Abstract. The study of representations of affine Hecke algebras has led to a new notion of shapes and standard Young tableaux which works for the root system of any finite Coxeter group. This paper is completely independent of affine Hecke algebra theory and is purely combinatorial. We define generalized shapes and standard Young tableaux and show that these new objects coincide with the classical ones for root systems of Type A. The classical notions of conjugation of shapes, ribbon shapes, axial distances, and the row reading and column reading standard tableaux, have natural generalizations to the root system case. In the final section we give an interpretation of the shapes and standard tableaux for root systems of Type C which is in a form similar to classical theory of shapes and standard tableaux.

0. Introduction

In my recent work on representations of affine Hecke algebras [Ra1] I have been led to a generalization of standard Young tableaux. These generalized tableaux are important in the context of representation theory because the standard tableaux model the internal structure of irreducible representations of the affine Hecke algebra. In fact, most of the time the number of tableaux of a given shape is the same as the dimension of the corresponding irreducible representation of the affine Hecke algebra.

In this paper I introduce and study generalized shapes and standard tableaux purely combinatorially. The main theorem is that the generalized standard tableaux of a given shape describe the connected components of a certain graph, the calibration graph. It is this graph which is intimately connected to the structure of representations of affine Hecke algebras.

In the Type A case the generalized shapes can be converted into “placed configurations of boxes”. This conversion is nontrivial and is the subject of Section 3. In the cases where this placed configuration of boxes is a placed skew shape the generalized standard tableaux coincide with the classical standard tableaux of a skew shape. The generalized skew shapes play a major role in the results on representations of affine Hecke algebras which are obtained in [Ra1].

In Section 1 I give definitions of

(a) skew shapes,
(b) ribbon shapes,
(c) axial distances,
(d) conjugation of shapes, and
(e) row reading and column reading tableaux,
in the generalized setting. In Section 4 it is shown that these definitions yield the classical versions
of these objects in the Type A case. The last section of this paper explains how one can convert
the generalized shapes and standard tableaux for the Type C case into configurations of boxes and
fillings. In this form the shapes and standard tableaux for Type C look similar to the classical
standard Young tableaux.

It is my hope that others will also take up the study of the generalized shapes and standard
tableaux introduced in this paper. There are many more problems than there is time for solving
them and every combinatorial fact which can be proved about these objects says something about
the structure of representations of affine Hecke algebras. One hopes that everything that is known
about classical standard Young tableaux will have an analogue in this more general setting. Al-
though I have uncovered some of these generalizations, there are many facets of classical tableau
theory which still need to be generalized.

From a representation theoretic point of view, one expects that there might exist generalizations of
(a) the Robinson-Schensted-Knuth correspondence,
(b) the Littlewood-Richardson coefficients,
(c) the Kostka-Foulkes polynomials,
(d) major index and descents of tableaux,
(e) charge of tableaux,
(f) Jacobi-Trudi formulas.

Any solutions to these problems would be extremely helpful for understanding the underlying
representation theory. It is possible that some of the generalizations might be obtained simply by
understanding how to do them for the Type C tableaux given in the last section of this paper.
This approach is attractive since the form of the Type C tableaux given in the last section looks
so similar to classical tableau theory.

Remarks on the results in this paper

(1) Recently, it has become clear to me that one of the reasons that skew Schur functions play
such a important role in the classical theory of symmetric functions is because the skew shapes
describe particularly well behaved irreducible representations of the affine Hecke algebra of
type A. Since these nice representations exist in all types it seems reasonable that the many
wonderful identities involving skew Schur functions should have general type analogues in
terms of the generalized skew shapes defined in this paper. An example of a skew Schur
function identity that has a particularly nice generalization to all types is the identity in [Mac
I §5 Ex. 21b].

(2) A remark similar to (1) can be made concerning ribbon shapes. This special class of shapes
has a good generalization to all types and the representation theory associated to ribbon
shapes has special features [Ro], [Mat, 4.3.5]. This fact seems to give some philosophical
“reason” why there is such an amazing theory of ribbon Schur functions. The theory of
ribbon Schur functions has been developed in the last decade by Lascoux, Leclerc, Thibon,
Krob, Reutenauer, Malvenuto and others [GK] [La1-2]. I am sure that there is much to learn
about the representation theory of affine Hecke algebras from what is already known in the
ribbon (and noncommutative) Schur function theory.

(3) The theory of generalized shapes gives rise to some strange new shapes even in the type A
case, see Section 3.6. To my knowledge these shapes have not been studied before but they do
retain many of the combinatorial properties that skew shapes have. In particular, standard
tableaux make perfectly good sense for these shapes and these strange standard tableaux do have representation theoretic meaning.

(4) There seem to be strong connections between the combinatorics in this paper, the combinatorics of the Shi arrangements (see [St1-2], [AL], [He], [ST]) and the combinatorics of sign types developed by Shi [Sh3]. These connections need to be better understood.

(5) I do not think that there is a connection between the generalized standard Young tableaux introduced in this paper and the generalized tableaux of P. Littelmann [Li1-2]. Littelmann’s analogue of tableaux are really a generalization of column strict tableaux not of standard tableaux. Column strict tableaux give information about the representations of \( GL_n(\mathbb{C}) \) and standard tableaux give information about the representations of the symmetric group \( S_n \). There is an analogous dichotomy in the generalized case; Littelmann’s generalized column strict tableaux model the representations of complex semisimple Lie groups and my generalized standard tableaux model the representations of affine Hecke algebras. At the moment, I do not believe that there is any connection between the representation theories of the complex Lie groups and the affine Hecke algebras which would allow one to transfer information from one side to the other (except in the type A case, where one has a Schur-Weyl type duality.)

Acknowledgements

This paper is only a part of a large project [Ra1-3] [RR1-2] on representations of affine Hecke algebras which I have been working on intensely for about a year. During that time I have benefited from conversations with many people. To choose only a few, there were discussions with S. Fomin, M. Vazirani, L. Solomon, F. Knop and N. Wallach which played an important role in my progress. There were several times when I tapped into J. Stembridge’s fountain of useful knowledge about root systems. G. Benkart was a very patient listener on many occasions. R. Simion, T. Halverson, H. Barcelo, P. Deligne, and R. Macpherson all gave large amounts of time to let me tell them my story and every one of these sessions was helpful to me in solidifying my understanding. I thank C. Kriloff for her amazing proofreading.

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1. Generalized shapes and standard Young tableaux

\( (1.1) \) Notations. Let \( W \) be a finite Coxeter group and let \( R \) be the root system of \( W \). The root system \( R \) spans a real vector space which we shall denote \( \mathbb{R}^n \). There is an inner product on \( \mathbb{R}^n \) via which \( W \) is a group generated by reflections. Fix a system \( R^+ \) of positive roots in \( R \) and write \( \alpha > 0 \) if \( \alpha \in R^+ \). If \( w \in W \) the inversion set of \( w \) is

\[
R(w) = \{ \alpha > 0 \mid wa < 0 \}.
\]

Let \( \{\alpha_1, \ldots, \alpha_n\} \) be the simple roots in \( R^+ \) and let \( s_1, \ldots, s_n \) denote the corresponding simple reflections in \( W \). The positive roots determine a fundamental chamber \( C = \{ x \in \mathbb{R}^n \mid (x, \alpha) > 0 \text{ for all } \alpha \in R^+ \} \). An element \( x \in \mathbb{R}^n \) is dominant if \( x \in \overline{C} \), the closure of the chamber \( C \).
(1.2) Placed shapes and standard tableaux. If $\gamma \in \overline{C}$ define
\[ Z(\gamma) = \{ \alpha \in R^+ \mid \langle \gamma, \alpha \rangle = 0 \} \quad \text{and} \quad P(\gamma) = \{ \alpha \in R \mid \langle \gamma, \alpha \rangle = 1 \}. \]
The set $Z(\gamma) \cup -Z(\gamma)$ is a (parabolic) root subsystem of $R$ and is generated by the simple roots that it contains. The stabilizer $W_\gamma$ of $\gamma$ in $W$ is the Weyl group of the subsystem $Z(\gamma)$ and the set $P(\gamma)$ is stable under the action of $W_\gamma$.

Remark. The sets $Z(\gamma)$ and $P(\gamma)$ appear in Heckman and Opdam [HO, Definition 2.1] and implicitly in other works [K1, Theorem 2.2].

A placed shape is a pair $(\gamma, J)$ where $\gamma \in \overline{C}$ and $J \subseteq P(\gamma)$. A standard tableau of shape $(\gamma, J)$ is an element $w \in W$ such that $R(w) \cap Z = \emptyset$ and $R(w) \cap P = J$.

In Theorem 3.5 we shall show that this is a generalization of the classical notion of standard Young tableau. Let
\[ F(\gamma, J) = \{ \text{standard tableaux of shape } (\gamma, J) \}. \]

Examples
(1) If $\gamma$ is a generic element of $\mathbb{R}^n$ then $Z(\gamma) = P(\gamma) = \emptyset$. In this case the only possibility for $J$ is $J = \emptyset$ and $F(\gamma, \emptyset) = W$.

(2) Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha^\vee$, where $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$. Then $Z(\rho) = \emptyset$ and $P(\rho) = \{ \alpha_1, \ldots, \alpha_n \}$. If $J \subseteq \{ \alpha_1, \ldots, \alpha_n \}$ we have
\[ F(\rho, J) = \{ w \in W \mid D(w) = J \}, \quad \text{where} \quad D(w) = \{ \alpha_i \mid ws_i < w \} \]
is the right descent set of $w \in W$.

(1.3) A nonemptiness condition. I feel that the following conjecture should have a simple slick proof. The $\Rightarrow$ direction is easy and it may be possible to prove the other direction by using [Bou, Ch. VI §1, Prop. 22]. For Type A the conjecture is true and is a consequence of Theorem 4.5 of section 4.

Conjecture. Let $(\gamma, J)$ be a shape. The set $F(\gamma, J)$ is non-empty if and only if $J$ satisfies the condition
\[
\text{If } \beta \in J, \alpha \in Z \text{ and } \beta - \alpha \in R^+ \text{ then } \beta - \alpha \in J.
\]

(1.4) Placed skew shapes. If $\gamma \in \mathbb{R}^n$ view $\gamma$ as the function on the root system $R$ given by
\[
\gamma: \quad R \quad \rightarrow \quad \mathbb{R} \quad \alpha \quad \mapsto \quad \langle \gamma, \alpha \rangle
\]
The element $\gamma$ is regular if $\langle \gamma, \alpha \rangle \neq 0$ for all $\alpha \in R$, and is integral if $\langle \gamma, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in R$. For each subset $K \subseteq \{ \alpha_1, \ldots, \alpha_n \}$ let $R_K$ be the root system generated by $K$ and let $W_K$ be the Weyl group of $R_K$. Let $\gamma|_K$ denote the function $\gamma$ restricted to $R_K$.

A placed shape $(\gamma, J)$ is a placed skew shape if for all $w \in F(\gamma, J)$,

(a) for each simple root $\alpha_i$, $w\gamma|_{\{\alpha_i\}}$ is regular, and
(b) for each pair of simple roots $\{\alpha_i, \alpha_j\}$

either $\ w_\gamma|_{\{\alpha_i, \alpha_j\}}$ is regular

or $\ w_\gamma|_{\{\alpha_i, \alpha_j\}}$ is in the $W_{\{\alpha_i, \alpha_j\}}$-orbit of

the function $\tau$ given by $\langle \tau, \alpha_i \rangle = 1$ and $\langle \tau, \alpha_j \rangle = 0$,

where $\alpha_i$ is long and $\alpha_j$ is short.

Remark. In the type A case, a definition of skew shape similar to the one given above has also been given by Fomin [Fo] in connection with his approach to the theory of seminormal representations of the symmetric group.

(1.5) Ribbon shapes. A placed shape $(\gamma, J)$ is a placed ribbon shape if $Z(\gamma) = \emptyset$. All placed ribbon shapes are placed skew shapes.

Example. Suppose that $(\gamma, J)$, is a ribbon shape and $\gamma$ is integral. Then $Z(\gamma) = \emptyset$, $P(\gamma) \subseteq \{\alpha_1, \ldots, \alpha_n\}$ and

$$\mathcal{F}^{(\gamma, J)} = \{w \in W \mid L(w) \cap P = J\},$$

where, as in Example 2 of (1.2), $L(w)$ is the left descent set of $w$.

(1.6) Although we shall not define the affine Hecke algebra or discuss its representation theory in this paper it is important to note that the definition of placed skew shape is motivated by the following theorem of [Ra1].

Theorem. Let $R$ be the root system of a finite Weyl group and let $\tilde{H}$ be the corresponding affine Hecke algebra. There is a one-to-one correspondence between placed skew shapes $(\gamma, J)$ and irreducible really calibrated representations $\tilde{H}^{(\gamma, J)}$ of the affine Hecke algebra $\tilde{H}$. Under this correspondence

$$\dim(\tilde{H}^{(\gamma, J)}) = (\# \text{ of standard tableaux of shape } (\gamma, J)).$$

(1.7) Conjugation. Let $(\gamma, J)$ be a placed shape and let $W_\gamma$ be the stabilizer of $\gamma$ in $W$. Define the conjugate placed shape to be

$$(\gamma, J)' = (-u\gamma, -u(P(\gamma) \setminus J)),$$

where $u$ is the minimal length coset representative of $w_0W_\gamma \in W/W_\gamma$ and $w_0$ is the longest element of $W$.

It will be useful to note the following:

(a) $P(-u\gamma) = -uP(\gamma)$, (b) $Z(-u\gamma) = uZ(\gamma)$, (c) $R(u) = R^+ \setminus Z(\gamma)$.

The proofs are as follows:

(a) Since $\gamma$ is dominant, $-u\gamma = w_0\gamma$ is dominant and thus $\langle -u\gamma, -u\alpha \rangle = 1$ only if $-u\alpha > 0$. With this in mind $P(-u\gamma) = -uP(\gamma)$ follows from the equation $\langle -u\gamma, -u\alpha \rangle = 1 \iff \langle \gamma, \alpha \rangle = 1$.

(b) Let $v \in W_\gamma$ such that $w_0 = uv$. (By [Bou IV §1 Ex. 3], $v$ is unique.) Then $R^+ \supseteq -w_0Z(\gamma) = -uvZ(\gamma) = uZ(\gamma)$, and it follows that

$$Z(-u\gamma) = R^+ \cap \{\alpha \in R \mid \langle u\gamma, \alpha \rangle = 0\} = R^+ \cap (uZ(\gamma) \cup -uZ(\gamma)) = uZ(\gamma).$$
(c) Let $R^- = -R^+$ be the set of negative roots in $R$. Let $v \in W_\gamma$ such that $w_0 = uv$. Then $v$ is the longest element of $W_\gamma$ and $R(v) = Z(\gamma)$. Thus
\[
R(u) = \{ \alpha \in R \mid \alpha \in R^+, w_0\alpha \in R^- \}
= \{ \alpha \in R \mid \alpha \in R^+, \alpha \in R^+, \text{ since } w_0R^- = R^+, \}
= R^+ \setminus R(v) = R^+ \setminus Z(\gamma).
\]

(1.8) Proposition. Conjugation $(\gamma, J) \leftrightarrow (\gamma, J)'$ is a well defined involution on placed shapes.

**Proof.**

(a) The weight $-u\gamma = -uv\gamma = -w_0\gamma$ is dominant (i.e. $-u\gamma \in \mathcal{C}$) and $-u(P(\gamma) \setminus J) \subseteq P(-u\gamma)$ since $-uP(\gamma) = P(-u\gamma)$. This shows that $(\gamma, J)'$ is well defined.

(b) Write $w_0 = uv$ where $v$ is the longest element of $W_\gamma$. Similarly, write $w_0 = u'v'$ where $u'$ is the minimal length coset representative of $w_0W_{w_0\gamma}$ and $v'$ is the longest element in $W_{w_0\gamma}$. Conjugation by $w_0$ is an involution on $W$ which takes simple reflections to simple reflections and $W_{w_0\gamma} = w_0W_\gamma w_0$. It follows that $v' = w_0v_0$. This gives
\[
u'u = (w_0v')(w_0v) = w_0w_0v_0w_0v = 1.
\]

Then, using fact (1.7a) above,
\[
-u'(P(-u\gamma) \setminus (-u(P(\gamma) \setminus J))) = -u'(-uP(\gamma) \setminus (-u(P(\gamma) \setminus J))) = P(\gamma) \setminus (P(\gamma) \setminus J) = J.
\]

In (4.4) we shall show that this involution is a generalization of the classical conjugation operation on partitions.

Remark. In type $A$, the conjugation involution seems to coincide with the duality operation for representations of $p$-adic $GL(n)$ defined by Zelevinsky [Ze]. Zelevinsky’s involution has been been studied further in [MW] and [KZ] extended to general Lie type by Kato [K2] and Aubert [Au].

(1.9) Proposition. The conjugation of shapes involution extends to an involution on standard tableaux given by
\[
\mathcal{F}(\gamma, J) \xrightarrow{1-1} \mathcal{F}(\gamma, J)'
\]
where $u$ is the minimal length coset representative of the coset $w_0W_\gamma$.

**Proof.** Let $w \in \mathcal{F}(\gamma, J)$ and let $w' = wu^{-1}$. We must show that $R(w') \cap Z(-u\gamma) = \emptyset$ and $R(w') \cap P(-u\gamma) = -u(P(\gamma \setminus J))$. Let $R^- = -R^+$ be the set of negative roots. We shall use the facts in (1.7) freely.

(a) Since $R(w) \cap Z(\gamma) = \emptyset$, we get
\[
u^{-1}R(uw^{-1}) \cap Z(\gamma) = \{ \beta \in R \mid u\beta \in R(uw^{-1}), \beta \in Z(\gamma) \}
= \{ \beta \in R \mid u\beta \in R^+, wu^{-1}u\beta \in R^-, \beta \in Z(\gamma) \}
= \{ \beta \in R \mid \beta \in u^{-1}R^+, w\beta \in R^-, \beta \in Z(\gamma) \}
= \{ \beta \in R \mid \beta \in u^{-1}R^+, \beta \in R(w), \beta \in Z(\gamma) \}, \text{ since } Z(\gamma) \subseteq R^+,
= \{ \beta \in R \mid \beta \in u^{-1}R^+, \beta \in R(w) \cap Z(\gamma) \}
= \emptyset.
\]
and it follows that
\[ R(w') \cap Z(-w\gamma) = R(wu^{-1}) \cap uZ(\gamma) = u (u^{-1} R(wu^{-1}) \cap Z(\gamma)) = \emptyset. \]

(b) Assume that \( w \in W \) is such that \( R(w) \cap P(\gamma) = J \). Then
\[
-u^{-1} R(wu^{-1}) \cap P(\gamma) = \{ \beta \in R \mid -u\beta \in R(wu^{-1}), \beta \in P(\gamma) \}
= \{ \beta \in R \mid -u\beta \in R^+, -wu^{-1}u\beta \in R^- , \beta \in P(\gamma) \}
= \{ \beta \in R \mid u\beta \in R^- , w\beta \in R^+, \beta \in P(\gamma) \}
= \{ \beta \in R \mid \beta \in R(u), \beta \in R^+ \setminus R(w), \beta \in P(\gamma) \}, \quad \text{since } P(\gamma) \subseteq R^+
= \{ \beta \in R \mid \beta \in R^+ \setminus Z(\gamma), \beta \in P(\gamma) \setminus J \}, \quad \text{since } R(w) \cap P(\gamma) = J,
= P(\gamma) \setminus J, \quad \text{since } Z(\gamma) \text{ and } P(\gamma) \text{ are disjoint.}
\]

This yields
\[
R(w') \cap P(-u\gamma) = R(wu^{-1}) \cap -uP(\gamma) = -u \left( -u^{-1} R(wu^{-1}) \cap P(\gamma) \right) = -u \left( P(\gamma) \setminus J \right). \quad \square
\]

**1.10 Axial distances.** Let \((\gamma, J)\) be a placed shape and let \( w \in \mathcal{F}(\gamma,J) \) be a standard tableaux of shape \((\gamma, J)\). Let \( \alpha \in R \) be a root. The \( \alpha \)-axial distance for \( w \) is the value
\[
d_\alpha(w) = \langle w\gamma, \alpha \rangle.
\]
These numbers are crucial to the construction of irreducible representations of the affine Hecke algebra which is given in [Ra1]. In (4.1) we shall see that they are analogues of the axial distances used by A. Young [Y] in his constructions of the irreducible representations of the symmetric group.

**1.11 Row reading and column reading tableaux.** If \( w \in W \) let \( R(w) \) be the inversion set of \( w \). The **weak Bruhat order** is the partial order on \( W \) given by
\[
v \leq w \quad \text{if} \quad R(v) \subseteq R(w).
\]
This definition is not the usual definition of the weak Bruhat order but is equivalent to the usual one by [Bj, Prop. 2].

Let \((\gamma, J)\) be a placed shape. A **column reading tableau** of shape \((\gamma, J)\) is a minimal element of \( \mathcal{F}(\gamma,J) \) in the weak Bruhat order. A **row reading tableau** of shape \((\gamma, J)\) is a maximal element of \( \mathcal{F}(\gamma,J) \) in the weak Bruhat order.

A set of positive roots \( K \) is **closed** if \( \alpha, \beta \in K \), \( \alpha + \beta \in R^+ \) implies that \( \alpha + \beta \in K \). The **closure** \( K^c \) of a subset \( K \subseteq R^+ \) is the smallest closed subset of \( R^+ \) containing \( K \).

The following conjecture says that if \( \gamma \) is integral and \( \mathcal{F}(\gamma,J) \) is nonempty then \( \mathcal{F}(\gamma,J) \) contains a unique column reading tableau and a unique row reading tableau. A proof of the conjecture in the type A case is given in Theorem 4.5.

**Conjecture.** Let \((\gamma, J)\) be a placed shape such that \( \gamma \) is dominant and integral. If \( \mathcal{F}(\gamma,J) \neq \emptyset \) then
\[
R(w_{\min}) = J, \quad R(w_{\max}) = (P(\gamma) \setminus J) \cup Z(\gamma), \quad \text{and} \quad \mathcal{F}(\gamma,J) = [w_{\min}, w_{\max}],
\]
where \( K^c \) denotes the complement of \( K \) in \( R^+ \) and \([w_{\min}, w_{\max}]\) denotes the interval between \( w_{\min} \) and \( w_{\max} \) in the weak Bruhat order.

2. Calibration graphs
The following graphs arise naturally in the study of representations of affine Hecke algebras, see [Ro], [Rg, Prop. 3.5], and [Ra1]. Let $\gamma \in \mathbb{R}^n$. The *calibration graph* $\Gamma(\gamma)$ is the graph with

Vertices: $W_\gamma$

Edges: $w_\gamma \leftrightarrow s_i w_\gamma$, if $\langle w_\gamma, \alpha_i \rangle \neq \pm 1$.

(2.1) Theorem. Assume that $\gamma \in \mathbb{R}^n$ is dominant and let $\Gamma(\gamma)$ be the corresponding calibration graph. The connected components of $\Gamma(\gamma)$ are described by the sets

$F^{(\gamma,J)}$ such that $J \subseteq P(\gamma)$ and $F^{(\gamma,J)} \neq \emptyset$,

where $F^{(\gamma,J)}$ is the set of standard tableaux of shape $(\gamma, J)$ defined in (1.2).

(2.2) This theorem will become almost obvious once we change our point of view. The root system $R$ determines a (central) hyperplane arrangement

$A = \{H_\alpha \mid \alpha \in R\}$, where $H_\alpha = \{x \in \mathbb{R}^n \mid \langle x, \alpha \rangle = 0\}$.

The set of chambers (connected components) of $\mathbb{R}^n \setminus (\bigcup_\alpha H_\alpha)$ is

$C = \{wC \mid w \in W\}$, where $C = \{x \in \mathbb{R}^n \mid \langle x, \alpha \rangle > 0$ for all $\alpha > 0\}$

is the fundamental chamber. A chamber $\tilde{C} \in C$ is on the positive side (resp. negative side) of the hyperplane $H_\alpha$, $\alpha \in R^+$, if $\langle x, \alpha \rangle > 0$ (resp. $\langle x, \alpha \rangle < 0$) for all $x \in \tilde{C}$. The following Proposition allows us to view the calibration graph $\Gamma(\gamma)$ in terms of chambers in $\mathbb{R}^n$. Example 2.7 at the end of this section illustrates the conversion.

Proposition. Assume that $\gamma \in \mathbb{R}^n$ is dominant. Let $Z(\gamma)$ and $P(\gamma)$ be as defined in (1.2) and let $\Gamma(\gamma)$ be the calibration graph containing $\gamma$.

(a) The map

$W_\gamma \xrightarrow{\cong} \left\{ \tilde{C} \in C \mid \tilde{C} \text{ is on the positive side of } H_\alpha \text{ for every } \alpha \in Z(\gamma) \right\}$

$w_\gamma \leftrightarrow w^{-1} C$

is a bijection.

(b) Two vertices $w_\gamma$ and $v_\gamma$ in $\Gamma(\gamma)$ are connected by an edge if and only if the corresponding chambers $w^{-1} C$ and $v^{-1} C$ share a face and the hyperplane $H_\alpha$ containing this face satisfies $\alpha \notin P(\gamma)$.

(c) The map

$F^{(\gamma,J)} \xrightarrow{\cong} \left\{ \tilde{C} \in C \mid \tilde{C} \text{ is on the positive side of } H_\alpha \text{ for } \alpha \in Z(\gamma), \tilde{C} \text{ is on the positive side of } H_\alpha \text{ for } \alpha \in P(\gamma) \setminus J, \right.$

$\tilde{C} \text{ is on the negative side of } H_\alpha \text{ for } \alpha \in J, \left. \right\}$

$w \leftrightarrow w^{-1} C$

is a bijection.
Proof. (a) Since $\gamma$ is dominant the stabilizer $W_\gamma$ of $\gamma$ is a parabolic subgroup of $W$. It is generated by the simple reflections $s_i$ for $\alpha_i \in Z(\gamma)$. The statement will follow from the following identifications:

$$
\begin{align*}
W_\gamma &\xrightarrow{\ell^{-1}} W/W_\gamma \\
&\xrightarrow{\ell^{-1}} \{\text{minimal length coset representatives of } W/W_\gamma\} \\
&\xrightarrow{\ell^{-1}} \{wC \mid w \text{ minimal length coset representative of } W/W_\gamma\} \\
&\xrightarrow{\ell^{-1}} \{wC \mid \ell(ws_i) > \ell(w) \text{ for every } \alpha_i \in Z(\gamma)\} \\
&\xrightarrow{\ell^{-1}} \{wC \mid w^{-1}C \text{ is on the positive side of } H_\alpha \text{ for every } \alpha \in Z(\gamma)\} \\
&\xrightarrow{\ell^{-1}} \{w^{-1}C \mid w^{-1}C \text{ is on the positive side of } H_\alpha \text{ for every } \alpha \in Z(\gamma)\},
\end{align*}
$$

where $C = \{x \in E \mid \langle x, \alpha \rangle > 0 \text{ for all } \alpha > 0\}$ is the fundamental chamber. Let us give a step by step explanation.

1. There is a natural bijection between $W_\gamma$ and $W/W_\gamma$.
2. Since each coset in $W/W_\gamma$ has a unique coset representative of minimal length (see [Bou, IV.1, Exercise 3] or [Hum, Proposition 1.10c]) we may identify $W_\gamma$ with the set of minimal length coset representatives of $W/W_\gamma$.
3. The chambers of $\mathbb{R}^n \setminus (\cup_\alpha H_\alpha)$ are the regions $wC, w \in W$ and thus we can identify $W_\gamma$ with the set $\{wC \mid w \text{ is a minimal length coset representative of } W/W_\gamma\}$.
4. An element $w \in W$ is a minimal length coset representative of $W/W_\gamma$ if $\ell(ws_i) > \ell(w)$ for every $i$ such that $\alpha_i \in Z(\gamma)$.
5. So $w$ is a minimal length coset representative if and only if, for any $x \in C$, $\langle w^{-1}x, \alpha_i \rangle > 0$ for every $\alpha_i \in Z(\gamma)$. Since $Z(\gamma) \cup -Z(\gamma)$ is a root subsystem generated by the simple roots that it contains we have that $w$, a minimal length coset representative if and only if, for any $x \in C$, $\langle w^{-1}x, \alpha \rangle > 0$ for every $\alpha \in Z(\gamma)$. Thus, we may identify $W_\gamma$ with the set $\{wC \mid w^{-1}C \text{ is on the positive side of } H_\alpha \text{ for every } \alpha \in Z(\gamma)\}$.
6. The final step is to replace each chamber $wC$ by the chamber $w^{-1}C$.

(b) Recall that $u\gamma$ and $v\gamma$ are connected by an edge in $\Gamma(\gamma)$ if $u\gamma = s_i v\gamma$ and $\langle v\gamma, \alpha_i \rangle \neq \pm 1$. If $u$ and $v$ are minimal length coset representatives of $W/W_\gamma$, then the chambers $u^{-1}C$ and $v^{-1}C$ should be connected by an edge if $u = s_i v$ and $\langle \gamma, v^{-1}C \rangle \neq \pm 1$. The condition that $u = s_i v$ means that $u^{-1}C = v^{-1}s_i C$. Since $C$ and $s_i C$ share a face, it follows that $u^{-1}C = v^{-1}s_i C$ and $v^{-1}C$ share a face. This face is contained in the hyperplane $H_{v^{-1}C} = H_{u^{-1}C}$, since the face that $C$ and $s_i C$ share is contained in the hyperplane $H_{s_i C}$. Finally, the condition $\langle \gamma, v^{-1}C \rangle \neq \pm 1$ is the same as saying that $v^{-1}C \notin P(\gamma)$ and $u^{-1}C \notin P(\gamma)$.

(c) If $w \in W$, $\alpha > 0$, $x \in C$ and $w\alpha < 0$ then $\langle w^{-1}x, \alpha \rangle = \langle x, w\alpha \rangle < 0$. Thus

$$R(w) = \{\alpha > 0 \mid w^{-1}C \text{ is on the negative side of } H_\alpha\}.$$ 

So the condition $R(w) \cap Z(\gamma) = \emptyset$ is equivalent to the condition that $w^{-1}C$ is on the positive side of $H_\alpha$ for all $\alpha \in Z(\gamma)$. Similarly the condition $R(w) \cap P(\gamma) = J$ is equivalent to the condition that $w^{-1}C$ be on the negative side of $H_\alpha$ for all $\alpha \in J$ and on the positive side of $H_\alpha$ for all $\alpha \in P(\gamma) \setminus J$. □

Parts (a) and (b) of Proposition 2.2 allow us to view the calibration graph in terms of chambers in $\mathbb{R}^n \setminus (\cup_\alpha H_\alpha)$. The vertices correspond to the chambers on the positive side of the hyperplanes $H_\alpha$, $\alpha \in Z(\gamma)$. The edges of the graph are now the walls between the chambers. The only time that a wall between two chambers does not form an “edge of the graph” connecting the two chambers (vertices) is when that wall is contained in a hyperplane $H_\alpha$ with $\alpha \in P(\gamma)$. 
(2.3) Proof of Theorem 2.1  From parts (a) and (b) of Proposition 2.2 we get that the connected components of the graph $\Gamma(\gamma)$ correspond to the connected components of the intersection of

$$\left\{ \tilde{C} \mid \tilde{C} \text{ is on the positive side of } H_\alpha \text{ for } \alpha \in Z(\gamma) \right\} \quad \text{with} \quad \mathbb{R}^n \setminus \left( \bigcup_{\alpha \in P(\gamma)} H_\alpha \right).$$

Proposition 2.2 (c) says that the standard tableaux of shape $(\gamma, J)$ correspond to the chambers which are on the positive side of $H_\alpha$ for $\alpha \in Z(\gamma) \cup (P(\gamma) \setminus J)$ and on the negative side of $H_\alpha$ for $\alpha \in J$. The points in the chambers which satisfy these conditions form a connected subset of $\mathbb{R}^n \setminus \left( \bigcup_{\alpha \in P(\gamma)} H_\alpha \right)$ since the conditions describe them as the points in the intersection of half-spaces in $\mathbb{R}^n$. Finally, one only needs to note that the connected components $\mathbb{R}^n \setminus \left( \bigcup_{\alpha \in P(\gamma)} H_\alpha \right)$ are determined by which subset of hyperplanes in $P(\gamma)$ they are on the negative side of. This completes the proof of Theorem 2.1. ☐

(2.4) Invariance properties for calibration graphs.

Invariance property (1): As unlabeled graphs, $\Gamma(\gamma) = \Gamma(w\gamma)$ for all $w \in W$.

Invariance property (2): If $Z(\gamma) = Z(\kappa)$ and $P(\gamma) = P(\kappa)$ then $\Gamma(\gamma) = \Gamma(\kappa)$.

Property (1) is immediate from the definition of $\Gamma(\gamma)$. From the definition of the calibration graph one sees that the edges of $\Gamma(\gamma)$ are controlled by the set $P(\gamma)$. Since the vertices $W\gamma$ can be identified with the set $W/W_\gamma$, where $W_\gamma$ is the stabilizer of $\gamma$, it follows that the graph $\Gamma(\gamma)$ depends only on $W_\gamma$ and the set $P(\gamma)$. Since $W_\gamma$ is the group generated by the reflections $s_\alpha$ for $\alpha \in Z(\gamma)$, it follows that the structure of $\Gamma(\gamma)$ is dependent only on the sets $Z(\gamma)$ and $P(\gamma)$. This establishes invariance property (2).

(2.5) Intersections and shapes. Let $A$ be the arrangement of (affine) hyperplanes given by

$$H_\alpha = \{ x \in \mathbb{R}^n \mid \langle x, \alpha \rangle = 0 \}, \quad H_{\alpha + \delta} = \{ x \in \mathbb{R}^n \mid \langle x, \alpha \rangle = 1 \}, \quad H_{\alpha - \delta} = \{ x \in \mathbb{R}^n \mid \langle x, \alpha \rangle = -1 \}.$$

The intersection lattice $L(A')$ is the set of intersections, $I = \bigcap_{H_\beta \in B} H_\beta$, $B \subseteq A'$, partially ordered by inclusion (as subsets of $\mathbb{R}^n$). Since $A'$ is symmetric under the Weyl group $L(A')$ also carries a Weyl group symmetry. The quotient $L(A')/W$ is constructed by identifying intersections $I_1$ and $I_2$ if there is a $w \in W$ such that $wI_1 = I_2$. It follows from the invariance properties of the calibration graphs that the distinct calibration graphs are in one-to-one correspondence with the elements of the quotient $L(A')/W$. In particular, the number of distinct calibration graphs is finite.

Let $(\gamma, J)$ and $(\eta, K)$ be two placed shapes. We shall say that $(\gamma, J)$ and $(\eta, K)$ have the same underlying shape if there is a $w \in W$ such that

$$Z(w\gamma) = Z(\eta), \quad P(w\gamma) = P(\eta), \quad \text{and} \quad wJ = K.$$

When $(\gamma, J)$ and $(\eta, K)$ are placed shapes with the same underlying shape then the calibration graphs $\Gamma(\gamma)$ and $\Gamma(\kappa)$ are the same and there is a natural bijection

$$\mathcal{F}(\gamma, J) \overset{\sim}{\rightarrow} \mathcal{F}(\eta, K).$$
(2.6) It would be interesting to determine the size of $L(A')/W$, i.e. the number of distinct calibration graphs. A. Postnikov has explained that in the Type A case the numbers $f(n) = \text{Card}(L(A')/S_n)$ have the following generating function

$$\prod_{k \geq 1} (1 - q^k)^{2k-1} = \sum_{n \geq 1} f(n)q^n.$$ 

One can prove this by counting the number of ways of constructing the configurations of boxes described in Section 3.

The arrangement $A'$ is very similar to the Shi arrangement

$$A'' = \{ H_\alpha, H_{\alpha + \delta} \mid \alpha \in R^+ \}.$$ 

C. Athanasiadis has told me that, for the type A case, the number of elements of the intersection lattice $L(A'')$ which contain a dominant weight is equal to

$$\sum_{k=1}^{n} \binom{n-1}{k-1} F_{2k-1}$$

where $F_1 = 1$, $F_3 = 2$, $F_5 = 5$, ... are the odd Fibonacci numbers. The Shi arrangement has been an object of intense recent study, see [Sh3], [St1], [AL]. There are many indications [Sh3], [Xi,1.11, 2.6] that there should be a strong connection between the work in this paper and the combinatorics of the Shi arrangement.

(2.7) Example. Consider the root system of type $C_2$ where $R^+ = \{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2 \}$. Realize this root system in $\mathbb{R}^2$ by letting $\alpha_1 = 2\varepsilon_1$ and $\alpha_2 = \varepsilon_2 - \varepsilon_1$, where $\{\varepsilon_1, \varepsilon_2\}$ is an orthonormal basis of $\mathbb{R}^2$. Let $\gamma \in \mathbb{R}^2$ be given by $\langle \gamma, \alpha_1 \rangle = 0$ and $\langle \gamma, \alpha_2 \rangle = 1$. Then $\gamma$ is dominant (i.e. in $\mathcal{C}$) and integral and

$$Z(\gamma) = \{ \alpha_1 \} \quad \text{and} \quad P(\gamma) = \{ \alpha_2, \alpha_1 + \alpha_2 \}.$$
The Weyl group is $\gamma$. Type (3.1) The root system.
Let $\gamma$ be an orthonormal basis of $\mathbb{R}^n$ so that each sequence $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ is identified with the vector $\gamma = \sum_i \gamma_i \varepsilon_i$. The root system of type $A_{n-1}$ is given by the sets

$$R = \{ \varepsilon_j - \varepsilon_i \mid 1 \leq i, j \leq n \} \quad \text{and} \quad R^+ = \{ \varepsilon_j - \varepsilon_i \mid 1 \leq i < j \leq n \}.$$ 

The Weyl group is $W = S_n$, the symmetric group, acting by permutations of the $\varepsilon_i$. 

The calibration graph $\Gamma(\gamma)$

When we convert to regions in $\mathbb{R}^2 \setminus (\bigcup_{\alpha} H_{\alpha})$ we get the picture

The dashed line is the hyperplane corresponding to the root in $Z(\gamma)$ and the solid lines are the hyperplanes corresponding to the roots in $P(\gamma)$.

3. Type A and configurations of boxes

(3.1) The root system. Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be an orthonormal basis of $\mathbb{R}^n$ so that each sequence $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ is identified with the vector $\gamma = \sum_i \gamma_i \varepsilon_i$. The root system of type $A_{n-1}$ is given by the sets

$$R = \{ \varepsilon_j - \varepsilon_i \mid 1 \leq i, j \leq n \} \quad \text{and} \quad R^+ = \{ \varepsilon_j - \varepsilon_i \mid 1 \leq i < j \leq n \}.$$ 

The Weyl group is $W = S_n$, the symmetric group, acting by permutations of the $\varepsilon_i$. 

$\gamma$ 

$J = \emptyset$

$s_1s_2\gamma$ 

$s_2\gamma$ 

$J = \{\alpha_2\}$

$s_2s_1s_2\gamma$ 

$J = \{\alpha_2, \alpha_1 + \alpha_2\}$

$H_{\alpha_1 + \alpha_2}$ 

$H_{\alpha_1}$ 

$J = \emptyset$ 

$H_{\alpha_2}$ 

$C$ 

$s_2C$ 

$s_2s_1C$ 

$s_2s_1s_2C$ 

$J = \{\alpha_2, \alpha_1 + \alpha_2\}$

$H_{\alpha_1 + 2\alpha_2}$
(3.2) **Partitions, skew shapes, and (classical) standard tableaux.** A partition $\lambda$ is a collection of $n$ boxes in a corner. We shall conform to the conventions in [Mac] and assume that gravity goes up and to the left.

Any partition $\lambda$ can be identified with the sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ where $\lambda_i$ is the number of boxes in row $i$ of $\lambda$. The rows and columns are numbered in the same way as for matrices. In the example above we have $\lambda = (553311)$.

If $\lambda$ and $\mu$ are partitions such that $\mu_i \leq \lambda_i$ for all $i$ we write $\mu \subseteq \lambda$. The skew shape $\lambda/\mu$ consists of all boxes of $\lambda$ which are not in $\mu$. Let $\lambda/\mu$ be a skew shape with $n$ boxes. Number the boxes of each skew shape $\lambda/\mu$ along major diagonals from southwest to northeast and write box $i$ to indicate the box numbered $i$.

See Example 3.4 below. A (classical) **standard tableau of shape** $\lambda/\mu$ is a filling of the boxes in the skew shape $\lambda/\mu$ with the numbers $1, \ldots, n$ such that the numbers increase from left to right in each row and from top to bottom down each column. Let $F^{\lambda/\mu}$ be the set of standard tableaux of shape $\lambda/\mu$. Given a standard tableau $t$ of shape $\lambda/\mu$ define the **word** of $t$ to be permutation

$$w_t = \begin{pmatrix} 1 & \cdots & n \\ t(\text{box}_1) & \cdots & t(\text{box}_n) \end{pmatrix}$$

where $t(\text{box}_i)$ is the entry in box $i$ of $t$.

(3.3) **Placed skew shapes** Let $\lambda/\mu$ be a skew shape with $n$ boxes. Imagine placing $\lambda/\mu$ on a piece of infinite graph paper where the diagonals of the graph paper are indexed consecutively (with elements of $\mathbb{Z}$) from southeast to northwest. The **content** of a box $b$ is the number $c(b)$ of the diagonal that $b$ is on. Let

$$\gamma = \sum_{i=1}^{n} c(\text{box}_i) \varepsilon_i,$$

which we identify with the sequence $\gamma = (c(\text{box}_1), c(\text{box}_2), \ldots, c(\text{box}_n))$. The pair $(\gamma, \lambda/\mu)$ is a placed skew shape. It follows from the definitions in (1.2) that

$$Z(\gamma) = \{ \varepsilon_j - \varepsilon_i \mid j > i \text{ and box}_j \text{ and box}_i \text{ are in the same diagonal} \}, \quad \text{and} \quad P(\gamma) = \{ \varepsilon_j - \varepsilon_i \mid j > i \text{ and box}_j \text{ and box}_i \text{ are in adjacent diagonals} \}.$$ 

Define

$$J = \left\{ \varepsilon_j - \varepsilon_i \mid j > i \right\},$$

where *northwest* means strictly north and weakly west.
Example. The following diagrams illustrate standard tableaux and the numbering of boxes in a skew shape $\lambda/\mu$.

The word of the standard tableau $t$ is the permutation $w_t = (11, 6, 8, 2, 7, 1, 13, 5, 14, 3, 10, 4, 9, 12)$ (in one-line notation).

The following picture shows the contents of the boxes in the placed skew shape $(\gamma, \lambda/\mu)$ with $\gamma = (-7, -6, -5, -2, 0, 1, 1, 2, 2, 3, 3, 4, 5, 6)$.

In this case $J = \{\varepsilon_2 - \varepsilon_1, \varepsilon_6 - \varepsilon_5, \varepsilon_8 - \varepsilon_7, \varepsilon_{10} - \varepsilon_8, \varepsilon_{10} - \varepsilon_9, \varepsilon_{11} - \varepsilon_9, \varepsilon_{12} - \varepsilon_1\}$.

The following theorem shows how the generalized standard tableaux defined in (1.2) reduce to the classical standard Young tableaux in the type $A$ case.

Theorem. Let $(\gamma, \lambda/\mu)$ be a placed skew shape and let $J$ be as defined in (3.3). Let $\mathcal{F}^{\lambda/\mu}$ be the set of standard tableaux of shape $\lambda/\mu$ and let $\mathcal{F}^{(\gamma,J)}$ be the set of generalized standard tableaux of shape $(\gamma, J)$ as defined in (1.2). Then the map

$$
\mathcal{F}^{\lambda/\mu} \xrightarrow{w_t} \mathcal{F}^{(\gamma,J)}
$$

where $w_t$ is as defined in (3.2), is a bijection.

Proof. If $w = (w(1) \cdots w(n))$ is a permutation in $S_n$ then

$$
R(w) = \{\varepsilon_j - \varepsilon_i \mid j > i \text{ such that } w(j) < w(i) \}.
$$

The theorem is a consequence of the following chain of equivalences:

The filling $t$ is a standard tableau if and only if for all $1 \leq i < j \leq n$

(a) $t(\text{box}_i) < t(\text{box}_j)$ if $\text{box}_i$ and $\text{box}_j$ are on the same diagonal,

(b) $t(\text{box}_i) < t(\text{box}_j)$ if $\text{box}_j$ is immediately to the right of $\text{box}_i$, and
(c) \(t(\text{box}_i) > t(\text{box}_j)\) if box\(_j\) is immediately above box\(_i\).

These conditions hold if and only if
\[
\begin{align*}
(a) & \quad \varepsilon_j - \varepsilon_i \notin R(w(t)) \text{ if } \varepsilon_j - \varepsilon_i \notin Z(\gamma), \\
(b) & \quad \varepsilon_j - \varepsilon_i \notin R(w(t)) \text{ if } \varepsilon_j - \varepsilon_i \in P(\gamma) \setminus J, \\
(c) & \quad \varepsilon_j - \varepsilon_i \in R(w(t)) \text{ if } \varepsilon_j - \varepsilon_i \in J,
\end{align*}
\]

which hold if and only if
\[
\begin{align*}
(a) & \quad \alpha \notin R(w(t)) \text{ if } \alpha \in Z(\gamma), \quad (b) \quad \alpha \notin R(w(t)) \text{ if } \alpha \in P(\gamma) \setminus J, \quad \text{and} \quad (c) \quad \alpha \in R(w(t)) \text{ if } \alpha \in J.
\end{align*}
\]

Finally, these are equivalent to the conditions \(R(w(t)) \cap Z(\gamma) = \emptyset\) and \(R(w(t)) \cap P(\gamma) = J\).

(3.6) Placed configurations  We have described how one can identify placed skew shapes \((\gamma, \lambda/\mu)\) with pairs \((\gamma, J)\). One can extend this conversion to associate placed configurations of boxes to more general pairs \((\gamma, J)\). The resulting configurations are not always skew shapes.

Let \((\gamma, J)\) be a pair such that \(\gamma = (\gamma_1, \ldots, \gamma_n)\) is a dominant integral weight and \(J \subseteq P(\gamma)\). (The sequence \(\gamma\) is a dominant integral weight if \(\gamma_1 \leq \cdots \leq \gamma_n\) and \(\gamma_i \in \mathbb{Z}\) for all \(i\).) If \(J\) satisfies the condition
\[
\text{If } \beta \in J, \ \alpha \in Z(\gamma), \ \text{and} \ \beta - \alpha \in R^+ \text{ then } \beta - \alpha \in J
\]
then \((\gamma, J)\) will determine a placed configuration of boxes. As in the placed skew shape case, think of the boxes as being placed on graph paper where the boxes on a given diagonal all have the same content. (The boxes on each diagonal are allowed to slide along the diagonal as long as they don’t pass through the corner of a box on an adjacent diagonal.) The sequence \(\gamma\) describes how many boxes are on each diagonal and the set \(J\) determines how the boxes on adjacent diagonals are placed relative to each other. We want
\[
\gamma = \sum_{i=1}^{n} c(\text{box}_i) \varepsilon_i,
\]
and
\[
\begin{align*}
(a) & \quad \text{If } \varepsilon_j - \varepsilon_i \in J \text{ then box}_j \text{ is northwest of box}_i, \quad \text{and} \\
(b) & \quad \text{If } \varepsilon_j - \varepsilon_i \in P(\gamma) \setminus J \text{ then box}_j \text{ is southeast of box}_i,
\end{align*}
\]

where the boxes are numbered along diagonals in the same way as for skew shapes, southeast means weakly south and strictly east, and northwest means strictly north and weakly west.

If we view the pair \((\gamma, J)\) as a placed configuration of boxes then the standard tableaux are fillings \(t\) of the \(n\) boxes in the configuration with \(1, 2, \ldots, n\) such that for all \(i < j\)
\[
\begin{align*}
(a) & \quad t(\text{box}_i) < t(\text{box}_j) \text{ if box}_i \text{ and box}_j \text{ are on the same diagonal,} \\
(b) & \quad t(\text{box}_i) < t(\text{box}_j) \text{ if box}_i \text{ and box}_j \text{ are on adjacent diagonals and box}_j \text{ is southeast of box}_i, \quad \text{and} \\
(c) & \quad t(\text{box}_i) > t(\text{box}_j) \text{ if box}_i \text{ and box}_j \text{ are on adjacent diagonals and box}_j \text{ is northwest of box}_i.
\end{align*}
\]

As in Theorem 3.5 the permutation in \(\mathcal{F}(\gamma, J)\) which corresponds to the standard tableau \(t\) is \(w(t) = (t(\text{box}_1), \ldots, t(\text{box}_n))\). The following example illustrates the conversion.

Example. Suppose \(\gamma = (-1, -1, -1, 0, 0, 0, 1, 1, 1, 2, 2, 2)\) and
\[
J = \{\varepsilon_4 - \varepsilon_1, \varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_2, \varepsilon_5 - \varepsilon_3, \varepsilon_7 - \varepsilon_5, \varepsilon_7 - \varepsilon_6, \varepsilon_8 - \varepsilon_6, \varepsilon_{10} - \varepsilon_9, \varepsilon_{10} - \varepsilon_8, \\
\varepsilon_{10} - \varepsilon_7, \varepsilon_{11} - \varepsilon_9, \varepsilon_{11} - \varepsilon_8, \varepsilon_{11} - \varepsilon_7, \varepsilon_{12} - \varepsilon_9\}.\]
The placed configuration of boxes corresponding to \((\gamma, J)\) is as given below.

$$
\begin{array}{ccc}
2 & 2 & 10 \\
0 & 1 & 11 \\
-1 & 0 & 4 \\
1 & 1 & 7 \\
-1 & 0 & 1 \\
\end{array}
$$

contents of boxes  

numbering of boxes  

a standard tableau

(3.7) Books of placed configurations. The most general case to consider is when \(\gamma = (\gamma_1, \ldots, \gamma_n)\) is an arbitrary element of \(\mathbb{R}^n\) and \(J \subseteq P(\gamma)\). This case is handled as follows. First group the entries of \(\gamma\) according to their \(\mathbb{Z}\)-coset in \(\mathbb{R}\). Each group of entries in \(\gamma\) can be arranged to form a sequence

\[
\beta + C_\beta = \beta + (z_1, \ldots, z_k) = (\beta + z_1, \ldots, \beta + z_k), \quad \text{where } 0 \leq \beta < 1, \ z_i \in \mathbb{Z} \text{ and } z_1 \leq \cdots \leq z_k.
\]

Fix some ordering of these groups and let

\[
\vec{\gamma} = (\beta_1 + C_{\beta_1}, \ldots, \beta_r + C_{\beta_r})
\]

be the rearrangement of the sequence \(\gamma\) with the groups listed in order. Since \(\vec{\gamma}\) is a rearrangement of \(\gamma\), the calibration graphs \(\Gamma(\gamma)\) and \(\Gamma(\vec{\gamma})\) are the same (see Invariance Property (1) in (2.4)). This means that it is sufficient to understand the standard tableaux corresponding to \(\vec{\gamma}\).

The decomposition of \(\vec{\gamma}\) into groups induces decompositions

\[
Z(\vec{\gamma}) = \bigcup_{\beta_i} Z_{\beta_i}, \quad P(\vec{\gamma}) = \bigcup_{\beta_i} P_{\beta_i}, \quad \text{and, if } J \subseteq P(\vec{\gamma}), \text{ then } J = \bigcup_{\beta_i} J_{\beta_i},
\]

where \(J_{\beta_i} = J \cap P_{\beta_i}\). Each pair \((C_\beta, J_\beta)\) is a placed shape of the type considered in the previous subsection and we may identify \((\vec{\gamma}, J)\) with the book of placed shapes \(((C_{\beta_1}, J_{\beta_1}), \ldots, (C_{\beta_r}, J_{\beta_r}))\).

We think of this as a book with pages numbered by the values \(\beta_1, \ldots, \beta_r\) and with the placed configuration determined by \((C_{\beta_i}, J_{\beta_i})\) on page \(\beta_i\). In this form the standard tableaux of shape \((\vec{\gamma}, J)\) are fillings of the \(n\) boxes in the book with the numbers \(1, \ldots, n\) such that the filling on each page satisfies the conditions for a standard tableau in (3.6).

Example. If \(\gamma = (1/2, 1/2, 1, 1, 1, 3/2, -2, -2, -1/2, -1, -1, -1/2, 1/2, 0, 0, 0)\) then one possibility for \(\vec{\gamma}\) is

\[
\vec{\gamma} = (-2, -2, -1, -1, 0, 0, 0, 1, 1, 1, -1/2, -1/2, 1/2, 1/2, 1/2, 1/2, 3/2).
\]

In this case \(\beta_1 = 0, \beta_2 = 1/2, \beta_1 + C_{\beta_1} = (-2, -2, -1, -1, 0, 0, 0, 1, 1, 1) \) and \(\beta_2 + C_{\beta_2} = (-1/2, -1/2, 1/2, 1/2, 1/2, 3/2)\).
If \( J = J_{\beta_1} \cup J_{\beta_2} \) where \( J_{\beta_2} = \{ \varepsilon_{14} - \varepsilon_{13}, \varepsilon_{17} - \varepsilon_{16} \} \) and

\[
J_{\beta_1} = \{ \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_2, \varepsilon_5 - \varepsilon_2, \varepsilon_6 - \varepsilon_3, \varepsilon_6 - \varepsilon_4, \varepsilon_6 - \varepsilon_5, \varepsilon_9 - \varepsilon_7, \varepsilon_9 - \varepsilon_8, \varepsilon_{10} - \varepsilon_7, \varepsilon_{10} - \varepsilon_8 \}
\]
then the book of shapes is

where the numbers in the boxes are the contents of the boxes. The filling

is a standard tableau of shape \((\vec{\gamma}, J)\). This filling corresponds to the permutation

\[
w = (2, 12, 4, 5, 9, 1, 13, 15, 8, 11, 17, 3, 7, 6, 10, 16, 14) \quad \text{in } \mathcal{F}(\vec{\gamma}, J) \subseteq S_{16}.
\]

4. Skew shapes, ribbons, conjugation, etc. in Type A

As in the previous section let \( R \) be the root system of Type \( A_{n-1} \) as given in (3.1). For clarity, we shall state all of the results in this section for placed shapes \((\gamma, J)\) such that \( \gamma \) is dominant and integral, i.e. \( \gamma = (\gamma_1, \ldots, \gamma_n) \) with \( \gamma_1 \leq \cdots \gamma_n \) and \( \gamma_i \in \mathbb{Z} \). This simplification is mathematically unimportant, the reason for it is that it allows us to avoid the notational difficulties which arise when one wants to use books of placed shapes with several pages.

**4.1 Axial distance** Let \((\gamma, J)\) be a placed shape such that \( \gamma \) is dominant and integral. Let \( w \in \mathcal{F}(\gamma, J) \) and let \( t \) be the corresponding standard tableau as defined by the map in Theorem 3.5. Then it follows from the definitions of \( \gamma \) and \( w \) in (3.2) and (3.3) that

\[
\langle w\gamma, \varepsilon_i \rangle = \langle \gamma, w^{-1}\varepsilon_i \rangle = c(\text{box}_{w^{-1}(i)}) = c(t(i)),
\]
where \( t(i) \) is the box of \( t \) containing the entry \( i \).

In classical standard tableau theory the axial distance between two boxes in a standard tableau is defined as follows. Let \( \lambda \) be a partition and let \( t \) be a standard tableau of shape \( \lambda \). Let \( 1 \leq i, j \leq n \) and let \( t(i) \) and \( t(j) \) be the boxes which are filled with \( i \) and \( j \) respectively. Let \((r_i, c_i)\) and \((r_j, c_j)\) be the positions of these boxes, where the rows and columns of \( \lambda \) are numbered in the same way as for matrices. Then the axial distance from \( j \) to \( i \) in \( t \) is

\[
d_{ji}(t) = c_j - c_i + r_i - r_j,
\]

(see [Wz]). This may seem confusing at first but it is simpler if we rewrite it in terms of the corresponding placed shape \((\gamma, J)\) where \( \gamma \) is the sequence in \( \mathbb{R}^n \) determined by some placing of \( \lambda \) on infinite graph paper. Then one gets

\[
d_{ji}(t) = c(t(j)) - c(t(i)) = \langle w_{\gamma}, \varepsilon_j - \varepsilon_i \rangle = d_{\varepsilon_j - \varepsilon_i}(w),
\]

where \( w \in F(\gamma, J) \) is the permutation corresponding to the standard tableau \( t \) (see (3.2)) and \( d_{\alpha}(w) \) is the generalized axial distance defined in (1.10). This shows that the axial distance defined in (1.10) is a generalization of the classical notion of axial distance.

(4.2) Skew shapes. The following proposition shows that, in the case of a root system of type \( A \), the definition of generalized skew shape coincides with the classical notion of a skew shape.

**Proposition.** Let \((\gamma, J)\) be a placed shape such that \( \gamma \) is dominant and integral. Then the configuration of boxes associated to \((\gamma, J)\) is a placed skew shape if and only if \((\gamma, J)\) is a skew shape as defined in (1.4).

**Proof.** \( \iff \): We shall show that if the placed configuration corresponding to the pair \((\gamma, J)\) has any \(2 \times 2\) blocks of the forms

\[
\begin{array}{ccc}
\text{Case (1)} & \text{Case (2)} & \text{Case (3)} \\
\begin{array}{ccc}
 a & b & c \\
\end{array}
\end{array}
\]

then \( w\gamma \mid R \) is not regular for some appropriate \( w \in F(\gamma, J) \) and subsystem \( R_K \) in \( R \). This will show that the placed configuration must be a placed skew shape if \((\gamma, J)\) is a generalized skew shape. In the pictures above the shaded regions indicate the absence of a box and, for notational reference, we have labeled the boxes with \( a, b, c \).

**Case (1):** Create a standard tableau \( t \) such that the \(2 \times 2\) block is filled with

\[
\begin{array}{cc}
i - 1 & i \\
& i + 1
\end{array}
\]

by filling the region of the configuration strictly north and weakly west of box \( c \) in row reading order (sequentially left to right across the rows starting at the top), putting the next entry in box \( c \), and filling the remainder of the configuration in column reading order (sequentially down the
columns beginning at the leftmost available column). Let \( w = w(t) \) be the permutation in \( F(\gamma, J) \) which corresponds to the standard tableau \( t \). Let \( t(i) \) denote the box containing \( i \) in \( t \). Then, using the first identity (4.1),

\[
\langle w\gamma, \alpha_i + \alpha_{i+1} \rangle = \langle w\gamma, \varepsilon_{i+1} - \varepsilon_{i-1} \rangle = c(t(i + 1)) - c(t(i - 1)) = 0,
\]

since the boxes \( t(i + 1) \) and \( t(i - 1) \) are on the same diagonal. It follows that \( w\gamma|_{\{\alpha_{i+1}, \alpha_i\}} \) is not regular.

**Case (2):** Create a standard tableau \( t \) such that the \( 2 \times 2 \) block is filled with

\[
\begin{array}{cc}
  i - 1 & \\
  i & i + 1 \\
\end{array}
\]

by filling the region weakly north and strictly west of box \( c \) in column reading order, putting the next entry in box \( c \), and filling the remainder of the configuration in row reading order. Using this standard tableau \( t \), the remainder of the argument is the same as for case (1).

**Case (3):** Create a standard tableau \( t \) such that the \( 2 \times 2 \) block is filled with

\[
\begin{array}{cc}
  i - 1 & \\
  & i \\
\end{array}
\]

by filling the region strictly north and strictly west of box \( b \) in column reading order, putting the next entry in box \( b \), and filling the remainder of the configuration in row reading order. Let \( w = w(t) \) be the permutation in \( F(\gamma, J) \) corresponding to \( t \) and let \( t(i) \) denote the box containing \( i \) in \( t \). Then

\[
\langle w\gamma, \alpha_i \rangle = \langle w\gamma, \varepsilon_i - \varepsilon_{i-1} \rangle = c(t(i)) - c(t(i - 1)) = 0,
\]

since \( t(i) \) and \( t(i - 1) \) are on the same diagonal. It follows that \( w\gamma|_{\{\alpha_i\}} \) is not regular.

\[\Rightarrow\]: Let \( \gamma \in \mathbb{Z}^n \) and \( \lambda/\mu \) describe a placed skew shape (a skew shape placed on infinite graph paper). Let \( (\gamma, J) \) be the corresponding (generalized) placed shape as defined in (3.3). Let \( w \in F(\gamma, J) \) and let \( t \) be the corresponding standard tableau of shape \( \lambda/\mu \). Consider a \( 2 \times 2 \) block of boxes of \( t \). If these boxes are filled with

\[
\begin{array}{cc}
  i & j \\
  k & \ell \\
\end{array}
\]

then either \( i < j < k < \ell \) or \( i < k < j < \ell \). In either case we have \( i < \ell - 1 \) and it follows that \( \ell - 1 \) and \( \ell \) are not on the same diagonal. Thus

\[
\langle w\gamma, \alpha_\ell \rangle = c(t(\ell)) - c(t(\ell - 1)) \neq 0,
\]
and so \( w_\gamma \mid_{\{\alpha_i\}} \) is regular.

The same argument shows that one can never get a standard tableau in which \( \ell \) and \( \ell - 2 \) occur in adjacent boxes of the same diagonal and thus it follows that \( w_\gamma \mid_{\{\alpha_{i-1}, \alpha_i\}} \) is regular for all \( w \in \mathcal{F}(\gamma, J) \).

Thus \((\gamma, J)\) is a placed skew shape in the sense of (1.4).

\[\quad\]

**4.3 Ribbon Shapes.** In classical tableaux theory a *border strip* (or *ribbon*) is a skew shape which contains at most one box in each diagonal. Although the convention, [Mac, I §1 p. 5], is to assume that border strips are connected skew shapes we shall *not* assume this.

Recall from (1.5) that a placed shape \((\gamma, J)\) is a placed *ribbon* shape if \( \gamma \) is regular, i.e. \( \langle \gamma, \alpha \rangle \neq 0 \) for all \( \alpha \in R \).

**Proposition.** Let \((\gamma, J)\) be a placed ribbon shape such that \( \gamma \) is dominant and integral. Then the configuration of boxes corresponding to \((\gamma, J)\) is a placed border strip.

**Proof.** Let \((\gamma, J)\) be a placed ribbon shape with \( \gamma \) dominant and regular. Since \( \gamma = (\gamma_1, \ldots, \gamma_n) \) is regular, \( \gamma_i \neq \gamma_j \) for all \( i \neq j \). In terms of the placed configuration \( \gamma_i = c(\text{box}_i) \) is the diagonal that \( \text{box}_i \) is on. Thus the configuration of boxes corresponding to \((\gamma, J)\) contains at most one box in each diagonal.

**Example.** If \( \gamma = (-6, -5, -4, 0, 1, 3, 4, 5, 6, 7) \) and \( J = \{\varepsilon_2 - \varepsilon_1, \varepsilon_5 - \varepsilon_4, \varepsilon_7 - \varepsilon_6, \varepsilon_9 - \varepsilon_8, \varepsilon_{10} - \varepsilon_9\} \) then the placed configuration of boxes corresponding to \((\gamma, J)\) is the placed border strip

\[
\begin{array}{cccccc}
7 & 6 \\
4 & 5 & \\
1 & 3 & \\
& 0 & \\
-5 & -4 & \\
-6 & \\
\end{array}
\]

where we have labeled the boxes with their contents.

**4.4 Conjugation of Shapes.** Let \((\gamma, J)\) be a placed shape with \( \gamma \) dominant and integral (i.e. \( \gamma = (\gamma_1, \ldots, \gamma_n) \) with \( \gamma_1 \leq \cdots \leq \gamma_n \) and \( \gamma_i \in \mathbb{Z} \)) and view \((\gamma, J)\) as a placed configuration of boxes. In terms of placed configurations, conjugation of shapes is equivalent to transposing the placed configuration across the diagonal of boxes of content 0. The following example illustrates this.
Example. Suppose $\gamma = (-1, -1, -1, 0, 0, 1, 1)$ and $J = (\varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_5)$. Then the placed configuration of boxes corresponding to $(\gamma, J)$ is

\[
\begin{array}{ccc}
-1 & 0 & 1 \\
-1 & & 1 \\
-1 & & 0
\end{array}
\]

in which the shaded box is not a box in the configuration.

The minimal length representative of the coset $w_0 W_\gamma$ is the permutation

\[u = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 3 & 4 & 1 & 2 \end{pmatrix}.
\]

We have $-u\gamma = -w_0\gamma = (-1, -1, 0, 0, 1, 1)$ and

\[-u(P(\gamma) \setminus J) = -u \{\varepsilon_4 - \varepsilon_1, \varepsilon_5 - \varepsilon_1, \varepsilon_5 - \varepsilon_2, \varepsilon_5 - \varepsilon_3, \varepsilon_6 - \varepsilon_4, \varepsilon_7 - \varepsilon_4\} \\
= -\{\varepsilon_3 - \varepsilon_5, \varepsilon_4 - \varepsilon_5, \varepsilon_4 - \varepsilon_6, \varepsilon_4 - \varepsilon_7, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\} \\
= \{\varepsilon_5 - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \varepsilon_6 - \varepsilon_4, \varepsilon_7 - \varepsilon_4, \varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_2\}.
\]

Thus the configuration of boxes corresponding to the placed shape $(\gamma, J)'$ is

\[
\begin{array}{ccc}
1 & & \\
0 & 1 & \\
-1 & & 1 \\
-1 & & 0
\end{array}
\]

(4.5) **Row reading and column reading tableaux.** Let $(\gamma, J)$ be a placed shape such that $\gamma$ is dominant and integral and consider the placed configuration of boxes corresponding to $(\gamma, J)$. The *minimal box* of the configuration is the box such that

- $(m_1)$ there is no box immediately above,
- $(m_2)$ there is no box immediately to the left,
- $(m_3)$ there is no box northwest in the same diagonal, and
- $(m_4)$ it has the minimal content of the boxes satisfying $(m_1)$, $(m_2)$ and $(m_3)$.

There is at most one box in each diagonal satisfying $(m_1)$, $(m_2)$, and $(m_3)$. Thus, $(m_4)$ guarantees that the minimal box is unique. It is clear that the minimal box of the configuration always exists.

The *column reading* tableaux of shape $(\gamma, J)$ is the filling $t_{\text{min}}$ which is created inductively by

(a) filling the minimal box of the configuration with 1, and

(b) if $1, 2, \ldots, i$ have been filled in then fill the minimal box of the configuration formed by the unfilled boxes with $i + 1$. 


The row reading tableau of shape \((\gamma, J)\) is the standard tableau \(t_{\text{max}}\) whose conjugate \((t_{\text{max}})'\) is the column reading tableaux for the shape \((\gamma, J)'\) (the conjugate shape to \((\gamma, J)\)).

Recall the definitions of the weak Bruhat order and closed subsets of roots given in (1.11).

**Theorem.** Let \((\gamma, J)\) be a placed shape such that \(\gamma\) is dominant and integral (i.e. \(\gamma = (\gamma_1, \ldots, \gamma_n)\) with \(\gamma_1 \leq \cdots \leq \gamma_n\) and \(\gamma_i \in \mathbb{Z}\)). Let \(t_{\text{min}}\) and \(t_{\text{max}}\) be the column reading and row reading tableaux of shape \((\gamma, J)\), respectively, and let \(w_{\text{min}}\) and \(w_{\text{max}}\) be the corresponding permutations in \(\mathcal{F}(\gamma, J)\). Then

\[
R(w_{\text{min}}) = J, \quad R(w_{\text{max}}) = (P(\gamma) \setminus J) \cup Z(\gamma)^c, \quad \text{and} \quad \mathcal{F}(\gamma, J) = [w_{\text{min}}, w_{\text{max}}],
\]

where \(K^c\) denotes the complement of \(K\) in \(R^+\) and \([w_{\text{min}}, w_{\text{max}}]\) denotes the interval between \(w_{\text{min}}\) and \(w_{\text{max}}\) in the weak Bruhat order.

**Proof.** (a) Consider the configuration of boxes corresponding to \((\gamma, J)\). If \(k > i\) then either \(c(\text{box}_k) > c(\text{box}_i)\), or \(\text{box}_k\) is in the same diagonal and southeast of \(\text{box}_i\). Thus when we create \(t_{\text{min}}\) we have that

If \(k > i\) then \(\text{box}_k\) gets filled before \(\text{box}_i\) if and only if \(\text{box}_k\) is northwest of \(\text{box}_i\),

where the northwest is in a very strong sense: There is a sequence of boxes

\[\text{box}_i = \text{box}_{i_0}, \text{ box}_{i_1}, \ldots, \text{ box}_{i_r} = \text{box}_k\]

such that \(\text{box}_{i_m}\) is either directly above \(\text{box}_{i_{m-1}}\) or in the same diagonal and directly northwest of \(\text{box}_{i_{m-1}}\). In other words,

If \(k > i\) then \(t_{\text{min}}(\text{box}_k) < t_{\text{min}}(\text{box}_i) \iff \text{box}_k\) is northwest of \(\text{box}_i\).

So, from the formula for \(w_t\) in (3.2) we get

If \(k > i\) then \(w_{\text{min}}(k) < w_{\text{min}}(i) \iff \varepsilon_k - \varepsilon_i \in J,\)

where \(w_{\text{min}}\) is the permutation in \(\mathcal{F}(\gamma, J)\) which corresponds to the filling \(t_{\text{min}}\) and \(J\) is the closure of \(J\) in \(R\). It follows that

\[R(w_{\text{min}}) = J.\]

(b) There are at least two ways to prove that \(R(w_{\text{max}}) = (P(\gamma) \setminus J) \cup Z(\gamma)^c\). One can mimic the proof of part (a) by defining the maximal box of a configuration and a corresponding filling. Alternatively one can use the definition of conjugation and the fact that \(R(w_0w) = R(w)^c\). The permutation \(w_{\text{min}}\) is the unique minimal element of \(\mathcal{F}(\gamma, J)\) and the conjugate of \(w_{\text{max}}\) is the unique minimal element of \(\mathcal{F}(\gamma, J)'\). We shall leave the details to the reader.

(c) An element \(w \in W\) is an element of \(\mathcal{F}(\gamma, J)\) if and only if \(R(w) \cap P(\gamma) = J\) and \(R(w) \cap Z(\gamma) = \emptyset\). Thus \(\mathcal{F}(\gamma, J)\) consists of those permutations \(w \in W\) such that

\[J \subseteq R(w) \subseteq (P(\gamma) \setminus J) \cup Z(\gamma)^c.\]

Since the weak Bruhat order is the ordering determined by inclusions of \(R(w)\), it follows that \(\mathcal{F}(\gamma, J)\) is the interval between \(w_{\text{min}}\) and \(w_{\text{max}}\).
Example. Suppose \( \gamma = (-1,-1,-1,0,0,1,1) \) and \( J = \{ \varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_5 \} \). The minimal and maximal elements in \( \mathcal{F}(\gamma,J) \) are the permutations\

\[
\begin{array}{c}
w_{\text{min}} = (1, 2, 3, 4, 5, 6, 7) \\
w_{\text{max}} = (1, 2, 3, 4, 5, 6, 7)
\end{array}
\]

The permutations correspond to the standard tableaux

\[
\begin{array}{cccc}
1 & 2 & 5 \\
3 & & 6 \\
4 & 7
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
1 & 2 & 3 \\
5 & & 4 \\
6 & 7
\end{array}
\]

5. Standard tableaux for type \( C \) in terms of boxes

5.1 The root system. Let \( \{ \varepsilon_1, \ldots, \varepsilon_n \} \) be an orthonormal basis of \( \mathbb{R}^n \) so that each sequence \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n \) is identified with the vector \( \gamma = \sum_i \gamma_i \varepsilon_i \). The root system of type \( C_n \) is given by the sets\n
\[
R = \{ \pm 2\varepsilon_i, \varepsilon_j \pm \varepsilon_i \mid 1 \leq i, j \leq n \} \quad \text{and} \quad R^+ = \{ 2\varepsilon_i, \varepsilon_j - \varepsilon_i \mid 1 \leq i < j \leq n \}.
\]

The simple roots are given by \( \alpha_1 = 2\varepsilon_1, \alpha_i = \varepsilon_i - \varepsilon_{i-1}, 2 \leq i \leq n \). The Weyl group \( W = WC_n \) is the hyperoctahedral group of permutations of \( -n, \ldots, -1, 1, \ldots, n \) such that \( w(-i) = -w(i) \). This group acts on the \( \varepsilon_i \) by the rule \( w\varepsilon_i = \varepsilon_{w(i)} \), with the convention that \( \varepsilon_{-i} = -\varepsilon_i \).

5.2 Rearranging \( \gamma \). The analysis in this case is analogous to the method that was used in (3.7) to create books of placed configurations in the type A case. For clarity, we recommend that the reader compare the machinations below with the case done in (3.7).

Let \( \gamma \in \mathbb{R}^n \). By applying an element of the Weyl group to \( \gamma \) we can rearrange the entries of \( \gamma \) in increasing order (\( 0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n \)). Then, if \( \gamma_i \in (x, x+1/2) \) for some integer \( x \), replace \( \gamma_i \) with \( -\gamma_i \). Next group the elements of \( \gamma \) in terms of their \( \mathbb{Z} \)-cosets and rearrange each group to be in increasing order. There are three kinds of groups which can occur:

\[
\begin{align*}
\beta + C_\beta &= (\beta + z_1, \beta + z_2, \ldots, \beta + z_k), & \text{with } \beta \in (x+1/2, x+1) \text{ for some } x \in \mathbb{Z}, \\
C_0 &= (z_1 \leq z_2 \leq \cdots \leq z_k), & \text{with } z_i \in \mathbb{Z}_{\geq 0} \text{ and } z_1 \leq \cdots \leq z_k, \\
1/2 + C_{1/2} &= (1/2 + z_1 \leq 1/2 + z_2 \leq \cdots \leq 1/2 + z_k), & \text{with } z_i \in \mathbb{Z}_{\geq 0} \text{ and } z_1 \leq \cdots \leq z_k.
\end{align*}
\]

Choose some ordering on the groups and let

\[
\tilde{\gamma} = (\beta_1 + C_{\beta_1}, \ldots, \beta_r + C_{\beta_r}).
\]

Because these changes are obtained by applying elements of the Weyl group, the calibration graphs corresponding to \( \tilde{\gamma} \) and the \( \gamma \) are the same. Thus it is sufficient to study the standard tableaux corresponding to \( \tilde{\gamma} \).
(5.3) Books of placed configurations. As in the type $A$ case, we have set partitions

$$Z(\vec{\gamma}) = \bigcup_{\beta_i} Z_{\beta_i}, \quad P(\vec{\gamma}) = \bigcup_{\beta_i} P_{\beta_i},$$

and,

for any $J \subseteq P(\vec{\gamma}),$

$$J = \bigcup_{\beta_i} J_{\beta_i}, \quad \text{where } J_{\beta_i} = J \cap P_{\beta_i}.$$

Each pair $(\beta + C_\beta, J_\beta)$ is a placed shape and we may identify $(\vec{\gamma}, J)$ with the book of placed shapes $((\beta_1 + C_{\beta_1}, J_{\beta_1}), \ldots, (\beta_r + C_{\beta_r}, J_{\beta_r})).$

We think of this as a book with pages numbered by the values $\beta_1, \ldots, \beta_r$ and with a placed configuration determined by $(\beta_i + C_{\beta_i}, J_{\beta_i})$ on page $\beta_i.$ One determines the placed configurations as follows.

(5.4) Page $\beta, \beta \neq \frac{1}{2}, 0:$ As in the type $A$ case we will place boxes on a page of infinite graph paper which has the diagonals numbered consecutively with the elements of $\mathbb{Z},$ from bottom left to top right. For each $1 \leq i \leq n,$ place box $i$ on diagonal $\vec{\gamma}_i - \beta.$ The boxes on each diagonal are arranged in increasing order from top left to bottom right. Then

$$P_{\beta} = \{\varepsilon_j - \varepsilon_i \mid j > i \text{ and box}_i \text{ and box}_j \text{ are in adjacent diagonals}\} \quad \text{and}$$

$$Z_{\beta} = \{\varepsilon_j - \varepsilon_i \mid j > i \text{ and box}_i \text{ and box}_j \text{ are in the same diagonal}\}.$$

If $J_\beta \subseteq P_\beta$ arrange the boxes on adjacent diagonals according to the rules

(a) if $\varepsilon_j - \varepsilon_i \in J_\beta$ place box $j$ northwest of box $i,$ and

(a') if $\varepsilon_j - \varepsilon_i \in P_\beta \setminus J_\beta$ place box $j$ southeast of box $i.$

A standard tableau $t$ is a filling of the boxes with distinct entries from the set $\{-n, \ldots, -1, 1, \ldots, n\}$ such that if $i$ appears then $-i$ does not appear and

(a) if $j > i$ and box$_j$ and box$_i$ are in the same diagonal then $t(\text{box}_i) < t(\text{box}_j),$

(b) if $j > i,$ box$_i$ and box$_j$ are in adjacent diagonals and box$_j$ is northwest of box$_i$ then $t(\text{box}_i) > t(\text{box}_j),$

(c) if $j > i,$ box$_i$ and box$_j$ are in adjacent diagonals and box$_j$ is southeast of box$_i$ then $t(\text{box}_i) < t(\text{box}_j).$

Example. Suppose $\beta + C_\beta = \beta + (0, 0, 0, 1, 1, 2, 2, 2, 3, 3, 3)$ and

$$J_\beta = \{\varepsilon_4 - \varepsilon_1, \varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_2, \varepsilon_5 - \varepsilon_3, \varepsilon_7 - \varepsilon_5, \varepsilon_7 - \varepsilon_6, \varepsilon_8 - \varepsilon_6, \varepsilon_9 - \varepsilon_8, \varepsilon_{10} - \varepsilon_9, \varepsilon_{10} - \varepsilon_8, \varepsilon_{10} - \varepsilon_7, \varepsilon_{11} - \varepsilon_9, \varepsilon_{11} - \varepsilon_8, \varepsilon_{11} - \varepsilon_7, \varepsilon_{12} - \varepsilon_9\}.$$

The placed configuration of boxes corresponding to $(\beta + C_\beta, J_\beta)$ is as given below.

```
\begin{array}{|c|c|c|}
\hline
3 & 3 & 10 \\
1 & 2 & 11 \\
0 & 1 & 4 \\
0 & 2 & 7 \\
\hline
\end{array}
```

contents of boxes

```
\begin{array}{|c|c|c|}
\hline
10 & 11 & -10 \\
7 & 4 & -12 \\
5 & 1 & -9 \\
2 & 3 & 6 \\
1 & 2 & 9 \\
\hline
\end{array}
```

numbering of boxes

```
\begin{array}{|c|c|c|}
\hline
-7 & 1 & -5 \\
3 & 6 & 4 \\
\hline
\end{array}
```

a standard tableau
(5.5) Page $\frac{1}{2}$: We will place boxes on a page of infinite graph paper which has the diagonals numbered consecutively with the elements of $\frac{1}{2} + \mathbb{Z}$, from bottom left to top right. For each $1 \leq i \leq n$ such that $\vec{\gamma}_i \in \frac{1}{2} + \mathbb{Z}$, place box $i$ on diagonal $\vec{\gamma}_i$ and box $-i$ on diagonal $-\vec{\gamma}_i$. The boxes on each diagonal are arranged in increasing order from top left to bottom right. With this placing of boxes we have

$$P_{\frac{1}{2}} = \left\{ \begin{array}{l}
\varepsilon_j - \varepsilon_i, \text{ such that } j > i \text{ and box } j \text{ and box } i \text{ are in adjacent diagonals} \\
\varepsilon_j + \varepsilon_i, \text{ such that box } j \text{ and box } i \text{ are both in diagonal } 1/2 \\
2\epsilon_i, \text{ such that box } i \text{ is in diagonal } 1/2
\end{array} \right\},$$

$$Z_{\frac{1}{2}} = \{ \varepsilon_j - \varepsilon_i, \text{ such that } j > i \text{ and box } i \text{ and box } j \text{ are in the same diagonal} \}.$$ 

If $J_{\frac{1}{2}} \subseteq P_{\frac{1}{2}}$ arrange the boxes on adjacent diagonals according to the rules:

(a) If $\varepsilon_j - \varepsilon_i \in J_{\frac{1}{2}}$ place box $j$ northwest of box $i$ and box $-i$ northwest of box $-j$.

(b) If $\varepsilon_j - \varepsilon_i \in P_{\frac{1}{2}} \setminus J_{\frac{1}{2}}$ place box $j$ southeast of box $i$ and box $-i$ southeast of box $-j$.

(c) If $\varepsilon_j - \varepsilon_i \in J_{\frac{1}{2}}$ (j $\geq$ i) place box $j$ northwest of box $-i$ and box $i$ northwest of box $-j$.

(d) If $\varepsilon_j - \varepsilon_i \in P_{\frac{1}{2}} \setminus J_{\frac{1}{2}}$ (j $\geq$ i) place box $j$ southeast of box $-i$ and box $i$ southeast of box $-j$.

A standard tableau $t$ is a filling of the boxes with distinct entries from the set $\{ -n, \ldots, -1, 1, \ldots, n \}$ such that

$$t(\text{box } i) = -t(\text{box } -i)$$

and

(a) If $j > i$ and box $j$ and box $i$ are in the same diagonal then $t(\text{box } i) < t(\text{box } j)$,

(b) If $j > i$, box $j$ and box $i$ are in adjacent diagonals and box $j$ is northwest of box $i$ then $t(\text{box } i) > t(\text{box } j)$,

(c) If $j > i$, box $j$ and box $i$ are in adjacent diagonals and box $j$ is southeast of box $i$ then $t(\text{box } i) < t(\text{box } j)$.

Example. Suppose $\frac{1}{2} + C_{\frac{1}{2}} = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2} \right)$ and

$$J_{\frac{1}{2}} = \{ \varepsilon_{11} - \varepsilon_{10}, \varepsilon_{10} - \varepsilon_9, \varepsilon_9 - \varepsilon_8, \varepsilon_8 - \varepsilon_7, \varepsilon_7 - \varepsilon_6, \varepsilon_6 - \varepsilon_5, \varepsilon_5 - \varepsilon_4, \varepsilon_4 - \varepsilon_3, \varepsilon_3 - \varepsilon_2, \varepsilon_2 - \varepsilon_1, \varepsilon_1 - \varepsilon_3, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3, \varepsilon_1 + \varepsilon_4, 2 \varepsilon_1 \}.$$ 

The placed configuration of boxes corresponding to $(\frac{1}{2} + C_{\frac{1}{2}}, J_{\frac{1}{2}})$ is as given below.
(5.6) **Page 0:** We will place boxes on a page of infinite graph paper which has the diagonals numbered consecutively with the elements of $\mathbb{Z}$, from bottom left to top right. For each $1 \leq i \leq n$ such that $\vec{\gamma}_i \in C_0$,

1. place box$_i$ on diagonal 0 if $\vec{\gamma}_i = 0$, and
2. if $\vec{\gamma}_i \neq 0$ place box$_i$ on diagonal $\vec{\gamma}_i$ and box$_{-i}$ on diagonal $-\vec{\gamma}_i$.

The boxes on each diagonal are arranged in increasing order from top left to bottom right. With this placing of boxes we have

$$P_0 = \left\{ \begin{array}{ll}
\varepsilon_j - \varepsilon_i, & \text{such that } j > i \text{ and box}_i \text{ and box}_j \text{ are in adjacent diagonals} \\
\varepsilon_j + \varepsilon_i, & \text{such that box}_j \text{ is in diagonal 1 and box}_i \text{ is in diagonal 0}
\end{array} \right\},$$

$$Z_0 = \left\{ \begin{array}{ll}
\varepsilon_j - \varepsilon_i, & \text{such that } j > i \text{ and box}_i \text{ and box}_j \text{ are in the same diagonal} \\
2\varepsilon_i, & \text{such that box}_i \text{ is in diagonal 0}
\end{array} \right\}.$$

If $J_0 \subseteq P_0$ then arrange the boxes on adjacent diagonals according to:

(a) if $\varepsilon_j - \varepsilon_i \in J_0$ place box$_j$ northwest of box$_i$ and if box$_i$ is not on diagonal 0 place box$_{-i}$ northwest of box$_{-j}$,

(a) if $\varepsilon_j - \varepsilon_i \in P_0 \setminus J_0$ place box$_j$ southeast of box$_i$ and if box$_i$ is not on diagonal 0 place box$_{-i}$ southeast of box$_{-j}$,

(b) if $\varepsilon_j + \varepsilon_i \in J_0$ place box$_i$ northwest of box$_{-j}$,

(b) if $\varepsilon_j + \varepsilon_i \in P_0 \setminus J_0$ place box$_i$ southeast of box$_{-j}$.

A **standard tableau** $t$ is a filling of the boxes with distinct entries from the set \{-n, \ldots, -1, 1, \ldots, n\} such that

$$t(\text{box}_i) = -t(\text{box}_{-i}), \quad \text{if box}_i \text{ is not on the zero diagonal},$$

$$t(\text{box}_i) > 0, \quad \text{if box}_i \text{ is on the zero diagonal},$$

and

(a) if $j > i$ and box$_j$ and box$_i$ are in the same diagonal then $t(\text{box}_i) < t(\text{box}_j)$,

(b) if $j > i$, box$_i$ and box$_j$ are in adjacent diagonals and box$_j$ is northwest of box$_i$ then $t(\text{box}_i) > t(\text{box}_j)$,

(c) if $j > i$, box$_i$ and box$_j$ are in adjacent diagonals and box$_j$ is southeast of box$_i$ then $t(\text{box}_i) < t(\text{box}_j)$.

**Example.** Suppose $C_0 = \{0, 0, 0, 1, 1, 1, 2\}$ and

$$J_0 = \{\varepsilon_4 - \varepsilon_1, \varepsilon_4 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_1, \varepsilon_5 - \varepsilon_2, \varepsilon_5 - \varepsilon_3, \varepsilon_6 - \varepsilon_1, \varepsilon_6 - \varepsilon_2, \varepsilon_6 - \varepsilon_3, \varepsilon_7 - \varepsilon_6, \varepsilon_6 + \varepsilon_1, \varepsilon_5 + \varepsilon_1, \varepsilon_5 + \varepsilon_2, \varepsilon_4 + \varepsilon_1, \varepsilon_4 + \varepsilon_2\}.$$

The placed configuration of boxes corresponding to $(C_0, J_0)$ is as given below.

![Contents of boxes](image1.png)

![Numbering of boxes](image2.png)

![A standard tableau](image3.png)

contents of boxes  | numbering of boxes  | a standard tableau
Using the above rules one produces a book of placed configurations corresponding to 
\((\vec{\gamma}, J) = ((\beta_1 + C_{\beta_1}, J_{\beta_1}), \ldots, (\beta_r + C_{\beta_r}, J_{\beta_r}))\). A standard tableau \(t\) for this book of configurations is a filling of the boxes with distinct elements of \([-n, \ldots, -1, 1, \ldots, n]\) such that the filling on each page satisfies the conditions for a standard tableau for that page. Let 
\[F((C_{\beta_1}, J_{\beta_1}), \ldots, (C_{\beta_r}, J_{\beta_r}))\]
denote the set of such fillings.

The proof of the following Theorem is similar to the proof of Theorem (3.5).

**Theorem.** Given a standard tableau \(t\) for the book of configurations \(((C_{\beta_1}, J_{\beta_1}), \ldots, (C_{\beta_r}, J_{\beta_r}))\) define \(w_t \in WC_n\) by \(w_t(i) = t(\text{box}_i)\). Then the map 
\[
F((C_{\beta_1}, J_{\beta_1}), \ldots, (C_{\beta_r}, J_{\beta_r})) \leftrightarrow F(\vec{\gamma}, J)
\]
t \[\mapsto\]
w_t

is a bijection.
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