The Steiner 4-diameter of a graph *

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Abstract

The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou in 1989, is a natural generalization of the concept of classical graph distance. For a connected graph $G$ of order at least 2 and $S \subseteq V(G)$, the *Steiner distance* $d_G(S)$ among the vertices of $S$ is the minimum size among all connected subgraphs whose vertex sets contain $S$. Let $n, k$ be two integers with $2 \leq k \leq n$. Then the *Steiner $k$-eccentricity* $e_k(v)$ of a vertex $v$ of $G$ is defined by $e_k(v) = \max\{d(S) | S \subseteq V(G), |S| = k, \text{ and } v \in S\}$. Furthermore, the *Steiner $k$-diameter* of $G$ is $\text{sdiamp}_k(G) = \max\{e_k(v) | v \in V(G)\}$. In 2011, Chartrand, Okamoto and Zhang showed that $k - 1 \leq \text{sdiamp}_k(G) \leq n - 1$. In this paper, graphs with $\text{sdiamp}_4(G) = 3, 4, n - 1$ are characterized, respectively.

Keywords: Diameter, Steiner tree, Steiner $k$-diameter

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to [21] for graph theoretical notation and terminology not described here. For a graph \( G \), let \( V(G) \), \( E(G) \), \( e(G) \), and \( \delta(G) \) denote the set of vertices, the set of edges, the size, minimum degree, and the complement of \( G \), respectively. In this paper, we let \( K_n \), \( P_n \), \( K_{1,n-1} \) and \( C_n \) be the complete graph of order \( n \), the path of order \( n \), the star of order \( n \), and the cycle of order \( n \), respectively. For any subset \( X \) of \( V(G) \), let \( G[X] \) denote the subgraph induced by \( X \); similarly, for any subset \( F \) of \( E(G) \), let \( G[F] \) denote the subgraph induced by \( F \). We use \( G \setminus X \) to denote the subgraph of \( G \) obtained by removing all the vertices of \( X \) together with the edges incident with them from \( G \); similarly, we use \( G \setminus F \) to denote the subgraph of \( G \) obtained by removing all the edges of \( F \) from \( G \). If \( X = \{v\} \) and \( F = \{e\} \), we simply write \( G - v \) and \( G \setminus e \) for \( G - \{v\} \) and \( G \setminus \{e\} \), respectively. For two subsets \( X \) and \( Y \) of \( V(G) \) we denote by \( E_G[X,Y] \) the set of edges of \( G \) with one end in \( X \) and the other end in \( Y \). If \( X = \{x\} \), we simply write \( E_G[x,Y] \) for \( E_G[\{x\},Y] \). We divide our introduction into the following four subsections to state the motivations and our results of this paper.

1.1 Distance and its generalizations

Distance is one of the most basic concepts of graph-theoretic subjects. For a graph \( G \), let \( V(G) \), \( E(G) \), and \( e(G) \) denote the set of vertices, the set of edges, and the size of \( G \), respectively. If \( G \) is a connected graph and \( u, v \in V(G) \), then the distance \( d_G(u,v) \) between \( u \) and \( v \) is the length of a shortest path connecting \( u \) and \( v \). If \( v \) is a vertex of a connected graph \( G \), then the eccentricity \( e(v) \) of \( v \) is defined by \( e(v) = \max \{d_G(u,v) \mid u \in V(G)\} \). Furthermore, the radius rad\((G)\) and diameter diam\((G)\) of \( G \) are defined by \( \text{rad}(G) = \min \{e(v) \mid v \in V(G)\} \) and \( \text{diam}(G) = \max \{e(v) \mid v \in V(G)\} \). These last two concepts are related by the inequalities \( \text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G) \). The center \( C(G) \) of a connected graph \( G \) is the subgraph induced by the vertices \( u \) of \( G \) with \( e(u) = \text{rad}(G) \). Recently, Goddard and Oellermann gave a survey paper on this subject, see [20].

The distance between two vertices \( u \) and \( v \) in a connected graph \( G \) also equals the minimum size of a connected subgraph of \( G \) containing both \( u \) and \( v \). This observation suggests a generalization of distance. The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou in 1989, is a natural and nice generalization of the concept of classical graph distance. For a graph \( G(V,E) \) and a set \( S \subseteq V(G) \) of at least two vertices, an \( S \)-Steiner tree or a Steiner tree connecting \( S \) (or simply, an \( S \)-tree) is a subgraph \( T(V',E') \) of \( G \) that is a tree with \( S \subseteq V' \). Let \( G \) be a connected
are sharp.

Theorem 1

Observation 1 Let $srad$ be the Steiner $3$-eccentricity, so that $d_{rad}(G)$ is the minimum size among all connected subgraphs whose vertex sets contain $S$. Note that if $H$ is a connected subgraph of $G$ such that $S \subseteq V(H)$ and $|E(H)| = d_{rad}(S)$, then $H$ is a tree. Observe that $d_{rad}(S) = \min\{e(T) \mid S \subseteq V(T)\}$, where $T$ is subtree of $G$. Furthermore, if $S = \{u, v\}$, then $d_{rad}(S) = d(u, v)$ is the classical distance between $u$ and $v$. Set $d_{rad}(S) = \infty$ when there is no $S$-Steiner tree in $G$.

Let $n$ and $k$ be two integers with $2 \leq k \leq n$. The Steiner $k$-eccentricity $e_k(v)$ of a vertex $v$ of $G$ is defined by $e_k(v) = \max\{d(S) \mid S \subseteq V(G), |S| = k, \text{ and } v \in S\}$. The Steiner $k$-radius of $G$ is $srad_k(G) = \min\{e_k(v) \mid v \in V(G)\}$, while the Steiner $k$-diameter of $G$ is $sdiam_k(G) = \max\{e_k(v) \mid v \in V(G)\}$. Note for every connected graph $G$ that $e_2(v) = e(v)$ for all vertices $v$ of $G$ and that $srad_2(G) = rad(G)$ and $sdiam_2(G) = diam(G)$. Each vertex of the graph $G$ of Figure 1 (c) is labeled with its Steiner 3-eccentricity, so that $srad_3(G) = 4$ and $sdiam_3(G) = 6$.

Observation 1 Let $k, n$ be two integers with $2 \leq k \leq n$.

1. If $H$ is a spanning subgraph of $G$, then $sdiam_k(H) \leq sdiam_k(G)$.

2. For a connected graph $G$, $sdiam_k(G) \leq sdiam_{k+1}(G)$.

In [3], Chartrand, Okamoto, Zhang obtained the following result.

Theorem 1 [3] Let $k, n$ be two integers with $2 \leq k \leq n$, and let $G$ be a connected graph of order $n$. Then $k - 1 \leq sdiam_k(G) \leq n - 1$. Moreover, the upper and lower bounds are sharp.

In [13], Dankelmann, Swart and Oellermann obtained a bound on $sdiam_k(G)$ for a graph $G$ in terms of the order of $G$ and the minimum degree $\delta$ of $G$, that is, $sdiam_k(G) \leq \frac{3n}{\delta+1} + 3k$. Later, Ali, Dankelmann, Mukwembi [2] improved the bound of $sdiam_k(G)$ and showed that $sdiam_k(G) \leq \frac{3n}{\delta+1} + 2k - 5$ for all connected graphs $G$. Moreover, they constructed graphs to show that the bounds are asymptotically best possible.

As a generalization of the center of a graph, the Steiner $k$-center $C_k(G)$ ($k \geq 2$) of a connected graph $G$ is the subgraph induced by the vertices $v$ of $G$ with $e_k(v) = srad_k(G)$. Oellermann and Tian [11] showed that every graph is the $k$-center of some graph. In particular, they showed that the $k$-center of a tree is a tree and those trees that are $k$-centers of trees are characterized. The Steiner $k$-median of $G$ is the subgraph of $G$ induced by the vertices of $G$ of minimum Steiner $k$-distance. For Steiner centers and Steiner medians, we refer to [39, 40, 41].
The average Steiner distance $\mu_k(G)$ of a graph $G$, introduced by Dankelmann, Oellermann and Swart in [11], is defined as the average of the Steiner distances of all $k$-subsets of $V(G)$, i.e.

$$\mu_k(G) = \left(\binom{n}{k}\right)^{-1} \sum_{S \subseteq V(G), |S|=k} d_G(S).$$

For more details on average Steiner distance, we refer to [11, 12].

Let $G$ be a $k$-connected graph and $u, v$ be any pair of vertices of $G$. Let $P_k(u, v)$ be a family of $k$ internally vertex-disjoint paths between $u$ and $v$, i.e., $P_k(u, v) = \{P_1, P_2, \cdots, P_k\}$, where $p_1 \leq p_2 \leq \cdots \leq p_k$ and $p_i$ denotes the number of edges of path $P_i$. The $k$-distance $d_k(u, v)$ between vertices $u$ and $v$ is the minimum $p_k$ among all $P_k(u, v)$ and the $k$-diameter $d_k(G)$ of $G$ is defined as the maximum $k$-distance $d_k(u, v)$ over all pairs $u, v$ of vertices of $G$. The concept of $k$-diameter emerges rather naturally when one looks at the performance of routing algorithms. Its applications to network routing in distributed and parallel processing are studied and discussed by various authors including Chung [9], Du, Lyuu and Hsu [16], Hsu [24, 25], Meyer and Pradhan [34].

1.2 Application background of Steiner distance

Let $G$ be a $k$-connected graph and $u, v$ be any pair of vertices of $G$. Let $P_k(u, v)$ be a family of $k$ internally vertex-disjoint paths between $u$ and $v$, i.e., $P_k(u, v) = \{P_{p_1}, P_{p_2}, \cdots, P_{p_k}\}$, where $p_1 \leq p_2 \leq \cdots \leq p_k$ and $p_i$ denotes the number of edges of path $P_{p_i}$. The $k$-distance $d_k(u, v)$ between vertices $u$ and $v$ is the minimum $|p_k|$ among all $P_k(u, v)$ and the $k$-diameter $d_k(G)$ of $G$ is defined as the maximum $k$-distance $d_k(u, v)$ over all pairs $u, v$ of vertices of $G$. The concept of $k$-diameter emerges rather naturally when one looks at the performance of routing algorithms. Its applications to network routing in distributed and parallel processing are studied and discussed by various authors including Chung [9], Du, Lyuu and Hsu [16], Hsu [24, 25], Meyer and Pradhan [34].

The Wiener index $W(G)$ of the graph $G$ is defined as $W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v)$. Details on this oldest distance–based topological index can be found in numerous surveys, e.g., in [37, 38, 15, 42]. Li et al. [28] put forward a Steiner–distance–based generalization of the Wiener index concept. According to [28], the $k$-center Steiner Wiener index $SW_k(G)$ of the graph $G$ is defined by

$$SW_k(G) = \sum_{S \subseteq V(G), |S|=k} d(S).$$

(1.1)
For $k = 2$, the above defined Steiner Wiener index coincides with the ordinary Wiener index. It is usual to consider $SW_k$ for $2 \leq k \leq n - 1$, but the above definition would be applicable also in the cases $k = 1$ and $k = n$, implying $SW_1(G) = 0$ and $SW_n(G) = n - 1$. A chemical application of $SW_k$ was recently reported in [22]. Gutman [21] offered an analogous generalization of the concept of degree distance. Later, Furtula, Gutman, and Katanić [17] introduced the concept of Steiner Harary index and gave its chemical applications. For more details on Steiner distance indices, we refer to [17, 22, 21, 28, 29, 31, 32, 33].

1.3 Our results

From Theorem 1, we have $k - 1 \leq sdiam_k(G) \leq n - 1$. In [30], Mao characterized the graphs with $sdiam_3(G) = 2, 3, n - 1$, respectively, and studied the Nordhaus-Gaddum-type problem of the parameter $sdiam_k(G)$.

In this paper, graphs with $sdiam_4(G) = 3, 4, n - 1$ are characterized, respectively.

**Theorem 2** Let $G$ be a connected graph of order $n$ ($n \geq 4$).

(i) If $n = 4$, then $sdiam_4(G) = 3$;

(ii) If $n \geq 5$, then $sdiam_4(G) = 3$ if and only if $n - 3 \leq \delta(G) \leq n - 1$ and $C_4$ is not a subgraph of $G$.

A graph $H_1$ is defined as a connected graph of order $n$ ($n \geq 5$) obtained from a $K_4$ with vertex set $\{u_1, u_2, u_3, u_4\}$ and four stars $K_{1,a}, K_{1,b}, K_{1,c}, K_{1,d}$ by identifying the center of one star and one vertex in $\{u_1, u_2, u_3, u_4\}$, where $0 \leq a \leq b \leq c \leq d$, $d \geq 1$, and $a + b + c + d = n - 4$; see Figure 1.3.

A graph $H_2$ is defined as a connected graph of order $n$ ($n \geq 5$) obtained from $K_4 - e$ with vertex set $\{u_1, u_2, u_3, u_4\}$, $e = u_1u_4$ and two stars $K_{1,a}, K_{1,b}$ by identifying the center of a star and one vertex in $\{u_2, u_3\}$, and then adding the paths $u_1z_iu_4$ ($1 \leq i \leq c$), where $0 \leq a \leq b$, $b \geq 0$, $c \geq 0$ and $a + b + c = n - 4$; see Figure 1.3.

A graph $H_3$ is defined as a connected graph of order $n$ ($n \geq 5$) obtained from a cycle $C_4 = u_1u_2u_3u_4u_1$ by adding the paths $u_1x_iu_2$ ($1 \leq i \leq a$) and the paths $u_3y_ju_4$ ($1 \leq j \leq b$), where $0 \leq a \leq b$, $b \geq 1$ and $a + b = n - 4$; see Figure 1.3.

A graph $H_4$ is defined as a connected graph of order $n$ ($n \geq 5$) obtained from a star $K_{1,3}$ with vertex set $\{u_1, u_2, u_3, u_4\}$ and a star $K_{1,a}$ by identifying $u_3$ and the center of $K_{1,3}$, where $u_3$ is the center of $K_{1,3}$, and then adding the vertices $y_i$ and the edges $y_iu_j$ ($1 \leq i \leq b$, $j = 1, 2, 4$), where $0 \leq a \leq b$, $b \geq 1$ and $a + b = n - 4$; see Figure 1.3.
Theorem 3 Let $G$ be a connected graph of order $n$ $(n \geq 5)$. Then $sdiam_4(G) = 4$ if and only if $G$ satisfies one of the following conditions.

(i) $\delta(G) = n - 3$ and $C_4$ is a subgraph of $\overline{G}$;

(ii) $\delta(G) \leq n - 4$ and each $H_i$ $(1 \leq i \leq 4)$ is not a spanning subgraph of $\overline{G}$ (see Figure 1.3).

We now define some graph classes.

- Let $T_{a,b,c,d}$ $(0 \leq a, b, c, d \leq n - 1, a + b + c + d \leq n - 1)$ be a tree of order $n$ $(n \geq 5)$ obtained from three paths $P_1, P_2, P_3$ of length $n - b - c - 1, b, c$ respectively by identifying the $(a+1)$-th vertex of $P_1$ and one endvertex of $P_2$, and then identifying the $(n - b - c - d)$-th vertex of $P_1$ and one endvertex of $P_3$ (Note that $u$ and $v$ can be the same vertex);

- Let $\triangle_{a,b,c,d}$ $(0 \leq a, b, c, d \leq n - 2, a + b + c + d \leq n - 2)$ be an unicyclic graph of order $n$ $(n \geq 5)$ obtained from three paths $P_1, P_2, P_3$ of length $n - b - c - 1, b+1, c$ respectively by identifying the $(a+1)$-th vertex of $P_1$ and one endvertex of $P_2$, and then identifying the $(n - b - c - d)$-th vertex of $P_1$ and one endvertex of $P_3$, and then adding an edge $u_{b+1}v_{a+2}$ (Note that $v_{a+2}$ and $v$ can be the same vertex).
• Let $\Delta'_{a,b,c,d}$ ($0 \leq a, b, c, d \leq n-3$, $a+b+c+d \leq n-3$) be an bicyclic graph of order $n$ ($n \geq 5$) obtained from three paths $P_1, P_2, P_3$ of length $n-b-c-1, b+1, c+1$ respectively by identifying the $(a+1)$-th vertex of $P_1$ and one endvertex of $P_2$, and then identifying the $(n-b-c-d)$-th vertex of $P_1$ and one endvertex of $P_3$, and then adding two edges $v_{b+1}v_{a+2}$ and $w_{c+1}x_{d+2}$ (Note that $v_{a+2}$ and $v$ can be the same vertex).

• Let $G_2$ be a graph of order $n$ ($n \geq 5$) obtained from a cycle of order 4 and four paths $P_1, P_2, P_3, P_4$ of length $a, b, c, d$ ($0 \leq a, b, c, d \leq n-4$, $a+b+c+d = n-4$) respectively by identifying each vertex of this cycle with an endvertex of one of the four paths.

[Diagrams of $T_{a,b,c,d}$, $\Delta_{a,b,c,d}$, $\Delta'_{a,b,c,d}$, $G_1$, $G_2$, $G_3$]

Figure 1.1 Graphs for Theorem 4

• Let $G_3$ be a graph of order $n$ ($n \geq 5$) obtained from $K_4^-$ and four paths $P_1, P_2, P_3, P_4$ of length $a, b, c, d$ ($0 \leq a, b, c, d \leq n-4$, $a+b+c+d = n-4$) respectively by identifying each vertex of $K_4^-$ with an endvertex of one of the four paths, where $K_4^-$ denotes the graph obtained from a clique of order 4 by deleting one edge.

**Theorem 4** Let $G$ be a connected graph of order $n$ ($n \geq 5$). Then $sdiam_4(G) = n-1$ if and only if $G = T_{a,b,c,d}$ or $G = \Delta_{a,b,c,d}$ or $G = \Delta'_{a,b,c,d}$ or $G = G_1$ or $G = G_2$ or $G = G_3$. 

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2 Proofs of Theorem 2 and 3

In this section, we characterize graphs with $sdiam_4(G) = 3, 4$ and give the proofs of Theorems 2 and 3.

Lemma 1 Let $G$ be a connected graph of order $n$, and let $k$ be an integer with $3 \leq k \leq n-1$. Then $sdiam_k(G) = n-1$ if and only if the number of non-cut vertices in $G$ is at most $k$.

Proof. Let $r$ be the number of non-cut vertices in $G$. Suppose $sdiam_k(G) = n-1$. We claim that $r \leq k$. Assume, to the contrary, that $r \geq k+1$. For any $S \subseteq V(G)$ with $|S| = k$, there exists a non-cut vertex in $G$, say $u$, such that $u \in V(G) \setminus S$. Then $G \setminus u$ is connected, and hence $G \setminus u$ contains a spanning tree of size $n-2$. From the arbitrariness of $S$, we have $sdiam_k(G) \leq d_T(S) \leq n-2$, a contradiction. So $r \leq k$, as desired.

Conversely, we suppose $r \leq k$. Let $v_1, v_2, \ldots, v_r$ be all the non-cut vertices in $G$. Then the remaining vertices are all cut vertices of $G$. Choose $v_i, v_{i+1}, \ldots, v_{i+k-r} \in V(G) \setminus \{v_1, v_2, \ldots, v_r\}$. Set $S = \{v_1, v_2, \ldots, v_r, v_{i+1}, \ldots, v_{i+k-r}\}$. Note that each vertex in $V(G) \setminus S$ is a cut vertex of $G$. Therefore, any $S$-Steiner tree $T$ occupies all the vertices of $G$, and hence $sdiam_k(G) \geq d_T(S) \geq n-1$. From Theorem 1 we have $sdiam_k(G) = n-1$, as desired. 

The following corollary is immediate from the above lemma.

Corollary 1 Let $G$ be a connected graph of order $n$, and let $k$ be an integer with $3 \leq k \leq n-2$. Then $sdiam_k(G) \leq n-2$ if and only if the number of non-cut vertices in $G$ is at least $k+1$.

Mao [30] obtained the following result, which will be used later.

Lemma 2 [30] Let $n, k$ be two integers with $2 \leq k \leq n$, and let $G$ be a connected graph of order $n$. If $sdiam_k(G) = k-1$, then $0 \leq \Delta(G) \leq k-2$, namely, $n-k+1 \leq \delta(G) \leq n-1$.

Proof of Theorem 2 If $n = 4$, then $sdiam_4(G) = 3$. So we assume that $n \geq 5$. Suppose $sdiam_4(G) = 3$. For Lemma 2 if $sdiam_4(G) = 3$, then $n-3 \leq \delta(G) \leq n-1$. We claim that $C_4$ is not a subgraph of $G$. Assume, to the contrary, that $C_4$ is a subgraph of $G$. Choose $S = V(C_4)$. Since $G[S]$ is not connected, it follows that any $S$-Steiner tree must contain one vertex in $V(G) \setminus S$, and hence $sdiam_4(G) \geq d_G(S) \geq 4$, a contradiction. So $C_4$ is not a subgraph of $G$. 

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Conversely, we suppose that \( n - 3 \leq \delta(G) \leq n - 1 \) and \( C_4 \) is not a subgraph of \( \overline{G} \). Since \( n - 3 \leq \delta(G) \leq n - 1 \), it follows that \( G \) is a graph obtained from the complete graph of order \( n \) by deleting some independent paths and cycles. For any \( S \subseteq V(G) \), since \( C_4 \) is not a subgraph of \( \overline{G} \), it follows that \( \overline{G}[S] = 4K_1 \) or \( \overline{G}[S] = K_2 \cup 2K_1 \) or \( G[S] = 2K_2 \) or \( G[S] = P_4 \) or \( G[S] = K_3 \cup K_1 \) or \( \overline{G}[S] = P_3 \cup K_1 \). Then \( G[S] = K_4 \) or \( G[S] = K_4 \setminus e \) or \( G[S] = C_4 \) or \( G[S] = P_4 \) or \( G[S] = K_{1,3} \) or \( G[S] = K_{1,3}^+ \), where \( K_{1,3}^+ \) is the graph obtained from a star \( K_{1,3} \) by adding an edge. Since \( G[S] \) is a connected graph, it follows that \( d_G(S) \leq 3 \). From the arbitrariness of \( S \), we have \( sdiam_4(G) \leq 3 \) and hence \( sdiam_4(G) = 3 \) by Theorem 1. The proof is complete.

**Proof of Theorem 3.** Suppose that \( G \) is a graph with \( sdiam_4(G) = 4 \). From Theorem 2, we have \( \delta(G) = n - 3 \) and \( C_4 \) is a subgraph of \( \overline{G} \), or \( \delta(G) \leq n - 4 \). For the former, we have \( \delta(G) = n - 3 \) and \( C_4 \) is a subgraph of \( \overline{G} \), as desired. Suppose \( \delta(G) \leq n - 4 \). It suffices to prove that each \( H_i \) (\( 1 \leq i \leq 4 \)) is not a spanning subgraph of \( \overline{G} \), and we have the following claims.

**Claim 1.** \( H_1 \) is not a spanning subgraph of \( \overline{G} \).

**Proof of Claim 1.** Assume, to the contrary, that \( H_1 \) is a spanning subgraph of \( \overline{G} \). Choose \( S = \{u_1, u_2, u_3, u_4\} \subseteq V(H_1) = V(G) \). Then the subgraph in \( \overline{G} \) induced by the vertices in \( S \) is a complete graph of order 4, and hence \( G[S] = 4K_1 \) is not connected. Therefore, any \( S \)-Steiner tree \( T \) must occupy a vertex in \( V(G) \setminus S \), say \( x \). Because \( H_1 \) is a spanning subgraph of \( \overline{G} \), we have \( xu_1 \notin E(G) \) or \( xu_2 \notin E(G) \) or \( xu_3 \notin E(G) \) or \( xu_4 \notin E(G) \). Thus, the \( S \)-Steiner tree \( T \) must occupy another vertex in \( V(G) \setminus S \), and hence the tree \( T \) must occupy at least two vertices in \( V(G) \setminus S \). Then \( d_G(S) \geq 5 \), and hence \( sdiam_4(G) \geq 5 \), a contradiction. So \( H_1 \) is not a spanning subgraph of \( \overline{G} \), as desired.

**Claim 2.** \( H_2 \) is not a spanning subgraph of \( \overline{G} \).

**Proof of Claim 2.** Assume, to the contrary, that \( H_2 \) is a spanning subgraph of \( \overline{G} \). Choose \( S = \{u_1, u_2, u_3, u_4\} \subseteq V(H_2) = V(G) \). Since \( G[S] \) is not connected, it follows that any \( S \)-Steiner tree \( T \) must occupy a vertex in \( V(G) \setminus S \), say \( x \). From the structure of \( H_2 \), since \( H_2 \) is a spanning subgraph of \( \overline{G} \), we have \( xu_1, xu_4 \in E(\overline{G}) \) or \( xu_2 \in E(\overline{G}) \) or \( xu_3 \in E(\overline{G}) \). If \( xu_1, xu_4 \in E(\overline{G}) \), then there are at most three edges in \( \{xu_2, xu_3, u_1u_4\} \) belonging to \( G[S \cup \{x\}] \). In order to connect to \( u_1 \) or \( u_4 \), the \( S \)-Steiner tree \( T \) uses at least two vertex of \( V(G) \setminus S \). If \( xu_2 \in E(\overline{G}) \), then there are at most three edges in \( \{xu_1, xu_3, xu_4, u_1u_4\} \) belonging to \( G[S \cup \{x\}] \). In order to connect to \( u_2 \), the \( S \)-Steiner tree \( T \) must use at least two vertex of \( V(G) \setminus S \). The same is true for \( xu_3 \in E(\overline{G}) \). Therefore, \( e(T) \geq 5 \) and \( d_G(S) \geq 5 \), which results in \( sdiam_4(G) \geq 5 \), a contradiction. So \( H_2 \) is not a spanning subgraph of \( \overline{G} \).
Claim 3. $H_3$ is not a spanning subgraph of $\overline{G}$.

Proof of Claim 3. Assume, to the contrary, that $H_3$ is a spanning subgraph of $\overline{G}$. Choose $S = \{u_1, u_2, u_3, u_4\} \subseteq V(H_3) = V(\overline{G})$. Since $G[S] = 2K_2$ is not connected, it follows that any $S$-Steiner tree $T$ must occupy a vertex in $V(G) \setminus S$, say $x$. From the structure of $H_3$, since $H_3$ is a spanning subgraph of $\overline{G}$, we have $xu_1, xu_2 \in E(\overline{G})$ or $xu_3, xu_4 \in E(\overline{G})$. If $xu_1, xu_2 \in E(\overline{G})$, then there are at most four edges in $\{xu_3, xu_4, u_1u_2, u_3u_4\}$ belonging to $G[S \cup \{x\}]$. In order to connect to $u_1$ or $u_2$, the $S$-Steiner tree $T$ uses at least two vertex of $V(G) \setminus S$. If $xu_3, xu_4 \in E(\overline{G})$, then there are at most four edges in $\{xu_1, xu_2, u_1u_2, u_3u_4\}$ belonging to $G[S \cup \{x\}]$. In order to connect to $u_3$ or $u_4$, the $S$-Steiner tree $T$ must use at least two vertex of $V(G) \setminus S$. Therefore, $e(T) \geq 5$ and $d_G(S) \geq 5$, which results in $sdiam_4(G) \geq 5$, a contradiction. So $H_3$ is not a spanning subgraph of $\overline{G}$. $\blacksquare$

Claim 4. $H_4$ is not a spanning subgraph of $\overline{G}$.

Proof of Claim 4. Assume, to the contrary, that $H_4$ is a spanning subgraph of $\overline{G}$. Choose $S = \{u_1, u_2, u_3, u_4\} \subseteq V(H_4) = V(\overline{G})$. Since $G[S] = K_3 \cup K_1$ is not connected, it follows that any $S$-Steiner tree $T$ must occupy a vertex in $V(G) \setminus S$, say $x$. From the structure of $H_4$, since $H_4$ is a spanning subgraph of $\overline{G}$, we have $xu_3 \in E(\overline{G})$ or $xu_1, xu_2, xu_4 \in E(\overline{G})$. If $xu_3 \in E(\overline{G})$, then there are at most six edges in $\{xu_1, xu_2, xu_4, u_1u_2, u_1u_4, u_2u_4\}$ belonging to $G[S \cup \{x\}]$. In order to connect to $u_3$, the $S$-Steiner tree $T$ uses at least two vertex of $V(G) \setminus S$. If $xu_1, xu_2, xu_4 \in E(\overline{G})$, then there are at most four edges in $\{xu_3, u_1u_2, u_1u_4, u_2u_4\}$ belonging to $G[S \cup \{x\}]$. In order to connect to $u_1$ or $u_2$ or $u_4$, the $S$-Steiner tree $T$ uses at least two vertex of $V(G) \setminus S$. Therefore, $e(T) \geq 5$ and $d_G(S) \geq 5$, which results in $sdiam_4(G) \geq 5$, a contradiction. So $H_4$ is not a spanning subgraph of $\overline{G}$. $\blacksquare$

From the above argument, we know that the result holds.

Conversely, suppose that $G$ is a connected graph satisfying one of the following conditions.

1. $\delta(G) = n - 3$ and $C_4$ is a subgraph of $\overline{G}$;

2. $\delta(G) \leq n - 4$ and $H_i$ $(1 \leq i \leq 4)$ is not a spanning subgraph of $\overline{G}$.

Suppose that $\delta(G) = n - 3$ and $C_4$ is a subgraph of $\overline{G}$. Since $\delta(G) = n - 3$, it follows that $G$ is a graph obtained from the complete graph of order $n$ by deleting some pairwise independent paths and cycles. Then $\overline{G}$ is a union of pairwise independent paths, cycles, and isolated vertices. For any $S = \{u, v, w, z\} \subseteq V(G)$, since $\overline{G}$ contains $C_4$ as its subgraph, it follows that $\overline{G}[S] = C_4$ or $G[S] = 4K_1$ or $\overline{G}[S] = K_2 \cup 2K_1$ or $\overline{G}[S] = 2K_2$ or $G[S] = P_4$ or $G[S] = K_3 \cup K_1$ or $\overline{G}[S] = P_3 \cup K_1$. Then $G[S] = 2K_2$ or $G[S] = K_4$ or $G[S] = K_4 \setminus e$ or $G[S] = C_4$ or $G[S] = P_4$ or $G[S] = K_{1,3}$ or $G[S] = K_{1,3}^+$.  

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Combining this with any symmetry, we only to consider the former case. Clearly, we have $sdiam_4(G) = 4$, as desired.

Suppose $\delta(G) \leq n - 4$ and each $H_i (1 \leq i \leq 4)$ is not a spanning subgraph of $G$. For any $S \subseteq V(G)$ and $|S| = 4$, if there exists a vertex $x \in V(G) \setminus S$ such that $|E(G[x,S])| = 0$, then $|E(G[x,S])| = 4$, and hence the tree $T$ induced by the four edges in $E(G[x,S])$ is an $S$-Steiner tree in $G$, and hence $d_G(S) \leq 4$, as desired. From now on, we assume for any $S \subseteq V(G)$ and $|S| = 4$, and any $x \in V(G) \setminus S$, $|E(G(x,S))| \geq 1$. From the definition of $sdiam_4(G)$ and Theorem 2, it suffices to show that $d_G(S) \leq 4$ for any set $S \subseteq V(G)$ and $|S| = 4$. It is clear that $0 \leq |E(G[S])| \leq 6$. If $4 \leq |E(G[S])| \leq 6$, then $G[S]$ is connected, and hence $G[S]$ contains a spanning tree, which is an $S$-Steiner tree in $G$. So $d_G(S) = 3 < 4$, as desired. From now on, we assume $0 \leq |E(G[S])| \leq 3$.

If $|E(G[S])| = 0$, then $\overline{G}[S] = K_4$. Since $|E(\overline{G}[x,S])| \geq 1$ for any $x \in V(G) \setminus S$, it follows that $H_1$ is a spanning subgraph of $\overline{G}$, a contradiction.

Suppose $|E(G[S])| = 1$. Set $S = \{u_1, u_2, u_3, u_4\}$. Without loss of generality, let $u_1u_4 \in E(G)$ and $u_1u_2, u_1u_3, u_2u_3, u_2u_4, u_3u_4 \notin E(G)$. Then $u_1u_2, u_1u_3, u_2u_3, u_2u_4, u_3u_4 \in E(\overline{G})$, and hence $\overline{G}[S]$ is a graph obtained from $K_4$ by deleting one edge. Since $H_2$ is not a spanning subgraph of $\overline{G}$, it follows that there exists a vertex $x \in V(G) - S$ such that $xu_1 \in E(\overline{G})$ but $xu_2, xu_3, xu_4 \notin E(\overline{G})$ or $xu_4 \in E(\overline{G})$ but $xu_2, xu_3, xu_1 \notin E(\overline{G})$. By symmetry, we only to consider the former case. Clearly, $xu_2, xu_3, xu_4 \in E(G)$. Combining this with $u_1u_4 \in E(G)$, the tree $T$ induced by the edges in $\{u_1u_4, xu_2, xu_3, xu_4\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) \leq 4$, as desired.

Suppose $|E(G[S])| = 2$. Without loss of generality, we can assume that $u_1u_2, u_3u_4 \in E(G)$ or $u_1u_2, u_1u_4 \in E(G)$. First, we consider the case $u_1u_2, u_3u_4 \in E(G)$. Clearly, $u_1u_3, u_1u_4, u_2u_3, u_2u_4 \notin E(G)$, and hence $u_1u_3, u_1u_4, u_2u_3, u_2u_4 \in E(\overline{G})$. Note that for any $S \subseteq V(G)$ and $|S| = 4$, and any $x \in V(G) \setminus S$, $|E(\overline{G}[x,S])| \geq 1$. Since $H_3$ is not a spanning subgraph of $\overline{G}$, it follows that there exists a vertex $x \in V(G) \setminus S$ satisfying one of the following.

1. $xu_1 \in E(\overline{G})$ but $xu_2, xu_3, xu_4 \notin E(\overline{G})$;
2. $xu_1, xu_4 \in E(\overline{G})$ but $xu_2, xu_3 \notin E(\overline{G})$;
3. $xu_1, xu_3 \in E(\overline{G})$ but $xu_2, xu_4 \notin E(\overline{G})$;
4. $xu_2 \in E(\overline{G})$ but $xu_1, xu_3, xu_4 \notin E(\overline{G})$;

where $K_{1,3}^+$ is the graph obtained from $K_{1,3}$ by adding an edge. If $G[S] = 2K_2$, then $|E_G[x,S]| = 4$ for any $x \in V(G) \setminus S$, since $\delta(G) = n - 3$. Thus, we have $d_G(S) = 4$. For the other cases, $G[S]$ is connected, and so $d_G(S) = 3$. From the arbitrariness of $S$, we have $sdiam_4(G) = 4$, as desired.
(5) $xu_2, xu_4 \in E(\overline{G})$ but $xu_1, xu_3 \notin E(\overline{G})$;
(6) $xu_2, xu_3 \in E(\overline{G})$ but $xu_1, xu_4 \notin E(\overline{G})$;
(7) $xu_3 \in E(\overline{G})$ but $xu_1, xu_2, xu_4 \notin E(\overline{G})$;
(8) $xu_3, xu_1 \in E(\overline{G})$ but $xu_2, xu_4 \notin E(\overline{G})$;
(9) $xu_2, xu_3 \in E(\overline{G})$ but $xu_1, xu_4 \notin E(\overline{G})$;
(10) $xu_4 \in E(\overline{G})$ but $xu_1, xu_2, xu_3 \notin E(\overline{G})$;
(11) $xu_1, xu_4 \in E(\overline{G})$ but $xu_2, xu_3 \notin E(\overline{G})$;
(12) $xu_2, xu_4 \in E(\overline{G})$ but $xu_1, xu_3 \notin E(\overline{G})$.

By symmetry, we only consider the first three cases, and other cases can be similarly proved. If $xu_1 \notin E(G)$ but $xu_2, xu_3, xu_4 \in E(G)$, then the tree $T$ induced by the edges in $\{xu_2, xu_3, u_1u_2, u_3u_4\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) \leq 4$. If $xu_1, xu_4 \notin E(G)$ but $xu_2, xu_3 \in E(G)$, then the tree $T$ induced by the edges in $\{u_1u_2, u_3u_4, xu_2, xu_3\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) \leq 4$. If $xu_1, xu_3 \notin E(G)$ but $xu_2, xu_4 \in E(G)$, then the tree $T$ induced by the edges in $\{u_1u_2, u_3u_4, xu_2, xu_4\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) \leq 4$, as desired.

Next, we consider the case $u_1u_2, u_1u_4 \in E(G)$. Clearly, $u_1u_3, u_2u_3, u_2u_4, u_3u_4 \notin E(G)$, and hence $u_1u_3, u_2u_3, u_2u_4, u_3u_4 \in E(\overline{G})$. Note that for any $S \subseteq V(G)$ and $|S| = 4$, and any $x \in V(G) \setminus S$, $|E(\overline{G}[x,S])| \geq 1$. Since $H_4$ is not a spanning subgraph of $\overline{G}$, it follows that there exists a vertex $x \in V(G) \setminus S$ satisfying one of the following.

(1) $xu_1 \in E(\overline{G})$ but $xu_2, xu_3, xu_4 \notin E(\overline{G})$;
(2) $xu_1, xu_4 \in E(\overline{G})$ but $xu_2, xu_3 \notin E(\overline{G})$;
(3) $xu_1, xu_2 \in E(\overline{G})$ but $xu_4, xu_3 \notin E(\overline{G})$;
(4) $xu_2 \in E(\overline{G})$ but $xu_4, xu_3, xu_4 \notin E(\overline{G})$;
(5) $xu_2, xu_4 \in E(\overline{G})$ but $xu_1, xu_3 \notin E(\overline{G})$;
(6) $xu_1, xu_2 \in E(\overline{G})$ but $xu_4, xu_3 \notin E(\overline{G})$;
(7) $xu_4 \in E(\overline{G})$ but $xu_1, xu_2, xu_3 \notin E(\overline{G})$;
(8) $xu_1, xu_4 \in E(\overline{G})$ but $xu_2, xu_3 \notin E(\overline{G})$;
(9) $xu_2, xu_4 \in E(\overline{G})$ but $xu_1, xu_3 \notin E(\overline{G})$. 

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By symmetry, we only consider the first three cases. Note that $u_1u_2, u_1u_4 \in E(G)$. If $xu_1 \notin E(G)$ but $xu_2, xu_3, xu_4 \in E(G)$, then the tree $T$ induced by the edges in $\{u_1u_2, u_1u_4, xu_2, xu_3\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) \leq 4$. If $xu_1, xu_4 \notin E(G)$ but $xu_2, xu_3 \in E(G)$, then the tree $T$ induced by the edges in $\{u_1u_2, u_1u_4, xu_2, xu_3\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) \leq 4$. If $xu_1, xu_2 \notin E(G)$ but $xu_4, xu_3 \in E(G)$, then the tree $T$ induced by the edges in $\{u_1u_2, u_1u_4, xu_4, xu_3\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) \leq 4$, as desired.

Suppose $|E(G[S])| = 3$. Without loss of generality, let $u_1u_2, u_1u_4, u_2u_3 \in E(G)$ or $u_1u_2, u_1u_4, u_2u_4 \in E(G)$. If $u_1u_2, u_1u_4, u_2u_3 \in E(G)$, then the tree $T$ induced by the edges in $\{u_1u_2, u_1u_4, u_2u_3\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) = 3 \leq 4$, as desired. If $u_1u_2, u_1u_4, u_2u_4 \in E(G)$, then $u_1u_3, u_2u_4, u_2u_3 \notin E(G)$, and hence $u_1u_3, u_2u_4, u_2u_3 \in E(\overline{G})$. Since $H_4$ is not a spanning subgraph of $\overline{G}$, it follows that there exists a vertex $x \in V(G) \setminus S$ satisfying one of the following.

1. $xu_1 \in E(\overline{G})$ but $xu_2, xu_3, xu_4 \notin E(\overline{G})$;
2. $xu_1, xu_4 \in E(\overline{G})$ but $xu_2, xu_3 \notin E(\overline{G})$;
3. $xu_1, xu_2 \in E(\overline{G})$ but $xu_3, xu_4 \notin E(\overline{G})$;
4. $xu_2 \in E(\overline{G})$ but $xu_1, xu_3, xu_4 \notin E(\overline{G})$;
5. $xu_2, xu_4 \in E(\overline{G})$ but $xu_1, xu_3 \notin E(\overline{G})$;
6. $xu_1, xu_2 \in E(\overline{G})$ but $xu_3, xu_4 \notin E(\overline{G})$;
7. $xu_4 \in E(\overline{G})$ but $xu_2, xu_3, xu_1 \notin E(\overline{G})$;
8. $xu_1, xu_4 \in E(\overline{G})$ but $xu_2, xu_3 \notin E(\overline{G})$;
9. $xu_4, xu_2 \in E(\overline{G})$ but $xu_3, xu_1 \notin E(\overline{G})$.

By symmetry, we only consider the first three cases. Recall that $u_1u_2, u_1u_4, u_2u_4 \in E(G)$. If $xu_1 \notin E(G)$ but $xu_2, xu_3, xu_4 \in E(G)$, then the tree $T$ induced by the edges in $\{u_1u_2, u_1u_4, xu_2, xu_3\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) \leq 4$. If $xu_1, xu_4 \notin E(G)$ but $xu_2, xu_3 \in E(G)$, then the tree $T$ induced by the edges in $\{u_1u_2, u_1u_4, xu_2, xu_3\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) \leq 4$. If $xu_1, xu_2 \notin E(G)$ but $xu_3, xu_4 \in E(G)$, then the tree $T$ induced by the edges in $\{u_1u_2, u_1u_4, xu_3, xu_4\}$ is an $S$-Steiner tree in $G$ and hence $d_G(S) \leq 4$.

From the arbitrariness of $S$, we have $sdiam_4(G) \leq 4$. Since $\delta(G) = n - 3$ and $C_4 \in \overline{G}$, or $\delta(G) \leq n - 4$, it follows from Theorem 2 that $sdiam_4(G) = 4$. The proof is now complete.
3 Proof of Theorem 4

The following lemma is a preparation of our main result.

**Proposition 1** Let $G$ be a connected graph, and $H$ be a connected subgraph of $G$. Then the number of non-cut vertices of $G$ is not less than the number of non-cut vertices of $H$.

**Proof.** It suffices to show that there exists an injective mapping $f$ from the set of non-cut vertices of $H$ to the set of non-cut vertices of $G$. We define such a mapping $f$ as follows. Let $v$ be a non-cut vertex of $H$. If $v$ is a non-cut vertex of $G$, then let $f(v) = v$. If $v$ is a cut-vertex of $G$, then let $G_1$ be a component of $G \setminus v$ not containing any vertex of $H$. Let $T_1$ be a spanning tree of $G_1$, and let $w$ be an end-vertex of $T_1$ distinct from $v$. Then $w$ is a non-cut vertex of $G$, and we define $f(v) = w$. Now $f$ maps non-cut vertices of $H$ to non-cut vertices of $G$, and $f$ is injective since either $f(v) = v$ or $f(v)$ is in a component of $G \setminus V(H)$ which is (in $G$) attached only to $v$, and to no other vertex in $V(H)$. 

From Proposition 1, the following corollaries are immediate.

**Corollary 2** Let $G$ be a connected graph of order $n$ ($n \geq 3$), and let $c(G)$ be the circumference of the graph $G$. If $c(G) \leq n$, then there are at least $c(G)$ non-cut vertices in $G$.

**Corollary 3** Let $G$ be a connected graph of order $n$ ($n \geq 3$). Let $C_1, C_2, \cdots, C_r$ ($r \geq 2$) are cycles of the graph $G$ with $|V(C_i)| = n_i$ ($1 \leq i \leq r$). If $|V(C_i) \cap V(C_j)| \leq 1$ for any $i, j$ ($1 \leq i, j \leq r$, $i \neq j$), then the graph $G$ has at least $n_1 + n_2 + \cdots + n_r - 2(r - 1)$ are non-cut vertices in $G$.

We are now in a position to give the proof of Theorem 4.

**Proof of Theorem 4** Suppose $G = T_{a,b,c,d}$ or $G = \triangle_{a,b,c,d}$ or $G = \triangle'_{a,b,c,d}$ or $G = G_1$ or $G = G_2$ or $G = G_3$. Since there are at most four non-cut vertices in $G$, it follows from Lemma 1 that $sdiam_4(G) = n - 1$.

Conversely, suppose $sdiam_4(G) = n - 1$. If $G$ is a tree, then it follows from Lemma 1 that $G$ contains at most non-cut four vertices, and hence $G = T_{a,b,c,d}$. Now, we assume that $G$ contains cycles. Recall that $c(G)$ is the circumference of the graph $G$. Obviously, $3 \leq c(G) \leq n$. If $5 \leq c(G) \leq n$, then it follow from Corollaries 1 and 2 that $sdiam_4(G) \leq n - 2$, a contradiction. Therefore, $c(G) = 3$ or $c(G) = 4$. If $c(G) = 4$, then it follows from Lemma 1 and Corollaries 1 and 3 that $G$ contains four non-cut
vertices, and from Corollaries 3 that $G$ contains no two cycles $C_1, C_2$ with $|V(C_1)| = 4$ or $|V(C_2)| = 4$ such that $|V(C_1) \cap V(C_2)| \leq 1$. From Proposition 1, we have the following facts.

- $G \setminus V(C_i)$ ($i = 1, 2$) is a union of pairwise independent paths;
- The number of these paths are at most four;
- The endvertices of each pair of these paths share the different neighbors in $C_i$.

From these facts, we have $G = G_1$ or $G = G_2$ or $G = G_3$. If $c(G) = 3$, then it follows from Lemma 1 and Corollaries 1 and 3 that $G$ contains at most four non-cut vertices, and $G$ contains exactly one triangle or at most two cycles $C_1$ and $C_2$ with $|V(C_1)| = 3$ and $|V(C_2)| = 3$ such that $|V(C_1) \cap V(C_2)| \leq 1$. Suppose $G$ contains only one triangle. Let $K_{1,3}^*$ be the subdivision of star $K_{1,3}$ of order $t$. Then we have the following facts.

- The graph obtained from $G$ by deleting this triangle is $P_r \cup P_s \cup K_{1,3}^*$ ($r + s + t = n - 3$) or $P_r \cup K_{1,3}^*$ ($r + t = n - 3$) or $P_r \cup P_s (r + s = n - 3)$ or $P_{n-3} \cup K_{1,3}^*$ ($t = n - 3$) or $P_r \cup P_s \cup P_p (r + s + p + q = n - 3)$ or $P_r \cup P_s \cup P_p \cup P_q (r + s + p + q = n - 3)$;

- If the graph obtained from $G$ by deleting this triangle is not $P_r \cup P_s \cup P_p \cup P_q (r + s + p + q = n - 3)$, then the endvertices of each pair of these paths share the different neighbors in the triangle.

- If the graph obtained from $G$ by deleting this triangle is $P_r \cup P_s \cup P_p (r + s + p + q = n - 3)$ or $P_r \cup P_s \cup P_p \cup P_q (r + s + p + q = n - 3)$, then each vertex of this triangle share at least one common neighbor of each path.

Clearly, we have $G = \Delta_{a,b,c,d}$. If $G$ contains at most two cycles $C_1$ and $C_2$ with $|V(C_1)| = 3$ and $|V(C_2)| = 3$ such that $|V(C_1) \cap V(C_2)| \leq 1$, then $G \setminus (V(C_1) \cup V(C_2))$ is a union of pairwise independent paths, and the number of these path is at most five. So $G = \Delta_{a,b,c,d}$. The proof is complete.

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