ABUNDANCE THEOREM FOR SEMI LOG CANONICAL SURFACES IN POSITIVE CHARACTERISTIC

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Abstract. We prove the abundance theorem for semi log canonical surfaces in positive characteristic.

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0. Introduction

A semi log canonical (for short, slc) n-fold is a generalization of log canonical (for short, lc) n-folds. In this paper, we prove the abundance theorem for slc surfaces in positive characteristic. We use the same definition of slc varieties as the one of [Kollár].

Theorem 0.1. Let \((X, \Delta)\) be a projective slc surface over an algebraically closed field of positive characteristic. If \(K_X + \Delta\) is nef, then \(K_X + \Delta\) is semi-ample.

Let us briefly review the history of the semi log canonical varieties in characteristic zero. The notion of semi log canonical singularities is introduced in [KSB] for a moduli problem. The abundance theorem for slc surfaces is proved in [AFKM] and [KeKo]. [Fujino] generalizes 2010 Mathematics Subject Classification. Primary 14E30; Secondary 14J10.

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this result to dimension three. Moreover, [Fujino1] shows that the abundance theorem for slc n-folds follows from the two parts:

1. The abundance theorem for lc n-folds.
2. The finiteness theorem of the pluri-canonical representation for (n − 1)-folds.

[FG] shows that (2) holds for each n ∈ Z > 0. If n = 3, then (1) follows from [KeMaMc]. If n ≥ 4, then (1) is an open problem. For a recent development of the theory of slc varieties in characteristic zero, see [Fujino2], [FG], [Gongyo] and [HX]. For related topics, see [Kawamata], [AFM] and [KoMaMc].

In this paper, we use the strategy of [Fujino1]. Hence, we must prove (1) and (2) in the case where n = 2 and char k > 0. In this case, (1) is a known result by [Fujita]. It is not difficult to prove (2). However, [Fujino1] uses many fundamental results based on the minimal model theory and the Kawamata–Viehweg vanishing theorem. We can freely use the minimal model theory for surfaces in positive characteristic by [T1] (cf. [Fujita], [KoKo]). Although there exist counterexamples to the Kodaira vanishing theorem in positive characteristic ([Raynaud]), we can use some weaker vanishing theorems obtained in [T2] (cf. [KoKo], [Xie]).

In characteristic two, some new phenomena happen. For example, in characteristic zero, the Whitney umbrella \{x^2 = yz^2\} ⊂ A^3 is a typical example of slc surfaces (cf. [AFKM, Definition 12.2.1]). In characteristic two, this is slc but not normal crossing in codimension one. Moreover, [Fujino1] uses the following fact: if a field extension L/K satisfies [L : K] = 2 and its characteristic is zero, then L/K is a Galois extension. But, in characteristic two, this field extension L/K may be purely inseparable. Thus, some proofs are more complicated.

0.2 (Overview of contents). In Section 1, we summarize the notations. The normalization of an slc surface is an lc surface. Therefore, we should investigate lc surfaces. Every lc surface is birational to a dlt surface. Thus, in Section 2, we consider a dlt surface (X, Δ). More precisely, we consider \(\llcorner Δ \lrcorner\) because \(\llcorner Δ \lrcorner\) has the patching data of the normalization. In Section 3, we calculate the normalization of nodal singularities. In Section 4, we prove the main theorem. In Section 5, we summarize fundamental results on dlt surfaces. These results may be well-known but the author cannot find a good reference.

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1. Notations

We will not distinguish the notations invertible sheaves and Cartier divisors. For example, we will write $L + M$ for invertible sheaves $L$ and $M$.

Throughout this paper except for Section 3, we work over an algebraically closed field $k$ of positive characteristic and let $	ext{char } k = p$.

In this paper, a variety means a pure dimensional reduced scheme which is separated and of finite type over $k$. A curve or a surface means a variety whose dimension is one or two, respectively. Note that varieties, curves and surfaces may be reducible.

Let $X$ be a noetherian reduced scheme and let $X = \bigcup X_i$ be the irreducible decomposition. Let $Y_i \to X_i$ be the normalization of $X_i$. Then we define the normalization of $X$ by $\coprod Y_i \to \coprod X_i \to X$. We say $X$ is normal if the normalization morphism is an isomorphism.

Let $X$ be a variety. We say $\Delta$ is a $\mathbb{Q}$-divisor on $X$ if $\Delta$ is a finite sum $\Delta = \sum \delta_i \Delta_i$ where $\delta_i \in \mathbb{Q}$ and $\Delta_i$ is an irreducible and reduced closed subscheme of codimension one which is not contained in the singular locus $\text{Sing}(X)$. Note that, in this case, the local ring $\mathcal{O}_{X, \Delta_i}$ is a discrete valuation ring.

We will freely use the notation and terminology in [Kollár]. In the definition in [Kollár, Definition 2.8], for a pair $(X, \Delta)$, $\Delta$ is not necessarily effective. But, in this paper, we assume $\Delta$ is an effective $\mathbb{Q}$-divisor. For a reducible normal variety $X$ and an effective $\mathbb{Q}$-divisor $\Delta$, we say $(X, \Delta)$ is lc (resp. dlt, klt) if each irreducible component is lc (resp. dlt, klt).

For the definition of (nodes and) slc varieties, see Definition 3.1 and Definition 4.1. These definitions are the same as [Kollár] 1.41, 5.10.

2. Boundaries of dlt surfaces

In this section, we investigate dlt surfaces. First, we consider the case of curves. The main result of this section is Proposition 2.7. Proposition 2.7 is the surface version of Proposition 2.1.

Proposition 2.1. Let $(X, \Delta)$ be an irreducible lc curve. Let $f : X \to R$ be a projective surjective morphism such that $f_* \mathcal{O}_X = \mathcal{O}_R$. Assume that $S := \cup \Delta_i \neq 0$ and let $T := f(S)$. If $K_X + \Delta \equiv_f 0$, then one of the following assertions holds.

1. $f_* \mathcal{O}_S = \mathcal{O}_T$. 

(2) $f_*\mathcal{O}_S \neq \mathcal{O}_T$. $X \simeq \mathbb{P}^1$ and $\dim R = 0$. Moreover, $\Delta = S$ and $S$ is two distinct points.

**Proof.** If $\dim R = 1$, then we see $X \simeq R$ and we obtain (1). We may assume $\dim R = 0$. Since $\deg(K_X + \Delta) = 0$ and $\Delta \neq \emptyset$, we see $X \simeq \mathbb{P}^1$ and $S$ has at most two points. If $S$ is one point, then we obtain (1). □

In the above proposition, (1) is a good case. Hence, we classify the other case (2) as above. For this, we want sufficient conditions for $f_*\mathcal{O}_S = \mathcal{O}_T$.

**Proposition 2.2.** Let $f : X \to Y$ be a projective surjective morphism between irreducible normal varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume the following conditions.

1. $(X, \Delta)$ is a $\mathbb{Q}$-factorial lc surface such that $(X, \{\Delta\})$ is klt.
2. $S := \Delta \neq \emptyset$ and let $T := f(S)$.
3. $-(K_X + \Delta)$ is $f$-nef and $f$-big.

Then, $f_*\mathcal{O}_S = \mathcal{O}_T$. In particular, for every $y \in Y$, $S \cap f^{-1}(y)$ is connected or an empty set.

**Proof.**

**Step 1.** In this step, we assume $\dim Y \geq 1$ and we prove the assertion. Consider the exact sequence:

$$0 \to \mathcal{O}_X(-\Delta) \to \mathcal{O}_X \to \mathcal{O}_{\Delta} \to 0.$$ 

Take the push-forward by $f$:

$$0 \to f_*\mathcal{O}_X(-\Delta) \to \mathcal{O}_Y \to f_*\mathcal{O}_{\Delta} \to R^1f_*\mathcal{O}_X(-\Delta).$$

It is sufficient to prove that the last term $R^1f_*\mathcal{O}_X(-\Delta)$ vanishes. Since

$$-\Delta = K_X + \{\Delta\} - (K_X + \Delta),$$

we have $R^1f_*\mathcal{O}_X(-\Delta) = 0$ by [T2, Theorem 2.12].

**Step 2.** In this step, we assume $\dim Y = 0$ and we prove the assertion. It is sufficient to prove that $S$ is connected. Suppose the contrary and let us derive a contradiction. We may assume that $X$ is smooth by replacing $(X, \Delta)$ with $(Z, \Delta_Z)$ in Proposition 5.9. Since $X$ is a ruled surface, we have a surjective morphism

$$h : X \to B$$

onto a smooth projective curve $B$ or $X \simeq \mathbb{P}^2$. The latter case is clear, hence we can assume the existence of $h$. 


Since $-(K_X + \Delta)$ is $h$-nef and $h$-big, we can apply Step 1 to $h$. If $\Delta_X$ contains an $h$-horizontal prime divisor, then we see that $\Delta_X$ is connected by Step 1. Thus we may assume that every irreducible component of $\Delta_X$ is $h$-vertical. Again, by Step 1, we can assume that $h(\Delta_X) = \{Q_1, \ldots, Q_r\}$ with $r \geq 2$ and the intersection $\text{Supp}(\Delta_X) \cap h^{-1}(Q_i)$ is connected for every $i$.

We can run a $(K_X + \{\Delta\})$-MMP over $B$ that does not change the number of the connected components of $\Delta_X$. Indeed, if $\text{Supp} h^{-1}(Q_i) \subset \text{Supp} \Delta_X$, then we just contract a curve $C$ in $h^{-1}(Q_i)$ with $(K_X + \{\Delta\}) \cdot C < 0$. If $\text{Supp} h^{-1}(Q_i) \not\subset \text{Supp} \Delta_X$, then we contract a curve $C \subset \text{Supp} h^{-1}(Q_i)$ such that $C \not\subset \text{Supp} \Delta_X$ and $(K_X + \{\Delta\}) \cdot C < 0$. Repeating this, we end with a Mori fiber space:

$$h : X \xrightarrow{\varphi} X' \xrightarrow{h'} B$$

where $\varphi : X \rightarrow X'$ is a composition of extremal birational contractions and $h' : X' \rightarrow B$ is a Mori fiber space. Note that $(X', \Delta' := \varphi_* \Delta)$ is $\mathbb{Q}$-factorial lc, $(X', \{\Delta'\})$ is klt, and $-(K_{X'} + \Delta')$ is nef and big. Moreover, $\Delta'$ contains at least two fibers of $h'$.

Assume that $-(K_{X'} + \Delta')$ is ample. Since $\rho(X') = 2$ by [T1, Theorem 3.27(4)(b)], $X'$ has the two $(K_{X'} + \Delta')$-negative extremal rays. The extremal ray, not corresponding to $h'$, induces a morphism $\psi : X' \rightarrow X''$ that is a birational contraction or another Mori fiber space structure to a curve. For each case, we can apply Step 1 to $\psi$ and obtain a contradiction.

Therefore we may assume that $-(K_{X'} + \Delta')$ is not ample. Then, by Nakai–Moishezon criterion, there exists a curve $C'$ on $X'$ such that $(K_{X'} + \Delta') \cdot C' = 0$. This implies $C'^2 < 0$ and $h'(C') = B$. Since $\rho(X') = 2$, we obtain $-(K_{X'} + \Delta') \cdot \Gamma > 0$ for a curve $\Gamma \neq C'$. If $-(K_{X'} + \Delta')$ is semi-ample, then it induces the contraction of $C'$: $\psi : X' \rightarrow X''$ and $\psi^{-1}(C') \cap \text{Supp} \Delta_X = C' \cap \text{Supp} \Delta_X$ has at least two points. Applying Step 1 to $\psi$, we obtain a contradiction. Thus it suffices to show that $-(K_{X'} + \Delta')$ is semi-ample. We have

$$0 = (K_{X'} + \Delta') \cdot C' > (K_{X'} + C') \cdot C'$$

because $\Delta'$ contains an $h'$-horizontal curve. Then $C' \simeq \mathbb{P}^1$ by [T1, Theorem 3.19(1)]. Thus, by Keel’s theorem ([Keel, Theorem 0.2]), $-(K_{X'} + \Delta')$ is semi-ample. We are done.

\[\square\]

**Lemma 2.3.** Let

$$f : X \xrightarrow{q} X' \xrightarrow{f'} R$$
be projective morphisms between normal varieties such that \( q \) is birational and \( f_* O_X = O_Y \). Assume the following conditions.

1. \((X, \Delta)\) is a \(\mathbb{Q}\)-factorial lc surface such that \((X, \{\Delta\})\) is klt.
2. \(\text{Ex}(q) =: E\) is an irreducible curve.
3. \(-(K_X + \Delta)\) is \(q\)-nef.
4. \(\mathcal{H}\) is \(q\)-nef.

Then, for every \( r \in R \), the number of connected components of \( \mathcal{H} \cap f^{-1}(r) \) is equal to the number of connected components of \( \cup_{q} \Delta \cap f^{-1}(r) \).

**Proof.** Let \( q(E) =: x'_0 \) and \( f'(x'_0) =: r_0 \). If \( E \cap \text{Supp}_q \Delta = \emptyset \), then the assertion is clear. Thus, we may assume \( E \cap \text{Supp}_q \Delta \neq \emptyset \).

We claim \( q(\text{Supp}_q \Delta) = \text{Supp}_q \cup \Delta \). The inclusion \( q(\text{Supp}_q \Delta) \supset \text{Supp}_q \cup \Delta \) is clear. Then, it is enough to show \( q(E) = \text{Supp}_q \cup \Delta \). If \( E \not\subset \text{Supp}_q \cup \Delta \), then \( E \cap \text{Supp}_q \cup \Delta \neq \emptyset \) implies \( q(E) = \text{Supp}_q \cup \Delta \). On the other hand, if \( E \subset \text{Supp}_q \cup \Delta \), then the \(q\)-nefness implies that there exists a prime component \( C \neq E \) of \( \cup \Delta \) with \( C \cap E \neq 0 \). We see

\[ q(E) \in q(C) \subset \text{Supp}_q \cup \Delta. \]

In each case, we obtain the claim.

For every \( r \in R \), we obtain

\[
q(\text{Supp}_q \cup \Delta \cap f^{-1}(r)) = q(\text{Supp}_q \cup \Delta \cap q^{-1}(f^{-1}(r))) = q(\text{Supp}_q \cup \Delta \cap f'^{-1}(r)) = \text{Supp}_q \cup \Delta \cap f'^{-1}(r).
\]

Assume that the numbers of connected components are different. Then there exist at least two connected components \( X_1 \) and \( X_2 \) of \( \text{Supp}_q \cup \Delta \cap f^{-1}(r_0) \) such that \( x'_0 \in q(X_1) \) and \( x'_0 \in q(X_2) \). We take the intersection

\[ \text{Supp}_q \cup \Delta \cap f^{-1}(r_0) = X_1 \amalg X_2 \amalg \cdots \]

with \( q^{-1}(x'_0) \) and we obtain the following equation

\[ \text{Supp}_q \cup \Delta \cap q^{-1}(x'_0) = (X_1 \cap q^{-1}(x'_0)) \amalg (X_2 \cap q^{-1}(x'_0)) \amalg \cdots. \]

Thus, in order to derive a contradiction, it is sufficient to prove that \( \text{Supp}_q \cup q^{-1}(x'_0) \) is connected. Since \( -(K_X + \Delta) \) is \( q\)-nef and \( q\)-big, we can apply Proposition 2.2. Thus \( \text{Supp}_q \cup q^{-1}(x'_0) \) is connected. \(\square\)

**Proposition 2.4.** Let \( f : X \to Y \) be a projective surjective morphism between irreducible normal varieties such that \( f_* O_X = O_Y \). Assume the following conditions.

1. \((X, \Delta)\) is a \(\mathbb{Q}\)-factorial lc surface such that \((X, \{\Delta\})\) is klt.
2. \( S := \mathcal{H} \neq 0 \) and let \( T := f(S) \).
3. \( K_X + \Delta \equiv_f 0 \).
(4) $T = f(S) \subsetneq Y$.

Then, $f_*\mathcal{O}_S = \mathcal{O}_T$. In particular, for every $y \in Y$, $S \cap f^{-1}(y)$ is connected or an empty set.

**Proof.** By (4), we have $\dim Y \neq 0$. If $\dim Y = 2$, then the assertion follows from Proposition 2.2. Thus we may assume $\dim Y = 1$. It is sufficient to prove that $\mathcal{O}_Y = f_*\mathcal{O}_X \to f_*\mathcal{O}_S$ is surjective. Since the problem is local, by shrinking $Y$, we may assume that $f(S) = P \in Y$. If $S$ is connected, then $f_*\mathcal{O}_S \cong \mathcal{O}_P$ and $\mathcal{O}_S \to f_*\mathcal{O}_S$ is surjective. Therefore, it is sufficient to prove that $S$ is connected. We define a reduced divisor $D$ by

$$S + D = \text{Supp}(f^*P).$$

If $D = 0$, then $S$ is connected since $S = \text{Supp}(f^*P)$. Therefore, we assume that $D \neq 0$. Then, there exists an irreducible curve $E \subset \text{Supp}D$ such that $E \cap S \neq 0$. We see $(K_X + \{\Delta\}) \cdot E < 0$. Thus, we obtain a birational morphism $q : X \to X'$ such that $\text{Ex}(q) = E$. Let $\Delta' := q_*\Delta$. By Lemma 2.3 if $\text{Supp}_s \Delta' \cup$ is connected, then so is $\text{Supp}_s \Delta' \cup$. We can repeat this argument and we obtain a projective morphisms

$$f : X \xrightarrow{\tilde{q}} X'' \xrightarrow{f''} Y$$

where $\tilde{q}$ is a birational morphism such that $\text{Ex}(\tilde{q}) = \text{Supp}D$. Let $\Delta'' := \tilde{q}_*\Delta$. It is sufficient to show that $\text{Supp}_s \Delta'' \cup$ is connected. This follows from $\text{Supp}_s \Delta'' \cup = \text{Supp}(f''*P).$ \hfill \Box

In Proposition 2.7 the most complicated case is the Mori fiber space to a curve. Thus we investigate this case in the following lemma.

**Lemma 2.5.** Let $f' : X' \to R$ be a projective surjective morphism between normal varieties such that $f'_*\mathcal{O}_{X'} = \mathcal{O}_R$. Assume the following conditions.

1. $(X', \Delta')$ is a $\mathbb{Q}$-factorial lc surface such that $(X', \{\Delta'\})$ is klt.
2. $S' := \cup \Delta' \cup \neq 0$.
3. $K_{X'} + \Delta' \equiv_{f'} 0$.
4. There is a $(K_{X'} + \{\Delta'\})$-negative extremal contraction $g' : X' \to V$ over $R$ such that $\dim V = 1$.

Then the $g'$-horizontal part $(S')^h$ of $S'$ satisfies one of the following assertions.

a) $(S')^h = S'_1$, which is a prime divisor, and $[K(S'_1) : K(V)] = 2$.

b) $(S')^h = S'_2$, which is a prime divisor, and $[K(S'_2) : K(V)] = 1$.

c) $(S')^h = S'_1 + S'_2$, where each $S'_i$ is a prime divisor, and $[K(S'_i) : K(V)] = 1$. 


Furthermore, there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D_V$ on $V$ such that $K_X + \Delta = g^*(D_V)$.

In the case (b), $f'_s\mathcal{O}_{S'} = \mathcal{O}_{f'(S')}$. 

Proof. The assumption (3) means $K_{X'} + \Delta' \equiv g'_0$. Thus, by (4), $\Delta' \cup g$ is $g'$-ample. We see $(S')^h \neq 0$.

We prove that general fibers of $g' : X' \to V$ are $\mathbb{P}^1$. The dimension of every fiber is one. Since $\dim V = 1$ and $f'_s\mathcal{O}_{X'} = \mathcal{O}_V$, the field extension $K(X')/K(V)$ is algebraically closed and separable (cf. [Badescu Lemma 7.2]). Therefore general fibers are geometrically integral. Let $F$ be a general fiber of $g'$, that is, $F$ is a fiber which is a proper integral curve such that $F \cap \text{Sing}(X) = \emptyset$. The adjunction formula implies

$$(K_{X'} + F) \cdot F = K_{X'} \cdot F = -\Delta' \cdot F \leq -(S')^h \cdot F < 0.$$ 

This means $F \simeq \mathbb{P}^1$.

By $(K_{X'} + F) \cdot F = -2$, we have $(S')^h \cdot F \leq 2$ for a general fiber $F$. Therefore one of (a), (b) and (c) holds. By the abundance theorem ([TI Theorem 6.7]), we see $K_{X'} + \Delta' \sim_{\mathbb{Q},g'} 0$. This means $m(K_{X'} + \Delta') = g'^*(D)$ for some integer $m$ and some $\mathbb{Z}$-divisor $D$ on $V$. We define a $\mathbb{Q}$-divisor $D_V$ by $D = mD_V$.

Assume (b) and let us prove $f'_s\mathcal{O}_{S'} = \mathcal{O}_{f'(S')}$. Since $\dim V = 1$, we have $\dim R = 0$ or $\dim R = 1$. Assume $\dim R = 0$. It is sufficient to prove that $S'$ is connected. This holds because all of the fibers of $g'$ are irreducible and $(S')^h \neq 0$. Assume $\dim R = 1$. Then, we see $f'_s(S') = V \simeq R$. Since $S'_1$ and $R$ are birational, the morphism $f|_{S'_1} : S'_1 \to R$ is an isomorphism. We can write

$$S' = S'_1 + F_1 + \cdots + F_r$$

where each $F_i$ is the reduced subscheme whose support is a fiber of $g'$.

We prove $f'_s\mathcal{O}_{S'} = \mathcal{O}_R$ by the induction on $r$. If $r = 0$, then the assertion follows from $S'_1 \simeq R$. Assume $r > 0$. Consider the exact sequence:

$$0 \to \mathcal{O}_{S'} \to \mathcal{O}_{S'} - F_r \oplus \mathcal{O}_{F_r} \to \mathcal{O}_{(S' - F_r) \cap F_r} \to 0.$$ 

The last map defined by the difference. Note that the last term is the scheme-theoretic intersection. It is easy to see that $(S' - F_r) \cap F_r \simeq S'_1 \cap F_r$. Then $(S' - F_r) \cap F_r$ is reduced because $S'_1 \simeq R$. Consider the push-forward of the above exact sequence:

$$0 \to f'_s\mathcal{O}_{S'} \to f'_s\mathcal{O}_{S' - F_r} \oplus f'_s\mathcal{O}_{F_r} \to f'_s\mathcal{O}_{(S' - F_r) \cap F_r} \to R^1 f'_s\mathcal{O}_{S'}.$$ 

We see $R^1 f'_s\mathcal{O}_{X'} = 0$ by [T2 Theorem 2.12]. This implies $R^1 f'_s\mathcal{O}_{S'} = 0$. Since $F_r$ and $(S' - F_r) \cap F_r$ are reduced, we have $f'_s\mathcal{O}_{F_r} \simeq f'_s\mathcal{O}_{(S' - F_r) \cap F_r} \simeq \mathcal{O}_{F_r}$.
\( \mathcal{O}_{f(F_\ast)} \). This means \( f'_\ast \mathcal{O}_{S'} \to f'_\ast \mathcal{O}_{S'-F_r} \) is an isomorphism. By the induction hypothesis, we obtain \( f'_\ast \mathcal{O}_{S'} \simeq f'_\ast \mathcal{O}_{S'-F_r} \simeq \mathcal{O}_R \).

**Remark 2.6.** In the last argument in the above proof, we use the following fact. Let \( A \) be a ring and let \( M, N, L \) and \( P \) are \( A \)-modules. Assume the exact sequence
\[
0 \to M \xrightarrow{(\varphi_1, \varphi_2)} N \oplus L \xrightarrow{\psi} P \to 0.
\]
If \( \theta : L \to P \) is an isomorphism, then \( \varphi_1 : M \to N \) is also an isomorphism.

We can prove the following main result in this section.

**Proposition 2.7.** Let \( (X, \Delta) \) be an irreducible dlt surface. Let \( f : X \to R \) be a projective surjective morphism such that \( f_*\mathcal{O}_X = \mathcal{O}_R \). Assume that \( S := \langle \Delta \rangle \neq 0 \) and let \( T := f(S) \). If \( K_X + \Delta \equiv_f 0 \), then one of the following assertions holds.

1. \( f_*\mathcal{O}_S = \mathcal{O}_T \).
2. \( f_*\mathcal{O}_S \not\simeq \mathcal{O}_T \). There exist a projective surjective \( R \)-morphism \( g : X \to V \) to a smooth curve \( V \) and a \( \mathbb{Q} \)-divisor \( D_V \) on \( V \) such that \( g_*\mathcal{O}_X = \mathcal{O}_V \) and that \( K_X + \Delta = g^*(D_V) \) as \( \mathbb{Q} \)-divisors. Every connected component of \( S \) intersects the \( g \)-horizontal part \( S^h \) of \( S \). Moreover, the \( g \)-horizontal part \( S^h \) satisfies one of the following assertions.

\[(2.1s) \quad S^h = S_1, \text{ which is a prime divisor, and } [K(S_1) : K(V)] = 2. \text{ This field extension is separable.}\]
\[(2.1i) \quad S^h = S_1, \text{ which is a prime divisor, and } [K(S_1) : K(V)] = 2. \text{ This field extension is purely inseparable.}\]
\[(2.2) \quad S^h = S_1 + S_2, \text{ where } S_i \text{ is a prime divisor, and } g|_{S_i} : S_i \to V \text{ is an isomorphism for } i = 1, 2.\]

**Proof.** If \( f \) is birational, then Proposition \[2.4\] implies (1). Thus we may assume that \( R < \dim X \). We run a \((K_X + \{\Delta\})\)-MMP on \( X \) over \( R \). The end result is a proper birational morphism \( q : X \to X' \) over \( R \). Let \( f' : X' \to R \) be the induced morphism. Since \( K_X + \Delta \equiv_f 0 \), we obtain \( K_{X'} + \Delta' \equiv_f 0 \) where \( \Delta' := q_*\Delta \). Let \( S' := \langle \Delta' \rangle \). Then it is easy to see that \((X', \Delta')\) is a \( \mathbb{Q} \)-factorial lc pair and \((X, \{\Delta\})\) is klt.

**Step 1.** Assume that \((X', \{\Delta'\})\) is a minimal model over \( R \). Then \( K_{X'} + \{\Delta'\} \) is \( f' \)-nef and \( K_{X'} + \Delta' \equiv_f 0 \). So \( -\langle \Delta' \rangle \) is \( f' \)-nef. If \( \dim R = 0 \), then \( -\langle \Delta' \rangle = 0 \) because \( X' \) is projective. Lemma \[2.3\] implies \( \langle \Delta \rangle = 0 \). This case is excluded. Assume \( \dim R = 1 \). Since \( -\langle \Delta' \rangle \) is \( f' \)-nef, we see \( f'(-\langle \Delta' \rangle) \subseteq R \). Therefore, by Proposition \[2.4\] we obtain (1).
Step 2. Assume that there exists a Mori fiber space structure $g' : X' \to V$ over $R$. Let

$$g : X \xrightarrow{q} X' \xrightarrow{g'} V.$$ 

Then $-(K_{X'} + \{\Delta'\})$ is $g'$-ample. Note that, if $\dim V = 1$, then we can apply Lemma 2.5 and every connected component of $S$ intersects $S^h$ by Lemma 2.3.

First, assume that $\dim R = 0$. If $\cup \Delta'$ is connected, then we have (1) by Lemma 2.3. Thus we may assume that $\cup \Delta'$ is not connected.

We show $\dim V = 1$. Assume $\dim V = 0$. Then $\cup \Delta'$ is ample. Thus its suitable multiple is an effective ample Cartier divisor. This must be connected by the Serre vanishing theorem. This case is excluded.

Thus we can apply Lemma 2.5. Since all of the fibers of the Mori fiber space $g' : X' \to V$ are irreducible, we see $\cup \Delta' = S_1' + S_2'$. This implies (2).

Second, assume that $\dim R = 1$. Then we have $\dim V = 1$. Note that $T = R \simeq V$. We can apply Lemma 2.5. Thus we obtain (a), (b) or (c) of Lemma 2.5. If (a) or (c) holds, then (2) holds. Thus we may assume that (b) of Lemma 2.5 holds. We have $f'_* O_{S'} = O_T$. Lemma 2.3 implies $q(S) = S'$. By Proposition 2.2, we have $f_* O_S = f'_* O_{S'} = O_T$.

Example 2.8. Let $\Char k = 2$. Then, there exists a projective dlt surface $(X, \Delta)$ and smooth projective curve $R$ which satisfy Proposition 2.7(2.1i).

Construction. Let $X_0 := \mathbb{A}^2$ and let $C_0 := \{(x, y) \in \mathbb{A}^2 \mid x = y^2\}$. Note that the restriction of the first projection to $C_0$ is purely inseparable of degree two. Let $X_0 \subset X := \mathbb{P}^1 \times \mathbb{P}^1$ be the natural open immersion and let $C$ be the closure of $C_0$ in $X$. Let $g : X \to \mathbb{P}^1 =: V =: R$ be the first projection. It is easy to see that $C$ is smooth and $K_X + C \sim g^* O_{\mathbb{P}^1}(-1)$. Thus, we see that $(X, \Delta := C)$ is dlt and that $(X, \Delta)$ satisfies Proposition 2.7(2.1i).

3. Normalization of nodes

In this section, we calculate the normalization of nodal singularities to reduce problems for slc varieties to ones for dlt varieties. The main theorem of this section is Theorem 3.7. In this section, we do not work over a field and we treat noetherian or excellent schemes.

First we recall the definition of the nodal singularities in the sense of [Kollár 1.41].
Definition 3.1. Let \((R, m)\) be a noetherian local ring. We say \(R\) has a node (or \(R\) is nodal) if there exists an isomorphism \(R \cong S/(f)\) where \((S, l)\) is a two-dimensional regular local ring such that \(f \in l^2\) and that \(f\) is not a square in \(l^2/l^3\).

We mainly use the following notations.

Notation 3.2. Let \((R, m)\) be a nodal noetherian local ring. By definition, we can write \(R \cong S/(f)\) where \((S, l)\) is a two-dimensional regular local ring such that \(f \in l^2\) and that \(f\) is not a square in \(l^2/l^3\). Take a generator \(l = (x, y)\). We can write 

\[ f = ax^2 + bxy + cy^2 + g \]

where \(a, b, c \in \{0\} \cup S^\times\) and \(g \in l^3\). We set \(\overline{x} := x + (f) \in R/(f)\) and \(\overline{y} := y + (f) \in R/(f)\).

Remark 3.3. We use the same notations as Notation 3.2. We show that we may assume 

\[ c \in S^\times \]

by replacing a generator \(\{x, y\}\) of \(l\). If \(c \in S^\times\), then there is nothing to show. If \(a \in S^\times\), then we exchange \(x\) and \(y\). Since \(a, c \in \{0\} \cup S^\times\), we assume \(a = c = 0\). By \(f \not\in l^3\), we see \(b \not\in l\), that is, \(b \in S^\times\). Taking another generator \(X := x - y, Y := y\) of \(l = (x, y) = (X, Y)\), we obtain 

\[ f = bxy + g \]
\[ = b(X + Y)Y + g \]
\[ = bXY + bY^2 + g. \]

By \(b \in S^\times\), we may assume \(c \in S^\times\).

We calculate the normalization of nodes. We divide the proof into the following two cases: \(R\) is an integral domain or not. In Lemma 3.4, we treat the case where \(R\) is not an integral domain. In Lemma 3.5, we treat the case where \(R\) is an integral domain.

Lemma 3.4. Let \((R, m)\) be a nodal noetherian local ring. We use the same notations as Notation 3.2. Assume that \(R\) is not an integral domain. Then the following assertions hold.

1. \(f\) has a decomposition \(f = l_1l_2\) with \(l_1, l_2 \in S\) which satisfies the following properties.
   - \(l_1S \neq l_2S\).
   - For each \(i\), \(l_i \in l \setminus l^2\).
   - For each \(i\), \(l_i\) is a prime element of \(S\), that is, \(l_iS\) is a prime ideal.

2. \(l_1\) and \(l_2\) satisfies \(l = (l_1, l_2)\).
(3) For each $i$, $S/(l_i)$ is regular.

(4) The natural homomorphism

$$
\nu : R = S/(f) = S/(l_1l_2) \to S/(l_1) \times S/(l_2) =: T
$$

is the normalization.

(5) $m$ is the conductor of the normalization $\nu : R \hookrightarrow T$, that is,

$$
m = \{ r \in R \mid rT \subset R \}.
$$

(6) The normalization $\nu : R \hookrightarrow T$ induces

$$
\theta : k(m) = R/m \to T/mT \simeq k(m) \times k(m),
$$

where $p_i \circ \theta$ is the identity map for the projection $p_i$ to the $i$-th factor.

**Proof.** (1) Since $S$ is a unique factorization domain, we obtain a decomposition of $f$ into prime elements:

$$
f = ul_1^{n_1} \cdots l_r^{n_r}
$$

where $u \in S^\times$, $n_i \in \mathbb{Z}_{>0}$ and $l_i$ is a prime element of $S$. In particular, $l_i \in \mathfrak{l}$. Then, $f \notin \mathfrak{l}^3$ implies $n_1 + \cdots + n_r \leq 2$. Since $n_1 + \cdots + n_r = 1$ implies that $R$ is an integral domain, we see $n_1 + \cdots + n_r = 2$. Thus, we obtain one of the following two cases: $f = ul_1^2$ or $f = ul_1l_2$ where $l_1S \neq l_2S$. By $f \notin \mathfrak{l}^3$ and $l_i \in \mathfrak{l}$, we see $l_i \notin \mathfrak{l}^3$. Then, it is enough to show that the case $f = ul_1^2$ does not occur. Suppose $f = ul_1^2$. We can write

$$
l_1 = \alpha x + \beta y + h
$$

where $\alpha, \beta \in \{0\} \cup S^\times$ and $h \in \mathfrak{l}^2$. We obtain

$$
f = ul_1^2 = u(\alpha x + \beta y + h)^2 = u(\alpha x + \beta y)^2 + (\text{an element of } \mathfrak{l}^3).
$$

By replacing $f$ with $u^{-1}f$, this contradicts the definition of nodes: Definition 3.1

(2) Since $R$ is nodal, $(l_1, l_2)$ generates $\mathfrak{l}/\mathfrak{l}^2$. Then Nakayama’s lemma implies the assertion.

(3) The assertion follows from (2).

(4) The assertion follows from (3).

(5) Let $I \subset R$ be the conductor. The inclusion $m \supset I$ is clear. We show the inverse inclusion $(l_1, l_2) = m \subset I$. By the symmetry, it suffices to prove $l_1 \in I$. Take $\xi = (s_1 + (l_1), s_2 + (l_2)) \in S/(l_1) \times S/(l_2) = T$. Then, we obtain $l_1\xi = (0 + (l_1), l_1s_2 + (l_2))$. Therefore, $l_1\xi = \nu(l_1s_2)$. This is what we want to show.

(6) By $\nu(l_1 + l_2) = (l_2 + (l_1), l_1 + (l_2))$, we see $mT = m/(l_1) \times m/(l_2)$. This implies the assertion. □
Lemma 3.5. Let \((R, m)\) be a nodal noetherian local ring. We use the same notations as Notation 3.2. Suppose \(c \in S^\times\) (cf. Remark 3.3). Assume that \(R\) is an integral domain. Consider the following natural injective ring homomorphism

\[ \varphi : R \hookrightarrow R \left[ \frac{y}{x} \right] =: T. \]

Then the following assertions hold.

1. The ring homomorphism \(\theta : S \left[ \frac{y}{x} \right]/(f/x^2) \to R \left[ \frac{y}{x} \right] = T, \quad \frac{y}{x} \mapsto \frac{y}{x}\)
is an isomorphism.
2. \(T\) is a regular ring.
3. One of the following assertions holds.
   (a) \(T/mT \cong k(m) \times k(m)\) and the composition homomorphism

   \[ k(m) = R/m \to T/mT \cong k(m) \times k(m) \xrightarrow{p_i} k(m) \]
is the identity map for \(i = 1, 2\) where \(p_i\) is the projection to the \(i\)-th factor.
   (b) \(T/mT\) is a field and the natural homomorphism

   \[ k(m) = R/m \to T/mT \]

is a field extension with \([T/mT : k(m)] = 2\).
4. The equation \((\frac{y}{x})^2 + r_1 \frac{y}{x} + r_2 = 0\) holds in \(R \left[ \frac{y}{x} \right] = T\) for some \(r_1, r_2 \in R\). In particular, \(T\) is a finitely generated \(R\)-module.
5. \(T\) is the integral closure of \(R\) in the quotient field \(K(R)\).
6. The maximal ideal \(m\) is the conductor of the normalization, that is, \(m = \{ r \in R \mid rT \subset R \}\).

Proof. We use the same notations as Notation 3.2.

1. Set \(z := \frac{y}{x} \in K(S)\). Let us check \(f/x^2 \in S[\frac{y}{x}] = S[z]\). Since \(f \in I = (x, y)^2\), we can write \(f = \alpha x^2 + \beta xy + \gamma y^2\) for some \(\alpha, \beta, \gamma \in S\). Then we see \(f/x^2 \in S[z]\) by the following calculation:

\[ f = \alpha x^2 + \beta xy + \gamma y^2 = \alpha x^2 + \beta x(xz) + \gamma (xz)^2 = x^2(\alpha + \beta z + \gamma z^2). \]

Consider the natural homomorphism

\[ \theta : S \left[ \frac{y}{x} \right]/(f/x^2) \to R \left[ \frac{y}{x} \right], \quad \frac{y}{x} \mapsto \frac{y}{x}. \]

We prove that \(\theta\) is an isomorphism. For the time being, we show this assuming the following two assertions.
(A) The $S$-algebra homomorphism $S/(f) \rightarrow S \left[ \frac{y}{x} \right]/(f/x^2)$ is injective.

(B) $S \left[ \frac{y}{x} \right]/(f/x^2)$ is an integral domain.

Consider the following commutative diagram of $S$-algebras:

$$
\begin{array}{ccc}
S/(f) & \longrightarrow & R \\
\downarrow \text{inj} & & \downarrow \text{inj} \\
S \left[ \frac{y}{x} \right]/(f/x^2) & \longrightarrow & R \left[ \frac{y}{x} \right] \\
\end{array}
$$

Note that $R \left[ \frac{y}{x} \right] \subset K(R) = K(S/(f)) \subset K(S \left[ \frac{y}{x} \right]/(f/x^2))$. All of the four rings in the above diagram are contained in the quotient field $K(S \left[ \frac{y}{x} \right]/(f/x^2))$. In $K(S \left[ \frac{y}{x} \right]/(f/x^2))$, the element $\frac{y}{x} + (f/x^2)$ is the same as $\frac{y}{x} \in R \left[ \frac{y}{x} \right]$. Therefore we obtain

$$S \left[ \frac{y}{x} \right]/(f/x^2) = R \left[ \frac{y}{x} \right].$$

(A) We show that the natural map $S/(f) \rightarrow S \left[ \frac{y}{x} \right]/(f/x^2)$ is injective. For this, consider the following natural composition map

$$\psi : S \rightarrow S \left[ \frac{y}{x} \right] \rightarrow S \left[ \frac{y}{x} \right]/(f/x^2)$$

and we show Ker($\psi$) = $fS$. The inclusion Ker($\psi$) $\supset$ $fS$ is obvious. Let us prove the inverse inclusion Ker($\psi$) $\subset$ $fS$. Take an element $s \in S$ such that $\psi(s) = 0$, that is, $s \in fS \left[ \frac{y}{x} \right]$. We have

$$s = \frac{f}{x^2} \left( t_0 + t_1 \frac{y}{x} + \cdots + t_m \frac{y^m}{x^m} \right)$$

where $t_i \in S$. Let us show that we can assume $m = 0$. Assume $m \geq 1$. Moreover assume $t_m \in xS$, that is, $t_m = x \tilde{t}_m$ with $\tilde{t}_m \in S$. Then, by the following calculation:

$$t_m \frac{y^m}{x^m} = x \tilde{t}_m \frac{y^m}{x^m} = y \tilde{t}_m \frac{y^{m-1}}{x^{m-1}},$$

we obtain another expression: $s = \frac{f}{x^2} (t_0 + \cdots + t_{m-2} \frac{y^{m-2}}{x} + t_{m-1} \frac{y^{m-1}}{x})$ for some $t_{m-1} \in S$. Thus, we assume $m \geq 1$ and $t_m \notin xS$. Taking the multiplication with $x^{m+2}$, we obtain

$$sx^{m+2} = f(t_0 x^m + \cdots + t_{m-1} xy^{m-1} + ty^m).$$

This implies $ft_m y^m \in xS$. But, both the elements $t_m$ and $y$ are not in $xS$. Since $xS$ is a prime ideal, we obtain $f \in xS$. Then we can write $f = xg$ with $g \in S$. $f \in \mathfrak{l}^2$ implies $g \in \mathfrak{l}$. Therefore, $f$ is not a prime
element, which contradicts that $R$ is an integral domain. Therefore, we may assume $m = 0$ and we obtain
\[ s = \frac{f}{x^2}t_0. \]
Since $f \notin xS$, we see $t_0 \in xS$. Repeating this, we see $t_0 \in x^2S$, which implies $s \in fS$. This is what we want to show.

(B) First we prove that $S[\frac{x}{x}]$ is a unique factorization domain. We see that $xS[\frac{x}{x}]$ is a prime ideal because
\[ S[\frac{y}{x}] / xS[\frac{y}{x}] \simeq S[Z] / (x, xZ - y) \simeq (S / (x,y))[Z] \]
is an integral domain. By Nagata’s criterion (Nagata Lemma 1), $S[\frac{x}{x}]$ is a unique factorization domain if so is
\[ \left( S[\frac{y}{x}] \right)[\frac{1}{x}] = S[\frac{1}{x}]. \]
This ring $S[\frac{1}{x}]$ is a unique factorization domain because so is $S$.

We show that $S[\frac{x}{x}] / (f/x^2)$ is an integral domain. Since $S[\frac{x}{x}]$ is a unique factorization domain, let us check that $f/x^2$ is a prime element. Assume that there exists a decomposition
\[ f/x^2 = \left( s_0 + s_1 \frac{y}{x} + \cdots + s_k \frac{y^k}{x^k} \right) \left( t_0 + t_1 \frac{y}{x} + \cdots + t_l \frac{y^l}{x^l} \right) \]
where $s_i, t_j \in S$ and both the factors in the right hand side are not in $(S[\frac{x}{x}])^\times$. We may assume that, if $k \geq 1$ (resp. $l \geq 1$), then $s_k$ (resp. $t_l$) is not in $xS$. We show that $k = 0$ or $l = 0$ holds. Assume $k \geq 1$ and $l \geq 1$. We consider the following two cases: $k = l = 1$ and $k + l \geq 3$. If $k = l = 1$, then we obtain $f = (s_0x + s_1y)(t_0x + t_1y)$. This contradicts that $f$ is a prime element. If $k + l \geq 3$, then taking the multiplication with $x^{k+l}$, we see $s_k t_l y^{k+l} \in xS$. By $s_k \notin xS$ and $t_l \notin xS$, we have $y^{k+l} \in xS$, which is a contradiction. Therefore, $k = 0$ or $l = 0$ holds. By the symmetry, we may assume $l = 0$ and we obtain
\[ f/x^2 = \left( s_0 + s_1 \frac{y}{x} + \cdots + s_k \frac{y^k}{x^k} \right) t_0. \]
If $k \geq 1$ and $t_0 \in xS$, then, for $t_0 = xt_0'$, we obtain another expression:
\[ f/x^2 = (xs_0 + \cdots + x^{k-1} \frac{y^k}{x^k} + y^k s_k) t_0'. \]
Thus we may assume that $k = 0$ or $t_0 \notin xS$ holds. If $k = 0$, then we obtain the following contradiction: $f = x^2 s_0 t_0$. Assume $t_0 \notin xS$. Taking the multiplication with $x^k$, we see $k \leq 2$. This implies $f = (s_0 x^2 + s_1 x y + s_2 y^2) t_0$. Since $s_0 x^2 + s_1 x y + s_2 y^2 \in \mathfrak{m}$ and $f \in S$ is a prime element, we have $t_0 \in S^\times \subset (S[\frac{x}{x}])^\times$. This is a contradiction.
(2) Set $z = \frac{y}{x}$. First, we calculate the ring $(S[\frac{y}{x}]/(f/x^2))/(\mathfrak{p})$. The element $f/x^2$ can be written
\[
\frac{f}{x^2} = \frac{ax^2 + bxy + cy^2 + g}{x^2} = \frac{ax^2 + bx(xz) + c(xz)^2 + x^3\tilde{g}}{x^2} = a + bz + cz^2 + x\tilde{g}
\]
for some $\tilde{g} \in S[z]$. Here, since $(S, (x, y))$ is a regular local ring, we can check that the homomorphism
\[S[Z]/(xZ - y) \to S \left[ \frac{y}{x} \right], \quad Z \mapsto \frac{y}{x}\]
is an isomorphism. Then, we see
\[(S \left[ \frac{y}{x} \right]/(f/x^2))/(\mathfrak{p}) \simeq S \left[ \frac{y}{x} \right]/((f/x^2) + (x))
\simeq S[Z]/(xZ - y, a + bZ + cZ^2, x)
\simeq k(m)[Z]/(\overline{a} + \overline{b}Z + \overline{c}Z^2).
\]
Fix a maximal ideal $n$ of $S[\frac{y}{x}]/(f/x^2)$ and we show that the local ring $(S[\frac{y}{x}]/(f/x^2))_n$ is regular.

We show $\mathfrak{p} \in n$. Assume $\mathfrak{p} \notin n$. Then $n$ corresponds to a maximal ideal of $R[\frac{y}{x}]/(\mathfrak{p}) = R[\frac{y}{x}]$, that is, $n = (nR[\frac{y}{x}]/(\mathfrak{p})) \cap R[\frac{y}{x}]$. Since $(R, m)$ is one dimensional local integral domain and $\mathfrak{p} \in m$, $R[\frac{y}{x}]$ is a field. It implies $n = (0)$. Then $S[\frac{y}{x}]/(f/x^2)$ is a field. On the other hand, by the above isomorphism
\[(S \left[ \frac{y}{x} \right]/(f/x^2))/(\mathfrak{p}) \simeq k(m)[Z]/(\overline{a} + \overline{b}Z + \overline{c}Z^2)
\]and $\overline{c} \neq 0$, there exists a non-zero ideal $(\mathfrak{p})$ of $S[\frac{y}{x}]/(f/x^2)$. Thus $S[\frac{y}{x}]/(f/x^2)$ is not a field and we obtain a contradiction.

Therefore, $\mathfrak{p} \in n$. To show that the local ring $(S[\frac{y}{x}]/(f/x^2))_n$ is regular, it is enough to prove that the ring
\[(S \left[ \frac{y}{x} \right]/(f/x^2))/(\mathfrak{p}) \simeq k(m)[Z]/(\overline{a} + \overline{b}Z + \overline{c}Z^2)
\]is regular. If $\overline{a} + \overline{b}Z + \overline{c}Z^2$ is irreducible over $k(m)$, then the ring
$k(m)[Z]/(\overline{a} + \overline{b}Z + \overline{c}Z^2)$
is a field. Assume that $\overline{a} + \overline{b}Z + \overline{c}Z^2$ is not irreducible over $k(m)$. We have $\overline{c} \neq 0$. There are $\alpha, \beta \in R$ such that
$\overline{a} + \overline{b}Z + \overline{c}Z^2 = \overline{c}(Z + \alpha)(Z + \beta)$.

Since $R$ is nodal, we see $\overline{a} \neq \overline{\beta}$. Therefore,
\[(S \left[ \frac{y}{x} \right]/(f/x^2))/(\mathfrak{p}) \simeq k(m)[Z]/(\overline{a} + \overline{b}Z + \overline{c}Z^2) \simeq k(m) \times k(m).
\]This is what we want to show.
(3) Let us calculate \( T/\mathfrak{m}T \). By

\[ \mathfrak{m}T = \mathfrak{m}R \left[ \frac{y}{x} \right] = (\overline{x}, \overline{y})R \left[ \frac{y}{x} \right] = \overline{x}R \left[ \frac{y}{x} \right], \]

we obtain \( T/\mathfrak{m}T \simeq (S[\frac{y}{x}]/(f/x^2))/ (\overline{x}) \). By the proof of (2), we obtain

\[ (S \left[ \frac{y}{x} \right] / (f/x^2))/ (\overline{x}) \simeq k(\mathfrak{m})[Z]/(\overline{x} + \overline{b}Z + \overline{c}Z^2). \]

If \( \overline{x} + \overline{b}Z + \overline{c}Z^2 \) is irreducible, then we obtain (b). Assume that \( \overline{x} + \overline{b}Z + \overline{c}Z^2 \) is not irreducible. Then, we can write

\[ \overline{x} + \overline{b}Z + \overline{c}Z^2 = \overline{c}(Z + \overline{a})(Z + \overline{b}). \]

Since \( R \) is nodal, we see \( \overline{a} \neq \overline{b} \). This implies (a).

(4) By Notation 3.2, we have

\[ f = ax^2 + bxy + cy^2 + g \]

where \( a, b, c \in \{0\} \cup S^\times \) and \( g \in I^3 = (x, y)^3 \). Moreover, we have \( c \in S^\times \). For some \( \alpha, \beta, \gamma, \delta \in S \), we obtain

\[ f = ax^2 + bxy + cy^2 + \alpha x^3 + \beta x^2y + \gamma xy^2 + \delta y^3, \]

which implies

\[ \frac{f}{x^2} = a + b\frac{y}{x} + c(\frac{y}{x})^2 + \alpha x + \beta y + \gamma \frac{y}{x} + \delta(\frac{y}{x})^2 \]

\[ = (c + \delta y)(\frac{y}{x})^2 + (b + \gamma y)\frac{y}{x} + (a + \alpha x + \beta y). \]

By \( c \in S^\times \) and \( \delta y \in I \), we see \( c + \delta y \in S^\times \). Therefore the assertion follows from (1).

(5) The assertion follows from (2) and (4).

(6) Let \( I := \{ r \in R \mid rT \subset R \} \) be the conductor ideal. By this definition, \( I \) is an ideal of \( R \). Note that \( I \) is also an ideal of \( T \). Since \( R \neq T \), we obtain \( 1 \notin I \). In particular, \( I \subset \mathfrak{m} \). Let us show \( I \supset \mathfrak{m} \). By (4), we obtain

\[ T = R \left[ \frac{y}{x} \right] = R + R\frac{y}{x}. \]

This implies \( \overline{x}T \subset R \). Thus, \( \overline{x} \in I \). Since \( I \) is an ideal of \( T = R[\frac{y}{x}] \), we see \( \overline{y} = \overline{x}\frac{y}{x} \in I \). Therefore, \( I \supset \overline{x}R + \overline{y}R = \mathfrak{m} \). \( \square \)

We say a scheme \( X \) is **excellent** if \( X \) is covered by open affine schemes whose corresponding rings are excellent.

Combining Lemma 3.3 and Lemma 3.4, we obtain the following result.
Proposition 3.6. Let $X$ be a quasi-compact excellent reduced scheme and let $\eta$ be a scheme-theoretic point whose local ring $\mathcal{O}_{X,\eta}$ is nodal. Let $S := \{\eta\}$ be the reduced scheme. Let $\nu : Y \to X$ be the normalization, $D \subset X$ the closed subscheme defined by the conductor and $C \subset Y$ its scheme-theoretic inverse image:

$$
C := \nu^{-1}(D) \quad \xrightarrow{\text{closed immersion}} \quad Y
$$

Then, there exists an open subset $\eta \in X'$ in $X$ which satisfies the following properties.

1. Set $Y' := \nu^{-1}(X')$, $D' := D \cap X'$, $C' := C \cap Y'$ and $S' := S \cap X'$.
2. $D'$ is reduced and $S' = D'$. In particular, $D'$ is an integral scheme.
3. $\nu|_{C'} : C' \to D'$ satisfies one of the following conditions
   - $C' \cong D'_1 \amalg D'_2$ with $D'_i \cong D$ and each morphism $D'_i \hookrightarrow C' \xrightarrow{\nu|_{C'}} D'$ are isomorphism.
   - $C'$ is an integral scheme and the field extension $K(C')/K(D')$ satisfies $[K(C') : K(D')] = 2$.

Proof. We may assume $X = \text{Spec } A$, $Y = \text{Spec } B$, $D = \text{Spec } A/I$ and $C = \text{Spec } B/J$ where $I = J$. Let $S_\eta := A \setminus \eta$ where we consider $\eta$ as a prime ideal of $A$. There are the following two cases.

1. $\mathcal{O}_{X,\eta} = A_\eta = S_\eta^{-1}A$ is not an integral domain.
2. $\mathcal{O}_{X,\eta} = A_\eta = S_\eta^{-1}A$ is an integral domain.

1. Assume that $S_\eta^{-1}A$ is not an integral domain. We can apply Lemma 3.4 to $S_\eta^{-1}A$. Then, by shrinking $\eta \in \text{Spec } A$, we obtain the following commutative diagram:

$$
A \quad \xrightarrow{\phi} \quad A/\mathfrak{p}_1 \times A/\mathfrak{p}_2 \\
\downarrow \quad \downarrow
S_\eta^{-1}A \quad \xrightarrow{\psi} \quad S_\eta^{-1}(A/\mathfrak{p}_1) \times S_\eta^{-1}(A/\mathfrak{p}_2),
$$

where $(0) = \mathfrak{p}_1 \cap \mathfrak{p}_2$. Since $A$ is excellent, for each $i$, the regular locus $U_i$ of $\text{Spec } A/\mathfrak{p}_i$ forms an open subset of $\text{Spec } A/\mathfrak{p}_i$. Since $S_\eta^{-1}(A/\mathfrak{p}_i)$ is regular, we obtain $\eta \in U_i$. Therefore, by shrinking $\eta \in \text{Spec } A$, we may assume that each $A/\mathfrak{p}_i$ is regular. In particular, the homomorphism
A \to A/p_1 \times A/p_2 coincides with the normalization. Since $S^{-1}_\eta (A/I)$ is reduced and $A$ is noetherian, we may assume that $A/I$ is reduced by shrinking $\text{Spec} A$. This implies (1). We show (2). We have the induced homomorphism

$$\theta_i : A/I \to (A/(I + p_1)) \times (A/(I + p_2)) \to A/(I + p_i),$$

where the latter map is the projection to the $i$-th factor. By Lemma 3.4, $S^{-1}_\eta \theta_i$ is an isomorphism. Since $X = \text{Spec} A$ is noetherian and the kernel and the cokernel of $\theta$ is a finitely generated $A$-modules, we obtain the assertion.

(β) Assume that $S^{-1}_\eta A$ is an integral domain. We can apply Lemma 3.5 to $S^{-1}_\eta A$. We obtain the following commutative diagram:

$$\begin{array}{ccc}
A & \xrightarrow{\nu} & B \\
\downarrow & & \downarrow \\
S^{-1}_\eta A & \longrightarrow & S^{-1}_\eta B.
\end{array}$$

By Lemma 3.3, $S^{-1}_\eta (A/I)$ is reduced. This implies (1). By Lemma 3.5, there are the following two cases:

(a) $S^{-1}_\eta (B/J) \cong S^{-1}_\eta (A/I) \times S^{-1}_\eta (A/I)$ and the composition homomorphism

$$S^{-1}_\eta (A/I) \to S^{-1}_\eta (B/J) \cong S^{-1}_\eta (A/I) \times S^{-1}_\eta (A/I) \xrightarrow{p_i} S^{-1}_\eta (A/I)$$

is the identity map for $i = 1, 2$ where $p_i$ is the projection to the $i$-th factor.

(b) $S^{-1}_\eta (B/J)$ is a field and the natural homomorphism

$$S^{-1}_\eta (A/I) \to S^{-1}_\eta (B/J)$$

is a field extension with $[S^{-1}_\eta (B/J) : S^{-1}_\eta (A/I)] = 2$.

For each case, we obtain (2) by a similar argument to (α). □

The following theorem is the main result in this section.

**Theorem 3.7.** Let $k$ be a field. Let $X$ be a pure-dimensional reduced separated scheme of finite type over $k$. Assume that $X$ is $S_2$ and, for every codimension one scheme-theoretic point $\eta \in X$, the local ring $\mathcal{O}_{X,\eta}$ is regular or nodal. Let $\nu : Y \to X$ be the normalization, $D \subset X$
the closed subscheme defined by the conductor and \( C \subset Y \) its scheme-theoretic inverse image:

\[
C := \nu^{-1}(D) \xrightarrow{\text{closed immersion}} Y
\]

\[
\downarrow \nu|_C \quad \downarrow \nu
\]

\[
D \xrightarrow{\text{closed immersion}} X.
\]

Let \( L \) be an invertible sheaf on \( X \) and fix \( s \in H^0(Y, \nu^*L^{\otimes 2}) \). Let \( C = \bigcup C_i \) be the irreducible decomposition where each \( C_i \) is an integral scheme. Assume the following conditions.

1. The equation \( g^*(s|_{C_i}) = s|_{C_j} \) holds for every birational map \( g : C_i \dasharrow C_j \) such that \( \nu|_{C_i} = \nu|_{C_j} \circ g \) holds as rational maps. Note that \( g^*(s|_{C_j}) = s|_{C_i} \) means that there exist non-empty open subsets \( C'_i \subset C_i, C'_j \subset C_j \) and an isomorphism \( g' : C'_i \to C'_j \) induced by \( g \) such that \( g'^*(s|_{C'_j}) = s|_{C'_i} \).

2. For every \( i \), there exists \( t_i \in H^0(C_i, \nu^*L) \) such that \( s|_{C_i} = t_i^{\otimes 2} \).

Then there exists an element \( u \in H^0(X, L^{\otimes 2}) \) such that \( \nu^*u = s \).

Proof. Consider the exact sequence:

\[
0 \to \mathcal{O}_X \to \nu_*\mathcal{O}_Y \oplus \mathcal{O}_D \to \nu_*\mathcal{O}_C \to 0,
\]

which implies

\[
0 \to H^0(X, L^{\otimes 2}) \to H^0(Y, \nu^*L^{\otimes 2}) \oplus H^0(D, L^{\otimes 2}|_D) \to H^0(C, \nu^*L^{\otimes 2}|_C).
\]

It suffices to show that there exists \( t \in H^0(D, L^{\otimes 2}|_D) \) such that \( (\nu|_C)^*t = s|_C \). Since \( X \) is \( S_2 \), we can replace \( X \) with arbitrary open subscheme \( X' \) with \( \text{codim}_X (X \setminus X') \geq 2 \). Thus, we may assume that \( C \) and \( D \) are regular and of pure codimension one. We can apply Proposition [3.6]. Then, by replacing \( X \) with its open subscheme, \( C \to D \) satisfies one of the following properties.

(a) \( C \) is two copies of \( D \), that is, \( C \simeq D \amalg D \).

(b) \( C \to D \) is a finite surjective morphism between integral schemes such that \( [K(C) : K(D)] = 2 \) and that \( K(C)/K(D) \) is separable.

(c) \( C \to D \) is a finite surjective morphism between integral schemes such that \( [K(C) : K(D)] = 2 \) and that \( K(C)/K(D) \) is purely inseparable.

If (a) or (b) holds, then the condition (1) implies that \( s|_C \) descends to \( D \). If (c) holds, then the condition (2) implies that \( s|_C \) descends to \( D \). □
Remark 3.8. By the above proof, if the characteristic of \( k \) is not equal to 2, then we can drop the second condition (2) in Theorem 3.7.

4. Abundance theorem for slc surfaces

The following definition of slc varieties is the same as Definition-Lemma 5.10 in [Kollár]. For more details, see also [Kollár, 1.41, 5.1, 5.9, 5.10]. Moreover, we define sdlt varieties.

Definition 4.1. Let \( X \) be a variety. Assume that \( X \) is regular or nodal in codimension one. Let \( \Delta \) be an effective \( \mathbb{Q} \)-divisor such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. Let \( \nu : Y \to X \) be the normalization and we define \( \Delta_Y \) by \( K_Y + \Delta_Y = \nu^*(K_X + \Delta) \). We say \((X, \Delta)\) is slc if \((Y, \Delta_Y)\) is lc. We say \((X, \Delta)\) is sdlt variety if \((Y, \Delta_Y)\) is dlt and every irreducible component of \( X \) is normal.

Remark 4.2. (1) Note that sdlt in Definition 4.1 and semi-dlt in the sense of [Kollár, Definition 5.19] are different. There is an sdlt variety which is not semi-dlt (see the example after [Kollár, Definition 5.19]).

(2) In characteristic zero, semi-dlt varieties are sdlt by [Kollár, Definition 5.20]. In positive characteristic, we do not know whether the notions of semi-dlt and sdlt have some relations.

We recall the \( B \)-birational maps introduced in [Fujino1].

Definition 4.3. Let \((X, \Delta_X)\) and \((Y, \Delta_Y)\) be lc varieties (may be reducible). We say \( \sigma : (X, \Delta_X) \dasharrow (Y, \Delta_Y) \) is a \( B \)-birational map if \( \sigma : X \dasharrow Y \) is a birational map and there exist proper birational morphisms \( \alpha : W \to X \) and \( \beta : W \to Y \) from a normal variety \( W \) such that \( \beta = \sigma \circ \alpha \) and \( \alpha^*(K_X + \Delta_X) = \beta^*(K_Y + \Delta_Y) \). Note that \( B \)-birational maps may permute the irreducible components. We define

\[
\text{Aut}(X, \Delta_X) := \{ \sigma \in \text{Aut}(X) \mid K_X + \Delta_X = \sigma^*(K_X + \Delta_X) \}.
\]

To obtain sections on slc varieties, we consider the following sections on sdlt varieties. The idea is very similar to the admissible sections in [Fujino1].

Definition 4.4. Let \((X, \Delta)\) be an \( n \)-dimensional projective sdlt variety with \( n \leq 2 \). Let \( X = \bigsqcup X_i \) be the irreducible decomposition and let \( \nu : \bigsqcup X_i \to X \) be the normalization. We define \( \Delta_i \) by \( K_{X_i} + \Delta_i = (\nu^*(K_X + \Delta))|_{X_i} \). Note that \((X_i, \Delta_i)\) is dlt. Let \( m \) be a positive integer such that \( m(K_X + \Delta) \) is Cartier. We define \( B \)-invariant sections and separably gluable sections as follows.

(1) We say \( s \in H^0(X, m(K_X + \Delta)) \) is \( B \)-invariant if \( \psi^*(s|_{X_i}) = s|_{X_i} \) for every \( B \)-birational map \( \psi : (X_i, \Delta_i) \dasharrow (X_j, \Delta_j) \).
Moreover, we define $SG(X, m(K_X + \Delta))$ as separably gluable if $s|_{\bigcup_i \Delta_i \cup \Delta}$ is $B$-invariant.

We define vector subspaces

$$BI(X, m(K_X + \Delta)) := \{ s \in B \text{ is } B \text{-invariant} \subset H^0(X, m(K_X + \Delta)) \}$$

$$SG(X, m(K_X + \Delta)) := \{ s \text{ is separably gluable} \subset H^0(X, m(K_X + \Delta)) \}.$$

Moreover, we define

$$BI^{(2)}(X, 2m(K_X + \Delta)) := \{ t^2 \mid t \in BI(X, m(K_X + \Delta)) \}$$

$$G(X, 2m(K_X + \Delta)) := \{ s \mid s|_{\bigcup_i \Delta_i \cup \Delta} \in BI^{(2)}(\bigcup_i \Delta_i \cup \Delta, 2m(K_X + \Delta)) \}$$

We say $s \in H^0(X, 2m(K_X + \Delta))$ is gluable if $s \in G(X, 2m(K_X + \Delta))$.

**Remark 4.5.** In characteristic $p \neq 2$, we do not need $BI^{(2)}(X, 2m(K_X + \Delta))$ and $G(X, 2m(K_X + \Delta))$. For more details, see Remark 3.3 and the proof of Proposition 4.9.

The following lemma teaches us that, in order to obtain sections on an slc surface, we should consider gluable sections on a dlt surface.

**Lemma 4.6.** Let $(X, \Delta)$ be a projective slc surface. Let $\nu : Y \to X$ be the normalization and let $K_Y + \Delta_Y := \nu^*(K_X + \Delta)$. Let $\mu : (Z, \Delta_Z) \to (Y, \Delta_Y)$ be a birational morphism from a projective dlt surface $(Z, \Delta_Z)$ such that $K_Z + \Delta_Z = \mu^*(K_Y + \Delta_Y)$. Then the following assertions hold. If $s \in G(Z, 2m(K_Z + \Delta_Z))$, then $s = \mu^* \nu^* t$ for some $t \in H^0(X, 2m(K_X + \Delta)).$

**Proof.** The assertion holds by Theorem 3.7.

We summarize the basic properties of $B$-invariant sections and (separably) gluable sections.

**Lemma 4.7.** Let $(X, \Delta)$ be an n-dimensional projective sdlt variety with $n \leq 2$. Let $m$ be a positive integer such that $m(K_X + \Delta)$ is Cartier.

1. If $s \in BI(X, m(K_X + \Delta))$, then $s^2 \in BI^{(2)}(X, 2m(K_X + \Delta))$.
2. If $t \in BI^{(2)}(X, 2m(K_X + \Delta))$, then $t \in BI(X, 2m(K_X + \Delta))$.
3. The vector space $BI(X, m(K_X + \Delta))$ generates $O_X(m(K_X + \Delta))$ if and only if $BI^{(2)}(X, 2m(K_X + \Delta))$ generates $O_X(2m(K_X + \Delta))$.
4. If $s \in SG(X, m(K_X + \Delta))$, then $s^2 \in G(X, 2m(K_X + \Delta))$.
5. If $t \in G(X, 2m(K_X + \Delta))$, then $t \in SG(X, 2m(K_X + \Delta))$.
6. If the vector space $SG(X, m(K_X + \Delta))$ generates $O_X(m(K_X + \Delta))$, then $G(X, 2m(K_X + \Delta))$ generates $O_X(2m(K_X + \Delta))$. 
Assume that \( X \) is normal and let \( S := \triangle \neq 0 \). If the map
\[
SG(X, m(K_X + \Delta)) \to BI(S, m(K_X + \Delta)|_S)
\]
is surjective, then so is the map
\[
G(X, 2m(K_X + \Delta)) \to BI^{(2)}(S, 2m(K_X + \Delta)|_S).
\]

**Proof.** (1)(2)(3) These assertions follow from the definition.

(4) The assertion follows from \((\nu^*s^2)|_{\triangle} = ((\nu^*s)|_{\triangle})^2\).

(5) The assertions follows from (2).

(6)(7) The assertions follow from (4). \(\square\)

**Lemma 4.8.** Let \((X, \Delta)\) be a proper lc curve or a proper lc surface such that \(K_X + \Delta\) is semi-ample and \(S := \triangle \neq 0\). Let \( f := \varphi_{|K_X + \Delta|} : X \to R \) be a surjective morphism to a projective variety \(R\) such that \(f_*\mathcal{O}_X = \mathcal{O}_R\). Let \(T := f(S)\). Assume the following conditions.

(a) \( f_*\mathcal{O}_S = \mathcal{O}_T \).

(b) There exist sections \( \{s_i\}_{i=1}^q \subset H^0(S, m(K_X + \Delta)|_S) \) without common zeros for some \(m\).

Then, for some \(r > 0\), there exist sections \( \{u_i\}_{i=1}^l \subset H^0(X, rm(K_X + \Delta)) \) which satisfy the following conditions.

(1) \( u_i|_S = s_i \) for \(1 \leq i \leq q\) and \( u_i|_S = 0 \) for \(q+1 \leq i \leq l\).

(2) \( \{u_i\}_{i=1}^l \) have no common zeros.

**Proof.** There is an ample \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(H\) on \(R\) such that \(K_X + \Delta \sim \mathbb{Q} f^*H\). For \(r \gg 0\), we have the following commutative diagram.
\[
\begin{array}{ccc}
H^0(X, rm(K_X + \Delta)) & \longrightarrow & H^0(S, rm(K_X + \Delta)|_S) \\
\uparrow \simeq & & \uparrow \simeq \\
H^0(R, rmH) & \underset{\text{surjection}}{\longrightarrow} & H^0(T, rmH|_T)
\end{array}
\]

Let \(u_1, \ldots, u_q \in H^0(X, rm(K_X + \Delta))\) be lifts of \(s_1^*, \ldots, s_q^*\) and let us consider the following corresponding sections.
\[
u_i \longrightarrow s_i^* \\
\uparrow \\
u_i' \longrightarrow s_i'
\]

We may assume that \(r\) is so large that \(I_T \otimes \mathcal{O}_R(rmH)\) is generated by global sections where \(I_T\) is the corresponding ideal to the closed subscheme \(T\). Let \(t_{q+1}', \ldots, t_l'\) be the basis of \(H^0(R, I_T \otimes \mathcal{O}_R(rmH))\) and let \(u_{q+1}', \ldots, u_l'\) be its image to \(H^0(R, rmH)\). Then \(u_1', \ldots, u_l'\) have no common zeros. Thus the corresponding sections \(u_1, \ldots, u_l\) satisfy the desired properties. \(\square\)
The following proposition is the key to prove the abundance theorem for slc surfaces.

**Proposition 4.9.** Let \((X, \Delta)\) be a projective dlt surface such that \(S := \Delta \neq 0\). Let \(m\) be a sufficiently large and divisible integer such that \(m \in 2\mathbb{Z}_{>0}\). If \(K_X + \Delta\) is nef, then the following assertions hold.

(a) The following map is surjective:
\[
G(X, 2m(K_X + \Delta)) \to BI^{(2)}(S, 2m(K_X + \Delta)|_S).
\]

(b) Assume that \(BI(S, m(K_X + \Delta)|_S)\) generates \(O_S(m(K_X + \Delta)|_S)\). Then \(G(X, 2m(K_X + \Delta))\) generates \(O_X(2m(K_X + \Delta))\).

**Proof.** We may assume that \(X\) is irreducible. By the abundance theorem (cf. [Fujita]), we obtain \(f := \varphi|_{k(K_X + \Delta)} : X \to R\) such that \(f_*O_X = O_R\). Let \(f(S) =: T\). Then (1) or (2) holds.

(1) \(f_*O_S = O_T\).
(2) \(f_*O_S \neq O_T\).

(1) Assume \(f_*O_S = O_T\). By the diagram of the proof of Lemma 4.8, the map
\[
H^0(X, m(K_X + \Delta)) \to H^0(S, m(K_X + \Delta)|_S)
\]
is surjective. Thus the map
\[
SG(X, m(K_X + \Delta)) \to BI(S, m(K_X + \Delta)|_S)
\]
is also surjective. Thus assertion (a) follows from Lemma 4.7(7). We prove (b). Since \(BI(S, m(K_X + \Delta)|_S)\) generates \(O_S(m(K_X + \Delta)|_S)\), \(SG(X, m(K_X + \Delta))\) also generates \(O_X(m(K_X + \Delta))\) by Lemma 4.8. The assertion follows from Lemma 4.7(6).

(2) Assume \(f_*O_S \neq O_T\). We can apply Proposition 2.7 and we obtain Proposition 2.7(2). Then, we have projective morphisms
\[
f : X \dashrightarrow V \to R
\]
where \(V\) is a smooth projective curve.

**Case (2.1s).** Assume Proposition 2.7(2.1s) holds. By Lemma 4.7(7), it is sufficient to prove (a)' and (b)'.

(a)' The following map is surjective:
\[
SG(X, m(K_X + \Delta)) \to BI(S, m(K_X + \Delta)|_S).
\]

(b)' Assume that \(BI(S, m(K_X + \Delta)|_S)\) generates \(O_S(m(K_X + \Delta)|_S)\). Then \(SG(X, m(K_X + \Delta))\) generates \(O_X(m(K_X + \Delta))\).
First we prove (a). Note that there is a Galois involution \( \iota : S_1 \to S_1 \) and \( \iota \) is \( B \)-bidental. Let \( s \in BI(S, m(K_X + \Delta) |_S) \). Since \( s \) is \( B \)-invariant, this section \( s \) is invariant for \( \iota \). Thus \( s|_{S_1} \) is the pull-back of a section \( t \in H^0(V, m(D_V)) \). Let \( u := g^* t \in H^0(X, m(K_X + \Delta)) \). We prove that \( u|_S = s \). Let \( S = \bigcup S_i \) be the irreducible decomposition. Since \( S \) is reduced, we obtain the exact sequence:

\[
0 \to \mathcal{O}_S \to \bigoplus_i \mathcal{O}_{S_i}.
\]

Therefore it is sufficient to prove that \( u|_{S_i} = s|_{S_i} \) for every \( i \). For \( i = 1 \), this is clear by the construction. Thus we may assume that \( S_i \) is \( g \)-vertical. We take a proper birational morphism \( \lambda : X' \to X \) in Lemma 5.10. Let \( g'' : X'' \to X \) where \( S'' := \iota \Delta'' \). Thus it is sufficient to prove that \( u''|_{S''_i} = s''|_{S''_i} \) where \( u'' := \lambda^* u \). Since \( S''_i \) is an irreducible component of \( S'' \) such that \( g'' \)-vertical. Let \( S'''_i \) be the proper transform of \( S_i \). Assume \( S'''_i \cap S''_j \neq \emptyset \). Note that, since \( (X'', \Delta'') \) is dlt, the scheme-theoretic intersection \( S'''_i \cap S''_j \) is reduced. Hence, Lemma 5.10 implies \( g''|_{S'''_i} \simeq g''|_{S''_j} \). Since \( m(K_{X''} + \Delta'') \) is the pull-back of \( mD_V \), this means

\[
H^0(S'''_i, m(K_{X''} + \Delta'') |_{S''_i}) \simeq H^0(S''_i \cap S''_j, m(K_{X''} + \Delta'') |_{S''_i \cap S''_j}).
\]

By \( u''|_{S''_i} = s''|_{S''_i} \), we have \( u''|_{S''_i \cap S''_j} = s''|_{S''_i \cap S''_j} \). Therefore, by the above isomorphism, we see \( u''|_{S''_i \cap S''_j} = s''|_{S''_i \cap S''_j} \). If \( S''_j \cap S''_i \neq \emptyset \), then \( u''|_{S''_j} = s''|_{S''_j} \) by the same argument above. By the inductive argument, if a vertical irreducible component \( S''_i \) is contained in a connected component of \( S''_i \) which intersects \( S''_j \), then \( u''|_{S''_j} = s''|_{S''_j} \). By Lemma 2.3 and Proposition 2.7, every vertical irreducible component \( S''_i \) satisfies this property. Therefore, we see \( u \in SG(X, m(K_X + \Delta)) \) such that \( u|_S = s \).

Second, we prove (b). We prove that \( SG(X, m(K_X + \Delta)) \) generates \( \mathcal{O}_X(m(K_X + \Delta)) \). Let \( s_1, \cdots, s_r \in BI(S, m(K_X + \Delta) |_S) \) be a basis and let \( u_1, \cdots, u_r \in SG(X, m(K_X + \Delta)) \) be their lifts. Let \( t_1, \cdots, t_r \in H^0(V, mD_V) \) be the corresponding sections. Since \( BI(S, m(K_X + \Delta) |_S) \) generates \( \mathcal{O}_S(m(K_X + \Delta) |_S) \) and \( S \to V \) is surjective, \( t_1, \cdots, t_r \) have no common zeros. Thus the corresponding sections \( u_1, \cdots, u_r \) generates \( \mathcal{O}_X(m(K_X + \Delta)) \).

\textbf{Case} (2.2). Assume Proposition 2.7 (2.2) holds. It is sufficient to prove the above assertions (a)' and (b)'.

We prove (a)'. Note that there is a \( B \)-birational morphism \( \iota : S_2 \to S_1 \) obtained by \( S_2 \simeq V \simeq S_1 \). Let \( s \in BI(S, m(K_X + \Delta) |_S) \). Since \( s \) is \( B \)-invariant, we see \( \iota^*(s|_{S_1}) = s|_{S_2} \). Since \( S_1 \simeq V \), \( s|_{S_1} \) is the pull-back.
of a section $t \in H^0(V, mD_V)$. Let $u := g^*t \in H^0(X, m(K_X + \Delta))$. We would like to prove that $u|_S = s$. It is sufficient to prove that $u|_{S_i} = s|_{S_i}$ for every irreducible component $S_i$ of $S$. By the same argument as (2.1s), it is sufficient to prove this equality only for $i = 1, 2$. It is clear in the case where $i = 1$. Since $\iota^*(u|_{S_1}) = u|_{S_2}$, it is also clear in the case where $i = 2$. The assertion (b) holds by the same argument as (2.1s).

**Case (2.1i).** Assume Proposition [2.7](2.1i) holds. We see $p = \text{char } k = 2$.

We prove (a). Let $s \in BI^{(2)}(S, 2m(K_X + \Delta)|_S)$. Then we have $s = \tilde{s}^2$ where $\tilde{s} \in BI(S, m(K_X + \Delta)|_S)$. Note that $g|_{S_1} : S_1 \to V$ is the relative Frobenius morphism. Thus the absolute Frobenius morphism $F : S_1 \to S_1$ factors through $V$:

$$F : S_1 \stackrel{g|_{S_1}}{\to} V \stackrel{G}{\to} S_1.$$  

Note that $G$ is a non-$k$-linear isomorphism as schemes and that, for an invertible sheaf $L$ on $V$,

$$G^*(g|_{S_1})^*L \simeq G^*(g|_{S_1})^*G^*(G^{-1})^*L \simeq G^*F^*(G^{-1})^*L \simeq L^\otimes 2.$$

We show $\mathcal{O}_V(2mD_V) \simeq G^*\mathcal{O}_{S_1}(m(K_X + \Delta)|_{S_1})$. Since $m \in 2\mathbb{Z}$, we can write $m = 2m'$ where $m' \in \mathbb{Z}$. First, we see

$$(g|_{S_1})^*\mathcal{O}_V(2m'D_V) \simeq \mathcal{O}_{S_1}(2m'(K_X + \Delta)|_{S_1}) \simeq (g|_{S_1})^*G^*\mathcal{O}_{S_1}(m'(K_X + \Delta)|_{S_1}).$$

Then, for an invertible sheaf

$$M := (G^{-1})^*\mathcal{O}_V(2m'D_V) \otimes \mathcal{O}_{S_1}(-m'(K_X + \Delta)|_{S_1}),$$

we obtain $F^*M = (g|_{S_1})^*G^*M \simeq \mathcal{O}_{S_1}$. This implies

$$\mathcal{O}_V(2mD_V) \simeq G^*(g|_{S_1})^*\mathcal{O}_V(2m'D_V)$$

$$\simeq G^*F^*(G^{-1})^*\mathcal{O}_V(2m'D_V)$$

$$\simeq G^*\mathcal{O}_{S_1}(m'(K_X + \Delta)|_{S_1})$$

$$\simeq G^*\mathcal{O}_{S_1}(m(K_X + \Delta)|_{S_1}).$$

Therefore, the section $s$ is the pull-back of

$$t := G^*\tilde{s} \in H^0(V, 2mD_V).$$

Let $u := g^*t \in H^0(X, 2m(K_X + \Delta))$. Then, by the same argument as (2.2s), we see $u|_S = s$. This means $u \in G(X, 2m(K_X + \Delta))$.

We prove (b), that is, we prove that $G(X, 2m(K_X + \Delta))$ generates $\mathcal{O}_X(2m(K_X + \Delta))$. Let $s_1, \cdots, s_r \in BI^{(2)}(S, 2m(K_X + \Delta)|_S)$ be a basis and let $u_1, \cdots, u_r \in G(X, 2m(K_X + \Delta))$ be their lifts. Let $t_1, \cdots, t_r \in H^0(V, 2mD_V)$ be the corresponding sections. Here, $BI^{(2)}(S, 2m(K_X + \Delta)|_S)$ generates $\mathcal{O}_S(m(K_X + \Delta)|_S)$ by Lemma [1]. Therefore, $S \to V$ is surjective, $t_1, \cdots, t_r$ have no common zeros. Thus the corresponding sections $u_1, \cdots, u_r$ generates $\mathcal{O}_X(2m(K_X + \Delta))$. □
In order to construct $B$-invariant sections, we consider the following finiteness theorem.

**Theorem 4.10.** Let $(C, \Delta)$ be a projective lc curve and let $m$ be a positive integer such that $m(K_C + \Delta)$ is Cartier. Then $\rho_m(\text{Aut}(C, \Delta))$ is a finite group where $\rho_m$ is a group homomorphism defined by

$$\rho_m : \text{Aut}(C, \Delta) \to \text{Aut}(H^0(C, m(K_C + \Delta)))$$

$$\sigma \mapsto (s \mapsto \sigma^* s).$$

**Proof.** We may assume that $C$ is irreducible. If the genus $g(C) \geq 2$, then $\text{Aut}(C)$ is a finite group. Therefore, $\rho_m(\text{Aut}(C, \Delta))$ is a finite group since $\text{Aut}(C, \Delta) \subset \text{Aut}(C)$.

If $g(C) = 1$ and $\Delta \neq 0$, then $\text{Aut}(C, \Gamma \Delta \cap)$ is a quasi-projective scheme and $H^0(C, T_C \otimes \mathcal{O}_C(-\Gamma \Delta \cap)) = 0$. Therefore, $\text{Aut}(C, \Gamma \Delta \cap)$ is a finite group. Thus, $\rho_m(\text{Aut}(C, \Delta))$ is a finite group because $\text{Aut}(C, \Delta) \subset \text{Aut}(C, \Gamma \Delta \cap)$.

Assume that $g(C) = 1$ and $\Delta = 0$. Let $0 \in C$ be the origin of the elliptic curve $C$. Then $T_{-\sigma(0)} \circ \sigma \in \text{Aut}(C, [0])$ for any $\sigma \in \text{Aut}(C)$, where $T_{-\sigma(0)}$ is the translation of $C$ by $-\sigma(0)$. Note that $H^0(C, \mathcal{O}_C(K_C)) \cong k$ is spanned by a translation invariant 1-form on $C$ and that $\text{Aut}(C, [0])$ is a finite group. Therefore, $\rho_1(\text{Aut}(C))$ is a finite group. Since $\rho_m = \rho_1 \otimes m$, $\rho_m(\text{Aut}(C))$ is finite for every $m > 0$.

Finally, we assume that $C = \mathbb{P}^1$. If $|\text{Supp}\Delta| \geq 3$, then $\text{Aut}(C, \Delta)$ is a finite group. If $\deg(K_C + \Delta) < 0$, then there is nothing to prove. Therefore, we can reduce the problem to the case when $\Delta = \Gamma \Delta \cap = \{\text{two points}\}$. In this case, we can easily check that $\rho_m(\text{Aut}(C, \Delta))$ is finite for every $m > 0$. Moreover, $\rho_m(\text{Aut}(C, \Delta))$ is trivial if $m$ is an even positive integer. \hfill $\square$

The following proposition shows that the assumption of (b) in Proposition 4.9 holds.

**Proposition 4.11.** Let $(X, \Delta)$ be a projective lc curve. If $K_X + \Delta$ is nef, then $BI(X, m'(K_X + \Delta))$ generates $\mathcal{O}_X(m'(K_X + \Delta))$ for some integer $m' > 0$.

**Proof.** We see that $H^0(X, m(K_X + \Delta))$ generates $\mathcal{O}_X(m(K_X + \Delta))$ for some integer $m > 0$. Let $G := \rho_m(\text{Aut}(X, \Delta))$. Note that this group is finite by Theorem 4.10. Let $N := |G|$ and let $G = \{g_1, \ldots, g_N\}$. For $1 \leq i \leq N$, let $\sigma_i$ be the $N$-variable elementary symmetric polynomial of degree $i$. If $s \in H^0(X, m(K_X + \Delta))$, then

$$(\sigma_i(g_1^* s, \ldots, g_N^* s))^{N/i} \in BI(X, N! m(K_X + \Delta)).$$
Since
\[
\bigcap_{j=1}^N \{ g_j^* s = 0 \} = \bigcap_{i=1}^N \{ \sigma_i(g_1^* s, \cdots, g_N^* s) = 0 \},
\]
\(BI(X, N!m(K_X + \Delta))\) generates \(O_X(N!m(K_X + \Delta))\). \(\square\)

Let us prove the main theorem of this paper.

**Theorem 4.12.** Let \((X, \Delta)\) be a projective slc surface. If \(K_X + \Delta\) is nef, then \(K_X + \Delta\) is semi-ample.

**Proof.** Let \(\nu : Y \to X\) be the normalization and we define \(\Delta_Y\) by \(K_Y + \Delta_Y = \nu^*(K_X + \Delta)\). There exists a birational morphism \(\mu : Z \to Y\) from a projective dlt surface \((Z, \Delta_Z)\) where \(K_Z + \Delta_Z = \mu^*(K_Y + \Delta_Y)\). By Lemma 4.6, it is sufficient to prove that \(G(Z, m_0(K_Z + \Delta_Z))\) generates \(O_Z(m_0(K_Z + \Delta_Z))\) for some \(m_0 > 0\). This follows from Proposition 4.9(b) and Proposition 4.11. \(\square\)

5. **Appendix: Fundamental properties of dlt surfaces**

We summarize fundamental properties for dlt surfaces. In this section, we assume that all surfaces are irreducible. The results in this section may be well-known for experts.

First, we recall the definition of dlt surfaces. It is easy to see that the following definition is equivalent to [Kollár, Definition 2.8] and [Kollár-Mori, Definition 2.37].

**Definition 5.1.** Let \(X\) be a normal surface and let \(\Delta\) be a \(\mathbb{Q}\)-divisor such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier and \(0 \leq \Delta \leq 1\). Let
\[
S(X, \Delta) := \text{Sing}(X) \cup \{ x \in \text{Reg}(X) \mid \text{Supp} \Delta \text{ is not simple normal crossing at } x \}.
\]
We say \((X, \Delta)\) is dlt if \(a(E, X, \Delta) > -1\) for every proper birational morphism \(f : Y \to X\) and every \(f\)-exceptional prime divisor \(E \subset Y\) such that \(f(E) \in S(X, \Delta)\).

**Proposition 5.2.** Let \(X\) be a normal surface and let \(\Delta\) be a \(\mathbb{Q}\)-divisor such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier and \(0 \leq \Delta \leq 1\). The following assertions are equivalent:

1. \((X, \Delta)\) is dlt.
2. There exists a projective birational morphism \(\mu : X' \to X\) from a smooth surface such that \(\text{Ex}(\mu) \cup \text{Supp} \mu^*(\Delta)\) is a simple normal crossing divisor and each \(\mu\)-exceptional prime divisor \(E_i\) satisfies \(a(E_i, X, \Delta) > -1\).
Proof. Note that $S(X, \Delta)$ is a finite set.

Assume (1), that is, assume that $(X, \Delta)$ is dlt. Let $f : Y \to X$ be a log resolution of $(X, \Delta)$. Let

$$\text{Ex}(f) := E_1 \amalg \cdots \amalg E_r \amalg F_1 \amalg \cdots \amalg F_s$$

be the decomposition into the connected components where $P_i := f(E_i) \in S(X, \Delta)$ and $Q_j := f(F_j) \not\in S(X, \Delta)$. There exists a proper birational morphisms

$$Y \xrightarrow{g} Z \xrightarrow{h} X$$

such that $Z$ is a normal surface and $\text{Ex}(g) = F_1 \amalg \cdots \amalg F_s$. Indeed, $Z$ is obtained by glueing the varieties $X \setminus \{P_1, \cdots, P_r\}$ and $Y \setminus (F_1 \cup \cdots \cup F_s)$. Note that this morphism $h : Z \to X$ is projective because $Z$ is smooth. Thus this morphism satisfies (2).

Assume (2). Let $f : Y \to X$ be a proper birational morphism and let $E \subset Y$ be a prime divisor such that $f(E) \in S(X, \Delta)$. We prove $a(E, X, \Delta) > -1$. We may assume that there exists a proper birational morphism $Y \xrightarrow{f'} X'$ and $Y$ is smooth by replacing $Y$ with a desingularization of a resolution of indeterminacy $Y \to X'$. There are two cases: $\dim f'(E) = 0$ and $\dim f'(E) = 1$. The latter case is clear by (2). Thus we may assume $f'(E)$ is one point. Let $K_{X'} + \Delta' := \mu^*(K_X + \Delta)$. Since $f(E) \in S(X, \Delta)$, there exists an $\mu$-exceptional curve $E_i$ such that $f'(E) \in E_i$. We can write the prime decomposition

$$\Delta' := b_1 E_i + \cdots$$

where $b_i < 1$. Then we see that $a(E, X, \Delta) > -1$ since $\Delta'$ is simple normal crossing and since the morphism $f' : Y \to X'$ is a sequence of blow-ups. \hfill $\square$

**Proposition 5.3.** Let $(X, \Delta)$ be a dlt surface. Then $X$ is $\mathbb{Q}$-factorial.

*Proof.* See, for example, [111, Theorem 5.3]. \hfill $\square$

**Proposition 5.4.** Let $(X, \Delta)$ be a dlt surface. If a $\mathbb{Q}$-divisor $\Delta'$ satisfies $0 \leq \Delta' \leq \Delta$, then $(X, \Delta')$ is dlt.

*Proof.* Since $X$ is $\mathbb{Q}$-factorial, the assertion immediately follows from Definition 5.1. \hfill $\square$

**Proposition 5.5.** Let $(X, \Delta)$ be a dlt surface. Then the following assertions are equivalent.

1. $(X, \Delta)$ is plt.
2. $\downarrow \Delta$ is smooth.
3. Each connected component of $\downarrow \Delta$ is irreducible.
Proof. See [Kollár-Mori, Proposition 5.51]. Note that the proof of [Kollár-Mori, Proposition 5.51] needs the relative Kawamata–Viehweg vanishing theorem for a resolution of singularities $Y \to X$. This follows from [T2].

Corollary 5.6. Let $(X, \Delta)$ be a dlt surface. Then each prime component of $\Delta$ is smooth.

Proof. Let $C$ be a prime component of $\Delta$. Then $(X, C)$ is plt by Proposition 5.5.

Proposition 5.7. Let $(X, C + \Delta')$ be a dlt surface where $C$ is a smooth curve in $X$. Let $(K_X + C + \Delta')|_C =: K_C + \Delta_C$. Then $(C, \Delta_C)$ is lc, that is, $0 \leq \Delta_C \leq 1$.

Proof. Let $f : Y \to X$ be an arbitrary resolution and let $C_Y$ be the proper transform of $C$. Let $f^*(K_X + C + \Delta') =: K_Y + C_Y + \Delta_Y$. Note that $C \simeq C_Y$. Consider the following commutative diagram.

$$
\begin{array}{ccc}
C & \xrightarrow{\text{closed immersion}} & Y \\
\downarrow \simeq & & \downarrow f \\
C & \xrightarrow{\text{closed immersion}} & X
\end{array}
$$

We prove that $\Delta_C$ is effective. Let $f$ be the minimal resolution. Then $\Delta_Y$ is effective and $C_Y$ is not a prime component of $\Delta_Y$. Thus we have $0 \leq \Delta_C \leq 1$.

Corollary 5.8. Let $(X, \Delta)$ be a dlt surface. Assume $S := \Delta \neq 0$ and let $K_S + \Delta_S := (K_X + \Delta)|_S$. Then $S$ is normal crossing and $(S, \Delta_S)$ is sdlt.

Proof. By [Kollár-Mori, Theorem 4.15], $S$ is normal crossing. Thus, the assertion follows from Proposition 5.7.

Proposition 5.9. Let $(X, \Delta)$ be an lc surface. Then there exists a proper birational morphism $h : Z \to X$ from a smooth surface $Z$ such that $(Z, \Delta_Z)$ is dlt where $\Delta_Z$ is defined by $K_Z + \Delta_Z = h^*(K_X + \Delta)$.

Proof. Let $f : Y \to X$ be a log resolution of $(X, \Delta)$ and let $K_Y + \Delta_Y := f^*(K_X + \Delta)$. Let $\Delta_Y = \Delta_Y^+ - \Delta_Y^-$ where $\Delta_Y^+$ and $\Delta_Y^-$ are effective and $\Delta_Y^+$ and $\Delta_Y^-$ have no common irreducible components. Since $K_Y \cdot \Delta_Y^- < 0$ and each irreducible component of $\Delta_Y^-$ is $f$-exceptional, there exists
a \((-1)\)-curve \(C\) such that \(C \subset \text{Supp}\Delta^{-}\). Contract this \((-1)\)-curve \(Y \to Y'\). We repeat this procedure and we obtain morphisms

\[ f : Y \xrightarrow{g} Z \xrightarrow{h} X. \]

Then we see that \(Z\) is smooth and \(0 \leq \Delta_Z \leq 1\) where \(K_Z + \Delta_Z = h^*(K_X + \Delta)\). We prove that \((Z, \Delta_Z)\) is dlt. Let \(l : W \to Z\) be a proper birational morphism and \(E \subset W\) be an \(l\)-exceptional prime divisor such that \(l(E) \in S(Z, \Delta)\). We prove \(a(Z, \Delta_Z, E) > -1\). We may assume that \(W\) is smooth and \(l : W \to Z\) factors through \(Y\). We obtain four surfaces:

\[ W \xrightarrow{p} Y \xrightarrow{g} Z \xrightarrow{h} X. \]

Note that \(p(E) \subset \text{Supp}\Delta^{-}\).

There are two cases: (0) \(\dim p(E) = 0\) and (1) \(\dim p(E) = 1\).

(0) Assume \(\dim p(E) = 0\). Note that \(p\) is a composition of blow-ups. Since \(\Delta_Y\) is simple normal crossing and \(p(E) \in \text{Supp}\Delta^{-}\), we obtain \(a(Z, \Delta_Z, E) > 0\) by a direct calculation.

(1) Assume \(\dim p(E) = 1\). Since \(p(E) \subset \text{Supp}\Delta^{-}\), we obtain the inequality \(a(Z, \Delta_Z, E) > 0\). \(\square\)

**Lemma 5.10.** Let \((X, \Delta)\) be a dlt surface. Then there exists a proper birational morphism \(\lambda : X'' \to X\) from a normal surface \(X''\) which satisfies the following properties.

1. For \(K_{X''} + \Delta'' := \lambda^*(K_X + \Delta)\), the pair \((X'', \Delta'')\) is dlt.
2. If \(S_i''\) and \(S_j''\) are prime components of \(\Delta''\) such that \(S_i'' \neq S_j''\) and \(S_i'' \cap S_j'' \neq \emptyset\), then \(S_i'' \cap S_j''\) is one point.
3. \(\lambda_* (\langle \Delta'' \rangle) = \langle \Delta \rangle\) and \(\lambda_* \mathcal{O}_{\Delta''} = \mathcal{O}_{\Delta}\).

**Proof.** If \((X, \Delta)\) satisfies the condition (2), then the assertion is clear. Thus we may assume that there exists prime components \(S_i\) and \(S_j\) of \(\Delta\) such that \(S_i \neq S_j\) and \(S_i \cap S_j\) has at least two points. Let \(P \in S_i \cap S_j\). Note that, since \((X, \Delta)\) is dlt, \(P \in \text{Reg}(X)\) and \(\text{Supp}\Delta\) is simple normal crossing at \(P\). Let \(\mu : Y \to X\) be the blowup at \(P\) and let \(K_Y + \Delta_Y := \mu^*(K_X + \Delta)\). We apply this argument to \((Y, \Delta_Y)\) and we repeat the same procedure. Then, by a direct calculation and Lemma 2.2, we obtain the desired morphism \(X'' \to X\). \(\square\)

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