We show how the flat spacetime Galileon field theories (FSGFT) in arbitrary dimensions can be obtained through a Born-Infeld type structure. This construction involves a brane metric and non-linear combinations of derivatives of a scalar field. Our setup gives rise to some Galileon tensors and vectors useful for the variational analysis which are related to the momentum density of the probe Lovelock branes floating in a $N$-dimensional flat bulk. We find further that the Noether currents associated to these Galileon theories may be written in terms of such tensors.

Keywords: Galileons; Born-Infeld; Branes

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1. Introduction

Recently there has been a lot of interest in Galileon field theories because they may have implications for particle physics and cosmology. They have the ability to produce an accelerated expansion scenario in the absence of any type of matter interaction\cite{1,2,3} as well as to exhibit the Vainshtein screening mechanism\cite{4,5} at short distances. On theoretical grounds, these scalar field models with derivative self-interactions are second-order derivative theories which, surprisingly, are completely healthy non-higher derivative theories because their equations of motion remain
second-order. They have been covariantized and extended to $p$-forms. Besides, these theories are anchored to the former Horndeski’s scalar-tensor theories in curved 4-dimensional spacetimes and were discovered independently for flat spacetimes by Fairlie.

Following a geometric viewpoint the Galileons can be derived from the perspective of a brane probing a background spacetime as was developed in. In such brane prescription Galileons follow from the Lovelock invariants defined on the worldvolume swept out by a spacelike brane evaluated in an unitary gauge. For a $N$-dimensional background spacetime an interesting subclass of these theories is provided when such bulk is flat Minkowski, $\mathcal{M}_N$. There are two possible foliations for the $\mathcal{M}_N$ viewed as a maximally symmetric space by ($N-1$)-dimensional maximally symmetric timelike slices; that is, $\mathcal{M}_N$ can be foliated by flat $\mathcal{M}_{N-1}$ slices or by $dS_{N-1}$ slices, namely. Such flat spacetime Galileon field theories (FSGFT) are known as DBI Galileons and type II $dS$ DBI Galileons, respectively. Another constructions are possible but they are in dependence on the curved nature of the background spacetime.

In this paper it is shown how the flat spacetime Galileon field theories can be obtained from a Born-Infeld (BI) type action. This action is in connection with a Born-Infeld-Lovelock (BIL) framework developed for describing Lovelock brane models and is written close in the spirit to the one developed in. Our construction contains contribution from a brane metric and non-linear combinations of the derivatives of the scalar field defined on the brane. Despite the bulk is Minkowski this is a case of sufficient complexity that still deserves further explorations. By expanding this determinantal BI Lagrangian in terms of traces one finds a finite series containing all the Galileon terms for a given dimension. We may, additionally, make a variational analysis in order to obtain the equations of motion (eom) and to study the associated Noether currents.

The paper is organized as follows. The aim of Section 2 is to acquaint the reader with basic covariant facts of the FSGFT. In Section 3 we introduce a BI type action which, when expanded, contains all the FSGFT for any arbitrary dimension. This number is in dependence of the dimension of the worldvolume. In addition, we provide another BI-like structures pursuing the same claim. In Section 4 we introduce some Galileon tensors and vectors useful to express the Noether currents associated to these models and discuss some of their properties. Section 5 is devoted to the forthright derivation of the equations of motion and the Noether currents. We conclude in Section 6 with some comments. Appendices gather information about the brane geometry in the unitary gauge and some mathematical relations useful for expanding the BI-like structures. The notation used in the paper is the usual one. When working with the scalar field $\phi$ in curved spacetime with metric $g_{ab}$ and covariant derivative $\nabla_a$, we use the notation $\pi_a = \nabla_a \phi$ and $\Pi_{ab} := \nabla_a \nabla_b \phi$. For trace of powers of the matrix $\Pi_{ab}$ we have $[\Pi^a] := \text{Tr}(\Pi^a)$, e. g., $[\Pi] = \nabla^a \nabla_a \phi$, $[\Pi^2] = \nabla_a \nabla_b \phi \nabla^a \nabla^b \phi$ where the brane indices $a, b$ are raised with the inverse metric.
The contractions of powers of $\pi$ with $\Pi$ are usually denoted by $[\pi^n] = \pi \cdot \Pi^{n-2} \cdot \pi$, e.g., $[\pi^2] = \pi_\alpha \pi^\alpha$, $[\pi^3] = \pi_{ab} \Pi^{ab} \pi_b$.

2. Flat spacetime Galileons in arbitrary dimension

Consider the action governing the dynamical evolution of a single scalar field $\phi(x^a)$ living in an orientable codimension one brane, $\Sigma$, with local coordinates $x^a$ and metric $q_{ab}$ ($a, b = 0, 1, \ldots, p$) embedded in a $N = (p + 2)$-dimensional Minkowski background spacetime with metric $\eta_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, \ldots, p + 1$)

$$S[\phi] = \int_\Sigma d^{p+1} x \sqrt{-q} \sum_{n=0}^{p} \alpha_n L_n(\phi, \nabla_a \phi, \nabla_a \nabla_b \phi),$$

(1)

where

$$L_n = \gamma^{-1} f^{p} \delta_{b_1 b_2 b_3 \ldots b_n} h^{b_1 a_1} h^{b_2 a_2} h^{b_3 a_3} \ldots h^{b_n a_n},$$

(2)

and $\delta_{b_1 b_2 b_3 \ldots b_n}$ being the generalized Kronecker delta (gKd), $q = \det(q_{ab})$ and $\gamma = (1 + f^{-2} \nabla_a \phi \nabla_a \phi)^{-1/2}$. Here and in what follows, $f = f(\phi)$ such that $f = 1$ for the DBI Galileons whereas $f = \phi$ for the type II $dS$ DBI Galileons. In Planck units the coupling constants $\alpha_n$ have dimensions $[\alpha_n] = L^{n-p-1}$. Moreover, we have introduced the matrix

$$h_{ab} := \gamma f^{-2} \left[ -\Pi_{ab} + f^{-2} \gamma^2 (\pi_a \Pi_b \pi_c + f f' \pi_a \pi_b) + f f' q_{ab} \right],$$

(3)

with $f' = \partial f / \partial \phi$. The brane indices $a, b$ are raised and lowered with $q^{ab}$ and $q_{ab}$, respectively. Only the first $n$ of these Galileon Lagrangians are non-trivial in $n \leq p + 1$ dimensions. In addition, $\nabla_a$ denotes the covariant derivative compatible with $q_{ab}$. It is worth commenting on the origin of the matrix (3). This arise from the contraction between the inverse of the induced metric $g^{ab}$ and the extrinsic curvature $K_{ab}$ of the $\Sigma$ brane when they are expressed in terms of the unitary gauge (see (A.5) and (A.6)). Thus, $h^a_b = g^{ac} K_{cb}$. In connection with this fact the brane induced metric on $\Sigma$ is

$$g_{ab} = f^2 q_{ab} + \pi_a \pi_b,$$

(4)

such that $g^{ac} g_{cb} = \delta^a_b$.

With regards the structure (2), we set $L_0 = \gamma^{-1} f^{p+1}$. Another elegant framework for obtaining other type of Galileon field theories using the anti-symmetric Levi-Civita symbol is provided in [9]. Geometrically speaking, $\Sigma$ represents the world-volume swept out by a dynamical Lovelock $p$-brane which can be described locally by $y^\mu = X^\mu(x^a)$ evaluated in an unitary gauge where $y^\mu$ are local coordinates for the bulk and $X^\mu$ are the embedding functions. By expanding out Eq. (2) in terms of the traces associated to the matrix (3) (see Appendix B) for details on the traces
of the $h_{ab}$ matrix), at the first few orders we have

$$L_0 = \gamma^{-1} f^{p+1},$$

$$L_1 = \gamma^{-1} f^{p+1} \text{Tr}(h),$$

$$L_2 = \gamma^{-1} f^{p+1} \text{Tr}^2(h),$$

$$L_3 = \gamma^{-1} f^{p+1} \text{Tr}^3(h),$$

(5)

$$= f^{p-1} \left\{-[\Pi] + \frac{\gamma^2}{f^2} [\pi^3] + ff'(p + 2 - \gamma^2)\right\},$$

(6)

$$= f^{p-3} \left\{[\Pi]^2 - [\Pi^2] + 2\frac{\gamma^2}{f^2} ([\Pi][\pi^3] + [\pi^4])
+ 2ff' \left(-[p+1][\Pi] + \gamma^2 \left(f^2[\Pi] + (p + 1)[\pi^3]\right)\right)
+ f^2 f'^2 p(p + 3 - 2\gamma^2)\right\},$$

(7)

$$L_3 = \gamma^{-1} f^{p+1} \left\{[\Pi]^3 - 3\text{Tr}(h)\text{Tr}(h^2) + 2\text{Tr}(h^3)\right\},$$

$$= \gamma^2 f^{p-5} \left\{-([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi]^3)
- 3\frac{\gamma^2}{f^2} \left([-\pi^2]([\Pi]^2 - [\Pi^2]) + 2[\Pi][\pi^4] - 2[\pi^5]\right)
+ 3ff' \left([\Pi]^2 - [\Pi^2]\right) - \gamma^2 \left(f^2 ([\Pi]^2 - [\Pi^2]) + 2p ([\Pi][\pi^3] - [\pi^4])\right)
- 3f^2 f'^2 (p - 1) \left[(p + 2)[\Pi] - \frac{\gamma^2}{f^2} (2f^2[\Pi] + (p + 2)[\pi^3])\right]
+ f^3 f'^3 p(p - 1) (p + 4 - 3\gamma^2)\right\},$$

(8)

where we have considered that $[\pi^2] = \pi_a \pi^a = f^2 \gamma^{-2}(1 - \gamma^2)$. Regarding the so-called tadpole terms, $L_T$, they are not constructed in the present way but as the $N$-dimensional proper volume bounded by the worldvolume (see the discussion in $\text{[2,13]}$). The equations of motion derived from any of these Lagrangians will contain no more than two derivatives on the scalar field, ensuring that no extra degrees of freedom propagate around the background spacetime as we will show below.

3. Flat spacetime Galileons from Born-Infeld type structures

The setup outlined above allows us to find another general expression involving the flat spacetime Galileon field theories. Consider now the local Born-Infeld type action

$$S[\phi] = \int \sum_{p=1}^{\infty} d^{p+1}x \gamma^{-1} f^{p+1} \sqrt{-\det(q_{ab} + X_{ab})},$$

(9)

where

$$X_{ab} := 2\alpha h_{ab} + \alpha^2 h_{ac} h^{c} b,$$

(10)

and $h_{ab}$ being defined in $\text{[3]}$ and $\alpha$ is a concomitant constant characterizing the relative weight of the nonlinear terms in this model.
The BI type volume element form in (9) may be written in terms of the flat spacetime Galileon Lagrangians for a given dimension \( p \). Indeed, it follows from the fact that the combined matrix \( G_{ab} := q_{ab} + X_{ab} \) can be written as \( G_{ab} = q_{cd} (q^{ca} + \alpha h^{ca}) (q^{db} + \alpha h^{db}) \). This entails that the action (9) can be re-expressed as

\[
S[\phi] = \int_{\Sigma} d^{p+1}x \sqrt{-g} \gamma^{-1} f^{p+1} [\det (q_{ab} + \alpha h_{ab})].
\]

(11)

Now we turn to expand the characteristic polynomial of \( h_{ab} \) by using the identity (B.4). To do this we need once again the traces of \( h_{ab} \) (see Appendix B). It therefore follows, from Eqs. (B.5-B.8) specialized to \( f_{ab} = \alpha h_{ab} \), that after a lengthy but straightforward computation that

\[
f^{(1)} = \alpha f^{p-1} \left[ -[\Pi] + \frac{\gamma^2}{f^2} [\pi^3] + (p+2-\gamma^2) \right],
\]

(12)

\[
f^{(2)} = \frac{\alpha^2}{2} \gamma f^{p-3} \left\{ [\Pi]^2 - [\Pi]^2 + 2 \frac{\gamma^2}{f^2} (-[\Pi] [\pi^3] + [\pi^4]) + 2 f f' \left[ -(p+1) [\Pi] + \frac{\gamma^2}{f^2} (f^2 [\Pi] + (p+1) [\pi^3]) \right] + f^2 f'^2 (p+3-2\gamma^2) \right\},
\]

(13)

\[
f^{(3)} = \frac{\alpha^3}{6} \frac{\gamma^2}{f^2} f^{p-5} \left\{ -([\Pi]^3 - 3 [\Pi] [\Pi]^2 + 2 [\Pi]^3) - \frac{3 \gamma^2}{f^2} ([\Pi]^2 - [\Pi]^2) + 2 [\Pi] [\pi^4] - 2 [\pi^5]) + 3 f f' \left[ p ( [\Pi]^2 - [\Pi]^2) - \frac{\gamma^2}{f^2} (f^2 ( [\Pi]^2 - [\Pi]^2) + 2p ( [\Pi] [\pi^3] - [\pi^4])) \right] - 3 f^2 f'^2 (p-1) \left[ (p+2) [\Pi] - \frac{\gamma^2}{f^2} (2f^2 [\Pi] + (p+2) [\pi^3]) \right] + f^3 f'^3 (p-1) (p+4-3\gamma^2) \right\},
\]

(14)

\[
\vdots
\]

\[
f^{(s)} = \frac{\alpha^s}{s!} L_s,
\]

(15)

where \( L_s \) is given by Eq. (2). If we set \( f^{(0)} = 1 \) and then substitute all these results in (B.4) and therefore into the action (11) we obtain that the BI type action (9) can be expressed as

\[
S[\phi] = \int_{\Sigma} d^{p+1}x \sqrt{-g} \sum_{n=0}^{p} \left( \frac{\alpha^n}{n!} \right) L_n,
\]

(16)

where the Galileon Lagrangian functions (5-8) have been invoked and so on. It should be stressed that this action is similar to the action (1) whenever \( \alpha_n = \alpha^n / n! \). This structure is a particular case of the action (1). With regards this point, a
non-trivial question is how to find the correct couplings $\alpha_n$ that yield a viable physical theory but, for the moment, such point is beyond the scope of this work.

An interesting proposal on this subject relating the unitary analysis of BI gravity actions is developed in. Further, as in the gravitational case, one could relate this type of BI-like actions to a Chern-Simons limit of the Galileon theory by a particular choice of the couplings $\alpha_n$ in the series $\{1\}$. Hence, for our purposes, when the flat spacetime Galileon field theories are configured as in $\{1\}$, these admit a Born-Infeld type structure.

3.1. Other BI-like structures

The combination $X_{ab}$ is not constrained to have the form $\{10\}$ but it seems reasonable to think of some other possible choices for $X_{ab}$ in order to reproduce the flat spacetime Galileon Lagrangian terms. Some of them are more complex while others are limited but even so all of them contain interesting information. Consider for example

$$X_{ab} = (\alpha + \beta h) h_{ab}, \quad (17)$$

where $h = \text{Tr}(h_{ab})$ and $\beta$ being another constant. The action $\{9\}$ can be rewritten as

$$S[\phi] = \int \Sigma d^p x \sqrt{-q}^{-1} f^{p+1} \sqrt{\det(\delta_{ab} + X_{ab})}. \quad (18)$$

By using the expansion $(B.10)$ we note that if $\beta = \alpha^2/4$ then

$$\sqrt{\det(\delta_{ab} + X_{ab})} = 1 + \frac{\alpha}{2} h + \frac{\alpha^2}{4} \left[ h^2 - \text{Tr}(h^2) \right] + \frac{\alpha^3}{12} \left[ h^3 - 3h\text{Tr}(h^2) + 2\text{Tr}(h^3) \right]$$

$$+ \frac{\alpha^4}{192} \left[ 5h^4 - 27h^2\text{Tr}(h^2) + 40h\text{Tr}(h^3) + 6\text{Tr}(h^2)^2 - 24\text{Tr}(h^4) \right]$$

$$+ O(h^5). \quad (19)$$

When we substitute this expression into the action $\{18\}$ by considering the matrix $(3)$, we obtain

$$S[\phi] = \int \Sigma d^{p+1} x \sqrt{-q} \left[ L_0 + \frac{\alpha}{2} L_1 + \frac{\alpha^2}{4} L_2 + \frac{\alpha^3}{12} L_3 + O(h^4) \right]. \quad (20)$$

In other words, in the small value of $X_{ab}$ given by $(17)$ the action $(18)$ casts out only the first four terms of the Galileon Lagrangians but not beyond this. Again, we have invoked the expressions $(5-8)$ and so on. This action might be useful if one is interested only in $(3 + 1)$ dimensions. In addition, the choice $X_{ab} = \alpha h_{ab}$ results very limited because the corresponding expansion for small values of $X_{ab}$ only reproduce the first two Galileon Lagrangian terms.

4. Galileon tensors

For future reference we introduce the following tensors for each $n$ value

$$J^{ab}_{(n)} := \delta_{b_1 b_2 \cdots b_n}^{a_1 a_2 \cdots a_n} \delta_{b_1}^{b_1} h^{b_2}_{a_2} \cdots h^{b_n}_{a_n}. \quad (21)$$
which are symmetric. This fact can be proved with the aid of the gKd properties and
the definition $h^{ab} = g^{ac}K_{cb}$. By expanding out this expression in terms of minors
we have a recursion relation

$$
J^{ab}_{(n)} = \left[ \delta_0 a_1 a_2 a_3 \cdots a_n - \delta_{b_1} a_2 a_3 \cdots a_n + \cdots + (-1)^n \delta_{b_n} a_1 a_2 \cdots a_{n-1} \right] \times
\frac{g^{bb_0} h^{b_1} h^{b_2} \cdots h^{b_n}}{a_1 a_2 \cdots a_n},
$$

where we have used the expression defining the Lagrangians (2) and the definition (21). In addition, when we contract the expression (21) with the extrinsic curvature (A.6) we obtain the important identity

$$
J^{ab}_{(n)} K_{ab} = \gamma f^{-(p+1)} L_n g^{ab} - nh^{ab} J^{cb}_{(n-1)},
$$

(22)

where we have used the relation (2) again. Similarly, when we contract the tensors (21) with the brane induced metric (4) we have the relation

$$
J^{ab}_{(n)} g^{ab} = \gamma f^{-(p+1)} (p + 1 - n) L_n.
$$

(24)

We will refer hereafter these tensors as Galileon tensors.

We introduce now the worldvolume vectors

$$
J^a_{(n)} := \gamma^{-1} J^{ab}_{(n)} \pi_b.
$$

(25)

Taking into account Eq. (22) and the definition (24) we obtain a recursion relation

$$
J^a_{(n)} = \gamma^2 f^{-(p+3)} L_n \pi^a - nh^{ab} J^b_{(n-1)},
$$

where we have used expression (A.5) as well as expressing $[\pi^2]$ in favor of $\gamma^{-1} = (1 + f^{-2}[\pi^2])^{1/2}$. A convenient form for this relation is obtained when we insert the definition of the $h_{ab}$ matrix, Eq. (3),

$$
J^a_{(n)} = \gamma^2 f^{-(p+3)} L_n \pi^a + n\gamma f^{-2} \Pi^a_{b} J^b_{(n-1)} - n\gamma^3 f^{-4} \pi^a \pi^b \Pi_{bc} J^c_{(n-1)}
- n\gamma^3 f^{-4} \pi^a \pi^b J^c_{(n-1)} - n\gamma f^{-1} f' J^a_{(n-1)}.
$$

(26)

These vectors are to be referred to as the Galileon vectors. For illustration, in Appendix C we give some values of the tensors (21) and the vectors (25). Moreover, from Eqs. (A.4) and (24) we have

$$
\gamma^{-1} f^{p+1} \Pi_{ab} J^b_{(n)} = (p + 1 - n) f^{-1} f' L_n - \gamma^{-1} L_{n+1} + f'' f' \pi_a J^a_{(n)}.
$$

(27)

Alike from Eqs. (4) and (24) we have that the trace of the Galileon tensors is

$$
\text{Tr}(J^a_{(n)}) = J^a_{(n)} g^{ab} = (p + 1 - n) \gamma f^{-(p+3)} L_n - \gamma f^{-2} \pi_a J^a_{(n)}.
$$

(28)

The conservation of the Galileon tensors on regards the geometry provided by the induced metric (4) is $D_a J^{ab}_{(n)} = 0$ where $D_a$ is the covariant derivative compatible with $g_{ab}$ (see [15] for details). In terms of the geometry provided by $g_{ab}$ this relation becomes $\nabla_a (\gamma^{-1} J^a_{(n)}) - f^{-2} \pi^b K_{ac} J^{ac}_{(n)} + \gamma^{-1} f^{-1} f'(p+3) J^a_{(n)} \pi_a = 0$ where once again
we have made use of the formulae in Appendix A. Hence, by using the identity (23) and the definition of the Galileon vectors (25) we obtain
\[ \nabla_a \left( \gamma^{-1} f^{p+1} J^{ab}_{(n)} \right) = \gamma f^{-2} L_{n+1} \pi^b - 2 f^{p+1} J^{h}_{(n)} \cdot (29) \]
In this spirit, it remains to obtain a divergence expression for the Galileon vectors. The contraction of (29) with \( \pi^b \) and the use of Eq. (25) yields
\[ \nabla_a \left( f^{p+1} J^a_{(n)} \right) = \gamma f^{-2} L_{n+1} + \gamma^{-1} f^{p+1} \Pi_{ab} J^{ab}_{(n)} - 2 f^{p+1} f' \pi_a J^a_{(n)}. \]
(30)
Finally, by using Eqs. (27), (30) and (28) we have another useful identity given by
\[ \gamma L_{n+1} = -\nabla_a (f^{p+1} J^a_{(n)}) + \gamma^{-1} f^{p+2} f' \text{Tr}(J^{ab}_{(n)}). \]
(31)

4.1. Auxiliary Galileon tensors
In analogy with the relation (21) and for convenience in the obtaining of the eom and the Noether currents it will be useful to introduce some auxiliary Galileon tensors
\[ J^a_{(n)} := \delta^{a_1 a_2 a_3 \cdots a_n}_{b_1 b_2 b_3 \cdots b_n} h_{b_1} h_{b_2} h_{b_3} \cdots h_{b_n}. \]
(32)
In general these are not symmetric. Obviously \( J^a_{(n)} = J^{ac}_{(n)} g_{cb} \). In terms of the Galileon tensors and vectors these explicitly reads
\[ J^a_{(n)} = f^2 \gamma J^a_{(n)} + \gamma J^a_{(n)} \pi^b. \]
(33)
Finally, these tensors satisfy
\[ J^a_{(n)} h^{ba} = \gamma f^{-(p+1)} L_{n+1}. \]
(34)

5. Equations of motion and Noether currents in arbitrary dimensions
Let us consider an infinitesimal deformation \( \phi \rightarrow \phi + \delta \phi \) of the scalar field describing the normal deformation for the worldvolume \( \Sigma \). Under this deformation the basic quantities that characterize \( \Sigma \) change according to
\[ \delta \pi^a = \nabla_a \delta \phi, \]
\[ \delta \Pi_{ab} = \nabla_a \nabla_b \delta \phi, \]
\[ \delta \gamma = f^{-1} f' (1 - \gamma^2) \delta \phi - \gamma^3 f^{-2} \pi^a \nabla_a \delta \phi, \]
\[ \delta h_{ab} = -\gamma^3 f^{-2} \left[ h_{ab} + \nabla_b (\gamma f^{-2} \pi_a) - 2 \gamma^{-1} f^{-3} f' \pi_a \pi_b \right] \delta \phi - \gamma^2 f^{-2} [h_{ab} + \nabla_b (\gamma f^{-2} \pi_a)] \pi^c \nabla_c \delta \phi - 2 \gamma f^{-3} f' \pi_a \nabla_b \delta \phi - \nabla_b (\gamma f^{-2} \pi_a), \]
(38)
where in addition we have considered that $\delta f = f'\delta \phi$ and $\delta f' = 0$. Using these expressions the variation of the Galileon action reads

$$
\delta S[\phi] = \int dV \sum_{n=0}^{p} \beta_n \left[ \gamma L_n \delta \gamma^{-1} + (p + 1) f' L_n \delta \phi + n \gamma^{-1} f^{p+1} \mathcal{J}^{ab}_{(n-1)} \delta h_{ba} \right],
$$

Taking into account (38) and (34) the variation of the action reads

$$
\delta f = \int dV \sum_{n=0}^{p} \beta_n \left[ f^{-1} f'(p + \gamma^2) L_n \delta \phi + \gamma^2 f^{-2} L_n \pi^a \nabla_a \delta \phi 
+ n \gamma^{-1} f^{p+1} \mathcal{J}^{ab}_{(n-1)} \delta h_{ba} \right],
$$

where we have written $dV := dt + x \sqrt{-\tilde{g}}$ and $\beta_n := \alpha_n / n!$ for the sake of brevity. Taking into account (38) and (41) the variation of the action reads

$$
\delta S = \int dV \sum_{n=0}^{p} \beta_n \left\{ f^{-1} f'(p + \gamma^2) L_n \delta \phi + \nabla_a \left( \gamma^2 f^{-2} L_n \pi^a \delta \phi \right) 
- n f^{p-2} f' \mathcal{J}^{ba}_{(n-1)} \pi_b \nabla_a \delta \phi - n \gamma^2 f^{-1} f' L_n \delta \phi - n \gamma^2 f^{-2} L_n \pi^a \nabla_a \delta \phi 
+ 2n f^{p-1} f' \gamma^{-1} J^a_{(n-1)} \pi_a \delta \phi - n \gamma^{-1} f f' J^a_{(n-1)} \nabla_a \delta \phi 
- n f f' \mathcal{J}^{ab}_{(n-1)} \nabla_a \left( \gamma f^{-2} \pi_b \right) \delta \phi 
- n f f' \mathcal{J}^{ab}_{(n-1)} \nabla_a \left( \gamma f^{-2} \pi_b \right) \delta \phi 
\right\}.
$$

After integrations by parts and considering the identities and definitions provided by Section 3.4 a rather long but straightforward computation leads to

$$
\delta S[\phi] = \int dV \sum_{n=0}^{p} \beta_n \left[ \mathcal{E}_n \delta \phi + \nabla_a Q^a_{(n)}(\delta \phi, \nabla_b \delta \phi) \right],
$$

where

$$
\mathcal{E}_n = f^{-1} f'(p + \gamma^2) L_n - \gamma (1 + \gamma^2) f^{-3} \mathcal{J}^{ba}_{(n-1)} \nabla_a (n \gamma^{-1} f^{p+1} \mathcal{J}^{ab}_{(n-1)}) 
+ 2n \gamma^{-1} f^{p-1} f' \gamma^{-1} J^a_{(n-1)} \nabla_a (\gamma f^{-2} \pi_b) - n \gamma^2 f^{-1} f' L_n 
+ \nabla_a \left\{ -\gamma^2 f^{-2} \pi^a L_n + n \gamma^2 f^{-2} \pi^a L_n + n \gamma f^{p-1} \mathcal{J}^{bc}_{(n-1)} \nabla_b (\gamma f^{-2} \pi_c) \pi^a 
+ n \gamma^{-1} f f' J^a_{(n-1)} + \gamma^3 f^{-4} \pi^a \nabla_c (n \gamma^{-1} f^{p+1} \mathcal{J}^{cb}_{(n-1)}) 
- \gamma f^{-2} \nabla_b (n \gamma^{-1} f^{p+1} \mathcal{J}^{ba}_{(n-1)}) + n f^{p-2} f' \mathcal{J}^{ba}_{(n-1)} \pi_b \right\},
$$

and

$$
Q^a_{(n)} = \left[ \gamma^2 f^{-2} L_n \pi^a - n \gamma^2 f^{-2} L_n \pi^a - n \gamma f^{p-1} \mathcal{J}^{bc}_{(n-1)} \nabla_b (\gamma f^{-2} \pi_c) \pi^a 
- n f^{p-2} f' \mathcal{J}^{ba}_{(n-1)} \pi_b - n \gamma^{-1} f f' J^a_{(n-1)} + \gamma f^{-2} \nabla_b (n \gamma^{-1} f^{p+1} \mathcal{J}^{ba}_{(n-1)}) 
- \gamma^3 f^{-4} \pi^a \nabla_c (n \gamma^{-1} f^{p+1} \mathcal{J}^{cb}_{(n-1)}) \right] \delta \phi - n \gamma^{-1} f^{p+1} \mathcal{J}^{ab}_{(n-1)} \delta \phi. (41)
$$
To write down a shorter expression for the eom we first reduce the form of the divergence term in (30). Repeated application of Eqs. (A.7) and (33) in (40) implies that the divergence term reduces to $\nabla_a(-f^{p+1}J^a_{(n)}) + n\gamma^{-1}f^p f' J^a_{(n-1)}$). This expression can be written in a somewhat different form when we use the divergence relation for the Galileon vectors (30). Hence

$$\nabla_a(-f^{p+1}J^a_{(n)}) + n\gamma^{-1}f^p f' \Pi_{ab} J^b_{(n-1)} = \gamma L_{n+1} - (p + 1 - n\gamma^{-2})f^{-1}f'L_n + f^p f' \pi_a J^a_{(n)}$$

$$+ n\gamma^{-2}f^p f' \Pi_{ab} J^b_{(n-1)},$$

(42)

It is straightforward to prove now that by repeatedly making the substitution of Eqs. (25), (26), (29), (33) and (42) into (40), the equations of motion reduce to

$$\mathcal{E}_n = (p - n + \gamma^2)f^{-1}f'L_n - f^p f' \pi_a J^a_{(n)} + (1 - \gamma^2)f^{-1}f'L_n + 2f^p f' \pi_a J^a_{(n)}$$

$$- \gamma^{-1}(1 - \gamma^2)L_{n+1} - \gamma^{-1}f^{p+1}\Pi_{ab} J^b_{(n)}.$$ 

Now, by using the divergence identity for the Galileon vectors (30) we obtain that the equation of motion may be expressed as a conservation law

$$\mathcal{E}_n = -\nabla_a(f^{p+1} J^a_{(n)}) + \gamma^{-1}f^{p+2} f' \operatorname{Tr}(J^b_{(n)}) = 0.$$ 

(43)

In addition, guided by the identity (31) we may write that

$$\mathcal{E}_n = \gamma L_{n+1} = 0.$$ 

(44)

Clearly, we have only one equation of motion of second order in $\phi$ for each value of $n$ and we therefore see that there is only one physical degree of freedom. This last compact form is independent of the value of the function $f$ and is fully equivalent to the one for the equations of motion arising for the Lovelock branes Lagrangians (10) when they are expressed in the unitary gauge. Certainly, for DBI Galileons the equation of motion takes the form

$$\mathcal{E}_n = -\partial_a J^a_{(n)} = 0.$$ 

(45)

With regards the Noether currents, by similar manipulations on all the terms in (11) with the use of the identity (29) and the recursive relation for the Galileon vectors (30) we obtain

$$Q^a_{(n)} = \left[f^{p+1} J^a_{(n)} + n\gamma^{-1}f^p f' J^a_{(n-1)}\right] \delta \phi - n f^{p+1} J^b_{(n-1)} \nabla_b \delta \phi.$$ 

(46)

These are the general geometric Noether currents for these Galileon theories associated with the Poincaré symmetry of the background. Evidently, for the most simple case, $n = 0$, these specialize to $Q^a_{(0)} = f^{p+1} J^a_{(0)} \delta \phi$. Similarly, note that (46) specializes to a more simple expression for DBI Galileons

$$Q^a_{(n)} = J^a_{(n)} \delta \phi - n J^b_{(n-1)} \partial_b \delta \phi.$$ 

(47)

It should be noted that only the $J^a_{(n)}$'s and $J^a_{(n)}$'s are present in (46) which have a nice physical interpretation. These are the contractions along the tangent vectors.
to $\Sigma$ (A.3) of the linear momentum density, $f^a\mu$, associated to the Lovelock brane invariants (see Eq. (12) in [10]). The derivation and application of linear momentum and angular momentum for this type of branes remains to be developed and discussed elsewhere. In fact, in former Galileon field theories an alternative approach for obtaining Noether currents and charges was developed in [24].

6. Conclusions

In this work, we have first reviewed the flat spacetime Galileon field theories for any arbitrary dimension in a covariant form. Then we propose a Born-Infeld type framework for describing the flat spacetime Galileon Lagrangians. When this is expanded, such action becomes a finite series. In fact, the action (18) describes the dynamics of a Born-Infeld-Lovelock $p-$brane under the unitary gauge where $\phi$ is the brane position relative to the foliation i.e., the Goldstone field associated with spontaneously broken $N$-dimensional Poincaré invariance [19]. By considering a variational process we obtained the Noether currents associated to these models in terms of some tensors, $J^{ab}_{(n)}$, and vectors $J^a_{(n)}$, which are in relation with a conserved stress tensor in the Lovelock brane prescription [16]. We believe that these tensors will play an important role in the Hamiltonian development for the FSGFT which will be reported elsewhere. A non-trivial question is how to find the correct couplings $\alpha_n$ leading to a viable physical theory but it remains to be explored. In fact, a related subject has been recently posed by Hinterbichler et al. [25]. Another interesting proposal on this subject relating to the unitary analysis of BI gravity theories is developed in Ref [23]. The results here are so far confined to a Minkowski bulk which is a case of sufficient complexity but it remains to modify this BI type action in order to consider non-trivial interesting maximally symmetric ambient spacetimes in order to include curved Galileon field theories defined on the worldvolume. In any case, our approach suggests further explorations in order to understand how much this BI-like structure can be used to analyze issues related to acceleration of branes.

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Appendix A. Brane geometry in the unitary gauge

Consider a $N = (p + 2)$-dimensional flat Minkowski background spacetime, with isometry algebra the $N$-dimensional Poincaré algebra $p(N - 1, 1)$. We will assume that the bulk adopt a Gaussian normal foliation of the form

$$dS_{p+2}^2 = f^2(\phi) q_{ab}(x^c) dx^a dx^b + d\phi^2,$$

(A.1)

where $q_{ab}$ is a worldvolume metric and $X^{p+1} = \phi$ denotes a Gaussian normal transverse coordinate. For the embedding, named unitary gauge,

$$y^\mu = X^{\mu}(x^a) = \left(x^a, \phi(x^a)\right),$$

(A.2)

the tangent vectors to $\Sigma$ are given by

$$e^\mu_a = \partial_a X^\mu = \begin{cases} 
\delta^b_a & \text{when } \mu = b, \\
\nabla_a \phi & \text{when } \mu = p + 1.
\end{cases}$$

(A.3)

The normal vector $n^\mu$ to $m$ can be obtained from its intrinsic definition, namely, $e \cdot n = \eta_{\mu \nu} e^\mu_a n^\nu = 0$ and $n \cdot n = \eta_{\mu \nu} n^\mu n^\nu = 1$ where $\eta_{\mu \nu}$ is the background spacetime metric given by (A.1). Thus,

$$n^\mu = \gamma \left( -f^{-2} \nabla^a \phi \right) 1 \quad \text{and} \quad n_\mu = \gamma \left( -\nabla_a \phi 1 \right),$$

(A.4)

where $\gamma = (1 + f^{-2} \nabla^a \phi \nabla_a \phi)^{-1/2}$. From the embedding (A.2) the induced metric $g_{ab}$ and its inverse specialize to

$$g_{ab} = f^2 q_{ab} + \pi_a \pi_b \quad \text{and} \quad g^{ab} = f^{-2} \left( q^{ab} - \gamma^2 f^{-2} \pi^a \pi^b \right),$$

(A.5)

respectively, where $\nabla_a$ is the covariant derivative compatible with $q_{ab}$ and we have used the notation introduced in Sec. I. In addition, $\sqrt{g} = \sqrt{-q} \gamma^{-1} f^{p+1}$. Similarly, the extrinsic curvature of the worldvolume is

$$K_{ab} = \gamma \left( -\Pi_{ab} + 2 f^{-1} f' \pi_a \pi_b + f f' q_{ab} \right),$$

(A.6)

where $f' = \partial f / \partial \phi$. The rate change of $\gamma$ is given by

$$\nabla_a \gamma = \gamma f^{-1} f'(1 - \gamma^2) \pi_a - \gamma^3 f^{-2} \pi^b \Pi_{ab}. $$

(A.7)

With this expression we obtain a interesting relation involving the matrix

$$h_{ab} = \gamma f^{-1} f'(q_{ab} - f^{-2} \pi_a \pi_b) - \nabla_b (\gamma f^{-2} \pi_a).$$

(A.8)

The geometries provided by $G_{\mu \nu}$, $g_{ab}$ and $q_{ab}$ are closely connected. For instance, the connection symbols $\Gamma^\mu_{\alpha \beta}$ associated to $G_{\mu \nu}$ are related to the connection coefficients $\gamma^\alpha_{bc}$ associated to $q_{ab}$ as

$$\Gamma^\mu_{bc} = \gamma^\alpha_{bc},$$

(A.9)

$$\Gamma^{p+1}_{ab} = -f f' q_{ab},$$

(A.10)

$$\Gamma^a_{p+1} = f^{-1} f' \delta^a_b,$$

(A.11)
The characteristic determinant of the matrix $M$ is given by

$$\det(M) = \gamma_{bc} + \gamma^2 f^{-2} \pi^a \Pi_{bc} + f^{-1} f'(2\pi_b \delta^a_c - \gamma^2 \pi^a q_{bc} - 2\gamma^2 \pi^a \pi_b \pi_c).$$

(4)

Appendix B. Expansions

It is useful to consider the first traces of the matrix $h_{ab}$, Eq. (4)

$$\text{Tr}(h) = \gamma f^{-2} \left\{ -[\Pi] + \frac{\gamma^2}{f^2}[\pi^3] + f f'(p + 2 - \gamma^2) \right\},$$

(5)

$$\text{Tr}(h^2) = \gamma^2 f^{-4} \left\{ [\Pi^2] + 2\gamma^2 \frac{f}{f^2} \left\{ -[\pi^4] + f f'(1 - \gamma^2)[\pi^3] \right\} - 2 f f'[\Pi] + \frac{\gamma^4}{f^4}[\pi^3]^2 + f^2 f'(p + 4 - 4\gamma^2 + \gamma^4) \right\},$$

(6)

$$\text{Tr}(h^3) = \gamma^3 f^{-6} \left\{ -[\Pi^3] + 3 f f'[\Pi^2] - 3 f^2 f'^2 [\Pi] + 3\gamma^2 f^{-2}[\pi^5] + \gamma^6 f^{-6}[\pi^3]^3 - 3\gamma^2 f^{-1} f'(2 - \gamma^2)[\pi^4] - 3\gamma^4 f^{-4}[\pi^4][\pi^3] + 3\gamma^2 f^{-2}(1 - 3\gamma^2 + \gamma^4)[\pi^3] + 3\gamma^4 f^{-3} f'(1 - \gamma^2)[\pi^3]^2 + f^3 f' \left\{ p + 8 - 12\gamma^2 + 6\gamma^4 - \gamma^6 \right\} \right\}. $$

(7)

The characteristic determinant of the matrix $M^a b = \delta^a b + h^a b$ may be expressed as

$$\det(M^a b) = 1 + \frac{1}{h(1)} + \sum_{s=1}^{n} \prod_{a_1 \ldots a_s} h_{a_1 a_2 \ldots a_s},$$

(8)

where $s! h_{(s)} = \delta_{a_1 \ldots a_s} \cdot h_{a_1} h_{a_2} \ldots h_{a_s}$ denotes the determinant of the $s$-rowed minor. These minors can be expressed in terms of the traces of the $h^a b$

$$h_{(1)} = \text{Tr}(h),$$

(9)

$$h_{(2)} = \frac{1}{2} \left[ \text{Tr}(h^2) - \text{Tr}(h^3) \right],$$

(10)

$$h_{(3)} = \frac{1}{6} \left[ \text{Tr}(h^3) - 3\text{Tr}(h^2) \text{Tr}(h) + 2\text{Tr}(h^3) \right],$$

(11)

$$h_{(4)} = \frac{1}{24} \left[ \text{Tr}(h^4) + 8\text{Tr}(h^3) \text{Tr}(h) - 6\text{Tr}(h^2) \text{Tr}(h^2) + 3\text{Tr}(h^2)^2 - 6\text{Tr}(h^4) \right].$$

(12)

In some cases it will be useful to obtain the Taylor expansion of the square root of the characteristic determinant $\left[ \text{det}(\delta^a b + h^a b) \right]^{1/2}$ which may be obtained by using the well-known expansion $(1 + x)^{1/2} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 - \cdots$ for $|x| \leq 1$. Hence,

$$\left[ \text{det}(\delta^a b + h^a b) \right]^{1/2} = 1 + \frac{1}{2} \sum_{s=1}^{n} h_{(s)} - \frac{1}{8} \left( \sum_{s=1}^{n} h_{(s)} \right)^2 + \frac{1}{16} \left( \sum_{s=1}^{n} h_{(s)} \right)^3 - \cdots.$$
Thus, up to $O(h^6)$ we have

$$\det(\delta^a_b + h^a_b) = 1 + \frac{1}{2} \text{Tr}(h) - \frac{1}{8} \left[ 2\text{Tr}(h^2) - \text{Tr}(h) \right] + \frac{1}{48} \left[ 8\text{Tr}(h^3) - 6\text{Tr}(h^2)\text{Tr}(h) + \text{Tr}(h)^3 \right] - \frac{1}{384} \left[ 48\text{Tr}(h^4) - 32\text{Tr}(h^2)\text{Tr}(h) + 12\text{Tr}(h^2)^2 - \text{Tr}(h)^4 \right] + \frac{1}{3840} \left[ 384\text{Tr}(h^5) - 240\text{Tr}(h^4)\text{Tr}(h) + 80\text{Tr}(h^3)\text{Tr}(h)^2 - 20\text{Tr}(h)^2\text{Tr}(h)^3 \right] + 60\text{Tr}(h)\text{Tr}(h)^2 - 160\text{Tr}(h^2)\text{Tr}(h^3) + \text{Tr}(h)^5 \right] - O(h^6). \quad (B.10)$$

### Appendix C. Galileon tensors and vectors

At first few order we have the Galileon tensors

$$J^{ab}_{(0)} = f^{-2} (q^{ab} - \gamma^2 f^{-2} \pi^a \pi^b), \quad (C.1)$$

$$J^{ab}_{(1)} = \gamma f^{-4} \left\{ \left( \begin{array}{c} -[\Pi] + \frac{\gamma^2}{f^2} [\pi^3] + ff'(p+1) - \gamma^2 \pi^a \pi^b - \frac{\gamma^2}{f^2} \pi^{(a} \Pi^{b)c} \pi^c \end{array} \right\}, \quad (C.2)$$

$$J^{ab}_{(2)} = \gamma^2 f^{-6} \left\{ \left( \begin{array}{c} [\Pi]^2 - [\Pi^2] - 2 \frac{\gamma^2}{f^2} ([\Pi][\pi^3] - [\pi^4]) + 2 \frac{\gamma^2}{f^2} ff'(f^2[\Pi] + p[\pi^3]) - 2ff'[\Pi] + f^2 f'^2(p-1)(p+2-2\gamma^2) \right) q^{ab} + 2 \Pi^{(a} \Pi^{b)c} \right\} + 2 \left( [\Pi] + \frac{\gamma^2}{f^2} [\pi^3] + ff'(p-\gamma^2) \right) \Pi^{ab} + 4 \frac{\gamma^2}{f^2} ([\Pi] - pf') \pi^{(a} \Pi^{b)c} \pi^c \right\}$$

$$\left\{ \left( [\Pi]^2 - [\Pi^2] - 2ff'[\Pi] + f^2 f'^2(p-1)(p+2) \right) \pi^a \pi^b - 4 \frac{\gamma^2}{f^2} \pi^{(a} \Pi^{b)c} \Pi_{cd} \pi^d - 2 \frac{\gamma^2}{f^2} \Pi^{a} \Pi^{b} \pi^c \pi^d \right\} \right. \quad (C.3)$$

Similarly, at first few order we have the Galileon vectors

$$J^a_{(0)} = \gamma f^{-2} \pi^a, \quad (C.4)$$

$$J^a_{(1)} = \gamma^2 f^{-4} \left\{ ([\Pi] + ff'[p] \pi^a + \Pi^a_b \pi^b) \right\} \quad (C.5)$$

$$J^a_{(2)} = \gamma^3 f^{-6} \left\{ \left( \begin{array}{c} [\Pi]^2 - [\Pi^2] - 2ff'[p-1][\Pi] + f^2 f'^2(p-1)p \right) \pi^a - 2 ([\Pi] - ff'[p-1]) \Pi^a_b \pi^b + 2 \Pi^a_b \Pi^b_c \pi^c \end{array} \right\}. \quad (C.6)$$
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