Curve Diagrams, Laminations, and the Geometric Complexity of Braids

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Abstract

Braids can be represented geometrically as curve diagrams. The geometric complexity of a braid is the minimal complexity of a curve diagram representing it. We introduce and study the corresponding notion of geometric generating function. We compute explicitly the generating function for the group of braids on three strands and prove that it is not rational.

1 Introduction

Braid groups can be approached from various points of view, including algebraic and geometric ones.

The algebraic point of view is based on finite presentations of the group of braids, involving several possible generating families, the most famous being the families of Artin generators [1] and Garside generators [10]. Artin generators are usually considered as the most “natural” generators, but Garside generators have proved to be more tractable in answering several algorithmic and combinatorial questions.

Given a finite presentation, a braid is identified with a set of words. Each word has a complexity, which is its length, and the complexity of the braid is the minimal complexity of the words that represent it. Then, the generating function, or growth series, of the group $\mathcal{B}_n$ of $n$-strand braids, is defined by $\mathcal{B}_n(z) = \sum_{\beta \in \mathcal{B}_n} z^{||\beta||}$, where $||\beta||$ is the complexity of the braid $\beta$.

Both the complexity of the braid and the corresponding generating function depend on the generators. A natural question is to compute the generating function. Each group $\mathcal{B}_n$ has a rational generating function for Garside generators [3]. It is also known that the group $\mathcal{B}_3$ has a rational generating function for Artin generators [13] [14], but no such result is known to hold for the groups $\mathcal{B}_n$ with $n \geq 4$.

The above-mentioned results are obtained by using clever normal forms. A normal form consists in selecting a representative word for each braid. Several questions appear immediately: does there exist computable normal forms? regular normal forms? regular and geodesic normal forms? Whereas the answers to the first two questions can be shown to be intrinsic i.e. independent of the generating family, the answer to the third question is specific to each family.

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of generators [15]. When a regular and geodesic normal form exists, the generating function is rational and computing it is straightforward when considering an automaton recognising the normal form [8].

For the braid groups, the symmetric Garside normal form is a well-known regular geodesic normal form for the Garside generators [10]. On the contrary, the existence of a regular geodesic normal form for the Artin generators is a famous open question.

At first glance, the geometric approach seems quite different from its algebraic counterpart: braids are no longer considered as sets of words but as sets of drawings. In the geometric world, each drawing has a complexity (e.g. the number of intersections it has with some fixed set of curves), and the complexity of a braid is the minimal complexity of the drawings that represent it [7]. Like in the algebraic case, the group $B_n$ has a “geometric generating function”, or geometric growth series, defined by $B_n(z) = \sum_{\beta \in B_n} z^{||\beta||}$, where $||\beta||$ is the geometric complexity of the braid $\beta$.

The general goal of this paper is to study the notion of geometric generating function, which, to the best of our knowledge, has not been explored yet in the literature. The first step is to identify a geodesic normal form. Indeed, the notions of normal form and of geodicity are well-defined in the geometric context: choosing a normal form consists in picking one drawing per braid, and the normal form is geodesic if its elements are representatives of their respective braids with the minimal complexity. Here, the relevant geodesic normal form will be related to the notion of tight curve diagram (see Figure 1), which was already studied in [7].

The geometric complexity is arguably easier to compute than the length complexity. Indeed, consider a fixed braid group $B_n$ and the Artin generators. On the one hand, computing the length complexity of a braid represented by a word of length $k$ requires up to $2^{O(k)}$ operations: one has to check braid equality for all the words of length less than $k$. On the other hand, there exist algorithms that compute in $O(k)$ operations the geometric complexity of a braid represented by a drawing with complexity $k$ [12].

The main result of this paper goes in the opposite direction: we show that, even in the simple case of the group $B_3$, the geometric generating function $B_3(z)$ is not rational (see Theorem 5.1). This is in sharp contrast with the length-based generating functions of $B_3$ for Artin and Garside generators, which are rational, as recalled above. In addition, we also estimate the series $B_n(z)$ with $n \geq 4$.

To obtain these results, we use several tools that also do have an intrinsic interest. First, we show the connection between the complexities associated to two dual geometric representations of the braids, the braid laminations and the curve diagrams, and we prove that both complexities yield the same generating functions. Second, we design a system of integer-valued coordinates that capture directly the geometric representations of braids, and from which computing the geometric complexity of braids is straightforward. This system of coordinates is analogous to that of Dynnikov [6], and the algorithms for computing both kinds of coordinates have similar flavours, hence have comparable (low) complexities. However, our new system has specific advantages, since it is better suited for computing braid complexities, hence for computing geometric generating functions.
Figure 1: Tight lamination (left) and tight curve diagram (right) representing the same braid
2 Braids, Integral Laminations and Curve Diagrams

In this first part, we introduce several definitions of the group of braids on \( n \) strands. These definitions come from algebraic topology as well as from discrete group theory. Then, we relate braids to integral laminations and to curve diagrams, which are graphical representations of braids, stemming from the topological definition of braids.

2.1 Braids

The group of braids on \( n \) strands was originally introduced by Artin [1], who came with the following algebraic description.

**Definition 2.1 (Braid group).**

The group of braids on \( n \) strands is the group

\[
B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2 \rangle.
\]

This finite presentation of the group of braids comes along with the representation of braids as an isotopy class of braid diagrams, as illustrated in Figure 2.

![Figure 2: Braid diagram of the generator \( \sigma_i \) (\( 1 \leq i \leq n - 1 \))](image)

However, in this paper we focus on another, equivalent, approach of the group of braids. Indeed, the group of braids on \( n \) strands can also be defined as the mapping class group (also called modular group) of the unit disk with \( n \) punctures. Let us make this statement more precise.

Let \( D^2 \subseteq \mathbb{C} \) be the closed unit disk, let \( \partial D^2 \) be the unit circle (i.e. the boundary of \( D^2 \)), and let \( P_n \subseteq (-1, 1) \) be a set of size \( n \). We will refer below to the elements of \( P_n \) as being punctures in the disk \( D^2 \), and number them from left to right: \( P_n = \{ p_i : 1 \leq i \leq n \} \), with \( p_1 < \ldots < p_n \). We also call left point the point \(-1\), which we will also denote by \( p_0 \); and call right point the point \( +1 \), which we will also denote by \( p_{n+1} \).

Then, let \( H_n \) be the group of orientation-preserving homeomorphisms \( h : \mathbb{C} \to \mathbb{C} \) such that \( h(P_n) = P_n \), \( h(\partial D^2) = \partial D^2 \) and \( h(1) = 1 \), \( h(-1) = -1 \), i.e. the homeomorphisms fixing \( \partial D^2 \) and \( P_n \) setwise, and \( \pm 1 \) pointwise.
Theorem 2.2 (see [2]).
The group $B_n$ of braids on $n$ strands is isomorphic to the mapping class group of the punctured disk $D^2 \setminus P_n$, i.e. isomorphic to the quotient group of $H_n$ by the isotopy relation. ■

It is remarkable that this definition does not depend on which set $P_n$ of punctures we chose. In addition, each braid appears as a class of homeomorphisms of the unit disk $D^2$, which conveys the idea of giving a graphical representation of the braid.

According to standard notations for braids, we will denote below the class of the homeomorphism $h$ by $[h]$, and the class of the homeomorphism $h \circ h'$ by $[h \circ h'] = [h'][h]$: composition to the left gives rise to a braid multiplication to the right, and vice-versa.

For flexibility reasons, we introduce here a slightly different characterization of the group of braids, analogous to that of Theorem 2.2. Here, instead of considering the group $H_n$, we denote by $H^*_n$ the group of orientation-preserving homeomorphisms $h : \mathbb{C} \to \mathbb{C}$ such that $h(P_n) = P_n$, $h(-1) = -1$ and $h(1) = 1$, i.e. fixing $\{-1\}, \{1\}$ and $P_n$ setwise.

Theorem 2.3.
The group $B_n$ of braids on $n$ strands is isomorphic to the quotient group of $H^*_n$ by the isotopy relation. ■

Theorem 2.3 identifies braids to isotopy classes of self-homeomorphisms of $\mathbb{C}$. More precisely, let $S$ be a subset of the complex plane and let $\beta$ be a braid, and consider the isotopy class $\beta(S) = \{ h(S) : h \text{ is an homeomorphism that represents } \beta \}$. The group of braids $B_n$ acts transitively on the set $\{ \beta(S) : \beta \in B_n \}$, which induces an equivalence relation on the group $B_n$ itself.

We focus below on two subsets of $\mathbb{C}$: We call respectively trivial lamination and trivial curve diagram the sets

$$L := \left\{ \frac{1}{2}(p_j + p_{j+1}) + i \mathbb{R} : j \leq 1 \leq n-1 \right\} \text{ and } D := [-1, 1].$$

If $S = L$ or $S = D$, then the action of $B_n$ on $\{ \beta(S) : \beta \in B_n \}$ is free (see [2] for details). This means that the sets $\beta(S)$ and $\gamma(S)$ are disjoint as soon as $\beta \neq \gamma$: hence, each set $h(S)$ belongs to the set $\beta(S)$ for one unique braid $\beta$: we say that $h(S)$ represents the braid $\beta$.

2.2 Laminations

Definition 2.4 (Lamination).
Let us consider the set $P_n$ of $n$ punctures inside the disk $D^2$. We call lamination, and denote by $\mathcal{L}$, the union of $n-1$ non-intersecting open curves $\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}$ such that each curve $\mathcal{L}_j$

- contains two vertical half-lines with opposite directions (i.e. sets $z_j + i \mathbb{R}_{\leq 0}$ and $z_j' + i \mathbb{R}_{\geq 0}$);
- splits the plane $\mathbb{C}$ into one left region that contains the left point and $j$ punctures, and one right region that contains the right point and $n-j$ punctures.

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Figure 3 represents two laminations, including \( L \), the trivial one. In all subsequent figures, punctures are indicated by white dots, and the left and right points are indicated by black dots; the grey area represents the unit disk \( D^2 \), the curves of the lamination are drawn in black, and the segment \([−1, 1]\) is drawn in white.

![Figure 3: Laminations](image)

(a) Trivial lamination     (b) Non-trivial lamination

Then, following Dynnikov and Wiest [7], we define the norm of a lamination, and the \textit{laminated norm} of a braid.

**Definition 2.5** (Laminated norm and tight lamination).

Let \( \beta \in B_n \) be a braid on \( n \) strands, and let \( \mathcal{L} \) be a lamination representing \( \beta \).

The \textit{laminated norm} of \( \mathcal{L} \), which we denote by \( \|\mathcal{L}\|_\ell \), is the cardinality of the set \( \mathcal{L} \cap [−1, 1] \), i.e. the number of intersection points between the segment \([−1, 1]\) and the \( n−1 \) curves of the lamination \( \mathcal{L} \).

Moreover, if, among all the laminations that represent \( \beta \), the lamination \( \mathcal{L} \) has a minimal laminated norm, then we say that \( \mathcal{L} \) is a \textit{tight} lamination. In this case, we also define the \textit{laminated norm} of the braid \( \beta \), which we denote by \( \|\beta\|_\ell \), as the norm \( \|\mathcal{L}\|_\ell \).

![Figure 4: Identifying braids to tight laminations](image)

(a) \( \varepsilon \)     (b) \( \sigma_2^{-1} \)     (c) \( \sigma_2^{-1}\sigma_1 \)

Note that, although we call the mapping \( \beta \mapsto \|\beta\|_\ell \) a norm, following the seminal paper of Dynnikov and Wiest [7], this mapping does not satisfy standard properties of norms on metric spaces, such as separation axioms (i.e. that \( \beta = \varepsilon \) iff \( \|\beta\|_\ell = 0 \)) or sub-additivity axioms (i.e. that \( \|\beta \cdot \gamma\|_\ell \leq \|\beta\|_\ell + \|\gamma\|_\ell \) for all \( \beta, \gamma \in B_n \)). Counterexamples to those properties are provided.
by the fact that \( \| \varepsilon \|_\ell = n - 1 \) and that \( \| (\sigma_1 \sigma_2)^k \|_\ell = 2(F_{2k+3} - 1) \), where \( F_k \) denotes the \( k \)-th Fibonacci number.

However, Dynnikov and Wiest prove in [7] that the mapping \( \beta \mapsto \log \| \beta \|_\ell \) is comparable to a norm, i.e. that there exists positive constants \( m_n \) and \( M_n \) and a norm \( \mathcal{N} \) of \( B_n \) such that \( m_n(\mathcal{N}(\beta) - 1) \leq \log \| \beta \|_\ell \leq M_n(\mathcal{N}(\beta) + 1) \) for all \( \beta \in B_n \).

Finally, observe that our notions of lamination and of laminated norm follow the ones used in [7] but are slightly different from the ones defined in previous work (see [5, 9]), which we call closed laminations.

A closed lamination is the union of \( n \) non-intersecting closed curves \( \mathcal{L}_1, \ldots, \mathcal{L}_n \) such that each curve \( \mathcal{L}_j \) splits the plane \( \mathbb{C} \) into one inner region that contains the left point and \( j \) punctures, and one outer region that contains the right point and \( n - j \) punctures. Informally, closed laminations may be easily obtained from (non-closed) laminations as follows: for each integer \( i \leq n - 1 \), bend both half-line of the curve \( \mathcal{L}_i \) to the left, in order to transform \( \mathcal{L}_i \) into a closed line; then, add a circle \( \mathcal{L}_n \) that will enclose all punctures and all lines \( \mathcal{L}_i \) but not the right point.

Figure 5 presents the closed laminations corresponding to the laminations displayed in Figure 3.

From now on, we exclusively use laminations.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{closed_laminations.png}
\caption{Closed laminations}
\end{figure}

2.3 Curve Diagrams

As mentioned in Section 2.1, curve diagrams are an alternative to laminations.

Definition 2.6 (Curve diagram).
Let us consider the set \( P_n \) of \( n \) punctures inside the disk \( D^2 \). We call curve diagram, and denote by \( \mathcal{D}_n \), each non-intersecting open curve, with endpoints \(-1\) and \(+1\), that contains each puncture of the disk.

\[ \square \]

Since both laminations and curve diagrams consist in drawings on the complex plane, we represent them in analogous ways (see Figure 6).

We adapt the notion of laminated norm to curve diagrams, and thereby define the norm of a curve diagram, and the diagrammatic norm of a braid.
Definition 2.7 (Diagrammatic norm and tight curve diagram).

Let $\beta \in B_n$ be a braid on $n$ strands, and let $D$ be a curve diagram representing $\beta$. Recall that $L$ is the trivial lamination.

The diagrammatic norm of $D$, which we denote by $\|D\|_d$, is the cardinality of the set $D \cap L$, i.e. the number of intersection points between the curve diagram $D$ and the $n-1$ vertical lines of the lamination $L$.

Moreover, if, among all the curve diagrams that represent $\beta$, the curve diagram $D$ has a minimal diagrammatic norm, then we say that $D$ is a tight curve diagram. In this case, we also define the diagrammatic norm of the braid $\beta$, which we denote by $\|\beta\|_d$, as the norm $\|D\|_d$. 

Figure 6: Curve diagrams

Figure 7: Identifying braids to tight curve diagrams
3 Norm-Preserving Transformations

Counting intersections between the curves of a lamination $\beta(L)$ and the trivial curve diagram $D$ was the basic idea that led to the norm introduced by Dynnikov and Wiest in [7].

A natural question the comparison between the norm defined on laminations and the norm defined on curve diagrams (Definitions 2.5 and 2.7). We demonstrate here a simple connection between laminated and diagrammatic norms.

**Proposition 3.1.**

Let $\beta$ be a braid on $n$ strands. We have: $\|\beta\|_\ell = \|\beta^{-1}\|_d$, i.e. the laminated norm of the braid $\beta$ is equal to the diagrammatic norm of the braid $\beta^{-1}$.

**Proof.** Let $L$ be the trivial lamination and let $h \in H_\omega^n$ be a representative of the braid $\beta$ such that $h(L)$ is a tight lamination. Since the curve $h^{-1}(D)$ is a curve diagram of the braid $\beta^{-1}$, it follows that

$$\|\beta\|_\ell = |h(L) \cap D| = |h^{-1}(h(L) \cap D)| = |L \cap h^{-1}(D)| \geq \|\beta^{-1}\|_d.$$ 

One proves similarly that $\|\beta^{-1}\|_d \geq \|\beta\|_\ell$, which completes the proof.

From Proposition 3.1 follow directly Corollary 3.2 and Theorem 3.3.

**Corollary 3.2.**

Let $n$ and $k$ be positive integers. We have: $|\{\beta \in B_n : \|\beta\|_\ell = k\}| = |\{\beta \in B_n : \|\beta\|_d = k\}|$, i.e. the braids on $n$ strands with laminated norm $k$ are as numerous as those with diagrammatic norm $k$.

**Theorem 3.3.**

For all positive integers $n$, the geometric generating functions $\sum_{\beta \in B_n} z^{||\beta||_\ell}$ and $\sum_{\beta \in B_n} z^{\|\beta\|_d}$ are equal.

Geometrical symmetries induce some additional invariance properties of the laminated and diagrammatic norms. Indeed, consider the group morphisms $S_v$, $S_h$ and $S_c$ such that $S_v : \sigma_i \mapsto \sigma_{n-i}^{-1}$, $S_h : \sigma_i \mapsto \sigma_i^{-1}$ and $S_c : \sigma_i \mapsto \sigma_{n-i}$. Observe that $S_h \circ S_v = S_v \circ S_h = S_c$. is the morphism $\beta \mapsto \Delta^{-1} \cdot \beta \cdot \Delta$. 

![Figure 8: From $\|\sigma_{-1}^{-1}\|_\ell \text{ to } \|\sigma_{-1}^{-1}\|_d$](image)
Lemma 3.4.  
For all braids $\beta \in B_n$, we have: 

\[ \| \beta \|_\ell = \| S_v(\beta) \|_\ell = \| S_h(\beta) \|_\ell = \| S_c(\beta) \|_\ell \quad \text{and} \quad \| \beta \|_d = \| S_v(\beta) \|_d = \| S_h(\beta) \|_d = \| S_c(\beta) \|_d, \]  

i.e. the laminated and diagrammatic norms are invariant under $S_v$, $S_h$ and $S_c$.  

\begin{proof} 
From a geometric point of view, the braid morphisms $S_v$, $S_h$ and $S_c$ respectively induce vertical, horizontal and central symmetries on the laminations and the curve diagrams. More precisely, if $L$ and $D$ are a lamination and a curve diagram representing some braid $\beta \in B_n$, then:

- their vertically symmetric lamination $L_v$ and curve diagram $D_v$ represent the braid $S_v(\beta)$;
- their horizontally symmetric lamination $L_h$ and curve diagram $D_h$ represent the braid $S_h(\beta)$;
- their centrally symmetric lamination $L_c$ and curve diagram $D_c$ represent the braid $S_c(\beta)$.
\end{proof} 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{symmetries.png}
\caption{Vertical, horizontal and central symmetries}
\end{figure}
4 Counting Braids With a Given Norm

We aim now at counting directly tight curve diagrams with a given (diagrammatic) norm. Henceforth, we will denote by \( N_{n,k} \) the number of braids on \( n \) strands and (laminated or diagrammatic) norm \( k \), i.e.
\[
N_{n,k} = |\{ \beta \in B_n : \|\beta\|_d = k \}| = |\{ \beta \in B_n : \|\beta\|_\ell = k \}|
\]
and we will focus on computing the generating function \( B_n(p) := \sum_{k \geq 0} N_{n,k} x^k \). In order to achieve this goal, we provide here intrinsic characterizations of tight curve diagrams, then introduce discrete combinatorial structures that will be in bijection with tight curve diagrams of a given norm.

4.1 A Characterization of Tight Curve Diagrams

In the literature, most characterizations of tightness hold for laminations. Here, we express them in terms of curve diagrams.

**Definition 4.1 (Arcs and neighbour endpoints).**

Let \( L \) be a lamination and let \( D \) be a curve diagram such that \( L \cap D \) is a finite set, and such that \( L \) and \( D \) actually cross each other at each point of \( L \cap D \): we say that the sets \( D \) and \( L \) are compatible with each other.

We call arc of \( L \) with respect to \( D \) (or \( (L,D) \)-arc) a connected component of \( L \setminus D \). The endpoints of a \( (L,D) \)-arc necessarily lie on \( D \); moreover, if the arc is bounded, i.e. if it has two endpoints, then we call these endpoints neighbour endpoints in \( L \) with respect to \( D \) (or \( (L,D) \)-neighbour endpoints).

Similarly, we call arc of \( D \) with respect to \( L \) (or \( (D,L) \)-arc) a connected component of \( D \setminus L \). Such an arc has two endpoints, which we call neighbour endpoints in \( D \) with respect to \( L \) (or \( (D,L) \)-neighbour endpoints).

From this notion of arcs and neighbour endpoints follows a standard intrinsic characterization of tight laminations(see [9, 5]).

**Proposition 4.2.**

Let \( L \) be a lamination and let \( D \) be the trivial curve diagram. The lamination \( L \) is tight if and only if, for every pair \( (u,v) \) of elements of \( L \cap D \), with \( u \leq v \), that are both \( (L,D) \)- and \( (D,L) \)-neighbour endpoints, the real interval \( ]u,v[ = \{ x \in \mathbb{R} : u < x < v \} \) contains at least one puncture among \( p_0, p_1, \ldots , p_n \).

From Proposition 4.2 follows a characterization of tight curve diagrams.

**Proposition 4.3.**

Let \( D \) be a curve diagram. The curve diagram \( D \) is tight if and only if \( D \) is compatible with \( L \) and if, for every pair \( (u,v) \) of elements of \( L \cap D \) that are both \( (L,D) \)- and \( (D,L) \)-neighbour endpoints, the arc of \( D \) with endpoints \( u \) and \( v \) contains exactly one point among \( p_1, \ldots , p_n \).

**Proof.** First, note that, if \( L \) and \( D \) have a common point \( p \) at which \( D \) does not cross a curve of \( L \), then \( D \) is certainly not tight: indeed, \( p \) cannot be one of the points \( p_1, \ldots , p_n \) and therefore...
one can modify slightly \( \mathcal{D} \) in order to obtain a curve diagram \( \mathcal{D}' \) isotopic to \( \mathcal{D} \) (i.e., representing the same braid) and such that \( L \cap \mathcal{D}' \subseteq (L \cap \mathcal{D}) \setminus \{p\} \). Henceforth, we assume that \( \mathcal{D} \) is compatible with \( L \).

Let \( h \) be an homeomorphism of \( \mathbb{C} \) such that \( h(D) = D', h(P_n) = P_n, h(1) = 1 \) and \( h(-1) = -1 \). Then, let \( L \) be the lamination such that \( h(L') = L \). According to Proposition \( \PageIndex{5.1} \) the diagram \( \mathcal{D} \) is tight if and only if \( L \) is tight. According to Proposition \( \PageIndex{4.2} \) this also means that, for every pair \((u, v)\) of elements of \( L \cap \mathcal{D} \) that are both \((L, D)\)- and \((D, L)\)-neighbour endpoints, the arc of \( D \) with endpoints \( u \) and \( v \) contains at least one point among \( p_1, \ldots, p_n \). Clearly, \((u, v)\) are both \((L, D)\)- and \((D, L)\)-neighbour endpoints if and only if \((h(u), h(v))\) are both \((L, D')\)- and \((D, L')\)-neighbour endpoints: Proposition \( \PageIndex{4.3} \) follows.

### 4.2 From Drawings to Combinatorics

Proposition \( \PageIndex{4.3} \) provides an intrinsic characterisation of tight curve diagrams. However, such a characterization is not directly suitable for counting tight curve diagrams. We are going to introduce new tools that will indeed allow us to count tight curve diagrams.

**Definition 4.4** (Endpoints ordering).

Let \( \mathcal{D} \) be curve diagram compatible with \( L \), and let \( L_i \) be a curve of the lamination \( L \), with \( 1 \leq i \leq n - 1 \).

The punctures \( p_1 \) and \( p_n \) do belong to distinct connected components of \( \mathbb{C} \setminus L_i \), hence the curve \( L_i \) intersects the curve diagram \( \mathcal{D} \) at least once. We orient each curve \( L_i \) from bottom to top and thereby induce a linear ordering on \( L_i \cap \mathcal{D} \): we denote by \( L_i^j \) the \( j \)-th smallest element of \( L_i \cap \mathcal{D} \).

For the sake of coherence, we also define the sets \( L_0 := \{-1\} \) and \( L_n := \{+1\} \). Then, we denote by \( L_0^i \) the left point \(-1\), and by \( L_n^i \) the right point \(+1\).

We also orient \( \mathcal{D} \) from \(-1\) to \(+1\). We thereby induce a linear ordering on \( \mathcal{D} \): we denote by \( \overline{v} \) the \( k \)-th smallest element of \( L \cap \mathcal{D} \). In addition, we denote by \( \overline{v}^0 \) the left point \(-1\), and by \( \overline{v}^{n+1} \) the right point \(+1\), where \( m = |L \cap \mathcal{D}| \).

The orderings on \( L_i \) (when \( 1 \leq i \leq n - 1 \)) and \( \mathcal{D} \) give rise to the notion of neighbour relation.

**Definition 4.5** (Witnesses).

Let \( \mathcal{D} \) be a curve diagram compatible with \( L \).

We call endpoint witness, neighbour witness and puncture witness of \( \mathcal{D} \) the respective sets

\[
E_D^w := \{(u, v) : 0 \leq u \leq n, 1 \leq v \leq |D \cap L_i|\};
\]

\[
N_D^w := \{\{(u, v), (u', v')\} : \text{the endpoints } L_u^w \text{ and } L_{u'}^w \text{ are } (D, L)\text{–neighbours}\};
\]

\[
P_D^w := \{\{(u, v), (u', v')\} : \text{the } (D, L)\text{–arc with endpoints } L_u^w \text{ and } L_{u'}^w \text{ contains a puncture}\}.
\]

Finally, we call diagram witness the ordered triple \( D^w := (E_D^w, N_D^w, P_D^w) \).

As their name suggests, diagram witnesses indeed determine entirely the curve diagrams they are associated to.
Proof. Let us denote by $L_i$ the area lying to the left of $L$; we also define $Z_i$ as the area lying to the right of $L$. Since $E_D^w = E_D^w$, we have $|L_i \cap D| = |L_i \cap D'|$ for each curve $L_i$; therefore, up to some homeomorphism of $\mathcal{C}$ that leaves each line $L_i$ globally invariant, we may assume that $L \cap D = L \cap D'$. The notations $L_i^j$ are then unambiguous. Moreover, since $N_D^w = N_D^w$, an immediate induction on $k$ shows that the notations $\mathcal{L}_i^j$ are unambiguous too.

In addition, since $D$ is compatible with $L$, an immediate induction on $k$ shows that the $(D, L)$-arc and the $(D', L)$-arc with endpoints $\mathcal{L}_i^j$ and $\mathcal{L}_i^{j+1}$ are contained in the same zone $Z_i$. Therefore, and since $P_D^w = P_D^w$, the diagrams $D$ and $D'$ are equal up to some homeomorphism of $\mathcal{C}$ that preserves $L \cup P_i$. This means that $D$ and $D'$ indeed represent the same braid.

Henceforth, we focus on the second part of Proposition 4.6 and assume that $D$ is tight. If $D_D^w = D_D^w$, then the first part of Proposition 4.6 states that $D$ and $D'$ represent the same braid. Moreover, note that $\|D\|_d = |E_D^w| - 2 = |E_D^w| - 2 = \|D'\|_d$. Therefore, if $D$ is tight, so is $D'$.

Conversely, let us assume that $D'$ is tight and represents the same braid as $D$. Then, the diagrams $D$ and $D'$ are related by an isotopy that preserves the lamination $L$ (see [5, 7, 9] for details). This shows that $D_D^w = D_D^w$.

Therefore, it seems convenient to see braids, and in particular their tight curve diagrams, as specific diagram witnesses. Indeed, the latter objects, being discrete by nature, are likely to be easier to count.

**Figure 10:** Endpoints ordering and diagram witness

| $E_D^w$ | $\{(0, 1), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (3, 1)\}$ |
| $N_D^w$ | $\{\{(0, 1), (1, 5)\}, \{(1, 5), (1, 4)\}, \{(1, 4), (1, 1)\}, \{(1, 1), (2, 1)\}, \{(2, 1), (2, 2)\}, \{(2, 2), (2, 1)\}, \{(1, 2), (1, 3)\}, \{(1, 3), (2, 3)\}, \{(2, 3), (3, 1)\}\}$ |
| $P_D^w$ | $\{\{(1, 5), (1, 4)\}, \{(2, 1), (2, 2)\}, \{(1, 2), (1, 3)\}\}$ |
**Definition 4.7** (Left and right neighbours, limit box and neighbouring box).

Let $D$ be a curve diagram compatible with $L$, and let $i$ be an integer with $0 \leq i \leq n$. Assume that $L_i^j$ and $L_i^k$ are $(D, L)$-neighbour endpoints (with $j < k$), and let $A$ be the $(D, L)$-arc whose endpoints are $L_i^j$ and $L_i^k$.

We say that $L_i^j$ and $L_i^k$ are left (respectively, right) neighbours if $A$ lies to the left (respectively, right) of the curve $L_i$. In addition, we say that the set $\{j, j+1, \ldots, k\}$ is a left (respectively, right) limit box with rank $i$.

Finally, we call left (respectively, right) neighbouring box with rank $i$, and denote by $B_i^l$ (respectively, $B_i^r$), the set of all left (respectively, right) limit boxes with rank $i$.

Finally, let us consider the sets $S_{a,b} = \{\{i, i+1, \ldots, j\} : a < i < j < a + 2b + 1, i + j = 2(a + b) + 1\}$, where $a$ and $b$ are non-negative integers.

**Proposition 4.8.**

Let $D$ be a tight curve diagram compatible with $L$, and let $i$ be an integer with $1 \leq i \leq n - 1$. There exists non-negative integers $a, b, a', b'$ such that $B_i^l = S_{a,b}$ and $B_i^r = S_{a',b'}$.

**Proof.** We first show that there exist non-negative integers $a$ and $b$ such that $B_i^l = S_{a,b}$. If $B_i^l$ is empty, then $B_i^l = \emptyset = S_{0,0}$. We assume henceforth that $B_i^l$ is not empty.

We show then that the inclusion relation $\subseteq$ induces a forest on the elements of the set $B_i^l$. Indeed, if $\{u, \ldots, v\}$ and $\{u', \ldots, v'\}$ are two left limit boxes with rank $i$ such that $u' < u < v$, then let $A$ and $A'$ be the $(D, L)$-arcs that lie in the zone $Z_i$ and whose endpoints are respectively $(L_i^u, L_i^v)$ and $(L_i^{u'}, L_i^{v'})$. Since $A$ and $A'$ cannot intersect each other, we must have $u' < u < v < v'$.

Moreover, if $\{u, \ldots, v\}$ is a limit box with rank $i$ and $u + 2 \leq v$, let $w$ be an element of the set $\{u + 1, \ldots, v - 1\}$. Let $A''$ be the $(D, L)$-arcs that lies in the zone $Z_i$ and with endpoint $L_i^w$. Since $A''$ cannot intersect $A$, its second endpoint must be some point $L_i^u$, hence we know that $u < w, x < v$ and that $w$ is one extremum of some left limit box with rank $i$ and strictly contained inside $\{u, \ldots, v\}$.

In particular, the limit box $\{u, \ldots, v\}$ cannot be a leaf of the forest if $u + 2 \leq v$. Hence, every leaf of the forest is of the form $(y, y + 1)$. In addition, Proposition 4.3 then states that the arc $A$ with endpoints $L_i^y$ and $L_i^{y+1}$ contains the unique puncture of the zone $Z_i$. Hence, the forest has at most one leaf: consequently, the forest is a branch, and the inclusion relation $\subseteq$ induces a total order on the neighbouring box $B_i^l$.

Furthermore, let $\{\alpha, \ldots, \beta\}$ be the maximal element of $B_i^l$. We proved above that each element of the set $\{\alpha, \ldots, \beta\}$ is an extremum of some left limit box with rank $i$. Hence, an immediate induction on $j$ shows that each set $\{\alpha + j, \ldots, \beta - j\}$ (with $0 \leq j \leq \frac{\beta - \alpha}{2}$) must be a left limit box with rank $i$. This proves that the integers $\alpha$ and $\beta$ have opposite parities, and that $B_i^l = \{\{\alpha + j, \ldots, \beta - j\} : 0 \leq 2j < \beta - \alpha\} = S_{\alpha-1, (\beta-\alpha+1)/2}$.

The proof for the set $B_i^r$ is completely analogous. □
Corollary 4.9.
Let $\mathcal{D}$ be a curve diagram compatible with $L$, and let $i$ be an integer with $1 \leq i \leq n - 2$. At least one of the sets $B_i^r$ and $B_{i+1}^r$ is empty.

Proof. If neither $B_i^r$ nor $B_{i+1}^r$ is empty, let $\{u, u + 1\}$ and $\{v, v + 1\}$ their respective minimal elements: the zone $Z_{i+1}$ contains two arcs $A$ and $A'$ whose endpoints are respectively $(L_i^u, L_i^{u+1})$ and $(L_{i+1}^v, L_{i+1}^{v+1})$. According to Proposition 4.8, both $A$ and $A'$ must contain the unique puncture of the zone $Z_{i+1}$, which is impossible, proving Corollary 4.9.

Proposition 4.10 allows us to consider representations of curve diagrams that will be more compact than the diagram identifiers.

Definition 4.10 (Curve diagram coordinates and braid coordinates).
Let $\mathcal{D}$ be a tight curve diagram, and let $\beta$ be the braid represented by $\mathcal{D}$. According to Corollary 4.9, three cases are mutually exclusive:

(i) if $B_{i-1}^r$ is not empty, we define the pair $(a_i, b_i)$ such that $B_{i-1}^r = S_{a_i - b_i}$ (so that $b_i < 0$);
(ii) if $B_i^r$ is not empty, we define the pair $(a_i, b_i)$ such that $B_i^r = S_{a_i - b_i}$ (so that $b_i > 0$);
(iii) if both $B_{i-1}^r$ and $B_i^r$ are empty, we define the pair $(a_i, b_i)$ such that $b_i = 0$ and such that the puncture $p_i$ belongs to a $(\mathcal{D}, L)$-arc with endpoint $L_{i-1}^{r+1}$.

In addition, consider the sequence $s_0, \ldots, s_n$ such that $s_0 = 0$ and $s_{i+1} = s_i + b_{i+1}$ for $i \geq 0$. The tuple $(s_0, a_1, s_1, a_2, \ldots, a_n, s_n)$ is called coordinates of the diagram $\mathcal{D}$ and of the braid $\beta$.

![Figure 11: Curve diagram and associated coordinates](image)

Coordinates will play a crucial role below, as they characterize the (diagrammatic) complexity of braids.

Theorem 4.11.
Let $\mathcal{D}$ be a tight curve diagram, and let $(s_0, a_1, \ldots, s_n)$ be the coordinates of $\mathcal{D}$. We have $|\mathcal{D} \cap L_i| = 2s_i + 1$ for each integer $i \in \{0, \ldots, n\}$. In addition, $|\mathcal{D}|_d = n - 1 + 2 \sum_{i=0}^{n} s_i$. 

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Proof. The first part of Theorem 4.11 follows from an immediate induction on \( i \). Consequently, 
\[
|\mathcal{D}|_d + 2 = \sum_{i=0}^{n} |\mathcal{D} \cap L_i| = 2 \sum_{i=0}^{n} s_i + n + 1,
\] 
which completes the proof. \( \square \)

In addition, let us step back slightly from the system of coordinates introduced in Definition 4.10 and let us consider an entire family of tuples of integers.

**Definition 4.12 (Virtual coordinates).**

Let \( \mathbf{s} \mathbf{a} : = (s_0, a_1, \ldots, s_n) \) be non-negative integers. We say that \( \mathbf{s} \mathbf{a} \) are virtual coordinates if \( s_0 = s_n = 0 \) and if the relations \( 0 \leq a_i \leq 2 \min\{s_{i-1}, s_i\} + 1 \) hold for each integer \( i \in \{1, \ldots, n\} \).

**Corollary 4.13.**

Let \( \mathcal{D} \) be a tight curve diagram. The coordinates of \( \mathcal{D} \) are virtual coordinates.

Proof. Theorem 4.11 states that \( 2s_i + 1 = |\mathcal{D} \cap L_i| \) whenever \( 0 \leq i \leq n \). Hence, whenever \( s_{i-1} = s_i \), Definition 4.10 directly implies that \( 0 \leq a_i \leq 2s_i \). However, if \( s_{i-1} \neq s_i \), then we also showed in the proof of Theorem 4.17 that the points \( L_{i-1}^j \) and \( L_i^j \) are \( (\mathcal{D}, L) \)-neighbours whenever \( j \leq a_i \). This proves that

\[
0 \leq a_i \leq \min\{|\mathcal{D} \cap L_{i-1}|, |\mathcal{D} \cap L_i|\} = 2 \min\{s_{i-1}, s_i\} + 1.
\]

\( \square \)

The notion of virtual coordinates generalizes the notion of coordinates of a tight curve diagram. However, not all virtual coordinates are the coordinates of some braid. For instance, one may check, as shown in Section 5.1, that no braid has coordinates \( (0, 0, 1, 0, 0) \). Henceforth, we will call actual coordinates the coordinates of a braid, which form a strict subset of the set of virtual coordinates.

In addition, having generalised the notion of coordinates, we also generalize the notion of curve diagram and the notion of witnesses.

**Definition 4.14 (Virtual curve diagram).**

Let \( \mathbf{s} \mathbf{a} : = (s_0, a_1, \ldots, s_n) \) be virtual coordinates. Consider the integers \( b_i = s_i - s_{i-1} \), for \( i \in \{1, \ldots, n\} \). We call virtual curve diagram of the coordinates \( \mathbf{s} \mathbf{a} \) the drawing obtained as follows.

First, let us draw the vertical lines \( L_1, \ldots, L_{n-1} \) of the trivial lamination \( L \). Then, on each line \( L_i \), we draw points \( L_i^1, \ldots, L_i^{s_i+1} \) from bottom to top. We also place one point \( L_i^0 \) on \( p_0 \) and one point \( L_i^n \) on \( p_{n+1} \). We continue by drawing lines between some of the points \( L_i^j \):

- for all integers \( u, v \) and \( w \) such that \( b_u > 0 \) and \( \{v, \ldots, w\} \in S_{u,v,w} \), we draw, inside the region \( Z_u \), a line with endpoints \( L_u^v \) and \( L_u^w \);
- for all integers \( u, v \) and \( w \) such that \( b_u = 0 \) and \( \{v, \ldots, w\} \in S_{u,v,w} \), we draw, inside the region \( Z_u \), a line with endpoints \( L_u^v \) and \( L_u^w \);
- for all integers \( u, v \) and \( w \) such that \( \langle u, v \rangle \in (u + 1, w) \), we draw, inside the region \( Z_{u+1} \), a line with endpoints \( L_u^v \) and \( L_{u+1}^w \).

Due to the definition of the relation \( n_{\mathbf{s} \mathbf{a}} \), we can draw these lines so that they do not intersect each other: indeed, the lines we drew are precisely lines with endpoints \( L_u^v \) and \( L_u^w \) such that \( (i_1, j_1) \in n_{\mathbf{s} \mathbf{a}} (i_2, j_2) \). Finally, place a puncture on each line whose endpoints \( L_u^v \) and \( L_u^w \) are such that \( \{(u, v), (u', v')\} \in p_{\mathbf{s} \mathbf{a}} \). Each zone \( Z_i \) will contain exactly one such puncture. \( \square \)
Lemma 4.15.
Let $D$ be a tight curve diagram, and let $s_a$ be the coordinates of $D$. The curve diagram $D$ is a virtual curve diagram of $s_a$. Moreover, each virtual curve diagram of $s_a$ is equal to $D$, up to some homeomorphism of $\mathbb{C}$ that fixes $L$ and $\{+1,-1\}$ pointwise.

![Virtual witnesses — Virtual curve diagram (fragment)](image)

**Definition 4.16 (Virtual witnesses).**

Let $s_a := (s_0, a_1, \ldots, s_n)$ be virtual coordinates. Let us consider the integers $b_i := s_i - s_{i-1}$, for $i \in \{1, \ldots, n\}$.

We call virtual endpoint witness, virtual neighbour witness and virtual puncture witness of $s_a$
The respective sets

\[ E_w^{sa} := \{(i, j) : 0 \leq i \leq n, 1 \leq j \leq 2s_i + 1\}; \]
\[ N_w^{sa} := \{(i - 1, j), (i, j) : 1 \leq i \leq n, 1 \leq j \leq a_i\} \cup \]
\[ \{(i - 1, j), (i, k) : 1 \leq i \leq n, j - k = 2(s_i - s_i), \min\{j, k\} > a_i\} \cup \]
\[ \{(i, j), (i, k) : 1 \leq i \leq n - 1, a_i < j, j + k = 2(a_i + b_i + 1)\} \cup \]
\[ \{(i, j), (i, k) : 1 \leq i \leq n - 1, a_i + 1 < j, j + k = 2(a_i + b_{i+1} + 1)\}; \]
\[ P_w^{sa} := \{(i - 1, a_i + 1), (i, a_i + 1) : 1 \leq i \leq n, s_{i-1} = s_i\} \cup \]
\[ \{(i, a_i + b_i), (i, a_i + b_i + 1) : 1 \leq i \leq n, s_{i-1} < s_i\} \cup \]
\[ \{(i - 1, a_i - b_i), (i - 1, a_i - b_i + 1) : 1 \leq i \leq n, s_{i-1} > s_i\}. \]

Finally, we call virtual diagram witness the ordered triple \( D_w^{sa} := (E_w^{sa}, N_w^{sa}, P_w^{sa}) \).

Let us provide some intuition about the meaning of the different types of virtual neighbour witnesses and of virtual puncture witnesses, illustrated by Figure 1.2 Consider a tight curve diagram \( D \), with coordinates \( sa = (s_0, a_1, \ldots, s_n) \) and diagram witness \( D_w^{sa} = (E_w^{sa}, N_w^{sa}, P_w^{sa}) \).

The elements \( \{(i, j), (i', j')\} \) of \( N_w^{sa} \) represent pairs of points \( L_i^j \) and \( L_{i'}^{j'} \) related by some arc of \( D \) relatively to the lamination \( L \). As a consequence of Proposition 4.8 and Definition 4.10 we can split such arcs in 4 types:

1. arcs between two points \( L_i^j \) and \( L_{i+1}^{j+1} \), with \( j \leq a_i \);
2. arcs between two points \( L_i^j \) and \( L_{i+1}^{b_i} \), with \( k - j = 2b_i \) and \( \min\{j, k\} > a_i \);
3. arcs between two left neighbour points \( L_i^j \) and \( L_{i}^{b_i} \), with \( b_i > 0 \) and \( \{j, \ldots, k\} \in S_{a_i, b_i} \);
4. arcs between two right neighbour points \( L_i^j \) and \( L_{i}^{b_i} \), with \( b_{i+1} < 0 \) and \( \{j, \ldots, k\} \in S_{a_i, b_i} \).

Similarly, the elements \( \{(i, j), (i', j')\} \) of \( P_w^{sa} \) represent pairs of points \( L_i^j \) and \( L_{i'}^{j'} \) such that the arc joining \( L_i^j \) and \( L_{i'}^{j'} \) contains a puncture. Here too, Proposition 4.8 and Definition 4.10 allow us to split these arcs in 3 types:

5. arcs between two points \( L_i^{a_i + 1} \) and \( L_{i+1}^{a_i + 1} \);
6. arcs between two left neighbour points \( L_i^{a_i + b_i} \) and \( L_{i}^{a_i + b_i + 1} \), with \( b_i > 0 \);
7. arcs between two right neighbour points \( L_i^{a_i + 1 - b_{i+1}} \) and \( L_{i}^{a_i + 1 - b_{i+1}} \), with \( b_{i+1} < 0 \).

Theorem 4.17.

Let \( D \) be a tight curve diagram, and let \( sa \) be the coordinates of \( D \). We have: \( D_w^{sa} = D_w^{sa} \).

Proof. It follows from Theorem 4.14 that \( E_w^{sa} = E_w^{sa} \).

Then, an immediate induction on \( i \) proves that two endpoints \( L_i^{j-1} \) and \( L_i^{k} \) are \( (D, L) \)-neighbours if and only if either \( j = k \leq a_i \) or \( (j = k + 2b_i \) and \( \min\{j, k\} > a_i \). Consequently, Definition 4.10 shows that \( N_w^{sa} = N_w^{sa} \).

Finally, for each integer \( i \) such that \( B_{i-1}^j = B_i^j = \emptyset \), the puncture \( p_i \) must belong to a \( (D, L) \)-arc with endpoints \( L_i^{a_i + 1} \) and \( L_{i+1}^{a_i + 1} \). It follows that \( P_w^{sa} = P_w^{sa} \), which proves Theorem 4.17. 

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Corollary 4.18.
Tight curve diagrams represent the same braid if and only if they have the same coordinates. ■

Finally, observe that the set \( n_{sa} \) induces a symmetric binary relation on \( e_{sa} \). Hence, we denote this relation by \( sa \), and by \( ssa \) the transitive reflexive closure of \( sa \).

Theorem 4.19.
Let \( sa = (s_0, a_1, \ldots, s_n) \) be virtual coordinates, and let \( dw_{sa} = (e_{sa}, n_{sa}, p_{sa}) \) be the associated virtual diagram witness. There exists a tight curve diagram whose coordinates are \( sa \) if and only if the relation \( sa \) has exactly one equivalence class. ■

Proof. First, if \( sa \) are the coordinates of a tight curve diagram \( D \), then Theorem 4.17 proves that \( D_{sa} = D_{sa} \). Let \( (u, v) \) and \( (u', v') \) be two elements of \( e_{sa} = E_{sa} \). There exists integers \( k \) and \( \ell \) such that \( \sigma^k = L_{u}^v \) and \( \sigma^\ell = L_{u'}^{v'} \). Without loss of generality, we assume here that \( k \leq \ell \). Then, consider the pairs \( (u_i, v_i) \in e_{sa} = E_{sa} \) such that \( L_{u_i}^{v_i} = \sigma^i \).

Since \( N_{D} = n_{sa} \), we have \( (u_i, v_i) \sim (u_{i+1}, v_{i+1}) \) for each integer \( i \in \{k, \ldots, \ell - 1\} \). It follows that \( (u, v) = (u_k, v_k) \) and \( (u', v') = (u_\ell, v_\ell) \) belong to the same equivalence class of \( sa \). This proves that \( sa \) has exactly one equivalence class.

Conversely, let \( D \) be a virtual curve diagram of \( sa \), chosen so that the punctures of \( D \) coincide with the punctures \( p_1, \ldots, p_n \). If \( sa \) has exactly one equivalence class, then \( D \) must consist of one open line, whose endpoints are necessarily \( L_0^1 \) and \( L_1^1 \). It follows that \( D \) is precisely a tight curve diagram, whose coordinates are \( sa \). This completes the proof of Theorem 4.19. ■

The above coordinates are analogous to the Dynnikov coordinates (see [4, 5] for details) in several respects. First, both arise from counting intersection points between different collections of lines. Second, both provide an injective mapping from the braid group \( B_n \) into the set \( \mathbb{Z}^{2n} \) or \( \mathbb{Z}^{2n+1} \). Finally, both systems of coordinates come with very efficient algorithms, whose complexities are of the same order of magnitude. However, the coordinates used here are very closely linked with the notion of (diagrammatic or laminated) norm, whereas the process of computing the norm of a braid from its Dynnikov coordinates is less immediate.
5 Actually Counting Braids

It is now time to complete the task we set at the beginning of Section 4, we study here the integers \( N_{n,k} := |\{ \beta \in B_n : \| \beta \|_d = k \}| = |\{ \beta \in B_n : \| \beta \|_d = k \}| \) and the geometric generating functions \( B_n(z) := \sum_{\beta \in B_n} z^{\| \beta \|_d} \) provided by Corollary 3.2 and Theorem 3.3.

Due to Theorem 4.11, it is convenient to study the integers \( g_{n,k} := N_{n,2k+n-1} \) and the generating function \( G(x) := \sum_{k \geq 0} g_{n,k} x^k \). Observe that \( B_n(x) = x^{n-1} G_n(x^2) \).

We will use the system of coordinates introduced in Definition 4.10 and, following Theorem 4.19, count the tuples \( \mathbf{s}_a \) of virtual coordinates with a given sum \( \sum_{i=1}^{\ell} s_i \) such that the relation \( \mathbf{s} \equiv \mathbf{s}_a \) has exactly one equivalence class.

In addition, aiming to reduce the number of cases to look at, we will use the symmetries mentioned in Section 5. If a braid \( \beta \) has coordinates \( (s_0, a_1, s_1, \ldots, a_n, s_n) \), then its horizontally symmetric braid \( S_h(\beta) \) has coordinates \( (s_n, a_n, s_{n-1}, \ldots, a_1, s_0) \) and its vertically symmetric braid \( S_v(\beta) \) has coordinates \( (s_0, a_1, s_1, \ldots, a_n, s_n) \), where \( a'_i = 2 \min\{s_i, s_i\} + \mathbf{1}_{s_i-1} \neq s_i - a_i \).

5.1 An Introductory Example: The Braid Group \( B_2 \)

In the braid group \( B_2 \), everything is obvious. Indeed, the group \( B_2 \) is isomorphic to \( \mathbb{Z} \), and generated by the Artin braid \( \sigma_1 \). Since \( \| \sigma_1^k \|_d = 1 + 2|k| \) for all integers \( k \in \mathbb{Z} \), it follows that

\[
g_{2,k} = 1_{k=0} + 2 \cdot 1_{k \geq 1}, \quad G_2(x) = \frac{1+x}{1-x}. \]

Let us recover this result with the tools introduced above: we details computations as a warm-up. A braid \( \beta \) with norm \( 2k+1 \) has coordinates of the form \( (0, a_1, k, a_2, 0) \), with \( k \geq 0 \) and \( a_1, a_2 \in \{0, 1\} \).

If \( k = 0 \), then we must have \( a_1 = a_2 = 0 \), and Theorem 4.19 ensures that \( (0,0,0,0,0) \) are actual coordinates, whence \( g_{2,0} = 0 \).

We consider now the case \( k \geq 1 \). Using the vertical symmetry, we focus on the case \( a_1 \leq a_2 \), counting half of the actual coordinates. If \( a_1 = 1 \), then \( a_2 = 1 \), hence \( \{(0,1),(1,1),(2,1)\} \) is an equivalence class of \( \mathbb{m} \) that does not contain \( (1,2k+1) \). Similarly, if \( a_2 = 0 \), then \( a_1 = 0 \), hence \( \{(0,1),(1,2k+1),(2,1)\} \) is an equivalence class of \( \mathbb{m} \) that does not contain \( (1,1) \). Therefore, we

![Figure 13: \( \| \sigma_1^k \|_d = 2k+1 \)]
must have \( a_1 = 0 \) and \( a_2 = 1 \), and Theorem 4.1 ensures that \((0, 0, k, 1, 0)\) are actual coordinates, whence \( g_{2,k} = 1 \).

This second proof may seem longer and more convoluted than the direct proof obtained by enumerating the braids in the group \( B_2 \). However, enumerating the braids in \( B_3 \) seems out of reach, whereas considering virtual coordinates and identifying which are actual will be possible, as shown in Section 5.2.

### 5.2 A Challenging Example: The Braid Group \( B_3 \)

Our central result is the following one.

**Theorem 5.1.**

The integers \( g_{3,k} \) and the generating function \( G_3(x) \) are given by:

\[
G_3(x) = \frac{2 + 2x - x^2}{x^2(1 - x^2)} \left( \sum_{n \geq 3} \varphi(n)x^n \right) + \frac{-1 + 3x^2}{1 - x^2},
\]

\[
g_{3,k} = 1_{k=0} + 2 \left( \varphi(k + 2) - 1_{k \in 2\mathbb{Z}} + 2 \sum_{i=1}^{\lfloor k/2 \rfloor} \varphi(k + 3 - 2i) \right) 1_{k \geq 1},
\]

where \( \varphi \) denotes the Euler totient. In particular, \( G_3(x) \) is not rational.

The following subsections are devoted to the proof of Theorem 5.1.

#### 5.2.1 Proof of Theorem 5.1 – Step #1: Rationality

We first prove the last part of Theorem 5.1, i.e. that the generating function

\[
G_3(x) = \frac{2 + 2x - x^2}{x^2(1 - x^2)} \left( \sum_{n \geq 3} \varphi(n)x^n \right) + \frac{-1 + 3x^2}{1 - x^2}
\]

is not rational.

For the sake of contradiction, let us assume henceforth that this function is rational. Then, so is the generating function \( \sum_{n \geq 3} \varphi(n)x^n \). Consequently, the sequence \( \langle \varphi(k) \rangle \) is recurrent linear, hence is cancelled by some polynomial \( A(X) = \sum_{i=0}^{d} a_i X^i \) with integer coefficients, with \( a_d \neq 0 \).

Consider some prime number \( m \geq 3 \) such that \( m \equiv a_d = 1 \), and set \( Z_m := \mathbb{Z}/m\mathbb{Z} \). Up to multiplying \( A(X) \) by some integer constant, we may assume that \( a_d = -1 \) (mod \( m \)). Then, consider the tuples \( T_k = (\varphi(k), \varphi(k + 1), \ldots, \varphi(k + d - 1)) \) reduced modulo \( m \): the sequence \( (T_k)_{k \geq 0} \) has values in the finite set \( Z_m^d \). Moreover, the function

\[
\lambda: \ Z_m^d \rightarrow Z_m^d, \quad (x_0, x_1, \ldots, x_{d-1}) \rightarrow (x_1, \ldots, x_{d-1}, \sum_{i=0}^{d-1} a_i x_i)
\]

sends each tuple \( T_k \) onto \( T_{k+1} \). The sequence \( (T_k)_{k \geq 0} \) is therefore ultimately periodic: there exist integers \( \ell > 0 \) and \( K \geq 0 \) such that \( T_{k+\ell} = T_k \) for all integers \( k > K \). In particular, it follows that \( \varphi(k + \ell) \equiv \varphi(k) \) (mod \( m \)) whenever \( k > K \).
The integer $\ell mK! - 1$ is not congruent to 1 (mod $m$), hence cannot be a product of prime numbers congruent to 1 (mod $m$), and must have some prime factor $p$ such that $p \neq 1$ (mod $m$). Note that $p \wedge \ell = p \wedge m = 1$, and that $p \wedge K! = 1$, whence $p > K$. Similarly, the integer $pmK! - 1$ must have some prime factor $q$ such that $q \neq 1$ (mod $m$): note that $q > K$ and that $p \wedge q = 1$, whence $p \neq q$.

Then, let us define $\alpha = \varphi(\ell)$: since $\varphi$ is the Euler totient and since $p \wedge \ell = 1$, we have $p^\alpha = 1$ (mod $\ell$). We know that $p^{\alpha+1} = p$ (mod $\ell$) and that $p^{\alpha+1} > p > K$, whence $p^{\alpha}(p - 1) = \varphi(p^\alpha+1) = \varphi(p) = p - 1$ (mod $m$). Similarly, we know that $p^\alpha q \equiv q$ (mod $\ell$) and that $p^\alpha q > q > K$, whence $p^{\alpha-1}(p - 1)(q - 1) = \varphi(p^\alpha q) = \varphi(q) = q - 1$ (mod $m$). Since both $p - 1$ and $q - 1$ are invertible in $\mathbb{Z}_m$, it follows that $p^\alpha = 1 = p^{\alpha-1}(p - 1)$ (mod $m$), hence that $p^{\alpha-1} = p^{\alpha} - p^{\alpha-1}(p - 1) \equiv 0$ (mod $m$), which cannot be true since $p \wedge m = 1$. This contradiction shows that the generating function (▲) is not rational.

5.2.2 Proof of Theorem 5.1 – Step #2: Simple Cases

We need to focus on identifying which tuples of the form $\mathbf{sa} = (0, a_1, k, a_2, \ell, a_3, 0)$ satisfy the requirements of Theorem 1.19. Hence, we will take $\ell$ and $k$ as parameters, and compute the integer $C_{k, \ell} = |\{(a_1, a_2, a_3) : (0, a_1, k, a_2, \ell, a_3, 0) \text{ are actual coordinates}\}|$. Using the vertical symmetry, we know that $C_{k, \ell} = C_{\ell, k}$, and henceforth decide to focus on the case where $\ell \leq k$, and proceed to a disjunction of cases.

First, if $k = \ell = 0$, then $a_1 = a_2 = a_3 = 0$, and $\mathbf{sa}$ are the coordinates of the trivial braid $\varepsilon \in B_3$. Therefore, $C_{0,0} = 1$.

If $k = 0 < \ell$, then $a_1 = 0$, and $\mathbf{sa}$ are actual coordinates if and only if $(0, a_2, \ell, a_3, 0)$ are also actual coordinates. Indeed, as illustrated by Figure 14a, the coordinates $(0, a_2, \ell, a_3, 0)$ can be obtained from $(0, a_1, k, a_2, \ell, a_3, 0)$ by “shrinking” the edge between $(0, 1)$ and $(1, 1)$. Hence, $\mathbf{sa}$ are actual coordinates if and only if $\{a_2, a_3\} = \{0, 1\}$. Therefore, $C_{0,\ell} = C_{\ell,0} = 2$.

![Figure 14: Shrinking edges of virtual curve diagrams when $k = 0$ and $k = \ell$](image)

Figure 14: Shrinking edges of virtual curve diagrams when $k = 0$ and $k = \ell$
Similarly, if $1 \leq k = \ell$, then $sa$ are actual coordinates if and only if $0 \leq a_2 \leq 2k$ and if $(0, a_1, k, a_3, 0)$ are also actual coordinates: as illustrated by Figure 14b, the coordinates $(0, a_1, k, a_3, 0)$ can be obtained from $(0, a_1, k, a_2, k, a_3, 0)$ by shrinking each edge between $(1, j)$ and $(2, j)$, when $1 \leq j \leq 2k + 1$. We therefore have $2k + 1$ ways of choosing $a_2$ and $2$ ways of choosing $(a_1, a_3)$, whence $C_{k,k} = 2(2k + 1)$.

5.2.3 Proof of Theorem 5.1 – Step #3: Towards Cyclic Permutations

We consider now the case where $1 \leq k < \ell$, and set $m = \ell - k$. Using the horizontal symmetry, we may focus on the case where $a_1 = 1$: doing so, we will find exactly half of the actual coordinates $(0, a_1, k, a_2, \ell, a_3, 0)$.

Moreover, since $0 \leq k \leq \ell$, observe that:

- $(0, 1) \sim (1, 1)$;
- for all $j \in \{1, \ldots, 2k + 1\}$, either $(1, j) \sim (2, j)$, if $j \leq a_2$, or $(1, j) \sim (2, j + 2m)$, if $j > a_2$;
- either $(3, 1) \sim (2, 1)$, if $a_3 = 1$, or $(3, 1) \sim (2, 2\ell + 1)$, if $a_3 = 0$.

Therefore, for each point $(i, j)$, there exists some point $(2, m)$ such that $(i, j) \sim (2, m)$. Hence, each equivalence class of the relation $\sim$ contains points of the type $(2, m)$, as illustrated by Figure 15.

![Figure 15](image-url)

(a) First cases: $a_2 = 0$

(b) Last cases: $a_2 > 0$

Figure 15: Four different cases: $a_1 = 1$, $a_2 \geq 0$ and $a_3 \geq 0$

In addition, it makes sense to introduce additional symmetric relations on the set $\{0, \ldots, 2\ell + 1\}$. When $u$ and $v$ are elements of $\{0, \ldots, 2\ell + 1\}$ with $u < v$, we write $u \approx u'$ and $v \approx v'$ if either:
• $u = 0$ and $(2, v) \nsimeq (1, 1) \nsimeq (0, 1);$  
• $u \geq 1$ and $\{u, \ldots, v\} \in S_{a_2, m};$  
• $u \geq 1$ and $(2, u) \nsimeq (1, j_1) \nsimeq (1, j_2) \nsimeq (2, v)$ for some integers $j_1$ and $j_2.$

We also write $u \nsimeq v$ and $v \nsimeq u$ if either:

• $u = 0$ and $(2, u) \nsimeq (3, 1);$  
• $u \geq 1$ and $\{u, \ldots, v\} \in S_{a_3, f}.$

The relations $\nsimeq$ and $\nsim$ obey the following intuition, illustrated by Figure 16. Consider the virtual braid diagram of $sa.$ Starting from one point $L^q_0,$ two lines are drawn, one going to the left and the other one to the right. Follow the line that goes to the left (respectively, to the right) until you reach some point on the line $L_2;$ this will be the point $L^q_2$ such that $u \nsim v$ (respectively $u \nsim v$); if you could not reach such a point, this means you must have reached some point $L^q_0$ or $L^q_1,$ in which case we just have $u \nsim v$ (respectively $u \nsim v$). It follows immediately that the integers $v$ and $w$ uch that $u \nsim v$ and $u \nsim w$ exist and are unique.

![Figure 16: $u \nsim v$ and $u \nsim w$](image)

In addition, one shows easily that $u$ must have opposite parities to $v$ and $w.$ Hence, we define two additional relations:

• for $u, v \in \{0, \ldots, 2\ell + 1\},$ we write $u \nsim v$ if $u$ is even and $u \nsim v,$ or if $u$ is odd and $u \nsim v;$  
• for $x, y \in \{0, \ldots, \ell\},$ we write $x \nsim y$ if there exists some integer $z \in \{0, \ldots, \ell\}$ such that $2x + 1 \nsim 2z \nsim 2y + 1.$

Observe that $u$ and $v$ must have opposite parities whenever $u \nsim v.$ Hence, the relation $\nsim$ is a permutation of $\{0, \ldots, 2\ell + 1\}.$ In addition, for $u, v \in \{0, \ldots, 2\ell + 1\},$ we have $u \nsim v$ or $u \nsim v$ (i.e. $u \nsim v$ or $v \nsim u$) if and only if the points $(2, u)$ and $(2, v)$ are nearest neighbours on the set $\{(2, j): 0 \leq j \leq 2\ell + 1\}$ for the graph induced by the relation $\nsim.$

Therefore, the relation $\nsim$ has one unique equivalence class if and only if the permutation $\nsim$ has one unique orbit. Moreover, note that the permutation $\nsim$ maps the set $\{0, 2, \ldots, 2\ell\}$ on the set $\{1, 3, \ldots, 2\ell + 1\}$ and vice-versa. It follows that $\nsim$ has one unique orbit if and only if the permutation of $\{1, 3, \ldots, 2\ell + 1\}$ induced by $\nsim^2$ has one unique orbit, or, equivalently, if $\nsim$ is a cyclic permutation of the set $\{0, \ldots, \ell\}.$

We identify below the set $\{0, \ldots, \ell\}$ to the set $\mathbb{Z}_{\ell+1} := \mathbb{Z}/(\ell + 1)\mathbb{Z}.$ Hence, and following Theorem 1.19 we know that $sa$ will be actual coordinates if and only if the relation $\nsim$ is a cyclic permutation of $\mathbb{Z}_{\ell+1}.$
5.2.4 Proof of Theorem 5.1 – Step #4: Which Permutations are Cyclic?

Finally, and \( \alpha = \frac{q}{2} \); note that \( a_2 = [\alpha] + [\alpha] \). Then, let us consider separately various cases.

\( \triangleright \) If \( a_2 > 0 \) and \( a_3 = 1 \), then \( 0 \to 0 \), as shown in Figure 15[5]. Hence, the relation \( \to \) does not induce a cyclic permutation of the set \( \mathbb{Z}_{\ell+1} \).

\( \triangleright \) If \( a_2 = 0 \), then one checks easily, as illustrated in Figure 15a that

- if \( 0 \leq u < m \), then \( 2u + 1 \approx 2(m - u) \approx 2u + 1 + 2(k + a_3) \);
- if \( u = m \), then \( 2u + 1 \gg 0 \approx 2\ell + 1 \);
- if \( m < u \leq \ell \), then \( 2u + 1 \approx 2(\ell + 1 + m - u) \approx 2u + 1 - 2(m + 1 - a_3) \).

Hence, it follows that \( u \to u + (k + a_3) \) for all \( u \in \mathbb{Z}_{\ell+1} \).

\( \triangleright \) If \( k + 1 \gg a_2 > a_3 = 0 \), then one checks that

- if \( u = 0 \), then \( 2u + 1 \approx 0 \approx 2\ell + 1 \);
- if \( 1 \leq u < \alpha \), then \( 2u + 1 \approx 2(\ell + 1 - u) \approx 2u - 1 \);
- if \( \alpha \leq u < m + \alpha \), then \( 2u + 1 \approx 2(m + a_2 - u) \approx 2u + 1 + 2(k - a_2) \);
- if \( m + \alpha \leq u < \ell + 1 - \alpha \), then \( 2u + 1 \approx 2(\ell + 1 + m - u) \approx 2u + 1 - 2(m + 1) \);
- if \( \ell + 1 - \alpha \leq u \leq \ell \), then \( 2u + 1 \approx 2(\ell + 1 - u) \approx 2u - 1 \).

Hence, it follows that:

(i) \( u \to u - 1 \) if \( 0 \leq u < \alpha \);
(ii) \( u \to u + (k - a_2) \) if \( \alpha \leq u < m + \alpha \);
(iii) \( u \to u - (m + 1) \) if \( m + \alpha \leq u < \ell + 1 - \alpha \);
(iv) \( u \to u - 1 \) if \( \ell + 1 - \alpha \leq u \leq \ell \).

\( \triangleright \) If \( a_2 > k + 1 \) and \( a_3 = 0 \), then one checks that

- if \( u = 0 \), then \( 2u + 1 \approx 0 \approx 2\ell + 1 \);
- if \( 1 \leq u < k + 1 - \alpha \), then \( 2u + 1 \approx 2(\ell + 1 - u) \approx 2u - 1 \);
- if \( k + 1 - \alpha \leq u < \alpha \), then \( 2u + 1 \approx 2(k + 1 - u) \approx 2u + 1 + 2(m - 1) \);
- if \( \alpha \leq u < m + \alpha \), then \( 2u + 1 \approx 2(a_2 + m - u) \approx 2u + 1 - 2(a_2 - k) \);
- if \( m + \alpha \leq u \leq \ell \), then \( 2u + 1 \approx 2(\ell + 1 - u) \approx 2u - 1 \).

Hence, it follows that:

(i) \( u \to u - 1 \) if \( 0 \leq u < k + 1 - \alpha \);
(ii) \( u \to u + (m - 1) \) if \( k + 1 - \alpha \leq u < \alpha \);
(iii) \( u \to u - (a_2 - k) \) if \( \alpha \leq u < m + \alpha \);
(iv) \( u \to u - 1 \) if \( m + \alpha \leq u \leq \ell \).

Overall, in each case, we observe that each permutation \( \ell \) has a specific structure, which we call translation and translated cut.
Definition 5.2 (Cut and sliding cut).
Let \( a, b, c \) and \( n \) be non-negative integers such that \( a + b + c \leq n \), and set \( \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} \).
We call translation, and denote by \( T_{n,a} \), the permutation of \( \mathbb{Z}_n \) such that \( T_{n,a} : k \mapsto k - a \).
We call translated cut, and denote by \( TC_{n,a,b,c} \), the permutation of \( T_{n,1} \circ C_{n,a,b,c} \), where \( C_{n,a,b,c} : k \mapsto k + a \) if \( k \in \{0, \ldots, a - 1, a + b + c, \ldots, n - 1\} \), \( k + c \) if \( k \in [a, \ldots, a + b - 1] \), and \( k - b \) if \( k \in [a + b, \ldots, a + b + c - 1] \).

We proved above that

- if \( a_2 = 0 \), then the \( \mapsto \) is the translation \( T_{\ell+1,m+1-a_3} \);
- if \( k + 1 \geq a_2 > a_3 = 0 \), then \( \mapsto \) is the translated cut \( TC_{\ell+1, [\alpha], m, k+1-a_2} \);
- if \( a_2 > k + 1 \) and \( a_3 = 0 \), then \( \mapsto \) is the translated cut \( TC_{\ell+1,k+1-[\alpha],a_2-k-1,m} \).

Hence, it remains to check which translations and translated cuts are cyclic permutations. The first case is immediate, whereas the second one is not.

Lemma 5.3.
Let \( a \) and \( n \) be integers such that \( 0 \leq a \leq n \). The translation \( T_{n,a} : \mathbb{Z}_n \mapsto \mathbb{Z}_n \) is cyclic if and only if \( a \wedge n = 1 \).

Lemma 5.4.
Let \( a, b, c \) and \( n \) be non-negative integers such that \( a + b + c \leq n \). The translated cut \( TC_{n,a,b,c} : \mathbb{Z}_n \mapsto \mathbb{Z}_n \) is cyclic if and only if \( (c - 1) \wedge (b + 1) = 1 \).

Proof. First, observe that \( T_{n,a} \circ TC_{n,a,b,c} = TC_{n,0,b,c} \circ T_{n,a} \). This means that the translated cuts \( TC_{n,a,b,c} \) and \( TC_{n,0,b,c} \) are conjugate to each other. Therefore, the permutation \( TC_{n,a,b,c} \) is cyclic if and only if \( TC_{n,0,b,c} \) is cyclic too, and we henceforth assume that \( a = 0 \).
Second, observe that \( b \cdot \text{TC}_{n,0,b,c} \cdot n - 1 = \text{TC}_{n,0,b,c} \cdot n - 2 = \text{TC}_{n,0,b,c} \cdot \ldots = \text{TC}_{n,0,b,c} \cdot b + c - 1 \). Hence, the permutation \( \text{TC}_{n,0,b,c} \) is cyclic if and only if \( \text{TC}_{b+c,0,b,c} \) is cyclic too, and we henceforth assume that \( n = b + c \).

Third, observe that \( \text{TC}_{b+c,0,b,c} \) is simply the translation \( \text{T}_{b+c,1} \). Consequently, the permutation \( \text{TC}_{b+c,0,b,c} \) is cyclic if and only if \((b + c) \land (b + 1) = 1\), i.e. if \((c - 1) \land (b + 1) = 1\). This completes the proof.

Remember that sa are actual coordinates if and only if \( \rightarrow \) induces a cyclic permutation of \( \mathbb{Z}_{d+1} \). Hence, Lemmas 5.3 and 5.4 prove that sa are actual coordinates if and only if we are in the following cases:

(i) \( a_2 = 0, a_3 = 1 \) and \( m \land (k + 1) = 1 \);
(ii) \( a_2 = 0, a_3 = 0 \) and \( (m + 1) \land (k - a_2) = 1 \);
(iii) \( k + 1 \geq a_2 > 0, a_3 = 0 \) and \( (m + 1) \land (k - a_2) = 1 \);
(iv) \( 2k + 1 \geq a_2 \geq k + 2, a_3 = 0 \) and \( (m - 1) \land (a_2 - k) = 1 \).

In particular, it follows that, whenever \( k \geq 1 \) and \( m \geq 1 \), we have

\[
C_{k,k+m} = C_{k+m,k} = 2 \left( \sum_{a_2=0}^{k+1} \text{H}_{m \land (k+1)-1} + \sum_{a_2=0}^{k+1} \text{H}_{(m+1) \land (k-a_2)=1} + \sum_{a_2=k+2}^{2k+1} \text{H}_{(m-1) \land (a_2-k)=1} \right). \quad (*)
\]

5.2.5 Proof of Theorem 5.1 – Step #5: Generating Functions

Focus now on the generating function \( G_3(x) = \sum_{k\geq0} g_3,k x^k \). For the sake of clarity and conciseness, we only indicate the main steps of our computations, which are mainly based on rearranging terms. First, a direct consequence of Section 5.2.4 is

\[
G_3(x) = \sum_{k,\ell \geq 0} C_{k,\ell} x^{k+\ell} = 1 + \frac{4x}{1-x} - \frac{2x^2(x-3)}{(1-x)^2} + 4H(x),
\]

where

\[
H(x) = \frac{1}{2} \sum_{k,m \geq 1} C_{k,k+m} x^{2k+m}.
\]

Using the formula for \( C_{k,k+m} \) given in (*), we decompose the function \( H(x) \) as:

\[
H(x) = H_1(x) + H_2(x) + H_3(x) + H_4(x),
\]

where

\[
H_1(x) = \sum_{k,m \geq 1} \text{H}_{m \land (k+1)-1} x^{2k+m}, \quad H_2(x) = \sum_{k,m \geq 1} \sum_{a=1}^{k} \text{H}_{(m+1) \land a=1} x^{2k+m},
\]

\[
H_3(x) = \sum_{k,m \geq 1} x^{2k+m} \quad \text{and} \quad H_4(x) = \sum_{k,m \geq 1} \sum_{a=2}^{k+1} \text{H}_{(m-1) \land a=1} x^{2k+m}.
\]

Using simple term manipulation, we get:

\[
H_1(x) = \frac{1}{x^3} F(x) - \frac{x}{1-x} \quad \text{and} \quad H_2(x) = \frac{1}{x(1-x^2)} F(x) - \frac{x^2}{(1-x^2)^2},
\]

\[
H_3(x) = \frac{x^3}{(1-x)(1-x^2)} \quad \text{and} \quad H_4(x) = \frac{1}{x(1-x^2)} F(x) - \frac{x^2}{(1-x)(1-x^2)}.
\]

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We first use Theorem 5.1 to estimate precisely the generating function $G_3(x)$ associated to the group $B_3$. We will then focus on the groups $B_n$ for $n \geq 4$.

**Proposition 5.5.**

When $n \to +\infty$, we have:

$$g_{3,k} \sim 4 \left(1 + \frac{1}{k \in 2\mathbb{Z}}\right) \frac{k^2}{\pi^2}.$$

**Proof.** For the sake of simplicity, we first introduce some notation. We define $\alpha = \frac{1}{\pi}$ and $\phi_k = \sum_{i=0}^{k/2} \phi(k - 2i)$, as well as real numbers $\varepsilon_k$, $\theta_k$ and $\eta_k$ such that $\phi_{2k} = (\alpha + \varepsilon_k)k^2$, $\phi_{2k-1} = (2\alpha + \theta_k)k^2$ and $\eta_k = \varepsilon_k + \theta_k$.

It is a standard result that

$$\sum_{k=1}^{n} \phi(k) \sim \frac{3}{\pi^2} n^2$$

when $n \to +\infty$ (see [11] Theorem 330); this means exactly that $\eta_k \to 0$ when $k \to +\infty$.

Then, let $A$ be some positive constant, and let $K$ be some positive integer such that $\frac{\alpha^2}{(2k+1)^2} \leq A$ and such that $|\eta_k| \leq A$ whenever $k \geq K$. In addition, for each integer $\ell \geq \log_2(K)$, we define $M_\ell = \max(|\varepsilon_k| : 2^\ell \leq k \leq 2^{\ell+1})$.

If $2^\ell \leq k \leq 2^{\ell+1}$, then

$$\phi_{4k} = \sum_{i=0}^{k} \phi(4i) + \sum_{i=0}^{k-1} \phi(4i+2) = 2 \sum_{i=0}^{k} \phi(2i) + \sum_{i=0}^{k-1} \phi(2i+1) = 2\phi_{2k} + \phi_{2k-1},$$

i.e. $\varepsilon_{2k} = \frac{\varepsilon_k + \varepsilon_{k+1}}{2}$. It follows that

$$|\varepsilon_{2k}| \leq \frac{M_\ell + A}{4} \leq 2A + \frac{3M_\ell}{4}.$$

Similarly, if $2^\ell \leq k < 2^{\ell+1}$, then $\phi_{4k+2} = \phi_{4k} + \phi(4k+2) = 2\phi_{2k} + \phi_{2k+1}$, i.e.

$$\varepsilon_{2k+1} = \frac{\alpha}{(2k+1)^2} + \frac{2k^2}{(2k+1)^2} \varepsilon_k - \frac{(k+1)^2}{(2k+1)^2} \varepsilon_{k+1} + \frac{(k+1)^2}{(2k+1)^2} \eta_{k+1}.$$
Since $k \geq 2\ell \geq K$, we know that \( \frac{a}{(2k+1)^2} \leq A \). Moreover, note that \( \frac{2k^2+(k+1)^2}{(2k+1)^2} = \frac{3}{4} - \frac{4\ell}{4(2k+1)^2} \leq \frac{3}{4} \).

It follows that
\[
|\varepsilon_{2k+1}| \leq A + \frac{2k^2 + (k + 1)^2}{(2k + 1)^2} M_\ell - \frac{(k + 1)^2}{(2k + 1)^2} A \leq 2A + \frac{3M_\ell}{4}.
\]

Overall, \( |\varepsilon_m| \leq 2A + \frac{3M_\ell}{4} \) whenever \( 2^\ell_{+1} \leq m \leq 2^{\ell+2} \), whence \( M_{\ell+1} \leq 2A + \frac{3M_\ell}{4} \). It follows that \( \limsup M_\ell \leq 8A \) and, since \( A \) is an arbitrary positive constant, that \( M_\ell \to 0 \) when \( \ell \to +\infty \).

Recall that \( |\varepsilon_k| \leq M_\ell \) whenever \( 2^\ell \leq k \leq 2^{\ell+1} \): this proves that \( \varepsilon_k \to 0 \) and that \( \theta_k = \eta_k - \varepsilon_k \to 0 \), i.e. that
\[
\phi_k \sim (1 + 1_{k \in 2\mathbb{Z}+1}) \frac{k^2}{\pi^2}
\]
when \( k \to +\infty \).

With the above notations, and according to Theorem 5.1, we have \( g_{3,k} = 4\phi_{k+1} = \mathcal{O}(k) \). Moreover, we just showed that \( k^2 = \mathcal{O}(\phi_k) \). It follows that \( g_{3,k} \sim 4\phi_{k+1} \) when \( k \to +\infty \), which proves Proposition 5.5. \( \square \)

We did not manage to compute explicitly the generating functions \( G_n(x) \) nor the integers \( g_{n,k} \) for \( n \geq 4 \). Hence, we settle for upper and lower bounds.

First, we find an upper bound on the number of virtual coordinates \( (s_0, a_1, s_1, \ldots, a_n, s_n) \) such that \( \sum_{i=0}^{n} s_i = k \); of course, this will also provide us with an upper bound on the integers \( g_{n,k} \).

**Proposition 5.6.**

Let \( n \geq 1 \) and \( k \geq 0 \) be integers. Then, \( g_{n,k} \leq 2^n \left( \frac{k+n-1}{n-1} \right)^{n-2} \binom{k+n-2}{n-2} \).

**Proof.** First, there are exactly \( \binom{k+n-2}{n-2} \) ways of choosing non-negative integers \( s_1, \ldots, s_{n-1} \) whose sum is \( k \).

Second, we know that \( 0 \leq a_i \leq 2\min\{s_{i-1}, s_i\} + 1 \) for all integers \( i \in \{1, 2, \ldots, n\} \). Then, let \( u \) be an integer such that \( s_u = \max\{s_1, \ldots, s_{n-1}\} \): we know that \( 0 \leq a_j \leq 2s_j + 1 \) when \( 1 \leq j \leq u \) and that \( 0 \leq a_j \leq 2s_j + 1 \) when \( u + 1 \leq j \leq n \). Therefore, let \( S_u \) denote the set \( \{1, 2, \ldots, n-1\} \setminus \{u\} \): the tuple \( (a_1, \ldots, a_n) \) must belongs to the Cartesian product \( \prod_{j \in S_u} \{0, 1, \ldots, 2s_j + 1\} \times \{0, 1\} \), whose cardinality is \( P = 2^n \prod_{j \in S_u} (s_j + 1) \).

By arithmetic-geometric inequality, it follows that
\[
P \leq 2^n \left( \frac{\sum_{j \in S_u} (s_j + 1)}{n-2} \right)^{n-2} \leq 2^n \left( \frac{\sum_{j=1}^{n-1} (s_j + 1)}{n-1} \right)^{n-2} = 2^n \left( \frac{k+n-1}{n-1} \right)^{n-2},
\]
which completes the proof. \( \square \)

In order to compute a lower bound, we also prove a combinatorial result which is interesting in itself.

**Proposition 5.7.**

Let \( (s_0, s_1, \ldots, s_n) \) be non-negative integers, with \( s_0 = s_n = 0 \). There exists integers \( a_1, \ldots, a_n \) such that \( (s_0, a_1, s_1, \ldots, a_n, s_n) \) are actual coordinates. \( \blacksquare \)
Proof. We just need to choose \( a_i = s_{i-1} \) if \( s_{i-1} \leq s_i \), and \( a_i = s_i + 1 \) if \( s_{i-1} > s_i \). Indeed, with such a choice, we prove by induction, for all \( i \leq n \), the following property \( P_i \):

\[
\forall j \leq s_i, (i, j) \sqsupseteq (i, 2s_i + 1 - j) \quad \text{and} \quad \forall k \leq i, \forall l \leq 2s_k + 1, \exists m \leq 2s_i + 1 \text{ such that } (k, l) \sqsupseteq (i, m).
\]

First, \( P_0 \) is vacuously true. Now, let \( i \in \{1, \ldots, n\} \) be some integer such that \( P_{i-1} \) is true, and let us prove \( P_i \): it is enough to prove the first part of \( P_i \) and to prove that the second part of \( P_i \) holds for \( k = i - 1 \).

If \( s_{i-1} \leq s_i \), then \( P_i \) follows immediately from the fact that

- \((i - 1, j) \sqsupseteq (i, j) \) and \((i - 1, 2s_i + 1 - j) \sqsupseteq (i, 2s_i + 1 - j)\) whenever \( 1 \leq j \leq s_{i-1} \);
- \((i, s_i + 1 - j) \sqsupseteq (i, s_i + j)\) whenever \( 1 \leq j \leq s_i - s_{i-1} \);
- \((i - 1, 2s_i + 1) \sqsupseteq (i, 2s_i + 1)\).

However, if \( s_{i-1} > s_i \), the proposition \( P_{i-1} \) states that \((i - 1, j) \sqsupseteq (i - 1, 2s_{i-1} + 1 - j)\) when \( 1 \leq j \leq s_{i-1} \). Since \( B_i \) is a set of \( s_{i-1} - s_i \), it also follows that, whenever \( s_i + 2 \leq j \leq 2s_{i-1} + 2 - s_i \),

\[
(i - 1, j) \sqsupseteq (i - 1, 2s_{i-1} + 3 - j), \quad \text{whence} \quad (i - 1, j) \sqsupseteq (i - 1, 2s_{i-1} + 3 - j) \sqsupseteq (i - 1, j - 2).
\]

This proves immediately that \((i - 1, s_i) \sqsupseteq (i - 1, s_i + 2k)\) and that \((i - 1, s_i + 2k) \sqsupseteq (i - 1, s_i + 2k + 1)\) whenever \( 0 \leq k \leq s_{i-1} - s_i \). Since \( P_{i-1} \) states that \((i - 1, s_{i-1}) \sqsupseteq (i - 1, s_{i-1} + 1)\), it even follows

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that \((i-1, s_i) \cong (i-1, s_i+1)\). Hence, we know that \((i, s_i) \cong (i-1, s_i) \cong (i-1, s_i+1)\) and that \((i-1, j) \cong (i-1, s_i) \cong (i, s_i)\) whenever \(s_i \leq j \leq 2s_{i-1} + 1 - s_i\).

Moreover, like in the case \(s_i \leq j \leq 2s_{i-1}\), we know that
\[
\bullet (i, j) \cong (i, j) \text{ and }(i-1, 2s_{i-1} + 1 - j) \cong (i, 2s_i + 1 - j) \text{ whenever } 1 \leq j \leq s_i - 1;
\]

\[
\bullet (i-1, 2s_{i-1} + 1) \cong (i, 2s_i + 1).
\]

This proves \(P_i\) and completes the induction.

Therefore, we know that \(P_n\) holds, which, according to Theorem 4.19, means exactly that \(s a\) are actual coordinates.

**Corollary 5.8.**

Consider integers \(n \geq 1\) and \(k \geq 0\). Then, \(g_{n,k} \geq \binom{k+n-2}{n-2}\).

### 5.4 Experimental Data and Conjectures

Proposition 5.6 and Corollary 5.8 prove that \(\binom{k+n-2}{n-2} \leq g_{n,k} \leq 2^n \left(\frac{k+n-1}{n-1}\right)^{n-2} \binom{k+n-2}{n-2}\). Unfortunately, these lower and upper bounds do not match, since their ratio is equal to \(2^n \left(\frac{k+n-1}{n-1}\right)^{n-2}\), hence grows arbitrarily when \(n\) and \(k\) grow. Therefore, aiming to identify simple asymptotic estimations of \(g_{n,k}\) when \(n\) is fixed and \(k \to +\infty\), we look for experimental data,

![Figure 19: Estimating \(g_{n,k}\) — experimental data for \(n = 4\) and \(n = 5\)](image)

Figure 19 presents the ratios \(g_{n,k}/k^{2(n-2)}\) (in black) and \(g_{n,k}/(k+n)^{2(n-2)}\) (in grey). We computed \(g_{n,k}\) by enumerating all the virtual coordinates, then checking individually which of them were actual coordinates (up to refinements such as using the above-mentioned symmetries to reduce the number of cases to look at).
The two series of points suggest the following conjecture, which was already proven to be true when $n = 2$ and $n = 3$.

**Conjecture 5.9.**
Let $n \geq 2$ be some integer. There exists two positive constants $\alpha_n$ and $\beta_n$ such that $\alpha_n k^{2(n-2)} \leq g_{n,k} \leq \beta_n k^{2(n-2)}$ for all integers $k \geq 1$.

Figure 19 also suggests that the ratios $g_{n,k}/k^{2(n-2)}$ might be split into convergent clusters, according to the value of $k$ mod 6 (when $n = 4$) or $k$ mod 2 (when $n = 5$). Once again, this is coherent with the patterns noticed for $n = 2$ and $n = 3$, and therefore suggests a stronger conjecture.

**Conjecture 5.10.**
Let $n \geq 2$ be some integer. There exists some positive integer $\rho_n$ such that, for every integer $\ell \in \{0, 1, \ldots, \rho_n - 1\}$, the sequence of ratios $g_{n,k\rho_n}/k^{2(n-2)}$ has a positive limit $\lambda_{n,\ell}$ when $k \to +\infty$.

Assuming Conjecture 5.10, a natural further step would be to compute the limits $\lambda_{n,\ell}$ or to study more precisely the asymptotic behaviour of the ratios $g_{n,k}/k^{2(n-2)}$. In particular, we hope that computing arbitrarily precise approximations of the constants $\lambda_{n,\ell}$ for small values of $n$ might help us guess analytic values of $\lambda_{n,\ell}$, thereby providing insight about the underlying combinatorial or number-theory-related structure of the integers $g_{n,k}$.
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