Non-negative submodular stochastic probing via stochastic contention resolution schemes

Marek Adamczyk *1

1Department of Computer, Control, and Management Engineering, Sapienza University of Rome, Italy, adamczyk@dis.uniroma1.it.

September 4, 2015

Abstract

In a stochastic probing problem we are given a universe $E$, where each element $e \in E$ is active independently with probability $p_e \in [0, 1]$, and only a probe of $e$ can tell us whether it is active or not. On this universe we execute a process that one by one probes elements — if a probed element is active, then we have to include it in the solution, which we gradually construct. Throughout the process we need to obey inner constraints on the set of elements taken into the solution, and outer constraints on the set of all probed elements. The objective is to maximize a function of successfully probed elements.

This abstract model was presented by Gupta and Nagarajan (IPCO’13), and provides a unified view of a number of problems. Adamczyk, Sviridenko, Ward (STACS’14) gave better approximation for matroid environments and linear objectives. At the same time this method was easily extendable to settings, where the objective function was monotone submodular. However, the case of non-negative submodular function could not be handled by previous techniques.

In this paper we address this problem, and our results are twofold. First, we adapt the notion of contention resolution schemes of Chekuri, Vondrák, Zenklusen (SICOMP’14) to show that we can optimize non-negative submodular functions in this setting with a constant factor loss with respect to the deterministic setting. Second, we show a new contention resolution scheme for transversal matroids, which yields better approximations in the stochastic probing setting than the previously known tools. The rounding procedure underlying the scheme can be of independent interest — Bansal, Gupta, Li, Mestre, Nagarajan, Rudra (Algorithmica’12) gave two seemingly different algorithms for stochastic matching and stochastic $k$-set packing problems with two different analyses, but we show that our single technique can be used to analyze both their algorithms.

*Supported by the ERC StG project PAAI no. 250515.
1 Introduction

Stochastic variants of optimization problems were considered already in 1950 [5, 10], but only in recent years a significant attention was brought to approximation algorithms for stochastic variants of combinatorial problems. In this paper we consider adaptive stochastic optimization problems in the framework of Dean et al. [12] who presented a stochastic knapsack problem. Since the work of Dean et al. a number of problems in this framework were introduced [9, 16, 17, 18, 2, 19, 11]. Gupta and Nagarajan [20] presented an abstract framework for a subclass of adaptive stochastic problems giving a unified view for stochastic matching [9] and sequential posted pricing [7]. Adameczyk et al. [1] generalized the framework by also considering monotone submodular functions in the objective. In this paper we generalize the framework even further by showing that also maximizing a non-negative submodular function can be considered in the probing model. On the way we develop a randomized procedure for transversal matroids which can be used to improve approximation for the $k$-set packing problem [4].

Below paper enhances the iterative randomized rounding for points from matroid polytopes that was presented in [1]. The analysis from [1] does not easily carry over when the objective submodular function is non-monotone. To handle non-monotone objectives we are making use of contention resolution schemes introduced by Chekuri et al. [8]. Contention resolution schemes in the context of stochastic probing already were used by Gupta and Nagarajan [20]. Recently, Feldman et al. [14] presented online version of contention resolution schemes which, on top of applications for online settings, yield good approximations for stochastic probing problem for a broader set of constraints than before — most notably, for inner knapsack constraints and deadlines.

Our paper fills the gaps between and merges results from basically four different paper [1, 8, 20, 13]. That is the reason why this paper comes with diverse contributions: we improve the bound on measured greedy algorithm of Feldman et al. [13]; adjust contention resolution schemes to stochastic probing setting in a way in which submodular optimization is be possible; use iterative randomized rounding technique to develop contention resolution schemes; moreover, we revisit the algorithms of Bansal et al. [4].

Below we present the necessary background.

The probing model

We describe the framework following [20]. We are given a universe $E$, where each element $e \in E$ is active with probability $p_e \in [0, 1]$ independently. The only way to find out if an element is active, is to probe it. We call a probe successful if an element turns out to be active. On universe $E$ we execute an algorithm that probes the elements one-by-one. If an element is active, the algorithm is forced to add it to the current solution. In this way, the algorithm gradually constructs a solution consisting of active elements.

Here, we consider the case in which we are given constraints on both the elements probed and the elements included in the solution. Formally, suppose that we are given two independence systems of downward-closed sets: an outer independence system $(E, I^{\text{out}})$ restricting the set of elements probed by the algorithm, and an inner independence system $(E, I^{\text{in}})$, restricting the set of elements taken by the algorithm. We denote by $Q^t$ the set of elements probed in the first $t$ steps of the algorithm, and by $S^t$ the subset of active elements from $Q^t$. Then, $S^t$ is the partial solution constructed by the first $t$ steps of the algorithm. We require that at each time $t$, $Q^t \in I^{\text{out}}$ and $S^t \in I^{\text{in}}$. Thus, at each time $t$, the element $e$ that we probe must satisfy both $Q^{t-1} \cup \{e\} \in I^{\text{out}}$ and $S^{t-1} \cup \{e\} \in I^{\text{in}}$. The goal is to maximize expected value $\mathbb{E}[f(S)]$ where $f : 2^E \to \mathbb{R}_{\geq 0}$ and $S$ is the set of all successfully probed elements.

We shall denote such a stochastic probing problem by $(E, p, I^{\text{in}}, I^{\text{out}})$ with function $f$ stated on the side, if needed.
Submodular Optimization

A set function $f : 2^E \rightarrow \mathbb{R}_{\geq 0}$ is submodular, if for any two subsets $S, T \subseteq E$ we have $f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$. Without loss of generality, we assume also that $f(\emptyset) = 0$.

The multilinear extension $F : [0, 1]^E \rightarrow \mathbb{R}_{\geq 0}$ of $f$, whose value at a point $y \in [0, 1]^E$ is given by

$$F(y) = \sum_{A \subseteq E} f(A) \cdot \prod_{e \in A} y_e \prod_{e \notin A} (1 - y_e).$$

Note that $F(1_A) = f(A)$ for any set $A \subseteq E$, so $F$ is an extension of $f$ from discrete domain $2^E$ into a real domain $[0, 1]^E$. The value $F(y)$ can be interpreted as the expected value of $f$ on a random subset $A \subseteq E$ that is constructed by taking each element $e \in E$ with probability $y_e$.

Contention Resolution Schemes

Consider a ground set of elements $E$ and an down-closed family $\mathcal{I} \subseteq 2^E$ of $E$’s subsets — we call $(E, \mathcal{I})$ an independence system. Let $\mathcal{P}(\mathcal{I})$ be the convex hull of characteristic vectors of sets from $\mathcal{I}$. Given $x \in \mathcal{P}(\mathcal{I})$ we define $R(x)$ to be a random set in which every element $e \in E$ is included in $R(x)$ with probability $x_e$; set $R(x)$ defined like that is used frequently throughout the paper.

Chekuri et al. [8] presented a framework of contention resolution schemes (CR schemes) that allows to maximize non-negative submodular functions for various constraints. The following definition and theorem come from [8].

**Definition 1**

Let $(E, \mathcal{I})$ be independence system. For $b, c \in [0, 1]$, a $(b, c)$-balanced CR scheme $\pi$ for $\mathcal{P}(\mathcal{I})$ is a randomized procedure that for every $x \in b \cdot \mathcal{P}(\mathcal{I})$ and $A \subseteq E$, returns a random set $\pi_x(A)$ such that:

1. always $\pi_x(A) \subseteq A \cap \text{supp}(x)$ and $\pi_x(A) \in \mathcal{I}$,
2. $\mathbb{P}[e \in \pi_x(A_1)] \geq \mathbb{P}[e \in \pi_x(A_2)]$ whenever $e \in A_1 \subseteq A_2$,
3. for all $e \in \text{supp}(x)$, $\mathbb{P}[e \in \pi_x(R(x))|e \in R(x)] \geq c$.

**Theorem 1**

Let $(E, \mathcal{I})$ be an independence system. Let $f : 2^E \rightarrow \mathbb{R}$ be a non-negative submodular function with multilinear relaxation $F$, and $x$ be a point in $b \cdot \mathcal{P}(\mathcal{I})$. Let $\pi$ be a $(b, c)$-balanced CR scheme for $\mathcal{P}(\mathcal{I})$, and let $S = \pi_x(R(x))$. If $f$ is monotone then $\mathbb{E}[f(S)] \geq c \cdot F(x)$. Furthermore, there is a function $\eta_f : 2^E \rightarrow 2^E$ that depends on $f$ and can be evaluated in linear time, such that even for $f$ non-monotone $\mathbb{E}[f(\eta_f(S))] \geq c \cdot F(x)$.

Function $\eta_f(S)$ represents a pruning operation that removes from $S$ some elements. To prune a set $S$ with pruning function $\eta_f$, an arbitrary ordering of the elements of $E$ is fixed: for simplicity of notation let $E = \{1, ..., |E|\}$ which gives a natural ordering. Starting with $S^{\text{prun}} = \emptyset$ the final set $S^{\text{prun}} = \eta_f(S)$ is constructed by going through all elements of $E$ in the given order. When considering an element $e$, $S^{\text{prun}}$ is replaced by $S^{\text{prun}} + e$ if $f(S^{\text{prun}} + e) - f(S^{\text{prun}}) \geq 0$.

Note that a pruning operation like that is not possible to execute in the probing model since we commit to elements. We address this issue in Section 2 where we show how to perform on-the-fly pruning.

**Stochastic $k$-set packing**

We are given $n$ elements/columns, where each element $e \in [n]$ has a random profit $v_e \in \mathbb{R}_+$, and a random $d$-dimensional size $S_e \in \{0, 1\}^d$. The sizes are independent for different elements, but
allows to get a \( S \) when they assume that the outcomes of size vectors \( S \) column \( 2 \leq a \). By act \( b \), define a stochastic contention resolution scheme (stoch-CR scheme). To obtain our results we extend the framework of CR schemes into the probing model, and we define a stochastic probing, captured in Theorem 4 and new analyses of algorithms from [1] for stochastic \( k \)-set packing and stochastic matching. It is based on the iterative randomized rounding technique of [1].

### 1.1 Our contributions at a high level

There are two main contributions of the paper. First is Theorem 2 which implies that non-negative submodular optimization in the probing model is possible if we are given CR schemes. Our second contribution is the improved insight for transversal matroids in the context of stochastic probing, captured in Theorem 4 and new analyses of algorithms from [1] for stochastic \( k \)-set packing and stochastic matching. It is based on the iterative randomized rounding technique of [1].

#### 1.1.1 Non-negative submodular optimization via CR schemes in the probing model

To obtain our results we extend the framework of CR schemes into the probing model, and we define a stochastic contention resolution scheme (stoch-CR scheme). Define a polytope

\[
\mathcal{P}(\mathcal{I}^\text{in}, \mathcal{I}^\text{out}) = \left\{ x \left| x \in \mathcal{P}(\mathcal{I}^\text{out}), p \cdot x \in \mathcal{P}(\mathcal{I}^\text{in}), x \in [0,1]^E \right\} \right. 
\]

By \( \text{act}(S) \) we denote the subset of active elements of set \( S \). Note that event \( e \in \text{act}(R(x)) \) means both that \( e \in R(x) \) and that \( e \) is active; this event has probability \( p_e x_e \).

**Definition 2**

Let \( (\mathcal{E}, p, \mathcal{I}^\text{in}, \mathcal{I}^\text{out}) \) be a stochastic probing problem. For \( b, c \in [0,1] \), a \((b,c)\)-balanced stoch-CR scheme \( \bar{\pi} \) for a polytope \( \mathcal{P}(\mathcal{I}^\text{in}, \mathcal{I}^\text{out}) \) is a probing strategy that for every \( x \in b \cdot \mathcal{P}(\mathcal{I}^\text{in}, \mathcal{I}^\text{out}) \) and \( A \subseteq \mathcal{E} \), obeys outer constraints \( \mathcal{I}^\text{out} \), and the returned random set \( \bar{\pi}_x(A) \) satisfies the following:

1. \( \bar{\pi}_x(A) \) consists only of active elements,
2. \( \bar{\pi}_x(A) \subseteq A \cap \text{supp}(x) \) and \( \bar{\pi}_x(A) \in \mathcal{I}^\text{in} \),
3. \( \mathbb{P}[e \in \bar{\pi}_x(R(x)) | e \in \text{act}(R(x))] \geq c \),
4. \( \mathbb{P}[e \in \bar{\pi}_x(A_1)] \geq \mathbb{P}[e \in \bar{\pi}_x(A_2)] \) whenever \( e \in A_1 \subseteq A_2 \).

In Section 2 we present a mathematical program that models our problem. Solving the program, getting \( x^+ \), and running the stoch-CR scheme \( \bar{\pi}_{x^+} \) on \( R(x^+) \) constitute the algorithm from the below Theorem.

**Theorem 2**

Consider a stochastic probing problem \( (\mathcal{E}, p, \mathcal{I}^\text{in}, \mathcal{I}^\text{out}) \), where we need to maximize a non-negative submodular function \( f : 2^\mathcal{E} \rightarrow \mathbb{R}_{\geq 0} \). If there exists a \((b,c)\)-balanced stoch-CR scheme \( \bar{\pi} \) for \( \mathcal{P}(\mathcal{I}^\text{in}, \mathcal{I}^\text{out}) \), then there exists a probing strategy whose expected outcome is at least \( c(b \cdot e^{-b} - o(1)) \cdot \mathbb{E}[f(OPT)] \).
To put this Theorem into perspective. There exists a \( b, \left(\frac{1-x^{-k}}{b}\right)^k \)-balanced scheme for intersection of \( k \) matroids [8], and there exists a \( b, (1-b)^k \)-balanced ordered scheme for intersection of \( k \) matroids [7, 14]. For example when \( k_{in} = k_{out} = 1 \), Lemma 10 and the above Theorem, after plugging appropriate number \( b \), yield approximation of 0.13. See Section 2 for discussion on other types of constraints.

1.1.2 Optimization for transversal matroids

**Stochastic probing on intersection of transversal matroids**  We improve upon the approximation described in the previous Section, if we assume the constraints are intersections of transversal matroids. We do it by developing a new stoch-CR scheme. This scheme is direct in the sense that we do not construct it by applying first a scheme for outer and then for inner constraints, as in Lemma 10.

**Lemma 3**
There exists a \( b, \left(\frac{1}{1+b(k_{in}+k_{out})}\right)^k \)-balanced stoch-CR scheme when constraints are intersections of \( k_{in} \) inner and \( k_{out} \) outer transversal matroids.

For \( k_{in} = k_{out} = 1 \) the above Lemma together with Theorem 2 give approximation factor of 0.15, so a modest improvement over 0.13, but it gets significantly better for larger values of \( k_{in} + k_{out} \). With \( k = k_{in} + k_{out} \) we plug \( b = \frac{2}{\sqrt{4+4k+1}} \) and we use Theorem 2 to conclude the following Theorem.

**Theorem 4**
For maximizing non-negative submodular function in the probing model with \( k_{in} \) inner and \( k_{out} \) outer transversal matroids there exists an algorithm with approximation ratio of

\[
\frac{1}{k + \sqrt{k + \frac{1}{2} + \frac{1}{2}} \left(1 - \Theta\left(\frac{1}{\sqrt{k}}\right)\right)}.
\]

There are further applications of the techniques used in Lemma 3.

**Regular CR scheme for transversal matroids**  When \( k_{in} = 0 \) this scheme yields \( b, \frac{1}{1+b} \)-balanced CR scheme for deterministic setting. So far only a \( b, (1-b)^k \)-balanced ordered scheme and a \( b, ((1-e^k)/b)^k \)-balanced scheme were known [8]; see section 2 for the definition of ordered scheme. Our scheme can be seen as an improvement when one looks at the max \( b \cdot c \) — first one yields \( \frac{1}{c(k+1)} \), second \( \frac{2}{e(k+1)} \) (for \( k \) big), and we get \( \frac{1}{k+1} \).

**Stochastic \( k \)-set packing and stochastic matching**  Here we want to argue that the martingale-based analysis of the algorithm for transversal matroids can be of independent interest. In the bipartite stochastic matching problem [4] the inner constraints define bipartite matchings and outer constraints define general \( b \)-matchings — both are intersections of 2 transversal matroids. Stochastic \( k \)-set packing problem does not belong to our framework, but we have already defined it in previous subsection.

Bansal et al. [4] presented an algorithm for the \( k \)-set packing, and have proven that it yields a \( (k + 1) \)-approximation but when assuming that the column outcomes are monotone. They also presented a 3-approximation algorithm for the stochastic matching problem. Both algorithms first choose a subset of elements: for \( k \)-set packing the elements are chosen independently, and for stochastic matching the elements are chosen using a dependent rounding procedure of

\[1\text{The precise value is smaller than } \frac{1}{k+1} \text{ starting } k \geq 4.\]
Gandhi et al. [15]. After that they probe elements in a random order. Bansal et al. [4] gave two quite different analyses of the two algorithms, while we show how both algorithms can be analyzed using our martingale-based technique for transversal matroids. First, we show that their algorithm for set packing is in fact a \((k+1)\)-approximation even without the monotonicity assumption (see Appendix F). Second, we also show that the our technique of analysis provides a proof of 3-approximation of their algorithm for stochastic matching (see Appendix G).

2 Optimization via stoch-CR schemes

Mathematical program

Another extension of \(f\) studied in [8] is given by:

\[
f^+(y) = \max \left\{ \sum_{A \subseteq E} \alpha_A f(A) \left| \sum_{A \subseteq E} \alpha_A \leq 1, \forall j \in E \sum_{A \subseteq E, A \ni j} \alpha_A \leq y_j, \forall A \subseteq E \alpha_A \geq 0 \right. \right\}.
\]

Intuitively, the solution \((\alpha_A)_{A \subseteq E}\) above represents the distribution over \(2^E\) that maximizes the value \(\mathbb{E}[f(A)]\) subject to the constraint that its marginal values satisfy \(\mathbb{P}[i \in A] \leq y_i\). The value \(f^+(y)\) is then the expected value of \(\mathbb{E}[f(A)]\) under this distribution, while the value of \(F(y)\) is the value of \(\mathbb{E}[f(A)]\) under the particular distribution that places each element \(i\) in \(A\) independently. This relaxation is important for our applications because the following mathematical programming relaxation gives an upper bound on the expected value of the optimal feasible strategy for the related stochastic probing problem:

\[
\max \left\{ f^+(x \cdot p) \mid x \in \mathcal{P}(I^{in}, I^{out}) \right\}.
\]

Lemma 5

Let \(OPT\) be the optimal feasible strategy for the stochastic probing problem in our general setting, then \(\mathbb{E}[f(OPT)] \leq f^+(x^* \cdot p)\).

Proof of the following Lemma can be found in [1], and for sake of completeness we put it in the Appendix [3].

However, the framework of Chekuri et al. [8] uses multilinear relaxation \(F\) and not \(f^+\). The Lemma below allows us to make a connection. Note that \(F(x)\) is exactly equal to \(\mathbb{E}[F(R(x))]\), i.e., it corresponds to sampling each point \(e \in E\) independently with probability \(x_e\), while definition of \(f^+(x)\) involves the best possible, most likely correlated, distribution of \(E\)’s subsets. It follows immediately that for any point \(x\) we have \(f^+(x) \geq F(x)\), and therefore the following Lemma states a stronger lower-bound for the measured greedy algorithm of Feldman et al. [13]. Details are presented in the next paragraph.

Lemma 6

Let \(b \in [0, 1]\), let \(f\) be a submodular function with multilinear extension \(F\), and let \(\mathcal{P}\) be any downward closed polytope. Then, the solution \(x \in [0,1]^E\) produced by the measured greedy algorithm satisfies 1) \(x \in b \cdot \mathcal{P}\), 2) \(F(x) \geq \left( b \cdot e^{-b} - o(1) \right) \cdot \max_{y \in \mathcal{P}} f^+(y)\).

Stronger bound for measured continuous greedy

We now briefly review the measured continuous greedy algorithm of Feldman et al. [13]. The algorithm runs for \(\delta^{-1}\) discrete time steps, where \(\delta\) is a suitably chosen, small constant. Let \(y(t)\) be the algorithm’s current fractional solution at time \(t\). At time \(t\), the algorithm selects vector \(I(t) \in \mathcal{P}\) given by \(\arg\max_{x \in \mathcal{P}} \sum_{e \in E} x_e \cdot (F(y(t) \lor 1_{\{e\}}) - F(y(t)))\) (where \(\lor\) denotes element-wise maximum). Then, it sets \(y_e(t + \delta) = y_e(t) + \delta I_e(t) \cdot (1 - y_e(t))\) and continues to time \(t + \delta\).
The analysis of Feldman et al. shows that if, at every time step
\[ F(y(t + \delta)) - F(y(t)) \geq \delta \cdot [e^{-t} \cdot f(OPT) - F(y(t))] - O(n^3 \delta^2 f(OPT)), \]
then we must have \( F(y(T)) \geq [Te^{-T} - o(1)] \cdot f(OPT) \). We note that, in fact, this portion of their analysis works even if \( f(OPT) \) is replaced by any constant value. Thus, in order to prove our claim, it suffices to derive an analogue of (2) in which \( f(OPT) \) is replaced by \( f^+(x^+) \), where \( x^+ = \arg\max_{x \in P} f^+(y) \). The remainder of the proof then follows as in [13].

Lemma 7 below contains the required analogue of (2). Hence it implies Lemma 6. Proof of below Lemma is placed in the Appendix A.

**Lemma 7**

For every time \( 0 \leq t \leq T \)
\[ F(y(t + \delta)) - F(y(t)) \geq \delta \cdot [e^{-t} \cdot f^+(x^+) - F(y(t))] - O(n^3 \delta^2) \cdot f^+(x^+). \]

**Using stoch-CR schemes**

The following Lemma is implied by Theorem 11—it follows just from the fact that set \( act(R(x)) \) is distributed exactly as \( R(x \cdot p) \).

**Lemma 8**

Let \((E, p, I^{in}, I^{out})\) be a probing problem. Let \( f : 2^E \rightarrow \mathbb{R}_{\geq 0} \) be a non-negative submodular function with multilinear relaxation \( F \), and \( x \) be a point in \( b \cdot \mathcal{P}(I^{in}, I^{out}) \) for \( b \in [0, 1] \). Let \( \pi_x \) be a \((b,c)\)-balanced stoch-CR scheme for \( \mathcal{P}(I^{in}, I^{out}) \). Let \( \pi_x(R(x)) \) be the output of the CR scheme, and let \( \eta_f(\pi_x(R(x))) \) be a pruned subset of \( \pi_x(R(x)) \). It holds that
\[ \mathbb{E}[f(\eta_f(\pi_x(R(x))))] \geq c \cdot F(x \cdot p). \]

However, we cannot apply yet the above Lemma to reason about a probing strategy, because here the pruning \( \eta_f \) of \( S \) is done after the process. In the probing model we commit to an element once we successfully probe it, and therefore we cannot do the pruning operation after the execution of a stoch-CR scheme. However, since a probing strategy inherently includes elements one-by-one, we can naturally add to any stoch-CR scheme the pruning operation done on the fly. The idea is to simulate the probes of elements that would be rejected by the pruning criterion. To simulate a probe of \( e \) means not to probe \( e \) and to toss a coin with probability \( p_e \) — in case of success to behave afterwards as if \( e \) was indeed taken into the solution, and in case of failure to behave as if \( e \) was not taken. During the execution of a stoch-CR scheme \( \pi_x \) we construct two sets: \( S^{prun} \) consists of elements successfully probed, and \( S^{virt} \) consists of elements whose simulation was successful. If in a step we want to probe an element \( e \) such that \( f(S^{prun} + S^{virt} + e) - f(S^{prun} + S^{virt}) < 0 \), then we simulate the probe of \( e \) and if successful \( S^{virt} \leftarrow S^{virt} + e \); otherwise we really probe \( e \) and \( S^{prun} \leftarrow S^{prun} + e \) if successful. We can see that at any step, it holds that \( S^{prun} = \eta_f(S^{prun} + S^{virt}) \). Also, the final random set \( S^{prun} + S^{virt} \) is distributed exactly as \( \pi_x(R(x)) \). Hence, the outcome of such a probing strategy is \( \mathbb{E}[f(S^{prun})] = \mathbb{E}[f(\eta_f(S^{prun} + S^{virt}))] = \mathbb{E}[f(\eta_f(\pi_x(R(x))))] \geq c \cdot F(x \cdot p) \), where the inequality comes from Lemma 8. Thus we have proven what follows.

**Lemma 9**

Let \( f, x, \pi_x \) be as in Lemma 8. There exists a probing strategy whose expected outcome is \( \mathbb{E}[f(\eta_f(\pi_x(R(x))))] \geq c \cdot F(x \cdot p) \).

Now we can finish the proof of Theorem 2. Consider the following algorithm. First, use Lemma 6 to find a point \( x^* \) such that
\[ F(x^* \cdot p) \geq (b \cdot e^{-b} - o(1)) \cdot \max_{x \in \mathcal{P}(I^{in}, I^{out})} f^+(x \cdot p). \]
Second, run the probing strategy based on a stoch-CR scheme $\tilde{\pi}_{x^*}$ as described in Lemma 9. This yields, together with Lemma 8, that the outcome of such a probing strategy is

$$\mathbb{E}[f(\eta_f(\tilde{\pi}_{x^*}(R(x))))] \geq c \cdot f(x^* \cdot p) \geq c \cdot \left( b \cdot e^{-b} - o(1) \right) \cdot \max_{x \in \mathcal{P}(I^{in}, I^{out})} f^+(x \cdot p) \geq c \cdot \left( b \cdot e^{-b} - o(1) \right) \cdot \mathbb{E}[f(OPT)].$$

In [8] also an alternative approach than pruning was used. They defined a strict contention resolution scheme where the approximation guarantee $\mathbb{P}[e \in \pi_x(R(x)) | e \in R(x)] \geq c$ holds with equality rather than inequality. Since the pruning operation depends on an objective function, resigning from it allows for the algorithm to be used in maximizing many submodular functions at the same time. In our stochastic setting we can also skip the pruning operation if we have a strict scheme. No proof of this fact is needed, since we can directly use an appropriate analog of Lemma 8 (Theorem 4.1 from [1]).

### Stochastic probing for various constraints

Gupta and Nagarajan [20] introduced a notion of ordered CR scheme, for which there exists a (possibly random) permutation $\sigma$ of $E$, so that for each $A$ the set $\pi_{\sigma}(A)$ is the maximal independent subset of $I$ obtained by considering elements in the order of $\sigma$. Ordered schemes are required to implement probing strategies, because of the commitment to the elements. CR schemes exist for various types of constraints, e.g., matroids, sparse packing integer programs, constant number of knapsacks, unsplittable flow on trees (UFP), $k$-systems (including intersection of $k$ matroids, and $k$-matchoids). Ordered schemes exist for $k$-systems and UFP on trees. See Theorem 4 in [20] for a listing with exact parameters.

The following Lemma is based on Lemma 1.6 from [8]. The proof basically carries over, the only thing we have to do is to again incorporate probes’ simulations as in the proof of Lemma 9.

**Proof of Lemma 10** is placed in Appendix C. Theorem 3.4 from [20] yields a similar result but would imply a $(b, c_{out} + c_{in} - 1)$-balanced stoch-CR scheme. Thus the below Lemma can be considered as a strengthening of Theorem 3.4 from [20], because an FKG inequality is used in the proof instead of a union-bound.

#### Lemma 10

Consider a probing problem $(E, p, I^{in}, I^{out})$. Suppose we have a $(b, c_{out})$-balanced CR-scheme $\pi^{out}$ for $\mathcal{P}(I^{out})$, and a $(b, c_{in})$-balanced ordered CR scheme $\pi^{in}$ for $\mathcal{P}(I^{in})$. Then there exists a $(b, c_{out} \cdot c_{in})$-balanced stoch-CR scheme for $\mathcal{P}(I^{in}, I^{out})$.

In light of the above Lemma, one can question Definition 2 of a stoch-CR scheme — why do we need to define it at all, if we can just be using two separate classic CR schemes $\pi^{out}, \pi^{in}$. The reason is that there may exist stoch-CR schemes that are not convolutions of two deterministic schemes, and that yield better approximations than corresponding convoluted ones. Such a stoch-CR scheme is presented in Section 3.

In a recent paper, Feldman et al. [14] presented a variant of online contention resolution schemes. They enriched the set of constraints possible to use in the stochastic probing problem — previously inner knapsack constraints were not possible to incorporate, as well as deadlines (element $e$ can be taken only after $d_e$ steps) for weighted settings. Their results can be extended to monotone submodular settings by making use of a stronger bound for continuous greedy algorithm [9] presented in [1]. The stronger bound for measured greedy algorithm — which works for non-monotone functions — that we give in this paper can also be used in [14] enhancing their result by the possibility of handling non-monotone functions as well.
3 Stoch-CR scheme for intersection of transversal matroids

In this section we prove Lemma 3. We assume we have only one inner matroid $M^{in}$ to keep the presentation simple. Also, we shall present a $(1, \frac{1}{2})$-balanced CR scheme, instead of $(b, \frac{1}{b+1})$.

In the Appendix C we present a full scheme with arbitrary $b, k^{in}$, and $k^{out}$.

Our stoch-CR scheme on the input is given a point $x$ such that $p \cdot x \in P(M^{in})$ and a set $A \subseteq E$, and on the output it returns set $\bar{\pi}_x(A)$. The procedure is divided in two phases. First, the preprocessing phase, depends only on the point $x$. Second, the random selection phase, depends on the set $A \subseteq E$ and the outcome of the preprocessing phase.

Matroid properties

Let $M^{in} = (E, I^{in})$. We know [21] that the convex hull of $\{1_A | A \in I\}$, i.e., characteristic vectors of all independent sets of $M^{in}$, is equivalent to the matroid polytope $P(M^{in}) = \{x \in \mathbb{R}^E_{\geq 0} | \sum_{A \in I^{in}} x_A \cdot x_e \leq r_{M^{in}}(A)\}$, where $r_{M^{in}}$ is the rank function of $M^{in}$. Thus for any $x \in P(M^{in})$ in polynomial time we can find representation $p \cdot x = \sum_{i=1}^m \beta_i \cdot 1_{B_i}$, where $B_1, \ldots, B_m \in I$ and $\beta_1, \ldots, \beta_m$ are non-negative weights such that $\sum_{i=1}^m \beta_i = 1$. We shall call sets $B_1, \ldots, B_m$ the support of $p \cdot x$ in $M^{in}$, and denote it by $B$.

In the remainder of the section we assume that we know the graph representation $(E \cup V, \subseteq E \times V)$ of the matroid described below. This assumption is quite natural and common, e.g., [3].

Definition 3

Consider bipartite graph $(E \cup V, \subseteq E \times V)$. Let $I$ be a family of all subsets $S$ of $E$ such that there exists a matching between $S$ and $V$ of size exactly $|S|$. Then $M = (E, I)$ is a matroid, called transversal matroid.

Preprocessing

In what follows we shall write superscripts indicating the time in which we are in the process.

We start by finding the support $B^0$ of vector $p \cdot x \in P(M^{in})$, i.e., $p \cdot x = \sum_i \beta_i \cdot 1_{B_i}$. For every two sets $B, A \in B^0$ we find a mapping $\varphi^0[B, A] : B \rightarrow A \cup \{\bot\}$, which we call transversal mapping. This mapping will satisfy three properties.

Property 1

For each $a \in A$ there is at most one $b \in B$ for which $\varphi^0[B, A](b) = a$.

Property 2

For $b \in B \setminus A$, if $\varphi^0[B, A](b) = \bot$, then $A + b \in I$, otherwise $A - \varphi^0[B, A](b) + b \in I$.

Note that unlike in standard exchange properties of matroids, we do not require that $\varphi^0[B, A](b) = b$, if $b \in A \cap B$. Property 3 will be presented in a moment. Once we find the family $\varphi^0$ of transversal mappings, for each element $e \in E$ we choose one set among $B_i^0 : e \in B_i^0$ with probability $\beta_i / p_e x_e$; since $\sum_{B_i^0 : e \in B_i^0} \beta_i = p_e x_e$ this is properly defined. Denote by $c(e)$ the index of the chosen set, and call $e$-critical the set $B_{c(e)}^0$ for any $t$ (note that $c(e)$ is fixed throughout the process). We concisely denote indices of critical sets by $C = (c(e))_{e \in E}$. For each element $e$ we define $\Gamma^0(e) = \{f | f \neq e \wedge \varphi^0[B_{c(f)}^0, B_{c(e)}^0](f) = e\}$ — the blocking set of $e$. The Lemma below follows from Property 1, and its proof is in the Appendix C.

Lemma 11

If $p \cdot x \in P(M^{in})$, then for any element $e$, it holds that

$\mathbb{E}_{C \subseteq E} \left[ \sum_{f \in \Gamma^0(e)} b_{f \cdot \chi[f \in R(x) \setminus e]} \right] \leq 1$, where the expectation is over $R(x)$ and the choice of critical sets $C$; here $\chi[\mathcal{E}]$ is a 0-1 indicator of random event $\mathcal{E}$. 

Algorithm 1 Stoch-CR scheme π_π (A)

1: find support B^0 of p · x in M^{in} and family φ^0; choose critical sets C
2: remove from A all e : x_e = 0; mark all e ∈ A as available
3: S ← ∅
4: while there are still available elements in A do
5:  pick element e uniformly at random from A
6:  if e is available then
7:    probe e
8:    if probe of e successful then
9:      S ← S ∪ {e}
10:     for each set B^t_i of support B^t do
11:        B^t_i ← B^t_i + e
12:     else simulate the probe of e
13:       if probe or simulation was successful then
14:        for each set B^t_i of support B^t do
15:          f ← φ \left[B^t_{c(e)}, B^t_{c(f)}\right](e)
16:          if f ≠ e then B^t_{c(f)} ← B^t_{c(f)} − f and call f unavailable
17:        compute the family φ^{t+1}
18:        for each i do B^{t+1}_i ← B^t_i
19:      t ← t + 1;
20:  return S as π_π (A)

Random selection procedure

The whole stoch-CR scheme is presented in Algorithm 1. During the algorithm we modify sets of support B^t after each step, but we keep the weights β_i unchanged. We preserve an invariant that each B^t_i from B^t is an independent set of matroid M^{in}. At the end of the algorithm the set π_π (A) belongs to every set B^t_i ∈ B^t. Hence, the final set π_π (A) is independent in every matroid.

Now we define Property 3 of transversal mappings. Suppose that in the first step we update the support B^0 according to the for loop in line 15 and we obtain B^1. Different support B^1 requires a different family of mappings φ^1, and so in step 2, the elements that can block e are Γ^1 (e). If it happens that Γ^1 (e) ≠ Γ^0 (e), then we cannot show the monotonicity property of stoch-CR scheme. However, we can require from the transversal mappings to keep the blocking sets Γ^t (e) unchanged, as long as e is available. In the Appendix B we show how to find such a family of transversal mappings φ^0 and how to construct φ^{t+1} given φ^t.

Property 3

Let φ^t be a family of transversal mappings for B^t. Suppose we update the support B^t and obtain B^{t+1}. Then we can find a family φ^{t+1} of transversal mappings such that Γ^t (e) = Γ^{t+1} (e) for any element e that is still available after step t.

Analysis

First, an explanation. We allow to pick in line 5 elements that we have once probed and simulate their probe. This guarantees that the probability that an available element is blocked is equal for every step. Otherwise, again, we would not be able to guarantee the monotonicity.

In the analysis we deploy martingale theory. In particular Doob’s Stopping Theorem which states that if a martingale \left(Z^t\right)_{t≥0} and stopping time τ are both “well-behaving”, then E [Z^τ] = E [Z^0]. In the Appendix D we put necessary definitions, statement of Doob’s Stopping Theorem, and an explanation of why we can use this Theorem.
The random process executed in the while loop depends on the critical sets chosen in the preprocessing phase. Therefore, when we analyze the random process we condition on the choice of critical sets $C$.

We say $e$ is available, if it is still possible to probe, i.e., it is not yet blocked, and it was not yet probed. Define $X_e = \mathbb{1}[e \in A]$; in Iverson notation $[false] = 0 = 1 - [true]$.

Let $Y^t_e$ for $t = 0, 1, ..., \infty$ be a random variable indicating if $e$ is still available after step $t$. Initially, $Y^0_e = X_e$. Let $P^t_e$ be a random variable indicating if $e$ was probed in one of steps $0, 1, ..., t$; we have $P^0_e = 0$ for all $e$. Variable $P^{t+1}_e - P^t_e$ indicates if $e$ was probed at step $t+1$. Given the information $\mathcal{F}^t$ about the process up to step $t$, the probability of this event is

$$ \mathbb{E} \left[ P^{t+1}_e - P^t_e \mid \mathcal{F}^t, C \right] = \frac{Y^t_e}{|A|} $$

because if element $e$ is still available after step $t$ (i.e., $Y^t_e = 1$), then with probability $\frac{1}{|A|}$ we choose it in line 5 and proceed (with probability $\frac{Y^t_e}{|A|}$) and otherwise (i.e., $Y^t_e = 0$) we cannot probe it.

Variable $Y^t_e - Y^{t+1}_e$ indicates whether element $e$ stopped being available at step $t+1$, i.e., we either have picked it in line 5 and probed (with probability $\frac{Y^t_e}{|A|}$), or some $f \in \Gamma^t(e)$ has blocked $e$ in line 7 (with probability $\frac{Y^t_e}{|A|} \cdot \sum_{f \in \Gamma^t(e)} p_f X_f$). So in total

$$ \mathbb{E} \left[ Y^t_e - Y^{t+1}_e \mid \mathcal{F}^t, C \right] = \frac{Y^t_e}{|A|} \left( 1 + \sum_{f \in \Gamma^t(e)} p_f X_f \right), $$

and we can say that

$$ \mathbb{E} \left[ (P^{t+1}_e - P^t_e) \cdot \left( 1 + \sum_{f \in \Gamma^t(e)} p_f X_f \right) - (Y^t_e - Y^{t+1}_e) \mid \mathcal{F}^t, C \right] = 0. $$

This means that the sequence $\left( \left( 1 + \sum_{f \in \Gamma^t(e)} p_f X_f \right) \cdot P^t_e + Y^t_e \right)_{t \geq 0}$ is a martingale. Let $\tau = \min \{ t \mid Y^t_e = 0 \}$ be the step in which edge $e$ became unavailable. It is clear that $\tau$ is a stopping time (definition in Appendix D). Thus from Doob’s Stopping Theorem we get that

$$ \mathbb{E}_\tau \left[ \left( 1 + \sum_{f \in \Gamma^\tau(e)} p_f X_f \right) \cdot P^\tau_e + Y^\tau_e \mid C \right] = \mathbb{E} \left[ \left( 1 + \sum_{f \in \Gamma^0(e)} p_f X_f \right) \cdot P^0_e + Y^0_e \mid C \right], $$

and this is equal to $X_e$, because $P^0_e = 0$ and $Y^0_e = X_e$. We have $Y^\tau_e = 0$, and expression $1 + \sum_{f \in \Gamma^\tau(e)} p_f X_f$ is in fact equal to $1 + \sum_{f \in \Gamma^0(e)} p_f X_f$ (Property 3 of the transversal mapping, as $e$ was available before step $\tau$), which depends solely on $C$ and $A$. Hence

$$ \left( 1 + \sum_{f \in \Gamma^0(e)} p_f X_f \right) \cdot \mathbb{E}_\tau [P^\tau_e \mid C] = X_e. $$

Now just note that $\mathbb{E}_\tau [P^\tau_e \mid C]$ is exactly the probability that $e$ is probed, so we conclude that

$$ \mathbb{P} \left[ e \in \pi_x(A) \mid C \right] = p_e \cdot \mathbb{E}_\tau [P^\tau_e \mid C] = p_e X_e \left( 1 + \sum_{f \in \Gamma^0(A)} p_f X_f \right). $$

Monotonicity Set $\Gamma^0(e)$ does not depend on $A$, but only on the vector $p \cdot x$ and $C$, so for $A_1 \subseteq A_2$ we have $\sum_{f \in \Gamma^0(A)} |f \in A_1| \leq \sum_{f \in \Gamma^0(A)} p_f \cdot |f \in A_2|$.
**Approximation guarantee** In the identity

$$P[e \in \bar{\pi}_x(A) | C] = \frac{p_e X_e}{1 + \sum_{f \in \Gamma^0(A)} p_f X_f}$$

we place random set $R(x)$ instead of $A$; now $X_f = \chi[f \in R(x)]$ is a random variable. Let us condition on $e \in R(x)$, take expected value $E_{C,R(x)}[\cdot | e \in R(x)]$, and apply Jensen’s inequality to convex function $x \mapsto \frac{1}{x}$ to get:

$$P[e \in \bar{\pi}_x(R(x)) | e \in R(x)] = E_{C,R(x)}[P[e \in \bar{\pi}_x(R(x)) | C, R(x)] | e \in R(x)]$$

$$= \frac{p_e X_e}{1 + \sum_{f \in \Gamma^0(e)} p_f X_f} \left| e \in R(x) \right\} \geq \frac{p_e}{E_{C,R(x)} \left[ 1 + \sum_{f \in \Gamma^0(e)} p_f X_f \right] | e \in R(x) \right\} .$$

Since $E_{C,R(x)} \left[ \sum_{f \in \Gamma^0(e)} p_f X_f \right] | e \in R(x) \right\] \leq 1$ from Lemma 11 we conclude that

$$P[e \in \bar{\pi}_x(R(x)) | e \in R(x)] \geq \frac{p_e}{2},$$

and therefore $P[e \in \bar{\pi}_x(R(x)) | e \in \text{act}(R(x))] \geq \frac{1}{2}$, which is exactly Property 3 from the definition of stoch-CR scheme.

**Acknowledgments**

I thank Justin Ward for his help in proving Lemma 7, and also for valuable suggestions that helped to improve the presentation of the paper.
References

[1] Marek Adamczyk, Maxim Sviridenko, and Justin Ward. Submodular stochastic probing on matroids. In STACS, 2014.

[2] Arash Asadpour, Hamid Nazerzadeh, and Amin Saberi. Stochastic submodular maximization. In WINE, pages 477–489, 2008.

[3] Moshe Babaioff, Nicole Immorlica, and Robert Kleinberg. Matroids, secretary problems, and online mechanisms. In SODA, 2007.

[4] Nikhil Bansal, Anupam Gupta, Jian Li, Julián Mestre, Viswanath Nagarajan, and Atri Rudra. When LP is the cure for your matching woes: Improved bounds for stochastic matchings. Algorithmica, 63, 2012.

[5] E. M. L. Beale. Linear programming under uncertainty. Journal of the Royal Statistical Society. Series B, 17:173–184, 1955.

[6] Gruia Calinescu, Chandra Chekuri, Martin Pál, and Jan Vondrák. Maximizing a Submodular Set Function Subject to a Matroid Constraint. SIAM JoC, 40, 2011.

[7] Shuchi Chawla, Jason D. Hartline, David L. Malec, and Balasubramanian Sivan. Multi-parameter mechanism design and sequential posted pricing. In STOC, 2010.

[8] Chandra Chekuri, Jan Vondrák, and Rico Zenklusen. Submodular function maximization via the multilinear relaxation and contention resolution schemes. SIAM J. Comput., 43(6):1831–1879, 2014.

[9] Ning Chen, Nicole Immorlica, Anna R. Karlin, Mohammad Mahdian, and Atri Rudra. Approximating matches made in heaven. In ICALP, 2009.

[10] G.B. Dantzig. Linear programming under uncertainty. Management Science, 1:197–206, 1955.

[11] Brian C. Dean, Michel X. Goemans, and Jan Vondrák. Adaptivity and approximation for stochastic packing problems. In SODA, pages 395–404, 2005.

[12] Brian C. Dean, Michel X. Goemans, and Jan Vondrák. Approximating the stochastic knapsack problem: The benefit of adaptivity. Math. Oper. Res., 33, 2008.

[13] Moran Feldman, Joseph Naor, and Roy Schwartz. A unified continuous greedy algorithm for submodular maximization. In FOCS, 2011.

[14] Moran Feldman, Ola Svensson, and Rico Zenklusen. Online contention resolution schemes. CoRR, abs/1508.00142, 2015.

[15] Rajiv Gandhi, Samir Khuller, Srinivasan Parthasarathy, and Aravind Srinivasan. Dependent rounding and its applications to approximation algorithms. J. ACM, 53, 2006.

[16] Michel X. Goemans and Jan Vondrák. Stochastic covering and adaptivity. In LATIN, pages 532–543, 2006.

[17] Sudipto Guha and Kamesh Munagala. Approximation algorithms for budgeted learning problems. In STOC, pages 104–113, 2007.

[18] Sudipto Guha and Kamesh Munagala. Model-driven optimization using adaptive probes. In SODA, pages 308–317, 2007.
[19] Anupam Gupta, Ravishankar Krishnaswamy, Marco Molinaro, and R. Ravi. Approximation algorithms for correlated knapsacks and non-martingale bandits. In FOCS, pages 827–836, 2011.

[20] Anupam Gupta and Viswanath Nagarajan. A stochastic probing problem with applications. In IPCO, 2013.

[21] A. Schrijver. Combinatorial Optimization - Polyhedra and Efficiency. 2003.
A Stronger bound for measured continuous greedy

Recall, we need to prove Lemma A. To do so, we shall require the following additional facts from the analysis of [13].

Lemma 12 (Lemma 3.3 in [13])
Consider two vectors \(x, x' \in [0, 1]^E\), such that for every \(e \in E\), \(|x_e - x'_e| \leq \delta\). Then, \(F(x') - F(x) \geq \sum_{e \in E} (x'_e - x_e) \cdot \partial_e F(x) - O(n^3 \delta^2) \cdot f(OPT)\).

Lemma 13 (Lemma 3.5 in [13])
Consider a vector \(x \in [0, 1]^E\). Assuming \(x_e \leq a\) for every \(e \in E\), then for every set \(S \subseteq E\), \(F(x \vee 1_S) \geq (1 - a) f(S)\).

Lemma 14 (Lemma 3.6 in [13])
For every time \(0 \leq t \leq T\) and element \(e \in E\), \(y_e(t) \leq 1 - (1 - \delta)^{t/\delta} \leq 1 - \exp(-t) + O(\delta)\).

Proof. Applying Lemma 12 to the solutions \(y(t + \delta)\) and \(y(t)\), we have

\[
F(y(t + \delta)) - F(y(t)) \geq \sum_{e \in E} \delta \cdot I_e(t) (1 - y(t)) \cdot \partial_j F(y(t)) - O(n^3 \delta^2) \cdot f(OPT) \quad \text{(3)}
\]

\[
= \sum_{e \in E} \delta \cdot I_e(t) (1 - y(t)) \cdot \frac{F(y(t \vee 1_j)) - F(y(t))}{1 - y(t)} - O(n^3 \delta^2) \cdot f(OPT)
\]

\[
= \sum_{e \in E} \delta \cdot I_e(t) \cdot [F(y(t \vee 1_j)) - F(y(t))] - O(n^3 \delta^2) \cdot f(OPT)
\]

\[
\geq \sum_{e \in E} \delta \cdot x_e^+ [F(y(t \vee 1_j)) - F(y(t))] - O(n^3 \delta^2) \cdot f(OPT) \quad \text{(4)}
\]

where the last inequality follows from our choice of \(I(t)\).

Moreover, we have \(f^+(x^+) = \sum_{A \subseteq E} \alpha_A f(A)\) for some set of values \(\alpha_A\) satisfying \(\sum_{A \subseteq E} \alpha_A = 1\) and \(\sum_{A \subseteq E : e \in A} \alpha_A = x_e^+\). Thus, \(\sum_{e \in E} x_e^+ [F(y(t \vee 1_j)) - F(y(t))] = \sum_{A \subseteq E} \alpha_A \sum_{j \in A} [F(y(t \vee 1_j)) - F(y(t))] \geq \sum_{A \subseteq E} \alpha_A [F(y(t \vee 1_A)) - F(y(t))] \geq \sum_{A \subseteq E} \alpha_A \left[ (e^{-t} - O(\delta)) \cdot f(A) - F(y(t)) \right] = (e^{-t} - O(\delta)) \cdot f^+(x^+) - F(y(t)).\)

where the first inequality follows from the fact that \(F\) is concave in all positive directions, and the second from Lemmas 13 and 14. Combining this with the inequality (4), and noting that \(f^+(x^+) \geq f^+(OPT) = f(OPT)\), we finally obtain \(F(y(t + \delta)) - F(y(t)) \geq \delta \cdot [e^{-t} \cdot f^+(x^+) - F(y(t))] - O(n^3 \delta^2) \cdot f^+(x^+)\).

\[\square\]

B Transversal mappings

Recall the definition of a transversal matroid.
Definition 4
Consider a bipartite graph \((E \cup V, \subseteq E \times V)\). Let \(\mathcal{I}\) be a family of all subsets \(S\) of \(E\) such that there exists an injection from \(S\) to \(V\). Then \(M = (E, \mathcal{I})\) is a matroid, called transversal matroid.

We assume we know the graph \((E \cup V, \subseteq E \times V)\) of the matroid.

Let \(B^0\) be the initial support. Let \(A \in B^0\) be an independent set. From the definition of the transversal matroid, there exists an injection \(v^A : A \rightarrow V\); we shall say that \(a \in A\) is matched to \(v^A\). There can be many such injections for a given set, but we initially pick one for every \(A \in B^0\). When a set of the support will be changed we shall explicitly define an injection. In fact, only for added elements we will define a new match, for all other elements they will be matched all the time to the same vertex, as long as they are available.

For any two \(A,B \in B^0\) we define the mapping \(\phi^0[B,A] : B \rightarrow A \cup \{\perp\}\) as follows. Let \(v^A\) be the injection of \(A\), and let \(v^B\) be the injection of \(B\). If there exists element \(a\) such that \(v^A(a) = v^B(b)\), then we set \(\phi^0[B,A](b) = a\); if not, we set \(\phi^0[B,A](b) = \perp\).

Let us verify that such a definition satisfies first two properties.

Property 1. For each \(a \in A\) there is at most one \(b \in B\) for which \(\phi^0[B,A](b) = a\).

This one is trivially satisfied because there can be at most one \(b \in B\) that according to \(v^B\) is matched to \(v^A(a)\).

Property 2. For \(b \in B \setminus A\), if \(\phi^0[B,A](b) = \perp\), then \(A+b \in \mathcal{I}\), otherwise \(A-\phi^0[B,A](b)+b \in \mathcal{I}\).

Suppose \(\phi^0[B,A](b) = \perp\). It means that \(b\) is matched to \(v^B(b)\) to which no element \(a \in A\) is matched to. Therefore when we add edge \((b,v^B(b))\) to the injection \(\{(a,v^A(a))\}_{a \in A}\) it is still a proper injection, since \(b \notin A\), and so \(A+b \in \mathcal{I}\). Now suppose \(\phi^0[B,A](b) = a' \neq \perp\). Now the set \(A\) changes to \(A - a' + b\) and the underlying injection is \(\{(a,v^A(a))\}_{a \in A \setminus a'} \cup \{(b,v^B(b))\}\), which is a valid injection since \(b \notin A\), and if so, then \(A - a' + b\) is indeed an independent set. So Property 2 also holds.

Now let us move to the most technically demanding property.

Property 3. Let \(\phi^t\) be a family of transversal mappings for \(B^t\). Suppose we update the support \(B^t\) and obtain \(B^{t+1}\). Then we can find a family \(\phi^{t+1}\) of transversal mappings such that \(\Gamma^t(a) = \Gamma^{t+1}(a)\) for any element \(a\) that is still available after step \(t\).

Suppose that in step \(t\) we have chosen element \(c\), and we update the support \(B^t\) as described in the for loop of the algorithm in line 15. First of all assume that \(c \neq a\), otherwise \(a\) becomes unavailable so there is nothing to prove. Let \(C^t\) be the critical set of \(c\) and let \(v^{C^t}(c)\) be the vertex to which \(c\) is matched according to \(v^{C^t}\). Consider set \(B^t \in B^t\) and let us describe how \(c\) affects \(B^{t+1}\) and injection \(v^{B^{t+1}}\).

Case 1, \(c \in B^t\) : If it is \(c\) from \(B^t\) that is matched to \(v^{C^t}(c)\), i.e., \(v^{B^t}(c) = v^{C^t}(c)\), then we do not have to change anything, we set \(B^{t+1} := B^t\) and \(v^{B^{t+1}} = v^{B^t}\). If \(v^{B^t}(c) \neq v^{C^t}(c)\), then let \(b_1\) be such that \(v^{B^t}(b_1) = v^{C^t}(c)\). We remove \(b_1\) from \(B^t\), i.e., \(B^{t+1} := B^t \setminus b_1\) (we do not have to add \(c\) to \(B^{t+1}\) because it is already there). For every \(b_3 \in B^{t+1}\) we set \(v^{B^{t+1}}(b_3) = v^{B^t}(b_3)\).

See Figure 1.
Figure 1: Illustration of Case 1, \( c \in B^t \)
Case 2, $c \notin B^t$: Let $b_1$ be such that $v^{B^t}(b_1) = v^{C^t}(c)$. We remove $b_1$ from $B^t$ and add $c$ instead, i.e., $B^{t+1} = B^t - b_1 + c$. The injection is defined as: $v^{B^{t+1}}(c) = v^{C^t}(c)$, and $v^{B^{t+1}}(b_3) = v^{B^t}(b_3)$ for $b_3 \in B^t \setminus b_1$. See Figure 2.

---

Figure 2: Illustration of Case 2, $c \notin B^t$
Given sets $A^{t+1}, B^{t+1}$ with corresponding injections $v^{A^{t+1}}, v^{B^{t+1}}$ we define mapping $\phi^{t+1}$ as before, i.e., $\phi^{t+1}\left[B^{t+1}, A^{t+1}\right](b) = a$, if $v^{A^{t+1}}(a) = v^{B^{t+1}}(b)$.

Now we need to show that the set $\Gamma^{t+1}(a) = \left\{ b \middle| b \neq a \land \phi\left[B_{c(b)}, B_{c(a)}^{t}\right](b) = a \right\}$ is equal to $\Gamma^t(a) = \left\{ b \middle| b \neq a \land \phi\left[B_{c(b)}^{t+1}, B_{c(a)}^{t+1}\right](b) = a \right\}$, if $a$ is still available.

Consider again sets $A^{t+1}, B^{t+1}$ and suppose that $A^t$ is the critical set of $a$ and that $B^t$ is the critical set of $b$. Suppose that both $a, b$ are matched to $v_{ab} = v^{A^t}(a) = v^{B^t}(b)$, i.e., $b \in \Gamma^t(a)$. If it happened that in step $t$ element $c$ removed $a$ and $b$, i.e., $v(C^t)(c) = v_{ab}$, then elements $a, b$ are blocked and not available, so there is nothing to prove here. If $v(C^t)(c) \neq v$, then from the reasoning in Case 1 and 2, we know that $a$ and $b$ are still matched to $v_{ab}$, i.e., $v_{ab} = v^{A^{t+1}}(a) = v^{B^{t+1}}(b)$. But if so, then $\phi^{t+1}\left[B^{t+1}, A^{t+1}\right](b) = a$, and $b \in \Gamma^{t+1}(a)$ still, because $A^{t+1}, B^{t+1}$ join critical sets of $a, b$. Conversely, if $b$ is not matched to $v^{A^t}(a)$, i.e., $v^{B^t}(b) \neq v^{A^t}(a)$, then $b \notin \Gamma^t(a)$. But if $c$ during the update does not block $b$, then $b$ does not change its matched vertex so we still have $v^{B^{t+1}}(b) \neq v^{A^{t+1}}(a)$, and still $b \notin \Gamma^{t+1}(a)$.

Illustration is given in Figure 3

![Figure 3: Illustration of change in $\Gamma$. We have blocked $b_1$, and if $b_1 \in \Gamma^t(a_1)$, then it does not matter anyway, because we have also blocked $a_1$. Element $c$ was matched (w.r.t. $B_t$) to the same vertex as $a_2$, but $B_t$ is not the critical set of $c$, so $c \notin \Gamma^t(a_2)$. Assume $B^t$ is a critical set of $b_3$: we have $\phi^t[B^t, A^t](b_3) = a_3$ and so $b_3 \in \Gamma^t(a_3)$; after the update we still have $\phi^{t+1}[B^{t+1}, A^{t+1}](b_3) = a_3$, so $b_3 \in \Gamma^{t+1}(a_3)$. Element $a_4$ did not have any element $b' \in B^t$ in $\Gamma^t(a_4)$, so it does not have any $b' \in B^{t+1}$ in $\Gamma^{t+1}(a_4)$ as well.](image)

C Omitted proofs

**Lemma 5.** Let $OPT$ be the optimal feasible strategy for the stochastic probing problem in our general setting, then $E\left[f\left(OPT\right)\right] \leq f^+(x^+ \cdot p)$.

**Proof.** We construct a feasible solution $x$ of the following program

$$\text{maximize } \left\{ f^+(x \cdot p) \middle| x \cdot p \in \bigcap_{j=1}^{k_{\text{in}}} P\left(M_{j,\text{in}}^\text{in}\right), \ x \in \bigcap_{j=1}^{k_{\text{out}}} P\left(M_{j,\text{out}}^\text{out}\right) \right\},$$

by setting $x_e = P\left[OPT \text{ probes } e\right]$. First, we show that this is indeed a feasible solution. Since $OPT$ is a feasible strategy, the set of elements $Q$ probed by any execution of $OPT$ is always an independent set of each outer matroid $M_{j,\text{out}}^\text{out} = \left(E, \mathcal{T}_{j,\text{out}}^\text{out}\right)$, i.e. $Q \in \bigcap_{j=1}^{k_{\text{out}}} \mathcal{T}_{j,\text{out}}^\text{out}$. Thus the vector
In what follows let us skip writing $\mathbf{1}_A \mid A \in \bigcap_{j=1}^{k_{\text{out}}} \mathcal{I}_j^{\text{out}}$, and so $x \in \mathcal{P}\left(\mathcal{M}_j^{\text{out}}\right)$ for any $j \in \{1, \ldots, k_{\text{out}}\}$. Analogously, the set of elements $S$ that were successfully probed by $\text{OPT}$ satisfy $S \in \bigcap_{j=1}^{k_{\text{in}}} \mathcal{I}_j^{\text{in}}$ for every possible execution of $\text{OPT}$. Hence, the vector $\mathbb{E}[\mathbf{1}_S] = x \cdot p$ may be represented as a convex combination of vectors from $\mathbf{1}_A \mid A \in \bigcap_{j=1}^{k_{\text{out}}} \mathcal{I}_j^{\text{out}}$ and so $x \cdot p \in \mathcal{P}\left(\mathcal{M}_j^{\text{in}}\right)$ for any $j \in \{1, \ldots, k_{\text{in}}\}$. The value $f^+(x \cdot p)$ gives the maximum value of $\mathbb{E}_{\mathcal{S},\mathcal{D}}[f(S)]$ over all distributions $\mathcal{D}$ satisfying $\mathbb{P}_{S \sim \mathcal{D}}[e \in S] \leq x_e p_e$. The solution $S$ returned by $\text{OPT}$ satisfies $\mathbb{P}[e \in S] = \mathbb{P}[\text{OPT probes } e] p_e = x_e p_e$. Thus, $\text{OPT}$ defines one such distribution, and so we have $\mathbb{E}[f(\text{OPT})] \leq f^+(x \cdot p) \leq f^+(x^+ \cdot p)$. \hfill \Box

Lemma 11. If $p \cdot x \in \mathcal{P}(\mathcal{M}^{\text{in}})$, then for any element $e$, it holds that

$$\mathbb{E}_{\mathcal{C},R(x)}\left[\sum_{f \in \Gamma(e)} p_f \cdot \chi[f \in R(x)]\right] \leq 1,$$

where the expectation is over $R(x)$ and the choice of critical sets $\mathcal{C}$; here $\chi[\mathcal{E}]$ is a 0-1 indicator of random event $\mathcal{E}$.

Proof. In what follows let us skip writing 0 in the superscript of bases $B_i^0$, mappings $\phi^0$, and set $\Gamma^0(e)$.

Let us condition for now on the critical set $B_{c(e)}$ of element $e$. For $f$ to belong to $\Gamma(e)$ it has to be the case that $\phi[B_{c(f)}, B_{c(e)}](f) = e$. Therefore

$$\sum_{f \in \Gamma(e)} p_f \cdot \chi[f \in R(x)] = \sum_{f \in E \setminus \Gamma(e)} p_f \cdot \chi[f \in R(x)] \cdot \left(\sum_{i \phi[B_i, B_{c(e)}](f) = e} \chi[B_i \text{ is } f\text{-critical}]\right),$$

and by changing the order of summation it is equal to

$$\sum_{i} \sum_{f \in B_i \setminus e : \phi[B_i, B_{c(e)}](f) = e} p_f \cdot \chi[f \in R(x)] \cdot \chi[B_i \text{ is } f\text{-critical}].$$

Consider $f$ such that $f \in B_i \setminus e : \phi[B_i, B_{c(e)}](f) = e$. Since $\chi[f \in R(x)]$ and $c(f)$ (the index critical set of $f$) are independent, and $\mathbb{E}[\chi[f \in R(x)]] = x_f$ and $\mathbb{P}[B_i \text{ is } f\text{-critical}|B_{c(e)}] = \mathbb{P}[i = c(f)|B_{c(e)}] = \frac{\beta_i}{p_f x_f}$, we get that

$$\mathbb{E}\left[p_f \cdot \chi[f \in R(x)] \cdot \chi[B_i^f \text{ is } f\text{-critical}] \mid B_{c(e)}\right] = p_f x_f \cdot \frac{\beta_i}{p_f x_f} = \beta_i,$$

and hence

$$\mathbb{E}\left[\sum_{f \in \Gamma(e)} p_f \cdot \chi[f \in R(x)] \mid B_{c(e)}\right] = \sum_{i} \sum_{f \in B_i \setminus e : \phi[B_i, B_{c(e)}](f) = e} \beta_i^f \leq \sum_{i} \beta_i^f = 1,$$

where the inequality follows from the fact that for each $B_i$ there can be at most one element $f \in B_i$ such that $\phi[B_i, B_{c(e)}](f) = e$. \hfill \Box

Lemma 10 Consider a probing problem $(E, p, \mathcal{I}^{\text{in}}, \mathcal{I}^{\text{out}})$. Suppose we have a $(b, c_{\text{out}})$-balanced CR-scheme $\pi^{\text{out}}$ for $\mathcal{P}(\mathcal{I}^{\text{out}})$, and a $(b, c_{\text{in}})$-balanced ordered CR scheme $\pi^{\text{in}}$ for $\mathcal{P}(\mathcal{I}^{\text{in}})$. Then there exists a $(b, c_{\text{out}} \cdot c_{\text{in}})$-balanced stock-CH scheme for $\mathcal{P}(\mathcal{I}^{\text{in}}, \mathcal{I}^{\text{out}})$.

Proof. First let us recall a Lemma from [8].

Lemma 1.6 from [8] Let $\mathcal{I} = \bigcap_i \mathcal{I}_i$ and $\mathcal{P}_\mathcal{I} = \bigcap_i \mathcal{P}_i$. Suppose each $\mathcal{P}_\mathcal{I}_i$ has a monotone $(b, c_i)$-balanced CR scheme. Then $\mathcal{P}_\mathcal{I}$ has a monotone $(b, \prod_i c_i)$-balanced CR scheme defined as $\pi_x(A) = \bigcap_i \pi_{x_i}^i(A)$ for $A \subseteq N, x \in b \mathcal{P}_\mathcal{I}$.\hfill \Box

C OMMITTED PROOFS
Suppose we have a CR scheme \( \pi_{\text{out}} \) for \( \mathcal{P}(I_{\text{out}}) \) and an ordered CR scheme \( \pi_{\text{in}} \) for \( \mathcal{P}(I_{\text{in}}) \). We would like to define the stochastic contention resolution scheme for \( \mathcal{P}(I_{\text{in}}, I_{\text{out}}) \) just as in the Lemma above, i.e., as \( \pi_x(A) = \pi_{\text{out}}(A) \cap \pi_{\text{in}}^p(\text{act}(A)) \). However, we cannot just simply run \( \pi_{\text{out}} \) on \( A \), and then \( \pi_{\text{in}}^p \) on \( A \) again, and take the intersection, because that does not constitute a feasible probing strategy. Once again, we need to make use of simulated probes to get a stoch-CR scheme that will have a probability distribution of \( \pi_{\text{out}}(A) \cap \pi_{\text{in}}^p(\text{act}(A)) \). How to implement such a strategy? We first run \( \pi_{\text{out}}(A) \) on the set \( A \). Later we use \( \pi_{\text{in}}^p \) to scan elements of \( A \) in the order given from the definition of an ordered scheme. If \( \pi_{\text{in}}^p \) considers element \( e \) such that \( e \in A \setminus \pi_{\text{out}} \), then we simulate the probe of \( e \); if the \( e \in \pi_{\text{out}}(A) \), then we just probe it. Therefore, the CR-scheme \( \pi_{\text{in}}^p \) works in fact on the set \( \text{act}(\pi_{\text{out}}(A) \cup (A \setminus \pi_{\text{out}}(A))^{\text{virt}}) \), where \( (A \setminus \pi_{\text{out}}(A))^{\text{virt}} \) represents simulated probes of elements in \( A \setminus \pi_{\text{out}}(A) \). Assuming \( e \in \pi_{\text{out}}(A) \), it is easy to see that elements in \( \text{act}(A) \) and elements in \( \text{act}(\pi_{\text{out}}(A) + (A \setminus \pi_{\text{out}}(A))^{\text{virt}}) \) have the same probability distribution. Therefore,

\[
\mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(A)) \right] = \\
\mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(\pi_{\text{out}}(A) \cup (A \setminus \pi_{\text{out}}(A))^{\text{virt}})) \mid \pi_{\text{out}}(A), e \in \pi_{\text{out}}(A) \right]. 
\]

And the RHS corresponds to a second phase of a feasible probing strategy. Thus we have:

\[
\mathbb{P} \left[ e \in \pi_x(A) \right] = \\
\mathbb{P} \left[ e \in \pi_{p,x}^p(\pi_{\text{out}}(A) \cup (A \setminus \pi_{\text{out}}(A))^{\text{virt}}) \right] = \\
\mathbb{E}_{\pi_{p,x}^p} \left[ \chi \left[ e \in \pi_{p,x}^p(\pi_{\text{out}}(A) \cup (A \setminus \pi_{\text{out}}(A))^{\text{virt}}) \right] \mid \pi_{\text{out}}(A), e \in \pi_{\text{out}}(A) \right].
\]

Now just note that

\[
\mathbb{E}_{\pi_{p,x}^p} \left[ \chi \left[ e \in \pi_{p,x}^p(\pi_{\text{out}}(A) \cup (A \setminus \pi_{\text{out}}(A))^{\text{virt}}) \right] \mid \pi_{\text{out}}(A), e \in \pi_{\text{out}}(A) \right] = \\
\mathbb{P} \left[ e \in \pi_{p,x}^p(\pi_{\text{out}}(A) \cup (A \setminus \pi_{\text{out}}(A))^{\text{virt}}) \mid \pi_{\text{out}}(A), e \in \pi_{\text{out}}(A) \right] = \\
\mathbb{P} \left[ e \in \pi_{p,x}^p(\pi_{\text{out}}(A)) \right] \cdot \mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(A)) \right],
\]

from line [8], and so we can simplify the previous expression to

\[
\mathbb{P} \left[ e \in \pi_x(A) \right] = \mathbb{E}_{\pi_{p,x}^p} \left[ \chi \left[ e \in \pi_{p,x}^p(\pi_{\text{out}}(A)) \right] \cdot \mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(A)) \right] \right] = \\
\mathbb{E}_{\pi_{p,x}^p} \left[ \chi \left[ e \in \pi_{p,x}^p(\pi_{\text{out}}(A)) \right] \right] \cdot \mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(A)) \right] = \\
\mathbb{P} \left[ e \in \pi_{p,x}^p(\pi_{\text{out}}(A)) \right] \cdot \mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(A)) \right].
\]

Now the analysis just follows the lines of Lemma 1.6 from [3]. We plug \( R(x) \) for \( A \), and apply expectation on \( R(x) \) conditioned on \( e \in R(x) \) to get:

\[
\mathbb{E}_{R(x)} \left[ \mathbb{P} \left[ e \in \pi_x(R(x)) \mid e \in R(x) \right] \right] = \mathbb{E}_{R(x)} \left[ \mathbb{P} \left[ e \in \pi_{p,x}^p(R(x)) \right] \cdot \mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(R(x))) \mid e \in R(x) \right] \right].
\]

From the fact that both \( \pi_{\text{out}}, \pi_{\text{in}} \) are monotone, and \( \text{act}(R(x)) \) is an increasing function of \( R(x) \) we get that \( \mathbb{P} \left[ e \in \pi_{\text{out}}(R(x)) \right] \) and also \( \mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(R(x))) \right] \) are increasing functions of \( R(x) \). Thus applying FKG inequality gives us that

\[
\mathbb{E}_{R(x)} \left[ \mathbb{P} \left[ e \in \pi_x(R(x)) \mid e \in R(x) \right] \right] = \mathbb{E}_{R(x)} \left[ \mathbb{P} \left[ e \in \pi_{p,x}^p(R(x)) \right] \cdot \mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(R(x))) \mid e \in R(x) \right] \right] \geq \mathbb{E}_{R(x)} \left[ \mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(R(x))) \mid e \in R(x) \right] \right] \cdot \mathbb{E}_{R(x)} \left[ \mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(R(x))) \mid e \in R(x) \right] \right] \geq b \cdot c_{\text{out}} \cdot \mathbb{E}_{R(x)} \left[ \mathbb{P} \left[ e \in \pi_{p,x}^p(\text{act}(R(x))) \right] \right].
\]
Now applying also expectation on $\text{act}(R(x))$, we get finally

$$
\mathbb{E}_{R(x),\text{act}(R(x))} [\mathbb{P} [e \in \pi_x(R(x)) | e \in R(x)] \\
\geq c_{\text{out}} \cdot \mathbb{E}_{R(x),\text{act}(R(x))} [\mathbb{P} [e \in \pi_x^{\text{in}}(\text{act}(R(x))) | e \in R(x)] \geq p_c \cdot c_{\text{out}} \cdot c_{\text{in}}.
$$

(7)

Also, directly from equation $\mathbb{P} [e \in \pi_x(A)] = \mathbb{P} [e \in \pi_x^{\text{out}}(A)] \cdot \mathbb{P} [e \in \pi_x^{\text{in}}(\text{act}(A))]$ we get the monotonicity of the stochastic CR-scheme $\pi_x$, since both $\pi_x^{\text{out}}$ and $\pi_x^{\text{in}}(\text{act}(\cdot))$ are monotone. □

## D Martingale Theory

### Definition 5

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\Omega$ is a sample space, $\mathcal{F}$ is a $\sigma$-algebra on $\Omega$, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$. Sequence $\{\mathcal{F}_t : t \geq 0\}$ is called a filtration if it is an increasing family of sub-$\sigma$-algebras of $\mathcal{F}$: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}$. Intuitively speaking, when considering a stochastic process, $\sigma$-algebra $\mathcal{F}_t$ represents all information available to us right after making step $t$. In our case $\sigma$-algebra $\mathcal{F}_t$ contains all information about each randomly chosen element to probe, about outcome of each probe, and about each support update for every matroid, that happened before or at step $t$.

### Definition 6

A process $(Z_t)_{t \geq 0}$ is called a martingale if for every $t \geq 0$ all following conditions hold:

1. random variable $Z_t$ is $\mathcal{F}_t$-measurable,
2. $\mathbb{E}[|Z_t|] < \infty$,
3. $\mathbb{E}[Z_{t+1} | \mathcal{F}_t] = X_t$.

### Definition 7

Random variable $\tau : \Omega \mapsto \{0, 1, \ldots\}$ is called a stopping time if $\{\tau = t\} \in \mathcal{F}_t$ for every $t \geq 0$.

Intuitively, $\tau$ represents a moment when an event happens. We have to be able to say whether it happened at step $t$ given only the information from steps $0, 1, 2, \ldots, t$. In our case we define $\tau$ as the moment when element became unavailable, i.e., it was chosen to be probed or it was blocked by other elements. It is clear that this is a stopping time according to the above definition.

### Theorem 15 (Doob’s Optional-Stopping Theorem)

Let $(Z_t)_{t \geq 0}$ be a martingale. Let $\tau$ be a stopping time such that $\tau$ has finite expectation, i.e., $\mathbb{E}[\tau] < \infty$, and the conditional expectations of the absolute value of the martingale increments are bounded, i.e., there exists a constant $c$ such that $\mathbb{E}[|Z_{t+1} - Z_t| | \mathcal{F}_t] \leq c$ for all $t \geq 0$. If so, then $\mathbb{E}[Z_{\tau}] = \mathbb{E}[Z_0]$.

In our case $\mathbb{E}[\tau] \leq |A|$, because we just pick an element at random from $A$, so the expected value of picking $e$ to be probed is exactly $|A|$, and since it can be earlier blocked by other elements, we have $\mathbb{E}[\tau] \leq |A|$. Also the martingale we use is $\left(1 + \sum_{f \in \Gamma_x(e)} P_f X_f\right) \cdot P_{\ell}^{\text{out}} + Y_{\ell}^{\text{in}} \geq 0$, and since $P_{\ell}^{\text{out}}, Y_{\ell}^{\text{in}} \in \{0, 1\}$ then it means that for any $t$ we have $\left|1 + \sum_{f \in \Gamma_x(e)} P_f X_f\right| \cdot P_{\ell}^{\text{out}} + Y_{\ell}^{\text{in}} \leq |A| + 1$, and so $\mathbb{E}[|Z_{t+1} - Z_t| | \mathcal{F}_t] \leq |A| + 1$. Therefore, we can use Doob’s Optional-Stopping Theorem in our analysis.
Algorithm 2 Stoch-CR scheme $\pi_x(A)$ for $\mathcal{M}^\text{in}_1, \ldots, \mathcal{M}^\text{in}_{k\text{in}}$ inner matroids and $\mathcal{M}^\text{out}_1, \ldots, \mathcal{M}^\text{out}_{k\text{out}}$ outer matroids, and $x \in b \cdot \mathcal{P}(\mathcal{I}^\text{in}, \mathcal{I}^\text{out})$.

1: //Preprocessing:
2: find support $\mathcal{B}^\text{in}_{i,j}$ of $\frac{1}{b}p \cdot x \in \mathcal{P}(\mathcal{M}^\text{in}_j)$ for each $\mathcal{M}^\text{in}_j$;
3: find support $\mathcal{B}^\text{out}_{i,j}$ of $\frac{1}{b}x \in \mathcal{P}(\mathcal{M}^\text{out}_j)$ for each $\mathcal{M}^\text{out}_j$;
4: find family $\phi^\text{in}_{i,j}$ for every $\mathcal{M}^\text{in}_j$ and find family $\phi^\text{out}_{i,j}$ for every $\mathcal{M}^\text{out}_j$;
5: choose critical sets $\mathcal{C}^\text{in}_{i,j}$ for each $\mathcal{M}^\text{in}_j$ and $\mathcal{C}^\text{out}_{i,j}$ for each $\mathcal{M}^\text{out}_j$;

6: //Random selection phase:
7: remove from $A$ all $e$ such that $x_e = 0$; mark all $e \in A$ as available; $S \leftarrow \emptyset$
8: while there are still available elements in $A$ do
9: if $e$ is available then
10: probe $e$
11: if probe of $e$ successful then
12: $S \leftarrow S \cup \{e\}$
13: for each matroid $\mathcal{M}^\text{in}_j$ do
14: for each set $B^t_i$ of support $\mathcal{B}^\text{in}_{i,j}$ do
15: $B^t_i \leftarrow B^t_i + e$
16: call $e$ unavailable
17: else simulate the probe of $e$
18: if probe or simulation was successful then
19: for each set $B^t_i$ of support $\mathcal{B}^t$ do
20: $f \leftarrow \phi \left[ B^t_{c^\text{in}(e)}(e), B^t_{c^\text{in}(f)}(e) \right]$
21: if $f \neq e$ then $B^t_{c^\text{in}(e)} \leftarrow B^t_{c^\text{in}(e)} - f$ and call $f$ unavailable
22: for each matroid $\mathcal{M}^\text{out}_j$ do
23: for each set $B^t_i$ of support $\mathcal{B}^t_{\text{out}[j]}$ do
24: $B^t_i \leftarrow B^t_i + e$
25: $f \leftarrow \phi \left[ B^t_{c^\text{out}(e)}, B^t_{c^\text{out}(f)} \right]$
26: if $f \neq e$ then $B^t_{c^\text{out}(e)} \leftarrow B^t_{c^\text{out}(e)} - f$ and call $f$ unavailable
27: compute families $\phi^{t+1}_{\text{in}[j]}$, $\phi^{t+1}_{\text{out}[j]}$,
28: $t \leftarrow t + 1$
30: return $S$ as $\pi_x(A)$
E Full proof of Lemma 3

The full scheme is presented on Figure 2. Let us concisely denote by $C$ all the critical sets chosen, i.e., $C = \left( \binom{\mathcal{M}_{in}^0}{j} \right)_{j \in [k_{in}]} \times \left( \binom{\mathcal{M}_{out}^0}{j} \right)_{j \in [k_{out}]}$. Transversal mappings $\phi (\mathcal{M})$ are found in exactly the same manner as in the single matroid version.

There are two main differences with respect to what was presented in the main body. First, since $p \cdot x \in b \cdot \mathcal{P} (\mathcal{M}_{in}^0)$, then $p \cdot x \in \mathcal{P} (\mathcal{M}_{out}^0)$, and the support $\mathcal{B}_{in[j]}^0$ is found by decomposing $\frac{1}{b}p \cdot x$. Thus when element $f$ chooses a critical set in matroid $\mathcal{M}_{in}^0$ it chooses with probability $\beta_{in[j]}^0 / (\frac{1}{b}p_f x_f) = b \cdot \beta_{in[j]}^0 / (p_f x_f)$. This results in the following modification in Lemma 11. The proof is completely analogous, so we skip it.

**Lemma 16**

If $\frac{1}{b}p \cdot x \in \mathcal{P} (\mathcal{M}_{in}^0)$, then for any element $e$, it holds that

$$
\mathbb{E}_{\mathcal{C}_{in}, X} \left[ \sum_{f \in \Gamma_{in[j]}^0} p_f \cdot \chi [f \in R(x)] \right] \leq b, \text{ where the expectation is over } R(x) \text{ and the choice of critical sets } C.
$$

Now we also need to deal with outer matroids. Again, the proof is completely analogous, so we skip it.

**Lemma 17**

If $\frac{1}{b}x \in \mathcal{P} (\mathcal{M}_{out}^0)$, then for any element $e$, it holds that

$$
\mathbb{E}_{\mathcal{C}_{out}, X} \left[ \sum_{f \in \Gamma_{out[j]}^0} \chi [f \in R(x)] \right] \leq b, \text{ where the expectation is over } R(x) \text{ and the choice of critical sets } C.
$$

Second. Let $Y^t_e$ for $t = 0, 1, \ldots$, be a random variable indicating if $e$ is still available after step $t$. Initially $Y^0_e = X_e$. Let $P^t_e$ be a random variable indicating, if $e$ was probed in one of the steps $0, 1, \ldots, t$. In step $t + 1$ element $e$ can be blocked if, for some $j \in [k_{in}]$, we pick element $f \in \Gamma_{in[j]}^0 (e)$ and successfully probe it (or successfully simulate), or if we just pick element $f \in \Gamma_{out[j]}^0 (e)$, and probe it or simulate its probe, disregarding of the outcome. Let $\Gamma_{in}^t (e) = \bigcup_{j \in [k_{in}]} \Gamma_{in[j]}^t (e)$ and let $\Gamma_{out}^t (e) = \bigcup_{j \in [k_{in}]} \Gamma_{out[j]}^t (e) \setminus \Gamma_{in}^t (e)$ — this subtraction is to not count an element twice, because if element $f$ belongs to both $\Gamma_{in[j]}^t (e)$ and $\Gamma_{out[j]}^t (e)$, then just probing $f$ (or simulating its probe) automatically blocks $e$, disregarding of $f$’s probe (simulation) outcome. Hence, we do not account for the excessive $p_f X_f$ influence of $f$ on $e$. Therefore, the probability that $e$ stops to be available at step $t + 1$ is equal to

$$
\mathbb{E} \left[ Y^t_e - Y^{t+1}_e \bigg| \mathcal{F}^t, C \right] = \frac{Y^t_e}{|A|} + \frac{Y^t_e}{|A|} \sum_{f \in \Gamma_{in}^t (e)} p_f X_f + \frac{Y^t_e}{|A|} \sum_{f \in \Gamma_{out}^t (e)} X_f.
$$

As in the single matroid case, the probability of probing $e$ is just equal to $\mathbb{E} \left[ P^{t+1}_e - P^t_e \bigg| \mathcal{F}^t, C \right] = \frac{Y^t_e}{|A|}$. Thus the martingale we use now in the analysis is

$$
\left( 1 + \sum_{f \in \Gamma_{in}^t (e)} p_f X_f + \sum_{f \in \Gamma_{out}^t (e)} X_f \right) \cdot P^t_e + Y^t_e
$$

for $t \geq 0$.

Let $\tau = \min \left\{ t \big| Y^t_e = 0 \right\}$ be the step in which edge $e$ became unavailable. It is clear that $\tau$ is a stopping time. Thus from Doob’s stopping theorem we get that
\[ \mathbb{E}_x \left[ \left( 1 + \sum_{f \in \Gamma_{in}^0(e)} p_f X_f + \sum_{f \in \Gamma_{out}^0(e)} X_f \right) \cdot P_e x + Y_e \right] \]
\[ = \mathbb{E} \left[ \left( 1 + \sum_{f \in \Gamma_{in}^0(e)} p_f X_f + \sum_{f \in \Gamma_{out}^0(e)} X_f \right) \cdot P_e x + Y_e \right] , \]

And as before, since the transversal mappings are controlled per matroid, we have that \( \Gamma_{in}^t(e) = \Gamma_{in}^0(e) \) and \( \Gamma_{out}^t(e) = \Gamma_{out}^0(e) \) for \( t \leq \tau \). Thus

\[ P \left[ e \in \bar{\pi}_x (A) \middle| C \right] = p_e \cdot \mathbb{E}_x \left[ P_e x \right] = p_e X_e \left( 1 + \sum_{f \in \Gamma_{in}^0(e)} p_f X_f + \sum_{f \in \Gamma_{out}^0(e)} X_f \right) . \quad (8) \]

Monotonicity follows as before. The approximation guarantee similarly from Jensen.

We take random set \( R(x) \) instead of \( A \); now \( X_f = \chi \left[ f \in R(x) \right] \) is a random variable. Let us condition on \( e \in R(x) \), take expected value \( \mathbb{E}_{C,R(x)} \left[ \cdot \middle| e \in R(x) \right] \) on both sides of (8) and apply Jensen’s inequality to convex function \( x \mapsto \frac{1}{x} \) to get:

\[ P \left[ e \in \bar{\pi}_x (R(x)) \middle| e \in R(x) \right] = \mathbb{E}_{C,R(x)} \left[ P \left[ e \in \bar{\pi}_x (R(x)) \middle| C, R(x) \right] \middle| e \in R(x) \right] = \mathbb{E}_{C,R(x)} \left[ p_e X_e \left( 1 + \sum_{f \in \Gamma_{in}^0(e)} p_f X_f + \sum_{f \in \Gamma_{out}^0(e)} X_f \right) \middle| e \in R(x) \right] \geq p_e \mathbb{E}_{C,R(x)} \left[ 1 + \sum_{f \in \Gamma_{in}^0(e)} p_f X_f + \sum_{f \in \Gamma_{out}^0(e)} X_f \middle| e \in R(x) \right] . \]

Since

\[ \mathbb{E}_{C,R(x)} \left[ \sum_{f \in \Gamma_{in}^0(e)} p_f X_f \middle| e \in R(x) \right] \leq b \]

from Lemma 16 and

\[ \mathbb{E}_{C,R(x)} \left[ \sum_{f \in \Gamma_{out}^0(e)} X_f \middle| e \in R(x) \right] \leq b \]

from Lemma 17 we conclude that

\[ \mathbb{E}_{C,R(x)} \left[ 1 + \sum_{f \in \Gamma_{in}^0(e)} p_f X_f + \sum_{f \in \Gamma_{out}^0(e)} X_f \middle| e \in R(x) \right] \leq \mathbb{E}_{C,R(x)} \left[ 1 + \sum_{j \in [k^{in}]} \sum_{f \in \Gamma_{in}^0(e)} p_f X_f + \sum_{j \in [k^{out}]} \sum_{f \in \Gamma_{out}^0(e)} X_f \middle| e \in R(x) \right] \leq 1 + \sum_{j \in [k^{in}]} \mathbb{E}_{C,R(x)} \left[ \sum_{f \in \Gamma_{in}^0(e)} p_f X_f \middle| e \in R(x) \right] + \sum_{j \in [k^{out}]} \mathbb{E}_{C,R(x)} \left[ \sum_{f \in \Gamma_{out}^0(e)} X_f \middle| e \in R(x) \right] \leq 1 + k^{in} \cdot b + k^{out} \cdot b. \]
And therefore
\[ P \left[ e \in \pi_x (R(x)) \mid e \in R(x) \right] \geq \frac{p_e}{b(2k^n + k^{out}) + 1}, \]
which yields
\[ P \left[ e \in \pi_x (R(x)) \mid e \in act (R(x)) \right] \geq \frac{1}{b(2k^n + k^{out}) + 1}, \]
which is exactly Property 3 from the definition of stoch-CR scheme. Lemma 3 follows.

## F Stochastic $k$-set packing

In this section we are showing a $(k + 1)$-approximation algorithm for Stochastic $k$-Set Packing.

We are given $n$ elements/columns, where each item $e \in E = [n]$ has a profit $v_e \in \mathbb{R}_+$, and a random $d$-dimensional size $S_e \in \{0, 1\}^d$. The sizes are independent for different items. Additionally, for each item $e$, there is a set $C_e$ of at most $k$ coordinates such that each size vector $S_e$ takes positive values only in these coordinates, i.e., $S_e \subseteq C_e$ with probability 1. We are also given a capacity vector $b \in \mathbb{Z}_+^d$ into which items must be packed. We assume that $v_e$ is a random variable that can be correlated with $S_e$. The coordinates of $S_e$ also might be correlated between each other.

Important thing to notice is that in this setting, unlike in the previous ones, here when we probe an element, there is no success/failure outcome. The size $S_e$ of an element $e$ materializes, and the reward $v_e$ is just drawn.

Let $p^j_e = \mathbb{E} [S_e (j)]$ be the expected size of the $j$ coordinate of column $e$. The following is an LP that models the problem. Here $U (e)$ denotes a uniform matroid of rank $c$.

\[
\begin{align*}
\max & \quad \sum_{e=1}^{n} \mathbb{E} [v_e] \cdot x_e \\
\text{s.t.} & \quad p^j \cdot x \in \mathcal{P} (U (b_j)) \quad \forall j \in [d] \\
& \quad x_e \in [0, 1] \quad \forall e \in [n].
\end{align*}
\]

Where, as usual, $x_e$ stands for $\mathbb{P} [OPT \text{ probes column } e]$. We are going to present a probing strategy in which for every element $e$ probability that we will probe $e$ will be at least $\frac{p_e}{k+1}$. From this the Theorem will follow.

The algorithm is presented on Figure 3.

Constraint for row $j$ is in fact given be a uniform matroid in which we can take at most $b_j$ elements from subset $\{e \mid j \in C_e\} \subseteq E$. Therefore, we can decompose $p^j \cdot x = \sum_l \beta^j_l \cdot B^j_l$. Uniform matroid is a transversal matroid, so we use the transversal mapping $\phi^j_l$ between sets $B^j_l$, also let $C = (C^j)_{j \in [d]}$ be the vector indicating the critical sets. We define in the same way as we did already in Lemma 3 the sets $\Gamma^j_l (e)$ of blocking elements, i.e.,

\[ \Gamma^j_l (e) = \left\{ f \mid f \neq e \wedge \phi^j_l [B^j_l (f), B^j_l (e)] \right\}. \]

As before, let us from now on condition on $C$. Let us analyze the impact of $f \in \Gamma^j_l (e)$ on $e$. Element $f \in \Gamma^j_l (e)$ blocks $e$ when $f$ is chosen and $S_f (j) = 1$. However, right now $f$ can belong to $\Gamma^j_l (e)$ for many $j \in C_e$. Therefore if $f$ is chosen in line 9 of the Algorithm 3 then the probability that $f$ blocks $e$ is equal to $\mathbb{P} \left[ \bigvee_{j \in \Gamma^j_l (e)} (S_f (j) = 1) \right] | C].$

Let us now repeat the steps of Lemma 1. Let $X_e = \chi [e \in R (x)]$. Let $Y^t_e$ for $t = 0, 1, ..., $ be a random variable indicating if $e$ is still available after step $t$. Initially $Y^0_e = X_e$. Let $P^t_e$ be a random variable indicating, if $e$ was probed in one of steps $0, 1, ..., \text{ or } t$, we have $P^t_e = 0$ for all $e$. 

Algorithm 3 Algorithm for stochastic $k$-set packing

1: //Preprocessing:
2: for each $j \in [d]$ do
3: find support $B^0_j$ of $p^j \cdot x$ in $P(U(b_j))$
4: family $\phi^0_j$
5: critical sets $C = (C^j)_{j \in [d]}$
6: //Rounding:
7: let $A \leftarrow R(x)$; mark all $e \in A$ as available; $S \leftarrow \emptyset$
8: while there are still available elements in $A$ do
9: pick element $e$ uniformly at random from $A$
10: if $e$ is available then
11: probe $e$
12: $S \leftarrow S + e$
13: for each $j \in C_e$ such that $S_e(j) = 1$ do
14: for each set $B^j_{i,t}$ of support $B^{j,t}$ do
15: $B^j_{i,t} \leftarrow B^j_{i,t} + e$
16: call $e$ unavailable
17: else simulate the probe of $e$
18: for each $j \in C_e$ such that $S_e(j) = 1$ (whether we probe or simulate) do
19: for each set $B^j_{i,t}$ of support $B^{j,t}$ do
20: $f \leftarrow \phi \left[ B^j_{i,t} \phi(e), B^j_{i,t} \phi(f) \right] (e)$
21: if $f \neq e$ then $B^j_{i,t} \phi(f) \leftarrow B^j_{i,t} \phi(f) - f$ and call $f$ unavailable
22: compute the family $\phi^j_{i,t+1}$
23: for each $i$ do $B^j_{i,t+1} \leftarrow B^j_{i,t}$
24: $t \leftarrow t + 1$
25: return $S$
Variable $P_{e}^{t+1} - P_{e}^{t}$ indicates if $e$ was probed at step $t + 1$. Given the information $F^t$ about the process up to step $t$, the probability of this event is $\mathbb{E} \left[ P_{e}^{t+1} - P_{e}^{t} \mid F^t, \mathcal{C} \right] = \frac{Y_{e}^{t}}{|A|}$, because if element $e$ is still available after step $t$ (i.e., $Y_{e}^{t} = 1$), then with probability $\frac{1}{|A|}$ we choose it in line 9 and otherwise (i.e., $Y_{e}^{t} = 0$) we cannot probe it.

Variable $Y_{e}^{t} - Y_{e}^{t+1}$ indicates whether element $e$ stopped being available at step $t + 1$. For this to happen we need to pick $f \in \Gamma_{j}^{t}(0)$ and the probe (or simulation) of $f$ needs to result in a vector $S_{f}$ such that $S_{f} (j) = 1$. However, as already noted, there can be many $j$ for which $f \in \Gamma_{j}^{t}(0)$. Therefore, probability that $e$ stops being available in step $t + 1$ is equal to

$$\mathbb{E} \left[ Y_{e}^{t} - Y_{e}^{t+1} \mid F^t, \mathcal{C} \right] = \frac{Y_{e}^{t}}{|A|} + \frac{Y_{e}^{t}}{|A|} \cdot \sum_{f} X_{f} \cdot \mathbb{P} \left[ \bigvee_{j:f \in \Gamma_{j}^{t}(e)} (S_{f} (j) = 1) \mid \mathcal{C} \right].$$

Here we stress the condition on $\mathcal{C}$ because sets $\Gamma_{j}^{t}$ are constructed given the choice of critical sets. Again we can reason that

$$\left( 1 + \sum_{f} X_{f} \cdot \mathbb{P} \left[ \bigvee_{j:f \in \Gamma_{j}^{t}(e)} (S_{f} (j) = 1) \mid \mathcal{C} \right] \right) \cdot P_{e}^{t} + Y_{e}^{t}$$

is a martingale. Let $\tau = \min \{ t \mid Y_{e}^{t} = 0 \}$ be the step in which edge $e$ became unavailable. It is clear that $\tau$ is a stopping time. Thus from Doob’s Stopping Theorem we get that

$$\mathbb{E}_{\tau} \left[ \left( 1 + \sum_{f} X_{f} \cdot \mathbb{P} \left[ \bigvee_{j:f \in \Gamma_{j}^{t}(e)} (S_{f} (j) = 1) \mid \mathcal{C} \right] \right) \cdot P_{e}^{\tau} + Y_{e}^{\tau} \mid \mathcal{C} \right] = \mathbb{E}_{\tau} \left[ \left( 1 + \sum_{f} X_{f} \cdot \mathbb{P} \left[ \bigvee_{j:f \in \Gamma_{j}^{t}(e)} (S_{f} (j) = 1) \mid \mathcal{C} \right] \right) \cdot P_{e}^{\tau} + Y_{e}^{\tau} \mid \mathcal{C} \right] = X_{e}.$$

We argue again using the properties of transversal mapping $\phi_{j}^{t}$ that we have $\Gamma_{j}^{t} (e) = \Gamma_{j}^{0} (e)$ for each $j \in C_{e}$, since $e$ was available before step $\tau$. And if so, then

$$\mathbb{E}_{\tau} \left[ \left( 1 + \sum_{f} X_{f} \cdot \mathbb{P} \left[ \bigvee_{j:f \in \Gamma_{j}^{t}(e)} (S_{f} (j) = 1) \mid \mathcal{C} \right] \right) \cdot P_{e}^{\tau} + Y_{e}^{\tau} \mid \mathcal{C} \right] = X_{e}$$

and since $\left( 1 + \sum_{f} X_{f} \cdot \mathbb{P} \left[ \bigvee_{j:f \in \Gamma_{j}^{t}(e)} (S_{f} (j) = 1) \mid \mathcal{C} \right] \right)$ is just a number depending on $\mathcal{C}$, we can say that

$$\left( 1 + \sum_{f} X_{f} \cdot \mathbb{P} \left[ \bigvee_{j:f \in \Gamma_{j}^{t}(e)} (S_{f} (j) = 1) \mid \mathcal{C} \right] \right) \cdot \mathbb{E}_{\tau} \left[ P_{e}^{\tau} \mid \mathcal{C} \right] = X_{e}.$$

Note that $\mathbb{E}_{\tau} \left[ P_{e}^{\tau} \mid \mathcal{C} \right] = \mathbb{P} \left[ e \text{ is probed} \mid \mathcal{C} \right]$, and conclude that

$$\mathbb{P} \left[ e \text{ is probed} \mid \mathcal{C} \right] = X_{e} / \left( 1 + \sum_{f} X_{f} \cdot \mathbb{P} \left[ \bigvee_{j:f \in \Gamma_{j}^{t}(e)} (S_{f} (j) = 1) \mid \mathcal{C} \right] \right).$$
At this point we reason as follows:

$$\sum_f X_f \cdot \mathbb{P} \left[ j: f \in \Gamma_j^0(e) \right] (S_f(j) = 1) \mid C \right]$$

$$\sum_f X_f \cdot \mathbb{P} \left[ S_f(j) = 1 \mid C \right]$$

$$= \sum_f X_f \cdot \sum_{j: f \in \Gamma_j^0(e)} p_f^j$$

$$= \sum_{j \in C_e} \sum_{f \in \Gamma_j^0(e)} p_f^j X_f,$$

where inequality just follows simply from the union-bound. Then we have an identity since event \((S_f(j) = 1)\) is independent of \(C\) and its probability is just equal to \(p_f^j\). Later we just change the order of summation. Therefore we have shown that

$$\mathbb{P} \left[ e \text{ is probed} \mid C \right] \geq X_e \left( 1 + \sum_{j \in C_e} \sum_{f \in \Gamma_j^0(e)} p_f^j X_f \right).$$

We apply expectation \(E_{C,R(x)} \left[ \cdot \mid e \in R(x) \right] \) to both sides, use Jensen’s inequality to get

$$\mathbb{P} \left[ e \text{ is probed} \mid e \in R(x) \right] \geq 1 \left/ \left( 1 + E_{C,R(x)} \left[ \sum_{j \in C_e} \sum_{f \in \Gamma_j^0(e)} p_f^j X_f \mid e \in R(x) \right] \right) \right..$$

Now we can say that

$$E_{C,R(x)} \left[ \sum_{j \in C_e} \sum_{f \in \Gamma_j^0(e)} p_f^j X_f \mid e \in R(x) \right]$$

$$= \sum_{j \in C_e} E_{C,R(x)} \left[ \sum_{f \in \Gamma_j^0(e)} p_f^j X_f \mid e \in R(x) \right] \leq \sum_{j \in C_e} 1 = |C_e| \leq k,$$

where the last inequality \(E_{C,R(x)} \left[ \sum_{f \in \Gamma_j^0(e)} p_f^j X_f \mid e \in R(x) \right] \leq 1\) for each \(j\) comes from Lemma 11. Hence

$$\mathbb{P} \left[ e \text{ is probed} \mid e \in R(x) \right] \geq \frac{1}{1 + k},$$

which gives

$$\mathbb{P} \left[ e \text{ is probed} \right] = \mathbb{P} \left[ e \in R(x) \right] \cdot \mathbb{P} \left[ e \text{ is probed} \mid e \in R(x) \right] \geq \frac{x_e}{1 + k}$$

as desired.

**G  Stochastic Matching and handling negative correlation**

In the Stochastic Matching problem we are given an undirected graph \(G = (V,E)\). Each edge \(e \in E\) is assigned a probability \(p_e \in (0,1]\) and a weight \(w_e > 0\), and each node \(v \in V\) is assigned
a patience $t_v \in \mathbb{N}^+$. Each time an edge is probed and it turns out to be present with probability $p_e$, in which case it is (irrevocably) included in the matching we gradually construct and it gives profit $w_e$. Therefore the inner constraints are given by intersection of two partition matroids. We can probe at most $t_u$ edges that are incident to node $u$ — these are outer constraints which also can be described by intersection of two partition matroids. Our goal is to maximize the expected weight of the constructed matching.

Let us consider the bipartite case where $V = (A \cup B)$ and $E \subseteq A \times B$. In this case Bansal et al. [4] provided an LP-based 3-approximation. We shall also obtain this approximation factor. Here we present a variant of our scheme from Lemma 3 that does not use transversal mappings, because in this case they are trivial. Moreover, previously we required the input of the stoch-CR scheme to be a set of elements sampled independently, i.e., $R(x)$. Now we shall apply the ideas from Lemma 3, but we use it on a set of edges that are negatively correlated. Such a set is returned by the algorithm of Gandhi et al. [15]. Also, since the objective is linear, and not submodular, we do not have to take care of the monotonicity of the scheme, and therefore we do not draw edges with repetitions. In fact we just scan rounded edges according to a random permutation, and therefore the below algorithm is the same that was presented by Bansal et al. [4]. Thus what follows is an alternative analysis of the algorithm from [4].

Consider the following LP for stochastic matching problem:

$$\max \sum_e w_e p_e x_e$$

(9)

$$\sum_{e \in \delta(v)} p_e x_e \leq 1 \quad \forall v \in V$$

(10)

$$\sum_{e \in \delta(v)} x_e \leq t_v \quad \forall v \in V$$

(11)

$$0 \leq x_e \leq 1 \quad \forall e \in E.$$  

(12)

Suppose that $(x_e)_{e \in E}$ is the optimal solution to this LP. We round the solution with dependent rounding of Gandhi et al. [15]: we call the algorithm GKPS. Let $(\hat{X}_e)_{e \in E}$ be the rounded solution, and denote $\hat{E} = \{ e \in E \mid \hat{X}_e = 1 \}$. From the definition of dependent rounding we know that:

1. (Marginal distribution) $\mathbb{P}[\hat{X}_e = 1] = x_e$;

2. (Degree preservation) For any $v \in V$ it holds that

$$\sum_{e \in \delta(v)} \hat{X}_e \leq \left[ \sum_{e \in \delta(v)} x_e \right] \leq t_v;$$

3. (Negative correlation) For any $v \in V$ and any subset $S \subseteq \delta(v)$ of edges incident to $v$ it holds that:

$$\forall b \in \{0, 1\} \mathbb{P}\left[ \bigwedge_{e \in S} \left( \hat{X}_e = b \right) \right] \leq \prod_{e \in S} \mathbb{P}[\hat{X}_e = b].$$

Negative correlation property and constraint (11) imply that

$$\mathbb{E}\left[ \sum_{f \in \delta(e)} p_f \hat{X}_f \mid e \in \hat{E} \right] \leq \mathbb{E}\left[ \sum_{f \in \delta(e)} p_f \hat{X}_f \right] = \sum_{f \in \delta(e)} p_f x_f \leq 2 - 2p_e x_e. \quad (13)$$
Algorithm 4 Algorithm for Stochastic Matching

1: Solve the LP; let $x$ an optimal solution;
2: let $\hat{X} \in \{0,1\}^E$ be a solution rounded using GKPS; let $\hat{E} = \{e \mid \hat{X}_e = 1\}$; call every $e \in \hat{E}$ safe
3: while there are still safe elements in $\hat{E}$ do
4: pick element $e$ uniformly at random from safe elements of $\hat{E}$
5: probe $e$
6: if probe successful then
7: $S \leftarrow S \cup \{e\}$
8: call every $f \in \hat{E} \cap \delta(e)$ blocked
9: return $S$

Given the solution $(\hat{X}_e)_{e \in E}$ we execute the selection algorithm presented on Figure 4. Because of the Degree preservation property we will not exceed the patience of any vertex. We say that an edge $e$ is safe if no other edge adjacent to $e$ was already successfully probed; otherwise edge is blocked. Initially all edges are safe.

The expected outcome of our algorithm is

$$\sum_e \mathbb{P}[e \text{ probed}] \cdot p_e \cdot w_e = \sum_e \mathbb{P}[\hat{X}_e = 1] \cdot \mathbb{P}[e \text{ probed} | \hat{X}_e = 1] \cdot p_e \cdot w_e$$

$$= \sum_e w_e p_e x_e \cdot \mathbb{P}[e \text{ probed} | \hat{X}_e = 1],$$

and we shall show that $\mathbb{P}[e \text{ probed} | \hat{X}_e = 1] \geq \frac{1}{3}$ for any edge $e$, which will imply 1/3-approximation of the algorithm.

From now let us condition that we know the set of edges $\hat{E}$ and we know that $\hat{X}_e = 1$.

Consider a random variable $Y^t_e$ which indicates if edge $e$ is still in the graph after step $t$. We consider variable $Y^t_f$ for any edge $f \in E$. Initially we have $Y^0_f = 1$ for any $f \in \hat{E}$, and $Y^0_f = 0$ for $f \notin \hat{E}$. Let variable $P^t_e$ denote if edge $e$ was probed in one of steps $0,1,\ldots,t$; we have $P^0_e = 0$.

Let $\Sigma^t_e$ be the number of edges that are left after $t$ steps. Variable $P^{t+1}_e - P^t_e$ indicates whether edge $e$ was probed in step $t+1$. Given the information $\mathcal{F}^t$ about the process up to step $t$, probability of this event is $\mathbb{E}[(P^{t+1}_e - P^t_e) | \mathcal{F}^t, \hat{E}, \hat{X}_e = 1] = \frac{Y^t_e}{\Sigma^t_e}$, i.e., if edge $e$ still exists in the graph after step $t$ (i.e. $Y^t_e = 1$), then the probability is $\frac{1}{\Sigma^t_e}$, otherwise it is 0.

Variable $Y^t_e - Y^{t+1}_e$ indicates whether edge $e$ was blocked from the graph in step $t+1$. Given $\mathcal{F}^t$, probability of this event is $\mathbb{E}[(Y^t_e - Y^{t+1}_e) | \mathcal{F}^t, \hat{E}, \hat{X}_e = 1] = \frac{Y^t_e}{\Sigma^t_e} \cdot \left( \sum_{f \in \delta(e)} p_f Y^t_f + 1 \right)$.

It is immediate to note that $Y^t_f \leq \hat{X}_f$ for any edge $f$, and that $P^{t+1}_e - P^t_e$ is always nonnegative. Hence

$$\mathbb{E} \left[ \left( \sum_{f \in \delta(e)} p_f \hat{X}_f + 1 \right) \cdot (P^{t+1}_e - P^t_e) - (Y^t_e - Y^{t+1}_e) \right] | \mathcal{F}^t, \hat{E}, \hat{X}_e = 1 \geq 0,$$

which means that the sequence

$$\left( \left( \sum_{f \in \delta(e)} p_f \hat{X}_f + 1 \right) \cdot P^t_e - (1 - Y^t_e) \right)_{t \geq 0}$$
is a super-martingale.

Let \( \tau = \min \{ t \mid Y_t^e = 0 \} \) be the step in which edge \( e \) was either blocked or probed. It is clear that \( \tau \) is a stopping time. Thus from Doob’s Stopping Theorem — this time in the variant for super-martingales, i.e., if \( \mathbb{E} \left[ Z^{t+1} - Z^t \mid \mathcal{F}^t \right] \geq 0 \), then \( \mathbb{E} [Z^\tau] \geq \mathbb{E} [Z^0] \) — we get that

\[
\mathbb{E}_\tau \left[ \left( \sum_{f \in \delta(e)} p_f \hat{X}_f + 1 \right) \cdot P_e^\tau - (1 - Y_e^\tau) \bigg| \hat{E}, \hat{X}_e = 1 \right] \geq \mathbb{E} \left[ \left( \sum_{f \in \delta(e)} p_f \hat{X}_f + 1 \right) \cdot P_e^0 - (1 - Y_e^0) \bigg| \hat{E}, \hat{X}_e = 1 \right],
\]

where the expectation above is over the random variable \( \tau \) only. Since \( P_e^0 = 0, Y_e^0 = 1, Y_e^\tau = 0 \) the above inequality implies that

\[
\mathbb{E}_\tau \left[ \left( \sum_{f \in \delta(e)} p_f \hat{X}_f + 1 \right) \cdot P_e^\tau \bigg| \hat{E}, \hat{X}_e = 1 \right] \geq 1.
\]

Since we condition all the time on \( \hat{E} \) and \( \hat{X}_e = 1 \) we can write that

\[
\left( \sum_{f \in \delta(e)} p_f \hat{X}_f + 1 \right) \cdot \mathbb{E}_\tau \left[ P_e^\tau \bigg| \hat{E}, \hat{X}_e = 1 \right] \geq 1.
\]

Let us notice that \( \mathbb{E}_\tau \left[ P_e^\tau \bigg| \hat{E}, \hat{X}_e = 1 \right] \) is exactly equal to \( \mathbb{P} \left[ e \text{ probed} \mid \hat{E}, \hat{X}_e = 1 \right] \). Thus we can write that

\[
\mathbb{P} \left[ e \text{ probed} \mid \hat{E}, \hat{X}_e = 1 \right] \geq \frac{1}{\sum_{f \in \delta(e)} p_f \hat{X}_f + 1}.
\]

Now we can apply to both sides of the above inequality expectation over \( \hat{E} \) but still conditioned on \( \hat{X}_e = 1 \):

\[
\mathbb{P} \left[ e \text{ probed} \mid \hat{X}_e = 1 \right] = \mathbb{E}_{\hat{E}} \left[ \mathbb{P} \left[ e \text{ probed} \mid \hat{E}, \hat{X}_e = 1 \right] \bigg| \hat{X}_e = 1 \right] \geq \mathbb{E}_{\hat{E}} \left[ \frac{1}{\sum_{f \in \delta(e)} p_f \hat{X}_f + 1} \bigg| \hat{X}_e = 1 \right],
\]

and from Jensen’s inequality, and the fact that \( x \mapsto \frac{1}{x} \) is convex, we get that

\[
\mathbb{E}_{\hat{E}} \left[ \frac{1}{\sum_{f \in \delta(e)} p_f \hat{X}_f + 1} \bigg| \hat{X}_e = 1 \right] \geq \mathbb{E}_{\hat{E}} \left[ \frac{1}{\sum_{f \in \delta(e)} p_f \hat{X}_f + 1} \bigg| \hat{X}_e = 1 \right].
\]

From inequality (13) we get \( \mathbb{E}_{\hat{E}} \left[ \sum_{f \in \delta(e)} p_f \hat{X}_f + 1 \bigg| \hat{X}_e = 1 \right] \leq \mathbb{E}_{\hat{E}} \left[ \sum_{f \in \delta(e)} p_f \hat{X}_f + 1 \right] \leq 3 - 2\lambda_e \rho_e \leq 3 \) and we conclude that

\[
\mathbb{P} \left[ e \text{ probed} \mid \hat{X}_e = 1 \right] \geq \frac{1}{3}.
\]