A rigid cone in the truth-table degrees with jump

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Abstract

The automorphism group of the truth-table degrees with order and jump is fixed on the set of degrees above the fourth jump, $0^{(4)}$.

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1 Introduction

A cone in a partial order \((D, \leq)\) is a set of the form \(D(\geq a) := \{x \in D : x \geq a\}\) for some \(a \in D\). A subset of \(S\) of \(D\) is rigid if it is fixed under the action of the automorphism group \(\text{Aut}(D, \leq)\), i.e., for each \(x \in S\) and each \(\pi \in \text{Aut}(D, \leq)\), \(\pi(x) = x\). We will also be interested in the case of structures \((D, \leq, j)\) where \(j\) is a unary function on \(D\). In that case, rigidity of \(S \subseteq D\) is defined with respect to \(\text{Aut}(D, \leq, j)\) rather than \(\text{Aut}(D, \leq)\).

It is not known whether the structure of the Turing degrees is rigid, but it is known [JS] that the structure of the Turing degrees with jump contains a rigid cone. This is shown by applying a jump inversion theorem and results on initial segments. Here we show that also the structure of truth-table degrees with jump \((D_{tt}, \leq, j)\) contains a rigid cone. For definitions relating to initial segments we refer the reader to the author’s doctoral dissertation [KH2], survey article [KH1], and forthcoming article [KH3].

Our main result is that each automorphism of the truth-table degrees with jump is equal to the identity on the cone above \(0^{(4)}\). This contrasts with the results of Anderson [A] that each automorphism of the truth-table degrees (not necessarily jump invariant) is equal to the identity on some cone, and each automorphism that preserves \(0^{(3)}\) and \(0^{(5)}\) is equal to the identity on the cone above \(0^{(5)}\). It is still open whether non-trivial automorphisms of these structures exist at all.

2 Steps of the proof

In this section we describe the global structure of the proof of our main theorem 2.8; further recursion-theoretic and lattice-theoretic details are given in the subsequent sections.

Definition 2.1. In the \(tt\)-degrees we denote the order by \(\leq\). If \(x, y\) are \(tt\)-degrees, we say that \(x \equiv_T y\) if for some \(X \in x\) and \(Y \in y\), we have \(X \equiv_T Y\).

The following theorem is due to Mohrherr [M3].

Theorem 2.2. Let \(n \geq 1\) and \(a \geq 0^{(n)}\). Then for some \(b, a = b^{(n)}\).

Proposition 2.3. For each \(g\), \([0, g]\) is \(\Sigma_3^0(g)\)-presentable.

Proof. An analysis of the definition of \(tt\)-reducibility. \(\square\)

Corollary 2.4. Each upper semilattices with least and greatest element that can be realized as initial segments \([0, g]\) with \(g^{(2)} \leq y^{(3)}\) is \(\Sigma_3^0(y)\)-presentable.

Theorem 2.5. For any \(y\), the upper semilattices with least and greatest element that can be realized as initial segments \([0, g]\) with \(g^{(2)} \leq y^{(3)}\) are exactly the \(\Sigma_3^0(y)\)-presentable ones.
Proof. By Corollary 2.4 and Theorem 4.4.

Theorem 2.6. Let \( \pi \) be an automorphism of the truth-table degrees with jump and let \( x \geq 0^{(3)} \). Then \( \pi(x) \equiv_T x \).

Proof. By Theorem 2.2 there is a \( y \) such that \( x = y^{(3)} \). The initial segments \([0, y']\) and \([0, \pi(y')]\) are jump-isomorphic via \( \pi \), so by Theorem 2.5 the \( \Sigma^0_4(y)\)- and \( \Sigma^0_4(\pi(y))\)-presentable bounded usls coincide. Hence by Proposition 5.8,

\[
\pi(y)^{(3)} \equiv_T y^{(3)}
\]

and so

\[
\pi(x) = \pi(y^{(3)}) = \pi(y)^{(3)} \equiv_T y^{(3)} = x.
\]

Lemma 2.7. \( a \equiv_T b \Rightarrow a' = b' \).

Theorem 2.8. Let \( \pi \) be an automorphism of the truth-table degrees with jump and let \( x \geq 0^{(4)} \). Then \( \pi(x) = x \).

Proof. By Theorem 2.2 there is a \( y \) such that \( x = y^{(4)} \). Let \( z = y^{(3)} \), so \( x = z' \) and \( z \geq 0^{(3)} \). By Theorem 2.6 \( \pi(z) \equiv_T z \) and by Lemma 2.7 \( a \equiv_T b \Rightarrow a' = b' \).

Hence

\[
\pi(x) = \pi(z') = \pi(z) = z' = x.
\]

3 Mal’tsev homogeneous lattice tables

If \((L, \leq)\) is a partial order (transitive, reflexive, antisymmetric relation) such that greatest lower bounds \( \alpha \wedge \beta \) of all \( \alpha, \beta \in L \) exist then \((L, \leq, \wedge)\) is called a lower semilattice; if least upper bounds \( \alpha \vee \beta \) of all pairs \( \alpha, \beta \in L \) exist, then \((L, \leq, \vee)\) is called an upper semilattice (usl). If \( L \) is both an lower semilattice and an upper semilattice then \( L \) is a lattice. \( L \) is called bounded if there exist elements \( 0, 1 \in L \) such that for all \( \alpha \in L \), \( 0 \leq \alpha \leq 1 \). In particular every finite lattice is bounded. If \( L \) has more than one element (so in the bounded case, \( 0 \neq 1 \)) then we say that \( L \) is nontrivial. A unary algebra is a collection of functions \( f : X \to X \) on a set \( X \), closed under composition. The partition lattice \( \text{Part}(X) \) on a set \( X \) consists of all equivalence relations (considered as sets of ordered pairs) on \( X \), ordered by inclusion. We will be interested in the case where \( X \) is finite or countably infinite.

A lattice table (see \[L\]) \( \Theta \) is (1) a set \( X \) together with (2) a finite set of equivalence relations \( \alpha_1, \ldots, \alpha_n \) on \( X \), and (3) an order \( \leq \) given by \( \alpha_i \leq \alpha_j \leftrightarrow \alpha_i \supseteq \alpha_j \) (reverse inclusion of sets of ordered pairs), such that \( \{\alpha_1, \ldots, \alpha_n\} \) ordered by inclusion is a 0-1 sublattice of \( \text{Part}(X) \). We write \( \hat{\Theta} = \{\alpha_1, \ldots, \alpha_n\} \). We think of \( \Theta \) as equal to \( X \), but endowed with additional structure. So \( x \in \Theta \) means
$x \in X$, etc. but for emphasis we may write $|\Theta|$ for $X$. Note that $\Theta$ is determined by $\widehat{\Theta}$.

Elements of $|\Theta|$ are denoted by lower-case Roman letters such as $u, v, w, x, y, z$, and elements of semilattices in general and $\widehat{\Theta}$ in particular by lower-case Greek letters such as $\alpha, \beta, \gamma$.

If $\alpha \in \widehat{\Theta}$ and $(x, y) \in \alpha$ then we write $x \sim_\alpha y$. If $\Theta$ is a lattice table then an endomorphism of $\Theta$ is a map from $\Theta$ to $\Theta$ preserving all equivalence relations in $\widehat{\Theta}$. That is, $(\forall x, y \in \Theta)(\forall \alpha \in \widehat{\Theta})(x \sim_\alpha y \rightarrow f(x) \sim_\alpha f(y))$. End $\Theta$ denotes the unary algebra consisting of all endomorphisms of $\Theta$.

$C_{\Theta}(x, y)$ denotes the principal equivalence relation in $\Theta$ generated by $(x, y)$, i.e.

$$C_{\Theta}(x, y) = \cap\{\alpha \in \widehat{\Theta} : (x, y) \in \alpha\}.$$

We define $\text{End}_{\Theta}(x, y)$ to be the principal congruence relation in $\Theta$ generated by $(x, y)$, i.e. the equivalence relation generated by all pairs $(f(x), f(y))$ for $f \in \text{End} \Theta$.

**Lemma 3.1.** $\text{End}_{\Theta}(x, y) \subseteq C_{\Theta}(x, y)$.

**Proof.** If $(u, v) \in \text{End}_{\Theta}(x, y)$ then $(c, d)$ is in the transitive closure of

$$\{(f(x), f(y)) \mid f \in \text{End} \Theta\},$$

so it suffices to show each such $(f(x), f(y)) \in C_{\Theta}(x, y)$. For this it suffices to show $(f(x), f(y)) \in \alpha$ provided that $(x, y) \in \alpha$ for $\alpha \in \widehat{\Theta}$; this holds since $f \in \text{End} \Theta$.

**Definition 3.2.** Let $\Theta$ be a lattice table. We say that $\Theta$ is Mal’tsev homogeneous if for all $x, y, u, v \in \Theta$ satisfying

$$(\forall \alpha \in \widehat{\Theta})(x \sim_\alpha y \rightarrow u \sim_\alpha v),$$

there exist $n \in \omega = \{0, 1, 2, \ldots\}, z_1, \ldots, z_n \in \Theta$ and $f_0, \ldots, f_n \in \text{End} \Theta$ such that

$$(\forall i \leq n)(\{f_i(x), f_i(y)\} = \{z_i, z_{i+1}\})$$

where $z_0 = u$ and $z_{n+1} = v$.

The $z_i$ are called homogeneity interpolants.

This notion of homogeneity is more general (weaker) than those considered in [L].

Note that if $\alpha \wedge \beta = \gamma$ in $\widehat{\Theta}$ then $\alpha$ and $\beta$ generate $\gamma$. That is, if $x \sim_\gamma y$ then there exist meet interpolants $z_1, \ldots, z_n$ for $x, y$ such that $x \sim_\alpha z_1 \sim_\beta z_2 \cdots \sim_\alpha z_n \sim_\beta y$.
Definition 3.4. If Θ is a lattice table and Y ⊆ |Θ|, then for each α ∈ ̂Θ, α ∣ Y = {(x, y) ∈ Y × Y : (x, y) ∈ α}. Let ̂Θ ∣ Y = {α ∣ Y : α ∈ ̂Θ}.

If Θ₀ and Θ₁ are lattice tables, then we say that Θ₀ ≤ Θ₁ if |Θ₀| ⊆ |Θ₁| and ̂Θ₁ ∣ |Θ₀| = ̂Θ₀. Note that if Θ₀ ≤ Θ₁ then ̂Θ₀ and ̂Θ₁ are isomorphic (nontrivial, finite) lattices.

If Θₙ, n ∈ ω are lattice tables such that Θₙ ≤ Θₙ₊₁ for each n, then ∪ₙ∈ω Θₙ is the lattice table Θ such that |Θ| = ∪ₙ∈ω |Θₙ| and ̂Θ ∣ |Θₙ| = ̂Θₙ for each n.

In particular Θₙ ⊆ Θ and ̂Θₙ and ̂Θ are isomorphic lattices for each n.

Definition 3.5. Θ is a sequential lattice table if there exist Θₙ, n ∈ ω, such that Θ = ∪ₙ∈ω Θₙ, and

1. each Θₙ is a (0, 1, ∨)-substructure of Part (|Θₙ|) (Θₙ is an usl table),
2. Θ is a lattice table, and
3. for each n, meet interpolants for elements of Θₙ exist in Θₙ₊₁.

Then Θ is a sequential Mal’tsev homogeneous lattice table if in addition

4. Θ is Mal’tsev homogeneous, with homogeneity interpolants for elements of Θₙ appearing in Θₙ₊₁ (compare VII.1.1, 1.3 of [U]).

Definition 3.6 (Direct limit). Let a sequence (Lᵢ, φᵢ)ᵢ∈ω be given, where each Lᵢ is a finite lattice, φᵢ : Lᵢ → Lᵢ₊₁ is a (0, 1, ∨) homomorphism, and Lᵢ ∩ Lⱼ = ∅ for i ≠ j.

Let L' = ∪ᵢ∈ω Lᵢ as a set. Let ≈ be the equivalence relation on L' generated by a ≈ φᵢ(a) for a ∈ Lᵢ. Then L = L'/ ≈ is an upper semilattice called the direct limit of the sequence (Lᵢ, φᵢ)ᵢ∈ω.

Definition 3.7. Fix finite lattices L⁰, L¹ and a (0, 1, ∨) homomorphism φ : L⁰ → L² ⊆ L¹, and lattice tables Θ⁰, Θ¹. Suppose Ψᵢ : Lᵢ → ̂Θᵢ, i = 0, 1, are isomorphisms. For a ∈ Lᵢ, we write ∼ₐ for ∼ₜₐ.

We say that Θ¹ embeds in Θ⁰ with respect to φ and Ψ₀, Ψ₁ if there is a function Θ(φ) : Θ¹ → Θ⁰ such that for all x, y ∈ Θ¹, and all a ∈ L⁰,

\[ x ∼ₐ y ⇔ Θ(φ)(x) ∼ₐ Θ(φ)(y). \]

In the characterization of intervals |0, g| for g < o', Definition 3.7 plays a key role which we will now describe.

Suppose a bounded countable upper semilattice L is given such that the ordering ≤ of L is computably enumerable but not necessarily computable. That is, there is a computable sequence that consists of all pairs (α, β) such that α ≤ β, but if a given pair (α, β) does not appear anywhere in the list then this cannot be determined effectively.
For reasons whose explanation would take us too far afield (but see [KH1]), we need a computable sequence of lattice tables \( \Theta^0, \Theta^1, \ldots \) such that \( \hat{\Theta}^s \) is isomorphic to our approximation to \( L \) at stage \( s \). (We will start with a sequence \( (L^i, \varphi_i)_{i \in \omega} \) having \( L \) as direct limit, and our approximation to \( L \) at stage \( s \) will be \( L^s \).) Suppose we discover at stage \( s + 1 \) that \( \alpha \leq \beta \), whereas at stage \( s \) we knew that \( \beta \leq \alpha \) but thought that \( \alpha \nleq \beta \). Further suppose that we cannot ignore what was done using \( \Theta^s \) at stage \( s \), but we can let \( \Theta^{s+1} \) be a subset of \( \Theta^s \). If \( \Theta^{s+1} \) embeds into \( \Theta^s \) with respect to \( \varphi \) (the homomorphism mapping our approximation to \( L \) at stage \( s \) to our approximation to \( L \) at stage \( s + 1 \)) then by thinning \( \Theta^0 \) to \( \Theta(\varphi)\Theta^1 \), we eliminate all elements \( x, y \) that are witnesses to the fact that \( \alpha \neq \beta \). This allows us to identify \( \alpha \) and \( \beta \), even though so far we have been working under the assumption that \( \alpha \neq \beta \).

We mention for the reader who is a computability theorist that in the characterization of lattices isomorphic to \([0, g]\) for \( g < 0' \), the ordering of \( L \) is computably enumerable only relative to the Turing degree \( 0'' \), and the “stages” above are really levels of a priority tree, the true path of which it requires \( 0'' \) to identify.

The full result needed for the application to the Turing degrees is contained in Theorem 3.8 and Proposition 3.9.

**Theorem 3.8.** Let \( L \) be a bounded countable nontrivial usl and let \( (L^i, \varphi_i)_{i \in \omega} \) be any system of nontrivial finite lattices having \( L \) as direct limit in the sense of Definition 3.6. Then there exists

1. a function \( h : \omega \to \omega \),
2. a double sequence of finite lattice tables \( (\Theta^i_j)_{i,j \in \omega} \) with \( \Theta^i_j \subseteq \Theta^i_{j+1} \) for each \( i \in \omega, j \geq h(i) \), and
3. for each \( i \in \omega \) an increasing function \( m_i : \omega \to \omega \) with \( m_i(0) = h(i) \), such that
   1. letting \( \Theta^i = \bigcup_{j \in \omega} \Theta^i_j \), we have \( |\Theta^i| \geq |\Theta^{i+1}| \) for each \( i \in \omega \),
   2. for each \( i, j \geq h(i) \) and \( k \) such that \( m_i(j) \leq k < m_i(j + 1) \), we have
      \[ \Theta^i_k = \Theta^{m_i(j)}_{m_i(j)} \],
3. for each \( i \in \omega \), \( (\Theta^i_{m_i(j)})_{j \in \omega} \) is a sequential Mal’tsev homogeneous lattice table,
4. for each \( i \in \omega \), \( \hat{\Theta}_i \) is isomorphic to \( L^i \), and
5. there exist isomorphisms \( \Psi_i : L^i \to \hat{\Theta}_i \) such that \( \Theta^{i+1} \) embeds in \( \Theta^i \) with respect to \( \varphi_i \) and \( \Psi_i, \Psi_{i+1} \), and the embedding is the identity map. In other words, for all \( x, y \in \Theta^{i+1} \) and \( \alpha \in L^i \), we have
   \[ x \sim_{\Psi_i \alpha} y \iff x \sim_{\Psi_{i+1} \varphi_i \alpha} y. \]
The essential property in Theorem 3.8 and the one that goes beyond those of [LL], is (5). The following Proposition can be proved by inspecting the proof of Theorem 3.8.

**Proposition 3.9** (Computability-theoretic properties). Let $a$ be a Turing degree and let $L$ be a $\Sigma^0_1(a)$-presentable usl, as in [KH1]. Then in Theorem 3.8 we may assume that $h$ is $a$-computable; the array $\{\Theta^i_j \mid j \geq h(i)\}$ is $a$-computable; each $m_i$ is computable; for each $i < \omega, (\Theta^i_{m_i(j)})_{j \in \omega}$ is a computable sequence; each $\Theta^i$ is computable; and there is a computable function taking $L^0, \ldots, L^i$ to $\Theta^i$.

We now begin the development that will lead to a proof of Theorem 3.8. If $A$ is a unary algebra then $\text{Con } A$ denotes the congruence lattice of $A$, i.e. the lattice of all equivalence relations $E$ on $X$ preserved by all $f \in A$, ordered by inclusion.

The following observation can be traced back to Mal’tsev [G1, M1, M2].

**Proposition 3.10.** For any unary algebra $A$, the dual of $\text{Con } A$ is a Mal’tsev homogeneous lattice table.

**Proof.** Suppose $A$ is a unary algebra on a set $X$. Let $\Theta$ be the lattice table such that $\hat{\Theta}$ is the dual of $\text{Con } A$. Since $\text{Con } A$ is a 0-1 sublattice of $\text{Part}(A)$, $\Theta$ is a lattice table.

If $f$ is an operation in $A$ and $\alpha \in \hat{\Theta}$ then $\alpha$ is a congruence relation on $A$ and hence $\forall x, y(x \sim_\alpha y \rightarrow f(x) \sim_\alpha f(y))$, which means that $f \in \text{End } \Theta$. So

$$A \subseteq \text{End } \Theta.$$ 

Clearly for any unary algebras $A, B$ on the same underlying set, we have $A \subseteq B \Rightarrow \text{Con } A \supseteq \text{Con } B$. Hence Con End $\Theta \subseteq \text{Con } A = \hat{\Theta}$.

If $u, v, x, y \in X$, $f \in \text{End } \Theta$ and $(x, y) \in \text{End}_\Theta(u, v)$ then there exist $z_1, \ldots, z_k$ such that $(z_{i}, z_{i+1}) = (g_i(u), g_i(v))$ for $g_i \in \text{End } \Theta$ with $z_1 = x, z_k = y$, hence letting $w_i = f(z_i)$ and $h_i = f \circ g_i$ we have $(w_i, w_{i+1}) = (h_i(u), h_i(v)) \in \text{End}_\Theta(u, v)$, $w_1 = f(x), w_k = f(y)$, and so $(f(x), f(y)) \in \text{End}_\Theta(u, v)$. Hence we have shown $\text{End}_\Theta(u, v) \in \text{Con } \text{End } \Theta \subseteq \Theta$. Since End $\Theta$ contains the identity map, $\text{End}_\Theta(u, v)$ contains $(u, v)$. Hence $\text{End}_\Theta(u, v)$ is in $\Theta$ and contains $(u, v)$, so it contains $C_\Theta(u, v)$. So $\Theta$ is Mal’tsev homogeneous.

We recall the construction of [P].

**Definition 3.11.** Let $L$ be a nontrivial lattice. $A = (A, r, h)$ is called an $L - \{1\}$-colored graph if $A$ is a set, $r$ is a set of size-two subsets of $A$, i.e. $(A, r)$ is an undirected graph without loops, and $h : r \rightarrow L - \{1\}$ is a mapping of the set $r$ of the edges of the graph into $L - \{1\}$.

The map $e : L \rightarrow \text{Part}(A)$ is defined by: for $\alpha \in L, e(\alpha)$ is the equivalence relation on $A$ generated by identifying points $x, y$ if there is a path from $x$ to $y$. 

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in the graph consisting of edges all of which have color $\geq \alpha$. In this case we say that $x, y$ are connected with color $\geq \alpha$.

**Definition 3.12** ($\alpha$-cells). Let $L$ be a nontrivial lattice and let $\alpha \in L - \{1\}$. An $\alpha$-cell $B_\alpha = (B_\alpha, s_\alpha, k_\alpha)$ is an $L - \{1\}$-colored graph consisting of (1) a base edge $\{x, y\}$ colored $\alpha$, and (2) for each pair $(\alpha_1, \alpha_2)$ of elements of $L$ such that $\alpha_1 \land \alpha_2 \leq \alpha$, a chain of edges $\{x, u_1\}, \{u_1, u_2\}, \{u_2, u_3\}, \{u_3, y\}$, colored $\alpha_1, \alpha_2, \alpha_1, \alpha_2$, respectively. Here $x, y, u_1, u_2, u_3$ are distinct elements of $B_\alpha$. The base edge and chain of edges corresponding to a particular inequality $\alpha_1 \land \alpha_2 \leq \alpha$ is referred to as a pentagon. So an $\alpha$-cell consists of several pentagons, intersecting only in a common base edge.

**Definition 3.13** (Pudlák graphs). Let $L$ be a nontrivial lattice. The Pudlák graph $[\mathcal{P}]$ of $L$ is an $L - \{1\}$-colored graph $\mathcal{A}$, defined as follows.

1. $\mathcal{A}_0^P$ consists of a single edge colored by $0 \in L$. (In fact, how we choose to color this one edge has no impact on later proofs.)
2. $\mathcal{A}_{n+1}^P$ contains $\mathcal{A}_n^P$ as a subgraph and is obtained by attaching to each edge of $\mathcal{A}_n^P$ of any color $\alpha$ an $\alpha$-cell.
3. $\mathcal{A}^P = \bigcup_{n \in \omega} \mathcal{A}_n^P$.

We will use the following modification, which contains infinitely many copies of each edge in Pudlák’s graph.

1. $\mathcal{A}_0^{(i)} = \mathcal{A}_0^P$, for each $i \in \omega$.
2. $\mathcal{A}_j^{(i)}$ is obtained by attaching to each edge of $\mathcal{A}_{j-1}^{(i)}$ of any color $\alpha$, $i$ many $\alpha$-cells.
3. $\mathcal{A}_j = \mathcal{A}_j^{(j)}$.
4. $\mathcal{A} = \bigcup_{n \in \omega} \mathcal{A}_n = \mathcal{A}(L)$ is called the homogenized Pudlák graph of $L$.

The underlying set of $\mathcal{A}_n$ is denoted by $A_n$.

Let $\Theta = \Theta(L)$ be the lattice table with $|\Theta| = A$, and $\hat{\Theta} = \{e(\alpha) : \alpha \in L\}$.

Note that by definition of $\Theta$ being a lattice table, $\hat{\Theta}$ is ordered by reverse inclusion. Similarly let $\hat{\Theta}_n$ be the lattice table with $|\Theta_n| = A_n$,

$$\hat{\Theta}_n = \{e(\alpha) \mid \Theta_n \mid \alpha \in L\}.$$  

**Lemma 3.14.** Let $B_0 \subseteq B_1 \subseteq A$, where $A$ is the underlying set of $\Theta(L)$.

For $i = 0, 1$, let $\Xi_i$ be the usl table whose underlying set is $B_i$, and whose equivalence relations are computed using graph points belonging to $B_i$ only.

Then $\Xi_0 \subseteq \Xi_1$ in the sense of Definition [14].
Proof. We have to show that if \( x \sim_\alpha y \) holds in \( \Xi_1 \) then \( x \sim_\alpha y \) holds in \( \Xi_0 \). The only way this could fail is if there is a path of edges between \( x \) and \( y \) leading out of \( \Xi_0 \) and then back in. We may assume that the path does not leave and re-enter \( \Xi_0 \) via the same node. So it suffices to show that any path that goes around a pentagon not contained in \( \Xi_0 \) but whose base is in \( \Xi_0 \) may be replaced by the edge \( x, y \) cutting across which has equal or greater color, i.e. with no loss of equivalence.

Lemma 3.15. \( \Theta_n \subseteq \Theta_{n+1} \) for each \( n \), so \( \Theta = \bigcup_{n \in \omega} \Theta_n \).

Proof. Let \( \Xi_i = \Theta_{n+i} \) for \( i = 0, 1 \) and apply Lemma 3.14.

Theorem 3.16. Let \( L \) be a nontrivial finite lattice. \( L \) is dual isomorphic to the congruence lattice of \( \text{End} \Theta(L) \). In fact, \( e : L \rightarrow \hat{\Theta}(L) \) is an isomorphism, and \( \hat{\Theta}(L) = \text{Con} \text{End} \Theta(L) \).

Proof. Pudlák [P] assumes that \( L \) is an algebraic lattice [G2], defines a certain algebra \( S \subseteq \text{End} \Theta^p(L) \), and shows that \( e : L \rightarrow \hat{\Theta^p}(L) \) is an isomorphism, and \( \hat{\Theta^p}(L) = \text{Con} S \). Now trivially \( \hat{\Theta^p}(L) \subseteq \text{Con} \text{End} \Theta^p(L) \) holds, and \( S \subseteq \text{End} \Theta^p(L) \) implies \( \text{Con} \text{End} \Theta^p(L) \subseteq \text{Con} S \). So we have \( \hat{\Theta^p}(L) = \text{Con} \text{End} \Theta^p(L) \).

In fact Pudlák’s proof works for our graph \( \Theta \) as well, i.e. it shows that \( e : L \rightarrow \hat{\Theta}(L) \) is an isomorphism, and \( \hat{\Theta}(L) = \text{Con} \text{End} \Theta(L) \).

Now let \( L \) be a finite lattice. Since every finite lattice is algebraic, \( L \) is an algebraic lattice. Hence \( e : L \rightarrow \hat{\Theta}(L) \) is an isomorphism and \( \hat{\Theta}(L) = \text{Con} \text{End} \Theta(L) \).

Lemma 3.17. The sequence \( \Theta_n(L), n \in \omega \) has a subsequence which is a computable Mal’tsev homogeneous sequential lattice table.

Proof. Since \( \Theta \) is a congruence lattice, \( \Theta \) is a Mal’tsev homogeneous lattice table. Hence as \( \Theta = \bigcup_{n \in \omega} \Theta_n \), a subsequence of \( \Theta_n, n \in \omega \) will be a sequential Mal’tsev homogeneous lattice table. The sequence is computable since to compute an equivalence relation on elements of \( \Theta_n \), it is sufficient to consider paths in \( \Theta_n \), since \( \Theta_n \subseteq \Theta_{n+1} \) by Lemma 3.15.

From now on we will assume that in fact \( \Theta_n, n \in \omega \) is itself the subsequence from Lemma 3.17. Fix nontrivial finite lattices \( L^0, L^1 \) and a \((0, 1, \lor)\)-isomorphisms \( \varphi : L^0 \rightarrow \varphi(L^0) \subseteq L^1 \). Let the \( \land \)-isomorphism \( \varphi^* : L^1 \rightarrow L^0 \) be defined by \( \varphi^* \beta = \bigvee \{ \alpha \in L^0 \mid \varphi(\alpha) \leq \beta \} \). This \( \varphi^* \) is known as the Galois adjoint of \( \varphi \) [GHK].

The map \( \varphi^* \) has many nice properties; we list the ones we need in the following lemma.
Lemma 3.18. 1. $\varphi^*$ is a $(\land, 1)$-homomorphism.

2. If $\beta < 1$ then $\varphi^*\beta < 1$.

3. $\varphi^*$ is injective on $\varphi L^0$.

4. $\alpha \leq \varphi^*\beta \iff \varphi^*\varphi\alpha \leq \varphi^*\beta$.

Proof. These all follow easily from the definition of $\varphi^*$ and the fact that $\{\alpha \in L^0 \mid \varphi(\alpha) \leq \beta\}$ is the principal ideal generated by $\varphi^*(\beta)$, i.e. $\{\alpha \in L^0 \mid \alpha \leq \varphi^*(\beta)\}$.

Lemma 3.19. Let $C(\varphi)A L^1$ be the graph obtained from $A L^1$ by replacing each color $\beta$ by $\varphi^*\beta$. Then $C(\varphi)A L^1$ is isomorphic to a subgraph of $A L^0$.

Proof. Each pentagon of $C(\varphi)A L^1$ represents an inequality of the form

$$\varphi^*\beta_1 \land \varphi^*\beta_2 \leq \varphi^*\beta,$$

for $\beta_1, \beta_2, \beta \in L^1$ satisfying $\beta_1 \land \beta_2 \leq \beta$. Then $\varphi^*\beta_1 \land \varphi^*\beta_2 = \varphi^*(\beta_1 \land \beta_2) \leq \varphi^*\beta$, so the represented inequality $\varphi^*\beta_1 \land \varphi^*\beta_2 \leq \varphi^*\beta$ holds in $L^0$.

Hence we can obtain an isomorphic copy of $C(\varphi)A L^1$ within $A L^0$ by running through the construction of $A L^0$, omitting every pentagon that represents an inequality involving members of $L^0 - \varphi^*L^1$, and omitting pentagons for inequalities that are true in $L^0$ but not in $L^1$. If an edge becomes disconnected from $A_0$ by such omissions then it too is omitted. Since $L^1$ may have many more elements than $L^0$, we make use of the fact that $A$ contains infinitely many copies of each edge from Pudlák’s original graph $A^P$. Since $\varphi^*(\beta) = 1 \rightarrow \beta = 1$ by Lemma 3.18, recoloring of points is never identification of points.

Lemma 3.20. Let $\Theta(\varphi)$ be the isomorphism from Lemma 3.19 sending $A L^1$ to a subgraph of $A L^0$ isomorphic to $C(\varphi)A L^1$.

Then $\Theta(\varphi)A L^1 \subseteq \Theta L^0$ in the sense of Definition 3.2.

Proof. Let $\Xi_0 = \Theta(\varphi)A L^1$ and $\Xi_1 = \Theta L^0$ and apply Lemma 3.14.

Lemma 3.21. Let $\Psi_i$ be the map $e$ of Definition 3.14 for $L = L^i, i = 0, 1$.

Then $\Theta(L^1)$ embeds in $\Theta(L^0)$ with respect to $\varphi$ and $\Psi_0, \Psi_1$.

Proof. Let $x, y$ be points in $\Theta L^1$, i.e. in $A L^1$, and let $\alpha \in L^0$. Then obviously $x \sim_{\varphi\alpha} y \rightarrow \Theta(\varphi)x \sim_{\varphi^*\varphi\alpha} \Theta(\varphi)y$. Now suppose $\Theta(\varphi)x \sim_{\varphi^*\varphi\alpha} \Theta(\varphi)y$. Then there is a path witnessing this, which by Lemma 3.19 we may assume lies within $\Theta(\varphi)A L^1$. Hence the path has an inverse image path under $\Theta(\varphi)^{-1}$. This is then a path from $x$ to $y$ with colors $\beta$ for all of which $\varphi^*\beta \geq \varphi^*\varphi\alpha$. But then $\alpha \leq \varphi^*\beta$ by Lemma 3.18(4), and so $\varphi\alpha \leq \beta$, so $x \sim_{\varphi\alpha} y$. So in fact $x \sim_{\varphi\alpha} y \leftarrow \Theta(\varphi)x \sim_{\varphi^*\varphi\alpha} \Theta(\varphi)y$. Colors $\gamma$ of edges in $\Theta(\varphi)A L^1$ are all of the form $\varphi^*(\beta)$ for some $\beta$. So $\Theta(\varphi)x \sim_{\varphi^*\varphi\alpha} \Theta(\varphi)y$ iff there is a path from $\Theta(\varphi)x$ to $\Theta(\varphi)y$, all
edges of which are colored $\gamma \geq \varphi^* \alpha$, or equivalently by Lemma 3.18(4) (using $\gamma = \varphi^* \beta$), colored $\gamma \geq \alpha$. Hence equivalently $\Theta(\varphi)x \sim_\alpha \Theta(\varphi)y$. □

Proof of Theorem 3.8. Let $m_i(n)$ be the least $m$ such that $\Theta(\varphi_i)\Theta^{i+1}_n \subseteq \Theta^m_n$. Let $h(i) = m_i(0)$, for $i \in \omega$. Let $\Theta^0 = \Theta(L^0)$ and for $i \geq 1$, denoting composition by juxtaposition,

$$\Theta^i = \Theta(\varphi_0) \cdots \Theta(\varphi_{i-1})\Theta(L^i).$$

Let $\Theta^i_k = \Theta(\varphi_0) \cdots \Theta(\varphi_{i-1})\Theta_j(L^i)$ if $k = m_0 m_1 \cdots m_{i-1}(j)$ for some $j$; otherwise, let $\Theta^i_k = \Theta^i_{k-1}$. The Theorem now follows easily. □

4 Initial segments of the tt-degrees

Lemma 4.1. Suppose for each $e$, $g$ lies on a tree $T_e$ which is $e$-splitting for some $c$ for some tables with the properties of Proposition 3.9 in the sense of [LL]. Then $g$ is hyperimmune-free.

Proof. For each $e \in \omega$ there exists $e^* \in \omega$ such that for all stages $s$ and all oracles $g$, if $\{e^*\}^g(x) \downarrow$ then $\{e\}^g(x) = \{e\}^g(x)$ and $\{e\}^g(y) \downarrow$ for all $y \leq x$. If $g$ lies on $T_{e^*}$ then it follows that $\{e\}^g$ is total and $\{e^*\}^{T(\sigma)}(x) \downarrow$ for each $\sigma$ of length $x + 1$. Hence $\{e\}^g = \{e^*\}^g$ is dominated by the recursive function $f(x) = \max\{\{e\}^{T(\sigma)}(x) : |\sigma| = x + 1\}$.

Proposition 4.2. Let $L$ be a $\Sigma_4^0(y)$-presentable upper semilattice with least and greatest element. Then there exist $t, i, g$ such that

1. $t : \omega \to 2$ is $0''$-computable,
2. $i$ is the characteristic function of a set $I$ such that $I \leq_m y^{(3)}$,
3. $g^{(2)}(e) = t(i(0), \ldots, i(e))$ for all $e \in \omega$,
4. $[0, g]$ is isomorphic to $L$, and
5. $g$ is hyperimmune-free.

Proof. The proof in [LL] must be modified to employ the lattice tables of Proposition 3.9.

By Proposition 3.9, for all $x, y \in \Theta^{k+1}$ and $\alpha \in L^k$, we have [identifying the isomorphism between $L^i$ and $\Theta^i$ with the identity]

$$x \sim_{\varphi^* \alpha} y \leftrightarrow x \sim_\alpha y.$$

Lemma 4.1 of [LL] is modified so that $\psi_{T_e}^{c}$ is $\psi_{T_e, \varphi_e^c}$. The equivalence

$$uF_{m(i)}(c)v \leftrightarrow uG_i(c)v$$
now becomes
\[ uF_{m(i)}(c)v \leftrightarrow uG_i(\varphi_k c)v \]

Just as in Lemma 3.1 it is shown that \( \psi_{T,c} \) is Turing equivalent to \( \psi_{T_0,c} \), it now follows that \( \psi_{T,\varphi_k c} \) is Turing equivalent to \( \psi_{T_0,c} \), which is what we want.

\[ i(e) = 1 \text{ iff the answers to the } \Pi_0^1(\text{y}^{(2)})\text{-question about } L^c \text{ is yes.} \]

By Lemma 4.1, \( g \) is hyperimmune-free.

**Lemma 4.3.** If \( t, i, A, q \) satisfy

1. \( t : \omega \to 2 \) is \( q \)-computable,
2. \( i \) is the characteristic function of a set \( I \) such that \( I \leq_m q' \),
3. \( A(e) = t(i(0), \ldots, i(e)) \) for all \( e \in \omega \),

then \( A \leq_{tt} q' \).

**Proof.** The value of \( A(e) \) is determined by the following \( e + 2 \) many yes-or-no questions to \( q' \):

Is \( i(0) = 0? \) · · · Is \( i(e) = 0? \) and, using the answers to the first \( e + 1 \) many questions: Is \( t(i(0), \ldots, i(e)) = 0? \)

**Theorem 4.4.** Each \( \Sigma_0^4(\text{y}) \)-presentable upper semilattice with least and greatest element can be realized as an initial segment \([0, g]_T \) with \( g^{(2)} \leq y^{(3)} \).

**Proof.** Let \( g \) be as in Proposition 4.2. By Lemma 4.3 with \( q = y^{(2)} \) and \( A = g^{(2)} \), we have \( g^{(2)} \leq_{tt} y^{(3)} \).

By Proposition 4.2, \( L \) is isomorphic to \([0, g]_T \). Since \( g \) is hyperimmune-free, \([0, g]_T = [0, g]_{tt} \).

5 Coding a set into a lattice

**Definition 5.1.** Let \( L \) be an upper semilattice and suppose \( G = \{ g_i \mid i < \omega \} \subseteq L \). If there exist \( p, q \in L \) such that

\[ \{ g_i \mid i \in \omega \} \subseteq \{ x \mid x \vee p \geq q \ & \ (\forall y < x)(y \vee p \not\geq q) \} \]

then \( G \) is called a Slaman-Woodin set (SW-set) for \( p, q \) in \( L \).

If there exist \( e_0, e_1, f_0, f_1 \in L \) such that for each \( i < \omega \),

\[ g_{2i+1} = (g_{2i} \vee e_1) \land f_1 \ & \ g_{2i+2} = (g_{2i+1} \vee e_0) \land f_0, \]

then the function \( i \mapsto g_i \) is called a Shore sequence for \( e_0, e_1, f_0, f_1 \) in \( L \).
Lemma 5.2. Let \( a \) be a Turing degree. Let \( L \) be a \( \Sigma_1^0(a) \)-presented upper semilattice containing elements \( p, q, e_0, e_1, f_0, f_1 \), and atoms \( g_i \) for \( i \in \omega \), such that \( G = \{ g_i \mid i < \omega \} \) is a Slaman-Woodin set for \( p, q \) and \( i \mapsto g_i \) is a Shore sequence for \( e_0, e_1, f_0, f_1 \). Then \( \{ y, i \mid y = g_i \} \leq_T a \).

Proof.
\[
y = g_{2i+1} \iff \exists x(x = g_{2i} \land y \leq x \lor e_1 \land y \leq f_1 \land y \lor p \geq q)
y = g_{2i+2} \iff \exists x(x = g_{2i+1} \land y \leq x \lor e_0 \land y \leq f_0 \land y \lor p \geq q)
\]
Note that the matrices of the formulas on the right hand side are positive formulas in the language with \( \lor \) and \( \leq \). The function \( \lor \) is \( a \)-recursive and the relation \( \leq \) is \( \Sigma_0^0(a) \). Hence the entire right hand sides are \( \Sigma_0^0(a) \). So starting with \( g_0 \) we can find \( g_n, a \)-recursively. \( \square \)

Definition 5.3. Let \( a \in \mathcal{D} \). An usl \( L \) is said to be of degree \( a \) if (1) \( L \) is \( a \)-presentable, and (2) if \( b \in \mathcal{D} \) and \( L \) is \( b \)-presentable then \( a \leq b \).

Definition 5.4. Given \( U \subseteq \omega \) we define a lattice \( L(U) \).
It consists of \( 0, 1 \), atoms \( \{ g_i : i \in \omega \} \), more atoms \( e_0, e_1, p, s \) and non-atoms \( f_0, f_1 < 1 \) with the properties of Lemma 5.2 (taking \( q = 1 \)) and an additional element \( s \) with the following property for each \( n \in \omega \):
\[
n \in U \iff g_n \lor s = 1.
\]

Remark 5.5. Historically, the technique of enumerating the \( g_n \) \( a' \)-recursively was first done in [S2]. The idea of the improvement can be seen in [S1] Lemma 1.11. The Slaman-Woodin conditions used to combine these ideas to get the above lemma were presented in [NSS1] with a proof appearing in [NSS2] Lemma 2.13(i). The construction of \( L(U) \) was presented to the author by Slaman; see also Theorem 3.7 of [NSS1].

Remark 5.6. Here are some details for the proof that such a lattice \( L(U) \) exists (thanks to assistance from participants in a 2008 seminar at the University of Hawai’i). We make \( L(U) \) a height-three lattice, i.e., every element is either 0, 1, an atom or a co-atom. The atoms are \( e_0, e_1, s, \) and the \( g_i \). The element \( p \) may be either an atom or a co-atom, and is incomparable with all other elements except that \( 0 \leq p \leq 1 \). The co-atoms are \( e_0 \land g_{2n+1}, \) and \( e_1 \land g_{2n}, f_0, f_1, \) and \( g_i \land s \) whenever \( i \not\in U \). These elements are incomparable except as forced by the above conditions. The point of including \( p \) and \( q \) is that \( y \lor p \geq q \) is a positive statement that implies \( y \leq 0 \).

The following lemma will have many applications:

Lemma 5.7. Let \( U \subseteq \omega \).

1. \( L(U) \) has degree \( u \).
2. If $L(U)$ is $\Sigma^0_1(b)$-presentable, then $U \in \Sigma^0_1(b)$.

3. If $U \in \Sigma^0_1(b)$ then $L(U)$ is $\Sigma^0_1(b)$-presentable.

**Proof.** 1. The definition of $L(U)$ appeals to an oracle of degree $u$ only and so $L(U)$ is $u$-presentable. Suppose $L(U)$ is presented with degree $v$. By Lemma 5.2 the relation $y = g_i$ is recursive in $v$. Now $i \in U \iff g_i \lor s \geq 1$, so since $\lor$ and $\geq$ are recursive in $v$, $u \leq v$. 2. We have

$$n \in U \iff \exists x(x = g_n \land x \lor s \geq 1) \iff \forall x(x = g_n \rightarrow x \lor s \geq 1).$$

By Lemma 5.2 $U$ is of the form $\exists x (\Delta^0_1(b) \land \Sigma^0_1(b)) U \in \Sigma^0_1(b)$.

3. Immediate from the fact that all clauses of the definition of $L(U)$ except “$n \in U \iff g_n \lor s \geq 1$” are recursive. \qed

**Proposition 5.8.** If each $\Sigma^0_1(x)$-presentable bounded usl is $\Sigma^0_1(y)$-presentable, then $x^{(3)} \leq_T y^{(3)}$.

**Proof.** Let $b = x^{(3)}$. Since $L(B \oplus \overline{B})$ is $\Sigma^0_1(B)$-presentable, it is $\Sigma^0_1(y^{(3)})$-presentable. Thus by Lemma 5.7 2), $B \oplus \overline{B}$ is $\Sigma^0_1(y^{(3)})$ and hence $B \leq_T y^{(3)}$. \qed

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