Design of Ciphers based on the Geometric Structure of the Laguerre and Minkowski Planes

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Abstract

Till now geometric structures don’t play a major role in cryptography. Gilbert, MacWilliams and Sloane [GMS74] introduced an authentication scheme in the projective plane and showed its perfectness in the sense of Shannon [Sha49]. In [Cap21] we introduced an encryption scheme in the Möbius plane and showed that it fulfills Shannon’s requirement of perfectness in first approximation and also the requirement of completeness according to Kam and Davida [KD79]. In this paper we will apply a similar approach to define encryption schemes in the geometries of the Laguerre plane and the Minkowski plane.

We will show that the encryption scheme in the Laguerre geometry meets Shannon’s requirement of perfectness sharp and that the encryption scheme in the Minkowski geometry meets this requirement in first approximation. The Laguerre cipher also fulfills the requirement of completeness according to Kam and Davida.

Keywords: circle geometry, Laguerre, Minkowski, cryptography, complete, perfect

1 Introduction

A cryptographic transformation can be understood as an incidence relation, whereby messages $m$ and ciphertexts $c$ are represented as points. The cryptographic transformation $f$ that maps $m$ to $c$ is then described by a geometric object that incises with these two points. In this paper we will design new encryption transformations in the geometries of the Laguerre plane and the Minkowski plane and analyze their properties.

The basic property of cycles in the Laguerre and Minkowski plane, being that three points incise with a cycle, allow to associate one point with a message, another point with the ciphertext and still have a degree of freedom for a secret key. We will analyze encryption methods in the Laguerre and Minkowski plane, namely the criteria of perfectness according to Shannon [Sha49] and the completeness in the case that the geometry is defined
over a field of characteristic 2. Shannon’s requirement of perfectness states that basically the result of the encryption process cannot be distinguished from a noisy channel. Completeness according to Kam and Davida [KD79] means that, assumed that input and output of a transformation are represented as bit vectors, there is at least one input vector for which a change in the \( i \)-th bit results in a change of the \( j \)-th bit of the output vector for arbitrary \( i \) and \( j \).

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\section{Cryptographic Transformations and Cryptographic Schemes}

This and the following section repeat definitions and theorems from [Cap21]. They will be required to define cryptographic schemes in the Laguerre and Minkowski plane and to discuss their properties.

\textbf{Definition 2.1.} Let \( M \) and \( C \) be sets, where \( M \) is called a set of messages and \( C \) a set of ciphertexts. If there is a nonempty set of functions \( F \) of the form \( F : M \rightarrow C \) with the property that every \( f \in F \) is reversible, then the elements \( f \in F \) are called cryptographic transformations and \( (M, C, F) \) is called a cryptographic scheme.

\textbf{Theorem and Definition 2.2.} A cryptographic scheme \( (M, C, F) \) is called a countable infinite cryptographic scheme, if the sets \( M, C \) and \( F \) are countable infinite. If the set of ciphertexts \( C' \) of a cryptographic scheme \( (M', C', F') \) is finite then \( (M', C', F') \) is called a finite cryptographic scheme.

The proof can be found in [Cap21].

\section{Properties of Cryptographic Schemes}

\textbf{Definition 3.1.} If \( (M, C, F) \) is a cryptographic scheme, then the probability of the occurrence of a message \( m \in M \) is denoted with \( \mu(m) = P(m) \) and is called a-priori probability of the message \( m \). Similarly the probability of the occurrence of the message \( m \) under the condition that \( m \) is mapped to \( c \) by any \( f \in F \) is denoted with \( \nu(m, c) = P_{c=f(m)}(m) \) and is called a-posteriori probability of the message \( m \) for a given ciphertext \( c \).

\textbf{Remark 3.2.} In the case that a cryptographic system \( (M, C, F) \) is finite, relative frequencies can be used to calculate the a-priori and a-posteriori
probabilities. Then \( \mu(m) = \frac{H(m)}{|M|} \) and \( \nu(m,c) = \frac{|\{f \in F : c = f(m)\}|}{\sum_{m \in M} |\{f \in F : c = f(m)\}|} \). Here \( H(m) \) means the frequency of the occurrence of the message \( m \).

**Definition 3.3.** Let \((M, C, F)\) be a cryptographic scheme for which every \( c \in C \) is a possible ciphertext and let \( \mu \) and \( \nu \) be its a-priori and a-posteriori probabilities. Then \((M, C, F, \mu, \nu)\) is called perfect according to Shannon [Sha49] as long as \( \mu(m) = \nu(m,c) \) for any \( m \in M \) and for any \( c \in C \).

**Remark 3.4.** If \((M, C, F)\) is finite and if \( \mu(m) = \mu(m_0) \) for any \( m \in M \) then \((M, C, F, \mu, \nu)\) is perfect as long as \( \frac{1}{|M|} = \frac{|\{f \in F : c = f(m)\}|}{\sum_{m \in M} |\{f \in F : c = f(m)\}|} \) for any \( m \in M \) and for any \( c \in C \).

**Definition 3.5.** Let \((M, C, F)\) be a cryptographic scheme with \( M = \mathbb{Z}_r^2 \) and \( C = \mathbb{Z}_s^2 \). Then the cryptographic transformation \( f \in F \) is called complete according to Kam and Davida [KD79], if there is at least one message \( m_0 = m_{0_1}, m_{0_2}, ..., m_{0_r} \in M \) for every pair of indices \( i \leq r \) and \( j \leq s \), where a change in the \( i \)-th bit of \( m_0 \in M \) results in a change of the \( j \)-th bit of \( c_0 = c_{0_1}, c_{0_2}, ..., c_{0_s} = f(m_0) \in C \). A cryptographic scheme \((M, C, F)\) that consists exclusively of complete cryptographic transformations is called a complete cryptographic scheme.

### 4 Brief Introduction of the Geometry of the Laguerre Plane

**Definition 4.1.** The triple \((P, G, X)\) of points \( P \), generators \( G \) and cycles \( X \) is called Laguerre plane, if it satisfies following properties:

(L1) \( \forall a \in P \exists [a] \in G : a \in [a] \). \([a]\) is the generator that passes through the point \( a \).

(L2) \( \forall a, b, c \in P, \text{ with } [a] \neq [b] \neq [c] \neq [a] \exists C \in X : a, b, c \in C \).

(L3) \( \forall G \in G, \forall C \in X : |G \cap C| = 1 \).

(L4) **Touch axiom:** \( \forall C \in X, a \in C \text{ and } b \in P \setminus \{C \cup [a]\} \exists D \in X : a, b \in D \text{ and } C \cap D = \{a\} \).

(L5) \( \exists C \in X : |C| \geq 3 \), there is \( |P| > 3 \) and \( \forall G \in G : |G| \geq 2 \).

The Laguerre geometry can be defined as the points \( P \) on the surface of a circular cylinder in the Euclidian space. The cycles \( X \) of the Laguerre geometry are then the cone cuts of the circular cylinder with all planes that are not parallel to the axis of the cylinder. The tangents of the circular cylinder
with planes touch the circular cylinder in lines, which are the generators of the Laguerre geometry.

The Laguerre geometry of the circular cylinder can be projected into the Euclidian plane by the use of a stereographic projection. The center of the projection is chosen as a point of the circular cylinder, the projection plane shall be in parallel to the axis of the cylinder and shall not contain the projection center. If the coordinate system of the Euclidian plane is chosen properly, the cycles of the Laguerre geometry that pass the center of the projection are mapped to the lines in the Euclidian plane that are not parallel to the $y$-axis. All other cycles of the cylinder are transferred to parabolas in the Euclidian plane with axes in parallel to the $y$-axis. The points of the Laguerre geometry that lie on the line $L_0$ passing the center of the projection are mapped to distant points. The set of distant points form the distant generator. Since $L_0$ is a line of the Euclidian space, the distant generator consists of as many points as a line of the Euclidian space. Figure 1 illustrates the stereographic projection of the Laguerre geometry.

In order to find an analytical representation of the Laguerre plane, we want to introduce the dual numbers:

$$\mathbf{D} := \{a + b\epsilon : a, b \in \mathbf{F}\}, \epsilon^2 = 0$$

$\mathbf{D}$ is a local ring with $\mathbf{F}$ as a sub-field and $\mathbf{F}\epsilon$ as a maximum ideal. In $\mathbf{D}$ exists an involutorial automorphism $\cdot : \mathbf{D} \to \mathbf{D}$ that has the elements of $\mathbf{F}$ as fixpoints. We obtain the Laguerre plane by closing the dual numbers.
with a the distant generator. We will use the notation \( F := F \cup \{ \infty \} \) for the closed field \( F \) and \( \overline{D} := D \cup \{ \infty + F \epsilon \} \) for the closure of the dual numbers.

5 Describing the Laguerre Plane Using Equations

With the right choice of the coordinate system, the cycles of the Laguerre plane are represented as parabolas and lines in \( D \). A cycle \( C \in X \) is determined by the parameters \( a, b, c \in F \) with \( (a, b) \neq (0, 0) \) and is represented by the following set:

\[
\{(x, y) \in F^2 : y = ax^2 + bx + c\} \cup \{\infty - a\epsilon\}
\]

6 Describing the Laguerre Plane by the Use of Double Ratios

A cycle in the Laguerre plane that includes the points \( a, b, c \in D \) with \( [a] \neq [b] \neq [c] \neq [a] \) can be represented with the following set:

\[
\{z \in D : [z] \neq [a] \text{ and } Dr(a, b, c, z) := \frac{a}{a-c} / \frac{b-c}{b-c} \in F\} \cup \{a\} \cup \{\infty + \frac{a-c}{b-c} \epsilon\}
\]

7 Describing the Laguerre Plane Using Fractional Linear Functions

For the representation of the Laguerre plane with the use of fractional linear functions we will use a representation based on the closed dual numbers \( \overline{D} \) with coordinates of the field \( F \). Then the cycles in the Laguerre plane are determined as the pictures of a reference cycle, e.g. the set \( \{z \in F\} \) regarding following mappings \( \gamma \).

\[
\gamma : z \mapsto \frac{az + b}{cz + d}, z \in F, a, b, c, d \in D, ad - bc \in D \setminus F \epsilon
\]

\[
\gamma(\infty) = \frac{a}{c}, \gamma\left(-\frac{d}{c}\right) = \infty, \text{ if } \frac{d}{c} \in F
\]

8 Combinatorial Aspects of the Laguerre Plane

In the following we will focus on a Laguerre plane \( (P, G, X) \) based on a finite field \( F \) with \( |F| = q \). Let be \( C \in X \) with \( |C| \geq 3 \) (see (L5)), \( A, B \in G \) with \( A \neq B \) and \( c \in C \setminus (A \cup B) \). We assign every point \( a \in A \) a cycle
Figure 2: Bijection between two generators.

$D_a \in X$ with $a \in D_a$ in the following way: If $a \in C$, then let $D_a = C$. If $a \notin C$, then $D_a$ shall be uniquely defined by $D_a \cap C = c$ (acc. to (L4)). Due to properties (L3) and (L4) the mapping

$$
\gamma : \begin{cases} 
A \in B \\
da \mapsto D_a \cap B
\end{cases}
$$

is bijective (see figure 2).

Since the stereographic projection maps a generator of the Laguerre geometry $(P, G, X)$ to a line of the affine plane over the field $F$, there are $q$ points on every generator. Furthermore there is $|q| = |A| = |B|$ for every $A, B \in G$. With (L3) follows $|P| = |A| \cdot |C|$ and hence $|C| = |D|$ for every $C, D \in X$.

Let $A, B, C$ be as defined above, and let $b$ be a point with $b \in B \setminus C$ and $E \in X$ be a cycle with $b \in E$. We assign every point $c \in C \setminus B$ the cycle $F_c \in X$ with $c \in F_c$ and $F_c \cap E = \{b\}$. Basically this means that $F_c$ touches $E$ in $b$. If $c \in E$, then let $F_c := C$. Figure 3 illustrates this construction.

According to (L4) the cycle $F_c$ is uniquely determined by the cycle $C$ and the points $b$ and $c$. Because of (L3) the cycle $F_c$ has a common point with the generator $A$. Furthermore there is $F_c \cap A \neq F_d \cap A$ for any $c \neq d$. This provides a bijection from the set of points of $C \setminus \{C \cap B\}$ to $A$. Hence $|C| = |A| + 1 = q + 1$, $\forall C \in X$. Together with the results shown above we get $P = q(q + 1)$. Since a cycle is defined by exactly three points from three different generators (L2), we get together with (L1) as result for the number of cycles $|X| = q^3$. 

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9 Encryption in the Laguerre Plane

In [Cap94] an encryption scheme based on the Möbius plane has been introduced. It is quite logical to transfer these results to the Laguerre plane. Again, we will use the incidence properties, in this case in the Laguerre plane, to create a mapping between three points representing message, cipher text and a key. To ensure that such an incidence exists, we choose messages, cipher texts and keys from different generators. Hence these points can never be parallel.

We want to use a Laguerre plane over a finite field $\mathbb{F}$ with $|\mathbb{F}| \geq 3$ and hence $|G| \geq 4$. So we select four generators $G_1, G_2, G_3, G_4$ of the Laguerre plane and assign them with the different sets of a cipher system. In this context we would like to use the notation $(a, b, c)^\circ$ for the cycle through the three points $a, b, c$ of the Laguerre plane which are pairwise not parallel. Also, as a small modification to definition 2.1 we don’t expect the transformations $F$ of the cryptographic scheme to be reversible, but introduce a set of decryption transformations $G$ instead.

**Definition 9.1.** A cipher system $(M, C, F, G)$ is defined in the Laguerre plane $(P, G, X)$ as follows:

Messages: $M := G_1$

Cipher texts: $C := G_2$

Keys: $K := G_3 \times G_4$

Encryption functions: $F : \begin{cases} K \times M \to C \\ (k, l, m) \mapsto c := (k, l, c)^\circ \cap G_2 \end{cases}$
Decryption functions: \( G : \begin{cases} K \times C \to M \\ (k, l, c) \mapsto m := (k, l, c)^\circ \cap G_1 \end{cases} \)

The cipher system \((M, C, F, G)\) is called Laguerre cipher.

Due to property (L2) of the Laguerre plane there is exactly one cycle \( C \in X \) for any points \( m \in M \) and \( k, l \in K \). Because of (L3) we know that \(|C \cap C| = 1\). Hence the cryptographic transformation \( F \) is unique. Since the decryption function is defined analogue to the encryption function, it is also unique.

10 Cryptoanalysis of the Laguerre Cipher

Let \((P, G, X)\) be a Laguerre plane over a finite field \( F \). To examine the property of perfectness we want to assume that all messages are equally distributed. The probability measures \( \mu \) and \( \nu \) shall be defined as described in definition 3.3. The probability of the occurrence of a message \( m \in M \) is \( \mu(m) = 1/|F| \), since there are \(|F|\) points on a generator.

The probability \( \nu(m, c) \) that a message \( m \) belongs to a certain ciphertext \( c \) is determined by the number of elements in the set of cycles \( K' := \{C \in X : m, c \in C\} \) that incide with the points \( m, c \) divided by the number of elements in the set of cycles \( K'' := \{C \in X : m \in C\} \) that incide with \( m \). Since a generator \( G' \) with \( m, c \notin G' \) has a unique common point with every cycle from \( K' \) and since \( G' \) has \(|F|\) points, we get \(|K'| = |F|\). The corresponding considerations about the number of cycles that pass \( m \) lead to \(|K''| = |F|^2\). This leads to a probability for the occurrence of a message of \( \mu(m) = 1/|F| \) equaling the probability \( \nu(m, c) = |F|/|F|^2 \) that the message \( m \) belongs to a certain ciphertext \( c \). Hence the Laguerre cipher fulfills the requirement of perfectness according to Shannon.

To examine the property of completeness according to Kam and Davida we define a Laguerre cipher as introduced in definition 9.1 in a Laguerre plane \((P, G, X)\) over a finite field \( F := \mathbb{Z}_2^n \). The points in \( P \) have following coordinates:

\[
\begin{align*}
m &= m_1 + m_2 \epsilon \\
c &= c_1 + c_2 \epsilon \\
k &= k_1 + k_2 \epsilon \\
l &= l_1 + l_2 \epsilon
\end{align*}
\]

We choose the generators of \((P, G, X)\) such that \( m_1 + F \epsilon = E_1 \) and \( c_1 + F \epsilon = E_2 \). Let also \( E_1, E_2 \) be in the finite. Otherwise the Laguerre plane can be transformed accordingly. Since the Laguerre cipher introduced in definition 9.1 uses only points from \( E_1 \) as messages and only points from \( E_2 \)
as ciphertexts, the cryptographic function \( f : (m, k, l) \mapsto c \) effectively maps the 2nd coordinate \( m_2 \) of \( m \) to the 2nd coordinate \( c_2 \) of \( c \). We can describe these coordinates as binary vectors of length \( n \).

To show the property of completeness we select two indices \( i, j \in 1, 2, ..., n \) and define \( e_i \) and \( e_j \) as the respective unit vectors in \( \mathbb{Z}_n^2 \). For two points \( k \) and \( l \) on the generators \( E_3 \) and \( E_4 \) we have to find a \( m \) and \( c \) that hold following conditions:

\[
(k, l, m) \cap E_2 = \{c\} \quad (1)
\]

\[
(k, l, m') \cap E_2 = \{c'\}
\]

\[
\begin{align*}
m' &= m_1 + (m_2 + e_i)e \\
c' &= c_1 + (c_2 + e_j)e
\end{align*}
\quad (2)
\]

When we use double ratios to describe cycles in the Laguerre plane, the equations (1) and (2) can be presented as follows:

\[
\text{Dr}(k, l, m, c) \in F \\
\text{Dr}(k, l, m', c') \in F
\]

or

\[
\frac{k - m}{k - c} = c \cdot \frac{1 - m}{1 - c}, c \in F \\
\frac{k - m'}{k - c'} = d \cdot \frac{1 - m'}{1 - c'}, d \in F
\]

This leads to the following conditions for \( m_2, c_2, c \) and \( d \):

\[
\begin{align*}
(k_1 - m_1)(l_1 - c_1) &= c \cdot (k_1 - c_1)(l_1 - m_1) \\
(k_1 - m_1)(l_2 - c_2) + (k_2 - m_2)(l_1 - c_1) &= c \cdot ((k_1 - c_1)(l_2 - m_2) + (k_2 - c_2)(l_1 - m_1)) \\
(k_1 - m_1)(l_1 - c_1) &= d \cdot (k_1 - c_1)(l_1 - m_1) \\
(k_1 - m_1)(l_2 - c_2 - e_j) + (k_2 - m_2 - e_i)(l_1 - c_1) &= \\
&= k \cdot ((k_1 - c_1)(l_2 - m_2 - e_i) + (k_2 - c_2 - e_j)(l_1 - m_1 - e_i))
\end{align*}
\]

Values can be chosen for \( m_2, c_2, c \) and \( d \) to solve this system of linear equations.
Another circle geometry is given with the Minkowski plane. Examples are hyperboloids in the three-dimensional projective space $P$ over a commutative field $F$. A hyperboloid can be represented as the set of points $p \in P$ with $p := F^4(x_1, x_2, x_3, x_4)^\top$ and $(x_1, x_2, x_3, x_4) \in (F^4)^*$ that satisfy the quadratic equation $x_1 x_4 - x_2 x_3 = 0$. In comparison to the circular cylinder of the Laguerre plane the hyperboloid of the Minkowski plane has two distinguished classes of straight lines $G_1$ and $G_2$ as generators. Two generators that are different from each other don’t intersect, when they belong to the same class of generators, and they share exactly one common point, when they belong to different classes. The generator $G_i \in G_i$, $i = 1, 2$ that contains the point $p$ is denoted with $[p]_i$. Two points $p_1$ and $p_2$ are called connectable, when $[p_1]_i \neq [p_2]_i$ for $i = 1, 2$, otherwise they are called parallel.

The set of cycles of the Minkowski plane is determined by the figures that result from intersections with the planes of the three-dimensional projective space $P$ that are no tangential planes of the hyperboloid. The following definition summarizes the properties of the Minkowski plane [KK88].

**Definition 11.1.** The sets of points $H$, generators $G_1 \cup G_2$ and cycles $X$ constitute the Minkowski plane $(H, G_1 \cup G_2, X)$, if following properties are met:

(N1) $\forall p \in H \exists G_i : G_i \in G_i, [p]_i = G_i, i \in \{1, 2\}$

(N2) $\forall G_1 \in G_1, \forall G_2 \in G_2 \exists p \in H : p \in G_i, i = 1, 2$

(N3) $\forall G \in G_1 \cup G_2 : |G| \geq 2$

(N4) $\forall G \in G_1 \cup G_2, \forall C \in X : |G \cap C| = 1$

(N5) For three pairwise connectable points $p_1, p_2, p_3 \in H$ exists exactly one $C \in X$ with $p_1, p_2, p_3 \in C$.

(R) Rectangle axiom: For $A, B, C \in X$ the set $\{[[a]_1 \cap B]_2 \cap [[a]_2 \cap C]_1 : a \in A\}$ is a cycle in $X$.

(T) Touch axiom: $\forall A \in X, a \in A, b \in H \setminus (A \cup [a]_1 \cup [a]_2) \exists B \in X : b \in B, A \cap B = \{a\}$

(S) Symmetry axiom: If $C, D \in X$ are two cycles and, if the point $p \in C \setminus D$ fulfills the condition $[[p]_1 \cap D]_2 \cap [[p]_2 \cap D]_1 \in C$ then $[[x]_1 \cap D]_2 \cap [[x]_2 \cap D]_1 \in C$ holds for any $x \in C$. 

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Figure 4: Stereographic projection of the Minkowski geometry.

The hyperboloid $H$ can be projected into the Euclidean plane $\mathbb{R}^2$ by means of a central projection. Therefore a point $n \in H$ is chosen as the center of the projection and a plane $P$ of the projective space with $n \notin P$ as projection plane. If $T_n$ is the tangential plane in the point $n$, we consider $E := E \setminus T_n$. The generators of $H$ are mapped to two sets of parallel lines in $E$. The cycles $C \in X$ that do not pass the point $n$ are transformed to hyperbolas with the images of the generators in $H$ as asymptotes. The cycles in $X$ that pass the projection center $n$ are mapped to the lines of $E$ that are different from the images of the two sets of generators. Figure 4 visualizes this projection.

Let $F$ be the field of the coordinates of the three-dimensional projective space $P$. The points of this projective space can be written as 2x2-matrices $M_{22}$ over the field $F$. The set of points can be written as $P := \{F^*x : x \in M_{22}\}$. The subset $H := \{F^*x : \det x = 0\}$ defines a hyperboloid in $P$.

For every point $F^*a \in P$ we define $(F^*a)^\perp := \{F^*x \in P : \det(x + a) = \det x + \det a\}$ as the polar plane of the point $F^*a$ regarding the quadric $H$. The plane $(F^*a)^\perp$ is a tangential plane of $H$ only, if $\det a = 0$. Since every
plane of the projective space is a polar plane for some point in the projective
space, the sets \((F^* a)^\perp \cap H\) with \(\det a \neq 0\) determine all sections of \(H\) with
planes that are not tangential.

Concluding, the Minkowski plane \((H, G_1 \cup G_1, X)\) over the commutative
field \(F\) can be represented with following coordinates:

\[
H = \{F^* x : \det x = 0\}
\]
\[
G_1 = \left\{\left. F^* \begin{pmatrix} g_1 x_1 \\
g_1 x_2 \\
g_2 x_1 \\
g_2 x_2 \end{pmatrix} \right| (x_1, x_2) \in F^{2*} \right\} : (g_1, g_2) \in F^{2*}
\]
\[
G_2 = \left\{\left. F^* \begin{pmatrix} x_1 g_1 \\
x_1 g_2 \\
x_2 g_1 \\
x_2 g_2 \end{pmatrix} \right| (x_1, x_2) \in F^{2*} \right\} : (g_1, g_2) \in F^{2*}
\]
\[
X = \{(F^* a)^\perp \cap H : a \in M_{22}, \det a \neq 0\}
\]

12 Describing the Minkowski Plane using Equations

The set \(Q\) defines a one-sheeted hyperboloid in the three-dimensional real
projective space \(P\).

\[
Q := \{(x, y, z, t) \in P : x^2 - y^2 + z^2 - t^2 = 0\}
\]

The stereographic projection \(\sigma\) with projection center \(n = F^* (0, 0, 1, 1)\)
maps \(Q\) to the plane \(P : z = 0\). The generators of the Minkowski plane are
mapped to two crossing bundles of parallel lines. If \(T_n\) is the tangent plane
to the hyperboloid \(Q\) through the point \(n\), then \(T_n\) and \(Q\) intersect in two
lines that meet in the point \(n\). These two lines have no image in the plane \(P\).
By this reason the projection \(\sigma\) is analytically continued by identifying the
image of these two lines with the two distant generators of the Minkowski
plane.

The cycles of the Minkowski plane can be described in the plane \(P\) by
replacing the Euclidean metric that is defined by the quadratic form \(x^2 + y^2\)
with the indefinite form \(x^2 - y^2\). If a suitable coordinate system is chosen
for \(P\), a cycle \(C\) can be described in the finit as the following set of points:

\[
C = \{(x, y) \in \mathbb{R}^2 : (x - a)(y - b) = c, a, b, c \in \mathbb{R}\}
\]  

Here the cycles of the Minkowski plane are of the shape of hyperbolas
with asymptotes which are parallel to the lines \(y = c\) and \(x = c\) with \(c \in \mathbb{R}\).

This model of the Minkowski plane can be generalized by using a field
\(F\) instead of the real numbers \(\mathbb{R}\). In this case the cycles of the Minkowski
plane fulfill following condition:

\[
C = \{(x, y) \in F^2 : (x - a)(y - b) = c, a, b, c \in F\}
\]
13 Combinatorial Aspects of the Minkowski Plane

We consider a Minkowski plane based on a finite field $\mathcal{F}$ with $|\mathcal{F}| = q$ elements. From the analytical representation of the Minkowski plane introduced in section 11 follows for the set of generators $\mathcal{G}_1 \cup \mathcal{G}_2$ that $|\mathcal{G}_i| = q + 1$ for $i = 1, 2$. With (N2) follows for the number of points in the Minkowski plane $|\mathcal{H}| = (q + 1)^2$. Furthermore we gain the number of cycles $|\mathcal{X}| = (q + 1)q(q - 1)$ from properties (N1), (N2), (N4) and (N5).

14 Encryption based on the Minkowski Plane

The encryption scheme introduced in [Cap21] for the Möbius plane can be transferred to the Minkowski plane $(\mathcal{H}, \mathcal{G}_1 \cup \mathcal{G}_2, \mathcal{X})$. Messages and cipher texts can be identified with the set of triples of points in $\mathcal{H}$. Here we have to consider the case that the point $m$ representing the message and the point $k$ representing the key are parallel. Then these two points are not connectable. In this case the cipher text can be determined using following construction: Lets consider that $[m]_1 = [k]_1$, then we define the associated ciphertext with $c := M \cap [k]_2$, where $M$ is determined by the three messages $m, n, o$. According to (N4) there is always such a point of intersection (see figure 5). The corresponding practice can be applied in case of $[m]_2 = [k]_2$.

With these preparations we can define a ciphersystem in the Minkowski plane. Therefore a point $\infty$ shall be distinguished in $(\mathcal{H}, \mathcal{G}_1 \cup \mathcal{G}_2, \mathcal{X})$. Then we consider the derivation of the Minkowski plane in that point:

![Figure 5: Encryption in the Minkowski plane with parallel key.](image)
\[ H^\infty := H \setminus ([\infty]_1 \cup [\infty]_2) \]
\[ X^\infty := \{ C \setminus \{\infty\} : C \in X(\infty) \}, \]
\[ G_1^\infty := \{ G \setminus [\infty]_2 : G \in G_1 \setminus \{[\infty]_1\}\} \]
\[ G_2^\infty := \{ G \setminus [\infty]_1 : G \in G_2 \setminus \{[\infty]_2\}\} \]

Then \((H^\infty, G_1^\infty \cup G_2^\infty \cup X^\infty)\) is an affine plane. For two given points \(a, b \in H^\infty\) with \(a \neq b\) we denote \(a, b\) as the straight line uniquely determined by these two points. Furthermore the sets of degenerated, ordinary and all hyperbolas in the affine plane shall be defined as follows:

\[ [x] := [x]_1 \cup [x]_2 \forall x \in H^\infty \]
\[ [H^\infty] := \{ [x] \setminus [\infty] : x \in H^\infty \} \]
\[ X^\infty := \{ X \setminus [\infty] : X \in X \setminus X(\infty) \} \]
\[ \overline{X} := X^\infty \cup [H^\infty] \]

With this preparation we can define a cipher system in the Minkowski plane. For this cipher we will use straight lines as keys, however the following definition can be generalized in a way that cycles of the Minkowski plane are used as keys \[\text{[Cap94]}\].

**Definition 14.1.** Let \((H^\infty, G_1^\infty \cup G_2^\infty \cup X^\infty)\) be the derivation of the Minkowski plane \((H, G_1 \cup G_2, X)\) in the point \(\infty\) and let \(m_1, m_2, m_3 \in H^\infty\) be three different points of the affine plane that cannot be connected with one straight line. Then we want to make following definitions:

**Messages:** \(\binom{H^\infty}{3}\) shall denote the set of triples of points in \(H^\infty\). Then the set of messages can be defined as follows:

\[ M := \{ (m_1, m_2, m_3) \in \binom{H^\infty}{3} : [m_1] \neq [m_2] \neq [m_3] \neq [m_1] \neq L \in X^\infty : \{m_1, m_2, m_3\} \in L \} \]

**Cipher texts:** \(C := M\)

**Keys:** \(K := \{(k_1, k_2, k_3) \in (H^\infty)^3\}\) with following properties:

\[ k_i \notin (m_1, m_2, m_3)^\circ \] and
\[ (m_1, m_2, m_3)^\circ \cap m_i, k_i \cap m_j, k_j = \emptyset \text{ for } i, j \in \{1, 2, 3\} \text{ and } i \neq j \]

**Encryption Function:** \(F : K \times M \rightarrow C\)
15 Cryptoanalysis of the Minkowski Cipher

To analyze the property of perfectness according to definition 3.3 for the Minkowski cipher, we consider the Minkowski plane \((H, G_1 \cup G_2, X)\) over a finite field \(F\) with \(q := |F|\) elements. When we select a point \(\infty \in X\), we get an affine plane \((H^{\infty}, G_1^{\infty} \cup G_2^{\infty} \cup X^{\infty})\) as the derivation of the Minkowski plane in the point \(\infty\). Let \((m_1, m_2, m_3) \in M\) be a message from the set of messages as defined in definition 14.1. Then there exists exactly one cycle \((m_1, m_2, m_3)^{\circ} \in X^{\infty}\).

To show the perfectness of the Minkowski cipher, we look at the encryption of the message points \(m_1, m_2, m_3\) that all lie on the same cycle \(C\) of the Minkowski plane. Probability measures \(\mu\) and \(\nu\) as introduced in definition 3.1 shall be defined. We will further assume that all messages occur with the same probability. The number of possible keys that can be used to encrypt \(m_i\) to \(c_i\) is determined by the conditions for the set of keys \(K\) as defined in definition 14.1.

We look at the encryption of the message point \(m_1\). The message \((m_1, m_2, m_3)\) determines the hyperbola \(H := (m_1, m_2, m_3)^{\circ} \in X^{\infty}\). The message point \(m_1\) can be one of the \(q - 1\) points of the hyperbola \(H\). Hence the probability for the occurrence of \(m_1\) is \(\mu(m_1) = 1/(q-1)\).

The set \(K_1\) of keys of the Minkowski cipher that can be used to encrypt \(m_1\) is described by the set of all points in the affine plane that are not on the hyperbola \(H\), i.e. \(X^{\infty} \setminus H\). Hence \(|K_1| = q^2 - (q - 1)\). To find the number of keys that can encrypt the message \(m_1\) to the cipher text \(c_1\) we have to distinguish two cases.
m_1 = c_1\ Every\ point\ of\ the\ tangent\ to\ the\ hyperbola\ H\ through\ the\ point\ c_1 \\
eq c_1\ All\ points\ of\ the\ line\ m_1, c_1\ except\ m_1\ and\ c_1\ and\ the\ points\ [m_1]_1\cap[c_1]_2 \\
and\ [m_1]_2\cap[c_1]_1\ are\ possible\ keys.\ Hence\ \{f \in F : (m_1, c_1) \subset f\} = q - 1.

The following table summarizes the resulting a-priori and a-posteriori probabilities for the encryption of m_2:

| Case | A-priori probability | A-posteriori probability |
|------|----------------------|--------------------------|
| m_1 = c_1, m_2 = c_2 | \mu(m_2) = \frac{1}{q-2} | \nu(m_2, c_2) = \frac{1}{q-1} |
| m_1 = c_1, m_2 \neq c_2 | \mu(m_2) = \frac{1}{q-2} | \nu(m_2, c_2) = \frac{q-1}{q^2-3q+1} |
| m_1 \neq c_1, m_2 = c_2 | \mu(m_2) = \frac{1}{q-2} | \nu(m_2, c_2) = \frac{1}{q^2-3q+1} |
| m_1 \neq c_1, m_2 \neq c_2 | \mu(m_2) = \frac{1}{q-2} | \nu(m_2, c_2) = \frac{q-1}{q^2-3q+1} |

Hence the encryption of the message point m_2 is also approximately perfect for large q.

The a-priori probability for the encryption of m_3 is \mu(m_3) = \frac{1}{q-3}. The following table shows also the a-posteriori probabilities:
Also the last encryption step is approximately perfect for large $q$.

To show the completeness of the Minkowski cipher in the sense of definition 3.5, we will look at the special case of a number field $F$ with char $F = 2$. The points $p \in H^\infty$ can then be described in the form $p(x, y)$ and $x \in \mathbb{Z}_2^n$, $y \in \mathbb{Z}_2$. A message of $2n$ bits length can be understood as the point $p(x, y)$ with coordinates $x$ and $y$. The $i$-th bit of the message shall be the $i$-th position in the representation of the point $p(x, y)$, which would be the $i$-th component of the vector $x$ for $i = 1, ..., n$ and the $(i - n)$-th component of the vector $y$ for $i = n + 1, ..., 2n$.

Let $m$ be a message point and $c$ be the corresponding ciphertext point when using the key point $k$. The encryption is complete in the sense of definition 3.5, if there is a representation of $m$ where a change in position $i$ of $c$ is caused by a change in position $j$ of $c$.

Two cases have to be considered. The indices $i$ and $j$ may affect the same coordinate of both $m$ and $c$. Without restricting generality we assume the $x$-coordinate. Alternatively, the first index $i$ may affect one coordinate, say the $x$-coordinate of $m$, the second index $j$ may then affect the $y$-coordinate of $c$.

We have a look at the second case first. Let $m(x, y)$ and $c(u, v)$ be the message and ciphertext points and their coordinates. Furthermore, let $e_i$, $i \in \{1, ..., n\}$ be the unit vectors in $\mathbb{Z}_2^n$. We transform the index $j$ to $j \rightarrow j - n$. Then $m'(x + e_i, y)$ and $c'(u, v + e_j)$ are the message and ciphertext points with changes in positions $i$ and $j$.

It has to be shown that for any key $k$ it is possible to find two points $m, c$ as message and ciphertext in a way that the encryption function $f \in F$ holds $f(k, m) = c$ and $f(k, m') = c'$. Due to the definition of the encryption function in definition 3.5, the cases $|m| \cap |k| \leq 2$, $|m_1| = |k|$ and $|m_2| = |k|$ need to be distinguished. Without restricting generality we assume that $k = (0, 0)$, otherwise the Minkowski plane can be transformed accordingly.

Due to the way, the keys of the Minkowski cipher have been chosen in definition 3.5, we have to find two lines $K, K'$ in $G_1^\infty \cup G_2^\infty \cup X^\infty$ over the field $\mathbb{Z}_2^3$ with $m, c, k \in K$ and $m', c', k \in K'$.
Our assertion is equal to

\[
\text{det}(m, c) = 0 \\
\text{det}(m', c') = 0
\]  

Substituting the coordinates for \( m, m', c, c' \) results in:

\[
xv - uy = 0 \\
xv + xe_j + ve_i + e_i e_j - uy = 0
\]  

Or:

\[
xv - uy = 0 \\
xv - uy = 0
\]

It is easy to provide coordinates for \( m(x, y) \) and \( c(u, v) \) that satisfy these equations.

\( [m]_1 = [k]_1 \) According to the construction of the encryption function in definition 14.1 the cipher text points \( c, c' \) are both on the line \( [k]_2 \). So the conditions in equation (5) are met.

\( [m]_2 = [k]_2 \) It follows that \( c \) is on \( [k]_1 \). So the first condition of equation (5) is met. To fulfill also the second one, we have to find a line \( K' \) with \( m', c', k \in K' \). Again values can be found for \( m \) and \( c \) so that following equation is fulfilled:

\[
\text{det}(m', c') = 0
\]  

This concludes the examination of the case that the indices \( i \) and \( j \) affect different coordinates of \( m \) and \( c \).

To show the first case we again assume that the two points \( m(x, y) \) and \( c(u, v) \) and their coordinates as given. Let now \( m'(x + e_i, y) \) and \( c'(u + e_j, v) \) be the altered points. Again the three different ways of encryption need to be distinguished:

\( [m] \cap [k] \leq 2 \) Once again it has to be shown that condition (5) is fulfilled. In analogy to the procedure shown above we reach following conditions:

\[
xv - uy = 0 \\
xv + e_i v - yu - ye_j = 0
\]  

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Which can be simplified to:

\[ xv - uy = 0 \]
\[ e_i v - ye_j = 0 \]  (10)

Again it is easy to find suitable points \( m \) and \( c \) to meet these conditions for all possible \( i, j \).

\([m]_1 = [k]_1\) In this case the message has the coordinates \( m(x, 0) \). From definition \[14.1\] of the Minkowski cipher follows that the cipher text has the coordinates \( c(0,v) \). Hence the condition \[5\] leads to the following:

\[ xv = 0 \]
\[ xv + e_i v = 0 \]  (11)

Since both \( x \) and \( v \) cannot be 0 due to the requirements for the keys of the Minkowski cipher in definition \[14.1\] equation (11) has no legitimate solution.

\([m]_2 = [k]_2\) In this case the message has the coordinates \( m(0,y) \). From definition \[14.1\] of the Minkowski cipher follows that the cipher text has the coordinates \( c(u,0) \). Hence the condition \[5\] leads to the following:

\[ (0 + e_i)0 - y(u + e_j) = 0 \]  (12)

Since \( y \neq 0 \) and \( u \neq -e_j \), as \( c' \neq k \) due to the requirements for the keys of the Minkowski cipher in definition \[14.1\] equation (12) has no legitimate solution.

As a summary we showed that the Minkowski cipher with the encryption functions introduced in definition \[14.1\] is not complete in accordance with definition \[3.5\].

16 Resume and Outlook

We introduced encryption functions on the geometric structures of the Laguerre and the Minkowski geometry and showed that the former fulfills both the requirements of perfectness in the sense of Shannon \[Sha49\] and completeness according to \[KD79\]. The latter is also perfect in first approximation but doesn’t have the property of completeness, at least when using the encryption functions as defined in this paper. Further research can be done in the Minkowski geometry to find improved encryption functions. One way would be to apply a similar approach to that one used in the Laguerre geometry, i.e. select generators as sets of possible messages and ciphertexts.
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