Asymptotically Optimal Sampling Policy for Quickest Change Detection with Observation-Switching Cost

Tze Siong Lau and Wee Peng Tay, Senior Member, IEEE

Abstract—We consider the problem of quickest change detection (QCD) in a signal where its observations are obtained using a set of actions, and switching from one action to another comes with a cost. The objective is to design a stopping rule consisting of a sampling policy to determine the sequence of actions used to observe the signal and a stopping time to quickly detect for the change, subject to a constraint on the average observation-switching cost. We propose a sampling policy of finite window size and a generalized likelihood ratio (GLR) Cumulative Sum (CuSum) stopping time for the QCD problem. We show that the GLR CuSum stopping time is asymptotically optimal with a properly designed sampling policy and formulate the design of this sampling policy as a quadratic programming problem. We prove that it is sufficient to consider policies of window size not more than one when designing policies of finite window length and propose several algorithms that solve this optimization problem with theoretical guarantees. Finally, we apply our approach to the problem of QCD of a partially observed graph signal.

Index Terms—Quickest change detection, GLR CuSum, sampling policy, graph sampling

I. INTRODUCTION

Quickest change detection (QCD) is the problem of detecting an abrupt change in a system while keeping the detection delay to a minimum. In a usual scenario, a sequence of independent and identically distributed (i.i.d.) observations \( \{x_t : t \in \mathbb{N} \} \) with distribution \( f_0 \) up to an unknown change point \( \nu \), and i.i.d. with distribution \( f_1 \neq f_0 \) after \( \nu \) is obtained. The objective is to detect for the change at \( \nu \) as quickly as possible while maintaining false alarm constraints [1]–[3]. QCD has applications across diverse fields, including quality control [4]–[7], fraud detection [8], cognitive radio [9], [10], network surveillance [11]–[14], structural health monitoring [15], spam detection [16]–[18], bioinformatics [19], power system line outage detection [20], and sensor networks [21]–[24].

In many applications, the signal of interest \( x_t \) may be high dimensional. For example, \( x_t \) may consist of observations from many correlated sensors. Due to the large number of sensors in the network, bandwidth and power constraints prevent us from observing the entire network at any time instance, and we may only obtain sensor readings from a small subset of sensors at any time instance [25], [26]. While it may seem optimal to observe the maximum number of sensors allowed by the network, this sampling policy may not be feasible due to power and communication bandwidth considerations. Furthermore, the action of switching from one subset to another subset of sensors also incurs power and communication costs. In this paper, we consider both of these costs collectively as the observation-switching cost, and we study the problem of QCD while maintaining an average observation-switching cost (AOSC) constraint. To be more general, we consider the case where the signal can only be observed using an action selected from a set of permissible actions with observation-switching costs associated with the sequence of actions chosen. We assume that the pre- and post-change distributions as well as their conditional distributions given the actions are known to the observer. The objective is to design a sampling policy together with a stopping time that satisfies both the QCD false alarm and AOSC requirements. To solve this problem, we propose a sampling policy that determines which action to perform based on the previous actions and a generalized likelihood ratio (GLR) Cumulative Sum (CuSum) stopping time for the QCD problem. We show that the GLR CuSum stopping time is asymptotically optimal with a properly designed sampling policy and formulate the design of the sampling policy as a quadratic programming problem.

A. Related Work

Existing works in QCD where the signal is not entirely available to the decision maker or the fusion center can be categorized into three main categories. In the first category, the papers [27]–[30] consider the problem of distributed or decentralized QCD where each node observes and processes its signal locally, with some memory of its previous messages, before sending a message to the fusion center. The authors of [28] consider the problem where each sensor only has access to the local information at that node and would process the signal to send a quantized message to the fusion center for further processing.

The second category of papers [31]–[34] considers a QCD problem where there is an additional cost for sensing at each node, and a control policy that determines whether a given observation is taken. In [31], the authors developed a data-efficient scheme that allows for optional on-off sampling of the
observations in the case where either the post-change family of distributions is finite, or both the pre- and post-change distribution belong to a one parameter exponential family.

In the third category, the papers [35]–[37] consider QCD where the observer only has access to compressed or incomplete measurements. The authors of [35] study the problem of sequential change point detection where a randomly generated linear projection is used to reduce the dimensions of a high dimensional signal for the purpose of QCD. In [37], the authors consider QCD with a closed-loop control of the actions where the current nodes to observe is determined by the maximal likelihood estimate of the post change hypothesis. In [38], we discussed the QCD problem where the observer is only able to obtain a partial observation of the signal through an action with an open-loop control of the actions.

Unlike the papers mentioned above, in this paper, we provide a general framework by considering random decision rules to select the current actions. We also do not give a fixed cost to the sampling of observation. Instead, we consider a more general approach where we use a set of permissible actions to model the practical sampling constraints and a cost is associated to the switching of actions to model observation-switching costs. In this paper, we consider the case where the decision maker is given a finite set of pre-defined actions, and the observed signal is a function of the action and the full signal. We also do not make any assumptions about the pre- and post-change distributions.

B. Our Contributions

To the best of our knowledge, there are no existing work that considers the QCD problem while taking observation-switching costs into account. In this paper, we consider the problem QCD while maintaining an AOSC constraint. The objective is to design a sampling policy together with a stopping time that satisfies both the QCD false alarm average run length (ARL) and AOSC requirements. Our main contributions are as follows:

1) We formulate the QCD problem with an AOSC constraint with sampling policy of window size \( W < \infty \), where the current action is allowed to be dependent only on the previous \( W \) actions. We show that the GLR CuSum stopping time together with a properly designed sampling policy is asymptotically optimal. We formulate the design of the sampling policy of window size \( W \) as a quadratic programming problem with an additional combinatorial constraint.

2) We derive expressions for the AOSC and the asymptotic worst-case average detection delay (WADD) for the GLR CuSum stopping time together with a sampling policy of window size \( W < \infty \). Using these expressions, we prove that under the constraint that \( W \geq 1 \), there exists an asymptotically optimal sampling policy of window size \( W = 1 \).

3) For the case when the window size of the sampling policy is \( W = 0 \), we provide derivations to transform the policy design problem into a linear programming problem for the special case where all observation-switching costs are equal and also an iterative rank minimization (IRM) algorithm to obtain a locally optimal solution for the general case.

4) For the case when the window size of the sampling policy is \( W = 1 \), we provide relaxations and derivations to transform the policy design problem into a convex programming problem.

The rest of this paper is organized as follows. In Section II, we present our signal model and problem formulation. In Section III, we present properties of the AOSC and the asymptotic ARL-WADD trade-off. In Section IV, we present the GLR CuSum stopping time and formulate the design of the sampling policy as a quadratic programming problem. We present algorithms to solve the policy design problems in Section V. Numerical results are presented in Section VI. We conclude in Section VII.

Notations: The operator \( \mathbb{E}_f \) denotes mathematical expectation with respect to (w.r.t.) the probability density (pdf) \( f \), and \( X \sim f \) means that the random variable \( X \) has distribution with pdf \( f \). If the change point is at \( \nu \) and post-change distribution is \( f_{\nu+1} \), we let \( P_{-\operatorname{NoValue}} \) and \( P_{\nu,m} \) be the probability measure and mathematical expectation, respectively. The Gaussian distribution with mean \( \mu \) and covariance \( \Sigma \) is denoted as \( N(\mu, \Sigma) \). Almost-sure convergence under the probability measure \( P_{-\operatorname{NoValue}} \) is denoted as \( \frac{P_{\nu,m}}{\operatorname{as}} \). We use \( 1_E \) as the indicator function of the set \( E \), and \( \mathbb{N} \) to denote the set of positive integers. We use \( \mathbb{R} \) to denote the set of real numbers. We also use the notation \( a^{k:t} \) to denote the sequence \( (a_k, a_{k+1}, \ldots, a_t) \). For \( \alpha = (a_1, a_2, \ldots, a_M) \), we use the notation \( \alpha[j] = a_j \) to denote its \( j \)-th entry. For a probability transition matrix \( T \), we use the notation \( T[i,j] \) to denote the probability of moving to a state \( j \) given that it is currently at state \( i \). For a probability mass function \( f \), \( \operatorname{supp}(f) \) denotes the support of \( f \).

II. Problem Formulation: Quickest Change Detection with a Cost for Switching Actions

Let \( f_0 \) and \( f_1, \ldots, f_M \) be \( M + 1 \) distinct distributions on \( \mathbb{R}^N \), and \( X_1, X_2, \ldots \) be a sequence of vector-valued random variables satisfying the following:

\[
\begin{cases}
X_t \sim f_0 \text{ i.i.d. for all } t < \nu, \\
X_t \sim f_m \text{ i.i.d. for all } t \geq \nu.
\end{cases}
\] (1)

where \( \nu \geq 0 \) and \( m \in \{1, \ldots, M\} \) are unknown but deterministic constants. The QCD problem is to detect the change in distribution as quickly as possible by observing \( X_1, X_2, \ldots \), while keeping the false alarm rate low.

In this paper, we assume that the observer is only able to obtain a partial observation \( (A_t, Y_t) \) of \( X_t \), where \( Y_t \equiv A_t(X_t) \) is a function of the random variable \( X_t \) under the action \( A_t \) at each time \( t \). Let \( A \) be the collection of permissible actions. We assume that the set \( A = \{1, 2, \ldots, |A|\} \) is finite. We also assume that at each time \( t \), under the distribution \( f_m \), the observation \( Y_t \) is conditionally independent of \( Y_1, \ldots, Y_{t-1} \) and \( A_1, \ldots, A_{t-1} \) given the action \( A_t \). Some examples of \( A \) that arise in practical applications include:
1) Network Sampling. The set of rank $n < N$ transformations with
\[ A = \{ \mathbf{M} \mid \mathbf{M} = [e_{i_1}, \ldots, e_{i_n}]^T, i_1 < i_2 < \ldots < i_n \}, \]
where $e_i$ is an $N \times 1$ column vector with all zeros except a 1 at the $i$-th position, and $\mathbf{M}(X_t) = \mathbf{M}X_t$.

2) 1-bit Quantization. The set of functions on $\mathbb{R}$ for a fixed $n \in \mathbb{N}$ with
\[ A = \{ q : \mathbb{R} \to \mathbb{R} \mid q(x) = 1_{\{x \geq q\}}, \ q \in \{1, 2, \ldots, 2^n\} \}, \]
where $1_A$ is the indicator function of the set $A$.

In our sequential change detection problem, we obtain observations $(A_1, Y_1) = (a_1, y_1), (A_2, Y_2) = (a_2, y_2), \ldots$ sequentially and aim to detect the change in distribution from observations $(A_t, Y_t)$ to $(A_{t+1}, Y_{t+1})$ as quickly as possible. This is determined by a stopping time. At each time instant, we detect delay using the worst case average detection delay $q$.

The set of functions on $\mathbb{R}$ to action $\pi$ is a Markov chain of order $f$ sequentially and aim to detect the change in distribution from observations $A_t$ and $Y_t$.

For a stopping time $\tau$, we let $\tau, \pi$ be a Markov chain of order $f$ and probability transition matrix $T$. For a stopping time $\tau$, we recall a relation between the average number of visits and the stationary distributions of a Markov chain. Let $N_t(\alpha; \beta) = \sum_{i=1}^{\tau} \mathbb{P}(\tau \in [i,i+1))$, the expected proportion of visits to each of the states, denoted as $\mathbb{P}(\tau \in [i,i+1))$.

For the rest of this section, we present results that relate the AOSC and WADD of sampling policies with different initial distributions but equal probability transition matrices. First, we recall a relation between the average number of visits and the stationary distributions of a Markov chain. Let $N_t(\alpha; \beta) = \sum_{i=1}^{\tau} \mathbb{P}(\tau \in [i,i+1))$, the expected proportion of visits to each of the states, denoted as $\mathbb{P}(\tau \in [i,i+1))$.

For any state $\alpha \in A^W$, the Markov chain defined by the transition matrix $T$ has at least one recurrence class. Let $R$ be the number of recurrence classes and $U$ be the number of transient states. By the Ergodic Theorem for finite state Markov chains [40], for a finite state Markov Chain with $R$ recurrent classes $\{R_1, R_2, \ldots, R_R\}$, there exists $R$ stationary distributions $\xi_1, \ldots, \xi_R$ where $\xi_r(\alpha) = 0$ if the state $\alpha \notin R_r$, and for recurrent states $\beta \in R_r$, we have
\[ \lim_{t \to \infty} \frac{N_t(\alpha; \beta)}{t} = \xi_r(\alpha) \text{ a.s.,} \]
\[ \lim_{t \to \infty} \frac{\mathbb{E}_{- \text{NoValue}}[N_t(\alpha; \beta)]}{t} = \xi_r(\alpha), \]
for any state $\alpha \in A^W$. For transient states $\beta$,
\[ \lim_{t \to \infty} \frac{\mathbb{E}_{- \text{NoValue}}[N_t(\alpha; \beta)]}{t} = \sum_{r=1}^{R} f_{\beta,r} \xi_r(\alpha), \]
where $f_{\beta,r}$ is the first-passage probability of initializing at state $\beta$ and entering the recurrence class $R_r$ before any other recurrence classes.

For any state $\beta$, denoting $\xi$ as the vector of expected proportion of visits to each of the states initializing at state $\beta$ such that $\xi(\alpha) = \lim_{t \to \infty} \frac{\mathbb{E}_{- \text{NoValue}}[N_t(\alpha; \beta)]}{t}$, we can see that $\xi$ is a stationary distribution of the probability transition matrix $T$ as it is a convex linear combination of stationary distributions. Thus, for any initial distribution $q$, the expected proportion of visits to each of the states, denoted as $\mathbb{E}_{- \text{NoValue}}[N_t(\alpha; \beta)]$, is a convex linear combination of stationary distributions.

III. PROPERTIES OF THE AOSC

In this section, we present results that relate the AOSC and WADD of a sampling policy. When the window size of the sampling policy is zero, the actions used to observe the signal are generated i.i.d. with respect to the distribution $q$. In this case, the observations $(A_t, Y_t) = (a_t, y_t)$ are also generated i.i.d.

When the window size of the sampling policy is positive, unlike the former case, it is possible that the actions $(A_t)_{t \in \mathbb{N}}$ and observations $(A_t, Y_t)_{t \in \mathbb{N}}$ are not generated i.i.d. We denote the joint probability distribution function of $(A_t, Y_t)$ under $f_m$ as
\[ p_m(a^{1:t}, y^{1:t}) = q(a^{1:M}) \prod_{j=1}^{t} p_m(a_j | a^{k-M:1-k}), \]
where $p_m$ is the conditional probability mass function of $A_k$ given $A^{k-M:1-k}$ induced by the probability transition matrix $T$ and $p_m(y|a)$ is the conditional probability density function of an observation $Y_t = y$ given the action $A_t = a$ under the distribution $f_m$.

A sampling policy $\pi = (q, T)$ with window size $W$ can also be written as a Markov chain $\pi' = (q', T')$ of order 1 where $T'$ satisfies $T'[\alpha, \beta] = 0$ whenever $\beta[i] \neq \alpha(i-1)$ for some $i \in \{2, \ldots, W\}$ and $\alpha, \beta \in A^W$. For the rest of this paper, we switch between either representation of a sampling policy $\pi$ to simplify the computations in the proofs. We denote the observation-switching costs associated with the latest two actions $C[\alpha|W-1], \alpha[W]$, where $\alpha[W] = \alpha[W-1]$. For the rest of this section, we consider that the AOSC and WADD of sampling policies with different initial distributions but equal probability transition matrices.

In order to take the observation-switching costs of a policy $\pi$ into consideration, we let $C$ be a $|A| \times |A|$ matrix where its $(i,j)$-th entry $C[i,j]$ denotes the cost of switching from action $i$ to action $j$. Inspired by a similar cost first proposed in [33], we define the AOSC of the policy $\pi$ as
\[ \text{AOSC}(\pi) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_{- \text{NoValue}}[\sum_{t=2}^{n+1} C[A_{t-1}, A_t]]. \]
stationary distribution of $T$. In the next lemmas, we see that the AOSC and asymptotic log likelihood ratios depend only on $T$ and the expected proportion of visits to each of the states.

**Lemma 1.** Let $\pi_1 = (q, T)$ be a policy of finite window size $W$ and $\pi_2 = (\overline{q}, T)$, then we have

$$\text{AOSC}(\pi_1) = \text{AOSC}(\pi_2).$$

**Proof:** See Appendix A.

Next, we let

$$\Lambda_m(k, t) = \frac{\log p_m(A^{k:t}, Y^{k:t})}{p_0(A^{k:t}, Y^{k:t})} = \sum_{i=0}^{t} \log \frac{p_m(Y_i|A_i)}{p_0(Y_i|A_i)}.$$

We show that for any policy $\pi = (q, T)$ where $\overline{q}$ supports only one recurrence class, the following results hold.

**Lemma 2.** For any policy $\pi = (q, T)$ of finite window size $W$ where $q$ has support in only one recurrence class $R$, and any $m \in \{1, \ldots, M\}$ and change-point $\nu < \infty$, we have

$$P_{\text{No Value}}\left(\lim_{t \to \infty} \frac{1}{t-\nu} \Lambda_m(\nu, \nu + t) = I_{m, \pi}\right) = 1,$$

with

$$I_{m, \pi} = \sum_{\alpha \in R} \overline{q}(\alpha) D(p_m(\cdot | \alpha[W]) || p_0(\cdot | \alpha[W])).$$

**Proof:** See Appendix B.

**Lemma 3.** Let $\pi = (q, T)$ be a policy of finite window size $W$. Suppose $\overline{q}$ supports only one recurrence class $R$. For any $\epsilon > 0$ and $m \in \{1, \ldots, M\}$, we have

$$\lim_{t \to \infty} P_{\text{No Value}}\left(\left|\frac{1}{t} \Lambda_m(\nu, \nu + t) - I_{m, \pi}\right| > \epsilon\right) = 0,$$

for $0 \leq \nu < \infty$, and

$$\sup_{0 \leq \nu < \infty} \frac{1}{t} \Lambda_m(\nu, \nu + j) - I_{m, \pi} \rightarrow 0 \quad \text{as} \quad t \to \infty.$$

**Proof:** See Appendix C.

IV. STRUCTURE OF ASYMPTOTICALLY OPTIMAL STOPPING TIME

In this section, we present the GLR CuSum test for the QCD problem with AOSC constraints and study its asymptotic properties as $\gamma \to \infty$.

A. Asymptotic Properties of GLR CuSum

For a fixed policy $\pi = (q, T)$, the GLR CuSum stopping time $\tau_{\pi}$ w.r.t. the observed sequence $(A_t = a_t, Y_t = y_t)_{t \in \mathbb{N}}$ is defined as:

$$\tau_{\pi} = \inf \{ t \mid S(t, \pi) > \log(M\gamma) \},$$

$$S(t, \pi) = \max_{1 \leq m \leq M} \max_{1 \leq i \leq t} \sum_{j=i}^{t} \log \frac{p_m(y_j|a_j)}{p_0(y_j|a_j)},$$

where $\gamma \in \mathbb{R}$ is a preselected threshold. The GLR CuSum stopping time $\tau_{\pi}$ can be re-written as

$$\tau_{\pi} = \min_{1 \leq m \leq M} \tau_{m, \pi}, \quad S(t, \pi) = \max_{1 \leq m \leq M} S_m(t, \pi),$$

$$\tau_{m, \pi} = \inf \{ t \mid S_m(t, \pi) > \log(M\gamma) \},$$

$$S_m(t, \pi) = S_m(t - 1, \pi) + \log \frac{p_m(y_t|a_t)}{p_0(y_t|a_t)}$$

for $t > 0$, $S_m(0, \pi) = 0$ for $m \in \{1, \ldots, M\}$

where $x^+ \equiv \max(x, 0)$. We note that for $m \in \{1, \ldots, M\}$, $S_m(t, \pi)$ is the CuSum statistic corresponding to the post-change distribution $f_m$ and policy $\pi$. Thus, the GLR CuSum statistic $S(t, \pi)$ is the maximum of the CuSum statistics for each of the post-change distributions $f_m$. Next, we present some asymptotic properties of $S(t, \pi)$ and $\tau_{\pi}$ for a policy $\pi$.

In this paper, we use $\approx$ to denote the notion of asymptotic equivalence [41]:

$$f \approx g \quad \text{if and only if} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

For a fixed policy $\pi = (q, T)$ such that the expected proportion of visits to each of the states, $\overline{q}$, has support in one recurrence class, we apply Lemma 3 together with [1, Theorem 8.2.3] to obtain the following proposition.

**Proposition 1.** For a fixed policy $\pi = (q, T)$ such that $\overline{q}$ has support in one recurrence class, we have the following asymptotic ARL-WADD trade-off for any $1 \leq m \leq M$:

$$\text{ARL}(\tau_{m, \pi}, \pi) \geq M\gamma, \quad \text{WADD}(m, \pi, \pi) \leq \log \frac{\gamma}{I_{m, \pi}} (1 + o(1)),$$

as $\gamma \to \infty$.

Thus, when the signal is $\{X_t : t \in \mathbb{N}\}$ is sampled using the policy $\pi$ such that $\overline{q}$ has support in one recurrence class, using similar techniques from [2, Theorem 6.16] together with Proposition 1, we know that the GLR CuSum stopping time $\tau_{\pi}$ gives us a stopping time satisfying $\text{ARL}(\tau_{\pi}, \pi) \geq \gamma$ and $\tau_{\pi}$ is asymptotically optimal for the following problem as $\gamma \to \infty$:

$$\min_{\tau} \text{WADD}(\tau, \pi) \quad \text{s.t.} \quad \text{ARL}(\tau, \pi) \geq \gamma.$$

In the next proposition, we discuss the case where $\overline{q}$ has support in multiple recurrence classes.

**Proposition 2.** Let the policy $\pi = (q, T)$ be such that $\overline{q}$ has support in multiple recurrence classes. Then, there exists a policy $\pi' = (q', T)$ where $\overline{q'}$ has support in only one recurrence class such that for any stopping time $\tau$,

$$\text{AOSC}(\pi') \leq \text{AOSC}(\pi) \text{ and } \text{WADD}(\tau, \pi') \leq \text{WADD}(\tau, \pi).$$

**Proof:** See Appendix D.

Using this proposition, we obtain a result regarding asymptotically optimal solutions of Problem (4).

**Theorem 1.** There exists a policy $\pi = (q', T')$ with $\overline{q'}$ having support in one recurrence class such that $(\tau_{\pi}, \pi)$ is asymptotically optimal for Problem (4) as $\gamma \to \infty$, where $\tau_{\pi}$ is the GLR CuSum stopping time.

**Proof:** See Appendix E.
Using Theorem 1, the minimization in Problem (4), over the sampling policy \( \pi \) and stopping time \( \tau \), can be decoupled. When the signal is sampled using a sampling policy \( \pi = (q,T) \) with \( q \) having support in one recurrence class, satisfying AOSC(\( \pi \)) \( \leq \alpha_{\text{AOSC}} \), the GLR CuSum \( \tau_\pi \) is asymptotically optimal with the asymptotic WADD-ARL trade-off given as WADD(\( \tau_\pi, \pi \)) = \( \frac{\log \gamma}{\min_{m \geq 1} I_{m,\pi}} (1 + o(1)) \) as \( \gamma \to \infty \).

**Definition 2.** We call \( \min_{1 \leq m \leq M} I_{m,\pi} \) the asymptotic ARL-WADD trade-off rate.

Let \( \pi^* \) be an optimal solution to the following problem:

\[
\min \pi \quad \max_{1 \leq m \leq M} I_{m,\pi}^{-1}
\text{s.t.} \quad \text{AOSC}(\pi) \leq \alpha_{\text{AOSC}}, \\
\text{supp}(q) \subseteq \text{one recurrence class}.
\]

(11)

By the argument above, \( (\tau_\pi^*, \pi^*) \) is asymptotically optimal for Problem (4). We call Problem (11) the policy design problem.

V. OPTIMAL SAMPLING POLICY

In this section, we investigate the sampling policy design Problem (11) under the cases where the switching costs from one action to another are all equal or not.

A. Equal Switching Costs

In this subsection, we propose a method to solve the policy design problem in which the switching costs are constant, i.e., \( \mathcal{C}[a,b] = c \), for a fixed \( c \in \mathbb{R} \) and any \( a,b \in \mathcal{A} \). First, we note that Problem (11) is feasible if and only if \( \alpha_{\text{AOSC}} \geq c \). Furthermore, if \( \alpha_{\text{AOSC}} \geq c \) then Problem (4) reduces to

\[
\min \pi \quad \text{WADD}(\tau, \pi)
\text{s.t.} \quad \text{ARL}(\tau, \pi) \geq \gamma,
\]

(12)

where the AOSC constraint is automatically satisfied. Next, we show that for the case when all action-switching costs are equal, there exists a memoryless policy \( \pi \) (i.e., \( W = 0 \)) for which the GLR CuSum \( \tau_\pi \) achieves asymptotic optimality.

**Proposition 3.** Suppose Problem (11) is feasible and \( (\tau_\pi, \pi) \) is an asymptotically optimal solution for Problem (4). There exists a policy \( \pi_0 \) with window size \( W = 0 \) such that

\[
\text{WADD}(\tau_{\pi_0}, \pi_0) \approx \text{WADD}(\tau_\pi, \pi) \quad \text{as} \quad \gamma \to \infty.
\]

**Proof:** See Appendix F.

From Proposition 3, to solve the policy design problem Problem (11) for some \( W \in \mathbb{N} \), it suffices to solve Problem (11) for the case where \( W = 0 \).

When \( W = 0 \), Problem (11) becomes

\[
\min \pi \quad \max_{1 \leq m \leq M} \left( \sum_{a \in \mathcal{A}} q(a) D(p_0(\cdot | a) \| p_m(\cdot | a)) \right)^{-1}
\text{s.t.} \quad \sum_{a \in \mathcal{A}} q(a) = 1, \quad q(a) \geq 0 \quad \text{for all} \quad a \in \mathcal{A},
\]

(13)

This is equivalent to solving the linear optimization problem:

\[
\min_{q,x} \quad x
\text{s.t.} \quad \sum_{a \in \mathcal{A}} q(a) = 1, \quad q(a) > 0 \quad \text{for all} \quad a \in \mathcal{A},
\]

\[
\sum_{a \in \mathcal{A}} q(a) D(p_0(\cdot | a) \| p_m(\cdot | a)) + x \geq 0
\quad \text{for all} \quad m \in \{1, \ldots, M\}.
\]

(14)

Let \( q_0 \) be the solution for Problem (14) and \( T_0 \) be the probability transition matrix with rows equal to \( q_0 \). Using similar techniques from [2, Theorem 6.16] together with Proposition 1, we know that the GLR CuSum algorithm with \( \pi_0 = (q_0, T_0) \) as the sampling policy gives us a stopping time satisfying ARL(\( \tau_{\pi_0} \) \( \geq \gamma \) with asymptotically optimal ARL-WADD trade-off as \( \gamma \) tends to infinity.

B. Unequal Switching Costs

In this subsection, we propose a method to solve the policy design problem in which the switching costs are not all equal. First, we present a proposition regarding the structure of asymptotically optimal solutions of Problem (4).

**Proposition 4.** Suppose \( (\tau_\pi, \pi) \) is an asymptotically optimal solution for Problem (4) with window size at least 1. There exists a policy \( \pi_1 \) with window size \( W = 1 \) such that AOSC(\( \pi_1 \)) = AOSC(\( \pi \)) and

\[
\text{WADD}(\tau_{\pi_1}, \pi_1) \approx \text{WADD}(\tau_\pi, \pi) \quad \text{as} \quad \gamma \to \infty.
\]

**Proof:** See Appendix G.

From Proposition 4, the policy design problem for window size \( W \in \mathbb{N} \) can be reduced to a problem of window size \( \min(W, 1) \). Thus, we only need to study the cases where \( W = 0 \) or \( W = 1 \). In the following, we present algorithms to solve Problem (11) for each of these cases.

1) Window size \( W = 0 \): Using a similar argument from Section V-A, we can see that for any optimal sampling policy \( \pi = (q,T) \), we have \( \overline{\mathcal{A}} = q \) and \( T \) has only one recurrence class. Thus, we have

\[
I_{m,\pi} = \sum_{a \in \mathcal{A}} q(a) D(p_m(\cdot | a) \| p_0(\cdot | a)),
\]

\[
\text{AOSC}(\pi) = \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{A}} q(a) q(b) \mathcal{C}[a,b],
\]

and Problem (11) becomes

\[
\min \quad \max_{1 \leq m \leq M} \left( \sum_{a \in \mathcal{A}} q(a) D(p_m(\cdot | a) \| p_0(\cdot | a)) \right)^{-1}
\text{s.t.} \quad \sum_{a \in \mathcal{A}} q(a) = 1, \quad q(a) \geq 0 \quad \text{for all} \quad a \in \mathcal{A},
\]

(15)

\[
\sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{A}} q(a) q(b) \mathcal{C}[a,b] \leq \alpha_{\text{AOSC}}.
\]
Using the same argument from Problem (13), we can introduce a new variable $x$ to obtain a linear cost function
\[
\begin{align*}
\min_{q,x} & \quad x \\
\text{s.t.} & \quad \sum_{a \in \mathcal{A}} q(a) = 1, \quad q(a) > 0 \quad \text{for all } a \in \mathcal{A}, \\
& \quad \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{A}} q(a)q(b)C[a,b] \leq \alpha_{\text{AOSC}}, \\
& \quad \sum_{a \in \mathcal{A}} q(a)D(p_0(a)\|p_m(a)) + x \geq 0 \\
& \quad \text{for all } m \in \{1, \ldots, M\},
\end{align*}
\]
(16)
This is a quadratically constrained quadratic program (QCQP), and we may assume that $\mathcal{C}$ is symmetric without loss of generality. However, without additional assumptions, the problem is NP-hard.

First, we discuss some special cases where the global optimum can be obtained easily. When $\mathcal{C}$ is positive semi-definite, Problem (16) is a convex programming problem. A convex program solver [42], [43] can be used to obtain globally optimal solutions. For the case where there are only cost of observations rather than cost of switching (i.e., $\mathcal{C}[a,b] = h(b)$ for some function $h : \mathcal{A} \rightarrow \mathbb{R}$), the quadratic constraint in Problem (16) reduces to
\[
\sum_{b \in \mathcal{A}} q(b)h(b) \leq \alpha_{\text{AOSC}}.
\]
(17)
In this case, Problem (16) becomes a linear programming problem, which can be solved by a linear program solver [42].

For the general case, we use the IRM algorithm [44] to obtain a locally optimal solution. In order to apply the IRM algorithm, we first rewrite the quadratic constraint
\[
\sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{A}} q(a)q(b)C[a,b] = q^T \mathcal{C}q = \text{Tr}(\mathcal{C}qq^T) \leq \alpha_{\text{AOSC}}
\]
and Problem (16) is equivalent to
\[
\begin{align*}
\min_{q,x} & \quad x \\
\text{s.t.} & \quad \sum_{a \in \mathcal{A}} q(a) = 1, \quad q(a) > 0 \quad \text{for all } a \in \mathcal{A}, \\
& \quad \text{Tr}(\mathcal{C}q) \leq \alpha_{\text{AOSC}}, \\
& \quad QI = q, \\
& \quad Q \succeq 0 \quad \text{and} \quad \text{rank}(Q) = 1, \\
& \quad \sum_{a \in \mathcal{A}} q(a)D(p_0(a)\|p_m(a)) + x \geq 0 \\
& \quad \text{for all } m \in \{1, \ldots, M\},
\end{align*}
\]
(18)
where $I$ is a $n^2 \times 1$ vector of ones. We note that Problem (18) becomes a convex programming problem when the constraint $\text{rank}(Q) = 1$ is ignored. We are now ready to present the IRM algorithm [44]. Fix $\omega > 1$.

First, we solve the convex problem
\[
\begin{align*}
\min_{Q,q,x} & \quad x + \omega^k r \\
\text{s.t.} & \quad \sum_{a \in \mathcal{A}} q(a) = 1, \quad q(a) > 0 \quad \text{for all } a \in \mathcal{A}, \\
& \quad \text{Tr}(\mathcal{C}q) \leq \alpha_{\text{AOSC}}, \\
& \quad QI = q, \\
& \quad Q \succeq 0, \\
& \quad \sum_{a \in \mathcal{A}} q(a)D(p_0(a)\|p_m(a)) + x \geq 0 \\
& \quad \text{for all } m \in \{1, \ldots, M\},
\end{align*}
\]
(19)
to obtain a solution $(Q_0, q_0, x_0)$ and let $Q_0 = VDV^T$ be the eigen-decomposition of $Q_0$. Let $V_0$ be the eigenvectors corresponding to the $n - 1$ smallest eigenvalues of $Q_0$.

At each step $k$, we solve the following convex problem:
\[
\begin{align*}
\min_{Q,q,x,r} & \quad x + \omega^k r \\
\text{s.t.} & \quad \sum_{a \in \mathcal{A}} q(a) = 1, \quad q(a) > 0 \quad \text{for all } a \in \mathcal{A}, \\
& \quad \text{Tr}(\mathcal{C}q) \leq \alpha_{\text{AOSC}}, \\
& \quad QI = q, \\
& \quad Q \succeq 0, \\
& \quad rI_{n-1} - V_{k-1}^T XV_{k-1} \succeq 0, \\
& \quad \sum_{a \in \mathcal{A}} q(a)D(p_0(a)\|p_m(a)) + x \geq 0 \\
& \quad \text{for all } m \in \{1, \ldots, M\},
\end{align*}
\]
(20)
to obtain a solution $(Q_k, q_k, x_k, r_k)$ and let $V_k$ be the eigenvectors corresponding to the $n - 1$ smallest eigenvalues of $Q_k$. We iterate until $r_k < \epsilon$, where $\epsilon$ is a small threshold chosen as a stopping criterion. Following similar methods from [44], it can be shown that $r_k \to 0$ at a linear rate and that $q_k$ converges to a locally optimal solution for Problem (18) if Problem (20) is feasible for all $k$.

2) Window size $W = 1$: Unlike the case when $W = 0$, not every distribution $q$ is a stationary distribution of $T$. Furthermore, when $W > 0$, it is possible that more than one recurrence class exists. In this case, Problem (11) becomes
\[
\begin{align*}
\min_{T,q} & \quad \max_{1 \leq m \leq M} \left( \sum_{a \in \mathcal{A}} q(a)D(p_0(a)\|p_m(a)) \right)^{-1} \\
\text{subject to} & \quad \sum_{a \in \mathcal{A}} q(a) = 1, \quad q(a) > 0 \quad \text{for all } a \in \mathcal{A}, \\
& \quad \sum_{b \in \mathcal{A}} T[a,b] = 1, \quad \text{for all } a \in \mathcal{A}, \\
& \quad \sum_{b \in \mathcal{A}} T[a,b] \succeq 0, \quad \text{for all } a, b \in \mathcal{A}, \\
& \quad q^T T = q^T, \\
& \quad \sum_{b \in \mathcal{A}} a \in \mathcal{A} T[a,b]C[a,b]q(a) \leq \alpha_{\text{AOSC}}, \\
& \quad \text{supp}(q) \subseteq \text{one recurrence class}.
\end{align*}
\]
Using the same argument from Problem (13), Problem (21) is equivalent to

$$
\begin{align*}
\min_{T,q,x} & \quad x \\
\text{s.t.} & \quad \sum_{a \in A} q(a) = 1, \quad q(a) > 0, \\
& \quad \sum_{b \in A} T[a,b] = 1, \\
& \quad 0 \leq T[a,b] \leq 1, \\
& \quad q^T T = q^T, \\
& \quad \sum_{b \in A} \sum_{a \in A} T[a,b] C[a,b] q(b) \leq \alpha_{AOSC}, \\
& \quad \text{supp}(q) \subseteq \text{one recurrence class}, \\
& \quad \sum_{a \in A} q(a) D(p_m(\cdot | b) \parallel p_0(\cdot | b)) + x \geq 0, \\
& \quad \text{for all } m \in \{1, \ldots, M\}, a, b \in A.
\end{align*}
$$

(22)

Problem (22) has two quadratic constraints $Tq = q$ and $\sum_{b \in A} \sum_{a \in A} T[a,b] C[a,b] q(b) \leq \alpha_{AOSC}$. Thus, it is a QCQP with an additional combinatorial constraint that supp$(q)$ is contained in a single recurrence class of $T$. Even without the combinatorial constraint, finding the global optimal for Problem (22) would be difficult without additional assumptions on $C$.

By considering a similar construction used in the proof of Proposition 4, we show that any policy $\pi_1 = (q, T)$ with window size 1 can be expressed as a policy $\pi_2 = (q_2, T_2)$ of window size 2 satisfying the following:

$$
\begin{align*}
\sum_{a \in A} q_2(a, b) & = \sum_{c \in A} q_2(c, b) \quad \text{for any } a, b \in A, \\
T_2[(a, b), (c, d)] & = \begin{cases} 
\frac{q_2(c,d)}{\sum_{a \in A} q_2(a,c)} & \text{if } b = c, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
$$

(23)

(24)

Furthermore, any policy $\pi_2 = (q_2, T_2)$ with window size 2 that satisfies (23) and (24) can be expressed as a policy $\pi_1$ of window size 1. When we consider a policy $\pi_2 = (q_2, T_2)$ with window size 2 that satisfies (23) and (24), the constraint $T_2 q_2 = q_2$ becomes automatically satisfied and the AOSC constraint is linearized to $\sum_{(a,b) \in A^2} C[a,b] q(a, b) \leq \alpha_{AOSC}$. Hence, Problem (22) is equivalent to the following problem which is a linear programming problem with an additional combinatorial constraint. In order to handle the combinatorial constraint, we can solve Problem (25) without the combinatorial constraint supp$(q)$ $\subseteq$ one recurrence class and if the solution satisfies it, we have found the globally optimal solution. Alternatively, we can select a sufficiently small $\epsilon > 0$, and require that $q(a, b) > \epsilon$ for all $a, b \in A$. The new constraint ensures that there is only one recurrence class for any feasible policy and thus, the recurrence class constraint is satisfied. The relaxed problem becomes

$$
\begin{align*}
\min_{q,x} & \quad x \\
\text{s.t.} & \quad \sum_{(a,b) \in A^2} q(a,b) = 1, \quad q(a, b) > \epsilon \text{ for all } a, b \in A, \\
& \quad \sum_{(a,b) \in A^2} C[a,b] q(a, b) \leq \alpha_{AOSC}, \\
& \quad \sum_{a \in A} q(a, b) = \sum_{c \in A} q(c, b), \\
& \quad \text{supp}(q) \subseteq \text{one recurrence class}, \\
& \quad \sum_{(a,b) \in A} q(a,b) D(p_m(\cdot | b) \parallel p_0(\cdot | b)) + x \geq 0, \\
& \quad \text{for all } m \in \{1, \ldots, M\}, a, b \in A.
\end{align*}
$$

(26)

(27)

VI. NUMERICAL RESULTS

In this section, we consider the QCD problem with AOSC based on partially observed graph signals under the various conditions discussed in this paper. We consider the problem of quickest detection of a rogue node in a graph signal. We assume that our graph $G$ is a connected graph with $N$ nodes. We model the graph signal [45] in the pre-change regime with a zero-mean Gaussian distribution with covariance $\Sigma = L^1 + \epsilon I$, where $L^1$ is the psuedo-inverse of the graph Laplacian matrix $L$, $I$ is a $N \times N$ identity matrix and $\epsilon$ is the noise power in the graph signal. Thus, in the pre-change regime, we have

$$X_t \sim f_0 = N(0, \Sigma).
$$

For the post-change regime, we assume that the signal obtained at the rogue node follows the same distribution as the pre-change distribution. However, this signal becomes independent of signals obtained from the rest of the graph. Thus, in the post-change regime, we have

$$X_t \sim f_m = N(0, \Sigma_m)
$$

for $m \in \{1, \ldots, M\}$, where the covariance matrix $\Sigma_m$ is given as

$$
\Sigma_m[i,j] = \begin{cases} 
\Sigma[i,j] & \text{for } i \neq m \text{ and } j \neq m, \\
\Sigma[m,m] & \text{for } i = j = m, \\
0 & \text{otherwise}.
\end{cases}
$$

(27)

For the set of actions, we assume that the infrastructure constrains us to observe at most $N - 2$ nodes at any instance. The set of actions $A$ is the set of all partial observations where

$$
A = \bigcup_{n=2}^{N-2} \{ \mathbf{M} | \mathbf{M} = [e_{i_1}, e_{i_2}, \ldots, e_{i_n}]^T, i_1 < i_2 < \ldots < i_n \}.
$$
In our experiments, we consider the graph $\mathcal{G}$, as shown in Fig. 1. In this case, there is a total of 50 actions that can be used to observe the signal on the graph $\mathcal{G}$.

![Topo Fig.](image)

**Fig. 1.** Graph $\mathcal{G}$ generated using the Erdős-Rényi (ER) random graph model with $N = 6$ nodes and probability of an edge $p = 0.2$.

We consider three possible cases of observation-switching costs $C$:

1) no costs are involved with $C^1 = 0$;
2) there is only activation cost where $C^2[i,j]$ is the number of nodes observed using action $i$;
3) there are both switching and activation costs with $C^3 = C^2 + C'$ where $C'[i,j]$ is the number of nodes in the symmetric difference between the set of nodes observed by action $i$ and set of nodes observed by action $j$.

Note that each $C^1, C^2, C^3 \in \mathbb{R}^{50 \times 50}$.

When $C = C^1$, using results from Section V-A, we only need to design a policy of window size 0. By considering the problem for $W = 0$, we obtain the optimal policy $\pi_{1,W=0}$ by solving Problem (14). Similarly, for the cases where $C = C^2, C^3$, using results from Section V-B, we only need to consider the cases $W = 0, 1$. When $C = C^2$ and $W = 0$, we obtain the optimal policy $\pi_{2,W=0}$ by solving Problem (16) with a linear AOSC constraint (17). For the case $C = C^3$ with $W = 0$, we obtain a locally optimal policy $\pi_{3,W=0}$ by performing the IRM algorithm described in Section V-B. In the case where $C = C^2$ or $C^3$ with $W = 1$, we obtain an approximately optimal policy $\pi_{2,W=1,\epsilon}$ or $\pi_{3,W=1,\epsilon}$ respectively by solving Problem (26) with $\epsilon \in \{10^{-4}, 10^{-5}, 10^{-6}\}$.

When no costs are involved (i.e., $C = 0$), the optimal asymptotic ARL-WADD trade-off rate (cf. Definition 2) achieved by the stopping rule and policy $(\pi_{1,W=0}, \pi_{1,W=0})$ is $\bar{I} = 1.5586$. This is an upper bound for the asymptotic ARL-WADD trade-off rates for all other choices of $C$.

In Fig. 2, we compare the asymptotic ARL-WADD trade-off rate of $\pi_{2,W=0}$ and $\pi_{2,W=1,\epsilon}$. First, we observe that the asymptotic ARL-WADD trade-off is below the upper bound $\bar{I}$ across the range of achievable AOSC. In the case where $C = C^2$, the AOSC constraint in Problem (16) reduces to

$$\text{AOSC}(\pi) = \sum_{h \in \mathcal{A}} \left( \sum_{a \in \mathcal{A}} q(a, b) \right) h(b)$$

where $h(b)$ is the number of nodes observed using action $b$. Using similar arguments from the proof of Proposition 3, we can see that there is an asymptotically optimal trade-off for a policy window of size $W = 0$ is equal to the asymptotically optimal trade-off for a policy of window size $W = 1$. Hence, the performance between $\pi_{2,W=0}$ and $\pi_{2,W=1,\epsilon}$ becomes more similar as $\epsilon$ tends to zero.

In Fig. 3, we compare the asymptotic ARL-WADD trade-off rate of $\pi_{3,W=0}$ and $\pi_{3,W=1,\epsilon}$. Similarly, we observe that the asymptotic ARL-WADD trade-off is below the upper bound $\bar{I}$ across the range of achievable AOSC. In this case, we observe that the optimal policies of window size $W = 1$ significantly outperforms the optimal policies of window size $W = 0$. When we are using a policy of window size $W = 0$, the lowest AOSC achievable while Problem (4) remains feasible is about 4. However, using a policy of window size $W = 1$, we are able to reduce AOSC to about 2 while Problem (4) remains feasible. These can be seen be comparing the lowest AOSC achieved by the respective curves in Fig. 3.

**VII. CONCLUSION AND FUTURE WORK**

In this paper, we discussed the problem of QCD while taking sampling and switching costs into consideration. We formulated the QCD problem with an additional AOSC constraint. Asymptotically optimal stopping times were proposed and the design of optimal sampling policies were formulated as quadratic optimization problems. We showed that sampling policies of window size $W > 1$ can be reduced to a sampling
policy of window size $W = 1$ while maintaining the same asymptotic ARL-WADD trade-off and AOSC. Thus, it is sufficient to solve the policy design problem for $W = 0, 1$. We applied the IRM algorithm to the policy design problem to obtain locally optimal solutions. For cases with additional assumptions on the observation-switching cost matrix $C$, globally optimal solutions can be obtained. The methods developed are for the case when the window $W$ is finite. The results regarding the structure of asymptotically optimal stopping times such as Lemma 1, Proposition 2 and Theorem 1, do not hold in general when $W = \infty$ and would be an interesting direction for future work.

In some Internet of Things (IoT) applications, the nodes collecting observations have some computing resource to perform local computations and observations. In these applications, the individual sensors can make use of aggregated information to decide whether it is necessary to sample while taking into account of sampling and communication costs. This would require for us to consider active rather than static sampling policies for QCD and would be an interesting area for further research.

**APPENDIX A**

**PROOF OF Lemma 1**

Each time a state $\alpha$ is visited, a switching cost $C_{\alpha}$ is incurred. Thus, the AOSC can be written as

$$\text{AOSC}(\pi_1) = \lim_{t \to \infty} \sum_{\beta \in A^W} \sum_{\alpha \in A^W} \mathbb{E}_{-\text{NoValue}} \left[ \frac{C_{\alpha} N_t(\alpha; \beta)}{t} \right] q(\beta)$$

$$= \sum_{\beta \in A^W} \sum_{\alpha \in A^W} q(\beta) C_{\alpha} \lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \beta)}{t} \right].$$

Similarly, for $\pi_2$, we have

$$\text{AOSC}(\pi_2) = \sum_{\beta \in A^W} \sum_{\alpha \in A^W} \bar{q}(\beta) C_{\alpha} \lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \beta)}{t} \right].$$

Suppose there are $R$ recurrence classes $\mathcal{R}_1, \ldots, \mathcal{R}_R$. For any $r \in \{1, \ldots, R\}$ with both $\beta, \delta \in \mathcal{R}_r$, we have

$$\lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \beta)}{t} \right] = \sum_{r=1}^R \lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \delta)}{t} \right].$$

Let $\beta_r \in \mathcal{R}_r$ for $r = 1, \ldots, R$ and for any transient state $\delta$, we have

$$\lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \delta)}{t} \right] = \sum_{r=1}^R \sum_{\alpha \in A^W} f_{\delta,r} \lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \delta)}{t} \right].$$

Let $\{\delta_j | j = 1, \ldots, U\}$ be the transient states. For any $r = 1, \ldots, R$, we have

$$\sum_{\beta \in \mathcal{R}_r} \bar{q}(\beta) = \sum_{\beta \in \mathcal{R}_r} q(\beta) + \sum_{j=1}^U f_{\delta_j,r} q(\delta_j).$$

Putting (28) to (32) together, we have

$$\text{AOSC}(\pi_2) = \sum_{\beta \in A^W} \sum_{\alpha \in A^W} \bar{q}(\beta) C_{\alpha} \lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \beta)}{t} \right].$$

$$= \sum_{r=1}^R \sum_{\alpha \in A^W} \bar{q}(\beta) C_{\alpha} \lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \beta)}{t} \right].$$

$$= \sum_{r=1}^R \sum_{\alpha \in A^W} \bar{q}(\beta) C_{\alpha} \lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \beta)}{t} \right].$$

$$= \sum_{r=1}^R \sum_{\alpha \in A^W} \bar{q}(\beta) C_{\alpha} \lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \beta)}{t} \right].$$

$$= \sum_{r=1}^R \sum_{\alpha \in A^W} \bar{q}(\beta) C_{\alpha} \lim_{t \to \infty} \mathbb{E}_{-\text{NoValue}} \left[ \frac{N_t(\alpha; \beta)}{t} \right].$$
\[ \sum_{r=1}^{R} \sum_{\beta \in R_r} \sum_{\alpha \in A^W} q(\delta_j)C_\alpha \lim_{t \to \infty} \mathbb{E}_{-Novalue-} \left[ \frac{N_t(\alpha; \beta)}{t} \right] \]
\[ + \sum_{r=1}^{R} \sum_{j=1}^{U_r} f_{\delta_j,r} q(\delta_j)C_\alpha \xi_r[\alpha], \]  
(34)

where (33) and (34) are due to \( \xi_r[\alpha] = 0 \) for \( \alpha \notin R_r \). Thus, we have \( AOSC(\pi_1) = AOSC(\pi_2) \) and the proof is complete.

**APPENDIX B**

**PROOF OF LEMMA 2**

Since \( \text{supp}(q) \) lies in the recurrence class \( \mathcal{R} \), for any \( t \geq \nu \) and \( \beta \in A^W \) such that \( q(\beta) > 0 \), we have \( \beta \in \mathcal{R} \) and

\[ \frac{1}{t - \nu + 1} \sum_{\alpha \in A^W} \log p_m(Y_t|\alpha[W]) \]

\[ = \sum_{\alpha \in A^W} \sum_{i \text{ s.t. } \nu \leq i \leq t} \log \frac{p_m(Y_t|\alpha[W])}{p_0(Y_t|\alpha[W])}, \]

\[ = \sum_{\alpha \in \mathcal{R}} \sum_{i \text{ s.t. } \nu \leq i \leq t} \log \frac{p_m(Y_t|\alpha[W])}{p_0(Y_t|\alpha[W])}, \]

(35)

where the last equality follows because for any \( \alpha \notin \mathcal{R} \), \( \{ i \text{ s.t. } \nu \leq i \leq t, A_i^0 = A_i^1 \} = \emptyset \). We have

\[ \frac{N_t(\alpha; \beta)}{t - \nu + 1} \xrightarrow{\text{a.s.}} \mathbb{E}(\alpha) \quad \text{as} \quad t \to \infty, \]

and

\[ \frac{1}{N_t(\alpha; \beta)} \sum_{i \text{ s.t. } \nu \leq i \leq t} \log \frac{p_m(Y_t|\alpha[W])}{p_0(Y_t|\alpha[W])} \]

\[ \xrightarrow{\text{a.s.}} D(p_m(\cdot | \alpha[W]) || p_0(\cdot | \alpha[W])) \quad \text{as} \quad t \to \infty. \]

Thus from (35), we obtain

\[ \frac{1}{t - \nu + 1} A_m(k, t) \]

\[ \xrightarrow{\text{a.s.}} \mathbb{E}(\alpha) D(p_m(\cdot | \alpha[W]) || p_0(\cdot | \alpha[W])) = I_m, \pi. \]

The proof is now complete.

**APPENDIX C**

**PROOF OF LEMMA 3**

Suppose \( \text{supp}(\eta) \subseteq \mathcal{R} \) for a recurrence class \( \mathcal{R} \), then \( \text{supp}(q) \subseteq \mathcal{R} \cup \mathcal{U} \), where \( \mathcal{U} \) is a set of transient states such that the first-passage probability of entering \( \mathcal{R} \) from each \( \beta \in \mathcal{U} \) is one. Then, from Lemma 2, (7) follows. Next, we have

\[ \sup_{0 \leq \nu < \infty} \mathbb{E}_{\nu,m} \left( t^{-1} \max_{0 \leq j < t} A_m(\nu, \nu + j) \right) \]

\[ > (1 + \epsilon) I_{m, \pi} \left| A^{1-\nu-1}, Y^{1-\nu-1} \right| \]

(36)

\[ = \sup_{0 \leq \nu < \infty} \mathbb{E}_{\nu,m} \left( t^{-1} \max_{0 \leq j < t} A_m(\nu, \nu + j) \right) \]

\[ > (1 + \epsilon) I_{m, \pi} \left| A^{\nu-W;\nu-1} \right| \]

(37)

\[ = \sup_{0 \leq \nu < \infty} \mathbb{E}_{\nu,m} \left( t^{-1} \max_{0 \leq j < t} A_m(\nu, \nu + j) \right) \]

\[ > (1 + \epsilon) I_{m, \pi} \left| A^{\nu-W;\nu-1} = \alpha \right| \]

(38)

where (37) is because \( A_m(\nu, \nu + j) \) is independent of \( A^{1-\nu-W-1} \) and \( Y^{1-\nu-1} \) given \( A^{\nu-W;\nu-1} \) for each \( 0 \leq j < t \), and (38) is because each of the terms within the set \( \{ \alpha \in A^W \mid q(\alpha) > 0 \} \) that we take maximum over converges to zero due to Lemma 2.

**APPENDIX D**

**PROOF OF PROPOSITION 2**

Without loss of generality, we consider the case where \( \eta \) has support in the recurrence classes \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \). Let \( q_1, q_2 \) be stationary distributions on \( A^W \) such that \( q_i \) has support only in \( \mathcal{R}_i \) for \( i \in \{1, 2\} \) and that there exists \( 0 < \lambda < 1 \) such that \( \eta = \lambda q_1 + (1 - \lambda) q_2 \). Let \( \pi_1 = (q_1, T) \) and \( \pi_2 = (q_2, T) \). From Lemma 1, we obtain

\[ AOSC(q, T) = AOSC(\eta, T) = \lambda AOSC(q_1, T) + (1 - \lambda) AOSC(q_2, T) \]

\[ \geq \min \{ AOSC(\pi_1), AOSC(\pi_2) \}. \]

Let \( E_1 \) be the event that \( \mathcal{R}_1 \) is visited before \( \mathcal{R}_2 \) and \( E_2 \) be the event that \( \mathcal{R}_2 \) is visited before \( \mathcal{R}_1 \). We have

\[ \text{WADD}(\tau, \pi) = \sup_{1 \leq m \leq M} \mathbb{E}_{-Novalue-} \left[ (\tau - \nu + 1)^+ \mid A^{1-\nu-1}, Y^{1-\nu-1} \right] \]

\[ = \sup_{1 \leq m \leq M} \mathbb{E}_{-Novalue-} \left[ (\tau - \nu + 1)^+ \mid A^{1-\nu-1}, Y^{1-\nu-1}, E_i \right] \]

\[ = \max_{i=1,2} \sup_{1 \leq m \leq M} \mathbb{E}_{-Novalue-} \left[ (\tau - \nu + 1)^+ \mid A^{1-\nu-1}, Y^{1-\nu-1}, E_i \right] \]

Setting \( \pi' = \pi_j \) where \( j = \arg \min_{i=1,2} AOSC(\pi_i) \) completes the proof.
APPENDIX E
PROOF OF THEOREM 1

We first note that Problem (4) is feasible and an optimal solution exists. Let the policy \( \pi^* = (q, T) \) and stopping-time \( \tau^* \) such that \((\tau^*, \pi^*)\) is optimal for Problem (4). From Proposition 2, we can find a policy \( \pi = (q', T') \) such that \( q' \) has support in only one recurrence class and

\[
AOSC(\pi) \leq AOSC(\pi^*),
\]

\[
WADD(\tau^*, \pi) \leq WADD(\tau^*, \pi^*).
\]

From the discussion before (10), the GLR CuSum stopping time \( \tau_{\pi} \) is asymptotically optimal for the following problem:

\[
\min \{ \tau \} \quad WADD(\tau, \pi) \quad \text{s.t.} \quad ARL(\tau, \pi) \geq \gamma.
\]

(39)

Thus, the GLR CuSum stopping time \( \tau_{\pi} \) satisfies

\[
WADD(\tau_{\pi}, \pi) \approx WADD(\tau^*, \pi) \leq WADD(\tau^*, \pi^*),
\]

as \( \gamma \to \infty \). Hence, \((\tau_{\pi}, \pi)\) is also an asymptotically optimal solution to Problem (4).

APPENDIX F
PROOF OF PROPOSITION 3

If \( \pi \) is equivalent to a policy of window size 0, then \( \pi_0 = \pi \) proves the proposition. If \( \pi = (q, T) \) is a policy of window size \( W > 0 \), by Theorem 1, it suffices to consider the case where the support of \( \overline{q} \) is a subset of a recurrence class of \( T \).

Let \( \pi_0 = (q_0, T_0) \) such that \( q_0 = \overline{q}^0 \) and \( T_0 \) be the probability transition matrix with rows equal to \( q_0 \) where

\[
q_0(a) = \sum_{b \in A : A \subseteq \{b \}} \bar{q}(a)
\]

for any \( a \in A \). For any \( m \in \{1, \ldots, M\} \), we have

\[
I_{m,\pi} = \sum_{a \in A} \bar{q}(a) D(p_m(\cdot | a) \| p_0(\cdot | a)) = \sum_{a \in A} \bar{q}(a) \sum_{a \in A : a \subseteq \{b \}} D(p_m(\cdot | a) \| p_0(\cdot | a)) = \sum_{a \in A} q_0(a) D(p_m(\cdot | a) \| p_0(\cdot | a)) = I_{m,\pi_0}, \tag{40}
\]

where (40) is due to the fact that \( \overline{q}^0 = q_0 \) as \( \pi_0 \) is a policy of window size 0. By Proposition 1, we obtain \( WADD(\tau_{\pi}, \pi_0) \approx WADD(\tau_{\pi_0}, \pi_0) \) as \( \gamma \to \infty \). The proof is now complete.

APPENDIX G
PROOF OF PROPOSITION 4

If \( \pi \) is equivalent to a policy of window size 1, we let \( \pi_1 = \pi \) and we have proved the proposition. If \( \pi \) is a policy of window size \( W > 1 \), by Theorem 1, it suffices to consider the case where the support of \( \overline{q} \) is a subset of a recurrence class \( R \). The AOSC of \( \pi \) can be expressed as

\[
AOSC(\pi) = AOSC(\overline{q}, T)
\]

\[
= \limsup_{n \to \infty} \frac{1}{n} \sum_{t=2}^{n+1} C(A_{t-1}, A_t) - \sum_{t=2}^{n+1} C(A_{t-1}, A_t) \bigg| A^{1:W} = \beta
\]

\[
= \sum_{\beta \in A^W} \sum_{\alpha \in A^W} \bar{q}(\beta) \bar{q}(\alpha) C_{\alpha \beta} = \sum_{\alpha \in A^W} \sum_{\beta \in A^W} \bar{q}(\beta) \bar{q}(\alpha).
\]

Let \( \pi_2 = (q_2, T_2) \) be a policy of window size 2 and it can be similarly shown that \( AOSC(\pi_2) = \sum_{\alpha \in A} C_{\alpha \pi_2}(\alpha) \). Thus, it can be seen that the AOSC of \( \pi_2 \) depends only on \( \overline{q}_2 \). Let \( q_2 = \overline{q}_1 \) be the projection of \( \overline{q} \) onto \( A^2 \) where

\[
q_1(a, b) = \sum_{\alpha \in A^W : |A - (a, \alpha | A| = b} \bar{q}(\alpha),
\]

for any \( a, b \in A \) and \( T_2 \) be defined as

\[
T_2[(a, b), (c, d)] = \begin{cases} \frac{q_1^1(c, d)}{\sum_{d \in A} q_1^1(b, d)} & \text{if } c = b, \\ 0 & \text{otherwise.} \end{cases}
\]

and hence, \( T_2 \) is a probability transition matrix. Since the support of \( \overline{q} \) is a subset of a single recurrence class of \( T \), the support of \( q_2 \) also lies in a single recurrence class of \( T_2 \). Next, we show that \( q_2 \) is a stationary distribution of \( T_2 \):

\[
T_2 q_2[(c, d)] = \sum_{(a, b) \in A^2} T_2[(a, b), (c, d)] q_1^1(a, b)
\]

\[
= \sum_{a \in A} \bar{q}_2(c, a) \sum_{d \in A} \bar{q}_2(b, d) = \sum_{d \in A} \bar{q}_2(b, d)
\]

Therefore, \( q_2 \) is a stationary distribution of \( T_2 \) which has support contained in one recurrence class. Hence, \( \overline{q}_2 = q_2 \) and \( AOSC(\pi) = AOSC(\pi_2) \). Using similar arguments from the proof of Proposition 3, we have \( I_{m,\pi} = I_{m,\pi_1} \) for \( m = \{1, \ldots, M\} \) and \( WADD(\tau_{\pi}, \pi) \approx WADD(\tau_{\pi_2}, \pi_2) \) as \( \gamma \to \infty \).
Furthermore, a quick computation shows that \( \tau_2 = (q_2, T_2) \) is equivalent to the policy \( \pi_3 = (q_1, T_1) \) of window size 1 with
\[
q_1(a) = \sum_{b \in A} q_2(a, b) \quad \text{and} \quad T_1(a, b) = \sum_{a, b \in A} q_2(a, b).
\]
Thus, we have \( \text{AOSC}(\pi_3) = \text{AOSC}(\pi_1) \) and \( \text{WADD}(\pi_3, \pi) \approx \text{WADD}(\pi_{13}, \pi_1) \) as \( \gamma \to \infty \). The proof is complete.

REFERENCES

[1] A. Tartakovsky, I. Nikiforov, and M. Basseville, *Sequential analysis: Hypothesis testing and change point detection*. CRC Press, 2014.

[2] H. V. Poor and O. Hadjiliadis, *Quickest detection*. Cambridge University Press Cambridge, 2009, vol. 40.

[3] V. V. Veeravalli and T. Banerjee, “Quickest change detection,” *Academic Press Library in Signal Processing*, vol. 3, pp. 209–255, 2014.

[4] W. H. Woodall, D. J. Spitzner, D. C. Montgomery, and S. Gupta, “Using control charts to monitor process and product quality profiles,” *J. of Quality Technology*, vol. 36, no. 3, p. 309, 2004.

[5] T. L. Lai, “Sequential change point detection in quality control and dynamical systems,” *J. of the Roy. Statistical Soc.*, pp. 613–658, 1995.

[6] L. Lai, Y. Fan, and H. V. Poor, “Quickest detection in cognitive radio: A sequential change detection framework,” in *IEEE Conf. Global Telecommun.* IEEE, 2008, pp. 1–5.

[7] Y. Zhang, W. P. Tay, K. H. Li, M. Esseghir, and D. Gaiti, “Learning temporal-spatial spectrum reuse,” *IEEE Trans. Commun.*, vol. 64, no. 7, pp. 3092 – 3103, Jul. 2016.

[8] L. Akoglu and C. Faloutsos, “Event detection in time series of mobile communication graphs,” in *Proc. Army Sci. Conf.*, 2010, pp. 77–79.

[9] K. Sequeira and M. Zaki, “ADMIT: anomaly-based data mining for intrusions,” in *Proc. Conf. Knowl. Discovery and Data Mining*. ACM, 2002, pp. 386–395.

[10] A. G. Tartakovsky, B. L. Rozovskii, R. B. Blazek, and H. Kim, “A novel approach to detection of intrusions in computer networks via adaptive sequential and batch-sequential change-point detection methods,” *IEEE Trans. Signal Process.*, vol. 54, no. 9, pp. 3372–3382, 2006.

[11] A. G. Tartakovsky, “Rapid detection of attacks in computer networks by quickest change-point detection methods,” *Data anal. for network security*, pp. 33–70, 2014.

[12] H. Sohn, J. A. Cramercki, and C. R. Farrar, “Structural health monitoring using statistical process control,” *J. Structural Eng.*, vol. 126, no. 11, pp. 1536–1566, 2000.

[13] S. Xie, G. Wang, S. Lin, and P. S. Yu, “Review spam detection via temporal pattern discovery,” in *Proc. Conf. Knowl. Discovery and Data Mining*. ACM, 2012, pp. 823–831.

[14] F. Ji, W. P. Tay, and L. Varshney, “An algorithmic framework for estimating rumor sources with different stabilities,” *IEEE Trans. Signal Process.*, vol. 65, no. 10, pp. 2517 – 2530, May 2017.

[15] W. Tang, F. Ji, and W. P. Tay, “Estimating infection sources in networks using partial timestamps,” *IEEE Trans. Inf. Forensics Security*, vol. 13, no. 2, pp. 3035 – 3049, Dec. 2018.

[16] V. M. R. Muggeo and G. Adelfio, “Efficient change point detection for genomic sequences of continuous measurements,” *Biostatistics*, vol. 27, no. 2, pp. 161–166, 2011.

[17] T. Banerjee, Y. C. Chen, A. D. Dominguez-Garcia, and V. V. Veeravalli, “Power system line outage detection and identification-A quickest change detection approach,” in *Proc. IEEE Int. Conf. Acoust., Speech, and Signal Process*. IEEE, 2014, pp. 3450–3454.

[18] Y.-W. Hong and A. Scaglione, “Distributed change detection in large scale sensor networks through the synchronization of pulse-coupled oscillators,” in *Proc. IEEE Int. Conf. Acoustics, Speech, and Signal Process.*, vol. 1, May 2004, pp. III–889.

[19] J. Yang, X. Zhong, and W. P. Tay, “A dynamic Bayesian nonparametric model for blind calibration of sensor networks,” *IEEE Internet Things J.*, vol. 5, no. 5, pp. 3942 – 3953, Oct. 2018.