Coulomb interaction effects in spin-polarized transport

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Abstract

We study the effect of the electron-electron interaction on the transport of spin polarized currents in metals and doped semiconductors in the diffusive regime. In addition to well-known screening effects, we identify two additional effects, which depend on many-body correlations and exchange and *reduce* the spin diffusion constant. The first is the “spin Coulomb drag” - an intrinsic friction mechanism which operates whenever the average velocities of up-spin and down-spin electrons differ. The second arises from the decrease in the longitudinal spin stiffness of an interacting electron gas relative to a noninteracting one. Both effects are studied in detail for both degenerate and non-degenerate carriers in metals and semiconductors, and various limiting cases are worked out analytically. The behavior of the spin diffusion constant at and below a ferromagnetic transition temperature is also discussed.

72.25.Dc, 72.25.-b, 75.40.Gb
I. INTRODUCTION

The theory of spin-polarized transport is attracting the attention of the physics community both for its potential applications to the emerging field of “spintronics” \[1\] and in relation to the intriguing field of quantum computation \[2\]. In this context Kikkawa et al. \((3,4)\) generated much excitement by demonstrating optical injection and subsequent control of packets of spin polarization in the conduction band of n-doped GaAs. Such packets could in principle be used to transport information between separate regions of a system. Unlike ordinary electron-hole packets, whose mobility is limited by strong scattering in the hole component, these unipolar electronic spin packets are both long-lived (with lifetime \(\tau_s \nsim 10\ ns\)) and highly mobile \[4\].

From the theoretical point of view, most recent work on spin transport phenomena is based on an independent electron model, where, in general, the coupling between different spin channels is completely neglected. Flatt`e and Byers \[5\] have recently discussed the behavior of spin packets in semiconductors in the framework of the Hartree approximation, where the Coulomb interaction is taken into account only through the imposition of a charge neutrality constraint. This constraint establishes an electrostatic coupling between different spin channels. They neglect, however, all further many-body effects.

In this paper we go beyond the treatment by Flatt`e and Byers by considering more subtle correlation (spin-drag) and exchange (spin-softening) effects. Our objective is the derivation of drift-diffusion equations for spin packets in a full many-body context, i.e., allowing for correlation between different spin channels. These equations contain the spin-packet mobility and the diffusion constant as key parameters. We show that both exchange interactions and the drag effect between different spin populations \[6\] concur in reducing the value of the diffusion constant by a sizable amount, leaving at the same time the mobility basically unaffected. The microscopic quantities responsible for this effect are the longitudinal spin stiffness (the second derivative of the free energy with respect to magnetization, not to be confused with the transverse spin stiffness of the ferromagnetic state) and the spin-drag transresistivity. We discuss in detail the behavior of these quantities in various regimes and show how important the Coulomb effects are when dealing with doped semiconductors (see Sec. III).

This paper is organized as follows:

In Section II we review the basic ideas underlying the spin-resolved drift diffusion equations and make use of the Landau transport equation to elucidate the structure of the homogeneous spin resistivity matrix. In particular, we show that the off-diagonal element of the spin-resistivity matrix (the spin trans-resistivity) is almost exclusively controlled by the Coulomb interaction: the contribution from spin-flip scattering, while finite, is utterly negligible for short-range scatterers.

In Section III we present the calculation of the Coulomb contribution to the spin transresistivity. The theory of Ref. \[6\] is extended in various directions. First we study the spin drag effect as a function of temperature going from the degenerate regime (which is appropriate for ordinary metals) to the nondegenerate regime, which is appropriate for low-density/high-temperature doped semiconductors. Then, we calculate the spin drag in the “mixed” case in which one spin component is degenerate while the other is nondegenerate: this is relevant to situations in which a strong spin polarization exists.
Section IV is devoted to a description of the behavior of the longitudinal spin stiffness of the homogeneous electron gas as a function of density and temperature.

In Section V we present a detailed derivation of the drift-diffusion equation for a macroscopic spin packet, and give explicit expressions for the mobility and diffusion constant in terms of microscopic quantities such as the spin stiffness and the spin transresistivity.

Section VI suggests experiments aimed at directly measuring the spin transresistivity and the spin diffusion constant in metals or semiconductors.

Electron gas theory predicts that, at sufficiently low temperatures and densities, the homogeneous electron gas undergoes a ferromagnetic transition. Such a transition could be in principle observed in a doped semiconductor. Section VII examines the behavior of spin diffusion constant in the paramagnetic phase and how the occurrence of ferromagnetism as a second order phase transition would affect it. Our calculations show that the diffusion constant vanishes at the transition temperature and increases as the system becomes fully spin polarized with decreasing temperature.

II. GENERAL THEORY I

A. The drift-diffusion equation

The theory of diffusive transport in metals and semiconductors is based on the assumption of a local linear relationship between the current densities of up and down spin electrons and the gradient of the local electro-chemical potentials:

\[-e\vec{J}_\alpha(\vec{r}) = -\sum_\beta \sigma_{\alpha\beta}(\vec{r}) \vec{\nabla} \psi_\beta(\vec{r}).\]  

(1)

Here \(e\) is the absolute value of the electron charge, \(\vec{J}_\alpha(\vec{r}) (\alpha = \uparrow \text{ or } \downarrow)\) are number currents (1) and the electro-chemical potentials \(\psi_\alpha(\vec{r})\) are given by the electrostatic potential \(\phi(\vec{r})\) plus the local chemical potential, which can be spin-dependent,

\[\psi_\alpha(\vec{r}) = \phi(\vec{r}) - \frac{1}{e} \frac{\partial f(n_\uparrow, n_\downarrow, T)}{\partial n_\alpha}.\]  

(2)

Here \(f(n_\uparrow, n_\downarrow, T)\) is the free energy per unit volume of a homogeneous interacting electron gas evaluated at the local spin densities \(n_\alpha(\vec{r})\) and uniform temperature \(T\); \(\sigma_{\alpha\beta} \equiv \lim_{\omega \to 0} \sigma_{\alpha\beta}(q = 0, \omega)\) is the homogeneous conductivity matrix of the electron gas, whose structure, in the presence of interactions, will be elucidated below.

Substituting (2) in (1) and writing

\[\vec{\nabla} \frac{\partial f(n_\uparrow, n_\downarrow, T)}{\partial n_\alpha} = \sum_\beta \frac{\partial^2 f(n_\uparrow, n_\downarrow, T)}{\partial n_\alpha \partial n_\beta} \vec{\nabla} n_\beta,\]  

(3)

we obtain

\[e\vec{J}_\alpha(\vec{r}) = \sum_\beta \left( \sigma_{\alpha\beta}(\vec{r}) \vec{\nabla} \phi(\vec{r}) - eD_{\alpha\beta}(\vec{r}) \vec{\nabla} n_\beta(\vec{r}) \right)\]  

(4)

where the diffusion matrix \(D_{\alpha\beta}\) is given by
\[ e^2 D_{\alpha\beta} = \sum_{\gamma} \sigma_{\alpha\gamma} S_{\gamma\beta} \]  
(5)

and

\[ S_{\alpha\beta} = \frac{\partial^2 f(n_\uparrow, n_\downarrow, T)}{\partial n_\alpha \partial n_\beta} \]  
(6)

is the static longitudinal spin-stiffness matrix - the inverse of the spin susceptibility matrix \( \chi_{\alpha\beta} \). Eq. (5) is the well known Einstein relation between the mobility and the diffusion constant, generalized here to the case of spin polarized transport. The first term of Eq. (4) is the drift current associated with the electrostatic field, the second is the diffusion current associated with the gradient of the electronic densities. These two terms cancel out exactly in a situation of local equilibrium, due to the Einstein relation and the fact that \([\chi^{-1}]_{\alpha\beta} = S_{\alpha\beta}\).

On a formal level the main effect of the Coulomb interaction is the appearance of non-vanishing off-diagonal elements of the conductivity and spin-stiffness matrices. \( \sigma_{\uparrow\downarrow} \neq 0 \) implies that an electric field acting only on the up-spin electrons must necessarily drag along a current of down-spin electrons. Conversely, a current of up-spin electrons \( J_\uparrow \neq 0 \) flowing against a background of stationary down-spin electrons \( J_\downarrow = 0 \) will necessarily induce a gradient of spin-down electro-chemical potential \( \tilde{E}_\downarrow = -\nabla \psi_\downarrow = e\rho_\downarrow \bar{J}_\uparrow \), where \( \rho_{\alpha\beta} \) is the resistivity matrix, inverse to \( \sigma_{\alpha\beta} \). We shall see later how one can make use of these effects for a direct experimental measure of the spin Coulomb drag. The other noteworthy feature is \( S_{\uparrow\downarrow} \neq 0 \): this means that the chemical potential of up-spins \( \partial f(n_\uparrow, n_\downarrow, T)/\partial n_\uparrow \) is a function of both up and down spin densities. Thus, a disturbance acting on one of the two spin populations will affect the other through Coulomb correlation.

### B. Structure of the resistivity matrix

Although the homogeneous resistivity matrix can be calculated from first-principle Kubo formulas and/or transport equations its general structure (including the off-diagonal terms, due mostly to the Coulomb interaction) is best understood at the phenomenological level. We first present the phenomenological picture and then justify it from a more formal consideration of the Landau transport equation.

Let \( \tilde{E}_\uparrow(t) \) and \( \tilde{E}_\downarrow(t) \) be uniform effective electric fields, \( \tilde{E}_\alpha = -\nabla \psi_\alpha \), that couple to up- and down-spins respectively. We restrict ourselves to the linear response regime. If \( \bar{v}_\alpha \) is the velocity of the center of mass of electrons of spin \( \alpha \), and \( N_\alpha \) the number of such electrons, then the equation of motion for \( \bar{v}_\alpha \) has the form

\[ m^* N_\alpha \dot{\bar{v}}_\alpha = -eN_\alpha \tilde{E}_\alpha + \tilde{F}_{\alpha\alpha} - \frac{m^*}{\tau_\alpha} N_\alpha \bar{v}_\alpha + \frac{m^*}{\tau'_\alpha} N_\alpha \bar{v}_\bar{\alpha}, \]  
(7)

where \( m^* \) is the effective mass of the carriers and \( \bar{\alpha} \equiv -\alpha \).

Let us examine the meaning of the various terms in (7).

The first term on the right hand side is the net force exerted by the electric field on spin-\( \alpha \) electrons.

The second term, \( \tilde{F}_{\alpha\alpha} \), is the Coulomb force exerted by spins of the opposite orientation \( \bar{\alpha} \) on spin-\( \alpha \) electrons. Notice that the net force exerted by spins of the same orientation

\[ e^2 D_{\alpha\beta} = \sum_{\gamma} \sigma_{\alpha\gamma} S_{\gamma\beta} \]  
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vanishes by virtue of Newton’s third law. For exactly the same reason we must have \( F_{\alpha\alpha}^C = -F_{\alpha\alpha}^C \), and by Galilean invariance this force can only depend on the relative velocity of the two components. Hence, in the linear approximation, we can write

\[
\vec{F}_{\alpha\alpha}^C = -\gamma m^* n_{\alpha} \frac{n_{\alpha}}{n} (\vec{v}_{\alpha} - \vec{v}_a),
\]

where \( n = n_\uparrow + n_\downarrow \) is the total density. Eq. (8) defines the spin drag coefficient \( \gamma \).

The third term on the right hand side of (7) combines two distinct physical effects. One is the net force exerted on spin-\( \alpha \) electrons by electron-impurity collisions that do not flip the spin of the incoming electrons. The other is the rate at which momentum is lost to the \( \alpha \) component as a result of electron-impurity collisions that flip the spin from \( \alpha \) to \( \bar{\alpha} \). Accordingly, the momentum relaxation rate \( \tau_{\alpha}^{-1} \) is written as the sum of non-spin-flip and spin-flip contributions: \( \tau_{\alpha}^{-1} = \tau_{n\alpha}^{-1} + \tau_{sf,\alpha}^{-1} \). Spin-flip times have been found to be very long both in metals (\( \tau_{sf,\alpha} \approx 10^{-1} n s \)) and in semiconductors (\( \tau_{sf,\alpha} \approx 10 n s \)), whereas non-spin-flip times are usually much shorter (of the order \( 10^{-3} - 10^{-4} n s \)). Thus, for most practical purposes, \( \tau_{\alpha}^{-1} \approx \tau_{n\alpha}^{-1} \).

Finally, the last term on the right hand side of (7) represents the rate at which momentum is fed into the \( \alpha \) component by electron-impurity collisions that flip the spin from \( \bar{\alpha} \) to \( \alpha \). Because electrons emerging from collisions have a randomized momentum distribution with nearly zero average, we expect \( (\tau_{\alpha}'^{-1})^{-1} \) to be smaller than even the already small spin-flip rate \( \tau_{sf,\alpha}^{-1} \). Indeed, we shall see in the next section that \( (\tau_{\alpha}')^{-1} \) vanishes for short-range scatterers in the Born approximation.

Fourier-transforming the equation of motion (7) for the current density \( \vec{j}_\alpha(\omega) = -e n_{\alpha} \vec{v}_\alpha(\omega) \) we find

\[
i\omega \vec{j}_\alpha(\omega) = -\frac{n_\alpha e^2}{m^*} \vec{E}_\alpha(\omega) + \left( \frac{n_{\bar{\alpha}}}{n} \gamma + \frac{1}{\tau_{\alpha}} \right) \vec{j}_\alpha(\omega) - \left( \frac{n_{\bar{\alpha}}}{n} \gamma + \frac{1}{\tau_{\alpha}'} \right) \vec{j}_{\bar{\alpha}}(\omega).
\]

The resistivity matrix \( \rho_{\alpha\beta} \) is defined as the coefficient of proportionality between the electric field and the current, i.e. \( \vec{E}_\alpha = \sum_{\alpha'} \rho_{\alpha\alpha'} \vec{j}_{\alpha'} \). A quick comparison between this definition and Eq. (9) shows that the complete form of the resistivity matrix \( \rho_{\alpha\beta} \) is

\[
\rho = \begin{pmatrix}
-i\omega \frac{m^*}{e n_\uparrow} + \frac{m^*}{n_\uparrow e^2 \tau_\uparrow} + \frac{n_\downarrow m^*}{n_\downarrow e^2 \tau_\downarrow} & - \frac{m^*}{n_\uparrow e^2 \tau_\uparrow} - \frac{m^*}{n_\downarrow e^2 \tau_\downarrow} \\
-\frac{m^*}{n_\uparrow e^2 \tau_\uparrow} - \frac{m^*}{n_\downarrow e^2 \tau_\downarrow} & -i\omega \frac{m^*}{e n_\downarrow} + \frac{m^*}{n_\downarrow e^2 \tau_\downarrow} + \frac{n_\uparrow m^*}{n_\uparrow e^2 \tau_\uparrow}
\end{pmatrix}.
\]

Notice that this matrix is symmetric, due to the relation \( 1/n_\downarrow \tau_\downarrow' = 1/n_\uparrow \tau_\uparrow' \), which will be proved in the next section.

Due to the extreme smallness of the spin-flip rates \( 1/\tau_{\alpha}' \), the off-diagonal resistivity (also referred to as “spin-transresistivity”) is controlled almost entirely by the Coulomb interaction term, i.e. we can safely assume

\[
\rho_{\uparrow\downarrow} = -\frac{m^*}{n e^2} \gamma
\]

and \( \gamma \) is directly proportional to the spin trans-resistivity. But then, Galilean invariance and Newton’s third law demand that the same \( \rho_{\uparrow\downarrow} \) appear also as a correction to the ordinary diagonal resistivity. This effect is quite distinct from the “trivial” renormalizations due to the electronic screening of these interactions.
C. Derivation from Landau transport equation

In this section we provide a microscopic justification of the phenomenological Eq. (7) and give explicit expressions for the electron-impurity relaxation rates.

We start from the linearized transport equation for the quasiparticle distribution function in Landau theory of Fermi liquid [12]. This is simply the classical Boltzmann equation for quasiparticles in a self-consistent field described by Landau parameters. In the homogeneous case the self-consistent field coincides with the classical electric field and the distribution function $f_{\alpha}(\vec{k},t)$ obeys the well-known kinetic equation

$$\frac{\partial f_{\alpha}(\vec{k},t)}{\partial t} - e\vec{E}_{\alpha} \cdot \frac{\partial f_{\alpha}(\vec{k},t)}{\partial \vec{k}} = \left( \frac{\partial f_{\alpha}(\vec{k},t)}{\partial \vec{k}} \right)_{\text{coll}}, \tag{12}$$

where $f_{\alpha}^{(0)}(\vec{k})$ is the equilibrium distribution function.

The collision term $\left( \frac{\partial f_{\alpha}(\vec{k},t)}{\partial t} \right)_{\text{coll}}$ has contributions from the Coulomb interaction as well as spin-flip and non spin-flip electron-impurity interactions. The various contributions are listed below

- **Coulomb collisions**
  $$\left( \frac{\partial f_{\alpha}(\vec{k},t)}{\partial t} \right)_{\text{coll}}^C = - \sum_{\vec{p}\vec{k}',\beta,\alpha',\beta'} W^C(\vec{k}\alpha,\vec{p}\beta;\vec{k}'\alpha',\vec{p}'\beta') \delta_{\vec{k}+\vec{p},\vec{k}'+\vec{p'}} \delta_{\alpha+\beta,\alpha'+\beta'} \{ f_{\alpha}(\vec{k}) f_{\beta}(\vec{p}) [1 - f_{\alpha'}(\vec{k}')] - f_{\alpha}(\vec{k}) f_{\beta}(\vec{p}) f_{\alpha'}(\vec{k}') f_{\beta'}(\vec{p}') \} \delta(\epsilon_{k\alpha} + \epsilon_{p\beta} - \epsilon_{k'\alpha'} - \epsilon_{p'\beta'}), \tag{13}$$
  where $W^C(\vec{k}\alpha,\vec{p}\beta;\vec{k}'\alpha',\vec{p}'\beta')$ is the probability of the Coulomb scattering process $\vec{k}\alpha,\vec{p}\beta \rightarrow \vec{k}'\alpha',\vec{p}'\beta'$ and $\epsilon_{k\alpha}$ is the energy of a particle of momentum $\vec{k}$ and spin $\alpha$ relative to the chemical potential. The conservations of momentum, energy, and spin are explicitly displayed.

- **Non-spin-flip electron-impurity collisions**
  $$\left( \frac{\partial f_{\alpha}(\vec{k},t)}{\partial t} \right)_{\text{coll}}^{nf} = - \sum_{\vec{k}'} W^{nf}(\vec{k}\alpha,\vec{k}'\alpha') [f_{\alpha}(\vec{k}) - f_{\alpha}(\vec{k}')] \delta(\epsilon_{k\alpha} - \epsilon_{k'\alpha'}), \tag{14}$$
  where $W^{nf}(\vec{k}\alpha,\vec{k}'\alpha')$ is the probability of the non-spin flip (nf) scattering process $\vec{k}\alpha \rightarrow \vec{k}'\alpha'$.

- **Spin-flip electron-impurity collisions**
  $$\left( \frac{\partial f_{\alpha}(\vec{k},t)}{\partial t} \right)_{\text{coll}}^{sf} = - \sum_{\vec{k}'} W^{sf}(\vec{k}\alpha,\vec{k}'\bar{\alpha}) [f_{\alpha}(\vec{k}) - f_{\bar{\alpha}}(\vec{k}')] \delta(\epsilon_{k\alpha} - \epsilon_{k'\bar{\alpha}}), \tag{15}$$
  where $W^{sf}(\vec{k}\alpha,\vec{k}'\bar{\alpha})$ is the probability of the spin flip (sf) scattering process $\vec{k}\alpha \rightarrow \vec{k}'\bar{\alpha}$.
In order to obtain a closed equation of motion for the currents, such as Eq. (12), we must multiply both sides of Eq. (12) by $-e\vec{k}/m^*$, sum over $k$, and then express the integrated collision term

$$\vec{F}_\alpha = \sum_\vec{k} \vec{k} \left( \frac{\partial f_\alpha(\vec{k}, t)}{\partial t} \right)_{\text{coll}}$$

back in terms of the currents. Of course, this cannot be done rigorously, but for an isotropic system slightly perturbed from equilibrium one can assume [13] that the distribution function of the state with currents $\vec{f}_\alpha = -en_\alpha \vec{v}_\alpha$ is given by

$$f_\alpha(\vec{k}, t) = f^{(0)}_\alpha(\epsilon_{\vec{k}\alpha}) - \frac{\partial f^{(0)}_\alpha(\epsilon_{\vec{k}\alpha})}{\partial \epsilon_{\vec{k}\alpha}} \vec{v}_\alpha(t) \cdot \vec{k}. \quad (17)$$

Substituting this into Eqs. (13), (14), and (15), and linearizing with respect to the currents wherever needed, we arrive, after tedious but straightforward manipulations, at the desired equation of motion (9), with the following expressions for the various relaxation times:

$$\gamma = \frac{n}{m^* N_n n_\alpha \sum_{\vec{k}\vec{k}'\vec{p}\vec{p}'} \frac{(\vec{k} - \vec{k}')^2}{2dk_B T} W^C(\vec{k}\alpha, \vec{p}\alpha; \vec{k}'\alpha, \vec{p}'\alpha) \delta_{\vec{k} + \vec{p}, \vec{k}' + \vec{p}'} \delta(\epsilon_{\vec{k}\alpha} + \epsilon_{\vec{p}\alpha} - \epsilon_{\vec{k}'\alpha} - \epsilon_{\vec{p}'\alpha})$$

$$f^{(0)}_\alpha(\epsilon_{\vec{k}\alpha}) f^{(0)}_\alpha(\epsilon_{\vec{p}\alpha}) f^{(0)}_\alpha(-\epsilon_{\vec{k}'\alpha}) f^{(0)}_\alpha(-\epsilon_{\vec{p}'\alpha}), \quad (18)$$

where $d$ is the number of spatial dimensions;

$$\gamma = \frac{1}{\tau_{n_\alpha f,\alpha}} = \sum_\vec{k} \frac{\partial f^{(0)}_\alpha(\epsilon_{\vec{k}\alpha})}{\partial \epsilon_{\vec{k}\alpha}} \frac{k^2}{n_\alpha d} \sum_{\vec{k}'} W^{nf}(\vec{k}\alpha, \vec{k}'\alpha)(1 - \vec{k} \cdot \vec{k}') \delta(\epsilon_{\vec{k}\alpha} - \epsilon_{\vec{k}'\alpha}), \quad (19)$$

where $\vec{k}$ and $\vec{k}'$ are unit vectors in the directions of $\vec{k}$ and $\vec{k}'$;

$$\frac{1}{\tau_{s_\alpha f,\alpha}} = -\sum_\vec{k} \frac{\partial f^{(0)}_\alpha(\epsilon_{\vec{k}\alpha})}{\partial \epsilon_{\vec{k}\alpha}} \frac{k^2}{n_\alpha d} \sum_{\vec{k}'} W^{sf}(\vec{k}\alpha, \vec{k}'\alpha) \delta(\epsilon_{\vec{k}\alpha} - \epsilon_{\vec{k}'\alpha}), \quad (20)$$

$$\frac{1}{\tau_{\alpha}} = -\frac{1}{n_\alpha d} \sum_\vec{k} \frac{\partial f^{(0)}_\alpha(\epsilon_{\vec{k}\alpha})}{\partial \epsilon_{\vec{k}\alpha}} \sum_{\vec{k}'} W^{s_\alpha f}(\vec{k}\alpha, \vec{k}'\alpha) \vec{k} \cdot \vec{k}' \delta(\epsilon_{\vec{k}\alpha} - \epsilon_{\vec{k}'\alpha}). \quad (21)$$

Notice that $1/\tau_{s_\alpha f,\alpha}$ and $1/\tau_{\alpha}$ arise, respectively, from the first and the second term on the right side of Eq. (13). The key difference between these two relaxation rates is that the expression for the latter involves an angular average of the scattering probability with weight factor $\vec{k} \cdot \vec{k}' = \cos(\theta)$. This average vanishes in the Born approximation for short-range scatterers, since the scattering probability becomes isotropic (independent of $\vec{k}$ and $\vec{k}'$) in this special case. Quite generally, one can expect $1/\tau_{\alpha}$ to be much smaller than $1/\tau_{s_\alpha f,\alpha}$ in agreement with the qualitative arguments given in the previous section. This means that the spin transresistivity is almost entirely a Coulomb interaction effect, and therefore its measurement can shed light on the nature of the Coulomb correlation between up- and down-spin electrons. This is one of the main points we wanted to make in this section.

Finally, notice that Eq. (21) implies the identity

$$\frac{1}{n_\downarrow \tau_{\downarrow}^f} = \frac{1}{n_\uparrow \tau_{\uparrow}^f} \quad (22)$$

which guarantees the symmetry of the resistivity matrix Eq. (10).
III. CALCULATION OF THE SPIN TRANSRESISTIVITY

The theory of the spin transresistivity has been worked out in [6]. This theory closely parallels the theory of the ordinary Coulomb drag between parallel two-dimensional electron or hole-gas layers [14] but differs in some important details, as the fact that electrons of opposite spin interact with the same set of impurities, so that certain electron-impurity terms which appear in the Kubo formulation of the transresistivity do not vanish upon disorder averaging. Fortunately, it turned out that these terms cancel out exactly at low frequency ($\omega << E_F$) and to leading order in the electron-electron and electron-impurity interactions [6].

In this section we first review for completeness the derivation of Ref. [6], and then present an RPA calculation of $\rho_{\uparrow\downarrow}$ at finite temperature [15]. At variance with the calculation of Ref. [6] we present our results not only in the low temperature limit (which is relevant to metals and where $\gamma \sim T^2$), but also in the non-degenerate $k_B T >> E_F$ and quasi-degenerate $k_B T \sim E_F$ regimes (which are relevant to doped semiconductors), where $k_B$ is the Boltzmann constant and $E_F = \hbar^2 (3\pi^2 n)^{2/3} / 2m^*$ is the Fermi energy [16].

We start from the Kubo formula [17] for the uniform conductivity matrix

$$\sigma_{\alpha,\alpha'}(\omega) = -\frac{1}{i\omega} \frac{e^2}{m^*} \left( n_\alpha \delta_{\alpha,\alpha'} + \frac{\langle\langle \vec{P}_{\alpha} \cdot \vec{P}_{\alpha'} \rangle\rangle_\omega}{m^*} \right), \quad (23)$$

where $\langle\langle A; B \rangle\rangle_\omega$ represents, as usual [17], the retarded response function for the expectation value of $A$ under the action of a field that couples linearly to $B$. We assume to be in the diffusive, weak scattering regime characterized by $\hbar / \tau_D << k_B T$, where $\tau_D = (n_\uparrow / n_\downarrow) \tau_\uparrow + (n_\downarrow / n) \tau_\downarrow$. Because of the “high” temperature, weak localization effects are negligible. In this regime the resistivity is essentially independent of frequency for $\omega << k_B T / \hbar$. It is therefore legitimate to take the limit of weak electron-impurity and electron-electron scattering before taking the limit of $\omega \to 0$ [18]. When the limits are carried out in this order the $\vec{P}_{\alpha}$’s are almost constants of motion and therefore the second term in the square bracket of Eq. (23) is a small correction to the first. Inverting Eq. (23) to first order in $\langle\langle \vec{P}_{\alpha} \cdot \vec{P}_{\alpha'} \rangle\rangle_\omega$ and selecting the $\uparrow\downarrow$ matrix element we obtain

$$\rho_{\uparrow\downarrow}(\omega) = \frac{i\omega}{e^2} \frac{\langle\langle \vec{P}_{\uparrow}; \vec{P}_{\downarrow} \rangle\rangle_\omega}{n_\uparrow n_\downarrow}. \quad (24)$$

It is convenient to recast this equation in a form that emphasizes the importance of the non conservation of $P_\uparrow$ and $P_\downarrow$. To this end we make use twice of the general equation of motion

$$\langle\langle A; B \rangle\rangle_\omega = \frac{1}{\omega} \langle\langle [A, B] \rangle\rangle_\omega + i\langle\langle \hat{A}; B \rangle\rangle_\omega, \quad (25)$$

where $\hat{A} \equiv -i[A, H]$ is the time derivative of the operator $A$, and $\langle\langle \cdot \rangle\rangle$ denotes the thermal average. Thus, Eq. (24) can be rewritten as

$$\rho_{\uparrow\downarrow}(\omega) = \frac{i}{e^2 n_\uparrow n_\downarrow} \frac{\langle\langle \hat{\vec{P}}_{\uparrow}; \hat{\vec{P}}_{\downarrow} \rangle\rangle_\omega}{\omega} + i\langle\langle [\hat{\vec{P}}_{\uparrow}, \vec{P}_{\downarrow}] \rangle\rangle_\omega. \quad (26)$$
The commutator term controls the high frequency behavior of $\rho_{\uparrow\downarrow}(\omega)$ and can be expressed in terms of ground-state properties [19]. This term however gives a purely imaginary contribution to the trans-resistivity. Our present interest is in the real part of the trans-resistivity, which is controlled by the imaginary part of the force-force response function.

The force operator is given by

$$\dot{\vec{P}}_\alpha = -\frac{i}{V} \sum_\vec{q} \vec{q} v_{\vec{q}} \rho_{\vec{q} \alpha} \rho_{-\vec{q} \alpha} - \frac{i}{V} \sum_\vec{q} \vec{q} v_{\vec{q}} e^{-i\omega} \rho_{\vec{q} \alpha} \rho_{-\vec{q} \alpha},$$

(27)

where $v_{\vec{q}} = \frac{4\pi e^2}{q^2} \epsilon$ is the Fourier transform of the Coulomb interaction, $\epsilon$ the dielectric constant of the material, $v_{\vec{q}} e^{-i\omega}$ is the Fourier transform of the electron-impurity interaction, $\rho_{\vec{q} \alpha}$ is the electronic spin density fluctuation operator, $\rho_{\vec{q} \alpha}$ is the Fourier transform of the impurity density (a number), and $V$ is the volume of the system.

As explicitly shown in [6], the contribution of correlated impurity scattering to the trans-resistivity, in the low frequency $\hbar\omega << E_F$ limit and to leading order in the electron-electron and electron-impurity interactions, vanishes since the Coulomb-impurity term exactly cancels the impurity-impurity contribution. Thus, in this limit, the real part of the spin trans-resistivity takes the form

$$\text{Re} \rho_{\uparrow\downarrow}(\omega) = \frac{1}{n_{\uparrow} n_{\downarrow} e^2 V} \sum_{\vec{q} \vec{q}'} \frac{\vec{q} \cdot \vec{q}'}{3} v_{\vec{q}} v_{\vec{q}'} \cdot$$

$$\text{Im} \langle \langle \rho_{-\vec{q} \downarrow} \rho_{\vec{q} \uparrow} ; \rho_{\vec{q} \downarrow} \rho_{-\vec{q} \uparrow} \rangle \rangle_\omega.$$

(28)

We proved the cancellation of correlated impurity scattering effects within the frame of the Drude-Boltzmann theory defined by Eq. (24) which is the result of interchanging the natural order of the $\omega \to 0$ limit and the weak scattering limit. This approach is only adequate in the "classical" regime $k_B T >> \hbar/\tau_D$. A more sophisticated treatment of quantum effects in correlated impurity scattering [20] suggests that the spin drag would be even larger than predicted by the present theory at temperatures $k_B T << \hbar/\tau_D$. The temperature range in which these quantum corrections are important shrinks to zero in the limit of weak impurity scattering.

We have calculated the four point response function

$$\chi_4(\vec{q}, \vec{q}', \omega) \equiv \langle \langle \rho_{-\vec{q} \downarrow} \rho_{\vec{q} \uparrow} ; \rho_{\vec{q} \downarrow} \rho_{-\vec{q} \uparrow} \rangle \rangle_\omega$$

at finite temperature in a generalized Random Phase Approximation (RPA). Because of its infinite range, the Coulomb interaction must be treated to infinite order, even when weak. The sum of the RPA diagrams [6] has been evaluated by standard methods [21] with the following result:

$$\text{Re} \rho_{\uparrow\downarrow}(\omega, T) = \frac{1}{n_{\uparrow} n_{\downarrow} e^2 V} \sum_{\vec{q}} \frac{q^2}{3} v_{\vec{q}}^2 \cdot \frac{(e^{-\beta \omega} - 1)}{\omega}$$

$$\int_{-\infty}^{\infty} d\omega' \frac{\chi''_{\uparrow\downarrow}(q, \omega') \chi''_{\downarrow\uparrow}(q, \omega - \omega') - \chi''_{\downarrow\uparrow}(q, \omega') \chi''_{\uparrow\downarrow}(q, \omega - \omega')}{(e^{-\beta \omega'} - 1)(e^{-\beta (\omega - \omega')} - 1)}.$$  

(29)

Here $\beta = 1/k_B T$, $\chi''_{\alpha\alpha'}(q, \omega)$ is the imaginary part of the RPA spin-resolved density-density response function, which is related to the noninteracting response function $\chi_{0\alpha}(q, \omega)$ as follows

$$[\chi^{-1}(q, \omega)]_{\alpha\alpha'} = [\chi_{0\alpha}]^{-1}(q, \omega) \delta_{\alpha\alpha'} - v_q.$$

(30)
It is possible to show by simple but tedious algebraic calculations that Eq. (29) for the spin trans-resistivity $\rho_{\uparrow\downarrow}(\omega, T)$ reduces, in the case of finite temperature and $\omega = 0$, to the well known result of memory function and diagrammatic theories for the conventional Coulomb drag [22], [23].

$$Re \rho_{\uparrow\downarrow}(0, T) = \frac{\beta}{n_\uparrow n_\downarrow e^2} \sum_q \frac{q^2}{V} \frac{1}{3} \frac{1}{2} \int_0^\infty \frac{d\omega'}{\pi} \frac{\chi''_0(q, \omega') \chi''_0(q, -\omega')}{|\epsilon(q, \omega')|^2 \sinh^2(\beta\omega'/2)}. \quad (31)$$

Furthermore, for $T = 0$ and $\omega \neq 0$, the RPA is equivalent to the decoupling approximation for the four-point response function used in [24] to calculate the dynamical exchange-correlation kernel. In this limit, the real part of the spin transresistivity takes the form

$$Re \rho_{\uparrow\downarrow}(\omega, 0) = \frac{1}{n_\uparrow n_\downarrow e^2 \omega} \sum_q \frac{q^2}{3} v_q^2$$

$$\int_0^\omega \frac{d\omega'}{\pi} \left[ \chi''_{\uparrow\uparrow}(q, \omega') \chi''_{\downarrow\uparrow}(-q, \omega - \omega') - \chi''_{\uparrow\downarrow}(q, \omega') \chi''_{\downarrow\downarrow}(-q, \omega - \omega') \right] \quad (32)$$

Thus our calculation demonstrates that those two approximations, quite different at a first sight, are simply RPAs performed in different limits.

From now on, for simplicity of notation, we will refer to $Re \rho_{\uparrow\downarrow}$ simply as $\rho_{\uparrow\downarrow}$.

**A. Numerical evaluation**

To calculate $\rho_{\uparrow\downarrow}$ at finite temperature, we have used in Eq. (29) the temperature dependent expression for the three-dimensional noninteracting spin-resolved density-density response function

$$\chi''_{\alpha\alpha}(q, \omega; T) = -\frac{1}{16\pi \bar{q} a^3 Ry} \left[ \beta Ry \ln \left( 1 + e^{\beta \left( \frac{h\omega + \epsilon_q}{2} - \xi_\alpha \right)} \right) \right],$$

$$\chi''_{\alpha\beta}(q, \omega; T) = \frac{\pi}{V} \sum_k (n_{k\alpha} - n_{k+q\alpha}) \delta(h\omega + \epsilon_k - \epsilon_{k+q}), \quad (34)$$

where $a^*$ is the effective Bohr radius, $\bar{q} = qa^*$, $\bar{\omega} = h\omega / Ry$, $Ry = e^2/2a^*$ is the effective Rydberg, $\xi_\alpha$ is the chemical potential for the $\alpha$ spin population and $\epsilon_q = h^2q^2/2m^*$. Eq. (33) follows directly from the definition

$$\frac{\pi}{V} \sum_k (n_{k\alpha} - n_{k+q\alpha}) \delta(h\omega + \epsilon_k - \epsilon_{k+q})$$

where $n_{\bar{q}\alpha} = 1/[\exp(\beta(\epsilon_q - \xi_\alpha)) + 1]$ is the average number of $\alpha$-spin electrons with energy $\epsilon_q$.

Fig.1 shows $|\rho_{\uparrow\downarrow}|$ as a function of temperature and density [23]. The data are calculated in the paramagnetic phase and for semiconductor parameters (GaAs), i.e. $m^* = 0.067$, $\epsilon = 12$, and carrier density $n_1 = 1.5 \times 10^{16} \text{ cm}^{-3}$, $n_2 = 1.5 \times 10^{17} \text{ cm}^{-3}$, and $n_3 = 1.5 \times 10^{18} \text{ cm}^{-3}$. $\rho_{\uparrow\downarrow}$
peaks at about the Fermi temperature $T_F$, underlying the crossing between the degenerate and the non degenerate regimes. As can be seen, $\rho_{\uparrow\downarrow}$ is strongly enhanced as the density decreases, mainly due to the prefactor dependence $\sim 1/n^2$. In fact its maximum increases of almost two orders of magnitude, passing from 0.3 milli$\Omega \times cm$ for $n = n_1$ to 14 milli$\Omega \times cm$ for $n = n_3$. In the calculations of the following sections, we will mainly focus on the density value $n = n_2$, corresponding to a Fermi temperature $T_F = 178K$. The inset of Fig. 1 presents for this density value the comparison between $\rho_{\uparrow\downarrow}$ and its non-degenerate analytical approximation (dashed lines) discussed in details in Sec.III C.

We now turn to a quantitative assessment of the relevancy of the spin Coulomb drag. First of all it is necessary to underline that the spin drag is an intrinsic effect of spin-polarized transport: that is, while impurity scattering could in principle be suppressed in a perfect crystal, the spin Coulomb drag will always be present, even in the purest sample, and dominate over phonon scattering at sufficiently low temperature. However, since available samples are usually far from perfection, it is reasonable to ask how the spin transresistivity compares to the more familiar Drude resistivity. In metals, as we shall show in detail in the next section, one finds, at most, $\rho_{\uparrow\downarrow} \sim 10^{-2} \mu\Omega \times cm$ so that $\rho_{\uparrow\downarrow}/\rho_D$ is of the order of few percents. The situation is very different for semiconductors: since both the Fermi temperature (at which $\rho_{\uparrow\downarrow}$ peaks) and the carrier density are considerably lower than in metals, $\rho_{\uparrow\downarrow}$ can become comparable and even greater than $\rho_D$. This strong variation depends on specific semiconductor characteristics, such as the effective mass, mobility, and density of the carriers.

In Fig. 2, we show the effect of the carrier mobility $\mu$ on the ratio $\rho_{\uparrow\downarrow}/\rho_D$. We plot $\rho_{\uparrow\downarrow}/\rho_D$ in respect to temperature for an n-doped semiconductor as GaAs ($m^* = 0.067 m_e$, $\epsilon = 12$ and $n = 1.5 \times 10^{17} \text{cm}^{-3}$; upper panel) and for a p-doped semiconductor, as (Ga,Mn)As ($m^* = 0.5 m_e$, $\epsilon = 12$, $n = 1.2 \times 10^{19} \text{cm}^{-3}$; lower panel). Each curve corresponds to a different mobility value as reported in the figure caption. The values increase from “A” to ”D”. In particular the value $\mu = 3 \times 10^{3} \text{cm}^2/\text{Vs}$ (labels as ”C”), corresponds to the value measured for a spin packet in [4]. As can be seen, changing the material it is possible to increase the ratio $\rho_{\uparrow\downarrow}/\rho_D$ by an order of magnitude, to the point that the spin transresistivity can become greater than the Drude resistivity.

In Fig. 3, we show the dependence on the ratio $\rho_{\uparrow\downarrow}/\rho_D$ on the carrier density for GaAs. The results are presented for two different temperatures ($T = 20K$, dashed lines and $T = 300K$, solid line) and two different values of the mobility (as labelled in the figure). Each curve peaks at a density such that $T_F = T$.

In conclusion, our calculations demonstrate that in semiconductors, at $T \approx T_F$, the spin Coulomb drag must definitely be considered an important contribution to the resistivity for spin polarized currents.

B. The degenerate limit

Let us now focus on the low-temperature ($k_B T << E_F$) and low-frequency ($\hbar \omega << E_F$) regime. This is the only regime of practical importance in ordinary high-density metals. The spin-Coulomb drag coefficient is controlled by a subset of all the processes that lead to the finite lifetime of a quasi-particle at the Fermi surface, namely, the processes in which the quasi-particle in question exchanges momentum with an electron of opposite spin (scattering
processes between parallel spin electrons do not cause relaxation of the spin-current. Since
the inverse quasi-particle life-time at the Fermi surface is known in Fermi liquid theory to
scale as \((k_B T/E_F)^2\) we expect the same scaling to hold for the spin drag coefficient at low
temperature.

This prediction is confirmed by the detailed calculation as follows. At low temperature,
the exponential factors in Eq. \((23)\) restrict the region of integration to \(\omega \sim k_B T/h\). The
low-frequency form of the density fluctuation spectra \(\chi''_{\alpha \alpha'}(q, \omega)\) is a linear function of \(\omega\).
In the limit of vanishing impurity concentration \(\chi_{0\alpha}(q, \omega)\) is simply the Lindhard function,
whose imaginary part, at low frequency, is given by \(\chi''_{0\alpha}(\vec{q}, \omega \to 0) = -(m^*/4\pi)(\omega/q)\) and
whose real part can be approximated by its value at \(\omega = 0\). Making use of these limiting
forms, the calculation of \(\rho_{\uparrow \downarrow}\) can be carried in an essentially analytical fashion. The result is

\[
\text{Re} \rho_{\uparrow \downarrow}(\omega, T) = -\frac{\hbar a^* 4\pi^2(k_B T)^2 + h^2 \omega^2}{6(Ry)^2} \cdot \frac{1}{24\pi^3\bar{n}_s\bar{n}_\uparrow} \int_0^{2k_F a^*} \frac{dq}{q^2} \frac{1}{|\epsilon(\bar{q}/a^* 0)|^2},
\]

\((35)\)

where \(k_F \equiv \min(k_{F\uparrow}, k_{F\downarrow})\), with \(k_{F\alpha}\) the \(\alpha\) spin population Fermi wave-vector, \(\bar{n}_\alpha \equiv n_\alpha a^{*3}\)
and \(\epsilon(q, \omega) = 1 - v_F \chi_{\alpha}(\bar{q}, \omega) - v_q \chi_{\alpha}(\bar{q}, \omega)\) is the RPA dielectric function. Eq. \((35)\)
shows that, in the absence of impurities, \(\rho_{\uparrow \downarrow}(\omega, T)\) is proportional to \(\omega^2\) for \(k_B T \ll h\omega\) and to \(T^2\)
for \(h\omega \ll k_B T\).

Modifications in the form of \(\chi_{0\alpha}(q, \omega)\) due to the presence of impurities can be taken
into account through Mermin’s approximation scheme \([13]\). These modifications amount
to replacing \(\omega/qv_F\) by \(\omega/Dq^2\) \((D = v_F^2\tau/3\) being the diffusion constant) for \(\omega < 1/\tau\) and
\(q < 1/v_F\tau\) where \(v_F\) is the Fermi velocity and \(\tau\) is the electron-impurity mean scattering
time. The \(\omega\) and \(T\) dependencies of Eq. \((35)\) are not affected.

Writing explicitly in Eq. \((35)\) the dependence on \(r_{sa}\) (where \(r_{sa} = (4\pi n_\alpha a^{*3}/3)^{-1/3}\) is the
value of \(r_s\) for spin \(\alpha\)) one can also see that \(\rho_{\uparrow \downarrow}(\omega, T) \sim r_s^{-1}r_{s\uparrow}^{3}\) \(r_{s\downarrow}^{3}\) so that \(\rho_{\uparrow \downarrow}\) will strongly increase with decreasing electron density. In Fig. \([\,]\) we plot \(|\rho_{\uparrow \downarrow}(\omega = 0, T)|\) as a function of the
temperature, for \(n_\uparrow = n_\downarrow\) and in the density range \(1 < r_s < 7\). The Figure shows that,
for metallic densities correspondent to \(r_s \approx 5\) and temperatures of the order of \(40 - 60K\) (at
which for example experiments on spin relaxation time using spin polarized currents have
been performed \([\,]\)), the spin trans-resistivity is appreciable \((|\rho_{\uparrow \downarrow}(\omega = 0, T)| \approx 0.01\mu\Omega cm\).
The corresponding Coulomb scattering time is \(\gamma^{-1}\) from Eq. \((11)\). For \(r_s = 5\) we obtain
\(\gamma^{-1} \approx 10^{-13}s\) and \(\delta_s/v_F \approx 10^{-10}s\); \(\gamma^{-1}\) is indeed several orders of magnitude smaller
than the spin-flip time. This demonstrates that neglecting spin-flip processes is indeed a good
approximation for this kind of metals.

C. The non-degenerate limit

The non-degenerate limit is characterized by \(T >> T_F\). First of all we calculate the non-degenerate limit of the non-interacting temperature-dependent spectral function
Eq. \((23)\): starting from the definition Eq. \((14)\), we use the classical expression for the fugacity
\(\exp(\beta\xi_\alpha) = n_\alpha \cdot 8\pi^3(\beta/2m^*\pi)^{3/2}\), and obtain
\[ \chi''_{0\alpha}(q,\omega;T) = -\frac{\sqrt{2\pi}\beta m^* n_\alpha}{\hbar q} \exp\left(-\frac{\beta \epsilon_q}{4}\right) \exp\left(-\frac{\beta \hbar^2 \omega^2}{4\epsilon_q}\right) \sinh\left(\frac{\beta \hbar \omega}{2}\right). \] (36)

In addition, in order to calculate the non-degenerate limit \( \rho_{\uparrow \downarrow}(T,0) \), we have used the classical limit for the dielectric constant \( \epsilon(q,\omega) = 1 + (4\pi e^2/\epsilon q^2)(n/k_BT) \). The final result is

\[
\rho_{\uparrow \downarrow}(0,T) = \frac{8e^2\sqrt{m^*}}{\sqrt{2\pi^3(k_BT)^2}} \left( \int_0^{\infty} dx \frac{x \exp(-x)}{(x + \lambda)^2} \right) \\
\approx \frac{8e^2\sqrt{m^*}}{\sqrt{2\pi^3(k_BT)^2}} \left[ -1 - C - \ln(\lambda) \right],
\] (37)

where the second expression, Eq. (38), is valid in the limit \( \lambda \ll 1, \lambda = h^2 k_D^2/k_BT 4m^* \), \( k_D^2 = 4\pi e^2 n_\epsilon/k_BT \) is the inverse of the squared Debye screening length and \( C \approx 0.577 \) is the Euler’s constant. Notice that, in the non-degenerate limit, \( \rho_{\uparrow \downarrow} \) becomes almost independent of the total density \( n \) and independent of the spin density components \( n_\alpha \), while a quantum mechanical dependence on \( n \) survives even in this regime. \( \rho_{\uparrow \downarrow} \) tends to zero as \((k_BT)^{-3/2} \ln(k_BT) \) as \( T \to \infty \). The inset of Fig. 1 illustrates the comparison between \( \rho_{\uparrow \downarrow} \) and its asymptotic form. This approximation becomes valid for \( T \gg T_F \), but, since \( T_F \sim n^{2/3} \), such limit is fulfilled only at very low carrier densities.

D. The mixed (degenerate/nondegenerate) case

A very interesting limit is the one corresponding to a spin polarization process, for which \( n_\uparrow \to n \) and \( n_\downarrow \to 0 \). This is indeed relevant for one of the problems we want to analyze, i.e. a semiconductor with strongly spin-polarized carriers. This system is in a very peculiar state: its minority down-spin population is non-degenerate, i.e. \( k_BT \gg E_{F\downarrow}, E_{F\uparrow} = h^2(6\pi^2 n_\alpha)^{2/3}/2m^* \), while, for low enough temperatures, the majority up-spin population is degenerate, i.e. \( k BT \ll E_{F\uparrow} \). The expression for the non-interacting spin-resolved density-density response functions entering the spin transresistivity Eq. (29) can then be taken from the previous sub-sections and are given by

\[ \chi_{0\downarrow}''(q,\omega';T) = -\frac{m^* \omega'}{4\pi \hbar^3} \quad \text{for } 0 < q < 2k_F, \] (39)

\[ = 0 \quad \text{otherwise} \] (40)

and

\[ \chi_{0\uparrow}''(q,\omega';T) = -\left( \frac{\pi m^*}{2} \right)^{\frac{1}{2}} \beta^2 n_\downarrow \exp\left(-\frac{\beta m^* \omega^2}{2q^2}\right) \frac{\omega'}{q}, \] (41)

where Eq. (40) is valid up to first order in \( \omega' \) and Eq. (41) represents the classical limit \((\hbar \to 0)\) of Eq. (39). Using Eqs. (40), (41) and the approximation, due to the small density of down-spin carriers \( \epsilon(q,\omega') \approx \epsilon_\uparrow(q,0) \), the expression for the spin transresistivity becomes

\[ \text{Re} \rho_{\uparrow \downarrow}(0,T) = -\frac{\hbar a^* 2\sqrt{\pi}}{9} \frac{\sqrt{(k_BT)^{\frac{1}{2}}}}{n_\uparrow} \int_0^{2k_F a^*} \frac{d\bar{q}}{\bar{q}} \frac{1}{|\epsilon_\uparrow(\bar{q}/a^*,0)|^2}. \] (42)
Eq. (42) is very similar to Eq. (35), the result obtained when both components are degenerate. This is a consequence of the fact that, in the appropriate regime, the non-degenerate spectral function Eq. (41) presents the same dependence in $\omega'$ and $q$ as the degenerate one. We want to underline that, in this limit, $\rho_{\uparrow\downarrow}(0, T)$ is independent of $n_\downarrow$ and that in any case $\rho_{\uparrow\downarrow} \to 0$ for $T \to 0$.

IV. SPIN STIFFNESS OF AN INTERACTING SPIN-POLARIZED ELECTRON GAS

The other ingredient entering the drift-diffusion expression for the current, Eq. (4), is the longitudinal spin stiffness matrix $S_{\alpha\beta}$. In particular we are interested in the combination

$$S = \partial^2 f(n, m, T)/\partial m^2 = (S_{\uparrow\uparrow} - S_{\uparrow\downarrow} + S_{\downarrow\downarrow} - S_{\downarrow\uparrow})/4,$$

where $m = n_\uparrow - n_\downarrow$, which gives the curvature of the free energy with respect to the magnetization at constant density. This quantity coincides with the inverse of the longitudinal spin susceptibility of the uniform electron gas.

We evaluated $S$ numerically starting from the formulas provided by Tanaka and Ichimaru [26], who calculated the free energy density of the three-dimensional electron gas as a function of temperature, density, and spin polarization.

Fig. 5 shows $S$ divided by its non-interacting value $S_{ni}$ as a function of the dimensionless temperature $T/T_F$ for various densities, starting with $n = 4.2 \times 10^{17} \text{ cm}^{-3}$ for the upper curve down to $n = 4.2 \times 10^{11} \text{ cm}^{-3}$ for the lowest one, decreasing by an order of magnitude from one curve to the next.

Two regimes are clearly visible. For densities larger than a critical value $n_c \approx 4.2 \times 10^{13} \text{ cm}^{-3}$ the spin stiffness decreases monotonically with decreasing temperature settling to a finite value in the ground-state (top four curves). For densities lower than $n_c$ a second-order ferromagnetic transition occurs: the critical temperature $T_c$ raises from $\sim 0$ at $n = n_c$ to a sizeable fraction of the Fermi temperature at $n = 4.2 \times 10^{11} \text{ cm}^{-3}$. As in any second-order transition, the spin stiffness vanishes at the transition temperature [27]. For $T < T_c$ the spontaneous magnetization $\bar{m}$ is given by the stable minimum of the free energy, which satisfies the conditions

$$\frac{\partial f(n, m, T)}{\partial m} \bigg|_{m=\bar{m}} = 0,$$

$$\bar{S} = \frac{\partial^2 f(n, m, T)}{\partial m^2} \bigg|_{m=\bar{m}} > 0. \quad (43)$$

Obviously, ferromagnetism shows up only at extremely low densities, and the electron gas model may break down well before getting to such densities: it is nevertheless instructive to study the repercussions of the behavior of $S$ on the spin diffusion constant both above and below the critical density.

Let us now consider the limiting behaviors of the spin stiffness at high ($T >> T_F$) and low ($T << T_F$) temperatures.
A. The high temperature limit

In the high-temperature limit \((T \gg T_F(n))\) the free energy density of the electron gas has the following expansion:

\[
f(n_{\uparrow}, n_{\downarrow}, T) \simeq k_B T \sum_{\alpha} n_\alpha \left[ \ln(n_\alpha \lambda_T^3) - 1 \right] \\
+ \frac{k_B T}{2^{3/2}} \sum_{\alpha} n_\alpha^2 \lambda_T^3 \\
- 2\pi^2 e^2 \lambda_T^2 \sum_{\alpha} n_\alpha^2 \\
- \frac{2\pi^{1/2}}{3} \frac{e^3 n^{3/2}}{(k_B T)^{1/2}}, \tag{44}
\]

where \(\lambda_T \equiv (2\pi \hbar^2/m^* k_B T)^{1/2}\) is the thermal wavelength. The first term is the free energy of the classical ideal gas, the second term is the leading quantum correction for noninteracting Fermions \([28]\); the third term is the leading quantum/interaction correction, namely, the high temperature exchange free-energy \([29]\), the last term is the leading classical interaction correction from Debye-Huckel theory. Only the first three terms depend on the magnetization, and therefore contribute to the spin stiffness. Taking a second derivative with respect to magnetization we find, after simple calculations,

\[
\frac{S}{S_c} = 1 + \frac{n_{\uparrow} n_{\downarrow}}{n^2} \left[ \frac{n \lambda_T^3}{2^{1/2}} - \frac{8\pi^2 e^2 n^{1/3}}{k_B T (n \lambda_T^3)^{2/3}} \right], \tag{45}
\]

where \(S_c\) is the Curie spin stiffness of an ideal classical gas of density \(n\):

\[
S_c = \frac{k_B T n}{4 n_{\uparrow} n_{\downarrow}}. \tag{46}
\]

Notice that the leading interaction correction to the noninteracting spin stiffness is negative, in agreement with the behavior seen in Fig. 6.

B. The low temperature limit

Let us now examine what happens in the limit \(T \to 0\). Above the critical density \(n_c\) the spin stiffness simply tends to a constant zero-temperature limit, smaller than the noninteracting value, in agreement with the Landau theory of Fermi liquids.

For \(n < n_c\) the low temperature phase is ferromagnetic, and in this case the density of majority spin electrons \(n_{\uparrow}\) approaches the total density, while the density of minority spin electrons \(n_{\downarrow}\) tends to zero for \(T \to 0\). To understand the behavior of the spin stiffness shown in Fig. 6 we assume that, in the nearly 100% polarized limit the free energy can be written as the sum of the ground-state energy of the degenerate interacting up-spin gas plus the free energy of an infinitely dilute noninteracting down spin gas:

\[
f(n_{\uparrow}, n_{\downarrow}, T) \simeq \epsilon_0(n_{\uparrow}) - k_B T n_{\downarrow} \ln(1 - \zeta) + \text{constant}, \tag{47}
\]
where \( \epsilon_0(n) \) is the ground-state energy density of a degenerate liquid of up-spin electrons, and \( \zeta = (n_\uparrow - n_\downarrow)/(n_\uparrow + n_\downarrow) \) is the degree of spin polarization. Obviously, this approximation ignores the correlation between down- and up-spin electrons, or, more precisely, presumes that this correlation is smaller than the entropic term \( k_B T n_\downarrow \ln(1 - \zeta) \) for \( \zeta \to 1 \).

Starting from Eq. (47) it is trivial to show that the minority spin density vanishes for \( T \to 0 \) as

\[
n_\downarrow \sim n e^{\epsilon'_0(n)/k_B T} \tag{48}
\]

where \( \epsilon'_0(n) = d\epsilon_0(n)/dn < 0 \) at low density, while the spin stiffness goes as

\[
S \sim \frac{k_B T}{4n_\downarrow}, \tag{49}
\]

which diverges exponentially for \( T \to 0 \). This is precisely what our numerical calculations, based on the formulas of Ref. (26), indicate. As we shall see, this result is important in understanding the behavior of the diffusion constant of a unipolar spin packet when the system is fully spin polarized.

V. THE EVOLUTION OF A SPIN PACKET

We now examine the motion of a spin packet under the effect of a uniform electric field. Let us apply Eq. (4) to calculate the time evolution of a spin packet obtained by injecting an excess spin density \( \Delta m(\vec{r},0) = M \delta(\vec{r}) \) near the origin at time \( t = 0 \). We denote by \( m(\vec{r},t) = n_\uparrow(\vec{r},t) - n_\downarrow(\vec{r},t) \) the net spin density at point \( \vec{r} \) and time \( t \), by \( m(0) = n_\uparrow(0) - n_\downarrow(0) \) the uniform value of the spin density at thermodynamic equilibrium, and by \( \Delta m(\vec{r},t) = m(\vec{r},t) - m(0) \) the excess spin density following spin injection. To solve this problem we combine the expression for the current density Eq. (4) with the generalized continuity equations for the spin-density components

\[
\frac{\partial \Delta n_\alpha(\vec{r},t)}{\partial t} = -\frac{\Delta n_\alpha(\vec{r},t)}{\tau_{sf,\alpha}} + \frac{\Delta n_\alpha(\vec{r},t)}{\tau_{sf,\bar{\alpha}}} - \vec{\nabla} \cdot \vec{J}_\alpha(\vec{r}), \tag{50}
\]

where \( \tau_{sf,\alpha} \) is the spin-flip relaxation time for the \( \alpha \) component. Substituting in this equation the drift-diffusion expression for the current Eq. (4) we obtain the two equations

\[
\frac{\partial \Delta n_\alpha(\vec{r},t)}{\partial t} = -\frac{\Delta n_\alpha(\vec{r},t)}{\tau_{sf,\alpha}} + \frac{\Delta n_\alpha(\vec{r},t)}{\tau_{sf,\bar{\alpha}}} + \frac{E}{e} \sum_\beta \frac{\partial \bar{\sigma}_\alpha}{\partial n_\beta} \nabla(\Delta n_\beta) + \bar{\sigma}_\alpha \frac{\nabla \cdot E}{e} + \sum_\beta \left[ \nabla D_{\alpha\beta} \nabla n_\beta + D_{\alpha\beta} \nabla^2(\Delta n_\beta) \right], \tag{51}
\]

(\( \alpha = \uparrow \) or \( \downarrow \)) where

\[
\bar{\sigma}_\alpha = \sum_\beta \sigma_{\alpha\beta}. \tag{52}
\]

Eq. (51) includes the term \( \bar{\sigma}_\alpha \nabla \cdot E/e = \bar{\sigma}_\alpha (\Delta n_\uparrow + \Delta n_\downarrow)/\epsilon \). This term is “dangerous” because it contains the product of a large quantity \( \bar{\sigma}_\alpha \) times a small quantity, the space charge
\( \Delta n_\uparrow + \Delta n_\downarrow \), the product itself being of the order of the other quantities of interest in the calculation.

It is tempting, but wrong, to invoke the local charge neutrality constraint
\[
\Delta n_\uparrow(r) = -\Delta n_\downarrow(r)
\]
at this point. Instead, we will first combine the two component of Eqs. (51) to eliminate the space charge \((\nabla \cdot E)\) term, and only after doing that can we impose, without serious loss of accuracy, the charge neutrality constraint. To eliminate the \(\nabla \cdot E\) term we multiply each component of Eq. (51) by the conductivity \(\tilde{\sigma}_\alpha\) of the opposite channel, and, in order to get the equation of motion for \(\Delta m\), we subtract the equation for \(\alpha\)-spins from the equation for \(\bar{\alpha}\)-spins. Only at this point we impose the local charge neutrality constraint Eq. (53). With this procedure we obtain the correct drift-diffusion equation for \(\Delta m(\vec{r}, t)\)
\[
\frac{\partial \Delta m(\vec{r}, t)}{\partial t} = -\frac{\Delta m(\vec{r}, t)}{\tau_s} + \frac{\sum_\alpha \tilde{\sigma}_\alpha \nabla(\tilde{D}_\alpha \nabla \Delta m(\vec{r}, t))}{\sum_\alpha \tilde{\sigma}_\alpha} + \frac{\sum_\alpha \tilde{\sigma}_\alpha \tilde{\mu}_\alpha}{\sum_\alpha \tilde{\sigma}_\alpha} \vec{E} \cdot \nabla \Delta m(\vec{r}, t)
\]
(54)
where \(\tau_s = (1/\tau_{sf,\uparrow} + 1/\tau_{sf,\downarrow})^{-1}\) is the spin relaxation time, which is very long [4][11],
\[
\tilde{D}_\alpha = D_{\alpha\alpha} - D_{\alpha\bar{\alpha}},
\]
(55)
\[
\tilde{\mu}_\alpha = \mu_{\alpha\alpha} + \mu_{\alpha\bar{\alpha}},
\]
(56)
\(\vec{E}\) is an externally applied electric field, and the matrix \(\mu_{\alpha\beta}\) is defined as
\[
e_{\mu_{\alpha\beta}} \equiv \frac{\partial \sigma_{\alpha\beta}}{\partial n_\alpha} - \frac{\partial \sigma_{\alpha\beta}}{\partial n_\bar{\alpha}}
\]
(57)
\[
= \pm 2 \frac{\partial \sigma_{\alpha\beta}}{\partial m}, \text{ plus if } \alpha = \uparrow, \text{ minus otherwise.}
\]
(58)
This is a generalization of the familiar relation between mobility and conductivity: it takes into account the dependence of the mobility \(\mu_{\alpha\beta}\) on both spin density components. The second term in Eq. (57) accounts for the reduction of the mobility in the \(\alpha\) channel due to the drag of the \(\alpha\) spin population on the \(\bar{\alpha}\) population [4].

The fact that the mobilities enter Eq. (54) as a spin symmetric combination (Eq. (56)) while the diffusion constants are in a spin antisymmetric combination (Eq. (55)), reflects the fact that the electrostatic field has the same sign for both spin components, while the density gradients have opposite signs (see Eq. (53)).

If we consider the linear regime - i.e. we neglect terms of the order of \((\nabla n_\alpha)^2\) - Eq. (57) reduces to the more familiar
\[
\sigma_{\alpha\beta} = e n_\alpha \mu_{\alpha\beta}
\]
(59)
and Eq. (54) can be written as
\[
\frac{\partial \Delta m(\vec{r}, t)}{\partial t} = -\frac{\Delta m(\vec{r}, t)}{\tau_s} + D_s \nabla^2 \Delta m(\vec{r}, t) + \mu_s \vec{E} \cdot \nabla \Delta m(\vec{r}, t),
\]
(60)
where
\[ \mu_s = \frac{\sum_\alpha \tilde{\sigma}_\alpha \bar{\mu}_\alpha}{\sum_\alpha \tilde{\sigma}_\alpha} \]  
and
\[ D_s = \frac{\sum_\alpha \tilde{\sigma}_\alpha \tilde{D}_\alpha}{\sum_\alpha \tilde{\sigma}_\alpha} \]

are the effective mobility and diffusion constants \[30\]. We underline that in reality the range of validity of Eq. \((60)\) extends beyond the linear approximation into the classical regime, i.e. to high carrier densities or high temperatures, since in that regime the relationship between density and conductivity is linear. Eqs \((61)\) and \((62)\) show that the mobility and the diffusion constants of the packet are weighted averages of, respectively, the mobilities \(\tilde{\mu}_\alpha\) and diffusion constants \(\tilde{D}_\alpha\) of the two spin channels, the weight being the conductivity \(\tilde{\sigma}_\alpha\) of the opposite channel. This is due to the local charge neutrality constraint Eq. \((53)\), that forces the two components of the disturbance to travel together, so that the conductivity of each spin channel is strongly influenced by the motion of the disturbance in the other channel. In the non interacting limit Eqs \((61)\) and \((62)\) reduce to the expressions presented in \[31\].

The solution of Eq. \((60)\) is
\[ \Delta m(\vec{r},t) = \Delta M e^{-t/\tau_s} \frac{1}{(4\pi D_s t)^{3/2}} e^{-|\vec{r} + \mu_s \vec{E} t|^2/4D_s t}. \]  
Eq. \((63)\) has the form of a Gaussian packet that drifts under the effect of the electric field \(\vec{E}\) with a pace determined by \(\mu_s\), and spreads in time at a rate determined by \(D_s\). The mobility and diffusion constants of electron-hole packets of similar shape can be measured through the Haynes-Shockley experiment \[31\]. Thus a similar experiment can in principle determine \(\mu_s\) and \(D_s\) independently, provided that \(\tau_s\) is sufficiently long.

With the help of some algebra, Eq. \((61)\) can be rewritten as
\[ \mu_s = \frac{1}{e n_\uparrow n_\downarrow} \frac{1}{\sum_\alpha (1/\tilde{\sigma}_\alpha)} \]  
where \(\tilde{\sigma}_\alpha\) is given by Eq. \((52)\), and Eq. \((62)\) as
\[ D_s = \frac{S}{e^2 \sum_\alpha \rho_\alpha} \frac{4}{\sum_\alpha \rho_\alpha} = \frac{k_B T}{e^2 n} \frac{S}{S_c} \frac{1}{1/n_\uparrow n_\downarrow (\rho_{D\uparrow} + \rho_{D\downarrow} - \rho_{\uparrow\downarrow})}, \]

where
\[ \tilde{\rho}_\alpha \equiv \rho_{\alpha\alpha} - \rho_{\alpha\bar{\alpha}}, \]
\(S_c\) is given by Eq. \((46)\), \(\rho_{D\alpha} = m^*/ne^2 \tau_\alpha\) is the ordinary Drude resistivity associated with the \(\alpha\) spin channel, and \(\rho_{\uparrow\downarrow}\) - a negative number - is the spin drag transresistivity discussed in Sec. \[III\].
Eq. (64) can be seen as the generalization to a spin packet of the ordinary relationship between mobility and conductivity, while Eq. (65) is the corresponding generalization of the Einstein relation. To derive Eq. (66), we made use explicitly of the structure of the matrix $\rho_{\alpha\beta}$ discussed in Sec. II B.

The calculation of $\mu_s$ and $D_s$ is simplified under the assumption that the scattering times for the two spin components are not too different, i.e., $\tau_{\uparrow} = \tau_{\downarrow} = \tau_D$. This assumption is well justified for non-degenerate carriers. In the degenerate case, $1/\tau_\alpha$ has a weak $n_\alpha^{1/3}$ dependence on the density. From Eqs. (61) and (66) we obtain

$$\mu_s = \frac{e\tau_D}{m^*},$$

and

$$D_s = \frac{\mu_s k_B T}{e S_c} \frac{1}{1 - \rho_{\uparrow\downarrow}/\rho_D},$$

where $\rho_D = m^*/ne^2\tau_D$ is now the ordinary Drude resistivity.

Eq. (68) tells us that the mobility of the packet is not explicitly modified by Coulomb interaction and in fact coincides with the ordinary homogeneous mobility. Strictly speaking this result is only valid under the assumption that up-spin and down-spin electrons have equal mobilities and thus drift at the same speed in an applied electric field. Coulomb interactions, being Galilean-invariant, cannot change the total momentum of such a uniformly drifting electron gas.

The situation is completely different for the diffusion constant. As the spin packet spreads out the up- and down-spin currents are directed in opposite directions and friction arises: for this reason the expression for $D_s$ contains the spin-drag resistivity as a factor that reduces the diffusion. In addition, the Coulomb interaction together with the Pauli exclusion principle reduces the energy cost of spin-density fluctuations (i.e., the spin stiffness) decreasing further the rate of diffusion of a spin packet.

VI. EXPERIMENTAL OBSERVATION OF INTERACTION EFFECTS IN SPIN-POLARIZED TRANSPORT

A. Direct observation of spin Coulomb drag

We will now propose an experiment aimed at detecting the effect of the spin Coulomb drag and measuring the spin trans-resistivity. We describe the experiment as it could be done on metals, but the same scheme could be applied to semiconductors (in which the drag effect is larger), provided that an appropriate method of injecting spin current is used.

The setup is shown in Fig. 6: a paramagnetic metal film of thickness $L$ is sandwiched between two ferromagnets polarized in the same direction. A battery is connected to the ferromagnets inducing a spin-polarized current $\vec{j}_\uparrow$ from the first ferromagnet (“injector”) through the paramagnet and toward the second ferromagnet (“receiver”). The injector and receiver are chosen to be half-metals, i.e., they have only electron states of spin $\uparrow$ at the Fermi level (see Fig. 6). It follows that the injected current $\vec{j}_\uparrow$ is carried only by spin $\uparrow$ electrons
If we choose $L << \delta_s$, where $\delta_s$ is the spin relaxation length, we can safely neglect spin-flip processes. Spin relaxation lengths are relatively large in some materials ($\delta_s \approx 100 \mu m$ in Al [11]), so the condition $L << \delta_s$ is not particularly restrictive. At any rate, we have seen in Section II that the coupling between spin channels induced by spin-flip scattering is expected to be much weaker than the one due to the Coulomb interaction. Due to the spin Coulomb drag, the injected $\vec{j}_\uparrow$ will drag spin $\downarrow$ electrons toward the junction with the receiver. But, since there is no conduction band available in the receiver for spin $\downarrow$ electrons the circuit will behave as an open circuit for spin $\downarrow$ electrons, i.e., $\vec{j}_\downarrow = 0$. The vanishing of $\vec{j}_\downarrow$ is an indication that the Coulomb drag force is exactly balanced by the gradient of the electro-chemical potential for spin down electrons

$$\bar{E}_\downarrow = \frac{\nabla \psi_\downarrow}{e} = \rho_{\uparrow\downarrow} \vec{j}_\uparrow$$

What Eq. (70) tells us is that, due to the spin Coulomb drag, there will be a measurable electro-chemical potential difference $eE_\downarrow l = e\rho_{\uparrow\downarrow} j_\uparrow l$ for spin $\downarrow$ electrons between two points within the metal separated by a distance $l$ along the direction of the current.

To measure this potential difference a second circuit including a voltmeter of very large resistance is connected to the regions of the paramagnet close to the junctions (See Fig. 6). Our purpose is to measure $E_\downarrow$, so this second circuit must be driven by the spin $\downarrow$ electro-chemical potential only. In order to accomplish this, we propose to use as contacts two half-metallic ferromagnetic electrodes (“detectors”), similar to the injector and the receiver, but polarized in the opposite direction. In this way, for the same reasons explained before, the detection circuit will be “open” as far as spin $\uparrow$ electrons are concerned, and the current flowing in the voltmeter will be exclusively driven by the electro-chemical potential difference of spin $\downarrow$ electrons. The spin trans-resistivity will then be given by $\rho_{\uparrow\downarrow} = (\Delta V_D/I_\uparrow)(A/l)$, where $\Delta V_D$ is the voltage measured by the meter, $A$ is the cross-section of the paramagnetic metal, $l$ is the distance between the detectors, and $I_\uparrow$ the current flowing between injector and receiver. As shown by our calculations (see Fig. 4), we expect, in metals, a resistivity of the order of $10^{-2} \mu \Omega \text{cm}$ that is proportional to $T^2$ for $k_B T >> \omega$.

B. Haynes-Shockley experiment

The Haynes-Shockley experiment [31] demonstrated the drift and diffusion of minority carriers in a doped semiconductor. The experiment allows a direct and independent measure of the minority carrier diffusion and mobility coefficients. After a pulse of excess carriers is created at some point in the semiconductor, it drifts under the action of an electric field, for a known distance $L$, after which it is monitored. By measuring the drift time and the width of the packet, it is then possible to compute both the mobility and the diffusion constant of the packet, which coincide with those of the minority carriers.

The experiments of Ref. (4) can be seen as a Haynes-Shockley-type experiment, since they are based on a direct monitoring in space and time of unipolar spin packets. In these experiments, Kikkawa and Awschalom were able to measure independently the diffusion and the mobility of the spin packets. In the next section we are going to compare our theoretical results with their experimental findings.
VII. PARAMAGNETIC SEMICONDUCTORS AND SPONTANEOUS FERROMAGNETIC TRANSITION

We will now focus on the results of our calculations for the diffusion constant of the spin packet. First of all we will analyze the paramagnetic regime in which \( n^\uparrow = n^\downarrow \). In Fig. 7 we plot the ratio \( D_s/D_{ni} \), where \( D_s \) represents the fully interacting calculation according to Eq. (69) and \( D_{ni} \) is the non-interacting diffusion constant, as in [5]. The figure shows results for n-doped GaAs, at three different temperatures and in a range of densities that is relevant to the experiments of Ref. [4]. We see that the interaction correction is quite significant, and, in the paramagnetic regime, reduces the value of the diffusion constant as expected, i.e. \( D_s/D_{ni} < 1 \) always. The solid lines correspond to the calculations performed at a temperature \( T = 300K \), the dashed lines to \( T = 20K \) and the dot lines to \( T = 1.6K \). The curves marked with "SD" correspond to the case in which interactions in \( D_s \) are taken into account only through the spin Coulomb drag effect. The figure shows clearly that at low temperatures the most important many-body contribution to the diffusion is due to the softening of the spin stiffness, while, already at 20K the spin drag contribution becomes relevant, to represent most of the interaction effects at room temperature. We see that \( D_s/D_{ni} \to 1 \) for high densities. This is due to the enhancement of screening in this regime, so that the particles tend to behave as non-interacting ones. In the high density regime \( S \approx S_{ni} \) as can be seen in Fig. 5. If the temperature is high enough, the spin drag still reduces the diffusion constant by a sizable amount, but eventually, even this contribution disappears with increasing density, and \( D \to D_{ni} \).

At low density, the system enters the non-degenerate regime, so that both \( D \to D_c \) and \( D_{ni} \to D_c \), where \( D_c = \mu_s k_B T/e \) is the classical non-interacting diffusion constant. This limit will be discussed in greater detail below.

In Fig. 8 we plot \( D_s/D_c \) for the same parameters of Fig. 7. The solid lines represent the fully interacting diffusion constant \( D_s \) while the dashed lines represent the corresponding non-interacting approximation \( D_{ni} \). We see that, despite the significant reduction due to the interaction correction, \( D_s \) remains still considerably larger than \( D_c \), consistent with experimental observations [4].

As mentioned above, in the non-degenerate limit \( D/D_c \) approaches 1. It is worthwhile to examine how this limit is approached. In the non-interacting theory [4] the non-degenerate limit is approached from above because the leading correction to \( D_c \) is due to the spin stiffness and comes from the quantum kinetic energy, i.e. it is positive. In the interacting theory there is an additional exchange correction to \( S \), which is negative, and competes with the quantum kinetic one (see Sec. [VA]). However, the leading correction to \( D_c \) does not come from the spin stiffness term, but from the spin Coulomb drag, and it is always negative (see Eq. (69)). In fact, for \( T >> T_F \), \( \rho_{\uparrow\downarrow}/\rho_D \sim [n/(k_B T)^{3/2}] \ln(n/(k_B T)^2) \) (see Eq. (38)), and the logarithmic term dominates over corrections entering the spin stiffness in both the \( n \to 0 \) and \( T \to \infty \) limits. Thus, due to interactions, \( D_s/D_c \to 1 \) from below always. This is evident in Fig. 7 where, for a fixed temperature, \( D_s/D_c \) becomes negative as the density decreases under a certain threshold, while for a fixed density the ratio becomes negative when shifting to curves calculated at higher temperature.

We want to stress that \( D_s \) also displays a marked dependence on the sample mobility that affects the diffusion constant through Drude resistivity \( \rho_D \). The higher the mobility,
the more important becomes the factor containing the spin transresistivity $1/(1 - \rho_{\uparrow\downarrow}/\rho_D)$ (see Eq. (39)). As we already underlined in Sec. IIIA the ratio $\rho_{\uparrow\downarrow}/\rho_D$ can become very relevant and even greater than 1 (see Fig. 2 and 3). The diffusion constant in these cases is then regulated by the spin drag effect that cannot be neglected.

Perhaps the most interesting feature of Eq. (39) is the possibility of a large variation in $D_s$ when the electron gas undergoes a ferromagnetic transition. From the curves in Fig. 5 and Fig. 9 we see that $S$ and $D_s$ vanish at the transition temperature and increase sharply as the system settles in the fully polarized state. In the case of intrinsic ferromagnetism, the critical behavior of $D_s$ is completely due to Coulomb interactions among carriers.

As can be seen from Fig. 9, $D_s/D_c \to 1$ as the system fully polarizes. In this limit $n_\downarrow \to 0$, $\rho_{\uparrow\downarrow}/\rho_D \to 0$, and $S/S_c \to 1$, as demonstrated in Sec. IVB, so that $D_s$ reduces to the diffusion constant of carriers of minority orientation (which are non-degenerate), i.e. to the classical value $D_c = k_B T \mu/e$.

Unfortunately, in an ordinary electron liquid, the ferromagnetic transition is predicted to occur only at extremely low densities. There is, however, an interesting variant: semiconductors doped with magnetic impurities (for example Mn) can undergo a ferromagnetic transition at rather high carrier densities \cite{33,34}, $n \sim 10^{20} \text{cm}^{-3}$ for (Ga,Mn)As, and temperatures $T < T_c \sim 110K$ \cite{33}. Our theory on the dynamics of a spin packet can be extended to these systems with similar results \cite{33}. This extension will not be pursued here.

VIII. CONCLUSIONS

In this paper, we have tried to demonstrate the importance of many-body effects in spin polarized transport. We have discussed in detail the spin Coulomb drag effect, an intrinsic source of friction in spin transport that can limit spin currents even in the purest materials. We have worked out the behavior of the spin transresistivity $\rho_{\uparrow\downarrow}$ in different physical regimes and shown that it ranges from $10^{-8} \Omega\text{cm}$ in metals to $10^{-3} - 10^{-2} \Omega\text{cm}$ in semiconductors. Moreover, the ratio $\rho_{\uparrow\downarrow}/\rho_D$, which is only a few percents in metals, becomes comparable to, or even larger than unity in semiconductors. We hope that an experimental group will soon take up the challenge of measuring the spin transresistivity, for example through the experiment we suggest, in order to confirm the theory.

We have also demonstrated the importance of including Coulomb interactions in a quantitative theory of spin diffusion, and shown that a measure of $D_s$ for a unipolar spin packet would be a sensitive probe of many-body effects such as the spin-Coulomb drag and the Coulomb enhancement of the spin susceptibility.

Finally we have studied the behavior of $D_s$ at and below a spontaneous ferromagnetic ordering transition and found that $D_s$ exhibits a critical behavior.

IX. ACKNOWLEDGEMENTS

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FIGURES

FIG. 1. Spin transresistivity $\rho_{\uparrow\downarrow}$ as a function of temperature (rescaled by $T_F$) for GaAs parameters ($m^* = 0.067 m_e$, $\epsilon = 12$, $\mu = 3 \times 10^3 \text{cm}^2/\text{Vs}$). Each curve corresponds to a different density: $n_1 = 1.5 \times 10^{16} \text{cm}^{-3}$, $n_2 = 1.5 \times 10^{17} \text{cm}^{-3}$, $n_3 = 1.5 \times 10^{18} \text{cm}^{-3}$. Inset: comparison between $\rho_{\uparrow\downarrow}$ and its analytical approximation in the non-degenerate regime vs temperature (rescaled by $T_F$) for $n = n_2$.

FIG. 2. Upper panel: $\rho_{\uparrow\downarrow}/\rho_D$ as a function of temperature for GaAs parameters ($m^* = 0.067 m_e$, $\epsilon = 12$, $n = 1.5 \times 10^{17} \text{cm}^{-3}$). Each curve corresponds to a different mobility: $A = 10^2 \text{cm}^2/\text{Vs}$, $B = 10^3 \text{cm}^2/\text{Vs}$, $C = 3 \times 10^3 \text{cm}^2/\text{Vs}$, $D = 10^4 \text{cm}^2/\text{Vs}$, as labelled. Lower panel: same as upper panel but for (Ga,Mn)As parameters ($m^* = 0.5 m_e$, $\epsilon = 12$, $n = 1.2 \times 10^{19}$).

FIG. 3. $\rho_{\uparrow\downarrow}/\rho_D$ as a function of carrier density for GaAs parameters ($m^* = 0.067 m_e$, $\epsilon = 12$). The solid curves are calculated at the temperature $T = 300 K$, while the dashed curves at $T = 20 K$. For each temperature two different mobilities are considered, $C = 3 \times 10^3 \text{cm}^2/\text{Vs}$ and $D = 10^4 \text{cm}^2/\text{Vs}$, as labelled.

FIG. 4. Temperature and density dependence of $\rho_{\uparrow\downarrow}$ in the degenerate regime. The top line corresponds to $r_s = 7$. The electron-gas parameter is decremented by 1 starting from the top.

FIG. 5. Spin stiffness $S$, rescaled by its non-interacting approximation $S_{ni}$ vs $T/T_F$. The carrier density is $n = 4.2 \times 10^{11} \text{cm}^{-3}$ for the lower curve and increases by a factor 10 for each line starting from the bottom. The cusps correspond to the onset of ferromagnetism.

FIG. 6. (a) Experimental setup to detect the spin Coulomb drag effect: the voltage $\Delta V$ is applied between two parallel half-metallic ferromagnets (injector (inj.) and receiver (rec.)) that sandwich a paramagnet (P). The voltage $\Delta V_D$ is detected using two ferromagnetic electrodes (d) similar to the injector and the receiver, but polarized in the opposite direction. (b) Schematic band-structure of injector, receiver, detectors and paramagnet P. (c) Schematic behavior of the chemical ($\xi_\downarrow$) and electro-chemical potentials $\psi_\alpha(r)$ (see Eq. [3]). For this setup $\xi_\uparrow$ is basically constant (not shown in the figure).

FIG. 7. The interacting diffusion constant of a spin-packet $D_s$ rescaled by its non-interacting approximation $D_{ni}$ vs density for different temperatures: solid lines correspond to $T = 300 K$, dashed lines to $T = 20 K$ and dotted lines to $T = 1.6 K$. For each temperature, we plot also the curve obtained considering interactions only through the spin Coulomb drag effect (labeled as SD). In all the calculations the dielectric constant of the semiconductor is $\epsilon = 12$ and the mobility is $\mu = 3 \times 10^3 \text{cm}^2/\text{Vs}$.
FIG. 8. The interacting (solid line) and non-interacting (dashed line) diffusion constant of a spin-packet rescaled by its classical non-interacting approximation $D_c$ vs density for different temperatures. In all the calculations the dielectric constant of the semiconductor is $\epsilon = 12$ and the mobility is $\mu = 3 \times 10^3 \text{cm}^2/\text{Vs}$.

FIG. 9. The diffusion constant of a spin packet rescaled by its classical non-interacting approximation $D_c$ vs temperature in a low-density electron gas with no magnetic impurities.
\[ n = n_2 \]

\[ n = n_3 \]

\[ \text{GaAs} \]

\[ -\rho \] (ohm cm \( \times 10^{-3} \))

\[ \frac{T}{T_F} \]

\[ \frac{T}{T_F} \]

\[ n_1 \]

\[ n_2 \]

\[ n_3 \]
Fig. 2

\[ T = 178 \text{K} \]

\[ \frac{\rho}{\rho_D} \]

\[ T_F = 178 \text{K} \]

\[ (\text{Ga, Mn})\text{As} \]

\[ T_F = 444 \text{K} \]

\[ -\frac{\rho_N}{\rho_D} \]

\[ -\frac{\rho_N}{\rho_D} \]

\[ T (\text{K})\]

Graph showing the temperature dependence of \( \frac{\rho}{\rho_D} \) for GaAs and \((\text{Ga, Mn})\text{As}\).
Fig. 3

GaAs

$T = 300K$

$T = 20K$

$-\rho_{\varepsilon} / \rho_D$ vs $n$ (cm$^{-3}$)

$D$ and $C$ curves for different temperatures.
\[ r_s = 1-7 \]

\[ r_s = 7 \]
Fig. 31

\[ \frac{S}{S_{ni}} \] vs. \( \frac{T}{T_F} \)

- \( n = 4.2 \times 10^{11} \) cm\(^{-3} \)
Fig. 6
Fig. 7
$D_s / D_c$ vs $n$ (cm$^{-3}$) for different temperatures: $T = 1.6K$, $T = 20K$, $T = 300K$.
Fig. 9

$n = 4.2 \times 10^{12} \text{ cm}^{-3}$

$m^* = 0.067 \text{ } m_e$