta = (\eta_t, t \in [0, T]) \) such that \( \mathbb{E}|\eta_t| < +\infty \), the notation \( \pi_t(\eta) \) is used for the conditional expectation of \( \eta_t \) given the \( \sigma \)-field \( \mathcal{Y}_t \),

\[
\pi_t(\eta) = \mathbb{E}(\eta_t / \mathcal{Y}_t).
\]

One question that we want to discuss in this paper is the possible “equivalence” of the problems (3) and (4). In our paper [5], we have proved that when the quadratic functional involved in the exponential is \textit{singular}, namely when matrices \( \Lambda_s \) are singular, i.e., \( \Lambda_{11} = \Lambda_{22} = -\Lambda_{12} \), the equality \( \bar{h} = \hat{h} \) holds, even in a non Markovian setting. Here below a simple example where \( \bar{h} \neq \hat{h} \) is proposed which shows that if the quadratic functional is \textit{nonsingular} then the answer may be negative even for the Markovian model (1)-(2).

The paper is organized as follows. Preparing for the analysis of the filtering problems, in Section 2 a Cameron-Martin type formula for the \textit{conditional Laplace transform} of a quadratic functional of the involved signal process is derived. Then in Section 3 the LEG and RS filtering problems in the nonsingular setting are solved. Finally, Section 4 is devoted to the analysis of the announced example which shows the discrepancy between the two filtering problems.
2 Conditional version of a Cameron-Martin formula

Actually, the resolution of the LEG and RS filtering problems is based on a conditional version of a Cameron-Martin formula and we follow the same lines as in our paper [5].

In the present Section, the process $X = (X_t, t \geq 0)$ is an arbitrary continuous Gaussian process with mean function $m = (m_t, t \geq 0)$ and covariance function $K = (K(t, s), t \geq 0, s \geq 0)$, i.e.,

$$\mathbb{E}X_t = m_t, \quad \mathbb{E}(X_t - m_t)(X_s - m_s) = K(t, s), \quad t \geq 0, \ s \geq 0.$$

We are interested in the explicit representation of

$$\mathcal{I}_T = \mathbb{E}\left[ \mu \exp \left\{ \frac{\mu}{2} (X_T g) M \left( \frac{X_T}{g} \right) + \frac{\mu}{2} \int_0^T (X_s h_s) \Lambda_s \left( \frac{X_s}{h_s} \right) ds \right\} / \mathcal{Y}_T \right],$$

for any variable $g \in \mathcal{Y}_T$ and process $h \in \mathcal{H}$, and with symmetric deterministic nonnegative definite matrices $M$ and $\Lambda_s$.

Let us formulate the condition $(C_{\mu})$:

$(C_{\mu})$ the Riccati-Volterra equation

$$\bar{\gamma}(t, s) = K(t, s) - \int_0^s \bar{\gamma}(t, r)[A_r^2 - \mu \Lambda_{11}(r)]\bar{\gamma}(s, r)dr, \quad 0 \leq s \leq t \leq T,$$

has a unique and bounded solution on $\{(t, s) : 0 \leq s \leq t \leq T\}$, such that $\bar{\gamma}(t, t) \geq 0$ for $0 \leq t \leq T$ and moreover

$$1 - \mu M_{11}\bar{\gamma}(T, T) > 0.$$

Notice that for all $\mu$ negative the condition $(C_{\mu})$ is satisfied and if $\mu$ is positive, the condition $(C_{\mu})$ is satisfied for $\mu$ sufficiently small, for example, those such that for any $t \leq T$ $A_t^2 - \mu \Lambda_{11}(t)$ is nonnegative (cf. Lemma 2 [5]).

Now we claim the following extension of the 1–D version of Proposition 2 [5]:


Proposition 1. Suppose that the condition \((C_\mu)\) is satisfied. Let \(Z^h = (Z^h_s, 0 \leq s \leq T)\) be the unique solution of the Itô-Volterra equation
\[
Z^h_t = m_t + \int_0^t \bar{\gamma}(t, s) \mu [\Lambda_{11}(s) Z^h_s + \Lambda_{12}(s) h_s] ds + \int_0^t \bar{\gamma}(t, s) A_s [dY_s - A_s Z^h_s ds],
\]
and \(\bar{\gamma}_{XX}(t) = \bar{\gamma}(t, t), 0 \leq t \leq T\) where \(\bar{\gamma}\) is the unique solution of equation \((6)\). Then the following equality holds:
\[
\mathcal{I}_T = (1 - \mu M_{11} \bar{\gamma}_{XX}(T))^{-1/2} \exp \left\{ \frac{\mu}{2} \int_0^T \bar{\gamma}_{XX}(s) \Lambda_{11}(s) ds \right\} \times
\times \exp \left\{ \frac{\mu}{2} (Z^h_T g) G_T \left( \frac{Z^h_T}{g} \right) + \frac{\mu}{2} \int_0^T (Z^h_s h_s) \Lambda_s \left( \frac{Z^h_s}{h_s} \right) ds \right\} \times
\times \exp \left\{ \int_0^T A_s (Z^h_s - \pi_s(X)) d\nu_s - \frac{1}{2} \int_0^T |A_s (Z^h_s - \pi_s(X))|^2 ds \right\}, \quad (8)
\]
where
\[
G_T = (1 - \mu M_{11} \bar{\gamma}_{XX}(T))^{-1} \begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} - \mu \bar{\gamma}_{XX}(T) \det(M) \end{pmatrix}, \quad (9)
\]
and \((\nu_t, t \geq 0)\) is the innovation process associated to \(Y\), i.e.,
\[
d\nu_t = dY_t - A_t \pi_t(X) dt, \quad \nu_0 = 0. \quad (10)
\]
Remark 1. (i) Note that in the singular case where \(M_{11} = M_{22} = -M_{12}\) and \(\Lambda_{11} = \Lambda_{22} = -\Lambda_{12}\) Proposition \(1\) reduces to the \(1 - D\) version of Proposition 2 [5].

(ii) Note also that the condition \((C_\mu)\) implies that \(G_T\) is nonnegative definite.

Proof of the Proposition [1] The proof is based on the ideas developed in the proof of Propositions 1 and 2 [3]. Actually, it is sufficient to work with \(\mu < 0\) since the result will be valid for sufficiently small \(\mu > 0\) because of the analytical properties of the involved functions. To simplify the notations we work with \(\mu = -1\); then for the general situation it is sufficient to replace \(M\) and \(\Lambda\) by \(-\mu M\) and \(-\mu \Lambda\) respectively.
Let us introduce the auxiliary observations \((\bar{Y}_t, 0 \leq t \leq T)\) such that:

\[
\begin{align*}
\{ & d\bar{Y}^1_t = dY_t, \\
& d\bar{Y}^{2,3}_t = dB_t + \Lambda^\frac{1}{2}_t (X_t) dt,
\}
\end{align*}
\]

(11)

where \(\bar{B} = (\bar{B}_t, t \geq 0)\) denotes a 2-D standard Brownian motion, independent of \((X, \tilde{B})\).

Below, for any process \(\eta = (\eta_t, t \in [0, T])\) such that \(\mathbb{E}|\eta_t| < +\infty\), the notation \(\bar{\pi}_t(\eta)\) is used for the conditional expectation of \(\eta_t\) given the auxiliary \(\sigma\)-field \(\bar{\mathcal{Y}}_t = \sigma(\{\bar{Y}_s, 0 \leq s \leq t\})\), \(\bar{\pi}_t(\eta) = \mathbb{E}(\eta_t / \bar{\mathcal{Y}}_t)\).

Let also \(\xi_t\) be defined by

\[
d\xi_t = (X_t h_t) \Lambda^\frac{1}{2}_t d\bar{Y}^{2,3}_t, \quad \xi_0 = 0.
\]

(12)

We see that the conditional distribution of \((X_t, \xi_t)\) given \(\bar{\mathcal{Y}}_t\) is Gaussian with the conditional expectation \((\bar{\pi}_t(X), \bar{\pi}_t(\xi))\) and the conditional covariance

\[
\begin{pmatrix}
\bar{\gamma}_{XX}(t) & \bar{\gamma}_{X\xi}(t) \\
\bar{\gamma}_{X\xi}(t) & \bar{\gamma}_{\xi\xi}(t)
\end{pmatrix},
\]

where

\[
\begin{align*}
\bar{\gamma}_{XX}(t) &= \mathbb{E}[(X_t - \bar{\pi}_t(X))^2 / \bar{\mathcal{Y}}_t], \\
\bar{\gamma}_{X\xi}(t) &= \mathbb{E}[(X_t - \bar{\pi}_t(X))(\xi_t - \bar{\pi}_t(\xi)) / \bar{\mathcal{Y}}_t],
\end{align*}
\]

(13)

and

\[
\bar{\gamma}_{\xi\xi}(t) = \mathbb{E}[(\xi_t - \bar{\pi}_t(\xi))^2 / \bar{\mathcal{Y}}_t].
\]

(14)

Proceeding as in [5] Section 2.2, we obtain that

- the conditional variance \(\bar{\gamma}_{XX}(t)\) is deterministic and actually nothing but the variance of the filtering error, i.e.,

\[
\bar{\gamma}_{XX}(t) = \mathbb{E}[(X_t - \bar{\pi}_t(X))^2],
\]

(15)

given by \(\bar{\gamma}_{XX}(t) = \bar{\gamma}(t, t)\), where \(\bar{\gamma}(t, s)\) is the unique solution of the equation (6) with \(\mu = -1\),

- the difference

\[
Z^h_t = \bar{\pi}_t(X) - \bar{\gamma}_{X\xi}(t)
\]

(16)

is \(\mathcal{Y}_t\)-measurable and is the unique solution of the equation (7) with \(\mu = -1\).
\[ \bar{\pi}_t(\xi) = \int_0^t \bar{\gamma}_{XX}(s)\Lambda_{11}(s)ds + \int_0^t (\bar{\pi}_s(X), h_s)\Lambda_s \left( \begin{array}{c} \bar{\pi}_s(X) \\ h_s \end{array} \right) ds + \int_0^t \left[ (\bar{\pi}_s(X), h_s) + \bar{\gamma}_{XX}(s)(1, 0) \right] \Lambda_s \bar{\nu}_s^{2,3} + \int_0^t \bar{\gamma}_{XX}(s)A_s d\bar{\nu}_s^1, \]  
(17)

- the conditional variance \( \bar{\gamma}_{xx}(t) \) satisfies the equation:

\[ \bar{\gamma}_{xx}(t) = \int_0^t \bar{\gamma}_{XX}(s)\Lambda_{11}(s)ds + 2\int_0^t \bar{\gamma}_{XX}(s)(1, 0)\Lambda_s^{\frac{1}{2}}d\bar{\nu}_s^{2,3} - \int_0^t \bar{\gamma}_{XX}(s)[\Lambda_{11}(s) + A_s^2]\bar{\gamma}_{xx}(s)ds + 2\int_0^t (\bar{\pi}_s(X), h_s)\Lambda_s \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \bar{\gamma}_{XX}(s)ds, \]  
(18)

where \( \bar{\nu}_t = (\bar{\nu}_t^1, [\bar{\nu}_t^{2,3}]') \) is the 3 - D innovation process associated to the auxiliary observations \( \bar{Y} \), i.e.,

\[ d\bar{\nu}_t = d\bar{Y}_t - \begin{pmatrix} A_t & 0 \\ \Lambda_t^{\frac{1}{2}} & h_t \end{pmatrix} \begin{pmatrix} \bar{\pi}_t(X) \\ h_t \end{pmatrix} dt, \bar{\nu}_0 = 0. \]  
(19)

Now we turn to the proof of equality (8) for \( \mu = -1 \).

Let \( \rho_t \) be defined by:

\[ \rho_t = \exp \left\{ -\int_0^t (X_s, h_s)\Lambda_s^{\frac{1}{2}}dB_s - \frac{1}{2} \int_0^t (X_s, h_s)\Lambda_s \left( \begin{array}{c} X_s \\ h_s \end{array} \right) ds \right\}. \]  
(20)

At first we note that the same arguments that we have used in the proof of the Proposition 1 [5] give the equality:

\[ \mathcal{I}_T = \frac{\mathbb{E}[\exp(-\frac{1}{2}(X_T, g)M_{0}^{-1}(X_T, g) - \xi_T)/\mathcal{Y}_T]}{\mathbb{E}[\rho_T/\mathcal{Y}_T]} = \frac{\varphi_1(T)}{\pi_{T}(\rho)}, \]  
(21)

where \( \xi_T \) and \( \rho_T \) are defined by (12) and (20) respectively. But the conditional Gaussian properties of the pair \((X, \xi)\) given \( \mathcal{Y}_T \) gives the following (see for example [6], Lemma 11.6):

\[ \ln \varphi_1(T) = -\frac{1}{2} \ln(1 + M_{11}\bar{\gamma}_{XX}(T)) - \frac{1}{2} (Z_T, g)G_T \left( \begin{array}{c} Z_T \\ g \end{array} \right) - \bar{\pi}_T(\xi) + \frac{1}{2} \bar{\gamma}_{\xi\xi}(T), \]  
(22)
where the terms \( G_T, \bar{\gamma}_{\xi\xi}(T), \bar{\gamma}_{XX}(T) \) and \( Z_T^h \) are defined by the equations (9), (14), (15) and (16) respectively.

Now it follows from (22) that to prove the statement of the Proposition it is sufficient to write the expression for

\[
\Psi_T = \exp \left( -\bar{\pi}_T(\xi) + \frac{1}{2} \bar{\gamma}_{\xi\xi}(T) \right) / \bar{\pi}_T(\rho).
\]

Since

\[
d\rho_t = -\rho_t(X_t h_t)\Lambda_t^{\frac{1}{2}} dB_t, \quad \rho_0 = 1,
\]

thanks to the general filtering theorem [6, Theorem 7.16] we can write

\[
d\bar{\pi}_t(\rho) = \bar{\pi}_t(\rho) \left\{ -(\bar{\pi}_t(X), h_t)\Lambda_t^{\frac{1}{2}} d\nu_t^{2,3} + \left[ \frac{\bar{\pi}_t(\rho X)}{\bar{\pi}_t(\rho)} - \bar{\pi}_t(X) \right] A_t d\nu_t^1 \right\}.
\]

We note that the classical Bayes formula gives that

\[
\frac{\bar{\pi}_t(\rho X)}{\bar{\pi}_t(\rho)} = \pi_t(X).
\]

Hence

\[
d\bar{\pi}_t(\rho) = \bar{\pi}_t(\rho) \left\{ -(\bar{\pi}_t(X), h)\Lambda_t^{\frac{1}{2}} d\nu_t^{2,3} + [\pi_t(X) - \bar{\pi}_t(X)] A_t d\nu_t^1 \right\},
\]

or, equivalently:

\[
\bar{\pi}_T(\rho) = \exp \left\{ -\int_0^T (\bar{\pi}_s(X), h_s)\Lambda_s^{\frac{1}{2}} d\nu_s^{2,3} + \int_0^T [\pi_s(X) - \bar{\pi}_s(X)] A_s d\nu_s^1 
- \frac{1}{2} \int_0^T |A_s [\pi_s(X) - \bar{\pi}_s(X)]|^2 ds - \frac{1}{2} \int_0^T \| (\pi_s(X), h_s)\Lambda_s^{\frac{1}{2}} \|^2 ds \right\}.
\]

The equalities (17), (18) and (24) imply:

\[
\ln(\Psi_t) = -\frac{1}{2} \int_0^T \bar{\gamma}(s, s)\Lambda_{11}(s) ds - \frac{1}{2} \int_0^T (\bar{\pi}_s(X), h_s)\Lambda_s \left( \begin{array}{c} \bar{\gamma}_s(X) \\ h_s \end{array} \right) ds
+ \int_0^T (\bar{\pi}_s(X), h_s)\Lambda_s \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \bar{\gamma}_{XX}(s) ds - \frac{1}{2} \int_0^t \bar{\gamma}_{XX}(s)\Lambda_{11}(s)\bar{\gamma}_{XX}(s) ds.
\]
\begin{align*}
\frac{1}{2} \int_0^T [\pi_s(X) - \bar{\pi}_s(X)] A_s d\bar{\nu}_s^1 \\
-\frac{1}{2} \int_0^T \bar{\pi}_X(s) A^2 \bar{\gamma}_X(s) ds + \frac{1}{2} \int_0^T |A_s [\pi_s(X) - \bar{\pi}_s(X)]|^2 ds.
\end{align*}

Replacing $d\bar{\nu}_1^t$ by $d\bar{\nu}_1^t = d\nu_t + A_t [\pi_t(X) - \bar{\pi}_t(X)]$ we obtain that:

\begin{align*}
\ln(\Psi_t) = & -\frac{1}{2} \int_0^T \bar{\gamma}(s, s) \Lambda(s) ds - \\
& -\frac{1}{2} \int_0^T (\bar{\pi}_s(X) - \bar{\gamma}_X(s), h_s) A_s \left( \frac{\bar{\pi}_s(X) - \bar{\gamma}_X(s)}{h_s} \right) ds \\
& + \int_0^T [\pi_s(X) - \bar{\pi}_s(X) - \bar{\gamma}_X(s)] A_s d\nu_s - \frac{1}{2} \int_0^T |A_s (\pi_s(X) - \bar{\pi}_s(X) - \bar{\gamma}_X(s))|^2 ds,
\end{align*}

and it gives the statement of the Proposition.

**Remark 2.** (i) Let us observe that for $\mu$ negative the proof of Proposition 1 clarifies the probabilistic interpretation of the ingredients $\bar{\gamma}_X(t)$ and $\bar{\gamma}_h(t)$ in terms of an auxiliary risk-neutral filtering problem. They are nothing else but the filtering error $\bar{\gamma}_X(t)$ (see equation (15)) and the difference $\bar{\gamma}_h(t) = \bar{\pi}_t(X) - \bar{\gamma}_X(t)$ (see equation (16)). It is worth mentioning that $\bar{\pi}_t(X)$ and $\bar{\gamma}_X(t)$ are only $\bar{\gamma}_t$-measurable variables but that the difference $\bar{\gamma}_h(t) = \bar{\pi}_t(X) - \bar{\gamma}_X(t)$ is actually a $\bar{\gamma}_t$-measurable variable.

(ii) If $\mu$ is positive, but sufficiently small in order that the condition $(C_\mu)$ is satisfied, due to analytical properties of involved functions with respect to $\mu$, equality (8) is still valid. But it is worth emphasizing that there is no connection anymore between functions $Z^h$ and $\bar{\gamma}_X$ and a risk-neutral filtering problem.

### 3 Solution of the filtering problems with exponentials of integral functionals criteria

Actually, in the particular Markov model (1)–(2), the equations (6)–(7) can be transformed. Indeed, due to the specific structure of the covariance function $K$ of the signal process $X$, the solution of equation (6) is obtained in the form $\bar{\gamma}(t, s) = \Pi_t \Pi_s^{-1} \bar{\gamma}_X(s), 0 \leq s \leq t$, where $\Pi_s$ is the solution of the
differential equation $\dot{\Pi}_s = a(s)\Pi_s$, $s \geq 0$, $\Pi_0 = 1$ and $\bar{\gamma}_{xx}(s)$ satisfies the following differential equation:

$$
\dot{\bar{\gamma}}_{xx} = 2a\bar{\gamma}_{xx} + 1 - \bar{\gamma}_{xx}^2 [A^2 - \mu \Lambda_{11}], \bar{\gamma}_{xx}(0) = 0.
$$

(25)

Moreover, this particular form of $\bar{\gamma}(t, s)$ leads also to a differential equation for the solution $Z^h$ of (7):

$$
dZ^h_t = [a - \bar{\gamma}_{xx}(A^2 - \mu \Lambda_{11})]Z^h_t dt + \mu \bar{\gamma}_{xx} \Lambda_{12} h_t dt + \bar{\gamma}_{xx} A dY_t, Z^h_0 = 0.
$$

(26)

3.1 Solution of the LEG filtering problem

Let us formulate the following condition ($C^*_\mu$):

($C^*_\mu$) the forward and backward Riccati equations:

$$
\dot{\bar{\gamma}}_{xx} = 2a\bar{\gamma}_{xx} + 1 - \bar{\gamma}_{xx}^2 [A^2 - \mu \Lambda_{11}], \bar{\gamma}_{xx}(0) = 0,
$$

(27)

$$
\dot{\Gamma} = -\frac{\text{det}(\Lambda)}{\Lambda_{22}} - 2(a + \mu \bar{\gamma}_{xx} \frac{\text{det}(\Lambda)}{\Lambda_{22}})\Gamma - \mu \Gamma^2 \bar{\gamma}_{xx}^2 [A^2 - \mu \frac{\Lambda_{12}^2}{\Lambda_{22}}], \Gamma(T, T) = 0,
$$

(28)

have unique nonnegative and bounded solutions ($\bar{\gamma}_{xx}(t), 0 \leq t \leq T$) and $(\Gamma(T, t), 0 \leq t \leq T)$.

Notice that for all $\mu$ negative the condition ($C^*_\mu$) is satisfied and if $\mu$ is positive, it is satisfied for sufficiently small.

**Proposition 2.** Suppose that the condition ($C^*_\mu$) is satisfied. Let $\bar{h} = (\bar{h}_t, 0 \leq t \leq T)$ such that:

$$
\bar{h}_t = -\frac{\Lambda_{12}(t)}{\Lambda_{22}(t)} (1 + \mu \bar{\gamma}_{xx}(t) \Gamma(T, t)) Z^h_t,
$$

(29)

where $\Gamma(T, \cdot)$ is the solution of the backward Riccati equation [28] and $Z^h = (Z^h_t, 0 \leq t \leq T)$ is the solution of the following equation:

$$
dZ^h_t = [a + \mu \bar{\gamma}_{xx}(\text{det}(\Lambda) - \mu \Lambda_{12}^2 \bar{\gamma}_{xx} \Gamma)]Z^h_t dt + A\bar{\gamma}_{xx} [dY_t - AZ^h_t dt].
$$

(30)
Then \( \bar{h} \) is the solution of the LEG filtering problem (3) and moreover, the corresponding optimal risk is given by

\[
\mathbb{E}
\left[
\mu \exp \left\{ \frac{\mu}{2} \int_0^T (X_s \bar{h}_s)\Lambda_s \begin{pmatrix} X_s \\ \bar{h}_s \end{pmatrix} \, ds \right\}
\right]
= \mu \exp \left\{ \frac{\mu}{2} \int_0^T \bar{\gamma}_{XX}(s)\Lambda_{11}(s) \, ds + \frac{\mu}{2} \int_0^T \Gamma(T, s)A_s^2\bar{\gamma}^2_{XX}(s) \, ds \right\}.
\]

**Proof** Of course, since we assume that condition \((C^*_\mu)\) is satisfied, condition \((C^\mu)\) with \(M = 0\) is also fulfilled. Then we can apply the Cameron-Martin formula (8) with \(M = 0\) (and hence in particular \(G_T = 0\)). It gives that

\[
\mathbb{E}_\mu \exp \left\{ \frac{\mu}{2} \int_0^T (X_s \bar{h}_s)\Lambda_s \begin{pmatrix} X_s \\ \bar{h}_s \end{pmatrix} \, ds \right\} = \exp \left\{ \frac{\mu}{2} \int_0^T \bar{\gamma}_{XX}(s)\Lambda_{11}(s) \, ds \right\} \times
\]

\[
\mathbb{E}_\mu \exp \left\{ \frac{\mu}{2} \int_0^T (Z_s^0 \bar{h}_s)\Lambda_s \begin{pmatrix} Z_s^h \\ \bar{h}_s \end{pmatrix} \, ds \right\} + \int_0^T A_s(Z_s^h - \pi_s(X)) \, d\nu_s - \frac{1}{2} \int_0^T |A_s(Z_s^h - \pi_s(X))|^2 \, ds \right\}, \quad (31)
\]

where for arbitrary \(h \in \mathcal{H}\) the process \(Z^h\) is the solution of equation (26). To find the solution of LEG filtering problem we propose to follow the ideas of [1] and [8], developed for the LEG control problem. Let us apply the Itô formula to \(\mu \Gamma(T, t)(Z_t^h)^2\), where \(\Gamma(T, \cdot)\) and \(Z^h\) are the solutions of the equations (28) and (26) respectively. We see that

\[
\mathbb{E}_\mu \exp \left\{ \frac{\mu}{2} \int_0^T (X_s \bar{h}_s)\Lambda_s \begin{pmatrix} X_s \\ \bar{h}_s \end{pmatrix} \, ds \right\} =
\]

\[
\exp \left\{ \frac{\mu}{2} \int_0^T \bar{\gamma}_{XX}(s)\Lambda_{11}(s) \, ds + \frac{\mu}{2} \int_0^T \Gamma(T, s)A_s^2\bar{\gamma}^2_{XX}(s) \, ds \right\}
\]

\[
\times \mathbb{E}_\mu \exp \left\{ \int_0^T A_s(Z_s^h - \pi_s(X) + \mu \Gamma(T, s)\bar{\gamma}_{XX}(s)Z_s^h) \, d\nu_s -
\right.
\]

\[
- \frac{1}{2} \int_0^T |A_s(Z_s^h - \pi_s(X) + \mu \Gamma(T, s)\bar{\gamma}_{XX}(s)Z_s^h)|^2 \, ds \right\}
\]

\[
\times \exp \left\{ \frac{\mu}{2} \int_0^T \Lambda_{22}(s) \left[ \bar{h}_s + \frac{\Lambda_{12}(s)}{\Lambda_{22}(s)} (1 + \mu \bar{\gamma}_{XX}(s)\Gamma(T, s)) Z_s^h \right]^2 \, ds \right\}. \quad (32)
\]

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Proceeding as in the proof of Theorem 1 \[5\] we see that Equation \(32\) implies that the cost function \(3\) (see also \(31\)) has a uniform lower bound which is attained for \(\bar{h}\) defined by the equation \(29\).

**Remark 3.**

(i) It is clear that in the singular case where \(\Lambda_{11} = \Lambda_{22} = -\Lambda_{12}\), equation \(28\) implies that \(\Gamma \equiv 0\) and therefore \(Z^h = \bar{h}\) (cf.\[5\]).

(ii) But in the general case \(\Gamma\) may depend on \(T\) and as a consequence, \(\bar{h}_t\) may also depend on \(T\). An example of such a dependence will be given below. Of course, by its definition \(\bar{h}_t\) does not depend on \(T\) and hence \(\bar{h} \neq \hat{h}\) in this example.

### 3.2 Solution of the RS filtering problem.

Let us formulate the following condition \((C_{***})\):

\((C_{***})\) the Riccati equation \(27\) has a unique, nonnegative and bounded solution on \([0, T]\) such that for \(0 \leq t \leq T\)

\[1 - \mu \tilde{\gamma}_{XX}(t) \Lambda_{11}(t) > 0.\]

**Proposition 3.** Suppose that the condition \((C_{***})\) is satisfied. Let \(\hat{h} = (\hat{h}_t, 0 \leq t \leq T)\) such that:

\[
\hat{h}_t = -\frac{\Lambda_{12}(t)}{\Lambda_{22}(t)} [1 - \mu \tilde{\gamma}_{XX}(t) \Lambda_{22}^{-1}(t) \det(\Lambda_t)]^{-1} Z^\hat{h}_t,
\]

where \(Z^\hat{h}_t = (Z^\hat{h}_t, 0 \leq t \leq T)\) is the solution of the following equation:

\[
dZ^\hat{h}_t = [a + \mu \tilde{\gamma}_{XX} \det(\Lambda) \frac{1 - \mu \tilde{\gamma}_{XX} \Lambda_{11} \det(\Lambda)}{\Lambda_{22} - \mu \tilde{\gamma}_{XX} \det(\Lambda)} Z^\hat{h}_t] dt + A \tilde{\gamma}_{XX} [dY_t - AZ^\hat{h}_t dt].
\]

Then \(\hat{h}\) is the solution of the RS filtering problem \(4\).

**Proof** Again, since we assume that condition \((C_{***})\) is satisfied, for any fixed \(t \leq T\), we can apply the Cameron-Martin formula \(8\) with \(t\) in place
of $T$ and $\Lambda_t$ in place of $M$. It gives that

\[
\mathcal{I}_t = (1 - \mu \Lambda_{11}(t) \bar{\gamma}_{XX}(t))^{-\frac{1}{2}} \exp \left\{ \frac{\mu}{2} \int_0^t \bar{\gamma}_{XX}(s) \Lambda_{11}(s) \, ds \right\} \times \\
\times \exp \left\{ \frac{\mu}{2} (Z_t^\hat{h} g) G_t \left( \begin{array}{c} Z_t^\hat{h} \\ g \end{array} \right) + \frac{\mu}{2} \int_0^t (Z_s^\hat{h} \hat{h}_s) \Lambda_s \left( \begin{array}{c} Z_s^\hat{h} \\ \hat{h}_s \end{array} \right) \, ds \right\} \times \\
\times \exp \left\{ \int_0^t A_s (Z_s^\hat{h} - \pi_s(X)) \, d\nu_s - \frac{1}{2} \int_0^t |A_s(Z_s^\hat{h} - \pi_s(X))|^2 \, ds \right\}, \quad (35)
\]

where

\[
G_t = (1 - \mu \Lambda_{11}(t) \bar{\gamma}_{XX}(t))^{-1} \left( \begin{array}{cc} \Lambda_{11}(t) & \Lambda_{12}(t) \\ \Lambda_{12}(t) & \Lambda_{22}(t) - \mu \bar{\gamma}_{XX}(t) \det(\Lambda_t) \end{array} \right). \quad (36)
\]

Since $\hat{h}_s, 0 \leq s < t$ is fixed, the optimization of the quadratic form $(Z_t^\hat{h} g) G_t \left( \begin{array}{c} Z_t^\hat{h} \\ g \end{array} \right)$ in the equality (35) with respect to $g$ gives the value of $\hat{h}_t$: $\hat{h}_t = -\frac{G_{12}}{G_{11}} Z_t^\hat{h}$, or equivalently

\[
\hat{h}_t = -\frac{\Lambda_{12}(t)}{\Lambda_{22}(t)} [1 - \mu \bar{\gamma}_{XX}(t) \Lambda_{22}^{-1}(t) \det(\Lambda)]^{-1} Z_t^\hat{h}, \quad (37)
\]

where $Z_t^\hat{h}$ is defined by the equation (26) with $h = \hat{h}$ and so by the equation (34).

**Remark 4.** (i) It is clear that for the singular case where $\Lambda_{11} = \Lambda_{22} = -\Lambda_{12}$, equalities (33) and (34) imply that $\hat{h} = Z^\hat{h} = Z^\hat{h} = \hat{h}$ (cf. [5]).

(ii) Let us emphasize that, of course, $\hat{h}_t$ does not depend on $T$ and so generally $\hat{h}_t \neq \bar{h}_t$.

Of course the RS filtering problem can be solved for an arbitrary continuous Gaussian process $X = (X_t, t \geq 0)$ with mean function $m = (m_t, t \geq 0)$ and covariance function $K = (K(t, s), t \geq 0, s \geq 0)$. To complete this section we propose the following generalization of Proposition 3 which can be proved by the same way.

**Proposition 4.** Suppose that equation (6) has a unique and bounded solution $\bar{\gamma} = (\bar{\gamma}(t, s), 0 \leq s \leq t \leq T)$ such that $\bar{\gamma}(t, t) \geq 0$ and $1 - \mu \bar{\gamma}(t, t) \Lambda_{11}(t) > 0$
for \(0 \leq t \leq T\).

Let \(\widehat{h} = (\widehat{h}_t, 0 \leq t \leq T)\) such that:

\[
\widehat{h}_t = -\frac{\Lambda_{12}(t)}{\Lambda_{22}(t)} \left[1 - \mu \hat{\gamma}_{XX}(t) \Lambda_{22}(t) \det(\Lambda_t)\right]^{-1} Z_t^\widehat{h},
\]

where \(\hat{\gamma}_{XX}(t) = \hat{\gamma}(t, t), 0 \leq t \leq T\) and \(Z_t^\widehat{h} = (Z_t^\widehat{h}_t, 0 \leq t \leq T)\) is the unique solution of the Itô-Volterra equation:

\[
Z_t^\widehat{h} = m_t + \mu \int_0^t \hat{\gamma}(t, s) \det(\Lambda_s) \left[\frac{1 - \Lambda_{11}(s) \hat{\gamma}(s, s) \det(\Lambda_s)}{\Lambda_{22}(s) - \mu \hat{\gamma}(s, s) \det(\Lambda_s)}\right] Z_s^\widehat{h} ds \\
+ \int_0^t \hat{\gamma}(t, s) A_s[dY_s - A_s Z_s^\widehat{h} ds].
\]

Then \(\widehat{h}\) is the solution of the RS filtering problem \([1]\).

\section{Discrepancy between LEG and RS filtering problems: an example}

To show the possible dependence of the solution of the LEG filtering problem on \(T\) and so the discrepancy between LEG and RS filtering problems we propose to take

\[
\Lambda = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad a = 0, \quad A = 1, \quad \mu = -1.
\]

In this case equations (27) and (28) reduce to the following:

\[
\dot{\hat{\gamma}}_{XX} = 1 - 3\hat{\gamma}^2_{XX}, \quad \hat{\gamma}_{XX}(0) = 0, \\
\dot{\Gamma} = 2\hat{\gamma}_{XX} \Gamma - 1 + 2\hat{\gamma}^2_{XX} \Gamma^2, \quad \Gamma(T, T) = 0.
\]

Thus

\[
\hat{\gamma}_{XX}(t) = \frac{1}{\sqrt{3}} \text{th} \sqrt{3}t.
\]

Equation (40) can be also solved explicitly using the classical linearization method for Riccati equations:

\[
\Gamma = \varphi_1^{-1} \varphi_2, \quad \varphi_2(T) = 0, \quad \varphi_1(T) = 1,
\]
\[
\begin{align*}
\dot{\varphi}_1 &= -\bar{\gamma}_{xx} \varphi_1 - 2\bar{\gamma}_{xx}^2 \varphi_2; \\
\dot{\varphi}_2 &= -\varphi_1 + \bar{\gamma}_{xx} \varphi_2. \\
\end{align*}
\]

(41)

Hence

\[
\dot{\varphi}_2 = \varphi_2 (\dot{\bar{\gamma}}_{xx} + 3\bar{\gamma}_{xx}^2) = \varphi_2,
\]

and thus

\[
\begin{align*}
\varphi_2(t) &= \text{sh}(T - t), \\
\varphi_1(t) &= \text{ch}(T - t) + \frac{1}{\sqrt{3}} \text{th} \sqrt{3t} \cdot \text{sh}(T - t), \\
\Gamma(T, t) &= \frac{\text{sh}(T - t)}{\text{ch}(T - t) + \frac{1}{\sqrt{3}} \text{th} \sqrt{3t} \cdot \text{sh}(T - t)}.
\end{align*}
\]

Representation (29) gives that

\[
\bar{h}_t = (1 - \bar{\gamma}_{xx}(t)\Gamma(T, t))\bar{Z}_t,
\]

or

\[
\bar{h}_t = \int_0^t \bar{H}(T, t, s) \, dY_s,
\]

where

\[
\bar{H}(T, t, s) = (1 - \bar{\gamma}_{xx}(t)\Gamma(T, t, s))\bar{\Psi}(T, t, s),
\]

with

\[
\begin{align*}
\bar{\Psi}(T, s, s) &= \bar{\gamma}_{xx}(s), \\
\dot{\bar{\Psi}} &= (-2\bar{\gamma}_{xx} - \bar{\gamma}_{xx}^2 \Gamma)\bar{\Psi}.
\end{align*}
\]

Equation (41) gives that

\[
(\ln \bar{\Psi})' = \frac{1}{2}(\ln \varphi_1)' - \frac{3}{2} \bar{\gamma}_{xx} = \frac{1}{2}(\ln \varphi_1)' - \frac{1}{2} (\ln \text{ch} \sqrt{3t})',
\]

Finally

\[
\bar{\Psi}(T, t, s) = c(s) \sqrt{\frac{\varphi_1(t)}{\text{ch} \sqrt{3t}}},
\]

and

\[
\bar{H}(T, t, s) = \frac{\text{sh} \sqrt{3s} \cdot \text{ch}(T - t)}{\sqrt{\alpha_t \alpha_s}},
\]

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with
\[ \alpha_t = \frac{\sqrt{3} + 1}{2} \text{ch}(T + (\sqrt{3} - 1)t) + \frac{\sqrt{3} - 1}{2} \text{ch}(T - (\sqrt{3} + 1)t). \]

The same calculations based on the equality (33) give the representation
\[ \hat{h}_t = \int_0^t \hat{H}(t, s) dY_s, \]

where
\[ \hat{H}(t, s) = \frac{1}{\sqrt{3}} \cdot \frac{(\text{ch} \sqrt{3}t)^{1/3} \text{sh} \sqrt{3}s}{(\text{ch} \sqrt{3}s)^{2/3}}. \]

References

[1] A. Bensoussan and J.H. van Schuppen, Optimal control of partially observable stochastic systems with an exponential of integral performance index, SIAM J. Optimization and Control, 23 (1985), 599-613.

[2] C.D. Charalambus, S. Dey and R.J. Elliott, New finite-dimensional risk sensitive filters: Small-noise limits, IEEE Trans. Automat. Control 43 (10) (1997), 1424-1429.

[3] S. Dey, R.J. Elliott and J.B. Moore, Finite dimensional risk-sensitive estimation for continuous time nonlinear systems, Proceedings of the European Control Conference, Brussels, 1997.

[4] R.J. Elliott, L. Aggoun and J.B. Moore, Hidden Markov Models: Estimation and Control, Springer, Berlin, 1994.

[5] M.L. Kleptsyna, A. Le Breton and M.Viot, On the linear-exponential filtering problem for general Gaussian processes, SIAM J. Optimization and Control 47 (6) (2008), 2886 - 2911.

[6] R.S. Liptser and A.N. Shiryaev, Statistics of Random Processes I - General Theory, Springer-Verlag, New-York, 1977.
[7] J.L. Speyer, C. Fan and R.N. Banavar, *Optimal Stochastic Estimation with Exponential Criteria*, Proceedings of the 31st Conference on Decision and Control, 2 (1992), 2293-2298.

[8] P. Whittle, *Risk-sensitive optimal control*, John Wiley and Sons, New York, 1990.

[9] P. Whittle, *A risk-sensitive maximum principle: the case imperfect state observations*, IEEE Trans. Automat. Control 36 (7) (1991), 793-801.