Entanglement criteria from multiple observables entropic uncertainty relations

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We investigate quantum entanglement for both bipartite and multipartite systems by using entropic uncertainty relations (EUR) defined in terms of the joint Shannon entropy of probabilities of local measurement outcomes. We introduce state-independent and state-dependent entropic inequalities whose violation certifies the presence of quantum correlations. We show that the additivity of EUR holds only for EUR that involve two observables, while inequalities that consider more than two observables or the addition of the Von Neumann entropy of a subsystem enable to detect quantum correlations. Furthermore, we study their detection power for bipartite systems and for several classes of states of a three-qubit system.

Entropic uncertainty relations (EUR) are inequalities that express preparation uncertainty relations (UR) as sums of Shannon entropies of probability distributions of measurement outcomes. First introduced for continuous variables systems [1–4], they were then generalized for pair of observables with discrete spectra [5–9] (see [10] for a review of the topic). Conversely to the most known UR defined for product of variances [11,12], which are usually state-dependent, EUR provide lower bounds, which quantify the knowledge trade-off between the different observables, that are state-independent. Variance-based UR for the sum of variances [13] in some cases also provide state-independent bounds [14–16]. EUR, due to their simple structure, allow to consider UR for more than two observables in a natural way by simply adding more entropies, a task that is not straightforward for product of variances UR. However, tight bounds for multiple observables EUR are known only for small dimensions and for restricted sets of observables, typically for complementary observables [19,25], namely the ones that have mutually unbiased bases as eigenbases, and for angular momentum observables [26,27].

Besides their importance from a fundamental point of view as preparation uncertainty relations, EUR have recently been used to investigate the nature of correlations in composite quantum systems, providing criteria that enable to detect the presence of different types of quantum correlations, both for bipartite and multipartite systems. Entanglement criteria based on EUR were defined in [27,30], while steering inequalities in [32–37].

The paradygmatic example of EUR for observables with a discrete non-degenerate spectra is due to Maassen and Uffink [17], and it states that for any two observables $A_1$ and $A_2$, defined on a $d$-dimensional system, the following inequality holds:

$$H(A_1) + H(A_2) \geq -2 \log_2 d = q_{MU},$$

(1)

where $H(A_1)$ and $H(A_2)$ are the Shannon entropies of the measurement outcomes of two observables $A_1 = \sum_j a^1_j |a^1_j\rangle\langle a^1_j|$ and $A_2 = \sum_j a^2_j |a^2_j\rangle\langle a^2_j|$, namely $H(A_I) = -\sum_j p(a^I_j) \log p(a^I_j)$ being $p(a^I_j)$ the probability of obtaining the outcome $a^I_j$ of $A_I$, and $c = \max_{x,k} |\langle a^1_x|a^2_k\rangle|$ is the maximum overlap between their eigenstates. The bound (1) is known to be tight if $A_1$ and $A_2$ are complementary observables. We remind that two observables $A_1$ and $A_2$ are said to be complementary iff their eigenbases are mutually unbiased, namely iff $|\langle a^1_j|a^2_k\rangle| = \frac{1}{\sqrt{d}}$ for all eigenstates, where $d$ is the dimension of the system (see [39] for a review on MUBs). In this case $q_{MU} = \log_2 d$, hence we have:

$$H(A_1) + H(A_2) \geq \log_2 d.$$  

(2)

The above relation has a clear interpretation as uncertainty relation: let us suppose that $H(A_1) = 0$, which
means that the state of the system is an eigenstate of $A_1$, then the other entropy $H(A_2)$ must be maximal, hence if we have a perfect knowledge of one observable the other must be completely undetermined. For arbitrary observables stronger bounds, that involve the second largest term in $|\langle a_j|b_k\rangle|$, were derived in [8, 9].

An interesting feature of EUR is that they can be generalized to an arbitrary number of observables in a straightforward way from Maassen and Uffink’s EUR. Indeed, let us consider for simplicity the case of three observables $A_1, A_2$ and $A_3$, which they mutually satisfy the following EURs:

$$H(A_i) + H(A_j) \geq q_{MU}^i$$

where $i, j = 1, 2, 3$ labels the three observables. Then, we have:

$$\sum_{k=1}^{3} H(A_k) = \frac{1}{2} \sum_{k=1}^{3} \sum_{j \neq k} H(A_k) + H(A_j) \geq \frac{1}{2} (q_{MU}^1 + q_{MU}^2 + q_{MU}^3)$$

where we have applied (3) to each pair. If we have $L$ observables, the above inequality becomes:

$$\sum_{k=1}^{L} H(A_k) \geq \frac{1}{(L-1)} \sum_{t \in T_2} q_{MU}^t$$

where $t$ varies between the set $T_2$, made by labels of all possible $\frac{L(L-1)}{2}$ pairs of observables. For example if $L = 4$, then $T_2 = \{12, 13, 14, 23, 24, 34\}$.

However EUR in the form (5) are usually not tight, i.e. in most cases the lower bounds can be improved. Tight bounds are known only for small dimensions and for complementary or angular momentum observables. For the sake of simplicity, henceforth all explicit examples will be discussed only for complementary observables. The maximal number of complementary observables for any given dimension is an open problem [59], which finds its roots in the classification of all complex Hadamard matrices. However if $d$ is a power of a prime then $d + 1$ complementary observables always exist. However for any $d$, even if it is not a power of a prime, it is possible to find at least three complementary observables [59]. The method that we will define in the next section can be therefore used in any dimension. The qubit case, where at most three complementary observables exist, which are in correspondence with the three Pauli matrices, was studied in [25], while for systems with dimension three to five tight bounds for an arbitrary number of complementary observables were derived in [25]. For example in the qubit case, where the three observables $A_1, A_2$ and $A_3$ correspond to the three Pauli matrices $\sigma_x, \sigma_y$ and $\sigma_z$, we have:

$$H(A_1) + H(A_2) + H(A_3) \geq 2$$

and the minimum is achieved by the eigenstates of one of the $A_i$. In the case of a qutrit, where there exist four complementary observables, we instead have:

$$H(A_1) + H(A_2) + H(A_3) \geq 3,$$

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) \geq 4.$$

The minimum values are achieved by:

$$\frac{e^{i\varphi} |0\rangle + |1\rangle}{\sqrt{2}}, \quad \frac{e^{i\varphi} |0\rangle + |2\rangle}{\sqrt{2}}, \quad \frac{e^{i\varphi} |1\rangle + |2\rangle}{\sqrt{2}},$$

where $\varphi = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}$. Another result, for $L < d + 1$, can be found in [22], where it has been shown that if the Hilbert space dimension is a square, that is $d = r^2$, then for $L < r + 1$ the inequality (5) is tight, namely:

$$\sum_{i=1}^{L} H(A_i) \geq \frac{L}{2} \log_2 d = q_{BW}.$$

In order to have a compact expression to use, we express the EUR for $L$ observables in the following way:

$$\sum_{i=1}^{L} H(A_i) \geq f(A, L),$$

where $f(A, L)$ indicates the lower bound, which can be tight or not, and it depends on the set $A = \{A_1, \ldots, A_L\}$ of $L$ observables considered. Here we also point out in the lower bound how many observables are involved. When we refer explicit to tight bounds we will use the additional label $T$, namely $f^T(A, L)$ expresses a lower bound that we know is achievable via some states.

II. BIPARTITE ENTANGLEMENT CRITERIA

In this section we discuss bipartite entanglement criteria based on EUR, defined in terms of joint Shannon entropies. The framework consists in two parties, says Alice and Bob, that share a quantum state $\rho_{AB}$, and they want to establish if their state is entangled. Alice and Bob can perform $L$ measurements each that we indicate respectively as $A_1, \ldots, A_L$ and $B_1, \ldots, B_L$. Alice and Bob measure the observable $A_i \otimes B_j$ and they want to have a criterion defined in terms of the joint Shannon entropies $H(A_i, B_j)$ which certifies the presence of entanglement. As a reminder, in a bipartite scenario we say that the state $\rho_{AB}$ is entangled iff it cannot be expressed as a convex combination of product states, which are represented by separable states, namely iff:

$$\rho_{AB} \neq \sum_i p_i \rho_A^i \otimes \rho_B^i,$$

where $p_i \geq 0$, $\sum_i p_i = 1$, and $\rho_A^i, \rho_B^i$ are Alice and Bob’s states respectively.

**Proposition 1.** If the state $\rho_{AB}$ is separable, then the following EUR must hold:

$$\sum_{i=1}^{L} H(A_i, B_i) \geq f(A, L) + f(B, L),$$

where

$$f(A, L) = \sum_{i=1}^{L} H(A_i) \geq f(A, L),$$

and

$$f(B, L) = \sum_{i=1}^{L} H(B_i) \geq f(B, L).$$
where \( f(A, L) \) and \( f(B, L) \) are the lower bounds of the single system EUR, namely

\[
\sum_{i=1}^{L} H(A_i) \geq f(A, L),
\]

(14)

\[
\sum_{i=1}^{L} H(B_i) \geq f(B, L).
\]

(15)

**Proof.** Let us focus first on \( H(A_i, B_i) \) which, for the properties of the Shannon entropy, can be expressed as:

\[
H(A_i, B_i) = H(A_i) + H(B_i|A_i).
\]

(16)

We want to bound \( H(B_i|A_i) \) which is computed over the state \( \rho_{AB} = \sum_j p_j \rho_A^j \otimes \rho_B^j \). Through the convexity of the relative entropy, one can prove that the conditional entropy \( H(B|A) \) is concave in \( \rho_{AB} \). Then we have:

\[
H(B_i|A_i) \sum_j p_j \rho_A^j \otimes \rho_B^j \geq \sum_j p_j H(B_i|A_i) \rho_A^j \otimes \rho_B^j,
\]

(17)

thus, since the right-hand side of the above is evaluated on a product state, we have:

\[
H(B_i|A_i) \sum_j p_j \rho_A^j \otimes \rho_B^j \geq \sum_j p_j H(B_i) \rho_B^j.
\]

(18)

Therefore, considering \( \sum_{i=1}^{L} H(A_i, B_i) \), we derive the following:

\[
\sum_{i=1}^{L} H(A_i, B_i) \geq \sum_{i=1}^{L} H(A_i) + \sum_j p_j \sum_{i=1}^{L} H(B_i) \rho_B^j.
\]

(19)

We can then observe that: \( \sum_{i=1}^{L} H(A_i) \geq f(A, L) \) and \( \sum_i H(B_i) \rho_B^j \geq f(B, L) \), indeed the second inequality holds since EUR are state-independent bounds, therefore we have:

\[
\sum_{i=1}^{L} H(A_i, B_i) \geq f(A, L) + \sum_j p_j f(B, L)
\]

\[
= f(A, L) + f(B, L),
\]

(20)

since \( \sum_j p_j = 1 \).

Any state that violates the inequality \( \sum_{i=1}^{L} H(A_i, B_i) \geq f(A, L) + f(B, L) \) must be therefore entangled. However this is not sufficient to have a proper entanglement criteria. Indeed, if we consider the observables \( A_i \otimes B_i \) as ones of the bipartite system then they must satisfy an EUR for all states, even the entangled ones, which can be expressed as:

\[
\sum_{i=1}^{L} H(A_i, B_i) \geq f(AB, L),
\]

(21)

where the lower bound now depends on the observables \( A_i \otimes B_i \), while \( f(A, L) \) and \( f(B, L) \) depend respectively on \( A_i \) and \( B_i \) individually. In order to have a proper entanglement criterion then we should have that

\[
f(AB, L) < f(A, L) + f(B, L),
\]

(22)

which means that the set of entangled states that violate the inequality is not empty. As it was shown in [33], for \( L = 2 \), we have \( f(AB, 2) = f(A, 2) + f(B, 2) \) for any observables, which expresses the additivity of EUR for pair of observables. A counterexample of this additivity property for \( L > 3 \) is provided by the complete set of complementary observables for two qubits, indeed we have:

\[
H(A_1, B_1) + H(A_2, B_2) + H(A_3, B_3) \geq 3,
\]

(23)

and the minimum is attained by the Bell states while \( f(A, 3) + f(B, 3) = 4 \), which provides the threshold that enables to entanglement detection in the case of two qubits.

Let us now clarify the difference of this result with respect to ones defined in terms of EUR based on conditional entropies, in particular to entropic steering inequalities. Indeed, if one looks at the proof of Proposition 1, it could be claimed that there is no difference at all since we used the fact that \( \sum_i H(B_i|A_i) \geq f(B, L) \), which is a steering inequality, namely violation of it witnesses the presence of quantum steering from Alice to Bob. However the difference is due to the symmetric behavior of the joint entropy, which contrasts with the asymmetric of quantum steering. To be more formal, the joint Shannon entropy \( H(A_i, B_i) \) can be rewritten in two forms:

\[
H(A_i, B_i) = H(A_i) + H(B_i|A_i)
\]

(24)

\[
= H(B_i) + H(A_i|B_i),
\]

then:

\[
\sum_i H(A_i, B_i) = \sum_i (H(A_i) + H(B_i|A_i)),
\]

(25)

and

\[
\sum_i H(A_i, B_i) = \sum_i (H(B_i) + H(A_i|B_i)).
\]

(26)

If now the state is not steerable from Alice to Bob, we have \( \sum_i H(B_i|A_i) \geq f(B, L) \), which implies \( \sum_{i=1}^{L} H(A_i, B_i) \geq f(A, L) + f(B, L) \). Note that in this case if we look at \( \sum_i H(A_i|B_i) \) no bound can be derived, apart from the trivial bound \( \sum_i H(A_i|B_i) \geq 0 \), since there are no assumptions on the conditioning from Bob to Alice. Conversely, if the state is not steerable from Bob to Alice, i.e. we exchange the roles, we have \( \sum_i H(B_i|A_i) \geq 0 \) and \( \sum_i H(A_i|B_i) \geq f(A, L) \), which implies again \( \sum_{i=1}^{L} H(A_i, B_i) \geq f(A, L) + f(B, L) \) . Therefore if we just look at the inequality (13), we cannot distinguish between entanglement, or the two possible
forms of quantum steering and since the presence of steering, for bipartite systems, implies entanglement it is more natural to think about Eq. (13) as an entanglement criterion, while if we want to investigate steering properties of the state we should look at the violation of the criteria
\[ \sum_i H(B_i|A_i) \geq f(B, L) \] and
\[ \sum_i H(A_i|B_i) \geq f(A, L). \]

State-dependent bounds

A stronger entanglement criteria can be derived by considering the state-dependent EUR:
\[ \sum_{i=1}^{L} H(A_i) \geq f(A, L) + S(\rho_A), \tag{27} \]
or the corresponding version for Bob’s system
\[ \sum_{i=1}^{L} H(B_i) \geq f(B, L) + S(\rho_B), \]
where \( S(\rho_A) \) and \( S(\rho_B) \) represent the Von Neumann entropies of the marginal states of \( \rho_{AB} \).

Proposition 2. If the state \( \rho_{AB} \) is separable, then the following EUR must hold:
\[ \sum_{i=1}^{L} H(A_i, B_i) \geq f(A, L) + f(B, L) + \max(S(\rho_A), S(\rho_B)). \tag{28} \]

Proof. The proof is the same of Proposition 1 where we use (27) in (19), instead of the state-dependent bound (11). The same holds if we use the analogous version for Bob. Then, aiming at the strongest criterion, we can take the maximum between the two Von Neumann entropies.

The edge in using these criteria, instead of the one defined in Proposition 1, is such that even for \( L = 2 \) the bound is meaningful. Indeed a necessary condition to the definition of a proper criterion is that:
\[ f^T(AB, 2) < f(A, 2) + f(B, 2) + S(\rho_X), \tag{29} \]
where \( X = A, B \) with the additional requirement that the bound on the left is tight, i.e. there exist states the violate the criterion. As an example we can consider a two-qubit system, the observables \( X_{AB} = \sigma_X^A \otimes \sigma_X^B \) and \( Z_{AB} = \sigma_Z^A \otimes \sigma_Z^B \), which for all states of the whole system satisfy \( H(X_{AB}) + H(Z_{AB}) \geq 2 \), and the state \( \rho_{AB} = |\phi^+\rangle \langle \phi^+| \), indeed for this scenario the entanglement criterion reads:
\[ H(X_{AB}) + H(Z_{AB}) \geq 3, \tag{30} \]
which is actually violated since the left hand side is equal to 2. Note that in general the condition \( f^T(AB, L) < f(A, L) + f(B, L) + S(\rho_X) \) is necessary to the usefulness of the corresponding entanglement criteria.

III. MULTIPARTITE ENTANGLEMENT CRITERIA

We now extend the results of Proposition 1 and 2 for multipartite systems, where the notion of entanglement has to be briefly discussed since it has a much richer structure than the bipartite case, indeed we can distinguish between different levels of separability. First, we say that a state \( \rho_{V_1...V_n} \) of \( n \) systems \( V_1, ..., V_n \) is fully separable if it can be written in the form:
\[ \rho_{V_1...V_n}^{FS} = \sum_i p_i \rho_{V_i} \otimes ... \otimes \rho_{V_n}, \tag{31} \]
with \( \sum_i p_i = 1 \), namely it is a convex combination of product states of the single subsystems. As a case of study we will always refer to tripartite systems, where there are three parties, say Alice, Bob and Charlie. In this case a fully separable state can be written as:
\[ \rho^{FS}_{ABC} = \sum_i p_i \rho_A^i \otimes \rho_B^i \otimes \rho_C^i. \tag{32} \]

Any state does not admit such a decomposition contains entanglement among some subsystems. However, we can define different levels of separability. Hence, we say that the state \( \rho_{V_1...V_n} \) of \( n \) systems is separable with respect to a given partition \( \{I_1, ..., I_k\} \), where \( I_i \) are disjoint subsets of the indices \( I = \{1, ..., n\} \), such that \( \bigcup_{j=1}^{k} I_j = I \), iff it can be expressed as:
\[ \rho_{V_1...V_n}^{1...k} = \sum_i p_i \rho_A^i \otimes ... \otimes \rho_C^i, \tag{33} \]

namely some systems share entangled states, while the state is separable with respect to the partition considered. For tripartite system we have three different possible bipartitions: 1/23, 2/13 and 3/12. As an example, if the state \( \rho_{ABC} \) can be expressed as:
\[ \rho_{ABC}^{123} = \sum_i p_i \rho_A^i \otimes \rho_B^i \otimes \rho_C^i, \tag{34} \]
then there is no entanglement between Alice and Bob+Charlie, while these last two share entanglement. If a state does not admit such a decomposition, it is entangled with respect to this partition. Finally, we say that \( \rho_{V_1...V_n} \) of \( n \) systems can have at most \( m \)-system entanglement iff it is a mixture of all states such that each of them is separable with respect to some partition \( \{I_1, ..., I_k\} \), where all sets of indices \( I_k \) have cardinality \( N \leq m \). For tripartite systems this corresponds to the notion of biseparability, namely the state can have at most 2-system entanglement. A biseparable state can be written as:
\[ \rho_{ABC} = \sum_i p_i \rho_A^i \otimes \rho_B^i + \sum_j q_j \rho_B^j \otimes \rho_{AC}^j + \sum_k m_k \rho_C^k \otimes \rho_{AB}^k, \tag{35} \]
with \( \sum_i p_i + \sum_j q_j + \sum_k m_k = 1 \). For \( n = 3 \) a state is then said to be genuine tripartite entangled if it is 3-system entangled, namely if it does not admit such a decomposition.
Full separability

Let us clarify the scenario: in each system $V_i$ we consider a set of $L$ observables $V_{i1}^1, ..., V_{iL}^i$ that we indicate as $V_i$. The single system EUR is expressed as:

$$\sum_{j=1}^{L} H(V_j^i) \geq f(V_i,L).$$

We are interested in defining criteria in terms of $\sum_{j=1}^{L} H(V_j^i)$. A first result regards the notion of full separability.

**Proposition 3.** If the state $\rho_{V_1,...,V_n}$ is fully separable, then the following EUR must hold:

$$\sum_{j=1}^{L} H(V_j^i) \geq \sum_{i=1}^{n} f(V_i,L).$$

**Proof.** Let us consider the case $n=3$. For a given $j$ we have:

$$H(V_j^1, V_j^2, V_j^3) = H(V_j^1) + H(V_j^2 | V_j^1) + H(V_j^3 | V_j^1, V_j^2).$$

Since the state is separable with respect to the partition 23|1, due to concavity of the Shannon entropy, we have:

$$H(V_j^2 V_j^3 | V_j^1) \geq \sum_{i} p_i H(V_j^2 V_j^3 | \rho_{j1} \otimes \rho_{j2}).$$

By using the chain rule of the Shannon entropy, the above right hand side can be rewritten as:

$$\sum_{i} p_i H(V_j^2 V_j^3 | \rho_{j1} \otimes \rho_{j2}) = \sum_{i} p_i H(V_j^2 | \rho_{j1}),$$

where the last term can be lower bounded by exploiting the separability of the state and the concavity of the Shannon entropy, namely:

$$\sum_{i} p_i H(V_j^2 | \rho_{j1}) \geq \sum_{i} p_i H(V_j^2 | \rho_{j2}).$$

By summing over $j$ we arrive at the thesis:

$$\sum_{j=1}^{L} H(V_j^1, V_j^2, V_j^3) \geq \sum_{i=1}^{3} f(V_i,L),$$

since $\sum_j H(V_j^1) \geq f(V_1,L), \sum_i p_i \sum_j H(V_j^2 | \rho_{j1}) \geq f(V_2,L)$ and $\sum_i p_i \sum_j H(V_j^3 | \rho_{j1}) \geq f(V_3,L)$ because of the state-independent EUR. The extension of the proof to $n$ systems is straightforward.

The following proposition follows directly by considering the state-dependent bound:

$$\sum_{j=1}^{L} H(V_j^i) \geq f(V_i,L) + S(\rho_i).$$

**Proposition 4.** If the state $\rho_{V_1,...,V_n}$ is fully separable, then the following EUR must hold:

$$\sum_{j=1}^{L} H(V_j^i, V_j^2, V_j^3) \geq \sum_{i=1}^{n} f(V_i,L) + \max (S(\rho_1), ..., S(\rho_n)).$$

**Genuine multiparticle entanglement**

We now analyze the strongest form of multiparticle entanglement in the case of three systems, say Alice, Bob, and Charlie. We make the further assumptions that the three systems have the same dimension and in each system the parties perform the same set of measurements, which implies that there is only one bound of the single system EUR that we indicate as $F_1(L)$. We indicate the bound on a pair of systems as $F_2(L)$, namely $\sum_{i=1}^{L} H(A_i, B_i) \geq F_2(L)$ and the same by permuting the three systems. This will contribute to the readability of the paper. With this notation the criterion defined in Proposition 1 for three systems reads as: $\sum_{j=1}^{L} H(V_j^1, V_j^2, V_j^3) \geq 3 F_1(L),$ that must be satisfied by all fully separable states.

**Proposition 5.** If $\rho_{ABC}$ is not genuine multiparticle entangled, namely it is biseparable, then the following EUR must hold:

$$\sum_{j=1}^{L} H(V_j^1, V_j^2, V_j^3) \geq \frac{1}{3} F_1(L) + \frac{1}{3} F_2(L).$$

**Proof.** Let us assume that $\rho_{ABC}$ is biseparable, that is:

$$\rho_{ABC} = \sum_{i} p_i \rho_{A} \otimes \rho_{BC} + \sum_{l} q_l \rho_{B} \otimes \rho_{AC} + \sum_{k} m_k \rho_{C} \otimes \rho_{AB}.$$ 

The joint Shannon entropy $H(V_j^1, V_j^2, V_j^3)$ can be expressed as:

$$H(V_j^1, V_j^2, V_j^3) = \frac{1}{3} H(V_j^1) + H(V_j^2 | V_j^1) + \frac{1}{3} H(V_j^3 | V_j^1, V_j^2),$$

since $\sum_j H(V_j^1) \geq f(V_1,L), \sum_i p_i \sum_j H(V_j^2 | \rho_{j1}) \geq f(V_2,L)$ and $\sum_i p_i \sum_j H(V_j^3 | \rho_{j1}) \geq f(V_3,L)$ because of the state-independent EUR. The extension of the proof to $n$ systems is straightforward.

The following proposition follows directly by considering the state-dependent bound:

$$\sum_{j=1}^{L} H(V_j^i) \geq f(V_i,L) + S(\rho_i).$$
\[ H \left( V_1^j, V_2^j \mid V_3^j \right) \geq \sum_l p_l H \left( V_1^j \right)_{\rho_A} + \sum_l q_l H \left( V_2^j \right)_{\rho_B} + \sum_l m_k H \left( V_3^j \right)_{\rho_{AB}}; \]

Then, by considering the sum over \( j \) of the sum of the above entropies, and using EUR, we find:

\[ \sum_j H \left( V_1^j, V_2^j \mid V_3^j \right) + H \left( V_1^j, V_2^j \mid V_3^j \right) + H \left( V_1^j, V_2^j \mid V_3^j \right) \geq 2F_1 \left( L \right) + F_2 \left( L \right). \]  

The thesis (45) is now implied by combining the expression above, Eq. (47), and the following EUR:

\[ \sum_j H(V_1^j) + H(V_2^j) + H(V_3^j) \geq 3F_1 \left( L \right). \]  

\[ \square \]

**Proposition 6.** If \( \rho_{ABC} \) is not genuine multipartite entangled, namely it is biseparable, then the following EUR must hold:

\[ \sum_{j=1}^L \left( V_1^j, V_2^j, V_3^j \right) \geq \frac{5}{3} F_1 \left( L \right) + \frac{1}{3} F_2 \left( L \right) + \frac{1}{3} \sum_{x=A,B,C} S \left( \rho_X \right). \]  

The above proposition follows from the proof of Prop. 5, where we consider the single system state-dependent EUR.

**IV. ENTANGLEMENT DETECTION**

We now very briefly discuss how our criteria behave for bipartite and multipartite systems. We will mainly focus on pure states and multi-qubit systems.

**A. Bipartite systems**

Let us start with the easiest case of two qubits. In this scenario we will consider complementary observables and the tight EUR [20], [25], hence the considered criteria read as:

\[ H(A_1, B_1) + H(A_2, B_2) < 2 + \max(S(\rho_A), S(\rho_B)); \]

\[ \sum_{i=1}^3 H(A_i, B_i) < 4, \]  

where \( A_1 = Z_1, A_2 = X_1 \) and \( A_3 = Y_1 \) being \( Z_1, X_1 \) and \( Y_1 \) the usual Pauli matrices for the first qubit; the same holds for the second qubit. In the case of two qubits we have already shown in Section II that maximally entangled states are detected by the above criteria.

We can then consider the family of entangled two-qubit states given by:

\[ |\psi_\epsilon\rangle = \epsilon |00\rangle + \sqrt{1-\epsilon^2} |11\rangle, \]  

where \( \epsilon \in (0, 1) \). We first note that for this family we have \( S(\rho_A) = S(\rho_B) = -\epsilon^2 \log_2 \epsilon^2 - (1-\epsilon^2) \log_2 (1-\epsilon^2) \), which is the equal to \( H(A_1, B_1) \). Conversely, we have instead \( H(A_2, B_2) = -\frac{1}{2} \log_2 \left( \frac{1}{2} \right) - \frac{1}{2} \left( 1+\epsilon \right) \log_2 \left( \frac{1}{2} \left( 1+\epsilon \right) \right) \), with \( \bar{\epsilon} = 2\sqrt{1-\epsilon^2} \) and \( H(A_2, B_2) = H(A_3, B_3) \).

The family of states (47) is then completely detected by (48) and (56), since \( H(A_2, B_2) < 2 \), while (55) fails to detect all states (see Fig. 1).

Let us consider now the entangled two-qudit states given by:

\[ |\psi_\lambda\rangle = \sum_{i=0}^{d-1} \lambda_i |ii\rangle, \]  

where \( \lambda_i^2 = 1 \) and \( 0 < \lambda_i < 1 \). As an entanglement criterion we consider:

\[ H(A_1, B_1) + H(A_2, B_2) \geq 2 \log_2 d + \max(S(\rho_A), S(\rho_B)), \]  

where the first observable \( A_1 \) is the computational basis and \( A_2 \) is its Fourier transform, which is well-defined in any dimension, and the same for \( B_1 \) and \( B_2 \). First, we can observe that for these states we have \( S(\rho_A) = S(\rho_B) = \sum_{i=0}^{d-1} \lambda_i^2 \log_2 \lambda_i^2 \). Moreover, since \( A_1 \) and \( B_1 \) are respectively represented by the computational bases, we have \( H(A_1, B_1) = \sum_i -\lambda_i^2 \log_2 \lambda_i^2 \). Hence, the entanglement condition becomes:

\[ H(A_2, B_2) < 2 \log_2 d. \]  

However, for any two-qudit states we have \( H(A_2, B_2) \leq 2 \log_2 d \) and the maximum is achieved by states that give uniform probability distributions for \( A_2 \otimes B_2 \). Since \( A_2 \) and \( B_2 \) are the Fourier transformed bases of the computational ones, the family (58) cannot give a uniform probability distribution due the definition of the Fourier transform. The maximum value could be attained only by states of the form \( |ii\rangle \), hence by separable states which are excluded in (58). Thus, our criterion (59) detects all two-qudit entangled states of the form (58).

We now want to consider how these criteria perform for
mixed bipartite states. As an example, we consider the two-qubit Werner state given by:

\[
\rho_{AB} = \frac{1-p}{4} \mathbb{I}_A \otimes \mathbb{I}_B + \frac{p}{2} |\psi^-\rangle \langle \psi^-|,
\]

where \(0 \leq p \leq 1\) and \(|\psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)\). It is known that the Werner states are entangled for \(p > \frac{1}{2}\). As measurements we consider the three complementary observables, which in this case give the same joint entropies:

\[
H(A_i, B_i) = -\left(\frac{1-p}{2}\right) \log_2 \left(\frac{1-p}{2}\right) - \left(\frac{1+p}{2}\right) \log_2 \left(\frac{1+p}{2}\right)
\]

for \(i = 1, 2, 3\). Hence, \(65\) is violated for \(p > 0.78\), \(66\) for \(p > 0.88\) and \(67\) for \(p > 0.65\). As expected, the state-dependent bound achieves the best results since it involves more resources.

Finally, to test how the criteria perform on mixed state we run a simulation over 50000 randomly generated states. The results of this simulation, which can be seen in Fig. 2, show that over 18428 entangled states the state-dependent criterion \(65\) detects 938 states (5.09\%), while the state-dependent criterion \(66\) identifies only 20 states (0.1\%). This can be explained by the fact that random uniform mixed states have high entropy due to the classical randomness naturally present in such states.

### B. Multipartite systems

As an example of multipartite systems we focus on the case of a three qubit system. In this case a straightforward generalization of the Schmidt decomposition is not available, the pure states can be however parameterized and classified in terms of five real parameters:

\[
|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\phi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle,
\]

where \(\sum \lambda_i^2 = 1\). In particular we are interested in two classes of entangled states, the GHZ states, given by

\[
|GHZ\rangle = \lambda_0 |000\rangle + \lambda_4 |111\rangle,
\]

and the W-states, which are

\[
|W\rangle = \lambda_0 |000\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle.
\]

As in the previous, the three observables considered in each system are the three Pauli matrices, hence \(A_1 = Z_1, A_2 = X_1\) and \(A_3 = Y_1\) and the same for the other subsystems. The criteria for detecting the presence of
Figure 4. Tripartite W-States: The above shows in which areas in the plane $\lambda_0 \times \lambda_2$ the sum of the three entropies $\sum_{i=1}^{3} H(A_i, B_i, C_i)$ is below the thresholds that represent the state-independent bounds (65) and (67). The green area represents non-separable states, the blue one states that are not identified by our criteria. No states is identified as genuine multipartite entangled.

Figure 5. W-states Non-Separability: the plot shows the effectiveness of the state-dependent criterion (66) on the W-states, indeed almost all states are correctly detected (green area) as non-separable. Only a small area (blue) close to the origin is not identified.

Figure 6. W-State Genuine MultPartite Entanglement: the plot shows the performance of state-dependent criterion (68) on the W-states. A small set of states is identified as genuine multipartite entangled (red area). Most W-states are not seen (blue area).

entanglement, namely states that are not fully separable, in this case read as:

$$\sum_{i=1}^{3} H(A_i, B_i, C_i) < 6, \quad (65)$$

while the criteria for genuine multipartite entanglement are:

$$\sum_{i=1}^{3} H(A_i, B_i, C_i) < \frac{13}{3}, \quad (67)$$

For the class of GHZ states the sum of the three entropies $\sum_{i=1}^{3} H(A_i, B_i, C_i)$ is plotted as a function of $\lambda_0$ in Fig. 3 with respect to the state independent and dependent bounds. We can therefore see that the state-independent bounds fail to detect even the weakest form of entanglement. Conversely, the state-dependent bounds identify all states as non-separable but none as genuine multipartite entangled.

For the class of the W states the effectiveness of our criteria is shown in Fig. 4, 5, and 6. Since the W-states depend on two parameters we decided to use contour plots in the plane $\lambda_0 \times \lambda_2$ showing which subsets of W states are detected as non-fully separable or genuine multipartite entangled. As we can see the state-independent bounds (Fig. 4) detect the non-separable character of the W-states for a large subset of them. Conversely, no state is identified as genuine multipartite entangled. By using the state-dependent bounds (Fig 4. and 6.) we are able to detect almost all non separable W-states and, above all, we can also identify a small subset of W states as genuine multipartite entangled.

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V. CONCLUSIONS

In conclusion, we derived and characterized a number of entropic uncertainty inequalities, defined in terms of the joint Shannon entropy, whose violation guarantees the presence of entanglement. On a theoretical level, which was the main aim of this work, we clarified that EUR entanglement criteria for the joint Shannon entropy require at least three different observables or, if one considers only two measurements, the addition of the Von Neumann entropy of a subsystem, showing thus that the additivity character of the EUR holds only for two measurements \( [38] \). We also managed to extend our criteria to the case of multipartite systems, which enable us to discriminate between different types of multipartite entanglement. We then showed how these criteria perform for both bipartite and multipartite systems, providing several examples of states that are detected by the proposed criteria.

This material is based upon work supported by the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Superconducting Quantum Materials and Systems Center (SQMS) under contract number DEAC02-07CH11359 and by the EU H2020 QuantERA ERA-NET Cofund in Quantum Technologies project QuICHE.

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