Deformation Quantization of Almost Kähler Models and Lagrange–Finsler Spaces

Sergiu I. Vacaru

The Fields Institute for Research in Mathematical Science,
222 College Street, 2d Floor, Toronto M5T 3J1, Canada

November 15, 2007

Abstract

Finsler and Lagrange spaces can be equivalently represented as almost Kähler manifolds endowed with a metric compatible canonical distinguished connection structure generalizing the Levi Civita connection. The goal of this paper is to perform a natural Fedosov–type deformation quantization of such geometries. All constructions are canonically derived for regular Lagrangians and/or fundamental Finsler functions on tangent bundles.

Keywords: Deformation quantization, Finsler and Lagrange geometry, nonlinear connections, almost Kähler geometry.

MSC: 81S10, 53D55, 53B40, 53B35, 53D50, 53Z05

PACS: 02.40.-k, 02.90.+g, 02.40.Yy

1 Introduction

An almost Kähler manifold (space) is a Riemannian manifold \( \mathbb{K} = (M^{2n}, g) \) of even dimension \( 2n \) and metric \( g \) together with a compatible almost complex structure \( J \) such that the symplectic form \( \theta \triangleq g(J\cdot,\cdot) \) is closed. Therefore, a space \( \mathbb{K} \) defines a symplectic geometry with preferred metric and almost complex structures. In a more general case of almost Hermitian manifolds \( \mathbb{H} \), the form \( \theta \) is not integrable (not closed), see details in [1] [2].

Let us consider a smooth \( n \)-dimensional real manifold \( M \), its tangent bundle \( TM = (TM, \pi, M) \) with surjective projection \( \pi : TM \to M \), total
space $TM$ and base $M$, and denote $\overline{TM} = TM \setminus \{0\}$, where 0 means the image of the null cross-section of $\pi$. A Finsler space $F^n = (M, F)$ consists of a Finsler metric (fundamental function) $F(x,y)$ defined as a real valued function $F : TM \rightarrow \mathbb{R}$ with the properties that the restriction of $F$ to $\overline{TM}$ is a function 1) positive; 2) of class $C^\infty$ and $F$ is only continuous on $\{0\}$; 3) positively homogeneous of degree 1 with respect to $y^i$, i.e. $F(x, \lambda y) = |\lambda|F(x,y)$, $\lambda \in \mathbb{R}$; and 4) the Hessian $F_{ij} = (1/2)\partial^2 F / \partial y^i \partial y^j$, defined on $\overline{TM}$, is positive definite. There is a classical result by M. Matsumoto [3] that a Finsler geometry $F^n$ can be modelled as an almost Kähler space $F_K = (TM, F\theta)$, where $F\theta = F_g(J\cdot, \cdot)$ for $J$ adapted to a canonical nonlinear connection structure.

A Lagrange space $L^n = (M, L)$ is defined by a Lagrange fundamental function $L(x,y)$, i.e. a regular real function $L : TM \rightarrow \mathbb{R}$, for which the Hessian

$$L_{ij} = (1/2)\partial^2 L / \partial y^i \partial y^j$$

is not degenerate. The concept was introduced by J. Kern [4] and developed as a model of geometric mechanics by using methods of Finsler geometry in a number of works of R. Miron’s school on Lagrange–Finsler geometry and generalizations [5, 6, 7, 8, 9, 10, 11]. For researches interested in applications to modern physics, we note that in our approach the review [12] is a reference for everything concerning the geometry of nonholonomic manifolds and generalized Lagrange and Finsler spaces and applications to standard theories of gravity and gauge fields. It should be emphasized here that a Lagrange space $L^n$ is a Finsler space $F^n$ if and only if its fundamental function $L$ is positive and two homogeneous with respect to variables $y^i$, i.e. $L = F^2$.

For simplicity, in this paper we shall work in the bulk with Lagrange spaces, considering the Finsler ones to consist of a more particular, homogeneous, subclass.

This paper is motivated by the results of V. Oproiu [13, 14, 15] who proved that any Lagrange space can be realized as an almost Kähler model $L_K = (TM, L\theta)$, where $L\theta = L_g(J\cdot, \cdot)$ for $J$ adapted to a canonical nonlinear connection structure. This allows us to elaborate a Fedosov quantization scheme [16, 17, 18, 19] for Lagrange and Finsler spaces. We shall develop for nonholonomic tangent bundles the approach from Ref. [20] in order to prove that any Lagrange and/or Finsler space can be quantized by geometric deformation methods. The geometric constructions will be performed

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1 For coordinates $u^a = (x^i, y^a)$ on $TM$, when indices $i, j, a, b, \ldots$ run values $1, 2, \ldots n$, we get also coordinates on $TM$ if not all fiber coordinates $y^a$ vanish; in brief, we shall write $u = (x, y)$. 

2
for canonical nonlinear and linear connections in Lagrange and Finsler geometry, see similar results in Ref. [21], for Lagrange–Fedosov spaces and Fedosov nonholonomic manifolds provided with almost symplectic connection adapted to the nonlinear connection structure.

For simplicity, we shall work on tangent bundles even the results have a natural extension for N–anholonomic manifolds, i.e. manifolds provided with nonintegrable distributions defining nonlinear connection structures. Such generalizations of Lagrange–Finsler methods to geometric and (in our cases) nonholonomic deformations will be used for elaborating certain models of quantum gravity for lifts of the Einstein gravity on the tangent bundle [22] and almost Kähler models of the Einstein gravity constructed on nonholonomic semi–Riemannian manifolds [23].

The work is organized as follows: In Section 2 we establish our notations and remember the basic definitions and results on Lagrange and Finsler geometry, canonical nonlinear and distinguished connections and metric structures. We show how such geometric models on tangent bundles can be reformulated equivalently as almost Kähler nonholonomic structures. Section 3 is devoted to basic constructions in nonholonomic deformation and quantization. We consider star products for symplectic manifolds provided with nonlinear connection structure, define canonical distinguished connections adapted to the nonlinear connection and related almost complex structure. There are defined the Fedosov operators for Lagrange–Finsler spaces. This allows us, in Section 4, to formulate and sketch the proofs of Fedosov’s theorems for such nonholonomic tangent bundles and provide a deformation quantization of Lagrange spaces. Finally, we compute the important coefficient $c_0$ of zero degree of cohomology classes of quantized Lagrange spaces.

Conventions: We shall use "left–up" and "left–low" labels like $L^D$ and $L^g$ in order to emphasize that the geometric object $g$ is defined canonically by a regular Lagrange function $L$. A tensor analysis in Lagrange–Finsler spaces requires a more sophisticate system of notations, see details in [12]. Moreover, we use Einstein’s summation convention in local expressions. The system of notations is a general one used in a series of our works on nonholonomic Einstein spaces, generalized Ricci–Lagrange flows and nonholonomic deformation quantization.

Acknowledgments: The author is grateful to Professor Vasile Oproiu for very important discussions and references on almost Kähler models of Lagrange spaces.
2 Almost Kähler Lagrange Structures

In this section, we outline briefly the almost Kähler model of Lagrange and Finsler spaces [3, 13, 14, 15, 5, 6, 12].

A nonlinear connection (N–connection) \( N \) on a tangent bundle \( TM \) can be defined by a Whitney sum (nonholonomic distribution)

\[
TTM = hTM \oplus vTM,
\]

given locally to by a set of coefficients \( N^a_i(x, y) \) defined with respect to a coordinate basis \( \partial_\alpha = \partial / \partial u^\alpha, \partial_a = \partial / \partial y^a \) and its dual \( du^\beta = (dx^j, dy^b) \). In a particular case, we get linear connections for \( N^a_i = \Gamma^a_{ib}(x)y^b \).

The curvature of N–connection is defined as the Neijenhuis tensor

\[
\Omega^a_{ij} = \frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b}.
\]

Let \( L(x, y) \) be a regular Lagrangian with nondegenerate \( g_{ij}(1) \) and action integral

\[
S(\tau) = \int_0^1 L(x(\tau), y(\tau))d\tau
\]

for \( y^k(\tau) = dx^k(\tau)/d\tau \), for \( x(\tau) \) parametrizing smooth curves on a manifold \( M \) with \( \tau \in [0, 1] \). We can formulate certain very important results on geometrization of Lagrange mechanics.

- The Euler–Lagrange equations \( \frac{d}{d\tau} \frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0 \) are equivalent to the "nonlinear geodesic" (equivalently, semi–spray) equations

\[
\frac{d^2 x^k}{d\tau^2} + 2G^k(x, y) = 0,
\]

where

\[
G^k = \frac{1}{4}g^{kj} \left( y^i \frac{\partial^2 L}{\partial y^j \partial x^i} - \frac{\partial L}{\partial x^j} \right)
\]

defines the canonical N–connection (for Lagrange spaces)

\[
L N^a_j = \frac{\partial G^a(x, y)}{\partial y^j}.
\]

\(^2\)proofs consist straightforward computations
• The regular Lagrangian $L(x, y)$ defines a canonical (Sasaki type) metric structure on $\tilde{T}M$,

$$Lg = Lg_{ij}(x, y) \, e^i \otimes e^j + Lg_{ij}(x, y) \, e^i \otimes e^j; \tag{4}$$

where the preferred frame structure (defined linearly by $L_N a_i$) is $e_\nu = (e_i, e_a)$, where

$$e_i = \frac{\partial}{\partial x^i} - L_N a_i(u) \frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a}, \tag{5}$$

and the dual frame (coframe) structure is $e^\nu = (e_i, e_a)$, where

$$e^i = dx^i \quad \text{and} \quad e^a = dy^a + L_N a_i(u) dx^i, \tag{6}$$

satisfying nontrivial nonholonomy relations

$$[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W_i^{\gamma \beta} e_\gamma \tag{7}$$

with (antisymmetric) nontrivial anholonomy coefficients $W_i^{ab} = \partial_a N_i^b$ and $W_i^{ja} = \Omega_i^{ja}$.  

• We get a Riemann–Cartan canonical model $RC\, L$ on $T\, M$ of Lagrange space $L^n$ if we choose the canonical metrical distinguished connection $D = (hD, vD) = (\hat{L}^i_{jk}, \hat{C}^i_{jc})$ (in brief, $d$–connection, which is a linear connection preserving under parallelism the splitting $\tilde{(2)}$)

$$\hat{\Gamma}^i_{\ jk} = \hat{\Gamma}^i_{\ j\gamma} e^\gamma = \hat{L}^i_{\ jk} e^k + \hat{C}^i_{\ jc} e^c; \tag{8}$$

for $\hat{L}^i_{\ jk} = \hat{L}^a_{\ bk}, \hat{C}^i_{\ jc} = \hat{C}^a_{\ bc}$ in $\hat{\Gamma}^a = \hat{\Gamma}^a_{\ b\gamma} e^\gamma = \hat{L}^a_{\ bc} e^k + \hat{C}^a_{\ bc} e^\nu, \quad \text{and} \quad \hat{L}^a_{\ bc} = \frac{1}{2} g^{ib} (e_k g_{jh} + e_j g_{kh} - e_h g_{jk}), \quad \hat{C}^a_{\ bc} = \frac{1}{2} g^{ae} (e_b g_{ec} + e_c g_{eb} - e_e g_{bc}),$$

which are just the generalized Christoffel indices.  

We note that $RC\, L$ contains a nonholonomically induced torsion structure defined by 2–forms

$$\Omega^i = \hat{C}^i_{\ jc} e^j \wedge e^c \quad \text{and} \quad \Omega^a = -\frac{1}{2} \Omega_i^{aj} e^i \wedge e^j + (e_b N^a_i - \hat{L}_{bi}^a) e^i \wedge e^b \tag{9}$$

3for simplicity, in this work, we shall omit left labels $L$ in formulas if that will not result in ambiguities; we shall use boldface indices for spaces and objects provided or adapted to a N–connection structure.

4we contract ”horizontal” and ”vertical” indices following the rule: $i = 1$ is $a = n + 1$; $i = 2$ is $a = n + 2$; ... $i = n$ is $a = n + n$.
computed from Cartan’s structure equations

\[
\begin{align*}
d e^i - e^k \wedge \hat{\Gamma}^i_{jk} &= -\Omega^i_k, \\
d e^a - e^b \wedge \hat{\Gamma}^a_{ab} &= -\Omega^a_j,
\end{align*}
\]

in which the curvature 2–form is denoted \( \Omega^i_j \), see explicit formulas for coefficients in \([5, 6, 12]\) and formula \((22)\).

- In principle, we can work also with the torsionless Levi Civita connection \( L\nabla \), constructed for the same metric \((1)\) which with respect to \(N\)-adapted bases \((5)\) and \((6)\) is given by the same coefficients \((8)\) but subjected to the condition that they must solve the structure equations \((10)\) with \( \nabla \Omega^i = \nabla \Omega^a = 0 \) and \( \Omega^i_j \neq \nabla \Omega^i_j \). This provides a Riemann type geometrization \( RL \) on \( TM \) of Lagrange space \( L^n \). We note that \( \nabla \) does not preserve under parallelism the \(N\)-connection splitting \((2)\), i.e. it is not adapted to the \(N\)-connection structure defined canonically by a regular Lagrangian \( L \). From a formal point of view, we can work equivalently with both type of connections because \( \hat{\nabla} \) and \( L\nabla \) are uniquely defined by the same data \((Lg, L N^a_j)\), i.e. by the same fundamental function \( L \), in metric compatible forms, \( \hat{\nabla} Lg = hD Lg = vD Lg = 0 \) and \( L\nabla Lg = 0 \); even \( \hat{\nabla} \) has a nonzero torsion, it is induced canonically by the same \( Lg \) and \( L N^a_j \).

The canonical \(N\)-connection \( L N^a_j \) \((3)\) induces an almost Kähler structure defined canonically by a regular \(L(x, y) \) \([13, 14, 15]\) (in this paper, we use the constructions from \([5, 6, 12]\) ). We introduce an almost complex structure for \( L^n \) as a linear operator \( J \) acting on the vectors on \( TM \) following formulas

\[
J(e_i) = -e_i \quad \text{and} \quad J(e_i) = e_i,
\]

where the superposition \( J \circ J = -I \), for \( I \) being the unity matrix. The operator \( J \) reduces to a complex structure \( \mathcal{J} \) if and only if the distribution \((2)\) is integrable.

A regular Lagrangian \(L(x, y) \) induces a canonical 1–form

\[
L\omega = \frac{1}{2} \frac{\partial L}{\partial y^i} e^i
\]

and metric \( Lg \) \((1)\) induces a canonical 2–form

\[
L\theta = Lg_{ij}(x, y) e^i \wedge e^j.
\]

associated to \( J \) following formulas \( L\theta(X, Y) \overset{\Delta}{=} Lg(JX, Y) \) for any vectors \( X \) and \( Y \) on \( TM \) decomposed with respect to a \(N\)-adapted basis \((5)\).

We can prove the results:
1. A regular $L$ defines on $TM$ an almost Hermitian (symplectic) structure $L\theta$ for which $d L\omega = L\theta$;

2. The canonical N–connection $L N^a_j$ and its curvature have the properties

$$\sum_{ijk} L g_{i(i} L \Omega^k_{jk)} = 0, \quad L g_{ij,k} - L g_{ik,j} = 0, \quad e_k L g_{ij} - e_j L g_{ik} = 0,$$

where $(ijk)$ means symmetrization of indices and

$$L g_{ij,k} = e_k L g_{ij} - L B^s_{ik} L g_{sj} - L B^s_{jk} L g_{is},$$

for $L B^s_{ik} = e_i L N^s_k$, which means that the almost Hermitian model of a Lagrange space $L^n$ is an almost Kähler manifold with $d L\theta = 0$. We conclude that the triad $\mathcal{E}^{2n}(\tilde{T}M, L g, J)$ defines an almost Kähler space (see details in [13, 14, 15]).

Proofs of properties 1 and 2 follow from computation

$$d L\theta = \frac{1}{6} \sum_{(ijk)} L g_{is} L \Omega^s_{jk} e^i \wedge e^j \wedge e^k + \frac{1}{2} (L g_{ij,k} - L g_{ik,j}) e^i \wedge e^j \wedge e^k + \frac{1}{2} (e_k L g_{ij} - e_i L g_{kj}) e^k \wedge e^i \wedge e^j.$$

The next step is to define the concept of almost Kähler d–connection $\hat{\theta} D$, which is compatible both with the almost Kähler ($L\theta, J$) and N–connection structures $L N$, and satisfies the conditions

$$\hat{\theta} D_X L g = 0 \quad \text{and} \quad \hat{\theta} D_X J = 0,$$

for any vector $X = X^i e_i + X^a e_a$. By a straightforward computation, we prove (see details in [5, 6]):

**Theorem 2.1** The canonical d–connection $\hat{\Gamma}^a_{\beta\gamma} = (\hat{L}^a_{bk}, \hat{\Omega}^a_{ij})$ with coefficients \( \hat{L}^a_{bk} \) defines also a (unique) canonical almost Kähler d–connection $\hat{\theta} D = \hat{D}$ for which, with respect to N–adapted frames (5) and (6), the coefficients $\hat{T}^a_{jk} = 0$, torsion vanishes on $hTM$, and $\hat{T}^a_{bc} = 0$, torsion vanishes on $vTM$, but there are cross non–zero coefficients of type (4), $\hat{T}^i_{jc} = \hat{\Omega}^i_{jc}$, $\hat{T}^a_{ij} = \Omega^a_{ij}$ and $\hat{T}^a_{ib} = e_b N^a_i - L^a_{bi}$.

There are two important particular cases: If $L = F^2$, for a Finsler space, we get a almost Kähler model of Finsler space \( \mathcal{E} \), when $\hat{\theta} D = \hat{D}$ transforms in the so–called Cartan–Finsler connection [24]. We get a Kählerian model of a Lagrange, or Finsler, space if the respective almost complex structure $J$ is integrable.
3 Nonholonomic Deformations and Quantization

The geometry of Lagrange–Fedosov manifolds was investigated in Ref. [21]. The aim of this section is to provide a nonholonomic modification of Fedosov’s constructions in order to perform a geometric quantization of Lagrange (in particular, Finsler) spaces provided with canonical metric and nonlinear and linear connection structures defined by respective fundamental Lagrange (Finsler) functions, see next section. We shall use the approach to Fedosov quantization of geometries with arbitrary metric compatible affine connections on almost Kähler manifolds and related symplectic structures for a manifolds $M$ elaborated in Ref. [20]. We shall redefine the constructions from $M$ and $TM$, respectively, on $TM$ and $TTM$ endowed with canonical N–connection, metric, symplectic and almost Kähler structures uniquely defined by fundamental Lagrange (Finsler) functions.

3.1 Star products for symplectic manifolds

Let us denote by $C^\infty(V)[[v]]$ the spaces of formal series in variable $v$ with coefficients from $C^\infty(V)$ on a Poisson manifold $(V, \{\cdot, \cdot\})$. Following Refs. [25, 26, 27], a deformation quantization is an associative algebra structure on $C^\infty(V)[[v]]$ with a $v$–linear and $v$–adically continuous star product

$$1f \star 2f = \sum_{r=0}^{\infty} rC(1f, 2f) v^r,$$

where $rC, r \geq 0$, are bilinear operators on $C^\infty(V)$ with $0C(1f, 2f) = 1f 2f$ and $1C(1f, 2f) - 1C(2f, 1f) = i\{1f, 2f\}$, with $i$ being the complex unity. Following conventions from [12], we use ”up” and ”low” left labels which are convenient to be introduced on Finsler like spaces in order to not create confusions with a number of ”horizontal” and ”vertical” indices and labels which must be distinguished if the manifolds are provided with N–connection structure. We note that, in our further constructions, the manifold $V$ will be a tangent bundle, $V = TM$, or a nonholonomic manifold $V = V$, (for instance, a Riemann–Cartan manifold) provided with a nonholonomic distribution defining a N–connection.

If all operators $rC, r \geq 0$ are bidifferential, a corresponding star product $\star$ is called differential. We can define different star products on a $(V, \{\cdot, \cdot\})$. Two differential star products $\star$ and $\star'$ are equivalent if there is an isomorphism of algebras $A : (C^\infty(V)[[v]], \star) \to (C^\infty(V)[[v]], \star')$, where $A = \sum_{r \geq 1} rA v^r$, for $0A$ being the identity operator and $rA$ being differential
operators on $C^\infty(V)$.

For a particular case of Poisson manifolds, when $(V, \theta)$ is a symplectic manifold, each differential star product $*$ on $V$ belongs to its characteristic class $cl(*) \in (1/iv)[\theta] + H^2(V, \mathbb{C})[[v]]$, where $\mathbb{C}$ is the field of complex numbers, and $(1/iv)[\theta] + H^2(V, \mathbb{C})[[v]]$ is an affine vector space, see details in [18, 28, 29]. For symplectic structures, the equivalence classes of differential star products on $(V, \theta)$ can be bijectively parametrized by the elements of the corresponding affine vector space using the map $* \rightarrow cl(*)$.

The bibliography on existence proofs, methods and descriptions of equivalent classes for the first examples of star products (Moyal–Weyl and Wick star products, asymptotic expansions with a numerical parameter, Planck constant, $\hbar \rightarrow 0$, Berezin–Toeplitz deformation quantization) is outlined in Refs. [20, 30, 31, 32, 33, 34]. A very important conclusion following from the above–mentioned works is that a natural deformation quantization can be constructed on an arbitrary compact almost Kähler manifold. The question of existence of deformation quantization and corresponding geometric formalism on general Poisson manifolds were solved in the Kontsevich’s work [35, 36]. In this work, we are interested to describe this deformation quantization for Lagrange–Finsler spaces defined by corresponding nonholonomic structures on arbitrary almost Kähler spaces.

### 3.2 Canonical d–connections and complex structures

The deformation quantization using Fedosov’s machinery can be also constructed for a natural class of affine connections, in general, considered by Yano [37]. For Lagrange–Finsler spaces and their almost Kähler models, we work with the canonical d–connection (8): the constructions will be re–defined with respect to $N$–adapted bases in a form when the results from [20] will hold true for nontrivial nonholonomic structures.

Let $\mathbb{K}^{2n} = (\hat{T}M, Lg, J)$ defines an almost Kähler model of Lagrange space with canonical d–connection $\theta \hat{\nabla} = \hat{\nabla}$. For a chart $U \subset TM$, we set the local coordinates $x^a = (x^i, y^a)$ and parametrize $J(e_i) = -e_i$ and $J(e_i) = e_i$, for $e_\alpha = (e_i, e_a)$, and denote

\[ J(e_\alpha) = J^{\alpha'}{}_{\alpha} e_{\alpha'}, \quad \text{or} \quad J(e_i) = J_i{}^i e_i \quad \text{and} \quad J(e_a) = J_a{}^a e_a. \]  

(13)

We also write

\[ \theta_{\alpha\beta} = \theta(e_\alpha, e_\beta), \quad \text{or} \quad \theta_{ij} = \theta(e_i, e_j) \quad \text{and} \quad \theta_{ab} = \theta(e_a, e_b), \]  

(14)

corresponding to metric (4) with

\[ Lg_{\alpha\beta} = Lg(e_\alpha, e_\beta), \quad \text{or} \quad Lg_{ij} = Lg(e_i, e_j) \quad \text{and} \quad Lg_{ab} = Lg(e_a, e_b). \]  

(15)
Using the inverse matrices correspondingly to those ones considered above, we can write
\[
J_{\alpha}' = L_{\gamma} g_{\alpha \beta} g_{\beta \gamma} = L_{\gamma} g_{\alpha \beta} \theta_{\beta \gamma}
\]
or in horizontal and vertical (in brief, h- and v-) components,
\[
J_i' = L_{g_{ij}} g_{ij} \theta_{ji} = L_{g_{ij}} g_{ij} \theta_{ji}
\]
and
\[
J_a' = L_{g_{ab}} g_{ab} \theta_{ba} = L_{g_{ab}} g_{ab} \theta_{ba}.
\]
We can define \(J_{\alpha}'\) as the inverse to \(J_{\alpha}\). In our further considerations, we shall omit decompositions into h- and v-components if it would be possible to write formulas in a more compactified form with Greek indices not creating ambiguities in distinguishing the nonholonomic N–connections structure.

The Nijenhuis tensor \(\Omega\) for the complex structure \(J\) on \(TM\) provided with N–connection \(N\) is defined in the form
\[
\Omega(X, Y) = [JX, JY] - J[JX, Y] - J[X, Y] - [X, Y]
\]
where \(X\) and \(Y\) are vector fields on \(TM\).

Let \(D = \{\Gamma_{\alpha \beta}^\gamma\}\) be any metric compatible d–connection on \(TM\) (it is any affine connection preserving the splitting \((2)\) and satisfying \(D(g) = 0\) for a given metric structure \(g\)). A component calculus with respect to N–adapted bases \((5)\) and \((6)\), for \(\Omega(e_\alpha, e_\beta) = \Omega_{\alpha \beta}^\gamma e_\gamma\) results in
\[
\Omega_{\alpha \beta}^\gamma = 4 T_{\alpha \beta}^\gamma
\]
where \(T_{\alpha \beta}^\gamma\) is the torsion of \(\Gamma_{\alpha \beta}^\gamma\). For the case of Lagrange spaces with canonical d–connection \(\hat{D}\) \((8)\), the non–trivial components \(\hat{T}_{\alpha \beta}^\gamma\) of the are given by nonzero components of 2–forms \((9)\) being defined completely by \(L_{g_{ij}}\) and \(L_{N_{\alpha i}}\) in their turn, generated by a regular \(L(x, y)\).

**Proposition 3.1** Any metric compatible d–connection \(D\) with the torsion given by formula \((16)\) respects the symplectic form \(\theta(X, Y) = g(JX, Y)\) and therefore the complex structure \(J\).

**Proof.** It consists a straightforward verification of conditions \(D_X g = 0\) and \(D_X J = 0\) which is a N–adapted calculus with respect to \((5)\) and \((6)\), see similar computations proving the Proposition 2.1 in Ref. [20]. In our case,

5 We chose a definition of this tensor as in Ref. [6] which is with minus sign comparing to the definition used in [20]; we also use a different letter for this tensor, like for the N–connection curvature, because in our case the symbol "N" is used for N–connections.

6 This formula is a nonholonomic analog, for our conventions, with inverse sign, of the formula (2.9) from [20].
we work not with affine metric compatible connections but with respective d–connections.

As a consequence of Theorem 2.1 and Proposition 3.1, we have:

**Corollary 3.1** The unique metric compatible canonical d–connection \( \hat{\Gamma}^\alpha_{\beta\gamma} \)
\[ = \left( \hat{L}_b^a, \hat{C}^a_{bc} \right) \] (8), with torsion components \( \hat{T}^i_{jk} = 0 \), \( \hat{T}^a_{bc} = 0 \), \( \hat{T}^i_{jk} = \hat{C}^i_{jc}, \hat{T}^a_{ij} = \Omega^a_{ij} \) and \( \hat{T}^a_{ib} = c_bN^a_i - \hat{L}_b^a \) computed with respect to N–adapted bases (3) and (7), satisfies the formula \( \Omega_i^\gamma = 4\hat{T}_i^\gamma_{\alpha\beta} \) and respects the canonical symplectic structure \( L^0 (11) \) constructed for the canonical N–connection \( LN^a_i (3) \) and metric \( Lg (4) \). In a particular case, for \( L = F^2 \), similar results hold true for the Cartan connection on Finsler spaces and respective canonical Finsler symplectic, metric and N–connection structures.

We conclude that geometrizing a Lagrange, or Finsler, space in terms of geometric objects of a nonholonomic almost Kähler manifold we can perform directly a natural deformation quantization by adapting the constructions from Ref. [20] to the canonical N–connection, and for respective canonical geometrical objects (like metric, symplectic form, d–connection, induced by fundamental Lagrange, or Finsler, functions).

Our further considerations will consist from a generalization to the nonholonomic tangent bundles \( TM \) of the results and methods elaborated in the above–mentioned reference for usual manifolds \( M \) endowed with metric compatible affine connections and respective almost Kähler structures. In other turn, we shall emphasize the constructions only for the case of canonical Lagrange, Finsler, geometric objects. For simplicity, we shall omit details for local computations and proofs if they will be certain "N–adapted" Lagrange–Finsler analogs of formulas and results obtained in [20, 37] or [16, 17, 18, 19].

### 3.3 Fedosov operators for Lagrange–Finsler spaces

In this section, we modify Fedosov’s constructions to provide all data necessary for deformation quantization of almost Kähler models of Lagrange–Finsler spaces.

On \( TM \) endowed with canonical Lagrange structures, we introduce the tensor \( L^A_{\alpha\beta} L^\alpha\beta = Lg^\alpha\beta - i Lg^\alpha\beta \), see related formulas (4), (11), (14) and (15). The local coordinates on \( TM \) are parametrized in the form \( u = \{ u^\alpha \} \) and the local coordinates on \( T_u TM \) are labelled \( (u, z) = (u^\alpha, z^\beta) \), where \( z^\beta \) are
the second order fiber coordinates. We shall work with formal series

\[ a(v, z) = \sum_{r \geq 0, |\alpha| \geq 0} a_{r, \alpha}(u)z^{\alpha}v^r, \quad (17) \]

where \(\alpha\) is a multi-index, defining the formal Wick algebra \(W_u\) (we use a boldface letter in order to emphasize what we perform our constructions for spaces provided with \(N\)-connection structure). We use the formal Wick product on \(W_u\), for two elements \(a\) and \(b\) defined by formal series of type (17),

\[ a \circ b (z) \equiv \exp \left( i \frac{v}{2} L \Lambda^{\alpha\beta} \frac{\partial^2}{\partial z^\alpha \partial z_1^\beta} \right) a(z) b(z_1) |_{z=z_1}. \quad (18) \]

It is possible to construct a nonholonomic bundle \(W = \bigcup_u W_u\) of formal Wick algebras defined as a union of algebras \(W_u\) distinguished by the \(N\)-connection structure, see Refs. [39, 40] on such "d-algebras", for instance, on gauge and spinor field geometries adapted to \(N\)-connection structures. The fibre product (18) can be trivially extended to the space of \(W\)-valued \(N\)-adapted differential forms \(LW \otimes \Lambda\) by means of the usual exterior product of the scalar forms \(\Lambda\), where \(LW\) denotes the sheaf of smooth sections of \(W\) (we put the left label \(L\) in order to emphasize that the constructions are adapted to the canonical \(N\)-connection structure induced by a regular Lagrangian). There is a standard grading on \(\Lambda\), denoted \(\text{deg}_\lambda\). It is possible to introduce grading \(\text{deg}_v, \text{deg}_s, \text{deg}_a\) on \(LW \otimes \Lambda\) defined on homogeneous elements \(v, z^\alpha, e^\alpha\) as follows: \(\text{deg}_v(v) = 1, \text{deg}_s(z^\alpha) = 1, \text{deg}_a(e^\alpha) = 1\), and all other gradings of the elements \(v, z^\alpha, e^\alpha\) are set to zero. In this case, the product \(\circ\) from (18) on \(LW \otimes \Lambda\) is bigraded, we write w.r.t. the grading \(\text{Deg} = 2 \text{deg}_v + \text{deg}_s\) and the grading \(\text{deg}_a\), see also conventions from [20].

The canonical d-connection \(L\hat{\nabla} = \{ L\hat{\Gamma}^{\gamma}_{\alpha\beta} \}\) can be extended to an operator on \(LW \otimes \Lambda\) following the formula

\[ L\hat{\nabla} (a \otimes \lambda) \equiv \left( e_\alpha(a) - u^\beta L\hat{\Gamma}^{\gamma}_{\alpha\beta} z e_\alpha(a) \right) \otimes (e^\alpha \wedge \lambda) + a \otimes d\lambda, \quad (19) \]

where \(e_\alpha\) and \(e^\alpha\) are defined respectively by formulas (5) and (6) and \(z e_\alpha\) is similar to \(e_\alpha\) on \(N\)-anholonomic fibers of \(TTM\), depending on \(z\)-variables (for holonomic second order fibers, we can take \(z e_\alpha = \partial/\partial z^\alpha\)). For second order mechanical, or Finsler, models, \(z e_\alpha\) can be constructed canonically from higher order Lagrangians and respective semi-spray configurations and \(N\)-connections [7, 8, 9, 10, 11]. In superstring theory and nonholonomic (super)gravity and higher order spinor structures, such effective higher order
N–connections have to be defined from (super) vielbein configurations \([41, 42]\). It should be noted that the operator \([19]\) can be similarly defined for arbitrary metric compatible d–connection \(D = \{\Gamma_{\alpha\beta}\}\) and arbitrary N–connection structures on \(TTM\), but for purposes of this paper we consider only the case of geometric objects induced canonically by a fundamental function \(L\), or \(F\). Using formulas \([18]\) and \([19]\), we can show that \(\hat{L}\hat{D}\) is a N–adapted deg\(_s\)–graded derivation of the distinguished algebra \((L^W \otimes \Lambda, \circ)\), in brief, one call d–algebra.

Now, we can introduce on \(L^W \otimes \Lambda\) the Fedosov operators \(L\delta\) and \(L\delta^{-1}\) (we put additional left labels in order to emphasize that in this work they are completely generated by a regular Lagrange, or Finsler, canonical structure on \(TM\)):

\[
L\delta(a) = e^\alpha \wedge z e_\alpha(a),
\]

\[
L\delta^{-1}(a) = \begin{cases} 
\frac{1}{p+q} z^0 e_\alpha(a), & \text{if } p+q > 0, \\
0, & \text{if } p = q = 0,
\end{cases}
\]

where \(a \in L^W \otimes \Lambda\) is homogeneous w.r.t. the grading \(\deg_s\) and \(\deg_a\) with \(\deg_s(a) = p\) and \(\deg_a(a) = q\). We get the formula

\[
a = (L\delta L\delta^{-1} + L\delta^{-1} L\delta + \sigma)(a)
\]

where \(a \mapsto \sigma(a)\) is the projection on the \((\deg_s, \deg_a)\)–bihomogeneous part of \(a\) of degree zero, \(\deg_s(a) = \deg_a(a) = 0\). We can verify that \(L\delta\) is also a deg\(_a\)–graded derivation of the d–algebra \((L^W \otimes \Lambda, \circ)\).

The Lagrange canonical d–connection \(\hat{L}\hat{D}\) \([5]\) on \(TM\) induces respective torsion and curvature on \(L^W \otimes \Lambda\),

\[
\hat{T} = \frac{z^0}{2} L\theta_{\gamma\tau} \hat{T}_{\alpha\beta}^\tau(u) e^\alpha \wedge e^\beta
\]

and

\[
\hat{R} = \frac{z^0 z^0}{4} L\theta_{\gamma\tau} \hat{R}_{\varphi\alpha\beta}^\tau(u) e^\alpha \wedge e^\beta
\]

where the torsion \(\hat{T}_{\alpha\beta}^\tau\) \([9]\) has nontrivial components \(\hat{T}_{jk}^i = \hat{\gamma}_{jk}^i, \hat{T}_{ij}^a = \Omega_{ij}^a\) and \(\hat{T}_{ib}^a = e_b N_i^a - \hat{L}_b^a\) and the curvature \(\hat{R}_{\varphi\alpha\beta}^\tau\) with nontrivial components

\[
\hat{R}_{hjk}^i = e_k \hat{L}^i_{hj} - e_j \hat{L}^i_{hk} + \hat{L}^m_{hj} \hat{L}^i_{mk} - \hat{L}^m_{hk} \hat{L}^i_{mj} - \hat{\gamma}_{ha} \Omega_{kj}^a,
\]

\[
\hat{R}_{jka} = e_a \hat{L}^i_{jk} - \hat{D}_k \hat{C}_{ija}^i, \quad \hat{S}_{bcd} = e_d \hat{C}_{abc} - e_c \hat{C}_{abd} + \hat{C}_{bc} \hat{C}_{ed} - \hat{C}_{bd} \hat{C}_{ec},
\]

all computed in a form when there are solved the structure equations \([10]\).
Using the formulas (17) and (18) and the identity
\[ L_{\theta^\tau} \hat{\mathcal{R}}^\tau_{\gamma \alpha \beta} = L_{\theta^\gamma} \hat{\mathcal{R}}^\tau_{\phi \alpha \beta}, \]
(23)
we prove the formulas
\[ \left[ L_{\hat{\mathcal{D}}}, L_\delta \right] = \frac{i}{v} \text{ad}_{\text{Wick}}(\hat{T}) \quad \text{and} \quad L_{\hat{\mathcal{D}}}^2 = -\frac{i}{v} \text{ad}_{\text{Wick}}(\hat{R}), \]
(24)
where \([·, ·]\) is the deg\(_a\)-graded commutator of endomorphisms of \( L^W \otimes \Lambda \) and \( \text{ad}_{\text{Wick}} \) is defined via the deg\(_a\)-graded commutator in \( (L^W \otimes \Lambda, \circ) \).

4 Fedosov Quantization of Lagrange Spaces

We generalize the standard statements of Fedosov’s theory for the case of Lagrange–Finsler spaces provided with canonical N–connection and d–connection structures. The class \( c_0 \) of the deformation quantization of Lagrange geometry is calculated.

4.1 Fedosov’s theorems for Lagrange–Finsler spaces

Using the formalism of Fedosov operators on Lagrange spaces, we formulate and sketch the proof of two theorems generalizing similar ones to the case of N–connection and metric compatible d–connection geometries on \( TM \), induced by fundamental Lagrange (Finsler) functions.

Let us denote the Deg–homogeneous component of degree \( k \) of an element \( a \in L^W \otimes \Lambda \) by \( a^{(k)} \).

**Theorem 4.1** For any regular Lagrangian \( L \) on \( \tilde{T}M \), there is a flat canonical Fedosov d–connection
\[ L_{\hat{\mathcal{D}}} \doteq -L_\delta + L_{\hat{\mathcal{D}}} - \frac{i}{v} \text{ad}_{\text{Wick}}(r) \]
satisfying the condition \( L_{\hat{\mathcal{D}}}^2 = 0 \), where the unique element \( r \in L^W \otimes \Lambda \), deg\(_a\)(\( r \)) = 1, \( L_\delta^{-1} r = 0 \), solves the equation
\[ L_\delta r = \hat{T} + \hat{R} + L_{\hat{\mathcal{D}}}r - \frac{i}{v} \circ r \]

\(^7\)It should be noted that formulas (20), (21), (22), (23) and (24) can be written for any metric \( g \) and metric compatible d–connection \( D, Dg = 0 \), on \( TM \), provided with arbitrary N–connection \( N \) (we have to omit “hats” and labels \( L \)). It is a more sophisticated problem to define such constructions for Finsler geometries with the so-called Chern connection which are metric noncompatible [13]. For applications in standard models of physics, we chose the variants of Lagrange–Finsler spaces defined by metric compatible d–connections, see discussion in [12].
and this element can be computed recursively with respect to the total degree $\text{Deg}$ as follows:

$$
\begin{align*}
\tau^{(0)} &= 0, \quad \tau^{(1)} = 0, \quad \tau^{(2)} = L \delta^{-1} \hat{T}, \\
\tau^{(3)} &= L \delta^{-1} \left( \hat{R} + L \hat{D} \tau^{(2)} - \frac{i}{v} \tau^{(2)} \circ \tau^{(2)} \right), \\
\tau^{(k+3)} &= L \delta^{-1} \left( L \hat{D} \tau^{(k+2)} - \frac{i}{v} \sum_{l=0}^{k} \tau^{(l+2)} \circ \tau^{(l+2)} \right), \quad k \geq 1.
\end{align*}
$$

**Proof.** We sketch the idea of proof which is similar to the standard Fedosov constructions but $N$–adapted. By induction, we use the identities

$$
L \delta \hat{T} = 0 \quad \text{and} \quad L \delta \hat{R} = L \hat{D} \hat{T}.
$$

In Ref. [20], these identities were proved for arbitrary affine connections with torsion and almost Kähler structures on $M$. In our case, we work with a particular class of geometric objects, induced canonically from $L^a$, on $TM$. In another turn, the constructions are generalized to nonholonomic bundles. □

We note that the canonical Fedosov d–connection $L \hat{D}$ is a deg$_a$–graded derivation of the algebra $(L \mathcal{W} \otimes \Lambda, \circ)$. This means that $L \mathcal{W} _\hat{D} \cong \ker (L \hat{D}) \cap L \mathcal{W}$ is $N$–adapted subalgebra of $(L \mathcal{W}, \circ)$.

The next theorem gives a rule how to define and compute the star product induced by a regular Lagrangian.

**Theorem 4.2** A star–product on the canonical almost Kähler model of Lagrange (Finsler) spaces $\mathbb{K}^{2n} = (TM, L \mathbf{g}, J)$ is defined on $C^\infty(\hat{T}M)[[v]]$ by formula

$$1f \ast 2f = \sigma(\tau(1f)) \circ \sigma(2f),$$

where the projection $\sigma : L \mathcal{W} _\hat{D} \rightarrow C^\infty(\hat{T}M)[[v]]$ onto the part of deg$_a$–degree zero is a bijection and the inverse map $\tau : C^\infty(\hat{T}M)[[v]] \rightarrow L \mathcal{W} _\hat{D}$ can be calculated recursively w.r.t. the total degree $\text{Deg}$,

$$
\begin{align*}
\tau(f)^{(0)} &= f \quad \text{and, for } k \geq 0, \\
\tau(f)^{(k+1)} &= L \delta^{-1} \left( L \hat{D} \tau(f)^{(k)} - \frac{i}{v} \sum_{l=0}^{k} \text{ad}_{\text{Wick}}(\tau^{(l+2)}) \tau(f)^{(l)} \right),
\end{align*}
$$

\hspace{10cm} (15)
Proof. We note that the connection $L\hat{D}$ and its almost Kähler version defined by Proposition 3.1, Theorems 2.1 and 4.1 in the case of almost Kähler manifolds is a special N–adapted case of the star–product constructed in Ref. [38]. □

The statements of the above presented Fedosov’s theorems generalized for Lagrange–Finsler spaces can be extended for arbitrary metric compatible d–connections on $\tilde{T}M$. For Finsler spaces, we can use the so–called R. Miron’s procedure of computing all metric compatible d–connections for a given metric $g$, see Refs. [5, 6] (from a formal point of view, we shall have the same formulas without ”hats” and $L$–labels, but with arbitrary d–torsions and corresponding curvatures). It should be noted that there is also the so–called Kawaguchi metrization procedure, which allows to work with metric noncompatible d–connections, described in details in the above–mentioned Miron and Anastasiei monographs. In Ref. [44], such constructions were elaborated for nonholonomic manifolds with the aim to apply Finsler methods in modern gravity theories.

4.2 Cohomology classes of quantized Lagrange spaces

It follows from the results obtained in [34] that the characteristic class of the star product from [38] is $(1/iv)[\theta] - (1/2i)\varepsilon$, where $\varepsilon$ is the canonical class for an underlying Kähler manifold. This canonical class can be defined for any almost complex manifold. In this section, we calculate the crucial part of the characteristic class $cl$ of the star product $*$ which we have constructed in Theorem 4.2 for an almost Kähler model of Lagrange (Finsler) spaces, i.e. we shall compute the coefficient $c_0$ at the zeroth degree of $v$. Only the coefficient $c_0(*)$ of the class $cl(*) = (1/iv)[L\theta] + c_0(*) + ...$ can not recovered from Deligne’s intrinsic class [28]. Here we also note that the cohomology class of the formal Kähler form parametrizing a quantization with separation of variables on a Kähler manifold differs from the characteristic class of this quantization only in the coefficient $c_0$ as it is proved in [34].

Let us recall the rigorous definition of the class $c_0$ of a star–product (12) (see details, for instance, in Ref. [19]) adapting the constructions for tangent bundles provided with N–connection structure. One denotes by $f\xi$ the corresponding Hamiltonian vector field corresponding to a function $f \in C^\infty(TM)$ on a symplectic tangent bundle $(TM, \theta)$ and considers the antisymmetric part

$$-C(\,^1f, \,^2f) = \frac{1}{2} (C(\,^1f, \,^2f) - C(\,^2f, \,^1f))$$

16
of bilinear operator $C(1f, 2f)$. A star–product (12) is normalized if

$$1C(1f, 2f) = \frac{i}{2}\{1f, 2f\},$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. For normalized $*$ the bilinear operator $\frac{1}{2}C$ is a de Rham–Chevalley 2–cocycle. In this case, there is a unique closed 2–form $L\kappa$, induced by a regular Lagrangian $L$, such that

$$2C(1f, 2f) = \frac{1}{2}L\kappa(f_1\xi, f_2\xi)$$

for all $1f, 2f \in C^\infty(TM)$. The class $c_0$ of a normalized star–product $*$ is stated as the equivalence class $c_0(\star) \equiv [L\kappa]$.

A fiberwise equivalence operator on $L\mathcal{W}$ can be defined by the formula

$$L\hat{G} \equiv \exp(-v L\triangle),$$

where

$$L\triangle = \frac{1}{8} Lg^{\alpha\beta}(ze_\alpha ze_\beta + ze_\beta ze_\alpha)$$

for nonholonomic configurations on the second order fibers on $TTM$, or

$$L\triangle = \frac{1}{4} Lg^{\alpha\beta} \frac{\partial^2}{\partial z^\alpha \partial z^\beta}$$

if we elaborate a model with trivial N–connection for the second order fibres on $TTM$. We can check directly the formulas

$$\left[ L\hat{D}, L\triangle \right] = \left[ L\hat{D}, L\hat{G} \right] = 0 \text{ and } \left[ L\delta, L\triangle \right] = \left[ L\delta, L\hat{G} \right] = 0, \quad (26)$$

which allows us to define a fibrewise star–product on $L\mathcal{W}$,

$$a \circ' b \equiv L\hat{G}(L\hat{G}^{-1}a \circ L\hat{G}^{-1}b),$$

which is the Weyl star–product

$$a \circ' b(z) = \exp\left(\frac{iv}{4} L\theta^{\alpha\beta}(e_\alpha z e_\beta - ze_\beta e_\alpha)\right);$$

$$= \exp\left(\frac{iv}{2} L\theta^{\alpha\beta}\left(e_\alpha \frac{\partial}{\partial z^\beta} - \frac{\partial}{\partial z^\beta} e_\alpha\right)\right),$$

for holonomic 2d order fibres.
The next step is to push forward the Fedosov d–connection \( L\hat{D} \) from Theorem 4.1 using formulas (26). We get a new canonical d–connection operator

\[
L\hat{D}' = LGL\hat{D}L^{-1} = L\delta + L\hat{D} - \frac{i}{\nu}ad_{\text{Weyl}}(r'),
\]

where \( r' = LG r \) and \( ad_{\text{Weyl}} \) is calculated with respect to the \( \sigma' \)–commutator.

For symplectic manifolds, it is well known that each star–product is equivalent to a normalized one. The class \( c_0(\ast) \) of a star product \( \ast \) is defined as the cohomology class \( c_0(\ast') \) of an equivalent normalized star–product \( \ast' \). We have first to construct an equivalent normalized star–product in order to calculate the class \( c_0(\ast) \) for \( \ast \) from Theorem 4.2. This procedure is described in details in section 4 of Ref. [20] for arbitrary affine metric compatible connection on a manifold \( M \). In our case, those formulas have to be redefined with respect to \( N \)–adapted bases and canonical d–connection, \( N \)–connection and metric structures. For simplicity, we omit in this work such tedious but trivial generalizations but present only the most important formulas and definitions.

A straightforward computation of \( _2C \) from (25), using statements of Theorem 4.1, results in a proof of

\textbf{Lemma 4.1} The unique 2–form \( L\kappa \) can be expressed in the form

\[
L\kappa = -\frac{i}{8} J_{\tau}^{\alpha'} \hat{R}_{\alpha'\beta} \tau e^{\alpha} \wedge e^{\beta} - i L\lambda ,
\]

where the exact \( N \)–adapted 1–form \( L\mu = d L\mu \), for

\[
L\mu = \frac{1}{6} J_{\tau}^{\alpha'} \hat{T}_{\alpha'\beta} \tau e^{\beta} ,
\]

with nontrivial components of curvature and torsion defined by the canonical d–connection computed following formulas (9) and (22).

For trivial \( N \)–connection structures and arbitrary metric and metric compatible affine connections, the Lemma 4.1 is equivalent to statements of Lemma 4.1 from [20], in our case, redefined for Riemann–Cartan geometries modelled on \( TM \). We reformulated the results in a form when generalizations for arbitrary metric \( g \) compatible d–connection and \( N \)–connection structures, \( D \) and \( N \) on \( TM \), can be performed following the formal rule of omitting ”hats” and ”\( L \)–labels".
Let us recall the definition of the canonical class $\varepsilon$ of an almost complex manifold $(M, \mathbb{J})$ and redefine it for $N^{TTM} = hTM \oplus vTM$ stating a $N$–connection structure $\mathbb{N}$. The distinguished complexification of such second order tangent bundles is introduced in the form

$$T_C\left(N^{TTM}\right) = T_C(hTM) \oplus T_C(vTM).$$

For such nonholonomic bundles, the class $N\varepsilon$ is the first Chern class of the distributions $T_C\left(N^{TTM}\right) = T_C(hTM) \oplus T_C(vTM)$ of couples of vectors of type $(1, 0)$ both for the $h$– and $v$–parts. Our aim is to calculate the canonical class $L\varepsilon$ (we put the label $L$ for the constructions canonically defined by a regular Lagrangian $L$) for the almost Kähler model of a Lagrange space $L^n$. We take the canonical $d$–connection $\hat{\mathbf{D}}$ that it was used for constructing $*$ and considers $h$– and $v$–projections

$$h\Pi = \frac{1}{2}(\text{Id}_h - iJ_h)$$

and

$$v\Pi = \frac{1}{2}(\text{Id}_v - iJ_v),$$

where $\text{Id}_h$ and $\text{Id}_v$ are respective identity operators and $J_h$ and $J_v$ are defined by formulas (13), which are projection operators onto corresponding $(1, 0)$–subspaces. It follows from (23) that $\text{Tr}\hat{\mathbf{R}} = \text{Tr}(\Omega^\alpha_\beta) = 0$, see (10).

The matrix $(h\Pi, v\Pi)\hat{\mathbf{R}}(h\Pi, v\Pi)^T$, where $(...)^T$ means transposition, is the curvature matrix of the $N$–adapted restriction of the connection $\hat{\mathbf{D}}$ to $T_C\left(N^{TTM}\right)$. Now, we can compute the Chern–Weyl form

$$L_\gamma = -i\text{Tr}\left[(h\Pi, v\Pi)\hat{\mathbf{R}}(h\Pi, v\Pi)^T\right] = -i\text{Tr}\left[(h\Pi, v\Pi)\hat{\mathbf{R}}\right]$$

$$= -\frac{1}{4}J_{\alpha'}\hat{\mathbf{R}}^\alpha_{\alpha'\beta}e^\alpha \wedge e^\beta$$

to be closed. By definition, the canonical class is $L\varepsilon = [L_\gamma]$. It follows from Lemma 4.1 and the above presented considerations the proof of

**Theorem 4.3** The zero–degree cohomology coefficient $c_0(*)$ for the almost Kähler model of Lagrange space is computed

$$c_0(*) = -(1/2i)\ L\varepsilon,$$

where the value $L\varepsilon$ is canonically defined by a regular Lagrangian $L(u)$.

Finally we note that the formula from this Theorem can be directly applied for the Cartan connection in Finsler geometry with $L = F^2$. In our partner works [22, 23], we consider its extensions respectively for generalized Lagrange spaces, or canonical nonholonomic lifts of semi–Riemannian metrics on $TM$, and nonholonomic deformations of the Einstein gravity into almost Kähler models on nonholonomic manifolds.
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