A new weakly universal cellular automaton in the 3D hyperbolic space with two states

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Abstract — In this paper, we show a construction of a weakly universal cellular automaton in the 3D hyperbolic space with two states. Moreover, based on a new implementation of a railway circuit in the dodecagrid, the construction is a truly 3D-one.

Key words: cellular automata, weak universality, hyperbolic spaces, tilings.

1 Introduction

In this paper, we construct a weakly universal cellular automaton in the 3D hyperbolic space with two states. Moreover, based on a new implementation of a railway circuit in the dodecagrid, the construction is a truly 3D-one.

The dodecagrid is the tiling \( \{5,3,4\} \) of the 3D hyperbolic space, and we refer the reader to [15, 6] for an algorithmic approach to this tiling. We also refer the reader to [5, 7] for an implementation of a railway circuit in the dodecagrid which yields a weakly universal cellular automaton with 5 states. The circuit is the one used in other papers by the author, alone or with co-authors, inspired by the circuit devised by Ian Stewart, see [18]. The notion of weak universality is discussed in previous papers, see for instance [20, 3, 10, 5] and comes from the fact that the initial configuration is infinite. However, it is not an arbitrary configuration: it has to be regular at large according to what was done previously, see [9, 17, 16, 12].

In [14], I found a way to implement 1D cellular automata in two grids of the hyperbolic plane and one grid of the 3D hyperbolic space: the pentagrid, the heptagrid and the dodecagrid respectively. In this paper, I proved that such an implementation is possible without increasing the number of states of the automaton which is implemented when this is performed in the heptagrid or in the dodecagrid. I proved that such an implementation is still possible in the pentagrid when the cellular automaton satisfies an additional condition. It turns out that rule 110 of elementary cellular automata, see for instance [11, 19], satisfies this condition. This proves that there are weakly universal cellular automata with two states, in the pentagrid, in the heptagrid and in the dodecagrid.

However, such a construction is not fully satisfying for the reason that it does not make use of the geometrical properties of the plane or the space. Also, the result is obtained by application of a very strong result on 1D cellular automata,
the weak universality of rule 110, which requires very huge configurations and enormous computations. In this paper, we provide a construction which is far more simpler and requires very few resources for verification.

As we definitely changed the implementation of the railway circuit, we pay a new visit to the description of the circuit in Section 2 as well as to the previous implementations in Section 3. In Section 4 we define the elements of the new implementation and, in Section 5 we describe the scenario of the simulation. In Section 6 we give the rules of the automaton and we give a brief account of the computer program which checked the rules and the simulation, allowing us to prove:

**Theorem 1** There is a weakly universal cellular automaton in the dodecagrid which is weakly universal and which has two states exactly, one state being the quiescent state. Moreover, the cellular automaton is rotation invariant and the set of its cells changing their state is a truly 3D-structure.

In Section 7 we look at the remaining tasks.

## 2 The railway circuit

As initially devised in [18] and then mentioned in [4, 2, 17, 16, 7], the circuit uses tracks represented by lines and quarters of circles and switches. There are three kinds of switches: the fixed, the memory and the flip-flop switches. They are represented by the schemes given in Fig. 1.

![Figure 1](image)

*Figure 1* The three kinds of switches. From left to right: fixed, flip-flop and memory switches.

Note that a switch is an oriented structure: on one side, it has a single track u and, on the the other side, it has two tracks a and b. This defines two ways of crossing a switch. Call the way from u to a or b active. Call the other way, from a or b to u passive. The names comes from the fact that in a passive way, the switch plays no role on the trajectory of the locomotive. On the contrary, in an active crossing, the switch indicates which track, either a or b will be followed by the locomotive after running on u: the new track is called the selected track.

As indicated by its name, the fixed switch is left unchanged by the passage of the locomotive. It always remains in the same position: when actively crossed by the locomotive, the switch always sends it onto the same track. The flip-flop switch is assumed to be crossed actively only. Now, after each crossing by the locomotive, it changes the selected track. The memory switch can be crossed by the locomotive actively and passively. In an active passage, the locomotive
is sent onto the selected track. Now, the selected track is defined by the track of the last passive crossing by the locomotive. Of course, at initial time, the selected track is fixed.

Figure 2 The elementary circuit.

With the help of these three kinds of switches, we define an elementary circuit as in [18], which exactly contains one bit of information. The circuit is illustrated by Fig. 2 below and it is implemented in the Euclidean plane. It can be remarked that the working of the circuit strongly depends on how the locomotive enters it. If the locomotive enters the circuit through $E$, it leaves the circuit through $O_1$ or $O_2$, depending on the selected track of the memory switch which stands near $E$. If the locomotive enters through $U$, the application of the given definitions shows that the selected track at the switches near $E$ and $U$ are both changed: the switch at $U$ is a flip-flop which is changed by the actual active passage of the locomotive and the switch at $E$ is a memory one which is changed because it is passively crossed by the locomotive and through the non-selected track. The actions of the locomotive just described correspond to a read and a write operation on the bit contained by the circuit which consists of the configurations of the switches at $E$ and at $U$. It is assumed that the write operation is triggered when we know that we have to change the bit which we wish to rewrite.

From this element, it is easy to devise circuits which represent different parts of a register machine. As an example, Fig. 3 illustrates an implementation of a unit of a register.

Other parts of the needed circuitry are described in [4, 2]. The main idea in these different parts is to organize the circuit in possibly visiting several elementary circuits which represent the bits of a configuration which allow the whole system to remember the last visit of the locomotive. The use of this technique is needed for the following two operations.

When the locomotive arrives to a register $R$, it arrives either to increment $R$ or to decrement it. As can be seen on Fig. 3 when the instruction is performed,
Figure 3 Here, we have two consecutive units of a register. A register contains infinitely many copies of units. Note the tracks $i$, $d$, $r$, $j_1$, and $j_2$. For incrementing, the locomotive arrives at a unit through $i$ and it leaves the unit through $r$. For decrementing, it arrives though $d$ and it leaves also through $r$ if decrementing the register was possible, otherwise, it leaves through $j_1$ or $j_2$.

Figure 4 An example of the implementation of a small program of a register machine. On the left-hand side of the figure, the part of the sequencer. It can be noticed how the tracks are attached to each instruction of the program. Note that there are four decrementing instructions for $W$; this is why a selector gathers the arriving tracks before sending the locomotive to the control of the register. On the way back, the locomotive is sent on the right track.

the locomotive goes back from the register by the same track. Accordingly, we need somewhere to keep track of the fact whether the locomotive incremented $R$. 


or it decremented $R$. This is one type of control. The other type comes from the fact that several instructions usually apply to the same register. Again, when the locomotive goes back from $R$, in general it goes back to perform a new instruction which depends on the one it has just performed on $R$. Again this can be controlled by what we called the selector in [4, 2].

At last, the dispatching of the locomotive on the right track for the next instruction is performed by the sequencer, a circuit whose main structure looks like its implementation in the classical models of cellular automata such as the game of life or the billiard ball model. The reader is referred to the already quoted papers for full details on the circuit. Remember that this implementation is performed in the Euclidean plane, as clear from Fig. 4 which illustrates the case of a few lines of a program of a register machine.

Now, we turn to the implementation in the hyperbolic plane. We refer the reader to [6, 14] for the few required features of hyperbolic geometry. As announced in the introduction, first we look at the previous implementations.

### 3 The previous hyperbolic implementations

The first implementations were based on the following idea. The tracks are realized by a 1D-structure and the switches are realized by the meeting of the tracks at a cell. There were no other ingredients and this explains why the first obtained automaton had 22 states, see [2]. In this paper, we indicated the implementation of the tracks and of the circuit with many details. These details are reproduced and completed in [7] from which Figures 5, 6, 7, 8 and 9 are taken.

![Figure 5](image.png)

**Figure 5** Crossing of paths: the basic pattern.

*Note the nodes which are circled. Both paths must avoid these nodes in order to observe the following rule: a node coloured with $B$ has at most two neighbours, also coloured with $B$. This condition entails the shape of the basic pattern ef.*

Thanks to the 3D representation, as this was already noticed in [5], we can avoid crossings and so, we have no more to bother with the complications which arise from this situation as witnessed by Figure 5. However, the other figures remain valid and, as witnessed by Figure 7, we can see that complicated patterns cannot be avoided.

Now, as Figures 8 and 9 suggest, we can define an analogue of the segments of lines and the quarters of circles used in the Euclidean implementation. Here, we
replace the segments of lines by what we called \textit{vertical} in \cite{16 11} in different tilings, and we replace the quarter of circles by what we called \textit{horizontal} in the same papers.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{Implementation of the elementary unit in the Fibonacci tree.}
\begin{itemize}
\item Note the implementation of the three kinds of switches.
\item Also note the implementation of the various switches, depending on the desired direction of the path: the additional nodes represent the exact neighbours of the intersection node which the paths exactly go through.
\end{itemize}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{Implementation of parallel paths in the Fibonacci tree.}
\begin{itemize}
\item In the figure, forbidden nodes are circled.
\item Note the reproduction of the shape of the broken line \textit{ab} on the broken lines \textit{bc} and \textit{bd}. The reproduction is the result of a simple shift.
\end{itemize}
\end{figure}

In fact, the implementation of \cite{11} has taken benefit of a new implementation
of horizontals with respect to \([10]\).

Already in \([5]\), the idea of marking the intersection point of three tracks meeting at a switch by a colour corresponding to the type of switch was replaced by signals placed around the considered cell. This was easy in the 3D space, which explains that there we first obtained a weakly universal cellular automaton with 5 states. Moreover, the rules of this automaton satisfy a very strong property: they are not only rotation invariant, they are also different of each other by the Parikh vector associated to the rules. This vector is obtained by taking each state considered as a letter, and to write the rule as a word consisting of blocks of the same letter, each letter being repeated as many times as it appears among the neighbours of the cell. We said that the rules were pairwise **lexicographically** different.

![Figure 8](image)

**Figure 8** *Delimitation of a block in a Fibonacci tree \(F^*\).*

In the figure, we represent the levels between which the block is delimited. The branch which delimits the block on its left-hand side is supported by the leftmost branch of \(F^*\). The branch which delimits the right-hand side of the block is supported by the rightmost branch of a sub-tree of \(F^*\).

In \([12]\), a new idea is introduced in the context of the heptagrid. The idea consists in replacing the linear structure of the tracks by a new one: the orbit of the locomotive remains a 1D-structure, but it is not materialized by cells which are different from the quiescent state. The track are materialized by cells which are immediate neighbours of this orbit. This allowed to obtain a weakly cellular automaton in the heptagrid with 4 states only, the rules being also rotation invariant. Moreover, the implementation is a true planar one: there are a lot of cycles in the orbit of the locomotive. These immediate neighbours of the orbit were called **milestones** in \([12]\). The milestone technique was used again in \([11]\), which allowed to obtain a weakly universal cellular with three states only, the rules being again rotation invariant. However, the pairwise lexicographic difference could not be kept.

Now, \([11]\) introduces a new representation with respect to \([5]\). Indeed, remark that the trace of the dodecagrid on a plane \(\Pi_0\) which is supported by a face of
one of the dodecahedra of the tiling is a copy of the pentagrid. Then, by using
the Schlegel diagrams used in [5], we can project each dodecahedron which is in
contact with $\Pi_0$ onto $\Pi_0$ itself. This allows us to obtain a representation which
is illustrated by the figures of the next section.

\begin{figure}
\centering
\includegraphics{figure9}
\caption{Schematic representation of the whole circuit which corresponds to Figure 4.}
\end{figure}

In the figure, register 1 is on the leftmost branch while register 2 is on the rightmost
one. The instructions which increment register 2 are in orange, while the instruction
which decrements it is in purple. Similarly, for register 1: the incrementing instruc-
tions is in dark green and the decrementing one is in light green. The return track
which corresponds to jumps of decrementing instructions are in yellow for register 2,
in dark blue for register 1.

It is probably the place to introduce here the convention which is used with
this representation: if not otherwise mentioned, in this projection, a face $F$ of
a dodecahedron $D$ takes the colour of the state taken by the neighbour of $D$
which shares with it the face $F$.

4 A new implementation in the dodecagrid

Now, we start the new implementation by a new analysis of the railway circuit.

This new implementation was motivated by the remark that the motion
rules of the locomotive work also if the locomotive is reduced to one cell which
is of the same colour of the milestones: the black colour as now we have only
two colours, white and black. But they work in one direction only. The same
pattern cannot be used for moving the locomotive, which we call from now on particle, in both directions. One direction must be fixed for this pattern, the other direction being forbidden.

The main idea is to introduce tracks by pairs, as this is mostly the case in real life. There is a track for one direction and another one for the opposite direction. Now, we can see in Figure 7 how complex is the construction of a ‘parallel’ line in the hyperbolic plane. We shall see later that thanks to the third dimension, we can find a more efficient solution.

Now, before turning to this solution, we have to look at how the switches work in this new setting.

4.1 The new switches

For this analysis, we shall again use the tracks $u$, $a$ and $b$ defined in Section 2. We remind that the active passage goes from $u$ to either $a$ or $b$ and that the passive crossing goes from either $a$ or $b$ to $u$. As we split the ways into two tracks, we shall denote them by $u_d$, $u_r$, $a_d$, $a_r$, $b_d$ and $b_r$ respectively, where the subscript $d$ indicates the active direction and $r$ indicates the return one. A priori, this defines two switches: the first one from $u_d$ to $a_d$ or $b_d$ and the second one from $a_r$ or $b_r$ to $u_r$. We shall call the first one the active switch and the second one the passive switch. Note that each of these new switches deals with one-way tracks only. This can be illustrated by the right-hand side picture of Figure 10.

As the flip-flop switch is used in an active passage only, it makes use of single tracks only, in the direction to the switch for the way $u$ defined in Section 2, in the direction coming from the switch on the two ways $a$ and $b$ defined in Section 2 too. Accordingly, the flip-flop switch is only an active switch. There is no passive switch in this case.

And so, we remain with the fixed and the memory switches.

First, let us look at the fixed switch. As the switch is fixed, we may assume that $u$ always goes to $a$. This means that the track $b_d$ is useless. Now, it is plain that $u_d$ and $a_d$ constitute the two rays of a line issued from a point of the line. Consequently, there is no active fixed switch. The switch is concerned by the

![fixed switch and memory switch](image-url)
return tracks only: \( u_r, a_r \text{ and } b_r \), it is a passive switch. It is easy to see that it works as a collector: it collects what comes from both \( a_r \text{ and } b_r \) and send this onto \( u_r \). Now, as at any time there is at most one particle arriving at the switch from the union of \( a_r \text{ and } b_r \), the collector receives the particle from \( a_r \text{ or } b_r \) alternately, never at the same time, see the left-hand side picture of Figure 10.

Now, let us look at the memory switch. It is clear that, in this case, we have both an active and a passive switch. However, a closer look at the situation shows that the passive memory switch has a tight connection with the fixed switch and that the active memory switch has a tight connection with the flip-flop switch. We shall return to this point in Section 5.

For the moment, we go back to the implementation of the tracks in the dodecagrid.

### 4.2 Implementing two tracks

As already announced, we can exploit the third dimension in order to get a more efficient way. We first take the same general principle as in [11]. This means that a vertical segment is defined by a line \( \ell \) of the pentagrid which is a line which supports a side of a pentagon of the tiling. We say that \( \ell \) is the guideline of the vertical segment and it consists of a sequence of sides. Now, in [11], we fixed a face of a dodecahedron which fixes a plane \( \Pi_0 \). As already mentioned, \( \Pi_0 \) consists of pentagons which are faces of dodecahedra of the tiling and these faces realize a copy of the pentagrid on \( \Pi_0 \). In [11] a vertical segments is a sequence of dodecahedra which follows the guideline and which have a face on \( \Pi_0 \) in a such way that all the dodecahedra of the vertical segment lie in the same half-space defined by \( \Pi_0 \). We say that the vertical segment is above \( \Pi_0 \). We can see such a vertical line in Figure 11 realized by the yellow tiles which can be seen there. We can see in the figure that each dodecahedron which is above \( \Pi_0 \) is projected as a Schlegel diagram on the face which lies on \( \Pi_0 \).

Our implementation for vertical lines is based on the one we did in [11]. Remark that, in this implementation, the dodecahedra of the line have a milestone which is below \( \Pi_0 \). We can see three milestones on the projection of each dodecahedron, but as a dodecahedron of a vertical line has four milestones around itself, the fourth one is the neighbour which can be seen from the face which lies on \( \Pi_0 \). Fix \( \Delta \) a dodecahedron of the vertical segment \( V \) and let \( F \) be its face on \( \Pi_0 \). Denote by \( [F] \) the reflection of \( \Delta \) in \( F \). We know that \( [F] \) is a milestone of \( \Delta \). As \( \Delta \) is above \( \Pi_0 \), \( [F] \) is below \( \Pi_0 \). Now, consider the face \( G \) of \( \Delta \) which has a side on the guideline and which is not \( F \). We can see that \( [G] \) is also a milestone which has a face \( H \) on \( \Pi_0 \), so that \( G \) is also above \( \Pi_0 \). As there are four dodecahedra around an edge in the dodecagrid, the four dodecahedra which are around the side \( s \) of the guideline shared by \( F \) and \( G \) are \( \Delta, [F], [G] \) and \( [H] \). Now, we can see that \( \Delta \) and \( [H] \) are the reflection of each other under the reflection in \( s \). This reflection also shows that the situation of \( [F] \) and \( [G] \) with respect to \( \Delta \) is the same as their situation with respect to \( [H] \). Let us call \( [F] \) the ballast of \( \Delta \). Then, we can see that \( [G] \) can be considered as the ballast of \( [H] \). The reflection in \( s \) transforms \( \Delta \) into \( [H] \) and the milestones of \( \Delta \).
into milestones of $[H]$. This can be seen in the below picture of the right-hand side of Figure 11. The reflection of $V$ in $s$ defines a vertical line $W$ which is this time below $\Pi_0$. But the directions of $V$ and $W$ are the same. So that keeping $[F]$ and $[G]$ as milestones of $[H]$, we have to change the two others in order to get the opposite direction along $W$ with respect to that of $V$: call this transformation of $\Delta$ into this new pattern around $[H]$ quasi-reflection of $\Delta$ into $[H]$.

Figure 11 Left-hand side: The new vertical ways with two tracks. In yellow, one direction; in brown, the opposite direction.
Right-hand side:
up: the numbering of the faces in a dodecagrid of the tracks;
down: a cut of the tracks in the plane of the face 10 of the central cell.

To do that, number the faces of $\Delta$ as indicated in the above picture of the right-hand side of Figure 11. The face which is almost surrounded by the milestones is always face 1 and the ballast is on the face 5 of $\Delta$, as well as of $[H]$. Moreover, the numbers of the faces around face 11 are increasing while clockwise turning around face 11. Accordingly, the milestones of $\Delta$ are on faces 5, 2, 6 and 7. Taking the same conventions for $[H]$ and taking into account that $[H]$ is below $\Pi_0$, we can see that the milestones of $[H]$ are on its faces 0, 2, 5 and 7. Applying this transformation onto each dodecahedron of $W$, we get the track in the opposite direction which is illustrated by the left-hand side of Figure 11. In the figure, we imagine that $\Pi_0$ is transparent, so that $W$ can be seen through
it. We also imagine that the milestones associated with the faces $G$ are also transparent: otherwise, the tiles of $W$ could not be seen.

For the horizontal tracks, we apply the same idea, but its realization is more complex. We already know from [11] that the implementation of a horizontal line is more complex than that of a vertical line. Remember that there are two kinds of tiles in a horizontal segment: we have straight elements, those which are present in a vertical segment, and corners. In Figure 12, one of the two tracks consist of yellow and green tiles. The yellow tiles are the straight elements and the green ones are the corners. The corner is represented by the right-hand side picture of Figure 12. However, we have four kinds of corners. Some of them have a milestone under face 0 and the others do not have it. Also in some of them, face 9 has no milestone but face 10 does have while in the others, the situation is opposite: there is a milestone on face 9 but there is none on face 10.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{The new horizontal ways with two tracks. In yellow and green, one direction. In brown and purple, the opposite direction. Note that here we have straight elements and corners. We also have two kinds of horizontal elements as well as two kinds of corners.}
\end{figure}

We shall soon see the reasons of these distinctions.

As for a vertical segment, the ballast of the return track of a horizontal segment consists of the milestones of the direct track which are in contact with the plane of their ballast. Now, this requires to carefully define the corners and the straight elements.
From the study of [11], a horizontal segment, we shall later say a direct horizontal segment, is a word of the form \((SQ)^a\), with \(a\) a positive integer, where \(Q\) is a corner and \(S\) is a straight element. We say that \(SQ\) is the basic unit. In such a sequence of units, the quarter is connected to its neighbours in the chain by its faces 1 and 2, while the straight element is connected by its faces 1 and 4 or its faces 1 and 10. Whether it is 4 or 10 depends on the status of the face 5 in the pentagrid defined on \(\Pi_0\) by the dodecagrid: it is face 4 when the face 5 is a white node, it is a face 10 when the face 5 is a black node. This distinction is relevant in the construction of the return track.

First, consider the case when the face 10 of the straight element is in contact with the corner of the basic unit. Denote by \(A\) the straight element and by \(B\) the corner. As the face 10 of \(A\) is in contact with \(B\), consider the side \(s_6\) of face 6 which is the single side of this face lying on \(\Pi_0\). The quasi reflection of \(A\) in \(s_6\) defines a dodecahedron \(A_6\) surrounded by milestones as in a straight element and the face 1 of \(A_6\) defines a direction which is opposite to that defined by the face 1 of \(A\). Accordingly, we consider \(A_6\) as belonging to the expected return track. Besides its face 5, no other face of \(A\) than its face 6 is in contact with \(\Pi_0\). Consider \(B\). Outside its face 0, three of its faces are in contact with \(\Pi_0\): they are faces 3, 4 and 5. The intersections with \(\Pi_0\) are sides which we call \(s_3\), \(s_4\) and \(s_5\) respectively. Let \(B_3\), \(B_4\) and \(B_5\) the images of \(B\) in the quasi-reflection in \(s_3\), \(s_4\) and \(s_5\) respectively. As the face 0 of \(B\) is shared by a milestone, the same analysis as previously holds, and we can put around each \(B_i\), \(i \in \{3, 4, 5\}\) four milestones. In the cases when \(i\) is in \(\{3, 5\}\), the face \(i\) of \(B\) is shared by a milestone which we can define as the ballast of \(B_i\) and then the ballast of \(B\) is another milestone of \(B_i\). We define the two remaining milestones of \(B_i\) thanks to the numbering which we already defined. The situation for \(B_4\) is different: the face 4 of \(B\) is not shared by a milestone and so, the ballast of \(B_4\) should not be a milestone. In order to keep the constraint which we just noticed for \(B_4\), we define the face of \(B_4\) in contact with \(\Pi_0\) to be face 0. Then we put milestones in the faces 2, 5, 6 and 7 of \(B_4\), its face 2 being in the plane of the face 4 of \(B\). Now, we can see that \(B_3\) and \(B_4\) are not in contact with each other, and this is the same for \(B_4\) and \(B_5\). However, \(B_5\) is in contact with \(A_6\): taking into account the construction of \(B_5\), the face 1 of \(A_6\) is in contact with the face 3 of \(B_5\).

Now, it is not difficult to see that the faces of \(B\), \(B_3\) and \(B_4\) which lie on \(\Pi_0\) share a common point \(v_1\) ad that there is a fourth pentagon around \(v_1\) which is the reflection of the face 0 of \(B\) in \(v_1\). We can construct on this face a dodecahedron \(Q_1\) which is in contact with both \(B_5\) and \(B_4\). We define it as a corner which will be defined in a direction which is the same as those of \(B_5\) and \(B_4\). We may decide that the ballast of \(Q_1\) is not a milestone.

Now, we can make a similar remark with \(B\), \(B_4\) and \(B_3\) whose faces on \(\Pi_0\) share a common point \(v_2\). We can construct another corner \(Q_2\) in between \(B_3\) and \(B_4\) which is constructed in the same way as \(Q_1\).

And so, in this situation, when the face 10 of \(A\) is in contact with \(B\), we can see that this defines a part of the return track which consists of six dodecahedra: four straight elements and two corners. Moreover, among the straight elements, three of them have a ballast which is a milestone while this is not the case for
the fourth one.

We can remark that the considered segment is a word \( s_1s_1cs_0cs_0s_1 \), where \( s_1 \) defines a straight element with a ballast which is a milestone and \( s_0 \) defines a straight element where the ballast is not a milestone. We can also notice from the construction that \( s_0 \) is simply a rotated image of \( s_1 \) around face 1.

Second, consider the case when the face of \( A \) in contact with \( B \) is face 4. In this case, besides face 5, \( A \) has two faces in contact with \( \Pi_0 \): faces 6 and 10. For \( B \), the situation is the same as previously, with three faces, 3, 4 and 5 in contact with \( \Pi_0 \). Applying the quasi-reflection in the sides of the considered face lying on \( \Pi_0 \), this defines dodecahedra \( A_6 \), \( A_{10} \), \( B_3 \), \( B_5 \) and \( B_4 \) which can be defined as straight elements of the return track, having milestones as their ballast which is in fact the milestone of the source-dodecahedron which lies on \( \Pi_0 \), above the plane. The numbering of the faces and the distribution of the milestones are exactly as those described in the previous case for \( A_6 \), \( B_5 \) and \( B_3 \). Now, for \( A_{10} \) and \( B_4 \), we have the same situation as \( B_4 \) in the previous case of a basic unit: for these dodecahedra, their ballast should not be a milestone. Accordingly, for \( A_6 \) and \( B_4 \), we define the same numbering and the same distribution of the milestones as for \( B_4 \) in the previous case. Now, we can see that \( A_6 \) and \( A_{10} \) are not in contact and, similarly, as in the previous case, \( B_3 \) and \( B_4 \) as well as \( B_5 \) and \( B_4 \) are not in contact. We apply the same construction as previously, yielding three corners these time, \( Q_1 \), \( Q_2 \) and \( Q_3 \): \( Q_1 \) in between \( B_4 \) and \( B_5 \), \( Q_2 \) in between \( B_4 \) and \( B_3 \), and \( Q_3 \) in between \( A_6 \) and \( A_{10} \), with exactly the same characteristics as the corners which we obtained in the previous case.

Accordingly, when the face 4 of \( A \) is in contact with \( B \), we can see that this defines a segment of the return track which consists of eight dodecahedra: five straight segments and three corners. Moreover, among the five straight elements, four of them have a ballast which is a milestone while this is not the case for the fifth one.

As in the previous case, we can remark that the considered segment is a word \( s_1cs_0s_1cs_0cs_0s_1 \), where \( s_1 \) and \( s_0 \) as defined as previously.

Now, it is plain that the face 1 of \( B_3 \) is always in contact with the \( A_6 \) defined by the next basic unit. Consequently, a direct segment of the form \( (SQ)^n \) defines a return segment of the form \( (s_1cs_0)^n \) \( s_1cs_0cs_1)^n \), where the occurrence of \( cs_0 \) is dictated by the number of the face of \( S \) which is in contact with \( Q \).

It is now time to deal with the motion of the particle.

### 4.3 The motion of the particle

In order to do that, we consider the case of a vertical segment which is the easiest one.

We have to define what is the direction of the motion. From the previous analysis, define the numbering of the faces as in the right-hand side picture of Figure 12. For a straight element of a direct track, the milestones are on faces 0, 5, 6 and 7. They are on faces 2, 5, 6 and 7 for a straight element of a return track. This also holds for horizontal tracks. In a vertical track, direct or return, the face on \( \Pi_0 \) is always face 5. In a horizontal track, it is also most often face 5.
but, from time to time, it is face 0. We have described when this happens in a precise way.

From now on, we define that face 1 of a straight element is an exit for the particle. The entry may be either face 4 or face 10. Of course, we might decide to choose the opposite convention. However, we shall see that this solution makes the things a bit easier. In some occasions, as we shall see a bit later, the straight element may be rotated around face 1 so that face 0 lies on $\Pi_0$. In this case, the entries may be either face 4 or face 3. Face 10 remains possible but it will be never used. At last, another rotation is also use in which face 2 lies on $\Pi_0$. In this latter case, the entries are face 3 or 8. Face 8 will be mostly used.

For a corner, the situation is simple. There are two possible tiles as the pattern of the milestones around the cell of the track is not symmetric. And so, there is a pattern as the one which is presented by the right-hand side picture of Figure [12] and the symmetric one with respect to the reflection which exchanges faces 9 and 10 in the same figure, leaving faces 0, 4, 7 and 11 globally invariant. The particle passes through faces 1 and 2. As the just indicated two patterns are not a rotated image of each other, we can use one pattern for the direction from face 1 to face 2 and the other for the direction from face 2 to face 1. We decide that the milestone which is on one of the faces 9 or 10 is put on the side of the entry. Accordingly, when face 9 is not covered by a milestone, the entry is face 1 and the exit is face 2. When face 10 is not covered by a milestone, face 1 is now the exit and face 2 is the entry.

Before turning to the switches, we have to pay a new visit to the bridges which were introduced in [5] and which were adapted to the new definition of tracks in [11].

### 4.4 The bridges

In [11], we have indicated how to perform the crossing of two vertical segments, as we may always assume that crossings can be performed in this way. To avoid crossings, we decide that one track will pass in another plane of the 3D space in order to avoid the other track and to go back to its way. As each segment consists of two tracks, and although the tracks are very close to each other, we cannot represent a whole bridge within a single figure.

Consider two segments $S_1$ and $S_2$. We may assume that in both segments, the face 5 of the dodecahedra involved in both tracks are all in the same plane $\Pi_0$. Let $\ell_1$, $\ell_2$, be the guidelines of $S_1$, $S_2$ respectively. Assume that $\ell_1$ and $\ell_2$ meet at some point $P$. This point is a vertex of a pentagon of $\Pi_0$ as $\ell_1$ and $\ell_2$ are two lines of the pentagrid defined on $\Pi_0$ by the dodecagrid. There are four pentagons around $P$. From this, we can see that if we keep one segment unchanged, the other has to avoid four dodecahedra of the other segment: two of them above $\Pi_0$ and the two others below. Let $S_2$ be unchanged and let us consider the change needed for $S_1$.

Let us introduce coordinates from $\mathbb{Z}$ on both tracks of $S_1$. As one track is above $\Pi_0$ and the other is below the plane, they will be denoted by $T_u$ and $T_l$ respectively. A dodecahedron of a track is thus numbered $T_u[n]$ or $T_l[n]$. To
fix things, we decide that $P$ is shared by $T_u[0]$, $T_u[-1]$, $T_\ell[0]$ and $T_\ell[-1]$, the projection of $T_u[0]$ onto $\Pi_0$ being a neighbour of the projection of $T_\ell[-1]$ onto the same plane. Now, we shall have two bridges $b_u$ and $b_\ell$: $b_u$ for the upper track will pass above $\Pi_0$ and $b_\ell$ will pass below $P_0$. In order to define the bridges, we consider the plane $\Pi_1$ which contains $\ell_1$ and which is perpendicular to $\Pi_0$.

The dodecahedra of the tracks will mostly have their face 5 on $\Pi_1$, and the tracks will be in different half-spaces with respect to $\Pi_1$. The dodecahedra of $b_u$ replace the dodecahedra $T_u[i]$ with $i \in [-3, 2]$ and, similarly, the dodecahedra of $b_\ell$ replace the dodecahedra $T_\ell[i]$, with the same values for $i$. The dodecahedra of $b_u$ and $b_\ell$ follow a kind of half-circle in $\Pi_1$ which is somehow truncated at the ends. Most of these dodecahedra are at a distance 6 or 7 from $P$, $T_u[0]$, $T_u[1]$, $T_\ell[0]$ and $T_\ell[1]$ being at distance 0 by definition. A slight modification allows us to reduce a bit the number of needed dodecahedra. In order to better see this point, Figures 13 and 14 represent the ends of both $b_u$ and $b_\ell$. Figure 13 is a projection onto $\Pi_0$ of the ends of $b_u$ and $b_\ell$: we can see there the last dodecahedron of the tracks of $S_1$. Figure 14 is a projection of these ends onto $\Pi_1$. On this figure, we have represented four more dodecahedra of the upper bridge at each end in order to better see the situation.

![Figure 13](image)

**Figure 13** The ends of the bridges: projection onto $\Pi_0$. The numbers given to the faces around their common point are enough to find out all the numbers of the faces.

Denote by $b_u[1]$ the dodecahedron of the bridge which replaces $T_u[-3]$ and similarly by $b_\ell[1]$ the one which replaces $T_\ell[2]$. For the other end, denote by $b_u[-1]$ and $b_\ell[-1]$ the dodecahedra of $b_u$ and $b_\ell$ respectively which replace $T_u[2]$ and $T_\ell[-3]$ respectively. This will allow us to number the dodecahedra of $b_u$ and $b_\ell$ which are close to their ends. Both faces on $\Pi_0$ of these dodecahedra share a point $L$ or $R$ with their neighbours of $T_u$ and $T_\ell$ respectively. They are corners which are placed with face 5 on $\Pi_0$ instead of face 0. For both $b_u[1]$ and $b_\ell[1]$, face 0 is on $\Pi_1$ so that these corners send the particle onto $\Pi_1$ or they
expect it from this plane.

Figures 13 and 14 describe the departure/arrival of each end of the bridges. Figure 13 shows the projections of \( b_u[1] \) and the corresponding dodecahedron of \( b_l \) onto \( \Pi_0 \). Note that on this projection, \( b_u[1] \) is seen from above while \( b_l[1] \) is seen from below, as if \( \Pi_0 \) would be transparent. We have a similar phenomenon in Figure 14. The tiles of \( T_u \) and \( b_u \) which are projected onto \( \Pi_1 \) are projected from the front of \( \Pi_1 \) with respect to the reader. The tiles of \( T_l \) and \( b_l \) are projected from behind the plane. The consequence of these characteristics of the projections is that the numbering of the faces is clockwise on the projections corresponding to \( T_u \) and \( b_u \). They are counter-clockwise on the projections corresponding to \( T_l \) and \( b_l \).

![Figure 14 The ends of the bridges: projection onto \( \Pi_1 \). The numbers given to the faces around their common point are enough to find out all the numbers of the faces.](image)

It is not difficult to see that it is possible to join the two ends represented by Figure 14. Indeed, \( b_u[5] \) and \( b_u[-5] \) are at distance 5 from \( P \). Accordingly, they are on level 5 of two Fibonacci trees which are rooted at the pentagons which share \( P \) in the half-plane of \( \Pi_0 \) which contains \( b_u \). The horizontal segment which we defined in this section follows such a level. In this case, the yellow tiles are on level 5 while the green ones are on level 6. We also know that the track going from \( b_u[5] \) to \( b_u[-5] \) along \( b_u \) is a word of the form \( S(QS)^+ \), where \( S \) represents a straight element and \( Q \) represents a corner. Accordingly this portion of half-circles completes the part of the bridge which is not represented on the figure. By symmetry, we do the same for \( b_l \).

We are now ready to study the implementation of the switches.
5 The scenario of the simulation

We shall examine the three kinds of switch successively. We start with the trivial case of the fixed switch. We go on with the rather easy case of the flip-flop switch and we complete the study by the rather involved case of the memory switch.

5.1 Fixed switches

As known from Section 2, we have a passive switch only to implement. The definition of the motion of the particle by a passage from face 4 or 10 to face 1 shows that there is nothing to do but abutting the two arriving tracks to the exiting one. This is illustrated by Figure 15. It is enough to make the central cell symmetric, which is easy to realize: it is enough to rotate a straight element around its face 1 in such a way that the face lying on $\Pi_0$ is now face 0.

![Figure 15](image)

Figure 15 Two fixed switch: left-hand side and right-hand side. Note the symmetry of the figure. Also note that this requires straight elements only.

Now, it is easy to put the return track, either along one arriving track or along the other: Figure 15 shows that the orientation of the milestones presented in the figure makes both constructions possible. The changes needed to transform one picture of the figure into the other are easy to perform. Note that we have here an easy realization of two versions of a fixed switch while in many previous papers we had only a right-hand side fixed switch, the other fixed switch being simulated by the right-hand side one followed by a crossing.

We shall see that this easy implementation will help us to realize a rather simple implementation of the passive memory switch.
5.2 Flip-flop switches

Now, we have to realize the flip-flop switch. We can easily see that it is not enough to reverse the straight elements with respect to the previous figure in order to realize the switch. This has to be done for the tracks but it does not work for the central cell, the cell which the three tracks abut. The central cell must have a specific pattern. We decide to append just one additional milestone and we change a bit the pattern, making it symmetric and significantly different from that of the fixed switch.

This fixes the frame but now, we have to implement the mechanism which first, forces the particle to go to one side and not to the other one and then, to change the selected track.

![Figure 16](image)

**Figure 16** The flip-flop switch: here the selected track is the right-hand side one.

Note that three cells have a particular pattern: the central cell and the two tracks marking the abutting of the ways $a$ and $b$.

Let us remember notations and let us define new ones. We remember that three tracks abut a switch. In section 2 we called them $u$, $a$ and $b$, where, in an active passage, $u$ is before the switch and $a$ with $b$ are together after the switch, $a$ on the right-hand side, $b$ on the left-hand one. Also number the cells in the following way, followed by the computer program: 1, 2 and 3 are the cells of $u$, 3 being the cell which abuts the central cell which receives number 4. Next, 5, 6 and 7 are the cells of $b$ and 8, 9 and 10 are those of $a$, with 5 and 8 abutting 4. Cell 5 is called the entry of $b$ and cell 8 is called that of $a$. At last, we distinguish three other cells called 11, 12 and 13. To locate them, we number
the faces of 4, 5 and 8. In all these cells, face 1 is the yellow outer face whose both neighbouring faces inside the same pentagon are brown. Then, the other faces are increasingly numbered by clockwise turning around the cell, first the outside ring of faces and then the inside one, completing with the innmost face. Considering this numbering, cells 11, 12 and 13 are the cells which are put on the face 9 of the cells 5, 8 and 4 respectively. Cells 11 and 12 are the sensors of cells 5 and 8 respectively.

Now, cells 11 and 12 signalize the selected track: one of them is white and the other is black. The selected track is the one for which the sensor of its entry is white. Cell 13 is called the controller. It is usually white and when it flashes to black, cells 11 and 12 simultaneously change their state to the opposite one. Cell 13 is black just for this action: after that it goes back to the white state.

This scenario is illustrated by Figure 18 as well as by the trace of execution given by Table 1.

Table 1 Trace of execution performed by the computer program to check the crossing of a right-hand side flip-flop switch by the particle.

| active crossing of a right-hand side flip-flop switch | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|------------------------------------------------------|---|---|---|---|---|---|---|---|---|----|----|----|----|
| time 0                                               | W | B | W | W | W | W | W | W | W | B | W | W | W |
| time 1                                               | W | W | B | W | W | W | W | W | W | B | W | W | W |
| time 2                                               | W | W | W | B | W | W | W | W | W | B | W | W | W |
| time 3                                               | W | W | W | W | W | W | W | B | B | W | W | B | W |
| time 4                                               | W | W | W | W | W | W | B | W | W | B | W | B | W |
| time 5                                               | W | W | W | W | W | W | W | B | B | W | B | W | B |

How can cell 13 detect the situation? In fact, cells 11, 12 and 13 are not fixed cells. So, to change states according to the indicated scenario, they are themselves decorated by an appropriate pattern. It is the same pattern for cells 11 and 12 as they play the same role, it is another pattern for cell 13. The patterns are illustrated by Figure 17.

![Figure 17](image_url)

**Figure 17** The cells which control the working of a flip-flop switch. Cells 11, 12 and 13 are placed on the face 9 of cells 5, 8 and 4 respectively. Cells 11 and 12 are mere signals. Cell 13 is a control unit. When it flashes, cells 11 and 12 change their states. Here, cell 11 is represented with darker colours in order to indicate that cell 11 is black while cell 12 is black.
In order to understand the working of these cells, we have to take into consideration that cells 4, 5 and 8 share a common vertex, that cell 4 and 5 share a common face as well as cells 4 and 8. More precisely, the face 4 of cell 5 and the face 3 of cell 4 are the same. Similarly, the face 3 of cell 8 and the face 4 of cell 4 are the same. We shall say that cell 4 can see cells 5 and 8 through its faces 3 and 4 respectively and that cells 5 and 8 can see cell 4 through their face 4 and 3 respectively. As a consequence, as cells 11, 12 and 13 share also a common vertex due to their position on cells 5, 8 and 4 respectively, we have that cell 13 can see cells 11 and 12 through its faces 3 and 4 respectively and that cells 11 and 12 can see cell 13 from their faces 4 and 3 respectively. This property is represented in Figure 17 where in cell 13, face 3 has another colour: this simply indicates that the neighbour of this cell is black and the neighbour is cell 11. This holds for a flip-flop where the right-hand side track is selected. When the left-hand side track is selected, cell 11 is white, cell 12 is black and, accordingly, in cell 13, face 3 has a white neighbour and face 4 has a black one.

The trace of execution given in Table 1 shows that the depicted scenario takes place actually. Figure 18 illustrates the motion of the particle. It represents the action of the controller by materializing its different states as additional states. This is to underline the fact that although cell 13 is by itself either black or white, the presence of its pattern as shown by Figure 17 make it possible to view these states as two new additional states.

Of course, we have the symmetric transformation when the particle crosses a flip-flop where the selected track is the left-hand side one.

![Figure 18](image-url) *Passage of the particle across a flip-flop switch. Note the changes of states on the fourth and fifth pictures.*

5.3 Memory switches

Now, we arrive to the most complex structure in our implementation. This is not due to the fact that we have to implement two single-track switches: a passive one and an active one. It is mainly because these two single-track switches must be connected. This comes from the definition of the memory switch: the
selected track of the active switch is defined by the last crossing of the passive switch.

The name of the passive switch is a bit misleading. As will be seen, although discreet, the passive switch has a definite action. If the particle happens to cross it through the track which corresponds to the selected one in the active switch, it sends a signal which triggers the change of the selected track in the active switch. This can be performed thanks to a careful study of the previous switches to which we now turn.

Figure 19 represents the passive memory switch. Here, the selected track is the left-hand side one.

![Figure 19: The passive memory switch. In this figure, the selected track is the left-hand side one. Nothing happens if the particle comes through this track. If it comes through the other one, it will trigger the change of selection, both in the passive and in the active switch.](image)

As in the case of the flip-flop switch, we number the cells involved in the passive switch by taking the number they received in the computer program.

This time, the cells from 1 to 3 are the cells of u, 4 is again the central cell, the cells from 5 to 7 are b and those from 8 to 11 are a. Cell 4 can see cells 5 and 8 through its faces 3 and 4 respectively and, conversely, cells 5 and 8 can see cell 4 through their face 4 and 3 respectively. Cells 11, 12 and 13 are very different from the cells with the same numbers in a flip-flop switch. Here, cell 13 can be characterized as follows: let A be the common vertex of cells 4, 5 and 8. Above Π₀, there are four dodecahedra sharing A. We just mentioned three of
them. The fourth one is cell 13, which is obtained from cell 4 by reflection in the plane orthogonal to $H_0$ which passes through $A$ and which cuts the faces 1 of cell 5 and 8 perpendicularly. Now, we can number the faces of cell 13 in such a way that cell 13 can see cells 5 and 8 through its faces 4 and 3. Cells 11 and 12 are put on cell 13, on its faces 10 and 8. Cells 11 and 12 are the reflection of cells 5 and 8 respectively in the edge which is shared by both the concerned dodecahedra. If the selected track is $b$, cell 12 is black and cell 11 is white. If the selected track is $a$, cell 11 is black and cell 12 is white. Now we can describe what happens more clearly.

Table 2 Trace of execution performed by the computer program to check the crossing of a left-hand side passive memory switch by the particle. Here, the particle runs over the non-selected track.

| time | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 0    | W | W | W | W | W | B | W | W | W | B  | W  | W  | W  | W  | W  |
| 1    | W | W | W | W | B | W | W | W | W | W  | B  | W  | W  | W  | W  |
| 2    | W | W | W | B | W | W | W | W | W | W  | B  | B  | W  | W  | W  |
| 3    | W | W | B | W | W | W | W | W | W | B  | B  | W  | W  | B  | W  |
| 4    | W | B | W | W | W | W | W | W | W | B  | W  | W  | B  | W  | W  |
| 5    | B | W | W | W | W | W | W | W | W | B  | W  | W  | W  | W  | W  |
| 6    | W | W | W | W | W | W | W | W | W | B  | W  | W  | W  | W  | W  |
| 7    | W | W | W | W | W | W | W | W | W | B  | W  | W  | W  | W  | W  |

Figure 20 Passage of the particle across a passive memory switch. Note the changes of states on the third and fourth pictures.

As required by the working of a memory switch, if the particle crosses the passive memory switch through the selected track, nothing happens.

And so, consider the case when the particle crosses the passive switch through the non-selected track. When the particle is in cell 8, cell 13 can see the particle and, as its cell 12 is black, it knows that it must flash. This means that
it becomes black for the next time and then returns to the white state at the following time. This can be seen in the trace of execution displayed by Table 2 as well as in Figure 20. When cell 13 flashes, cells 11 and 12 exchange their states: if it was black it becomes white and conversely.

But this is not the single change. The trace of Table 2 mentions two additional cells: 14 and 15. They are the first two cells of a path which conveys the flash emitted by cell 13 to the active switch in order that the selected track of the active switch should also be changed. Cell 14 is put on cell 13, on the face 9 of this cell precisely. This can be seen on the fourth picture of Figure 20: we can see that this cell becomes black just after the flash of cell 13.

This path introduces a new feature which does not exist in the simulations of the previous papers. As long as the flash does not reach the active switch, we have two particles in the circuit: the one which plays the role of the locomotive introduced in Section 2, and the one which is emitted by the sensor of the passive memory switch. This new particle can be seen as a temporary copy of the first one.

Let us now describe the path. But, to better understand this point, we have to indicate how the active memory switch is implemented. In fact, it is implemented as a flip-flop switch and, accordingly, cells are numbered in the same way as for the flip-flop switch and cells 11, 12 and 13 are in the same connections with the other cells. However, in the active memory switch cell 13 is very different from the cell 13 of the flip-flop switch. This is materialized by the pattern of milestones around the cell which is very different, see Figure 21.

If we consider cells 1 to 10 in a passive memory switch and in an active one, up to a possible change of some numbers, their projection on $\Pi_0$ is the same: this is clear from the previous descriptions and from Figures 16 and 19. Call this projection the basis of the memory switch. Consider the point $P$ which is the common point of cells 4, 5 and 8. It is also shared by cell 13 in the passive memory switch. Call $\Delta$ the line which is perpendicular to $\Pi_0$ and which passes through the point $A$ of $\Pi_0$ which is shared by the face 4 and 5 of cell 4. It is
a line of the dodecagrid: it consists of edges of dodecahedra belonging to the tiling.

We fix two points \( A_p \) and \( A_a \) on \( \Delta \) such that \( A_p \) and \( A_a \) are vertices of dodecahedra of the tiling and that \( A \) is the mid-point of \( [A_pA_a] \). We shall consider that \( A_p \) is above \( A_a \). Now, we consider the planes \( \Pi^a_0 \) and \( \Pi^g_0 \) which are perpendicular to \( \Delta \) and which pass through \( A_p \) and \( A_a \) respectively. By our definitions, \( A_p \) is in the half-space which is above \( \Pi^a_0 \). We put a copy of the active memory switch onto \( \Pi^g_0 \). This means that the face 0 of cell 4 is on \( \Pi^g_0 \) and that cell 4 is in the half-space which is above \( \Pi^g_0 \). Consider the planes \( \Pi^a_4 \) and \( \Pi^g_5 \) which are perpendicular to \( \Pi^g_0 \) and which contain the faces 4 and 5 of cell 4 respectively. The intersection of these planes is the line \( \Delta \). Remember that \( \Pi^a_0 \) and \( \Pi^g_3 \) are not parallel: as they have a common perpendicular they have no point in common, neither in the space nor at infinity. Next, we put a copy of the passive memory switch onto \( \Pi^g_0 \) but in such a way that the cell 4 of the passive switch is in the half-space which is below \( \Pi^g_0 \); in this way, the cells 4 of both switches are contained in the intersection of two half-spaces defined by \( \Pi^a_0 \) and \( \Pi^g_0 \). Also, we put the cell in such a way that its face 2 is on \( \Pi^g_5 \) so that its face 3 is on \( \Pi^a_4 \).

![Figure 22](image-url)

Figure 22 is a projection of the passive memory switch on \( \Pi_4 \). The traces
of $\Pi_5$ and $\Pi_0^p$ are indicated in the figure. Only four cells have a face on $\Pi_4$: cells 4, 5, 8 and 13. This is why these numbers only appear in the figure for cells of the memory switch. Now, the common face of cells 4 and 5 is on $\Pi_4$ and it is the same for the common face of cells 5 and 13: this explains why the numbers of the cells sharing a face appear in the same yellow pentagons in the figure. We cannot see the other faces of the switch as they are not in contact with $\Pi_4$. Cells 14 and 15 are the first cells of the path from the passive switch to the active one. We can see them in Figure 19. Most cells of the path of the figure are in orange and in purple: this is to indicate that there are behind $\Pi_4$ if the memory switch is mainly in front of the plane. However, the path has to arrive to the dodecahedron which is on the face 9 of cell 4 of the active memory switch. Accordingly, at some point, the path must cross the plane $\Pi_4$ in order to be in front of it. This is performed when the path reaches $\Delta$: the path turns around the line and goes down further along $\Delta$ by two dodecahedra. This is indicated in the figure by the green cell which belongs to the path and which has this colour as it stands in front of $\Pi_4$.

![Figure 23](image)

**Figure 23** Zoom on the part of the path where the path performs a half-turn around $\Delta$ in order to arrive on the appropriate face of the cell 13 of the active memory switch. On the left-hand side: projection onto $\Pi_0^p$ from above. In the middle: projection onto $\Pi_5$: the path goes down by two dodecahedra. On the right-hand side: again projection onto $\Pi_5$, but from the other side as the path has passed across the plane on its other side.

The path goes down along $\Delta$, but it does not go down until reaching cell 4. At a distance of three tiles from cell 4, the path follows a horizontal segment in $\Pi_4$ until it reaches $\zeta$, the line of $\Pi_4$ which passes through $P_0^a$, the point of $\Pi_0^p$ which is shared by the faces 3 and 4 of cell 4 in the active memory switch. Then, the path goes down along $\zeta$, until it reaches the cell which is on the face 1 of cell 13. Denote this cell by $[1]_{13}$. More generally, we recursively denote by $[i]_j$ the dodecahedron which is put on the face $i$ of the dodecahedron denoted by $j$ which we denote by $i_j$ if needed. The last index in $j$ is a number already fixed.

In Figures 24 and 25 we have a numbering of the faces of the cell. Face 1 is the face which is the closest to cell $[0]_{13}$, the neighbour of cell 13 which shares its face 0. In the computer program, cell $[0]_{13}$ is cell 14. In Figure 24 it seems that the path goes from one face to the next one for faces 9, 8, 7 and 1. In fact, the
path consists of the neighbours of cell $[1]_{13}$ which are put on the just indicated faces together with corners which allow to go from one cell to the other. Indeed, the faces $9_{[1]_{13}}$ and $8_{[1]_{13}}$ are perpendicular.

Figure 24 The dodecahedra use of the path from $\xi$ to the cell 13 of the active memory switch. The last five cells in the figure are straight elements: in between them, from the first one to the fifth one, there is a corner which may be of one or another of the two possible forms. Note that the last three cells have exactly the same pattern, rotated in the same way. Cell $[1]_{[9]_{13}}$ is a neighbour of cell $[0]_{13}$. Remember that cell $[0]_{13}$ is cell 14. For all these cells, the bottom is in contact with a black cell as this is the state of cell $[1]_{13}$.

Figure 25 Illustration of the end of the path on cell $[1]_{13}$. The green and purple faces sharing an edge represent the corner joining the dodecahedra of the path which possess these faces. They can also be interpreted as the entry and the exit of a cell, in purple and green respectively.

Accordingly, their common edge, which is also shared by $[1]_{13}$ is also shared by a fourth dodecahedron which is a neighbour of both $[9]_{[1]_{13}}$ and $[8]_{[1]_{13}}$. This fourth dodecahedron can indifferently be numbered by $[a]_{[9]_{[1]_{13}}}$ or $[b]_{[8]_{[1]_{13}}}$ where $a$ and $b$ are appropriate faces of $[9]_{[1]_{13}}$ and $[8]_{[1]_{13}}$ respectively. This fourth dodecahedron has its faces 1 and 2 shared by $[9]_{[1]_{13}}$ and $[8]_{[1]_{13}}$ resep-
tively. Which face is associated with which of these dodecahedra is determined by the orientation of the numbering: face 2 follows face 1 when clockwise turning around the projection of the dodecahedron on its bottom face.

And so, we can see that this end of the path can be represented as a word of the form $(SQ)^4Q$, the penultimate element being a corner: the one which joins the cell $[1][1]_{13}$ to cell 14. Now, cell 14 is also a corner as the exit from cell $[1][1]_{13}$ is a face which is perpendicular to one of cell 14. With this end of the path, we can notice that sequences of cells of the form $(SQ)^+$ provide us with a very flexible tool allowing to construct a path joining any pair of cells in the dodecagrid.

Before looking at the further connections between the active memory switch and the passive one, it is important to note the patterns of the straight elements used in the path connecting the main sensor of the passive memory switch with the controller of the active one. Indeed, in Figures 24 and 25, we can see that the just mentioned cells have two or three milestones. In fact, for all of them, the bottom of the cell is face 5 or 6. Now, these faces must be in contact with a milestone. Now, as cell $[1][1]_{13}$ is always black, it behaves like a milestone and so, for the cells which have three milestones in the figures, they have in fact four of them at the requested places with respect to the entry and the exit. We remain with the single case of two milestones in the figures, it is the cell $[9][1]_{13}$. Now, the face of this cell in contact with cell $[1][1]_{13}$ is its face 6, so that we have three milestones on the following faces: 2, 6 and 7. We shall see in Section 6 that with this pattern we can still establish rules ensuring the motion of the particle.

To conclude this point, we have to indicate how the tracks attached to the active and the passive memory switches are connected. Denote by $u_a$, $a_a$ and $b_a$, $u_p$, $a_p$ and $b_p$ the tracks meeting at the switches with easy notations: $u$, $a$ and $b$ where already introduced, and the indices $a$ and $p$ refers to the active and passive memory switches respectively. We remain with explaining how $u_a$ and $u_p$ meet, and with the same question with $a_a$, $b_a$ and $a_p$, $b_p$ similarly.

In Figure 16, we may notice that the dodecahedra belonging to $u_a$ and $a_a$ have a face belonging to the plane which supports the face 5 of the central cell. This means that, in the setting which we considered in this section, the dodecahedra belonging to $u_a$ and $a_a$ have a face lying on $\Pi_5$. The same property occurs in the passive memory switch for the dodecahedra belonging to $u_p$ and $a_p$. At this point, we have to notice that the active and passive switches are put in some sense face to face: the sensors of both switches belongs to the intersection of two half-spaces delimited by $\Pi_a^5$ and $\Pi_p^5$. This means that, if $a_a$ denotes the right-hand side track leaving the switch in an active crossing, then $a_p$ denotes the left-hand side track. A similar convention is assumed for $b_a$ and $b_p$. Now, the dodecahedra belonging to $b_a$ and $b_p$ do not possess the same property with respect to $\Pi_4$. Nevertheless, we can continue the tracks in such a way that, starting from a certain point, the dodecahedra of the tracks have a face on $\Pi_4$. It is enough to make the track follow a horizontal segment starting from the third cell after the central one, until the track meets the plane.

Now, assuming this point, we can define the route followed by the tracks
designed by the same first letter as a route similar to the one used for the bridges. In the planes $\Pi_4$ and $\Pi_5$, we can define the levels of four Fibonacci trees rooted at $A$ and take one of them which will guide the tracks, call this level the border. The border meet the plane $P_0$ at some point. This means that in the plane $P_0$, when the tracks meet the border, they are separated. One way go up to the plane $P_0^0$ and, there, it follows the plane $P_4$ or $P_5$, depending on which track we consider, until the central cell of the passive switch is met. The other track goes down along the border until it meets the plane $P_0^0$ and from there, it goes to the central cell of the active switch. The needed length for the radius of the border is defined by the number of cells needed by the path from the controller of the passive switch until that of the active switch. A radius of 10 tiles is more than enough for our purpose, as can be seen from Figures 23, 24 and 25.

From this, we have to notice that outside the level defined by the border, the tracks $u_a, u_p, a_a, a_p, b_a$ and $b_p$ need not to go on following $P_4$ or $P_5$. We have now presented all the elements of the scenario defining the simulation of our railway circuit. We can now turn to the study of the rules and the computer program.

6 The rules and the computer program

In this section, we shall follow the guidelines of the previous section and, at each step, we shall formulate the rules corresponding to the considered part of the scenario. In each step, we shall have two kinds of rules: conservative rules and motion ones. Conservative rules are characterized by the fact that the new state is the same as the current one. In motion rules, the new state is different from the current one. This syntactic difference expresses the presence or not of the particle nearby the cell or within it. However, in the motion rules, we shall include rules which look like conservative. As these rules involve the particle, they are considered as motion rule too. We shall have an illustration of this point in the flip-flop switch for instance.

Here, we shall take the same format for the rules as the one used in [11]. The rule will be presented as a word $\eta^0\eta_0..\eta_11\eta^1$ of length 14, where $\eta^0$ is the current state of the cell, $\eta_i$ is the cell of the $i^{th}$ neighbour of the cell and $\eta^1$ is the new state of the cell after the application of the rule. In this paper, we shall display the rules according to the display used in the file treated by the computer program. As can be seen, the notations are straightforward from the just indicated formalism.

Here we present this new display by Table 6 which defines a few samples of the basic conservative rules defined by the following principle: when a cell has at most three black neighbours, its state remains unchanged. In all the tables where the rules are displayed, we also display a numbering of the faces which allows the reader to easily make the correspondence with the states of a face.
6.1 The rules for the tracks

For the rules concerning the tracks, we remember that the patterns are the same than those of [11]. Now, the rules are different. We remind that for the straight element, the faces which are shared with a milestone are faces 2, 5, 6 and 7 with, in this context, the entry through cell 3, 4, 8 or 10 and the exit through face 1. Most often, the bottom of the cell is face 0 or face 5. But, in principle, other rotations are possible. For the straight element, the rules are given by Table 4.

**Table 3** The basic conservative rules: the state of a cell which has at most three black neighbours remains unchanged.

```
| adress | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|--------|----|---|---|---|---|---|---|---|---|---|---|---|---|---|
|0 (0)   |     | W | W | W | W | W | W | W | W | W | W | W | W | W |
|0 (0)   |     | B | W | W | W | W | W | W | W | W | W | W | W | B |
|0 (0)   |     | B | B | B | W | W | W | W | W | W | W | W | W | B |
|0 (0)   |     | W | B | W | W | W | W | W | W | W | W | W | W | W |
|0 (0)   |     | W | B | B | B | W | W | W | W | W | W | W | W | W |
|0 (0)   |     | B | B | B | B | W | W | W | W | W | W | W | W | B |
```

**Table 4** The rules for the straight elements. The conservative rules are labelled with 0. Note that the motion is always in the same direction: the exit is through face 1. Also note the special element with three milestones only.

```
| adress | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|--------|----|---|---|---|---|---|---|---|---|---|---|---|---|---|
|-- straight element, direct track: | | | | | | | | | | | | | | | |
| (0)    | W | W | W | B | W | W | B | B | B | W | W | W | W | W |
|-- presence of the particle: | | | | | | | | | | | | | | | |
| (1-3)  | W | W | W | B | B | B | B | W | W | W | W | W | B |
| (1-4)  | W | W | W | B | B | B | B | W | W | W | W | B |
| (1-10) | W | W | W | B | B | B | B | W | W | B | B | B | W | W |
| (1-8)  | W | W | W | B | B | B | B | W | W | B | B | B | W | W |
| (2)    | B | W | W | B | B | B | B | W | W | W | W | W | W | W |
| (3)    | W | W | B | B | W | B | B | W | W | W | W | W | W | W |
|-- special straight element, memory switch: | | | | | | | | | | | | | | | |
| (0)    | W | W | W | B | W | W | B | B | W | W | W | W | W | W |
| (1)    | W | W | W | B | B | W | B | B | W | W | W | W | W | B |
| (2)    | B | W | W | B | W | W | W | W | B | B | W | W | W | W |
| (3)    | W | W | B | B | W | W | B | B | W | W | W | W | W | W |
```

In Table 4 we can notice that the rules labelled with (0) are conservative and that the others are concerned with the motion of the particle.
An interesting feature of the rules given in Table 4 is that the rules associated with the return track are indeed rotated forms of the other rules. This comes from the symmetry of the milestone pattern with respect to a certain plane reflection leaving the dodecahedron invariant: it is namely the bisector of the angles defined by the two 'inner' faces shared with a milestone.

Table 5 *The rules for the four kinds of corners.*

| --adress | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|----------|----|---|---|---|---|---|---|---|---|---|---|----|----|----|
|-- corners:-- |
| (0) | W | W | W | W | B | W | B | B | B | B | W | B | B | W |
| (0) | W | W | W | W | B | W | B | B | B | B | W | B | B | W |
|-- version with B upon face 0:-- |
| (0) | W | B | W | W | B | W | B | B | B | B | B | W | B | W |
| (0) | W | B | W | W | B | W | B | B | B | B | W | B | B | W |
|-- when the particle is nearby:-- |
|-- direction : 1 -> 2-- |
| (1) | W | W | B | W | B | W | B | B | B | B | W | B | B | B |
| (2) | B | W | W | W | B | W | B | B | B | B | W | B | B | W |
| (3) | W | W | W | W | B | W | B | B | B | B | W | B | B | W |
|-- version with B upon face 0:-- |
| (1) | W | B | B | W | B | W | B | B | B | B | B | W | B | B |
| (2) | B | B | W | W | B | W | B | B | B | B | B | W | B | B |
| (3) | W | B | W | B | B | W | B | B | B | B | B | W | B | B |
|-- direction : 2 -> 1-- |
| (1) | W | W | W | B | W | B | B | B | B | B | B | W | B | B |
| (2) | B | W | W | W | B | W | B | B | B | B | B | W | B | W |
|-- version with B upon face 0:-- |
| (1) | W | B | W | B | B | W | B | B | B | B | B | W | B | B |
| (2) | B | B | W | W | B | W | B | B | B | B | B | W | B | W |
| (3) | W | B | B | W | B | W | B | B | B | B | B | W | B | W |
|--adress | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

At last, we have also the rules for the special motion of the temporary particle conveying the signal to the controller of the active memory switch. Here there
are only three milestones put onto face 2, 6 and 7, the exit still being face 1 and the entry being face 4 or 3.

As indicated in Subsection 4.2 there is another important element for constructing the tracks: the corners. In Subsection 4.2, the corners are given in two versions which are symmetric images of one another in a reflection through a plane. And so, these two versions are not rotated images of each other. In one version, if the entry is face 1 and the exit is face 2, then the milestones are put onto faces 2, 3, 5, 6, 7, 8, 10 and 11. If we consider the bisector of the angle defined by the planes of faces 1 and 2, the corner is almost symmetric with respect to the reflection in this plane, but faces 9 and 10 are exchanged by this reflection and one face is white while the other is black. We take advantage of this situation to consider the symmetric image as defining the reverse motion: the entry is now face 2 and the exit is face 1.

But later, especially for the passive memory switch and for defining the path from its sensor to the controller of the active memory switch, we have two introduce another version of both corners: the new pattern consists in putting an additional milestone on face 0. The necessity of this can be remarked from Figure 23: it is clear that in several cases, face 0 of a corner must be shared by a black cell. Table 5 shows the rules for all these versions of the corners.

6.2 The rules for the flip-flop switch

Before turning to the flip-flop switch, it should be remarked that, from the structure of a straight element, there is no need to introduce additional rules for the working of the fixed switch: the rules defined in Subsection 6.1 are enough to ensure the correct working of this type of switches.

Now, let us have a look to the flip-flop switch, which is a purely active switch. As mentioned in Subsection 5.2, it is needed to introduce new patterns. The first one deals with the central cell. As noticed in Subsection 5.2, the central cell has a special pattern. Consider a numbering where the track a abuts face 1 while a and b abut faces 4 and 3 respectively, the face in contact with Π₀ being face 0. Then the milestones are put onto faces 2, 5, 8, 10 and 11, see Figure 16.

The other patterns are a consequence of the mechanism introduced for the working of the switch as a flip-flop one. In Figure 17 we have seen the three patterns used for this purpose with the appropriate numbering of each new cell. In these numberings, cells 11 and 12 have only three milestones which are on faces 1, 6 and 7. Note that the conservative rules associated to these cells are not rotated forms of the rules given in Table 3. Cell 13 has a more complex pattern: its milestones are on faces 1, 6, 7, 8 and 10.

However, we can see on the conservative rules of cell 13 that exactly 6 black cells occur in both rules. This comes from the fact cell 13 is in contact with both cells [9]₁ and [9]₈. Always one exactly of these two cells is black, but which one depends on the motion of the particle. In fact, in the configuration of a flip-flop switch, outside the cells of the track, exactly three cells among all heir neighbours can change their state: cells 11, 12 and 13. We have seen the
Table 6 *The rules for the flip-flop.*

| --adress | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|----------|----|---|---|---|---|---|---|---|---|---|---|----|----|----|
| central cell: |
| (0) | W | W | W | B | W | W | W | B | B | W | B | B |
| (0) | W | W | W | B | W | W | W | B | B | B | B | B |
| -- when it is crossed by the particle: -- |
| (1) | W | W | B | B | W | W | B | W | B | B | B | B |
| (2) | B | W | W | B | W | W | W | B | W | B | B | B |
| (3-3) | W | W | W | B | B | W | B | W | B | W | B | B |
| (3-4) | W | W | W | B | B | B | W | W | B | W | B | B |
| -- cells 5 and 8: -- |
| (0) | W | W | W | B | W | W | B | W | W |
| -- when a particle passes nearby: -- |
| (1) | W | W | W | B | B | B | W | B | W | W |
| (1) | W | W | W | B | B | B | B | W | B | W |
| (0) | W | W | W | B | B | B | B | W | B | W |
| -- cell 13, the controller: -- |
| (0) | W | W | W | B | W | B | W | W | B | W | B |
| (0) | W | W | W | B | W | W | W | B | W | B |
| -- activation of cell 13 (the particle is seen through face 0): -- |
| (1-3) | W | B | B | W | W | B | W | B | B | B | W | B |
| (1-4) | W | B | B | W | W | B | W | B | B | B | W | B |
| (2) | B | W | B | W | B | B | B | W | B | W | W |
| (1-B) | B | W | B | B | W | W | B | B | W | W |
| -- cells 11 and 12: -- |
| (0) | W | W | W | B | W | W | W | B | W | W |
| (0) | B | W | B | W | W | W | B | W | W |
| -- change of state in cells 11 and 12: -- |
| (1-W) | W | B | B | W | W | B | W | B | W | W |
| (1-W) | W | B | B | W | W | B | W | B | W | W |
| (1-B) | B | W | B | W | B | W | B | W | W |
| (1-B) | B | W | B | W | W | B | B | W | W |

reason of these changes and what is their meaning in Section 5. We can see in
Table 6 how the rules implement the scenario, very directly. In particular, we can see that the presence of a black neighbour on the face 9 of cell 5 or cell 8 is enough to prevent the particle to enter the track which follows cell 5 or cell 8 respectively. When cell 11 or 12 is white, cell 5 or 8 respectively behaves like an ordinary straight element and so, the particle can cross the cell and go on along the corresponding track. We can also see that the flash of cell 13 is triggered by the occurrence of two black neighbours: one through face 3 or 4 and the other through face 0. The first one is the marker of the non selected track and the second is the particle itself which is passing through cell 4.

6.3 The rules for the memory switch

Now, we come to the rules needed by the memory switch. We know that this switch is split into two components: the passive memory switch which deals with passive crossings only and the active memory switch which deals with active passages only.

The rules concerning the passive memory switch are displayed by Table 7. We can see that there is no special rule for the central cell: it is a straight element with face 0 on Π₀, exactly as in a fixed switch.

Now, we can see that, as explained in Subsection 5.3, the cells 11, 12 and 13 of the passive memory switch are very different from those of a flip-flop. In particular, for the passive memory switch, cells 11 and 12 have three milestones two, but they are placed on faces 1, 8 and 10, which makes this pattern different from that of the cells numbered in the same way for the flip-flop switch, even under any positive displacement keeping a dodecahedron globally invariant. This is entailed by the fact that cells 11 and 12 do not react in the same way as their homonyms. In the flip-flop switch, cells 11 and 12 see a neighbour of the central cell: cell 13 which is cell [9]₄. Here, this is not the case. Cells 11 and 12 can see cell 13 only on which they are put. They can see a neighbour of cell 5 or 8 respectively, but this neighbour is always white.

Now, cell 13 is also very different: it has 5 milestones on faces 0, 1, 6, 7 and 11. Moreover, one of its two neighbours seen through faces 8 and 10, is black and the other is white. These neighbours are namely cell 11 and 12, on faces 10 and 8 respectively. The conservative rules reflect this structure. By its position on cell 13, The black cell indicates the side which is the non selected track, this is why we say that these cells are markers. The motion rules show that cell 13 becomes black as soon as the particle is seen on the track which is on the same side as its black marker. Now, cell 13 sees the particle before it enters the central cell while, in the case of the flip-flop, it can see it one time later only: this is due to to the position of the central cell.

Table 8 indicate the rules for the active memory switch. This time, what is common with the flip-flop memory switch also applies here. This is the case, in particular, for cells 11 and 12 which have the same pattern as in the case of the flip-flop switch. Note too that here, they have the same position as in this latter switch. They play exactly the same role as in the flip-flop switch.
However, their change of state does not occur in the same way as in the flip-flop switch, where the change is caused by the presence of the particle in the central cell. In the active switch, the change occurs much later than the time when the particle visits the central cell. However, the particle is not present neither in cell 5 nor in cell 8 when cell 13 flashes for cells 11 and 12 to change their state. This makes it necessary to append three new rules only: the corresponding rules when one of the cells is black are already present in Table 6 and, accordingly, they also apply here.

Table 7  The rules for the passive memory switch.

| Address | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| Cell 13: |   |   |   |   |   |   |   |   |   |   |    |    |    |
| (0)     | W | B | B | W | W | W | W | B | B | W | W | B | B | W |
| (0)     | W | B | B | B | W | W | W | B | B | B | W | W | B | W |
| When the particle passes, seen through face 3 or 4, |   |   |   |   |   |   |   |   |   |   |    |    |    |
| It flashes, if cell 8 or 10 respectively is black: |   |   |   |   |   |   |   |   |   |   |    |    |    |
| (1)     | W | B | B | W | W | B | B | W | W | B | B | B | B | B |
| (1)     | W | B | B | B | W | W | B | B | B | B | W | W | B | B |
| (2)     | B | B | B | W | W | W | W | B | B | W | W | B | B | W |
| (2)     | B | B | B | W | W | W | W | B | B | B | W | W | B | W |
| It does not flash, if cell 8 or 10 respectively is white: |   |   |   |   |   |   |   |   |   |   |    |    |    |
| (0)     | W | B | B | W | W | B | B | B | B | W | W | B | W |
| (0)     | W | B | B | B | W | W | B | W | B | W | W | B | W |
| When cell [9] flashes after the flash of cell 13: |   |   |   |   |   |   |   |   |   |   |    |    |    |
| (0)     | W | B | B | W | W | W | W | B | B | B | B | W | B | W |
| (0)     | W | B | B | B | W | W | W | B | B | W | B | W | B | W |
| Cells 11 and 12: |   |   |   |   |   |   |   |   |   |   |    |    |    |
| (0)     | W | W | B | W | W | W | W | W | W | B | W | B | W | W |
| (0)     | B | W | B | W | W | W | W | W | W | B | W | B | W | B |
| (0)     | W | B | B | W | W | W | W | W | W | B | W | B | B | W |
| (0)     | B | B | B | W | W | W | W | W | W | B | W | B | W | W |
| Effect on cells 5 and 8 when cell 13 flashes |   |   |   |   |   |   |   |   |   |   |    |    |    |
| (0)     | W | B | W | B | W | W | B | B | B | W | W | W | W |
| (0)     | W | B | B | B | W | W | B | B | W | W | W | W | W |

Now, cell 13 has also a different working than its homonym of the flip-flop
switch. The main difference is that here, cell 13 remains white just after the visit of the particle. Accordingly, it is triggered to flash in a different way. We can see the rules corresponding to this new situation in Table 8.

### Table 8 The rules for the active memory switch.

| adress | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---------|----|---|---|---|---|---|---|---|---|---|---|----|----|----|
| cell 13: |    |   |   |   |   |   |   |   |   |   |   |    |    |    |
| (0)     | W  | W | B | B | B | B | B | W | B | B | B | W | W |
| (0)     | W  | W | B | B | B | B | B | W | B | B | B | W | W |
| no change when the particle passes under face 11: |    |   |   |   |   |   |   |   |   |   |   |    |    |    |
| (0)     | W  | W | B | B | B | B | B | W | B | B | B | W | W |
| (0)     | W  | W | B | B | B | B | B | W | B | B | B | W | W |
| (1)     | W  | B | B | B | B | B | B | W | B | B | B | W | B |
| (1)     | W  | B | B | B | B | B | B | W | B | B | B | W | B |
| (2)     | B  | W | B | B | B | B | B | W | B | B | B | W | W |
| (2)     | B  | W | B | B | B | B | B | W | B | B | B | W | W |
| cells 11 and 12 in the active memory switch: |    |   |   |   |   |   |   |   |   |   |   |    |    |    |
| (2)     | W  | B | B | w | w | w | W | B | B | W | W | W | W |
| (3)     | W  | W | B | W | W | W | W | B | B | W | W | W | W |
| (3)     | W  | W | B | W | W | W | W | B | B | W | W | W | W |

#### 6.4 A word about the computer program

The computer program is based on the same one which was used to check the rules of the weakly universal cellular automaton on the dodecagrid with three states which was used in [11]. However, it was adapted to this automaton in the part computing the initial configuration and, also in the main function as the control steps are a bit different from those of the previous program. As already mentioned, all the traces given in Section 5 are taken from a general trace computed by the program.

Now, from all the features proved in Section 5 and from the computations performed in this one, we can conclude that the proof of Theorem 1 is now complete.

#### 7 Conclusion

With this result, we reached the frontier between decidability and weak universality for cellular automata in hyperbolic spaces: starting from 2 states there are weakly universal such cellular automata, with 1 state, there are none, which
is trivial. Moreover, the set of cells run over by the particle is a true spatial structure. We can see that the third dimension is much more used in this implementation than in the one considered in [11].

What can be done further?

In fact, the question is not yet completely closed. In [13] we proved that it is possible to implement a 1D-cellular automaton with \( n \) states into the pentagrid, the heptagrid and the dodecagrid and also a whole family of tilings of the hyperbolci plane\( \text{i} \) with the same number of states. For the pentagrid, it was needed to append an additional condition which is satisfied, in particular, by the elementary cellular automaton defined by rule 110. Consequently, as stated in [13]:

**Theorem 2** (M. Margenstern, [13]). There is a weakly universal rotation invariant cellular automaton in the pentagrid, in the heptagrid and in the dodecagrid.

However, this is a general theorem based on the very complicate proof of a deep result involving a number of computations in comparison with which those of this paper are quasi-nothing. Also, the implementation provides a structure which is basically a linear one. This is why, it seems to me that the construction of the present paper is worth of interest: it is a truly spatial construction. Moreover, the construction is very elementary.

Now, there are still a few questions. What can be done in the plane with a true planar construction? At the present moment, the smaller number of states is 4, in the heptagrid, see [12], while it is 9 in the pentagrid, see [17]. And so, there is some definite effort before closing this question.

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