AN ANALOGUE OF THE LÉVY-HINČIN FORMULA FOR BI-FREE INFINITELY DIVISIBLE DISTRIBUTIONS

YINZHENGYING GU‡, HAO-WEI HUANG‡, AND JAMES A. MINGO‡

Abstract. In this paper, we derive the bi-free analogue of the Lévy-Hinčin formula for compactly supported planar probability measures which are infinitely divisible with respect to the additive bi-free convolution introduced by Voiculescu. We also provide examples of bi-free infinitely divisible distributions with their bi-free Lévy-Hinčin representations. Furthermore, we construct the bi-free Lévy processes and the additive bi-free convolution semigroups generated by compactly supported planar probability measures.

1. Introduction

Around thirty years ago, Voiculescu introduced free probability theory in order to attack some problems in the theory of operator algebras. He introduced free independence, an analogue of the classical notion of independence, with the intention of studying these problems in a probabilistic framework. The (additive) free convolution ⊞, an analogue of the classical convolution ∗, is a binary operation on the set of compactly supported probability measures on R which corresponds to the sum of free random variables in a non-commutative probability space. This operation was later generalized to the set MR of Borel probability measures on R by Bercovici and Voiculescu [2]. One of the essential functions in the theory is the free R-transform of measures in MR, which linearizes the additive free convolution ⊞ [2][3]. The combinatorial apparatus of free cumulants and the lattice of non-crossing partitions, introduced by Speicher [12], also play important roles in free probability theory for the study of sums and products of free n-tuples of random variables.

Either in classical or free probability theory, infinitely divisible probability distributions play a central role. A probability measure μ ∈ MR is said to be ∗-infinitely divisible (resp. ⊞-infinitely divisible) if, for every n ∈ N, it can be represented as an n-fold classical (resp. free) convolution of some probability measure in MR. Measures which are ∗-infinitely divisible were first studied by de Finetti, Kolmogorov, Lévy, and Hinčin as they arise as the limit distributions of sums of independent random variables within a triangular array. The logarithm of the Fourier transform of a ∗-infinitely divisible distribution permits an integral representation called the Lévy-Hinčin representation. The

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free infinitely divisible distributions were first investigated by Voiculescu, and since then the theory has been well developed. The theory of free infinitely divisible distributions generalize the free central limit theorem. Likewise, the free Lévy-Hinčin formula gives a complete description of the free infinitely divisible distributions. We refer the reader to [1][2][9] for more details.

Infinite divisibility of probability measures is closely related to stationary processes with independent increments. From the theoretical and applied points of view, Lévy processes form a very important research area in classical probability. Such processes in free probability theory also receive a lot of attention. By analogy with classical probability theory, the distribution \( \mu_t \) of \( X_t \) in an (additive) free Lévy process \( (X_t)_{t \geq 0} \) satisfies the properties that \( \mu_0 = \delta_0 \) (the point mass at 0), the weak convergence of \( \mu_t \) to \( \delta_0 \) as \( t \to 0^+ \) and the semigroup property relative to the free convolution:

\[
\mu_s \boxplus \mu_t = \mu_{s+t}, \quad s, t \geq 0.
\]

As in the classical case, the distribution \( \mu_1 \) is free infinitely divisible.

For \( n \in \mathbb{N} \) and \( \mu \in M_{\mathbb{R}} \), denote by \( \mu_n \) the \( n \)-fold free convolution of \( \mu \). One pecu-
liarity of free convolution is that the discrete free convolution semigroup \( (\mu_n)_{n \in \mathbb{N}} \) can be embedded in a continuous family \( (\mu_t)_{t \geq 1} \) which satisfies the semigroup property (1.1) for \( s, t \geq 1 \). This elegant result for a compactly supported measure \( \mu \), proved by Nica and Speicher [9], has no parallel in classical probability theory. The exhibition of explicit random variables whose distributions are measures in \( (\mu_t)_{t \geq 1} \) has several applications in random matrix theory.

Recently, Voiculescu [14][15] introduced bi-free probability theory in order to study algebras of left operators and algebras of right operators simultaneously. This gives rise to the notions of bi-free cumulants, bi-free \( \mathcal{R} \)-transforms, and the operation of (additive) bi-free convolution. Infinitely divisible distributions originate from the generalization of the central limit theorem where, roughly speaking, each such distribution appears in the limit as \( N \to \infty \) of the sum \( X_{N,1} + \cdots + X_{N,N} \) of (classically) independent random variables within a triangular array. In the free probability setting, the free counterpart of infinitely divisible distributions appear in the limits of triangular arrays of free random variables. In this paper, we prove the bi-free counterpart of the limit distribution theorem for sums of bi-free pairs of random variables within a triangular array. Analogously, one can define the bi-free infinite divisibility of a planar probability measure in a similar manner. One of the main goals of this paper is to characterize the bi-free \( \mathcal{R} \)-transforms of bi-free infinitely divisible distributions with compact supports in \( \mathbb{R}^2 \) and derive their bi-free Lévy-Hinčin representations. With the help of a bi-free limit distribution theorem, we are able to provide some examples, such as the bi-free Gaussian and bi-free (compound) Poisson distributions, which are bi-free infinitely divisible. A natural object in the study of the bi-free infinite divisibility of distributions is the extension of the notions of free Lévy process and free convolution semigroup to the bi-free setting. Another goal of this paper is to prove the existence of the bi-free convolution semigroups generated by
and results from \[7\][8][14]. An ordered pair \((B, \pi)\) in a non-commutative probability space \((Bi-free independence and bi-free cumulants.

2.1. Bi-free independence and bi-free cumulants. First, we review some definitions and results from \[7\][8][14]. An ordered pair \((B, C)\) is said to be a pair of (included) faces in a non-commutative probability space \((A, \varphi)\) if \(B\) and \(C\) are unital subalgebras of \(A\), in which \(B\) and \(C\) are called the left and right face, respectively. In \[14\], Definition 2.6, Voiculescu defined bi-free independence for pairs of faces as follows.

**Definition 2.1.** A family \(\pi = \{(B_k, C_k)\}_{k \in K}\) of pairs of faces in \((A, \varphi)\) is said to be bi-free if there exists a family of vector spaces with specified vector states \(\{(X_k, \lambda_k, \xi_k)\}_{k \in K}\) and unital homomorphisms \(l_k : B_k \to L(X_k), r_k : C_k \to L(X_k)\) such that the joint distribution of \(\pi\) with respect to \(\varphi\) is equal to the joint distribution of \(\pi = \{((\lambda_k \circ l_k(B_k), \rho_k \circ r_k(C_k)))\}_{k \in K}\) with respect to the vacuum state on \(L(X)\), where \((X, \hat{X}, \xi) = \ast_{k \in K}(X_k, \hat{X}_k, \xi_k)\), \(\lambda_k\) and \(\rho_k\) are the left and right representations of \(L(X_k)\) on \(L(X)\).

Let \(I\) and \(J\) be index sets. If \((b_i', i \in I), (b_j'', j \in J), (c_i', i \in I), (c_j'', j \in J)\) are elements of \(A\), then the two-faced families of non-commutative random variables \((b', c') = ((b_i', i \in I), (c_j', j \in J))\) and \((b'', c'') = ((b_j'', j \in J), (c_i'', i \in I))\) are said to be bi-free if the associated pairs of faces \((C(b_i' : i \in I), C(c_j' : j \in J))\) and \((C(b_j'' : j \in J), C(c_i'' : i \in I))\) are bi-free (see \[14\], Section 2). Moreover, if \((b', c')\) and \((b'', c'')\) are bi-free with joint distributions \(\mu'\) and \(\mu''\), respectively, then the joint distribution of \(((b_i' + b_j'')_i \in I, (c_i' + c_j'')_j \in J)\) is called the additive bi-free convolution of \(\mu'\) and \(\mu''\), and is denoted by \(\mu' \boxplus \mu''\) (see \[14\], Section 4).

It was shown in \[14\], Section 5, that there exist universal polynomials, called bi-free cumulants, on the mixed moments of bi-free pairs of two-faced families of non-commutative random variables which linearize the additive bi-free convolution. However, there were no explicit formulas for the bi-free cumulants. Later, Mastnak and Nica in \[8\], Definition 5.2 defined \((l, r)\)-cumulants and combinatorial-bi-free independence as follows.

**Definition 2.2.** Let \((A, \varphi)\) be a non-commutative probability space. There exists a family of multilinear functionals

\[
(\kappa_{\chi} : A^n \to \mathbb{C})_{n \geq 1, \chi : [n] \to \{l, r\}}
\]

which is uniquely determined by the requirement that

\[
\varphi(a_1, \ldots, a_n) = \sum_{\pi \in \mathcal{P}(\chi \cup \{0\})} \left( \prod_{V \in \pi} \kappa_{\chi|V}((a_1, \ldots, a_n)|V) \right)
\]
for every $n \geq 1$, $\chi : [n] \rightarrow \{l, r\}$, and $a_1, \ldots, a_n \in \mathcal{A}$. These $(\kappa_\chi)_{n \geq 1, \chi : [n] \rightarrow \{l, r\}}$ are called the $(l, r)$-cumulants of $(\mathcal{A}, \varphi)$.

Given $\chi : [n] \rightarrow \{l, r\}$, $n \geq 1$, such that $\chi^{-1}(\{l\}) = \{i_1 < \cdots < i_p\}$ and $\chi^{-1}(\{r\}) = \{j_1 < \cdots < j_{n-p}\}$, the set of partitions $\mathcal{P}(\chi)(n)$ appearing in the above definition is obtained by applying the permutation $\sigma_\chi \in S_n$ to the elements of $\text{NC}(n)$, the set of non-crossing partitions of $[n]$, where $\sigma_\chi$ is defined by

$$
\sigma_\chi(k) = \begin{cases}
  i_k, & \text{if } k \leq p, \\
  j_{n-k+1}, & \text{if } k > p.
\end{cases}
$$

**Definition 2.3.** (Section 1). Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $a_1, \ldots, a_d, b_1, \ldots, b_d$ be elements of $\mathcal{A}$. Denoting $c_{i:l} = a_i$ and $c_{i:r} = b_i$ for $1 \leq i \leq d$, then the two-faced pairs $(a_1, b_1), \ldots, (a_d, b_d)$ are said to be combinatorially-bi-free if

$$
\kappa_\chi(c_{i_1:\chi(i_1)}, \ldots, c_{i_n:\chi(i_n)}) = 0
$$

whenever $n \geq 2$, $\chi : [n] \rightarrow \{l, r\}$, $i_1, \ldots, i_n \in [d]$, and there exist $1 \leq p < q \leq n$ such that $i_p \neq i_q$.

After giving the above definition, Mastnak and Nica asked the question of whether combinatorial-bi-free independence was equivalent to bi-free independence, and it was answered affirmatively by Charlesworth, Nelson, and Skoufranis in [7] using bi-non-crossing partitions. We refer to [7, Section 2] for details. For $\chi : [n] \rightarrow \{l, r\}$, $n \geq 1$, such that $\chi^{-1}(\{l\}) = \{i_1 < \cdots < i_p\}$ and $\chi^{-1}(\{r\}) = \{j_1 < \cdots < j_{n-p}\}$, the set of bi-non-crossing partitions $\text{BNC}(\chi)$ defined in [7, Section 2] coincides with $\mathcal{P}(\chi)(n)$ which, from another diagrammatic point of view, consists of the non-crossing partitions of $[n]$ such that the numbers $1, \ldots, n$ are rearranged according to the total order $<_\chi$ on $[n]$ defined by

$$
i_1 <_\chi \cdots <_\chi i_p <_\chi j_{n-p} <_\chi \cdots <_\chi j_1.
$$

For this reason, we also denote $\mathcal{P}(\chi)(n) = \text{BNC}(\chi)$ by $\text{NC}_\chi(n)$. As lattices with respect to reverse refinement order, $\text{NC}_\chi(n)$ is isomorphic to $\text{NC}(n)$, thus the Möbius function $\mu_\chi$ on $\text{NC}_\chi(n)$ is given by

$$
\mu_\chi(\tau, \pi) = \mu(\sigma_\chi^{-1} \cdot \tau, \sigma_\chi^{-1} \cdot \pi),
$$

where $\mu$ denotes the Möbius function on $\text{NC}(n)$. Finally, as shown in [7, Sections 3, 4], the $(l, r)$-cumulants are the same as the bi-free cumulants, and we have the moment-cumulant formulas

$$
\varphi(a_1 \cdots a_n) = \sum_{\pi \in \text{NC}_\chi(n)} \kappa_\chi^\pi(a_1, \ldots, a_n)
$$

and

$$
\kappa_\chi^\pi(a_1, \ldots, a_n) = \sum_{\pi \in \text{NC}(n)} \varphi_\pi(a_1, \ldots, a_n) \mu_\chi(\pi, 1_n)
$$

for all $a_1, \ldots, a_n \in \mathcal{A}$, where $\kappa_\chi^\pi(a_1, \ldots, a_n) = \kappa_\chi(a_1, \ldots, a_n)$ and $\kappa_\chi^\pi(a_1, \ldots, a_n)$ factors according to the blocks of $\pi$ by the multiplicativity of the family $(\kappa_\chi^\pi)_{n \geq 1, \chi : [n] \rightarrow \{l, r\}}$.\]
2.2. Free and bi-free $\mathcal{R}$-transforms. Recall that the joint distribution of a family $(a_i)_{i \in I}$ of random variables in a non-commutative probability space $(\mathcal{A}, \varphi)$ is the linear functional $\mu$ on the algebra $\mathbb{C}(X_i : i \in I)$ of non-commutative polynomials in $|I|$ variables satisfying $\mu(P) = \varphi(h(P))$ for all $P \in \mathbb{C}(X_i : i \in I)$, where $h : \mathbb{C}(X_i : i \in I) \to \mathcal{A}$ is the unital algebra homomorphism such that $h(X_i) = a_i$.

If $a$ is a self-adjoint random variable in a $C^*$-probability space $(\mathcal{A}, \varphi)$, then its distribution $\mu_a$ belongs to $\mathcal{M}_\mathbb{R}$. The Cauchy transform (or one-variable Green’s function) of $a$ is defined as

$$G_a(z) = \varphi((z1 - a)^{-1}),$$

while the $\mathcal{R}$-transform of $a$ is defined as

$$\mathcal{R}_a(z) = \sum_{n \geq 0} \kappa_{n+1}(a, \ldots, a) z^n,$$

where $(\kappa_n)_{n \in \mathbb{N}}$ are the free cumulants of $(\mathcal{A}, \varphi)$. It turns out that the functions $G_a$ and $\mathcal{R}_a$ are analytic in a neighborhood of $\infty$ and $0$, respectively, and the function

$$K_a(z) = \mathcal{R}_a(z) + \frac{1}{z}$$

satisfies $G_a(K_a(z)) = z$. One of the most important properties of the $\mathcal{R}$-transform is that it linearizes the additive free convolution in the sense that

$$\mathcal{R}_{a+b}(z) = \mathcal{R}_a(z) + \mathcal{R}_b(z)$$

if $a$ and $b$ are free self-adjoint random variables in $(\mathcal{A}, \varphi)$ or, equivalently,

$$\mathcal{R}_{\mu_a \boxplus \mu_b}(z) = \mathcal{R}_{\mu_a}(z) + \mathcal{R}_{\mu_b}(z),$$

which holds in a neighborhood of $0$ [10]. In addition, if $[a, b] = 0$, i.e. $a$ and $b$ commute, then the distribution $\mu_{(a,b)}$ of $(a,b)$ is a Borel probability measure on $\mathbb{R}^2$ and the two-dimensional Cauchy transform (or two-variable Green’s function) of $(a,b)$ is defined as

$$G_{(a,b)}(z, w) = \varphi((z1-a)^{-1}(w1-b)^{-1}),$$

which is an analytic function in a neighborhood of $(0,0)$.

Note that if $(a,b)$ is a general two-faced pair in a $C^*$-probability space $(\mathcal{A}, \varphi)$, then the bi-free cumulant $\kappa^\chi_{n,l}$ of $(a,b)$ depends on $\chi : [n] \to \{l, r\}$. Since we are interested in the case where $a$ and $b$ are commuting self-adjoint random variables, it turns out that all the bi-free cumulants of $(a,b)$ are real, and $\kappa^\chi_{n,l}$ depends on $\chi$ only through $|\chi^{-1}(\{l\})|$ and $|\chi^{-1}(\{r\})|$. Moreover, the commutativity of $a$ and $b$ implies that every bi-free cumulant of $(a,b)$ is a special free cumulant.

**Lemma 2.4.** Let $(a, b)$ be a two-faced pair in a $C^*$-probability space $(\mathcal{A}, \varphi)$ such that $a = a^*$, $b = b^*$, and $[a, b] = 0$. We denote the free and bi-free cumulants of $(a, b)$ by

$$\kappa_{m,n}(a, b) = \kappa_{m+n}(a, \ldots, a, b, \ldots, b)_{m \text{ times } n \text{ times}},$$

and

$$\kappa^\chi_{N}(a, b) = \kappa^\chi_{N}(c_\chi(1), \ldots, c_\chi(N)).$$
respectively, where \( \chi : \{1, \ldots, N\} \to \{l, r\} \) and \( c_{\chi(k)} = a \) or \( b \) depending on whether \( \chi(k) = l \) or \( r \) for \( 1 \leq k \leq N \). Then \( \kappa_{m,n}(a,b) = \kappa^\chi_{m+n}(a,b) \) for all \( \chi : \{1, \ldots, m+n\} \to \{l, r\} \) such that \( |\chi^{-1}\{\{l\}\}| = m \) and \( |\chi^{-1}\{\{r\}\}| = n \).

**Proof.** By the moment-cumulant formulas, we have

\[
\kappa_{m,n}(a,b) = \sum_{\pi \in \text{NC}(m+n)} \varphi_\pi(a, \ldots, a, b, \ldots, b) \mu_\pi(1_{m+n})
\]

and

\[
\kappa^\chi_{m+n}(a,b) = \sum_{\pi \in \text{NC}_\chi(m+n)} \varphi_\pi(c_\chi(1), \ldots, c_\chi(m+n)) \mu_\chi(1_{m+n}),
\]

where \( \mu \) and \( \mu_\chi \) denote the Möbius functions on NC\((m+n)\) and NC\(_\chi(m+n)\), respectively.

For each permutation \( \pi \in \text{NC}(m+n) \), which has a linear non-crossing diagram associated to it, the linear diagram of the corresponding partition \( \tilde{\pi} = \sigma_\chi \cdot \pi \in \text{NC}_\chi(m+n) \) under the bijection \( \sigma_\chi : \text{NC}(m+n) \to \text{NC}_\chi(m+n) \) is obtained by relabelling the numbers \( 1, \ldots, m+n \) in the linear diagram of \( \pi \) with \( i_1, \ldots, i_m, j_1, \ldots, j_n \) where \( \{i_1 < \cdots < i_m\} = \chi^{-1}\{\{l\}\} \) and \( \{j_1 < \cdots < j_n\} = \chi^{-1}\{\{r\}\} \). Since \( a \) and \( b \) commute, we have \( \varphi_\pi(a, \ldots, a, b, \ldots, b) = \varphi_{\tilde{\pi}}(c_\chi(1), \ldots, c_\chi(m+n)) \) for every \( \pi \in \text{NC}(m+n) \). Moreover, since \( \mu_\chi(\tilde{\pi}, 1_{m+n}) = \mu(\sigma_\chi^{-1} \cdot \tilde{\pi}, 1_{m+n}) = \mu(\pi, 1_{m+n}) \) for every \( \pi \in \text{NC}_\chi(m+n) \), the assertion follows.

**Notation 2.5.** Let \((a,b)\) be as above and \(m,n \geq 0\) such that \(m+n \geq 1\). We extend the notations used in the above lemma for the free and bi-free cumulants of \((a,b)\) to all of NC\((m+n)\) and NC\(_\chi(m+n)\), where \( \chi : [m+n] \to \{l, r\} \) such that \( |\chi^{-1}\{\{l\}\}| = m \) and \( |\chi^{-1}\{\{r\}\}| = n \). That is, for \( \pi \in \text{NC}(m+n) \) or NC\(_\chi(m+n)\), we have

\[
\kappa_\pi(a,b) = \kappa_\pi(a, \ldots, a, b, \ldots, b)
\]

and

\[
\kappa^\chi_\pi(a,b) = \kappa^\chi_\pi(c_\chi(1), \ldots, c_\chi(m+n)),
\]

where \( c_{\chi(k)} = a \) or \( b \) depending on whether \( \chi(k) = l \) or \( r \) for \( 1 \leq k \leq m+n \). Similarly, we let

\[
\varphi_\pi(a,b) = \varphi_\pi(a, \ldots, a, b, \ldots, b)
\]

for \( \pi \) in NC\((m+n)\).

For a two-faced pair \((a,b)\) in a C*-probability space where \(a\) and \(b\) are commuting self-adjoint random variables, let

\[
\mathcal{R}_{(a,b)}(z,w) = \sum_{m,n \geq 0} \kappa^\chi_{m+n}(a,b) z^m w^n = \sum_{m,n \geq 0} \kappa_{m,n}(a,b) z^m w^n,
\]

where \( \chi : [m+n] \to \{l, r\} \) such that \( |\chi^{-1}\{\{l\}\}| = m \) and \( |\chi^{-1}\{\{r\}\}| = n \), be the bi-free \( \mathcal{R} \)-transform of \((a,b)\). Then we have the following relation for bi-free \( \mathcal{R} \)-transforms [15, Theorem 2.4].
The following equality of germs of holomorphic functions holds in a neighborhood of $(0,0)$ in $\mathbb{C}^2$:

$$R_{(a,b)}(z, w) = 1 + zR_a(z) + wR_b(w) - \frac{zw}{G_{(a,b)}(K_a(z), K_b(w))}.$$  

The bi-free $R$-transform is the analogue of the free $R$-transform in the bi-free setting. More precisely, if $(a', b')$ and $(a'', b'')$ are bi-free two-faced pairs, then for $(z, w)$ near $(0, 0)$,

$$R_{(a'+a'', b'+b'')}(z, w) = R_{(a', b')}(z, w) + R_{(a'', b'')}(z, w),$$

which is equivalent to

$$R_{\mu(a', b')} R_{\mu(a'', b'')}(z, w) = R_{\mu(a', b')}(z, w) + R_{\mu(a'', b'')}(z, w).$$

2.3. Moment sequences. Let $\mathbb{Z}_+^2 = \{(m,n) : m, n \in \mathbb{N} \cup \{0\}\}$. Given a 2-sequence $R := \{R_{m,n}\}_{(m,n) \in \mathbb{Z}_+^2}$ with $R_{0,0} > 0$, one can then equip the algebra $\mathbb{C}[s,t]$ of polynomials in commuting variables $s$ and $t$ with a sesquilinear form $[\cdot, \cdot]_R$ satisfying

$$[s^{m_1}t^{n_1}, s^{m_2}t^{n_2}]_R = R_{m_1+m_2, n_1+n_2}$$

for $(m_1, n_1), (m_2, n_2) \in \mathbb{Z}_+^2$. Note that if $p = \sum_{j=1}^l c_j s^{m_j}t^{n_j} \in \mathbb{C}[s,t]$, then

$$[p, p]_R = \sum_{j,k=1}^l c_j c_k R_{m_j+m_k, n_j+n_k}.$$

Recall that the 2-sequence $\{\kappa_{m,n}\}_{(m,n) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}}$ of free cumulants of a pair of commuting self-adjoint random variables $(a, b)$ in some $C^*$-probability space contain the full information about $(a, b)$. It turns out that the study of such 2-sequences is closely related to the two-parameter moment problems [6][11]. The following result is from [3].

**Theorem 2.7.** A 2-sequence $R = \{R_{m,n}\}_{(m,n) \in \mathbb{Z}_+^2}$ with $R_{0,0} > 0$ is a moment 2-sequence, i.e. there exists a finite positive Borel measure $\rho$ on $\mathbb{R}^2$ such that

$$R_{m,n} = \int_{\mathbb{R}^2} s^m t^n \, d\rho(s,t), \quad (m,n) \in \mathbb{Z}_+^2,$$

if there exists a finite number $L > 0$ with the following properties: for all $p \in \mathbb{C}[s,t]$,

1. $[p, p]_R \geq 0$,
2. $[sp, p]_R \leq L \cdot [p, p]_R$ and $[tp, p]_R \leq L \cdot [p, p]_R$ hold.

If these conditions hold, then the representing measure $\rho$ of $R$ is compactly supported in $[-L,L]^2$ and uniquely determined.

3. Bi-free infinitely divisible distributions

In this section, we study the bi-free infinite divisibility of compactly supported probability measures on $\mathbb{R}^2$ and provide the bi-free analogue of the Lévy-Hinčin formula.
3.1. A bi-free limit theorem for bipartite systems. The non-commutative probability spaces considered throughout the paper are assumed to be bipartite, i.e. all left variables commute with all right variables.

**Theorem 3.1.** For each $N \in \mathbb{N}$, let $((a_{N,k}, b_{N,k}))_{k=1}^{N}$ be a triangular array of two-faced pairs in some non-commutative probability space $(\mathcal{A}_N, \varphi_N)$. Furthermore, assume that each $\mathcal{A}_N$ is bipartite and that the two-faced pairs $(a_{N,1}, b_{N,1}), \ldots, (a_{N,N}, b_{N,N})$ are bi-free and identically distributed. Then the following two statements are equivalent.

1. There is a two-faced pair $(a, b)$ in some non-commutative probability space $(\mathcal{A}, \varphi)$ such that $[a, b] = 0$ and

\[
\left( \sum_{k=1}^{N} a_{N,k}, \sum_{k=1}^{N} b_{N,k} \right) \xrightarrow{\text{dist}} (a, b).
\]

2. For all $m, n \geq 0$ such that $m + n \geq 1$, the limits $\lim_{N \to \infty} N \cdot \varphi_N(a_{N,k}^{m} b_{N,k}^{n})$, which are independent of $k$, exist. Furthermore, if (1) and (2) hold, then the bi-free cumulants of $(a, b)$ are given by

\[
\kappa_{m+n}^\chi(a, b) = \lim_{N \to \infty} N \cdot \varphi_N(a_{N,k}^{m} b_{N,k}^{n}),
\]

where $\chi : \{1, \ldots, m + n\} \to \{l, r\}$ satisfies $|\chi^{-1}(\{l\})| = m$ and $|\chi^{-1}(\{r\})| = n$.

**Remark 3.2.** (1) We follow the usual notion of convergence in distribution in the free probability context. That is, the assertion (1) in Theorem 3.1 holds if and only if

\[
\lim_{N \to \infty} \varphi_N \left( \left( \sum_{k=1}^{N} a_{N,k} \right)^{m} \left( \sum_{k=1}^{N} b_{N,k} \right)^{n} \right) = \varphi(a^{m} b^{n})
\]

for all $(m, n) \in \mathbb{Z}_+^2$ and all mixed moments $\varphi(a^{m} b^{n})$ exist.

2. By Lemma 2.4, the bi-free cumulants $\kappa_{m+n}^\chi(a, b)$ are the same as the free cumulants $\kappa_{m,n}(a, b)$ for all $m, n \geq 0$ such that $m + n \geq 1$.

To prove the above theorem, we need the following lemma which relates convergence of moments to convergence of cumulants.

**Lemma 3.3.** For each $N \in \mathbb{N}$, let $(\mathcal{A}_N, \varphi_N)$ be a non-commutative probability space and let $\kappa_N$ be the corresponding free cumulants. Let $(a_N, b_N)$ be a two-faced pair in $(\mathcal{A}_N, \varphi_N)$ such that $a_N$ and $b_N$ commute. Then the following two statements are equivalent.

1. For all $m, n \geq 0$ with $m + n \geq 1$, the limits $\lim_{N \to \infty} N \cdot \varphi_N(a_N^{m} b_N^{n})$ exist.

2. For all $m, n \geq 0$ with $m + n \geq 1$, the limits $\lim_{N \to \infty} N \cdot \kappa_{m,n}^N(a_N, b_N)$ exist.

Furthermore, if these conditions hold, then the limits in (1) and (2) are the same.

**Proof.** By the moment-cumulant formulas, we have

\[
\lim_{N \to \infty} N \cdot \varphi_N(a_N^{m} b_N^{n}) = \lim_{N \to \infty} N \cdot \sum_{\pi \in \text{NC}(m+n)} \kappa_{\pi}^N(a_N, b_N)
\]
and
\[
\lim_{N \to \infty} N \cdot \kappa_{m,n}^{N}(a_{N}, b_{N}) = \lim_{N \to \infty} N \cdot \sum_{\pi \in \mathrm{NC}(m+n)} (\varphi_N)_{\pi}(a_{N}, b_{N}) \mu(\pi, 1_{m+n}).
\]

If the first statement (resp. second statement) is true, then the only non-vanishing term on the right-hand side of the second equation (resp. first equation) above corresponds to \(\pi = 1_{m+n} \), and the assertion follows. \(\Box\)

**Proof of Theorem 3.1.** Assume that the first statement is true. Since we will not be using the bi-free cumulants of \((\mathcal{A}, \varphi)\) until the end of the proof, we let \(\kappa^{\chi}\) denote the bi-free cumulants of \((\mathcal{A}_{N}, \varphi_{N})\) for now. For \(m, n \geq 0\) such that \(m + n \geq 1\), we have
\[
\varphi(a^{m}b^{n}) = \lim_{N \to \infty} \varphi_{N} \left( \sum_{k=1}^{N} a_{N;k} \right)^{m} \left( \sum_{k=1}^{N} b_{N;k} \right)^{n}
\]
\[
= \lim_{N \to \infty} \sum_{\tau(1), \ldots, \tau(m) \in P(m)} \varphi_{N}(a_{N:1} \cdots a_{N:1}) \cdot \cdots \cdot \kappa_{N}^{\chi}(a_{N:1} \cdots a_{N:1}, b_{N:1} \cdots b_{N:1})
\]
\[
= \lim_{N \to \infty} \sum_{\tau(1), \ldots, \tau(m) \in P(m)} \kappa_{N}^{\chi}(a_{N:1} \cdots a_{N:1}, b_{N:1} \cdots b_{N:1})
\]
\[
= \lim_{N \to \infty} \sum_{\pi \in \mathcal{P}(m+n) \tau \in \text{NC}(m+n)} N(N - 1) \cdots (N - |\pi| + 1) \cdot \kappa_{N}^{\chi}(a_{N;1}, b_{N;1}),
\]
where the last expression, which is independent of \(k\), follows from the fact that mixed bi-free cumulants vanish. It remains to show that the limits
\[
\lim_{N \to \infty} N(N - 1) \cdots (N - |\pi| + 1) \cdot \kappa_{N}^{\chi}(a_{N;1}, b_{N;1})
\]
exist for all \(\pi \in \mathcal{P}(m+n)\) and \(\tau \in \text{NC}(m+n)\) with \(\tau \leq \pi\). Then the special case \(\pi = \tau = 1_{m+n}\) would give us the existence of
\[
\lim_{N \to \infty} N \cdot \kappa_{m+n}^{\chi}(a_{N;1}, b_{N;1}),
\]
which is equal to \(\lim_{N \to \infty} N \cdot \kappa_{m,n}^{N}(a_{N;1}, b_{N;1})\) by Lemma 2.3 and the existence of \(\lim_{N \to \infty} N \cdot \varphi_{N}(a_{N;1}^{m}b_{N;1}^{n})\) would follow from Lemma 3.3. We proceed by induction on \(m\) and \(n\). If \(m = 1\) and \(n = 0\), then
\[
\lim_{N \to \infty} N \cdot \kappa_{1}^{N}(a_{N;1}) = \lim_{N \to \infty} N \cdot \varphi_{N}(a_{N;1}) = \lim_{N \to \infty} \varphi_{N}(a_{N;1} + \cdots + a_{N;N}) = \varphi(a)
\]
events. The case \(m = 0\) and \(n = 1\) is similar. If \(m = n = 1\), then we have
\[
\varphi(ab) = \lim_{N \to \infty} (N(N - 1) \cdot \kappa_{1}^{N}(a_{N;1}b_{N;1}) + N \cdot \kappa_{1,1}^{N}(a_{N;1}, b_{N;1})).
\]
Since \(\varphi(ab)\), \(\lim_{N \to \infty} N \cdot \kappa_{1}^{N}(a_{N;1})\), and \(\lim_{N \to \infty} N \cdot \kappa_{1,1}^{N}(a_{N;1}, b_{N;1})\) all exist, we obtain the existence of \(\lim_{N \to \infty} N \cdot \kappa_{1,1}^{N}(a_{N;1}, b_{N;1})\). For the inductive step, assume the assertion is true for all
\( m \leq r \) and \( n \leq s \) such that \( r + s \geq 1 \). If \( m = r + 1 \) and \( n = s \), then we have
\[
\varphi(a^{r+1}b^s) = \lim_{N \to \infty} \sum_{\pi \in \mathcal{P}(r+s+1)} \sum_{\tau \in \text{NC}_\chi(r+s+1)} N(N-1) \cdots (N - |\pi| + 1) \cdot \kappa_\chi^N(a_{N;k}, b_{N;k})
\]
\[
= \lim_{N \to \infty} \left( \sum_{\tau \in \text{NC}_\chi(r+s+1)} N \cdot \kappa_\chi^N(a_{N;k}, b_{N;k}) + L \right),
\]
where \( L = \sum_{\pi \in \mathcal{P}(r+s+1)} \sum_{\tau \in \text{NC}_\chi(r+s+1), \tau \neq 1_{r+s+1}} N(N-1) \cdots (N - |\pi| + 1) \cdot \kappa_\chi^N(a_{N;k}, b_{N;k}) \). By the induction hypothesis, the limits
\[
\lim_{N \to \infty} N(N-1) \cdots (N - |\pi| + 1) \cdot \kappa_\chi^N(a_{N;k}, b_{N;k})
\]
exist for all \( \pi \neq 1_{r+s+1} \) and \( \tau \in \text{NC}_\chi(r+s+1) \) with \( \tau \leq \pi \), thus \( \lim_{N \to \infty} L \) exists. On the other hand, if \( \tau \in \text{NC}_\chi(r+s+1) \) such that \( \tau \neq 1_{r+s+1} \), then \( \tau \) has at least two blocks, thus \( \lim_{N \to \infty} N \cdot \kappa_\chi^N(a_{N;k}, b_{N;k}) \) vanishes. Hence, \( \lim_{N \to \infty} N \cdot \kappa_\chi^N(a_{N;k}, b_{N;k}) \) exists as required. The case \( m = r \) and \( n = s + 1 \) is similar.

Conversely, assume the second statement is true. For \( m, n \geq 0 \) such that \( m + n \geq 1 \), we have
\[
\lim_{N \to \infty} \varphi_N \left( \left( \sum_{k=1}^{N} a_{N;k} \right)^m \left( \sum_{l=1}^{N} b_{N;l} \right)^n \right)
\]
\[
= \lim_{N \to \infty} \sum_{\pi \in \mathcal{P}(m+n)} \sum_{\tau \in \text{NC}_\chi(m+n), \tau \leq \pi} N(N-1) \cdots (N - |\pi| + 1) \cdot \kappa_\chi^N(a_{N;k}, b_{N;k}).
\]
By assumption, the only non-vanishing terms on the right-hand side of the above equation correspond to \( \pi = \tau \). Thus we have
\[
\lim_{N \to \infty} \varphi_N \left( \left( \sum_{k=1}^{N} a_{N;k} \right)^m \left( \sum_{l=1}^{N} b_{N;l} \right)^n \right)
\]
\[
= \lim_{N \to \infty} \sum_{\tau \in \text{NC}_\chi(m+n)} N(N-1) \cdots (N - |\tau| + 1) \cdot \kappa_\chi^N(a_{N;k}, b_{N;k}).
\]
By Lemmas 2.3 and 3.3 we have
\[
\lim_{N \to \infty} N(N-1) \cdots (N - |\tau| + 1) \cdot \kappa_\chi^N(a_{N;k}, b_{N;k})
\]
\[
= \lim_{N \to \infty} N(N-1) \cdots (N - |\tau| + 1) \cdot (\varphi_N)_\tau(a_{N;k}, b_{N;k})
\]
for all \( \tau \in \text{NC}_\chi(m+n) \), thus they all exist. Hence, there is a two-faced pair \((a, b)\) in a non-commutative probability space \((\mathcal{A}, \varphi)\) such that
\[
\varphi(a^m b^n) = \lim_{N \to \infty} \varphi_N \left( \left( \sum_{k=1}^{N} a_{N;k} \right)^m \left( \sum_{k=1}^{N} b_{N;k} \right)^n \right),
\]
which also shows that $a$ and $b$ commute. Finally, let $\kappa^\chi$ denote the bi-free cumulants of $(\mathcal{A}, \varphi)$, and change the notations for the free and bi-free cumulants of $(\mathcal{A}, \varphi_N)$ to $c^N$ and $c^\chi$, respectively. Then we have

$$\varphi(a^mb^n) = \sum_{\tau \in NC_\chi(m+n)} \kappa^\chi_\tau(a, b)$$

$$= \lim_{N \to \infty} \sum_{\tau \in NC_\chi(m+n)} N(N-1) \cdots (N-|\tau|+1) \cdot c^\chi_\tau(a_{N^k}, b_{N^k})$$

$$= \sum_{\tau \in NC_\chi(m+n)} \lim_{N \to \infty} N(N-1) \cdots (N-|\tau|+1) \cdot c^\chi_\tau(a_{N^k}, b_{N^k}).$$

By induction on the number of arguments in the bi-free cumulants, we have

$$\kappa^\chi_\tau(a, b) = \lim_{N \to \infty} N(N-1) \cdots (N-|\tau|+1) \cdot c^\chi_\tau(a_{N^k}, b_{N^k})$$

for all $\tau \in NC_\chi(m+n)$. In particular, when $\tau = 1_{m+n}$, we have

$$\kappa^\chi_{m+n}(a, b) = \lim_{N \to \infty} N \cdot c^\chi_{m+n}(a_{N^k}, b_{N^k})$$

$$= \lim_{N \to \infty} N \cdot c^\chi_{m,n}(a_{N^k}, b_{N^k})$$

$$= \lim_{N \to \infty} N \cdot \varphi_N(a_{N^k}b_{N^k})$$

by Lemmas 2.4 and 3.3 again.

3.2. **Operators on full Fock spaces.** Recall that the full Fock space over a Hilbert space $\mathcal{H}$ is defined as

$$\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes n},$$

where $\Omega$ is a distinguished vector of norm one, called the vacuum vector. This gives us a $C^*$-probability space $(B(\mathcal{F}(\mathcal{H})), \tau_\mathcal{H})$, where $\tau_\mathcal{H}$, called the vacuum expectation state, is defined as $\tau_\mathcal{H}(T) = \langle T\Omega, \Omega \rangle$ for $T \in B(\mathcal{F}(\mathcal{H}))$. For our purposes, we are mainly interested in the creation, annihilation, and gauge operators on $\mathcal{F}(\mathcal{H})$, defined as follows.

**Definition 3.4.** Let $f \in \mathcal{H}$ and $T \in B(\mathcal{H})$.

1. The **left creation operator given by the vector $f$**, denoted $l(f) \in B(\mathcal{F}(\mathcal{H}))$, is determined by the formulas $l(f)(\Omega) = f$ and

$$l(f)(\xi_1 \otimes \cdots \otimes \xi_n) = f \otimes \xi_1 \otimes \cdots \otimes \xi_n$$

for all $n \geq 1$ and all $\xi_1, \ldots, \xi_n \in \mathcal{H}$. The adjoint $l(f)^*$ of $l(f)$ is called the left annihilation operator given by the vector $f$.

2. The **right creation operator given by the vector $f$**, denoted $r(f) \in B(\mathcal{F}(\mathcal{H}))$, is determined by the formulas $r(f)(\Omega) = f$ and

$$r(f)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_1 \otimes \cdots \otimes \xi_n \otimes f$$

for all $n \geq 1$ and all $\xi_1, \ldots, \xi_n \in \mathcal{H}$. The adjoint $r(f)^*$ of $r(f)$ is called the right annihilation operator given by the vector $f$. 

(3) The left gauge operator associated to $T$, denoted $\Lambda_l(T) \in B(\mathcal{F}(\mathcal{H}))$, is determined by the formulas $\Lambda_l(T)(\Omega) = 0$ and

$$\Lambda_l(T)(\xi_1 \otimes \cdots \otimes \xi_n) = (T\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n$$

for all $n \geq 1$ and all $\xi_1, \ldots, \xi_n \in \mathcal{H}$.

(4) The right gauge operator associated to $T$, denoted $\Lambda_r(T) \in B(\mathcal{F}(\mathcal{H}))$, is determined by the formulas $\Lambda_r(T)(\Omega) = 0$ and

$$\Lambda_r(T)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_1 \otimes \cdots \otimes \xi_{n-1} \otimes (T\xi_n)$$

for all $n \geq 1$ and all $\xi_1, \ldots, \xi_n \in \mathcal{H}$.

**Remark 3.5.** Let $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$, and let $\mathcal{B}_i$ and $\mathcal{C}_i$ be the $C^*$-algebras generated by $\{l(f) : f \in \mathcal{H}_i\} \cup \{\Lambda_l(T) : T\mathcal{H}_i \subset \mathcal{H}_i \text{ and } T|_{\mathcal{H}_i \otimes \mathcal{H}_i} = 0\}$ and $\{r(f) : f \in \mathcal{H}_i\} \cup \{\Lambda_r(T) : T\mathcal{H}_i \subset \mathcal{H}_i \text{ and } T|_{\mathcal{H}_i \otimes \mathcal{H}_i} = 0\}$, respectively. Then $((\mathcal{B}_i, \mathcal{C}_i))_{i \in I}$ is bi-free in $\mathcal{F}(\mathcal{H}), \tau_\mathcal{H}.$ [2] Section 6.

**Proposition 3.6.** Let $\mathcal{H}$ be a Hilbert space. For any $f, g \in \mathcal{H}$, $T_1 = T_1^*, T_2 = T_2^* \in B(\mathcal{H})$, and $\lambda_1, \lambda_2 \in \mathbb{R}$, the self-adjoint operators

$$a = l(f) + l(f)^* + \Lambda_l(T_1) + \lambda_1 \cdot 1 \quad \text{and} \quad b = r(g) + r(g)^* + \Lambda_r(T_2) + \lambda_2 \cdot 1$$

commute if and only if $\langle f, g \rangle = 0$, $T_1g = T_2f$, and $[T_1, T_2] = 0$. Moreover, if $a$ and $b$ commute, then the distribution $\mu_{(a,b)}$ of $(a,b)$ is bi-free infinitely divisible, i.e. for every $n \in \mathbb{N}$, $\mu_{(a,b)}$ can be written as an $n$-fold bi-free convolution of some planar probability measure.

**Proof.** Without loss of generality, we may assume $\lambda_1 = \lambda_2 = 0$. First, we consider the commutativity between $a$ and $b$. Since $l(f)r(g)\Omega = r(g)l(f)\Omega$ and $l(f)r(g)(\xi_1 \otimes \cdots \otimes \xi_n) = r(g)(l(f))(\xi_1 \otimes \cdots \otimes \xi_n)$ for all $\xi_1, \ldots, \xi_n \in \mathcal{H}$ with $n \geq 1$, it follows that

$$l(f)r(g) = r(g)l(f) \quad \text{and} \quad l(f)^*r(g)^* = r(g)^*l(f)^*.$$  \hspace{1cm} (3.3)

Next, note that we have $l(f)r(g)^*\Omega = 0$, $r(g)^*l(f)\Omega = \langle f, g \rangle \Omega$, and $l(f)r(g)^* = r(g)^*l(f)$ on $\mathcal{F}(\mathcal{H}) \otimes \mathbb{C}\Omega$. On the other hand, we have $r(g)(l(f)^*\Omega = 0$, $l(f)^*r(g)\Omega = \langle g, f \rangle \Omega$, and $l(f)^*r(g) = r(g)(l(f)^*\Omega$ on $\mathcal{F}(\mathcal{H}) \otimes \mathbb{C}\Omega$, which imply that the operator

$$A = l(f)r(g)^* + l(f)^*r(g) - r(g)^*l(f) - r(g)l(f)^*$$

vanishes on $\mathcal{F}(\mathcal{H}) \otimes \mathbb{C}\Omega$ and satisfies

$$A\Omega = -i2\langle f, g \rangle \Omega.$$  \hspace{1cm} (3.4)

Similarly, the conditions $l(f)\Lambda_r(T_2)\Omega = 0$, $\Lambda_r(T_2)l(f)\Omega = T_2f$, and $l(f)\Lambda_r(T_2) = \Lambda_r(T_2)l(f)$ on $\mathcal{F}(\mathcal{H}) \otimes \mathbb{C}\Omega$ combining with the conditions $r(g)\Lambda_l(T_1)\Omega = 0$, $\Lambda_l(T_1)r(g)\Omega = T_1g$, and $r(g)\Lambda_l(T_1) = \Lambda_l(T_1)r(g)$ on $\mathcal{F}(\mathcal{H}) \otimes \mathbb{C}\Omega$ imply that the operator

$$B = l(f)\Lambda_r(T_2) + \Lambda_l(T_1)r(g) - \Lambda_r(T_2)l(f) - r(g)\Lambda_l(T_1)$$

vanishes on $\mathcal{F}(\mathcal{H}) \otimes \mathbb{C}\Omega$ and satisfies

$$B\Omega = T_1g - T_2f.$$  \hspace{1cm} (3.5)
One can also see from the identities \( l(f)^* \Lambda_c(T_2) \xi = (T_2 \xi, f) \), \( \Lambda_c(T_2) l(f)^* \xi = 0 \) for \( \xi \in \mathcal{H} \), and \( l(f)^* \Lambda_r(T_2) = \Lambda_r(T_2) l(f)^* \) on \( \mathcal{F}(\mathcal{H}) \ominus \mathcal{H} \), and the identities \( \Lambda_l(T_1) r(g)^* \xi = 0 \), \( r(g)^* \Lambda_l(T_1) \xi = (T_1 \xi, g) \), and \( \Lambda_l(T_1) r(g)^* = r(g)^* \Lambda_l(T_1) \) that the operator

\[
C = l(f)^* \Lambda_r(T_2) + \Lambda_l(T_1) r(g)^* - \Lambda_r(T_2) l(f)^* - r(g)^* \Lambda_l(T_1)
\]

vanishes on \( \mathcal{F}(\mathcal{H}) \ominus \mathcal{H} \) and

\[
(3.6) \quad C \xi = \langle \xi, (T_2 f - T_1 g) \rangle, \quad \xi \in \mathcal{H}.
\]

Finally, the conditions that \( \Lambda_l(T_1) \Lambda_r(T_2) \xi = T_1 T_2 \xi \), \( \Lambda_r(T_2) \Lambda_l(T_1) \xi = T_2 T_1 \xi \) for \( \xi \in \mathcal{H} \), and \( \Lambda_l(T_1) \Lambda_r(T_2) = \Lambda_r(T_2) \Lambda_l(T_1) \) on \( \mathcal{F}(\mathcal{H}) \ominus \mathcal{H} \) imply that the operator

\[
D = \Lambda_l(T_1) \Lambda_r(T_2) - \Lambda_r(T_2) \Lambda_l(T_1)
\]

vanishes on \( \mathcal{F}(\mathcal{H}) \ominus \mathcal{H} \) and

\[
(3.7) \quad D \xi = (T_1 T_2 - T_2 T_1) \xi, \quad \xi \in \mathcal{H}.
\]

The desired result now follows from the fact that \( ab - ba = A + B + C + D \) and the established identities \( \text{[3.3]} - \text{[3.7]} \).

For the second assertion, fix \( n \in \mathbb{N} \) and let

\[
\mathcal{H}_n = \mathcal{H} \oplus \cdots \oplus \mathcal{H}.
\]

Furthermore, let

\[
\tilde{a} = l \left( \frac{f \oplus \cdots \oplus f}{\sqrt{n}} \right) + l \left( \frac{f \oplus \cdots \oplus f}{\sqrt{n}} \right)^* + \Lambda_l(T_1 \oplus \cdots \oplus T_1)
\]

and

\[
\tilde{b} = r \left( \frac{g \oplus \cdots \oplus g}{\sqrt{n}} \right) + r \left( \frac{g \oplus \cdots \oplus g}{\sqrt{n}} \right)^* + \Lambda_r(T_2 \oplus \cdots \oplus T_2).
\]

Then \( [\tilde{a}, \tilde{b}] = 0 \) and the distribution of \((a, b)\) in \((B(\mathcal{F}(\mathcal{H})), \tau_{\mathcal{H}})\) is same as the distribution of \((\tilde{a}, \tilde{b})\) in \((B(\mathcal{F}(\mathcal{H}_n)), \tau_{\mathcal{H}_n})\). Note that \( \tilde{a} \) and \( \tilde{b} \) can be written as

\[
\tilde{a} = \left[ l \left( \frac{f \oplus 0 \oplus \cdots \oplus 0}{\sqrt{n}} \right) + l \left( \frac{f \oplus 0 \oplus \cdots \oplus 0}{\sqrt{n}} \right)^* + \Lambda_l(0 \oplus \cdots \oplus 0) \right] + \cdots + \left[ l \left( \frac{0 \oplus \cdots \oplus 0 \oplus f}{\sqrt{n}} \right) + l \left( \frac{0 \oplus \cdots \oplus 0 \oplus f}{\sqrt{n}} \right)^* + \Lambda_l(0 \oplus \cdots \oplus 0 \oplus T_1) \right]
\]

and

\[
\tilde{b} = \left[ r \left( \frac{g \oplus 0 \oplus \cdots \oplus 0}{\sqrt{n}} \right) + r \left( \frac{g \oplus 0 \oplus \cdots \oplus 0}{\sqrt{n}} \right)^* + \Lambda_r(0 \oplus \cdots \oplus 0 \oplus 0) \right] + \cdots + \left[ r \left( \frac{0 \oplus \cdots \oplus 0 \oplus g}{\sqrt{n}} \right) + r \left( \frac{0 \oplus \cdots \oplus 0 \oplus g}{\sqrt{n}} \right)^* + \Lambda_r(0 \oplus \cdots \oplus 0 \oplus T_2) \right].
\]

Denote the \( n \) summands of \( \tilde{a} \) (respectively, of \( \tilde{b} \)) in the above summations by \( \tilde{a}_1, \ldots, \tilde{a}_n \) (respectively, \( \tilde{b}_1, \ldots, \tilde{b}_n \)), then \( \{\tilde{a}_k, \tilde{b}_k\}_{1 \leq k \leq n} \) are bi-free and identically distributed by Remark 3.3. Moreover, for every \( 1 \leq k \leq n \), \( \tilde{a}_k \) and \( \tilde{b}_k \) are commuting and self-adjoint random variables, thus the distribution of \((\tilde{a}_k, \tilde{b}_k)\) is a Borel probability measure on \( \mathbb{R}^2 \).
where

By Theorem 3.1, we have

\[ \text{If } [a, b] = 0, \text{ then the bi-free cumulants of } (a, b) \text{ are given as follows:} \]

\[ \kappa_{1,0}(a, b) = \lambda_1, \quad \kappa_{0,1}(a, b) = \lambda_2, \]

\[ \kappa_{m,0}(a, b) = \kappa_m(l(f)^*, \Lambda_1(T_1), \ldots, \Lambda_1(T_1), l(f)) = \langle T_1^{m-2} f, f \rangle, \quad m \geq 2, \]

\[ \kappa_{0,n}(a, b) = \kappa_n(r(g)^*, \Lambda_r(T_2), \ldots, \Lambda_r(T_2), r(g)) = \langle T_2^{n-2} g, g \rangle, \quad n \geq 2, \]

and

\[ \kappa_{m,n}(a, b) = \langle \Lambda_{l}(T_1)^{m-1} l(f) \Omega, \Lambda_r(T_2)^{n-1} r(g) \Omega \rangle = \langle T_1^{m-1} f, T_2^{n-1} g \rangle \]

for \( m, n \geq 1 \).

**Proof.** The equalities concerning \( \kappa_1(a), \kappa_1(b), \kappa_m(a), \) and \( \kappa_n(b) \) are known results in one-variable free probability theory, see, for instance, \([10, \text{Proposition 13.5}]\). For the equality concerning \( \kappa_{m,n}(a, b) \), we use the same setup as in the proof of Proposition 3.6.

Observe that the random variables

\[ l(f), l(f)^*, r(g), r(g)^*, \Lambda_{l}(T_1), \Lambda_{r}(T_2) \]

in \((B(F(H)), \tau_H)\) have the same joint distribution as the random variables

\[ l\left(\frac{f \oplus \cdots \oplus f}{\sqrt{n}}\right), l\left(\frac{f \oplus \cdots \oplus f}{\sqrt{n}}\right)^*, r\left(\frac{g \oplus \cdots \oplus g}{\sqrt{n}}\right), r\left(\frac{g \oplus \cdots \oplus g}{\sqrt{n}}\right)^*, \]

\[ \Lambda_{l}(T_1 \oplus \cdots \oplus T_1), \Lambda_{r}(T_2 \oplus \cdots \oplus T_2) \]

in \((B(F(H_n)), \tau_{H_n})\). Moreover, the latter random variables are the sums of \( n \) random variables, with the summands have the same joint distribution as the random variables

\[ \frac{1}{\sqrt{n}} l(f), \frac{1}{\sqrt{n}} l(f)^*, \frac{1}{\sqrt{n}} r(g), \frac{1}{\sqrt{n}} r(g)^*, \Lambda_{l}(T_1), \Lambda_{r}(T_2) \]

in \((B(F(H)), \tau_H)\). Expanding \( \kappa_{m,n}(a, b) \) using the multilinearity of the cumulants and the commutativity of the random variables [\( l(f) + l(f)^*/\sqrt{n} + \Lambda_{l}(T_1) \) and \( r(g) + r(g)^*/\sqrt{n} + \Lambda_{r}(T_2) \)], we see that each summand is of the form \( \kappa_{m+n}(c_1, \ldots, c_{m}, d_1, \ldots, d_{n}) \), where

\[ c_i \in \{l(f), l(f)^*, \Lambda_{l}(T_1)\} \quad \text{and} \quad d_j \in \{r(g), r(g)^*, \Lambda_{r}(T_2)\}. \]

By Theorem 3.1, we have

\[ \kappa_{m+n}(c_1, \ldots, c_{m}, d_1, \ldots, d_{n}) = \lim_{n \to \infty} n \cdot (\tilde{c}_1 \cdots \tilde{c}_m \tilde{d}_1 \cdots \tilde{d}_n \Omega, \Omega) \]

where

\[ \tilde{c}_i \in \left\{\frac{l(f)}{\sqrt{n}}, \frac{l(f)^*}{\sqrt{n}}, \Lambda_{l}(T_1)\right\} \quad \text{and} \quad \tilde{d}_j \in \left\{\frac{r(g)}{\sqrt{n}}, \frac{r(g)^*}{\sqrt{n}}, \Lambda_{r}(T_2)\right\}. \]
Since $m, n \geq 1$, and by the definitions of how creation, annihilation, and gauge operators act on the full Fock spaces, the only non-vanishing summand of $\kappa_{m,n}(a, b)$ is

$$\kappa_{m+n}(l(f)^*, \Lambda_l(T_1), \ldots, \Lambda_l(T_1), \Lambda_r(T_2), \ldots, \Lambda_r(T_2), r(g))$$

which is equal to

$$\lim_{n \to \infty} n \cdot \left( \frac{1}{\sqrt{n}} l(f)^* \Lambda_l(T_1)^{m-1} \Lambda_r(T_2)^{n-1} R \frac{1}{\sqrt{n}} r(g) \Omega, \Omega \right) = (T_1^{m-1} f, T_2^{n-1} g)$$

by the hypothesis that $a$ and $b$ commute. \qed

3.3. **Bi-free Lévy-Hinčin representations.** The next lemma establishes the relations between the bi-free $\mathcal{R}$-transforms of a two-faced pair of commuting self-adjoint random variables in some $C^*$-probability space and free $\mathcal{R}$-transforms.

**Lemma 3.8.** For any two-faced pair $(a, b)$ of commuting self-adjoint random variables in a $C^*$-probability space $(\mathcal{A}, \varphi)$, we have

$$\mathcal{R}_{(a,b)}(z, 0) = z\mathcal{R}_a(z) \quad \text{and} \quad \mathcal{R}_{(a,b)}(0, w) = w\mathcal{R}_b(w)$$

for $z$ and $w$ in a small neighborhood of 0.

**Proof.** By the representation of $\mathcal{R}_{(a,b)}(z, w)$ given in Theorem 2.6 it suffices to show that

$$\lim_{w \to 0} \frac{zw}{G_{(a,b)}(K_a(z), K_b(w))} = 1$$

for small $z \neq 0$. For $(z, w)$ in a deleted neighborhood of $(0, 0)$, we have

$$G_{(a,b)}(z^{-1}, w^{-1}) = w\varphi((z^{-1} - 1a)^{-1}(1 - wb)^{-1}),$$

thus

$$\frac{zw}{G_{(a,b)}(K_a(z), K_b(w))} = \frac{z(w\mathcal{R}_b(w) + 1)}{\varphi((K_a(z)1 - a)^{-1}(1 - b/K_b(w))^{-1})}.$$ 

Since $\lim_{w \to 0} 1/K_b(w) = 0$ and $\lim_{w \to 0} w\mathcal{R}_b(w) = 0$, it follows that

$$\lim_{w \to 0} \frac{zw}{G_{(a,b)}(K_a(z), K_b(w))} = \varphi((K_a(z)1 - a)^{-1}) = \frac{z}{G_a(K_a(z))} = 1$$

as desired. The second equality can be shown in a similar way. \qed

Recall that a probability measure $\nu$ on $\mathbb{R}$ is $\mathbb{R}$-infinitely divisible if and only if its $\mathcal{R}$-transform is of the form

$$(3.8) \quad \mathcal{R}_\nu(z) = \kappa_\nu^* + \int_\mathbb{R} \frac{z}{1 - zs} d\rho_\nu(s),$$

where $\rho_\nu$, called the Lévy measure of $\nu$, is a finite positive Borel measure on $\mathbb{R}$. The integral representation (3.8) is usually referred to as the free Lévy-Hinčin representation for $\nu$.

Denote by $\mathcal{C}_0[s, t]$ the algebra of polynomials in $\mathbb{C}[s, t]$ with vanishing constant term, i.e. $p(0, 0) = 0$ if $p \in \mathcal{C}_0[s, t]$. As indicated in (2.2), given a real 2-sequence $R = \{R_{m,n}\}_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}}$, one can equip $\mathcal{C}_0[s, t]$ with a sesquilinear form $[\cdot, \cdot]_R$ defined by

$$(3.9) \quad [s^{m_1}t^{n_1}, s^{m_2}t^{n_2}]_R = R_{m_1+m_2,n_1+n_2}$$
for \((m_1, n_1), (m_2, n_2) \neq (0, 0)\).

**Definition 3.9.** A real 2-sequence \(R = \{R_{m,n}\}_{(m,n) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}}\) is said to be conditionally positive semi-definite if

\[
[p, p]_R \geq 0
\]

for all \(p \in \mathbb{C}_0[s,t]\), where \([\cdot, \cdot]_R\) is the sesquilinear form defined above. The 2-sequence \(R\) is said to be conditionally bounded if there exists a finite number \(L > 0\) such that

\[
|[s^m t^n p, p]_R| \leq L^{m+n} \cdot [p, p]_R
\]

for all \(p \in \mathbb{C}_0[s,t]\) and \(m, n \geq 0\).

Note that conditional positive semi-definiteness and conditional boundedness of \(R\) do not depend on the values of \(R_{1,0}\) and \(R_{0,1}\) as they do not appear in any summand of \([p, p]_R\) and \([s^m t^n p, p]_R\) for any \(p \in \mathbb{C}_0[s,t]\).

We are now ready to characterize bi-free infinitely divisible distributions and provide their bi-free Lévy-Hinčin representations.

**Theorem 3.10.** Let \(\mu\) be a compactly supported planar probability measure and let \(\kappa = \{\kappa^n_{m,n}\}_{(m,n) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}}\) be the bi-free cumulants of \(\mu\), then the following statements are equivalent.

1. The measure \(\mu\) is bi-free infinitely divisible.
2. The 2-sequence \(\kappa\) is conditionally positive semi-definite and conditionally bounded in the sense defined in Definition 3.9.
3. There exist finite positive Borel measures \(\rho_1\) and \(\rho_2\) with compact supports on \(\mathbb{R}^2\) and a finite Borel measure \(\rho\) on \(\mathbb{R}^2\) satisfying \(td\rho_1(s,t) = s\rho(s,t)\) and \(sd\rho_2(s,t) = t\rho(s,t)\) such that

\[
\mathcal{R}_\mu(z,w) = z\mathcal{R}_1(z) + w\mathcal{R}_2(w) + \int_{\mathbb{R}^2} \frac{zw}{(1-zs)(1-wt)} \, d\rho(s,t)
\]

holds for \((z, w)\) in a neighborhood of \((0,0)\), where

\[
\mathcal{R}_1(z) = \kappa^n_{1,0} + \int_{\mathbb{R}^2} \frac{z}{1-zs} \, d\rho_1(s,t) \quad \text{and} \quad \mathcal{R}_2(w) = \kappa^n_{0,1} + \int_{\mathbb{R}^2} \frac{w}{1-wt} \, d\rho_2(s,t).
\]

Moreover, if conditions (1)-(3) hold, then the functions \(\mathcal{R}_1\) and \(\mathcal{R}_2\) in (3) are the \(\mathcal{R}\)-transforms of free infinitely divisible distributions.

**Proof.** Suppose first that assertion (1) holds, i.e. for each \(N \in \mathbb{N}\), there are commuting self-adjoint random variables \(a_N\) and \(b_N\) in some \(C^*\)-probability space \((\mathcal{A}_N, \varphi_N)\) with \(\mu_N \boxplus \cdots \boxplus \mu_N = \mu\). Then, by Theorem 3.1, the bi-free cumulants of \(\mu\) are given by

\[
\kappa^n_{m,n} = \lim_{N \to \infty} N \cdot \varphi_N(a_N^n b_N^m), \quad (m, n) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}.
\]
Observe that for any polynomial \( p = p(s, t) = \sum_{j=1}^{l} c_j s^{m_j} t^{n_j} \in \mathbb{C}_0[s, t] \), we have

\[
[p, p]_\kappa = \sum_{j,k=1}^{l} c_j \bar{\kappa}^n_{m_j,m_k + m_k,m_j + n_k}
= \lim_{N \to \infty} N \cdot \varphi_N(a_N^{m_j+m_k} b_N^{n_j+n_k})
= \lim_{N \to \infty} \varphi_N \left( \sum_{j,k=1}^{l} c_j \bar{\mu}^n_{a_N + b_N} \right)
= \lim_{N \to \infty} \varphi_N (p(a_N, b_N) p(a_N, b_N)^*) \geq 0,
\]

which yields the conditional positive semi-definiteness of \( \kappa \). On the other hand, viewing \( \mu \) as the distribution of a two-faced pair \((a, b)\) of commuting self-adjoint random variables in some \( C^*\)-probability space and using Lemma \ref{lemma8.5} we have for \( N \in \mathbb{N} \) and for \( z \) in a small deleted neighborhood of 0,

\[
z \mathcal{R}_a(z) = \mathcal{R}_{(a,b)}(z, 0) = N \cdot \mathcal{R}_{(a_N,b_N)}(z, 0) = N \cdot z \mathcal{R}_{a_N}(z) = z \mathcal{R}_{\mu_{a_N}^{\otimes}}(z),
\]

where \( \mu_{a_N}^{\otimes} = \mu_{a_N} \otimes \cdots \otimes \mu_{a_N} \). This shows that the distribution of \( a \) is \( \otimes \)-infinitely divisible and \( \mu_{a_N}^{\otimes} = \mu_a \). We thus conclude from \ref{lemma8.5} Lemma 8.5] the existence of a finite number \( L > 0 \) such that \( \text{supp}(\mu_{a_N}) \subset [-L, L] \) for all \( N \). Similarly, the distribution of \( b \) is \( \otimes \)-infinitely divisible, \( \mu_{b_N}^{\otimes} = \mu_b \), and \( \text{supp}(\mu_{b_N}) \subset [-L, L] \) for all \( N \) if \( L \) is chosen large enough. This implies that the supports of the distributions \( \{\mu_{(a_N,b_N)}\}_{N \in \mathbb{N}} \) are uniformly bounded. Indeed, for all \( c > L \) and \( m, n \in \mathbb{N} \), we have

\[
c^{2m} \mu_{(a_N,b_N)}(\{(s,t) : |s| \geq c\}) \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} s^{2m} d\mu_{(a_N,b_N)}(s, t)
= \varphi_N(a_N^{2m}) = \int_{\mathbb{R}^2} s^{2m} d\mu_{a_N}(s) \leq L^{2m},
\]

and similarly \( \mu_{(a_N,b_N)}(\{(s,t) : |t| \geq c\}) \leq (L/c)^{2n} \), which allow us to conclude that \( \text{supp}(\mu_{(a_N,b_N)}) \subset [-L, L]^2 \). Hence for \( p \in \mathbb{C}_0[s, t] \) as before, we have

\[
|s^m t^n p, p|_\kappa = \sum_{j,k=1}^{l} c_j \bar{\kappa}^n_{m_j,m_k + m_k,m_j + n_k}
= \lim_{N \to \infty} N \cdot \varphi_N(a_N^{m_j} b_N^{n_j} p(a_N, b_N)p(a_N, b_N)^*)
\leq \lim_{N \to \infty} \int_{\mathbb{R}^2} |s|^m |t|^n |p(s, t)|^2 d\mu_N(s, t)
\leq L^{m+n} \cdot \lim_{N \to \infty} N \cdot \int_{\mathbb{R}^2} |p(s, t)|^2 d\mu_N(s, t) = L^{m+n} \cdot |p, p|_\kappa,
\]
as desired. Hence, assertion (2) holds.
Moreover, the inequality $H$ indicates that $\mathcal{I} = \{p_0 \in \mathbb{C}_0[s,t] : [p_0,p_0]_\kappa = 0\}$ is an ideal of $\mathbb{C}_0[s,t]$ as

$$0 \leq [s^m t^n p_0, s^m t^n p_0]_\kappa \leq L^{2(m+n)} \cdot [p_0,p_0]_\kappa = 0$$

for all $p_0 \in \mathcal{I}$. Let $\mathcal{H}_0$ be the quotient vector space $\mathbb{C}_0[s,t]/\mathcal{I}$. If $h = p+\mathcal{I}$ and $g = q+\mathcal{I}$ are in $\mathcal{H}_0$, then

$$\langle h, g \rangle_0 := [p, q]_\kappa$$

can be verified to be a well-defined inner product on $\mathcal{H}_0$ by the Cauchy-Schwarz inequality. Let $\mathcal{H}$ be the Hilbert space obtained by completing $\mathcal{H}_0$ with respect to the norm defined by the inner product (3.10). Consider now the linear transformations $T_1, T_2 : \mathcal{H}_0 \rightarrow \mathcal{H}$ defined by

$$T_1 h = sp(s,t) + \mathcal{I} \quad \text{and} \quad T_2 h = tp(s,t) + \mathcal{I}$$

for $h = p+\mathcal{I} \in \mathcal{H}_0$. Note that both $T_1$ and $T_2$ are well-defined since $\mathcal{I}$ is an ideal. Moreover, the inequality

$$\|T_1 h\|^2 = \langle T_1 h, T_1 h \rangle_0 = [sp, sp]_\kappa \leq L^2 \cdot \|h\|^2$$

shows that $T_1$ can be extended to a bounded linear operator on $\mathcal{H}$. On the other hand, for $k = q+\mathcal{I}$, where $q = \sum_{j=1}^l d_j s^{u_j} t^{v_j} \in \mathbb{C}_0[s,t]$, we have

$$\langle T_1^* h, k \rangle_0 = \langle h, T_1 k \rangle_0 = \sum_{j,k=1}^l c_j d_k \kappa_{m_j+u_k+1, n_j+v_k}$$

$$= \sum_{j,k=1}^l c_j d_k \kappa_{m_j+1, u_k, n_j+v_k} = \langle T_1 h, k \rangle_0,$$

from which we see that $T_1$ is self-adjoint. Similarly, one can show that $T_2$ extends to a self-adjoint operator in $B(\mathcal{H})$. To finish the proof of (2) $\Rightarrow$ (1), let us consider the operators

$$a = l(f) + l(f)^* + \Lambda_l(T_1) + \kappa_{l, 0}^\mu \cdot 1 \quad \text{and} \quad b = r(g) + r(g)^* + \Lambda_r(T_2) + \kappa_{0, 1}^\mu \cdot 1,$$

where $f = s+\mathcal{I}$ and $g = t+\mathcal{I}$, in the $C^*$-probability space $B(\mathcal{F}(\mathcal{H}), \tau_\mathcal{H})$. Clearly, $\exists \langle f, g \rangle = \exists \kappa_{l, 0}^\mu = 0$, $T_1 g = st + I = T_2 f$, and $T_1 T_2 = T_2 T_1$ as

$$T_2 T_1 h = tsp(s,t) + I = T_1 T_2 h$$

for $h = p+\mathcal{I}$ in the dense subspace $\mathcal{H}_0$ of $\mathcal{H}$. Hence, by Proposition 3.6 we see that $(a, b)$ is a two-faced pair of commuting self-adjoint random variables in $B(\mathcal{F}(\mathcal{H}), \tau_\mathcal{H})$ whose distribution $\mu_{(a, b)}$ is infinitely divisible. Next, we show that $\mu_{(a, b)} = \mu$ by claiming
that all the bi-free cumulants of \((a, b)\) agree with the corresponding bi-free cumulants \(\kappa\) of \(\mu\). The claim follows directly from Proposition \ref{prop:3.7}. Indeed, for all \(m, n \geq 1\), we have

\[
\kappa_{m,n}^{(a,b)} = \kappa_{m+n}(a, \ldots, a, b, \ldots, b) = \langle T_1^{m-1} f, T_2^{n-1} g \rangle_0 = [s^m, t^n]_\kappa = \kappa_{m,n}^\mu.
\]

If \(m \geq 2\) and \(n = 0\), then we have

\[
\kappa_{m,0}^{(a,b)} = \kappa_m(a) = \langle T_1^{m-2} f, f \rangle_0 = [s^{m-1}, s]_\kappa = \kappa_{m,0}^\mu,
\]

and similarly we have \(\kappa_{0,n}^{(a,b)} = \kappa_{0,n}^\mu\) for all \(n \geq 2\). Clearly, we also have

\[
\kappa_{1,0}^{(a,b)} = \kappa_1(a) = \kappa_{1,0}^\mu \quad \text{and} \quad \kappa_{0,1}^{(a,b)} = \kappa_1(b) = \kappa_{0,1}^\mu.
\]

Thus the two distributions \(\mu_{(a,b)}\) and \(\mu\) coincide, and hence assertion (1) holds.

In what follows, we show that assertions (2) and (3) are equivalent. Suppose first that assertion (2) holds. Since the 2-sequences \(\{\theta_{m,n}^{(1)}(m,n)\in\mathbb{Z}_+^2\} \) and \(\{\theta_{m,n}^{(2)}(m,n)\in\mathbb{Z}_+^2\} \) defined by \(\theta_{m,n}^{(1)} = \kappa_m^\mu + 2_n\) and \(\theta_{m,n}^{(2)} = \kappa_m^\mu + n\) are positive semi-definite and bounded, it follows from Theorem \ref{thm:2.7} that there exist two finite positive Borel measures \(\rho_1\) and \(\rho_2\) with compact supports on \(\mathbb{R}^2\) such that

\[
\kappa_{m+2,n}^\mu = \int_{\mathbb{R}^2} s^m t^n d\rho_1(s,t), \quad m, n \geq 0
\]

and

\[
\kappa_{m,n+2}^\mu = \int_{\mathbb{R}^2} s^m t^n d\rho_2(s,t), \quad m, n \geq 0.
\]

Observe that for \(z\) and \(w\) small enough so that \(\|zT_1\| < 1\) and \(\|wT_2\| < 1\), following the notations introduced in the proof of (2) \(\Rightarrow\) (1), we have

\[
\sum_{m,n \geq 1} \kappa_{m,n}^\mu z^m w^n = zw \sum_{m,n \geq 0} \langle (zT_1)^m f, (wT_2)^n g \rangle_0
\]

\[
= zw \langle (1 - T_1)^{-1} f, (1 - T_2)^{-1} g \rangle_0.
\]

Let the spectral resolutions of \(T_1\) and \(T_2\) be

\[
T_1 = \int_{\mathbb{R}} s dE_1(s) \quad \text{and} \quad T_2 = \int_{\mathbb{R}} t dE_2(t).
\]

Since \(E_1(s)\) commutes with \(E_2(t)\) for all \(s\) and \(t\), it follows from the spectral theorem that

\[
\sum_{m,n \geq 1} \kappa_{m,n}^\mu z^m w^n = \int_{\mathbb{R}^2} \frac{zw}{(1 - zs)(1 - wt)} d\rho(s,t),
\]

where \(\rho(s,t) = \langle E_1(s)E_2(t)f, g \rangle_0\) is a finite compactly supported Borel measure on \(\mathbb{R}^2\). A simple calculation shows that

\[
\kappa_{m+1,n+1}^\mu = \int_{\mathbb{R}^2} s^m t^n d\rho(s,t), \quad m, n \geq 0.
\]

Note that for all \(m, n \geq 0\), we have

\[
\int_{\mathbb{R}^2} (s^m t^n) t d\rho_1(s,t) = \kappa_{m+2,n+1}^\mu = \int_{\mathbb{R}^2} (s^m t^n) s d\rho(s,t)
\]
and
\[
\int_\mathbb{R}^2 (s^m t^n) s \, d\rho_2(s, t) = \kappa_{m+1, n+2}^\mu = \int_\mathbb{R}^2 (s^m t^n) t \, d\rho(s, t),
\]
from which, along with the Stone-Weierstrass theorem, we deduce that
\[
t d\rho_1(s, t) = s d\rho(s, t) \quad \text{and} \quad s d\rho_2(s, t) = t d\rho(s, t).
\]
Now, for \(|z|\) sufficiently small, we have
\[
\sum_{m \geq 2} \kappa_{m, 0}^\mu z^m = z^2 \sum_{m \geq 0} \kappa_{m+2, 0}^\mu z^m = z^2 \sum_{m \geq 0} (sz)^m \int_\mathbb{R}^2 d\rho_1(s, t) = \int_\mathbb{R}^2 \frac{z^2}{1 - sz} \, d\rho_1(s, t),
\]
and, similarly, for \(|w|\) sufficiently small, we have
\[
\sum_{n \geq 2} \kappa_{0, n}^\mu w^n = \int_\mathbb{R}^2 \frac{w^2}{1 - wt} \, d\rho_2(s, t).
\]
Combining the above conclusions with the characterization of \(\mathfrak{B}\)-infinitely divisible distributions, we conclude that assertion (3) holds. Conversely, if assertion (3) holds, then one can easily see that the bi-free cumulants of \(\mu\) are given by the formulas (3.11)–(3.13).

Hence, for any polynomial \(p = \sum_{j=1}^l c_j s^{m_j} t^{n_j} \in \mathbb{C}[s, t]\), we have
\[
\langle p, p \rangle_\kappa = \sum_{j, k=1}^l c_j c_k \kappa_{m_j + m_k, n_j + n_k} \int_\mathbb{R}^2 |p(s, t)|^2 \frac{d\rho(s, t)}{st} \geq 0,
\]
where \(d\rho(s, t)/(st) = d\rho_1(s, t)/s^2\) is a positive measure on \(\mathbb{R}^2\) (in the integrand of the integral above, the products of the forms \(s \times (1/s)\) and \(t \times (1/t)\) are both interpreted as 1 if \(s\) or \(t\) is zero). Indeed, for \(I_1 = \{j \in I : m_j = 0\}\) and \(I_2 = \{j \in I : n_j = 0\}\), where \(I = \{1, \cdots, l\}\), we have the following cases: (i) \((j, k) \in I_1 \times I_2\); (ii) \((j, k) \in I_2 \times I_1\); (iii) \((j, k) \in (I_1 \cup I_2)^c \times I_1\). In (i),
\[
\int_\mathbb{R}^2 s^{m_j + m_k} t^{n_j + n_k} \frac{d\rho(s, t)}{st} = \int_\mathbb{R}^2 t^{n_j - 1} s^{m_k} t^{n_k} \frac{d\rho(s, t)}{s}
\]
can be simplified to
\[
\int_\mathbb{R}^2 t^{n_j - 1} s^{m_k} t^{n_k - 1} \frac{d\rho_2(s, t)}{st} = \kappa_{m_k, n_j + n_k} = \kappa_{m_j + m_k, n_j + n_k}
\]
if \(m_k = 0\), and simplified to
\[
\int_\mathbb{R}^2 t^{n_j - 1} s^{m_k - 1} t^{n_k} \frac{d\rho(s, t)}{s} = \kappa_{m_j + m_k, n_j + n_k}
\]
if \(n_k = 0\). Similarly, in (ii), we have
\[
\int_\mathbb{R}^2 s^{m_j + m_k} t^{n_j + n_k} \frac{d\rho(s, t)}{st} = \kappa_{m_j + m_k, n_j + n_k}
\]
In case (iii), we have
\[
\int_\mathbb{R}^2 s^{m_j + m_k} t^{n_j + n_k} \frac{d\rho(s, t)}{st} = \int_\mathbb{R}^2 s^{m_j - 1} t^{n_j - 1} s^{m_k} t^{n_k} \frac{d\rho(s, t)}{s} = \kappa_{m_j + m_k, n_j + n_k}.
\]
Thus one has the desired equality (3.14). Moreover, by a similar argument, for \( m, n \geq 0 \) we have
\[
|s^{m+n}| p \rho | = \int_{\mathbb{R}^2} s^m t^n |p(s, t)|^2 \frac{d\rho(s, t)}{st},
\]
from which, along with (3.14), we deduce that \( |s^{m+n}| p \rho | \leq L^{m+n} |p| \rho | \), where \( L = \sup \{|s|, |t|: (s, t) \in \text{supp}(\rho)\} < \infty \). This yields assertion (2) and completes the proof.

**Example 3.11.** Let us see some examples of \( \mathbb{B} \)-infinitely divisible distributions and their bi-free Lévy-Hinčin representations.

1. Let \( a = l(f) + l(f)^* \) and \( b = r(g) + r(g)^* \) with \( \Im(f, g) = 0 \) be two semi-circular random variables in some \( C^* \)-probability space \( (B(\mathcal{F}(\mathcal{H})), \tau_\mathcal{H}) \). Such a two-faced pair \( (a, b) \) is called a bi-free Gaussian pair (see [14, Section 7]). By Propositions 3.6 and 3.7, the only non-vanishing bi-free cumulants of \( (a, b) \) are
\[
\kappa_{2,0}^\mu = \|f\|^2, \quad \kappa_{0,2}^\mu = \|g\|^2, \quad \text{and} \quad \kappa_{1,1}^\mu = \langle f, g \rangle,
\]
where \( \mu \) denotes the distribution of \( (a, b) \). Hence,
\[
\mathcal{R}_\mu(z, w) = \|f\|^2 z^2 + \langle f, g \rangle zw + \|g\|^2 w^2,
\]
and we conclude from Theorem 3.10 that \( \mu \) is a \( \mathbb{B} \)-infinitely divisible distribution with \( \rho_1 = \|f\|^2 \delta_{(0,0)} \), \( \rho_2 = \|g\|^2 \delta_{(0,0)} \), and \( \rho = \langle f, g \rangle \delta_{(0,0)} \).

2. Let \( \lambda > 0 \) and \( (\alpha, \beta) \in \mathbb{R}^2 \). For \( N \in \mathbb{N} \), let
\[
\mu_N = \left( 1 - \frac{\lambda}{N} \right) \delta_{(0,0)} + \frac{\lambda}{N} \delta_{(\alpha, \beta)}
\]
and let \( \{(a_{N,k}, b_{N,k})\}_{k=1}^N \) be pairs of commuting self-adjoint random variables which are bi-free and identically distributed with distribution \( \mu_N \) in some \( C^* \)-probability space \( (\mathcal{A}_N, \varphi_N) \). Note that for \( m, n \geq 0 \) with \( m + n \geq 1 \), we have
\[
\lim_{N \to \infty} N \cdot \varphi_N(a_{N,1}^m b_{N,1}^n) = \lim_{N \to \infty} N \cdot \frac{\lambda}{N} \alpha^m \beta^n = \lambda \alpha^m \beta^n,
\]
thus
\[
(a_{N,1} + \cdots + a_{N,N} b_{N,1} + \cdots + b_{N,N}) \overset{\text{dist}}{\to} (a, b),
\]
where \( (a, b) \) is a two-faced pair of commuting self-adjoint random variables in some \( C^* \)-probability space \( (\mathcal{A}, \varphi) \). Hence,
\[
\mathcal{R}_{(a, b)}(z, w) = \sum_{m,n \geq 0, m+n \geq 1} \lambda (\alpha z)^m (\beta w)^n
\]
\[
= z\mathcal{R}_{P_\alpha}(z) + w\mathcal{R}_{P_\beta}(w) + \int_{\mathbb{R}^2} \frac{zw}{(1-zs)(1-wt)} d\rho(s, t),
\]
where \( P_\alpha \) and \( P_\beta \) are the free Poisson distributions with rate \( \lambda \) and jump sizes \( \alpha \) and \( \beta \), respectively, and \( \rho = \lambda st \delta_{(\alpha, \beta)} \). Note that \( \rho_1 = \lambda s^2 \delta_{(\alpha, \beta)} \) and \( \rho_2 = \lambda t^2 \delta_{(\alpha, \beta)} \), which shows that the distribution \( \mu \) of \( (a, b) \) is \( \mathbb{B} \)-infinitely divisible by Theorem 3.10. We call \( \mu \) the **bi-free Poisson distribution** with rate \( \lambda \) and jump size \( (\alpha, \beta) \).
(3) For any \( \lambda > 0 \) and compactly supported planar probability distribution \( \nu \), consider the distribution
\[
\mu_N = \left(1 - \frac{\lambda}{N}\right) \delta_{(0,0)} + \frac{\lambda}{N} \nu.
\]
Since for any \((m, n) \in \mathbb{Z}^2_+\), the limit
\[
\kappa_{m,n} := \lim_{N \to \infty} N \cdot \int_{\mathbb{R}^2} s^m t^n \, d\mu_N(s,t) = \lambda \int_{\mathbb{R}^2} s^m t^n \, d\nu(s,t)
\]
exists, we conclude by Theorem 3.1 that there exists a planar probability distribution \( \mu \) so that \( \kappa_{m,n}^{\mu} = \kappa_{m,n} \) for any \((m, n) \in \mathbb{Z}^2_+\). Then simple calculations show that
\[
R_\mu(z, w) = \sum_{m \geq 1} \kappa_{m,0}^\mu \frac{z^m}{m!} + \sum_{n \geq 1} \kappa_{0,n}^\mu \frac{w^n}{n!} + \sum_{m,n \geq 1} \kappa_{m,n}^\mu \frac{z^m w^n}{m! n!}
\]
holds in a neighborhood of \((0,0)\), where
\[
R_1(z) = \kappa_{1,0} + \int_{\mathbb{R}} \frac{z}{1 - zs} \lambda s^2 \, d\nu(s,t)
\]
and
\[
R_2(w) = \kappa_{0,1} + \int_{\mathbb{R}} \frac{w}{1 - wt} \lambda t^2 \, d\nu(s,t),
\]
and \( dp(s,t) = std\nu(s,t) \). We call this \( \boxplus \)-infinitely divisible distribution \( \mu \) a compound bi-free Poisson distribution.

4. BI-FREE LÉVY PROCESSES

In classical probability theory, there is an important class of stochastic processes, called Lévy processes, where each process has independent and stationary increments. The non-commutative analogues of these processes are called free Lévy processes, first studied by Biane. We refer to [5] for details. In this section, we shall discuss the relation between bi-free infinitely divisible distributions and stationary processes with bi-free increments.

**Definition 4.1.** A bi-free Lévy process \( (Z_t)_{t \geq 0} \) is a family of pairs of commuting self-adjoint random variables in some \( C^* \)-probability space, that is, \( Z_t = (X_t, Y_t) \) where \( X_t = X_t^*, Y_t = Y_t^* \), and \([X_t, Y_t] = 0\), with the following properties:

1. \( Z_0 = (0,0) \);
2. for any set of times \( 0 \leq t_0 < t_1 < \cdots < t_n \), the increments
   \[
   Z_{t_0}, Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \ldots, Z_{t_n} - Z_{t_{n-1}}
   \]
   are bi-freely independent, where \( Z_t - Z_s := (X_t - X_s, Y_t - Y_s) \);
3. for all \( s \) and \( t \) in \([0, \infty)\), the distribution of \( Z_{s+t} - Z_s \) depends only on \( t \);
4. the distribution of \( Z_t \) tends to \( \delta_{(0,0)} \) weakly as \( t \to 0^+ \).

We have the following relation between bi-free infinitely divisible distributions and bi-free Lévy processes.
Theorem 4.2. (1) Let \((Z_t)_{t \geq 0}\) be a bi-free Lévy process and let \(\mu_t\) be the distribution of \(Z_t, t \geq 0\), then \(\mu_1\) is bi-free infinitely divisible and for any \(T > 0\), the distributions \((\mu_t)_{0 \leq t \leq T}\) have uniformly bounded supports. Moreover, the family \((\mu_t)_{t \geq 0}\) satisfies the bi-free semigroup property

\[
\mu_s \boxplus \boxplus \mu_t = \mu_{s+t}, \quad s, t \geq 0,
\]

and for any \(t \geq 0\), the identity

\[
R_{\mu_t}(z, w) = tR_{\mu_1}(z, w)
\]

holds in a neighborhood of \((0, 0)\).

(2) For any bi-free infinitely divisible compactly supported planar measure \(\mu\), there exists a bi-free Lévy process \((Z_t)_{t \geq 0}\) such that the distribution of \(Z_1\) is equal to \(\mu\).

Proof. First, we prove assertion (2). Using the Hilbert space \(\mathcal{H}\), the vectors \(f, g\) in \(\mathcal{H}\), and the operators

\[
a = l(f) + (l(f))^* + \Lambda_1(T_1) + \kappa^\mu_{1,0} \quad \text{and} \quad b = r(g) + (r(g))^* + \Lambda_\tau(T_2) + \kappa^\mu_{0,1}
\]
in \(B(\mathcal{H})\) constructed in the proof of Theorem 3.10, the bi-free cumulants of the pair of random variables \((a, b)\) coincide with the corresponding bi-free cumulants of \(\mu\). Let \(K = L^2(\mathbb{R}_+, dx) \otimes \mathcal{H}\), where \(\mathbb{R}_+ = [0, \infty)\). For any Borel set \(I\) in \(\mathbb{R}_+\), denote by \(\chi_I\) the characteristic function of \(I\) and \(M_I\) the multiplication operator by \(\chi_I\) in \(B(L^2(\mathbb{R}_+, dx))\). Furthermore, let

\[
f_t = \chi_I \otimes f, \quad g_t = \chi_I \otimes g, \quad A_I = M_I \otimes T_1, \quad \text{and} \quad B_I = M_I \otimes T_2.
\]

Consider now the family \((Z_t)_{t \geq 0} = ((X_t, Y_t))_{t \geq 0}\), where

\[
X_t = l(f_{[0,t]}) + l(f_{[0,t]})^* + \Lambda_1(A_{[0,t]}) + t \cdot \kappa^\mu_{1,0}
\]

and

\[
Y_t = r(g_{[0,t]}) + r(g_{[0,t]})^* + \Lambda_\tau(B_{[0,t]}) + t \cdot \kappa^\mu_{0,1}
\]

are commuting self-adjoint random variables in the C* probability space \((B(\mathcal{F}(K)), \tau_K)\) (the property \([X_t, Y_t] = 0\) follows from Proposition 3.6). Clearly, conditions (1) and (2) in Definition 4.1 hold for the family \((Z_t)_{t \geq 0}\) constructed above by the fact that \(\{L^2((t_j, t_{j+1}], dx) \otimes \mathcal{H}_j\} \) are pairwise orthogonal Hilbert spaces and Remark 4.5. Moreover, Proposition 5.1 shows that the bi-free cumulants of \(Z_{s+t} - Z_s\) are given by

\[
\kappa_{m,n} = \langle (M_{(s,s+t]} \otimes T_1)^{m-1} \chi_{(s,s+t]} \otimes f), (M_{(s,s+t]} \otimes T_2)^{n-1} \chi_{(s,s+t]} \otimes g) \rangle
\]

\[
= \langle \chi_{(s,s+t]} \otimes T_1^{m-1} f, \chi_{(s,s+t]} \otimes T_2^{n-1} g \rangle
\]

\[
= t \langle T_1^{m-1} f, T_2^{n-1} g \rangle = t \kappa_{m,n}^{Z_1}
\]

for \(m, n \geq 1\), where \(\kappa_{m,n}^{Z_1}\) denotes the bi-free cumulants of \(Z_1\), and

\[
\kappa_{m,0} = \langle (M_{(s,s+t]} \otimes T_1)^{m-2} \chi_{(s,s+t]} \otimes f), \chi_{(s,s+t]} \otimes f \rangle
\]

\[
= \langle \chi_{(s,s+t]} \otimes T_1^{m-2} f, \chi_{(s,s+t]} \otimes f \rangle = t \kappa_{m,0}^{Z_1}
\]

for \(m \geq 2\). Similarly, we have \(\kappa_{0,n} = t \kappa_{0,n}^{Z_1}\) for \(n \geq 2\). By Proposition 5.7, again we obtain \(\kappa_{1,0} = t \kappa_{1,0}^{Z_1}\) and \(\kappa_{0,1} = t \kappa_{0,1}^{Z_1}\). Since the bi-free cumulants of \(Z_{s+t} - Z_s\) depends only
on \( t \), so does the distribution of \( Z_{s+t} - Z_s \). Note that the above calculations also show that the bi-free cumulants of \( Z_1 \) coincide with the corresponding bi-free cumulants of \( \mu \) and the identity \((1.16)\) holds for the family \((\mu_t)_{t \geq 0}\). To finish the proof, it remains to show that \( \mu_t \to \delta_{(0,0)} \) weakly as \( t \to 0^+ \). Observing that \( \sup_{0 \leq t \leq 1} \left( \|X_t\|, \|Y_t\| \right) < \infty \), it is equivalent to showing that the mixed moments of \( \mu_t \) converge to 0 as \( t \to 0^+ \) because the supports of \( \mu_t, 0 \leq t \leq 1 \), are uniformly bounded. Using the fact that \( \kappa_{m,n}^{Z_t} = t \kappa_{m,n}^{Z_t} \to 0 \) as \( t \to 0^+ \) for any \( m, n \geq 0 \) with \( m + n \geq 1 \) and the existence of universal polynomials on the relations of bi-free cumulants and mixed moments of planar probability distributions [14], the claim holds. Hence, the family \((Z_t)_{t \geq 0}\) constructed above is a bi-free Lévy process with distributions \((\mu_t)_{t \geq 0}\).

Next, we prove assertion (1). Let \((Z_t)_{t \geq 0}\) be a bi-free Lévy process with distributions \((\mu_t)_{t \geq 0}\). Then \( Z_s - Z_0 \) and \( Z_{s+t} - Z_s \) are bi-free, and \((Z_s - Z_0) + (Z_{s+t} - Z_s) = Z_{s+t}\). Thus we have shown that \( \mu_s \boxplus \boxtimes \mu_t = \mu_{s+t} \). By the semigroup property \( \mu_t \) is bi-free infinitely divisible, thus by assertion (2) there exists a bi-free semigroup \((\nu_t)_{t \geq 0}\) of planar probability distributions such that \( \nu_t = \mu_1, \nu_t \to \delta_{(0,0)} \) weakly as \( t \to 0^+ \), and \( \mathcal{R}_{\nu_t} = t \mathcal{R}_{\mu_1} \). By the semigroup property \( \mu_t = \nu_t \) for \( t \in \mathbb{Q} \cap [0, \infty) \). Again by the semigroup property continuity at 0 implies continuity at any \( t \), which yields \( \mu_t = \nu_t \) for all \( t \geq 0 \). Hence \( \mathcal{R}_{\mu_t} = t \mathcal{R}_{\mu_1} \) as claimed. This finishes the proof.

5. Bi-free convolution semigroups

Let \( \mu \) be a compactly supported probability measure on \( \mathbb{R} \), then we have a free convolution semigroup \((\mu_t)_{t \geq 1}\), where the existence of \( \mu_t \) for large \( t \) was shown by Bercovici and Voiculescu [4], and later extended to all \( t \geq 1 \) by Nica and Speicher [9]. In the bi-free setting, we will use the method of Nica and Speicher to show the existence of the bi-free convolution semigroup generated by a compactly supported probability measure on \( \mathbb{R}^2 \). Let us first recall some definitions and results regarding free compressions. We refer the reader to [10] Lecture 14 for details.

**Definition 5.1.** Let \((A, \varphi)\) be a non-commutative probability space and \( p \in A \) a projection (i.e. \( p^2 = p \)) such that \( \varphi(p) \neq 0 \), then we have the compression \((p \mathcal{A}p, \varphi^{p \mathcal{A}p})\), where
\[
p \mathcal{A}p = \{ pap : a \in A \}
\]
and
\[
\varphi^{p \mathcal{A}p}(\cdot) = \frac{1}{\varphi(p)} \varphi(\cdot)
\]
restricted to \( p \mathcal{A}p \). The compression is also a non-commutative probability space with unit element \( p = p \cdot 1 \cdot p \). Moreover, if \((\kappa_n)_{n \geq 1}\) denote the free cumulants corresponding to \( \varphi \), then the free cumulants corresponding to \( \varphi^{p \mathcal{A}p} \) will be denoted by \((\kappa_n^{p \mathcal{A}p})_{n \geq 1}\).

Suppose that \((A, \varphi)\) is a non-commutative probability space and \( p, a_1, \ldots, a_m \in A \) such that \( p \) is a projection with \( \varphi(p) \neq 0 \) and \( p \) is free from \( \{a_1, \ldots, a_m\} \). Then recall from [10] Theorem 14.10 that we have
\[
\kappa_n^{p \mathcal{A}p}(p a_{i(1)} p, \ldots, p a_{i(n)} p) = \frac{1}{\varphi(p)} \kappa_n(\varphi(p) a_{i(1)}, \ldots, \varphi(p) a_{i(n)})
\]
for all $n \geq 1$ and all $1 \leq i(1), \ldots, i(n) \leq m$. The above result can be extended to the bi-free setting as follows. If $p, a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathcal{A}$ are self-adjoint random variables such that $\varphi(p) \neq 0$, $p$ is free from $\{a_1, \ldots, a_r, b_1, \ldots, b_r\}$, and $[a_i, b_j] = 0$ for all $1 \leq i, j \leq r$, then we have

$$
\kappa_{m,n}^{p,M}(p_{a(1)}p, \ldots, p_{a(m)}p, p_{b(1)}p, \ldots, p_{b(n)}p) = \frac{1}{\varphi(p)} \kappa_{m+n}(\varphi(p)a_{1}, \ldots, \varphi(p)a_{m}, \varphi(p)b_{1}, \ldots, \varphi(p)b_{n})
$$

for all $m, n \geq 0$ with $m + n \geq 1$ and all $1 \leq i(1), \ldots, i(m), j(1), \ldots, j(n) \leq r$.

This gives us the existence of the bi-free convolution semigroups.

**Theorem 5.2.** Let $\mu$ be a compactly supported probability measure on $\mathbb{R}^2$, then there exists a convolution semigroup $(\mu_t)_{t \geq 1}$ of compactly supported probability measures on $\mathbb{R}^2$ such that

$$
\mu_1 = \mu \quad \text{and} \quad \mu_{s+t} = \mu_s \boxplus \mu_t
$$

for all $s, t \geq 1$, and the mapping $t \mapsto \mu_t$ is continuous with respect to the weak$^*$ topology on planar probability measures.

**Proof.** Let $(a, b)$ be a two-faced pair in a $C^*$-probability space $(\mathcal{A}, \varphi)$ such that $a = a^*$, $b = b^*$, $[a, b] = 0$, and the distribution of $(a, b)$ is $\mu$. Let $p \in \mathcal{A}$ be a projection such that $\varphi(p) = 1/t$ and $p$ is free from $\{a, b\}$. Consider the two-faced pair $(p(ta)p, p(tb)p)$ in the compressed space $(pAp, \varphi_pAp)$. Note that the distribution of $(p(ta)p, p(tb)p)$, denoted by $\mu_t$, is also a compactly supported probability measure on $\mathbb{R}^2$. Since $p(ta)p$ and $p(tb)p$ also commute, the bi-free cumulants of $(p(ta)p, p(tb)p)$ are given by the free cumulants as

$$
\kappa_{m,n}^{\mu_t} = \kappa_{m,n}^{p,A}(p(ta)p, p(tb)p) = t\kappa_{m,n}(a, b)
$$

for all $m, n \geq 0$ such that $m + n \geq 1$. Thus

$$
\kappa_{m,n}^{\mu_{s+t}} = (s + t)\kappa_{m,n}^{\mu_s} = \kappa_{m,n}^{\mu_s} + \kappa_{m,n}^{\mu_t}
$$

for all $s, t \geq 1$. This shows that $\mu_{s+t} = \mu_s \boxplus \mu_t$. Moreover, it is clear that $\mu_1 = \mu$, and the mapping $t \mapsto t\kappa_{m,n}^{\mu_t}$ is continuous, hence all moments and cumulants of $\mu_t$ are continuous in $t$.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN’S UNIVERSITY, JEFFERY HALL, KINGSTON, ONTARIO, K7L 3N6, CANADA

E-mail address: gu.y@queensu.ca, huhuang@mast.queensu.ca, mingo@mast.queensu.ca