Special Relativity: Einstein’s Spherical Waves versus Poincaré’s Ellipsoidal Waves

Dr. Yves Pierseaux
Physique des particules, Université Libre de Bruxelles (ULB)
ypiersea@ulb.ac.be
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Abstract

We show that the image by the Lorentz transformation of a spherical (circular) light wave, emitted by a moving source, is not a spherical (circular) wave but an ellipsoidal (elliptical) light wave. Poincaré’s ellipsoid (ellipse) is the direct geometrical representation of Poincaré’s relativity of simultaneity. Einstein’s spheres (circles) are the direct geometrical representation of Einstein’s convention of synchronisation. Poincaré adopts another convention for the definition of space-time units involving that the Lorentz transformation of an unit of length is directly proportional to Lorentz transformation of an unit of time. Poincaré’s relativistic kinematics predicts both a dilation of time and an expansion of space as well.

1 Introduction: Einstein’s Spherical Wavefront & Poincaré’s Ellipsoidal Wavefront

Einstein writes in 1905, in the third paragraph of his famous paper:

At the time $t = \tau = 0$, when the origin of the two coordinates (K and k) is common to the two systems, let a spherical wave be emitted therefrom, and be propagated -with the velocity $c$ in system K. If $x, y, z$ be a point just attained by this wave, then

$$x^2 + y^2 + z^2 = c^2 t^2$$

(1)

Transforming this equation with our equations of transformation (see Einstein’s LT, 29), we obtain after a simple calculation

$$\xi^2 + \eta^2 + \zeta^2 = c^2 \tau^2$$

(2)

The wave under consideration is therefore no less a spherical wave with velocity of propagation $c$ when viewed in the moving system k. [Einstein A.1905]
Poincaré writes in 1908 in his second paper on ”La dynamique de l’´ electron” with the subtitle ”Le principe de relativité”:

Imagine an observer and a source involved together in the transposition. The wave surfaces emanating for the source will be spheres, having as centre the successive positions of the source. The distance of this centre from the present position of the source will be proportional to the time elapsed since the emission - that is to say, to the radius of the sphere. But for our observer, on account of the contraction, all these spheres will appear as elongated ellipsoids. The compensation is now exact, and this is explained by Michelson’s experiments. [Poincaré H. (1908)]

We can further find in Poincaré’s text the equation (two dimensions) of an elongated light ellipse whose observer at rest (let us call him: O) is situated at the centre and whose source S (with ”our observer”, let us call him O’) in moving is situated at the focus F of the ellipse.

The contrast between both great relativists, Einstein et Poincaré, about an experiment that seems to be the same (the image of a spherical wave emitted by a moving source) is very clear: according to Einstein, the image of a spherical wave is a spherical wave whilst according to Poincaré the image of a spherical wave (around O) is an ellipsoidal wave. Does the latter not know the invariance of the quadratic form? Not at all because he does demonstrate, with the structure of group, in his first paper on ,”La dynamique de l’´ electron” [Poincaré H. 1905], that the Lorentz Transformation (LT) ”doesn’t modify the quadratic form $x^2 + y^2 + z^2 - c^2 t^2$”. We must point out that Poincaré’s lengthened light waves has been completely ignored for a whole century by the scientific community 1. We note also that Poincaré doesn’t use LT in the previous quotation and directly deduces the ellipsoidal shape of the light wavefront from the principle of contraction of (the unit) of length (see conclusion).

So who is right: Einstein or Poincaré? The best thing that we can do, to solve this dilemma, is to apply a LT to a spherical wavefront.

2 Image by LT of the Object ”Circular Wave”

What is the image (the shape) in K of a spherical wave emitted in $t' = t = 0$ by a source S at rest in the origin O’ of K’? The LT defined by Poincaré is:

$$x' = k(x - \varepsilon t) \quad y' = y \quad t' = k(t - \varepsilon x)$$

We keep Poincaré’s notations where $\varepsilon, k$ correspond to Einstein-Planck’s notations $\beta, \gamma$ because, according to Poincaré in his 1905 work about the theory of relativity, ”I shall choose the units of length and of time in such a way that the velocity of light is equal to unity” [Poincaré H. 1905]. The deep meaning of Poincaré’s choice of space-time units with $c = 1$ will be specified in the conclusion. In order to have one only wavefront, we have

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1Poincaré’s ellipsoidal wavefront was in fact mentioned in his course of 1905-1906 ”Les limites de la loi de Newton” [Poincaré H. (1906)]. We also find them in ”La Mécanique Nouvelle” (1909) [Poincaré H. (1909)]. In fact it was in 1904 at a talk in Saint Louis that Poincaré first introduced the elongated ellipsoidal wavefront as an alternative and not as a consequence of the contraction of the unit of length [Poincaré H. (1904)].
to define a time $t'$ as unit of time $1_{t'}$. The equation of the circular wave front in $K'$ (the geometrical locus of the object-points in $K'$) in $t' = 1$ is:

$$x'^2 + y'^2 = t'^2 = 1_{t'}$$  \hspace{1cm} (4)

The unprimed coordinates of the image-points are given by the inverse LT:

$$x = k(x' + \varepsilon t') \quad y = y' \quad t = k(t' + \varepsilon x')$$  \hspace{1cm} (5)

The coordinates $(0, 0, 1)$ in $K$ of the source in $t' = 1$ are $(k\varepsilon, 0, k)$ and $(k\varepsilon t', 0, kt')$ in $t' \neq 1$.

Let us determine the images $(x, y, t)$ in $K$ of different object-points in $t' = 1$: $(1, 0, 1), (-1, 0, 1), (0, 1, 1), (\sqrt{2}, \sqrt{2}, 1)$ etc. (see figure 1 in annex). The image-point E, $k(1 + \varepsilon), 0, k(1 + \varepsilon)$, is on the large dotted circle $x^2 + y^2 = t^2$ with radius $r = t = k(1 + \varepsilon)$.

Let us now determine Poincaré's right and all the image-points of the circular wavefront in $K'$ are on an elliptical wavefront. By introducing, in the system $K'$, the angle $\theta'$ determined by both the radius vector $r'$ and the Ox' axes, we have $x' = r' \cos \theta'$ et $y' = r' \sin \theta'$. So with $r' = t' \neq 1$ we have:

$$t = k(t' + \varepsilon \cos \theta')$$  \hspace{1cm} (6)

which is the temporal LT (5), $t = k(t' + \varepsilon x')$, with $x' = r' \cos \theta' = t' \cos \theta'$. We can also write $(r = t)$ the locus of the images-points:

$$r = kr'(1 + \varepsilon \cos \theta')$$  \hspace{1cm} (7)

If $r' = t' = 1$ (figure 1), we then have:

$$t = r = k(1 + \varepsilon \cos \theta')$$  \hspace{1cm} (8)

We will show now that this ”temporal equation” (6) or ”normal equation” (7) is the equation of an ellipse in polar coordinates if we define the polar angle $\theta$ (see figure 2 in annex) as the relativistic transformation of the angle $\theta'$ (paragraph 3).

3 Poincaré's Elongated Ellipse and the Relativity of Simultaneity

Let us first determine Poincaré’s elongated ellipse in Cartesian coordinates. We are seeking for the space shape of the wavefront $t' = 1$ in $K$, given the invariance of the quadratic form:

$$x^2 + y^2 = t^2$$  \hspace{1cm} (9)
If the time \( t \) were fixed (see paragraph 4 on Einstein’s synchronisation), we would obviously have a circular wavefront; but \( t \) depends by LT on \( x' \). If \( t' \) is written in function of \( x' \), we would not have the image of the wave in \( K \). We must write \( t \) in function of \( x \).

By using the first and the third (\( x \) and \( t \)) LT (5), we have respectively if \( r' = t' \neq 1 \) and \( r' = t' = 1 \):

\[
t = k^{-1}t' + \varepsilon x \quad t = k^{-1} + \varepsilon x
\]  

We immediately obtain the Cartesian equation of Poincaré’s elongated ellipse respectively if \( r' = t' \neq 1 \) and \( r' = t' = 1 \) (figure 2):

\[
x^2 + y^2 = (k^{-1}t' + \varepsilon x)^2 \quad x^2 + y^2 = (k^{-1} + \varepsilon x)^2
\]  

At once we check that Poincaré’s ellipse, by replacing \( x' \)

\[
x' = k^{-1}x - \varepsilon t'
\]

in \( x'^2 + y'^2 = t'^2 = 1_e \) (4), respectively if \( r' = t' \neq 1 \) and \( r' = t' = 1 \), can be also written thus:

\[
(k^{-1}x - \varepsilon t')^2 + y^2 = t'^2 \quad (k^{-1}x - \varepsilon)^2 + y^2 = 1_e
\]

The image-points (figure 2) are situated on Poincaré’s elongated ellipse, with Observer \( O \) at the focus \( F \) and Source \( S \) at the centre \( C \). The eccentricity of the ellipse is \( \varepsilon = \frac{k\varepsilon}{k} \) where \( k \) is the length of the great axis (we choose, in figure 2, the small axis of the ellipse \( r' = t' = 1 \)). The equation of Poincaré’s ellipse can be written in polar coordinates with pole \( O \), focus \( F \) and the polar angle \( \theta \) defined in \( K \) (with both standard parameters of the ellipse \( e, p \)):

\[
r = \frac{p}{1 - e \cos \theta}
\]

with the small axe of the ellipse \( b = r' = 1 \)

\[p = a(1 - \varepsilon^2) = ak^{-2} = kk^{-2} = k^{-1}\]

we immediately deduce the polar equation of Poincaré’s ellipse

\[
r = \frac{\sqrt{1 - e^2}}{1 - e \cos \theta} = \frac{1}{k(1 - e \cos \theta)}
\]

with eccentricity \( e = \frac{F}{a} = \frac{k\varepsilon}{k} = \varepsilon \) and with the two standard parameters of the special relativity \( \varepsilon, k \) :

\[a^2 - f^2 = b^2 \quad k^2 - \varepsilon^2 k^2 = 1\]

If \( r' = t' \neq 1 \), we have the equation of the ellipse

\[
r = \frac{\sqrt{1 - e^2}}{1 - e \cos \theta} = \frac{r'}{k(1 - e \cos \theta)}
\]

with \( r'^2(k^2 - \varepsilon^2 k^2) = r'^2 \).

It should be reminded that the "normal equation" (7) of the ellipse is

\[2\text{It is the inverse case that is explicitly considered by Poincaré (historical introduction).}\\

\[ r = kr'(1 + \varepsilon \cos \theta') \]  

(17)

Thus we obtain from (16 and 7) the formula of relativistic transformation of angle

\[ \cos \theta = \frac{\cos \theta' + \varepsilon}{1 + \varepsilon \cos \theta'} \]  

(18)

So it is now utterly demonstrated that Poincaré is right and that the geometrical image by LT of a circular wavefront is an elongated ellipse its polar equation being (16) and its Cartesian equation being (12). Poincaré’s ellipse gives the other formulae of aberration, in particular:

\[ \sin \theta = \frac{\sqrt{1 - \varepsilon^2}}{1 + \varepsilon \cos \theta'} \sin \theta' \]  

(19)

It is now essential to interpret the historical case (see introduction and footnote 2) considered by Poincaré (in connection with Michelson’s experiment where the source is on the Earth, see conclusion): the circular light wavefronts are developed around O (the ether is now by definition at rest relative to K): \( r = t = 1 \). What is the image of the circular locus of the points (determined now by \( \theta \)) seen from O’ (where the source is at rest in K’, ”system of the Earth”? Given that Poincaré’s ellipse I, in the first case, is directly inscribed in LT, it is easy to define Poincaré’s ellipse II, in the second case, both by inverting in (7) the primed and the unprimed and by changing the sign of \( \varepsilon \). The ”normal” equation of Poincaré’s (historical) ellipse II is therefore:

\[ r' = kr(1 - \varepsilon \cos \theta) \]  

(20)

The polar equation of Poincaré’s ellipse II, its source S (in O’, ”on the Earth” ) in moving occupies the focus F* (see figure 2) and the observer O occupies the centre C is (with 18):

\[ r' = r \frac{1}{k(1 + \varepsilon \cos \theta')} \]  

(21)

The ”normal” (temporal) equation of the ellipse I then is the polar equation of the ellipse II. The Cartesian equation of Poincaré’s ellipse II is:

\[ (k^{-1}x' + \varepsilon t)^2 + y^2 = t^2 \]  

(22)

So in Poincaré’s relativistic kinematics we can have, with no contradiction at all, an elliptical wavefront in the system of the source.

What is now the physical interpretation of Poincaré’s elongated ellipse? We underline, at this stage, three points (for the relativistic Doppler effect, see conclusion):

1) Poincaré’s elongated ellipse is the direct translation of the ”headlight effect”: the isotropic emission of a moving source is not isotropic seen from the rest system (relativistic transformation of angles \( \theta' \) into \( \theta \)). In three dimensions of space the reduction of the angle of aperture of the cone of emission of a moving source is physically (synchrotron radiation,
bremsstrahlung...) represented, on the whole (from any angle), by the ellipsoidal shape of the wavefront.

2) Poincaré’s elongated ellipse is the direct translation of the **relativity of simultaneity**: the set of simultaneous events in K’ of the spherical wavefront in time t’ is not a set of simultaneous events in K, in time t. In particular, if the two events (1, 0, 1) et (−1, 0, 1) are simultaneous in K’, they are not simultaneous (6), k(1 + ε), 0, k(1 + ε) and k(1 − ε), 0, k(1 − ε), in K. Let us also note that the image of the distance "2" between these two events in K’ is elongated "2k" in K. These two fundamental points put the emphasis on the fact that Poincaré’s ellipse is not only a geometrical image but also a physical shape of the wavefront.

3) Poincaré’s elongated ellipse is the direct translation of Poincaré’s completely relativistic3 ether: *put the ether at rest in one* (K) *or in the other system* (K’) *is exactly equivalent to define the ellipse with the direct LT or the inverse LT*. So in Poincaré’s own words: if t’ is the true time (”circular” time), t then is the local time (”elliptical” time) and inversely (by LT) if t is the true time (”circular” time), t’ then is the local time (”elliptical” time). That is completely relativistic and Poincaré’s elongated ellipse is ”the immediate interpretation of Michelson experimental result” (see conclusion).

Poincaré’s ether is relativistic but *not deleted* (as Einstein’s one) because it remains the relativistic definition of state of rest: when we choose by definition ether at rest in one system (spheres or true time), it is not at rest in the other system (ellipsoids or local time).

Objectively we have two possibilities to choose the criterion of the relativistic state of rest of a system: the source of light or the medium of light. In Einstein-Minkowski’s relativistic kinematics, the criterion is clearly the source (the proper system, see paragraph 4). In Poincaré’s relativistic kinematics, the criterion is clearly the ether (”circular waves”).

That is a paramount difference because in Einstein-Minkowski’s proper system (see paragraph 4) we always have by definition spherical waves or in other words, the equality between forth travel time and back travel time. It is not the case with Poincaré’s definition of units where we can have without any contradiction, an elliptical wavefront (a local time) in the system of the source (see conclusion). Poincaré’s relativistic duality between true time and local time *doesn’t correspond* to Einstein-Minkowski’s relativistic duality between proper time and improper time (paragraph 4).

4 **Einstein’s Kinematics: identical Spheres, identical rigid Rods and Convention of Synchronisation**

If according to Einstein, the object (1) and the image (2), are both spherical and concentric within the two systems, then two simultaneous events in K, for example (1, 0, 1) and (−1, 0, 1), must be also simultaneous in k.

That seems in contradiction not only with Poincaré’s ellipse but also with Einstein’s well known definition of relativity of simultaneity. Therefore the image by LT in k of a spherical wave in K **cannot be a spherical wave**. So it could appear at this stage that Poincaré is right and *Einstein is wrong*.

However the question is: ”What in Einstein’s reasoning is true?” Let us return to

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3Poincaré is philosophically very anti-absolutist as well [Poincaré H. (1907)].
Einstein’s 1905 quotation (paragraph 1). The two quadratic forms \( x^2 + y^2 + z^2 = c^2 t^2 \) and \( \xi^2 + \eta^2 + \zeta^2 = c^2 \tau^2 \) are the geometrical equations of two spheres (two circles in two dimensions or two equidistant points respectively from \( O \) and \( O' \) in one dimension). Let us note that the young Einstein doesn’t specify, in the previous quotation, in which system the source is at rest. So if we consider now that we have two identical sources, in \( O' \) and \( O \), emitting a signal of light simultaneously at the time,

\[
\tau = t = 0
\]

the physical situation is perfectly identical in each system (Einstein’s deletion of ether\(^5\)). So we must have two identical spherical wavefronts, \( x^2 + y^2 + z^2 = c^2 \) around \( O \) and \( \xi^2 + \eta^2 + \zeta^2 = c^2 \) around \( O' \), simultaneously at the time

\[
\tau = t = 1_t = 1_\tau
\]

It immediately follows from the latter choice of two identical units of time that we have two identical units of length \( 1_x = 1_\xi = c1_t = c1_\tau \). Let us point out that the travel time of the circular wavefront either to the right or to the left (on the \( x, \xi \) axis) are identical within the two systems. Einstein’s definition of identical units of time is therefore completely coherent with Einstein’s identical rigid rods \((1_t = 1_\tau \) is on the other hand incompatible with Poincaré’s definition of units, see conclusion). Einstein writes in this sense in 1905:

Let there be given a stationary rigid rod; and let its length be \( L_0 \) as measured by a measuring-rod which is also stationary. In accordance with the principle of relativity the length of the rod in the moving system \( \) must be equal to the length \( L_0 \) of the stationary rod. [Einstein A.1905]

In this respect, M. Born is perhaps the only physicist who underlined that the young Einstein introduces in fact a tacit assumption (1921):

A fixed rod that is at rest in the system \( K \) and is of length 1 cm, will, of course, also have the length 1 cm, when it is at rest in the system \( k \). We may call this tacit assumption of Einstein’s theory the principle of the physical identity of the units of measure. [Born M.]

Einstein’s principle of identity (see also [Weisskopf V. (1)]) stipulates \( L_0 = 1_x = 1_\xi = c1_t = c1_\tau \), and therefore (24) becomes

\[
\tau = t = \frac{L_0}{c}
\]

\(^4\)We can also consider that the emission is an event in a strong Einstein’s meaning and an event has no velocity. We find in Einstein’s introduction the enigmatic sentence “ The introduction of a “lumineferous ether” will prove to be superfluous inasmuch as the view here to be developed will not require an “absolutely stationary space provided with special properties nor assign a velocity-vector to a point of the empty space in which electromagnetic processes take place.” The emission by a source of light is an event that has no velocity (see note 7) and therefore everything happens as if the source were at rest within each system. [Pierseaux Y., concept d’événement]

\(^5\)Einstein’s deletion of ether is completely inseparable of Einstein’s photon (1905)[Pierseaux Y. (Ph.D, ULB)].
There are two Einstein’s spherical waves and each spherical wave defines, in one dimension, two simultaneous events, (-1, 0, 1) and (-1, 0, 1), within each system (K and k). In other words: **Einstein’s rigid rod** \((2L_0)\) is defined by **two simultaneous events within each system** (K and k). Einstein’s spherical waves are not in contradiction with "Einstein’s relativity of simultaneity (after LT)" because it is "Einstein’s convention of simultaneity (before LT)" or in other words "Einstein’s convention of synchronisation of identical clocks in A and B with the exchange of a signal of light in K (before LT)". Let us now demonstrate that point by rigorously distinguishing the two stages of Einstein’s deduction: before LT and after LT.

4.1 Before LT (the proper Systems)

According to the young Einstein, "It is essential to have time defined by means of stationary clocks in stationary system". Einstein’s famous repetition of the concept "stationary" is essential because he notices about his second system \(k (\xi, \eta, \zeta, \tau)\):

To do this [deduce LT] we have to express in equations that \(\tau\) is nothing else than the set of data of clocks at rest in system k, which have been synchronized \([A'B']\) according to the rule given in paragraph 1 \([AB]\) [Einstein A.1905].

Without any loss of generality we make \(A \equiv O\) (respectively \(A' \equiv O'\)) in young Einstein’s notations (and then \(t_A = \tau_{A'} = 0\)). We have \(2t_B = t_O^*\) in K (respectively \(2\tau_{B'} = \tau_{O'}^*\) in k ) and \(c = 2\frac{OB}{t_O}\) in K (respectively \(c = 2\frac{OB'}{\tau_{O'}^*}\) in k), with \(L_0 = OB = O'B'\), where \(t_O = 0\) (respectively \(\tau_{O'} = 0\)) is the time of emission of the light signal in K (respectively in k) and \(t_O^*\) (respectively \(\tau_{O'}^*\)) is the time of reception of the light signal in B in K (respectively in B’ k).

\[
t_O^* = \tau_{O'}^* = 2\frac{L_0}{c} = T_0
\]

(26)

Given that forth travel time and back travel time are identical with

\[
t_O = \tau_{O'} = 0
\]

(27)

we finally have

\[
t_B = \tau_{B'} = \frac{L_0}{c} = \frac{1}{2}T_0.
\]

(28)

Einstein’s interpretation of the invariant quadratic form as **two physical spherical wavefronts** (1 & 2) is therefore exactly the same concept (see equations 23 and 27, 25 and 28) as Einstein’s 1905 **convention of synchronisation within the two systems** (in one dimension where forth wavefront becomes forth time travel and back wavefront becomes back time travel). The proper time \(T_0\) (index "zero" means "proper"), the duration between two events at the same place, in young Einstein’s notation is \(t_A^*\) (respectively \(\tau_{A'}^*\)) or \(t_O^*\) (respectively \(\tau_{O'}^*\)) with the identity between forth travel time and back travel time (factor \(\frac{1}{2}\)). This is Einstein’s synchronisation **without contraction** :\(OB = O'B'\) (For Poincaré’s convention of synchronisation with contraction see [Reignier J.] and [Pierseaux Y. (Ph.D, ULB)]).

\(^6\)The ether is at rest (or stationary) within the two systems and it is therefore superfluous. The deletion of the medium of light involves the relativistic state of rest is defined, by Einstein’s repetition with respect to the source of light.
We point out that Einstein’s convention is not based on the choice of only one unit of length in one system (see Poincaré, conclusion) but on two identical units of length (B’ is not the image by LT of B) within each system.

Historically Einstein deduced the identical units of time $T_0$ from the identical units of length $L_0$ and from invariance of "one way speed of light" (see forth travel time and back travel time or the circular waves). We can as well, like Minkowski [Minkowski H.], reverse that situation by defining first a proper time $T_0$ and next the proper length $L_0$. This inversion seems avoid the rigid rods but in fact nothing is changed because Einstein’s two spherical waves, Einstein’s convention of synchronisation, Einstein’s one way speed of light [Selleri F.], Einstein’s rigid rods $L_0$ and Minkowski’s ”Eigenzeit” are completely inseparable. Without Einstein’s isotropy (spheres 1 & 2), there is no Minkowski’s Eigenzeit because the relativistic state of rest is defined relatively to the source (see paragraph 3, third point and conclusion): ”identical units within each system” and ”isotropy in each proper system” are exactly the same concept.

The main result in the framework of this paper concerns Einstein’s ”one way speed of light” definition of proper length $L_0$ with half of proper time, $\frac{1}{2}T_0$. What does it mean? Einstein defines first the simultaneity of two events at the same place ($A = B$). Secondly he defines the simultaneity of two events at different places ($A \neq B$). But the departure (from A), the arrival (in B) and the return (in A) of the light are 3 successive events. What are finally Einstein’s two simultaneous events in A and B? These two events are in k: $(\xi_A, \frac{1}{2}T_0)$ and $(\xi_B, \frac{1}{2}T_0)$. These are the two ends of the rigid rod at the same time. So we have Einstein’s relativistic (one way) definition of rigidity:

**Definition 1** The proper length, $L_0 = \xi_B - \xi_A$, is defined by two events at the same time, $\tau = \frac{1}{2}T_0$, in the proper system k (respectively, $L_0 = x_2 - x_1$, at the same time, $t = \frac{1}{2}T_0$, in the proper system K).

### 4.2 After LT (the improper Systems)

Einstein’s construction of invariant quadratic form as physical spherical wavefront means that Einstein (and Minkowski) defines the units before taking in account the relative velocity $v$: Einstein’s units are completely independent on the relative velocity $v$ (and also $\beta$ and $\gamma$). Therefore there is no contradiction with LT, because the definition of space-time units is a preparation of the two systems not only prior to the use of LT and even, more fundamentally, prior to the deduction of LT (see note 4). Einstein’s 1905 deduction of LT is very complicated but Einstein’s immediate 1907 deduction of LT from the invariance of the quadratic form, i.e. the invariance of (one way [Selleri F.]) speed of light, has become a classic [Einstein A.1907]:

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7Einstein’s spheres are not necessarily identical. Fock V. underlined about the scale factor: ”We have $c^2r^2 - \xi^2 - \eta^2 - \zeta^2 = \varphi^2(x, y, z, t) (c^2t^2 - x^2 - y^2 - z^2)$. The factor $\varphi^2$, or rather $\varphi$, evidently characterises the ratio of the scales of measurement in the primed and unprimed frames. Further, it follows that this factor cannot depend on the relative velocity. It is usually said, following Einstein, that the scale factor can ”evidently” depend on nothing but the relative velocity, and it is subsequently proved that, in fact, it does not have any dependence but is equal to 1: $\varphi(x, y, z, t) = 1$. ” [Pierceaux Y. (principle of identity), principe d’identit]. In Poincaré’s relativistic kinematics the dependance of the scale factor on the velocity, $l(\varepsilon)$, is paramount. The group property of LT proves that the velocity is a relative velocity.
\[\xi = \gamma(x - vt) \quad \eta = y \quad \tau = \gamma(t - \frac{v}{c^2}x)\] (29)

If Einstein’s definition of space-time units is not in contradiction with LT, it requires on the other hand a specific use of LT. In young Einstein’s words: “We will call the length to be discovered \(L\) the length of the (moving) rod in the stationary system” [Einstein A.1905]. In current words, the length to be discovered by LT is the improper length \(L\), and also the improper time \(T\), respectively relatively to the proper length \(L_0\) or the proper time \(T_0\). The role of the LT consists fundamentally of introducing the velocity \(v\) or defining the improper moving system \(k\) relative to proper \(K\) or inversely. In all standard books we can find Einstein’s deduction, with the use of LT, of the dilation of proper time \(T = \gamma T_0\) and the contraction of proper length \(\gamma^{-1}L_0\). Therefore the improper time and the improper length (in the moving system) are inversely proportional. Let us examine Einstein’s use of LT (the standard deduction) in details.

4.2.1 DILATION OF PROPER TIME

The proper time, \(T_0 = \tau_2 - \tau_1\), is the duration between this two events at the same place \((\xi_1 = \xi_2 = \xi)\) in \(k\). We find the duration \(T\) in \(K\) by the second LT:

\[t_1 = \gamma(\tau_1 - \frac{v}{c^2}\xi) \quad \quad t_2 = \gamma(\tau_2 - \frac{v}{c^2}\xi)\]

The duration, \(T = t_2 - t_1\), in the moving system \(K\) is

\[T = \gamma T_0\] (30)

With the first LT we remark that the two considered events are not at the same place in \(K\)

\[x_1 = \gamma(\xi - v\tau_1) \quad \quad x_2 = \gamma(\xi - v\tau_2)\] (31)

This is a very well known result: Einstein(-Poincaré’s, see conclusion) dilation is the consequence of the fact that we must use two clocks in different places \((\Delta x = vT_0)\) of the moving system.

4.2.2 CONTRACTION OF PROPER LENGTH

According to Einstein (as in all standard books on SR), the proper length \(L_0 = \xi_2 - \xi_1\) is the length at rest in \(k\) \((\xi_2, \xi_1\) are the coordinates of the ends of the rod in \(k\)). The length of the moving rod is then defined as the distance between the two ends of the rod at the same time \((t = t_1 = t_2)\) in \(K\). We immediately find this length, \(L = x_2 - x_1\), by the inverse first LT:

\[\xi_1 = \gamma(x_1 + vt) \quad \quad \xi_2 = \gamma(x_2 + vt)\]

and therefore we obtain Einstein’s famous contraction:

\[L = \gamma^{-1}L_0\] (32)
Both Einstein’s deductions, dilation of time and contraction of length, are presented in all standards books as perfectly symmetric: two events at the same place (in k) for the dilation of duration and two events at the same time (in K) for the contraction of length. Nevertheless: what are the complete coordinates of the two events (ends of the rods) in the proper system k? In order to have the complete symmetry, we must consider the other LT not only in the case of dilation of duration (31) but also in the case of contraction of length. The second LT is:

$$\tau_1 = \gamma(t + \frac{v}{c^2}x_1)$$  \hspace{1cm}  $$\tau_2 = \gamma(t + \frac{v}{c^2}x_2)$$  \hspace{1cm}  (33)

This is a completely ignored result. The second LT determines obviously the times, $\tau_1$ and $\tau_2$, of the ends of the rods $\xi_1$ and $\xi_2$ in the proper system k and thus the complete coordinates of the two events ($\xi_1, \tau_1$) and ($\xi_2, \tau_2$): the simultaneous events in k are, obviously by LT, not simultaneous events ($\Delta\tau = \frac{v}{c^2}L_0$) in k. This is in contradiction with Einstein’s definition of identical RIGID rods (see above Definition 1 in 4-1) that implies that the proper length must be defined by simultaneous events in the proper system (the ends of the rigid rods are defined at the same time $\tau$). So Einstein’s contraction is not deduced directly from LT: it is a supplementary hypotheses (this is not the case in Poincaré’s kinematics, see conclusion).

**Definition 2**  The proper length $L_0$ is defined by two simultaneous events ($\xi_1, \tau$) and ($\xi_2, \tau$) in k (definition 1) and the improper length is defined by two simultaneous events ($x_1, t$) and ($x_2, t$) in K. But these events are not the images by LT one another.

Einstein’s inverse ($\gamma^{-1}$) contraction (32) or ”Einstein’s breaking of symmetry” is therefore clearly in opposition with Poincaré’s direct proportionality of the transformation of time and length in the moving system (see conclusion).

5 Conclusion: Definition of space-time Units in Poincaré’s Relativistic Kinematics.

Poincaré writes in 1911 in ”L’espace et le temps” on the special theory of relativity:

”Today some physicists want to adopt a new convention. This is not that they have to do it; they consider that this convention is easier, that’s all; and those who have another opinion may legitimately keep the old assumption in order not to disturb their old habits.”[Poincaré H. 1912]

”Some physicists” is a clear allusion to Einstein and Minkowski. What is the difference, according to Poincaré, between the ”old convention” and the ”new convention”? Let us examine Poincaré’s old (tacit) assumption in detail. What happens if we place another source in the second system K in Poincaré’s relativistic kinematics? Suppose that the relativistic ether is by definition at rest (spheres around O’) relative to the first source in K’. Poincaré’s relativistic ether is then moving relative to the second source at O in K and so we rediscover the second case with an ellipsoidal wave in the system of the source.
And reciprocally, with inverse LT, the role of the ether (the criterion of relativistic rest) is inverted. Logically in Poincaré’s SR, with one source or two sources, we always have a sphere in one system and an ellipsoid in the other system and never two spheres in the two systems (paragraph 4).

If historically Poincaré deduced directly the ellipsoid from the contraction of unit, we must now deduce the contraction of unit from the ellipsoid directly provided by the LT. From the main property of an elongated ellipse \( r^+ + r^- = 2kr' \) (see figure 2 the forth distance \( r^+ \) and the back distance \( r^- \) with respect to the second focus \( F^* \) or the forth travel time \( t^+ \) and the back travel time \( t^- \) with respect to the second focus \( F^* \))\(^8\) where M means "mean (average)", "round trip" or "two ways", we obtain:

\[
\begin{align*}
  r_M &= \frac{r^+ + r^-}{2} = kr' \\
  t_M &= \frac{t^+ + t^-}{2} = kt'
\end{align*}
\]

(34)

We have by definition in the system K' \( t' = 1_{t'} \) and \( r' = 1_{r'} \) (the choice of only one length unit). So if the elongated ellipse is an alternative definition of the units we must be able to deduce immediately Poincaré’s "round trip" units in K. Indeed we have:

\[
1_{r} = k1_{r'} \quad 1_{t} = k1_{t'}
\]

(35)

The unit of local time ("elliptical time") \( 1_{t} \) is always dilated in relation with the unit of true time ("circular time") \( 1_{r} \).

For the unit of space, we must first show that there is no transversal contraction :

\[
1_{y} = 1_{r} \sin \theta \text{ et } 1_{y'} = 1_{r'} \sin \theta'
\]

with \( \theta' = \frac{\pi}{2} \), we have (19) \( \sin \theta = k^{-1} \) and thus

\[
1_{y} = 1_{y'}
\]

(36)

We immediately have for the longitudinal component \( \cos \theta = \cos \theta' = 1 \), with \( r^+ = k(1 + \varepsilon) \) and \( r^- = k(1 - \varepsilon) \)

\[
1_{x} = k1_{x'}
\]

(37)

Let us call \( 1_{x'} \) ”the unit at (relativistic) rest” and \( 1_{x} \) ”the unit in (relativistic) moving”. So the unit at rest \( 1_{x'} \) is seen purely longitudinally elongated (by a factor \( k \)) by the observer O in moving in K. This is an unusual language but if we inverse the situation (if \( A > B \implies B < A \)) we then have

\[
1_{x'} = k^{-1}1_{x}
\]

(38)

The unit \( 1_{x} \) in moving is seen longitudinally contracted (by a factor \( k^{-1} \)) from the observer at rest O’. This is a more usual language\(^9\). We rediscover therefore the initial postulate of Poincaré about the contraction of a moving unit (see historical introduction).

\(^8\)Poincaré’s exact synchronisation (at the second order) is developed with Poincaré’s elongated ellipsoid in [Pierseaux Y. (Ph.D, ULB)] (1999). Poincaré’s elongated ellipse \( (t^+ \neq t^-) \) is Poincaré’s convention of synchronisation.

\(^9\)In Einstein’s relativistic logic \( A > B \) has no meaning because there are identical rigid rods (A=B) within the two systems (see definitions 1 and 2 in paragraph 4). In Poincaré’s relativistic logic it is completely equivalent to say ”the image of the unit at rest is always elongated” and ”the image of an unit in moving is always contracted".
We underline that Poincaré’s deduction of dilated units is based, and only based, on the application of LT (the “old convention”). He doesn’t need like Einstein a supplementary hypotheses (see paragraph 4). Einstein-Minkowski’s definition of identical units within both systems is clearly beyond the LT (the “new convention”, see paragraph 4). At the end of the deduction of his elongated ellipse Poincaré writes:

This hypothesis of Lorentz and FitzGerald will appear most extraordinary at first sight. All that can be said in its favour for the moment is that it is merely the immediate interpretation of Michelson experimental result, if we define (in italics in the text) distances by the time taken by light to traverse them.[Poincaré H. (1908)]

So with the ellipse we see immediately that the time of the round trip is the same in all directions (and therefore for the two Michelson’s perpendicular directions). So Poincaré’s historical ellipse is the immediate interpretation of Michelson’s null result\(^{10}\) without postulating that the source in the system (proper) of the Earth emits spherical waves.

And this is not all: according to Poincaré the distances are defined by the dilated time taken by light to traverse them. We can deduce this fundamental point directly from LT. The usual definition of the length of a rod implies that we consider at the same time the two ends of the rod. So we consider the two ends of the unit of length \(1_x'\) in K’ at the same time \(t' = 0\) (the primed coordinates are \(0,0\) and \(1,0\)). What is the length of the rod in the other system K (the moving system) according to Poincaré, i.e. according to LT? The calculation with (5) gives immediately \(1_x = k_1x'\). The elongation in the moving system of the stationary rod is a direct consequence of the fact that two simultaneous events in K’ are not simultaneous events in K (see paragraph 4) .

We conclude by remarking that Poincaré’s relativistic kinematics is based on a fundamental space-time proportionality (a dilation by a factor \(k\)) in perfect harmony with the invariance of the speed of light.

\[
\frac{r_M}{t_M} = \frac{kr'}{kt'} = \frac{1_v'}{1_v} = \frac{k_1v'}{k_1v} = c = 1
\]  

(39)

Poincaré’s direct space-time proportionality (35 &37) (very strange in Einstein’s kinematics\(^{11}\), see paragraph 4, (30 & 32)) characterizes fundamental Poincaré’s choice of space-time units in relativistic kinematics I shall choose the units of length and of time in such a way that the velocity of light is equal to unity \((X'\nu' = \lambda\nu = c = 1\)) . This is the reason why we kept Poincaré’s notations \(\varepsilon (\beta)\) and \(k (\gamma)\): behind Poincaré’s notations, there is not only Poincaré’s perfectly symmetrical representation of LT (5) but also Poincaré’s ”old convention” about the metric (in the sense of space-time units of measure) underlying the invariance of quadratic form in SR.

\(^{10}\)And also an immediate explanation of Sagnac non null result (1913). According to Selleri one of the main problems of rotating platform with Einstein’s kinematics is precisely Einstein’s invariance of one way speed of light, \(t^+ = t^-\), in the proper system [Selleri F.]. In Poincaré’s relativistic kinematics we can have in the system of the source \(t^+ \neq t^-\) (see figure 2). With Poincaré’s elongated ellipse [Pierseaux Y. (Ph.D, ULB)] and Poincaré’s group with rotations [Reignier J.], we predict immediately (at the second order \(k\)) the experimentally measured difference of time \(\frac{t^+ - t^-}{2} = k\varepsilon L\) \((L = 2\pi R, R\) being the radius of the platform).

\(^{11}\)In Einstein’s kinematics, the image by (30 & 32 ) of a purely longitudinal light clock in the proper system implies that the velocity of light in the moving system is \(\gamma^{-2} c\). This is perhaps the reason why we only find, in Einstein’s texts the purely transversal light clocks (without contraction)!
The existence of a "fine structure" of SR (two very close but not merged theories) is therefore demonstrated [Pierseaux Y. (Ph.D, ULB)].

According to Minkowski (1908), Einstein’s SR was not a local theory but a theory of "the world" or "the Universe" (worldline, worldpoint, worldinterval and even world principle, which is Minkowski’s name for the principle of relativity [Minkowski H.]).

The problem is that Minkowski’s metric is incompatible with an expansion of the Universe (Hubble 1929). Finally we point out that Poincaré’s metric involves not only a dilation of time but also an expansion of space. We will show in another paper that Poincaré’s completely relativistic expansion is directly connected with the deduction of the relativistic Doppler formulae from Poincaré’s ellipse.

6 Annex: Penrose’s and Poincaré’s elongated ellipsoid

The question under discussion is directly connected to another question: Penrose-Terrel’s analysis on "The Apparent Shape of a Relativistic Moving Sphere" (1959) or "The Invisibility of Lorentz Contraction" (1959) in Einstein’s SR. If we search the apparent shape for one observer of a moving material sphere, according to Penrose [Penrose R.], we have to send a signal of light that is reflected on the surface of the sphere and that finally returns to the observer. Penrose shows that we have to take into account Einstein’s 1905 relativistic formulae of aberration and Doppler effect. Terrel writes thus:

The factor M is the magnification, the ratio between subtended angles as seen by the observers O’ and O, or the ratio of apparent distances of the objects from the two observers. It is interesting that M is precisely the Doppler shift factor becoming $\sqrt{\frac{1-v^2}{1+v^2}}$ for $\theta = 0 = \theta'$. [Terrel J.]

In one dimension we avoid the question of aberration (for $\theta = 0 = \theta'$), which is the main problem of Penrose-Terrel and not under discussion in the present paper. If we try to measure a moving contracted rod $L = \gamma^{-1}L_0$ with the mean time travel of the signal of light (forth + and back –travel), Lampa [Lampa A.], before Penrose and Terrel, shows that, the longitudinal Doppler effect is respectively $\nu^+ = \sqrt{\frac{1-v^2}{1+v^2}}$ and $\nu^- = \sqrt{\frac{1+v^2}{1-v^2}}$. We have the mean travel time $t^+ = \sqrt{\frac{1+v^2}{1-v^2}}$ and $t^- = \sqrt{\frac{1-v^2}{1+v^2}} = \gamma T$. The mean "apparent distance" from O is, according to Lampa, $L_{app} = \gamma L$. Penrose explains how his elongated ellipsoid disappears:

The length of the image of the sphere in the direction of motion is thus greater than might otherwise be expected so that if it were not for the flattening the sphere would appear to be elongated. [Penrose R.]

And also Rindler:

This shows that a moving sphere presents a circular outline to all observers in spite of length contraction (or rather: because of length contraction; for without length contraction the outline would be distorted). [Rindler W.]

\[12\] In the deduction of Poincaré’s ellipse we have immediately by LT: $t^+ = k(1 + \varepsilon)$ and $t^- = k(1 - \varepsilon)$. 

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So according to Lampa-Penrose-Terrel the image of the rigid rod is, by compensation with Einstein’s contraction, a rigid rod. This enigmatic compensation, \( L_{\text{app}} = \gamma \gamma^{-1} L_0 = L_0 \), might be true (the image of a sphere, not by LT but ”by Doppler and aberration”, would be a sphere but only for ”sufficiently small subtended solid angle” [Terrel J., Weisskopf V. (2)]). However it is clear that Einstein’s convention is different to Poincaré’s one: in Poincaré’s SR the elongated light ellipsoid \textit{appears} because of the contraction of unit of length (see conclusion) whilst in Einstein-Minkowski-Penrose’s SR the elongated material ellipsoid \textit{disappears} because of Einstein’s contraction. Let us remark that in this scientific tradition (the relativistic shape of a sphere of matter), which begins in 1924 with Lampa, nobody made the slightest reference to Poincaré’s 1906 elongated ellipse (the relativistic shape of a sphere of light).

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