On the phase transition in the scalar theory

J. R. Espinosa\textsuperscript{1,2}
M. Quirós\textsuperscript{1,3}
and
F. Zwirner\textsuperscript{4}

Theory Division, CERN,
Geneva, Switzerland

Abstract

The basic tool for the study of the electroweak phase transition is $V_{\text{eff}}(\phi, T)$, the one-loop finite-temperature effective potential, improved by all-loop resummations of the most important infrared contributions. In this paper we perform, as a first step towards a full analysis of the Standard Model case, a detailed study of the effective potential of the scalar theory. We show that subleading corrections to the self-energies lead to spurious terms, linear in the field-dependent mass $m(\phi)$, in the daisy-improved effective potential. Consistency at subleading order requires the introduction of superdaisy diagrams, which prevent the appearance of linear terms. The resulting $V_{\text{eff}}(\phi, T)$ for the scalar theory hints at a phase transition which is either second-order or very weakly first-order.
It was argued long ago by Kirzhnits and Linde [1] that the electroweak gauge symmetry should be restored at sufficiently high temperature. Soon after this, Weinberg [2] and Dolan and Jackiw [3] laid the foundations for a quantitative treatment of this problem (for reviews of later developments, see e.g. refs. [4]). Recently, the observation [5] that the rate of anomalous $B$-violating processes is unsuppressed at high temperatures has revived the interest in the subject: a detailed discussion of the electroweak phase transition is required, if one wants to confront any model of particle interactions with the observed cosmological baryon asymmetry. In the Standard Model (SM), the amount of CP violation appears to be too small for baryogenesis at the electroweak scale (for recent reviews, see e.g. refs. [6]). However, the requirement that a pre-existing baryon asymmetry be not washed out by anomalous $B$-violating interactions, just after the electroweak phase transition, can in principle put a stringent upper bound on the SM Higgs mass (tentative bounds have been given in refs. [7], and possible ways out discussed in refs. [8]). In extensions of the SM, one can aim at the construction of consistent and phenomenologically acceptable models for baryogenesis at the electroweak scale, even if a detailed description of the latter has to face a number of technical difficulties [3].

The basic tool for the discussion of the electroweak phase transition is $V_{\text{eff}}(\phi, T)$, the finite-temperature one-loop effective potential, improved by all-loop resummations of the most important infrared contributions. A lively theoretical debate [9–14] on the structure of $V_{\text{eff}}(\phi, T)$ has recently taken place, with a certain amount of disagreement. In particular, the existence and the nature of terms linear in the field-dependent masses, in the high-temperature expansion of $V_{\text{eff}}(\phi, T)$ and after the inclusion of subleading infrared corrections, are rather controversial: a positive linear term was found in [10], whilst a negative linear term was obtained in [11]. It was subsequently argued in [13], on the basis of general arguments, that there are no linear terms, but no explicit proof of this statement at subleading order was given. To better understand the origin of these conflicting results and the nature of the electroweak phase transition in the SM, we discuss here some of the issues in a simpler context, the theory of a single real scalar. The present analysis will be extended to the more complicated case of the Standard Model in a longer paper, currently in preparation. After reviewing some well-known results, we explicitly compute the combinatorics of tadpole and vacuum daisy diagrams, which was recently questioned in [13]. We show that by integrating the tadpoles one precisely finds the one-loop effective potential, with the field-dependent mass $m^2(\phi)$ replaced, in the infrared-dominated terms, by an effective $T$-dependent mass $\overline{m}^2(\phi, T)$, in agreement with the result obtained by direct calculation of the vacuum daisy diagrams. We also improve over the previous computations by showing how to include subleading terms in the computation of the effective mass $\overline{m}^2(\phi, T)$.

We show that, to include them consistently, avoiding at the same time the problem of negative squared masses, one must go to the superdaisy approximation, where $\overline{m}^2(\phi, T)$ is computed as the self-consistent solution of a corresponding gap equation. We finally solve the gap equation and study the behaviour of the resulting effective potential: within the limits of our improved perturbative expansion, we find that the phase transition is either second-order or very weakly first-order.

We consider the theory of a real scalar field $\phi$, with a tree-level potential (invariant under the discrete $\mathbb{Z}_2$ symmetry $\phi \rightarrow -\phi$)

$$V_{\text{tree}}(\phi) = -\frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4,$$

(1)

and positive $\lambda$ and $\mu^2$. At the tree level, the field-dependent mass of the scalar field is $m^2(\phi) = 3\lambda \phi^2 - \mu^2$, and the minimum of $V_{\text{tree}}$ corresponds to $\phi^2 = \mu^2/\lambda \equiv v^2$, so that $m^2(v) = 2\lambda v^2 = 2\mu^2$. 

1
The basic tool for the study of the vacuum state of the theory is the effective potential \[15\]. To perform a systematic loop expansion of the effective potential at finite temperature, it is convenient, following ref. \[3\], to shift the scalar field around a constant background value \( \bar{\phi} : \phi = \bar{\phi} + \phi' \). Since the procedure is well known, and there is no risk of confusion, in the following we shall use the symbol \( \phi \) also for \( \bar{\phi} \). All our calculations will be performed in the imaginary-time formalism.

The one-loop contribution to the effective potential, \( V_{\text{eff}} = V_{\text{tree}} + V^{(1)} + V^{(2)} + \ldots \), comes from the quadratic terms in \( \phi' \) in the effective action, and is given, at the temperature \( T \), by \[3\]

\[
V^{(1)}[m^2(\phi), T] = \frac{T}{2} \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} \log \left[ \omega_n^2 + \vec{p}^2 + m^2(\phi) \right],
\]

where \( \omega_n = 2\pi n T \ (n \in \mathbb{Z}) \) are the bosonic Matsubara frequencies, and field-independent contributions will be neglected here and in the following. It was stressed in refs. \[13\] that \( V^{(1)} \) can be easily obtained as

\[
V^{(1)} = \int d\phi \mathcal{T}^{(1)},
\]

where \( \mathcal{T}^{(1)} \) is the one-loop tadpole diagram. Indeed, using the Feynman rules associated to \( (1) \) and including the appropriate symmetry factor, we can write

\[
\mathcal{T}^{(1)}[m^2(\phi), T] = 6\lambda \phi \frac{T}{2} \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\omega_n^2 + \vec{p}^2 + m^2(\phi)},
\]

which, upon integration over \( \phi \), gives eq. \( 4 \).

Before moving to the discussion of higher-loop contributions to the effective potential, we recall some well-known results, which will be useful later on. Following ref. \[3\], we can split the one-loop contribution of eq. \( 2 \) into a \( T = 0 \) and a \( T \neq 0 \) part

\[
V^{(1)}[m^2(\phi), T] = V^{(1)}[m^2(\phi), 0] + \Delta V^{(1)}[m^2(\phi), T].
\]

After introducing an appropriate set of counterterms, the \( T = 0 \) contribution reads

\[
V^{(1)}[m^2(\phi), 0] = \frac{m^4(\phi)}{64\pi^2} \left[ \log \frac{m^2(\phi)}{2\mu^2} - \frac{3}{2} \right].
\]

The specific form of the counterterms, and therefore of \( V^{(1)}[m^2(\phi), 0] \), depends on the renormalization prescription. Eq. \( 6 \) corresponds to the \( \overline{MS} \) scheme \[16\], with a renormalization scale \( Q^2 = 2\mu^2 \). The \( T \neq 0 \) contribution is given by

\[
\Delta V^{(1)}[m^2(\phi), T] = \frac{T^4}{2\pi^2} J_+(y^2),
\]

where

\[
y^2 = \frac{m^2(\phi)}{T^2}, \quad J_+(y^2) = \int_0^\infty dx \ x^2 \log \left( 1 - e^{-\sqrt{x^2+y^2}} \right).
\]

In the high-temperature limit \( y^2 \ll 1 \), we can write the expansion

\[
\Delta V^{(1)}[m^2(\phi), T] = \frac{m^2(\phi)T^2}{24} - \frac{m^3(\phi)T}{12\pi} - \frac{m^4(\phi)}{64\pi^2} \left[ \log \frac{m^2(\phi)}{T^2} - 5.4076 \right] + \ldots,
\]
where the dots stand for terms $O[m^6(\phi)/T^2]$ or higher. Observe the cancellation of the $m^4 \log m^2$ terms between the $T = 0$ and the $T \neq 0$ contributions in the high-temperature expansion.

The simplest approximation to the $T$-dependent effective potential, $V_{\text{eff}}(\phi, T)$, consists in adding to $V_{\text{tree}}$ only the leading (field-dependent) contribution to $V^{(1)}[m^2(\phi), T]$ in the high-temperature expansion. As a result, we obtain

$$V_{\text{eff}}(\phi, T) = \left(-\mu^2 + \frac{\lambda T^2}{4}\right) \frac{\phi^2}{2} + \frac{\lambda}{4} \phi^4,$$

(10)

whose behaviour is illustrated in fig. 1a, for the representative choice of parameters $\lambda = 0.1$, $v = 250$ GeV. One can easily see that eq. (10) describes a second-order phase transition, with the critical temperature given by $T_0 = 2\mu/\sqrt{\lambda} = 2v$, and the $T$-dependent minimum $v_T$ evolving as $v_T^2 = v^2 - T^2/4$ for $T < T_0$.

If, instead of the approximation (10), one wants to use the full one-loop effective potential, $V_{\text{eff}}(\phi, T) = V_{\text{tree}}(\phi) + V^{(1)}[m^2(\phi), T]$, one runs into several difficulties. To begin with, for $\phi < v/\sqrt{3}$ the field-dependent mass $m^2(\phi)$ is negative, and some expressions in eqs. (2)–(3) become ill-defined. For $T$ small enough, the minimum $v_T$ of the one-loop effective potential is still large enough that in its neighbourhood $m^2(\phi) > 0$. However, for temperatures close to $T_0$ this cannot be true, and the naive expression of the one-loop effective potential is of no use. The other well-known problem of the naive one-loop potential is the breakdown of the perturbative expansion near the critical temperature, which is due to the presence of higher-loop infrared-divergent diagrams in the limit $m^2(\phi) \to 0$. Both of these problems can be alleviated by the use of resummation techniques, which take into account the most important infrared contributions to the effective potential at all orders in the perturbative expansion. These resummations correct the part of the one-loop effective potential which is associated to the modes of zero Matsubara frequency, $\omega_n = 0$: a quick glance at eqs. (3) and (4) shows that it corresponds precisely to the $m^3$ term in the high-temperature expansion of eq. (2).

We now recall how to perform the resummation of the most important class of infrared-dominated diagrams, the so-called daisy and superdaisy diagrams. In contrast to ref. [1] (and to most of the subsequent refinements [17,18]), we shall work with just one scalar field $\phi$ and for an arbitrary constant background field configuration, not only for $\phi = 0$. We shall justify a posteriori our procedure, after having properly identified the parameters of the improved perturbative expansion.

As a first step, we need to compute the one-loop scalar self-energy at finite temperature, corresponding to the two diagrams depicted in fig. 2, in the infrared limit $p^0 = 0$, $\vec{p} \to 0$, where $p = (p^0, \vec{p})$ is the external momentum. For the temperature-dependent self-energy we find

$$\Pi[m^2(\phi), T] = \Pi^{(a)}[m^2(\phi), T] + \Pi^{(b)}[m^2(\phi), T],$$

(11)

where, after (MS) subtraction of the appropriate $T = 0$ counterterms[4]

$$\Pi^{(a)}[m^2(\phi), T] = \frac{1}{32\pi^2} \left[6\lambda m^2(\phi) \left(\log \frac{m^2(\phi)}{2\mu^2} - 1\right) + \frac{3\lambda T^2}{2\pi^2} \cdot I_+(y^2)\right],$$

(12)

$$\Pi^{(b)}[m^2(\phi), T] = \frac{1}{32\pi^2} \left[36\lambda^2 \phi^2 \log \frac{m^2(\phi)}{2\mu^2} + \frac{9\lambda^2 \phi^2}{\pi^2} \cdot I_+(y^2)\right],$$

(13)

\footnote{Notice that to obtain $\Pi[m^2(\phi), T]$ it is not necessary to perform any new diagrammatic calculation, since to compute the self-energy at zero external momentum one can simply take the second derivative of $V^{(1)}[m^2(\phi), T]$ in eq. (4).}
and \( I_+ (y^2) = 2[dJ_+ (y^2) / dy^2] \), \( I_+ ' (y^2) = [dI_+ (y^2) / dy^2] \).

In the calculation of the effective potential, the second step of the daisy resummation procedure consists in replacing, in the one-loop terms originating from the zero-frequency modes, i.e. in the \( m^3 \) term of the high-temperature expansion (9), the field-dependent mass \( m^2 (\phi) \) by an effective, \( \phi \)- and \( T \)-dependent mass,

\[
\mathcal{m}^2 (\phi, T) = m^2 (\phi) + \Pi [m^2 (\phi), T], \tag{14}
\]

where \( \Pi [m^2 (\phi), T] \) is the one-loop self-energy in the infrared limit, given by eqs. (11)–(13). The standard way of implementing this substitution has been recently questioned in ref. [13], so we give here some details of the calculation. Both techniques previously discussed at the one-loop level can be applied to obtain the effective potential at any order in the perturbative expansion, and in particular to the resummation of daisy diagrams.

The daisy diagrams contributing to the tadpole \( T_0 \) (where the subscript zero means that only zero-frequency modes are propagating in big bubbles) are depicted in figs. 3a–d. Considering all inequivalent permutations of bubbles, we can write

\[
\mathcal{T}^{daisy} \equiv \sum_{i=1}^{4} \mathcal{T}^{(d_i)} = \sum_{i=1}^{4} \sum_{j,l} C^{(d_i)}_{j,l} T^{(d_i)}_{j,l}, \tag{15}
\]

where the combinatorial factors are given by

\[
C^{(d_1)}_{j,l} = \binom{j + l}{l} 2^{-2(j+l+1)}, \quad C^{(d_2)}_{j,l} = \binom{j + l - 1}{l} 2^{-2(j+l+1)},
\]

\[
C^{(d_3)}_{j,l} = C^{(d_4)}_{j,l} = \binom{j + l - 1}{j} 2^{-2(j+l)} . \tag{16}
\]

Using the Feynman rules associated to (1) and the self-energies of fig. 2, we can write

\[
\mathcal{T}^{(d_1)} = \sum_{j,l} C^{(d_1)}_{j,l} T^{(d_1)}_{j,l} = \mathcal{T}^{(1)}_0 (m^2 + \Pi) - \mathcal{T}^{(1)}_0 (m^2), \tag{17}
\]

where \( \mathcal{T}^{(1)}_0 \) is the zero-frequency part of the one-loop tadpole defined in eq. (11). We then see that adding the class of diagrams in fig. 3a to the one-loop tadpole diagram amounts to shifting \( m^2 \) by \( \Pi \) in the zero-frequency part of the propagator. Eq. (17) cannot be integrated to give a logarithm, because \( \Pi \) depends in general on \( \phi \). However, the previous result changes when one considers the additional contributions to \( \mathcal{T}^{daisy} \) coming from figs. 3b–d. It is easy to check that

\[
\mathcal{T}^{(d_2)} = \frac{d\Pi^{(a)}}{d\phi} T \frac{1}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\vec{p}^2 + m^2 (\phi) + \Pi};
\]

\[
\mathcal{T}^{(d_3)} + \mathcal{T}^{(d_4)} = \frac{d\Pi^{(b)}}{d\phi} T \frac{1}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\vec{p}^2 + m^2 (\phi) + \Pi}. \tag{19}
\]

Adding up the one-loop and daisy contributions to the tadpole, eqs. (11) and (17)–(19), we obtain

\[
\mathcal{T}^{(1)}_0 + \mathcal{T}^{daisy} = \left( 6\lambda \phi + \frac{d\Pi}{d\phi} \right) \frac{T}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\vec{p}^2 + m^2 (\phi) + \Pi}, \tag{20}
\]

\(^2\)Only by making an approximation where \( \Pi \) does not depend on \( \phi \), as in refs. [12,13], eq. (17) can be integrated in \( \phi \), giving as a result a logarithmic function.
which yields for the effective potential, upon integration over $\phi$,

$$V_{\text{eff}}(\phi, T) = V_{\text{tree}}(\phi) + V^{(1)},$$

(21)

$$V^{(1)}(\phi, T) = V(1) \left[ m^2(\phi), T \right] - \frac{T}{12\pi} \left[ m^3(\phi, T) - m^3(\phi) \right],$$

(22)

where $V^{(1)}$ was given in eq. (2), $m^2$ in eq. (14), and $V^{(1)}$ is the daisy-improved one-loop contribution to the effective potential. We would like to stress again two important points: 1) only the $m^3$ term, which arises from the infrared-dominated diagrams, receives the shift $m^2 \to \overline{m}^2$; 2) the diagrams in figs. 3b–d are essential to obtain the result of eq. (21), unless one is working in an approximation where $\Pi$ does not depend on $\phi$, in which case they vanish\(^3\).

The result of eq. (21) can be obtained directly from the diagrammatic expansion of the vacuum daisy diagrams shown in fig. 3e, where the small bubbles are taken at zero external momentum, and only zero-frequency modes are propagating in the big bubbles. Adding all inequivalent permutations of bubbles,

$$V^{\text{daisy}} = \sum_{j,l} C_{j,l} V_{j,l},$$

(23)

where the combinatorial factors are

$$C_{j,l} = \frac{(j+l-1)!}{j!l!} 2^{-(j+l+1)},$$

(24)

so that

$$V^{\text{daisy}} = -\frac{T}{2} \sum_{N=1}^{\infty} \frac{1}{N} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{(-\Pi)^N}{[\vec{p}^2 + m^2(\phi)]^N}.$$  

(25)

More explicitly, $V_{\text{eff}} = V_{\text{tree}} + V^{(1)}(m^2) + V^{\text{daisy}} = V_{\text{tree}} + V^{(1)}$, which is precisely eq. (24).

The combinatorics of two-loop graphs, both for vacuum and for tadpole diagrams, do not fit the general rules given in (24) and (16), respectively. However, they do so if only the zero frequency mode propagates in the main loop, as in the case under consideration. In particular, the first two-loop diagram in the table comes with a combinatorial factor of 1/8, whilst eq. (24) gives a factor of 1/4. The extra factor of 2 comes from the symmetry of the diagram, since both bubbles are equivalent and the propagation of the zero-frequency mode can be considered for each of the two. The second two-loop diagram in the table comes with a combinatorial factor of 1/12, whilst again 1/4 is required by eq. (24). The extra factor of 3 comes again from the symmetry of the diagram, since the three propagators are equivalent and each of them can carry the zero-frequency mode.

We can now study the effective potential at the level of the daisy approximation, eq. (21). In particular, we can study the effects of the $m^3$ term in the high-temperature expansion of eq. (9), which hints at the possibility of a first-order phase transition. The simplest improvement over the naïve one-loop effective potential consists in keeping only the leading term in the high-temperature expansion of the self-energy, i.e. to take $\Pi(m^2) = (\lambda/4)T^2$, as was done in ref. [12]. In that case the effective mass to be substituted in eq. (21) reads

$$\overline{m}^2(\phi, T) = m^2(\phi) + \frac{\lambda}{4} T^2,$$

(26)

\(^3\)We believe that this is the origin of the apparent discrepancy found in ref. [13]. This discrepancy disappears when all diagrams are included.
which becomes $3\lambda\phi^2$ at $T_0 = 2\nu$. At that temperature there is a $\phi^3$ term (with negative coefficient) in the high-temperature expansion of (21), while the coefficient of the $\phi^2$ term vanishes. This is the signal of the onset of a first-order phase transition: for $T$ just above $T_0$, the potential has a positive curvature at $\phi = 0$, but the effect of the $\phi^3$ term bends down the potential at moderate values of $\phi$. We show the effective potential corresponding to this approximation in fig. 1b, and the values of the effective mass (26) in fig. 4a.

To further improve over the approximation (26), we could try to perform in eq. (21) the full shift defined by eq. (14). However, in the calculation of $\Pi[m^2(\phi), T]$ we still have to face the problem of negative squared masses, which does not allow us to explore $\phi^2 < \nu^2/3$. Disregarding this problem for a moment, we can look at the behaviour of the $m^3$ term in the high-temperature expansion (9) of $V^{(1)}$. Performing the shift $m^2 \to \overline{m}^2$, with $\overline{m}^2$ given by (14), the $m^3$ term in $V^{(1)}$ would become

$$-rac{T}{12\pi} \left[ m^2 + \Pi(m^2) \right]^{3/2},$$

which would give rise, in the high-temperature expansion

$$\Pi[m^2(\phi), T] = \frac{\lambda}{4} T^2 - \frac{3\lambda}{4\pi} mT - \frac{9\lambda^2}{4\pi} \frac{\phi^2 T}{m} + \ldots,$$

(28)

to a term linear in $m$. This linear term originates from subleading contributions to the self-energy: similar terms have recently been obtained in the SM effective potential [10,11], and will be discussed in that context in our forthcoming paper. Both in the scalar theory and in the SM, the linear term is an artefact of the resummation of daisy diagrams. As discussed in more detail later, when including subleading terms in the infrared improvement of the effective potential, we should also include the so-called superdaisy diagrams [3]. Including the superdaisy diagrams amounts to solving the gap equation

$$m^2(\phi, T) = m^2(\phi) + \Pi[\overline{m}^2(\phi, T), T],$$

(29)

and to substitute its solution in eq. (21).

Before proceeding, a few words about the expansion parameters corresponding to the different approximations are in order. The leading behaviour of two- and three-loop corrections to the zero-frequency part of the one-loop effective potential is presented in the table. The reader can easily extend the classification to higher loop orders. The relevant parameters are [17]

$$\alpha \equiv \lambda \frac{T^2}{m^2}, \quad \beta \equiv \frac{\lambda}{m} T.$$

(30)

Subleading corrections have an extra suppression factor $(\beta/\alpha)$ with respect to the leading behaviour. Since we are working at arbitrary values of $\phi$, we must also introduce

$$\gamma \equiv \frac{\phi^2}{T^2},$$

(31)

which comes from the trilinear vertices. The validity of the (unimproved) loop expansion requires not only $\lambda < 1$, but also the additional condition $\alpha < 1$. For values of $\phi$ and $T$ that are relevant for the description of the phase transition, $\alpha \gtrsim 1$ and the perturbative expansion in $\alpha$ breaks down [13,17], but even in this case $\beta$ can still be a small expansion parameter. The daisy

---

4Independent arguments for the absence of linear terms, based on effective-theory methods, have been given in [13].
resummation amounts to replacing $m^2$ by $\overline{m}^2$, given by eq. (31), in the perturbative expansion, in which case $\bar{\alpha} \equiv \lambda T^2 / \overline{m}^2 \ll 1$ in the region of interest for the phase transition, while $\bar{\beta}$ may remain as a good expansion parameter. As for the parameter $\gamma$ in eq. (31), if we consider values of $\phi$ such that $\phi \ll T$, then $\gamma \ll 1$ and diagrams proportional to $\gamma^n$ $(n \geq 1)$ can be neglected. Here we first assume that $\bar{\alpha} \bar{\beta} \bar{\gamma} \ll 1$ (as could be the case in the region of interest for the study of the phase transition), so that they can be consistently neglected in the resummation procedure. The latter prescription amounts to dropping the $\Pi^{(b)}$ term in the second member of the gap equation (29). From the table we can see that diagrams not belonging to the daisy or superdaisy classes are suppressed by factors of order at least $\bar{\beta}^2$, so that we can keep consistently in the gap equation terms of order $\bar{\beta}$, i.e. subleading terms from $\Pi^{(a)}$. Keeping in general the subleading terms in the self-energy (28) amounts to working to $O(\bar{\beta})$, as we are doing in this paper, whereas the approximation of eq. (26) amounts to working to $O(\bar{\beta}^0)$. Working to $O(\bar{\beta})$ is necessary to include linear terms in the self-energy, and it is the best one can afford without considering the non-daisy, non-superdaisy diagrams (see table). The validity of both approximations relies upon the condition $\bar{\beta} < 1$.

To obtain an approximate analytical solution to the gap equation (29), we can write its high-temperature expansion (it is not restrictive to assume $\overline{m} \geq 0$)

$$\overline{m}_0^2(\phi, T) = m^2(\phi) + \frac{\lambda}{4} T^2 - \frac{3\lambda T}{4\pi} \overline{m}_0(\phi, T),$$

where the zero subscript reminds us that we are neglecting the $\phi^2$ term in (28). Eq. (32) admits the obvious solution, valid up to $O(\bar{\beta}^2)$ corrections,

$$\overline{m}_0(\phi, T) = - \frac{3\lambda T}{8\pi} + \sqrt{\left(\frac{3\lambda T}{8\pi}\right)^2 + \frac{\lambda T^2}{4} + m^2(\phi)}.$$  \hspace{1cm} (33)

At $T \geq T_0$, $\overline{m}_0(0, T) \geq 0$, while at $T < T_0$ there is no solution for values of $\phi$ such that $\phi^2 < (\nu^2 - T^2/4)/3$. From the explicit solution (33) we can see that $\overline{m}_0$ is an analytic function of $m^2$ near $m^2 = 0$, in contrast with the daisy approximation (14), so that there can be no terms linear in $m(\phi)$ in the effective potential. Therefore, as anticipated, the absence of linear terms from the improved one-loop effective potential (at subleading order) is ensured by the consistent inclusion of the superdaisy infrared-dominated diagrams.

The exact solution to the gap equation (29), with $\Pi = \Pi^{(a)}$, is plotted in fig. 4b [the approximation (33) gives results which are indistinguishable on the scale of fig. 4b], for different temperatures close to the critical one. We also display in fig. 4c the corresponding values of the parameters $\bar{\alpha}/(2\pi)$ (dashed lines) and $\bar{\beta}/(2\pi)$ (solid lines). They give indications on the values of $\phi$ and $T$ at which our (superdaisy-improved) perturbative expansion breaks down. The resulting effective potential (21) is displayed in fig. 1c.

A word of caution should be said here, since the validity of our last result relies upon that of the $\bar{\beta}$ expansion. As shown in fig. 4c, $\bar{\beta}/(2\pi)$ is of order one for the values of $T$ and $\phi$ associated to the apparent first-order structure. Furthermore, we have also plotted in fig. 4c the parameter $\bar{\alpha} \bar{\beta} \bar{\gamma}$ (dash-dotted lines), which measures how much the diagrams involving trilinear couplings, and so the $\Pi^{(b)}$ term in eq. (29), are negligible. We can see from fig. 4c that the condition $\bar{\alpha} \bar{\beta} \bar{\gamma} \ll 1$ is not fulfilled in the relevant region, this not being unrelated to the fact that also the condition $\bar{\beta} \ll 1$ is not fulfilled in that region.

\footnote{This is the case of diagrams involving trilinear couplings. These diagrams were not considered in previous calculations, performed at $\phi = 0$.}
Although we know, according to our previous arguments, that consistency of the superdaisy approximation, which is an expansion in the parameter $\bar{\beta}$ neglecting $O(\bar{\beta}^2)$ corrections, requires $\Pi^{(b)}$ not to be relevant in the gap equation (29), we now include it for the sake of comparison. We are then led to solving the gap equation

$$m^2(\phi, T) = m^2(\phi) + \frac{\lambda}{4} T^2 - \frac{3\lambda T}{4\pi} \bar{m}(\phi, T) - \frac{9\lambda^2}{4\pi} \frac{\phi^2 T}{m(\phi, T)},$$

(34)

where a high-temperature expansion has already been performed. Since our expansions hold up to corrections $O(\bar{\beta}^2)$, we can consistently give the analytic solution of eq. (34), to order $\bar{\beta}$, as

$$m^2(\phi, T) = m_0^2(\phi, T) - \frac{9\lambda^2}{4\pi} \frac{\phi^2 T}{\sqrt{m_0^2(\phi) + \frac{\lambda T}{4}}},$$

(35)

where $m_0(\phi, T)$ is the solution (33). The solution (35) to the gap equation is plotted in fig. 4d, and the resulting effective potential is displayed in fig. 1d. From eq. (35) we can see, as was the case from eq. (33), that the absence of terms linear in $m(\phi)$ from the effective potential follows from the resummation of the superdaisy diagrams. From fig. 4d we can see that the gap equation (34) does not have a real solution for a range of $\phi$ values which depends on the temperature $T$. Inside this range the solution (34) becomes purely imaginary, and so the $m^3$ term of the effective potential becomes purely imaginary as well. This situation was discussed in general in [19], where it was shown that the imaginary part of the effective potential has a natural interpretation as a decay rate per unit volume of a well-defined quantum state. We shall disregard the contribution from those imaginary-frequency modes and plot the real part of the effective potential in fig. 1d. Solid (dash-dotted) lines correspond to the region where the gap equation does (does not) have a real solution and the effective potential does not (does) get an imaginary part. Fig. 1d hints, as was the case in the approximation of fig. 1c, at a very weakly first-order phase transition. However, this structure appears again in the region where even our improved perturbative expansion is not reliable, as one can easily read from fig. 4c. Therefore we can only conclude that the phase transition is either second-order or extremely weakly first-order. Our findings are in agreement with general results based on the $\epsilon$-expansion [20], which suggest second-order phase transitions in theories without an asymptotically-free coupling.

In conclusion, we have analysed the structure of the one-loop effective potential for the scalar theory, when higher-order infrared-dominated diagrams are resummed to leading and subleading order. We have explicitly shown that the combinatorics of higher-loop diagrams is appropriate to shift the value of $m^2(\phi)$ in the zero-frequency part of the one-loop effective potential. We have identified the parameters of the improved expansion, and proved that including subleading terms in the self-energy is not consistent with keeping only the daisy class of diagrams: such a procedure would produce spurious linear terms in $m(\phi)$ in the effective potential. Consistency at subleading order requires introducing superdaisy diagrams, which prevent the appearance of linear terms. In the scalar theory, the improved effective potential at subleading order suggests, within the limits of the improved perturbative expansion, a phase transition which is either second-order or very weakly first-order.

### Acknowledgements

We acknowledge discussions with T. Altherr, A. Brignole, J. Orloff, A. Rebhan and M.E. Shaposhnikov.
References

1. D.A. Kirzhnits and A.D. Linde, Phys. Lett. 72B (1972) 471.

2. S. Weinberg, Phys. Rev. D9 (1974) 3357.

3. L. Dolan and R. Jackiw, Phys. Rev. D9 (1974) 3320.

4. D.J. Gross, R.D. Pisarski and L.G. Yaffe, Rev. Mod. Phys. 53 (1981) 43;
   J.I. Kapusta, *Finite-temperature Field Theory* (Cambridge University Press, 1989);
   A.D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood, New York, 1990).

5. F.R. Klinkhammer and N.S. Manton, Phys. Rev. D30 (1984) 2212;
   V.A. Kuzmin, V.A. Rubakov and M.E. Shaposhnikov, Phys. Lett. B155 (1985) 36;
   P. Arnold and L. McLerran, Phys. Rev. D36 (1987) 581 and D37 (1988) 1020;
   M. Dine, O. Lechtenfeld, B. Sakita, W. Fischler and J. Polchinski, Nucl. Phys. B342 (1990) 381.

6. A.D. Dolgov, Kyoto preprint YITP/K-940 (1991);
   M.E. Shaposhnikov, preprint CERN-TH.6304/91;
   M. Dine, Santa Cruz preprint SCIPP 91/27.

7. M.E. Shaposhnikov, JETP Lett. 44 (1986) 465, Nucl. Phys. B287 (1987) 757 and B299 (1988) 797;
   A.I. Bochkarev, S.Yu. Khlebnikov and M.E. Shaposhnikov, Nucl. Phys. B329 (1990) 490;
   A.I. Bochkarev, S. Kuzmin and M.E. Shaposhnikov, Phys. Lett. B244 (1990) 27.

8. M. Fukugita and T. Yanagida, Phys. Rev. D42 (1990) 1285;
   J.A. Harvey and M.S. Turner, Phys. Rev. D42 (1990) 3344;
   A.E. Nelson and S.M. Barr, Phys. Lett. B246 (1991) 141;
   B.A. Campbell, S. Davidson, J. Ellis and K.A. Olive, Phys. Lett B256 (1991) 457;
   W. Fishler, G.F. Giudice, R.G. Leigh and S. Paban, Phys. Lett. B258 (1991) 45;
   L.E. Ibáñez and F. Quevedo, preprint CERN-TH.6433/92;
   H. Dreiner and G.G. Ross, Oxford preprint OUP-T-92-08P.

9. G.W. Anderson and L.J. Hall, Phys. Rev. D45 (1992) 2685.

10. M.E. Shaposhnikov, Phys. Lett. B277 (1992) 324.

11. D.E. Brahm and S.D.H. Hsu, Caltech preprints CALT-68-1762, HUTP-91-A064 and CALT-68-1705, HUTP-91-A063.

12. M.E. Carrington, Phys. Rev. D45 (1992) 2933.

13. M. Dine, R.G. Leigh, P. Huet, A. Linde and D. Linde, Santa Cruz preprint SCIPP-92-06, SLAC-PUB-5740, SU-ITP-92-6 and SLAC preprint SLAC-PUB-5741, SCIPP-92-07, SU-ITP-92-7.

14. P. Arnold, Univ. of Washington preprint UW/PT-92-06, NUHEP-TH-92-06.
15. J. Iliopoulos, C. Itzykson and A. Martin, Rev. Mod. Phys. 47 (1975) 165, and references therein;
S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888;
S. Weinberg, Phys. Rev. D7 (1973) 2887;
R. Jackiw, Phys. Rev. D9 (1974) 1686.

16. W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D18 (1978) 3998.

17. P. Fendley, Phys. Lett. B196 (1987) 175.

18. K. Takahashi, Z. Phys. C26 (1985) 601;
T. Altherr, Phys. Lett. B238 (1990) 360;
N. Banerjee and S. Mallik, Phys. Rev. D43 (1991) 3368;
R.R. Parwani, Stony Brook preprint ITP-SB-91-64;
V. Jain, Max-Planck-Institute preprint MPI-Ph/92-41.

19. E.J. Weinberg and A. Wu, Phys. Rev. D36 (1987) 2474.

20. P. Ginsparg, Nucl. Phys. B170 (1980) 388.

**Table caption**

Leading infrared contributions to $V_{\text{eff}}/T^4$, in units of $m^3/T^3$, for two- and three-loop vacuum
diagrams. Big bubbles correspond to loops where only zero-frequency modes are considered.

**Figure captions**

Fig.1: The temperature-dependent effective potential for $\lambda = 0.1$, $v = 250$ GeV and some
representative values of the temperature $T$. Case (a) corresponds to the approximation of
eq. (10); case (b) to that of eqs. (21), (22) and (26); case (c) corresponds to the solution
of the gap equation (29), with $\Pi = \Pi^{(a)}$; case (d) corresponds to the solution (33) of the
gap equation (14): when $\overline{m}(\phi, T)$ becomes purely imaginary, the dash-dotted line gives
the real part of $V_{\text{eff}}(\phi, T)$. In all cases, the potential is (arbitrarily) normalized to zero at
the origin.

Fig.2: One-loop self-energy diagrams.

Fig.3: Tadpole (a–d) and vacuum (e) daisy diagrams.

Fig.4: Solutions of the gap equation and expansion parameters for the same choice of parameters
as in fig. 1: a) the approximate solution (29); b) the exact solution of the gap equation (29),
with $\Pi = \Pi^{(a)}$; c) the expansion parameters $\alpha(\phi, T)/(2\pi)$ (dashed lines), $\beta(\phi, T)/(2\pi)$
(solid lines) and $\bar{\alpha}\beta\gamma$ (dash-dotted lines) corresponding to case b); d) the solution (35)
of the gap equation (14). In a), b) and d), the solid lines correspond to $T = 490, 495$ GeV,
the dashed lines to $T_0 = 500$ GeV, the dash-dotted lines to $T = 505, 510$ GeV. In c), for
each of the three parameters, the temperature $T$ increases by 5 GeV steps going from the
right curves to the left ones.