ESTIMATION OF CHANGE-POINT MODELS

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Abstract. We consider the testing and estimation of change-points, locations where the distribution abruptly changes, in a sequence of observations. Motivated by this problem, in this contribution we first investigate the extremes of Gaussian fields with trend, which then help us to give the asymptotic p-value approximations of the likelihood ratio statistics from change-point models.

1. Introduction

Change-point problems appear to have arisen originally in problems of quality control, where one observes the output of a production process sequentially and wants to signal any departure of the average output, from some known target value $\mu_0$. Early outstanding contributions in a long line of papers on the sequential detection are [19, 21, 23, 28]. For recent reviews imbedded in otherwise original research articles see [15, 16]. Another paper [22] emphasizes tentative selection of several sets of candidate change-points followed by model selection to make the final choice.

Next we give a description of the change-point model (see [18, 20, 29] for more details). To simplify the discussion, assume that $X_i, i = 1, 2, \ldots, m$, are independent, normally distributed random variables with means $\mu_i$ and variance 1. Consider the problem of testing

$H_0: \mu_1 = \mu_2 = \cdots = \mu_m (= \mu_0)$

against

$H_1: \exists 1 \leq \rho_1 < \rho_2 \leq m, \mu_1 = \cdots = \mu_{\rho_1} = \mu_0, \mu_{\rho_1+1} = \cdots = \mu_{\rho_2} = \mu_0 + \delta, \mu_{\rho_2+1} = \cdots = \mu_m = \mu_0.$

Let us denote

$S_i = \sum_{j=1}^i X_i, \quad i = 1, \ldots, m.$

As in [18], if it is assumed that $\mu_0$ and $\delta$ are known, then the log likelihood ratio statistic for testing $H_0$ against $H_1$ is given by

$Z_1 = \delta \max_{0 \leq i < j \leq m} \left[ S_j - j\mu_0 - (S_i - i\mu_0) - (j - i)\frac{\delta}{2} \right] = \max_{0 \leq i < j \leq m} [\tilde{S}_j - \tilde{S}_i],$

where

$\tilde{S}_i = \delta \left[ S_i - i \left( \mu_0 + \frac{\delta}{2} \right) \right].$

When $\mu_0$ is unknown one possible course, by [20], is to replace $\mu_0$ with its estimate under $H_0$, $S_m/m$ which leads to the test statistic

$Z_2 = \delta \max_{0 \leq i < j \leq m} \left[ S_j - j \frac{S_m}{m} - \left( S_i - i \frac{S_m}{m} \right) - (j - i)\frac{\delta}{2} \right].$

The paper [20] is concerned with Bernoulli and Poisson random variables rather than normal. Since $\mu_0$ is a nuisance parameter, they suggest to calculate the conditional distribution of $Z_2$ given $S_m$. The
Consider the self-similar property of a Gaussian random walk, we make transform of this problem, 

\[ Z_3 = \delta \max_{0 \leq i < j \leq m} \left[ S_j - S_i - (j - i) \frac{S_m}{m} - \frac{1}{2} \delta (j - i) \times \left( 1 - \frac{j - i}{m} \right) \right]. \] (1)

When \( \delta \) is also unknown, one might use either \( Z_2 \) or \( Z_3 \) based on some value \( \delta_0 \), the smallest difference in means that is considered important to detect, or proceed to the full log likelihood ratio statistic by maximizing (1) over \( \delta \), obtaining

\[ Z_4 = \max_{0 \leq i < j \leq m} \frac{[S_j - S_i - (j - i)S_m / m]^+}{[(j - i) \times (1 - (j - i)/m)]^{1/2}}, \]

where \( x^+ = \max(x, 0) \). Each of these statistics is the maximum of a Gaussian random field. In order to approximate the \( p \)-value, it is important to give the tail distributions of the maximums of these Gaussian fields.

Considering the self-similar property of a Gaussian random walk, we make transform of this problem, such as for \( Z_2 \) with \( d, n > 0 \)

\[ \mathbb{P}\{Z_2 > dn\} = \mathbb{P}\left\{ \delta \max_{0 \leq i < j \leq m} \left[ S_j - j \frac{S_m}{m} - \left( S_i - i \frac{S_m}{m} \right) - (j - i) \frac{\delta}{2} \right] > dn \right\} \]

\[ = \mathbb{P}\left\{ \max_{(s,t) \in S_d} \left[ (S_t - S_s) - (t - s)S_1 - \frac{\delta}{2} (t - s) \sqrt{m} \right] > \frac{dn}{\delta \sqrt{m}} \right\}, \]

where

\[ S_d = \left\{ (s,t) : s = i \frac{m}{n}, t = j \frac{m}{n}, i,j = 1, \ldots, m \right\}. \]

Hence we can estimate the problem as

\[ p_2(n) := \mathbb{P}\left\{ \sup_{(s,t) \in S} \left( (B(t) - B(s)) - (t - s)B(1) - c(t - s) \sqrt{m} \right) > d \frac{n}{\sqrt{m}} \right\}, \]

for large enough \( n \), where \( B(t) \) is the standard Brownian motion, \( c, d \) are positive constants and

\[ S = \{(s,t) : 0 \leq s \leq t \leq 1\}. \]

Considering \( S \supseteq S_d \), \( p_2(n) \) with continuous time interval is, in fact, an upper bound for \( \mathbb{P}\{Z_2 > dn\} \).

The problem corresponding to \( Z_1 \) is

\[ p_1(d) := \mathbb{P}\left\{ \sup_{(s,t) \in S} (B(t) - B(s)) - c(t - s) > d \right\} = \mathbb{P}\left\{ \sup_{0 \leq t \leq 1} (B(t) - ct) > d \right\} = \Psi(d + c) + e^{-2cd} \Psi(d - c), \]

where \( \Psi(\cdot) \) is the survival function of \( \mathcal{N}(0,1) \). In the last equality we use a well-known result (see [11]).

Similarly, the problems corresponding to \( Z_3 \) and \( Z_4 \) are, respectively,

\[ p_3(n) := \mathbb{P}\left\{ \sup_{(s,t) \in S} (B(t) - B(s)) - (t - s)B(1) - c(t - s) \sqrt{m} > d \frac{n}{\sqrt{m}} \right\}, \]

and

\[ p_4(d) := \mathbb{P}\left\{ \sup_{(s,t) \in S} \frac{(B(t) - B(s)) - (t - s)B(1)}{\sqrt{(t - s) \times (1 - (t - s))}} > d \right\}. \]

In Sec. 3, we give the asymptotic estimations of \( p_i(n) \), \( i = 2, 3 \), for large \( n \) in two different scenarios: when \( n = m \) and \( n \) is independent of \( m \), and finally, \( p_4(d) \) for large \( d \).
As we have noted, the distribution of \( Z_i, i = 1, 2, 3, 4, \) is determined by solving a first passage problem for the Gaussian random field with trends. First, we give the general results about extremes of two-dimensional Gaussian fields with trends in Sec. 2.

Organization of the remaining paper is the following. In Sec. 2, the tail asymptotics of the supremum of a family of Gaussian fields with trends are given. The applications concerning change-point models are displayed in Sec. 3. Finally, we present all the proofs in Sec. 4.

2. Main Results

First, we introduce some notation that plays significant role in the following theorem. Define for \( \lambda, \lambda_1 > 0, \) and some continuous function \( f(t), t \in \mathbb{R}, \)
\[
\mathcal{P}_\alpha^{f(s-t)} := \lim_{\lambda \to \infty} \frac{1}{\lambda} \mathcal{P}_\alpha^{f(s-t)}(\lambda, \lambda) \in (0, \infty),
\]
\[
\mathcal{Q}_\alpha := \lim_{\lambda \to \infty} \frac{1}{\lambda^2} \mathcal{Q}_\alpha(\lambda, \lambda) \in (0, \infty),
\]
\[
\mathcal{H}_\alpha := \lim_{\lambda \to \infty} \frac{1}{\lambda} \mathcal{H}_\alpha(\lambda) \in (0, \infty),
\]
with
\[
\mathcal{P}_\alpha^{f(s-t)}(\lambda, \lambda_1) := \mathbb{E}\left\{ \sup_{0 \leq s \leq \lambda, |s-t| \leq \lambda_1} e^{|x|} \right\},
\]
\[
\mathcal{Q}_\alpha(\lambda, \lambda_1) := \mathbb{E}\left\{ \sup_{0 \leq s \leq \lambda, 0 \leq s-t \leq \lambda_1} e^{|x|} \right\},
\]
\[
\mathcal{H}_\alpha(\lambda) := \mathbb{E}\left\{ \sup_{0 \leq t \leq \lambda} e^{|x|} \right\},
\]
where \( B_\alpha^{(1)}(t), B_\alpha^{(2)}(t), B_\alpha(t), t \in \mathbb{R}, \) are mutually independent standard fractional Brownian motion with Hurst index \( \alpha \in (0, 2] \) (see [2–7, 10, 12–14, 17, 24, 25, 27] for various properties including the positive finite property of \( \mathcal{Q}_\alpha, \mathcal{H}_\alpha, \) and \( \mathcal{P}_\alpha^{f} \)). Hereafter \( \sim \) means asymptotic equivalence, \( (x)_+ = \max(x, 0), \) and \( \mathbb{I}_\{ \cdot \} \) is the indicator function.

**Theorem 1.** Let \( X(s, t), (s, t) \in \mathcal{E}, \mathcal{E} = \{(s, t): s \in [S_1, S_2], |s-t| < T\}, 0 < T \leq S_1 < S_2, \) be a centered Gaussian random field with continuous sample paths, variance function \( \sigma^2 \) and correlation function \( r. \) Suppose that \( \sigma(s, t) \) attains its maximum equal to 1 over \( \mathcal{E} \) at \( (s, t) \in \mathcal{L} = \{(s, t): (s, t) \in \mathcal{E}, s-t = 0\}, \) and
\[
1 - \sigma(s, t) \sim b|s-t|^\beta, |s-t| \to 0,
\]
holds for some \( b > 0, \beta \in (0, 2]. \) Further assume that
\[
1 - r(s, t, s', t') \sim a(|s-s'|^\alpha + |t-t'|^\alpha), |s-s'|, |t-t'|, |s-t|, |s' - t'| \to 0,
\]
holds for some \( a > 0 \) and \( \alpha \in (0, 2] \) and
\[
r(s, t, s', t') < 1
\]
holds for \( (s, t), (s', t') \in \mathcal{E}, (s, t) \neq (s', t'). \) Then for \( c \in \mathbb{R} \) as \( u \to \infty, \) we have
\[
\mathbb{P}\left\{ \sup_{(s, t) \in \mathcal{E}} (X(s, t) - c(s-t)) > u \right\} \sim \mathcal{C}_1 u^{2/\alpha+(2/\alpha-2/\beta)+\Psi(u)},
\]
where \( \mathcal{C}_1 = \begin{cases} 
2(S_2 - S_1) a^{2/\alpha}(\mathcal{H}_\alpha)^2 b^{-1/\beta} \Gamma \left( \frac{1}{\beta} + 1 \right) e^{\left( c^2/(4b) \right) \mathbb{E}(\beta = 2)} & \text{if } \alpha < \beta, \\
(S_2 - S_1) a^{1/\alpha} \mathcal{P}_\alpha^{f(s-t)} & \text{if } \alpha = \beta, \\
2^{1/\alpha} u^{2/\alpha}(S_2 - S_1) a^{1/\alpha} \mathcal{H}_\alpha & \text{if } \alpha > \beta,
\end{cases}\)
and
\[ f(t) = \frac{b}{a} |t|^\alpha + \frac{c}{\sqrt{a}} tH_1(\alpha = 2). \]

Further, as \( u \to \infty \)
\[
\mathbb{P}\left\{ \sup_{(s,t)\in E} (X(s,t) - c(s-t))^2 > u \right\} \sim C_2 u^{2/\alpha + (2/\alpha - 2/\beta)} + \Psi(u),
\]
where
\[
C_2 = \begin{cases} 
2(S_2 - S_1)a^{2/\alpha}(H_\alpha)^2b^{-1/\beta}\Gamma\left(\frac{1}{\beta} + 1\right) & \text{if } \alpha < \beta, \\
(S_2 - S_1)a^{1/\alpha}P_{\alpha}^{(s-t)} & \text{if } \alpha = \beta, \\
2^{1/\alpha}(S_2 - S_1)a^{1/\alpha}H_\alpha & \text{if } \alpha > \beta,
\end{cases}
\]
and
\[ f(t) = \frac{b}{a} |t|^\alpha. \]

**Remark 1.** It follows from the proof of Theorem 1 that the shape of \( E \) is not necessarily a parallelogram. If \( L \) is in \( E \), which means all points except the two endpoints of \( L \) are inner points of \( E \), then only the length of \( L \) matters.

### 3. Applications

In this section, we return to our original problems in Sec. 1. First, we consider the scenario \( n = m \) in \( p_i(n) \), \( i = 2, 3 \), and \( p_4(d) \). Then we denote
\[ Y(s,t) = B(t) - B(s) - (t-s)B(1), \quad u = \sqrt{n}. \]

**Proposition 1.**

1. For \( c, d > 0 \), we have
\[
\mathbb{P}\left\{ \sup_{(s,t)\in S} (Y(s,t) - c(t-s)) > du \right\} \sim 32d^2(d+c)^3u^2e^{-2d(c+d)u^2}
\]
as \( u \to \infty \).

2. For \( c > 4d > 0 \), we have
\[
\mathbb{P}\left\{ \sup_{(s,t)\in S} (Y(s,t) - c(t-s) \times (1-(t-s))) > du \right\} \sim \frac{32cd}{\sqrt{c(c-4d)}}u^2e^{-2cdu^2}
\]
as \( u \to \infty \).

3. We have
\[
\mathbb{P}\left\{ \sup_{(s,t)\in S} \frac{Y(s,t)}{\sqrt{(t-s) \times (1-(t-s))}} > d \right\} \sim 2d^4\Psi(d)
\]
as \( d \to \infty \).

Next we consider the scenario where \( n \) is independent of \( m \) in \( p_i(n) \), \( i = 2, 3 \). These problems can be expressed as follows by the help of
\[ Y(s,t) = B(t) - B(s) - (t-s)B(1). \]

**Proposition 2.** For \( c \in \mathbb{R} \), we have
\[
\mathbb{P}\left\{ \sup_{(s,t)\in S} (Y(s,t) - c(t-s)) > u \right\} \sim 4u^2e^{-2u^2-2cu}
\]
as \( u \to \infty \) and

\[
P \left\{ \sup_{(s, t) \in S} \left( Y(s, t) - c(t - s) \times (1 - (t - s)) \right) > u \right\} \sim 4u^2 e^{-(1/2)(2u+c/2)^2}.
\]

4. Proofs

In this section, we give the proofs for the main theorem and for the propositions in Sec. 3.

Proof of Theorem 1. Hereafter, \( Q_i, \ i \in \mathbb{N} \), denote some positive constants that may differ from line to line. In the following proof, without loss of generality, we assume that \( c \geq 0 \). Let us denote

\[
E(\delta) = \left\{ (s, t) : |t - s| \leq \frac{\delta}{3}, \ s \in [S_1, S_2] \right\}
\]

and

\[
E(u) = \left\{ (s, t) : |t - s| \leq \left( \frac{\ln u}{u} \right)^{2/\beta}, \ s \in [S_1, S_2] \right\}.
\]

By (2), for any \( \varepsilon \in (0, 1) \), there exists \( \delta \in (0, 1) \) such that for \( (s, t) \in E(\delta) \),

\[
1 + (1 - \varepsilon)b|s - t|^\beta \leq \frac{1}{\sigma(s, t)} \leq 1 + (1 + \varepsilon)b|s - t|^\beta.
\]  \( (7) \)

Further, by (3), we can take \( \delta \in (0, 1) \) small enough such that for \( (s, t), (s', t') \in E(\delta) \) and \( |s - s'| \leq \delta \),

\[
\frac{1}{2}(a|s - s'|^\alpha + a|t - t'|^\alpha) \leq 1 - r(s, t, s', t') \leq 2(a|s - s'|^\alpha + a|t - t'|^\alpha).
\]  \( (8) \)

Below we set for \( \Delta_1, \Delta_2 \subseteq \mathbb{R}^2 \),

\[
P_u(\Delta_1) := P \left\{ \sup_{(s, t) \in \Delta_1} (X(s, t) - c(s - t)) > u \right\},
\]

\[
P_u(\Delta_1, \Delta_2) := P \left\{ \sup_{(s, t) \in \Delta_1} (X(s, t) - c(s - t)) > u, \sup_{(s, t) \in \Delta_2} (X(s, t) - c(s - t)) > u \right\}.
\]

Then we have

\[
P \left\{ \sup_{(s, t) \in E} (X(s, t) - c(s - t)) > u \right\} = P_u(E)
\]

and

\[
P_u(E(u)) \leq P_u(E) \leq P_u(E(u)) + P_u(E(\delta) \setminus E(u)) + P_u(E \setminus E(\delta)).
\]  \( (9) \)

Since

\[
\sigma_m := \sup_{(s, t) \in E \setminus E(\delta)} \sigma(s, t) < 1,
\]

the Borell–TIS inequality as in [1] implies that

\[
P_u(E \setminus E(\delta)) \leq e^{-(u-Q_1)^2/(2\sigma_m^2)} = o(\Psi(u)), \ \ u \to \infty,
\]  \( (10) \)

where

\[
Q_1 = \mathbb{E} \left\{ \sup_{(s, t) \in E \setminus E(\delta)} X(s, t) \right\} < \infty.
\]

Denote

\[
D_k(\delta) = \left\{ (s, t) : s \in S_1 + [k\delta, (k + 1)\delta], \ |s - t| \leq \delta, \ k \in \mathbb{N} \right\}, \ M(\delta) = \left\lfloor \frac{S_2 - S_1}{\delta} \right\rfloor + 1.
\]

In light of (7) for \( u \) large enough, we have

\[
\inf_{(s, t) \in E(\delta) \setminus E(u)} \frac{1}{\sigma(s, t)} \geq 1 + Q_2 \left( \frac{\ln u}{u} \right)^2,
\]

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and by (8) for \((s,t), (s',t') \in D_k(\delta)\) with \(0 \leq k \leq M(\delta)\),

\[
E\{(X(s,t) - X(s',t'))^2\} = 2(1 - r(s,t,s',t')) \leq 4\alpha(|s - s'|^\alpha + |t - t'|^\alpha).
\]

Consequently, by [26, Theorem 8.1] for \(u\) large enough,

\[
P_u(E(\delta) \setminus E(u)) \leq \mathbb{P}\left\{ \sup_{(s,t) \in (E(\delta) \setminus E(u))} X(s,t) > u \right\}
\]
\[
\leq \mathbb{P}\left\{ \sup_{(s,t) \in (E(\delta) \setminus E(u))} X(s,t) > u \left(1 + \frac{Q_2\left(\frac{l\mu}{u}\right)}{u}\right)\right\}
\]
\[
\leq \sum_{k=0}^{M(\delta)} \mathbb{P}\left\{ \sup_{(s,t) \in D_k(\delta)} X(s,t) > u \left(1 + \frac{Q_2\left(\frac{l\mu}{u}\right)}{u}\right)\right\}
\]
\[
\leq Q_3 M(\delta) u^{2\alpha} \Psi\left(2\left(1 + \frac{Q_2\left(\frac{l\mu}{u}\right)}{u}\right)^2\right) = o(\Psi(u)), \quad u \to \infty. \tag{11}
\]

This estimate combined with (9), (10), and the fact that

\[
P_u(E(u)) \geq \mathbb{P}\{X(S_1,S_1) > u\} = \Psi(u)
\]

leads to

\[
P_u(E) \sim P_u(E(u)), \quad u \to \infty. \tag{12}
\]

Further we focus on \(P_u(E(u))\).

**Case 1: \(\alpha < \beta\).** For \(\lambda > 0\), we introduce the following notation:

\[
D_{k,l}(u) = \left[\frac{k\lambda}{u^{2/\alpha}} + (k + 1)\frac{\lambda}{u^{2/\alpha}}\right] \times \left[\frac{l\lambda}{u^{2/\alpha}} + (l + 1)\frac{\lambda}{u^{2/\alpha}}\right], \quad k, l \in \mathbb{Z},
\]
\[
M_1(u) = \left[\frac{S_1 u^{2/\alpha}}{\lambda}\right] - 1, \quad M_2(u) = \left[\frac{S_2 u^{2/\alpha}}{\lambda}\right] + 1, \quad N(u) = \left[\frac{u^{2/\alpha - 2/\beta} \ln u}{\beta}\right] + 1,
\]
\[
\mathcal{J}_1(u) = \{(k,l) : D_{k,l}(u) \subset E(u)\}, \quad \mathcal{J}_2(u) = \{(k,l) : D_{k,l}(u) \cap E(u) \neq \emptyset\},
\]
\[
\mathcal{K}_1(u) = \{(k,l,k_1,l_1) : (k,l), (k_1,l_1) \in \mathcal{J}_1(u), (k,l) \neq (k_1,l_1), \ k \leq k_1, \ D_{k,l}(u) \cap D_{k_1,l_1}(u) \neq \emptyset\},
\]
\[
\mathcal{K}_2(u) = \{(k,l,k_1,l_1) : (k,l), (k_1,l_1) \in \mathcal{J}_1(u), \ k \leq k_1, \ D_{k,l}(u) \cap D_{k_1,l_1}(u) = \emptyset, \ u^{-2/\alpha}|k - k_1|\lambda \leq \frac{\delta}{2}\},
\]
\[
\mathcal{K}_3(u) = \{(k,l,k_1,l_1) : (k,l), (k_1,l_1) \in \mathcal{J}_1(u), \ k \leq k_1, \ D_{k,l}(u) \cap D_{k_1,l_1}(u) = \emptyset, \ u^{-2/\alpha}|k - k_1|\lambda \geq \frac{\delta}{2}\},
\]
\[
u_{k,l}^+ = \left(u + c(k - l + 1)\frac{\lambda}{u^{2/\alpha}}\right) \left(1 + (1 + \epsilon) b(|k - l| + 1)\beta\frac{\lambda^\beta}{u^{2\beta/\alpha}}\right),
\]
\[
u_{k,l}^- = \left(u + c(k - l - 1)\frac{\lambda}{u^{2/\alpha}}\right) \left(1 + (1 - \epsilon) b(\max(|k - l| - 1, 0))\beta\frac{\lambda^\beta}{u^{2\beta/\alpha}}\right).
\]

For large \(u\),

\[
\bigcup_{(k,l) \in \mathcal{J}_1(u)} D_{k,l}(u) \subseteq E(u) \subseteq \bigcup_{(k,l) \in \mathcal{J}_2(u)} D_{k,l}(u).
\]

The Bonferroni inequality implies

\[
\sum_{(k,l) \in \mathcal{J}_1(u)} P_u(D_{k,l}(u)) - \sum_{i=1}^3 A_i(u) \leq P_u(E(u)) \leq \sum_{(k,l) \in \mathcal{J}_2(u)} P_u(D_{k,l}(u)), \tag{13}
\]

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where for $i = 1, 2, 3$,

$$
\mathcal{A}_i(u) = \sum_{(k,l,k_1,l_1) \in \mathcal{K}_i(u)} P_u(D_{k,l}(u), D_{k_1,l_1}(u))
\leq \sum_{(k,l,k_1,l_1) \in \mathcal{K}_i(u)} P \left\{ \sup_{(s,t) \in D_{k,l}(u)} \bar{X}(s,t) > u_{k,l}^{-\epsilon}, \sup_{(s,t) \in D_{k_1,l_1}(u)} \bar{X}(s,t) > u_{k_1,l_1}^{-\epsilon} \right\}.
$$

We set

$$
X^{(1)}_{u,k,l}(s,t) = \bar{X}(ku^{-2/\alpha} + s, lu^{-2/\alpha} + t), \quad (s,t) \in D_{0,0}(u), \quad (k,l) \in \mathcal{J}_2(u).
$$

Then (3) and Lemma 1 imply that

$$
\lim_{u \to \infty} \sup_{(k,l) \in \mathcal{J}_2(u)} \left| \frac{P \left\{ \sup_{(s,t) \in D_{0,0}(u)} X^{(1)}_{u,k,l}(s,t) > u_{k,l}^{-\epsilon} \right\}}{\Psi(u_{k,l}^{-\epsilon})} - \left( \mathcal{H}_\alpha (a^{1/\alpha} \lambda)^2 \right) \right| = 0. \quad (14)
$$

Further, as $u \to \infty$, $\lambda \to \infty$, $\epsilon \to 0$,

$$
\sum_{(k,l) \in \mathcal{J}_2(u)} P_u(D_{k,l}(u)) \leq \sum_{(k,l) \in \mathcal{J}_2(u)} P \left\{ \sup_{(s,t) \in D_{0,0}(u)} \bar{X}(s,t) > u_{k,l}^{-\epsilon} \right\}
\sim \left( \mathcal{H}_\alpha (a^{1/\alpha} \lambda)^2 \Psi(u) \sum_{(k,l) \in \mathcal{J}_2(u)} \sum_{k=M_1(u)}^{M_2(u)} \sum_{l=-N(u)}^{N(u)} e^{-\epsilon b|k-l|^{1/\beta} + c(k-l)u_4^{1/\alpha} - ct\Pi_{(\beta=2)}} \int_{-\infty}^{\epsilon} e^{-(1-\epsilon)b|t|^{1/\beta} - ct\Pi_{(\beta=2)}} dt
\sim \left( \frac{\mathcal{H}_\alpha (a^{1/\alpha} \lambda)^2}{\lambda} \right) \Psi(u)(S_2 - S_1) u^{1/\alpha + 2/\beta} \int_{-\infty}^{\epsilon} e^{-(1-\epsilon)b|t|^{1/\beta} - ct\Pi_{(\beta=2)}} dt
\sim (a^{1/\alpha} \mathcal{H}_\alpha)^2 (S_2 - S_1) \int_{-\infty}^{\epsilon} e^{-b|t|^{1/\beta} - ct\Pi_{(\beta=2)}} dt u^{1/\alpha + 2/\beta} \Psi(u).
$$

Similarly, as $u \to \infty$, $\lambda \to \infty$, $\epsilon \to 0$,

$$
\sum_{(k,l) \in \mathcal{J}_1(u)} P_u(D_{k,l}(u)) \geq \sum_{(k,l) \in \mathcal{J}_1(u)} P \left\{ \sup_{(s,t) \in D_{k,l}(u)} \bar{X}(s,t) > u_{k,l}^{+\epsilon} \right\}
\sim 2(S_2 - S_1) a^{2/\alpha} (a^{1/\alpha} \mathcal{H}_\alpha)^2 b^{1-1/\beta} \Gamma \left( \frac{1}{\beta} + 1 \right) e^{2/(4b)^2_{(\beta=2)}} u^{1/\alpha + 2/\beta} \Psi(u). \quad (16)
$$

Next we will show that $\mathcal{A}_i(u)$, $i = 1, 2, 3$, are all negligible in comparison with

$$
\sum_{(k,l) \in \mathcal{J}_1(u)} P_u(D_{k,l}(u)).
$$
For any \((k, l, k_1, l_1) \in K_1(u)\), without loss of generality, we assume that \(k + 1 = k_1\). Let

\[
D_{k,l}^1(u) = \left[ \frac{\lambda}{u^{2/\alpha}}, \left( (k + 1)\lambda - \sqrt{\lambda} \right) \frac{1}{u^{2/\alpha}}, \left( (k + 1)\lambda + \sqrt{\lambda} \right) \right],
\]

\[
D_{k,l}^2(u) = \left[ \frac{\lambda}{u^{2/\alpha}}, \left( (k + 1)\lambda - \sqrt{\lambda} \right) \frac{1}{u^{2/\alpha}}, \left( (k + 1)\lambda + \sqrt{\lambda} \right) \right].
\]

For \((k, l, k_1, l_1) \in K_1(u)\),

\[
P_u(D_{k,l}(u), D_{k_1,l_1}(u)) \leq P_u(D_{k,l}^1(u), D_{k_1,l_1}(u)) + P_u(D_{k,l}^2(u)).
\]

By analogy with (14) and (15) we have

\[
\lim_{u \to \infty} \sup_{(k,l) \in J_1(u)} \left| \mathbb{P}\left\{ \sup_{(s,t) \in D_{k,l}^1(u)} X(s,t) > u_{k,l}^{-\varepsilon} \right\} - \mathcal{H}_\alpha(a^{1/\alpha} \sqrt{\lambda}) \mathcal{H}_\alpha(a^{1/\alpha} \lambda) \right| = 0
\]

and

\[
A_{11}(u) := \sum_{(k,l) \in J_1(u)} P_u(D_{k,l}^2(u)) \leq \sum_{(k,l) \in J_1(u)} \mathbb{P}\left\{ \sup_{(s,t) \in D_{k,l}^2(u)} X(s,t) > u_{k,l}^{-\varepsilon} \right\}
\]

\[
\leq \mathcal{H}_\alpha(a^{1/\alpha} \sqrt{\lambda}) \mathcal{H}_\alpha(a^{1/\alpha} \lambda) \sum_{(k,l) \in J_1(u)} \Psi(u_{k,l}^{-\varepsilon})
\]

\[
\sim \frac{\mathcal{H}_\alpha(a^{1/\alpha} \sqrt{\lambda}) \mathcal{H}_\alpha(a^{1/\alpha} \lambda)}{\lambda^2} \left( S_2 - S_1 \right) \int_{-\infty}^{\infty} e^{-(1-\varepsilon)|t|^\beta - c_2(t_{\beta = 2})} dt \ u^{4/\alpha - 2/\beta} \Psi(u)
\]

\[
\sim (a^{1/\alpha} \mathcal{H}_\alpha)^2 \frac{1}{\sqrt{\lambda}} (S_2 - S_1) \int_{-\infty}^{\infty} e^{-(1-\varepsilon)|t|^\beta - c_2(t_{\beta = 2})} dt \ u^{4/\alpha - 2/\beta} \Psi(u)
\]

\[
= o(u^{4/\alpha - 2/\beta} \Psi(u)), \quad u \to \infty, \quad \lambda \to \infty, \quad \varepsilon \to 0.
\]

Since \(D_{k,l}(u)\) has at most eight neighbors, in the light of (3) and [8, Lemma 5.4] we see that for \(u\) large enough,

\[
A_{12}(u) \leq \sum_{(k,l,k_1,l_1) \in K_1(u)} \mathbb{P}\left\{ \sup_{(s,t) \in D_{k_1,l_1}(u)} X(s,t) > u_{k_1,l_1}^{-\varepsilon}, \sup_{(s,t) \in D_{k,l}(u)} X(s,t) > u_{k,l}^{-\varepsilon} \right\}
\]

\[
\leq Q_4 \lambda^4 e^{-Q_5 \lambda^{\alpha/2}} \sum_{(k,l,k_1,l_1) \in K_1(u)} \Psi(\min(u_{k,l}^{-\varepsilon}, u_{k_1,l_1}^{-\varepsilon}))
\]

\[
\leq 8Q_4 \lambda^4 e^{-Q_5 \lambda^{\alpha/2}} \sum_{(k,l) \in J_1(u)} \Psi(u_{k,l}^{-\varepsilon}) = O(u^{4/\alpha - 2/\beta} \Psi(u)), \quad u \to \infty, \quad \lambda \to \infty, \quad \varepsilon \to 0,
\]

and

\[
A_2(u) \leq \sum_{(k,l,k_1,l_1) \in K_2(u)} \mathbb{P}\left\{ \sup_{(s,t) \in D_{k_1,l_1}(u)} X(s,t) > u_{k_1,l_1}^{-\varepsilon}, \sup_{(s,t) \in D_{k,l}(u)} X(s,t) > u_{k,l}^{-\varepsilon} \right\}
\]

\[
\leq Q_6 \sum_{(k,l,k_1,l_1) \in K_2(u)} \lambda^4 e^{-Q_7((|k-k_1|-1)^\alpha + (|l-l_1|-1)^\alpha)\lambda^{\alpha}} \Psi(\min(u_{k,l}^{-\varepsilon}, u_{k_1,l_1}^{-\varepsilon}))
\]

\[
\leq Q_6 \lambda^4 \sum_{(k_1,l_1) \in \mathbb{N}^2} e^{-Q_7((k_1)^\alpha + (l_1)^\alpha)\lambda^{\alpha}} \sum_{(k,l) \in J_1(u)} \Psi(u_{k,l}^{-\varepsilon})
\]

\[
= o(u^{4/\alpha - 2/\beta} \Psi(u)), \quad u \to \infty, \quad \lambda \to \infty, \quad \varepsilon \to 0.
\]
Then we have
\[ A_1(u) \leq 2A_1(u) + A_{12}(u) = o(u^{4/\alpha - 2/\beta} \Psi(u)), \quad u \to \infty, \quad \lambda \to \infty. \quad (18) \]
For \((k, l, k_1, l_1) \in \mathcal{K}_3(u)\), the inequality \(|s - s'| \geq \delta/3\) holds with \((s, t) \in D_{k,l}(u), (s', t') \in D_{k_1,l_1}(u)\). Then by (8), for \(u\) large enough,
\[ \text{Var}(\bar{X}(s, t) + \bar{X}(s', t')) = 2(1 + r(s, t, s', t')) \leq 2 + 2 \sup_{|s-s'| \geq \delta/3} r(s, t, s', t') \leq 4 - a \left( \frac{\delta}{3} \right)^\alpha. \]
The inequality holds for \((k, l, k_1, l_1) \in \mathcal{K}_3(u), (s, t) \in D_{k,l}(u), (s', t') \in D_{k_1,l_1}(u)\). Further, Borell–TIS inequality implies that
\[ A_3(u) \leq \sum_{(k,l,k_1,l_1) \in \mathcal{K}_3(u)} \mathbb{P}\left\{ \sup_{(s,t,s_1,t_1) \in D_{k,l}(u) \times D_{k_1,l_1}(u)} \bar{X}(s, t) + \bar{X}(s_1, t_1) > 2u \right\} \leq \sum_{(k,l,k_1,l_1) \in \mathcal{K}_3(u)} e^{-(2u - Q_8)^2/(2(4-a(\delta/3)^\alpha))} \leq Q_8 u^{8/\alpha} e^{-2u - Q_8^2/(2(4-a(\delta/3)^\alpha))} = o(u^{4/\alpha - 2/\beta} \Psi(u)), \quad u \to \infty, \quad (19) \]
where
\[ Q_8 = 2E\left\{ \sup_{(s,t) \in \mathcal{E}} \bar{X}(s, t) \right\} < \infty. \]
Inserting (15)–(19) into (13) yields the relation
\[ P_u(E(u)) \sim 2(S_2 - S_1)a^{2/\alpha} (\mathcal{H}_a)^2 b^{-1/\beta} \Gamma \left( \frac{1}{\beta} + 1 \right) e^{2/(4b)^{\beta=2}} u^{4/\alpha - 2/\beta} \Psi(u), \quad u \to \infty, \]
which compared with (12) implies the final result.

Case 2: \(\alpha = \beta\). For \(\lambda > 0\), we introduce the following notation:
\[ M(u) = \left[ \frac{(S_2 - S_1)u^{2/\alpha}}{\lambda} \right], \quad N(u) = \left[ \frac{(\ln u)^{2/\beta}}{\lambda} \right] + 1, \]
\[ D_{k,l}(u) = \left\{ (s, t) : s \in S_1 + \left[ k \frac{\lambda}{u^{2/\alpha}}, (k + 1) \frac{\lambda}{u^{2/\alpha}} \right], (s - t) \in \left[ l \frac{\lambda}{u^{2/\alpha}}, (l + 1) \frac{\lambda}{u^{2/\alpha}} \right] \right\}, \]
\[ D_k(u) = \left\{ (s, t) : s \in S_1 + \left[ k \frac{\lambda}{u^{2/\alpha}}, (k + 1) \frac{\lambda}{u^{2/\alpha}} \right], |s - t| \leq \frac{\lambda}{u^{2/\alpha}} \right\}, \quad k, l \in \mathbb{Z}, \]
\[ \mathcal{K}_1(u) = \{(k, k_1) : 0 < k < k_1 < M(u), k_1 = k + 1\}, \]
\[ \mathcal{K}_2(u) = \{(k, k_1) : 0 < k < k_1 < M(u), k_1 > k + 1, u^{-2/\alpha}|k - k_1| \lambda \leq \frac{\delta}{2}\}, \]
\[ \mathcal{K}_3(u) = \{(k, k_1) : 0 < k < k_1 < M(u), k_1 > k + 1, u^{-2/\alpha}|k - k_1| \lambda \geq \frac{\delta}{2}\}, \]
\[ u^+_{l, \alpha} = \left( u + c(l + 1) \frac{\lambda}{u^{2/\alpha}} \right) \left( 1 + (1 + \varepsilon) b l + \mathbb{I}_{\{l \geq 0\}} \right) \frac{\lambda^\alpha}{u^2}, \]
\[ u^-_{l, \alpha} = \left( u + c l \frac{\lambda}{u^{2/\alpha}} \right) \left( 1 + (1 - \varepsilon) b l + \mathbb{I}_{\{l < 0\}} \right) \frac{\lambda^\alpha}{u^2}. \]
For large \(u\), we have
\[ \bigcup_{k=0}^{M(u)-1} D_k(u) \subseteq E(u) \subseteq \left( \bigcup_{k=0}^{M(u)} D_k(u) \bigcup \bigcup \bigcup_{l=-N(u)}^{N(u)} D_{k,l}(u) \right). \]
The Bonferroni inequality implies
\[
\sum_{k=0}^{M(u)-1} P_u(D_k(u)) - \frac{3}{u} \sum_{i=1}^{A_i(u)} \leq \sum_{k=0}^{M(u)} P_u(D_k(u)) + \sum_{k=0}^{M(u)} \sum_{l \neq N(u)} P_u(D_{k,l}(u)),
\]
where
\[
A_i(u) = \sum_{(k,l,k_1,l_1) \in K_i(u)} P_u(D_{k,l}(u), D_{k_1,l_1}(u)), \quad i = 1, 2, 3.
\]
Let for \(0 \leq k \leq M(u),\)
\[
X^{(2)}_{u,k}(s,t) = \bar{X} \left( S_1 + k \frac{\lambda}{u^{2/\alpha}} + s, S_1 + k \frac{\lambda}{u^{2/\alpha}} + t \right),
\]
where
\[
(s,t) \in D^{(2)}(u) = \left\{ (s,t) : s \in \left[ 0, \frac{\lambda}{u^{2/\alpha}} \right], \ |s-t| \leq \frac{\lambda}{u^{2/\alpha}} \right\}.
\]
Then by Lemma 1
\[
\mathbb{P} \left\{ \sup_{(s,t) \in D^{(2)}(u)} \frac{X^{(2)}_{u,k}(s,t)}{(1+(c/u)(s-t))(1+(1-\varepsilon)b|s-t|^{\alpha})} > u \right\} \sim \Psi(u) P_{\alpha}^{f^{-\varepsilon}(s-t)(a^{1/\alpha} \lambda, a^{1/\alpha} \lambda)},
\]
as \(u \to \infty\) uniformly over \(0 \leq k \leq M(u),\) and
\[
\sum_{k=0}^{M(u)} P_u(D_k(u)) \leq \sum_{k=0}^{M(u)} \mathbb{P} \left\{ \sup_{(s,t) \in D_k(u)} \frac{\bar{X}(s,t)}{(1+(c/u)(s-t))(1+(1-\varepsilon)b|s-t|^{\alpha})} > u \right\}
\]
\[
= \sum_{k=0}^{M(u)} \mathbb{P} \left\{ \sup_{(s,t) \in D^{(2)}(u)} \frac{X^{(2)}_{u,k}(s,t)}{(1+(c/u)(s-t))(1+(1-\varepsilon)b|s-t|^{\alpha})} > u \right\}
\]
\[
\sim \sum_{k=0}^{M(u)} P_{\alpha}^{f^{-\varepsilon}(s-t)(a^{1/\alpha} \lambda, a^{1/\alpha} \lambda)} \Psi(u) \sim \frac{(S_2 - S_1)u^{2/\alpha}}{\lambda} P_{\alpha}^{f^{-\varepsilon}(s-t)(a^{1/\alpha} \lambda, a^{1/\alpha} \lambda)} \Psi(u) \sim (S_2 - S_1)u^{2/\alpha} P_{\alpha}^{f(s-t)} u^{2/\alpha} \Psi(u), \quad u \to \infty, \ \lambda \to \infty, \ \varepsilon \to 0,
\]
where
\[
f^{-\varepsilon}(t) = (1-\varepsilon) \frac{b}{a} t^{1/\alpha} + \frac{c}{\sqrt{\alpha}} t^{1/2},
\]
and
\[
f(t) = \frac{b}{a} t^{1/\alpha} + \frac{c}{\sqrt{\alpha}} t^{1/2}.
\]
Similarly,
\[
\sum_{k=0}^{M(u)-1} P_u(D_k(u)) \geq \sum_{k=0}^{M(u)-1} \mathbb{P} \left\{ \sup_{(s,t) \in D_k(u)} \frac{\bar{X}(s,t)}{(1+(c/u)(s-t))(1+(1+\varepsilon)b|s-t|^{\alpha})} > u \right\}
\]
\[
\sim (S_2 - S_1)a^{1/\alpha} P_{\alpha}^{f(s-t)} u^{2/\alpha} \Psi(u), \quad u \to \infty, \ \lambda \to \infty, \ \varepsilon \to 0.
\]
Let for \(0 \leq k \leq M(u)\) and \(-N(u) \leq l \leq N(u),\)
\[
X^{(3)}_{u,k,l}(s,t) = \bar{X} \left( S_1 + k \frac{\lambda}{u^{2/\alpha}} + s, S_1 + k \frac{\lambda}{u^{2/\alpha}} - l \frac{\lambda}{u^{2/\alpha}} + t \right),
\]
where
\[
(s,t) \in D^{(3)}(u) = \left\{ (s,t) : s \in \left[ 0, \frac{\lambda}{u^{2/\alpha}} \right], \ 0 \leq s-t \leq \frac{\lambda}{u^{2/\alpha}} \right\}.
\]
Then by Lemma 1,

$$\lim_{u \to \infty} \sup_{0 \leq k \leq M(u)} \sup_{-N(u) \leq l \leq N(u)} \left| \mathbb{P} \left\{ \sup_{(s,t) \in D^{(3)}(u)} X_{u,k,l}^{(3)}(s,t) > u_t^{-\varepsilon} \right\} \Psi(u_t^{-\varepsilon}) - Q_\alpha(a^{1/\alpha} \lambda, a^{1/\alpha} \lambda) \right| = 0.$$ 

Then we have

$$\sum_{k=0}^{M(u)} \sum_{l=-N(u)}^{N(u)} \mathbb{P}_u(D_{k,l}(u)) \leq \sum_{k=0}^{M(u)} \sum_{l=-N(u)}^{N(u)} \mathbb{P} \left\{ \sup_{(s,t) \in D^{(3)}(u)} X_{u,k,l}^{(3)}(s,t) > u_t^{-\varepsilon} \right\} \Psi(u_t^{-\varepsilon})$$

$$= \sum_{k=0}^{M(u)} \sum_{l=-N(u)}^{N(u)} \mathbb{P} \left\{ \sup_{(s,t) \in D^{(3)}(u)} X_{u,k,l}^{(3)}(s,t) > u_t^{-\varepsilon} \right\} \sim \sum_{k=0}^{M(u)} \sum_{l=-N(u)}^{N(u)} Q_\alpha(a^{1/\alpha} \lambda, a^{1/\alpha} \lambda) \Psi(u_t^{-\varepsilon})$$

$$\sim \frac{(S_2 - S_1)u^{2/\alpha}}{\lambda} Q_\alpha(a^{1/\alpha} \lambda, a^{1/\alpha} \lambda) \sum_{l=-N(u)}^{N(u)} \Psi(u_t^{-\varepsilon})$$

$$\sim \frac{(S_2 - S_1)u^{2/\alpha}}{\lambda} Q_\alpha(a^{1/\alpha} \lambda, a^{1/\alpha} \lambda) \Psi(u) \sum_{l=-\infty}^{\infty} e^{-(1-\varepsilon)b[l+u]} \alpha^{-a} - cl\alpha^{-a}$$

$$\leq (S_2 - S_1)u^{2/\alpha} \alpha^{-a} \Psi(u) \sum_{l=-\infty}^{\infty} e^{-(1-\varepsilon)b[l+u]} \alpha^{-a} - cl\alpha^{-a}$$

$$= o(u^{2/\alpha} \Psi(u)), \quad u \to \infty, \lambda \to \infty, \varepsilon \to \infty. \quad (23)$$

Further, by using similar arguments as in (17)–(19), we obtain that

$$A_i(u) = o(u^{2/\alpha} \Psi(u)), \quad u \to \infty, \lambda \to \infty, i = 1, 2, 3,$$

which combined with (20)–(23) implies

$$\mathbb{P}_u(E(u)) \sim (S_2 - S_1)a^{1/\alpha} u^{2/\alpha} \mathbb{P}_\alpha^{(s-t)}(u), \quad u \to \infty.$$

**Case 3:** $\alpha > \beta$. For $\lambda, \lambda_1 > 0$, we introduce the same notation as in Case 2 except

$$D_k(u) = \left\{ (s,t) : s \in \left[ S_1 + k \frac{\lambda}{u^{2/\alpha}}, S_1 + (k + 1) \frac{\lambda}{u^{2/\alpha}} \right], |s-t| \leq \frac{\lambda_1}{u^{2/\alpha}} \right\}, \quad k \in \mathbb{N}.$$ 

Hence by $\alpha > \beta$, for large $u$, we have

$$\mathcal{L} \subseteq E(u) \subseteq \bigcup_{k=0}^{M(u)} D_k(u).$$

The Bonferroni inequality implies

$$\mathbb{P}_u(\mathcal{L}) \leq \mathbb{P}_u(E(u)) \leq \sum_{k=0}^{M(u)} \mathbb{P}_u(D_k(u)). \quad (24)$$

By (2)–(4), we see that for $s \in [S_1, S_2]$,

$$\sigma(s, s) \equiv 1,$$
and for \(s, s' \in [S_1, S_2]\),
\[
r(s, s', s') = 1 - 2a|s - s'|^\alpha(1 + o(1)), \quad |s - s'| \to 0,
\]
and
\[
r(s, s', s') < 1, \quad s \neq s'.
\]
Let \(Y(s), s \in [S_1, S_2]\), be a homogeneous Gaussian process with continuous trajectories, unit variance, and correlation function \(r_Y(s)\) satisfying for some \(\varepsilon_1 \in (0, 1)\) the following relations:
\[
r_Y(s) = 1 - 2(1 - \varepsilon_1)a|s|^\alpha(1 + o(1)), \quad |s| \to 0
\]
and
\[
r_Y(s) < 1, \quad s \neq 0.
\]
Thus, the Slepian inequality (see, e.g., [1]) and [26, Theorem 7.1] imply
\[
\mathbb{P}_u(\mathcal{L}) = \mathbb{P}\left\{ \sup_{(s, t) \in \mathcal{L}} X(s, t) > u \right\} = \mathbb{P}\left\{ \sup_{s \in [S_1, S_2]} X(s, s) > u \right\} \geq \mathbb{P}\left\{ \sup_{s \in [S_1, S_2]} Y(s) > u \right\} \sim (S_2 - S_1)(2 - \varepsilon_1)a^{1/\alpha} H_a u^{2/\alpha} \Psi(u) \sim (S_2 - S_1)(2a)^{1/\alpha} H_a u^{2/\alpha} \Psi(u), \quad u \to \infty, \quad \varepsilon_1 \to 0.
\]
Let for \(0 \leq k \leq M(u)\),
\[
X_{u,k}^{(4)}(s, t) = \bar{X} \left( S_1 + k \frac{\lambda}{u^{2/\alpha}} + s, S_1 + k \frac{\lambda}{u^{2/\alpha}} + t \right),
\]
where
\[
(s, t) \in D^{(4)}(u) = \left\{ (s, t) : s \in \left[ 0, \frac{\lambda}{u^{2/\alpha}} \right], |s - t| \leq \frac{\lambda_1}{u^{2/\alpha}} \right\}.
\]
Then Lemma 1 implies
\[
\lim_{u \to \infty} \sup_{0 \leq k \leq M(u)} \left| \mathbb{P}\left\{ \sup_{(s, t) \in D^{(4)}(u)} X_{u,k}^{(4)}(s, t) > u \right\} \Psi(u) - \mathcal{P}_0(a^{1/\alpha} \lambda, a^{1/\alpha} \lambda_1) \right| = 0.
\]
Then we have
\[
\sum_{k=0}^{M(u)} \mathcal{P}_u(D_k(u)) = \sum_{k=0}^{M(u)} \mathbb{P}\left\{ \sup_{(s, t) \in D_k(u)} \bar{X}(s, t) > u \right\}
\]
\[
= \sum_{k=0}^{M(u)} \mathbb{P}\left\{ \sup_{(s, t) \in D^{(4)}(u)} X_{u,k}^{(4)}(s, t) > u \right\} \sim \sum_{k=0}^{M(u)} \mathcal{P}_0(a^{1/\alpha} \lambda, a^{1/\alpha} \lambda_1) \Psi(u)
\]
\[
\sim (S_2 - S_1)u^{2/\alpha} \lambda \mathcal{P}_0(a^{1/\alpha} \lambda, a^{1/\alpha} \lambda_1) \Psi(u) \sim (S_2 - S_1)u^{2/\alpha} H_a(2^{1/\alpha} a^{1/\alpha} \lambda) \Psi(u), \quad u \to \infty, \quad \lambda_1 \to 0, \quad \lambda \to \infty,
\]
where in (25) we use the fact that
\[
\lim_{\lambda_1 \to 0} \mathcal{P}_0(a^{1/\alpha} \lambda, \lambda_1) = \lim_{\lambda_1 \to 0} \mathbb{E}\left\{ \sup_{0 \leq s \leq \lambda, |s - t| \leq \lambda_1} \exp(\sqrt{2} B_\alpha^{(1)}(s) + \sqrt{2} B_\alpha^{(2)}(t) - |s|^\alpha - |t|^\alpha) \right\}
\]
\[
= \mathbb{E}\left\{ \sup_{0 \leq s \leq \lambda} \exp(\sqrt{2} B_\alpha^{(1)}(s) + \sqrt{2} B_\alpha^{(2)}(s) - 2|s|^\alpha) \right\}
\]
\[
= \mathbb{E}\left\{ \sup_{0 \leq s \leq \lambda} \exp(2 B_\alpha^{(1)}(s) - 2|s|^\alpha) \right\} = H_\alpha(2^{1/\alpha} \lambda).
\]
Then for $0 \leq \varepsilon \leq 1$ and for any $(s, t) \in S$, we have

\[
\sigma^2_Y(s, t) = (t - s) - (t - s)^2,
\]

and the correlation function of $Y(s, t)$ satisfies

\[
1 - r_Y(s, t, s', t') \sim 2(|t - t'| + |s - s'|), \quad |t - t'|, |s - s'| \to 0,
\]

and for any $(s, t), (s', t') \in S$,

\[
1 - r_Y(s, t, s', t') \leq 2\mathbb{E}\left\{ (Y(s, t) - Y(s', t'))^2 \right\} \leq \mathcal{Q}|t - t'| + \mathcal{Q}|s - s'|,
\]

where $\mathcal{Q}$ is a positive constant. Thus, $r_Y(s, t, s', t') < 1$, $(s, t) \neq (s', t')$.

(1) We have for any $u > 0$

\[
P\left\{ \sup_{(s, t) \in S} (Y(s, t) - c(t - s)u) > du \right\} = P\left\{ \sup_{(s, t) \in S} \frac{Y(s, t)}{d + c(t - s)} > u \right\}.
\]

We note that the variance function of $Y(s, t)/(d + c(t - s))$

\[
\frac{(t - s) - (t - s)^2}{(d + c(t - s))^2}
\]

attains its maximum at $t - s = d/(2d + c)$, and the maximum is equal to $1/(4d(c + d))$. Then

\[
P\left\{ \sup_{(s, t) \in S} \frac{Y(s, t)}{d + c(t - s)} > u \right\} = P\left\{ \sup_{0 \leq s - d/(2d + c) \leq t \leq 1} Z(s, t) > \sqrt{4d(c + d)}u \right\},
\]

where

\[
Z(s, t) = \frac{B(t) - B(s)}{d + c(t - s + d/(2d + c))}.
\]

Then for $0 \leq s - d/(2d + c) < t \leq 1$, the standard deviation of $Z(s, t)$ denoted as $\sigma_Z(s, t)$ attains its maximum at $s = t$ and satisfies

\[
1 - \sigma_Z(s, t) \sim \frac{(2d + c)^4}{8(d^2 + cd)^2} (t - s)^2, \quad |t - s| \to 0.
\]

Its correlation function satisfies

\[
1 - r_Z(s, t, s', t') \sim 2(|t - t'| + |s - s'|), \quad |t - t'|, |s - s'|, |t - s|, |t' - s'| \to 0,
\]
and

\[ r_{Z}(s, t, s', t') < 1, \quad (s, t) \neq (s', t'). \]

Thus, the result follows from Theorem 1.

(2) For any \( u > 0 \), we have

\[
\mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}} \left( Y(s, t) - c(t - s) \times (1 - (t - s))u \right) > u \right\} = \mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}} \frac{Y(s, t)}{d + c(t - s) \times (1 - (t - s))} > u \right\}.
\]

The variance function of

\[
Y(s, t) \quad \text{is equal to}
\]

\[
\frac{(t - s) - (t - s)^2}{(d + c(t - s) \times (1 - (t - s)))^2}
\]

which attains its maximum \( 1/\sqrt{4cd} \) at \( t - s = (1 \pm \sqrt{1 - 4d/c})/2 \) which are two parallel lines in the area \( 0 \leq s < t \leq 1 \). Following along the same lines of [26][Corollary 8.2], we have

\[
\mathbb{P}\left\{ \sup_{0 \leq s < t \leq 1} \frac{B(t) - B(s) - (t - s)B(1)}{d + c(t - s) \times (1 - (t - s))} > u \right\} \sim \mathbb{P}\left\{ \sup_{0 \leq s - (1 + \sqrt{1 - 4d/c})/2 < t \leq 1} Z^+(s, t) > \sqrt{4cd}u \right\} + \mathbb{P}\left\{ \sup_{0 \leq s - (1 - \sqrt{1 - 4d/c})/2 < t \leq 1} Z^-(s, t) > \sqrt{4cd}u \right\},
\]

where

\[
Z^\pm(s, t) = \sqrt{4cd} \times \frac{B(t) - B(s) - (t - s + (1 \pm \sqrt{1 - 4d/c})/2)B(1)}{d + c(t - s + (1 \pm \sqrt{1 - 4d/c})/2) \times (1 - (t - s + (1 \pm \sqrt{1 - 4d/c})/2)).
\]

Then the standard deviation of \( Z^\pm(s, t) \) satisfies

\[ 1 - \sigma_{Z}(s, t) \sim \frac{c(c - 4d)}{8d^2} (t - s)^2, \quad |t - s| \to 0, \]

and its correlation function satisfies

\[ 1 - r_{Z}(s, t, s', t') \sim 2(|t - t'| + |s - s'|), \quad |t - t'|, |s - s'|, |t - s|, |t' - s'| \to 0. \]

Moreover,

\[ r_{Z}(s, t, s', t') < 1, \quad (s, t) \neq (s', t'). \]

Thus, the result follows from Theorem 1.

(3) We note that for

\[ Z(s, t) := \frac{B(t) - B(s) - (t - s)B(1)}{(t - s) \times (1 - (t - s))}, \quad 0 \leq s < t \leq 1, \]

the variance function of \( Z(s, t) \) is

\[ \sigma_{Z}^2(s, t) \equiv 1, \]

and its correlation function satisfies for \( (s, t) \in \mathcal{S}(\delta) \),

\[ 1 - r_{Z}(s, t, s', t') \sim 2(|t - t'| + |s - s'|), \quad |t - t'|, |s - s'| \to 0. \]

We note that

\[
\mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}(\delta)} Z(s, t) > d \right\} \leq \mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}} Z(s, t) > d \right\} \leq \mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}(\delta)} Z(s, t) > d \right\} + \mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S} \setminus \mathcal{S}(\delta)} Z(s, t) > d \right\}.
\]
By [26, Theorem 7.1],
\[ \mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}(\delta)} Z(s,t) > d \right\} \sim 2(1 - \delta)^2 d^4 \Psi(d), \quad d \to \infty. \]

Let \( W(s, t), (s, t) \in \mathbb{R}^2 \), be a homogeneous Gaussian field with unit variance and correlation function
\[ r_W(s, t) = \exp(-Q|t - t'| - Q|s - s'|). \]

Then the Slepian inequality (see, e.g., [1]) and [26, Theorem 7.1], implies that
\[ \mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}(\delta)} Z(s,t) > d \right\} \leq \mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}(\delta)} W(s,t) > d \right\} \sim 4\delta(2 - \delta)d^4\Psi(d) \]

as \( d \to \infty \). Thus letting \( \delta \to \infty \), we have
\[ \mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}} Z(s,t) > d \right\} \sim 2d^4\Psi(d), \quad d \to \infty. \]

**Proof of Proposition 2.** For \( 0 \leq s < t \leq 1 \), the variance function of \( Y(s,t) \) is
\[ \sigma_Y^2(s,t) = (t-s)(t-s') \]
which attains its maximum equal to \( 1/4 \) at \( t-s = 1/2 \).

Further, if we set
\[ Z(s,t) = 2\left( B(t) - B\left( s - \frac{1}{2} \right) - \left( t-s + \frac{1}{2} \right) B(1) \right), \]
then for \( 0 \leq s - 1/2 < t \leq 1 \), the variance function of \( Z(s,t) \) is
\[ \sigma_Z^2(s,t) = 4\left( t-s + \frac{1}{2} \right) \left[ 1 - \left( t-s + \frac{1}{2} \right) \right] \]
which attains its maximum at \( t-s = 0 \) with \( \sigma_Z(s,t)|_{t=s=0} = 1 \). Further, the standard deviation satisfies
\[ 1 - \sigma_Z(s,t) \sim 2(t-s)^2, \quad |t-s| \to 0, \]
and its correlation function satisfies
\[ 1 - r_Z(s,t,s',t') \sim 2(|t-t'| + |s-s'|), \quad |t-t'|, |s-s'|, |t-s|, |t'-s'| \to 0, \]
so,
\[ r_Z(s,t,s',t') < 1, \quad (s,t) \neq (s',t'). \]

We have
\[ \mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}} (Y(s,t) - c(t-s)) > u \right\} = \mathbb{P}\left\{ \sup_{0 \leq s - 1/2 < t \leq 1} (Z(s,t) - 2c(t-s)) > 2u + c \right\}. \]

Application of Theorem 1 yields the first claim.

Since
\[ 2c\left( t-s + \frac{1}{2} \right) \times \left( 1 - \left( t-s + \frac{1}{2} \right) \right) = \frac{c}{2} - 2c(t-s)^2, \]
we have
\[ \mathbb{P}\left\{ \sup_{(s,t) \in \mathcal{S}} (Y(s,t) - c(t-s) \times (1 - (t-s))) > u \right\} = \mathbb{P}\left\{ \sup_{0 \leq s - 1/2 < t \leq 1} (Z(s,t) + 2c(t-s)^2) > 2u + \frac{c}{2} \right\}. \]

Again application of Theorem 1 yields the claim. \( \square \)

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5. Appendix

Lemma 1. Let $X_{u,k}(s,t)$, $k \in K_u$, $(s,t) \in \mathbb{R}^2$, be a family of centered Gaussian fields with continuous sample paths. Let further $u_k$, $k \in K_u$, be given positive constants satisfying

$$\lim_{u \to \infty} \sup_{k \in K_u} \frac{|u_k - 1|}{u} = 0. \tag{26}$$

If $X_{u,k}$ has unit variance and correlation function $r_k$ (not depending on $u$) satisfying (3) uniformly over $k \in K_u$, then for some $\lambda_1, \lambda_2 > 0$, we have

$$\lim_{u \to \infty} \sup_{k \in K_u} \left| \frac{\mathbb{P}\left\{ \sup_{(s,t) \in D_1(u)} X_{u,k}(s,t) > u_k \right\}}{\Psi(u_k)} - \mathcal{H}_\alpha(a^{1/\alpha} \lambda_1) \mathcal{H}_\alpha(a^{1/\alpha} \lambda_2) \right| = 0,$$

where

$$D_1(u) = [0, \lambda_1 u^{-2/\alpha}] \times [0, \lambda_2 u^{-2/\alpha}]$$

and for $b \geq 0$, $c \in \mathbb{R}$,

$$\lim_{u \to \infty} \sup_{k \in K_u} \left| \frac{\mathbb{P}\left\{ \sup_{(s,t) \in D_2(u)} \frac{X_{u,k}(s,t)}{(1+(c/u)|s-t|^{\alpha})(1+b|s-t|^{\alpha})} > u \right\}}{\Psi(u)} - \mathcal{P}_\alpha f(s-t) (a^{1/\alpha} \lambda_1, a^{1/\alpha} \lambda_2) \right| = 0,$$

where

$$D_2(u) = \{(s,t) : s \in [0, \lambda_1 u^{-2/\alpha}], |s-t| \leq \lambda_2 u^{-2/\alpha}, f(t) = \frac{b}{a} |t|^{\alpha} + \frac{c}{\sqrt{a}} t^{\alpha=2} \}.$$

Moreover,

$$\lim_{u \to \infty} \sup_{k \in K_u} \left| \frac{\mathbb{P}\left\{ \sup_{(s,t) \in D_3(u)} X_{u,k}(s,t) > u_k \right\}}{\Psi(u_k)} - \mathcal{Q}_\alpha (a^{1/\alpha} \lambda_1, a^{1/\alpha} \lambda_2) \right| = 0,$$

where

$$D_3(u) = \{(s,t) : s \in [0, \lambda_1 u^{-2/\alpha}], 0 \leq s-t \leq \lambda_2 u^{-2/\alpha} \}.$$

Proof. It follows along the same lines as [9, Theorem 2.1]. \qed

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