REGULARIZING PROPERTIES OF (NON-GAUSSIAN) TRANSITION SEMIGROUPS IN HILBERT SPACES

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ABSTRACT. Let $\mathcal{X}$ be a separable Hilbert space with norm $\|\cdot\|$ and let $T > 0$. Let $Q$ be a linear, self-adjoint, positive, trace class operator on $\mathcal{X}$, let $F : \mathcal{X} \to \mathcal{X}$ be a (smooth enough) function and let $W(t)$ be a $\mathcal{X}$-valued cylindrical Wiener process. For $\alpha \in [0,1/2)$ we consider the operator $A := -(1/2)Q^{2\alpha - 1} : Q^{1-2\alpha}(\mathcal{X}) \subseteq \mathcal{X} \to \mathcal{X}$. We are interested in the mild solution $X(t,x)$ of the semilinear stochastic partial differential equation
\[
\begin{align*}
dX(t,x) &= (AX(t,x) + F(X(t,x)))dt + Q^\alpha dW(t), \quad t \in (0,T]; \\
X(0,x) &= x \in \mathcal{X},
\end{align*}
\]
and its associated transition semigroup
\[
P(t)\varphi(x) := E[\varphi(X(t,x))], \quad \varphi \in B_b(\mathcal{X}), \quad t \in [0,T), \quad x \in \mathcal{X};
\]
where $B_b(\mathcal{X})$ is the space of the bounded and Borel measurable functions. We will show that under suitable hypotheses on $Q$ and $F$, $P(t)$ enjoys regularizing properties, along a continuously embedded subspace of $\mathcal{X}$. More precisely there exists $K := K(F,T) > 0$ such that for every $\varphi \in B_b(\mathcal{X})$, $x \in \mathcal{X}$, $t \in (0,T]$ and $h \in Q^\alpha(\mathcal{X})$ it holds
\[
|P(t)\varphi(x + h) - P(t)\varphi(x)| \leq Kt^{-1/2}\|Q^{-\alpha}h\|.
\]

1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space. We denote by $E[\cdot]$ the expectation with respect to $\mathbb{P}$. Let $\mathcal{X}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $Q$ be a linear, self-adjoint, positive, trace class operator on $\mathcal{X}$. For $\alpha \in [0,1/2)$ we consider the operator $A := -(1/2)Q^{2\alpha - 1} : Q^{1-2\alpha}(\mathcal{X}) \subseteq \mathcal{X} \to \mathcal{X}$, and a suitable (smooth enough) function $F : \mathcal{X} \to \mathcal{X}$. Let $W(t)$ be a $\mathcal{X}$-valued cylindrical Wiener process (see Remark 6).

For $T > 0$ we consider the mild solution $(X(t,x),t \in [0,T])$ of the semilinear stochastic partial differential equation
\[
\begin{align*}
dX(t,x) &= (AX(t,x) + F(X(t,x)))dt + Q^\alpha dW(t), \quad t \in (0,T]; \\
X(0,x) &= x \in \mathcal{X},
\end{align*}
\]
and its associated transition semigroup
\[
P(t)\varphi(x) := E[\varphi(X(t,x))] \quad t \in [0,T], \quad x \in \mathcal{X};
\]
where $\varphi \in B_b(\mathcal{X})$ (the space of the real-valued, bounded and Borel measurable functions). By mild solution of (1.1) we mean that for every $x \in \mathcal{X}$ there exists a $\mathcal{X}$-valued adapted stochastic process $(X(t,x))_{t \geq 0}$ satisfying the mild form of (1.1), namely for $x \in \mathcal{X}$ and $t \in [0,T]$ it holds
\[
X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s,x))ds + \int_0^t e^{(t-s)A}Q^\alpha dW(s),
\]
and such that $\mathbb{P}(\int_0^T \|X(s,x)\|^2 ds < +\infty) = 1$, for any $x \in \mathcal{X}$. The aim of this paper is to show that, under suitable assumptions, the semigroup $P(t)$, defined in (1.2), maps $B_b(\mathcal{X})$ into the space of Lipschitz continuous functions along an appropriate continuously embedded subspace of $\mathcal{X}$. To

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be more precise we introduce some notations and hypotheses. We say that a function \( \varphi : X \to \mathbb{R} \)
\( Y \)-Lipschitz, where \( Y \) is a continuously embedded subspace of \( X \) with norm \( \| \cdot \|_Y \), if there exists \( L \) \( > 0 \) such that for every \( x \in X \) and \( y \in Y \)
\[
|\varphi(x + y) - \varphi(x)| \leq L\|y\|_Y.
\]

**Hypotheses 1.** Let \( T > 0 \) and let \( \alpha \in [0, 1/2] \). Let \( X \) be a real separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). We assume that

(i) \( Q \) is a linear, self-adjoint, (strictly) positive, trace class operator on \( X \) and let

\[
A := -(1/2)Q^{2\alpha - 1} : Q^{1 - 2\alpha}(X) \subseteq X \to X;
\]

(ii) there exists \( \gamma \in (0, 1) \) such that for any \( t \in (0, T) \)
\[
\int_0^t s^{-\gamma}\text{Tr}|e^{sA}Q^{2\alpha}|ds < +\infty,
\]

where \( \text{Tr} \) denotes the trace operator (see (2.1)).

We remark that by Hypothesis [11], [24 Section II Corollary 4.7] and [24 Theorem 2.3.15], \( A \)
\( \) the infinitesimal generator of a strongly continuous, analytic and contraction semigroup \( e^{tA} \)
on \( X \). Hypothesis [11] is standard in the literature, since it guarantees that the mild solution
\( \) of Hypotheses 1 is path-continuous. We remark that this condition may appear different from the one in
\( \) [24 Theorem 5.11], because the authors of [24] use a Hilbert–Schmidt norm for operators from
\( \) \( Q^{1/2}(X) \) \( \) to \( \) \( X \). Their condition becomes (1.4) in our case. We stress that since \( Q^{\alpha} \)
can be written as a negative power of the operator \( -A \), then (1.1) can be rewritten in the following way
\[
\begin{cases}
    dX(t, x) = (AX(t, x) + F(X(t, x)))dt + (-2A)^{(2\alpha - 1)/2}dW(t), & t \in (0, T];
    
    X(0, x) = x \in X,
\end{cases}
\]

where the smoothing effect of the diffusion on the noise is more evident.

Hypotheses 1 and the assumption of Lipschitz continuity of \( F \) are classical for the study of the existence and uniqueness of the solution of (1.1). Moreover, for \( \alpha \in [0, 1/2] \), if some further conditions are assumed, it is possible to prove that for any \( t \in (0, T] \)
\[
P(t) (B_0(X)) \subseteq \text{Lip}_b(X)
\]

where \( \text{Lip}_b(X) \) is the space of the bounded and Lipschitz continuous functions on \( X \). There is a vast literature dealing with similar types of smoothing properties. See for example [30] [11] [40]
\( \) for an overview on the finite dimensional case and [3] [19] [43] [77] for the infinite dimensional case.

The main result of this paper is a regularization result similar to (1.5) for the transition semigroup \( P(t) \), defined in (1.2), with some non-standard hypotheses on \( F \). Let \( H_\alpha := Q^{\alpha}(X) \)
and for every \( h, k \in H_\alpha \)
\[
\langle h, k \rangle_\alpha := \langle Q^{-\alpha}h, Q^{-\alpha}k \rangle,
\]

then \( (H_\alpha, \langle \cdot, \cdot \rangle_\alpha) \) is a Hilbert space continuously embedded in \( X \). We denote by \( \| \cdot \|_\alpha \) the norm induced by \( \langle \cdot, \cdot \rangle_\alpha \) on \( H_\alpha \). Our setting is similar to the one of [26], although there a different problem (existence of an invariant measure for a stochastic Cahn–Hilliard type equation) was considered.

**Definition 2.** Let \( Y, Z \) be two Hilbert spaces, endowed with the norms \( \| \cdot \|_Y \) and \( \| \cdot \|_Z \) respectively, and let \( \Phi : X \to Z \) be a Borel measurable function. Assume that \( Y \) \( \) is continuously embedded in \( X \). We say that \( \Phi \) is \( Y \)-Lipschitz when there exists \( C > 0 \) such that for every \( x \in X \) and \( y \in Y \)
\[
\|\Phi(x + y) - \Phi(x)\|_Z \leq C\|y\|_Y.
\]

We denote by \( \text{Lip}_Y(X; Z) \) the sets of Borel measurable, \( Z \)-valued and \( Y \)-Lipschitz functions, and by \( \text{Lip}_b(Y; X) \) the subset of \( \text{Lip}_Y(X; Z) \) consisting of bounded functions. If \( Z = \mathbb{R} \) we simply write \( \text{Lip}_b(Y; X) \). We call \( Y \)-Lipschitz constant of \( \Phi \) the infimum of all the constants \( C > 0 \) verifying (1.6).

Now we state the hypotheses we will use throughout the paper.
Hypotheses 3. Let Hypotheses hold true and let $F : \mathcal{X} \to \mathcal{X}$ be a Borel measurable (possibly unbounded) function such that

(i) $F(\Omega) \subseteq H_\alpha$ and $F$ is $H_\alpha$-Lipschitz, with $H_\alpha$-Lipschitz constant $L_{F, \alpha}$;

(ii) if $\alpha \in [1/4, 1/2]$, then we assume that $F : \mathcal{X} \to H_\alpha$ is locally bounded.

Let us make some considerations about these assumptions. The requirement that $F(\Omega)$ is contained in $H_\alpha$ is not uncommon, for example the case $F = -Q^{2b}D^2$ where $U : \mathcal{X} \to \mathbb{R}$ is a suitable convex function, often appears in the literature (see [11, 12, 7] for $\alpha = 1/2$, [11, 14, 17] for $\alpha = 0$ and [13] for general $\alpha$). A condition similar to Hypothesis 3 was already considered in [15]. We stress that Hypothesis 3 does not imply that $F$ is continuous. In Section 6 we show some examples of functions $F$ satisfying Hypotheses. Now we state the main result of this paper.

Theorem 4. Assume that Hypotheses hold. Then, for any $t \in (0, T]$, the semigroup $P(t)$ maps the space $B_b(\mathcal{X})$ to $\text{Lip}_b(H_\alpha)(\mathcal{X})$. More precisely for every $\varphi \in B_b(\mathcal{X})$, $x \in \mathcal{X}$, $h \in H_\alpha$ and $t \in (0, T]$ it holds

$$|P(t)\varphi(x + h) - P(t)\varphi(x)| \leq \frac{e^{L_{F, \alpha}T}}{\sqrt{t}}\|\varphi\|_\infty\|h\|_\alpha.$$ 

For $\alpha \in [0, 1/2)$, the regularization result of Theorem 4 is weaker than [15], but we emphasize that we do not assume that $F$ is Lipschitz continuous on $\mathcal{X}$. We were also interested to see what result can be obtained by assuming more standard assumptions on $F$. In Section 5 we are going to use the same techniques as in the proof of Theorem 4 to the case in which $F$ is Lipschitz continuous, and we prove the following result.

Theorem 5. Assume that Hypotheses hold. Let $F : \mathcal{X} \to \mathcal{X}$ be a function such that $F(\Omega) \subseteq H_\alpha$ and $Q^{-\alpha}F$ is Lipschitz continuous with Lipschitz constant $K_{F, \alpha}$. Then

(a) for any $t \in (0, T]$, $P(t)(B_b(\mathcal{X})) \subseteq \text{Lip}_b(\mathcal{X})$, if $\alpha \in [0, 1/2)$;

(b) for any $t \in (0, T]$, $P(t)(B_b(\mathcal{X})) \subseteq \text{Lip}_{b, H_\alpha/ \sqrt{t}}(\mathcal{X})$, if $\alpha = 1/2$.

Statement 3 of Theorem 5 was already proved in [3, 31]. However, for $\alpha \in [0, 1/4)$, our proof is simpler, because we can exploit the identity $A = -(1/2)Q^{2\alpha} - 1$ and the analyticity of the semigroup $e^{tA}$. Instead the case $\alpha = 1/2$ is not covered by [3, 31]. In the papers [25, 29, 38, 39] the case $\alpha = 1/2$ is considered, but, as we shall see in Section 5 the authors consider a different concept of derivative compared to the one presented in this paper in Section 3.2.

Before proceeding we want to make some considerations about the results of this paper and some of the results already appeared in the literature. In [19, Section 7.7], [25] and [14] the authors study a more general stochastic partial differential equation than (1.1), but in our case their assumptions imply that $Q^\alpha$ has a continuous inverse, and in infinite dimension it makes sense only when $\alpha = 0$, since $Q^{-\theta}$ is unbounded for every $\theta > 0$. In [9] the author proves (1.5) in an important case, namely when $A$ is the realization of a second order differential operator in $L^2(\Omega, d\xi)$ ($\Omega$ is an appropriate domain of $\mathbb{R}^n$, for some $n \in \mathbb{N}$, and $d\xi$ is the Lebesgue measure), and $F$ satisfies some technical conditions. In [5, 31, 40] the authors work in a more general setting. However, the case $\alpha = 1/2$ is not covered by their theory. Indeed one of the fundamental hypotheses assumed in [5, 31, 40] is the following: for any $t \in (0, T]$ 

$$e^{tA}(\xi) \subseteq Q^{1/2}(\xi),$$

(1.7)

where $Q_t \varphi = \int_0^t e^{sA}Q^{2a} s \varphi ds$. If Hypotheses hold true and $\alpha \in [0, 1/2)$, then (1.7) is verified. Indeed, in our case, $Q_t = Q(\text{Id} - e^{tA})$ and recalling that by the analyticity of $e^{tA}$ it holds that for any $t \in (0, T]$, the range of $e^{tA}$ is contained in the domain of $A^k$ for every $k \in \mathbb{N}$ (see [37, Proposition 2.1.1(ii)]), it is sufficient to prove that $(\text{Id} - e^{tA})$ is invertible. Since $2A$ is negative, we have $\|e^{2A}\|_{L^2(\xi)} < 1$, and so $(\text{Id} - e^{tA})$ is invertible. In particular $Q^{1/2}(\xi) = Q^{1/2}(\xi)$ and so we get (1.7). Instead for $\alpha = 1/2$ condition (1.7) is not verified, because $A = -(1/2)\text{Id}$ and so $e^{-(1/2)\text{Id}A}(\xi) = \text{Id} \neq Q(\xi)$, for any $t \in (0, T]$. Moreover in [5, 31, 40] the authors assume that $F$ is Lipschitz continuous, while our Hypotheses do not imply Lipschitz continuity of $F$. 


2. Notation and preliminary results

Let $H_1$ and $H_2$ be two real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{H_1}$ and $\langle \cdot, \cdot \rangle_{H_2}$ respectively. We denote by $\mathcal{B}(H_1)$ the family of the Borel subsets of $H_1$ and by $B_0(H_1; H_2)$ the set of the $H_2$-valued, bounded and Borel measurable functions. We denote by $C^k_0(H_1; H_2)$, $k \geq 0$ the set of the $k$-times Fréchet differentiable functions from $H_1$ to $H_2$ with bounded derivatives up to order $k$. If $H_2 = \mathbb{R}$ we simply write $C^k_0(H_1)$. For a function $\Phi \in C^k_0(H_1; H_2)$ we denote by $D\Phi(x)$ the derivative operator of $\Phi$ at the point $x \in H_1$. If $f \in C^k_0(H_1)$, for every $x \in H_1$ there exists a unique $k \in H_1$ such that for every $h \in H_1$

$$Df(x)(h) = \langle h, k \rangle_{H_1}.$$ 

We let $Df(x) := k$. If $\Phi : H_1 \rightarrow H_2$ is Gateaux differentiable we denote by $D^G\Phi(x)$ the Gateaux derivative operator of $\Phi$ at the point $x \in H_1$. See [27] Chapter 7.

Let $B \in \mathcal{L}(X)$ (the set of bounded linear operators from $X$ to itself). We say that $B$ is non-negative (positive) if for every $x \in X \setminus \{0\}$

$$\langle Bx, x \rangle \geq 0 \ (> 0).$$

In the a same way we define the non-positive (negative) operators. We recall that a non-negative and self-adjoint operator $B \in \mathcal{L}(X)$ is a trace class operator whenever

$$\text{Tr}[B] := \sum_{n=1}^{+\infty} \langle Be_n, e_n \rangle < +\infty, \quad (2.1)$$

for some (and hence, every) orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $X$. We recall that the trace is independent of the choice of the basis. See [21] Section XI.6 and XI.9.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete, filtered probability space. We denote by $\mathbb{E}[\cdot]$ the expectation with respect to $\mathbb{P}$. Let $Y$ be a Banach space. If $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \to (Y, \mathcal{B}(Y))$ is a random variable, we denote by

$$\mathcal{L}(\xi) := \mathbb{P} \circ \xi^{-1}$$

the law of $\xi$ on $(Y, \mathcal{B}(Y))$. Throughout the paper when we refer to a process we mean a process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Now we make a remark about the notion of Wiener process in a Hilbert space.

Remark 6. Let $E$ be a separable Hilbert space and let $S$ be a self-adjoint and positive operator from $E$ to itself. If $S$ is a trace class operator, we call genuine $E$-valued Wiener process with $S$ as covariance operator a $E$-valued adapted process $\{W(t)\}_{t \geq 0}$ such that

1. $W(0) = 0$ and $\mathcal{L}(W(t) - W(s))$ is the Gaussian measure with mean zero and covariance operator $(t-s)S$ on $E$, where $S$ is an operator satisfying Hypothesis [70];
2. for $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \cdots < t_n$ the random variables $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent;
3. for $\mathbb{P}$-almost every $\omega \in \Omega$, $W(\cdot, \omega)$ is a continuous function on $[0, +\infty)$. This condition is called path-continuity.
If \( S \) is not a trace class operator, it is however possible to define a generalized \( E \)-valued Wiener process with \( S \) as covariance matrix (see \[20\] Section 4.1.2, \[25\] Section 2.5.1 and \[41\] Section 1). In this paper we call \( E \)-valued \( S \)-Wiener process both genuine \( E \)-valued Wiener process and generalized \( E \)-valued Wiener process. In particular we call \( E \)-valued cylindrical Wiener process a generalized \( E \)-valued Wiener process with covariance operator \( \mathbb{I} \).

We remark that if Hypotheses \([1]\) hold, then the right hand side of \( (2.3) \) is well defined. Indeed it is enough to show that the process

\[
\{W_A(t)\}_{t \geq 0} := \left\{ \int_0^t e^{(t-s)A}Q^\alpha dW(s) \right\}_{t \geq 0}
\]

is well defined. By \[20\] Theorem 5.2 and Theorem 5.11, if Hypothesis \([1]\) holds true, then \( \{W_A(t)\}_{t \geq 0} \) is Gaussian, continuous in mean square and it has a continuous and predictable version (see \[20\] Section 3.3).

**Definition 7.** For \( T > 0 \) and \( p \geq 1 \), we denote by \( \mathcal{X}^p([0,T]) \) the space of the progressively measurable (see \[20\] Section 3.3) \( X \)-valued processes \( \{\psi(t)\}_{t \in [0,T]} \) such that

\[
\|\psi\|_{\mathcal{X}^p([0,T])} := \sup_{t \in [0,T]} \mathbb{E}[\|\psi(t)\|^p] < +\infty.
\]

We now state a general result in the theory of stochastic partial differential equations with Lipschitz continuous nonlinearities.

**Theorem 8.** Let Hypotheses \([1]\) hold. Let \( T > 0 \) and let \( \Phi : [0,T] \times X \rightarrow X \) be measurable as a function from the \( \sigma \)-field \( \{[0,T] \times \Omega \times \mathcal{G}_T \times \mathcal{B}(X)\} \) to \( (X, \mathcal{B}(X)) \), where \( \mathcal{G}_T \) is the restriction to \( [0,T] \times \Omega \) of the \( \sigma \)-field generated by the sets

\[
(v,w) \times \mathcal{F}, \quad 0 \leq v \leq w < +\infty, \quad J \in \mathcal{F}_w.
\]

Assume that there exists \( y \in X \) such that the map \( t \mapsto \Phi(t, y) \) from \([0,T]\) to \( X \) is \( L^2 \)-summable, namely

\[
\int_0^T \|\Phi(t, y)\|^2 dt < +\infty,
\]

and that \( \Phi \) is a Lipschitz continuous function on \( X \) uniformly with respect to \( t \in [0,T] \), i.e. for every \( x, y \in X \) and \( t \in [0,T] \), it holds

\[
\|\Phi(t, x) - \Phi(t, y)\| \leq L_\Phi \|x - y\|
\]

where \( L_\Phi > 0 \) is a constant independent of \( t, x \) and \( y \). Consider the stochastic partial differential equation

\[
\begin{align*}
  dX(t,x) &= \left\{ AX(t,x) + \Phi(t, X(t,x)) \right\} dt + Q^\alpha dW(t), \quad t \in (0,T]; \\
  X(0,x) &= x \in X.
\end{align*}
\]

For each \( x \in X \), \([22]\) has unique mild solution \( X(t,x) \) in \( \mathcal{X}^2([0,T]) \) such that

(a) \( X(\cdot, x) \) is \( \mathbb{P} \)-a.s. continuous in \([0,T]\);

(b) The map \( x \mapsto X(\cdot, x) \) from \( X \) to \( \mathcal{X}^2([0,T]) \) is Lipschitz continuous.

Condition \((2.2)\) is weaker than the one assumed in \[20\] Theorem 7.5, namely there exists a constant \( C_F > 0 \) such that, for every \( t \in (0,T] \) and \( x \in X \), we have

\[
\|\Phi(t, x)\| \leq C_F (1 + \|x\|).
\]

Instead \((2.2)\) is enough to prove the same results of \[20\] Theorem 7.5 which are used in this paper. The only difference is that the mild solution belongs to \( \mathcal{X}^2([0,T]) \) and not to all the \( \mathcal{X}^p([0,T]) \) space, with \( p \geq 2 \). We will give a proof of Theorem 8 in Appendix \( A \) and we will give an example of a function \( \Phi \) satisfying \((2.2)\), but not satisfying the hypotheses of \[20\] Theorem 7.5, in Section \( 6.2 \).
If $F: \mathcal{X} \to \mathcal{X}$ is Lipschitz continuous, then, by Theorem 8, the transition semigroup
\[ P(t)\varphi(x) := \mathbb{E}[^\varphi(X(t,x))], \quad t \in [0,T], \ x \in \mathcal{X}, \ \varphi \in B_b(\mathcal{X}), \]
is well defined, where $X(t,x)$ is the mild solution of (1.1). Now we state a regularity result for the spatial derivative of the mild solution of (1.1).

**Theorem 9.** In addition to the assumptions of Theorem 8 assume $\Phi : [0,T] \times \mathcal{X} \to \mathcal{X}$ is such that the map $x \mapsto \Phi(t,x)$ is Gateaux differentiable for every $t \in [0,T]$ and there exists $C > 0$ such that for every $t \in [0,T]$ and $x,y \in \mathcal{X}$ it holds
\[ \|D^G \Phi(t,x) y\| \leq C\|y\|. \]

Let $X(t,x)$ be the mild solution of (1.1). Then the map $x \mapsto X(\cdot,x)$ from $\mathcal{X}$ to $\mathcal{X}^2([0,T])$ is Gateaux differentiable at $x_0 \in \mathcal{X}$ with bounded and continuous directional derivatives. Moreover, for every $x_0, h \in \mathcal{X}$ and $t \in [0,T]$, the process $Y(t,h) = D^G X(t,x_0)h$ is the unique mild solution of
\[
\begin{align*}
\left\{ 
\begin{array}{l}
dY(t,h) = (AY(t,h) + (D^G \Phi(t,X(t,x_0)))Y(t,h))dt, \quad t \in (0,T]; \\
Y(0,h) = h \in \mathcal{X}.
\end{array}
\right.
\end{align*}
\]

For a proof of Theorem 9 we refer to [20, Theorem 9.8] and the arguments used in Appendix A. We conclude the section by recalling a result that we will use in the next sections (see [44, Lemma 2.3]).

**Lemma 10.** Assume Hypotheses 4 hold true. Let $F \in C^2_b(\mathcal{X}; \mathcal{X})$ and $\varphi \in C^2_b(\mathcal{X})$. If $X(t,x)$ is the mild solution of (1.1) and $P(t)$ is the transition semigroup defined in (1.2), then for each $t \in (0,T]$, $P(t)\varphi \in C^2_b(\mathcal{X})$ and
\[ \varphi(X(t,x)) = P(t)\varphi(x) + \int_0^t (DP(t-s)\varphi(X(s,x)), Q^\alpha dW(s)), \ \mathbb{P}\text{-a.s.} \quad (2.4) \]

From here on, all the results involving processes must be understood as valid $\mathbb{P}$-a.s. for $t$ fixed.

### 3. Regularization results

This section is devoted to the proof of Theorem 4. We start with some basic facts about the space $H_\alpha$.

**Proposition 11.** Assume that Hypotheses 4 hold true and let $H_\alpha := Q^\alpha(\mathcal{X})$. For every $h, k \in H_\alpha$ we set
\[ \langle h, k \rangle_\alpha := \langle Q^{-\alpha}h, Q^{-\alpha}k \rangle. \quad (3.1) \]
Then $(H_\alpha, \|\cdot\|_\alpha)$ is a separable Hilbert space continuously embedded in $\mathcal{X}$, where $\|\cdot\|_\alpha$ is the norm associated to the inner product in (3.1) and
\[ \|h\| \leq \|Q^\alpha\|_{C(\mathcal{X})}\|h\|_\alpha, \quad (3.2) \]
for every $h \in H_\alpha$. Furthermore the following holds
(a) $Q^\alpha$ is linear and bounded from $H_\alpha$ to itself;
(b) $e^{\lambda A}$ is a contraction semigroup in $H_\alpha$;
(c) $H_\alpha$ is dense in $\mathcal{X}$.
(d) $H_\alpha$ is a Borel subset of $\mathcal{X}$.
(e) $W_\alpha(t) := Q^\alpha W(t)$ is a $H_\alpha$-valued $Q^{2\alpha}$-Wiener process.

**Proof.** Statements (a)–(d) are standard (e.g. [14]) and their proofs are left to the reader. Statement (e) follows noting that $Q^\alpha : \mathcal{X} \to H_\alpha$ is continuous and the Borel subsets of $H_\alpha$ are Borel subsets of $\mathcal{X}$ (see [20, Remark 5.1]).
We remark that if $\alpha = 0$ then $H_\alpha = \mathcal{X}$. The study of the mild solution of (1.1) and of the transition semigroup (1.2), when $\alpha = 0$, is already present in the literature, see for example [13, 37]. Instead $H_{1/2} = Q^{1/2}(\mathcal{X})$ is the Cameron–Martin space associated to the Gaussian measure with mean zero and covariance operator $Q$ on $\mathcal{X}$. This space is of fundamental interest for the Malliavin calculus, see for example [3, 20].

We conclude this introductory section with a lemma about a function that will be important throughout the rest of the paper.

**Lemma 12.** Let Hypotheses 3 hold true. For every $x \in \mathcal{X}$ and $t \in (0, T]$ the function $F_{x,t} : H_\alpha \to H_\alpha$ defined as

$$F_{x,t}(h) := F(h + e^{tA}x), \quad h \in H_\alpha,$$

is Lipschitz continuous, and

$$\int_0^T \|F_{x,t}(h)\|_{H_\alpha}^2 ds < +\infty. \quad (3.3)$$

**Proof.** The Lipschitz continuity is an easy consequence of Hypothesis 3. If $\alpha \in [1/4, 1/2]$, condition (5) follows by Hypothesis 3. Instead, if $\alpha \in [0, 1/4)$, by Proposition 2.1.1 and recalling that $A = -(1/2)Q^{2\alpha - 1}$ and that $e^{\alpha A}x$ belongs to $H_\alpha$ for every $s > 0$ and $x \in \mathcal{X}$ (due to the analyticity of $e^{\alpha A}$), we have

$$\int_0^T \|F_{x,t}(h)\|_{H_\alpha}^2 ds = \int_0^T \|F(e^{\alpha A}x)\|_{H_\alpha}^2 ds$$

$$\leq 2 \max \left\{ L_{F,\alpha}, \|F(0)\|_{H_\alpha}^2 \right\} \int_0^T (1 + \|e^{\alpha A}x\|_{H_\alpha}^2) ds$$

$$= 2 \max \left\{ L_{F,\alpha}, \|F(0)\|_{H_\alpha}^2 \right\} \left( T + \int_0^T \|e^{\alpha A}x\|_{H_\alpha}^2 ds \right)$$

$$= 2 \max \left\{ L_{F,\alpha}, \|F(0)\|_{H_\alpha}^2 \right\} \left( T + \int_0^T \|Q^{-\alpha}e^{\alpha A}x\|_{H_\alpha}^2 ds \right)$$

$$\leq 2 \max \left\{ L_{F,\alpha}, \|F(0)\|_{H_\alpha}^2 \right\} \left( T + \|x\|_{H_\alpha}^2 \int_0^T \frac{C_\alpha}{t(2\alpha)/(1-2\alpha)} ds \right)$$

$$\leq 2 \max \left\{ L_{F,\alpha}, \|F(0)\|_{H_\alpha}^2 \right\} \left( T + \|x\|_{H_\alpha}^2 \int_0^T \frac{C_\alpha}{t(2\alpha)/(1-2\alpha)} ds \right)$$

$$\leq 2 \max \left\{ L_{F,\alpha}, \|F(0)\|_{H_\alpha}^2 \right\} \left( T + C_\alpha \frac{2\alpha - 1}{4\alpha - 1} \int_0^T \|x\|_{H_\alpha}^2 ds \right)< +\infty,$$

for some positive constant $C_\alpha$. \( \square \)

3.1. **Existence and uniqueness.** Now we want to show that Hypotheses 3 are sufficient to guarantee the existence and uniqueness of the mild solution of (1.1). We remark that the existence and the uniqueness of the mild solution of equation (1.1), if $F$ lacks continuity, was already studied in [14] and [15], under a set of hypotheses that differ from ours. We cannot use the results seen in Section 2 since $F$ is not a Lipschitz continuous function on $\mathcal{X}$. Instead, similarly to [20], we take $H_\alpha$ as the underlying Hilbert space. For $\alpha \in [0, 1/2)$ we stress that by Hypotheses 3 and the analyticity of the semigroup $e^{tA}$, the mild solution $X(t,x)$ of (1.1) belongs to $H_\alpha$ for $t \in (0, T]$, but not for $t = 0$, because $X(0,x) = x \in \mathcal{X}$. Instead, for $\alpha = 1/2$, we cannot state that $X(t,x)$ belongs to $H_\alpha$ not even for $t \in (0, T]$, because in this case the condition $e^{tA} \mathcal{X} \subseteq H_\alpha$ is not verified ($A = -(1/2)\text{Id}$). Hence, in order to work on $H_\alpha$, it is necessary to define an auxiliary stochastic partial differential equation associated to (1.1) whose mild solution is a $H_\alpha$-valued process.

To do so we observe that, at least formally, the process $\{X(t,x) - e^{tA}x\}_{t \in [0,T]}$, for $x \in \mathcal{X}$, solves the equation

$$\left\{ \begin{array}{l}
    dY(t,0) = (AY(t,0) + F_{x,t}(Y(t,0)))dt + Q^\alpha dW(t), \quad t \in (0, T]; \\
    Y(0,0) = 0,
\end{array} \right.$$
and $Y(t,0)$ belongs to $H_\alpha$ for every $t \in [0,T]$. Indeed, still formally,

$$d(X(t,x) - e^{tA}x) = (AX(t,x) + F(X(t,x)))dt + Q^a dW(t) - Ae^{tA}x$$

and $X(0,x) - x = 0$. This procedure was the main idea behind the techniques used in section. Indeed, let $T > 0$ and for every $x \in \mathcal{X}$ we consider the stochastic partial differential equation

$$\begin{cases}
dZ_x(t,h) = (AZ_x(t,h) + F_x(t)(Z_x(t,h)))dt + Q^a dW(t), & t \in (0,T];

Z_x(0,h) = h \in H_\alpha,
\end{cases}$$

(3.4)

and its mild solution, namely the process $\{Z_x(t,h)\}_{t \geq 0}$ such that for $t \in [0,T],$

$$Z_x(t,h) = e^{tA}h + \int_0^t e^{(t-s)A}F_x(s,Z_x(s,h))ds + \int_0^t e^{(t-s)A}Q^a dW(s).$$

The reason to study the behaviour of the mild solution of (3.4) for every $h \in H_\alpha$, and not only for $h = 0$, will become clear in Section 3.2. In order to show that (3.4) has a unique mild solution we introduce the spaces $H_\alpha^2([0,T])$ defined as in Definition 4, with $H_\alpha$ replacing $\mathcal{X}$, endowed with the norm

$$\|\psi\|_{H_\alpha^2([0,T])} := \sup_{t \in [0,T]} \mathbb{E}[|\psi(t)|^2_{H_\alpha^2}] .$$

Proposition 13. Assume Hypotheses hold true and let $x \in \mathcal{X}$. For each $h \in H_\alpha$, (3.4) has unique mild solution $Z_x(t,h)$ in $H_\alpha^2([0,T])$ such that

(a) $Z_x(\cdot,h)$ is $\mathbb{P}$-a.s. continuous in $[0,T]$;

(b) The map $h \mapsto Z_x(\cdot,h)$ from $H_\alpha$ to $H_\alpha^2([0,T])$ is Lipschitz continuous.

Proof. It is enough to observe that, by Lemma 12 the hypotheses of Theorem 8 for equation (3.4) are satisfied with $H_\alpha$ replacing $\mathcal{X}$. \hfill \Box

We are now ready to state and prove the main theorem of this subsection.

Theorem 14. If Hypotheses hold true, then for every $x \in \mathcal{X}$ the stochastic partial differential equation (1.1) has a unique mild solution $X(t,x)$ belonging to $\mathcal{X}^2([0,T])$ and $\mathbb{P}$-a.s. path-continuous. Furthermore

$$X(t,x) = Z_x(t,0) + e^{tA}x,$$

where $Z_x(t,0)$ is the unique mild solution of (3.4) with $h = 0$.

Proof. We start by proving the uniqueness statement. Let $X(t,x)$ and $Y(t,x)$ be two mild solutions of (1.1) in $\mathcal{X}^2([0,T])$. Then

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s,x))ds + \int_0^t e^{(t-s)A}Q^a dW(s),$$

and

$$X(t,x) - e^{tA}x = \int_0^t e^{(t-s)A}F(X(s,x) + e^{sA}x - e^{sA}x)ds + \int_0^t e^{(t-s)A}Q^a dW(s)$$

$$= \int_0^t e^{(t-s)A}F_x(s)(X(s,x) - e^{sA}x)ds + \int_0^t e^{(t-s)A}Q^a dW(s).$$

Since $e^{(t-s)A}F_x(s)(X(s,x) - e^{sA}x)$ and $\int_0^t e^{(t-s)A}Q^a dW(s)$ belong to $H_\alpha$, we get that the process $X(t,x) - e^{tA}x$ has values in $H_\alpha$. In the same way $Y(t,x) - e^{tA}x$ has values in $H_\alpha$. So $\{X(t,x) - Y(t,x)\}_{t \in [0,T]}$ is a $H_\alpha$-valued process. Observe that by Proposition 11(b), we get

$$\mathbb{E} \left[ \|X(t,x) - Y(t,x)\|^2_{H_\alpha} \right] \leq TL_F \mathbb{E} \left[ \int_0^T \|e^{(t-s)A}(F(X(s,x) - F(Y(s,x)))\|^2_{H_\alpha} ds \right.$$}

$$\leq TL_F \int_0^T \mathbb{E} \left[ \|X(s,x) - Y(s,x)\|^2_{H_\alpha} \right] ds.$$
Remark 15. In Theorem 14 we have shown that for every $X \in \mathcal{X}^2([0,T])$ we get $E\left[\|X(t,x) - Y(t,x)\|_{\alpha}^2\right] = 0$ for every $t \in [0,T]$ and $x \in X$. We stress that we cannot make the same arguments in $\mathcal{X}^2([0,T])$, since the $F$ is $H_\alpha$-Lipschitz and may not be Lipschitz continuous on $X$. By [33, p. 188] we get $E\left[\|X(t,x) - Y(t,x)\|_{\alpha}^2\right] = 0$ for every $p \geq 2$, $t \in [0,T]$ and $x \in X$

\[
\|X(t,x) - Y(t,x)\|_{\mathcal{X}^2([0,T])}^2 = \sup_{t \in [0,T]} E\left[\|X(t,x) - Y(t,x)\|^2\right] \leq \|Q^\alpha\|_{\mathcal{L}(H_\alpha)} \sup_{t \in [0,T]} E\left[\|X(t,x) - Y(t,x)\|_{\alpha}^2\right] = 0.
\]

This concludes the proof of the uniqueness in $\mathcal{X}^2([0,T])$.

Now we show the existence of the mild solution. We have already noted that, when Hypotheses (3.5) hold, by Proposition 13 the stochastic partial differential equation

\[
\begin{cases}
  dZ_x(t,0) = (AZ_x(t,0) + F_{x,t}(Z_x(t,0)))dt + Q^\alpha dW(t), & t \in (0,T]; \\
  Z_x(0,0) = 0,
\end{cases}
\]

has a unique mild solution $Z_x(t,0)$ in $H^2_\alpha([0,T])$. We claim that the process $X(t,x) = Z_x(t,0) + e^{tA}x$ is the mild solution of (1.1). Indeed

\[
X(t,x) = e^{tA}x + Z_x(t,0) = e^{tA}x + \int_0^t e^{(t-s)A}F_{x,s}(Z_x(s,0))ds + \int_0^t e^{(t-s)A}Q^\alpha dW(s)
\]

\[
= e^{tA}x + \int_0^t e^{(t-s)A}F(X(s,x))ds + \int_0^t e^{(t-s)A}Q^\alpha dW(s).
\]

Now we check that $e^{tA}x + Z_x(t,0)$ belongs to $\mathcal{X}^2([0,T])$. Indeed, by the contractivity of $e^{tA}$, Proposition 11 and (3.2) we have

\[
\|e^{tA}x + Z_x(t,0)\|_{\mathcal{X}^2([0,T])}^2 = \sup_{t \in [0,T]} E\left[\|e^{tA}x + Z_x(t,0)\|^2\right] = 2\|x\|^2 + 2\|Q^\alpha\|_{\mathcal{L}(H_\alpha)}\|Z_x(t,0)\|_{H^2_\alpha([0,T])}
\]

Since $\{Z_x(t,0)\}_{t \geq 0}$ belongs to $H^2_\alpha([0,T])$ by Proposition 13 the claim follows. The property $P(\int_0^T \|e^{tA}x + Z_x(t,0)\|^2 ds < +\infty) = 1$ is an easy consequence of the fact that $e^{tA}x + Z_x(t,0)$ belongs to $\mathcal{X}^2([0,T])$. Therefore $X(t,x) = Z_x(t,0) + e^{tA}x$ is a mild solution of (1.1). Furthermore, by (3.2), $X(t,x)$ is $\mathbb{P}$-a.s. path-continuous.

**Remark 15.** In Theorem 14 we have shown that for every $x \in X$ and $t \in (0,T]$ the process $X(t,x) = Z_x(t,0) + e^{tA}x$ is the unique mild solution of (1.1) in $\mathcal{X}^2([0,T])$. So, for every $x \in X$, $t \in (0,T]$ and $h \in H_\alpha$ the process $Z_x(t,0) + e^{tA}(x + h)$ is the unique mild solution of the stochastic partial differential equation

\[
\begin{cases}
  dX(t,x + h) = (AX(t,x + h) + F(X(t,x + h)))dt + Q^\alpha dW(t), & t \in (0,T]; \\
  X(0,x + h) = x + h,
\end{cases}
\]

that belongs to $\mathcal{X}^2([0,T])$. However in some cases it is more useful to represent the mild solution of (3.5) by another process, as it will become apparent in the next subsection. For any $h \in H_\alpha$ and $x \in X$, the process $Z_x(t,h) + e^{tA}x$ is the mild solution of (3.5). Indeed

\[
Z_x(t,h) + e^{tA}x = e^{tA}x + e^{tA}h + \int_0^t e^{(t-s)A}F_{x,s}(Z_x(s,h))ds + \int_0^t e^{(t-s)A}Q^\alpha dW(s)
\]

\[
= e^{tA}(x + h) + \int_0^t e^{(t-s)A}F(Z_x(s,h) + e^{sA}x)ds + \int_0^t e^{(t-s)A}Q^\alpha dW(s).
\]

In the same way, as in the proof of Theorem 14, it is possible to prove that $Z_x(t,h) + e^{tA}x$ belongs to $\mathcal{X}^2([0,T])$. So $X(t,x + h) = Z_x(t,h) + e^{tA}x$ almost surely with respect to $\mathbb{P}$. 

3.2. Space regularity. In this subsection we will show that the mild solution of (1.1) constructed in Section 3.1 is Gateaux differentiable along \( H_\alpha \). Now we clarify what we mean by “differentiable along \( H_\alpha \).”

**Definition 16.** Let \( Y \) be a Hilbert space endowed with the norm \( \| \cdot \|_Y \) and let \( \Phi : \mathcal{X} \to Y \).

(i) We say that \( \Phi \) is differentiable along \( H_\alpha \) at the point \( x \in \mathcal{X} \), if there exists \( L \in \mathcal{L}(H_\alpha, Y) \) such that

\[
\lim_{\| h \|_\alpha \to 0} \frac{\| \Phi(x + h) - \Phi(x) - Lh \|_Y}{\| h \|_\alpha} = 0.
\]

When it exists, the operator \( L \) is unique and we set \( D_\alpha \Phi(x) := L \). If \( Y = \mathbb{R} \), then \( L \in H_\alpha^* \) and so there exists \( k \in H_\alpha \), such that \( Lh = \langle h, k \rangle_\alpha \) for any \( h \in H_\alpha \). We set \( D_\alpha \Phi(x) := k \) and we call it \( H_\alpha \)-gradient of \( \Phi \) at \( x \in \mathcal{X} \).

(ii) We say that \( \Phi \) is two times differentiable along \( H_\alpha \) at the point \( x \in \mathcal{X} \) if it is differentiable along \( H_\alpha \) at every point of \( \mathcal{X} \) and there exists \( T \in \mathcal{L}(H_\alpha, \mathcal{L}(H_\alpha, Y)) \) such that

\[
\lim_{\| k \|_\alpha \to 0} \frac{\| (D_\alpha \Phi(x + k) - D_\alpha \Phi(x))h - (Thk)_\alpha \|_Y}{\| k \|_\alpha} = 0.
\]

uniformly for \( h \in H_\alpha \) with norm 1. When it exists, the operator \( T \) is unique and we set \( D^2_\alpha \Phi(x) := T \). If \( Y = \mathbb{R} \), then \( T \in \mathcal{L}(H_\alpha, H_\alpha^*) \), so there exists \( S \in \mathcal{L}(H_\alpha) \) such that \( (Thk)_\alpha = \langle Sh, k \rangle_\alpha \), for any \( h, k \in H_\alpha \). We set \( D^2_\alpha \Phi(x) := S \) and we call it \( H_\alpha \)-Hessian of \( \Phi \) at \( x \in \mathcal{X} \).

(iii) We say that \( \Phi \) is Gateaux differentiable along \( H_\alpha \) at the point \( x \in \mathcal{X} \) if there exists \( L \in \mathcal{L}(H_\alpha, Y) \) such that

\[
Lh = \frac{\Phi(x + th) - \Phi(x)}{t} \quad \text{for} \quad t \to 0.
\]

When it exists, the operator \( L \) is unique and we set \( D^G_\alpha \Phi(x) := L \).

For simplicity sake we will write \( D_\alpha \Phi(x)h := D_\alpha \Phi(x)(h) \) and \( D^G_\alpha \Phi(x)h := D^G_\alpha \Phi(x)(h) \). For \( k = 1, 2 \), we denote by \( C_b, H_\alpha (\mathcal{X}; Y) \) the set of the \( k \)-times differentiable functions along \( H_\alpha \) such that the operator \( D_\alpha \Phi \), if \( k = 1 \), and the operators \( D_\alpha \Phi \) and \( D^2_\alpha \Phi \), if \( k = 2 \), are bounded. If \( Y = \mathbb{R} \) we will simply write \( C_b, H_\alpha (\mathcal{X}) \).

We remark that if \( \Phi : \mathcal{X} \to Y \) is differentiable along \( H_\alpha \) at \( x \in \mathcal{X} \), then it is Gateaux differentiable along \( H_\alpha \) at \( x \in \mathcal{X} \) and it holds \( D_\alpha \Phi(x) = D^G_\alpha \Phi(x) \). The derivative operators defined in Definition 13 are related to the ones presented in [33] and [34]. We will do a detailed comparison in Section 5.2.

We now prove some basic consequences of the above definition.

**Proposition 17.** Let \( Y \) be a Hilbert space endowed with the norm \( \| \cdot \|_Y \) and let \( \Phi : \mathcal{X} \to Y \). If \( \Phi \) is Fréchet differentiable at \( x \in \mathcal{X} \), then it is differentiable along \( H_\alpha \) and, for every \( h \in H_\alpha \),

\[
D_\alpha \Phi(x)h = D \Phi(x)h. \tag{3.6}
\]

Furthermore if \( \varphi : \mathcal{X} \to \mathbb{R} \) is Fréchet differentiable at \( x \in \mathcal{X} \), then we have

\[
D_\alpha \varphi(x) = Q^{2\alpha} D \varphi(x).
\]

Proof. By the Fréchet differentiability of \( \Phi \) we know that for every \( x \in \mathcal{X} \)

\[
\lim_{\| h \| \to 0} \frac{\| \Phi(x + h) - \Phi(x) - D \Phi(x)h \|_Y}{\| h \|} = 0.
\]

By (3.2) we have that \( \| h \| \to 0 \), whenever \( \| h \|_\alpha \to 0 \) and

\[
0 \leq \lim_{\| h \|_\alpha \to 0} \frac{\| \Phi(x + h) - \Phi(x) - D \Phi(x)h \|_Y}{\| h \|_\alpha} = \lim_{\| h \|_\alpha \to 0} \frac{\| \Phi(x + h) - \Phi(x) - D \Phi(x)h \|_Y}{\| h \|} \cdot \frac{\| h \|_\alpha}{\| h \|}.
\]
We want to apply the contraction mapping theorem to $V$. Consider the linear operator $D\Phi(x)h = Q^\alpha D\phi(x)h$.

Proof. Let $\Phi$ be the mild solution of $0 = Q^\alpha D\phi(x)$ for every $x \in X$ and $h \in H_\alpha$. Hence $D\alpha\phi(x)h = Q^\alpha D\phi(x)h$.

Lemma 18. Assume that Hypotheses 3 hold true and let $k = 1, 2$. If $F$ belongs to $C^k_{\beta,H_\alpha}(X; H_\alpha)$ then for any $x \in X$ and $t \in (0, T)$ the function $F_{x,t} : H_\alpha \to H_\alpha$ defined as

$$F_{x,t}(h) := F(h + e^{tA}x), \quad h \in H_\alpha,$$

belongs to $C^k_{\beta}(H_\alpha; H_\alpha)$. Furthermore $\|D\alpha F_{x,t}\|_{\mathcal{L}(H_\alpha)} \leq L_{F,\alpha}$, where $L_{F,\alpha}$ is the $H_\alpha$-Lipschitz constant of $F$.

Proof. We only prove the statement in the case $k = 1$, since the proof for $k = 2$ is similar. By the definition of the space $C^1_{\beta,H_\alpha}(X; H_\alpha)$ for every $x \in X$

$$\lim_{\|h\|_\alpha \to 0} \frac{\|F(y + h) - F(y) - D\alpha F(y)h\|_\alpha}{\|h\|_\alpha} = 0. \quad (3.7)$$

Now letting $y = e^{tA}x + h_0$ in (3.7) we get

$$\lim_{\|h\|_\alpha \to 0} \frac{\|F(e^{tA}x + h_0 + h) - F(e^{tA}x + h_0) - D\alpha F(e^{tA}x + h_0)h\|_\alpha}{\|h\|_\alpha} = 0.$$

So $D\alpha F_{x,t}(h_0) = D\alpha F(e^{tA}x + h_0)$. The furthermore part is a standard consequence of the identity we just showed.

Theorem 19. Assume that Hypotheses 3 hold true and let $F \in C^1_{\beta,H_\alpha}(X; H_\alpha)$. For every $x \in X$ and $h \in H_\alpha$ let $Z_x(t, h)$ be the mild solution of

$$\begin{cases} dY(t, h_0) = (AY(t, h_0) + D\alpha F_{x,t}(Z_x(t, h_0))Y(t, h_0))dt, \quad t \in (0, T]; \\ Y(0, h_0) = h_0 \in H_\alpha, \end{cases} \quad (3.8)$$

admits a unique mild solution $Y(\cdot, h_0)$ in $H^2(0, T)$. Furthermore for every $t \in [0, T]$,

$$\|Y(t, h_0)\|_\alpha \leq e^{T L_{F,\alpha}}\|h_0\|_\alpha. \quad (3.9)$$

Finally the map $h \mapsto Z_x(\cdot, h)$ is Gateaux differentiable with values in $H^2(0, T)$ and for any $t \in [0, T]$ and $h_0 \in H_\alpha$, it holds $Y(t, h_0) = D\alpha Z_x(t, h_0)$.

Proof. Consider the linear operator $V$ defined on $H^2(0, T)$ as

$$V(Y)(t) := e^{tA}h_0 + \int_0^t e^{(t-s)A}D\alpha F_{x,t}(Z_x(s, h))Y(s)ds, \quad t \in [0, T].$$

We want to apply the contraction mapping theorem to $V$, since a fixed point of $V$ is a mild solution of (3.8). First we check that $V(H^2(0, T)) \subseteq H^2(0, T)$. If $Y \in H^2(0, T)$, then by Proposition 11 and Lemma 18 we have by standard computations

$$\|V(Y)\|^2_{H^2(0, T)} \leq 2\|h_0\|^2_\alpha + 2 T^2 L^2_{F,\alpha} \|Y\|^2_{H^2(0, T)} < +\infty. \quad (3.10)$$

Now we show that $V$ is Lipschitz continuous on $H^2(0, T)$. Let $Y_1, Y_2 \in H^2(0, T)$, then by standard calculations we get

$$\|V(Y_1) - V(Y_2)\|^2_{H^2(0, T)} \leq T \sup_{t \in [0, T]} \left( \int_0^t \left( \int_0^s \left( e^{(t-s)A}D\alpha F_{x,t}(Z_x(s, h))(Y_1(s) - Y_2(s)) \right)^2 ds \right)^{1/2} \right)^2.$$
Using the same arguments as in (3.10) we obtain
\[ \|V(Y_1) - V(Y_2)\|_{\mathcal{H}_2^2([0,T])}^2 \leq T^2 L_{F,\alpha}^2 \|Y_1 - Y_2\|_{\mathcal{H}_2^2([0,T])}^2. \]
So there exists \( T^* > 0 \) such that \( V \) is a contraction on \( \mathcal{H}_2^2([0,T^*]) \).

Let \( Y(t) := \begin{cases} Y_r(y), & t \in [rT^*, (r+1)T^*], \ r = 0, \ldots, n, \\ Y_{a}(t), & t \in [nT^*, T]; \end{cases} \)

By a classical arguments we have that \( Y \) is the unique mild solution in \( \mathcal{H}_2^2([0,T]) \).

To prove (3.9) we start by observing that by Proposition 11(b) and Lemma 18
\[ \parallel Y(t,h_0) \parallel_{\alpha} \leq \parallel h_0 \parallel_{\alpha} + L_{F,\alpha} \int_0^t \parallel Y(s,h_0) \parallel_{\alpha} ds. \]
Recalling that the functions \( Y(\cdot,h_0), Z_\alpha(\cdot,h) \) and \( D\mathcal{F}_{x,t} \) are continuous, the Gronwall inequality yields (3.9).

By Proposition 13 and Lemma 18 if \( F \in C^1_{b,H_\alpha}(\mathcal{X};H_\alpha) \), then for each \( T > 0 \), the map \( h \mapsto Z_\alpha(\cdot,h) \) from \( H_\alpha \) to \( \mathcal{H}_2^2([0,T]) \) is Gateaux differentiable with bounded and continuous directional derivatives, for every \( x \in \mathcal{X} \). Moreover, for any \( h_0 \in H_\alpha \), the process \( Y(t,h_0) := D^G Z_\alpha(t,h_0) \) is the unique mild solution in \( H_\alpha([0,T]) \) of (3.8).

Now we want to study the process \( D^G Z_\alpha(t,x,h) \) with \( x \in \mathcal{X} \) and \( h_0 \in H_\alpha \).

**Theorem 20.** Assume that Hypotheses 3 hold true, and let \( F \in C^1_{b,H_\alpha}(\mathcal{X};H_\alpha) \). The map \( x \mapsto X(\cdot,x) \) from \( \mathcal{X} \) to \( \mathcal{X}^2([0,T]) \) is Gateaux differentiable along \( H_\alpha \) and for \( x \in \mathcal{X}, t \in [0,T] \) and \( h \in H_\alpha \) it holds
\[ D^G_{\alpha} X(t,x)h = D^G Z_\alpha(t,0)h. \quad (3.11) \]

**Proof.** Let \( x \in \mathcal{X}, h \in H_\alpha, t \in [0,T] \) and \( s > 0 \). By Remark 15 we know that \( X(t,x + sh) = Z_\alpha(t,sh) + e^{tA_x}s \), so
\[ X(t,x + sh) - X(t,x) = Z_\alpha(t,sh) - Z_\alpha(t,0). \]
Hence by (3.2) and Proposition 13 we have
\[
\begin{align*}
0 &\leq \lim_{s \to 0} \frac{\parallel X(s,x + sh) - X(s,x) - D^G Z_\alpha(\cdot,0)h \parallel_{\mathcal{X}^2([0,T])}^2}{s} \\
&= \lim_{s \to 0} \sup_{t \in [0,T]} \mathbb{E} \left[ \frac{\parallel X(t,x + sh) - X(t,x) - D^G Z_\alpha(t,0)h \parallel_{\mathcal{X}^2([0,T])}^2}{s} \right] \\
&= \lim_{s \to 0} \sup_{t \in [0,T]} \mathbb{E} \left[ \frac{\parallel Z_\alpha(t,sh) - Z_\alpha(t,0) - D^G Z_\alpha(t,0)h \parallel_{\mathcal{X}^2([0,T])}^2}{s} \right] \\
&\leq \|Q^o\|_{\mathcal{L}(H_\alpha)} \lim_{s \to 0} \sup_{t \in [0,T]} \mathbb{E} \left[ \frac{\parallel Z_\alpha(t,sh) - Z_\alpha(t,0) - D^G Z_\alpha(t,0)h \parallel_{\mathcal{X}^2([0,T])}^2}{s} \right] \\
&= \|Q^o\|_{\mathcal{L}(H_\alpha)} \lim_{s \to 0} \parallel Z_\alpha(s,sh) - Z_\alpha(s,0) - D^G Z_\alpha(s,0)h \parallel_{\mathcal{H}_2^2([0,T])}^2 = 0.
\end{align*}
\]
Linearity and continuity in \( H_\alpha \) of \( h \mapsto D^G_{\alpha} X(t,x)h \) follows from the linearity and continuity of \( h \mapsto D^G Z_\alpha(t,0)h \).

To end this subsection we state and prove the following corollary.

**Corollary 21.** Assume that Hypotheses 3 hold true, let \( T > 0 \) and let \( F \in C^1_{b,H_\alpha}(\mathcal{X};H_\alpha) \). If \( g : \mathcal{X} \to \mathbb{R} \) is a function belonging to \( C^1_{b,H_\alpha}(\mathcal{X}) \) and \( h \in H_\alpha \), then for any \( x \in \mathcal{X} \) and \( t \in [0,T] \)
\[ ((D^G_{\alpha}(g \circ X))(t,x))h = \langle (D_0 g)(X(t,x)), D^G_{\alpha} X(t,x)h \rangle_{\alpha}, \quad (3.12) \]
Proof. Since \( g \in C_{b,H_o}(\mathcal{X}) \), then for every \( x \in \mathcal{X} \) and \( h \in H_o \)
\[
g(x+\varepsilon h) = g(x) + \varepsilon \langle D_o g(x), h \rangle_o + o(\varepsilon) \quad \varepsilon \to 0.
\]
We define for \( x \in \mathcal{X}, h \in H_o, t \in [0,T] \) and \( \varepsilon > 0 \)
\[
K_\varepsilon(t,x,h) := X(t,x+\varepsilon h) - X(t,x) - \varepsilon D^G Z_\varepsilon(t,0)h = Z_\varepsilon(t,0)h - Z_\varepsilon(t,0) - \varepsilon D^G Z_\varepsilon(t,0)h.
\]
Observe that by Proposition 13 we have that \( \|K_\varepsilon(\cdot,x,h)\|_{H^2([0,T])}^2 = o(\varepsilon) \), when \( \varepsilon \) goes to zero. Hence for \( \varepsilon \to 0 \)
\[
g(X(t,x+\varepsilon h)) = g(X(t,x)) + \varepsilon D^G Z_\varepsilon(t,0)h + K_\varepsilon(t,x,h)
= g(X(t,x)) + \varepsilon \langle (D_o g)(X(t,x)), D^G Z_\varepsilon(t,0)h + \varepsilon^{-1} K_\varepsilon(t,x,h) \rangle_o + o(\varepsilon)
= g(X(t,x)) + \varepsilon \langle (D_o g)(X(t,x)), D^G Z_\varepsilon(t,0)h \rangle_o + (\langle (D_o g)(X(t,x)), K_\varepsilon(t,x,h) \rangle_o + o(\varepsilon).
\]
So for \( \varepsilon \to 0 \) we get
\[
0 \leq \sup_{t \in [0,T]} \mathbb{E} \left[ \left| g(X(t,x+\varepsilon h)) - g(X(t,x)) - \varepsilon \langle (D_o g)(X(t,x)), D^G Z_\varepsilon(t,0)h \rangle_o \right|^2 \right] \\
\leq \sup_{t \in [0,T]} \mathbb{E} \left[ \left| g(X(t,x+\varepsilon h)) - g(X(t,x)) - \varepsilon \langle (D_o g)(X(t,x)), D^G Z_\varepsilon(t,0)h \rangle_o \right|^2 \right] \\
= \left( \sup_{x \in \mathcal{X}} \|D_o g(x)\|_{L(H_o)} \right) \left( \sup_{t \in [0,T]} \mathbb{E} \left[ \|K_\varepsilon(t,x,h)\|_{H^2([0,T])}^2 \right] \right) + o(\varepsilon)
\leq \left( \sup_{x \in \mathcal{X}} \|D_o g(x)\|_{L(H_o)} \right) \left( \|K_\varepsilon(\cdot,x,h)\|_{H^2([0,T])}^2 \right) + o(\varepsilon).
\]
This implies that \( \mathbb{P}\)-a.s it holds \( \langle (D_o g)(X(t,x)), D^G Z_\varepsilon(t,0)h \rangle_o = \langle (D_o g)(X(t,x)), D^G Z_\varepsilon(t,0)h \rangle_o \) and the proof is concluded recalling Theorem 20. \( \square \)

3.3. Proof of Theorem 4. Throughout this subsection \( X(t,x) \) will denote the mild solution of (3.14), while \( P(t) \) is its associated transition semigroup, defined in (3.12). To prove Theorem 4 we will use a similar procedure to the one used in [13] section 7.7 and [14]. First we are going to prove Theorem 4 for sufficiently regular functions \( F \) and \( \varphi \). Note that Lemma 22 is an adaptation of Lemma 11 to our situation.

**Lemma 22.** Assume Hypotheses hold true. Let \( F \in C_{b,H_o}^2(\mathcal{X};H_o), \varphi \in C_{b,H_o}^2(\mathcal{X}) \) and \( x \in \mathcal{X} \). Then for each \( t \in (0,T], P(t)\varphi \in C_{b,H_o}^2(\mathcal{X}) \) and
\[
\varphi(X(t,x)) = P(t)\varphi(x) + \int_0^t \langle D_o P(t-s)\varphi(X(s,x)), Q^o dW(s) \rangle_o.
\]
(3.13)

**Proof.** Let \( x \in \mathcal{X} \) and consider the transition semigroup
\[
T_x(t)\psi(h) := \mathbb{E} [\psi(Z_x(t,h))], \quad t \in [0,T], \ h \in H_o, \ \psi \in B_b(H_o);
\]
where \( Z_x(t,h) \) is the mild solution of (3.14). Let \( \varphi \in C_{b,H_o}^2(\mathcal{X}) \) and consider the function \( \tilde{\varphi}(h) := \varphi(e^{tA}x + h) \) on \( H_o \). Proceeding in the same way as in Lemma 18 we have \( \tilde{\varphi} \in C_{b}^2(H_o) \). Moreover since by Theorem 4 it holds \( X(t,x) = e^{tA}x + Z_x(t,0) \) then
\[
P(t)\varphi(x) = T_x(t)\tilde{\varphi}(0).
\]
We recall that, by Lemma 11 and Lemma 18 \( T_x(t)\tilde{\varphi} \in C_{b}^2(H_o) \). Moreover, by Remark 13 if \( x \in \mathcal{X}, h \in H_o \) and \( t \in [0,T] \) then
\[
P(t)\varphi(x + h) = \mathbb{E} [\varphi(e^{tA}x + Z_x(t,h))] = T_x(t)\tilde{\varphi}(h).
\]
We claim that \( P(t)\varphi \) is differentiable along \( H_\alpha \). Indeed for every \( x \in X \)
\[
\lim_{\|h\|_\alpha \to 0} \frac{|P(t)\varphi(x + h) - P(t)\varphi(x) - \langle DT_x(t)\hat{\varphi}(0), h \rangle_\alpha|}{\|h\|_\alpha} = \lim_{\|h\|_\alpha \to 0} \frac{|T_x(t)\hat{\varphi}(h) - T_x(t)\hat{\varphi}(0) - \langle DT_x(t)\hat{\varphi}(0), h \rangle_\alpha|}{\|h\|_\alpha} = 0. \tag{3.14}
\]
So \( P(t)\varphi \) belongs to \( C^1_{b,H_\alpha}(X) \). A similar argument gives \( P(t)\varphi \in C^2_{b,H_\alpha}(X) \). By Lemma 10 for each \( t \in (0,T) \), \( x \in X \) and \( h \in H_\alpha \) we have
\[
\hat{\varphi}(Z_x(t,h)) = T_x(t)\hat{\varphi}(h) + \int_0^t \langle DT_x(t-s)\hat{\varphi}(Z_x(s,h)), Q^\alpha dW(s) \rangle_\alpha. \tag{3.15}
\]
So (3.13) follows by (3.14) with \( h = 0 \) and (3.14).

Now we prove a variant of the Bismut–Elworthy–Li formula.

**Proposition 23.** Assume that Hypotheses 3 hold. Let \( F \in C^2_{b,H_\alpha}(X;H_\alpha) \) and \( \varphi \in C^2_{b,H_\alpha}(X) \). For every \( x \in X \), \( h \in H_\alpha \) and \( t \in (0,T] \)
\[
\langle DaP(t)\varphi(x),h \rangle_\alpha = \frac{1}{t} E \left[ \varphi(X(t,x)) \int_0^t \langle D^G_x Z_x(s,0)h,Q^\alpha dW(s) \rangle_\alpha \right]. \tag{3.16}
\]
Furthermore
\[
|\langle DaP(t)\varphi(x),h \rangle_\alpha|^2 \leq \frac{1}{12} \|\varphi\|_{\infty}^2 E \left[ \int_0^t \|D^G_x Z_x(s,0)h\|_{H_\alpha}^2 ds \right]. \tag{3.17}
\]

**Proof.** (3.17) is a standard consequence of (3.16) and the Itô isometry (see [12] Lemma 3.1.5) so we will only show (3.16). We recall that, by Theorem 20 \( D^G_x X(t,x) = DZ_x(t,0)h \). Let \( h \in H_\alpha \), \( t \in (0,T] \) and \( x \in X \). Multiplying both sides of (3.13) by
\[
\int_0^t \langle D^G_x Z_x(s,0)h,Q^\alpha dW(s) \rangle_\alpha,
\]
and taking the expectations we get
\[
E \left[ \varphi(X(t,x)) \int_0^t \langle D^G_x Z_x(s,0)h,Q^\alpha dW(s) \rangle_\alpha \right] = E \left[ P(t)\varphi(x) \int_0^t \langle D^G_x Z_x(s,0)h,Q^\alpha dW(s) \rangle_\alpha \right] + E \left[ \int_0^t \langle DaP(t-s)\varphi(X(s,x)), Q^\alpha dW(s) \rangle_\alpha \int_0^t \langle D^G_x Z_x(s,0)h,Q^\alpha dW(s) \rangle_\alpha \right].
\]
We recall that the process \( \{Q^\alpha W(s)\}_{s \geq 0} \) is a \( H_\alpha \)-valued Wiener process (see Proposition 11(c)). By [22] Remark 2, the process \( \{ \int_0^t \langle D^G_x Z_x(s,0)h,Q^\alpha dW(s) \rangle_\alpha \}_{t \geq 0} \) is a martingale provided that for every \( t \in [0,T] \) and \( h \in H_\alpha \)
\[
\int_0^t E \left[ \|D^G_x Z_x(s,0)h\|_{H_\alpha}^2 \right] ds < +\infty.
\]
By Theorem 10 we know that \( D^G_x Z_x(\cdot,0) \in H^2_{\alpha}(0,T) \), then for any \( t \in [0,T] \) and \( h \in H_\alpha \)
\[
\int_0^t E \left[ \|D^G_x Z_x(s,0)h\|_{H_\alpha}^2 \right] ds \leq T \|D^G_x Z_x(\cdot,0)h\|_{H^2_{\alpha}(0,T)}^2 < +\infty.
\]
Hence \( t \mapsto \int_0^t \langle D^G_x Z_x(s,0)h,Q^\alpha dW(s) \rangle_\alpha \) is a martingale and we have for every \( t \in [0,T] \), \( x \in X \) and \( h \in H_\alpha \)
\[
E \left[ P(t)\varphi(x) \int_0^t \langle D^G_x Z_x(s,0)h,Q^\alpha dW(s) \rangle_\alpha \right] = 0.
\]
Hence by (3.12), with $G = P(t-s)\varphi$, and the Itô isometry we obtain
\[
E \int_0^t \langle (D_\alpha P(t-s)\varphi)(X(s,x)), Q^\alpha dW(s) \rangle_{\alpha} = E \int_0^t \langle (D_\alpha P(t-s)\varphi)(X(s,x)), D_\alpha^G Z_x(s,0)h, Q^\alpha dW(s) \rangle_{\alpha}
\]

\[
= E \int_0^t \langle (D_\alpha P(t-s)\varphi)(X(s,x)), D_\alpha^G Z_x(s,0)h \rangle_{\alpha} ds
\]

\[
= \int_0^t \langle D_\alpha^G E[(P(t-s)\varphi \circ X)(s,x)] \rangle ds.
\]

By the very definition of $P(t)$ we know that $E[(P(t-s)\varphi \circ X)(s,x)] = (P(s)P(t-s)\varphi)(x) = P(t)\varphi(x)$. Recalling that $P(t)\varphi$ belongs to $C_{b,H_\alpha}^2(\mathcal{X})$ it holds $D_\alpha^G P(t)\varphi(x) = D_\alpha P(t)\varphi(x)$. So, by Lemma 22 we conclude
\[
E \left[ \varphi(X(t,x)) \int_0^t \langle D_\alpha^G Z_x(s,0)h, Q^\alpha dW(s) \rangle_{\alpha} \right] = \int_0^t \langle D_\alpha P(t)\varphi(x), h \rangle_{\alpha} ds
\]

\[
= t \langle D_\alpha P(t)\varphi(x), h \rangle_{\alpha}.
\]

Recalling (3.11) we get the thesis. □

The last step before proving Theorem 4 is the following corollary.

**Corollary 24.** Assume that Hypotheses 3 hold. Let $F \in C_{b,H_\alpha}^2(\mathcal{X}; H_\alpha)$ and $\varphi \in C_{b,H_\alpha}^2(\mathcal{X})$. For every $t \in (0,T]$, $x \in \mathcal{X}$ and $h \in H_\alpha$

\[
|P(t)\varphi(x + h) - P(t)\varphi(x)| \leq \frac{e^{L_{F,\alpha}T}}{\sqrt{T}} \|\varphi\|_{\infty} \|h\|_{\alpha},
\]

(3.18)

**Proof.** Taking into account (3.9), (3.11) and (3.17) we obtain the gradient estimate

\[
\|D_\alpha P(t)\varphi(x)\|_{\alpha} \leq \frac{e^{L_{F,\alpha}T}}{\sqrt{T}} \|\varphi\|_{\infty}, \quad t \in (0,T], \quad x \in \mathcal{X}.
\]

(3.19)

Let $x \in \mathcal{X}$ and $h \in H_\alpha$, then by the mean value theorem, there exists $c_h > 0$ such that

\[
P(t)\varphi(x + h) - P(t)\varphi(x) = \langle D_\alpha P(t)\varphi(x), h \rangle_{\alpha}.
\]

So, by (3.11), the thesis follows. □

Now we can prove Theorem 4 of Theorem 4. We start by assuming that $F \in C_{b,H_\alpha}^2(\mathcal{X}; H_\alpha)$ and we show that since (3.18) is verified for $\varphi \in C_{b,H_\alpha}^2(\mathcal{X})$ then it also holds for $\varphi \in B_b(\mathcal{X})$. We recall that by [17] Theorem 5.4., if $\varphi \in C_b(\mathcal{X})$ then there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}} \subseteq C_{b}^2(\mathcal{X})$ such that, for every $x \in \mathcal{X}$,

\[
\lim_{n \to +\infty} \varphi_n(x) = \varphi(x), \quad \|\varphi_n\|_{\infty} \leq \|\varphi\|_{\infty}.
\]

Since $C_{b}^2(\mathcal{X}) \subseteq C_{b,H_\alpha}^2(\mathcal{X})$ by Proposition 17 (3.18) yields

\[
|P(t)\varphi_n(x + h) - P(t)\varphi_n(x)| \leq \frac{e^{L_{F,\alpha}T}}{\sqrt{T}} \|\varphi_n\|_{\infty} \|h\|_{\alpha}, \quad n \in \mathbb{N}, \quad t \in (0,T], \quad h \in H_\alpha.
\]

Observe that by the dominated convergence theorem $P(t)\varphi_n(x + h)$ and $P(t)\varphi_n(x)$ converge to $P(t)\varphi(x + h)$ and $P(t)\varphi(x)$, respectively. Therefore (3.18) is verified also for $\varphi \in C_b(\mathcal{X})$. By the Riesz representation theorem and (3.18), for every $x \in \mathcal{X}$, $h \in H_\alpha$ and $t \in (0,T]$, we have the following estimate for the total variation of the finite measure $\mathcal{L}(X(x + h), t) - \mathcal{L}(X(x), t)$

\[
\text{Var}(\mathcal{L}(X(t,x + h)) - \mathcal{L}(X(t,x))) := \sup_{\|\varphi\|_{\infty} \leq 1} \int_{\mathcal{X}} \varphi d\left(\mathcal{L}(X(t,x + h)) - \mathcal{L}(X(t,x))\right)
\]
\[
\frac{\|P(t)\varphi(x) - P(t)\varphi(x)\|}{\|x - h\|_\infty} \leq \frac{c_{L,F,\alpha} T}{\sqrt{t}} \|h\|_\alpha.
\]

Let \( \varphi \in B_b(\mathcal{X}) \), then for \( t \in (0, T] \), \( x \in \mathcal{X} \) and \( h \in H_\alpha 
\[
|P(t)\varphi(x + h) - P(t)\varphi(x)| = \left| \int_\mathcal{X} \varphi d\left( \mathcal{L}(X(t, x + h)) - \mathcal{L}(X(t, x)) \right) \right| 
\leq \|\varphi\|_\infty \frac{c_{L,F,\alpha} T}{\sqrt{t}} \|h\|_\alpha.
\]

As a second step, we prove that (5.15) is verified for \( \varphi \in C^2_{b,H,\alpha}(\mathcal{X}) \) if \( F \) satisfies Hypotheses 3. We recall that by Lemma 12, \( F_{x,t} \) is Lipschitz continuous on \( H_\alpha \), so it is possible to construct a sequence \( \{F_{x,t}^{(n)}\}_{n \in \mathbb{N}} \subseteq C^2_{b}(H_\alpha; H_\alpha) \) (see [44] Lemma 2.5) such that the functions \( F_{x,t}^{(n)} \) are Lipschitz continuous with Lipschitz constant less or equal than \( L_{F,\alpha} \), and
\[
\lim_{n \to +\infty} \|F_{x,t}^{(n)}(h) - F_{x,t}(h)\|_\alpha = 0, \quad h \in H_\alpha.
\]

We consider the transitions semigroups
\[
P^{(n)}(t)\varphi(x) := \mathbb{E}[\varphi(X^{(n)}(t, x))], \quad \varphi \in C_b(\mathcal{X}),
\]
where \( X^{(n)}(t, x) := Z_x^{(n)}(t, 0) + e^{tA} x \) and \( Z_x^{(n)}(t, 0) \) is the mild solution of
\[
dZ_x(t, 0) = (AZ_x(t, 0) + F_{x,t}^{(n)}(Z_x(t, 0)))dt + Q^a dW(t), \quad t \in (0, T];
\]
\[
Z_x(0, 0) = 0.
\]

Fix \( \varphi \in C^2_{b,H,\alpha}(\mathcal{X}) \). Then by (5.15) for every \( x \in \mathcal{X} \), \( h \in H_\alpha \) and \( t \in (0, T] \), we get
\[
|P^{(n)}(t)\varphi(x + h) - P^{(n)}(t)\varphi(x)| \leq \frac{c_{L,F,\alpha} T}{\sqrt{t}} \|\varphi\|_\infty \|h\|_\alpha.
\]

By [44] Theorem A.1 there exists a subsequence \( \{Z_x^{(n_k)}(t, 0)\}_{k \in \mathbb{N}} \) such that \( X^{(n_k)}(t, x) = Z_x^{(n_k)}(t, 0) + e^{tA} x \to Z_x(t, 0) + e^{tA} x = X(t, x) \), where the convergence is almost surely with respect to \( P \). Since \( \varphi \) is bounded and continuous then
\[
P^{(n_k)}(t)\varphi(x) = \mathbb{E}\left[ \varphi(X^{(n_k)}(t, x)) \right] = \mathbb{E}\left[ \varphi(Z_x^{(n_k)}(t, 0) + e^{tA} x) \right] 
= \mathbb{E}\left[ \varphi(Z_x(0, 0) + e^{tA} x) \right] = \mathbb{E}\left[ \varphi(X(t, x)) \right] = P(t)\varphi(x).
\]

So for every \( x \in \mathcal{X} \), \( h \in H_\alpha \) and \( t \in (0, T] \),
\[
|P(t)\varphi(x + h) - P(t)\varphi(x)| \leq \frac{c_{L,F,\alpha} T}{\sqrt{t}} \|\varphi\|_\infty \|h\|_\alpha.
\]

By the first step we conclude the proof. \( \square \)

**Remark 25.** We stress that the \( H_\alpha \)-Lipschitzianity of \( F \) in Hypotheses 3 can be replaced by a weaker condition: for every \( x \in \mathcal{X} \) there exists \( L_{F,\alpha}(x) > 0 \) such that for every \( x \in \mathcal{X} \) and \( h \in H_\alpha \)
\[
\|F(x + h) - F(x)\|_\alpha \leq L_{F,\alpha}(x)\|h\|_\alpha.
\]

Clearly, with this condition, whenever the constant \( L_{F,\alpha} \) appears in the paper it has to be replaced with \( L_{F,\alpha}(x) \). So the semigroup \( P(t) \) does not map \( B_b(\mathcal{X}) \) in \( \text{Lip}_b H_\alpha(\mathcal{X}) \), but for every \( \varphi \in B_b(\mathcal{X}) \), we have that for every \( t \in (0, T] \), \( x \in \mathcal{X} \) and \( h \in H_\alpha \),
\[
|P(t)\varphi(x + h) - P(t)\varphi(x)| \leq \frac{c_{L,F,\alpha (x)} T}{\sqrt{t}} \|\varphi\|_\infty \|h\|_\alpha.
\]
4. Proof of Theorem 5

This section is devoted to the proof of Theorem 5. First of all we stress that $F$ is Lipschitz continuos, since $Q^{-\alpha}F$ is Lipschitz continuos. Indeed, let $x, y \in X$, we have

$$\|F(x) - F(y)\| = \|Q^\alpha Q^{-\alpha}(F(x) - F(y))\|$$

$$\leq \|Q^\alpha\|_{L(X)} \|Q^{-\alpha}F(x) - Q^{-\alpha}F(y)\| \leq \|Q^\alpha\|_{L(X)} K_{F,\alpha} \|x - y\|,$$

where $K_{F,\alpha}$ is the Lipschitz constant of $Q^{-\alpha}F$. We set $L_{F,\alpha} := \|Q^\alpha\|_{L(X)} K_{F,\alpha}$. We can, and do, assume $Q^{-\alpha}F \in C_0^1(X;X)$, the general case follows by standard approximation arguments as in the proof of Theorem 4.

We will show some preliminary results which will be useful. By Theorem 8, the stochastic partial differential equation (4.1) has a unique mild solution $X(t,x)$ for every $p \geq 2$, and the process $Y(t,k) = D^G X(t,x)k$ is the unique mild solution of

$$\begin{cases}
    dY(t,k) = (AY(t,k) + DF(X(t,x))Y(t,k))dt, & t \in (0,T);
    
    Y(0,k) = k \in X.
\end{cases}$$

In the same way as in the proof of Theorem 11 using the contraction mapping theorem in the space $X^2([0,T])$ and the Gronwall inequality, we obtain that $D^G X(t,x)k$ belongs to $X^2([0,T])$ and for every $t \in [0,T]$ and $x,k \in X$,

$$\|D^G X(t,x)k\| \leq e^{\int_0^t \|DF(X(t,s))\|_{L(X)} ds} \leq e^{L_{F,\alpha} T} \|k\|.$$  \hspace{1cm} (4.2)

Now let us prove some results that will be useful in case $\alpha \in [0,1/2)$.

**Lemma 26.** Assume Hypotheses 4 hold true and let $F : X \to X$ be such that $F(X) \subseteq H_\alpha$, $Q^{-\alpha}F$ is Lipschitz continuos and $F \in C_0^1(X;X)$. If $X(t,x)$ is the mild solution of (4.1), then the following hold true:

(a) if $\alpha \in [0,1/2)$, then $D^G X(s,x)k$ belongs to $H_\alpha$;
(b) for $\alpha \in [0,1/2)$, $k \in X$ and $t \in [0,T]$ it holds

$$\int_0^t \|Q^{-\alpha} e^{-\tau} A D^G X(r,x)\|_{L(X)}^2 dr \leq T L_{F,\alpha}^2 e^{2L_{F,\alpha} T} \|k\|^2.$$  \hspace{1cm} (4.3)

**Proof.** We remark that by Theorem 8 the process $\{D^G X(t,x)k\}_{t \in [0,T]}$ is well defined and it is the mild solution of (4.1). This means that

$$D^G X(t,x)k = e^{tA}k + \int_0^t e^{(t-s)A} D^G X(s,x)(\mathcal{D}^G X(t,x)) ds.$$  \hspace{1cm} (4.4)

We start by proving (a). By 37 Proposition 2.1.1, if $\alpha \in [0,1/2)$ then for any $t > 0$ and $\beta \geq 0$

$$e^{tA}(X) \subseteq Q^\beta(X).$$

By (4.4), $D^G X(s,x)k$ belongs to $H_\alpha$, for every $s \in (0,T]$ and $x,k \in X$. Observe that (4.3) is a consequence of the Lipschitzianity of $Q^{-\alpha}F$ and (4.2).

The main idea behind the proof of Theorem 5 is to obtain an estimate for $\|DP(x)\|$, for $\alpha \in [0,1/2)$, and for $\|DP(x)\|$, for $\alpha = 1/2$, independent of $x$. When such an estimate is found, then we proceed as in the proof of Theorem 4. When $\alpha \in [0,1/2)$ there is a difference between the case $\alpha \in [0,1/4)$ and the case $\alpha \in [1/4,1/2)$. In the first case we can use (4.1) which allows us to get a sharper gradient estimate. Instead, for the second case, we are forced to use other results. We now split the proof of Theorem 5 in three cases.

For $\alpha \in [0,1/4)$, as we have mentioned in Section 8, we present a simpler proof than the one in 5, 37, that exploits the identity $A = -(1/2)Q^{2\alpha-1}$ and the analyticity of the semigroup $e^{tA}$. 

of Theorem 5 for $\alpha \in [0, 1/4)$. First of all we prove a preliminary result. Recalling that $Q^{-\alpha} = A^{\alpha/((1-2\alpha))}$, then by [37] Formula (2.1.2) for $\alpha \in [0, 1/4)$, there exists $C_\alpha > 0$ such that for any $k \in X$ and $t \in [0, T]$ it holds

$$
\int_0^t \|Q^{-\alpha}e^{sA}k\|^2 ds \leq C_\alpha T^{(1-4\alpha)/(1-2\alpha)} \frac{2\alpha-1}{4\alpha-1} ||k||^2.
$$

(4.5)

Now we proceed in the same way as in the proof of Proposition 23. By Lemma 26(a),

$$
\int_0^t \langle Q^{-\alpha}D^G X(s,x)k, dW(s) \rangle
$$

(4.6)

is well defined. By (4.5)-(4.3) we have

$$
\|Q^{-\alpha}D^G X(s,x)k\|^2 \leq \frac{1}{L^2} \|\phi\|_{L^2}^2 \left[ \int_0^t \|Q^{-\alpha}D^G X(s,x)k\|^2 ds \right],
$$

and so, by [22] Remark 2, (4.6) is a martingale. Multiplying both sides of (4.4) by (4.6), and using the same arguments used in the proof of Theorem 5 we obtain, for any $k \in H_\alpha$,

$$
\langle DP(t)\phi(x), k \rangle = \frac{1}{t} E \left[ \phi(X(t,x)) \int_0^t \langle Q^{-\alpha}D^G X(s,x)k, dW(s) \rangle \right],
$$

and by the Itô isometry

$$
\|DP(t)\phi(x)\|^2 \leq \frac{1}{tL^2} \|\phi\|_{L^2}^2 \left[ \int_0^t \|Q^{-\alpha}D^G X(s,x)k\|^2 ds \right],
$$

and so by (4.5)-(4.3)

$$
\|DP(t)\phi(x)\| \leq C \sqrt{\|\phi\|_{L^2}},
$$

(4.7)

where $C^2 = 2\max \left\{ C_\alpha T^{(1-4\alpha)/(1-2\alpha)} \frac{2\alpha-1}{4\alpha-1}, TL^2 \sigma^2 e^{2L_0T} \right\}$. With the aid of (4.7) we conclude using arguments similar to those of the proof of Theorem 1.

If $\alpha \in [1/4, 1/2)$, then (4.5) is not verified, so we have to obtain an analogous of (4.7) in another way. However, we cannot get this estimate from (2.4). So we need the same results used in [5, 31]. We will give just give a hint of the proof.

of Theorem 3 for $\alpha \in [1/4, 1/2)$. In view of Hypotheses 1 and Lemma 26(b), by [5] Proposition 6 and the chain rule, we have

$$
\langle DP(t)\phi, k \rangle = E \left[ \langle D\phi(X(t,x)), D^G X(t,x)k \rangle \right]
$$

$$
= E \left[ \phi(X(t,x)) \left\langle \int_0^t e^{(t-s)A}Q_\alpha^\frac{1}{2}dW(s), Q^{-2\alpha}D^G \left( X(t,x)k \right) \right\rangle \right]
$$

$$
- E \left[ \phi(X(t,x)) \int_0^t \left\langle Df(X(s,x))e^{(t-s)A}D^G X(t,x)k, dW(s) \right\rangle \right],
$$

(4.8)

where $Q_1 = Q(Id - e^{2tA})$. We remark that $e^{tA}(X) \subseteq Q_1^{1/2}(X) \subseteq Q_1^{-\alpha}(X)$ for every $t \in (0, T]$. Recalling that by Hypothesis [11] $\int_0^T \text{Tr}[e^{2(t-s)A}Q_1^{2\alpha}] ds < +\infty$, by [12], Lemma 26(b) and (4.8) we get that there exists $C(t, F) > 0$ such that for every $x \in X$

$$
\|DP(t)\phi(x)\| \leq C(t, F) \|\phi\|_{L^2}.
$$

(4.9)

Using (4.9) we conclude the proof in a same way as in the case $\alpha \in [0, 1/4]$.

Remark 27. We note that in the case $\alpha \in [1/4, 1/2)$ it is not possible to obtain an explicit estimate as the one in (3.15). Indeed even in [3] Theorem 8 the dependence on $t$ of the constant $C(t, F)$ is implicit.

We just need to show Theorem 5 in the case $\alpha = 1/2$. 

□
of Theorem [3] for $\alpha = 1/2$. In the same way as in the proof of Proposition [23] multiplying both sides of (4.4) by
\[
\int_0^t \langle D^G X(s, x), Q^{1/2} dW(s) \rangle,
\]
we obtain, for any $h \in H_\alpha$,
\[
\langle Q^{1/2} D_P(t) \varphi(x), h \rangle = \frac{1}{t} E \left[ \varphi(X(t, x)) \int_0^t \langle Q^{1/2} D^G X(s, x), h, dW(s) \rangle \right],
\]
and
\[
|\langle Q^{1/2} D_P(t) \varphi(x), h \rangle|^2 \leq \frac{1}{t^2} \|\varphi\|_\infty^2 E \left[ \int_0^t \|Q^{1/2} D^G X(s, x)h\|^2 ds \right] \leq \frac{1}{t^2} \|\varphi\|_\infty^2 \|Q^{1/2}\|_{L(X)}^2 E \left[ \int_0^t \|D^G X(s, x)h\|^2 ds \right].
\]
By Proposition [17] we have
\[
\|Q^{1/2} D_P(t) \varphi(x)\| = \|QDP(t) \varphi(x)\|_{1/2} = \|D_{1/2}P(t) \varphi(x)\|_{1/2} \quad (4.10)
\]
and so by (4.9)-(4.10) we obtain
\[
\|D_{1/2}P(t) \varphi(x)\|_{1/2} \leq \frac{e^{L \varphi T}}{\sqrt{t}} \|Q^{1/2}\|_{L(X)} \|\varphi\|_\infty. \quad (4.11)
\]
With the aid of (4.11) we conclude using arguments similar to those of the proof of Theorem [4].

5. Comparisons with some results in the literature

In this section we will relate the results of this paper to those already known in the literature. We stress that the commutation between $A$ and $Q$ helps us to simplify some calculations appearing in this paper.

5.1. Comparisons with [5, 31]. In [5] and [31] the transition semigroup $P(t)$ of the stochastic partial differential equation
\[
\begin{align*}
\frac{dX(t, x)}{dt} &= (AX(t, x) + RF(X(t, x)))dt + RdW(t), \quad t \in (0, T]; \\
X(0, x) &= x \in X,
\end{align*}
\]
(5.1)
is studied and it is shown that
\[
P(t) (B_\varnothing(X)) \subseteq \text{Lip}_\varnothing(X), \quad \forall t \in (0, T] \quad (5.2)
\]
under the following hypotheses.

Hypotheses 28.
(i) $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup $e^{tA}$;
(ii) $R$ is a linear and continuous operator on $X$, and
\[
Q_t x := \int_0^t e^{sA}RR^*e^{sA^*}x ds,
\]
are trace class operator, for any $t \in [0, T]$;
(iii) for every $t \in [0, T]$, the semigroup $e^{tA}$ is a Hilbert–Schmidt operator and there exists $k > 0$ such that
\[
\int_0^t s^{-k} \text{Tr}[e^{sA}RR^*e^{sA^*}]ds < +\infty;
\]
(iv) for $t \in [0, T]$ the range of $e^{tA}$ is contained in the range of $Q^{1/2}$;
(v) $F : X \to X$ is Fréchet differentiable function with bounded gradient.
To prove [52], in [5] and [31] (and in many other papers, see for example [9] [42] [39]) the authors use the Girsanov theorem to make a change of variable in order to exploit the regularity results of the transition semigroup $T(t)$ associated to (5.1) with $F = 0$. Indeed we recall that, for any $t > 0$, we have
\[ T(t)(B_b(X)) \subseteq \text{Lip}_b(X). \] (5.3)

Hypothesis [28][51] is needed to guarantee [33][40] (see, for example [5] Section 8.3.1, [19] Section 10.3) and [32]. Clearly the hypotheses on $F$ of Theorem [4] are significantly different from Hypothesis [25][39] and consequently also the results on the transition semigroup $P(t)$ that are obtained are different. Instead, for $\alpha \in [0,1/2)$ the hypotheses of Theorem [5] are covered by Hypotheses [28]. Indeed it is enough to set
\[ R = Q^\alpha, \quad A = -\frac{1}{2} Q^{2\alpha - 1}. \]
and to recall that, by [37] Proposition 2.1.1(i), for any $\beta \geq 0$ and $t > 0$, we have
\[ e^{tA}(X) \subseteq Q^{\beta(1-2\alpha)}(X). \]

Of course, in this paper the relation $A = -(1/2)Q^{2\alpha - 1}$ simplifies the calculus. However, our approach is different since we do not use the Girsanov theorem. Finally, as we just said in Section [11] the case $\alpha = 1/2$ is not covered by the Hypotheses [28] In particular Hypothesis [28][51] is not verified, since $A = -(1/2)\text{Id}$. This lack of regularity is not somethig related to the function $F$. Indeed, it is known that the transition semigroup $M(t)$ associated to
\[ \begin{align*}
    dY(t,x) &= -\frac{1}{2}Y(t,x)dt + Q^{1/2}dW(t), \quad t \in (0,T]; \\
    Y(0,x) &= x \in X,
\end{align*} \]
is an Ornstein–Uhlenbeck semigroup defined by the Mehler formula
\[ M(t)f(x) := \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y)\gamma(dy), \quad x \in X, f \in B_b(X); \]
where $\gamma$ is the Gaussian measure on $X$ with mean zero and covariance operator $Q$ and it regularizes only along $Q^{1/2}(X)$ (see for example [10] Proposition 2.3]). Hence with $\alpha = 1/2$ we cannot hope to achieve a result similar to (6.2).

5.2. Comparisons with [28][29][38][39]. In [28][29][38][39] the authors work in a very general setting: $Q$ and $A$ are not linked by any relationship and $X$ is a separable Banach space with a Schauder basis. They define the following differential operator.

**Definition 29.** Let $f : X \to \mathbb{R}$ be a continuous function, the $Q^\alpha$-directional derivative $\nabla^{Q^\alpha} f(x;y)$ at a point $x \in X$ in the direction $y \in X$ is defined as:
\[ \nabla^{Q^\alpha} f(x;y) := \lim_{s \to 0} \frac{f(x + sQ^\alpha y) - f(x)}{s}, \]
provided that the limit exists and the map $y \mapsto \nabla^{Q^\alpha} f(x;y)$ belongs to $X^\ast$.

Furthermore the authors of [28][29][38][39] assume the following.

**Hypotheses 30.** Let $f : X \to \mathbb{R}$ be a continuous function such that $A + f - \eta$ is dissipative on $X$ for some $\eta \in \mathbb{R}$ and there exists $k \geq 0$ such that $\|f(x)\| \leq c(1 + \|x\|^k)$ for some positive constant $c$. Moreover assume that $f(x) \in Q^\alpha(X)$ and let $F(x) = Q^{-\alpha}f(x)$. Finally assume that $F : X \to X$ is a continuous and Gateaux differentiable function with continuous directional derivatives, and there exists $j \geq 0$ such that for every $x, y \in X$
\[ \|F(x)\| \leq c(1 + \|x\|^j), \quad \|(D^j F(x))y\| \leq c(1 + \|x\|^j)\|y\|; \]
for some positive constant $c$. 

Using Hypotheses [28, 29] (and hypotheses on \( Q \) similar to Hypotheses 28, 29, 35, 39) prove that, for every \( \varphi \in B_b(X) \), the function \( P(t)\varphi \) admits \( Q^\alpha \)-directional derivatives in every direction \( y \in X \). We stress that if \( f \) is differentiable along \( H_\alpha \), then its \( Q^\alpha \)-directional derivatives exists and

\[
\nabla^Q f(x; y) = \langle Q^{-\alpha} D_\alpha f(x), y \rangle.
\]

Instead if \( f \) admits \( Q^\alpha \)-directional derivatives, it may be not differentiable along \( H_\alpha \). We remark that the derivatives operator defined in Definition 16 is a sort of Fréchet derivative along \( H_\alpha \), while (29) are Gateaux derivatives along the direction of \( H_\alpha \). Finally we stress that in this paper we obtain a Lipschitzianity result (see Theorem 14 and Theorem 3), instead, in [28, 29, 35, 39], the authors cannot achieve a similar result, with Definition 29.

6. Examples

In this section we will give some examples to which the results of this paper can be applied.

6.1. An example for \( \alpha \in [0, 1/2) \). Let \( \alpha \in [0, 1/2) \) and let \( \Pi \in \mathcal{L}(H_\alpha) \). Let \( \beta \geq \alpha \) and consider the map \( F : X \to X \) defined as

\[
F(x) := \begin{cases} 
\Pi(Q^\alpha x), & x \in H_\alpha; \\
0, & x \in X \setminus H_\alpha.
\end{cases}
\]  

(6.1)

We claim that \( F \) satisfies Hypotheses 3. Indeed, since \( F|_{H_\alpha} \) is continuous, then recalling Proposition 11(b) we obtain that \( F \) is Borel measurable. If \( x, h \in H_\alpha \) then

\[
\|F(x + h) - F(x)\|_\alpha = \|\Pi(Q^\alpha x + Q^\alpha h) - \Pi(Q^\alpha x)\|_\alpha \leq \|\Pi\|_{\mathcal{L}(H_\alpha)}\|Q^\alpha\|_{\mathcal{L}(H_\alpha)}\|h\|_\alpha.
\]

While if \( x \in X \setminus H_\alpha \) and \( h \in H_\alpha \), recalling that \( x + h \in X \setminus H_\alpha \), we get

\[
\|F(x + h) - F(x)\|_\alpha = 0.
\]

So \( F \) is \( H_\alpha \)-Lipschitz. Now if \( t \in (0, T] \), \( x \in X \) and \( h \in H_\alpha \) since \( \alpha \in [0, 1/2) \) we know that \( e^{tA}x \) belongs to \( H_\alpha \) and by Proposition 11(b) so

\[
\|F(e^{tA}x + h)\|_\alpha^2 = \|\Pi(Q^\alpha(e^{tA}x) + Q^\alpha h)\|_\alpha^2
\]

\[
= \|\Pi(e^{tA}(Q^\alpha x) + Q^\alpha h)\|_\alpha^2
\]

\[
\leq \|\Pi\|_{\mathcal{L}(H_\alpha)}\|e^{tA}(Q^\alpha x) + Q^\alpha h\|_\alpha^2
\]

\[
\leq 2\|\Pi\|_{\mathcal{L}(H_\alpha)}\|e^{tA}(Q^\alpha x)\|_\alpha^2 + 2\|\Pi\|_{\mathcal{L}(H_\alpha)}\|Q^\alpha h\|_\alpha^2
\]

\[
\leq 2\|\Pi\|_{\mathcal{L}(H_\alpha)}\|Q^\alpha x\|_\alpha^2 + 2\|\Pi\|_{\mathcal{L}(H_\alpha)}\|Q^\alpha h\|_\alpha^2
\]

\[
\leq 2\|\Pi\|_{\mathcal{L}(H_\alpha)}\max\{\|Q^\alpha x\|_\alpha^2, \|Q^\alpha h\|_\alpha^2\}(1 + \|h\|_\alpha^2).
\]

This concludes the proof of our claim. So if we assume Hypotheses 1 then for every \( x \in X \), by Theorem 13 the stochastic partial differential equation

\[
\begin{align*}
&dX(t, x) = (AX(t, x) + F(X(t, x)))dt + Q^\alpha dW(t), \quad t \in (0, T]; \\
&X(0, x) = x \in X,
\end{align*}
\]

has a unique mild solution \( X(t, x) \) in \( \mathcal{C}^2([0, T]) \). In particular, applying Theorem 3, the transition semigroup \( P(t)\varphi(x) = \mathbb{E}[\varphi(X(t, x))] \), defined for \( \varphi \in B_b(X) \), maps the space \( B_b(X) \) in the space \( \text{Lip}_{b,H_\alpha}(X) \) for every \( t \in (0, T] \). We stress that the function \( F \) defined in (6.1), is not continuous on \( X \) so the classical theory of stochastic partial differential equation cannot be used.
6.2. An example for $\alpha = 1/2$. Consider the space $X = L^2([0, 1], d\xi)$ where $d\xi$ denotes the Lebesgue measure on $[0, 1]$ and let $Q : L^2([0, 1], d\xi) \to L^2([0, 1], d\xi)$ be the positive and self-adjoint operator defined as

$$Qf(\xi) = \int_{0}^{1} \max \{\xi, \eta\} f(\eta) d\eta.$$  

We emphasize that we have assumed as $Q$ the covariance operator of the Wiener measure on $L^2([0, 1], d\xi)$, but we could consider any $Q$ such that $Q(L^2([0, 1], d\xi)) \subseteq W^{1,2}_0([0, 1], d\xi)$, where $W^{1,2}_0([0, 1], d\xi)$ is the set of the real-valued functions $f$ defined on $[0, 1]$ such that $f$ is absolutely continuous, $f' \in L^2([0, 1], d\xi)$ and $f(0) = 0$. If $\alpha = 1/2$, it is known that Hypotheses\[11\] hold true and $H_{1/2}$ is the space $W^{1,2}_0([0, 1], d\xi)$. Moreover the norm $\|\cdot\|_{1/2}$ is equivalent to the norm $\|f\|_{W^{1,2}_0([0, 1], d\xi)} := \|f'\|_{L^2([0, 1], d\xi)}$.

For all these results see [13, Remark 2.3.13 and Lemma 2.3.14].

Let $T > 0$ and let $g : [0, 1] \to [0, 1]$ be a non-decreasing and Lipschitz continuous function with Lipschitz constant $L_g$. Consider the stochastic partial differential equation

$$\begin{cases} 
    dX(t, f) = \left(-\frac{1}{2}X(t, f) + F(X(t, f))\right) dt + Q^{1/2}dW(t), & t \in (0, T]; \\
    X(0, f) = f \in L^2([0, 1], d\xi),
\end{cases}$$

where $F : L^2([0, 1], d\xi) \to L^2([0, 1], d\xi)$ is defined as

$$F(f) := \begin{cases} 
    f \circ g - f(g(0)), & f \in W^{1,2}_0([0, 1], d\xi); \\
    0, & \text{otherwise.}
\end{cases}$$

We claim that $F$ satisfies Hypotheses\[11\]. Indeed by [13] Proposition 129 the function $f \circ g - f(g(0))$ is absolutely continuous and it maps zero to itself. Moreover

$$\|f \circ g - f(g(0))\|^2_{W^{1,2}_0([0, 1], d\xi)} = \int_{0}^{1} |(f \circ g)'(\xi)|^2 d\xi = \int_{0}^{1} |f'(g(\xi))g'(\xi)|^2 d\xi \leq L_g \int_{0}^{1} |f'(g(\xi))| |g'(\xi)| d\xi = L_g \int_{0}^{1} |f'(\eta)|^2 d\eta < +\infty.$$ 

So $F(L^2([0, 1], d\xi))$ is contained in $W^{1,2}_0([0, 1], d\xi)$. If $f, h \in W^{1,2}_0([0, 1], d\xi)$ we get

$$\|F(f + h) - F(f)\|^2_{W^{1,2}_0([0, 1], d\xi)} = \int_{0}^{1} |((f + h) \circ g)'(\xi) - (f \circ g)'(\xi)|^2 d\xi = \int_{0}^{1} |f'(g(\xi))g'(\xi) + h'(g(\xi))g'(\xi) - f'(g(\xi))g'(\xi)|^2 d\xi = \int_{0}^{1} |h'(g(\xi))g'(\xi)|^2 d\xi \leq L_g \int_{0}^{1} |h'(g(\xi))|^2 |g'(\xi)| d\xi = L_g \int_{0}^{1} |h'(\eta)|^2 d\eta = L_g \|h\|^2_{W^{1,2}_0([0, 1], d\xi)}.$$ 

While if $f \in L^2([0, 1], d\xi) \setminus W^{1,2}_0([0, 1], d\xi)$ and $h \in W^{1,2}_0([0, 1], d\xi)$ we get

$$\|F(f + h) - F(f)\|^2_{W^{1,2}_0([0, 1], d\xi)} = 0.$$
Then, $F$ is $W^{1,2}_0([0,1],d\xi)$-Lipschitz. Now let $t \in [0,T]$, $f \in L^2([0,1],d\xi)$ and $h \in W^{1,2}_0([0,1],d\xi)$. If $f$ belongs to $L^2([0,1],d\xi) \setminus W^{1,2}_0([0,1],d\xi)$ then
\[
\|F(e^{-t/2}f + h)\|_{W^{1,2}_0([0,1],d\xi)}^2 = 0,
\]
while if $f \in W^{1,2}_0([0,1],d\xi)$ then
\[
\|F(e^{-t/2}f + h)\|_{W^{1,2}_0([0,1],d\xi)}^2 = \int_0^1 |(e^{-t/2}f + h)\circ g)'(\xi)|^2 d\xi \\
= \int_0^1 |(e^{-t/2}f'(g(\xi)) + h'(g(\xi)))g'(\xi)|^2 d\xi \\
\leq L_g \int_0^1 |(e^{-t/2}f'(g(\xi)) + h'(g(\xi)))|^2 |g'(\xi)| d\xi \\
= L_g \int_0^1 |e^{-t/2}f'(\eta) + h'(\eta)|^2 d\eta \\
\leq 2L_g \int_0^1 |e^{-t/2}f'(\eta)|^2 d\eta + 2L_g \int_0^1 |h'(\eta)|^2 d\eta \\
\leq 2L_g \|f\|_{W^{1,2}_0([0,1],d\xi)}^2 + 2L_g \|h\|_{W^{1,2}_0([0,1],d\xi)}^2 \\
\leq 2L_g \max\left\{1, \|f\|_{W^{1,2}_0([0,1],d\xi)}^2 \right\} \left(1 + \|h\|_{W^{1,2}_0([0,1],d\xi)}^2 \right).
\]
Finally using the same arguments as in Section 6.1, we obtain that $F$ is Borel measurable. So $F$ satisfies Hypotheses 3.

For every $x \in \mathbb{X}$ and $T > 0$, by Theorem 13, the stochastic partial differential equation (6.2) has a unique mild solution $X(t,x)$ in $X^2([0,T])$. In particular, applying Theorem 4, the transition semigroup $P(t)\varphi(x) = \mathbb{E}[\varphi(X(t,x))]$, defined for $\varphi \in B_0(L^2([0,1],d\xi))$, maps the space $B_0(L^2([0,1],d\xi))$ in the space $\text{Lip}_{b,W^{1,2}_0([0,1],d\xi)}(L^2([0,1],d\xi))$ for every $t \in (0,T]$. We stress that the function $F$ defined in (6.1) is not continuous on $\mathbb{X}$ so the classical theory of stochastic partial differential equations cannot be used. Furthermore the results of [9] and [31] cannot be used since $\alpha = 1/2$ as we already remarked in Section 5.

6.3. An example for Remark 25. We consider the same setting of Section 6.2. Let $Y$ be the set of absolutely continuous functions $f : [0,1] \rightarrow \mathbb{R}$ such that $f'$ is bounded and $f(0) = 0$. Let $T > 0$ and let $g : [0,1] \rightarrow \mathbb{R}$ be a Lipschitz continuous function with Lipschitz continuous derivative. We denote by $L_g$ and $L_{g'}$ the Lipschitz constants of $g$ and $g'$, respectively. Consider the stochastic partial differential equation
\[
\begin{aligned}
&dX(t,f) = \left(-\frac{1}{2}X(t,f) + F(X(t,f))\right)dt + Q^{1/2}dW(t), \quad t \in (0,T]; \\
&X(0,f) = f \in L^2([0,1],d\xi),
\end{aligned}
\tag{6.3}
\]
where $F : L^2([0,1],d\xi) \rightarrow L^2([0,1],d\xi)$ is defined as
\[
F(f) := \begin{cases} 
\quad g \circ f - g(f(0)), & f \in Y; \\
\quad 0, & \text{otherwise}.
\end{cases}
\]
We claim that $F$ satisfies the conditions of Remark 25. Indeed by [45] Exercise 17 of Section 5.4 the function $g \circ f - g(f(0))$ is absolutely continuous and it maps zero to itself. Moreover
\[
\|g \circ f - g(f(0))\|_{W^{1,2}_0([0,1],d\xi)}^2 = \int_0^1 |(g \circ f)'(\xi)|^2 d\xi \\
= \int_0^1 |g'(f(\xi))f'(\xi)|^2 d\xi \\
\leq L_g^2 \int_0^1 |f'(\eta)|^2 d\eta = L_g^2 \|f\|_{W^{1,2}_0([0,1],d\xi)}^2.
\]
So $F(L^2([0,1],d\xi))$ is contained in $W_0^{1,2}([0,1],d\xi)$. If $f \in Y$ and $h \in W_0^{1,2}([0,1],d\xi)$ we get

$$
\|F(f + h) - F(f)\|_{W_0^{1,2}([0,1],d\xi)}^2 = \int_0^1 |(g \circ (f + h))(\xi) - (g \circ f)(\xi)|^2 d\xi
$$

\[
= \int_0^1 |g'(f(\xi) + h(\xi))f'(\xi) + h'(\xi)) - g'(f(\xi))f'(\xi)|^2 d\xi
\]

\[
\leq 2 \int_0^1 |g'(f(\xi) + h(\xi))f'(\xi) + h'(\xi)) - g'(f(\xi))f'(\xi)|^2 d\xi
\]

\[
+ 2 \int_0^1 |g'(f(\xi) + h(\xi)) - g'(f(\xi))|f'(\xi)|^2 d\xi
\]

\[
\leq 2L_2^2 \int_0^1 |h'(\xi)|^2 d\xi + 2L_2^2 \int_0^1 |h(\xi)f'(\xi)|^2 d\xi
\]

\[
\leq 2L_2^2 \|h\|^2_{W_0^{1,2}([0,1],d\xi)} + 2L_2^2 \|f'\|^2_{L^2} \int_0^1 |h(\xi)|^2 d\xi
\]

\[
\leq 2L_2^2 \|h\|^2_{W_0^{1,2}([0,1],d\xi)} + 2L_2^2 \|f'\|^2_{L^2} \int_0^1 \left| \int_0^\xi h'(\eta)d\eta \right|^2 d\xi
\]

\[
\leq 2L_2^2 \|h\|^2_{W_0^{1,2}([0,1],d\xi)} + 2L_2^2 \|f'\|^2_{L^2} \int_0^1 \int_0^\xi |h'(\eta)|^2 d\eta d\xi
\]

\[
\leq 2L_2^2 \|h\|^2_{W_0^{1,2}([0,1],d\xi)} + 2L_2^2 \|f'\|^2_{L^2} \|h\|^2_{W_0^{1,2}([0,1],d\xi)}
\]

\[
\leq 2 \max \left\{ L_2^2, L_2^2 \|f'\|^2_{L^2} \right\} \|h\|^2_{W_0^{1,2}([0,1],d\xi)}
\]

While if $f \in L^2([0,1],d\xi) \setminus Y$ and $h \in W_0^{1,2}([0,1],d\xi)$ we get

$$
\|F(f + h) - F(f)\|_{W_0^{1,2}([0,1],d\xi)} = 0.
$$

These imply that $F$ is $W_0^{1,2}([0,1],d\xi)$-Lipschitz. Now let $t \in [0,T]$, $f \in L^2([0,1],d\xi)$ and $h \in W_0^{1,2}([0,1],d\xi)$. If $f$ belongs to $L^2([0,1],d\xi) \setminus W_0^{1,2}([0,1],d\xi)$ then

$$
\|F(e^{-t/2}f + h)\|_{W_0^{1,2}([0,1],d\xi)}^2 = 0.
$$

While if $f \in W_0^{1,2}([0,1],d\xi)$ then

$$
\|F(e^{-t/2}f + h)\|_{W_0^{1,2}([0,1],d\xi)}^2 = \int_0^1 |(g \circ (e^{-t/2}f + h))(\xi)|^2 d\xi
$$

\[
= \int_0^1 |g'(e^{-t/2}f(\xi) + h(\xi))(e^{-t/2}f'(\xi) + h'(\xi))|^2 d\xi
\]

\[
\leq L_2^2 \int_0^1 |e^{-t/2}f'(\eta) + h'(\eta)|^2 d\eta
\]

\[
\leq 2L_2^2 \int_0^1 |e^{-t/2}f'(\eta)|^2 d\eta + 2L_2^2 \int_0^1 |h'(\eta)|^2 d\eta
\]

\[
\leq 2L_2^2 \|f\|^2_{W_0^{1,2}([0,1],d\xi)} + 2L_2^2 \|h\|^2_{W_0^{1,2}([0,1],d\xi)}
\]

\[
\leq 2L_2^2 \max \left\{ 1, \|f\|^2_{W_0^{1,2}([0,1],d\xi)} \right\} \left( 1 + \|h\|^2_{W_0^{1,2}([0,1],d\xi)} \right)
\]

Finally using the same arguments as in Section 6.3 we obtain that $F$ is Borel measurable. So $F$ satisfies the conditions of Remark 25.

For every $x \in X$ and $T > 0$, by Theorem 11, the stochastic partial differential equation (3.3) has a unique mild solution $X(t,x)$ in $X^2([0,T])$. In particular, by Remark 25, the transition
semigroup $P(t)\varphi(f) = \mathbb{E}[\varphi(X(t, f))]$, defined for $\varphi \in B_b(L^2([0,1], d\xi))$, satisfies

$$|P(t)\varphi(f + h) - P(t)\varphi(f)| \leq e^{\sqrt{7} \max \left\{L_x, L_{\varphi}, f', L_{\varphi} \right\}} \|\varphi\|_\infty \|h\|_\alpha,$$

whenever $t \in (0,T]$, $f \in Y$ and $h \in W^{1,2}_0([0,1], d\xi)$, while if $f \in L^2([0,1], d\xi) \setminus Y$, $t \in (0,T]$ and $h \in W^{1,2}_0([0,1], d\xi)$, then

$$|P(t)\varphi(f + h) - P(t)\varphi(f)| \leq \frac{1}{\sqrt{t}} \|\varphi\|_\infty \|h\|_\alpha.$$

6.4. A gradient type perturbation. Assume that Hypotheses 1 hold true and consider the function $F : \mathcal{X} \to \mathcal{X}$ defined by

$$F(x) = Q^\alpha DU(x)$$

for some convex, Fréchet differentiable with Lipschitz continuous Fréchet derivative function $U : \mathcal{X} \to \mathbb{R}$. This type of function $F$ is pretty common in the literature (see for example [1, 2, 3, 4, 7, 11, 17, 18, 19, 30]). It is easy to see that the hypotheses of Theorem 5 are satisfied. Indeed, it is obvious that $F(\mathcal{X}) \subseteq H_\alpha$. Moreover

$$\|F(x) - F(y)\| = \|Q^\alpha DU(x) - Q^\alpha DU(y)\|$$

$$\leq \|Q^\alpha\|_{\mathcal{L}(\mathcal{X})}\|DU(x) - DU(y)\|$$

$$\leq \|Q^\alpha\|_{\mathcal{L}(\mathcal{X})}L_{DU}\|x - y\|,$$

where $L_{DU}$ is the Lipschitz constant of $DU$. So for every and $x \in \mathcal{X}$, by Theorem 8 the stochastic partial differential equation

$$\begin{cases}
  dX(t, x) = (AX(t, x) + F(X(t, x)))dt + Q^\alpha dW(t), & t \in (0,T]; \\
  X(0, x) = x \in \mathcal{X},
\end{cases}$$

has a unique mild solution $X(t, x)$ in $\mathcal{X}^2([0, T])$. In particular, applying Theorem 8 the transition semigroup $P(t)\varphi(x) = \mathbb{E}[\varphi(X(t, x))]$, defined for $\varphi \in B_b(\mathcal{X})$, maps the space $B_b(\mathcal{X})$ in the space $\text{Lip}_{b, H_\alpha}(\mathcal{X})$ for every $t \in (0,T]$. We remark that this result, when $\alpha \in [0, 1/2)$, was already proved in [31] and [35], while if $\alpha = 1/2$ it is new.

6.5. Cahn–Hilliard type equations. Cahn–Hilliard stochastic equations such as

$$du(t, x) = (\Delta^2 u(t, x) - \Delta f(u(t, x)))dt + dW(t), \quad (t, x) \in \mathbb{R}^+ \times [0, \pi]^d$$

where $d \in \mathbb{N} \setminus \{0\}$, $f$ is a polynomial of odd degree with positive leading coefficient and $u : \mathbb{R}^+ \times [0, \pi]^d \to \mathbb{R}$, where considered in [8, 12]. Here we consider an abstract generalization already studied in [13] and [29].

$$\begin{cases}
  dX(t, x) = (AX(t, x) + (-A)^{1/2}F(X(t, x)))dt + dW(t), & t \in (0,T]; \\
  X(0, x) = x \in \mathcal{X}.
\end{cases} \quad (6.4)$$

We assume that Hypotheses 1 hold with $\alpha = 0$. Let $F : \mathcal{X} \to \mathcal{X}$ be such that $F(\mathcal{X}) \subseteq H_{1/2}$, and for every $x, k \in \mathcal{X}$, there exists a constant $L_F > 0$ such that

$$\|F(x + k) - F(x)\|_{1/2} \leq L_F\|k\|.$$

Then, recalling that $A = -(1/2)Q^{-1}$, we apply Theorems 11 on the transition semigroup $P(t)$ associated to (6.4), and so $P(t)$ maps the space $B_b(\mathcal{X})$ in Lip$_b(\mathcal{X})$ for every $t \in (0,T]$. 

6.6. A classical example. Again, we consider the setting of Section 6.2. Let $F : L^2([0,1],d\xi) \rightarrow L^2([0,1],d\xi)$ defined by choosing $x_1,\ldots,x_n \in L^2([0,1],d\xi)$ and a function $f : [0,1] \times \mathbb{R}^n \rightarrow \mathbb{R}$, $(\xi,y_1,\ldots,y_n) \mapsto f(\xi,y_1,\ldots,y_n)$ and setting
\[
(F(g))(\xi) := f\left(\xi, \int_0^1 g(\eta)x_1(\eta)d\eta, \ldots, \int_0^1 g(\eta)x_n(\eta)d\eta\right).
\]
Assume that $x_1,\ldots,x_n$ are orthonormal and for every $i = 1,\ldots,n$
\[
f \frac{\partial f}{\partial \xi} \frac{\partial f}{\partial y_i} \text{ are bounded and continuous on } [0,1] \times \mathbb{R}^n;
f(0,y_1,\ldots,y_n) = 0, \text{ for every } y_1,\ldots,y_n \in \mathbb{R}.
\]
$F$ satisfies the hypotheses of Theorem 5. Indeed $F(L^2([0,1],d\xi)) \subseteq W^{1,2}_0([0,1],d\xi)$, since
\[
(F(g))(0) = f\left(0, \int_0^1 g(\eta)x_1(\eta)d\eta, \ldots, \int_0^1 g(\eta)x_n(\eta)d\eta\right) = 0
\]
and
\[
(F(g))'(\xi) = \frac{\partial f}{\partial \xi}\left(\xi, \int_0^1 g(\eta)x_1(\eta)d\eta, \ldots, \int_0^1 g(\eta)x_n(\eta)d\eta\right) \leq \left\| \frac{\partial f}{\partial \xi} \right\|_\infty.
\]
Moreover for $g_1,g_2 \in L^2([0,1],d\xi)$
\[
\|F(g_1) - F(g_2)\|_{L^2([0,1],d\xi)} = \int_0^1 \left| f\left(\xi, \int_0^1 g_1(x_1)d\eta, \ldots, \int_0^1 g_1(x_n)d\eta\right) - f\left(\xi, \int_0^1 g_2(x_1)d\eta, \ldots, \int_0^1 g_2(x_n)d\eta\right) \right| \\
\leq n^2 \sup_{i=1,\ldots,n} \left\{ \left\| \frac{\partial f}{\partial \xi} \right\|^2_\infty, \left\| \frac{\partial f}{\partial y_i} \right\|^2_\infty \right\} \sum_{i=1}^n 2 \left( \int_0^1 (g_1 - g_2)x_i d\eta \right)^2 \\
\leq n^2 \sup_{i=1,\ldots,n} \left\{ \left\| \frac{\partial f}{\partial \xi} \right\|^2_\infty, \left\| \frac{\partial f}{\partial y_i} \right\|^2_\infty \right\} \|g_1 - g_2\|^2_{L^2([0,1],d\xi)}.
\]
By Theorem 8 for every $x \in X$, the stochastic partial differential equation
\[
\begin{cases}
\underline{\text{d}X(t,x)} = \left(-\frac{1}{2}X(t,x) + F(X(t,x))\right)dt + Q^{1/2}dW(t), & t \in (0,T]; \\
X(0,x) = x \in X,
\end{cases}
\]
has a unique mild solution $X(t,x)$ in $X^2([0,T])$. By Theorem 8 the transition semigroup maps the space $B_b(X)$ in $\text{Lip}_b_{H^{1/2}}(X)$ for every $t \in (0,T]$. So we get an improvement of 31 Section 4], since there the case $\alpha = 1/2$ was not considered.

\textbf{Appendix A. Proof of Theorem 8}

The aim of this section is to look for pathwise uniqueness of for (2.3).

of Theorem 8. We first prove uniqueness. Let $X_1(t,x)$, $X_2(t,x)$ be two mild solutions of (2.3). Recall that by definition a mild solution solves
\[
X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}(\Phi(s,X(s,x)) + \int_0^s e^{(t-s)A}Q^\alpha dW(s).
\]
Hence, we have
\[
E[\|X_1(t,x) - X_2(t,x)\|^2] = E\left[\left\|\int_0^t e^{(t-s)A}(\Phi(s,X_1(s,x)) - \Phi(s,X_2(s,x))) ds\right\|^2\right] \\
\leq tE\left[\int_0^t \left\|e^{(t-s)A}(\Phi(s,X_1(s,x)) - \Phi(s,X_2(s,x)))\right\|^2 ds\right] \\
\leq tE\left[\int_0^t \left\|\Phi(s,X_1(s,x)) - \Phi(s,X_2(s,x))\right\|^2 ds\right]
\]
\[ \leq tL_\Phi^2 \int_0^t \mathbb{E} \left[ \|X_1(v, x) - X_2(v, x)\|^2 \right] ds. \]

From the Gronwall inequality and the same arguments of the proof of [20] Theorem 7.5, uniqueness follows.

The proof of existence is based on the contraction mapping theorem. We define the Volterra operator

\[ V(Y)(t) := e^{tA}x + \int_0^t e^{(t-s)A} \Phi(s, Y(s))ds + \int_0^t e^{(t-s)A}Q^\alpha dW(s), \]

in the space \( X^2([0, T]) \), and first of all we show that \( V \) maps \( X^2[0, T] \) into itself. Indeed, for any \( Y \in X^2[0, T] \) we have

\[ \|V(Y)\|_{X^2[0, T]}^2 \leq 3\|e^{tA}x\|^2_{X^2[0, T]} + 3\int_0^t \|e^{(t-s)A} \Phi(s, Y(s))\|^2_{X^2[0, T]} ds + 3\int_0^t \|e^{(t-s)A}Q^\alpha dW(s)\|^2_{X^2[0, T]}. \]  

(A.1)

We recall that \( \|e^{tA}x\|^2_{X^2[0, T]} \leq \|x\|^2 \), for \( t > 0 \). Let \( y \in X \) be such that (2.2) holds, then

\[ \left\| \int_0^t e^{(t-s)A} \Phi(s, Y(s))ds \right\|^2_{X^2[0, T]} \leq 2T \mathbb{E} \left[ \left\| \int_0^T \Phi(s, Y(s))ds \right\|^2 \right] \]

\[ \leq 2TE \left[ \int_0^T \|\Phi(s, Y(s))\|^2 ds \right] + 2TE \left[ \int_0^T \|\Phi(s, Y(s)) - \Phi(s, y)\|^2 ds \right] \]

\[ \leq 4T^2 L_\phi^2 \mathbb{E} \left[ \int_0^T \|y\|^2 ds \right] + 2TE \left[ \int_0^T \|\Phi(s, y)\|^2 ds \right] \]

Since \( Y \in X^2[0, T] \) and recalling that (2.2) holds,

\[ \left\| \int_0^t e^{(t-s)A} \Phi(s, Y(s))ds \right\|^2_{X^2[0, T]} < +\infty. \]

By [20] Theorem 4.36 and Hypothesis \( H \) the third summand in (A.1) is finite. In the same way as the proof of uniqueness, we have

\[ \|V(Y_1) - V(Y_2)\|^2_{X^2[0, T]} \leq T^2 L_\phi^2 \|Y_1 - Y_2\|^2_{X^2[0, T]}. \]

So the existence follows by the contraction mapping theorem (using similar arguments as the one used in Theorem [19] and same arguments of proof of [20] Theorem 7.5).

\[ \square \]

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