Local Coordinate Spaces: a proposed unification of manifolds and fiber bundles, and associated machinery*

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Abstract

This paper presents a unified view of manifolds and fiber bundles, which, while superficially different, have strong parallels. It introduces the notions of an m-atlas and of a local coordinate space, and shows that special cases are equivalent to fiber bundles and manifolds. Along the way it defines some convenient notation, defines categories of atlases, and constructs potentially useful functors.

1 Introduction

Historically, the concept of pseudo-groups allowed unifying manifolds and manifolds with boundary. The definitions of fiber bundles and manifolds have strong parallels, and can be unified in a similar fashion; there are several ways to do so. The central part of this paper, section 9 (Local Coordinate Spaces) on page 53, defines an approach using categories and commutative diagrams that is designed for easy exposition at the possible expense of abstractness and generality. In particular, I have chosen to assume the Axiom of Choice (AOC).

This paper treats atlases as objects of interest in their own right, although it does not give them primacy. It introduces notions that are convenient for use here and others that, while not used here, may be useful for future work. It defines the new notions of model space, m-atlas and of m-atlas morphism. Informally, a model space is a topological space with a category specifying a family of open sets and functions satisfying specified conditions.

Although this paper incidentally defines partial equivalents to manifolds and fiber bundles using model spaces and model atlases, it proposes the more general Local Coordinate Space (LCS) in order to explicitly reflect the role of the group in fiber bundles.

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1 The phrase has been used before, but with a different meaning.
A local coordinate space (LCS) is a space (total space) with some additional structure, including a coordinate model space and an atlas whose transition functions are restricted to morphisms of the coordinate model space; one can impose, e.g., differentiability restrictions, by appropriate choice of the coordinate category. There is an equivalent paradigm that avoids explicit mention of the total space by imposing compatibility conditions on the transition functions, but that approach is beyond the scope of this paper.

This paper defines functors among categories of atlases, categories of model spaces, categories of local coordinate spaces, categories of manifolds and categories of fiber bundles; it constructs more machinery than is customary in order to facilitate the presentation of those categories and functors.

Sections 2 to 7 present nomenclature and give basic results. Section 8 defines m-atlases, m-atlas morphisms and categories of them; lemma 8.25 proves that the defined categories are indeed categories. Section 9 defines local coordinate spaces and categories of them; theorem 9.9 (LCS(\(\mathcal{M}, \Sigma\)) is a category) on page 58 proves that the defined categories are indeed categories.

Section 9.2 (Examples) on page 58 gives some examples of structures that can be represented as local coordinate spaces; section 10 (Equivalence of manifolds) on page 60 and section 11 (Equivalence of fiber bundles) on page 86 present two of the examples in detail, showing the equivalence of manifolds and fiber bundles with special cases of local coordinate spaces by explicitly exhibiting functors to and from local coordinate spaces.

**Remark** 1.1. The unconventional definitions of manifold and fiber bundle are intended to make their relationship to local coordinate spaces more natural.

Most of the lemmata, theorems and corollaries in this paper should be substantially identical to results that are familiar to the reader. What is novel is the perspective and the material directly related to local coordinate spaces. The presentation assumes only a basic knowledge of Category Theory, such as may be found in the first chapter of [Mac Lane, 1998] or [Adámek, Herrlich, Strecker, 1990].

### 1.1 New concepts and notation

This paper introduces a significant number of new concepts and some modifications of the definitions for some conventional concepts. It also introduces some notation of lesser importance. The following are the most important.

1. **Nearly commutative diagram (NCD)**, NCD at a point, locally NCD and special cases with related nomenclature
2. **Model space** and related concepts
3. **Model topology** and M-paracompactness
4. **Signature, \(\Sigma\)-commutation** and related concepts
5. **Local Coordinate Space (LCS)** and related concepts
2 Conventions

A diagram arrow with an Equal-Tilde (A \rightarrow\sim f \rightarrow B) represents an isomorphism.

One with a hook (A \hookrightarrow i \rightarrow B) represents an inclusion map. One with a double arrowhead (A \pi \rightarrow \rightarrow \rightarrow B) represents a surjection.

All diagrams shown are commutative; none are exact.

Blackboard bold upper case will denote specific sets, e.g., the Naturals.

Bold lower case italic letters will refer to sets, sequences and tuples of functions, e.g., f \text{def} = (f_1, f_2).

Bold lower case Latin letters will refer to sequence valued functions of sequences and tuple valued functions of tuples, e.g., \text{range} yields the sequence of ranges of a sequence of functions.

Bold upper case calligraphic (script) letters will refer to sequences of categories, e.g., \text{A} \text{def} = (A_\alpha, \alpha \in A).

Bold upper case italic letters will refer to sequences or tuples, e.g., A = (x, y, z), to sets of them, to sets of topological spaces or to sets of open sets.

Fraktur will refer to topologies and to topology-valued functions, e.g., \mathfrak{Top}.

Functions have a range, domain and relation, not just a relation. Unless otherwise stated, they are assumed to be continuous.

Groups are assumed to be topological groups. The ambiguous notation x^{-1} will be used when it is obvious from context what the group operation \star and the group identity 1_G are.

Lower case Greek letters other than \pi, \rho, \sigma, \phi and \psi will refer to ordinals, possibly transfinite, and to formal labels. A letter with a Greek superscript and a letter with a Latin or numeric superscript always refer to distinct variables.

Lower case \pi will refer to a projection operator

Lower case \rho will refer to a continuous effective group action, i.e., a continuous representation of a group in a homeomorphism group.

Lower case \sigma will refer to a sequence of ordinals, referred to as a signature.

Lower case \phi will refer to a coordinate function.

Lower case Latin letters will refer to

1. elements of a set or sequence
2. functions
3. natural numbers

\text{2Similar to coordinate bundles}
Upper case calligraphic (script) Latin letters will refer to categories and functors. Due to font limitations the special form \( Triv \) will be used instead of lower case calligraphic letters to refer to constructed categories.

Upper case Greek letters other than \( \Sigma \) may refer to

1. ordinal used as the limit of a sequence of consecutive ordinals, e.g., \( x_\alpha, \alpha \preceq A \)
2. ordinal used as the order type of a sequence of consecutive ordinals, e.g., \( x_\alpha, \alpha \prec A \)

Upper case \( \Sigma \) will refer to a sequence of signatures

Upper case Latin letters will refer to

1. Natural numbers
2. Topological spaces
3. Open sets
4. Elements of a sequence or tuple of functions, e.g., \( f_E \) might be \( f_0: E_1 \rightarrow E_2 \).

Upright Latin letters will be used for long names.

The term \( C^k \) includes \( C^\infty \) (smooth) and \( C^\omega \) (analytic).

This paper uses the term morphism in preference to arrow, but uses the conventional Ar.

The term sequence without an explicit reference to \( \mathbb{N} \) will refer to a general ordinal sequence, possibly transfinite.

Sequence numbering, unlike tuple numbering, starts at 0 and the exposition assumes a von Neumann definition of ordinals, so that \( \alpha \in \beta \equiv \alpha < \beta \).

Except where explicitly stated otherwise, all categories mentioned are small categories with underlying sets, but the morphisms will often not be set functions between the objects and there will not always be a forgetful function to \textbf{Set} or \textbf{Top}. By abuse of language no distinction will be made between a category \( \mathcal{A} \) of topological spaces and the concrete category \( (\mathcal{A}, \mathcal{U}) \) over \textbf{Top}. Similarly, no distinction will be made among the object \( U \in \text{Ob}(\mathcal{A}) \), the topological space \( U \) and the underlying set.

When defining a category, the Ordered pair \( (O, M) \) refers to the smallest concrete category over \textbf{Set} or \textbf{Top} whose objects are in \( O \), whose morphisms from \( o^1 \in O \) to \( o^2 \in O \) are functions \( f: o^1 \rightarrow o^2 \) in \( M \) and whose composition is function composition.

When defining a category, the Ordered triple \( (O, M, C) \) refers to the small category whose objects are in \( O \), whose morphisms are in \( M \), whose \( \text{Hom} \) is

\[
\text{Hom}_{(O, M, C)}(o_1 \in O, o_2 \in O) \overset{\text{def}}{=} \{(f, o_1, o_2) \in M\}
\]

and whose composition is \( C \).

By abuse of language I may write “\( S \)” for \( \text{Ob}(S) \), “\( A \in \mathcal{A} \)” for \( A \in \text{Ob}(\mathcal{A}) \), “\( A \subset \mathcal{A} \)” for \( A \subset \text{Ob}(\mathcal{A}) \), “\( A \in \mathcal{A} \subset B \in \mathcal{B} \)” for “the underlying set of \( A \) is
contained in the underlying set of \( B \) and the inclusion \( i: x \in A \mapsto x \in B \) is a morphism” and “\( f: A \rightarrow B \)” for \( f \in \text{Hom}_\mathcal{C}(A, B) \), where \( \mathcal{C} \) is understood by context.

By abuse of language I shall use the same nomenclature for sequences and tuples.

By abuse of language I shall use the \( \times \) and \( \times \) symbols for both Cartesian products of sets and Cartesian products of functions on those sets.

By abuse of language, and assuming AOC, I shall refer to some sets as ordinal sequences, e.g., “\( (C_\alpha, \alpha \in A) \)” for “\( \{C_\alpha \mid \alpha \in A\} \)”, in cases where the order is irrelevant.

By abuse of language, I may omit universal quantifiers in cases where the intent is clear.

In some cases I define a notion similar to a conventional notion and also need to refer to the conventional notion. In those cases I prefix a letter or phrase to the term, e.g., m-paracompact versus paracompact.

3 General notions

This section describes nomenclature used throughout the paper. In some cases this reflects new nomenclature or notions, in others it simply makes a choice from among the various conventions in the literature.

Definition 3.1 (Operations on categories). If \( \mathcal{C} \) is a category then \( x \in \text{Ob}_\mathcal{C} \) iff \( x \) is an object of \( \mathcal{C} \) and \( y \in \text{Ar}_\mathcal{C} \) iff \( y \) is a morphism of \( \mathcal{C} \).

If \( \mathcal{S} \) and \( \mathcal{T} \) are categories then \( \mathcal{S} \subseteq \mathcal{T} \) iff \( \mathcal{S} \) is a subcategory of \( \mathcal{T} \) and \( \mathcal{S} \subseteq \mathcal{T} \) iff \( \mathcal{S} \) is a full subcategory of \( \mathcal{T} \).

If \( \mathcal{S} \) and \( \mathcal{T} \) are categories then the category union of \( \mathcal{S} \) and \( \mathcal{T} \), abbreviated \( \mathcal{S} \cup \mathcal{T} \), is the category whose objects are in \( \mathcal{S} \) or in \( \mathcal{T} \) and whose morphisms are in \( \mathcal{S} \) or in \( \mathcal{T} \).

Definition 3.2 (Identity). \( \text{Id}_\mathcal{S} \) is the identity function on the space \( \mathcal{S} \), \( \text{Id}_o \) is the identity morphism for the object \( o \in \mathcal{S} \), \( \text{Id}_{U, V} \), for \( U \subseteq V \), is the inclusion map, \( \text{Id}_\mathcal{C} \) is the identity functor on the category \( \mathcal{C} \).

\( \text{Id}_{\mathcal{S}_i}, i = 1, 2, \) is the sequence of identity functions for the elements of the sequence \( \mathcal{S}^i \) \( \text{def} = (\mathcal{S}^1, \alpha \prec A) \). Let \( \mathcal{S}^{(1)} \subseteq \mathcal{S}^2 \). Then \( \text{Id}_{\mathcal{S}_1, \mathcal{S}_2} \) is the sequence of inclusion maps \( (\text{Id}_{\mathcal{S}_1, \mathcal{S}_2})_\alpha, \alpha \prec A \) for the elements of the sequences \( \mathcal{S}^i \).

The subscript may be omitted when it is clear from context.

Definition 3.3 (Images). \( f[U] \text{def} = \{f(x) \mid x \in U\} \) is the image of \( U \) under \( f \) and \( f^{-1}[V] \text{def} = \{x \mid f(x) \in V\} \) is the inverse image of \( V \) under \( f \).

\( \text{The object is often expressed as a tuple, e.g., } \text{Id}_{(A, B)} \text{ is the identity morphism for the object } (A, B) \)
Remark 3.4. This notation, adopted from [Kelley, 1955], avoids the ambiguity in the traditional $f(U)$ and $f^{-1}(V)$.

**Definition 3.5** (Projections). $\pi_\alpha$ is the projection function that maps a sequence into element $\alpha$ of the sequence. $\pi_i$ is also the projection function that maps a tuple into element $i$ of the tuple.

**Definition 3.6** (Topological category). A topological category is a small subcategory of $\text{Top}$ or its concrete category over $\text{Set}$.

$\mathcal{T}$ is a full topological category iff it is a topological category and whenever $U^i, V^i \in \mathcal{T}$, $i = 1, 2$, $V^i \subseteq U^i$, $f: U^1 \rightarrow U^2 \in \mathcal{T}$ and $f[V^1] \subseteq V^2$ then $f: V^1 \rightarrow V^2 \in \mathcal{T}$.

**Lemma 3.7** (Inclusions in topological categories are morphisms). Let $\mathcal{T}$ be a full topological category, $S^i \in \mathcal{T}$, $i = 1, 2$, and $S^1 \subseteq S^2$. Then $\text{Id}_{S^1, S^2}$ is a morphism of $\mathcal{T}$.

**Proof.** $\text{Id}_{S^1, S^2} \in \mathcal{T}$, $S^1 \subseteq S^2$ by hypothesis and $S^1 \subseteq S^2$, so $\text{Id}_{S^1, S^2} \in \mathcal{T}$ by definition 3.6. □

**Definition 3.8** (Local morphisms). Let $\mathcal{T}^i$, $i = 1, 2$, be a full topological category and $S^i \in \mathcal{T}^i$. A continuous function $f: S^1 \rightarrow S^2$ is locally a $\mathcal{T}^1\cdot \mathcal{T}^2$ morphism of $S^1$ to $S^2$ iff $\mathcal{T}^1 \subseteq \mathcal{T}^2$ and for every $u \in S^1$ there is an open neighborhood $U_u$ for $u$ and an open neighborhood $V_u$ for $v \equiv f(u)$ such that $f[U_u] \subseteq V_u$ and $f: U_u \rightarrow V_u$ is a morphism of $\mathcal{T}^2$.

Let $\mathcal{T}$ be a full topological category and $S^i \in \mathcal{T}$, $i = 1, 2$. A continuous function $f: S^1 \rightarrow S^2$ is locally a $\mathcal{T}$ morphism of $S^1$ to $S^2$ if it is locally a $\mathcal{T}\cdot \mathcal{T}$ morphism of $S^1$ to $S^2$.

**Lemma 3.9** (Local morphisms). Let $\mathcal{T}^i$, $i = 1, 2, 3$, be a full topological category, $\mathcal{T}^i \subseteq \mathcal{T}^{i+1}$ and $S^i \in \mathcal{T}^i$.

If $f^i: S^i \rightarrow S^{i+1} \in \mathcal{T}^{i+1}$ then $f^i$ is locally a $\mathcal{T}^i\cdot \mathcal{T}^{i+1}$ morphism of $S^i$ to $S^{i+1}$.

**Proof.** Let $u \in S^i$ and $v \equiv f^i(v) \in S^{i+1}$. $S^i$ is an open for $u$, $S^{i+1}$ is an open neighborhood for $v$ and $f^i: S^i \rightarrow S^{i+1} \in \mathcal{T}^{i+1}$ by hypothesis. □

If each $f^i: S^i \rightarrow S^{i+1}$, is locally a $\mathcal{T}^i\cdot \mathcal{T}^{i+1}$ morphism of $S^i$ to $S^{i+1}$ then $f^2 \circ f^1: S^1 \rightarrow S^3$ is locally a $\mathcal{T}^1\cdot \mathcal{T}^3$ morphism of $S^1$ to $S^3$.

**Proof.** Since $\mathcal{T}^1 \subseteq \mathcal{T}^2$ and $\mathcal{T}^2 \subseteq \mathcal{T}^3$, $\mathcal{T}^1 \subseteq \mathcal{T}^3$. Let $u \in S^1$, $v \equiv f^1(u)$ and $w \equiv f^2(v)$. There exist an open neighborhood $U_u$ for $u$, open neighborhoods $V_u$, $V'_v$ for $v$ and an open neighborhood $W_w$ of $w$ such that $f^1[U_u] \subseteq V_u$, $f^1: U_u \rightarrow V_u$ is a morphism of $\mathcal{T}^1$, $f^2[V'_v] \subseteq W_v$ and $f^2: V'_v \rightarrow W_v$ is
a morphism of \( \mathcal{T}^3 \). Then \( \hat{\mathcal{V}}_u \defeq \mathcal{V}_u \cap V'_v \neq \emptyset \), \( \hat{\mathcal{V}}_u \) is an open neighborhood of \( v \) and \( \hat{U}_u \defeq f_i^{-1}[\hat{\mathcal{V}}_u] \) is an open neighborhood for \( u \). \( f^1 \colon \hat{U}_u \longrightarrow \hat{\mathcal{V}}_u \) and \( f^2 \colon \hat{\mathcal{V}}_u \longrightarrow W_v \) are morphisms of \( \mathcal{T}^3 \) by definition 3.6 (Topological category) on page 6 and thus \( f^2 \circ f^1 \colon \hat{U}_u \longrightarrow W_v \) is a morphism of \( \mathcal{T}^3 \). \( \square \)

**Corollary 3.10** (Local morphisms). Let \( \mathcal{T}^i \), \( i = 1, 2 \), be a full topological category, \( \mathcal{T}^i \subseteq \mathcal{T}^{i+1} \), \( S^1 \subseteq S^2 \). Then \( \text{Id}_{S^1,S^2} \) is locally a \( \mathcal{T}^1 \)-\( \mathcal{T}^2 \) morphism of \( S^1 \) to \( S^2 \) and \( \text{Id}_{S^1} \) is locally a \( \mathcal{T} \) morphism of \( S^1 \) to \( S^1 \).

**Proof.** \( S^1 \subseteq S^2 \) because \( S^1 \subseteq S^1 \) and \( S^2 \subseteq S^2 \) by hypothesis, so \( \text{Id}_{S^1,S^2} \) is locally a \( \mathcal{T} \) morphism of \( S^1 \) to \( S^2 \) by hypothesis, so \( \text{Id}_{S^1} \) is locally a \( \mathcal{T} \) morphism of \( S^1 \) to \( S^1 \).

**Definition 3.11** (Sequence functions). Let \( S \defeq (s_\alpha, \alpha \preceq A) \) be a sequence of functions. Then

\[
\text{domain}(S) \defeq (\text{domain}(s_\alpha), \alpha \preceq A) \quad (3.1)
\]

\[
\text{range}(S) \defeq (\text{range}(s_\alpha), \alpha \preceq A) \quad (3.2)
\]

Let \( T \defeq (t_\alpha, \alpha \preceq A) \) be a sequence of functions with \( \text{range}(S) = \text{domain}(T) \). Then their composition is the sequence \( T \circ S \defeq (t_\alpha \circ s_\alpha, \alpha \preceq A) \).

Let \( S \defeq (s_\gamma, \gamma \preceq \Gamma) \), then these functions extract information about the sequence:

\[
\text{head}(S, \Omega) \defeq (s_\gamma, \gamma \preceq \Omega) \quad (3.3)
\]

\[
\text{head}(S) \defeq \text{head}(S, \Gamma) \quad (3.4)
\]

\[
\text{length}_0(S) \defeq \Gamma \quad (3.5)
\]

\[
\text{tail}(S) \defeq S_\Gamma \quad (3.6)
\]

Let \( S \defeq (s_\gamma, \gamma \preceq \Gamma) \), then

\[
\text{length}(S) \defeq \Gamma \quad (3.7)
\]

**Remark 3.12.** If \( \text{length}_0(S) \) is defined then \( \text{length}(S) = \text{length}_0(S) + 1 \). \( \text{length}_0(S) \) is the ordinal type of head(\( S \)), not the ordinal type of \( S \).

Let \( S \defeq (S_\alpha, \alpha \preceq A) \) and \( T \defeq (T_\alpha, \alpha \preceq A) \) be sequences of categories. Then \( S \) is a subcategory sequence of \( T \), abbreviated \( S \subseteq T \), iff every category in \( S \) is a subcategory of the corresponding category in \( T \), i.e., \( (\forall \alpha \preceq A) S_\alpha \subseteq T_\alpha \), and \( S \) is a full subcategory sequence of \( T \), abbreviated \( S \subseteq \text{full}-\text{cat} T \), iff every...
category in $S$ is a full subcategory of the corresponding category in $T$, i.e.,
\[ \left( \forall \alpha \prec A \right) S_{\alpha}^{\text{full-cat}} \subseteq T_{\alpha}. \]

The category sequence union of $S$ and $T$, abbreviated $S^{\text{cat}} \cup T^{\text{cat}}$, is the sequence of category unions of corresponding categories in $S$ and $T$, i.e., $(S_{\alpha} \cup T_{\alpha})$.

**Lemma 3.13 (Sequence functions).** Let $f_i \overset{\text{def}}{=} (f_i^{\alpha}, \alpha \prec A)$, $i = 1, 2, 3$ be sequences of functions with $\text{domain}(f_i) = \text{range}(f_{i-1})$ and $\text{domain}(f_3) = \text{range}(f_2)$. Then $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$.

**Proof.**
\[
(f_3 \circ f_2) \circ f_1 = \\
(f_3^{\alpha} \circ f_2^{\alpha}, \alpha \prec A) = \\
(f_3^{\alpha} \circ (f_2^{\alpha} \circ f_1^{\alpha}), \alpha \prec A) = \\
f_3^{\alpha} \circ (f_2^{\alpha} \circ f_1^{\alpha})
\]

\[\square\]

Let $f \overset{\text{def}}{=} (f_\alpha, \alpha \prec A)$ be a sequence of functions, $D = \text{domain}(f)$ and $R = \text{range}(f)$. Then $\text{ID}_R$ is a left $\overset{0}{\circ}$ identity for $f$ and $\text{ID}_D$ is a right $\overset{0}{\circ}$ identity for $f$.

**Proof.**
\[
\text{Id}_R \overset{0}{\circ} f = \\
(\text{Id}_{\text{range}(f_\alpha)} \circ f_\alpha, \alpha \prec A) = \\
(f_\alpha, \alpha \prec A) = \\
f
\]
\[
f \overset{0}{\circ} \text{Id}_D = \\
(f_\alpha \circ \text{Id}_{\text{domain}(f_\alpha)}, \alpha \prec A) = \\
(f_\alpha, \alpha \prec A) = \\
f
\]

\[\square\]

**Definition 3.14 (Tuple functions).** Let $S \overset{\text{def}}{=} (s_n, n \in [1, N])$ be a tuple of functions. Then
\[
\text{domain}(S) \overset{\text{def}}{=} (\text{domain}(s_n), n \in [1, N]) \quad (3.11)
\]
\[
\text{range}(S) \overset{\text{def}}{=} (\text{range}(s_n), n \in [1, N]) \quad (3.12)
\]

8
Let $T \overset{\text{def}}{=} (t_n, n \in [1, N])$ be a tuple of functions with $\text{range}(S) = \text{domain}(T)$.

Then their composition is the tuple $T \circ S \overset{\text{def}}{=} (t_n \circ s_n, n \in [1, N])$

Let $S \overset{\text{def}}{=} (s_m, m \in [1, M])$ and $T \overset{\text{def}}{=} (t_n, n \in [1, N])$ be tuples. Then the following are tuple functions

\[
\text{head}(S, I) \overset{\text{def}}{=} (s_m, m \in [1, I]) \quad (3.13)
\]
\[
\text{head}(S) \overset{\text{def}}{=} \text{head}(S, M - 1) \quad (3.14)
\]
\[
\text{tail}(T, I) \overset{\text{def}}{=} (t_n, n \in [I, N]) \quad (3.15)
\]
\[
\text{tail}(T) \overset{\text{def}}{=} t_N \quad (3.16)
\]
\[
\text{join}(S, T) \overset{\text{def}}{=} (s_1, \ldots, s_M, t_1, \ldots, t_N) \quad (3.17)
\]

**Lemma 3.15** (Tuple functions). Let $f^i \overset{\text{def}}{=} (f^i_n, n \in [1, N]), i = 1, 2, 3$ be tuples of functions with $\text{domain}(f^2) = \text{range}(f^1)$ and $\text{domain}(f^3) = \text{range}(f^2)$.

Then $(f^3 \circ f^2) \circ f^1 = f^3 \circ (f^2 \circ f^1)$.

Proof.

\[
(f^3 \circ f^2) \circ f^1 =
\]
\[
((f^3_n \circ f^2_n) \circ f^1_n, n \in [1, N]) =
\]
\[
(f^3_n \circ (f^2_n \circ f^1_n), n \in [1, N]) =
\]
\[
f^3 \circ (f^2 \circ f^1)
\]

\[\square\]

Let $f \overset{\text{def}}{=} (f_\alpha, \alpha \prec A)$ be a sequence of functions, $D \overset{\text{def}}{=} \text{domain}(f)$ and $R \overset{\text{def}}{=} \text{range}(f)$. Then $\text{Id}_R$ is a left $\overset{\text{def}}{\circ}$ identity for $f$ and $\text{Id}_D$ is a right $\overset{\text{def}}{\circ}$ identity for $f$.

Proof.

\[
\text{Id}_R \circ f =
\]
\[
(\text{Id}_{\text{range}(f_\alpha)} \circ f_\alpha, \alpha \prec A) =
\]
\[
(f_\alpha, \alpha \prec A) =
\]
\[
f
\]
\[
f \circ \text{Id}_D =
\]
\[
(f_\alpha \circ \text{Id}_{\text{domain}(f_\alpha)}, \alpha \prec A) =
\]
\[
(f_\alpha, \alpha \prec A) =
\]
\[
f
\]

\[\square\]
**Definition 3.16** (Tuple composition for labeled morphisms). Let $M_i \overset{\text{def}}{=} (f^i, o^i_1, o^i_2)$, $i = 1, 2$, be tuples such that each $f^i$ is a sequence of functions or each $f^i$ is a tuple of functions, $\text{range}(f^1) = \text{domain}(f^2)$ and $o^i_2 = o^i_1$. Then

$$M^2 \overset{A}{\circ} M^1 \overset{\text{def}}{=} (f^2 \overset{A}{\circ} f^1, o^1_1, o^2_2) \quad (3.21)$$

**Lemma 3.17** (Tuple composition for labeled morphisms). Let $M_i \overset{\text{def}}{=} (f^i, o^i_1, o^i_2)$, $i = 1, 2, 3$, be tuples such that $f^i$ are sequences or tuples of functions, $\text{range}(f^i) = \text{domain}(f^{i+1})$ and $o^i_2 = o^{i+1}_1$, $i = 1, 2$. Then

$$M^3 \overset{A}{\circ} (M^2 \overset{A}{\circ} M^1) = (M^3 \overset{A}{\circ} M^2) \overset{A}{\circ} M^1 \quad (3.22)$$

**Proof.** From definition 3.16 (Tuple composition for labeled morphisms), lemma 3.13 (Sequence functions) on page 8 and lemma 3.15 (Tuple functions), we have

$$M^3 \overset{A}{\circ} (M^2 \overset{A}{\circ} M^1) =
M^3 \overset{A}{\circ} (f^2 \overset{A}{\circ} f^1, o^1_1, o^2_2) =
(f^3 \overset{A}{\circ} f^2 \overset{A}{\circ} f^1, o^1_1, o^2_2) =
(f^3 \overset{A}{\circ} f^2, o^2_1, o^2_2) \overset{A}{\circ} M^1 =
(M^3 \overset{A}{\circ} M^2) \overset{A}{\circ} M^1 \quad (3.23)$$

**Proof.** From definition 3.16 (Tuple composition for labeled morphisms), lemma 3.13 (Sequence functions) on page 8 and lemma 3.15 (Tuple functions), we have

$$M^3 \overset{A}{\circ} (M^2 \overset{A}{\circ} M^1) =
M^3 \overset{A}{\circ} (f^2 \overset{A}{\circ} f^1, o^1_1, o^2_2) =
(f^3 \overset{A}{\circ} f^2 \overset{A}{\circ} f^1, o^1_1, o^2_2) =
(f^3 \overset{A}{\circ} f^2, o^2_1, o^2_2) \overset{A}{\circ} M^1 =
(M^3 \overset{A}{\circ} M^2) \overset{A}{\circ} M^1 \quad (3.24)$$

Let $D^i \overset{\text{def}}{=} \text{domain}(f^i)$ and $R^i \overset{\text{def}}{=} \text{range}(f^i)$. Then $(\text{Id}_{D^i}, o^2_2, o^2_1)$ is a left $A$ identity for $M^i$ and $(\text{Id}_{D^i}, o^2_1, o^1_1)$ is a right $A$ identity for $M^i$.

**Proof.**

$$\begin{align*}
(\text{Id}_{D^i}, o^2_1, o^2_2) \overset{A}{\circ} M^i &=
(\text{Id}_{D^i} \overset{A}{\circ} f^i, o^2_1, o^2_2) =
(f^i, o^2_1, o^2_2) =
M^i \\
M^i \overset{A}{\circ} (\text{Id}_{D^i}, o^2_1, o^2_2) &=
(f^i \overset{A}{\circ} \text{Id}_{D^i}, o^1_1, o^2_2) =
(f^i, o^1_1, o^2_2) =
M^i
\end{align*} \quad (3.25)$$
Definition 3.18 (Cartesian product of sequence). Let $S^i = (S^i_\alpha, \alpha \prec A)$, $i = 1, 2$, be a sequence and $f = (f_\alpha: S^1_\alpha \rightarrow S^2_\alpha, \alpha \prec A)$ be a sequence of functions, then $\prod S^i = \prod_{\alpha \prec A} S^i_\alpha$ is the generalized Cartesian product of the sequence $S^i$ and $\prod f: S^1 \rightarrow S^2 = \prod_{\alpha \prec A} f_\alpha$ is the generalized Cartesian product of the function sequence $f$.

Definition 3.19 (underline). Let $S^1 = (S^1_\alpha, \alpha \preceq A)$, $S^2 = (S^2_\alpha, \alpha \preceq A)$ be sequences and $f = (f_\alpha: S^1_\alpha \rightarrow S^2_\alpha, \alpha \preceq A)$ be a sequence of functions, then

$$f: head(S^1) \rightarrow head(S^2) \equiv \prod_{\alpha \prec A} head(f) = \prod_{\alpha \prec A} f_\alpha$$

(3.26)

is the function mapping $(s_\alpha \in S^1_\alpha, \alpha \prec A)$ into $(f_\alpha(s_\alpha), \alpha \prec A)$.

Definition 3.20 (Head and tail compositions). Let $f^1: (S^1_\alpha, \alpha \prec A) \rightarrow S^1_A$, $f^2: (S^2_\alpha, \alpha \prec A) \rightarrow S^2_A$ and $g = (g_\alpha: S^1_\alpha \rightarrow S^2_\alpha, \alpha \preceq A)$. Then (see figs. 1 and 2)

$$f^2 \odot g \equiv f^2 \circ g$$

(3.27a)

$$g \cdot f^1 \equiv \text{tail}(g) \circ f^1$$

(3.27b)

Definition 3.21 (Topology functions). Let $S$ be a topological space and $Y$ a subset. Then

1. $\mathcal{Top}(S)$ is the topology of $S$.

2. $\mathcal{Top}(Y, S) \equiv \{ U \cap Y | U \in \mathcal{Top}(S) \}$ is the relative topology of $Y$.

3. $\text{Top}(Y, S) \equiv (Y, \mathcal{Top}(Y, S))$ is $Y$ with the relative topology.
Let $S$ be a set of topological spaces. Then $S_{\text{op}} \defeq \bigcup_{S \in S} S$ is the set of open subspaces of $S$. Let $S$ and $T$ be spaces, $S' \subseteq S$ and $T' \subseteq T$ be subspaces and $f: S \to T$ a function such that $f(S') \subseteq T'$. Then $f: S' \to T'$, also written $f \restriction S'$, is $f$ considered as a function from $S'$ to $T'$.

**Definition 3.22 (Truth space).** The truth set is $\mathbb{T} \defeq \{\text{False}, \text{True}\}$, where $\text{False} \defeq \emptyset$ and $\text{True} \defeq \{\emptyset\}$. $\mathbb{TruthTop} \defeq \{\emptyset, \mathbb{T}\}$ and the truth space $\mathbb{TruthSpace} \defeq (\mathbb{T}, \mathbb{TruthTop})$ is $\mathbb{T}$ with the indiscrete topology.

**Definition 3.23 (Truth category).** The truth category is $\mathbb{T} \defeq (\mathbb{TruthSpace}, \{\text{False} \longrightarrow \text{False}, \text{False} \longrightarrow \text{True}, \text{True} \longrightarrow \text{True}\})$. The truth model space is $\mathbb{TruthSpace} \defeq (\mathbb{TruthSpace}, \mathbb{T})$.

**Definition 3.24 (Constraint functions).** A constraint function is a continuous function with range $\mathbb{TruthSpace}$ or a model function with range $\mathbb{TruthSpace}$.

**Definition 3.25 (Sequence inclusion).** Let $S \defeq (S_\alpha, \alpha < A)$ and $T \defeq (T_\alpha, \alpha < A)$ be sequences. $S \subseteq T$ if $(\forall_{\alpha < A}) S_\alpha \subseteq T_\alpha$ or $(\forall_{\alpha < A}) S_\alpha^{\text{Ob}} \subseteq T_\alpha$.

**Lemma 3.26 (Sequence inclusion).** Let $S \defeq (S_\alpha, \alpha < A)$ be a sequence, $\mathcal{T}_i \defeq (T^i_\alpha, \alpha < A)$, $i = 1, 2$, a sequence of categories, $\mathcal{T}_1^{\text{cat}} \subseteq \mathcal{T}_2$ and $S^0 \subseteq \mathcal{T}_1$. Then $S^0 \subseteq \mathcal{T}_2$. 

![Figure 2: $g \cdot f^1$](image)
Proof. If \( \left( \forall_{\alpha \prec A} \right) S_\alpha \in \mathcal{T}_\alpha^1 \) then \( \left( \forall_{\alpha \prec A} \right) S_\alpha \in \mathcal{T}_\alpha^2 \).

4 Nearly commutative diagrams

The notion of commutative diagrams is very useful in, e.g., Algebraic Topology. Often one encounters commutative diagrams in which two outgoing terminal nodes can be connected by a bridging function such that the resulting diagram is still commutative. This paper uses the term nearly commutative to describe a restricted class of such diagrams.

Let \( \mathcal{C} \) be a full topological category and \( D \) a tree with two branches, whose nodes are topological spaces \( U_i \) and \( V_j \) and whose links are continuous functions \( f_i: U_i \to U_{i+1} \) and \( f'_j: U_j \to U_{j+1} \) between the sets:

\[
D = \{ f_0: U_0 = V_0 \to U_1, \ldots, f_{m-1}: U_{m-1} \to U_m, \quad f'_0: U_0 = V_0 \to V_1, \ldots, f'_{n-1}: V_{m-1} \to V_n \}
\]

with \( U_0 = V_0, U_m \in \mathcal{C} \) and \( V_n \in \mathcal{C} \), as shown in fig. 3.

**Definition 4.1** (Nearly commutative diagrams in category \( \mathcal{C} \)). \( D \) is nearly commutative in category \( \mathcal{C} \) iff the two final nodes are in \( \mathcal{C} \) and there is an isomorphism \( \hat{f}: U_m \xrightarrow{\cong} V_n \) in \( \mathcal{C} \) making the graph a commutative diagram, as shown in fig. 4.

**Definition 4.2** (Nearly commutative diagrams in category \( \mathcal{C} \) at a point). Let \( \mathcal{C} \) and \( D \) be as above and \( x \) be an element of the initial node. \( D \) is nearly
commutative in $\mathcal{C}$ at $x$ iff there are subobjects of the nodes such that the tree formed by replacing the nodes is nearly commutative in $\mathcal{C}$ and $x$ is in the new initial node, as shown in Fig. 5 (Local nearly commutative diagram) on page 15: $x \in U'_{0} = V'_{0}$, $U'_{i} \subseteq U_{i}$, $V'_{j} \subseteq V_{j}$, $f \mid_{U'_{i}}: U'_{i} \rightarrow U_{i+1}$, $f' \mid_{V'_{j}}: V'_{j} \rightarrow V_{j+1}$.

**Definition 4.3** (Locally nearly commutative diagrams in category $\mathcal{C}$). Let $\mathcal{C}$ and $D$ be as above. $D$ is locally nearly commutative in $\mathcal{C}$ iff it is nearly commutative in $\mathcal{C}$ at $x$ for every $x$ in the initial node.

**Lemma 4.4** (Locally nearly commutative diagrams in category $\mathcal{C}$). Let $\mathcal{C}$ and $D$ be as above. If $U_{0} = \emptyset$ then $D$ is locally nearly commutative in $\mathcal{C}$.

**Proof.** $D$ is vacuously locally nearly commutative at every $x \in U_{0}$ since there is no such $x$. 

Let $D$ be locally nearly commutative in $\mathcal{C}$ and let $\hat{U}_{0} = \hat{V}_{0} \subseteq U_{0}$ and $\hat{D}$ be $D$ with $U_{0} = \hat{V}_{0}$ replaced by $\hat{U}_{0} = \hat{V}_{0}$. Then $\hat{D}$ is locally nearly commutative in $\mathcal{C}$.

**Proof.** If $x \in \hat{U}_{0}$ then $x \in U_{0}$ and hence $D$ is locally nearly commutative in $\mathcal{C}$ at $x$. Replacing $U'_{0}$ with $U'_{0} \cap \hat{U}_{0}$ in the definition shows that $\hat{D}$ is locally nearly commutative in $\mathcal{C}$ at $x$.

**Remark 4.5.** It will often be clear from context what the relevant categories are. This paper may use the term “nearly commutative” without explicitly identifying the categories in which the modes are found.
5 Model spaces

Let $S$ be a topological space. We need to formalize the notions of an open cover by sets that are "well behaved" in some sense, e.g., convex, sufficiently small, and of "well behaved" functions among those sets, e.g., preserving fibers, smooth. We do this by associating a category of acceptable sets and functions.

Remark 5.1. Using pseudo-groups, as in [Kobayashi, 1996, p. 1], would not allow restricting model neighborhoods to, e.g., convex sets.

Definition 5.2 (Model spaces). Let $S$ be a topological space and $\mathcal{S}$ a small category whose objects are open subsets of $S$ and whose morphisms are continuous functions. $M = (S, \mathcal{S})$ is a model space for $S$ iff

1. $\text{Ob}(S)$ is an open cover for $S$. Note that it need not be a basis for $S$.
2. $\text{Ob}(S)$ is closed under finite intersections.
3. The morphisms of $\mathcal{S}$ are continuous functions in $S$.
4. If $f: A \rightarrow B$ is a morphism, $A' \in \text{Ob}(S) \subseteq A \in \text{Ob}(S), \ B' \in \text{Ob}(S) \subseteq B \in \text{Ob}(S)$ and $f[A'] \subseteq B'$ then $f \ |_{A'}: A' \rightarrow B'$ is a morphism.
5. If $A' \in \mathcal{S} \subseteq A \in \text{Ob}(S)$ then the inclusion map $\text{Id}_{A',A}: A' \rightarrow A$ is a morphism.
Remark 5.3. This is actually a consequence of item 4, but it is convenient to give it here.

6. Restricted sheaf condition: informally, consistent morphisms can be glued together. Whenever

(a) \( U_\alpha \) and \( V_\alpha, \alpha \prec A \), are objects of \( S \).
(b) \( f_\alpha: U_\alpha \rightarrow V_\alpha \) are morphisms of \( S \).
(c) \( U \overset{\text{def}}{=} \bigcup_{\alpha \prec A} U_\alpha \in \text{Ob}(S) \),
(d) \( V \overset{\text{def}}{=} \bigcup_{\alpha \prec A} V_\alpha \in \text{Ob}(S) \),
(e) \( f: U \rightarrow V \) is a continuous function and for every \( \alpha \prec A \), \( f \) agrees with \( f_\alpha \) on \( U_\alpha \)

then \( f \) is a morphism of \( S \).

\[ \text{Top}(M) \overset{\text{def}}{=} S = \pi_1(M). \]

Let \( U \in \text{Ob} \). Then \( \text{Top}(U, M) \overset{\text{def}}{=} \text{Top}(U, S) \) is \( U \) with the relative topology.

\[ \mathcal{T}\text{op}(M) \overset{\text{def}}{=} \left\{ \text{Top}(U, S) \mid U \in \text{Ob} \right\}, \text{Ar}(S) \]

is the topological category of \( M \).

By abuse of language we write \( U \subseteq M \).

Lemma 5.4 (The topological category of a model space is a full topological category). Let \( M \overset{\text{def}}{=} (S, S) \) be a model space for \( S \). Then \( \mathcal{T}\text{op}(M) \) is a full topological category.

Proof. \( \mathcal{T}\text{op}(M) \) is a small subcategory of \( \textbf{Top} \) by construction. \( \mathcal{T}\text{op}(M) \) is a full topological category by item 4 of definition 5.2. \( \square \)

Definition 5.5 (Model neighborhoods). Let \( S \overset{\text{def}}{=} (S, S) \) be a model space for \( S \). Then the objects of \( S \) are model neighborhoods of \( S \). If \( u \in U \in S \) then \( U \) is a model neighborhood for \( u \). If \( U_i \in S, i = 1, 2 \), and \( U_1 \subseteq U_2 \) then \( U_1 \) is a model subneighborhood of \( U_2 \).

Definition 5.6 (Model subspaces). \( M \overset{\text{def}}{=} (X, \mathcal{X}) \) is a model subspace of \( N \overset{\text{def}}{=} (Y, \mathcal{Y}) \), abbreviated \( M \overset{\text{mod}}{\subseteq} N \), iff

1. \( X \) is a subspace of \( Y \)
2. \( \mathcal{X} \overset{\text{full-cat}}{\subseteq} \mathcal{Y} \)
3. Every object of \( \mathcal{Y} \) contained in \( X \) is an object of \( \mathcal{X} \)
4. When \( f: o_1^{\text{Ob}} \rightarrow o_2^{\text{Ob}} \) is a morphism of \( \mathcal{Y} \) then \( f: o_1 \rightarrow o_2 \) is a morphism of \( \mathcal{X} \).
Let $M \overset{\text{def}}{=} (X, \mathcal{X})$ be a model space and $Y$ a model neighborhood of $M$. Then $\text{Mod}(Y, M)$, the relative model space of $Y$, is $(Y, \mathcal{Y})$, where $\mathcal{Y}$ is the full subcategory of $\mathcal{X}$ containing all model subneighborhoods of $Y$.

5.1 M-nearly commutative diagrams

Let $M \overset{\text{def}}{=} (M, \mathcal{M})$ be a model space and $D$ a tree with two branches, whose nodes are topological spaces $U_i$ and $V_j$ and whose links are continuous functions $f_i: U_i \longrightarrow U_{i+1}$ and $f'_j: U_j \longrightarrow U_{j+1}$ between the spaces:

$$D = \{ f_0: U_0 = V_0 \longrightarrow U_1, \ldots, f_{m-1}: U_{m-1} \longrightarrow U_m, \quad f'_0: U_0 = V_0 \longrightarrow V_1, \ldots, f'_{m-1}: V_{m-1} \longrightarrow V_n \}$$

with $U_0 = V_0, U_m \in \mathcal{M}$ and $V_n \in \mathcal{M}$, as shown in fig. 3 (Uncompleted nearly commutative diagram) on page 13.

Definition 5.7 (M-nearly commutative diagrams). $D$ is M-nearly commutative in model space $M$ iff $D$ is nearly commutative in category $\mathcal{T}_{\text{op}}(M)$.

Definition 5.8 (M-nearly commutative diagrams at a point). Let $M$ and $D$ be as above and $x$ be an element of the initial node. $D$ is M-nearly commutative in $M$ at $x$ iff $D$ is nearly commutative in category $\mathcal{T}_{\text{op}}(M)$ at $x$.

Definition 5.9 (M-locally nearly commutative diagrams). Let $\mathcal{C}$ and $D$ be as above. $D$ is M-locally nearly commutative in $M$ iff $D$ is nearly commutative in category $\mathcal{T}_{\text{op}}(M)$ at every point.

5.2 Trivial model spaces

Informally, a trivial model space of a specific type is one that does not restrict the potential objects and morphisms of its type.

Definition 5.10 (Trivial model spaces). Let $S$ be a topological space and $\mathcal{S}$ the category of all continuous functions between open sets of $S$. Then $S_{\text{triv}} \overset{\text{def}}{=} (S, S)$ is the trivial model space of $S$ and $S_{\text{triv}}$ is a trivial model space.

Let $\mathcal{S}$ be a set of topological spaces. Then $S_{\text{triv}} = \left\{ S' \mid S' \in \mathcal{S} \right\}$ is the set of all trivial model spaces in $\mathcal{S}$ and $S_{\text{triv}}$, the category of all continuous functions between objects of $S_{\text{triv}}$, is the trivial model category of $\mathcal{S}$.

Lemma 5.11 (The trivial model space of $S$ is a model space). Let $S$ be a topological space. Then $S_{\text{triv}}$ is a model space.
Proof. Let \( S \overset{\text{def}}{=} \pi_2(S) \) be a morphism of \( S \), \( A' \in \text{Ob}(S) \subseteq A \) and \( B' \in \text{Ob}(S) \subseteq B \). Then the definition of continuity and \( S \) imply each of the following.

1. \( \text{Ob}(S) = \mathcal{Top}(S) \) and thus is an open cover of \( S \).
2. \( \text{Ob}(S) = \mathcal{Top}(S) \) and thus is closed under finite intersections.
3. All morphisms in \( \text{Ar}(S) \) are continuous.
4. Since \( f: A \rightarrow B \) is a morphism of \( S \), \( f \) is continuous. If \( A' \in \text{Ob}(S) \) then \( A' \) is open. If \( B' \in \text{Ob}(S) \) then \( B' \) is open. If \( f: A \rightarrow B \) is continuous then \( f \mid_{A'}: A' \rightarrow B \) is continuous. If \( f[A'] \subseteq B' \) then \( f \mid_{A'}: A' \rightarrow B' \) is well defined and continuous, hence a morphism.
5. If \( A' \in \text{Ob}(S) \subseteq A \in \text{Ob}(S) \) then
   
   (a) \( A \) and \( A' \) are open.
   
   (b) The inclusion map \( \text{Id}_{A'}: A' \hookrightarrow A \) is continuous.
   
   (c) The inclusion map \( i: A' \hookrightarrow A \) is a morphism \( S \) by the definition of \( S \).

\[ \square \]

5.3 Minimal model spaces

Even if a set of open sets fails one of items 1 and 2 or a set of functions among them fails one of items 3 to 6 in the definition of a model space, there is a minimal model space containing them.

**Definition 5.12** (Minimal model spaces). Let \( S \) be a topological space, \( O \) a set of open sets in \( S \), \( f \) a set of continuous functions between elements of \( O \) and \( S \) the smallest concrete category over \( \text{Top} \) having all sets in \( O \) as objects, having all functions in \( f \) as morphisms and satisfying items 3 to 6 of definition 5.2 (Model spaces) on page 15.

Then

\[
\text{Mod}_{\text{min}}(S, O, f) \overset{\text{def}}{=} (\bigcup O, S) \tag{5.1}
\]

is the minimal model space of \( S \) with neighborhoods \( O \) and neighborhood mappings \( f \).

**Remark 5.13.** The trivial model space \( S \) is a special case.

**Lemma 5.14** (Minimal model spaces are model spaces). Let \( S \) be a topological space, \( O \) a set of open sets in \( S \) and \( f \) a set of continuous functions between elements of \( O \). Then \( (C, C) \overset{\text{def}}{=} \text{Mod}_{\text{min}}(S, O, f) \) is a model space.

**Proof.** \((C, C)\) satisfies the conditions of definition 5.2 (Model spaces) on page 15.
1. Finite intersections of open sets are open and \( C = \bigcup_{U \in \mathcal{O}} U \) by construction

2. \( \text{Ob}(\mathcal{C}) \) is closed under finite intersections by construction

3. Compositions of continuous functions, inclusion maps and restrictions of continuous functions are continuous

4. Restrictions of morphisms are morphisms by construction

5. Inclusion maps are morphisms by construction

6. The restricted sheaf condition holds by construction

5.4 M-paracompact model spaces

Paracompactness is an important property for topological spaces because of partitions of unity. There is an analogous property for model spaces.

Definition 5.15 (Model topology). Let \( M \overset{\text{def}}{=} (S, \mathcal{S}) \) be a model space for \( S \). Then the model topology \( \mathcal{M}^* \) for \( M \) is the topology generated by \( \text{Ob}(\mathcal{S}) \).

Remark 5.16. \( \mathcal{M}^* \) is not guaranteed to be T0 even if \( S \) is T4. However, \( \mathcal{M}^* \) may be normal or regular even if \( S \) is not.

Definition 5.17 (m-paracompactness). A model space \( M \overset{\text{def}}{=} (S, \mathcal{S}) \) is m-paracompact iff \( \mathcal{M}^* \) is regular and every cover of \( S \) by model neighborhoods has a locally finite refinement by model neighborhoods.

Remark 5.18. This is a stronger condition than merely requiring \( \mathcal{M}^* \) to be paracompact.

Theorem 5.19 (m-paracompactness and paracompactness). If \( M = (S, \mathcal{S}) \) is m-paracompact and the model neighborhoods form a basis for \( S \) then \( S \) is paracompact.

Proof. Let \( R \) be an open cover of \( S \). Since the model neighborhoods form a basis, every set in \( R \) is a union of model neighborhoods and thus there is a refinement \( R_1 \) by model neighborhoods. Since by hypothesis \( M \) is m-paracompact, \( R_1 \) has a locally finite refinement \( R_2 \) by model neighborhoods. Since model neighborhoods are open sets, \( R_2 \) is a locally finite refinement of \( R \) in the conventional sense.

5.5 Model functions and model categories

It is convenient to have a notion of mappings between model spaces that are well behaved in some sense, e.g., fiber preserving; using that notion it is then possible to group model spaces into categories.
Definition 5.20 (Model functions). Let $M^i \defeq (S^i, S^i)$, $i = 1, 2$, be a model space and $f: S^1 \to S^2$ be a continuous function. $f$ is a model function iff the inverse images of model neighborhoods are model neighborhoods and the images of model neighborhoods are contained in model neighborhoods.

\[
\left( \forall V \in S^2 \right) f^{-1}[V] \in S^1 \tag{5.2}
\]

\[
\left( \forall U \in S^1 \right) \left( \exists V \in S^2 \right) f[U] \subseteq V \tag{5.3}
\]

By abuse of language we write $f: M^1 \to M^2$ both for $f$ considered as a model function and for $f$ considered as a continuous function.

Let $M^n \defeq (S^n, S^n)$ be model spaces, $M^i \subseteq M^i$ and $f[S^1] \subseteq S^2$. Then by abuse of language we write $f: M^1 \to M^2$ for $f | S^1, S^2$.

Lemma 5.21 (Composition of model functions). Let $M_i \defeq (S_i, S_i)$, $i = 1, 2, 3$, be model spaces and $f_1: M_1 \to M_2$, $f_2: M_2 \to M_3$ model functions. Then $f_2 \circ f_1$ is a model function.

Proof. Let $U_i \in S_i$, $i = 1, 2, 3$. Since $f_2$ is a model function, $f_2^{-1}[U_3] \in S_2$. Since $f_1$ is a model function, $(f_2 \circ f_1)^{-1}[U_3] = f_1^{-1}[f_2^{-1}[U_3]] \in S_1$. Since $f_1$ is a model functions, $f_1[U_1]$ is contained in a model neighborhood $V_2$. Since $f_2$ is a model functions, $f_2[V_2]$ is contained in a model neighborhood $V_3$. Then $(f_2 \circ f_1)[U_1] \subseteq V_3$.

Definition 5.22 (Model homeomorphisms). Let $M^i \defeq (S^i, S^i)$ and $M^2 \defeq (S^2, S^2)$ be model spaces and $f: S^1 \to S^2$ be a model function. $f$ is a model homeomorphism iff it is also invertible and its inverse is a model function.

Definition 5.23 (Model categories). A category $\mathcal{M}$ is a model category iff

1. the objects of $\mathcal{M}$ are model spaces
2. the morphisms of $\mathcal{M}$ are model functions.
3. composition is functional composition.
4. Every model subspace of an object in $\mathcal{M}$ is in $\mathcal{M}$ and the inclusion map is a morphism.
5. If $S^i \defeq (S^i, S^i)$, $i = 1, 2$, are subspaces of $S \defeq (S, S)$ in $\mathcal{M}$ and $f: S^1 \to S^2$ is a morphism of $S$ then $f: S^1 \to S^2$ is a morphism of $\mathcal{M}$.

Definition 5.24 (Trivial model categories). Let $M$ be a set of model spaces. Then $\text{Mod}_{\text{triv}}(M)$ is the category of all model functions between model subspaces of model spaces in $M$.
**Definition 5.25** (Local m-morphisms). Let $\mathcal{M}^i$, $i = 1, 2$, be model categories and $S^i \overset{\text{def}}{=} (S^i, S^i)^{\text{Ob}} \in \mathcal{M}^i$.

A continuous function $f: S^1 \to S^2$ is locally an m-morphism of $S^1$ to $S^2$ iff $S^1 \subseteq S^2$ and for every $u \in S^1$ there is a model neighborhood $U_u$ for $u$ and a model neighborhood $V_u$ for $v \overset{\text{def}}{=} f(u)$ such that $f[U_u] \subseteq V_u$ and $f: U_u \to V_u$ is a morphism of $S^2$.

A continuous function $f: S^i \to S^i$ is locally an m-morphism of $S^i$ iff it is locally an m-morphism of $S^i$ to $S^i$.

A continuous function $f: S^1 \to S^2$ is locally an $\mathcal{M}^1$-$\mathcal{M}^2$ m-morphism of $S^1$ to $S^2$ iff $\mathcal{M}^1 \subseteq \mathcal{M}^2$ and for every $u \in S^1$ there is a model neighborhood $U_u$ for $u$ and a model neighborhood $V_u$ for $v \overset{\text{def}}{=} f(u)$ such that $f[U_u] \subseteq V_u$ and $f: U_u \to V_u$ is a morphism of $S^2$.

**Lemma 5.26** (Local m-morphisms). Let $\mathcal{M}^i$, $i = 1, 2, 3$, be model categories and $S^i \overset{\text{def}}{=} (S^i, S^i)^{\text{Ob}} \in \mathcal{M}^i$.

If $f_j: S^i \to S^i$, $j = 1, 2$, is locally an m-morphism of $S^i$ then $f_2 \circ f_1: S^i \to S^i$ is locally an m-morphism of $S^i$.

**Proof.** Let $u \in S^i$, $v \overset{\text{def}}{=} f_1(u)$ and $w \overset{\text{def}}{=} f_2(v)$. There exist a model neighborhood $U_u$ for $u$, model neighborhoods $V_u$, $V_v$, and a model neighborhood $W_v$ of $w$ such that $f_1[U_u] \subseteq V_v$, $f_2[V_v] \subseteq W_v$. $f_1: U_u \to V_v$ is a morphism of $S^i$ and $f_2: V_v \to W_v$ is a morphism of $S^i$. Then $V_u \overset{\text{def}}{=} V_v \cap V_v \neq \emptyset$, $V_u$ is a model neighborhood of $v$ and $U_u \overset{\text{def}}{=} f_1^{-1}[V_v]$ is a model neighborhood for $u$. $f_1: U_u \to V_v$ and $f_2: V_v \to W_v$ are morphisms of $S^i$ by item 4 of definition 5.2. (Model spaces) on page 15 and thus $f_2 \circ f_1: U_u \to W_v$ is a morphism of $S^i$. \qed

If $S$ is a model neighborhood of $S \overset{\text{def}}{=} (S, S)$ then every function $f: S \to S$ that is locally an m-morphism of $S$ is a morphism of $S$.

**Proof.** For every $u \in S$ there is a model neighborhood $U_u$ for $u$ and a model neighborhood $V_u$ for $v \overset{\text{def}}{=} f(u)$ such that $f[U_u] \subseteq V_u$ and $f: U_u \to V_u$ is a morphism of $S$. $\text{Id}_{V_u, S} \in S$ so $\text{Id}_{V_u, S} \circ (f: U_u \to V_u)$ is a morphism of $S$. Since $\bigcup_{u \in S} U_u = S$ and $\bigcup_{u \in S} S$ are model neighborhoods, the result follows by item 6 of definition 5.2 (Model spaces) on page 15. \qed

If each $f_i: S^i \to S^{i+1}$ is locally an $\mathcal{M}^i$-$\mathcal{M}^{i+1}$ m-morphism of $S^i$ to $S^{i+1}$ then $f_2 \circ f_1: S^1 \to S^3$ is locally an $\mathcal{M}^1$-$\mathcal{M}^3$ m-morphism of $S^1$ to $S^3$.

**Proof.** Since $\mathcal{M}^1 \subseteq \mathcal{M}^2$ and $\mathcal{M}^2 \subseteq \mathcal{M}^3$, $\mathcal{M}^1 \subseteq \mathcal{M}^3$. Let $u \in S^1$, $v \overset{\text{def}}{=} f_1(u)$ and $w \overset{\text{def}}{=} f_2(v)$. There exist a model neighborhood $U_u$ for $u$, model neighborhoods $V_v$, $V_w$ for $v$ and $w$ such that $f_1[U_u] \subseteq V_v$, $f_1: U_u \to V_v$ is a morphism of $\mathcal{M}^2$, $f_2[V_v] \subseteq W_v$, $f_2: V_v \to W_v$ is
a morphism of \( M^3 \). Then \( \hat{V}_u \) is a model neighborhood of \( v \) and \( \hat{U}_u \) is a model neighborhood for \( u \). \( f^1: \hat{U}_u \longrightarrow \hat{V}_u \) and \( f^2: \hat{V}_u \longrightarrow W_v \) are morphisms of \( M^3 \) by item 5 of definition 5.23 (Model categories) on page 20 and thus \( f^2 \circ f^1: \hat{U}_u \longrightarrow W_v \) is a morphism of \( M^3 \).

If \( S^i \mod \subseteq S^{i+1} \) then \( \text{Id}_{S^i, S^{i+1}} \) is locally an \( m \)-morphism of \( S^i \). If each \( S^i \) is a model neighborhood of \( S^i \) then \( \text{Id}_{S^i, S^{i+1}} \) is a morphism of \( S^{i+1} \).

**Proof.** Let \( u \in S^1 \) and \( U_u \) a model neighborhood of \( S^i \) for \( u \). Since \( S^i \mod \subseteq S^{i+1} \), \( U_u \) is also a model neighborhood of \( S^{i+1} \) for \( u \), and hence \( \text{Id}_{U_u} \in S^{i+1} \).

If each \( S^i \) is a model neighborhood of \( S^i \) then each \( \text{Id}_{S^i} \) is a morphism of \( S^i \).

Since \( S^i \mod \subseteq S^{i+1} \), \( S^i \in S^{i+1} \) and \( \text{Id}_{U_u, S^{i+1}} \) is a morphism of \( S^{i+1} \) by item 5 of definition 5.23 (Model categories) on page 20. Since \( \bigcup_{u \in S^1} U_u = S^1 \) and \( \bigcup_{u \in S^1} S^2 \) are model neighborhoods, the result follows by item 6 of definition 5.2 (Model spaces) on page 15.

If \( M^i \cat \subseteq M^{i+1} \) and \( S^i \mod \subseteq S^{i+1} \) then \( \text{Id}_{S^i, S^{i+1}} \) is a morphism of \( M^{i+1} \).

**Proof.** \( S^i \in M^{i+1} \) because \( M^i \subseteq M^{i+1} \). \( \text{Id}_{S^{i+1}, S^{i+1}} \in M^{i+1} \) by item 5 of definition 5.23 (Model categories) on page 20.

**Corollary 5.27** (Local \( m \)-morphisms). Let \( M^i, i = 1, 2 \), be model categories and \( S^i := (S, S') \in \text{Ob} \).

If \( S^i \) is a model neighborhood of \( S^i \) then \( \text{Id}_{S^i} \) is a morphism of \( S^i \).

**Proof.** \( S^i \mod \subseteq S^i \).

### 5.6 Spaces and proper functions

Several of the following definitions involve spaces of a restrictive character and specific types of mappings among them. Except where otherwise qualified, the word space will have this restricted meaning.

**Definition 5.28.** A space is a topological space, a model space or either with an additional associated structure.

**Definition 5.29.** Let \( S^1, S^2 \) be spaces and \( f: S^1 \longrightarrow S^2 \) a continuous function; \( f \) need not preserve any associated algebraic structure. \( f \) is a proper function.

---

4 However, in practice commutation relations will often enforce the preservation of algebraic structures.

5 In the spirit of [Kelley, 1955, footnote, p. 112] This nomenclature is an excellent example of the time-honored custom of referring to a problem we cannot handle as abnormal, irregular, improper, degenerate, inadmissible, and otherwise undesirable.
iff

1. $S^1$ and $S^2$ are both Truthspace or both Truthspace, and $f(\text{True}) = \text{True}$.

2. $S^1$, $S^2$ are topological spaces other than Truthspace.

3. $S^1$ is a topological space other than Truthspace, $S^2$ is a model space other than Truthspace and the images of open sets are contained in model neighborhoods.

4. $S^1$ is a model space other than Truthspace, $S^2$ is a topological spaces other than Truthspace and the inverse images of open sets are model neighborhoods.

5. $S^1$, $S^2$ are both model spaces other than Truthspace and $f$ is a model function.

**Definition 5.30.** Let $S$ be a topological space. Then the singleton category of $S$, abbreviated $S_{\text{Sing}}$, is the category whose sole object is $S$ and whose morphisms are all of the continuous functions from $S$ to itself.

Let $S$ be a model space. Then the singleton category of $S$, abbreviated $S_{\text{Sing}}$, is the category whose sole object is $S$ and whose morphisms are all of the model functions from $S$ to itself that are locally morphisms of $S$.

Let $S = (S^\alpha, \alpha \prec A)$ be a sequence of spaces. Then the singleton category sequence of $S$, abbreviated $S_{\text{Sing}}$, is $(S^\alpha, \alpha \prec A)$.

Let $S$ be a set of spaces. Then the singleton category of $S$, abbreviated $S_{\text{Sing}}$, is $\bigcup_{S \in S} S_{\text{Sing}}$.

**Remark 5.31.** By abuse of language the notation $S_{\text{Sing}}$ will be used to name sequences of categories constructed with this and similar functions.

6 Signatures

Given a sequences of spaces, we need a way to characterise a function that takes arguments in those spaces and has a value in them. For this purpose we use a sequence of ordinals where the last ordinal in the sequence identifies the space containing the function’s value.

**Definition 6.1 (Signature).** Let $S = (S_\alpha, \alpha \prec A)$ be a sequence and $\sigma = (\alpha_\beta \prec A, \beta \preceq B)$ be a sequence of ordinals in $A$. Then $\sigma$ is a signature over $S$.

**Definition 6.2 (S-signature of function).** Let $S = (S_\alpha, \alpha \prec A)$ be a sequence, $\sigma = (\alpha_\beta \prec A, \beta \preceq B)$ be a signature over $S$ and $f: (S_{\alpha_\beta}, \beta \prec B) \to S_{\alpha_\beta}$ continuous. Then $\sigma$ is the $S$-signature for $f$ and $f$ has $S$-signature $\sigma$.

**Remark 6.3.** If $n \in \mathbb{N}$ and $\sigma = (\alpha, \beta \preceq n)$ then $f$ is an $n$-ary operation over $S_\alpha$. 

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$f$ is proper iff either
1. $S_{\alpha, \beta} \leq B$ are all topological spaces.
2. $S_{\alpha, \beta} \leq B$ are all model spaces.

**Definition 6.4 ($S$-signature of function).** Let $S^\text{def} = (S_\alpha, \alpha < A)$ be a sequence, $S^\text{def} = (S_\alpha, \alpha < A)$ be a sequence of categories with $\left( \bigwedge_{\alpha < A} \right) S_\alpha \in S_\alpha$, $\sigma^\text{def} = (\alpha_\beta < A, \beta \leq B)$ be a signature over $S$ and $f: (S_{\alpha, \beta}, \beta < B) \rightarrow S_{\alpha, \beta}$ continuous. Then $\sigma$ is the $S$-signature for $f$ and $f$ has $S$-signature $\sigma$.

**Definition 6.5 ($S$-signature of function sequence).** Let $\Sigma^\text{def} = (\sigma_{\gamma, \gamma < \Gamma}, \gamma < \Gamma)$ be a sequence of signatures over $S$ and $F = (F_\gamma, \gamma < \Gamma)$ a sequence of functions where $F_\gamma$ has $S$-signature $\sigma_\gamma$, then $\Sigma$ is the $S$-signature for $F$ and $F$ has $S$-signature $\Sigma$.

**Definition 6.6 ($S$-signature of function sequence).** Let $\Sigma^\text{def} = (\sigma_{\gamma, \gamma < \Gamma}, \gamma < \Gamma)$ be a sequence of signatures over $S$ and $F = (F_\gamma, \gamma < \Gamma)$ a sequence of functions where $F_\gamma$ has $S$-signature $\sigma_\gamma$, then $\Sigma$ is the $S$-signature for $F$ and $F$ has $S$-signature $\Sigma$.

### 6.1 Functions associated with signatures and function sequences

When characterizing functions with signatures, several auxiliary functions are helpful.

**Definition 6.7 (Signature operators).** Let $S^i = (S_\alpha, \alpha < A)$, $i = 1, 2$, be sequences of categories with $S^i = (S_\alpha, \alpha < A)$ sequences of sets in those categories, $f^i, i = 1, 2$ functions with $S^i$-signatures $\sigma = (\alpha_\beta < A, \beta \leq B)$, $g^\text{def} = (g_\alpha: S^1_\alpha \rightarrow S^2_\alpha, \alpha < A)$ a sequence of functions then

\[
\langle \sigma | S^i \rangle^\text{def} = S^i = (S_{\alpha, \beta}, \beta \leq B) 
\]

selects sets from $S^i$ according to signature $\sigma$.

\[
\langle \sigma | g \rangle^\text{def} = g^\text{def} = (g_{\alpha, \beta}, \beta \leq B) 
\]

selects functions from $g$ according to signature $\sigma$.

\[
\underline{\beta < B} S^1_{\alpha, \beta} \times S^2_{\alpha, \beta} = (\sigma | g) = \times_{\beta < B} g_{\alpha, \beta} 
\]

is the product of the functions selected by all but the last ordinal in signature $\sigma$

\[
g^\sigma f^1 = \text{tail}(\langle \sigma | g \rangle) \circ f^1 
\]
composes the last function selected from \( g \) by signature \( \sigma \) with \( f^1 \).

\[
f^2 \circ \sigma \ g \defeq f^2 \circ \sigma \ g \tag{6.5}
\]
composes \( f \) with all but the last function in \( g \) selected by signature \( \sigma \).

### 6.2 Commutative diagrams with signatures

The conventional usage of the word commutative diagram is for functions of a single argument. It is convenient to extend the nomenclature to functions of multiple arguments and to sequences of such functions. This is convenient, e.g., as a means of specifying that functions preserve algebraic structures.

**Definition 6.8** (\( \Sigma \)-commutation). Let \( \mathcal{S}^i \defeq (\mathcal{S}^i_\alpha, \alpha \prec A) \), \( i = 1, 2 \), be a sequence of categories, with \( \mathcal{S}^i \defeq (\mathcal{S}^i_\alpha, \alpha \prec A) \) a sequence of spaces, \( F^i \defeq (F^i_\gamma, \gamma \prec \Gamma) \) be a sequence of functions with \( \mathcal{S}^i \)-signature \( \Sigma \defeq (\sigma_\gamma \prec \Gamma) \)
\[(\sigma_{\gamma, \alpha \beta} \prec A, \beta \preceq B, \gamma \prec \Gamma) \rightarrow f \text{ def } (f: S_{1}^{\gamma} \longrightarrow S_{2}^{\gamma}, \alpha \prec A) \] a sequence of functions satisfying \(f \circ \sigma_{\gamma} = F_{1}^{\gamma} \circ f, \gamma \prec \Gamma\), then \(f \Sigma\)-commutes with the function sequences \(F_{1}, F_{2}\).

**Remark 6.9.** If all of the \(\sigma_{\gamma, \alpha \beta}\) are equal to the same ordinal \(\hat{\alpha}\) and each \(B_{\gamma}\) is finite then \(\{(S_{1}^{\hat{\alpha}}(i), \sigma_{i})| \gamma \prec \Gamma\}\) is an \(\Omega\)-algebra and \(f\) is an \(\Omega\)-homomorphism.

**Lemma 6.10** (\(\Sigma\)-commutation). Let \(S_{1}^{i} = (S_{1}^{i}, \alpha \prec A), i \in [1, 3]\), be a sequence of categories, with \(S_{1}^{i} = (S_{1}^{i}, \alpha \prec A)\) a sequence of spaces and \(F_{1}^{i} = (F_{1}^{i}, \gamma \prec \Gamma)\) be a sequence of functions with \(S_{1}^{i}\)-signature \(\Sigma \text{ def } (\sigma_{\gamma, \gamma} \prec \Gamma) \text{ def } ((\sigma_{\gamma, \alpha \beta}, \alpha \prec A, \beta \preceq B, \gamma \prec \Gamma)\).

Let \(f_{1} = (f_{i}: S_{1}^{i} \longrightarrow S_{1}^{i+1}, \alpha \prec A), i = 1, 2, \Sigma\)-commute with \(F_{1}^{i}, F_{2}^{i+1}\). Then \(f_{1} F^{2} \Sigma\)-commutes with \(F_{1}, F_{2}\).

**Proof.** The result follows by diagram chasing in fig. 9.

Let \(S_{1}^{1} \subseteq S_{2}\) and \(F_{1}^{\gamma} = F_{2}^{\gamma} \times \sigma_{\gamma} \longrightarrow \text{tail}\left(S_{1}^{\gamma}\right)\). Then \(\text{Id}_{S_{1}}, S_{2}\) \(\Sigma\)-commutes with \(F_{1}, F_{2}\).

**Proof.** The result follows from definitions 6.7 and 6.8 on pages 24 and 25:
\[
\text{Id}_{S_{1}, S_{2}} \circ \sigma_{\gamma} \cdot F_{1}^{\gamma} = \text{Id}_{S_{1}, \alpha \beta, S_{2}^{\gamma, \alpha \beta}} \circ F_{1}^{\gamma} = F_{2}^{\gamma} \uparrow \sigma_{\gamma} = F_{2}^{\gamma} \circ \times_{\beta \prec B, \gamma} \text{Id}_{S_{1}, S_{2}^{\gamma}} = F_{2}^{\gamma} \circ \text{Id}_{S_{1}, S_{2}}
\]

**Corollary 6.11** (\(\Sigma\)-commutation). Let \(S_{1} = (S_{1}, \alpha \prec A)\), be a sequence of categories, with \(S_{1} = (S_{1}, \alpha \prec A)\) a sequence of spaces and \(F = (F_{1}, \gamma \prec \Gamma)\) be a sequence of functions with \(S_{1}\)-signature \(\Sigma \text{ def } (\sigma_{\gamma, \gamma} \prec \Gamma) \text{ def } ((\sigma_{\gamma, \alpha \beta}, \alpha \prec A, \beta \preceq B, \gamma \prec \Gamma)\). Then \(\text{Id}_{S} \Sigma\)-commutes with \(F, F\).
Proof. \( S \subseteq S, F_\gamma = F_\gamma; \times \sigma S \rightarrow \text{tail}(\sigma S) \) and \( \text{Id}_S = \text{Id}_{S,S} \). □

7 Prestructures

Prestructures and prestructure morphisms are generalizations of Ω-algebras and Ω-homomorphisms, and are notational conveniences to simplify imposing commutation relations on multiple unrelated functions of multiple variables.

**Definition 7.1 (Prestructures).** Let \( S \overset{\text{def}}{=} (S_\alpha, \alpha \prec A) \) be a sequence of categories, \( S \overset{\text{def}}{=} (S_\alpha, \alpha \prec A) \) a sequence of spaces, \( F = (F_\gamma, \gamma \prec \Gamma) \) a sequence of continuous functions and \( \Sigma \overset{\text{def}}{=} (\sigma_\gamma, \gamma \prec \Gamma) \overset{\text{def}}{=} (\Sigma_\gamma, \alpha \prec \beta \leq B_\gamma, \gamma \prec \Gamma) \) a sequence of signatures. Then \( P \overset{\text{def}}{=} (S, S, \Sigma, F) \) is a \( S - \Sigma \) prestructure iff

1. \( S_\alpha \) is either a topological category or a model category.
2. \( S \subseteq S \).
3. \( F \) has \( S \)-signature \( \Sigma \).

**Lemma 7.2 (Prestructures).** Let \( S^i, i = 1, 2 \) be a sequence of categories, \( S^1 \overset{\text{cat}}{\subseteq} S^2, S \subseteq S^1 \), \( F \) a sequence of continuous functions with \( S^1 \)-signatures \( \Sigma \).
and \( P^1 \equiv (\mathcal{S}^1, \mathcal{S}, \Sigma, F) \) a \( \mathcal{S}^1 - \Sigma \) prestructure. Then \( P^2 \equiv (\mathcal{S}^2, \mathcal{S}, \Sigma, F) \) is a \( \mathcal{S}^2 - \Sigma \) prestructure.

Proof. \( S^1 \subseteq S^2 \) by lemma 3.26 (Sequence inclusion) on page 12. The other conditions do not depend on the category sequence.

Let \( \mathcal{S}^i, i = 1, 2, \) be a sequence of categories, \( \mathcal{S}^1 \subseteq \mathcal{S}^2 \), \( \mathcal{S}^i \subseteq \mathcal{S}^i \), \( \mathcal{S}^1 \subseteq \mathcal{S}^2 \), \( F^2 \equiv (F^2, \gamma \propto \Gamma) \) a sequence of continuous functions with \( \mathcal{S}^2 \)-signatures \( \Sigma \equiv (\sigma_{\gamma}, \gamma \propto \Gamma) \equiv ((\sigma_{\gamma} \circ \alpha, \beta \leq B_{\gamma}), \gamma \propto \Gamma), F^2_\gamma \times \mathcal{S}^1 \subseteq \text{tail}(\mathcal{S}^1), F^1 \equiv (F^1_{\gamma}, \mathcal{S}^1 \rightarrow \text{tail}(\mathcal{S}^1), \gamma \propto \Gamma) \) and \( P^2 \equiv (\mathcal{S}^2, \mathcal{S}^2, \Sigma, F^2) \) a \( \mathcal{S}^2 - \Sigma \) prestructure. Then \( P^1 \equiv (\mathcal{S}^1, \mathcal{S}^1, \Sigma, F^1) \) is a \( \mathcal{S}^1 - \Sigma \) prestructure.

Proof. \( S^1 \subseteq S^1 \) by hypothesis. \( F^1 \) is continuous and has \( S^1 \)-signatures \( \Sigma \) by construction.

Definition 7.3 (Morphisms of prestructures). Let \( \mathcal{S}^i = (\mathcal{S}^i, \alpha \propto \lambda), i = 1, 2, \) be a sequence of categories, \( \mathcal{S} \equiv (\mathcal{S}^i, \alpha \propto \lambda) \subseteq \mathcal{S}^i \) be a sequence of spaces, \( F^i = (F^i_\gamma, \gamma \propto \Gamma) \) be a sequence of continuous functions with \( \mathcal{S}^i \)-signatures \( \Sigma \), \( f \equiv (f_\alpha : \mathcal{S}^1 \rightarrow \mathcal{S}^2, \alpha \propto \lambda) \) be a sequence of proper functions and \( P^i \equiv (\mathcal{S}^i, \mathcal{S}^i, \Sigma, F^i) \) be a \( \mathcal{S}^i - \Sigma \) prestructure. \( f \) is a (strict) prestructure morphism of \( P^1 \) to \( P^2 \) iff it \( \Sigma \)-commutes with \( F^1, F^2 \), as shown in fig. 8 (\( f \Sigma \)-commutes with the function sequences \( F^1, F^2 \)) on page 26.

It is a semi-strict prestructure morphism iff \( \mathcal{S}^1 \subseteq \mathcal{S}^2 \) and for each \( \alpha \)

1. Each \( \mathcal{S}^1, i = 1, 2, \) is a model category: \( f_\alpha \) is locally an \( \mathcal{S}^1_{\alpha} - \mathcal{S}^2_{\alpha} \) m-morphism of \( \mathcal{S}^1_{\alpha} \) to \( \mathcal{S}^2_{\alpha} \).
2. Each \( \mathcal{S}^1, i = 1, 2, \) is a topological category: \( f_\alpha \) is locally a \( \mathcal{S}^1_{\alpha} - \mathcal{S}^2_{\alpha} \) morphism of \( \mathcal{S}^1_{\alpha} \) to \( \mathcal{S}^2_{\alpha} \).
3. Otherwise \( f_\alpha \) is a morphism of \( \mathcal{S}^2_{\alpha} \).

It is a strict prestructure morphism iff \( \mathcal{S}^1 \subseteq \mathcal{S}^2 \) and each function \( f_\alpha \) is a morphism of \( \mathcal{S}^2_{\alpha} \).

Remark 7.4. It is not sufficient to require that each \( f_\alpha \) be a morphism of \( \mathcal{S}^1_{\alpha} \) \( \mathcal{S}^2_{\alpha} \) because that would not ensure that the composition of strict prestructure morphisms is strict.
If $S^1 \subseteq S^2$ and $F^1_\gamma = F^2_\gamma \times S^1_i \to \text{tail} (S^1_i)$ then $\text{Id}_{P_i, P^i} \overset{\text{def}}{=} \text{Id}_{S^1_i, S^2_i}$.

\text{Lemma 7.5} (Prestructure morphisms). Let $S^i = (S^i_\alpha, \alpha \prec A)$, $i \in [1, 4]$, be a sequence of categories, $S^i \overset{\text{def}}{=} (S^i_\alpha, \alpha \prec A) \subseteq S^i$, a sequence of spaces, $F^i = (F^i_\gamma, \gamma \prec \Gamma)$, be a sequence of continuous functions with $S^i$-signatures $\Sigma$ and $P^i \overset{\text{def}}{=} (S^i, S^i, \Sigma, F^i)$ be a $S^i - \Sigma$ prestructure.

If $S^i \subseteq S^{i+1}$ and each $F^i_\gamma = F^{i+1}_\gamma \times S^i_i \to \text{tail} (S^i_i)$ then $\text{Id}_{P_i, P^{i+1}}$ is a prestructure morphism from $P^i$ to $P^{i+1}$.

$\text{Id}_{S^1_i, S^{i+1}}$ is a semi-strict prestructure morphism from $P^i$ to $P^{i+1}$ iff $S^i \subseteq S^{i+1}$.

$\text{Id}_{S^1_i, S^{i+1}}$ is a strict prestructure morphism from $P^i$ to $P^{i+1}$ iff each $S^1_i \subseteq S^{i+1}$ and $S^i \subseteq S^{i+1}$.

\text{Proof.} $\text{Id}_{S^1_i, S^{i+1}}$ $\Sigma$-commutes with $F^i, F^{i+1}$ by \text{lemma 6.10} ($\Sigma$-commutation) on page 26.

The hypotheses for strictness are precisely those of definition 7.3 (Morphisms of prestructures).

$\square$

Let $f^i \overset{\text{def}}{=} (f^i_\alpha, S^i_\alpha \to S^{i+1}_\alpha, \alpha \prec A)$, $i = 1, 2, 3$, be a sequence of proper functions.

If $S^i = (S^i_\alpha, \alpha \prec A)$, $i \in [1, 4]$, is a sequence of categories, $S^i \subseteq S^i$ and $P^i \overset{\text{def}}{=} (S^i_\alpha, S^i, \Sigma, F^i)$, then $f^i$ is a prestructure morphism of $P^i$ to $P^{i+1}$ iff $f^i$ is a prestructure morphism of $P^i$ to $P^{i+1}$.

If $S^i \subseteq S^i$ and $f^i$ is a semi-strict (strict) prestructure morphism of $P^i$ to $P^{i+1}$ then $f^i$ is a semi-strict (strict) prestructure morphism of $P^i$ to $P^{i+1}$.

\text{Proof.} The commutation relations do not depend on the choice of categories.

If $S^{i+1} \subseteq S^i$ and $f^i$ is a semi-strict prestructure morphism of $P^i$ to $P^{i+1}$ then for each $\alpha$, if

1. Each $S^i_\alpha$, $j = i, i + 1$, is a model category: $f_\alpha$ is locally an $S^i_\alpha$-$S^{i+1}_\alpha$ morphism of $S^i_\alpha$ to $S^{i+1}_\alpha$ and thus locally an $S^i_\alpha$-$S^{i+1}_\alpha$ morphism of $S^i_\alpha$ to $S^{i+1}_\alpha$.

2. Each $S^i_\alpha$, $j = i, i + 1$, is a topological category: $f_\alpha$ is locally a $S^i_\alpha$-$S^{i+1}_\alpha$ morphism of $S^i_\alpha$ to $S^{i+1}_\alpha$ and thus locally a $S^i_\alpha$-$S^{i+1}_\alpha$ morphism of $S^i_\alpha$ to $S^{i+1}_\alpha$.

3. Otherwise $f_\alpha$ is a morphism of $S^{i+1}_\alpha$ and thus a morphism of $S^{i+1}_\alpha$.  

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If \( S^i \subseteq S'^i \) and \( \hat{f}^i \) is a strict prestructure morphism of \( P^i \) to \( P'^i \), then each \( \hat{f}_\alpha^i \in S'^i_{\alpha} \subseteq S'^{i+1}_\alpha \), \( \alpha < A \), and thus \( \hat{f}^i \) is a strict prestructure morphism of \( P^i \) to \( P'^{i+1} \).

\[
\hat{f}^{i+1} \circ \hat{f}^i, \ i \in \{1, 3\}, \text{ is a prestructure morphism and } \hat{f}^3 \circ (\hat{f}^2 \circ \hat{f}^1) = (\hat{f}^3 \circ \hat{f}^2) \circ \hat{f}^1.
\]

**Proof.** If \( \hat{f}^1 \Sigma \)-commutes with \( F^1 \), \( F^2 \) and \( \hat{f}^2 \Sigma \)-commutes with \( F^2 \), \( F^3 \) then by lemma 6.10 (\( \Sigma \)-commutation) on page 26 \( \hat{f}^2 \circ \hat{f}^1 \Sigma \)-commutes with \( F^1 \), \( F^3 \). Thus \( \hat{f}^2 \circ \hat{f}^1 \) is a morphism.

Associativity is just lemma 3.13 (Sequence functions) on page 8.

\( \text{Id}_{P^i} \) is an identity morphism of \( P^i \).

**Proof.** \( \hat{f}^i \circ \text{Id}_{S^i} = \hat{f}^i \) and \( \text{Id}_{S^{i+1}} \circ \hat{f}^i = \hat{f}^i \)

Let each \( \hat{f}^i \) be a semi-strict prestructure morphism. Then \( \hat{f}^2 \circ \hat{f}^1 \) is a semi-strict prestructure morphism.

**Proof.** If \( S^1 \subseteq S^2 \subseteq S^3 \) then \( S^1 \subseteq S^2 \subseteq S^3 \). For each \( \alpha < A \):

1. Each \( S^i_{\alpha} \), \( i = 1, 2, 3 \), is a model category:
   - Each \( f^i_{\alpha} \) is locally an \( S^i_{\alpha} \)-\( S^{i+1}_{\alpha} \) m-morphism of \( S^i_{\alpha} \) to \( S^{i+1}_{\alpha} \). Then \( \hat{f}^2_\alpha \circ \hat{f}^1_\alpha \) is locally an \( S^i_{\alpha} \)-\( S^{i+2}_{\alpha} \) m-morphism of \( S^i_{\alpha} \) to \( S^{i+2}_{\alpha} \) by lemma 5.26 (Local m-morphisms) on page 21.

2. Each \( S^i_{\alpha} \), \( i = 1, 2, 3 \), is a topological category:
   - Each \( f^i_{\alpha} \) is locally a \( S^i_{\alpha} \)-\( S^{i+1}_{\alpha} \) morphism of \( S^i_{\alpha} \) to \( S^{i+1}_{\alpha} \). Then \( \hat{f}^2_\alpha \circ \hat{f}^1_\alpha \) is locally an \( S^i_{\alpha} \)-\( S^{i+2}_{\alpha} \) morphism of \( S^i_{\alpha} \) to \( S^{i+2}_{\alpha} \) by lemma 3.9 (Local morphisms) on page 6.

3. Otherwise
   - Each \( f^i_{\alpha} \) is a morphism of \( S^{i+1}_{\alpha} \). Then \( \hat{f}^2_\alpha \circ \hat{f}^1_\alpha \) is a morphism of \( S^{i+2}_{\alpha} \).

Let each \( \hat{f}^i \) be a strict prestructure morphism. Then \( \hat{f}^2 \circ \hat{f}^1 \) is a strict prestructure morphism.

**Proof.** If \( S^1 \subseteq S^2 \subseteq S^3 \) then \( S^1 \subseteq S^2 \subseteq S^3 \). If \( f^1_{\alpha} \) is a morphism of \( S^2_{\alpha} \subseteq S^3_{\alpha} \) and \( f^2_{\alpha} \) is a morphism of \( S^3_{\alpha} \) then \( \hat{f}^2_\alpha \circ \hat{f}^1_\alpha \) is a morphism of \( S^3_{\alpha} \).
**Corollary 7.6** (Prestructure morphisms). Let \( \mathcal{S} = (\mathcal{S}_\alpha, \alpha \prec A) \) be a sequence of categories, \( \mathcal{S} \defeq (\mathcal{S}_\alpha, \alpha \prec A) \subseteq \mathcal{S} \) be a sequence of spaces, \( \mathcal{F} = (\mathcal{F}_\gamma, \gamma \prec \Gamma) \) be a sequence of continuous functions with \( \mathcal{S} \)-signatures \( \Sigma \) and \( \mathcal{P} \defeq (\mathcal{S}, \mathcal{S}, \Sigma, \mathcal{F}) \) be a \( \mathcal{S} - \Sigma \) prestructure. Then \( \mathrm{Id}_\mathcal{P} \) is a strict prestructure morphism from \( \mathcal{P} \) to \( \mathcal{P} \).

**Proof.** \( \mathcal{S} \subseteq \mathcal{S} \) and \( \mathcal{F}_\gamma = \mathcal{F}_\gamma \upharpoonright \mathcal{S}_{\mathrm{tail}}(\mathcal{S}) \).

Each \( \mathrm{Id}_{\mathcal{S}_\alpha, \mathcal{S}_\alpha} = \mathrm{Id}_{\mathcal{S}_\alpha} \in \mathcal{S}_\alpha \) and \( \mathcal{S} \subseteq \mathcal{S} \). \( \Box \)

### 8 M-charts and m-atlases

The conventional definitions of fiber bundles and manifolds use the language of charts, atlases and transition functions, with slight technical differences among them. This paper continues that usage, but modifies the definitions of charts in order to make them fit more naturally into the context of local coordinate spaces. It adds a prefix, e.g., \( C^k \), in order to avoid confusion with the conventional definitions. An m-atlas based on a topological spaces is closer to the conventional definitions of an atlas for a manifold while an m-atlas based on a model space is suitable for defining both manifolds and fiber bundles. Although simple manifolds and fiber bundles could both be defined directly in terms of maximal m-atlases, this paper has a different perspective, and uses the m-atlases as part of the more general Local Coordinate Space (LCS), presented in section 9 (Local Coordinate Spaces) on page 53.

This section defines M-atlases, categories of M-atlases and functors, and proves some basic results.

#### 8.1 M-charts

**Definition 8.1** (M-charts). Let \( E \defeq (E, \mathcal{E}) \) and \( C \defeq (C, \mathcal{C}) \) be model spaces. An m-chart \( (U, V, \phi) \) of \( E \) in the coordinate space \( C \) consists of

1. A model neighborhood \( U \in \mathcal{E} \), known as a coordinate patch
2. A model neighborhood \( V \in \mathcal{C} \)
3. A model homeomorphism \( \phi : U \xrightarrow{\cong} V \), known as a coordinate function

**Remark 8.2.** I consider it clearer to explicate the range, rather than the conventional usage of specifying only the domain and function or the minimalist usage of specifying only the function.

Let \( E \) be a topological space and \( C \defeq (C, \mathcal{C}) \) a model space. An m-chart \( (U, V, \phi) \) of \( E \) in the coordinate space \( C \) consists of
1. An open set $U$ of $E$, known as a coordinate patch

2. A model neighborhood $V \in \text{Ob} C$

3. A homeomorphism $\phi: U \xrightarrow{\sim} V$, known as a coordinate function, that maps open sets into model neighborhoods

Lemma 8.3 (M-charts). Let $E \overset{\text{def}}{=} (E, \mathcal{E})$ and $C \overset{\text{def}}{=} (C, \mathcal{C})$ be model spaces.

Let $(U, V, \phi)$ be an m-chart of $E$ in the coordinate space $C$ such that every open subset of $V$ is a model neighborhood of $C$. Then $(U, V, \phi)$ is an m-chart of $E$ in the coordinate space $C$.

Proof. $U$, $V$ and $\phi$ satisfy the conditions of the definition:

1. Since $U$ is a model neighborhood, it is open.

2. $V$ is a model neighborhood of $C$,

3. Since $\phi: U \xrightarrow{\sim} V$ is a model homeomorphism, it is a homeomorphism. If $U' \subseteq U$ is open then $\phi[U'] \subseteq V$ is open and thus a model neighborhood of $(C, \mathcal{C})$ by hypothesis. Thus $\phi$ is a homeomorphism that maps open sets into model neighborhood of $(C, \mathcal{C})$.

Let $(U, V, \phi)$ be an m-chart of $E$ in the coordinate space $C$. $(U, V, \phi)$ is an m-chart of $E$ in the coordinate space $C$ iff every open subset of $U$ is a model neighborhood of $E$.

Proof. If $(U, V, \phi)$ is an m-chart of $E$ in the coordinate space $C$ and also an m-chart of $E$ in the coordinate space $C$ then let $U' \subseteq U$ be open. Since $(U, V, \phi)$ is an m-chart of $E$ in the coordinate space $C$, $\phi[U']$ is a model neighborhood of $C$. Since $(U, V, \phi)$ is an m-chart of $E$ in the coordinate space $C$, $\phi$ is a model function and thus $\phi^{-1}[\phi[U']]$ is a model neighborhood of $E$.

If $(U, V, \phi)$ is an m-chart of $E$ in the coordinate space $C$ and every open $U' \subseteq U$ is a model neighborhood of $E$ then

1. Since $U \subseteq U$ is open, $U$ is a model neighborhood of $E$.

2. $V$ is a model neighborhood of $C$.

3. $\phi$ is a model homeomorphism: If $U' \subseteq U$ is a model neighborhood of $E$ then $U'$ is open and thus $\phi[U']$ is a model neighborhood of $C$. If $V' \subseteq V$ is a model neighborhood of $C$ then $\phi^{-1}[V']$ is open and thus a model neighborhood of $E$.

By definition, every open set of $E$ is a model neighborhood of $E$.  

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Corollary 8.4 (M-charts of $E_{\text{triv}}$). Let $E$ be a topological space. $(U,V,\phi)$ is an m-chart of $E$ in the coordinate space $C$ iff $(U,V,\phi)$ is an m-chart of $E_{\text{triv}}$ in the coordinate space $C$.

Proof. Let $(U,V,\phi)$ be an m-chart of $E$ in the coordinate space $C$. Every open subset of $E$ is a model neighborhood of $E$ by definition, hence $(U,V,\phi)$ is an m-chart of $E_{\text{triv}}$ in the coordinate space $C$.

Let $(U,V,\phi)$ be an m-chart of $E_{\text{triv}}$ in the coordinate space $C$ and let $V' \subseteq V_{\text{triv}}$ be open. $\phi^{-1}[V'] \subseteq U$ is open, hence by definition a model neighborhood of $E_{\text{triv}}$. Then $\phi[\phi^{-1}[V']] = V'$ is a model neighborhood of $C$.

Definition 8.5 (Subcharts). Let $(U,V,\phi)$ be an m-chart of $E_{\text{def}} = (E,E)$ in the coordinate space $(C,C)$ and $U' \subseteq U$ a model neighborhood of $(E,E)$. Then $(U',V',\phi') \equiv (U',\phi[U'],\phi|_{U',V'})$ is a subchart of $(U,V,\phi)$.

Let $(U,V,\phi)$ be an m-chart of $E$ in the coordinate space $(C,C)$ and $U' \subseteq U$ an open set of $E$. Then $(U',V',\phi') \equiv (U',\phi[U'],\phi|_{U',V'})$ is a subchart of $(U,V,\phi)$.

Lemma 8.6 (Subcharts). Let $E_{\text{def}} = (E,E)$ and $C_{\text{def}} = (C,C)$ be model spaces.

Let $(U,V,\phi)$ be an m-chart of $E$ in the coordinate space $C$ and $(U',V',\phi')$ a subchart of $(U,V,\phi)$. Then $(U',V',\phi')$ is an m-chart of $E$ in the coordinate space $C$.

Proof. $U'$, $V'$ and $\phi'$ satisfy the conditions of the definition of an m-chart:

1. $U'$ is a model neighborhood of $E$ by the definition of subchart.
2. Since $U'$ is a model neighborhood and $\phi$ is a model homeomorphism, $V' = \phi[U']$ is a model neighborhood.
3. Since $\phi$ and $\phi^{-1}$ are model functions, so are $\phi' = \phi|_{U',V'}$ and $\phi'^{-1} = \phi^{-1}|_{V',U'}$. Thus $\phi'$ is a model homeomorphism.

Let $(U,V,\phi)$ be an m-chart of $E$ in the coordinate space $C$ and $(U',V',\phi')$ a subchart of $(U,V,\phi)$. Then $(U',V',\phi')$ is an m-chart of $E$ in the coordinate space $C$.

Proof. $U'$, $V'$ and $\phi'$ satisfy the conditions of the definition of an m-chart:

1. $U'$ is open by the definition of subchart.
2. Since $\phi$ is a homeomorphism and $U'$ is open, $V' = \phi[U']$ is open.
3. Since $\phi$ is a homeomorphism that maps open sets into model neighborhoods, $\phi|_{U',V'}$ is a homeomorphism that maps open sets into model neighborhoods.
**Definition 8.7** (M-compatibility). Let \((U, V, \phi)\) and \((U', V', \phi')\) be m-charts of \((E, \mathcal{E})\) in the coordinate space \((C, \mathcal{C})\). Then \((U, V, \phi)\) is m-compatible with \((U', V', \phi')\) iff either

1. \(U\) and \(U'\) are disjoint
2. The transition function \(t = \phi' \circ \phi^{-1} \downharpoonright_{\phi(U \cap U')}\) is an isomorphism of \(C\).

**Lemma 8.8** (Symmetry of m-compatibility). Let \((U, V, \phi)\) and \((U', V', \phi')\) be m-charts of \((E, \mathcal{E})\) in the coordinate space \((C, \mathcal{C})\). Then \((U, V, \phi)\) is m-compatible with \((U', V', \phi')\) in the coordinate space \((C, \mathcal{C})\) iff \((U', V', \phi')\) is m-compatible with \((U, V, \phi)\) in the coordinate space \((C, \mathcal{C})\).

**Lemma 8.9** (M-compatibility of subcharts). Let \((U_i, V_i, \phi_i), i = 1, 2,\) be m-charts of \((E, \mathcal{E})\) in the coordinate space \((C, \mathcal{C})\), \((U'_i, V'_i, \phi'_i)\) be subcharts and \((U_1, V_1, \phi_1)\) be m-compatible with \((U_2, V_2, \phi_2)\). Then \((U'_1, V'_1, \phi'_1)\) is m-compatible with \((U'_2, V'_2, \phi'_2)\).

**Proof.** Since subcharts are charts, \(U'_i\) and \(V'_i\) are model neighborhoods. If \(U_1 \cap U_2 = \emptyset\) then \(U'_1 \cap U'_2 = \emptyset\). If \(U'_1 \cap U'_2 = \emptyset\) then \((U'_1, V'_1, \phi'_1)\) is m-compatible with \((U'_2, V'_2, \phi'_2)\). Otherwise, the transition function \(t'_2 \triangleq \phi_2 \circ \phi_1^{-1} \downharpoonright_{\phi_1(U_1 \cap U_2)}\) is a model homeomorphism and hence \(t'_2 \downharpoonright_{\phi_1(U_1 \cap U_2)} \rightsquigarrow \phi_2(U'_1 \cap U'_2)\) is a model homeomorphism.

Let \((U_i, V_i, \phi_i), i = 1, 2,\) be m-charts of \(E\) in the coordinate space \((C, \mathcal{C})\) and \((U'_i, V'_i, \phi'_i)\) subcharts. Then \((U'_1, V'_1, \phi'_1)\) is m-compatible with \((U_1, V_1, \phi_1)\).
Proof. Since subcharts are charts, $U'_1$ and $V'_2$ are open. If $U_1 \cap U_2 = \emptyset$ then $U'_1 \cap U'_2 = \emptyset$. If $U'_1 \cap U'_2 = \emptyset$ then $(U'_1, V'_1, \phi'_1)$ is m-compatible with $(U'_2, V'_2, \phi'_2)$. Otherwise, the transition function $t_2 \equiv \phi_2 \circ \phi_1^{-1} \mid_{\phi_1[U_1 \cap U_2]}$ is a model homeomorphism and hence $t_2 \mid_{\phi_1[U'_1 \cap U'_2]}: \phi_1[U'_1 \cap U'_2] \xrightarrow{\text{def}} \phi_2[U'_1 \cap U'_2]$ is a model homeomorphism.

Corollary 8.10 (M-compatibility with subcharts). Let $(U, V, \phi)$ be an m-chart of $(E, \mathcal{E})$ in the coordinate space $(C, \mathcal{C})$ and $(U', V', \phi')$ a subchart. Then $(U', V', \phi')$ is m-compatible with $(U, V, \phi)$.

Proof. $(U, V, \phi)$ is m-compatible with itself and is a subchart of itself.

Let $(U, V, \phi)$ be an m-chart of $E$ in the coordinate space $(C, \mathcal{C})$ and $(U', V', \phi')$ a subchart. Then $(U', V', \phi')$ is m-compatible with $(U, V, \phi)$.

Proof. $(U, V, \phi)$ is m-compatible with itself and is a subchart of itself.

Definition 8.11 (Covering by m-charts). Let $A$ be a set of charts of the topological space $E$ in the coordinate space $C = (C, \mathcal{C})$. $A$ covers $E$ iff $\pi_1[A]$ covers $E$.

Let $A$ be a set of charts of the model space $E = (E, \mathcal{E})$ in the coordinate space $C = (C, \mathcal{C})$. $A$ covers $E$ iff $\pi_1[A]$ covers $E$.

8.2 M-atlases

A set of charts can be atlases for different coordinate model spaces even if it is for the same total model space. In order to aggregate atlases into categories, there must be a way to distinguish them. Including the two\(^6\) spaces in the definitions of the categories serves the purpose.

Definition 8.12 (M-atlases). Let $A$ be a set of mutually m-compatible m-charts of $E = (E, \mathcal{E})$ in the coordinate space $C = (C, \mathcal{C})$. Then $A$ is an m-atlas of $E$ in the coordinate space $C$, abbreviated $\text{isAt}_{\text{Ob}}(A, E, C)$, iff $A$ covers $E$. $A$ is a full atlas of $E$ in the coordinate space $C$, abbreviated $\text{isAt}_{\text{Ob}}(A, E, C)_{\text{full}}$, iff $A$ covers $E$ and $\pi_2[A]$ covers $C$. The triple $(A, E, C)$ refers to $A$ considered as an m-atlas of $E$ in the coordinate space $C$.

Let $E = (E, \mathcal{E})$ and $C = (C, \mathcal{C})$ be model spaces. Then

$$\text{At}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \{(A, E, C) | \text{isAt}_{\text{Ob}}(A, E, C)\}$$

(8.1)

$$\text{At}_{\text{Ob}}^{\text{full}}(E, C) \overset{\text{def}}{=} \left\{(A, E, C) | \text{isAt}_{\text{Ob}}(A, E, C)_{\text{full}}\right\}$$

(8.2)

Let $E$ be a topological space, $C = (C, \mathcal{C})$ a model space and $A$ be a set of mutually m-compatible m-charts of $E$ in the coordinate space $C$. Then $A$ is an m-atlas of $E$ in the coordinate space $C$, abbreviated as $\text{isAt}_{\text{Ob}}(A, E, C)$, iff

---

\(^6\) The spaces are redundant, but convenient.
\(\pi_1[A]\) covers \(E\). \(A\) is a full atlas of \(E\) in the coordinate space \(C\), abbreviated is\(\text{Atl}_\text{Ob}(A,E,C)\), iff \(\pi_1[A]\) covers \(E\) and \(\pi_2[A]\) covers \(C\). The triple \((A,E,C)\) refers to \(A\) considered as an m-atlas of \(E\) in the coordinate space \(C\).

Let \(E\) be a topological space and \(C = (C,C)\) a model space. Then

\[
\text{Atl}_\text{Ob}(E,C) \overset{\text{def}}{=} \{(A,E,C)\mid \text{isAtl}_\text{Ob}(A,E,C)\}
\]

(8.3)

\[
\text{Atl}_\text{Ob}(E,C)_{\text{full}} \overset{\text{def}}{=} \left\{(A,E,C)\mid \text{isAtl}_\text{Ob}(A,E,C)\right\}_{\text{full}}
\]

(8.4)

By abuse of language we write \(U \in A\) for \(U \in \pi_1[A]\).

**Definition 8.13** (M-compatibility with atlases). An m-chart \((U,V,\phi)\) of \((E,\mathcal{E})\) is m-compatible with an m-atlas \(A\) iff it is m-compatible with every chart in the m-atlas.

An m-chart \((U,V,\phi)\) of \(E\) is m-compatible with an m-atlas \(A\) iff it is m-compatible with every chart in the m-atlas.

**Lemma 8.14** (M-compatibility of subcharts with atlases). Let \(A\) be an m-atlas of \((E,\mathcal{E})\) in the coordinate space \((C,C)\) and \(C_1 = (U_1, V_1, \phi_1)\) an m-chart in \(A\). Then any subchart of \(C_1\) is m-compatible with \(A\).

Let \(A\) be an m-atlas of \(E\) in the coordinate space \((C,C)\) and \(C_1 = (U_1, V_1, \phi_1)\) an m-chart in \(A\). Then any subchart of \(C_1\) is m-compatible with \(A\).

**Proof.** The same proof applies in both cases. Let \(C' = (U', V', \phi')\) be a subchart of \(C_1\) and \(C_2 = (U_2, V_2, \phi_2)\) another chart in \(A\).

1. If \(U_1 \cap U_2 = \emptyset\), then \(U' \cap U_2 = \emptyset\)

2. If \(U' \cap U_2 = \emptyset\) then \(C'\) is m-compatible with \(C_2\)

3. Otherwise the transition function \(t_2^1 \overset{\text{def}}{=} \phi_2 \circ \phi_1^{-1} \mid_{\phi_1[U_1 \cap U_2]}\) is an isomorphism of \((C,C)\). Since \(\phi_1[U_1 \cap U_2] \text{ and } V'\) are model neighborhoods, so is \(\phi_1[U' \cap U_2]\) and thus \(t_2^1 \mid_{\phi_1[U' \cap U_2]}\) is an isomorphism of \((C,C)\).

\[\square\]

**Lemma 8.15** (Extensions of m-atlases). Let \(E = (E,\mathcal{E})\) and \(C = (C,C)\) be model spaces, \(A\) an m-atlas of \(E\) in the coordinate space \(C\), and \((U,V,\phi), (U',V',\phi')\) m-charts of \(E\) in the coordinate space \(C\) m-compatible with \(A\) in the coordinate space \(C\). Then \((U,V,\phi)\) is m-compatible with \((U',V',\phi')\) in the coordinate space \(C\).

**Proof.** If \(U \cap U' = \emptyset\) then \((U,V,\phi)\) is m-compatible with \((U',V',\phi')\). Otherwise, \(\phi\) is a model homeomorphism, \(U \cap U'\) is a model neighborhood of \(E\), \(\phi[U \cap U']\) is a model neighborhood of \(C\), \(\phi'[U \cap U']\) is a model neighborhood of \(C\) and \(\phi' \circ \phi^{-1} \mid_{\phi(U \cap U')} \overset{\text{def}}{=} \phi' \circ \phi^{-1} \mid_{\phi(U \cap U')}\) is a model homeomorphism. It remains to show that \(\phi' \circ \phi^{-1} \mid_{\phi(U \cap U')}\) is a model homeomorphism. Let \((U_\alpha,V_\alpha,\phi_\alpha), \alpha < A\), be charts in
A such that $U \cap U' \subseteq \bigcup_{\alpha \in A} U_\alpha$ and $U \cap U' \cap U_\alpha \neq \emptyset$, $\alpha \prec A$. Since the charts are m-compatible with $(U_\alpha, V_\alpha, \phi_\alpha)$, \( \phi' \circ \phi^{-1}_\alpha \mid _{\phi_\alpha^{-1}[U \cap U'] \cap U_\alpha} \) and $\phi_\alpha \circ \phi^{-1}_\alpha$ [\( \phi_\alpha^{-1}[U \cap U'] \cap U_\alpha \)] are model homeomorphisms and thus $\phi' \circ \phi^{-1}_\alpha \mid _{\phi_\alpha^{-1}[U \cap U'] \cap U_\alpha} = \phi' \circ \phi^{-1}_\alpha \circ \phi_\alpha \circ \phi^{-1}_\alpha$ is a model homeomorphism. Then by item 6 of definition 5.2 (Model spaces) on page 15, $\phi' \circ \phi^{-1}$ is a model homeomorphism.

**Definition 8.16** (Maximal m-atlases). Let $E = (E, \mathcal{E})$ and $C = (C, \mathcal{C})$ be model spaces and $A$ an m-atlas of $E$ in the coordinate space $C$. $A$ is a maximal m-atlas of $E$ in the coordinate space $C$, abbreviated isAtl\(_{\text{max}}\)(A, E, C), iff $A$ cannot be extended by adding an additional m-compatible chart. $A$ is a semi-maximal m-atlas of $E$ in the coordinate space $C$, abbreviated isAtl\(_{\text{S-max}}\)(A, E, C), iff whenever $(U, V, \phi) \in A$, $U' \subseteq U$, $V' \subseteq V$ and $V'' \in \mathcal{C}$ are model neighborhoods, $\phi[U'] = V'$ and $\phi': V' \xrightarrow{\sim} V''$ is an isomorphism of $C$ then $(U', V'', \phi' \circ \phi \mid _{U'}) \in A$.

Let $E$ be a topological space, $C = (C, \mathcal{C})$ be a model space and $A$ an m-atlas in the coordinate space $C$. $A$ is a maximal m-atlas of $E$ in the coordinate space $C$, abbreviated isAtl\(_{\text{max}}\)(A, E, C), iff $A$ is an m-atlas that cannot be extended by adding an additional m-compatible chart. $A$ is a semi-maximal m-atlas of $E$ in the coordinate space $C$, abbreviated isAtl\(_{\text{S-max}}\)(A, E, C), iff whenever $(U, V, \phi) \in A$, $U' \subseteq U$, $V' \subseteq V$ and $V'' \in \mathcal{C}$ are model neighborhoods, $\phi[U'] = V'$ and $\phi': V' \xrightarrow{\sim} V''$ is an isomorphism of $C$ then $(U', V'', \phi' \circ \phi \mid _{U'}) \in A$.

**Remark 8.17.** There is no sheaf condition\(^7\) the union of a set of coordinate patches in the maximal atlas whose coordinate functions match on the intersections need not be a coordinate patch in the atlas.

Let $(E, \mathcal{E})$ and $(C, \mathcal{C})$ be model spaces. Then

\[
isAtl_{\text{max-full}}(A, E, C) \overset{\text{def}}{=} \text{isAtl}_{\text{full}}(A, E, C) \land \text{isAtl}_{\text{max}}(A, E, C) \tag{8.5}
\]

\[
isAtl_{\text{max-full}}(A, E, C) \overset{\text{def}}{=} \text{isAtl}_{\text{full}}(A, E, C) \land \text{isAtl}_{\text{max}}(A, E, C) \tag{8.6}
\]

\[
isAtl_{\text{S-max-full}}(A, E, C) \overset{\text{def}}{=} \text{isAtl}_{\text{full}}(A, E, C) \land \text{isAtl}_{\text{S-max}}(A, E, C) \tag{8.7}
\]

\[
isAtl_{\text{S-max-full}}(A, E, C) \overset{\text{def}}{=} \text{isAtl}_{\text{full}}(A, E, C) \land \text{isAtl}_{\text{S-max}}(A, E, C) \tag{8.8}
\]

\[
\text{Atl}_{\text{max}}((E, \mathcal{E}), (C, \mathcal{C})) \overset{\text{def}}{=} \left\{ (A, (E, \mathcal{E}), (C, \mathcal{C})) \mid \text{isAtl}_{\text{max}}(A, E, (C, \mathcal{C})) \right\} \tag{8.9}
\]

\(^7\) However, note item 6 (restricted sheaf condition) of definition 5.2 (Model spaces) on page 15.
\[
\mathcal{A}_{\text{Ob}}((E, \mathcal{E}), (C, \mathcal{C})) \overset{\text{def}}{=} \left\{ (A, (E, \mathcal{E}), (C, \mathcal{C})) \mid \text{isAtl}_{\text{Ob}}(A, E, (C, \mathcal{C})) \right\}_{\text{S-max}} \tag{8.10}
\]
\[
\mathcal{A}_{\text{Ob}}((E, \mathcal{E}), (C, \mathcal{C})) \overset{\text{def}}{=} \left\{ (A, (E, \mathcal{E}), (C, \mathcal{C})) \mid \text{isAtl}_{\text{Ob}}(A, E, (C, \mathcal{C})) \right\}_{\text{max-full}} \tag{8.11}
\]
\[
\mathcal{A}_{\text{Ob}}((E, \mathcal{E}), (C, \mathcal{C})) \overset{\text{def}}{=} \left\{ (A, (E, \mathcal{E}), (C, \mathcal{C})) \mid \text{isAtl}_{\text{Ob}}(A, E, (C, \mathcal{C})) \right\}_{\text{S-max-full}} \tag{8.12}
\]

Let \( E \) be a topological space and \((C, \mathcal{C})\) be a model space. Then
\[
\mathcal{A}_{\text{Ob}}(E, (C, \mathcal{C})) \overset{\text{def}}{=} \left\{ (A, E, (C, \mathcal{C})) \mid \text{isAtl}_{\text{Ob}}(E, (C, \mathcal{C})) \right\}_{\text{max}} \tag{8.13}
\]
\[
\mathcal{A}_{\text{Ob}}(E, (C, \mathcal{C})) \overset{\text{def}}{=} \left\{ (A, E, (C, \mathcal{C})) \mid \text{isAtl}_{\text{Ob}}(E, (C, \mathcal{C})) \right\}_{\text{S-max}} \tag{8.14}
\]
\[
\mathcal{A}_{\text{Ob}}(E, (C, \mathcal{C})) \overset{\text{def}}{=} \left\{ (A, E, (C, \mathcal{C})) \mid \text{isAtl}_{\text{Ob}}(E, (C, \mathcal{C})) \right\}_{\text{max-full}} \tag{8.15}
\]
\[
\mathcal{A}_{\text{Ob}}(E, (C, \mathcal{C})) \overset{\text{def}}{=} \left\{ (A, E, (C, \mathcal{C})) \mid \text{isAtl}_{\text{Ob}}(E, (C, \mathcal{C})) \right\}_{\text{S-max-full}} \tag{8.16}
\]

Let \( E \) and \( C \) be sets of model spaces. Then
\[
\mathcal{A}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \left\{ (A, (E, \mathcal{E}), (C, \mathcal{C})) \mid (E, \mathcal{E}) \in E \land (C, \mathcal{C}) \in C \land \text{isAtl}_{\text{Ob}}(A, (E, \mathcal{E}), (C, \mathcal{C})) \right\}_{\text{max}} \tag{8.17}
\]
\[
\mathcal{A}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \left\{ (A, (E, \mathcal{E}), (C, \mathcal{C})) \mid (E, \mathcal{E}) \in E \land (C, \mathcal{C}) \in C \land \text{isAtl}_{\text{Ob}}(A, (E, \mathcal{E}), (C, \mathcal{C})) \right\}_{\text{S-max}} \tag{8.18}
\]
\[
\mathcal{A}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \left\{ (A, (E, \mathcal{E}), (C, \mathcal{C})) \mid (E, \mathcal{E}) \in E \land (C, \mathcal{C}) \in C \land \text{isAtl}_{\text{Ob}}(A, (E, \mathcal{E}), (C, \mathcal{C})) \right\}_{\text{max-full}} \tag{8.19}
\]
\[
\mathcal{A}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \left\{ (A, (E, \mathcal{E}), (C, \mathcal{C})) \mid (E, \mathcal{E}) \in E \land (C, \mathcal{C}) \in C \land \text{isAtl}_{\text{Ob}}(A, (E, \mathcal{E}), (C, \mathcal{C})) \right\}_{\text{S-max-full}} \tag{8.20}
\]

Let \( E \) be a set of topological spaces and \( C \) a set of model spaces. Then
\[
\mathcal{A}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \left\{ (A, E, (C, \mathcal{C})) \mid E \in E \land (C, \mathcal{C}) \in C \land \text{isAtl}_{\text{Ob}}(A, E, (C, \mathcal{C})) \right\}_{\text{max}} \tag{8.21}
\]
\[ \mathcal{A}_{\text{Ob}}(E, C) \]  
\[ \left( \begin{array}{l} \mathcal{A}_{\text{Ob}}^\text{S}_{\text{max}} \left( (A, E, C) \right) \\ E \in E \wedge (C, C) \in C \wedge \text{isAtl}_{\text{Ob}}(A, E, C) \end{array} \right) \]  
\[ \text{(8.22)} \]

\[ \mathcal{A}_{\text{Ob}}^\text{S}_{\text{max,full}} (E, C) \]  
\[ \left( \begin{array}{l} \mathcal{A}_{\text{Ob}}^\text{S}_{\text{max,full}} (E, C) \left( A, (E, C), (C, C) \right) \\ (E, C) \in E \wedge (C, C) \in C \wedge \text{isAtl}_{\text{Ob}}(A, (E, C), (C, C)) \end{array} \right) \]  
\[ \text{(8.23)} \]

**Lemma 8.18** (Maximal m-atlas are semi-maximal m-atlases). Let \( E = (E, \mathcal{C}) \) and \( C = (C, C) \) be model spaces and \( A \) a maximal m-atlas of \( E \) in the coordinate space \( C \). Then \( A \) is a semi-maximal m-atlas of \( E \) in the coordinate space \( C \).

**Proof.** Let \( (U, V, \phi) \in A \), \( U' \subseteq U \), \( V' \subseteq V \) and \( V'' \in \mathcal{C} \) be model neighborhoods, \( \phi[U'] \equiv V' \) and \( \phi' : V' \xrightarrow{\simeq} V'' \) be an isomorphism of \( C \). \( (U', V', \phi) \) is a subchart of \( (U, V, \phi) \) and by lemma 8.14 (M-compatibility of subcharts with atlases) on page 36 is m-compatible with the charts of \( A \). Since \( \phi' \) is a model homeomorphism, \( (U', V'', \phi' \circ \phi) \) is m-compatible with the charts of \( A \). Since \( A \) is maximal, \( (U', V'', \phi' \circ \phi) \) is a chart of \( A \).

Let \( E \) be a topological space, \( C = (C, C) \) a model space and \( A \) a maximal m-atlas of \( E \) in the coordinate space \( C \). Then \( A \) is a semi-maximal m-atlas of \( E \) in the coordinate space \( C \).

**Theorem 8.19** (Existence and uniqueness of maximal m-atlases). Let \( A \) be an m-atlas of \( E \equiv (E, \mathcal{C}) \) in the coordinate space \( C \equiv (C, C) \). Then there exists a unique maximal m-atlas \( \text{Atlas}_m(A, E, C) \) of \( E \) in the coordinate space \( C \) m-compatible with \( A \).

**Proof.** Let \( P \) be the set of all m-atlas of \( E \) in the coordinate space \( C \) containing \( A \) and m-compatible in the coordinate space \( C \) with all of the m-charts in \( A \). Let \( \bigcup_{\text{max}} P \) be a maximal chain of \( A \). Then \( A' = \bigcup_{\text{max}} P \) is a maximal m-atlas of \( E \) in the coordinate space \( C \) m-compatible with \( A \). Uniqueness follows from lemma 8.15 (Extensions of m-atlases) on page 36.

Let \( A \) be an m-atlas of \( E \) in the coordinate space \( C \equiv (C, C) \). Then there exists a unique maximal m-atlas \( \text{Atlas}_m(A, E, C) \) of \( E \) in the coordinate space \( C \) m-compatible with \( A \).
Proof. Since $A$ is an m-atlas of $E$ in the coordinate space $C \overset{\text{def}}{=} (C, C)$, then by corollary 8.4 (M-charts of $E$) on page 33 $A$ is an m-atlas of $E$ in the coordinate space $C$. Let $A'$ be a maximal m-atlas of $E$ in the coordinate space $C$. Then $A$ is an m-atlas of $E$ in the coordinate space $C$. 

8.3 M-atlas morphisms and functors

Definition 8.20 (M-atlas morphisms). Let $E^i, C^i, i = 1, 2$ be model categories, $E^i \in \text{Ob}(E^i), C^i \in \text{Ob}(C^i)$ and $A^i$ be m-atlas of $E^i$ in the coordinate spaces $C^i$. $f \overset{\text{def}}{=} (f_0: E^1 \rightarrow E^2, f_1: C^1 \rightarrow C^2)$ is a (strict) $E^1$-$E^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1, C^2$, abbreviated as isAtl$_{\text{triv}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$, and a (strict) $E^1$-$E^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate model categories $C^1, C^2$, abbreviated as isAtl$_{\text{triv}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$, iff for any $(U^1, V^1, \phi^1: U^1 \overset{\sim}{\rightarrow} V^1) \in A^1,$ $(U^2, V^2, \phi^2: U^2 \overset{\sim}{\rightarrow} V^2) \in A^2$, the diagram $D \overset{\text{def}}{=} \{(I \overset{\text{def}}{=} U^1 \cap f_0^{-1}(U^2), V^1, E^2, U^2, V^2), \{f_0, \phi^2, \phi^1, f_1\}\}$ is M-locally nearly commutative in $C^2$, i.e., for any $x \in I$ there are objects $U^1 \subseteq I, V^1 \subseteq V^1, U^2 \subseteq U^2, V^2 \subseteq V^2, \beta^1 \subseteq C^2$ and an isomorphism $\tilde{f}: V^2 \overset{\sim}{\rightarrow} V^2$ such that eqs. (8.24) to (8.30) below hold. The triple $(f, (A^1, E^1, C^1), (A^2, E^2, C^2))$ will refer to $f$ considered as an $E^1$-$E^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1, C^2$.

It is a semi-strict $E^1$-$E^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1, C^2$, abbreviated as isAtl$_{\text{semi-strict}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$, iff $E^1 \overset{\text{mod}}{\subseteq} E^2, C^1 \overset{\text{mod}}{\subseteq} C^2, f_0$ is locally an m-morphism of $E^2$ and $f_1$ is locally an m-morphism of $C^2$.

It is a semi-strict $E^1$-$E^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate space $C^1$, abbreviated as isAtl$_{\text{semi-strict}}(A^1, E^1, C^1, A^2, E^2, f_0, f_1)$, if it is a semi-strict $E^1$-$E^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1, C^1$.
It is a semi-strict $E^1$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate space $C^1$, abbreviated as $\text{isAtl}_{A^1} (A^1, E^1, C^1, A^2, f_0, f_1)$, iff it is a semi-strict $E^1-E^1$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^1$.

It is a semi-strict $E^1-E^2-C^1-C^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$, abbreviated as $\text{isAtl}_{A^1} (A^1, E^1, C^1, A^2, f_0, f_1, C^1, C^2)$, iff $E^1 \subseteq E^2$, $C^1 \subseteq C^2$, $f_0$ is locally an m-morphism of $E^1$ to $E^2$ and $f_1$ is locally a $C^1-C^2$ morphism of $C^1$ to $C^2$.

It is a semi-strict $E^1-E^2-C^1-C^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$, abbreviated as $\text{isAtl}_{A^1} (A^1, E^1, C^1, A^2, E^2, f_0, f_1, C^1, C^2)$, iff $E^1 \subseteq E^2$, $C^1 \subseteq C^2$, $f_0$ is locally an $E^1-E^2$ morphism of $E^1$ to $E^2$ and $f_1$ is locally an m-morphism of $C^1$ to $C^2$.

It is a semi-strict $E^1-E^2-C^1-C^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate space $C^1$, abbreviated as $\text{isAtl}_{A^1} (A^1, E^1, C^1, A^2, E^2, f_0, f_1, C^1, C^2)$, iff $E^1 \subseteq E^2$, $C^1 \subseteq C^2$, $f_0$ is locally an $E^1-E^2$ morphism of $E^1$ to $E^2$ and $f_1$ is locally a $C^1-C^2$ morphism of $C^1$ to $C^2$.

It is a semi-strict $E^1-E^2-C^1-C^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$, abbreviated as $\text{isAtl}_{A^1} (A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1, C^1, C^2)$, iff $E^1 \subseteq E^2$, $C^1 \subseteq C^2$, $f_0$ is locally an $E^1-E^2$ morphism of $E^1$ to $E^2$ and $f_1$ is locally a $C^1-C^2$ morphism of $C^1$ to $C^2$.

It is a strict $E^1-E^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$, abbreviated as $\text{isAtl}_{A^1} (A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$, iff $E^1 \subseteq E^2$. 

Figure 11: Completed atlas morphism
\( C^1 \subseteq C^2, f_0 \) is a morphism of \( E^2 \) and \( f_1 \) is a morphism of \( C^2 \).

It is a strict \( E^1 \) m-atlas morphism of \( A^1 \) to \( A^2 \) in the coordinate space \( C^1 \), abbreviated as \( \text{isAtl}_\text{str}(A^1, E^1, C^1, A^2, f_0, f_1) \), iff it is a strict \( E^1 \)-\( E^1 \) m-atlas morphism of \( A^1 \) to \( A^2 \) in the coordinate spaces \( C^1, C^1 \).

It is a strict \( E^1 \)-\( E^2 \)-\( C^1 \)-\( C^2 \) m-atlas morphism of \( A^1 \) to \( A^2 \) in the coordinate spaces \( C^1, C^2 \), abbreviated as

\[
\text{isAtl}_\text{str}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1, C^1, C^2),
\]

iff

\[
E^1 \subseteq E^2, C^1 \subseteq C^2, f_0 \text{ is a morphism of } E^2 \text{ and } f_1 \text{ is a morphism of } C^2.
\]

It is a strict \( E^1 \)-\( E^1 \)-\( C^1 \)-\( C^2 \) m-atlas morphism of \( A^1 \) to \( A^2 \) in the coordinate spaces \( C^1, C^2 \), abbreviated as

\[
\text{isAtl}_\text{str}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1, C^1, C^2),
\]

iff

\[
E^1 \subseteq E^2, C^1 \subseteq C^2, f_0 \text{ is a morphism of } E^2 \text{ and } f_1 \text{ is a morphism of } C^2.
\]

It is a strict \( E^1 \)-\( E^2 \)-\( E^1 \)-\( E^2 \)-\( C^1 \)-\( C^2 \) m-atlas morphism of \( A^1 \) to \( A^2 \) in the coordinate spaces \( C^1, C^2 \), abbreviated as

\[
\text{isAtl}_\text{str}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1, E^1, E^2, C^1 C^2),
\]

iff

\[
E^1 \subseteq E^2, C^1 \subseteq C^2, f_0 \text{ is a morphism of } E^2 \text{ and } f_1 \text{ is a morphism of } C^2.
\]

\[
x \in U'^1
\]
\[
f_1 \circ \phi^1(x) \in \hat{V}'^2
\]
\[
f_0[U'^1] \subseteq U'^2
\]
\[
\phi^1[U'^1] \subseteq \hat{V}'^1
\]
\[
f_1[V'^1] \subseteq \hat{V}'^2
\]
\[
\phi^2[U'^2] \subseteq V'^2
\]
\[
\hat{f} \circ \phi^2 \circ f_0 = f_1 \circ \phi^1
\]

The identity morphism of \( (A^i, E^i, C^i) \) is

\[
\text{Id}_{(A^i, E^i, C^i)} \overset{\text{def}}{=} ((\text{Id}_{E^i}, \text{Id}_{C^i}), (A^i, E^i, C^i), (A^i, E^i, C^i))
\]

Let \( C^i, i = 1, 2 \) be model categories, \( C^i \text{Ob} \subseteq C^i, E^i \) be topological spaces and \( A^i \) be m-atlas of \( E^i \) in the coordinate spaces \( C^i \). A pair of functions \( \hat{f} \overset{\text{def}}{=} (f_0: E^1 \longrightarrow E^2, f_1: C^1 \longrightarrow C^2) \), is a (strict) \( E^1 \)-\( E^2 \) m-atlas morphism of \( A^1 \) to \( A^2 \).
Lemma 8.21 (M-atlas morphisms). Let $E^i, C^i, C'^i, i = 1, 2, 3$, be model categories, $E^i \subseteq E'^i \subseteq C'^i$. $E^i \in E^i, C^i \in C^i, A^i$ an $m$-morphism of $E^i$ in the coordinate spaces $C^i, A^2$ semi-maximal and $f_1^i \overset{\text{def}}{=} (f_0^i, E^i \longrightarrow E'^i, f_1^i, C^i \longrightarrow C'^i+1)$
a (strict) $E^i\cdot E^{i+1}$ m-atlas morphism of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$.

If $\mathcal{E}^i \subseteq \mathcal{E}^{i+1} \subseteq C^i$ and $A^i = A^{i+1}$ then $\text{Id}_{\mathcal{E}^i \cdot C^i} \cdot (E^{i+1}, C^{i+1})$ is a strict $\mathcal{E}^i \cdot \mathcal{E}^{i+1} \cdot E^i \cdot E^{i+1} \cdot C^i \cdot C^{i+1}$ m-atlas morphism of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$.

Proof. A commutative diagram is $M$-locally commutative. $\text{Id}_{\mathcal{E}^i} \subseteq \mathcal{E}^{i+1}$ and $\text{Id}_{C^i} \subseteq C^{i+1}$. □

If $f^i$ is a semi-strict (strict) $\mathcal{E}^i \cdot \mathcal{E}^{i+1} \cdot E^i \cdot E^{i+1} \cdot C^i \cdot C^{i+1}$ m-atlas morphism of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$ then $f^1$ is a semi-strict (strict) $\mathcal{E}^i \cdot \mathcal{E}^{i+1} \cdot E^i \cdot E^{i+1} \cdot C^i \cdot C^{i+1}$ m-atlas morphism of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$.

Proof. If $f_0^i$ is locally a morphism of $\mathcal{E}^i$ then $f_0^i$ is locally a morphism of $\mathcal{E}^i$. If $f_1^i$ is locally a morphism of $C^i$ then $f_1^i$ is locally a morphism of $C^i$. If $f_0^i \in C^i$ then $f_0^i \in C^i$. If $f_1^i \in C^i$ then $f_1^i \in C^i$.

$f^2 \circ f^1 = (f_2^i \circ f_0^i, f_1^i \circ f_0^i)$ is a (strict) $E^1 \cdot E^3$ m-atlas morphism of $A^1$ to $A^3$ in the coordinate spaces $C^1$, $C^3$.

Let $C^i$, $i = 1, 2, 3$, be a model category, $E^i$ be a topological space, $C^i \subseteq \mathcal{E}^i$, $A^i$ an m-atlas of $E^i$ in the coordinate spaces $C^i$, $A^2$ semi-maximal and $f^i \defeq (f_0^i: E^i \to E^{i+1}, f_1^i: C^i \to C^{i+1})$ a (strict) $E^i \cdot E^{i+1}$ m-atlas morphism of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$. Then $f^2 \circ f^1 = (f_2^i \circ f_0^i, f_1^i \circ f_0^i)$ is a (strict) $E^1 \cdot E^3$ m-atlas morphism of $A^1$ to $A^3$ in the coordinate spaces $C^1$, $C^3$.

Proof. If each $f^i$ is a (strict) $E^i \cdot E^{i+1}$ m-atlas morphism of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$, then $f_0^i \circ f_1^i$ and $f_2^i \circ f_1^i$ are model functions by lemma 5.21 (Composition of model functions) on page 20. If each $f^i$ is a (strict) $E^i \cdot E^{i+1}$ m-atlas morphism of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$, then $f_0^i \circ f_1^i$ is continuous and $f_2^i \circ f_1^i$ is a model function by lemma 5.21.

Let $(U^i, V^i, \phi^i)$ be charts in $A^i$, $I_1 = U^1 \cap f_0^1 \cap f_0^2 \cap f_0^3 \cap U^3$ and $I_2 = U^1 \cap f_0^2 \cap f_0^3 \cap U^3$. $I_1 \subseteq U^1 \cap f_0^1 \cap f_0^2 \cap U^3$. If $I_1 \cdot 3 = \emptyset$ then $D_1 \cdot 3 \defeq (I_1 \cdot 3, V^1, E^3, U^3, V^3, \{f_0, \phi^2, \phi^1, f_1\})$ is vacuously M-locally nearly commutative. Otherwise, since $(f_0^i: E^i \to E^{i+1}, f_1^i: C^i \to C^{i+1}, A^1, A^2)$ is a $E^i \cdot E^{i+1}$ m-atlas morphism of $C^1$, $C^2$, $D \defeq (\{f_0^i, \phi^2, \phi^1, f_1\})$, fig. 12 (Uncompleted atlas morphisms), is M-locally nearly commutative in $C^2$ and for any $x \in I_1 \subseteq I_1 \cdot 3$ there are objects $U^1 \subseteq I_1 \cdot 2$, $V^1 \subseteq V^1$, $U^2 \subseteq U^2$, $V^2 \subseteq V^2$, $\text{Uncompleted atlas morphisms}$ and an isomorphism $\phi: V^2 \to \hat{V}^2$ such that eqs. (8.24) to (8.30) in definition 8.20 (M-atlas morphisms) hold.
Figure 12: Uncompleted atlas morphisms
Figure 13: Partially completed atlas morphisms
Let $y = f_0^1(x)$, $I^{r,3} = U^{r,2} \cap I^{2,3}$ and $\phi'^2; U^{r,2} \xrightarrow{\sim} \hat{V}^{r,2} \overset{\phi}{\approx} f_0^1 \circ \phi^2$. Since $A^2$ is semi-maximal, $(U^{r,2}, \hat{V}^{r,2}, \phi'^2)$ is a chart of $A^2$.

Since $(f_0^1; E^2 \xrightarrow{f_2}; C^2 \xrightarrow{f_3}; C^3), (A^2, A^3)$ is an $E^2\dashrightarrow E^3$ $m$-atlas morphism of $C^2, C^3$, the diagram $D \overset{\text{def}}{=} (\{ I^{r,2}, \hat{V}^{r,2}, E^3, U^3, \hat{V}^3 \}, \{ f_0^2, \phi, \phi'^2, f_2 \})$ is M-locally nearly commutative in $C^3$ and hence there are objects $U^{m,2} \subseteq I^{r,2}$, $V^{m,2} \subseteq \hat{V}^{r,2}$, $U^{m,3} \subseteq U^3$, $V^{m,3} \subseteq \hat{V}^3$, $V^{m,3} \subseteq C^3$ and an isomorphism $\hat{f}^3; V^{m,3} \xrightarrow{\sim} \hat{V}^{m,3}$ such that eqs. (8.33) to (8.39) below hold.

\[ y = f_0^1(x) \in U^{m,2} \quad (8.33) \]
\[ f_0^2 \circ \phi'^2(y) \in V^{m,3} \quad (8.34) \]
\[ f_0^2[U^{m,2}] \subseteq U^{m,3} \quad (8.35) \]
\[ \phi'^2[U^{m,2}] \subseteq V^{m,2} \quad (8.36) \]
\[ f_2^3[V^{m,2}] \subseteq \hat{V}^{m,3} \quad (8.37) \]
\[ \phi'[U^{m,3}] \subseteq V^{m,3} \quad (8.38) \]
\[ \hat{f}^3 \circ \phi^3 \circ f_0^2 = f_0^1 \circ \phi'^2 \quad (8.39) \]

Let $I^{m,3} \overset{\text{def}}{=} U^{m,1} \cap (f_0^2 \circ f_0^1)^{-1}[U^{m,3}]$ and $V^{m,1} \overset{\text{def}}{=} V^{m,1} \cap (f_0^2 \circ f_0^1)^{-1}[V^{m,3}]$ Then $\phi^1; U^{m,1} \xrightarrow{\sim} V^{m,1}, \hat{f}^2; \hat{V}^{m,2} \xrightarrow{\sim} V^{m,2}, \hat{f}^3; \hat{V}^{m,3} \xrightarrow{\sim} V^{m,3}, \phi'^2; U^{m,2} \xrightarrow{\sim} V^{m,2}$ and $\phi^3; U^{m,3} \xrightarrow{\sim} V^{m,3}$ are isomorphisms and

\[ x \in U^{m,1} \quad (8.40) \]
\[ f_0^2 \circ f_0^1 \circ \phi^1(x) \in \hat{V}^{m,3} \quad (8.41) \]
$f_0^2 \circ f_0^1[U'^3] \subseteq U'^3$  
(8.42)

$\phi^1[U'^n] \subseteq V'^n$  
(8.43)

$f_2^1 \circ f_1^1[V'^n] \subseteq V'^n$  
(8.44)

$\phi^3[U'^n] \subseteq V'^n$  
(8.45)

$\tilde{f}^3 \circ \phi^3 \circ f_2^1 \circ f_0^1 = f_2^1 \circ \phi^2 \circ f_0^1 = f_1^1 \circ f_1^1 \circ \phi^1$  
(8.46)

Definition 8.20 has several definitions for a semi-strict and strict m-atlas morphism, differing in the restrictions on $f_j^i$.

If each $E_i^\text{mod} \subseteq E_i^{i+1}$ and each $f_j^i$ is locally a morphism of $E_i^i$ to $E_i^{i+1}$, then by lemma 5.26 (Local m-morphisms) on page 21, $f_j^{i+1} \circ f_0^i$ is locally a morphism of $E_i^i$ to $E_i^{i+3}$.

If each $E_i^\text{full-cat} \subseteq E_i^{i+1}$ and each $f_j^i$ is a morphism of $E_i^{i+1}$, then $f_j^{i+1} \circ f_0^i$ is a morphism of $E_i^{i+2}$.

If each $f_0^i$ is continuous then $f_j^{i+1} \circ f_0^i$ is continuous.

If each $C_i^\text{mod} \subseteq C_i^{i+1}$ and each $f_1^i$ is locally a morphism of $C_i^i$ to $C_i^{i+1}$, then by lemma 5.26, $f_j^{i+1} \circ f_1^i$ is locally a morphism of $C_i^i$ to $C_i^{i+2}$.

If each $C_i^\text{full-cat} \subseteq C_i^{i+1}$ and each $f_1^i$ is a morphism of $C_i^i$, then $f_j^{i+1} \circ f_1^i$ is a morphism of $C_i^{i+2}$.

The identity morphism of $(A^i, E^i, C^i)$ is an identity morphism.

Proof. The result follows from lemma 3.17 (Tuple composition for labeled morphisms) on page 10.

Corollary 8.22 (M-atlas morphisms). Let $C^i$, $i = 1, 2, 3$, be model categories, $C^i \in \text{Ob } C^i$, $E^i$ be topological spaces, $A^i$ be m-atlases of $E^i$ in the coordinate spaces $C^i$, $A^2$ semi-maximal and $f^i \overset{\text{def}}{=} (f_0^i: E^i \longrightarrow E^{i+1}, f_1^i: C^i \longrightarrow C^{i+1})$ be (strict) $E^i$-$E^{i+1}$ m-atlas morphisms of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$.

Then $f_j^{i+1} \circ f_i^j = (f_0^i \circ f_0^j, f_1^i \circ f_1^j)$ is a (strict) $E^i$-$E^{i+1}$ m-atlas morphisms of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$.

Proof. Since $(f_0^i: E^i \longrightarrow E^{i+1}, f_1^i: C^i \longrightarrow C^{i+1})$ is a (strict) $E^i$-$E^{i+1}$ m-atlas morphisms of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$, then $(f_0^i \circ f_0^j, f_1^i \circ f_1^j)$ is a (strict) $E^i$-$E^{i+1}$ m-atlas morphisms of $A^i$ to $A^{i+1}$ in the coordinate spaces $C^i$, $C^{i+1}$.

Let $E^i$, $C^i$, $i = 1, 2, 3$ be model categories, $E^i \in \text{Ob}(E^i)$, $C^i \in \text{Ob}(C^i)$, $A^i$ m-atlases of $E^i$ in the coordinate spaces $C^i$, $A^2$ maximal and $f^i \overset{\text{def}}{=} (f_0^i: E^i \longrightarrow E^{i+1}, f_1^i: C^i \longrightarrow C^{i+1})$ (strict) $E^i$-$E^{i+1}$ m-atlas morphisms of $A^i$ to
A^{i+1} in the coordinate spaces C^i, C^{i+1}. Then f^{i+1} \circ f^i = (f_0^2 \circ f_0^1, f_2^2 \circ f_1^1) is a (strict) E^1-E^3 m-atlas morphism of A^1 to A^3 in the coordinate spaces C^1, C^3.

Proof. Since A^2 is maximal it is semi-maximal. □

Let C^i, i = 1, 2, 3, be model categories, C^i ∈ C^i, E^i be topological spaces, A^i be m-atlasses of E^i in the coordinate spaces C^i, A^2 maximal and f^i \text{ def } \{(f_0^1: E^i \rightarrow E^{i+1}, f_1^1: C^i \rightarrow C^{i+1}) be (strict) E^1-E^{i+1} m-atlas morphisms of A^i to A^{i+1} in the coordinate spaces C^i, C^{i+1}. Then f^{i+1} \circ f^i = (f_0^2 \circ f_0^1, f_2^2 \circ f_1^1) is a (strict) E^1-E^3 m-atlas morphisms of A^1 to A^3 in the coordinate spaces C^1, C^3.

Proof. Since A^2 is maximal it is semi-maximal. □

Definition 8.23 (Sets of m-atlas morphisms). Let E^i and C^i. i = 1, 2, be model spaces. Then

\[ \text{Atl}_{\text{Ax}}(E^i, C^1, E^2, C^2) \text{ def } \{(f_0^i, (A^1, E^1, C^1), (A^2, E^2, C^2)) | \text{isAtl}_{\text{Ax}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1) \} \]  \hspace{1cm} (8.47)

Let E^i, i = 1, 2 be topological spaces and C^i = (C^i, C^i), i = 1, 2 be model spaces. Then

\[ \text{Atl}_{\text{Ax}}(E^1, C^1, E^2, C^2) \text{ def } \{(f_0^i, (A^1, E^1, C^1), (A^2, E^2, C^2)) | \text{isAtl}_{\text{Ax}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1) \} \]  \hspace{1cm} (8.48)

Definition 8.24 (Category Atl(E, C)). Let E and C be sets of model spaces. Let P \text{ def } E \times C. Then

\[ \text{Atl}_{\text{Ob}}(E, C) \text{ def } \bigcup_{(E^\mu, C^\mu) \in P} \text{Atl}_{\text{Ob}}(E^\mu, C^\mu) \]  \hspace{1cm} (8.49)

\[ \text{Atl}_{\text{Ax}}(E, C) \text{ def } \bigcup_{(E^\mu, C^\mu) \in P} \text{Atl}_{\text{Ax}}(E^\mu, C^\mu, E^\nu, C^\nu) \]  \hspace{1cm} (8.50)

\[ \text{Atl}(E, C) \text{ def } (\text{Atl}_{\text{Ob}}(E, C), \text{Atl}_{\text{Ax}}(E, C), \circ) \]  \hspace{1cm} (8.51)

Lemma 8.25 (Atl(E, C) is a category). Let E and C be sets of model spaces. Then Atl(E, C) is a category and the identity morphism for an object (A^i, E^i, C^i) of Atl_{\text{Ob}}(E, C) is Id_{(A^i, E^i, C^i)}.
Proof. Let \((A^i, E^i, C^i), i = 1, 2, 3\) be objects of \(\text{Atl}(E, C)\) and let 
\[ m_i \overset{\text{def}}{=} ((f^i_0, f^i_1), (A^i, E^i, C^i), (A^{i+1}, E^{i+1}, C^{i+1})) \] 
be morphisms of \(\text{Atl}(E, C)\). Then

1. Composition:
\[(f^2 \circ f^1) \overset{\text{def}}{=} (f^2_0 \circ f^1_0, f^2_1 \circ f^1_1, (A^1, E^1, C^1), (A^2, E^2, C^2))\] 
is a morphism of \(\text{Atl}(E, C)\) by lemma 8.21 (M-atlas morphisms) on page 43.

2. Associativity:
Composition is associative by lemma 3.17 (Tuple composition for labeled morphisms) on page 10.

3. Identity:
\[ \text{Id}_{(A^1, E^1, C^1)} \] 
is an identity morphism by lemma 3.17.

\[ \square \]

Definition 8.26 (Category \(\text{Atl}(E, C)\)). Let \(E\) and \(C\) be model categories. Let 
\[ P \overset{\text{def}}{=} \text{Ob}(E) \times \text{Ob}(C) \] 
Then
\[ \text{Atl}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \bigcup_{E \in \text{Ob } E, C \in \text{Ob } C} \text{Atl}_{\text{Ob}}(E, C) \] 
(8.52)

\[ \text{Atl}_{\text{Str}}(E, C) \overset{\text{def}}{=} \left\{ ((f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2)) \mid (E^i, C^i) \in P \wedge \text{isAtl}_{\text{Str}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1, E, C) \right\} \] 
(8.53)

\[ \text{Atl}(E, C) \overset{\text{def}}{=} (\text{Atl}_{\text{Ob}}(E, C), \text{Atl}_{\text{Str}}(E, C), \text{Id}_{\text{Atl}(E, C)}) \] 
(8.54)

Let \((A^1, E^1, C^1) \in \text{Atl}_{\text{Ob}}(E, C)\).

\[ \text{Id}_{(A^1, E^1, C^1)} \overset{\text{def}}{=} (\text{Id}_{E^1}, \text{Id}_{C^1}, (A^1, E^1, C^1), (A^1, E^1, C^1)) \] 
(8.55)

Lemma 8.27 (\(\text{Atl}(E, C)\) is a category). Let \(E\) and \(C\) be model categories. Then 
\(\text{Atl}(E, C)\) is a category and the identity morphism for an object \((A^i, E^i, C^i)\) of 
\(\text{Atl}_{\text{Ob}}(E, C)\) is \(\text{Id}_{(A^i, E^i, C^i)}\).

Proof. Let \((A^i, E^i, C^i), i = 1, 2, 3\) be objects of \(\text{Atl}(E, C)\) and let 
\[ m_i \overset{\text{def}}{=} ((f^i_0, f^i_1), (A^i, E^i, C^i), (A^{i+1}, E^{i+1}, C^{i+1})) \] 
be morphisms of \(\text{Atl}(E, C)\). Then

1. Composition:
\[ f^2 \circ f^1 \text{ is a morphism of } E, \text{ and } f^2 \circ f^1 \text{ is a morphism of } C \text{ and } f^2 \circ f^1 \text{ is an } E^1-E^2 \text{ m-atlas morphism of } A^1 \text{ to } A^2 \text{ in the coordinate spaces } C^1, C^2 \] 
by lemma 8.21 (M-atlas morphisms) on page 43.
2. Associativity:
Composition is associative by lemma 3.17 (Tuple composition for labeled morphisms) on page 10.

3. Identity:
Id_{(A^i, E^i, C^i)} is an identity morphism by lemma 3.17.

\[ \text{Definition 8.28 (Category \( \text{Atl}(E, C) \) of topological spaces). Let \( E \) be a set of topological spaces and \( C \) a set of model spaces. Let \( P \overset{\text{def}}{=} E \times C \). Then} \]

\[ \text{Atl}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \bigcup_{E^1 \in E, C^1 \in C} \text{Atl}_{\text{Ob}}(E^1, C^1) \quad (8.56) \]

\[ \text{Atl}_{\text{Ar}}(E, C) \overset{\text{def}}{=} \bigcup_{(E^1, C^1) \in P} \text{Atl}_{\text{Ar}}(E^1, C^1) \quad (8.57) \]

\[ \text{Atl}(E, C) \overset{\text{def}}{=} (\text{Atl}_{\text{Ob}}(E, C), \text{Atl}_{\text{Ar}}(E, C), \circ) \quad (8.58) \]

Let \( (A^1, E^1, C^1) \) be an object in \( \text{Atl}_{\text{Ob}}(E, C) \).

\[ \text{Id}_{(A^1, E^1, C^1)} \overset{\text{def}}{=} (f_{E^1}, f_{C^1}, (A^1, E^1, C^1), (A^1, E^1, C^1)) \quad (8.59) \]

\[ \text{Lemma 8.29 (\( \text{Atl}(E, C) \) of spaces is a category). Let \( E \) be a set of topological spaces and \( C \) a set of model spaces. Then \( \text{Atl}(E, C) \) is a category and the identity morphism for an object \( (A^1, E^1, C^1) \) is \( \text{Id}_{(A^1, E^1, C^1)} \).} \]

\[ \text{Proof. Let} \ (A^i, E^i, C^i), \ i = 1, 2, 3 \text{ be objects of} \ \text{Atl}(E, C) \text{, and let} \ m_i \overset{\text{def}}{=} (f^0_i, f^1_i, (A^i, E^i, C^i), (A^i, E^i, C^i), (A^i, E^i, C^i)) \text{ be morphisms of} \ \text{Atl}(E, C) \text{. Then} \]

1. Composition:
\[ (f^0_0 \circ f^1_0, f^2_0 \circ f^1_0, (A^1, E^1, C^1), (A^3, E^3, C^3)) \text{ is a morphism of} \ \text{Atl}(E, C) \text{ by corollary 8.22 (M-atlas morphisms) on page 48.} \]

2. Associativity:
Composition is associative by lemma 3.17 (Tuple composition for labeled morphisms) on page 10.

3. Identity:
\( \text{Id}_{(A^i, E^i, C^i)} \) is an identity morphism by lemma 3.17.
Definition 8.30. Let $E$ be a set of topological spaces and $C$ a set of model spaces. Then
\[
E^{\text{triv}} := \left\{ E^\mu | E^\mu \in E \right\}
\]
(8.60)

\[
\text{Atl}_{\text{triv}}(E, C) \equiv \text{Atl}_{\text{triv}}(E, C)
\]
(8.61)

Definition 8.31 ($\mathcal{F}_{\text{Top}}$ on objects). Let $E$ be a topological space, $C$ a model space and $A$ an $m$-atlas of $E$ in the coordinate space $C$. Then
\[
\mathcal{F}_{\text{Top}}(A, E, C) \equiv (A, E, C)
\]
(8.62)

Definition 8.32 ($\mathcal{F}_{\text{Top}}$ on morphisms). Let $C^i, i = 1, 2$, be model categories, $C^i_{\text{Ob}} \subset C^i$, $E^i$ topological spaces, $A^i$ $m$-atlas of $E^i$ in the coordinate spaces $C^i$, $(f_0: E^1 \to E^2$ and $f_1: C^1 \to C^2)$. Then
\[
\mathcal{F}_{\text{Top}}((f_0^1, E^1 \to E^2), f_1^1: C^1 \to C^2), (A^1, E^1, C^1), (A^2, E^2, C^2)) \equiv
\]
\[
\left((f_0^2 \circ f_0^1, f_1^2 \circ f_1^1), (A^1, E^1, C^1), (A^2, E^2, C^2)\right)
\]
(8.63)

Theorem 8.33 ($\mathcal{F}_{\text{Top}}$ is a functor). Let $E$ be a set of topological spaces and $C$ a set of model spaces. $\mathcal{F}_{\text{Top}}$ is a functor from $\text{Atl}(E, C)$ to $\text{Atl}_{\text{triv}}(E, C)$.

Proof. Let $C^1 = (A^1, E^1, C^1)$ be an object of $\text{Atl}(E, C)$ and
\[
((\text{Id}_{E^1}, \text{Id}_{C^1}), (A^1, E^1, C^1), (A^1, E^1, C^1)) \text{ be an identity morphism of}
\]
\[
(A^1, E^1, C^1). \text{ Then } ((\text{Id}_{E^1}, \text{Id}_{C^1}), (A^1, E^1, C^1), (A^1, E^1, C^1)) \text{ is an identity morphism of}
\]
\[
(A^1, E^1, C^1).
\]

Let $m_1 \equiv (f_0^1, E^1 \to E^2), (A^1, E^1, C^1)$ and
\[
m_2 \equiv (f_0^2, f_1^2), (A^2, E^2, C^2), (A^3, E^3, C^3)) \text{ be morphisms of } \text{Atl}(E, C).
\]

\[
\mathcal{F}_{\text{Top}}(m_2 \circ m_1) \equiv
\]
\[
\mathcal{F}_{\text{Top}}((f_0^2 \circ f_0^1, f_1^2 \circ f_1^1), (A^1, E^1, C^1), (A^2, E^2, C^2)) =
\]
\[
((f_0^3 \circ f_0^2, f_1^3 \circ f_1^2), (A^1, E^1, C^1), (A^3, E^3, C^3)) =
\]
\[
((f_0^3, f_1^3), (A^1, E^1, C^1), (A^2, E^2, C^2)) \circ ((f_0^1, f_1^1), (A^2, E^2, C^2), (A^3, E^3, C^3)) =
\]
\[
\mathcal{F}_{\text{Top}}((f_0^3, f_1^3), (A^2, E^2, C^2), (A^3, E^3, C^3)) =
\]
\[
\mathcal{F}_{\text{Top}}((f_0^1, f_1^1), (A^1, E^1, C^1), (A^2, E^2, C^2)) =
\]
\[
\mathcal{F}_{\text{Top}}(m_2) \circ \mathcal{F}_{\text{Top}}(m_1)
\]
(8.64)

\[\square\]
9 Local Coordinate Spaces

This section of the paper defines local coordinate spaces, morphisms among them and categories of them.

**Definition 9.1** (Local $\ast - \Sigma$ coordinate spaces). Let $\mathcal{M} \overset{\text{def}}{=} (\mathcal{M}_\alpha, \alpha \prec A)$ be a sequence of categories, $\mathcal{M} \overset{\text{def}}{=} (\mathcal{M}_\alpha, \alpha \prec A)$ be a sequence of spaces and $\mathcal{F} \overset{\text{def}}{=} (F_\gamma, \gamma \prec \Gamma)$ a sequence of functions. $L \overset{\text{def}}{=} (\mathcal{M}, \mathcal{M}, A, F, \Sigma)$ is a local $\mathcal{M} - \Sigma$ coordinate space, abbreviated isLCS$_{\text{Ob}}$ $L$, a local $\mathcal{M} - \Sigma$ coordinate space, abbreviated isLCS$_{\text{Ob}}$ $(L, A)$, and a local $A - \Sigma$ coordinate space, abbreviated isLCS$_{\text{Ob}}$ $(L, A)$, iff

1. $\mathcal{M}_1$ is a model category
2. $(\mathcal{M}, \mathcal{M}, \Sigma, F)$ is a $\mathcal{M}$-$\Sigma$ prestructure
3. $A$ is a maximal m-atlas of $\mathcal{M}_0$ in $\mathcal{M}_1$.
4. $\forall \gamma \prec \Gamma$, if $F_\gamma: (M_\sigma, \alpha, \beta \prec B_\gamma) \rightarrow \mathcal{T}$ is a constraint function then $F_\gamma[\bigwedge_{\beta < B_\gamma} M_\sigma, \alpha, \beta] = \{\text{True}\}$, i.e.,

\[
\bigwedge_{(s_\beta \in M_\sigma, \alpha, \beta \prec B_\gamma)} F_\gamma(s_\beta, \beta \prec B_\gamma) = \text{True}.
\]

Define the total space $(E, E)$ to be $\mathcal{M}_0$, the Coordinate space $(C, C)$ to be $\mathcal{M}_1$, the adjunct spaces, if any, to be $\mathcal{M}_\alpha, \alpha > 1$, and the adjunct functions if any, to be $F_\alpha, \alpha \prec A$.

**Remark 9.2.** There are alternative approaches that are beyond the scope of this paper, e.g.,

1. Requiring the atlas to be full would eliminate certain pathologies.
2. A more complex definition would more easily accommodate structures with more than one atlas, e.g.,
   (a) A fiber bundle with a differential structure
   (b) Multiple fiber bundles on the same base space.

**Lemma 9.3** (Local $\ast - \Sigma$ coordinate spaces). Let $\mathcal{M}^i$, $i = 1,2$, be a sequence of categories, $\mathcal{M}^2_j$, $j = 0, 1$, model categories, $\mathcal{M}^1 \overset{\text{full-cat}}{\subseteq} \mathcal{M}^2$ and $L^1 \overset{\text{def}}{=} (\mathcal{M}^1, \mathcal{M}, A, F, \Sigma)$ a local $\mathcal{M}^1 - \Sigma$ coordinate space, Then $L^2 \overset{\text{def}}{=} (\mathcal{M}^2, \mathcal{M}, A, F, \Sigma)$ is a local $\mathcal{M}^2 - \Sigma$ coordinate space,

**Proof.** $L^2$ satisfies the conditions of definition 9.1 (Local $\ast - \Sigma$ coordinate spaces):

1. $\mathcal{M}_0^2$ and $\mathcal{M}_1^2$ are model categories by hypothesis.
2. \( (\mathcal{M}^2, M, \Sigma, F) \) is a \( \mathcal{M}^2-\Sigma \) prestructure by lemma 7.2 (Prestructures) on page 27.

3. \( A \) is a maximal \( m \)-atlas of \( M_0 \) in \( M_1 \) by hypothesis.

4. \( \forall \gamma \prec \Gamma, \) if \( F_{\gamma}: (M_{\sigma, \alpha, \beta}, \beta \prec B_\gamma) \rightarrow T \) is a constraint function then
   \( F_{\gamma}[X_{\beta \prec B_\gamma, M_{\sigma, \alpha, \beta}}] = \{\text{True}\}, \) i.e.,
   \[ \bigvee (s_{\beta \in M_{\sigma, \alpha, \beta}, \beta \prec B_\gamma}) F_{\gamma}(s_{\beta}, \beta \prec B_\gamma) = \text{True}. \]

\[ \square \]

**Definition 9.4** (LCSOb).

\[ \text{LCSOb}(\mathcal{M}, \Sigma) \equiv \left\{ L \mid M \in \mathcal{M} \land \text{isLCSOb } L \right\} \quad (9.1) \]

A variety of conditions can be imposed on the transition functions \( t_\beta^\alpha = \phi_\beta \circ \phi_\alpha^{-1} \) by appropriate choice of category.

**Remark 9.5.** While some of cases in section 9.2 (Examples) on page 58 require constraint functions, the specific examples worked out in sections 10 and 11 below do not.

### 9.1 Morphisms of local coordinate spaces

Informally, a morphism between two local coordinate spaces is a sequence of functions that is compatible with the functions and \( m \)-atlases of the two local coordinate spaces. The definition implicitly uses commutation relations, which allows a variety of properties to fall out automatically, e.g., preservation of fibers, being a homomorphism.

**Definition 9.6** (morphisms of local coordinate spaces). Let \( \mathcal{M} \overset{i}{\leftarrow} (M_i, \alpha \prec A), i = 1, 2, \) be a sequence of categories, \( M_i \overset{i}{\leftarrow} (M_i, \alpha \prec A) \) a sequence of spaces, \( L_i \overset{i}{\leftarrow} (\mathcal{M}_i, M_i, A_i, F_i, \Sigma), \) a local \( M_i-\Sigma \) coordinate space and \( f \overset{i}{\leftarrow} (f_\alpha: M_1 \overset{i}{\leftarrow} M_2, \alpha \prec A). \)

\( f \) is a morphism from \( L_1 \) to \( L_2 \), abbreviated

\( \text{isLCSAr}(\mathcal{M}_1, M_1, A_1, F_1, \Sigma, \mathcal{M}_2, M_2, A_2, F_2, f), \) iff

1. \( f \) is a morphism from the prestructure \( P^1 \overset{i}{\leftarrow} (\mathcal{M}_1, M_1, \Sigma, F_1^1) \) to the prestructure \( P^2 \overset{i}{\leftarrow} (\mathcal{M}_2, M_2, \Sigma, F_2^2). \)

2. \( (f_0, f_1) \) is a \( M_0^1-M_0^2 \) \( m \)-atlas morphism from \( A_1 \) to \( A_2 \) in the coordinate model categories \( \mathcal{M}_1^1-\mathcal{M}_2^2. \)

\( f \) is a semi-strict morphism from \( L_1 \) to \( L_2 \), abbreviated

\( \text{isLCSAr}_s(\mathcal{M}_1, M_1, A_1, F_1, \Sigma, \mathcal{M}_2, M_2, A_2, F_2, f), \) iff
1. $f$ is a semi-strict morphism from the prestructure $P^1 \defeq (\mathcal{M}^1, M^1, \Sigma, F^1)$ to the prestructure $P^2 \defeq (\mathcal{M}^2, M^2, \Sigma, F^2)$.

2. $(f_0, f_1)$ is a semi-strict $M^2_0$-$M^2_0$ m-atlas morphism from $A^1$ to $A^2$ in the coordinate model categories $M^2_1$-$M^2_2$.

$f$ is a strict morphism from $(\mathcal{M}^1, M^1, A^1, F^1, \Sigma)$ to $(\mathcal{M}^2, M^2, A^2, F^2, \Sigma)$, abbreviated isLCS$(\mathcal{M}^1, M^1, A^1, F^1, \Sigma, \mathcal{M}^2, M^2, A^2, F^2, f)$, iff

1. $f$ is a strict morphism from the prestructure $P^1 \defeq (\mathcal{M}^1, M^1, \Sigma, F^1)$ to the prestructure $P^2 \defeq (\mathcal{M}^2, M^2, \Sigma, F^2)$.

2. $(f_0, f_1)$ is a strict $M^2_0$-$M^2_0$-$M^2_0$-$M^2_0$-$M^2_1$-$M^2_2$ m-atlas morphism from $A^1$ to $A^2$ in the coordinate spaces $M^2_1$, $M^2_2$.

Lemma 9.7 (morphisms of local coordinate spaces). Let $\mathcal{M}^i \defeq (M^i_0, \alpha \prec A)$, $i \in [1, 4]$, be a sequence of categories, $\mathcal{M}^i \defeq (M^i_0, \alpha \prec A) \in \mathcal{M}^i$ a sequence of spaces, $P^i \defeq (\mathcal{M}^i, M^i, F^i, \Sigma)$, $L^i \defeq (\mathcal{M}^i, M^i, A^i, F^i, \Sigma)$ a local $M^i$-$\Sigma$ coordinate space and $f^i \defeq f^i_0: M^i_0 \longrightarrow M^i_0^{i+1}, \alpha \prec A)$, $i = 1, 2, 3$, a (strict) morphism from $L^i$ to $L^{i+1}$.

If $\mathcal{M}^i \subseteq \mathcal{M}^{i+1}, S^i = S^{i+1}, A^i = A^{i+1}$ and each $F^i_\gamma: F^i_\gamma \times S^i \longrightarrow \text{tail}(S^i)$ then $\text{Id}_{\mathcal{M}^i, \Sigma}$ is a strict morphism from $L^i$ to $L^{i+1}$.

Proof. $\text{Id}_{\mathcal{M}^i, \Sigma}$ is a strict morphism from $P^i$ to $P^{i+1}$ by lemma 7.5 (Prestructure morphisms) on page 29.

$\text{Id}_{\mathcal{M}^i, \Sigma}$ is a strict $M^i_0$-$M^i_0^{i+1}$-$M^i_0$-$M^i_0^{i+1}$-$M^i_1$-$M^i_1^{i+1}$ m-atlas morphism of $A^i$ to $A^{i+1}$ in the coordinate spaces $M^i_1$, $M^i_1^{i+1}$. by lemma 8.21 (M-atlas morphisms) on page 43.

Let $\mathcal{M}^i \defeq (M^i_0, \alpha \prec A)$, $i \in [1, 4]$, be a sequence of categories and $L^i \defeq (\mathcal{M}^i, M^i, A^i, F^i, \Sigma)$ be a local coordinate space. Then $f^i$ is a morphism from $L^i$ to $L^{i+1}.$

Proof. The commutation relations do not depend on the categories.

Let $\mathcal{M}^i \defeq (M^i_0, \alpha \prec A)$, $i \in [1, 4]$, be a sequence of categories, $M^i_0$ and $\mathcal{M}^i$ be model categories, $\mathcal{M}^i \subseteq \mathcal{M}^i$, $P^i \defeq (\mathcal{M}^i, M^i, F^i, \Sigma)$ and $L^i \defeq (\mathcal{M}^i, M^i, A^i, F^i, \Sigma)$. Then

1. $L^i$ is a local $M^i$-$\Sigma$ coordinate space and $f^i$ is a (strict) morphism from $L^i$ to $L^{i+1}$.

2. If $f^i$ is a semi-strict morphism from $L^i$ to $L^{i+1}$ then $f^i$ is a semi-strict morphism from $L^i$ to $L^{i+1}$.
3. If $f^i$ is a strict morphism from $L^i$ to $L^{i+1}$ then $f^i$ is a strict morphism from $L^i$ to $L^{i+1}$.

Proof. Each $L^i$ is a local $M^i$-$\Sigma$ coordinate space:

1. $M^n_i$ and $M^n_1$ are model categories by hypothesis.

2. $P^i$ is a $M^n_i$-$\Sigma$ prestructure by lemma 7.2 (Prestructures) on page 27.

3. $A^i$ is a maximal m-atlas of $M^i_0$ in $M^i_1$ by hypothesis.

4. All constraint functions evaluate to True by hypothesis.

Each $f^i$ is a (strict) morphism from $L^i$ to $L^{i+1}$:

1. $f^i$ is a morphism from the prestructure $P^i$ to the prestructure $P^{i+1}$ by lemma 7.5 (Prestructure morphisms) on page 29.

If $f^i$ is a semi-strict (strict) morphism from the prestructure $P^i$ to the prestructure $P^{i+1}$ then $f^i$ is a semi-strict (strict) morphism from the prestructure $P^i$ to the prestructure $P^{i+1}$ by lemma 7.5.

2. $(f^i_0, f^i_1)$ is a $M^i_0$-$M^i_0^{i+1}$ m-atlas morphism from $A^i$ to $A^{i+1}$ in the coordinate model spaces $M^i_1$, $M^{i+1}_1$ by hypothesis.

If $(f^i_0, f^i_1)$ is a semi-strict (strict) $M^i_0$-$M^i_0^{i+1}$-$M^i_0^{i+1}$-$M^i_1$-$M^{i+1}_1$ m-atlas morphism from $A^i$ to $A^{i+1}$ in the coordinate model spaces $M^i_1$, $M^{i+1}_1$ then $(f^i_0, f^i_1)$ is a semi-strict (strict) $M^i_0$-$M^i_0^{i+1}$-$M^i_0^{i+1}$-$M^i_1$-$M^{i+1}_1$ m-atlas morphism from $A^i$ to $A^{i+1}$ in the coordinate model spaces $M^i_1$, $M^{i+1}_1$ by lemma 8.21 (M-atlas morphisms) on page 43.

\[ f^{i+1} \circ f^i \text{ is a (strict) morphism from } L^i \text{ to } L^{i+2}. \]

Proof. 1. If each $f^j$ is a (strict) morphism from the prestructure $P^j$ to the prestructure $P^{j+1}$ then $f^{j+1} \circ f^j$ is a (strict) morphism from the prestructure $P^j$ to the prestructure $P^{j+2}$ by lemma 7.5 (Prestructure morphisms) on page 29.

If each $f^j$ is a semi-strict (strict) morphism from the prestructure $P^j$ to the prestructure $P^{j+1}$ then $f^{j+1} \circ f^j$ is a semi-strict (strict) morphism from the prestructure $P^j$ to the prestructure $P^{j+2}$ by lemma 7.5.

2. $(f^{i+1}_0 \circ f^i_0, f^{i+1}_1 \circ f^i_1)$ is a semi-strict (strict) $M^i_0$-$M^i_0^{i+1}$-$M^i_0^{i+1}$-$M^i_1$-$M^{i+1}_1$ m-atlas morphism from $A^i$ to $A^{i+2}$ in the coordinate model spaces $M^i_1$, $M^{i+2}_1$ by lemma 8.21 (M-atlas morphisms) on page 43. If $(f^{i+1}_0 \circ f^i_0, f^{i+1}_1 \circ f^i_1)$ is a semi-strict (strict) $M^i_0$-$M^i_0^{i+1}$-$M^i_0^{i+1}$-$M^i_1$-$M^{i+1}_1$ m-atlas morphism from $A^i$ to $A^{i+2}$ in the coordinate model spaces $M^i_1$, $M^{i+2}_1$ then $(f^{i+1}_0 \circ f^i_0, f^{i+1}_1 \circ f^i_1)$ is a semi-strict (strict) $M^i_0$-$M^i_0^{i+1}$-$M^i_0^{i+1}$-$M^i_1$-$M^{i+1}_1$ m-atlas morphism from $A^i$ to $A^{i+2}$ in the coordinate model spaces $M^i_1$, $M^{i+2}_1$ by lemma 8.21.

\[ \square \]
Definition 9.8 (LCS\textsubscript{Ar}). Let $\mathcal{M}^i \overset{\text{def}}{=} (\mathcal{M}^i \alpha, \alpha \prec A)$, $i = 1, 2$, be sequences of categories and $\Sigma \overset{\text{def}}{=} (\sigma, \gamma \prec \Gamma)$, $\gamma \prec \Gamma$) a signature over $\mathcal{M}^i$.

\[
\text{LCS}_{\text{Ar}}(\mathcal{M}^1, \Sigma, \mathcal{M}^2) \overset{\text{def}}{=} \left\{ (f, (\mathcal{M}^1, \mathcal{M}^1, A^1, F^1, \Sigma), (\mathcal{M}^2, \mathcal{M}^2, A^2, F^2, \Sigma)) \mid M^i \in \mathcal{M}^i \wedge \text{isLCS}_{\text{Ar}}(\mathcal{M}^1, \mathcal{M}^1, A^1, F^1, \Sigma, \mathcal{M}^2, \mathcal{M}^2, A^2, F^2, f) \right\} \tag{9.2}
\]

If $\mathcal{M}^1 \overset{\text{full-cat}}{\subseteq} \mathcal{M}^2$ then

\[
\text{LCS}_{\text{Ar}}(\mathcal{M}^1, \Sigma, \mathcal{M}^2) \overset{\text{semi-strict}}{=} \left\{ (f, (\mathcal{M}^1, \mathcal{M}^1, A^1, F^1, \Sigma), (\mathcal{M}^2, \mathcal{M}^2, A^2, F^2, \Sigma)) \mid M^i \in \mathcal{M}^i \wedge \text{isLCS}_{\text{Ar}}(\mathcal{M}^1, \mathcal{M}^1, A^1, F^1, \Sigma, \mathcal{M}^2, \mathcal{M}^2, A^2, F^2, f) \right\} \tag{9.4}
\]

\[
\text{LCS}_{\text{Ar}}(\mathcal{M}^1, \Sigma, \mathcal{M}^2) \overset{\text{strict}}{=} \left\{ (f, (\mathcal{M}^1, \mathcal{M}^1, A^1, F^1, \Sigma), (\mathcal{M}^2, \mathcal{M}^2, A^2, F^2, \Sigma)) \mid M^i \in \mathcal{M}^i \wedge \text{isLCS}_{\text{Ar}}(\mathcal{M}^1, \mathcal{M}^1, A^1, F^1, \Sigma, \mathcal{M}^2, \mathcal{M}^2, A^2, F^2, f) \right\} \tag{9.5}
\]

\[
\text{LCS}_{\text{Ar}}(\mathcal{M}^1, \Sigma, \mathcal{M}^2) \overset{\text{semi-strict}}{=} \left\{ (f, (\mathcal{M}^1, \mathcal{M}^1, A^1, F^1, \Sigma), (\mathcal{M}^2, \mathcal{M}^2, A^2, F^2, \Sigma)) \mid M^i \in \mathcal{M}^i \wedge \text{isLCS}_{\text{Ar}}(\mathcal{M}^1, \mathcal{M}^1, A^1, F^1, \Sigma, \mathcal{M}^2, \mathcal{M}^2, A^2, F^2, f) \right\} \tag{9.6}
\]

\[
\text{LCS}_{\text{Ar}}(\mathcal{M}^1, \Sigma, \mathcal{M}^2) \overset{\text{strict}}{=} \left\{ (f, (\mathcal{M}^1, \mathcal{M}^1, A^1, F^1, \Sigma), (\mathcal{M}^2, \mathcal{M}^2, A^2, F^2, \Sigma)) \mid M^i \in \mathcal{M}^i \wedge \text{isLCS}_{\text{Ar}}(\mathcal{M}^1, \mathcal{M}^1, A^1, F^1, \Sigma, \mathcal{M}^2, \mathcal{M}^2, A^2, F^2, f) \right\} \tag{9.7}
\]

Let $M^i \overset{\text{def}}{=} (\mathcal{M}^i \alpha, \alpha \prec A) \in \mathcal{M}^i$, $i = 1, 2$, be a sequence of spaces, $F^i \overset{\text{def}}{=} (\mathcal{M}^i, \mathcal{M}^i, F^i, \Sigma)$ and $L^i \overset{\text{def}}{=} (\mathcal{M}^i, \mathcal{M}^i, A^i, F^i, \Sigma)$ be a local $\mathcal{M}^i$-coordinate space.

If $\mathcal{M}^1 \overset{\text{full-cat}}{\subseteq} \mathcal{M}^2$ then $S^i \overset{\text{def}}{=} (\mathcal{M}^1, \mathcal{M}^2, A^i, \sigma^i)$ and each $F^i_\gamma = F^i_\gamma \times_{\mathcal{S}^1} \text{tail}^{\sigma^i}$

\[
\text{Id}_{L^1, L^2} \overset{\text{def}}{=} (\text{Id}_{M^1, M^2}, L^1, L^2) \tag{9.8}
\]

\[
\text{Id}_{L^1} \overset{\text{def}}{=} \text{Id}_{L^1, L^1} \tag{9.9}
\]

This nomenclature is justified below.
Theorem 9.9 (LCS($\mathcal{M}, \Sigma$) is a category). Let $\mathcal{M} \overset{\text{def}}{=} (\mathcal{M}_\alpha, \alpha < A)$ be a sequence of categories and $\Sigma \overset{\text{def}}{=} (\sigma_\gamma, \gamma < \Gamma) \overset{\text{def}}{=} ((\sigma_\alpha, \alpha < \beta), \gamma < \Gamma)$ a signature over $\mathcal{M}$. Then LCS($\mathcal{M}, \Sigma$), LCS\_{semi-strict}($\mathcal{M}, \Sigma$) and LCS\_{strict}($\mathcal{M}, \Sigma$) are categories.

Let $L^i \in \text{Ob} \text{ LCS}(\mathcal{M}, \Sigma)$. Then $\text{Id}_{L^i}$ is the identity morphism of $L^i$.

Proof. Let $L^i \overset{\text{def}}{=} (M^i, A^i, F^i, \Sigma)$ be objects of LCS($\mathcal{M}, \Sigma$), let $P^i \overset{\text{def}}{=} (\mathcal{M}, M^i, \Sigma, F^i)$ and let $m^i \overset{\text{def}}{=} (f^i \overset{\text{def}}{=} (f^i_\alpha: M^i_\alpha \longrightarrow M^i_{\alpha + 1}, \alpha < A), L^i, L^i + 1)$ be morphisms.

1. Composition:

$f^{i+1} \circ f^i$ is a prestructure morphism from $M^i$ to $M^{i+2}$ by lemma 7.5 (Prestructure morphisms) on page 29 and $(f^{i+1}_0 \circ f^i_0, f^{i+1}_1 \circ f^i_1)$ is an m-atlas morphism from $A^i$ to $A^{i+1}$ by lemma 8.21 (M-atlas morphisms) on page 43.

If each $f^i$ is a strict prestructure morphism from $P^i$ to $P^{i+1}$ then $f^{i+1} \circ f^i$ is a strict prestructure morphism from $M^i$ to $M^{i+2}$ by lemma 7.5

If each $(f^i_0, f^i_1)$ is a strict (semi-strict) $\mathcal{M}_0, M_0 \rightarrow M^i_0 - M^{i+1}_0 \rightarrow M_1$ m-atlas morphism from $A^i$ to $A^{i+1}$ in the coordinate spaces $M^i_1, M^{i+1}_1$ then $(f^{i+1}_0 \circ f^i_0, f^{i+1}_1 \circ f^i_1)$ is a strict (semi-strict) $\mathcal{M}_0, M_0 - M^i_0 \rightarrow M^{i+1}_0 - M_1$ m-atlas morphism from $A^i$ to $A^{i+1}$ in the coordinate spaces $M^i_1, M^{i+1}_1$ by lemma 8.21.

2. Associativity:

Composition is associative by lemma 3.17 (Tuple composition for labeled morphisms) on page 10.

3. Identity:

$\text{Id}_{L^i}$ is the identity morphism of $L^i$ by lemma 3.17.

9.2 Examples

Functors can be constructed among special cases of local coordinate spaces and many mathematical structures, although this paper only gives details for two of them.

Example 9.10 (Manifolds). Choosing the coordinate category as open subsets of a Banach space or more generally a Fréchet space, yields a manifold; the definitions and results of section 10 (Equivalence of manifolds) on page 60, also cover manifolds with boundaries and manifolds with tails.
Example 9.11 (Fiber bundles). Choosing the coordinate category as the space of products of open sets in a topological space (base) with a fixed fiber and the morphisms as fiber-preserving maps yields a fiber bundle; section 11 (Equivalence of fiber bundles) on page 86 covers the more general case of a restricted group of transition functions on fibers.

Example 9.12 (Lie groups). Let $G$ be a topological group with group operation $\ast$ and identity $1_G$, $C$ a linear space \(^8\) and $A$ a maximal $C^k$-atlas\(^9\) of $G$ in the coordinate space $C$. Define $\hat{\ast} : G \times G \to G$, $\text{isCk}_*,A : G \times G \to \text{Truthspace}$ and $L$ by:

$$\hat{\ast}(g_1, g_2) \overset{\text{def}}{=} g_1 \ast g_2^{-1}$$  \hspace{1cm} (9.10)

$$\text{isCk}_*,1_G,A(g_1, g_2) \overset{\text{def}}{=} \begin{cases} \text{True} & \hat{\ast} \text{ is } C^k \text{ at } (g_1, g_2) \\ \text{False} & \hat{\ast} \text{ is not } C^k \text{ at } (g_1, g_2) \end{cases}$$  \hspace{1cm} (9.11)

$$M \overset{\text{def}}{=} \left( \left( \begin{array}{c} G \\ \text{triv} \end{array} \right), \left( \begin{array}{c} C \\ \text{C}^k\text{-triv} \end{array} \right), \text{Truthspace} \right)$$  \hspace{1cm} (9.12)

$$\mathcal{M} \overset{\text{def}}{=} M_{\text{Sing}}$$  \hspace{1cm} (9.13)

$$F \overset{\text{def}}{=} (\ast, \text{isCk}_*,A)$$  \hspace{1cm} (9.14)

$$\Sigma \overset{\text{def}}{=} ((0, 0, 0), (0, 0, 2))$$  \hspace{1cm} (9.15)

$$L \overset{\text{def}}{=} (\mathcal{M}, M, A, F, \Sigma)$$  \hspace{1cm} (9.16)

Then $L$ is a local coordinate space iff $\hat{\ast}$ is $C^k$; the constrain function $\text{isCk}_*,A$ expresses this condition. For $k = \infty$, that LCS is equivalent to a Lie group.

Example 9.13 (Fiber bundle with global sections). Let $B \overset{\text{def}}{=} (E, X, Y, \pi, G, \rho, A)$ be a fiber bundle\(^{10}\), $s : X \to E$ and

$$\text{isSection}_{E,X,\pi,s}(x) \overset{\text{def}}{=} \begin{cases} \text{True} & \pi \circ s(x) = x \\ \text{False} & \pi \circ s(x) \neq x \end{cases}$$  \hspace{1cm} (9.17)

a constraint function. Then a simple modification of definition 11.59 (Functor from fiber bundles to Local Coordinates) on page 109 gives a local coordinate space equivalent to $B$ with the global section $s$.

Example 9.14 (Minkowsky manifolds with foliations). Similarly to the other examples, a local coordinate space can represent a manifold, a global section of the tensor bundle, a constraint function for the signature, a global time coordinate, constraint functions enforcing differentiability and a constraint function enforcing a foliation.

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\(^{8}\)See definition 10.1 (Linear spaces) on page 60 and definition 10.5 (Trivial $C^k$ linear model spaces) on page 61.

\(^{9}\)See definition 10.24 (Maximal $C^k$-atlases) on page 65.

\(^{10}\)See definition 11.56 (fiber bundles) on page 108.
10 Equivalence of manifolds

For a manifold\textsuperscript{11} the coordinate category is open subsets of a Banach space or more generally a Fréchet space, with an appropriate choice of morphisms. Choosing a separating hyperplane and half space, with open sets in the chosen half space, allows manifolds with boundary. Similarly, choosing a ball with tails allows a manifold with tails.

For differentiable manifolds the coordinate category is similar, but the morphisms are limited to those sufficiently differentiable, in order to impose a differentiability constraint on the transition functions $t^\alpha_\beta = \phi_\beta \circ \phi^{-1}_\alpha$.

This section defines $C^k$-atlases, $C^k$-manifolds, local coordinate spaces equivalent to $C^k$-manifolds, categories of them and functors, and gives some basic results.

10.1 Linear spaces and linear model spaces

**Definition 10.1** (Linear spaces). Let $S$ be a locally arcwise connected topological subspace, with non-void interior, of a real (complex) Banach or Fréchet space. Then $S$ is a real (complex) linear space.

**Remark 10.2.** This paper uses the term ball to refer to balls of the underlying space but uses the terms open set and neighborhoods to refer to the relative topology.

Let $S$ be a small category whose objects are real (complex) linear spaces and whose morphisms are $C^k$ functions. Then $S$ is a $C^k$ linear category.

Let $S$ be a real (complex) linear space. Then $S_{C^k-Triv}$, the category of all $C^k$ functions between open subspaces of $S$, is the trivial $C^k$ linear category of $S$.

**Lemma 10.3** (Open subsets of linear spaces are locally arcwise connected). Let $U$ be an open subset of the real (complex) linear space $S$. Then $U$ is locally arcwise connected.

**Proof.** An open subset of a locally connected space is locally connected.  \hfill $\Box$

**Definition 10.4** (Linear model spaces). Let $S$ be a real (complex) linear space and $S = (S, S)$ a model space for $S$. Then $S$ is a real (complex) linear model space.

Let $S = (S, S)$ be a real (complex) linear model space such that every morphism of $S$ is a $C^k$ function. Then $S$ is a real (complex) $C^k$ linear model space.

Let $S$ be a small category whose objects are real (complex) $C^k$ linear model spaces and whose morphisms are $C^k$ model functions. Then $S$ is a $C^k$ linear model category.

\textsuperscript{11} The literature has several different definitions of a manifold. This paper uses one chosen for ease of exposition.
Definition 10.5 (Trivial $C^k$ linear model spaces). Let $S$ be a real (complex) linear space and $\mathcal{S}$ the category of all $C^k$ functions between open sets of $S$. Then $S_{C^k - \text{triv}} \overset{\text{def}}{=} (S, S)$ is the trivial $C^k$ linear model space of $S$ and $S_{C^k - \text{triv}}$ is a real (complex) trivial $C^k$ linear model space.

Let $\mathcal{S}$ be a set of real (complex) linear spaces.

The category of open trivial $C^k$ model spaces in $\mathcal{S}$, abbreviated $\mathcal{S}_{C^k - \text{op-triv}}$, is the category whose objects are $\{U_{C^k - \text{triv}} \mid U \in \mathcal{S}_{\text{op}}\}$, the trivial $C^k$ linear model spaces of non-null open sets of spaces in $\mathcal{S}$, and whose morphisms are all the $C^k$ functions among them.

The set of trivial $C^k$ linear model spaces of $\mathcal{S}$ is $\mathcal{S}_{C^k - \text{triv}} \overset{\text{def}}{=} \{S'_{C^k - \text{triv}} \mid S' \in \mathcal{S}\}$.

The category of trivial $C^k$ linear model spaces of $\mathcal{S}$, abbreviated $\mathcal{S}_{C^k - \text{Triv}}$, is the category whose objects are $\mathcal{S}_{C^k - \text{triv}}$ and whose morphisms are all the $C^k$ functions among them.

Lemma 10.6 (The trivial $C^k$ linear model space of $S$ is a linear model space). Let $S$ be a real (complex) linear space. Then $S_{C^k - \text{triv}} = (S, S)$ is a linear $C^k$ model space.

Proof. $S_{C^k - \text{triv}}$ satisfies the conditions in definition 5.2 (Model spaces) on page 15:

1. $\text{Ob}(\mathcal{S})$ is an open cover for $S$.
2. $\text{Ob}(\mathcal{S})$ is closed under finite intersections.
3. The morphisms of $\mathcal{S}$ are $C^k$, hence continuous.
4. If $f: A \to B$ is a morphism, $A' \in \text{Ob}(\mathcal{S}) \subseteq A \in \text{Ob}(\mathcal{S})$, $B' \in \text{Ob}(\mathcal{S}) \subseteq B \in \text{Ob}(\mathcal{S})$ and $f[A'] \subseteq B'$ then $f \mid_{A'}: A' \to B'$ is continuous and thus a morphism.
5. If $A' \in \text{Ob}(\mathcal{S}) \subseteq A \in \text{Ob}(\mathcal{S})$ then the inclusion map $i: A' \hookrightarrow A$ is $C^k$ and thus a morphism.
6. Restricted sheaf condition: Whenever
   (a) $U_\alpha$ and $V_\alpha$, $\alpha \prec A$, are objects of $\mathcal{S}$.
   (b) $f_\alpha: U_\alpha \to V_\alpha$ are morphisms of $\mathcal{S}$.
   (c) $U \overset{\text{def}}{=} \bigcup_{\alpha \prec A} U_\alpha \in \text{Ob}(\mathcal{S})$,
   (d) $V \overset{\text{def}}{=} \bigcup_{\alpha \prec A} V_\alpha \in \text{Ob}(\mathcal{S})$
   (e) $f: U \to V$ is a continuous function and for every $\alpha \prec A$, $f$ agrees with $f_\alpha$ on $U_\alpha$.
then $f$ is $C^k$ and thus a morphism of $S$. \hfill $\Box$

**Definition 10.7** ($C^k$ singleton categories). Let $C$ be a $C^k$ linear model space. Then the $C^k$ singleton category of $C$, abbreviated $C_{C^k-Sing}$, is the category whose sole object is $C$ and whose morphisms are all of the $C^k$ model functions from $C$ to itself.

### 10.2 $C^k$-nearly commutative diagrams

Let $C$ be a linear space, $C \triangleq S_{C^k-Triv}$ and $D$ a tree with two branches, whose nodes are topological spaces $U_i$ and $V_j$ and whose links are continuous functions $f_i: U_i \to U_{i+1}$ and $f'_j: U_j \to U_{j+1}$ between the spaces:

$$D = \{f_0: U_0 = V_0 \to U_1, \ldots, f_{m-1}: U_{m-1} \to U_m, \quad f'_0: U_0 = V_0 \to V_1, \ldots, f'_{m-1}: V_{m-1} \to V_n \}$$

with $U_0 = V_0, U_m \subseteq C$ and $V_n \subseteq C$, as shown in fig. 3 (Uncompleted nearly commutative diagram) on page 13.

**Definition 10.8** ($C^k$-nearly commutative diagrams). $D$ is $C^k$-nearly commutative in linear space $C$ iff $D$ is nearly commutative in category $C$.

**Definition 10.9** ($C^k$-nearly commutative diagrams at a point). Let $C$, $C$ and $D$ be as above and $x$ be an element of the initial node. $D$ is $C^k$-nearly commutative in $C$ at $x$ iff $D$ is nearly commutative in $C$ at $x$.

**Definition 10.10** ($C^k$-locally nearly commutative diagrams). Let $C$, $C$ and $D$ be as above. $D$ is $C^k$-locally nearly commutative in $C$ if $D$ is locally nearly commutative in $C$.

### 10.3 $C^k$ charts

**Definition 10.11** ($C^k$ charts). Let $C$ be a linear space and $E$ a topological space. A $C^{k,12}$ chart $(U, V, \phi)$ of $E$ in the coordinate space $C$ consists of

1. An open subset $U \subseteq E$, known as a coordinate patch
2. An open subset $V \subseteq C$
3. A homeomorphism $\phi: U \cong V$, known as a coordinate function

**Remark** 10.12. I consider it clearer to explicate the range, rather than the conventional usage of specifying only the domain and function or the minimalist usage of specifying only the function.

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12 With $k \in \mathbb{N} \cup \{\infty, \omega\}$. 

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**Definition 10.13** (C^k subcharts). Let \((U, V, \phi)\) be a C^k chart of E in the coordinate space \(C\) and \(U' \subseteq U\) open. Then \((U', V', \phi') \overset{\text{def}}{=} (U', \phi[U'], \phi|_{U', V'})\) is a subchart of \((U, V, \phi)\).

By abuse of language we will write \((U', V', \phi)\) for \((U', V', \phi')\).

**Lemma 10.14** (C^k subcharts). Let \((U, V, \phi)\) be a C^k chart of E in the coordinate space \(C\) and \((U', V', \phi')\) a subchart of \((U, V, \phi)\). Then \((U', V', \phi')\) is a C^k chart of E in the coordinate space \(C\).

**Proof.** \((U', V', \phi')\) satisfies the conditions of definition 10.11

1. \(U'\) is open by hypothesis.

2. \(\phi\) is a homeomorphism, so \(V' = \phi[U']\) is also open.

3. \(\phi\) is a homeomorphism, so \(\phi|_{U'}: U' \rightarrow \phi[U']\) is also.

**Definition 10.15** (C^k compatibility). Let \((U, V, \phi)\) and \((U', V', \phi')\) be C^k charts of E in the coordinate space \(C\). Then \((U, V, \phi)\) is C^k compatible with \((U', V', \phi')\) iff either

1. \(U\) and \(U'\) are disjoint

2. The transition function \(t = \phi' \circ \phi^{-1} |_{\phi[U \cap U']}\) is a C^k diffeomorphism.

**Lemma 10.16** (Symmetry of C^k compatibility). Let \((U, V, \phi)\) and \((U', V', \phi')\) be C^k charts of E in the coordinate space \(C\). Then \((U, V, \phi)\) is C^k compatible with \((U', V', \phi')\) iff \((U', V', \phi')\) is C^k compatible with \((U, V, \phi)\).

**Proof.** It suffices to prove the implication in only one direction.

1. \(U \cap U' = U' \cap U\).

2. Since the transition function \(t = \phi' \circ \phi^{-1} |_{\phi[U \cap U']}\) is a C^k diffeomorphism of \(C\), so is \(t^{-1} = \phi \circ \phi'^{-1} |_{\phi'[U \cap U']}\).

**Lemma 10.17** (C^k compatibility of subcharts). Let \((U_i, V_i, \phi_i), i = 1, 2, \) be C^k charts of E in the coordinate space \(C\), \((U'_1, V'_1, \phi'_1)\) be subcharts and \((U_1, V_1, \phi_1)\) be C^k compatible with \((U_2, V_2, \phi_2)\). Then \((U'_1, V'_1, \phi'_1)\) is C^k compatible with \((U'_2, V'_2, \phi'_2)\).

**Proof.** If \(U_1 \cap U_2 = \emptyset\) then \(U'_1 \cap U'_2 = \emptyset\). If \(U'_1 \cap U'_2 = \emptyset\) then \((U'_1, V'_1, \phi'_1)\) is C^k compatible with \((U'_2, V'_2, \phi'_2)\). Otherwise, the transition function \(t_2 \overset{\text{def}}{=} \phi_2 \circ \phi_1^{-1} |_{\phi_1[U_1 \cap U_2]}\) is a C^k diffeomorphism and hence \(t_2 \overset{\text{def}}{=} \phi_1|_{U'_1 \cap U'_2} \circ \phi_1^{-1} |_{\phi_1[U_1 \cap U_2]} \circ t_1 |_{\phi_1[U_1 \cap U_2]} \circ \phi_2 |_{\phi_2[U'_1 \cap U'_2]}\) is a C^k diffeomorphism.

\[ \Box \]
Corollary 10.18 (C^k compatibility with subcharts). Let \((U,V,\phi)\) be a C^k charts of \(E\) in the coordinate space \(C\) and \((U',V',\phi')\) a subchart. Then \((U',V',\phi')\) is C^k compatible with \((U,V,\phi)\).

Proof. (\(U,V,\phi\)) is C^k compatible with itself and is a subchart of itself. □

Definition 10.19 (Covering by C^k charts). Let \(A\) be a set of charts of \(E\) in the coordinate space \(C\). \(A\) covers \(E\) iff \(\pi_1[A]\) covers \(E\), i.e., \(E = \bigcup \pi_1[A]\).

10.4 C^k-atlases

A set of charts can be atlases for different coordinate spaces even if it is for the same total space. In order to aggregate them into categories, there must be a way to distinguish them. Including the two^13 spaces in the definitions of the categories serves the purpose.

Definition 10.20 (C^k-atlases). Let \(A\) be a set of mutually C^k compatible charts of \(E\) in the coordinate space \(C\). \(A\) is a C^k-atlas of \(E\) in the coordinate space \(C\), abbreviated isAtl_{Ob}^{C^k}(A, E, C), iff

1. \(A\) covers \(E\)
2. There is at least one chart \((U,V,\phi) \in A\) where \(V\) contains a ball of the underlying Banach or Fréchet space.

\(A\) is a full C^k-atlas of \(E\) in the coordinate space \(C\), abbreviated isAtl_{Ob}^{C^k} full(A, E, C), iff

1. \(\pi_1[A]\) covers \(E\)
2. \(\pi_2[A]\) covers \(C\).
3. There is at least one chart \((U,V,\phi) \in A\) where \(V\) contains a ball of the underlying Banach or Fréchet space.

By abuse of language we write \(U \in A\) for \(U \in \pi_1[A]\).

Let \(E\) be a set of topological spaces and \(C\) a linear space. Then

\[ \text{At}_{Ob}^{C^k}(E,C) \overset{\text{def}}{=} \left\{ (A, E, C) \middle| \text{isAtl}_{Ob}^{C^k}(A, E, C) \right\} \]  \hspace{1cm} (10.1)

\[ \text{At}_{Ob}^{C^k}_{\text{full}}(E,C) \overset{\text{def}}{=} \left\{ (A, E, C) \middle| \text{isAtl}_{Ob}^{C^k}_{\text{full}}(A, E, C) \right\} \]  \hspace{1cm} (10.2)

Let \(E\) be a set of topological spaces and \(C\) a set of linear spaces. Then

\[ \text{At}_{Ob}^{C^k}(E,C) \overset{\text{def}}{=} \bigcup_{C_\mu \in C} \text{At}_{Ob}^{C^k}(E_\mu, C_\mu) \]  \hspace{1cm} (10.3)

^13The total space is redundant, but convenient.
\[ \mathcal{A}_{\text{full}}^{\text{ck}}(E, C) \seteq \left\{ (A, E, C) \left| \is\mathcal{A}_{\text{full}}^{\text{ck}}(A, E, C) \right. \right\} \quad (10.4) \]

**Definition 10.21** (Compatibility of charts with \( C^k \)-atlases). A chart \((U, V, \phi)\) of \( E \) in the coordinate space \( C \) is \( C^k \) compatible with a \( C^k \)-atlas \( A \) iff it is \( C^k \) compatible with every chart in the atlas.

**Lemma 10.22** (Compatibility of subcharts with \( C^k \)-atlases). Let \( A \) be a \( C^k \)-atlas of \( E \) in the coordinate space \( C \) and \( C_1 = (U_1, V_1, \phi_1) \) a \( C^k \) chart in \( A \). Then any subchart of \( C_1 \) is \( C^k \) compatible with \( A \).

**Proof.** Let \( C' = (U', V', \phi') \) be a subchart of \( C_1 \) and \( C_2 = (U_2, V_2, \phi_2) \) another chart in \( A \).

1. If \( U_1 \cap U_2 = \emptyset \), then \( U' \cap U_2 = \emptyset \).
2. If \( U' \cap U_2 = \emptyset \) then \( C' \) is \( C^k \) compatible with \( C_2 \).
3. Otherwise the transition function \( t_2^1 \eqldef \phi_2 \circ \phi_1^{-1} \mid_{\phi_1[U_1 \cap U_2]} \) is a \( C^k \) diffeomorphism and thus \( t_2^1 \mid_{\phi_2[U' \cap U_2]} \) is a \( C^k \) diffeomorphism.

**Lemma 10.23** (Extensions of \( C^k \)-atlases). Let \( A \) be a \( C^k \) atlas of \( E \) in the coordinate space \( C \) and \((U_i, V_i, \phi_i), i = 1, 2 \) be \( C^k \) charts of \( E \) in the coordinate space \( C \) \( C^k \) compatible with \( A \) in the coordinate space \( C \). Then \((U_1, V_1, \phi_1)\) is \( C^k \) compatible with \((U_2, V_2, \phi_2)\) in the coordinate space \( C \).

**Proof.** If \( U_1 \cap U_2 = \emptyset \) then \((U_1, V_1, \phi_1)\) is \( C^k \) compatible with \((U_2, V_2, \phi_2)\).

Otherwise, \( \phi_2 \circ \phi_1^{-1} \mid_{\phi_1[U_1 \cap U_2]}: \phi_1[U_1 \cap U_2] \xrightarrow{\sim} \phi_2[U_1 \cap U_2] \) is a homeomorphism. It remains to show that \( \phi_2 \circ \phi_1^{-1} \mid_{\phi_1[U_1 \cap U_2]} \) is a \( C^k \) diffeomorphism.

Let \((U'_\alpha, V'_\alpha, \phi'_\alpha), \alpha < A\), be charts in \( A \) such that \( U_1 \cap U_2 \subseteq \bigcup_{\alpha < A} U'_\alpha \) and \( U_1 \cap U_2 \cap U'_\alpha \neq \emptyset \), \( \alpha < A \). Since the charts are \( C^k \) compatible with \((U'_\alpha, V'_\alpha, \phi'_\alpha), \phi_2 \circ \phi_1^{-1} \mid_{U_1 \cap U_2 \cap U'_\alpha} = \phi'_\alpha \circ \phi_1^{-1} \mid_{U_1 \cap U_2 \cap U'_\alpha} \) are \( C^k \) diffeomorphisms and thus \( \phi_2 \circ \phi_1^{-1} = \phi_2 \circ \phi_1^{-1} \circ \phi'_\alpha \circ \phi_1^{-1} \) is a \( C^k \) diffeomorphism.

**Definition 10.24** (Maximal \( C^k \)-atlases). Let \( E \) be a topological spaces and \( C \) a linear space. Then \( A \) is a maximal \( C^k \)-atlas of \( E \) in the coordinate space \( C \), abbreviated \( \is\mathcal{A}_{\text{full}}^{\text{ck}}(A, E, C) \), iff \( A \) is a \( C^k \)-atlas that cannot be extended by adding an additional \( C^k \) compatible chart. \( A \) is a semi-maximal \( C^k \)-atlas of \( E \) in the coordinate space \( C \), abbreviated \( \is\mathcal{A}_{\text{full}}^{\text{ck}}(A, E, C) \), iff whenever \((U, V, \phi)\) \( \in \is\mathcal{A}_{\text{max}}^{\text{ck}}(A, E, C) \), \( A \) contains \( (U', V', \phi') \), \( U' \subseteq U, V' \subseteq V \) and \( V'' \subseteq C \) are open, \( \phi[U'] = V' \) and \( \phi': V' \xrightarrow{\sim} V'' \) is a \( C^k \) diffeomorphism then \((U', V', \phi' \circ \phi) \in A\).

\[
\is\mathcal{A}_{\text{full}}^{\text{ck}}(A, E, C) \eqldef \is\mathcal{A}_{\text{full}}^{\text{ck}}(A, E, C) \land \is\mathcal{A}_{\text{max}}^{\text{ck}}(A, E, C) \quad (10.5)
\]
\[
\text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \overset{\text{def}}{=} \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \wedge \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C})
\]

(10.6)

\[
\mathcal{A}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \overset{\text{def}}{=} \left\{ (\mathbf{E}, \mathbf{C}) \mid \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \wedge \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \right\}
\]

(10.7)

\[
\mathcal{A}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \overset{\text{def}}{=} \left\{ (\mathbf{E}, \mathbf{C}) \mid \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \wedge \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \right\}
\]

(10.8)

\[
\mathcal{A}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \overset{\text{def}}{=} \left\{ (\mathbf{E}, \mathbf{C}) \mid \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \wedge \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \right\}
\]

(10.9)

Let \( \mathbf{E} \) be a set of topological spaces and \( \mathbf{C} \) a set of linear spaces. Then

\[
\mathcal{A}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \overset{\text{def}}{=} \left\{ (\mathbf{E}, \mathbf{C}) \mid \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \wedge \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \right\}
\]

(10.10)

\[
\mathcal{A}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \overset{\text{def}}{=} \left\{ (\mathbf{E}, \mathbf{C}) \mid \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \wedge \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \right\}
\]

(10.11)

\[
\mathcal{A}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \overset{\text{def}}{=} \left\{ (\mathbf{E}, \mathbf{C}) \mid \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \wedge \text{isAtl}_{\text{Ob}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \right\}
\]

(10.12)

**Lemma 10.25** (Maximal \( C^k \)-atlases are semi-maximal \( C^k \)-atlases). Let \( \mathbf{E} \) be a topological space, \( \mathbf{C} \) a \( C^k \)-linear space and \( \mathcal{A} \) a maximal \( C^k \)-atlas of \( \mathbf{E} \) in the coordinate space \( \mathbf{C} \). Then \( \mathcal{A} \) is a semi-maximal \( C^k \)-atlas of \( \mathbf{E} \) in the coordinate space \( \mathbf{C} \).

**Proof.** Let \((U, V, \phi) \in \mathcal{A}, U' \subseteq U, V' \subseteq V \) and \( V'' \subseteq \mathbf{C} \) be open, \( \phi[U'] = V' \) and \( \phi': V' \overset{\sim}{\longrightarrow} V'' \) be a \( C^k \) diffeomorphism. \((U', V', \phi)\) is a subchart of \((U, V, \phi)\) and by lemma 10.22 (Compatibility of subcharts with \( C^k \)-atlases) on page 65 is \( C^k \) compatible with the charts of \( \mathcal{A} \). Since \( \phi' \) is a \( C^k \) diffeomorphism, \((U', V'', \phi' \circ \phi)\) is \( C^k \) compatible with the charts of \( \mathcal{A} \). Since \( \mathcal{A} \) is maximal, \((U', V'', \phi' \circ \phi)\) is a chart of \( \mathcal{A} \).

**Theorem 10.26** (Existence and uniqueness of maximal \( C^k \)-atlases). Let \( \mathcal{A} \) be a \( C^k \)-atlas of \( \mathbf{E} \) in the coordinate space \( \mathbf{C} \). Then there exists a unique maximal \( C^k \)-atlas \( \text{Atlas}_{\text{max}}^{\mathcal{A}}(\mathbf{E}, \mathbf{C}) \) of \( \mathbf{E} \) in the coordinate space \( \mathbf{C} \) compatible with \( \mathcal{A} \).

**Proof.** Let \( \mathcal{P} \) be the set of all \( C^k \)-atlases of \( \mathbf{E} \) in the coordinate space \( \mathbf{C} \) containing \( \mathcal{A} \) and \( C^k \) compatible in the coordinate space \( \mathbf{C} \) with all of the \( C^k \) charts in \( \mathcal{A} \). Let \( \mathcal{P} \) be a maximal chain of \( \mathcal{A} \). Then \( \mathcal{A}' = \bigcup_{\mathcal{P}} \mathcal{P} \) is a maximal \( C^k \) atlas of \( \mathbf{E} \) in the coordinate space \( \mathbf{C} \) compatible with \( \mathcal{A} \). Uniqueness follows from lemma 10.23 (Extensions of \( C^k \)-atlases) on page 65.
10.5 $C^k$-atlas morphisms and functors

This section defines categories of $C^k$-atlases $(\text{Atl}^C_{\text{full}}(E,C), \text{Atl}^C_{\text{max}}(E,C))$, constructs functors $(\text{F}_{\text{C}^k,M-\text{atlas}}^C)$ between them and categories of m-atlases $(\text{Atl}(E,C_{\text{triv}}, C_{\text{c-triv}}))$ and constructs inverse functors $(\text{F}_{\text{C}^k,\text{atlas,C}^k})$.

**Definition 10.27 ($C^k$-atlas morphisms).** Let $E^i$, $i = 1, 2$, be topological spaces, $C^i$ linear spaces and $A^i$ $C^k$-atlases of $E^i$ in the coordinate spaces $C^i$. A pair of functions $(f_0, f_1)$ is a (full) $E^1$-$E^2 C^k$-morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$, abbreviated as $\text{isAtl}^C_{\text{full}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$, iff

1. $f_0: E^1 \longrightarrow E^2$ is a continuous function.
2. $f_1: C^1 \longrightarrow C^2$ is a $C^k$ function.

3. for any $(U^1, V^1, \phi^1: U^1 \cong V^1) \in A^1$, $(U^2, V^2, \phi^2: U^2 \cong V^2) \in A^2$, the diagram $D_{\text{def}}(\{I \subseteq U^1 \cap f_0^{-1}[U^2], V^1, E^2, U^2, V^2\}, \{f_0, \phi^2, \phi^1, f_1\})$ is $C^k$-locally nearly commutative in $C^2$, i.e., for any $x \in I$ there are open sets $U^1 \subseteq I, V^1 \subseteq V^1, U^2 \subseteq U^2, V^2 \subseteq V^2$, and a $C^k$ diffeomorphism $\hat{f}: V^2 \cong V^2$ such that eqs. (8.24) to (8.30) on pages 42 to 42 in definition 8.20 (M-atlas morphisms) on page 40 hold.

$(f_0, f_1)$ is also a full $E^1$-$E^2 C^k$-morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$, abbreviated as $\text{isAtl}^C_{\text{max}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$ iff $\bigcup \pi_2[A^1] = C^1 \land \bigcup \pi_2[A^2] = C^2$.

$(f_0, f_1)$ is also a maximal $E^1$-$E^2 C^k$-morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$, abbreviated as $\text{isAtl}^C_{\text{max}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$ iff $\text{isAtl}^C_{\text{full}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$ is $\text{isAtl}^C_{\text{max}}(A^1, E^1, C^1)$ and $\text{isAtl}^C_{\text{max}}(A^2, E^2, C^2)$.

$(f_0, f_1)$ is also a semi-maximal $E^1$-$E^2 C^k$-morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$, abbreviated as $\text{isAtl}^C_{\text{max}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$ iff $\text{isAtl}^C_{\text{max}}(A^1, E^1, C^1)$ and $\text{isAtl}^C_{\text{max}}(A^2, E^2, C^2)$.

$(f_0, f_1)$ is also a full maximal $E^1$-$E^2 C^k$-morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$, abbreviated as $\text{isAtl}^C_{\text{max}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$ iff $\text{isAtl}^C_{\text{max}}(A^1, E^1, C^1)$, $\text{isAtl}^C_{\text{max}}(A^2, E^2, C^2)$ and $\bigcup \pi_2[A^1] = C^1 \land \bigcup \pi_2[A^2] = C^2$.

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14 The conventional definition uses only the first of the two functions and a slightly different compatibility condition.
$(f_0, f_1)$ is also a full semi-maximal $E^1$-$E^2$ $C^k$-morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$, abbreviated as $\text{isAtl}^{C^k}_{S\text{-max\-full}}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1)$ if

$$\text{isAtl}^{C^k}_{S\text{-max\-full}}(A^1, E^1, C^1), \text{isAtl}^{C^k}_{S\text{-max\-full}}(A^2, E^2, C^2) \text{ and } \bigcup_{\max}\pi_2[A^1] = C^1 \land \bigcup_{\max}\pi_2[A^2] = C^2.$$  

The identity morphism of $(A^i, E^i, C^i)$ is

$$\text{Id}_{(A^i, E^i, C^i)} \overset{\text{def}}{=} ((\text{Id}_{A^i, E^i}, \text{Id}_{C^i}), (A^i, E^i, C^i), (A^i, E^i, C^i))$$  

(10.13)

This nomenclature will be justified later.

Let $E^i$, $i = 1, 2$, be topological spaces and $C^i$ be linear spaces. Then

$$\text{Atl}^{C^k}_{A^1}(E^1, C^1, E^2, C^2) \overset{\text{def}}{=} \{(f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2)\}$$  

(10.14)

$$\text{Atl}^{C^k}_{\text{full}}(E^1, C^1, E^2, C^2) \overset{\text{def}}{=} \{(f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2)\}$$  

(10.15)

$$\text{Atl}^{C^k}_{\text{max}}(E^1, C^1, E^2, C^2) \overset{\text{def}}{=} \{(f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2)\}$$  

(10.16)

$$\text{Atl}^{C^k}_{S\text{-max\-full}}(E^1, C^1, E^2, C^2) \overset{\text{def}}{=} \{(f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2)\}$$  

(10.17)

$$\text{Atl}^{C^k}_{S\text{-max\-full}}(E^1, C^1, E^2, C^2) \overset{\text{def}}{=} \{(f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2)\}$$  

(10.18)

$$\text{Atl}^{C^k}_{S\text{-max\-full}}(E^1, C^1, E^2, C^2) \overset{\text{def}}{=} \{(f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2)\}$$  

(10.19)
Lemma 10.28 (C^k-atlas morphisms). Let E be a set of topological spaces, C a set of C^k linear spaces, E^i ∈ E, i = 1, 2, C^i ∈ C and A^i C^k-atlas of E^i in the coordinate spaces C^i. A pair of functions f \text{def} (f_0, f_1) is an E^1-E^2 C^k-morphism of A^1 to A^2 in the coordinate spaces C, C^2 iff f is a strict E^1_{\text{Triv}} - E^1_{\text{Triv}}. C_{\text{max}} - C_{\text{max}} C^k-\text{triv} C^k-\text{triv} morphism of A^1 to A^2 in the coordinate spaces C^1, C^2.

Proof. The model neighborhoods of C^i are the open sets of C^i and the morphisms of C_{\text{max}} are the C^k functions between spaces in C.

Corollary 10.29 (C^k-atlas morphisms). Let E^i, i = 1, 2, 3, be topological spaces, C^i linear spaces, A^i C^k-atlas of E^i in the coordinate spaces C^i and (f_0^i, f_1^i) E^i-E^{i+1} C^k-atlas morphisms of A^i to A^{i+1} in the coordinate spaces C^i, C^{i+1}. Then (f_2^i \circ f_0^1, f_2^i \circ f_1^1) is an E^1-E^3 C^k-morphism of A^1 to A^3 in the coordinate spaces C^1, C^3.

Proof. The result follows from Lemma 8.21 (M-atlas morphisms) on page 43.

Definition 10.30 (Categories of C^k atlases). Let E be a set of topological spaces and C a set of linear spaces. Let P \text{def} E \times C. Then

\[
\mathcal{A}l_{\text{full}}^{C_k}(E, C) \text{def} \bigcup_{(E^\mu, E^\nu, C^\mu, C^\nu) \in P} \mathcal{A}l_{\text{full}}^{C_k}(E^\mu, C^\mu, E^\nu, C^\nu) \tag{10.20}
\]

\[
\mathcal{A}l^{C_k}(E, C) \text{def} \left(\mathcal{A}l_{\text{full}}^{C_k}(E, C), \mathcal{A}l_{\text{max}}^{C_k}(E, C), \mathcal{A}l_{\text{def}}^{C_k}(E, C), \mathcal{A}l_{\text{full}}^{C_k}(E, C) \right) \tag{10.21}
\]

\[
\mathcal{A}l_{\text{full}}^{C_k}(E, C) \text{def} \bigcup_{(E^\mu, E^\nu, C^\mu, C^\nu) \in P \text{ full}} \mathcal{A}l_{\text{full}}^{C_k}(E^\mu, C^\mu, E^\nu, C^\nu) \tag{10.22}
\]

\[
\mathcal{A}l_{\text{full}}^{C_k}(E, C) \text{def} \left(\mathcal{A}l_{\text{full}}^{C_k}(E, C), \mathcal{A}l_{\text{full}}^{C_k}(E, C), \mathcal{A}l_{\text{full}}^{C_k}(E, C) \right) \tag{10.23}
\]

\[
\mathcal{A}l_{\text{max}}^{C_k}(E, C) \text{def} \bigcup_{(E^\mu, E^\nu, C^\mu, C^\nu) \in P \text{ max}} \mathcal{A}l_{\text{max}}^{C_k}(E^\mu, C^\mu, E^\nu, C^\nu) \tag{10.24}
\]

\[
\mathcal{A}l_{\text{max}}^{C_k}(E, C) \text{def} \left(\mathcal{A}l_{\text{max}}^{C_k}(E, C), \mathcal{A}l_{\text{max}}^{C_k}(E, C), \mathcal{A}l_{\text{max}}^{C_k}(E, C) \right) \tag{10.25}
\]
\[
\operatorname{Atl}_{\text{Ar}}^{C_k}(E, C) \overset{\text{def}}{=} \bigcup_{(E^\mu, C^\mu) \in P} \operatorname{Atl}_{\text{Ar}}^{C_k} \left( (E^\mu, C^\mu), (E^\nu, C^\nu) \right) \quad (10.26)
\]
\[
\operatorname{Atl}_{\text{Ob}}^{C_k}(E, C) \overset{\text{def}}{=} \left( \operatorname{Atl}_{\text{Ob}}^{C_k}(E, C), \operatorname{Atl}_{\text{Ar}}^{C_k}(E, C), A^1 \right) \quad (10.27)
\]
\[
\operatorname{Atl}_{\text{max} \rightarrow \text{full}}^{C_k}(E, C) \overset{\text{def}}{=} \bigcup_{(E^\mu, C^\mu) \in P} \operatorname{Atl}_{\text{Ar}}^{C_k} \left( (E^\mu, C^\mu), (E^\nu, C^\nu) \right) \quad (10.28)
\]
\[
\operatorname{Atl}_{\text{max} \rightarrow \text{full}}^{C_k}(E, C) \overset{\text{def}}{=} \left( \operatorname{Atl}_{\text{Ob}}^{C_k}(E, C), \operatorname{Atl}_{\text{Ar}}^{C_k}(E, C), A^1 \right) \quad (10.29)
\]
\[
\operatorname{Atl}_{\text{full} \rightarrow \text{max} \rightarrow \text{full}}^{C_k}(E, C) \overset{\text{def}}{=} \bigcup_{(E^\mu, C^\mu) \in P} \operatorname{Atl}_{\text{Ar}}^{C_k} \left( (E^\mu, C^\mu), (E^\nu, C^\nu) \right) \quad (10.30)
\]
\[
\operatorname{Atl}_{\text{max} \rightarrow \text{full} \rightarrow \text{max}}^{C_k}(E, C) \overset{\text{def}}{=} \left( \operatorname{Atl}_{\text{Ob}}^{C_k}(E, C), \operatorname{Atl}_{\text{Ar}}^{C_k}(E, C), A^1 \right) \quad (10.31)
\]

**Lemma 10.31** (\(\operatorname{Atl}^{C_k}(E, C)\) is a category). Let \(E\) be a set of topological spaces and \(C\) a set of linear spaces. Then \(\operatorname{Atl}^{C_k}(E, C)\), \(\operatorname{Atl}_{\text{Ar}}^{C_k}(E, C)\) and \(\operatorname{Atl}_{\text{max} \rightarrow \text{full}}^{C_k}(E, C)\) are categories.

Let \((A^i, E^i, C^i)\) \(\in \operatorname{Atl}_{\text{Ob}}^{C_k}(E, C)\). Then \(\operatorname{Id}_{(A^i, E^i, C^i)}\) is the identity morphism for \((A^i, E^i, C^i)\).

**Proof.** Let \((A^i, E^i, C^i), i = 1, 2, 3\) be objects of \(\operatorname{Atl}^{C_k}(E, C)\) and let \(m^i \overset{\text{def}}{=} ((f_0^i, f_1^i), (A^i, E^i, C^i), (A^{i+1}, E^{i+1}, C^{i+1}))\) be morphisms of \(\operatorname{Atl}^{C_k}(E, C)\). Then

1. **Composition:**
   \((f_0^i \circ f_1^i, f_1^i \circ f_1^i), (A^1, E^1, C^1), (A^3, E^3, C^3))\) is a morphism of \(\operatorname{Atl}^{C_k}(E, C)\) by corollary 10.29 (\(C^k\)-atlas morphisms) on page 69.

2. **Associativity:**
   Composition is associative by lemma 3.17 (Tuple composition for labeled morphisms) on page 10.

3. **Identity:**
   \(\operatorname{Id}_{(A^i, E^i, C^i)}\) is an identity morphism by lemma 3.17.

The proofs for \(\operatorname{Atl}_{\text{full} \rightarrow \text{max} \rightarrow \text{full}}^{C_k}(E, C)\) and \(\operatorname{Atl}_{\text{max} \rightarrow \text{full} \rightarrow \text{max}}^{C_k}(E, C)\) are the same. \(\square\)
Definition 10.32 (Functors from $\mathcal{C}^k$ atlases to $m$-atlases). Let $E^i, i = 1, 2$, be topological spaces, $C^i$ linear spaces, $\mathcal{A}^i \mathcal{C}^k$-atlases of $E^i$ in the coordinate space $C^i$, $f_0: E^1 \rightarrow E^2$ continuous and $f_1: C^1 \rightarrow C^2 \mathcal{C}^k$. Then

$$\mathcal{F}_{\mathcal{C}^k,M-atlas}(\mathcal{A}^i, E^i, C^i) \overset{\text{def}}{=} (\mathcal{A}^i, E^i, C^i, \text{triv}_{\mathcal{C}^k-\text{triv}})$$

(10.32)

$$\mathcal{F}_{\mathcal{C}^k,M-atlas}((f_0, f_1), (\mathcal{A}^1, E^1, C^1), (\mathcal{A}^2, E^2, C^2)) =$$

$$\left(\left((f_0, f_1), (\mathcal{A}^1, E^1, C^1), (\mathcal{A}^2, E^2, C^2)\right)\right)$$

(10.33)

Theorem 10.33 (Functors from $\mathcal{C}^k$ atlases to $m$-atlases). Let $E$ be a set of topological spaces and $C$ a set of linear spaces. Then $\mathcal{F}_{\mathcal{C}^k,M-atlas}$ is a functor from $\mathcal{A}^k(E, C)$ to $\mathcal{A}^k(E, C)$

Proof. Let $o^i \overset{\text{def}}{=} (\mathcal{A}^i, E^i, C^i), i \in [1, 3]$, be objects of $\mathcal{A}^k(E, C)$ and $m^i \overset{\text{def}}{=} ((f_0^i, f_1^i), o^i, o^{i+1}), i = 1, 2$, be morphisms from $o^i$ to $o^{i+1}$.

$\mathcal{F}_{\mathcal{C}^k,M-atlas}(m^1)$ is a morphism from $\mathcal{F}_{\mathcal{C}^k,M-atlas}^1 \mathcal{C}^k \rightarrow \mathcal{F}_{\mathcal{C}^k,M-atlas}^2 \mathcal{C}^k$:

$$\mathcal{F}_{\mathcal{C}^k,M-atlas}(m^1) =$$

$$\mathcal{F}_{\mathcal{C}^k,M-atlas}((f_0, f_1), (\mathcal{A}^1, E^1, C^1), (\mathcal{A}^2, E^2, C^2)) =$$

$$\left(\left((f_0^1, f_1^1), (\mathcal{A}^1, E^1, C^1), (\mathcal{A}^2, E^2, C^2)\right)\right)$$

(10.34)

$\mathcal{F}_{\mathcal{C}^k,M-atlas}$ maps identity functions to identity functions:

$$\mathcal{F}_{\mathcal{C}^k,M-atlas}(\text{Id}_{\mathcal{A}^i, E^i, C^i}) =$$

$$\mathcal{F}_{\mathcal{C}^k,M-atlas}(\text{Id}_{\mathcal{C}^i})(\mathcal{A}^i, E^i, C^i) =$$

$$\left(\left((\text{Id}_{\mathcal{E}^i}, \text{Id}_{\mathcal{C}^i}), (\mathcal{A}^i, E^i, C^i), (\mathcal{A}^i, E^i, C^i)\right)\right)$$

(10.35)

$\mathcal{F}_{\mathcal{C}^k,M-atlas}(\mathcal{F}_{\mathcal{C}^k,M-atlas}(\mathcal{A}^i, E^i, C^i), \mathcal{F}_{\mathcal{C}^k,M-atlas}(\mathcal{A}^i, E^i, C^i)) = \text{Id}_{\mathcal{F}_{\mathcal{C}^k,M-atlas}(\mathcal{A}^i, E^i, C^i)}$

$\mathcal{F}_{\mathcal{C}^k,M-atlas}(m^2 \circ m^1) = \mathcal{F}_{\mathcal{C}^k,M-atlas}(m^2 \circ m^1)$:

1. $m^2 \circ m^1 = (f_2 \circ f_0, f_2 \circ f_1), (\mathcal{A}^1, E^1, C^1), (\mathcal{A}^3, E^3, C^3)$

2. $\mathcal{F}_{\mathcal{C}^k,M-atlas}((\mathcal{A}^i, E^i, C^i)) = (\mathcal{A}^i, E^i, C^i)$
Proof. from model spaces and the coordinate spaces of model spaces, $\mathcal{F}_{C^k}\text{M-atlas}(m^i) = ((f_0^i, f_1^i), (A^i, E^i_{\text{triv}}, C^i_{\text{triv}}), (A^{i+1}, E^{i+1}_{\text{triv}}, C^{i+1}_{\text{triv}}))$

4. $\mathcal{F}_{C^k}\text{M-atlas}(m^2) \circ \mathcal{F}_{C^k}\text{M-atlas}(m^1) = ((f_0^3 \circ f_1^3, f_1^3 \circ f_1^1), (A^1, E^1_{\text{triv}}, C^1_{\text{triv}}), (A^3, E^3_{\text{triv}}, C^3_{\text{triv}}))$

5. $\mathcal{F}_{C^k}\text{M-atlas}(m^2 \circ m^1) = ((f_0^3 \circ f_0^1, f_1^3 \circ f_1^1), (A^1, E^1_{\text{triv}}, C^1_{\text{triv}}), (A^3, E^3_{\text{triv}}, C^3_{\text{triv}}))$

Definition 10.34 (Functors from m-atlas to $C^k$ atlases). Let $E^i, i = 1, 2, 3$ be model spaces, $C^k$ be linear model spaces, $A^i$ a maximal m-atlas of $E^i$ in the coordinate space $C^i$ and $(f_0^i, f_1^i)$ an $E^1$-$E^2$ m-atlas morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1, C^2$. Then

$$\mathcal{F}_{C^k}\text{M-atlas,C}(A^i, E^i, C^i) \overset{\text{def}}{=} (A^i, \pi_1(E^i), \pi_1(C^i)) \quad (10.36)$$

$$\mathcal{F}_{C^k}\text{M-atlas,C}((f_0^i, f_1^i), (A^1, E^1, C^1), (A^2, E^2, C^2)) \overset{\text{def}}{=} \left( (f_0^i, f_1^i), (A^1, \pi_1(E^1), \pi_1(C^1)), (A^2, \pi_1(E^2), \pi_1(C^2)) \right) \quad (10.37)$$

Theorem 10.35 (Functors from m-atlas to $C^k$ atlases). Let $E$ be a set of model spaces and $C$ a set of linear model spaces. Then $\mathcal{F}_{C^k}\text{M-atlas,C}$ is a functor from $\text{Atl}(E, C)$ to $\text{Atl}^{C^k}(\pi_1[E], \pi_1[C])$.

Proof. Let $o^i \overset{\text{def}}{=} (A^i, (E^i, E^i), (C^i, C^i)), i \in [1, 3]$, be objects of $\text{Atl}^{C^k}(E, C)$ and $m^i \overset{\text{def}}{=} ((f_0^i, f_1^i) \circ o^i, o^{i+1})$, $i = 1, 2$, be morphisms from $o^i$ to $o^{i+1}$.

$\mathcal{F}_{C^k}\text{M-atlas,C}(m^1)$ is a morphism from $\mathcal{F}_{C^k}\text{M-atlas,C} o^1$ to $\mathcal{F}_{C^k}\text{M-atlas,C} o^2$:

$$\mathcal{F}_{C^k}\text{M-atlas,C}(m^1) = \mathcal{F}_{C^k}\text{M-atlas,C}((f_0^1, f_1^1), (A^1, E^1, C^1), (A^2, E^2, C^2))$$

$$\left( (f_0^1, f_1^1), (A^1, \pi_1(E^1), \pi_1(C^1)), (A^2, \pi_1(E^2), \pi_1(C^2)) \right) \quad (10.38)$$

$\mathcal{F}_{C^k}\text{M-atlas,C}$ maps identity functions to identity functions:

$$\mathcal{F}_{C^k}\text{M-atlas,C} \text{Id}_{(A^i, E^i, C^i)} = \mathcal{F}_{C^k}\text{M-atlas,C}((\text{Id}_{E^i}, \text{Id}_{C^i}), (A^i, E^i, C^i), (A^i, E^i, C^i)) = \left( (\text{Id}_{\pi_1(E^i)}, \text{Id}_{\pi_1(C^i)}), (A^i, \pi_1(E^i), \pi_1(C^i)), (A^i, \pi_1(E^i), \pi_1(C^i)) \right) \quad (10.39)$$

$$\mathcal{F}_{C^k}\text{M-atlas,C}(A^i, E^i, C^i), \mathcal{F}_{C^k}\text{M-atlas,C}(A^i, E^i, C^i), \mathcal{F}_{C^k}\text{M-atlas,C}(A^i, E^i, C^i) = \text{Id}_{\mathcal{F}_{C^k}\text{M-atlas,C}(A^i, E^i, C^i)}$$

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A continuous function and Definition 10.36

10.6 Associated model spaces and functors

1. $m^2 \circ m^1 = ((f_0^2 \circ f_0^1, f_1^2 \circ f_1^1), (A^1, E^1, C^1), (A^2, E^2, C^2))$

2. $F_{M-\text{atlas}, \mathcal{C}_k}((A^1, E^i, C^i)) = (A^i, \pi_1(E^i), \pi_1(C^i))$

3. $F_{M-\text{atlas}, \mathcal{C}_k}(m^i) = ((f_0^i, f_1^i), (A^i, \pi_1(E^i), \pi_1(C^i)), (A^{i+1}, \pi_1(E^{i+1}), \pi_1(C^{i+1})))$

4. $F_{M-\text{atlas}, \mathcal{C}_k}(m^2 \circ m^1) = ((f_0^2 \circ f_0^1, f_1^2 \circ f_1^1), (A^1, \pi_1(E^1), \pi_1(C^1), (A^3, \pi_1(E^3), \pi_1(C^3)))$

5. $F_{M-\text{atlas}, \mathcal{C}_k}(m^2 \circ m^1) = ((f_0^2 \circ f_0^1, f_1^2 \circ f_1^1), (A^1, \pi_1(E^1), \pi_1(C^1), (A^3, \pi_1(E^3), \pi_1(C^3)))$

\[ \square \]

10.6 Associated model spaces and functors

**Definition 10.36 (Coordinate model spaces associated with $\mathcal{C}_k$-atlases).** Let $A^i, i = 1, 2$, be a $\mathcal{C}_k$-atlas of $E^i$ in the coordinate space $C^i$, $f_0: E^1 \to E^2$ a continuous function and $f_1: C^1 \to C^2$ a $\mathcal{C}_k$ function. Then

\[
\begin{align*}
\mathcal{F}_{\min}^{\mathcal{C}_k}(A^1, E^i, C^i) &\overset{\text{def}}{=} \operatorname{Mod}_{\min}(C^i, \pi_2[A^i], \left\{ \phi' \circ \phi^{-1} \left| \left( \bigcup_{(U', V', \phi') \in A^i} U' \cap U' \neq \emptyset \right) \right. \right\}) \tag{10.40} \\
\mathcal{F}_{\min}^{\mathcal{C}_k}(f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2) &\overset{\text{def}}{=} f_1: \mathcal{F}_{\min}^{\mathcal{C}_k}(A^1, E^1, C^1) \to \mathcal{F}_{\min}^{\mathcal{C}_k}(A^2, E^2, C^2) \tag{10.41}
\end{align*}
\]

The minimal coordinate $\mathcal{C}_k$ model space with neighborhoods in the $\mathcal{C}_k$-atlas $A^i$ of $E^i$ in the coordinate space $C^i$ is $\mathcal{F}_{\min}^{\mathcal{C}_k}(A^1, E^i, C^i)$.

The coordinate mapping associated with the $E^1-E^2\mathcal{C}_k$-atlas morphism $(f_0, f_1)$ of $A^1$ to $A^2$ in the coordinate spaces $C^1, C^2$ is $f_1: \mathcal{F}_{\min}^{\mathcal{C}_k}(A^1, E^1, C^1) \to \mathcal{F}_{\min}^{\mathcal{C}_k}(A^2, E^2, C^2)$. If it is a model function then the it is also the coordinate $m$-atlas morphism associated with the $E^1-E^2\mathcal{C}_k$-atlas morphism $(f_0, f_1)$ of $A^1$ to $A^2$ in the coordinate spaces $C^1, C^2$.
Lemma 10.37 (Coordinate model spaces associated with $\mathcal{C}^k$-atlases). Let $\mathcal{A}$ be a $\mathcal{C}^k$-atlas of $E$ in the coordinate space $\mathcal{C}$. Then $\mathcal{F}^\mathcal{C}_2(\mathcal{A}, E, C)$ is a $\mathcal{C}^k$ linear model space iff $\bigcup \pi_2[\mathcal{A}]$ contains a ball.

Proof. $\mathcal{F}^\mathcal{C}_2(\mathcal{A}, E, C)$ satisfies the conditions for a model space.

1. Since $\pi_2[\mathcal{A}]$ is an open cover of $\bigcup \pi_2[\mathcal{A}]$, the set of finite intersections is also an open cover.

2. Finite intersections of finite intersections are finite intersections.

3. Restrictions of continuous functions are continuous.

4. If $f: A \rightarrow B$ is a morphism of $\mathcal{F}^\mathcal{C}_2(\mathcal{A}, E, C)$, $A, A', B, B'$ model neighborhoods of $\mathcal{F}^\mathcal{C}_2(\mathcal{A}, E, C)$, $A' \subseteq A$, $B' \subseteq B$ and $f[A'] \subseteq B'$ then since $f: A \rightarrow B$ is a morphism it is a restriction of a transition function between its restrictions to sets in $\pi_2[\mathcal{A}]$ and its restrictions are also, hence morphisms, and thus $f \mid A': A' \rightarrow B'$ is a morphism.

5. If $(U, V, \phi) \in \mathcal{A}$ then $\text{Id}_V = \phi \circ \phi^{-1}A$ is a transition function and hence a morphism of $\mathcal{F}^\mathcal{C}_2(\mathcal{A}, E, C)$. If $A, A'$ objects of $\mathcal{F}^\mathcal{C}_2(\mathcal{A}, E, C)$ and $A' \subseteq A$ then the inclusion map $i: A' \hookrightarrow A$ is a restriction of an identity morphism of $\mathcal{F}^\mathcal{C}_2(\mathcal{A}, E, C)$ and hence a morphism.

6. Restricted sheaf condition: let

(a) $U_\alpha, V_\alpha, \alpha \prec A$, be objects of $\pi_2(\mathcal{F}^\mathcal{C}_2(\mathcal{A}, E, C))$

(b) $f_\alpha: U_\alpha \rightarrow V_\alpha$ be morphisms of $\pi_2(\mathcal{F}^\mathcal{C}_2(\mathcal{A}, E, C))$

(c) $U \overset{\text{def}}{=} \bigcup_{\alpha \prec A} U_\alpha$

(d) $V \overset{\text{def}}{=} \bigcup_{\alpha \prec A} V_\alpha$

(e) $f: U \rightarrow V$ be continuous and $\left( \bigvee_{\alpha \prec A} f(x) = f_\alpha(x) \right)$

Then $f$ is $\mathcal{C}^k$ and hence a morphism of $\pi_2(\mathcal{F}^\mathcal{C}_2(\mathcal{A}, E, C))$.

If $\bigcup \pi_2[\mathcal{A}]$ is a linear space, then by definition 10.1 (Linear spaces) on page 60, $\bigcup \pi_2[\mathcal{A}]$ it contains a ball. Conversely, $\bigcup \pi_2[\mathcal{A}]$ is locally connected so if $\bigcup \pi_2[\mathcal{A}]$ contains a ball then the conditions of definition 10.1 (Linear spaces) on page 60 are met. □
Let $A^i$, $i = 1, 2$, be a semi-maximal $C^k$-atlas of $E^i$ in the coordinate space $C^i$, $f_0: E^1 \rightarrow E^2$ a continuous function, $f_1: C^1 \rightarrow C^2$ a $C^k$ function and $(f_0, f_1)$ a $C^k$-atlas morphism from $A^1$ to $A^2$. Then $f_1: \mathcal{F}^{C^k}_{\min}(A^1, E^1, C^1) \rightarrow \mathcal{F}^{C^k}_{\min}(A^2, E^2, C^2)$ is well defined.

Proof. Let $v^1 \in \mathcal{F}^{C^k}_{\min}(A^1, E^1, C^1)$, $(U^i, V^i, \phi^i) \in A^i$, $i = 1, 2$, be a chart with $u^1 \in U^1$, $\phi^1(u^1) = v^1$ and $f_0(u^1) \in U^2$. Then there are open sets $U^i \subseteq U^1 \cap f_0^{-1}(U^2)$, $V^i \subseteq V^1$, $U^2 \subseteq U^2$, $V^2 \subseteq V^2$, $V^2 \subseteq C^2$ and a $C^k$ diffeomorphism $\hat{f}: \hat{V}^2 \rightarrow V^2$ such that eqs. (8.24) to (8.30) on pages 42 to 42 in definition 8.20 (M-atlas morphisms) on page 40 hold with $x^{\text{def}} = u^1$. Since $A^2$ is semi-maximal, $(U^2, \hat{V}^2, \hat{f} \circ \phi^2)$ is a chart of $A^2$ and by eq. (8.25) $f_1(v^1) \in \hat{V}^2$.

Definition 10.38 (Model spaces associated with $C^k$-atlas). Let $A^i$, $i = 1, 2$ be $C^k$-atlas of $E^i$ in the coordinate spaces $C^i$, $f_0: E^1 \rightarrow E^2$ a continuous function and $f_1: C^1 \rightarrow C^2$ a $C^k$ function. Then

$$\mathcal{F}^{C^k}_{\min}(A^i, E^i, C^i) \overset{\text{def}}{=} \text{Mod}_{\text{min}} \left( E^i, \pi_i[A^i], \left\{ \phi^{i-1} \circ \phi \left| \left( \bigcup_{(U^i, V^i, \phi^i) \in A^i} U \cap U' \neq \emptyset \right) \left( \bigcup_{(U', V', \phi') \in A'} U' \cap U'' \neq \emptyset \right) \right. \right\} \right)$$

(10.42)

$$\mathcal{F}^{C^k}_{\min}(f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2)) \overset{\text{def}}{=} f_0: \mathcal{F}^{C^k}_{\min}(A^1, E^1, C^1) \rightarrow \mathcal{F}^{C^k}_{\min}(A^2, E^2, C^2)$$

(10.43)

The minimal $C^k$ model space with neighborhoods in the $C^k$-atlas $A^i$ of $E^i$ in the coordinate space $C^i$ is $\mathcal{F}^{C^k}_{\min}(A^i, E^i, C^i)$.

The mapping associated with the $E^1$-$E^2$ $C^k$-atlas morphism $(f_0, f_1)$ of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$ is $f_0: \mathcal{F}^{C^k}_{\min}(A^1, E^1, C^1) \rightarrow \mathcal{F}^{C^k}_{\min}(A^2, E^2, C^2)$. If it is a model function then it is also the m-atlas morphism associated with the $E^1$-$E^2$ $C^k$-atlas morphism $(f_0, f_1)$ of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$.

Lemma 10.39 (Model spaces associated with $C^k$-atlases). Let $A$ be a $C^k$-atlas of $E$ in the coordinate space $C$. Then $\mathcal{F}^{C^k}_{\min}(A, E, C)$ is a model space.

Proof. Lemma 5.14 (Minimal model spaces are model spaces) on page 18.
Theorem 10.40 (Functors from $C^k$ atlases to model spaces). Let $E$ be a set of topological spaces and $C$ a set of linear spaces. Then $\mathcal{F}_1^{\text{min}}$ is a functor from $\text{Atl}^{C^k}(E,C)$ to $E$, $\mathcal{F}_1^{\text{min}}$ is a functor from $\text{Atl}^{C^k}(E,C)$ to $C$, and $\mathcal{F}_2^{\text{min}}$ is a functor from $\text{Atl}^{C^k}(E,C)$ to $C_{\text{op}}$. The proof for $\mathcal{F}_1^{\text{min}}: \text{Atl}^{C^k}(E,C) \to E$ is identical.

Proof. Let $\mathcal{F}_1^{\text{min}}: \text{Atl}^{C^k}(E,C) \to E$ be a set of morphisms in $\text{Atl}^{C^k}(E,C)$ and let $\mathcal{F}_1^{\text{min}}: \text{Atl}^{C^k}(E,C) \to E$ be a set of morphisms in $\text{Atl}^{C^k}(E,C)$.

1. $\mathcal{F}(f): A \to B): \mathcal{F}(A) \to \mathcal{F}(B);$

\[ \mathcal{F}_1^{\text{min}}(m^i) = f_i; \quad \mathcal{F}_1^{\text{min}}(o^i) \to \mathcal{F}_1^{\text{min}}(o^{i+1}) \]

2. $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f);$

\[ \mathcal{F}_1^{\text{min}}(m^2 \circ m^1) = \mathcal{F}_1^{\text{min}}((f^2 \circ f^1, f^3 \circ f^1)(A^1, E^1, C^1), (A^3, E^3, C^3)) = f_0^2 \circ f_1^1; \quad \mathcal{F}_1^{\text{min}}(o^1) \to \mathcal{F}_1^{\text{min}}(o^3) = (f_0^2, f_1^1)(\mathcal{F}_1^{\text{min}}(o^2) \to \mathcal{F}_1^{\text{min}}(o^3)) \circ (f_1^1, \mathcal{F}_1^{\text{min}}(o^1) \to \mathcal{F}_1^{\text{min}}(o^2)) = \mathcal{F}_1^{\text{min}}((f_0^2, f_1^1)(A^2, E^2, C^2), (A^3, E^3, C^3)) \circ \mathcal{F}_1^{\text{min}}((f_0^2, f_1^1)(A^1, E^1, C^1), (A^2, E^2, C^2)) = \mathcal{F}_1^{\text{min}}(m^2) \circ \mathcal{F}_1^{\text{min}}(m^1) \]

3. $\mathcal{F}(\text{Id}_A) = \text{Id}_{\mathcal{F}(A)};$

(a) $\mathcal{F}_1^{\text{min}}(\text{Id}_E) = \mathcal{F}_1^{\text{min}}((\text{Id}_E, \text{Id}_C), (A^i, E^i, C^i), (A^i, E^i, C^i)) = \text{Id}_E; \quad \mathcal{F}_1^{\text{min}}(o^i) \to \mathcal{F}_1^{\text{min}}(o^i)$

(b) $\text{Id}_{\mathcal{F}_1^{\text{min}}}(o^i) = \text{Id}_{\mathcal{F}_1^{\text{min}}}(o^i) \to \mathcal{F}_1^{\text{min}}(o^i)$

The proof for $\mathcal{F}_2^{\text{min}}: \text{Atl}^{C^k}(E,C) \to C$ is identical.
2. \( \mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f) : \mathcal{F}^{\text{C}_k}_{\text{min}}(m^2 \circ m^1) = \)
\[ \mathcal{F}^{\text{C}_k}_{\text{min}}((f_0^2 \circ f_0^1, f_1^2 \circ f_1^1) (A^1, E^1, C^1), (A^3, E^3, C^3)) = \]
\[ f_0^2 \circ f_0^1 : \mathcal{F}^{\text{C}_k}_{\text{min}}(o^1) \rightarrow \mathcal{F}^{\text{C}_k}_{\text{min}}(o^1) = \]
\[ (f_1^2 : \mathcal{F}^{\text{C}_k}_{\text{min}}(o^2) \rightarrow \mathcal{F}^{\text{C}_k}_{\text{min}}(o^2)) \circ (f_1^1 : \mathcal{F}^{\text{C}_k}_{\text{min}}(o^1) \rightarrow \mathcal{F}^{\text{C}_k}_{\text{min}}(o^2)) = \]
\[ \mathcal{F}^{\text{C}_k}_{\text{min}}((f_0^2, f_1^2)(A^2, E^2, C^2), (A^3, E^3, C^3)) \circ \]
\[ \mathcal{F}^{\text{C}_k}_{\text{min}}((f_0^1, f_1^1)(A^1, E^1, C^1), (A^2, E^2, C^2)) = \]
\[ \mathcal{F}^{\text{C}_k}_{\text{min}}(m^2) \circ \mathcal{F}^{\text{C}_k}_{\text{min}}(m^1) \]
3. \( \mathcal{F}(\text{Id}_A) = \text{Id}_{\mathcal{F}(A)} : \)
\[ (a) \quad \mathcal{F}^{\text{C}_k}_{\text{min}}(\text{Id}_{o^i}) = \mathcal{F}^{\text{C}_k}_{\text{min}}((\text{Id}_{E^i}, \text{Id}_{C^i}), (A^i, E^i, C^i), (A^i, E^i, C^i)) = \]
\[ \text{Id}_{C^i} : \mathcal{F}^{\text{C}_k}_{\text{min}}(o^i) \rightarrow \mathcal{F}^{\text{C}_k}_{\text{min}}(o^i) \]
\[ (b) \quad \text{Id}_{\mathcal{F}^{\text{C}_k}_{\text{min}}(o^i)} = \text{Id}_{\mathcal{F}^{\text{C}_k}_{\text{min}}(o^i)} : \mathcal{F}^{\text{C}_k}_{\text{min}}(o^i) \rightarrow \mathcal{F}^{\text{C}_k}_{\text{min}}(o^i) \]

The proof for \( \mathcal{F}^{\text{C}_k}_{\text{min}} : \text{Atl}^{\text{C}_k}_{\text{full}}(E, C) \rightarrow C^{\text{C}_k-\text{triv}} \) is identical. \( \square \)

10.7 \( \text{C}^k \) manifolds

Conventionally a manifold is different from its atlases, but \( \text{Atl}^{\text{C}_k}_{\text{max}}(E, C) \) in definition 10.30 (Categories of \( \text{C}^k \) atlases) on page 69 encourages treating them on an equal footing. All of the results for maximal atlases carry directly over to results for manifolds.

Remark 10.41. The variations of the \( \text{LCS}^{\text{C}_k} \) constructors in definition 10.46 (Functors from \( \text{C}^k \) manifolds to Local 2-∅ coordinate spaces) on page 78 are very similar, as are the variations of the \( \mathcal{F}_{\text{Man,LCS}}^{\text{C}_k} \) functors; likewise the variations of the \( \mathcal{F}_{\text{LCS,Man}}^{\text{C}_k} \) functors in definition 10.48 (Functor from Local 2-∅ coordinate spaces to \( \text{C}^k \) manifolds) on page 84 are very similar.

The corresponding results and proofs in theorem 10.47 (Functors from \( \text{C}^k \) manifolds to Local 2-∅ coordinate spaces) on page 80 are likewise very similar, as are the corresponding results and proofs in theorem 10.49 (Functor from Local 2-∅ coordinate spaces to \( \text{C}^k \) manifolds) on page 84.

Definition 10.42 (\( \text{C}^k \) manifolds). Let \( E \) be a topological space, \( C \) a linear space and \( A \) a maximal\(^{15} \) \( \text{C}^k \)-atlas of \( E \) in the coordinate space \( C \). Then \( (E, C, A) \) is a \( \text{C}^k \) manifold.

\(^{15}\) Requiring that the atlas be full would eliminate some pathologies.
Let $E$ be a set of topological spaces and $C$ be a set of linear spaces. Then

$$\text{Man}^{ck}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \text{Atl}^{ck}_{\text{max}}(E, C)$$ (10.44)

**Remark 10.43.** The manifold $(E, C, A)$ corresponds to the object $(A, E, C)$.

**Definition 10.44** (C^k manifold morphisms). Let $S_i \overset{\text{def}}{=} (E^i, C^i, A^i)$, $i = 1, 2$, be C^k manifolds and $(f_0; E^1 \longrightarrow E^2, f_1; C^1 \longrightarrow C^2)$ be an $E^1$-$E^2$ C^k-morphism of $A^1$ to $A^2$ in the coordinate spaces $C^1$, $C^2$. Then $(f_0, f_1)$ is a C^k morphism of $S^1$ to $S^2$.

Let $E^i$, $i = 1, 2$ be topological spaces and $C^i$ be linear spaces. Then

$$\text{Man}^{ck}_{\text{Ar}}(E^1, C^1, E^2, C^2) \overset{\text{def}}{=} \text{Atl}^{ck}_{\text{Ar}}(E^1, C^1, E^2, C^2)$$ (10.45)

Let $E$ be a set of topological spaces and $C$ a set of linear spaces. Then

$$\text{Man}^{ck}_{\text{Ar}}(E, C) \overset{\text{def}}{=} \text{Atl}^{ck}_{\text{max}}(E, C)$$ (10.46)

$$\text{Man}^{ck}(E, C) \overset{\text{def}}{=} \text{Atl}^{ck}_{\text{max}}(E, C)$$ (10.47)

**Theorem 10.45** (Categories of C^k manifolds). Let $E$ be a set of topological spaces and $C$ a set of linear spaces. Then $\text{Man}^{ck}(E, C)$ is a category and the identity morphism of $(A, E, C)$ is an identity morphism.

**Proof.** The result follows directly from definitions 10.42 and 10.44 above and lemma 10.31 ($\text{Atl}^{ck}(E, C)$ is a category) on page 70 .

**Definition 10.46** (Functors from C^k manifolds to Local 2-∅ coordinate spaces). Let $E$ be a set of topological spaces, $C$ be a set of C^k linear spaces, $E$ and $\hat{E}$ model categories, $\mathcal{C}$ and $\hat{C}$ C^k linear model categories, $\mathcal{M} \overset{\text{def}}{=} (\mathcal{E}, \mathcal{C}), \hat{\mathcal{M}} \overset{\text{def}}{=} (\hat{\mathcal{E}}, \hat{\mathcal{C}})$,

$$\mathcal{M}^{\text{triv}} \overset{\text{def}}{=} \left( \mathcal{E}_{\text{triv}}, C_{\text{triv}} \right), \mathcal{M}^{\text{op-triv}} \overset{\text{def}}{=} \left( \mathcal{E}_{\text{triv}}, C_{\text{op-triv}} \right), \mathcal{M}^{\text{full-cat}} \subseteq \mathcal{M}^{\text{full-cat}}$$

Let $E^i \in E$, $i = 1, 2$, $C^i \in C$, $M^i \overset{\text{def}}{=} \left( E^i_{\text{triv}} C^i_{\text{triv}} \right)$,

$$\mathcal{M}^{\text{sing}} \overset{\text{def}}{=} \left( M_{\text{Sing}}^i, M_{\text{Sing}}^i \right), A^i \overset{\text{def}}{=} \left( F^1_{\text{sing}} A^i, E^i, C^i \right) \in \hat{\mathcal{E}}, \mathcal{F}^{\text{Ck}}_2(A^i, E^i, C^i) \in \hat{\mathcal{C}}, (E^i, C^i, A^i) a C^k manifold, \mathcal{S}^{\text{def}}(A^i, E^i, C^i), M^{\text{def}} \overset{\text{min}}{=} \left( F^1_{\text{min}}(A^i, E^i, C^i), \mathcal{F}^{\text{Ck}}_2(A^i, E^i, C^i) \right).$$
\( M^i_{\text{triv}} \overset{\text{def}}{=} (M^0_{\text{triv}}, M^1_{\text{triv}}) \), \( f_0: E^1 \rightarrow E^2 \) continuous and \( f_1: C^1 \rightarrow C^2 \). Then

\[
F^c_{\text{Man}, \text{LCS}} S^i \overset{\text{def}}{=} \left( M^i_{\text{Sing}}, M^i_{\text{Sing}}, A^i, 0, 0 \right) \quad (10.48)
\]

\[
F^c_{\text{Man}, \text{LCS}} M S^i \overset{\text{def}}{=} \left( M, M^i_{\text{Sing}}, A^i, 0, 0 \right) \quad (10.49)
\]

\[
F^c_{\text{Man}, \text{LCS}} \hat{M} S^i \overset{\text{def}}{=} \left( M^i_{\text{Sing}}, M^i_{\text{Sing}}, A^i, 0, 0 \right) \quad (10.50)
\]

\[
F^c_{\text{Man}, \text{LCS}} \hat{M} S^i \overset{\text{def}}{=} \left( M, M^i_{\text{Sing}}, A^i, 0, 0 \right) \quad (10.51)
\]

\[
F^c_{\text{Man}, \text{LCS}} ((f_0, f_1), S^1, S^2) \overset{\text{def}}{=} \left( (f_0, f_1), F^c_{\text{Man}, \text{LCS}} S^1, F^c_{\text{Man}, \text{LCS}} S^2 \right) \quad (10.52)
\]

\[
F^c_{\text{Man}, \text{LCS}} ((f_0, f_1), S^1, S^2) \overset{\text{def}}{=} \left( (f_0, f_1), F^c_{\text{Man}, \text{LCS}} S^1, F^c_{\text{Man}, \text{LCS}} S^2 \right) \quad (10.53)
\]

\[
F^c_{\text{Man}, \text{LCS}} ((f_0, f_1), S^1, S^2) \overset{\text{def}}{=} \left( (f_0, f_1), F^c_{\text{Man}, \text{LCS}} S^1, F^c_{\text{Man}, \text{LCS}} S^2 \right) \quad (10.54)
\]

\[
F^c_{\text{Man}, \text{LCS}} ((f_0, f_1), S^1, S^2) \overset{\text{def}}{=} \left( (f_0, f_1), F^c_{\text{Man}, \text{LCS}} S^1, F^c_{\text{Man}, \text{LCS}} S^2 \right) \quad (10.55)
\]

\[
\text{LCS}^c_{\text{Ob}} (E, C) \overset{\text{def}}{=} \left\{ F^c_{\text{Man}, \text{LCS}} (A, E, C) \right\} \quad (10.56)
\]

\[
E \in E \land C \in C \land \text{isAtt}_{\text{max}}^c (A, E, C) \}
\]

\[
\text{LCS}^c_{\text{Ar}} (E, C) \overset{\text{def}}{=} \left\{ F^c_{\text{Man}, \text{LCS}} ((f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2)) \right\} \quad (10.57)
\]

\[
E^i \in E \land C^i \in C \land \text{isAtt}_{\text{max}}^c (A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1) \}
\]

\[
\text{LCS}^c (E, C) \overset{\text{def}}{=} \left\{ \text{LCS}^c_{\text{Ob}} (E, C), \text{LCS}^c_{\text{Ar}} (E, C), \delta \right\} \quad (10.58)
\]

\[
\text{LCS}^c_{\text{Ob}} (E, C) \overset{\text{def}}{=} \left\{ F^c_{\text{Man}, \text{LCS}} (A, E, C) \right\} \quad (10.59)
\]

\[
E \in E \land C \in C \land \text{isAtt}_{\text{max}}^c (A, E, C) \}
\]
$\text{LCS}^\text{ck}_{\text{Ar}}(E, C) \overset{\text{def}}{=} \left\{ F_{\text{Man}, \text{LCS}}^\text{ck}(f_0, f_1), (A^1, E^1, C^1), (A^2, E^2, C^2) \right\} \bigg| \begin{array}{l} E^i \in E \land C^i \in C \land \text{isAtl}\text{ck}_{\text{Ar}}^\text{max}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1) \end{array} \right\} \quad (10.60)$

$LCS^\text{ck}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \left\{ \text{LCS}^\text{ck}_{\text{Ob}}(E, C), \text{LCS}^\text{ck}_{\text{Ar}}(E, C), \hat{\circ} \right\} \quad (10.61)$

$LCS^\text{ck}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \left\{ F_{\text{Man}, \text{LCS}}^\text{ck} S \right\} \quad (10.62)$

$LCS^\text{ck}_{\text{Ar}}(E, C) \overset{\text{def}}{=} \left\{ F_{\text{Man}, \text{LCS}}^\text{ck} \bigg| \begin{array}{l} E^i \in E \land C^i \in C \land \text{isAtl}\text{ck}_{\text{Ar}}^\text{max}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1) \end{array} \right\} \quad (10.63)$

$LCS^\text{ck}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \left\{ \text{LCS}^\text{ck}_{\text{Ob}}(E, C), \text{LCS}^\text{ck}_{\text{Ar}}(E, C), \hat{\circ} \right\} \quad (10.64)$

$LCS^\text{ck}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \left\{ F_{\text{Man}, \text{LCS}}^\text{ck} S \right\} \quad (10.65)$

$LCS^\text{ck}_{\text{Ar}}(E, C) \overset{\text{def}}{=} \left\{ F_{\text{Man}, \text{LCS}}^\text{ck} \bigg| \begin{array}{l} E^i \in E \land C^i \in C \land \text{isAtl}\text{ck}_{\text{Ar}}^\text{max}(A^1, E^1, C^1, A^2, E^2, C^2, f_0, f_1) \end{array} \right\} \quad (10.66)$

$LCS^\text{ck}_{\text{Ob}}(E, C) \overset{\text{def}}{=} \left\{ \text{LCS}^\text{ck}_{\text{Ob}}(E, C), \text{LCS}^\text{ck}_{\text{Ar}}(E, C), \hat{\circ} \right\} \quad (10.67)$

**Theorem 10.47** (Functors from $C^k$ manifolds to Local 2-$\emptyset$ coordinate spaces).

Let $E$ be a set of topological spaces, $C$ be a set of $C^k$ linear spaces, $E$ and $\hat{E}$ model categories, $C$ and $\hat{C}$ $C^k$ linear model categories, $\mathcal{M} = (E, \hat{C}), \hat{\mathcal{M}} = (\hat{E}, \hat{C})$, 80
\[ \mathcal{M} \overset{\text{def}}{=} \left( E \overset{\text{triv}}{\rightarrow} C_k \overset{\text{triv}}{\rightarrow} C_k \right), \quad \mathcal{M} \overset{\text{op-triv}}{=} \left( E \overset{\text{triv}}{\rightarrow} C_k \overset{\text{op-triv}}{\rightarrow} C_k \right), \quad \mathcal{M} \subseteq \mathcal{M} \subseteq \mathcal{M} \subseteq \mathcal{M} \]

Then LCS\(^{ck}\)(C, E), LCS\(^{ck}\)\(\mathcal{M}(C, E)\), LCS\(^{ck}\)\(\text{min}(C, E)\) and LCS\(^{ck}\)\(\mathcal{M}(C, E)\) are categories and the identity morphism of \(L^\alpha \overset{\text{def}}{=} ((C, E), (C^\alpha, E^\alpha), A^\alpha, 0, 0)\) is that given in definition 9.8 (LCS\(_{Ar}\)) on page 57 : \(\text{Id}_{L^\alpha} \overset{\text{def}}{=} \left( (\text{Id}_{E^\alpha}, \text{Id}_{C^\alpha}), (L^\alpha, L^\alpha) \right)\).

**Proof.** LCS\(^{ck}\)(C, E), LCS\(^{ck}\)\(\mathcal{M}(C, E)\), LCS\(^{ck}\)\(\text{min}(C, E)\) and LCS\(^{ck}\)\(\mathcal{M}(C, E)\) satisfy the definition of a category:

1. Composition:

   Let \(m^i \overset{\text{def}}{=} F\(^{ck}\)\(\text{Man}_{\text{LCS}}\left( (f_0^i, f_1^i), S^i, S^i + 1 \right)\) be morphisms of LCS\(^{ck}\)(C, E). \(m^i \overset{\alpha}{\rightarrow} m^i \)

   \[ m^1 = F\(^{ck}\)\(\text{Man}_{\text{LCS}}\left( (f_0^2 \circ f_0^1, f_1^2 \circ f_1^1), S^1, S^3 \right) \).

   By lemma 8.21 (M-atlas morphisms) on page 43 , \((f_0^2 \circ f_0^1, f_1^2 \circ f_1^1)\) is an m-atlas morphism.

2. Associativity: Composition is associative by lemma 3.17 (Tuple composition for labeled morphisms) on page 10 .

3. Unit: \(\text{Id}_{L^\alpha}\) is an identity morphism by lemma 3.17.

The same proof applies to LCS\(^{ck}\)\(\mathcal{M}(C, E)\), LCS\(^{ck}\)\(\text{min}(C, E)\) and LCS\(^{ck}\)\(\mathcal{M}(C, E)\).

Let \(E^i \in E, i \in [1, 3] \), \(C^i \in C \), \(M^i \overset{\text{def}}{=} (E^i \overset{\text{triv}}{\rightarrow} C^i \overset{\text{triv}}{\rightarrow} C^i) \), \(\mathcal{M}^i \overset{\text{def}}{=} (M^i_0, M^i_1) \), \(A^i\) a maximal \(C^i\)-atlas of \(E^i\) in \(C^i\), \(F_1^\text{min}(A^i, E^i, C^i) \in \mathcal{E}, F_2^\text{min}(A^i, E^i, C^i) \in \mathcal{C}\), \((E^i, C^i, A^i)\) a \(C^i\) manifold, \(S^i \overset{\text{def}}{=} (A^i, E^i, C^i) \), \(M^i \overset{\text{def}}{=} (M^i_0, M^i_1) \), \(L^i \overset{\text{def}}{=} F_\text{Man}_{\text{LCS}} S^i =

\[ \left( \left( E^i \overset{\text{triv}}{\rightarrow} C^i \overset{\text{triv}}{\rightarrow} C^i \right), \left( E^i \overset{\text{triv}}{\rightarrow} C^i \overset{\text{triv}}{\rightarrow} C^i \right), A^i, 0, 0 \right), \]

\[ L^i \overset{\text{def}}{=} \left( \left( E^i \overset{\text{triv}}{\rightarrow} C^i \overset{\text{triv}}{\rightarrow} C^i \right), \left( E^i \overset{\text{triv}}{\rightarrow} C^i \overset{\text{triv}}{\rightarrow} C^i \right), A^i, 0, 0 \right), \]

\[ L^i \overset{\text{def}}{=} \left( \left( E^i \overset{\text{triv}}{\rightarrow} C^i \overset{\text{triv}}{\rightarrow} C^i \right), \left( E^i \overset{\text{triv}}{\rightarrow} C^i \overset{\text{triv}}{\rightarrow} C^i \right), A^i, 0, 0 \right), \]

\[ f_0^i : C^i \overset{\text{triv}}{\rightarrow} C^i + 1 \text{ continuous and } f_1^i : C^i \overset{\text{triv}}{\rightarrow} C^i + 1 \text{ } C^i \text{ coordinate space and } F_\text{Man}_{\text{LCS}} S^i \text{ is a local } \mathcal{M}^i - \emptyset \text{ coordinate space and } F_\text{Man}_{\text{LCS}} S^i \text{ is a local } \mathcal{M} - \emptyset \text{ coordinate space.}

**Proof.** \(F_\text{Man}_{\text{LCS}} S^i\) satisfies the conditions of definition 9.1 (Local * – Σ coordinate spaces) on page 53 :

1. Model categories: \(E^i\) and \(C^i\) are model spaces
2. \( \left( \mathcal{M}, \left( \mathcal{E}^i, \mathcal{C}^i_{\text{triv}}, \mathcal{C}^i_{\text{triv}} \right), \emptyset, \emptyset \right) \) satisfies the conditions of definition 7.1 (Prestructures) on page 27.

3. Maximal \( m \)-atlas: \( \mathcal{A}^i \) is a maximal \( \mathcal{C}^k \) atlas by hypothesis and a maximal atlas of \( \mathcal{E}^i \) in the coordinate space \( \mathcal{C}^i \) by construction.

4. Constraint functions: There are no adjunct functions so there are no constraint functions.

\[ \mathcal{M}^\text{cat}_{\text{Sing}} \subseteq \mathcal{M}, \text{ so } \mathcal{F}^\mathcal{C}^k_{\text{Man,LCS}} \mathcal{S}^i \text{ is a local } \mathcal{M}-\emptyset \text{ coordinate space by lemma 9.3 (Local } * - \Sigma \text{ coordinate spaces) on page } 53. \]

Proof. \( \mathcal{F}^\mathcal{C}^k_{\text{Man,LCS}} \mathcal{S}^i \) is a local \( \mathcal{M}^\text{cat}_{\text{Sing}} - \emptyset \) coordinate space and \( \mathcal{F}^\mathcal{C}^k_{\text{Man,LCS}} \mathcal{S}^i \) is a local \( \mathcal{M}^\text{cat}_{\text{Sing}} - \emptyset \) coordinate space.

\[ \mathcal{F}^\mathcal{C}^k_{\text{Man,LCS}} \mathcal{S}^i \text{ satisfies the conditions of definition 9.1 (Local } * - \Sigma \text{ coordinate spaces) on page } 53: \]

1. Model categories: \( \mathcal{E}^i \) and \( \mathcal{C}^i_{\text{triv}} \) are model spaces.

2. \( \left( \left( \mathcal{E}^i, \mathcal{C}^i_{\text{triv}}, \mathcal{C}^i_{\text{op-triv}} \right), \emptyset, \emptyset \right) \) satisfies the conditions of definition 7.1 (Prestructures) on page 27.

3. Maximal \( m \)-atlas: \( \mathcal{A}^i \) is a maximal \( \mathcal{C}^k \) atlas by hypothesis and a maximal atlas of \( \mathcal{E}^i \) in the coordinate space \( \mathcal{C}^i \) by construction.

4. Constraint functions: There are no adjunct functions so there are no constraint functions.

\[ \mathcal{M}^\text{cat}_{\text{triv, Sing}} \subseteq \mathcal{M}, \text{ so } \mathcal{F}^\mathcal{C}^k_{\text{Man,LCS}} \mathcal{S}^i \text{ is a local } \mathcal{M}^\text{cat}_{\text{Sing}} - \emptyset \text{ coordinate space by lemma 9.3.} \]

\[ (f^1_0, f^1_1) \text{ is a morphism from } \mathcal{L}^1 \text{ to } \mathcal{L}^2 \text{ and a strict morphism from } \mathcal{L}^1, \mathcal{M} \text{ to } \mathcal{L}^2, \mathcal{M}. \]

Proof. It satisfies the conditions of definition 9.6 (morphisms of local coordinate spaces) on page 54:

1. Prestructure morphism: \( f^1 \Sigma \)-commutes with \( \emptyset, \emptyset \). \( f^1_0 \) is continuous, hence a morphism of \( \mathcal{E} \). \( f^1_1 \) is \( \mathcal{C}^k \), hence a morphism of \( \mathcal{C} \).
2. m-atlas morphism: \((f_1^1, f_1^1)\) is a \(E^1\)-\(E^2\) m-atlas morphism of \(A^1\) to \(A^2\) in the coordinate spaces \(C^1\) \(\mapsto C^i\) \(\mapsto C^\text{triv}\) \(\mapsto C^\text{\text{triv}}\).

\(\,(f_0^1, f_1^1)\) is a strict \(E^1\)-\(E^2\) m-atlas morphism of \(A^1\) to \(A^2\) in the coordinate spaces \(C^1\) \(\mapsto C^i\) \(\mapsto C^\text{triv}\) \(\mapsto C^\text{\text{triv}}\) (\(C^\text{\text{triv}}\)-atlas morphisms) on page 69.

\[ \mathcal{F}^\text{\text{triv}}_{\text{Man,LCS}}((f_0^1, f_1^1), S^1, S^2) \] is a morphism from \(L^1\) to \(L^2_{\text{triv}}\) and

\[ \mathcal{F}^\text{\text{triv}}_{\text{Man,LCS}}((f_0^1, f_1^1), S^1, S^2) \] is a strict morphism from \(L^1_{\text{triv}}\) to \(L^2_{\text{triv}}\).

Proof. It satisfies the conditions of definition 9.6 (morphisms of local coordinate spaces) on page 54:

1. Prestructure morphism: \((f_0^1, f_1^1)\) \(\Sigma\)-commutes with \(0, \emptyset\).

2. m-atlas morphism: \((f_0^1, f_1^1)\) is an m-atlas morphism from \((E^1, C^1)\) to \((E^2, C^2)\).

\[ \mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}} \] is a functor from \(\text{Man}^{C^i}(E, C)\) to \(\text{LCS}^{C^i \otimes \text{triv}}(E, C)\).

Proof. \(\mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}}\) satisfies the definition of a functor:

1. \(F(f: A \rightarrow B) = F(f): F(A) \rightarrow F(B)\):

\[ \begin{align*}
&\text{(a) } \mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}}((f_0^1, f_1^1), S^1, S^2) = ((f_0^1, f_1^1), L^i, L^i) \\
&\text{(b) } \mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}}(S^1) = \left(\left(\,(E^1, C^i)\,\right), \,0, \,0\,\right) = L^i
\end{align*} \]

2. Composition: \(\mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}}\left((f_0^2 \circ f_0^1, f_2^1 \circ f_1^1), S^2, S^3\right) = \mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}}\left((f_0^2 \circ f_0^1, f_2^1 \circ f_1^1), S^1, S^3\right) = \mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}}\left((f_0^1, f_1^1), S^2, S^3\right) \circ \mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}}\left((f_0^2, f_2^1), S^2, S^3\right)\)

3. Identity:

\[ \mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}}(\text{Id}_{S^i}) = \mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}}\left((\text{Id}_{E^i}, \, \text{Id}_{C^i}), \, S^i, \, S^i\right) = \left(\,(\text{Id}_{E^i}, \, \text{Id}_{C^i})\,\right), \,L^i, \,L^i\right) = \text{Id}_{L^i} = \text{Id}_{\mathcal{F}_{\text{Man,LCS}}^{C^i \otimes \text{triv}}(S^i)} \]
\( \mathcal{F}_{\text{Man}, \text{LCS}}^{c_k} \) is a functor from \( \text{Man}^{c_k}(E, C) \) to \( \text{LCS}_{\text{min}}^{c_k}(E, C) \).

**Proof.** \( \mathcal{F}_{\text{Man}, \text{LCS}}^{c_k} \) satisfies the definition of a functor:

1. \( \mathcal{F}(f: A \to B) = \mathcal{F}(A) \to \mathcal{F}(B) \):
   - (a) \( \mathcal{F}_{\text{Man}, \text{LCS}}^{c_k}((f_0^i, f_1^i), S^i_1, S^i_2) = ((f_0^i, f_1^i), L^i, L^i) \)
   - (b) \( \mathcal{F}_{\text{Man}, \text{LCS}}^{c_k}(S^i) \equiv \left( \left( E^i_{\text{triv}}, C^i_{\text{op-triv}} \right), A^i, \emptyset, \emptyset \right) = L^i \)

2. Composition: \( \mathcal{F}_{\text{Man}, \text{LCS}}^{c_k}((f_0^2 \circ f_1^2, f_1^2 \circ f_1^1), S^i_1, S^i_2) \)
   - \( \mathcal{F}_{\text{Man}, \text{LCS}}^{c_k}((f_0^1, f_1^1), S^i_1, S^i_2) \)
   - \( \mathcal{F}_{\text{Man}, \text{LCS}}^{c_k}((f_0^2, f_1^2), S^i_2, S^i_3) \)
   - \( \mathcal{F}_{\text{Man}, \text{LCS}}^{c_k}(S^i_3) \)

3. Identity:
   - \( \mathcal{F}_{\text{Man}, \text{LCS}}^{c_k}((\text{Id}^i, \text{Id}^i), S^i_1, S^i_2) \)
   - \( \mathcal{F}_{\text{Man}, \text{LCS}}^{c_k}(S^i_1) \)

---

**Definition 10.48** (Functor from Local 2-∅ coordinate spaces to \( C^k \) manifolds). Let \( C^i \equiv (C^i, C^i), i = 1, 2 \) be a \( C^k \) linear model space, \( E^i \) a model space, \( \mathcal{M}_0^i \) a model category, \( \mathcal{M}_1^i \) a \( C^k \) linear model category, \( \mathcal{M}^i_0 \equiv (\mathcal{M}_0^i, \mathcal{M}_1^i), M^i \equiv (E^i, C^i), M^i \in \mathcal{M}^i, A^i_1 \text{ maximal m-atlas of } E^i \text{ in } C^i, f_0: E^1 \rightarrow E^2, f_1: C^1 \rightarrow C^2, f \equiv (f_0, f_1) \) and \( L^1 \equiv (\mathcal{M}^i, M^i, A^i, 0) \) a local 2-∅ coordinate spaces. Then

\[
\mathcal{F}_{\text{LCS}, \text{Man}}^{c_k}(f, L^i) \equiv (f, (E^i, C^i, A^i, (E^2, C^2, A^2)) \text{ (10.68)}
\]

**Theorem 10.49** (Functor from Local 2-∅ coordinate spaces to \( C^k \) manifolds).

Let \( C^i \equiv (C^i, C^i), i = 1, 2 \) be \( C^k \) linear model spaces, \( E^i \equiv (E^i, C^i) \) model spaces, \( A^i_1 \text{ maximal } C^k \text{-atlas of } E^i \text{ in } C^i, M^i_0 \equiv (E^i, C^i), \mathcal{M}^i_0 \equiv (M^i_0, f_0, E^i \rightarrow E^{i+1}, f_1: C^i \rightarrow C^{i+1}) \) and \( L^i \equiv (\mathcal{M}^i, M^i, A^i_1, 0) \) a local 2-∅ coordinate space. Then
1. $\mathcal{F}_{\text{LCS,Man}}^{C_k} \overset{\text{def}}{=} (E^i, C^i, A^i)$ is a $C^k$ manifold.

Proof. $(E^i, C^i, A^i)$ satisfies the conditions of definition 10.42 ($C^k$ manifolds) on page 77: $(E^i)$ is a topological space, $C^i$ is a $C^k$ linear space and $A^i$ is a maximal $C^k$-atlas $E^i$ in $C^i$. □

2. $\mathcal{F}_{\text{LCS,Man}}^{C_k}$ is a functor from $\text{LCS}^{C_k} \mathcal{M}(E, C)$ to $\text{Man}^{C_k}(E, C)$.

Proof. $\mathcal{F}_{\text{LCS,Man}}^{C_k}$ satisfies the definition of a functor: Let $S^i = (A^i, E^i, C^i)$.

(a) $\mathcal{F}(f): A \to B$: $\mathcal{F}(A) \to \mathcal{F}(B)$:

i. $\mathcal{F}_{\text{LCS,Man}}^{C_k}((f_0^i, f_1^i), L^i, L^{i+1}) = (f_0^i, f_1^i), S^i, S^{i+1})$

ii. $\mathcal{F}_{\text{LCS,Man}}^{C_k}(L^i) = S^i$

(b) Composition:

$\mathcal{F}_{\text{LCS,Man}}^{C_k}((f_0^2, f_3^2), L^2, L^3) \circ ((f_0^1, f_1^1), L^1, L^2) =$

$\mathcal{F}_{\text{LCS,Man}}^{C_k}(((f_0^2 \circ f_0^1, f_2^1 \circ f_1^1), L^1, L^3)) =$

$\left((f_0^2 \circ f_0^1, f_2^1 \circ f_1^1), S^1, S^3\right) =$

$\left((f_0^2, f_1^1), S^2, S^3\right) \circ S\left((f_0^1, f_1^1), S^1, S^2\right) =$

$\mathcal{F}_{\text{LCS,Man}}^{C_k}(((f_0^2, f_1^1), L^2, L^3)) \circ \mathcal{F}_{\text{LCS,Man}}^{C_k}(((f_0^1, f_1^1), L^1, L^2))$

(c) Identity:

$\mathcal{F}_{\text{LCS,Man}}^{C_k}(\text{Id}_{L^1}) = \mathcal{F}_{\text{LCS,Man}}^{C_k}(\text{Id}_{E^i}, \text{Id}_{C^i}, L^i, L^i) =$

$(\text{Id}_{E^i}, \text{Id}_{C^i}, S^i, S^i) = \text{Id}_{\mathcal{F}_{\text{LCS,Man}}^{C_k}}(L^i)$

□

3. $\mathcal{F}_{\text{LCS,Man}}^{C_k} \circ \mathcal{F}_{\text{Man,LCS}}^{C_k} = \text{Id}$

Proof. $\mathcal{F}_{\text{LCS,Man}}^{C_k} \circ \mathcal{F}_{\text{Man,LCS}}^{C_k}(S^i) = \mathcal{F}_{\text{LCS,Man}}^{C_k}(L^i) = S^i$  □

4. $\mathcal{F}_{\text{Man,LCS}}^{C_k} \circ \mathcal{F}_{\text{LCS,Man}}^{C_k} = \text{Id}$

Proof. $\mathcal{F}_{\text{Man,LCS}}^{C_k} \circ \mathcal{F}_{\text{LCS,Man}}^{C_k}(L^i) = \mathcal{F}_{\text{Man,LCS}}^{C_k}(S^i) = L^i$  □
11 Equivalence of fiber bundles

For fiber bundles, the adjunct spaces are the base space \( X = (X, \mathcal{X}) \), the fiber \( Y = (Y, \mathcal{Y}) \) and the group \( G \); the category of the coordinate space is the category of Cartesian products \( \{ U \times Y | U \in \mathcal{X} \} \) of model neighborhoods in the base space with the entire fiber, with morphisms \( t: U \times Y \rightarrow U \times Y \) that preserve the fibers, i.e., \( \pi_1 \circ t = \pi_1 \), and are generated by the group action on the fiber (Equation (11.16)).

The sole adjunct functions are the projection \( \pi: E \rightarrow X \), the group operation and the group action on the fiber.

This section defines bundle atlases, fiber bundles, local coordinate spaces equivalent to fiber bundles, categories of them and functors, and gives basic results

**Definition 11.1** (Trivial group category of groups). Let \( G \) be a set of topological groups. The trivial group category of \( G \), abbreviated \( \text{group}_{\text{triv}} \), is the category of all continuous homomorphisms between groups of \( G \). By abuse of language it will be shortened to \( G_{\text{triv}} \) when the meaning is clear from context.

**Definition 11.2** (Group actions). Let \( Y \) be a topological space, \( G \) a topological group, \( \rho \) an effective group action of \( G \) on \( Y \), \( y \in Y \) and \( g \in G \). Then \( y \ast g \), defined as \( \rho(y, g) \).

Let \( X \) be a topological space and \( x \in X \). Then \( (x, y) \ast g = (x, y \ast g) \).

A \( \ast \) with a subscript or superscript refers to the group action \( \rho \) with the same subscript or superscript.

**Remark 11.3.** This notation is only used when it is clear from context what the group action is.

**Definition 11.4** (Protobundles). Let \( B \) defined as \((E, X, Y, G, \pi, \rho)\), where \( E, X \) and \( Y \) are topological spaces, \( G \) a topological group, \( \pi: E \rightarrow X \) a continuous surjection and \( \rho \) an effective group action of \( G \) on \( Y \). Then \( B \) is a protobundle.

**Remark 11.5.** While this definition does not itself require \( E \) to have a local product structure nor require \( \pi \) to have the Covering Homotopy Property, only those protobundles having an atlas are of interest, and for them definition 11.30 (Bundle atlases) on page 95 imposes additional constraints.

11.1 \( G-\rho \) model spaces

**Definition 11.6** (\( G-\rho \)-model spaces). Let \( XY \) be defined as \((X \times Y, \mathcal{Y})\) be a model space, \( G \) a topological group and \( \rho \) an effective group action of \( G \) on \( Y \) such that the

---

16 The literature has several definitions of fiber bundle. This paper uses one chosen for clarity of exposition. It differs from [Steenrod, 1999, p. 8] in that, e.g., it uses the machinery of maximal atlases rather than equivalence classes of coordinate bundles, the nomenclature differs in several minor regards.
objects of $\mathcal{X}\mathcal{Y}$ are products of open sets with $Y$ and the morphisms are fiber-preserving automorphisms generated by the group action, i.e.,

\[
\left(\forall U \in \mathcal{X}\mathcal{Y}\right) \left(\exists V \in \mathcal{X}\mathcal{Y}\right) U = V \times Y
\]  

(11.1)

\[
\left(\forall f \in \mathcal{X}\mathcal{Y}: \mathcal{V} \times \mathcal{Y} \xrightarrow{\sim} \mathcal{V}' \times \mathcal{Y}\right) V = V'
\]  

(11.2)

\[
\left(\forall f \in \mathcal{X}\mathcal{Y}: \mathcal{V} \times \mathcal{Y} \xrightarrow{\sim} \mathcal{V} \times \mathcal{Y}\right) \left(\exists g \in G\right) \left(\forall (x,y) \in \mathcal{V} \times \mathcal{Y}\right) f(x,y) = (x, y \ast g(x))
\]  

(11.3)

Then $\mathcal{X}\mathcal{Y}$ is a $G$-$\rho$ model space of $X \times Y$, abbreviated $\text{isG}_\rho(\mathcal{X}\mathcal{Y}, Y, G, \rho), G_{\mathcal{X}\mathcal{Y}, f}$ and $G_{\mathcal{X}\mathcal{Y}} \overset{\text{def}}{=} \bigcup_{f \in \mathcal{X}\mathcal{Y}} G_{\mathcal{X}\mathcal{Y}, f}$.

**Lemma 11.7** ($G$-$\rho$-model spaces). Let $\mathcal{X}\mathcal{Y}$ be a $G$-$\rho$ model space of $X \times Y$ and $f \in \mathcal{X}\mathcal{Y}: V \times Y \xrightarrow{\sim} V \times Y$.

The function $g$ in eq. (11.3) is unique.

**Proof.** The group action is effective. □

$G_{\mathcal{X}\mathcal{Y}}$ is unique.

**Proof.** The function $g$ is unique. □

**Definition 11.8** (Morphisms of $G$-$\rho$-model spaces). Let $X_i,Y_i$, $i = 1, 2, 3$, be topological spaces, $G^i$ a topological group, $\rho^i$ an effective group action on $Y^i$ and $\mathcal{X}^i \overset{\text{def}}{=} (X_i \times Y_i, \mathcal{X}^i)$ a $G^i$-$\rho^i$ model space of $X_i \times Y_i$. Then a model function $f_{C}: \mathcal{X}^1 \rightarrow \mathcal{X}^2$ is a $G^1$-$G^2$-$\rho^1$-$\rho^2$ morphism of $X^1 \times Y^1$ to $X^2 \times Y^2$, abbreviated $\text{isG}_\rho\text{morph}(\mathcal{X}^1, G^1, \rho^1, \mathcal{X}^2, G^2, \rho^2, f_C)$ if it preserves the group action, i.e., there is a continuous homomorphism $f_C: G^1 \rightarrow G^2$ such that fig. 15 (Preserving group action) is commutative, i.e., eq. (11.4) holds.

\[
\left(\exists f_C: G_1 \rightarrow G_2\right) \left(\forall (x,y) \in \mathcal{X} \times \mathcal{Y}\right) f_C((x,y) \ast g) = f_C((x,y)) \ast^2 f_C(g)
\]  

(11.4)

**Lemma 11.9** (Morphisms of $G$-$\rho$-model spaces). Let $X_i,Y_i$, $i = 1, 2, 3$, be topological spaces, $G^i$ a topological group, $\rho^i$ an effective group action on $Y^i$, $\mathcal{X}^i \overset{\text{def}}{=} (X_i \times Y_i, \mathcal{X}^i)$ a $G^i$-$\rho^i$ model space of $X_i \times Y_i$ and $f_{C_1}: \mathcal{X}^i \rightarrow \mathcal{X}^{i+1}$ a $G^i$-$G^{i+1}$-$\rho^i$-$\rho^{i+1}$ morphism of $X^i \times Y^i$ to $X^{i+1} \times Y^{i+1}$.

The function $f_{C_1}$ in eq. (11.4) is unique.

**Proof.** The group action is effective. □
Figure 15: Preserving group action

\[ f_i^C \] preserves fibers, i.e., \( \pi_i(f_i^C(x, y)) = \pi_i(f_i^C(x, y')) \) for \( x \) in \( X^i \) and \( y, y' \) in \( Y^i \).

**Proof.** Since \( \rho^i \) is effective, there is a \( g \in G^i \) such that \( y' = y \ast^i g \) and thus by definition 11.8

\[
\begin{align*}
\pi_i(f_i^C(x, y')) &= \\
\pi_i(f_i^C(x, y \ast^i g)) &= \\
\pi_i(f_i^C(x, y) \ast^{i+1} f_i^G(g)) &= \\
\pi_i(f_i^C(x, y)) &
\end{align*}
\]

(11.5)

There exists a unique function \( f_X^i : X^i \longrightarrow X^{i+1} \) such that \( f_X^i \circ \pi_1 = \pi_1 \circ f_i^C \).

**Proof.** Define \( f_X^i(x) = \pi_1(f_i^C(x, y)) \), where \( y \) is an arbitrary point of \( Y^i \). It does not depend on the choice of \( y \) because \( f_i^C \) preserves fibers.

**Remark 11.10.** \( f_i^C \) may be a twisted product: there need not exist \( f_Y^i : Y^i \longrightarrow Y^{i+1} \) such that \( f_i^C = f_X^i \times f_Y^i \).

\( f_2^i \circ f_1^i \) is a \( G^1 \cdot G^3 \cdot \rho^1 \cdot \rho^3 \) morphism of \( X^1 \times Y^1 \) to \( X^3 \times Y^3 \).

**Proof.** Let \( f_i^G : G^i \longrightarrow G^{i+1} \) be a continuous homomorphism such that

\[
\left( \bigwedge_{(x^1, y^1) \in X^1 \times Y^1, g^1 \in G^i} f_i^G((x^1, y^1) \ast g^1) = f_i^G((x^1, y^1)) \ast f_i^G(g^1) \right) (11.6)
\]

Let \( (x^1, y^1) \in X^1 \times Y^1 \) and \( g^1 \in G^1 \). Then fig. 16 (Preserving group actions)
is commutative and
\[
\begin{align*}
  f_2^3 \circ f_1^3 ((x^1, y^1) \ast g^1) &= \\
  f_2^3 (f_1^3 ((x^1, y^1)) \ast f_1^3 (g^1)) &= \\
  f_2^3 (f_1^3 ((x^1, y^1))) \ast f_1^3 (g^1) &= \\
  f_2^3 \circ f_1^3 ((x^1, y^1)) \ast f_1^3 (g^1) &= \tag{11.7}
\end{align*}
\]

\[\Box\]

**Definition 11.11** (Categories of \(G\)-\(\rho\)-model spaces). Let \(X^\alpha, Y^\alpha, \alpha \prec A\), be topological spaces, \(G^\alpha\) a topological group, \(\rho^\alpha\) an effective group action on \(Y^\alpha\), \(XY^\alpha \overset{\text{def}}{=} (X^\alpha \times Y^\alpha, X\overline{Y}^\alpha)\) a \(G^\alpha\)-\(\rho^\alpha\) model space of \(X^\alpha \times Y^\alpha\), \(\mathcal{X}\mathcal{Y}\mathcal{G}_{\rho \text{Ob}} \overset{\text{def}}{=} \{(X^\alpha, Y^\alpha, G^\alpha, \rho^\alpha) | \alpha \prec A\}\), \(\mathcal{X}\mathcal{Y}\mathcal{G}_{\rho \text{Ar}} \overset{\text{def}}{=} \{f_c: X^\alpha \times Y^\alpha \longrightarrow X^\beta \times Y^\beta | \text{is}\rho\text{morph}(XY^\alpha, G^\alpha, \rho^\alpha, XY^\beta, G^\beta, \rho^\beta, f_c)) \land \alpha \prec A \land \beta \prec A\}\) and \(\mathcal{X}\mathcal{Y}\mathcal{G}_{\rho} \overset{\text{def}}{=} (\mathcal{X}\mathcal{Y}\mathcal{G}_{\rho \text{Ob}}, \mathcal{X}\mathcal{Y}\mathcal{G}_{\rho \text{Ar}})\). Then any subcategory of \(\mathcal{X}\mathcal{Y}\mathcal{G}_{\rho}\) is a \(G\)-\(\rho\) model category.

**Definition 11.12** (Trivial \(G\)-\(\rho\)-model spaces). Let \(X\) and \(Y\) be topological spaces, \(G\) a topological group, \(\rho\) an effective group action of \(G\) on \(Y\) and \(\mathcal{X}\mathcal{Y}\) the category of all products of open subsets of \(X\) with \(Y\) and all homeomorphisms induced by the group action, i.e.,

\[
\text{Ob}(\mathcal{X}\mathcal{Y}) \overset{\text{def}}{=} \left\{V \times Y | V \in X_{\text{op}}\right\} \tag{11.8}
\]
Then the trivial $G$-model space of $X,Y$, abbreviated $X,Y^{G-\rho}_{-\text{triv}}$, is $(\times \times, X \times Y)$ and $X,Y^{G-\rho}_{-\text{triv}}$ is a trivial $G$-model space of $X,Y$.

**Remark 11.13.** Let $G'$ be a topological group and $\rho'$ an effective group action of $G'$ on $Y$ such that $X,Y^{G-\rho}_{-\text{triv}} = X,Y^{G'-\rho'-\text{triv}}$. Although $G'$ must be isomorphic to $G$, it need not have the same topology.

The identity morphism of $X,Y^{G-\rho}_{-\text{triv}}$ is $Id_{X,Y^{G-\rho}_{-\text{triv}}} \defeq Id_{X \times Y}$.

Let $B^{\alpha} \defeq (E^{\alpha}, X^{\alpha}, Y^{\alpha}, G^{\alpha}, \pi^{\alpha}, \rho^{\alpha}), \alpha \prec A$, be a protobundle and $B \defeq \{B^{\alpha} | \alpha \prec A\}$ be a set of protobundles.

The trivial coordinate model category of $B$, $B^{G_{-\text{Triv}}}$, is the category with objects all trivial $G^{\alpha}_{-\rho_{\alpha}}$ model spaces of $X^{\alpha}, Y^{\alpha}, \alpha \prec A$ and morphisms all continuous functions compatible with the group action:

\[
B^{G_{-\text{Triv}}}_{\text{Ob}} \defeq \left\{ X,Y^{G_{-\rho}_{-\text{triv}}} | (E, X, Y, G, \pi, \rho) \in B \right\} 
\]

(11.10)

\[
B^{G_{-\text{Triv}}}_{\text{Ar}} \defeq \left\{ f_C: X^1 \times Y^1 \longrightarrow X^2 \times Y^2 \left| \exists (E', X', Y', G', \pi', \rho') \in B \right\} \right\}
\]

isG\rho\text{morph}(X^1, Y^1, G^1, \rho^1, X^2, Y^2, G^2, \rho^2, f_C)

(11.11)

\[
B^{G_{-\text{Triv}}}_{\text{Ob}} = \left( B^{G_{-\text{Triv}}}_{\text{Ob}}, B^{G_{-\text{Triv}}}_{\text{Ar}} \right) 
\]

(11.12)

**Remark 11.14.** The morphisms $f_C$ may be twisted products: there need not exist $f_Y: Y^1 \longrightarrow Y^2$ such that $f_C = f_X \times f_Y$.

The trivial product coordinate model category of $B$, $B^{G_{-\text{prod-triv}}}$, is the category with objects all trivial $G^{\alpha}_{-\rho_{\alpha}}$ model spaces of $X^{\alpha}, Y^{\alpha}, \alpha \prec A$ and morphisms all products of continuous functions compatible with the group action:
\[
B_{\text{Bun-prod-\TrivOb}} \overset{\text{def}}{=} B_{\text{Bun-\TrivOb}}
\]  
(11.13)

\[
B_{\text{Bun-prod-\TrivAr}} \overset{\text{def}}{=} \left\{ f_C \overset{\text{def}}{=} f_X \times f_Y : X^1 \times Y^1 \longrightarrow X^2 \times Y^2 \right\}
\]
\[
\left( \exists (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \in B \right) \text{isG\rho morph}( X^1, Y^1, G^1, \rho^1, X^2, Y^2, G^2, \rho^2, f_C ) \}
\]  
(11.14)

\[
B_{\text{Bun-prod-\Triv}} \overset{\text{def}}{=} \left( B_{\text{Bun-prod-\TrivOb}}, B_{\text{Bun-prod-\TrivAr}} \right)
\]  
(11.15)

Any subcategory of \( B_{\text{Bun-\Triv}} \) is a trivial coordinate model category and any subcategory of \( B_{\text{Bun-prod-\Triv}} \) is a trivial product coordinate model category.

**Lemma 11.15** (The trivial coordinate model category of \( B \) is a category). Let \( B^\alpha \overset{\text{def}}{=} (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \), \( \alpha \prec A \), be a protobundle and \( B \overset{\text{def}}{=} \{ B^\alpha | \alpha \prec A \} \) be a set of protobundles.

Then \( B_{\text{Bun-\Triv}} \) is a category and the identity morphism for object \( B^\alpha \) is \( \text{Id}_{X^\alpha \times Y^\alpha} \).

**Proof.** \( B_{\text{Bun-\Triv}} \) satisfies the definition of a category:

1. Composition:
   The composition of G-\( \rho \) morphisms is a G-\( \rho \) morphism by lemma 11.9 (Morphisms of G-\( \rho \)-model spaces) on page 87.  
2. Associativity:
   Morphisms are simply functions and composition of morphisms is simply composition of functions.
3. Unit:
   The identity morphisms are simply identity functions and composition of morphisms is simply composition of functions.

\( B_{\text{Bun-prod-\Triv}} \) is a category and the identity morphism for object \( B^\alpha \) is \( \text{Id}_{X^\alpha \times Y^\alpha} \).

**Proof.** \( B_{\text{Bun-prod-\Triv}} \) satisfies the definition of a category:
1. Composition:
The composition of $G$-$\rho$ morphisms is a $G$-$\rho$ morphism by lemma 11.9 (Morphisms of $G$-$\rho$-model spaces) on page 87. Let $X, Y \in \mathcal{B}^{G\rho\text{-triv}}$. Then

$$f_{C}^{i+1} \circ f_{C}^{i} = (f_{X}^{i+1} \circ f_{X}^{i}) \times (f_{Y}^{i+1} \circ f_{Y}^{i}).$$

2. Associativity:
Morphisms are simply functions and composition of morphisms is simply composition of functions.

3. Unit:
The identity morphisms are simply identity functions and composition of morphisms is simply composition of functions.

Lemma 11.16 (The trivial $G$-$\rho$ model space of $X, Y$ is a $G$-$\rho$ model space of $X, Y$). Let $X$ and $Y$ be topological spaces, $G$ a topological group and $\rho$ an effective group action on $Y$. Then $X, Y \in G\rho\text{-triv}$ is a $G$-$\rho$ model space of $X \times Y$.

Proof. $X, Y \in G\rho\text{-triv}$ satisfies the conditions of definition 11.6 ($G$-$\rho$-model spaces) on page 86:

1. Product with fiber:

$$\left( \forall U \in \mathcal{Ob}(X, Y \in G\rho\text{-triv}) \right) \left( \exists V \in \mathcal{X}_{G\rho\text{-triv}} \right) U = V \times Y$$

by definition 11.12.

2. Generated by $\rho$: Let $f \in \pi_{2}(X, Y \in G\rho\text{-triv}): V \times Y \xrightarrow{\cong} V \times Y$. Then

$$\left( \exists g \in G \right) \left( \forall (x, y) \in V \times Y \right) f(x, y) = (x, y \ast g(x))$$

by definition 11.12.

11.2 $G$-$\rho$-nearly commutative diagrams
Let $X$ and $Y$ be topological spaces, $G$ a topological group, $\rho$ an effective group action on $Y$, $C = (C, C) \in G\rho\text{-triv}$, and $D$ a tree with two branches, whose
nodes are topological spaces $U_i$ and $V^j$ and whose links are continuous functions $f_i: U_i \longrightarrow U_{i+1}$ and $f^j_i: U_j \longrightarrow U_{j+1}$ between the spaces:

$$D = \{ f_0: U_0 = V_0 \longrightarrow U_1, \ldots, f_{m-1}: U_{m-1} \longrightarrow U_m, \quad f'_0: U_0 = V_0 \longrightarrow V_1, \ldots, f'_{m-1}: V_{m-1} \longrightarrow V_n \}$$

with $U_0 = V_0, U_m \subseteq C$ open and $V_n \subseteq C$ open, as shown in fig. 3 (Uncompleted nearly commutative diagram) on page 13.

**Definition 11.17** ($G$-$\rho$-nearly commutative diagrams). $D$ is nearly commutative in $X,Y,\pi,\rho$ iff $D$ is nearly commutative in category $C$.

**Definition 11.18** ($G$-$\rho$-nearly commutative diagrams at a point). Let $C$ and $D$ be as above and $x$ be an element of the initial node. $D$ is nearly commutative in $X,Y,\pi,\rho$ at $x$ iff $D$ is nearly commutative in $C$ at $x$.

**Definition 11.19** ($G$-$\rho$-locally nearly commutative diagrams). Let $C$ and $D$ be as above. $D$ is locally nearly commutative in $X,Y,\pi,\rho$ iff $D$ is locally nearly commutative in $C$.

### 11.3 Bundle charts

**Definition 11.20** ($Y$-$\pi$-bundle charts). Let $E, X, Y$ be topological spaces and $\pi: E \longrightarrow X$. A $Y$-$\pi$-bundle chart $(U, V \times Y, \phi)$ of $E$ in the coordinate space $X \times Y$ consists of

1. An open set $U \subseteq E$, known as a coordinate patch
2. An open set $V \times Y \subseteq X \times Y$
3. A homeomorphism $\phi: U \xrightarrow{\pi} V \times Y$, known as a coordinate function, that preserves fibers. i.e., $\pi \circ \phi = \pi$.

**Lemma 11.21** (Properties of projection). Let $(U, V \times Y, \phi)$ be a $Y$-$\pi$-bundle chart of $E$ in the coordinate space $X \times Y$ and $v \in V$. Then $\pi|_U$ is a surjection.

**Proof.** Let $v \in V$, $y \in Y$ and $u \overset{\text{def}}{=} \phi^{-1}((v, y)) \in U$. Then $\pi(u) = v$. $\square$

$$\pi^{-1}\{v\} \text{ is homeomorphic to } Y.$$ 

**Proof.** $\phi$ and $\phi^{-1}$ are homeomorphisms, so their restrictions are homeomorphisms and thus $\phi^{-1}\{v\}$ is homeomorphic to $\{v\} \times Y$, which is homeomorphic to $Y$. $\square$

**Definition 11.22** ($Y$-$\pi$ subcharts). Let $(U, V \times Y, \phi)$ be a $Y$-$\pi$-bundle chart of $E$ in the coordinate space $X \times Y$ and $U' \subseteq U$ open. Then $(U', V' \times Y, \phi') \overset{\text{def}}{=} (U', \phi[U'], \phi|_{U'})$ is a subchart of $(U, V \times Y, \phi)$.
Lemma 11.23 (Y-π subcharts). Let \((U, V \times Y, \phi)\) be a \(Y\)-\(\pi\)-bundle chart of \(E\) in the coordinate space \(X \times Y\) and \((U', V' \times Y, \phi')\) a subchart of \((U, V \times Y, \phi)\). Then \((U', V' \times Y, \phi')\) is a \(Y\)-\(\pi\)-bundle chart of \(E\) in the coordinate space \(X \times Y\).

**Proof.** \((U', V' \times Y, \phi')\) satisfies the conditions of definition 11.20:

1. \(U' \subseteq E\) is open.
2. \(\phi[U']\) is open since \(\phi\) is a homeomorphism.
3. \(\phi \restriction_{U'}: U' \to V' \times Y\) is the restriction of a homeomorphism and thus a homeomorphism. \(\phi \restriction_{U'}\) preserves fibers because \(\phi\) does.

\[\phi[U'] = \phi \circ \phi^{-1}[U \cap U'] \quad (11.16)\]

**Definition 11.24 (G-\(\rho\)-compatibility).** Let \(E, X, Y\) be topological spaces, \(G\) a topological group, \(\rho: Y \times G \to Y\) an effective right action of \(G\) on \(Y\), \(\pi: E \to X\) surjective and \((U, V \times Y, \phi)\), \((U', V' \times Y, \phi')\) \(Y\)-\(\pi\)-bundle charts. \((U, V \times Y, \phi)\) and \((U', V' \times Y, \phi')\) are \(G\)-\(\rho\)-compatible if either

1. \(U\) and \(U'\) are disjoint
2. The transition function
   \[
   t^n_{\rho} = \phi_\rho \circ \phi'^{-1}|_{\phi'[U \cap U']} \quad (11.16)
   \]
   is generated by the group action, i.e., there is a continuous function \(g^n_{\rho}: \pi_1[\phi'^*[U \cap U']] \to G\) such that
   \[
   \bigwedge_{(x, y) \in \phi'^*[U \cap U']} t^n_{\rho}(x, y) = (x, y \ast g^n_{\rho}(x))
   \]

**Lemma 11.25 (Symmetry of G-\(\rho\) compatibility).** Let \((U, V \times Y, \phi)\) and \((U', V' \times Y, \phi')\) be \(Y\)-\(\pi\)-bundle charts of \(E\) in the coordinate space \(X \times Y\). Then \((U, V \times Y, \phi)\) is \(G\)-\(\rho\)-compatible with \((U', V' \times Y, \phi')\) iff \((U', V' \times Y, \phi')\) is \(G\)-\(\rho\)-compatible with \((U, V \times Y, \phi)\).

**Proof.** It suffices to prove the result in one direction. If \((U, V \times Y, \phi)\) \(\cap\) \((U', V' \times Y, \phi')\) then \((U, V \times Y, \phi)\) \(\cap\) \((U', V' \times Y, \phi')\). Otherwise, let \(g^n_{\rho}: \pi_1[\phi'^*[U \cap U']] \to G\) be a continuous function such that
   \[
   \bigwedge_{(x, y) \in \phi'^*[U \cap U']} t^n_{\rho}(x, y) = (x, y \ast g^n_{\rho}(x))
   \]
and the inverse transition function is also generated by the group action:
   \[
   \bigwedge_{(x, y) \in \phi'^*[U \cap U']} t^n_{\rho}(x, y) = (x, y \ast g^n_{\rho}(x)^{-1})
   \]

**Lemma 11.26 (G-\(\rho\)-compatibility of subcharts).** Let \((U, V \times Y, \phi)\) be a \(Y\)-\(\pi\)-bundle chart of \(E\) in the coordinate space \(X \times Y\), \((U', V' \times Y, \phi')\) a subchart and \((U^1, V^1, \phi^1)\) be \(G\)-\(\rho\)-compatible with \((U^2, V^2, \phi^2)\). Then \((U, V \times Y, \phi)\) is \(G\)-\(\rho\)-compatible with \((U^2, V^2 \times Y, \phi^2)\).
Proof. If $U^1 \cap U^2 = \emptyset$ then $U^1 \cap U^2 = \emptyset$. If $U^1 \cap U^2 = \emptyset$ then $(U^1, V^1 \times Y, \phi^1)$ is $G$-$\rho$-compatible with $(U^2, V^2 \times Y, \phi^2)$. Otherwise, the transition function $t_2 \overset{\text{def}}{=} \phi^1 \circ \phi^{2-1} \mid_{\phi^2[U^1 \cap U^2]}$ is generated by the group action and hence $t_2 \mid_{\phi^2[U^1 \cap U^2]} : \phi^1[U^1 \cap U^2] \overset{\cong}{\to} \phi^1[U^1 \cap U^2]$ is generated by the group action.

Corollary 11.27 (G-$\rho$-compatibility with subcharts). Let $(U, V \times Y, \phi)$, be a $Y$-$\pi$-bundle chart of $E$ in the coordinate space $X \times Y$ and $(U', V' \times Y, \phi')$ a subchart. Then $(U', V' \times Y, \phi')$ is $G$-$\rho$-compatible with $(U, V \times Y, \phi)$.

Proof. $(U, V \times Y, \phi)$ is $G$-$\rho$-compatible with itself and is a subchart of itself.

Definition 11.28 (Covering by $Y$-$\pi$-bundle charts). Let $A$ be a set of $Y$-$\pi$-bundle charts of $E$ in the coordinate space $X \times Y$. $A$ covers $E$ iff $E = \bigcup \pi_1[A]$.

Lemma 11.29. Let $A$ be a set of $Y$-$\pi$-bundle charts of $E$ in the coordinate space $X \times Y$ that covers $E$ and $x \in X$. Then $\pi^{-1}[\{x\}]$ is homeomorphic to $Y$.

Proof. Since $A$ covers $E$, there is a chart $(U, V, \phi)$ in $A$ containing $x$. Then $\pi^{-1}[\{x\}]$ is homeomorphic to $Y$ by lemma 11.21 (Properties of projection) on page 93.

11.4 Bundle atlases

A set of charts can be atlases for different fiber bundles even if it is for the same total model space, base space and fiber. In order to aggregate atlases into categories, there must be a way to distinguish them. Including the spaces\footnote{The spaces are redundant, but convenient.}, group and group action in the definitions of the categories serves the purpose.

Definition 11.30 (Bundle atlases). Let $B \overset{\text{def}}{=} (E, X, Y, G, \pi, \rho)$, be a protobundle. Then $A$ is a bundle atlas of $B$, abbreviated isAtl\textsuperscript{Bun}$\overset{\text{def}}{=} (A, B)$ and $A$ is a $\pi$-$G$-$\rho$-bundle atlas of $E$ in the coordinate space $X \times Y$, abbreviated isAtl\textsuperscript{Bun}$\overset{\text{def}}{=} (A, E, X, Y, G, \pi, \rho)$, iff it consists of a set of mutually $G$-$\rho$-compatible $Y$-$\pi$-bundle charts of $E$ in the coordinate space $X \times Y$ that covers $E$\footnote{There is no need to introduce the concept of a full $\pi$-$G$-$\rho$-bundle atlas because a $\pi$-$G$-$\rho$-bundle atlas is automatically full.}.

By abuse of language we write $U \in A$ for $U \in \pi_1[A]$.

Remark 11.31. The definition of a $\pi$-$G$-$\rho$-bundle atlas is by design similar to the definition of a coordinate bundle in [Steenrod, 1999, p. 7], but there are significant differences. This paper will use the term bundle atlas to avoid confusion.

Let $B^\alpha \overset{\text{def}}{=} (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha)$, $\alpha \in A$, be a protobundle and $B \overset{\text{def}}{=} \{B^\alpha | \alpha \in A\}$ be a set of protobundles. Then

\[
\text{Atl}^\text{Bun}_\text{Ob}(B^\alpha) \overset{\text{def}}{=} \{(A, B^\alpha) | (A, B^\alpha) \in \text{Atl}^\text{Bun}_\text{Ob}(A, E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha)\} \quad (11.17)
\]
Proof. Let $A$ be a $\pi$-$G$-$\rho$-bundle atlas of $E$ in the coordinate space $X \times Y$ then

1. Every chart in $A$ is a $Y$-$\pi$-bundle chart of $E$ in the coordinate space $X \times Y$, and hence its coordinate function preserves fibers.

2. The charts in $A$ are mutually $G$-$\rho$-compatible; hence the transition functions are generated by the group action and are morphisms of $X,Y$.

If $A$ is an $m$-atlas of $E$ in the coordinate space $X \times Y$ then the transition functions are generated by the group action and thus the charts are mutually $G$-$\rho$-compatible.

If every coordinate function preserves fibers then the $m$-charts of $A$ are $Y$-$\pi$-bundle charts.

Definition 11.33 (Compatibility of charts with bundle atlases). A $Y$-$\pi$-bundle chart $(U,V \times Y,\phi)$ of $E$ in the coordinate space $X \times Y$ is $G$-$\rho$-compatible with a $\pi$-$G$-$\rho$-bundle atlas $A$ iff it is $G$-$\rho$-compatible with every chart in the atlas.

Lemma 11.34 (Compatibility of subcharts with bundle atlases). Let $A$ be a $\pi$-$G$-$\rho$-bundle atlas of $E$ in the coordinate space $X \times Y$ and $C^1 = (U,V \times Y,\phi^1)$ a $Y$-$\pi$-bundle chart in $A$. Then any subchart of $C^1$ is $G$-$\rho$-compatible with $A$.

Proof. Let $C'^1 = (U'^1,V'^1 \times Y,\phi'^1)$ be a subchart of $C^1$ and $C'^2 = (U'^2,V'^2 \times Y,\phi'^2)$ another chart in $A$.

1. If $U'^1 \cap U'^2 = \emptyset$, then $U' \cap U^2 = \emptyset$.
2. If $U' \cap U^2 = \emptyset$ then $C'^1$ is $G$-$\rho$-compatible with $C'^2$.
3. Otherwise the transition function $t^1 \overset{\text{def}}{=} \phi^1 \circ \phi'^{-1} \mid_{\phi^1[U'^1 \cap U'^2]}$ is generated by the group action and thus $t^2 \mid_{\phi'^1[U'^1 \cap U'^2]}$ is generated by the group action.

Lemma 11.35 (Extensions of bundle atlases). Let $A$ be a $\pi$-$G$-$\rho$-atlas of $E$ in the coordinate space $X \times Y$ and $(U_i,V_i,\phi_i)$, $i = 1,2$ be $\pi$-$G$-$\rho$-charts of $E$ in the coordinate space $X \times Y$ $G$-$\rho$-compatible with $A$ in the coordinate space $X \times Y$. Then $(U_1,V_1,\phi_1)$ is $G$-$\rho$-compatible with $(U_2,V_2,\phi_2)$ in the coordinate space $X \times Y$. 

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Proof. If \( U_1 \cap U_2 = \emptyset \) then \( (U_1, V_1, \phi_1) \) is \( G\)-\( \rho \)-compatible with \( (U_2, V_2, \phi_2) \).

Otherwise, \( \phi_2 \circ \phi_1^{-1} \mid_{\phi_1(U_1 \cap U_2)} : \phi_1(U_1 \cap U_2) \xrightarrow{\sim} \phi_2(U_1 \cap U_2) \) is a homeomorphism.

It remains to show that \( \phi_2 \circ \phi_1^{-1} \mid_{\phi_1(U_1 \cap U_2)} \) is generated by the group action.

Let \( (U'_\alpha, V'_\alpha, \phi'_\alpha), \alpha < A \), be charts in \( A \) such that \( U_1 \cap U_2 \subseteq \bigcup_{\alpha < A} U'_\alpha \) and \( U_1 \cap U_2 \cap U'_\alpha \neq \emptyset, \alpha < A \). Since the charts are \( G\)-\( \rho \)-compatible with \( (U'_\alpha, V'_\alpha, \phi'_\alpha) \), \( \phi_2 \circ \phi'^{-1}_1 \mid_{U_1 \cap U_2 \cap U'_\alpha} \) and \( \phi'_1 \circ \phi^{-1}_1 \mid_{U_1 \cap U_2 \cap U'_\alpha} \) are generated by the group action and thus \( \phi_2 \circ \phi'^{-1}_1 = \phi^2 \circ \phi'^{-1}_1 \circ \phi'^{-1}_1 \) is generated by the group action. \( \square \)

**Definition 11.36 (Maximal bundle atlases).** Let \( A \) be a \( \pi\)-\( G\)-\( \rho \)-bundle atlas of \( E \) in the coordinate space \( X \times Y \). \( A \) is a maximal \( \pi\)-\( G\)-\( \rho \)-bundle atlas, abbreviated \( \text{isAt}^{\text{Bun}}_{\text{max}} (A, E, X, Y, G, \pi, \rho) \), iff it cannot be extended by adding an additional \( G\)-\( \rho \)-compatible \( Y\)-\( \pi \)-bundle chart.

\( A \) is a semi-maximal \( \pi\)-\( G\)-\( \rho \)-bundle atlas of \( E \) in the coordinate space \( C \), abbreviated \( \text{isAt}^{\text{Bun}}_{\text{max}} (A, E, X, Y, G, \pi, \rho) \), iff whenever \( (U, V \times Y, \phi) \in A, U' \subseteq U, V' \subseteq V \times Y \) and \( V'' \times Y \subseteq X \times Y \) are open sets, \( \phi(U') = V' \times Y \), \( \phi' : V' \times Y \xrightarrow{\sim} V'' \times Y \) is a fiber preserving homeomorphism generated by the group action then \( (U', V'' \times Y, \phi' \circ \phi) \in A \).

Let \( B^\alpha \overset{\text{def}}{=} \{ E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha \}, \alpha < A \), be a protobundle and \( B \overset{\text{def}}{=} \{ B^\alpha \mid \alpha < A \} \) be a set of protobundles. Then

\[
\text{At}^{\text{Bun}}_{\text{max}}(B^\alpha) = \left\{ (A, B^\alpha) \mid \text{isAt}^{\text{Bun}}_{\text{max}} (A, E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \right\} \quad (11.19)
\]

\[
\text{At}^{\text{Bun}}_{\text{max}}(B) = \bigcup_{\alpha < A} \text{At}^{\text{Bun}}_{\text{max}}(B^\alpha) \quad (11.20)
\]

\[
\text{At}^{\text{Bun}}_{\text{S-max}}(B^\alpha) = \left\{ (A, B^\alpha) \mid \text{isAt}^{\text{Bun}}_{\text{S-max}} (A, E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \right\} \quad (11.21)
\]

\[
\text{At}^{\text{Bun}}_{\text{S-max}}(B) = \bigcup_{\alpha < A} \text{At}^{\text{Bun}}_{\text{S-max}}(B^\alpha) \quad (11.22)
\]

**Lemma 11.37 (Maximal \( \pi\)-\( G\)-\( \rho \)-bundle atlases are semi-maximal \( \pi\)-\( G\)-\( \rho \)-bundle atlases).** Let \( E, X, Y \) be topological spaces, \( G \) a topological group, \( \pi : E \longrightarrow X \) surjective, \( \rho : Y \times G \longrightarrow Y \) an effective right action of \( G \) on \( Y \) and \( A \) a maximal \( \pi\)-\( G\)-\( \rho \)-bundle atlas of \( E \) in the coordinate space \( X \times Y \). Then \( A \) is a semi-maximal \( \pi\)-\( G\)-\( \rho \)-bundle atlas of \( E \) in the coordinate space \( X \times Y \).

Proof. Let \( (U, V \times Y, \phi) \in A, U' \subseteq U, V' \times Y \subseteq V \times Y \) and \( V'' \times Y \subseteq X \times Y \) be open sets and \( \phi[U'] = V' \times Y \), \( \phi' : V' \times Y \xrightarrow{\sim} V'' \times Y \) be a fiber preserving homeomorphism generated by the group action. \( (U', V', \phi) \) is a subchart of
Let $(U, V, \phi)$ and by Lemma 11.34 (Compatibility of subcharts with bundle atlases) on page 96 is $G$-$\rho$-compatible with the charts of $A$. Since $\phi'$ is a fiber preserving homeomorphism generated by the group action, $(U', V' \times Y, \phi' \circ \phi)$ is $G$-$\rho$-compatible with the charts of $A$. Since $A$ is maximal, $(U', V'', \phi' \circ \phi)$ is a chart of $A$.

**Theorem 11.38** (Existence and uniqueness of maximal $\pi$-$G$-$\rho$-bundle atlases). Let $B = (E, X, Y, G, \pi, \rho)$ be a protobundle and $A$ a $\pi$-$G$-$\rho$-bundle atlas of $E$ in the coordinate space $X \times Y$. Then there exists a unique maximal $\pi$-$G$-$\rho$-bundle atlas $\text{Atlas}^\text{Bun}_\text{max}(A, B)$ of $E$ in the coordinate space $X \times Y$ $G$-$\rho$-compatible with $A$.

**Proof.** Let $P$ be the set of all $\pi$-$G$-$\rho$-bundle atlases $E$ in the coordinate space $X \times Y$ containing $A$ and $G$-$\rho$ compatible in the coordinate space $X \times Y$ with $A$. Let $P$ be a maximal chain of $P$. Then $A' = \bigcup P$ is a maximal $\pi$-$G$-$\rho$-bundle atlas of $E$ in the coordinate space $X \times Y$ $G$-$\rho$ compatible with $A$. Uniqueness follows from Lemma 11.35 (Extensions of bundle atlases) on page 96.

**Lemma 11.39** (Existence and uniqueness of projection for atlases in $G$-$\rho$-model spaces). Let $E$, $X$ and $Y$ be topological spaces, $G$ a topological group, $\rho$ an effective action of $G$ on $Y$, $C = (X \times Y, X')$ a $G$-$\rho$ model space of $X \times Y$ and $A$ an $m$-atlas of $E$ in the coordinate model space $C$. Then there exists a unique function $\pi: E \longrightarrow X$ such that for any chart $(U, V, \phi)$ in $A$, $\pi |_{U} = \pi \circ \phi$. If $A$ is full then $\pi$ is surjective.

**Proof.** Let $(U, V, \phi)$ be an arbitrary chart in $A$ and define $\pi(e \in U) = \pi \circ \phi(e)$. $\pi(e)$ does not depend on the choice of chart because the morphisms of a $G$-$\rho$ model space preserve fibers. $\pi$ is continuous because it is continuous on each coordinate patch.

Let $x \in X$. If $A$ is full then there exists a chart $(U, V, \phi)$ in $A$ such that $x \in \pi_1[V]$. Let $u$ be an arbitrary point in $\phi^{-1}([x] \times Y)$. Then $\pi(u) = \pi_1(\phi(u)) = x$.

### 11.5 Bundle atlas morphisms and functors

This section defines categories of bundle atlases, $\text{Atl}^\text{Bun}_B$ and constructs functors from them to categories of $m$-atlases, $\text{Bun}_-T_{\text{riv}}$ and $\text{Bun}_-\text{prod}-T_{\text{riv}}$. It only constructs reverse functors for $\text{Bun}_-\text{prod}-T_{\text{riv}}$.

**Definition 11.40** (Bundle-atlas morphisms). Let $B^i = (E^i, X^i, Y^i, G^i, \pi^i, \rho^i)$, $i = 1, 2$, be a protobundle and let $A^i$ be a $\pi^i$-$G^i$-$\rho^i$-atlas of $E^i$ in the coordinate space $C^i = X^i \times Y^i$. Then $f = (f_E: E^1 \longrightarrow E^2, f_X: X^1 \longrightarrow X^2, f_Y: Y^1 \longrightarrow Y^2, f_G: G^1 \longrightarrow G^2)$ is a $B^1$-$B^2$ bundle-atlas morphism from $A^1$ to $A^2$, abbreviated is $\text{Atl}^\text{Bun}_{\text{Ar}}(A^1, B^1, A^2, B^2, f)$, iff

1. all four functions are continuous
2. \( f_G \) is a homomorphism

3. \( f \) commutes with \( \pi^i \) and \( \rho^i \), i.e.,
   
   (a) \( \pi^2 \circ f_E = f_X \circ \pi^1 \)
   
   (b) \( \left( \bigvee_{y \in Y^1} f_Y(y) \right) \circ f_G = f_Y \left( \bigvee_{y \in Y^1} g \right) \)

4. for any \( (U^1, V^1, \phi^1; U^1, \xrightarrow{\sim} V^1) \in A^1 \), \( (U^2, V^2, \phi^2; U^2, \xrightarrow{\sim} V^2) \in A^2 \),
   
   the diagram \( D \equiv \left( \left( I \equiv U^1 \cap f_0^{-1}[U^2], V^1, E^2, U^2, V^2 \right) \right) \) is
   
   locally nearly commutative in \( X, Y, \pi, \rho \).

If \( A^1 \) and \( A^2 \) are maximal atlases then \( f \) is also a maximal \( B^1 \rightarrow B^2 \) bundle-atlas morphism from \( A^1 \) to \( A^2 \), abbreviated isAtl_{max}^{Bun}(A^1, B^1, A^2, B^2, f)

The identity morphism of \( (A^i, B^i) \) is

\[
Id_{\langle A^i, B^i \rangle} \equiv \left( \left( Id_{E^i}, Id_{X^i}, Id_{Y^i}, Id_{G^i} \right) (A^i, B^i), (A^i, B^i) \right) 
\tag{11.23}
\]

This nomenclature will be justified later.

Let \( B^i \equiv (E^i, X^i, Y^i, G^i, \pi^i, \rho^i), i = 1, 2 \), be a protobundle. Then

\[
\text{Atl}^{Bun}_{\text{max}}(B^1, B^2) \equiv \left\{ (f, (A^1, B^1), (A^2, B^2)) \mid \text{isAtl}^{Bun}_{\text{max}}(A^1, B^1, A^2, B^2, f) \right\} 
\tag{11.24}
\]

\[
\text{Atl}^{Bun}_{\text{max}}(B^1, B^2) \equiv \left\{ (f, (A^1, B^1), (A^2, B^2)) \mid \text{isAtl}^{Bun}_{\text{max}}(A^1, B^1, A^2, B^2, f) \right\} 
\tag{11.25}
\]

\[
\text{Atl}^{Bun}_{\text{max}}(B^1, B^2) \equiv \left\{ (f, (A^1, B^1), (A^2, B^2)) \mid \text{isAtl}^{Bun}_{\text{max}}(A^1, B^1, A^2, B^2, f) \right\} 
\tag{11.26}
\]

\[
\text{Atl}^{Bun}(B^i) \equiv (\text{Atl}^{Bun}_{\text{Ob}}(B^i), \text{Atl}^{Bun}_{\text{Ar}}(B^i, B^i), \circ) 
\tag{11.27}
\]

\[
\text{Atl}^{Bun}(B^i) \equiv (\text{Atl}^{Bun}_{\text{Ob}}(B^i), \text{Atl}^{Bun}_{\text{Ar}}(B^i, B^i), \circ) 
\tag{11.28}
\]

**Lemma 11.41** (Bundle-atlas morphisms). Let \( B^i \equiv (E^i, X^i, Y^i, G^i, \pi^i, \rho^i), i = 1, 2 \), be a protobundle and let \( A^i \) be a \( \pi^i-G^i-\rho^i \)-atlas of \( E^i \) in the coordinate space \( C^i = X^i \times Y^i \). Then \( f \equiv (f_E; E^1 \rightarrow E^2, f_X; X^1 \rightarrow X^2, f_Y; Y^1 \rightarrow Y^2, f_G; G^1 \rightarrow G^2) \) is a \( B^1 \rightarrow B^2 \) bundle-atlas morphism from \( A^1 \) to \( A^2 \) iff \( f_X \times f_Y \) is a \( G^1-G^2-\rho^1-\rho^2 \)-morphism from \( X^1 \times Y^1 \) to \( X^2 \times Y^2 \) and \( (f_E, f_X \times f_Y) \) is a \( E^1-E^2 \) \( m \)-atlas morphism from \( A^1 \) to \( A^2 \) in the coordinate spaces \( X^1, Y^1 \) and \( X^2, Y^2 \) respectively.
Proof. If \( f \) is a \( \mathbf{B}^1\)-\( \mathbf{B}^2 \) bundle-atlas morphism then \( f_X \times f_Y \) is a model function and \( f_G \) is the function asserted to exist in eq. (11.4) (preservation of group action) of definition 11.8 (Morphisms of \( G\)-\( \rho \)-model spaces) on page 87, so \( f_X \times f_Y \) is a \( G_1\)-\( G^2\)-\( \pi^1\)-\( \pi^2 \) morphism.

A diagram is locally nearly commutative in \( X^2, Y^2, \pi^2, \rho^2 \) if it is \( m \)-locally nearly commutative in \( \pi_2( X^2, Y^2 ) \), thus \( (f_E, f_X \times f_Y) \) is a \( m \)-atlas morphism in the coordinate space \( X^2, Y^2 \), so \( (f_E, f_X \times f_Y) \) is an \( m \)-atlas morphism from \( A^1 \) to \( A^2 \) in the coordinate space \( X^2, Y^2 \).

If \( f_X \times f_Y \) is a \( G_1\)-\( G^2\)-\( \pi^1\)-\( \pi^2 \) morphism from \( X^1, Y^1 \) to \( X^2, Y^2 \) then \( f \) commutes with \( \rho^i \). If \( (f_E, f_X \times f_Y) \) is a \( E \)-\( E \) \( m \)-atlas morphism from \( A^1 \) to \( A^2 \) in the coordinate space \( X^2, Y^2 \) then \( f \) commutes with \( \pi^i \).

A diagram is locally nearly commutative in \( X^2, Y^2, \pi^2, \rho^2 \) if it is \( m \)-locally nearly commutative in \( \pi_2( X^2, Y^2 ) \), thus \( (f_E, f_X \times f_Y) \) is a \( m \)-atlas morphism in the coordinate space \( X^2, Y^2 \), so \( f \) is a \( \mathbf{B}^1\)-\( \mathbf{B}^2 \) bundle-atlas morphism from \( A^1 \) to \( A^2 \).

\[ \square \]

Corollary 11.42 (Bundle-atlas morphisms). Let \( B^i \) be a protobundle, \( A^i \) be a \( \pi^i \)-\( G^i \)-\( \rho^i \)-atlas of \( E^i \) in the coordinate space \( C^i = X^i \times Y^i \) and \( f^i \) be a \( \pi^i \)-\( G^i \)-\( \rho^i \)-morphism from \( C^i \) to \( X^+ \times Y^+ \). Then \( f^i \) is a \( \mathbf{B}^1 \)-\( \mathbf{B}^2 \) bundle-atlas morphism from \( A^1 \) to \( A^2 \).

Proof. \( f^i_X \times f^i_Y \) is a \( \mathbf{B}^1 \)-\( \mathbf{B}^2 \) model function in the coordinate space \( X^3, Y^3 \) by Lemma 8.21 (M-atlas morphisms) on page 43 and \( (f^i_X \times f^i_Y)^{G^1 \rightarrow G^2} \) a \( \mathbf{B}^1 \)-\( \mathbf{B}^2 \) bundle-atlas morphism from \( A^1 \) to \( A^2 \) in the coordinate space \( X^3, Y^3 \) by Lemma 11.9 (Morphisms of \( G\)-\( \rho \)-model spaces) on page 87.

\[ \square \]

Definition 11.43 (Categories of bundle atlases). Let \( B^\alpha \) be a protobundle and \( B = \{ B^\alpha \} \) a set of protobundles. Then

\[
\mathcal{A}_{\text{Bun}}^{\text{Bun}} B \text{ is the set of } \mathcal{A}_{\text{Bun}}^{\text{Bun}} B \text{ bundle-atlas morphisms from } B \text{ to } B.
\]

\[ \mathcal{A}_{\text{Bun}}^{\text{Bun}} B \text{ is } \mathcal{A}_{\text{Bun}}^{\text{Bun}} B \text{ bundle-atlas morphisms from } B \text{ to } B. \]

\[ \mathcal{A}_{\text{Bun}}^{\text{Bun}} B \text{ is } \mathcal{A}_{\text{Bun}}^{\text{Bun}} B \text{ bundle-atlas morphisms from } B \text{ to } B. \]

\[ \mathcal{A}_{\text{Bun}}^{\text{Bun}} B \text{ is } \mathcal{A}_{\text{Bun}}^{\text{Bun}} B \text{ bundle-atlas morphisms from } B \text{ to } B. \]
\[ \text{Atl}_{\text{max}}^\text{Bun} \stackrel{\text{def}}{=} \bigcup_{B^\mu \in B} \text{Atl}_{\text{max}}^\text{Bun}(B^\mu, B^\nu) \quad (11.31) \]

\[ \text{Atl}_{\text{max}}^\text{Bun} \stackrel{\text{def}}{=} (\text{Atl}_{\text{max}}^\text{Bun}(B), \text{Atl}_{\text{max}}^\text{Ar}(B), \odot) \quad (11.32) \]

\[ \text{Atl}_{\text{max}}^\text{Bun} \stackrel{\text{def}}{=} \bigcup_{B^\mu \in B} \text{Atl}_{\text{max}}^\text{Bun}(B^\mu, B^\nu) \quad (11.33) \]

\[ \text{Atl}_{\text{max}}^\text{Bun} \stackrel{\text{def}}{=} (\text{Atl}_{\text{Ob}}^\text{Bun}(B), \text{Atl}_{\text{max}}^\text{Ar}(B), \odot) \quad (11.34) \]

**Lemma 11.44** ($\text{Atl}_{\text{Bun}}^\text{B}$ is a category). Let \( B^\alpha \stackrel{\text{def}}{=} (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \), \( \alpha \prec A \), be a protobundle and \( B \stackrel{\text{def}}{=} \{ B^\alpha | \alpha \prec A \} \) be a set of protobundles. Then \( \text{Atl}_{\text{Bun}}^\text{B} \) is a category.

Let \((A^\circ, B^\alpha) \in \text{Atl}_{\text{Ob}}^\text{Bun} \). Then \( \text{Id}_{(A^\circ, B^\alpha)} \) is the identity morphism for \((A^\circ, B^\alpha) \).

**Proof.** Let \((A^i, B^i), i = 1, 2, 3\) be objects of \( \text{Atl}_{\text{Bun}}^\text{B} \) and let \( m^i \stackrel{\text{def}}{=} (f^i, (A^i, B^i), (A^{i+1}, B^{i+1})) \) be morphisms of \( \text{Atl}_{\text{Bun}}^\text{B} \). Then

1. Composition:
   \((m^2 \circ m^1, (A^1, E^1, C^1), (A^2, E^2, C^2)) \) is a morphism of \( \text{Atl}_{\text{Bun}}^\text{B} \) by corollary 11.42 (Bundle-atlas morphisms) on page 100.

2. Associativity:
   Composition is associative by lemma 3.17 (Tuple composition for labeled morphisms) on page 10.

3. Identity:
   \( \text{Id}_{(A^i, B^i)} \) is an identity morphism by lemma 3.17.

**Definition 11.45** (Functor from Bundle atlases to m-atlases). Let \( B^i \stackrel{\text{def}}{=} (E^i, X^i, Y^i, G^i, \pi^i, \rho^i), i = 1, 2, \) be a protobundle, let \( A^i \) be a \( \pi^i-G^i-\rho^i \)-atlas of \( E^i \) in the coordinate space \( C^i = X^i \times Y^i \) and let \( f^i \stackrel{\text{def}}{=} (f_E: E^1 \longrightarrow E^2, f_X: X^1 \longrightarrow X^2, f_Y: Y^1 \longrightarrow Y^2, f_G: G^1 \longrightarrow G^2) \) be a \( B^1-B^2 \) bundle-atlas morphism from \( A^1 \) to \( A^2 \). Then

\[ \text{F}^\text{Bun, M-atlas}(A^i, B^i) \stackrel{\text{def}}{=} (A^i, E^i, X^i, Y^i) \]

(11.35)
Theorem 11.46 (Functor from Bundle atlases to m-atlases). Let $E$ be a set of topological spaces, $B^\alpha \overset{\text{def}}{=} \{ E, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha \}$ be a protobundle, $B \overset{\text{def}}{=} \{ B^\alpha | \alpha \prec A \}$ be a set of protobundles and $C^\alpha \overset{\text{def}}{=} \{ X^\alpha, Y^\alpha \}$. Then $F_{\text{Bun},M-\text{atlas}}^\text{Bun}$ is a functor from $\text{Atl}^{\text{Bun}}(B)$ to $\text{Atl}(E, B)$ and a functor from $\text{Atl}^{\text{Bun}}(B)$ to $\text{Atl}(E, B_{\text{triv}})$.

Proof. Let $\alpha^i \overset{\text{def}}{=} \{ A^i, B^i \}$, $i \in [1, 3]$, be objects of $\text{Atl}^{\text{Bun}}(B)$ and $m^i \overset{\text{def}}{=} \{ f^i, o^i, o^{i+1} \}$, $i = 1, 2$, be morphisms. $F_{\text{Bun},M-\text{atlas}}^\text{Bun}(m^i)$ is a morphism from $F_{\text{Bun},M-\text{atlas}}^\text{Bun} o^i$ to $F_{\text{Bun},M-\text{atlas}}^\text{Bun} o^{i+1}$:

$$\begin{align*}
F_{\text{Bun},M-\text{atlas}}^\text{Bun}(m^i) &= F_{\text{Bun},M-\text{atlas}}^\text{Bun}(f^i, A^i, B^i, (A^{i+1}, B^{i+1})) \\
&= ((f^i, f_X \times f_Y), (A^i, E_i^{triv}, G^\rho-triv), (A^{i+1}, E^{i+1}, G^{i+1}) \overset{\text{def}}{=} (11.37))
\end{align*}$$

$F_{\text{Bun},M-\text{atlas}}^\text{Bun}$ maps identity functions to identity functions:

$$\begin{align*}
F_{\text{Bun},M-\text{atlas}}^\text{Bun}(\text{Id}(A^i, B^i)) &= F_{\text{Bun},M-\text{atlas}}^\text{Bun}  \\
F_{\text{Bun},M-\text{atlas}}^\text{Bun}(\text{Id}(A^i, B^i)) &= (11.38)
\end{align*}$$

1. $m^2 \circ m^1 = ((f_0^2 \circ f_1^2, f_X^2 \circ f_Y^2, f_G^2 \circ f_G^2), (A^1, B^1), (A^3, B^3))$

2. $F_{\text{Bun},M-\text{atlas}}^\text{Bun}(A^i, B^i) = (A^i, E_i^{triv}, X^i, Y^i)_{G^\rho-triv}$

3. $F_{\text{Bun},M-\text{atlas}}^\text{Bun}(m^1) = (f_0^1, f_Y^1, (A^1, E_i^{triv}, X^i, Y^i)_{G^\rho-triv}, (A^{i+1}, E^{i+1}, G^{i+1}) \overset{\text{def}}{=} (11.38))$

4. $F_{\text{Bun},M-\text{atlas}}^\text{Bun}(m^2) \circ F_{\text{Bun},M-\text{atlas}}^\text{Bun}(m^1)$

$$\begin{align*}
F_{\text{Bun},M-\text{atlas}}^\text{Bun}(m^2) \circ F_{\text{Bun},M-\text{atlas}}^\text{Bun}(m^1) &= ((f_0^2 \circ f_0^1, f_Y^2 \circ f_Y^1), (A^1, E_1^{triv}, G^1_{G^\rho-triv}), (A^3, E^3, C^3) \overset{\text{def}}{=} (11.38))
\end{align*}$$
5. \( \mathcal{F}_{\text{Bun}, \text{M-atlas}}(m^2 \circ m^1) = (f_0^1 \circ f_0^1 \circ f_1^1, \beta_{\text{triv}}^1, \gamma_{\text{triv}}^1), (A^1, \beta_{\text{triv}}^1, \gamma_{\text{triv}}^1), (A^3, \beta_{\text{triv}}^3, \gamma_{\text{triv}}^3) \) 

\[ \square \]

**Lemma 11.47** (Base space functions derived from bundle-atlas morphisms). Let \( E^i, i = 1, 2, \) be a model space, \( X^i, Y^i \) topological spaces, \( G^i \) a topological group, \( \rho^i \) an effective action of \( G^i \) on \( Y^i \), \( C^i \) a \( G^i \)-\( \rho^i \) model space of \( X^i \times Y^i \), \( f_C: C^1 \longrightarrow C^2 \) a \( G^1 \)-\( \rho^1 \)-\( \rho^2 \) morphism of \( X^1 \times Y^1 \) to \( X^2 \times Y^2 \), i.e., a model function that preserves group action, \( A^i \) an \( m \)-atlas of \( E^i \) in the coordinate model space \( C^i \) and \( f \equiv (f_E: E^1 \longrightarrow E^2, f_C: C^1 \longrightarrow C^2) \) an \( E^1 \)-\( E^2 \) \( m \)-atlas morphism of \( A^1 \) to \( A^2 \) in the coordinate spaces \( C^1, C^2 \).

Then there exists a unique function \( f_X: X^1 \rightarrow X^2 \) such that \( f_X \circ \pi_1 = \pi_1 \circ f_C \).

**Proof.** Let \( x \in X^1, y, y' \in Y^1 \), Then \( f_C \) preserves fibers, i.e., \( \pi_1(f_C(x, y)) = \pi_1(f_C(x, y')) \). By lemma 11.9 (Morphisms of \( G \)-\( \rho \)-model spaces) on page 87 . Define \( f_X(x) \equiv \pi_1(f_C(x, y)) \).

\[ \square \]

**Remark 11.48.** There need not exist \( f_Y: Y^1 \rightarrow Y^2 \) such that \( f_C = f_X \times f_Y \).

**Definition 11.49** (Functor from \( m \)-atlases to Bundle atlases). Let \( \mathcal{E} \) be a trivial model category, \( \mathcal{C} \) be a trivial product coordinate model category with objects \( \{ C^\alpha = (X^\alpha \times Y^\alpha, X Y^\alpha) | \alpha \in \mathcal{A} \} \), \( G \) a group valued function on \( \text{Ob}(C) \) and \( \rho \) a function valued function on \( \text{Ob}(C) \) such that for every \( C^\alpha \in \mathcal{C}, \rho(C^\alpha) \) an effective action of \( G(C^\alpha) \) on \( Y^\alpha \) and \( X^\alpha \in G(C^\alpha) - \rho(C^\alpha) \) \( \text{-triv} \) \( C^\alpha \).

Let \( E^i \equiv (E^i, E^i) \in \mathcal{E}, i = 1, 2, \) be a trivial model space, \( C^i \equiv (X^i \times Y^i, X Y^i) \in \mathcal{C} \) \( G^i \equiv G(C^i), \rho^i = \rho(C^i), \) \( A^i \) a full \( m \)-atlas of \( E^i \) in the coordinate space \( C^i, \) \( \pi^i: E^i \rightarrow X^i \) the unique function asserted in lemma 11.39 and \( B^i \equiv (E^i, X^i, Y^i, G^i, \pi^i, \rho^i) \). Then define

\[ \mathcal{F}_{\text{M-atlas}-G, \rho, \text{Bun}}(A^i, E^i, C^i) \equiv (A^i, B^i) \]  

(11.39)

Let \( f \equiv (f_C: E^1 \longrightarrow E^2, f_C \equiv f_X \times f_Y: C^1 \longrightarrow C^2) \) be an \( E^1 \)-\( E^2 \) \( m \)-atlas morphism of \( A^1 \) to \( A^2 \) in the coordinate spaces \( C^1, C^2 \). Then the unique function asserted to exist in eq. (11.4) (preservation of group action) of definition 11.8 (Morphisms of \( G \)-\( \rho \)-model spaces) on page 87.

Then

\[ \mathcal{F}_{\text{M-atlas}-G, \rho, \text{Bun}}(f, (A^1, E^1, C^1), (A^2, E^2, C^2)) \equiv ((f_E, f_X, f_Y), (A^1, B^1), (A^2, B^2)) \]  

(11.40)
Theorem 11.50 (Functor from m-atlases to bundle atlases). Let \( \mathcal{C} \) be a trivial product coordinate model category, \( \mathcal{E} \) a model category, \( G \) a group valued function on \( \text{Ob}(\mathcal{C}) \) and \( \rho \) a function valued function on \( \text{Ob}(\mathcal{C}) \) such that for every \( \mathcal{C}^\alpha \overset{\text{def}}{=} (X^\alpha \times Y^\alpha, \lambda^\alpha) \in \mathcal{C}, \rho(\mathcal{C}^\alpha) \) is an effective action of \( G(\mathcal{C}^\alpha) \) on \( Y^\alpha \) and \( X^\alpha, Y^\alpha = \mathcal{C}^\alpha \).

\[ G(\mathcal{C}^\alpha)-\rho(\mathcal{C}^\alpha)\text{-triv} \]

Let \( \pi \) be the unique function valued function on \( \text{Atl}_{\text{Ob}}(\mathcal{E}, \mathcal{C}) \) such that for every \( (\mathcal{A}^\alpha, (E^\alpha, \mathcal{E}^\alpha), (\mathcal{C}^\alpha, \mathcal{C}^\alpha)) \in \text{Atl}_{\text{Ob}}(\mathcal{E}, \mathcal{C}) \) and \( (U, V, \phi) \in \mathcal{A}^\alpha, \pi_1 \circ \phi = \pi(\mathcal{A}^\alpha) |_U \) and let \( \mathcal{B}^\alpha \overset{\text{def}}{=} (E^\alpha, X^\alpha, Y^\alpha, G(\mathcal{C}^\alpha), \pi(\mathcal{A}^\alpha), \rho(\mathcal{C}^\alpha)) \)

\[ (\mathcal{A}^\alpha, (E^\alpha, \mathcal{E}^\alpha), (\mathcal{C}^\alpha \overset{\text{def}}{=} X^\alpha \times Y^\alpha, \mathcal{C}^\alpha)) \in \text{Atl}_{\text{Ob}}(\mathcal{E}, \mathcal{C}) \].

Then \( \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho} \) is a functor from \( \text{Atl}(E^\alpha) \xrightarrow{\text{Bun prod}} \text{Bun} \) to \( \text{Atl}_{\text{Bun}}(B) \).

Proof. Let \( \mathcal{O}^i \overset{\text{def}}{=} (A^i, E^i, C^i \overset{\text{def}}{=} (X^i \times Y^i, \lambda^i)) \in \mathcal{M}, i \in [1, 3], G^i \overset{\text{def}}{=} G(\mathcal{O}^i), \pi^i \overset{\text{def}}{=} \pi(\mathcal{O}^i), \rho^i \overset{\text{def}}{=} \rho(\mathcal{O}^i), A^i \) an m-atlas of \( E^i \) in the coordinate model space \( C^i \), \( B^i \overset{\text{def}}{=} (E^i, X^i, Y^i, G^i, \pi^i, \rho^i), m^i \overset{\text{def}}{=} (f^i, \mathcal{O}^i, \mathcal{O}^{i+1}) \in \mathcal{M}, i = 1, 2, \) an \( E^i \times E^{i+1} \) m-atlas morphism of \( A^i \) to \( A^{i+1} \) in the coordinate spaces \( C^i, C^{i+1} \) that preserves the group action, \( f_G: G^i \rightarrow G^{i+1} \) the unique function asserted to exist in eq. (11.4) (preservation of group action) of definition 11.8 (Morphisms of \( G-\rho \)-model spaces) on page 87 and \( \pi^i: E^i \rightarrow \Pi^i \) the unique function asserted in lemma 11.39.

Let \( f_G: G^i \rightarrow G^{i+1} \) be the function asserted by eq. (11.4) (\( G-\rho \) model spaces) on page 87; let \( f^i_1 = f^i_1 \times f^i_1 \) be the decomposition given by definition 11.12 (Trivial \( G-\rho \)-model spaces) on page 89.

\( \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho} \) is a morphism from \( \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i} \) to \( \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^{i+1}} \):

\[
\mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i} : \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho} \mathcal{O}^i \rightarrow \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho} \mathcal{O}^{i+1}.
\]

\[ \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i}(f^i_1) = \]

\[
\mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i}(f^i_1, \mathcal{O}^{i+1}) =
\]

\[
((f^i_1, f^i_1, f^i_1, f^i_1, f^i_1, f^i_1), (A^i, B^i, (A^{i+1}, B^{i+1})) =
\]

\[
((f^i_1, f^i_1, f^i_1, f^i_1, f^i_1, f^i_1), \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i}, \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^{i+1}})
\]

\( \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i} \) maps identity functions to identity functions:

\[
\mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i}(\text{Id}_{\mathcal{O}^i}) =
\]

\[
\mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i}((\text{Id}_{E^1}, \text{Id}_{\mathcal{E}^1}), \mathcal{O}^i, \mathcal{O}^i) =
\]

\[
((\text{Id}_{E^1}, \text{Id}_{X^1}, \text{Id}_{Y^1}, \text{Id}_{G^1}), (A^i, B^i), (A^i, B^i)) =
\]

\[
((\text{Id}_{E^1}, \text{Id}_{X^1}, \text{Id}_{Y^1}, \text{Id}_{G^1}), \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i}, \mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i}) =
\]

\[
\text{Id}_{\mathcal{F}^\text{Bun}_{\text{M-atlas}-G-\rho, \mathcal{O}^i}}.
\]

(11.42)
\[ \mathcal{F}_{\text{Bun}}^\text{M-atlas-}G-\rho, \text{Bun}(m^2) \circ \mathcal{F}_{\text{M-atlas-}G-\rho, \text{Bun}}(m^1) = \mathcal{F}_{\text{Bun}}^\text{M-atlas-}G-\rho, \text{Bun}(m^2 \circ m^1). \]

1. \( m^2 \circ m^1 = ((f_0^2 \circ f_0^1, f_1^2 \circ f_1^1), o^1, o^3) \)
2. \( \mathcal{F}_{\text{Bun}}^\text{M-atlas-}G-\rho, \text{Bun}(o^i) = (A^i, B^i) \)
3. \( \mathcal{F}_{\text{Bun}}^\text{M-atlas-}G-\rho, \text{Bun}(m^i) = ((f_0^1, f_1^1, f_2^1), (A^i, B^i), (A^{i+1}, B^{i+1})) \)
4. \( \mathcal{F}_{\text{Bun}}^\text{M-atlas-}G-\rho, \text{Bun}(m^2) \circ \mathcal{F}_{\text{Bun}}^\text{M-atlas-}G-\rho, \text{Bun}(m^1) = ((f_0^2 \circ f_0^1, f_2^2 \circ f_2^1, f_2^1 \circ f_1^1), (A^1, B^1), (A^3, B^3)) \)
5. \( \mathcal{F}_{\text{Bun}}^\text{M-atlas-}G-\rho, \text{Bun}(m^2 \circ m^1) = ((f_0^2 \circ f_0^1, f_2^2 \circ f_2^1, f_2^1 \circ f_1^1), (A^1, B^1), (A^3, B^3)) \)

\[ \mathcal{F}_{\text{Bun}, \text{M-atlas}} \circ \mathcal{F}_{\text{Bun}}^\text{M-atlas-}G-\rho, \text{Bun} = \text{Id}. \]

**Proof.** Expanding the definitions, we have

1. \( \mathcal{F}_{\text{Bun}}^\text{M-atlas-}G-\rho, \text{Bun}(A^i, E^i, C^i) = (A^i, B^i) \)
2. \( \mathcal{F}_{\text{Bun}, \text{M-atlas}}(A^i, B^i) = (A^i, E^i, X^i, Y^i) (A^i, E^i, C^i) \), since by definition 11.49, \( E^i \) and \( C^i \) are trivial.
3. \( \mathcal{F}_{\text{Bun}}^\text{M-atlas-}G-\rho, \text{Bun}(f_0^1, f_1^1, f_2^1), (A^1, E^1, C^1), (A^2, E^2, C^2) \) = \( ((f_0^1, f_1^1, f_2^1), (A^1, B^1), (A^2, B^2)) \)
4. \( \mathcal{F}_{\text{Bun}, \text{M-atlas}}(f_0^1, f_1^1, f_2^1), (A^1, B^1), (A^2, B^2) \) = \( ((f_0^1, f_1^1, f_2^1), (A^1, E^1, C^1), (A^2, E^2, C^2)) \) = \( ((f_0^1, f_1^1, f_2^1), (A^1, E^1, C^1), (A^2, E^2, C^2)) \), since by definition 11.49, \( E^i \) and \( C^i \) are trivial.

\[ \square \]

### 11.6 Associated model spaces and functors

**Definition 11.51** (Coordinate model spaces associated with bundle atlases). Let \( B^i \overset{\text{def}}{=} (E^i, X^i, Y^i, G^i, \pi^i, \rho^i) \), \( A^i \) be a \( \pi^i\text{-}G^i\text{-}\rho^i\)-bundle atlas of \( E^i \) in the coordinate space \( C^i = X^i \times Y^i \) and \( f \)

\[ \overset{\text{def}}{=} (f_E : E^1 \longrightarrow E^2, f_X : X^1 \longrightarrow X^2, f_Y : Y^1 \longrightarrow Y^2, f_G : G^1 \longrightarrow G^2) \] such that \( \text{isAtl}^\text{Bun}_{\text{Oh}}(A^1, E^i, X^i, Y^i, G^i, \pi^i, \rho^i) \) and \( \text{isAtl}^\text{Bun}_{\text{Ar}}(A^1, B^1, A^2, B^2, f) \). Then
Lemma 11.52 (Coordinate model spaces associated with bundle atlases). Let \( B \equiv (E, X, Y, G, \pi, \rho) \) and let \( A \) be a \( \pi - G - \rho \)-bundle atlas of \( E \) in the coordinate space \( C = X \times Y \). Then \( \mathcal{F}^\text{Bun}_2(A, B) \) is a model space.

Proof. \( \mathcal{F}^\text{Bun}_2(A, B) \) satisfies the conditions for a model space. for a model space. Let \( C \equiv \pi_2(\mathcal{F}^\text{Bun}_2(A, B)) \).

1. Since \( \pi_2(A) \) is an open cover of \( \bigcup \pi_2(A) \), the set of finite intersections is also an open cover.

2. Finite intersections of finite intersections are finite intersections

3. Restrictions of continuous function are continuous

4. If \( f : A \to B \) is a morphism of \( \mathcal{F}^\text{Bun}_2(A, B) \) \( A, A', B, B' \) objects of \( C \) \( A' \subseteq A \), \( B' \subseteq B \) and \( f[A'] \subseteq B' \) then since \( f : A \to B \) is a morphism it is a restriction of a transition function between its restrictions to sets in \( \pi_2(A) \) and its restrictions are also, hence morphisms, and thus \( f \mid_{A'} : A' \to B' \) is a morphism.

5. If \( (U, V, \phi) \in A \) then \( \text{Id}_V = \phi \circ \phi^{-1} A \) is a transition function and hence a morphism of \( \mathcal{F}^\text{Bun}_2(A, B) \). If \( A, A' \) objects of \( C \) and \( A' \subseteq A \) then the inclusion map \( i : A' \hookrightarrow A \) is a restriction of an identity morphism of \( \mathcal{F}^\text{Bun}_2(A, B) \) and hence a morphism.

6. Restricted sheaf condition: let
   (a) \( U_\alpha, V_\alpha, \alpha \prec A \), be objects of \( C \)
   (b) \( f_\alpha : U_\alpha \to V_\alpha \) be morphisms of \( C \)
   (c) \( U \equiv \bigcup_{\alpha \prec A} U_\alpha \)
   (d) \( V \equiv \bigcup_{\alpha \prec A} V_\alpha \)

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(e) \( f: U \rightarrow V \) be continuous and \( \left( \bigvee_{\alpha \in A} \bigwedge_{x \in U_{\alpha}} f(x) = f_{\alpha}(x) \right) \)

Then \( f \) is generated by the group action and hence a morphism of \( C \)

\[ \square \]

**Definition 11.53** (Model spaces associated with bundle atlas). Let \( B^i \overset{\text{def}}{=} (E^i, X^i, Y^i, G^i, \pi^i, \rho^i), \) \( i = 1, 2, \) and \( A^i \) be a \( \pi^i-G^i-\rho^i \)-bundle of \( E^i \) in the coordinate space \( C^i = X^i \times Y^i \). Then

\[
F_1^{\text{Bun}}(B^i, A^i) \overset{\text{def}}{=} \text{Mod}_{\text{min}}(C^i, \pi_1[A^i], \{ \phi^{-1} \circ \phi \mid \left( \bigvee_{(U, V, \phi) \in A^i} \left( U \cap U' \neq \emptyset \right) \right) \}) \quad (11.45)
\]

\[
F_1^{\text{Bun}}(f, (A^1, B^1), (A^2, B^2)) \overset{\text{def}}{=} f_E: F_1^{\text{Bun}}(A^1, B^1) \rightarrow F_1^{\text{Bun}}(A^2, B^2) \quad (11.46)
\]

The minimal model spaces with neighborhoods in the atlas \( A^i \) is \( F_1^{\text{Bun}}(E^i, X^i, Y^i, \pi^i, G^i, \rho^i, A^i) \).

The mapping associated with the \( B^1-B^2 \) bundle-atlas morphism \( f \) from \( A^1 \) to \( A^2 \) is \( f_E: F_1^{\text{Bun}}(A^1, B^1) \rightarrow F_1^{\text{Bun}}(A^2, B^2) \). If it is a model function then it is also the \( m \)-atlas morphism associated with the \( B^1-B^2 \) bundle-atlas morphism \( f \) from \( A^1 \) to \( A^2 \).

**Lemma 11.54** (Model spaces associated with bundle atlas). Let \( B \overset{\text{def}}{=} (E, X, Y, G, \pi, \rho) \) and let \( A \) be a \( \pi-G-\rho \)-bundle of \( E \) in the coordinate space \( C = X \times Y \). Then \( F_1^{\text{Bun}}(A, B) \) is a model space.

**Proof.** Lemma 5.14 (Minimal model spaces are model spaces) on page 18 \( \square \)

**Theorem 11.55** (Functors from bundle atlases to model spaces). Let \( B^\alpha \overset{\text{def}}{=} (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha), \) \( \alpha \prec A, \) be a protobundle, and \( B \overset{\text{def}}{=} \{ B^\alpha \mid \alpha \prec A \} \) be a set of protobundles. Then \( F_1^{\text{Bun}} \) is a functor from \( \text{Atl}^{\text{Bun}} B \) to \( B_{\text{triv}} \) and \( F_2^{\text{Bun}} \) is a functor from \( \text{Atl}^{\text{Bun}} B \) to \( B_{\text{Bun}-\text{triv}} \).

**Proof.** Let \( A^i \overset{\text{def}}{=} (A^i, E^i, C^i), \) \( i \in [1, 3], \) be objects in \( \text{Bun}(E, X, Y, \pi, G, \rho) \) and let \( m^i \overset{\text{def}}{=} (f^i_0, f^i_1), o^i, o^{i+1} \) be morphisms in \( \text{Atl}^{\text{Bun}}(E, C) \).

\[
F_1^{\text{Bun}}, \text{Atl}^{\text{Bun}} B \rightarrow B_{\text{triv}}:
\]

1. \( F(f: A \rightarrow B); F(A) \rightarrow F(B): F_1^{\text{Bun}}(m^i) = f^i_0; F_1^{\text{Bun}}(o^i) \rightarrow F_1^{\text{Bun}}(o^{i+1}) \)

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2. $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$:

\[
\mathcal{F}^\text{Bun}_1(m^2 \rtimes m^1) = \\
\mathcal{F}^\text{Bun}_1((f^2_0 \circ f^1_0, f^2 \circ f^1_1)(A^1, E^1, C^1), (A^3, E^3, C^3)) = \\
f^2_0 \circ f^1_0 : \mathcal{F}^\text{Bun}_1(o^1) \longrightarrow \mathcal{F}^\text{Bun}_1(o^3) = \\
(f^2_0, \mathcal{F}^\text{Bun}_1(o^2)) \circ (f^1_0, \mathcal{F}^\text{Bun}_1(o^1) \longrightarrow \mathcal{F}^\text{Bun}_1(o^2)) = \\
\mathcal{F}^\text{Bun}_1((f^2_0, f^2_1)(A^2, E^2, C^2), (A^3, E^3, C^3)) \circ \\
\mathcal{F}^\text{Bun}_1((f^2_0, f^2_1)(A^1, E^1, C^1), (A^2, E^2, C^2)) = \\
\mathcal{F}^\text{Bun}_1(m^2) \circ \mathcal{F}^\text{Bun}_1(m^1)
\]

3. $\mathcal{F}(\text{Id}_A) = \text{Id}_{\mathcal{F}(A)}$:

(a) $\mathcal{F}^\text{Bun}((\text{Id}_E, \text{Id}_C)) = \mathcal{F}^\text{Bun}((\text{Id}_E^i, \text{Id}_C^i), (A^i, E^i, C^i), (A^i, E^i, C^i)) = \\
\text{Id}_{\mathcal{F}^\text{Bun}(o^i)} : \mathcal{F}^\text{Bun}_1(o^i) \longrightarrow \mathcal{F}^\text{Bun}_1(o^i)$

(b) $\text{Id}_{\mathcal{F}^\text{Bun}(o^i)} = \text{Id}_{\mathcal{F}^\text{Bun}_1(o^i)} : \mathcal{F}^\text{Bun}_2(o^i) \longrightarrow \mathcal{F}^\text{Bun}_2(o^i)$

3. $\mathcal{F}(\text{Id}_A)$:

(a) $\mathcal{F}^\text{Bun}_2((\text{Id}_E, \text{Id}_C)) = \mathcal{F}^\text{Bun}_2((\text{Id}_E^i, \text{Id}_C^i), (A^i, E^i, C^i), (A^i, E^i, C^i)) = \\
\text{Id}_{\mathcal{F}^\text{Bun}_2(o^i)} : \mathcal{F}^\text{Bun}_2(o^i) \longrightarrow \mathcal{F}^\text{Bun}_2(o^i)$

(b) $\text{Id}_{\mathcal{F}^\text{Bun}_2(o^i)} = \text{Id}_{\mathcal{F}^\text{Bun}_2(o^i)} : \mathcal{F}^\text{Bun}_3(o^i) \longrightarrow \mathcal{F}^\text{Bun}_3(o^i)$

\[\square\]

11.7 Fiber bundles

Conventionally a fiber bundle is different from its atlases, but definition 11.43 (Categories of bundle atlases) on page 100 encourages treating them on an equal footing. All of the results for maximal bundle atlases carry directly over to results for fiber bundles.

**Definition 11.56** (fiber bundles). Let $E$, $X$ and $Y$ be topological spaces, $\pi: E \longrightarrow X$ surjective, $G$ a topological group, $\rho: Y \times G \longrightarrow Y$ an effective right
action of $G$ on $Y$ and $A$ a maximal $\pi$-$G$-$\rho$-bundle atlas of $E$ in the coordinate space $X \times Y$. Then $(E, X, Y, \pi, G, \rho, A)$ is a fiber bundle.

Let $B \overset{\text{def}}{=} \{ B^\alpha \overset{\text{def}}{=} (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \mid \alpha \prec \beta \}$, where $E^\alpha, X^\alpha, Y^\alpha$ are topological spaces, $G^\alpha$ a topological group, $\pi^\alpha : E^\alpha \to X^\alpha$ surjective and $\rho^\alpha : Y^\alpha \times G^\alpha \to Y^\alpha$ an effective right action of $G^\alpha$ on $Y^\alpha$. Then

$$\text{Bun}_{\text{Ob}} B \overset{\text{def}}{=} \text{At}_{\text{Ob}}^\text{Bun} B \quad (11.47)$$

**Definition 11.57** (Bundle maps). Let $B \overset{\text{def}}{=} \{ B^\alpha \overset{\text{def}}{=} (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \mid \alpha \prec \beta \}$, where $E^\alpha, X^\alpha, Y^\alpha$ are topological spaces, $G^\alpha$ a topological group, $\pi^\alpha : E^\alpha \to X^\alpha$ surjective and $\rho^\alpha : Y^\alpha \times G^\alpha \to Y^\alpha$ an effective right action of $G^\alpha$ on $Y^\alpha$. Then

$$\text{Bun}_{A^\beta} B \overset{\text{def}}{=} \text{At}_{A^\beta}^\text{Bun} B \quad (11.48)$$

$$\text{Bun} B \overset{\text{def}}{=} (\text{Bun}_{\text{Ob}} B, \text{Bun}_{A^\beta} B, \delta) \quad (11.49)$$

Let $(A^i, B^i) \in \text{Bun}_{\text{Ob}} B$, $i = 1, 2$. Then $f \overset{\text{def}}{=} (f_E : E^1 \to E^2, f_X : X^1 \to X^2, f_Y : Y^1 \to Y^2, f_G : G^1 \to G^2)$ is a bundle map from $(A^1, B^1)$ to $(A^2, B^2)$ if it is a bundle-atlas morphism from $A^1$ to $A^2$. The identity morphism for $(A^i, B^i)$ is

$$\text{Id}_{(A^i, B^i)} \overset{\text{def}}{=} ((\text{Id}_E, \text{Id}_X, \text{Id}_Y, \text{Id}_G), (A^i, B^i), (A^i, B^i)) \quad (11.50)$$

**Theorem 11.58** (Categories of fiber bundles). Let $B \overset{\text{def}}{=} \{ B^\alpha \overset{\text{def}}{=} (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \mid \alpha \prec \beta \}$, where $E^\alpha, X^\alpha, Y^\alpha$ are topological spaces, $G^\alpha$ a topological group, $\pi^\alpha : E^\alpha \to X^\alpha$ surjective and $\rho^\alpha : Y^\alpha \times G^\alpha \to Y^\alpha$ an effective right action of $G^\alpha$ on $Y^\alpha$. Then $\text{Bun} B$ is a category and $\text{Id}_{B^\alpha}$ is the identity morphism for $B^\alpha$.

**Proof.** The result follows directly from definition 11.56 (fiber bundles), definition 11.57 (Bundle maps) and lemma 11.44 ($\text{At}_{A^\beta}^\text{Bun} B$ is a category) on page 101.

**Definition 11.59** (Functor from fiber bundles to Local Coordinate Spaces). Let $E$, $X$ and $Y$ be sets of topological spaces, $G$ be a set of topological groups, $B^\alpha \overset{\text{def}}{=} (E^\alpha \in E, X^\alpha \in X, Y^\alpha \in Y, G^\alpha \in G, \pi^\alpha, \rho^\alpha)$, $\alpha \prec \beta$, be a protobundle, $B \overset{\text{def}}{=} \{ B^\alpha \mid \alpha \prec \beta \}$ be a set of protobundles, $G^\alpha \overset{\text{def}}{=} X^\alpha \times Y^\alpha, \mathcal{E}, \mathcal{X}, \mathcal{Y}$ and $\mathcal{G}$ be model categories, $\mathcal{X} \mathcal{Y} \mathcal{G} \rho$ be a $G$-$\rho$-model category, $\mathcal{M} \overset{\text{def}}{=} \left( \mathcal{E}, \mathcal{X} \mathcal{Y} \mathcal{G} \rho, \mathcal{X}, \mathcal{Y}, \mathcal{G} \right)$.

$$\mathcal{M} \overset{\text{def}}{=} \left( \begin{array}{c} E \\ \text{Triv} \end{array} \right) \text{Bun}_{\text{Triv}} \left( \begin{array}{c} X \\ \text{Triv} \end{array} \right) \left( \begin{array}{c} Y \\ \text{Triv} \end{array} \right) \left( \begin{array}{c} G \\ \text{Triv} \end{array} \right) \quad \text{full-cat}$$

$\quad \in \mathcal{M}, B^i \overset{\text{def}}{=} (E^i, X^i, Y^i, G^i, \pi^i, \rho^i)$.
\( B \ i = 1, 2 \), with group operation \(*_i\): \( G^i \times G^i \rightarrow G^i \), \( A^i \) a maximal \( \pi^i\)-\( G^i \) -\( \rho^i \)-bundle atlas of \( E^i \) in the coordinate space \( C^i_{\text{def}} \ = X^i \times Y^i, \pi^i_{\text{def}} \ = \pi_1 \colon X^i \times Y^i \rightarrow X^i, \pi^i_Y \ = \pi_2 \colon X^i \times Y^i \rightarrow Y^i \), \( F^i_{\text{def}} = (\pi^i, \pi^i_X, *_i, \rho^i) \), \( \mathcal{M}^i_{\text{def}} = \)

\[
\left( E^i, X^i, Y^i, G^i \right), \mathcal{M}^i_{\text{def}} = \left( E^i, X^i, Y^i, X^i, Y^i, G^i \right), \Sigma_{\text{def}} = \left( (0, 2), (1, 2), (1, 3), (4, 4, 4), (3, 4, 3) \right), L^i_{\text{def}} = \left( \mathcal{M}^i, M^i, A^i, F^i, \Sigma \right), L^{i, \mathcal{M}}_{\text{def}} = \]

\( \mathcal{F} \)\text{Bun}_{\text{Fib}, \text{LCS}}(A^i, B^i)_{\text{def}} = L^i \quad (11.51)\]

\[
\mathcal{F} \)\text{Bun}_{\text{Fib}, \text{LCS}}(f, (A^1, B^1), (A^2, B^2))_{\text{def}} = \left( f_\bullet E \rightarrow E^2, f_X : X^1 \rightarrow X^2, f_Y : Y^1 \rightarrow Y^2, f_G : G^1 \rightarrow G^2 \right) \text{a } B^1 \text{-} B^2 \text{ bundle-atlas morphism from } A^1 \text{ to } A^2. \text{ Then}
\]

\[
\mathcal{F} \)\text{Bun}_{\text{Fib}, \text{LCS}}(f, (A^1, B^1), (A^2, B^2))_{\text{def}} = \left( (f_\bullet E \times f_X \times f_Y, f_X, f_Y, f_G), L^1, L^2 \right) \quad (11.52)\]

\[
\mathcal{F} \)\text{Bun}_{\text{Fib}, \text{LCS}}(f, (A^1, B^1), (A^2, B^2))_{\text{def}} = \left( (f_\bullet E \times f_X \times f_Y, f_X, f_Y, f_G), L^{1, \mathcal{M}}, L^{2, \mathcal{M}} \right) \quad (11.53)\]

\[
\mathcal{F} \)\text{Bun}_{\text{Fib}, \text{LCS}}(f, (A^1, B^1), (A^2, B^2))_{\text{def}} = \left( (f_\bullet E \times f_X \times f_Y, f_X, f_Y, f_G), L^{1, \mathcal{M}}, L^{2, \mathcal{M}} \right) \quad (11.54)\]

\[
\text{LCS}_{\text{Ob}}^\mathcal{B} = \left\{ \mathcal{F} \)\text{Bun}_{\text{Fib}, \text{LCS}}(A, (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha)) \right\}
\]

\[
(E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \in \mathcal{B} \land \text{isAtl}_{\text{Ob}}^\mathcal{B}(A, E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \quad (11.55)\]

\[
\text{LCS}_{\text{Ar}}^\mathcal{B} = \left\{ \mathcal{F} \)\text{Bun}_{\text{Fib}, \text{LCS}}(f, (A^1, B^1), (A^2, B^2)) \right\}
\]

\[
B' \in \mathcal{B} \land \text{isAtl}_{\text{Ar}}^\mathcal{B}(A^1, B^1, A^2, B^2, f) \quad (11.56)\]

\[
\text{LCS}_{\text{Bun}}^\mathcal{B} = (\text{LCS}_{\text{Ob}}^\mathcal{B}, \text{LCS}_{\text{Ar}}^\mathcal{B}, A) \quad (11.57)\]
Theorem 11.60

Let $E$, $X$ and $Y$ be sets of topological spaces, $G$ be a set of topological groups, $B^\alpha \triangleq \{ E^\alpha \in E, X^\alpha \in X, Y^\alpha \in Y, G^\alpha \in G, \pi^\alpha, \rho^\alpha \}$, $\alpha \prec A$, be a protobundle, $B \triangleq \{ B^\alpha | \alpha \prec A \}$ be a set of protobundles, $C^\alpha \triangleq X^\alpha \times Y^\alpha, \mathcal{E}, \mathcal{X}, \mathcal{Y}, \mathcal{G}$ be model categories, $\mathcal{X}(\mathcal{Y}(\mathcal{G}(\rho, \Sigma)))$ be a $\mathcal{G}$-$\rho$-model category, $\mathcal{M} \triangleq (E, \mathcal{X}(\mathcal{Y}(\mathcal{G}(\rho, \Sigma))))$.

\[ (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha) \in B \land \pi^\alpha \text{ is a local } G^\alpha \text{-} \rho^\alpha \text{-atlas of } (A_i, B_i, A^2, B^2, f) \] (11.58)

\[ B' \in B \land \pi^\alpha \text{ is a local } G^\alpha \text{-} \rho^\alpha \text{-atlas of } (A_i, B_i, A^2, B^2, f) \] (11.59)

\[ \mathcal{M} \triangleq \{ \mathcal{M}(B_i \in B, LCS_{\text{max}}(B_i, B^2, f) \} \] (11.60)

**Theorem 11.60 (Functor from fiber bundles to Local Coordinate Spaces).** Let $E$, $X$ and $Y$ be sets of topological spaces, $G$ be a set of topological groups, $B^\alpha \triangleq \{ E^\alpha \in E, X^\alpha \in X, Y^\alpha \in Y, G^\alpha \in G, \pi^\alpha, \rho^\alpha \}$, $\alpha \prec A$, be a protobundle, $B \triangleq \{ B^\alpha | \alpha \prec A \}$ be a set of protobundles, $C^\alpha \triangleq X^\alpha \times Y^\alpha, \mathcal{E}, \mathcal{X}, \mathcal{Y}, \mathcal{G}$ be model categories, $\mathcal{X}(\mathcal{Y}(\mathcal{G}(\rho, \Sigma)))$ be a $\mathcal{G}$-$\rho$-model category, $\mathcal{M} \triangleq (E, \mathcal{X}(\mathcal{Y}(\mathcal{G}(\rho, \Sigma))))$.

Then:

1. $\mathcal{M}^i$ and $\mathcal{M}$ are sequences of categories by construction.

Proof. They satisfy the criteria in definition 9.1 (Local $* \rightarrow \Sigma$ coordinate spaces) on page 53:

- $\mathcal{M}^i$ is a local $\mathcal{M}^i \cdot \Sigma$ coordinate space and $L^i \mathcal{M}$ is a local $\mathcal{M} \cdot \Sigma$ coordinate space.
2. \( M^i \in \mathcal{M}^i_{\text{triv}} \subseteq \mathcal{M}^i_{\text{Sing}} \) full-c\text{-}\text{cat} \subseteq \mathcal{M}^i_{\text{triv}} \) by construction and \( \mathcal{M}^i_{\text{triv}} \subseteq \mathcal{M} \) by hypothesis.

3. \( \mathcal{E}^i_{\text{triv}} \) is a model category by definition 5.10 (Trivial model spaces) on page 17, \( X^i, Y^i \) is a model category by definition 11.12 (Trivial \( G^i \)-\( \rho^i \)-model spaces) on page 89, \( \mathcal{E} \) is a model category by hypothesis and \( \mathcal{X} \mathcal{Y} \mathcal{G}^i \rho \) is a model category by definition 11.11 (Categories of \( G^i \)-\( \rho^i \)-model spaces) on page 89.

4. \( \left( \mathcal{M}^i_{\text{triv}}, \mathcal{M}^i, \Sigma, F^i \right) \) and \( \left( \mathcal{M}, \mathcal{M}^i, \Sigma, F^i \right) \) are prestructures: \( F^i \) has \( \mathcal{M}^i_{\text{triv}} \)-signatures \( \Sigma \) and \( F^i \) has \( \mathcal{M} \)-signatures \( \Sigma \).

5. \( A^i \) is a maximal m-atlas of \( E^i \) in \( G^i \)-\( \rho^i \)-triv \( \mathcal{X} \mathcal{Y} \mathcal{G}^i \rho \) by lemma 11.41 (Bundle-atlas morphisms) on page 99.

6. There are no constraint functions.

\( \text{LCS}^\text{Bun}_B \) and \( \text{LCS}^\text{Bun, M}_B \) are categories and the identity morphism for \( L^i \) is \( \text{Id}_{L^i} \) \( \text{def} = \left( \text{Id}_{E^i}, \text{Id}_{X^i}, \text{Id}_{X^i \times Y^i}, \text{Id}_{Y^i}, \text{Id}_{G_X}, L^i, L^i \right) \).

**Proof.** \( \text{LCS}^\text{Bun}_B \) and \( \text{LCS}^\text{Bun, M}_B \) satisfy the definition of a category:

1. Composition:
   \( f^2 \circ f^1 \) is a bundle map from \( (A^1, B^1) \) to \( (A^1, B^1) \) by corollary 11.42 (Bundle-atlas morphisms) on page 100. Then
   \[ \mathcal{F}^\text{Bun, LCS} \left( f^2 \circ f^1, (A^2, B^2), (A^3, B^3) \right) \circ \mathcal{F}^\text{Bun, LCS} \left( f^1, (A^1, B^1), (A^2, B^2) \right) = \mathcal{F}^\text{Bun, LCS} \left( f^2 \circ f^1, (A^1, B^1), (A^2, B^2) \right) \] and
   \[ \mathcal{F}^\text{Bun, M} \left( f^2 \circ f^1, (A^1, B^1), (A^2, B^2) \right) \circ \mathcal{F}^\text{Bun, LCS} \left( f^1, (A^1, B^1), (A^2, B^2) \right) = \mathcal{F}^\text{Bun, LCS} \left( f^2 \circ f^1, (A^1, B^1), (A^2, B^2) \right). \]

2. Associativity:
   Composition is associative by lemma 3.17 (Tuple composition for labeled morphisms) on page 10.

3. Unit:
   \( \text{Id}_{L^i} \) is an identity morphism by lemma 3.17 (Tuple composition for labeled morphisms) on page 10.

\( \square \)
\( \mathcal{F}_{\text{Bun}, \text{LCS}}^\text{Bun} \left( f^i, (A^i, B^i), (A^{i+1}, B^{i+1}) \right) \) is a morphism from LCS\(^{\text{Bun}}\) \( B^i \) to LCS\(^{\text{Bun}}\) \( B^{i+1} \) and \( \mathcal{F}_{\text{Fib}, \text{LCS}}^\text{Bun} \left( f^i, (A^i, B^i), (A^{i+1}, B^{i+1}) \right) \) is a strict morphism from LCS\(^{\text{Bun}}\) \( M^i B^i \) to LCS\(^{\text{Bun}}\) \( M^{i+1} B^{i+1} \).

**Proof.** It satisfies the conditions of definition 9.6 (morphisms of local coordinate spaces) on page 54:

1. Prestructure morphism:
   \( f^1 \Sigma \) commutes with \( F^1, F^2 \).

2. m-atlas morphism:
   \( (\hat{f}_0, \hat{f}_1) \) is an m-atlas morphism from \( (E^1, C^1) \) to \( (E^2, C^2) \) by lemma 11.41 (Bundle-atlas morphisms) on page 99.

\( \mathcal{F}_{\text{Bun}, \text{LCS}}^\text{Bun} \) is a functor from \( \text{Bun} B \) to LCS\(^{\text{Bun}}\) \( B \) and \( \mathcal{F}_{\text{Fib}, \text{LCS}}^\text{Bun} \) is a functor from \( \text{Bun} B \) to LCS\(^{\text{Bun}}\) \( M B \).

**Proof.** \( \mathcal{F}_{\text{Bun}, \text{LCS}}^\text{Bun} \) and \( \mathcal{F}_{\text{Fib}, \text{LCS}}^\text{Bun} \) satisfy the definition of a functor:

1. F(f: A to B) = F(f): F(A) to F(B):
   - \( \mathcal{F}_{\text{Bun}, \text{LCS}}^\text{Bun} \left( f^i(A^i, B^i), (A^{i+1}, B^{i+1}) \right) = \left( (f^i_E, f^i_X, f^i_Y, f^i, f^i_Y, f^i), L^i, L^{i+1} \right) \)
   - \( \mathcal{F}_{\text{Fib}, \text{LCS}}^\text{Bun} \left( f^i(A^i, B^i), (A^{i+1}, B^{i+1}) \right) = \left( (f^i_E, f^i_X, f^i_Y, f^i, f^i_Y, f^i), L^i, L^{i+1}, M \right) \)

2. Composition: F(f g) = F(f) F(g)
   This follows from the proof above that LCS\(^{\text{Bun}}\) \( B \) and LCS\(^{\text{Bun}}\) \( M B \) are categories.

3. Identity:
   \( \mathcal{F}_{\text{Bun}, \text{LCS}}^\text{Bun}(\text{Id}(A^i, B^i)) = \text{Id} \)
   \( \mathcal{F}_{\text{Fib}, \text{LCS}}^\text{Bun}(\text{Id}(A^i, B^i)) = \text{Id} \)
   \( \mathcal{F}_{\text{Bun}, \text{LCS}}^\text{M}(\text{Id}(A^i, B^i)) = \text{Id} \)
   \( \mathcal{F}_{\text{Fib}, \text{LCS}}^\text{M}(\text{Id}(A^i, B^i)) = \text{Id} \)

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Definition 11.61 (Functor from local coordinate spaces to fiber bundles). Let \( \mathcal{E} \) be a model category, \( \mathcal{XYYG} \rho \) a \( G^\rho \)-model category, \( \mathcal{X}, \mathcal{Y} \) and \( \mathcal{G} \) topological categories, \( \mathcal{M} \) a sequence of categories, \( M^i, i = 1, 2, \) a sequence, \( E^i_{\text{Ob}} \in \mathcal{E}, \)
\[
C^i = (C^i, C^i)_{\text{Ob}} \in \mathcal{XYYG} \rho, X^i_{\text{Ob}} \in \mathcal{X}, Y^i_{\text{Ob}} \in \mathcal{Y}, G^i_{\text{Ob}} \in \mathcal{G}, \pi^i_1: E^i \rightarrow X^i, \pi^i_2: C^i \rightarrow X^i,\]
\[
\pi^i_Y: C^i \rightarrow Y^i, \times: G^i \times G^i \rightarrow G^i, \rho^i: Y^i \times G^i \rightarrow Y^i, \mathcal{M}^i \] a sequence of categories,

\[
L^i_{\text{def}} = \left( \mathcal{M}^i, M^i, A^i, F^i, \Sigma \right) \text{ and } L^i, M^i_{\text{def}} = (\mathcal{M}^i, M^i, A^i, F^i, \Sigma) \text{ local coordinate spaces, } B^i_{\text{def}} = (E^i, X^i, Y^i, G^i, \pi^i, \rho^i) \text{ and } f = (f_\beta, \beta \prec \text{length}(M^i)) \text{ a morphism from } L^1 \text{ to } L^2, \text{ satisfying}
\]
1. head(\( M^5, 5 \)) = \( (\mathcal{E}, \mathcal{XYYG}, \mathcal{X}, \mathcal{Y}, \mathcal{G}) \)
2. head(\( \mathcal{M}^5, 5 \)) = \( (E^i_{\text{triv}}, X^i_{\text{Sing}}, Y^i_{\text{Sing}}, G^i_{\text{Sing}}, \Sigma) \)
3. head(\( M^5, 5 \)) = \( (E^i, C^i, X^i, Y^i, G^i) \)
4. \( M^i \in M \)
5. head(\( F^5, 5 \)) = \( (\pi^i_1, \pi^i_2, \pi^i_Y, \rho^i) \)
6. \( C^i = X^i \times Y^i \)
7. \( G^i \) is a topological group with group operation \( \ast^i \). Subsequent references to \( G^i \) should be read as referring to the group rather than just the underlying topological space.
8. \( \pi^i \) is surjective.
9. \( \pi^i_X = \pi_1: X^i \times Y^i \rightarrow X^i \).
10. \( \pi^i_Y = \pi_2: X^i \times Y^i \rightarrow Y^i \).
11. \( \rho^i \) is an effective right action of \( G^i \) on \( Y^i \).
12. head(\( \Sigma, 5 \)) = \( ((0, 2), (1, 2), (1, 3), (4, 4, 4), (3, 4, 3)) \)

Remark 11.62. Due to the commutation requirement, specifying \( \mathcal{M}_4 = \mathcal{G}_{\text{group}-\text{triv}} \) is not necessary in order to ensure that \( f_G \) is a homomorphism.

Then
\[
\mathcal{F}^{\text{Bun}}_{\text{LCS, Fib}} L^i_{\text{def}} = (E^i, X^i, Y^i, \pi^i, G^i, \rho^i, A^i) \tag{11.61}
\]
\[
\mathcal{F}^{\text{Bun}}_{\text{LCS, Fib}} (f, (A^1, B^1), (A^2, B^2))_{\text{def}} = (f_0, f_2, f_3, f_4), \mathcal{F}^{\text{Bun}}_{\text{LCS, Fib}} L^1, \mathcal{F}^{\text{Bun}}_{\text{LCS, Fib}} L^2 \tag{11.62}
\]
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Theorem 11.63 (Functor from local coordinate spaces to fiber bundles). Let $\mathcal{E}$ be a model category, $\mathcal{X}\mathcal{Y}\mathcal{G}\rho$ a $\mathcal{G}$-$\rho$-model category, $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{G}$ topological categories, $\mathcal{M}$ a sequence of categories, $E^\alpha \equiv (E^\alpha, \mathcal{E}^\alpha) \in \mathcal{E}$, $\alpha \prec A$, $C^\alpha \equiv (C^\alpha, \mathcal{C}^\alpha) \in \mathcal{C}$, $X^\alpha \in \mathcal{X}$, $Y^\alpha \in \mathcal{Y}$, $G^\alpha \in \mathcal{G}$ a topological group with group operation $*^\alpha$, $\pi^\alpha: E^\alpha \rightarrow X^\alpha$ surjective and $\pi_X^\alpha: C^\alpha \rightarrow X^\alpha$, $\pi_Y^\alpha: C^\alpha \rightarrow Y^\alpha$, $*^\alpha: G^\alpha \times G^\alpha \rightarrow G^\alpha$, $\rho^\alpha: Y^\alpha \times G^\alpha \rightarrow Y^\alpha$ is an effective right action of $G^\alpha$ on $Y^\alpha$, $\mathcal{M}^\alpha$ a sequence of categories, $M^\alpha$ a sequence, $L^\alpha \equiv \left( \mathcal{M}^\alpha, M^\alpha, A^\alpha, F^\alpha, \Sigma \right)$ and $L^\alpha, M^\alpha \equiv \left( \mathcal{M}^\alpha, M^\alpha, A^\alpha, F^\alpha, \Sigma \right)$ local coordinate spaces, $B^\alpha \equiv (E^\alpha, X^\alpha, Y^\alpha, G^\alpha, \pi^\alpha, \rho^\alpha)$ and $B \equiv \{ B^\alpha | \alpha \prec A \}$, satisfying

1. $\text{head}(\mathcal{M}, 5) = (E, \mathcal{X}\mathcal{Y}\mathcal{G}\rho, \mathcal{X}, \mathcal{Y}, \mathcal{G})$

2. $\text{head}(\mathcal{M}^\alpha, 5) = \left( \mathcal{M}^\alpha, M^\alpha, A^\alpha, F^\alpha, \Sigma \right)$

3. $\text{head}(M^\alpha, 5) = (E^\alpha, C^\alpha, X^\alpha, Y^\alpha, G^\alpha)$

4. $M^\alpha \in \mathcal{M}$

5. $\text{head}(F^\alpha, 5) = (\pi^\alpha, \pi_X^\alpha, \pi_Y^\alpha, \star^\alpha, \rho^\alpha)$

6. $C^\alpha = X^\alpha \times Y^\alpha$

7. $G^\alpha$ is a topological group with group operation $*^\alpha$. Subsequent references to $G^\alpha$ should be read as referring to the group rather than just the underlying topological space.

8. $\pi^\alpha$ is surjective.

9. $\pi_X^\alpha = \pi_1: X^\alpha \times Y^\alpha \rightarrow X^\alpha$.

10. $\pi_Y^\alpha = \pi_2: X^\alpha \times Y^\alpha \rightarrow Y^\alpha$.

11. $\rho^\alpha$ is an effective right action of $G^\alpha$ on $Y^\alpha$.

12. $\text{head}(\Sigma, 5) = ((0, 2), (1, 2), (1, 3), (4, 4, 4), (3, 4, 3))$

Let $\alpha^i \prec A$, $i = 1, 2$, $M^i \equiv M^\alpha_i$, $E^i = (E^\alpha, E^\alpha_i) \equiv E^\alpha_i$, $C^i = (C^\alpha, C^\alpha_i) \equiv C^\alpha_i$, $X^i = X^\alpha_i$, $Y^i = Y^\alpha_i$, $G^i = G^\alpha_i$, $\pi^i = \pi^\alpha_i$, $\pi_X^i = \pi_X^\alpha_i$, $\pi_Y^i = \pi_Y^\alpha_i$, $*^i \equiv *^\alpha_i$, $\rho^i = \rho^\alpha_i$, $M^i \equiv M^\alpha_i$, $L^i \equiv L^\alpha_i$, $L^1.\mathcal{M}^i = L^\alpha_i.\mathcal{M}^\alpha_i$, $B^i = B^\alpha_i$, $f^i \equiv f^\alpha_i$

$(f^i_\beta, \beta \prec \text{length}(M^i))$ a morphism from $L^1$ to $L^2$, $f_E \equiv f_0$, $f_C \equiv f_1$, $f_X \equiv f_2$, $f_Y \equiv f_3$ and $f_G \equiv f_4$. Then:

$\mathcal{F}_{\text{LCS,Fib}} L^1$ and $\mathcal{F}_{\text{LCS,Fib}} L^\alpha_i.\mathcal{M}^i$ are fiber bundles.
Proof. \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}} \) \( L^i \) \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}} \) \( L^i,M \) satisfy the conditions of definition 11.56 (fiber bundles) on page 108 by hypothesis. \( f_c = f_X \times f_Y \)

Proof. Since \( f \) is a morphism from \( L^1 \) to \( L^2 \), it \( \Sigma \) commutes with \( F^1, F^2 \), \( \pi_1 \circ f_c = \pi_X^\Sigma \circ f_c = f_X \circ \pi_1 \) and \( \pi_2 \circ f_c = \pi_Y^\Sigma \circ f_c = f_Y \circ \pi_2 \). \( \square \)

\( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}}(f, L^1, L^2) \) and \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}}(f, L^1,M, L^2,M) \) are bundle maps.

Proof. \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}}(f, L^1, L^2) \) and \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}}(f, L^1,M, L^2,M) \) satisfy the conditions of definition 11.40 (Bundle-atlas morphisms) on page 98.

1. \( f_E, f_X, f_Y \) and \( f_G \) are continuous
2. \( f_G \) is a homomorphism due to commutation
3. \( f \) commutes with \( \pi^i \) and \( \rho^i \)
4. for any \((U^1, V^1, \phi^1; U^1 \xrightarrow{\sim} V^1) \in A^1, (U^2, V^2, \phi^2; U^2 \xrightarrow{\sim} V^2) \in A^2 \), the diagram \( \{f_0, \phi^2, \phi^1, f_1\} \) is locally nearly commutative in \( X, Y, \pi, \rho \) because \( X \mathcal{Y} \mathcal{G} \rho \) is a \( G, \rho \)-model category by hypothesis, \( X^\alpha, Y^\alpha \) is a \( G, \rho \)-model category by construction and \( (f_0, f_1) \) is an \( m \)-atlas morphism from \( A^1 \) to \( A^2 \), \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}} f \) satisfies the conditions of definition 11.57 (Bundle maps) on page 109.

\( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}} \) is a functor from \( \text{LCS}^{\text{Bun}} B \) to \( \text{Bun} B \) and is a functor from \( \text{LCS}^{\text{Bun},M} B \) to \( \text{Bun} B \).

Proof. \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}} \) satisfies the definition of a functor:

1. \( F(f: A \to B) = F(f): F(A) \to F(B) \):
   \( \text{a}) \) \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}}(f_1, L^1, L^2) = (\{U^1_B, f_1^U, f_Y^1, f_Y^2\}, B^1, B^2) \)
   \( \text{b}) \) \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}}(M^1, M^i, A^i, F^i, \Sigma) = B^i \)
2. Composition:
   \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}}\left(\left(f^2 \circ f^1, L^1, L^2\right)\right) = \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}}\left(\left(f^2 \circ f^1, L^1, L^2\right)\right) = (\{f_2^\Sigma \circ f_Y^2 \circ f_Y^1, f_Y^2 \circ f_Y^1, f_Y^2 \circ f_Y^1\}, B^1, B^2) \)
   \( \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}}(f^2, L^2, L^3) \circ \mathcal{F}^{\text{Bun}}_{\text{LCS,Fib}}(f^1, L^1, L^2) \)

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3. Identity:
\[ F_{\text{LCS}, \text{Fib}}^\text{Bun}(\text{Id}_L) = F_{\text{LCS}, \text{Fib}}^\text{Bun}(\text{Id}_L^i, L^i, L^i) = \]
\[ (\text{Id}_E^i, \text{Id}_X^i, \text{Id}_Y^i, \text{Id}_G^i, (A^i, B^i), (A^i, B^i)) = \text{Id}_{(A^i, B^i)} = \text{Id}_{F_{\text{LCS}, \text{Fib}}^\text{Bun} L^i} \]

The proof does not depend on the category, so it applies to LCS$^\text{Bun, M}$B as well.

\[ F_{\text{LCS}, \text{Fib}}^\text{Bun} \circ F_{\text{Fib}, \text{LCS}}^\text{Bun} = \text{Id} \text{ and } F_{\text{LCS}, \text{Fib}}^\text{Bun} \circ F_{\text{Fib}, \text{LCS}}^\text{Bun, M} = \text{Id}. \]

**Proof.** Expanding the definitions, we have

1. \[ F_{\text{Fib, LCS}}^\text{Bun, M}(A^i, B^i) = L^i \]
2. \[ F_{\text{LCS, Fib}}^\text{Bun} L^i = (A^i, B^i) \]
3. \[ F_{\text{Fib, LCS}}^\text{Bun}((f_E^i, f_X^i, f_Y^i, f_G^i), (A^1, B^1), (A^2, B^2)) = \]
\[ ((f_E^i, f_X^i, f_Y^i, f_G^i), L^1, L^2) \]
4. \[ F_{\text{Fib, LCS}}^\text{Bun}((f_E^i, f_X^i, f_Y^i, f_G^i), L^1, L^2) = \]
\[ ((f_E^i, f_X^i, f_Y^i, f_G^i), (A^1, B^1), (A^2, B^2)) \]

The proof does not depend on the category, so it applies to LCS$^\text{Bun, M}$B as well.

\[ F_{\text{Fib, LCS}}^\text{Bun} \circ F_{\text{Fib, LCS}}^\text{Bun} \text{ is the identity functor on LCS}^\text{Bun} \text{B and } F_{\text{Fib, LCS}}^\text{Bun, M} \circ F_{\text{Fib, LCS}}^\text{Bun} \text{ is the identity functor on LCS}^\text{Bun, M}. \]

**Proof.** Expanding the definitions, we have

1. \[ F_{\text{LCS, Fib}}^\text{Bun} L^i = (A^i, B^i) \]
2. \[ F_{\text{Fib, LCS}}^\text{Bun}(A^i, B^i) = L^i \]
3. \[ F_{\text{Fib, LCS}}^\text{Bun}((f_E^1, f_X^1, f_Y^1, f_G^1), (A^1, B^1), (A^2, B^2)) = \]
\[ ((f_E^1, f_X^1, f_Y^1, f_G^1), L^1, L^2) \]
4. \[ F_{\text{Fib, LCS}}^\text{Bun}((f_E^1, f_X^1, f_Y^1, f_G^1), L^1, L^2) = \]
\[ ((f_E^1, f_X^1, f_Y^1, f_G^1), (A^1, B^1), (A^2, B^2)) \]

The same proof applies for $F_{\text{Fib, LCS}}^\text{Bun, M}$ with $L^i, M$ in place of $L^i$.  

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12 Future directions

If this paradigm proves useful, it can be extended to include a set of admissible functions on the model neighborhoods of the charts, possibly using the language of sheaves. That might be desirable for coordinate spaces more general than Fréchet spaces.

Further work is needed to determine whether it is productive to allow a local coordinate space to have more than one atlas, e.g., for more than one bundle structure on the same base space.

The extension of paracompactness to model spaces is intended to be useful for partitions of unity on fiber bundles. Further work is needed to determine whether that is actually the case.

The definitions given here include some fairly strong conditions, e.g., AOC. Further work is needed to determine whether they should be relaxed for applications beyond manifolds and fiber bundles.

Further work is needed to determine whether the concepts of category-based atlases\textsuperscript{19} of model spaces and prestructures have general utility.

If the concept of nearly commutative diagrams proves useful, further work is needed to determine whether a more general definition has utility.

Further work is needed to devise a definition of constraints that expresses global properties, e.g., compactness, and is both clear and rigorous.

Further work is needed to determine conditions for mappings associated with atlas morphisms to be model functions.

This paper uses the language of category theory as an organizing principle, but defines various notions concretely with sets. It may be desirable to abstract away some of the details, in the spirit of, e.g., topoi.

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\textsuperscript{19} As opposed to pseudogroup based