Integrand Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division

Pierpaolo Mastrolia
Max-Planck Institut für Physik, Föhringer Ring, 6, D-80805 München, Germany
Dipartimento di Fisica e Astronomia, Università di Padova, and INFN Sezione di Padova, via Marzolo 8, 35131 Padova, Italy
E-mail: ppaolo@mppmu.mpg.de

Edoardo Mirabella
Max-Planck Institut für Physik, Föhringer Ring, 6, D-80805 München, Germany
E-mail: mirabell@mppmu.mpg.de

Giovanni Ossola
Physics Department, New York City College of Technology, City University of New York, 300 Jay Street, Brooklyn, NY 11201, USA
Kavli Institute for Theoretical Physics, University of California, Kohn Hall, Santa Barbara, CA 93106, USA
E-mail: GOssola@citytech.cuny.edu

Tiziano Peraro
Max-Planck Institut für Physik, Föhringer Ring, 6, D-80805 München, Germany
E-mail: peraro@mppmu.mpg.de

Abstract: We describe the application of a novel approach for the reduction of scattering amplitudes, based on multivariate polynomial division, which we have recently presented. This technique yields the complete integrand decomposition for arbitrary amplitudes, regardless of the number of loops. It allows for the determination of the residue at any multiparticle cut, whose knowledge is a mandatory prerequisite for applying the integrand-reduction procedure. By using the division modulo Gröbner basis, we can derive a simple integrand recurrence relation that generates the multiparticle pole decomposition for integrands of arbitrary multiloop amplitudes. We apply the new reduction algorithm to the two-loop planar and nonplanar diagrams contributing to the five-point scattering amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA in four dimensions, whose numerator functions contain up to rank-two terms in the integration momenta. We determine all polynomial residues parametrizing the cuts of the corresponding topologies and subtopologies. We obtain the integral basis for the decomposition of each diagram from the polynomial form of the residues. Our approach is well suited for a seminumerical implementation, and its general mathematical properties provide an effective algorithm for the generalization of the integrand-reduction method to all orders in perturbation theory.
1. Introduction

The unitarity of the $S$ matrix encodes the most profound property of a quantum system, namely the probability conservation. The optical theorem, that relates the difference between the transition amplitude and its complex conjugate to their product, is the direct consequence of unitarity. Hence, at a given order in perturbation theory, it connects the discontinuity of the amplitude across a given branch cut to the sum of all the Feynman diagrams sharing that specific cut, which factorize into two lower-order amplitudes. By
elaborating on the role of the optical theorem, and introducing the concept of generalised cuts [1–4], unitarity has been inspiring a novel organization of the perturbative calculus, where Feynman diagrams are grouped according to their multiparticle factorization channels.

Scattering amplitudes in quantum field theories are analytic functions of the momenta of the interacting particles; hence they are determined by their singularities. The singularity structure is retrieved when virtual particles go on shell, under the effect of complex deformations of the kinematic, as needed for solving multiple on-shell conditions simultaneously.

The investigation of the mathematical properties of the residues at the singularities led to the discovery of new relations involving scattering amplitudes, such as the BCFW recurrence relation [4], its link to the leading singularity of one-loop amplitudes [2], and the OPP integrand-decomposition formula [5].

Automating the evaluation of one-loop multiparticle amplitudes, for an accurate description of scattering processes that were considered prohibitive, has become feasible. Motivated by the challenging experimental program of the LHC, where the ubiquity of QCD manifests itself through the production of multijet events, several codes have been developed with the goal of reaching the next-to-leading order level of accuracy for the cross sections [6–14].

On the more mathematical side, it became clear that within the on-shell and unitarity-based methods, the theory of multivariate complex functions could play an important role in order to compute the generalized cuts efficiently. The holomorphic anomaly [15, 16] and the spinor integration [17, 18], as well as, Cauchy’s residue theorem [2, 4], Laurent series expansion [19–21], Stokes’ Theorem [22, 23], and Global residue theorem [24] have been employed for carrying out the integration of the phase-space integrals, left over after applying the on-shell cut conditions to the loop integrals.

Progress on the unitarity-based methods and the vivid research activity spun off has been recently reviewed in [25] and [26–32].

At two loops, generalized unitarity techniques have been introduced for supersymmetric amplitudes [33] and later for QCD amplitudes [34]. The multiple cuts of two-loop amplitudes were proposed to extend the simplicity of the one-loop quadruple cut [2] to the leading singularity techniques [35,36] and to the maximal-cuts method [37]. The maximal-unitarity approach developed by Kosower, Larsen, Caron-Huot, and Johansson [38–41] has refined this technique by a systematic application of the global residue theorem.

The singularity structure of multiloop scattering amplitudes can be also exposed in the integrand, before integrating over the loop momenta. The integrand-reduction methods use the singularity structure of the integrands to decompose the (integrated) amplitudes in terms of Master Integrals (MIs). The multiparticle pole expansion of the integrand is equivalent to the decomposition of the numerator in terms of products of denominators, multiplied by polynomials. These latter correspond to the residues at the multiple-cuts. In general, the coefficients of the MIs are a subset of the coefficients appearing in the polynomial residues. Therefore the complete determination of the residues leads to the
complete decomposition of the amplitudes in terms of MI's. The final result is then obtained by evaluating the latter.

The parametric form of the polynomial residues is process independent and it can be determined \textit{a priori}, from the topology of the corresponding on-shell diagram, namely from the graph identified by the denominators that go simultaneously on shell. The actual value of the coefficients is clearly process dependent, and its determination is indeed the goal of the integrand reduction. Integrand-reduction methods determine the (unknown) coefficients by \textit{polynomial fitting}, through the evaluation of the (known) integrand at values of the loop momenta fulfilling the cut conditions. The integrands contributing to the amplitude, which have to be evaluated in correspondence to the solutions of the on-shell conditions, are the only input required. They can be provided either as a product of tree-level amplitudes, like in unitarity-based approaches, or as a combination of Feynman diagrams, retaining the full loop-momentum dependence. In the former case the on-shell diagram represents a cut of the amplitude while in the latter case it is simply the cut of an integral where the on-shell conditions are applied to its numerator. The integrand-reduction methods have been originally developed at one loop \cite{5}. Extensions beyond one loop were proposed in \cite{42,43}. A key point of the higher-loop extension is the proper parametrization of the residues of the multiparticle poles. Each residue is a multivariate polynomial in the \textit{irreducible scalar products} (ISPs) among the loop momenta and either external momenta or polarization vectors constructed out of them. ISPs cannot be expressed in terms of denominators, thus any monomial formed by ISPs is the numerator of an integral which may be a MI appearing in the final result.

Both the numerator and the denominators of any integrand are multivariate polynomials in the components of the loop variables. As recently shown in \cite{44,45}, the decomposition of the integrand can be obtained using basic principles of algebraic geometry, by performing the \textit{multivariate polynomial division} between the numerator and the Gröbner basis generated by (a subset of) the denominators. Moreover, the multivariate polynomial divisions give a systematic classification of the polynomial structures of the residues, leading to both the identification of the MIs and the determination of their coefficients.

In \cite{42} it was observed that the set of independent integrals which emerge from the integrand-reduction algorithms is not minimal. Integration-by-parts \cite{46}, Lorentz-invariance \cite{47}, and Gram-determinant \cite{48} identities may constitute additional, independent relations which can further reduce the number of MI's that have to be actually evaluated, after the reduction stage. Badger, Frellesvig and Zhang have explicitly shown that the number of independent ten-denominator integrals identified through the integrand decomposition of the three-loop four-point ladder box diagram is significantly reduced by using integration-by-parts identities \cite{49}. In the case of one- and two-loop amplitudes, an alternative technique for counting the numbers of tensor structures and of the independent coefficients has been presented by Kleiss, Malamos, Papadopoulos and Verheyen in \cite{50}. During the completion of this work, Feng and Huang \cite{51} have shown that, by using multivariate polynomial division \cite{44,45}, a systematic classification of a four-dimensional integral basis for two-loop integrands is doable.

In \cite{45}, we have set the mathematical framework for the multiloop integrand-reduction
algorithm. We have shown that the residues are uniquely determined by the denominators involved in the corresponding multiple cut. We have derived a simple integrand recurrence relation generating the multiparticle pole decomposition. The algorithm is valid for arbitrary amplitudes, irrespective of the number of loops, the particle content (massless or massive), and of the diagram topology (planar or nonplanar). Interestingly, at one loop our algorithm allows for a simple derivation of the OPP reduction formula [5]. The spurious terms, when present, naturally arise from the structure of the denominators entering the generalized cuts.

In the same work [45], we gave the proof of the maximum-cut theorem. The theorem deals with cuts where the number of on-shell conditions is equal to the number of integration variables and therefore the loop momenta are completely localized. The theorem ensures that the number of independent solutions of the maximum cut is equal to the number of coefficients parametrizing the corresponding residue. The maximum-cut theorem generalizes at any loop the simplicity of the one-loop quadruple cut [2, 5], where the two coefficients parametrizing the residue are determined by the two solutions of the cut.

In this paper, we apply our algorithm to the two-loop five-point planar and nonplanar diagrams contributing to amplitudes in $\mathcal{N} = 4$ super Yang-Mills (SYM) and $\mathcal{N} = 8$ Supergravity (SUGRA) in four dimensions [52, 53]. We use the numerator functions computed in [53], which contain up to rank-two terms in each integration momenta. In particular, we derive the generic polynomial residues which are required by the reduction procedure. Later, we show that the integrand reduction can be performed both seminumerically, by polynomial fitting, and analytically. The latter computation has been performed generalizing the method of integrand reduction through Laurent expansion [54], which has been recently introduced to improve the integrand reduction of one-loop amplitudes.

All the numerical and analytic computations presented in this paper have been performed using C++, FORM [55] and the mathematica package S@M [56].

2. Integrand reduction

In this Section we describe the general strategy for the reduction of scattering amplitudes at the integrand level, following [42, 45]. In dimensional regularization, an $\ell$-loop amplitude can be written as a linear combination of $n$-denominator integrals of the form

$$\mathcal{A}_n = \int d^d \tilde{q}_1 \ldots \int d^d \tilde{q}_\ell \mathcal{I}_{i_1 \ldots i_n}(\tilde{q}_1, \ldots, \tilde{q}_\ell)$$

$$= \int d^d \tilde{q}_1 \ldots \int d^d \tilde{q}_\ell \frac{\mathcal{N}_{i_1 \ldots i_n}(\tilde{q}_1, \ldots, \tilde{q}_\ell)}{D_i(\tilde{q}_1, \ldots, \tilde{q}_\ell) \cdot \cdot \cdot D_n(\tilde{q}_1, \ldots, \tilde{q}_\ell)},$$

$$D_i = \left(\sum_a \alpha_{i,a} \tilde{q}_a + p_i\right)^2 - m_i^2$$ (2.1)

where $q_1, \ldots, q_\ell$ are integration momenta and $\alpha_{i,a} \in \{0, \pm 1\}$. Objects living in $d = 4 - 2\epsilon$ are denoted by a bar. We use the notation $\tilde{q}_a^\mu = q_a^\mu + \lambda_{qa}$, where $q_a^\mu$ is the four-dimensional part of $\tilde{q}_a$, while $\lambda_{qa}$ is its $(-2\epsilon)$-dimensional part [57]. In the following we will limit
ourselves to the four-dimensional case. Extensions to higher-dimensional cases, according to the chosen dimensional regularization scheme, can be treated analogously.

The integrand-reduction methods [5, 42, 43, 54, 58–65] trade the decomposition of the loop integrals in terms of MIs with the algebraic problem of building a general relation, at the integrand level, for the numerator functions of each integral contributing to the amplitude. In this paper we use the method introduced in [45]. The algorithm relies solely on general properties of the loop integrand, i.e. on the maximum power of the loop momenta present in the numerator, and on the quadratic form of Feynman propagators. The residue of each multiparticle pole is determined by the on-shell conditions corresponding to the simultaneous vanishing of the denominators it is sitting on. In particular, we obtain the multipole decomposition of the integrand of Eq. (2.1) using an integrand recurrence relation based on multivariate polynomial division together with a criterion for the reducibility of the integrands. In the following subsections we will briefly review the two ingredients of the method.

2.1 Integrand recurrence relation

The four-dimensional version of the integrand of Eq. (2.1) is

\[ I_{i_1 \ldots i_n} = \frac{N_{i_1 \ldots i_n}(q_1, \ldots, q_\ell)}{D_{i_1}(q_1, \ldots, q_\ell) \cdots D_{i_n}(q_1, \ldots, q_\ell)}. \]  

(2.2)

The numerator \( N_{i_1 \ldots i_n} \) and any of the denominators \( D_i \) are polynomial in the components of the loop momenta, say \( z = (z_1, \ldots, z_4) \), i.e.

\[ I_{i_1 \ldots i_n} = \frac{N_{i_1 \ldots i_n}(z)}{D_{i_1}(z) \cdots D_{i_n}(z)}. \]  

(2.3)

We construct the ideal generated by the \( n \) denominators

\[ J_{i_1 \ldots i_n} = (D_{i_1}, \ldots, D_{i_n}) \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(z)D_{i_\kappa}(z) : h_\kappa(z) \in P[z] \right\}, \]

where \( P[z] \) is the set of polynomials in \( z \). The common zeros of the elements of \( J_{i_1 \ldots i_n} \) are exactly the common zeros of the denominators. We chose a monomial order and we construct a Gröbner basis generating the ideal \( J_{i_1 \ldots i_n} \)

\[ G_{i_1 \ldots i_n} = \{g_1(z), \ldots, g_m(z)\}. \]  

(2.4)

The \( n \)-ple cut conditions \( D_{i_1} = \ldots = D_{i_n} = 0 \) are equivalent to \( g_1 = \ldots = g_m = 0 \). The multivariate division of \( N_{i_1 \ldots i_n} \) modulo \( G_{i_1 \ldots i_n} \) leads to

\[ N_{i_1 \ldots i_n}(z) = \Gamma_{i_1 \ldots i_n} + \Delta_{i_1 \ldots i_n}(z), \]

(2.5)

where \( \Gamma_{i_1 \ldots i_n} = \sum_{i=1}^{m} Q_i(z)g_i(z) \) is a compact notation for the sum of the products of the quotients \( Q_i \) and the divisors \( g_i \). The polynomial \( \Delta_{i_1 \ldots i_n} \) is the remainder of the division. Since \( G_{i_1 \ldots i_n} \) is a Gröbner basis, the remainder is uniquely determined once the monomial
order is fixed. The term $\Gamma_{i_1\ldots i_n}$ belongs to the ideal $\mathcal{J}_{i_1\ldots i_n}$, thus it can be expressed in terms of denominators, as

$$\Gamma_{i_1\ldots i_n} = \sum_{\kappa=1}^{n} N_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n}(z) D_{i_\kappa}(z). \quad (2.6)$$

The explicit form of $N_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n}$ can be found by expressing the elements of the Gröbner basis in terms of the denominators. Using Eqs. (2.5) and (2.6), we cast the numerator in the suggestive form

$$N_{i_1\ldots i_n}(z) = \sum_{\kappa=1}^{n} N_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n}(z) D_{i_\kappa}(z) + \Delta_{i_1\ldots i_n}(z). \quad (2.7)$$

Plugging Eq. (2.7) in Eq. (2.3), we get a nonhomogeneous recurrence relation for the $n$-denominator integrand,

$$\mathcal{I}_{i_1\ldots i_n} = \sum_{\kappa=1}^{n} \mathcal{I}_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n} + \frac{\Delta_{i_1\ldots i_n}}{D_{i_1}\ldots D_{i_n}}. \quad (2.8)$$

According to Eq. (2.8), $\mathcal{I}_{i_1\ldots i_n}$ is expressed in terms of $(n-1)$-denominator integrands,

$$\mathcal{I}_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n} = \frac{N_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n}}{D_{i_1}\ldots D_{i_{\kappa-1}}D_{i_{\kappa+1}}\ldots D_{i_n}}. \quad (2.9)$$

The nonhomogeneous term contains the remainder of the division (2.3). By construction, it contains only irreducible monomials with respect to $\mathcal{G}_{i_1\ldots i_n}$, and it is identified with the residue of the cut $(i_1 \ldots i_n)$.

The integrands $\mathcal{I}_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n}$ can be decomposed repeating the procedure described in Eqs. (2.3)-(2.5). In this case the polynomial division of $N_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n}$ has to be performed modulo the Gröbner basis of the ideal $\mathcal{J}_{i_1\ldots i_{\kappa-1}i_{\kappa+1}\ldots i_n}$, generated by the corresponding $(n-1)$ denominators. The complete multi-pole decomposition of the integrand $\mathcal{I}_{i_1\ldots i_n}$ is obtained by successive iterations of Eqs. (2.3)-(2.5).

### 2.2 Reducibility criterion

An integrand $\mathcal{I}_{i_1\ldots i_n}$ is said to be reducible if it can be written in terms of lower-point integrands, i.e. when the numerator can be written as a linear combination of denominators. Eqs. (2.5) and (2.6) allow one to characterize the reducibility of the integrands:

**Proposition 2.1** The integrand $\mathcal{I}_{i_1\ldots i_n}$ is reducible if and only if the remainder of the division modulo a Gröbner basis vanishes, i.e. $N_{i_1\ldots i_n} \in \mathcal{J}_{i_1\ldots i_n}$.

A direct consequence of the Proposition 2.1 is

**Proposition 2.2** An integrand $\mathcal{I}_{i_1\ldots i_n}$ is reducible if the cut $(i_1 \ldots i_n)$ leads to a system of equations with no solution.
Indeed if the system of equations $D_{i_1}(z) = \cdots = D_{i_n}(z) = 0$ has no solution, the weak Nullstellensatz theorem ensures that $1 \in J_{i_1} \cdots i_n$, i.e. $J_{i_1} \cdots i_n = P[z]$. Therefore any polynomial in $z$ is in the ideal. Any numerator function $N_{i_1} \cdots i_n$ is polynomial in the integration momenta, thus $N_{i_1} \cdots i_n \in J_{i_1} \cdots i_n$ and it can be expressed as a combination of the denominators $D_{i_1}(z), \ldots, D_{i_n}(z)$ \cite{45,50}. In this case Eq. (2.8) becomes

$$I_{i_1} \cdots i_n = \sum_{\kappa=1}^{n} I_{i_1} \cdots i_{\kappa-1} i_{\kappa+1} i_n.$$  \hspace{1cm} (2.10)

The reducibility criterion and the recurrence relation (2.8) are the two mathematical properties underlying the integrand decomposition of scattering amplitudes, at any order in perturbation theory. If the $n$ denominators cannot vanish simultaneously, the corresponding integral is reducible, namely it can be written in terms of integrands with $(n - 1)$ denominators. If the $n$-ple cut leads to a consistent system of equations, we extract the polynomial form of the residue as the remainder of the division of the numerator modulo the Gröbner basis associated to the $n$-ple cut. The quotients of the polynomial division generate integrands with $(n - 1)$ denominators which should undergo the same decomposition. The algorithm will stop when all cuts are exhausted, and no denominator is left. Upon integration, the nonvanishing terms present in each residue may give rise to master integrals.

Each residue $\Delta_{i_1} \cdots i_n$ belongs to a vector subspace $Q_{i_1} \cdots i_n[z]$ of $P[z]$. Its dimension is independent of the choice of the basis. The residue can be expressed in terms of the ISPs writing the components $z$ in terms of scalar products. A suitable choice of the bases of the loop momenta allow one to write the ISPs as multivariate monomials in $z$ generating $Q_{i_1} \cdots i_n[z]$. When the number of external legs of the cut diagram is less than five, then the ISPs may involve spurious terms. As in the one-loop case \cite{5,42,43,54,58–64}, they originate from the components of the loop momenta belonging to the orthogonal space, i.e. the space orthogonal to the one spanned by the independent external momenta of the cut diagram.

### 2.3 Maximum-cut Theorem

Following \cite{45}, we define Maximum cut an $n$-ple cut $D_{i_1}(z) = \cdots = D_{i_n}(z) = 0$ fully constraining all the components $z$ of the loop momenta. Examples of maximum cuts for the one-loop case are the 4-ple (5-ple) cut in 4 ($d = 4 - 2\epsilon$) dimensions. We assume that, at nonexceptional phase-space points, a maximum cut has a finite number $n_s$ of solutions, each with multiplicity one. Under these assumptions one can prove the following theorem \cite{45}

**Theorem 2.1 (Maximum cut)** The residue at the maximum cut is a polynomial paramatre by $n_s$ coefficients, which admits a univariate representation of degree $(n_s - 1)$.

The maximum-cut theorem guarantees that the maximum number of terms needed to parametrize the residue of the maximum cut is exactly equal to $n_s$. Therefore it guarantees
the full reconstruction of the residue by sampling the integrand on the \( n_s \) solutions of the maximum cut. Theorem 2.1 generalizes at any loop the simplicity of the one-loop maximum cuts \([2,5]\). Indeed, in \( d = 4 - 2\epsilon \) dimensions, the residue of the quintuple cut is parametrized by one coefficient, and can be reconstructed by sampling on the single solution of the cut itself. Similarly the two coefficients of the residue of the quadruple cut in four dimensions can be determined by sampling the integrand on the two solutions of the cut.

### 2.4 Two-loop integrand reduction

In four dimensions, the generic two-loop integral \( A_n \) reads as follows

\[
A_n = \int d^4q \int d^4k \mathcal{I}_n(q,k) = \int d^4q \int d^4k \frac{N_{1\ldots n}(q,k)}{D_1 D_2 \cdots D_n}, \tag{2.11}
\]

Every integrand with more than eight denominators \( D_i \) leads to a system of equations with no solution for its cut\(^1\). Proposition 2.2 implies that such an integrand is reducible and can therefore be expressed in terms of integrands with eight or less denominators. The recursive procedure described in Section 2.2 leads to the following multipole decomposition

\[
\mathcal{I}_n = \sum_{i_1 < < i_8 = 1}^{n} \frac{\Delta_{i_1 \ldots i_8}}{D_{i_1} \cdots D_{i_8}} + \sum_{i_1 < i_7 = 1}^{n} \frac{\Delta_{i_1 \ldots i_7}}{D_{i_1} \cdots D_{i_7}} + \cdots + \sum_{i_1 < i_2 = 1}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{i = 1}^{n} \frac{\Delta_i}{D_i} + \mathcal{Q}_\varnothing \tag{2.12}
\]

where \( i_a << i_b \) stands for a lexicographic order \( i_a < i_{a+1} < \ldots < i_{b-1} < i_b \). Equivalently, the numerator decomposition formula reads

\[
N_{1\ldots n} = \sum_{i_1 < < i_8 = 1}^{n} \Delta_{i_1 \ldots i_8} \prod_{j \neq i_1, \ldots, i_8} D_j + \sum_{i_1 < < i_7 = 1}^{n} \Delta_{i_1 \ldots i_7} \prod_{j \neq i_1, \ldots, i_7} D_j + \cdots + \sum_{i_1 < i_2 = 1}^{n} \Delta_{i_1 i_2} \prod_{j \neq i_1, i_2} D_j + \sum_{i = 1}^{n} \Delta_i \prod_{j \neq i} D_j + \mathcal{Q}_\varnothing \prod_{j = 1}^{n} D_j. \tag{2.13}
\]

The residue \( \Delta_{i_1 \ldots i_k} \) is obtained from the corresponding rank \( r_{i_1 \ldots i_k} \) integrand \( \mathcal{I}_{i_1 \ldots i_k} \) using the following procedure:

1. Decompose the loop momenta in using two bases \( \{\tau_i\}_{i=1,\ldots,4} \) and \( \{e_j\}_{j=1,\ldots,4} \):

\[
q^\mu = -p_0^\mu + \sum_{i=1}^{4} x_i \tau_i^\mu, \quad k^\mu = -r_0^\mu + \sum_{i=1}^{4} y_j e_j^\mu. \tag{2.14}
\]

In this case \( z \equiv (y_1, \ldots, y_4, x_1, \ldots, x_4) \).

2. Consider a generic rank \( r_{i_1 \ldots i_k} \) polynomial in \( z \)

\[
N_{i_1 \ldots i_k}(z) = \sum_{j \in J(r_{i_1 \ldots i_k})} \alpha_j \left( \prod_{i=1}^{8} z_i^j \right), \quad J(r) \equiv \{ j \in \mathbb{N}^8 : \sum_{i=1}^{8} j_i \leq r \}. \tag{2.15}
\]

---

\(^1\)A potential ambiguity may arise in topologies with nine denominators two of which are degenerate. However in this case the one-loop subtopology contains at least six denominators yielding thus a system of equations with no solution.
3. Choose a monomial order and construct a Gröbner basis $G_{i_1\ldots i_k} = \{g_1(z), \ldots, g_m(z)\}$, generating the ideal $J_{i_1\ldots i_k} = \langle D_{i_1}, \ldots, D_{i_k} \rangle$.

4. Divide $N_{i_1\ldots i_k}$ modulo $G_{i_1\ldots i_k}$ holding the remainder $\Delta_{i_1\ldots i_k}$.

The integrand decomposition (2.12) allows one to express the amplitude in terms of MIs, associated to diagrams with 8, 7,\ldots, 2 denominators. Depending on the powers of the integration momenta appearing in the numerator, the multivariate division may also generate the single-cut residues $\Delta_i$, and the quotients of the last divisions, $Q_\emptyset$. These two contributions generate spurious terms only but they are needed for the complete reconstruction of the numerator away from the solutions of the multiple cuts.

We expect that the integrand-reduction formula could be extended to $d$ dimensions, where additional degrees of freedom related to $\tilde{\lambda}_q$ and $\tilde{\lambda}_k$ enter [66].

3. Five-point amplitudes in $\mathcal{N} = 4$ SYM

The five-point amplitude in $\mathcal{N} = 4$ SYM can be expressed in terms of six diagrams [53]. The color ordered amplitude is given by a sum over the cyclic permutations of the external momenta. We apply the integrand reduction only to the three diagrams depicted in Fig. [4].

The other three diagrams are trivially expressed in terms of scalar integrals, since their numerator is independent of the loop momenta. We consider one integrand at a time and we obtain its decomposition by evaluating its numerator on solutions of multiple cuts, i.e. on values of the loop momenta such that some of its denominators vanish.

We introduce the notation $\mathcal{I}^{(4,i)}(\mathcal{N}^{(4,i)})$ to denote the integrand (numerator) of the diagram in Fig. [4] (i) in $\mathcal{N} = 4$ SYM. The integrand of diagrams in Fig. [4] are

$$
\mathcal{I}^{(4,a)}_{1\ldots 8}(q,k) = \frac{N^{(4,a)}_{1\ldots 8}(q,k)}{D_1 \cdots D_8}, \quad N^{(4,a)}_{1\ldots 8}(q,k) = 2q \cdot u_1 + \beta_1 , \quad (3.1)
$$

$$
\mathcal{I}^{(4,b)}_{1\ldots 8}(q,k) = \frac{N^{(4,b)}_{1\ldots 8}(q,k)}{D_1 \cdots D_8}, \quad N^{(4,b)}_{1\ldots 8}(q,k) = 2q \cdot u_2 + \beta_2, \quad (3.2)
$$

$$
\mathcal{I}^{(4,c)}_{1\ldots 8}(q,k) = \frac{N^{(4,c)}_{1\ldots 8}(q,k)}{D_1 \cdots D_8}, \quad N^{(4,c)}_{1\ldots 8}(q,k) = 2q \cdot u_3 + 2k \cdot u_3 + \beta_3, \quad (3.3)
$$

The vectors $u_1^\mu$, $u_2^\mu$, and $u_3^\mu$ and the constants $\beta_i$ are defined as [53]

$$
u_1^\mu = \frac{1}{4} \left( \gamma_{35124}(p_5^\mu - p_3^\mu) + \gamma_{34125}(p_4^\mu - p_3^\mu) + \gamma_{45123}(p_5^\mu - p_4^\mu) + 2 \gamma_{12345}(p_5^\mu - p_4^\mu) \right), \quad (3.4)
$$

$$
u_2^\mu = \frac{1}{4} \left( \gamma_{23145}(p_2^\mu - p_3^\mu) + \gamma_{24135}(p_4^\mu - p_3^\mu) + \gamma_{34125}(p_3^\mu - p_4^\mu) + 2 \gamma_{15234}(p_3^\mu - p_5^\mu) \right), \quad (3.5)
$$

$$
u_3^\mu = \frac{1}{4} \left( \gamma_{12345}(p_1^\mu - p_3^\mu) + \gamma_{25134}(p_2^\mu - p_5^\mu) + \gamma_{15234}(p_1^\mu - p_5^\mu) + 2 \gamma_{34125}(p_5^\mu - p_4^\mu) \right), \quad (3.6)
$$

$$
\beta_1 = \frac{1}{4} \left( \gamma_{35124}(s_{34} + s_{12} + s_{35}) + 2 \gamma_{34125} \cdot s_{12} + \gamma_{45123}(s_{34} + s_{12} + s_{35}) + 2 \gamma_{12345}(s_{23} - s_{13}) \right), \quad (3.7)
$$
Figure 1: Five-point diagrams entering the amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA. They are the pentabox diagram (a), the crossed pentabox diagram (b) and the double pentagon diagram (c). For each diagram, the definition of the denominators is shown as well.

\begin{align}
D_1 &= k^2 \\
D_2 &= (k + p_3)^2 \\
D_3 &= (k - p_1)^2 \\
D_4 &= q^2 \\
D_5 &= (q + p_5)^2 \\
D_6 &= (q - p_3)^2 \\
D_7 &= (q - p_4 - p_5)^2 \\
D_8 &= (q + k + p_2 + p_3)^2 .
\end{align} 

$\beta_2 = \frac{1}{4} \left( - (\gamma_{23145} + \gamma_{24135}) s_{23} + \gamma_{34125} s_{15} + s_{34} + 2 s_{23} - \gamma_{15234} (s_{13} - s_{35}) \right), \quad (3.8)$
Figure 2: Cut diagrams of the sevenfold cuts. Starting from the left, we show the diagram of the cut \((1234568)\), \((1234678)\), \((1234578)\), and \((1235678)\).

\[
\beta_3 = \frac{1}{4} \left( (\gamma_{12345} - \gamma_{25134}) s_{12} + \gamma_{15234} (s_{34} + s_{15} + 2 s_{12}) - 2 \gamma_{34125} (s_{13} - s_{14}) \right). \tag{3.9}
\]

where the kinematic invariants \(s_{ij}\) and the functions \(\gamma\) read as follows

\[
s_{ij} \equiv (p_i + p_j)^2 = 2 (p_i \cdot p_j) \tag{3.10}
\]

\[
\gamma_{12345} \equiv \left( \frac{[12][23][34][45][51]}{[14][23][12][34] - [12][34][14][23]} \right) - (1 \leftrightarrow 2). \tag{3.11}
\]

In \(\mathcal{N} = 4\) SYM, given the simple form of the numerators, the multipole decomposition of the integrands only requires one iteration. The numerators can be decomposed as

\[
\mathcal{N}_1^{(4,x)}(q,k) = \Delta_{12345678} + \sum_{i=1}^{7} \Delta_{1\cdots(i-1)(i+1)\cdots8} D_i, \quad x = a, b, c. \tag{3.12}
\]

The number of 7-ple residues of the integrands \(\mathcal{N}^{(4,a)}\) and \(\mathcal{N}^{(4,b)}\) is almost halved since the numerator depends on \(q\) only, thus \(\Delta_{1\cdots(i-1)(i+1)\cdots8} = 0\) for \(i \neq 4, 5, 6, 7\).

In the next subsections, we list the parametrization of the residues entering our computation, namely of the residues in Eq. (3.12). All the eightfold residues are related to maximum cuts. According to the maximum-cut theorem [45], the number of coefficients needed to parametrize the residue of a maximum cut is finite and equal to the number of the solutions of the corresponding cut. We found the most general parametrization of the eightfold residue, which is process-independent and valid for numerators of any rank in both \(q\) and \(k\). The parametrization of the sevenfold residues is also process-independent and it is given for the case of renormalizable numerators of rank six at most, which is more than we need for the applications presented in this paper. The parametrization of higher rank numerators can be obtained including additional terms with higher powers of the loop momenta in the residues [54]. The number of coefficients required to parametrize the residues agrees with an independent computation performed using a technique based on Gram determinants.

3.1 Residue of the planar pentabox

The decomposition of the pentabox diagram in Fig. [a] requires the parametrization of the residues of the eightfold cut \((12345678)\) and the sevenfold cuts depicted in Fig. [4]. The
| cut      | bases                                | $z$                                                                 | Monomials in the residue                                      |
|---------|--------------------------------------|----------------------------------------------------------------------|--------------------------------------------------------------|
| (12345678) | Eq. (3.13)                           | $(y_4, y_5, y_2, y_1, x_4, x_3, x_2, x_1)$                          | $S_{12345678} = \{ 1, x_1, y_1, y_2 \}$                     |
| (1234568)  | Eq. (3.13)                           | $(y_4, y_5, y_2, y_1, x_4, x_3, x_2, x_1)$                          | $S_{1234568} = \{ 1, x_1, x_2^2, x_3^2, x_4^2, x_2, x_1, x_2, x_3, x_4, x_1, x_3, x_4, x_2, x_1 \}$ |
| (1234678)  | Eq. (3.13)                           | $(y_4, y_5, y_2, y_1, x_4, x_3, x_2, x_1)$                          | $S_{1234678} = S_{1234568}$                                  |
| (1234578)  | Eq. (3.13)                           | $(y_4, y_5, y_2, y_1, x_4, x_3, x_2, x_1)$                          | $S_{1234578} = \{ 1, x_1, x_2^2, x_3^2, x_4^2, x_2, x_1, x_2, x_3, x_4, x_1, x_3, x_4, x_2, x_1 \}$ |
| (1235678)  | Eq. (3.15)                           | $(y_4, y_5, y_2, y_1, x_4, x_3, x_2, x_1)$                          | $S_{1235678} = \{ 1, x_1, x_2^2, x_3^2, x_4^2, x_2, x_1, x_2, x_3, x_4, x_1, x_3, x_4, x_2, x_1 \}$ |

| cut      | bases                                | $z$                                                                 | Monomials in the residue                                      |
|---------|--------------------------------------|----------------------------------------------------------------------|--------------------------------------------------------------|
| (12345678) | Eq. (3.13)                           | $(y_4, y_5, y_2, y_1, x_4, x_3, x_2, x_1)$                          | $S_{12345678} = \{ 1, x_1, y_1, y_2 \}$                     |
| (1234568)  | Eq. (3.13)                           | $(y_4, y_5, y_2, y_1, x_4, x_3, x_2, x_1)$                          | $S_{1234568} = \{ 1, x_1, x_2^2, x_3^2, x_4^2, x_2, x_1, x_2, x_3, x_4, x_1, x_3, x_4, x_2, x_1 \}$ |
| (1234678)  | Eq. (3.13)                           | $(y_4, y_5, y_2, y_1, x_4, x_3, x_2, x_1)$                          | $S_{1234678} = S_{1234568}$                                  |
| (1234578)  | Eq. (3.13)                           | $(y_4, y_5, y_2, y_1, x_4, x_3, x_2, x_1)$                          | $S_{1234578} = \{ 1, x_1, x_2^2, x_3^2, x_4^2, x_2, x_1, x_2, x_3, x_4, x_1, x_3, x_4, x_2, x_1 \}$ |
| (1235678)  | Eq. (3.15)                           | $(y_4, y_5, y_2, y_1, x_4, x_3, x_2, x_1)$                          | $S_{1235678} = \{ 1, x_1, x_2^2, x_3^2, x_4^2, x_2, x_1, x_2, x_3, x_4, x_1, x_3, x_4, x_2, x_1 \}$ |

Table 1: Set of monomials parametrizing the residues entering the decomposition of the five-point pentabox diagram. They have all been found using degree lexicographic monomial ordering. For each cut the bases and the chosen ordering for loop variables are shown as well.

parametrization is obtained using the procedure described in Section 2. The relevant bases are

\[
\begin{align*}
    r_0 & = 0^\mu, \quad e_1^\mu = p_3^\mu, \quad e_2^\mu = p_4^\mu, \quad e_3^\mu = \frac{3|\gamma^\mu|4}{2}, \quad e_4^\mu = \frac{4|\gamma^\mu|3}{2}, \\
    p_0 & = 0^\mu, \quad r_1^\mu = p_2^\mu, \quad r_2^\mu = p_3^\mu, \quad r_3^\mu = \frac{1|\gamma^\mu|2}{2}, \quad r_4^\mu = \frac{1|\gamma^\mu|2}{2}, \\
    x_1 & = \frac{(q-p_1)}{(p_1-p_2)}, \quad x_2 = \frac{(q-p_2)}{(p_1-p_2)}, \quad y_1 = \frac{(k-p_4)}{(p_4-p_4)} , \quad y_2 = \frac{(k-p_3)}{(p_3-p_4)} , \\
\end{align*}
\]

\[
\begin{align*}
    r_0 & = 0^\mu, \quad e_1^\mu = p_3^\mu, \quad e_2^\mu = p_3^\mu, \quad e_3^\mu = \frac{3|\gamma^\mu|4}{2}, \quad e_4^\mu = \frac{4|\gamma^\mu|3}{2}, \\
    p_0 & = 0^\mu, \quad r_1^\mu = p_1^\mu, \quad r_2^\mu = p_3^\mu, \quad r_3^\mu = \frac{3|\gamma^\mu|3}{4} + \frac{1|\gamma^\mu|3}{4}, \quad r_4^\mu = \frac{1|\gamma^\mu|2}{4}, \\
    x_1 & = \frac{(q-p_3)}{(p_1-p_3)}, \quad x_4 = \frac{(q-p_3)}{(p_1-p_3)}, \quad y_1 = \frac{(k-p_3)}{(p_1-p_3)} , \quad y_4 = \frac{(k-p_3)}{(p_1-p_3)} , \\
\end{align*}
\]

\[
\begin{align*}
    r_0 & = 0^\mu, \quad e_1^\mu = \frac{3|\gamma^\mu|4}{4} + \frac{1|\gamma^\mu|3}{4}, \quad e_2^\mu = p_3^\mu, \quad e_3^\mu = p_1^\mu, \\
    p_0 & = -p_4^\mu, \quad r_1^\mu = \frac{3|\gamma^\mu|3}{4} + \frac{1|\gamma^\mu|3}{4}, \quad r_2^\mu = p_3^\mu, \quad r_3^\mu = p_1^\mu, \\
    x_1 & = \frac{(q-p_4)}{e_1}, \quad x_2 = \frac{(q-p_4-p_1)}{(p_1-p_3)}, \quad y_1 = \frac{(k-r_1)}{r_1} , \quad y_3 = \frac{(k-p_3)}{(p_1-p_3)} .
\end{align*}
\]
Figure 3: Cut diagrams of the sevenfold cuts. Starting from the left, we show the diagram of the cut (1234568), (1234678), (1234578), and (1235678).

Table 2: The same as Table 1, but for the five-point crossed pentabox diagram.

The residue of each cut \((i_1 \ldots i_k)\) is written in terms of a set of monomials \(S_{i_1 \ldots i_k}\), collected in Table 1.

### 3.2 Residue of the crossed pentabox

The diagram in Fig. (b) is decomposed in terms of the residue of the eightfold cut (12345678) and of the residue of the sevenfold cuts in Fig. 3. Each residue can be expressed
The eightfold cut is a maximum cut. It exhibits eight solutions and it is depicted in Fig. 4. The remaining sevenfold cuts can be obtained using the transformation (3.16). The decomposition of the double pentagon diagram requires the parametrization of the residues of the eightfold cut (12345678) and all the sevenfold cuts. However, the topology in Fig. 4 (c) is invariant under the transformation

\[ \begin{align*}
    p_1^\mu & \leftrightarrow p_3^\mu, & p_4^\mu & \leftrightarrow p_5^\mu, & q^\mu & \leftrightarrow k^\mu; \\
\end{align*} \]

(3.16)

thus the only sevenfold cut needed are (1345678), (1245678), (2345678), and (1234567), depicted in Fig. 4. The remaining sevenfold cuts can be obtained using the transformation (3.16). The eightfold cut is a maximum cut. It exhibits eight solutions and it is parametrized by eight coefficients, in accordance with the maxim-cut theorem.

The sets of monomials parametrizing the relevant residues are collected in Table 3.

| cut         | bases                                              | Monomials in the residue |
|-------------|----------------------------------------------------|--------------------------|
| (12345678)  | Eq. (3.17)                                         | \( S_{12345678} = \{1, y_1, x_2, y_1 x_2, x_2^2, x_3, x_1, x_2 x_1\} \) |
| (1345678)   | Eq. (3.17) \((y_4, y_3, y_2, y_1, x_2, y_3, x_3, x_1, x_2)\) | \( S_{1345678} = \{1, y_1, y_2^2, y_1^2, y_1^3, y_1^4, y_2, y_1 y_2, y_1^2 y_2, y_1^3 y_2, y_1^4 y_2, x_1, y_1 x_1, y_1^2 x_1, y_1^3 x_1, y_1^4 x_1, y_2 x_1, y_1 y_2 x_1, y_1^2 y_2 x_1, y_1^3 y_2 x_1, y_1^4 y_2 x_1, y_1^5 y_2 x_1, x_2^2, y_1 x_2^2, y_1^2 x_2^2, y_1^3 x_2^2, y_1^4 x_2^2, y_2 x_2^2, x_1^2, y_1 x_1^2, y_2 x_1^2, x_1 x_2, x_2 x_1^2, x_2^2 x_1, x_1^2 x_2^2, x_2^2 x_1^2 x_2\} \) |
| (1245678)   | Eq. (3.17)                                         | \( S_{1245678} = S_{1345678} \) |
| (2345678)   | Eq. (3.17)                                         | \( S_{2345678} = \{1, y_1, y_1^2, y_1^3, y_1^4, y_2, y_1 y_4, y_1^2 y_4, y_1^3 y_4, x_2, y_1 x_2, y_1^2 x_2, y_1^3 x_2, y_1^4 x_2, y_1 x_4, y_1^2 x_4, y_1^3 x_4, y_1^4 x_4, y_4 x_4, y_1 y_4 x_4, y_1^2 y_4 x_4, y_1^3 y_4 x_4, y_1^4 y_4 x_4, y_1^5 y_4 x_4, x_2 x_4, x_1 x_2 x_4, x_2^2 x_1 x_2, y_1 x_1^2 x_2, y_1^2 x_1^2 x_2, y_1^3 x_1^2 x_2, y_1^4 x_1^2 x_2, y_1^5 x_1^2 x_2, x_1^2 x_2^2, x_2^2 x_1^2 x_2\} \) |
| (1234567)   | Eq. (3.17)                                         | \( S_{1234567} = \{1, y_1, y_1^2, y_1^3, y_1^4, y_2, y_1 y_2, y_2 y_1, y_2^2 y_1, y_1^2 y_2 x_1, y_1^3 y_2 x_1, y_1^4 y_2 x_1, y_1^5 y_2 x_1, y_2 x_1, y_1 y_2 x_1, y_1^2 y_2 x_1, y_1^3 y_2 x_1, y_1^4 y_2 x_1, y_1^5 y_2 x_1, y_2 x_1^2, x_1^2, y_1 x_1^2, y_2 x_1^2, x_1 x_2, x_2 x_1^2, x_2^2 x_1, x_1^2 x_2^2, x_2^2 x_1^2 x_2\} \) |

Table 3: The same as Table 1 but for the five-point double pentagon diagram in Fig. 4 (c).
They are obtained by multivariate polynomial division using the following bases

\[
\begin{cases}
  r_0^\mu = 0^\mu, \\
  e_1^\mu = p_1^\mu, \\
  e_2^\mu = p_2^\mu, \\
  e_3^\mu = \frac{\langle 4|\gamma^\mu|3 \rangle}{2}, \\
  e_4^\mu = \frac{\langle 3|\gamma^\mu|4 \rangle}{2},
\end{cases}
\]

\[
\begin{cases}
  P_0^\mu = 0^\mu, \\
  \tau_1^\mu = p_1^\mu, \\
  \tau_2^\mu = p_1^\mu, \\
  \tau_3^\mu = \frac{\langle 5|\gamma^\mu|1 \rangle}{2}, \\
  \tau_4^\mu = \frac{\langle 1|\gamma^\mu|5 \rangle}{2},
\end{cases}
\]  \tag{3.17}

\[
\begin{cases}
  r_0^\mu = 0^\mu, \\
  e_1^\mu = p_1^\mu, \\
  e_2^\mu = p_2^\mu, \\
  e_{3,4}^\mu = \frac{\langle 1|4|3|\gamma^\mu|1 \rangle \pm \langle 3|4|1|\gamma^\mu|3 \rangle}{4},
\end{cases}
\]

\[
\begin{cases}
  P_0^\mu = -p_3^\mu, \\
  \tau_1^\mu = p_1^\mu, \\
  \tau_2^\mu = p_3^\mu, \\
  \tau_{3,4}^\mu = \frac{\langle 1|4|3|\gamma^\mu|1 \rangle \pm \langle 3|4|1|\gamma^\mu|3 \rangle}{4},
\end{cases}
\]  \tag{3.18}

4. Seminumerical integrand reduction

In the previous section we illustrated how to determine the general structure of the residues by means of the multivariate polynomial division. Knowing this structure, we can proceed and numerically perform the integrand reduction to extract the values of all process-dependent coefficients which appear in the residues. The decomposition can be checked by verifying the identity between the original numerator and its reconstruction, i.e. between l.h.s. and r.h.s. of Eq. (3.12), for arbitrary values of the integration momenta \( q \) and \( k \). This procedure is known as global \( N = N \) test of the integrand reduction.

4.1 Planar pentabox diagram

**Eightfold cut.** The residue of the eightfold cut can be parametrized using the monomials in Table I

\[ \Delta_{12345678} = c_{12345678} q \cdot p_1 + c_{12345678} 1(q \cdot p_1) + c_{12345678} 2(k \cdot p_3) + c_{12345678} 3(k \cdot p_4). \]  \tag{4.1}

The number of solutions equals the number of coefficients, in accordance with the maximum-cut theorem. Therefore the four coefficients appearing in Eq. (4.1) can be obtained by
sampling the numerator on the four solutions of the eightfold cut, where the decomposition (3.12) becomes
\[ N^{(4,a)}_1 = \Delta_{12345678} . \] (4.2)

In our case we find that only \( c_{12345678,0} \) and \( c_{12345678,1} \) are nonvanishing.

**Sevenfold cut.** The residue of the generic sevenfold cut \((i_1 \cdots i_7)\) appearing in Eq. (3.12) can be parametrized using the results listed in Table 1. For the process at hand, the structure of the numerator ensures that residue can be parametrized just by a constant term:
\[ \Delta_{i_1 \cdots i_7} = c_{i_1 \cdots i_7,0} . \] (4.3)

The actual value of \( c_{i_1 \cdots i_7,0} \) is obtained by sampling the numerator and the residue of the eightfold cut in correspondence of one solution of the sevenfold cut, where
\[ \Delta_{i_1 \cdots i_7} = \frac{N^{(4,a)}_1(q,k) - \Delta_{12345678}}{\prod_{h \neq i_1 \cdots i_7} D_h} . \] (4.4)

The multipole decomposition of the integrand \( I^{(4,a)}_{1 \cdots 8} \) becomes
\[ I^{(4,a)}_{1 \cdots 8}(q,k) = \frac{c_{12345678,0} + c_{12345678,1}(q \cdot p_1)}{D_1 \cdots D_8} + \sum_{i=4}^{7} \frac{c_{i-(i-1)(i+1) \cdots 8,0}}{\prod_{h \neq i} D_h} . \] (4.5)

This result also shows the decomposition of the integral as linear combination of two MIs with eight denominators and four MIs with seven denominators.

**4.2 Crossed pentabox diagram**

**Eightfold cut.** The residue of the eightfold cut is parametrized as (cf. Table 2)
\[ \Delta_{12345678} = c_{12345678,0} + c_{12345678,1}(q \cdot p_1) + c_{12345678,2}(k \cdot p_3) + c_{12345678,3}(k \cdot p_4) . \] (4.6)

The coefficients are obtained sampling the numerator at the four solutions of the maximum cut \((12345678)\), where
\[ N^{(4,b)}_{1 \cdots 8} = \Delta_{12345678} . \] (4.7)

The only nonvanishing coefficients are \( c_{12345678,0} \) and \( c_{12345678,1} \).

**Sevenfold cut.** The generic sevenfold cut appearing in the multipole decomposition of \( I^{(4,b)}_{1 \cdots 8} \) is \((i_1 \cdots i_7) \in \{(1234568), (1234578), (1234678), (1235678)\}\). The structure of \( N^{(4,b)}_{1 \cdots 8} \) guarantees that the only nonvanishing coefficient is the one of the monomial 1, i.e.
\[ \Delta_{i_1 \cdots i_7} = c_{i_1 \cdots i_7,0} . \] (4.8)
We sample the numerator and the residue of the eightfold cut at a solution of the cut \((i_1 \cdots i_7)\) and we get \(c_{i_1 \cdots i_7,0}\) by using the relation

\[
\Delta_{i_1 \cdots i_7} = \frac{\mathcal{N}^{(4,oa)}_{1-8}(q,k) - \Delta_{12345678}}{\prod_{h \neq i_1 \cdots i_7} D_h} .
\] (4.9)

Within the numerical precision, the coefficients of the crossed pentabox turn out to be equal to the ones of the planar pentabox. This is expected since the two diagrams share the same numerator and also the denominators appearing in its decomposition are in common between the two.

The integrand \(T_{1-8}^{(4,b)}\) is decomposed as follows:

\[
T_{1-8}^{(4,b)}(q,k) = \frac{c_{12345678,0} + c_{12345678,1}(q \cdot p_1)}{D_1 \cdots D_8} + \sum_{i=4}^{7} \frac{c_{1 \cdots (i-1)(i+1) \cdots 8,0}}{\prod_{h \neq i} D_h} .
\] (4.10)

As for the previous diagram, this result yields the decomposition of the integral as linear combination of two MIs with eight denominators and four MIs with seven denominators.

### 4.3 Double pentagon diagram

**Eightfold cut.** The parametrization of the residue of the eightfold cut is given in Table 3 and can be written as

\[
\Delta_{12345678} = c_{12345678,0} + c_{12345678,1}(k \cdot p_3) + c_{12345678,2}(q \cdot p_5) \\
+ c_{12345678,3}(q \cdot p_1) + c_{12345678,4}(q \cdot p_5)^2 + c_{12345678,5}(q \cdot p_5)(q \cdot p_1) \\
+ c_{12345678,6}(k \cdot p_3)(q \cdot p_5) + c_{12345678,7}(q \cdot p_5)^3 .
\] (4.11)

The eightfold cut is a maximum cut, thus the eight solutions of the eightfold cut allow one to determine the coefficients in Eq. (4.11) using the relation

\[
\mathcal{N}^{(4,b)}_{1-8} = \Delta_{12345678} ,
\] (4.12)

which holds at the solutions of the eightfold cut. The nonvanishing coefficients are \(c_{12345678,i}\) for \(i \leq 4\).

**Sevenfold cut.** The numerator has rank one and it is easy to see that the generic sevenfold residue \(\Delta_{i_1 \cdots i_7}\) entering Eq. (3.12) can be parametrized by a constant term, i.e.

\[
\Delta_{i_1 \cdots i_7} = c_{i_1 \cdots i_7,0} .
\] (4.13)

The value of \(c_{i_1 \cdots i_7,0}\) is obtained by sampling the numerator and the residue of the eightfold cut at a solution of the cut \((i_1 \cdots i_7)\), where the relation

\[
\Delta_{i_1 \cdots i_7} = \frac{\mathcal{N}^{(4,c)}_{1-8}(q,k) - \Delta_{12345678}}{\prod_{h \neq i_1 \cdots i_7} D_h} .
\] (4.14)
holds. The multipole decomposition of the integrand of the double pentagon reads as follows

\[
I^{(4,c)}_{1\ldots 8}(q, k) = \frac{c_{12345678,0} + c_{12345678,1}(k \cdot p_3)}{D_1 \cdots D_8} + \frac{c_{12345678,2}(q \cdot p_5) + c_{12345678,3}(q \cdot p_1)}{D_1 \cdots D_8} + \sum_{i=1}^{8} \frac{c_{1\ldots(i-1)(i+1)\ldots,8,0}}{\prod_{h \neq i} D_h}.
\]

(4.15)

The corresponding decomposition of the integral is a linear combination of four MIs with eight denominators and eight MIs with seven denominators.

4.4 Unitarity-based construction

In the previous subsections as well as in the following sections, we apply the multiloop integrand-reduction method in the case of integrands provided by a diagrammatic representation of the scattering amplitude, where the full dependence on the loop momenta is known. The method can, however, be applied also using a unitarity-based representation of the integrands, where the latter is known only in correspondence to multiple cuts in physical channels as a state sum over the product of tree-level amplitudes.

In this case, the integrand of the (color ordered) amplitude in \( \mathcal{N} = 4 \) SYM reads as

\[
I^{(4)}(q, k) = \sum_{(i_1\ldots i_8)} \frac{\Delta_{i_1\ldots i_8}}{D_{i_1} \cdots D_{i_8}} + \sum_{(i_1\ldots i_7)} \frac{\Delta_{i_1\ldots i_7}}{D_{i_1} \cdots D_{i_7}},
\]

(4.16)

where the first (second) sum runs over all the eightfold (sevenfold) cuts.

The residue of the generic eightfold cut is given by

\[
\Delta_{i_1\ldots i_8} = \text{Res}_{i_1\ldots i_8} \left\{ I^{(4)} \right\},
\]

(4.17)

where the (maximum-cut) residue \( \text{Res}_{i_1\ldots i_8} \{ I^{(4)} \} \) is the state sum over the product of seven three-point tree-level amplitudes.

The residue of the generic sevenfold cut reads instead

\[
\Delta_{i_1\ldots i_7} = \text{Res}_{i_1\ldots i_7} \left\{ I^{(4)} - \sum_{(i_1\ldots i_8)} \frac{\Delta_{i_1\ldots i_8}}{D_{i_1} \cdots D_{i_8}} \right\},
\]

(4.18)

where \( \text{Res}_{i_1\ldots i_7} \{ I^{(4)} \} \) is the state-sum product of six tree-level amplitudes, while the sum runs over all the sets \( (i_1 \cdots i_8) \) containing \( (i_1 \cdots i_7) \) as a subset.

The extension of the algorithm to lower cuts, if needed, is straightforward.

5. Five-point amplitudes in \( \mathcal{N} = 8 \) SUGRA

The five-point amplitude in \( \mathcal{N} = 8 \) SUGRA can be expressed in terms of the same six diagrams as in \( \mathcal{N} = 4 \) SYM [53]. Again, the color ordered amplitude is given by a sum
over the cyclic permutations of the external momenta. In $\mathcal{N} = 8$ SUGRA, the numerator of each integrand is obtained by squaring the corresponding numerator in $\mathcal{N} = 4$ SYM, as shown in [53]. We apply the integrand reduction only to the three diagrams depicted in Fig. [4] whose numerator exhibits a nontrivial dependence on the loop momenta. In the following we denote the integrand (numerator) of the diagram in Fig. [4] (i) by $\mathcal{I}^{(8,i)}(\mathcal{N}^{(8,i)})$.

The numerators are of rank two in the loop momenta. Following the same machinery as in the case of $\mathcal{N} = 4$ SYM, we show that their decomposition can be expressed in terms of 8-, 7-, and 6-denominator integrands:

$$
\mathcal{N}^{(8,x)}_{1\cdots 8}(q,k) = \Delta_{12345678} + \sum_{i=1}^{8} \Delta_{1\cdots (i-1)(i+1)\cdots 8} D_i \\
+ \sum_{i<j=1}^{8} \Delta_{1\cdots (i-1)(i+1)\cdots (j-1)(j+1)\cdots 8} D_i D_j, \quad x = a, b, c .
$$

The corresponding decomposition for the integrands $\mathcal{I}^{(4,a)}_{1\cdots 8}$, $\mathcal{I}^{(4,b)}_{1\cdots 8}$ and $\mathcal{I}^{(4,c)}_{1\cdots 8}$ reads

$$
\mathcal{I}^{(4,x)}_{1\cdots 8}(q,k) = \frac{\Delta_{12345678}}{D_1 \cdots D_8} + \sum_{i=1}^{8} \frac{\Delta_{1\cdots (i-1)(i+1)\cdots 8}}{\prod_{h \neq i} D_h} \\
+ \sum_{i<j=1}^{8} \frac{\Delta_{1\cdots (i-1)(i+1)\cdots (j-1)(j+1)\cdots 8}}{\prod_{h \neq i,j} D_h}, \quad x = a, b, c .
$$

Since the numerators $\mathcal{N}^{(8,a)}_{1\cdots 8}$ and $\mathcal{N}^{(8,b)}_{1\cdots 8}$ are of rank two in $q$ and independent of $k$, their decomposition is significantly simplified. Indeed in these cases $\Delta_{1\cdots (i-1)(i+1)\cdots 8} = 0$ for $i \neq 4, 5, 6, 7$ and $\Delta_{1\cdots (i-1)(i+1)\cdots (j-1)(j+1)\cdots 8} = 0$ for $i, j \neq 4, 5, 6, 7$.

### 5.1 Seminumerical computation

In this section we briefly describe the numerical decomposition of the three numerators. We checked the decomposition by verifying that at arbitrary values of $q$ and $k$ the numerator and its reconstruction are equal.

### Planar pentabox

**Eightfold cut.** The computation of the residue of the eightfold cut $\Delta_{12345678}$ follows the same pattern as the $\mathcal{N} = 4$ SYM planar pentabox, described in Section [1]. The nonvanishing coefficients are $c_{12345678,0}$ and $c_{12345678,1}$.

**Sevenfold cut.** The residue of the generic sevenfold cut $(i_1 \cdots i_7)$ in Eq. (5.1) can be parametrized in terms of the monomials collected in Table 1. The structure of the numerator guarantees that the residue contains rank-one terms at most:

$$
\Delta_{i_1 \cdots i_7} = c_{i_1 \cdots i_7,0} + c_{i_1 \cdots i_7,1}(q + w_0) \cdot w_1 + c_{i_1 \cdots i_7,2}(q + w_0) \cdot w_2 \\
+ c_{i_1 \cdots i_7,3}(k + w_3) \cdot w_4 + c_{i_1 \cdots i_7,4}(k + w_3) \cdot w_5 .
$$

\[5.3\]
The momenta $w_i^\mu$ depend on the cut $(i_1 \cdots i_7)$. The actual value of the coefficients can be obtained by sampling on five independent solutions of the sevenfold cut, where

$$\Delta_{i_1 \cdots i_7} = \frac{\mathcal{N}_{1 \cdots 8}^{(8,a)}(q, k) - \Delta_{12345678}}{\prod_{h \neq i_1, \ldots, i_7} D_h}.$$  (5.4)

In this case, all the coefficients but $c_{i_1 \cdots i_7, 0}$, $c_{i_1 \cdots i_7, 1}$, and $c_{i_1 \cdots i_7, 2}$ vanish.

**Sixfold cut.** The numerator $\mathcal{N}_{1 \cdots 8}^{(8,a)}$ has rank two in $q$ and is independent of $k$. The only nonvanishing term in the residue of the generic sixfold cut $(i_1 \cdots i_6)$ in (5.1) is the constant; therefore we have

$$\Delta_{i_1 \cdots i_6} = c_{i_1 \cdots i_6, 0}.$$  (5.5)

The actual value of the constant can be obtained evaluating the decomposition (5.1) at one solution of the sixfold cut, where

$$\Delta_{i_1 \cdots i_6} = \frac{\mathcal{N}_{1 \cdots 8}^{(8,a)}(q, k) - \Delta_{12345678}}{\prod_{h \neq i_1, \ldots, i_6} D_h} - \sum_{k=4}^{7} \frac{\Delta_{i_1 \cdots h \cdots i_6}}{D_h}.$$  (5.6)

After polynomial fitting of $\Delta_{12345678}$, $\Delta_{i_1 \cdots i_7}$ and $\Delta_{i_1 \cdots i_8}$, the resulting multipole decomposition of Eq. (5.2) contains 20 nonvanishing coefficients two of which are spurious (i.e. their contribution vanishes upon integration) while the others give rise to MIs, namely two with eight denominators, ten with seven denominators and six with six denominators.

**Crossed pentabox**

As in the $\mathcal{N} = 4$ SYM case, the crossed pentabox in Fig 1 (b) has the same numerator and the same decomposition as the planar pentabox. Therefore the coefficients of the former are exactly the same as the coefficients of the latter.

**Double pentagon**

**Eightfold cut.** The computation residue of the eightfold cut of the double pentagon follows the same lines of the $\mathcal{N} = 4$ SYM double pentagon (see Section 4.3). The parametrization of the residue is given in Eq. (4.11). In this case the only vanishing coefficient is $c_{12345678, 7}$.

**Sevenfold cut.** The residue $\Delta_{i_1 \cdots i_7}$ of the generic cut $(i_1 \cdots i_7)$ can be parametrized using Eq. (5.3). At the sevenfold cut $(i_1 \cdots i_7)$ the decomposition (5.1) reduces to

$$\Delta_{i_1 \cdots i_7} = \frac{\mathcal{N}_{1 \cdots 8}^{(8,e)}(q, k) - \Delta_{12345678}}{\prod_{h \neq i_1, \ldots, i_7} D_h}.$$  (5.7)

The coefficients are then computed by sampling Eq. (5.7) at five solutions of the sevenfold cut. The nonvanishing ones are those multiplying constant or linear terms in the loop momenta.
Sixfold cut. The residue $\Delta_{i_1 \cdots i_6}$ of the generic sixfold cut $(i_1 \cdots i_6)$ can be parametrized by a constant, as in Eq. (5.5). The constant is computed using one solution of the sixfold cut and the expression of the decomposition (5.1) at the sixfold cut:

$$
\Delta_{i_1 \cdots i_6} = \frac{N^{(8,a)}(q, k) - \Delta_{12345678}}{\prod_{h \neq i_1, \ldots, i_6} D_h} - \sum_{h \neq i_1, \ldots, i_6}^{8} \frac{\Delta_{i_1 \cdots h \cdots i_6}}{D_h}.
$$

(5.8)

We find that $\Delta_{123456} = 0$, while the residues of all the other sixfold cuts are nonvanishing.

After polynomial fitting of $\Delta_{12345678}$, $\Delta_{i_1 \cdots i_7}$ and $\Delta_{i_1 \cdots i_6}$, the resulting multipole decomposition of Eq. (5.2) contains in this case 74 nonvanishing coefficients, four of which are spurious. The integral can be decomposed as a linear combination of seven MIs with eight denominators, 36 MIs with seven denominators and 27 MIs with six denominators.

6. Analytic integrand reduction

In this section we perform the reduction of the five-point diagrams analytically. We apply a two-loop generalization of the integrand reduction through Laurent expansion formulated in [54]. As in the one-loop case, the Laurent expansion allows one to find simpler formulas for the coefficients entering the decomposition. Moreover, the subtraction of the higher residues can be performed at the coefficient level rather than at the integrand level. Indeed the Laurent expansion makes each function entering the reduction separately polynomial. Therefore the subtraction can be omitted during the reduction and accounted for correcting the reconstructed coefficients. For simplicity we will focus on the rank-one numerators in the five-point integrands of $\mathcal{N} = 4$ SYM. The method can, however, be extended to higher-rank numerators as described in Section 6.4 for the planar pentabox diagram in $\mathcal{N} = 8$ SUGRA.

6.1 Planar pentabox diagram

The analytic decomposition of the pentabox diagram in Fig. 1(a) we are about to discuss, is valid for any numerator of the type (3.1), i.e. for any rank-one numerator depending on $q$ only. Indeed our computation is carried out for generic $u_1^\mu$ and $\beta_1$; the results for $\mathcal{N} = 4$ SYM will be recovered at the very end, using Eqs. (3.4) and (3.7).

Eightfold cut. The four solutions of the eightfold cut (12345678) are

$$
\begin{align*}
(q_1^\mu, k_1^\mu) &= \left( \begin{array}{c} 4 \end{array} \right) \left( \begin{array}{c} 3 \end{array} \right) \mu \left( \begin{array}{c} 4 \end{array} \right) \left( \begin{array}{c} 1 \end{array} \right) \mu \left( \begin{array}{c} 2 \end{array} \right) \mu, \left( \begin{array}{c} 5 \end{array} \right) \left( \begin{array}{c} 3 \end{array} \right) \mu \left( \begin{array}{c} 3 \end{array} \right) \mu \left( \begin{array}{c} 2 \end{array} \right) \mu \end{align*}
$$

$$
\begin{align*}
(q_2^\mu, k_2^\mu) &= \left( \begin{array}{c} 4 \end{array} \right) \left( \begin{array}{c} 3 \end{array} \right) \mu \left( \begin{array}{c} 3 \end{array} \right) \mu \left( \begin{array}{c} 2 \end{array} \right) \mu, \left( \begin{array}{c} 5 \end{array} \right) \left( \begin{array}{c} 1 \end{array} \right) \mu \left( \begin{array}{c} 2 \end{array} \right) \mu \left( \begin{array}{c} 1 \end{array} \right) \mu \end{align*}
$$

$$
\begin{align*}
(q_3^\mu, k_3^\mu) &= \left( \begin{array}{c} 4 \end{array} \right) \left( \begin{array}{c} 4 \end{array} \right) \mu \left( \begin{array}{c} 3 \end{array} \right) \mu \left( \begin{array}{c} 3 \end{array} \right) \mu \left( \begin{array}{c} 3 \end{array} \right) \mu \left( \begin{array}{c} 2 \end{array} \right) \mu \left( \begin{array}{c} 2 \end{array} \right) \mu \end{align*}
$$

$$
\begin{align*}
(q_4^\mu, k_4^\mu) &= \left( \begin{array}{c} 4 \end{array} \right) \left( \begin{array}{c} 4 \end{array} \right) \mu \left( \begin{array}{c} 4 \end{array} \right) \mu \left( \begin{array}{c} 3 \end{array} \right) \mu \left( \begin{array}{c} 3 \end{array} \right) \mu \left( \begin{array}{c} 2 \end{array} \right) \mu \left( \begin{array}{c} 2 \end{array} \right) \mu \end{align*}
$$

The general parametrization of the residue $\Delta_{12345678}$ is given in Eq. (4.1). The simple form of the numerator $N^{(4,a)}_{1-8}$ implies that the coefficients $c_{12345678,2}$ and $c_{12345678,3}$ vanish. The nonvanishing coefficients are obtained by sampling at the solutions $(q_1, k_1)$ and $(q_3, k_3)$.
only. The outcome is
\[
c_{12345678,0} = -\frac{1}{\langle 54 \rangle \langle 31 \rangle \langle 53 \rangle \langle 41 \rangle - \langle 53 \rangle \langle 41 \rangle \langle 54 \rangle \langle 31 \rangle} \times (\langle 54 \rangle \langle 41 \rangle \langle 3 \rangle u_{1} \langle 4 \rangle \langle 54 \rangle \langle 31 \rangle - \langle 54 \rangle \langle 31 \rangle \langle 4 \rangle u_{1} \langle 3 \rangle \langle 54 \rangle \langle 41 \rangle - \beta_{1} \langle 54 \rangle \langle 31 \rangle \langle 53 \rangle \langle 41 \rangle + \beta_{1} \langle 53 \rangle \langle 41 \rangle \langle 54 \rangle \langle 31 \rangle) \tag{6.1}
\]

\[
c_{12345678,1} = -2 \frac{\langle 54 \rangle \langle 3 \rangle u_{1} \langle 4 \rangle \langle 53 \rangle \langle 41 \rangle - \langle 53 \rangle \langle 41 \rangle \langle 54 \rangle \langle 31 \rangle}{\langle 54 \rangle \langle 31 \rangle \langle 53 \rangle \langle 41 \rangle - \langle 53 \rangle \langle 41 \rangle \langle 54 \rangle \langle 31 \rangle} \tag{6.2}
\]

**Sevenfold cuts.** We discuss the generic sevenfold cut \((i_{1} \cdots i_{7})\) appearing the decomposition \((3.12)\). For later convenience, we define the uncut propagator

\[
D_{is}(q,k) = (q + P_{is})^{2}, \quad \text{with } i_{8} \in \{1, \ldots, 8\} \quad \text{and } i_{8} \neq i_{1}, \ldots, i_{7}. \tag{6.3}
\]

The momentum \(P_{is}^{\mu}\) is a linear combination of external momenta and it can be inferred from Fig. 1 (a), for instance \(P_{7}^{\mu} = -p_{4}^{\mu} - p_{5}^{\mu}\). The simplicity of the numerator allows one to parametrize the residue using the constant term \(c_{i_{1} \cdots i_{7},0}\); cf. Eq. (I.3).

Every sevenfold cut of this topology exhibits a \(t\)-dependent family of solutions of the type

\[
(q_{1}^{\mu},k_{1}^{\mu}) = (v_{q1,1}^{\mu} t + v_{q1,0}^{\mu}, v_{k1}^{\mu}). \tag{6.4}
\]

The coefficient \(c_{i_{1} \cdots i_{7},0}\) can be computed evaluating Eq. (I.4) at the solutions (6.4). The computation can be simplified performing a Laurent expansion around \(t = \infty\):

\[
\left[ \frac{\mathcal{N}_{1 \cdots 8}^{(4,a)}(q_{1},k_{1})}{D_{is}(q_{1},k_{1})} - \frac{\Delta_{12345678}(q_{1},k_{1})}{D_{is}(q_{1},k_{1})} \right] \bigg|_{t \to \infty} = c_{i_{1} \cdots i_{7},0} \tag{6.5}
\]

Indeed in general neither

\[
\left. \frac{\mathcal{N}_{1 \cdots 8}^{(4,a)}(q_{1},k_{1})}{D_{is}(q_{1},k_{1})} \right|_{t \to \infty} \quad \text{nor} \quad \left. \frac{\Delta_{12345678}(q_{1},k_{1})}{D_{is}(q_{1},k_{1})} \right|_{t \to \infty}
\]

are polynomial in \(t\) but only their difference is. However, their truncated Laurent expansion obtained neglecting \(\mathcal{O}(1/t)\) terms is polynomial in \(t\), namely in this case a constant

\[
\left. \frac{\mathcal{N}_{1 \cdots 8}^{(4,a)}(q_{1},k_{1})}{D_{is}(q_{1},k_{1})} \right|_{t \to \infty} = n_{i_{1} \cdots i_{7},0} + \mathcal{O} \left( \frac{1}{t} \right), \quad \left. \frac{\Delta_{12345678}(q_{1},k_{1})}{D_{is}(q_{1},k_{1})} \right|_{t \to \infty} = b_{i_{1} \cdots i_{7},0} + \mathcal{O} \left( \frac{1}{t} \right). \tag{6.7}
\]

Therefore, similarly to the one-loop case [54], the coefficients \(n_{i_{1} \cdots i_{7},0}\) and \(b_{i_{1} \cdots i_{7},0}\) can be computed separately obtaining the coefficient \(c_{i_{1} \cdots i_{7},0}\) by their difference:

\[
c_{i_{1} \cdots i_{7},0} = n_{i_{1} \cdots i_{7},0} - b_{i_{1} \cdots i_{7},0}. \tag{6.8}
\]

The subtraction can be performed at the coefficient level rather than at the integrand level. Moreover the known structure of \(\Delta_{12345678}\) allows one to compute the coefficient \(b_{i_{1} \cdots i_{7},0}\) once and for all, irrespective of the actual form of the numerator:

\[
b_{i_{1} \cdots i_{7},0} = \frac{c_{12345678,1}(p_{1} \cdot v_{q1,1})}{2(P_{is} + v_{q1,0}) \cdot v_{q1,1}}. \tag{6.9}
\]
The coefficients \( n_{i_1 \cdots i_7, 0} \) read as follows:

\[
\begin{align*}
n_{1234568, 0} &= -\frac{u_1 \cdot v_{q_1, 1}}{p_5 \cdot v_{q_1, 1}} \quad \text{with} \quad v_{q_1, 1}^{\mu} = \frac{\langle 3|\gamma^\mu|4 \rangle}{2}, \\
n_{1234678, 0} &= \frac{u_1 \cdot v_{q_1, 1}}{p_3 \cdot v_{q_1, 1}} \quad \text{with} \quad v_{q_1, 1}^{\mu} = \frac{\langle 5|\gamma^\mu|4 \rangle}{2}, \\
n_{1234578, 0} &= -\frac{u_1 \cdot v_{q_1, 1}}{p_4 \cdot v_{q_1, 1}} \quad \text{with} \quad v_{q_1, 1}^{\mu} = \frac{\langle 3|\gamma^\mu|P_{345} \rangle}{2}, \\
n_{1235678, 0} &= -\frac{u_1 \cdot v_{q_1, 1}}{p_4 \cdot v_{q_1, 1}} \quad \text{with} \quad v_{q_1, 1}^{\mu} = \frac{\langle 5|\gamma^\mu|P_{345} \rangle}{2}.
\end{align*}
\] (6.10)

The massless momentum \( P_{abc} \) is defined as

\[
P_{abc}^{\mu} = p_b^{\mu} + p_c^{\mu} - \frac{s_{bc}}{2(p_b + p_c) \cdot p_a^{\mu}}. \quad (6.11)
\]

As already stated, the foregoing discussion applies to any numerator of the form given in Eq. (6.11). By using the explicit expressions of \( u_1 \) and \( \beta_1 \) given in Eqs. (3.4) and (3.7), we can write down the results for the coefficients in \( N = 4 \) SYM in terms of the functions \( \gamma \) defined in Eq. (3.11)

\[
\begin{align*}
c_{1234568, 0} &= \frac{1}{2} (\gamma_{12345}(s_{23} - s_{13} - s_{45}) + s_{12}(\gamma_{34125} + \gamma_{35124} + \gamma_{45123})) \\
c_{1234568, 1} &= -2 \gamma_{12345} \\
c_{1234568, 0} &= \frac{1}{4}(\gamma_{12345} + s_{45123}) \\
c_{1234578, 0} &= \frac{1}{4}(\gamma_{12345} + s_{45123}) \\
c_{1235678, 0} &= \frac{1}{4}(\gamma_{12345} + s_{45123}) \\
c_{1235678, 0} &= \frac{1}{4}(\gamma_{12345} + s_{45123}). \quad (6.12)
\end{align*}
\]

The complete integrand decomposition is obtained plugging the coefficients of Eq. (6.12) in Eq. (4.3). These results are in agreement with the ones found in the numerical computation of Section 4.1.

### 6.2 Crossed pentabox diagram

As already noticed in Section 4.2, the crossed pentabox of Fig. 1 (b) and the planar pentabox have the same decomposition. Indeed the numerator \( N_{1 \ldots 8}^{(4, a)} \) and the numerator \( N_{1 \ldots 8}^{(4, b)} \), Eq. (6.2), are equal and are decomposed in terms of the same denominators; cf. Eqs. (4.11) and (4.10) multiplied by \( D_i \ldots D_{i_8} \). Therefore the coefficients appearing in the multipole decomposition (4.10) are equal to the corresponding ones appearing in the decomposition (4.11) of the planar pentabox.

### 6.3 Double pentagon diagram

The numerator \( N_{1 \ldots 8}^{(4, c)} \), Eq. (3.3), is highly symmetric in \( q \) and \( k \). Indeed the dependence on \( q \) and \( k \) can be disentangled and \( N_{1 \ldots 8}^{(4, c)} \) can be cast in the following form:

\[
N_{1 \ldots 8}^{(4, c)}(q, k) = R_{1 \ldots 8}^{(4, c)}(q) - T_{1 \ldots 8}^{(4, c)}(k), \quad (6.13)
\]
where
\[
\mathcal{R}^{(4,c)}_{1-8}(\ell) \equiv 2 \ell \cdot u_2 + \beta_2, \quad \mathcal{T}^{(4,c)}_{1-8}(k) = \mathcal{R}^{(4,c)}_{1-8}(k) \big|_{p_1 \leftrightarrow p_3, p_4 \leftrightarrow p_5} \tag{6.14}
\]

The vector \( u_2 \) and the constant \( \beta_2 \) are defined in Eqs. (3.5) and (3.8), respectively. Although in principle there are no additional issues in performing the reduction of the full numerator, the computation can be simplified by performing the reduction of the eightfold cut, e.g. The two remaining coefficients can be obtained by sampling the numerator on two solutions of the eightfold cut (12345678). They can be decomposed as follows:
\[
\mathcal{R}^{(4,c)}_{1-8}(q) = \Delta_{12345678} + \sum_{i=1}^{3} \Delta_{1-8(i-1)i(i+1)8} D_i. \tag{6.15}
\]

**Eightfold cut.** The solutions of the eightfold cut (12345678) are eight and they can be used to compute the eight coefficients parametrizing the residue
\[
\Delta_{12345678} = \hat{c}_{12345678,0} + \hat{c}_{12345678,1}(k \cdot p_3) + \hat{c}_{12345678,2}(q \cdot p_5) \\
+ \hat{c}_{12345678,3}(q \cdot p_1) + \hat{c}_{12345678,4}(q \cdot p_5)^2 + \hat{c}_{12345678,5}(q \cdot p_5)(q \cdot p_1) \\
+ \hat{c}_{12345678,6}(k \cdot p_3)(q \cdot p_5) + \hat{c}_{12345678,7}(q \cdot p_5)^3. \tag{6.16}
\]
The rank of \( \mathcal{R}^{(4,c)}_{1-8} \) implies that for \( i \geq 4 \) \( \hat{c}_{12345678,i} = 0 \). The simplicity of the numerator simplifies the computation even further. Indeed decomposing \( u_2^\mu \) in the basis \( \{ p_i \}_{i=1,\ldots,4} \), and using the conditions \( D_1 = D_2 = D_3 = 0 \) we get
\[
\hat{c}_{12345678,0} = \beta_2, \quad \hat{c}_{12345678,1} = 0. \tag{6.17}
\]
The two remaining coefficients can be obtained by sampling the numerator on two solutions of the eightfold cut, e.g.
\[
(k^\mu_1, q^\mu_1) = \left( \begin{array}{c} \langle 41 \rangle \langle 5 \rangle | \gamma^\mu | 1 \\ \langle 45 \rangle \end{array} \right), \quad \langle 35 \rangle \langle 4 \rangle | \gamma^\mu | 3 \\
\langle 45 \rangle \end{array} \right),
\]
\[
(k^\mu_2, q^\mu_2) = \left( \begin{array}{c} \langle 41 \rangle \langle 1 \rangle | \gamma^\mu | 5 \\ \langle 45 \rangle \end{array} \right), \quad \langle 35 \rangle \langle 3 \rangle | \gamma^\mu | 4 \\
\langle 45 \rangle \end{array} \right).
\]
The missing coefficients read as follows:
\[
\hat{c}_{12345678,2} = -2 \left( \frac{\langle 41 \rangle \langle 5 \rangle | 3 | u_2 | 4 \rangle - \langle 31 \rangle \langle 4 \rangle | u_2 | 3 \rangle}{\langle 45 \rangle \langle 3 \rangle | 1 \rangle | 4 \rangle - \langle 35 \rangle \langle 4 \rangle | 1 \rangle | 3 \rangle}, \\
\hat{c}_{12345678,3} = 2 \left( \frac{\langle 45 \rangle \langle 3 \rangle | u_2 | 4 \rangle - \langle 35 \rangle \langle 4 \rangle | u_2 | 3 \rangle}{\langle 45 \rangle \langle 3 \rangle | 1 \rangle | 4 \rangle - \langle 35 \rangle \langle 4 \rangle | 1 \rangle | 3 \rangle}. \tag{6.18}
\]

**Sevenfold cuts.** We consider the generic sevenfold cut \( (i_1 \ldots i_7) \) appearing in the decomposition (6.13). The solutions of the cut can be cast into one-parameter families. In particular each cut allows for a solution with the following asymptotic behavior
\[
(k^\mu_1, q^\mu_1) = \left( v^\mu_{k_1,1} t + v^\mu_{k_1,0} + \mathcal{O} \left( \frac{1}{t} \right), \right) v^\mu_{q_1,1} t + v^\mu_{q_1,0} + \mathcal{O} \left( \frac{1}{t} \right) \right) \quad \text{for } t \to \infty. \tag{6.19}
\]
We compute the coefficient $\hat{c}_{1345678,0}$ by evaluating the decomposition (6.15) at one solution of the sevenfold cut, and by expanding around $t = \infty$:

$$\left[ \frac{\mathcal{R}_{1\cdots 8}^{(4,c)}(q_1)}{D_{is}(q_1, k_1)} - \frac{\Delta_{12345678}(q_1, k_1)}{D_{is}(q_1, k_1)} \right]_{t \to \infty} = \hat{c}_{i_1 \cdots i_7, 0}.$$  (6.20)

The denominator $D_{is}$ is written in terms of $P_{is}^\mu$ as in Eq. (6.3). The actual form of $P_{is}^\mu$ is inferred from Fig. 1 (c). Also in this case the $t \to \infty$ limit makes both

$$\frac{\mathcal{R}_{1\cdots 8}^{(4,c)}(q_1)}{D_{is}(q_1, k_1)} \text{ and } \frac{\Delta_{12345678}(q_1, k_1)}{D_{is}(q_1, k_1)}$$

polynomial in $t$,

$$\left. \frac{\mathcal{R}_{1\cdots 8}^{(4,c)}(q_1)}{D_{is}(q_1, k_1)} \right|_{t \to \infty} = \hat{n}_{i_1 \cdots i_7, 0} + O\left(\frac{1}{t}\right), \quad \left. \frac{\Delta_{12345678}(q_1, k_1)}{D_{is}(q_1, k_1)} \right|_{t \to \infty} = \hat{b}_{i_1 \cdots i_7, 0} + O\left(\frac{1}{t}\right).$$  (6.22)

The coefficients $\hat{n}_{i_1 \cdots i_7, 0}$ and $\hat{b}_{i_1 \cdots i_7, 0}$ can be computed separately, and

$$\hat{c}_{i_1 \cdots i_7, 0} = \hat{n}_{i_1 \cdots i_7, 0} - \hat{b}_{i_1 \cdots i_7, 0}.$$  (6.23)

Therefore the subtraction can be performed at the coefficient level via the universal function

$$\hat{b}_{i_1 \cdots i_7, 0} = \frac{\hat{c}_{12345678,2}(v_{q_1,1} \cdot p_5) + \hat{c}_{12345678,3}(v_{q_1,1} \cdot p_1)}{2(v_{q_1,0} + P_{is}) \cdot v_{q_1,1}}.$$  (6.24)

The coefficients $\hat{n}_{i_1 \cdots i_7, 0}$ are given by

$$\hat{n}_{1345678,0} = \frac{u_2 \cdot v_{q_1}}{p_3 \cdot v_{q_1}}, \quad \text{with} \quad v_\mu = \eta_1 \frac{\langle 4 | \gamma^\mu | 3 \rangle}{2} + \eta_2 p_4^\mu,$$

$$\hat{n}_{1245678,0} = \frac{u_2 \cdot v_{q_1}}{p_4 \cdot v_{q_1}}, \quad \text{with} \quad v_\mu = \eta_3 \frac{\langle 4 | \gamma^\mu | 3 \rangle}{2} + \eta_4 p_3^\mu,$$

$$\hat{n}_{12345678,0} = \frac{u_2 \cdot v_{q_1}}{p_3 \cdot v_{q_1}}, \quad \text{with} \quad v_\mu = \eta_5 (p_3^\mu - p_4^\mu) + \eta_6 \frac{\langle 3 | \gamma^\mu | 4 \rangle}{2} + \eta_7 \frac{\langle 4 | \gamma^\mu | 3 \rangle}{2}.$$  (6.25)

We define

$$\eta_1 = -\frac{\langle 5 | 2 | 4 | 1 \rangle}{\langle 4 | 2 | 3 | 4 \rangle},$$

$$\eta_2 = \frac{\langle 5 | 2 | 3 | 1 \rangle}{\langle 4 | 2 | 3 | 4 \rangle},$$

$$\eta_3 = \frac{\langle 3 | 5 | 1 | 2 \rangle}{\langle 3 | 4 | 3 | 2 \rangle},$$

$$\eta_4 = \frac{\langle 4 | 5 | 1 | 2 \rangle}{\langle 3 | 4 | 3 | 2 \rangle},$$

$$\eta_5 = -\sigma_1 + \sqrt{\sigma_1^2 - 4 \sigma_2 \sigma_3},$$

$$\eta_6 = \frac{2 \sigma_2 \sigma_8 - \sigma_1 \sigma_4 + \sigma_4 \sqrt{\sigma_1^2 - 4 \sigma_2 \sigma_3}}{2 \sigma_2 \sigma_5},$$

$$\eta_7 = -\frac{2 \sigma_2 \sigma_7 - \sigma_1 \sigma_6 + \sigma_6 \sqrt{\sigma_1^2 - 4 \sigma_2 \sigma_3}}{2 \sigma_2 \sigma_5}.$$  (6.26)
in terms of

\[
\begin{align*}
\sigma_1 &= -4(\sigma_7\sigma_4 + \sigma_6\sigma_8)(p_1 \cdot p_2), \\
\sigma_2 &= -4(\sigma_6\sigma_4 - \sigma_2^2)(p_1 \cdot p_2), \\
\sigma_3 &= -4\sigma_7\sigma_8(p_1 \cdot p_2), \\
\sigma_4 &= 2(p_1 \cdot p_5)\langle 4 5 \rangle[3 1] - 2(p_2 \cdot p_5)\langle 4 5 \rangle[3 1] + \langle 3 5 \rangle\langle 4 2 \rangle[3 1][3 2] - \langle 4 5 \rangle\langle 4 2 \rangle[3 2][4 1], \\
\sigma_5 &= -\langle 3 5 \rangle\langle 4 2 \rangle[3 2][4 1] + (\langle 3 2 \rangle\langle 4 5 \rangle[3 1][4 2], \\
\sigma_6 &= 2(p_2 \cdot p_3)\langle 3 5 \rangle[4 1] - 2(p_2 \cdot p_5)\langle 3 5 \rangle[4 1] + \langle 3 5 \rangle\langle 3 2 \rangle[3 1][4 2] - \langle 3 2 \rangle\langle 4 5 \rangle[4 1][4 2], \\
\sigma_7 &= \langle 3 5 \rangle\langle 5 2 \rangle[4 1][1 2], \\
\sigma_8 &= \langle 4 5 \rangle\langle 5 2 \rangle[3 1][1 2].
\end{align*}
\]  

(6.27)

This completes the reduction of a generic numerator \( R^{(4,c)}_{1\ldots8} \) of the form given in Eq. (6.14). As previously stated, the reduction of the full numerator \( N^{(4,c)}_{1\ldots8} \) of \( N = 4 \) SYM can be recovered from the one discussed in this section, by means of Eqs. (6.13) and (6.14). To that purpose, we observe that the substitutions (3.16) give the one-to-one mapping between denominators

\[
D_1 \leftrightarrow D_4, \quad D_2 \leftrightarrow D_6, \quad D_3 \leftrightarrow D_5, \quad D_7 \leftrightarrow D_8.
\]

Putting everything together and using the definitions of \( u_2 \) and \( \beta_2 \) of Eqs. (3.3) and (3.8) we recover the full multipole decomposition of Eq. (4.15), with coefficients

\[
\begin{align*}
c_{12345678, 0} &= \frac{1}{4} \left( \gamma_{34125} (2s_{13} - 2s_{35} + 2s_{23} + s_{15} + s_{34}) \\
&\quad - \gamma_{15234} (2s_{13} - 2s_{35} + 2s_{12} + s_{15} + s_{34}) \\
&\quad - (\gamma_{23145} + \gamma_{24135}) s_{23} - (\gamma_{25134} - \gamma_{12345}) (s_{35} - s_{14} - s_{12}) \right), \\
c_{12345678, 1} &= -2 \gamma_{34125}, \\
c_{12345678, 2} &= \frac{1}{2} (2\gamma_{34125} - \gamma_{23145} - \gamma_{24135} - 2\gamma_{15234} + \gamma_{25134} - \gamma_{12345}), \\
c_{12345678, 3} &= \frac{1}{2} (2\gamma_{34125} - \gamma_{23145} - \gamma_{24135} + 2\gamma_{15234} + \gamma_{25134} - \gamma_{12345}), \\
c_{2345678, 0} &= \frac{1}{4} (2\gamma_{34125} - \gamma_{23145} + \gamma_{24135}), \\
c_{1345678, 0} &= \frac{1}{4} (-3\gamma_{34125} + 2\gamma_{23145} + \gamma_{24135} - \gamma_{25134} + \gamma_{12345}), \\
c_{1245678, 0} &= \frac{1}{4} (\gamma_{34125} - \gamma_{23145} - 2\gamma_{24135} + \gamma_{25134} - \gamma_{12345}), \\
c_{1235678, 0} &= \frac{1}{4} (-2\gamma_{15234} - \gamma_{25134} - \gamma_{12345}), \\
c_{1234678, 0} &= \frac{1}{4} (-2\gamma_{34125} + \gamma_{15234} + \gamma_{25134}), \\
c_{1234578, 0} &= \frac{1}{4} (2\gamma_{34125} + \gamma_{15234} + \gamma_{12345}), \\
c_{1234568, 0} &= \frac{1}{4} (2\gamma_{34125} + \gamma_{25134} - \gamma_{12345}).
\end{align*}
\]
The analytic reduction via Laurent expansion can be extended to numerators of higher rank. As an example we perform the computation of the sevenfold residues for the \( N = 8 \) SUGRA two-loop five-points planar pentabox, whose numerator has rank two in the loop momentum \( q \).

The most general parametrization of the residue of the generic sevenfold cut \((i_1 \cdots i_7)\) is

\[
\Delta_{i_1 \cdots i_7}(q, k) = c_{i_1 \cdots i_7, 0} + c_{i_1 \cdots i_7, 1}(q + w_0) \cdot w_1 + c_{i_1 \cdots i_7, 2}(q + w_0) \cdot w_2. \tag{6.29}
\]

The momenta \( w_0^\mu, w_1^\mu, \) and \( w_2^\mu \) are a linear combination of the external momenta and depend on the cut \((i_1 \cdots i_7)\). Their actual form is not relevant for this discussion but can be inferred from Table 2. We consider two \( t \)-dependent solutions of the sevenfold cut:

\[
(q^\mu_i, k^\mu_i) = \left( v^\mu_{q_i, 1} t + v^\mu_{q_i, 0}, v^\mu_{k_i} \right) \quad \text{with} \quad i = 1, 2. \tag{6.30}
\]

The coefficients in Eq. (6.29) can be computed evaluating (5.1) at the solutions (6.30)

\[
\frac{\mathcal{N}_{1s}^{(8, 0)}(q_i, k_i) - \Delta_{12345678}(q_i, k_i)}{D_{1s}(q_i, k_i)} = \Delta_{i_1 \cdots i_7}(q_i, k_i)
\]

\[
= c_{i_1 \cdots i_7, 0} + c_{i_1 \cdots i_7, 1}(v_{q_i, 0} + w_0) \cdot w_1
\]

\[
+ c_{i_1 \cdots i_7, 2}(v_{q_i, 0} + w_0) \cdot w_2 + c_{i_1 \cdots i_7, 1}(v_{q_i, 1} \cdot w_1) t
\]

\[
+ c_{i_1 \cdots i_7, 2}(v_{q_i, 1} \cdot w_2) t, \tag{6.31}
\]

where \( D_{1s} \) is defined in Eq. (5.3). The Laurent expansion around \( t = \infty \) simplifies the computation. Indeed in this limit both

\[
\frac{\mathcal{N}_{1s}^{(8, 0)}(q_i, k_i)}{D_{1s}(q_i, k_i)} \quad \text{and} \quad \frac{\Delta_{12345678}(q_i, k_i)}{D_{1s}(q_i, k_i)} \tag{6.32}
\]

have the same polynomial structure of the residue:

\[
\frac{\mathcal{N}_{1s}^{(8, 0)}(q_i, k_i)}{D_{1s}(q_i, k_i)} \bigg|_{t \to \infty} = n_{i_1 \cdots i_7, 0} + n_{i_1 \cdots i_7, 1} t + O \left( \frac{1}{t} \right)
\]

\[
\frac{\Delta_{12345678}(q_i, k_i)}{D_{1s}(q_i, k_i)} \bigg|_{t \to \infty} = b_{i_1 \cdots i_7, 0} + O \left( \frac{1}{t} \right). \tag{6.33}
\]

The expression of the coefficients is obtained by plugging the expansions (6.33) in Eq. (6.31) and by comparing both sides. In particular \( c_{i_1 \cdots i_7, 1} \) and \( c_{i_1 \cdots i_7, 2} \) are the solution of the system

\[
\begin{align*}
&c_{i_1 \cdots i_7, 1}(v_{q_1} \cdot w_1) + c_{i_1 \cdots i_7, 2}(v_{q_1} \cdot w_2) = n_{i_1 \cdots i_7, 1}^{[1]} \\
&c_{i_1 \cdots i_7, 1}(v_{q_2} \cdot w_1) + c_{i_1 \cdots i_7, 2}(v_{q_2} \cdot w_2) = n_{i_1 \cdots i_7, 1}^{[2]}. \tag{6.34}
\end{align*}
\]
The coefficient $c_{i_1 \cdots i_7,0}$ is given by

$$c_{i_1 \cdots i_7,0} = n_{i_1 \cdots i_7,0} - b_{i_1 \cdots i_7,0} - c_{i_1 \cdots i_7,1}(v_{q_1,0} + w_0) \cdot w_1 - c_{i_1 \cdots i_7,2}(v_{q_1,0} + w_0) \cdot w_2$$

$$= n_{i_1 \cdots i_7,0} - b_{i_1 \cdots i_7,0} - c_{i_1 \cdots i_7,1}(v_{q_2,0} + w_0) \cdot w_1 - c_{i_1 \cdots i_7,2}(v_{q_2,0} + w_0) \cdot w_2,$$

in terms of the functions

$$b_{i_1 \cdots i_7,0} = \frac{c_{12345678,1}(p_1 \cdot v_{q,1})}{2(v_{q,0} + P_{18}) \cdot v_{q,1}}.$$

Eq. (6.35) shows that the coefficient $c_{i_1 \cdots i_7,0}$ can be written as the constant term $n_{i_1 \cdots i_7,0}$ of the Laurent expansion of the integrand, corrected by two kinds of contributions. The first, $b_{i_1 \cdots i_7,0}$, implements the eightfold-cut subtraction as a correction at the coefficient level. The other terms are proportional to the higher-rank coefficients of the same cut found as solutions of the system in Eq. (6.34).

7. Conclusions

We recently proposed a new approach for the reduction of scattering amplitudes [45], based on multivariate polynomial division. This technique yields the complete integrand decomposition for arbitrary amplitudes, regardless of the number of loops. In particular it allows for the determination of (the polynomial form of) the residue at any multiparticle cut, whose knowledge is a mandatory prerequisite for applying the integrand-reduction procedure. We have also shown how the shape of the residues is uniquely determined by the on-shell conditions and, by using the division modulo Gröbner basis, we have derived a simple integrand recurrence relation generating the multiparticle pole decomposition for arbitrary multiloop amplitudes.

In the present paper, we applied the new reduction algorithm to planar and non planar diagrams appearing in the two-loop five-point amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA (in four dimensions), whose numerator functions contain up to rank-two terms in the integration momenta. We determined all polynomial residues parametrizing the cuts of the corresponding topologies and subtopologies. At the same time, the polynomial form of the residues defines the integral basis for the amplitude decomposition. For the considered cases, we found that the amplitude can be decomposed in terms of independent integrals with eight, seven, and six denominators.

Our presented approach is well suited for a seminumerical implementation. The mathematical framework it is based on is very general and provides an effective algorithm for the generalization of the integrand-reduction method to all orders in perturbation theory.

Acknowledgments

We thank Simon Badger, Yang Zhang, and Zhibai Zhang for useful discussions, and Ulrich Schubert for valuable comments on the manuscript. P.M. and T.P. are supported by the Alexander von Humboldt Foundation, in the framework of the Sofja Kovaleskaja Award.
endowed by the German Federal Ministry of Education and Research. The work of G.O. is supported in part by the National Science Foundation under Grant No. PHY-1068550. G.O. wishes to acknowledge the support of KITP, Santa Barbara under National Science Foundation Grant No. PHY-1125915, and the kind hospitality of the Max-Planck Institut für Physik in Munich during the completion of this project.

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