The supersingular locus of the Shimura variety of $\text{GU}(2, n-2)$

Maria Fox and Naoki Imai

Abstract

We study the supersingular locus of a reduction at an inert prime of the Shimura variety attached to $\text{GU}(2, n-2)$. More concretely, we realize irreducible components of the supersingular locus as closed subschemes of flag schemes over Deligne–Lusztig varieties defined by explicit conditions. Moreover we study the intersections of the irreducible components. A stratification of Deligne–Lusztig varieties defined using a power of Frobenius action appears in the description of the intersections.

1 Introduction

The Shimura varieties play important roles in the study of number theory. One way to approach the arithmetic of Shimura varieties is to construct integral models and study their reductions. Among other things, the geometry of the supersingular loci of reductions of Shimura varieties is an important topic. One of the striking results in this direction is the study of the supersingular locus of a reduction of the Shimura variety of $\text{GU}(1, n-1)$ at an inert prime by Vollaard–Wedhorn in [VW11], where they give a description of the supersingular locus and their intersections in terms of Deligne–Lusztig varieties. This result is crucially used in [KR11].

A long standing problem since [VW11] is to extend such a result to unitary groups of other signatures. The only result in this line is a work [HP14] of Howard–Pappas on the $\text{GU}(2, 2)$-case using an exceptional isomorphism. A source of difficulty is that the Shimura variety of $\text{GU}(2, n-2)$ is not fully Hodge–Newton decomposable in the sense of [GHN19, Definition 3.1] if $n \geq 5$. In such a case, we can not expect that the supersingular locus is a union of Deligne–Lusztig varieties by [GHN19, Theorem B].

On the other hand, the study of the supersingular locus is essentially reduced to a study of an affine Deligne–Lusztig variety via the Rapoport–Zink uniformization. Further, a construction of irreducible components of an affine Deligne–Lusztig variety under some unramified condition is given by Xiao–Zhu in [XZ17]. In their construction, we can rephrase the source of difficulty in the following way: Even though the affine Deligne–Lusztig variety related to the Shimura variety of $\text{GU}(2, n-2)$ is defined using a minuscule cocharacter, non-minuscule cocharacters appear in the construction of its irreducible components if $n \geq 5$.

The objective of this paper is to get an explicit description of the irreducible components of the affine Deligne–Lusztig variety related to the Shimura variety of $\text{GU}(2, n-2)$ in terms of Deligne–Lusztig varieties.

2020 Mathematics Subject Classification. Primary: 11G18; Secondary: 14M15
Let $F$ be a non-archimedean local field. We write $L$ for the completion of the maximal unramified extension of $F$. Let $G$ be the unramified general unitary group of degree $n$ over $F$. Let $\mu$ be a cocharacter of $G$ corresponding to $z \mapsto (\text{diag}(z, z, 1, \ldots, 1), z)$ under an isomorphism $G_L \simeq \text{GL}_n \times \mathbb{G}_m$. Let $X_{\mu^*}(\varpi^{-1})$ denote the affine Deligne–Lusztig variety for the dual $\mu^*$ of $\mu$ and $\varpi^{-1} \in G(L)$, where $\varpi$ is a uniformizer of $F$ and we regard $\varpi^{-1}$ as an element of $G(L)$ by embedding it into the $\mathbb{G}_m$-component. We put $r = [n/2]$. Then $X_{\mu^*}(\varpi^{-1})$ has $r$ isomorphism classes of irreducible components, whose representatives are given by $X^{b_i, x_0}_i(\tau^*_i)$ for $1 \leq i \leq r$ as explained in [XZ17]. If $i = 1$ or $i = n/2$, then $X^{b_i, x_0}_i(\tau^*_i)$ is isomorphic to the perfection of a Deligne–Lusztig variety as shown in Proposition 8.2 and Proposition 8.3. Assume that $2 \leq i \leq [(n - 1)/2]$. Then we construct a kind of Demazure resolution $X_i$ of $X^{b_i, x_0}_i(\tau^*_i)$. On the other hand, we construct a vector bundle $\mathcal{V}_i$ of rank $2i - 1$ over a perfection $Y_i$ of a Deligne–Lusztig variety. Let $\text{Par}_t(\mathcal{G}_i)$ denote the flag scheme parametrizing subvector bundles $\mathcal{W} \subset \mathcal{V}_i$ of rank $i - 1$.

**Theorem 1.1 (Theorem 8.5).** The scheme $X_i$ is isomorphic to the closed subscheme of $\text{Par}_t(\mathcal{G}_i)$ defined by an orthogonality condition on $\mathcal{W}$ (cf. (8.4)).

An open subscheme of $X_i$ is isomorphic to an open dense subscheme $\hat{X}^{b_i, x_0}_i(\tau^*_i)$ of $X^{b_i, x_0}_i(\tau^*_i)$. Therefore we can describe $\hat{X}^{b_i, x_0}_i(\tau^*_i)$ inside $\text{Par}_t(\mathcal{G}_i)$ as we do in Proposition 8.6. However, it is important to describe the entire $X_i$ to study the intersection of irreducible components of $X_{\mu^*}(\varpi^{-1})$. We give a description of the intersections of the irreducible components in most cases. As an interesting new phenomenon, we see that an intersection is isomorphic to the perfect closed subscheme of $(\mathbb{P}^n)^{pf}$ defined by two equations

$$\sum_{i=1}^n x_i^{q+1} = 0, \quad \sum_{i=1}^n x_i^{q+1} = 0.$$  

This is a stratification of a Deligne–Lusztig variety with respect to relative positions of parabolic subgroups and their twists by the third power of the Frobenius action. Such an intersection did not appear in the preceding researches in fully Hodge–Newton decomposable cases. Our study does not cover all the intersections in general because of some technical difficulty, but it does cover all the cases if $n \leq 6$.

The method in this paper should work for unitary groups of other signatures since the results in [XZ17] and equidimensionality of Satake cycle in [S5] are available also for other signatures. On the other hand, they will be more complicated for general signatures since the number of isomorphism classes of irreducible components of the affine Deligne–Lusztig varieties become larger. Those are our future problems.

We explain the contents of each section. In 2 we recall a terminology on relative positions in flag schemes. We also give some gluing constructions of reductive schemes. In 3 we recall Deligne–Lusztig varieties and their Bruhat stratifications. We give also a new stratification using twists by a power of Frobenius map. We study the irreducibility of the stratification in some unitary case. In 4 we recall affine Grassmannian and Satake cycles. In 5 we recall and generalize results on equidimensionality of Satake cycles in [Ha06]. In 6 we recall a construction of irreducible components of affine Deligne–Lusztig varieties in [XZ17]. In 7 we explain the setting of a unitary group and apply the result in 5 to the unitary case. We also give a negative answer to a question of Xiao–Zhu (cf. Remark 7.4). In 8 we give an explicit description of irreducible components. In 9 we study the intersection of irreducible components. In 10 we explain the results in
the \( n = 6 \) case as an example. In §11, we explain a relation between the affine Deligne–Lusztig varieties and the supersingular locus of reductions of Shimura varieties in our case.

**Acknowledgements**

The authors would like to thank Liang Xiao and Xinwen Zhu for answering questions on their work. The authors are grateful to Ben Howard for helpful comments. The contents of this paper grew out of a discussion at the AIM workshop “Geometric realizations of Jacquet–Langlands correspondences” in 2019. The authors are grateful to the organizers of the workshop for the invitations. Fox was partially supported by NSF MSPRF Grant 2103150. This work was supported by JSPS KAKENHI Grant Number 22H00093.

2 Flag scheme

2.1 Relative position

Let \( G \) be a reductive group scheme over a scheme \( \mathcal{S} \). Let \( \text{Par}(G) \) be the scheme of parabolic subgroups of \( G \). Let \( \text{Dyn}(G) \) be the scheme of Dynkin for \( G \) constructed in [SGA3-3, XXIV, 3.3].

**Remark 2.1.** If \((\mathcal{T}, M, R)\) is a splitting of \( G \) in the sense of [SGA3-3, XXII, Définition 1.13] and \( \Delta \) is a set of simple roots, then we have a canonical isomorphism

\[
\text{Dyn}(G) \simeq \Delta_{\mathcal{T}}.
\]

(2.1)

This is stated in [SGA3-3, XXIV, 3.4 (iii)] choosing a pinning, but the isomorphism actually depends only on \((\mathcal{T}, M, R)\) and \( \Delta \).

Let \( \text{Oc}(\text{Dyn}(G)) \) be the scheme of sets of open and closed subschemes of \( \text{Dyn}(G) \) (cf. [SGA3-3, XXVI, 3.1]). We have a projective smooth morphism

\[
t: \text{Par}(G) \to \text{Oc}(\text{Dyn}(G))
\]

of schemes as [SGA3-3, XXVI, Théorème 3.3]. For \( t, t' \in \text{Oc}(\text{Dyn}(G))(\mathcal{T}) \), we put

\[
\text{Par}_t(G) = t^{-1}(t), \quad \text{Par}_{t, t'}(G) = (t \times t)^{-1}(t, t').
\]

We recall results from [SGA3-3 XXVI, 4.5.3, 4.5.4]. Let \( \text{Stand}(G) \) be the scheme of pairs of parabolic subgroups of \( G \) in mutually standard positions. Let \( \text{TypeStand}(G) \) be the scheme of types of mutually standard positions in \( G \). The natural morphism

\[
t_2: \text{Stand}(G) \to \text{TypeStand}(G)
\]

is smooth and a quotient of \( \text{Stand}(G) \) by the conjugacy action of \( G \). We have a commutative diagram

\[
\begin{array}{ccc}
\text{Stand}(G) & \xrightarrow{t_2} & \text{TypeStand}(G) \\
\downarrow & & \downarrow q_# \\
\text{Par}(G) \times_{\mathcal{T}} \text{Par}(G) & \xrightarrow{t \times t} & \text{Oc}(\text{Dyn}(G)) \times_{\mathcal{T}} \text{Oc}(\text{Dyn}(G)).
\end{array}
\]
Let \( \mathcal{P} \) be a parabolic subgroup scheme of \( \mathcal{G} \). Let \( \text{Par}(\mathcal{G}; \mathcal{P}) \) be the scheme of parabolic subgroups of \( \mathcal{G} \) in standard positions relative to \( \mathcal{P} \). Let \( t \in \text{Oc}(\text{Dyn}(\mathcal{G}))(\mathcal{I}) \). We put

\[
\text{Par}_t(\mathcal{G}; \mathcal{P}) = \text{Par}(\mathcal{G}; \mathcal{P}) \cap \text{Par}_t(\mathcal{G}).
\]

Then we have a morphism

\[
t_{\mathcal{P}}: \text{Par}_t(\mathcal{G}; \mathcal{P}) \rightarrow q_{\mathcal{G}}^{-1}(t(\mathcal{P}), t)
\]

induced by \( t_2 \). For an \( \mathcal{I} \)-scheme \( \mathcal{I}' \) and \( r \in (q_{\mathcal{G}}^{-1}(t(\mathcal{P}), t))(\mathcal{I}') \), we define \( \text{Par}_t(\mathcal{G}; \mathcal{P})_r \) by the fiber product

\[
\begin{array}{ccc}
\text{Par}_t(\mathcal{G}; \mathcal{P}) & \xrightarrow{t_{\mathcal{P}}} & \mathcal{I}' \\
\downarrow & & \downarrow r \\
\text{Par}_t(\mathcal{G}; \mathcal{P})_r & \xrightarrow{q_{\mathcal{G}}^{-1}(t(\mathcal{P}), t)} & \mathcal{I}'
\end{array}
\]

**Remark 2.2.** Let \( \mathcal{D} \) be a parabolic subgroup scheme of \( \mathcal{G} \). Let \( \mathcal{I}' \) be an \( \mathcal{I} \)-scheme. We write \( \mathcal{G}' \), \( \mathcal{P}' \), \( \mathcal{D}' \) for the base change of \( \mathcal{G} \), \( \mathcal{P} \), \( \mathcal{D} \) to \( \mathcal{I}' \). Assume that a maximal torus \( \mathcal{T}' \) of \( \mathcal{G}' \) is contained in \( \mathcal{P}' \cap \mathcal{D}' \). Then we have a natural isomorphism

\[
W_{\mathcal{G}}(\mathcal{I}') \backslash W_{\mathcal{G}}(\mathcal{I}') / W_{\mathcal{G}}(\mathcal{I}') \simeq q_{\mathcal{G}}^{-1}(t(\mathcal{P}), t(\mathcal{D})) \times_{\mathcal{G}} \mathcal{I}'
\]

over \( \mathcal{I}' \) as in \([\text{SGA3-3}, \text{XXVI. 4.5.3}]\).

**Notation 2.3.** Assume that \( \mathcal{G} \) is split and \( \mathcal{I} \) is connected. Let \((\mathcal{I}, M, R)\) be a splitting of \( \mathcal{G} \) and \( \Delta \) be a set of simple roots. Let \((W, S)\) be the Coxeter system of \((M, R, \Delta)\). For \( I \subset S \), let \( W_I \) be the subgroup of \( W \) generated by \( I \), and let \( t(I) \) be the element of \( \text{Oc}(\text{Dyn}(\mathcal{G}))(\mathcal{I}) \) corresponding to \( I \) under \((2.1)\). Conversely, let \( I(t) \) be the subset of \( S \) corresponding to \( t \) under \((2.1)\) for \( t \in \text{Oc}(\text{Dyn}(\mathcal{G}))(\mathcal{I}) \). We simply write \( W_I \) for \( W_{I(t)} \).

### 2.2 Inner gluing

**Definition 2.4.** Let \( \mathcal{G}_0 \) be a reductive group scheme over a scheme \( \mathcal{I}_0 \). Let \( \mathcal{I} \) be a scheme over \( \mathcal{I}_0 \). An inner gluing over \( \mathcal{I} \) of \( \mathcal{G}_0 \) as a pair \((\mathcal{G}, \varphi)\), where \( \mathcal{G} \) is a reductive group scheme over \( \mathcal{I} \) and \( \varphi \) is a global section of the Zariski sheaf

\[
\text{Isom}_{\mathcal{I}}(\mathcal{G}_0 \times_{\mathcal{I}_0} \mathcal{I}, \mathcal{G}) / \text{Inn}_\mathcal{I}(\mathcal{G}_0 \times_{\mathcal{I}_0} \mathcal{I})
\]

on \( \mathcal{I} \).

**Remark 2.5.** Let \( \mathcal{V} \) be a vector bundle of rank \( n \) on \( \mathcal{I} \). We put \( \mathcal{G} = \text{Aut}_\mathcal{I}(\mathcal{V}) \). By taking Zariski local trivializations of \( \mathcal{V} \), we obtain an inner gluing \((\mathcal{G}, \varphi_\mathcal{V})\) over \( \mathcal{I} \) of \( \text{GL}_{n, \mathbb{Z}} \). This is independent of the choice of trivializations, because a difference of trivializations induces an inner automorphism of \( \text{GL}_n \).

**Lemma 2.6.** Let \( \pi: \mathcal{I} \rightarrow \mathcal{I}_0 \) be a morphism of schemes. Let \( \mathcal{G}_0 \) be a reductive group scheme over \( \mathcal{I}_0 \). Let \((\mathcal{G}, \varphi)\) an inner gluing over \( \mathcal{I} \) of \( \mathcal{G}_0 \).

1. The section \( \varphi \) induces isomorphisms

\[
\text{Oc}(\text{Dyn}(\mathcal{G}_0)) \times_{\mathcal{I}_0} \mathcal{I} \xrightarrow{\sim} \text{Oc}(\text{Dyn}(\mathcal{G})),
\]

\[
\text{TypeStand}(\mathcal{G}_0) \times_{\mathcal{I}_0} \mathcal{I} \xrightarrow{\sim} \text{TypeStand}(\mathcal{G})
\]

which are compatible with \( q_{\mathcal{G}_0} \) and \( q_{\mathcal{G}} \).
(2) Assume that $G_0$ is split and $\mathcal{X}_0$ is connected. Let $(\mathcal{X}_0, M, R)$ be a splitting of $G_0$ and $\Delta$ be a set of simple roots. Let $(W, S)$ be the Coxeter system of $(M, R, \Delta)$. Let $t_0, t'_0 \in \text{Oc}(\text{Dyn}(G_0))(\mathcal{X}_0)$. Let $t, t' \in \text{Oc}(\text{Dyn}(G))(\mathcal{X})$ denote the pullbacks to $\mathcal{X}$ of $t_0, t'_0$. Then $\varphi$ induces an isomorphism

$$(W_{t_0} \setminus W/W_{t'_0})_{\mathcal{X}} \xrightarrow{\sim} q_{\mathcal{X}}^{-1}(t, t').$$

**Proof.** There is a Zariski covering $\{U_{\lambda}\}_{\lambda \in \Lambda}$ of $S$ and a family of ismorphisms $\varphi_{\lambda}: G_0 \times_{\mathcal{X}_0} U_{\lambda} \xrightarrow{\sim} G \times_{\mathcal{X}} U_{\lambda}$ such that $\varphi_{\lambda}$ is compatible with $\varphi|_{U_{\lambda}}$. Then the family of ismorphisms $\varphi_{\lambda}$ induce isomorphisms

$$\text{Oc}(\text{Dyn}(G_0)) \times_{\mathcal{X}_0} U_{\lambda} \xrightarrow{\sim} \text{Oc}(\text{Dyn}(G \times_{\mathcal{X}} U_{\lambda})).$$

These isomorphisms glue together to give the first isomorphism in the claim [1] by [SGA3-3, XXIV, 3.4 (iv)].

The family of ismorphisms $\varphi_{\lambda}$ induce also isomorphisms

$$\text{Stand}(G_0) \times_{\mathcal{X}_0} U_{\lambda} \xrightarrow{\sim} \text{Stand}(G \times_{\mathcal{X}} U_{\lambda}).$$

By taking the quotients by the conjugacy actions of $G_0 \times_{\mathcal{X}_0} U_{\lambda} \cong G \times_{\mathcal{X}} U_{\lambda}$, we obtain isomorphisms

$$\text{TypeStand}(G_0) \times_{\mathcal{X}_0} U_{\lambda} \xrightarrow{\sim} \text{TypeStand}(G \times_{\mathcal{X}} U_{\lambda}).$$

These isomorphisms glue together to give the second isomorphism in the claim [1] because we take quotients by conjugacy actions. By the constructions, two isomorphisms in the claim [1] are compatible with $q_{G_0}$ and $q_{G}$. By [1] we have an isomorphism

$$q_{G_0}^{-1}(t_0, t'_0) \times_{\mathcal{X}_0} \mathcal{X} \xrightarrow{\sim} q_{G}^{-1}(t, t') \quad (2.3)$$

induced by $\varphi$. The claim [2] follows from [SGA3-3, XXII, Proposition 3.4] and (2.3). \hfill $\Box$

### 3 Stratification of Deligne–Lusztig variety

#### 3.1 Deligne–Lusztig variety

Let $G_0$ be a connected reductive group over $\mathbb{F}_q$. We take a maximal torus and a Borel subgroup $T_0 \subset B_0 \subset G_0$ over $\mathbb{F}_q$. We write $G$, $B$ and $T$ for the base changes to $\overline{\mathbb{F}}_q$ of $G_0$, $B_0$ and $T_0$. Let $(W, S)$ be the Coxeter system of $G$ with respect to $T$ and $B$. For $I, J \subset S$, we write $\text{Par}_I(G)$ and $\text{Par}_{I,J}(G)$ for $\text{Par}_{t(I)}(G)$ and $\text{Par}_{t(I),t(J)}(G)$.

For $I, J \subset S$ and $w \in W$, we put

$$\text{Par}_{I,J}(G)_{[w]} = t_2^{-1}(r_w),$$

where $r_w \in (q_{G}^{-1}(t(I), t(J)))(\overline{\mathbb{F}}_q)$ corresponds to $[w] \in W_1 \setminus W/W_J$ by Lemma 2.6[2]. Let $\text{Par}_{I,J}(G)_{[w],w}$ be the closed reduced subscheme of $\text{Par}_{I,J}(G)$ determined by

$$\bigcup_{[w'] \leq [w]} \text{Par}_{I,J}(G)_{[w']}.$$
Let $F$ be the $q$-th power Frobenius endomorphism of $G$ obtained from $G_0$. Let $I \subset S$ and $w \in W$. For $* \in \{[w], \leq [w]\}$ with $[w] \in W_I \backslash W/W_{F(I)}$, let $X_I^F(*)$ be the locally closed subscheme of $\text{Par}_I(G)$ defined by the fiber product

$$X_I^F(*) \longrightarrow \text{Par}_{I,F(I)}(G)_* \longrightarrow \text{Par}_I(G) \overset{(\text{id},F)}{\longrightarrow} \text{Par}_I(G) \times \text{Par}_{F(I)}(G).$$

If there is no confusion, we simply write $X_I(*)$ for $X_I^F(*)$.

For $I \subset J \subset S$, we have a natural morphism

$$\pi_{I,J}: X_I([w]) \rightarrow X_J([w])$$

which sends a parabolic subgroup $P$ of $G$ of type $I$ to a unique parabolic subgroup $P'$ of $G$ of type $J$ containing $P$.

### 3.2 Bruhat stratification

Let $I, J \subset S$ and $w \in W$. Let $P_J$ be a parabolic subgroup of $G$ of type $J$. For $* \in \{[w'], \leq [w']\}$ with $[w'] \in W_I \backslash W/W_J$, we let $X_I([w])_{P_J,*}$ be the locally closed subscheme of $X_I([w])$ defined by the fiber product

$$X_I([w])_{P_J,*} \longrightarrow \text{Par}_{I,J}(G)_* \longrightarrow \text{Par}_I(G) \overset{(\text{id},P_J)}{\longrightarrow} \text{Par}_I(G) \times \text{Par}_J(G).$$

### 3.3 Stratification relative to Frobenius twists

For $1 \leq i \leq m$, let $F_i$ be an Frobenius endomorphism of $G$ which descends it to an algebraic group over a finite filed. Let $w_1, \ldots, w_m \in W$. For $* \in \{[w_i], \leq [w_i]\}$ with $[w_i] \in W_I \backslash W/W_{F_i(I)}$ and $1 \leq i \leq m$, let $X_{I, \ldots, F_i}^m(*, \ldots, *)$ be the locally closed subscheme of $Par_I$ defined by the fiber product

$$X_{I, \ldots, F_i}^m(*, \ldots, *) \longrightarrow \prod_{1 \leq i \leq m} \text{Par}_{I,F_i}(G)_* \longrightarrow \text{Par}_I(G) \overset{\prod_{1 \leq i \leq m}(\text{id},F_i)}{\longrightarrow} \prod_{1 \leq i \leq m} (\text{Par}_I(G) \times \text{Par}_{F_i(I)}(G)).$$

Then $X_{I, \ldots, F_i}^m([w_1], \ldots, [w_m])$ for $[w_i] \in W_I \backslash W/W_{F_i(I)}$ and $2 \leq i \leq m$ give a stratification of $X_{I, F_i}([w_1])$.

### 3.4 Unitary case

We put $V_0 = F_q^d$ equipped with the hermitian form

$$F_q^d \times F_q^d \rightarrow F_q^d; \ ((a_i)_{1 \leq i \leq d}, (a'_i)_{1 \leq i \leq d}) \mapsto \sum_{i=1}^d a_i a'_{d+1-i}.$$
We put $G_0 = GU(V_0)$. By taking the first factor of the decomposition
\[ F_q^2 \otimes_{F_q} F_q \simeq F_q^6 \times F_q^2; \quad a \otimes b \mapsto (ab, ab^q), \]
we have an isomorphism
\[ G \simeq GL_d \times G_m. \quad (3.1) \]
Let $T \subset B \subset G$ be the maximal torus and the Borel subgroup determined by the diagonal torus $T_d$ and the upper triangular subgroup $B_d$ of $GL_d$ under (3.1). Let $(W_G, \{s_1, \ldots, s_{d-1}\})$ be the Coxeter system of $G$ with respect to $T$ and $B$, where $s_i$ corresponds to the simple root
\[ T_d \times G_m \to G_m; (\text{diag}(x_1, \ldots, x_d), z) \mapsto x_i x_{i+1}^{-1} \]
of $GL_d \times G_m$ under (3.1). For $1 \leq i_1 < \cdots < i_l \leq d - 1$, we put
\[ I_{i_1, \ldots, i_l}^d = \{ s_i \mid i \in \{1, \ldots, d-1\} \setminus \{i_1, \ldots, i_l\} \}. \]

**Lemma 3.1.** Assume that $2 \leq i \leq d/2$. The schemes $X_{i-1,d-i}^{F, F^2, F^3}([1], \leq [s_{i-1}], [1])$ and $X_{i-1,d-i}^{F, F^2}([1], \leq [s_{d-i}])$ are irreducible.

**Proof.** The scheme $X_{i-1,d-i}^{F, F^2}([1])$ is irreducible by [BR06, Theorem 1]. Hence, it suffices to show the following claims:

1. The image of
   \[ \pi_{i-1,d-i}^{F, F^2, F^3} : X_{i-1,d-i}^{F, F^2, F^3}([1]) \to X_{i-1,d-i}^{F, F^2, F^3}([1]) \]
on $F_q$-valued points is equal to $X_{i-1,d-i}^{F, F^2, F^3}([1], \leq [s_{i-1}], [1])(F_q).

2. The image of
   \[ \pi_{i-1,d-i}^{F, F^2} : X_{i-1,d-i}^{F, F^2}([1]) \to X_{i-1,d-i}^{F, F^2}([1]) \]
on $F_q$-valued points is equal to $X_{i-1,d-i}^{F, F^2}([1], \leq [s_{d-i}])((F_q)$.

We show the claim (1). We equip $F_q^d$ with the paring
\[ F_q^d \times F_q^d \to F_q; \quad ((x_i)_{1 \leq i \leq d}, (y_i)_{1 \leq i \leq d}) \mapsto \sum_{i=1}^{d} x_i y_{d+1-i}. \quad (3.2) \]
For an $F_q$-vector subspace $V \subset F_q^d$, let $V^\perp$ denote the orthogonal complement of $V$ with respect to the paring (3.2). The $q$-th power Frobenius element $F$ acts on $F_q^d$. A point of $X_{i-1,d-i}^{F, F^2}([1])(F_q)$ corresponds to a filtration $0 \subset V_1 \subset V_2 \subset F_q^d$ such that $\dim V_1 = i - 1$, $\dim V_2 = d - i$ and
\[ V_1 \subset F(V_1^\perp) \subset V_2 \subset F(V_1^\perp). \quad (3.3) \]
The condition (3.3) implies
\[ V_1 + F^2(V_1) \subset F(V_2^\perp). \quad (3.4) \]
Therefore we have
\[ F^3(V_1) \subset F(V_1 + F^2(V_1)) \subset F^2(V_2^\perp) \subset F(V_2) \subset V_1^\perp \cap F^2(V_1^\perp) \subset V_1^\perp. \quad (3.5) \]
The conditions (3.3), (3.4) and (3.5) imply that $V_1$ defines a point of
\[ X_{I_d}^{F,F^2,F^3}([1],[s_{i-1}],[1])(\overline{F_q}), \]
because $\dim F(V_1^\perp) = i$. To show the claim (1), it suffices to show that the image of $\pi_{I_d}^{-1,d-1,I_d}^{-1}$ on $\overline{F_q}$-valued points contains
\[ X_{I_d}^{F,F^2,F^3}([1],[s_{i-1}],[1])(\overline{F_q}), \] (3.6)
because $X_{I_d}^{-1,d-1}([1])$ is proper. A point of (3.6) gives an $\overline{F_q}$-vector subspace $V_1 \subset \overline{F_q}$ of dimension $i - 1$ such that
\[ V_1 \subset F(V_1^\perp), \quad \dim(V_1 + F^2(V_1)) = i, \quad F^3(V_1) \subset V_1^\perp. \] (3.7)
The condition implies
\[ F(V_1 + F^2(V_1)) \subset V_1^\perp \cap F^2(V_1^\perp) \]
and $\dim V_1^\perp \cap F^2(V_1^\perp) = d - i$. We take $V_2 \subset \overline{F_q}$ such that $F(V_2) = V_1^\perp \cap F^2(V_1^\perp)$. Then $(V_1, V_2)$ defines a point of $X_{I_d}^{-1,d-1}([1])(\overline{F_q})$ whose image under $\pi_{I_d}^{-1,d-1,I_d}^{-1}$ is the point of (3.6) corresponding to $V_1$. Therefore we obtain the claim (1).

The claim (2) is proved similarly. \(\square\)

4 Affine Grassmannian

Let $F$ be a non-archimedean local field with residue field $k = \mathbb{F}_q$. Let $\mathcal{O}$ be the ring of integers of $F$. For a perfect $k$-algebra $R$, we put
\[ W_\mathcal{O}(R) = \lim_{\leftarrow n} W(R) \otimes_{W(k)} \mathcal{O} / \varpi^n, \]
$D_R = \text{Spec}(W_\mathcal{O}(R))$ and $D_R^* = \text{Spec}(W_\mathcal{O}(R)[\frac{1}{\varpi}])$. For an affine group scheme $H$ of finite type over $\mathcal{O}$, we define the jet group $L^+H$ and the loop group $LH$ by
\[ L^+H(R) = H(W_\mathcal{O}(R)), \quad LH(R) = H(W_\mathcal{O}(R)[\frac{1}{\varpi}]). \]
We put $L = W_\mathcal{O}(\overline{k})[\frac{1}{\varpi}]$. We note that $LH(\overline{k}) = H(L)$.

Let $G$ be a reductive group scheme over $\mathcal{O}$. Let $T$ be the abstract Cartan subgroup. Let $\Phi \subset X^\bullet(T)$ denote the set of roots of $G$. We fix a Borel subgroup $B \subset G$, which determines the semi-group of dominant coweights $X^\bullet(T)^+ \subset X^\bullet(T)$. Let $U$ be the unipotent radical of $B$. Let $\rho \in X^\bullet(T)_0$ be the half sum of all positive roots.

Let $Gr_G$ denote the affine Grassmannian over $k$ of $G$ defined by $Gr_G = LG/L^+G$. For a finite etale extension $\mathcal{O}'$ of $\mathcal{O}$ with residue field $k'$, we have a natural isomorphism
\[ (Gr_G)_{k'} \cong Gr_{G_{\mathcal{O}'}} \] (4.1)
by the construction. We simply write $Gr$ for $Gr_G$ if there is no confusion. Then $Gr$ is an ind-perfectly projective scheme by [BS17, Corollary 9.6]. Let $\mathcal{E}^0$ denote the trivial $G$-torsor over $\mathcal{O}$. For a perfect $k$-algebra $R$, we have
\[ \text{Gr}(R) = \left\{ (\mathcal{E}, \beta) \bigg| \begin{array}{l} \mathcal{E} \text{ is a } G\text{-torsor on } D_R, \\
\beta : \mathcal{E}|_{D_R^*} \simeq \mathcal{E}^0|_{D_R^*} \text{ is a trivialization.} \end{array} \right\} \] (4.2)
(cf. [Zhu17, Lemma 1.3]). We sometimes write $\beta : E \rightarrow E^0$ for $\beta : E|_{D^+} \simeq E^0|_{D^+}$ in (1.2), and call it a modification. Given a point $(E, \beta)$, one can define a relative position invariant $\text{inv}(\beta) \in X_*(T)^+$. Hence we obtain the claim because $\text{Gr}_{\beta}$ is transitive and $\text{Gr}_{\beta}$ is a nonempty scheme stable under the action of $L\beta$. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be two sequences. We put

\[
\text{Gr}_{\lambda | \mu}^0 = \text{Gr}_{\lambda} \times_{\text{Gr}_{\mu}} \text{Gr}_{\mu}, \quad \hat{\text{Gr}}_{\lambda | \mu}^0 = \hat{\text{Gr}}_{\lambda} \times_{\hat{\text{Gr}}_{\mu}} \hat{\text{Gr}}_{\mu}.
\]

where the products are over the convolution maps $m_{\lambda} : \text{Gr}_{\lambda} \rightarrow \text{Gr}_{\lambda}$, $m_{\mu} : \text{Gr}_{\mu} \rightarrow \text{Gr}_{\mu}$ and their restrictions. We write

\[
m_{\lambda | \mu} : \text{Gr}_{\lambda | \mu}^0 \rightarrow \text{Gr}_{\mu}
\]

for the natural projection. We simply write $m$ for $m_{\lambda | \mu}$ if there is no confusion. For $1 \leq j \leq l$, we define

\[
\text{pr}_j : \text{Gr}_{\lambda | \mu}^0 \rightarrow \text{Gr}_{(\lambda_1, \ldots, \lambda_j)}
\]

by sending $((\alpha_i)_{1 \leq i \leq k}, (\beta_i)_{1 \leq i \leq n})$ to $(\alpha_j)_{1 \leq i \leq j}$.

An irreducible component of $\text{Gr}_{\lambda | \mu}^0$ of dimension $\langle \rho, |\lambda| + |\mu| \rangle$ is called a Satake cycle. Let $S_{\lambda | \mu}$ be the set of Satake cycles in $\text{Gr}_{\lambda | \mu}^0$. We sometimes write $\text{Gr}_{\lambda | \mu}^{0, a}$ instead of $a \in S_{\lambda | \mu}^0$ for the Satake cycle. We put

\[
\hat{\text{Gr}}_{\lambda | \mu}^{0, a} = \hat{\text{Gr}}_{\lambda | \mu}^0 \cap \text{Gr}_{\lambda | \mu}^{0, a}.
\]

Lemma 4.1. For $a \in S_{\lambda | \mu}$, the scheme $\hat{\text{Gr}}_{\lambda | \mu}^{0, a}$ is not empty.

Proof. The dimension of $\text{Gr}_{\lambda | \mu}^{0, a} \setminus \hat{\text{Gr}}_{\lambda | \mu}^{0, a}$ is less than $\langle \rho, |\lambda| + |\mu| \rangle$ by [XZ17, Proposition 3.1.10 (1)]. Hence we obtain the claim. \hfill \Box

We fix an embedding $T \subset B$. Let $\mu \in X_*(T)$. Let $O$ be $O_L = W_\mu(k)$ or a finite etale extension of $O$ which splits $G$. For $a \in \Phi$, let $U_{a, \sigma}$ denote the root subgroup of $G_{\sigma}$ corresponding to $a$. Let $P_{a, \sigma}$ denote the parabolic subgroup of $G_{\sigma}$ generated by $T_{\sigma}$ and $U_{a, \sigma}$ for $a \in \Phi$ such that $\langle a, \mu \rangle$.

We write $\varpi$ for $\mu(\varpi) \in G(L) = LG(k)$. Let $[\varpi^\mu]$ denote the point of $\text{Gr}_{\mu}$ determined by $\varpi$. For $\mu \in X_*(T)^+$, the Schubert cell $\text{Gr}_{\mu}$ is the $L^\mu$-orbit of $[\varpi]$. By [Zhu17, Proposition 1.23 (1)].

Lemma 4.2. For $a \in S_{\lambda | \mu}$, the natural morphism $\text{Gr}_{\lambda | \mu}^{0, a} \rightarrow \text{Gr}_{\mu}$ is surjective.

Proof. The natural morphism $\hat{\text{Gr}}_{\lambda | \mu}^{0, a} \rightarrow \text{Gr}_{\mu}$ is surjective, because the action of $L^\mu G$ on $\text{Gr}_{\mu}$ is transitive and $\hat{\text{Gr}}_{\lambda | \mu}^{0, a}$ is a nonempty scheme stable under the action of $L^\mu G$ by Lemma 4.1. Hence we obtain the claim because $\text{Gr}_{\lambda | \mu}^{0, a} \rightarrow \text{Gr}_{\mu}$ is perfectly proper and $\text{Gr}_{\mu} \subset \text{Gr}_{\mu}$ is Zariski dense by [Zhu17, Proposition 1.23 (3)]. \hfill \Box

9
For \( \lambda \in X_\bullet(T) \), let \( S_\lambda \) be the \((LU)\pi\)-orbit of \( \omega^\lambda \) in \( \hat{\text{Gr}}_\pi \). For \( \lambda \in X_\bullet(T) \) and \( \mu \in X_\bullet(T)^+ \), an irreducible component of \( S_\lambda \cap \text{Gr}_\mu \) is called a Mirković–Vilonen cycle after [MV07]. Let \( \mathcal{MV}_\mu(\lambda) \) be the set of the Mirković–Vilonen cycles in \( S_\lambda \cap \text{Gr}_\mu \). We sometimes write \((S_\lambda \cap \text{Gr}_\mu)^b\) instead of \( \mathcal{MV}_\mu(\lambda) \) for the Mirković–Vilonen cycle.

Let \((\hat{G}, \hat{B}, \hat{T})\) be the Langlands dual over \( \overline{K} \) of \((G, B, T)\). For \( \mu \in X_\bullet(T)^+ = X^\bullet(\hat{T})^+ \), let \( V_\mu \) denote the irreducible algebraic representation of \( \hat{G} \) of highest weight \( \mu \). For an algebraic representation \( V \) of \( G \) and \( \lambda \in X_\bullet(T) = X^\bullet(\hat{T}) \), let \( V(\lambda) \) denote the \( \lambda \)-weight space of \( V \). Then we have

\[
|\mathcal{MV}_\mu(\lambda)| = \dim V_\mu(\lambda) \tag{4.3}
\]

by [GHKR06] Proposition 5.4.2 and [Zhu17] Corollary 2.8].

For \( \nu, \mu \in X_\bullet(T)^+ \) and \( \lambda \in X_\bullet(T)^+ \) such that \( \nu + \lambda \in X_\bullet(T)^+ \), there is an injective map

\[
i^\mathcal{MV}_\nu : \mathcal{S}(\nu, \mu)^{\nu + \lambda} \to \mathcal{MV}_\mu(\lambda)
\]

constructed by [XZ17] Lemma 3.2.7.

## 5 Equidimensionality of Satake cycles

**Lemma 5.1.** The morphism \( m_{\mu*} : \text{Gr}_\mu \to \text{Gr}_\pi \) is Zariski-locally trivial on \( \hat{\text{Gr}}_\lambda \).

**Proof.** Taking the base change to an unramified extension of \( \mathcal{O} \), we may assume that \( G \) is split by [11]. As in the proof of [Hai06, Lemma 2.1], it suffices to show that

\[
L^+G \to L^+G/(L^+G \cap \omega^\lambda L^+G \omega^{-\lambda})
\]

has a section Zariski-locally. Since \( L^+U/(L^+U \cap \omega^\lambda L^+U \omega^{-\lambda}) \) is an open subscheme of \( L^+G/(L^+G \cap \omega^\lambda L^+G \omega^{-\lambda}) \), it suffice to show that

\[
L^+U \to L^+U/(L^+U \cap \omega^\lambda L^+U \omega^{-\lambda})
\]

has a section. We fix an identification \( \mathcal{G}_\alpha \simeq U_{\alpha, \mathcal{O}} \) for a positive root \( \alpha \). For a positive root \( \alpha \), let \( L^+_{\leq \langle \alpha, \lambda \rangle} U_{\alpha, \mathcal{O}} \) be a closed subscheme of \( L^+U_{\alpha, \mathcal{O}} \) defined by the condition \( x_i = 0 \) for \( i \geq \langle \alpha, \lambda \rangle \) for a point \( \sum_{i=0}^\infty \omega^i x_i \) of \( L^+U_{\alpha, \mathcal{O}} \). Then the composition

\[
\prod_{\alpha \leq \langle \alpha, \lambda \rangle} L^+_{\leq \langle \alpha, \lambda \rangle} U_{\alpha, \mathcal{O}} \to L^+U \to L^+U/(L^+U \cap \omega^\lambda L^+U \omega^{-\lambda})
\]

is an isomorphism. Hence we have a section. \( \square \)

**Lemma 5.2.** Assume that \( \mu \) is a dominant minuscule cocharacter and \( w \in W \). We have an isomorphism

\[
S_{w\mu} \cap \text{Gr}_\mu \simeq L^+U_{\pi}/((L^+U)_{\pi} \cap \omega^{\mu}(L^+U)_{\pi} \omega^{-\mu}).
\]

In particular, \( S_{w\mu} \cap \text{Gr}_\mu \) is a perfection of an affine space of dimension \( \langle \rho, \mu + w\mu \rangle \).

**Proof.** The first claim follows from [XZ17, (3.2.3)]. The second claim follows from the first one as in the proof of [Hai06, Lemma 3.2]. \( \square \)

**Theorem 5.3.** Assume that \( \mu_\bullet \) are minuscule. For a point \( y \) of \( \hat{\text{Gr}}_\lambda \), the fiber of \( m_{\mu*} : \text{Gr}_{\mu} \to \text{Gr}_{\pi} \) at \( y \) is equidimensional of dimension \( \langle \rho, |\mu_\bullet| - \lambda \rangle \).
Proof. This is proved in the same way as [Hai06 Theorem 3.1] using Lemma 5.1 and Lemma 5.2 instead of [Hai06 Lemma 2.1 and Lemma 3.2] respectively. \qed

Proposition 5.4. Assume that each $\mu_i$ is a sum of minuscule cocharacters. Then, for a point $y$ of $Gr_\lambda$, any irreducible component of the fiber $m_{\mu_i}^{-1}(y)$ whose generic point belongs to $Gr_{\mu_i}$ has dimension $\langle \rho, [\mu_i] - \lambda \rangle$.

Proof. This follows from Theorem 5.3 in the same way as [Hai06 Proposition 4.1]. \qed

6 Affine Deligne–Lusztig variety

Recall that $L = W_\mathcal{O}(\mathfrak{k})[\frac{1}{\mathfrak{m}}]$. Let $b \in G(L)$ and $\mu \in \mathfrak{X}_\bullet(T)$. Let $\sigma$ denote the $q$-th power Frobenius element. We define the affine Deligne–Lusztig variety $X_\mu(b)$ by

$$X_\mu(b) = \{ g(L^+G)_\sigma \in Gr_\sigma \mid g^{-1}b\sigma(g) \in (L^+G)_\sigma\mathfrak{o}((L^+G)_\sigma) \}.$$ 

Let $B(G)$ be the set of $\sigma$-conjugacy classes of $G(L)$. We define $B(G, \mu) \subset B(G)$ as [Kot97 6.2]. Then $X_\mu(b)$ is non-empty if and only if $[b] \in B(G, \mu)$ by [Gas10 Theorem 5.1].

An element of $B(G)$ is called unramified if it is contained in the image of the natural map $B(T) \to B(G)$. Let $(B(G))_{ur}$ denote the set of unramified elements of $B(G)$.

For $\chi \in \mathfrak{X}_\bullet(T)$, we put

$$\overline{\chi} = \frac{1}{|\langle \sigma \rangle|} \sum_{\chi' \in \mathfrak{X}_\bullet(T)_\sigma} \chi' \in \mathfrak{X}_\bullet(T)_\sigma^T.$$ 

The natural paring $\mathfrak{X}_\bullet(T) \times \mathfrak{X}_\bullet(T) \to \mathbb{Z}$ induces a paring $\langle , \rangle : \mathfrak{X}_\bullet(T)_{ur} \times \mathfrak{X}_\bullet(T)_\sigma^T \to \mathbb{Q}$. We put

$$\mathfrak{X}_\bullet(T)_\sigma^T = \{ [\lambda] \in \mathfrak{X}_\bullet(T)_\sigma \mid \langle [\lambda], \overline{\alpha} \rangle \geq 0 \text{ for every } \alpha \in \Delta \}.$$ 

Then we have a bijection

$$\mathfrak{X}_\bullet(T)_\sigma^T \cong B(G)_{ur}; \ [\lambda] \mapsto [\overline{\alpha}^\lambda]$$

as in [XZ17 Lemma 4.2.3].

For $\tau \in \mathfrak{X}_M(T)$, we write $X_\mu(\tau)$ for $X_\mu(\overline{\alpha}^\tau)$. We assume that $b = \overline{\alpha}^\tau$ for $\tau \in \mathfrak{X}_\bullet(T)$ such that $[\tau] \in \mathfrak{X}_\bullet(T)_\sigma^T$. We can define the twisted centralizer $J_\tau$ over $\mathcal{O}$ for $\overline{\alpha}^\tau$ as in [XZ17 4.2.13]. We note that $J_\tau = G$ if $[b] \in B(G)_{ur}$ is basic.

We assume that $G$ satisfies [XZ17 Hypothesis 4.4.1]. Further, we assume that $Z_G$ is connected.

Let $\lambda \in \mathfrak{X}_\bullet(T)$ such that $[\lambda] = [\tau] \in \mathfrak{X}_\bullet(T)_\sigma^T$. We take $\delta_\lambda \in \mathfrak{X}_\bullet(T)$ such that $\lambda = \tau + \delta_\lambda - \sigma(\delta_\lambda)$. Let $b \in \mathfrak{MV}_M(\lambda)$. We take $\nu_b \in \mathfrak{X}_\bullet(T)$ as in [XZ17 Lemma 4.4.3]. We put $\tau_b = \lambda + \nu_b - \sigma(\nu_b)$. Then we have an isomorphism

$$X_\mu(b) = X_\mu(\tau) \simeq X_\mu(\tau_b); \ gL^+G \mapsto \overline{\alpha}^{\delta_\lambda + \nu_b}g(L^+G). \ (6.1)$$

Let $a \in S_{(\nu_b, \nu)}^{\lambda + \nu_b}$ be the unique element such that $b = i_{\nu_b}^{\mathfrak{MV}}(a)$. 


We define $X_{\mu, \nu}^\mu, \nu(b)(\tau_b)$ by the fibre product
\[
X_{\mu, \nu}^\mu, \nu(b)(\tau_b) \to \text{Gr}_{\nu(b), \mu} \mid_{\tau_b + \sigma(\nu(b))} \times \text{Gr}_{\tau_b + \sigma(\nu(b))}.
\]
Further, we define $X_{\mu, \nu}^a, \nu(b)(\tau_b)$ by the fibre product
\[
X_{\mu, \nu}^a, \nu(b)(\tau_b) \to \text{Gr}_{0, a(\nu(b), \mu)} \mid_{\tau_b + \sigma(\nu(b))} \times X_{\mu, \nu}^\mu, \nu(b)(\tau_b) \to \text{Gr}_{0, \mu(b)} \mid_{\tau_b + \sigma(\nu(b))}.
\]
Let $x_0$ denote $[1] \in J_\tau(F)/J_\tau(O)$. We put
\[
\hat{X}_{\mu, \nu}^b, x_0(\tau_b) = X_{\mu, \nu}^a, \nu(b)(\tau_b) \cap \text{Gr}_{\nu(b)}.
\]
Let $X_{\mu, \nu}^b, x_0(\tau_b)$ denote the closure of $\hat{X}_{\mu, \nu}^b, x_0(\tau_b)$ in $X_{\mu, \nu}^a, \nu(b)(\tau_b)$.

By [Z17, Theorem 4.4.14], there is a bijection between the set
\[
\bigsqcup_{\lambda \in X_\tau(T)^+ \cap \text{MV}_\mu(\lambda) \times J_\tau(F)/J_\tau(O)}
\]
and the set of irreducible components of $X_{\mu, \nu}^b(\tau_b)$ given by
\[
(b, [g]) \mapsto X_{\mu, \nu}^b, [g](\tau_b) := gX_{\mu, \nu}^b, x_0(\tau_b),
\]
where we regard $X_{\mu, \nu}^b, [g](\tau_b)$ as a subscheme of $X_{\mu, \nu}^b(\tau_b)$ by (6.1).

7 Unitary group

7.1 Setting

Let $F_2$ be the quadratic unramified extension of $F$. Let $O_2$ denote the ring of integers of $F_2$. Let $\varpi$ be a uniformizer of $F$. We put $\Lambda = O_2^n$ equipped with the hermitian form
\[
O_2^n \times O_2^n \to O_2^n := ((a_i)_{1 \leq i \leq d}, (a'_i)_{1 \leq i \leq d}) \mapsto \sum_{i=1}^n \sigma(a_i)a'_{n+1-i}.
\]
We put $G = \text{GU}(\Lambda)$. By taking the first factor of the decomposition
\[
O_2 \otimes O_2 \simeq O_2 \times O_2; \quad a \otimes b \mapsto (ab, a\sigma(b)),
\]
we have an isomorphism
\[
G_{O_2} \simeq \text{GL}_n \times \mathbb{G}_m.
\]
We put $V = \Lambda \otimes O F$. Let $\hat{G} = \text{GL}_n \times \mathbb{G}_m$ denote the dual group over $\overline{\mathbb{Q}}_\ell$ with a maximal torus $\hat{T}$ and a Borel subgroup $\hat{B}$, which are the diagonal torus and the upper triangular
subgroup on the $\text{GL}_n$-component. For $\mu \in \mathbb{X}_\bullet(T)^+$, let $\mu^* \in \mathbb{X}_\bullet(T)^+$ be the element such that $V_{\mu^*} = V_{\mu^*}$. For an index $i \in \{1, \ldots, n\}$, we will use the notation $i' = n + 1 - i$. The group $\mathbb{X}^\bullet(\hat{T})$ has a basis $\{e_i\}_{i=0}^n$, where $e_0$ is the projection to the $\mathbb{G}_m$-component and $e_i$ is the character of $\hat{T}$ given by evaluating the $(i, i)$ entry for $i \geq 1$. In the following, all cocharacters of $T$ (equivalently, characters of $\hat{T}$) will be written according to this basis. We have $\sigma(e_0) = e_0$ and $\sigma(e_i) = -e_i$ for $1 \leq i \leq n$.}

7.2 Satake cycle

Let $\mu = e_0 + e_1 + e_2 \in \mathbb{X}_\bullet(T)$. We put $r = [n/2]$. We put

$$\nu_i = e_1 + \cdots + e_{i-1} - e_{i+1} - \cdots - e_n, \quad \tau_i = e_0$$

for $1 \leq i \leq [(n-1)/2]$, and

$$\nu_r = e_1 + \cdots + e_{r-1}, \quad \tau_r = e_0 + e_1 + \cdots + e_n$$

if $n$ is even. We put $\lambda_i = e_0 - e_i - e_{i'}$ for $1 \leq i \leq r$.

**Lemma 7.1.** For $\lambda \in e_0 + (1 - \sigma)\mathbb{X}_\bullet(T)$, we have $\mathbb{MV}_{\nu^*}^\bullet(\lambda) \neq \emptyset$ if and only if $\lambda \in \{\lambda_1, \ldots, \lambda_r\}$. Further, $\mathbb{MV}_{\nu^*}^\bullet(\lambda_i)$ is a singleton for $1 \leq i \leq r$.

**Proof.** For $\lambda \in \mathbb{X}_\bullet(T)$, we have $\dim V_{\nu^*}^\bullet(\lambda) \leq 1$, and $V_{\nu^*}^\bullet(\lambda)$ is nonzero if and only if $\lambda = e_0 - e_i - e_j$ for some $1 \leq i < j \leq n$. If $e_0 - e_i - e_j \in e_0 + (1 - \sigma)\mathbb{X}_\bullet(T)$ for some $1 \leq i < j \leq n$, we must have $j = i'$. Hence the claim follows from (4.3). \qed

Let $1 \leq i \leq r$. Note that $\sigma(\nu_i^*) = \lambda_i + \nu_i^* - \tau_i^*$. Let $b_i$ be the unique element of $\mathbb{MV}_{\nu^*}^\bullet(\lambda_i)$. There is $a_i \in S(\nu_i^*, \mu^*)|\lambda_i + \nu_i^*$ such that $i_{\nu_i^*}^\text{MV}(a_i) = b_i$. Since $i_{\nu_i^*}^\text{MV}$ is injective by [XZ17], Lemma 3.2.7, the set $S(\nu_i^*, \mu^*)|\lambda_i + \nu_i^*$ is also a singleton.

We study the Satake cycle $\text{Gr}^0_{\nu_i^* \mu^* \mid} |\sigma(\nu_i^*)$.

**Lemma 7.2.** We have

$$\text{Gr}^0_{\nu_i^* \mu^* \mid} |_{\nu_i} = \text{Gr}^0_{\nu_i^* \mu^* \mid} |_{\nu_i}.$$

**Proof.** By the definition, $\text{Gr}^0_{\nu_1^* \mu^* \mid} |_{\sigma(\nu_1^*)}$ is equal to the inverse image of $\text{Gr}_{\nu_1^* \mu^* \mid} |_{\sigma(\nu_1^*)}$ under the convolution morphism

$$m_{\nu_1^* \mu^*} : \text{Gr}_{\nu_1^* \mu^* \mid} \to \text{Gr}.$$

By Lemma 5.1 and Proposition 5.2, $\text{Gr}^0_{\nu_1^* \mu^* \mid} |_{\sigma(\nu_1^*)}$ is irreducible, since $S(\nu_1^*, \mu^*)|_{\lambda_i + \nu_1^*}$ is a singleton. Therefore, we obtain the claim. \qed

We do not use the following lemma in the sequel, but it shows that a study of intersections of irreducible components of affine Deligne–Lusztig varieties is more subtle than intersections of Satake cycles.

**Lemma 7.3.** (1) The actions of $L^+G$ on $\text{Gr}^0_{\nu_1^* \mu^* \mid} |_{\nu_1}$ and $\text{Gr}^0_{\nu_1^* \mu^* \mid} |_{\nu_2}$ are transitive.

(2) The Satake cycle $\text{Gr}^0_{\nu_2^* \mu^* \mid} |_{\nu_2}$ contains $\text{Gr}^0_{\nu_1^* \mu^* \mid} |_{\nu_1}$.  

13
Proof. We show [1] It suffices to show that the number of the orbits under the action of \( L^+G \) on \( Gr_{(\nu^*, \mu^*)}^0 \) is 2. Let \((L^+G)_{\nu_1}\) be the stabilizer of \([\nu^{-}\mu]\) in \( L^+G \). Since the action of \( L^+G \) on \( Gr_{\nu_1} \) is transitive, it suffices to show that the number of the orbits in \( m_{\nu_1}^{-1}(\nu^{-}\mu)_{\nu_1} \) under the action of \((L^+G)_{\nu_1}\) is 2. These orbits are in a bijection with \((P_{\nu_1}C_k)_{\cal T} \setminus (P_{\mu}C_k)_{\cal T} \). Hence the number of the orbits is 2.

We show [2] By Lemma 4.2, the natural morphism \( Gr_{(\nu_1^*, \mu^*)}^0 \to Gr_{\nu_2} \) is surjective. Hence the intersection of \( Gr_{(\nu_1^*, \mu^*)}^0 \) and \( Gr_{(\nu_2^*, \mu^*)}^0 \) is not empty.

If the intersection of \( Gr_{(\nu_1^*, \mu^*)}^0 \) and \( Gr_{(\nu_2^*, \mu^*)}^0 \) is not empty, then \( Gr_{(\nu_1^*, \mu^*)}^0 \) contains \( Gr_{(\nu_2^*, \mu^*)}^0 \) because \( L^+G \) acts transitively on \( Gr_{(\nu_1^*, \mu^*)}^0 \) and \( Gr_{(\nu_2^*, \mu^*)}^0 \) is stable under the action of \( L^+G \). Then \( Gr_{(\nu_1^*, \mu^*)}^0 \) contains \( Gr_{(\nu_2^*, \mu^*)}^0 \), since \( Gr_{(\nu_2^*, \mu^*)}^0 \) is dense in \( Gr_{(\nu_1^*, \mu^*)}^0 \).

If the intersection of \( Gr_{(\nu_1^*, \mu^*)}^0 \) and \( Gr_{(\nu_2^*, \mu^*)}^0 \) is empty, the intersection of \( Gr_{(\nu_1^*, \mu^*)}^0 \) and \( Gr_{(\nu_2^*, \mu^*)}^0 \) is not empty. Then \( Gr_{(\nu_1^*, \mu^*)}^0 \) contains \( Gr_{(\nu_2^*, \mu^*)}^0 \) because \( L^+G \) acts transitively on \( Gr_{(\nu_1^*, \mu^*)}^0 \) and \( Gr_{(\nu_2^*, \mu^*)}^0 \) is stable under the action of \( L^+G \).

Remark 7.4. Lemma 7.3 [2] shows that \( X_{\mu^*, \nu_2^*}(\tau_2) \) is not irreducible. This answers a question in [XZ17, Remark 4.4.6 (3)].

8 Irreducible Components

We note that \([\nu^{-}\mu]\) \( \in B(G, \mu^*) \) is the basic class. By results in [3] and Lemma 7.1, \( X_{\mu^*, \nu_0^*} \) has \( r \) isomorphism classes of irreducible components. The \( r \) isomorphism classes of irreducible components are given by \( X_{\mu^*, \nu_0^*}(\tau_i^*) \) for \( 1 \leq i \leq r \), where \( X_{\mu^*, \nu_0^*}(\tau_i^*) \) is the closures in \( X_{\mu^*, \nu_0^*}(\tau_i^*) \) of \( X_{\mu^*, \nu_0^*}(\tau_i^*) \) fitting in the cartesian diagram

\[
\begin{array}{ccc}
\tilde{X}_{\mu^*, \nu_0^*}(\tau_i^*) & \to & Gr_{(\nu_i^*, \mu^*)}^0, \tau_i^* + \sigma(\nu_i^*) \\
\downarrow & & \downarrow pr_1 \times m \\
\tilde{Gr}_{\nu_i^*} & \to & Gr_{\nu_i^*} \times Gr_{\nu_i^*} + \sigma(\nu_i^*).
\end{array}
\]

If \( i = 1 \), the above construction defines a Deligne–Lusztig variety. If \( i = 2 \) and \( n \geq 5 \), this defines a variety that is not a Deligne–Lusztig variety.

By (4.1) and (7.1), we have an isomorphism

\[
Gr_{G} \otimes_{\mathbb{F}_q} F_{q^2} \simeq Gr_{GL_n} \times \mathbb{G}_m.
\]

(8.1)

We put \( E^0 = \mathcal{O}_2 \) and \( L^0 = \mathcal{O}_2 \) and view them as trivial vector bundles on \( D_{F_{q^2}} \). For any perfect \( F_{q^2} \)-algebra \( R \),

\[
Gr_{GL_n} \times \mathbb{G}_m(R) = \left\{ (E, L, \beta, \beta') \mid \begin{array}{l}
E \text{ is a vector bundle on } D_{R} \text{ of rank } n, \\
L \text{ is a line bundle on } D_{R}, \\
\beta : E|_{D_{R}} \simeq E^0|_{D_{R}} \text{ and } \\
\beta' : L|_{D_{R}} \simeq L^0|_{D_{R}} \text{ are trivializations.}
\end{array} \right\}
\]

(8.2)

14
by (4.2). Under the identification by (8.1), the Frobenius endomorphism of $\text{Gr}_G \otimes F_q \mathbb{F}_q^2$ sends $(\mathcal{E}, \mathcal{L}, \beta, \beta')$ in (8.2) to

$$(F(\mathcal{E}^\vee) \otimes \mathcal{L}, F(\beta^\vee)^{-1} \otimes \beta', \beta')$$

where

$$F(\beta^\vee)^{-1} \otimes \beta': (F(\mathcal{E}^\vee) \otimes \mathcal{L})|_{D_r'} \simeq (F((\mathcal{E}^\vee)^0) \otimes \mathcal{L}^0)|_{D_r'} = E^0|_{D_r'}.$$  

We regard $\text{Gr}_{GL_n}$ as an open and closed sub-ind-scheme of $\text{Gr}_{GL_n \times G_m}$ by

$$(\mathcal{E}, \beta) \mapsto (\mathcal{E}, \mathcal{L}^0, \beta, \text{id}).$$

If $\lambda \in X_\bullet(T)$ is trivial on $G_m$-component under the identification (6.1), then we view $\text{Gr}_{G,A}$ as a subscheme of $\text{Gr}_{GL_n} \subset \text{Gr}_{GL_n \times G_m}$ under the identification (8.1). Under the identification by (8.1), the Frobenius endomorphism of $\text{Gr}_G \otimes F_q \mathbb{F}_q^2$ becomes

$$(\mathcal{E}, \beta) \mapsto (F(\mathcal{E}^\vee), F(\beta^\vee)^{-1})$$

in $\text{Gr}_{GL_n}$. We put $\mu_{GL} = \varepsilon_1 + \varepsilon_1$.

### 8.1 Component for $\nu_1$

An analogue of a component in [VW11].

**Proposition 8.1.** The irreducible component $X_{\mu_1, x_0}(\tau_1^*)$ is parametrized by $\mathcal{E} \rightarrow \mathcal{E}^0$ bounded by $\nu_1^*$ such that $\varpi F(\mathcal{E}^\vee) \subset \mathcal{E}$. In particular, it is isomorphic to $X_{I_{m-1}}([1])$.  

**Proof.** We have

$$\text{Gr}^{0,a_1}_{(\nu_1^*, \mu^*)}|_{\nu_1} = \text{Gr}^0_{(\nu_1^*, \mu^*)}|_{\nu_1} = \text{Gr}^0_{(\nu_1^*, \mu^*)}|_{\nu_1}$$

since $\nu_1$ is minuscule. Hence we have $X_{\mu_1, x_0}(\tau_1^*) = X_{\mu_1, x_0}(\tau_1^*) = X_{\mu_1, x_0}(\tau_1^*)$. By the definition, $X_{\mu_1, x_0}(\tau_1^*)$ is parametrized by $\mathcal{E} \rightarrow \mathcal{E}^0$ bounded by $\nu_1^*$ such that $F(\mathcal{E}^\vee) \rightarrow \mathcal{E}$ is bounded by $\mu_{GL}^*$. The condition that $F(\mathcal{E}^\vee) \rightarrow \mathcal{E}$ is bounded by $\mu_{GL}^*$ is equivalent to $\varpi F(\mathcal{E}^\vee) \subset \mathcal{E}$.

The last isomorphism in the claim is given by sending $\mathcal{E}$ to $\mathcal{E}^\vee/\mathcal{E}^0 \subset \mathcal{E}^0/\mathcal{E}^0$.  

### 8.2 Components for $\nu_r$ when $n$ is even.

A generalization of a component in [HP14].

**Proposition 8.2.** The irreducible component $X_{\mu_r, x_0}(\tau_r^*)$ is parametrized by $\mathcal{E} \rightarrow \mathcal{E}^0$ bounded by $\nu_r^*$ such that $\varpi \mathcal{E} \subset F(\mathcal{E}^\vee)$. In particular, it is isomorphic to $X_{I_{m-1}}([1])$.  

**Proof.** We have

$$\text{Gr}^{0,a_r}_{(\nu_r^*, \mu^*)}|_{\nu_r^* + \nu_r} = \text{Gr}^0_{(\nu_r^*, \mu^*)}|_{\nu_r^* + \nu_r} = \text{Gr}^0_{(\nu_r^*, \mu^*)}|_{\nu_r^* + \nu_r}$$

since $\nu_r^*$ and $\tau_r^* + \nu_r$ are minuscule. Hence we have $X_{\mu_r, x_0}(\tau_r^*) = X_{\mu_r, x_0}(\tau_r^*) = X_{\mu_r, x_0}(\tau_r^*)$. It is parametrized by $\mathcal{E} \rightarrow \mathcal{E}^0$ bounded by $\nu_r^*$ such that $\varpi \mathcal{E} \subset F(\mathcal{E}^\vee)$.

By the definition, $X_{\mu_r, x_0}(\tau_r^*)$ is parametrized by $\mathcal{E} \rightarrow \mathcal{E}^0$ bounded by $\nu_r^*$ such that $\varpi F(\mathcal{E}^\vee) \rightarrow \mathcal{E}$ is bounded by $\mu_{GL}^*$. The condition that $\varpi F(\mathcal{E}^\vee) \rightarrow \mathcal{E}$ is bounded by $\mu_{GL}^*$ is equivalent to $\varpi \mathcal{E} \subset F(\mathcal{E}^\vee)$.

The last isomorphism in the claim is given by sending $\mathcal{E}$ to $\mathcal{E}/\mathcal{E}^0 \subset \mathcal{E}^0/\mathcal{E}^0$.  

15
Example 8.3. Assume that \( n = 4 \). Then \( X_{\mu}^{\nu_{+},\nu_{-}}(\tau^*) \) is isomorphic to the perfection of the Fermat hypersurface defined by
\[
x_1x_4^2 + x_2x_3^2 + x_3x_2^2 + x_4x_1^2 = 0
\]
in \( \mathbb{P}^3 \). This is a component which appears in [HP14, p.1689].

8.3 Non-minuscule case
Let \( 2 \leq i \leq [(n - 1)/2] \). We put
\[
\nu_{i,+} = \varepsilon_1 + \cdots + \varepsilon_{i-1}, \quad \nu_{i,-} = -\varepsilon_i \cdots - \varepsilon_{1}\varepsilon_i.
\]
We put \( \xi_i = \varepsilon_1 + \cdots + \varepsilon_{2i-1} \). Let \((\text{Gr}_{\nu_{i,+}} \times \text{Gr}_{\nu_{i,-}})\xi_i\) be a subspace of \( \text{Gr}_{\nu_{i,+}} \times \text{Gr}_{\nu_{i,-}} \) defined by the condition that
\[
\mathcal{E}^- \xrightarrow{\beta} \mathcal{E}^0 \xrightarrow{\beta^{-1}} \mathcal{E}^+ \quad (\xi_i)
\]
is bounded by \( \xi_i \) for a point \((\mathcal{E}^- \xrightarrow{\beta} \mathcal{E}^0, \mathcal{E}^- \xrightarrow{\beta} \mathcal{E}^0)\) of \( \text{Gr}_{\nu_{i,+}} \times \text{Gr}_{\nu_{i,-}} \). Let
\[
(\text{Gr}_{\nu_{i,+},\nu_{i,-}} \times \text{Gr}_{\nu_{i,-},\nu_{i,+}})\xi_i
\]
be a subspace of \( \text{Gr}_{\nu_{i,+},\nu_{i,-}} \times \text{Gr}_{\nu_{i,-},\nu_{i,+}} \) defined by the condition that
\[
\mathcal{E}^- \xrightarrow{\beta} \mathcal{E}^0 \xrightarrow{\beta^{-1}} \mathcal{E}^+ \quad (\xi_i)
\]
is bounded by \( \xi_i \) for a point
\[
(\mathcal{E}^- \xrightarrow{\beta} \mathcal{E}^0, \mathcal{E}^- \xrightarrow{\beta} \mathcal{E}^0)
\]
of \( \text{Gr}_{\nu_{i,+},\nu_{i,-}} \times \text{Gr}_{\nu_{i,-},\nu_{i,+}} \). We have a natural morphism
\[
\pi_1 : (\text{Gr}_{\nu_{i,+},\nu_{i,-}} \times \text{Gr}_{\nu_{i,-},\nu_{i,+}})\xi_i \to (\text{Gr}_{\nu_{i,+}} \times \text{Gr}_{\nu_{i,-}})\xi_i.
\]
Let \( \mathcal{Y}_i \) be a vector bundle of rank \( 2i - 1 \) over \((\text{Gr}_{\nu_{i,+}} \times \text{Gr}_{\nu_{i,-}})\xi_i\), defined by \( \mathcal{E}^+ / \mathcal{E}^- \), where \((\mathcal{E}^+, \mathcal{E}^-)\) is a point of \((\text{Gr}_{\nu_{i,+}} \times \text{Gr}_{\nu_{i,-}})\xi_i\). We put \( \mathcal{Y}_i = \text{Aut}(\mathcal{Y}_i) \). Let \( t_{i,\mathbb{Z}} \in \text{Oc}(\text{Dyn}(\text{GL}_{2i-1,\mathbb{Z}}))(\mathbb{Z}) \) be the image under
\[
t(\mathbb{Z}) : \text{Par}(\text{GL}_{2i-1,\mathbb{Z}})(\mathbb{Z}) \to \text{Oc}(\text{Dyn}(\text{GL}_{2i-1,\mathbb{Z}}))(\mathbb{Z})
\]
of the parabolic subgroup of \( \text{GL}_{2i-1,\mathbb{Z}} \) defined as the stabilizer of \( \mathbb{Z}^{i-1} \subset \mathbb{Z}^{i-1} \oplus \mathbb{Z}^i = \mathbb{Z}^{2i-1} \). Let
\[
t_i \in \text{Oc}(\text{Dyn}(\mathcal{Y}_i))((\text{Gr}_{\nu_{i,+}} \times \text{Gr}_{\nu_{i,-}})\xi_i) \quad (8.3)
\]
the element determined from \( t_{i,\mathbb{Z}} \) by Remark 2.5 and Lemma 2.6 (2).

We define a morphism
\[
\Psi : (\text{Gr}_{\nu_{i,+},\nu_{i,-}} \times \text{Gr}_{\nu_{i,-},\nu_{i,+}})\xi_i \to \text{Par}_t(\mathcal{Y}_i)
\]
by sending
\[
(\mathcal{E}^- \xrightarrow{\beta} \mathcal{E}^0, \mathcal{E}^- \xrightarrow{\beta} \mathcal{E}^0)
\]
to the stabilizer of $E/E_\subset E_+/E_-$. Then $\Psi$ is an isomorphism. Note that a natural morphism

$$\pi_0: (\text{Gr}(\nu_{i+}^*\nu_{i-}^*) \times \text{Gr}(\nu_{i+}^*\nu_{i-}^*))\xi_i \to \text{Gr}(\nu_i^*)$$

is an isomorphism over $\text{Gr}(\nu_i^*)$.

Recall that $X_{n^*\nu_i^*}(\tau_i^*)$ and $X_{\mu^*\nu_i^*}(\tau_i^*)$ are closed subspaces of $\text{Gr}(\nu_i^*)$. The condition for the subspace

$$\pi_0^{-1}(X_{\mu^*\nu_i^*}(\tau_i^*)) \subset (\text{Gr}(\nu_{i+}^*\nu_{i-}^*) \times \text{Gr}(\nu_{i+}^*\nu_{i-}^*))\xi_i$$

is that $E \subset F(E') \subset \frac{1}{w}E$.

For a point $(E, E_+, E_-)$ of $(\text{Gr}(\nu_{i+}^*\nu_{i-}^*) \times \text{Gr}(\nu_{i+}^*\nu_{i-}^*))\xi_i$, we put $\mathcal{W} = E/E_- \subset E_+/E_-$, which is a subvector bundle of rank $i - 1$. Let $\mathcal{W} \subset E'/E_-'$ be the annihilator of $\mathcal{W}$. Then we have $\mathcal{W} = E'/E_-'$.

Let $Y_i$ be the closed subscheme of $(\text{Gr}(\nu_{i+}^*\nu_{i-}^*))\xi_i$ defined by the conditions

1. $E_+ \subset F(E')$,
2. $E_- \subset F(E'_+)$,
3. $\varpi F(E') \subset E_-$.

Then we have $Y_i = X_{\mathbb{I}_{i+}^*\nu_i^*}([1])^d$ under the identification given by sending $(E_+, E_-)$ to

$$0 \subset \varpi E_+/\varpi E^0 \subset E_-/\varpi E^0 \subset E^0/\varpi E^0.$$ 

Assume that $(E_+, E_-)$ is a point of $Y_i$. The condition $E \subset F(E')$ is equivalent to that the image of $\mathcal{W}$ under the natural morphism

$$\phi_1: E_+/E_- \to F(E')/F(E'_+) = F(E'/E'_+)$$

is contained in $F(\mathcal{W})$. The condition $F(E') \subset \frac{1}{w}E$ is equivalent to that the image of $F(\varpi \mathcal{W})$ under the natural morphism

$$\phi_2: E'/\varpi E^0 \to E_+/E_-$$

is contained in $\mathcal{W}$. We put

$$X_i = \pi_0^{-1}(X_{\mu^*\nu_i^*}(\tau_i^*)) \cap \pi_1^{-1}(Y_i).$$

Then $X_i$ is the subscheme of $\pi_1^{-1}(Y_i)$ cut out by the conditions

$$\phi_1(\mathcal{W}) \subset F(\mathcal{W}),$$

$$\phi_2(F(\varpi \mathcal{W})) \subset \mathcal{W}. \quad (8.4)$$

Let $\pi_0'$ and $\pi_1'$ be the restrictions of $\pi_0$ and $\pi_1$ to $X_i$ respectively. We have

$$\pi_0^{-1}(X_{\mu^*\nu_i^*}(\tau_i^*)) \xrightarrow{\pi_0'} X_i \xrightarrow{\pi_1'} \pi_1^{-1}(Y_i)$$

$$\xrightarrow{\pi_0} X_{\mu^*\nu_i^*}(\tau_i^*) \xrightarrow{\pi_1} Y_i.$$ 

We note that $\pi_0$ and $\pi_0'$ are isomorphisms over $X_{\mu^*\nu_i^*}(\tau_i^*)$. 

17
Lemma 8.4. The inverse image \( \pi_0^{-1}(\hat{X}_{\mu}^{h_{i,x_0}}(\tau_i^*)) \) is contained in \( X_i \).

Proof. Let \( (E, E_+, E_-) \) be a point of \( \pi_0^{-1}(\hat{X}_{\mu}^{h_{i,x_0}}(\tau_i^*)) \). Then we have \( E_- = E \cap E^0 \). By the condition \( F(E^\vee) \subset \varpi E \), we have

\[
\varpi F(E^\vee) = \varpi F((E \cap E^0)^\vee) = \varpi (F(E^\vee) + E^0) \subset E.
\]

Hence we have

\[
\varpi F(E^\vee) \subset E \cap E^0 = \varnothing.
\]

This means that \( (E_+, E_-) \) is a point of \( Y_i \).

Let \( \mathscr{G}_i \) denote the restriction of \( \mathscr{G}_i \) to \( Y_i \). We have an isomorphism

\[
\Psi_{Y_i} : \pi_1^{-1}(Y_i) \simeq \text{Par}_i(\mathscr{G}_i) \tag{8.6}
\]

induced by \( \Psi \).

Theorem 8.5. The closed subscheme \( X_i \subset \pi_1^{-1}(Y_i) \simeq \text{Par}_i(\mathscr{G}_i) \) is defined by the condition \( \phi_1(\mathscr{W}) \subset F(\mathscr{W}^\perp) \).

Proof. It suffices to show that the condition \( \mathbb{S} \mathbb{S} \) is automatic. The condition \( \mathbb{S} \mathbb{S} \) is equivalent to \( \varpi F(E^\vee) \subset E \) under \( \mathbb{S} \mathbb{S} \). Let \( (E, E_+, E_-) \) be a point of \( \pi_1^{-1}(Y_i) \). Then we have

\[
\varpi F^{-1}(E^\vee) \subset E_- \subset E.
\]

By taking the dual, we have \( \varpi F(E^\vee) \subset E_- \). Hence the condition \( \varpi F(E^\vee) \subset E \) is satisfied.

Proposition 8.6. The scheme \( \hat{X}_{\mu}^{h_{i,x_0}}(\tau_i^*) \) is isomorphic to the subscheme of \( \pi_1^{-1}(Y_i) \) defined by the condition \( E \subset F(E^\vee) \) and \( E \cap E^0 = \varnothing \).

Proof. The natural morphism \( \pi_1^{-1}(\hat{X}_{\mu}^{h_{i,x_0}}(\tau_i^*)) \to \hat{X}_{\mu}^{h_{i,x_0}}(\tau_i^*) \) is an isomorphism. Hence the claim follows from Lemma 8.4 and Theorem 8.5.

9 Intersections

Let \( x, x' \in J_*(F)/J_*(\mathcal{O}) \). Let \( \Lambda_x \) and \( \Lambda_{x'} \) be the lattices of \( V \) determined by \( x \) and \( x' \). We put

\[
l_{x,x'} = \text{length}_\mathcal{O}(\Lambda_x/(\Lambda_x \cap \Lambda_{x'})).
\]

Let \( E_x \) and \( E_{x'} \) be the modifications of \( E^0 \) corresponding to \( \Lambda_x \) and \( \Lambda_{x'} \). Let \( P_{x,x'} \) be the parabolic subgroup of \( G \) that is the stabilizer of the filtration

\[
\varpi \Lambda_x \subset \varpi \Lambda_{x'} + \varpi \Lambda_x \subset (\Lambda_x \cap \varpi \Lambda_{x'}) + \varpi \Lambda_x \subset (\Lambda_x \cap \Lambda_{x'}) + \varpi \Lambda_x \subset \Lambda_x.
\]

We note that \( \varpi \Lambda_x \subset \Lambda_{x'} \) if and only if \( \varpi \Lambda_{x'} \subset \Lambda_x \) by taking dual with respect to the hermitian paring. We put

\[
\begin{align*}
d_1 &= \text{dim}((\varpi \Lambda_{x'} + \varpi \Lambda_x)/\varpi \Lambda_x), \\
d_2 &= \text{dim}(((\Lambda_x \cap \varpi \Lambda_{x'}) + \varpi \Lambda_x)/\varpi \Lambda_x).
\end{align*}
\]

We note that \( d_1 + d_2 = l_{x,x'} \).

18
9.1 Intersection of components for $\nu_i$ and $\nu_{i'}$, where $i, i' \neq r$ if $n$ is even.

9.1.1 Different hyperspecial subgroups

We assume that $x \neq x'$. For a subscheme $X$ of $X_{\mu,x}^\ast(\tau^\ast_1)$, let $X_{P_{x,x'},[w]}^{\text{pf}}$ be the inverse image of $X_{P_{x,x'},[w]}^{\text{pf}}$ under $X \hookrightarrow X_{\mu,x}^\ast(\tau^\ast_1) \to X_{P_{x,x'},[w]}^{\text{pf}}$. We recall that

$$X_{\mu,x,x'}^\ast(\tau^\ast_1) = X_{\mu,x,x'}^\ast(\tau^\ast_1) \setminus X_{\mu,x,x'}^\ast(\tau^\ast_1).$$

Assume that $i \leq i'$. For $j_1, j_2 \in \mathbb{N}$ such that $i - 1 - d_2 \leq j_1 \leq i - 1$ and $d_2 - i \leq j_2 \leq n - i - d_2 - j_1$, we define $w_{j_1,j_2} \in S_n$ by

$$w_{j_1,j_2}(j) = \begin{cases} j + j_1 & \text{if } i - j_1 \leq j \leq d_2, \\ j + i - j_1 - d_2 - 1 & \text{if } d_2 + 1 \leq j \leq d_2 + j_1, \\ j + j_2 & \text{if } n - j_2 - i + 1 \leq j \leq n - d_2, \\ j + d_2 - i - j_2 & \text{if } n - d_2 + 1 \leq j \leq n - d_2 + j_2, \\ j & \text{otherwise.} \end{cases}$$

We put $\mathcal{E}^+_x = (\mathcal{E}_x + \mathcal{E}_{x'}) \cap \mathcal{E}_x$ and $\mathcal{E}^-_x = (\mathcal{E}_x \cap \mathcal{E}_{x'}) + \mathcal{E}_x$. Let $\mathcal{E}^+$ and $\mathcal{E}^-$ be the universal vector bundles on $X_{P_{x,x'},[w_{j_1,j_2}]}^{\text{pf}}$. We note that

$$\text{length}(\mathcal{E}^+_x/\mathcal{E}^+_x) = j_1, \quad \text{length}(\mathcal{E}^+_x/\mathcal{E}^-_x) = j_2.$$ 

We put $\mathcal{E}^+_{x,x'} = \mathcal{E}^+_x \cap (\mathcal{E}^+_{x,x'} + \mathcal{E}_{x'})$ and $d_{j_1,j_2} = j_2 - j_1 + 2i - 1 - d_2$. We note that

$$\mathcal{E}^+_{x,x'} = \mathcal{E}^-_{x,x'} + \mathcal{E}^+_x \cap \mathcal{E}_{x'} \subset F(\mathcal{E}^+_x) \cap (F(\mathcal{E}^+_x + \mathcal{E}_{x'}) = F(\mathcal{E}^+_{x,x'}).$$

Lemma 9.1. We have $\text{length}(\mathcal{E}^+_{x,x'}/\mathcal{E}^-_{x,x'}) = d_{j_1,j_2}$. Further $\mathcal{E}^+_{x,x'}/\mathcal{E}^-_{x,x'}$ is a vector bundle on $X_{P_{x,x'},[w_{j_1,j_2}]}^{\text{pf}}$.

Proof. We have

$$\text{length}(\mathcal{E}^+_{x,x'}/\mathcal{E}^-_{x,x'}) = \text{length}(\mathcal{E}^+_x + \mathcal{E}_{x'})/\mathcal{E}^-_{x,x'})$$

$$= \text{length}(\mathcal{E}^+_x + \mathcal{E}_{x'})/(\mathcal{E}^-_{x,x'}) - \text{length}(\mathcal{E}^+_x + \mathcal{E}_{x'})/(\mathcal{E}^-_{x,x'})$$

$$= \text{length}(\mathcal{E}^+_x + \mathcal{E}_{x'})/\mathcal{E}^-_{x,x'}) + \text{length}(\mathcal{E}^+_x + \mathcal{E}_{x'})/(\mathcal{E}^-_{x,x'}) - j_2$$

$$= j_1 + \text{length}(\mathcal{E}^+_x + \mathcal{E}_{x'})/(\mathcal{E}^-_{x,x'}) - j_2 = j_1 + d_2 - j_2.$$ 

Hence we obtain the first claim. By the above equalities, length($\mathcal{E}^+_{x,x'}/\mathcal{E}^-_{x,x'}$) is constant on $X_{P_{x,x'},[w_{j_1,j_2}]}^{\text{pf}}$. Hence $\mathcal{E}^+_{x,x'}/\mathcal{E}^-_{x,x'}$ is a vector bundle on $X_{P_{x,x'},[w_{j_1,j_2}]}^{\text{pf}}$ by [BS17] Lemma 7.3. Therefore $\mathcal{E}^+_{x,x'}/\mathcal{E}^-_{x,x'}$ is also a vector bundle. \(\square\)

Let $\mathcal{G}_{j_1,j_2}$ be the restriction of $\mathcal{G}$ to $X_{P_{x,x'},[w_{j_1,j_2}]}^{\text{pf}}$. Let

$$t_{j_1,j_2} \in \text{Oc}(\text{Dyn}(\mathcal{G}_{j_1,j_2}))(X_{P_{x,x'},[w_{j_1,j_2}]})$$

19
denote the restriction of $t_i$ in \([8.3]\). Let $\mathcal{P}_{j_1,j_2}$ be the parabolic subgroup of $\mathfrak{g}_{j_1,j_2}$ determined by

$$
\mathcal{E}_- \subset \mathcal{E}_+ \cap (\mathcal{E}_- \cup \mathcal{E}_+) \subset \mathcal{E}_+.
$$

We put $l_{j_1,j_2} = i' - 1 - j_2 - d_1$. We define $s_{j_1,j_2} \in S_{2i-1}$ by

$$
s_{j_1,j_2}(j) = \begin{cases} 
  j + l_{j_1,j_2} & \text{if } i - l_{j_1,j_2} \leq j \leq d_{j_1,j_2}, \\
  j + i - 1 - d_{j_1,j_2} - l_{j_1,j_2} & \text{if } d_{j_1,j_2} + 1 \leq j \leq d_{j_1,j_2} + l_{j_1,j_2}, \\
  j & \text{otherwise}.
\end{cases}
$$

Let $r_{j_1,j_2}$ be the element of

$$
(d_{j_1,j_2}^{-1}(t(\mathcal{P}_{j_1,j_2}, t_{j_1,j_2}))(X_{P_{i_1-1,n}}^\mu)_{[\mathfrak{p}_x,x',[w_{j_1,j_2}]}^\mu)
$$

corresponding to $[s_{j_1,j_2}]$ by Lemma \[2.6\][2].

**Proposition 9.2.** Assume that $\hat{X}_{\mu^t}^b(t^*_x) \cap \hat{X}_{\mu^t}^{b',x'}(t^*_x)$ is not empty. Then we have $1 \leq l_{x,x'} \leq i + i' - 1$.

The subscheme $\hat{X}_{\mu^t}^b(t^*_x) \cap \hat{X}_{\mu^t}^{b',x'}(t^*_x)$ is the locus defined by the condition that $\mathcal{E}_x \cap \mathcal{E}_{x'} \subset \mathcal{E}_x \subset \mathcal{E}_+ \cap \mathcal{E}_{x'}$, $\mathcal{E}_+ \cap \mathcal{E}_{x'} \subset \mathcal{E}_x \cap \mathcal{E}_{x'}$,

$$
\text{length}((\mathcal{E}_- \cup \mathcal{E}_-)/\mathcal{E}_{x,x'}) \leq [(i' - i + d_2 - d_1)/2]
$$

and

$$
\text{length}((\mathcal{E}_+ \cup \mathcal{E}_-)/\mathcal{E}_{x,x'}) = [i' - i + d_2 - d_1].
$$

In particular, $\hat{X}_{\mu^t}^b(t^*_x) \cap \hat{X}_{\mu^t}^{b',x'}(t^*_x)$ is the union of $\left(\hat{X}_{\mu^t}^b(t^*_x) \cap \hat{X}_{\mu^t}^{b',x'}(t^*_x)\right)_{[\mathfrak{p}_x,x',[w_{j_1,j_2}]}$ for $j_1, j_2 \in \mathbb{N}$ such that $j_1 + d_2 - i \leq j_2 \leq j_1 + d_2 - i + 1$,

$$
i - 1 - d_2 \leq j_1 \leq i - 1 - d_1, \\
i' - i - d_1 \leq j_2 \leq \min\{[(i' - i + d_2 - d_1)/2], n - i - d_2 - j_1\}.
$$

Further we have

$$
\left(\hat{X}_{\mu^t}^b(t^*_x) \cap \hat{X}_{\mu^t}^{b',x'}(t^*_x)\right)_{[\mathfrak{p}_x,x',[w_{j_1,j_2}]} = \hat{X}_{\mu^t}^b(t^*_x)_{[\mathfrak{p}_x,x',[w_{j_1,j_2}]} \cap \operatorname{Par}_{j_1,j_2}(\mathcal{P}_{j_1,j_2} \cap \mathfrak{p}_x,x',[w_{j_1,j_2}]).
$$

**Proof.** The intersection $\hat{X}_{\mu^t}^b(t^*_x) \cap \hat{X}_{\mu^t}^{b',x'}(t^*_x)$ is parametrized by $\mathcal{E} \rightarrow \mathcal{E}_x$ which is equal to $\nu^*_t$ such that $\mathcal{E} \subset F(\mathcal{E}^*) \subset \mathcal{E}$ and $\mathcal{E} \rightarrow \mathcal{E}_{x'}$ is equal to $\nu^*_t$. Let $\mathcal{E}$ be a point of $\hat{X}_{\mu^t}^b(t^*_x) \cap \hat{X}_{\mu^t}^{b',x'}(t^*_x)$. We put

$$
\mathcal{E}_+ = \mathcal{E} + \mathcal{E}_x, \\
\mathcal{E}_- = \mathcal{E} \cap \mathcal{E}_x, \\
\mathcal{E}'_+ = \mathcal{E} + \mathcal{E}_{x'}, \\
\mathcal{E}'_- = \mathcal{E} \cap \mathcal{E}_{x'}.
$$

Then we have

$$
\text{length}(\mathcal{E}_x/\mathcal{E}_-) = i, \\
\text{length}(\mathcal{E}/\mathcal{E}_-) = i - 1, \\
\text{length}(\mathcal{E}_x/\mathcal{E}'_-) = i', \\
\text{length}(\mathcal{E}/\mathcal{E}'_-) = i' - 1.
$$

Hence we have $\text{length}(\mathcal{E}_-/(\mathcal{E}_- \cap \mathcal{E}'_-)) \leq i' - 1$ and $\text{length}(\mathcal{E}_x/(\mathcal{E}_- \cap \mathcal{E}'_-)) \leq i + i' - 1$. Therefore the inclusion $\mathcal{E}_- \cap \mathcal{E}'_- \subset \mathcal{E}_x \cap \mathcal{E}_{x'}$ implies that

$$
1 \leq l_{x,x'} \leq i + i' - 1.
$$
We have \( \mathcal{E}_x + \omega \mathcal{E}_x \subset \mathcal{E}_+ \) and \( \omega \mathcal{E}^+_{x,x'} = \omega \mathcal{E}_x + (\mathcal{E}_x \cap \omega \mathcal{E}_x') \subset \mathcal{E}_- \), since \( \omega \mathcal{E}_x \subset \mathcal{E}_- \). We have \( \omega \mathcal{E}_x \subset \mathcal{E}_x \cap (\mathcal{E}_x' + \omega \mathcal{E}_x) = \mathcal{E}_x^- \) and \( \mathcal{E}_- \subset \mathcal{E}_x \cap \frac{1}{\omega} \mathcal{E}_x' \), since \( \omega \mathcal{E}_+ \subset \mathcal{E}_x \) and \( \omega \mathcal{E} \subset \mathcal{E}_x' \).

We put \( j_1 = \text{length}((\mathcal{E}_+ + \mathcal{E}^+_{x,x'})/\mathcal{E}^+_{x,x'}) \) and \( j_2 = \text{length}((\mathcal{E}_- + \mathcal{E}^-_{x,x'})/\mathcal{E}^-_{x,x'}) \). We have

\[
\text{length}(\mathcal{E}_-/(\mathcal{E}_- \cap \mathcal{E}_-')) = \text{length}(\mathcal{E}_-/(\mathcal{E}_- \cap \mathcal{E}_x')) = j_2 + \text{length}((\mathcal{E}_- + \mathcal{E}^-_{x,x'})/(\mathcal{E}_- \cap \mathcal{E}_x')) = j_2 + \text{length}(\mathcal{E}^-_{x,x'}/(\mathcal{E}_- \cap \mathcal{E}_x')) = j_2 + d_1.
\]

We have

\[
j_2 + i - d_2 = \text{length}((\mathcal{E}_- + \mathcal{E}^-_{x,x'})/\mathcal{E}_-) \leq 1 + \text{length}((F(\mathcal{E}^+_{x,x'}) + \mathcal{E}^-_{x,x'}/F(\mathcal{E}^+_{x,x'}))
\]

since \( \text{length}(F(\mathcal{E}^+_{x,x'})/\mathcal{E}_-) = 1 \). Further we have

\[
\text{length}((F(\mathcal{E}^+_{x,x'}) + \mathcal{E}^-_{x,x'}/F(\mathcal{E}^+_{x,x'})) = \text{length}(\mathcal{E}_+/(\mathcal{E}_- \cap \mathcal{E}^+_{x,x'})) \leq \text{length}(\mathcal{E}_+/\mathcal{E}_- + \mathcal{E}^+_{x,x'})) \leq \text{length}(\mathcal{E}/(\mathcal{E}_+ + \mathcal{E}^-_{x,x'})) = i' - 1 - \text{length}(\mathcal{E}_-/(\mathcal{E}_- \cap \mathcal{E}^-_{x,x'})) = i' - 1 - j_2 - d_1.
\]

Therefore we obtain \( j_2 \leq \lfloor (i' - i + d_2 - d_1)/2 \rfloor \).

Further, \( j_1 + d_2 - d_1 \leq j_2 \leq j_1 + d_2 - i + 1 \) follows from \( \text{length}(F(\mathcal{E}^+_{x,x'})/\mathcal{E}_+) = 1 \). This implies \( i - 1 - d_2 \leq j_1 \) and \( d_2 - i \leq j_2 \). We have \( j_1 \leq i - 1 - d_1 \) and \( j_2 \leq n - i - d_2 - j_1 \) by the inclusions \( \mathcal{E}_x + \omega \mathcal{E}_x' \subset \mathcal{E}_+ \cap \mathcal{E}^+_{x,x'} \) and \( \mathcal{E}_+ + \mathcal{E}^+_{x,x'} \subset \mathcal{E}_- \cap \mathcal{E}^-_{x,x'} \). The equality

\[
\text{length}((\mathcal{E} + \mathcal{E}^-_{x,x'}/\mathcal{E}^-_{x,x'}) + \text{length}((\mathcal{E} + \mathcal{E}^-_{x,x'})/\mathcal{E}^-_{x,x'}) = i' - 1 - d_1.
\]

and \( \text{length}((\mathcal{E} + \mathcal{E}^-_{x,x'}/\mathcal{E}_-) \leq i - 1 \) imply \( j_2 \geq i' - i - d_1 \).

We have

\[
\text{length}((\mathcal{E} + \mathcal{E}^-_{x,x'}/\mathcal{E}^-_{x,x'}) = \text{length}(\mathcal{E}^-(\mathcal{E} \cap \mathcal{E}^-_{x,x'})) = \text{length}(\mathcal{E}^-(\mathcal{E} \cap \mathcal{E}^-_{x,x'})) = \text{length}((\mathcal{E} + \mathcal{E}^-_{x,x'})/\mathcal{E}^-_{x,x'}) = \text{length}((\mathcal{E} + \mathcal{E}^-_{x,x'}/\mathcal{E}^-_{x,x'}) - \text{length}(\mathcal{E}^-_{x,x'}/\mathcal{E}_-).
\]

Hence, \( \text{length}((\mathcal{E} + \mathcal{E}^-_{x,x'}/\mathcal{E}_-) = i' - 1 - j_2 - d_1 \) if and only if \( \text{length}((\mathcal{E} + \mathcal{E}^-_{x,x'}/\mathcal{E}^-_{x,x'}) = i' - 1 \). This implies the last claim. \( \square \)

9.1.2 Same hyperspecial subgroup

Assume that \( x = x' \). It suffices to consider the case where \( x = x' = x_0 \), since all the hyperspecial subgroups are conjugate.

Let \( 2 \leq i \leq \lfloor (n - 1)/2 \rfloor \). Let \( (\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-) \) be a point of \( X_i \). Let \( s \) be the rank of \( (\mathcal{E} \cap \mathcal{E}^0)/\mathcal{E}_- \). We put \( \mathcal{V}_1 = \mathcal{E}/\mathcal{E}_- \) and take \( \mathcal{V}_2 \subset \mathcal{E}^0/\mathcal{E}_- \) and \( \mathcal{V}_3 \subset \mathcal{E}_+/(\mathcal{E} \cap \mathcal{E}^0) \) such that projections induce isomorphisms \( \mathcal{V}_2 \simeq (\mathcal{E} + \mathcal{E}^0)/\mathcal{E} \) and \( \mathcal{V}_3 \simeq \mathcal{E}_+/(\mathcal{E} + \mathcal{E}^0) \). An open neighbourhood of \( (\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-) \) in \( \text{Gr}(i - 1, \mathcal{V}_1) \) under \( \text{S.6} \) is given by \( \text{Hom}(\mathcal{V}_1, \mathcal{V}_2 \oplus \mathcal{V}_3) \) sending \( f \in \text{Hom}(\mathcal{V}_1, \mathcal{V}_2 \oplus \mathcal{V}_3) \) to the inverse image \( \mathcal{E}_f \) of

\[
\{ v + f(v) \mid v \in \mathcal{V}_1 \} \subset \mathcal{E}_+ \cap \mathcal{E}_-
\]

in \( \mathcal{E}_+ \). By Theorem \text{S.6}, the condition that \( \mathcal{E}_f \) belongs to \( X_i \) is equivalent to

\[
\langle v + f(v), F(v' + f(v')) \rangle = 0 \quad (9.1)
\]
in $\varpi^{-1}W_\mathcal{O}(R)/W_\mathcal{O}(R)$ for $v, v' \in \mathcal{V}_1$. We write $f$ as $f_2 + f_3$ for $f_2 \in \text{Hom}(\mathcal{V}_1, \mathcal{V}_2)$ and $f_3 \in \text{Hom}(\mathcal{V}_1, \mathcal{V}_3)$. For $v, v' \in (\mathcal{E} \cap \mathcal{E}^0)/\mathcal{E}_-$, the condition (9.1) is equivalent to

$$\langle v + f_2(v), F(f_3(v')) \rangle + \langle f_3(v), F(v' + f_2(v') + f_3(v')) \rangle = 0$$

in $\varpi^{-1}W_\mathcal{O}(R)/W_\mathcal{O}(R)$.

Take a basis $v_1, \ldots, v_{s-1}$ of $\mathcal{V}_1$ such that $v_1, \ldots, v_s$ form a basis of $(\mathcal{E} \cap \mathcal{E}^0)/\mathcal{E}_-$. Take a basis $v_{s+1}, \ldots, v_{2i-1}$ of $\mathcal{V}_2$ and a basis $v_{2i-1}, \ldots, v_{2i-s-1}$ of $\mathcal{V}_3$. Write $f(v_i)$ as $x_{l,i}v_i + \cdots + x_{l,2i-1}v_{2i-1}$. Then the condition (9.2) is equivalent to

$$\langle v_l + \sum_{j=1}^{2i-1} x_{l,j}v_j, F(\sum_{k=2i-s}^{2i-1} x_{m,k}v_k) \rangle + \langle \sum_{k=2i-s}^{2i-1} x_{l,k}v_k, F(v_m + \sum_{j=1}^{2i-1} x_{m,j}v_j) \rangle = 0$$

for $1 \leq l, m \leq s$. We can write this as

$$\langle (v_l + \sum_{j=1}^{2i-1} x_{l,j}v_j, F(v_k)) \rangle_{l,k} = -(x_{l,k})_{l,k}((v_k, F(v_m + \sum_{j=1}^{2i-1} x_{m,j}v_j))_{k,m}.$$ 

Taking the determinant, we obtain

$$\det(x_{l,k})_{l,k} \left( \det((v_l + \sum_{j=1}^{2i-1} x_{l,j}v_j, F(v_k))_{l,k}(\det(x_{l,k})_{l,k})^{q-1} - (-1)^s \det((v_k, F(v_m + \sum_{j=1}^{2i-1} x_{m,j}v_j))_{k,m} = 0. $$

The condition $\mathcal{E}_{l} \cap \mathcal{E}^0 = \mathcal{E}_-$ is equivalent to $\det(x_{l,k})_{l,k} \neq 0$. Hence, if $(\mathcal{E}, \mathcal{E}_+, \mathcal{E}_-)$ belongs to the closure of $\pi_0^{-1}(X^{b_0,x_0}(\tau^*_s))$, then we have $\det((v_k, F(v_m))_{k,m} = 0$. This means $F^{-1}(\mathcal{E}_+^0) \subset \mathcal{E}$. Hence we have obtained the following proposition:

**Proposition 9.3.** The intersection

$$\pi_0^{-1}(X^{b_0,x_0}(\tau^*_s)) \cap \pi_0^{-1}(X^{b_0,x_0}(\tau^*_r))$$

is contained in the locus defined by the condition $F^{-1}(\mathcal{E}_+^0) \subset \mathcal{E}$.

Conversely, we assume that $F^{-1}(\mathcal{E}_+^0) \subset \mathcal{E}$. Then we may assume that $v_1$ is a basis of $F^{-1}(\mathcal{E}_+^0)/\mathcal{E}_-$, $v_i$ is an element of $(F(\mathcal{E}_+^0)/\mathcal{E}_-)$ lifting a basis of $(\mathcal{E} \cap \mathcal{E}^0)/(\mathcal{E} \cap \mathcal{E}_0)$ such that $v_i \notin F^{-1}(\mathcal{E}_+^0)/\mathcal{E}_-$ and $v_{2i-s}$ is an element of $F(\mathcal{E}_+^0)/\mathcal{E}_-$ lifting a basis of $(\mathcal{E} \cap \mathcal{E}^0)/\mathcal{E}_0$. Further, we may assume that $v_2, \ldots, v_{s-1}$ and $v_{2i-1}, \ldots, v_{2i-1-s}, v_{2i-s}, \ldots, v_{i+1}$ form dual base with respect to the paring

$$\mathcal{E}/(F^{-1}(\mathcal{E}_+^0) \times \mathcal{E}_+/(F(\mathcal{E}_+^0)); (v, v') \mapsto \langle F(v), v' \rangle.$$ 

and that $\langle v_j, F(v_k) \rangle = 0$ for $i + 1 < j < 2i - 1$ and $1 \leq k \leq 2i - 1$. Then the condition (9.1) is equivalent to

$$\langle \sum_{j=2i-s}^{2i-1} x_{l,j}v_j, F(\sum_{k=2i-s}^{2i-1} x_{m,k}v_k) \rangle + \left\{ \begin{array}{ll}
\langle v_l + x_{l,i}v_i, F(\sum_{k=2i-s}^{2i-1} x_{m,k}v_k) \rangle & \text{if } 1 \leq l \leq r, \\
\langle v_l + x_{l,i}v_i, F(\sum_{k=i}^{2i-1} x_{m,k}v_k) \rangle & \text{if } s + 1 \leq l \leq i - 1,
\end{array} \right.

\left\{ \begin{array}{ll}
0 & \text{if } m = 1, \\
\langle x_{l,(2i+1-m)} \rangle & \text{if } 2 \leq m \leq s, \\
x_{l,(2i-m)} & \text{if } s + 1 \leq m \leq i - 1,
\end{array} \right. $$

(9.3)
for $1 \leq l, m \leq i - 1$. We put

$$y = \det(x_{i,j})_{1 \leq s, 2i-s \leq 2i-1}.$$ 

We want to show that the quotient of $k[[x_{i,j}]]_{1 \leq i \leq i-1, i \leq j \leq 2i-1}$ by the relation (9.3) is nonzero after inverting $y$.

**Proposition 9.4.** (1) The intersection

$$\pi_0^{-1}(X_{\mu}^{b_1,x_0}(\tau_1^*)) \cap \pi_0^{-1}(X_{\mu}^{b_2,x_0}(\tau_2^*))$$

is equal to the locus defined by the condition $F^{-1}(E_{\nu}^\circ) = \mathcal{E}$.

(2) We have an isomorphism $X_{\mu}^{b_1,x_0}(\tau_1^*) \cap X_{\mu}^{b_2,x_0}(\tau_2^*) \cong X_{I_1}^{F,F^3}(1,1)^{\text{pf}}$ given by $\mathcal{E} \mapsto \mathcal{E}/\mathcal{E}_\nu^\circ$.

**Proof.** In this case, (9.3) becomes

$$\langle x_{1,3}v_3, F(x_{1,2}v_2 + x_{1,3}v_3) \rangle + \langle v_1 + x_{1,2}v_2, F(x_{1,3}v_3) \rangle = 0.$$ 

If the quotient of $k[[x_{1,2}, x_{1,3}]]$ by this relation is zero after inverting $x_{1,3}$, there is a positive integer $N$ such that $x_{1,3}^N$ is divisible by

$$\langle x_{1,3}v_3, F(x_{1,2}v_2 + x_{1,3}v_3) \rangle + \langle v_1 + x_{1,2}v_2, F(x_{1,3}v_3) \rangle$$
in $k[[x_{1,2}, x_{1,3}]]$. This does not happen because $\langle v_3, F(v_2) \rangle \neq 0$, which follows from $v_2 \notin F^{-1}(E_{\nu}^\circ)/E_{\nu}$. Hence we have (1) The claim (2) follows from (1). \[\square\]

By Proposition 9.4, $X_{\mu}^{b_1,x_0}(\tau_1^*) \cap X_{\mu}^{b_2,x_0}(\tau_2^*)$ is isomorphic to the perfect closed subscheme of $(\mathbb{P}^{n-1})^{\text{pf}}$ defined by two equations

$$\sum_{i=1}^{n} x_i x_{n+1-i}^q = 0, \quad \sum_{i=1}^{n} x_i x_{n+1-i}^3 = 0.$$ 

Since all non-degenerate hermitian forms on $\mathbb{F}_q^n$ are isomorphic, the above scheme is isomorphic to the perfect closed subscheme of $(\mathbb{P}^{n-1})^{\text{pf}}$ defined by two equations

$$\sum_{i=1}^{n} x_i^{q+1} = 0, \quad \sum_{i=1}^{n} x_i^{3+1} = 0.$$ 

### 9.2 Intersection of components for $\nu_i$ and $\nu_r$ when $n$ is even.

**Proposition 9.5.** If $i \neq r$, then $X_{\mu}^{b_i,x}(\tau_i^*) \cap X_{\mu}^{b_r,x}(\tau_r^*)$ is empty.

**Proof.** Assume $\mathcal{E}$ is a point of $X_{\mu}^{b_i,x}(\tau_i^*) \cap X_{\mu}^{b_r,x}(\tau_r^*)$. Then $\mathcal{E} \mapsto F(\mathcal{E}^\nu)$ and $\mathcal{E} \mapsto F(\mathcal{E}^\nu)$ are bounded by $\mu$. This is a contradiction. \[\square\]

Assume that $x \neq x'$. 

23
Proposition 9.6. Assume that $X_{\nu^1}^{b\times} (\tau^\times_\nu) \cap X_{\nu^1}^{b\times} (\tau^\times_\nu)$ is not empty. Then we have $1 \leq l_{x,x'} \leq r - 1$ and $\mathcal{E} \subset X_{\nu^1}^{b\times} (\tau^\times_\nu) \cap X_{\nu^1}^{b\times} (\tau^\times_\nu)$ is parametrized by $\mathcal{E} \to \mathcal{E}_x$ bounded by $\nu^1$ such that $\mathcal{E} \subset \mathcal{E}_x$. In particular, it is isomorphic to $\mathcal{E}_x$. 

Proof. Assume that $\mathcal{E} \subset \mathcal{E}_x$ is a point of $X_{\nu^1}^{b\times} (\tau^\times_\nu) \cap X_{\nu^1}^{b\times} (\tau^\times_\nu)$. Since $\mathcal{E} \subset \mathcal{E}_x$, both $\mathcal{E} \to \mathcal{E}_x$ and $\mathcal{E} \to \mathcal{E}_{x'}$ are bounded by $\nu^1$. We have the following chain conditions:

\[
\mathcal{E}_x \subset \mathcal{E} \subset \mathcal{E}_x, \\
\mathcal{E}_{x'} \subset \mathcal{E} \subset \mathcal{E}_{x'}.
\]

The inclusion follows from $\mathcal{E}_x \subset \mathcal{E} \subset \mathcal{E}_{x'}$. Note that $\text{length}(\mathcal{E}_x/\mathcal{E}_{x'}) = 2$, while both $\text{length}(\mathcal{E}/\mathcal{E}_x)$ and $\text{length}(\mathcal{E}/\mathcal{E}_{x'})$ are $r - 1$. Then $\mathcal{E}_x \cap \mathcal{E}_{x'}$ and $\mathcal{E}$ are related by

\[
\mathcal{E}_x + \mathcal{E}_{x'} \subset \mathcal{E}_x \subset \mathcal{E}_{x'} \subset \mathcal{E}_x/\mathcal{E}_{x'}.
\]

Since $l_{x,x'} = \text{length}(\mathcal{E}_x + \mathcal{E}_{x'})/\mathcal{E}_x$, we have

\[
l_{x,x'} = r - 1 - \text{length}(\mathcal{E}/(\mathcal{E}_x + \mathcal{E}_{x'})).
\]

Hence we have $1 \leq l_{x,x'} \leq r - 1$. The isomorphism in the claim is given by sending $\mathcal{E}$ to $\mathcal{E}/(\mathcal{E}_x + \mathcal{E}_{x'}) \subset \mathcal{E}_x \cap \mathcal{E}_{x'}/(\mathcal{E}_x + \mathcal{E}_{x'})$. \hfill \square

10 Example

In this section, we study in details the case where $n = 6$. We identify a moduli parametrizing modification $\mathcal{E} \subset \mathcal{E}_x$ bounded by $\nu_1$ with $(\mathbb{P}^5)^{pf}$ by taking a basis of $\Lambda_x$ such that the Hermitian paring is the standard one. Let $\mathbb{P}_{x,x'}^{\pm}$ be the projective space of $(\mathbb{P}^5)^{pf}$ defined by the condition $\mathcal{E} \subset \mathcal{E}_x^{\pm} \subset \mathcal{E}$. Let $\mathbb{P}_{x,x'}^{-}$ be the projective space of $(\mathbb{P}^5)^{pf}$ defined by the condition $\mathcal{E} \subset \mathcal{E}_x^{-} \subset \mathcal{E}$. We note that $\mathbb{P}_{x,x'}^{\pm}$ and $\mathbb{P}_{x,x'}^{-}$ are isomorphic to $(\mathbb{P}^{d_2})^{pf}$ and $(\mathbb{P}^{d_1-1})^{pf}$ respectively.

10.1 Intersection of components for $\nu_1$

We may assume that $x \neq x'$. The intersection is not empty only if $l_{x,x'} = 1$. In this case, $d_1 = 0$, $d_2 = 1$, $j_1 = j_2 = 0$. The intersection is $\mathbb{P}_{x,x'}^{-}$, which is a point given by $\mathcal{E}_x \cap \mathcal{E}_{x'}$.

10.2 Intersection of components for $\nu_1$ and $\nu_2$

If $x = x'$, then the intersection is isomorphic to the perfect closed subscheme of $(\mathbb{P}^5)^{pf}$ defined by two equations

\[
\sum_{i=1}^{6} x_i x_{i-j}^{q} = 0, \quad \sum_{i=1}^{6} x_i x_{i-j}^{p} = 0.
\]

We assume that $x \neq x'$.
10.2.1 \( d_1 = 0, d_2 = 1 \)

In this case, \( j_1 = 0 \) and \( j_2 = 1 \). The intersection is equal to the perfect closed subscheme of \( \mathbb{P}_{x,x'} \) defined by equation

\[
\sum_{i=1}^{5} x_i x_{6-i} = 0.
\]

10.2.2 \( d_1 = 0, d_2 = 2 \)

In this case, \( j_1 = 0 \) and \( j_2 = 1 \). The intersection is \( \mathbb{P}_{x,x'}, \) which is isomorphic to \( (\mathbb{P}^1)^{\nu} \).

Remark 10.1. If \( d_1 = d_2 = 1 \), then there is no \( j_2 \in \mathbb{N} \) satisfying the condition in Proposition 7.3.

10.3 Intersection of components for \( \nu_2 \)

Let \((\mathcal{E}_+, \mathcal{E}_-)\) be a point of \( X_{I_0}^4([1])^{\nu} \). The hermitian paring on \( V \) induces a paring on \( \mathcal{E}_+/\mathcal{E}_- \) since we have \( \mathcal{E}_+ \subset F(\mathcal{E}_-) \) and \( \mathcal{E}_- \subset F(\mathcal{E}_+) \). We take a basis \( v_1, v_2, v_3 \) of \( \mathcal{E}_+/\mathcal{E}_- \) such that \( v_1 \in F^{-1}(\mathcal{E}_+)/\mathcal{E}_-, \) \( v_2 \in \mathcal{E}_+ \), \( v_3 \in \mathcal{E}_- \). Let \( E \) be a point of \( X_{I_0}^4(\tau_2') \) in the fiber of \((\mathcal{E}_+, \mathcal{E}_-)\) under

\[
\pi: X_{H}^4(\tau_2') \rightarrow X_{I_0}^4([1])^{\nu}.
\]

We can take a generator \( v = x_1 v_1 + x_2 v_2 + x_3 v_3 \) of \( \mathcal{E}/\mathcal{E}_- \) for \( x_1, x_2 \in k \), since \( \mathcal{E} \not\subset \mathcal{E}_x \). Then we have

\[
\langle v, F(v) \rangle = x_1 \langle v_1, F(v_3) \rangle + x_2 \langle v_2, F(v_3) \rangle + x_3 \langle v_3, F(v_2) \rangle + \langle v_3, F(v_3) \rangle \]

because \( \langle w, F(w') \rangle = 0 \) for \( w, w' \in \mathcal{E}_x/\mathcal{E}_- \) and \( \langle v_3, F(v_1) \rangle = 0 \). Hence the fiber of \((\mathcal{E}_+, \mathcal{E}_-)\) under \( \pi \) is defined by

\[
x_1 \langle v_1, F(v_3) \rangle + x_2 \langle v_2, F(v_3) \rangle + x_3 \langle v_3, F(v_2) \rangle + \langle v_3, F(v_3) \rangle = 0
\]

in \((\mathbb{A}^2)^{\nu}\). We note that \((\langle v_1, F(v_3) \rangle, \langle v_2, F(v_3) \rangle) \neq (0, 0) \) because \( v_3 \notin \mathcal{E}_x/\mathcal{E}_- \).

We describe the fiber of

\[
\pi_{j_1,j_2}: X_{I_0}^4(\tau_2')_{\mathbb{P}_{x,x'}([w_{j_1,j_2}])} \cap \text{Par}_{j_1,j_2}(\mathscr{G}_{j_1,j_2}, \mathscr{P}_{j_1,j_2})_{\tau_{j_1,j_2}} \rightarrow X_{I_0}^4([1])^{\nu}_{\mathbb{P}_{x,x'}([w_{j_1,j_2}])}
\]

when

\[
\left( X_{I_0}^4(\tau_2') \cap X_{I_0}^4(\tau_2') \right)_{\mathbb{P}_{x,x'}([w_{j_1,j_2}])}
\]

is not empty.

10.3.1 \( d_1 = 0, d_2 = 1 \)

In this case, \( 0 \leq j_1 \leq 1 \) and \( j_2 = 0 \). We have \( d_{j_1,j_2} = 2 - j_1 \). The fiber of \( \pi_{j_1,0} \) is given by the condition \( \mathcal{E} \not\subset \mathcal{E}_{+,-} \).

10.3.2 \( d_1 = 0, d_2 = 2 \)

In this case, \( 0 \leq j_1 \leq 1 \) and \( 0 \leq j_2 \leq 1 \). We have \( d_{j_1,j_2} = 1 - j_1 + j_2 \). The fiber of \( \pi_{j_1,0} \) is given by the condition \( \mathcal{E} \not\subset \mathcal{E}_{+,-} \). The fiber of \( \pi_{j_1,1} \) is given by the condition \( \mathcal{E} \subset \mathcal{E}_{+,-} \).

25
10.3.3 \( d_1 = 1, \ d_2 = 1 \)
In this case, \( j_1 = 0 \) and \( j_2 = 0 \). We have \( d_{j_1,j_2} = 2 \). The fiber of \( \pi_{0,0} \) is given by the condition \( E \subset E_{+,+} \).

10.3.4 \( d_1 = 0, \ d_2 = 3 \)
In this case, \( j_1 = 0 \) and \( j_2 = 1 \). We have \( d_{j_1,j_2} = 1 \). The fiber of \( \pi_{0,1} \) is given by the condition \( E = E_{+,+} \).

10.3.5 \( d_1 = 1, \ d_2 = 2 \)
In this case, \( j_1 = 0 \) and \( j_2 = 0 \). We have \( d_{j_1,j_2} = 1 \). The fiber of \( \pi_{0,0} \) is given by the condition \( E = E_{+,+} \).

10.4 Intersection of components for \( \nu_3 \)
10.4.1 \( l_{x,x'} = 1 \)
The intersection is isomorphic to the perfection of the Fermat hypersurface defined by
\[
x_1 x_4^q + x_2 x_3^q + x_3 x_2^q + x_4 x_1^q = 0
\]
in \( \mathbb{P}^3 \).

10.4.2 \( l_{x,x'} = 2 \)
The intersection is a point given by \( E_x + E_{x'} \).

11 Shimura variety

Let \( E \) be a quadratic imaginary field, and let \( V \) be an \( n \)-dimensional Hermitian space over \( E \) with signature \((2, n - 2)\) at infinity. Fix a prime \( p \neq 2 \) inert in \( E \). Further assume that \( V \otimes_E \mathbb{Q}_p^2 \) contains a self-dual \( \mathbb{Z}_p^2 \) lattice \( \Lambda \). Let \( G = GU(V) \) be the general associated unitary group. We put \( G = GU(\Lambda) \) as before.

We take a basis of \( V_\mathbb{C} = V \otimes_E \mathbb{C} \) over \( \mathbb{C} \) such that the Hermitian form is given by the matrix diag\((1^2, -1^{n-2})\). Let \( h: \text{Res}_{\mathbb{C}/\mathbb{R}} G_{m,\mathbb{C}} \to G_\mathbb{R} \) be the morphism of algebraic groups over \( \mathbb{R} \) such that \( h(z) \) corresponds to diag\((z \cdot 1_2, \overline{z} \cdot 1_{n-2})\) for \( z \in \mathbb{C}^\times \) under

\[
G(\mathbb{R}) \subset \text{Aut}_\mathbb{C}(V_\mathbb{C}) \cong \text{GL}_n(\mathbb{C}),
\]
where the last isomorphism is given by the basis taken above. Let \( X \) be the \( G(\mathbb{R}) \)-conjugacy class of \( h \). Then \((G, X)\) is a Shimura datum.

We have an isomorphism

\[
(\text{Res}_{\mathbb{C}/\mathbb{R}} G_{m,\mathbb{C}})_\mathbb{C} \cong G_{m,\mathbb{C}} \times G_{m,\mathbb{C}}
\]
of algebraic groups over \( \mathbb{C} \) induced by the isomorphism \( \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C} ; \ a \otimes b \mapsto (ab, \overline{a}b) \).
We define \( \mu_h \) by the composition

\[
G_{m,\mathbb{C}} \hookrightarrow G_{m,\mathbb{C}} \times G_{m,\mathbb{C}} \cong (\text{Res}_{\mathbb{C}/\mathbb{R}} G_{m,\mathbb{C}})_\mathbb{C} \xrightarrow{h_\mathbb{C}} G_\mathbb{C},
\]

26
where the first morphism is the inclusion into the first factor. Let \( \mu : G_{mE} \to G_E \) be the morphism of algebraic over \( E \) such that \( \mu(z) \) corresponds to \((\text{diag}(z \cdot 1_2, 1_{n-2}), z)\) for \( z \in E^\times \) under the isomorphism
\[
G_E \simeq GL_n(E) \times G_{mE}
\]
given by taking a basis of \( V \) over \( E \). Then \( \mu_h \) and \( \mu_C \) are in the same \( G(\mathbb{C}) \)-conjugacy class. We note that the reflex field \( E(G,X) \) of \((G,X)\) is \( E \) if \( n \neq 4 \) and \( \mathbb{Q} \) if \( n = 4 \).

Let \( K^p \subset G(\mathbb{A}^p_f) \) be a sufficiently small open compact subgroup. Let \( K_p \subset G(\mathbb{Q}_p) \) be a hyperspecial subgroup. We put \( K = K^p K_p \subset G(\mathbb{A}_f) \). Let \( \text{Sh}_K(G,X) \) be the canonical model over \( E(G,X) \) of the Shimura variety attached to \((G,X)\) and \( K \). Let \( \mathcal{S}_K(G,X) \) be the canonical integral model of \( \text{Sh}_K(G,X) \) over \( \mathcal{O}_{E(G,X)_1(p)} \) constructed in \([\text{Kis10}]\).

Let \( S_K(G,X) \) be the perfection of \( \mathcal{S}_K(G,X) \otimes \overline{\mathbb{F}}_p \). We have the Newton map
\[
N : S_K(G,X)(\overline{\mathbb{F}}_p) \to B(G, \mu^*)
\]
as in \([\text{XZ17}]\ 7.2.7\). Let \([b] \in B(G, \mu^*)\) be the basic element. We write \( S_K(G,X)_{[b]} \) for the closed perfect subscheme of \( S_K(G,X) \) defined by \( N^{-1}([b]) \). We call \( S_K(G,X)_{[b]} \) the supersingular locus of \( S_K(G,X) \).

**Remark 11.1.** In \([\text{Kot92}]\), a moduli space of abelian schemes with additional structures is constructed. It is isomorphic to a finite union of integral models of Shimura varieties. Under the isomorphism, a point of \( S_K(G,X)_{[b]} \) corresponds to a supersingular abelian variety.

We take a point \( x \in S_K(G,X)_{[b]}(\overline{\mathbb{F}}_p) \). We put \( L = W(\overline{\mathbb{F}}_p)[\frac{1}{2}] \). Then we have a basic element \( b_x \in G(L) \) and an algebraic group \( I_x \) over \( \mathbb{Q} \) as in \([\text{XZ17}]\ 7.2.9\). We have an embeddings \( I_x(\mathbb{Q}) \subset G(\mathbb{A}_f^p) \) and \( I_x(\mathbb{Q}) \subset J_{b_x}(\mathbb{Q}_p) \) as in \([\text{XZ17}]\ 7.2.13\). Then we have the isomorphism
\[
I_x(\mathbb{Q}) \backslash X_{\mu^*}(b_x) \times G(\mathbb{A}_f^p)/K^p \simeq S_K(G,X)_{[b]} \tag{11.1}
\]
by \([\text{XZ17}]\) Corollary 7.2.16. By the isomorphism \( \text{(11.1)} \) and results in the previous sections, we obtain a description of irreducible components of \( S_K(G,X)_{[b]} \) and their intersections if \( K^p \) is sufficiently small.

**References**

[BR06] C. Bonnafé and R. Rouquier, On the irreducibility of Deligne-Lusztig varieties, C. R. Math. Acad. Sci. Paris 343 (2006), no. 1, 37–39.

[BS17] B. Bhatt and P. Scholze, Projectivity of the Witt vector affine Grassmannian, Invent. Math. 209 (2017), no. 2, 329–423.

[Gas10] Q. R. Gashi, On a conjecture of Kottwitz and Rapoport, Ann. Sci. Éc. Norm. Supér. (4) 43 (2010), no. 6, 1017–1038.

[GHKR06] U. Görtz, T. J. Haines, R. E. Kottwitz and D. C. Reuman, Dimensions of some affine Deligne-Lusztig varieties, Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 3, 467–511.
[GHN19] U. Görtz, X. He and S. Nie, Fully Hodge-Newton decomposable Shimura varieties, Peking Math. J. 2 (2019), no. 2, 99–154.

[Hai06] T. J. Haines, Equidimensionality of convolution morphisms and applications to saturation problems, Adv. Math. 207 (2006), no. 1, 297–327.

[HP14] B. Howard and G. Pappas, On the supersingular locus of the GU(2, 2) Shimura variety, Algebra Number Theory 8 (2014), no. 7, 1659–1699.

[Kis10] M. Kisin, Integral models for Shimura varieties of abelian type, J. Amer. Math. Soc. 23 (2010), no. 4, 967–1012.

[Kot92] R. E. Kottwitz, Points on some Shimura varieties over finite fields, J. Amer. Math. Soc. 5 (1992), no. 2, 373–444.

[Kot97] R. E. Kottwitz, Isocrystals with additional structure. II, Compositio Math. 109 (1997), no. 3, 255–339.

[KR11] S. Kudla and M. Rapoport, Special cycles on unitary Shimura varieties I. Unramified local theory, Invent. Math. 184 (2011), no. 3, 629–682.

[MV07] I. Mirković and K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings, Ann. of Math. (2) 166 (2007), no. 1, 95–143.

[SGA3-3] Schémas en groupes. III: Structure des schémas en groupes réductifs, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153, Springer-Verlag, Berlin-New York, 1970.

[VW11] I. Vollaard and T. Wedhorn, The supersingular locus of the Shimura variety of GU(1, n − 1) II, Invent. Math. 184 (2011), no. 3, 591–627.

[XZ17] L. Xiao and X. Zhu, Cycles on Shimura varieties via geometric Satake, 2017, arXiv:1707.05700

[Zhu17] X. Zhu, Affine Grassmannians and the geometric Satake in mixed characteristic, Ann. of Math. (2) 185 (2017), no. 2, 403–492.

Maria Fox
Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA
mariafox@uoregon.edu

Naoki Imai
Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
naoki@ms.u-tokyo.ac.jp