Analysis of approaches to spline interpolation of functions with large gradients in the boundary layer

I.A. Blatov¹, A I Zadorin²
¹ Volga Region State University of Telecommunications and Informatics, ul. L’va Tolstogo,23, 443010, Samara, Russia
² Sobolev Institute of Mathematics, pr. Ak. Koptyuga,4, 630090, Novosibirsk, Russia
E-mail: blatow@mail.ru, zadorin@ofim.oscsbras.ru

Abstract. The problem of cubic spline interpolation of functions with large gradients in the boundary layer is considered. It is assumed that the decomposition in the form of the sum of the regular and singular components is valid for the interpolated function. This decomposition is valid for the solution of a boundary value problem for a second order ordinary differential equation with a small parameter ε at the highest derivative. An overview of approaches to the construction of splines, the error of which is uniform with respect to the small parameter ε, is given. The approaches are based on the use of Shishkin and Bakhvalov meshes, which are thickened in the boundary layer region. An alternative approach based on a modification of a cubic spline with fitting to the singular component of the interpolated function is also considered. The comparison of the accuracy of the applied approaches with the performance of computational experiments is carried out.

1. Introduction
On the basis of boundary value problems for equations with a small parameter ε at higher derivatives, convective-diffusion processes with predominant convection are simulated. Solutions of such problems have large gradients in the boundary layer region, which affects the accuracy of classical difference schemes. For the numerical solution of such problems, special difference schemes are constructed, the error of which is uniform in the parameter ε. Two approaches are widely known for constructing ε-uniformly converging difference schemes: fitting the scheme to the boundary layer component [1] and using the classical difference schemes on grids, condensed in the boundary layer [2], [3].

The development of difference schemes for singularly perturbed problems has been the subject of the work of a number of authors. At the same time, the problem of developing spline interpolation methods for functions corresponding to the solution of singularly perturbed problems is also relevant. It is of interest to develop interpolation methods on grids that provide uniform convergence of difference schemes.

The spline theory has been extensively developed in the works of a number of authors. Note, for example, the monographs [4] [5], [6]. However, the development of splines for functions with a feature corresponding to the presence of a boundary layer is relevant. In accordance with [7] and other works, the application of polynomial interpolation formulas to functions with large gradients in the boundary layer can lead to errors of the order of O(1). Cubic splines are widely...
used to construct spline-collocation difference schemes for the numerical solution of singularly perturbed problems [8]. In this case, the error of splines between mesh nodes is not estimated.

In this paper, we will carry out a comparative analysis of the developed approaches to the solution of a singularly perturbed boundary value problem: the solution of the problem (3) has a boundary layer region

$$\Phi(x) \in C([0,1],\mathbb{R})$$ is valid. If in (1) $\Phi(x) = e^{-\alpha x/\varepsilon}$, then it has place lower estimate

$$\| u(x) - S_3(x, u) \|_{C[0,1]} \geq C_1 \min\{(N\varepsilon)^{-1}, (N\varepsilon)^{-4}\}. $$

From Theorem 1 it follows that in the case of a uniform grid, the error of interpolation by a cubic spline grows unboundedly if for a given $N$, $\varepsilon \to 0$. So, the problem of constructing a spline with $\varepsilon$-uniform error for a function of the form (1) is relevant.
3. Shishkin mesh

The analysis of the cubic spline error on the Shishkin mesh [3] was carried out in [10]. Let’s set the Shishkin mesh $\Omega$:

$$h_n = h = \frac{\sigma}{N/2}, \quad n = 1, \ldots, N/2, \quad h_n = H = \frac{1 - \sigma}{N/2}, \quad n = N/2 + 1, \ldots, N,$$  \hspace{1cm} (6)

$$\sigma = \min \left\{ \frac{1}{2}, \frac{4\varepsilon}{\alpha \ln N} \right\}. \hspace{1cm} (7)$$

**Theorem 2** There are constants $C$ and $\beta > 0$ that do not depend on $\varepsilon, N$, such that the error estimates hold

$$\| S_3(x, u) - \Phi(x) \|_{C[x_n, x_{n+1}]} \leq C_5 \left\{ \begin{array}{l} N^{-4} \ln^4 N, 0 \leq n \leq N/2 - 1, \\
N^{-5} / \varepsilon e^{-\beta(n-N/2)}, N/2 \leq n \leq N - 1. \end{array} \right. \hspace{1cm} (8)$$

It follows from (8) that this estimate is nonuniform with respect to the parameter $\varepsilon$, if

$$x \in [x_n, x_{n+1}], \quad \frac{N}{2} \leq n \leq \left[ \frac{N}{2} + \frac{1}{\beta \ln \frac{1}{N\varepsilon}} \right], \quad N\varepsilon \leq 1.$$  

On the intervals $[x_n, x_{n+1}]$ for other values of $n$ or for $N\varepsilon \geq 1$, the error estimate is of the order of $O((\ln(N)/N)^4)$ uniformly in the parameter $\varepsilon$.

The following theorem shows that the estimates (8) are unimprovable.

**Theorem 3** Let be $\Phi(x) = e^{-x/\varepsilon}$. Then there are $C, \beta > 0$ that do not depend on $\varepsilon, N$ such that the lower bounds hold

$$\| S_3(x, u) - \Phi(x) \|_{C[x_n, x_{n+1}]} \geq C N^{-5} / \varepsilon e^{-\beta(n-N/2)}, \quad \frac{N}{2} \leq n \leq N - 1. \hspace{1cm} (9)$$

In accordance with (9), on the Shishkin mesh, the spline error on some grid intervals grows indefinitely for $\varepsilon \to 0$.

In accordance with [10], we modify the cubic spline on the Shishkin mesh by shifting one interpolation node. We put $x_{N/2} = (x_{N/2} + x_{N/2+1})/2$, $\bar{x}_n = x_n, n \in [0, N/2 - 1] \cup [N/2 + 1, N]$. Let $S_M(x, u)$ be an interpolation cubic spline determined from the conditions

$$S_M(\bar{x}_n, u) = u(\bar{x}_n), \quad n \in [0, N], \quad S'_M(0, u) = u'(0), \quad S'_M(1, u) = u'(1).$$

The only difference between the spline $S_M(x, u)$ and $S_3(x, u)$ is that the interpolation node $x_{N/2}$ is replaced by the node $\bar{x}_{N/2}$. In this case, the nodes of the spline itself do not change and coincide with the nodes of $\Omega$. In accordance with [11], the following theorem is true.

**Theorem 4** Let the function $u(x)$ have the representation (1), (2), $\Omega$ is the Shishkin mesh (6). Then, for some constant $C$, the following estimate holds

$$\| u(x) - S_M(x, u) \|_{C[x_n, x_{n+1}]} \leq C \left\{ \begin{array}{l} N^{-4} \ln^4 N, 0 \leq n \leq N/2 - 1, \\
1/N^4, \quad N/2 \leq n \leq N - 1. \end{array} \right. \hspace{1cm} (10)$$

**Calculation of derivatives.** It is known [4] that in the regular case, when the function $u(x)$ has uniformly bounded derivatives, it is possible to calculate the derivatives based on the differentiation of the cubic spline, and the error estimates $|u^{(j)}(x) - S_M^{(j)}(x, u)| \leq C h^{4-j}, \ j = 0, 1, 2$ are valid, where $h$ is the step of the uniform grid. Let us proceed to the analysis of this approach in the case when the function $u(x)$ has large gradients and the decomposition (1) is valid for it. In accordance with [11], the following theorem is true.
Theorem 5 Let the function $u(x)$ have the representation (1), $\Omega$ is the Shishkin mesh (6), (7). Then, for some constant $C$, the error estimates are valid:

$$\varepsilon |u'(x) - S_M'(x,u)| \leq C\frac{\ln^3 N}{N^3}, \quad x \leq \sigma,$$

$$|u'(x) - S_M'(x,u)| \leq C\left[\frac{1}{N^3} + \frac{1}{\varepsilon N^4} e^{-\alpha(x-\sigma)/\varepsilon}\right], \quad x > \sigma,$$

$$\varepsilon^2 |u''(x) - S_M''(x,u)| \leq C\frac{\ln^2 N}{N^2}, \quad x \leq \sigma,$$

$$|u''(x) - S_M''(x,u)| \leq C\left[\frac{1}{N^2} + \frac{1}{\varepsilon^2 N^4} e^{-\alpha(x-\sigma)/\varepsilon}\right], \quad x > \sigma.$$ 

In accordance with this theorem, relative estimates of the errors in the region of large gradients of the function and absolute estimates of the error outside the region of the boundary layer are obtained. Close estimates for the error in calculating derivatives on the Shishkin mesh are also valid for the cubic spline $S_3(x,u)$.

4. Bakhvalov mesh

The Bakhvalov mesh was developed in [2] for constructing an $\varepsilon$ - uniformly converging difference scheme for the numerical solution of a singularly perturbed problem. It is of interest whether this mesh is applicable to function interpolation in the presence of a boundary layer. Let’s define the Bakhvalov mesh of the interval $[0,1]$, based on the [2] approach with the [12] specification. We take into account the decomposition (1) for the interpolated function $u(x)$.

Let $\sigma = \min \left\{ \frac{1}{2}, \frac{4\varepsilon}{\alpha} \ln \frac{1}{\varepsilon} \right\}$ for $\varepsilon \leq e^{-1}$ and $\sigma = 1/2$ for $\varepsilon > e^{-1}$.

In the case of $\sigma < \frac{1}{2}$, we define the nodes of the $\Omega$ by the formula $x_n = g(n/N)$, $n = 0, 1, \ldots, N$, where

$$g(t) = \left\{ \begin{array}{ll}
-\frac{4\varepsilon}{\alpha} \ln \left[ 1 - 2(1 - \varepsilon)t \right], & 0 \leq t \leq \frac{1}{2}, \\
\sigma + (2t - 1)(1 - \sigma), & 1/2 \leq t \leq 1.
\end{array} \right. \quad (11)$$

Thus, for $\sigma < \frac{1}{2}$ the grid $\Omega$ is uniform on the interval $[\sigma,1]$ with step $h = 2(1 - \sigma)/N$. In the region of the boundary layer, for $0 \leq n \leq N/2$, the grid nodes in accordance with (11) are set in the form

$$x_n = -\frac{4\varepsilon}{\alpha} \ln \left[ 1 - 2(1 - \varepsilon)\frac{n}{N} \right], \quad \sigma = x_{N/2} = -\frac{4\varepsilon}{\alpha} \ln \varepsilon.$$ 

In the case $\sigma = \frac{1}{2}$, the grid $\Omega$ is defined as uniform with step $h = 1/N$.

So, the Bakhvalov mesh $\Omega$ is set. Define a cubic spline $S_3(x,u) \in S(\Omega, 3, 1)$. Define a spline on the mesh $\Omega$ from the interpolation conditions (4).

The following theorem is proved in [13].

Theorem 6 There are constants $C, C_1$ and $\beta > 0$ that do not depend on $\varepsilon, N$ such that for $\varepsilon \leq C/N$ the following estimate holds

$$\| u(x) - S_3(x,u) \|_{C[x_n,x_{n+1}]} \leq C_1 \left\{ \begin{array}{ll}
N^{-4}, & 0 \leq n \leq N/2 - 2, \\
N^{-4} \ln \left( 1 + \frac{1}{\varepsilon N} \right) + 1/N^4, & n = N/2 - 1, \\
\frac{N^{-5}}{\varepsilon^2} e^{-\beta(n-N/2)} + 1/N^4, & N/2 \leq n \leq N - 1.
\end{array} \right. \quad (12)$$

The resulting error estimate (12) is nonuniform with respect to the small parameter $\varepsilon$ on mesh intervals when leaving the boundary layer.

In the case of $\varepsilon \geq C/N$, in accordance with [14], the following theorem is true.
Theorem 7 For an arbitrary constant $C$, there is a constant $C_1$ such that, for $\varepsilon \geq C/N$, the following error estimate holds

$$\| u(x) - S_3(x, u) \|_{[0,1]} \leq C_1/N^4.$$ 

Due to the nonuniform convergence in $\varepsilon$ of the cubic spline $S_3(x, u)$ according to estimates (12), by analogy with [10], we define a modified interpolation spline built on the basis of the shift of two interpolation nodes. With such a modification, the matrix of the system of equations for finding the coefficients in the expansion of the spline through the basis splines becomes strictly diagonally dominant. This leads to a $\varepsilon$-uniform estimate of the spline error.

So, put

$$\bar{x}_n = (x_n + x_{n+1})/2, \ n \in [N/2 - 1, N/2], \ \bar{x}_n = x_n, \ n \in [0, N/2 - 2] \cup [N/2 + 1, N].$$

Let $\tilde{S}_3(x, u)$ be an interpolation cubic spline determined from the conditions

$$\tilde{S}_3(\bar{x}_n, u) = u(\bar{x}_n), \ n \in [0, N], \tilde{S}_3'(0, u) = u'(0), \ \tilde{S}_3'(1, u) = u'(1).$$

Theorem 8 There are constants $C, C_1$, independent of $\varepsilon, N$ such that the following estimate holds for $\varepsilon \leq C/N$

$$\| u(x) - \tilde{S}_3(x, u) \|_{[0,1]} \leq C_1 N^{-4}.$$ 

Remark 1. We can assume that in Theorems 7 and 8 the constant $C$ is the same. Otherwise, it is enough to take the minimum of these constants as $C$.

Remark 2. By virtue of Theorems 7 and 8, the application of the interpolation spline $\tilde{S}_3(x, u)$ for $\varepsilon = O(1/N)$ and interpolation spline $S_3(x, u)$ for $1/N = O(\varepsilon)$ allows to obtain uniform in $\varepsilon$ error estimates of order $O(N^{-4})$.

Calculation of derivatives. In [15] estimates of the error in calculating derivatives on the basis of constructing a cubic spline $S_3(x, u)$ on a Bakhvalov mesh are obtained.

Theorem 9 Suppose that the function $u(x)$ satisfies the representation (1), (2), $\Omega$ is a given Bakhvalov mesh. Then, for some constant $C$, the error estimate for $x \in [x_{n-1}, x_n]$ is valid depending on the value of $n$:

$$\varepsilon |S_3'(x, u) - u'(x)| \leq \frac{C}{N^3}, \ n < \frac{N}{2},$$

$$\varepsilon |S_3''(x, u) - u''(x)| \leq \frac{C}{N^4} \ln^3 \left( 1 + \frac{K}{N\varepsilon} \right) + \frac{C\varepsilon}{N^3}, \ n = \frac{N}{2},$$

$$|S_3'(x, u) - u'(x)| \leq \frac{C}{N^4 \varepsilon} e^{-\beta(n-N/2)} + \frac{C}{N^3}, \ n > \frac{N}{2}, \ \beta > 0, \ n = \frac{N}{2}.$$
5. Fitting a spline to a singular component
Consider a modification of a cubic spline based on fitting to the singular component responsible for large gradients of the function in the boundary layer. We will assume that the following decomposition is valid for the function \( u(x) \):

\[
u(x) = q(x) + \gamma \Phi(x), \quad |q^{(j)}(x)| \leq C_1, \quad 0 \leq j \leq 4, \quad \Phi(x) = e^{-\alpha x/\varepsilon}, \quad \alpha > 0, \quad x \in [0, 1]. \tag{13}\]

We assume that in the representation (13) the regular component \( q(x) \) and the constant \( \gamma \) are not specified. Derivatives of function \( \Phi(x) \) grow unboundedly near the boundary \( x = 0 \) if \( \varepsilon \to 0 \). According to [16], the decomposition (13) is valid for the solution of singularly perturbed problem (3), with \( \alpha = a_1(0) \). In the general case, in accordance with [16], the derivatives of the function \( q(x) \), starting from the second, can grow with decreasing value of the parameter \( \varepsilon \). This case requires additional investigation.

Let \( \Omega \) be a uniform grid of the interval \([0, 1]\) with nodes \( x_n, \ n = 0, 1, \ldots, N \) and step \( h \). Let us define the space of \( L \)-splines exact on the singular component \( \Phi(x) \) from (13):

\[
SL(\Omega, 3, 1) = \{ S(x) \in C^2[0, 1] : S(x) = a_n + b_n x + c_n x^2 + d_n e^{-\alpha x/\varepsilon}, \quad x \in [x_n, x_{n+1}], 0 \leq n \leq N-1 \}.
\]

We define the interpolation \( L \)-spline \( S_L(x, u) \in SL(\Omega, 3, 1) \) from the conditions

\[
S_L(x_n, u) = u(x_n), \quad 0 \leq n \leq N, \quad S_L''(0, u) = u''(0), \quad S_L''(1, u) = u''(1).
\]

According to [17], the following theorem is true.

**Theorem 10** Let the function \( u(x) \) have the decomposition (13). Then for the spline \( S_L(x, u) \) the error estimates are valid

\[
\| S_L(x, u) - u(x, \varepsilon) \|_{C[0,1]} \leq C \begin{cases} \min \{ h^3, \frac{h^4}{\varepsilon} \}, & \varepsilon \in (0, 1], \\ h^4, & \varepsilon \in (1, +\infty). \end{cases} \tag{14}\]

In accordance with the estimate (14), for \( \varepsilon = 1 \), the error of the spline is of the order of \( O(h^4) \). This is consistent with the fact that as \( \varepsilon/h \) increases, the spline becomes cubic [17]. For \( \varepsilon \leq h \), the spline error becomes of the order of \( O(h^3) \).

**Calculation of derivatives.** In [18], estimates of the error in calculating derivatives based on the differentiation of the spline \( S_L(x, u) \) are obtained.

**Theorem 11** Suppose that the function \( u(x) \) satisfies the decomposition (13), the grid is uniform. Then, for some constant \( C \), the following error estimates hold

\[
\| S_L^{(j)}(x, u) - u^{(j)}(x) \|_{C[0,1]} \leq C \min \{ h^{3-j}, \frac{h^{4-j}}{\varepsilon} \}, \quad j = 1, 2, \quad \varepsilon \in (0, 1].
\]

6. Numerical results
Let’s define a function of the form (1)

\[
u(x) = \cos \frac{\pi x}{2} + e^{-\frac{x}{\varepsilon}}, \quad x \in [0, 1].
\]

The tables show the maximum errors of spline interpolation calculated in nodes of the condensed grid, obtained from the original computational grid dividing each of its grid intervals into 10 equal parts.
Table 1. Cubic spline error on a uniform mesh

| $\varepsilon$ | 16   | 32   | 64   | 128  | 256  | 512  |
|---------------|------|------|------|------|------|------|
| 1             | $2.82 \cdot 10^{-7}$ | $1.76 \cdot 10^{-8}$ | $1.16 \cdot 10^{-9}$ | $1.02 \cdot 10^{-10}$ | $4.30 \cdot 10^{-12}$ | $2.68 \cdot 10^{-13}$ |
| $10^{-1}$     | $3.43 \cdot 10^{-4}$ | $2.33 \cdot 10^{-5}$ | $1.51 \cdot 10^{-6}$ | $9.58 \cdot 10^{-8}$  | $6.03 \cdot 10^{-9}$  | $4.11 \cdot 10^{-10}$ |
| $10^{-2}$     | 0.43  | 8.38  | $10^{-2}$ | 9.72  | $10^{-3}$ | 8.00  | $10^{-4}$ |
| $10^{-3}$     | 0.98  | 5.48  | 1.93  | 0.66  | 0.15  | 2.03  | $10^{-2}$ |
| $10^{-4}$     | 1.05  | 5.23  | $10^{1}$ | 2.58  | $10^{1}$ | 1.25  | $10^{1}$ |
| $10^{-5}$     | 0.76  | 5.30  | $10^{2}$ | 2.64  | $10^{2}$ | 1.32  | $10^{2}$ |
| $10^{-6}$     | 0.66  | 5.30  | $10^{3}$ | 2.65  | $10^{3}$ | 1.33  | $10^{3}$ |
| $10^{-7}$     | 0.66  | 5.30  | $10^{4}$ | 2.65  | $10^{4}$ | 1.33  | $10^{4}$ |
| $10^{-8}$     | 0.66  | 5.30  | $10^{5}$ | 2.65  | $10^{5}$ | 1.33  | $10^{5}$ |

Table 2. The error of a cubic spline on the Shishkin mesh

| $\varepsilon$ | 16   | 32   | 64   | 128  | 256  | 512  |
|---------------|------|------|------|------|------|------|
| 1             | $2.82 \cdot 10^{-7}$ | $1.76 \cdot 10^{-8}$ | $1.16 \cdot 10^{-9}$ | $1.02 \cdot 10^{-10}$ | $4.30 \cdot 10^{-12}$ | $2.68 \cdot 10^{-13}$ |
| $10^{-1}$     | $3.43 \cdot 10^{-4}$ | $2.33 \cdot 10^{-5}$ | $1.51 \cdot 10^{-6}$ | $9.58 \cdot 10^{-8}$  | $6.03 \cdot 10^{-9}$  | $4.11 \cdot 10^{-10}$ |
| $10^{-2}$     | 0.43  | 8.38  | $10^{-2}$ | 9.72  | $10^{-3}$ | 8.00  | $10^{-4}$ |
| $10^{-3}$     | 0.98  | 5.48  | 1.93  | 0.66  | 0.15  | 2.03  | $10^{-2}$ |
| $10^{-4}$     | 1.05  | 5.23  | $10^{1}$ | 2.58  | $10^{1}$ | 1.25  | $10^{1}$ |
| $10^{-5}$     | 0.76  | 5.30  | $10^{2}$ | 2.64  | $10^{2}$ | 1.32  | $10^{2}$ |
| $10^{-6}$     | 0.66  | 5.30  | $10^{3}$ | 2.65  | $10^{3}$ | 1.33  | $10^{3}$ |
| $10^{-7}$     | 0.66  | 5.30  | $10^{4}$ | 2.65  | $10^{4}$ | 1.33  | $10^{4}$ |
| $10^{-8}$     | 0.66  | 5.30  | $10^{5}$ | 2.65  | $10^{5}$ | 1.33  | $10^{5}$ |

Table 3. The error of the modified cubic spline on the Shishkin mesh

| $\varepsilon$ | 16   | 32   | 64   | 128  | 256  | 512  |
|---------------|------|------|------|------|------|------|
| 1             | $3.1 \cdot 10^{-7}$ | $2.0 \cdot 10^{-8}$ | $1.3 \cdot 10^{-9}$ | $1.1 \cdot 10^{-10}$ | $4.9 \cdot 10^{-12}$ | $3.1 \cdot 10^{-13}$ |
| $10^{-1}$     | $3.4 \cdot 10^{-4}$ | $2.3 \cdot 10^{-5}$ | $1.5 \cdot 10^{-6}$ | $9.6 \cdot 10^{-8}$  | $6.0 \cdot 10^{-9}$  | $4.1 \cdot 10^{-10}$ |
| $10^{-2}$     | 0.43  | 8.38  | $10^{-2}$ | 9.72  | $10^{-3}$ | 8.00  | $10^{-4}$ |
| $10^{-3}$     | 0.98  | 5.48  | 1.93  | 0.66  | 0.15  | 2.03  | $10^{-2}$ |
| $10^{-4}$     | 1.05  | 5.23  | $10^{1}$ | 2.58  | $10^{1}$ | 1.25  | $10^{1}$ |
| $10^{-5}$     | 0.76  | 5.30  | $10^{2}$ | 2.64  | $10^{2}$ | 1.32  | $10^{2}$ |
| $10^{-6}$     | 0.66  | 5.30  | $10^{3}$ | 2.65  | $10^{3}$ | 1.33  | $10^{3}$ |
| $10^{-7}$     | 0.66  | 5.30  | $10^{4}$ | 2.65  | $10^{4}$ | 1.33  | $10^{4}$ |
| $10^{-8}$     | 0.66  | 5.30  | $10^{5}$ | 2.65  | $10^{5}$ | 1.33  | $10^{5}$ |

Table 1 shows the errors for the cubic spline on a uniform grid. The estimates (5) of theorem 1 and inadmissibility of using a cubic spline on a uniform mesh for small $\varepsilon$ are confirmed.

Table 2 shows the errors of the cubic spline on the Shishkin mesh. It follows from the table that the error increases with decreasing $\varepsilon$ for fixed $N$, which corresponds to the estimate (9).
The results of table 3 for a cubic spline with one shifted interpolation node of the Shishkin mesh demonstrate uniform in $\varepsilon$ convergence, which corresponds to the error estimates in theorem 4.
The errors of calculating the derivatives are shown in tables 6-8.

Table 6. Error in calculating the first derivative on a uniform grid

| $\varepsilon$ | $N$ | 16 | 32 | 64 | 128 | 256 | 512 |
|---------------|-----|----|----|----|-----|-----|-----|
| $10^{-1}$     |     | 3.84e-5 | 4.81e-6 | 6.01e-7 | 7.52e-8 | 9.40e-9 | 1.17e-9 |
| $10^{-2}$     |     | 4.61e-3 | 6.29e-4 | 8.18e-5 | 1.04e-5 | 1.32e-6 | 1.65e-7 |
| $10^{-3}$     |     | 8.85e-1 | 2.59e-1 | 5.36e-2 | 8.59e-3 | 1.20e-3 | 1.58e-4 |
| $10^{-4}$     |     | 1.22e+1 | 6.09 | 2.92 | 1.23 | 4.00e-1 | 9.21e-2 |

Table 7. The error in calculating the first derivative on the Shishkin mesh

| $\varepsilon$ | $N$ | 16 | 32 | 64 | 128 | 256 | 512 |
|---------------|-----|----|----|----|-----|-----|-----|
| $10^{-1}$     |     | 3.84e-5 | 4.81e-6 | 6.01e-7 | 7.52e-8 | 9.40e-9 | 1.17e-9 |
| $10^{-2}$     |     | 4.61e-3 | 6.29e-4 | 8.18e-5 | 1.04e-5 | 1.32e-6 | 1.65e-7 |
| $10^{-3}$     |     | 3.16e-1 | 2.45e-2 | 3.65e-3 | 4.94e-4 | 6.41e-5 | 8.15e-6 |
| $10^{-4}$     |     | 5.37e-1 | 1.35e-1 | 2.45e-2 | 3.65e-3 | 4.94e-4 | 6.41e-5 |

Table 8. The error in calculating the first derivative on the Bakhvalov mesh

| $\varepsilon$ | $N$ | 16 | 32 | 64 | 128 | 256 | 512 |
|---------------|-----|----|----|----|-----|-----|-----|
| $10^{-1}$     |     | 3.84e-5 | 4.81e-6 | 6.01e-7 | 7.52e-8 | 9.40e-9 | 1.17e-9 |
| $10^{-2}$     |     | 4.61e-3 | 6.29e-4 | 8.18e-5 | 1.04e-5 | 1.32e-6 | 1.65e-7 |
| $10^{-3}$     |     | 2.86e-3 | 3.52e-4 | 4.36e-5 | 5.43e-6 | 6.78e-7 | 8.47e-8 |
| $10^{-4}$     |     | 2.87e-3 | 3.53e-4 | 4.37e-5 | 5.45e-6 | 6.80e-7 | 8.49e-8 |

Table 4 shows the error and the calculated order of accuracy of a cubic spline on a Bakhvalov mesh with a shift of two interpolation nodes. The calculation results correspond to the fourth order of accuracy for all $\varepsilon$ and $N$ and agree with the estimate in theorem 8.

Table 5 shows the error and the order of accuracy of the exponential spline on a uniform grid. The calculation results show that as $\varepsilon$ decreases, the order of accuracy decreases from the fourth to the third, which is consistent with the estimate (14).

When calculating the derivatives of the function specified at the grid nodes, based on the differentiation of the spline, the following results are obtained. When the spline has the error that is not uniform in the parameter $\varepsilon$, then its application for calculating derivatives also leads to significant errors. Applying a cubic spline on the Shishkin and Bakhvalov meshes and the use of an exponential spline on a uniform grid gives the error in the calculation of derivatives uniform in the parameter $\varepsilon$. The greatest accuracy is obtained by using the Bakhvalov mesh. The errors of calculating the derivatives are shown in tables 6-8.
7. Conclusion
A comparison of the developed approaches to the interpolation of functions with large gradients in the exponential boundary layer is carried out. The use of a cubic spline on a uniform grid is unacceptable in the presence of an exponential boundary layer. The use of a cubic spline on the Shishkin and Bakhvalov meshes does not provide a uniform spline error in a small parameter. The error grows indefinitely at $\varepsilon \to 0$ when leaving the region of high gradients. A shift of one interpolation node for the Shishkin mesh and two interpolation nodes for the Bakhvalov mesh leads to the fact that the error of a cubic spline becomes uniform in the parameter $\varepsilon$. In the case of a uniform grid, the use of an analog of a cubic spline, constructed on the basis of fitting to the singular component of the interpolated function, gives an error of the order $O(N^{-3})$, uniform in the parameter $\varepsilon$. When $\varepsilon \gg 1/N$, the exponential spline becomes cubic and the error becomes of the order of $O(N^{-4})$. In accordance with the error estimates and the results of numerical experiments the cubic spline on the Bakhvalov mesh is more accurate.

Estimates of the error in calculating the derivatives of a function given at the grid nodes, based on spline interpolation, are given. It was found that if the error of the spline is uniform in the small parameter, then the error in calculating the derivatives is also uniform in the small parameter. The use of a uniform grid for calculating derivatives in the case of a cubic spline is unacceptable if the function has large gradients.

Acknowledgments
The research was funded in accordance with the state task of the IM SB RAS, project FWNF-2022-0016, and by RFBR (project 20-01-00650).

References
[1] Il’in A M 1969 Differencing scheme for a differential equation with a small parameter affecting the highest derivative Math. Notes 6 (2) pp 596-602
[2] Bakhvalov N S 1969 The optimization of methods of solving boundary value problems with a boundary layer USSR Comput. Math. Math. Phys. 9 pp 139-166
[3] Shishkin G I 1992 Grid Approximations of Singularly Perturbed Elliptic and Parabolic Equations (Yekaterinburg: Ural. Otd. Ross. Akad. Nauk) (in Russian)
[4] Zavyalov Y S, Kvasov B N and Miroshnichenko V L 1981 Methods of Spline Functions (Moscow: Nauka) (in Russian)
[5] Ahlberg J H, Nilson EN and Walsh J L 1967 The Theory of Splines and Their Applications (New York: Academic Press)
[6] Boor C De 1985 Practical Guide to Splines (New York: Springer-Verlag)
[7] Lin J T. 2010 Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems ( Berlin: Springer)
[8] Flaherty J E and Mathon W 1980 Collocation with polynomial and tension splines for singularly-perturbed boundary value problems Siam Journal on Scientific and Statistical Computing 1-2 pp 260-289
[9] Lin T 2001 The necessity of Shishkin decompositions Applied Mathematics Letters 14 pp 891-896
[10] Blatov I A, Zadorin A I and Kitaeva E V 2017 Cubic spline interpolation of functions with high gradients in boundary layers Comput. Math. Math. Phys. 57 (1) pp 7-25
[11] Blatov I A, Zadorin A I and Kitaeva E V 2019 Approximation of a function and its derivatives on the basis of cubic spline interpolation in the presence of a boundary layer Comput. Math. Math. Phys. 59 (3) pp 343-354
[12] Roos H G 2019 Layer-adapted meshes: milestones in 50 years of history Appl. Math. arXiv:1909.08273v1
[13] Blatov Igor, Kitaeva Elena and Zadorin Nikita 2021 Cubic spline on a Bakhvalov mesh in the presence of a boundary layer Lecture Notes in Computational Science and Engineering 141 pp 39-54
[14] Blatov I A, Dobrobo N V and Kitaeva N V 2021 The cubic interpolation spline for functions with boundary layer on a Bakhvalov mesh J. Phys.: Conf. Ser. 1715 012001
[15] Blatov I A and Zadorin A I 2021 Application a cubic spline to calculate derivatives in the presence of a boundary layer J. Phys.: Conf. Ser. 1791 012009
[16] Kellog R B and Tsan A 1978 Analysis of some difference approximations for a singular perturbation problem without turning points Math. Comput. 32 (144) pp 1025-1039
[17] Blatov I A, Zadorin A I and Kitaeva E V 2018 On the parameter-uniform convergence of exponential spline interpolation in the presence of a boundary layer Comput. Math. Math. Phys. 58 (3) pp 348-363
[18] Blatov I A, Zadorin A I and Kitaeva E V 2018 An application of the exponential spline for the approximation of a function and its derivatives in the presence of a boundary layer  
*J. Phys.: Conf. Ser.* **1050** 012012