NON-SPHERICAL SOURCES OF STATIC GRAVITATIONAL FIELDS: INVESTIGATING THE BOUNDARIES OF THE NO-HAIR THEOREM

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ABSTRACT. A new, globally regular model describing a static, non spherical gravitating object in General Relativity is presented. The model is composed by a vacuum Weyl–Levi-Civita special field - the so called gamma metric - generated by a regular static distribution of mass-energy. Standard requirements of physical reasonableness such as, energy, matching and regularity conditions are satisfied. The model is used as a toy in investigating various issues related to the directional behavior of naked singularities in static spacetimes and the blackhole (Schwarzschild) limit.

1. INTRODUCTION

Analytical modelling of astrophysical gravitating systems in General Relativity is a poorly developed field, especially due to the difficulties occurring in the field equations when the - obviously unrealistic - hypothesis of spherical symmetry is abandoned. As a matter of fact we do not have any analytic model which could resemble a realistic (i.e. rotating, axially symmetric) star. This problem has an interesting theoretical counterpart which is unsolved as well. It is the problem of finding sources for the rotating blackhole solution under conditions of physical reasonableness. Besides its obvious intrinsic interest, finding this kind of solutions would allow physicist to investigate those interesting phenomena which can be expected to occur “at the boundary” between the singularity theorems and the blackhole no-hair theorem. For instance, we do not know anything about those processes which should produce the progressive “cuts of the hairs” required to bring a non-vacuum, non singular rotating and collapsing configuration to settle down to a Kerr state before (naked) singularity formation. Of course the same question arises in the problem of collapse of a non-rotating source of a Schwarzschild black hole, which according to the Israel theorem [1] is the only static asymptotically flat vacuum solution to the Einstein’s equations with a regular horizon. In order to delve deeper into these questions, we present here an interior solution which satisfies all physically relevant requirements and matches smoothly on the boundary surface to one of the static axially symmetric vacuum solutions [2] known as the gamma metric [3]. This metric belongs to the family of Weyl’s solutions, and is continuously linked to the Schwarzschild space-time through one of its parameters. The motivation for this choice stems from the fact that the exterior gamma metric corresponds to a solution of the Laplace equation (in cylindrical coordinates) with the same singularity structure as the Schwarzschild solution (a line segment). In this sense the gamma metric appears as the “natural” generalization of Schwarzschild space-time to the axisymmetric case. This new interior adds to the few other known sources of the gamma metric [4] and would allow to study the behavior of very compact self-gravitating systems, which according to the Israel theorem are very sensitive to small fluctuations of spherical symmetry.

2. STATIC, AXIALLY SYMMETRIC GRAVITATIONAL FIELDS AND THE GAMMA METRIC

The static, axially symmetric metric can be written, in full generality, as

\[ ds^2 = -\chi^2 dt^2 + \alpha^2 \left( (dx^1)^2 + (dx^2)^2 \right) + \beta^2 d\varphi^2 \]
with \( \alpha = \alpha(x^1, x^2), \beta = \beta(x^1, x^2), \chi = \chi(x^1, x^2) \). The components of the Ricci tensor satisfy to

\[
(2.2) \quad R^t_t + R^\varphi_\varphi = \frac{\Delta(\beta \chi)}{\alpha^2 \beta \chi}
\]

where \( \Delta f = \frac{\partial^2 f}{\partial (x^1)^2} + \frac{\partial^2 f}{\partial (x^2)^2} \) is the flat Laplace operator in the fictitious two-dimensional space parameterized by cartesian coordinates \( x^1 \) and \( x^2 \).

Thus, if we are considering vacuum gravitational fields (so that Ricci is zero) or if the matter source satisfies to \( T^t_t + T^\varphi_\varphi - T^\mu_\mu = 0 \) then one can make use of the so called Weyl gauge. This consists in changing variables \( (x^1, x^2) \rightarrow (\varrho, z) \) where \( \varrho : = \beta \chi \) is harmonic and \( z \) is its harmonic conjugate. It follows that the metric can be written as follows:

\[
(2.3) \quad ds^2 = -e^{2\lambda} dt^2 + e^{2\nu-2\lambda} [(d\varrho)^2 + (dz)^2] + \varrho^2 e^{-2\lambda} d\varphi^2
\]

where

\[
(2.4) \quad \alpha = e^{\nu-\lambda}, \chi = e^{\lambda}.
\]

In the present section we shall consider only metrics satisfying the Weyl gauge (the physical content of this condition will be discussed in next section). The field equations can, in this case, be written as follows:

\[
(2.5) \quad \kappa T^t_t = 2\varrho \lambda_{,\varrho\varrho} + 2\varrho \lambda_{,zz} + 2\lambda_{, \varrho} - \varrho \lambda_{,\varrho}^2 - \varrho \lambda_{,zz}^2 - \varrho \nu_{,\varrho\varrho} - \varrho \nu_{,zz}
\]

\[
(2.6) \quad \kappa T^\varrho_\varrho = \varrho (\lambda_{,\varrho}^2 - \lambda_{,zz}^2) - \nu_{,\varrho}
\]

\[
(2.7) \quad \kappa T^z_z = -\varrho (\lambda_{,\varrho}^2 - \lambda_{,zz}^2) + \nu_{,\varrho}
\]

\[
(2.8) \quad \kappa T^\varphi_\varphi = 2\varrho \lambda_{,\varrho} \lambda_{,z} - \nu_{,z}
\]

\[
(2.9) \quad \kappa T^z_\varphi = -\varrho (\lambda_{,\varrho}^2 + \lambda_{,zz}^2 + \nu_{,\varrho\varrho} + \nu_{,zz})
\]

where we put

\[
\kappa : = 8\pi \varrho e^{2\nu-2\lambda}.
\]

It is worth recalling the structure of the vacuum solutions. For vanishing energy-momentum tensor, we get

\[
(2.10) \quad \Delta \lambda = 0,
\]

\[
(2.11) \quad \nu_{,\varrho} = \varrho (\lambda_{,\varrho}^2 - \lambda_{,zz}^2), \nu_{,z} = 2\varrho \lambda_{,\varrho} \lambda_{,z},
\]

\[
(2.12) \quad \Delta \nu + \lambda_{,\varrho}^2 + \lambda_{,zz}^2 = 0,
\]

where \( \Delta \) is the flat Laplace operator on the fictitious two-dimensional space parameterized by the cylindrical coordinates \( \varrho \) and \( z \) (i.e. \( \Delta f = f_{,\varrho\varrho} + f_{,zz} + \frac{f_{,z}}{\varrho} \)). Once a solution of the Laplace equation has been chosen, the second equation becomes an exact differential and allows the calculation of \( \nu \) by a quadrature:

\[
(2.13) \quad \nu = \int_{\varrho_0}^{\varrho} \varrho [(\lambda_{,\varrho}^2 - \lambda_{,zz}^2) d\varrho + 2\lambda_{,\varrho} \lambda_{,z} dz] + c
\]

with \( \Gamma \) any open curve in the \((\varrho, z)\) plane, and it can be verified that the third equation identically holds. There is, therefore, a one to one correspondence between solutions of the Laplace equation on flat two dimensional space and vacuum, static, axially symmetric spacetimes. Asymptotic flatness requires vanishing of \( \lambda \) at space infinity, and the general solution of Laplace equation is thus of the form \( 5^5 \):

\[
(2.14) \quad \lambda = \sum_{n=0}^{\infty} \frac{a_n}{R^{n+1}} P_n(\cos \psi)
\]
where
\[(2.15) \quad R = \sqrt{\rho^2 + z^2}, \quad \cos \psi = \frac{z}{R}\]
and \(P_n(\cos \psi)\) are Legendre’s polynomials. It then follows:
\[(2.16) \quad \nu = \sum_{n,k=0}^{\infty} \frac{(n+1)(k+1)}{n+k+2} \frac{a_n a_k}{R^{n+k+2}} (P_{n+1} P_{k+1} - P_n P_k).\]

The real constants \(a_n\) are the so-called Weyl moments. The Weyl moments are not the multipole moments measured by an observer at space infinity \([6]\). For instance, the Schwarzschild solution of mass \(m\) has the following Weyl moments:
\[(2.17) \quad a_{2n} = -\frac{m^{2n+1}}{2n+1}, a_{2n+1} = 0\]
and thus it is not the monopole-Weyl solution. It is however easy to relate the \(a_n\) (Weyl) and \(M_n\) (multipole) moments \([7]\); for instance it can be shown that
\[(2.18) \quad M_0 = M = -a_0,\]
\[(2.19) \quad M_2 = Q = a_2 - \frac{1}{3} a_0^3.\]

In what follows we shall concentrate on that particular metric corresponding to the solution of the Laplace equation for a source of constant density \(\gamma\) uniformly distributed on a segment of length \(2m\) on the \(z\) axis centered at \(z = 0\) \([3]\):
\[(2.20) \quad \lambda = \frac{\gamma}{2} \ln \left( \frac{R_+ + R_- - 2m}{R_+ + R_- + 2m} \right)\]
\[(2.21) \quad \nu = \frac{\gamma^2}{2} \ln \left( \frac{R_+ + R_- - 2m)(R_+ + R_- + 2m)}{4R_+ R_-} \right)\]
where \(R_+ = \sqrt{\rho^2 + (z + m)^2}, R_- = \sqrt{\rho^2 + (z - m)^2}\). This metric is at best visualized in the so-called Erez-Rosen coordinates \((\rho, z) \to (r, \vartheta)\):
\[(2.22) \quad \rho^2 = (r^2 - 2mr) \sin^2 \vartheta, z = (r - m) \cos \vartheta\]
in which:
\[(2.23) \quad \lambda = \frac{\gamma}{2} \ln \left( 1 - \frac{2m}{r} \right)\]
\[(2.24) \quad \nu = \frac{\gamma^2}{2} \ln \left( \frac{1 - \frac{2m}{r} + \frac{m^2}{r^2} \sin^2 \vartheta}{1 - \frac{2m}{r}} \right)\]
so that
\[(2.25) \quad ds^2 = \Delta^{\gamma - 1} \Sigma^{\gamma - 1} dr^2 + r^2 \Delta^{\gamma - 1} \Sigma^{\gamma - 1} d\vartheta^2 + r^2 \sin^2 \vartheta \Delta^{1 - \gamma} d\varphi^2 - \Delta^\gamma dt^2\]
where
\[(2.26) \quad \Delta = \left( 1 - \frac{2m}{r} \right), \quad \Sigma = \left( 1 - \frac{2m}{r} + \frac{m^2}{r^2} \sin^2 \vartheta \right).\]
This metric is usually called \textit{gamma metric}. It has many interesting properties which we are going to discuss below.
3. PHYSICAL PROPERTIES OF THE GAMMA METRIC

The gamma metric coincides with the Schwarzschild metric of mass \( m \) when \( \gamma = 1 \). One of the advantages of the Erez-Rosen frame is that in this limit the gamma reduces to Schwarzschild in Schwarzschild coordinates. The mass and the quadrupole moment of the gamma metric are given by [3]:

\[
M = \gamma m,
\]

\[
Q = (\gamma^2 - 1)\frac{\gamma}{3} m^3.
\]

In the \( \gamma = 0 \) limit the metric becomes flat, whilst the Curzon solution (i.e. the Weyl monopole solution) is obtained in the limit \( \gamma \rightarrow \infty, m \rightarrow 0 \) with \( \gamma m = \text{const} \) [9]. As we have seen the gamma metric is originated - formally - by the solution of the Laplace equation for a line bar of constant density. However this a rather formal identification, and we are interested here in properties of the gamma metric as a vacuum field observed at space infinity. First of all, one can study the surface of revolution analogue to Schwarzschild spheres [10].

To an \( r = r_b \) surface in Schwarzschild pertains an area

\[
A_S = \int_0^\pi \int_0^{2\pi} r_b^2 \sin \vartheta d\vartheta d\varphi = 4\pi r_b^2.
\]

Calculating the same area for \( r = r_b \) surfaces in the gamma case, we can classify such surfaces as oblate or prolate with respect to the Schwarzshild sphere. Obviously, this property of the revolution surfaces is also the expected “shape” of static axially symmetric sources of the metric.

The 2-dimensional metric on the slices \( r = r_b, t = \text{const} \) is

\[
d\Omega^2 = r_b^2 \Delta_b^{\gamma^2 - \gamma-1} d\vartheta^2 + r_b^2 \sin^2 \vartheta \Delta_b^{1-\gamma} d\varphi^2
\]

where a subscript denotes evaluation at \( r = r_b \). One has, to lowest order in \( \epsilon = \gamma - 1 \),

\[
det\bar{g} = r_b^4 \sin^2 \vartheta \Delta_b^{(1-\gamma)^2} \Sigma_b^{1-\gamma^2} = r_b^2 \sin \vartheta (1 - \epsilon \ln \Sigma_b).
\]

From which follows:

\[
A_\gamma = A_S \left(1 - \frac{1}{2} \epsilon \int_0^\pi \sin \vartheta \ln \Sigma_b d\vartheta \right).
\]

Since \( \ln \Sigma_b < 0 \) we get \( A_\gamma > A_S \) for \( \epsilon > 0 \ (\gamma > 1) \); in such cases the slices are oblate (respectively prolate for \( \gamma < 1 \)). If the metric has to be interpreted has the vacuum field generated by a symmetric object then the physically interesting case is obviously that of oblate sources (e.g. galaxies). Stability has, for instance, to be expected only for such objects. This result is in agreement with the physical shape since the expansion of the newtonian potential is given by [8]:

\[
g_{00} = -1 + 2 \left[ \frac{M_0}{r} + \frac{Q}{r^3} P_2(\cos \vartheta) + o \left( \frac{1}{r^3} \right) \right]
\]

\[
(3.7) \quad = -1 + 2 \left[ \frac{\gamma m}{r} + \frac{1}{r^3} \frac{\gamma^3}{3} (\gamma^2 - 1) P_2(\cos \vartheta) + o \left( \frac{1}{r^3} \right) \right]
\]

so that the newtonian potential of the source is that of a disk if \( \gamma > 1 \).
4. The nature of the singularity of the gamma metric

In the present section we investigate on the nature of the singularity of the gamma metric, extending and completing previous results by Virbhadra \[11\]. The Kretschmann scalar $K = R_{abcd}R^{abcd}$ is

$$K(r, \theta) = \frac{16m^2\gamma^2N(r, \vartheta)}{r^{10}\Delta^{2\gamma^2 - 2\gamma + 2\Sigma3 - 2\gamma}}$$

where

$$N(r, \vartheta) = m^2\sin^2 \vartheta [3m\gamma(\gamma^2 + 1)(m - r) + \gamma^2(4m^2 - 6mr + 3r^2) + m^2(\gamma^4 + 1)] + 3r(m\gamma + m - r)^2(r - 2m).$$

For $\gamma = 1$ we get the Schwarzschild result $K = \frac{48m^2}{r^6}$, which is obviously regular at $r = 2m$. In what follows we consider only the case $\gamma \neq 1$. The $r = 2m$ surface is then a true singularity of the spacetime and the no-hair theorem assures that it will be a naked singularity in the sense that it is not covered by an event horizon.

The diverging behavior of $K$ in $r = 2m$ obviously depends on $\vartheta$. For $\vartheta = 0$ one has $\Sigma = \Delta$ and therefore:

$$K(r, 0) = \frac{48m^2\gamma^2(\gamma m - m - r)^2}{r^8\Delta^{1 - 2\gamma}},$$

this function diverges at $r = 2m$ only if $\gamma < 2$.

If $\vartheta = \vartheta_0 \neq 0 \ K(r, \vartheta_0) \approx \Delta^{-\gamma^2 + \gamma - 1}$. Since $\gamma^2 - \gamma + 1 > 0 \ \forall \gamma \neq 0$, the Kretschmann scalar diverges in $r = 2m$ if the singularity is approached on any plane different from the polar one, for any value of $\gamma$. It is therefore interesting to investigate the visibility of such singularity in dependence of the direction in which it is approached.

Consider a null “radial” ($\vartheta = \text{const}$) geodesic. A faraway observer can receive a signal from the $r = 2m$ surface if

$$\lim_{\varepsilon \to 0} \int_{2m + \varepsilon}^{R} dt = \lim_{\varepsilon \to 0} \int_{2m + \varepsilon}^{R} \Delta(\frac{2 - 2\gamma - 1}{\Sigma^{1 - 2\gamma}}) dr < \infty.$$

On any plane different from the polar one ($\vartheta \neq 0$) there exists a positive constant $C$ such that

$$\lim_{\varepsilon \to 0} \int_{2m + \varepsilon}^{R} \Delta(\frac{2 - 2\gamma - 1}{\Sigma^{1 - 2\gamma}}) dr < \lim_{\varepsilon \to 0} C \int_{2m + \varepsilon}^{R} \Delta(\frac{2 - 2\gamma - 1}{\Sigma^{1 - 2\gamma}}) dr < \infty$$

since the integral on the right converges for any $\gamma \neq 1$ (if $\gamma = 1$ divergence of the integral obviously signals the presence of the event horizon).

If however $\vartheta = 0$:

$$\lim_{\varepsilon \to 0} \int_{2m + \varepsilon}^{R} \Delta(\frac{2 - 2\gamma - 1}{\Sigma^{1 - 2\gamma}}) dr = \lim_{\varepsilon \to 0} \int_{2m + \varepsilon}^{R} \Delta^{-\gamma} dr$$

and the integral converges if and only if $\gamma < 1$. The situation is summarized in the table 1.

**:Table 1. Directional behavior of the singularity of the gamma metric**

| $\vartheta$ | $\gamma < 1$ | $\gamma > 1$ |
|-------------|--------------|--------------|
| $\vartheta = 0$ | visible | not visible |
| $\vartheta \neq 0$ | visible | visible |
5. Sources of the Gamma Metric

To construct physically viable sources of the gamma in order to investigate their behavior “at the boundary of the no-hair theorem” one needs to chose a matter model and then integrate the coupled Einstein-matter field equations. However, this proved by now to be impossible in the non-spherical case (also in the case of the Schwarzschild field few exact interior solutions are known). We thus need a simplifying assumption which however should restrain as little as possible the physical significance of the results. There is, in fact, the famous argument known as “Synge trick” which proposes to “guess” the metric from general considerations and then to evaluate the matter content from Einstein equations; however, usually such “Synge solutions” have nothing to do with physics. The idea we follow here traces back to work by Hernandez[12] and it is based on deformations of physically valid spherically symmetric solutions. We interpret here these deformations as actual changes in the shape of the objects, i.e. we use $\epsilon$ as the deformation parameter.

As far as we are aware the unique known examples of interior solution for the gamma are that given by Stewart et al. [4]. One of this solution is based on the Schwarzschild constant density interior solution as a “seed”, but the resulting density is not strictly decreasing from the center at higher orders and therefore the solution cannot be stable with respect to radial perturbations. The second one is based on the Adler interior solution.

In order to construct a new interior solution we take here a different point of view. Rather than searching for a solution generated by a “seed” to lowest order, we consider a “variation of the mass” of the gamma metric, namely, that family of interior solutions which is obtained by (a priori) any inhomogeneous distribution of mass which generates the vacuum gamma field at a fixed boundary. It is obvious, that not all the mass distributions will be physically viable, and indeed we shall see that the space of allowed functions is severely restricted by physical conditions.

It is well known that an interior solution generating the Schwarzschild field can be obtained from the vacuum metric putting $m = \mu(r)$. The Einstein tensor then becomes:

\[
G^{0}_{0} = 2 \frac{\mu'(r)}{r^2} \\
G^{1}_{1} = 2 \frac{\mu'(r)}{r^2} \\
G^{2}_{2} = \frac{\mu''(r)}{r} \\
G^{3}_{3} = \frac{\mu''(r)}{r}
\]

So that the matching, regularity and energy conditions

\[
\mu(r_b) = m \\
\lim_{r \to 0} \frac{\mu(r)}{r^2} = 0 \\
2 \frac{\mu'}{r^2} \geq 0 \\
2 \frac{\mu'}{r^2} - \frac{\mu''}{r} \geq 0
\]

can easily be satisfied, for instance, by a matter distribution of the form

\[
\mu(r) = \frac{4}{3} \pi E_0 r^3 \left(1 - \frac{3}{4} \frac{r}{r_b}\right).
\]

Note that the energy condition $G^{0}_{0} - G^{1}_{1} \geq 0$ holds identically for all $r$.

We now apply the same reasoning to the gamma. Considering $\gamma = 1 + \epsilon$ to first order in $\epsilon$
the vacuum metric becomes:

\begin{align}
(5.10) & \quad g_{00} = -\Delta - \epsilon \Delta \ln \Delta \\
(5.11) & \quad g_{11} = \Delta^{-1} + \epsilon \Delta^{-1} (\ln \Delta - 2 \ln \Sigma) \\
(5.12) & \quad g_{22} = r^2 + \epsilon r^2 (\ln \Delta - 2 \ln \Sigma) \\
(5.13) & \quad g_{33} = r^2 \sin^2 \vartheta - \epsilon r^2 \sin^2 \vartheta \ln \Delta
\end{align}

So that taking the Schwarzschild interior solution as the order zero interior solution is possible to build an interior of the form

\begin{align}
(5.14) & \quad \tilde{g}_{00} = -\left(1 - \frac{2\mu(r)}{r}\right) - \epsilon \left(1 - \frac{2\mu(r)}{r}\right) F(r) \\
(5.15) & \quad \tilde{g}_{11} = \left(1 - \frac{2\mu(r)}{r}\right)^{-1} + \epsilon \left(1 - \frac{2\mu(r)}{r}\right)^{-1} (F(r) + G(r, \vartheta)) \\
(5.16) & \quad \tilde{g}_{22} = r^2 + \epsilon r^2 (F(r) + G(r, \vartheta)) \\
(5.17) & \quad \tilde{g}_{33} = r^2 \sin^2 \vartheta - \epsilon r^2 \sin^2 \vartheta F(r)
\end{align}

Matching conditions require

\begin{align}
(5.18) & \quad \mu(r_b) = m \\
(5.19) & \quad \mu'(r_b) = 0 \\
(5.20) & \quad F(r_b) = \ln \left(1 - \frac{2m}{r_b}\right) \\
(5.21) & \quad F'(r_b) = \frac{2m}{r_b^3} \left(1 - \frac{2m}{r_b}\right)^{-1} \\
(5.22) & \quad G(r_b, \vartheta) = -2 \ln \left(1 - \frac{2m}{r_b} + \frac{m^2}{r_b^2} \sin^2 \vartheta\right) = A(\vartheta) \\
(5.23) & \quad G'(r_b, \vartheta) = -\frac{4m}{r_b^2} \frac{1 - \frac{m}{r_b} \sin^2 \vartheta}{1 - \frac{2m}{r_b} + \frac{m^2}{r_b^2} \sin^2 \vartheta} = B(\vartheta)
\end{align}

while regularity conditions for $r \to 0$ require

\begin{align}
(5.24) & \quad \frac{\mu(r)}{r^2} \to 0 \\
(5.25) & \quad \frac{G(r, \vartheta)}{r^2} \to 0
\end{align}

Now the energy conditions have to be imposed. It turns out that a suitable solution which satisfies the weak energy conditions can be found only for $\epsilon > 0$, i.e., for oblate sources.

The first energy condition, namely $G^0_0 \geq 0$, is satisfied automatically for all $r \neq r_b$ if the corresponding condition for Schwarzschild is satisfied. In $r = r_b$ the order zero condition vanishes and so $[G^0_0]_{r=r_b} \geq 0$ must be imposed. From this follows:

\begin{align}
(5.26) & \quad G''(r_b, \vartheta) \leq \left(1 - \frac{2m}{r_b}\right)^{-1} \left(\frac{4m}{r_b^3} - \frac{1}{r_b^2} G_{,\vartheta}(r_b, \vartheta) - \frac{1}{r_b} \left(1 - \frac{m}{r_b}\right) G'(r_b, \vartheta)\right).
\end{align}

In the same way conditions $G^0_0 - G^2_2 \geq 0$ and $G^0_0 - G^3_3 \geq 0$ are satisfied automatically for all $r \neq 0$ and so $[G^0_0 - G^2_2]_{r=0} \geq 0$ and $[G^0_0 - G^3_3]_{r=0} \geq 0$ must be imposed. The first is however satisfied by means of the regularity condition for $G(r, \vartheta)$ while the second leads to

\begin{align}
(5.27) & \quad \lim_{r \to 0} \left(\frac{2F'(r)}{r} + F''(r)\right) = c \geq 0.
\end{align}
Finally the condition that must be satisfied for every \( r \) and \( \vartheta \) is

\[
G_0^0 - G_1^1 = \left( 1 - \frac{2\mu(r)}{r} \right) \left( \frac{2}{r} F'(r) - \frac{1}{2} G''(r, \vartheta) \right) + \frac{1}{2r^2} \left( G_{\vartheta\vartheta}(r, \vartheta) + \frac{\cos \vartheta}{\sin \vartheta} G_{\vartheta}(r, \vartheta) \right) \geq 0.
\]

(5.28)

It is easy to check that the latter is satisfied in \( r = 0 \) and \( r = r_b \) if the previous conditions are satisfied and so, by continuity it must be satisfied in a neighborhood of those two points. Taking an appropriate function \( G(r, \vartheta) \) so that \( G''(r, \vartheta) \) and \( G_{\vartheta\vartheta}(r, \vartheta) + \frac{\cos \vartheta}{\sin \vartheta} G_{\vartheta}(r, \vartheta) \) are small compared to \( F'(r) \) for all \( \vartheta \) satisfies the condition everywhere. Explicit examples of choices of \( F(r) \) and \( G(r, \vartheta) \) satisfying the conditions are given in the appendix. However we show in the figures the behavior of the energy density and the critical energy condition for one of such examples describing a compact object with \( m = 1.4M_\odot \) and \( r_b = 10Km \).

![Figure 1](image.png)

**Figure 1.** In figure are shown (a) the condition for the energy density \( T_{00}^0 \geq 0 \) and (b) the condition \( T_{00}^0 - T_{11}^1 \geq 0 \) (in the adimensional unit obtained multiplying by \( r_b^2 \)) for a fixed value of \( \epsilon = \frac{1}{100} \) in dependence of \( \vartheta \) and the adimensional radial variable \( \frac{r}{r_b} \).

### 6. Discussion and Conclusions

General Relativity is in principle the “final theory” for the description of macroscopic gravitating objects. It is therefore odd enough that after nearly a century from its invention we do not have even one sound model of a star within the theory. In fact no solution of stationary axially symmetric Einstein field equation in matter is known, which is able to represent a finite non singular rotating object. Even in the simpler case of static, but not spherical, gravitating objects very little is known. Of course this case is the non rotating limit of the previous one, since it is only in the case of the perfect fluid that the static object must be spherical, and rotating stars are not to be expected to be made out of perfect fluids. For instance it is very well known that anisotropy plays a relevant role \(^{13}\) in the structure of compact objects such as white dwarfs and neutron stars.

So motivated we investigated here a simple static but not spherically symmetric model of a compact object, showing the existence of physically valid solutions of Einstein field equations describing it.

This model should allow for studying the dynamic behavior of the compact object close to the surface \( r_b = 2m \). This may be done, either by considering the dynamic equation just after the system leaves the equilibrium \(^{14}\) or by calculating the active gravitational (Tolman) mass as the system leaves the equilibrium \(^{15}\). In either case the bifurcation between the exactly spherically symmetric case and the gamma solution, should appear...
even for very small values of \( \gamma \), if \( \frac{m}{r_b} \) is sufficiently close to \( \frac{1}{2} \).

However, observe that specific constraints appear in our model on the value of \( \frac{m}{r_b} \). Indeed, the central radial pressure is given by

\[
P(0) = E_0 + \epsilon \left( -E_0 \ln \left( 1 - \frac{2m}{r_b} \right) + \frac{F'(r_b)}{r_b} \left( \frac{1}{2} E_0 r_b^2 - \frac{1}{8\pi} \right) + \frac{C}{r_b^2} \left( \frac{1}{6} E_0 r_b^2 - \frac{1}{8\pi} \right) \right)^2
\]

which is consistent with the small departure from spherical symmetry approximation if

\[
\epsilon \left( -E_0 \ln \left( 1 - \frac{2m}{r_b} \right) + \frac{F'(r_b)}{r_b} \left( \frac{1}{2} E_0 r_b^2 - \frac{1}{8\pi} \right) + \frac{C}{r_b^2} \left( \frac{1}{6} E_0 r_b^2 - \frac{1}{8\pi} \right) \right) \ll E_0.
\]

Since both \( \ln \left( 1 - \frac{2m}{r_b} \right) \) and \( \frac{F'(r_b)}{r_b} \left( \frac{1}{2} E_0 r_b^2 - \frac{1}{8\pi} \right) \) diverge as \( \frac{m}{r_b} \to \frac{1}{2} \), the pressure blows up in that limit. We must therefore consider an upper limit in the surface gravitational potential for which the model is acceptable. The same reasoning can be applied to the “arbitrary” positive constant \( C \). In fact once the scale given by \( \epsilon \) is fixed the value of \( C \) must not exceed a limit value for which the approximation holds.

Given the fact that \( \frac{m}{r_b} = \frac{1}{2} \pi E_0 r_b^2 \) maybe it is better to say that once we fix the value of \( \frac{m}{r_b} \) (and so \( E_0 \)) the possible values of \( \epsilon \) become fixed as well. So that for \( \frac{m}{r_b} \to \frac{1}{2} \) we must have \( \epsilon \to 0 \), and the more we depart from spherical symmetry the less the value allowed for \( \frac{m}{r_b} \) becomes.

Finally we would like to mention that work aimed to extend these results to slow rotation scenario is in progress.

**Appendix A. Example**

The Einstein tensor for the interior metric results:

\[
G^0_0 = 2 \mu'(r) \frac{r}{r^2} + \epsilon \left[ -2 \mu'(r) \left( F + G \right) - \frac{G_{\phi\phi}}{2r^2} - \frac{1}{2r} \left( 1 - \frac{2\mu(r)}{r} \right) G'' \right]
\]

\[
G^1_1 = 2 \mu'(r) \frac{r}{r^2} + \epsilon \left[ -2 \mu'(r) \left( F + G \right) + \frac{\cos \theta G_{\phi \phi}}{\sin \theta 2r^2} + \right.
\]

\[
\left. + \frac{1}{2r} \left( \mu'(r) - \frac{\mu(r)}{r} \right) G'' - \frac{1}{2r} \left( 1 - \frac{2\mu(r)}{r} \right) \left( 2F' + G' \right) \right]
\]

\[
G^2_2 = \mu''(r) \frac{r}{r} + \epsilon \left[ -\mu''(r) \left( F + G \right) + \frac{G_{\phi\phi}}{\sin \theta 2r^2} \right]
\]

\[
\left. - \frac{1}{2r} \left( \mu'(r) - \frac{\mu(r)}{r} \right) G'' + \frac{1}{2r} \left( 1 - \frac{2\mu(r)}{r} \right) \left( 2F' + G' \right) \right]
\]

\[
G^3_3 = \mu''(r) \frac{r}{r} + \epsilon \left[ -\mu''(r) \left( F + G \right) - \frac{G_{\phi\phi}}{2r^2} - \frac{1}{2} \left( 1 - \frac{2\mu(r)}{r} \right) \left( 2F'' + G'' \right) + \right.
\]

\[
\left. + \frac{1}{2r} \left( \mu'(r) - \frac{\mu(r)}{r} \right) \left( 4F' + G' \right) - \frac{1}{2r} \left( 1 - \frac{2\mu(r)}{r} \right) \left( 2F' + G' \right) \right]
\]

A simple choice of \( F(r) \) satisfying all conditions is

\[
F(r) = \frac{1}{2} \frac{F'(r_b)}{r_b} r^2 + \frac{1}{2} \frac{C}{r_b^2} \left( 1 - \frac{2}{3} \frac{r}{r_b} \right) + D
\]

with \( C \) an arbitrary positive constant and \( D \) given by the condition for \( F(r) \) at the boundary.

Furthermore considering \( G(r, \vartheta) \) of the form

\[
G(r, \vartheta) = g(r) A(\vartheta) + h(r) B(\vartheta)
\]
with \( g(r) \geq 0 \) so that \( g(r_b) = 1 \) and \( g'(r_b) = 0 \) and \( h(r) \leq 0 \) so that \( h(r_b) = 0 \) and \( h'(r_b) = 1 \) satisfies the matching conditions. The regularity requires:

\[
\lim_{r \to 0} \frac{g(r)}{r^2} = \lim_{r \to 0} \frac{h(r)}{r^2} = 0.
\]

The condition \( G_0^0 - G_1^1 \geq 0 \) is then satisfied if \( g(r) \), \( g''(r) \), \( h(r) \) and \( h''(r) \) are small enough with respect to \( F(r) \) which is always possible due to the degree of freedom given by the choice of the positive constant \( C \). The condition \( [G_0^0]_{r=r_b} \geq 0 \) finally implies

\[
G''(r_b, \vartheta) \leq \left( 1 - \frac{2m}{r_b} \right)^{-1} \left( \frac{4m}{r_b} - \frac{A_{\vartheta\vartheta}(\vartheta)}{r_b^2} - \left( 1 - \frac{m}{r_b} \right) \frac{B(\vartheta)}{r_b} \right)
\]

which is easily satisfied.

If we consider an object with \( m = 1.4M_\odot \) and \( r_b = 10Km \) (i.e. a neutron star) in geometrized units we get \( \frac{m}{r_b} = \frac{1}{5} \) and \( E_0 = \frac{3}{5\pi r_b^2} \) (which corresponds to \( 2.6 \cdot 10^{15} \frac{g}{cm^3} \)) then we have a wide variety of choices for the functions \( F(r) \) and \( G(r, \vartheta) \) and the central pressure doesn’t diverge for every small value of \( \epsilon \) so that the reasonable choice \( \epsilon = \frac{1}{100} \) leads to a central pressure \( \frac{1}{1000} \) times greater than the corresponding pressure for the spherically symmetric case.

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