On the Complexity of Polytopes in $LI(2)$

Komei Fukuda ∗ May Szedlák†

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Abstract

In this paper we consider polytopes given by systems of $n$ inequalities in $d$ variables, where every inequality has at most two variables with nonzero coefficient. We denote this family by $LI(2)$. We show that despite of the easy algebraic structure, polytopes in $LI(2)$ can have high complexity. We construct a polytope in $LI(2)$, whose number of vertices is almost the number of vertices of the dual cyclic polytope, the difference is a multiplicative factor of depending on $d$ and in particular independent of $n$. Moreover we show that the dual cyclic polytope can not be realized in $LI(2)$.

1 Introduction

Throughout, we assume that we are given a bounded polytope by a system of $n$ inequalities in $d$ variables, of form $P = \{x \in \mathbb{R}^d \mid Ax \leq b\}$, where $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$. In the feasibility problem we want to find a solution $x \in P$. In general no strongly polynomial time algorithm (polynomial in $d$ and $n$) to solve the feasibility problem is known. Although the simplex algorithm runs fast in practice, in general it can have exponential running time \cite{3, 10}. On the other hand the ellipsoid method runs in polynomial time on the encoding of the input size, but is not practical \cite{9}. A first practical polynomial time algorithm, the interior-point method, was introduced in \cite{8}, and has been modified in many ways since \cite{12}.

We denote by $LI(2)$ the family of systems $Ax \leq b$, that have at most two variables per inequality with nonzero coefficient. In this family, Hochbaum and Naor’s algorithm finds a feasible point or a certificate for infeasibility in time $O(d^2n \log n)$ \cite{6}, i.e., it solves the feasibility problem in strongly polynomial time. Using this result and Clarkson’s redundancy removal algorithm \cite{2}, it was shown that in $LI(2)$ all redundancies can be detected in strongly polynomial time $O(nd^2s \log s)$, where $s$ denotes the number

∗Department of Mathematics and Department of Computer Science, Institute of Theoretical Computer Science, ETH Zürich, CH-8092 Zürich, Switzerland. komei.fukuda@math.ethz.ch
†Department of Computer Science, Institute of Theoretical Computer Science, ETH Zürich, CH-8092 Zürich, Switzerland. may.szedlak@inf.ethz.ch
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of nonredundant constraints [5]. Because of this difference in running time, it is hence natural to ask, whether polytopes LI(2) have a simpler structure than general polytopes. In particular we are interested to know how many vertices a polytope of this family can have.

It is known that in general the dual cyclic polytope maximizes the number of vertices for a polytope given by \( n \) constraints (see Theorem 1). In this paper we construct a polytope in LI(2), that has almost the same complexity as the dual cyclic polytope. This polytope was already introduced in [1] in the context of deformed products. In this polytope the number of vertices is smaller by a factor that only depends on the dimension \( d \) and not on \( n \), (see Lemma 3). A similar result can be shown not only for vertices but for all \( k \)-faces (see Theorem 6). This shows that polytopes in LI(2) can have high complexity; if \( d \) is constant, then even the same complexity as the dual cyclic polytope.

We will also show in Theorem 8 that the dual cyclic polytope can not be realized in LI(2) for \( d \geq 4 \). In particular in the dual cyclic polytope any pair of the \( n \) facets are adjacent, however in LI(2), there are \( \Omega(n^2/d^2) \) pairs that are not adjacent.

2 Definitions and Known Results

Let \( P = \{ x \in \mathbb{R}^d \mid Ax \leq b \} \) be a convex polytope in \( \mathbb{R}^d \), where \( A \in \mathbb{R}^{n\times d} \) and \( b \in \mathbb{R}^n \). The rows of \( Ax \leq b \) are called the constraints. The dimension of \( P \), denoted \( \text{dim}(P) \), is defined as the number of affinely independent points in \( P \) minus one. A \( k \)-dimensional subset \( F \subseteq P \) is a \( k \)-face of \( P \), if \( F \) has dimension \( k \) and if there exists a hyperplane \( h : ax \leq b \), such that \( ax^* = b \) for all \( x^* \in F \) and \( ax^* < b \) for all \( x^* \in P \setminus F \). The 0-dimensional faces are called the vertices of \( P \), the \((d-1)\)-dimensional faces are called facets. If \( F \) is a \( k \)-face, then \( ax = b \) for at least \( d - k \) constraints of \( Ax \leq b \).

For \( 0 \leq k \leq d \) we denote by \( f_k := f_k(P) \) the number of \( k \)-dimensional faces of \( P \). The \( f \)-vector of \( P \) is defined by \( f(P) := (f_0, f_1, \ldots, f_d) \).

**Theorem 1** (McMullen’s Upper Bound Theorem [11, 4]). The maximum number of \( k \)-faces in a \( d \)-dimensional polytope with \( n \) constraints is attained by the dual cyclic polytope \( c^*(n, d) \) and is given by

\[
f_k(c^*(n, d)) = \sum_{r=\min(k, \lfloor d/2 \rfloor)}^{\lfloor d/2 \rfloor - 1} \binom{n - d - 1 + r}{r} \binom{r}{k} + \sum_{r=\max(k, \lfloor d/2 \rfloor)}^{d} \binom{n - r - 1}{d - r} \binom{r}{k}.
\]

In particular the number of vertices is given by

\[
f_0(c^*(n, d)) = \binom{n - \lfloor d/2 \rfloor}{n - d} + \binom{n - \lfloor d/2 \rfloor - 1}{n - d}.
\]

**Remark.** For \( \lfloor d/2 \rfloor \leq k \leq d \) the formula can be simplified to

\[
f_k(c^*(n, d)) = \binom{n}{d - k}.
\]
This means that any \((d - k)\) constraints define a \(k\)-face.

For our calculation we will make use of the following well known formulas. Stirling’s formula says that
\[
n! = \Theta \left( \sqrt{n \frac{n^n}{e^n}} \right),
\]
as \(n\) goes to infinity. It follows that
\[
\binom{n}{k} \leq O(1) \cdot \frac{n^n}{k^k (n-k)^{n-k}}.
\]
(1)

Furthermore we need the well known inequality
\[
1 + x \leq e^x, \text{ for all } x \in \mathbb{R}.
\]
(2)

We conclude that
\[
\binom{n}{k} \leq O(1) \cdot \left( \frac{n}{k} \right)^k \left( 1 + \frac{k}{n-k} \right)^{-n-k} \leq O(1) \cdot \left( \frac{n}{k} \right)^k e^k.
\]
(3)

3 Lower Bound on Maximum Complexity of \(LI(2)\)

In the following we always assume that \([d/2]\) is a divisor of \(n\) (if \(d\) is even) or \(n - 1\) (if \(d\) is odd). All results naturally extend to any \(d < n\), but we would like to avoid to have even more floors and ceilings in the notation. We want to construct a polytope in \(LI(2)\), that has high complexity, i.e., with an \(f\)-vector of order close to the \(f\)-vector of the dual cyclic polytope.

In a first part let us assume that \(d\) is even. We pair the set of variables and define an \(n/(d/2)\) polygon on each of the pairs. Formally, for \(1 \leq i \leq d/2\), let
\[
A_i \left( \begin{array}{c} x_{2i-1} \\ x_{2i} \end{array} \right) \leq b_i,
\]
be a polygon in the \((x_{2i-1}, x_{2i})\)-plane, given by \(n/(d/2)\) constraints with \(n/(d/2)\) vertices. We denote \(P_i^* := P_i^*(n, d) = \{x \in \mathbb{R}^2 \mid A_i(x_{2i-1}, x_{2i})^T \leq b_i\}\) and by \(G_i\) the set of constraints of \(P_i^*\).

Now \(P^*(n, d)\) is defined as the \(d\)-dimensional polytope that we obtain from the union of \(G_i, 1 \leq i \leq d/2\). Since the \(P_i^*\)'s do not share any variables,
\[
P^*(n, d) = \{x \in \mathbb{R}^d \mid (x_{2i-1}, x_{2i}) \in P_i, \text{ for all } 1 \leq i \leq d/2\}.
\]

For \(d\) odd, we pair the first \(d-1\) variables and use the construction as above. Moreover we add the constraint \(x_d \geq 0\), i.e.,
\[
P^*(n, d) = \{x \in \mathbb{R}^d \mid (x_1, \ldots, x_{d-1}) \in P^*(n-1, d-1) \land x_d \geq 0\}.
\]
Theorem 2. [1] For \( d \) even, the polytope \( P^*(n, d) \) in \( LI(2) \) has the following number of vertices:

\[
\left( \frac{n}{d/2} \right)^{d/2}.
\]

For \( d \) odd it is

\[
\left( \frac{n-1}{[d/2]} \right)^{[d/2]}.
\]

The proof of [1] is given in a much more general setting of deformed products, we will here give the proof for our special case.

Proof. Let us assume first that \( d \) is even. For \( 1 \leq i \leq d/2 \) let \( G_i = \{ g^1_i, \ldots, g^n_{i(d/2)} \} \), where the \( g^j_i : a^j_i(x_{2i-1}, x_{2i})^T \leq b^j_i \), ordered in such a way that \( g^j_i \) and \( g^{j+1}_i \), \( 1 \leq j \leq d/2 \), define a vertex of \( P_i^* \). Throughout the proof, \( j + 1 \) is always considered modulo \( n/(d/2) \).

We will show that if for every \( P_i^* \) we choose two consecutive constraints \( g^j_i \) and \( g^{j+1}_i \), these \( d \) constraints define a vertex of \( P^*(n, d) \) and those are the only sets of \( d \) constraints that define vertices (see also Figure 1). Let us denote the set of vertices of \( P_i^* \) by \( V(P_i^*) \).

Formally we show that

\[
V(P^*) = \{ x \in \mathbb{R}^d | \exists (j_1, \ldots, j_{d/2}) : a^j_{i_1}(x_{2i_1-1}, x_{2i_1})^T = b^j_{i_1} \land a^j_{i_2}(x_{2i_2-1}, x_{2i_2})^T = b^j_{i_2} \forall i \}.
\]

Let us first show that the set on the right hand side is a subset of \( V(P^*) \). We show that

\[
V(P^*) = \{ x \in \mathbb{R}^d | \exists (j_1, \ldots, j_{d/2}) : a^j_{i}(x_{2i_1-1}, x_{2i_1})^T = b^j_{i} \land a^{j+1}_{i}(x_{2i_1-1}, x_{2i_1})^T = b^{j+1}_{i} \forall i \}.
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\]

Figure 1: \( d \) constraints that define vertex in \( P^*(12, 6) \)

the \( g^1_i, g^2_i \), \( 1 \leq i \leq d/2 \), define a vertex, the rest follows from symmetry. Let us denote those \( d \) constraints by \( G' \) and \( x^* \) the intersection point of their boundaries. It follows that \( x^* \in P^* \) because \( (x^*_{2i-1}, x^*_{2i}) \in P^*_i \) for all \( i \). We define the halfspace \( h \) by

\[
h : \sum_{i=1}^{d/2} (a^1_i(x_{2i-1}, x_{2i})^T + a^2_i(x_{2i-1}, x_{2i})^T) \leq \sum_{i=1}^{d/2} (b^1_i + b^2_i),
\]

the halfspace obtained by the sum of all constraints in \( G' \). Let us denote this halfspace by \( h : a^*x \leq b^* \). Then by definition it follows that \( a^*x^* = b^* \). Now let \( y \in P^* \setminus x^* \). Since \( y \in P^* \) it follows that \( a^1_i(x_{2i-1}, x_{2i})^T \leq b^1_i \) and \( a^2_i(x_{2i-1}, x_{2i})^T \leq b^2_i \) for all \( i \). Moreover
since \( y \neq x^* \) there exists some \( k \) such that \( a_k^1(x_{2i-1}, x_{2i})^T < b_k^1 \) or \( a_k^2(x_{2k-1}, x_{2k})^T < b_k^2 \). It follows that \( a'x < b' \), hence by definition of a 0-face, \( x^* \) is a vertex.

For the other direction we need to show that no other \( d \) constraints define a vertex (see also Figure 2). If we choose more than two constraints from some \( G_i \), then the intersection of their boundaries is empty. If we choose two constraints in \( G_i \) that are not adjacent, the point it defines in \( P_i^* \) violates some constraints of \( G_i \). Hence, we need to choose two consecutive constraints. The case where \( d \) is odd is similar. The vertices of \( P^*(n, d) \) are given by the constraints defining the vertices of \( P^*(n-1, d-1) \) together with \( x_d \geq 0 \). \( P^*(n-1, d-1) \) is the \( d-1 \) dimensional polytope defined by the constraints \( G \setminus \{ x_d \geq 0 \} \). The proof now follows by simple counting.

We will compare the number of vertices between \( P^*(n, d) \) and the dual cyclic polytope. Since we do not compare the exact values, but only the leading terms, we will not exactly compute the polynomial terms in \( d \), but denote them by \( \text{poly}(d) \).

**Lemma 3.** The dual cyclic polytope has a factor \( O(e^{|d/2|}) \) more vertices than \( P^*(n, d) \), i.e., \( f_0(c^*(n, d)) \leq O(e^{|d/2|}) \cdot f_0(P^*(n, d)) \).

We see that this factor is independent of \( n \), hence if \( d \) is constant then the number of vertices of \( P^*(n, d) \) is asymptotically equal to the number of vertices of the dual cyclic polytope.

**Proof.** Considering only the leading term of \( f_0(c^*(n, d)) \) and using inequality (3) we get

\[
f_0(c^*(n, d)) = \binom{n - [d/2]}{n-d} + \binom{n - [d/2] - 1}{n-d}
\leq 2 \cdot \binom{n - [d/2]}{|d/2|}
\leq O(1) \cdot e^{[d/2]} \cdot \left( \frac{n - [d/2]}{|d/2|} \right)^{|d/2|}
\leq O(1) \cdot e^{[d/2]} \cdot \left( \frac{n}{|d/2|} \right)^{|d/2|}.
\]
Therefore \( f_0(c^*(n, d)) \leq O(e^{d/2}) \cdot f_0(P^*(n, d)). \)

In the following we do not only compare the number of vertices between \( P^*(n, d) \) and \( c^*(n, d) \), but also their \( f \)-vectors. We will see that if \( k \leq \lceil d/2 \rceil - 1 \), then \( f_k(P^*(n, d)) \) is by a factor at most \( e^{d/2} \) larger than \( f_k(c^*(n, d)) \). If \( k \geq \lceil d/2 \rceil \), then the factor is at most \( e^{d-k} \).

**Theorem 4.** For \( d \) even

\[
f_k(P^*(n, d)) = \sum_{r=\max\{0,d/2-k\}}^{d/2-[k/2]} \binom{d/2}{r} \binom{d/2 - r}{d - k - 2r} \binom{n}{d/2}^{d-k-r}.
\]

For \( d \) odd and \( 0 < k < d \)

\[
f_k(P^*(n, d)) = f_{k-1}(P^*(n-1, d-1)) + f_k(P^*(n-1, d-1))
\]

\[
= \sum_{r=\max\{0,d/2-(k-1)\}}^{[d/2]-(k-1)/2} \binom{d/2}{r} \binom{d/2 - r}{d - k - 2r} \binom{n-1}{d/2}^{d-k-r}
\]

\[
+ \sum_{r=\max\{0,[d/2]-k\}}^{[d/2]-[k/2]} \binom{d/2}{r} \binom{d/2 - r}{(d-1) - k - 2r} \binom{n-1}{d/2}^{(d-1)-k-r}.
\]

The value of \( f_0(P^*(n, d)) \) follows from Theorem 2 and obviously \( f_d(P^*(n, d)) = 1 \).

**Proof.** The proof is similar to the proof of Theorem 2, we only give the main idea. Assume that \( d \) is even and \( 0 \leq k \leq d \). The \( k \)-faces of \( P^*(n, d) \) are induced by certain intersections of \( d-k \) constraints of \( G \) with \( P^*(n, d) \). Let \( K \) be \( d-k \) constraints from \( G \) such that the following holds. For every \( 1 \leq i \leq d/2 \), \( G_i \) contains at most two constraints of \( K \). If it contains two constraints \( h_\ell \) and \( h_m \) then they are consecutive, i.e., they define a vertex in \( P^*_i \) (see also Figure 3). The intersection of the boundaries of the constraints \( K \) with \( P^*(n, d) \) are in one to one correspondence with the \( k \)-faces. This works with a similar argument as in the proof of Theorem 2.

![Figure 3: Example of 3-face in \( P^*(12,6) \)](attachment:image.png)
It remains to count the number of faces that are induced by constraints of form $K$. Let us consider the sets in $K$, such that there are exactly $r$ many $G_i$'s that contain two constraints of $K$. There are

$$\binom{d/2}{r} \binom{d/2 - r}{(d - k) - 2r} \left( \frac{n}{d/2} \right)^{(d-k) - r}$$

of those. Now if $(d - k) \leq d/2$, then $r$ can be in $\{0, \ldots, \lfloor (d - k)/2 \rfloor \}$ and if $(d - k) > d/2$, then $r$ is in $\{d/2 - k, \ldots, \lfloor (d - k)/2 \rfloor \}$. The claim for even $d$ follows.

The case where $d$ is odd is similar. We will not go into detail but only give the main idea. With the same kind of argumentation as above one can show that the $k$-dimensional faces are induced by $(d - k)$ constraints $K$ of $P^*$ in one of the following ways. In the first case $K$ does not contain the constraint $x_d \geq 0$. Then the constraints in $K$ must induce a $k-1$ face in $\mathbb{R}^{d-1}$, which then induces a $k$-face in $\mathbb{R}^d$. There are $f_{k-1}(P^*(n-1,d-1))$ constraints of this form. In the second case $K$ contains the constraint $x_d \geq 0$. Then the remaining $(d-k) - 1$ constraints must induce a $k$ face in $P^{d-1}$. There are $f_k(P^*(n-1,d-1))$ constraints of this form.

**Lemma 5.** The following tables show the leading terms of $P^*(n,d)$ and $c^*(n,d)$, if $d = o(n)$.

| $k$ | $P^*(n,d)$ | $c^*(n,d)$ |
|-----|------------|------------|
| $k \leq d/2$ | $\binom{d/2}{k} \cdot \left( \frac{n}{d/2} \right)^{d/2}$ | $\binom{d/2}{k} \cdot \left( \frac{n}{d/2} - \frac{d-k}{d/2} \right)$ |
| $k > d/2$ | $\binom{d/2}{d-k} \cdot \left( \frac{n}{d/2} \right)^{d-k}$ | $\binom{n-k-1}{d-k}$ |

$d$ even

| $P^*(n,d)$ | $c^*(n,d)$ |
|------------|------------|
| $k \leq \lfloor d/2 \rfloor$ | $\left( \binom{\lfloor d/2 \rfloor}{k} \right) + \left( \binom{\lfloor d/2 \rfloor}{k+1} \right) \cdot \left( \frac{n-1}{\lfloor d/2 \rfloor} \right)^{\lfloor d/2 \rfloor}$ | $\binom{\lfloor d/2 \rfloor}{k} \cdot \left( \frac{n-\lfloor d/2 \rfloor-1}{\lfloor d/2 \rfloor} \right)$ |
| $k \geq \lfloor d/2 \rfloor$ | $\binom{\lfloor d/2 \rfloor}{d-k} \cdot \left( \frac{n-1}{\lfloor d/2 \rfloor} \right)^{d-k}$ | $\binom{n-k-1}{d-k}$ |

$d$ odd

The proof of the lemma follows by checking the formulas of $P^*(n,d)$ and $c^*(n,d)$.

**Theorem 6.** For $k < \lfloor d/2 \rfloor$

$$f_k(c^*(n,d)) = O(poly(d) \cdot c^{\lfloor d/2 \rfloor} \cdot f_k(P^*(n,d))),$$

and for $k \geq \lfloor d/2 \rfloor$

$$f_k(c^*(n,d)) = O(poly(d) \cdot e^{d-k} \cdot f_k(P^*(n,d))),$$

where $poly(d)$ is some polynomial in $d$. 

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Proof. We only consider the case where \( k \leq \lfloor d/2 \rfloor - 1 \), as the case where \( k \geq \lfloor d/2 \rfloor \) is similar. We will prove the statement for odd \( d \), the case where \( d \) is even follows immediately by replacing all \( \lfloor d/2 \rfloor \) and \( \lceil d/2 \rceil \) by \( d/2 \). First note that the leading term of \( P^*(n, d) \) can be written as

\[
\left( \left( \frac{\lfloor d/2 \rfloor}{k} \right) + \left( \frac{\lceil d/2 \rceil }{k+1} \right) \right) \cdot \left( \frac{n-1}{\lfloor d/2 \rfloor} \right)^{\lfloor d/2 \rfloor} = \text{poly}(d) \cdot \left( \frac{\lfloor d/2 \rfloor}{k} \right) \cdot \left( \frac{n-1}{\lceil d/2 \rceil} \right)^{\lfloor d/2 \rfloor}.
\]

For the term of \( c^*(n, d) \) we have

\[
\left( \frac{\lfloor d/2 \rfloor}{k} \right) \cdot \left( \frac{n-\lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor} \right) \leq O(1) \cdot e^{\lfloor d/2 \rfloor} \cdot \left( \frac{\lfloor d/2 \rfloor}{k} \right) \cdot \left( \frac{n-\lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor} \right)^{\lfloor d/2 \rfloor} \leq O(1) \cdot e^{\lfloor d/2 \rfloor} \cdot \left( \frac{\lfloor d/2 \rfloor}{k} \right) \cdot \left( \frac{n-1}{\lfloor d/2 \rfloor} \right)^{\lfloor d/2 \rfloor}.
\]

Therefore if \( k \leq \lfloor d/2 \rfloor - 1 \), it holds that \( f_k(c^*(n, d)) = O(\text{poly}(d) \cdot e^{\lfloor d/2 \rfloor}) \cdot f_k(P^*(n, d)) \). \( \square \)

4 Upper Bound on Maximum Complexity of \( LI(2) \)

In this section we show that no polytope in \( LI(2) \) can achieve the complexity of the dual cyclic polytope. To our knowledge, this is the first time such bounds are given. In Lemma \( \text{[4]} \) we show that for all polytopes \( P \) in \( LI(2) \), \( d \geq 4 \) and \( \lfloor d/2 \rfloor \leq k \leq d - 2 \) it holds that \( f_k(P) < f_k(c^*(n, d)) \). Using this result in Theorem \( \text{[8]} \) we show that this holds for all \( k \leq d - 2 \).

Lemma 7. Let \( P \) be any polytope in \( LI(2) \) given by \( n \) nonredundant constraints, \( d \geq 4 \) and denote by \( n' \) the number of constraints that contain exactly two variables per inequality. As for each index \( i \in [d] \) there are at most two inequalities that contain only \( x_i \) it follows that \( n - 2d \leq n' \leq n \). Then for all \( \lfloor d/2 \rfloor \leq k \leq d - 2 \) we have

\[
f_k(P) < f_k(c^*(n, d)).
\]

In particular

\[
f_{d-2}(P) \leq \binom{n}{2} - \binom{n'}{d} + n' < \binom{n}{2} = f_{d-2}(c^*(n, d)).
\]

Proof. Let us focus on the case of \( f_{d-2}(P) \). In the dual cyclic polytope we know that any two facets are adjacent, i.e., their intersection defines a \((d-2)\)-face. In \( LI(2) \) however, not every two facets can be adjacent. Assume \( P \) is given by \( n \) constraints with index set \( E \). For \( i < j \in [d] \) let \( E_{ij} \) be the indices of the constraints that contain \( x_i \) and \( x_j \) and denote \( |E_{ij}| = n_{ij} \). As in the proof of Theorem \( \text{[4]} \) we know that out of the \( \binom{n}{2} \) pairs only \( n_{ij} \) pairs are adjacent. Summing over all \( i < j \) it follows that at least

\[
\sum_{i<j} \left( \binom{n_{ij}}{2} - n_{ij} \right)
\]
pairs of facets in $P$ are not adjacent. Now using that $\sum_{i<j} n_{ij} = n'/\binom{d}{2}$, we get that
\[
\sum_{i<j} \left(\binom{n_{ij}}{2} - n_{ij}\right) \geq \binom{d}{2} \cdot \frac{n'}{\binom{d}{2}} - n'
\]
\[
= \frac{1}{\binom{d}{2}} \cdot \frac{n' \cdot (n' - \frac{1}{\binom{d}{2}})}{2} - n'
\]
\[
\geq \frac{\binom{n'}{2}}{\binom{d}{2}} - n'.
\]
The claim for $k = d - 2$ follows. For other values of $k$ one can similarly show that not all $(d - k)$-tuples of constraints define a $k$-face in $P$. \qed

**Theorem 8.** Let $P$ be any $d$-dimensional polytope in $LI(2)$ given by $n$ nonredundant constraints, where $d \geq 4$. Then for all $k \leq d - 2$ we have
\[
f_k(P) < f_k(c^*(n,d)).
\]
In particular
\[
f_k(P) \leq f_k(c^*(n,d)) - \binom{d-2}{k} \cdot \left(\frac{n'}{\binom{d}{2}} + n'\right),
\]
where $n'$ is defined as in Lemma 7.

Although asymptotically the bounds that we prove are the same as the bounds of the dual cyclic polytope, this shows that the dual cyclic polytope is not realizable in $LI(2)$.

Before proving this theorem we introduce a few notions used in the proof of McMullen’s Upper Bound Theorem (for more details see [11, 7, 4]). From now on we only consider simple $d$-dimensional polytopes given by $n$ nonredundant constraints. A polytope $P$ is called simple, if every vertex of $P$ satisfies exactly $d$ inequalities with equality. We observe that by small perturbations, for any $d$-dimensional $P'$ in $LI(2)$ given by $n$ inequalities there exists a simple polytope $P$ in $LI(2)$ with $f_k(P') \leq f_k(P)$ for all $k \in [d]$. Let us denote the family of simple $d$-dimensional polytopes in $LI(2)$ by $SLI(2)$.

Let $P$ be any polytope in $SLI(2)$, given by $n$ nonredundant constraints. We consider a linear program with objective value $c^T x$, subject to those constraints. We assume that $c$ is generic, i.e., no edge of $P$ is parallel to the hyperplane given by $c^T x = 0$. We now orient every edge of $P$ w.r.t. $c^T x$, towards the vertex with higher objective value. Let us denote the graph defined by those directed edges by $\vec{G}(P)$. Now for $i = 0, \ldots, d$ we denote by $h_i(\vec{G}(P))$ the number of vertices with indegree $i$.

By double counting one can show that $h_i(\vec{G}(P))$ is independent of the objective value, hence we can write $h_i(\vec{G}(P)) = h_i(P)$ Let $k$ be fixed, we count the pairs $(F, v)$
of $k$ faces $F$ with unique sink $v$. By definition of $\G(P)$ every face has a unique sink, hence there are exactly $f_k(P)$ many such pairs. On the other hand by properties of simple polytopes it holds that for any $k$ distinct edges to $v$, there exists a unique $k$-face containing the $k$ edges. Let $v$ be fixed and let $r$ be the indegree of $v$. Summing over all indegrees $r \geq k$ it follows that for all $k = 0, \ldots, d$,

$$\sum_{r=k}^{d} h_r(\G(P)) \binom{r}{k} = f_k(P). \quad (4)$$

Solving this system of linear equalities one can show that for all $i = 0, \ldots, d$,

$$h_i(P) := h_i(\G(P)) = \sum_{k=i}^{d} (-1)^{k-i} \binom{k}{i} f_k(P). \quad (5)$$

Hence $h_i(P)$ is independent of the objective value.

To prove Theorem 8 we use the following strengthened version of McMullen’s theorem, which holds for any simple polytopes. This strengthening was first given by Kalai in [7] with a small correction made by Fukuda in [4, Chapter 7]. Note that Theorem 9 implies McMullen’s theorem, since by (4) we know that each $f_k(P)$ is a nonnegative linear combination of the $h_r(P)$’s.

**Theorem 9 (Strengthened Upper Bound Theorem [11, 4]).** Let $P$ be a simple polytope given by $n$ nonredundant constraints. Then for all $i = 0, \ldots, d$ it holds that

$$h_i(P) \leq h_i(c^*(n, d)).$$

**Proof of Theorem 8** Let $P$ be any polytope in $SLI(2)$. By Lemma 7 the theorem holds for $\lceil d/2 \rceil \leq k \leq d - 2$ (since $d \geq 4$ it holds in particular for $k = d - 2$). We claim that

$$h_{d-2}(P) \leq h_{d-2}(c^*(n, d)) - \binom{n'}{2} \frac{1}{d} + n'.$$

By equation (5)

$$h_{d-2}(P) = f_{d-2}(P) - (d-1)f_{d-1}(P) + \binom{d}{d-2} f_d(P).$$

We know that

$$f_{d-1}(P) = f_{d-1}(c^*(n, d)) = n$$

and $f_d(P) = f_d(c^*(n, d)) = 1$.

Furthermore by Lemma 7 we know

$$f_{d-2}(P) \leq f_{d-2}(c^*(n, d)) - \binom{n'}{2} \frac{1}{d} + n'.$$
It follows that
\[
    h_{d-2}(P) = f_{d-2}(P) - (d-1)f_{d-1}(P) + \left( \begin{array}{c} d \\ d-2 \end{array} \right) f_d(P)
\]
\[
    \leq f_{d-2}(c^*(n,d)) - \left( \begin{array}{c} n' \\ d \end{array} \right) + n' - (d-1)f_{d-1}(c^*(n,d)) + \left( \begin{array}{c} d \\ d-2 \end{array} \right) f_d(c^*(n,d))
\]
\[
    = h_{d-2}(c^*(n,d)) - \left( \begin{array}{c} n' \\ d \end{array} \right) + n',
\]
which shows the claim. By equation (4) and Theorem 3 for \( k \leq d-2 \) it follows that
\[
f_k(P) = \sum_{r=k}^{d} \binom{r}{k} h_r(P)
\]
\[
\leq \sum_{r=k}^{d} \binom{r}{k} h_r(c^*(n,d)) - \binom{d-2}{k} \left( \frac{n'-2}{d} \right) + n'
\]
\[
= f_k(c^*(n,d)) - \binom{d-2}{k} \left( \frac{n'-2}{d} \right) + n'.
\]

Remark. One can show that for \( d = 3 \), the bounds of McMullen’s Upper Bound Theorem can be achieved in \( LI(2) \). Let \( P \) be a polytope given by \( n-2 \) constraints in variables \( x_1 \) and \( x_2 \), such that they define a polygon with \( n-2 \) vertices in two dimensions. We furthermore add the constraints \( x_3 \geq 0 \) and \( x_3 \leq 1 \). We can easily observe that \( f_0 = 2n-4 \) and \( f_1 = 3n-6 \). Those are exactly the bounds achieved by the dual cyclic polytope.

5 Discussion and Open Questions

We saw that \( f_k(P^*(n,d)) \) differs from \( f_k(c^*(n,d)) \) by a factor \( O(e^{d/2}) \) if \( k < \lceil d/2 \rceil \) and \( O(e^{d-k}) \) otherwise. In particular, if \( d \) is constant, then \( f_k(P^*(n,d)) \) is of the same order as \( f_k(c^*(n,d)) \). The high complexity of \( P^*(n,d) \) shows us that although \( LI(2) \) has a much simpler structure than general linear programs, it is still a powerful and complex tool. We also showed that the dual cyclic polytope is not realizable in \( LI(2) \). However in the upper bound we showed, the asymptotic complexity remains the same. It would be interesting to get a deeper understanding of \( LI(2) \) and how it is different from general linear programs. The main open question that remains is how large the complexity of \( f(P) \) can be for a polytope \( P \) in \( LI(2) \). Is it possible to have higher complexity than the complexity of \( P^*(n,d) \)? If yes, what is the maximum complexity that can be achieved? Is it asymptotically the same as the complexity of the a dual cyclic polytope? This is an interesting direction for future research.
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