Tetrahedron equation and quantum $R$ matrices for infinite-dimensional modules of $U_q(A_1^{(1)})$ and $U_q(A_2^{(2)})$

Atsuo Kuniba$^1$ and Masato Okado$^2$

$^1$ Institute of Physics, Graduate School of Arts and Sciences, University of Tokyo, Komaba, Tokyo 153-8902, Japan
$^2$ Department of Mathematics, Osaka City University, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

E-mail: atsuo@gokutan.c.u-tokyo.ac.jp and okado@sci.osaka-cu.ac.jp

Received 4 September 2013, in final form 16 October 2013
Published 12 November 2013
Online at stacks.iop.org/JPhysA/46/485203

Abstract

From the $q$-oscillator solution to the tetrahedron equation associated with a quantized coordinate ring, we construct solutions to the Yang–Baxter equation by applying a reduction procedure formulated earlier by Sergeev and the first author. The results are identified with the quantum $R$ matrices for the infinite-dimensional modules of $U_q(A_1^{(1)})$ and $U_q(A_2^{(2)})$ corresponding to an affinization of Verma modules of their subalgebras isomorphic to $U_q(sl_2)$ and $U_q(sl_{2+1})$.

PACS numbers: 02.20.Uw, 02.30.Ik
Mathematics Subject Classification: 81R50, 17B37, 16T25

1. Introduction

The tetrahedron equation [13] is a three-dimensional (3d) extension of the Yang–Baxter equation [1]. Among its many formulations, the homogeneous version of vertex type is a quartic equation on the matrix $R$ acting on the tensor cube of a vector space $F$ representing the ‘single spin’ states: see (2.1). The tetrahedron equation possesses the structure that if one of the components in $R \in \text{End}(F^\otimes 3)$ is evaluated away appropriately, the resulting object in $\text{End}(F^\otimes 2)$ satisfies the Yang–Baxter equation. The eliminated space, let us call it the third component, serves as the internal degrees of freedom of the local spins in the resulting two-dimensional (2d) vertex model. This reduction works naturally also for the compositions of $R$s in the third component, which implies that a single solution to the tetrahedron equation generates an infinite sequence of solutions to the Yang–Baxter equation. They correspond to the optional size in the third direction of the 3d lattice on which an integrable vertex model associated with $R$ is defined. This kind of connection between the 2d and 3d integrable systems has been studied from various viewpoints since [2] for example. In particular, it has been shown recently that the reduction scheme reproduces the conventional quantum $R$ matrices for some reducible representations of $U_q(A_n^{(1)})$ [4] and the spin representations of $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$, $U_q(D_{n+1}^{(2)})$ [10]. In the former, the reduction of the third space is due to the trace, whereas in the latter it is done by taking the matrix elements between special ‘boundary
It is an open problem to clarify the grand picture of such 2d–3d connections. See section 5 for a unified formulation of the problem under which the preceding results [4, 10] are allocated and explained more precisely.

In this paper, we add a further solution of the problem as a modest step toward a thorough understanding of the 2d–3d connection. We apply the reduction scheme based on the special boundary vectors [10] to the solution \( R \) of the tetrahedron equation studied in [3, 4, 6, 8, 12]. It acts on the tensor cube of the \( q \)-oscillator Fock space \( F = \bigoplus_{m \geq 0} \mathbb{Q}(q) m \). It is the same \( R \) as the one used in the trace construction in [4]. For simplicity, we concentrate on the simplest ‘single-site’ situation along the third direction. Then the resulting solution to the Yang–Baxter equation is the linear operator acting on \( F \otimes F \), which defines an integrable 2d vertex model accommodating the \( q \)-oscillator Fock states on each edge. Besides its explicit formula (proposition 2), our main result is the identification with the quantum \( U_q(\mathfrak{sl}_2) \) and \( D_{11}^{(2)} \). The layout of the paper is as follows. In section 2, we recall the reduction procedure based on the boundary vectors [10] along the simplest one-site situation. An explicit formula (proposition 2), our main result is the identification with the quantum \( U_q(\mathfrak{sl}_2) \) and \( D_{11}^{(2)} \). It is an open problem to clarify the grand picture of such 2d–3d connections. See section 2 for more accounts on the origin of the solution.

In this paper, we shall explain the basic construction and only state the main results. Some further results are announced. A full solution of them will bring an important insight into the 2d–3d connection in integrable systems in a broad sense.
We recall the prescription in [10] that reduces a solution to the tetrahedron equation to the one for the Yang–Baxter equation. We restrict ourselves to the simplest ‘single-site’ situation relevant to this paper. See [10] for a more general treatment.

Suppose there are vectors
\begin{align}
|\chi_{s}(x, y)| = |\chi_{s}(x)| \otimes |\chi_{s}(xy)| \otimes |\chi_{s}(y)| \in F \otimes F \otimes F, \tag{2.2}
\end{align}
where \(x, y\) are extra (spectral) parameters such that
\begin{align}
\mathcal{R}|\chi_{s}(x, y)| = |\chi_{s}(x, y)|. \tag{2.3}
\end{align}
The index \(s\) is a label of possibly more than one such vector. Suppose also similar vectors exist in the dual space:
\begin{align}
\langle \chi_{s}(x, y)| = \langle \chi_{s}(x) | \otimes \langle \chi_{s}(xy) | \otimes \langle \chi_{s}(y) | \in F^{*} \otimes F^{*} \otimes F^{*}, \tag{2.4}
\end{align}
with the property
\begin{align}
\langle \chi_{s}(x, y)| \mathcal{R} = \langle \chi_{s}(x, y)|. \tag{2.5}
\end{align}
Then evaluating the tetrahedron equation (2.1) between \(|\chi_{s}(x, y)|\) and \(|\chi_{s}(1, 1)|\) on the \((4, 5, 6)\)th components, one obtains the Yang–Baxter equation
\begin{align}
\mathcal{R}_{12}(x)\mathcal{R}_{31}(xy)\mathcal{R}_{23}(y) = \mathcal{R}_{23}(y)\mathcal{R}_{13}(xy)\mathcal{R}_{12}(x) \in \text{End}(F \otimes F \otimes F), \tag{2.6}
\end{align}
where indices again signify the non-trivially acting components in \(F^{\otimes 3}\), and
\begin{align}
\mathcal{R}_{12}(z) = \langle \chi_{t}(z)| \mathcal{R}_{123}|\chi_{t}(1)\rangle \in \text{End}(F \otimes F) \otimes 1 \tag{2.7}
\end{align}
for example. Here the bracket means the evaluation with respect to the third component. Denoting (2.7) just by \(\mathcal{R}(z) \in \text{End}(F \otimes F)\), it is also convenient to introduce
\begin{align}
\tilde{\mathcal{R}}(z) = \varrho(z)P\mathcal{R}(z), \tag{2.8}
\end{align}
where \(P(u \otimes v) = v \otimes u\) is the linear operator exchanging the components and \(\varrho(z)\) is an arbitrary scalar function. Then the Yang–Baxter equation takes another familiar form:
\begin{align}
(\tilde{\mathcal{R}}(z) \otimes 1)(1 \otimes \tilde{\mathcal{R}}(xy))(1 \otimes \tilde{\mathcal{R}}(y)) = (1 \otimes \tilde{\mathcal{R}}(y))(\tilde{\mathcal{R}}(xy) \otimes 1)(1 \otimes \tilde{\mathcal{R}}(x)). \tag{2.9}
\end{align}
Note the degree of freedom to choose \(s\) and \(t\) in (2.7) although it has temporarily been suppressed in the notation. In fact, it will allow us to cover the quantum affine algebras for \(A_{1}^{(1)}\) and \(A_{2}^{(2)}\) in our main theorem 10.

Now we proceed to a concrete realization of the above scheme in this paper. We will always take \(F\) to be an infinite-dimensional space \(F = \bigoplus_{m \geq 0} \mathbb{Q}(q)|m\). The dual space will be denoted by \(F^{*} = \bigoplus_{m \geq 0} \mathbb{Q}(q)|m\) with the bilinear pairing \((m|n) = (q^{2})^{m\delta_{m,n}}\). For simplicity, vectors like \(|i| \otimes |j| \otimes |k|\) \(\in F^{\otimes 3}\) and \(|i| \otimes |j| \in (F^{*})^{\otimes 2}\) etc will be abbreviated to \(|i, j, k|\) and \(|i, j|\) etc.

The solution \(\mathcal{R}\) of the tetrahedron equation we are concerned with is the one obtained as the intertwiner of the quantum coordinate ring \(A_{q}(sl_{3})\) [6] 6, which was also found from a quantum geometry consideration in a different gauge including square roots [3, 4]. They were shown to be essentially the same object and to constitute the solution of the 3d reflection equation [8]. It can also be identified with the transition matrix of the PBW bases of the nilpotent subalgebra of \(U_{q}(sl_{3})\) [9, 12]. Here we simply call it 3d \(\mathcal{R}\). It is given by
\begin{align}
\mathcal{R}|i, j, k\rangle = \sum_{a,b,c} q_{i,j,k}^{a,b,c}|a, b, c\rangle, \tag{2.10}
\end{align}
\footnote{In general, \(|\chi_{s}(x', y')\rangle\) can be used. However, in our examples treated later, such freedom is absorbed elsewhere.}
\footnote{The dual space \(F^{*}\) and this pairing will only be used in this section and section 5.}
\footnote{The formula for it on page 194 in [6] unfortunately contains a misprint. Equation (2.11) here is a correction of it.}
Remark 1. The prescription [10] applied to the n-site setting leads to the four solutions of the Yang–Baxter equation \( R(z) = \mathcal{R}^{1,2}(z) \) on \( F \otimes F \otimes F \) whose elements are given by

\[
\mathcal{R}^{i,j}_{a,b} = \sum_{c_{1},...,c_{n} = 0} a \left( \frac{q^{2}}{q^{2}_{a}} \right)_{c_{0}} \prod_{l=1}^{n} \mathcal{R}^{a,b}_{c_{l}c_{l+1}} \chi_{1}^{c_{l}c_{l+1}}.
\]

Let us turn to the vectors \(|\chi_{1}(x, y)\rangle\) and \(|\chi_{2}(x, y)\rangle\) in (2.2)–(2.5). We use two such vectors obtained in [10]. In the present notation, they read as

\[
|\chi_{1}(z)\rangle = \sum_{m \geq 0} \frac{\zeta_{m}}{(q^{2})_{m}} |m\rangle, \quad |\chi_{2}(z)\rangle = \sum_{m \geq 0} \frac{\zeta_{m}}{(q^{2})_{m}} |2m\rangle,
\]

where \(|\chi_{1}(z)\rangle\) and \(|\chi_{2}(z)\rangle\) are the matrices acting on \( F \otimes F \) whose elements are given by

\[
\mathcal{R}^{i,j}_{a,b} = \sum_{c,k \geq 0} \frac{\zeta^{c}_{k} (q^{2})_{a}}{(q^{2})^{c}_{k}} \mathcal{R}^{a,b}_{i,j,k}.
\]

Due to (2.11), this is zero unless \( a + b = i + j \) and the sum is actually a single one due to the constraint \( b + sc = j + tk \). It follows that \( \mathcal{R}^{1,2} \) is decomposed as

\[
\mathcal{R}^{1,2}(z) = \mathcal{R}^{1,2}(z_{+}) \oplus \mathcal{R}^{1,2}(z_{-}) \oplus \mathcal{R}^{1,2}(z_{+}) \oplus \mathcal{R}^{1,2}(z_{-}),
\]

\[
\mathcal{R}^{1,2}(z) \in \text{End} (F^{1} \otimes F^{2}), \quad F^{k} = \bigoplus_{m \geq 0, c \geq -1} (q^{2})_{m} |m\rangle.
\]

It implies \( \mathcal{R}^{1,2}(z) : F^{1} \otimes F^{2} \to F^{2} \otimes F^{1} \). For example \( \mathcal{R}^{1,2}(z) \) is just the submatrix of \( \mathcal{R}^{1,2}(z_{a,b}) \) with the indices \( a, j \) restricted to be odd and \( b, i \) to be even. Another notable fact is that

\[
\mathcal{R}^{1,2}(z_{a,b}) = \left( q^{2}_{a} \right)^{i} (q^{2})_{b} \left( q^{2}_{a} \right)^{j} \mathcal{R}^{1,2}(z_{a,b})
\]

which can easily be derived from the property of \( \mathcal{R}^{1,2}(z_{a,b}) \) mentioned after (2.11). Henceforth, we concentrate on \( \mathcal{R}^{1,2}(z) \) and \( \mathcal{R}^{1,2}(z) \) in the rest of the paper.

Remark 1. The prescription [10] applied to the n-site setting leads to the four solutions of the Yang–Baxter equation \((s, t = 1, 2)\) acting on \( F^{s,n} \otimes F^{t,n} \) whose elements are given by

\[
\mathcal{R}^{s,t}(z)_{a,b} = \sum_{c_{0},...,c_{n} = 0} \frac{\zeta^{c_{0}}_{0} (q^{2})_{c_{0}}}{(q^{2})_{c_{0}}} \prod_{l=1}^{n} \mathcal{R}^{a,b}_{c_{l}c_{l+1}}.
\]
where $a = (a_1, \ldots, a_n)$ etc. The quantity (2.15) corresponds to the $n = 1$ case. In the more general problem formulated in section 5, this corresponds to $R^{\alpha,\beta}(z|\epsilon_1, \ldots, \epsilon_n)$ (5.8) with $\epsilon_1 = \cdots = \epsilon_n = 0$.

By a direct calculation, we have

**Proposition 2.** For $(s, t) \in \{(1, 1), (2, 2), (1, 2)\}$, the following formula is valid:

$$R^{\alpha,\beta}(z)_{ij} = \begin{pmatrix} \delta^{(1)}_{i+j} & \delta^{(1)}_{i+j} \end{pmatrix} R^{\alpha,\beta}(z|\epsilon_1, \ldots, \epsilon_n) \sum_{m,n,\lambda,\mu,\lambda+\mu=b} (-1)^{(k-1)(m+\lambda+n+\mu)} q^\phi \times \left( \binom{b-j+i}{m} q^\phi \left( \min(b, j) - \lambda \right) q^\phi \left( i \right) q^\phi \left( j \right) \right) \times \left( (-1)^{z} q^{(s+zt)} \epsilon(\lambda-\mu+\lambda+\mu+2n+\epsilon); q^\phi \right)_\infty,$$

$$\phi = \frac{m}{2} (k^2 m - \kappa + 2) + n(n + 2(b - j) + 1) - i \min(b, j) + \mu^2 + \lambda + (\lambda - \mu)(b - j) + \varepsilon(4m + 2n + \lambda - \mu + i),$$

$$\kappa = \begin{cases} 2 & \text{if } (s, t) = (1, 2), \ b \geq j, \ \varepsilon = 1 \text{ if } (s, t) = (1, 2), \ b - j \in 2\mathbb{Z}_{>0}, +1, \\ 0 & \text{otherwise,} \end{cases}$$

The sum in (2.19) is over $m, n, \lambda, \mu \geq 0$ with the constraint $\lambda + \mu = b$. It is a finite sum due to the support property of the q-binomial coefficients. The second Kronecker delta postulates $b \equiv j \pmod{2}$ when $s = 2$, which guarantees $(j - b - \epsilon)/s, \ (b - j - \varepsilon)/k \in \mathbb{Z}$.

Denoting the scalar function $\varrho(z)$ in (2.8) and (2.12) for $R^{\alpha,\beta}(z)$ by $q^{\varrho}(z)$, we choose it as

$$q^{\varrho}(z) = \begin{pmatrix} z^1; \ q^\phi \end{pmatrix}_{\infty} q^{\epsilon_1,\epsilon_2},$$

where for $(s, t) = (2, 2)$, the signs $\epsilon_1, \epsilon_2$ are to be taken according to the four components in (2.16). Then it follows from Proposition 2 that the matrix elements of $R^{\alpha,\beta}(z)$ are rational functions of $q$ and $z$. They are useful for computer checks of the Yang–Baxter equation (2.9).

**Example 3.** Let $M^{\alpha,\beta}_{ij} = R^{\alpha,\beta}(z|\epsilon_i,\epsilon_j)$ be the matrix where $i$ and $j$ are the row and the column indices taking $(0, 0)$ at the top left corner. One has $M^{\alpha,\beta}_{00} = (1)$ for any $s, t$:

$$M^{1,1} = \begin{pmatrix} \frac{(q^{\alpha+1})(q^{\beta+1})}{(q^{\alpha+1})(q^{\beta+1})} & \frac{1 - z}{q^{\alpha+1}} \\ \frac{q^{\alpha+1}}{q^{\beta+1}} & \frac{1}{q^{\alpha+1}} \end{pmatrix}, \quad M^{1,2} = \begin{pmatrix} \frac{(q^{\alpha+1})(q^{\beta+1})}{(q^{\alpha+1})(q^{\beta+1})} & \frac{1 - z}{q^{\alpha+1}} \\ \frac{q^{\alpha+1}}{q^{\beta+1}} & \frac{1}{q^{\alpha+1}} \end{pmatrix}, \quad M^{2,1} = \begin{pmatrix} \frac{q^{\alpha+1}}{q^{\beta+1}} & \frac{q^{\alpha+1}}{q^{\beta+1}} \\ \frac{q^{\alpha+1}}{q^{\beta+1}} & \frac{1}{q^{\alpha+1}} \end{pmatrix}, \quad M^{2,2} = \begin{pmatrix} \frac{1}{q^{\alpha+1}} & \frac{1}{q^{\alpha+1}} \\ 0 & \frac{1}{q^{\alpha+1}} \end{pmatrix}.$$
Proposition 4. \[ M_2^{2,2} = \begin{pmatrix} \frac{q(z^2-1)}{q(z^2-1)} & 0 & \frac{z-1}{q(z^2-1)} \\ 0 & \frac{q(z-1)}{q(z^2-1)} & 0 \\ \frac{q(z^2-1)}{q(z^2-1)} & 0 & \frac{z-1}{q(z^2-1)} \end{pmatrix}. \]

\[ M_3^{2,2} = \begin{pmatrix} 0 & \frac{z-q^2}{(z-1)(q^2z-1)} & 0 \\ \frac{q(z^2-1)}{(z-1)(q^2z-1)} & 0 & \frac{z-q^2}{(z-1)(q^2z-1)} \\ 0 & \frac{q(z^2-1)}{(z-1)(q^2z-1)} & 0 \end{pmatrix}. \]

3. Quantum R matrices for infinite-dimensional modules

The quantum affine algebras (without the derivation operator) \( U_q(A_1^{(1)}) \) and \( U_q(A_2^{(2)}) \) are the Hopf algebras generated by \( e_i, f_i, k_i^{\pm 1} \) (\( i = 0, 1 \)) satisfying the relations

\[ k_0 k_1^{-1} = k_1^{-1} k_0 = 1, \quad [k_i, k_j] = 0, \]

\[ k_i e_i k_i^{-1} = q_i^{a_{ij}} e_i, \quad k_i f_i k_i^{-1} = q_i^{-a_{ij}} f_i, \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \]

\[ \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu e_i^{(1-a_{ij}-\nu)} e_j^{(\nu)} = 0, \quad \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu f_j^{(1-a_{ij}-\nu)} f_i^{(\nu)} = 0 \quad (i \neq j), \]

where \( e_i^{(\nu)} = e_i^{\nu} / [\nu]_0 !, \quad f_i^{(\nu)} = f_i^{\nu} / [\nu]_0 !, \quad (q_0, q_1) = (q, q) \) for \( A_1^{(1)} \) and \( (q^2, q) \) for \( A_2^{(2)} \). The \((a_{ij})_{0\leq i, j\leq 1}\) is the Cartan matrix:

\[ (a_{ij})_{0\leq i, j\leq 1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \text{ for } A_1^{(1)}, \quad \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \text{ for } A_2^{(2)}. \]

We use the coproduct of the form

\[ \Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta e_i = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i. \]

Let us introduce a \( U_q(A_1^{(1)}) \)-module structure on the space \( F \).

Proposition 4. The following defines a \( U_q(A_1^{(1)}) \)-module structure on \( F \):

\[ e_0 |m\rangle = -|\alpha q^{-m-1}|m-1\rangle, \quad f_0 |m\rangle = |m+1\rangle, \quad k_0 |m\rangle = q^{-m} |m\rangle, \]

\[ e_0 = x f_1, \quad f_0 = x^{-1} e_1, \quad k_0^{\pm 1} = k_i^{\pm 1}, \]

\[ (3.1) \]

where \( \alpha \) and \( x \) are nonzero parameters.

For generic \( \alpha \) it is irreducible, which will be denoted by \( V_\alpha(\alpha) \).

Remark 5. As a module over \( U_q(sl_2) \) generated by \( e_1, f_1, k_1 \), the space \( F \) is identified with a Verma module, namely, it is generated by \( f_1 \) from an eigenvector \( |0\rangle \) of \( k_1 \) killed by \( e_1 \). Such a module is already known in [7], although eigenvalues are special there.

Proposition 6. The following defines an irreducible \( U_q(A_2^{(2)}) \)-module structure on \( F \):

\[ e_0 |m\rangle = -\varepsilon_0 \frac{|2m| |2m-2|}{[4]^2} |m-2\rangle, \quad f_0 |m\rangle = |m+2\rangle, \quad k_0 |m\rangle = \varepsilon_0 q^{-2m-2} |m\rangle, \]

\[ e_1 |m\rangle = x (m+1), \quad f_1 |m\rangle = \varepsilon_1 (-1)^m x^{-1} \frac{|2m|}{[2]} |m-1\rangle, \quad k_1 |m\rangle = \varepsilon_1 (-1)^m q^{2m+1} |m\rangle \]

\[ (3.2) \]

where \( \varepsilon_0^3 = \varepsilon_1^2 = 1 \) and \( x \) is a nonzero parameter.
Remark 7. As a module over $U_q(sl_2)$ generated by $e_0, f_0, k_0$, the space $F$ is decomposed into two components $F^+$ and $F^-$ in (2.17). Both are Verma modules but with different eigenvalues.

In what follows we will be exclusively concerned with the case $\epsilon_0 = \epsilon_1 = 1$, which will be denoted by $V_x$. Note that we have assigned $x$ (the spectral parameter) to the color 1 generators rather than color 0 for $U_q(A_2^{(2)})$. As the vector spaces, $V_x(\alpha)$ and $V_x$ are the same as $F$.

The quantum $R$ matrix $\tilde{R}(z) = \tilde{R}(z, q|\alpha, \beta)$ for our modules of $U_q(A_1^{(1)})$, $U_q(A_2^{(2)})$ is a linear operator

$$\tilde{R}(z) : V_x(\alpha) \otimes V_y(\beta) \to V_y(\beta) \otimes V_x(\alpha) \quad (z = x/y),$$

(3.3)

where $\alpha, \beta$ dependence should be dropped for $U_q(A_2^{(2)})$. The $R$ matrix is characterized by the intertwining relations [5]

$$\Delta(g) \tilde{R}(z) = \tilde{R}(z) \Delta(g) \quad (\forall g \in U_q)$$

(3.4)

and the normalization which we choose as $\tilde{R}(z) |0, 0\rangle = |0, 0\rangle$. Define its matrix elements by

$$\tilde{R}(z)_{a, b}^{(i, j)} = \sum_{a, b} \tilde{R}(z)_{a, b}^{(i, j)}|a, b\rangle,$$

(3.5)

where the sum is over those $a, b \geq 0$ satisfying $a + b = i + j$ due to the weight conservation implied by (3.4) with $g = k_0, k_1$.

We include a description of the quantum $R$ matrices in terms of their spectral decomposition. For $U_q(A_1^{(1)})$, we set

$$v_{a, b}^{(d)} = \sum_{j=0}^{d} (-q_1^{-d}b^{-1})^j \left(\frac{d}{d}ight) \prod_{k=1}^{j} \left(\frac{a_1^{1-d}}{\beta q^{1-d}}\right) (d - j, j) \in V_x(\alpha) \otimes V_y(\beta).$$

(3.6)

Up to an overall constant, this vector is characterized by the conditions

$$\Delta k_1 v_{a, b}^{(d)} = (a_1^{-1}q q_{d} a_{1^{-1}}^{1-d}) v_{a, b}^{(d)}, \quad \Delta e_1 v_{a, b}^{(d)} = 0.$$ 

(3.7)

Proposition 8 ($U_q(A_1^{(1)})$ case). The following direct sum decomposition holds as the module over $U_q(sl_2)$ mentioned in remark 5:

$$V_x(\alpha) \otimes V_y(\beta) = \bigoplus_{d \geq 0} V^{(d)} \otimes |\beta\rangle,$$

(3.8)

On each subspace $V^{(d)}$, $\tilde{R}(z) = \tilde{R}(z, q|\alpha, \beta)$ acts diagonally as $(z = x/y)$

$$\tilde{R}(z)_{a, b}^{(d)} v_{a, b}^{(d)} = \delta_{a, b}^{(d)} \alpha_{a, b}^{(d)} (\Delta f_1)^{d} v_{a, b}^{(d)}, \quad \alpha_{a, b}^{(d)} (z) = \left(\frac{\beta z}{\alpha} \right)^{d} (2q_1, q_1^{d} a_{1^{-1}}^{1-d}) a_1^{1-d} (2q_1^{d} \beta z^{1-d}, q_1^{d}).$$

(3.9)

The $R$ matrix satisfies the Yang–Baxter equation (common $q$-dependence is suppressed)

$$\tilde{R}(y/z|\beta, \gamma) \otimes 1 \cdot (1 \otimes \tilde{R}(x/y|\alpha, \beta)) \cdot (\tilde{R}(x/z|\alpha, \gamma) \otimes 1)$$

$$= (1 \otimes \tilde{R}(x/y|\alpha, \beta)) \cdot (\tilde{R}(x/z|\alpha, \gamma) \otimes 1) \cdot (1 \otimes \tilde{R}(y/z|\beta, \gamma)),$$

(3.10)

which is an equality of the maps $V_x(\alpha) \otimes V_y(\beta) \otimes V_z(\gamma) \to V_z(\gamma) \otimes V_y(\beta) \otimes V_x(\alpha)$. The inversion relation $\tilde{R}(z|\alpha, \beta) \tilde{R}(z^{-1}|\beta, \alpha) = Id$ is valid.

For $U_q(A_2^{(2)})$, we set

$$u_{-}^{(d)} = \sum_{j=0}^{d} (-1)^{j}(j+1)/2 \left(\frac{d}{d}j\right) (j, d - j) \in V_x \otimes V_y.$$ 

(3.11)

Thus, $u_{+}^{(0)} = u_{-}^{(0)} = |0, 0\rangle$. They are characterized by the conditions

$$\Delta k_0 u_{+}^{(d)} = q^{d-4} u_{-}^{(d)}, \quad \Delta e_0 u_{+}^{(d)} = 0, \quad \tilde{R}(z) u_{+}^{(d)} = (\pm 1)^{d} \alpha^{(d)}(\pm z) u_{\pm}^{(d)},$$

(3.12)

where $\alpha^{(d)}(z)$ is specified in (3.14) and $z = x/y$. 

7
Proposition 9 $(U_q(A_2^{(2)}))$ case. The following direct sum decomposition holds as the module over $U_q(sl_2)$ mentioned in remark 7 ($V_+^{(0)} = V_-^{(0)}$ is denoted by $V^{(0)}$):

\[
V_+ \otimes V_- = V^{(0)} \oplus \bigoplus_{d \geq 1} (V^{(d)}_+ \oplus V^{(d)}_-), \quad V^{(d)}_{\pm} = \bigoplus_{r \geq 0} \mathbb{C}(q)(\Delta f_0) y^{(d)}_{\pm}, \tag{3.13}
\]

On each subspace $V^{(d)}_{\pm}$, $\bar{R}(z) = \tilde{R}(z, q)$ acts diagonally as ($z = x/y$)

\[
\tilde{R}(z)(\Delta f_0) y^{(d)}_{\pm} = (\pm 1)^d \sigma^{(d)}(\pm z)(\Delta f_0) y^{(d)}_{\pm}, \quad \sigma^{(d)}(z) = \prod_{m=1}^{d} q_{2m-1} + (\pm 1)^{m-1} z^2. \tag{3.14}
\]

The $R$ matrix satisfies the Yang–Baxter equation (3.10) without dependence on $\alpha, \beta, \gamma$. The inversion relation $\bar{R}(z) \bar{R}(z^{-1}) = \text{Id}$ is valid.

4. Main theorem

In section 2, we derived the solutions $\tilde{R}^{\alpha, \beta}(z) = \tilde{R}^{\alpha, \beta}(z, q)$ to the Yang–Baxter equation by a reduction of the 3d $\mathcal{R}$. In section 3, the quantum $R$ matrices of the rank 1 quantum affine algebras were determined in terms of their spectral decompositions. Both of these matrices act on the infinite-dimensional space $F \otimes F$. Our main theorem presented below identifies them up to a scalar multiple and a similarity transformation.

Theorem 10.

(i) The $\tilde{R}^{1,1}(z)$ equals a similarity transformation of the $U_q(A_1^{(1)})$ $R$ matrix $\tilde{R}(z, q|\alpha, \beta)$ specialized as

\[
\tilde{R}^{1,1}(z, q^2)_{i,j} = (-iq)^{j-i} \tilde{R}(z, q)_{i,j}, \quad (i = in - iq) \text{ in } \mathbb{C}(q), \tag{4.1}
\]

where $i$ in $-iq$ means $\sqrt{-1}$ and is unrelated to the matrix indices.

(ii) The components $\tilde{R}^{\pm,\pm}(z)$ and $\tilde{R}^{\pm,\mp}(z)$ of $\tilde{R}^{\alpha, \beta}(z)$ are proportional to a similarity transformation of the $U_q(A_1^{(1)})$ $R$ matrix $\tilde{R}(z, q|\alpha, \beta)$ specialized as

\[
\tilde{R}^{\pm,\pm}(z, q)_{i,j} = \tilde{R}^{\pm,\pm}(z, q^2)_{i,j}, \tag{4.2}
\]

\[
r^{\pm,\pm} = 1, \quad r^{\pm,\mp} = \frac{q}{z-1}, \quad r^{-,\pm} = \frac{1}{1-z}, \quad r^{-,\mp} = \frac{q^2}{q^2z-1}, \tag{4.3}
\]

where $\tilde{n}$ denotes the largest integer not exceeding $\frac{n}{2}$.

(iii) The $\tilde{R}^{1,2}(z)$ equals a similarity transformation of the $U_q(A_2^{(2)})$ $R$ matrix $\tilde{R}(z, q)$ as

\[
\tilde{R}^{1,2}(z, -q^{\mp})_{i,j} = q^{j-i} \tilde{R}(z, q)_{i,j}, \tag{4.4}
\]

In (ii) of the theorem, the general formula for $r^{\epsilon_1, \epsilon_2}$ is $\tilde{R}^{\epsilon_1, \epsilon_2}(z, q), \tilde{R}^{\epsilon_1, \epsilon_2}(q^2)$, which can be found in example 3. The factors of the form $p^{j-h}$ are attributed to the similarity transformation by the operator $1 \otimes K_p$ with $K_p(q) = p^m(m)$, which does not spoil the Yang–Baxter equation. Combined with proposition 2, theorem 10 provides an explicit formula for the quantum $R$ matrices of our infinite-dimensional modules of the rank 1 quantum affine algebras.

Let $M_d^{(j)}(q)$ be the $(d+1) \times (d+1)$ matrix $M_d^{(j)}$ introduced in example 3 exhibiting the $q$-dependence. We close the section with a corollary of theorem 10 giving the eigenvalues of $M_d^{1,1}(q^2)$, $M_d^{2,2}(q)$, and $M_d^{1,1}(q^2)$.

7 The eigenvalues of $M_d^{2,2}(q)$ with odd $d$ are not directly derivable from (3.9) since the matrix consists of the sub-matrices $\tilde{R}^{\epsilon_1, \epsilon_2}(z)$ with $\epsilon_1 \epsilon_2 = -1$ which correspond, due to (4.2), to the situation $\alpha \neq \beta$ in (3.9).
Corollary 11. (i) The eigenvalues of $M_{d}^{1,1}(q^{2})$ are given by the specialization of $\sigma^{(j)}_{\alpha,\beta}(z)$ (3.9) as
\[
\{1, \sigma^{(1)}_{-1q,-1q}(z), \ldots, \sigma^{(d)}_{-1q,-1q}(z)\}, \quad \sigma^{(j)}_{-1q,-1q}(z) = \prod_{m=1}^{j} \frac{z + q^{2m}}{1 + zq^{2m}},
\]
(ii) The eigenvalues of $M_{2d}^{2,2}(q)$ are given by the specialization of $\sigma^{(j)}_{\alpha,\beta}(z)$ (3.9) as
\[
\{1, \tilde{\sigma}^{(1)}(z)^{\times 2}, \ldots, \tilde{\sigma}^{(d)}(z)^{\times 2}\}, \quad \tilde{\sigma}^{(j)}(z) = \sigma^{(j)}_{\alpha,\beta}(z)|_{q=1,\alpha=\beta=q} = \prod_{m=1}^{j} \frac{z - q^{4m-2}}{1 - zq^{4m-2}}.
\]
where the superscript $\times 2$ stands for twofold degeneracy. (iii) The eigenvalues of $M_{d}^{1,2}(-q^{2})$ are given by $\sigma^{(j)}(z)$ (3.14) as
\[
\{\pm \sigma^{(1)}(z), \pm \sigma^{(2)}(z), \ldots, \pm \sigma^{(d)}(z)\} \quad \text{if } d \text{ is odd,}
\]
\[
\{1, \sigma^{(2)}(z), \sigma^{(4)}(z), \ldots, \sigma^{(d)}(z)\} \quad \text{if } d \text{ is even.}
\]
These results can be directly checked for small $d$ by using example 3.

5. Generalizations

The result in this paper is regarded as the solution of a special case of a more general problem, which we shall now explain. First, we recall the 3d $L$ operator [4] in the form adapted to the present context. Let $F = \bigoplus_{m \geq 0} \mathbb{Q}(q)|m\rangle$ be the Fock space and $F^{*}$ be its dual as before. Set $V = \mathbb{Q}(q)v_{0} \oplus \mathbb{Q}(q)v_{1}$. By the 3d $L$ operator we mean the following:
\[
\mathcal{L} = (\mathcal{L}^{\alpha,\beta}_{\gamma,\delta}) \in \text{End}(V \otimes V \otimes F), \quad \mathcal{L}(v_{\gamma} \otimes v_{\delta} \otimes |m\rangle) = \sum_{\gamma,\delta} v_{\gamma} \otimes v_{\delta} \otimes \mathcal{L}^{\alpha,\beta}_{\gamma,\delta}|m\rangle,
\]
where there are six nonzero $\mathcal{L}^{\alpha,\beta}_{\gamma,\delta} \in \text{End}(F)$ given by
\[
\mathcal{L}^{0,0}_{0,0} = 1, \quad \mathcal{L}^{0,1}_{0,1} = k, \quad \mathcal{L}^{0,0}_{1,0} = -qk, \quad \mathcal{L}^{1,0}_{0,0} = a^{-}, \quad \mathcal{L}^{1,0}_{0,1} = a^{+}.
\]
The operators $a^{\pm}, k \in \text{End}(F)$ are called $q$-oscillators and act on $F$ by
\[
a^{+}|m\rangle = |m + 1\rangle, \quad a^{-}|m\rangle = (1 - q^{2m})|m - 1\rangle, \quad k|m\rangle = q^{m}|m\rangle.
\]
In short, the 3d $L$ operator $\mathcal{L}$ represents a 6-vertex model having the $q$-oscillator valued Boltzmann weights. It satisfies the tetrahedron equation [4]:
\[
\mathcal{R}_{1,2,3}\mathcal{L}_{b,c,2}\mathcal{L}_{a,c,3}\mathcal{L}_{a,b,1} = \mathcal{L}_{a,b,1}\mathcal{L}_{a,c,2}\mathcal{L}_{b,c,3}\mathcal{R}_{1,2,3}.
\]
This is an equality in $\text{End}(V \otimes V \otimes V \otimes F \otimes F \otimes F)$, where $V, V, V$ are the copies of $V$ and $F, F, F$ are those for $F$. The indices of $\mathcal{R}$ and $\mathcal{L}$ signify the components of the tensor product on which these operators act non-trivially. Viewed as an equation on $\mathcal{R}$, (5.4) is equivalent [8] to the intertwining relation of the irreducible representations of the quantized coordinate ring $A_{q}(sl_{3})$ [6] in the sense that both lead to the same solution given in (2.11) up to an overall normalization.

Next, we introduce the notation allowing us to treat $\mathcal{R}$ and $\mathcal{L}$ on an equal footing:
\[
W^{(0)} = F, \quad W^{(1)} = V, \quad S^{(0)} = \mathcal{R}, \quad S^{(1)} = \mathcal{L}.
\]
The $S^{(r)}$ act on $W^{(r)} \otimes W^{(r)} \otimes F$ for $r = 0, 1$. In what follows the copy of the Fock space $F$ that plays a role analogous to the auxiliary space of the transfer matrices will be designated by ‘3’. Define $h_{3}$ acting on it by $h_{3}|m\rangle = m|m\rangle$. 

J. Phys. A: Math. Theor. 46 (2013) 485203
Let $n$ be any positive integer. Consider the copies of $W^{(i)}$ denoted by $\mathcal{R}^{(i)}$ and $\tilde{\mathcal{R}}^{(i)}$ for $i = 1, 2, \ldots, n$. We write their tensor product as

$$\mathcal{R}^{(i)} = \mathcal{R}^{(i)}_1 \otimes \cdots \otimes \mathcal{R}^{(i)}_n$$

and similarly for $\tilde{\mathcal{R}}^{(i)}$ and $\tilde{\mathcal{R}}^{(i)}$. The labels $\alpha$, $\beta$ and $\gamma$ of the copies are just for distinction and these spaces are the same as $W^{(1)} \otimes \cdots \otimes W^{(n)}$ as vector spaces.

We introduce the five families of $R$ matrices each consisting of the $2^n$ members labeled with $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{0, 1\}^n$ as follows:

$$\mathcal{R}^{(i)}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \text{Tr}_3 \left( Z^{\varepsilon_1}_{s_1, s_2, s_3} Z^{\varepsilon_2}_{s_4, s_5, s_6} \cdots Z^{\varepsilon_n}_{s_{2n-1}, s_{2n-2}, s_{2n-3}} \right).$$

Here, the bracket is evaluated in the Fock space 3 by $\langle m|m' \rangle = \delta_{m,m'}(q^n)_m$ and the trace is $\text{Tr}_3(X) = \sum_{m,n} \langle m|X|n \rangle (q^n)_m$. They are linear operators acting on $\mathcal{R}^{(i)} \otimes \tilde{\mathcal{R}}^{(i)}$. By construction and the tetrahedron equations (2.1) and (5.4) together with the properties (2.3) and (2.5), we have

**Theorem 12.** Denote any one of (5.7) and (5.8) with a fixed $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ by $\mathcal{R}_{\alpha, \beta}(z)$. Then it satisfies the Yang–Baxter equation in $\text{End}(\mathcal{R}^{(i)} \otimes \tilde{\mathcal{R}}^{(i)} \otimes \tilde{\mathcal{R}}^{(i)})$:

$$\mathcal{R}_{\alpha, \beta}(x)\mathcal{R}_{\alpha, \gamma}(xy)\mathcal{R}_{\beta, \gamma}(y) = \mathcal{R}_{\beta, \gamma}(y)\mathcal{R}_{\alpha, \gamma}(xy)\mathcal{R}_{\alpha, \beta}(x).$$

These solutions $\mathcal{R}_{\alpha, \beta}(z)$ to the Yang–Baxter equation are rational functions of the parameter $q$ and the spectral parameter $z$ up to an overall scalar function of $z$. Thus, it is natural to seek their origin in conventional quantum group theory. We formulate it as

**Problem.** Find the appropriate quantum affine algebra and its (possibly infinite-dimensional) representation by which $\mathcal{R}_{\alpha, \beta}(z)$ is characterized as the intertwiner of the tensor product up to a normalization. So far it has been studied in the following cases. (The two $q$s entering $\mathcal{R}_{\alpha, \beta}(z)$ and $U_q$ are not necessarily the same.)

(i) $\mathcal{R}^{(i)}(z, 0, 0, \ldots, 0)$ was claimed to be the direct sum $\oplus_{\varepsilon, \delta, \omega} R_{\varepsilon, \delta, \omega}$ of the $U_q(A_{n-1}^{(1)})$ quantum $R$ matrices for the symmetric representations (relative normalization of the summands left unspecified) [4].

(ii) $\mathcal{R}^{(i)}(z, 1, 1, \ldots, 1)$ with $(s, t) = (2, 1), (2, 2)$ and $(1, 1)$ were identified [10] with the quantum $R$ matrices for the spin representations of $U_q(B_n^{(1)})$, $U_q(D_{n-1}^{(1)})$ [11] and $U_q(D_{n-1}^{(1)})$, respectively. It was also suggested [10] that the case $(s, t) = (1, 2)$ is the quantum $R$ matrix of the spin representation of $U_q(B_n^{(1)})$ corresponding to the realization of $B_n^{(1)}$ as an affinization of its classical subalgebra $D_n$ rather than the standard $B_n$.

(iii) $\mathcal{R}^{(i)}(z, 0)$ with $(s, t) = (1, 1), (2, 2)$ (resp. $(1, 2), (2, 1)$) are identified with the $U_q(A_{n-1}^{(1)})$ (resp. $U_q(A_n^{(1)})$) quantum $R$ matrices for the infinite-dimensional representations corresponding to an affinization of the Verma module of their classical subalgebras $U_q(B_n)$. It is consistent with (4.1) in view of $D_2^{(1)} = A_1^{(1)}$ and is also natural from [10, remark 7.2].
It is interesting to investigate the effect of the mixture of $\mathcal{R}$ and $\mathcal{L}$ in (5.7) and (5.8) which is first formulated here explicitly. We finish by presenting an explicit formula of $\mathcal{R}^\mu(z|0,1)$ as the simplest example.

Note the decomposition $W^{(0,1)} = \bigoplus_{d \geq 0} W^{(d)}$, where $W^{(0)} = \mathbb{Q}(q)|0\rangle \otimes v_0$ and $W^{(d)} = \mathbb{Q}(q)|d\rangle \otimes v_0 \otimes \mathbb{Q}(q)|d-1\rangle \otimes v_1$ for $d \geq 1$. Accordingly, $\mathcal{R}^\mu(z|0,1)$ splits into the direct sum of the matrices acting on $W^{(d)} \otimes W^{(d')}$. It turns out that they are zero unless $d = d' = 0$ or $dd' \geq 1$. Thus, $\mathcal{R}^\mu(z|0,1)$ consists of a $1 \times 1$ matrix and infinitely many $4 \times 4$ matrices corresponding to $\text{End}(W^{(d)} \otimes W^{(d')})$ with $dd' \geq 1$. Explicitly, it is expressed as

$$\mathcal{R}^\mu(z|0,1) = \frac{1}{1 - z} \text{Id}_{0,0} \otimes \bigoplus_{d,d' \geq 1} z^{d'-1}(q^{d-d'} + z')^{-1} (1 \otimes K^{-1}) \mathcal{R}_{d',d}^{\mu}(z)(K \otimes 1),$$

(5.10)

where $\text{Id}_{0,0}$ denotes the $1 \times 1$ identity matrix and $K(|m\rangle \otimes v_0) = q^{1-2m}|m\rangle \otimes v_0$ causes just a gauge transformation. $\mathcal{R}_{d',d}^{\mu}(z) \in \text{End}(W^{(d)} \otimes W^{(d')})$ is given by the specialization of

$$\mathcal{R}_{\mu,\nu}(z) = \begin{pmatrix} z - \mu \nu & 0 & 0 & 0 \\ 0 & v - \mu z & (1 - \nu^2)z & 0 \\ 0 & 1 - \mu z & \mu - vz & 0 \\ 0 & 0 & 0 & 1 - \mu vz \end{pmatrix}.$$

(5.11)

By this we mean the linear operator acting as $\xi_0 \otimes \eta_0 \mapsto (z - \mu \nu)\xi_0 \otimes \eta_0$, $\xi_0 \otimes \eta_1 \mapsto (v - \mu z)\xi_0 \otimes \eta_1 + (1 - \mu^2)\xi_1 \otimes \eta_0$, etc in terms of the basis $\xi_i \otimes \eta_j$ of $W^{(d)} \otimes W^{(d')}$ taken as $\xi_i = |d - i\rangle \otimes v_i$ and $\eta_j = |d' - j\rangle \otimes v_j$ for $i, j = 0, 1$. The case $\mu = \nu$ is known to be the intertwiner of the quantum affine super algebra $U_q(\widehat{\mathfrak{sl}(1|1)})$. The $\mathcal{R}_{\mu,\nu}(z)$ satisfies the Yang–Baxter equation $\mathcal{R}_{\lambda,\mu}(x)\mathcal{R}_{\lambda,\nu}(xy)\mathcal{R}_{\mu,\nu}(y) = \mathcal{R}_{\mu,\nu}(y)\mathcal{R}_{\lambda,\nu}(xy)\mathcal{R}_{\lambda,\mu}(x)$. Further results, including the detailed derivation and the proof of this paper, will appear elsewhere.

Acknowledgments

The authors thank Yasuhiko Yamada for collaboration in the previous work, Kailash C Misra and Yoshihisa Saito for communications on literature and Tatsuya Toyoda for a careful reading of the manuscript. AK thanks Vladimir Bazhanov, Vladimir Mangazeev and Sergey Sergeev for kind interest during his stay in Canberra in March 2012. In particular, he benefited from the collaboration with Sergeev in [10]. This work is supported by Grants-in-Aid for Scientific Research no. 23340007, no. 24540203 and no. 23654007 from JSPS.

References

[1] Baxter R J 2007 Exactly Solved Models in Statistical Mechanics (New York: Dover)
[2] Bazhanov V V and Stroganov Y G 1982 Conditions of commutativity of transfer matrices on a multidimensional lattice Theor. Math. Phys. 52 685–91
[3] Bazhanov V V, Mangazeev V V and Sergeev S M 2008 Quantum geometry of 3-dimensional lattices J. Stat. Mech. P07004
[4] Bazhanov V V and Sergeev S M 2006 Zamolodchikov’s tetrahedron equation and hidden structure of quantum groups J. Phys. A: Math. Gen. 39 3295–310
[5] Jimbo M 1985 A q-difference analogue of $U(q)$ and the Yang–Baxter equation Lett. Math. Phys. 10 63–9
[6] Kapranov M M and Voevodsky V A 1994 2-Categories and Zamolodchikov tetrahedron equations Proc. Symp. Pure Math. 56 177–259
[7] Kulish P P and Damaskinsky E V 1990 On the q oscillator and the quantum algebra, su_q(1, 1) J. Phys. A: Math. Gen. 23 L415–9
[8] Kuniba A and Okado M 2012 Tetrahedron and 3D reflection equations from quantized algebra of functions J. Phys. A: Math. Theor. 45 465206
[9] Kuniba A, Okado M and Yamada Y 2013 A common structure in PBW bases of the nilpotent subalgebra of $U_q(g)$ and quantized algebra of functions SIGMA 9 049

[10] Kuniba A and Sergeev S 2013 Tetrahedron equation and quantum $R$ matrices for spin representations of $B^{(1)}_n$, $D^{(1)}_n$ and $D^{(2)}_{n+1}$ Commun. Math. Phys. at press (arXiv:1203.6436)

[11] Okado M 1990 Quantum $R$ matrices related to the spin representations of $B_n$ and $D_n$ Commun. Math. Phys. 134 467–86

[12] Sergeev S M 2008 Tetrahedron equations and nilpotent subalgebras of $U_q(sl_n)$ Lett. Math. Phys. 83 231–5

[13] Zamolodchikov A B 1980 Tetrahedra equations and integrable systems in three-dimensional space Sov. Phys.—JETP 79 641–64