Electron spin correlations: a geometric representation

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Abstract

An analysis is made within the quantum formalism of the probabilistic features of the electron spin correlation, with the purpose of clarifying the concepts of contextuality and measurement dependence. The quantum formulas for the spin correlation are then derived on the basis of a probability distribution function and its associated geometric representation, both for a pair of projections of a single-particle spin and for the bipartite system in singlet spin state. This endows the spin correlation with a clear probabilistic meaning and leaves the door open for a possible physical image of the electron spin, as discussed at the end of the paper.

1 Introduction

The question of whether the mathematical formalism of quantum mechanics implies necessarily a different kind of probabilities from those used in classical statistical mechanics, is a subject of continuing debate. Clarification of the issue is not only of fundamental importance for a better understanding of quantum theory and a demistification of the quantum phenomenon, including issues such as nonlocality, acausality or the absence of realism; it also has important implications for the development and extension of probability theory with a view to its applications in other areas, as complex and diverse as epidemiology, finances, game theory and cognitive science (see, e. g., [1], [2] and references therein).

The present paper is an attempt to contribute to this clarification, by addressing the question: is there a quantum probability that is unique in the sense that it does not apply outside of quantum systems? In other words, is the unusualness of the quantum formalism rooted in its probabilistic framework, and does it imply the need to renounce basic principles that hold for the rest of physics? For this purpose we focus on the electron spin correlation, and make an analysis of the various probabilistic features provided by the quantum formalism.
Two conceptual elements that are shown to play a central role in the present analysis are the context, and the conditional probabilities. A distinction is made between the notion of context used here to refer to the measurement that is carried out—i.e., what is being measured—as opposed to the notion of contextuality frequently used in quantum measurement theory to refer to the result of a measurement being dependent on which other quantity has been measured. By the same token, conditional probabilities as discussed here are probabilities conditioned by the context. Such context-conditioning is connected with the specific partitioning of the probability space, as has been shown in previous work [3].

The electron spin has the advantage of being amenable to a geometric representation and an associated description in terms of a probability distribution function involving random variables, or what is usually called a hidden-variable description. This allows us to reproduce the probabilistic features [4] and derive the quantum result for the single spin and the bipartite singlet spin correlation. That both cases can be dealt with following a similar approach is due to the use of conditional probabilities in calculating the respective correlations. Further to endowing the probabilities with a concrete meaning, the result leaves open the possibility of an understanding of the physics that underlies the quantum description. A proposal in this regard is advanced at the end of the present paper, in the light of recent experimental evidence pointing to a finer dynamics of the spinning electron, which requires further investigation.

The present paper is structured as follows. Section 2 starts with the introduction of an algebraic representation of the spin projection probabilities for the single-spin case, which serves to discuss the notions of contextuality and conditioned probabilities. This representation is shown briefly to reproduce the basic probabilistic properties predicted by the quantum formalism for the electron spin correlation. A central feature of this algebraic approach is the clear separation of the context (what is being measured) from the state of the system (in which it is measured). Section 3 focuses on the bipartite singlet spin state. The quantum description of the spin correlation is shown to imply a context-dependent disaggregation of the probability space into mutually exclusive subspaces. In section 4, a probability distribution function \( \rho(\phi) \) is introduced that reproduces the quantum probabilistic results. This distribution function is shown in section 5 to be amenable to a geometric representation that gives meaning to the random variable \( \phi \). The paper concludes with a discussion on the possibility of a physical image for the electron spin.

2 The spin-1/2 particle

2.1 Analysis of contextuality

In a recent article [5], Grangier introduces a “principle of contextual quantization”, meaning that whatever the context, a measurement on a given system gives one modality among \( N \) possible ones, where the value of \( N \) characterizes
the system. These $N$ modalities are mutually exclusive, i.e., only one can be realized at a time.

Thus for example the projection of an electron spin along an arbitrary direction $a$ gives either +1 or -1. Since ± are the only possible outcomes, $N = 2$. Assume first that the result of the projection along $a$ is +1; if the spin is measured again along $a$, the result +1 is again obtained. If however the projection is measured along a different direction $b$, one gets −1 with a certain probability. This can be expressed by means of a $2 \times 2$ matrix of probabilities that depends on the two directions $a$ and $b$, the rows referring to the possible signs of $a$ and the columns to those of $b$:

$$P(b, a) = \begin{pmatrix} P_{ab}(+ | +) & P_{ab}(− | +) \\ P_{ab}(+ | −) & P_{ab}(− | −) \end{pmatrix},$$

(1)

with $P_{ab}(b | a)$ the probability of $b$ conditioned by the value of $a$. Thus for instance, $P_{ab}(− | +)$ is the probability that, given a +1 projection along $a$, the projection along $b$ is −1. Clearly, since the projection along $b$ must be either +1 or −1,

$$P_{ab}(+ | +) + P_{ab}(− | +) = 1,$$

(2a)

and

$$P_{ab}(+ | −) + P_{ab}(− | −) = 1.$$

(2b)

The probabilities are invariant under an inversion of the sense of the directions $a$ and $b$ that interchanges all the plus and minus signs

$$P_{ab}(+ | +) = P_{ab}(− | −),$$

(3a)

and

$$P_{ab}(+ | −) = P_{ab}(− | +).$$

(3b)

The matrix $P(a, b)$ is therefore symmetric, i.e., $P(a, b) = P(b, a)$, whence $a$ and $b$ may be freely interchanged. Moreover, it is doubly stochastic [2], because both the rows and the columns add to 1.

Notice that the matrix coefficients represent conditional probabilities, the upper ones referring to the (+ or -) projections along $b$ conditioned by the +1 projection along $a$ and the lower ones by the −1 projection along $a$. The corresponding joint probabilities are given by expressions of the form [6]

$$P_{ab}(++) = P_{a}(+)P_{ab}(+ | +), \quad P_{ab}(−+) = P_{a}(+)P_{ab}(− | +),$$

(4)

where $P_{a}(+)$ is the probability of the projection along $a$ being +1, and similarly for the lower pair. Thus the conditional probabilities $P_{ab}(+ | +)$, $P_{ab}(− | +)$ restrict the sample space to the situation in which the projection along $a$ is +1, and similarly for $P_{ab}(− | −)$, $P_{ab}(+ | −)$. This will be important for the discussion in Section 5. The total probability is the sum of the respective joint probabilities; thus for instance

$$P_{b}(+) = P_{a}(+)P_{ab}(+ | +) + P_{a}(−)P_{ab}(+ | −).$$

(5)
Clearly,
\[ P_a(+) + P_a(-) = 1, \quad P_b(+) + P_b(-) = 1. \]  
(6)

The correlation of the projections is given by the formula
\[ C(a,b) = \frac{P_{ab}(++) + P_{ab}(--) - P_{ab}(+-) - P_{ab}(-+)}{P_{ab}(++) + P_{ab}(--) + P_{ab}(+-) + P_{ab}(-+)}. \]  
(7)

On account of Eqs. (2)-(6), the sum of the joint probabilities in the denominator gives 1, and Eq. (7) simplifies into
\[ C(a,b) = P_{ab}(+|+) - P_{ab}(-|-) = P_{ab}(-|-) - P_{ab}(+|--). \]  
(8)

Notice that, by involving the conditional probabilities only, this result is independent of the total probabilities \( P_a(\pm) \), \( P_b(\pm) \). This is an important feature of the matrix of probabilities, as it means that it applies to any joint measurement along \( a \) and \( b \) as described above, regardless of the spin state, i.e., of the preparation of the spin to be measured. Briefly, one may say that \( P(a,b) \) refers to the contextuality of the measurements, viz the arrangement of the measuring devices, in line with the meaning of the term 'context' used in Refs. [2], [5].

### 2.2 Spin projection probabilities

To calculate the conditional probabilities for the single spin case, we use the standard expressions for the bases of spin state vectors along two arbitrary directions \( a \) and \( b \) lying on the same vertical plane and forming angles \( \theta_a \) and \( \theta_b \), respectively, with the \( z \) axis. In terms of \( \vartheta_{a,b} \equiv \theta_{a,b}/2 \),
\[ |+\rangle_a = \begin{pmatrix} \cos \vartheta_a \\ -\sin \vartheta_a \end{pmatrix}, \quad |\rangle_a = \begin{pmatrix} \sin \vartheta_a \\ \cos \vartheta_a \end{pmatrix}, \]  
(9)

and similarly for \( |\pm\rangle \). This gives, with \( \vartheta_{ba} = \vartheta_b - \vartheta_a \),
\[ b \langle + | + \rangle_a = b \langle - | - \rangle_a = \cos \vartheta_{ba}, \]  
(10a)
\[ b \langle + | - \rangle_a = b \langle - | + \rangle_a = -\sin \vartheta_{ba}. \]  
(10b)

The conditional probabilities are therefore given by
\[ P_{ab}(+|+) = P_{ab}(-|-) = \cos^2 \vartheta_{ba}, \]  
(11a)
\[ P_{ab}(+|-) = P_{ab}(-|+) = \sin^2 \vartheta_{ba}, \]  
(11b)
whence Eq. (11) becomes
\[ P(b,a) = \begin{pmatrix} \cos^2 \vartheta_{ba} & \sin^2 \vartheta_{ba} \\ \sin^2 \vartheta_{ba} & \cos^2 \vartheta_{ba} \end{pmatrix}. \]  
(12)

From Eq. (8) we obtain for the correlation of the spin projections
\[ C_{\hat{Q}}(a,b) = \langle \psi | (\hat{\sigma} \cdot b)(\hat{\sigma} \cdot a) | \psi \rangle \]
the well-known result for the quantum correlation,
\[ C_Q(a,b) = \cos^2 \vartheta_{ba} - \sin^2 \vartheta_{ba} = \cos \theta_{ba}, \] (13)
regardless of the spin state \(|\psi\rangle\).

2.3 On the ‘quantumness’ of spin probabilities

The mathematical element represented by Eq. (1), with its associated properties discussed above, is, according to Grangier [5], a ‘fundamentally quantum idea’, because with a couple of simple consistency arguments it leads to the inevitable conclusion that the only possible theory is quantum mechanics.

The first consistency argument refers to the sum of the projectors, which must be equal to 1, as indicated in Eqs. (2), for any measurement context. The appeal made in [5] to Gleason’s theorem does not apply to the present case, in which we are dealing with a two-dimensional Hilbert space ([7] [8]). It would seem, therefore, that we need to resort to the Kochen-Specker theorem [9], which excludes any non-contextual hidden-variable theory able to reproduce the quantum results, thus assigning a seal of uniqueness to quantum probabilities. This points to the relevance of establishing a clear definition of what is meant by contextual, a point to which we will return in the following sections. The second consistency argument in [5] refers to the unitarity of the transformations between projectors, which is necessary to preserve the mutually exclusive character of events in each context [10]. That this condition is satisfied can be proved by associating to the probability matrix \( P(b,a) \) given by (12), an orthogonal matrix \( F_{ba} \) whose elements are the square roots of the coefficients of \( P(b,a) \),

\[ F_{ba} = \begin{pmatrix} \cos \vartheta_{ba} & -\sin \vartheta_{ba} \\ -\sin \vartheta_{ba} & -\cos \vartheta_{ba} \end{pmatrix}. \] (14)

Indeed, a change of measuring context, from \((a, b)\) to \((a, c)\), with \( \vartheta_{ca} = \vartheta_c - \vartheta_a = \vartheta_{cb} + \vartheta_{ba} \), changes \( F_{ba} \) into \( F_{ca} \) via a unitary transformation,

\[ F_{ca} = U_{cb} F_{ba}, \] (15)

with the matrix \( U_{cb} \) given by

\[ U_{cb} = \begin{pmatrix} \cos \vartheta_{cb} & \sin \vartheta_{cb} \\ -\sin \vartheta_{cb} & \cos \vartheta_{cb} \end{pmatrix}, \] (16)

and \( U_{cb} U_{cb}^\dagger = 1 \). In terms of Pauli matrices, Eqs. (15) and (16) take the form

\[ F_{ba} = \cos \vartheta_{ba} \sigma_z - \sin \vartheta_{ba} \sigma_x, \] (17)
\[ U_{cb} = \cos \vartheta_{cb} I + i \sin \vartheta_{cb} \sigma_y. \] (18)

Notice that when operating on \( F_{ba} \), the matrix \( U_{cb} \) leaves the right subindex \( a \) unchanged. This can be understood by noting that \( U_{cb} \), being an orthogonal matrix, describes a rotation by an angle \( \theta_{cb} \) around the \( a \) axis. Since

\[ U_{db} = U_{dc} U_{cb}, \] (19)
successive application of $U$ on $F_{ba}$ gives
\[ U_{dc}U_{cb}F_{ba} = U_{dc}F_{ca} = U_{db}F_{ba} = F_{da}. \] (20)

The same matrix $U$, when operating over a vector basis, transforms it into a new basis. Take, e. g., the initial basis of state vectors along $b$, given by Eq. (9) (with $a \rightarrow b$), and apply to them the transformation $U_{cb}$,
\[ U_{cb} \begin{pmatrix} \cos \vartheta_b \\ -\sin \vartheta_b \end{pmatrix} = \begin{pmatrix} \cos \vartheta_c \\ -\sin \vartheta_c \end{pmatrix}, \quad U_{cb} \begin{pmatrix} \sin \vartheta_b \cos \vartheta_b \\ \cos \vartheta_b \end{pmatrix} = \begin{pmatrix} \sin \vartheta_c \cos \vartheta_c \\ \cos \vartheta_c \end{pmatrix}. \] (21)

Therefore, the change of measuring context from $(a, b)$ to $(a, c)$ implies also a change of vector basis, from $|\pm\rangle_b$ to $|\pm\rangle_c$.

Notice that this transformation does not have any effect on the state of the system. It does, however, introduce a change in the partitioning of the probability space, reflected in the coefficients of the probability matrix (12).

3 The entangled (singlet) bipartite system

3.1 Separating the contributions to the spin correlation

Let us now consider a system made of two $\frac{1}{2}$–spin particles in the (entangled) singlet state
\[ |\Psi^0\rangle = \frac{1}{\sqrt{2}} (|+\rangle_a |+\rangle_b - |+\rangle_a |-\rangle_b - |-\rangle_a |+\rangle_b), \] (22)
in terms of the standard notation $|\phi\rangle |\chi\rangle = |\phi\rangle \otimes |\chi\rangle$, with $|\phi\rangle$ a vector in the Hilbert space of spin 1, and $|\chi\rangle$ a vector in the Hilbert space of spin 2. The direction $r$ is arbitrary since the singlet state is spherically symmetric. The projection of the spin 1 operator along $a$ is described by $(\hat{\sigma} \cdot a) \otimes I$, and the projection of the spin 2 operator along $b$ is described by $I \otimes (\hat{\sigma} \cdot b)$, whence the correlation is given by
\[ C_Q(a, b) = \langle \Psi^0 | (\hat{\sigma} \cdot a) \otimes (\hat{\sigma} \cdot b) |\Psi^0\rangle. \] (23)

For the calculation of $C_Q$ we make use of the individual spin state vectors (11) to construct an orthonormal basis for the bipartite system:
\[ |\phi^1\rangle_{ab} = |+\rangle_a |-\rangle_b, \quad |\phi^2\rangle_{ab} = |-\rangle_a |+\rangle_b, \]
\[ |\phi^3\rangle_{ab} = |+\rangle_a |+\rangle_b, \quad |\phi^4\rangle_{ab} = |-\rangle_a |-\rangle_b, \] (24)
and write
\[ C_Q(a, b) = \langle \Psi^0 | (\hat{\sigma} \cdot a) \left( \sum_{k=1}^4 |\phi^k\rangle_{ab} \langle \phi^k|_{ab} \right) (\hat{\sigma} \cdot b) |\Psi^0\rangle. \] (25)

The operators
\[ \hat{P}^k(a, b) = |\phi^k\rangle_{ab} \langle \phi^k|_{ab} \] (26)
appearing in (25) are the projection operators in the product space of the individual spin spaces, 
\( S = S_1 \otimes S_2 \), with respective eigenvalues \( A_k \) corresponding to the bipartite states \( |\phi^k\rangle_{ab} \) and given according to (24) by

\[
A_1 = A_2 = -1 \equiv A^-, \quad A_3 = A_4 = +1 \equiv A^+.
\] (27)

This allows us to rewrite Eq. (25) in the form

\[
C_Q(a, b) = \sum_{k=1}^{4} A_k(a, b) C_k(a, b),
\] (28)

which is the appropriate spectral decomposition of the spin correlation. In terms of the projection operators (26), we may write the spin correlation operator in the form

\[
\hat{C}_Q(a, b) = \sum_{k=1}^{4} A_k(a, b) \hat{P}_k(a, b) \equiv \sum_{k=1}^{4} \hat{C}_k(a, b),
\] (29)

with \( A_k \) the eigenvalues given by Eqs. (27). The coefficients appearing in (28)

\[
C_k(a, b) = |\langle \phi^k |_{ab} | \Psi^0 \rangle|^2,
\] (30)

which are the relative weights of the eigenvalues \( A_k \), are calculated with the help of Eqs. (23) and (24),

\[
C_1(a, b) = C_2(a, b) = \frac{1}{2} \cos^2 \vartheta_{ba},
\] (31a)

\[
C_3(a, b) = C_4(a, b) = \frac{1}{2} \sin^2 \vartheta_{ba}.
\] (31b)

The conditional probabilities are therefore given in this case by

\[
P_{ab}(+|-) = P_{ab}(-|+) = \cos^2 \vartheta_{ba},
\] (32a)

\[
P_{ab}(+|+) = P_{ab}(-|-) = \sin^2 \vartheta_{ba},
\] (32b)

whence Eq. (1) becomes

\[
P(b, a) = \begin{pmatrix}
\sin^2 \vartheta_{ba} & \cos^2 \vartheta_{ba} \\
\cos^2 \vartheta_{ba} & \sin^2 \vartheta_{ba}
\end{pmatrix}.
\] (33)

Eqs. (31) inserted into Eq. (28) reproduce the quantum result,

\[
C_Q(a, b) = - \cos \theta_{ba}.
\] (34)
3.2 Context-dependent partitioning of the probability space

It is important to observe that each term in the sum (29) projects onto one and only one of the four mutually orthogonal subspaces $U_k(a, b)$ that add to form space $S$ \(^{(11)}\),

$$S = U^1 \oplus U^2 \oplus U^3 \oplus U^4. \quad (35)$$

In operational terms (\(^{(12)}\), Ch. 2), this means that the result of every (joint) measurement falls under one and only one of these (eigen)subspaces. Each of the coefficients $C_k$ is therefore identified with a probability measure, namely the probability of obtaining $A_k$ as the result of a measurement, in accordance with the Born rule (\(^{(13)}\), Ch. 1).

Let us now consider the observable $C_Q(a, b')$ with $b' \neq b$. The corresponding projection operators are

$$\hat{P}^k(a, b') = |\phi_k\rangle_{ab'} \langle \phi_k|_{ab'}, \quad (36)$$

where $|\phi_k\rangle_{ab'}$ is defined as in (24) with $b$ replaced by $b'$. Therefore, instead of the partitioning of $S$ given by (35), the spectral decomposition involves now the partitioning into four mutually orthogonal subspaces $U^k(a, b')$, such that every (joint) measurement falls under one and only one of these new subspaces. In other words, the probability subspaces are specific to the observable being measured, i.e., to the measurement setting. This assigns an unambiguous meaning to the term measurement dependence that has been introduced in the context of the Bell-type inequalities (see e.g. \(^{(14)}\)): Contrary to a widespread notion of the term as implying a (functional) dependence of a set of hidden variables common to the entire probability space on the measurement setting, according to the present discussion it refers to the dependence of the partitioning of the probability space on the measurement setting.

This calculation carried out within the quantum Hilbert-space formalism \(^{(3)}\) confirms that the context must in general be considered when calculating quantum-mechanical probabilistic quantities. Specifically, the context—in this case, the directions $a$ and $b$—is shown to entail the division of the entire probability space into mutually exclusive, complementary probability subspaces. The two concepts, measurement dependence and contextual probabilities, are thus seen to be closely linked.

4 Probability distribution for the electron spin

In a recent article \(^{(4)}\) a general probability distribution $\rho(\phi)$ has been proposed for the electron spin projection problem, which serves to reproduce the conditional probabilities and the correlation $C(a, b)$, for both the single spin and the bipartite singlet state. This probability distribution has the form\(^{(4)}\)

$$\rho(\phi) = \frac{1}{2} \sin \phi, \quad 0 \leq \phi \leq \pi, \quad (37)$$

\(^{(4)}\) The same formula for the distribution, Eq. (37), has been previously obtained by other authors, also within the standard framework of quantum mechanics; see, e.g., \(^{(15)}\).
with
\[ \int_{\Phi} \rho(\phi)d\phi = 1. \] (38)
The partitioning of the probability space \( \Phi \) into \( \Phi^+_ab, \Phi^-ab \) must be such that, according to Eqs. (11) in the single-spin case,
\[ \int_{\Phi^+_ab} \rho(\phi)d\phi = \cos^2 \vartheta_{ab}, \int_{\Phi^-ab} \rho(\phi)d\phi = \sin^2 \vartheta_{ab}. \] (39)
With \( \rho(\phi) \) given by Eq. (37), the subdivision is (recall that \( \vartheta_{ab} = \theta_{ab}/2 \))
\[ \int_{\Phi^+_ab} \rho(\phi)d\phi = \frac{1}{2} \int_{\theta_{ab}}^{\pi} \sin \phi d\phi = \cos^2 \frac{\theta_{ab}}{2}, \] (40a)
\[ \int_{\Phi^-ab} \rho(\phi)d\phi = \frac{1}{2} \int_{0}^{\theta_{ab}} \sin \phi d\phi = \sin^2 \frac{\theta_{ab}}{2}. \] (40b)
The correlation \( C(a, b) \) is given accordingly by
\[ C(a, b) = \left( \int_{\Phi^+_ab} - \int_{\Phi^-ab} \right) \rho(\phi)d\phi = \cos \theta_{ab}, \] (41)
in agreement with Eq. (13). Equation (37) can therefore be considered to represent a bona fide hidden-variable distribution for the single electron spin. It is important to keep in mind that the contextuality resides in the partitioning of the sample space, not in the outcomes of measurements. In other words, the same function \( \rho(\phi) \) applies to different settings; but the set of values of \( \phi \) realized in each case to give either +1 or −1, depends on the setting.

In the bipartite case the signs of the spin projection along \( b \) are inverted with respect to the single-spin case, so that instead of Eq. (39) we have
\[ \int_{\Phi^-ab} \rho(\phi)d\phi = \cos^2 \vartheta_{ab}, \int_{\Phi^+_ab} \rho(\phi)d\phi = \sin^2 \vartheta_{ab}, \] (42)
the corresponding subdivision is
\[ \int_{\Phi^-ab} \rho(\phi)d\phi = \frac{1}{2} \int_{\theta_{ab}}^{\pi} \sin \phi d\phi, \] (43a)
\[ \int_{\Phi^+_ab} \rho(\phi)d\phi = \frac{1}{2} \int_{0}^{\theta_{ab}} \sin \phi d\phi, \] (43b)
and the correlation is given accordingly by \( C(a, b) = -\cos \theta_{ab} \).
5 Geometric model for the electron spin

The form of the probability distribution (37), along with the partitioning of the sample space indicated in Eqs. (40) and (43), is suggestive of a geometric representation that can be explored as a basis for a model for the spinning electron [4]. We shall discuss the single-spin case, and restrict the analysis to both vectors \( a \) and \( b \) lying on the \( xz \) plane for simplicity in the discussion.

In line with the probabilistic description, we are considering an element pertaining to an ensemble of realizations. Assume we want to determine \( b \), given a certain value for \( a \), say \( a = +1 \). Take for simplicity the \( +z \) axis along \( a \). We know for sure that a second spin projection along \( a \) gives again \( a = +1 \).

In terms of the conditional probabilities introduced in Section 2,

\[
P_{aa}(+ | +) = 1, \quad P_{aa}(+ | −) = 0.
\]

(44)

This means that the spin vector must lie in the upper half space (or northern hemisphere), forming in principle any angle measured on the \( xz \) plane. We propose to identify the variable \( φ \) with that angle; then \( φ \) lies in the interval \( 0 \leq φ \leq π \), with the origin of \( φ \) along the \(+x\) axis and \( φ \) increasing counterclockwise.

Conversely, given \( a = −1 \), the spin vector must lie in the lower half space, forming any angle \( φ \) on the \( xz \) plane such that \( 0 \leq φ \leq π \), with the origin of \( φ \) along the \( −x \) axis, i.e., \( P_{aa}(− | −) = 1, \quad P_{aa}(− | +) = 0 \). (The argument is of course reversible, in the sense that if \( b \) is given, the angle variable \( φ \) is measured with reference to the direction of \( b \).)

When \( a = +1 \), the sign of the projection along the direction \( b \) forming an angle \( \theta_{ab} \) with the \( +z \) axis is \( b = +1 \) for any angle \( φ \) on that plane such that \( \theta_{ab} \leq φ \leq π \), whilst it is negative for \( 0 \leq φ \leq \theta_{ab} \). This gives a concrete geometrical meaning to Eqs. (39)-(41), and justifies the partitioning of the probability space into the complementary subspaces \( Φ_{+ab}(0,\theta_{ab}), \quad Φ_{−ab}(\theta_{ab},π) \). What determines in each individual instance the specific value of the variable \( φ \), is not known here; \( φ \) may vary at random between realizations, within the entire interval \((0,π)\).

What is the source of such randomness and the mechanism that gives rise to the distribution function \( ρ(φ) \), is also unknown at this stage. What is important here is that a probability distribution exists that reproduces the desired conditional probabilities and correlations, without additional assumptions.

To make the context dependence more explicit, one may rewrite Eq. (41) as

\[
C(a,b) = \frac{1}{2} \int_{0}^{\pi} [\text{sign} \sin(φ − \theta_{ab})] \sin φ dφ = \cos \theta_{ab}.
\]

(45)

When the direction is changed from \( b \) to \( b' \), the geometry changes and the probability space is subdivided accordingly, so that one gets instead

\[
C(a,b') = \frac{1}{2} \int_{0}^{\pi} [\text{sign} \sin(φ' − \theta_{ab'})] \sin φ' dφ' = \cos \theta_{ab'}.
\]

(46)
A prime has been added to the integration variable $\phi$ in Eq. (46) to stress that, although the distribution function $\rho(\phi)$ is the same, its realization is independent from the previous one. This means that the individual results obtained in one context may not be transferred to the other.

The observation just made has important implications: it ascribes an unavoidable random character to the variable $\phi$. If the behavior of the system were deterministic, one could label every individual element of the ensemble and assign to it a fixed value of $\phi$, regardless of which projection (whether along $b$ or $b'$) is being measured.

An analogous approach, mutatis mutandis, can be followed in the entangled bipartite case: the spin projections along $a$ and $b$ are now those of particles 1 and 2, and the probabilities are conditioned by the outcome of spin 1, spin 2 being in this case antiparallel to spin 1. The corresponding change of sign of $b$ is reflected in the final outcome, $C(a, b) = -\cos \theta_{ab}$ (cf. Eq. (45)).

In the conventional terminology, the conclusion is that the ‘hidden’ variable $\phi$ with its associated distribution $\rho(\phi)$ does not serve to complete the quantum description, since the random element is still present. It does serve, however, the purpose of offering a better understanding of the probabilistic features of spin within the context of standard probability theory, and a geometric explanation for such features.

To demonstrate that there is indeed no need to abandon classical probability has been also the motivation behind different computer simulations that produce results in violation of Bell-type inequalities (e.g. [16, 17, 18, 19]; see also [20] and refs. therein). As indicated in Ref. [17], ‘one should not try to explain away the strange features of quantum mechanics as some kind of defect of classical probabilistic thinking, but one should use classical probabilistic thinking to pinpoint these features’. The present work offers a contribution in this direction.

6 Final comment. A possible physical image of spin

At this point one may ask whether a physical image of the electron spin can be made compatible with the geometric representation just discussed, under the condition that $\rho(\phi)$, with $0 \leq \phi \leq \pi$, represents a distribution of random variables. Such image would have to be consistent with the physical notion of spin as a dynamical quantity, with an associated intrinsic angular momentum $s$ of fixed magnitude and a magnetic moment roughly given by $\mu = (e/m)s$.

In the presence of a constant magnetic field $H$, a classical, frictionless magnetic spinning body is known to regularly precess around the direction of $H$ with constant angular frequency as a result of the torque exerted by the field (see e.g. Ref. [21]). A similar image has been conventionally associated with the electron, in which case the frequency of precession or Larmor frequency is given by $\omega_L = (e/m)H$. Even for intense magnetic fields, this frequency is many orders of magnitude smaller than the spinning frequency, which accord-
ing to Dirac’s theory is estimated to be of the order of Compton’s frequency, \( \omega_C = mc^2/h \sim 10^{21}\, \text{s}^{-1} \).

This crude image does not seem to leave any space for the additional inclination variable represented by \( \phi \) in our geometric model, and even less for the possible random character of this variable. However, such picture may change in the light of recent experimental evidence. Observations made with ferromagnetic materials in the pico- and femtosecond scales ([22], see also [23]), provide evidence of a spin dynamics far richer than previously assumed, due to effects of damping and inertia. This makes the study of the dynamics also quite more complicated, owing to the nonlinearity of the dynamical equations, which are impossible to solve analytically. Analysis of the detailed dynamics of the spinning electron is clearly outside the scope of the present paper. What is relevant, however, to our discussion, is the theoretical possibility of spin nutations, similar to the ones of a spinning top, and their experimentally observed appearance, at a characteristic frequency \( \omega_N \) much higher than the usual Larmor precession, yet much smaller than Compton’s frequency. These apparently intrinsic nutations have been established experimentally thanks to the use of an intense, transient magnetic field from a superradiant source of frequency close to \( 10^{12}\, \text{s} \), to which the nutating spin resonates. The lack of such sources had previously hampered the observation of this nutation dynamics.

Take now the geometric model described in the previous section, and consider the dynamics of the electron spinning around its own axis plus the spin angular momentum precessing around the direction of the magnetic field, along the \( z \) axis, which was defined as the direction \( a \). If in addition the spin vector is allowed to nutate, and it does so in a highly complex and irregular manner due to the nonlinearity of the dynamics, it may in principle scan the entire range of values of \( \phi \), from 0 to \( \pi \). As long as we cannot observe this nutation, because of its extremely high frequency, the angle \( \phi \) remains as a ‘hidden variable’. We are not able to determine the variations of \( \phi \) that occur with such high resolution, we only know that on the average they must be described by a distribution function such as \( \rho(\phi) \). Whether the randomness of \( \phi \) is due to the permanent interaction of the spinning electron with the fluctuating vacuum, or whether it is a product of the chaotic behavior of spin at this scale, is an open question; in any case, there is no need to think that randomness is an inherent element of physics.

With this discussion we hope to have provided elements in favor of the plausibility of a physical explanation for the probabilistic description of the electron spin given by quantum mechanics, thereby avoiding the need to resort to arguments of an unphysical or spooky nature. To conclude, we may briefly say that, although the electron spin itself is a quantum property, whose dynamics is still in need of a more complete theory, the current probability theory seems well suited for an explanation of its probabilistic features.
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