Topical Review

Towards a mathematical theory of the Madelung equations: Takabayasi’s quantization condition, quantum quasi-irrotationality, weak formulations, and the Wallstrom phenomenon

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Abstract

Even though the Madelung equations are central to many ‘classical’ approaches to the foundations of quantum mechanics such as Bohmian and stochastic mechanics, no coherent mathematical theory has been developed so far for this system of partial differential equations. Wallstrom prominently raised objections against the Madelung equations, aiming to show that no such theory exists in which the system is well-posed and in which the Schrödinger equation is recovered without the imposition of an additional ‘ad hoc quantization condition’—like the one proposed by Takabayasi. The primary objective of our work is to clarify in which sense Wallstrom’s objections are justified and in which sense they are not, with a view on the existing literature. We find that it may be possible to construct a mathematical theory of the Madelung equations which is satisfactory in the aforementioned sense, though more mathematical research is required. More specifically, this work makes five main contributions to the subject: First, we rigorously prove that Takabayasi’s quantization condition holds for arbitrary $C^1$-wave functions. Nonetheless, we explain why...
there are serious doubts with regards to its applicability in the general theory of quantum mechanics. Second, we argue that the Madelung equations need to be understood in the sense of distributions. Accordingly, we review a weak formulation due to Gasser and Markowich and suggest a second one based on Nelson’s equations. Third, we show that the common examples that motivate Takabayasi’s condition do not satisfy one of the Madelung equations in the distributional sense, leading us to introduce the concept of ‘quantum quasi-irrotationality’. This terminology was inspired by a statement due to Schönberg. Fourth, we construct explicit ‘non-quantized’ strong solutions to the Madelung equations in two dimensions, which were claimed to exist by Wallstrom, and provide an analysis thereof. Fifth, we demonstrate that Wallstrom’s argument for non-uniqueness of solutions of the Madelung equations, termed the ‘Wallstrom phenomenon’, is ultimately due to a failure of quantum mechanics to discern physically equivalent, yet mathematically inequivalent states—an issue that finds its historic origins in the Pauli problem.

Keywords: Madelung equations, Schrödinger equation, quantum potential, quantum vortices, stochastic mechanics

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1. Introduction and main results

The Madelung equations [117] are known for having sparked a variety of ‘classical’ approaches to the Schrödinger theory of quantum mechanics. Arguably, the most common ones are the de Broglie–Bohm theory [24, 27, 52, 53, 88, 127, 153, 154] and stochastic mechanics [45, 56, 128–130, 148]. For one body of mass \( m \) the three (formal) equations in 3-dimensional space may be stated as follows:

\[
\begin{align*}
    m \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) &= -\nabla V + \frac{\hbar^2}{2m} \nabla \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \\
    \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) &= 0 \\
    \nabla \times \vec{v} &= 0.
\end{align*}
\]

The above equations can be generalized to the many-body case (cf e.g. section 7.1.2 in [88]) and there also exist analogue equations within the 1-body Pauli theory (see [96] and chapter 11 in [22]).

Though the Madelung equations and the Schrödinger equation are closely related, the mathematically precise relationship between the two systems of partial differential equations (PDEs) is still a matter of controversy (see e.g. [5, 71, 75])—a controversy that can be traced back to a variety of objections raised by Wallstrom [174–176] and that has recently been linked to a major open problem in the field of ‘geometric hydrodynamics’ (cf problem 37 in [103]). The discussion mostly concerns itself with the 1-body equations, but it is also of importance to the more general \( N \)-body case (see [145]). We shall elaborate on Wallstrom’s contribution later in this section. The importance of the mathematical theory of distributions in this context will be established in section 2.2 and discussed throughout the article.

The central, underlying question is the following:

Under which conditions are the Madelung equations and the Schrödinger equation mathematically equivalent and at which point does this equivalence break down?

So far, only three works [67, 140, 173] have made mathematically rigorous progress in addressing the above question, all with serious limitations. In lemma 2.1 of [67] Gasser and Markowich considered reformulated, ‘weak’ Madelung equations and showed that, under very mild regularity assumptions on the wave function and various other conditions, the equations were indeed implied by the Schrödinger equation\(^3\). The caveat is that the authors did not show how to reobtain the Schrödinger equation and made overly restrictive assumptions on the potential \( V \) (cf equation (A1) therein). In contrast, von Renesse took an optimal transport approach in order to find a new reformulation of the Madelung equations via the tools of Wasserstein geometry [173]. In the smooth setting he showed local equivalence between this formulation, the so called Hamilton–Jacobi–Madelung equations (equation (1.4) therein), and the Schrödinger equation (theorem 2.1 and corollary 2.2 in [173]). By imposing strong assumptions of regularity and restricting himself to time-independent simply-connected domains,

\(^3\) The reader is referred to section 2.2 for an explanation of the word ‘weak’. In this context, it also noteworthy that in the 1980s Carlen derived a ‘weak’ continuity equation from the Schrödinger equation under rather general assumptions (cf theorem 2.1(iii) in [39]).
there was no need to account for Wallstom’s objections\(^4\). In a similar vein, Reddiger showed local equivalence between the Madelung equations and the Schrödinger equation in the smooth setting (theorem 3.2 of [140]). To assert global equivalence, however, he required an additional topological assumption: Namely, that at each fixed time every connected component of the domain of \(\vec{v}\) is simply connected.

Some approaches—like stochastic mechanics and the theory based on Kolmogorovian probability theory laid out in [140], assert that the Madelung equations are physically more fundamental than the Schrödinger equation. Providing an answer to the above question is thus not merely of mathematical interest, but is possibly central to the historical and still current debate on the foundations of quantum mechanics\(^5\): Depending on how the mathematical question is resolved, it could shift the physical discussion away from the confines of quantum mechanics towards the more general question of the foundations of non-relativistic quantum theory.

The importance of Wallstrom’s objections [174–176] to this foundational discussion is highlighted by the following statement in section 2.6 of a recent book by de la Peña et al [45]:

[...] Wallstrom’s work has been considered by many as the definitive blow against Nelson’s [stochastic mechanics] and similar theories.

In the same sentence the authors state that they consider the issue as resolved, but we disagree with this assertion: As several other researchers\(^6\), de la Peña et al aim to address Wallstrom’s objections by imposing assumptions on the stochastic processes they consider (cf equation (4.37) and section 4.7 in [45]). While, for the sake of transparency, we support the idea of founding non-relativistic quantum theory on the mathematical theory of stochastic processes, addressing Wallstrom’s objections in this manner seems unlikely to convince contemporary critical voices in the wider physics community\(^7\). Contrarily, those voices might fall silent, if it was shown to be possible to address Wallstrom’s objections on the phenomenological level of the Schrödinger theory—that is, without the \textit{a priori} introduction of stochastic processes. As we will argue, the futility of such an approach has not been established.

Wallstrom’s substantial objections to non-relativistic quantum theories based on the Madelung equations [174–176] are twofold\(^8\): First, in [174, 175] he argued that

\(^{4}\) In a subsequent work [101] Khesin et al allowed for less regular probability densities and wave functions. However, their work asks for the domain to be compact and connected—which excludes the physically most common cases of \(\mathbb{R}^3\) and its unbounded, open subsets.

\(^{5}\) We refer, for instance, to p 143 in [10], p 153 sq. in [76], p 327 in [180], section 3.5.1 in [9], and [8].

\(^{6}\) See equation (4.4) in [184], equation (26) in [47, 48, 72], note 37 in [9]) as well as [70].

\(^{7}\) Such stochastic approaches generally belong to the class of ‘hidden variable’ theories. There is a large body of literature criticizing the latter. See, for instance, [7, 14, 60]. Moreover, motivated by specific interpretations of Bell’s theorem [15] and the CHSH-inequality [43] ‘non-local hidden variable theories’ have been developed and even those have come under scrutiny in recent decades [32, 33, 69, 112, 132]. The topic of hidden variables remains a lively and controversial subject of scholarly debate.

\(^{8}\) In section 6 in [175] the further objection was made that particle-based approaches fail to explain wave-like quantum behavior. This argument ought not to be given much weight, for it is the statistics of particles that exhibit wave-like behavior, not the particles themselves. This was already noted by Landé [109] in 1969 (see also p 341 in [16] as well as [110]):

[W]ave-like material phenomena result from the quantum mechanics of matter particles, as we know already since 1927 from Born’s famous statistical particle interpretation of the de Broglie–Schrödinger waves.

As long as said statistics is described correctly by the theory, the supposed counterargument is devoid of physical content. We also refer to section 2.5.1 in [45] and references therein.
an additional, supposedly ad-hoc ‘quantization condition’ needs to be imposed for the equations to be equivalent. Second, in [176] he asserted that the Madelung equations fail to yield a unique time-evolution from initial data, at least for the case that the connected components on which the density does not vanish merge over time.

The main goal of this article is to elaborate on those objections and to encourage further research—with a focus on the first objection and in particular with regards to the central, underlying question above. The physical relevance of the question means that a satisfactory resolution thereof has to form the cornerstone of any mathematical theory of the Madelung equations.

The precise contribution of this work to the controversy and to the development of such a mathematical theory will be given after we have explained the first objection in more detail.

In order to clarify Wallstrom’s first objection, recall that at fixed time a $C^1$-wave function $\Psi$, which does not vanish on its domain, gives rise to the drift field $\vec{v}$ above via

$$\vec{v} = \frac{\hbar}{m} \text{Im} \left( \frac{\nabla \Psi}{\Psi} \right) = \frac{\hbar}{2m} \left( \frac{\nabla \Psi}{\Psi} - \frac{\nabla \Psi^*}{\Psi^*} \right).$$

(1.2)

The aforementioned ‘quantization’ condition then states that $\vec{v}$ has to satisfy

$$\frac{m}{2\pi \hbar} \oint \vec{v} \cdot d\vec{r} \in \mathbb{Z}$$

(1.3)

at any given time for it to be physically acceptable.

While Takabayasi [165] is often credited with the discovery of condition (1.3), in the literature on quantum vortices [20, 21, 83–85, 90, 145, 150, 151, 158, 167, 182] this credit is mostly given to Dirac (cf p 67 in [50]).[9][10] We shall nonetheless call condition (1.3) Takabayasi’s quantization condition, for he was the first to suggest that it ought to be added to the Madelung equations, equations (1.1a)–(1.1c) above (cf p 155 in [165]).

With regards to the greater discussion on the foundations of non-relativistic quantum theory, Takabayasi [165] already voiced the following criticism:

[Equation (1.3) above] is so to speak the ‘quantum condition’ for fluidal motion

and of ad hoc and compromising character for our formulation, just as the

[Bohr–Sommerfeld] quantum condition for old quantum theory.

[reference omitted]

In essence, the above also constitutes Wallstrom’s central objection voiced in [174, 175].

At this point in time, Takabayasi’s condition has become ubiquitous in the literature surrounding the Madelung equations (see e.g. section 3.2.2 in [88], section 3.1 in [28], section 1.2.1 in [127], and [94, 125, 139, 151, 158, 159, 181]) and it appears to have become largely assimilated without putting much weight on the aforementioned conceptual objection.

9 Though there may be links on a mathematical [80, 98] and even on a physical level [116], physically one needs to distinguish between quantum vortices in fundamental physics, as studied here, and those occurring in superfluids. The latter is a macroscopic quantum phenomenon. See e.g. [171].

10 Contrary to what is suggested on p 278 in [92] and by [89], the article [179] by Wentzel should not be credited with the discovery of the condition. Therein, Wentzel considers the 1-dimensional time-independent Schrödinger equation and a meromorphic continuation of the wave function in order to be able to apply the residue theorem—a wholly different mathematical context.

11 As pointed out by a referee, the condition was already a topic at the 1927 Solvay conference—at least indirectly in the context of Sommerfeld’s quantization condition for the old quantum theory. The reader is referred to the discussions on de Broglie’s and Bohr’s reports in [54, 123].
The five main contributions of this article to the subject matter are as follows:

1) We rigorously prove that equation (1.3) holds for $C^1$-wave functions (proposition 2.1). Nonetheless, we argue on the basis of the mathematical theory of quantum mechanics that the condition is too restrictive—as already observed by Smolin in this context (cf section IV in [162]), not all wave functions are this regular. As we explain in section 2.2 and remark 2.7 in particular, there is serious doubt whether the condition can be appropriately extended.

2) In section 2.2 we argue that the relation between the Madelung equations and the Schrödinger equation needs to be understood in an appropriate ‘distributional’ sense. While this is rather obvious for those acquainted with the mathematical theory of quantum mechanics or the modern theory of PDEs, this central point has been largely ignored in the related physics literature. Therefore, we briefly review the relevant mathematical formalism. In section 3 we discuss what constitutes such a distributional approach to the Madelung equations, which in the mathematics literature is known as a ‘weak’ or ‘variational’ formulation. We discuss two such formulations, one given in seminal work by Gasser and Markowich [67] and one based on Nelson’s equations [128]. There may, however, be other weak formulations of physical relevance. Using this distributional approach, we further resolve an objection to the Bohmian theory due to Hall [75].

3) Consider the standard solutions $\Psi_{nl\mu}$ of the 1-body (time-independent) Schrödinger equation with attractive Coulomb potential in 3 dimensions—a generally accepted physical model to obtain the gross structure of the electromagnetic absorption spectrum of atomic hydrogen and of ‘hydrogen-like atoms’. We show rigorously that—using ‘physics notation’ for the Dirac delta—the distributional curl of the drift field $\vec{v}$ corresponding to $\Psi_{nl\mu}$ is given by

$$\left(\nabla \times \vec{v}\right)(t,x,y,z) = \frac{2\pi \mu \hbar}{m} \begin{pmatrix} 0 \\ 0 \\ \delta(x) \delta(y) \end{pmatrix} \quad (1.4)$$

(cf corollary 2.11).

That is, those solutions exhibit quantum vorticity in the sense that the third Madelung equation, equation (1.1c), is in general not satisfied, if the curl is understood in the distributional sense (see definitions 2.8 and 2.10, as well as corollary 2.11). Motivated by a statement made by Schönberg [158] in the 1950s, we call such drift fields quasi-irrotational (definition 2.10). It is also worth noting that here the quantum vorticity depends on the magnetic quantum number $\mu$—whose ‘quantization’ forces the other quantum numbers $n$ and $l$ to take integer values as well.

4) In section 4 we explicitly construct and analyze ‘non-quantized’, stationary, strong solutions of the Madelung equations for the case of the 2-dimensional isotropic harmonic oscillator (cf theorem 4.1 and corollary 4.2). Wallstrom claimed that such solutions exist [174]. Yet he neither gave any explicit examples nor is the existence thereof mathematically trivial.

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12 In the mathematics literature those two terms are often used interchangeably. Though for particular classes of PDEs there is indeed a relationship to the calculus of variations (cf section 6.1.2 and remark 2 of section 8.1.2 in [55]), we do not mean to suggest that the theory should necessarily be based on a variational principle. See e.g. [72, 183, 184] for the latter approach in the context of the stochastic theory.
since the need to account for other boundary conditions could in principle force the ‘quantization’ of the quantum numbers. We further provide a mathematical analysis of those solutions, taking various perspectives. In particular, we show rigorously that the ‘stationary states’ corresponding to those ‘non-quantized’ solutions are indeed incompatible with Schrödinger time evolution (proposition 4.4). While our results seem to undergird Wallstrom’s objection, we argue that it is ultimately the failure to account for point 2) above that makes the argument mathematically problematic. In particular, the respective wave functions are not weakly differentiable in most cases (proposition 4.3)—which suggests that the argument might not hold up, if one takes a more restrictive view on the Schrödinger equation (cf remark 3.5 and appendix A) and relates it to an appropriate weak formulation of the Madelung equations.

5) With regards to Wallstrom’s second objection [176], we argue in remark 3.4 of section 3 that it is the failure of the quantum-mechanical Schrödinger theory to discern physically equivalent, yet mathematically inequivalent states that leads to this Wallstrom phenomenon. We also show how it is related to the historical Pauli problem (see [5, 164, 177, 178])—whose negative resolution may be viewed as a theoretical anomaly within quantum mechanics.

With regards to the implication of those results for the general analysis of the precise relation between the Schrödinger equation and the Madelung equations, we refer to section 5 at the end of this article. There we also provide a more in-depth discussion of prior attempts in the literature to resolve the controversy.

In this work, we have taken care to provide rigorous proofs for the mathematical statements made. In order to put the focus on the arguments in the main text, those proofs have been pushed to the appendix.

For the convenience of the reader, we shall clarify some notation and conventions: \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \) denote the set of natural numbers, integers, real numbers, and complex numbers, respectively. \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( \mathbb{R}_+ \) is the open interval \((0, \infty) \subset \mathbb{R} \). For \( \alpha \in \mathbb{C} \), the quantities \(|\alpha|, \arg \alpha, \text{Re} \alpha, \text{Im} \alpha \) denote the absolute value, the principal value of the argument, the real part, and the imaginary part of \( \alpha \), respectively. By convention, we have \( \arg \alpha \in [0, 2\pi) \), \( \pi \) is the imaginary unit and the symbol \( \sim \) means ‘asymptotic to’. As already indicated, we distinguish between the (strictly positive) particle mass and the magnetic quantum number by using \( m \) for the former and \( \mu \) for the latter. Similarly, we use the symbol \( \rho \) for the probability density and \( \rho \) for the radial distance in polar coordinates. With regards to special functions, we use \( (n, \mu, x) \rightarrow L_n^\mu(x) \) for the associated Laguerre polynomials\(^{13} \), \( (n, \mu, x) \rightarrow P_n^\mu(x) \) for the associated Legendre polynomials, \( (l, \mu, x) \rightarrow Y_l^\mu(x) \) for the spherical harmonics, \( (a, b, x) \rightarrow {}_1F_1(a, b; x) \) and \( (a, b, x) \rightarrow U(a, b; x) \) for the confluent hypergeometric function of first and second kind, respectively, and \( x \rightarrow \Gamma(x) \) for the gamma function. As for function spaces, for \( n, m \in \mathbb{N}, k \in \mathbb{N}_0, \) and \( \mathbb{K} \in \{\mathbb{R, C}\} \), we use \( C^k(\mathbb{R}^n, \mathbb{K}^m) \) for the space of \( k \) times continuously differentiable \( \mathbb{K}^m \)-valued functions in \( \mathbb{R}^n \), \( C^0_0(\mathbb{R}^n, \mathbb{K}^m) \) if they are smooth and compactly supported, and \( \mathcal{S}(\mathbb{R}^n, \mathbb{K}^m) \) for the respective space of Schwartz functions (all spaces are equipped with their usual topology). \( \mathcal{H} \) is a general or specified Hilbert space with inner product \( \langle \cdot, \cdot \rangle \), antilinear in the first argument and linear in the second. We use common multi-variable calculus notation, like \( \Delta \) for the Laplacian, \( \int_U d^n r \) for an integral over \( U \subseteq \mathbb{R}^n \), \( \nabla \) for the del

\(^{13} \) Note that we use Slater’s convention for the associated Laguerre polynomials \( L_n^\mu(x) \) (cf section 5.5 in [161]). The latter coincide with the ones given by Messiah divided by \( (n + \mu)! \) (cf appendix B, section I.2 in [124]).
operator, etc. $\partial^\alpha$ is a multivariate (strong or weak) derivative with respect to the multi-index $\alpha$ and $|\alpha|$ denotes its order. We also use the SI-unit system throughout the article, denoting by $\varepsilon_0$ the electric constant, and by $e$ the magnitude of the electron charge.

2. On Takabayasi’s quantization condition

2.1. The condition for strong solutions

Consider the Schrödinger equation for one body in 3-dimensional space,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi,$$

(2.1)

with initial data $\Psi(t, \vec{r})|_{t=0} = \Psi_0(\vec{r})$ for $\vec{r}$ in some open subset of $\mathbb{R}^3$ and some given, sufficiently ‘regular’ potential $V$ thereon. Naively, we find that both $\Psi_0$ and $\Psi$ must be two-times differentiable in the variables $\vec{r}$ and $\Psi$ must be once differentiable in the variable $t$ for the equation to make sense on an elementary mathematical level. Though this view is not only naive but runs counter to the axioms of quantum mechanics, we shall nevertheless consider such so called ‘strong’ or ‘classical’ solutions here (as opposed to ‘weak’ or ‘distributional’ solutions, see section 2.2 below).\textsuperscript{14}

Consequently, if we fix the time $t$ and drop the respective dependence in our notation in the remainder of this subsection, the wave function $\Psi$ will be twice differentiable in each remaining variable—if not on $\mathbb{R}^3$ itself, then on some open subset $D \subseteq \mathbb{R}^3$ thereof. As differentiability at a point implies continuity there, we find that $\Psi$ is necessarily an element of $C^1(D, \mathbb{C})$, i.e. $\Psi$ is continuous on $D$, all first order partial derivatives exist thereon and are also continuous.

We shall show that it is this property alone that forces Takabayasi’s condition, equation (1.3), to hold. While for the mathematical reader acquainted with the concept of the winding number as well as degree theory this may not be too surprising, no proof of this fact or reference thereto has so far been provided in the literature on Takabayasi’s condition. The most detailed ‘derivation’ was given by Hirschfelder et al [85], which, due to the nature of the work, lacked mathematical rigor. As we shall see in the coming sections, the mathematics surrounding Takabayasi’s condition can become quite subtle, so that it is important to clarify when it holds and when it may fail.

Proposition 2.1. For $n \in \mathbb{N}$ with $n \geq 2$, let $D$ be open in $\mathbb{R}^n$ and let $\Psi$ be in $C^1(D, \mathbb{C})$ such that $\Psi$ does not vanish on $D$. Define $\vec{v}$ as in equation (1.2) above.

If the integral in equation (1.3) is taken over any $C^1$-curve $\gamma$ in $D$ defined on some closed interval $[a, b]$ with $\gamma(a) = \gamma(b)$, then the condition, equation (1.3), holds true.\textsuperscript{□}

Weakening the assumptions of proposition 2.1 above is non-trivial: Clearly, $\Psi$ is not allowed to vanish for $\vec{v}$ from equation (1.2) to be generally well-defined. Furthermore, continuous differentiability of $\Psi$ assures that $\vec{v}$ is continuous, which in turn allows us to apply the fundamental theorem of calculus without any further complications. Furthermore, not every $C^1$-wave function $\Psi$ is a so-called ‘WKB-state’ or ‘JWKB-state’, named after the authors of the historical

\textsuperscript{14} The word ‘classical solution’ is a term used in the mathematical theory of partial differential equations and does not have any relation to the term ‘classical physics’, apart from signifying historical developments of the respective subject areas.
As shown by countless examples, it is in general not possible to find even a continuous function $S$ on $D$ such that $\Psi = |\Psi| e^{i\phi}/\hbar$. That there is no such function $S$ appearing in the proof of proposition 2.1 is therefore not accidental.

Next consider the domain $D$, as given in proposition 2.1, and assume that $\vec{v}$ is a $C^1$-vector field. If $D$ is simply-connected (or consists of countably many simply-connected components) and $\gamma$ is a smooth curve in $D$ (as above) which is also the boundary of some (oriented) bounded, connected smooth surface, then we may apply Stokes’ theorem to show that the integral in equation (1.3) vanishes (cf e.g. section 13.3.3.2 in [38]). The former is the topological condition that Reddiger imposed in his proof of the (local) equivalence of the Schrödinger equation and the Madelung equations (theorem 3.2 in [140]). Contrarily, if this topological assumption does not hold—for instance due to a ‘line singularity’ of $\vec{v}$—then Stokes’ theorem cannot be employed, as the respective surface is not compact. Roughly speaking, this is how the integral in equation (1.3) can fail to be zero.

The paramount examples for Takabayasi’s condition are the standard solutions $\Psi_{nl\mu}$ of the aforementioned model for atomic hydrogen and of ‘hydrogen-like atoms’. In that instance, the integer in equation (1.3) is always given by the quantum number $\mu$.

In order to illustrate how the condition can hold despite the possibility to superpose the respective states, we shall consider the following example.

**Example 2.2.** The textbook result for the above model is that the energy eigenfunctions $\Psi_{nl\mu}$ in spherical coordinates $(r, \theta, \phi)$ centered at the position of the proton read as

$$\Psi_{nl\mu}(r, \theta, \phi) = R_{nl}(r) Y_{\mu}^l(\theta, \phi), \quad (2.2a)$$

with $n \in \mathbb{N}$, $l \in \{0, \ldots, n - 1\}$ and $\mu \in \{-l, \ldots, l\}$. The energy eigenvalues $E_n$ are degenerate in the sense that they do not depend on $l$ or $\mu$. Therefore, any linear combination of the possible $\Psi_{nl\mu}$ with same $n$ is an energy eigenfunction and consequently gives rise to a stationary state.

Fixing $n > 1$, we may now consider the family of energy eigenfunctions $\Phi_a$ with $a \in [-1, 1]$ and

$$\Phi_a(r, \theta, \phi) = R_{nl}(r) \left( \frac{1-a}{\sqrt{2(1+a^2)}} Y_{\mu}^{-1}(\theta, \phi) + \frac{1+a}{\sqrt{2(1+a^2)}} Y_{\mu}^{+1}(\theta, \phi) \right). \quad (2.2b)$$

As one easily checks, all $\Phi_a$ are normalized and they reduce to $\Psi_{n1\pm 1}$ for $a = \pm 1$. As in proposition 2.1, denote by $D_a$ the respective maximal domain on which $\Phi_a$ is $C^1$ and nowhere vanishing.

It is worthwhile to check for which values of $a$ the respective drift field $\vec{v}_a$ is well-defined and continuous. For convenience, set $\hbar$ and $m$ equal to 1. As $R_{nl}$ is real-valued, $\vec{v}_a$ does not depend thereon. As the angular part of $\Phi_a$ is well-defined and smooth on $\mathbb{R}^3 \setminus \{0\}$, we only

15 In the mathematical literature a function $S$ satisfying this equality (up to the constant $1/\hbar$) is called a lifting of the $S^1$-valued map $\Psi/|\Psi|$. See e.g. [31, 36].

16 Note that there exist several examples of wrongful application of Stokes’ theorem in the literature related to Takabayasi’s condition. Though there exist modern generalizations of Stokes’ theorem (see [78, 79] and theorem 8.9 in [77]), they are also not applicable to the special cases considered in section 2.3 below.
Figure 1. A plot of 100 equidistant, numerically evaluated values of the integral $I(a)$ from equation (2.2e) in the interval $[-1, 1]$ is shown. For $a \in [-1, 0)$ we have $I(a) = -1$. For $a = 0$ the wave function $\Phi_a$ from equation (2.2b) becomes real and a nodal surface develops in the $x$-$z$-plane. Though $\vec{v}_0 = 0$ and thus $I(0)$ formally evaluates to 0, $\gamma$ does not lie in $D_0$ due to the nodal surface, so that, strictly speaking, $I(0)$ is not defined. For $a \in (0, 1]$ the value $I(a)$ then jumps to $+1$.

need to check where it vanishes. The latter happens if and only if both its real part and its imaginary part vanishes. Using

$$Y_{11}^{-1}(\theta, \phi) = -\frac{3}{8\pi} \sin \theta e^{i\phi} = -\left(Y_{11}^{-1}(\theta, \phi)\right)^*,$$  \hspace{1cm} (2.2c)

and excluding $\theta \in \{0, \pi\}$, we find that this is only the case for $a = 0$ in the limits of $\phi$ tending to an element of the set $\{0, \pi, 2\pi\}$.

In this instance, $\Phi_0$ is real and the $x$-$z$-plane is a nodal surface. If this nodal surface had not ‘cut off’ the curve

$$\gamma: \quad [0, 2\pi] \to D_0 : \quad \phi \mapsto (r_0 \cos \phi, r_0 \sin \phi, 0)$$  \hspace{1cm} (2.2d)

for some admissible $r_0 \in \mathbb{R}_+, \text{ then the function }$

$$I: \quad [-1, 1] \to \mathbb{R} : \quad a \mapsto I(a) = \frac{m}{2\pi \hbar} \int_{\gamma} \vec{v}_a \cdot d\vec{r}$$  \hspace{1cm} (2.2e)

would have been well-defined and continuous. In turn, we could have applied the intermediate value theorem to show that $I$ takes every value between $-1$ and $+1$—thus contradicting proposition 2.1.

Due to the formation of the nodal surface for $a = 0$, this is, of course, not what occurs. Numerically calculated values of the function $I$ are depicted in figure 1. \hfill \Box

2.2. On distributions in quantum theory

Mathematically, Takabayasi’s condition, equation (1.3), suffers from the problem that not all wave functions are as regular as the ones considered in the prior section. This was already observed by Smolin [162]. While the fact that not all wave functions are $C^1$ is insufficient to adequately address Wallstrom’s criticism [174, 175], it is nonetheless a central point we wish to make in this work.
In this spirit, this section serves to motivate the use of distributions and distributional derivatives in the context of this article. We shall find that this distributional point of view is necessary due to the mathematical axioms of quantum mechanics.

To begin with, consider the first order, linear differential equation

\[ u' = 0 \]  

(2.3)

for a function \( u : \mathbb{R} \rightarrow \mathbb{R} \). This is arguably the simplest equation one can consider as an introduction to the modern mathematical theory of PDEs\(^{17}\). Nonetheless, we shall see that there are a number of \textit{a priori} inequivalent ways in which this formal equation can be mathematically interpreted.

Approaching equation (2.3) from a naive perspective, the perhaps most obvious solution is \( u(x) = c \) for some constant \( c \in \mathbb{R} \). However, any real-valued step function solves (2.3), if we are free to take out isolated points from the domain \( \mathbb{R} \) so that \( u \) is defined on \( \Omega \subset \mathbb{R} \) instead. The derivative will then satisfy the PDE on \( \Omega \)—and we may even smoothly extend it to \( \mathbb{R} \). Of course, one could force the former solution by asking \( u \) to be well-defined and smooth (or \( C^1 \)) on the entirety of \( \mathbb{R} \). Yet for more complicated PDEs an analogous requirement may be too rigid, as we often do not know an appropriate maximal domain for each solution beforehand.

The common resolution to this problem, that is employed in the modern theory of PDEs, is to first consider equation (2.3) as a distributional differential equation.

To do this, we first need to posit the existence of a distribution \( T \). The precise definition of distributions depends on the choice of space of ‘test functions’ \( D \) as well as the topology on it, we shall only define it for the standard choice of \( D = C_0^\infty (\mathbb{R}, \mathbb{R}) \) that is appropriate to equation (2.3) above and we refer the reader to the literature for other cases. For this choice of \( D \), we say that a sequence \((\phi_k)_{k \in \mathbb{N}}\) of test functions, each of which is supported on a common compact subset of \( \mathbb{R} \), converges to \( \phi \) in \( D \), if for every multi-index \( \alpha \) the sequence \((\partial^\alpha \phi_k)_{k \in \mathbb{N}}\) converges to \( \partial^\alpha \phi \) uniformly. A distribution \( T \) is then a linear functional

\[ T: \ D \rightarrow \mathbb{R} \quad : \quad \phi \rightarrow T(\phi) \]  

(2.4)

that is continuous in the sense that for every such sequence \((\phi_k)_{k \in \mathbb{N}}\) converging to \( \phi \) in \( D \) we have

\[ \lim_{k \to \infty} T(\phi_k) = T(\phi) \]  

(2.5)

(cf section 0.E in \([58]\)).

In order to express equation (2.3) in the language of distributions, we also recall that the derivative \( T' \) of a distribution \( T \) is defined via

\[ T'(\phi) = -T(\phi') \]  

(2.6)

for all \( \phi \in D \). As we shall see below, this definition is motivated by the integration by parts formula.

Using this definition, equation (2.3) simply reads \( T' = 0 \) or, equivalently,

\[ \forall \phi \in D: \quad T'(\phi) = 0. \]  

(2.7)

It is noteworthy that, at least for our choice of \( D \) above, derivatives of distributions are always well-defined—which need not be the case for ‘ordinary functions’ \( u \).

\(^{17}\) We refer, for instance, to \([35, 58, 113]\) for an introduction to this theory. More advanced material can be found in \([11]\), which also defines common notation employed in this field.
As a solution ansatz to equation (2.7), we may make the further assumption that $T$ is a regular distribution, i.e. that there exists a locally integrable function\(^{18}\) $u$ such that

$$T(\varphi) = \int_{\mathbb{R}} u(x) \varphi(x) \, dx$$

(2.8)

for all $\varphi \in \mathcal{D}$. Due to the implicit dependence on $u$, we shall use the notation $T_u$ instead of $T$.\(^{19}\)

The differential equation, equation (2.3), then reads

$$T_u(\varphi) = -\int_{\mathbb{R}} u(x) \varphi'(x) \, dx = 0.$$  

(2.9)

If it were possible to integrate by parts, we would obtain

$$\forall \varphi \in \mathcal{D} : \int_{\mathbb{R}} u'(x) \varphi(x) \, dx = 0.$$  

(2.10)

The fundamental theorem of calculus of variations implies that the only functions $u'$ that satisfy (2.10) are those that vanish almost everywhere—that is, $u'$ can only be non-zero on a subset of Lebesgue measure zero\(^{20}\). Indeed, it can be shown that, up to a set of measure zero, $u(x) = c$ is the only solution to equation (2.9) (cf lemma 8.1 in [35]), so that the step to obtain equation (2.10) is justified.

Reformulating equation (2.3) into equation (2.9) or equation (2.10) has two advantages:

i) It is enough for $u$ to be weakly differentiable in the sense that there exists some locally integrable function $v$ such that $T_u = T_v$. In that case, we may use the notation $u'$ for $v$. In particular, the function $u'$ may differ from $c$ on an arbitrary set of measure zero.

ii) Equation (2.9) excludes step functions as a solution. In fact, the distributional derivative of a step function is not a regular distribution, i.e. step functions are not weakly differentiable. Roughly speaking, even though at isolated points the values of a function are not of relevance in this distributional framework, the distributional derivative can still ‘see’ some discontinuities and singularities.

The example above exemplifies that, in the context of solving differential equations, the distinction between interpreting the differential equation in the strong sense versus interpreting it in the distributional sense can be crucial in terms of what counts as a solution and what does not. This applies, in particular, to the dynamical equations of quantum theory.

Of course, while the theory of distributions may provide a mathematically more satisfying—as well as more elaborate—way of formulating and solving PDEs, its relevance in the context of this article and quantum mechanics in general still requires justification.

---

\(^{18}\) For $p \in [1, \infty)$ the function $u : \mathbb{R} \to \mathbb{R}$ is locally $L^p$-integrable, if for any compact subset $K$ of $\mathbb{R}$ the integral

$$\int_{K} |u(x)|^p \, dx$$

is well-defined and converges. Then the integral in equation (2.8) converges as well due to Hölder’s inequality. The case $p = 1$ is the most important one here, so ‘local integrability’ refers to local $L^1$-integrability. The condition as well as the argument is easily generalized to higher dimensions.

\(^{19}\) Not all distributions are regular. The most common counterexample is the Dirac (delta) distribution: In 1 dimension it is defined via the equation $\delta(x) = \delta(x)$ for any $x \in \mathbb{R}$ and $\varphi \in \mathcal{D}$.

\(^{20}\) Intuitively, those are the sets that the (Lebesgue) integral ‘cannot see’.
Before providing such a justification, we shall recall some central aspects of the mathematical theory of quantum mechanics:

The reader may recall that in quantum mechanics observables, such as the Hamiltonian $\hat{H}$, are given by linear operators on a separable Hilbert space $\mathcal{H}$. This sentence is, however, only true in general if we allow linear operators $\hat{A}$ to be unbounded in the sense that $\hat{A}$ is only defined on a linear subspace of $\mathcal{H}$. This definition of an unbounded operator has historical origins and can be somewhat confusing, for it paradoxically allows $\hat{A}$ to be bounded. By definition an ‘unbounded’ operator $\hat{A}$ is bounded, if its domain $\text{dom}\hat{A}$ is $\mathcal{H}$ and its operator norm

$$\|\hat{A}\| = \sup_{\Psi \in \mathcal{H}} \frac{\|\hat{A}\Psi\|}{\|\Psi\|}$$

(2.11)

is finite. Contrarily, the stereotypical example of an unbounded operator is defined on a dense linear subspace $\text{dom}\hat{A}$ of $\mathcal{H}$ and

$$\|\hat{A}\| = \sup_{\Psi \in \text{dom}\hat{A}} \frac{\|\hat{A}\Psi\|}{\|\Psi\|}$$

(2.12)

does not exist.

**Example 2.3.** We consider a single free body in $n$-dimensional space, $n \in \mathbb{N}$. The Hilbert space $\mathcal{H}$ is $L^2(\mathbb{R}^n, \mathbb{C})$, the space of complex-valued, square integrable functions on $\mathbb{R}^n$. While the mathematical definition of $L^2(\mathbb{R}^n, \mathbb{C})$ is slightly more subtle, as we shall address below, this is the appropriate space to consider due to the fact that square-integrability assures that any wave function $\Psi$ in $\mathcal{H}$ gives rise to a corresponding probability density. Moreover, it is equipped with the inner product that the quantum-mechanical Born rule implicitly relies on.

The free body Hamiltonian $\hat{H}_0$ is proportional to the (negative) Laplacian and, accordingly, the expression $\hat{H}_0\Psi$ is not well-defined for arbitrary $\Psi \in \mathcal{H}$—even if the derivatives are understood in the weak sense. It is, of course, possible to choose a variety of different domains $\text{dom}\hat{H}_0$ on which $\hat{H}_0$ is well-defined. A natural choice is the space of Schwartz functions $S(\mathbb{R}^n, \mathbb{C})$.

As $S(\mathbb{R}^n, \mathbb{C})$ contains $C^\infty(\mathbb{R}^n, \mathbb{C})$ and the latter is dense in $L^2(\mathbb{R}^n, \mathbb{C})$, $S(\mathbb{R}^n, \mathbb{C})$ is also dense (see e.g. corollary 4.23 in [35]). Accordingly, $\hat{H}_0$ is a densely defined linear operator on $\mathcal{H}$. Moreover, by considering a sequence $(\Psi_k)_{k \in \mathbb{N}}$ of normalized Gaussians with standard deviation $1/k$ one finds that the expression $\|\hat{H}_0\Psi_k\| / \|\Psi_k\|$ diverges as $k \to \infty$. That is, $\hat{H}_0$ cannot be bounded.

As a theory of mechanics, a central question in quantum mechanics is how to find the time evolution of an appropriate, given initial state $\Psi_0 \in \mathcal{H}$. If we naively view the generalized Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi_t = H\Psi_t$$

(2.13)

as a matrix equation and assume that $\hat{H}$ does not depend on $t$ explicitly, we find that the state $\Psi_t = U_t\Psi_0$, with $U_t = \exp(-iHt/\hbar)$ provides a solution for all $t \in \mathbb{R}$. While this illustrates the general idea, the actual mathematical situation is, of course, more subtle since $\hat{H}$ is in general unbounded. In particular, even if $\Psi_0$ is in the domain of $\hat{H}$, $H\Psi_0$ need not be, so that it is not *a priori* clear how to interpret $U_t$. 


The mathematical machinery needed to resolve this problem was mainly developed by Stone and von Neumann in the years around 1930: It is possible to define $U_t$, on the entirety of $\mathcal{H}$, provided that $\hat{H}$ is not only symmetric, i.e.
\begin{equation}
\forall \Psi, \Phi \in \text{dom} \hat{H}: \quad \langle \Psi, \hat{H} \Phi \rangle = \langle \hat{H} \Psi, \Phi \rangle,
\end{equation}
but self-adjoint, i.e. that in addition $\text{dom} \hat{H}$ is equal to the domain of its adjoint\footnote{Following the definition in equation (2.15), the adjoint $\hat{H}^\dagger$ of $\hat{H}$ is then defined by setting $\hat{\Psi} = \hat{H}^\dagger \Phi$.}
\begin{equation}
\text{dom} \hat{H}^\dagger = \{ \Psi \in \mathcal{H} \mid \exists \Phi \in \mathcal{H} \forall \Phi \in \text{dom} \hat{H}: \quad \langle \Psi, \hat{H} \Phi \rangle = \langle \hat{H}^\dagger \Psi, \Phi \rangle \},
\end{equation}
(cf sections VIII.1 and VIII.4 in [142]).

The generalized Schrödinger equation, equation (2.13), is then only a formal PDE and should not be taken literally. While there do exist other mathematical interpretations of equation (2.13),\footnote{One may also consider weak formulations (see e.g. p. 106 in [67, 74]). Moreover, mathematically literal interpretations which do not understand the equation in the strong sense do exist (cf appendix A).} this is how it is generally understood in the mathematical theory of quantum mechanics.

In principle this reduces the problem of solving equation (2.13) to determining the operator $U_t$. Yet in practice one faces a number of hurdles.

First, one needs to show that $\hat{H}$ is not merely symmetric but also self-adjoint on an appropriately chosen domain $\text{dom} \hat{H}$. Heuristically, the smaller one chooses $\text{dom} \hat{H}$, the larger $\text{dom} \hat{H}^\dagger$ will become, so that the domain on which $\hat{H}$ is self-adjoint—if it exists—is the one on which $\hat{H}$ can still be sensibly defined and $\text{dom} \hat{H}^\dagger$ has become small enough to be contained by $\text{dom} \hat{H}$. Finding this domain might be too difficult, but, fortunately, there exist a number of theorems that allow one to prove the weaker condition of essential self-adjointness on a much smaller domain\footnote{Due to its applicability to Coulomb potentials, the most important one is arguably the Kato–Rellich Theorem. We refer to theorems X.12 and X.15 in [141], [99, 144] as well as [100]. Note that there is an analogue for quadratic forms, the KLMN Theorem (cf theorem X.17 in [141] or theorem 6.1.17 in [46]).}. If a symmetric operator is essentially self-adjoint, then it can be uniquely extended to a larger domain on which it is self-adjoint. As this is only a sketch of the mathematical theory of quantum mechanics, we shall neither define essential self-adjointness here nor provide any such theorems. The reader is referred, for instance, to section VIII.2 in [142] for the former and [141] for the latter.

**Example 2.4.** In example 2.3, $\hat{H}_0$ is essentially self-adjoint on $\text{dom} \hat{H}_0 = \mathcal{S}(\mathbb{R}^3, \mathbb{C})$. We refer to theorem IX.27 in [141]. The domain on which $\hat{H}_0$ is self-adjoint will be given in remark 2.7 below.

The second hurdle one faces is that of finding ‘eigenfunctions’ of $\hat{H}$, so that one can indeed compute $U_t$. The word is put in quotation marks, since—even for self-adjoint $\hat{H}$—a maximal collection of linearly independent (orthonormal) eigenfunctions $\Phi_k \in \text{dom} \hat{H}$,
\begin{equation}
\hat{H} \Phi_k = E_k \Phi_k
\end{equation}
with $E_k \in \mathbb{R}$ and $k$ in some countable index set $I$, need not be a (Schauder) basis for $\mathcal{H}$—i.e. they need not ‘span’ the entire Hilbert space. Only if this is the case, the time evolution of a given $\Psi_0$ in $\mathcal{H}$ can be determined via
\begin{equation}
\Psi_t \equiv U_t \Psi_0 = \sum_{k \in I} \langle \Phi_k, \Psi_0 \rangle e^{-iE_k t/\hbar} \Phi_k.
\end{equation}
Still, in the general case of a self-adjoint, unbounded Hamiltonian \( \hat{H} \) the space \( \mathcal{H} \) can be decomposed into three, mutually orthogonal Hilbert subspaces of \( \mathcal{H} \):

\[
\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_{a.c.} \oplus \mathcal{H}_{s.c.}.
\]

The respective subspaces correspond to the decomposition of the spectrum of \( \hat{H} \) into its point spectrum, singularly continuous spectrum, and absolutely continuous spectrum, respectively (see e.g. section 28.2 in [23]). If the Hamiltonian \( \hat{H} \) is ‘well-behaved’ in some mathematical sense—as it is the case for the model of ‘hydrogen-like atoms’—then \( \mathcal{H}_{a.c.} = \{0\} \), the space \( \mathcal{H}_p \) may be identified with the bound states, and \( \mathcal{H}_{a.c.} \) with the scattering states in \( \mathcal{H} \) (see e.g. section 28.3 in [23]).

**Example 2.5.** Continuing example 2.4, we first recall that, due to Plancherel’s theorem, the Fourier transform \( \mathcal{F} \) is a continuous, linear automorphism of \( \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}) \) that is also unitary. Since for all \( \Psi \in \text{dom} \hat{H}_0 \) and \( \vec{p} \in \mathbb{R}^3 \) we have

\[
\left( \mathcal{F} (\hat{H}_0 \Psi) \right)(\vec{p}) = \frac{\vec{p}^2}{2m} \left( \mathcal{F} \Psi \right)(\vec{p}),
\]

the free Hamiltonian \( \hat{H}_0 \) is unitarily equivalent to a multiplication operator. Since the spectrum of the latter is easily shown to be \([0, \infty)\), this is also the spectrum of \( \hat{H}_0 \). By theorem 28.7 in [23], we have \( \mathcal{H} = \mathcal{H}_{a.c.} \). In fact, all elements of \( L^2(\mathbb{R}^3, \mathbb{C}) \) are scattering states of \( \hat{H}_0 \). □

Even if \( \hat{H} \) is not ‘well-behaved’, the restriction of the self-adjoint Hamiltonian to the intersection of its domain with the respective Hilbert subspaces again defines a self-adjoint operator on said subspace (cf theorem 28.3 in [23]). Furthermore, the time evolution operator \( U_t \) preserves those subspaces and its restriction thereto is the time-evolution operator of the respective restricted Hamiltonian. Only in the subspace \( \mathcal{H}_p \) this time evolution is given by equation (2.17) above—a fact that is sometimes glossed over in the physics literature (cf section XI.II in [124] and section 5.2 in part 1 of [168]).

So assuming that \( \hat{H} \) is well-behaved in the above sense, how do we determine \( U_t \) for the scattering states? One possible approach is via the use of so-called **generalized eigenfunctions** (see e.g. section 1.6 and supplement 1.2 in [17] as well as chapter 29 in [23]).

This forces one to go beyond standard Hilbert space theory and to consider **rigged Hilbert spaces** or **Gelfand triples** instead—thus giving the aforementioned link to the mathematical theory of distributions. Without going too much into detail, the general idea is that one chooses an ‘appropriate space’ of test functions \( \mathcal{D} \) contained in the domain of \( \hat{H} \), which in turn gives rise to a space of distributions \( \mathcal{D}' \). Ignoring topological subtleties, the Gelfand triple is then given by \((\mathcal{D}, \mathcal{H}, \mathcal{D}')\). Each former space is viewed as being contained in the latter: \( \mathcal{D} \) is trivially contained in \( \mathcal{H} \), and \( \mathcal{H} \) is contained in \( \mathcal{D}' \), provided we view elements of the former as regular distributions in the latter. A generalized eigenfunction for a real number \( \lambda \) in the spectrum of \( \hat{H} \) is then a distribution \( T_\lambda \in \mathcal{D}' \) such that for all \( \varphi \in \mathcal{D} \) we have

\[
T_\lambda \left( \hat{H} \varphi \right) = \lambda T_\lambda \left( \varphi \right).
\]

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24 We refer to [135] for a physical interpretation of \( \mathcal{H}_{a.c.} \).

25 As it is common in quantum mechanics, on \( L^2(\mathbb{R}^3, \mathbb{C}) \) we define the Fourier transform via

\[
(\mathcal{F} \Psi)(\vec{p}) = \frac{1}{\sqrt{2\pi \hbar}} \int_{\mathbb{R}^3} d^3 r \, e^{-i\vec{p} \cdot \vec{r}/\hbar} \Psi(\vec{r}).
\]

The (sign) convention is that the Fourier transform takes ‘position space wave functions’ to ‘momentum space wave functions’ (cf chapter IV, section 2 in volume 1 of [124]).
This construction allows one to allow for ‘eigenfunctions’ that are distributions as opposed to merely elements of $\mathcal{H}$.

We shall continue example 2.5 above to illustrate the central ideas.

**Example 2.6.** We may formally compute

$$\hat{H}_0 \tilde{\varphi}/\hbar = \frac{p^2}{2m} \tilde{\varphi}/\hbar. \tag{2.21a}$$

This suggests that for all $\tilde{p} \in \mathbb{R}^3$ the function

$$\Phi_{\tilde{p}}: \quad \mathbb{R}^3 \to \mathbb{C} : \quad \vec{r} \mapsto \Phi_{\tilde{p}}(\vec{r}) = e^{i\tilde{p} \cdot \vec{r}/\hbar} \tag{2.21b}$$

is a kind of ‘eigenfunction’ of $\hat{H}_0$. Yet $\Phi_{\tilde{p}}$ is not square-integrable, so it cannot be an eigenfunction in the strict sense. This means, in particular, that $\Phi_{\tilde{p}}$ is not a physically realized state.

Yet we may use the complex conjugate of $\Phi_{\tilde{p}}$ to define a regular distribution $\tilde{\Phi}_{\tilde{p}}$ on $\mathcal{D} = \text{dom} \hat{H}_0 = \mathcal{S} (\mathbb{R}^3, \mathbb{C})$. The complex conjugate is taken, as we heuristically think of the function as being put in the first slot of $\langle \ldots \rangle$. Up to a constant factor, $\tilde{\Phi}_{\tilde{p}}$ is the restriction of $\mathcal{F} (\ldots) (\tilde{p})$ to $\mathcal{S} (\mathbb{R}^3, \mathbb{C})$. Equation (2.19) then shows that $\tilde{\Phi}_{\tilde{p}}$ is a generalized eigenfunction in the sense of equation (2.20) with eigenvalue $E(\tilde{p}) = \tilde{p}^2/2m$.

Using the above generalized eigenfunctions, for any ‘initial’ $\varphi_0 \in \mathcal{D}$ we may compute

$$\tilde{\Phi}_{\tilde{p}}(\varphi_t) = \tilde{\Phi}_{\tilde{p}} \left( \exp \left(-i t \hat{H}_0/\hbar \right) \varphi_0 \right) = \exp \left(-i t E(\tilde{p})/\hbar \right) \tilde{\Phi}_{\tilde{p}}(\varphi_0). \tag{2.21c}$$

Recalling the relation to the Fourier transform, we thus find for all $\vec{r} \in \mathbb{R}^3$ that

$$\varphi_t(\vec{r}) = \frac{1}{\sqrt{2\pi \hbar}} \int_{\mathbb{R}^3} d^3\vec{p} \left( \mathcal{F} \varphi_0 \right) (\vec{p}) e^{-i \tilde{p} \cdot \vec{r}/\hbar} e^{i\tilde{p} \cdot \vec{r}/\hbar}. \tag{2.21d}$$

As equations (2.21c) and (2.21d) also make sense for $\varphi_0 \in \mathcal{H}$ and the respective extension is continuous, we find that equation (2.21d) defines the time evolution on $\mathcal{H} = \mathcal{H}_{\text{a.c.}}$ in analogy to equation (2.17) (cf section 1.6 in [17]).

We refer to supplement 1 in [17], chapter 29 in [23], as well as the works [118–120] by de la Madrid for further reading. Sections 8 and 9 in chapter V of [124] provide a heuristic introduction to the concept of generalized eigenfunctions.

Apart from the need for generalized eigenfunctions in scattering problems, we would like to mention two more points that underscore the statement that quantum mechanics relies on the theory of distributions in its mathematical formulation:

First, one may recall that the use of so called ‘delta-potentials’ is common in quantum-mechanical models (see e.g. sections 2.5 and 5.3.2 as well as problems 2.50, 6.1, 6.8, and 11.4 in [68]). Though such potentials are certainly idealizations of more realistic (still idealized) potentials, such as the one in the Kronig–Penney model [107], they nevertheless require a mathematically rigorous treatment.

The second argument is that in quantum theory weak and distributional derivatives—not strong derivatives—are the appropriate notion of derivative to use:

Recall that the common Hilbert spaces in quantum mechanics are $L^2 (\mathbb{R}^n, \mathbb{C})$ and its relatives, as exemplified by example 2.3 above. Strictly speaking, in order to assure that $L^2 (\mathbb{R}^n, \mathbb{C})$ is indeed complete with respect to the natural norm (and thus a Hilbert space), elements of $L^2 (\mathbb{R}^n, \mathbb{C})$ are not functions $\Psi$ but equivalence classes thereof, denoted by $[\Psi]$. By definition, we consider two square-integrable functions $\Psi$ and $\Psi'$ to be equivalent—if they correspond

\[26\] Note that the Fourier transform is also a continuous linear automorphism on the space of Schwartz functions.
to the same element \([\Psi] = [\Psi']\) in \(L^2(\mathbb{R}^n, \mathbb{C})\)—if they differ at most on a set of measure zero. The importance of this mathematical subtlety can for instance be observed in the theory of Fourier series, where the Fourier series of a function need not equal the function at every point—the so called Gibbs phenomenon. Similarly, an energy eigenfunction expansion of a square integrable function in quantum mechanics, if it exists, may differ from the original function on a set of measure zero.

Indeed, the consideration of equivalence classes of functions as opposed to functions themselves is not merely mathematical pedantry: Events of probability zero, that is changes of the probability density on a set of measure zero, should neither change our statistical description nor how it evolves in time. Considering equivalence classes of functions is an appropriate mathematical expression thereof.

Therefore, if a wave function is supposed to be an element of \(H\), it is—strictly speaking—not admissible to speak of the value of a wave function at an individual point, let alone evaluate its strong derivative. Yet, provided we have specified an appropriate space \(D\) of ‘test functions’, we may consider its distributional derivative, and, if it exists, its weak derivative.

Surely, in many cases of practical interest the aforementioned mathematical subtlety is not of much relevance. Yet it cannot be glossed over in discussions concerning the mathematical structure of quantum mechanics. In this spirit, we shall make the following remark on Takabayasi’s condition from section 2.1.

**Remark 2.7 (Limitations of Takabayasi’s quantization condition).** In section 2.1 we explained why the generalization of proposition 2.1 to wave functions that are less regular than \(C^1\) is not trivial. We shall show here why there are serious doubts on whether the condition in equation (1.3) can be appropriately extended to be sensible within the general mathematical theory of quantum mechanics.

We shall start off by noting that the above statement that wave functions may not be evaluated at a point does have an important limitation: It may happen that an equivalence class \([\Psi] \in H\) has a representative \(\Psi\) that satisfies certain continuity or even differentiability assumptions. Generally speaking, such representatives are unique. In that case, we can, of course, evaluate the function \(\Psi\) and ask if its strong derivatives exist. If those derivatives indeed exist everywhere, then \([\Psi]\) is weakly differentiable up to the same order and the (the equivalence classes of the) strong derivatives of \(\Psi\) are equal to the respective weak derivatives (as equivalence classes).

An important mathematical result that should be mentioned in this context is the Sobolev embedding theorem. We refer, for instance, to chapter V in [1]. Theorem 1.1 in supplement 2 of [17] provides a result on special cases of interest to quantum mechanics, namely for the Sobolev spaces \(H^k(\mathbb{R}^n, \mathbb{C})\):

For \(k \in \mathbb{N}_0\) and \(n \in \mathbb{N}\) the space \(H^k(\mathbb{R}^n, \mathbb{C})\) is defined as the set of all equivalence classes \([\Psi]\) of functions \(\Psi : \mathbb{R}^n \to \mathbb{C}\) such that both \([\Psi]\) and all of its \(k\)th order weak derivatives are in \(L^2(\mathbb{R}^n, \mathbb{C})\). Each \(H^k(\mathbb{R}^n, \mathbb{C})\) is equipped with the inner product

\[
\langle \Phi, \Psi \rangle_{H^k} = \sum_{|\alpha| \leq k} \langle \partial^\alpha \Phi, \partial^\alpha \Psi \rangle,
\]

\(\partial^\alpha\) denoting the partial derivative with respect to the multi-index \(\alpha\). With respect to this inner product and the induced norm, every \(H^k(\mathbb{R}^n, \mathbb{C})\) is a separable Hilbert space (cf section 1.4.1 in [11]).

It is, however, important to understand that the inner product in equation (2.22a) is only of direct interest in quantum mechanics for \(k = 0\), i.e. for the case of \(H^0(\mathbb{R}^n, \mathbb{C}) = L^2(\mathbb{R}^n, \mathbb{C})\).
While for \( k \geq 1 \) the \( H^k \)-spaces are not appropriate Hilbert spaces of quantum-mechanical states, they are nonetheless of use for mathematical analysis within the theory. This is due to the fact that, for the case of 1-body in \( \mathbb{R}^3 \), the respective free body Hamiltonian \( \hat{H}_0 \) with domain \( H^2 (\mathbb{R}^3, \mathbb{C}) \) is self-adjoint on \( L^2 (\mathbb{R}^3, \mathbb{C}) \) (cf theorem IX.27 in [141]) and that, in many cases, Hamiltonians with potentials may be viewed as perturbations of \( \hat{H}_0 \) that preserve self-adjointness (cf section X.2 in [141]).

The Sobolev embedding theorem now states that for every \( [\Psi] \in H^2 (\mathbb{R}^3, \mathbb{C}) \) there exists a representative \( \Psi \) that is bounded and Lipschitz-continuous (cf theorem 5.4 in [1], Part I Case C and Part II Case). It does, however, not imply that a \( C^1 \) representative exists—which would be needed to establish Takabayasi’s quantization condition for elements in the domain of such perturbed Hamiltonians.

If we are considering general elements of the Hilbert space \( L^2 (\mathbb{R}^3, \mathbb{C}) \), the situation is even worse: Functions whose equivalence class lies in that space are generally very ill-behaved and not even continuous. Moreover, they can be changed on an arbitrary set of measure zero, so even if the integral in Takabayasi’s condition, equation (1.3), is well-defined for a given representative it can be chosen to yield any number in \( \mathbb{R} \) without changing the equivalence class.

In conclusion and returning to the main subject of this work, if we intend to partially reproduce or even generalize quantum mechanics on the basis of the Madelung equations, then we need to consider the latter as a system of PDEs that is to be understood in the sense of distributions. The existence of solutions to the latter as well as the (in)equivalence to the Schrödinger equation therefore fundamentally depends on how we formulate the equations, our choice of solution spaces for the quantities in question as well as for their initial values, and also the boundary conditions we may wish to encode in those solution spaces. In section 3 below we shall give explicit examples of such ‘distributional Madelung equations’.

### 2.3. Quantum vorticity and quantum quasi-irrotationality

As argued in the previous section, the Madelung equations need to be understood as distributional differential equations. Due to a lack of regularity of the respective drift fields or wave functions—depending, of course, on the choice of function spaces—Takabayasi’s condition, equation (1.3), may therefore not be universally applicable (cf remark 2.7).

One may, however, ask the question, if one can generalize Takabayasi’s condition in a manner that does not depend on the regularity of those (stationary) solutions of the Madelung equations it was motivated by. Given the discussion in section 2.2 above, a distributional approach to this question seems natural. Furthermore, the considerations in section 2.1 imply a close relationship between Takabayasi’s condition and the third Madelung equation, equation (1.1c). We are thus lead to investigate whether those model solutions indeed satisfy the third Madelung equation in a distributional sense or not.

The particular model solutions we consider here are stationary solutions of the isotropic harmonic oscillator in 2 dimensions, as well as the functions \( \Psi_{nlm} \) in the aforementioned model for hydrogen-like atoms. Consideration of the former model in this context was suggested by Wallstrom in [174], the latter model we chose as we view it to be of more direct physical relevance.

We will find that for both of those classes of solutions the third Madelung equation does in general not hold. Moreover, the results suggest the possibility to generalize Takabayasi’s
condition to the wider class of drift fields $\vec{v}$ whose (Euclidean) components are locally $(L^1)$-integrable.

In the first model we consider the time-independent Schrödinger equation for a single isotropic harmonic oscillator in two dimensions with frequency $\omega \in \mathbb{R}_+$ and in polar coordinates $(\rho, \phi)$:

$$E \Psi = -\frac{\hbar^2}{2m} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Psi}{\partial \phi^2} \right) + \frac{m \omega^2}{2} \rho^2 \Psi. \quad (2.23)$$

Though the problem of finding energy eigenstates is arguably easier to solve in Cartesian coordinates, one commonly approaches the above problem via a separation ansatz

$$\Psi(\rho, \phi) = R(\rho) \Phi(\phi). \quad (2.24)$$

The latter leads to an angular equation

$$\frac{d^2 \Phi}{d\phi^2} + \mu^2 \Phi = 0 \quad (2.25)$$

as well as a radial equation

$$ER = -\frac{\hbar^2}{2m} \left( \frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} - \frac{\mu^2}{\rho^2} R \right) + \frac{m \omega^2}{2} \rho^2 R \quad (2.26)$$

for some constant $\mu \in \mathbb{C}$.

It is not the focus of this section to consider the question of if and why $\mu$ has to be ‘quantized’ here, i.e. if and why $\mu \in \mathbb{Z}$. We refer to section 4 for a discussion of this point. Instead we use stationary solutions of the respective Schrödinger equation with $\mu \in \mathbb{Z}$, take the respective densities $\rho$ and drift fields $\vec{v}$ and then use those to look at the third Madelung equation from a distributional point of view.

For $n \in \mathbb{N}_0$, $\mu \in \mathbb{Z}$, and some normalization factor $1/A_{nm}$ the functions $\rho$ and $\vec{v}$, taking values

$$\rho(x, y) = \frac{1}{A_{nm}} \left( x^2 + y^2 \right)^{|\mu|} \left( \frac{m \omega}{\hbar} \left( x^2 + y^2 \right) \right)^{\frac{1}{2}} e^{-\frac{m \omega}{\hbar} (x^2 + y^2)} \quad (2.27)$$

and

$$\vec{v}(x, y) = \frac{\mu \hbar}{m} \frac{1}{x^2 + y^2} \left( -y \right) \quad (2.28)$$

for $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, provide strong, stationary solutions to the Madelung equations for this problem\(^{28}\). We refer to section 4.1 below for the corresponding solutions of the Schrödinger equation.

To analyze the above strong solutions from a distributional perspective and also to prepare for a similar analysis of the physically more interesting, aforementioned hydrogen model, we shall make the following definition:

**Definition 2.8.** 1) Let $\vec{v}$ be a (real-valued) vector field on $\mathbb{R}^2$ (or an equivalence class thereof, as discussed in section 2.2) such that each component, $v^1$ and $v^2$, is locally $L^1$-integrable. Let

---

\(^{27}\) Note that the radial equation, equation (3.1) in [174], is incorrect.

\(^{28}\) To avoid cluttered notation and also due to the lack of relevance of the trivial time-dependence here, we drop the latter.
$\mathcal{D}$ be either $C_0^\infty(\mathbb{R}^2, \mathbb{R})$ or $\mathcal{S}(\mathbb{R}^2, \mathbb{R})$. The distributional curl of $\vec{v}$ is the distribution curl $\vec{v}$ defined via

$$ (\text{curl } \vec{v})(\varphi) = -\int_{\mathbb{R}^2} dx \, dy \left( v^2 \frac{\partial}{\partial x} \varphi - v^1 \frac{\partial}{\partial y} \varphi \right) $$

(2.29a)

for all $\varphi \in \mathcal{D}$.

2) Let $\vec{v}$ be a (real-valued) vector field on $\mathbb{R}^3$ (or an equivalence class thereof) such that each component is locally $L^1$-integrable. Let $\mathcal{D}$ be either $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ or $\mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$. Then the distributional curl of $\vec{v}$ is the distribution

$$ \nabla \times \vec{v} : \mathcal{D} \to \mathbb{R}^3 $$

(2.29b)

defined via

$$ (\nabla \times \vec{v})(\vec{\varphi}) = \int_{\mathbb{R}^3} d^3 r \, (\vec{v} \times \nabla) \vec{\varphi} $$

(2.29c)

for all $\vec{\varphi} \in \mathcal{D}$. □

Of course, the definitions in definition 2.8 were chosen such that they provide an adequate distributional generalization of the respective curl operator. That this is indeed the case can be checked by assuming that $\vec{v}$ is weakly differentiable in the sense that all component functions $v^i$ are weakly differentiable (‘integrating by parts’).

Let us now compute the distributional curl of $\vec{v}$ from equation (2.28) above. We shall find that it is proportional to the 2-dimensional Dirac distribution at the origin $\delta_0$ (cf footnote 19).

**Proposition 2.9.** The distributional curl of $\vec{v}$ from equation (2.28) above equals

$$ \text{curl} \vec{v} = \frac{2\pi \mu \hbar}{m} \delta_0. $$

(2.30)

In particular, $\vec{v}$ is not weakly differentiable. □

Generally, we say that a drift field $\vec{v}$ exhibits quantum vorticity (at a given time), if its distributional curl does not vanish (at that time).

According to proposition 2.9, the drift field $\vec{v}$ from equation (2.28) does indeed exhibit quantum vorticity. This vorticity is concentrated at the origin in the sense that the distributional support of curl $\vec{v}$ is the singleton $\{0\} \subset \mathbb{R}^2$.\footnote{For $n \in \mathbb{N}$ and a given space $\mathcal{D}$ of test functions on $\mathbb{R}^n$ a distribution $T$ is said to vanish on an open set $U \subset \mathbb{R}^n$, if for all $\varphi \in \mathcal{D}$ with support supp $\varphi$ contained in $U$ we have $T(\varphi) = 0$. Then the support of a distribution is defined as the complement of the union of all open sets on which $T$ vanishes.}

This observation motivates the following definition.

**Definition 2.10.** A locally integrable vector field $\vec{v}$, as given in definition 2.8 above, is quasi-irrotational, if its distributional curl is supported on a non-empty set of Lebesgue measure zero. It is irrotational, if its distributional curl vanishes entirely. □

The above terminology was inspired by a statement made by Schönberg [158] in the 1950s:

The presence or absence of vorticity is not the fundamental fact, since the Schrödinger equation may be applicable even when there is vorticity, provided the motion be quasi-irrotational.

\footnote{For $n \in \mathbb{N}$ and a given space $\mathcal{D}$ of test functions on $\mathbb{R}^n$ a distribution $T$ is said to vanish on an open set $U \subset \mathbb{R}^n$, if for all $\varphi \in \mathcal{D}$ with support supp $\varphi$ contained in $U$ we have $T(\varphi) = 0$. Then the support of a distribution is defined as the complement of the union of all open sets on which $T$ vanishes.}
While definition 2.10 does not capture Schönberg’s original intentions behind those words, our results suggest that the statement nonetheless holds true if reinterpreted in this sense (even in the absence of magnetic fields, see [158]).

Given a quasi-irrotational drift field $\vec{v}$ in the sense of definition 2.10, we call a connected component of the support of its distributional curl a *quantum vortex*. Note that the above definition does not imply any restriction on the number of such connected components, so, at least in principle, $\vec{v}$ could have infinitely many quantum vortices.

The consequences of proposition 2.9 for the 2-dimensional (1-body) Madelung equations are as follows:

If we view the above solutions $(\rho, \vec{v})$ from equations (2.27) and (2.28) as physically acceptable—which is not beyond dispute—then we cannot simply carry the third Madelung equation over to the distributional setting. As proposition 2.9 shows then, it does not hold for such solutions, for $\vec{v}$ is only quasi-irrotational.

Of course, one could have constructed a basis of real-valued wave functions for this problem instead and still obtained corresponding stationary solutions of the Madelung equations. As the respective drift field is zero in this instance, there are not any quantum vortices and the third Madelung equation is also satisfied in the distributional sense. In fact, in the absence of magnetic fields a given basis of energy eigenfunctions can always be transformed into a basis of real-valued energy eigenfunctions, so that this behavior is generic (cf section 5.6 in [160]). We shall elaborate on this point below as well as in section 4.2.

With regards to the issue of ‘quantization’, we observe that curl $\vec{v}$ is proportional to the magnetic quantum number $\mu$ in this instance—just like the respective integral in Takabayasi’s condition, equation (1.3). Therefore, in this distributional setting the expression for curl $\vec{v}$ allows one to distinguish between the different stationary solutions $(\rho, \vec{v})$ above on the level of the Madelung equations—albeit the expressions for $\vec{v}$ and curl $\vec{v}$, equations (2.28) and (B.6), in principle also make sense for non-integer $\mu$.

We shall not address the general question of how the distributional curl behaves under superposition, as this would go beyond the scope of this work. Whether it is possible to generalize proposition 2.9 to general 2-dimensional, not necessarily stationary, states therefore remains to be shown.

There remains the question of the physical meaning of the distribution curl $\vec{v}$. As this is not a purely mathematical question, we shall move on to an arguably more physical example instead and take up this discussion there again.

As already indicated, this example is the 1-body Schrödinger theory with attractive Coulomb interaction for ‘hydrogen-like atoms’. While it is true that this is more adequately treated as a 2-body problem, it is known that the high mass of the proton allows one to treat it as effectively fixed in space, so that we may consider the physical situation as a 1-body problem instead.

Denote by $n$, $l$, and $\mu$ the principal, angular momentum, and magnetic quantum number, respectively, and by

$$a_0 = \frac{4\pi \varepsilon_0 \hbar^2}{me^2}$$

the Bohr radius (in SI units) for this model. $a_0$ is the Bohr-radius for hydrogen, for hydrogen-like atoms one needs to further divide this number by the number of protons in the nucleus.

\[30\] In the presence of magnetic fields the third Madelung equation does not even hold in the strong sense. See e.g. [93, 95, 158].
For \( n \in \mathbb{N}_0, \ l \in \{0, \ldots, n - 1\}, \ \mu \in \{-l, \ldots, l\}, \) and \( \vec{r} = (x, y, z) \) in \( \mathbb{R}^3 \) excluding the \( z \)-axis, we set

\[
\varrho(\vec{r}) = \frac{1}{A_{nl\mu}} (\lambda(\vec{r}))^{2l} \left( \mathcal{L}_{n-l-1}^{2l+1}(\lambda(\vec{r})) \right)^2 e^{-\lambda(\vec{r}) \left( \mathcal{P}_{l}^{n | l}(z/|\vec{r}|) \right)^2}
\]

with \( \lambda(\vec{r}) = \frac{2|\vec{r}|}{(na_0)} \) and some normalization factor \( 1/A_{nl\mu} \in \mathbb{R}_+ \), and we set

\[
\vec{v}(\vec{r}) = \frac{\mu \hbar}{m} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.
\]

If one makes the quantities \( \varrho \) and \( \vec{v} \) trivially dependent on time (see footnote 28), then they provide stationary strong solutions of the respective Madelung equations (cf volume I, chapter XI, section 6 and appendix B, section I.2 in \[124\], equations (7.34) and (10.31) in \[13\], equations (A.5.7) and (A.6.3) in \[147\]).

Given the close relationship between the drift fields from equations (2.28) and (2.33), we can easily compute the distributional curl of the latter.

**Corollary 2.11.** Let \( \vec{v} \) be as in equation (2.33). Denote by \( \varphi^3 \) the third component of a given vector-valued test function \( \vec{\varphi} \in \mathcal{D} \).

Then the distributional curl of \( \vec{v} \) is given by

\[
\nabla \times \vec{v} = \frac{2\pi \mu \hbar}{m} \begin{pmatrix} 0 \\ 0 \\ \xi \end{pmatrix},
\]

with

\[
\xi(\vec{\varphi}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi^3(0, 0, z) \, dz
\]

for all \( \vec{\varphi} \in \mathcal{D} \). In particular, \( \vec{v} \) is not weakly differentiable. \( \square \)

Corollary 2.11 implies that the drift field from equation (2.33) also exhibits quantum vorticity and that this vorticity is concentrated on the \( z \)-axis. Therefore, the drift field is again quasi-irrotational in the sense of definition 2.10.

Thus, in this physical model and for the solutions above, the third Madelung equation does not hold—as it was the case for the previous 2-dimensional model. Much of the above discussion on the 2-dimensional model carries over to this case, so we shall not repeat those points here.

A notable difference is that the additional degree of freedom in the 3-dimensional case suggests a larger diversity of quantum vortices compared to the 2-dimensional case: According to Bialynicki-Birula et al \[21\], the drift field of the normalized state

\[
\frac{1}{\sqrt{2}} \left( \Psi_{200} + e^{i\pi/2} \Psi_{210} \right)
\]

has a ‘vortex ring’ of radius \( 2a_0 \) in the \( z = 0 \) plane. Whether this is indeed a quantum vortex in the sense of definition 2.10 would need to be shown. In an earlier article \[20\] the same first two authors suggest that all quantum vortices in 3 dimensions are ‘singularities of the \( \delta^{(2)} \) type’, yet this, too, remains to be rigorously defined and proven. In \[83\] Hirschfelder implicitly suggested that quantum vortices in 3 dimensions are topological 1-manifolds (cf p 5478 therein), on which he based the assertion that every such quantum vortex is homeomorphic to either the
real line (‘axial vortex’) or to the unit circle $\mathbb{S}^1$ (‘toroidal vortex’). Due to the Level Set Theorem (cf theorem 1.2.1 in [146]), this is indeed the case if 0 is a regular value of a $C^1$ wave function (possibly restricted in domain). It is an open question whether it holds, for instance, if we significantly weaken the assumption on the wave function—say, if $[\Psi]$ is an element of the Sobolev spaces $H^1(\mathbb{R}^3, \mathbb{C})$ or $H^2(\mathbb{R}^3, \mathbb{C})$.

Let us now consider the physical interpretation of the distributional curl in this 3-dimensional model. We suggest two closely related interpretations.

First we may view equation (2.34a) as an idealized expression representing a limit case. A good analogy is the view of a Dirac delta as an idealization of a Gaussian function with a very narrow peak in the limit where the width tends to zero and the volume integral over the function is kept constant. Indeed, in an affirmation to the above comment made by Bialynicki-Birula and Bialynicka-Birula [20], one may view the distribution $\xi$ from equation (2.34b) as a Dirac delta on the $z$-axis, so that in ‘physics notation’ one formally obtains equation (1.4). If one is willing to accept the interpretation of equation (1.4) as a limiting expression, then it forces one to view the third Madelung equation, interpreted in the distributional sense, not as a general equation, but one that only holds in particular cases.

This view is indeed suggested by the first Madelung equation, equation (1.1a): If, in the smooth setting, one takes the curl thereof and denotes the vorticity $\nabla \times \vec{v}$ by $\vec{\omega}$, then the following evolution equation is obtained:

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega})$$

(see equations (26) and (27) in [169] and equation (92) in [131]). An equivalent formulation is given by

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \nabla) \vec{\omega} = (\vec{\omega} \cdot \nabla) \vec{v} - (\nabla \cdot \vec{v}) \vec{\omega}.$$  (2.37)

As the constraint $\omega = 0$ trivially solves equation (2.36), it is at least consistent with the first Madelung equation. Yet, in principle—even in the strong sense—the initial data $\vec{v}_0$ does not need to be vorticity-free, and neither does $\vec{v}$ as part of a (strong) solution $(\rho, \vec{v})$ of the first two Madelung equations, equations (1.1a) and (1.1b). Furthermore, if one takes the heuristic view [140] that the Madelung equations are evolutionary equations for ensembles of single point masses with probability density $\rho(t, \cdot)$ at time $t$ with the evolution of $\rho$ being governed by the flow of the vector field $\vec{v}$, then the constraint imposed by the third Madelung equation, equation (1.1c), seems somewhat ad hoc. Why should only irrotational drift fields be allowed?

The idea of dropping the third Madelung equation is far from novel: Takabayasi already suggested such ‘vorticial flow’ in 1952 (cf section 13 in [165] and appendix E in [166]). Freistadt has credited a 1954 article by Bohm and Vigier [29] for suggesting ‘that only the <<=smeared out>> flow might be irrotational […] while there might be vortices on a small scale’ (cf p 14. in [61]). Schönberg also suggested the consideration of rotational drift fields in 1954 (cf p 119 in [157] and p 567 in [158]).

Since there are no universally agreed upon ‘distributional Madelung equations’, it is an open question, whether an equation analogous to equation (2.36) holds in a distributional sense. With regards to the results of this section, it is possibly the case, that a consideration of the vorticity

31 Hoffmann-Ostenhof et al [86] have treated the question of the regularity of the nodal sets of stationary states for rather general classes of potentials.
of locally integrable drift fields $\vec{v}$ will play a role in establishing a precise mathematical relationship between such distributional Madelung equations and the Schrödinger equation.

In this respect, it is worth mentioning that an even more general approach was suggested by Loffredo and Morato in the context of stochastic mechanics (see [115, 116] as well as [126]): They suggested that in the case of non-vanishing vorticity $\vec{\omega}$ an additional $\vec{\omega}$-dependent term needs to be added to the first Madelung equation. The authors state that ‘any solution to these equations, with a generic rotational velocity field, relaxes towards a standard solution with irrotational velocity field associated to a quantum state’ (cf p 209 in [116]). While they did allow drift fields such as the ones in equation (2.33), they did not elaborate on Takabayasi’s condition. Nonetheless, their work shows that allowing for ‘vorticial flows’ may not be as simple as dropping equation (1.1c), so that a modification of the first Madelung equation, equation (1.1a), may need to be considered as well\textsuperscript{32}.

As further way to generalize the third Madelung equation is to account for spin and magnetic fields (cf e.g. [25, 26, 96] and section 9.3.2 in [88]). Indeed, Gurtler and Hestenes [73] noted that for electrons or other particles with non-zero magnetic moment it is generally more appropriate to view the Schrödinger equation as a special case of the Pauli equation. In the non-relativistic limit the latter describes such ‘spin-1/2 particles’ in the presence of general electromagnetic fields. We shall, however, not dive into this any further.

We shall return to the question of the physical interpretation of the distributional curl for the 3-dimensional example considered in this section. A second such interpretation would be to view the distributional curl and other distributions in this context as generalized random variables in the sense that one can apply the respective distribution to the probability density $\varrho$ and interpret the result as the corresponding expectation value. If $\tilde{A}$ is a distribution on say $\mathcal{S}(\mathbb{R}^3, \mathbb{C})$ and one is willing to employ ‘physics notation’ to formally construct a ‘random variable’ $A$, then this translates to the formal equation\textsuperscript{33}

$$\tilde{A}(\varrho) = \int_{\mathbb{R}^3} \varrho \, d^3 r \, A.$$ (2.38)

Equation (2.38) justifies the view of $\tilde{A}(\varrho)$ as the expectation value of the random variable $A$.

A mathematical problem one faces with this approach is that a given probability density $\varrho$—especially in quantum mechanics—need not be a Schwartz function.

In the 2-dimensional example above, $\varrho$ is indeed a Schwartz function, so that curl $\vec{v}(\varrho)$ is well-defined. One easily checks that the corresponding expectation value vanishes.

In our 3-dimensional example we find, however, that $\varrho$ from equation (2.32) fails to be smooth at the origin. Thus $\varrho$ is not a test function in any of the senses above and so, strictly speaking,

$$(\nabla \times \vec{v})(\varrho) := (\nabla \times \vec{v})(\varrho, \varrho, \varrho)$$ (2.39)

is not a mathematically sensible expression. Still, one may ask if one can canonically extend the domain of $\nabla \times \vec{v}$ to include $\varrho$.

As the next lemma shows, this is indeed the case.

\textsuperscript{32} The second Madelung equation assures probability conservation. At least in the absence of particle creation and annihilation, there is no justification for modifying it (see also section 6 in [140]).

\textsuperscript{33} The expression is formal because not every distribution $\tilde{A}$ is regular. Still, such formal expressions are abundant in physics, as exemplified by the formula

$$\delta_0(\varphi) = \int_{-\infty}^{\infty} dx \, \delta(x) \varphi(x) = \varphi(0).$$
Lemma 2.12. Let \( \vec{v} \) be given by equation (2.33).

Then its distributional curl \( \nabla \times \vec{v} \) can be canonically extended to the domain of all \( \varphi \in C^1(\mathbb{R}^3 \setminus \{0\}, \mathbb{R}) \) for which \( \varphi \) is \( L^1 \)-integrable, vanishes at infinity, and admits a continuous extension to \( \mathbb{R}^3 \) for each \( i \in \{1, 2, 3\} \). Equations (2.34) still hold on this extended domain.

In particular, the expression (2.39) is well-defined for any \( \varphi \) given by equation (2.32). □

Lemma 2.12 also provides us with a simple method to compute the respective expectation values.

Proposition 2.13. Let \( \nabla \times \vec{v} \) be the distributional curl of the vector field \( \vec{v} \), as given by equation (2.33), and let \( \varphi \) be given by equation (2.32).

Then the quantity \( \langle \nabla \times \vec{v} \rangle(\varphi) \) vanishes for all admissible \( n, l, \) and \( \mu \). □

Therefore, even though in this model the distribution \( \nabla \times \vec{v} \) does not vanish for \( \mu \neq 0 \), the corresponding expectation value always does—as it was the case for the 2-dimensional example above.

Physically, this suggests that, at least for those two models and for the solutions considered here, the appearance of quantum vorticity cannot be measured (directly) even when the solutions exhibit it on a mathematical level.

If one subscribes to the prior described view that those models represent limiting cases of more general models in which the quantum vorticity is supported on a set of positive Lebesgue measure, then in such more general models the vorticity is measurable ‘in principium’.

3. Remarks on the distributional Madelung equations

The purpose of this section is to clarify the loose term ‘distributional Madelung equations’ and make some remarks thereon in relation to the published literature. While the topic has not received much attention in the mathematical literature, we do provide some suggestions and considerations that we expect to be of assistance in future considerations of the problem. Nonetheless, it should be kept in mind that other weak formulations of the Madelung equations may exist and that there might even be physical reasons to prefer such a formulation over the ones presented here.

In section 1 we already mentioned that Gasser and Markowich [67], with some caveats, showed that the Schrödinger equation with initial data in the Sobolev space \( H^1(\mathbb{R}^3, \mathbb{C}) \) implies a reformulated version of the Madelung equations in the distributional sense:

\[
\frac{\partial \overrightarrow{j}}{\partial t} + \nabla \cdot \left( \frac{\overrightarrow{j} \otimes \overrightarrow{j}}{\varphi} \right) = -\varphi \nabla V + \frac{\hbar^2}{4m} \left( \Delta \varphi - \nabla \cdot \left( \frac{\nabla \varphi \otimes \nabla \varphi}{\varphi} \right) \right) \tag{3.1a}
\]

\[
\frac{\partial \varphi}{\partial t} + \nabla \cdot \overrightarrow{j} = 0. \tag{3.1b}
\]

More specifically, let \( \Psi \) in

\[
L^\infty \left( [0, \infty) ; H^1(\mathbb{R}^3, \mathbb{C}) \right) = \left\{ \Psi : [0, \infty) \to H^1(\mathbb{R}^3, \mathbb{C}) : \epsilon \mapsto \Psi, \left| \epsilon \in [0, \infty) \right| \text{ess sup} \| \Psi \|_{H^1} < \infty \right\} \tag{3.2}
\]

See e.g. p 55 in [11] for an explanation of this notation. We note that, strictly speaking, an element \( \Psi \) in \( L^\infty \left( \mathbb{R} ; H^1(\mathbb{R}^3, \mathbb{C}) \right) \) is an equivalence class of an equivalence class of functions.
be a solution of the Schrödinger equation with potential $V$ and initial data $\Psi_0 \in H^1(\mathbb{R}^3, \mathbb{C})$, where $\Psi_0$ and $V$ satisfy some further technical assumptions\(^{35}\) (cf lemma 2.1 in [67]). Denoting the weak gradient of $\Psi$ by $\nabla \Psi$, define the quantities
\[
\varrho = |\Psi|^2 \quad \text{and} \quad \vec{j} = \frac{\hbar}{m} \text{Im}(\Psi^* \nabla \Psi).
\] (3.3)

Then $\varrho$ and $\vec{j}$, with initial data $\varrho_0$ and $\vec{j}_0$ obtained from $\Psi_0$, solve the integral equations\(^{36}\)
\[
0 = \int_0^\infty dt \int_{\mathbb{R}^3} d^3 r \left( \vec{j} \cdot \frac{\partial \vec{\varphi}}{\partial t} + \left( \frac{\vec{j} \otimes \vec{j}}{\varrho} \right) \cdot \nabla \vec{\varphi} - \varrho \nabla V \vec{\varphi} \right.

\[
+ \frac{\hbar^2}{4m} \left( \nabla \varrho \cdot \Delta \vec{\varphi} + \left( \frac{\nabla \varrho \otimes \nabla \varrho}{\varrho} \right) \cdot \nabla \vec{\varphi} \right) + \int_{\mathbb{R}^3} d^3 r \vec{j}_0 \cdot \vec{\varphi}(0, .) \right)
\)](3.4a)

and
\[
0 = \int_0^\infty dt \int_{\mathbb{R}^3} d^3 r \left( \vec{\varphi} \frac{\partial \vec{\xi}}{\partial t} + \vec{j} \cdot \nabla \vec{\xi} \right) + \int_{\mathbb{R}^3} d^3 r \varrho_0 \vec{\xi}(0, .) \right)
\)](3.4b)

for all $\vec{\varphi} \in C_0^\infty \left( (0, \infty) \times \mathbb{R}^3, \mathbb{R}^3 \right)$ and all $\vec{\xi} \in C_0^\infty \left( (0, \infty) \times \mathbb{R}^3, \mathbb{R} \right)$.

Equations (3.4) constitute a weak formulation of the (formal) equations (3.1). That is, if one states that some given $\varrho \in L^\infty \left( [0, \infty), H^{1,1} (\mathbb{R}^3, \mathbb{R}) \right)$ and $\vec{j} \in L^\infty \left( [0, \infty), L^1 (\mathbb{R}^3, \mathbb{R}^3) \right)$ solve equations (3.1) in a distributional sense\(^{37}\), then it is implicit that each function in the respective equivalence classes satisfies the respective weak formulation, equations (3.4), for all $\vec{\varphi} \in C_0^\infty \left( (0, \infty) \times \mathbb{R}^3, \mathbb{R}^3 \right)$ and all $\vec{\xi} \in C_0^\infty \left( (0, \infty) \times \mathbb{R}^3, \mathbb{R} \right)$.

Heuristically, one obtains a weak formulation for a given set of formal PDEs by multiplying the equations with an appropriate test function, integrating over the space-time region of interest, and applying integration by parts and the divergence theorem to ‘shift’ the derivatives to the test function. The idea is that, if given weak (or even distributional) solutions are ‘sufficiently regular’ on the entire domain, one can reverse those steps and apply the fundamental theorem of the calculus of variations (cf corollary 4.24 in [35]) to show that one has in fact strong solutions of the original set of PDEs. We refer to section 3.4 in [55] for an introduction to weak formulations of initial value problems. Section 5.1 in [11] discusses the weak formulation of the incompressible Navier–Stokes equations, which is a system of PDEs that is mathematically similar to the ones considered here.

The significance of the article [67] by Gasser and Markowich is twofold: First, to our knowledge, the authors provided the first weak formulation of the Madelung equations, equations (1.1), in the literature. Second, they used this formulation to make major contributions to the mathematical theory of the classical limit (see also section 3 in [66]).

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\(^{35}\) It is worthy of note that potentials can be time-dependent, though they are often of electromagnetic origin so that the magnetic field and the spin may need to be taken into account as well in that instance. See e.g. [136] for a mathematical treatment of time-dependent potentials in the mathematical theory of quantum mechanics.

\(^{36}\) For sufficiently regular vector fields $\vec{a}, \vec{b}$ on $\mathbb{R}^3$, we set $(\vec{a} \otimes \vec{b}) \cdot \nabla \vec{a} := \vec{a} \cdot \left( (\vec{a} \cdot \nabla) \vec{b} \right)$.

\(^{37}\) For $p \in [1, \infty], k, n, m \in \mathbb{N}$, and $K \in \{ \mathbb{R}, \mathbb{C} \}$, the space $W^{k,p}(\mathbb{R}^n, K^m)$ denotes the Sobolev space of equivalence classes of $K^m$-valued $L^p$-functions on $\mathbb{R}^n$ whose weak derivative up to $k$th order are also $L^p$. Every $W^{k,p}(\mathbb{R}^n, K^m)$ is a Banach space, and for finite $p$ it is separable.
Remark 3.1 (On the classical limit). In the contemporary physics literature the theory of ‘(quantum) decoherence’ is a popular approach to the question of the classical limit of quantum mechanics. We refer, for instance, to [9, 156, 186] for an introduction.

Contrarily, the work by Gasser and Markowich [67] shows that for JWKB-initial data and in the limit \( \hbar \to 0 \), the time evolution of the probability density \( \varrho \) and the drift field \( \vec{v} \) is governed by solutions of the respective Newtonian point particle equations (cf equations (3.13) and (3.17), as well as lemmas 3.2 and 3.3 therein)\(^{38}\).

To understand the significance of their result, recall that, in accordance with the ensemble interpretation of quantum mechanics [12], \( \varrho \) is to be understood as a particle detection probability density. Moreover, in the stochastic approach mentioned in the introduction, \( \varrho \) is a phenomenological quantity obtained from a stochastic process that describes the more fundamental evolution of samples (i.e. mass points) in the ensemble. Roughly speaking, if the characteristic lengths and masses of the physical system are large as compared to \( \hbar \), then the system is approximately governed by equations in the mathematical limit \( \hbar \to 0 \). In this sense, the work by Gasser and Markowich [67] may be understood as saying that, in this instance, the samples behave in accordance with the laws of Newtonian mechanics and that their motion, in turn, determines the evolution of the ensemble quantities \( \varrho \) and \( \vec{v} \). No additional ‘theory of measurement’ is needed to achieve this remarkable result.

So far, the significant contribution of [67] to the literature on the classical limit has not received much acknowledgement in the physics community.

It is also worthy of note that their work lends further credence to the hypothesis expressed in postulate 1 of [140]: Namely, that Kolmogorovian probability theory ought to be used as an axiomatic basis of non-relativistic quantum theory, not the Dirac-von Neumann axioms. The reason is that, in passing to the classical limit, there is nothing in the formalism that would justify a change of the theory of probability employed.

With regards to Takabayasi’s condition, we note that in [67] Gasser and Markowich did not consider the question of whether (a variant of) the Schrödinger equation can be recovered from equations (3.4). Indeed, one does not expect this to be the case without any further assumptions: In the smooth setting the equations are equivalent to equations (1.1a) and (1.1b), yet the former one only implies the vorticity equation, equation (2.36), not the third Madelung equation, equation (1.1c).

Hence, if one wanted to show that equations (3.1a) and (3.1b) are equivalent to the Schrödinger equation in some sense, we expect an additional constraint on the vorticity to be required.

In the literature on super-fluidity such a condition was recently suggested by Antonelli et al [4]:

The authors have shown that in any dimension \( n \in \mathbb{N} \) any (equivalence class of) wave function(s) \( \Psi \) in the Sobolev space \( H^1(\mathbb{R}^n, \mathbb{C}) \) can be split into \( \Psi = |\Psi| \tilde{Q} \), that \( \sqrt{\tilde{\rho}} := |\Psi| \) is in \( H^1(\mathbb{R}^n, \mathbb{C}) \), that the vector field

\[
\vec{\lambda} = \frac{\hbar}{m} \text{Im}(Q^* \nabla \Psi)
\]

\[(3.5)\]

\(^{38}\) Note that on p 106 the authors define \( \tilde{\rho} \) as the Radon–Nikodym derivative of the vector-valued measure \( A \to \int_{\mathbb{R}^n} \tilde{\rho} \, d^3x \) with respect to the measure \( A \to \int_{\mathbb{R}^n} \rho \, d^3x \).
is square-integrable, and—most notably—that the current density
\[ \vec{j} = \sqrt{\rho} \vec{\lambda} = \frac{\hbar}{m} \Im (\Psi^* \nabla \Psi) \]  
(3.6)
satisfies a generalized irrotationality condition
\[ \nabla \times \vec{j} = 2 (\nabla \sqrt{\rho}) \times \vec{\lambda} \]  
(3.7)
in the distributional sense (cf lemma 3.1 therein).

The condition, equation (3.7), as well as the ‘polar decomposition’ \( \Psi = |\Psi| \Omega \) was first described in a prior article by Antonelli and Marcati [5]. In particular, for an elaboration on the authors’ polar decomposition method, see section 3 in [5], the appendix of [6], as well as the foundational article by Brenier [34].

In remark 1.3 in [4] the authors state the following:

In the case of a smooth solution \((\rho, \vec{j})\), for which we can write \( \vec{j} = \rho \vec{v} \), for some smooth velocity field \( \vec{v} \), the Generalized Irrotationality condition defined above (equation (3.7)) is equivalent to \( \rho \nabla \times \vec{v} = 0 \), i.e. the velocity field \( \vec{v} \) is irrotational \( \rho \partial^3 r \) almost everywhere. It shows that the previous definition is the right weak formulation of the classical irrotationality condition \( \nabla \times \vec{v} = 0 \) valid away from vacuum [...] [notation adapted]

As we do not expect the condition (3.7) to yield any ‘quantization condition’, we consider it an open question of whether it is ‘the right weak formulation of the classical irrotationality condition’ in this different, foundational context. In any case, such an additional constraint needs to be shown to be compatible with the evolution equations, so as to not make the full system of PDEs overdetermined.

There might, however, be another approach.

A potential problem that equations (3.1a) and (3.1b) exhibit is that this choice of formulation of the Madelung equations was motivated by the study of similar equations in the mathematical theory of fluid-dynamics. Yet, contrary to Madelung’s original interpretation [117], the interpretation of the Madelung equations as describing the evolution of a ‘quantum fluid’ has been discarded by the vast majority of physicists. According to the Born rule—which is today considered a standard part of quantum mechanics—at some given time \( t \) the function \( \rho(t, .) \) is a probability density for detecting the body in an arbitrary region of space\(^{39}\). It is therefore a misinterpretation to view \( \rho \partial^3 r \) as the mass density of a fluid.

Hence, at least on physical grounds, there is a substantial difference between the Madelung equations and fluid-dynamical equations, be they ‘classical’ or ‘quantum’ fluids. Carrying over ideas and methods from the mathematical theory of fluid dynamics in a one-to-one manner can therefore lead to incorrect physics. A particular example is that in the context of microscopic physics, as described for instance by the (linear) Schrödinger equation, the potential \( V \) is not directly coupled to the density \( \rho \) (as it would be for charged fluids). Also, fluid-dynamical concepts such as temperature, pressure, and constitutive equations have no place here, as those are of macroscopic nature\(^{40}\).

\(^{39}\) See section 5.1 in [140] for a more detailed discussion of this point in the smooth setting.

\(^{40}\) Such considerations do not apply to drift-diffusion equations. A relationship between the latter and the Madelung equations is suggested by the theories of stochastic mechanics and stochastic electrodynamics. See e.g. [148] for a recent review.
Still, using equation (3.1) for a weak formulation of the Madelung equations does have mathematical justification: Roughly speaking, if one does not multiply equation (1.1a) by \( \rho \) to obtain equation (3.1a), the division by \( \sqrt{\rho} \) in the quantum force term makes it difficult to find a workable weak formulation.

In spite of this, it is possible to find such a weak formulation—though, as we shall see, it is not entirely without limitations.

First, we reformulate the problematic term with the help of Nottale’s identity

\[
\frac{1}{\alpha} \nabla \left( \frac{\Delta \varphi^\alpha}{\varphi^\alpha} \right) = \Delta (\nabla \ln \varphi) + 2\alpha \left( (\nabla \ln \varphi) \cdot \nabla \right) (\nabla \ln \varphi),
\]

which holds for all \( \alpha > 0 \) and sufficiently regular \( \varphi \) (see e.g. equation (153) in [131]). Thereafter, one defines the osmotic velocity

\[
\vec{u}(\rho) = \frac{\hbar}{2m} \nabla \rho
\]

(cf equations (2.27) and (2.64) as well as p 123 in [45]) and reformulates the continuity equation in terms of \( \vec{u}(\rho) \) and \( \vec{v} \).

Ultimately, one arrives at Nelson’s equations (cf equation (34) in [128] and equation (1) in [41]):

\[
m \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla V + m (\vec{u} \cdot \nabla) \vec{u} + \frac{\hbar}{2} \Delta \vec{u}
\]

\[
\frac{\partial \vec{v}}{\partial t} + \nabla \left( \vec{v} \cdot \vec{u} + \frac{\hbar}{2m} \nabla \cdot \vec{v} \right) = 0.
\]

The advantage of Nelson’s reformulation of the Madelung equations is that it allows one to express the latter entirely in terms of \( \vec{v} \) and \( \vec{u} \), so that, via the equation (3.9), one can treat \( \rho \) as a derived quantity. Note that one does not require irrotationality of \( \vec{v} \) to derive equations (3.10).

In looking for a weak formulation of Nelson’s equations, we need to find a way to reformulate the two terms of the form \( \vec{v} \cdot \nabla \vec{w} \) in equation (3.10a). The terms are inconvenient, since, following the aforementioned procedure, we ultimately intend to ‘shift’ as many derivatives as possible to the test functions. For the incompressible Navier–Stokes equations the term is handled using the divergence-freeness of \( \vec{w} \) (cf section 5.1 in [11]). While this does not work in this instance, we can impose the condition of irrotationality instead, which in turn allows us to use the identity

\[
(\vec{v} \cdot \nabla) \vec{w} = \nabla \left( \frac{\vec{w}^2}{2} \right).
\]

We make use of equation (3.11) to suggest a weak formulation of the Madelung equations that differs from equations (3.4):

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41 Equation (3.8) can be shown by pulling \( \alpha \) into the exponent of each logarithm and then using various vector identities like equation (3.11) below.

42 Different sign conventions are used in the literature. The terms ‘diffusive velocity’ and ‘stochastic velocity’ are also in use. Typically, the osmotic velocity is understood as a measure of ‘diffusion’ due to the analogy to Fick’s first law (cf p 42 and section 4.5.1 in [45]).

43 The fact that the Madelung equations can be reformulated in this manner suggests that the introduction of \( \vec{u} \) via (3.9) is not merely a mathematical convenience, but that the quantity is of potential physical significance as well.

44 Using a common physics notation, we set \( \vec{w}^2 = \vec{w} \cdot \vec{w} \).
Given initial data $\tilde{u}_0$ and $\tilde{v}_0$, we require that $\tilde{u}$ and $\tilde{v}$ satisfy

$$0 = \int_0^\infty dt \int_{\mathbb{R}^3} d^3r \left( m \ddot{v} + V - \frac{m}{2} \dddot{\varphi} \right) \nabla \cdot \varphi + \frac{\hbar}{2} \tilde{u} \Delta \varphi + \int_{\mathbb{R}^3} d^3r m \tilde{v}_0 \cdot \varphi(0, .) \tag{3.12a}$$

$$0 = \int_0^\infty dt \int_{\mathbb{R}^3} d^3r \left( \dddot{\varphi} + \frac{\hbar}{2} \tilde{u} \Delta \varphi - \frac{h}{2} \tilde{v} \Delta \varphi \right) \nabla \cdot \varphi + \int_{\mathbb{R}^3} d^3r \tilde{u}_0 \cdot \varphi(0, .) \tag{3.12b}$$

for all $\varphi, \tilde{\varphi} \in C_0^\infty((0, \infty) \times \mathbb{R}^3, \mathbb{R}^3)$. Now let $\tilde{u}_0$, $\tilde{v}_0$ be in in $L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$, and $\tilde{u}$, $\tilde{v}$ be in $L^\infty_{\text{loc}}((0, \infty); L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3))$, the latter being given by

$$\left\{ \tilde{w} : \forall t \in [0, \infty) \forall \text{ compact } K \subset \mathbb{R}^3 : \tilde{w}(t, .) \in L^2(K, \mathbb{R}^3) \right\} \tag{3.13}$$

It can be checked by a straightforward estimate of the integrals\(^{45}\), that equations (3.12) are mathematically sensible in that instance.

**Remark 3.2 (Quasi-irrotational solutions are not in $L^2_{\text{loc}}$).** The alert reader may object, that by using the identity (3.11) we implicitly disposed of solutions such as the ones in equations (2.32) and (2.33).

Indeed, those solutions (including their ‘non-quantized’ siblings) are not admissible in the above weak formulation of the Madelung equations, though for other reasons:

It is easy to check that $\tilde{v}$ from equation (2.33) is not in $L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ (or $L^\infty_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$), respectively). Therefore, there is no solution $(\tilde{u}, \tilde{v})$ of equations (3.12) for which $\tilde{v}$ is given by equation (2.33). This problem was, however, not entirely caused by the use of the identity (3.11) in setting up the weak formulation. It is also the $\tilde{u} \cdot \tilde{v}$ term in the weak formulation of Nelson’s formulation of the continuity equation, equation (3.12b), that suggests use of the space $L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ for fixed time.

Note that the weak formulation of the Madelung equations by Gasser and Markowich, equations (3.4) above, does not suffer from this issue.

We also point out that there may be a mathematical link between quasi-irrotationality of $\tilde{v}$ from equation (2.32) and the fact that it is in $L^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ yet not in $L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$. Investigating this possible link may provide an answer to the question, whether there are (quasi-)rotational solutions of equations (3.12) or not.

\(^{45}\) For $\varphi \in C^\infty_0((0, \infty) \times \mathbb{R}^3, \mathbb{R})$ denote by $I$ the (compact) support of the function $t \mapsto \|\varphi(t, .)\|_{L^\infty(I, \mathbb{R})}$. Then for all $\tilde{w} \in L^\infty_{\text{loc}}((0, \infty); L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}))$ we have

$$\left| \int_0^\infty dt \int_{I} d^3r \tilde{w}(t, .) \varphi(t, .) \right| \leq \int_0^\infty dt \| \tilde{w}(t, .) \varphi(t, .) \|_{L^1(I, \mathbb{R})}$$

$$\leq \int_I \| \varphi(t, .) \|_{L^\infty(I, \mathbb{R})} \| \tilde{w}(t, .) \|_{L^2(\text{supp } \varphi(t, .), \mathbb{R}^3)}$$

$$\leq \left( \int_I dt \right) \left( \sup_{t \in I} \| \varphi(t, .) \|_{L^\infty(I, \mathbb{R})} \right)$$

$$\times \left( \text{ess sup}_{t \in I} \left( \| \tilde{w}(t, .) \|_{L^2(\text{supp } \varphi(t, .), \mathbb{R}^3)} \right) \right)^2.$$
Regarding the physical (in)acceptability of ‘throwing out solutions’ such as the ones in equations (2.32) and (2.33), we refer back to section 2.3.

In the weak formulation of Nelson’s equations, equations (3.12) above, the initial data \( \vec{u}_0 \) for \( \vec{u} \) is to be determined from the initial data \( \vartheta_0 \) for \( \vartheta \). As \( \vartheta(t, .) \) is supposed to be a probability density for all \( t, \vartheta_0 \), needs to be in \( L^1(\mathbb{R}^3, \mathbb{R}) \) and (every representative in the equivalence class is) strictly positive almost everywhere. If we understand the derivative in equation (3.9) in the weak sense, we furthermore require that \( \vartheta_0 \) is weakly differentiable and that the quantity \( \nabla \vartheta_0/\vartheta_0 \) is in \( L^2(\mathbb{R}^3, \mathbb{R}^3) \) (as an equivalence class of vector fields) so that \( \vec{u}_0 \) is, too.

Apart from providing a formulation involving the vector field \( \vec{v} \) directly, equations (3.12) can also easily be generalized to distributional potentials. If, for instance, we wanted to consider \( V(t, \vec{r}) = \alpha \delta^3(\vec{r}) \) for some \( \alpha \in \mathbb{R} \setminus \{0\} \), then we would make the following replacement in equation (3.12a):

\[
\int_0^\infty dt \int_{\mathbb{R}^3} d^3r \nabla \cdot \vec{\varphi} \to \alpha \int_0^\infty dt (\nabla \cdot \vec{\varphi}) (t, 0).
\]

(3.14)

It is not immediate how to generalize equations (3.1) above, so that they can also account for distributional potentials.

Under appropriate conditions of regularity of the quantities involved, every solution of the Schrödinger equation will also be a solution of equations (3.12)—with the restriction expressed in remark 3.2 above. As, to our knowledge, equations (3.12) have not been considered in the literature before, it is unknown whether the initial value problem is well-posed. Accordingly, it is also not known how precisely the system of equations relates to the Schrödinger equation—though the condition that \( \vec{v}(t, .) \in L^2(\mathbb{R}^3, \mathbb{R}^3) \) for almost all \( t \) does introduce an additional restriction.

To show that there are indeed non-trivial solutions to equations (3.12), we shall consider a simple example.

**Example 3.3.** For the (stationary) ground state of the isotropic harmonic oscillator of frequency \( \omega \) in 3 dimensions we have

\[
\vec{u}(t, \vec{r}) = -\omega \vec{r} \quad \text{and} \quad \vec{v}(t, \vec{r}) = 0
\]

for all \( t \in \mathbb{R} \) and \( \vec{r} \in \mathbb{R}^3 \). One easily checks that \( \vec{u} \in L^\infty_{loc}([0, \infty); L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)) \).

Given a solution \( (\vec{u}, \vec{v}) \) of equation (3.12), we may then call \( (\vartheta, \vec{v}) \) a solution of the Madelung equations, if for almost all \( t \in \mathbb{R} \) the (equivalence class of) functions \( \vartheta(t, .) \) solves the boundary value problem\(^{46}\)

\[
0 = \int_{\mathbb{R}^3} d^3r \left( \vec{u}(t, .) \cdot \vec{\varphi} + \frac{\hbar}{2m} \ln \vartheta(t, .) \nabla \cdot \vec{\varphi} \right) \quad (3.16a)
\]

\[
1 = \int_{\mathbb{R}^3} d^3r \vartheta(t, .) \quad (3.16b)
\]

for all \( \varphi \in C^\infty_0(\mathbb{R}^3, \mathbb{R}) \). Such solutions \( \vartheta(t, .) \), if they exist, satisfy the same assumptions as stated for \( \vartheta_0 \) above. If this boundary value problem does not have a solution \( \vartheta \), \( \vec{u} \) itself needs to be discarded.

\(^{46}\) Strictly speaking, \( \vartheta \) is not a dimensionless quantity so that \( \ln \vartheta \) is not well-defined. Yet this can be easily addressed with the introduction of a constant ‘reference value’ (cf p 42 in [45]).
That is, the full system of equations is given by equation (3.12) with
\[ \vec{u}_0 = \frac{\hbar}{2m} \nabla \ln \varrho_0 \] (3.17)
in the weak sense, along with equation (3.16).

The two distinct weak formulations of the Madelung equations given here should clarify what is meant by the somewhat vague term ‘distributional Madelung equations’.

In the remainder of this section we will provide some general remarks that may be of use in formulating ‘the right’ distributional Madelung equations. We shall not go into the question of whether (quasi-)rotational solutions are acceptable or not. For the latter question we refer the reader to sections 2.3 and 5.

We shall begin by noting that for the above two weak formulations, we have assumed that there are no further boundary conditions given. That those can be of relevance is shown by the examples of the ‘infinite potential well’ as well as the single/double slit experiment. Both are examples of homogeneous Dirichlet boundary conditions, which can be handled via the use of trace operators (see e.g. section 5.5 in [55]).

This brings us to the more general problem of the support of \( \varrho(t, .) \), which is defined by viewing \( \varrho(t, .) \) as a regular distribution (cf footnote 29).

In the second weak formulation proposed here we implicitly require that \( \text{supp} \varrho(t, .) = \mathbb{R}^3 \) for almost all \( t \in [0, \infty) \), as otherwise \( \ln \varrho(t, .) \) is not well-defined. While a result due to Galindo [65] and later Hegerfeldt [81] justifies this assumption to some degree, the weak formulation by Gasser and Markowich does not suffer from this limitation.

The case that \( \text{supp} \varrho(t, .) \) is a proper subset of \( \mathbb{R}^3 \) should generally be allowed. The physical reason is that the bodies in the ensemble are typically localized in a small region of space. Commonly used probability densities such as Gaussians are in actuality an idealization, though in many quantum-mechanical problems of physical relevance stationary states are supported on the entirety of \( \mathbb{R}^3 \) and there is some justification in viewing this as the generic case [65, 81].

Yet, if we want to allow for the case that \( \text{supp} \varrho(t, .) \) is a proper subset of \( \mathbb{R}^3 \) (for some \( t \in \mathbb{R} \)), we have to ask whether we ought to place any further constraints on \( \text{supp} \varrho(t, .) \) or, better to say, on \( \text{supp} \varrho_0 \).

Indeed, physical considerations demand that the support of \( \varrho_0 \) (and of \( \Psi_0 \)) is connected:

Consider a simple experiment (in vacuum and without external forces) in which two mutually parallel particle guns are directed at a detector screen at an appropriate distance away. After each triggering of the apparatus, a single particle gets ejected from one of the guns, the chance of the first gun getting triggered being \( a \in (0, 1) \) and the chance of the other one getting triggered being \( 1 - a \). For each run, \( t = 0 \) is fixed to be the point in time after which the gun was triggered and a brief, given moment has passed to give the particle time to leave the apparatus. In this experiment, one can model \( \varrho_0 \) as the sum of two bump functions with mutually disconnected support and one can idealize \( \vec{v}_0 \) as a constant vector field on \( \text{supp} \varrho_0 \) pointing in the direction of the detector and vanishing otherwise. Correspondingly, choose the values of the initial wave function \( \Psi_0 \) to be \( \sqrt{\varrho_0(\vec{r})} \exp(ik_0 \cdot \vec{r}) \) with \( k_0 = m\vec{v}_0(\vec{r}_0)/\hbar \) for some arbitrary \( \vec{r}_0 \in \text{supp} \varrho_0 \).

47 In physics textbooks the double slit experiment is often incorrectly depicted as a stationary problem and, in turn, used to justify the ‘superposition principle’. Viewing it as a dynamical scattering problem is clearly more physical. Such an approach was taken, for instance, by Sanz and Miret-Artés in [152]—though the authors used Gaussians for their analysis and did not impose any additional boundary conditions.
The respective solution of the free Schrödinger equation yields that the disconnected supports of \( \varrho_0 \) merge immediately and that there is interference—thus contradicting the impact statistics observed on the detector.

The failure of the Schrödinger theory to predict the correct behavior in this instance is due to a misapplication of the theory. The correct prediction is obtained by finding the Schrödinger evolution for each individual bump function \( \varrho_{1,0} \) and \( \varrho_{2,0} \) (both normalized to unity) with their associated drift fields, so that at each \( t > 0 \) the probability density is given by

\[
\varrho(t, \cdot) = a \varrho_1(t, \cdot) + (1 - a) \varrho_2(t, \cdot).
\]

(3.18)

The determination of \( \varrho(t, \cdot) \) is analogous to the experiment being done with ‘classical’ projectiles (whose statistics follows a different time evolution). The underlying reason is that, as in the case of projectiles, the time evolution of the individual sample in the ensemble is independent from the time evolution of the other samples.

Mathematically, this physically incorrect prediction can be prevented by requiring that the support of any initial probability density \( \varrho_0 \), or initial wave function \( \Psi_0 \), is connected. While the result by Galindo [65] and Hegerfeldt [81] constitutes an important step in showing that to some degree this property is preserved under time evolution, more research is needed in this regard.

**Remark 3.4 (The Wallstrom phenomenon).** The above requirement that the initial probability density ought to have connected support is closely related to Wallstrom’s second objection [176] that was outlined in section 1. We will explain the objection, its relation to the so-called Pauli problem in quantum mechanics, and provide some potential avenues for resolution.

In [176] Wallstrom considered a (sufficiently regular) initial wave function \( \Psi_0 \) with associated density \( \varrho_0 \) and drift field \( \vec{v}_0 \). He argued that, if the set

\[
\Omega_0 = \left\{ \vec{r} \in \mathbb{R}^n \mid \varrho_0(\vec{r}) \neq 0 \right\}
\]

(3.19a)

is disconnected, then changing the phase factors for \( \Psi_0 \) on each connected component of \( \Omega_0 \) does not change the initial conditions \( (\varrho_0, \vec{v}_0) \) for the respective Madelung equations while the corresponding Schrödinger equation can in principle lead to different solutions of the Madelung equations whenever the components ‘merge’ under time evolution. Thus, Wallstrom argued, the Madelung equations cannot provide unique solutions in general.

Indeed, in a recent article [121], Markowich and Sierra independently rediscovered the described effect, which we shall call the ‘Wallstrom phenomenon’ hereafter. While the assumptions of the main mathematical statement in [176] are too strong to allow for the phenomenon—as the corresponding time-dependent flow of \( \vec{v} \) is continuous—Markowich’s and Sierra’s results leave no doubt that it exists (cf theorem 3.2 and corollary 1 in [121]).

In the beginning of their article Markowich’s and Sierra [121] gave a 1-dimensional example of an initial \( H^2 \)-wave function with connected support for which this phenomenon occurs. Thus, the assumption of connected support alone is insufficient to prevent the occurrence of the Wallstrom phenomenon. Moreover, while the above operation can change the regularity of the initial wave function, it does also not seem to be sufficient to impose more restrictive assumptions of regularity—with the important exception that, if the flow of \( \vec{v} \) exists and is continuous, such a ‘merging’ of the connected components of \( \Omega_0 \) cannot occur (cf proposition 3.6 in [140]).

While Wallstrom viewed the phenomenon as a problem of the Madelung equations, we wish to point out that it is also a problem of the Schrödinger equation: With the exception of comparing the time evolution of the probability density after the supports have merged, it is not
possible to physically discern the different initial states—all predicted detection probabilities and expectation values of observables\(^{48}\) at \(t = 0\) are the same. Accepting all of those initial states as physical would undermine the predictive power of the Schrödinger theory.

As Antonelli and Marcati had already suggested in [5], the above problem is closely related to the ”Pauli problem”. As the name suggests, this refers to a historical question initially posed by Pauli [134]: Is it possible to reconstruct a wave function \(\Psi_0\) from its position probability density \(|\Psi_0|^2\) and its ”momentum probability density” \(|\mathcal{F}\Psi_0|^2\)? This question was answered negatively, since a variety of counterexamples had been found over the decades. The first one was given by Bargmann in Reichenbach’s book [143], for later ones we refer to section 2 in [178] as well as [164]. Therein the reader also finds a historical review of the Pauli problem.

The Wallstrom phenomenon and the Pauli problem are connected by the overarching question of how to discern quantum-mechanical states that are initially physically equivalent—i.e. with regards to measurement\(^{49}\) —yet nonetheless lead to physically inequivalent states under time evolution. In one dimension and for wave functions with first order strong derivatives this question was studied by Weigert [177], leading to the conclusion that it is precisely Wallstrom’s phenomenon that leads to this problem\(^{50}\).

The underlying problem exhibited by the Wallstrom phenomenon and the negative answer to the Pauli problem therefore seems to be the failure of quantum mechanics to specify which (initial) states are physically acceptable and which ones are not. Phrased differently, the Schrödinger theory in \(L^2\) is physically incomplete without a further restriction on the set of (initial) states.

We shall provide some considerations for how to resolve the problem.

While one may ask for the support of initial wave functions or probability densities to be connected, one cannot simply ask that the same holds for \(\Psi_0\), for that would exclude any stationary state with nodes and thus be physically unacceptable. Moreover, for \(L^1\)- or \(H^1\)-wave functions the nodal set is not even defined. For \(H^2\)-wave functions the Sobolev embedding theorem does provide for a consistent definition, however (cf remark 2.7).

Wallstrom’s suggested resolution [176] is to ask for JWKB-initial data \(\Psi_0 = \sqrt{\det} \exp(iS_0/\hbar)\) and for \(S_0\) to ”be continuous across the nodal boundaries.” This condition, however, seems arbitrary, for even for continuous \(\Psi_0\) the function \(S_0\) cannot be assumed to be continuous everywhere (cf section 2.3). He also suggests that one could require \(S_0\) to be computed from the respective line integral over a given drift field \(\vec{v}_0\). That is, in 3 dimensions we have

\[
S_0(\vec{r}) - S_0(\vec{r}_0) = m \int_{\gamma} \vec{v}_0 \cdot d\vec{r}
\]

for any (smooth or \(C^1\)) curve \(\gamma\) from \(\vec{r}_0\) to \(\vec{r}\). Yet here \(\vec{v}_0\) needs to be sufficiently regular for the integral to be defined—as it is the case for Takabayasi’s condition—and the same issue occurs whenever the connected components of the domain of \(\vec{v}_0\) are multiply-connected.

\(^{48}\) The ”operator” suggested in equation (23) of [177] is not an observable in the quantum-mechanical sense, since there is no non-trivial domain for which it is an operator on \(\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})\). Still, one could in principle consider operators \(\hat{A}\) for which there exists a \(\Psi \in \text{dom}\hat{A}\) such that the support of \(\hat{A}\Psi\) is not contained in the support of \(\Psi\). In that case the discussed phase factor could change the expectation value of \(\hat{A}\) with regards to the full wave function. This is, however, not the case for the physically important operators of energy, momentum, position, and angular momentum.

\(^{49}\) Without aiming to address the question of measurement in quantum theory in any depth here, this statement is meant to refer to expectation values of physical operators. See also footnote \(^{48}\) above.

\(^{50}\) The author of [177] did not mention the connection to Wallstrom’s work [176].
In the example Markowich and Sierra [121] provide, it is intuitively clear that the wave function without the artificial phase factor provides the physically correct time evolution. As one can make this choice, neither Wallstrom’s phenomenon nor the Pauli problem undermine the predictive power of the Schrödinger equation here. The mathematical problem is therefore to formalize this intuitive choice for the general case and to place appropriate assumptions on \( \rho_0 \) and \( \vec{v}_0 \) (or \( \vec{J}_0 \)) to assure that Wallstrom’s phenomenon does not lead to non-uniqueness for the given weak formulation of the Madelung equations.

Not only would this be an important contribution to the mathematical theory of the Madelung equations, it could also provide a satisfactory conceptual resolution to Pauli’s original question.

We shall provide some general considerations on the regularity on \( \rho(t, .) \) and \( \vec{v}(t, .) \) that one might want to impose in finding an appropriate weak formulation of the Madelung equations.

As Teufel and Tumulka indicated (cf p 352 in [170]), the hydrogen ground state in the 3-dimensional Schrödinger theory places limits on the differentiability of \( \rho(t, .) \). Recalling equation (2.32), in spherical polar coordinates \( (r, \theta, \phi) \) the respective probability density is proportional to \( e^{-r/a_0} \). By integrating by parts in those coordinates, one checks that this density is twice weakly differentiable, but its third distributional derivative includes a Dirac delta at the origin.

In 3-dimensional models one should therefore not assume existence of weak derivatives of \( \rho(t, .) \) of order higher than two.

Regarding the constraints put on the time-dependent vector field \( \vec{v} \), one may want to—but need not—impose the almost everywhere existence of its integral curves, in turn yielding respective assumptions on \( \vec{v} \) itself. Berndl et al [18] as well as Teufel and Tumulka [170] have studied this question in the context of the Bohmian theory, where it is of relevance to the conceptual structure of the theory. One of the major assumptions in [18] is that the initial wave function should be a \( C^\infty \)-vector of the respective Hamiltonian \( \hat{H} \), i.e. an element of \( \text{dom} \hat{H}^n \) for all \( n \in \mathbb{N} \). This is a very strong assumption. In [170] Teufel and Tumulka weakened this assumption, requiring \( \vec{v} \) to be a (time-dependent) \( C^1 \)-vector field. Furthermore, they argue that ‘the general existence theory for first order ODEs that are not Lipschitz but only in some Sobolev space’ is not applicable, for the divergence of \( \vec{v} \) ‘typically diverges at the nodes of the wave function.’ We note that no examples were given to support this claim. Regarding this general mathematical theory, we refer the reader to the 2014 review article [3] by Ambrosio and Crippa. Sections 4 and 5 therein discuss the general link between the continuity equation and the integral curve equation for Sobolev vector fields and those of bounded variation, respectively. The respective sections are mainly based on [2, 49]. Section 2.2 in [105] also provides a comprehensible summary of mathematical results related on the existence theory of integral curves for the vector field \( \vec{v} \).

**Remark 3.5 (Hall’s criticism and time-invariance of \( \text{dom} \hat{H} \)).** We shall make some comments on Hall’s criticism [75] of the Bohmian theory. With respect to the discussion in this section, its importance lies in pointing out that a mathematically precise relationship between the Schrödinger equation and the respective distributional Madelung equations need not respect the Hilbert space formalism of quantum mechanics. That is, one does not need to constrain oneself to the respective \( L^2 \)-space or a closed (Hilbert) subspaces thereof—as the Dirac-von Neumann axioms indirectly call for in this instance. The subsequent discussion as well as the one in appendix A should clarify those words.

In [75] Hall gives an example that supposedly illustrates ‘the breakdown of the velocity equation’, equation (1.2) (cf p 9554 in [75]): Based on a prior work by Berry [19], he considers
the 1-dimensional single ‘particle in a box’ problem on the open interval $\Omega = (0, L)$ for some $L \in \mathbb{R}_+$ and the (equivalance class of the) constant wave function $\Psi_0$ with $\Psi_0(x) = 1/\sqrt{L}$ for all $x \in \Omega$. Berry [19] has shown that the evolution of $\Psi_0$ via the usual energy eigenfunction decomposition yields wave functions at time $t$ whose graph is a fractal for almost all $t \geq 0$.

As an attempt of resolution, both Sanz [149] and later Grübl and Penz [71] have suggested that it may be possible to construct ‘quantum trajectories’ by considering a sequence of so-called $C^\infty$-vectors, computing a sequence of such trajectories for the time-evolved wave functions, and then taking the limit in some appropriate sense. While this might be a fruitful approach to the construction of ‘quantum trajectories’ for general, ‘suitably irregular’ drift fields $\vec{v}$, we take a different perspective on the issue here.

We shall first recapitulate the mathematics of the problem on the basis of the articles by Grübl and Penz [71] and Bonneau et al [30].

First one observes that one needs to satisfy the homogeneous Dirichlet boundary condition at fixed times:

$$\forall x \in \partial \Omega: \quad \Psi(x) = 0 \iff \Psi(0) = \Psi(L) = 0. \quad (3.20a)$$

In order to account for equation (3.20a), one considers the Sobolev spaces $H^0_0(\Omega, C)$. For $k \in \mathbb{N}$ those are defined as the closure of $C^0_k(\Omega, C)$ with respect to the norm in $H^k(\Omega, C)$, i.e. $H^0_0(\Omega, C)$ consists of all elements $\Phi$ such that there exists a sequence of $(\Phi_n)_{n \in \mathbb{N}}$ in $C^\infty(\Omega, C)$ that converges to $\Phi$ in the $H^k$-norm as $n \to \infty$. Each $H^0_0(\Omega, C)$ is a linear subspace of $H^0(\Omega, C)$, the latter consisting of all equivalence classes of $C$-valued, absolutely continuous functions satisfying the boundary conditions (cf theorems 8.2 and 8.12 in [35]).

To account for the boundary condition, equation (3.20a), it is thus sufficient that $\Psi \in H^0_0(\Omega, C)$, viewing $H^0_0(\Omega, C)$ as a linear subspace of $L^2(\Omega, C)$.\footnote{We refer, for instance, to the beginning of section 8.4 in Brezis’ book [35] for an example of how homogeneous Dirichlet boundary conditions are treated within the mathematical theory of PDEs.}

Furthermore, we may interpret the derivatives in the free-body Hamiltonian $\hat{H}$ in the weak sense [30, 71], leading us to consider the linear subspace

$$\text{dom} \hat{H} = H^0_0(\Omega, C) \cap H^2(\Omega, C) \quad (3.20b)$$

of $L^2(\Omega, C)$ as a natural domain for the Hamiltonian $\hat{H}$ (cf section 3 in [71]). On this domain $\hat{H}$ is indeed self-adjoint, since it is proportional to the ‘Dirichlet Laplacian’ (cf example 1 in section X.3 of [141]).\footnote{Note that in [30] the authors considered the case $\text{dom} \hat{H} = H^0_0(\Omega, C)$ instead. However, if $\Psi \in H^0_0(\Omega, C)$, then its weak derivative $\Psi'$ is an element of $H^0_0(\Omega, C)$. Thus, we can choose an absolutely continuous representative for which $\Psi'(0) = \Psi'(L) = 0$. By considering $H^0_0(\Omega, C)$ we have therefore implicitly imposed an additional boundary condition that is not derived from any physical considerations. Accordingly, in [30] the authors find that $\hat{H}$ is not self-adjoint for this choice of domain. If one considers Grübl’s and Penz’ choice of domain, equation (3.20b), instead, one does therefore not need to answer the question which self-adjoint extensions of $\hat{H}$ on $H^0_0(\Omega, C)$ are physically preferred.}

In particular, the time evolution $U_t \Psi_0$ of Berry’s initially constant wave function $\Psi_0$ is well-defined in this sense.

Having clarified the mathematics, let us now return to Hall’s claim of the ‘breakdown of the velocity equation’ [75]: The author is correct to criticize that it is far from clear how equation (1.2) is to be mathematically interpreted for general $L^2$-wave functions. Though Berry’s example [19] is particularly illustrative, it is a well-known fact that not all $L^2$-functions are weakly differentiable. Moreover, as neither $\text{dom} \hat{H}$, $H^0_0(\Omega, C)$, or even $H^1(\Omega, C)$ are closed...
subspaces of $L^2(\Omega, \mathbb{C})$, one cannot simply resolve the issue by choosing a smaller Hilbert space. Keeping the approaches suggested in [71, 149] out of this discussion, it is therefore not possible to address the problem within the strict confines of quantum mechanics.

However, neither the Bohmian theory nor theories based on the distributional Madelung equations aim to reproduce quantum mechanics in every detail. Rather, the overarching question is whether they constitute internally consistent physical theories in agreement with empirical data within their domain of applicability.

Indeed, Berry’s example exhibits one major defect: Since $\Psi_0$ does not lie in $H^1_0(\Omega, \mathbb{C})$, it does not satisfy the boundary condition for the problem, equation (3.20a). There is thus a physical argument for its rejection as an acceptable initial condition.

On the other hand, Grübl and Penz [71] have observed that $U_t$ preserves the domain of $\hat{H}$ in the sense that for any initial wave function $\Phi_0 \in \text{dom}\hat{H}$ and any $t \in \mathbb{R}$ we have $U_t\Phi_0 \in \text{dom}\hat{H}$ here (cf proposition A.1 in appendix A; also note the result by Carlen [39] described in appendix A).

Instead of considering the time evolution on $L^2(\Omega, \mathbb{C})$—which ignores the boundary conditions—we may therefore restrict ourselves to the time evolution on the natural domain of the Hamiltonian $\hat{H}$ (see also appendix A). In that case the velocity equation, equation (1.2), is well-defined on the (distributional) support of all wave functions $\Phi_0 \in \text{dom}\hat{H}$ since $\text{dom}\hat{H}$ is a linear subspace of $H^1(\Omega, \mathbb{C})$. At least with respect to the drift field itself, there is thus no such ‘breakdown of the velocity equation’. Note that the Sobolev embedding theorem does, however, not imply that the drift field is Lipschitz continuous in $x$ (cf theorem 5.4, part II, case C’ in [1]), so that the Picard–Lindelöf theorem is not applicable to its integral curves.

It also worthy of note that $\text{dom}\hat{H}$ is dense in $L^2(\Omega, \mathbb{C})$, since this is the case for $C_0^\infty(\Omega, \mathbb{C})$. Thus, for an arbitrary $\varepsilon > 0$ there exists a $\Psi'_0 \in \text{dom}\hat{H}$ such that $\|\Psi_0 - \Psi'_0\| < \varepsilon$. That is, $\Psi'_0$ approximates Berry’s initial wave function to arbitrary precision $\varepsilon$. As $\|\Psi_0 - \Psi'_0\| = \|U_t\Psi_0 - U_t\Psi'_0\|$, the same holds for all times $t$. It is thus impossible to make an empirical argument against the restriction to the subspace $\text{dom}\hat{H}$.

To summarize this section, the current state of research does not provide a satisfactory answer to the question of what precisely is meant with the term ‘distributional Madelung equations’. Nonetheless, we hope that the general considerations expressed here contribute to further research, so that more results will be established on the general relationship between such Madelung equations and the Schrödinger equation.

4. Some explicit ‘non-quantized’ strong solutions of the 2-dimensional Madelung equations

In [174] Wallstrom claimed that the isotropic harmonic oscillator in two dimensions provides an explicit example in support of the claim that there are solutions of the Madelung equations that are not solutions of the Schrödinger equation. In the following section we will explicitly construct such strong solutions for the Madelung equations. As we shall see, their existence is not trivial. Thereafter, we analyze them from a distributional point of view.

4.1. Construction and analysis of strong solutions

To construct such ‘non-quantized’ strong solutions of the Madelung equations, consider again the time-independent Schrödinger equation in equation (2.23). As above, we apply the separation ansatz, equation (2.24), in order to obtain an angular equation, equation (2.25), and a radial equation, equation (2.26). However, we shall not constrain $\mu$ to the integers $a$ priori.
We first consider the question of how the ‘quantization’ of $\mu$, that is the condition $\mu \in \mathbb{Z}$, may be obtained when one looks for strong solutions of the Schrödinger equation.

To recapitulate the relevant statements in [174], Wallstrom argues that in this instance $\Psi$ has to be ‘single-valued’: Supposedly, the condition

$$\Psi(\rho, \phi) = \Psi(\rho, \phi + 2\pi k)$$

(4.1)

has to hold for all $k \in \mathbb{Z}$. Then, since $\Phi(\phi) = e^{\pm i \mu \phi}$ yields two linearly independent fundamental solutions to the angular equation (2.25), he then states that condition (4.1) implies that $\mu \in \mathbb{Z}$. The author then goes on to claim that, without the above condition—or another ‘quantization condition’ such as the one in equation (1.3)—one obtains (stationary) solutions of the Madelung equations that are not solutions of the Schrödinger equation.

Upon inspection, condition (4.1) turns out to be mathematically problematic: In looking for strong solutions of equation (2.23), it is important to keep in mind that $\Psi$ is actually a function defined on a subset of $\mathbb{R}^2$. Yet in this instance we consider its coordinate representation in polar coordinates, the latter being only defined on the coordinate chart codomain $(0, 1) \times (0, 2\pi)$. While the choice of this codomain is a matter of convention, bijectivity of the coordinate map demands that equation (4.1) ought not be accepted as a mathematical condition.

One may, however, ask for the global function to be continuous on the entirety of $\mathbb{R}^2$, so that for the coordinate representation $\Psi$ we require

$$\lim_{\phi \to 0} \Psi(\rho, \phi) = \lim_{\phi \to 2\pi} \Psi(\rho, \phi)$$

(4.2)

for all $\rho > 0$. Indeed, equation (4.2) then implies that $\mu \in \mathbb{Z}$.

The problematic equation (4.1) may therefore be replaced by the condition of continuity, equation (4.2), to yield the desired result. Note that this is a weaker condition than the one considered in proposition 2.1.

But is there an a priori justification for imposing continuity on the positive x-axis, equation (4.2)? At least under certain regularity conditions away from this set, quantum mechanics does provide such a reason:

Consider the Hamiltonian $\hat{H}$ defined by equation (2.23) in polar coordinates. If we require $\hat{H}$ to be symmetric on $C' := \text{dom} \hat{H} \cap C^2(\mathbb{R}_+ \times (0, 2\pi), \mathbb{C})$, then for all $\Psi, \Phi$ in $C'$ we have that

$$\int_0^{2\pi} d\phi \int_0^\infty \rho d\rho \Psi^*(\rho, \phi) \big( \hat{H} \Phi \big) (\rho, \phi) = \int_0^{2\pi} d\phi \int_0^\infty \rho d\rho \big( \hat{H} \Psi \big)^* (\rho, \phi) \Phi (\rho, \phi).$$

(4.3)

Looking at the angular part of $\hat{H}$ in equation (4.3) only, integration by parts shows that any $\Psi \in \text{dom} \hat{H}$ does indeed need to satisfy equation (4.2) as well as

$$\lim_{\phi \to 0} \frac{\partial \Psi}{\partial \phi} (\rho, \phi) = \lim_{\phi \to 2\pi} \frac{\partial \Psi}{\partial \phi} (\rho, \phi)$$

(4.4)

for all $\rho \in \mathbb{R}_+$.

Therefore, at least on $C'$ symmetry of $\hat{H}$ implies equation (4.2) and thus ‘quantization’ of $\mu$.

Let us move on to Wallstrom’s claim that ‘explicit solutions to the Madelung equations not satisfying the Schrödinger equation may be constructed’ [174].

We first observe that $\mu$ has to be real for wave functions satisfying the separation ansatz, equation (2.24), if we also ask for the respective probability density to be continuous on

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53 The definition of a coordinate map/chart may be found in any textbook on topological or differentiable manifolds.
$\mathbb{R}^2 \setminus \{0\}$: $\Phi(\phi) = e^{\pm i\mu \phi}$ for $\mu \in \mathbb{C}$ are the only linearly independent solutions of the angular equation, equation (2.25). Due to the separation ansatz, equation (2.24), $\Phi \mapsto |\Phi|^2 = e^{\pm 2 \text{Im}(\mu) \phi}$ has to extend to a continuous function in Cartesian coordinates. Thus $\mu \in \mathbb{R}$, indeed.

Similarly, $E$ has to be real, as otherwise the factor $e^{-iE \tau / \hbar}$ in the $(t$-dependent) wave function would contradict probability conservation.

Using reality of $\mu$, one easily checks that the corresponding drift field $\vec{v}$ can be smoothly extended to $\mathbb{R}^2 \setminus \{0\}$ and for $\Phi(\phi) = e^{\mp i\mu \phi}$ the field $\vec{v}$ is given by equation (2.28) above, with an additional minus sign for $\Phi(\phi) = e^{-i\mu \phi}$.

In the absence of other constraints, this is the mathematical argument that underlies Wallstrom’s claim above.

We remark that, even if one looks for strong solutions of the Madelung equations, the drift field $\vec{v}$ is ‘irregular’ in the sense of having a singularity at the origin. Moreover, as all other singularities and nodes of the density and the drift field $\vec{v}$ are radial, none of the connected components of $\text{dom} \vec{v}$ are simply connected (unless we make the cut at e.g. $\phi = 0$). The topological condition required in the statement of theorem 3.2 in [140], which shows the (local) equivalence of the Schrödinger equation and the Madelung equations in the strong sense, is therefore violated here.

Still, the situation is not quite as simple as Wallstrom suggested (cf section III in [174]): In order to solve the radial equation (2.26), we also need to satisfy the radial integrability condition

$$\int_0^\infty |R(\rho)|^2 \rho \, d\rho < \infty. \quad (4.5)$$

As we will show in the proof of theorem 4.1 below, it is not true that all solutions of the radial equation with $\mu \in \mathbb{R}$ automatically satisfy equation (4.5). In particular, that non-trivial solutions exist for non-integer $\mu$ is not given a priori, but needs to be checked explicitly. This point does not seem to have been taken into account in [174].

Nonetheless, the following theorem shows that strong solutions of the radial equation with $\mu \in \mathbb{R} \setminus \mathbb{Z}$ that also satisfy the integrability condition do indeed exist.

**Theorem 4.1.** For arbitrary $\mu \in \mathbb{R} \setminus \mathbb{Z}$, $E \in \mathbb{R}$ and up to a constant factor, all non-trivial (strong) solutions $R$ of equation (2.26), that are integrable in the sense of equation (4.5), fall under one of the following two cases:

i) For $a \in \mathbb{R} \setminus \mathbb{N}_0$ and $\mu \in (-1, 0) \cup (0, 1)$ we have

$$R(\rho) = \rho^a U \left(a, \mu + 1; \frac{m \omega}{\hbar} \rho^2 \right) e^{-\frac{m \omega}{\hbar} \rho^2} \quad (4.6a)$$

with

$$E = \hbar \omega (-2a + \mu + 1). \quad (4.6b)$$

ii) For $n \in \mathbb{N}_0$ and $\mu \in (-1, \infty) \setminus \mathbb{Z}$ we have

$$R(\rho) = \rho^n L_n^\mu \left( \frac{m \omega}{\hbar} \rho^2 \right) e^{-\frac{m \omega}{\hbar} \rho^2} \quad (4.6c)$$

54 Do note, however, the analogy to the example of differentiating a step function and smoothly extending its derivative, which we discussed at the beginning of section 3.
It is important to keep in mind that, due to linearity of the Schrödinger equation, superpositions of wave functions of same energy $E$ constructed from the solutions in theorem 4.1 also need to be considered. We shall not consider the question here in which cases solutions constructed in this manner also give rise to strong solutions of the Madelung equations, but only look at those arising from wave functions satisfying the separation ansatz, equation (2.24).

**Corollary 4.2.** The functions $\varrho$ and $\vec{v}$, defined as follows, provide smooth, stationary solutions to the 2-dimensional Madelung equations with potential

$$V: \mathbb{R}^3 \to \mathbb{R} : \quad (t,x,y) \mapsto V(t,x,y) = \frac{m}{2} \omega^2 (x^2 + y^2),$$

(4.7a)

satisfying the normalization condition

$$\forall t \in \mathbb{R}: \quad 1 = \int_{\mathbb{R}^2} \text{d}x \text{d}y \varrho(t,x,y).$$

(4.7b)

i) For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, $\mu \in (-1,0) \cup (0,1)$, $(t,x,y)$ in

$$\left\{(t,x,y) \in \mathbb{R}^3 \mid (x,y) \neq 0 \quad \text{and} \quad U \left( a, \mu + 1; \frac{m \omega}{\hbar} (x^2 + y^2) \right) \neq 0 \right\},$$

(4.7c)

and $A_{\alpha\mu} \in \mathbb{R}^+$ chosen such that equation (4.7b) is satisfied, set

$$\varrho(t,x,y) = \frac{1}{A_{\alpha\mu}} (x^2 + y^2)^{\mu} \left( U \left( a, \mu + 1; \frac{m \omega}{\hbar} (x^2 + y^2) \right) \right)^2 e^{-\frac{m \omega}{\hbar} (x^2 + y^2)}$$

(4.7d)

and

$$\vec{v}(t,x,y) = \pm \frac{\mu \hbar}{m} \frac{1}{x^2 + y^2} \left( -\frac{y}{x} \right).$$

(4.7e)

ii) For $n \in \mathbb{N}_0$, $\mu \in (-1,\infty) \setminus \mathbb{Z}$, $(t,x,y)$ in

$$\left\{(t,x,y) \in \mathbb{R}^3 \mid (x,y) \neq 0 \quad \text{and} \quad L_n^\mu \left( \frac{m \omega}{\hbar} (x^2 + y^2) \right) \neq 0 \right\},$$

(4.7f)

and $A_{n\mu} \in \mathbb{R}^+$ chosen such that equation (4.7b) is satisfied, set

$$\varrho(t,x,y) = \frac{1}{A_{n\mu}} (x^2 + y^2)^{\mu} \left( L_n^\mu \left( \frac{m \omega}{\hbar} (x^2 + y^2) \right) \right)^2 e^{-\frac{m \omega}{\hbar} (x^2 + y^2)}$$

(4.7g)

and $\vec{v}(t,x,y)$ as above.

□

As Wallstrom anticipated correctly [174], from a purely mathematical perspective the solutions of the respective Madelung equations in corollary 4.2 are perfectly acceptable—at least
if we look at strong solutions only. Though the densities $\rho$, as given by equation (4.7d) or equation (4.7g), fail to be differentiable at the origin, the origin is not in the domain of $\phi(t, \cdot)$ due to the singularity of $\vec{v}$ there and the requirement that the domains of $\rho$ and $\vec{v}$ coincide (cf p 1351 in [140]).

From a physical point of view, it would, of course, be more appropriate to consider a physically more realistic model in 3 dimensions, such as the one of the ‘hydrogen-like atom’ considered in section 2.3. For without the ultimate ability to argue on the basis of empirical data, the discussion of whether the solutions in corollary 4.2 are physically acceptable is misplaced. Do note, however, that such data is commonly obtained from spectroscopy, while the 3-dimensional model in section 2.3 describes only hydrogenlike atoms in the absence of external radiation.

Nonetheless, apart from the fact that for the solutions in corollary 4.2 $\mu$ is not ‘quantized’, it is noteworthy that there exists no minimum of the energy $E$ in either one of the two sets of solutions. For the solutions constructed from point ii) in theorem 4.1, there is only an infimum of the energy, $E = 0$, but no minimum. For solutions from point i), $E$ is not even bounded from below. There is no doubt that most physicists would find this fact alone objectionable. That $E$ is indeed the physical energy of the system, even in theories based on the Madelung equations (see e.g. equation (4.22) in [140]), can be easily derived from the so called ‘Hamilton–Jacobi–Madelung equation’.

Whether analogous ‘non-quantized’ solutions exist in more physical models, as for instance in the aforementioned model of hydrogen-like atoms, is beyond the scope of this article.

4.2. Analysis of strong solutions from the point of view of distributions

In this section we analyze the solutions of theorem 4.1 and corollary 4.2 from a distributional perspective. Though, for a lack of an established weak formulation of the Madelung equations (cf section 3), this analysis will be necessarily incomplete, the results here may nonetheless be of use for providing an answer to the precise relationship between the Schrödinger equation and the Madelung equations in this distributional framework.

We begin by recalling that the drift fields in corollary 4.2—which were computed using strong derivatives and smooth extensions—do not satisfy the third Madelung equation, equation (1.1c), in the distributional sense. We refer back to section 2.3, in particular definition 2.8 and proposition 2.9. There we have discussed the option of requiring the third Madelung equation to hold in a distributional sense as well, with the caveat that this would require further physical justification.

Clearly, if we do impose this condition on the strong solutions in corollary 4.2, they will have to be thrown out—just like the ones for integer $\mu$.

For the case that $\mu$ is an integer, however, we noted that we can still get stationary solutions of the Madelung equations for every admissible energy by considering the respective real-valued wave functions instead: That is, we choose

$$\Psi(\rho, \phi) = R(\rho) \sin(\mu \phi), \quad \text{or}$$

$$\Psi(\rho, \phi) = R(\rho) \cos(\mu \phi)$$

with the respective function $R$ and the trivial $t$-dependence omitted.$^{55}$

Contrarily, if $\mu$ is real yet not an integer, we are faced with the following problem.

$^{55}$ Still, this approach comes at the price of requiring the angular momentum around the origin to vanish.

41
Proposition 4.3. Let $R$ and $\mu$ be as defined by points i) or ii) in theorem 4.1. Set

$$\text{dom}\,\Psi = \{ (x,y) \in \mathbb{R}^2 | x < 0 \text{ whenever } y = 0 \}.$$  \hspace{1cm} (4.9)

Then the following holds.

i) The function $\Psi$ with values

$$\Psi(x,y) = R(\sqrt{x^2 + y^2}) \exp(i\mu\arg(x+iy))$$  \hspace{1cm} (4.10)

for $(x,y) \in \text{dom}\,\Psi$ is not weakly differentiable.

ii) If $\Psi$ is given by

$$\Psi(x,y) = R(\sqrt{x^2 + y^2}) \cos(\mu\arg(x+iy))$$  \hspace{1cm} (4.11)

for any $(x,y) \in \text{dom}\,\Psi$, then it is also not weakly differentiable.

iii) If $\Psi$ is given by

$$\Psi(x,y) = R(\sqrt{x^2 + y^2}) \sin(\mu\arg(x+iy))$$  \hspace{1cm} (4.12)

for any $(x,y) \in \text{dom}\,\Psi$, then it is weakly differentiable if and only if $\mu$ is also half-integer. \hfill \Box

We expect that $\Psi$ in proposition 4.3iii) is not twice weakly differentiable (with respect to $y$), even if $\mu$ is half-integer. A rigorous proof would be rather laborious, however, as in some cases one simultaneously needs to handle the lack of integrability of the strong derivative on $\text{dom}\,\Psi$ and the discontinuity due to the cosine.

Regarding the respective densities $\varrho$ in points ii) and iii), the discontinuities in $\Psi$ are not removed by computing $|\Psi|^2$, so that we also lose weak differentiability in this manner. For the special case of $\mu \in \mathbb{Z}/2$ in point iii), we also expect that $\varrho$ is only once weakly differentiable.

So requiring both irrotationality of $\vec{v}$ in the distributional sense and weak differentiability up to second order of the $L^1$-function $\varrho$ would exclude the aforementioned solutions, equations (4.9) and (4.8), for non-integer $\mu$, while still allowing for the real-valued solutions from equation (4.8) for integer $\mu$. As discussed in section 3, requiring weak differentiability of $\varrho$ (and $\Psi$) up to second order is a reasonable request—as long as it is compatible with the respective dynamics (see also remark 3.5).

We have to leave unaddressed the question of whether this result generalizes to general stationary superpositions of the wave functions determined by theorem 4.1.

Surely, one may object to the exclusion of the quasi-irrotational, strong solutions of the Madelung equations for integer $\mu$, since these correspond to commonly accepted solutions of the respective Schrödinger equation. Yet the need for these solutions—that is for non-vanishing angular momentum—requires a physical argument. Moreover, the respective drift fields are ‘pathological’ in the sense that they are not locally $L^2$-integrable (cf remark 3.2).

In the remainder of this section we shall consider the question of what the mathematical theory of quantum mechanics has to say about the wave functions from theorem 4.1. Though we have already commented on this matter in section 4.1 to some degree, in this section we will not make any a priori assumption on the regularity of the wave functions apart from those imposed by the theory.
We shall follow Berezin and Shubin in recalling the mathematical situation for the problem (cf section 1.8 in [17]):

For the 1-dimensional problem the eigenfunctions \( \Phi_n \) of the respective Hamiltonian belong to the space of Schwartz functions \( S(\mathbb{R}, \mathbb{C}) \), considered as a subspace of the Hilbert space \( L^2(\mathbb{R}, \mathbb{C}) \). The corresponding energy eigenvalues are

\[
E_n = \hbar \omega \left( n + \frac{1}{2} \right). \tag{4.13}
\]

Due to equation (4.13), the Hamiltonian is well-defined on the domain

\[
\left\{ \Psi \in L^2(\mathbb{R}, \mathbb{C}) \mid \sum_{n=0}^{\infty} |\langle \Phi_n, \Psi \rangle n|^2 \text{ converges} \right\}. \tag{4.14}
\]

For this choice of domain the operator is indeed self-adjoint.

For the 2-dimensional case the appropriate Hilbert space is \( \mathcal{H} = L^2(\mathbb{R}^2, \mathbb{C}) \). As, roughly speaking, the Hamiltonian

\[
\hat{H} = -\frac{\hbar^2}{2m} \Delta + \frac{m}{2} \omega^2 (x^2 + y^2) \tag{4.15}
\]

is the sum of two Hamiltonians for the 1-dimensional problem, we consider the energy eigenfunctions

\[
\Phi_{n_1,n_2} : (x, y) \mapsto \Phi_{n_1}(x)\Phi_{n_2}(y) \tag{4.16}
\]

with corresponding energy eigenvalue \( E_{n_1,n_2} = E_{n_1} + E_{n_2} \). Accordingly, on the domain

\[
\text{dom}\hat{H} = \left\{ \Psi \in L^2(\mathbb{R}^2, \mathbb{C}) \mid \sum_{n_1,n_2=0}^{\infty} |\langle \Phi_{n_1,n_2}, \Psi \rangle (n_1 + n_2)|^2 \text{ converges} \right\} \tag{4.17}
\]

the operator \( \hat{H} \) is self-adjoint.

Since the Schrödinger time evolution is well-defined for any \( \Psi \in \mathcal{H} \), Smolin was correct to note that solutions to the Schrödinger equation need not be continuous but only square-integrable (cf section IV in [162]). Still, one has to ask if the wave functions from section 4.1 follow the time evolution determined by the respective Hamiltonian—i.e. whether they are indeed ‘stationary states’.

The next proposition answers this question in the negative.

**Proposition 4.4.** Let \( R, \mu, \) and \( E \) be given by points i) or ii) in theorem 4.1. Let \( \Psi \) have values

\[
\Psi_t(x, y) = R(\sqrt{x^2 + y^2}) e^{i(\mu \arg(x+iy) - Et/\hbar)} \tag{4.18}
\]

for all \( t \in \mathbb{R} \) and \( (x, y) \) in the set defined by equation (4.9).

Then

i) for all \( t \in \mathbb{R} \) (the equivalence class of) \( \Psi_t \) is an element of \( \mathcal{H} = L^2(\mathbb{R}^2, \mathbb{C}) \), and

ii) (the family of equivalence classes of) \( \Psi \) is not an \( L^2 \)-solution of the Schrödinger equation

for the operator \( \hat{H} \) on \( \text{dom}\hat{H} \), as given by equations (4.15) and (4.17), respectively. □

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56 In footnote 2 of [172] Smolin’s observation was criticized, implying that one should not allow for discontinuous wave functions. The necessity to look at the problem from the perspective of the mathematical theory of quantum mechanics resolves those concerns (cf section 2.2).

57 That is, if \( U_t \) is the respective time evolution operator, then the curves \( t \mapsto \Psi_t \) and \( t \mapsto U_t \Psi_0 \) in \( \mathcal{H} \) do not coincide.
In a more colloquial language proposition 4.4 states the following: Even though the wave functions from section 4.1 are admissible as ‘initial’ wave functions, the corresponding ‘stationary solutions’ are not quantum-mechanical solutions of the respective Schrödinger equation. Wallstrom’s argument, stated in the beginning of section 4.1, therefore requires the following modification to hold true: The solutions from section 4.1 do not satisfy the Schrödinger equation in the quantum-mechanical sense, while still yielding strong solutions of the respective Madelung equations.

Remark 4.5 (Further clarification of Wallstrom’s first objection). We shall comment on two works by Fritsche and Haugk [62, 63] that are of relevance in this context. According to [45], those provide ‘[d]etailed rebuttals of Wallstrom’s argument’. We find that this is not the case.

In [62] the authors state the following:

As \( \varrho(\vec{r}, t) \) and \( \vec{v}(\vec{r}, t) \) will in general be unique (and smooth) functions of \( \vec{r} \) we require \( \Psi(\vec{r}, t) \) to have the same property.

[Notation adapted]

This clearly does not address Wallstrom’s criticism, for his argument is precisely that there exist strong solutions of the Madelung equations that are not solutions of the Schrödinger equation (in the quantum-mechanical sense).

The relevant argument in section VIII of [63] is more substantial: Therein the authors consider the 3-dimensional 1-body Schrödinger equation with spherically symmetric potential and try to argue from the linearity of the Schrödinger equation that the respective magnetic quantum numbers have to be integer. While their argument is incomplete—it fails for the case that the functions \( F \) they consider are orthogonal—it may be viewed as a precursor to proposition 4.4ii) above. However, even if the respective statement in [63] were corrected and appropriately generalized, it would fail to address Wallstrom’s criticism.

However, as argued in section 2.2 and elaborated upon in remark 3.5, the question is not how the Schrödinger equation in the quantum-mechanical sense relates to the Madelung equations in the strong sense. Rather, the question is how the Schrödinger equation is related to an appropriate weak formulation of the Madelung equations within the modern theory of PDEs.

As we observed in remark 3.5 above, the quantum-mechanical space of initial wave functions, \( L^2(\mathbb{R}^2, \mathbb{C}) \) in this instance, may be too large to obtain an equivalence between the two systems of PDEs and we might need to restrict ourselves to the domain of the Hamiltonian instead (cf proposition A.1 in appendix A).

We may therefore ask, whether the solutions \( \Psi \) found in section 4.1 are contained in \( \text{dom} \hat{H} \). To answer the question, one would need to compute all coefficients \( \langle \Phi_{n,m}, \Psi \rangle \) and check whether the respective series ‘\( \hat{H}\Psi \)’ converges or not. Instead we shall provide the reader with the following sufficient condition (which might even be necessary).

**Lemma 4.6.** Let \( \hat{H} \) with domain \( \text{dom} \hat{H} \) be given by equations (4.15) and (4.17), respectively. Then the set of equivalence classes

\[
\left\{ [\Psi] \in \mathcal{H}^2(\mathbb{R}^2, \mathbb{C}) \mid \forall \Psi' \in [\Psi] : \left. (x, y) \mapsto (x^2 + y^2) \Psi'(x, y) \right\} \subset L^2(\mathbb{R}^2, \mathbb{C}) \right\}
\]

is a linear subspace of \( \text{dom} \hat{H} \). □
By proposition 4.3i), the ‘non-quantized’ \( \Psi \) found in section 4.1 do not meet the sufficient condition of lemma 4.6 for being contained in \( \text{dom} \hat{H} \). Since the condition is arguably weak, it is at least doubtful whether the necessary condition is met.

From proposition 4.3i) it does follow, however, that the (initial values of the) ‘non-quantized’ solutions are not in \( H^1(\mathbb{R}^2, \mathbb{C}) \). They are therefore not acceptable, if we consider the 2-dimensional analog of the result by Gasser and Markowich [67] discussed in section 3.

Furthermore, proposition 4.3i) shows that is not even clear how to define the drift field for the respective wave functions. The procedure of computing it from equation (1.2) almost everywhere in the strong sense and then smoothly extending the result to \( \mathbb{R}^2 \setminus \{0\} \) is not acceptable in this context, for one has to consider distributional derivatives instead and, by propositions 4.3 and i), the gradient of \( \Psi \) is not a regular distribution.

As argued in section 2.2, it is precisely this error in not approaching the relation between the Schrödinger equation and the Madelung equations from the perspective of the modern theory of PDEs that constitutes the biggest gap in Wallstrom’s argument. Establishing a mathematically rigorous relationship between the two systems of PDEs requires a specification of the respective function spaces as well as a clarification of how the equations ought to be mathematically interpreted. Without such, the claim that the Madelung equations and the Schrödinger equation are inequivalent is devoid of mathematical content.

5. Conclusion

The central contributions of this work were already listed in section 1, so we shall not repeat those here. Instead, we shall provide the reader with some general comments as well as a few clarifying remarks on the related literature.

Most importantly, we wish to emphasize that this work does not provide a full resolution of Wallstrom’s objections. What it does provide is a clarification to what extent those objections are justified as well as various possible avenues for finding such a resolution.

In this respect, the central message is that the relation between the Schrödinger equation and the Madelung equations needs to be clarified in the context of the mathematical theory of distributions (see [4, 5, 39, 67]). Interpreting the respective PDEs in the strong sense only would already defy the theory of quantum mechanics, so we should not expect this approach to take us beyond established theory. In relating the two systems of PDEs the mathematical foundations of quantum mechanics need to be taken into account, even if it is not the aim of a given weak formulation of the Madelung equations to reproduce the quantum-mechanical Schrödinger theory in every aspect.

Therefore, at this point in time the task of resolving the controversy is primarily one of functional analysis. Given the geometric nature of the equations, however, prior works that took a more geometric approach might be of use as well. We refer to [101–104, 108, 114, 140, 173]. Similarly, one should note that there exist approaches in the ‘Lagrangian picture’ [51, 87, 138, 155, 166] and approaches based on the calculus of variations [64, 72, 108, 183, 184].

Based on our research, the vast amount of articles citing Wallstrom’s works on Takabayasi’s condition [174, 175] confirm that his objections have been viewed as a discreditation of theories based on the Madelung equations—thus confirming the statement by de la Peña et al [45]

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58 The existence of this picture requires ‘Bohmian trajectories’ to be well-defined. See e.g. [71].
quoted in the introduction. As Wallstrom’s objections failed to properly account for the mathematical structure of quantum mechanics, we hope to have convinced the reader that casting such a judgment on the viability of the Madelung equations as fundamental laws of nature is premature at this point.

Still, the view that Wallstrom effectively ended the scientific discussion deserves some sympathy, for there have been several works in the literature that have attempted to address Wallstrom’s objections without giving a full convincing resolution:

As stated in section 1, addressing Wallstrom’s objections on the level of stochastic processes only [45, 47, 48, 70, 72, 184] is unlikely to convince the wider physics community.

The proposals made in [40, 42, 62, 63, 82, 82] that attempt to address Wallstrom’s objections on the phenomenological level we view as largely unsuccessful: The argument in [40] is misplaced, for the space $C_\alpha$ the authors define only corresponds to smooth functions on the circle, if $\alpha \in \mathbb{Z}$. Still, the authors deserve credit for discussing Wallstrom’s objections in relation to the actual mathematical formalism of quantum mechanics. In [82] the respective authors state that ‘In bound states, the superposition principle by itself guarantees single-valuedness.’ and that this is supposedly ‘illustrated’ in [91]. While no argument was given that could establish this general assertion, the argument seems to be similar to the one given in [63]. The latter we already addressed in remark 4.5. Finally, in [42] the authors tried to use an (outdated) argument in a 1939 article by Pauli [133] to exclude wave functions such as the ones found in section 4.1. The counterargument voiced in remark 4.5 applies here as well.

Apart from the suggestions of resolution provided in this article, potentially promising resolutions of the matter have also been given by Zak [185] and Loffredo and Morato [115]: Zak argued that the ‘solutions of the Schrödinger and Madelung equations may have different criteria of instability’. While the stability of solutions of the Schrödinger equation follows from linearity, the Madelung equations are inherently non-linear, independent of the chosen weak formulation. Stability is one of the criteria of well-posedness in the sense of Hadamard, so that Zak’s suggestion may be of relevance in the mathematical study of a given weak formulation of the Madelung equations. The suggestion by Loffredo and Morato is more radical in that the authors suggest a modification of the first two Madelung equations, equations (1.1a) and (1.1b), for the case that the drift field has non-vanishing vorticity. We have discussed this suggestion in section 2.3, noting, in particular, that it is indeed difficult to justify the irrotationality condition (1.1c) on physical grounds.

We shall finish with the remark that nowadays there is significant historical evidence that with regards to the Madelung equations the debate on the foundations of quantum theory has had a strongly ideological component [16, 44, 59]. We hope that our work contributes to shifting the debate more towards mathematical facts and a discussion of the actual empirical evidence.

**Data availability statement**

No new data were created or analysed in this study.

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Appendix A. Invariance of domain of self-adjoint Hamiltonians under time evolution

The following result is of potential importance for determining the relevant function spaces of a given weak formulation of the Madelung equations. We make no claims of originality, since it is commonly used as an argument in proving Stone’s theorem (cf theorem VIII.8 in [142] and theorem 23.2 in [23]) and it was also used in [71]. Nonetheless, its potential relevance justifies an explicit statement.

Proposition A.1. Let $\hat{H}$ be a self-adjoint operator with domain $\text{dom}\hat{H}$ in a Hilbert space $\mathcal{H}$. Let $t \mapsto U_t = \exp(-i\hat{H}/\hbar)$ be the corresponding strongly continuous one-parameter unitary group.

Then for all $t \in \mathbb{R}$ we have

$$U_t(\text{dom}\hat{H}) = \text{dom}\hat{H}.$$  

□

While one can use commutativity of $\hat{H}$ and $U_t$ to show the assertion (cf corollary 3.16 in [137]), we prove it here as a corollary of theorem VIII.7 in [142].

Proof. Denote by $\| \cdot \|$ the norm on $\hat{H}$. For any $\Psi_0 \in \text{dom}\hat{H}$ and $t \in \mathbb{R}$ set $\Psi_t = U_t\Psi_0$. Observe that for all $\varepsilon > 0$ we have

$$\left\| \frac{U_t\Psi_0 - \Psi_0}{\varepsilon} + i\hat{H}\Psi_0/\hbar \right\| = \left\| \frac{U_t\Psi_t - \Psi_t}{\varepsilon} + iU_t\hat{H}\Psi_0/\hbar \right\|. \quad (A.1)$$

By point (c) in the aforementioned theorem, the limit on the left hand side as $\varepsilon \to 0$ is 0. By point (d) therein and equality with the right hand side, $\Psi_t$ is in $\text{dom}\hat{H}$. □

In section 3 we noted that in quantum mechanics the Schrödinger equation is generally interpreted in a formal sense, i.e. it merely specifies the time evolution operator $U_t = \exp(-i\hat{H}/\hbar)$ defined on $\mathcal{H}$ for all $t \in \mathbb{R}$.

In the following we show that proposition A.1 allows for another mathematical interpretation of equation (2.13).

Let $\Psi : \mathbb{R} \to \mathcal{H} : t \mapsto \Psi_t$ be any continuous curve in $\mathcal{H}$. If for every $t \in \mathbb{R}$ there exists a $\Phi_t$ in $\mathcal{H}$ such that

$$\lim_{\varepsilon \to 0} \left\| \frac{\Psi_{t+\varepsilon} - \Psi_t}{\varepsilon} - \Phi_t \right\| = 0,$$  

(A.2)

then $\Phi_t$ may be understood as a derivative of $\Psi$ at $t$ and we may hence use the notation $\partial\Psi_t/\partial t$ for $\Phi_t$.

The Schrödinger equation, equation (2.13), may then be understood as stating that for a given initial state $\Psi_0 \in \mathcal{H}$ there exists a continuous curve $t \mapsto \Psi$ in $\mathcal{H}$ with $\Psi_{t=0} = \Psi_0$ such that for all $t \in \mathbb{R}$ the following holds:

$$\lim_{\varepsilon \to 0} \left\| i\hbar \frac{\Psi_{t+\varepsilon} - \Psi_t}{\varepsilon} - \hat{H}\Psi_t \right\| = 0.$$  

(A.3)
Using theorem VIII.7 in [142] in conjunction with proposition A.1 above, one shows that for any \( \Psi_0 \in \text{dom } \hat{H} \) the curve \( t \mapsto \Psi_t = U_t \Psi_0 \) indeed satisfies the Schrödinger equation in the above sense. Moreover by point (d) in the theorem, the derivative of this curve exists only if \( \Psi_0 \) is an element of \( \text{dom } \hat{H} \) — i.e. \( \Psi_0 \in \mathcal{H} \) is not sufficient in this formulation of the Schrödinger equation.

Apart from \( \text{dom } \hat{H} \), Carlen’s results [39] indicate that another, larger invariant subspace of \( \mathcal{H} \) may be of relevance to a weak/distributional formulation of the Madelung equations: the so-called form domain of \( \hat{H} \). Roughly speaking, this is the largest linear subspace \( Q(\mathcal{H}) \) of \( \mathcal{H} \) on whose product \( Q(\hat{H}) \times Q(\hat{H}) \) the quadratic form \( (\Phi, \Psi) \mapsto \langle \Phi, \hat{H} \Psi \rangle \) can be sensibly defined. We refer to example 2 in section VIII.6 of [142] or section 9.3 in [46] for a rigorous definition. For Rellich class potentials in \( \mathbb{R}^3 \) (cf footnote 23) the form domain of the respective Hamiltonian \( \hat{H} \) is \( H^1(\mathbb{R}^3, \mathbb{C}) \), and Carlen was able to show that for such \( \hat{H} \) a weak continuity equation holds thereon (cf theorem 2.1(iii) in [39]).

### Appendix B. Proofs

#### B.1. Proof of proposition 2.1

Under the respective assumptions, \( \vec{v} \) is well-defined and the integral exists. Setting \( S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \), consider the \( C^1 \)-function

\[
Q: D \to S^1: x \mapsto Q(x) = \Psi(x)/|\Psi(x)|. \tag{B.1}
\]

Now restrict \( Q \circ \gamma \) to \( (a, b) \) to obtain the map \( \xi, \xi \) is a \( C^1 \)-map between manifolds. Arguing as in the proof of theorem 17.35 in [111], we may apply Sard’s theorem to pick a regular value \( z_0 \) of \( \xi \) in \( S^1 \) (cf theorem 1.5.18 in [146]). Then \( \xi^{-1}(\{z_0\}) \) is a (possibly empty) properly embedded 0-dimensional submanifold of \( (a, b) \) (cf corollary 1.8.3 in [146], proposition A.53(c), as well as the proof of corollary 5.14 in [111]). Thus, \( \xi^{-1}(\{z_0\}) \) is a compact 0-dimensional manifold and, as such, it is finite (or empty). So we may write \( \xi^{-1}(\{z_0\}) = \bigcup_{j=0}^{N} \{ t_j \} \) with \( N \in \mathbb{N}_0 \) and \( t_j < t_{j+1} \) for all \( j \in \{1, \ldots, N-1\} \), the case \( N = 0 \) referring to \( \xi^{-1}(\{z_0\}) = \emptyset \).

We shall make a branch cut at \( z_0 \in S^1 \) to redefine the complex logarithm: For \( \varphi_0 \in [0, 2\pi) \) let \( z_0 = e^{i\varphi_0} \) and then

\[
\ln: S^1 \to [\varphi_0, \varphi_0 + 2\pi) \subset \mathbb{C} \tag{B.2}
\]

is a (‘single-valued’) complex logarithm on the manifold \( S^1 \), discontinuous at \( z_0 \) and smooth otherwise.

Setting \( t_0 = a \) and \( t_{N+1} = b \), a straightforward calculation shows that the left hand side of equation (1.3) is equal to

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{\nabla Q}{Q} \cdot d\vec{r} = \frac{1}{2\pi i} \sum_{j=0}^{N+1} \int_{t_j}^{t_{j+1}} \frac{d}{dt} \left( \ln(Q \circ \gamma(t)) \right) dt \tag{B.3}
\]

\[
= \frac{1}{2\pi i} \sum_{j=0}^{N+1} \left( \lim_{t \to t_j^+} \ln(Q \circ \gamma(t)) - \lim_{t \to t_j^-} \ln(Q \circ \gamma(t)) \right). \tag{B.4}
\]

59 Also note that the Friedrichs extension of a positive symmetric operator is defined via a quadratic form of this type (cf theorem X.23 in [141]).
As $z_0$ is a regular value of $\xi$, for each $j \in \{1, \ldots, N\}$ the derivative of $Q \circ \gamma$ at $t_j$ is non-zero and its sign determines the value of the respective one-sided limit in equation (B.4). If it is positive, the two summands with the limits $t \to t_j^+$ together yield $2\pi i$. Else they yield $-2\pi i$. Regarding the endpoints, if $(Q \circ \gamma)(a) = z_0$, we may apply the same argument since $\gamma(a) = \gamma(b)$. Otherwise, due to continuity of the logarithm on $\mathbb{S}^1 \setminus \{z_0\}$, the respective limits will cancel each other. The assertion follows by summing all terms.

B.2. Proof of proposition 2.9

Using polar coordinates $(\rho, \phi)$, one easily shows that the (Euclidean) components of $\vec{v}$ are locally integrable.

For all $\varphi \in \mathcal{D}$ we find that $(\text{curl} \, \vec{v})(\varphi)$ equals

$$-rac{\mu \hbar}{m} \int_{\rho_1}^{\rho_2} \int_{\phi_1}^{\phi_2} \left( \frac{\cos \phi}{\rho} \left( \cos \phi \frac{\partial \varphi}{\partial \rho}(\rho, \phi) + \sin \phi \frac{\partial \varphi}{\partial \phi}(\rho, \phi) \right) + \sin \phi \frac{\partial \varphi}{\partial \phi}(\rho, \phi) \right) d\phi d\rho$$

Thus

$$(\text{curl} \, \vec{v})(\varphi) = -\frac{\mu \hbar}{m} \int_{\rho_1}^{\rho_2} \int_{\phi_1}^{\phi_2} \left( \frac{2\pi \mu \hbar}{m} \varphi(0) = \frac{2\pi \mu \hbar}{m} \delta_0(\varphi) \right).$$

The last statement follows from the fact that the Dirac delta is not a regular distribution.

B.3. Proof of corollary 2.11

The first two components in equation (2.34a) vanish, due to the fact that $v^1$ and $v^3$ do not depend on $z$, $v^3 = 0$, and due to the (generalized) fundamental theorem of calculus (cf theorem 8.2 in [35]). The third component follows from direct comparison of the (Cartesian) integral with the one from proposition 2.9 above.

It remains to prove that there does not exist any locally $L^1$-integrable function $\chi$ such that for all $\varphi \in \mathcal{D}$ we have

$$\xi(\varphi) = -\int_{\mathbb{R}^3} \chi \varphi^3. \quad (B.7)$$

To simplify notation, we write $\xi(\varphi)$ for the above expression, so that $\xi$ is viewed as a distribution acting on an $\mathbb{R}$-valued test function $\varphi$ instead.

Consider a sequence of bump functions $(\chi_k)_{k \in \mathbb{N}}$ in $C^\infty_0(\mathbb{R}, \mathbb{R})$ such that for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$ we have $0 \leq \chi_{k+1}(x) \leq \chi_k(x)$ and $\chi_k(0) = 1$. Furthermore, as $k \to \infty$ we require that $\chi_k(x)$ tends to 1 for $x = 0$ and to 0 else. Such a sequence exists. Employing a standard line of reasoning, we shall use this sequence to construct a contradiction.

Let $\eta$ be any real-valued, positive, nonzero function such that for all $k \in \mathbb{N}$ the function

$$\varphi_k : (x, y, z) \mapsto \varphi_k(x, y, z) = \chi_k(x) \eta(y, z) \quad (B.8)$$

is in $C^\infty_0(\mathbb{R}^3, \mathbb{R})$. Then for all $(x, y, z) \in \mathbb{R}^3$ we find that $\varphi_k(x, y, z)$ is bounded by $\chi_k(x, y, z) \eta(y, z)$. The respective bound is integrable and as $k \to \infty$ the sequence $\varphi_k$ converges pointwise to the zero function almost everywhere. Therefore, by Lebesgue’s dominated convergence theorem, we have

$$\lim_{k \to \infty} \int_{\mathbb{R}^3} d^3x \chi(x, y, z) \varphi_k(x, y, z) \eta(y, z) = 0. \quad (B.9)$$
Yet we also find that
\[
\lim_{k \to \infty} \xi(\varphi_k) = \int_{-\infty}^{\infty} dz \eta(0, z) > 0, \tag{B.10}
\]
thus yielding the contradiction.

### B.4. Proof of lemma 2.12

We may drop the index and continuously extend \(\varphi\) to \(\mathbb{R}^3\).

For the first component of the distributional curl, we show that \(\int_{-\infty}^{\infty} dz \partial \varphi / \partial z\) vanishes. Denote by \(\mathbb{I}_j\) the indicator function on an interval \(I\) in \(\mathbb{R}\). For \((x, y, z) \in \mathbb{R}^3\) and all \(n \in \mathbb{N}\) define
\[
f_n(x, y, z) = \frac{\partial \varphi}{\partial z}(x, y, z) \mathbb{I}_{(-n, n)}(z). \tag{B.11}
\]
Since \(f_n(x, y, .)\) is also continuous at the origin, we find
\[
\left| \int_{\mathbb{R}} dz f_n(x, y, z) \right| = |\varphi(x, y, n) - \varphi(x, y, -n)| \leq 2 \sup_{n \in \mathbb{Z}} |\varphi(x, y, n)| < \infty. \tag{B.12}
\]
As for all \(n \in \mathbb{N}\) we have \(f_n(x, y, .) \leq f_{n+1}(x, y, .)\), Beppo Levi’s theorem implies that \(\partial \varphi / \partial z(x, y, .)\) is in \(L^1\). Since \(\varphi\) vanishes at infinity, the respective integral of \(\partial \varphi / \partial z\) with respect to \(z\) vanishes, too.

Then, by Tonelli’s and Fubini’s theorem, the first component of the distributional curl vanishes. The same holds true for the second component.

Regarding the third component, we observe that, if the integral exists, it is equal to the corresponding integral in cylindrical polar coordinates \((\rho, \phi, z)\). Recalling the expression in equation (B.5), we therefore consider the formal integral \(\int_0^{\infty} d\rho \partial \varphi / \partial \rho(\rho, \phi, z)\). Upon defining
\[
g_n(\rho, \varphi, z) = \frac{\partial \varphi}{\partial \rho}(\rho, \phi, z) \mathbb{I}_{(0, \infty)}(\rho), \tag{B.13}
\]
we may apply an analogous argument to the one above to conclude that
\[
\int_0^{\infty} \frac{\partial \varphi}{\partial \rho}(\rho, \phi, z) = -\varphi(0, 0, z), \tag{B.14}
\]
referring to the function in Cartesian coordinates on the right hand side.

Finally, we consider \(\varrho\) from equation (2.32). We have \(\varrho \in C^1(\mathbb{R}^3 \setminus \{0\}, \mathbb{R})\). Moreover, \(\varrho\) is integrable and vanishes at infinity, independent of the values of \(n, l, \) and \(\mu\).

To show continuity at the origin, consider \(\varrho\) in spherical polar coordinates \((r, \theta, \phi)\). For \(n = 1\) we find that \(l = 0, \mu = 0, \) and thus \(P_l^{(\mu)} = 1\). The limit \(r \to 0\) is independent of the angles and thus \(\varrho\) is continuous at the origin. For \(n > 1\) and \(l = 0\) the argument is analogous. For \(n > 1\) and \(l > 0\) the function vanishes as \(r \to 0\) and is therefore continuous at the origin as well.

### B.5. Proof of proposition 2.13

As stated in lemma 2.12, we may apply equation (2.34). For \(\mu = 0\) the expression trivially vanishes. For \(\mu \neq 0\) the integrand has a factor of
\[
\left( P_l^{(\mu)}(z/|z|) \right)^2 = \left( P_l^{(\mu)}(\text{sgn} z) \right)^2. \tag{B.15}
\]
Due to the Rodrigues’ formula for the associated Legendre polynomials (see e.g. equation (7.36) in [13]), the latter expression vanishes for all \(z \neq 0, l, \) and \(\mu \neq 0\).
B.6. Proof of theorem 4.1

We solve equation (2.26) by the known procedure of reducing it to the confluent hypergeometric equation:

First, non-dimensionalize equation (2.26) by setting \( E = E/(h\omega/2) \) and \( r = \sqrt{m\omega/h\rho} \).

This yields

\[
\frac{r^2}{2} \frac{d^2R}{dr^2}(r) + \frac{dR}{dr}(r) - \frac{E}{r} R(r) + \frac{\epsilon^2}{r^2} R(r) - \mu^2 R(r) = 0.
\] (B.16)

As \( r \in R_+ \), we may set \( x = r^2 \) and formulate equation (B.16) in terms of the new variable \( x \).

By an abuse of notation, denote the new function by \( R(x) \):

\[
R(x) = x^{\mu/2} e^{-x/2} \xi(x).
\] (B.17)

This indeed yields the confluent hypergeometric equation

\[
x \frac{d^2\xi}{dx^2}(x) + (b - x) \frac{d\xi}{dx}(x) - a \xi(x) = 0
\] (B.18)

with

\[
a = \frac{\mu + 1 - E/2}{2} \quad \text{and} \quad b = \mu + 1.
\] (B.19)

By assumption, both \( a \) and \( b \) are real with \( b \notin \mathbb{Z} \).

The remaining proof consists of looking up known solutions \( \xi \) of equation (B.18) and assuring that the integrability condition

\[
\int_0^\infty x^a e^{-x} |\xi(x)|^2 \, dx < \infty
\] (B.20)

is satisfied. The former requires a careful consideration of different sets of linearly independent solutions for the respective values of the parameters \( a \) and \( b \) (cf table 1 in [122]), the latter a study of the asymptotic behavior of those solutions in the limits as \( x \to 0 \) and \( x \to \infty \) (cf chapter 4 in [161])—as well as a proof of the con- or divergence of the respective integral.

i) The first case we consider is \( a \in \mathbb{R} \setminus \mathbb{N}_0 \) and \( b \in \mathbb{R} \setminus \mathbb{Z} \).

Denoting by \( _1F_1 \) the confluent hypergeometric function of first kind, we have

\[
\xi(x) = c_1 \, _1F_1(a,b;x) + c_2 \, U(a,b;x)
\] (B.21)

for arbitrary \( c_1, c_2 \in \mathbb{C} \).

As \( x \to \infty \), the first solution satisfies

\[
_1F_1(a,b;x) \sim \frac{\Gamma(b)}{\Gamma(a)} x^{a-b} e^x
\] (B.22)

(cf equation (4.1.7) in [161]), and the second one obeys

\[
U(a,b;x) \sim x^{-a},
\] (B.23)
(cf equation (4.1.11) in [161]). As we require
\[
\lim_{x \to \infty} x^\mu e^{-x} |\xi(x)|^2 = 0 \tag{B.24}
\]
for the integral in equation (B.20) to converge, we find that \( c_1 = 0 \).

As \( x \to 0 \), the definition of \( U \) in terms of \( _1F_1 \), equation (1.3.1) in [161], yields
\[
U(a, b; x) \sim \begin{cases} 
\frac{\Gamma(b-1)x^{-b}}{\Gamma(a)}, & 1 - b < 0 \\
\frac{\Gamma(1-b)}{\Gamma(1-b+a)}, & 1 - b > 0 
\end{cases} \tag{B.25}
\]
Recalling equation (B.20), for \( \mu > 0 \) we thus obtain the restriction that \( -\mu > -1 \), while for \( \mu < 0 \) we similarly obtain \( \mu > -1 \).

In order to prove convergence or divergence of the respective integral, we need some estimates for \( x \to 0 \) and \( x \to \infty \): By equation (B.23) and the definition of the limit, for every \( \varepsilon > 0 \) there exists \( x_\infty \in \mathbb{R}_+ \) such that for all \( x > x_\infty \) we have
\[
|U(a, b; x) - x^{-a}| < \varepsilon x^{-a}. \tag{B.26}
\]
Analogously, for \( \mu < 0 \) equation (B.25) implies that for our choice of \( \varepsilon \) there exists \( x_- \in \mathbb{R}_+ \) such that for all \( x \in (0, x_-) \) we have
\[
|U(a, b; x) - d_-| < \varepsilon d_- \tag{B.27}
\]
where we set \( d_- = \Gamma(-\mu)/\Gamma(a-\mu) \). For \( \mu > 0 \) equation (B.25) implies that there exists \( x_+ \in \mathbb{R}_+ \) such that for all \( x \in (0, x_+) \) we have
\[
|U(a, b; x) - d_+ x^{-\mu}| < \varepsilon d_+ x^{-\mu}, \tag{B.28}
\]
setting \( d_+ = \Gamma(\mu)/\Gamma(a) \).

We are now in a position to prove convergence of the integral in equation (B.20) for \( \xi(x) = U(a, b; x) \) and \( \mu \in (-1, 0) \cup (0, 1) \): We split the integral into three, going from 0 to \( d_\pm \), \( d_\pm \) to \( d_\infty \), and \( d_\infty \) to \( \infty \), respectively. For the third integral we derive the estimate
\[
(U(a, \mu + 1; x))^2 = \left( |U(a, \mu + 1; x) - x^{-a} + x^{-a}| \right)^2 \leq |U(a, \mu + 1; x) - x^{-a} + x^{-a}|^2 \leq (1 + \varepsilon)^2 x^{-2a}. \tag{B.29}
\]
Thus,
\[
\int_{x_\infty}^{\infty} x^\mu e^{-x} |U(a, \mu + 1; x)|^2 \, dx < (1 + \varepsilon)^2 \int_{x_\infty}^{\infty} x^{\mu-2a} e^{-x} \, dx, \tag{B.30}
\]
\[\]
60 Note that \( z \to 1/\Gamma(z) \) is holomorphic on the entire complex plane.
so the former integral converges by the dominated convergence theorem. The first integral is handled analogously for each one of the two cases \( \mu \in (-1, 0) \) and \( \mu \in (0, 1) \). In the second integral we integrate a continuous function over a compact interval \( [x_-, x_+] \), so the integral converges. Finally, as the individual integrals converge, their sum converges.

Formally, we also need to show divergence of the integral for \( \mu \) not in \( (-1, 0) \cup (0, 1) \): For \( \mu < -1 \) and \( 0 < x < x_- \) equation (B.27) yields

\[
d_- = |d_- - U(a, \mu + 1; x) + U(a, \mu + 1; x)] = (B.33)
\]

\[
\leq |d_- - U(a, \mu + 1; x)| + |U(a, \mu + 1; x)| \quad (B.34)
\]

\[
< \varepsilon d_- + |U(a, \mu + 1; x)| \quad (B.35)
\]

and thus

\[
(1 - \varepsilon)d_- < |U(a, \mu + 1; x)|. \quad (B.36)
\]

In turn,

\[
\int_0^\infty x^\mu e^{-x} |U(a, \mu + 1; x)|^2 \, dx \geq \int_0^{d_-} x^\mu e^{-x} |U(a, \mu + 1; x)|^2 \, dx \quad (B.37)
\]

\[
> (1 - \varepsilon)^2 d_- \int_0^{d_-} x^\mu e^{-x} \, dx. \quad (B.38)
\]

For \( \mu < -1 \) the latter integral diverges. The case \( \mu > 1 \) is handled analogously using the estimate equation (B.28) for \( 0 < x < x_+ \).

Summing up, for \( a \in \mathbb{R} \setminus \mathbb{N}_0 \) equation (B.18) only has nontrivial solutions whenever \( \mu \in (-1, 0) \cup (0, 1) \). Those solutions are proportional to \( U(a, \mu + 1; x) \).

ii) There remains the case that \( -a = n \in \mathbb{N}_0 \) and \( b \in \mathbb{R} \setminus \mathbb{Z} \).

Following table I in [122] and using the definition of \( L_n^\mu \) in section 5.5 in [161], the general solution is given by

\[
\xi(x) = c_1 L_n^\mu(x) + c_2 x^{1-b} \, _1F_1(1 + a - b, 2 - b; x). \quad (B.39)
\]

for \( c_1, c_2 \in \mathbb{C} \).

As \( x \to \infty \), we conclude that \( c_2 = 0 \) by an argument analogous to the one above.

As \( x \to 0 \), the polynomial \( L_n^\mu(x) \) is asymptotic to a constant. Hence equation (B.20) requires that \( \mu > -1 \).

For those values of \( \mu \), the integral converges, because it can be written as a finite sum of integrals over \( x^{(k+\mu)}e^{-x} \) for different \( k \in \mathbb{N}_0 \). For the other values of \( \mu \) we show divergence of the integral as we did for \( U(a, \mu + 1; x) \) above.

In conclusion, for \( -a = n \in \mathbb{N}_0 \) we only obtain nontrivial solutions for \( \mu \in (-1, \infty) \setminus \mathbb{Z} \). Those solutions are proportional to \( L_n^\mu \).
B.7 Proof of proposition 4.3

i) Set \(\Phi(x,y) = R(\sqrt{x^2 + y^2})\). Extend \(\Phi\) to \(\mathbb{R}^2 \setminus \{0\}\). On this new domain, \(\Psi\) is \(C^1\). We shall use the notation \(\partial/\partial y\) for strong derivatives of functions on the respective domain.

Similar to the proof of corollary 2.11 above, we need to show that there does not exist a locally integrable \(\chi\) such that

\[
\int_{\mathbb{R}^2} dx dy \Psi(x,y) \frac{\partial \xi}{\partial y}(x,y) = - \int_{\mathbb{R}^2} dx dy \chi(x,y) \xi(x,y) \quad (B.40)
\]

holds for all \(\xi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})\). Again, the general procedure consists of two main steps: First, integrate by parts to create non-vanishing ‘boundary terms’. Second, assume existence of \(\chi\) and use the boundary terms in order to produce a contradiction.

The first main step is carried out by splitting the integral on the left hand side of equation (B.40) into integrals over \(\Omega_1 = (-\infty,0) \times \mathbb{R}, \Omega_2 = (0,\infty) \times (0,\infty)\) and \(\Omega_3 = (0,\infty) \times (-\infty,0)\). We then use Fubini’s theorem to rewrite each into an iterated integral (cf theorem 4.5 in [35]).

We shall only provide a full argument for the integral over \(\Omega_3\)—which is the most illustrative one—the argument for the other ones is analogous:

We first show that for every \(x \in (0,\infty)\) the restriction of \(\Psi\) to \(\Omega_3\) evaluated at \(x\) is in \(W^{1,1}((-\infty,0), \mathbb{C})\) (cf footnote 37). Clearly, it is in \(L^1((-\infty,0), \mathbb{C}) \cap C^1((-\infty,0), \mathbb{C})\), so we only need to show that the derivative is integrable over \((-\infty,0)\): For \(r > 0\) denote by \(y \mapsto 1_{(-r,-1/r)}(y)\) the indicator function for the interval \((-r,-1/r)\). Now consider the function

\[
(-\infty,0) \to \mathbb{C} : y \mapsto \frac{\partial \Psi}{\partial y}(x,y) 1_{(-r,-1/r)}(y). \quad (B.41)
\]

By the fundamental theorem of calculus, this function is integrable over \((-\infty,0)\). Since \(\Psi|_{\Omega_3}(x,)\) remains finite for both classes of solutions as \(y\) tends to 0 from below and the function tends to 0 as \(y \to \infty\), Beppo Levi’s theorem yields the result (cf theorem 4.1 in [35]).

We may now integrate by parts using corollary 8.10 in [35]: For all \(x \in (0,\infty)\) we find

\[
\int_{-\infty}^0 dy \Psi(x,y) \frac{\partial \xi}{\partial y}(x,y) = -\int_{-\infty}^0 dy \frac{\partial \Psi}{\partial y}(x,y) \xi(x,y) + \Phi(x,0) e^{2\pi i \mu} \xi(x,0). \quad (B.42)
\]

One finds that the integral over \(\Omega_2\) also yields two summands within the integral with respect to \(x\), while the one over \(\Omega_1\) only yields one. We wish to combine the expressions to conclude that the left hand side of equation (B.40) equals

\[
-\int_{\mathbb{R}^2} dx dy \frac{\partial \Psi}{\partial y}(x,y) \xi(x,y) + \left(e^{2\pi i \mu} - 1\right) \int_0^\infty dx \Phi(x,0) \xi(x,0). \quad (B.43)
\]

This, however, requires the use of the linearity of \(\int_0^\infty dx\), which is only admissible if the integral over either one of the two summands converges—since we have almost everywhere convergence of the initial integral by Fubini’s theorem.
It is sufficient to show that \( R \in L^1 ((0, \infty), \mathbb{R}) \). In full analogy to the respective arguments in the proof of theorem 4.1, this is done by considering the asymptotic behavior at 0. For the first class of solutions we use equation (B.25) to find that, modulo constants, \( R(x) \) is asymptotic to \( x^\mu \) for \( \mu \in (-1, 0) \) and to \( x^{\mu-2\mu} = x^{-\mu} \) for \( \mu \in (0, 1) \). For the second class of solutions the assertion is shown directly, thus proving the above claim.

Since, all integrals in equation (B.43) are convergent and equal to the left hand side of equation (B.40), the first main step is completed.

To tackle the second main step, we observe that \( \chi \) on the right hand side of equation (B.40) exists if and only if there exists a locally integrable function \( \chi' \) such that

\[
\int_0^\infty dx \Phi(x,0) \xi(x,0) = \int_{\mathbb{R}^2} dx dy \chi'(x,y) \xi(x,y) \tag{B.44}
\]

for all \( \xi \). The aforementioned contradiction is now obtained by a more or less standard argument:

First, consider a sequence of mollifiers \( x \mapsto \eta_n(x) \) in \( C_0^\infty (\mathbb{R}, \mathbb{R}) \), as given on p 108 sq. in [35]. Second, take the sequence \( (\zeta_k)_{k \in \mathbb{N}} \) of bump functions from the proof of corollary 2.11. Then, for arbitrary \( x_0 \neq 0 \) set \( \xi_{nk}(x,y) = \eta_n(x-x) \zeta_k(y) \) so that each \( \xi_{nk} \) is in \( C_0^\infty (\mathbb{R}^2, \mathbb{C}) \). We will show that for all \( x_0 \neq 0 \) we have

\[
\lim_{n \to \infty} \lim_{k \to \infty} \int_0^\infty dx \Phi(x,0) \xi_{nk}(x,0) = \Phi(x_0,0) \tag{B.45}
\]

Due to \( \zeta_k(0) \equiv 1 \), the left hand side equals

\[
\lim_{n \to \infty} \int_0^\infty dx R(x) \eta_n(x-x) = \lim_{n \to \infty} \int_{-\infty}^{x_0} dz R(x_0-z) \eta_n(z) \tag{B.46}
\]

We now follow proposition 4.21 in [35]: Since \( R \) is continuous at \( x_0 \), for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( z \) with \( |z| < \delta \) and \( (x_0) \) we have

\[
|R(x_0-z) - R(x_0)| < \varepsilon \tag{B.47}
\]

Now choose any natural number \( n_0 > 1/\delta \). Then for any \( n > n_0 \) we find

\[
\int_{-\infty}^{x_0} dz R(x_0-z) \eta_n(z) = \int_{-\infty}^{x_0} dz \left( R(x_0-z) - R(x_0) \right) \eta_n(z) \leq \int_{-1/n}^{1/n} dz |R(x_0-z) - R(x_0)| \eta_n(z) \tag{B.48}
\]

Due to equation (B.47) and normalization of \( \eta_n \), we conclude that for every \( \varepsilon > 0 \) there exists an \( n_0 \) such that for all \( n \geq n_0 \) the above expression is less than \( \varepsilon \).

Returning to equation (B.44), we will also show that

\[
\lim_{n \to \infty} \lim_{k \to \infty} \int_{\mathbb{R}^2} dx dy \chi'(x,y) \xi_{nk}(x,y) = 0 \tag{B.50}
\]
It is enough to consider the limit \( k \to \infty \) for fixed \( n \). Observe that the absolute value of the integrand is bounded by \( |x'| \xi n_1 \) almost everywhere. Furthermore, for almost every \((x,y) \in \mathbb{R}^2\) we have

\[
\lim_{k \to \infty} \chi'(x,y) \eta_n(x_0 - x) \zeta_k(y) = \begin{cases} 
\chi'(x,y) \eta_n(x_0 - x) , y = 0 \\
0 , y \neq 0.
\end{cases} \quad (B.51)
\]

That is, in the pointwise limit \( k \to \infty \) the functions \( \chi' \xi n \) tend to the zero function almost everywhere. Equation (B.50) then follows from the dominated convergence theorem.

Finally, the contradiction results from choosing an \( x_0 > 0 \) for which \( \Phi(x_0, 0) \neq 0 \).

ii) We use equation (B.43). Since it was the second term that prevented existence of a weak derivatives in that instance, the situations for the functions from equations (4.8a) and (4.8b) will be analogous, whenever the respective factor in front does not vanish. For \( \Psi \) in equation (4.8b) that factor is \( \cos(2\pi \mu) - 1 \), thus yielding the assertion.

iii) As in point ii), we need to look at the zeros of the factor \( \sin(2\pi \mu) \). This vanishes for \( 2\mu \in \mathbb{Z} \). Following the chain of arguments in point i) up to equation (B.43), one shows that the weak derivative of \( \Psi \) with respect to \( x \) exists as well. Thus \( \Psi \) is indeed weakly differentiable in those special cases and not weakly differentiable otherwise.

### B.8. Proof of proposition 4.4

i) This is a direct corollary of theorem 4.1.

ii) The general idea of proof is that one cannot write a discontinuous function as a finite linear combination of continuous eigenfunctions, which implies that the time evolution of \( \Psi_0 \) cannot be given by equation (4.18).

Define \( \Phi_{n_1n_2} \) as in equation (4.16) with corresponding energy eigenvalues \( E_{n_1n_2} \). Setting

\[
a_{n_1n_2} = \langle \Phi_{n_1n_2} , \Psi_0 \rangle ,
\]

(B.52)

the time evolution of \( \Psi_0 \) is given by

\[
\tilde{\Psi}_t = \sum_{n_1n_2 \in \mathbb{N}_0} a_{n_1n_2} e^{-iE_{n_1n_2}t/\hbar} \Phi_{n_1n_2} . \quad (B.53)
\]

We first show that there are infinitely many \((n_1, n_2) \in \mathbb{N}_0^2\) for which \( a_{n_1n_2} \neq 0 \): Aiming for a contradiction, assume there are only finitely many. It follows that \( \tilde{\Psi}_0 \) is continuous on \( \mathbb{R}^2 \). Yet, by construction, \( \Psi_0 \) and \( \tilde{\Psi}_0 \) belong to the same equivalence class in \( L^2(\mathbb{R}^2, \mathbb{C}) \), and continuous representatives are unique. Therefore, \( \tilde{\Psi}_0 \) is a continuous extension of \( \Psi_0 \) to \( \mathbb{R}^2 \)—an impossibility.

In the final main step of the proof we also aim for a contradiction: Assume that for all \( t > 0 \) the functions \( \Psi_t \) and \( \tilde{\Psi}_t \) belong to the same equivalence class. Consider

\[
\langle \Phi_{n_1n_2} , \tilde{\Psi}_t \rangle = a_{n_1n_2} e^{-iE_{n_1n_2}t/\hbar} . \quad (B.54)
\]

Due to linear independence of the \( \Phi_{n_1n_2} \), our assumption implies that

\[
0 = \left( e^{-iE_{n_1n_2}t/\hbar} - e^{-iE_{n_1n_2}t/\hbar} \right) a_{n_1n_2} . \quad (B.55)
\]
for all $n_1, n_2 \in \mathbb{N}_0$. But unless $a_{n_1 n_2} = 0$, we must have
\[ E \in E_{n_1 n_2} + 2\pi \hbar \mathbb{Z} / t. \] (B.56)

As this has to hold for all $t > 0$, we find that for any $n_1, n_2 \in \mathbb{N}_0$ either $a_{n_1 n_2} = 0$ or $E = E_{n_1 n_2}$. In particular, there must be infinitely many $(n_1, n_2)$ for which $E = E_{n_1 n_2}$. However, from the expression for $E_{n_1 n_2}$ we find that $E$ is equal to $E_{n_1 n_2}$ at most finitely often—the desired contradiction.

**B.9. Proof of lemma 4.6**

Denote the subspace in equation (4.19) by $V$. For $\Psi \in \text{dom} \hat{H} + V =: \text{dom} \hat{H}'$ we set $\hat{H}' \Psi$ equal to $\hat{H} \Psi$ whenever $\Psi \in \text{dom} \hat{H}$ and we interpret $\hat{H}' \Psi$ in the weak sense whenever $\Psi \in V$.

$\hat{H}'$ is well-defined: For $\Psi \in \text{dom} \hat{H} \cap V$ consider its energy eigenfunction expansion and apply $\hat{H}'$ as defined for vectors in $V$. Upon recalling that the weak Laplacian is symmetric on $H^2(\mathbb{R}^2, \mathbb{C})$, one finds that the two definitions of $\hat{H}' \Psi$ coincide. Thus $\hat{H}'$ is a well-defined linear extension of $\hat{H}$.

The assertion also follows from this calculation.

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