Mean Shrinkage Estimation for High-Dimensional Diagonal Natural Exponential Families

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Abstract

Shrinkage estimators have been studied widely in statistics and have profound impact in many applications. In this paper, we study simultaneous estimation of the mean parameters of random observations from a diagonal multivariate natural exponential family. More broadly, we study distributions for which the diagonal entries of the covariance matrix are certain quadratic functions of the mean parameter. We propose two classes of semi-parametric shrinkage estimators for the mean vector and construct unbiased estimators of the corresponding risk. Further, we establish the asymptotic consistency and convergence rates for these shrinkage estimators under squared error loss as both $n$, the sample size, and $p$, the dimension, tend to infinity. Finally, we consider the diagonal multivariate natural exponential families, which have been classified as consisting of the normal, Poisson, gamma, multinomial, negative multinomial, and hybrid classes of distributions. We deduce consistency of our estimators in the case of the normal, gamma, and negative multinomial distributions if $pn^{-1/3} \log^{4/3} n \to 0$ as $n, p \to \infty$, and for Poisson and multinomial distributions if $pn^{-1/2} \to 0$ as $n, p \to \infty$.

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1 Introduction

Shrinkage estimators have been studied widely in statistics and have profound impact in many applications. In 1956, Stein proved that in the case of the multivariate normal distribution on $p$-dimensional Euclidean space, $\mathbb{R}^p$, with $p \geq 3$, there exists a class of estimators of the population mean that dominates the sample mean under squared error loss. Subsequently, an explicit formula for such an estimator was given by James and Stein (1961). These results gave rise to shrinkage estimation, a novel approach to improved estimation of the mean.

Following on the work of Stein (1956), the Stein phenomenon or paradox, as named by Efron and Morris (1977), now incorporates many results on shrinkage estimation of mean parameters. Brown (1966) and Brown and Fox (1974) proved the admissibility of invariant estimators for multidimensional location parameters. Clevenson and Zidek (1975) derived minimax shrinkage estimators for estimating the means of Poisson populations. Efron and Morris (1975) analyzed several data sets using Stein’s estimator and its generalizations. Brandwein and Strawderman (1990) developed shrinkage estimators for non-normal location families, including spherically symmetric distributions. Gruber (1998) derived, using Bayesian and non-Bayesian approaches, a class of shrinkage estimators, for which the James-Stein estimator is a special case. Xie et al. (2012) proposed parametric and semi-parametric classes of mean shrinkage estimators for heteroskedastic hierarchical models and derived their asymptotic properties, and Xie et al. (2016) extended those results to families of distributions with quadratic variance functions, heteroskedastic location-scale families, and exponential families. Kou and Yang (2017) further extended the concept of optimal shrinkage estimation to the heteroskedastic linear model. In the non-parametric setting, Muandet et al. (2016) derived shrinkage estimators for mean functions in reproducing kernel Hilbert space, and Siapoutis (2019) extended some of those results to general Hilbert spaces.

In the present paper, we study simultaneous estimation of the mean parameters of random observations from a diagonal multivariate natural exponential family. These distributions include the multivariate normal, Poisson, gamma, multinomial, and negative multinomial. More broadly, we study distributions for which the diagonal entries of the covariance matrix are certain quadratic functions of the mean parameters. We propose two classes of semi-parametric shrinkage estimators for the mean vector and construct unbiased estimators of the corresponding risk, and we establish the asymptotic consistency and convergence rates for these shrinkage estimators under squared error loss as both $n$, the sample size, and $p$, the dimension, tend to infinity. We specialize these results for the diagonal multivariate natural exponential families and show the consistency for normal, gamma and negative multinomial distributions if $pn^{-1/3} \log^{4/3} n \to 0$ as $n, p \to \infty$, and for Poisson and multinomial distributions if $pn^{-1/2} \to 0$ as $n, p \to \infty$. We remark that our results extend the work of Xie et al.
(2016), who derived shrinkage estimators for the simultaneous estimation of the mean parameters of the univariate natural exponential families with quadratic variance functions.

The article is organized as follows. We begin by presenting in Section 2 the basic definitions for the multivariate natural exponential families and the diagonal multivariate natural exponential families. In Section 3, we construct a large class of semi-parametric shrinkage estimators for the mean parameter for the distributions for which the diagonal entries of the covariance matrix are certain quadratic functions of the mean parameter that shrink toward a given location and show their asymptotic properties under squared error loss. In Section 4, we construct another class of semi-parametric shrinkage estimators that shrinks toward the grand mean and study their properties. Finally, in Section 5, we study the distributions that belong to the diagonal multivariate natural exponential families and establish their asymptotic consistency.

2 The diagonal multivariate natural exponential families

We introduce some standard definitions for the multivariate natural exponential families. These definitions are also provided by Barndorff-Nielsen (1978), Brown (1986), Casalis (1990), Čencov (1982), Jørgensen (1987), Letac (1989), and Morris (1982).

Suppose that \( p > 1, \eta = (\eta_1, \ldots, \eta_p) \in \mathbb{R}^p, x = (x_1, \ldots, x_p) \in \mathbb{R}^p, \) and \( \mu \) is a positive measure on \( \mathbb{R}^p \). Define the Laplace transform of \( \mu \) as

\[
L_{\mu}(\eta) := \int_{\mathbb{R}^p} e^{\langle \eta, x \rangle} \mu(dx).
\]

(2.1)

Letting \( H(\mu) = \{ \eta \in \mathbb{R}^p : L_{\mu}(\eta) < \infty \} \), we denote by \( \text{Int}(H(\mu)) \) the interior of \( H(\mu) \).

Let \( \mathcal{M}_p \) be the set of all \( \mu \) that are not concentrated on a strict affine subspace of \( \mathbb{R}^p \) with \( \text{Int}(H(\mu)) \) being non-empty. If \( \mu \in \mathcal{M}_p \) and \( \eta \in \text{Int}(H(\mu)) \), then \( P(\eta, \mu)(dx) = (L_{\mu}(\eta))^{-1}e^{\langle \eta, x \rangle} \mu(dx), x \in \mathbb{R}^p, \) is a probability measure on \( \mathbb{R}^p \). The family of distributions \( F(\mu) = \{ P(\eta, \mu)(dx) : \eta \in \text{Int}(H(\mu)) \} \) is called the natural exponential family generated by \( \mu \).

We define the cumulant-generating function of the measure \( \mu \) by \( k_{\mu}(\eta) = \log L_{\mu}(\eta), \eta \in \text{Int}(H(\mu)). \) The mean function of \( \mu \) is

\[
m := (m_1, \ldots, m_p) = \int_{\mathbb{R}^p} xP(\eta, \mu)(dx).
\]

It is a consequence of (2.1) that

\[
m = \left( \frac{\partial k_{\mu}}{\partial \eta_1}, \ldots, \frac{\partial k_{\mu}}{\partial \eta_p} \right).
\]
Further, the covariance matrix of \( m \) is defined by

\[
\text{Cov}(m) := \int_{\mathbb{R}^p} (x - m)^\top (x - m) P(\eta, \mu)(dx).
\]

Again by (2.1), we have

\[
\text{Cov}(m) = \left( \frac{\partial^2 k_{\mu}}{\partial \eta_i \partial \eta_j} \right)_{i,j=1}^p.
\]

A natural exponential family \( F \) in \( \mathbb{R}^p \) is said to be \textit{diagonal} if there exists functions \( \alpha_j : \mathbb{R} \to \mathbb{R}, j = 1, \ldots, p \), such that the diagonal of the matrix \( \text{Cov}(m) \) is of the form

\[
\text{diag} \left( \text{Cov}(m) \right) = (\alpha_1(m_1), \ldots, \alpha_p(m_p)).
\]

The family \( F \) is also said to be \textit{irreducible} if it is not the product of two independent natural exponential families in \( \mathbb{R}^k \) and \( \mathbb{R}^{p-k} \), for some \( k = 1, \ldots, p-1 \).

For an irreducible, diagonal, natural exponential family \( F \), Bar-Lev et al. (1994) showed that there are only six such families in \( \mathbb{R}^p \) and for these families each function \( \alpha_j \) is a quadratic polynomial in \( m_j \), i.e., a polynomial of degree at most two. These families are the familiar multivariate normal, Poisson, gamma, multinomial, and negative multinomial distributions, and an additional exceptional family called the hybrid distributions. In this article, we focus on the first five most common and well-established distributions in the literature.

### 3 Shrinkage estimation toward a given location

Let \( \Theta \subseteq \mathbb{R} \) be the space of all possible values of our parameters. For \( i = 1, \ldots, n \), let \( Y_i = (Y_{i1}, \ldots, Y_{ip}) \in \mathbb{R}^p \) be a random vector, with distribution function \( F_i \) having finite mean and covariance matrix. We suppose that \( Y_1, \ldots, Y_n \) are mutually independent random vectors, we denote the mean of each \( Y_i \) by \( E(Y_i) := \theta_i = (\theta_{i1}, \ldots, \theta_{ip}) \), where the unknown parameters \( \theta_{ij} \in \Theta \) for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, p \). Further, we denote by \( \text{Cov}(Y_i) \) the covariance matrix of \( Y_i, i = 1, \ldots, n \).

For known constants \( \nu_0, \nu_1, \nu_2 \in \mathbb{R} \), define

\[
V(t) = \nu_0 + \nu_1 t + \nu_2 t^2,
\]

(3.1)

\( t \in \Theta \). We assume that \( \nu_0, \nu_1, \nu_2 \) are such that \( V(t) > 0 \) for all \( t \in \Theta \). Motivated by the structure of the covariance matrices for the five most common irreducible diagonal natural exponential families, we further assume that, for each \( i = 1, \ldots, n \), the diagonal entries of the covariance matrix of the distribution \( F_i \) are of the form

\[
\text{diag}(\text{Cov}(Y_i)) := (\text{Var}(Y_{i1}), \ldots, \text{Var}(Y_{ip})) = \left( \frac{V(\theta_{i1})}{\tau_{i1}}, \ldots, \frac{V(\theta_{ip})}{\tau_{ip}} \right),
\]
where the known constants $\tau_{ij} \in \mathbb{N}$.

In this article, we consider two classes of semi-parametric shrinkage estimators for the mean parameters $\theta_i$, $i = 1, \ldots, n$. One class of estimators will shrink $Y_{ij}$ toward a given location $\mu_j \in \mathbb{R}$, $j = 1, \ldots, p$, and the second class of estimators will provide shrinkage toward the mean vector $\bar{Y}_j = \frac{1}{n} \sum_{i=1}^{n} Y_{ij}$, for $j = 1, \ldots, p$.

Let $b = (b_1, \ldots, b_n)$ where $b_i \in [0, 1]$, $i = 1, \ldots, n$. Also, let $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{R}^p$.

In this section, we consider shrinkage estimators of the form

$$\hat{\theta}_{ij}^{b,\mu} = (1 - b_i) Y_{ij} + b_i \mu_j,$$  \hspace{1cm} (3.2)

$i = 1, \ldots, n$, $j = 1, \ldots, p$, that shrink $Y_{ij}$ toward a given location $\mu_j \in \mathbb{R}$. Define $\tau_i = \sum_{j=1}^{p} \tau_{ij}$, $i = 1, \ldots, n$. For any $i, k = 1, \ldots, n$ we assume that $b_i \leq b_k$ whenever $\tau_i \geq \tau_k$; this means that the larger the sum of the within-group sample sizes $\tau_{ij}$, the smaller the amount of shrinkage toward $\mu_j$. We also require that $|\mu_j| \leq \max\{ |Y_{il}| : i = 1, \ldots, n, l = 1, \ldots, p \}$ for all $j = 1, \ldots, p$, so that shrinkage will take place toward a vector $\mu_j$ that is within the range of the data.

The squared error loss of the estimators in (3.2) is

$$\ell_{b,\mu} = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} (\hat{\theta}_{ij}^{b,\mu} - \theta_{ij})^2,$$  \hspace{1cm} (3.3)

and we define its risk as the expected value of the loss function given by

$$R_{b,\mu} = E(\ell_{b,\mu}).$$  \hspace{1cm} (3.4)

We wish to find an optimal choice of $b$ and $\mu$ that minimizes the risk (3.4). However, this is not feasible since the risk depends on the unknown parameters $\theta_{ij}$. Similar to Xie et al. (2016), we propose and minimize an unbiased estimator of its risk, given in Proposition 3.1.

**Proposition 3.1.** An unbiased estimator of the risk $R_{b,\mu}$ in (3.4) is given by

$$\hat{R}_{b,\mu} = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ b_i^2 (Y_{ij} - \mu_j)^2 + (1 - 2b_i) \frac{V(Y_{ij})}{\tau_{ij} + \nu_2} \right].$$  \hspace{1cm} (3.5)

**Proof.** Taking the expectation of the estimator in (3.5), we obtain

$$E(\hat{R}_{b,\mu}) = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ b_i^2 E(Y_{ij} - \mu_j)^2 + (1 - 2b_i) E\left( \frac{V(Y_{ij})}{\tau_{ij} + \nu_2} \right) \right].$$  \hspace{1cm} (3.6)

The term $E(Y_{ij} - \mu_j)^2$ in (3.6) can be expressed as

$$E(Y_{ij} - \mu_j)^2 = \text{Var}(Y_{ij} - \mu_j) + (E(Y_{ij} - \mu_j))^2$$

$$= \text{Var}(Y_{ij}) + (E(Y_{ij}) - \mu_j)^2.$$
Further, the term $E\left(\frac{V(Y_{ij})}{\tau_{ij} + \nu_2}\right)$ in (3.6) can be simplified to

$$E\left(\frac{V(Y_{ij})}{\tau_{ij} + \nu_2}\right) = \frac{\nu_0 + \nu_1 E(Y_{ij}) + \nu_2 E(Y^2_{ij})}{\tau_{ij} + \nu_2}$$

$$= \frac{\nu_0 + \nu_1 E(Y_{ij}) + \nu_2 \text{Var}(Y_{ij}) + \nu_2 (E(Y_{ij}))^2}{\tau_{ij} + \nu_2}$$

$$= \frac{\tau_{ij} \text{Var}(Y_{ij}) + \nu_2 \text{Var}(Y_{ij})}{\tau_{ij} + \nu_2}$$

$$= \text{Var}(Y_{ij}),$$  \hspace{1cm} (3.7)

where we use the fact that

$$\tau_{ij} \text{Var}(Y_{ij}) = V(\theta_{ij}) = \nu_0 + \nu_1 E(Y_{ij}) + \nu_2 (E(Y_{ij}))^2.$$

On summing over all $i, j$, we find that (3.6) equals

$$E(\hat{R}_{b,\mu}) = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ b_i^2 (\text{Var}(Y_{ij}) + (E(Y_{ij}) - \mu_j)^2) + (1 - 2b_i) \text{Var}(Y_{ij}) \right]$$

$$= \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ b_i^2 (\theta_{ij} - \mu_j)^2 + (1 - b_i)^2 \text{Var}(Y_{ij}) \right].$$  \hspace{1cm} (3.8)

Now, note that

$$R_{b,\mu} = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E(\hat{\theta}_{ij}^b - \theta_{ij})^2$$

$$= \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E((1 - b_i)Y_{ij} + b_i \mu_j - \theta_{ij})^2$$

$$= \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E((1 - b_i)Y_{ij} + b_i \mu_j - \theta_{ij} + b_i \theta_{ij} - b_i \theta_{ij})^2.$$  \hspace{1cm} (3.9)

By rearranging the terms and applying the trinomial identity $(x + y + z)^2 \equiv x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ for $x, y, z \in \mathbb{R}$, we obtain (3.9) in the form

$$R_{b,\mu} = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ b_i^2(\theta_{ij} - \mu_j)^2 + (1 - b_i)^2 E(Y^2_{ij}) + (1 - b_i)^2 \theta_{ij}^2 - 2(1 - b_i)^2 \theta_{ij}^2 \right.\hspace{1cm}$$

$$- 2b_i (1 - b_i) \theta_{ij} (\theta_{ij} - \mu_j) + 2b_i (1 - b_i) \theta_{ij} (\theta_{ij} - \mu_j)]$$

$$= \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ b_i^2(\theta_{ij} - \mu_j)^2 + (1 - b_i)^2 \text{Var}(Y_{ij}) \right].$$  \hspace{1cm} (3.10)

It follows from (3.8) and (3.10) that $E(\hat{R}_{b,\mu}) = R_{b,\mu}$ and therefore the $\hat{R}_{b,\mu}$ estimator in (3.5) is an unbiased estimator of the risk in (3.4).
Consequently, we aim to find $\hat{b}^*$ and $\hat{\mu}^*$ that minimize (3.5) over the set

$$\Lambda = \left\{ (b, \mu) : b_i \in [0, 1], b_i \leq b_k \text{ for any } i, k = 1, \ldots, n \text{ whenever } \tau_i \geq \tau_k, \right. $$

and $|\mu_j| \leq \max \{|Y_{il}| : i = 1, \ldots, n, l = 1, \ldots, p\}$ for $j = 1, \ldots, p$, whenever $\tau_i \cdot \tau_k \geq \nu_2$. Further, in Theorem 3.3, we will provide the asymptotic behavior of our proposed estimators (3.11) among a large class of shrinkage estimators.

Now, we introduce the assumptions that we need for proving the main results of the article:

(A) $\limsup_{n, p \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij})/np < \infty$,

(B) $\limsup_{n, p \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij})\theta_{ij}^2/np < \infty$,

(C) $\limsup_{n, p \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}^2)/np < \infty$,

(D) $\sup_{i,j} \left( \tau_{ij} / (\tau_{ij} + \nu_2) \right)^2 < \infty$,

(E) $E(\max Y_{ij}^2)/n = O(n^{-\alpha}p^\beta)$ for some $\alpha > 0$ and $\beta \geq 0$ such that $n^{-\alpha}p^\beta \to 0$ as $n, p \to \infty$.

We now present the two main results on the first class of shrinkage estimators given in (3.11). The first result establishes the uniform convergence of the $\hat{R}_{b,\mu}$ estimator to the actual loss.

**Theorem 3.2.** Suppose that the assumptions (A)-(E) hold. Then, as $n, p \to \infty$,

$$E\left( \sup_{(b,\mu) \in \Lambda} |\hat{R}_{b,\mu} - \ell_{b,\mu}| \right) = O(n^{-1/2} + n^{-\alpha/2}p^{\beta/2}).$$

(3.12)

The second theorem shows that our proposed estimator is asymptotically optimal among a large class of shrinkage estimators.
Theorem 3.3. Suppose that the assumptions (A)-(E) hold, and consider any shrinkage estimator of the form

$$\hat{\theta}_i^{\hat{b},\hat{\mu}} = (1 - \hat{b}_i)Y_i + \hat{b}_i\hat{\mu},$$

$i = 1, \ldots, n$, where $(\hat{b}, \hat{\mu}) \in \Lambda$. Then, as $n, p \to \infty$, for any $\epsilon > 0$

$$\ell_{b^*,\hat{\mu}^*} \leq \ell_{b,\hat{\mu}} + O_p(n^{-1/2} + n^{-\alpha/2}p^{\beta/2}),$$

and

$$\limsup_{n,p \to \infty} (R_{b^*,\hat{\mu}^*} - R_{b,\hat{\mu}}) \leq 0.$$  \hspace{1cm} (3.14)$$

Because the proofs of Theorem 3.2 and 3.3 are lengthy, we will present them at the end of this section.

Remark 3.4. Interestingly, assumptions (A) and (B) hold if the condition

$$\limsup_{n \to \infty} \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{\theta_{ij}^k}{\tau_{ij}} < \infty,$$

$k = 0, 1, 2, 3, 4$ is satisfied. The above statement holds because

$$\sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}) = \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{\nu_0 + \nu_1 \theta_{ij} + \nu_2 \theta_{ij}^2}{\tau_{ij}}$$

$$= \nu_0 \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{1}{\tau_{ij}} + \nu_1 \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{\theta_{ij}}{\tau_{ij}} + \nu_2 \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{\theta_{ij}^2}{\tau_{ij}},$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}) \theta_{ij}^2 = \nu_0 \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{\theta_{ij}^2}{\tau_{ij}} + \nu_1 \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{\theta_{ij}^3}{\tau_{ij}} + \nu_2 \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{\theta_{ij}^4}{\tau_{ij}}.$$

Thus, assumptions (A) and (B) place restrictions on the growth of $\theta_{ij}^k/\tau_{ij}, k = 0, \ldots, 4$, for all $i, j$.

Remark 3.5. On the other hand, assumption (C), because it involves the fourth moment of each $Y_{ij}$, places restrictions on the growth of the kurtosis of $Y_{ij}$. This leads us to observe that the condition,

$$\limsup_{n \to \infty} \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E(Y_{ij}^4) < \infty,$$

$i = 1, \ldots, p$ implies conditions (A), (B), and (C). We prove this as follows.
By Jensen’s inequality, we have $\text{Var}(Y_{ij}) \leq E(Y_{ij}^2) \leq (E(Y_{ij}^4))^{1/2}$. Applying the Cauchy-Schwarz inequality, we obtain
\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}) \leq \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} (E(Y_{ij}^2))^{1/2} \leq \left( \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E(Y_{ij}^4) \right)^{1/2},
\]
and therefore (3.15) implies (A).

As for (B), we have
\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}) \theta_{ij}^2 \leq \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E(Y_{ij}^2) \theta_{ij}^2 \leq \left( \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E(Y_{ij}^4) \right)^{1/2} \left( \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij}^4 \right)^{1/2},
\]
where the second inequality follows from the Cauchy-Schwarz inequality. By Jensen’s inequality, as $(E(Y_{ij}^2))^2 \leq E(Y_{ij}^4)$, then we see that (3.15) implies (B).

Finally, since $\text{Var}(Y_{ij}^2) \leq E(Y_{ij}^4)$ then
\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}^2) \leq \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E(Y_{ij}^4),
\]
and therefore (3.15) implies (C).

**Remark 3.6.** Later in Lemma 5.1, we will derive sufficient conditions for which condition (E) holds in the special case where each $Y_{ij}$ belongs to a natural exponential family with quadratic variance functions.

Let us define the quantities
\[
T_1 = \frac{1}{np} \left| \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ \frac{V(Y_{ij})}{\tau_{ij} + v_2} - (Y_{ij} - \theta_{ij})^2 \right] \right|,
\]
\[
T_2 = \frac{2}{np} \sup_{(b, \mu) \in \Lambda} \left| \sum_{i=1}^{n} \sum_{j=1}^{p} b_i \left[ \frac{V(Y_{ij})}{\tau_{ij} + v_2} - (Y_{ij} - \theta_{ij})^2 \right] \right|,
\]
and
\[
T_3 = \frac{2}{np} \sup_{(b, \mu) \in \Lambda} \left| \sum_{i=1}^{n} \sum_{j=1}^{p} b_i (Y_{ij} - \theta_{ij})(\theta_{ij} - \mu_j) \right|.
\]

In proving the previously-stated theorems, we need the following lemma which provides the convergence properties of $T_1$, $T_2$, and $T_3$. 

Lemma 3.7. Suppose that the assumptions (A)-(E) hold. Then, \( E(|T_1|) = O(n^{-1/2}) \), \( E(|T_2|) = O(n^{-1/2}) \), and \( E(|T_3|) = O(n^{-1/2} + n^{-\alpha/2}p^{\beta/2}) \) as \( n, p \to \infty \).

Since the proof of Lemma 3.7 is lengthy, we provide it in Appendix A. Now, we have all the results needed to prove Theorems 3.2 and 3.3.

Proof of Theorem 3.2. By applying (3.2), (3.3), and (3.5), we obtain

\[
\hat{R}_{b, \mu} - \ell_{b, \mu} = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ b_i^2 (Y_{ij} - \mu_j)^2 + (1 - 2b_i) \frac{V(Y_{ij})}{\tau_{ij} + \nu_2} \right] - \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} [(1 - b_i)Y_{ij} + b_i\mu_j - \theta_{ij}]^2. \tag{3.16}
\]

Expanding the second term in (3.16), we have

\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} [(1 - b_i)Y_{ij} + b_i\mu_j - \theta_{ij}]^2
= \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} [(Y_{ij} - \theta_{ij})^2 + b_i^2(Y_{ij} - \mu_j)^2 - 2b_i(Y_{ij} - \theta_{ij})(Y_{ij} - \mu_j)]. \tag{3.17}
\]

Rearranging terms in (3.16) and applying (3.17), we find that (3.16) is given by

\[
\hat{R}_{b, \mu} - \ell_{b, \mu} = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} (1 - 2b_i) \frac{V(Y_{ij})}{\tau_{ij} + \nu_2} - \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} (Y_{ij} - \theta_{ij})^2 + \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} 2b_i(Y_{ij} - \theta_{ij})(Y_{ij} - \mu_j). \tag{3.18}
\]

For the third term of (3.18), we note

\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} 2b_i(Y_{ij} - \theta_{ij})(Y_{ij} - \mu_j)
= \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} 2b_i(Y_{ij} - \theta_{ij})(\theta_{ij} - \mu_j + Y_{ij} - \theta_{ij})
= \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} 2b_i(Y_{ij} - \theta_{ij})(\theta_{ij} - \mu_j) + \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} 2b_i(Y_{ij} - \theta_{ij})^2.
\]

Hence, we find that (3.18) equals

\[
\hat{R}_{b, \mu} - \ell_{b, \mu} = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} (1 - 2b_i) \left[ \frac{V(Y_{ij})}{\tau_{ij} + \nu_2} - (Y_{ij} - \theta_{ij})^2 \right] + \frac{2}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} b_i(Y_{ij} - \theta_{ij})(\theta_{ij} - \mu_j). \tag{3.19}
\]
Taking the supremum over $\Lambda$ of the absolute value of the eq. (3.19), we have

$$\sup_{(b, \mu) \in \Lambda} |\hat{R}_{b, \mu} - \ell_{b, \mu}| \leq T_1 + T_2 + T_3. \quad (3.20)$$

By taking expectations on both sides of (3.20) and applying Lemma 3.7, we find that (3.12) holds.

**Proof of Theorem 3.3.** Since $(\hat{b}^*, \hat{\mu}^*)$ is a minimizer of the estimator (3.5), we have that $\hat{R}_{b^*, \mu^*} \leq \hat{R}_{b, \mu}$ for any $b$ and $\mu$. For $\epsilon > 0$,

$$P(\ell_{b^*, \mu^*} \geq \ell_{b, \mu} + \epsilon) \leq P\left(\ell_{b^*, \mu^*} - \hat{R}_{b^*, \mu^*} \leq \ell_{b, \hat{\mu}} - \hat{R}_{b, \hat{\mu}} + \epsilon\right)$$

$$\leq P\left(|\ell_{b^*, \mu^*} - \hat{R}_{b^*, \mu^*}| \geq \epsilon/2\right)$$

$$+ P\left(|\ell_{b, \hat{\mu}} - \hat{R}_{b, \hat{\mu}}| \geq \epsilon/2\right). \quad (3.21)$$

Now, since

$$|\ell_{b^*, \mu^*} - \hat{R}_{b^*, \mu^*}| \leq \sup_{(b, \mu) \in \Lambda} |\ell_{b, \mu} - \hat{R}_{b, \mu}|$$

and

$$|\ell_{b, \hat{\mu}} - \hat{R}_{b, \hat{\mu}}| \leq \sup_{(b, \mu) \in \Lambda} |\ell_{b, \mu} - \hat{R}_{b, \mu}|,$$

then it follows that (3.21) is bounded above by

$$2P\left(\sup_{(b, \mu) \in \Lambda} |\ell_{b, \mu} - \hat{R}_{b, \mu}| \geq \epsilon/2\right).$$

By Markov’s inequality, we obtain

$$P(\ell_{b^*, \mu^*} \geq \ell_{b, \hat{\mu}} + \epsilon) \leq \frac{2E\left(\sup_{(b, \mu) \in \Lambda} |\ell_{b, \mu} - \hat{R}_{b, \mu}|\right)}{\epsilon/2}. \quad (3.22)$$

By applying Theorem 3.2 to (3.22), we obtain (3.13).

To prove (3.14), we have

$$\ell_{b^*, \hat{\mu}^*} - \ell_{b, \hat{\mu}} = (\ell_{b^*, \mu^*} - \hat{R}_{b^*, \mu^*}) + (\hat{R}_{b^*, \mu^*} - \hat{R}_{b, \hat{\mu}}) + (\hat{R}_{b, \hat{\mu}} - \ell_{b, \hat{\mu}}), \quad (3.23)$$

where we add and subtract the terms $\hat{R}_{b^*, \mu^*}$ and $\hat{R}_{b, \hat{\mu}}$. Again, using the fact that $\hat{R}_{b^*, \mu^*} \leq \hat{R}_{b, \hat{\mu}}$ for any $\hat{b}$ and $\hat{\mu}$, we find that (3.23) is bounded above by

$$(\ell_{b^*, \mu^*} - \hat{R}_{b^*, \mu^*}) + (\hat{R}_{b, \hat{\mu}} - \ell_{b, \hat{\mu}}),$$

Hence, we obtain

$$\ell_{b^*, \hat{\mu}^*} - \ell_{b, \hat{\mu}} \leq 2 \sup_{(b, \mu) \in \Lambda} |\hat{R}_{b, \mu} - \ell_{b, \mu}|. \quad (3.24)$$

By taking expectations in (3.24) and applying Theorem 3.2, we obtain the desired result.
4 Shrinkage estimation toward the grand mean

In the previous section, we consider the class of semi-parametric shrinkage estimators of the mean parameters \( \theta_i, i = 1, \ldots, n \), given by (3.2). That class of estimators shrinks each \( Y_{ij} \) toward a location \( \mu_j \) that is determined by solving an optimization problem; specifically, we minimize the unbiased estimator of the risk (3.5) over the set \( \Lambda \). Now, we consider the second class of shrinkage estimators of the mean parameter. We replace the given location \( \mu_j \in \mathbb{R} \) with the mean vector \( \bar{Y}_j = \frac{1}{n} \sum_{i=1}^{n} Y_{ij}, j = 1, \ldots, p \).

The second class of shrinkage estimators is of the form

\[
\hat{\theta}_{ij}^b, \bar{Y} = (1 - b_i)Y_{ij} + b_i \bar{Y}_j, \tag{4.1}
\]

\( i = 1, \ldots, n, \ j = 1, \ldots, p \). For any \( i, k = 1, \ldots, n \), we again assume that \( b_i \leq b_k \) whenever \( \tau_i \geq \tau_k \).

The squared error loss of the estimators in (4.1) is

\[
\ell_b = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} (\hat{\theta}_{ij}^b, \bar{Y} - \theta_{ij})^2, \tag{4.2}
\]

and therefore we define its risk as

\[
R_b = E(\ell_b). \tag{4.3}
\]

Again, we find an optimal choice of \( b \) by minimizing an unbiased estimator of the risk (4.3). In Proposition 4.1, we propose an unbiased estimator of the risk.

**Proposition 4.1.** An unbiased estimator of the risk \( R_b \) in (4.3) is given by

\[
\hat{R}_b = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ b_i^2 (Y_{ij} - \bar{Y}_j)^2 + \left( 1 - 2 \left( \frac{1}{n} - b_i \right) \right) V(Y_{ij}) \right]. \tag{4.4}
\]

**Proof.** Taking the expectation of the estimator in (4.4) and using (3.7), we obtain

\[
E(\hat{R}_b) = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ b_i^2 E(Y_{ij} - \bar{Y}_j)^2 + \left( 1 - 2 \left( \frac{1}{n} - b_i \right) \right) V(Y_{ij}) \right]
\]

\[
= \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ b_i^2 E(Y_{ij} - \bar{Y}_j)^2 + \text{Var}(Y_{ij}) - 2b_i \text{Var}(Y_{ij}) + \frac{2}{n} b_i E(Y_{ij}^2) - \frac{2}{n} b_i [E(Y_{ij})]^2 \right]. \tag{4.5}
\]

Now, note that

\[
R_b = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E(\hat{\theta}_{ij}^b, \bar{Y} - \theta_{ij})^2 = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E \left( b_i \bar{Y}_j + (1 - b_i)Y_{ij} - \theta_{ij} \right)^2
\]

\[
= \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} E \left( b_i \bar{Y}_j + Y_{ij} - b_i Y_{ij} - \theta_{ij} \right)^2 \tag{4.6}
\]
By rearranging the terms and applying the elementary identity, \((x+y)^2 \equiv x^2 + y^2 + 2xy\) for \(x, y \in \mathbb{R}\), we derive (4.6) in the form

\[
R_b = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} [b_i^2 E(Y_{ij} - \bar{Y}_j)^2 + E(\theta_{ij} - Y_{ij})^2 \\
+ 2E((Y_{ij} - \theta_{ij})(\bar{Y}_j b_i - Y_{ij} b_i))]
\]

\[
= \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} [b_i^2 E(Y_{ij} - \bar{Y}_j)^2 + \text{Var}(Y_{ij}) + 2E((Y_{ij} - \theta_{ij})(\bar{Y}_j b_i - Y_{ij} b_i))]. \tag{4.7}
\]

Expanding the third term of the equation (4.7), we obtain

\[
\sum_{i=1}^{n} \sum_{j=1}^{p} E((Y_{ij} - \theta_{ij})(\bar{Y}_j b_i - Y_{ij} b_i))
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{p} E(Y_{ij} \bar{Y}_j b_i - Y_{ij} Y_{ij} b_i - \theta_{ij} \bar{Y}_j b_i + \theta_{ij} Y_{ij} b_i)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ E(Y_{ij} \bar{Y}_j b_i - \theta_{ij} \bar{Y}_j b_i) - b_i \text{Var}(Y_{ij}) \right]. \tag{4.8}
\]

We further expand the first two terms of the equation (4.8). We split the first term into two sums for \(i = k\) and \(i \neq k\), thus

\[
\sum_{i=1}^{n} \sum_{j=1}^{p} E(Y_{ij} \bar{Y}_j b_i) = \frac{1}{n} \sum_{j=1}^{p} \sum_{i=1}^{n} b_i E(Y_{ij} Y_{kj})
\]

\[
= \frac{1}{n} \sum_{j=1}^{p} \sum_{i=1}^{n} b_i E(Y_{ij}^2) + \frac{1}{n} \sum_{j=1}^{p} \sum_{i \neq k} b_i E(Y_{ij} Y_{kj}),
\]

and since \(Y_{ij}\) are mutually independent for all \(i\), we get

\[
\sum_{i=1}^{n} \sum_{j=1}^{p} E(Y_{ij} \bar{Y}_j b_i) = \frac{1}{n} \sum_{j=1}^{p} \sum_{i=1}^{n} b_i E(Y_{ij}^2) + \frac{1}{n} \sum_{j=1}^{p} \sum_{i \neq k} b_i E(Y_{ij}) E(Y_{kj}). \tag{4.9}
\]

By using similar arguments, the second term in (4.8) becomes

\[
\sum_{i=1}^{n} \sum_{j=1}^{p} E(\theta_{ij} \bar{Y}_j b_i) = \frac{1}{n} \sum_{j=1}^{p} \sum_{i=1}^{n} \sum_{k=1}^{n} b_i E(Y_{ij}) E(Y_{kj})
\]

\[
= \frac{1}{n} \sum_{j=1}^{p} \sum_{i=1}^{n} b_i [E(Y_{ij})]^2 + \frac{1}{n} \sum_{j=1}^{p} \sum_{i \neq k} b_i E(Y_{ij}) E(Y_{kj}). \tag{4.10}
\]

Substituting (4.9) and (4.10) in (4.8), it follows from (4.5) and (4.7) that \(E(\bar{R}_b) = R_b\) and thus the estimator in (4.4) is an unbiased estimator of the risk in (4.3).
We are interested in finding $\bar{b}^*$ that minimizes (4.4) over the set

\[ \Psi = \{ b : b_i \in [0, 1] \text{ and } b_i \leq b_k \text{ for any } i, k = 1, \ldots, n \text{ whenever } \tau_i \geq \tau_k \}, \]

and the shrinkage estimator in (4.1) becomes

\[ \hat{\theta}^{\bar{b}^*, \bar{Y}}_i = (1 - \bar{b}^*_i) Y_i + \bar{b}^*_i \bar{Y}, \quad (4.11) \]

\[ i = 1, \ldots, n. \]

Next, we present our main results for the second class of shrinkage estimators given by (4.11). In Theorem 4.2, we show that the estimator $\hat{R}_b$ is uniformly close to the actual loss $\ell_b$.

**Theorem 4.2.** Suppose that the assumptions (A)-(E) hold. Then, as $n, p \to \infty$,

\[ E \left( \sup_{b \in \Psi} |\hat{R}_b - \ell_b| \right) = O(n^{-1/2} + n^{-\alpha/2}p^{\beta/2}). \quad (4.12) \]

Further, we show that our proposed estimator is asymptotically optimal among a large class of shrinkage estimators.

**Theorem 4.3.** Suppose that the assumptions (A)-(E) hold, and consider any shrinkage estimator of the form

\[ \hat{\theta}^{\bar{b}, \bar{Y}}_i = (1 - \bar{b}_i) Y_i + \bar{b}_i \bar{Y}, \]

\[ i = 1, \ldots, n, \] where $\bar{b} \in \Psi$. Then, as $n, p \to \infty$, for any $\epsilon > 0$

\[ \ell_{\bar{b}^*} \leq \ell_{\bar{b}} + O_p(n^{-1/2} + n^{-\alpha/2}p^{\beta/2}), \]

and

\[ \limsup_{n \to \infty} (R_{\bar{b}^*} - R_{\bar{b}}) \leq 0. \]

The proofs of Theorems 4.2 and 4.3 are presented at the end of this section. Let us define the quantities

\[ T_4 = \frac{2}{n^2p} \sum_{i=1}^{n} \sum_{j=1}^{p} (Y_{ij} - \theta_{ij})^2, \]

\[ T_5 = \frac{2}{np} \sup_{(b, \mu) \in \Psi} \sum_{j=1}^{p} |\bar{Y}_j| \left| \sum_{j=1}^{p} b_i (Y_{ij} - \theta_{ij}) \right|. \]

In the following result, which is needed to prove Theorem 4.2 and 4.3, we establish the convergence properties of $T_4$ and $T_5$. 
Lemma 4.4. Suppose that the assumptions (A)-(E) hold. Then, as \( n, p \to \infty \), we have \( E(|T_4|) = O(n^{-1/2}) \) and \( E(|T_5|) = O(n^{-\alpha/2}p^{\beta/2}) \).

Proof. In bounding \( T_4 \), we obtain
\[
E\left( \frac{2}{n^2p} \sum_{i=1}^{n} \sum_{j=1}^{p} (Y_{ij} - \theta_{ij})^2 \right) = \frac{2}{n^2p} \sum_{i=1}^{n} \sum_{j=1}^{p} E((Y_{ij} - \theta_{ij})^2)
= \frac{2}{n^2p} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}).
\]
Under the regularity condition (A), it follows that, as \( n, p \to \infty \), \( E(|T_4|) = O(n^{-1/2}) \).

In bounding \( T_5 \), as in the proof of Lemma 3.7, we have
\[
2n p E \left( \sup_{(b,\mu) \in \Psi} \left| \sum_{i=1}^{n} b_i \sum_{j=1}^{p} |Y_{ij} - \theta_{ij}| \right| \right)
\leq \frac{2}{n p} \sum_{j=1}^{p} E \left( \max_{i,j} |Y_{ij}| \sup_{(b,\mu) \in \Psi} \left| \sum_{i=1}^{n} b_i (Y_{ij} - \theta_{ij}) \right| \right)
\leq \frac{2}{n p} \sum_{j=1}^{p} \left[ E \left( \max_{i,j} Y_{ij}^2 \right) E \left( \sup_{(b,\mu) \in \Psi} \left| \sum_{i=1}^{n} b_i (Y_{ij} - \theta_{ij}) \right|^2 \right) \right]^{1/2},
\]
and thus \( E(|T_5|) = O(n^{-\alpha/2}p^{\beta/2}) \), which completes the proof.

We now provide the proofs of Theorems 4.2 and 4.3.

Proof of Theorem 4.2. By using (4.1), (4.2), and (4.4), we obtain
\[
\hat{R}_b - \ell_{b} = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left( 1 - 2 \left( 1 - \frac{1}{n} \right) b_i \right) \left[ \frac{V(Y_{ij})}{\tau_{ij} + v_2} - (Y_{ij} - \theta_{ij})^2 \right]
+ \frac{2}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} b_i \left( \theta_{ij} (Y_{ij} - \theta_{ij}) + \frac{1}{n} (Y_{ij} - \theta_{ij})^2 - (Y_{ij} - \theta_{ij}) Y_{j} \right).
\]
(4.13)

Taking the supremum over \( \Psi \) of the absolute value of the eq. (4.13) , we have
\[
\sup_{(b,\mu) \in \Psi} |\hat{R}_b - \ell_{b}| \leq T_1 + \left( 1 - \frac{1}{n} \right) T_2 + T_{31} + T_4 + T_5.
\]
By taking expectations on both sides of the above expression and applying Lemmas 3.7 and 4.4, it follows that (4.12) holds.

Proof of Theorem 4.3. With Theorem 4.2 established, the proof of Theorem 4.3 is almost identical to that of Theorem 3.3.
5 Shrinkage estimation for the diagonal multivariate natural exponential families

In this section, we focus on the diagonal multivariate natural exponential families and simplify conditions (A)-(E) for those families. Bar-Lev et al. (1994) showed that there are six irreducible, diagonal natural exponential families in $\mathbb{R}^p$. These families are the familiar multivariate normal, Poisson, gamma, multinomial, and negative multinomial distributions, and an additional exceptional family called the hybrid distributions. In this section, we focus on the first five of these families.

We remind the reader of the setting in which our results are derived. Let $\Theta \subseteq \mathbb{R}$ be the space of all possible values of our parameters. For $i = 1, \ldots, n$, let $Y_i = (Y_{i1}, \ldots, Y_{ip}) \in \mathbb{R}^p$ be a random vector, with distribution function $F_i$, that belongs to a diagonal multivariate natural exponential family and having finite mean and covariance matrix. We suppose that $Y_1, \ldots, Y_n$ are mutually independent random vectors, we denote the mean of each $Y_i$ by $E(Y_i) := \theta_i = (\theta_{i1}, \ldots, \theta_{ip})$, where the unknown parameters $\theta_{ij} \in \Theta$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, p$. For known constants $\nu_0, \nu_1, \nu_2 \in \mathbb{R}$, define the function $V(t) = \nu_0 + \nu_1 t + \nu_2 t^2$, $t \in \Theta$. We assume that $\nu_0, \nu_1, \nu_2$ are such that $V(t) > 0$ for all $t \in \Theta$. We further assume that, for each $i = 1, \ldots, n$, the diagonal entries of the covariance matrix of the distribution $F_i$ are of the form

$$\text{diag } (\text{Cov}(Y_i)) := (\text{Var}(Y_{i1}), \ldots, \text{Var}(Y_{ip})) = \left( \frac{V(\theta_{i1})}{\tau_{i1}}, \ldots, \frac{V(\theta_{ip})}{\tau_{ip}} \right),$$

where the known constants $\tau_{ij} \in \mathbb{N}$.

In simplifying conditions (A)-(E), we modify and apply a result of (Xie et al., 2016, Lemma A.1, p. 593). In stating the following result, we recall that if a random variable $X$ has mean $\mu$ and variance $\sigma^2$ then the skew of $X$ is $E(X - \mu)^3/\sigma^3$. We introduce the following assumptions:

(F) $\limsup_{n,p \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{p} |\theta_{ij}|^{2+\tilde{\epsilon}}/np < \infty$ for some $\tilde{\epsilon} > 0$,

(G) $\limsup_{n,p \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{p} (\text{Var}(Y_{ij}))^{2}/np < \infty$,

(H) $\sup_{i} \text{skew}(Y_{ij}) = \sup_{i} \left( (\nu_1 + 2\nu_2 \theta_{ij})/\left(\tau_{ij}^{1/2}(\nu_0 + \nu_1 \theta_{ij} + \nu_2 \theta_{ij}^2)^{1/2}\right) \right) < \infty$ for all $j$.

**Lemma 5.1.** Let $Y_1, \ldots, Y_n$ be mutually independent random vectors with $Y_{ij}$ coming from one of the five natural exponential families with quadratic variance functions. Then conditions (B) and (F)-(H), imply condition (E) if (i) $0 < \tilde{\epsilon} < 2$, $pn^{-\tilde{\epsilon}/2} \to 0$ as $n,p \to \infty$, or (ii) $\tilde{\epsilon} \geq 2$, $pn^{-1/3} \log^{4/3} n \to 0$ as $n,p \to \infty$.

**Proof.** The proof is similar to the proof of Lemma A.1 in Xie et al. (2016), where $Y_{ij} = \sigma_{ij}^2 Z_{ij} + \theta_{ij}$ with $\sigma_{ij}^2 = \text{Var}(Y_{ij})$ and $Z_{ij}$ are independent for all $i$ with mean zero.
and variance one. Since \( Y_{ij}^2 = \sigma_{ij}^2 Z_{ij}^2 + \theta_{ij}^2 + 2\sigma_{ij} \theta_{ij} Z_{ij} \), we get that

\[
\max_{i,j} Y_{ij}^2 = \max_{i,j} \sigma_{ij}^2 \cdot \max_{i,j} Z_{ij}^2 + \max_{i,j} \theta_{ij}^2 + 2 \max_{i,j} \sigma_{ij} |\theta_{ij}| \cdot \max_{i,j} |Z_{ij}|.
\]

Using the hypotheses of Lemma 5.1, as well as Lemma A.1 in Xie et al. (2016), we have

\[
E(\max_{i,j} Y_{ij}^2) = O((np)^{1/2} p \log^2 n + (np)^{2/(2+\tilde{\epsilon})} + (np)^{1/2} p \log n)
\]

\[
= \begin{cases} 
O((np)^{2/(2+\tilde{\epsilon})}), & 0 < \tilde{\epsilon} < 2 \\
O((np)^{1/2} p \log^2 n), & \tilde{\epsilon} \geq 2
\end{cases}
\]

For \( 0 < \tilde{\epsilon} < 2 \), we deduce that

\[
\frac{1}{n} E(\max_{i,j} Y_{ij}^2) = O(n^{-\tilde{\epsilon}/(2+\tilde{\epsilon})} p^{2/(2+\tilde{\epsilon})}).
\]

This means for (E) to hold, we require \( pn^{-\tilde{\epsilon}/2} \to 0 \) as \( n, p \to \infty \). For \( \tilde{\epsilon} \geq 2 \), we deduce that

\[
\frac{1}{n} E(\max_{i,j} Y_{ij}^2) = O(n^{-1/2} p^{3/2} \log^2 n),
\]

i.e., \( pn^{-1/3} \log^{4/3} n \to 0 \) implies (E). \( \square \)

For the five diagonal multivariate natural exponential families that we consider in this article, below we list the respective conditions, under which regularity conditions (A)-(E) are satisfied.

**Proposition 5.2.** For the normal distribution with \( E(Y_{ij}) = \theta_{ij}, \theta_{ij} \in \mathbb{R} \), and \( \text{Var}(Y_{ij}) = 1 \), conditions (A)-(E) reduce to: \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij}^4 = O(np) \).

For the Poisson distribution with \( E(Y_{ij}) = \text{Var}(Y_{ij}) = \theta_{ij} \) and \( \theta_{ij} > 0 \), conditions (A)-(E) reduce to: \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij}^3 = O(np) \) and \( \inf \theta_{ij} > 0, j = 1, \ldots, p \).

For the gamma distribution with \( E(Y_{ij}) = \theta_{ij}, \text{Var}(Y_{ij}) = \theta_{ij}^2 / \lambda, \theta_{ij} > 0, \) and \( \lambda > 0 \), conditions (A)-(E) reduce to: \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij}^4 = O(np) \).

For the multinomial distribution with \( E(Y_{ij}) = \theta_{ij}, \text{Var}(Y_{ij}) = (\theta_{ij} - \theta_{ij}^2) / N_i, \theta_{ij} \in (0,1), \) and \( N_i \geq 2 \), conditions (A)-(E) reduce to: \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij}^3 = O(np) \) and \( \inf \theta_{ij} > 0, j = 1, \ldots, p \).

For the negative multinomial distribution with \( E(Y_{ij}) = \theta_{ij}, \text{Var}(Y_{ij}) = (\theta_{ij} + \theta_{ij}^2) / N_i, \theta_{ij} > 0, \) and \( N_i \in \mathbb{N} \), conditions (A)-(E) reduce to: \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij}^4 = O(np) \) and \( \inf \theta_{ij} > 0, j = 1, \ldots, p \).

**Proof.** For the multivariate normal distribution, we have \( E(W_{ij}) = \theta_{ij}' \) and \( \text{Var}(W_{ij}) = \sigma_{ij}' \). Instead of the variables \( W_{ij} \), we consider the transformation \( Y_{ij} = W_{ij} / (\sigma_{ij}')^{1/2} \). Then, we have \( E(Y_{ij}) = \theta_{ij}' / (\sigma_{ij}')^{1/2} = \theta_{ij} \) for \( \theta_{ij} \in \mathbb{R} \) and \( \text{Var}(Y_{ij}) = 1 \), i.e., \( v_0 = 1 \).
\[ \nu_1 = \nu_2 = 0 \text{ and } \tau_{ij} = 1. \] Therefore, Condition (A) is satisfied directly. Using the Jensen’s inequality, we have

\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \theta^2_{ij} \leq \left( \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \theta^4_{ij} \right)^{1/2},
\]

and

\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y^2_{ij}) = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ E(Y^4_{ij}) - (E(Y^2_{ij}))^2 \right] = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} (2 + 4\theta^2_{ij}).
\]

Therefore, it is straightforward to verify that \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta^4_{ij} = O(np) \) imply conditions (B) and (C). Condition (D) is also immediately satisfied. Note each \( Y_{ij} \) follows a univariate normal distribution. Thus, condition (F) is trivially satisfied for \( \tilde{\epsilon} = 2 \). Also, conditions (G) and (H) are satisfied.

For the multivariate Poisson distribution, since \( E(Y_{ij}) = \text{Var}(Y_{ij}) = \theta_{ij} \) for \( \theta_{ij} > 0 \), we have \( \nu_1 = 1, \nu_0 = \nu_2 = 0, \) and \( \tau_{ij} = 1 \). By Jensen’s inequality, we obtain

\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}) = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij} \leq \left( \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \theta^3_{ij} \right)^{1/3}.
\]

Therefore, it is straightforward to see that \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta^3_{ij} = O(np) \) implies conditions (A). Condition (B) is satisfied directly. Since

\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y^2_{ij}) = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ E(Y^4_{ij}) - (E(Y^2_{ij}))^2 \right] = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} (\theta_{ij} + 6\theta^2_{ij} + 4\theta^3_{ij}),
\]

we obtain (C) by arguments similar to those in the proof of (A). Condition (D) is also immediately satisfied. Note that each \( Y_{ij} \) follows a univariate Poisson distribution. Thus, condition (F) is satisfied for \( \tilde{\epsilon} = 1 \). Also, condition (G) is satisfied. Since \( \sup_i \text{skew}(Y_{ij}) = \sup_i (1/\theta^3_{ij}) \), and by the assumption that \( \inf_i \theta_{ij} > 0 \), we obtain (H).

For the multivariate gamma distribution, since \( E(Y_{ij}) = \theta_{ij} \) and \( \text{Var}(Y_{ij}) = \theta^2_{ij}/\lambda \) for \( \theta_{ij}, \lambda > 0 \), we obtain \( \nu_0 = \nu_1 = 0, \nu_2 = 1/\lambda, \) and \( \tau_{ij} = 1 \). Using Jensen’s inequality, we obtain

\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}) = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{\theta^2_{ij}}{\lambda} \leq \left( \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \frac{\theta^4_{ij}}{\lambda^2} \right)^{1/2}.
\]
Therefore, it is straightforward to verify that \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij}^4 = O(np) \) implies conditions (A). Condition (B) is satisfied directly. Since
\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}^2) = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ (E(Y_{ij}^4) - (E(Y_{ij}^2))^2 \right] \\
\leq \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left( \frac{6}{\lambda^3} + \frac{3}{\lambda^2} + \frac{4}{\lambda} + 1 \right) \theta_{ij}^4,
\]
we can prove that condition (C) holds. Condition (D) is also immediately satisfied.

Note that each \( Y_{ij} \) follows a univariate gamma distribution. Thus, condition (F) is satisfied for \( \bar{\epsilon} = 2 \). Also, condition (G) is satisfied. Since \( \sup_i \text{skew}(Y_{ij}) = \sup_i (2/\lambda^{1/2}) \), and \( \lambda > 0 \), we obtain (H).

For the multivariate multinomial distribution, we have \( E(W_{ij}) = \theta_{ij}' \) and \( \text{Var}(W_{ij}) = \theta_{ij}'^{2}/N_i \). Instead of the variables \( W_{ij} \), we consider the transformation \( Y_{ij} = W_{ij}/N_i \). Then, we have \( E(Y_{ij}) = \theta_{ij}'/N_i = \theta_{ij} \) and \( \text{Var}(Y_{ij}) = (\theta_{ij} - \theta_{ij}'^2)/N_i \), i.e., \( \nu_0 = 0, \nu_1 = 1, \nu_2 = -1 \) and \( \tau_{ij} = N_i \). Using Jensen’s inequality, we can show condition (A). Condition (B) is also satisfied. Since
\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}^2) = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ (E(Y_{ij}^4) - (E(Y_{ij}^2))^2 \right] \\
\leq \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ \left( \frac{5}{N_i^3} \right) \theta_{ij} + \left( \frac{5}{N_i} - \frac{4}{N_i^2} + \frac{16}{N_i^3} \right) \theta_{ij}^3 \right],
\]
it is straightforward to verify that \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij}^3 = O(np) \) and \( N_i \geq 2 \) imply conditions (C). Condition (D) is also satisfied for \( N_i \geq 2 \). Note that each \( Y_{ij} \) follows a Binomial distribution. Thus, condition (F) is satisfied for \( \bar{\epsilon} = 1 \). Also, condition (G) is satisfied using similar arguments. Since
\[
\sup_i \text{skew}(Y_{ij}) = \sup_i \frac{1}{N_i^{1/2}} \frac{1 - 2\theta_{ij}}{(\theta_{ij} - \theta_{ij}'^2)^{1/2}} \leq \sup_i \frac{1}{(\theta_{ij} - \theta_{ij}'^2)^{1/2}},
\]
and by assuming that \( \inf_i \theta_{ij} > 0 \), we prove (H).

For the multivariate negative multinomial distribution, we have \( E(W_{ij}) = \theta_{ij}' \) and \( \text{Var}(W_{ij}) = \theta_{ij}' + (\theta_{ij}'^2/N_i) \). Instead of the variables \( W_{ij} \), we consider the transformation \( Y_{ij} = W_{ij}/N_i \). Then, we have \( E(Y_{ij}) = \theta_{ij}'/N_i = \theta_{ij} \) and \( \text{Var}(Y_{ij}) = (\theta_{ij} + \theta_{ij}'^2)/N_i \) for \( \theta_{ij} > 0 \) and \( N_i \in \mathbb{N} \), i.e., \( \nu_0 = 0, \nu_1 = 1, \nu_2 = 1 \) and \( \tau_{ij} = N_i \). We can prove (A) and (B) by using similar arguments as those in the multivariate multinomial distribution. Since
\[
\frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}^2) = \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ (E(Y_{ij}^4) - (E(Y_{ij}^2))^2 \right] \\
\leq \frac{1}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} \left[ a_1 \theta_{ij} + a_2 \theta_{ij}' + a_3 \theta_{ij}^3 + a_4 \theta_{ij}'^4 \right],
\]
where \( a_k > 0, k = 1, 2, 3, 4 \), it is straightforward to verify that \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij} = O(np) \) implies conditions \((C)\). Condition \((D)\) is satisfied directly. Note that each \( Y_{ij} \) follows a negative binomial distribution. Thus, condition \((F)\) is automatically satisfied for \( \tilde{\epsilon} = 2 \). Also, the condition \((G)\) is immediately satisfied. Since

\[
\sup_i \text{skew}(Y_{ij}) = \sup_i \frac{1}{N_i^{1/2}} \frac{1 + 2\theta_{ij}}{(\theta_{ij} + \theta_{ij}^2)^{1/2}},
\]

and by assuming that \( \inf_i \theta_{ij} > 0 \), we get condition \((H)\). \( \square \)

Since Proposition 5.2 ensures conditions \((A)-(E)\), it follows that Theorems 3.2, 3.3, 4.2, and 4.3 hold for the diagonal multivariate natural exponential families.

**Remark 5.3.** We note that for each one of the five diagonal multivariate natural exponential families in Proposition 5.2, the rate of convergence is controlled by the sum of the third or fourth power of the mean parameters. For the normal, gamma and negative multinomial distributions, we have that \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij} = O(np) \), i.e., \( \tilde{\epsilon} = 2 \). Therefore, the rate of convergence becomes \( O(n^{-1/2} + n^{-1/4}p^{3/4} \log n) \), i.e., \( p \) should grow slower than \( n^{1/3} \). For the Poisson and multinomial distributions, we have that \( \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij}^3 = O(np) \), i.e., \( \tilde{\epsilon} = 1 \). Therefore, the rate of convergence becomes \( O(n^{-1/2} + n^{-1/6}p^{1/3}) \), i.e., \( p \) should grow slower than \( n^{1/2} \).

## A Appendix

In this section, we establish Lemma 3.7. In proving that result, we will apply Doob’s \(L^r\) maximal inequality (Doob, 1990, Theorem 3.4, p. 317), which we state as follows.

**Lemma A.1.** (Doob’s \(L^r\) maximal inequality) Let \( \{M_n : n \geq 1\} \) be a martingale. If \( r > 1 \) and \( E(|M_j|^r) < \infty \) for all \( 0 \leq j \leq n \), then

\[
E\left( \max_{0 \leq j \leq n} |M_j| \right)^r \leq \left( \frac{r}{r-1} \right)^r E(|M_n|^r).
\]

**Proof of Lemma 3.7.** In bounding \( T_1 \), we define

\[
Z_i = \frac{1}{p} \sum_{j=1}^{p} \left[ \frac{V(Y_{ij})}{\tau_{ij} + \nu_2} - (Y_{ij} - \theta_{ij})^2 \right]. \tag{A.1}
\]

An alternative expression for \( Z_i \) is

\[
Z_i = \frac{1}{p} \sum_{j=1}^{p} \left[ -\frac{\tau_{ij}}{\tau_{ij} + \nu_2} (Y_{ij}^2 - E(Y_{ij}^2)) + \left( 2\theta_{ij} + \frac{\nu_1}{\tau_{ij} + \nu_2} \right) (Y_{ij} - \theta_{ij}) \right]. \tag{A.2}
\]
To prove this, we substitute in (A.1) the formula $V(Y_{ij}) = \nu_0 + \nu_1 Y_{ij} + \nu_2 Y_{ij}^2$ from (3.1); then the $i$th term in (A.1) is a quadratic polynomial in $Y_{ij}$. So to prove (A.2), we need only to verify that the coefficients of $Y_{ij}^k$, $k = 0, 1, 2$ in the $i$th terms in (A.1) and (A.2) are the same.

For $k = 1, 2$, it is simple to verify that the coefficient of $Y_{ij}^k$ in the $i$th terms in (A.1) and (A.2) are equal. For $k = 0$, i.e., the term which is free of $Y_{ij}$, we need to show that

\[
\frac{\tau_{ij}}{\tau_{ij} + \nu_2} E(Y_{ij}^2) - \left(2\theta_{ij} + \frac{\nu_1}{\tau_{ij} + \nu_2}\right) \theta_{ij} = \frac{\nu_0}{\tau_{ij} + \nu_2} - \theta_{ij}^2. \tag{A.3}
\]

Noting that $\theta^2_{ij} = (EY_{ij})^2 = E(Y_{ij}^2) - \text{Var}(Y_{ij})$ and $\tau_{ij} \text{Var}(Y_{ij}) = \nu_0 + \nu_1 \theta_{ij} + \nu_2 \theta_{ij}^2$, we find that the left-hand side of (A.3) equals

\[
\frac{\tau_{ij}}{\tau_{ij} + \nu_2} \left(\text{Var}(Y_{ij}) + \theta_{ij}^2\right) - 2\theta_{ij}^2 - \frac{\nu_1 \theta_{ij}}{\tau_{ij} + \nu_2} = \frac{\tau_{ij} \text{Var}(Y_{ij}) + \tau_{ij} \theta_{ij}^2}{\tau_{ij} + \nu_2} - 2\theta_{ij}^2 - \frac{\nu_1 \theta_{ij}}{\tau_{ij} + \nu_2} = \frac{\nu_0 + \nu_1 \theta_{ij} + \nu_2 \theta_{ij}^2 + \tau_{ij} \theta_{ij}^2}{\tau_{ij} + \nu_2} - 2(\tau_{ij} + \nu_2)\theta_{ij}^2 - \nu_1 \theta_{ij} = \frac{\nu_0 - (\tau_{ij} + \nu_2)\theta_{ij}^2}{\tau_{ij} + \nu_2},
\]

which equals the right-hand side of (A.3).

Returning to (A.2), noting that $E(Z_i) = 0$, and applying Jensen’s inequality, we obtain

\[
E(Z_i^2) \leq \frac{1}{p} \sum_{j=1}^{p} E\left(-\frac{\tau_{ij}}{\tau_{ij} + \nu_2} (Y_{ij}^2 - E(Y_{ij}^2)) + \left(2\theta_{ij} + \frac{\nu_1}{\tau_{ij} + \nu_2}\right) (Y_{ij} - \theta_{ij}) \right)^2. \tag{A.4}
\]

Applying the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ for all $a, b \in \mathbb{R}$, we have

\[
E(Z_i^2) \leq \frac{2}{p} \sum_{j=1}^{p} \left[\left(\frac{\tau_{ij}}{\tau_{ij} + \nu_2}\right)^2 \text{Var}(Y_{ij}^2) + \left(2\theta_{ij} + \frac{\nu_1}{\tau_{ij} + \nu_2}\right)^2 \text{Var}(Y_{ij})\right]. \tag{A.5}
\]

By applying the same inequality as in (A.4), we find that (A.5) is bounded above by

\[
\frac{2}{p} \sum_{j=1}^{p} \left(\frac{\tau_{ij}}{\tau_{ij} + \nu_2}\right)^2 \text{Var}(Y_{ij}^2) + 8\theta_{ij}^2 \text{Var}(Y_{ij}) + 2\left(\frac{\nu_1}{\tau_{ij} + \nu_2}\right)^2 \text{Var}(Y_{ij}) \equiv \frac{2}{p} \sum_{j=1}^{p} \left(\frac{\tau_{ij}}{\tau_{ij} + \nu_2}\right)^2 \text{Var}(Y_{ij}^2) + \frac{16}{p} \sum_{j=1}^{p} \theta_{ij}^2 \text{Var}(Y_{ij}) + \frac{4}{p} \sum_{j=1}^{p} \left(\frac{\nu_1}{\tau_{ij} + \nu_2}\right)^2 \text{Var}(Y_{ij}).
\]
Therefore,
\[
E\left(\frac{1}{n} \sum_{i=1}^{n} Z_i\right)^2 = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(Z_i)
\leq \frac{2}{n^2 p} \sum_{i=1}^{n} \sum_{j=1}^{p} \left(\frac{\tau_{ij}}{\tau_{ij} + \nu_2}\right)^2 \text{Var}(Y_{ij}) + \frac{16}{n^2 p} \sum_{i=1}^{n} \sum_{j=1}^{p} \theta_{ij}^2 \text{Var}(Y_{ij})
+ 4 \frac{1}{n^2 p} \sum_{i=1}^{n} \sum_{j=1}^{p} \left(\frac{\nu_1}{\tau_{ij} + \nu_2}\right)^2 \text{Var}(Y_{ij}).
\]

Since \(\tau_{ij} \geq 1\) for all \(i, j\), we have
\[
\frac{1}{(\tau_{ij} + \nu_2)^2} \leq \frac{\tau_{ij}^2}{(\tau_{ij} + \nu_2)^2},
\]
and by taking the supremum over \(i, j\), we further obtain that
\[
\sup_{i,j} \left(\frac{1}{\tau_{ij} + \nu_2}\right)^2 \leq \sup_{i,j} \left(\frac{\tau_{ij}}{\tau_{ij} + \nu_2}\right)^2.
\]
Hence, condition (D) implies that \(\sup_{i,j} (\nu_1/(\tau_{ij} + \nu_2))^2 < \infty\). Since,
\[
T_1 = \left|\frac{1}{n} \sum_{i=1}^{n} \frac{1}{p} \sum_{j=1}^{p} \left[ \frac{V(Y_{ij})}{\tau_{ij} + \nu_2} - (Y_{ij} - \theta_{ij})^2 \right] \right| = \left|\frac{1}{n} \sum_{i=1}^{n} Z_i \right|
\]
and under conditions (A) - (D), as \(n, p \to \infty\), \(E(T_1^2) = E(\sum_{i=1}^{n} Z_i/n)^2 = O(1/n)\). Hence, we obtain the desired result.

In bounding \(T_2\), we assume, without loss of generality, that \(\tau_1 \leq \cdots \leq \tau_n\), and thus \(b_1 \geq \cdots \geq b_n\). Therefore, we obtain
\[
T_2 = \sup_{1 \leq b_1 \geq \cdots \geq b_n \geq 0} \frac{2}{n p} \left| \sum_{i=1}^{n} \sum_{j=1}^{p} b_i \left[ \frac{V(Y_{ij})}{\tau_{ij} + \nu_2} - (Y_{ij} - \theta_{ij})^2 \right] \right|
= \max_{1 \leq k \leq n} \frac{2}{n p} \left| \sum_{i=1}^{k} \sum_{j=1}^{p} \left[ \frac{V(Y_{ij})}{\tau_{ij} + \nu_2} - (Y_{ij} - \theta_{ij})^2 \right] \right|
= \max_{1 \leq k \leq n} \frac{2}{n} \left| \sum_{i=1}^{k} Z_i \right|.
\]
Let \(M_k = \sum_{i=1}^{k} Z_i\). Then,
\[
E(M_{k+1}|M_1, \ldots, M_k) = E(Z_1 + \cdots + Z_{k+1}|Z_1, \ldots, Z_k)
= E(Z_1|Z_1, \ldots, Z_k) \cdots + E(Z_{k+1}|Z_1, \ldots, Z_k).
\]  \hspace{1cm} \text{(A.6)}

Using the fact that \(Z_i\) are independent and \(E(Z_i) = 0\) for all \(i = 1, \ldots, n\), we obtain
\[
E(M_{k+1}|M_1, \ldots, M_k) = Z_1 + \cdots + Z_k + E(Z_{k+1})
= Z_1 + \cdots + Z_k = M_k.
\]  \hspace{1cm} \text{(A.7)}
Therefore, \( \{M_k : k \geq 1\} \) forms a martingale. Applying Lemma A.1 for \( r = 2 \), we have

\[
E\left(\max_{1 \leq k \leq n} M_k^2\right) \leq 4E(M_n^2) = 4E\left(\sum_{i=1}^{n} Z_i\right)^2
= 4\operatorname{Var}\left(\sum_{i=1}^{n} Z_i\right) = 4 \sum_{i=1}^{n} \operatorname{Var}(Z_i) = 4 \sum_{i=1}^{n} E(Z_i^2),
\]

and thus again we obtain, as \( n, p \to \infty \), \( E(T_2^2) \leq 8E\left(\sum_{i=1}^{n} Z_i/n\right)^2 = O(1/n) \). Hence, we obtain the desired result.

In bounding \( T_3 \), we note that

\[
\frac{2}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} b_i(Y_{ij} - \theta_{ij})(\theta_{ij} - \mu_j)
= \frac{2}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} b_i\theta_{ij}(Y_{ij} - \theta_j) - \frac{2}{np} \sum_{i=1}^{n} \sum_{j=1}^{p} b_i\mu_j(Y_{ij} - \theta_{ij}). \tag{A.8}
\]

Taking the supremum over \( \Lambda \) of the absolute value of (A.8), we have that

\[
T_3 \leq T_{31} + T_{32}
\]

where

\[
T_{31} = \sup_{(b, \mu) \in \Lambda} \frac{2}{np} \left| \sum_{i=1}^{n} \sum_{j=1}^{p} b_i\theta_{ij}(Y_{ij} - \theta_{ij}) \right|
\]

\[
T_{32} = \sup_{(b, \mu) \in \Lambda} \frac{2}{np} \left| \sum_{i=1}^{n} \sum_{j=1}^{p} b_i\mu_j(Y_{ij} - \theta_{ij}) \right|
\]

For the term \( T_{31} \), since \( b_1 \geq \cdots \geq b_n \),

\[
T_{31} = \sup_{1 \geq b_1 \geq \cdots \geq b_n \geq 0} \frac{2}{np} \left| \sum_{i=1}^{n} \sum_{j=1}^{p} b_i\theta_{ij}(Y_{ij} - \theta_{ij}) \right|
= \max_{1 \leq k \leq n} \frac{2}{np} \left| \sum_{i=1}^{k} \sum_{j=1}^{p} \theta_{ij}(Y_{ij} - \theta_{ij}) \right|
\]

Let \( N_k = \sum_{i=1}^{k} U_i \), where \( U_i = \sum_{j=1}^{p} \theta_{ij}(Y_{ij} - \theta_{ij})/p \). By conditioning on \( U_1, \ldots, U_k \) and applying the same arguments as in eq. (A.6) and (A.7), we obtain that

\[
E(N_{k+1}|N_1, \ldots, N_k) = E(U_1 + \cdots + U_{k+1}|U_1, \ldots, U_k) = N_k,
\]

and therefore \( \{N_k : k \geq 1\} \) forms a martingale. Applying Lemma A.1 for \( r = 2 \), we obtain

\[
E\left(\max_{1 \leq k \leq n} N_k^2\right) \leq 4E(N_n^2) = 4 \sum_{i=1}^{n} E(U_i^2),
\]
and under condition (B), we obtain, as \( n, p \to \infty \),

\[
E(T_{31}^2) \leq \frac{8}{n} E \left( \sum_{i=1}^{n} U_i \right)^2 \leq \frac{8}{n^2 p} \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}) \theta_{ij}^2 = O(1/n).
\]

For the term \( T_{32} \), we have

\[
\frac{2}{np} E \left( \sup_{(b, \mu) \in \Lambda} \left| \sum_{i=1}^{n} \sum_{j=1}^{p} b_i \mu_j (Y_{ij} - \theta_{ij}) \right| \right)
\leq \frac{2}{np} E \left( \sup_{(b, \mu) \in \Lambda} \sum_{j=1}^{p} \left| \mu_j \right| \left| \sum_{i=1}^{n} b_i (Y_{ij} - \theta_{ij}) \right| \right)
\leq \frac{2}{np} E \left( \sum_{j=1}^{p} \sup_{(b, \mu) \in \Lambda} \left| \mu_j \right| \sup_{(b, \mu) \in \Lambda} \left| \sum_{i=1}^{n} b_i (Y_{ij} - \theta_{ij}) \right| \right). \tag{A.9}
\]

Since \(|\mu_j| \leq \max\{|Y_{il}| : i = 1, \ldots, n, l = 1, \ldots, p\}\) for all \( j = 1, \ldots, p \), it follows that (A.9) equals to

\[
\frac{2}{np} \sum_{j=1}^{p} E \left( \max_{i,j} |Y_{ij}| \sup_{(b, \mu) \in \Lambda} \left| \sum_{i=1}^{n} b_i (Y_{ij} - \theta_{ij}) \right| \right). \tag{A.10}
\]

Applying the Cauchy-Schwarz inequality, we derive that (A.10) is bounded above by

\[
\frac{2}{np} \left[ E \left( \max_{i,j} Y_{ij}^2 \right) E \left( \sup_{(b, \mu) \in \Lambda} \left| \sum_{i=1}^{n} b_i (Y_{ij} - \theta_{ij}) \right|^2 \right) \right]^{1/2}
\equiv \frac{2}{np} \left( E \left( \max_{i,j} Y_{ij}^2 \right) \right)^{1/2} \sum_{j=1}^{p} \left[ E \left( \sup_{(b, \mu) \in \Lambda} \left| \sum_{i=1}^{n} b_i (Y_{ij} - \theta_{ij}) \right|^2 \right) \right]^{1/2}.
\]

For \( b_1 \geq \cdots \geq b_n \), we apply Lemma A.1 as before to obtain

\[
\frac{2}{np} E \left( \sup_{(b, \mu) \in \Lambda} \left| \sum_{i=1}^{n} \sum_{j=1}^{p} b_i \mu_j (Y_{ij} - \theta_{ij}) \right| \right)
\leq \frac{2}{np} \left( E \left( \max_{i,j} Y_{ij}^2 \right) \right)^{1/2} \sum_{j=1}^{p} \left[ E \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (Y_{ij} - \theta_{ij}) \right|^2 \right) \right]^{1/2}
\leq \frac{2}{np} \left( E \left( \max_{i,j} Y_{ij}^2 \right) \right)^{1/2} \sum_{j=1}^{p} \left[ \sum_{i=1}^{n} \text{Var}(Y_{ij}) \right]^{1/2}. \tag{A.11}
\]

By again applying Jensen’s inequality, we find that (A.11) is bounded above by

\[
\frac{2}{np} \left( E \left( \max_{i,j} Y_{ij}^2 \right) \right)^{1/2} \left( p \sum_{i=1}^{n} \sum_{j=1}^{p} \text{Var}(Y_{ij}) \right)^{1/2}
\equiv 2 \left( \frac{1}{n} E \left( \max_{i,j} Y_{ij}^2 \right) \right)^{1/2} \left( \frac{1}{np} \sum_{j=1}^{p} \sum_{i=1}^{n} \text{Var}(Y_{ij}) \right)^{1/2};
\]
hence,
\[
\frac{2}{np} E \left( \sup_{(b, \mu) \in \Lambda} \left| \sum_{i=1}^{n} \sum_{j=1}^{p} b_{ij} \mu_{j} (Y_{ij} - \theta_{ij}) \right| \right) \\
\leq 2 \left( \frac{1}{n} E \left( \max_{i,j} Y_{ij}^2 \right) \right)^{1/2} \left( \frac{1}{np} \sum_{j=1}^{p} \sum_{i=1}^{n} \text{Var}(Y_{ij}) \right)^{1/2}.
\]

Therefore, under conditions (A) and (E), it follows that, as \( n, p \to \infty \), \( E(\|T_{32}\|) = O(n^{-\alpha/2} p^{\beta/2}) \).

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