Mean Field Game of Optimal Relative Investment with Contagious Risk

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Abstract

This paper studies a mean field game (MFG) in a market with a large population of agents. Each agent aims to maximize the terminal wealth under the CRRA relative performance, in which the interaction occurs by peer competition. We start from the model with $n$ heterogeneous agents, in which the underlying risky assets subject to a common noise and contagious jump risk modelled by a multi-dimensional Hawkes process. With a continuum of agents, we formulate the MFG problem and construct a deterministic mean field equilibrium in an analytical form under some sufficient conditions, allowing us to investigate some impacts of model parameters in the limiting model and discuss their financial implications. Moreover, it is shown that this mean field equilibrium can serve as an approximate Nash equilibrium for the $n$-player game when $n$ is sufficiently large. The explicit order of the approximation error is also derived.

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1 Introduction

Motivated by the fact that peer competition sometimes has notable impacts on the fund manager’s decision making, optimal investment under relative performance for a finite number of agents and a continuum of agents has been an important research topic in recent years. Espinosa and Touzi (2015) and Bielagk et al. (2017) study $n$-agent games under the CARA utility including the equilibrium pricing and portfolio constraints by investigating the associated quadratic BSDEs systems. Lacker and Zariphopoulou (2019) consider the $n$-player game in log-normal markets under both CARA and CRRA relative performance criteria with the asset specialization to each agent. The constant equilibrium is obtained therein, and the MFG problem is formulated and solved when the utility incorporates a mean field competition component. In a similar fashion, Lacker and Soret (2020) generalize the problems to account for the dynamic consumption. Fu et al. (2020)
extend the model to Itô-diffusion price dynamics and solve the $n$-player game and MFG under the CARA relative performance by using a FBSDE approach. Dos Reis and Platonov (2020) examine MFGs under forward utilities of CARA type. Kraft et al. (2020) formulate and solve some $n$-player games in the incomplete market model with unhedgeable stochastic factors. Hu and Zariphopoulou (2021) recently investigate the $n$-player game and MFG in the case of incomplete Itô-diffusion market model and also in the case with random risk tolerance coefficients.

To the best of our knowledge, the $n$-player games and MFGs under relative performance when the underlying price dynamics exhibit jumps have not been studied before. On the other hand, the importance of considering defaultable risky assets, especially after the systemic failure caused by the global financial crisis, has attracted a lot of attention; see, for example, Bélanger et al. (2004), Yu (2007) and references therein. To better understand the impact by systemic default risk on dynamic portfolio management, abundant recent works have considered optimal investment problems when jumps of risky assets are contagious. See, for example, Bo and Capponi (2018), Bo et al. (2019a), Bo et al. (2019b), Shen and Zou (2020), Bo et al. (2021) among others that are based on the interacting intensity framework, allowing the credit default in one risky asset to increase the default intensities of other surviving names. See also Jin et al. (2021) in the context of optimal dividend control for an insurance group.

The present paper aims to enrich the study of the aforementioned $n$-player games and MFGs under relative performance by featuring contagious jump risk across all risky assets. In particular, the jump risk in price dynamics is modelled by a $n$-dimensional mutually exciting Hawkes process, whose componentwise intensity process satisfies the specific form of (2.2). As a result, the contagion phenomenon can be depicted because the jump of one risky asset leads to a larger jump intensity of all other risky assets. Meanwhile, we adopt the asset specialization framework in Lacker and Zariphopoulou (2019) with a common noise and focus on the CRRA relative performance utility. As opposed to the previous works, the presence of contagious jumps in price dynamics give rise to the controlled jump component. The common noise and the controlled jumps together complicate the analysis of the $n$-player game and the MFG significantly.

Starting from the seminal works by Lasry and Lions (2007) and Huang et al. (2006), MFGs have been actively studied and widely applied in economics, finance and engineering. Giving a full list of references is beyond the scope of this paper. For some recent developments in theories and applications, we refer to Guéant et al. (2011), Bensoussan et al. (2013), Carmona (2016), Carmona and Delarue (2018) and references therein. However, we also note the majority of existing research has focused on models when the controlled state processes have continuous paths, and the study of MFGs with controlled jumps is relatively rare. In the simple setting of inhomogeneous Poisson process, Nutz and Zhang (2019) consider the rank-based mean field competition when each agent controls the intensity of the Poisson project process. Yu et al. (2021) further extended the model to some two-layer mean field competitions based on teamwork formulations, in which team members collaborate to control the intensity of the Poisson project process. Gomes et al. (2013) and Neumann (2020) examine some MFGs with continuous time Markov chains. Hafayed et al. (2014) deals the McKean-Vlasov stochastic control problems. Recently, Benazzoli et al. (2020) study MFGs with controlled jump-diffusion processes, in which the jump component is driven by a Poisson process with a time-dependent intensity function. The existence of a Nash equilibrium is obtained therein.
by using relaxed control and martingale problem arguments. Building upon results in Benazzoli et al. (2020), Benazzoli et al. (2019) further verify that the mean field Nash equilibria can be used as an approximate Nash equilibrium in the \( n \)-player game when \( n \) is large enough and the rate of convergence is obtained.

Although our targeted MFG is in the realm of MFGs with controlled jump-diffusion processes, our model and methodology differ substantially from the ones in Benazzoli et al. (2020) (see also Benazzoli et al. (2019)). To be precise, the MFG problem in Benazzoli et al. (2020) stems from a symmetric nonzero-sum \( n \)-player game, in which the Poisson jump process for each player has the same deterministic intensity and all players have the same objective function. In contrast, our \( n \)-player game is formulated for heterogeneous agents with different underlying processes and relative performance utilities. The more complicated contagious jump risk is a new feature of our \( n \)-player game and a common noise appears in all risky assets, which are not concerned by Benazzoli et al. (2020). Therefore, the relaxed control approach in Benazzoli et al. (2020) may not work in our new problem. The mathematical contribution of the present paper is two-fold. First, we model the contagious jump risk in the \( n \)-player game by a mutual-exciting Hawkes process, which enables us to formulate a tractable MFG problem with controlled jumps under the assumption of constant type vector in the limiting model. The strict control approach can be applied and we can characterize a deterministic mean field equilibrium in an analytical form by using FBSDE and the stochastic maximum principle arguments; see Theorem 3.2. Some quantitative properties of the obtained mean field equilibrium are examined, yielding some interesting financial implications. Second, despite the lost tractability in the \( n \)-player game, we show that the mean field equilibrium provides an \( \varepsilon_n \)-approximation to the Nash equilibrium in the model with finite \( n \) agents when \( n \) is sufficiently large. We highlight that the explicit convergence rate of the approximation error \( \varepsilon_n \) is also obtained; see Theorem 5.3. This simple form mean field equilibrium can efficiently help to reduce the dimensionality of the \( n \)-player game in practical applications.

The rest of the paper is organized as follows. In Section 2, we introduce the \( n \)-player game under CRRA relative performance preference when the underlying risky assets are affected by a common noise and the contagious default risk modelled by a multi-dimensional Hawkes process. In Section 3, we formulate the MFG problem in the limiting model and obtain a time-dependent deterministic mean field equilibria. Section 4 presents some quantitative properties and numerical sensitivity results on the mean field equilibrium. Section 5 establishes an approximate Nash equilibrium for a \( n \)-player game problem. Some conclusion remarks and future directions are given in Section 6. Finally, the proofs of some auxiliary results are reported in Appendix A.

2 Market Model and Relative Performance

We consider in this section the financial market model with \( n \) agents. Each agent \( i \) invests in a common riskless bond and one individual risky asset \( i \). The common time horizon for all agents is denoted by \( T > 0 \). For \( i = 1, \ldots, n \), the price process of the \( i \)th risky asset follows the SDE that

\[
\frac{dS^i_t}{S^i_t} = (r + b_i)dt + \sigma_idW^i_t + \sigma^0_i dW^0_t - dM^i_t, \quad t \in [0, T],
\]
with the given parameters $b_i \in \mathbb{R}_+$, $\sigma_i, \sigma_i^0 > 0$. Here, $r \geq 0$ represents the riskless interest rate, and $(W^0_t, W^1_t, \ldots, W^n_t)_{t \in [0,T]}$ is a $n+1$-dimensional Brownian motion under the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Brownian motion $W^0 = (W^0_t)_{t \in [0,T]}$ appears in all price dynamics, which represents a common noise in the financial market. The Brownian motion $W^i = (W^i_t)_{t \in [0,T]}$, specified to each individual risky asset, stands for the idiosyncratic market noise. In particular, we denote $\mathbf{N} := (N^1_t, \ldots, N^n_t)_{t \in [0,T]}$ as an $n$-dimensional mutually exciting point process modeled by a Hawkes process. The intensity process is defined by $\mathbf{\Lambda} := (\Lambda^1_t, \ldots, \Lambda^n_t)_{t \in [0,T]}$, and the compensated process of $\mathbf{N}$, defined by $\mathbf{M}_t := \mathbf{N}_t - \int_0^t \mathbf{\Lambda}_s ds$, $t \in [0,T]$, is a $(\mathbb{P}, \mathcal{G})$-martingale. The global market filtration $\mathcal{G} = (\mathcal{G}_t)_{t \in [0,T]}$ is defined by $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\mathbf{N}_s; s \leq t)$ as the right-continuous augmentation by null sets (see Karatzas and Shreve (1991), Definition 7.2 in Chapter 2). By Bo and Capponi (2018), the Brownian motion under $\mathbb{F}$ is also a Brownian motion under $\mathcal{G}$, i.e., the immersion property holds.

It is assumed in the present paper that the vector of intensity processes $\mathbf{\Lambda} = (\Lambda^1_t, \ldots, \Lambda^n_t)_{t \in [0,T]}$ is governed by the dynamics that

$$d\Lambda^i_t = \alpha_i(\lambda^i_{\infty} - \Lambda^i_t)dt + \frac{1}{n} \sum_{j=1}^n \beta_{ij} \Lambda^j_t dN^j_t, \quad \Lambda^i_0 = \lambda^i_0 > 0, \quad i = 1, \ldots, n, \quad (2.2)$$

where $\lambda^i_{\infty} > 0$ is the mean-reverting default intensity level of stock $i$ with the speed $\alpha_i > 0$ and the $j$-th entry of the weight vector $\beta_{(i)} := (\beta_{ij})_{i=1}^n \in \mathbb{R}^n_+$ measures the extent to which the jump risk of stock $j$ affects the intensity of stock $i$. The contagious risk can be captured because a downward jump of one risky asset will increase the jump intensity of all other risky assets, leading to a higher risk of default clustering.

For $t \in [0,T]$, let the $\mathcal{G}$-predictable process $\pi^i_t$ be the proportion of wealth that the agent $i$ allocates in the stock $S^i$ at time $t$. In view of the stock price dynamics (2.1), the self-financing wealth process of agent $i$ under the control $\pi^i = (\pi^i_t)_{t \in [0,T]}$ is given by

$$X^i_t = x_i + \int_0^t (rX^i_s + b_i \pi^i_s X^i_s) \, ds + \int_0^t \pi^i_s X^i_s \sigma_i \, dW^i_s + \int_0^t \pi^i_s X^i_s \sigma^0_i \, dW^0_s - \int_0^t \pi^i_s X^i_s \, dM^i_s, \quad (2.3)$$

where $x_i > 0$ denotes the initial wealth of the agent $i$. The portfolio vector is denoted by $\pi := (\pi^1_t, \ldots, \pi^n_t)_{t \in [0,T]}$.

Let us denote $\mathcal{A}^i$ the set of admissible controls for the agent $i$. We say a control process $\pi^i = (\pi^i_t)_{t \in [0,T]} \in \mathcal{A}^i$ is admissible if $\pi^i$ is $\mathcal{G}$-predictable and satisfies $D_0 \leq \pi^i_t \leq 1 - \epsilon_0$ for some constant $D_0 \in \mathbb{R}$ and positive constant $\epsilon_0 \ll 1$ (both $D_0$ and $\epsilon_0$ depend on the control) such that the non-bankruptcy condition $X^i_t > 0$ holds a.s. for $t \in [0,T]$. Note that the pure jump martingale $M^i = (M^i_t)_{t \in [0,T]}$ has the jump size of 1. In view of (2.3), the wealth process $X^i = (X^i_t)_{t \in [0,T]}$ must be positive a.s. if the initial wealth level $x_i > 0$ because the admissible control is constrained to satisfy $\pi^i_t < 1$ for all $t \in [0,T]$. Each agent in the market aims to maximize the expected utility with a competition component represented by the geometric average of the terminal wealth $X_T = (X^1_T, \ldots, X^n_T)$ from all peers. The objective function of the agent $i$ is given by

$$J_i(\pi^1, \ldots, \pi^n) := \mathbb{E} \left[ U_i \left( X^i_T, \bar{X}_T \right) \right], \quad \text{with} \quad \bar{X}_T := \left( \prod_{i=1}^n X^i_T \right)^{\frac{1}{n}}, \quad (2.4)$$
in which the utility function $U_i : \mathbb{R}_+^2 \to \mathbb{R}$ of the agent $i$ is of the CRRA type that

$$U_i(x, m) := U(xm^{-\theta_i}; \gamma_i),$$

and $U(x; \gamma)$ is a power utility that $U(x; \gamma) := \frac{1}{\gamma} x^\gamma$. It is assumed in the present paper that all risk aversion parameters $\gamma_i \in (0, 1)$ and all competition weight parameters $\theta_i \in [0, 1]$.

In the $n$-player game problem, let us give the definition of the approximate Nash equilibrium.

**Definition 2.1.** Let the objective functional $J_i$ be defined in (2.4). An admissible strategy $\pi^* = (\pi^{*1}, \ldots, \pi^{*n}) \in \mathcal{A} := \prod_{i=1}^{n} \mathcal{A}^i$ is called a Nash equilibrium if, for all $\pi^i \in \mathcal{A}^i$ with $i = 1, \ldots, n$, it holds that

$$J_i(\pi^*) \geq J_i(\pi^i, \pi^{*\cdot-i}), \quad \text{with} \quad \pi^{*\cdot-i} := (\pi^{*,1}, \ldots, \pi^{*,i-1}, \pi^{*,i+1}, \ldots, \pi^{*,n}).$$

(2.6)

If there exists a constant $\varepsilon_n > 0$ satisfying $\lim_{n \to \infty} \varepsilon_n = 0$, and it holds that

$$\sup_{\pi^i \in \mathcal{A}^i} J_i(\pi^i, \pi^{*\cdot-i}) \leq J_i(\pi^*) + \varepsilon_n,$$

(2.7)

we call $\pi^*$ an $\varepsilon_n$-Nash equilibrium.

The contagious jump risk and the common noise in the SDE (2.3) together complicate the analysis of the Nash equilibrium in the model with $n$ agents. In response to these challenges, we first consider the limiting model when the number $n$ grows to infinity and look for a Nash equilibrium strategy in the mean field game problem. Next, we use the mean field solution to construct an approximate Nash equilibrium in the $n$-player game when $n$ is sufficiently large.

### 3 Mean Field Game Problem

The goal of this section is to study the optimal relative investment problem with infinitely many agents as a mean field game problem.

For $i = 1, \ldots, n$, let us first denote the type vector $o^i := (x_i, \lambda_0^i, \alpha_i, \lambda_\infty^i, \beta_i, \varsigma_i, b_i, \sigma_i, \sigma_0^i, \gamma_i, \theta_i) \in O := \mathbb{R}_+^3 \times (0, 1) \times [0, 1]$ and the space $E := O \times \mathbb{R}_+^2$. Let $\mathcal{B}(E)$ (resp. $\mathcal{P}(O)$) be the Borel $\sigma$-algebra generated by the open sets of $E$ (resp. the set of probability measures on $O$).

For mathematical tractability, the following assumption is imposed throughout the paper.

**($A_0$):** There exists a probability measure $\nu_0 \in \mathcal{P}(O)$ such that

$$\nu_0^i := \frac{1}{n} \sum_{i=1}^{n} \delta_{o^i} \Rightarrow \nu_0 := \delta_0, \quad \text{as} \ n \to \infty,$$

where the convergence holds with the order of $O(n^{-1})$ and “$\Rightarrow$” denotes the weak convergence, i.e., $\int_O f \, \nu_0^i \to \int_O f \, \nu_0$ as $n \to \infty$ for every bounded continuous function $f$ on $O$, and $o := (x_0, \lambda_0, \alpha, \lambda_\infty, \beta, \varsigma, b, \sigma, \sigma_0, \gamma, \theta) \in O$ denotes the limiting constant vector in $O$. In addition, it is assumed that there exists a constant vector $o_1 := (b, \sigma, \sigma_0, \gamma, \theta) \in O_1 := \mathbb{R}_+^3 \times (0, 1) \times [0, 1]$ such that $(b_i, \sigma_i, \sigma_0^i, \gamma_i, \theta_i) \to o_1$ as $i \to \infty$. 

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The wealth process of a representative agent in the mean field model is governed by
\begin{equation}
\dot{X}_t^\pi = \lambda_t^\pi + \int_0^t \alpha (\lambda_\infty - \lambda_s^\pi) ds + \int_0^t \beta \epsilon \lambda_s^\pi ds.
\end{equation}

It follows that \( \lambda_t^\pi \) admits the explicit form that
\begin{equation}
\lambda_t^\pi = \frac{\alpha \lambda_\infty}{\alpha - \beta \epsilon} + \left( \lambda_0 - \frac{\alpha \lambda_\infty}{\alpha - \beta \epsilon} \right) e^{(\beta \epsilon - \alpha) t}.
\end{equation}

The wealth process of a representative agent in the mean field model is governed by
\begin{equation}
\begin{aligned}
dX_t^o &= (rX_t^o + b\pi_t X_t^o) dt + \pi_t X_t^o (\sigma dW_t + \sigma^0 dW_t^0 - dM_t), \\
X_0^o &= x_0 > 0,
\end{aligned}
\end{equation}
where \( W = (W_t)_{t \in [0,T]} \) is a 1-dimensional Brownian motion under \((\Omega, \mathcal{F}, \mathbb{P})\) that is independent of \((W^0, W^1, \ldots, W^n)\). The pure jump martingale \( M = (M_t)_{t \in [0,T]} \) satisfies the decomposition that
\begin{equation}
M_t = N_t^o - \int_0^t \lambda_s^\pi ds, \quad t \in [0, T],
\end{equation}
where \( N^o = (N_t^o)_{t \in [0,T]} \) is a Poisson point process with the deterministic intensity process \((\lambda_t^\pi)_{t \in [0,T]}\).

For a sufficiently large \( n \), in view of the common noise \( W^0 \) in the wealth process \( X_t^i \), we may approximate the geometric mean \( \bar{X} = (\bar{X}_t)_{t \in [0,T]} \) by a càdlàg \( \mathbb{F}^0 = (\mathcal{F}_t^0)_{t \in [0,T]} := (\sigma(W_s^0), s \leq t)_{t \in [0,T]} \)-adapted process \( m = (m_t)_{t \in [0,T]} \) (i.e., \( m \in \mathcal{D} \), where \( \mathcal{D} \) is the set of \( \mathbb{F}^0 \)-adapted processes that are right continuous with left limits). Let \( \mathcal{G}_{\text{MF}} = (\mathcal{G}_{\text{MF}}^t)_{t \in [0,T]} \) denote the smallest filtration satisfying the usual assumptions, in which \( W, W^0, N^o \) are adapted. We denote \( \mathcal{A}_{\text{MF}} \) the set of admissible controls when \( \pi = (\pi_t)_{t \in [0,T]} \) is \( \mathcal{G}_{\text{MF}} \)-predictable and satisfies \( D_0 \leq \pi_t \leq 1 - \epsilon_0 \) with some constant \( D_0 \in \mathbb{R} \) and positive constant \( \epsilon_0 \ll 1 \) (depending on the control) such that \( X^o = (X_t^o)_{t \in [0,T]} \) has no bankruptcy.

The mean field game problem is then solved by following two steps:

**Step 1.** For a fixed \( \mathbb{F}^0 \)-adapted process \( m = (m_t)_{t \in [0,T]} \in \mathcal{D} \), we solve a stochastic control problem for a representative agent against the fixed environment that
\begin{equation}
\sup_{\pi \in \mathcal{A}_{\text{MF}}} J(\pi) = \sup_{\pi \in \mathcal{A}_{\text{MF}}} \mathbb{E} \left[ U(X_T^\pi m_T^\theta; \gamma) \right] = \sup_{\pi \in \mathcal{A}_{\text{MF}}} \mathbb{E} \left[ \frac{1}{\gamma} (X_T^\pi)^\gamma m_T^{-\theta \gamma} \right],
\end{equation}
where the wealth process \( X_T^\pi \) satisfies (3.3) with the risk aversion parameter \( \gamma \in (0,1) \) and the competition parameter \( \theta \in [0,1] \). The best response strategy for the representative agent is denoted by \( \pi^{*,m} \in \mathcal{A}_{\text{MF}} \), and \( X^{*,m} \) stands for the wealth process in (3.3) under the control \( \pi^{*,m} \).

**Step 2.** We next derive a mean field equilibrium as a fixed point by the consistency condition that \( m_t = \exp(\mathbb{E}[\log(X_t^{*,m})|\mathcal{F}_t^0]) \) for all \( t \in [0, T] \).

Let us first introduce the definition of mean field equilibrium.

**Definition 3.1.** For a given \( \mathbb{F}^0 \)-adapted process \( m = (m_t)_{t \in [0,T]} \in \mathcal{D} \), let \( \pi^{*,m} \in \mathcal{A}_{\text{MF}} \) be the best response solution to the stochastic control problem (3.4). The strategy \( \pi^* := \pi^{*,m^*} \) is called a mean field equilibrium (MFE) if it is the best response to itself such that \( m_t^* = \exp(\mathbb{E}[\log(X_t^{*,m^*})|\mathcal{F}_t^0]) \), \( t \in [0, T] \), where \( (X_t^{*,m^*})_{t \in [0,T]} \) is the wealth process in (3.3) under the control \( \pi^{*,m^*} \). Moreover, if \( \pi^* \) is deterministic, we call \( \pi^* \) a deterministic MFE.
To facilitate the proof of the main theorem in this section, we first present a preparation by the next auxiliary lemma, whose proof is reported in Appendix A.

Lemma 3.1. Define the function \( \Phi(\pi, \lambda) : (-\infty, 1) \times \mathbb{R}_+ \to \mathbb{R} \) with \( \sigma + \sigma^0 > 0 \) and \( b > 0 \) that
\[
\Phi(\pi, \lambda) := (\gamma - 1)[\sigma^2 + (\sigma^0)^2]\pi - \theta \gamma (\sigma^0)^2 \pi - \lambda (1 - \pi)^{\gamma - 1} + \lambda + b. \tag{3.5}
\]
Then, for each fixed \( \lambda \in \mathbb{R}_+ \), there exists a unique \( \pi^* \in (-\infty, 1) \) such that
\[
\Phi(\pi^*, \lambda) = 0. \tag{3.6}
\]
Moreover, there exists an \( \epsilon_0 \in (0, 1) \) small enough such that \( \pi^* \in (0, 1 - \epsilon_0) \). Equivalently, for \( \pi^* \) satisfying (3.6), there exists a unique continuous and decreasing function \( \phi : \mathbb{R}_+ \to (0, 1 - \epsilon_0) \) such that
\[
\pi^* = \phi(\lambda), \tag{3.7}
\]
where \( \phi(\lambda) \) has a continuous partial derivative with respect to \( \lambda \).

We can now establish the main result of this section that gives a time-dependent deterministic mean field equilibrium in the MFG problem as a function of the deterministic limiting intensity process.

Theorem 3.2. There exists one deterministic MFE strategy \( \pi^* = (\pi^*_t)_{t \in [0,T]} \in \mathcal{A}_{MF} \) that satisfies
\[
\Phi(\pi^*_t, \lambda^*_0) = 0, \quad t \in [0,T]. \tag{3.8}
\]
Equivalently, this deterministic MFE strategy \( \pi^* = (\pi^*_t)_{t \in [0,T]} \) can be written by
\[
\pi^*_t = \phi(\lambda^*_t), \quad t \in [0,T]. \tag{3.9}
\]
Here, \( \Phi(\pi, \lambda) \) and \( \phi(\lambda) \) are given in Lemma 3.1. In addition, let \( X^* = (X^*_t)_{t \in [0,T]} \) be the wealth process under the deterministic MFE \( \pi^* \). The associated fixed point \( m^* = (m_t)_{t \in [0,T]} \) satisfying the consistency condition \( m^*_t = \exp\{\mathbb{E}[\log(X^*_t)|F^0_t] \} \) for \( t \in [0,T] \) is characterized by
\[
m^*_t = x_0 \exp\left\{ \int_0^t \left( \eta(s; \pi^*_s) - \frac{1}{2}(\sigma^0 \pi^*_s)^2 \right) ds + \int_0^t \sigma^0 \pi^*_s dW^0_s \right\}, \tag{3.10}
\]
where the function \( \eta(t; \pi) \) is given by, for \( (t, \pi) \in [0,T] \times (-\infty, 1) \),
\[
\eta(t; \pi) = r + (b + \lambda^0)\pi - \frac{1}{2} \sigma^2 \pi^2 + \lambda^0 \log(1 - \pi). \tag{3.11}
\]
Here, \( \lambda^0 = (\lambda^0_t)_{t \in [0,T]} \) is given by (3.2).

Remark 3.3. Note that if there is no contagious jump risk in price dynamics, i.e., \( \lambda^0_t \equiv 0 \), \( t \in [0,T] \), the function \( \Phi(\pi, 0) \) defined in (3.5) is reduced to
\[
\Phi(\pi, 0) = (\gamma - 1)[\sigma^2 + (\sigma^0)^2]\pi - \theta \gamma (\sigma^0)^2 \pi + b, \quad \pi \in \mathbb{R}. \tag{3.2}
\]
It follows that
\[
\pi^*_t = \phi(0) = \frac{b}{(1 - \gamma)[\sigma^2 + (\sigma^0)^2] + \theta \gamma (\sigma^0)^2},
\]
which is a constant mean field Nash equilibrium that coincides with the result in Theorem 3.6 of Lacker and Zariphopoulou (2019). Therefore, the obtained MFE given in (3.9) is a generalization of the constant MFE in Lacker and Zariphopoulou (2019) due to additional contagious jump risk.
**Proof of Theorem 3.2.** First, for a given càdlàg process \( m = (m_t)_{t \in [0,T]} \in \mathcal{D} \), we aim to solve the mean field stochastic control problem via stochastic maximum principle; see Øksendal and Sulem-Bialobroda (2005). To this end, we first note that the Hamiltonian function corresponding to the control problem (3.4) is given by

\[
H(t,x,\pi,p,q,q^0,y) := (rx + bx\pi)p + \sigma \pi x q + \sigma^0 \pi x q^0 - \pi x \lambda t y,
\]

for \((t,x,\pi,p,q,q^0,y) : [0,T] \times \mathbb{R}_+ \times U \times \mathbb{R}^4\) with the policy space \( U := (-\infty, 1 - \epsilon_0] \).

Let \( \pi^m = (\pi^m_t)_{t \in [0,T]} \in \mathcal{A}_{MF} \) be an arbitrary admissible strategy that may depend on \( m \), and \( X^m = (X^m_t)_{t \in [0,T]} \) be the corresponding wealth process under \( \pi^m \). Then, the adjoint forward-backward SDEs (corresponding to \((\pi^m, X^m)\)) are given by

\[
\begin{align*}
\begin{cases}
\quad dX^m_t = \partial_p H(t,x^m_t,\pi^m_t,P^m_t,Q^m_t,Q^0,m_t,Y^m_t)dt + \pi^m_t X^m_t \left( \sigma dW^m_t + \sigma^0 dW^0_t - dM_t \right), \\
\quad dP^m_t = -\partial_x H(t,x^m_t,\pi^m_t,P^m_t,Q^m_t,Q^0,m_t,Y^m_t)dt + \pi^m_t Y^m_t dW^m_t + Q^0,m_t dW^0_t + Y^m_t dM_t, \\
\quad X^m_0 = x_0, \\
\quad P^m_T = (X^m_T)^{\gamma - 1} m^{-\theta \gamma}_T,
\end{cases}
\end{align*}
\]

where \( \partial_p H \) (resp. \( \partial_x H \)) denotes the partial derivative w.r.t. \( p \) (resp. \( x \)). In view of (3.12), we get that

\[
\begin{align*}
\begin{cases}
\quad dX^m_t = (rX^m_t + b\pi^m_t X^m_t)dt + \pi^m_t X^m_t \left( \sigma dW^m_t + \sigma^0 dW^0_t - dM_t \right), \\
\quad dP^m_t = \left[ -r P^m_t + \pi^m_t (bP^m_t + \sigma Q^m_t + \sigma^0 Q^0,m^m_t - \lambda_t Y^m_t) \right]dt + Q^0,m_t dW^0_t + Y^m_t dM_t, \\
\quad X^m_0 = x_0, \\
\quad P^m_T = (X^m_T)^{\gamma - 1} m^{-\theta \gamma}_T.
\end{cases}
\end{align*}
\]

We next solve FBSDE (3.13) in terms of \((\pi^m, X^m)\) explicitly. To do it, it follows from (3.13) that

\[
\begin{align*}
\quad d \log X^m_t = \left[ r + (b + \lambda^0_t) \pi^m_t - \frac{1}{2}(\sigma^2 + (\sigma^0)^2)(\pi^m_t)^2 \right] dt + \pi^m_t \sigma dW^m_t + \pi^m_t \sigma^0 dW^0_t + \log \left( 1 - \pi^m_t \right) dN^0_t. \\
\end{align*}
\]

Recall that \( \mathbb{F}^0 \) is the filtration generated by the Brownian motion \( W^0 = (W^0_t)_{t \in [0,T]} \). Taking conditional expectations on both sides of (3.14) w.r.t. \( \mathcal{F}^0_t \) for \( t \in [0,T] \), and using Lemma B.2 in Giesecke et al. (2015), we can deduce that

\[
\begin{align*}
\quad d \mathbb{E} [\log X^m_t | \mathcal{F}^0_t] = \left( r + (b + \lambda^0_t) \mathbb{E} [\pi^m_t | \mathcal{F}^0_t] - \frac{1}{2}(\sigma^2 + (\sigma^0)^2) \mathbb{E} [(\pi^m_t)^2 | \mathcal{F}^0_t] \right) dt \quad + \sigma^0 \mathbb{E} [\pi^m_t | \mathcal{F}^0_t] dW^0_t + \lambda^0_t \mathbb{E} [\log (1 - \pi^m_t) | \mathcal{F}^0_t] dt.
\end{align*}
\]

Let us denote \( m^X_t := \mathbb{E} [\log X^m_t | \mathcal{F}^0_t] \) for \( t \in [0,T] \). It follows from Itô lemma that

\[
\begin{align*}
\quad dm^X_t = d \mathbb{E} [\log X^m_t | \mathcal{F}^0_t] \quad = m^X_t \left( r + (b + \lambda^0_t) \mathbb{E} [\pi^m_t | \mathcal{F}^0_t] - \frac{1}{2}(\sigma^2 + (\sigma^0)^2) \mathbb{E} [(\pi^m_t)^2 | \mathcal{F}^0_t] \right) dt.
\end{align*}
\]
where we have used the notation

\[
\hat{\eta}(t; \pi^m) := r + (b + \lambda_0^m) \mathbb{E}[\pi^m_t | \mathcal{F}_t^0] - \frac{1}{2} \left( \sigma^2 + (\sigma^0)^2 \right) \mathbb{E}[\pi^m_t | \mathcal{F}_t^0]
\]

+ \frac{1}{2} \left( \sigma^0 \mathbb{E}[\pi^m_t | \mathcal{F}_t^0] \right)^2 + \lambda_0^m \mathbb{E}[\log(1 - \pi^m_t) | \mathcal{F}_t^0].
\]

To solve the FBSDE (3.13), we consider the ansatz that

\[
P_t^m = (X_t^m)^{\gamma - 1}(m_{t}^{X})^{-\theta \gamma} \varphi_t, \quad t \in [0, T],
\]

where \( \varphi : [0, T] \to \mathbb{R} \) is a deterministic function of class \( C^1 \), which satisfies the terminal condition \( \varphi_T = 1 \). First, note that \( P_T^m = (X_T^m)^{\gamma - 1}(m_{T}^{X})^{-\theta \gamma} \) holds trivially. Applying Itô lemma to \( P_t^m \), we can obtain that

\[
dP_t^m = P_t^m \left\{ \frac{\dot{X}_t^m}{X_t^m} + (\gamma - 1) \left[ r + (b + \lambda_0^m) \pi_t^m \right] - \theta \gamma \hat{\eta}(t; \pi^m) - \theta \gamma (\gamma - 1)(\sigma^0)^2 \pi_t^m \mathbb{E}[\pi^m_t | \mathcal{F}_t^0] \right\}
\]

+ \frac{1}{2} \left( \gamma - 1 \right)(\gamma - 2) \left( \sigma^2 + (\sigma^0)^2 \right) \left( \pi_t^m \right)^2 + \frac{1}{2} \theta \gamma (\theta \gamma + 1)(\sigma^0)^2 \left( \mathbb{E}[\pi^m_t | \mathcal{F}_t^0] \right)^2
\]

+ \left[ (1 - \pi_t^m)^{\gamma - 1} - 1 \right] \lambda_0^m \right\} dt
\]

+ (\gamma - 1) P_t^m \pi_t^m \sigma dW_t + P_t^m \left[ (\gamma - 1) \pi_t^m \sigma^0 - \theta \gamma \sigma^0 \mathbb{E}[\pi^m_t | \mathcal{F}_t^0] \right] dW_t^0
\]

+ \left( P_t^m \left[ (1 - \pi_t^m)^{\gamma - 1} - 1 \right] \right) dM_t.
\]

Comparing the expressions of \( P_t^m \) in (3.13) and (3.18), we have that

\[
\begin{align*}
Q_t^m &= (\gamma - 1) \sigma P_t^m \pi_t^m, \\
Y_t^m &= P_t^m \left[ (1 - \pi_t^m)^{\gamma - 1} - 1 \right], \\
Q_t^{0,m} &= P_t^m \left\{ (\gamma - 1) \pi_t^m \sigma^0 - \theta \gamma \sigma^0 \mathbb{E}[\pi^m_t | \mathcal{F}_t^0] \right\}.
\end{align*}
\]

Let \( \pi, X \in [0, T] \) be a candidate optimal control that may depend on \( m \), and \( X_t^m = (X_t^m)_{t \in [0, T]} \) be the terminal process under \( \pi_t^m \). For the solution \( (P_t^m, Q_t^m, Q_t^{0,m}, Y_t^m) \) of FBSDE (3.13) with \( (m^0, X^0) \) replaced by \( (m^0, X^0) \), we have from (3.12) that, for any \( m \in A_{\text{MF}}, \)

\[
H \left( t, X_t^m, \pi_t^m, P_t^m, Q_t^m, Q_t^{0,m}, Y_t^m \right)
\]

\[
= (r X_t^m + b \pi_t^m X_t^m) P_t^m + \sigma^m X_t^m Q_t^m + \sigma^0_\pi X_t^m Q_t^{0,m} - \pi^m X_t^m \lambda_t^0 P_t^m,
\]

\[
= r X_t^m P_t^m + \pi^m \left( b X_t^m P_t^m + \sigma X_t^m Q_t^m + \sigma^0 X_t^m Q_t^{0,m} - \pi^m \lambda_t^0 P_t^m \right).
\]

It can be observed from (3.20) that \( H \) is linear in \( \pi^m \). It is then natural to make the coefficient of \( \pi^m \) vanishing, i.e., for \( t \in [0, T], \)

\[
b P_t^m + \sigma Q_t^m + \sigma^0 Q_t^{0,m} - \lambda_t^0 P_t^m = 0.
\]
We first apply the relation in (3.19) to have that

\[
\begin{aligned}
Q_t^{*, m} &= (\gamma - 1) \sigma P_t^{*, m} \pi_t^{*, m}, \\
Y_t^{*, m} &= P_t^{*, m} \left[ (1 - \pi_t^{*, m}) \gamma - 1 \right], \\
Q_t^{0, *, m} &= P_t^{*, m} \left\{ (\gamma - 1) \pi_t^{*, m} \sigma^0 - \theta \gamma \sigma^0 \mathbb{E}[\pi_t^{*, m} | \mathcal{F}_t^0] \right\}.
\end{aligned}
\tag{3.22}
\]

Plugging (3.22) into (3.21), we get that the candidate best response \( \pi_t^{*, m} \) satisfies the equation that

\[
(\gamma - 1) \left[ \sigma^2 + (\sigma^0)^2 \right] \pi_t^{*, m} - \theta \gamma (\sigma^0)^2 \pi_t^{*, m} - \lambda_t^\sigma \left( (1 - \pi_t^{*, m}) \gamma - 1 \right) + b = 0, \quad t \in [0, T]. \tag{3.23}
\]

Next, we focus on a deterministic MFE and assume that \( \pi_t^{*, m} = (\pi_t^{*, m})_{t \in [0, T]} \) is deterministic. Therefore, the condition (3.23) reduces to

\[
(\gamma - 1) \left[ \sigma^2 + (\sigma^0)^2 \right] \pi_t^{*, m} - \theta \gamma (\sigma^0)^2 \pi_t^{*, m} - \lambda_t^\sigma \left( (1 - \pi_t^{*, m}) \gamma - 1 \right) + b = 0. \tag{3.24}
\]

As \( \lambda_t^\sigma \) for \( t \in [0, T] \) in (3.1) is deterministic and bounded, by Lemma 3.1, we can easily get that, for \( t \in [0, T] \), there exists a unique \( \pi_t^{*, m} \in (0, 1 - \epsilon_0] \) such that \( \Phi(\pi_t^{*, m}, \lambda_t^\sigma) = 0 \). Equivalently, for \( t \in [0, T] \), we have that

\[
\pi_t^{*, m} = \phi(\lambda_t^\sigma) \in (0, 1 - \epsilon_0], \tag{3.25}
\]

where \( \phi : \mathbb{R}_+ \to \mathbb{R} \) is a Lipschitz continuous function. We can easily verify that \( \pi_t^{*, m} = (\pi_t^{*, m})_{t \in [0, T]} \in \mathcal{A}_{MF} \). This yields a best (deterministic) response control \( \pi_t^{*, m} = \phi(\lambda_t^\sigma) \) for \( t \in [0, T] \). We observe that as the coefficients in (3.24) do not depend on \( m \in D \) (so is the function \( \Phi \) defined by (3.5)), \( \pi_t^{*, m} \) is independent of \( m \). Therefore, we can write \( \pi_t^{*, m} \) as \( \pi_t^* \), i.e., \( \pi_t^* = \phi(\lambda_t^\sigma) \) for \( t \in [0, T] \).

Next, we just need to solve the function \( \varphi : [0, T] \to \mathbb{R} \) in the ansatz solution (3.17) so that the adjoint processes are well defined. Comparing the drift term of \( P_t^{*, m} \) (associated with \( \pi_t^{*, m} \)) in (3.13) and (3.18), we have that

\[
\begin{aligned}
P_t^{*, m} &\left\{ \frac{\dot{\varphi}_t}{\varphi_t} + (\gamma - 1) \left[ (\gamma - 1) \left( r + (b + \lambda_t^\sigma) \pi_t^* \right) - \theta \gamma (r; \pi_t^*) - \theta \gamma (\gamma - 1)(\sigma^0)^2(\pi_t^*)^2 \right. \\
&\quad + \frac{1}{2} (\gamma - 1)(\gamma - 2) \left( (\sigma^2 + (\sigma^0)^2) (\pi_t^*)^2 \right) + \frac{1}{2} \theta \gamma (\gamma + 1)(\sigma^0)^2(\pi_t^*)^2 \left. \right] - \left[ (1 - \pi_t^*) \gamma - 1 \right] \lambda_t^\sigma \\
&\quad = - \left[ r P_t^{*, m} + \pi_t^* \left( b P_t^{*, m} + \sigma Q_t^{*, m} + \sigma^0 Q_t^{0, *, m} - \lambda_t^\sigma Y_t^{*, m} \right) \right].
\end{aligned}
\tag{3.26}
\]

Plugging (3.17) and (3.22) in (3.26), we obtain that

\[
\frac{\dot{\varphi}_t}{\varphi_t} = - \gamma r - (\gamma - 1) \left( b + \lambda_t^\sigma \right) \pi_t^* + \theta \gamma \tilde{\eta}(t; \pi_t^*) + \theta \gamma (\gamma - 1)(\sigma^0)^2(\pi_t^*)^2 \\
- \frac{1}{2} (\gamma - 1)(\gamma - 2) (\sigma^2 + (\sigma^0)^2) (\pi_t^*)^2 - \frac{1}{2} \theta \gamma (\gamma + 1)(\sigma^0)^2(\pi_t^*)^2 \\
- \left[ (1 - \pi_t^*) \gamma - 1 \right] \lambda_t^\sigma \\
= : \rho(t) \tag{3.27}
\]
with the terminal condition \( \varphi_T = 1 \). We stress here that \( \rho(t) \) depends on \( \pi^*_t \) for \( t \in [0, T] \). We can then deduce that \( \hat{\eta}(t, \pi^*) = \eta(t; \pi^*) \), where \( \eta(t; \pi) \) for \( (t, \pi) \in [0, T] \times U \) is defined by (3.11), i.e.,

\[
\eta(t; \pi) = r + (b + \lambda^2_t) \pi - \frac{1}{2} \sigma^2 \pi^2 + \lambda^2_t \log(1 - \pi).
\]

By solving the ODE problem (3.27), we have that

\[
\varphi_t = e^{\int_t^T \rho(s)ds}, \quad t \in [0, T].
\] (3.28)

It then follows from (3.13) that the adjoint processes corresponding to \( \pi^* \) can be rewritten by

\[
\begin{align*}
P^*_t, m &= (X_t^*)^{\gamma - 1} m_t^{-\theta \gamma} e^{\int_t^T \rho(s)ds}, \\
Q^*_t, m &= (\gamma - 1) \sigma \pi^*_t(X_t^*)^{\gamma - 1} m_t^{-\theta \gamma} e^{\int_t^T \rho(s)ds}, \\
Y^*_t, m &= [(1 - \pi^*_t)^{\gamma - 1} - 1] (X_t^*)^{\gamma - 1} m_t^{-\theta \gamma} e^{\int_t^T \rho(s)ds}, \\
Q^{0, m}_t &= \sigma^0 [(1 - \theta) \gamma - 1] \pi^*_t(X_t^*)^{\gamma - 1} m_t^{-\theta \gamma} e^{\int_t^T \rho(s)ds},
\end{align*}
\] (3.29)

where \( X^* = (X^*_t)_{t \in [0, T]} \) is the wealth process under \( \pi^* = (\pi^*_t)_{t \in [0, T]} \).

Finally, using the consistency condition in \textbf{Step 2} that \( m^*_t = \exp\{\mathbb{E}[\log(X^*_t)|\mathcal{F}^0_t]\} \), \( t \in [0, T] \), we next derive the expression of \( m^* = (m^*_t)_{t \in [0, T]} \). To this purpose, let us recall the process \( m^X = (m^X_t)_{t \in [0, T]} \) in (3.16) satisfies \( m^X_t := \exp\{\mathbb{E}[\log(X^m_t)|\mathcal{F}^0_t]\} \) for \( t \in [0, T] \), where \( X^m = (X^m_t)_{t \in [0, T]} \) is the wealth process under an arbitrary strategy \( \pi^m \in \mathcal{A}_{MF} \). Then, we have that \( m^*_t = m^X_{\pi^*_t} = \exp\{\mathbb{E}[\log(X^*_t)|\mathcal{F}^0_t]\} \) for \( t \in [0, T] \) and it follows that \( m^* = (m^*_t)_{t \in [0, T]} \) is given by (3.10). We therefore conclude that \( \pi^*_t = \phi(\lambda^2_t) \) for \( t \in [0, T] \) is a deterministic MFE, which completes the proof.

\[\square\]

\textbf{Remark 3.4.} We emphasize that the assumption (\textit{A}\textsubscript{O}) with a constant limiting type vector \( o \in \text{O} \) is needed to guarantee the existence of a deterministic MFE strategy in \textbf{Theorem 3.2} in the model with both common noise \( W^0 \) and contagious jump risk. We focus on a deterministic MFE strategy not only because it exhibits clean and interpretable analytical form, but it also crucially simplifies some future proofs to show the validity of a constructed approximate Nash equilibrium in the n-player game and to analyze its explicit convergence rate.

\section{4 Discussions on the Deterministic Mean Field Equilibrium}

We present in this section some quantitative properties and sensitivity results of the deterministic MFE strategy \( \pi^*_t = \phi(\lambda^2_t) \) obtained in \textbf{Theorem 3.2}. First, the next lemma summarizes some straightforward monotonicity results on several model parameters, and its proof is reported in Appendix \textit{A}.

\textbf{Lemma 4.1.} For each fixed \( t \in [0, T] \), let us use the notation \( \pi^*_t(b, \sigma, \sigma^0, \gamma, \theta) \) to highlight the dependence of the deterministic MFE strategy \( \pi^* \) on the model parameters \( (b, \sigma, \sigma^0, \gamma, \theta) \). Then, we have that

(i) \( b \mapsto \pi^*_t(b, \sigma, \sigma^0, \gamma, \theta) \) is increasing;
(ii) both $\sigma \mapsto \pi^*_t(b, \sigma, \sigma^0, \gamma, \theta)$ and $\sigma^0 \mapsto \pi^*_t(b, \sigma, \sigma^0, \gamma, \theta)$ are decreasing;

(iii) both $\gamma \mapsto \pi^*_t(b, \sigma, \sigma^0, \gamma, \theta)$ and $\theta \mapsto \pi^*_t(b, \sigma, \sigma^0, \gamma, \theta)$ are decreasing.

Note that items (i) and (ii) in Lemma 4.1 are consistent with our intuition that the higher return and lower volatility in the limiting market model will incentivize the agent to invest more in the risky asset account in the mean field equilibrium strategy. When the representative agent is more risk averse, item (iii) implies that the representative agent becomes more conservative and invests less in the risky asset. It is also interesting to see that for $0 < \gamma < 1$, the mean field Nash equilibrium $\pi^*$ has no short-selling and a higher competition parameter $\theta$ leads to a lower investment proportion in the risky asset. That is, in the equilibrium state, a more competitive agent with the risk aversion $0 < \gamma < 1$ will prefer to invest more in the riskless bond account to avoid the possible falling behind due to volatility and contagious jump risk.

We next numerically illustrate the sensitivity results of $\pi^*$ with respect to parameters associated to the contagious jump risk.

Recall that the function $\phi(\lambda)$ is decreasing in $\lambda > 0$ by Lemma 3.1. It is also clear that the limiting intensity process $\lambda_t^0$ admitting the explicit form in (3.2) is increasing in time $t$ if $\beta \varsigma > \alpha(1 - \frac{\lambda_\infty}{\lambda_0})$ and it is decreasing in time $t$ otherwise. Thanks to the analytical structure of $\pi^*_t = \phi(\lambda_t^0)$, if model parameters satisfy that $\beta \varsigma > \alpha(1 - \frac{\lambda_\infty}{\lambda_0})$, the deterministic MFE strategy $\pi^*_t$ is decreasing in time $t$ indicating that the representative agent in the mean field game will reduce the portfolio in the risky asset as time evolves because of the increasing probability of the downward jump risk. To numerically illustrate this case, we choose and fix parameters $\epsilon_0 = 10^{-10}$, $\gamma = 0.4$, $\theta = 0.5$, $\sigma = 0.3$, $\sigma_0 = 0.2$, $b = 0.2$, $\lambda_0 = 0.1$, $\lambda_\infty = 0.6$, $\alpha = 0.5$, $\beta = 0.4$, $\varsigma = 0.2$ and plot the function $t \to \pi^*_t$ in Figure 1, which is a decreasing function of $t \in [0, 10]$.

Another interesting observation is that the limiting intensity process converges to a long-run steady level as $t$ tends to $\infty$ when the parameters satisfy $\beta \varsigma < \alpha$. In this case, it follows from (3.2) that

$$
\lim_{t \to \infty} \lambda_t^0 = \frac{\alpha \lambda_\infty}{\alpha - \beta \varsigma}.
$$

(4.1)

Thus, using the continuity of $\lambda \mapsto \phi(\lambda)$ (see Lemma 3.1), we can also derive the long-run behavior of the deterministic MFE (given the time horizon $T$ is sufficiently large) that

$$
\lim_{t \to \infty} \pi^*_t = \lim_{t \to \infty} \phi(\lambda_t^0) = \phi \left( \frac{\alpha \lambda_\infty}{\alpha - \beta \varsigma} \right).
$$

(4.2)

For the given parameters, it is observed from Figure 1 that this long run steady value of the deterministic MFE is approximately 0.31.
Let us then examine the sensitivity of the mean field equilibrium \( \pi^* \) w.r.t. the mean recovery speed parameter \( \alpha \) from the intensity process \( \lambda^0_t \) at a fixed time \( t \). To this end, we choose and fix parameters \( \epsilon_0 = 10^{-10}, \gamma = 0.4, \theta = 0.5, \sigma = 0.3, \sigma_0 = 0.2, b = 0.2, \lambda_0 = 0.1, \lambda_\infty = 0.6, \beta = 0.4, \zeta = 0.2, t = 3 \), and plot the function of \( \alpha \rightarrow \pi^*_t \) in terms of the parameter \( \alpha \) in Figure 2 on the interval \([0, 1]\). For the given \( t = 3 \), we can observe that the equilibrium \( \pi^* \) is decreasing in the parameter \( \alpha \). For the chosen parameters, it is easy to see that a larger \( \alpha \) leads to a larger default intensity of the risky asset. Consequently, to avoid the higher probability of default, the agent prefers to invest less in the risky asset account.

Similarly, we plot in Figure 3 the mean field equilibrium \( \beta \rightarrow \pi^*_t \) as a function in terms of the parameter \( \beta \) from the intensity process \( \lambda^0_t \). We choose and fix other parameters \( \epsilon_0 = 10^{-10}, \gamma = 0.4, \theta = 0.5, \sigma = 0.3, \sigma_0 = 0.2, b = 0.2, \lambda_0 = 0.1, \lambda_\infty = 0.6, \alpha = 0.5, \zeta = 0.2 \) and \( t = 3 \). For the given \( t = 3 \), the mean field equilibrium \( \pi^*_t(\beta) \) is decreasing in the parameter \( \beta \). We also note that \( \beta \) and \( \zeta \) are symmetric in the definition of \( \lambda^0_t \), the sensitivity result of the mean filed equilibrium \( \pi^*_t(\zeta) \) w.r.t. the parameter \( \zeta \) is similar to the case w.r.t. the parameter \( \beta \). Again, from the definition of \( \lambda^0_t \), one can see that the intensity value is increasing in terms of \( \beta \) and \( \zeta \). Therefore, as \( \beta \) or \( \zeta \)
increases, the agent invests less in the risky asset due to the higher probability of default.

At last, we numerically analyze the sensitivity of the mean field equilibrium $\pi^*_t(\lambda_\infty)$ in terms of the parameter $\lambda_\infty > 0$ in Figure 4. We choose and fix other parameters $\epsilon_0 = 10^{-10}$, $\gamma = 0.4$, $\theta = 0.5$, $\sigma = 0.3$, $\sigma_0 = 0.2$, $b = 0.2$, $\lambda_0 = 0.1$, $\beta = 0.4$, $\alpha = 0.5$, $\varsigma = 0.2$ and $t = 3$. For the given $t = 3$, Figure 4 shows that the mean field equilibrium $\pi^*_t(\lambda_\infty)$ is decreasing in the parameter $\lambda_\infty$. For the chosen parameters, one can check that a larger $\lambda_\infty$ indicates a higher probability of the default jump in the risky asset. Consequently, the agent will reduce the investment proportion in the risky asset.

5 Approximate Nash Equilibrium in the $n$-Player Game

The goal of this section is to show that the mean field equilibrium obtained in Theorem 3.2 can help us to construct an approximate Nash equilibrium in the game with finite $n$ agents when $n$ is sufficiently large. Furthermore, the explicit order of the approximation error can also be
derived, which will facilitate the practical implementations of the mean field approximation in finite population game applications.

Recall that the intensity process \((\Lambda_{i,n}^{t,n})_{t \in [0,T]}\) follows the dynamics that

\[
d\Lambda_{i,n}^{t,n} = \alpha_i(\lambda_{\infty} - \Lambda_{i,n}^{t,n})dt + \frac{1}{n} \sum_{j=1}^{n} \beta_{i,j} dN_{j,n}^{t}, \quad \Lambda_0^i = \lambda_0^i > 0, \quad i = 1, \ldots, n, \tag{5.1}
\]

with the speed parameter \(\alpha_i > 0\) and a mean-reverting level \(\lambda_{\infty}^i > 0\), and the jump risk contagion weight \(\beta_{i,j} > 0\). To avoid the possible ambiguity in notation, we will keep the superscript \(n\) in this section. It follows that (5.1) admits a closed-form solution given by

\[
\Lambda_{i,n}^{t,n} = e^{-\alpha_i t} \lambda_0^i + \lambda_{\infty}^i (1 - e^{-\alpha_i t}) + \frac{1}{n} \sum_{j=1}^{n} \beta_{i,j} \int_0^t e^{-\alpha_i (t-s)} dN_{j,n}^{s}.
\tag{5.2}
\]

From its expression in (5.2) we can see that the intensity process tends to the mean reverting level in the long run, and \(t \to \Lambda_{i,n}^{t,n}\) is positive and bounded uniformly in \((i, n)\) by the assumption \((A_0)\). Hereafter, \(\Xi_\Lambda \subset \mathbb{R}_+\) denotes the state space of \(\Lambda_{i,n}^{t,n} = (\Lambda_{i,n}^{t,n})_{t \in [0,T]}\), which is a bounded set and is independent of \((i, n)\).

Next, before constructing an \(\varepsilon\)-Nash equilibrium in the \(n\)-player game, we first introduce an auxiliary problem based on the limiting case, whose best response strategy will help us to construct an approximate Nash equilibrium. Let us define the auxiliary control problem \((P_n)\) by

\[
\sup_{\pi^i \in \mathcal{A}^i} J^i_n(\pi^i, m^* := \sup_{\pi^i \in \mathcal{A}^i} \mathbb{E} \left[ \frac{1}{\gamma_i} (X_t^{i,n} T) - \theta_i^* \gamma_i \right], \quad i = 1, \ldots, n, \tag{5.3}
\]

subjecting to

\[
\begin{aligned}
dx_t^{i,n} &= X_t^{i,n}(r + b_t \pi_t^i)dt + \pi_t^i X_t^{i,n}(\sigma_t dW_t^i + \sigma_t^0 dW_t^0 - dM_t^{i,n}), \\
\Lambda_t^{i,n} &= \alpha_i(\lambda_{\infty} - \Lambda_t^{i,n})dt + \frac{1}{n} \sum_{j=1}^{n} \beta_{i,j} dN_{t,n}^{j,n}, \\
dm_t^{i,n} &= m_t^* \eta(t; \pi_t^i)dt + \sigma_t^0 \pi_t^i dW_t^0.
\end{aligned}
\tag{5.4}
\]

Here, we recall that \(m^* = (m_t^i)_{t \in [0,T]}\) is the fixed point given by (3.10), and \(\pi^* = (\pi_t^i)_{t \in [0,T]} \in \mathcal{A}_{MF}\) is the deterministic MFE characterized by (3.8) in Theorem 3.2. The following lemma characterizes the optimal strategy of the auxiliary control problem \((P_n)\). The proof is reported in Appendix A.

**Lemma 5.1.** Let \(\tilde{\pi}^{i,n}(t, \lambda) \in (-\infty, 1)\) for \((t, \lambda) = (t, (\lambda_1, \ldots, \lambda_n)^\top) \in [0, T] \times \Xi_\Lambda\) be the optimal (feedback) strategy of the auxiliary control problem \((P_n)\). Then, we have that

\[
\Phi_i(t, \lambda, \tilde{\pi}^{i,n}) = O\left(\frac{1}{n}\right),
\tag{5.5}
\]

where the function \(\Phi_i(t, \lambda, \pi) : [0, T] \times \Xi_\Lambda \times (-\infty, 1) \to \mathbb{R}\) is defined by

\[
\Phi_i(t, \lambda, \pi) := (\gamma_i - 1)\sigma_t^2 + (\sigma_t^0)^2 \pi - \theta_i \gamma_i \sigma_t^0 \pi_t^* - \lambda(1 - \pi)^{\gamma_i - 1} + \lambda + b_t,
\tag{5.6}
\]

and \(\pi^* = (\pi_t^i)_{t \in [0,T]}\) is the deterministic MFE given in Theorem 3.2. Moreover, there exists a pair \((\bar{D}, \bar{\varepsilon}) \in \mathbb{R} \times (0, 1)\) independent of \((t, \lambda, i, n)\) such that \(\tilde{\pi}^{i,n}(t, \lambda) \in [\bar{D}, 1 - \bar{\varepsilon}]\).
Intuitively, it can be seen from (5.5) in Lemma 5.1 that, as \( n \) tends to infinity, the optimal feedback strategy \( \hat{\pi}^{i,n} = \hat{\pi}^{i,n}(t, \lambda) \) converges to some \( \hat{\pi}^i \), where \( \hat{\pi}^i \) satisfies \( \Phi_i(t, \lambda, \hat{\pi}^{i}) = 0 \). The following lemma characterizes the zero point of \( \pi \mapsto \Phi_i(t, \lambda, \pi) \). The proof is similar to that of Lemma 3.1, and we hence omit it.

**Lemma 5.2.** For the function \( \Phi_i(t, \lambda, \pi) \) defined by (5.6), we have that, for any \((t, \lambda) \in [0, T] \times \Xi_\lambda\), there exists a unique \( \hat{\pi} = \hat{\pi}(t, \lambda) \in [D_0, 1 - \epsilon_0] \) such that

\[
\Phi_i(t, \lambda, \hat{\pi}) = (\gamma_i - 1)[\sigma_i^2 + (\sigma_i^0)^2] \hat{\pi} - \theta_i \gamma_i \sigma_i^0 \sigma_i^0 \pi_i^* - \lambda (1 - \hat{\pi})^{\gamma_i - 1} + \lambda + b_i = 0, \tag{5.7}
\]

where \((D_0, \epsilon_0) \in \mathbb{R} \times (0, 1)\) is a pair of constants that are independent of \((t, \lambda, i, n)\). Moreover, there exists a unique continuous function \( \phi_i : [0, T] \times \Xi_\lambda \rightarrow \mathbb{R} \) such that

\[
\hat{\pi} = \phi_i(t, \lambda), \quad (t, \lambda) \in [0, T] \times \Xi_\lambda, \tag{5.8}
\]

where \( \phi_i \) also has a continuous partial derivative with respect to \( \lambda \).

The above function \( \phi_i \) in (5.8) also plays an important role in the construction of an approximating Nash equilibrium. More precisely, for \( i = 1, \ldots, n \), we introduce a strategy for the agent \( i \) that

\[
\pi_t^{s,i,n} := \phi_i(t, \Lambda_t^{i,n}), \quad t \in [0, T], \tag{5.9}
\]

where \( \Lambda_t^{i,n} = (\Lambda_t^{i,n})_{t \in [0,T]} \) is given in (5.4). It follows from Lemma 5.2 that \( \pi_t^{s,i,n} = \phi_i(t, \Lambda_t^{i,n}) \in [D_0, 1 - \epsilon_0] \) for some \((D_0, \epsilon_0) \in \mathbb{R} \times (0, 1)\) in view that \( \Lambda_t^{i,n} \in \Xi_\lambda \) is bounded. Then, the corresponding wealth process \( X_t^{s,i} = (X_t^{s,i})_{t \in [0,T]} \) of agent \( i \) under the strategy \( \pi_t^{s,i,n} = (\pi_t^{s,i,n})_{t \in [0,T]} \) is governed by

\[
dX_t^{s,i,n} = X_t^{s,i,n}(r + b_i \pi_t^{s,i,n})dt + \pi_t^{s,i,n} X_t^{s,i,n}(\sigma_i dW_t^i + \sigma_i^0 dW_t^0 - dM_t^{i,n}). \tag{5.10}
\]

We can now present the main result of this section, which gives the approximate Nash equilibrium for the \( n \)-player game.

**Theorem 5.3.** Let the assumption \((A_D)\) hold. Consider the objective function (2.4) for the \( i \)-th agent. Then \( \pi_t^{s,i,n} = (\pi_t^{s,i,n}, \ldots, \pi_t^{s,n,n})_{t \in [0,T]} \) is an \( \varepsilon_n \)-Nash equilibrium, where \( \pi_t^{s,i,n} = (\pi_t^{s,i,n})_{t \in [0,T]} \) is given by (5.9) for \( i = 1, \ldots, n \). That is, we have that

\[
\sup_{\pi_t^{s,i,n} \in A^i} J_i(\pi^i, \pi_t^{s,-i,n}) \leq J_i(\pi_t^{s,i,n}, \pi_t^{s,-i,n}) + \varepsilon_n, \tag{5.11}
\]

where \( \pi_t^{s,-i,n} := (\pi_t^{s,1,n}, \ldots, \pi_t^{s,i-1,n}, \pi_t^{s,i+1,n}, \ldots, \pi_t^{s,n,n}) \). Moreover, the order of the error term satisfies \( \varepsilon_n = O(n^{-\frac{1}{2}}) \).

To prove Theorem 5.3, we first introduce the following auxiliary results, whose proofs are given in Appendix A.

**Lemma 5.4.** Let the assumption \((A_D)\) hold. Then, we have that

(i) Consider \( \pi_t^{s,i,n} = (\pi_t^{s,i,n})_{t \in [0,T]} \) defined in (5.9) and \( X_t^{s,i,n} = (X_t^{s,i,n})_{t \in [0,T]} \) given by (5.10) for \( i = 1, \ldots, n \). For any \( p \in \mathbb{R} \), there exists a constant \( D_p \) independent of \((i, n)\) such that

\[
\sup_{t \in [0,T]} \mathbb{E} \left[ \left| X_t^{s,i,n} \right|^p \right] \leq D_p. \tag{5.12}
\]
(ii) Let $\hat{\Lambda}_t^n := \frac{1}{n} \sum_{i=1}^{n} \Lambda_{t,i}^n$ for $t \in [0, T]$, and the mean field intensity $\lambda^o = (\lambda_t^o)_{t \in [0, T]}$ be given by (3.1). We have that
\[ \sup_{t \in [0, T]} \mathbb{E} \left[ \left| \hat{\Lambda}_t^n - \lambda_t^o \right|^2 \right] = O \left( \frac{1}{n^2} \right). \quad (5.13) \]

(iii) Denote by $X_{t,i}^{*;n} = \left( \prod_{i=1}^{n} X_{t,i}^{*,i/n} \right)^{1/n}$ the geometric mean wealth process at $t \in [0, T]$, where $X_{t,i}^{*,i,n} = (X_{t,i}^{*,i,n})_{t \in [0, T]}$ is defined in (5.10), and $m^* = (m_t^*)_{t \in [0, T]}$ in the limiting model is obtained in Theorem 3.2. We have that, for all $t \in [0, T],
\[ \sup_{t \in [0, T]} \mathbb{E} \left[ |X_{t,i}^{*,i,n} - m_t^*|^2 \right] = O \left( \frac{1}{\sqrt{n}} \right). \quad (5.14) \]

Based on all previous preparations, we can give the proof of Theorem 5.3.

**Proof of Theorem 5.3.** Recall that $X_{t,i}^{*,i,n} = (X_{t,i}^{*,i,n})_{t \in [0, T]}$ is defined by (5.10). For ease of presentation, let us denote
\[ \bar{X}_{t,i}^{*,i,n} := \left( X_{t,i}^{*,i,n} \prod_{j \neq i} X_{t,j}^{*,i,n} \right)^{1/n}, \quad \bar{X}_{t,i}^{*,n} := \left( \prod_{j=1}^{n} X_{t,i}^{*,i,n} \right)^{1/n}, \quad t \in [0, T]. \]

Here, $X_{t,i}^{*,i,n} = (X_{t,i}^{*,i,n})_{t \in [0, T]}$ satisfies the wealth dynamics (2.3) under the strategy $\pi^i = (\pi_t^i)_{t \in [0, T]} \in \mathcal{A}^i$. We recall the objective functional $J_i$ defined by (2.4). Then, we have that
\[ J_i(\pi^i, \pi^{*,-i,n}) = \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_{T,i}^{*,i,n} \right)^{\gamma_i} \left( \bar{X}_{t,i}^{*,i,n} \right)^{-\theta_i \gamma_i} \right], \quad J_i(\pi^{*,n}) = \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_{T,i}^{*,i,n} \right)^{\gamma_i} \left( \bar{X}_{t,i}^{*,n} \right)^{-\theta_i \gamma_i} \right]. \]

Next, we proceed to prove (5.11) by using the introduced auxiliary problem (P_n) in (5.3)-(5.4). Note that
\[
\sup_{\pi^i \in \mathcal{A}^i} J_i(\pi^i, \pi^{*,i,n}) - J_i(\pi^{*,i,n}, \pi^{*,-i,n}) = \left( \sup_{\pi^i \in \mathcal{A}^i} J_i(\pi^i, \pi^{*,i,n}) \right) - \left( \sup_{\pi^i \in \mathcal{A}^i} J_i(\pi^{*,i,n}, \pi^{*,-i,n}) \right) \leq \sup_{\pi^i \in \mathcal{A}^i} \left( J_i(\pi^i, \pi^{*,i,n}) - J_i(\pi^{*,i,n}, \pi^{*,-i,n}) \right) + \left( \sup_{\pi^i \in \mathcal{A}^i} J_i(\pi^i, \pi^{*,i,n}) \right).
\]

For the first term of RHS of (5.15), we have that
\[ J_i(\pi^i, \pi^{*,i,n}) - \tilde{J}_i(\pi^i, m^*) = \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_{T,i}^{*,i,n} \right)^{\gamma_i} \left( \bar{X}_{t,i}^{*,i,n} \right)^{-\theta_i \gamma_i} \right] - \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_{T,i}^{*,i,n} \right)^{\gamma_i} \left( m_t^* \right)^{-\theta_i \gamma_i} \right] \leq \left\{ \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_{T,i}^{*,i,n} \right)^{\gamma_i} \left( \bar{X}_{t,i}^{*,i,n} \right)^{-\theta_i \gamma_i} \right] - \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_{T,i}^{*,i,n} \right)^{\gamma_i} \left( m_t^* \right)^{-\theta_i \gamma_i} \right] \right\} + \left\{ \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_{T,i}^{*,i,n} \right)^{\gamma_i} \left( \bar{X}_{t,i}^{*,i,n} \right)^{-\theta_i \gamma_i} \right] - \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_{T,i}^{*,i,n} \right)^{\gamma_i} \left( m_t^* \right)^{-\theta_i \gamma_i} \right] \right\} := I_1^i + I_2^i.
\]

For the term $I_1^i$, we deduce that
\[ \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_{T,i}^{*,i,n} \right)^{\gamma_i} \left( \bar{X}_{t,i}^{*,i,n} \right)^{-\theta_i \gamma_i} \right] - \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_{T,i}^{*,i,n} \right)^{\gamma_i} \left( m_t^* \right)^{-\theta_i \gamma_i} \right] \]
\[
= \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_T^{i,n} \right)^{\gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i} \right].
\]

Note that \( \gamma_i \in (0, 1) \) implies that \( p_i := \theta_i \gamma_i \in [0, 1) \). Using the inequality \(|a^n - b^n| \leq p_i |a - b| \max\{a^{p_i - 1}, b^{p_i - 1}\} \) for all \( a, b > 0 \), we can derive that

\[
\left| \left( X_T^{*,i,n} \right)^{\gamma_i} - \left( X_T^{*,i,n} \right)^{\gamma_i} \right| = \frac{1}{(X_T^{*,i,n})^p_i (X_T^{*,n})^{p_i}} \left| \left( X_T^{*,i,n} \right)^{\gamma_i} - \left( X_T^{*,i,n} \right)^{\gamma_i} \right|
\]

\[
\leq \frac{p_i}{n} \left( \prod_{j \neq i} X_T^{*,j,n} \right)^{\gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \max \left\{ \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i}, \left( X_T^{*,n} \right)^{\gamma_i - \gamma_i} \right\}
\]

We only give the detailed proof on the event \( \{X_T^{*,i,n} \leq X_T^{*,i,n}\} \), and skip similar arguments on the event \( \{X_T^{*,i,n} > X_T^{*,i,n}\} \). It follows from \( p_i/n \in (0, 1) \) that

\[
\left| \left( X_T^{*,i,n} \right)^{\gamma_i} - \left( X_T^{*,i,n} \right)^{\gamma_i} \right|
\]

\[
\leq \frac{p_i}{n} \left( \prod_{j \neq i} X_T^{*,j,n} \right)^{\gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \max \left\{ \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i}, \left( X_T^{*,n} \right)^{\gamma_i - \gamma_i} \right\}
\]

\[
= \frac{p_i}{n} \left( \prod_{j \neq i} X_T^{*,j,n} \right)^{\gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i}
\]

This yields that

\[
\mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_T^{i,n} \right)^{\gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i} \right] - \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_T^{i,n} \right)^{\gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i} \right]
\]

\[
\leq \mathbb{E} \left[ \frac{1}{\gamma_i} \left( X_T^{i,n} \right)^{\gamma_i} \frac{p_i}{n} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \right] = \frac{p_i}{\gamma_i n} \mathbb{E} \left[ X_T^{*,i,n} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \right].
\]

For \( j = 1, \ldots, n \), let \( q_j = 2n \). Then \( \frac{1}{4} + \frac{1}{4} + \sum_{j=1}^n q_j^{-1} = 1 \). We have from the generalized Hölder’s inequality and (5.12) in Lemma 5.4 that

\[
\mathbb{E} \left[ X_T^{*,i,n} \left( X_T^{*,i,n} \right)^{\gamma_i - \gamma_i} \left( \prod_{j=1}^n X_T^{*,j,n} \right)^{\gamma_i - \gamma_i} \right]
\]

\[
\leq \mathbb{E} \left[ \left( X_T^{*,i,n} \right)^4 \right] \frac{1}{q_j} \mathbb{E} \left[ \left( X_T^{*,i,n} \right)^{4(q_j - 1)} \right] \frac{4}{q_j} \prod_{j=1}^n \mathbb{E} \left[ X_T^{*,j,n} \right] \frac{q_j - 1}{q_j}.
\]
where we recall the definition of \( \pi \).

We next claim that

\[
\frac{1}{\gamma_i} \left[ X_T^{i,n} \right]^{\gamma_i} \left( X_T^{*,i,n} \right)^{-\theta_i \gamma_i} \leq \frac{1}{\gamma_i} \left[ \left( (X_T^{*,i,n})^{-\theta_i} - (m_T^{*,i,n})^{-\theta_i} \right) \right]^{\gamma_i} \leq \frac{1}{\gamma_i} \left[ \left( (m_T^{*,i,n})^{p_i} - (X_T^{*,i,n})^{p_i} \right) \right]^{\gamma_i} \leq \frac{1}{\gamma_i} \left[ \left( m_T^{*,i,n} - X_T^{*,i,n} \right) \max \left\{ (m_T^{*,i,n})^{-\theta_i}, (X_T^{*,i,n})^{-\theta_i} \right\} \right]^{\gamma_i} \leq C \left( \left[ m_T^{*,i,n} - X_T^{*,i,n} \right] \right)^{\gamma_i} = O \left( \frac{1}{n^2} \right).
\]

For the second term of r.h.s. of (5.15), we can follow similar argument in proving (5.18) to get that

\[
J_i^n(\pi^{*,i,n}; m^*) - J_i^n(\pi^{*,i,n}, \pi^{*,i,n}) = O \left( \frac{1}{n^2} \right).\]

We next claim that

\[
\sup_{\pi^{i,n} \in A_t} J_i^n(\pi^i; m^*) - J_i^n(\pi^{*,i,n}; m^*) = O \left( \frac{1}{n^2} \right),
\]

where we recall the definition of \( \pi^{*,i,n} = (\pi_t^{*,i,n})_{t \in [0,T]} \) in (5.9). It follows from (5.9) that, \( \mathbb{P} \)-a.s.

\[
(\gamma_i - 1) \left[ \sigma_i^2 + (\sigma_i^0)^2 \right] \pi_t^{*,i,n} - \theta_i \gamma_i \sigma_i^0 \sigma_i^0 \pi_t^{*,i,n} - \Lambda_t^{i,n} (1 - \pi_t^{*,i,n}) \gamma_i - 1 + \Lambda_t^{i,n} + b_i = 0.
\]

Recall that the best response solution \( \tilde{\pi}_t^{*,i,n} = \tilde{\pi}_t^{*,i,n}(t, \lambda) \) of the auxiliary control problem \((P_n)\) satisfies (5.5) in Lemma 5.1. Let \( \tilde{\pi}_t^{*,i,n} = \tilde{\pi}_t^{*,i,n}(t, \Lambda_t) \) and it holds that, \( \mathbb{P} \)-a.s.

\[
\Phi_i(t, \Lambda_t^{i,n}, \tilde{\pi}_t^{*,i,n}) = O \left( \frac{1}{n^2} \right), \quad \Phi_i(t, \Lambda_t^{i,n}, \tilde{\pi}_t^{*,i,n}) = 0.\]

For any \( \xi \in [\min\{\tilde{\pi}_t^{*,i,n}, \tilde{\pi}_t^{*,i,n}\}, \max\{\tilde{\pi}_t^{*,i,n}, \tilde{\pi}_t^{*,i,n}\}] \), by Lemma 5.1, we have \( \mathbb{P} \)-a.s. that \( \partial_{\pi} \Phi_i(t, \Lambda_t^{i,n}; \xi) \in [C_1, C_2] \) for constants \( -\infty < C_1 < C_2 < \infty \), which are independent of \( (t, \lambda, i, n) \). Using (5.21) and the mean value theorem, we arrive at

\[
\left| \tilde{\pi}_t^{*,i,n} - \pi_t^{*,i,n} \right| = O \left( \frac{1}{n^2} \right), \quad \forall t \in [0, T].
\]
Building upon the estimate \((5.22)\), we next focus on the proof of \((5.20)\). In fact, by virtue of \((5.4)\), it follows from Ito’s formula that, for any admissible strategy \(\pi^i = (\pi^i_t)_{t \in [0, T]} \in \mathcal{A}^i\),

\[
\frac{d(X^{i,n}_t)^\gamma_{i,n}}{X^{i,n}_t} = \gamma_i[r + (b_i + \Lambda^{i,n}_t)\pi_t^i] + \frac{1}{2}(\gamma_i - 1)(\sigma^2_i(\pi^i_t)^2 + (\sigma^0_i\pi^i_t)^2))dt + \gamma_i\pi^i_t\sigma_idW^i_t \\
+ \gamma_i\pi^i_t\sigma^0_i dW^0_t + [(1 - \pi^i_t)\gamma_i - 1)dN^{i,n}_t \\
:= F(t, \pi^i_t)dt + \gamma_i\pi^i_t\sigma_idW^i_t + \gamma_i\pi^i_t\sigma^0_idW^0_t + [(1 - \pi^i_t)\gamma_i - 1)dN^{i,n}_t, \tag{5.23}
\]

with \(F(t, \pi^i_t) := \gamma_i[r + (b_i + \Lambda^{i,n}_t)\pi_t^i] + \frac{1}{2}(\gamma_i - 1)(\sigma^2_i + (\sigma^0_i)^2)(\pi^i_t)^2\). This is equivalent to that

\[
(X^{i,n}_t)^\gamma = x^\gamma_t \exp \left\{ \int_0^T F(t, \pi^i_t)dt + \int_0^T \gamma_i\pi^i_t\sigma_idW^i_t + \int_0^T \gamma_i\pi^i_t\sigma^0_idW^0_t + \int_0^T \log(1 - \pi^i_t)\gamma_i dN^{i,n}_t \right\}.
\]

Let us recall that, for \(t \in [0, T]\),

\[
m^*_t = x_0 \exp \left\{ \int_0^t \left( \eta(s; \pi^*_{s}) - \frac{1}{2}(\sigma^0_\pi^*_{s})^2 \right) ds + \int_0^t \sigma^0_\pi^*_{s} dW^0_s \right\}.
\]

Therefore, for any \(\pi^i \in \mathcal{A}^i\), it holds that

\[
E \left[ \frac{1}{\gamma_i} \left( X^{i,n}_T \right)^\gamma (m^*_T)^{-\theta_i}\gamma_i \right] \\
= \frac{1}{\gamma_i} E \left[ x^\gamma_t x_0^{-\theta_i}\gamma_i \exp \left\{ \int_0^T F(t, \pi^i_t)dt + \int_0^T \gamma_i\pi^i_t\sigma_idW^i_t + \int_0^T \gamma_i\pi^i_t\sigma^0_idW^0_t \\
+ \int_0^T \log(1 - \pi^i_t)\gamma_i dN^{i,n}_t - \int_0^T \theta_\gamma_i(\eta(t; \pi^*_{t}) - \frac{1}{2}(\sigma^0_{\pi^*_{t}})^2)dt - \int_0^T \theta_\gamma_i\sigma^0_{\pi^*_{t}} dW^0_t \right\} \right] \\
= \frac{1}{\gamma_i} x^\gamma_t x_0^{-\theta_i}\gamma_i E \left[ \exp \left\{ \int_0^T G(t, \pi^i_t, \pi^*_{t})dt + \int_0^T \gamma_i\pi^i_t\sigma_idW^i_t + \int_0^T \gamma_i(\pi^i_t\sigma^0_{\pi^*_{t}} - \theta_\pi^*_{t}\sigma^0_{\pi^*_{t}})dW^0_t \\
+ \int_0^T \log(1 - \pi^i_t)\gamma_i dN^{i,n}_t \right\} \right] \\
\] (5.24)

with \(G(t, \pi^i_t, \pi^*_{t}) := F(t, \pi^i_t) - \theta_\gamma_i(\eta(t; \pi^*_{t}) - \frac{1}{2}(\sigma^0_{\pi^*_{t}})^2)\).

Let \(\tilde{X}^{i,n} = (X^{i,n}_t)_{t \in [0, T]}\) and \(X^{*,i,n} = (X^{*,i,n}_t)_{t \in [0, T]}\) be the wealth processes corresponding to the strategies \(\tilde{\pi}^{i,n}\) and \(\pi^{*,i,n}\), respectively. Using the representation \((5.24)\) with \(\pi^i\) replaced by \(\tilde{\pi}^{i,n}\) and \(\pi^{*,i,n}\) respectively, we deduce that

\[
\sup_{\pi^i \in \mathcal{A}^i} \tilde{J}_i^*(\pi^i_\cdot; m^*_{\cdot}) - \tilde{J}_i^*(\pi^{*,i,n}_\cdot; m^*_{\cdot}) = E \left[ \frac{1}{\gamma_i} \left( \tilde{X}^{i,n}_T \right)^\gamma (m^*_T)^{-\theta_i}\gamma_i \right] - E \left[ \frac{1}{\gamma_i} \left( X^{*,i,n}_T \right)^\gamma (m^*_T)^{-\theta_i}\gamma_i \right] \\
= \frac{1}{\gamma_i} x^\gamma_t x_0^{-\theta_i}\gamma_i E \left[ \exp \left\{ \int_0^T G(t, \tilde{\pi}^{i,n}_t, \pi^*_{t})dt + \int_0^T \gamma_i\tilde{\pi}^{i,n}_t\sigma_idW^i_t + \int_0^T \gamma_i(\tilde{\pi}^{i,n}_t\sigma^0_{\pi^*_{t}} - \theta_\pi^*_{t}\sigma^0_{\pi^*_{t}})dW^0_t \\
+ \int_0^T \log(1 - \tilde{\pi}^{i,n}_t)\gamma_i dN^{i,n}_t \right\} \right] - \exp \left\{ \int_0^T G(t, \pi^{*,i,n}_t, \pi^*_{t})dt + \int_0^T \gamma_i\pi^{*,i,n}_t\sigma_idW^i_t \\
+ \int_0^T \gamma_i(\pi^{*,i,n}_t\sigma^0_{\pi^*_{t}} - \theta_\pi^*_{t}\sigma^0_{\pi^*_{t}})dW^0_t + \int_0^T \log(1 - \pi^{*,i,n}_t)\gamma_i dN^{i,n}_t \right\} \right] := I_3^i \tag{5.25}
\]

20
After a straightforward calculation, the term $I_3^i$ can be further rewritten as

$$I_3^i = \frac{1}{\gamma_i} \int_0^T \exp \left\{ \int_0^T \left( F(t, \tilde{\pi}_t^{i,n}) - F(t, \pi_t^{*,i,n}) \right) dt + \int_0^T \gamma_i \sigma_i (\tilde{\pi}_t^{i,n} - \pi_t^{*,i,n}) dW_t^i ight\}$$

$$+ \int_0^T \gamma_i \sigma_t^0 (\tilde{\pi}_t^{i,n} - \pi_t^{*,i,n}) dW_t^0 + \int_0^T \log \left( \frac{1 - \tilde{\pi}_t^{i,n}}{1 - \pi_t^{*,i,n}} \right) - 1 \right\} \]$$

$$\times \exp \left\{ \int_0^T G(t, \pi_t^{*,i,n}, \pi_t^*) dt + \int_0^T \gamma_i \pi_t^{*,i,n} dW_t^i + \int_0^T \gamma_i (\pi_t^{*,i,n} \sigma_t^0 - \theta_i \sigma_t^0 \pi_t^*) dW_t^0 ight\}$$

$$\times \left( \int_0^T \log \left( \frac{1 - \pi_t^{*,i,n}}{1 - \pi_t^{*,i,n}} \right) dN_t^i \right) - 1 \right\} \]$$

$$= \frac{1}{\gamma_i} \left[ \left( X_T^{*,i,n} - m_T^* \right)^2 \right] \left[ \left( \int_0^T \left( F(t, \tilde{\pi}_t^{i,n}) - F(t, \pi_t^{*,i,n}) \right) dt + \int_0^T \gamma_i (\tilde{\pi}_t^{i,n} - \pi_t^{*,i,n}) (\sigma_t dW_t^i + \sigma_t^0 dW_t^0) ight) \right]$$

$$+ \left( \int_0^T \gamma_i (\tilde{\pi}_t^{i,n} - \pi_t^{*,i,n}) (\sigma_t dW_t^i + \sigma_t^0 dW_t^0) \right) + \int_0^T \log \left( \frac{1 - \pi_t^{*,i,n}}{1 - \pi_t^{*,i,n}} \right) dN_t^i \right) - 1 \right\} \]$$

Applying the Cauchy-Schwarz inequality to (5.26) results in that

$$I_3^i \leq \frac{1}{\gamma_i} \left[ \left( X_T^{*,i,n} - m_T^* \right)^2 \right] \left[ \left( \int_0^T \left( F(t, \tilde{\pi}_t^{i,n}) - F(t, \pi_t^{*,i,n}) \right) dt + \int_0^T \gamma_i (\tilde{\pi}_t^{i,n} - \pi_t^{*,i,n}) (\sigma_t dW_t^i + \sigma_t^0 dW_t^0) \right) \right]$$

$$+ \left( \int_0^T \gamma_i (\tilde{\pi}_t^{i,n} - \pi_t^{*,i,n}) (\sigma_t dW_t^i + \sigma_t^0 dW_t^0) \right) + \int_0^T \log \left( \frac{1 - \pi_t^{*,i,n}}{1 - \pi_t^{*,i,n}} \right) dN_t^i \right) - 1 \right\} \]$$

$$= \frac{1}{\gamma_i} \left[ \left( X_T^{*,i,n} - m_T^* \right)^2 \right] \left[ \left( \int_0^T \left( F(t, \tilde{\pi}_t^{i,n}) - F(t, \pi_t^{*,i,n}) \right) dt + \gamma_i (\tilde{\pi}_t^{i,n} - \pi_t^{*,i,n}) (\sigma_t dW_t^i + \sigma_t^0 dW_t^0) \right) \right]$$

$$+ \left( \int_0^T \gamma_i (\tilde{\pi}_t^{i,n} - \pi_t^{*,i,n}) (\sigma_t dW_t^i + \sigma_t^0 dW_t^0) \right) + \int_0^T \log \left( \frac{1 - \pi_t^{*,i,n}}{1 - \pi_t^{*,i,n}} \right) dN_t^i \right) - 1 \right\} \]$$

It follows from (5.12) in Lemma 5.4 that $\mathbb{E}[(X_T^{*,i,n})^2] \gamma_i (m_T^*)^{-2\theta_i \gamma_i}$ is bounded and the bound is independent of $(i, n)$. Then, by (5.27), in order to verify (5.20), we need to show that

$$\mathbb{E} \left[ \left( \int_0^T \left( F(t, \tilde{\pi}_t^{i,n}) - F(t, \pi_t^{*,i,n}) \right) dt + \int_0^T \gamma_i (\tilde{\pi}_t^{i,n} - \pi_t^{*,i,n}) (\sigma_t dW_t^i + \sigma_t^0 dW_t^0) \right) \right]$$

$$+ \left( \int_0^T \gamma_i (\tilde{\pi}_t^{i,n} - \pi_t^{*,i,n}) (\sigma_t dW_t^i + \sigma_t^0 dW_t^0) \right) + \int_0^T \log \left( \frac{1 - \pi_t^{*,i,n}}{1 - \pi_t^{*,i,n}} \right) dN_t^i \right) - 1 \right\} \]$$

We introduce the density process $L = (L_t)_{t \in [0,T]}$ satisfying the following SDE under the original
probability measure \( \mathbb{P} \) that
\[
\frac{dL_t}{L_{t-}} = \gamma_i \sigma_t(\tilde{\pi}^{i,n}_t - \pi^{*,i,n}_t) dW_t^i + \gamma_i \sigma^0_t(\tilde{\pi}^{i,n}_t - \pi^{*,i,n}_t) dW^0_t + \left\{ \frac{(1 - \tilde{\pi}^{i,n}_t)^{\gamma_i}}{(1 - \pi^{*,i,n}_t)^{\gamma_i}} - 1 \right\} dM^{i,n}_t. \tag{5.29}
\]
In view that \( \tilde{\pi}^{i,n}, \pi^{*,i,n} \in \mathcal{A}^i \), the density process \( L \) is in fact a martingale. We then define a probability measure \( \mathbb{Q} \sim \mathbb{P} \) by
\[
\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathbb{Q}_t} = L_t, \quad t \in [0, T]. \tag{5.30}
\]
Using the change of measure and (5.22), we obtain the existence of a constant \( C > 0 \) independent of \( n \) such that
\[
\mathbb{E}\left[ \exp \left\{ \int_0^T \left( F(t, \tilde{\pi}^{i,n}_t) - F(t, \pi^{*,i,n}_t) \right) dt + \int_0^T \gamma_i \sigma_t(\tilde{\pi}^{i,n}_t - \pi^{*,i,n}_t) dW_t^i \right. \right.
\]
\[
+ \left. \int_0^T \gamma_i \sigma^0_t(\tilde{\pi}^{i,n}_t - \pi^{*,i,n}_t) dW^0_t + \int_0^T \log \left( \frac{(1 - \tilde{\pi}^{i,n}_t)^{\gamma_i}}{(1 - \pi^{*,i,n}_t)^{\gamma_i}} \right) dN^{i,n}_t \right\} \right]
\]
\[
= \mathbb{E}^\mathbb{Q}\left[ \exp \left\{ \int_0^T \left( F(t, \tilde{\pi}^{i,n}_t) - F(t, \pi^{*,i,n}_t) \right) dt + \int_0^T \frac{1}{2} \gamma_i^2 \sigma_t^2 + (\sigma^0_t)^2 (\pi^{*,i,n}_t - \pi^{*,i,n}_t)^2 dt \right. \right.
\]
\[
+ \left. \int_0^T \left( \frac{(1 - \tilde{\pi}^{i,n}_t)^{\gamma_i}}{(1 - \pi^{*,i,n}_t)^{\gamma_i}} - 1 \right) \Lambda^{i,n}_t dt \right\} \right]
\]
\[
= \mathbb{E}^\mathbb{Q}\left[ \exp \left\{ \int_0^T \left[ \gamma_i (b_t + \Lambda^{i,n}_t) (\tilde{\pi}^{i,n}_t - \pi^{*,i,n}_t) + \frac{1}{2} \gamma_i (\gamma_i - 1) \sigma_t^2 + (\sigma^0_t)^2 (\pi^{*,i,n}_t - \pi^{*,i,n}_t)^2 \right. \right. \right.
\]
\[
+ \left. \left. \frac{1}{2} \gamma_i^2 \sigma_t^2 + (\sigma^0_t)^2 (\pi^{*,i,n}_t - \pi^{*,i,n}_t)^2 + \left( \frac{(1 - \tilde{\pi}^{i,n}_t)^{\gamma_i}}{(1 - \pi^{*,i,n}_t)^{\gamma_i}} - 1 \right) \Lambda^{i,n}_t \right) dt \right\} \right]
\]
\[
\leq \mathbb{E}^\mathbb{Q}\left[ \exp \left( C \int_0^T |\tilde{\pi}^{i,n}_t - \pi^{*,i,n}_t| dt \right) \right] \leq 1 + O \left( \frac{1}{n} \right).
\]
Here, \( \mathbb{E}^\mathbb{Q} \) denotes the expectation operator under \( \mathbb{Q} \). This shows the validity of (5.28). It then follows from (5.27) and (5.28) that (5.20) holds. Combining (5.17)-(5.20), we can conclude the desired estimation that \( \sup_{\pi^i \in \mathcal{A}^i} J_i(\pi^i, \pi^{*,i,n}_t) - J_i(\pi^{*,i,n}_t, \pi^{*,i,n}_t) \leq O(n^{-\frac{1}{4}}). \)

6 Conclusions

This paper revisits the MFG and the \( n \)-player game under CRRA relative performance by allowing risky assets to have contagious jumps, which are specifically modelled by a multi-dimensional mutually exciting Hawkes process. As a first attempt to such problems to accommodate controlled jumps, it is assumed for tractability in the present paper that the limiting model has constant parameters. By using the FBSDE and stochastic maximum principle arguments, a deterministic Nash equilibrium for the MFG can be characterized as a function of the deterministic limiting intensity.
process. Furthermore, it is shown that this deterministic MFE provides a good approximation of the Nash equilibrium for the large but finite population game and the order of the approximation error is explicitly obtained.

Based on our current study, some future research directions can be considered. First, it will be attractive to consider both $n$-agent model and the limiting model with general parameters that are random variables or stochastic processes. A deterministic mean field equilibrium may no longer exist. The existence of a mean field equilibria and the verification of an approximate Nash equilibrium for the $n$-player game will require some distinctive mathematical arguments. Second, our work may pave the way to consider other sophisticated default intensity processes. For example, the default intensity of each risky asset may depend on the asset price itself or other stochastic factors. Some novel analysis for the mean field FBSDE with jumps are in demand to tackle the MFG problem. Third, it is interesting, albeit challenging, to consider possible contagion mechanisms beyond the interacting intensity framework, such as network graphs in Amini and Minca (2016), Klages-Mundt and Minca (2020), etc. Reformulating and solving MFGs under relative performance in these settings to account for cascading failures will be some interesting open problems.

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A Proofs of some auxiliary results

Proof of Lemma 3.1. Recall the definition of $\Phi(\pi, \lambda) : (-\infty, 1) \times \mathbb{R}_+ \to \mathbb{R}$ in (3.5). As $0 < \gamma < 1$, the partial derivative is given by, for $(\pi, \lambda) \in (-\infty, 1) \times \mathbb{R}_+$,

$$\partial_\pi \Phi(\pi, \lambda) = (\gamma - 1)[\sigma^2 + (\sigma^0)^2] - \theta \gamma (\sigma^0)^2 + (\gamma - 1)\lambda(1 - \pi)^{\gamma - 2} < 0,$$

which implies that $\Phi(\pi, \lambda)$ is monotonically decreasing with respect to $\pi \in (-\infty, 1)$. Note that $\Phi(0, \lambda) = b > 0$ and $\lim_{\pi \uparrow 1} \Phi(\pi, \lambda) = -\infty$. It is deduced that there exists a unique $\pi^* \in (0, 1 - \epsilon_0]$ such that $\Phi(\pi^*, \lambda) = 0$, where the constant $\epsilon_0 \in (0, 1)$ is small enough. Moreover, as $\partial_\pi \Phi(\pi, \lambda) < 0$ for all $(\pi, \lambda) \in (-\infty, 1) \times \mathbb{R}_+$, it follows from the implicit function theorem that there exists a unique continuous function $\phi$ such that $\pi^* = \phi(\lambda)$, and $\phi$ has a continuous partial derivative with respect to $\lambda$. We also note that

$$\partial_\lambda \phi(\lambda) = -\frac{\partial_\pi \Phi(\pi^*, \lambda)}{\partial_\pi \Phi(\pi^*, \lambda)} = -\frac{1 - (1 - \pi^*)^{\gamma - 1}}{(\gamma - 1)[\sigma^2 + (\sigma^0)^2] - \theta \gamma (\sigma^0)^2 + (\gamma - 1)\lambda(1 - \pi^*)^{\gamma - 2}} < 0,$$

and $|\partial_\lambda \phi(\lambda)| < K$ for some constant $K$ independent of $\lambda$. The desired result follows that the function $\phi$ is decreasing in $\lambda$ and is Lipschitz continuous with respect to $\lambda$.

Proof of Lemma 4.1. Note that the mean filed equilibrium $t \mapsto \pi^*_t$ in Theorem 3.2 is a positive deterministic function that $\pi^*_t = \phi(\lambda^*_t) \in (0, 1 - \epsilon_0)$ for $0 < \gamma < 1$. For each fixed $t \in [0, T]$, by
straightforward computations, we can derive the derivatives that

\[
\begin{align*}
\partial_0 \phi &= - \frac{\partial \Phi}{\partial \phi} = - \frac{1}{(\gamma - 1)[\sigma^2 + (\sigma^0)^2] - \theta \gamma (\sigma^0)^2 + (\gamma - 1)\lambda_0^2(1 - \pi_t^*)^{-2}} > 0, \\
\partial_\eta \phi &= - \frac{\partial \Phi}{\partial \phi} = - \frac{2(\gamma - 1)\sigma \pi_t^* - 2 \theta \gamma \sigma_0 \pi_t^*}{(\gamma - 1)[\sigma^2 + (\sigma^0)^2] - \theta \gamma (\sigma^0)^2 + (\gamma - 1)\lambda_0^2(1 - \pi_t^*)^{-2}} < 0, \\
\partial_{\sigma_0} \phi &= - \frac{\partial \Phi}{\partial \phi} = - \frac{2(\gamma - 1)\sigma_0 \pi_t^* - 2 \theta \gamma \sigma_0 \pi_t^*}{(\gamma - 1)[\sigma^2 + (\sigma^0)^2] - \theta \gamma (\sigma^0)^2 + (\gamma - 1)\lambda_0^2(1 - \pi_t^*)^{-2}} < 0, \\
\partial_\gamma \phi &= - \frac{\partial \Phi}{\partial \phi} = - \frac{\sigma^2 + (1 - \theta)(\sigma^0)^2 \pi_t^* - \lambda_0^2 \log(1 - \pi_t^*)(1 - \pi_t^*)^{-\frac{1}{2}}}{(\gamma - 1)[\sigma^2 + (\sigma^0)^2] - \theta \gamma (\sigma^0)^2 + (\gamma - 1)\lambda_0^2(1 - \pi_t^*)^{-2}} < 0, \\
\partial_\phi &= - \frac{\partial \Phi}{\partial \phi} = - \frac{-\gamma (\sigma^0)^2 \pi_t^*}{(\gamma - 1)[\sigma^2 + (\sigma^0)^2] - \theta \gamma (\sigma^0)^2 + (\gamma - 1)\lambda_0^2(1 - \pi_t^*)^{-2}} < 0,
\end{align*}
\]

and the claimed monotonicity results follow directly. \(\square\)

**Proof of Lemma 5.1.** Let \(A_t^{i_1} \) be the admissible control set starting with any time \(t \in [0, T]\). Then, we can define the value function of the auxiliary control problem \((P_n)\) given by, for \((t, x, m, \lambda) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \Xi_{\Lambda,t}\),

\[
V(t, x, m, \lambda) := \sup_{\pi^i \in A_t^i} \mathbb{E} \left[ \frac{1}{\gamma_t}(X_{T}^{i,n}(m_t^*)^{\gamma_t}) V(t, x, m, \lambda) = x, m_t^* = m, A_t = \lambda \right], \quad (A.1)
\]

where \(A_t := (A_t^{i_1}, \ldots, A_t^{n,n})^T\) for \(t \in [0, T]\). The value function \((A.1)\) is then associated with the HJB equation that

\[
0 = \partial_t V(t, x, m, \lambda) + \sum_{j=1}^{n} \partial_{\lambda_j} V(t, x, m, \lambda) \alpha_j(\lambda_{\infty}^j - \lambda_j) + \sum_{j \neq i} \lambda_j \left( V(t, x, m, \lambda + \frac{\beta_i \gamma_i}{n} e_j^n) - V(t, x, m, \lambda) \right)
\]

\[
+ \sup_{\pi^i \in (-\infty, 1)} \left\{ \partial_x V(t, x, m, \lambda) x r + (b_i + \lambda_i) \pi_i^* \right\} + \partial_m V(t, x, m, \lambda) \eta(t; \pi_t^*) m
\]

\[
+ \frac{1}{2} \partial_{xx} V(t, x, m, \lambda) x^2 + \left( \sigma_0^2 + (\sigma_0^0)^2 \right) \pi_i^* + \frac{1}{2} \partial_{mm} V(t, x, m, \lambda) m^2 (\sigma_0 \pi_t^*)^2
\]

\[
+ \partial_{xm} V(t, x, m, \lambda) x m \sigma_0^2 \pi_i^* \sigma_0 \pi_t^* + \lambda_i \left( V(t, (1 - \pi_t^*) x, m, \lambda + \frac{\beta_i \gamma_i}{n} e_j^n) - V(t, x, m, \lambda) \right)
\]

with the terminal condition \(V(T, x, m, \lambda) = \frac{1}{\gamma_t} x^{\gamma_t} m^{\gamma_t} \). Here, \(e_j^n\) denotes the \(n\)-dimensional column vector whose \(i\)-th entry is 1 and remaining ones are 0. Let us consider the decoupled form that \(V(t, x, m, \lambda) = \frac{1}{\gamma_t} x^{\gamma_t} m^{\gamma_t} B(t, \lambda)\), where \(B(t, \lambda)\) solves the following equation:

\[
0 = \partial_t B(t, \lambda) + \frac{1}{2} \theta_i \gamma_i (\theta_i \gamma_i + 1)(\sigma_0 \pi_t^*)^2 B(t, \lambda) - \theta_i \gamma_i \eta(t; \pi_t^*) B(t, \lambda) + \sup_{\pi^i \in (-\infty, 1)} \mathcal{H}(t, \lambda; \pi_i^*)
\]

\[
+ \sum_{j=1}^{n} \partial_{\lambda_j} B(t, \lambda) \alpha_j(\lambda_{\infty}^j - \lambda_j) + \sum_{j \neq i} \lambda_j \left( B(t, \lambda + \frac{\beta_i \gamma_i}{n} e_j^n) - B(t, \lambda) \right), \quad (A.2)
\]
where the terminal condition is given by $B(T, \lambda) = 1$, and $\mathcal{H}(t, \lambda; \pi^i)$ corresponds to the Hamiltonian operator that

$$\mathcal{H}(t, \lambda; \pi^i) := \gamma_i[r + (b_i + \lambda_i)\pi^i]B(t, \lambda) + \frac{1}{2}\gamma_i(\gamma_i - 1)(\sigma_i^2 + (\sigma_i^0)^2)(\pi^i)^2B(t, \lambda) - \theta_i\gamma_i\sigma_i^0\pi^i\sigma^0\pi^* + B(t, \lambda) + \lambda_i \left(1 - \pi^i\right)H\left(t, \lambda, \frac{\gamma_i}{n}e_i^n\right) - B(t, \lambda)\right).$$

It follows from Theorem 4.1 in Bo et al. (2019a) and Proposition 4.3 in Delong and Klüppelberg (2008) that (A.2) admits a unique (positive) classical solution. By applying the first-order condition to $\mathcal{H}(t, \lambda; \pi^i)$ with respect to $\pi^i \in (-\infty, 1)$, we obtain that the optimum $\bar{\pi}^{i,n} = \bar{\pi}^{i,n}(t, \lambda) \in (-\infty, 1)$ in Eq. (A.2) satisfies

$$b_i + \lambda_i + (\gamma_i - 1)(\sigma_i^2 + (\sigma_i^0)^2)\bar{\pi}^{i,n} - \theta_i\gamma_i\sigma_i^0\sigma^0\pi^* - (1 - \bar{\pi}^{i,n})\gamma_i^{-1}\frac{B\left(t, \lambda, \frac{\gamma_i}{n}e_i^n\right)B(t, \lambda)}{B(t, \lambda)}\lambda_i = 0. \quad \text{(A.4)}$$

By the assumption $(A_0)$, it holds that

$$\left|\frac{B\left(t, \lambda, \frac{\gamma_i}{n}e_i^n\right)}{B(t, \lambda)} - 1\right| = \left|\frac{B\left(t, \lambda, \frac{\gamma_i}{n}e_i^n\right) - B(t, \lambda)}{B(t, \lambda)}\right| \leq \frac{\gamma_i}{n}\|\partial_\lambda B(\cdot, \cdot)\|_{\infty} = O\left(\frac{1}{n}\right).$$

This yields the desired result (5.5).

It follows from the assumption $(A_0)$ and (5.5) that, there exist a constant $K > 0$ independent of $(i, n)$ such that $|\Phi_i(t, \lambda_i, \bar{\pi}^{i,n})| \leq K$. Note that $\pi \mapsto \Phi_i(t, \lambda_i, \pi)$ is continuous and decreasing, $\lim_{\pi \to \phi} \Phi_i(t, \lambda_i, \pi) = -\infty$ and $\lim_{\pi \to \phi} \Phi_i(t, \lambda_i, \pi) = +\infty$. We get the existence of some $(D_i, \epsilon_i) \in \mathbb{R} \times (0, 1)$ independent of $(t, \lambda, n)$ such that $\bar{\pi}^{i,n} \in [D_i, 1 - \epsilon_i]$. We next prove that there exists a pair $(\tilde{D}, \tilde{\epsilon}) \in \mathbb{R} \times (0, 1)$ which is independent of $(t, \lambda, i, n)$ such that $D_i \geq \tilde{D}$ and $\epsilon_i \geq \tilde{\epsilon}$. In fact, by the monotonicity of $\pi \mapsto \Phi_i(t, \lambda_i, \pi)$, it follows that

$$\Phi(t, \lambda_i, D_i) = (\gamma_i - 1)[\sigma_i^2 + (\sigma_i^0)^2]D_i - \theta_i\gamma_i\sigma_i^0\sigma^0\pi^* - \lambda_i(1 - D_i)\gamma_i^{-1} + \lambda_i + b_i \leq K,$n

and

$$\Phi(t, \lambda_i, 1 - \epsilon_i) = (\gamma_i - 1)[\sigma_i^2 + (\sigma_i^0)^2](1 - \epsilon_i) - \theta_i\gamma_i\sigma_i^0\sigma^0\pi^* - \lambda_i(\epsilon_i)\gamma_i^{-1} + \lambda_i + b_i \geq -K.$n

Thanks to the assumption $(A_0)$, we have that

$$(\gamma_i - 1)[\sigma_i^2 + (\sigma_i^0)^2]D_i \leq K + \theta_i\gamma_i\sigma_i^0\sigma^0\pi^* + \lambda_i(1 - D_i)\gamma_i^{-1} - \lambda_i - b_i \leq K + \sup_{i \geq 1} \theta_i\gamma_i\sigma_i^0\sigma^0.$n

This yields that

$$-D_i \leq \frac{K + \sup_{i \geq 1} \theta_i\gamma_i\sigma_i^0\sigma^0}{(1 - \gamma_i)[\sigma_i^2 + (\sigma_i^0)^2]} \leq \frac{K + \sup_{i \geq 1} \theta_i\gamma_i\sigma_i^0\sigma^0}{\inf_{i \geq 1}(1 - \gamma_i)[\sigma_i^2 + (\sigma_i^0)^2]} := -\tilde{D}.$n

Similarly, we also have that

$$\lambda_i(\epsilon_i)\gamma_i^{-1} \leq K + (\gamma_i - 1)[\sigma_i^2 + (\sigma_i^0)^2](1 - \epsilon_i) - \theta_i\gamma_i\sigma_i^0\sigma^0\pi^* + \lambda_i + b_i \leq K (+ \gamma_i - 1)[\sigma_i^2 + (\sigma_i^0)^2] + \lambda_i + b_i \leq K + \sup_{i \geq 1}(\gamma_i - 1)[\sigma_i^2 + (\sigma_i^0)^2] + \sup_{i \geq 1}\lambda_i + \sup_{i \geq 1}b_i := C.$n

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It holds that \((\epsilon_t)^{n-1} \leq \frac{C}{\lambda_t} \leq \frac{C}{\inf_{i \geq 1} \lambda_i}\). Therefore, we obtain that
\[
\epsilon_t \geq \left(\inf_{i \geq 1} \lambda_i \right) \frac{1}{C} \geq \inf_{i \geq 1} \left(\lambda_i \frac{1}{C}\right) := \hat{\epsilon},
\]
which completes the proof.

**Proof of Lemma 5.4.** (i) By (5.9), we have that \(\pi_t^{*,i,n} = \phi_t(t, \Lambda_t^{i,n}) \in [D_0, 1 - \epsilon_0]\). Also recall that the wealth process \(X_t^{*,i,n} = (X_t^{*,i,n})_{t \in [0,T]}\) satisfies (5.10). The conclusion clearly holds when \(p = 0\). It suffices to prove the result when \(p \neq 0\). By Itô formula, we have that
\[
d(X_t^{*,i,n})^p = p(X_t^{*,i,n})^p(r + (b_i + \Lambda_t^{i,n})\pi_t^{*,i,n}) + \frac{1}{2}p(p-1)(\pi_t^{*,i,n})^2(\sigma_i^2 + (\sigma_t^0)^2)dt
\]
\[+ p(X_t^{*,i,n})^p\pi_t^{*,i,n}(\sigma_t dW_t^i + \sigma_t^0 dW_t^0) + (X_t^{*,i,n})^p \log(1 - \pi_t^{*,i,n})^p dN_t^{i,n}.\]  \(\tag{A.5}\)

It follows that
\[
\mathbb{E}\left[(X_t^{*,i,n})^p\right] = \mathbb{E}\left[p \int_0^t \left[r + (b_i + \Lambda_t^{i,n})\pi_s^{*,i,n} - \frac{1}{2}(\pi_s^{*,i,n})^2(\sigma_i^2 + (\sigma_t^0)^2)\right] ds
\]
\[+ p \int_0^t \pi_s^{*,i,n}(\sigma_t dW_s^i + \sigma_t^0 dW_s^0) + \int_0^t \log(1 - \pi_s^{*,i,n})^p dN_s^{i,n}\right]\]
\[= \mathbb{E}\hat{Q}\left[p \int_0^t \left[r + b_i \pi_s^{*,i,n} + \frac{1}{2}(p-1)(\pi_s^{*,i,n})^2(\sigma_i^2 + (\sigma_t^0)^2)\right]
\]
\[+ ((1 - \pi_s^{*,i,n})^p + p\pi_s^{*,i,n} - 1)\Lambda_s^{i,n}ds\right].\]  \(\tag{A.6}\)

Here, \(\hat{Q} \sim \mathbb{P}\) is defined by \(\frac{d\hat{Q}}{d\mathbb{P}}|_{\mathcal{G}_t} = \hat{L}_t\), where \(\hat{L} = (\hat{L}_t)_{t \in [0,T]}\) satisfies SDE under \(\mathbb{P}\) that
\[
\frac{d\hat{L}_t}{\hat{L}_{t-}} = p \sigma_i \pi_t^{*,i,n} dW_t^i + p \sigma_t^0 \pi_t^{*,i,n} dW_t^0 + [(1 - \pi_t^{*,i,n})^p - 1]dM_t^{i,n}, \quad \hat{L}_0 = 1.\]  \(\tag{A.7}\)

Note that both of \(\pi_t^{*,i,n} = (\pi_t^{*,i,n})_{t \in [0,T]}\) and \(\Lambda_t^{i,n} = (\Lambda_t^{i,n})_{t \in [0,T]}\) are bounded. The desired estimate (5.12) follows from (A.6).

(ii) Note that the mean field intensity process reduces to
\[
\lambda_t^i = \lambda_0 + \int_0^t \alpha(\lambda_\infty - \lambda_s^0)ds + \int_0^t \beta \varsigma \lambda_s^0 ds, \tag{A.8}\]
and the intensity process \(\Lambda_t^{i,n} = (\Lambda_t^{i,n})_{t \in [0,T]}\) takes the form that
\[
d\Lambda_t^{i,n} = \alpha_t(\lambda_\infty^i - \Lambda_t^{i,n})dt + \frac{1}{n} \sum_{j=1}^n \beta \varsigma_{ij} dN_t^{j,n}. \tag{A.9}\]

It follows from (A.8) and (A.9) that
\[
d(\Lambda_t^{i,n} - \lambda_t^0) = \left[\alpha_t(\lambda_\infty^i - \Lambda_t^{i,n}) - \alpha(\lambda_\infty - \lambda_t^0)\right] dt + \frac{1}{n} \sum_{j=1}^n \beta \varsigma_{ij} dN_t^{j,n} - \beta \varsigma \lambda_t^0 dt.\]
Applying Itô’s lemma, we obtain that
\[
\begin{align*}
\int_0^t \frac{1}{n} \sum_{j=1}^n \left( \Lambda_{i,n}^j - \Lambda_t^o \right)^2 &= \int_0^t \sum_{j=1}^n \left( \Lambda_{i,n}^j - \Lambda_t^o \right)^2 
\end{align*}
\]
Taking the integral from 0 to \( t \) and then taking expectations on both sides, we arrive at
\[
\begin{align*}
\mathbb{E}\left[ \left( \Lambda_{i,n}^j - \Lambda_t^o \right)^2 \right] &= \int_0^t \mathbb{E}\left[ \left( \Lambda_{i,n}^j - \Lambda_t^o \right)^2 \right] ds + O \left( \frac{1}{n^2} \right) \\
+ \frac{1}{n} \sum_{j=1}^n \int_0^t 2 \beta_i \mathbb{E}\left[ (\Lambda_{s,n}^j - \Lambda_t^o) \Lambda_{i,n}^j \right] ds
\end{align*}
\]
It follows from Jensen’s inequality that, there exists a constant \( C > 0 \) independent of \( n \) such that
\[
\mathbb{E}\left[ \left( \Lambda_{i,n}^j - \Lambda_t^o \right)^2 \right] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[ \left( \Lambda_{i,n}^j - \Lambda_t^o \right)^2 \right]
\]
\[
= \frac{1}{n} \int_0^t \mathbb{E}\left[ \left( \Lambda_{i,n}^j - \Lambda_t^o \right)^2 \right] ds + O \left( \frac{1}{n^2} \right)
\]
\[
+ \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \int_0^t 2 \beta_i \mathbb{E}\left[ (\Lambda_{s,n}^j - \Lambda_t^o) \Lambda_{i,n}^j \right] ds
\]
\[
\leq \frac{1}{n} \int_0^t \mathbb{E}\left[ \left( \Lambda_{i,n}^j - \Lambda_t^o \right)^2 \right] ds + O \left( \frac{1}{n^2} \right)
\]
By using Gronwall’s inequality, we conclude that
\[
\frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[ \left( \Lambda_{i,n}^j - \Lambda_t^o \right)^2 \right] = O \left( \frac{1}{n^2} \right).
\]
The desired claim (5.13) then holds.

(iii) Recall that \( X_{t,n}^{*,i,n} = \left( \prod_{i=1}^n X_{t,i,n}^{*,i,n} \right)^{\frac{1}{n}} \) where \( X_{t,i,n}^{*,i,n} \) is defined by (5.10), and the geometric mean process \( m^* = (m_t^*)_{t \in [0,T]} \) satisfies (3.10). Let us define \( Y_t := \log(X_{t,n}^{*,i,n}) \) and
\[
\hat{Y}_t := \frac{1}{n} \sum_{i=1}^n Y_t^i = \log(X_{t,n}^{*,i,n}) = \frac{1}{n} \sum_{i=1}^n \log(X_{t,i,n}^{*,i,n}).
\]
It follows from (5.10) that
\[
dY^i_t = \left[ r + (b_i + \Lambda_t^{i,n}) \pi_t^{s,i,n} - \frac{1}{2} \left( \sigma_t^2 (\pi_t^{s,i,n})^2 + (\pi_t^0)^2 \right) \right] dt + \pi_t^{s,i,n} \sigma_t dW_t^i + \pi_t^{s,i,n} \sigma_t^0 dW_t^0 \\
+ \log \left( 1 - \pi_t^{s,i,n} \right) dN_t^{i,n}.
\]
(A.10)

We denote \( Z_t := \log(m_t^s) \), which satisfies that
\[
dZ_t = \left\{ r + (b + \lambda_0^r) \pi_t^r - \frac{1}{2} (\sigma^2 + \sigma_0^2) (\pi_t^r)^2 + \lambda_0^r \log(1 - \pi_t^r) \right\} dt + \sigma^0 \pi_t^r dW_t^0.
\]
(A.11)

Note that
\[
\begin{align*}
\mathbb{E} \left[ (\tilde{X}_t^{s,n} - m_t^s)^2 \right] &= \mathbb{E} \left[ e^{2Z_t} \left( e^{Y_t-Z_t} - 1 \right)^2 \right] \\
&\leq \mathbb{E} \left[ e^{4Z_t} \right] \frac{1}{2} \mathbb{E} \left[ \left( e^{2(Y_t-Z_t)} - 2e^{Y_t-Z_t} + 1 \right)^2 \right]^{\frac{1}{2}} \\
&= \mathbb{E} \left[ e^{4Z_t} \right] \frac{1}{2} \mathbb{E} \left[ e^{3(Y_t-Z_t)} + 4e^{2(Y_t-Z_t)} + 1 - 4e^{3(Y_t-Z_t)} + 2e^{2(Y_t-Z_t)} - 4e^{Y_t-Z_t} \right]^{\frac{1}{2}}.
\end{align*}
\]

To prove the claim (5.14), by the boundedness of \( \tilde{X}_t^{s,n} = (\tilde{X}_t^{s,n})_{t \in [0,T]} \), it is sufficient to prove that, for any \( t \in [0,T] \)
\[
\lim_{n \to \infty} \mathbb{E} \left[ e^{Y_t-Z_t} \right] = 1.
\]
(A.12)

To this end, for \( i = 1, \ldots, n \), we introduce the auxiliary SDE that
\[
d\hat{Y}_t^i = \left[ r + (b_i + \Lambda_t^{i,n}) \pi_t^{s,i,n} - \frac{1}{2} \left( \sigma_t^2 + (\sigma_t^0)^2 \right) (\pi_t^{s,i,n})^2 + \Lambda_t^{i,n} \log(1 - \pi_t^{s,i,n}) \right] dt + \pi_t^{s,i,n} \sigma_t dW_t^i,
\]
and
\[
d\bar{Y}_t = \left[ r + (b + \Lambda_t^{i,n}) \pi_t^{s,i,n} - \frac{1}{2} \left( \sigma_t^2 + (\sigma_t^0)^2 \right) (\pi_t^{s,i,n})^2 + \Lambda_t^{i,n} \log(1 - \pi_t^{s,i,n}) \right] dt + \pi_t^{s,i,n} \sigma_t^0 dW_t^0.
\]

For some positive constants \( p_i \) for \( i = 1, 2, 3 \) satisfying \( \sum_{i=1}^3 \frac{1}{p_i} = 1 \), we can derive by generalized Hölder inequality that
\[
\begin{align*}
\mathbb{E} \left[ e^{Y_t-Z_t} \right] &= \mathbb{E} \left[ e^{(Y_t - \frac{1}{n} \sum_{i=1}^n \bar{Y}_t^i) + (\frac{1}{n} \sum_{i=1}^n \tilde{Y}_t^i - \frac{1}{n} \sum_{i=1}^n Y_t^i) + (\frac{1}{n} \sum_{i=1}^n \pi_t^{s,i,n})} \right] \\
&\leq \mathbb{E} \left[ e^{p_1(Y_t - \frac{1}{n} \sum_{i=1}^n \bar{Y}_t^i)} \right]^{1/p_1} \mathbb{E} \left[ e^{p_2(\frac{1}{n} \sum_{i=1}^n \tilde{Y}_t^i - \frac{1}{n} \sum_{i=1}^n Y_t^i)} \right]^{1/p_2} \mathbb{E} \left[ e^{p_3(\frac{1}{n} \sum_{i=1}^n \pi_t^{s,i,n})} \right]^{1/p_3}.
\end{align*}
\]
(A.13)

For the first term on the RHS of (A.13), we have that
\[
\mathbb{E} \left[ e^{\frac{1}{n} \sum_{i=1}^n (\bar{Y}_t^i - \tilde{Y}_t^i)} \right] = \mathbb{E} \left[ e^{\frac{1}{n} \sum_{i=1}^n \int_0^t (b_i - b) \pi_t^{s,i,n} - \frac{1}{2} \left( \sigma_t^2 + (\sigma_t^0)^2 - \sigma^2 - (\sigma_0^0)^2 \right) (\pi_t^{s,i,n})^2} \right]
\]
\[
= 1 + O \left( \frac{1}{n} \right).
\]

For the second term on the RHS of (A.13), we have that
\[
\mathbb{E} \left[ e^{\frac{1}{n} \sum_{i=1}^n (\tilde{Y}_t^i - \bar{Y}_t^i)} \right] = \mathbb{E} \left[ \exp \left\{ \frac{1}{n} \sum_{i=1}^n \int_0^t \left[ (b_i - b) \pi_t^{s,i,n} - \frac{1}{2} \left( \sigma_t^2 + (\sigma_t^0)^2 - \sigma^2 - (\sigma_0^0)^2 \right) (\pi_t^{s,i,n})^2 \right] ds \right\} \right]
\]
\[
= 1 + O \left( \frac{1}{n} \right).
\]

For the third term on the RHS of (A.13), we have that
\[
\mathbb{E} \left[ e^{\frac{1}{n} \sum_{i=1}^n (\pi_t^{s,i,n})^2} \right] = \mathbb{E} \left[ \exp \left\{ \frac{1}{n} \sum_{i=1}^n \int_0^t \left[ (b_i - b) \pi_t^{s,i,n} - \frac{1}{2} \left( \sigma_t^2 + (\sigma_t^0)^2 - \sigma^2 - (\sigma_0^0)^2 \right) (\pi_t^{s,i,n})^2 \right] ds \right\} \right]
\]
\[
= 1 + O \left( \frac{1}{n} \right).
\]
where \( C \) is a positive constant independent of \( n \). By the assumption \((A_0)\), we have that
\[
E \left[ e^{\frac{1}{n} \sum_{i=1}^{n} (\hat{Y}_i - Y_i)} \right] = 1 + O \left( \frac{1}{n} \right).
\]

For the third term on the RHS of (A.13), it holds that
\[
\begin{align*}
E \left[ e^{\frac{1}{n} \sum_{i=1}^{n} (\hat{Y}_i - Y_i)} \right] &= E \left[ \exp \left\{ \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{T} \left[ \sigma_i - \sigma^0 \right] \pi_s^{i,n} dW_s^{0} \right\} \right] \\
&\leq E \left[ \exp \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{T} \left| b_i - b \right| + |\sigma_i - \sigma^0| + |(\sigma_i^0)^2 - (\sigma^0)^2| \right. \right. \\
&\quad \left. \left. \times \int_{0}^{T} ds \right\} \right] \\
&\leq E \left[ \exp \left\{ C \int_{0}^{T} \frac{1}{n} \sum_{i=1}^{n} \left[ (\pi_s^{i,n} - \pi^*) + \log(1 - \pi_s^{i,n}) - \log(1 - \pi^*) \right] ds \right\} \right] \\
&= \left[ 1 + O \left( \frac{1}{n} \right) \right].
\end{align*}
\]

We then claim that
\[
\begin{align*}
E \left[ e^{\frac{1}{n} \sum_{i=1}^{n} (\hat{Y}_i - Y_i)} \right] &= 1 + O \left( \frac{1}{n} \right). 
\end{align*}
\]

Recall that \( \pi_s^{i,n} = \phi_i(t, \Lambda_t^{i,n}) \) in (5.9) satisfies the equation that
\[
(\gamma_i - 1)[\sigma_i^2 + (\sigma_i^0)^2] \pi_s^{i,n} - \theta_i \gamma_i \sigma_i^0 \sigma_i \pi_s^* - \Lambda_t^{i,n}(1 - \pi_s^{i,n}) \gamma_i - 1 + \Lambda_t^{i,n} + b_i = 0,
\]

and \( \pi_s^* = \phi(\lambda_s^0) \) given by Theorem 3.2 is the solution to
\[
(\gamma - 1)[\sigma^2 + (\sigma^0)^2] \pi^* - \theta \gamma (\sigma^0)^2 \pi^* - \Lambda^0(1 - \pi^*) \gamma - 1 + \Lambda^0 + b = 0.
\]

We introduce an auxiliary control \( \hat{\pi}_i^t := \phi_i(t, \lambda_t^0) \), which satisfies
\[
(\gamma_i - 1)[\sigma_i^2 + (\sigma_i^0)^2] \hat{\pi}_i^t - \theta_i \gamma_i \sigma_i^0 \sigma_i \pi^* - \Lambda_t^{0}(1 - \hat{\pi}_i^t) \gamma_i - 1 + \Lambda_t^{0} + b_i = 0.
\]

From the proof of Lemma 5.4-(ii) and the assumption \((A_0)\), we have that \( \hat{\pi}_i^t \to \pi_s^* \) as \( i \to \infty \), and there exists a constant \( C > 0 \) independent of \( n \) such that
\[
\begin{align*}
\frac{1}{n} \sum_{i=1}^{n} (\pi_s^{i,n} - \pi^*) &= \frac{1}{n} \sum_{i=1}^{n} (\pi_s^{i,n} - \hat{\pi}_i^t) + \frac{1}{n} \sum_{i=1}^{n} (\hat{\pi}_i^t - \pi^*) \\
&= \frac{1}{n} \sum_{i=1}^{n} (\phi_i(t, \Lambda_t^{i,n}) - \phi_i(t, \lambda_t^0)) + \frac{1}{n} \sum_{i=1}^{n} (\phi_i(t, \lambda_t^0) - \phi(\lambda_s^0)) \\
&\leq \frac{1}{n} \sum_{i=1}^{n} C \left| \Lambda_t^{i,n} - \lambda_t^0 \right| + O \left( \frac{1}{n} \right) \\
&= O \left( \frac{1}{n} \right).
\end{align*}
\]
Moreover, the order of the term $\frac{1}{n} \sum_{i=1}^{n} [\log(1 - \pi_{t}^{*,i,n}) - \log(1 - \pi_{t}^{*})]$ is the same to the order of $\frac{1}{n} \sum_{i=1}^{n} (\pi_{t}^{*,i,n} - \pi_{t}^{*})$ because the function $\log(1 - x)$ is Lipschitz continuous in $x \in [D_{0}, 1 - \epsilon_{0}]$, which proves that the claim (A.14) holds. Putting all the pieces together completes the proof. □

**Conflict of Interest**

The authors declare that they have no conflict of interest.

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