Calogero model revisited. Commuting Hamiltonians and Hurwitz numbers

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June 30, 2020

To Andrey Mironov and Alexei Morozov

Abstract

The generalized Mironov-Morozov-Natanson (MMN) equation includes a set of commuting operators, which can be considered as Hamiltonians for the quantum Calogero-Sutherland problem with a special value of the coupling constant (free fermion point). These Hamiltonians can be considered as the center of the enveloping algebra of the group $GL_N(\mathbb{C})$. Another commuting series of Hamiltonians is presented, parametrized by an arbitrary matrix $A \in GL_N$. These Hamiltonians are related to the Hurwitz numbers in the same way as in the case of the MMN equation and generate a generalized variant of the Calogero-Sutherland model.

Key words: multivariable commuting differential operators, Schur polynomials, Hurwitz numbers, cut-and-join equation, Calogero problem

2010 Mathematical Subject Classification: 05A15, 14N10, 17B80, 35Q51, 35Q53, 35Q55, 37K20, 37K30,

1 Introduction

The following quantum Hamiltonian

$$h(\alpha) = \sum_{n,m > 0} p_n p_m p_{-n-m} + p_{n+m} p_{-n} p_{-m} + (\alpha - 1) \sum_{l > 0} p_l p_{-l}$$

(1)

independently appears in different areas of mathematical physics [22], [3], [30], [1], [45], [46], [27]. Here $p_n$ is the bosonic field which satisfies canonical relation $[p_n, p_m] = \alpha \delta_{n+m,0}$, $\alpha$ is a parameter. The collection $p_n$, $n > 0$ presents creation operators while $p_n$, $n < 0$ are treated as annihilation ones. The eigenfunctions of Hamiltonian (1) turned out to be Jack polynomials $J^{(\alpha)}(p)$ [3]. In [27] it was found that (1) is the Hamiltonian of the quantum Benjamin-Ono equation at singular point.

In [15] the notion of the cut-and-join equation was introduced, where the differential operator (1) with $\alpha = 1$ plays an obvious role in splitting $p_{n+m}$ into a product $p_n p_m$ and join it back. This was important because of the correspondence between symmetric functions and a symmetric group, see [20], where each monomial $p_\mu = p_{\mu_1} p_{\mu_2} \cdots$ corresponds to cycle class $C_\mu$. Then Goulden-Jackson cut-and-join equation describes the product $C_{(21^N)} C_\mu$ in terms of symmetric functions. This equation was used in the problem of enumeration of nonequivalent coverings of the Riemann sphere: namely it describes the coalescence of a branch point with the simple one. [15] pointed out the connection with the KP theory, namely, experts will explain that $h(1)$ belongs to the symmetries of the KP equation, known as $W_{1+\infty}$ symmetries, which were first selected from the famous $\hat{gl}_\infty$ symmetries in [37]. In work [9] an interesting observation was made that $h(1)$ can be considered the Hamiltonian of the dispersionless quantum KdV equation. It is not so surprising because the KdV equation is a special limits of Benjamin-Ono one. In the future, everywhere $\alpha = 1$. This case already has a generalization.

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In [23], [24], [25], [26], [2] the equation which describes the product of two cycle class was written first in power sum variables and also in the variables $p_n = \text{tr}(X)^n$, where actual variables are the entries of a matrix $X$, though it can be re-written in terms of eigenvalues of $X$. I am going to use this point and consider the generalization of

$$W_\Delta(D) \cdot s_\lambda(X) = \tilde{\varphi}_\lambda(\Delta)s_\lambda(X)$$  \hfill (2)

suggested by Mironov, Morozov and Natanzon in [23] and called in [23] generalized cut-and-join equation, we will refer it as Mironov, Morozov, Natanzon (MMN) equation. Here $X$ is a $N \times N$ matrix, $s_\lambda$ is the Schur function (a certain polynomial in the entries of $X$ labeled by a partition (multiindex) $\lambda$. $D$ is the following matrix with entries which are differential operators

$$D_{i,j} = \sum_{1 \leq k \leq N} X_{k,j} \frac{\partial}{\partial X_{k,i}}.$$  \hfill (3)

$W_\Delta(D)$ is polynomial in the entries of the matrix $D$ labeled by another partion $\Delta$. This polynomial is proportional to the normal ordered power sum polynomial $p_\Delta(D)$.

$$W_\Delta(D) = \frac{1}{\zeta_\Delta} : p_\Delta(D) :$$  \hfill (4)

where

$$\zeta_\Delta = \prod_i m_i! i^{m_i}.$$  \hfill (5)

where the number $m_k$ indicates how many times the part equal to $k$ is included in the partition $\Delta$.

The power sum polynomial is defined as

$$p_\Delta(A) = \text{tr}(A)^{\Delta_1} \cdots \text{tr}(A)^{\Delta_k}, \quad \Delta = (\Delta_1, \ldots, \Delta_k)$$  \hfill (6)

where $A$ is a matrix. In case we know the eigenvalues of $A$ one can rewrite this polynomial in the entries of $A$ as the Newton sum polynomial in the eigenvalues of $A$, we are not going to do it in case $A = D$. In case $A = D$ is the matrix valued differential operator with the normal ordering

The eigenvalue $\tilde{\varphi}_\Delta(\lambda)$ is a rational number which is proportional to the character of the symmetric group in the representation $\lambda$. Details will be written down below.

MMN equation describes in a beautiful compact way the result of the coalescence of the two branch points with arbitrary profiles in the covering problem. G.Olshanski pointed out that the MMN cut-and-join equation can be treated as the eigenvalue problem for a Casimir operator, see [39].

I dedicate this note to Andrei Mironov and Alexei Morozov on the occasion of their 60th birthday and as a token of gratitude for their great contribution to the development of mathematical physics in Russia.

2 Results

2.1 Commuting Hamiltonians

First, instead of $X_{i,j}$ and $\frac{\partial}{\partial X_{i,j}}$ I would rather say that we are dealing with the algebra of $N^2$ oscillators

$$[Z_{i,j}^\dagger, Z_{i',j'}] = \delta_{i,i'} \delta_{j,j'}, \quad i,j = 1, \ldots, N$$  \hfill (7)

and

$$[Z_{i,j}, Z_{i',j'}] = 0 = [Z_{i,j}^\dagger, Z_{i',j'}^\dagger]$$  \hfill (8)

We simply re-denoted matrix $X$ as $Z$ and matrix $D$ as $Z^\dagger Z$, where $Z_{i,j}^\dagger = \frac{\partial}{\partial X_{i,j}}$. The Fock space $F$ is the space of all polynomials in the variables $Z_{i,j}$, $i,j = 1, \ldots, N$. The left vacuum vector $|0\rangle$ is related to the simplest monomial 1. The elements $Z_{i,j}$ and $Z_{i,j}^\dagger$ play the role of creation and annihilation operators, respectively.
For a given partition, say, $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ and a matrix $A \in GL_N$ we introduce the normal ordered operators (Hamiltonians)

$$
: p_\mu(Z^\dagger Z A) := \prod_{i=1}^{\ell(\mu)} \text{tr}(Z^\dagger Z A)^{\mu_i} : \quad (9)
$$

which act on the Fock space $\mathcal{F}$. We use colons "::" as parentheses, indicating that the expression inside them is normally ordered, that is, all annihilation operators are considered right-shifted relative to the creation operators, for example: $Z^\dagger_{i,j} Z_{a,b} Z_{j,i} := Z_{j,i} Z^\dagger_{i,j} Z_{a,b} =: Z^\dagger_{a,b} Z_{j,i} Z^\dagger_{i,j} ::$. (If we were to use differential operators $\frac{\partial}{\partial X_{i,j}}$ instead of $Z^\dagger_{i,j}$, we would say that normal ordering means that differential operators do not act on coefficients $X_{j,i} = Z_{j,i}$ that happen to be inside the normal ordering sign).

**Remark 1.** Note that under the sign of the normal order, one can permute the traces of operators, e.g. $: p_\mu(Z^\dagger Z) p_\nu(Z^\dagger Z) :: = : p_\nu(Z^\dagger Z) p_\mu(Z^\dagger Z) ::$. We can also, as in the commutative case, rearrange the factors under the trace sign, preserving their cyclic order. For a given partition, we get $\zeta_\mu$ equivalent ways to write $: p_\mu(A) ::$, where $A$ is an operator-valued matrix.

Let me remind that the set of commuting operators $p_\lambda(Z^\dagger Z)$ is actually the set of quantum Hamiltonians of the Calogero-Sutherland model at a singular point of the coupling constant (free fermionic point) which I call CS $\alpha = 1$ model, and is written in the language of matrix elements $X = Z$, not eigenvalues.

One can prove that for each pair of partitions $\lambda$ and $\mu$ the compatibility condition is satisfied:

**Proposition 1.**

$$
[: p_\lambda(Z^\dagger Z) :: p_\mu(Z^\dagger Z A) ::] = 0 \quad (10)
$$

Thus, for each individual Young diagram $\mu$, the operator $: p_\mu(Z^\dagger Z A) ::$ is a symmetry of the quantum Calogero model.

More interesting is

**Proposition 2.** For all partitions $\mu, \nu$ we have

$$
[: p_\nu(Z^\dagger Z A) :: p_\mu(Z^\dagger Z A) ::] = 0, \quad (11)
$$

where $Z, A \in GL_N$.

**Proof.**

First of all, note that a product of normal ordered operators can be written as a sum of ordered operators, which we write as follows:

$$
: p_\nu(Z^\dagger Z A) :: p_\mu(Z^\dagger Z A) :: = \sum_{k=0}^{\text{min}(|\mu|, |\nu|)} : Q_k : \quad (12)
$$

where the direct analogue of the Leibniz rule is used for the product: $k$ is the number of applied couplings of $Z^\dagger$ entries of the left factor to the related $Z$ entries of the right factor in left hand side of (12) ("related" means that each $Z^\dagger_{i,j}$ is coupled to $Z_{j,i}$). In particular, we obtain $Q_0 = p_\mu(Z^\dagger Z A) p_\nu(Z^\dagger Z A)$.

In the same way we write

$$
: p_\mu(Z^\dagger Z A) :: p_\nu(Z^\dagger Z A) :: = \sum_{k=0}^{\text{min}(|\mu|, |\nu|)} : Q'_k : \quad (13)
$$

We see that $Q_0 = Q'_0$, see Remark 1. Let us show that $Q_k = Q'_k$ for each $k$.

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1In fact, this means that we are dealing not only with the radial part of the Laplace operator $\text{tr}(D)^2$ and of each $p_\lambda(D)$. It is these Hamiltonians (related to $\alpha = 1$ case) that we will call the Hamiltonians of the Calogero-Sutherland $\alpha = 1$ model below.

2The first version of this text was posted by my mistake, and the proof was not entered there, instead of it there was a piece of unfinished text. In this version, only the scheme of the proof will be inscribed. The full proof will be posted either separately or in the third version of this text.
Remark 2. Note that each $p_\lambda(Z^\dagger ZA)$ is a polynomial in the entries of the matrices $Z^\dagger$, $Z$, $A$ and a monomial in these matrices themselves (a monomial in the symbols $Z^\dagger$, $Z$, and $C$). I note it in order not to create confusion in the use of the terms “monomial” and “polynomial” below: “monomial” we are talking about matrices, and “polynomial” is about entries. I also use “monomial in entries” as a member of the polynomial.

For clarity, we will place the matrices included in a monomial $p_\lambda$ inside $p_\lambda$ on a circle, denote it $S^\dagger(\lambda_1)$, arranging them clockwise in the order in which they are located under the trace sign, if you go from left to right. So, the get the collection $S^\dagger(\lambda_1), S^\dagger(\lambda_2), \ldots$ related to $p_\lambda$ and the related collection of cyclic products - each product is the clockwise product of all matrices around a circle.

Step (i).

Now we tag a group $k$ of matrices $Z^\dagger$ in the monomial $p_\lambda(Z^\dagger ZA)$ (or, in our visualized picture - in the collection $S^\dagger(\nu_1), S^\dagger(\nu_n)$), we give these matrices numbers and denote them as $Z_{-1}, \ldots, Z_{-k}$. We call these matrices favorite ones. Each clockwise directed arc which starts at some favorite $Z_{-i}$ and ends at the nearest clockwise neighboring favorite matrix, say, $Z_{-j}$ we give the number $(-i)$ and the related arc product of matrices we denote $C_{-i}$. We suppose that arc product does not contain favorite matrices.

We write
\[
P_\mu(Z^\dagger ZA) \equiv \text{tr}(W_1) \cdots \text{tr}(W_n) K_1,
\]
where $n$ ($n \leq k$) is the number of circles from the collection above and where each $W_a$ contains at least one favorite matrix. Here $K$ is the product of all factors that do not contain favorite matrices. Consider a given $W_a = (Z_{-i_1} C_{-h_1}) \cdots (Z_{-i_p} C_{-h_p})$. Each $C_{-i_p}$ is a product $Z A(Z^\dagger Z A)^{n_{i_p}}$ and will be called the arc product, $3n_a + 2$ being the length of the arc started at $Z_{-a}$ and ends at the nearest clockwise favorite $Z^\dagger$ (note that this nearest favorite $Z^\dagger$ can coincide with $Z_{-a}$ itself). Note that $Z^\dagger$ and $Z$ which enter arc products are not numbered.

Consider a matrix $X \in GL_N$. we will present each matrix entry $X_{ij}$ as an arrow whose starting point is labeled with $i$ and end point is labeled with $j$. A product of matrices will be drawn as a chain of arrows and a trace of the product as a closed chain which we will treat as a polygon.

To each $Z_{-a}$ we assign a dotted arrow labeled with $a$ and to each $C_{-a}$ we assign a solid arrow labeled by $a$. Then we associate each $\text{tr}(W_a)$ with $2f(a)$-gon with alternating dotted and solid edges. We call these polygons that emerged from $p_\mu(Z^\dagger Z A)$ black polygons.

Next we construct “white polygons” from the product $p_\mu(Z^\dagger Z A)$ and $k$ selected favorite matrices. We label the group of these $k$-matrices $Z$ in the monomial $p_\mu(Z^\dagger Z A)$ (or, equivalently, in the set $S^\dagger(\mu_1), S^\dagger(\mu_2), \ldots$), where we number them somehow and denote them $Z_1, \ldots, Z_k$. We consider the set of arcs between clockwise nearest favorite $Z$. We number an arc by $a$ if it starts at $Z_a$, $a = 1, \ldots, k$ and directed clockwise. We denote related arc products $C_a$.

To each $Z_a$ we associate white dotted arrows, and to each $C_a$ we associate white solid arrows.

If we pay attention only on favorite matrices we can rewrite $p_\mu$ as
\[
P_\mu(Z^\dagger ZA) \equiv 2K_2 \text{tr}(U_1) \cdots \text{tr}(U_m),
\]
where $K_2$ is the product of all $p_{\mu_i}$ which does not contain any of $Z_a$. We have $U_a = (Z_{j_1} C_{j_1}) \cdots (Z_{j_k} C_{j_k})$, where each $C_{j_h}$ is of form $A(Z^\dagger Z A)^{m_{j_h}} Z^\dagger$ with certain $m_a$ and $3m_a + 2$ is the length of the arc numbered by $a$.

To each $Z_a$ we associate white dotted arrows, and to each $C_a$ we associate white solid arrows. Then each is a white polygon with alternating dotted and solid arrows.

For a given selection of favorite matrices we have a set of black polygons emerged from $p_{\mu u}$ and a set of white polygons emerged from $p_{m u}$. The coupling of the favorite matrices $Z_{i_a}^\dagger$ to $Z_a$ ($a = 1, \ldots, k$) according to \[\text{[ ]}\] geometrically means the gluing of black and white polygons in a way that each black dotted arrow is glued to the oppositely directed white dotted arrow with the same number.

We get an orientable 2D surface $\Omega$ with the embedded ribbon graph with inflated vertices. Each inflated vertex is a circle (more precisely a negative orientated polygon) obtained as alternating black and white solid arrows: the beginning of each solid black arrow is glued to the end of the solid white arrow. Let us number the vertices by $i = 1, \ldots, p$. The cyclic product of matrices around a vertex $i$ will be denoted $V_i$. It’s Euler characteristic of $\Omega$ is $E = n + m - k + p$.

We have

**Lemma 1.** By $(\cdots)'$ we denote the expectation value only with respect to the matrices $Z_{\pm a}$, $a = 1, \ldots, k$ while all other matrices are treated as constant ones which are not coupled.
We get the following relation

$$\frac{1}{K} (p_\nu(Z^1ZA)p_\mu(Z^1ZA))' = (\text{tr}(W_1)\cdots\text{tr}(W_n)\text{tr}(U_1)\cdots\text{tr}(U_m))' = \text{tr}(V_1)\cdots\text{tr}(V_p)$$  \hspace{1cm} (14)$$

where the left hand side is related to glued polygons drawn on a Riemann surface $\Omega$: namely, where $E = n + m - k + p$ is the Euler characteristic of $\Omega$. The embedded graph drawn on $\Omega$ has $k$ edges, $n + m$ faces (polygons) and $p$ vertices. Here $V_i, i = 1, \ldots, p$ are cyclic matrix products around the vertex numbered by $i$.

Step (ii).

In the same way we consider the product \[13\].

Now we select a set of the favorite matrices $Z_1^1, \ldots, Z_k^1$ in the product $p_\nu$, repeating all the steps. In this case $C_a = ZA(Z^1ZA)^{m_i}$ and $U_i' = (Z_i^1C_{-i}) \cdots$. The point that each selected $Z^1_a$ we find in the white are numbered by $a$ in step (i).

Next we select a set of the favorite matrices $Z_1, \ldots, Z_k$ in the product $p_\nu$, where each $Z_a$ is chosen inside the white are numbered by $a$. In this case $C_a = A(Z^1ZA)^{m_i}Z^1$ and $W_i' = (Z_iC_{-i}) \cdots$. Then after gluing of obtained polygons we get the same $\Omega$ with the same embedded ribbon graph. The difference with the case (i) is the different choice of the arc lengths. However the sums of arc lengths around each inflated vertices of the ribbon graph is the same in both cases.

We get

\[ \text{Lemma 2.} \]

$$\frac{1}{K} (p_\nu(Z^1ZA)p_\mu(Z^1ZA))'' = (\text{tr}(U_1')\cdots\text{tr}(U_n')\text{tr}(W_1')\cdots\text{tr}(W_m'))'' = \text{tr}(V_1')\cdots\text{tr}(V_p')$$  \hspace{1cm} (15)$$

where $\{''\}$ denotes the pairing with respect to the selected matrices and where $E = n + m - k + p$ is the Euler characteristic of $\Omega$.

Step (iii).

We prove that for each choice of the sets arc lengths $\{n_i, i = 1, \ldots, n\}$ and $\{m_i, i = 1, \ldots, m\}$ we find dual sets of $\{n_i', i = 1, \ldots, n\}$ and $\{m_i', i = 1, \ldots, m\}$ where right hand sides () and () are equal.

Step (iv).

We prove that $Q_i = Q_i'$ who are obtained after the summation over $k$ and over each selection of the favorite matrices.

\[ \text{Symmetric functions}. \]

Recall that the so-called symmetric functions of variables $x_1, \ldots, x_N$ can always be expressed in terms of power sum variables $p_n(x_1, \ldots, x_N) = \sum_{i=1}^N x_i^n$ (Newtonian sums), see \[20\]. If $x_1, \ldots, x_N$ are eigenvalues of some matrix $X$, then power sums can also be expressed in terms of entries of this matrix, $p_n = \text{tr}(X)^n$, and we used just such a representation. What is the analog of eigenvalues of a matrix with operator-valued entries I do not know, nevertheless, let’s call any function in variables $p_i(Z^1ZA), i = 1, 2, \ldots$ symmetric function.

We have such an obvious consequence of Proposition\[2\]

\[ \text{Corollary 1. Symmetric functions, understood as functions rewritten in variables of power sums } p_n(ZAZ^1), \text{ commute. For example,} \]

$$\left\vert p_\nu(Z^1ZA) \right\vert \cdot s_\lambda(Z^1ZA) \right\vert = 0, \quad \left\vert s_\lambda(Z^1ZA) \right\vert \cdot s_\mu(Z^1ZA) \right\vert = 0$$  \hspace{1cm} (16)$$

2.2 Hurwitz numbers

Gluing coverings of the Riemann sphere from polygons. Hurwitz numbers, geometric approach. Let us draw a graph on the Riemann sphere with one vertex $O$ placed on a single edge that divides the surface into two 1-gones $P_1$ and $P_2$. Let’s call this graph $\Gamma$. We will denote the side of the edge bordering face $P_1$ with the label 1, and the other side we will denote with the label $-1$.

We can say that the sphere consists of two 1-gons $P_1$ and $P_2$ whose edges are labeled 1 and $-1$ respectively and glued.

Now consider coverings of the sphere of a given degree $D$. 

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Let’s choose points, denote them by \( O_1 \) and \( O_2 \) in each 1-gon and call them the "capitals" of the corresponding 1-gons \( P_1 \) and \( P_2 \). Consider coverings of the sphere with three branch points \( O_1, O_2, \) and \( O \).

Let the point \( O_1 \) (where \( i = 1, 2 \)) has \( \ell_i \) pre-images and let the preimage of 1-gon \( P_1 \) with the "capital" \( O_1 \) be the set consisting of \( \mu_1^{(1)}, \mu_2^{(1)}, \ldots, \mu_{\ell_i}^{(1)} \)-gons, where we agreed that \( \mu_1^{(1)} \geq \mu_2^{(1)} \geq \cdots \geq \mu_{\ell_i}^{(1)} > 0 \), i.e., we write this set as a partition \( \mu^{(1)} = (\mu_1^{(1)}, \mu_2^{(1)}, \ldots, \mu_{\ell_i}^{(1)}) \) (or as the Young diagram \( \mu^{(1)} \) with line lengths \( \mu_1^{(1)}, \mu_2^{(1)}, \ldots \)). This partition is called the ramification profile at the point \( O_1 \). The set of pre-images of the polygons \( P_1 \) we denote \( P_1(\mu) = (P_1(\mu_1^{(1)}), \ldots, P_1(\mu_{\ell_i}^{(1)})) \).

At last let the vertex \( O \) have \( t \) pre-images with the valences (i.e., the number of outgoing half-edges) \( 2\nu_1, 2\nu_2, \ldots, 2\nu_t \) which are on the covering surface, where we the pre-image of graph \( \Gamma \) which we denote \( \tilde{\Gamma} \) is drawn. Let us assume that \( \nu_1 \geq \cdots \geq \nu_t \). The partition \( \nu = (\nu_1, \ldots, \nu_t) \) is the ramification profile above the point \( O \).

Let us note that the weights of partitions \( \mu^{(1)}, \mu^{(2)} \) and \( \nu \) are equal to the degree of the covering \( D \):

\[
|\mu^{(1)}| = |\mu^{(2)}| = |\nu| = D
\]

Note that the graph \( \tilde{\Gamma} \) is not necessarily simply connected. Each connected component is a graph drawn on a connected orientable surface. The faces of each component consist of pre-images of \( P_1 \) and of \( P_2 \) and each edge of \( \Gamma \) is a pre-image of the edge of \( \Gamma \), therefore, it is the boundary between a polygon from the set \( P_1(\mu^{(1)}) \) and a polygon from the set \( P_2(\mu^{(2)}) \). A covering surface can be obtained by gluing edges of the polygons \( P_1(\mu_1^{(1)}), P_1(\mu_2^{(1)}), \ldots \) which are pre-images of the side 1 to edges to the polygons \( P_2(\mu_1^{(2)}), P_2(\mu_2^{(2)}), \ldots \) which are pre-images of the side \(-1\).

For given partitions \( \mu^{(1)} \) and \( \mu^{(2)} \) there is a finite number of ways of the gluings to get a given ramification profile \( \nu \). Denote this number by \( N(\mu^{(1)}, \mu^{(2)}, \nu) \). We will identify gluings obtained by permuting polygons from the set \( P_\nu(\mu_i) \) (here \( i = 1, 2 \)) with the same number of sides or obtained by rotating polygons (cyclic permutation of sides). If a partition \( \mu \) has \( m_k \) parts equal to \( k, k = 1, 2, \ldots \) we introduce the number \( \varsigma_{\mu} := \prod_{i=1}^{\infty} m_k!^{k^{m_k}} \), see (5), which can be called the order of the automorphisms of the Young diagram \( \mu \). The following number

\[
H_{\nu\nu}(\mu^{(1)}, \mu^{(2)}, \nu) = \frac{1}{\varsigma_\mu(\nu)} N(\mu^{(1)}, \mu^{(2)}, \nu)
\]

is called the Hurwitz number, where \( S^2 \) point out the base surface. It describes the number of non-isomorphic coverings of the sphere with three branch points at which the ramification profiles are given by partitions \( \mu^{(1)}, \mu^{(2)}, \nu \).

It is known that the Hurwitz number does not depend on the order of its arguments. We will reproduce this later.

Wick’s rule as gluing rule. If you think a little, then it is clear that gluing can be represented as a pairing of matrices \( Z^1 \) and \( Z \),

\[
((Z^1)_{i,j} (Z_{i',j'}) = \delta_{i,i'} \delta_{j,j'}, \quad i, j = 1, \ldots, N
\]

assuming that they are assigned to different sides of the edge of \( \Gamma \). We assign monodromy \( W_1 = ZC \) to the Northern semisphere and monodromy \( W_2 = Z^1 \) to the Southern one. The monodromy of the vertex of \( \Gamma \) is \( CF \). If we drawn a dual graph \( \Gamma^* \) to graph \( \Gamma \), see Figure 1, and assign matrices \( Z^1 \) and \( Z \) to half-edges of the dual graph and talk about pairing of half-edges according to Leibnitz rule. Constant matrices are assigned to the corners of the graph \( \Gamma \), that is, to the arcs of the extended vertices, they will also be the arcs of the extended vertices of the dual graph \( \Gamma^* \) in the Figure 1. For visual illustration: The edge of the dual graph is the meridian connecting the North and South Poles (that is, the capitals of the Northern and Southern hemispheres, depicted respectively as white and black circles in the Figure 1).

The collection \( (Z^1 F)^{m_1}, (Z^1 F)^{m_2}, \ldots \) describes the monodromy of the pre-image of the monodromy \( Z^1 F \) of the southern hemisphere, while the collection \( (ZC)^{m_1}, (ZC)^{m_2}, \ldots \) describes the monodromy of the pre-image of the monodromy \( ZC \) of the northern hemisphere.

Wick’s rule states that the expectation of the monomial in bosons \( Z_{i,j}^1 Z_{i',j'} \) is calculated as follows. We write the monomial as a product of \( m \) pairs, this can be done in many ways, namely in \( m(2m-1) \) ways, if the monomial has \( 2m \) factors. We consider the product of all pairwise expectations.
Below, we relate polygons $\mathcal{P}_i$ of a graph Gamma to the trace $\text{tr}W_i$ and the components of the pre-image to $\text{tr}(W_i)^{\mu_i^{(i)}}$

$$\mathcal{P}_i \leftrightarrow \text{tr}W_i, \quad \mathcal{P}_i(\mu_k^{(i)}) \leftrightarrow \text{tr}(W_i)^{\mu_k^{(i)}}, \quad k = 1, \ldots, \ell(\mu^{(i)}), \quad i = 1, \ldots, F$$

To do it a convenient visualization of this correspondence is as follows. To an entry $C_{i,j}$ (to an entry $F_{i,j}$) where $i, j = 1, \ldots, N$, we will associate a solid white arrow (a solid black arrow), and attribute the beginning of the arrow $i$, and the end $j$. To an entry $Z_{i,j}$ (to an entry $Z_{i,j}^\dagger$) we will associate a punctured white arrow (a punctured black arrow), we attribute the beginning of the arrow $i$, and the end $j$.

The product of matrices corresponds to a chain of arrows glued to each other, and the trace of the product of matrices corresponds to a closed chain: a polygon, which we consider positively oriented. The vertices of the polygon contain numbers from 1 to $N$ vertices of the polygon contain numbers from 1 to $N$. The trace of the product is associated with the formal sum of such polygons over these numbers. In case the expressions $\text{tr}W_i$ is described by the formal sum of polygons with, say, $k$ pairs of alternating solid-white arrors, then $\text{tr}(W_i)^{\mu_i^{(i)}}$

We get:

**Lemma 3.**

$$\frac{1}{z_{\mu^{(1)}}z_{\mu^{(2)}}}(p_{\mu^{(1)}}(Z^\dagger F)p_{\mu^{(2)}}(ZC)) = \sum_\nu \mathcal{H}_{S^2}(\mu^{(1)}, \mu^{(2)}, \nu)\varphi_\lambda(\nu), \quad C, F \in \mathbb{G}L_N(C)$$

where $C, F \in \mathbb{G}L_N(C)$ are independent of $Z$ and $Z^\dagger$.

The proof of Lemma 3 without tedious details is as follows. As one can see each factor in the product $p_{\mu^{(1)}}(FC)p_{\mu^{(2)}}(FC) \cdots = p_{\mu^{(1)}}(FC)$ is related to a component of the pre-image of the small face and in this way $\nu$ is the ramification profile of the point $O$ which is the vertex of the graph $\Gamma_A$. While $p_{\mu^{(1)}}(Z^\dagger F)$ and $p_{\mu^{(2)}}(ZC)$ are related to the pre-images of polygons $\mathcal{P}_1$ and $\mathcal{P}_1$ respectively and of the capitals $O_1$ and $O_2$ of these polygons and $\mu^{(1)}$ and $\mu^{(2)}$ are related ramification profiles.

Using (18), (32) and also the orthogonality of characters

$$\sum_\lambda \frac{\dim \lambda}{n!} \varphi_\lambda(\mu)\varphi_\lambda(\nu) = \delta_{\mu,\nu}$$

one obtains from (18) the following:

**Lemma 4.**

$$\langle s_\lambda(ZC)s_\mu(Z^\dagger F) \rangle = \delta_{\lambda,\mu} \frac{|\lambda|!s_\lambda(CF)}{\dim \lambda}, \quad C, F \in \mathbb{G}L_N(C)$$

**2.3 Hurwitz numbers and commuting Hamiltonians**

Using Lemma 3 the sign of normal ordering and choosing $F = ZA$, we obtain

**Proposition 3.** Let $|\Delta| = |\nu| = n$

$$p_{\Delta}(Z^\dagger ZA) : p_{\nu}(ZC)) = \sum_{|\mu| = n} \langle p_\mu(ZAC) \rangle \mathcal{H}_{S^2}(\Delta, \nu, \mu)$$

where $\mathcal{H}_{S^2}(\Delta, \nu, \mu)$ are three-point Hurwitz numbers:

$$\mathcal{H}_{S^2}(\Delta, \nu, \mu) = \sum_{|\lambda| = n} \frac{\dim \lambda}{n!} \varphi_\lambda(\Delta)\varphi_\lambda(\nu)\varphi_\lambda(\mu)$$

**Generalization of MMN cut-and-join equation.**

**Corollary 2.** Suppose $AC = C$ where $A, C \in \mathbb{G}L_N$. Then in case $|\Delta| = |\lambda|$ we have

$$p_{\Delta}(Z^\dagger ZA) : s_\lambda(ZC)) = s_\lambda(ZAC) \left(\frac{\dim \lambda}{|\lambda|!}\right)^{-1} \chi_\lambda(\mu)$$
Related eigenvalue problems

**Corollary 3.** Suppose $AC = C$ and $|\mu| = |\lambda|$. Then

$$p_\mu(Z^\dagger ZA) : s_\lambda(ZC) = \left( \frac{\dim \lambda}{|\lambda|!} \right)^{-1} \chi_\lambda(\mu)s_\lambda(ZC)$$  \hspace{1cm} (26)

The important difference with equation (2) is the restrictive condition $|\lambda| = |\mu|$. It means that at I present the only set of eigenfunctions as polynomials of the weight $|\mu|$ which is the set : $p_\mu(Z^\dagger ZA) : |s_\lambda(ZC)|$ with all $|\nu| \leq |\lambda|$.

3 Another Hamiltonians

4 Discussion

This note is the first part of the work on commuting Hamiltonians and Hutwitz numbers. It is actually related to the simplest bi-partite embedded graph. Fig. 1 corresponds to the simplest bipartite graph with two inflated vertices and disconnected half-edges. This graph is dual to the graph considered in Section 2.2. It clearly shows what the matrices $Z, Z^\dagger, F, C$ from the Section correspond to. The black vertex and its half-edge correspond to the Hamiltonian, and the white vertex corresponds to the Fock vector.

![Figure 1:](image)

The second part deals with arbitrary embedded bipartite graphs. The simplest example can be presented just now without drawing graphs, it is

$$\prod_{i=1}^n p_\mu(Z_i A_i Z_i^\dagger) : |s_\lambda(Z_1 C_1 \cdots Z_n C_n)| = \left( \frac{\dim \lambda}{|\lambda|!} \right)^n \prod_{i=1}^n \chi_\lambda(\mu^i)|s_\lambda(Z_1 A_1 C_1 \cdots Z_n A_n C_n)|$$  \hspace{1cm} (27)

which is a direct consequence of the above. Actually it is related to the star graph with $n$ rays. This is the eigenvalue problem in case $A_i C_i = C_i$.

Acknowledgements

The author is grateful to A.D.Mironov, A.Isaev, M. Matushko, V.Sokolov and G. Olshanski for stimulating discussions. He thanks L.Chekhov, N.Slavnov, S.Konstantinous-Rizos, A.E.Mironov for the invitation to the workshops in Moscow, Yaroslavl’ and Novosibirsk where I got useful talks and also Petr and Nadya Grinevich for help. The work was supported by the Russian Science Foundation (Grant No.20-12-00195).

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Let us recall that the characters of the unitary group \( U(N) \) are labeled by partitions and coincide with the so-called Schur functions \(^{20}\). A partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a set of nonnegative integers \( \lambda_i \) which are called parts of \( \lambda \) and which are ordered as \( \lambda_i \geq \lambda_{i+1} \). The number of non-vanishing parts of \( \lambda \) is called the length of the partition \( \lambda \), and will be denoted by \( \ell(\lambda) \). The number \( |\lambda| = \sum \lambda_i \) is called the weight of \( \lambda \). The set of all partitions will be denoted by \( P \).

The Schur function labelled by \( \lambda \) may be defined as the following function in variables \( x = (x_1, \ldots, x_N) \):

\[
s_\lambda(x) = \frac{\det [x_{ij}^{\lambda_i - i + N}]_{i,j}}{\det [x_{ij}^{\lambda_i + i + N}]_{i,j}}
\]

in case \( \ell(\lambda) \leq N \) and vanishes otherwise. One can see that \( s_\lambda(x) \) is a symmetric homogeneous polynomial of degree \( |\lambda| \) in the variables \( x_1, \ldots, x_N \), and \( \deg x_i = 1, i = 1, \ldots, N \).

**Remark 3.** In case the set \( x \) is the set of eigenvalues of a matrix \( X \), we also write \( s_\lambda(X) \) instead of \( s_\lambda(x) \).

There is a different definition of the Schur function as quasi-homogeneous non-symmetric polynomial of degree \( |\lambda| \) in other variables, the so-called power sums, \( p = (p_1, p_2, \ldots) \), where \( \deg p_m = m \).

For this purpose let us introduce

\[
s_{\{h\}}(p) = \det[s_{\{h_i+j-N\}}(p)]_{i,j},
\]

where \( \{h\} \) is any set of \( N \) integers, and where the Schur functions \( s_{\{i\}} \) are defined by \( e^{\sum_{m>0} \frac{1}{m} p_m z^m} = \sum_{m \geq 0} s_{\{i\}}(p) z^i \). If we put \( h_i = \lambda_i - i + N \), where \( N \) is not less than the length of the partition \( \lambda \), then

\[
s_\lambda(p) = s_{\{h\}}(p).
\]

The Schur functions defined by (28) and by (29) are equal, \( s_\lambda(p) = s_\lambda(x) \), provided the variables \( p \) and \( x \) are related by the power sums relation

\[
p_m = \sum_i x_i^m
\]
In case the argument of \( s_\lambda \) is written as a non-capital fat letter the definition (29), and we imply the definition (28) in case the argument is not fat and non-capital letter, and in case the argument is capital letter which denotes a matrix, then it implies the definition (28) with \( x = (x_1, \ldots, x_N) \) being the eigenvalues.

Relation (29) relates polynomials \( s_\lambda \) and \( p_\Delta \) of the same degree \( d = |\lambda| = |\Delta| \). Explicitly one can write

\[
P_\Delta = \sum_{\lambda \in \mathcal{T}_d} \frac{\dim \lambda}{d!} \zeta_\Delta \varphi_\lambda(\Delta) s_\lambda(p)
\]

and

\[
s_\lambda(p) = \frac{\dim \lambda}{d!} \sum_{\Delta \in \mathcal{T}_d} \varphi_\lambda(\Delta)p_\Delta.
\]

The last relation is called the character map relation. Here

\[
\dim \lambda := \prod_{i < j \leq N} (\lambda_i - \lambda_j - i + j)
\]

(see example 1 in sect 1 and example 5 in sect 3 of chapt I in [20]), where \( N \geq \ell(\lambda) \). As one can check, the right hand side does not depend on \( N \). (We recall that \( \lambda_i = 0 \) in case \( i > \ell(\lambda) \). The number \( \dim \lambda \) is an integer.

The factors \( \varphi_\lambda(\Delta) \) satisfy the following orthogonality relations

\[
\zeta_{\Delta} \sum_{\lambda \in \mathcal{T}_d} \left( \frac{\dim \lambda}{d!} \right)^2 \varphi_\lambda(\mu)\varphi_\lambda(\Delta) = \delta_{\Delta, \mu}
\]

and

\[
\left( \frac{\dim \lambda}{d!} \right)^2 \sum_{\Delta \in \mathcal{T}_d} \zeta_\Delta \varphi_\lambda(\Delta)\varphi_{\mu}(\Delta) = \delta_{\lambda, \mu}.
\]

### B Geometrical definition of Hurwitz Numbers

In this presentation, we follow article [28].

The Hurwitz number is a characterisation of the branched covering of a surface with critical values of a prescribed topological type. Hurwitz numbers of oriented surfaces without boundaries were introduced by Hurwitz at the end of the 19th century. Later it turned out that they are closely related to the study of moduli spaces of Riemann surfaces [10], to integrable systems [33, 34, 8] to modern models of mathematical physics [matrix models], and to closed topological field theories [8]. In this paper we consider only Hurwitz numbers of compact surfaces without boundary.

Consider a branched covering \( f : P \to \Sigma \) of degree \( d \) of a compact surface without boundary. In the neighborhood of each point \( z \in P \), the map \( f \) is topologically equivalent to the complex map \( u \mapsto u^p \), defined on a neighborhood \( u \sim 0 \) in \( \mathbb{C} \). The number \( p = p(z) \) is called the degree of the covering \( f \) at the point \( z \). The point \( z \in P \) is said to be a branch point or critical point if \( p(z) \neq 1 \). There are only a finite number of critical points. The image \( f(z) \) of a critical point \( z \) is called the critical value of \( f \) at \( z \).

Let us associate with a point \( s \in \Sigma \) all points \( z_1, \ldots, z_\ell \in P \) for which \( f(z_i) = s \). Let \( p_1, \ldots, p_\ell \) be the degrees of the map \( f \) at these points. Their sum \( d = p_1 + \cdots + p_\ell \) is equal to the degree \( d \) of \( f \). Thus, to each point \( s \in S \) there corresponds a partition \( d = p_1 + \cdots + p_\ell \) of the number \( d \). Having ordered the degrees \( p_1 \geq \cdots \geq p_\ell > 0 \) at each point \( s \in \Sigma \), we introduce the Young diagram \( \Delta^s = [p_1, \ldots, p_\ell] \) of weight \( d \) with \( \ell = \ell(\Delta^s) \) rows of length \( p_1, \ldots, p_\ell : \Delta^s \) is called the topological type of the value \( s \), and \( s \) is a critical value of \( f \) if and only if at least one of the row-lengths \( p_i \) is greater than 1.

Let us note that the Euler characteristics \( E(P) \) and \( E(\Sigma) \) of the surfaces \( P \) and \( \Sigma \) are related via the Riemann-Hurwitz relation:

\[
E(P) = E(\Sigma)d + \sum_{z \in P} (p(z) - 1)
\]

or, equivalently,

\[
E(P) = E(\Sigma)d + \sum_{i=1}^p (\ell(\Delta^s_i) - d).
\]
where $s_1, \ldots, s_r$ are critical values.

We say that coverings $f_1 : P_1 \to \Sigma$ and $f_2 : P_2 \to \Sigma$ are equivalent if there exists a homeomorphism $F : P_1 \to P_2$ such that $f_1 = f_2 F$; in case $P_1 = P_2$ and $f_1 = f_2$ the homeomorphism $F$ is called an automorphism of the covering. The set of all automorphisms of a covering $f$ form the group $\text{Aut}(f)$ of finite order $|\text{Aut}(f)|$. Equivalent coverings have isomorphic automorphism groups.

We present two illustrative examples.

Example 1. Let $\Sigma = \mathbb{C} = \{z \in \mathbb{C}\} \cup \infty$, $P = P_1 = P_2 = \mathbb{C} = \{u \in \mathbb{C}\} \cup \infty$ be Riemann spheres. Consider the branched covering $z(u) = f(u) = f_1(u) = f_2(u) = u^3$. This covering $f : P \to \Sigma$ has 2 critical values 0 and $\infty$ with Young diagrams from one row of length 3. Automorphisms of the covering have the form $F(u) = u^{3\tau}$. The group $\text{Aut}(f)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

Example 2. Let $\Sigma = \mathbb{C} = \{z \in \mathbb{C}\} \cup \infty$ and $P = P_1 = P_2$ - this is a pair of Riemann spheres; that is $P = P' \cup P''$, where $P' = \{u' \in \mathbb{C}\} \cup \infty$ and $P'' = \{u'' \in \mathbb{C}\} \cup \infty$. Consider the branched covering $z(u') = f_1(u') = f_2(u') = (u')^3$, $z(u'') = f(u'') = f_1(u'') = f_2(u'') = (u'')^3$. This covering $f : P \to \Sigma$ has two critical values 0 and $\infty$ with Young diagrams of two rows of length 3. Automorphisms of the covering are generated by the following mappings:

1. $F(u') = (u')^{3\tau}, F(u'') = u''$.
2. $F(u') = (u'')^{3\tau}, F(u') = u'$.
3. $F(u') = (u''), F(u'') = u'$.

The group $\text{Aut}(f)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z}) \otimes (\mathbb{Z}/3\mathbb{Z}) \otimes (\mathbb{Z}/2\mathbb{Z})$.

From now on, unless indicated otherwise, we will assume that the surface $\Sigma$ is connected. Let us choose points $s_1, \ldots, s_r \in \Sigma$ and corresponding Young diagrams $\Delta^1, \ldots, \Delta^r$ of weight $d$. Let $\Phi$ be the set of equivalence classes of the coverings for which $s_1, \ldots, s_r$ is the set of all critical values, and $\Delta^1, \ldots, \Delta^r$ are the topological types of these critical values. The Hurwitz number is the number

$$H^d_{\text{Rie}}(\Delta^1, \ldots, \Delta^r) = \sum_{f \in \Phi} \frac{1}{|\text{Aut}(f)|}.$$  \hspace{1cm} (37)

It is easy to prove that the Hurwitz number is independent of the positions of the points $s_1, \ldots, s_r$ on $\Sigma$. One can show that the right hand side of (37) depends only on the Young diagrams of $\Delta^1, \ldots, \Delta^r$ and the Euler characteristic $e = e(\Sigma)$. Because of this sometimes we write $H^d_{\text{Rie}}(\Delta^1, \ldots, \Delta^r)$ instead of $H^d_{\text{Rie}}(\Delta^1, \ldots, \Delta^r)_{\Sigma}$.

If $f = 0$ we get an unbranched covering. We denote such Hurwitz number $H_0 ((1^d))$.

Example 3. Let $f : \Sigma \to \mathbb{R}P^2$ be a covering without critical points. Then, if $\Sigma$ is connected, then $\Sigma = \mathbb{R}P^2$, $\deg f = 1$ or $\Sigma = S^2$, $\deg f = 2$. Therefore if $d = 3$, then $\Sigma = \mathbb{R}P^2 \coprod \mathbb{R}P^2 \coprod \mathbb{R}P^2$ or $\Sigma = \mathbb{R}P^2 \coprod S^2$. Thus $H_1 ((1^3)) = \frac{1}{3!} + \frac{1}{2!} = \frac{2}{3}$.

### C Combinatorial definition of Hurwitz numbers

Consider the symmetric group (equivalently, the permutation group) $S_d$ and the equation

$$\sigma_1 \cdots \sigma_r \rho_1^2 \cdots \rho_i^2 \omega_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_1 \beta_1 \omega_1^{-1} = 1,$$  \hspace{1cm} (38)

where $\sigma_1, \ldots, \sigma_r, \rho_1, \ldots, \rho_i, \omega_1, \beta_1, \ldots, \alpha_1, \beta_1 \in S_d$, and moreover $\sigma_i \in C_{D_i}$, $i = 1, \ldots, r$, where $C_{D_i}$ is the conjugacy class labeled by a partition $\Delta^i = (\Delta^i_1, \Delta^i_2, \ldots)$. The Hurwitz number is the number of solutions of equation (38) divided by $d!$ (by the order of $S_d$).

It can be proved that so introduced the (combinatorial) Hurwitz number coincides with the (geometric) Hurwitz number $H_{\text{Rie}}(\Delta^1, \ldots, \Delta^r)$ introduced in Section 3 where $e = 2 - 2\alpha - m$. One can look at the base surface $\Sigma$ as a result of gluing $H$ handles and $m$ Möbius stripes to a sphere.

Consider the simplest example: $H = 0$ and $m = 1$; that is $\Sigma = \mathbb{R}P^2$ (real projective plane). Suppose $f = 0$; that is we deal with an unbranched covering. Suppose $d = 3$; that is we consider 3-sheeted covering. Let us solve $\rho^2 = 1$, where $\rho \in S_3$. One gets 4 solutions: 3 transpositions of the set 1, 2, and 3 and one identity permutation. There are 3! permutations in $S_3$. As a result we get $H_1 ((1^3)) = 4/3! = 2/3$ as we got in the last example of the previous section.

In the same way one can consider Example 1 of the previous section. In this case $m = 0$; that is $e = 2$; one gets the Riemann sphere with two branch points ($f = 2$) and 3-sheeted covering with profiles
\( \Delta^1 = \Delta^2 = (3) \). We solve the equation \( \sigma_1 \sigma_2 = 1 \), where both \( \sigma_{1,2} \) consist of a single cycle of length 3.

There are two solutions \( \sigma_1 = \sigma_2^{-1} \): one sends 1, 2, 3 to 3, 1, 2, the other sends 1, 2, 3 to 2, 3, 1. We get \( H_2 ((3), (3)) = 2/3! = 1/3 \).

Example 2 corresponds to \( H_2 ((3, 3), (3, 3)) \), \( d = 6 \). One can complete the exercise and get an answer \( H_2 ((3, 3)) = 1/\zeta_{(3, 3)} = 1/18 \), where \( \zeta_\lambda \) is given by \( \ref{5} \). Actually, for any \( d \) and for any pair of profiles one gets \( H_2 (\Delta^1, \Delta) = \delta_{\Delta^1, \Delta} / \zeta_\Delta \).

In \[21\] (and also in \[18\]) it was found that \( H_e (\Delta^1, \ldots, \Delta^k) \) is given by formula

\[
H_e (\Delta^1, \ldots, \Delta^k) = \sum_{\lambda \in \Upsilon_d} \left( \frac{\dim \lambda}{d!} \right)^k \varphi_\lambda (\Delta^1) \cdots \varphi_\lambda (\Delta^k) \tag{39}
\]

### D Differential operators \[29\]

As G.I.Olshansky pointed out to us, this type of formula appeared in the works of Perelomov and Popov \[39, 40, 41\] and describe the actions of the Casimir operators in the representaion \( \lambda \), see also \[51\], Section 9.

(iv) Let us notice that if we take a dual graph to the sunflower graph with \( n = 1 \) (dual to one petal \( \Gamma \), which is just a line segment, see fig 1 ), in this case we have one face and two vertices, we get a version of the Capelli-type relation. Then it is a task to compare explicitly such relations with beautiful results \[31, 32\].

(v) There are several allusions to the existence of interesting structures related to quantum integrability. First, as noted in \[28\] by this appearance 2D Yang-Mills theory \[50\]. See also possible connection to \[11\]. Then the appearance of the Yangians in works \[35, 36\] which, we hope, can be related to our subject. And finally, the work \[9\].

(vi) There is a direct similarity between integrals over complex matrices and integrals over unitary matrices. However, from our point of view direct anologues of the relations in the present paper are more involved in the case of unitary matrices. In particular, Hurwitz numbers are replaced by a special combination of these numbers.