ON THE ORIGIN OF THE ROTATION CURVES OF DARK MATTER–DOMINATED GALAXIES

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ABSTRACT

Rotation curves of dark matter–dominated galaxies measure the mass profiles of galactic halos and thereby test theories of their cosmological origin. While attention has focused lately on the possible discrepancy at small galactocentric radii between observed rotation curves and the singular density profiles predicted by N-body simulations of the cold dark matter (CDM) model, the observed rotation curves nevertheless contain valuable additional information with which to test the theory and constrain the fundamental cosmological parameters, despite this uncertainty at small radii. An analytical model we derived elsewhere for the postcollapse equilibrium of cosmological halos as truncated, nonsingular, isothermal spheres (TISs) reproduces many of the average properties of halos in CDM simulations to good accuracy, including the density profiles outside the central region. The circular velocity profile of this TIS model, moreover, is in excellent agreement with the observed ones and yields the mass and formation epoch of an observed halo from the parameters of its rotation curve. This allows us to predict correlations among rotation curve parameters, such as the maximum velocity and the radius at which it occurs, for different mass halos forming at different epochs in the CDM model. As an example, we derive the observed \(v_{\text{max}}-r_{\text{max}}\) relation analytically, with preference for the flat \(\Lambda\)CDM model.

Subject headings: cosmology: theory — dark matter — galaxies: formation — galaxies: general — galaxies: halos — galaxies: kinematics and dynamics

1. INTRODUCTION

The rotation curves of dark matter–dominated galaxies probe the mass profile of their galactic halos. The observed rotation curves of dwarf and low surface brightness (LSB) disk galaxies, therefore, offer a relatively direct test of theories of cosmological halo formation, free of the complicating effects of the dynamical coupling of dissipationless dark matter and dissipative baryonic matter that affect the mass profiles of baryon-dominated galaxies. Recently, attention has focused on the apparent conflict between the singular halo density profiles predicted by cold dark matter (CDM) N-body simulations and the observed rotation curves, which favor a flat-density core (see Moore et al. 1999). This has led to intense scrutiny of both the observations and the CDM model, primarily in one of three directions: improvement in the numerical resolving power of the CDM N-body simulations to determine better the logarithmic slope of the predicted density profiles at small radii (e.g., Moore et al. 1999), ideas for modifying the microscopic properties of the dark matter so as to retain the more successful aspects of the CDM model while flattening the halo density profiles at small radii (e.g., Davé et al. 2001 and references therein), and suggestions that the rotation curve data lack sufficient spatial resolution near the center to distinguish unambiguously between a density profile with a flat-density core and the singular profiles predicted by CDM N-body simulations (e.g., van den Bosch & Swaters 2000). Despite these uncertainties, a meaningful comparison between the data and predictions for the global properties of the rotation curves of dark matter–dominated galaxies is possible that serves to test the CDM model and to discriminate among the different background cosmologies. To demonstrate this, we show how the mass and formation epoch of a halo can be extracted from its observed rotation curve, by application of an analytical model for halo formation that reproduces the empirical description of these rotation curves extremely well.

Burkert (1995) show that the observed rotation curves of several dark matter–dominated dwarf galaxies are consistent with a common density profile with a flat-density core, according to

\[\rho(r) = \frac{\rho_{0, B}}{(r/r_{0, B} + 1)(r^2/r_{0, B}^2 + 1)}. \tag{1}\]

Kravtsov et al. (1998) found that the rotation curves for a larger sample that included both dwarf and LSB galaxies are well fitted by a similar universal halo profile given by

\[\rho(r) = \frac{\rho_{0}}{(r/r_0)^{\alpha}(1 + (r/r_0)^2)^{\beta - \gamma/3}}, \tag{2}\]

with \((\alpha, \beta, \gamma) = (2, 3, 0.2)\), which is only slightly steeper than the Burkert profile in the core but shares the slope at large radii, \(\rho \propto r^{-3}\). The circular velocity profiles (i.e., \(v(r) \equiv [GM(r)]^{1/2}\)) for equations (1) and (2) are virtually indistinguishable at all radii, even near the center, where \(v \propto r^{1-\gamma/2}\), since \(1 - \gamma/2 = 1 \text{ or } 0.9\), respectively.

This contrasts with the halo profiles found using N-body simulations of the standard CDM model. The universal fitting formula for simulated halos reported by Navarro, Frenk, & White (1997, hereafter NFW) is equation (2) with \((\alpha, \beta, \gamma) = (1, 3, 1)\), while Moore et al. (1999) find \((\alpha, \beta, \gamma) = (1.5, 3, 1.5)\). However, the circular velocity profiles for these halos differ significantly from those implied by equations (1) and (2) only at very small radii where the uncertainties in the shape of the observed rotation curves currently make discrimination difficult. In the meantime, equation (1) generally fits the data...
better than does the NFW profile, even when the latter is also an acceptable fit.

The Burkert profile, then, continues to serve as a useful empirical description of the universal mass profiles of dark matter–dominated galactic halos. We show here that the truncated, nonsingular isothermal sphere (TIS) model that we derived elsewhere (Shapiro, Iliev, & Raga 1999, hereafter Paper I; I. T. Iliev & P. R. Shapiro 2000, in preparation, hereafter Paper II; Iliev 2000) has a density profile for which the circular velocity profile is essentially indistinguishable from that of the Burkert profile and, as such, provides a theoretical motivation for the latter. The TIS model goes well beyond the prediction of density profile and rotation curve, however, to provide the size, mass, velocity dispersion, and collapse epoch of the halo as well. This makes possible the further interpretation of observations of dark matter–dominated galaxies for comparison with the predictions of various cosmological models. At the same time, the TIS model can be shown to reproduce many of the average properties of the halos found in simulations of the standard CDM model, outside of the innermost region where the TIS halo has a flat-density core, unlike the CDM halos. As a result, a comparison of the analytical TIS predictions with observed properties of galaxies also offers insight into the standard CDM model. Finally, if suggestions like the self-interacting dark matter proposal of Spergel & Steinhardt (2000) are correct, that CDM might be more “collisional” as a way to eliminate the central cusp of standard CDM halos, then our TIS solution will also apply to these models, to the extent that the halo relaxation process makes the final equilibrium approximately isothermal.

In § 2, we briefly summarize the relevant properties of the TIS model and present a simple analytical formula for it with which to fit an observed rotation curve. In § 3, we demonstrate the excellent agreement between the TIS model rotation curve and that implied by equation (1), and we show how this allows us to deduce the total mass and collapse epoch of a given halo directly from the parameters of its observed rotation curve. In § 4, we describe how the dependence of the average formation epoch of a halo on its mass in the CDM model results in statistical correlations among the parameters of the observed rotation curves. As an example, we use this approach to derive analytically the known correlation between the maximum velocity of each rotation curve and the radius at which it occurs.

2. THE TRUNCATED ISOTHERMAL SPHERE MODEL

The TIS model is a particular solution of the Lane-Emden equation (suitably modified when $\Lambda \neq 0$) that results from the collapse and virialization of a top-hat density perturbation (see equation (suitably modified when ) that results from the analytically the known correlation between the maximum ve-

rotation curves. As an example, we use this approach to derive statistical correlations among the parameters of the observed epoch of a halo on its mass in the CDM model results in directly from the parameters of its observed rotation curve. In § 3, we demonstrate how to fit an observed rotation curve, and we show how this allows us to deduce the total mass and collapse epoch of a given halo directly from the parameters of its observed rotation curve. In § 4, we describe how the dependence of the average formation epoch of a halo on its mass in the CDM model results in statistical correlations among the parameters of the observed rotation curves. As an example, we use this approach to derive analytically the known correlation between the maximum velocity of each rotation curve and the radius at which it occurs.

2. THE TRUNCATED ISOTHERMAL SPHERE MODEL

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$$\rho(r) = \rho_0 \left(\frac{A}{a^2 + \xi^2} - \frac{B}{b^2 + \xi^2}\right),$$

(3)

where $\xi \equiv r / r_c$ and $(A, a^2, B, b^2) = (21.38, 9.08, 19.81, 14.62)$, accurate to within $3\%$ over the full range $0 \leq \xi \leq \xi_c \approx 30$ for both the EdS case and most low-density models of interest (i.e., $\Omega_c \geq 0.3$). (Note that our definition of the core radius is $r_{\text{TIS}} = r_{\text{King}} / 3$, where $r_{\text{King}}$ is the “King radius” defined in Binn-ney & Tremaine 1987, p. 228.) Equation (3) can be integrated to yield an analytical fitting formula for the TIS rotation curve, as well, given by

$$\frac{v(r)}{\sigma_v} = \left\{ A - B + \frac{1}{3} \left[ b B \tan^{-1} \left( \frac{\xi}{b} \right) - a A \tan^{-1} \left( \frac{\xi}{a} \right) \right] \right\}^{1/2},$$

(4)

where $\sigma_v = (4\pi G\rho_0 r_c^3)^{1/2}$. This fit has a fractional error of less than $1\%$ over the full range of radii $0 \leq r \leq r_c$ for all matter-dominated background models. With a nonzero cosmological constant, the circular velocity becomes $v(r) = [GM(r)L]\r^{3/2} \times [1 - 2\rho_0 v'(r)]^{1/2}$, where $\rho_0$ is the constant vacuum energy density associated with the cosmological constant. For cases of current interest for a flat universe with $\Lambda \neq 0$ (i.e., $\Omega_\Lambda \geq 0.3$), equation (4) can still be used, however, since it departs from the exact solution only slightly in the outer halo (i.e., at $r \sim r_c$) for halos collapsing even as late as $z = 0$; the fractional error is less than 6% at all radii and less than 1% for $r \leq 3r_c$.

The TIS model quantitatively reproduces to good accuracy the average structural properties of halos found in CDM simulations, suggesting that it is a useful analytical approximation for halos that form from more realistic initial conditions. An exception to this agreement is the very inner profile, where the TIS has a uniform-density core instead of a central cusp. Our TIS predictions agree to an astonishingly high accuracy (i.e., to an order of 1%) with the cluster mass-radius and radius-temperature relationships and integrated mass profiles derived from detailed CDM simulations of X-ray cluster formation by Evrard, Metzler, & Navarro (1996). Apparently, these simulation results are not sensitive to our disagreement in the core. A direct comparison of the TIS and NFW mass profiles reveals a very close agreement (with a fractional deviation of less than ~10%) at all radii outside of a few TIS core radii (i.e., about one King radius), for NFW concentration parameters $4 \leq c_{\text{NFW}} \leq 7$.

3. APPLICATION TO GALAXY ROTATION CURVES

The rotation curve for equation (1) is given by

$$\frac{v_B(r)}{v_{\text{e}*B}} = \left\{ \ln \left[ \left( \xi_B + 1 \right)^2 \left( \xi_B + 1 \right) \right] - 2 \tan^{-1} \left( \xi_B \right) \right\}^{1/2},$$

(5)

where $\xi_B \equiv r / r_{0,B}$ and $v_{\text{e}*B} \equiv (\pi G \rho_0 B r_{0,B}^2)^{1/2}$. To compare our TIS directly with equation (1), we solve for the ratios $\rho_{0,TIS}/\rho_{0,B}$ and $r_{0,TIS}/r_{0,B}$ that minimize the $\chi^2$ of the fit of equation (5) to equation (4) over radii from 0 to $r_c$. The result in Figure 1, with $\rho_{0,TIS}/\rho_{0,B} = 0.7790$ and $r_{0,TIS}/r_{0,B} = 0.3276$, shows extremely good agreement, with a fractional deviation below 10% from 0.01$r_c$ to $r_c$ and below 4% over the range $r > 0.03r_c \approx r_{0,TIS}$. For the TIS, the maximum circular velocity and its location are $v_{\text{max,TIS}} = 1.5867 \rho_0 v_{\text{e}*B}$ at $r_{\text{max,TIS}} = 8.999 r_{0,TIS}$ for all matter-dominated cosmologies. For a flat universe with $\Lambda \neq 0$ ($\Omega_c \geq 0.3$), these numbers depend weakly on the collapse redshift but are reduced by no more than 0.2% and 1.7%, respectively, for $z_{\text{coll}} = 0$, and by even less for an earlier collapse. For the Burkert profile, $v_{\text{max,B}} = 1.2143 v_{\text{e}*B}$ at $r_{\text{max,B}} = 3.2446 r_{0,B}$. Hence, our best fit finds that $v_{\text{max,TIS}}$ and $v_{\text{max,B}}$ for the TIS and Burkert profiles are extremely close, with $v_{\text{max,TIS}}/v_{\text{max,B}} = 1.02$ and $r_{\text{max,TIS}}/r_{\text{max,B}} = 0.91$.

In short, our TIS model provides a solid, theoretical underpinning for the empirical fitting formula of Burkert and, by extension, a self-consistent theoretical explanation of the observed galaxy rotation curves that it was invented to fit. In
addition, the TIS model allows us to calculate the total mass $M_v$, the collapse epoch $z_{\text{coll}}$, and the other parameters of each observed dark matter-dominated halo from its rotation curve as follows. Assuming $v_{\text{max}}$ and $r_{\text{max}}$ are provided by observation,$^1$ the mass of the galaxy can be shown to be

$$M_v = 6.329 \times 10^{10} \left( \frac{r_{\text{max}}}{10 \ h^{-1} \text{kpc}} \right) \left( \frac{v_{\text{max}}}{100 \ \text{km} \ \text{s}^{-1}} \right)^2 \ (6)$$

for any matter-dominated universe. When $\Lambda \neq 0$, $M_v \propto r_{\text{max}}^2 v_{\text{max}}^2$, but the coefficient depends on $z_{\text{coll}}$, and the background cosmology (Paper II). For any flat universe ($\Lambda \neq 0$) of current interest, equation (6) is still a very good approximation. For $\Omega_\gamma = 1 - \lambda_0 \geq 0.3$, equation (6) underestimates the mass by less than 6.3%, 2.8%, and 1.2% for $z_{\text{coll}} = 0, 0.5,$ and 1, respectively.

In terms of $r_{\text{max}}$ and $v_{\text{max}}$, $z_{\text{coll}}$ is given to good accuracy by the implicit equation

$$F(\Omega_\gamma, \lambda_0, z_{\text{coll}}) = 2.284 \left( \frac{v_{\text{max}}/100 \ \text{km} \ \text{s}^{-1}}{r_{\text{max}}/10 \ h^{-1} \text{kpc}} \right)^{2/3} \ (7)$$

where $F \equiv \left[ \Omega_\gamma / \Omega(z_{\text{coll}}) \right] \left[ \Delta_{\text{v,SUS}}/18 \pi^2 \right]^{1/3} (1 + z_{\text{coll}})$, $\Omega(z) = [\Omega_\gamma (1 + z)]/[(1 - \Omega_\gamma - \lambda_0)/(1 + z) + \Omega_\gamma (1 + z) + \lambda_0]$, and $\Delta_{\text{v,SUS}}$ is the density [in units of $\rho_{\text{crit}}(z_{\text{coll}})$] after the top-hat collapse and virialization for the standard uniform sphere approximation (SUS).$^2$ For the EdS case, $F = (1 + z_{\text{coll}})$, while for the open, matter-dominated, and flat cases, $F \rightarrow \Omega_{\gamma}^{-1/3} (1 + z_{\text{coll}})^{2/3}$. For the EdS case, $F = (1 + z_{\text{coll}})$, while for the open, matter-dominated, and flat cases, $F \rightarrow \Omega_{\gamma}^{-1/3} (1 + z_{\text{coll}})^{2/3}$.

$^1$ In practice, observations are generally restricted to the inner parts of galaxy rotation curves, often not extending to radii as large as $r_{\text{max}}$. In that case, $v_{\text{max}}$ and $r_{\text{max}}$ are inferred by fitting the observed rotation curve to the TIS rotation curve at other radii.

$^2$ This $\Delta_{\text{v,SUS}}$ is well approximated by $\Delta_{\text{v,SUS}} = 18 \pi^2 (1 + c_x x - c_x^2 x^2)$, where $x = \Omega(z_{\text{coll}}) - 1$ and where $c_x = 82$ (60) and $c_x = 39$ (32) for the flat (open) cases, $\Omega_{\gamma} + \lambda_0 = 1$ ($\Omega_{\gamma} < 1$, $\lambda_0 = 0$), respectively (Bryan & Norman 1998).

4. STATISTICAL CORRELATIONS AMONG THE PROPERTIES OF DWARF AND LSB GALAXIES: THE $v_{\text{max}}$-$r_{\text{max}}$ RELATION

The halos in our TIS model are fully described for a given background universe by their total mass $M_v$ and collapse epoch $z_{\text{coll}}$. In hierarchical models of structure formation like CDM, however, these are not statistically independent parameters. Smaller mass halos on average collapse earlier and are denser than larger mass halos. In terms of galaxy rotation curve parameters, this dependence should be observable, for example, as a correlation between $v_{\text{max}}$ and $r_{\text{max}}$. Mori & Burkert (2000) used the Burkert (1995) fits to rotation curves of dwarf galaxies to report such an observed correlation, expressed as follows:

$$v_{\text{max}}, B = 9.81(r_{\text{max}}, B/1 \text{kpc})^{2/3} \text{km s}^{-1}. \ (8)$$

Kravtsov et al. (1998) also found a correlation using fits to observed rotation curves. Their results for a sample of dwarf and LSB galaxies are shown in Figure 2, along with that in equation (8). Kravtsov et al. (1998) further showed that the results of their CDM $N$-body simulations agreed with these data points.$^3$ This suggests that if we can relate the mass and collapse epoch of our TIS model halos in a statistical way within the context of the CDM model, a comparison of our predicted $v_{\text{max}}$-$r_{\text{max}}$ relation with this observed one will further check the relevance of our TIS model to halo formation from realistic initial conditions and, at the same time, give a theoretical explanation for both the data and the simulation results.

To predict the $v_{\text{max}}$-$r_{\text{max}}$ correlation for a CDM universe using our TIS model, we apply the well-known Press-Schechter (PS) approximation to derive $z_{\text{coll}}(M_v)$—the typical collapse epoch for a halo with a given mass. Halos of mass $M$ that collapse when $\sigma(M) = \delta_{\text{crit}}/\sigma$ are referred to as $\nu$-$\sigma$ fluctuations, where $\sigma(M)$ is the standard deviation of the density fluctuations at $z_{\text{coll}}$ according to linear perturbation theory, after the density field is filtered on the scale $M$, and $\delta_{\text{crit}}$ is the amplitude of a top-hat perturbation according to linear theory at the epoch $z_{\text{coll}}$ at which the exact solution predicts infinite collapse. The typical collapse epoch for halos of a given mass is that for which $\nu = 1$, the $1 \sigma$ fluctuations.

Our results for $1 \sigma$ fluctuations are shown in Figure 2a for different background cosmologies. The flat, untilted and the open, slightly tilted ($n_s = 1.14$) models are in reasonable agreement with the observed $v_{\text{max}}$-$r_{\text{max}}$ relation, while the untilted and strongly tilted ($n_s = 1.3$) open models are not. The empirical Burkert scaling relation is closely approximated by the currently favored $\Lambda$CDM model and less well by other models.

In practice, observed galaxies should exhibit a statistical spread of halo properties in accord with the expectations of the Gaussian statistics of the density fluctuations that formed them. Since observed galaxies will not all be “typical,” the scatter of the data points in Figure 2 is natural. To probe this in our model, we calculate the masses and collapse redshifts for halos formed by $\nu$-$\sigma$ fluctuations for different values of $\nu$. As shown

$^3$ This result by Kravtsov et al. (1998) is not sensitive to the question of whether their simulations adequately resolved the halo density profiles at very small radii.
in Figure 2b for $\Lambda$CDM, the Burkert scaling relation is closest to the TIS model prediction for $1 \sigma$ fluctuations, while all the observed galaxy data points except one correspond to $0.7 \leq \nu \leq 1.5$. Hence, on average, the galaxies that constitute the observed $v_{\text{max}}^{\text{rms}}$ correlation correspond to halos that formed at close to the typical collapse time expected theoretically for objects of that mass. This means that our TIS model is a self-consistent explanation for the observed $v_{\text{max}}^{\text{rms}}$ correlation.

This success of the TIS model in explaining the observed $v_{\text{max}}^{\text{rms}}$ relation and, by extension, the CDM simulation results of Kravtsov et al. (1998) that follow it can be understood by a completely analytical argument, as follows. We approximate the density fluctuation power spectrum as a power law in wavenumber $k$, $P(k) \propto k^n$. If we define a mass $M$ that corresponds to $k$ according to $M \propto k^{-3}$, then $\sigma(M) \propto M^{(3+n)/6}$ if we set $n = n_{\text{eff}} \equiv -3/(2 + \nu_{\text{eff}} + 1)$, where $\nu_{\text{eff}} \equiv (d \ln \sigma / d \ln M)$ exact at the relevant mass scale $M$. Our results for the $\Lambda$CDM case indicate that the galaxies that make up the $v_{\text{max}}^{\text{rms}}$ data points in Figure 2 collapsed at redshifts $1 \leq z_{\text{coll}} \leq 6$ with masses in the range of $8 \times 10^8 \leq M_{\odot} / (M_{\odot} h^{-1}) \leq 3 \times 10^{11}$. Hence, fluctuations have approximately the EdS precollapse growth rate, and we can let $\Omega(z_{\text{coll}}) = 1$ in equations (6) and (7). In that case, $(1 + z_{\text{coll}}) \propto \sigma(M) \propto M^{-(3+n)/6}$, $r_{\text{max}} \propto M^{(3+n)/6} \Omega_{\Lambda}^{1/3}$, and $v_{\text{max}} \propto M^{(1-n)/2} \Omega_0^{1/6}$, which combine to yield

$$v_{\text{max}} = v_{\text{max},*} \left( r_{\text{max}}^{*}/r_{\text{max}} \right)^{(1-n)/2} \Omega_{\Lambda}^{1/3},$$

where $v_{\text{max},*}$, and $r_{\text{max},*}$ are for a $1 \sigma$ fluctuation of fiducial mass $M_*$, as given by equations (6) and (7) with $(1 + z_{\text{rec}}) = (1 + z_{\text{coll}}) \sigma(M_*, z_{\text{rec}}) \delta_{\text{crit}}$ where $\delta_{\text{crit}} = 1.6865$ and $\sigma(M_*, z_{\text{rec}})$ is the value of $\sigma(M_*)$ evaluated at the redshift of recombination $z_{\text{rec}}$ [i.e., early enough that $(1 + z)\sigma$ is independent of $z$]. Over the relevant mass range of $M = 10^{10} h^{-1} M_{\odot}$, $n_{\text{eff}} \approx -2.4 \pm 0.1$ for our $\Lambda$CDM case. For $M = 10^{10} h^{-1} M_{\odot}$, our $\Lambda$CDM case ($\Omega_{\Lambda} = 0.3$, $h = 0.65$) yields $(1 + z_{\text{rec}}) \sigma(M_*, z_{\text{rec}}) = 5.563$, so $(1 + z_{\text{coll}}) = 3.30, v_{\text{max},*} = 53.2 \text{ km s}^{-1}$, and $r_{\text{max},*} = 5.59 h^{-1} \text{ kpc}$. With these values and $n_{\text{eff}} = -2.4$, equation (9) yields the TIS model analytical prediction $v_{\text{max}} = \left(13.0 \text{ km s}^{-1}\right)^2 (r_{\text{max}}/1 \text{ kpc})^{0.65}$, which is remarkably close to the empirical relation in equation (8).

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