INVERSE RANDOM SOURCE PROBLEM FOR BIHARMONIC EQUATION IN TWO DIMENSIONS

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Abstract. The establishment of relevant model and solving an inverse random source problem are one of the main tools for analyzing mechanical properties of elastic materials. In this paper, we study an inverse random source problem for biharmonic equation in two dimension. Under some regularity assumptions on the structure of random source, the well-posedness of the forward problem is established. Moreover, based on the explicit solution of the forward problem, we can solve the corresponding inverse random source problem via two transformed integral equations. Numerical examples are presented to illustrate the validity and effectiveness of the proposed inversion method.

1. Introduction. Motivated by significant scientific and industrial applications, the field of inverse problems has undergone huge growth in the past few decades. Particularly, the inverse random source problem has progressed to a field of intense activity and are now in the forefront of mathematical research with far ranging applications. For instance, Bao et al initially investigated an inverse random source problem for one dimensional Helmholtz equation in [4] and generalized the result to several dimensions in [2]. The corresponding stability analysis was conducted by Li and Yuan in [12, 13] with multifrequencies data. Recently Bao et al investigated forward and inverse random source problems for elastic wave equations in [3]. Moreover, an inverse random source problem of the Euler-Bernoulli equation is utilized by Bao et al [5] to determine the unknown material properties, especially for those nanostructures, whose scale are from 1 to 100 nm. The corresponding convergence result was shown in [6]. In this paper, we will focus on an inverse random source problems on quantifying mechanical properties for two dimensional materials which can be viewed as an extension of Bao et al [5, 6].

Before proceeding to propose the new model, we review some progress on identifying the property of nanomaterials. Currently, the main direction of research for mechanical properties of the nanomaterials is on one dimension such as nanotubes or nanowires, since they are main functional components of complex devices. It is well known that small size not only brings magical improvement of the mechanical properties, such as exceptionally high Young’s modulus and ultralarge elastic deformation, but also a great challenge in developing and validating methods to evaluate...
these properties from direct measurements, see [7, 14, 17, 18, 19] and references therein. Recently, Mlinar [15] reviewed several inverse methods on quantifying these mechanical properties which could be utilized in designing semiconductors. These models and methods made a great success for one dimensional materials, however, to our best knowledge, little is known on for higher dimensional nanomaterials.

In this paper, we will investigate both the direct and inverse random source problems for biharmonic equation which describes the elastic deformation of two-dimensional materials. As a source, the force we give to the plane is considered as a random function driven by a colored noise which may degenerate to the white noise when the correlation function is delta function. Given the source, the direct problem is to solve a stochastic partial differential equation to determine the deformation of the plate. The inverse problem could be formulated to identify the coefficient in the equation and reconstruct the structure of the random source provided that additional measurement data could be collected on the displacement of the plate on source points. For the forward problem, constructing a sequence of regular processes to approach the colored noise, we show the existence of a unique mild solution to the two dimensional biharmonic equation. For the inverse problem, two Fredholm integral equations are derived by utilizing the expectation and variance of the mild solution. It is acknowledged that the first kind of the Fredholm integral equations are severely ill-posed, and to overcome the challenges of the ill-posedness, Tikhonov regularization is employed to solve the problem. Numerical experiments show that the proposed method is effective for solving the two-dimensional problems.

The outline of this paper is as follows. In section 2, we will introduce the two-dimensional biharmonic equation and discuss the solutions for the deterministic and stochastic problems. Sections 3 is devoted to the inverse problem, where Fredholm integral equations will be introduced and Tikhonov regularization is employed to reconstruct the mean and variance. Numerical experiments and results are presented in section 4 to illustrate the effectiveness of the proposed method. The paper is concluded with general remarks and further research in section 5.

2. Forward problem and formulation. In this section, we introduce the biharmonic equation in two dimensions and discuss the solutions of the problem under the circumstances of deterministic and stochastic direct source.

Before we discuss the two-dimensional biharmonic equation, we introduce the physical model about it. Consider a plate $\Omega$ with the width of $h$. The force acting on the plate is vertical, which results to the deflection $w$, just as Figure 1 shows below.

Assume $\varepsilon$ is the vector of the strain and $\sigma$ is the vector of the stress. $E, \nu$ are Young modulus and Poisson’s ratio respectively. By using the theory of elastic mechanics, we can get the relation between deflection, stress and strain as follows:

$$\sigma = c \varepsilon = -czPw,$$

where

$$P = \begin{bmatrix}
\frac{\partial^2}{\partial x^2} \\
\frac{\partial^2}{\partial y^2} \\
\frac{\partial^2}{\partial x \partial y}
\end{bmatrix}, \quad c = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1-\nu)/2
\end{bmatrix}.$$

Now we discuss one unit $dx \times dy$ with the width of $h$ of the plate $\Omega$. We know it is under the vertical force $F$ and the inertia force is 0 since the static state. Therefore, we can have the following equations.
The moment on the cross section of the unit, denoted as $M_p$, can be obtained by

$$M_p = \begin{cases} M_x \\ M_y \\ M_{xy} \end{cases} = \int_A \sigma z dA = -\frac{h^3}{12} c w,$$

where $A$ is the area of the cross section of the unit we discuss, and $h$ is the width.

Considering the static state, we have the following equilibrium equations of shear force $Q$:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + F = 0,$$

where $Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$, $Q_y = \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_y}{\partial y}$.

Combining (2.1) and (2.2) gives the equation that describes the relation between acting force $F$ and the deflection $w$:

$$F = \frac{\partial^2}{\partial x^2} \left[ D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right] + \frac{\partial^2}{\partial y^2} \left[ D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \right] + 2 \frac{\partial^2}{\partial x \partial y} \left[ D (1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \right],$$

where

$$D = \frac{E h^3}{12(1 - \nu^2)}.$$

For uniform materials, $E$, $\nu$ and hence $D$ are constants. Thus, we can re-written function(2.3) as follows when the plate is uniform:

$$D \left( \frac{\partial^4 w}{\partial x^4} + \frac{2 \partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = F,$$

or it can be simply denoted as $D \Delta^2 w = F$ which is usually called biharmonic equation.

When the plate is made of nano-materials, it can be easily affected by the environment. Therefore $F$ here is generally assumed to be a random function driven by an additive noise such as [5, 6]:

$$F(X) = f(X) + g(X) + h(X) dW_X, \quad X \in \Omega.$$
Here $X = (x, y)$, or $(r, \theta)$ when we are using the polar coordination. $f(X)$ is the given force; $g$ and $h \geq 0$ are two deterministic functions of real value which have compact supports contained in the bounded domain $\Omega \in \mathbb{R}^2$, and $W_X$ is a homogeneous colored noise. More details about the Brownian sheet, white noise, colored noise, and corresponding stochastic integrals can be found in Bao et al [2].

Taking the actual examples for nano-plate into consideration, the load force $F(X)$ is assumed to take the following form instead of (2.6):

\begin{equation}
F(X) = F_k \delta(X - X_0) + g(X) + h(X) dW_X.
\end{equation}

(2.7)

Here $F_k$ is the AFM contact force which is constant, and $X_0$ is the point where the force loaded.

In order to make it easier to analyze equation (2.5), the bounded domain $\Omega$ is assumed to be a disk with radius $R$, and the following numerical examples are calculated in the form of polar coordinates. In this circumstance, the laplace operator $\Delta$ could be denoted as

\begin{equation}
L u(r, \theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2},
\end{equation}

(2.8)

and (2.5) can be re-expressed as follows:

\begin{equation}
L^2 w = \frac{F}{D}.
\end{equation}

(2.9)

We assume the nano-plate is simply supported on the boundary, and therefore the moment

\begin{equation}
\begin{cases}
M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right), \\
M_y = -D \left( \nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right),
\end{cases}
\end{equation}

(2.10)

on the boundary in two directions should be vanishing and so should the displacement. Hence the boundary conditions are

\begin{equation}
\begin{aligned}
&w|_{X \in \partial \Omega} = 0, \\
&M_x + M_y = 0.
\end{aligned}
\end{equation}

(2.11)

Therefore two-dimensional biharmonic equation with boundary conditions can be expressed as:

\begin{equation}
\begin{cases}
L^2 w = \frac{F}{D}, & X \in \Omega, \\
\quad w = 0, & X \in \partial \Omega, \\
\quad M_x + M_y = 0, & X \in \partial \Omega.
\end{cases}
\end{equation}

(2.12)

For better analysis, we re-written (2.12) as follows:

\begin{equation}
\begin{cases}
L w = -\frac{M}{D}, \\
L M = -F, \\
\quad w|_{X \in \partial \Omega} = 0, \\
\quad M|_{X \in \partial \Omega} = 0,
\end{cases}
\end{equation}

(2.13)

with

\begin{equation}
M = \frac{1}{1 + \nu} (M_x + M_y),
\end{equation}

and therefore we can consider the problem as two coupled Poisson equations.
Before we discuss the analytical solution of the problem above, we make some justification of the equivalency between (2.12) and (2.13). It is clear that any solution of (2.13) is a solution of (2.12), so we only explain why the opposite holds.

We assume that \( w \) is the solution to the equation (2.12). We define \( \tilde{M} = Lw \), and consequently \( \tilde{M} = \frac{\partial}{\partial t} \). In addition, \( \tilde{M}|_{x \in \partial \Omega} = Lw|_{x \in \partial \Omega} = 0 \) for \( (M_x + M_y) = -D(1 + \nu) \Delta w = -D(1 + \nu)Lw \), and \( (M_x + M_y)|_{x \in \partial \Omega} = 0 \). It is obvious that \( M = -D\tilde{M} \), so we replace \( \tilde{M} \) with \( -\frac{M}{D} \) and have the exact form of the equations (2.13).

We begin with the solution for the deterministic direct problem, which means there is no randomness presented in the source, i.e., \( F(X) = F_k \delta(X - X_0) \).

The analytical solution of \( w(X) \) for the problem (2.12) can be obtained via the following Green function \( G(X,Y) \):

\[
\begin{align*}
  w(X) &= -\frac{1}{D} \int_{\Omega} G^*(X,Y)M(Y)\,dY \\
  &= \frac{1}{D} \int_{\Omega} G^*(X,Y) \int_{\Omega} G^*(Y,X')F(X')\,dX'\,dY \\
  &= \frac{1}{D} \int_{\Omega} \int_{\Omega} G^*(X,Y)G^*(Y',Y)dYF(X')\,dX' \\
  &= \frac{1}{D} \int_{\Omega} G(X,X')F(X')\,dX'.
\end{align*}
\]

The following Green function on a disk:

\[
G^*(X,Y) = \frac{1}{\pi} \left( \ln \frac{1}{|X-Y|} - \ln \frac{R}{\sqrt{R^4 + r_X^2 r_Y^2 - 2R^2 r_X r_Y \cos(\theta_X - \theta_Y)}} \right)
\]

(2.15)

Once we have the Green function, we could proceed to the stochastic case. Different from the one dimensional case, the explicit solution can not be generalized to a random source \( F(X) \) directly. Instead, we need to investigate the regularity result of Green’s function in advance which plays a significant important role in the subsequent analysis.

**Lemma 2.1.** Let \( \Omega \in \mathbb{R}^2 \) be a bounded domain. For fixed \( X' \in \Omega \), \( G(X,X') \in L^2(\Omega) \).

**Proof.** First we prove that it holds for any \( Y \in \Omega \) that \( G^*(X,Y) \in L^2(\Omega) \) for the fixed \( X \).

let \( \rho = \sup_{X,Y \in \Omega} |X - Y| \), and we have \( \tilde{\Omega} \subset B_\rho(Y) \). The Green function \( G^*(X,Y) \) can be expressed as:

\[
G^*(X,Y) = -\frac{1}{2\pi} \ln \frac{1}{|X-Y|} + V(X,Y),
\]

\[
V(X,Y) = \frac{1}{\pi} \ln \frac{1}{\sqrt{R^4 + r_X^2 r_Y^2 - 2R^2 r_X r_Y \cos(\theta_X - \theta_Y)}}.
\]
where $V$ is a Lipschitz continuous function. Hence, it suffices to show that
\[ \ln \frac{1}{|X - Y|} \in L^2(\Omega). \]

From some easy calculation we can obtain
\[ \int_\Omega |\ln \frac{1}{|X - Y|}|^2 dX \leq \int_{B_r(Y)} |\ln \frac{1}{|X - Y|}|^2 dX \lesssim \int_0^r |\ln \frac{1}{r}|^2 dr < \infty. \]

Throughout the paper, $a \lesssim b$ means $a \leq Cb$, where $C > 0$ is a constant. The specific value of $C$ is not required but should be clear from the context.

Now we can proceed to showing that $G(X, X') \in L^2(\Omega)$, $\forall X \in \Omega$, with a fixed $X'$. Direct calculation gives
\[ \int_\Omega |G(X, X')|^2 dX = \int_\Omega \left| \int_\Omega G^*(X, Y)G^*(X', Y)dY \right|^2 dX \]
\[ \leq \int_\Omega \left( \int_\Omega G^2(X, Y)dY \right) \left( \int_\Omega G^2(X', Y)dY \right) dX \]
\[ = \int_\Omega \left( \int_\Omega G^2(X, Y)dY \right) \left( \int_\Omega G^2(X', Y)dY \right) dX \]
\[ = \int_\Omega G^2(X', Y)dY \int_\Omega G^2(X, Y)dY dX. \]

Since $\Omega$ is a bounded domain, and $G^*(X', Y) \in L^2(\Omega)$, $\forall Y \in \Omega$. Consequently, $G(X, X') \in L^2(\Omega)$, $\forall X' \in \Omega$.

**Lemma 2.2.** Let $\Omega \in \mathbb{R}$ be a bounded domain. It holds for any $\alpha \in \left( \frac{1}{2}, \infty \right)$ that:
\[ (2.16) \quad \int_\Omega |G(X, Y) - G(X, Z)|^\alpha dX \lesssim |Y - Z|^\frac{\alpha}{2}, \quad \forall Y, Z \in \Omega. \]

**Proof.** In fact, this lemma holds for any $G^*(X, Y)$, which is:
\[ \int_\Omega |G^*(X, Y) - G^*(X, Z)|^\alpha dX \lesssim |Y - Z|^\frac{\alpha}{2}, \quad \forall Y, Z \in \Omega. \]

and the corresponding proof can be found in [2].

Now we proceed to proving Lemma 2.2. Direct computation gives
\[ |G(X, Y) - G(X, Z)| = \left| \int_\Omega G^*(X, X')|G^*(X', Y) - G^*(X', Z)|dX' \right| \]
\[ \leq \int_\Omega |G^*(X, X')| \times |G^*(X', Y) - G^*(X', Z)|dX' \]
\[ \leq \left( \int_\Omega |G^*(X, X')|^{\beta} dX' \right)^{\frac{1}{\beta}} \times \left( \int_\Omega |G^*(X', Y) - G^*(X', Z)|^{\alpha} dX' \right)^{\frac{1}{\alpha}}. \]

Here $\alpha$ and $\beta$ are a pair of associate numbers, i.e., $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

So we have
\[ \int_\Omega |G(X, Y) - G(X, Z)|^\alpha dX \]
\[ = \int_\Omega \left( \int_\Omega |G^*(X, X')|^{\beta} dX' \right)^{\frac{2}{\beta}} \times \left( \int_\Omega |G^*(X', Y) - G^*(X', Z)|^{\alpha} dX' \right)^{\frac{1}{\alpha}} dX \]
\[ \lesssim |Y - Z|^\frac{\alpha}{2} \int_\Omega \left( \int_\Omega |G^*(X, X')|^{\beta} dX' \right)^{\frac{2}{\beta}} dX. \]
For $\alpha \in (\frac{3}{2}, \infty)$, we have $\beta \in (1, 3)$. Hence, it is sufficient to show that $G^*(X, X') \in L^3(\Omega)$, $\forall X' \in \Omega$. From the Lemma 2.1, we know $G^*(X', Y) \in L^3(\Omega)$ if $\ln \frac{1}{|X-Y|} \in L^3(\Omega)$.

$$\int_\Omega \left| \ln \frac{1}{|X-Y|} \right|^3 dX \leq \int_{B_r(Y)} \left| \ln \frac{1}{|X-Y|} \right|^3 dX \lesssim \int_0^r \ln \frac{1}{r} dX < \infty.$$ 

For a bounded domain, we can easily obtain that:

$$\int_\Omega \left( \int_\Omega |G^*(X, X')|^2 dX' \right) \frac{2}{3} dX \leq M,$$

where $M$ is constant. Then we have:

$$\int_\Omega |G(X, Y) - G(X, Z)|^\alpha dX \lesssim |Y - Z|^2, \quad \forall Y, Z \in \Omega.$$ 

\[\square\]

Now we proceed to discussing the solution for the direct problem with random source. Consider the problem:

$$\begin{cases}
Lw = -\frac{M}{D}, \\
LM = -(Fk\delta(X - X_0) + g(X) + hdW_X),
\end{cases}$$

(2.17)

where the homogeneous colored noise $dW_x$ has a correlation function

$$c(X, Y) = \mathbb{E}(dW_X dW_Y) = c(X - Y), \quad \forall X, Y \in \mathbb{R}^2.$$ 

We assume that $c \in L^q_{loc}(\mathbb{R}^2)$ for some $q_0 \geq 1$.

Generally speaking, there are more than one type of commonly used correlation functions such as Delta kernel, Riesz kernel and heat kernel. More details can be referred to in the paper [2]. We make the following hypothesis on the coefficients $g$ and $h$ to guarantee the well-posedness of the solution of (2.17).

**Assumption 1.** Assume that $g \in L^2(\Omega)$ and $h \in L^p(\Omega)$, where $p \in (2, \infty]$. Moreover, we assume that $h \in C^{0, \eta}(\Omega)$, i.e., $\eta$-Hölder continuous, where $\eta \in (0, 1]$.

The assumption on $g \in L^2(D)$ should be motivated by the solution of the deterministic direct problem (2.14). The regularity of $h$ should be chosen to make the stochastic integral $\int_\Omega G(X, Y)h(Y)dW_Y$ well-posed, which means it satisfies

$$\mathbb{E}(\left| \int_\Omega G(X, Y)h(Y)dW_Y \right|^2)$$

(2.18)

$$= \int_\Omega \int_0^\infty G(X, Y)h(Y)c(Y - Z)G(X, Z)h(Z)dYdZ < \infty.$$ 

To show (2.18) is bounded from above, we introduce the following Young’s inequality for convolutions.

**Lemma 2.3.** Let $p, q, r \geq 1$ and suppose that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. It holds that

$$\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(X)v(X - Y)w(Y)dXdY \right| \leq \|u\|_p \|v\|_q \|w\|_r,$$

$$\forall u \in L^p(\mathbb{R}^2), v \in L^q(\mathbb{R}^2), w \in L^r(\mathbb{R}^2).$$
Applying Lemma 2.3 to the right hand side of (2.18) immediately yields:

$$\int_{\Omega} \int_{\Omega} G(X, Y) h(Y) c(Y - Z) G(X, Z) h(Z) dY dZ$$

$$\leq ||G(X, \cdot) h(\cdot)||_{L^{p_0}(\Omega)} ||c||_{L^{p_0}(B_{2R})},$$

where $p_0 = \frac{2q_0}{2q_0 - 1}$, i.e., satisfying $\frac{1}{p_0} + \frac{1}{p_0} + \frac{1}{q_0} = 2$. Because we have assumed $q_0 \geq 1$, then $p_0 \in [1, 2]$. Therefore, to show (2.18), it is sufficient to show that $||G(X, \cdot) h(\cdot)||_{L^{p_0}(\Omega)}$ is bounded from above. By utilizing Young’s inequality, it is clear that

$$||G(X, \cdot) h(\cdot)||_{L^{p_0}(\Omega)} = \int_{\Omega} G(X, Y)^{p_0} h^{p_0}(Y) dY$$

$$\lesssim \left( \int_{\Omega} (G^{p_0}(X, Y))^{q_1} dY \right)^{\frac{1}{q_1}} \left( \int_{\Omega} (h^{p_0}(Y))^{p_1} \right)^{\frac{1}{p_1}},$$

where $q_1 = \frac{p_1}{p_1 - 1}$. Since $h \in L^p(\Omega)$ in Assumption 1, we can take $p_1 = \frac{p}{p_0}$, and hence $q_1 = \frac{p}{p - p_0}$ in (2.19), which gives

$$||G(X, \cdot) h(\cdot)||_{L^{p_0}(\Omega)} \lesssim \left( \int_{\Omega} G^{\frac{p_0 p}{p - p_0}}(X, Y) dY \right)^{\frac{p - p_0}{p}} \left( \int_{\Omega} h^p(Y) dY \right)^{\frac{p_0}{p}}.$$

From the equation (2.20), it is obvious that we require $p > p_0$, so that along with logarithmic singularity of the Green function $G(X, Y)$, we can immediately obtain

$$\int_{\Omega} G^{\frac{p_0 p}{p - p_0}}(X, Y) dY < \infty,$$

which completes the argument of (2.18).

Moreover, we require in the Assumption 1 that $p > 2 > p_0$ to guarantee the existence of the weak solution of the equation (2.13).

**Remark 1.** If we choose the delta kernel, which means $c(X) = \delta(X)$ or equivalently, $q_0 = 1$. Then, we require that $p_0 = \frac{2q_0}{2q_0 - 1} = 2$. It is easy to check that the integral

$$\int_{\Omega} |G(X, Y)|^{p_0} h^{p_0}(Y) dY$$

is finite when $p > p_0$. More details can be found in [3].

Then we have the following assumption. Note that the Hölder continuity will be used in the analysis for the existence of the solution.

Then we have the following theorem.

**Theorem 2.4.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Under Assumption 1, there exist a unique continuous stochastic process $w : \Omega \to \mathbb{C}$ satisfying

$$w(X) = \frac{1}{D} \left[ F_k G(X, X_0) + \int_{\Omega} G(X, X') g(X') dX' + \int_{\Omega} G(X, X') h(X') dW_{X'} \right],$$

which is called the mild solution of the equation (2.17).

**Proof.** First we show that there exists a continuous modification of the random field

$$v(X) = \int_{\Omega} G(X, Y) h(Y) dW_{Y}, X \in \Omega.$$

According to Lemma 2.3 and the Hölder inequality, for any $X, Y \in \Omega$, we have:

$$\mathbf{E}(|v(X) - v(Z)|^2) \leq ||G(X, \cdot) - G(Y, \cdot)||_{L^{p_0}(\Omega)} ||h||_{L^{1/2}(\Omega)} ||c||_{L^{p_0}(B_{2R})}.$$
When $\frac{p_0 p}{p - p_0} > \frac{3}{2}$, it follows from (2.16) that
\[
\int_{\Omega} |G(X,Y) - G(Z,Y)|^{\frac{p_0 p}{p - p_0}} dY \lesssim |X - Z|^\frac{3}{2},
\]
which gives
\[
E(|v(X) - v(Z)|^2) \lesssim |X - Z|^\frac{3(p - p_0)}{p_0 p}.
\]
Since $v(X) - v(Z)$ is a random Gaussian variable, we have [9] for any integer $q$ that
\[
E((v(X) - v(Z))^{2q}) \lesssim (E(|v(X) - v(Z)|^2))^q \lesssim |X - Z|^\frac{3q(p - p_0)}{p_0 p}.
\]
We obtain from Kolmogorov’s continuity theorem that there exists a continuous modification of the random field $v$.

Clearly, the uniqueness of the mild solution comes from the solution representation (2.21), which depends only on the Green function $G$ and the source functions $g$ and $h$.

Next we present a constructive proof to show the existence. First we construct a sequence of processes $dW^n_X$ satisfying $hdW^n_X \in L^2(\Omega)$ and a sequence $v_n(X) = \int_{\Omega} G(X,Y) h(Y) dW^n_Y$, $X \in \Omega$;

which satisfies $v_n \to v$ in $L^2(\Omega)$ as $n \to \infty$.

Let $T_n = \bigcup_{j=1}^n K_j$ be a regular triangulation of $\Omega$. The piecewise constant approximation sequence is given by
\[
dW^n_X = \sum_{j=1}^n |K_j|^{-1} \int_{K_j} dW_X \chi_j(X),
\]
where $\chi_j$ is the characteristic function of $K_j$ and
\[
\int_{K_j} dW_X \sim \mathcal{N}(0,Var_j), \quad Var_j = \int_{K_j} \int_{K_j} c(X - Y) dXdY.
\]
Clearly we have for any $q \geq 1$ that
\[
E(||dW^n_X||_{L^q(\Omega)})^q = E\left(\int_{\Omega} \left| \sum_{j=1}^n |K_j|^{-1} \int_{K_j} dW_X \chi_j(X) \right|^q dX \right)
\lesssim \sum_{j=1}^n |K_j|^{1-q}(Var_j)^{\frac{q}{2}} < \infty,
\]
which shows that $dW^n_X \in L^q(\Omega)$, for any $q \geq 1$. It follows from the Hölder inequality that for a given $p$ meeting Assumption 1, we let $q$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, then
\[
||hdW^n_X||_{L^2(\Omega)} \lesssim ||h||_{L^r(\Omega)}^2 ||dW^n_X||_{L^q(\Omega)}^2 < \infty,
\]
which means $hdW^n_X \in L^2(\Omega)$.

Using Lemma 2.3, we have
\[
E\left(\int_{\Omega} \left| \int_{\Omega} G(X,Y) h(Y) dW_Y - \int_{\Omega} G(X,Y) h(Y) dW^n_Y \right|^2 dX \right)
= E\left[\int_{\Omega} \left| \sum_{j=1}^n \chi_j(Y) |K_j|^{-1} \int_{K_j} (G(X,Y) h(Y) - G(X,Z) h(Z)) dZ dW_Y \right|^2 dX \right]
\]
Then we have
\[ g(2.14) \] when random source (2.21) reduces to the solution of the deterministic direct problem
\[ \text{Remark 2.} \]

By using the triangle and Cauchy-Schwartz inequalities, we have
\[ \int_\Omega |G(X,Y)h(Y) - G(X,Z)h(Z)|^2 dX \]
\[ \leq \sum_{j=1}^n |K_j|^{-1} \int_{K_j} \int_\Omega \left| G(X,Y)h(Y) - G(X,Z)h(Z) \right|^2 dX dZ dY. \]

It follows from (2.16), Lemma 2.1, and the \( \eta - H"{o}lder \) continuity of \( h \) that
\[ \int_\Omega |G(X,Y)h(Y) - G(X,Z)h(Z)|^2 dX \leq h^2(Y)|Y - Z|^{2\eta} + |Y - Z|^{2\eta}. \]

Then we have
\[ \mathbb{E}\left( \int_\Omega \int_\Omega (G(X,Y)h(Y)dW_Y - \int_\Omega G(X,Y)h(Y) dW^n_Y)^2 dX \right) \]
\[ \lesssim \|h\|_{L^2(\Omega)}^2 \max_{1 \leq j \leq n} (diamK_j)^{2\eta} + |\Omega| \max_{1 \leq j \leq n} (diamK_j)^{2\eta} \rightarrow 0, \]
as \( n \rightarrow \infty \) since the diameter of \( K_j \rightarrow 0 \) as \( n \rightarrow \infty \).

For each \( n \in \mathbb{N} \), we consider the following problem
\[ \begin{cases} \mathbf{L}w_n &= -\frac{M_n}{D}, \\ \mathbf{L}M_n &= -(F_k \delta(X - X_0) + g(X) + h(X) dW^n_X), \\ w_n|_{X \in \partial \Omega} &= 0, \\ M_n|_{X \in \partial \Omega} &= 0. \end{cases} \]
(2.22)

It follows from \( h(X) dW^n_X \in L^2(\Omega) \) that the problem (2.22) has a unique solution given by
\[ w_n(X) = \frac{1}{D} \left[ F_k G(X, X_0) + \int_\Omega G(X, X') g(X') dX' + v_n(X) \right]. \]
(2.23)

Since \( \mathbb{E}(||v_n - v||^2_{L^2(\Omega)}) = \mathbb{E}(||v_n - v||^2_{L^2(\Omega)}) \rightarrow 0 \) as \( n \rightarrow \infty \), there exists a subsequence of \( w_n \) which converges to \( w \). Letting \( n \rightarrow \infty \) in (2.23), we obtain the mild solution (2.21) and complete the proof.

**Remark 2.** It is clear to note that the mild solution of the direct problem with random source (2.21) reduces to the solution of the deterministic direct problem (2.14) when \( g = 0, h = 0 \), i.e., no randomness is presented in the source.

### 3. Inverse random source problem

In this section, we consider the question that for the mild solution (2.21), with a given acting force \( F_k \) and \( w(X) \), how to calculate the stiffness \( D \) of the 2D nano-material and construct the expectation \( g(X) \) and the variance \( h(X) \) of the noise loaded on the material. Note that in this section, we consider the noise is white rather than colored, i.e., \( c(X - Y) = \delta(X - Y) \). We first derive two Fredholm integral equations based upon the explicit solution obtained in the previous section. The key ingredient of success is Itô isometry. However the derived Fredholm integral equation is ill-posed. Therefore, Tikhonov regularization in [16] is employed to solve the corresponding inverse problems.
For the equation (2.13), we have the mild solution:

\( w(X) = \frac{1}{D} \left[ F_k G(X, X_0) + \int_\Omega G(X, X') g(X') dX' + \int_\Omega G(X, X') h(X') dW_{X'} \right]. \) (3.24)

By using the property of Itô integral, we take the expectation on both sides of (3.24) and using \( \mathbb{E} \left[ \int_\Omega G(X, X') h(X') dW_{X'} \right] = 0 \), we have

\( \mathbb{E}[w(X)] = \frac{1}{D} \left[ F_k G(X, X_0) + \int_\Omega G(X, X') g(X') dX' \right]. \) (3.25)

Then taking the variance on both sides of (3.24) and using

\[
\mathbb{E} \left( \left\| \int_\Omega G(X, Y) h(Y) dW_Y \right\|^2 \right) = \int_\Omega \int_\Omega G(X, Y) h(Y) c(Y - Z) G(X, Z) h(Z) dY dZ = \int_\Omega |G(X, Y)|^2 h^2(Y) dY,
\]

we have

\[ \text{Var}[Dw(X)] = \text{Var} \left[ \int_\Omega G(X, X') h(X') dW_{X'} \right] = D^2 \int_\Omega G^2(X, X') h^2(X') dX'. \] (3.26)

For the equation (3.25), we put two different load forces \( F_1 \) and \( F_2 \) on the same point \( X_0 \), and it can be acquired that

\[ \mathbb{E}_1[w(X)] = \frac{1}{D} \left[ F_1 G(X, X_0) + \int_\Omega G(X, X') g(X') dX' \right], \]
\[ \mathbb{E}_2[w(X)] = \frac{1}{D} \left[ F_2 G(X, X_0) + \int_\Omega G(X, X') g(X') dX' \right]. \]

The noises are not influenced by the load forces, Therefore the coefficient \( D \) can be simply computed by:

\[ D = G(X, X_0) \frac{F_1 - F_2}{\mathbb{E}_1 - \mathbb{E}_2}, \quad \forall X \in \Omega. \] (3.27)

Assume the random source is driven by incoherent white noise. Substituting the equation (3.27) into the equation (3.25) yields:

\[ \int_\Omega G(X, X') g(X') dX' = \frac{\mathbb{E}_2 F_1 - \mathbb{E}_1 F_2}{\mathbb{E}_1 - \mathbb{E}_2} G(X, X_0). \] (3.28)

For the sake of simplicity, we denote the right hand side of (3.28) as \( \tilde{\mathbb{E}}_0[w(X)] \). It is easy to verify that \( G(X, X') \) is a Hilbert-Schmidt kernel and hence the above integral operator, i.e., the left hand side of (3.28), is a symmetric compact operator in \( L^2(\Omega) \). Therefore, for this operator, there exists a sequence of real eigenvalues and a unique accumulation point, i.e. zero. Consequently, the corresponding inverse operators are unbounded intrinsically, which will result in instability on solving the inverse problem. To overcome this instability, it is better to shift the accumulation point suitably. Hence, we employ a Tikhonov regularization method with a posterior...
strategy of choosing the regularization parameter based on $L$ curve method. Let $J$ be a cost functional defined as follows:

$$J(g) = \| \int \int_{\Omega} G(X, X') g(X') dX' - \hat{E}[w(X)] \|^2 + \alpha \| g \|^2_{L^2},$$

where $\alpha$ is the regularization parameter and plays an important role in seeking the minimizer. Taking the Frechet derivative with respect to $g$ we can obtain a normal equation for the minimizer as follows:

$$\alpha g(X) + \int_{\Omega} \int_{\Omega} G(X, Y) G(Y, X') g(X') dX' dY = \int_{\Omega} G(X, X') \hat{E}[w(X')] dX',$$

from which we can calculate the numerical solution of $g$. The solvability of this Fredholm integral equation of the second kind could be verified by the Fredholm alternative theorem. Here we will omit the details of this issue. Once we have solved the unknown function $g(X)$, we could proceed to solving $h(X)$. The procedure is similar to $g(X)$.

4. Numerical examples. In this section, the implementation of the proposed algorithms both for the direct and inverse random source problems is discussed. Two examples are presented to demonstrate the validity and effectiveness of the proposed method.

4.1. Numerical scheme for the forward problem. To avoid the inverse ‘crime’, we use a traditional numerical method, i.e., five-point finite difference scheme, to solve the (2.13). Note that we are using the polar coordinates, and there exist a singular point at $r = 0$ in (2.8). Therefore we need to add some supplementary condition here. Since we know the fact that:

$$w(0, \theta) = w(r, \theta) - rw_r(r, \theta) + O(r^2), \quad r \to 0$$

Therefore,

$$\lim_{r \to 0^+} r \frac{\partial u}{\partial r} = 0.$$
Inverse random source problem for biharmonic equation

Figure 2: The mesh generation under the polar coordination

\begin{equation}
\frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} \approx \frac{1}{r_i^2} \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{h_\theta^2}.
\end{equation}

Note that for the character of polar coordinate, when \( j = n \), we have \( w_{i,j+1} = w_{i,1} \). Besides, we can learn the difference equation \( Lw \) at the point \( w(r_1, \theta_j) \) by using finite volume method:

\begin{equation}
Lw(r_1, \theta_j) \approx \frac{1}{h_r} \frac{w_{3,j}w_{1,j}}{2h_r} + \frac{1}{h_\theta^2} \frac{w_{i,j+1} - 2w_{1,j} + w_{1,j-1}}{h_\theta^2}.
\end{equation}

Through the difference equations (4.29) and (4.30) we can establish a operator matrix for \( L \).

Here we consider the correlation function \( c \) as delta kernel, which means the colored noise \( dW_X \) in two dimensions decays to two dimensional white noise. In order to simulate the random source, we have the equation by using the property of Brownian sheet:

\[
\int_{\Omega} h(X) dW_X = (-1)^2 \int_{\Omega} W_X \frac{\partial^2 h(X)}{\partial x \partial y} d\Omega,
\]

which \( W_X \) is a \( \mathcal{N}(0, \mu(\Omega)) \) random variable.

Then we can calculate the numerical solution \( M \), and furthermore, the numerical solution of \( w \).

Let \( F_k, g \) and \( h \) are set as the ones in Example 2, one realization of the forward problem gives the solution \( w \) which is shown in Figure 3.

4.2. Inverse problems. Once we have numerical forward solutions at hand, we can deal with inverse problems. Starting from (3.27), we could first obtain the inversion result for the coefficient \( D \). Here we will utilize average of different realizations of the forward solutions to approximate the expectation \( E[w(X)] \). This approximation is reasonable and convergent to the exact expectation by the large number theorem. The quantitative version of the convergence analysis could be found in [6]. Meanwhile, we could solve the unknown component of the random source
g(X) via the following integral equation:

\[ \alpha g(Z) + \int_{\Omega} \int_{\Omega} G(Z, Y)G(Y, X)g(X) dX dY = \int_{\Omega} G(X, X) \hat{E}[w(X)] dX. \tag{4.31} \]

To discrete the above equation on the designated grid points, we need to use numerical quadratures, such as trapezoidal, Simpson or Gauss quadratures, which convert the integral equation (4.31) into a matrix form:

\[ (\alpha I + DHS)g = GE, \tag{4.32} \]

where \( I \) is the \((n + 1)\) by \((n + 1)\) identity matrix, \( GE \) is the obtained vector with length of \( n + 1 \) after discretizing \( \int_{\Omega} G(Z, X) \hat{E}[w(X)] dX \). \( DHS \) is a square matrix with elements defined by

\[ DHS_{ij,uv} = \left( \frac{h_r}{2} \right)^2 \left( \frac{h_\theta}{2} \right)^2 \omega_{ij} \times \text{inner}(Z_r(u), Z_\theta(v), X_r(i), X_\theta(j)) \times X_r(i), \tag{4.33} \]

where

\[ \omega_{ij} = \begin{cases} 1, & i = 0, j = 0, \\ 2, & i = 0, or j = 0, \\ 4, & i \neq 0, j \neq 0, \end{cases} \]

and

\[ \text{inner}(Z_r(u), Z_\theta(v), X_r(i), X_\theta(j)) \]

\[ = \sum_{i=1}^{n} \sum_{k=1}^{n} [G(Z_r(u), Z_\theta(v), Y_r(k-1), Y_\theta(l-1)) \times G(X_r(i), X_\theta(j), Y_r(k-1), Y_\theta(l-1)) \times Y_r(k-1) \\
+ G(Z_r(u), Z_\theta(v), Y_r(k-1), Y_\theta(l)) \times G(X_r(i), X_\theta(j), Y_r(k-1), Y_\theta(l)) \times Y_r(k-1). \]

Figure 3: The solution to direct problem with random source
\begin{align*}
+ G(Z_r(u), Z_\theta(v), Y_r(k), Y_\theta(l - 1)) \times G(X_r(i), X_\theta(j), Y_r(k), Y_\theta(l - 1)) \times Y_r(k) & \\
+ G(Z_r(u), Z_\theta(v), Y_r(k), Y_\theta(l)) \times G(X_r(i), X_\theta(j), Y_r(k), Y_\theta(l)) \times Y_r(k) \].
\end{align*}

**Example 1.** Let \( g \) and \( h \) be:
\begin{align*}
g(r, \theta) &= 10 \sin(2\pi \frac{r}{R}) \sin \theta, \\
h(r, \theta) &= \sin(\pi \frac{r}{R}) \cos(\theta/2)).
\end{align*}

(4.34)

In order to calculate the variance and expectation of \( w(X) \), we simulated the forward problem for 100, 500, 2000 times and compare the outcomes respectively. The relative error of \( g(X) \) and \( h(X) \) is decreasing slightly. The reason stays at the accuracy of forward solver and the discretization of the (4.31). Hence, we only show the construction of \( D, g \) and \( h \) in the circumstances of \( T = 2000 \) in Figure 4, 5 and 6, respectively. Figure 4 shows the inversion results of \( D \), where the value in the centre is coarser than other points since we used finite volume method at that point, as presented in (4.30). The left subfigure of Figure 5 shows the regularization parameter \( \alpha \) is chosen by L-curve and the right subfigure shows the inversion result of the mean component of the random source, i.e., \( g(X) \).
Example 2. Let $g$ and $h$ be:

\begin{align}
g(r, \theta) &= 5 \sin(\pi \frac{r}{R}) \\
h(r, \theta) &= \frac{1}{2}(1 - \cos(2\pi \frac{r}{R})) \sin(\theta/2);
\end{align}

Similar to the example 1, we simulated the forward problem for 100,500,2000 times and compare the outcomes. The constructions of $g$ and $h$ in the circumstances of $T = 2000$ are displayed in Figure 7 and 8, respectively.

5. Conclusion. We have studied an inverse random source problem for the two-dimensional biharmonic equation where the source is driven by an additive white noise or colored noise. Under some suitable regularity assumption of the source functions $g$ and $h$, the existence and uniqueness of a mild solution is proved. Based on the solution, Fredholm integral equations are deduced for the inverse problem to reconstruct the mean and the variance of the random source. Tikhonov regularization method is employed to solve the ill-posed integral equations and two effective results are obtained. Further discussions will be made on inhomogeneous and anisotropic materials.
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