ON THE FRACTIONAL SUSCEPTIBILITY FUNCTION OF PIECEWISE EXPANDING MAPS

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Abstract. We associate to a perturbation \((f_t)\) of a (stably mixing) piecewise expanding unimodal map \(f_0\) a two-variable fractional susceptibility function \(Ψ_{φ}(η,z)\), depending also on a bounded observable \(φ\). For fixed \(η \in (0,1)\), we show that the function \(Ψ_{φ}(η,z)\) is holomorphic in a disc \(D_η \subset \mathbb{C}\) centered at zero of radius > 1, and that \(Ψ_{φ}(η,1)\) is the Marchaud fractional derivative of order \(η\) of the function \(t \mapsto R_{φ}(t) := \int φ(x) dμ_t\), at \(t = 0\), where \(μ_t\) is the unique absolutely continuous invariant probability measure of \(f_t\). In addition, we show that \(Ψ_{φ}(η,z)\) admits a holomorphic extension to the domain \(\{(η,z) \in \mathbb{C}^2 \mid 0 < \Re η < 1, z \in D_η\}\). Finally, if the perturbation \((f_t)\) is horizontal, we prove that \(\lim_{η \in (0,1), η \to 1} Ψ_{φ}(η,1) = \partial_t R_{φ}(t)|_{t=0}\).

1. Introduction

1.1. Linear response and violation thereof. Response theory describes how the “physical measure” \(μ_t\) of a dynamical system \(f_0\) responds to perturbations \(t \mapsto f_t\), given a class of observables (test functions) \(φ\). Classically, one studies linear response, where the goal is to express the derivative of \(R_{φ}(t) := \int φ(x) dμ_t\), for a fixed \(φ\) in a suitable class, in terms of \(φ\) together with \(f_0\), the vector field \(v_0 := \partial_t f_t|_{t=0}\), and the measure \(μ_0\).

Linear response was first investigated [14, 23, 30, 19] for smooth hyperbolic dynamics (Anosov or Axiom A). In the smooth mixing hyperbolic case, the physical measure \(μ_t\) is the SRB measure [42] and corresponds to the fixed point of a transfer operator \(L_t\) (whose dual preserves Lebesgue measure) on a suitable Banach space \(B\) (see e.g. [5]). This fixed point is a simple isolated eigenvalue in the spectrum of \(L_t\), and linear response can be proved via perturbation theory for simple eigenvalues (see e.g. [5] §2.5 and §5.3).

To avoid technicalities, we write the key formulas in the easy case of smooth expanding circle maps (see e.g. [4] for more details): Then, the physical measure is the unique absolutely continuous invariant probability measure, that is, \(μ_t = ρ_t dx\), with \(ρ_t\) smooth. The operator \(L_t\) acts on smooth functions, and we have the trivial but key identity

\[ ρ_t - ρ_0 = (I - L_t)^{-1}(L_t - L_0)ρ_0. \]
Next, assuming that \( v_0 = X_0 \circ f_0 \), it is not hard to show that
\[
\lim_{t \to 0} \frac{(\mathcal{L}_t - \mathcal{L}_0) \rho_0}{t} = -(X_0 \rho_0)' .
\]
Then, since \( \int \phi \mathcal{L}_0^k(\psi) \, dx = \int (\phi \circ f_0^k) \psi \, dx \), we have\(^1\) for continuous \( \phi \) (say),
\[
\int \phi \cdot (1 - \mathcal{L}_0)^{-1}(X_0 \rho_0)' \, dx = \sum_{k=0}^{\infty} \int \phi \cdot \mathcal{L}_0^k(X_0 \rho_0)' \, dx = \sum_{k=0}^{\infty} \int (\phi \circ f_0^k)(X_0 \rho_0)' \, dx .
\]
Finally, using (1), we get the fluctuation–dissipation formula (expressing the derivative as a Green–Kubo \(^2\) sum of decorrelations)
\[
(2) \quad \partial_t \mathcal{R}_\phi(t)|_{t=0} = \partial_t (\int \phi \, d\mu_t)|_{t=0} = -\sum_{k=0}^{\infty} \int (\phi \circ f_0^k)(X_0 \rho_0)' \, dx ,
\]
and, if \( \phi \) is differentiable, integrating by parts, we get the linear response formula
\[
(3) \quad \partial_t \mathcal{R}_\phi(t)|_{t=0} = \partial_t (\int \phi \, d\mu_t)|_{t=0} = \sum_{k=0}^{\infty} \int (\phi \circ f_0^k)' \cdot (X_0 \rho_0) \, dx .
\]
Exponential decay of the series in the right-hand side of (3) is not as apparent as in (2), since the derivative \( (\phi \circ f_0^k)'(x) = \phi'(f_0^k(x)) \cdot (f_0^k)'(x) \) grows exponentially. However, the presence of this derivative should be expected\(^3\) when describing response, and Ruelle\(^4\) pointed out that it was meaningful to view the linear response formula (3) as the value at \( z = 1 \) of a natural power series, the susceptibility function \( \Psi_\phi(z) \) is the Fourier transform of the response function \( k \mapsto \int X_0(\phi \circ f_0^k)' \rho_0 \, dx \), that is:
\[
(4) \quad \Psi_\phi(z) := \sum_{k=0}^{\infty} z^k \int (\phi \circ f_0^k)'(X_0 \rho_0) \, dx .
\]
We have, integrating by parts,
\[
\Psi_\phi(z) = -\sum_{k=0}^{\infty} z^k \int (\phi \circ f_0^k)(X_0 \rho_0)' \, dx ,
\]
so that exponential mixing implies that the susceptibility function \( \Psi_\phi(z) \) is holomorphic in a disc of radius larger than one, using that \( X_0 \rho_0 \) is smooth.

In situations when the map \( t \mapsto \mathcal{R}_\phi(t) \) is not differentiable (either due to bifurcations in the dynamics \(^5\) or to singularities \(^7\) of the test function \( \phi \)), it is natural to consider fractional response, i.e., to investigate weaker moduli of continuity of this map. The simplest situation where linear response breaks down is that of piecewise expanding unimodal maps. In this case, the transfer operator has a spectral gap when acting on \( BV \). However, the derivative \( \rho_0' \) of the invariant density \( \rho_0 \) involves a sum of Dirac masses, and thus does not belong to \( BV \), or to a space on which decorrelations are summable. Keller\(^8\) showed in 1982 that \( |\rho_t - \rho_0|_{L^1} = O(||t|| \log ||t||) \). Examples

\(^1\)Note that \( \int \phi(X_0 \rho_0)' \, dx = 0 \) because the circle has no boundary.

\(^2\)See e.g. the heuristic argument leading to \(^3\) (2).

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of families \((f_t)\) and smooth functions \(\phi\) such that \(|R_\phi(t) - R_\phi(0)| \geq |t||\log|t||\) were described in \cite{27} and \cite[Theorem 6.1]{3}. (See also the previous work of Ershov \cite{17}.) Baladi and Smania \cite{9} showed that if the family \(f_t\) is tangential \footnote{We refer to the beginning of \cite{4} for a definition of tangentiality, which is equivalent to the horizontalitiy condition \cite{10} on \(v_0\).} the topological class of \(f_0\), then for any continuous function \(\phi\), the map \(R_\phi(t)\) is differentiable at \(t = 0\). They also showed that when horizontality does not hold, then there exist smooth observables \(\phi\) such that \(R_\phi(t)\) is not Lipschitz at \(t = 0\). More recently, de Lima and Smania \cite{16} showed central limit theorems which imply that for a generic map \(f_0\), if \(v_0 = \partial_t f_0|_{t=0}\) is not horizontal, then for a generic \(\phi\), the map \(t \mapsto \int \phi \, d\mu_t\) cannot be Lipschitz on a set of parameters \(t\) of positive Lebesgue measure. (See also \cite{15} for a related result.)

In the piecewise expanding unimodal case, the susceptibility function \(\Psi_\phi(z)\) can also be defined by the formal power series \(\psi(t)\), and it has been studied in \cite{22} \cite{3} \cite{8}. In particular \cite{8}, if the postcritical orbit is dense (a generic condition) then \(\Psi_\phi(z)\) has a strong natural boundary on the unit circle, while if the perturbation \(v_0 = X_0 \circ f\) is horizontal and, in addition (a generic condition) the postcritical orbit is Birkhoff typical, then the nontangential limit of \(\Psi_\phi(z)\) as \(z\) tends to 1 coincides with \(\partial_t R_\phi(t)|_{t=0}\).

In the present paper, we introduce and study a two-variable fractional susceptibility function \(\Psi_\phi(\eta, z)\), for \(\Re \eta \in (0, 1)\), in the setting of piecewise expanding unimodal maps. The initial motivation for this work comes from the following paradox:

In 2005, Ruelle and Jiang considered \cite{32} \cite{22} finite Misiurewicz–Thurston (MT) parameters \(t_0\) in the quadratic family \(f_t(x) = t - x^2\). By definition of MT, there exists a repelling periodic point \(x_0\) and \(M \geq 2\) such that \(f_0^M(0) = x_0\). Such parameters \(t_0\) are Collet–Eckmann, so \(f_{t_0}\) admits a unique absolutely continuous invariant probability measure \(\mu_{t_0}\). The map \(f_{t_0}\) also has a finite Markov partition, which simplifies the analysis. Ruelle and Jiang proved that, for any \(C^1\) observable \(\phi\), the susceptibility function \(\Psi_\phi(z)\) defined by \(\psi(t)\) is meromorphic in the whole complex plane, and that \(z = 1\) is not a pole. Since \(\Psi_\phi(1)\) is the natural candidate for the derivative of \(R_\phi(t)\), this raised the hope that there could exist a “large” subset \(\Omega\) of Collet–Eckmann parameters, containing \(t_0\), and such that \(t \mapsto R_\phi(t)\) would be differentiable in the sense of Whitney on \(\Omega\) at \(t_0\). However, Baladi, Benedicks, and Schnellmann \cite{6} later showed that for any mixing (non horizontal) \footnote{All MT parameters are non horizontal by \cite{1} or \cite{26}.} MT parameter \(t_0\), there exist a \(C^\infty\) observable \(\phi\), a sequence \(t_n \to t_0\) of Collet–Eckmann parameters (with bounded constants), and \(C > 1\) such that

\[
\frac{\sqrt{|t_n - t_0|}}{C} \leq |R_\phi(t_n) - R_\phi(t_0)| \leq C \sqrt{|t_n - t_0|}, \quad \forall n.
\]

It is not known whether \(t_0\) is a Lebesgue density point in the set of \(t_n\) such that \(\psi(1)\) holds. Also, the analogue of the de Lima–Smania \cite{16} central limit theorems is not known in this setting. However, we expect that a similar CLT holds, and in particular that \(\Psi_\phi(1)\) cannot be interpreted as the derivative of \(R_\phi(t)\) in the sense of Whitney on a set of parameters containing \(t_0\) as a Lebesgue density point. The fact that \(\Psi_\phi(z)\) is holomorphic at \(z = 1\) can thus be viewed as a paradox.

Baladi and Smania \cite{11} very recently introduced two-variable fractional susceptibility functions \(\Psi_\phi(\eta, z)\) for the quadratic family, with the goal of resolving this paradox. The
piecewise expanding case is a toy model for the quadratic setting, and indeed, several
ideas previously developed for piecewise expanding families \[9\] were crucial to obtain
the breakthrough result \([5]\) of \([6]\) for the quadratic family. Although we believe it is
interesting in its own right, the analysis carried out in the present paper for this toy
model can also be viewed as a “proof of concept,” establishing the feasibility of the
fractional susceptibility function approach.

1.2. Informal statement of the results. We next describe briefly our main results.
It is well-known that there is no canonical notion of a fractional derivative, see \([35]\) and
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that fractional susceptibility functions defined via (e.g.) Bessel, or Riemann–Liouville
fractional derivatives would enjoy similar properties as those we establish here using the
Marchaud derivative.

In both susceptibility functions, \(\psi(x)\) and the observable \(\phi \in L^\infty\), to be the formal power series

\[
\Xi(\phi)(\eta, z) := \sum_{k=0}^{\infty} z^k \int \frac{\eta}{2\Gamma(1-\eta)} \int_{-\infty}^{\infty} \left( \frac{(\mathcal{L}_t \cdot \mathcal{L}_0)(x)}{|t|^{1+\eta}} \right) \phi \circ f^k(x) \cdot \left( \frac{(\mathcal{L}_t \cdot \mathcal{L}_0)(x)}{|t|^{1+\eta}} \right) \text{sign}(t) \, dt \, dx,
\]

and the frozen fractional susceptibility function of \((f_t)\) and \(\phi\) to be the formal power series

\[
\Xi^f(\phi)(\eta, z) := \sum_{k=0}^{\infty} z^k \int \frac{\eta}{2\Gamma(1-\eta)} \phi \circ f^k(x) \int_{-\infty}^{\infty} \left( \frac{(\mathcal{L}_t \cdot \mathcal{L}_0)(x)}{|t|^{1+\eta}} \right) \text{sign}(t) \, dt \, dx.
\]

In both susceptibility functions, \(\mathcal{L}_t\) is the transfer operator \(\mathcal{L}_t \psi(x) = \sum_{f_t(y) = x} \frac{\psi(y)}{|f_t(y)|}\).

Our first main result (Theorem \([23]\)) says that if \(\psi \) is bounded, then for any \(\eta \in (0, 1)\)
the improper integrals in the two above formal power series are convergent, and that
\(\Psi(\phi, \eta, z)\) and \(\Psi^f(\phi, \eta, z)\) are holomorphic functions of \(z\) in a disc \(D_\eta\) of radius (which may
depend on \(f_0\)) larger than one. In addition,

\[
\Xi^f(\phi, \eta, z) = \int \left( I - z \mathcal{L}_0 \right)^{-1} M^\eta_t \left( \mathcal{L}_t \rho_0(x) \right) \mid_{t=0} \, dx = \sum_k z^k \int \left( \phi \circ f^k \right) M^\eta_t \left( \mathcal{L}_t \rho_0(x) \right) \mid_{t=0} \, dx,
\]

and

\[
\Psi(\phi, \eta, 1) = M^\eta_t \mathcal{R}(t) \mid_{t=0}.
\]
Remark 1.1. Our lower bound for the radius of $D_\eta$ tends to 1 as $\eta \to 1$. If the critical point of $f_0$ is preperiodic, then we expect $\Psi_\phi(\eta, z)$ to be holomorphic in a disc of radius strictly larger than 1 and meromorphic in the entire complex plane for all $0 \leq \eta \leq 1$. However, we believe that, generically, the radius of convergence of $\Psi_\phi(\eta, z)$ should tend to 1 as $\eta \to 1$.

As a corollary of our main theorem and the results of [9], we obtain (Corollary 2.4) that, if $v_0$ is horizontal and $\phi$ is continuous, then

$$\lim_{\eta \to 1} \Psi_\phi(\eta, 1) = \lim_{\eta \to 1} \Psi_{\phi}^I(\eta, 1) = \Psi_\phi(1) = \partial_t R_\phi(t)|_{t=0}.$$}

Our second result, Theorem 2.6, is about more general fractional moduli of continuity: We consider there weighted Marchaud derivatives, replacing $|t|^{-1-\eta}$ by $(\log |t|)^{-\beta}|t|^{-1-\eta}$. Applying Theorem 2.3 to $\beta > 1$ and $\eta = 1$ gives a modulus of continuity $|t| (\log |t|)^\beta$, almost reproducing the $|t| (\log |t|)$ estimates from [24, 27, 3]. Applying Theorem 2.3 to $\beta < 0$, we show (Corollary 2.7) that $\Psi_\phi(\eta, z)$ and $\Psi_{\phi}(\eta, z)$ are holomorphic in the domain $\{ (\eta, z) \in \mathbb{C}^2 | 0 < \Re \eta < 1, z \in D_\eta \}$.

The fractional susceptibility functions in (6) and (7) are of “fluctuation–dissipation” type. It is tempting to consider the response fractional susceptibility function

$$\Psi_{\phi}^{fr}(\eta, z) := -\sum_{k=0}^{\infty} z^k \int_I (\phi \circ f_0^k) \cdot M_x^\phi(X_0 \rho_0) \, d\tau = \sum_{k=0}^{\infty} z^k \int_I (M_x^\phi(\phi \circ f_0^k)) \cdot X_0 \rho_0 \, d\tau,$$

(8)

where the Marchaud derivative is now taken with respect to $x$. The arguments in the present paper easily show that for all $\eta \in (0, 1)$ the function $\Psi_{\phi}^{fr}(\eta, z)$ is holomorphic in a disc of radius larger than 1 for all bounded $\phi$, and that in the horizontal case we have $\lim_{\eta \to 1} \Psi_{\phi}^{fr}(\eta, 1) = \partial_t R_\phi|_{t=0}$. The function $\Psi_{\phi}^{fr}(\eta, z)$ is at first sight the most seductive fractional susceptibility function. However, $\Psi_{\phi}^{fr}(\eta, 1)$ has no reason to coincide in general with any fractional derivative of order $\eta$ of $R_\phi(t)$ (except in the limit $\eta \to 1$, even in the linear examples studied in [24, 27]). In this respect, $\Psi_{\phi}^{fr}(\eta, z)$ is not better than the frozen susceptibility function $\Psi_{\phi}^{fr}(\eta, z)$. Keeping also in mind that the goal of this fractional approach is to resolve the paradox described above for the quadratic family, we focus on the definitions (6) and (7) in the present paper, referring to [111] for more on the fractional response susceptibility function.

Observe also that fractional derivatives do not enjoy a Leibniz formula with finitely many terms for the derivative of a product, so we cannot expect fractional susceptibility functions to be as well-behaved as ordinary ones.

We next make a few comments about our method of proof. We shall consider transfer operators acting on Sobolev spaces $H^{\tau,p}$ with $p > 1$, close to 1, and $0 < \tau < 1/p$. These spaces give us more flexibility than the $BV$ spaces classically used for piecewise expanding interval maps. We exploit the bounds of Thomine [35] for the essential spectral radius of $L_\tau$ on such spaces $H^{\tau,p}$. In particular, we have $\rho_0 \in H^{\tau,p}$ (see the beginning of

\[5\] The equality (8) follows by integration by parts for the Marchaud derivative [35] (6.27) for $\phi \in C^1$.
so that, for each $\eta < 1$, the function $M^2_\rho_0$ belongs to a Sobolev space on which the transfer operator $L_\lambda$ associated to $f_i$ has a spectral gap (see (22)). In order to apply the stable exponential decorrelation result of Keller–Liverani [25], we need Lasota–Yorke bounds which are uniform in $t$. Such bounds were known for the BV norm, but we have to carry out the corresponding estimates for the Sobolev spaces.

The paper is organised as follows: Section 2 starts with definitions and formal statements. In 2.1 we state precisely Theorem 2.3 on fractional susceptibility functions, followed by its consequence (Corollary 2.4) in the horizontal case. In 2.2 we state Theorem 2.6 about other moduli of continuity and Corollary 2.7 about holomorphic extensions of the fractional susceptibility functions. In 2.3 we discuss two linear examples. Section 3 introduces the key tools. In 3.1, we recall properties of the transfer operator acting on Sobolev spaces, from the work of Thomine [38]. In 3.2 we discuss stability of mixing and mixing rates for good families, recalling in particular the results of Keller and Liverani [25] that we shall use, and stating the relevant technical lemmas (Lemma 3.2 and Lemma 3.1). Subsection 3.3 contains the proof of the perturbation Lemma 3.3 for Sobolev spaces which is needed to apply results of Keller and Liverani. In Section 4, we prove Theorems 2.3 and 2.6, as well as Corollaries 2.4 and 2.7, using the techniques presented in Section 3. Appendix A contains the simple proof that the limit of the Marchaud derivatives as $\eta \to 1$ is the ordinary derivative. In Appendix B, we show the uniform Lasota–Yorke estimates (Lemma 3.1) needed to apply the results of Keller and Liverani to Sobolev spaces.

Throughout, we shall use the notation $C$ for a finite positive constant which can vary from place to place.

2. Definitions and formal statement of results

2.1. Fractional susceptibility functions for piecewise expanding interval maps.

Let $I$ be the compact interval $[-1,1]$ and let $r \in \{2,3\}$. We say that a continuous map $f: I \to I$ is a piecewise $C^r$ unimodal map if there exists $-1 < c < 1$ such that $f$ is increasing on $I_c = [-1,c]$, decreasing on $I_r = [c,1]$, and $f|_{I_r}$ extends as a $C^r$ map denoted $f_\sigma$ to a neighbourhood $I_\sigma$ of $I_c$. If, in addition, $\lambda(f) := \inf_{x=\pm} |f_\sigma'(x)| > 1$, we say that $f: I \to I$ is a piecewise $C^r$ expanding unimodal map, and we define

$$\lambda_n(f) := \inf_{\sigma \in \{\pm\}^n} \inf_{x} |(f_\sigma^n)'(x)|, \quad \Lambda(f) = \left( \lim_{n \to \infty} \lambda_n(f)^{-1/n} \right)^{-1},$$

for all $n \geq 1$, where $f^n_\sigma$ is the composition of the maps $f_\sigma$, $i = 1, \ldots, n$, and the points $x$ in $I_\sigma$ are restricted to those $x \in I_{\sigma_i}$ for which the composition exists. Following [9], we say that a piecewise $C^r$ expanding unimodal map $f$ is good if, either $c$ is not periodic for $f$, or $|f^{P_f}(c)'| > 2$ where $P_f \geq 2$ is the minimal period of $c$. Finally, we say that a piecewise $C^r$ expanding unimodal map $f$ is mixing if $f$ is topologically mixing on $[f^2(c), f(c)]$.

The results of Thomine [38] hold for piecewise $C^{1+\eta_0}$ maps, up to replacing the condition $\tau < 1/p$ there by $\tau < \max(\eta_0, 1/p)$. It would be interesting to adapt our results to this setting, for $0 < \eta < \eta_0$.

Goodness will ensure the uniform Lasota–Yorke Lemma 3.1 and thus stable mixing. It is not the weakest possible condition, see e.g. [18 Example 5.6] but some assumption is needed [21 Remark 15]. See also [10] Proposition 2.1.
\textbf{Definition 2.1} (Perturbation \((f_t)\) of a piecewise \(C^r\) expanding unimodal map). Let \(r \in \{2, 3\}\) and let \(f\) be a piecewise \(C^r\) expanding unimodal map. Given \(\varepsilon > 0\) and a family \((f_t)_{|t| \leq \varepsilon}\) of piecewise \(C^r\) unimodal maps (for fixed \(I_\pm\)) is called a \(C^r\) perturbation of \(f_0 = f\) if the \(C^r\) norm of the extension \(f_{t, \sigma}\) of \(f_t\) to \(\tilde{I}_\pm\) is uniformly bounded, with \(\|(f - f_t)|_{\tilde{I}_\pm}\|_{C^{r-1}} = O(t)\) as \(t \to 0\) for \(\sigma = \pm\).

If all \(f_t\) are topologically conjugated, or if \(c\) is not periodic for \(f\), set \(\Lambda := \inf_{|t| < \varepsilon} \Lambda(f_t)\), otherwise, if \(f\) is good, with \(c\) periodic of minimal period \(P_f \geq 2\), set

\[
\Lambda^{-1} := \sup_{|t| < \varepsilon} \lim_{k \to \infty} \left(\frac{\lambda_k P(f_t)}{2^k}\right)^{-\frac{1}{2k}};
\]

in all these cases, up to taking a smaller \(\varepsilon\), we may and shall assume that \(\Lambda > 1\).

Any piecewise \(C^2\) expanding unimodal map \(f_t\) admits a unique absolutely continuous invariant probability measure \(\mu_t = \rho_t \, dx\). The measure \(\mu_t\) is ergodic, and it is mixing if \(f_t\) is mixing. The density \(\rho_t\) is the unique fixed point (see e.g. \cite{2}) of the transfer operator \(\mathcal{L}_t\) defined on \(BV\) by

\[
\mathcal{L}_t \varphi(x) = \sum_{f_t(y) = x} \frac{\varphi(y)}{|f_t'(y)|}.
\]

We now recall the definition of the Marchaud fractional derivatives in order to introduce fractional susceptibility functions. Let \(0 < \eta < 1\), and set \(\Gamma_\eta = \frac{\eta}{\Gamma(1-\eta)}\), where \(\Gamma\) is Euler's function. Let \(g: \mathbb{R} \to \mathbb{C}\) be a bounded globally \(\eta\)-Hölder function with \(\eta \in (0, 1)\).

Recall \cite{35} p. 110, Theorem 5.9\) that for any \(\eta \in (0, \bar{\eta})\), the left-sided and right-sided Marchaud derivatives of \(g\) at \(t_0 \in \mathbb{R}\) are

\[
M^\eta_+ g(t_0) = \Gamma_\eta \int_0^\infty \frac{g(t_0) - g(t_0 - t)}{t^{1+\eta}} \, dt, \quad M^\eta_- g(t_0) = \Gamma_\eta \int_0^\infty \frac{g(t_0) - g(t_0 + t)}{t^{1+\eta}} \, dt.
\]

The two-sided Marchaud derivative is then defined by

\[
M^\eta g(t_0) := \frac{1}{2} \left( M^\eta_+ g(t_0) - M^\eta_- g(t_0) \right)
\]

\[
= \frac{\Gamma_\eta}{2} \left( \int_0^\infty \frac{g(t_0) - g(t_0 - t)}{t^{1+\eta}} \, dt - \int_0^\infty \frac{g(t_0) - g(t_0 + t)}{t^{1+\eta}} \, dt \right)
\]

\[
= \frac{\Gamma_\eta}{2} \int_{-\infty}^\infty \frac{g(t_0 + t) - g(t_0)}{|t|^{1+\eta}} \, \text{sign}(t) \, dt.
\]

The choice of the normalisation \(\Gamma_\eta\) ensures the following key property:

\textbf{Lemma 2.2.} Assume that \(g\) is bounded on \(\mathbb{R}\) and differentiable at \(t_0\). Then

\[
\lim_{\eta \uparrow 1} M^\eta_+ g(t_0) = g'(t_0) \quad \text{and} \quad \lim_{\eta \uparrow 1} M^\eta_- g(t_0) = -g'(t_0), \quad \text{so that} \quad \lim_{\eta \uparrow 1} M^\eta g(t_0) = g'(t_0).
\]

(The above lemma is certainly well-known, we provide a proof in Appendix \[\underline{A}\].)

Given a \(C^2\) perturbation \((f_t)_{|t| \leq \varepsilon}\) of a piecewise \(C^2\) expanding unimodal map \(f\), and given \(\varepsilon_1 \leq \varepsilon\), we put

\[
f_t^{(1)} = f_t, \forall |t| \leq \varepsilon_1, \quad f_t^{(\varepsilon_1)} = f_{\varepsilon_1}, \forall \varepsilon \geq \varepsilon_1, \quad f_t^{(-\varepsilon_1)} = f_{-\varepsilon_1}, \forall t \leq -\varepsilon_1.
\]

\(^8\text{See also} [26] \text{for an alternative approach.}\)
Then, for any function \( \phi \in L^1(I) \), we define the \( \eta \)-fractional susceptibility function of \((f_t^{(\epsilon)})\) for the observable \( \phi \) to be the formal power series
\[
\Psi_\phi(\eta, z) = \sum_{k=0}^{\infty} \frac{\Gamma(q)}{2^k k!} \int_I \phi \int_{-\infty}^{\infty} L_t^{(k)} \left( \frac{(L_t - L_0)\rho_0}{|t|^{1+\eta}} \right) \text{sign}(t) \, dt \, dx,
\]
and we define the frozen \( \eta \)-fractional susceptibility function of \((f_t^{(\epsilon)})\) and \( \phi \) to be the formal series
\[
\Psi^f_\phi(\eta, z) = \sum_{k=0}^{\infty} \frac{\Gamma(q)}{2^k k!} \int_I \phi \int_{-\infty}^{\infty} L_t^{(k)} \left( \frac{(L_t - L_0)\rho_0}{|t|^{1+\eta}} \right) \text{sign}(t) \, dt \, dx.
\]

The coefficient of \( z^k \) in each of the two formal power series above is a sum of improper integrals, for \( t \in (-\infty, 0) \) and \( t \in (0, \infty) \). We shall see in the proof of Theorem 2.3 that each integral converges, if \( \phi \) belongs to \( L^q \) for large enough \( q \). In particular, using Fubini, we recover the formulas (6) and (7) stated in the introduction.

We are now ready to state our main result. Recall \( \Lambda > 1 \) from Definition 2.1. For a function \( g(x,t) \) of two real variables, we denote by \( M^q|g|_{t=t_0} \) the Marchaud derivative of \( g \) in the \( t \) variable, at \( t = t_0 \).

**Theorem 2.3** (Fractional susceptibility function). Let \((f_t)_{t \leq \epsilon}\) be a \( C^2 \) perturbation of a mixing piecewise \( C^2 \) expanding unimodal map \( f \). Assume that either \( f \) is good or that all the \( f_t \) are topologically conjugated to \( f \). Then there exist \( \kappa < 1 \) (depending only on \( f_0 \)) and \( \epsilon_1 \in (0, \epsilon) \) such that for any \( 0 < \eta < 1 \), and for any \( \phi \in L^q(I) \) with \( q > (1-\eta)^{-1} \), the following holds for the fractional susceptibility functions of \((f_t^{(\epsilon)})\) and \( \phi \):

(a) \( \Psi_\phi(\eta, z) \) and \( \Psi^f_\phi(\eta, z) \) are holomorphic in the open disc of radius \( \min(\Lambda^{1-\eta}, \kappa^{-1}) \).

(b) \( \Psi_\phi(\eta, 1) = M^q(\int_I \phi(x)\rho_t(x) \, dx)_{t=0} \).

(c) \( \Psi^f_\phi(\eta, z) = \int_I \phi(x) \cdot ((I - zL_0)^{-1}M^q(L_t\rho_0)|_{t=0})(x) \, dx \) for all \( |z| < 1 \).

As explained in (11.2) formula (10) in the above theorem can be viewed as the key result of this work. The proof of Theorem 2.3 will be given in Section 4 after we introduce some necessary tools in Section 3. We next make a few remarks about the statement:

Clearly, if \( \phi \in L^\infty \), then we can consider all values of \( \eta \in (0, 1) \).

The proof of Theorem 2.3 actually shows that for \( \phi \in L^q(I) \), the response function is Hölder continuous with exponent \( \eta \) provided that \( q > (1-\eta)^{-1} \).

We do not claim that the holomorphy radius given in claim a) is optimal. However, we expect that, generically, the maximal holomorphic extension radius of \( \Psi_\phi(\eta, z) \) tends to one as \( \eta \to 1 \). It is unclear whether the frozen fractional susceptibility function is holomorphic in a disc of radius larger than one, uniformly in \( \eta \to 1 \).

Claim c) implies that, for any \( \eta \in (0, 1) \), and all \( |z| < 1 \)
\[
\Psi^f_\phi(\eta, z) = \sum_{k=0}^{\infty} z^k \int_I \phi(\int_0^t (x) \cdot (M^q(L_t\rho_0)|_{t=0})(x) \, dx.
\]
For \( z = 1 \) this is reminiscent of the fluctuation–dissipation formula for linear response (see e.g. 7). Note, however, that we cannot integrate by parts (in spite of [33 (6.27)])
because the Marchaud derivative is with respect to the parameter $t$. See Corollary 2.4 for more information on the frozen susceptibility function in the “horizontal” case.

To state an interesting corollary of our Theorem 2.3, letting $H_u$ denote the Heaviside jump at $u \in \mathbb{R}$, i.e., $H_u(x) = -1$ if $x < u$, while $H_u(x) = 0$ if $x > u$, and $H_u(u) = -1/2$, and setting $c_k = f^k(c)$, we recall that Proposition 3.3] we may decompose $\rho_0$ as $\rho_0 = \rho_0^{\text{reg}} + \rho_0^{\text{sal}}$, where the regular part $\rho_0^{\text{reg}}$ is differentiable and supported in $I$, with derivative in $BV$, and the saltus (or singular) part $\rho_0^{\text{sal}} = \sum_{k=1}^{N_f} s_k H_{c_k}$ where $N_f = \#\{c_k \mid k \geq 1\} \in [2, \infty]$. In addition, if the critical point $c$ is not periodic, we have

$$\rho_0^{\text{sal}} = \sum_{k=1}^{\infty} s_k H_{c_k}, \quad \text{where} \quad s_1 = -\lim_{x\uparrow c_1} \rho_0(x) < 0, \quad s_k = \frac{s_1}{(f^{k-1})'(c_1)}.$$  

Note that if $N_f$ is finite but $c$ is not periodic (it is then preperiodic) then the jump $s_j$ at $c_j$ is given by the sum of all $s_k$ for $k$ such that $c_k = c_j$.

We say that a bounded function $v$ is horizontal for $f$ if, setting $P_f = N_f$ if $c$ is periodic with minimal period $P_f \geq 2$, and $P_f = \infty$ otherwise, we have

$$\sum_{k=0}^{P_f-1} v(c_k) \frac{1}{(f^j)'(c_1)} = 0.$$  

If $v$ is not horizontal, we say that $v$ is transversal. In the horizontal case, we have (Corollary 2.4) is proved in Section 4.3:

**Corollary 2.4** (Fractional susceptibility of horizontal perturbations). Let $(f_t)_{|t| \leq \varepsilon}$ be a $C^3$ perturbation of a mixing piecewise $C^3$ expanding unimodal map $f$. Assume that either $f$ is good or that all the $f_t$ are topologically conjugated to $f$. Assume in addition that $f_t(-1) = f_t(1) = -1$ for all $t$, that $v_0 = \partial_t f_t|_{t=0} = X_0 \circ f_0$ is horizontal for $f_0$, where $X_0$ is $C^2$ on $I(f)$, and, finally that $(x, t) \mapsto f_t(x)$ extends as a $C^2$ map to $(\overline{I} + \overline{\mathbb{L}}) \times [-\varepsilon, \varepsilon]$. Then, there exists $\varepsilon_2 \in (0, \varepsilon)$ such that for any $\phi \in C^0$, the following holds for the fractional susceptibility function of $(f^{(x)}_t)$ and $\phi$:

$$\lim_{\eta \downarrow 1} \Psi_{\phi}(\eta, 1) = \partial_t \left( \int \phi(x) \rho_t(x) \, dx \right) \bigg|_{t=0}$$

$$\begin{aligned}
&= -\sum_{j=1}^{P_f} \phi(c_j) \sum_{k=1}^{j} s_k X_0(c_k) - \int \phi \cdot (I - \mathcal{L}_0)^{-1} (X_0' \rho_0^{\text{sal}} + (X_0 \rho_0^{\text{reg}})'') \, dx.
\end{aligned}$$  

Assume furthermore that $c$ is not periodic for $f$. Then, we have

$$\lim_{\eta \downarrow 1} (\Psi_{\phi}(\eta, 1) - \Psi_{\phi}^{\text{fr}}(\eta, 1)) = 0.$$  

**Remark 2.5.** The first term of (17) can be rewritten as

$$\begin{aligned}
-\sum_{j=1}^{P_f} \phi(c_j) \sum_{k=1}^{j} s_k X_0(c_k) &= -\int \phi \alpha (\rho_0^{\text{sal}})'', \quad \text{where} \quad \alpha(x) := -\sum_{j=0}^{P_f-1} \frac{v(f^j(x))}{(f^{j+1})'(x)}.
\end{aligned}$$

This implies $v_0(-1) = v_0(1) = 0$ and if $f(c) = 1$ then $v_0(c) = 0$. 

Indeed, $\alpha$ is the solution (which Lemma 2.2, Remark 2.3, Proposition 2.4) is unique and continuous under the assumptions of the corollary of the twisted cohomological equation

\begin{equation}
X_0(f(x)) = v_0(x) = \alpha(f(x)) - f'(x)\alpha(x), \quad x \neq c, \quad X_0(c) = v(c) = \alpha(c),
\end{equation}

so that, for $j \leq P_f$, \(s\)

\begin{equation}
\sum_{k=1}^{j} s_k X_0(c_k) = s_1 \sum_{k=1}^{j} \frac{X_0(c_k)}{(f^j - 1)'(c_1)} = s_1 \left( X_0(c_1) - \alpha(c_1) + \frac{\alpha(c_j)}{(f^j - 1)'(c_1)} \right) = s_j \alpha(c_j).
\end{equation}

2.2. Fractional moduli of continuity and Keller’s $x \log x$ bound. In the definition of the Marchaud derivatives $M^\alpha_+ g$ and $M^\alpha_- g$ of a function $g$, we may replace $t^{1+\eta}$ with other weights. Suppose for instance that $\ell: [0, \infty) \to \mathbb{C}$ is a continuous function and that $\gamma \in [0, 1)$ is a constant such that

\begin{equation}
\ell(0) = 0, \quad \int_0^{1} \frac{t^\gamma}{|\ell(t)|} \, dt < \infty \quad \text{and} \quad \int_1^{\infty} \frac{1}{|\ell(t)|} \, dt < \infty.
\end{equation}

We then define the right and left-sided $\ell$-Marchaud derivatives of a bounded $\gamma$-Hölder function $g$ by

\begin{equation*}
M_+^{(\ell)} g(t_0) = \int_0^{\infty} \frac{g(t_0) - g(t_0 - t)}{\ell(t)} \, dt, \quad M_-^{(\ell)} g(t_0) = \int_0^{\infty} \frac{g(t_0) - g(t_0 + t)}{\ell(t)} \, dt.
\end{equation*}

We define the two-sided $\ell$-Marchaud derivative by $M^{(\ell)} g = \frac{1}{2} (M_+^{(\ell)} g - M_-^{(\ell)} g)$. Finally, we define the $\ell$-susceptibility function to be the formal power series

\begin{equation*}
\Psi_{\phi}^{(\ell)}(\ell, z) = \sum_{k=0}^{\infty} z^k \int_{\mathbb{R}} \phi \int_{-\infty}^{\infty} \left( \frac{L_t^k (L_t - L_0) \rho_0}{\ell(|t|)} \right) \text{sign}(t) \, dt \, dx.
\end{equation*}

Similarly, we define the frozen $\ell$-susceptibility function by the formal power series

\begin{equation*}
\Psi_{\phi}^{(\ell)}(\ell, z) = \sum_{k=0}^{\infty} z^k \int_{\mathbb{R}} \phi \int_{-\infty}^{\infty} \left( \frac{L_0^k (L_t - L_0) \rho_0}{\ell(|t|)} \right) \text{sign}(t) \, dt \, dx.
\end{equation*}

The following theorem is proved in almost the same way as Theorem 2.3 (see Section 4.2).

**Theorem 2.6** (Generalized fractional susceptibility function). Let $(f_t)_{|t| \leq \varepsilon}$ be a $C^2$ perturbation of a mixing piecewise $C^2$ expanding unimodal map $f$. Assume that either $f$ is good or all the $f_t$ are topologically conjugated to $f$. Let $\ell$ and $\gamma \geq 0$ be such that holds. Then, for any $q \geq (1 - \gamma)^{-1}$ and any $\phi \in L^q$, the following holds:

(a) the susceptibility functions $\Psi_{\phi}^{(\ell)}(\ell, z)$ and $\Psi_{\phi}^{(\ell)}(\ell, z)$ are well-defined and holomorphic in a disc of radius strictly larger than one.

(b) $\Psi_{\phi}^{(\ell)}((\ell, 1)) = M^{(\ell)} \left( \int_{\mathbb{R}} \phi(x) \rho_1(x) \, dx \right) |_{t=0}$.

(c) $\Psi_{\phi}^{(\ell)}((\ell, z)) = \int_{\mathbb{R}} \phi(x) \cdot ((I - zL_0)^{-1} M^{(\ell)}(L_t \rho_0)) |_{t=0} \, dx$ for all $|z| \leq 1$.

For instance, (cf. the power-logarithmic kernel operators discussed in §21) we may fix $0 < \eta < 1$ and $\beta \geq 0$ and set

\begin{equation*}
\ell(t) = \ell_{\beta}(t) = t^{1+\eta} |\min\{-1, \log t\}|^\beta.
\end{equation*}
Then, if \( \phi \in L^q \), \( \eta = 1 - \frac{1}{q} \) and \( \beta > 1 \), Theorem 2.6 for \( \gamma < \eta \) implies
\[
M^{(\ell, \beta)}\left( \int_I \phi(x) \rho_t(x) \, dx \right)_{t=0} = \Psi_\phi((\ell, \beta), 1).
\]

In particular, if \( q = \infty \), we may take \( \eta = 1 \) and \( \beta > 1 \) arbitrarily close to 1 which is reminiscent of Keller’s [24] \( t \| \log |t| \) modulus of continuity.

Another consequence of Theorem 2.6 is that we may consider the Marchaud derivative \( M^q \left( \int_I \phi(x) \rho_t(x) \, dt \right) \) and the susceptibility functions \( \Psi_\phi(\eta, z) \) and \( \Psi^*_\phi(\eta, z) \) for non-real values of \( \eta \). A corollary of Theorem 2.6 is the following result, which we prove in Section 4.

**Corollary 2.7** (Holomorphic extension in \( (\eta, z) \)). Let \( (f_t)_{|t| \leq \varepsilon} \) be a \( C^2 \) perturbation of a mixing piecewise \( C^2 \) expanding unimodal map \( f \). Assume that either \( f \) is good or all the \( f_t \) are topologically conjugated to \( f \), and let \( \kappa < 1 \) and \( \varepsilon_1 \) be from Theorem 2.3. Let \( \phi \in L^q \), for some \( q > 1 \). Then the function \( \eta \mapsto M^q \left( \int_I \phi(x) \rho_t(x) \, dt \right) \) is holomorphic in the strip \( 0 < \Re \eta < 1 - \frac{1}{q} \), and the functions \((\eta, \zeta) \mapsto \Psi_\phi(\eta, z)\)

are holomorphic in the domain \( \{ (\eta, z) \mid 0 < \Re \eta < 1 - \frac{1}{q} \}, |z| < \min(\kappa, \lambda^{-\frac{2}{q}}) \} \).

### 2.3. Two linear examples.

We illustrate our definitions with two families of linear tent maps. One simplifying feature is that \( \rho^{\text{reg}}_t \) vanishes identically for all \( t \) in both examples. To study \( \rho^{\text{sal}}_t \), we shall use that for any \( x \neq u \), we have
\[
M^\theta_t H_t(x)|_{t=0} = -\frac{1}{2\Gamma(1-\eta)} \frac{1}{|x-u|^{\eta}}.
\]

To prove (22), observe that if \( x-u > 0 \), then
\[
H_{t+u}(x) - H_u(x) = H_t(x-u) - H_0(x-u) = \begin{cases} 0 & \text{if } t < x-u \\ -1 & \text{if } t > x-u. \end{cases}
\]

Similarly, if \( x-u < 0 \), then
\[
H_{t+u}(x) - H_u(x) = H_t(x-u) - H_0(x-u) = \begin{cases} 1 & \text{if } t < x-u \\ 0 & \text{if } t > x-u. \end{cases}
\]

It follows that
\[
M^\theta_t H_t(x)|_{t=0} = \frac{\Gamma_\eta}{2} \int_{-\infty}^{\infty} \frac{H_{t+u}(x) - H_u(x)}{|t|^{1+\eta}} \text{sign}(t) \, dt
\]
\[
= -\frac{\Gamma_\eta}{2} \int_{x-u}^{\infty} \frac{1}{|t|^{1+\eta}} \, dt = -\frac{1}{2\Gamma(1-\eta)} \frac{1}{|x-u|^{\eta}},
\]
as claimed. Note also that since \( H_{u+t}(x) = H_u(x-t) \) we have
\[
M^\theta_y H_u(y)|_{y=x} = -M^\theta_y H_t(x)|_{t=0}, \quad \forall x \neq u.
\]

The first example is the family of tent maps with fixed slopes \( \lambda_0 \in (1, 2) \) given by \( f_t(x) = \lambda_0 x + \lambda_0 - 1 + t_0 + t \) for \( -1 \leq x \leq 0 \), and \( f_t(x) = -\lambda_0 x + \lambda_0 - 1 + t_0 + t \) for \( 0 \leq x \leq 1 \), where \( |t| < \varepsilon_1 \), for small enough \( \varepsilon_1 \in (0, \min\{|t_0|, 2 - \lambda_0 - t_0\}) \). (For \( |t| > \varepsilon_1 \) we set \( f_t(x) = f_{\pm \varepsilon_1}(x) \).) This family is horizontal because each \( f_t \) is topologically conjugated to \( f_0 \). (Indeed, setting, \( h_t(x) = (\lambda_0 - 1 + t_0 + t)x \), we have \( h_t \circ f_0(x) = f_t \circ h_t(x) \). We can also use that the topological entropy of \( f_t \) is log \( \lambda_0 \) for all \( t \)).
We assume that the orbit of \( c = 0 \) is infinite for the sake of simplicity. We have \( \tilde{X}_0 \equiv 1 \) on the support of \( \tilde{\rho}_0 \), so that (22) and (23) imply
\[
M^\eta_t(\tilde{X}_0\tilde{\rho}_0)(x) = \sum_{n=1}^{\infty} \tilde{s}_n M^\eta_n H_{c_n}(x) = \frac{1}{2\Gamma(1 - \eta)} \sum_{n=1}^{\infty} \frac{s_1}{(\tilde{f}_0^{n-1})'(c_1)} \frac{1}{|x - c_n|^\eta},
\]
where \((\tilde{f}_0^{n-1})'(c_1) = \sigma_n \lambda_0^{n-1}\), for \( \sigma_n = \text{sgn}(\tilde{f}_0^{n-1})'(c_1) \). It follows that
\[
\Psi_\phi^{\text{resp}}(\eta, z) = -\frac{1}{2\Gamma(1 - \eta)} \sum_{k=0}^{\infty} z^k \int_I \frac{1}{(\phi \circ \tilde{f}_0^k)} \sum_{n=1}^{\infty} \frac{s_1 \sigma_n}{\lambda_0^{n-1}} \frac{1}{|x - c_n|^\eta} \, dx.
\]
Next, observe that since \( \tilde{f}_t(y) = \tilde{f}_0(y) + t \) for \( |t| < \varepsilon_1 \), we have \( (\tilde{L}_t \phi)(x) = (\tilde{L}_0 \phi)(x - t) \) for such \( t \), so that, using \( \tilde{L}_0 \tilde{\rho}_0^{\text{sal}} = \tilde{\rho}_0^{\text{sal}} \), we find, for any \(|t| < \varepsilon_1\),
\[
(\tilde{L}_t \tilde{\rho}_0^{\text{sal}})(x) = \sum_{n=1}^{\infty} \frac{s_1}{(\tilde{f}_0^{n-1})'(c_1)} H_{c_n}(x - t) = \sum_{n=1}^{\infty} \frac{s_1 \sigma_n}{\lambda_0^{n-1}} H_t(x - c_n).
\]
Set \( H_t^{(\varepsilon_1)}(y) = H_t(y) \) if \(|t| < \varepsilon_1\), and \( H_t^{(\varepsilon_1)}(y) = H_{\pm \varepsilon_1}(y) \) if \(|t| > \varepsilon_1\). Then, we have,
\[
M^\eta_t \big|_{t=\varepsilon_1}(y) = \begin{cases} -\frac{1}{2\Gamma(1 - \eta)} \frac{1}{|y-u|^\eta} & \text{if } 0 < |y-u| < \varepsilon_1, \\ 0 & \text{if } |y-u| \geq \varepsilon_1. \end{cases}
\]
(The claims above are proved in the same way as (22).) Using (24) and (25), we get,
\[
M^\eta_t(\tilde{L}_t \tilde{\rho}_0)(x)|_{t=0} = -\frac{1}{2\Gamma(1 - \eta)} \sum_{n=1}^{\infty} \frac{s_1 \sigma_n}{\lambda_0^{n-1}} \left( \frac{|c_n - e_1, c_0 + \varepsilon_1|}{|x - c_n|^\eta} \right).
\]
Finally,
\[
\Psi_\phi^{\text{sal}}(\eta, z) = -\frac{1}{2\Gamma(1 - \eta)} \sum_{k=0}^{\infty} z^k \int_I \frac{1}{(\phi \circ \tilde{f}_0^k)} \sum_{n=1}^{\infty} \frac{s_1 \sigma_n}{\lambda_0^{n-1}} \frac{|c_n - e_1, c_0 + \varepsilon_1|}{|x - c_n|^\eta} \, dx.
\]
So we see that, in the horizontal linear case given by the family \((\tilde{f}_t)\), the frozen and response susceptibility function coincide if \( \varepsilon_1 \geq \sup_n (c_1 - c_n, c_0 - c_2) \). This does not seem possible, since \( \varepsilon_1 \leq \varepsilon_0 \), where \( \varepsilon_0 \) is given by the uniform Lasota–Yorke bound Lemma 3.1. However, the two susceptibility functions are qualitatively similar. (In fact, the relation \( \tilde{f}_t = \tilde{f}_0 + t \) for \(|t| < \varepsilon_1\) implies that the response and frozen susceptibilities differ by a function holomorphic in a disc of radius larger than one; this can be shown as in [11 Proposition 2.5]). In addition, if we replaced the definitions of all fractional susceptibility functions the Marchaud derivative by the truncated Marchaud derivative
\[
M^{\eta, (\varepsilon_1)}(t) := \frac{\Gamma(\eta)}{2} \int_{-\varepsilon_1}^{\varepsilon_1} \frac{g(t_0 + t) - g(t_0)}{|t|^{1+\eta}} \text{sign}(t) \, dt,
\]
then the frozen and response susceptibility functions would coincide for \( \tilde{f}_t \). (Lemma 2.2 and the integration by parts formula mentioned in footnote 5 both hold for \( M^{\eta, (\varepsilon_1)} \).)

The second example is the family of tent maps with varying slopes \( \tilde{f}_t(x) = \lambda_t x + \lambda_t - 1 \) for \(-1 \leq x \leq 0\) and \( \tilde{f}_t(x) = -\lambda_t x + \lambda_t - 1 \) for \( 0 \leq x \leq 1\), where \( \lambda_t = \lambda_0 + t \), for \( \lambda_0 \in (1, 2) \) and \(|t| < \varepsilon_1\), with small enough \( \varepsilon_1 < \min(2 - \lambda_0, \lambda_0 - 1) \). This family is not horizontal

\footnote{With this modification, the use of \( f_t^{(\varepsilon_1)} \) as defined in [11] would not be needed.}
3.1. Basic definition and properties. We will use the Sobolev spaces \( \mathcal{H}^{\tau,p} = \mathcal{H}^{\tau,p}(I) \) of functions \( \varphi \in L^p(\mathbb{R}) \) supported in \( I \) such that
\[
\|\varphi\|_{\mathcal{H}^{\tau,p}} = \|\mathcal{F}^{-1}((1 + |\xi|^2)^{\tau/2}(\mathcal{F}\varphi))\|_{L^p} < \infty,
\]
where \( p > 1 \) and \( \tau \in [0, 1/p) \) are real numbers, and \( \mathcal{F} \) denotes the Fourier transform. (Note that \( \mathcal{H}^{r,p}(I) \subset \mathcal{H}^{\tilde{r},p}(I) \) if \( \tilde{r} < \tau \), and that the embedding is compact.)

Let \( f_t \) be a piecewise \( C^2 \) expanding unimodal map. Thomine \([38]\) showed that the transfer operator \( L_t \) is bounded on \( \mathcal{H}^{\tilde{r},p} \), for any \( 1 < p < \infty \) and \( 0 \leq \tau < \frac{1}{p} \). More precisely, there exists a constant \( C = C(\sup |f_t^n|, \inf |f_t|) \) such that

\[
\|L_t\varphi\|_{\mathcal{H}^{\tilde{r},p}} \leq C\|\varphi\|_{\mathcal{H}^{\tilde{r},p}}, \quad \forall \varphi, \; \forall 0 \leq \tau < 1/p.
\]

Recall that the essential spectral radius \( r_{\text{ess}}(L|_{\mathcal{H}^{\tilde{r},p}(I)}) \) of a bounded operator \( L : \mathcal{B} \to \mathcal{B} \) on a Banach space \( \mathcal{B} \) is the smallest \( r \geq 0 \) such that the spectrum of \( L \) on \( \mathcal{B} \) in the complement of the disc of radius \( r \) consists of isolated eigenvalues of finite multiplicity. Thomine \([38]\) Theorem 1.3] proved that, for any \( 0 < \tau < \frac{1}{p} \), we have\[31\]

\[
r_{\text{ess}}(L|_{\mathcal{H}^{\tilde{r},p}(I)}) \leq \lim_{n \to \infty} \left| 2^{p} \cdot 2^{n(1-\frac{1}{r})}(f_t^n)'(\lambda_n(f_t))^{-\tau} \right|^{\frac{1}{n}}
\]

\[
\leq \lim_{n \to \infty} 2^{1-\frac{\tau}{p}}(\lambda_n(f_t))^{-\tau+n}/n.
\]

In particular, if \( \lambda \in (1, \Lambda(f_t)) \), we find \( p(f_t) = p(f_t, \lambda) > 1 \) such that

\[
r_{\text{ess}}(L|_{\mathcal{H}^{\tilde{r},p}}) < \lambda^{-\tau-1+\frac{1}{p}} < 1, \quad \forall 1 < p < p(f_t), \; \forall 0 < \tau < \frac{1}{p}.
\]

By standard arguments (see \([38]\) Theorem 1.6), using that \( f_t \) is unimodal and thus topologically transitive on \([f_t^n(c), f_t(c)]\), and that the dual of \( L_t \) preserves Lebesgue measure \( dx \) on \( I \), it follows that for such \( \tau \) and \( p \) the spectral radius of \( L_t \) on \( \mathcal{H}^{\tilde{r},p} \) is equal to one, that the invariant density \( \rho_t \) belongs to \( \mathcal{H}^{\tilde{r},p} \), (extending \( \rho_t \) by zero outside of its domain), that \( \rho_t \) is the unique fixed point of \( L_t \) in \( \mathcal{H}^{\tilde{r},p} \), and that the algebraic multiplicity of the eigenvalue 1 is equal to one.

If \( 0 < \tilde{r} < \tau \) then, outside of the closed disc of radius \( \lambda^{\tilde{r}-\tau-1+\frac{1}{p}} \), the spectrum of \( L_t \) on \( \mathcal{H}^{\tilde{r},p}(I) \) and \( \mathcal{H}^r(I) \) coincide, including multiplicity and generalised eigenspaces (this follows from the fact that both spaces are continuously embedded in \( L^p \) while \( C^1 \) functions are dense in both spaces \([31]\) 2.1.3, Proposition 1], applying the result in \([12]\) App. A], see also \([5]\) App. A.2]).

### 3.2. Stability of mixing and mixing rates for good families.

To prove stability of the spectrum of the perturbation \((f_t)\), the following uniform Lasota–Yorke lemma is crucial. (A version of Lemma \([4,\text{Lemma }3.1]\) is established for the BV and \( L^1 \) norm, see\[12\] e.g. \([25]\) comment above Remark 2]. The proof for our Sobolev spaces is given in Appendix \([3]\).

**Lemma 3.1 (Uniform Lasota–Yorke bound).** Let \((f_t)\) be a \( C^2 \) perturbation of a piecewise \( C^2 \) expanding unimodal map \( f = f_0 \). Assume that either \( f \) is good or all the \( f_t \) are topologically conjugated to \( f \), and recall \( \Lambda > 1 \) from Definition \([\ref{def:topconj}])

\[1\]

Then for \((f_t)\) bounded by 2, and his “\( n \)-complexity at the end” is bounded by 2\(^n\). Cf. the proof of our uniform Lasota–Yorke estimate Lemma \([4,\text{Lemma }3.1]\).

\[2\]

Above equation (26) in \([\ref{BMM}]) there is a mistaken reference to Remark 5 in \([25]\) instead.
any \(\tilde{\Lambda} < \Lambda\) there exist \(p_0 \in (1, p(f))\) and \(\varepsilon_0 \leq \varepsilon\) such that for any \(p \in (1, p_0)\) and \(0 < \tilde{\tau} < \tau < \frac{1}{p}\) there exist finite constants \(C_0\) and \(C\) such that

\[
\|L_t^n\varphi\|_{\mathcal{H}^\tau,p} \leq C\tilde{\Lambda}(-\tau + \frac{1}{p})n\|\varphi\|_{\mathcal{H}^\tau,p} + C\varepsilon_0^n\|\varphi\|_{\mathcal{H}^\tau,p}, \quad \forall |t| < \varepsilon_0, \quad \forall n \geq 0, \quad \forall \varphi.
\]

We shall also use the following perturbation estimate, proved in \([3, \text{Lemma 3.3}]\):

**Lemma 3.2** (Perturbation bound). Let \((f_t)_{|t| \leq \varepsilon}\) be a \(C^2\) perturbation of a piecewise \(C^2\) expanding unimodal map \(f_0\). For any \(p > 1\) and \(0 < \tilde{\tau} < \frac{1}{p}\) there exists \(C < \infty\) such that

\[
\|(L_t - L_0)\varphi\|_{\mathcal{H}^\tau,p} \leq C|t|^{\tilde{\tau}}\|\varphi\|_{\mathcal{H}^\tau,p}, \quad \forall |t| < \varepsilon, \quad \forall \tau < \tilde{\tau} < \frac{1}{p}, \quad \forall \varphi.
\]

Assume now that \(f_t = f\) is mixing. Then for any \(\mathcal{H}^\tau,p\) with \(p \in (p(f), 1)\) and \(\tau < 1/p\), we have that 1 is the only eigenvalue of the transfer operator \(L_0\) of \(f\) on the unit circle (adapting e.g. the proof \([2, \text{Theorem 3.5}]\) for \(L_t\) acting on \(BV\)). In other words, the operator \(L_0\) has a spectral gap on \(\mathcal{H}^\tau,p\). The following notation will be useful:

**Definition 3.3** (Maximal eigenvalue \(\kappa < 1\) of a mixing piecewise \(C^2\) expanding unimodal map \(f\)). Let \(f\) be a mixing piecewise \(C^2\) expanding unimodal map. If there exist \(p > 1\) and \(\tau < 1/p\) such that \(L_0\) on \(\mathcal{H}^\tau,p\) has an eigenvalue \(\zeta \neq 1\) with \(|\zeta| > \Lambda(f)^{-\tau + 1/p}\), then we set \(\kappa(f) < 1\) to be the maximal such modulus \(|\zeta|\). Otherwise we set \(\kappa(f) = 0\).

Recalling the notation from \([31]\), fix \(\tau < 1/p\) for \(p < p(f, \tilde{\Lambda})\). Then for any \(\kappa_0 > \max(\kappa(f), \tilde{\Lambda}^{-\tau + 1/p})\) there exists \(C\) such that

\[
\|L_t^n\varphi\|_{\mathcal{H}^\tau,p} \leq C\kappa_0^n\|\varphi\|_{\mathcal{H}^\tau,p}, \quad \forall n \geq 0, \quad \forall \varphi \in \mathcal{H}^\tau,p := \{ \varphi \in \mathcal{H}^\tau,p(I) \mid \int \varphi \, dx = 0 \}.
\]

Lemmas 3.3 and 3.2 put us in a position to apply the results of Keller–Liverani and show stability of \([3, \text{Lemma 3.3}]\): Since \(\mathcal{H}^\tau,p\) is invariant under each \(L_t\), \([2, \text{Corollary 2 (2)}]\) gives for any \(\kappa_1 > \max(\kappa(f), \tilde{\Lambda}^{-\tau + 1/p})\) constant, \(C = C(\tau, p) < \infty, \varepsilon_1 \in (0, \varepsilon_0)\) such that

\[
\|L_t^n\varphi\|_{\mathcal{H}^\tau,p} \leq C\kappa_1^n\|\varphi\|_{\mathcal{H}^\tau,p}, \quad \forall n \geq 1, \quad \forall |t| \leq \varepsilon_1, \quad \forall \varphi \in \mathcal{H}^\tau,p.
\]

The uniform mixing rate given by the above bound will be crucial to establish our main theorem.

### 3.3. Four basic lemmas and the proof of the perturbation Lemma 3.2

We end this section by showing the perturbation Lemma 3.2. The proof will use the following four standard lemmas (the first three lemmas are also instrumental in the proof of Lemma 3.1):

**Lemma 3.4** (Triebel \([40, \text{Section 4.2.2}]\)). Suppose that \(g \in C^\gamma\) where \(\gamma > \tau\), and let \(p > 1\). Then there exists \(C = C(\gamma, p, \tau) > 0\) such that \(\|g\varphi\|_{\mathcal{H}^\tau,p} \leq C\|g\|_{C^\gamma} \|\varphi\|_{\mathcal{H}^\tau,p}\).

**Lemma 3.5** (See e.g. \([38, \text{Lemma 3.3}]\)). Let \(T\) be a \(C^1\) diffeomorphism of \(\mathbb{R}\) such that \(\frac{A}{2} \leq |T'(x)| \leq 2A\) for some \(A > 0\) and all \(x \in \mathbb{R}\). Then, for all \(0 \leq \tau \leq 1\) and \(p > 1\) there exists \(C = C(\tau, p) > 0\) such that \(\|\varphi \circ T\|_{\mathcal{H}^\tau,p} \leq CA^{\tau - 1} \|\varphi\|_{\mathcal{H}^\tau,p} + CA^{\tau - 1} \|\varphi\|_{L^p}\).

\(13C(\tau, p)\) tends to infinity as \(\eta\) tends to 1.
**Lemma 3.6** (Strichartz [37 Corollary I.4.2]). Suppose that $0 \leq \tau < \frac{1}{p} < 1$. Then there is a constant $C = C(\tau, p)$ such that $\|1_I f\|_{L^\infty_t L^p_x} \leq C \|\varphi\|_{L^{1+\tau}_t L^{p\tau}_x}$ for any interval $I$.

**Lemma 3.7** (See e.g. [5] Lemma 2.39). For any $p > 1$ and any $C^1$ map $T: \mathbb{R} \to \mathbb{R}$ there exists $C = C(p, \|T\|_{C^1})$ such that $\|\varphi - \varphi \circ T\|_{L^{1+\tau}_t L^{p\tau}_x} \leq C \|I - T\|_{C^{1-\hat{\tau}}_t} \|\varphi\|_{L^{1+\tau}_t L^{p\tau}_x}$ for all $0 < \hat{\tau} < \tau < 1$ and all $\varphi$.

**Proof of Lemma 3.7.** Let $I_t = f_t(I)$, and put $J_t = I_t \setminus (I_0 \cap I_t)$ and $K_t = I_0 \setminus (I_0 \cap I_t)$. (It is possible that $J_t$ or $K_t$ is empty.) We have

$$L_t \varphi - L_0 \varphi = 1_{I_0 \cap I_t} (L_t \varphi - L_0 \varphi) + 1_{J_t} L_t \varphi - 1_{K_t} L_0 \varphi,$$

since $L_t \varphi$ is supported in $I_t = J_t \cup (I_0 \cap I_t)$ and $L_0 \varphi$ is supported in $I_0 = K_t \cup (I_0 \cap I_t)$. Hence

$$\|L_t \varphi - L_0 \varphi\|_{L^{1+\tau}_t L^{p\tau}_x} \leq \|1_{I_0 \cap I_t} (L_t \varphi - L_0 \varphi)\|_{L^{1+\tau}_t L^{p\tau}_x} + \|1_{J_t} L_t \varphi\|_{L^{1+\tau}_t L^{p\tau}_x} + \|1_{K_t} L_0 \varphi\|_{L^{1+\tau}_t L^{p\tau}_x}.$$

We consider first the term $\|1_{I_0 \cap I_t} (L_t \varphi - L_0 \varphi)\|_{L^{1+\tau}_t L^{p\tau}_x}$. If $x \in I_0 \cap I_t$, both $f_{t, \pm}^{-1}(x)$ and $f_{0, \pm}^{-1}(x)$ are defined, and we may write

$$\|1_{I_0 \cap I_t} (L_t \varphi - L_0 \varphi)\|_{L^{1+\tau}_t L^{p\tau}_x} \leq \|1_{I_0 \cap I_t} (f_{t, \pm}^{-1} - f_{0, \pm}^{-1}) \|_{L^{1+\tau}_t L^{p\tau}_x} + \|1_{I_0 \cap I_t} (f_{t, \pm}^{-1} - f_{0, \pm}^{-1}) \|_{L^{1+\tau}_t L^{p\tau}_x}.$$

We consider the first term on the right-hand side (the second one is treated in the same fashion). Since $(f_t)$ is a $C^2$ perturbation, we may extend $f_{t, -}: [-1, c] \to I_t$ to a $C^2$-diffeomorphism of $\mathbb{R}$, still denoted $f_{t, -}: \mathbb{R} \to \mathbb{R}$, such that $\inf_{\|f_t\|_{C^1} < \varepsilon} f_{t, -} : \mathbb{R} \to \mathbb{R}$, with $f_{t, -}$ has compact support and is bounded uniformly in $t$, and $\|f_{t, -} - f_{0, -}\|_{C^1}$ is $O(|t|)$ as $|t| \to 0$. Since $0 < \hat{\tau} < 1/p$, by the Strichartz Lemma 3.6

$$\|1_{I_0 \cap I_t} (\frac{\varphi}{f_{t, -}}) \circ f_{t, -}^{-1} - (\frac{\varphi}{f_{0, -}}) \circ f_{0, -}^{-1}\|_{L^{1+\tau}_t L^{p\tau}_x} \leq C \|\frac{\varphi}{f_{t, -}} \circ f_{t, -}^{-1} - (\frac{\varphi}{f_{0, -}}) \circ f_{0, -}^{-1}\|_{L^{1+\tau}_t L^{p\tau}_x},$$

where $\varphi$ is extended by zero outside of its domain $I$. We then split

$$\frac{\varphi}{f_{t, -}} \circ f_{t, -}^{-1} - (\frac{\varphi}{f_{0, -}}) \circ f_{0, -}^{-1} \leq \frac{\varphi}{f_{t, -}} \circ f_{t, -}^{-1} - (\frac{\varphi}{f_{0, -}}) \circ f_{0, -}^{-1} + ((\frac{1}{f_{t, -}} \circ f_{t, -}^{-1} - 1) \cdot (\varphi \circ f_{0, -}^{-1})) \|_{L^{1+\tau}_t L^{p\tau}_x}.$$

The first term in (37) is estimated using Lemmas 3.4, 3.5 and 3.7

$$\|\frac{\varphi}{f_{t, -}} \circ f_{t, -}^{-1} - (\frac{\varphi}{f_{0, -}}) \circ f_{0, -}^{-1}\|_{L^{1+\tau}_t L^{p\tau}_x} = \|((\frac{\varphi}{f_{t, -}}) \circ f_{t, -}^{-1} - (\frac{\varphi}{f_{0, -}}) \circ f_{0, -}^{-1}) \circ f_{0, -}^{-1}\|_{L^{1+\tau}_t L^{p\tau}_x} \leq C \|((\frac{\varphi}{f_{t, -}}) \circ f_{t, -}^{-1} - (\frac{\varphi}{f_{0, -}}) \circ f_{0, -}^{-1}) \circ f_{0, -}^{-1}\|_{L^{1+\tau}_t L^{p\tau}_x} \leq C \|I - f_{t, -}^{-1} \circ f_{0, -}^{-1}\|_{C^1} \|\varphi\|_{L^{1+\tau}_t L^{p\tau}_x},$$

if $0 < \hat{\tau} < \tau < 1$. It is easy to check that $\|f_{t, -} - f_{0, -}\|_{C^1} = O(|t|)$ implies that $\|I - f_{t, -} \circ f_{0, -}\|_{C^1} = O(|t|)$. 


Note that \( \| \frac{1}{|f'_{t,-}|} \circ f_{0,-1} - \frac{1}{|f'_{0,-}|} \circ f_{0,-1} \|_{C^1} = O(|t|) \) as \( t \to 0 \). Since \( 0 \leq \tilde{\tau} < \tau < 1 \), for the second term in (37) we then have by Lemmas 3.4 and 3.5 the upper bound
\[
\left\| \left( \frac{1}{|f'_{t,-}|} \circ f_{0,-1} - \frac{1}{|f'_{0,-}|} \circ f_{0,-1} \right) \cdot (\varphi \circ f_{0,-1}) \|_{\mathcal{H}^\tau,p} \leq C|t|\|\varphi\|_{\mathcal{H}^\tau,p} \leq C|t|\|\varphi\|_{\mathcal{H}^\tau,p}.
\]

We have shown that if \( 0 < \tilde{\tau} < \tau < \frac{1}{p} \) then
\[
\| I_{K_1} (L_\tau \varphi - L_0 \varphi) \|_{\mathcal{H}^\tau,p} \leq C|t|^{\tilde{\tau} - \tilde{\tau}}\|\varphi\|_{\mathcal{H}^\tau,p}.
\]

We return to (36) and consider \( \| I_{J_1} L_\tau \varphi \|_{\mathcal{H}^\tau,p} \).

If \( q \in (p, \infty) \), letting \( r \) be the conjugate of \( q/p \), then Hölder’s inequality gives for any interval \( J \subset I \) and any \( u \in L^q(I) \) that
\[
\| I_{J_1} u \|_{L^p(J)} = \| I_{J_1} u^p \|^\frac{1}{p} \| L^q(I) \|_F \leq \left( \| I_{J_1} \|_{L^p(J)} \| u^p \|_{L^q(I)} \right)^{\frac{1}{p}} = |J|^{\frac{1}{q}} \| u \|_{L^q(I)} = |J| \| \frac{1}{q} - \frac{1}{p} \| u \|_{L^q(I)}.
\]

Since \( |J_1| \leq C|t| \), it follows by the Sobolev embedding theorem that, taking \( q > p \) such that \( \tau = \frac{1}{p} - \frac{1}{q} \),
\[
\| I_{J_1} L_\tau \varphi \|_{\mathcal{H}^\tau,p} \leq C\|\varphi\|_{\mathcal{H}^\tau,p}.
\]

Finally, since \( \mathcal{H}^\tau,p = [L^p, \mathcal{H}^\tau,p]_{\theta/\tau} \) (where \([B_0, B_1]_\theta \) denotes complex interpolation) interpolating at \( \theta = \tilde{\tau}/\tau \) (see [34, §2.5.2] and [39, §1.9]) between \( \| K_1 \| \) and \( \| L_0 \| \), we get
\[
\| I_{J_1} L_\tau \varphi \|_{\mathcal{H}^\tau,p} \leq C|t|^{\tilde{\tau}(1 - \theta)}\|\varphi\|_{\mathcal{H}^\tau,p} = C|t|^{\tilde{\tau} - \tilde{\tau}}\|\varphi\|_{\mathcal{H}^\tau,p}.
\]

In the same way, since \( |K_1| \leq C|t| \), we get \( \| K_1 L_\tau \varphi \|_{\mathcal{H}^\tau,p} \leq C|t|^{\tilde{\tau} - \tilde{\tau}}\|\varphi\|_{\mathcal{H}^\tau,p} \). Recalling (33), we have proved \( \| L_\tau \varphi - L_0 \varphi \|_{\mathcal{H}^\tau,p} \leq C|t|^{\tilde{\tau} - \tilde{\tau}}\|\varphi\|_{\mathcal{H}^\tau,p} \) for \( 0 < \tilde{\tau} < \tau < \frac{1}{p} \).

4. Proofs of Theorems 2.3 and 2.6 and Corollaries 2.4 and 2.7

4.1. Proof of Theorem 2.3 on the fractional susceptibility function. Clearly,
\[
\int_I (L_\tau - L_0) \varphi(x) \, dx = 0, \quad \forall \varphi \in L^1.
\]

Since \( I \) is compact, we may assume that \( q \neq \infty \) (otherwise replace \( q \) by any finite number larger than \((1 - \eta)^{-1}\)). It suffices to consider the contribution of the improper integral \( \int_{-\infty}^{0} \Phi_\phi(\eta, z) \), the computation for the integral \( \int_{0}^{\infty} \) is exactly the same. We will prove claim (31) for \( \Phi_\phi(\eta, z) \), the proof for \( \Phi_\phi^\tau(\eta, z) \) is obtained by a slight simplification. Let \( q' > 1 \) be the conjugate of \( q \), that is \( \frac{1}{q} + \frac{1}{q'} = 1 \). Then, by Hölder’s inequality,
\[
\int_I |\tilde{\phi} \cdot L^\tau_k (L_\tau - L_0) \rho_0| \, dx \leq \| L^\tau_k (L_\tau - L_0) \rho_0 \|_{L^{q'}} \| \tilde{\phi} \|_{L^q}.
\]

Next, fixing \( p \in (1, q') \) (we shall need to take \( p \) close to 1 soon) and setting
\[
\tilde{\tau} := \frac{1}{p} - \frac{1}{q'} = \frac{1}{p} - 1 + \frac{1}{q} \in (0, \frac{1}{p} - \eta),
\]

it follows that
the Sobolev embedding theorem [20, Theorem 1.3.5], gives a constant $C_{p,q'}$ such that
\[
\int |\phi \cdot L_t^k(L_t - L_0)\rho_0| \, dx \leq C_{p,q'} \|L_t^k(L_t - L_0)\rho_0\|_{H^{\tau,p}} \|\phi\|_{L^q}.
\]
Thus, for each fixed $k \geq 0$, by Lemma 3.2 the improper integral defining the coefficient of $z^k$ in $\Psi(\eta, z)$ or $\Psi^R(\eta, z)$ is a well-defined complex number.

Assume from now on that $p \in (1, \min(p_0, q'))$ where $p_0$ is from Lemma 3.1 and let $\kappa$ be as in Definition 3.3.

Since $\Lambda^{-\tilde{\tau} - \frac{1}{p'}} = \Lambda^{-\frac{1}{q'}}$, using (40) for $\varphi = \rho_0$, we have for any $\kappa_1 > \max(\kappa, \Lambda^{-\frac{1}{q'}})$, by Keller and Liverani’s consequence (35) of the uniform Lasota–Yorke and perturbation bounds, constants $\varepsilon_1 > 0$ and $14 C_q < \infty$ such that
\[
\|L_t^k(L_t - L_0)\rho_0\|_{H^{\tau,p}} \leq C_{\eta,\kappa_1}^k \|L_t - L_0\|_{H^{\tau,p}}, \forall |t| \leq \varepsilon_1.
\]

For any $\tau \in (\tilde{\tau}, \frac{1}{q'})$, the bound (33) implies
\[
\|L_t^k(L_t - L_0)\rho_0\|_{H^{\tau,p}} \leq C |t|^{-\tilde{\tau} - \tau} \|\rho_0\|_{H^{\tau,p}}, \forall |t| \leq \varepsilon_1.
\]
We conclude that for any $0 \leq |t| < \varepsilon_1$
\[
\int |\phi \cdot L_t^k(L_t - L_0)\rho_0| \, dx \leq C_{\eta,\kappa_1}^k |t|^{-1 - \eta + \tau - \tilde{\tau}} \|\rho_0\|_{H^{\tau,p}} \|\phi\|_{L^q}.
\]

Since $q > (1 - \eta)^{-1}$, we may choose $\tau < 1/p$ and $\tilde{\tau} = 1/p - 1 + 1/q$ such that $\eta < \tau - \tilde{\tau}$.

For such $\tau$ and $\tilde{\tau}$, we may integrate (44) over $t \in (0, \varepsilon_1)$, and we get
\[
\int_0^{\varepsilon_1} \int |\phi(x)L_t^k(L_t - L_0)\rho_0| \, dx \, dt \leq C_{\eta,\kappa_1}^k \|\rho_0\|_{H^{\tau,p}} \|\phi\|_{L^q}.
\]

For $t > \varepsilon_1$, we have $L_t = L_{x \equiv 1}$, so that, using again the arguments above,
\[
\int_{t > \varepsilon_1} |\phi \cdot L_t^k(L_t - L_0)\rho_0| \, dx \, dt \leq \int_{t > \varepsilon_1} \frac{1}{|t|^{1+\eta}} \left( \int |\phi \cdot L_{\equiv 1}^k(L_{\equiv 1} - L_0)\rho_0| \, dx \right) \, dt
\]
\[
\leq C_{\eta,\kappa_1}^k \varepsilon_1^{-\tilde{\tau} - \tau} \|\rho_0\|_{H^{\tau,p}} \|\phi\|_{L^q}.
\]

By Fubini, the bounds (45) and (46) imply that $\Psi_\phi(\eta, z)$ and $\Psi^R_\phi(\eta, z)$ are holomorphic in the disc of radius $\kappa_1^{-1}$, if $\phi \in L^q$ for $q > (1 - \eta)^{-1}$. Since we may take $q$ arbitrarily close to $(1 - \eta)^{-1}$ and $\kappa_1$ arbitrarily close to $\min(\kappa, \Lambda^{-\frac{1}{q'}})$, this shows (33).

We next show claim (13) of the theorem. The bounds (40), (42), (45), and (46) together with Fubini and the dominated convergence theorem imply that, as holomorphic functions in the disc of radius $\kappa_1^{-1} > 1$,
\[
\Psi_\phi(\eta, z) = \frac{\Gamma_q}{2} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \int \phi \left( \frac{L_t^k(L_t - L_0)\rho_0}{|t|^{1+\eta}} \right) \text{sign}(t) \, dx \, dt
\]
\[
= \frac{\Gamma_q}{2} \int_{-\infty}^{\infty} \int \phi \left( (I - zL_t)^{-1} \frac{(L_t - L_0)\rho_0}{|t|^{1+\eta}} \right) \text{sign}(t) \, dx \, dt.
\]

As $\eta \to 1$, we have $C_\eta \leq C(1 - \eta)^{-1}$. 

In view of (10), the trivial identity \((L_t - L_0)\rho_0 = (I - L_t)(p_t - \rho_0)\) implies the key formula (11). In particular,
\[
\Psi_{\phi}(\eta, 1) = \frac{\Gamma_\eta}{2} \int_{-\infty}^\infty \frac{1}{|t|^{1+\eta}} \int_I \phi(x)(\rho_0(x) - \rho_t(x)) \, dx \, \text{sign}(t) \, dt
\]
\[
= M^n \left( \int_I \phi(x)p_t(x) \, dx \right) \bigg|_{t=0}.
\]
This proves claim (b).

To conclude the proof of Theorem 2.3, it only remains to prove (c). Note that \(L_0^k\) does not depend on \(t\) and acts on functions depending on \(x\). Since we have shown that every improper integral defining \(\Psi_{\phi}^k(\eta, z)\) is convergent, and since the sum over \(k\) in (13) converges absolutely for \(|z| \leq 1\), we may write
\[
\Psi_{\phi}^k(\eta, z) = \sum_{k=0}^\infty \int_I \phi^k \cdot L_0^k \left( \frac{\Gamma_\eta}{2} \int_{-\infty}^\infty \frac{L_0 - L_t}{|t|^{1+\eta}} \, \text{sign}(t) \, dt \right) \, dx
\]
\[
= \int_I \phi(I - zL_0)^{-1} \left( \frac{\Gamma_\eta}{2} \int_{-\infty}^\infty \frac{L_0 - L_t}{|t|^{1+\eta}} \, \text{sign}(t) \, dt \right) \, dx.
\]
Thus, (c) just follows from the definition of the Marchaud derivative \(M^n\).  

\[\square\]

4.2. Proof of Theorem 2.6 on the generalized fractional susceptibility function. The proof of Theorem 2.6 is the same as the proof of Theorem 2.3 up to equation (44). Instead of (44) we have
\[
\int_I \left| \phi - \frac{L_0 - L_t}{\ell(t)} \right| \, dx \leq C_{k,1} \frac{|t|^{\tau-\bar{\tau}}}{\ell(t)} \| \rho_0 \| \| \phi \|_{L^q}.
\]
We choose \(\tau < 1/p\) and \(\bar{\tau} = 1/p - 1 + 1/q\). Then \(\tau - \bar{\tau} = 1 - 1/q \geq \gamma\) and
\[
\int_0^{\varepsilon_1} \frac{t^{\tau-\bar{\tau}}}{\ell(t)} \, dt \leq \int_0^{\varepsilon_1} \frac{t^{\gamma}}{\ell(t)} \, dt < \infty,
\]
by the assumption on \(\gamma\). Hence we get, similar to (45), that
\[
\int_0^{\varepsilon_1} \int_I \left| \phi(x) \cdot L_0^k \frac{L_0 - L_t}{\ell(t)} \rho_0 \right| \, dx \, dt \leq C_{k,1} \| \rho_0 \| \| \phi \|_{L^q}.
\]
Similar to (46), we have
\[
\int_{t > \varepsilon_1} \int_I \left| \phi \cdot L_0^k \frac{L_0 - L_t}{\ell(t)} \rho_0 \right| \, dx \, dt \leq \int_{t > \varepsilon_1} \frac{1}{\ell(t)} \, dt \cdot \int_I \left| \phi \cdot L_0^k (L_{\varepsilon_1} - L_0) \rho_0 \right| \, dx
\]
\[
\leq C_{k,1} \| \rho_0 \| \| \phi \|_{L^q},
\]
where we used the assumption \(\int_0^1 1/\ell(t) \, dt < \infty\). We may choose \(\kappa_1\) arbitrary close to \(\min(\kappa, \Lambda^{1-\gamma})\), which proves (13).

Claims (5) and (13) are proved just as the corresponding claims in Theorem 2.3 replacing \(|t|^{1+\eta}\) by \(\ell(t)\) in the proofs.
4.3. Proof of Corollary 2.7 on holomorphic extensions.

**Remark 4.1** (Sketch of an alternative proof using Morera’s theorem). The proof below is by estimating the growth of derivatives. We sketch here a more conceptual proof: First extend the definition of $M^\eta$ and Theorem 2.3 to complex $\eta$ with $0 < \Re \eta < 1 - \frac{1}{q}$. This implies in particular that the double integrals appearing in $\Psi_\phi(\eta, z)$ are well-defined for such $\eta$, and Fubini is justified. Then, since Euler’s Gamma function and $t^{-1-\eta}$ are holomorphic in the domain considered, Morera’s theorem gives the desired holomorphic extension.

We first show that the function

$$\eta \mapsto M^\eta(\int_0^x \rho_t(x) \, dt)|_{t=0}$$

is holomorphic in the strip $0 < \Re \eta < 1 - \frac{1}{q}$. We give the argument for the right-sided derivative, the left-sided case is the same up to introducing signs. Let $0 < \gamma < 1 - \frac{1}{q}$ and take $\eta_0$ such that $0 < \Re \eta_0 < \gamma$. The right-sided fractional derivative with the parameter $\eta = \eta_0 + \zeta$ is given by

$$F_{\eta_0}(\zeta) := M^{\eta_0+\zeta}\left(\int_0^x \rho_t(x) \, dt\right)|_{t=0} = \Psi_\phi(\eta_0 + \zeta, 1)$$

$$= \sum_{k=0}^\infty \frac{\Gamma_{\eta_0+\zeta}}{2} \int_0^\infty \phi \int_0^\infty \left(L_k^\zeta(L_t - L_0)\rho_0 \sum_{n=0}^\infty \frac{(\log |t|)^n \zeta^n}{t^{1+\eta_0} n!}\right) \, dt \, dx.$$

Let $\ell_n(t) = |t|^{1+\eta_0}(\log |t|)^{-n}$. Using Theorem 2.6 and $|t|^{-\zeta} = \sum_{n=0}^\infty \frac{1}{n!}(\log |t|)^n \zeta^n$, we then have

$$F_{\eta_0}(\zeta) = \sum_{k=0}^\infty \frac{\Gamma_{\eta_0+\zeta}}{2} \int_0^\infty \phi \int_0^\infty \left(L_k^\zeta(L_t - L_0)\rho_0 \sum_{n=0}^\infty \frac{(\log |t|)^n \zeta^n}{t^{1+\eta_0} n!}\right) \, dt \, dx.$$

$$= \frac{\Gamma_{\eta_0+\zeta}}{2} \sum_{n=0}^\infty \frac{\Psi_\phi((\ell_n, 1))}{n!} \zeta^n.$$

To justify exchanging the order of sums and integrations above, we will estimate below

$$\left(\frac{\zeta^n}{n!} \int_0^\infty \left(L_k^\zeta(L_t - L_0)\rho_0 \frac{(\log |t|)^n}{|t|^{1+\eta_0}}\right) \, dt \, dx \right).$$

Since $F_{\eta_0}(\zeta)$ is a product of $\frac{\Gamma_{\eta_0+\zeta}}{2}$ and $\sum_{n=0}^\infty \frac{\Psi_\phi((\ell_n, 1))}{n!} \zeta^n$, which are both holomorphic in $\zeta$, provided $|\zeta|$ is sufficiently small, this will show that $F_{\eta_0}(\zeta)$ is holomorphic.

To estimate (47), we first use the proof of Theorem 2.6 to get

$$\left|\frac{\zeta^n}{n!} \int_0^\infty \left(L_k^\zeta(L_t - L_0)\rho_0 \frac{(\log |t|)^n}{|t|^{1+\eta_0}}\right) \, dt \, dx \right| \leq C_n k^\zeta n^n \|\rho_0\|_{L^\eta} \|\phi\|_{L^q},$$

where

$$C_n = \frac{C}{n!} \left(\int_0^{\frac{1}{1+\gamma}} \frac{t^{1+\gamma}}{\ell_n(t)} \, dt + \int_{\ell_n(t)}^{1} \frac{1}{\ell_n(t)} \, dt\right),$$
and $C$ is a constant that does not depend on $n$. Next, since $γ > ℜη_0$ we have
\[
\left| \int_{-ε_1}^{ε_1} \frac{|t|^γ}{t_n(t)} \, dt \right| = 2 \int_0^{ε_1} \frac{|\log t|^n}{t^{1-(γ-ℜη_0)}} \, dt \leq 2 \int_0^{1} \frac{|\log t|^n}{t^{1-(γ-ℜη_0)}} \, dt = 2 \frac{n!}{(γ-ℜη_0)^{n+1}}
\]
and
\[
\left| \int_{|t|>ε_1} \frac{1}{t_n(t)} \, dt \right| \leq 2|ε_1|^{1+ℜη_0} |\log ε_1|^n + 2 \int_1^{∞} \frac{(\log t)^n}{t^{1+ℜη_0}} \, dt
\]
\[
= 2|ε_1|^{1+ℜη_0} |\log ε_1|^n + 2 \frac{n!}{(ℜη_0)^{n+1}}.
\]
In conclusion, $C_n$ grows at most with an exponential speed. Hence, we may change order of integration if $ζ$ is in a sufficiently small disc. In particular, our estimates show that $\frac{Ψ_φ(ℓ_n, 1)}{n!}$ grows at most exponentially in $n$, so that
\[
F_{η_0}(ζ) = \frac{Γ_{η_0+ζ}}{2} \sum_{n=0}^{∞} \frac{Ψ_φ(ℓ_n, 1)}{n!} ζ^n,
\]
holds when $ζ$ is in a disc around the origin.
The proof that the function $(η, ζ) \mapsto Ψ_φ(η, z)$ is holomorphic in the domain
\[
\{ (η, z) \mid 0 < ℜη < 1 - \frac{1}{q}, |z| < \min(κ, Λ^{-ζ/q}) \}
\]
uses the same estimates, writing instead
\[
Ψ_φ(η_0 + ζ, z) = \sum_{k=0}^{∞} \frac{Γ_{η_0+ζ}}{2} z^k \int_{I} \int_{-∞}^{∞} \left( L_k^t(L_t - L_0)^{ρ_0} \right) \text{sign}(t) \, dt \, dx
\]
\[
= \sum_{k=0}^{∞} \frac{Γ_{η_0+ζ}}{2} z^k \int_{I} \int_{-∞}^{∞} \left( L_k^t(L_t - L_0)^{ρ_0} \sum_{n=0}^{∞} \left( \frac{|\log t|^n}{|t|^{1+η_0}} \right) \frac{ζ^n}{n!} \right) \text{sign}(t) \, dt \, dx.
\]

4.4. Proof of Corollary 2.4 on horizontal perturbations. By [9] Theorem 2.8 (see [10]), the horizontality assumption implies that $f_t$ is tangential to the topological class of $f_0$ that is, there exist a $C^2$ perturbation $(f_t)$ of $f_0$ with $f_t - f_0 = O(t^2)$ (as $|t| \to 0$) and homeomorphisms $h_t$ of $I$ with $h_t(c) = c$ and $f_t = h_t \circ f \circ h_t^{-1}$. Note that, letting $L_t$ be the operator associated to $f_t$, the proof of Lemma 3.2 gives $ε_2$ such that (cf. [9] (27) proof of Proposition 3.3) that
\[
\|(L_t - L_0^t)φ\|_{H^{ρ, p}} \leq C |t|^{2(τ - ̃τ)} \|(φ\|_{H^{ρ, p}}, \quad ∀|t| < ε_2,
\]
(this will be used to show (18)). Since we may choose $0 < ̃τ < τ < 1/p$ such that $2(τ - ̃τ) > 1$, it is not hard to see that we may replace $f_t$ by $f_̃t$ in the proof of Corollary 2.4.
The two first equalities claimed in Corollary 2.4 follow from statement (b) of Theorem 2.3 combined with Lemma 2.2 and [9] Proposition 4.3, Theorem. 5.1, respectively.
The last statement in the corollary, (18), can be deduced from statement (c) of Theorem 2.3 together with the following claim (the last term in the right-hand side is

\footnote{It would be interesting to have a more direct proof of (18).}
where $c$ is differentiable at $t$.

\[ \partial_t (\text{49}) \]

Therefore, since $|k,t| = t$, by

\[ \text{recall that } c \text{ is not periodic for } f \]

\[ \int \phi \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} L_k^j(\tilde{s}_j X_0(c_j) \delta_{c_j}) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \tilde{s}_j X_0(c_j) \phi(c_{j+k}) = \sum_{\ell=0}^{\infty} \phi_{\ell \ell} \sum_{j=1}^{\ell} \tilde{s}_j X_0(c_j). \]

Indeed, it suffices to note that (recall that $c$ is not periodic for $f$)

\[ \int \phi \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} L_k^j(\tilde{s}_j X_0(c_j) \delta_{c_j}) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \tilde{s}_j X_0(c_j) \phi(c_{j+k}) = \sum_{\ell=0}^{\infty} \phi_{\ell \ell} \sum_{j=1}^{\ell} \tilde{s}_j X_0(c_j). \]

The desired identity (48) will be an immediate consequence of the decomposition $\rho = \rho^\text{reg} + \rho^\text{sal}$, with (45), and Lemma 2.2 after we establish that $t \mapsto \int I \phi(x)(\mathcal{L}_t \rho_0)(x) \, dx$ is differentiable at $t = 0$ for any $\phi \in C^0$, with derivative

\[ \partial_t \left( \int I \phi(x)(\mathcal{L}_t \rho_0)(x) \, dx \right) \bigg|_{t=0} = - \int I \phi(x)(\rho_0) \, dx. \]

To show (49), we first recall Step 2 in the proof of [9, Theorem 5.1]. For this, letting $c_{k,t} = f^k_t(c)$ and denoting by $B_t$ the vector space of functions $\varphi = \varphi^\text{reg} + \sum_{k=1}^{\infty} a_k H_{c,k,t}$ where $\varphi^\text{reg} \in BV \cap C^0$ is supported in $I$ and is differentiable with derivative in $BV$, and the $a_k$ are complex numbers, we recall the invertible map $G_t : B_t \to B_0$ from [9], defined by

\[ G_t(\varphi) = \varphi^\text{reg} + \sum_{k=1}^{\infty} a_k H_{c,k}. \]

In this notation, Step 2 in the proof of [9, Theorem 5.1] (the proof uses $f_t = \tilde{f}_t$) says

\[ \partial_t (G_t(\mathcal{L}_t(G_t^{-1} \rho_0))) \bigg|_{t=0} = - X'_0(\rho_0) - X_0(\rho_0^\text{reg}') \in BV. \]

To use the above identity, we decompose $\mathcal{L}_t \rho_0 - \rho_0$ into

\[ (G_t(\mathcal{L}_t(G_t^{-1} \rho_0)) - \rho_0) + (\mathcal{L}_t(G_t^{-1} \rho_0)) - G_t(\mathcal{L}_t(G_t^{-1} \rho_0))) + (\mathcal{L}_t \rho_0 - \mathcal{L}_t(G_t^{-1} \rho_0)). \]

The first term above is handled by (49). For the other two terms, recall that $c$ is not periodic, so $P_f = \infty$. For the last term in (51), note that, since $c_{k,t} = h_t(c_k)$, we have (see [9, Lemma 2.2, Remark 2.5])

\[ \lim_{t \to 0} \frac{c_{k,t} - c_k}{t} = - \sum_{j=0}^{\infty} \frac{X_0(c_{j+k})}{(j+1)!}(c_k), \quad \forall k \geq 1. \]

Therefore, since $\mathcal{L}_0(\delta_{c_k}) = \delta_{c_{k+1}}$, using bounded distortion

\[ \lim_{t \to 0} \frac{\int \phi(\mathcal{L}_t \rho_0 - \mathcal{L}_t(G_t^{-1} \rho_0))) \, dx}{t} = \lim_{t \to 0} \frac{\int \phi \mathcal{L}_t(\sum_{k=1}^{\infty} \tilde{s}_k H_{c_k} - \sum_{k=1}^{\infty} \tilde{s}_k H_{c_{k,t}}) \, dx}{t} \]

\[ = \sum_{k=1}^{\infty} \tilde{s}_k \alpha(c_k) \phi(c_{k+1}) = \sum_{k=1}^{\infty} \tilde{s}_{k+1} \alpha(c_k) \phi(c_{k+1}). \]
Finally, for the second term in (51), we have

\[(\mathcal{L}_t(G_t^{-1}(\rho_0)) - G_t(\mathcal{L}_t(G_t^{-1}(\rho_0))))\]

\[= [\mathcal{L}_t(\rho_0^{\text{reg}} + \sum_{k=1}^{\infty} \tilde{s}_k H_{c_k,t})])^{\text{sal}} - G_t[\mathcal{L}_t(\rho_0^{\text{reg}} + \sum_{k=1}^{\infty} \tilde{s}_k H_{c_k,t})])^{\text{sal}}\]

\[= \left(\frac{1}{|f_{t,-}'(c)|} + \frac{1}{|f_{t,+}'(c)|}\right) \rho_0^{\text{reg}}(c)(H_{c_1,t} - H_{c_1}) + \sum_{k=1}^{\infty} \frac{\tilde{s}_k}{|f_{k,t}'(c_k,t)|}(H_{c_{k+1},t} - H_{c_{k+1}}).\]

Therefore, using \(\mathcal{L}_0 \rho_0(c_1) = \rho_0(c_1) = -\tilde{s}_1\) and \(\rho_0^{\text{sal}}(c) = 0\),

\[\lim_{t \to 0} \int \phi(\mathcal{L}_t(G_t^{-1}(\rho_0)) - G_t(\mathcal{L}_t(G_t^{-1}(\rho_0)))) dt = -\tilde{s}_1 X_0(c_1) \phi(c_1) - \sum_{k=1}^{\infty} \tilde{s}_{k+1} \alpha(c_{k+1}) \phi(c_{k+1}).\]

The twisted cohomological equation (19) implies that the sum of (53) and (54) is

\[-\tilde{s}_1 X_0(c_1) \phi(c_1) + \sum_{k=1}^{\infty} \tilde{s}_{k+1} (f'(c_k) \alpha(c_k) - \alpha(f(c_k))) \phi(c_{k+1})\]

\[= -\sum_{k=0}^{\infty} \tilde{s}_{k+1} X_0(c_{k+1}) \phi(c_{k+1}) = -\int \phi(X_0 \rho_0^{\text{sal}})'.\]

This concludes the proof of (49), and of Corollary 2.4. \(\square\)

**Appendix A. Proof of Lemma 2.2**

Without loss of generality, we set \(t_0 = 0\). Since \(g\) is bounded, there exists \(A\) such that

\[\left|\int_1^{\infty} \frac{g(0) - g(-t)}{t^{1+\eta}} dt \right| \leq \int_1^{\infty} \frac{2A}{t^{1+\eta}} dt < \infty.\]

Since \(g\) is differentiable at 0, there exists \(B\) such that

\[\left|\int_0^{1} \frac{g(0) - g(-t)}{t^{1+\eta}} dt \right| \leq \int_0^{1} \frac{B}{t^{\eta}} dt < \infty.\]

This shows that \(M_+^\eta g(0)\) exists for all \(\eta < 1\). Exchanging \(g\) with \(t \mapsto g(-t)\), we get that \(M_-^\eta g(0)\) exists for all \(\eta < 1\).

Let \(\varepsilon > 0\). We will prove \(g'(0) - \varepsilon \leq \liminf_{\eta \uparrow 1} M_+^\eta g(0) \leq \limsup_{\eta \uparrow 1} M_-^\eta g(0) \leq g'(0) + \varepsilon\). Since \(\varepsilon\) is arbitrary, this implies that \(\lim_{\eta \uparrow 1} M_+^\eta g(0) = g'(0)\).

Since the limit \(g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}\) exists, we can find \(\delta > 0\) such that

\[-\varepsilon \leq \frac{g(h) - g(0)}{h} - g'(0) \leq \varepsilon, \quad \forall |h| \leq \delta.\]
Next, since $0 < \eta < 1$, we can write
\[
M^\eta_+ g(0) = \frac{\eta}{\Gamma(1 - \eta)} \left( \int_0^\delta \frac{g(0) - g(-t)}{t} \frac{1}{t^\eta} \, dt + \int_\delta^\infty \frac{g(0) - g(-t)}{t^{1+\eta}} \, dt \right)
\]
\[
\leq \frac{\eta}{\Gamma(1 - \eta)} \left( \int_0^\delta (g'(0) + \varepsilon) \frac{1}{t^\eta} \, dt + \int_\delta^\infty \frac{g(0) - g(-t)}{t^{1+\eta}} \, dt \right)
\]
\[
\leq \frac{\eta}{(1 - \eta) \Gamma(1 - \eta)} \delta^{1-\eta}(g'(0) + \varepsilon) + \frac{\eta}{\Gamma(1 - \eta)} \int_\delta^\infty \frac{2A}{t^{1+\eta}} \, dt
\]
Using that $\Gamma(1) = 1$, $\lim_{\eta \uparrow 1} (1 - \eta) = \infty$ and $\lim_{\eta \uparrow 1} \delta^{1-\eta} = 1$, this shows that
\[
\limsup_{\eta \uparrow 1} M^\eta_+ g(0) \leq g'(0) + \varepsilon.
\]
In the same way, we get
\[
M^\eta_- g(0) \geq \frac{\eta}{\Gamma(2 - \eta)} \delta^{1-\eta}(g'(0) - \varepsilon) - \frac{1}{\Gamma(1 - \eta)} \frac{2A}{\delta^{1-\eta}},
\]
and thus $\liminf_{\eta \uparrow 1} M^\eta_+ g(0) \geq g'(0) - \varepsilon$. Using $t \mapsto g(-t)$, we get $\lim_{\eta \uparrow 1} M^\eta_- g(0) = -g'(0)$.

**APPENDIX B. PROOF OF THE UNIFORM LASOTA–YORKE ESTIMATE (LEMMA 3.1)**

Note that in view of (30) it is enough to show that for any $\bar{A} < A$, there exist $p_0 \in (1, p(f))$ and $\varepsilon_0 \leq \varepsilon$ such that for any $p \in (1, p_0)$ and $0 \leq \hat{\tau} < \tau < \frac{1}{p}$ there exist $N \geq 1$ and a constant $C(N) < \infty$ such that
\[
\|L^N_\tau \varphi\|_{H^r,p} \leq \bar{A}^{\frac{1}{r} - 1} \|\varphi\|_{H^r,p} + C(N) \|\varphi\|_{H^r,p}, \quad \forall |t| < \varepsilon_0, \forall \varphi.
\]
(To conclude for general $n = jN + k$, use a geometric series for $jN$ and (30) for $k \leq N - 1$.)

Our strategy to prove (56) is to follow Thomine’s argument [38], using a “zoom” [1]. For this, note that Lemma 3.5 gives $C_{zoom} = C(\tau, p) > 0$ such that for any positive number $Z > 0$, setting $Z(x) = Z \cdot x$,
\[
\|\varphi \circ Z\|_{H^r,p} \leq C_{zoom} Z^{\frac{1}{p} - 1} \|\varphi\|_{H^r,p} + C_{zoom} Z^{-\frac{1}{p}} \|\varphi\|_{L^p}, \forall n.
\]
Define $Z_n : \mathbb{R} \to \mathbb{R}$ by $Z_n(x) = Z_n \cdot x$, where [2] the sequence of zooms $Z_n > 1$, for $n \geq 1$, will be specified later. We define an auxiliary zoomed norm by $\|\varphi\|_n = \|\varphi \circ Z_n^{-1}\|_{H^r,p}$, which is equivalent with $\|\cdot\|_{H^r,p}$, for each fixed $n$.

Let $\psi : \mathbb{R} \to [0, 1]$ be a $C^\infty$-function supported in $(-1, 1)$ with $\sum_{m \in \mathbb{Z}} \psi_m = 1$, where $\psi_m(x) = \psi(x + m)$. We define
\[
\psi_{m,n}(x) = \psi_m \circ Z_n(x) = \psi_m(Z_n \cdot x), \quad m \in \mathbb{Z}, n \in \mathbb{Z}_+.
\]
\footnote{The zoom and localisation can perhaps be replaced by the fragmentation and reconstitution lemmas in [2, Chapter 2].}
\footnote{It is essential that $Z_n$ will be chosen uniformly in $t$ for each fixed $n$.}
Note that the support of $\psi_{m,n}$ becomes small if $Z_n$ is large. For any $x \in I$ and any $n \geq 1$, there are at most three integers $m$ such that $\psi_{m,n}(x) \neq 0$, and we may decompose

$$L_t^n \varphi(x) = \sum_{m \in \mathbb{Z}} C_t^m(\psi_{m,n}\varphi)(x) = \sum_{m \in \mathbb{Z}} \sum_{J \in \mathcal{A}(t,n)} \left(1_J \psi_{m,n} \frac{\varphi}{|f_t^n|'}\right) \circ f_{t,J}^{-n}(x),$$

where $\mathcal{A}(t,n)$ is the partition of monotonicity associated to $f_t^n$, while $f_{t,J}$ is the extension to $I$ of the inverse of $f_t^n|_J$.

For any $t$, the intersection multiplicity

$$\sup_x \# \{(m,J) \mid m \in \mathbb{Z}, J \in \mathcal{A}(t,n), x \in f_t^n(J \cap \text{supp} \psi_{m,n})\}$$

is bounded by $3 \cdot 2^n$ (this is called the “complexity at the end” in Thomine’s paper). Therefore, by Lemma 3.5 in [38] there is $C = C(\tau, p) > 0$ and for each $n$ there exists $C_n = C(\tau, p, \psi, n)$ such that

$$\|L_t^n \varphi\|_p^n \leq C \sum_{m \in \mathbb{Z}} \sum_{J \in \mathcal{A}(t,n)} 2^{n(p-1)} \left\|\left(1_J \psi_{m,n} \frac{\varphi}{|f_t^n|'}\right) \circ f_{t,J}^{-n}\right\|_n^p + C_n \sum_{m \in \mathbb{Z}} \sum_{J \in \mathcal{A}(t,n)} \left\|\left(1_J \psi_{m,n} \frac{\varphi}{|f_t^n|'}\right) \circ f_{t,J}^{-n} \circ Z_n^{-1}\right\|_n^p$$

$$\leq C \sum_{m \in \mathbb{Z}} \sum_{J \in \mathcal{A}(t,n)} 2^{n(p-1)} \left\|\left(1_J \psi_{m,n} \frac{\varphi}{|f_t^n|'}\right) \circ f_{t,J}^{-n}\right\|_n^p + C_n \|\varphi\|_L^p.$$  

To control the double sum in (59), we write

$$(1_J \psi_{m,n}\varphi) \circ f_{t,J}^{-n} \circ Z_n^{-1} = \bar{\varphi}_{m,n,J} \circ Z_n \circ f_{t,J}^{-n} \circ Z_n^{-1},$$

where $\bar{\varphi}_{m,n,J} = (1_J \psi_{m,n}\varphi) \circ Z_n^{-1}$. Since $\sup_t |f_t^n| < \infty$ and $\inf_t \inf \{f_t^n\} > 1$, a standard bounded distortion estimate gives for any $n$ a large enough $Z_n$ (uniformly in $t$, $m$, and $J$) such that, for each $t$ and $J \in \mathcal{A}(t,n)$, and for any $y \in \text{supp}(1_J \psi_{m,n})$, we have,

$$\frac{1}{2(|f_t^n|'(y))} \leq \frac{1}{(f_t^n)' \circ f_{t,J}^{-n} \circ Z_n^{-1}(x)} \leq \frac{3}{2(|f_t^n|'(y))}, \forall x \in \text{supp}(1_J \psi_{m,n} \circ f_{t,J}^{-n} \circ Z_n^{-1}).$$

Applying Lemma 3.5 to $Z_n \circ f_{t,J}^{-n} \circ Z_n^{-1}$, we obtain $C = C(\tau, p)$ independent of $n$ and $t$, and $C_n = C_n(p)$ independent of $t$ such that

$$\|L_t^n \varphi\|_{L^p} \leq C(|f_t^n|'(y))^{\frac{p}{2}} \|\varphi_{m,n,J}\|_{L^p} + C_n \|\varphi_{m,n,J}\|_{L^p}, \forall |t| \leq \varepsilon.$$

\footnote{We mean $|1 - (f_t^n)'(x)/(f_t^n)'(y)| \leq K(f_t^n, f_t^n) \cdot d(f_t^n(x), f_t^n(y))$, and the zoom $Z_n$ ensures $K(f_t^n, f_t^n) \cdot d(f_t^n(x), f_t^n(y)) < 1/2.$}
Since $1/p < 1$, the bound \( (60) \) together with Lemma 3.4 for \( \gamma = 1 \) imply
\[
\left\| \left( 1_J \psi_{m,n} \frac{\varphi}{|f^n_t|} \right) \circ f^{-n}_{t,J} \right\|_n \leq C\left( |(f^n_t)'(y)| \frac{1}{p}(\lambda_n(f_t))^{-\tau-1}\right) \| \varphi_{m,n,J} \|_{H^{\tau,p}} + C_n \| \varphi_{m,n,J} \|_{L^p} \\
\leq C(\lambda_n(f_t))^{-(\tau+1-\frac{1}{p})} \| \varphi_{m,n,J} \|_{H^{\tau,p}} + C_n \| \varphi_{m,n,J} \|_{L^p}.
\]

Therefore, by the convexity inequality \(|a| + |b|)^p \leq 2^{p-1}(|a|^p + |b|^p)\), we have
\[
\left\| \left( 1_J \psi_{m,n} \frac{\varphi}{|f^n_t|} \right) \circ f^{-n}_{t,J} \right\|_n^p \leq C(p)(\lambda_n(f_t))^{-(p(\tau+1)-1)} \| \varphi_{m,n,J} \|_{H^{\tau,p}} + C_n \| \varphi_{m,n,J} \|_{L^p}^p,
\]
where \( C(p) \) and \( C_n \) are independent of \( t \). By the Strichartz Lemma 3.3 \( (\text{we use here} \ 0 \leq \tau < 1/p) \) and the (classical) localisation Theorem 2.4.7 in \( (40) \) (or Lemma 3.4 in \( (38) \) there exists \( K = K(\psi, \tau, p) > 0 \) such that for all \( m, n \) and \( \varphi \)
\[
\sum_{m \in \mathbb{Z}} \| \varphi_{m,n,J} \|_{H^{\tau,p}}^p = \sum_{m \in \mathbb{Z}} \| (1_J \cdot (\psi_m \circ Z_n) \cdot \varphi) \circ Z_n^{-1} \|_{H^{\tau,p}}^p \\
\leq K \sum_{m \in \mathbb{Z}} \| (\psi_m \circ Z_n) \cdot \varphi \circ Z_n^{-1} \|_{H^{\tau,p}}^p \leq K^2 \| \varphi \circ Z_n^{-1} \|_{H^{\tau,p}}^p = K^2 \| \varphi \|_n^p.
\]

For a fixed map \( f_0 \) and for any \( n \), if \( Z_n \) is large enough, then for any \( m \) there are at most two intervals \( J \in \mathcal{A}(0, n) \) such that \( (1_J \psi_{m,n}) \circ f^{-n}_J \) is not identically zero. (This is the “complexity at the beginning” in Thomine’s argument.) For a perturbation \( (f_t) \), caution is needed, especially (but not only) if \( c \) is periodic for \( f \) (see footnote 7).

Assume first that \( c \) is not periodic. Fix \( \bar{A} < \bar{A} < A \), choose \( p_0 \in (1, p(f)) \) and \( n_0 \geq 1 \) such that
\[
2^{n_0-1} \cdot (\lambda_{n_0}(f_t))^{-(p\tau+1)-1} \leq \bar{A}^{-n_0(-p\tau+1)-1}, \ \forall |t| < \varepsilon_0.
\]
Next, for \( p \in (1, p_0) \), let \( C(p) \) be the constant from \( (61) \), let \( K(p) \) be the constant from \( (62) \). Then, there exists \( N \geq n_0 \) such that
\[
2 \cdot C(p) \cdot K(p)^2 \cdot 2^{N(p-1)} \cdot (\lambda_N(f_t))^{-(p(\tau+1)-1)} < \bar{A}^{-N(p\tau+1)-1}.
\]
Now, since \( c \) is not periodic, (see e.g. \( [13] \) p. 376) there exists \( \varepsilon' = \varepsilon'(N) < \varepsilon_0 \) such that for any \( |t| \leq \varepsilon'(N) \) there is a natural bijection \( J_t = J_{t,N} \) between \( \mathcal{A}(0, N) \) and \( \mathcal{A}(t, N) \).
In addition, we may ensure that \( |f^{N}(J_t(J))| \geq |f^{N}(J)|/2 \) for all \( J \in \mathcal{A}(0, N) \) and all \( |t| \leq \varepsilon'(N) \). By bounded distortion, there exists \( Z_N \) such that for all \( |t| \leq \varepsilon'(N) \) and any \( m \) there are at most two intervals \( J \in \mathcal{A}(t, N) \) such that \( (1_J \psi_{m,N} \varphi) \circ f^{-N}_{t,J} \) is not

\footnote{Since the \( C^1 \) norm of \( 1/(f^n_t)' \) is bounded uniformly in \( n \) and \( t \) by some \( D \), choosing a suitable zoom \( Z_n \) allows to bound the \( C^1 \) norm of \( (1_J \psi_{m,n} \circ (f^n_t)') \circ Z_n^{-1} \) by \( \lambda_n(f_t)^{-1} + D/Z_n \leq C\lambda_n(f_t)^{-1} \).}
identically zero. Therefore, it follows from \((61)\) and \((62)\) that

\[
\sum_{m \in \mathbb{Z}} \sum_{J \in \mathcal{A}(t, N)} \left\| \left( 1 \cdot \psi_{m,N} \right) \circ f_{t,J}^{-N} \right\|_p^p \leq \sum_{m \in \mathbb{Z}} \sum_{J \in \mathcal{A}(t, N)} \left( C(\lambda_N(f_t))^{-p} \right) \left\| \left( 1 \cdot \psi_{m,N} \right) \circ f_{t,J}^{-N} \right\|_{p}^p \leq C K^2 (\lambda_N(f_t))^{-p} \left\| \varphi \right\|_p^p + 2C_N \left\| \varphi \right\|_p^p,
\]

\((65)\) \((66)\)

Finally, we use \((67)\) twice to obtain

\[
\left\| \mathcal{L}^N_t \varphi \right\|_{\mathcal{H}^{r,p}} \leq 2 CK^2 2^{N(1-\frac{1}{p})} (\lambda_N(f_t))^{-\frac{1}{2}} < \varepsilon''(N).
\]

\((67)\)

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