INFINITESIMAL DEFORMATIONS OF SOME QUOT SCHEMES

INDRANIL BISWAS, CHANDRANANDAN GANGOPADHYAY, AND RONNIE SEBASTIAN

Abstract. Let $E$ be a vector bundle on a smooth complex projective curve $C$ of genus at least two. Let $Q(E,d)$ be the Quot scheme parameterizing the torsion quotients of $E$ of degree $d$. We compute the cohomologies of the tangent bundle $T_{Q(E,d)}$. In particular, the space of infinitesimal deformations of $Q(E,d)$ is computed. Kempf and Fantechi computed the space of infinitesimal deformations of $Q(O_C,d) = C^{(d)}$ ([Kem81], [Fan94]).

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1. Introduction

Let $C$ be a smooth projective curve over $\mathbb{C}$ of genus $g_C$, with $g_C \geq 2$. Let $E$ be a vector bundle on $C$ of rank $r \geq 1$. Fix an integer $d \geq 1$. Let $Q := Q(E,d)$ be the Quot scheme parameterizing all torsion quotients of $E$ of degree $d$. It is known that $Q$ is a smooth projective variety of dimension $rd$. This Quot scheme has various moduli theoretic interpretations, see [BGL94], [BDW96], [BFP20], [HPL21], [BRa13], [OP21], which have led
to extensive studies of it. Our aim here is to compute the cohomologies of the tangent bundle \( T_Q \), especially \( H^1(Q, T_Q) \) that parametrizes the infinitesimal deformations of \( Q \).

The group of holomorphic automorphisms \( \text{Aut}(Q) \) of \( Q \) is a complex Lie group whose Lie algebra is \( H^0(Q, T_Q) \) \cite[Lemma 1.2.6]{Ser06}. It is known that

\[
H^0(Q(\mathcal{O}^\oplus r, d), T_{Q(\mathcal{O}^\oplus r, d)}) = \mathfrak{sl}(r, \mathbb{C}) = H^0(X, \text{End}(\mathcal{O}^\oplus r))/\mathbb{C}
\]

for all \( r \geq 2 \) \cite{BDH15}. From this it follows that the maximal connected subgroup of \( \text{Aut}(Q(\mathcal{O}^\oplus r, d)) \) is \( \text{PGL}(r, \mathbb{C}) = \text{Aut}(\mathcal{O}^\oplus r)/\mathbb{C}^\ast \). More generally, if either \( E \) is semistable or \( r \geq 3 \), then

\[
H^0(Q(E, d), T_{Q(E,d)}) = H^0(X, \text{End}(E))/\mathbb{C}
\]

\cite{Gan19}, and hence the maximal connected subgroup of \( \text{Aut}(Q(E, d)) \) is \( \text{Aut}(E)/\mathbb{C}^\ast \).

Regarding the next cohomology \( H^1(Q, T_Q) \), first consider the case of \( r = 1 \). In this case, the Quot scheme \( Q(E, d) \) is identified with the \( d \)-th symmetric product \( C(d) \) of \( C \). The infinitesimal deformation space \( H^1(C(d), T_{C(d)}) \) was computed in \cite{Kem81} under the assumption that \( C \) is non-hyperelliptic, and it was computed in \cite{Fan94} when \( g \geq 3 \) (see also \cite[Remark 2.6]{Fan94} for the case \( g = 2 \)).

Henceforth, we will always assume that \( r = \text{rank}(E) \geq 2 \).

If \( d = 1 \), then \( Q \cong \mathbb{P}(E) \). Consequently, \( H^1(Q(E, 1), T_{Q(E,1)}) \) can be computed easily.

Associated to the vector bundle \( E \) there is the Atiyah bundle \( At(E) \) on \( C \) \cite[Theorem 1]{Ati57}. The infinitesimal deformations of the pair \( (C, E) \) are parametrized by \( H^1(X, At(E)) \) \cite[Proposition 4.2]{Che12}. For the natural homomorphisms \( \mathcal{O}_C \hookrightarrow \text{End}(E) \hookrightarrow At(E) \), the quotients \( \text{End}(E)/\mathcal{O}_C \) and \( At(E)/\mathcal{O}_C \) are vector bundles and will be denoted by \( \text{ad}(E) \) and \( at(E) \) respectively. Also, given any vector bundle \( V \) on \( C \) we can construct a natural bundle called the Secant bundle \( \text{Sec}^d(V) \) on \( C(d) \) (see \cite[Proposition 1]{Mat65}, \cite[Section 2]{BL11}). In particular we have a bundle \( \text{Sec}^d(at(E)) \) on \( C(d) \).

Recall that we have the Hilbert-Chow map \( \phi : Q \to C(d) \) (also called Quot-to-Chow morphism by some authors \cite[Section 2.4]{Ric20}). We refer to \cite[Section 2]{GS20} for the definition of this map.

We prove the following (see Theorem 9.9):

**Theorem 1.1.** Let \( r = \text{rank}(E) \geq 2 \). Then

1. \( \text{Sec}^d(at(E)) \cong \phi_* T_Q \) and
2. \( R^i \phi_* T_Q = 0 \) for all \( i > 0 \).

The cohomologies of \( \text{Sec}^d(at(E)) \) can be computed easily. Therefore, we can compute the cohomologies of \( H^1(Q, T_Q) \) using Theorem 1.1. We prove the following (see Theorem 9.10):

**Theorem 1.2.** Let \( \text{rank}(E), d \geq 2 \). Denote the genus of \( C \) by \( g_C \). The following three statements hold:

1. For all \( d - 1 \geq i \geq 0 \),

\[
H^i(Q, T_Q) = H^0(C, at(E)) \otimes \bigwedge^i H^1(C, \mathcal{O}_C) \oplus H^1(C, at(E)) \otimes \bigwedge^{i-1} H^1(C, \mathcal{O}_C).
\]
In particular,
\[ h^i(Q, T_Q) = \binom{g_C}{i} \cdot h^0(C, at(E)) + \binom{g_C}{i-1} \cdot h^1(C, at(E)). \]

(2) When \( i = d \),
\[ H^d(Q, T_Q) = \bigwedge^{d-1} H^1(C, \mathcal{O}_C) \otimes h^1(C, at(E)). \]

In particular,
\[ h^d(Q, T_Q) = \binom{g_C}{d-1} \cdot h^1(C, at(E)). \]

(3) For all \( i \geq d + 1 \),
\[ H^i(Q, T_Q) = 0. \]

By Theorem 1.2 when \( g_C \geq 2 \) we get
\[ H^0(Q, T_Q) = H^0(C, at(E)) = H^0(C, ad(E)). \]

Recall that \( Q \) is a fine moduli space, that is, there exists a certain universal quotient on \( C \times Q \). The kernel of this universal quotient, which is locally free, is denoted by \( \mathcal{A} \). We also compute \( H^1(C \times Q, \mathcal{E}nd(\mathcal{A})) \), which is the space of all infinitesimal deformations of \( \mathcal{A} \). More precisely, the following is proved (see Corollary 9.16):

**Theorem 1.3.** Let \( \text{rank}(E), g_C \), \( d \geq 2 \). Then we have
\[ H^1(C \times Q, \mathcal{E}nd(\mathcal{A})) = H^1(C \times Q, \mathcal{O}_{C \times Q}) = H^1(C, \mathcal{O}_C) \oplus H^1(Q, \mathcal{O}_Q). \]

It can be seen using Corollary 9.1 that \( H^1(Q, \mathcal{O}_Q) = H^1(C, \mathcal{O}_C) \).

In [Gan18] it was shown that when \( E \cong \mathcal{O}_C^n \) for some \( n \geq 1 \), then \( \mathcal{A} \) is slope stable with respect to some natural polarizations of \( C \times Q \). By [HL10, Corollary 4.5.2], the space \( H^1(C \times Q, \mathcal{E}nd(\mathcal{A})) \) is the tangent space at \([\mathcal{A}]\) of the Moduli space \( \mathcal{M} \) of sheaves on \( C \times Q \) with the same Hilbert polynomial (with respect to some fixed polarization on \( C \times Q \)) as \( \mathcal{A} \). Moreover, the differential of the determinant map \( \mathcal{M} \rightarrow \text{Pic}(C \times Q) \) at the point \([\mathcal{A}]\) is given by the trace map
\[ H^1(C \times Q, \mathcal{E}nd(\mathcal{A})) \xrightarrow{H^1(tr)} H^1(C \times Q, \mathcal{O}_{C \times Q}) \]
(see [HL10, Theorem 4.5.3]). Note that this map is onto, since the composition of the maps \( \mathcal{O}_{C \times Q} \rightarrow \mathcal{E}nd(\mathcal{A}) \xrightarrow{tr} \mathcal{O}_{C \times Q} \) is an isomorphism. Therefore, Theorem 1.3 implies that the determinant map \( \mathcal{M} \rightarrow \text{Pic}(C \times Q) \) induces an isomorphism at the level of tangent spaces at the point \([\mathcal{A}]\).

We briefly describe how the paper is organized. In Sections 2 to 5 we prove several preliminary results which we shall need later. In these sections we show that the relative adjoint Atiyah sequence associated to a vector bundle \( V \) on \( C \times X \) (see (4.4)), restricted to \( c \times X \) can be obtained in three different ways. In Section 6 we recall the construction of the space \( S_d \) from [GS20] and prove a result relating to its canonical divisor. In Section 7 we prove some results related to projective bundles. In Section 8, the results in Section 7 and Section 5 are used in computing cohomologies of some sheaves on \( S_d \). These cohomology computations are used in Section 9 to compute higher direct images, for the Hilbert-Chow map, of some natural vector bundles on \( Q \). These computations are then used to prove the main results.
2. SOME TANGENT BUNDLE SEQUENCES

The base field is $\mathbb{C}$. As before, $C$ is a smooth projective curve. Let $X$ be a smooth projective variety and $V \to C \times X$ a vector bundle. Let

$$\pi : \mathbb{P}(V) \to C \times X$$

be the projective bundle parametrizing the hyperplanes in the fibers of $V$. Denoting the natural projections of $C \times X$ to $C$ and $X$ by $q_C$ and $q_X$ respectively, define

$$\pi_C := q_C \circ \pi : \mathbb{P}(V) \to C \quad \text{and} \quad \pi_X := q_X \circ \pi : \mathbb{P}(V) \to X.$$ \hspace{1cm} (2.1)

Let $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}(V)}(1) \to \mathbb{P}(V)$ be the tautological line bundle, and let $\Omega_{\pi} \to \mathbb{P}(V)$ be the relative cotangent bundle for the morphism $\pi$. We have the relative Euler sequence on $\mathbb{P}(V)$:

$$0 \to \Omega_{\pi}(1) := \Omega_{\pi} \otimes \mathcal{O}(1) \to \pi^*V \overset{q'}{\to} \mathcal{O}(1) \to 0.$$ \hspace{1cm} (2.2)

Define

$$p_{1,2X} : C \times \mathbb{P}(V) \to C \times X, \quad (c, v) \mapsto (c, \pi_X(v)),$$

where $\pi_X$ is the map in (2.1). Let

$$i : \mathbb{P}(V) \hookrightarrow C \times \mathbb{P}(V), \quad v \mapsto (\pi_C(v), v)$$ \hspace{1cm} (2.3)

be the closed embedding, where $\pi_C$ is the map in (2.1). The composition of maps

$$\mathbb{P}(V) \overset{i}{\to} C \times \mathbb{P}(V) \overset{p_{1,2X}}{\to} C \times X$$

evidently coincides with $\pi$. Let

$$q : p_{1,2X}^*V \to i_*\mathcal{O}(1)$$

be the surjective homomorphism given by the following composition of homomorphism of sheaves

$$p_{1,2X}^*V \to i_*p_{1,2X}^*V \cong i_*\pi^*V \overset{i_*q'}{\to} i_*\mathcal{O}(1) \to 0$$

on $C \times \mathbb{P}(V)$, where $q'$ is the projection in (2.2). Denote

$$\mathcal{V} := \text{kernel}(q) \subset p_{1,2X}^*V;$$

so we have the short exact sequence

$$0 \to \mathcal{V} \to p_{1,2X}^*V \overset{q}{\to} i_*\mathcal{O}(1) \to 0.$$ \hspace{1cm} (2.4)

Restricting (2.4) to $i(\mathbb{P}(V)) = \mathbb{P}(V)$ we get a right exact sequence

$$i^*\mathcal{V} \to i^*p_{1,2X}^*V \cong \pi^*V \overset{q'}{\to} \mathcal{O}(1) \to 0,$$

where $i$ and $q'$ are the maps in (2.3) and (2.2) respectively. Therefore we get a surjective homomorphism from $i^*\mathcal{V}$ to the kernel of $q'$, which by (2.2) is given by $\Omega_{\pi}(1)$.

$$i^*\mathcal{V} \to \Omega_{\pi}(1) \to 0.$$ \hspace{1cm} (2.5)

**Proposition 2.1.** There is a natural isomorphism $i^*\mathcal{V}(-1) \cong \Omega_{\mathbb{P}(V)/X}$. The composition of this isomorphism with the tensor product of (2.5) with $\mathcal{O}(-1)$

$$\Omega_{\mathbb{P}(V)/X} = i^*\mathcal{V}(-1) \to \Omega_{\pi}$$

coincides with the natural surjection of cotangent bundles $\Omega_{\mathbb{P}(V)/X} \to \Omega_{\pi} \to 0$. 
Proof. The codimension of a subvariety \( A \subset B \) will be denoted by \( \text{codim}(A, B) \).

Let

\[ A \subset B \subset T \]

be smooth varieties such that \( A \) and \( B \) are closed in \( T \). Assume that there is a quadruple \((G, W, s, s')\), where

- \( G \) is a vector bundle on \( T \) with \( \text{rank}(G) = \text{codim}(A, T) \),
- \( s \in H^0(T, G) \) with the property that its vanishing defines \( A \),
- \( W \) is a vector bundle on \( B \) with \( \text{rank}(W) = \text{codim}(A, B) \), and
- \( s' \in H^0(B, W) \) with the property that its vanishing defines \( A \).

Further assume that there is a homomorphism \( W \to G \mid_B \) that takes the section \( s' \) to \( s \mid_B \).

Consequently, we have a commutative diagram

\[
\begin{array}{ccc}
G^{v} \mid_A & \xrightarrow{\sim} & I_{A/T}/I^2_{A/T} \\
\downarrow & & \downarrow \\
W^{v} \mid_A & \xrightarrow{\sim} & I_{A/B}/I^2_{A/B}.
\end{array}
\]

(2.6)

For convenience \( C \times X \) will be denoted by \( Y \). There is the natural inclusion map

\[ \mathbb{P}(V) \times_Y \mathbb{P}(V) \hookrightarrow \mathbb{P}(V) \times_X \mathbb{P}(V) \]

and also the diagonal embedding

\[ \mathbb{P}(V) \hookrightarrow \mathbb{P}(V) \times_Y \mathbb{P}(V), \quad z \mapsto (z, z). \]

For \( 1 \leq i < j \leq 3 \), let \( p_{ij} \) denote the projection of \( C \times \mathbb{P}(V) \times_X \mathbb{P}(V) \) to the product of its \( i \)-th and \( j \)-th factor. Consider the diagram of homomorphisms on sheaves \( C \times \mathbb{P}(V) \times_X \mathbb{P}(V) \)

\[
\begin{array}{c}
p_{12}^*V \quad p_{12}^*p_{1,2X}^*V \cong p_{13}^*p_{1,2X}^*V \quad p_{12}^*i_*O(1) \quad 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
p_{13}^*i_*O(1).
\end{array}
\]

(2.7)

The dotted arrow gives a global section

\[ s \in H^0(\mathbb{P}(V) \times_X \mathbb{P}(V), p_{23*}(p_{12}^*V \otimes p_{13}^*i_*O(1))), \]

(2.8)

It is straightforward to check that this \( p_{23*}(p_{12}^*V \otimes p_{13}^*i_*O(1)) \) is locally free. Let

\[ Z \subset \mathbb{P}(V) \times_X \mathbb{P}(V) \]

be the vanishing locus of \( s \) in (2.8).

We will prove that \( Z \) is the diagonal in \( \mathbb{P}(V) \times_X \mathbb{P}(V) \).

To prove this we will first show that \( Z \) is contained in \( \mathbb{P}(V) \times_Y \mathbb{P}(V) \). For this consider the commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}(V) \times_X \mathbb{P}(V) & \xrightarrow{j} & C \times \mathbb{P}(V) \times_X \mathbb{P}(V) \xrightarrow{p_{12}} C \times \mathbb{P}(V) \\
q_2 & \downarrow & p_{13} \\
\mathbb{P}(V) & \xrightarrow{i} & C \times \mathbb{P}(V) \xrightarrow{c} C \times X;
\end{array}
\]

\[
\begin{array}{c}
C \times \mathbb{P}(V) \\
\xrightarrow{c} C \times X;
\end{array}
\]
here \( j(a, b) = (\pi_C(b), a, b) \) and \( q_2(a, b) = b \). Pullback (2.7) along the map \( j \). It is easy to see in this pullback that \( p_{13}^* i_* \mathcal{O}(1) \) is a line bundle supported on \( \mathbb{P}(V) \times_X \mathbb{P}(V) \) and \( p_{12}^* i_* \mathcal{O}(1) \) is a line bundle supported on \( \mathbb{P}(V) \times_Y \mathbb{P}(V) \). Further restricting this pullback to \( Z \) we get a surjection

\[
p_{12}^* i_* \mathcal{O}(1) \big|_Z \longrightarrow p_{13}^* i_* \mathcal{O}(1) \big|_Z ,
\]

which shows that \( Z \) is contained \( \mathbb{P}(V) \times_Y \mathbb{P}(V) \) and the homomorphism in (2.9) is an isomorphism. Consequently, the restrictions to \( Z \) of the two projection maps

\[
\tau_1, \tau_2 : \mathbb{P}(V) \times_Y \mathbb{P}(V) \longrightarrow \mathbb{P}(V)
\]

actually coincide, which means that \( Z \) is contained in the diagonal.

Conversely, it is easily checked that the section \( s \) in (2.8) vanishes on the diagonal in \( \mathbb{P}(V) \times_X \mathbb{P}(V) \). This proves the assertion that \( Z \) is the diagonal in \( \mathbb{P}(V) \times_X \mathbb{P}(V) \).

Let \( q_1, q_2 : \mathbb{P}(V) \times_X \mathbb{P}(V) \longrightarrow \mathbb{P}(V) \) denote the two projections. The sheaf \( p_{12}^* \mathcal{V}^\vee \otimes p_{13}^* i_* \mathcal{O}(1) \) is supported on the image of \( j \) and it is locally free, in fact, it is isomorphic to \( j^* p_{12}^* \mathcal{V}^\vee \otimes q_2^* \mathcal{O}(1) \). Thus, we may identify \( p_{23}^* (p_{12}^* \mathcal{V}^\vee \otimes p_{13}^* i_* \mathcal{O}(1)) \) with \( j^* p_{12}^* \mathcal{V}^\vee \otimes q_2^* \mathcal{O}(1) \). Denote \( j^* p_{12}^* \mathcal{V}^\vee \otimes q_2^* \mathcal{O}(1) \) on \( \mathbb{P}(V) \times_X \mathbb{P}(V) \) by \( G \). It is easily checked that the restriction of \( s \in \mathbb{H}^0(\mathbb{P}(V) \times_X \mathbb{P}(V), G) \) (see (2.8)) to \( \mathbb{P}(V) \times_Y \mathbb{P}(V) \) factors as

\[
\tau_1^* \mathcal{O}(1) \stackrel{s'}{\longrightarrow} \tau_1^* \Omega^\vee_\mathcal{V}(-1) \longrightarrow \tau_1^* i^* \mathcal{V}^\vee
\]

(see (2.10)). Let \( W \) on \( \mathbb{P}(V) \times_Y \mathbb{P}(V) \) be the locally free sheaf \( \tau_1^* \mathcal{O}(1) \otimes \tau_1^* \Omega^\vee_\mathcal{V}(-1) \). The vanishing locus of the section \( s' \) is precisely the diagonal. Restricting to the diagonal, and using (2.6) it follows that there is a commutative diagram

\[
\begin{array}{ccc}
i^* \mathcal{V}(-1) & \longrightarrow & \Omega^\vee_{\mathcal{V} / X} \\
\downarrow & & \downarrow \\
\Omega_x & \longrightarrow & \Omega_x
\end{array}
\]

This proves the proposition. \( \square \)

3. Atiyah sequence

Let \( V \) be a locally free sheaf of rank \( r \) over a smooth variety \( X \). Its Atiyah bundle \( At(V) \longrightarrow X \) fits in the following Atiyah exact sequence

\[
0 \longrightarrow \mathcal{E}nd(V) \longrightarrow At(V) \longrightarrow T_X \longrightarrow 0
\]

(see [Ati57]). We recall a construction of (3.1) which will be used. Let \( P_V \stackrel{q}{\longrightarrow} X \) denote the principal \( \text{GL}_r(\mathbb{C}) \)-bundle associated to \( V \). The differential of \( q \) produces an exact sequence on \( P_V \)

\[
0 \longrightarrow K := T_{P_V / X} \longrightarrow T_{P_V} \stackrel{dq}{\longrightarrow} q^* T_X \longrightarrow 0
\]

Applying \( q_* \) to it and then taking \( \text{GL}_r(\mathbb{C}) \)-invariants we get (3.1).

We have \( \mathcal{O}_X \subset \mathcal{E}nd(V) \); the quotient \( \text{ad}(V) := \mathcal{E}nd(V) / \mathcal{O}_X \) is identified with the sheaf of endomorphisms of \( V \) of trace zero. Define \( at(V) := At(V) / \mathcal{O}_X \). Taking the pushout of (3.3) along the quotient map \( \mathcal{E}nd(V) \longrightarrow \text{ad}(V) \) we get an exact sequence

\[
0 \longrightarrow \text{ad}(V) \longrightarrow at(V) \longrightarrow T_X \longrightarrow 0.
\]
We will need an alternate description of (3.3). Consider the projective bundle $\mathbb{P}(V) \xrightarrow{\pi} X$ for $V$, and let
\[ 0 \rightarrow T_{\mathbb{P}(V)/X} \rightarrow T_{\mathbb{P}(V)} \xrightarrow{\pi} \pi^*T_X \rightarrow 0 \] (3.4)
be the exact sequence on $\mathbb{P}(V)$ given by the differential $d\pi$.

**Lemma 3.1.** The sequence (3.3) coincides with the one obtained by applying $\pi_*$ to (3.4).

**Proof.** Consider the commutative diagram
\[
\begin{array}{ccc}
P_V \times \mathbb{P}^{r-1} & \xrightarrow{~\sim~} & \mathbb{P}(q^*V) \\
\downarrow p_1 & & \downarrow \pi \\
P_V & \xrightarrow{q} & X \\
\end{array}
\]
($p_1$ is the projection to the first factor). Here the first isomorphism follows from the fact that $q^*V$ is a trivial vector bundle [HL10, Example 4.2.3 and 4.2.6]. The differentials of the maps in it produce the following commutative diagram (without the dotted arrow)
\[
\begin{array}{ccc}
\tilde{\pi}^*T_{\mathbb{P}(V)/X} & \xrightarrow{~\sim~} & \tilde{\pi}^*T_{\mathbb{P}(V)} \\
\downarrow & & \downarrow \\
\tilde{\pi}^*K & \xrightarrow{\gamma} & \tilde{\pi}^*T_{q^*V} \\
\downarrow & & \downarrow \\
\tilde{\pi}^*T_{P_V} & \xrightarrow{\beta} & \tilde{\pi}^*q^*T_X \\
\end{array}
\]
in which the rows and columns are short exact sequences. As $\mathbb{P}(q^*V) \cong P_V \times \mathbb{P}^{r-1}$, it follows that $\delta$ has a section $s : \tilde{\pi}^*T_{P_V} \rightarrow T_{\mathbb{P}(q^*V)}$. Define $\beta := \gamma \circ s$, and consider the commutative diagram of short exact sequences
\[
\begin{array}{ccc}
0 & \xrightarrow{\beta} & \tilde{\pi}^*K \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\beta} & \tilde{\pi}^*T_{P_V} \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\beta} & \tilde{\pi}^*q^*T_X \\
\end{array}
\]
\[ \xrightarrow{\sim} 0. \] (3.6)

We will prove that $\beta$ is surjective and $GL_r(\mathbb{C})$-equivariant.

For this it suffices to verify the assertion over an open subset $U \subset X$ on which $V$ is trivial. In this case (3.5) becomes
\[
\begin{array}{ccc}
U \times GL_r(\mathbb{C}) \times \mathbb{P}^{r-1} & \xrightarrow{\sim} & U \times \mathbb{P}^{r-1} \\
\downarrow p_1 \sim \tilde{\pi} & & \downarrow \pi \\
U \times GL_r(\mathbb{C}) & \xrightarrow{q} & U \\
\end{array}
\]
where $\tilde{q}(u, A, [w]) = (u, [Aw])$. Let $(t, X, 0)$ be an element in the fiber $\tilde{\pi}^*T_{P_V}\mid_{(u, A, [w])}$; it is sent to $(t, Xw)$ by $\tilde{q}$, where $Xw \in T_{\mathbb{P}^{r-1}}\mid_{[Aw]}$ is the image of $Xw \in T_{\mathbb{C}^r \setminus \{(0, 0, \ldots, 0)\}}$ under the natural map
\[
\mathbb{C}^r \setminus \{(0, 0, \ldots, 0)\} \rightarrow \mathbb{P}^{r-1}.
\]
It is straightforward to see that the map
\[
\beta|_{(u, A, [w])} : \tilde{\pi}^* T_P \big|_{(u, A, [w])} \longrightarrow \tilde{\varphi}^* T_{P}(V) \big|_{(u, A, [w])} \cong T_{P}(V) \big|_{(u, [Aw])}
\]
is surjective.

The GL$_r$(C) action on $U \times$ GL$_r$(C) $\times \mathbb{P}^{r-1}$ is given by
\[
(u, A, [w]) \cdot g = (u, Ag, [g^{-1}w]).
\]
This action sends any $(t, X, 0) \in \tilde{\pi}^* T_P \big|_{(u, A, [w])}$ to $(t, Xg, 0) \in \tilde{\varphi}^* T_{P}(V) \big|_{(u, A, g^{-1}w)}$. Clearly, both these get mapped to the same vector in $T_{P}(V) \big|_{(u, [Aw])}$. This completes the proof of the assertion that $\beta$ is GL$_r$(C)-equivariant and surjective.

Next we will show that $\tilde{\pi}_s(\beta)$ is surjective. Again, this would be done locally. For any $(u, A) \in U \times$ GL$_r$(C), the bundle $(\tilde{\varphi}^* T_{P}(V))|_{(u, A) \times \mathbb{P}^{r-1}}$ is simply $T_{U,u} \otimes \mathcal{O}_{P(V)} \oplus T_{P(V)}$. Therefore, the dimension
\[
h^0((u, A) \times \mathbb{P}^{r-1}, (\tilde{\varphi}^* T_{P}(V))|_{(u, A) \times \mathbb{P}^{r-1}})
\]
does not change. Thus, by Grauert’s theorem the canonical map
\[
(\tilde{\pi}_s(\tilde{\varphi}^* T_{P}(V)))|_{(u, A)} \sim h^0((u, A) \times \mathbb{P}^{r-1}, (\tilde{\varphi}^* T_{P}(V))|_{(u, A) \times \mathbb{P}^{r-1}})
\]
is an isomorphism. Since $\tilde{\pi}_s(\tilde{\varphi}^* T_{P}) \cong T_{P(V)}$, to prove that $\tilde{\pi}_s(\beta)$ is surjective it suffices to show that
\[
T_{P}(V)\big|_{(u, A)} \longrightarrow H^0((u, A) \times \mathbb{P}^{r-1}, (\tilde{\varphi}^* T_{P}(V))|_{(u, A) \times \mathbb{P}^{r-1}})
\]
is surjective. Take any $(t, X) \in T_{P}(V)|_{(u, A)}$; the map in (3.7) sends it to a pair consisting of $t$ and a vector field on $\mathbb{P}^{r-1}$. Computing as above, the vector field assigns to the point $[w] \in \mathbb{P}^{r-1}$ the tangent vector $X\lambda^{-1}w \in T_{\mathbb{P}^{r-1}, [w]}$. Vector fields on $\mathbb{P}(V_u)$ are naturally identified with $\text{End}(V_u)/\langle \lambda \cdot \text{Id} \rangle = \text{ad}(V_u)$. Thus, the map in (3.7) sends $(t, X)$ to $(t, X\lambda^{-1})$. Consequently, (3.7) is surjective and its kernel consists of pairs of the form $(0, \lambda A)$, which implies that the kernel is one dimensional. This proves that $\tilde{\pi}_s(\beta)$ is surjective with its kernel being a line bundle on $P_V$.

The locally free sheaf $K$ in (3.2) is identified with $P_V \times M_r(C)$; note that $M_r(C)$ is the Lie algebra of GL$_r$(C). The map $K \longrightarrow T_{P(V)}$ has the following local description. Consider an open set $U \subset X$ over which $V$ is trivialized. For any $(u, A) \in U \times$ GL$_r$(C), the map
\[
T_{GL_r(C), Id} \cong K|_{(u, A)} \longrightarrow T_{P(V)}|_{(u, A)} \cong T_{U,u} \oplus T_{GL_r(C), A}
\]
is identified with the map defined by $X \mapsto (0, AX)$. As $\tilde{\pi}_s(\beta)$ is surjective, it follows from Snake Lemma for $\tilde{\pi}_s$ applied on (3.6) that the kernels of $\tilde{\pi}_s(\beta_0)$ and $\tilde{\pi}_s(\beta)$ are the same. Consequently, the kernel of $\tilde{\pi}_s(\beta)$ is precisely the trivial bundle $P_V \times \mathbb{C}$ sitting inside $P_V \times M_r(C)$ as scalar matrices.

Apply $q_* \tilde{\pi}_s$ to (3.6) and take the GL$_r$(C) invariants. The top row of the resulting sequence is the one in (3.1), while the bottom row is obtained by applying $\pi_*$ to (3.4). The map $q_*(K) \longrightarrow \pi_*(T_{P(V)/X})$ is identified with the canonical map $\mathcal{E}nd(V) \longrightarrow \text{ad}(V)$. This proves the lemma.

Let
\[
[At(V)] \in \text{Ext}^1(T_X, \mathcal{E}nd(V))
\]
denote the class of the extension in (3.1).
Lemma 3.2. Let $f : V \to V'$ be a morphism between vector bundles $V, V'$ on $X$. Then the image of the cohomology class $[At(V)]$ in (3.8) under the natural map

$$\text{Ext}^1(T_X, \mathcal{E}nd(V)) \xrightarrow{f \otimes \text{Id}} \text{Ext}^1(T_X, \mathcal{E}nd(V, V'))$$

coincides with the image of $[At(V')]$ under the natural map

$$\text{Ext}^1(T_X, \mathcal{E}nd(V')) \xrightarrow{\phi_f} \text{Ext}^1(T_X, \mathcal{E}nd(V, V')) .$$

Proof. We will recall a description of the image of $[At(V)]$ under the isomorphism

$$\text{Ext}^1(T_X, \mathcal{E}nd(V)) \cong \text{Ext}^1(V, V \otimes \Omega_X), \tag{3.9}$$

where $\Omega_X = T^*_X$. The ideal sheaf of the diagonal

$$\Delta \subset X \times X$$

will be denoted by $\mathcal{I}$. Let $p_1, p_2 : X \times X \to X$ be the two natural projections. Tensoring the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{X \times X}/\mathcal{I}^2 \to \mathcal{O}_\Delta \to 0 .$$

with $p^*_1V$, and then applying $p_2*$, we get an exact sequence

$$0 \to \Omega_X \otimes V \to p_2*(p^*_1V \otimes \mathcal{O}_{X \times X}/\mathcal{I}^2) \to V \to 0 .$$

The extension class of this sequence is $-[At(V)]$ under the isomorphism in (3.9); see [Ati57, Theorem 5]. Now we have a natural diagram

$$\begin{array}{ccc}
0 & \to & V \otimes \Omega_X \\
\downarrow f \otimes \text{Id} & & \downarrow f \\
0 & \to & p_2*(p^*_1V \otimes \mathcal{O}_{X \times X}/\mathcal{I}^2) \\
\end{array} \hspace{1cm} \begin{array}{ccc}
0 & \to & V' \otimes \Omega_X \\
\downarrow f & & \downarrow f \\
p_2*(p^*_1V' \otimes \mathcal{O}_{X \times X}/\mathcal{I}^2) & \to & V' \to 0 .
\end{array} \tag{3.10}$$

where the top row (respectively, bottom row) corresponds to $-[At(V)] \in \text{Ext}^1(V, V \otimes \Omega_X)$ (respectively, $-[At(V')] \in \text{Ext}^1(V', V' \otimes \Omega_X)$). Consequently, the pushout of the top row of (3.10) by the morphism $\text{Id} \otimes f$ is same as the pullback of the bottom row by the morphism $f$. In other words, the image of the class of top row under the map

$$\text{Ext}^1(V, V \otimes \Omega_X) \xrightarrow{(f \otimes \text{Id})} \text{Ext}^1(V, V' \otimes \Omega_X)$$

coincides with the image of the class of the bottom row under the map

$$\text{Ext}^1(V', V' \otimes \Omega_X) \xrightarrow{\phi_f} \text{Ext}^1(V, V' \otimes \Omega_X) .$$

Using this together with the canonical identification

$$\text{Ext}^1(V, V' \otimes \Omega_X) \cong \text{Ext}^1(T_X, \mathcal{E}nd(V, V'))$$

the lemma follows. \qed
4. The relative Atiyah sequence

Let $X$ be a smooth projective variety, $C$ a smooth projective curve and $V$ a vector bundle on $C \times X$. Let

$$p_C : C \times X \to C \quad \text{and} \quad p_X : C \times X \to X$$  \hspace{1cm} (4.1)

be the natural projections. Pulling back, along the inclusion map $p_C^*T_C \to p_C^*T_C \oplus p_X^*T_X$, of the Atiyah exact sequence for $V$

$$0 \to \text{End}(V) \to \text{At}(V) \to p_C^*T_C \to 0$$  \hspace{1cm} (4.2)

we get the relative Atiyah sequence

$$0 \to \text{End}(V) \to \text{At}(C)(V) \to p_C^*T_C \to 0.$$  \hspace{1cm} (4.3)

The pushout of (4.3) along the projection $\text{End}(V) \to \text{ad}(V)$ produces an exact sequence

$$0 \to \text{ad}(V) \to \text{at}(C)(V) \to p_C^*T_C \to 0$$  \hspace{1cm} (4.4)

on $C \times X$. Henceforth, (4.4) will be referred to as the relative adjoint Atiyah sequence.

Let $\pi : \mathbb{P}(V) \to C \times X$ denote the projective bundle for $V$; define

$$\pi_C := p_C \circ \pi : \mathbb{P}(V) \to C,$$

where $p_C$ is the projection in (4.1).

The tangent bundle sequence for the maps $\mathbb{P}(V) \xrightarrow{\pi} C \times X \xrightarrow{p_X} X$ (see (4.1)) produced an exact sequence

$$0 \to T_\pi \to T_{\mathbb{P}(V)/X} \to \pi_C^*T_C \to 0.$$  \hspace{1cm} (4.5)

The following lemma is similar to Lemma 3.1.

**Lemma 4.1.** The sequence in (4.4) coincides with the one obtained by applying $\pi_*$ to the sequence in (4.5).

**Proof.** Applying $\pi_*$ to (4.5) we get the top row of the natural commutative diagram

$$0 \longrightarrow \text{ad}(V) \longrightarrow \pi_*T_{\mathbb{P}(V)/X} \longrightarrow p_C^*T_C \longrightarrow 0 \hspace{1cm} (4.6)$$

By Lemma 3.1 the lower sequence in (4.6) is the pushout of (4.2) by the quotient map $\text{End}(V) \to \text{ad}(V)$. Therefore, from the commutativity of (4.6) it follows that the top row of it is the pushout of (4.3) by the quotient map $\text{End}(V) \to \text{ad}(V)$. \qed

Let $f : Y \to X$ be a morphism of smooth projective varieties. Define

$$F := \text{Id}_C \times f : C \times Y \to C \times X.$$  

The following is a consequence of Lemma 4.1.

**Corollary 4.2.** The relative adjoint Atiyah sequence for $F^*V$ coincides with the one obtained by applying $F^*$ to (4.4).
Proof. Let 
\[ p_C : C \times Y \longrightarrow C \]
be the natural projection. Note that we have an exact sequence as in (4.5) for \( Y \). Consider the Cartesian square
\[ \begin{array}{ccc}
\mathbb{P}(F^*V) & \xrightarrow{F'} & \mathbb{P}(V) \\
\pi' & \downarrow & \pi \\
C \times Y & \xrightarrow{F} & C \times X \\
\end{array} \]  
(4.7)
It is easily checked that applying \( F^* \) to the sequence
\[ 0 \longrightarrow \pi_*(T_{\pi}) \longrightarrow \pi_*(T_{\mathbb{P}(V)/X}) \longrightarrow p_C^*T_C \longrightarrow 0 \]
the sequence
\[ 0 \longrightarrow \pi'_*(T_{\pi'}) \longrightarrow \pi'_*(T_{\mathbb{P}(F^*V)/Y}) \longrightarrow p_C^*T_C \longrightarrow 0 \]
is obtained. By Lemma 4.1 the first sequence is the relative adjoint Atiyah sequence for \( F^*V \) and the second one is the relative adjoint Atiyah sequence for \( V \). This completes the proof of the corollary. □

Let 
\[ [At_C(V)] \in \text{Ext}^1(p_C^*T_C, \mathcal{E}nd(V)) \]
be the class of (4.3). The next result follows immediately from Lemma 3.2.

**Corollary 4.3.** Let \( f : V \longrightarrow V' \) be a morphism between vector bundles \( V, V' \) on \( X \). Then the image of \([At_C(V)]\) under the natural map
\[ \text{Ext}^1(p_C^*T_C, \mathcal{E}nd(V)) \xrightarrow{\sim} \text{Ext}^1(p_C^*T_C, \mathcal{H}om(V, V')) \]
coincides with the image of \( A(V') \) under the natural map
\[ \text{Ext}^1(p_C^*T_C, \mathcal{E}nd(V')) \xrightarrow{\sim} \text{Ext}^1(p_C^*T_C, \mathcal{H}om(V, V')). \]

The following lemma will be used later. Consider the Cartesian square
\[ \begin{array}{ccc}
\mathbb{P}(g^*V) & \xrightarrow{g'} & \mathbb{P}((V) \\
\pi' & \downarrow & \pi \\
Z & \xrightarrow{g} & C \times X \\
\end{array} \]  
(4.8)

**Lemma 4.4.** Applying \( g^*\pi_* \) (see (4.8)) to the exact sequence
\[ 0 \longrightarrow T_{\pi} \longrightarrow T_{\mathbb{P}(V)/X} \longrightarrow \pi_*(p_C^*T_C) \longrightarrow 0 \]
yields the exact sequence
\[ 0 \longrightarrow \pi'_*(g'^*T_{\pi'}) \longrightarrow \pi'_*(g'^*T_{\mathbb{P}(V)/X}) \longrightarrow g'^*p_C^*T_C \longrightarrow 0, \]
where \( p_C \) is the projection in (4.1).

Proof. Since (4.5) is an exact sequence of vector bundles, after applying \( g'^* \) it remains exact
\[ 0 \longrightarrow g'^*T_{\pi} \longrightarrow g'^*T_{\mathbb{P}(V)/X} \longrightarrow g'^*\pi_*(p_C^*T_C) \cong \pi'_*(g'^*p_C^*T_C) \longrightarrow 0. \]
The fibres of \( \pi' \) are projective spaces and restriction of \( g'^*T_{\pi} \) to a fibre is the tangent bundle of the corresponding projective space. Therefore, its first cohomology vanishes, and by
cohomology and base change theorems we have $R^1\pi'_*(g^*T_\pi) = 0$. Applying $\pi'_*$ to this exact sequence, we get the exactness of the bottom row in the statement of the lemma.

Now note that we have the following diagram induced by base change maps:

$$
\begin{array}{c}
0 \longrightarrow g^*\pi_*T_\pi \longrightarrow g^*\pi_*T_{\mathbb{P}(V)/X} \longrightarrow g^*\pi_*\pi^*p_C^*T_C \longrightarrow 0 \\
0 \longrightarrow \pi'_*(g^*T_\pi) \longrightarrow \pi'_*(g^*T_{\mathbb{P}(V)/X}) \longrightarrow \pi'_*(g^*\pi^*p_C^*T_C) \longrightarrow 0
\end{array}
$$

For any $s \in C \times X$, the fibre of $\pi$ is a projective space $\mathbb{P}(\mathcal{V}|_s)$ and the restriction of the vector bundles $T_\pi$ and $\pi^*p_C^*T_C$ to this fibre are given by the tangent bundle $T_{\mathbb{P}(\mathcal{V}|_s)}$ and the trivial bundle $T_{C,\mathbb{P}(\mathcal{V})}(s) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}|_s)}$ respectively. In particular, the dimension of space of global sections of both bundles is independent of $s$, which implies that both the base change maps in the above diagram are isomorphisms. Applying five lemma, we get the required isomorphism of exact sequences. $\square$

5. **Infinitesimal deformation map**

We continue with the notation of Section 4.

Applying the functor $p_{C*}$ to (4.3) produces a homomorphism

$$
\rho : T_C \longrightarrow R^1p_{C*}\mathcal{E}nd(V)
$$

on $C$ which is called the *infinitesimal deformation* map. Take any $c \in C$, and define $V_c := \mathcal{V}|_{c \times X}$. Let

$$
\rho_c : T_{C,c} \longrightarrow H^1(X, \mathcal{E}nd(V_c))
$$

be the composition of the homomorphism $\rho|_{T_{C,c}} : T_{C,c} \longrightarrow R^1p_{C*}\mathcal{E}nd(V)|_c$, obtained by restricting $\rho$ in (5.1) to $c \in C$, with the natural map

$$
R^1p_{C*}\mathcal{E}nd(V)|_c \longrightarrow H^1(X, \mathcal{E}nd(V_c)).
$$

The homomorphism $\rho_c$ in (5.2) is called the *infinitesimal deformation* map, at $c$, for the family of vector bundles $\mathcal{V}$ on $X$ parametrized by $C$.

Let $\pi_c : \mathbb{P}(\mathcal{V}_c) \longrightarrow X$ be the projective bundle for the above vector bundle $\mathcal{V}_c$. Denote

$$
F := \text{Id}_C \times \pi_c : C \times \mathbb{P}(\mathcal{V}_c) \longrightarrow C \times X,
$$

and let $i_c : c \times \mathbb{P}(\mathcal{V}_c) \longrightarrow C \times \mathbb{P}(\mathcal{V}_c)$ be the natural inclusion map. Let $\mathcal{K}$ be the kernel of the quotient map

$$
F^*\mathcal{V} \longrightarrow i_c*\pi_c^*F^*\mathcal{V} \cong i_c*\pi_c^*\mathcal{V}_c \longrightarrow i_c*\mathcal{O}_{\mathbb{P}(\mathcal{V}_c)}(1) \longrightarrow 0,
$$

so it fits in the exact sequence

$$
0 \longrightarrow \mathcal{K} \longrightarrow F^*\mathcal{V} \longrightarrow i_c*\mathcal{O}_{\mathbb{P}(\mathcal{V}_c)}(1) \longrightarrow 0.
$$

From this one easily computes that $\det(\mathcal{K}) = \det(F^*\mathcal{V}) \otimes p_C^*\mathcal{O}_C(-c)$.

Applying $i_c^*$ to (5.3) we get a right exact sequence

$$
i_c^*\mathcal{K} \longrightarrow i_c^*F^*\mathcal{V} \cong \pi_c^*\mathcal{V}_c \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{V}_c)}(1) \longrightarrow 0.
$$

This produces a surjection

$$
i_c^*\mathcal{K} \longrightarrow \Omega_{\pi_c}(1).
$$
The kernel of this map is a line bundle. Taking determinants and using \( \det(K) = \det(F^*V) \otimes p_C^*O_C(-c) \) it is easily checked that the kernel in (5.4) is

\[
O_{\mathbb{P}(V)}(1) \otimes p_C^*O_C(-c)|_c = O_{\mathbb{P}(V)}(1) \otimes C \cdot T_{C,c}^{\vee}.
\]

Therefore, we get an exact sequence

\[
0 \longrightarrow O_{\mathbb{P}(V)}(1) \otimes C \cdot T_{C,c}^{\vee} \longrightarrow i^*K \longrightarrow \Omega_{\pi_c}(1) \longrightarrow 0.
\]  (5.5)

Dualizing (5.5) and twisting with \( O_{\mathbb{P}(V)}(1) \) produces an exact sequence

\[
0 \longrightarrow T_{\pi_c} \longrightarrow i^*K^{\vee}(1) \longrightarrow O_{\mathbb{P}(V)} \otimes C \cdot T_{C,c} \longrightarrow 0.
\]  (5.6)

Note that the restriction of \( T_{\pi_c} \) to any fibre is the tangent bundle of a projective space, and hence its first cohomology vanishes. Therefore by cohomology and base change theorems we get \( R^1\pi_{cs}T_{\pi_c} = 0 \). Applying Lemma 5.1 to (5.6) we get the sequence

\[
0 \longrightarrow ad(V_c) \longrightarrow \pi_{cs}i_c^*K^{\vee}(1) \longrightarrow O_X \otimes C \cdot T_{C,c} \longrightarrow 0
\]  (5.7)

on \( X \). The long exact sequence of cohomologies for (5.7) gives a map

\[
H^0(X, O_X) \otimes C \cdot T_{C,c} \longrightarrow H^1(X, ad(V_c)).
\]  (5.8)

**Lemma 5.1.** The image of the map in (5.8) coincides with the image of the following composition of homomorphisms:

\[
T_{C,c} \xrightarrow{\rho_c} H^1(X, \mathcal{E}^*nd(V_c)) \longrightarrow H^1(X, \mathcal{E}^*nd(V_c)/O_X) = H^1(X, ad(V_c)),
\]  (5.9)

where \( \rho_c \) is the homomorphism in (5.2).

**Proof.** The images of the maps in (5.8) and (5.9) correspond to extension classes determined by two short exact sequences on \( X \). Thus, to prove the Lemma, it suffices to show that the corresponding short exact sequences can be identified with each other. It is straightforward to see, using the definition of \( \rho_c \) in (5.2), that the image of the map in (5.9) corresponds to the restriction of the short exact sequence (4.4) to \( c \times X \). Thus, it suffices to show that the restriction of the short exact sequence (4.4) to \( c \times X \) is identified with the short exact sequence (5.7).

Using Lemma 4.1 the sequence (4.4) is identified with

\[
0 \longrightarrow \pi_*T_{\pi} \longrightarrow \pi_*T_{\mathbb{P}(V)/X} \longrightarrow \pi_*\pi^*_cT_C \longrightarrow 0.
\]

Applying Lemma 4.4 by taking \( g \) (in equation (4.8)) to be the inclusion map \( c \times X \hookrightarrow C \times X \), it follows that the restriction of (4.4) to \( c \times X \) is

\[
0 \longrightarrow \pi_{cs}(T_{\pi}|_{\mathbb{P}(V)}) \longrightarrow \pi_{cs}(T_{\mathbb{P}(V)/X}|_{\mathbb{P}(V)}) \longrightarrow \pi_{cs}(p^*_cT_C|_{\mathbb{P}(V)}) \longrightarrow 0.
\]  (5.10)

It now remains to identify this sequence with (5.7).

In Proposition 2.1 it was proved that on \( \mathbb{P}(V) \), the natural surjection of cotangent bundles \( \Omega_{\mathbb{P}(V)/X} \twoheadrightarrow \Omega_{\pi} \), coincides with the map of bundles \( i^*\mathcal{V}(-1) \longrightarrow \Omega_{\pi} \) in (2.5). It is easily checked that, the restriction of the map \( i^*\mathcal{V} \longrightarrow \Omega_{\pi}(1) \) in (2.5) to \( \mathbb{P}(V_c) \), is the map \( i_c^*K \longrightarrow \Omega_{\pi_c}(1) \), that was obtained in (5.4). Putting these two together, we see that the restriction, \( \Omega_{\mathbb{P}(V)/X}|_{\mathbb{P}(V)} \longrightarrow \Omega_{\pi}|_{\mathbb{P}(V)} \), coincides with the map \( i^*_cK^{\vee}(-1) \longrightarrow \Omega_{\pi_c} \), obtained after twisting (5.4). Taking dual we see that the map \( T_{\pi}|_{\mathbb{P}(V)} \longrightarrow T_{\mathbb{P}(V)/X}|_{\mathbb{P}(V)} \) is identified with \( T_{\pi_c} \longrightarrow i^*_cK^{\vee}(1) \), which appears in (5.6). Now applying \( \pi_{cs} \) we see that (5.10) is identified with (5.7). This completes the proof of the Lemma.

\[ \square \]
Remark 5.2. We summarize what we have done above as follows. Given a triple \((c, X, V)\), where \(V\) is a vector bundle on \(C \times X\), and \(c \in C\) is a closed point, we produced a short exact sequence (5.7) on \(X\). We shall refer to this short exact sequence as \(\text{SES}(c, X, V)\). In Lemma 5.1 we proved that \(\text{SES}(c, X, V)\) is the same as restricting the relative adjoint Atiyah sequence of \(V\) to \(c \times X\). We also produced another triple \((c, X_1, V_1)\), where \(X_1 = \mathbb{P}(V_1)\) and \(V_1 = \mathcal{K}\). The next Proposition combined with Lemma 5.1 shows that \(\text{SES}(c, X_1, V_1)\) is non-split on \(X_1\). We will use this in the proof of Lemma 8.3.

We have the following Proposition due to Narasimhan and Ramanan.

**Proposition 5.3.** The infinitesimal deformation map of \(\mathcal{K}\) (see (5.2)) at \(c\), that is, the map in (5.2)

\[
T_{C,c} \xrightarrow{\rho_c} H^1(\mathbb{P}(V_c), \mathcal{E}nd(i^*_c \mathcal{K}))
\]

is injective.

**Proof.** If the vector bundle \(\mathcal{F}\) is defined by the cocycle \(\{a_{\alpha\beta}\}\) on a variety \(X\), then the Atiyah bundle \(\mathcal{A}(\mathcal{F})\) is given by the cocycle \(\{-a_{\alpha\beta} da_{\alpha\beta}\}\); this is well known, for example, see [BR08, § 4.4]. It is easily checked that, after identifying \(\mathcal{E}nd(\mathcal{F})\) with \(\mathcal{E}nd(\mathcal{F}^\vee)\) using the transpose map, the Atiyah bundle \(\mathcal{A}(\mathcal{F}^\vee)\) is given by the cocycle \(\{a_{\alpha\beta} da_{\alpha\beta}\}\). Thus, the classes \(\mathcal{A}(\mathcal{F})\) and \(\mathcal{A}(\mathcal{F}^\vee)\) differ by a minus sign in \(H^1(X, \mathcal{E}nd(\mathcal{F}) \otimes \Omega_X)\).

If we take \(X = C \times \mathbb{P}(V_c)\) and apply the above to \(\mathcal{K}\), then it is clear that the images of \(\mathcal{A}(\mathcal{K})\) and \(\mathcal{A}(\mathcal{K}^\vee)\) differ by a minus sign in \(H^1(\mathbb{P}(V_c), \mathcal{E}nd(i^*_c \mathcal{K}))\). From this it follows that the infinitesimal deformation map at \(c\) for \(\mathcal{K}\) is injective if and only if the infinitesimal deformation map at \(c\) for \(\mathcal{K}^\vee\) is injective. The injectivity for \(\mathcal{K}^\vee\) is proved in [NR75, Proposition 4.4].

\[\square\]

6. **Canonical bundle of \(S_d\)**

In this section we begin by recalling a construction from [GS20, § 4]. The main result of this section is Lemma 6.2. All assertions in this section, before Lemma 6.2, can be proved by using minor modifications of the proofs in [GS20, § 4, § 5].

As before, \(C\) is a complex projective curve and \(E \rightarrow C\) a vector bundle of rank \(r\). Let \(D \in C^{(d)}\). We fix an ordering of the points of \(D\)

\[
(c_1, c_2, \ldots, c_d) \in C^d.
\]

We will use this ordering to inductively construct a variety \(S_j\) and a vector bundle \(A_j \rightarrow C \times S_j\) for all \(1 \leq j \leq d\).

Set \(S_0 = \text{Spec} \mathbb{C}\) and \(A_0 = E\). For \(j \geq 1\), we will define \((S_j, A_j)\) assuming that the pair \((S_{j-1}, A_{j-1})\) has been defined. Let

\[
\alpha_{j-1} : \{c_j\} \times S_{j-1} \hookrightarrow C \times S_{j-1}
\]

be the natural closed immersion, where \(c_j\) is the point in (6.1). Consider the projective bundle

\[
f_{j,j-1} : S_j := \mathbb{P}(\alpha_{j-1}^* A_{j-1}) \rightarrow S_{j-1}
\]

and define the map

\[
F_{j,j-1} := \text{Id}_C \times f_{j,j-1} : C \times S_j \rightarrow C \times S_{j-1}.
\]
We also have the natural closed immersion
\[ i_j : \{ c_j \} \times S_j \hookrightarrow C \times S_j. \]
Let \( p_{1,j} : C \times S_j \to C \) and \( p_{2,j} : C \times S_j \to S_j \) be the natural projections. For each \( j \), we have the following diagram
\[
\begin{array}{ccc}
\{ c_j \} \times S_j & \xrightarrow{i_j} & C \times S_j \\
\downarrow & & \downarrow F_{j,j-1} \\
S_j & \xrightarrow{f_{j,j-1}} & S_{j-1}
\end{array}
\]
Let \( O_j(1) \to S_j \) denote the universal line bundle. Then over \( C \times S_j \) we have the homomorphisms
\[
F_{j,j-1}^* A_{j-1} \to (i_j)_* i_j^* F_{j,j-1}^* A_{j-1} = (i_j)_* f_{j,j-1}^* A_{j-1} = (i_j)_* O_j(1)
\]
Define \( A_j \) to be the kernel of the composition of homomorphisms \( F_{j,j-1}^* A_{j-1} \to (i_j)_* O_j(1) \) in (6.4). Thus, we have the following short exact sequence on \( C \times S_j \)
\[
0 \to A_j \to F_{j,j-1}^* A_{j-1} \to i_j_*(O_j(1)) \to 0.
\]
For any \( x \notin \{ c_j \} \times S_j \subset C \times S_j \) the localisations of \( A_j \) and \( F_{j,j-1}^* A_{j-1} \) at \( x \) are same. In particular \( A_j \) is locally free at \( x \). Now assume \( x \in \{ c_j \} \times S_j \subset C \times S_j \) and let \( R \) be the local ring of \( C \times S_j \) at \( x \). Then the localisation of \( i_j_*(O_j(1)) \) is given by \( R/rR \) for some regular element \( r \in R \) and hence its depth is given by \( \dim R - 1 \). By Auslander–Buchsbaum formula we get that projective dimension of \( i_j_*(O_j(1)) \) is 1. This implies that \( A_j \) is locally free at \( x \). Therefore \( A_j \) is locally free on \( C \times S_j \). Thus, \( (S_j, A_j) \) is constructed.

For \( d \geq j > i \geq 0 \), define morphisms
\[
f_{j,i} = f_{j,j-1} \circ \ldots \circ f_{i+1,i} : S_j \to S_i,
F_{j,i} = \text{Id}_C \times f_{j,i} = F_{j,j-1} \circ \ldots \circ F_{i+1,i} : C \times S_j \to C \times S_i.
\]
Note that both the morphisms are flat.
Closed points of \( S_d \) are in bijective correspondence with the filtrations
\[
E_d \subset E_{d-1} \subset E_{d-2} \subset \cdots \subset E_1 \subset E_0 = E,
\]
where each \( E_j \) is a locally free sheaf of rank \( r \) on \( C \) and \( E_j / E_{j+1} \) is a skyscraper sheaf of degree one supported at \( c_{j+1} \in C \).

**Remark 6.1.** The following is not difficult to check and as we will not be using this fact, we omit the proof. Let \( (b_1, b_2, \ldots, b_d) \) be a permutation of \( (c_1, c_2, \ldots, c_d) \) and let \( (S'_d, A'_d) \) denote the space built inductively as above using the \( b_i \). Then there is an isomorphism \( \theta : S_d \sim S'_d \) such that under the isomorphism \( \text{Id}_C \times \theta : C \times S_d \to C \times S'_d \) the bundles \( A_d \) and \( A'_d \) are isomorphic. This shows that up to an isomorphism, the pair \( (S_d, A_d) \) depends only on the effective divisor \( D = \sum_{i=1}^d [c_i] \in C^{(d)} \) defined by \( (c_1, c_2, \ldots, c_d) \).
Let \( p_1 : C \times S_d \to C \) denote the natural projection. We will construct a quotient of \( p_1^*E \). Using the flatness of \( F_{d,i} \), and pulling back (6.5), for \( i = 0, \ldots, d-1 \), we get a sequence of inclusions
\[
A_d \subset F_{d,d-1}^*A_{d-1} \subset F_{d,d-2}^*A_{d-2} \subset \cdots \subset F_{d,1}^*A_1 \subset F_{d,0}^*A_0 = p_1^*E.
\] (6.7)
Define
\[
B_d^j := p_1^*E/F_{d,j}^*A_j \cong F_{d,j}^*(p_1^*E/A_j),
\]
so \( B_d^j \) fits in the exact sequence
\[
0 \to A_d \to p_1^*E \to B_d^d \to 0.
\] (6.8)
The sheaf \( B_d^d \) is \( S_d \)-flat.

Pulling back the exact sequence (6.5) on \( C \times S_j \) along \( F_{d,j} \) and using flatness of \( F_{d,j} \), we see that
\[
F_{d,j-1}^*A_{j-1}/F_{d,j}^*A_j \cong F_{d,j}^*(i_j)_*(O_j(1)) \cong f_{d,j}^*(O_j(1)).
\]
Thus, for each \( j \) there is an exact sequence on \( C \times S_d \)
\[
0 \to f_{d,j}^*(O_j(1)) \to B_d^d \to B_{d-1}^d \to 0.
\]
The support of \( B_d^d \) is finite over \( S_d \). Consider the natural projection \( p_2 : C \times S_d \to S_d \), and apply \( p_2^* \) to the above sequence; taking the top exterior products, we have
\[
\det(p_2^*(B_d^d)) = \bigotimes_{j=1}^{d} f_{d,j}^*O_j(1).
\] (6.9)

Consider the Hilbert-Chow map
\[
\phi : Q \to C^{(d)}.
\] (6.10)
The universal property of \( Q \) and (6.8) yields a morphism \( g_d : S_d \to Q \). Let \( Q_D \) denote the scheme theoretic fiber of \( \phi \) over the point \( D \in C^{(d)} \). It is straightforward to check that \( g_d \) factors as
\[
g_d : S_d \to Q_D
\] (6.11)
(the same notation \( g_d \) is used for the map it is factoring through).

**Lemma 6.2.** There is a line bundle \( L \) on \( Q_D \) (see (6.11)) such that the canonical bundle \( K_{S_d} \) of \( S_d \) is isomorphic to \( g_d^*L \).

**Proof.** We will compute \( K_{S_d} \) by induction on \( d \). Recall the exact sequence (6.5) on \( C \times S_j \) defining \( A_j \)
\[
0 \to A_j \to F_{j,j-1}^*A_{j-1} \to i_{j-*}(O_j(1)) \to 0.
\]
The rightmost sheaf is a line bundle on the divisor \( c_j \times S_j \) and so
\[
\det(A_j) = \det(F_{j,j-1}^*A_{j-1}) \otimes p_C^*O_C(-c_j),
\] (6.12)
where \( p_C : C \times S_j \to C \) denotes the projection. Since \( S_j \) is obtained as a tower of projective bundles, it follows that the restriction of a line bundle \( L \) on \( C \times S_j \) to \( \{c\} \times S_j \) is
independent of the point $c$. Restricting (6.12) to $c_j \times S_j$ we get that
\[
\det \left( A_j \big|_{c_j \times S_j} \right) = (F_{j,j-1}^* \det(A_{j-1})) \big|_{c_j \times S_j} \\
= f_{j,j-1}^* (\det(A_{j-1}) \big|_{c_j \times S_{j-1}}) \\
= f_{j,j-1}^* \left( \det(A_{j-1} \big|_{c_{j-1} \times S_{j-1}}) \right).
\]
The last equality makes sense only when $j > 1$. However, when $j = 1$,
\[
\det(A_1 \big|_{c_1 \times S_1}) = f_{1,0}^* (\det(A_0) \big|_{c_1 \times S_0}),
\]
and the bundle on the right-hand side is trivial. Thus, by descending induction on $j$ we conclude that $\det(A_j \big|_{c_j \times S_j})$ is trivial. This also shows that $\det(A_j \big|_{c \times S_j})$ is trivial for all $c$.

For a locally free sheaf of rank $r$ on a smooth variety $X$ and the corresponding projective bundle $\pi : \mathbb{P}(E) \rightarrow X$,
\[
K_{\mathbb{P}(E)} = \pi^* (\det(E) \otimes K_X) \otimes \mathcal{O}_{\mathbb{P}(E)}(-r).
\]
Setting $X = S_{j-1}$ and $E = A_{j-1} \big|_{c_j \times S_{j-1}}$, we conclude that
\[
K_{S_j} = f_{j,j-1}^* K_{S_{j-1}} \otimes \mathcal{O}_j(-r)
\]
(note that $\det(E)$ is trivial). Using induction we get the first equality in the following.
\[
K_{S_d} = \bigotimes_{j=1}^{d} f_{d,j}^* \mathcal{O}_j(-r) = (\det(p_{2*}(B_d^d))^{-1})^{\otimes r}.
\]
The second equality follows using equation (6.9). It follows from [GS21, Lemma 3.1] that $p_{2*}(B_d^d)$ is the pullback along $g_d$ (see (6.11)) of a locally free sheaf from $Q$. Thus, the line bundle $\det(p_{2*}(B_d^d))$ is the pullback along $g_d$ of a line bundle from $Q$, and hence it is the pullback of a line bundle from $Q_d$. This completes the proof of the Lemma. 

**Corollary 6.3.** Let $V$ be a locally free sheaf on $Q_D$. Then $H^i(Q_D, V) \cong H^i(S_d, g_d^* V)$.

*Proof. As $g_d$ is birational (see [GS20, Proposition 5.13]), and $Q_D$ is normal (see [GS20, Corollary 6.6]), it follows that $g_d^*(\mathcal{O}_{S_d}) = \mathcal{O}_{Q_D}$. From [Laz04, Theorem 4.3.9] it follows that $R^q g_d^*(K_{S_d}) = 0$ for all $q > 0$. Using Lemma 6.2 this shows that $R^q g_d^*(\mathcal{O}_{S_d}) \otimes L = 0$ for $q > 0$, that is, $R^q g_d^*(\mathcal{O}_{S_d}) = 0$ for $q > 0$. Now the result follows using the Leray spectral sequence. *}

7. Projective bundle computations

We collect here some facts which shall be used later. Let $E$ be a locally free sheaf of rank $r$ on a scheme $T$ and $\mathbb{P}(E) \xrightarrow{\pi} T$ the corresponding projective bundle. Then we have an exact sequence on $\mathbb{P}(E)$
\[
0 \rightarrow \Omega_{\pi}(1) \rightarrow \pi^* E \rightarrow \mathcal{O}_{\mathbb{P}(E)}(1) \rightarrow 0.
\] (7.1)

**Lemma 7.1.** With notation as above,

(1) $R^i \pi_*(\Omega_{\pi}(1)) = 0 \forall i$,

(2) $H^i(\mathbb{P}(E), \Omega_{\pi}(1)) = 0 \forall i$,

(3) $R^i \pi_*(\mathcal{O}_{\mathbb{P}(E)}(-1)) = 0 \forall i$.
(4) $H^i(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(-1)) = 0 \ \forall \ i$,
(5) $H^i(\mathbb{P}(E), \pi^*E(-1)) = 0 \ \forall \ i$,
(6) $H^i(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}) \overset{\sim}{\longrightarrow} H^{i+1}(\mathbb{P}(E), \Omega^1_\pi) \ \forall \ i$,
(7) $H^i(\mathbb{P}(E), \pi^*E^\vee(1) \otimes \Omega^1_\pi) = 0 \ \forall \ i$,
(8) $H^i(\mathbb{P}(E), \mathcal{E}nd(\Omega^1_\pi)) \overset{\sim}{\longrightarrow} H^{i+1}(\mathbb{P}(E), \Omega_\pi) \ \forall \ i$,

Proof. Note that for any $t \in T$ from the Euler sequence on the projective space $\mathbb{P}(E_t)$

$$0 \longrightarrow \Omega_{\mathbb{P}(E_t)}(1) \longrightarrow E_t \otimes \mathcal{O}_{\mathbb{P}(E_t)} \longrightarrow \mathcal{O}_{\mathbb{P}(E_t)}(1) \longrightarrow 0$$

it follows that

$$H^i(\mathbb{P}(E_t), \Omega_{\mathbb{P}(E_t)}(1)) = 0 \ \forall \ i \geq 0.$$ Also we have that $H^i(\mathbb{P}(E_t), \mathcal{O}_{\mathbb{P}(E_t)}(-1)) = 0 \ \forall \ i \geq 0$. By Grauert’s theorem we get that

$$R^i\pi_*(\Omega^1_\pi(1)) = R^i\pi_*(\mathcal{O}_{\mathbb{P}(E)}(-1)) = 0 \ \forall \ i \geq 0.$$ Therefore we get (1) and (3). Using the Leray spectral sequence we see that (2) and (4) follows immediately from (1) and (3) respectively. From projection formula and (3) we get that

$$R^i\pi_*(\pi^*E(-1)) = E \otimes R^i\pi_*(\mathcal{O}_{\mathbb{P}(E)}(-1)) = 0 \ \forall \ i \geq 0.$$ Therefore (5) again follows from the Leray spectral sequence. Twisting (7.1) by $\mathcal{O}_{\mathbb{P}(E)}(1)$ we get an exact sequence

$$0 \longrightarrow \Omega_\pi \longrightarrow \pi^*E(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(E)} \longrightarrow 0.$$ Considering the associated long exact sequence of cohomologies, and applying (5), we get (6). Using projection formula and (1) we get that

$$R^i\pi_*(\pi^*E^\vee(1) \otimes \Omega_\pi) = E^\vee \otimes R^i\pi_*(\Omega^1_\pi(1)) = 0 \ \forall \ i \geq 0.$$ Therefore (7) follows from the Leray spectral sequence. For the last assertion consider the exact sequence

$$0 \longrightarrow \Omega_\pi \longrightarrow \pi^*E^\vee(1) \otimes \Omega_\pi \longrightarrow \mathcal{E}nd(\Omega_\pi) \longrightarrow 0$$ obtained by tensoring the dual of (7.1) with $\Omega_\pi$. Taking the corresponding long exact sequence of cohomologies and using (7) we see that the statement follows.

Next let $G$ be a locally free sheaf on $C \times T$ with $C$ being a smooth projective curve. Set $G_c := G|_{c \times T}$, and consider the corresponding projective bundle $\mathbb{P}(G_c) \xrightarrow{\pi} T$. Define

$$F := \text{Id}_C \times \pi : C \times \mathbb{P}(G_c) \longrightarrow C \times T.$$ Let $i : \{c\} \times \mathbb{P}(G_c) \hookrightarrow C \times \mathbb{P}(G_c)$ be the inclusion map. On $C \times \mathbb{P}(G_c)$ we have the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow F^*G \longrightarrow i_*(\mathcal{O}_{\mathbb{P}(G_c)}(1)) \longrightarrow 0; \quad (7.2)$$
the map on the right is the composition as in (6.4). It is evident that

$$F^*G(-c \times \mathbb{P}(G_c)) := F^*G \otimes \mathcal{O}_{C \times \mathbb{P}(G_c)}(-c \times \mathbb{P}(G_c)) \subset \mathcal{K}.$$
Restricting (7.2) to \( \{c\} \times \mathbb{P}(G_c) \) we see that the quotient of the above inclusion is \( i_*(\Omega_\pi(1)) \). It, furthermore, fits in a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & F^*G(-c \times \mathbb{P}(G_c)) \\
\downarrow & & \downarrow \\
i_*(\Omega_\pi(1)) & \rightarrow & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{L} \\
\downarrow & & \downarrow \\
i_*\left(\mathcal{K}|_{c \times \mathbb{P}(G_c)}\right) & \rightarrow & i_*(\Omega_\pi(1))
\end{array}
\]

The kernel of the left and middle vertical arrows are both \( \mathcal{K}(-c \times \mathbb{P}(G_c)) \). From this description it is clear that the left vertical arrow in (7.3) is the cokernel of the inclusion map

\[
\mathcal{K}(-c \times \mathbb{P}(G_c)) \subset F^*G(-c \times \mathbb{P}(G_c))
\]

which is obtained by tensoring the inclusion map \( \mathcal{K} \subset F^*G \) with \( \mathcal{O}_{C \times \mathbb{P}(G_c)}(-c \times \mathbb{P}(G_c)) \). In particular,

- \( \mathcal{L} \cong i_*(\mathcal{O}_{\mathbb{P}(G_c)}(1)) \), and
- the left vertical arrow in (7.3) is identified with the map \( F^*G \rightarrow i_*(\mathcal{O}_{\mathbb{P}(G_c)}(1)) \) in (7.2).

**Lemma 7.2.** Using the above notation, for all \( i \geq 0 \), and \( t \in C \), the natural maps

1. \( H^i\left(\mathbb{P}(G_c), F^*G(-c \times \mathbb{P}(G_c))|_{tx\mathbb{P}(G_c)}\right) \rightarrow H^i\left(\mathbb{P}(G_c), \mathcal{K}|_{tx\mathbb{P}(G_c)}\right) \), and
2. \( H^i\left(\mathbb{P}(G_c), F^*G^\vee|_{tx\mathbb{P}(G_c)}\right) \rightarrow H^i\left(\mathbb{P}(G_c), \mathcal{K}^\vee|_{tx\mathbb{P}(G_c)}\right) \)

are isomorphisms.

**Proof.** The inclusions maps \( F^*G(-c \times \mathbb{P}(G_c)) \subset \mathcal{K} \subset F^*G \) are isomorphisms outside \( c \times \mathbb{P}(G_c) \). Thus, it suffices to prove the two assertions under the assumption that \( c = t \). From the above discussion it follows that the map in (1) is identified with the composition of homomorphisms

\[
\begin{array}{ccc}
H^i\left(\mathbb{P}(G_c), F^*G|_{c \times \mathbb{P}(G_c)}\right) & \rightarrow & H^i\left(\mathbb{P}(G_c), \mathcal{O}_{\mathbb{P}(G_c)}(1)\right) \\
& & \rightarrow H^i\left(\mathbb{P}(G_c), \mathcal{K}|_{c \times \mathbb{P}(G_c)}\right)
\end{array}
\]

From Lemma 7.1 it follows that both these maps are isomorphisms.

The map \( F^*G^\vee|_{c \times \mathbb{P}(G_c)} \rightarrow \mathcal{K}^\vee|_{c \times \mathbb{P}(G_c)} \) factors as

\[
F^*G^\vee|_{c \times \mathbb{P}(G_c)} \rightarrow T_\pi(-1) \rightarrow \mathcal{K}^\vee|_{c \times \mathbb{P}(G_c)}.
\]

Taking cohomology, the second assertion in the lemma follows using Lemma 7.1. \( \square \)

## 8. Cohomology of some sheaves on \( S_d \)

Recall from Section 6 that associated to a divisor \( D \in C^{(d)} \) and an ordering \( (c_1, c_2, \ldots, c_d) \) of points of \( D \) we have the schemes \( S_j \) for \( 1 \leq j \leq d \). In this section we again fix this divisor \( D \). We now choose the ordering \( (c_1, c_2, \ldots, c_d) \) of \( D \) in the following manner: Define \( c_1 \) to be any point in \( D \). Now suppose we have defined \( c_j \) for \( 1 \leq j \leq d - 1 \). Then we define \( c_{j+1} \) to be \( c_j \) if \( c_j \in D \setminus \{c_1, c_2, \ldots, c_j\} \). Otherwise define \( c_j \) to be any point in \( D \setminus \{c_1, c_2, \ldots, c_j\} \). Throughout this section, we fix this ordering of \( D \). To clarify, the above conditions on the ordering do not determine the ordering uniquely.
For every $1 \leq j \leq d$, let 
$$
\Omega_j \longrightarrow S_j
$$
be the relative cotangent bundle of the map $f_{j,j-1}$ in (6.6). We have the relative Euler sequence

$$
0 \longrightarrow \Omega_j(1) \longrightarrow f_{j,j-1}^*\alpha_{j-1}^*A_{j-1} \longrightarrow \mathcal{O}_j(1) \longrightarrow 0 \quad (8.1)
$$
on $S_j$. For the sequence in (6.5)

$$
0 \longrightarrow A_j \longrightarrow F_{j,j-1}^*A_{j-1} \longrightarrow (i_j)_*\mathcal{O}_j(1) \longrightarrow 0,
$$
since the quotient is supported on $c_j \times S_j$, there are the natural inclusion maps

$$
F_{j,j-1}^*A_{j-1}(-c_j \times S_j) \subset A_j \quad \text{and} \quad F_{j,j-1}^*A_{j-1}^y \subset A_j^y.
$$

For a locally free sheaf $V$ on $C \times S_j$, and an effective divisor $D'$ on $C$, the vector bundle $V \otimes p_C^*\mathcal{O}_C(-D')$ will be denoted by $V(-D')$ (this involves a mild abuse of notation).

**Lemma 8.1.** Let $D'$ be an effective divisor on $C$. For all $i \geq 0$, and $t \in C$, the two natural maps

1. $H^i\left(S_j, F_{j,j-1}^*A_{j-1}(\mathbb{C} - D') \times S_j\right)_{|_{txS_j}} \longrightarrow H^i\left(S_j, A_j(\mathbb{C} - D')|_{txS_j}\right)$
2. $H^i\left(S_j, F_{j,j-1}^*A_{j-1} ^y|_{txS_j}\right) \longrightarrow H^i\left(S_j, A_j ^y|_{txS_j}\right)$

are isomorphisms.

**Proof.** This follows from Lemma 7.2 by setting $T = S_{j-1}$, $G = A_{j-1}(\mathbb{C} - D')$ and $c = c_j$ in it. \hfill $\square$

Recall from (6.8) that we have an exact sequence

$$
0 \longrightarrow A_d \longrightarrow F_{d,0}^*A_0 \longrightarrow B_d^d \longrightarrow 0
$$
on $C \times S_d$, and a filtration

$$
A_d \subset F_{d,d-1}^*A_{d-1} \subset F_{d,d-2}^*A_{d-2} \subset \cdots \subset F_{d,1}^*A_1 \subset F_{d,0}^*A_0 = p_1^*E
$$

(see (6.7)). Since $B_d^d$ is supported on $D \times S_d$, there is an inclusion map

$$
F_{d,0}^*A_0(-D) \subset A_d.
$$

**Corollary 8.2.** For any $t \in C$, the following statements hold:

1. the natural map $H^i\left(S_d, F_{d,0}^*A_0(-D \times S_d)|_{txS_d}\right) \longrightarrow H^i\left(S_d, A_d|_{txS_d}\right)$ is an isomorphism,
2. the natural map $H^i\left(S_d, F_{d,0}^*A_0^y|_{txS_d}\right) \longrightarrow H^i\left(S_d, A_d^y|_{txS_d}\right)$ is an isomorphism,
3. $H^i\left(S_d, A_d|_{txS_d}\right) = 0 \quad \forall \ i > 0$, and
4. $H^i\left(S_d, A_d^y|_{txS_d}\right) = 0 \quad \forall \ i > 0$. 

Proof. First statement (2) will be proved. For \( i, j \geq 0 \), we have a commutative square

\[
\begin{array}{ccc}
H^i \left( S_d, F_{d,j}^* A_j^\vee|_{\times S_d} \right) & \longrightarrow & H^i \left( S_d, F_{d,j+1}^* A_{j+1}^\vee|_{\times S_d} \right) \\
\sim & \longrightarrow & \sim \\
H^i \left( S_{j+1}, F_{j+1,j}^* A_{j+1}^\vee|_{\times S_{j+1}} \right) & \longrightarrow & H^i \left( S_{j+1}, A_{j+1}^\vee|_{\times S_{j+1}} \right)
\end{array}
\]

The lower horizontal arrow is an isomorphism using Lemma 8.1. Therefore the upper horizontal arrow is also an isomorphism. The composition

\[
H^i \left( S_d, F_{d,0}^* A_0^\vee|_{\times S_d} \right) \longrightarrow H^i \left( S_d, F_{d,1}^* A_1^\vee|_{\times S_d} \right) \longrightarrow \cdots \longrightarrow H^i \left( S_d, A_d^\vee|_{\times S_d} \right)
\]

of these upper horizontal isomorphisms for \( j = 0, 1, \cdots, d-1 \) produces an isomorphism

\[
H^i \left( S_d, F_{d,0}^* A_0^\vee|_{\times S_d} \right) \longrightarrow H^i \left( S_d, A_d^\vee|_{\times S_d} \right).
\]

This proves (2).

Now note that we have

\[
F_{d,0}^* A_0|_{\times S_d} = p_1^* E|_{\times S_d} \cong E_t \otimes \mathcal{O}_{S_d}.
\]

This and (2) together prove statement (4).

For every \( 0 \leq j \leq d-1 \) define the divisor

\[
D_j := \sum_{t=j+1}^d c_t
\]
on \( C \). Define \( D_d = 0 \). For \( d \geq i, j \geq 0 \), we have a commutative square

\[
\begin{array}{ccc}
H^i \left( S_d, F_{d,j}^* (A_j(-D_j \times S_{j+1}))|_{\times S_d} \right) & \longrightarrow & H^i \left( S_d, F_{d,j+1}^* (A_{j+1}(-D_{j+1} \times S_{j+1}))|_{\times S_d} \right) \\
\sim & \longrightarrow & \sim \\
H^i \left( S_{j+1}, F_{j+1,j}^* (A_j(-D_j \times S_{j+1}))|_{\times S_{j+1}} \right) & \longrightarrow & H^i \left( S_{j+1}, A_{j+1}(-D_{j+1} \times S_{j+1})|_{\times S_{j+1}} \right)
\end{array}
\]

The lower horizontal arrow is an isomorphism using Lemma 8.1. Therefore the upper horizontal arrow is also an isomorphism. The composition of these upper horizontal arrows for \( j = 0, 1, \cdots, d-1 \)

\[
H^i \left( S_d, F_{d,0}^* (A_0(-D \times S_1))|_{\times S_d} \right) \longrightarrow H^i \left( S_d, F_{d,1}^* (A_j(-D_1 \times S_2))|_{\times S_d} \right) \longrightarrow \cdots \longrightarrow H^i \left( S_d, A_d|_{\times S_d} \right)
\]

produces an isomorphism

\[
H^i \left( S_d, F_{d,0}^* (A_0(-D \times S_1))|_{\times S_d} \right) \longrightarrow H^i \left( S_d, A_d|_{\times S_d} \right).
\]

This proves (1).

Again, (3) follows using (1) and the fact that \( F_{d,0}^* A_0(-D \times S_d)|_{\times S_d} \) is the trivial bundle. \( \Box \)

For convenience of notation, denote \( G_d := A_d|_{\times S_d} \).

**Lemma 8.3.** Using the above notation,

(1) \( h^1(S_d, \mathcal{E} \otimes d(G_d)) = 1 \),
\[ (2) \ h^i(S_d, \mathcal{E} \text{nd}(G_d)) = 0 \text{ for all } i \geq 2, \]
\[ (3) \ h^i(S_d, G_d^\vee(1)) = 0 \text{ for all } i \geq 1. \]

**Proof.** The lemma will be proved by induction on \( d \). First set \( d = 1 \). Then \( S_1 = \mathbb{P}(E_{c_1}) \) and \( G_1 \) sits in the short exact sequence
\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow G_1 \rightarrow \Omega_{\mathbb{P}^1}(1) \rightarrow 0. \]
Since \( H^1(\mathbb{P}^1, T_{\mathbb{P}^1}) = 0 \), this sequence splits. Using this, it is easily checked that all three assertions of the lemma are true when \( d = 1 \).

Let \( \delta \geq 2 \) and assume that the lemma holds for all \( 1 \leq d \leq \delta - 1 \). Set \( d = \delta \). We will first prove (3).

Set \( X = S_{d-1} \) and \( V = A_{d-1} \) on \( C \times X \). Then the quotient in (5.3) is exactly the quotient in (6.4) when \( j = d \). Thus, the short exact sequence in (5.3) is identified with the short exact sequence (6.5). In that case, (5.5) becomes
\[ 0 \rightarrow \mathcal{O}_d(1) \rightarrow G_d \rightarrow \Omega_d(1) \rightarrow 0. \] (8.2)

Let \( T_d \rightarrow S_d \) denote the relative tangent bundle for the map \( f_{d,d-1} : S_d \rightarrow S_{d-1} \) (take \( j = d \) in (6.2)). Dualizing (8.2) and after a twist we have an exact sequence on \( S_d \)
\[ 0 \rightarrow T_d \rightarrow G^\vee_d(1) \rightarrow \mathcal{O}_{S_d} \rightarrow 0. \] (8.3)

This sequence is identified with the short exact sequence in (5.6). Since
\[ R^i f_{d,d-1*}(T_d) = R^i f_{d,d-1*}(\mathcal{O}_{S_d}) = 0 \]
for all \( i \geq 1 \), it follows that \( R^i f_{d,d-1*}(G^\vee_d(1)) = 0 \) for \( i \geq 1 \). Therefore by Leray spectral sequence we get that
\[ H^i(S_d, G_d^\vee(1)) = H^i(S_{d-1}, f_{d,d-1*}(G^\vee_d(1))). \]
Thus, it suffices to compute dimensions of \( H^i(S_{d-1}, f_{d,d-1*}(G^\vee_d(1))) \). Recall that \( S_d \) is the projective bundle associated to the bundle \( A_{d-1}|_{c_d \times S_{d-1}} \). Therefore the sheaf \( f_{d,d-1*}(T_d) \) is \( ad(A_{d-1}|_{c_d \times S_{d-1}}) \). Applying \( f_{d,d-1*} \) to (8.3) we get an exact sequence
\[ 0 \rightarrow ad(A_{d-1}|_{c_d \times S_{d-1}}) \rightarrow f_{d,d-1*}(G^\vee_d(1)) \rightarrow \mathcal{O}_{S_{d-1}} \rightarrow 0. \] (8.4)

First consider the case where \( c_d \neq c_{d-1} \). By the choice of the ordering \((c_1, c_2, \ldots, c_d)\) made in the beginning of this section, this means that \( c_d \) appears in \( D \) with multiplicity 1, and hence we can write \( D = c_d + D' \) such that \( c_d \) does not appear in the support of \( D' \). The sheaf \( A_{d-1} \) on \( C \times S_{d-1} \) agrees with \( p_C^*E \) outside \( D' \times S_{d-1} \). It follows that \( A_{d-1}|_{c_d \times S_{d-1}} \) is the trivial bundle. Therefore
\[ H^i(S_{d-1}, ad \left( A_{d-1}|_{c_d \times S_{d-1}} \right)) = 0 \ \forall \ i > 0. \]

From the long exact sequence of (8.4) it follows that
\[ h^i(S_d, G_d^\vee(1)) = h^i(S_{d-1}, f_{d,d-1*}(G^\vee_d(1))) = 0 \]
for all \( i \geq 1 \).

Next consider the case where \( c_d = c_{d-1} \). Set \( Y = S_{d-2} \), \( W = A_{d-2} \) and \( c = c_{d-1} \). Then we have a triple \((c, Y, W)\) as in Remark 5.2. We observed in Remark 5.2 that we get a triple

\[ (h^i(S_d, G_d^\vee(1)) = 0 \ \forall \ i > 0. \]
(c, Y_1, W_1) such that SES(c, Y_1, W_1) is non-split. We leave it to the reader to check that SES(c, Y_1, W_1) is exactly (8.4).

As a consequence, it follows that the boundary map in the cohomology sequence of (8.4), that is,

$$H^0(S_{d-1}, \mathcal{O}_{S_{d-1}}) \rightarrow H^1(S_{d-1}, ad\left(A_{d-1}\right|_{c_d \times S_{d-1}}))$$

is an inclusion. By inductive hypothesis, using c_d = c_{d-1}, and the rationality of S_{d-1}, we get that

$$h^1(S_{d-1}, ad\left(A_{d-1}\right|_{c_d \times S_{d-1}})) = h^1(S_{d-1}, \mathcal{E}nd\left(A_{d-1}\right|_{c_d \times S_{d-1}}))$$

$$= h^1(S_{d-1}, \mathcal{E}nd\left(A_{d-1}\right|_{c_d \times S_{d-1}}))$$

$$= h^1(S_{d-1}, \mathcal{E}nd(G_{d-1})) = 1,$$

$$h^i(S_{d-1}, ad\left(A_{d-1}\right|_{c_d \times S_{d-1}})) = h^i(S_{d-1}, \mathcal{E}nd\left(A_{d-1}\right|_{c_d \times S_{d-1}}))$$

$$= h^i(S_{d-1}, \mathcal{E}nd\left(A_{d-1}\right|_{c_d \times S_{d-1}}))$$

$$= h^i(S_{d-1}, \mathcal{E}nd(G_{d-1})) = 0 \quad \forall \ i \geq 2.$$  

Consequently, the boundary map in (8.5) is an isomorphism. It follows from the long exact sequence of cohomologies associated to (8.4) that

$$h^i(S_{d-1}, f_{d,d-1*}(G_d^\vee(1))) = h^i(S_d, G_d^\vee(1)) = 0$$

for i \geq 1. This proves the third assertion of the lemma when d = \delta.

Tensoring (8.2) with G_d^\vee we get the sequence

$$0 \rightarrow G_d^\vee(1) \rightarrow \mathcal{E}nd(G_d) \rightarrow G_d^\vee \otimes \Omega_d(1) \rightarrow 0.$$  

Using (3) we see that h^i(S_d, \mathcal{E}nd(G_d)) = \mathcal{H}^i(S_d, G_d^\vee \otimes \Omega_d(1)) for i \geq 1. We have a short exact sequence obtained by tensoring (8.3) with \Omega_d

$$0 \rightarrow \mathcal{E}nd(\Omega_d) \rightarrow G_d^\vee \otimes \Omega_d(1) \rightarrow \Omega_d \rightarrow 0.$$  

Since f_{d,d-1} : S_d \rightarrow S_{d-1} in (6.2) is a projective bundle, using Lemma 7.1 it follows that

$$h^i(S_d, \mathcal{E}nd(\Omega_d)) = h^i(S_d, \mathcal{O}_{S_d}) = 0 \quad \forall \ i > 0.$$  

Therefore, (again using Lemma 7.1) we get

$$h^i(S_d, G_d^\vee \otimes \Omega_d(1)) = h^i(S_d, \Omega_d) = h^{i-1}(S_d, \mathcal{O}_{S_d})$$

for i \geq 1. Thus, h^i(S_d, \mathcal{E}nd(G_d)) = h^{i-1}(S_d, \mathcal{O}_{S_d}) for i \geq 1, which proves the first two assertions of the lemma for d = \delta. This completes the proof of the lemma.

9. ON THE GEOMETRY OF \(Q\)

9.1. Notation. As before, the d-th symmetric product of C is denoted by C^{(d)}. Let

$$p_1 : C \times Q \rightarrow C, \ p_2 : C \times Q \rightarrow Q,$$

$$q_1 : C \times C^{(d)} \rightarrow C, \ q_2 : C \times C^{(d)} \rightarrow C^{(d)}$$

denote the natural projections. Recall that there is a universal exact sequence on C \times Q

$$0 \rightarrow \mathcal{A} \rightarrow p_1^*E \rightarrow \mathcal{B} \rightarrow 0.$$  

(9.3)
Let
\[ \Sigma \subset C \times C^{(d)} \]  \hspace{1cm} (9.4)
be the universal divisor. Define
\[ \Phi := \text{Id}_C \times \phi : C \times \mathbb{Q} \longrightarrow C \times C^{(d)}, \]  \hspace{1cm} (9.5)
where \( \phi \) is the Hilbert-Chow map in (6.10).

9.2. Direct image of sheaves on \( C \times \mathbb{Q} \).

**Corollary 9.1.** The following statements hold:

1. \( \Phi_* \mathcal{O}_{C \times \mathbb{Q}} = \mathcal{O}_{C \times C^{(d)}} \) and \( R^i \Phi_* \mathcal{O}_{C \times \mathbb{Q}} = 0 \), \( \forall \ i > 0 \).
2. \( \phi_* \mathcal{O}_{\mathbb{Q}} = \mathcal{O}_{C^{(d)}} \) and \( R^i \phi_* \mathcal{O}_{\mathbb{Q}} = 0 \), \( \forall \ i > 0 \).

**Proof.** The fibers of \( \phi \) (respectively, \( \Phi \)) over any point \( D \in C^{(d)} \) (respectively, \( (c, D) \in C \times C^{(d)} \)) is isomorphic to \( \mathbb{Q}_D \). By Corollary 6.3 we have \( h^i(\mathbb{Q}_D, \mathcal{O}_{\mathbb{Q}_D}) = h^i(S_d, \mathcal{O}_{S_d}) \). Since \( S_d \) is a tower of projective bundles, it follows that \( h^0(S_d, \mathcal{O}_{S_d}) = 1 \) and \( h^i(S_d, \mathcal{O}_{S_d}) = 0 \) for all \( i > 0 \). As both \( \phi \) and \( \Phi \) are flat morphisms (see \([GS20, \text{Corollary 6.3}]\)), the result now follows from Grauert’s theorem \([Har77, \text{p. 288–289, Corollary 12.9}]\). \( \square \)

Let
\[ Z \subset C \times \mathbb{Q} \]  \hspace{1cm} (9.6)
be the zero scheme of the inclusion map \( \text{det}(A) \hookrightarrow \text{det}(p_1^*E) \), where \( p_1 \) is the map in (9.1). From the definition of \( \phi \) it follows immediately that \( \Phi^* \Sigma = Z \). In fact, \( Z \) sits in the following commutative diagram in which both squares are Cartesian
\[ \begin{array}{ccc}
Z & \longrightarrow & C \times \mathbb{Q} \\
\downarrow & & \downarrow \Phi \\
\Sigma & \longrightarrow & C \times C^{(d)}
\end{array} \]  \hspace{1cm} (9.7)
(see (9.4) and (9.6)) and the composition of the top horizontal maps is a finite morphism; the same holds for the composition of the bottom horizontal maps in (9.7). The ideal sheaf \( \mathcal{O}_{C \times \mathbb{Q}}(-Z) \) therefore annihilates \( B \) in (9.3), which in turn produces an inclusion map \( p_1^*E(-Z) \subset A \), where \( p_1 \) is the map in (9.1). Applying \( \Phi_* \) and using Corollary 9.1 an inclusion map
\[ q_1^*E(-\Sigma) \cong \Phi_*[p_1^*E(-Z)] \hookrightarrow \Phi_*A \]  \hspace{1cm} (9.8)
is obtained, where \( q_1 \) is the map in (9.2). Also, note that since the cokernel of \( A \rightarrow p_1^*E \) is \( B \), the kernel of the map \( p_1^*E^\vee \rightarrow A^\vee \) is \( B^\vee \). But \( B \) is torsion, hence \( B^\vee = 0 \). Therefore we also have the natural inclusion map \( p_1^*E^\vee \hookrightarrow A^\vee \). Applying \( \Phi_* \) and using Corollary 9.1 we get an inclusion map
\[ q_1^*E^\vee \hookrightarrow \Phi_*(A^\vee). \]

**Proposition 9.2.** The following statements hold:

1. The natural map \( q_1^*E(-\Sigma) \longrightarrow \Phi_*A \) is an isomorphism (see (9.8)).
2. The natural map \( q_1^*E^\vee \hookrightarrow \Phi_*(A^\vee) \) is an isomorphism.
3. \( R^i \Phi_*A = R^i \Phi_*(A^\vee) = 0 \) for all \( i > 0 \).
Proof. First consider the map \( \Phi^* [p_1^* E(-Z)] \rightarrow \Phi^* \mathcal{A} \). Fix \((c, D) \in C \times C^{(d)}\). We will show that the homomorphism
\[
H^0 \left( c \times \mathcal{Q}_D, \ p_1^* E(-Z) \big|_{c \times \mathcal{Q}_D} \right) \rightarrow H^0 \left( c \times \mathcal{Q}_D, \ A \big|_{c \times \mathcal{Q}_D} \right)
\]  
(9.9)
is an isomorphism.

In view of Corollary 6.3, showing that (9.9) is an isomorphism is equivalent to showing that the map
\[
H^0 \left( c \times S_d, \ p_1^* E(-D) \big|_{c \times S_d} \right) \rightarrow H^0 \left( c \times S_d, \ A \big|_{c \times S_d} \right)
\]
is an isomorphism. But this is precisely the content of Corollary 8.2 (1) when \(i = 0\). Since \(A\) is flat over \(C \times C^{(d)}\), using Grauert’s theorem, \([Har77, Corollary 12.9]\), it now follows that \(q_1^* E(-\Sigma) \rightarrow \Phi^* \mathcal{A}\) is an isomorphism.

The other two statements follow from Corollary 8.2 in the same way. \(\square\)

Corollary 9.3. The natural map
\[
q_1^* E|_{\Sigma} \rightarrow \Phi^* \mathcal{B}
\]
is an isomorphism, where \(\Sigma\) is defined in (9.4) and \(q_1\) (respectively, \(\Phi\)) is the map in (9.2) (respectively, (9.5)). Moreover,
\[
R^i \Phi^* \mathcal{B} = 0
\]
for all \(i > 0\).

Proof. Using projection formula and Corollary 9.1 it follows that
\[
\Phi^* p_1^* E = q_1^* E \quad \text{and} \quad R^i \Phi^* p_1^* E = q_1^* E \otimes R^i \Phi^* \mathcal{O}_{C \times \mathcal{Q}} = 0
\]
for all \(i > 0\). Therefore the statement follows immediately by applying \(\Phi^*\) to the universal exact sequence
\[
0 \rightarrow \mathcal{A} \rightarrow p_1^* E \rightarrow \mathcal{B} \rightarrow 0
\]
and using Proposition 9.2. \(\square\)

Lemma 9.4. The natural map \(\mathcal{O}_Z \rightarrow \mathcal{H}om(\mathcal{B}, \mathcal{B})\), where \(Z\) is defined in (9.6), is an isomorphism.

Proof. We will first show that \(Z\) is an integral and normal scheme. First let us show \(Z\) is irreducible. As the squares in (9.7) are Cartesian, it follows that the fibers of the map \(Z \rightarrow \Sigma\) are the same as the fibers of the \(\phi\). These fibers have the same dimension (see [GS20, Proposition 6.1]) and \(\Sigma\) is irreducible. It follows \(Z\) is irreducible.

Next consider the locus \(U\) where the map \(\phi\) is smooth. The set \(U\) meets each fiber of \(\phi\) in an open set whose complement has codimension at least two (see [GS20, Corollary 5.6, 5.7]). It follows that \(U_Z := (C \times U) \cap Z\) is an open set in \(Z\) where \(Z \rightarrow \Sigma\) is smooth and \(\text{codim}(Z \setminus U_Z, Z) \geq 2\).

As \(\Sigma\) is smooth, it follows that \(U_Z\) is smooth. Thus, \(Z\) satisfies Serre’s conditions \(R_0, R_1\). That the fibers of \(\Phi\) (and hence of \(Z \rightarrow \Sigma\)) are Cohen-Macaulay is proved in [GS20, Corollary 6.4] (this is not explicitly mentioned in the statement of the Corollary, but is mentioned in the proof). It follows from [Stk, Tag 045J] (or see Corollary to [Mat86, Theorem 23.3]) that the \(Z\) is Cohen-Macaulay. This shows that \(Z\) is reduced, integral and normal.
Next we show that \( \mathcal{B} \) in (9.3) is a torsionfree \( \mathcal{O}_Z \)-module. Indeed, if \( \mathcal{B}' \subset \mathcal{B} \) is a torsion \( \mathcal{O}_Z \)-module, then it is also torsion as a \( \mathcal{O}_Q \)-module. This is a contradiction as \( \mathcal{B} \) is a coherent and flat \( \mathcal{O}_Q \)-module. Hence \( \mathcal{B} \) is a torsionfree \( \mathcal{O}_Z \)-module.

Let \( \text{Spec}(A) \subset Z \) be an affine open set, and let \( \mathcal{B}_A \) denote the module corresponding to \( \mathcal{B} \). We have the inclusion maps

\[
A \subset \text{Hom}_A(\mathcal{B}_A, \mathcal{B}_A) \subset \text{Hom}_{K(A)}(\mathcal{B}_A \otimes_A K(A), \mathcal{B}_A \otimes_A K(A)) = K(A).
\]

As \( A \) is normal, and \( \text{Hom}_A(\mathcal{B}_A, \mathcal{B}_A) \) is a finite \( A \)-module, it follows that \( \text{Hom}_A(\mathcal{B}_A, \mathcal{B}_A) \) coincides with \( A \). This proves the lemma. \( \square \)

**Theorem 9.5.** There is a map \( \Xi \) that fits in a short exact sequence

\[
0 \longrightarrow \text{ad}(q_1^*E|_Σ) \longrightarrow \Phi_*\text{Hom}(\mathcal{A}, \mathcal{B}) \longrightarrow R^1\Phi_*\text{End}(A) \longrightarrow 0
\]
on \( Σ \).

For every \( i \geq 1 \), there is a natural isomorphism

\[
R^i\Phi_*\text{Hom}(\mathcal{A}, \mathcal{B}) \sim R^{i+1}\Phi_*\text{End}(A).
\]

**Proof.** Application of \( \text{Hom}(\mathcal{A}, -) \) to the exact sequence in (9.3) produces an exact sequence

\[
0 \longrightarrow \text{End}(A) \longrightarrow \text{Hom}(\mathcal{A}, p_1^*E) \longrightarrow \text{Hom}(\mathcal{A}, \mathcal{B}) \longrightarrow 0.
\]

(9.10)

Using the projection formula and Proposition 9.2 it follows that

\[
R^i\Phi_*\text{Hom}(\mathcal{A}, p_1^*E) = q_1^*E \otimes R^i\Phi_*(A^\vee) = 0
\]

for all \( i > 0 \). Therefore, applying \( \Phi_* \) to (9.10) produces an exact sequence

\[
0 \longrightarrow \Phi_*\text{End}(A) \longrightarrow \Phi_*\text{Hom}(\mathcal{A}, p_1^*E) \longrightarrow \Phi_*\text{Hom}(\mathcal{A}, \mathcal{B}) \longrightarrow R^1\Phi_*\text{End}(A) \longrightarrow 0.
\]

(9.11)

Moreover, we get that there is an isomorphism

\[
R^i\Phi_*\text{Hom}(\mathcal{A}, \mathcal{B}) \sim R^{i+1}\Phi_*\text{End}(A)
\]

for every \( i \geq 1 \). This proves the second part of the theorem.

To prove the first part of the theorem, it is enough to show that the image of the map

\[
\Phi_*\text{Hom}(\mathcal{A}, p_1^*E) \longrightarrow \Phi_*\text{Hom}(\mathcal{A}, \mathcal{B})
\]
is isomorphic to \( \text{ad}(q_1^*E|_Σ) \).

Consider the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Hom}(p_1^*E, A) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{End}(A)
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\longrightarrow & \longrightarrow & \longrightarrow \\
0 & \longrightarrow & \text{Hom}(\mathcal{A}, p_1^*E) \\
\end{array}
\longrightarrow \text{Hom}(\mathcal{A}, \mathcal{B}) \longrightarrow 0.
\]

Using projection formula together with Proposition 9.2 it follows that

\[
R^1\Phi_*\text{Hom}(p_1^*E, A) = q_1^*E^\vee \otimes R^1\Phi_*A = 0,
\]

\[
\Phi_*\text{Hom}(\mathcal{A}, p_1^*E) = q_1^*E \otimes \Phi_*(A^\vee) = \text{Hom}(q_1^*E, q_1^*E).
\]
Consequently, applying $\Phi_*$ to the preceding diagram we get a diagram
\[
0 \longrightarrow \Phi_*\mathcal{H}om(p_1^*E, \mathcal{A}) \longrightarrow \Phi_*\mathcal{H}om(p_1^*E, p_1^*E) \longrightarrow \Phi_*\mathcal{H}om(p_1^*E, \mathcal{B}) \longrightarrow 0
\]
\[
0 \longrightarrow \Phi_*\mathcal{E}nd(\mathcal{A}) \longrightarrow \Phi_*\mathcal{H}om(\mathcal{A}, p_1^*E) \longrightarrow \Phi_*\mathcal{H}om(\mathcal{A}, \mathcal{B}).
\]

Thus, the image of the homomorphism $\Phi_*\mathcal{H}om(\mathcal{A}, p_1^*E) \longrightarrow \Phi_*\mathcal{H}om(\mathcal{A}, \mathcal{B})$ coincides with the image of the homomorphism $\Phi_*\mathcal{H}om(p_1^*E, \mathcal{B}) \longrightarrow \Phi_*\mathcal{H}om(\mathcal{A}, \mathcal{B})$. Now consider the following commutative diagram in which the left vertical arrow is an isomorphism due to Lemma 9.4
\[
0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{E}nd(p_1^*E|_Z) \longrightarrow ad(p_1^*E|_Z) \longrightarrow 0 \quad (9.12)
\]
\[
0 \longrightarrow \mathcal{H}om(\mathcal{B}, \mathcal{B}) \longrightarrow \mathcal{H}om(p_1^*E, \mathcal{B}) \longrightarrow \mathcal{H}om(\mathcal{A}, \mathcal{B})
\]

Applying $\Phi_*$ to it and using Corollary 9.3 we get the diagram (the right vertical arrow is defined to be $\Xi$)
\[
0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow \mathcal{E}nd(q_1^*E|_\Sigma) \longrightarrow ad(q_1^*E|_\Sigma) \longrightarrow 0 \quad (9.13)
\]
\[
0 \longrightarrow \Phi_*\mathcal{H}om(\mathcal{B}, \mathcal{B}) \longrightarrow \Phi_*\mathcal{H}om(p_1^*E, \mathcal{B}) \longrightarrow \Phi_*\mathcal{H}om(\mathcal{A}, \mathcal{B})
\]

Therefore, the image of the homomorphism $\Phi_*\mathcal{H}om(p_1^*E, \mathcal{B}) \longrightarrow \Phi_*\mathcal{H}om(\mathcal{A}, \mathcal{B})$ is same as the image of the map $ad(q_1^*E|_\Sigma) \xrightarrow{\Xi} \Phi_*\mathcal{H}om(\mathcal{A}, \mathcal{B})$. But the above diagram shows that this map is an inclusion. Hence its image is isomorphic to $ad(q_1^*E|_\Sigma)$. This completes the proof of the theorem. \qed

Note that it follows from Theorem 9.5 that the sheaf $R^i\Phi_*\mathcal{E}nd(\mathcal{A})$ is supported on $\Sigma$ for every $i \geq 1$. By Corollary 9.1, the canonical map $R^1\Phi_*\mathcal{E}nd(\mathcal{A}) \longrightarrow R^1\Phi_*ad(\mathcal{A})$ is an isomorphism. Recall the relative adjoint Atiyah sequence (see (4.4)) for the locally free sheaf $\mathcal{A}$ on $C \times \mathcal{Q}$
\[
0 \longrightarrow ad(\mathcal{A}) \longrightarrow at_C(\mathcal{A}) \longrightarrow p_1^*T_C \longrightarrow 0 \quad (9.14)
\]

Applying $\Phi_*$ to (9.14) we get a map of sheaves
\[
q_1^*T_C \longrightarrow R^1\Phi_*ad(\mathcal{A}) \quad (9.15)
\]
on $C \times C^{(d)}$.

For ease of notation, let
\[
\overline{q}_1 : \Sigma \longrightarrow C \quad (9.16)
\]
be the composite $\Sigma \hookrightarrow C \times C^{(d)} \overset{q_1}{\longrightarrow} C$, where $q_1$ is the map in (9.2).

**Theorem 9.6.** The map in (9.15) induces an isomorphism
\[
q_1^*T_C|_\Sigma = \overline{q}_1^*T_C \xrightarrow{\sim} R^1\Phi_*ad(\mathcal{A}),
\]
where $\Phi$ is the map in (9.5). Moreover, $R^i\Phi_*\mathcal{E}nd(\mathcal{A}) = 0$ for all $i \geq 2$. 
Proof. We have already observed above that \( R^1\Phi_*ad(A) \) is supported on \( \Sigma \). Thus, the map \( q_!^iTC \to R^1\Phi_*ad(A) \) factors through \( q_!^iTC|_\Sigma \to R^1\Phi_*ad(A) \).

Let \( (c, D) \in \Sigma \) be a point. Then \( c \in D \). We fix an ordering of the points of \( D \) as mentioned at the beginning of Section \( 8 \). We also choose this ordering in such a way that \( c_d = c \). Associated to this ordering, we have the space \( S_d \) as constructed in section \( 6 \). Recall, from (6.11), the map \( g_d : S_d \to Q_D \). We used the same notation to denote the composite map \( S_d \to Q_D \to Q \). Consider the composite

\[
T_{c,d} = \frac{q_!^iT_C}{(c,D)} \to R^1\Phi_*ad(A)|_{(c,D)} \to H^1(Q,D, \text{ad}(A|_{c \times Q_D})) \to H^1(S_d, \text{ad}(A|_{c \times S_d})). \tag{9.17}
\]

The last map, which is induced by \( g_d \), is an isomorphism by Corollary 6.3.

We will prove that the composite in (9.17) is an inclusion.

Take \( v \in T_{c,d} \). Let

\[
\alpha \in H^1(Q,D, \text{ad}(A|_{c \times Q_D})) \quad \text{and} \quad \beta \in H^1(S_d, \text{ad}(A|_{c \times S_d})) \tag{9.18}
\]
denote the images of \( v \) along maps in (9.17). We need to show that \( \beta \neq 0 \).

Consider the following diagram in which the square is Cartesian

\[
c \times S_d \xrightarrow{g_d} c \times Q_D \xrightarrow{} C \times Q \xrightarrow{\phi} (c, D) \xrightarrow{} C \times C^{(d)}
\]

It is evident that \( \alpha \) in (9.18) is the extension class of the short exact sequence obtained by restricting (9.14) to \( c \times Q_D \). When we further pullback this short exact sequence using the map \( g_d \), we get the short exact sequence on \( S_d \) whose extension class is \( \beta \) in (9.18). Thus, \( \beta \) is the extension class of the short exact sequence obtained by pulling back (9.14) along the top horizontal row.

However, the top horizontal row \( c \times S_d \xrightarrow{g_d} c \times Q_D \to C \times Q \) factors as

\[
c \times S_d \to C \times S_d \xrightarrow{\text{Id}_C \times g_d} C \times Q.
\]

We conclude that \( \beta \) corresponds to the short exact sequence on \( S_d \) obtained by pulling back the sequence (9.14) along the map \( \text{Id}_C \times g_d \) and restricting it to \( c \times S_d \). Note that \( A|_{C \times S_d} = A_d \) (see (6.8)). By Corollary 4.2, the pullback of (9.14) along the map \( \text{Id}_C \times g_d \) is the relative adjoint Atiyah sequence for the bundle \( A_d \) on \( C \times S_d \), that is, the exact sequence

\[
0 \to \text{ad}(A_d) \to at_C(A_d) \to p_C^*T_C \to 0. \tag{9.19}
\]

We shall use Proposition 5.3 to show that the restriction of (9.19) to \( c \times S_d \) is a non-trivial extension. Set \( X = S_{d-1} \), \( V = A_{d-1} \) on \( C \times X \) and \( c = c_d \) in Proposition 5.3. Note that the quotient in (5.3) is exactly the quotient in (6.4) when we take \( j = d \). As a result, applying Proposition 5.3, we get that the infinitesimal deformation map of \( A_d \) at the point \( c_d \) is injective. This means that the class of the restriction of

\[
0 \to \mathcal{E}nd(A_d) \to At_C(A_d) \to p_C^*T_C \to 0
\]
to $c \times S_d$ is non-zero. As $H^1(S_d, \mathcal{O}_{S_d}) = 0$, it follows that the class of the restriction of (9.19) to $c \times S_d$ is non-zero. This proves that $\beta$ in (9.18) is nonzero. Thus the composite in (9.17) is an inclusion.

From Lemma 8.3 it follows that $h^1(S_d, \mathcal{E}nd(A_d|_{c \times S_d})) = 1$. Thus, $h^1(S_d, ad(A_d|_{c \times S_d})) = 1$. In view of the injectivity of the composite in (9.17), this implies that the composite map in (9.17) is actually an isomorphism.

Since the composite in (9.17) is an isomorphism, it follows that the map

$$R^1\Phi_*ad(A)\big|_{(c,D)} \longrightarrow H^1 \left( Q_D, ad \left( A \big|_{c \times Q_D} \right) \right)$$

is surjective. By the base change theorem [Har77, Chapter 3, Theorem 12.11] the surjectivity of this map implies that it is in fact an isomorphism. Moreover, this also implies that

$$\mathfrak{T}_C\big|_{(c,D)} \longrightarrow R^1\Phi_*ad(A)\big|_{(c,D)}$$

is an isomorphism. As $\Sigma$ is integral we easily conclude that $R^1\Phi_*ad(A)$ is a line bundle, and the first statement of the theorem follows easily.

Next we prove the second statement. Proceeding as above, it suffices to show that for $(c, D) \in \Sigma$

$$H^i \left( S_d, \mathcal{E}nd \left( A_d \big|_{c \times S_d} \right) \right) = 0$$

for $i \geq 2$. Now this is the content of Lemma 8.3. This completes the proof of the theorem.

Corollary 9.7.

(1) There is the following short exact sequence on $\Sigma$ ($\Xi$ is the map in (9.13))

$$0 \longrightarrow ad(q_1^*E\big|_{\Sigma}) \longrightarrow \Phi_*\mathcal{H}om(A, B) \longrightarrow q_1^*T_C\big|_{\Sigma} \longrightarrow 0.$$

(2) $R^i\Phi_*\mathcal{H}om(A, B) = 0$ for $i \geq 1$.

(3) $H^i(\Sigma, \Phi_*\mathcal{H}om(A, B)) \longrightarrow H^i(\mathcal{Z}, \mathcal{H}om(A, B))$ for all $i$.

Proof. Statement (1) follows by combining the short exact sequence in the statement of Theorem 9.5 with the isomorphism in Theorem 9.6. Statement (2) follows using the isomorphism in the statement of Theorem 9.5 and the second assertion in Theorem 9.6. Statement (3) follows using the Leray spectral sequence and (2).

9.3. The tangent bundle of $Q$. The following proposition describing the tangent bundle of $Q$ is standard.

Proposition 9.8. The tangent bundle of $Q$ is

$$T_Q \cong p_2*(\mathcal{H}om(A, B)),$$

where $p_2$ is the projection in (9.1).

Proof. The proof is same as that of [Str87, Theorem 7.1].
(1) There is a diagram
\[
\begin{array}{ccccccccc}
0 & \to & q_1^* \text{ad}(E) |_{\Sigma} & \to & q_1^* \text{at}(E) |_{\Sigma} & \to & q_1^* T_C |_{\Sigma} & \to & 0 \\
\parallel & & \Vert & & \Vert & & \Vert & & \Xi \\
0 & \to & q_1^* \text{ad}(E) |_{\Sigma} & \xrightarrow{\Xi} & \Phi_* \text{Hom}(A, B) & \to & R^1 \Phi_* \text{ad}(A) & \to & 0 \\
\end{array}
\] (9.20)
in which the squares commute up to a minus sign, where $q_1$ and $\Phi$ are the maps in (9.2) and (9.5) respectively. The right vertical arrow is the one coming from Theorem 9.6. In particular, the middle vertical arrow is an isomorphism.

(2) $\text{Sec}^d(\text{at}(E)) \xrightarrow{\sim} \phi_* T_Q$.

(3) $R^i \phi_* T_Q = 0$ for all $i > 0$.

Proof. We will first construct diagram (9.20). Denote by $F$ the pushout of the diagram
\[
\begin{array}{ccccccccc}
0 & \to & \text{ad}(A) & \to & \text{at}_C(A) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Hom}(A, p_1^* E)/\mathcal{O}_{C \times Q} & \to & F & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Hom}(A, B) & \to & \text{Hom}(A, B) & \to & 0 \\
\end{array}
\] (9.21)
So using Snake lemma the following diagram is obtained:
\[
\begin{array}{ccccccccc}
0 & \to & \text{ad}(A) & \to & \text{at}_C(A) & \to & p_1^* T_C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Hom}(A, p_1^* E)/\mathcal{O}_{C \times Q} & \to & F & \to & p_1^* T_C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Hom}(A, B) & \to & \text{Hom}(A, B) & \to & 0 \\
\end{array}
\] (9.22)
Recall that by Corollary 4.2 the relative adjoint Atiyah sequence of $p_1^* E$ on $C \times Q$, where $p_1$ is the projection in (9.1), is simply the pullback of the relative adjoint Atiyah sequence for $E$. From Corollary 4.3 it follows that the middle row in (9.22) coincides with the pushout of relative adjoint Atiyah sequence of $p_1^* E$ by the morphism $\text{ad}(p_1^* E) \to \text{Hom}(A, p_1^* E)/\mathcal{O}$, that is, we have a diagram
\[
\begin{array}{ccccccccc}
0 & \to & p_1^* \text{ad}(E) & \to & p_1^* \text{at}(E) & \to & p_1^* T_C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Hom}(A, p_1^* E)/\mathcal{O}_{C \times Q} & \to & F & \to & p_1^* T_C & \to & 0 \\
\end{array}
\] (9.23)
Combining (9.22) and (9.23) we get maps
\[ p_1^* \text{at}(E) \to F \to \text{Hom}(A, B). \]
Applying $\Phi_*$ we get a map

$$q^*_1 at (E) \longrightarrow \Phi_* \mathcal{H}om (A, B)$$ (9.24)

Next we show that in the following diagram the squares commute up to a minus sign:

$$
\begin{array}{cccccc}
0 & \longrightarrow & q^*_1 ad (E) & \longrightarrow & q^*_1 at (E) & \longrightarrow & q^*_1 T_C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & q^*_1 ad (E)|_\Sigma & \longrightarrow & \Phi_* \mathcal{H}om (A, B) & \longrightarrow & R^1 \Phi_* ad (A) & \longrightarrow & 0
\end{array}
$$ (9.25)

Here the bottom sequence is the one in Theorem 9.5. The middle vertical arrow is given by (9.24) while the right vertical arrow is given by the boundary map of the sequence obtained by applying $\Phi_*$ to the relative adjoint Atiyah sequence of $A$. The commutativity of the box in the left is evident as $\Phi_* \mathcal{H}om (A, B)$ is supported on $\Sigma$. For the commutativity of the box in the right, first recall that since $F$ is a pushout of the diagram (9.21) we have a diagram in which the squares commute up to a minus sign

$$
\begin{array}{cccccc}
0 & \longrightarrow & ad (A) & \longrightarrow & at_C (A) & \longrightarrow & p^*_1 T_C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ad (A) & \longrightarrow & at_C (A) \oplus \mathcal{H}om (A, p^*_1 E)/\mathcal{O}_{C \times Q} & \longrightarrow & F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ad (A) & \longrightarrow & \mathcal{H}om (A, p^*_1 E)/\mathcal{O}_{C \times Q} & \longrightarrow & \mathcal{H}om (A, B) & \longrightarrow & 0
\end{array}
$$

Applying $\Phi_*$ we get a diagram

$$
\begin{array}{cccccc}
q^*_1 T_C & \longrightarrow & R^1 \Phi_* ad (A) & & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
q^*_1 at (E) & \longrightarrow & \Phi_* F & \longrightarrow & R^1 \Phi_* ad (A) & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Phi_* \mathcal{H}om (A, B) & \longrightarrow & R^1 \Phi_* ad (A) & & & \\
\end{array}
$$

in which the squares commute up to a minus sign. This shows the commutativity of the right square in (9.25) up to a minus sign. The first assertion of the theorem is proved because all sheaves in the lower row of (9.20) are supported on $\Sigma$.

It follows that we have an isomorphism $q^*_1 at (E)|_\Sigma \cong \Phi_* \mathcal{H}om (A, B)$. Applying $q_2*$ to this and using Proposition 9.8 yields the second assertion. The third assertion is deduced from Corollary 9.7 using $q_2|_\Sigma$ and $p_2|_Z$ are finite maps. \hfill $\square$

9.4. **Computation of cohomologies of $T_Q$.** The following theorem is deduced using the fact that the middle vertical arrow in (9.20) is an isomorphism.

Theorem 9.10.
(1) Let \( g_C \) be the genus of \( C \). For all \( d-1 \geq i \geq 0 \),

\[
H^i(Q, T_Q) = H^0(C, \text{ad}(E)) \otimes \bigwedge^i H^1(C, \mathcal{O}_C) \\
\bigoplus H^1(C, \text{ad}(E)) \otimes \bigwedge^{i-1} H^1(C, \mathcal{O}_C). 
\]

In particular,

\[
h^i(Q, T_Q) = \left( \begin{array}{c} g_C \\ i \end{array} \right) \cdot h^0(C, \text{ad}(E)) + \left( \begin{array}{c} g_C \\ i-1 \end{array} \right) \cdot h^1(C, \text{ad}(E)). 
\]

(2) When \( i = d \),

\[
H^d(Q, T_Q) = \bigwedge^{d-1} H^1(C, \mathcal{O}_C) \otimes h^1(C, \text{ad}(E)). 
\]

In particular,

\[
h^d(Q, T_Q) = \left( \begin{array}{c} g_C \\ d-1 \end{array} \right) \cdot h^1(C, \text{ad}(E)). 
\]

(3) For all \( i \geq d + 1 \),

\[
H^i(Q, T_Q) = 0. 
\]

Proof. From Corollary 9.7, Proposition 9.8 and Theorem 9.9 it follows that

\[
H^i(Q, T_Q) \cong H^i(Z, \mathcal{H}\text{om}(A, B)) \cong H^i \left( \Sigma, q_1^* \text{ad}(E) \big|_{\Sigma} \right). 
\]

But \( \Sigma \cong C \times C^{(d-1)} \), and \( q_1|_\Sigma : \Sigma \rightarrow C \) is just the first projection [ACG11, Section 10, Chapter 11]. Therefore by Künneth formula we get that

\[
H^i(Q, T_Q) = H^0(C, \text{ad}(E)) \otimes H^i \left( C^{(d-1)}, \mathcal{O}_{C^{(d-1)}} \right) \\
\bigoplus H^1(C, \text{ad}(E)) \otimes H^{i-1} \left( C^{(d-1)}, \mathcal{O}_{C^{(d-1)}} \right). 
\]

Now by [Mac62, Equation 11.1] we have

\[
H^i \left( C^{(d-1)}, \mathcal{O}_{C^{(d-1)}} \right) = \bigwedge^i H^1(C, \mathcal{O}_C) 
\]

if \( 0 \leq i \leq d-1 \) and 0 otherwise. The statement of the theorem now follows immediately. \( \square \)

Theorem 9.11.

(1) If \( d, g_C \geq 2 \), then

\[
h^1(Q, T_Q) = g_C \cdot h^0(C, \text{ad}(E)) + h^1(C, \text{ad}(E)) + 3g_C - 3. 
\]

In particular, the dimension of the space \( H^1(Q, T_Q) \) depends only on \( C \) and \( E \) and is independent of \( d \).

(2) Let \( d, g_C \geq 2 \). Then there is an exact sequence

\[
0 \rightarrow H^1 \left( \Sigma, q_1^* \text{ad}(E) \big|_{\Sigma} \right) \rightarrow H^1(Q, T_Q) \rightarrow H^1(C, T_C) \rightarrow 0, 
\]

where \( H^1 \left( \Sigma, q_1^* \text{ad}(E) \big|_{\Sigma} \right) \cong H^0(C, \text{ad}(E)) \otimes H^1(C, \mathcal{O}_C) \bigoplus H^1(C, \text{ad}(E)). \)
Proof. Recall that we have an exact sequence on \( C \)
\[
0 \to \text{ad}(E) \to \text{at}(E) \to T_C \to 0.
\]  (9.26)
Since \( g_C \geq 2 \), it follows that \( H^0(C, T_C) = 0 \). Therefore, the long exact sequence of cohomologies for (9.26) gives \( H^0(C, \text{ad}(E)) = H^0(C, \text{at}(E)) \) together with an exact sequence
\[
0 \to H^1(C, \text{ad}(E)) \to H^1(C, \text{at}(E)) \to H^1(C, T_C) \to 0.
\]
This shows that \( h^1(C, \text{at}(E)) = h^1(C, \text{ad}(E)) + 3g_C - 3 \). The first statement now follows from Theorem 9.10.

We will prove the second statement. Using Künneth formula it follows that
\[
H^1(\Sigma, q_1^*T_C|_{\Sigma}) \cong H^1(C, T_C).
\]
Then using (9.20), we get an exact sequence
\[
H^1(\Sigma, q_1^*\text{ad}(E)|_{\Sigma}) \to H^1(Q, T_Q) \to H^1(C, T_C).
\]  (9.27)
It was shown in (1) that the dimension of the middle term equals the sum of the dimensions of the extreme terms. It follows that the sequence in (9.27) must be injective on the left and surjective on the right. The last isomorphism follows using Künneth formula and the isomorphism \( H^1(C^{(d-1)}, \mathcal{O}_{C^{(d-1)}}) \cong H^1(C, \mathcal{O}_C) \).

In [BDH15] it was shown that when \( E \cong \mathcal{O}_C \) and \( g_C \geq 2 \), then
\[
H^0(Q, T_Q) = \text{sl}(r, \mathbb{C}) = H^0(C, \text{ad}(\mathcal{O}_C))
\].
This was generalized in [Gan19] where it was shown that when \( g_C \geq 2 \) and either \( E \) is semistable or \( \text{rank}(E) \geq 3 \), then
\[
H^0(Q, T_Q) = H^0(C, \text{ad}(E)).
\]
It follows immediately from Theorem 9.10 that we can drop these assumptions on \( E \) to get the following general statement.

**Corollary 9.12.** Let \( g_C \geq 2 \). Then
\[
H^0(Q, T_Q) = H^0(C, \text{at}(E)) = H^0(C, \text{ad}(E)).
\]

**Corollary 9.13.** There is an isomorphism of sheaves
\[
\Phi_*\mathcal{E}nd(\mathcal{A}) \cong \mathcal{O}_{C \times C^{(d)}} \bigoplus \text{ad}(q_1^* E)(-\Sigma).
\]

**Proof.** Consider the exact sequence in (9.11). In it, from Proposition 9.2 we have
\[
\Phi_*\mathcal{H}om(\mathcal{A}, p_1^*E) \cong \mathcal{E}nd(q_1^* E).
\]
Also, from (9.13) we know that the image of the map \( \mathcal{E}nd(q_1^* E) \to \Phi_*\mathcal{H}om(\mathcal{A}, \mathcal{B}) \) is \( \text{ad}(q_1^* E)|_{\Sigma} \). Consequently, (9.11) produces an exact sequence
\[
0 \to \Phi_*\mathcal{E}nd(\mathcal{A}) \to \mathcal{E}nd(q_1^* E) \to \text{ad}(q_1^* E)|_{\Sigma} \to 0.
\]
Writing \( \mathcal{E}nd(q_1^* E) = \mathcal{O}_{C \times C^{(d)}} \bigoplus \text{ad}(q_1^* E) \), the result follows easily.

**Lemma 9.14.** The vanishing statements
\[
q_1^* \mathcal{O}(-\Sigma) = R^1 q_1^* \mathcal{O}(-\Sigma) = 0
\]
hold.
Proof. Consider the exact sequence
\[ 0 \to \mathcal{O}(-\Sigma) \to \mathcal{O}_{C \times C^{(d)}} \to \mathcal{O}_{\Sigma} \to 0. \tag{9.28} \]
Note that the map \( q_1_* \mathcal{O}_{C \times C^{(d)}} \to q_1_* \mathcal{O}_{\Sigma} \) is an isomorphism since both of these sheaves are isomorphic to \( \mathcal{O}_C \). Therefore we have \( q_1_* \mathcal{O}(-\Sigma) = 0 \) and an exact sequence
\[ 0 \to R^1q_1_* \mathcal{O}(-\Sigma) \to R^1q_1_* \mathcal{O}_{C \times C^{(d)}} \xrightarrow{\varpi} R^1q_1_* \mathcal{O}_{\Sigma} \tag{9.29} \]
from (9.28). In view of it, to prove the lemma it suffices to show that \( \varpi \) in (9.29) is an isomorphism.

The sheaves \( \mathcal{O}_{C \times C^{(d)}} \) and \( \mathcal{O}_{\Sigma} \) are flat over \( C \) and if \( c \in C \), the induced map
\[ \mathcal{O}_{C \times C^{(d)}}|_{c \times C^{(d)}} \to \mathcal{O}_{\Sigma}|_{c \times C^{(d)}} \]
coincides with the homomorphism \( \mathcal{O}_{C^{(d)}} \to \mathcal{O}_{C^{(d-1)}} \) corresponding to the inclusion map \( C^{(d-1)} \hookrightarrow C^{(d)} \) defined by \( D \mapsto D+c \). By \cite[Kem81, Corollary 1.5]{Kem81} and the remark following it we know that the induced map
\[ H^1(C^{(d)}, \mathcal{O}_{C^{(d)}}) \to H^1(C^{(d)}, \mathcal{O}_{C^{(d-1)}}) \]
is an isomorphism. Using Grauert’s theorem this implies that \( \varpi \) in (9.29) is an isomorphism. This completes the proof of the lemma. \( \square \)

In \cite{Gan18} it was shown that when \( E \cong \mathcal{O}_C^n \) for some \( n \geq 1 \), the vector bundle \( \mathcal{A} \) is stable with respect to certain polarizations on \( Q \). In particular, \( H^0(C \times Q, \mathcal{E}_{nd}(\mathcal{A})) = 1 \) in that case. In the following corollary we see that this is in fact true in general without any assumptions on \( E \).

Corollary 9.15. The equality \( h^0(C \times Q, \mathcal{E}_{nd}(\mathcal{A})) = 1 \) holds.

Proof. Combining Corollary 9.13 and Lemma 9.14 we have
\[ q_1_* \Phi_* \mathcal{E}_{nd}(\mathcal{A}) = q_1_* \left[ \mathcal{O}_{C \times C^{(d)}} \oplus q_1^* \mathcal{E}_{ad}(E)(-\Sigma) \right] = \mathcal{O}_C. \]
The corollary now follows immediately. \( \square \)

Corollary 9.16. Let \( g_C \geq 2 \). Then
\[ H^1(C \times Q, \mathcal{E}_{nd}(\mathcal{A})) = H^1(C \times Q, \mathcal{O}_{C \times Q}) = H^1(C, \mathcal{O}_C) \oplus H^1(Q, \mathcal{O}_Q). \]

Proof. From the Leray Spectral sequence, it follows that there is an exact sequence
\[ 0 \to H^1(C \times C^{(d)}, \Phi_* \mathcal{E}_{nd}(\mathcal{A})) \to H^1(C \times Q, \mathcal{E}_{nd}(\mathcal{A})) \to H^0(C \times C^{(d)}, R^1\Phi_* \mathcal{E}_{nd}(\mathcal{A})). \]
From Theorem 9.6 it follows that \( R^1\Phi_* \mathcal{E}_{nd}(\mathcal{A}) \cong q_1^*T_C|_{\Sigma} \). Therefore, we have
\[ H^0(C \times C^{(d)}, R^1\Phi_* \mathcal{E}_{nd}(\mathcal{A})) = H^0(C \times C^{(d)}, q_1^*T_C|_{\Sigma}) = 0. \]
Here the last equality follows from the assumption that \( g_C \geq 2 \). So it suffices to compute \( H^1(C \times C^{(d)}, \Phi_* \mathcal{E}_{nd}(\mathcal{A})). \)

By Lemma 9.14 we have
\[ q_1*(q_1^*\mathcal{E}_{ad}(E)(-\Sigma)) = 0 = R^1q_1*(q_1^*\mathcal{E}_{ad}(E)(-\Sigma)). \]
Using this and the Leray Spectral sequence it follows that

$$H^1(C \times C^{(d)}, q_1^* ad(E)(-\Sigma)) = 0.$$  

Therefore, using Corollary 9.13 it is deduced that

$$H^1(C \times C^{(d)}, \Phi_* E)(\mathcal{A}) = H^1(C \times C^{(d)}, \mathcal{O}_{C \times C^{(d)}})$$

$$= H^1(C, \mathcal{O}_C) \oplus H^1(C^{(d)}, \mathcal{O}_{C^{(d)}})$$

$$= H^1(C, \mathcal{O}_C) \oplus H^1(Q, \mathcal{O}_Q).$$

Here the last equality follows from Corollary 9.1 and the Leray spectral sequence. This completes the proof of the Corollary. □

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Department of Mathematics, Shiv Nadar University, NH91, Tehsil Dadri, Greater Noida, Uttar Pradesh 201314, India

Email address: indranil.biswas@snu.edu.in, indranil129@gmail.com

Department of Mathematics, Indian Institute of Science Education and Research, Pune, 411008, Maharashtra, India.

Email address: chandranandan@iiserpune.ac.in

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, Maharashtra, India

Email address: ronnie@math.iitb.ac.in