A VANISHING IDENTITY ON ADJOINT REIDEMEISTER TORSIONS

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Abstract. For an oriented compact 3-manifold with a torus boundary, the adjoint Reidemeister torsion is defined as a function on the $\text{SL}_2(\mathbb{C})$-character variety depending on a choice of a boundary curve. Under reasonable assumptions, the adjoint torsion conjecturally satisfies a certain vanishing identity. In this paper, we prove that the conjecture holds for all hyperbolic twist knot exteriors by using Jacobi’s residue theorem.

1. Introduction

1.1. Overview. Let $M$ be an oriented compact 3-manifold with a torus boundary and $X^{\text{irr}}(M)$ be the character variety of irreducible $\text{SL}_2(\mathbb{C})$-representations. We assume that every irreducible component of $X^{\text{irr}}(M)$ is of dimension 1. Note that there are many known examples satisfying the assumption: for instance, whenever $M$ does not contain closed incompressible surface [CCG+94, §2.4].

In [Por97] Porti defined the adjoint torsion, denoted by $T_\gamma$, as a function on a Zariski open subset of $X^{\text{irr}}(M)$ depending on a choice of a boundary curve $\gamma$. By a boundary curve we mean a simple closed curve on $\partial M$ with a non-trivial class in $H_1(\partial M; \mathbb{Z})$. Roughly speaking, the value $T_\gamma(\chi_\rho) \in \mathbb{C}^\ast$ at the character $\chi_\rho$ of an irreducible representation $\rho$ is the sign-refined Reidemeister torsion twisted by the adjoint representation associated to $\rho$. The choice of a boundary curve $\gamma$ involves the definition so as to specify a basis of the twisted (co-)homology. We briefly recall the definition in Section 2.1.

Since Witten’s monumental paper [Wit89], there have been several studies on adjoint torsion in terms of quantum field theory. Recent studies on the relation with the Witten index ([DGZ, FBZ, GKY]) suggest the following conjecture.

Conjecture 1.1. Suppose that every component of $X^{\text{irr}}(M)$ is of dimension 1 and that the interior of $M$ admits a hyperbolic structure. Then for any boundary curve $\gamma \subset \partial M$ we have

$$\sum_{\chi_\rho \in \text{tr}^{-1}(\gamma)} \frac{1}{T_\gamma(\chi_\rho)} = 0$$

for generic $C \in \mathbb{C}$. Here $\text{tr}_\gamma : X^{\text{irr}}(M) \to \mathbb{C}$ is the trace function of $\gamma$.

Remark 1.2. Conjecture 1.1 has a natural generalization to when $M$ has several boundary tori or no boundary.

A knot in $S^3$ with a diagram as in Figure 1 is called a twist knot. We denote by $K_n$ for $n \neq 0 \in \mathbb{Z}$ the twist knot having $|n|$ right-handed half twists in the box (left-handed, if $n$ is negative). We may focus on twist knots $K_n$, as $K_{2n+1}$ is equivalent to the mirror image of $K_{-2n}$.

![Figure 1. A diagram of a twist knot.](image)

It is known that every knot exterior of $K_{2n}$ except for $n = 1$ (the trefoil knot) satisfies the assumptions of Conjecture 1.1. Namely, every twist knot $K_{2n}$ is hyperbolic except for $n = 1$ (see e.g [Men84]) and has the character variety $X^{\text{irr}}(M)$ consisting of 1-dimensional components. The latter can be derived from explicit computations [Ril84, MPvL11] or the fact that every twist knot exterior has no closed incompressible surface [HT85]. Main aim of this paper is to prove that Conjecture 1.1 holds for all hyperbolic twist knots.
**Theorem 1.3.** Let $M$ be the knot exterior of the twist knot $K_{2n}$ for $n \neq 0, 1$ and $\gamma \subset \partial M$ be a boundary curve. Then we have
\[ \sum_{\chi_\rho \in \text{tr}_{\gamma}^{-1}(C)} \frac{1}{T_{\gamma}(\chi_\rho)} = 0 \]
for generic $C \in \mathbb{C}$.

The adjoint torsion is quite hard to compute in general and its concrete computation is known only in a few examples. For twist knots, the adjoint torsion $T_{\gamma}$ with respect to the canonical longitude $\lambda$ was first computed in [DHY09] by using the relation with the twisted Alexander polynomial [Yam08] and computations were remarkably simplified in [Tra14] (also in [Tra18]). To prove Theorem 1.3, we extend computations to the adjoint torsion $T_{\gamma}$ for an arbitrary boundary curve $\gamma$. We here summarize our computations for convenience of the reader. See Section 2.2 for details.

Let $S_k(z)$ be the Chebyshev polynomials defined by $S_0(z) = 0, S_1(z) = 1, \text{ and } S_{k+1}(z) = zS_k(z) - S_{k-1}(z)$ for all $k \in \mathbb{Z}$. Throughout the paper, the Chebyshev polynomials are always written in the variable $z$ and we often write $S_k(z)$ simply as $S_k$. Let $M$ be the knot exterior of the twist knot $K_{2n}$ for $n \neq 0$. Reformulating [Ril84], the character variety $X^{irr}(M)$ is given by the zero set of
\[ F(m, z) = S_n(S_n - S_n - 1)(m^2 + m^{-2}) - (z - 1)S_n^2 + S_n - 1 \]
in $\mathbb{C}^* \times \mathbb{C}$ with the quotient identifying $(m, z)$ and $(m^{-1}, z)$. For a boundary curve $\gamma \subset \partial M$ let $E_\gamma$ be a function on the zero set of $F$ given by
\[ E_\gamma(m, z) = m^p \left( \frac{(z - 2)(S_{n+1} - S_n - 1)S_n^2}{S_n - S_n - 1} m^2 + (z - 2)(S_n + S_n - 1)S_n + 1 \right)^q. \]
Here $p/q \in \mathbb{Q} \cup \{ \infty \}$ is the slope of $\gamma$.

**Theorem 1.4.** Let $(m, z) \in \mathbb{C}^* \times \mathbb{C}$ be a solution to the Laurent polynomial $F$ and let $\chi_\rho$ be the corresponding irreducible character. Then $E_\gamma(m^\pm 1, z)$ are the eigenvalues of $\rho(\gamma)$ and
\[ T_{\gamma}(\chi_\rho) = \frac{m}{2E_\gamma} \det \left( \frac{\partial(F, E_\gamma)}{\partial(m, z)} \right) \]
if $\chi_\rho$ is $\gamma$-regular.

**Remark 1.5.** Choosing a boundary curve $\gamma$ as a meridian $\mu$ (when $p/q = 1/0$), we obtain
\[ T_{\mu}(\chi_\rho) = \frac{1}{2} \frac{\partial F}{\partial z}. \]
It is interesting that the adjoint torsion with respect to a meridian is related to a derivative of the defining equation of the character variety. A similar observation in terms of the A-polynomial was pointed out in [DG13, Remark 4.5].

### 1.2. Global residue

We note some remarks on the relation between Theorem 1.3 and the global residue theorem, saying that any top-dimensional meromorphic form defined on a compact complex manifold has global residue zero. Here the global residue means the total sum of local residues. We refer to [GH78, Tsi92] for general references on residue theory.

It follows from the equations (1) and (2) that for a constant $c \in \mathbb{C}^*$ the set $\text{tr}_{\gamma}^{-1}(c + 1/c)$ is given by the common zero set $Z_{F, G}$ of $F(m, z)$ and $G(m, z) := E_\gamma(m, z) - c$ in $\mathbb{C}^* \times \mathbb{C}$ and that
\[ \sum_{\chi_\rho \in \text{tr}_{\gamma}^{-1}(c + 1/c)} \frac{1}{T_{\gamma}(\chi_\rho)} = \sum_{(m, z) \in Z_{F, G}} \frac{-2c}{m \det \left( \frac{\partial(F, G)}{\partial(m, z)} \right)}. \]
This shows that Theorem 1.3 is equivalent to that the global residue of the meromorphic 2-form
\[ \omega = \frac{-2c}{F \cdot G} \frac{dm}{m} \wedge dz \]
defined on $\mathbb{C}^* \times \mathbb{C}$ is zero (for generic $c \in \mathbb{C}^*$). In particular, the inverses of adjoint torsions are local residues.

If $F$ and $G$ are generic in some sense, then there exists a toric compactification of $\mathbb{C}^* \times \mathbb{C}$ so that one can deduce that the global residue of $\omega$ is zero, as an application of the global residue theorem. See [Kho78] and [VY01] for precise conditions of genericness; the condition in [VY01] relaxes that in [Kho78]. Unfortunately, we however do not know a direct way to obtain Theorem 1.3 as a consequence of [VY01] since checking the condition in [VY01] seems to require heavy complex analysis (see Remark 3.1). We shall take a practical detour to prove Theorem 1.3 by reducing the problem to an one-variable problem. See Section 3 for details.
A Vanishing Identity on Adjoint Reidemeister Torsions

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2. ADJOINT REIDEMEISTER TORSION FOR TWIST KNOTS

2.1. A brief review on definitions. We here briefly recall a definition of the adjoint torsion for a knot exterior. We refer to [Por97, Tur02, Dub03] for details.

Let $C_\ast = (0 \to C_n \to \cdots \to C_0 \to 0)$ be a chain complex of vector spaces with a base field $F$ and a boundary map $\partial$. For a basis $c_\ast$ of $C_\ast$ and a basis $h_\ast$ of the homology $H_\ast(C_\ast)$, the Reidemeister torsion $\text{Tor}(C_\ast, c_\ast, h_\ast)$ is defined as follows. For each $0 \leq i \leq n$ we choose a tuple $b_i$ of vectors in $C_i$ such that $\partial b_i$ is a basis of $\partial C_i$, and a representative $h_i$ of $h_i$ in $C_i$. Then one can check that a tuple $c_i' = (\partial b_{i+1}, h_i, b_i)$ is a basis of $C_i$. Letting $A_i$ be the transition matrix taking the basis $c_i$ to the other basis $c_i'$, the (sign-refined) Reidemeister torsion is defined as

$$\text{Tor}(C_\ast, c_\ast, h_\ast) := (-1)^{\sum_j \alpha_j \beta_j} \prod_{i=0}^n \det A_i^{(-1)^{i+1}} \in \mathbb{F}^*$$

where $\alpha_j = \sum_{i=0}^j \dim C_i$ and $\beta_j = \sum_{i=0}^j \dim H_i(C_\ast)$. Note that the torsion does not depend on the choices of $b_\ast$ and $h_\ast$.

Let $M$ be the knot exterior of a knot $K \subset S^3$. We fix a triangulation of $M$ and an orientation of each cell so that the cells, say $c_1, \ldots, c_m$, form a basis of $C_\ast(M; \mathbb{R})$. It is well-known that $\dim H_i(M; \mathbb{R}) = 1$ for $i = 0, 1$ and $\dim H_i(M; \mathbb{R}) = 0$, otherwise. We choose a basis $h_\ast$ of $H_\ast(M; \mathbb{R})$ as $h_\ast = \{[pt], [\mu]\}$ where $pt$ is a point in $M$ and $\mu$ is a meridian of $K$. Let

$$\tau = \text{Tor}(C_\ast(M; \mathbb{R}), c_\ast, h_\ast) \in \mathbb{R}^*$$

and denote by $\text{sgn}(\tau) \in \{\pm 1\}$ its sign.

Let $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ be an irreducible representation and $\tilde{M}$ be the universal cover of $M$ with the induced triangulation. Viewing the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ of $\text{SL}_2(\mathbb{C})$ as a $\mathbb{Z}[\pi_1(M)]$-module through the adjoint representation $\text{Ad}_\rho : \pi_1(M) \to \text{Aut}(\mathfrak{g})$ associated to $\rho$, we consider a chain complex

$$C_\ast(M; \mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{Z}[\pi_1(M)]} C_\ast(\tilde{M}; \mathbb{Z})$$

with a basis $\mathcal{C} = \{h, e, f\} \otimes \{\bar{c}_1, \ldots, \bar{c}_m\}$. Here $\{h, e, f\}$ is a basis of $\mathfrak{g}$ and $\bar{c}_i$ is any lift of $c_i$ to $\tilde{M}$. The homology $H_\ast(M; \mathfrak{g})$ of $C_\ast(M; \mathfrak{g})$ is non-trivial and we choose its basis $\mathcal{H}$ depending on a choice of a boundary curve $\gamma \subset \partial M$ as follows. Here we require that $\rho$ is $\gamma$-regular, i.e.

- $\dim H_1(M; \mathfrak{g}) = 1$;
- the inclusion $\gamma \hookrightarrow M$ induces an epimorphism $H_1(\gamma; \mathfrak{g}) \to H_1(M; \mathfrak{g})$;
- if $\text{tr}(\rho(\pi_1(\partial M))) \subset \{\pm 2\}$, then $\rho(\gamma) \neq \pm 1\text{Id}$.

Remark 2.1. The $\gamma$-regularity is invariant under conjugating $\rho$. In particular, the notion of $\gamma$-regular character is well-defined. Most of irreducible characters are $\gamma$-regular: non-$\gamma$-regular irreducible characters are contained in the zero set of the differential of the trace function $\text{tr}_\gamma : X^\ast(M) \to \mathbb{C}$. See [Por97, Proposition 3.26] and [DHY09, Remark 9].

From the Poincare duality, we have $\dim H_i(M; \mathfrak{g}) = 1$ for $i = 1, 2$ and $\dim H_i(M; \mathfrak{g}) = 0$, otherwise. We choose any non-zero element $v \in \mathfrak{g}$ invariant under $\text{Ad}_\rho(\mathfrak{g})$ for all $g \in \pi_1(\partial M)$ and let $\mathcal{H}$ consist of the images of $v \otimes \gamma$ and $v \otimes \partial M$ under the canonical maps $H_1(\gamma; \mathfrak{g}) \to H_1(M; \mathfrak{g})$ and $H_2(\partial M; \mathfrak{g}) \to H_2(M; \mathfrak{g})$, respectively. Finally, the adjoint torsion $T_\gamma(\chi_\rho)$ at the character $\chi_\rho$ is defined as

$$T_\gamma(\chi_\rho) := \text{sgn}(\tau) \cdot \text{Tor}(C_\ast(M; \mathfrak{g}), \mathcal{C}, \mathcal{H}) \in \mathbb{C}^*.$$ 

Note that

- Several choices are involved in the definition of $T_\gamma(\chi_\rho)$: a triangulation of $M$, an order/orientations of the cells of $M$, lifts of the cells of $M$ to $\tilde{M}$, a basis of $\mathfrak{g}$, and the vector $v$. However, it turns out that the adjoint torsion does not depend on these choices;
- Orientations of $K$, $\mu$, $\gamma$, and $\partial M$ are also involved in the definition of $T_\gamma(\chi_\rho)$. If we reverse one of them, the sign of $T_\gamma(\chi_\rho)$ changes. We thus fix these orientations once and for all. As far as we consider Conjecture 1.1, the choices of these orientations would not be essential.
2.2. Adjoint torsions for twist knots. Let $M$ be the knot exterior of the twist knot $K_{2n}$ and $X^{\text{irr}}(M)$ be the character variety of irreducible $\text{SL}_2(\mathbb{C})$-representations. Recall that $X^{\text{irr}}(M)$, as a set, is the set of conjugacy classes of irreducible representations $[\text{CS83}]$:

$$X^{\text{irr}}(M) = \{ \rho : \pi_1(M) \to \text{SL}_2(\mathbb{C}) : \text{irreducible} \}/\text{Conjugation}.$$  

We first recall some known facts. The fundamental group of $M$ has a presentation

$$\pi_1(M) = \langle a, b \mid w^n a = b w^n \rangle$$

where $w = b a^{-1} b^{-1} a$ and an irreducible representation $\rho : \pi_1(M) \to \text{SL}_2(\mathbb{C})$ is given by

$$\rho(a) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} m & 0 \\ -u & m^{-1} \end{pmatrix}$$

up to conjugation for a point $(m, u) \in (\mathbb{C}^*)^2$ satisfying the Riley polynomial

$$R(m, u) := S_{n+1}(z) - (u^2 - (u + 1)(m^2 + m^{-2} - 3)) S_n(z) = 0.$$  

Here $z$ is the trace of $\rho(w)$

$$z = \text{tr} \rho(w) = 2 + (2 - m^2 - m^{-2}) u + u^2$$

and $S_k(z)$ are the Chebyshev polynomials defined by $S_0(z) = 0$, $S_1(z) = 1$, and $S_{k+1}(z) = z S_k(z) - S_{k-1}(z)$ for all $k \in \mathbb{Z}$. We will write $S_k(z)$ simply as $S_k$. Two points $(m_1, u_1)$ and $(m_2, u_2)$ satisfying the Riley polynomial represent the same character if and only if $m_1 = m_2^\pm 1$ and $u_1 = u_2$. Therefore, we obtain

$$X^{\text{irr}}(M) = \{ (m, u) \in (\mathbb{C}^*)^2 : R(m, u) = 0 \}/(m, u) \sim (m^{-1}, u).$$

We refer to $[\text{Ril84}]$ for details.

For computational reasons, we use the variable $z$ instead of the variable $u$. This can be done by using a relation

$$(S_n - S_{n-1}) u = (z - 2) S_n$$

obtained from the equations (6) and (7) by eliminating the variable $m$. In the above equation, both $S_n - S_{n-1}$ and $(z - 2) S_n$ can not be zero, as they have no common factor and $u \neq 0$. It follows that the variable $z$ uniquely determines the variable $u$ by

$$u = \frac{(z - 2) S_n}{S_n - S_{n-1}}. $$

We obtain another defining equation of $X^{\text{irr}}(M)$ by replacing the variable $u$ in $R(m, u)$ by the variable $z$:

$$X^{\text{irr}}(M) = \{ (m, z) \in \mathbb{C}^* \times \mathbb{C} : F(m, z) = 0 \}/(m, z) \sim (m^{-1}, z)$$

where

$$F(m, z) := S_n(S_n - S_{n-1})(m^2 + m^{-2}) - (z - 1) S_n^2 + S_{n-1}^2.$$  

We fix a point $(m, z) \in \mathbb{C}^* \times \mathbb{C}$ satisfying $F(m, z) = 0$. Let $\rho$ be the irreducible representation given as in the equation (5) and denote its character by $\chi_\rho$. We choose a meridian $\mu = a$ and denote by $\lambda$ the canonical longitude with respect to $\mu$. A formula for computing the adjoint torsion $T_\lambda(\chi_\rho)$ with respect to $\lambda$ is given in $[\text{Tra14}]$ and we re-express the formula in terms of the variable $z$.

**Theorem 2.2 ([Tra14]).** If $\chi_\rho$ is $\lambda$-regular, then the adjoint torsion $T_\lambda(\chi_\rho)$ is given by

$$T_\lambda(\chi_\rho) = -(2n + 1) S_n^2 + 2n S_n S_{n-1} - \frac{2(S_n' - S_{n-1}')} {S_n - S_{n-1}}$$

where $S_k'$ denotes the derivative of $S_k$.

**Proof.** It is computed in $[\text{Tra14}]$ that

$$T_\lambda(\chi_\rho) = \frac{-1}{(y + 2 - x^2)(y^2 - y x^2 + x^2)} \left( \frac{(2n - 1) y^2 + y x^2 - 2 n x^2 (x^2 - 2)}{y^2 - y x^2 + 2 x^2} + 2 n \right)$$

where $x = m + m^{-1}$ and $y = u + 2$. A straightforward computation gives

$$y + 2 - x^2 = \frac{S_n - S_{n-1}}{S_n}, \quad y^2 - y x^2 + x^2 = \frac{1}{S_n(S_n - S_{n-1})}, \quad y^2 - y x^2 + 2 x^2 = z + 2.$$  

Using these equations with the identity $k S_{k-1} + (k - 1) S_k = (z + 2)(S_k' - S_{k-1}')$, we obtain the desired expression of $T_\lambda(\chi_\rho)$.

\[\Box\]
Let $\gamma \subset \partial M$ be a boundary curve of slope $p/q \in \mathbb{Q} \cup \{\infty\}$, that is, $\gamma = \mu^p \lambda^q$ for coprime integers $p$ and $q$. To compute the adjoint torsion $\mathbb{T}_\gamma(\chi_\rho)$ with respect to $\gamma$, we compute the eigenvalues of $\rho(\gamma)$. Note that a similar computation can be also found in [Tra16].

**Lemma 2.3.** The matrix $\rho(\gamma)$ is of the form

$$\rho(\gamma) = \rho(\mu^p \lambda^q) = \begin{pmatrix} m^p l^q & * \\ 0 & m^{-p} l^{-q} \end{pmatrix}$$

where

$$l = -\frac{(z - 2)(S_{n+1} - S_{n-1})S_n^2}{S_n - S_{n-1}} m^2 + (z - 2)(S_n + S_{n-1})S_n + 1. \tag{10}$$

**Proof.** It is enough to show that the $(1,1)$-entry of $\rho(\lambda)$ coincides with the given $l$. It is known that $\lambda = w_*^n w^m$ where $w_*$ is the word obtained by writing $w$ in the reversed order (see e.g. [Ril84]). From [Tra16, §3.2], we have

$$\rho(w^n) = \left( \frac{S_{n+1} - (1 + (2m - 2)u + u^2)S_n}{(m - m - 1)u + mu^2} S_n \right) \left( \begin{array}{c} m^{-1} - m - mu \end{array} \right) S_n + 1,$$

and

$$\rho(w^n) = \left( \frac{S_{n+1} - (1 - m^2 - 2m)uS_n}{(m - 1 - m - 1)(u + 2m - u^2)} S_n \right) \left( \begin{array}{c} m^{-1} - m - mu \end{array} \right) S_n + 1.$$ 

It follows that the $(1,1)$-entry of $\rho(\lambda)$ is

$$\begin{align*}
(S_{n+1} - (1 - m^2 - 2m)uS_n)(S_{n+1} - (1 + (2m - 2)u + u^2)S_n) \\
+ (m - 1 - m - 1)u S_n \cdot ((m - 1)(u + 2m - u^2)) S_n \\
= S_{n+1}^2 - (z + (2m - 2)u)S_n S_{n+1} \\
+ (u(u + 1)m^2 - (u^3 + u^2 - 1) - u(u^2 + u + 1)m^2 + u^2 - u^2 - 1) S_n^2 \\
= 1 - (m^2 - 2m - 1)(1 + u)m - 1 \cdot (z - 2) S_n^2,
\end{align*}$$

We used the identity $S_k^2 - zS_k S_{k-1} + S_{k-1}^2 = 1$ and the equation (8) (i.e. $uS_n = ((z - 1)u + z - 2)S_{n-1}$) for the third and fourth equalities, respectively. The desired expression (10) is obtained by eliminating $u$ by using the equation (8) and then simplifying it by using the equation $F(m, z) = 0$. \qed

We define a function $E_\gamma$ on the zero set of $F$ in $\mathbb{C}^* \times \mathbb{C}$ as

$$E_\gamma(m, z) = m^p \left( -\frac{(z - 2)(S_{n+1} - S_{n-1})S_n^2}{S_n - S_{n+1}} m^2 + (z - 2)(S_n + S_{n-1})S_n + 1 \right)^q. \tag{11}$$

It follows from Lemma 2.3 that $E_\gamma(m, z)$ is an eigenvalue of $\rho(\gamma)$. Moreover, a similar computation given as in the proof of Lemma 2.3 shows that

$$l^{-1} = -\frac{(z - 2)(S_{n+1} - S_{n-1})S_n^2}{S_n - S_{n+1}} m^{-2} + (z - 2)(S_n + S_{n-1})S_n + 1$$

and thus $E_\gamma(m^{-1}, z) = E_\gamma(m, z)^{-1}$.

**Remark 2.4.** Using the equation $F(m, z) = 0$, one can re-express

$$l^{\pm 1} = -(z - 2) S_n^2 m^\pm 4 - (z - 2)(S_n - S_{n-1}) m^\pm 2 + (S_n - S_{n-1})^2$$

so that $l^{\pm 1}$ become Laurent polynomials. In particular, using this expression, $E_\gamma$ becomes a Laurent polynomial.

**Theorem 2.5.** If $\chi_\rho$ is $\gamma$-regular, then the adjoint torsion $\mathbb{T}_\gamma(\chi_\rho)$ is given by

$$\mathbb{T}_\gamma(\chi_\rho) = -\frac{m}{2E_\gamma} \det \left( \frac{\partial(F, E_\gamma)}{\partial(m, z)} \right).$$

As a consequence, the right-hand side has the same value at $(m, z)$ and $(m^{-1}, z)$.

**Proof.** For simplicity we let $F(m, z) = f_1(z)(m^2 + m^2) + f_2(z)$ and $E_\gamma(m, z) = m^p (g_1(z)m^2 + g_2(z))^q$. See the equations (9) and (11). Also, we let $f_3 := f_2/f_1$ so that $m^2 + m^2 + f_3 = 0$ and $d z / d m = -2(m - m^{-1})/f_3$. Note that using the equation $m^2 + m^2 + f_3 = 0$, one can simplify a high $m$-degree term to lower $m$-degree terms.
Straightforward computations give that
\[
\frac{d}{dm} = \frac{d}{dm} (g_1 m^2 + g_2) = 2mg_1 + \frac{dz}{dm} (g'_1 m^2 + g'_2) = 2mg_1 - \frac{2(m - m^{-3})(g'_1 m^2 + g'_2)}{f'_3}
\]
and that
\[
\det \left( \frac{\partial (F/f_1, E_\gamma)}{\partial (m, z)} \right) = -\frac{pE_\gamma}{m} f'_3 + \frac{qE_\gamma}{l} (2(m - m^{-3})(g'_1 m^2 + g'_2) - 2mg_1 f'_3)
\]
\[
= \frac{2E_\gamma}{l} \left( (m - m^{-3})(g'_1 m^2 + g'_2) - mg_1 f'_3 \right) \left( \frac{l}{m} \frac{dm}{dl} + q \right).
\]
On the other hand, using the equation \(m^2 + m^{-2} + f_3 = 0\), we have
\[
(m - m^{-3})(g'_1 m^2 + g'_2) - mg_1 f'_3 = (-g'_1 f_3 + 2g'_2 - g_1 f'_3)m + (-2g'_1 + g'_2 f_3)m^{-1}.
\]
We claim that
\[
- g'_1 f_3 + 2g'_2 - g_1 f'_3 + g_1 T_\lambda(\chi_\rho)/f_1 = 0 \quad \text{and} \quad -2g'_1 + g'_2 f_3 + g_2 T_\lambda(\chi_\rho)/f_1 = 0.
\]
Note these equations are only in the variable \(z\) due to Theorem 2.2. From the equality \(S'_n^2 - zS_n S_{n-1} + S_{n-1}^2 = 1\), we obtain
\[
2S'_n S'_n - S_n S_{n-1} - zS'_n S_{n-1} = 2S_{n-1} S'_n - 0.
\]
Together with the equality \(nS_{n-1} + (n - 1)S_n = (z + 2)(S'_n - S_{n-1})\), we obtain
\[
S'_n = \frac{(n - 1)zS_n - 2nS_{n-1}}{z^2 - 4} \quad \text{and} \quad S'_{n-1} = \frac{2(n - 1)S_n - znS_{n-1}}{z^2 - 4}.
\]
Plugging the above equations into the equation (15) (together with \(S_{n+1} = zS_n - S_{n-1}\)), we obtain two equations, each of which consists of terms in \(S_n\) and \(S_{n-1}\). With the aid of Mathematica, one checks that both of them have a factor \(S'_n - S_n S_{n-1} + S_{n-1}^2 - 1\), which is identically zero. This proves the equation (15).

Combining the equations (13), (14), and (15), we have
\[
\det \left( \frac{\partial (F/f_1, E_\gamma)}{\partial (m, z)} \right) = \frac{2E_\gamma}{l} \left( -\frac{g_1 T_\lambda(\chi_\rho)}{f_1} m - \frac{g_2 T_\lambda(\chi_\rho)}{f_1} m^{-1} \right) \left( \frac{l}{m} \frac{dm}{dl} + q \right).
\]
Recall that the last equality follows from [Por97, Theorem 4.1]:
\[
T_\gamma(\chi_\rho) = T_\lambda(\chi_\rho) \frac{d \log (m^{p\rho})}{d \log l} = T_\lambda(\chi_\rho) \left( \frac{l}{m} \frac{dm}{dl} + q \right).
\]
This completes the proof, since we have
\[
\det \left( \frac{\partial (F/f_1, E_\gamma)}{\partial (m, z)} \right) = \frac{1}{f_1} \det \left( \frac{\partial (F, E_\gamma)}{\partial (m, z)} \right)
\]
for any point \((m, z)\) satisfying \(F(m, z) = 0\).

**Remark 2.6.** We use two variables for computational simplicity, but using three variables with the variable \(l\) seems natural. Precisely, if we let \(E_\gamma = m^{p\rho}\) and
\[
H(m, z, l) = l - \left( -\frac{(z - 2)(S_{n+1} - S_{n-1})S_n^2}{S_n - S_{n-1}} m^2 + (z - 2)(S_n + S_{n-1})S_n + 1 \right),
\]
then we have
\[
T_\gamma(\chi_\rho) = -\frac{m}{2E_\gamma} \det \left( \frac{\partial (F, E, H)}{\partial (m, z, l)} \right)
\]
for a point \((m, z, l) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^*\) satisfying \(F = H = 0\). \(\square\)
3. Proof of Theorem 1.3

Recall that $M$ is the knot exterior of the twist knot $K_{2n}$ ($n \neq 0,1$) and $\gamma \subset \partial M$ is a boundary curve of slope $p/q \in \mathbb{Q} \cup \{\infty\}$. We may assume that $q \geq 0$. In Section 2, we computed that for generic $c \in \mathbb{C}^*$

\[
\sum_{\chi \in \text{tr}^{-1}(c+1/c)} \frac{1}{T_M(p, \gamma)} = \sum_{(m,z) \in Z_{F,G}} m \det \left( \frac{\partial(F,G)}{\partial(m,z)} \right) \left( \frac{-2c}{\partial(m,z)} \right)
\]

where

\[
F(m, z) = S_n(S_n - S_{n-1})(m^2 + m^{-2}) - (z - 1)S_n^2 + S_{n-1}^2,
\]

\[
G(m, z) = m^p \left( \frac{-(z - 2)(S_n + S_{n-1})S_n m^2 + (z - 2)(S_n + S_{n-1})S_n + 1}{S_n - S_{n-1}} \right)^q - c.
\]

Here $Z_{F,G}$ denotes the common zero set of $F$ and $G$, and genericity on $c$ is required to guarantee that $\text{tr}^{-1}(c+1/c)$ consists of irreducible $\gamma$-regular characters (see Remark 2.1).

Remark 3.1. Recall Remark 2.4 that we can make $G$ into a Laurent polynomial. If the system $(F, G)$ of Laurent polynomials is $(\Delta_F, 0)$-proper (see [VY01] for the precise definition), where $\Delta_F$ is the Newton polygon of $F$, then Theorem 1.3 directly follows as an application of [VY01, Theorem 1.2].

3.1. For even $p$. For simplicity let $F(m, z) = f_1(z)(m^2 + m^{-2}) + f_2(z)$, $G(m, z) = m^p(g_1(z)m^2 + g_2(z))^q - c$, and $f_3(z) = f_2(z)/f_1(z)$. From the equation $m^2 + m^{-2} + f_3 = 0$, we recursively obtain

\[
m_{2k} = h_k m^2 - h_{k-1}(z)
\]

for all $k \in \mathbb{Z}$ where $h_0(z) = 0$, $h_1(z) = 1$, and $h_{k+1}(z) = -f_3(z)h_k(z) - h_{k-1}(z)$. Then for any point $(m, z) \in Z_{F,G}$ we have

\[
G(m, z) = m^p(g_1 m^2 + g_2)^q - c
\]

\[
= \sum_{k=0}^{q} \binom{q}{k} g_1^k g_2^{q-k} m^{2k+p} - c
\]

\[
= \sum_{k=0}^{q} \binom{q}{k} g_1^k g_2^{q-k} m^{2k+p} - \sum_{k=0}^{q} \binom{q}{k} g_2^k g_1^{q-k} m^{h_k - k - 1} - c
\]

\[
= : \alpha(z)m^2 - \beta(z).
\]

For generic $c \in \mathbb{C}^*$ both $\alpha(z)$ and $\beta(z)$ can not be zero on $Z_{F,G}$, since they have no common zero and $m \neq 0$. In particular,

\[
Z_{F,G} = \left\{ (m, z) \in \mathbb{C}^* \times \mathbb{C} : H(z) = 0, m = \pm \sqrt[\alpha(z)]{\beta(z)/\alpha(z)} \right\} \text{ where } H(z) := f_1(z) \left( \frac{\alpha(z)}{\beta(z)} + \frac{\beta(z)}{\alpha(z)} \right) + f_2(z).
\]

Lemma 3.2. For any point $(m, z) \in Z_{F,G}$ we have

\[
\det \left( \frac{\partial(F,G)}{\partial(m,z)} \right) = -2m\alpha\beta'.
\]

Proof. Recall that we have $m_{2k} = h_k m^2 - h_{k-1}$ from $F(m, z) = 0$. It follows that

\[
\det \left( \frac{\partial(F, m_{2k} - h_k m^2 + h_{k-1})}{\partial(m,z)} \right) = 0
\]

for all $k \in \mathbb{Z}$. Therefore, we have

\[
\det \left( \frac{\partial(F,G)}{\partial(m,z)} \right) = \det \left( \frac{\partial(F, \alpha m^2 - \beta)}{\partial(m,z)} \right) = \det \left( \frac{\partial(H, \alpha m^2 - \beta)}{\partial(m,z)} \right) = -2m\alpha\beta'.
\]

\[
\sum_{(m,z) \in Z_{F,G}} m \det \left( \frac{\partial(F,G)}{\partial(m,z)} \right) = \sum_{z : H(z) = 0} \frac{c}{m^2 \alpha \beta'} = \sum_{z : H(z) = 0} \frac{2c}{\beta H'}.
\]
We claim the above equation is zero due to Jacobi’s residue theorem, saying that any non-constant polynomial \( f \) with \( f(0) \neq 0 \) and no multiple zero satisfies
\[
\sum_{z: f(z) = 0} \frac{g(z)}{f'(z)} = 0
\]
for any polynomial \( g \) with \( \deg g \leq \deg f - 2 \).

**Lemma 3.3.** For generic \( c \in \mathbb{C}^* \) one has
\[
\sum_{z: H(z) = 0} \frac{1}{\beta H'} = 0.
\]

**Proof.** Recall that \( \alpha \) and \( \beta \) are rational polynomials in the variable \( z \). We let \( \alpha = \alpha_1/\alpha_2 \) and \( \beta = \beta_1/\beta_2 \) for some polynomials \( \alpha_1, \alpha_2, \beta_1, \) and \( \beta_2 \) with \( \gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1 \). Here we take the greatest common divisor in \( \mathbb{C}[z] \). Plugging these into \( H \), we have
\[
H = f_1(\alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2) + f_2\alpha_1\alpha_2\beta_1\beta_2 \alpha \beta
\]
with \( \gcd(H_1, H_2) = 1 \) where \( \beta \) is the greatest common divisor of the numerator and denominator. Clearly, we have
\[
\sum_{z: H_1(z)=0} \frac{1}{\beta H_1'} = \sum_{z: H_1(z)=0} \frac{\beta_1 H_2}{\beta_1 H_1'}.
\]
Note that \( \gcd(\beta_1, H_1d) = \gcd(\beta_1, f_1\alpha_1^2\beta_2) = 1 \) for generic \( c \in \mathbb{C}^* \) (: \( f_1\alpha_1^2\beta_2 \) does not depend on \( c \)) and therefore \( \beta_1 \) divides \( H_2 \).

For generic \( c \in \mathbb{C}^* \) we may assume that \( \beta_1 H_1 \) has no multiple zero and is non-zero at \( z = 0 \). Lemma 3.4 below shows that \( \deg \beta_2 H_2 \leq \deg \beta_1 H_1 - 2 \). From Jacobi’s residue theorem, we have
\[
0 = \sum_{z: \beta_1 H_1 = 0} \frac{\beta_2 H_2}{(\beta_1 H_1)'} = \sum_{z: \beta_1 H_1 = 0} \frac{\beta_2 H_2}{(\beta_1 H_1)'} + \sum_{z: H_1 = 0} \frac{\beta_2 H_2}{(\beta_1 H_1)'}
\]
\[
= \sum_{z: \beta_1 H_1 = 0} \frac{\beta_2 H_2}{\beta_1 H_1'} + \sum_{z: H_1 = 0} \frac{\beta_2 H_2}{\beta_1 H_1'}
\]
\[
= \sum_{z: H_1 = 0} \frac{\beta_2 H_2}{\beta_1 H_1'}.
\]
Note that the last equality follows from the fact that \( H_2(z) = 0 \) and \( H_1(z) \neq 0 \) for all zeros \( z \) of \( \beta_1 \). \( \square \)

**Lemma 3.4.** For generic \( c \in \mathbb{C}^* \) one has \( \deg \beta + \deg H \geq 2 \).

Here the degree of a rational polynomial means the degree difference of the numerator and denominator: for instance, \( \deg H = \deg H_1 - \deg H_2 \) and \( \deg \beta = \deg \beta_1 - \deg \beta_2 \).

**Proof.** Let us consider the case \( n > 1 \); the other case \( n < 0 \) can be proved similarly. Recall that \( H = (f_1(\alpha^2 + \beta^2) + f_2\alpha\beta)/(\alpha\beta) \). Expanding the numerator \( f_1(\alpha^2 + \beta^2) + f_2\alpha\beta \), some cancellation may occur but terms involved with the constant \( c \) would not canceled out generically. As \( c \) appears in the constant term of \( \beta \), we have
\[
\deg H \geq \deg(f_2\alpha) - \deg(\alpha\beta) = \deg f_2 - \deg \beta.
\]
This completes the proof, since \( \deg H + \deg \beta \geq \deg f_2 = 2n - 1 \geq 2 \). \( \square \)

### 3.2. For odd \( p \)

As in Section 3.1, we have
\[
G(m, z) = m^{-1}m^{p+1}(g_1m^2 + g_2)q - c
\]
\[
= \sum_{k=0}^q \binom{q}{k} g_1^k g_2^{q-k} h_{k+\frac{p+1}{2}} - \sum_{k=0}^q \binom{q}{k} g_1^k g_2^{q-k} h_{k+\frac{p+1}{2}} m^{-1} - c
\]
\[
= : \alpha(z)m - \beta(z)m^{-1} - c.
\]
on the zero set of $F(m, z) = 0$. Also, from the fact that $E_{\gamma}(m, z)E_{\gamma}(m^{-1}, z) = 1$, we have

$$1 = (am - \beta m^{-1})(am^{-1} - \beta m)$$

$$= \alpha^2 + \beta^2 - \alpha \beta (m^2 + m^{-2})$$

$$= \alpha^2 + \beta^2 + f_3 \alpha \beta$$
on the zero set of $F$. Clearly, $\alpha^2 + \beta^2 + f_3 \alpha \beta$ is an one-variable rational polynomial and therefore it should be identically 1.

It is not hard to solve the equations $F(m, z) = 0$ and $am - \beta m^{-1} - c = 0$ (see the proof of Lemma 3.5 below for details):

$$Z_{F,G} = \left\{ (m, z) \in \mathbb{C}^* \times \mathbb{C} : H(z) = 0, \quad m = \frac{c^2 \alpha + \beta}{c(\alpha^2 - \beta^2)} \right\}$$

where $H(z) := (f_3 + 2)(\alpha + \beta)^2 + (c - 1/c)^2$. Note that since $\alpha^2 - \beta^2$ has finite solutions independently on $c$, we may assume that $\alpha^2 - \beta^2$ is non-zero on $Z_{F,G}$ for generic $c \in \mathbb{C}^*$.

**Lemma 3.5.** For any point $(m, z) \in Z_{F,G}$ we have

$$\det \left( \frac{\partial (F,G)}{\partial (m,z)} \right) = \frac{c f_1}{m(\beta^2 - \alpha^2)} H'.$$

**Proof.** Using the Euclidean algorithm, one can find an equation linear in the variable $m$ among linear combinations of $F$ and $B := am - \beta m^{-1} - c$. Precisely, one has

$$D := \frac{(\alpha \beta^2)}{f_1} \cdot F - (B + 2c \alpha + (\beta^2 - \alpha^2)m) \cdot B$$

$$= c(\beta^2 - \alpha^2)m + c^2 \alpha + \beta.$$

In particular, we have

$$\det \left( \frac{\partial (F,G)}{\partial (m,z)} \right) = \det \left( \frac{\partial (F,B)}{\partial (m,z)} \right)$$

$$= \det \left( \begin{array}{cc} \alpha \beta^2 / f_1 & 0 \\ \alpha^2 + \beta & 1 \end{array} \right)$$

$$= \frac{c f_1 (\beta^2 - \alpha^2)}{\alpha \beta^2} E'.$$

Here the first equality follows from the equation (17). On the other hand, plugging the equation $D$ (which is linear in $m$) into $B$ to eliminate the variable $m$, we obtain

$$E := \frac{c \alpha \beta^2 ((f_3 + 2)(\alpha + \beta)^2 + (c - 1/c)^2)}{(\alpha^2 - \beta^2)(c^2 \alpha + \beta)}.$$

Note that we used the fact that $\alpha^2 + \beta^2 + f_3 \alpha \beta = 1$. It follows that

$$\det \left( \frac{\partial (D,B)}{\partial (m,z)} \right) = \det \left( \frac{\partial (D,E)}{\partial (m,z)} \right) = c(\beta^2 - \alpha^2)E'.$$

Combining the above computations, we have

$$\det \left( \frac{\partial (F,G)}{\partial (m,z)} \right) = \frac{f_1}{\alpha \beta^2} \det \left( \frac{\partial (D,B)}{\partial (m,z)} \right)$$

$$= \frac{c f_1 (\beta^2 - \alpha^2)}{\alpha \beta^2} E'$$

$$= - \frac{c^2 f_1}{c^2 \alpha + \beta} H'$$

$$= \frac{c f_1}{m(\beta^2 - \alpha^2)} H'.$$

Note that the third and fourth equality follows from the equations $H = 0$ and $D = 0$, respectively. $\square$

Rewriting the equation (16) in the variable $z$ by using Lemma 3.5, we have

$$\sum_{(m,z) \in Z_{F,G}} \frac{-2c}{m \det \left( \frac{\partial (F,G)}{\partial (m,z)} \right)} = \sum_{z : H(z) = 0} \frac{2(\alpha^2 - \beta^2)}{f_1 H'}.$$

As in Section 3.1, we claim that the above equation is zero due to Jacobi’s residue theorem.
Lemma 3.6. For generic $c \in \mathbb{C}^*$ one has
\[
\sum_{z:H(z)=0} \frac{\alpha^2 - \beta^2}{f_1 H'} = 0.
\]

Proof. Let $\alpha = \alpha_1/\alpha_2$ and $\beta = \beta_1/\beta_2$ for some polynomials $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1$. Plugging these into $H$, we have
\[
H = \frac{(f_2 + 2f_1)(\alpha_1 \beta_2 + \alpha_2 \beta_1)^2 + (c - 1/c)^2 f_1 \alpha_1^2 \beta_2^2}{\alpha_1^2 \beta_2^2} = \frac{H_1 d}{H_2 d}
\]
with $\gcd(H_1, H_2) = 1$ and
\[
\sum_{z:H(z)=0} \frac{\alpha^2 - \beta^2}{f_1 H'} = \sum_{z:H_1(z)=0} \frac{(\alpha_2^2 \beta_2^2 - \alpha_2^2 \beta_1^2) H_2}{\alpha_1^2 \beta_2^2 H_1} = \sum_{z:H_1(z)=0} \frac{(\alpha_2^2 \beta_2^2 - \alpha_2^2 \beta_1^2)/d}{H_1}.
\]
We claim that $d$ divides $\alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2$ and thus $(\alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2)/d$ is a polynomial. It follows from the definitions of $\alpha$ and $\beta$ that the denominators $\alpha_2$ and $\beta_2$ divide some power of $f_1 = S_n(S_n - S_{n-1})$. (Recall that the denominator of $g_1$ is $S_n - S_{n-1}$.) In particular, every zero $z_0$ of $d$ is a zero of $f_1$. Let $\nu$ be the discrete valuation counting the multiplicity at $z_0$. From the fact that $d$ divides $(f_2 + 2f_1)(\alpha_1 \beta_2 + \alpha_2 \beta_1)^2 = (H_1 + (c - 1/c)^2 H_2) d$ and $\gcd(f_1, f_2) = 1$, we have
\[
\nu(d) \leq 2(\nu(\alpha_1 \beta_2 + \alpha_2 \beta_1)).
\]
On the other hand, using the equality $\alpha^2 + \beta^2 + f_1 \alpha \beta = 1$, one can re-express $H$ as
\[
H = \frac{(f_2 - 2f_1)(\alpha_1 \beta_2 - \alpha_2 \beta_1)^2 + (c + 1/c)^2 f_1 \alpha_1^2 \beta_2^2}{\alpha_1^2 \beta_2^2} = \frac{H_1 d}{H_2 d}
\]
Similarly, we have $\nu(d) \leq 2(\nu(\alpha_1 \beta_2 - \alpha_2 \beta_1))$ and thus
\[
\nu(d) \leq \nu(\alpha_1 \beta_2 + \alpha_2 \beta_1) + \nu(\alpha_1 \beta_2 - \alpha_2 \beta_1) = \nu(\alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2).
\]
This proves that $d$ divides $\alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2$.

Lemma 3.7 below shows that $\deg(\alpha_1^2 \beta_2^2 - \alpha_2^2 \beta_1^2) - \deg d \leq \deg H_1 - 2$, which is equivalent to $\deg(\alpha^2 - \beta^2) \leq \deg H + \deg f_1 - 2$. Then the lemma follows from Jacobi’s residue theorem. \qed

Lemma 3.7. For generic $C \in \mathbb{C}^*$ one has $\deg H + \deg f_1 - 2 \geq \deg(\alpha^2 - \beta^2)$.

Proof. Let us consider the case $n > 1$; the other case $n < 0$ can be proved similarly. From the fact that $\deg S_k = k - 1$ for $k \geq 1$, we have $\deg f_1 = 2n - 2$, $\deg f_2 = 2n - 1$, $\deg g_1 = 2n$, and $\deg g_2 = 2n - 1$. It follows that $\deg f_3 = \deg f_2 - \deg f_1 = 1$ and $\deg h_k = k - 1$ for $k \geq 1$. In the definition of $\alpha$,
\[
\alpha = \sum_{k=0}^{q} \binom{q}{k} g^k g_2^{-k} h_{k+\frac{n+1}{2}},
\]
its maximal degree only appears at $k = q$ and thus $\deg \alpha = q \deg g_1 + \deg h_{q+\frac{n+1}{2}} = (2n + 1)q + (p - 1)/2$. Similarly, we have $\deg \beta = (2n + 1)q + (p - 3)/2$. In the definition of $H = (f_3 + 2)(\alpha + \beta)^2 + (c - 1/c)^2$, its maximal degree uniquely appears in the term $f_2 \alpha^2$ and thus $\deg H = 2 \deg \alpha + \deg f_3$. This completes the proof, since we have $\deg H + \deg f_1 = 2 \deg \alpha + \deg f_2 = \deg(\alpha^2 - \beta^2) + 2n - 1$. \qed

Remark 3.8. Lemmas 3.4 and 3.7 do not hold for $n = 1$. In particular, the equation (16) is non-zero for the trefoil knot. This shows that the hyperbolicity assumption in Conjecture 1.1 is essential.

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