Computing frequency response of non-parametric uncertainty model of MIMO systems using $\nu$-gap metric optimization

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Abstract
This paper presents the computation of the non-parametric uncertainty model for multi input multi output (MIMO) systems, which is described by normalized coprime factors (NCF) using the frequency response data of the system. This computation is accomplished by minimizing a $\nu$-gap metric criterion. For this purpose, the problem is formulated to a convex optimization context, such that a semidefinite programming (SDP) can be implemented. Minimization constraints and the normality constraints of coprime factors are converted to linear matrix inequalities (LMI). Thus, by convex optimization algorithms, the semidefinite programming will be optimized. The proposed method can also be used for non-square multi input multi output systems in a conservative assumption. So, through the first process of optimization, the frequency responses of the normalized coprime factors are derived. Finally, to evaluate the performance of the proposed method in the computation of the normalized coprime factors of a system, the simulated results of this method are compared with those obtained by the other methods for two types of systems.

1 | INTRODUCTION

For years, control-oriented system identification techniques have been presented to deliver the proper models including a nominal model and a form of uncertainty [1]. The behaviour of perturbed systems may be different when they are measured by an operator norm under feedback control. The models used in many robust control techniques, like loop shaping, are required to be in the form of normalized right or left coprime factors and direct identification or computation of these factors is very important and applicable. Based on frequency response data direct identification of these factors is possible by computation of the frequency response of coprime factors. On the other hand, the gap metric criterion is used to show that the problem of robust optimization is equivalent to robust optimization for normalized coprime factors (NCF) perturbations [2]. It is well known that a relatively small uncertainty in lightly damped poles and zeros can result in a large distance measured in the $\nu$-gap metric, leading to conservative robust stability and performance guarantees [3]. However, while many uncertainty models with additive and multiplicative forms have been presented, the NCF's model has more appreciable properties in representing the system dynamics, especially for an unstable system [4]. Since there is a clear relationship between the $\nu$-gap metric and NCF models, this model is more attractive for perturbed system identifications. However, compared to the identification of an additive perturbed transfer matrix model, more efforts are required to identify the NCFs model from experimental data. This is because the NCF’s model is a purely mathematical representation of plant dynamics, and their time or frequency domain response cannot be directly measured [5]. In some recently researches, some methods have been introduced for the identification of the NCF’s model, especially in a closed-loop for unstable systems, where it is required to have a controller in the loop [6], so, model identification and controller design are accomplished simultaneously. In such research and similar works, knowledge of the initial controller is needed for identification, and a certain form of NCF is identified with parameterization. The main goal of these researches, such as [4, 6–10], is to find an optimized controller that can minimize the performance index. Moreover, some presented methods are only applicable to SISO systems. Some other methods, like ones that use the orthogonal basis functions, cannot be applied to all systems, especially unstable systems [11]. The methods in [5, 12]...
and, which are close to our work, are not as straightforward as the method presented in our paper. In these two methods it is required to use a parametric model (matrix) to ensure convergence of the algorithms.

In addition to the stability of the nominal model, normality and coprimeness are needed to identify NCFs. To overcome these difficulties, it is suggested in [13] that in order to identify an NCFs model, it may be appropriate to, at first, estimate the time or frequency domain response of NCFs from measurable physical variables. When this estimate and its error characteristics are available, the estimation of a plant’s NCFs and validation of a priori information about them are possible. In [13], NCFs are identified with constrained curve fitting. But, in practice there is not a mathematically appreciable relation between NCFs and physical measurable variables [5].

However, based on our best knowledge, until now, the presented methods for identification of MIMO systems based on the general form of the perturbed NCFs model, are challenging. Generally, when an indirect approach is adopted in non-parametric estimation, the estimate is possibly with an infinite variance; when a direct approach is employed in deriving a parametric model, a noise model is required that can describe the actual disturbance characteristics etc. [5]. For example, [14] presented robust controller design and identification of the coprime factorization model for MIMO systems. In order to convexify the problem, they used Taylor expansion at the initial point, considering a closed-loop stable configuration for a given initial controller. A newer one [15] developed an FRD control design for stable or stabilized MIMO for any kind of control structure based on non-smooth optimization.

However, they (and other authors) identified coprime factors (in a parametric method) through control design, but these methods are out of our scope and we did not add them to our references and literature surveys.

In this paper, we will show that based on the $\nu$-gap metric and frequency responses of a system, we can introduce a semi-definite programming (SDP) whose output variables are right or left NCFs and the resulted models have the least difference with $\nu$-gap metric. So, in this new method, by a standard form of SDP, frequency responses of NCFs are computed. After computing the frequency response of the NCFs with presented methods (as in [13]), the NCFs model will be identified, showing that the derived nominal model has the least difference with a real model.

After preliminaries (Section 1.2), in Section 2, we will discuss mathematical problem formulation and extraction of a formulation based on frequency responses for the NCFs model and $\nu$-gap metric. In Section 3, the proposed approach for the computation of NCFs in the frequency domain is provided and the results are summarized in a main theorem. Moreover, at the end of this section, the steps for the problem solving, which is a common SDP problem, are provided in a proposed algorithm. In the Section 4, we discuss the simulation of two examples and compare the results of the proposed algorithm with the ones in the two references by some related graphs. Finally, in Section 5, we conclude the paper. In the annex, some relations and proofs which are needed for discussion, are delivered.

### 1.1 Notations

In this paper, capital and bold letters denote matrices, while small and bold letters denote vectors. For scalars, regular alphabets are used. We use * for the conjugate transpose of matrix. If we use $\sim$ in front of a matrix in the $s$ space, it means $A(\sim)(s) = A^T(-s)$. Also, for a matrix $A^{-*} = (A^{-1})^*$ and $[A]$ represents the determinant of $A$.

To show that a matrix $A$ is positive semi-definite or positive definite, we use $A \succeq 0$ or $A > 0$ respectively; for negative semi-definite or negative definite cases, we use $\preceq$ or $<$. Respectively. If we have $A \succeq B$ it means $A - B \succeq 0$.

### 1.2 Preliminaries

The transfer function matrix of any MIMO system with $n$ inputs and $m$ outputs is an $m \times n$ matrix, which can be written in the NCFs form, as follows.

$$G_{mn}(s) = P_{mn}(s)Q_{nm}^{-1}(s)$$

$$G_{mn}(s) = M_{mn}^{-1}(s)N_{mn}(s)$$

Here, $M_{mn}(s)$, $N_{mn}(s) \in RH_{\infty}$ are left coprime factors(LCFs) and $P_{mn}(s)$, $Q_{nm}(s) \in RH_{\infty}$ are right coprime factors (RCFs). If these matrices (factors) are normalized, they will satisfy $P^*(\omega)P(\omega) + Q^*(\omega)Q(\omega) = I_{mm}$ and $N(\omega)N^*(\omega) = I_{nn}$. Clearly, these relations are valid in the frequency domain when $s = j\omega$; for complex matrices, $M(\omega)$, $N(\omega)$, $P(\omega)$, $Q(\omega)$, as $P^*(\omega)P(\omega) + Q^*(\omega)Q(\omega) = I$ and $N(\omega)N^*(\omega) + M(\omega)M^*(\omega) = I$.

For the perturbed MIMO system, the normalized right coprime factorization is defined as

$$G(\omega) = \left(P(\omega) + \Delta p(\Delta W_p(\omega)) \right) \left(Q(\omega) + \Delta Q(\omega)W_Q(\omega) \right)^{-1} \left[ \Delta P \left\| \Delta Q \right. \right] \leq 1$$

where $P^*(\omega)P(\omega) + Q^*(\omega)Q(\omega) = I$, and $W_p(\omega)$ and $W_Q(\omega)$ are weighting functions matrices and $\left[ \Delta P \left\| \Delta Q \right. \right]$ is the uncertainty matrix. Also, the normalized left coprime factorization can be defined similarly for perturbed MIMO systems that also is used in this paper.

On the other hand, in the robust control context, $\nu$-gap metric is used for the determination of the maximum distance between two transfer functions; it is a useful tool in uncertainty analysis and feedback system’s robustness. To calculate the $\nu$-gap metric, assume that $G_{mn}$ is the transfer function matrix of an MIMO system described by RCFs and LCFs as $G(\omega) = P(\omega)Q^{-1}(\omega) = M^{-1}(\omega)N(\omega)$ which can be represented in the frequency domain. Also, assume $G_i$ is the transfer function matrix of another system described by RCFs and LCFs as $G_i(\omega) = P_i(\omega)Q_i^{-1}(\omega) = M_i^{-1}(\omega)N_i(\omega)$.

The $\nu$-metric criterion between these two systems, is defined as
as [16]

\[ \delta_p (G,G_i) = \begin{cases} \Psi(G,G_i)_\infty, & \text{if } |\Theta| \neq 0 \forall \omega \text{ wno } |\Theta| = 0 \\ 1, & \text{otherwise} \end{cases} \]  

(4)

where

\[ \Theta = P_i^*P + Q_i^*Q \]

\[ \Psi (G,G_i) = MP - N_iQ \]

In this definition, wno is the abbreviation representing the number of counter clockwise encirclements around origin by |\Theta| aevaluated on the Nyquist contour [16].

In this paper, three following theorems are used.

Theorem 1. [17] Given the complex matrices A, B and C of compatible dimensions, there exists a solution \( \Delta \) to the linear matrix equation \( A = B\Delta C \) with \( \delta (\Delta) \leq 1 \) if and only if

\[ \begin{bmatrix} BB^* & A \\ A^* & C^*C \end{bmatrix} \succeq 0 \]  

(5)

Theorem 2. [1] (Schur complement) Let \( A \succeq 0 \) and \( C > 0 \); then

\[ \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \succeq 0 \]  

if and only if \( A - BC^{-1}B^* \succeq 0 \).

Theorem 3. [17] Let \( F = F^* \), \( L, R \) and \( D \) be the complex matrices of appropriate sizes, with \( D \leq 1 \). Let \( \Delta \) be a complex perturbation that has the block diagonal structure \( \Delta = \text{diag}(\Delta_1, \ldots, \Delta_N) \). Partition \( L, R \) and \( D \) conformally with \( \Delta \) as \( L = [L_1, \ldots, L_N], R = [R_1, \ldots, R_N]^* \) and \( D = [D_1, \ldots, D_N] \). Then, a sufficient condition for

\[ F + L\Delta(I - \Delta D)^{-1}R + R^*(I - \Delta D)^{-*}\Delta^* L^* \succeq 0, \forall \Delta \leq 1 \]

is that there exists a matrix \( \Sigma = \text{diag}(\Sigma_1, \ldots, \Sigma_N) \), partitioned conformally with \( \Delta \) with \( \Sigma_j \) real scalars, such that

\[ \Sigma \succeq 0, \begin{bmatrix} F - LL^* & R^* - LD^* \\ R - D^* L^* & \Sigma - D^* D^* \end{bmatrix} \succeq 0 \]  

(7)

2 PROBLEM FORMULATION

In this section, the problem of computing the general form of the NCFs model for the MIMO perturbed system in the frequency domain is formulated, which is expressed in a problem of semi-definite programming (SDP) with a \( \psi \)-gap metric index. This problem can be solved by convex optimization algorithms.

Assuming the frequency responses of the system \( G \) are available, we are going to obtain the frequency response of the NCFs of system. For this, let us have the frequency responses of a system in some different frequencies \( (\omega_1, \ldots, \omega_N) \), which are taken at different operating conditions or of different samples of a product. Consequently, \( \{G_i(\omega_j)\} \) is a set of frequency responses of the system that are measured in the frequency \( \omega_j \) and \( i = 1 \ldots K \) conditions. So, in different operating conditions or for several system samples, there are different system models; therefore, it is needed to have a nominal and an uncertainty model whose behaviour has the least difference with available responses \( G_{\Delta} = \{G_i, \ldots, G_k\} \) according to some indexes. So, we will define this difference with \( \psi \)-gap metric. In mathematical expressions, we want to determine the normalized right coprime factors (NRCFs) in a general form as follows.

\[ G(j\omega) = (P(j\omega) + \Delta P(j\omega)W_P(j\omega)) \times (Q(j\omega) + \Delta Q(j\omega)W_Q(j\omega))^{-1} \]  

(8)

It is needed to compute the behaviour of factors \( P \) and \( Q \) and their weighting matrices for all frequencies \( (\omega_1, \ldots, \omega_N) \). So, in the worst condition, the distance between the nominal model \( G \) and \( G_{\Delta} \) is minimized based on the \( \psi \)-gap metric. The worst condition of the distance between the nominal model and \( G_{\Delta} \) set, according to the \( \psi \)-gap metric, is defined as:

\[ J(G) = \max \delta_p (G,G_i) \]  

(9)

Hence, the nominal model which has the least \( J(G) \) is the best [18]. According to the \( \psi \)-gap metric, minimization of \( J(G) \) means

\[ \begin{array}{ll}
\min_{\alpha > 0} & \alpha \\
\text{s.t.} & \begin{bmatrix} \min \Psi(G,G_i)_{\infty} \leq \sqrt{\alpha} \\
\text{and } X_{ii} = \begin{bmatrix} P \\ Q \end{bmatrix} \text{ is unitary} \end{bmatrix} \end{array} \]  

(10)

It is clear that if in each frequency there is \( \max \Psi(G,G_i)_{\infty} \leq \sqrt{\alpha} \), the above minimization constraint will be satisfied, which means that in each frequency, we have the following condition [19]

\[ \max \|\Psi(G,G_i)\|_{\infty} \leq \sqrt{\alpha} \Leftrightarrow \Psi(G,G_i) \Psi^*(G,G_i) \leq \alpha I, \forall i, \forall \omega \]  

(11)

So, if this minimization is done in each frequency, the amount of difference of \( J(G) \) index will be less than \( \alpha_{\max} = \max_{\omega_1,\ldots,\omega_N} \alpha \). Consequently, by solving the above optimization problem, not only the frequency response of the nominal model of \( G \), but also the
weighting functions matrices \((W_p(i) \text{ and } W_Q(i))\) and the \(\nu\)-gap metric of \(\|\delta_p(G,G)\|_2 = \sqrt{\alpha}\) will be obtained.

Since we want to compute the frequency response of NCFs \((P \text{ and } Q)\), the formulation will be rewritten according to these factors. This formulation is done for the left NCFs in the following.

Using the definition Equation (4) for \(\nu\)-gap metric and the following definitions

\[ X = \begin{bmatrix} \Delta P + \Delta Q W_p \Delta M, -N \end{bmatrix} \]

we have

\[ Y = \begin{bmatrix} \Delta P W_p \Delta Q W_Q \end{bmatrix} \]

where \(X_0 = \begin{bmatrix} P \cr Q \end{bmatrix}, \Delta = \begin{bmatrix} \Delta P & 0 \\ 0 & \Delta Q \end{bmatrix} \) and \(W = \begin{bmatrix} W_p \cr W_Q \end{bmatrix} \).

3 | MAIN RESULTS

According to the formulation (10), during the minimization problem in each frequency, \(\alpha\) will be minimized and \(X_0(\omega)\) and \(W(\omega)\) will be computed. To compute the frequency response of a model which has the minimum \(\alpha\), the problem is converted to a minimization problem with matrix inequality constraints by presenting the following lemmas.

Lemma 1. If the following matrix inequality is satisfied

\[ \Sigma \geq 0, \begin{bmatrix} \alpha I & Y^+_i X_i \cr X^+_i Y_i & I \end{bmatrix} - \begin{bmatrix} Y^+_i & 0 \\ 0 & W^+ \end{bmatrix} \Sigma \begin{bmatrix} Y_i & 0 \\ 0 & W^+ \end{bmatrix} \geq 0 \]

then

\[ \max_i \Psi_i(G,G_i) \Psi^T(G,G_i) \leq \alpha I. \tag{13} \]

Proof. By using the Theorem 2, Equation (11) can be converted to the following relation. For simplicity, \(\Psi \equiv \Psi(G,G_i)\).

\[ \begin{bmatrix} \alpha I & \Psi_i \\ \Psi^T & I \end{bmatrix} \geq 0, \forall i; \forall \omega \tag{14} \]

So the minimization constraint \(f(G)\) in Equation (10) is converted to the above constraint, which of course, is not linear. At each frequency, the constraint Equation (13) can be changed to the following linear matrix inequality (LMI)

\[ \begin{bmatrix} \alpha I & \Psi \cr \Psi^T & I \end{bmatrix} \geq \begin{bmatrix} \alpha I & Y^+_i X_i \cr X^+_i Y_i & I \end{bmatrix} - \begin{bmatrix} Y^+_i & 0 \\ 0 & W^+ \end{bmatrix} \Delta \begin{bmatrix} 0 & W^+ \end{bmatrix} \]

\[ + \begin{bmatrix} 0 \\ W^+ \end{bmatrix} \Delta^+ \begin{bmatrix} Y_i & 0 \end{bmatrix} \geq 0 \tag{15} \]

If we suppose \(F = [x^1 \cdots x^n] \quad \Delta = [y^1 \cdots y^n] \quad \Delta = [\Delta_1 \cdots \Delta_n] \quad R = [0 \ W] \quad \text{and} \quad D = 0\) then, according to Theorem 3 if

\[ F + L \Delta (1 - D \Delta)^{-1} R + R^* (1 - D \Delta)^{-1} \Delta^* L^* \geq 0 \]

then there exists a matrix \(\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_N)\), partitioned conformally with \(\Delta\), with \(\sigma\) real scalars, such that the inequalities in Equation (13) are approved.

However, the matrix \(X_0\) should have a special property that is made by right NCFs and satisfies \(X^+_i X_i = I\), which is defined for a unitary matrix. To define some LMI constraints to guarantee the unitary condition of \(X_0\), the following lemma is presented.

Lemma 2. For semi-definite matrices \(A\) and \(C\), and the invertible \(B\) with appropriate sizes,

\[ \text{if } \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0 \text{ and } C \geq A \geq 0, \text{ then } \begin{bmatrix} C & X \\ X & B \end{bmatrix} \geq 0. \]

Proof. According to the Theorem 2, if \( \begin{bmatrix} A & X \\ X & B \end{bmatrix} \) is positive semi-definite, there is \(A \succeq X^* B^{-1} X\). On the other hand, because of \(C \geq A \geq 0\), there is \(C \succeq X^* B^{-1} X\); because the condition in the Theorem 2 is sufficient and necessary, then \( \begin{bmatrix} C & X \\ X & B \end{bmatrix} \succeq 0 \).

The normality constraint of the coprime factors is equivalent to a unitary constraint of \(X_0\), which is in Equation (10). In many optimization algorithms, there is an iteration loop and in each step, the previous values of variables are used in the next step. So, in our SDP optimization loop, in each step, \(X_0\) will be computed and used in the next step. We want to minimize the cost function and have the new value of \(X_0\) converge to a unitary matrix simultaneously. For this purpose, the following two lemmas will be proved for the guarantee of convergence \(X_0\) to a unitary matrix in an iterative loop.

Lemma 3. The constraints

\[ \begin{bmatrix} X^* X_0 + X_0^* X - X^* X \cr I \end{bmatrix} \geq 0 \tag{16} \]

and

\[ \begin{bmatrix} I & X_i \\ X_i^* \end{bmatrix} \geq 0 \tag{17} \]

ensure that \(1 \leq \delta(X_0) \leq \varepsilon + 1\).
Proof. If \( X_0 \) is a unitary matrix then \( X_0^*X_0 = I \) which is not a convex constraint. According to the Theorem 1, the constraint

\[
\begin{bmatrix}
X_0^*X_0 & I \\
I & X_0X_0^*
\end{bmatrix} \succeq 0
\]

(18)
guarantees that there is a contractive matrix \( \Delta \), such that

\[
X_0^*X_0 = I
\]

(19)

On the other hand, for any matrix \( \tilde{X} \) we have

\[
(X_0 - \tilde{X})^*(X_0 - \tilde{X}) \succeq 0
\]

(20)

So,

\[
X_0^*X_0 \succeq \tilde{X}^*X_0 + X_0^*\tilde{X} - \tilde{X}^*\tilde{X}
\]

(21)

So, if we add \( \tilde{X}^*X_0 + X_0^*\tilde{X} - \tilde{X}^*\tilde{X} \succeq I \) then it satisfies that \( X_0^*X_0 \succeq I \). On the other hand, for any matrices \( A \) and \( B \), if \( A \succeq B \succeq 0 \), then \( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \succeq 0 \) [20]. So we can rewrite as

\[
\begin{bmatrix}
\tilde{X}^*X_0 + X_0^*\tilde{X} - \tilde{X}^*\tilde{X} & I \\
I & \tilde{X}^*X_0 + X_0^*\tilde{X} - \tilde{X}^*\tilde{X}
\end{bmatrix} \succeq 0
\]

(22)

Therefore, with the assumption of Equation (17), there will be

\[
1 \leq \sigma(X_0) \leq \sigma(\tilde{X}); \text{ in an iterative loop, and the matrix } \Delta \text{ in Equation (19) converges to a unit matrix.}
\]

**Lemma 4.** The following LMI

\[
\begin{bmatrix}
\tilde{X}^*X & X_0^* \\
X_0 & I
\end{bmatrix} \succeq 0
\]

(27)
guarantees that \( 1 \leq \sigma(X_0) \leq \sigma(\tilde{X}) \); in an iterative loop, and the matrix \( \Delta \) in Equation (19) converges to a unit matrix.

**Proof.** If we have the condition \( \tilde{X}^*X \geq X_0^*X_0 \), it guarantees that \( 1 \leq \sigma(X_0) \leq \sigma(\tilde{X}) \); in an iterative loop, the \( \sigma(X_0) \) will be small and smaller and finally, it tends to be 1. In this situation, the matrix \( X_0 \) also tends to a unitary matrix. According to the Theorem 2, the condition \( \tilde{X}^*X \geq X_0^*X_0 \) can be written as the LMI in Equation (23). This LMI will be true if and only if \( \tilde{X}^*X \geq X_0^*X_0 \) and consequently, \( \sigma(X_0) \leq \sigma(\tilde{X}) \). Since, for any non-singular matrices \( A \) or \( B \), with proper dimensions, there are

\[
\sigma(A)\sigma(B) \leq \sigma(AB) \leq \sigma(A)\sigma(B)
\]

(28)

Therefore, with the assumption of Equation (17), there will be

\[
\frac{1}{\sigma^2(X_0)} \leq \sigma(\Delta_0), \frac{1}{\sigma^2(X_0)} \leq \sigma(\Delta_0)
\]

(29)

So, according to Equation (27), in each iteration there is \( \sigma(X_0) \leq \sigma(\tilde{X}) \); according to Equations (28) and (29), the maximum and minimum singular values will be closer to 1 in each iteration. Since the maximum value of this singular value is 1, then the contractive matrix will tend to be a unit matrix too.

**3.1 Main theorem**

An optimal normalized right coprime factors perturbed model Equation (8) for the frequency response set \( \{G_i(\omega)\} \) of a MIMO system can be computed by solving the following semi-definite programming (SDP) in the complex variable matrices \( X_0 = [P] \) and \( W = [\Sigma^*] \) and real variable matrix \( \Sigma = \)
\[
\begin{align*}
\text{FIGURE 1} & \quad \text{Nominal model (\(-\) +), the average value of the frequency responses (\(-\) o) of the computed model (\(-\) ) of the first example} \\
\end{align*}
\]

\[
\begin{align*}
\text{Proof.} & \quad \text{At each frequency of } \omega_j, \text{in fact, the problem Equation (10) should be solved such that constraint in Equation (15) is satisfied for unitary matrix } X_0. \text{ According to the lemma 1, Equation (13) guarantees Equation (11) and by applying lemma 3, the other two LMI constraints, Equations (16) and (17), are added such that the matrix } X_0 \text{ will be normalized. According to Lemma 4, the constraint } [\bar{X}^* \bar{X}] \succeq 0 \text{ is added to the SDP, to guarantee convergence of optimization algorithm. Finally, because of } P(s), Q(s) \in \mathcal{RH}_\infty, \text{ the last two constraints are added.} \\
\end{align*}
\]

Except for the condition in \([\Delta P, \Delta Q] \leq 1 \text{ for uncertainty matrices, there is not any special condition or assumption for matrices and the transfer function matrix.} \]

**Note 1.** In the case of computing the general form of the normalized left coprime factors (NLCFs) perturbed model, which is as follows

\[
G(i) = \left( M(i) + W_M(i) \Delta M(i) \right)^{-1} \left( N(i) + W_N(i) \Delta N(i) \right)
\]
the formulation can be rewritten in the same manner and in Equation (12), instead of $X$, the variable $Y$ is as follows

$$
\Psi (G_i, G) = Y^* X_i = \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} P_i \\ Q_i \end{bmatrix} \\
+ \begin{bmatrix} W_M & W_N \end{bmatrix} \begin{bmatrix} \Delta_M \\ 0 \\ 0 \\ \Delta_N \end{bmatrix} \begin{bmatrix} P_i \\ Q_i \end{bmatrix} = Y_0 X_i + W \Delta X_i
$$

(31)

where $Y_0 = \begin{bmatrix} M & N \end{bmatrix}$, $\Delta = \begin{bmatrix} \Delta_M \\ 0 \\ 0 \\ \Delta_N \end{bmatrix}$ and $W = \begin{bmatrix} W_M & W_N \end{bmatrix}$.

### 3.2 Calculation algorithm

There are several software tools like YALMIP [21] for solving the SDP problem resulted in Equation (30). At each frequency, this SDP is solved, and $X_0(\omega_j)$ and $W(\omega_j)$ are computed.

In the following, we summarize the proposed method for computing frequency responses of NCFs, in six steps.

1. Initialization: All variables and required settings for the SDP algorithm and also, the frequency index are as $j = 1$;
2. Determine the initial $\bar{X}$ at each frequency. As the best initial value, the average value of $G_i(\omega_j)$ for $i = 1 \ldots K$ is selected (e.g. $\bar{G} = \frac{\sum_{i=1}^{K} G_i}{K}$).
3. Using Equation (A.5) in Appendix, change $\bar{G}$ to $\bar{X}$,
4. Calculate the constraints Equations (16), (17) and (27).
5. Optimize the SDP Equation (30). During this optimization at each frequency, $X_0 = \begin{bmatrix} P \\ Q \end{bmatrix}$ and $W = \begin{bmatrix} W_P \\ W_Q \end{bmatrix}$ are derived and saved.
6. Check the frequency index. If $j$ is not equivalent to $N$, then go to the step 2.

The algorithm will be finished if the frequency index reached $N$. 

**Figure 2** Metric criterion of the nominal (average frequency responses) and identified models
Notes:

- In each frequency, the available samples ($K$) can be varied.
- The calculation algorithm is independent of the total frequency sample ($N$) and these samples can be increased, especially in some regions in which the system has a high variation.
- After the computation of the frequency responses of the NCFs and weighting matrices, they can be identified by identification techniques.

4 | SIMULATION RESULTS

Demonstrating the ability of this algorithm, in this section, we analyse two MIMO systems as examples.

**Example 1.** In the first example, we show the results of the computation of the NRCFs of a high-speed positioning mechanism presented in [17]. Twenty frequency responses were used for this identification. The Bode magnitude diagram of the nominal transfer function matrix, the average value of the frequency responses and the nominal model identified by the proposed algorithm are shown in Figure 1. The nominal model ($G_0$) for the additive uncertainty model ($G = G_0 + WP\Delta W_o$) is usually the average value of the frequency responses. As shown, the computed NCFs model is closer to the nominal model. This is shown in Figure 2 with the $\nu$-gap metric. In this figure, the maximum value of the $\nu$-gap metric of the perturbed model and the nominal model identified by NCFs and the maximum value of the $\nu$-gap metric of the perturbed model and the average frequency responses (the nominal model in the additive uncertainty model) are compared at each frequency. In numerical expression, $\delta_\nu(G,G_i)$ for the model identified by NCFs and for the model identified by an additive uncertainty model is equal, to 0.122 and 1.35 (over 11 times) respectively. These values for the nominal models are equal to 0.122 and 0.185 (over 1.5 times) respectively.

Figure 3 shows the singular values of the two normalized right coprime factors $P$ and $Q$. As can be seen, the singular values of the model identified by the new algorithm and the system nominal model have similar behaviours.
Example 2. The second example, selected from [18], is a distillation column system with pure delay and ill-conditioned and a nearly singular transfer function matrix. In this example, frequency responses in six operating conditions are used for identification. Figure 4 shows Bode magnitude diagrams of the transfer function matrix of the average frequency response and the identified nominal model by NCFs. Also \( \sqrt{\alpha} \) of the model identified in [18] and the computed model by our algorithm are compared together, as shown in Figure 5, which are very close to each other. Of course, in this reference, several optimizations have been done for computing the final optimized model; however, in our algorithm, the optimized NCFs perturbed model is derived directly and by one SDP optimization process.

5 | CONCLUSION

In this paper, the perturbed model of MIMO systems have been computed with frequency responses based on the \( \nu \)-gap metric minimization. This computation has been performed by introducing a convex optimization problem subjected to some LMI constraints (SDP). Direct computation of the right or left NCFs without additional constraints or conditions can be regarded as the best advantage of this type of formulation; so by having frequency responses, the perturbed coprime factors model of the system and its nominal model can be identified easily.

Finally, the extracted algorithm of this formulation has been used for two examples of two references. The results of the perturbed model computed by new algorithm have been compared with those in the references. As shown in the simulated results, our proposed method could directly and optimally compute the frequency responses of NCFs, such that the derived model had the minimum difference with the frequency responses according to the \( \nu \)-gap metric criterion.

Since, in the presented method and algorithm, any special condition or assumption has not been considered for the transfer function matrix, this method can be used for a vast domain of system. Also, this model can be transformed into another type of perturbed model.
FIGURE 5  Metric criterion of the optimized model in [16] (–o), $\sqrt{\alpha}$ (–), frequency responses (–).

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APPENDIX

According to the singular values theorem, the transfer function matrix at each frequency can be expressed by its singular values as

\[ G(j\omega) = P(j\omega) Q^{-1}(j\omega) \]

\[ P(j\omega) Q^{-1}(j\omega) = U(j\omega) \Sigma(j\omega) V^*(j\omega) = U^*V^* \]  \hspace{1cm} (A.1)

\[ P = U\Sigma V^* Q \Rightarrow P^*P = Q^*V\Sigma^TU^*U\Sigma V^* Q \]

\[ = Q^*V\Sigma^T\Sigma V^* Q \overset{\Delta}{=} X^*\Sigma^TX \]

\[ P^*P + Q^*Q = I \Rightarrow X^*\Sigma^T\Sigma X = I - X^*X \]  \hspace{1cm} (A.2)

So, we have

\[ X^*\Sigma^T\Sigma X = I - X^*X \Rightarrow X^*(I + \Sigma^T\Sigma)^{-1/2} \]

\[ X = I \Rightarrow X = (I + \Sigma^T\Sigma)^{-1/2} \]  \hspace{1cm} (A.3)

Consequently, the matrices \( P \) and \( Q \) are equal to

\[ Q = V(I + \Sigma^T\Sigma)^{-1/2} \]

\[ P = U^*V^*Q = U^*(I + \Sigma^T\Sigma)^{-1/2} \]  \hspace{1cm} (A.4)

In a similar manner, for the left coprime factors, we have

\[ M = (I + \Sigma\Sigma^T)^{-1/2}U^* \]

\[ N = MU^*V^* = (I + \Sigma\Sigma^T)^{-1/2}\Sigma V^* \]  \hspace{1cm} (A.5)

Also, since coprime factors are not unique, we can write

\[ Q_1 = QT \]

\[ P_1 = PT, \forall T: T \text{ is a unitary matrix} \]

\[ \Rightarrow G = P_1Q_1^{-1}, P_1^*P_1 + Q_1^*Q_1 = I \]  \hspace{1cm} (A.6)

The advantage of this coprime decoupling is that \( Q_1 \) is diagonal and real and its diagonal elements are non-zero and positive (\( Q_1 > 0 \)).