On strongly continuous $\rho h$-semigroup

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Abstract.

In this paper, we introduce a semi group which it constructs the solution of the partial differential equation as the form:

$$\frac{\rho(t) \partial u(t,x)}{\partial t} = \frac{h(t) \partial u(t,x)}{\partial x}, \quad h(0) = 1, \quad \rho(0) = 1$$

First, we introduce the operator theory and the fundamental theorems of the semigroup and certain notions of strongly continuous operators. These concepts are particular types of operator semigroups of functional analytic. Using functional analytic tools and methods from ergodic theory, we describe various features of the On strongly continuous $\rho h$-semigroup.

Key Words. semigroup, strongly continuous, operators, generator operator, $C_0$-semigroup.

Mathematics subject classification. 47Dxx
1. Introduction.

Many Scientists ([1],[2],[3]) introduce several generations of analysts working in the area of operator semigroups. In particular, the progress has been made in the asymptotic theory of strongly continuous semigroups. One of the major results in this direction a strongly continuous semigroup on a Banach space with the norm of the resolvent of its generator A is uniformly bounded in the right half-plane.

We consider the following equations:

\[
\rho(t) \frac{\partial u(t,x)}{\partial t} = h(t) \frac{\partial u(t,x)}{\partial x}, \quad h(0) = 1, \quad \rho(0) = 1
\]

We introduce a new type of semigroup namely (Multiplicative Canonical semigroup) and its define by:

\[
U_{\rho h}(t) \phi(x) = \phi[h^{-1}(h(x), \rho(t))]
\]

And also we introduce strongly continuous generalized canonical semigroup defined by:

\[
T_{\rho h}(t) \phi(x) = \exp \left[ \int_{h(x) \otimes t}^x \rho(\xi) d\xi \right] \phi[h(x) \otimes t]
\]

2. Strongly continuous semigroup and its generator operator.

2.1 Definition [6].

The function \(f(t)\) is called continuous at a point \(t_0\) if \(\|f(t) - f(t_0)\| \rightarrow 0\), at \(t \rightarrow t_0\), continuous on the interval \([a,b]\), if it is continuous at each point of this segment.

2.2 Definition [5].

The function \(f(t)\) is called differentiable in point \(t_0\), if there is an element \(f' \in E\) such that:

\[
\left\| \frac{f(t_0 + h) - f(t_0)}{h} - f' \right\|_E \rightarrow 0, \quad at \ h \rightarrow 0
\]

The element \(f'\) is called the derivative of the function \(f(t)\) at point \(t_0\) and denoted by.

\[
f' = f'(t_0).
\]
2.3 Definition [7].

We will say that the operator function $A(t)$ is continuous in norm at point $t_0 \in [a,b]$ if

$$
\lim_{t \to t_0} \|A(t) - A(t_0)\|_E = 0.
$$

2.4 Definition [8].

The operator-function $A(t)$ is strongly continuous in a point $t_0 \in [a,b]$ if at any fixed $x \in E_1$

$$
\lim_{t \to t_0} \|A(t)x - A(t_0)x\|_{E_2} = 0.
$$

2.5 Theorem (Banach–Shteghaus) [8].

An operator–function $A(t)$ is strongly continuous at $t_0 \in [a,b]$ on all $E_1$ if its norms are bounded, i.e.

$$
\|A(t)\| \leq M
$$

2.6 Definition [9].

We say that an operator $A$ is closed if for every $x_n \in D(A)$, then $\|x_n - x_0\| \to 0$ and $Ax_0 = y_0$.

2.7 Definition [8].

A family of bounded operators $T(t)$ ($t > 0$), define on the Banach space $E$, is called strongly continuous semigroup of operators if $T(t)$ strongly continuous and satisfies the condition $T(t)T(s) = T(t+s)$ ($t, s > 0$).

2.8 Definition [6].

It is said that $T(t)$ is a semigroup of class $C_0$ if it is strongly continuous and the following condition holds

$$
\lim_{t \to 0} \|T(t)x - x\|_E = 0.
$$

for any $x \in E$. 

3
2.9 Theorem [4].

The linear operator A is a generating operator (generator) of a semigroup T(t) of class C_0 iff its closed with a dense in E.

2.10 Definition[6].

A family of bounded operators T(t) (t >0), define on the Banach space E, is called strongly continuous multiplicative semigroup of operators if T(t) strongly continuous and satisfies the conditions.

1) T(1)=I
2) T(t)T(s) = T(t.s) (t, s > 0).

3. Multiplicative Canonical semigroup.

3.1 Definition.

Let t ∈ (t_1, t_2) ⊂ R, x ∈ (a,b) ⊂ R, ρ(t) and h(t) are real functions with domains D (ρ) = (t_1, t_2), D (h) = (a, b), continuously differentiable and strictly monotone. In addition, h(x), ρ(t) ∈ D(h^{-1}) ∩ D(ρ^{-1}) , where h^{-1} and ρ^{-1} - inverse functions.

Consider the differential equation.

\[ \frac{\rho(t)}{\dot{h}(t)} \frac{\partial u(t,x)}{\partial t} = \frac{h(t)}{h(t)} \frac{\partial u(t,x)}{\partial x}, \quad h(0) = 1, \quad \rho(0) = 1 \quad \cdots \cdots \cdots (1) \]

It is easy to see that the general solution of this equation is.

\[ u(t,x) = \varphi(h(x),\rho(t)) \quad \cdots \cdots \cdots \cdots (2) \]

Where \( \varphi \) is an arbitrary differentiable function. we can assign the one-parameter equation (1) to a one-parameter family of operators.

\[ U_{\varphi h}(t)x = \varphi[h^{-1}(h(x),\rho(t))] \quad \cdots \cdots \cdots (3) \]

under the assumption that \( \varphi \) belongs to the space of continuous and bounded functions \( C(a,b) \) with the norm.

\[ \| \varphi \| = \sup_{x \in (a,b)} | \varphi(x) | \]
3.2 Definition.

We define a binary operation \( \bigcirc^h \) by.

\[
x \bigcirc^h t = h^{-1}(h(x), h(t)) \quad \ldots \quad (4)
\]

We will prove that \( U_{ph}(t) \) defined by (3) is a semigroup of linear and bounded in \( C(a, b) \).

3.3 Lemma.

The operational family \( U_{ph}(t) \) defined by (3) is a semigroup of linear and bounded in \( C(a, b) \) of operators with the binary operation in (4).

Proof.

We note that.

1. \( U_{ph}(0) = \emptyset[h^{-1}(h(x), \rho(0))] = \emptyset[h^{-1}(h(x))] = \emptyset[x] \)

2. \( U_{ph}(t)U_{ph}(s)\emptyset(x) = U_{ph}(t)\emptyset[h^{-1}(h(x), \rho(s))] = \emptyset[h^{-1}(h(x), \rho(t), \rho(s))] = \emptyset[h^{-1}(h(x), \rho(t \bigcirc^s))] = U_{ph}(t \bigcirc^s)\emptyset(x) \)

Therefore \( U_{ph}(t) \) is semigroup operator.

3.4 Remark.

The semigroup \( U_{ph}(t) \) is called a Multiplicative Canonical semigroup.

3.5 Remark.

The function \( \rho(t) \) which given in the semigroup \( U_{ph}(t) \) is invariant relative to the functions \( h(x) \) on \( h(x) + c \), where \( c \)-constant.
3.6 Definition.

The semigroup $U_{\rho h}(t)$ is called $\rho h$-semigroup and equation (1) is its generating equation. We note that the function $\rho(t)$ it is possible to select such that the equation (1) generates a family $\rho h$-semigroup.

In the following proposition we show that $\rho h$-semigroup has a fixed point.

3.7 Proposition.

Let $\rho h$-semigroup generated by the equation (1) then there exists a point $t_0 \in (t_1, t_2)$, such that $U_{\rho h}(t_0)\varphi(x) = \varphi(x)$.

Proof.

It follows from continuity and monotony of the function $\rho_c(t) = \rho(t)c$, which with the appropriate selection of constant $c$, it becomes zero at unique point $t_0 \in (t_1, t_2)$. In this case, it follows that the Cauchy problem for equation (1) with the initial condition $u(t_0, x) = \varphi(x)$ has a unique solution and it can be represented as

$$u(x, t) = U_{\rho h}(t_0)\varphi(x) \ldots \ldots \ldots (5)$$

3.8 Definition.

If $\rho(t) = h(t)$ then the semigroup $\rho h$-semigroup can be written by the form.

$$U_{hh}(t)\varphi(x) = \varphi(t \Box h x) \ldots \ldots \ldots (6)$$

And its called $h$-symmetric and the equation (1) is called symmetric generating equation.

In the following lemma we will prove that the family $hh$-semigroup has one symmetric semigroup.
3.9 Lemma.

the family of semigroups produced by symmetric differential equation, contains only one symmetric semigroup.

Proof.

Let $U_h(t)\Phi(x) = \emptyset[ h^{-1}(h(x).h(t)) ]$, for $c \neq 1$ and $h_c(t) = h(t)c$, thus we have.

$$U_{h_c}(t)\Phi(x) = \emptyset[ h_c^{-1}(h_c(x).h_c(t)) ] = \emptyset[ h^{-1}(h(x).h(t)c) ]$$

Therefore $U_{h_c}$ is not symmetric for $c \neq 1$.

There exists a special cases of partial differential equation can be solved by another method and we take some of these cases in the following examples.

3.10 Examples.

1. $(t+1)\frac{\partial u(t,x)}{\partial t} = (x + 1) \frac{\partial u(t,x)}{\partial x}$, $h(x) = x+1$

2. $\frac{\partial u(t,x)}{\partial t} = \frac{\partial u(t,x)}{\partial x}$, $h(x) = e^x$

3. $\cot t \frac{\partial u(t,x)}{\partial t} = \cot x \frac{\partial u(t,x)}{\partial x}$, $h(x) = \cos(x)$

3.11 Definition.

The semigroup $U_{\rho h}(t)$ is called a strongly continuous at a point $t_0 \in (t_1, t_2)$, if for all $\varphi \in E$ the inequality holds.

$$\lim_{t \to t_0} \| U_{\rho h}(t)\varphi - \varphi \|_E = 0 \quad \ldots \quad \ldots \quad \ldots \quad (7)$$
3.12 Definition.

The semigroup \( U_{\rho h}(t) \), \( \rho(t) = t \) is a class

\[
U_h^{(0)}(t) = U_{\rho h}(t) = U_{th}(t) = \phi[h^{-1}(h(x), t)]
\]

We note that \( U_h^{(0)}(t) \) with the binary operation \( x \bigcirc y = x \cdot y \) is called Arithmetic semigroup.

3.13 Definition.

Let \( f(t) \) be a vector function, define on \( t \in (t_1, t_2) \) with valued in \( E \). \( \mu(t) \) be a strictly monotonic function define on \( D(\mu) \subseteq R, R(\mu) = (t_1, t_2) \), then a function \( g(t) = f(\mu(t)) \) is called \( \mu \)-deformation of function \( f(t) \).

3.14 Remark.

Every \( \rho h \)-semigroup is \( \rho \)-deformation of semigroup \( U_h^{(0)}(t) \).

3.15 Definition.

The function \( \varphi \in C(a,b) \) is called uniformly continuous if its \( \mu^{-1} \)-deformation \( \psi = \phi(\mu(x)) \) is bounded and uniformly continuous function.

we note that

\[
\| \phi \|_{C(a,b)} = \sup_{x \in (a,b)} |\phi(x)| = \sup_{s \in (\mu^{-1}(a), \mu^{-1}(b))} |\phi(\mu(s))| = \| \psi \|_{C\mu}
\]

In the following proposition we prove that the function \( U_h^{(0)}(t) \) is continuous function.

3.16 Proposition.

Every \( U_h^{(0)}(t) \) strongly continuous semigroup in the space of \( h^{-1} \)-uniformly continuous functions.
Proof.

We note that

\[
\| U_h^{(0)}(t) \phi(x) - \phi(x) \| = \sup_{x \in (a,b)} |\phi[h^{-1}(h(x), t)] - \phi[h^{-1}(h(x))]| \\
= \sup_{\tau \in (h^{-1}(a), h^{-1}(b))} |\psi(\tau + t) - \psi(t)| = \| \psi(\tau + t) - \psi(t) \| \rightarrow 0, t \rightarrow 0
\]

Now we can get the form of A generator operator of the semigroup $U_h^{(0)}(t)$ as the following theorem.

3.1.7 Theorem.

A generator operator of the semigroup $U_h^{(0)}(t)$ given by the differential expression.

\[
L \phi(x) = \frac{h(x)}{h'(x)} \frac{\partial \phi}{\partial x}, h(x) = 0 ; 0 < x < 1, \lim_{x \to b} h(x) = \infty
\]

And a domain $D(A_h^{(0)}) = \{ \phi, \phi \in C_{h^{-1}}, L \phi \in C_{h^{-1}} \}$.

Proof.

We have:

\[
R(\lambda, A) = (\lambda I - A)^{-1}, \quad Re(\lambda) > \omega
\]

Thus.

\[
R(\lambda, A_h^{(0)}) = \int_0^\infty e^{-\lambda t} U_h^{(0)}(t) \phi(x) dt
\]

\[
\| R(\lambda, A_h^{(0)}) \| < \frac{1}{Re(\lambda)}, \quad Re(\lambda) > 0
\]

\[
\forall n \in \mathbb{N}, \quad J_n = nR(n, A_h^{(0)})
\]

Thus we have.

\[
A_h^{(0)} J_n = n(J_n - I)
\]
\[ y_n(x) = (J_n \phi)(x) = n \int_0^\infty e^{-nt} u_h^{(0)}(t) \phi(x) dt = n \int_1^\infty e^{-nt} u_h^{(0)}(t) \phi(x) dt = n \int_0^\infty e^{-nt} \phi[h^{-1}(h(x) \cdot t)] dt = \int_x^b e^{-n[h(y)/h(x)]} h'(\tau) \phi(\tau) d\tau \]

\[ y_n(x) = n(J_n - I) \phi(x) = A_h^{(0)} y_n(x) \]

3.18 Definition.

Let \((a, b) \subseteq R\) be an interval and let \(h(x)\) be a differentiable function such that \(\lim_{x \to b} h(x) = \infty\), we define a space \(L_{p,\omega,h}\) by

\[
L_{p,\omega,h} = \left\{ \phi \mid \|\phi\|_{p,\omega,h,g} = \left[ \int_a^b |\exp[\omega h(x)] g(x) \phi(x)|^p d\phi(x) \right]^\frac{1}{p} \right\}, \quad p \geq 1, \omega > 0, g(x) > 0, g'(x)
\]

On the space \(L_{p,\omega,h}\) we define the following family of operators.

\[ T_{\rho h}(t) \phi(x) = \exp \left[ \int_{h(x) \otimes t}^x \rho(\xi) d\xi \right] \phi[h(x) \otimes t] \]

In the following theorem we get the estimate of the operator \(T_{\rho h}(t)\).

3.19 Theorem.

The family of operators \(T_{\rho h}(t)\) is strongly continuous generalized canonical semigroup defines on the space \(L_{p,\omega,h}\) and the following estimation holds.

\[ \|T_{\rho h}(t)\| \leq |t|^{-p} \]
Proof.

\[ \|T_{ph}(t)\phi\|_{p,\omega,h}^p = \int_a^b \exp \left[ \int \rho(h(x)) \phi(h(x) \odot t) \right] g(x) |\phi(h(x) \odot t)|^p dh(x) \leq \]

\[ \leq \int_a^b \exp[\omega h(x)] g(x) |\phi(h(x) \odot t)|^p dh(x) \leq \frac{1}{|t|} \int_a^b \exp[\omega h(\tau)] g(h^{-1}(h(\tau), t)) |\phi(\tau)|^p dh(\tau) \]

\[ \leq \frac{1}{|t|} \int_a^b \exp[\omega h(\tau)] g(\tau) |\phi(\tau)|^p dh(\tau) \]

\[ \|T_{ph}(t)\phi\|_{p,\omega,h}^p \leq \frac{1}{|t|} \|\phi\|_{p,\omega,h}^p \]

\[ \|T_{ph}(t)\phi\| \leq |t|^{-p} \|\phi\| \]

Clearly \( T_{ph}(1)\phi(x) = \phi(x) \)

\[ T_{ph}(t)T_{ph}(s)\phi(x) = T_{ph}(t) \exp \left[ \int \rho(h(x)) \phi(h(x) \odot s) \right] \phi(h(x) \odot s) \]

\[ = \exp \left[ \int \rho(h(x)) \phi(h(x) \odot t) \right] \exp \left[ \int \rho(h(x)) \phi(h(x) \odot t) \odot s \right] \phi(h(x) \odot t \odot s) \]

\[ = \exp \left[ \int \rho(h(x)) \phi(h(x) \odot t \odot s) \right] \phi(h(x) \odot t \odot s) = T_{ph}(t \odot s)\phi(x) \]

Generating operator of semigroup \( T_{ph}(t) \). We can construction the generating of the semigroup \( T_{ph}(t) \) by.

\[ A_{ph}\phi(x) = \lim_{t \to 1} \frac{1}{t} [T_{ph}(t) - I]\phi(x) \]

And this means,

\[ A_{ph}\phi(x) = \frac{d}{dt} T_{ph}(t)\phi(x)|_{t=1} = h(x) \frac{d\phi(x)}{dh(x)} - \rho(x)\phi(x) \]
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