A METRIC DEFORMATION ON FIBER BUNDLES AND APPLICATIONS

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Abstract. We develop the concept of a Cheeger deformation on the context of fiber bundles with compact structure group. This procedure is used to provide metrics with positive Ricci curvature on the total space of such bundles, generalizing the results in Nash [10] and Poor [11].

1. Introduction

The metric deformation known as Cheeger deformation was firstly introduced on [3]. Its main goal was to produce metrics with non-negative sectional curvature on manifolds with symmetries. Since then, Cheeger deformations were used in [6, 5] to produce new examples of manifolds with non-negative and positive sectional curvature; in [12, 13, 7] to study curvature properties on homogeneous spaces and biquotients and on [14] to lift positive Ricci curvature from a metric quotient $M/G$ to $M$.

On the other hand Nash [10] and Poor [11] proved that the total space of some classes of fiber bundles admit metrics of positive Ricci curvature: for instance, linear sphere bundles, vector bundles and principal bundles over manifolds with positive Ricci curvature. In particular, their results provide metrics of positive Ricci curvature on exotic spheres. The main idea of the present paper is to establish an analogous to the Cheeger deformation on a specific class of metrics on fiber bundles. We use such deformation to unify and extend the works of Nash [10] and Poor [11], providing positive Ricci curvature on a broader class of fiber bundles:

**Theorem 1.1.** Let $F \to M \xrightarrow{\pi} B$ be a fiber bundle from a compact manifold $M$ with fiber $F$, compact structure group $G$ and base $B$. Suppose that

1. $B$ has a metric of positive Ricci curvature,

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(2) The principal orbit of the $G$-action on $F$ has finite fundamental group.

(3) $F$ has a $G$-invariant metric such that $F/G$, with the orbit distance metric, has $\text{Ric}_{F/G} \geq 1$.

Then $M$ carries a metric of positive Ricci curvature.

For the exposition, we follow [9, 16], introducing a suitable reparametrization of 2-planes and a non-decreasing formula for the deformed sectional curvature (see (3.1)). The class of admissible metrics for the present deformation includes any submersion metric whose fibers metrically are homogeneous spaces with respect to the structure group (which is assumed to be compact).

On Section 2 we recall the basic concepts and notation of Cheeger deformations following [16]. On Section 3 we introduce the new deformation and prove Theorem 1.1.

2. Cheeger deformations

Here we quickly recall the procedure according to Ziller [17] and M. M"{u}ter [9].

Let $(M, g)$ be a Riemannian manifold with an isometric $G$-action, where $G$ is a compact Lie group and $Q$ is a biinvariant metric on $g$, the Lie algebra of $G$. For each point $p \in M$ the orbit $Gp$ through $p$ is a submanifold of $M$ satisfying $Gp \cong G/G_p$, where $G_p$ is the isotropy subgroup of the $G$-action at $p$. Let $\mathfrak{m}_p$ be the $Q$-orthogonal complement of $\mathfrak{g}_p$, the Lie algebra of $G_p$. Observe that $\mathfrak{m}_p$ is isomorphic to the tangent space to the orbit $G_p$. The isomorphism is induced by computing action fields: given an element $U \in \mathfrak{g}$, define

$$U^*_p = \frac{d}{dt} |_{t=0} e^{tU} p.$$ 

The map $U \mapsto U^*_p$ is a linear morphism whose kernel is $\mathfrak{g}_p$. We call the tangent space $T_pGp$ as the vertical space at $p$, $V_p$, and its orthogonal complement, $H_p$, as the horizontal space. A tangent vector $\overline{X} \in T_pM$ can be uniquely decomposed as $\overline{X} = X + U^*_p$, where $X$ is horizontal and $U \in \mathfrak{m}_p$.

The main idea in the Cheeger deformation is to consider the product manifold $M \times G$, observing that the action

$$r(p, g) := (rp, gr^{-1})$$

is free and isometric for the product metric $g \times \frac{1}{t} Q$, $t > 0$. Its quotient space is diffeomorphic to $M$ since the projection

$$\pi' : M \times G \to M$$

$$(p, g) \mapsto gp$$

is a principal bundle for action (1). It induces a family of metrics $g_t$ on $M$.

Next, we define useful tensors:

(1) The orbit tensor at $p$ is the linear map $P : \mathfrak{m}_p \to \mathfrak{m}_p$ defined by

$$g(U^*, V^*) = Q(PU, V), \quad \forall U^*, V^* \in \mathfrak{v}_p$$

(2) For each $t > 0$ we define $P_t : \mathfrak{m}_p \to \mathfrak{m}_p$ as

$$g_t(U^*, V^*) = Q(P_tU, V), \quad \forall U^*, V^* \in \mathfrak{v}_p$$

(3) The metric tensor of $g_t$, $C_t : T_pM \to T_pM$ is defined as

$$g_t(\overline{X}, \overline{Y}) = g(C_t\overline{X}, \overline{Y}), \quad \forall \overline{X}, \overline{Y} \in T_pM$$
All the three tensors above are symmetric and positive definite. The next proposition shows how they are related:

**Proposition 2.1** (Proposition 1.1 in [17]). The tensors above satisfy:

1. \( P_t = (P^{-1} + tI)^{-1} = P(1 + tP)^{-1} \).
2. If \( \vec{x} = X + U^* \) then \( C_t(\vec{x}) = X + ((1 + tP)^{-1}U)^* \).

The metric tensor \( C^{-1} \) can be used to define a suitable reparametrization of 2-planes. Using this reparametrization we can observe that Cheeger deformations do not create ‘new’ planes with zero sectional curvature. More generally:

**Theorem 2.2.** Let \( \vec{x} = X + U^* \), \( \vec{y} = Y + V^* \) be tangent vectors. Then \( \kappa_t(\vec{x}, \vec{y}) := R_{gn}(C_t^{-1}X, C_t^{-1}Y, C_t^{-1}Y, C_t^{-1}X) \) satisfies

\[
\kappa_t(\vec{x}, \vec{y}) = \kappa_0(\vec{x}, \vec{y}) + \frac{t^3}{4} \|[PU, PV]\|_Q^2 + z_t(\vec{x}, \vec{y}),
\]

where \( z_t \) is non-negative.

(See [19] Proposition 1.3 for details.)

Next we provide a formula for the Ricci curvature of Cheeger deformed metric (compare with [15] Equation (57)).

2.1. Ricci curvature. Let \( \{v_1, \ldots, v_k\} \) be a \( Q \)-orthonormal basis of eigenvectors of \( P : m_p \to m_p \), with eigenvalues \( \lambda_1 \leq \ldots \leq \lambda_k \). Given a \( g \)-orthonormal basis \( \{e_{k+1}, \ldots, e_n\} \) for \( H_p \), we fix the \( g \)-orthonormal basis \( \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\} \) for \( T_p M \), where \( e_i = \lambda_i^{-1/2} v_i^* \) for \( i \leq k \).

It follows that:

**Claim 1.** The set \( \{C_t^{-1/2}e_i\}_{i=1}^n \) is a \( g_t \)-orthonormal basis for \( T_p M \). Moreover, \( C_t^{-1/2}e_i = (1 + t\lambda_i)^{1/2} v_i \) for \( i \leq k \) and \( C_t^{-1/2}e_i = e_i \) for \( i > k \).

Define the horizontal Ricci curvature as

\[
\text{Ric}^h(\vec{x}) := \sum_{i=k+1}^n R(\vec{x}, e_i, e_i, \vec{x}).
\]

**Lemma 1.** For \( \{e_1, \ldots, e_n\} \) as above,

\[
\text{Ric}_{g_t}(\vec{x}) = \text{Ric}^h(C_t^{-1} \vec{x}) + \sum_{i=1}^k z_t(C_t^{-1/2}e_i, C_t^{-1} \vec{x})
\]

\[
+ \sum_{i=1}^k \frac{1}{1 + t\lambda_i} \left( \kappa_0(\lambda_i^{-1/2} v_i^*, C_t^{-1} \vec{x}) + \frac{t^3}{4} \|[v_i, tP(1 + tP)^{-1} \vec{x}]\|_Q^2 \right).
\]

**Proof.** A straightforward computation following equation (2) gives

\[
\text{Ric}_{g_t}(C_t^{-1} \vec{x}) = \sum_{i=1}^n R_{gn}(C_t^{-1/2}e_i, C_t^{-1} \vec{x}, C_t^{-1} \vec{x}, C_t^{-1/2}e_i) = \sum_{i=1}^n \kappa_t(C_t^{-1/2}e_i, \vec{x})
\]

\[
= \sum_{i=1}^n \kappa_0(C_t^{-1/2}e_i, \vec{x}) + \sum_{i=1}^k z_t(C_t^{-1/2}e_i, \vec{x}) + \frac{t^3}{4} \sum_{i=1}^k \|[PC_t^{1/2} \lambda_i^{-1/2} v_i, P \vec{x}]\|_Q^2
\]

\[
= \text{Ric}^h(\vec{x}) + \sum_{i=1}^k z_t(C_t^{-1/2}e_i, \vec{x}) + \sum_{i=1}^k \frac{1}{1 + t\lambda_i} \left( \kappa_0(\lambda_i^{-1/2} v_i^*, \vec{x}) + \frac{t^3}{4} \|[v_i, P \vec{x}]\|_Q^2 \right).
\]
Equation (1) now follows by replacing $X$ by $C_t X$. \hfill \Box

3. Cheeger deformations on fiber bundles

We use Cheeger deformation to produce deformed metrics on fiber bundles with compact structure group.

Let $F \rightarrow M \rightarrow B$ be a fiber bundle from a compact manifold $M$, with fiber $F$, compact structure group $G$ and base $B$. We start by recalling that the structure group of a fiber bundle is the group where some choice of transition functions on $M$ take values. Precisely, if $G$ acts effectively on $F$, then $G$ is a structure group for $\pi$ if there is a choice of local trivializations $\{(U_i, \phi_i : \pi^{-1}(U_i) \rightarrow U_i \times F)\}$ such that, for every $i, j$ with $U_i \cap U_j \neq \emptyset$, there is a continuous function $\varphi_{ij} : U_i \cap U_j \rightarrow G$ satisfying

\[ \phi_i \circ \phi_j^{-1}(p, f) = (p, \varphi_{ij}(p)f), \]

for all $p \in U_i \cap U_j$.

The existence of $\{\varphi_{ij}\}$ allows us to construct a principal $G$-bundle over $B$ (see [8, Proposition 5.2] for details) that we shall denote by $\mathcal{P}$. Furthermore, there exists a principal $G$-bundle $\tilde{\pi} : \mathcal{P} \times F \rightarrow M$ whose principal action is given by

\[ r(p, f) := (rp, rf). \]

(For details see the construction on the proof of [4, Proposition 2.7.1].)

For each pair $g$ and $g_F$ of $G$-invariant metrics on $\mathcal{P}$ and $F$, respectively, there exists a metric $h$ on $M$ induced by $\tilde{\pi}$. Denote by $\mathcal{M}$ the set of all metrics obtained in this way (for instance, if $G$ is transitive on $F$ then every metric on $M$ such that the holonomy acts by isometries on each fiber belongs to $\mathcal{M}$). The set $\mathcal{M}$ is the set of admissible metrics for our deformation:

**Definition 1** (The deformation). Given $h \in \mathcal{M}$, consider $g \times g_F$ a product metric on $\mathcal{P} \times F$ such that $\tilde{\pi} : (\mathcal{P} \times F, g \times g_F) \rightarrow (M, h)$ is a Riemannian submersion. We define $h_t$ as the submersion metric induced by $g_t \times g_F$, where $g_t$ is the time $t$ Cheeger deformation associated to $g$.

As it can be seen, the deformation itself is well-defined for a broader class of metrics (for instance, $\mathcal{P} \times F$ could have any metric such that each slice $\{p\} \times F$ has a $G$-invariant metric), however, if the metric is not a product metric, the deformed curvature is harder to control and Theorem 3.1 below does not hold.

Fix $(p, f) \in \mathcal{P} \times F$. Any $X \in T_{(p,f)}(\mathcal{P} \times F)$ can be written as $X = (X + V^\vee, X_F + W^*)$, where $X$ is orthogonal to the $G$-orbit on $\mathcal{P}$, $X_F$ is orthogonal to the $G$-orbit on $F$ and, for $V, W \in \mathfrak{g}, V^\vee$ and $W^*$ are the action vectors relative to the $G$-actions on $\mathcal{P}$ and $F$ respectively. Let $P, P_F$ and $P_t$ be the orbit tensors associated to $g$, $g_F$ and $g_t$, respectively.

We claim that $X$ is $g_t \times g_F$-orthogonal to the $G$-orbit of $X$ if and only if

\[ X = (X - (P_t^{-1}P_F W)^\vee, X_F + W^*). \]

for some $W \in \mathfrak{m}_F$. A vector $(X + V^\vee, X_F + W^*)$ is horizontal if and only if, for every $U \in \mathfrak{g}$:

\[ 0 = g_t \times g_F((X + V^\vee, X_F + W^*), (U^\vee, U^*)) = g(V^\vee, U^\vee) + g_F(W^*, U^*) = Q(P_t V + P_F W, U). \]

Since $U$ is arbitrary, we conclude that $V = -P_t^{-1}P_F W$. 

Then, that it is, actually, non-decreasing with respect to $P$. First observe that the right-hand-side of (11) is of the form (7): take $\ker \tilde{d}$.

Claim 2. Let $\mathcal{L}_{\tilde{\pi}} : T_{(p,f)} M \to T_{(p,f)} (\mathcal{P} \times F)$ be the horizontal lift associated to $\tilde{\pi}$. Then,

\begin{equation}
\mathcal{L}_{\tilde{\pi}}(X + X_F + U^*) = (X - (P_t^{-1} \tilde{\pi} U)^\vee, X_F + (P_t^{-1} \tilde{\pi} U)^*) + \tilde{\pi}(\tilde{\pi} - P_t^{-1} \tilde{\pi} U^*) = X + X_F + U^*.
\end{equation}

Since $\ker d\tilde{\pi} = \{ (U^\vee, U^* ) | U \in \mathfrak{g} \}$, convention (8) gives $d\tilde{\pi}(U^\vee, 0) = -U^*$, thus $d\tilde{\pi}(X - (P_t^{-1} \tilde{\pi} U)^\vee, X_F + (P_t^{-1} \tilde{\pi} U)^*) = X + X_F + (P_t^{-1} \tilde{\pi} U)^* = X + X_F + U^*$ since $\tilde{\pi} = (P_t^{-1} + P_t^{-1})^{-1}$.

Next, we prove that the unreduced sectional curvature is non-decreasing for reparametrized planes (Theorem 3.1). For the reparametrization, extend $\tilde{C}_t$ to $T_{\tilde{\pi}(p,f)} M$ via

\begin{equation}
\tilde{C}_t(X + X_F + U^*) := X + X_F + (\tilde{C}_t U)^*.
\end{equation}

Denote $\kappa_t(\tilde{X}, \tilde{Y}) = R_{h_t}(\tilde{C}_t^{-1} X, \tilde{C}_t^{-1} Y, \tilde{C}_t^{-1} Y, \tilde{C}_t^{-1} X)$.

Theorem 3.1 (Sectional curvature). Let $h \in \mathcal{M}$ and $g_t \times g_F$ be as in Definition 7. Then, for every pair $\tilde{X} = X + X_F + U^*$, $\tilde{Y} = Y + Y_F + V^*$,

\begin{equation*}
\kappa_t(\tilde{X}, \tilde{Y}) = \kappa_t(X + U^\vee, Y + V^\vee) + K_{g_F}(X_F - (P_t^{-1} PU)^*, Y_F - (P_t^{-1} PV)^*) + \tilde{\kappa}_t(\tilde{X}, \tilde{Y}),
\end{equation*}

where $\kappa_t$ is as in Theorem 2 and $\tilde{\kappa}_t$ is non-negative.

Proof. The proof follows from a direct use of Gray–O’Neill curvature formula and Claim 2. Observe that

\begin{equation*}
\mathcal{L}_{\tilde{\pi}}(\tilde{C}_t^{-1} \tilde{X}) = (C_t^{-1}(X + U^\vee), X_F - (P_t^{-1} PU)^*).
\end{equation*}

Let $\tilde{z}_t$ be three times the norm squared of the integrability tensor of $\tilde{\pi}$ applied to $\mathcal{L}_{\tilde{\pi}} C_t^{-1} \tilde{X}$, $\mathcal{L}_{\tilde{\pi}} C_t^{-1} \tilde{Y}$ (see 4 for details). Then,

\begin{equation*}
R_{h_t}(\tilde{C}_t^{-1} X, \tilde{C}_t^{-1} Y, \tilde{C}_t^{-1} Y, \tilde{C}_t^{-1} X) = K_{g_F}(C_t^{-1}(X + U^\vee), C_t^{-1}(Y + V^\vee))
+ K_{g_F}(X_F - (P_t^{-1} PU)^*, Y_F - (P_t^{-1} PV)^*) + \tilde{\kappa}_t(\tilde{X}, \tilde{Y})
= \kappa_t(X + U^\vee, Y + V^\vee) + K_{g_F}(X_F - (P_t^{-1} PU)^*, Y_F - (P_t^{-1} PV)^*) + \tilde{\kappa}_t(\tilde{X}, \tilde{Y}).
\end{equation*}

Although, the term $\tilde{\kappa}_t$ plays no role on the applications in this paper, we claim that it is, actually, non-decreasing with respect to $t$. This is a crucial observation since $\tilde{z}_0$ is an essential part of the initial curvature: since $\tilde{\pi} : (\mathcal{P} \times F, g \times g_F) \to (M, h)$
is chosen to be a Riemannian submersion, \( \tilde{z}_0 \) is the \( A \)-tensor term on the submersion formula. Or, equivalently, taking \( t = 0 \) in Theorem 3.1.

\[
K_h(\tilde{X}, \tilde{Y}) = \kappa_0(\tilde{X}, \tilde{Y})
\]

\[
= \kappa_0(X+U^\vee, Y+V^\vee) + K_{gf}(X_F - (P_F^{-1}P_U)^*, Y_F - (P_F^{-1}P_V)^*) + \tilde{z}_0(\tilde{X}, \tilde{Y})
\]

\[
= K_g(X+U^\vee, Y+V^\vee) + K_{gf}(X_F - (P_F^{-1}P_U)^*, Y_F - (P_F^{-1}P_V)^*) + \tilde{z}_0(\tilde{X}, \tilde{Y}).
\]

### 3.1. Applications to Ricci curvature.

We use of the theory developed so far to prove Theorem 1.1.

We begin by constructing an appropriate basis for the horizontal space of \( \tilde{\pi} \) with respect to \( g_t \times g_F \). Consider a \( Q \)-orthonormal basis \( \{v_k(0)\} \) of \( m_f \) and define

\[
v_k(t) = \tilde{P}_t^{-1/2}v_k(0).
\]

**Lemma 2.** The set

\[
\{(-P_t^{-1}\tilde{P}_tv_k(t)^\vee, P_F^{-1}\tilde{P}_tv_k(t)^*)\} = \{(-P_t^{-1}\tilde{P}_t^{1/2}v_k(0)^\vee, P_F^{-1}\tilde{P}_t^{1/2}v_k(0)^*)\}
\]

is \( g_t \times g_F \)-orthonormal and \( g_t \times g_F \) orthogonal to \( (U^\vee, U^*) \), for every \( U \in \mathfrak{g} \).

**Proof.** Note that the elements in (13) are of the form (11). Thus, it is sufficient to show that the set (13) is orthonormal. A straightforward computation gives:

\[
g_t \times g_F((-P_t^{-1}\tilde{P}_tv_i(t)^\vee, P_F^{-1}\tilde{P}_tv_j(t)^*) = Q(\tilde{P}_tv_i(t), P_F^{-1}\tilde{P}_tv_j(t))
\]

where we have used that \( (P_t^{-1} + P_F^{-1})\tilde{P}_t = \tilde{P}_t^{-1} \) and that \( \tilde{P}_t \) is symmetric. \( \square \)

Let \( \{e^B_t\} \) and \( \{e^F_t\} \) be orthonormal bases for the spaces normal to the orbits on \( \mathcal{P} \) and on \( F \), respectively. We complete the set on Lemma 2 to a \( g_t \times g_F \)-orthonormal basis for the \( \tilde{\pi} \)-horizontal space:

\[
\mathcal{B}_t := \{e^B_t(0), (-P_t^{-1}\tilde{P}_t^{1/2}v_k(0)^\vee, P_F^{-1}\tilde{P}_t^{1/2}v_k(0)^*), (0, e^F_t)\}.
\]

Denote by \( e_1, \ldots, e_n \) the elements in \( \mathcal{B}_t \). Theorem 1.1 follows from the next two lemmas.

**Lemma 3.** For any \((p, f) \in \mathcal{P} \times F \) and \( X + X_F + U^* \in T_{\tilde{\pi}(p, f)}M \),

\[
\lim_{t \to \infty} \text{Ric}_{h_t}(X + X_F + U^*) \geq \text{Ric}_{g_B}^b(X) + \text{Ric}_{g_F}^b(X_F) + \sum_k \frac{1}{4} ||v_k(0), U||_Q^2.
\]

**Proof.** Using the basis \( \mathcal{B}_t \), from (14), and Theorem 3.1 we have:

\[
\text{Ric}_{h_t}(\tilde{X}) = \sum_{i=1}^n \kappa_i(\tilde{C}_i \tilde{X}, \tilde{C}_i e_i) \geq
\]

\[
\sum_i \kappa_i(X - (C_i P_t^{-1} \tilde{P}_t U)^\vee, e_i^B) + \sum_i \kappa_i(C_i X - (C_i P_t^{-1} \tilde{P}_t U)^\vee, -C_i P_t^{-1} \tilde{P}_t^{1/2} v_k(0)^\vee)
\]

\[
+ \sum_j K_{gf}(X_F + (P_F^{-1} \tilde{P}_t U)^*, e_j^F) + \sum_k K_{gf}(X_F + (P_F^{-1} \tilde{P}_t U)^*, P_F^{-1} \tilde{P}_t^{1/2} v_k(0)^*).
\]
On the other hand, the $\tilde{P}_t$ satisfy:

\[(16)\] \[\lim_{t \to \infty} t\tilde{P}_t = 1,\]

\[(17)\] \[\lim_{t \to \infty} P_t^{-1}\tilde{P}_t = 1.\]

In particular, $\tilde{P}_t \to 0$ as $t \to \infty$. Both \[(16)\] follows since $t\tilde{P}_t = P_t(P_t + P_t)^{-1}tP_t$, $P_t \to 0$ and $tP_t \to 1$. Equation \[(17)\] follows since

\[
\lim_{t \to \infty} P_t^{-1}\tilde{P}_t = \lim_{t \to \infty} (tP_t)^{-1} \lim_{t \to \infty} t\tilde{P}_t = 1.
\]

Using \[(16)\], we observe that

\[
\lim_{t \to \infty} \left\{ \sum_j K_{g_F}(X_F + (P_F^{-1}\tilde{P}_tU)^{\gamma}, e_j^F) + \sum_k K_{g_F}(X_F + (P_F^{-1}\tilde{P}_tU)^{\gamma}, P_F^{-1}\tilde{P}_t^{1/2}v_k(0)^{\gamma}) \right\} = \text{Ric}^h_{g_F}(X_F).
\]

Moreover, using equation \[(2)\] and that $C_tP_t^{-1} = P^{-1}$,

\[
\lim_{t \to \infty} \sum_i \kappa_t((C_tP_t^{-1}\tilde{P}_tU)^{\gamma}, -C_tP_t^{-1}\tilde{P}_t^{1/2}v_k(0)^{\gamma}) \geq \lim_{t \to \infty} \text{Ric}^h_g(X - (P^{-1}\tilde{P}_tU)^{\gamma}) = \text{Ric}^h_g(X).
\]

For the remaining term:

\[
\kappa_t((C_tX - (C_tP_t^{-1}\tilde{P}_tU)^{\gamma}, -C_tP_t^{-1}\tilde{P}_t^{1/2}v_k(0)^{\gamma}) \geq K_g(X - (P^{-1}\tilde{P}_tU)^{\gamma}, -P^{-1}\tilde{P}_t^{1/2}v_k(0)^{\gamma}) + \frac{t^3}{4}||[\tilde{P}_tU, \tilde{P}_t^{1/2}v_k(0)]||_Q^2.
\]

Using \[(16)\] and \[(17)\], we obtain

\[
\lim_{t \to \infty} \kappa_t((C_tX - (C_tP_t^{-1}\tilde{P}_tU)^{\gamma}, -C_tP_t^{-1}\tilde{P}_t^{1/2}v_k(0)^{\gamma}) \geq \frac{1}{4}||v_k(0), U||_Q^2.
\]

Lemma 4 follows by putting together \[(18)\], \[(19)\] and \[(20)\]. \qed

**Lemma 4.** Let $F$ be a $G$-manifold. Assume that the orbits of the $G$-action on $F$ have finite fundamental group and that the orbital distance metric on $F/G$ has Ricci curvature $\text{Ric}_{F/G} \geq 1$. Then there is a $G$-invariant $g_F$ on $F$ with $\text{Ric}^h_{g_F} > 0$.

**Proof.** Let $g_F$ be a $G$-invariant metric on $F$ where the orbits of $G$ have finite fundamental group. Then, $g_F$ has positive horizontal Ricci curvature if and only if the Ricci curvature of any arbitrarily large Cheeger deformation of $g_F$ is positive. This is decoded in the following way:

\[(21)\] \[\lim_{t \to \infty} \text{Ric}(g_F)_t(X) = \text{Ric}^h(X),\]

where $(g_F)_t$ is the Cheeger deformation of $g_F$.

Note that the hypothesis on Lemma 4 are the same as the ones in [14] Theorem A]. Under such hypothesis, the authors constructed a $G$-invariant metric $g_F$ whose Ricci curvature was positive after arbitrarily large Cheeger deformation (see [14] Theorems 6.1, 6.2]), proving Lemma 4. \qed

**Proof of Theorem 1.1.** Let $g_F$ be as in Lemma 4. Assume by contradiction that, for each $t > 0$, there is a point $x_t \in M$ and an $h$-unitary vector $X_t + (X_F)_t + U_t^*$
such that $\text{Ric}_h(X_t + (X_F)_t + U^*_t) \leq 0$. Then, considering $t = n \in \mathbb{N}$, we get a sequence $X_n + (X_F)_n + U^*_n$ such that

$$\lim_{n \to \infty} \text{Ric}_h(X_n + (X_F)_n + U^*_n) \leq 0.$$  

Since $\mathcal{P}$ and $F$ are compact and the horizontal lift is an isometry, there is a limit point $x$ and a non-zero vector $X + X_F + U^*$ such that

$$\text{Ric}_g(X) + \frac{1}{4} \sum_j \|v_j(0), U\|^2_Q + \text{Ric}_{grF}^h(X_F) \leq 0.$$  

Recall that the sum $\sum_k \frac{1}{4} \|[v_k(0), U]\|^2_Q$ is the Ricci curvature of the $Q$-normal homogeneous space $G/G_f$ in the direction $U$. We claim that $G/G_f$ has finite fundamental group, whenever $f$ is on the principal part or not.

Let $H < G$ be a principal isotropy group for the $G$-action on $F$. By hypothesis, $\pi_1(G/H)$ is finite. Following Bredon [2 section IV.3], we can assume (up to conjugation) that $H < G_f$. The long homotopy sequence of $G_f/H \hookrightarrow G/H \rightarrow G/G_f$ gives:

$$\cdots \rightarrow \pi_1(G/H) \rightarrow \pi_1(G/G_f) \rightarrow \pi_0(G_f/H) \rightarrow \cdots$$

Therefore, $\pi_1(G/G_f)$ is finite if and only if $G_f/H$ has a finite number of connected components. However, $G_f$ is a compact manifold, since it is a closed subgroup of the compact Lie group $G$. Therefore, $\pi_1(G/G_f)$ is finite.

We conclude that there is a $C > 0$ such that $\frac{1}{4} \sum_j \|[v_j(0), U]\|^2_Q \geq C\|U\|^2_Q$ (see e.g. Proposition 3.4 in Nash [10], Proposition 6.6 in [14] or Theorem 1 in [1]). Therefore,

$$\text{Ric}_g^h(X) + \text{Ric}_{grF}^h(X_F) + C\|U\|^2_Q \leq 0,$$

which is a contradiction. \hfill $\square$

Next, we list the results from Nash [10] and Poor [11] which are Corollary to Theorem 1.1.

**Corollary 3.2** ([10] Theorem 3.5.). Let $(E, M, G/H, G, \pi)$ be a fibre bundle such that $M$ is compact and admits a metric $g$ with $\text{Ric}_g > 0$. If $G$ is compact and $G/H$ admits a metric with $\text{Ric} \geq 1$ (equivalently, if $\pi_1(G/H)$ is finite), then $E$ admits a metric of positive Ricci curvature.

**Corollary 3.3** ([10] Corollary 3.6.). Let $(E, M, S^n, O(n+1), \pi)$, $n \geq 2$, be a sphere bundle whose basis is compact and possesses a metric of positive Ricci curvature. Then $E$ admits a metric of positive Ricci curvature.

**Corollary 3.4** ([10] Theorem 3.8.). Let $(E, M, F, G, \pi)$ be a bundle with $M$ compact admitting a metric $g$ with $\text{Ric}_g > 0$. If $G$ is compact, possesses finite fundamental group and acts freely on $F$, which is compact such that $F/G$ admits a metric with $\text{Ric} \geq 1$, then $E$ admits a metric of positive Ricci curvature.

**Corollary 3.5** ([11] Main Theorem). Let $M$ and $F$ be compact Riemannian manifolds such that the Ricci curvature of $M$ and the sectional curvature of $F$ are positive. Suppose $\pi : FM \rightarrow M$ is a bundle with fiber $F$ for which the structural group $G$ acts on $F$ by isometries. Then $FM$ admits a metric of positive Ricci curvature in which the fibers of $\pi$ are totally geodesic and mutually isometric in $FM$.  

8 A METRIC DEFORMATION ON FIBER BUNDLES AND APPLICATIONS
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