SMOOTHNESS DEPENDENT STABILITY IN CORROSION DETECTION

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Abstract. We consider the stability issue for the determination of a linear corrosion in a conductor by a single electrostatic measurement. We established a global log-log type stability when the corroded boundary is simply Lipschitz. We also improve such a result obtaining a global log stability by assuming that the damaged boundary is $C^{3,1}$-smooth.

1 Introduction

In this paper we study the stable determination of a corrosion coefficient on an inaccessible boundary by means of electrostatic measurements.

More precisely, we consider

\[
\begin{aligned}
\Delta u &= 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= g, & \text{on } \Gamma_A, \\
\frac{\partial u}{\partial \nu} + \gamma u &= 0, & \text{on } \Gamma_I,
\end{aligned}
\]

(1.1)

where $\Gamma_A$ and $\Gamma_I$ are two open, disjoint portions of $\partial\Omega$ such that $\partial\Omega = \Gamma_A \cup \Gamma_I$ and $\Omega \subset \mathbb{R}^n$, $n \geq 2$. The portion $\Gamma_A$ corresponds to the part of boundary which is accessible to measurements while $\Gamma_I$ is the portion which is out of reach and where the corrosion damage occurs. The function $\gamma$ is known as corrosion coefficient and it models the surface impedance of the conductor. The inverse problem we address here consists in the determination of such $\gamma$ by means of the current density $g$ prescribed on $\Gamma_A$ and the corresponding measured potential $u|_{\Gamma_A}$. In particular, we are interested in providing global stability estimates for $\gamma$, or namely avoiding the a priori hypothesis that the unknown corrosion coefficient is a small perturbation of a given and known one.

Our first aim is to investigate the continuous dependence of $\gamma$ upon the data when the corroded boundary $\Gamma_I$ is merely Lipschitz. To this purpose, we notice that by the impedance condition in (1.1) we can formally compute $\gamma$ as

\[
\gamma(x) = -\frac{1}{u(x)} \frac{\partial u(x)}{\partial \nu}.
\]

(1.2)

Since the potential $u$ may vanish in some points on $\Gamma_I$, it follows that the above quotient may be highly unstable. In this respect it is necessary to compute the local vanishing rate of $u$ on $\Gamma_I$. Indeed, we proved that such a rate can be controlled in an exponential manner as follows

\[
\int_{\Delta_r(x_0)} u^2 \geq \exp(-Kr^{-K})
\]

(1.3)

where $K > 0$ and $\Delta_r(x_0) = B_r(x_0) \cap \Gamma_I$ with $x_0 \in \Gamma_I^\gamma \subset \Gamma_I$ (see Subsection 2.1 for a precise definition) for sufficiently small radius $r$ (see Subsection 4.1). By combining such a control with a logarithmic stability estimate for the underlying Cauchy problem we are able to prove a global stability estimate for $\gamma$ with a log-log type modulus of continuity.

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The second purpose of this paper is to strengthen the hypothesis on the corroded boundary assuming that \( \Gamma_1 \) is \( C^{1,1} \)-smooth in order to obtain a better rate of stability. Indeed under such additional a priori smoothness hypothesis, we derive a surface doubling inequality of this sort for sufficiently small radius \( r \) (see Subsection 4.2).

\[
\int_{\Delta_r(x_0)} u^2 \leq \text{const.} \int_{\Delta_r(x_0)} u^2, \tag{1.4}
\]

which allows us to deduce that the vanishing rate of \( u \) at the boundary is at most polynomial, that is

\[
\int_{\Delta_r(x_0)} u^2 \geq \frac{1}{K} r^K, \tag{1.5}
\]

for sufficiently small radius \( r \) (see Subsection 4.2). Again by gathering a logarithmic stability estimate for the Cauchy problem and the above vanishing rate we provide a global stability estimate for \( \gamma \) with a single log.

In addition we also give an alternative proof of the above mentioned global logarithmic stability estimate. Such an alternative argument mostly relies on the application of the theory of the Muckenhoupt weights which justifies the computation in (1.2) in the \( L^p \) sense for some \( p > 1 \).

Indeed, such a dependence of the modulus of continuity upon the smoothness of the boundary have been already observed in other contexts. In [3], inverse problems for the determination of unknown defects with Dirichlet and Neumann condition have been studied. The authors proved that when the unknown boundary is smooth enough and hence a doubling inequality at the boundary is available then stability turns out to be of logarithmic type. On the contrary relaxing the regularity assumptions on the unknown domain the rate of stability degenerates into a log-log type one.

Let us mention that global stability estimates for unknown boundary impedance coefficients have been previously discussed under analogous boundary smoothness assumptions in [5] and [16] for an inverse acoustic scattering problem.

The present inverse problem has been studied in [4] and in [11] in a two dimensional setting where the authors provided a global logarithmic stability estimate for the corrosion coefficient for \( C^{1,\alpha} \) corroded boundary.

Similar inverse problems have been studied for the heat equation [9] and for the Stokes equations [10], where logarithmic stability estimates for the Robin coefficient \( \gamma \) have been provided. However in such papers the analysis on the local vanishing control of the solution has not been carried over and as a consequence the stability results are stated only on a compact set where the solution does not vanish.

The paper is organized as follows. In Section 2 we introduce notation and definition, the main assumptions and we state our main results in Theorem 2.1 and in Theorem 2.2. In Section 3 we preliminary analyse the direct problem recalling some regularity properties of the solution in Lemma 3.1 and Lemma 3.2. Moreover, in Theorem 3.3 we provide an a priori bound of the boundary trace of the solution in the \( H^1 \) norm. The proof of such a bound relies on the well-known Rellich’s identity. In Subsection 4.1 we discuss the inverse problem under the a priori hypothesis of a merely Lipschitz boundary. In Theorem 4.2 we recall a known stability result for the underlying Cauchy problem based on unique continuation tools, while in Corollary 4.3 we use the latter result in order to deduce the stability for negative norms of the normal derivative of \( u \). In Theorem 4.4 we provide a lower bound on the local vanishing rate of the solution \( u \). The main ingredient of the proof is the so called Lipschitz Propagation of Smallness, see also [5, 16]. Finally in Proposition 4.5 we state a weighted interpolation inequality which was previously introduced in [5] and we conclude by giving the proof of Theorem 2.1. In Subsection 4.2 we treat the inverse problem under the further \( C^{1,1} \) a priori smoothness assumption on \( \Gamma_1 \). In Theorem 4.6 we recall a stability result for the Dirichlet trace of the solution in \( C^1 \) norm. The increased smoothness regularity hypothesis on \( \Gamma_1 \) allows us to refine the analysis on the local vanishing control of the solution, indeed in Proposition 4.7 a surface doubling inequality is provided. We use such an inequality as tool to state in Theorem 4.8 the polynomial rate of decay of the solution at the boundary. The main argument of this proof again relies on Lipschitz Propagation of Smallness estimates, see also [9]. In Proposition 4.9 we state a weighted interpolation inequality for a weight satisfying a polynomial vanishing rate. We conclude by giving a proof of Theorem 2.2. As already mentioned, we also provide another way to obtain the logarithmic stability results which involves in Proposition 4.10 the notion of Muckenhoupt weights [12]. We complete Section 4 with an alternative proof of the Theorem 2.2 relying on the result achieved in Proposition 4.10.
2 Main Results

2.1 Notation and definitions

We introduce some notation that we shall use in the sequel.

For any \( x_0 \in \partial \Omega \) and for any \( \rho > 0 \) we shall denote

\[
\Gamma_\rho^A = \{ x \in \Gamma_A : \text{dist}(x, \Gamma_I) > \rho \} , \tag{2.1}
\]

\[
\Gamma_\rho^I = \{ x \in \Gamma_I : \text{dist}(x, \Gamma_A) > \rho \} , \tag{2.2}
\]

\[
\Gamma_\rho(x_0) = B_\rho(x_0) \cap \overline{\Omega} , \tag{2.3}
\]

\[
\Delta_\rho(x_0) = B_\rho(x_0) \cap \partial \Omega . \tag{2.4}
\]

**DEFINITION 2.1.** We shall say that a domain \( \Omega \) is of Lipschitz class with constants \( r_0, M > 0 \) if for any \( P \in \partial \Omega \), there exists a rigid transformation of coordinates under which we have \( P = 0 \) and

\[
\Omega \cap B_{r_0} = \{ (x', x_n) : x_n > \varphi(x') \} \tag{2.5}
\]

where

\[
\varphi : B'_r \subset \mathbb{R}^{n-1} \to \mathbb{R} \tag{2.6}
\]

is a Lipschitz function satisfying

\[
|\varphi(0)| = |\nabla \varphi(0)| = 0 \quad \text{and} \quad \|\varphi\|_{C^{0,1}(B'_r)} \leq Mr_0 , \tag{2.7}
\]

where we denote by

\[
\|\varphi\|_{C^{0,1}(B'_r)} = \|\varphi\|_{L^\infty(B'_r)} + r_0 \sup_{x,y \in B'_r} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \tag{2.8}
\]

and \( B'_r(x_0) \) denotes a ball in \( \mathbb{R}^{n-1} \).

**DEFINITION 2.2.** Given \( \alpha, 0 < \alpha \leq 1 \), we shall say that a domain \( \Omega \) is of class \( C^{1,\alpha} \) with constants \( r_0, M > 0 \) if for any \( P \in \partial \Omega \), there exists a rigid transformation of coordinates under which we have \( P = 0 \) and

\[
\Omega \cap B_{r_0} = \{ (x', x_n) : x_n > \varphi(x') \} \tag{2.9}
\]

where

\[
\varphi : B'_r \subset \mathbb{R}^{n-1} \to \mathbb{R} \tag{2.10}
\]

is a \( C^{1,\alpha} \) function satisfying

\[
|\varphi(0)| = |\nabla \varphi(0)| = 0 \quad \text{and} \quad \|\varphi\|_{C^{1,\alpha}(B'_r)} \leq Mr_0 , \tag{2.11}
\]

where we denote

\[
\|\varphi\|_{C^{1,\alpha}(B'_r)} = \|\varphi\|_{L^\infty(B'_r)} + r_0 \| \nabla \varphi \|_{L^\infty(B'_r)} + r_0^{1+\alpha} \sup_{x,y \in B'_r, x \neq y} \frac{|\nabla \varphi(x) - \nabla \varphi(y)|}{|x - y|^{\alpha}} . \tag{2.12}
\]
2.2 Assumptions and a-priori information

Assumption on the domain

Given $r_0, M > 0$ constants, we assume that $\Omega \subset \mathbb{R}^n$ and

\[ \Omega \text{ is of Lipschitz class with constants } r_0, M. \]  \hspace{1cm} (2.12)

Moreover, we assume that

\[ \text{the diameter of } \Omega \text{ is bounded by } d_0. \]  \hspace{1cm} (2.13)

Assumption on $\gamma$

Given $\gamma_0 > 0$ constant we assume that the Robin coefficient $\gamma \geq 0$ is such that $\text{supp } \gamma \subset \Gamma_I$ and

\[ \| \gamma \|_{C^{0,1} (\Gamma_I)} \leq \gamma_0. \]  \hspace{1cm} (2.14)

Assumption on $g$

Given $E, \hat{r}$ positive constants we assume that the current flux $g$ is such that $\text{supp } g \subset \Gamma_{\hat{r}}$ and

\[ \| g \|_{C^{0,\alpha} (\Gamma_{\hat{r}})} \leq E. \]  \hspace{1cm} (2.15)

From now on we shall refer to the a-priori data as the following set of quantities $r_0, M, d_0, \gamma_0, E, \hat{r}$.

In the sequel we shall denote with $\eta(\cdot)$ a positive increasing concave function defined on $(0, +\infty)$, that satisfies

\[ \eta(t) \leq C (\log(t))^{-\vartheta}, \quad \text{for every } 0 < t < 1, \]  \hspace{1cm} (2.16)

where $C > 0, \vartheta > 0$ are constants depending on the a priori data only.

Let us fix an open connected portion $\Gamma$ of the boundary of $\Omega$. We introduce the trace space $H^{1/2}_0(\Gamma)$ as the interpolation space $[H^1_0(\Gamma), L^2(\Gamma)]_\frac{1}{2}$, we refer to [15, Chap.1] for further details. The functions in $H^{1/2}_0(\Gamma)$ might be also characterized as the elements in $H^{1/2}(\partial \Omega)$ which are identically zero outside $\Gamma$, this identification shall be understood throughout. We denote with $H^{-1/2}_0(\Gamma)$ its dual space, which also can be interpreted as a subspace of $H^{-1/2}(\partial \Omega)$.

2.3 The main results

THEOREM 2.1. Let $\Omega$ be a Lipschitz domain and let $\gamma_1, \gamma_2$ satisfy (2.14). Let $u_i, i = 1, 2$ be the weak solution to the problem (1.1) with $\gamma = \gamma_i$ respectively. If for some $\varepsilon$, we have

\[ \| u_1 - u_2 \|_{L^2(\Gamma_{\hat{r}})} \leq \varepsilon \]  \hspace{1cm} (2.17)

then

\[ \| \gamma_1 - \gamma_2 \|_{L^\infty(\Gamma_{\hat{r}})} \leq \eta \circ \eta(\varepsilon) \]  \hspace{1cm} (2.18)

THEOREM 2.2. Let $\Omega$ be a $C^{1,\alpha}$ domain with $0 < \alpha \leq 1$ and let $\gamma_1, \gamma_2$ satisfy (2.14). Furthermore, we assume that $\Gamma_I$ is of class $C^{1,1}$ with constants $r_0, M$. Let $u_i, i = 1, 2$ be the weak solution to the problem (1.1) with $\gamma = \gamma_i$ respectively. If for some $\varepsilon$, we have

\[ \| u_1 - u_2 \|_{L^2(\Gamma_{\hat{r}})} \leq \varepsilon \]  \hspace{1cm} (2.19)

then

\[ \| \gamma_1 - \gamma_2 \|_{L^\infty(\Gamma_{\hat{r}})} \leq \eta(\varepsilon) \]  \hspace{1cm} (2.20)
3 The direct problem

**Lemma 3.1.** Let $\Omega$ be a Lipschitz domain. Let $u \in H^1(\Omega)$ be a solution to (1.1) with $\gamma$ and $g$ satisfying the a-priori assumptions stated above. Then there exists a constant $0 < \alpha < 1$ and a constant $C > 0$ depending on the a-priori data only, such that $u \in C^{\alpha}(\bar{\Omega})$, such that

$$\|u\|_{C^{\alpha}(\bar{\Omega})} \leq C.$$  \hfill (3.1)

**Proof.** This is a standard regularity estimate up to the boundary. The Moser iteration techniques [13, Theorem 8.18] fits to this task. More details can be found in [19]. Such arguments only require the Lipschitz regularity of $\partial \Omega$.

**Lemma 3.2.** Let $\Omega$ be a $C^{1,\alpha}$ domain with $0 < \alpha \leq 1$. Let $u \in H^1(\Omega)$ be a solution to (1.1) with $\gamma$ and $g$ satisfying the a-priori assumptions stated above. Then there exists a constant $0 < \alpha' < 1$ and a constant $C > 0$ depending on the a-priori data only, such that $u \in C^{1,\alpha'}(\bar{\Omega})$, such that

$$\|u\|_{C^{1,\alpha'}(\bar{\Omega})} \leq C.$$  \hfill (3.2)

**Proof.** Again the proof relies in a slight adaptation of the arguments developed in [19] based on the Moser iteration technique and by well-known regularity bounds for the Neumann problem [2, p.667].

**Theorem 3.3.** Let $\Omega$ be a Lipschitz domain and let $v \in H^1(\Omega)$ be a solution to

$$\Delta v = 0 \text{ in } \Omega.$$  \hfill (3.3)

If its trace $v|_{\partial \Omega} \in H^1(\partial \Omega)$ then $\frac{\partial u}{\partial \nu}|_{\partial \Omega} \in L^2(\partial \Omega)$ and we have

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial \Omega)}^2 \leq C \left( \|\nabla_T v\|_{L^2(\partial \Omega)}^2 + \|u\|_{H^1(\Omega)}^2 \right).$$  \hfill (3.4)

Conversely, if $\frac{\partial u}{\partial \nu}|_{\partial \Omega} \in L^2(\partial \Omega)$ then $v|_{\partial \Omega} \in H^1(\partial \Omega)$ and

$$\|\nabla_T v\|_{L^2(\partial \Omega)}^2 \leq C \left( \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial \Omega)}^2 + \|u\|_{H^1(\Omega)}^2 \right).$$  \hfill (3.5)

Here $\nabla_T v$ denotes the tangential gradient of $v$ on $\partial \Omega$ and $C$ depends on $M, r_0$ and $d_0$ only.

**Proof.** These inequalities follow from well-known Rellich’s identity [17]. Related estimates were first proven by Jerison and Kenig [14]. A detailed proof in the present form can be found in [5, Proposition 5.1].

**Theorem 3.4.** Let $u$ be as in Lemma 3.1, then

$$\|u\|_{H^1(\partial \Omega)} \leq C,$$  \hfill (3.6)

where $C > 0$ depends on the a priori data only.

**Proof.** The proof is a consequence of (3.5) in combination with the impedance condition in (1.1), the regularity assumption (2.13) on $g$ and standard estimates for solution to boundary value problem for the Laplace equation.

4 The inverse problem

In this section we shall discuss the desired stability estimates. For a sake of exposition we first discuss in Subsection 4.1 the case when the boundary $\Gamma_I$ is of Lipschitz class only. While the treatment of the case when $\Gamma_I$ is $C^{1,1}$-smooth will follow in Subsection 4.2.
4.1 The Lipschitz corroded boundary case

**Lemma 4.1.** Let \( u \in H^1(\Omega) \cap C^0(\Omega) \), be a solution to
\[
\Delta u = 0 \quad \text{in} \quad \Omega. \tag{4.1}
\]
We have
\[
\| \frac{\partial u}{\partial \nu} \|_{H^{-1}(\Gamma_I)} \leq C \| u \|_{L^\infty(\Omega)} \tag{4.2}
\]
where \( C \) depends on \( M, r_0 \) and \( d_0 \) only.

**Proof.** By standard result on elliptic boundary value problem, for any \( \zeta \in H^1_0(\Gamma_I) \) we can consider the unique solution \( \varphi \in H^1(\Omega) \) to the Dirichlet problem
\[
\begin{cases}
\Delta \varphi = 0 & \text{in } \Omega, \\
\varphi = \zeta & \text{on } \Gamma_I, \\
\varphi = 0 & \text{on } \Gamma_A. 
\end{cases} \tag{4.3}
\]
Moreover we have
\[
\| \varphi \|_{H^1(\Omega)} \leq C \| \zeta \|_{H^1_0(\Gamma_I)} \tag{4.4}
\]
with \( C > 0 \) only depending on the a priori data. By the Green’s identity we have that
\[
\int_{\Gamma_I} \zeta \frac{\partial u}{\partial \nu} = \int_{\partial \Omega} u \frac{\partial \varphi}{\partial \nu} \tag{4.5}
\]
hence
\[
\left| \int_{\Gamma_I} \zeta \frac{\partial u}{\partial \nu} \right| \leq \int_{\partial \Omega} \left| \frac{\partial \varphi}{\partial \nu} \right| \tag{4.6}
\]
applying (3.4) to \( \varphi \) and taking into account (3.1) and (4.4) we get
\[
\left| \int_{\Gamma_I} \zeta \frac{\partial u}{\partial \nu} \right| \leq C \| u \|_{L^\infty(\Omega)} \left( \| \zeta \|_{H^1_0(\Gamma_I)} + \| \varphi \|_{H^1(\Omega)} \right) \tag{4.7}
\]
and the thesis follows by duality.

**Theorem 4.2.** Let \( u_i, i = 1, 2 \) be as in Theorem 2.1. If for some \( \varepsilon (2.17) \) holds, then
\[
\| u_1 - u_2 \|_{L^\infty(\Gamma_I)} \leq \eta(\varepsilon) \tag{4.9}
\]
where \( \eta \) is the modulus of continuity introduced in (2.16).

**Proof.** The proof follows by a slight adaptation of the argument developed in Proposition 4.4 in [19].

**Corollary 4.3.** Let \( u_i, i = 1, 2 \) be as in Theorem 2.1 then we have that
\[
\left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{H^{-1/2}_0(\Gamma_I)} \leq \eta(\varepsilon). \tag{4.10}
\]

**Proof.** By interpolation and the impedance condition we have that
\[
\left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{H^{-1/2}_0(\Gamma_I)} \leq C \left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{H^{-1}(\Gamma_I)}^{\theta} \| \gamma_1 u_1 - \gamma_2 u_2 \|_{L^2(\Gamma_I)}^{1-\theta} \tag{4.11}
\]
where \( C > 0, 0 < \theta < 1 \) are constants depending on the a priori data only. Finally by Lemma 4.1 and Theorem 6.3 we get the thesis.
THEOREM 4.4. Let $u$ be a weak solution to $(1.1)$. For every $r$, $0 < r < r_1$ and for every $x_0 \in \Gamma^0_\rho$ we have that

$$\int_{\Delta_r(x_0)} u^2 \geq \exp(-Kr^K)$$

(4.12)

where $r_1 = \min\{\frac{\rho}{4}, r_0, \frac{1}{k_1^{1+\frac{1}{K}}}\}$ and $k_1, k_2, K > 0$ only depend on the a priori data.

Proof. By the local stability estimates for the Cauchy problem discussed in [7, Theorem 1.7] and the bounds established earlier in Theorem 3.1 and Theorem 3.4, we get that for any $x_0 \in \Gamma^0_\rho$ and any $0 < r < r_1$ we have

$$\|u\|_{L^2(\Gamma^r_\rho(x_0))} \leq C(\|u\|_{H^\frac{1}{2}(\Delta_r(x_0))} + \|\partial_\nu u\|_{H^{-\frac{1}{2}}(\Delta_r(x_0))})^\delta \left(\|u\|_{L^2(\Gamma^r_\rho(x_0))}\right)^{1-\delta}$$

(4.13)

where $C > 0, 0 < \delta < 1$ are constants depending on the a priori data only. Moreover, by the following interpolation inequality

$$\|u\|_{H^\frac{1}{2}(\Delta_r(x_0))} \leq C\|u\|_{H^2(\Delta_r(x_0))}^{\frac{1}{2}} \|u\|_{H^1(\Delta_r(x_0))}^{\frac{1}{2}}$$

(4.14)

where $C > 0$ depends on the a priori data only, by the a priori bound in Theorem 3.4 and the impedance boundary condition we have that

$$\left(\int_{\Delta_r(x_0)} u^2\right)^{\frac{1}{2}} \geq C \int_{\Gamma^r_\rho(x_0)} u^2$$

(4.15)

Let us consider $\bar{x} \in \Gamma_r(x_0)$ be such that $B_{\bar{r}}(\bar{x}) \subset \Gamma^{\rho}_r(x_0)$. We now recall that using the arguments of Lipschitz propagation of smallness developed in [16, Proposition 3.1] and the impedance boundary condition we have that

$$\int_{B_{\bar{r}}(\bar{x})} |\nabla u|^2 \geq C \exp(-k_1r^{-k_2}) \int_{\Omega} |\nabla u|^2$$

(4.16)

where $k_1$ and $k_2$ are positive constants depending on the a priori data only.

Combining the standard inequality  

$$\int_{\Omega} |\nabla u|^2 \geq C_1 \|g\|_{H^{-\frac{1}{2}}(\Gamma, A)}$$

(4.17)

and the Caccioppoli inequality

$$\int_{B_{\bar{r}}(\bar{x})} |\nabla u|^2 \leq C_2 r^{-2} \int_{B_{\bar{r}}(\bar{x})} |u|^2$$

(4.18)

where $C_1, C_2 > 0$ are constants depending on the a priori data only we have that

$$\int_{B_{\bar{r}}(\bar{x})} |u|^2 \geq Cr^2 \exp(-k_1 r^{-k_2})$$

(4.19)

where $C$ is a constant depending on the a priori data only.

We observe that if $r < \min\{\frac{\rho}{4}, r_0, \frac{1}{k_1^{1+\frac{1}{K}}}\}$ we have that

$$\int_{B_{\bar{r}}(\bar{x})} |u|^2 \geq C \exp(-2k_1 r^{-k_2})$$

(4.20)

Moreover, combining the trivial inequality $\int_{\Gamma^r_\rho(x_0)} u^2 \geq \int_{B_{\bar{r}}(\bar{x})} u^2$ with (4.15) we have that

$$\int_{\Delta_r(x_0)} u^2 \geq C \exp(-4k_1 r^{-k_2})$$

(4.21)

Finally, we observe that it is possible to find a number $K > 0$ depending on $C, k_1, k_2, \delta$ only such the thesis follows. \qed
PROPOSITION 4.5. Given $M, K > 0$, let $w \geq 0$ be a measurable function on $\Gamma_0$ satisfying the conditions
\[ \|w\|_{L^\infty(\Gamma_0)} \leq M \] (4.22)
and
\[ \|w\|_{L^2(\Delta, (x_0))] \geq \exp(-K r^{-K}) \text{ for every } x \in \Gamma_0 \text{ and } r \in (0, r_1) \] (4.23)
where $r_1$ is as in Theorem 4.4 with $\rho = r_0$. Let $f \in C^\alpha(\Gamma_0)$ such that
\[ |f(x) - f(y)| \leq E|x - y|^{\alpha} \text{ for every } x, y \in \Gamma_0. \] (4.24)
If
\[ \int_{\Gamma_0} |f| w \leq \epsilon \] (4.25)
then
\[ \|f\|_{L^\infty(\Gamma_0)} \leq E \eta\left(\frac{\epsilon}{E}\right) \] (4.26)
where $\eta$ satisfies (2.16) with constants only depending on $M, K, r_0, \alpha, k_1, k_2$.

Proof. The proof of such weighted interpolation inequality relies on slight adaptation of the arguments in [8, Proposition 1].

Proof of Theorem 2.1. By a standard interpolation result we have that
\[ \|u_1(\gamma_1 - \gamma_2)\|_{L^2(\Gamma_1)} \leq C \|u_1(\gamma_1 - \gamma_2)\|_{H^{1/2}(\Gamma_1)} \|u_1(\gamma_1 - \gamma_2)\|_{H^{-1/2}(\Gamma_1)} \] (4.27)
where $C > 0$ is a constant depending on the a priori data only.

We observe that
\[ \|u_1(\gamma_1 - \gamma_2)\|_{H^1(\Gamma_1)} \leq \|\gamma_1 - \gamma_2\|_{C^{0,1}(\Gamma_1)} \|u_1\|_{H^1(\Gamma_1)} \leq C \] (4.28)
where $C > 0$ is a constant depending on the a priori data only.

Moreover by the impedance condition on $\Gamma_1$ it follows that
\[ \|u_1(\gamma_1 - \gamma_2)\|_{H_{00}^{1/2}(\Gamma_1)} \leq \left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{H_{00}^{1/2}(\Gamma_1)} + C \|u_1 - u_2\|_{H_{00}^{1/2}(\Gamma_1)} \] (4.29)
where $C > 0$ is a constant depending on the a priori data only.

By combining the estimate in Theorem 4.2 and in Corollary 4.3 we obtain
\[ \|u_1(\gamma_1 - \gamma_2)\|_{H_{00}^{1/2}(\Gamma_1)} \leq \eta(\epsilon). \] (4.30)
Hence by (4.27) we have that
\[ \|u_1(\gamma_1 - \gamma_2)\|_{L^2(\Gamma_0)} \leq \eta(\epsilon). \] (4.31)
The conclusion follows by applying Proposition 4.5 with $w = |u_1|$ and $f = (\gamma_1 - \gamma_2)^2$. □
4.2 The $C^{1,1}$-smooth corroded boundary case

THEOREM 4.6. Let $u_i, i = 1, 2$ be as in Theorem 2.2. If for some $\varepsilon$, \(2.17\) holds we have that
\[
\|u_1 - u_2\|_{C^1(\Gamma)} \leq \eta(\varepsilon) \tag{4.32}
\]
where $\eta$ is given by \(2.16\).

Proof. The proof can be achieved along the lines of Proposition 4.4 in [19] and Theorem 4.2 in [18].

PROPOSITION 4.7. Let $\Gamma_I$ be of class $C^{1,1}$ with constants $r_0, M$. Let $u$ be the solution to the problem \(1.1\), then there exist constants $K_1 > 0, \bar{r} > 0$ depending on the a priori data only, such that for every $x_0 \in \Gamma_{r_0}$ and every $r \in (0, \bar{r})$ the following holds
\[
\int_{\Delta_{2r}(x_0)} u^2 \leq K_1 \int_{\Delta_r(x_0)} u^2. \tag{4.33}
\]

Proof. We provide here a sketch of the proof. Let $v \in H^1(\Omega)$ be the weak solution to the problem
\[
\begin{cases}
\Delta v = 0, & \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} = 1, & \text{on } \Gamma_A, \\
\frac{\partial v}{\partial \nu} + \gamma u = 0, & \text{on } \Gamma_I. 
\end{cases} \tag{4.34}
\]
Dealing as in the proof of Lemma 3.3 of [19] an relying on an iterated used of the Harnack inequality as well as the Giraud's maximum principle, we may infer that there exists a constant $C > 0$ depending on the a priori data only such that $v(x) \geq C$ in $\Omega$.

It is trivial to check that the function $z = \frac{v}{u} \in H^1(\Omega)$ satisfies
\[
\begin{cases}
\text{div}(v^2 \nabla z) = 0, & \text{in } \Omega, \\
v^2 \frac{\partial z}{\partial \nu} = gv - u, & \text{on } \Gamma_A, \\
v^2 \frac{\partial z}{\partial \nu} = 0, & \text{on } \Gamma_I. 
\end{cases} \tag{4.35}
\]
Let us observe that such change of variable allows us to treat a new boundary problem with an homogeneous Neumann condition on $\Gamma_I$ instead of the Robin one. By the arguments due to Adolffson and Escauriaza in [1] (see also [3, Proposition 3.5]) we have that $u \in H^1(\Omega)$ satisfies the so called doubling inequality at the boundary which can be stated as follows. There exists a radius $\bar{r}$ depending on the a priori data only such that for any $x_0 \in \Gamma_{r_0}$ the following holds
\[
\int_{\Gamma_{2r}(x_0)} z^2 \leq C\beta^K \int_{\Gamma_r(x_0)} z^2 \tag{4.36}
\]
for every $r, \beta$ such that $\beta > 1$ and $0 < \beta r < 4\bar{r}$.

Now, we observe that repeating the arguments in Theorem 4.5 and in Theorem 4.6 in [18] and mainly based on well-known stability estimate for the Cauchy problem we can reformulate the above volume doubling inequality into a surface doubling inequality. Indeed, we have that there exists a constant $K_1 > 0$ depending on the a priori data only, such that for any $x_0 \in \Gamma_{r_0}$ and for every $r \in (0, \bar{r})$ the following holds
\[
\int_{\Delta_{2r}(x_0)} u^2 \leq K_1 \int_{\Delta_r(x_0)} u^2, \tag{4.37}
\]
and the thesis follows.
THEOREM 4.8. Let $\Gamma_I$ be of class $C^{1,1}$ with constants $r_0, M$. Let $u$ be a weak solution to (4.1). For every $r$, $0 < r < r_2$ and for every $x_0 \in \Gamma_{r_0}$ we have that

$$\int_{\Delta_r(x_0)} u^2 \geq \frac{1}{K} r^K$$

where $r_2 = \min\{\bar{r}, r_1\}$ and $K > 0$ only depends on the a priori data.

Proof. Let $x_0 \in \Gamma_{r_0}$. Dealing as in [6, Remark 4.11], we have that

$$\int_{\Delta_{2^{-j}r_2}(x_0)} u^2 \leq K_{j-1} \int_{\Delta_{2^{-j-1}r_2}(x_0)} u^2, \quad \text{for every } j = 2, 3, \ldots$$

(4.39)

By iteration over $j$ we get

$$\int_{\Delta_r(x_0)} u^2 \geq \left(\frac{r}{r_2}\right)^q \int_{\Delta_{2^{-j}r_2}(x_0)} u^2$$

(4.43)

By (4.39) and (4.44) we find that

$$\int_{\Delta_r(x_0)} u^2 \geq C \left(\frac{r}{r_2}\right)^q \int_{\Gamma_{2^{-j}r_2}(x_0)} u^2$$

(4.44)

where $C > 0$ is a constant depending on the a priori data only.

Let $\bar{x} \subset \Gamma_{2^{-j}r_2}(x_0)$ be such that $B_{2^{-j}r_2}(\bar{x}) \subset \Gamma_{2^{-j}r_2}(x_0)$. By (4.40) with $r = \frac{r}{r_2}$ we have that

$$\left(\int_{\Gamma_{2^{-j}r_2}(x_0)} u^2\right)^{\frac{1}{2}} \geq C$$

(4.45)

where $C$ is a constant depending on the a priori data only. Combining (4.44) and (4.45) we have that

$$\int_{\Delta_r(x_0)} u^2 \geq C \left(\frac{r}{r_2}\right)^q$$

(4.46)

where $C > 0$ is a constant depending on the a priori data only.

We conclude by observing that we may find a constant $K > 0$ depending on the a priori data only such that the thesis follows.

PROPOSITION 4.9. Given $M, K > 0$, let $w \geq 0$ be a measurable function on $\Gamma_{r_0}$ satisfying the conditions

$$\|w\|_{L^\infty(\Gamma_{r_0})} \leq M$$

(4.47)
and
\[ \|w\|_{L^2(\Delta_r(x_0))} \geq \frac{1}{K} r^K \text{ for every } x \in \Gamma^r_0 \text{ and } r \in (0, r_2) \]  
where \( r_2 \) is as in Theorem 4.8. Let \( f \in C^\alpha(\Gamma^r_0) \) such that
\[ |f(x) - f(y)| \leq E|x - y|^\alpha \text{ for every } x, y \in \Gamma^r_0. \]  
If
\[ \int_{\Gamma^r_0} |f|^2 \leq \varepsilon \]  
then
\[ \|f\|_{L^\infty(\Gamma^r_0)} \leq C \left( \frac{\varepsilon}{E} \right)^{\delta'} \]  
where \( C > 0, 0 < \delta' < 1 \) are constants only depending on \( M, K, r_0, \alpha, r_2 \).

Proof. By the bound in (4.47) we have that
\[ \int_{\Delta_r(x)} w^2 \geq M^{-1} \frac{r^{2K}}{x^2} \text{ for every } x \in \Gamma^r_0 \text{ and } r \in (0, r_2). \]  
Let now \( \bar{x} \) be such that \( |f(\bar{x})| = \|f\|_{L^\infty(\Gamma^r_0)} \). By the Hölder regularity of \( f \) we have that for every \( r > 0 \) and \( x \in \Delta_r(\bar{x}) \) the following holds
\[ |f(\bar{x})| \leq |f(x)| + Er^\alpha. \]  
Multiplying the above inequality by the weight \( w \) and integrating both sides over \( \Delta_r(\bar{x}) \) we obtain that
\[ |f(\bar{x})| \int_{\Delta_r(\bar{x})} w \leq \int_{\Delta_r(\bar{x})} w |f| + Er^\alpha \int_{\Delta_r(\bar{x})} w, \]  
from which we deduce that
\[ \|f\|_{L^\infty(\Gamma^r_0)} \leq \frac{\varepsilon}{\int_{\Delta_r(\bar{x})} w} + Er^\alpha \leq \varepsilon M^{\kappa} r^{-2K} + Er^\alpha. \]  
Now minimizing over \( r \in (0, r_2) \), the thesis follows with \( \delta' = \frac{\alpha}{2K + \alpha} \).

Proof of Theorem 2.2. By the impedance condition we have that
\[ \|u_1(\gamma_1 - \gamma_2)\|_{L^2(\Gamma^r_0)} \leq \left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{L^2(\Gamma^r_0)} + C\|u_1 - u_2\|_{L^2(\Gamma^r_0)} \]  
where \( C > 0 \) is a constant depending on the a priori data only.

By Theorem 4.6 we obtain that
\[ \|u_1(\gamma_1 - \gamma_2)\|_{L^2(\Gamma^r_0)} \leq \eta(\varepsilon). \]  
By applying Proposition 4.9 the thesis follows with \( w = u_1 \) and \( \lambda = (\gamma_1 - \gamma_2)^2 \) up to a possible replacement of the constants \( C \) and \( \vartheta \) in (2.16).

We now follow a slightly different strategy in order to prove Theorem 2.2. The main difference is based on the introduction of the notion of Muckenhoupt weights in Proposition 4.10.
PROPOSITION 4.10. Let $\Gamma_I$ be of class $C^{1,1}$ with constants $r_0, M$. Let $u$ be the solution to the problem (1.11), then there exist constant $p > 1, A > 0$ depending on the a priori data only, such that for every $\Gamma_I^n$ and every $r \in (0, \bar{r})$ the following holds

$$
\left( \frac{1}{|\Delta r(u_0)|} \int_{\Delta r(u_0)} u^2 \right) \left( \frac{1}{|\Delta r(u_0)|} \int_{\Delta r(u_0)} u^{-\frac{\beta}{p-1}} \right)^{p-1} \leq A.
$$

(4.59)

Proof. For a detailed proof we refer to Corollary 4.7 in [18]. The main tools of the proof relies on the above mentioned surface doubling inequality (4.35) and the theory of Muckenhoupt weights [12] as well.

Alternative proof of Theorem 2.2. Let $x_0 \in \Gamma_I^n$. Let us choose $r = \frac{r}{\bar{r}}$, where $\bar{r}$ is the radius in Proposition 4.10. By the lower bound in (4.35) with $r = \frac{r}{\bar{r}}$ and with $u = u_2$ we have that

$$
\int_{\Delta \frac{r}{\bar{r}}(x_0)} u_2^2 \geq C,
$$

(4.60)

where $C > 0$ is a constant depending on the a priori data only.

Combining (4.60) and (4.61), we have that for every $x_0 \in \Gamma_I^n$ the following holds

$$
\left( \int_{\Delta \frac{r}{\bar{r}}(x_0)} \left| u_2 \right|^\beta \right)^{\frac{1}{\beta}} \leq C,
$$

(4.61)

where $C > 0$ is a constant depending on the a priori data only.

Let us now consider $x \in \Delta \frac{r}{\bar{r}}(x_0)$, then by Theorem 4.6 and by (2.14) we have that

$$
|\gamma_1(x) - \gamma_2(x)| \leq (\gamma_0 + 1)\eta(x)\frac{1}{\left| u_2(x) \right|}.
$$

(4.62)

Denoting with $\beta = \frac{2}{p-1}$ and combining (4.61) and (4.62) we find that

$$
\left( \int_{\Delta \frac{r}{\bar{r}}(x_0)} \left| \gamma_1(x) - \gamma_2(x) \right|^{\beta} \right)^{\frac{1}{\beta}} \leq \eta(x).
$$

(4.63)

By the a priori bound (2.14), we get that

$$
\|\gamma_1 - \gamma_2\|_{L^{2}(\Delta \frac{r}{\bar{r}}(x_0))} \leq (2\gamma_0)^{\frac{1}{2}} \left( \int_{\Delta \frac{r}{\bar{r}}(x_0)} \left| \gamma_1(x) - \gamma_2(x) \right|^{\beta} \right)^{\frac{1}{\beta}},
$$

(4.64)

which in turn combined with (4.63) implies that by a possible further replacement of the constants $C, \theta$ in (2.16) we have

$$
\|\gamma_1 - \gamma_2\|_{L^{2}(\Delta \frac{r}{\bar{r}}(x_0))} \leq \eta(x).
$$

(4.65)

By interpolation we have that

$$
\|\gamma_1 - \gamma_2\|_{L^{\infty}(\Delta \frac{r}{\bar{r}}(x_0))} \leq C\|\gamma_1 - \gamma_2\|_{L^{2}(\Delta \frac{r}{\bar{r}}(x_0))} \|\gamma_1 - \gamma_2\|_{C^{0,1}(\Delta \frac{r}{\bar{r}}(x_0))},
$$

(4.66)

where $C > 0$ is a constant depending on the a priori data only.

Hence by the a priori bound (2.14) and (4.66) we have that by a possible further replacement of the constants $C, \theta$ in (2.16) we have

$$
\|\gamma_1 - \gamma_2\|_{L^{\infty}(\Delta \frac{r}{\bar{r}}(x_0))} \leq \eta(x).
$$

(4.67)

By a covering argument we finally deduce the thesis.
References

[1] V. Adolfsson, L. Escauriaza, $C^{1,\alpha}$ domains and unique continuation at the boundary, Comm. Pure Appl. Math 50 (1997), 935-969.

[2] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solution of elliptic partial differential equations satisfying general boundary conditions I., Comm. Pure Appl. Math. 12 (1959), 623-727.

[3] G. Alessandrini, E. Beretta, E. Rosset, S. Vessella, Optimal stability for Inverse Elliptic Boundary Value Problems with Unknown Boundaries, Ann. Sc. Norm. Super. Pisa - Scienze Fisiche e Matematiche - Serie IV. Vol. XXXIX. Fasc. 4 (2000).

[4] G. Alessandrini, L. Del Piero, L. Rondi, Stable determination of corrosion by a single electrostatic boundary measurement, Inverse Problem, 19 (2003), no. 4, 973-984.

[5] G. Alessandrini, A. Morassi, E. Rosset, Detecting cavities by electrostatic boundary measurements, Inverse Problems 18, (2002), 1333-1353.

[6] G. Alessandrini, A. Morassi, E. Rosset, Size estimates, Inverse problems: Theory and Applications (Contemporary Mathematics vol. 333), ed. G. Alessandrini and G. Uhlmann (Providence, RI: American Mathematical Society), 1-33.

[7] G. Alessandrini, L. Rondi, E. Rosset, S. Vessella, The stability for the Cauchy problem for elliptic equations, Inverse Problems 25, (2009), 123004 (47pp).

[8] G. Alessandrini, E. Sincich, S. Vessella, Stable determination of surface impedance on a rough obstacle by far field data, Inverse Problems and Imaging 7, (2013), 341-351.

[9] M. Bellazzouedi, J. Cheng, M. Choulli, Stability estimate for an inverse boundary coefficient problem in thermal imaging, J. Math. Anal. Appl. 343 (2008), 328-336.

[10] M. Boulakia, A.C. Egloffe, C. Grandmont, stability estimates for a Robin coefficient in the two-dimensional Stokes problem, Mathematical control and related field, 3 (2012), 21-49.

[11] S. Chaabane, I. Fellah, M. Jaoua, J. Leblond, Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems, Inverse Problems, 20 (2004), no.1, 47-59.

[12] R.R. Coifman, C.L. Fefferman, Weighted norm inequalities for maximal function and singular integrals, Studia Math. , 51 (1976), 241-250.

[13] D. Gilbarg, N.S. Trudinger, “Elliptic Partial Differential Equations of Second Order”, Second edition, Springer-Verlag, Berlin, Heidelberg, New York, (1977).

[14] D.S. Jerison, C. E. Kenig, The Neumann problem on Lipschitz domains. Bull. Amer. Math. Soc. (N.S) 4 (1981), no. 2, 203-207. (1980), 181-189 (English. Russian original).

[15] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Vol.1, Springer-Verlag, 1972.

[16] A. Morassi, E. Rosset, Stable determination of cavities in elastic bodies, Inverse Problems 20, (2004), 453-480.

[17] F. Rellich, Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral, Math. Z. 46 (1940), 635-636.

[18] E. Sincich, Stable determination of the surface impedance of an obstacle by far field measurements, SIAM J. Math. Anal. 38, (2006), 434-451.

[19] E. Sincich, Stability for the determination of unknown boundary and impedance with a Robin boundary condition, SIAM J. Math. Anal. 6, (2010), 2922-2943.