Branching Ratio and CP Asymmetry of $B^0 \to \eta^{(t)}\eta^{(t)}$ Decays in the Perturbative QCD Approach

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(Dated: March 26, 2022)

Abstract

We calculate the CP averaged branching ratios and CP-violating asymmetries for $B^0 \to \eta\eta$, $\eta\eta'$ and $\eta'\eta'$ decays by employing the perturbative QCD (pQCD) factorization approach. The pQCD predictions for the CP-averaged branching ratios are $Br(B^0 \to \eta\eta) \approx 0.67 \times 10^{-7}$, $Br(B^0 \to \eta\eta') \approx 0.18 \times 10^{-7}$, and $Br(B^0 \to \eta'\eta') \approx 0.11 \times 10^{-7}$, which are consistent with currently available experimental upper limits. We also predict large CP-violating asymmetries for the considered three decay modes, which can be tested by the future B meson experiments.

PACS numbers: 13.25.Hw, 12.38.Bx, 14.40.Nd

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I. INTRODUCTION

The two-body charmless B meson decays provide a good place for testing the standard model (SM), studying CP violation of B meson system, exploring the rich quantum chromodynamics (QCD) of strong interaction and searching for the signal or evidence of new physics beyond the SM [1]. Up to now, many $B \to M_1 M_2$ decays (where $M_i$ refers to the light pseudo-scalar or vector mesons) have been measured experimentally [2, 3] with good accuracy, calculated and studied phenomenologically in the QCD factorization (QCDF) approach [4, 5, 6, 7], the perturbative QCD (pQCD) factorization approach [8, 9, 10, 11, 12, 13, 14, 15, 16], or the soft collinear effective theory (SCET) [17].

Among various $B \to M_1 M_2$ decay channels, the decays involving the isosinglet $\eta$ or $\eta'$ mesons in the final state have been studied extensively during the past decade because of the so-called $K\eta'$ puzzle or other special features. At present, we still do not know how large is the gluonic content of the $\eta'$ meson, and how to calculate reliably the gluonic contributions to the decay modes involving $\eta'$ meson. It is still difficult to explain the observed pattern of the branching ratios for $B \to K^{(*)}\eta^{(*)}$ decays.

In pQCD factorization approach, the $B \to K\eta^{(*)}$, $\rho\eta^{(*)}$ and $\pi\eta^{(*)}$ decays have been studied in Refs. [13, 14, 15, 16]. In this paper, we would like to calculate the branching ratios and CP asymmetries for the three $B \to \eta\eta$, $\eta\eta'$ and $\eta'\eta'$ decays by employing the low energy effective Hamiltonian [18] and the pQCD approach [19, 20, 21]. Besides the usual factorizable contributions, we here are able to evaluate the non-factorizable and the annihilation contributions to these decays. On the experimental side, the CP-averaged branching ratios of $B \to \eta^{(*)}\eta^{(*)}$ decays have been measured very recently [22] in units of $10^{-6}$ (upper limits at 90% C.L.):

\[
\begin{align*}
Br(B^0 \to \eta\eta) & = 1.1^{+0.5}_{-0.4} \pm 0.1 \quad (< 1.8), \\
Br(B^0 \to \eta\eta') & = 0.2^{+0.7}_{-0.5} \pm 0.4 \quad (< 1.7), \\
Br(B^0 \to \eta'\eta') & = 1.0^{+0.8}_{-0.6} \pm 0.1 \quad (< 2.4).
\end{align*}
\]

The accuracy of the data is still low, only the upper limits can be used to compare with the theoretical predictions.

This paper is organized as follows. In Sec. II we give a brief review for the pQCD factorization approach. In Sec. III we calculate analytically the related Feynman diagrams and present the various decay amplitudes for the studied decay modes. In Sec. IV we show the numerical results for the branching ratios and CP asymmetries of $B \to \eta^{(*)}\eta^{(*)}$ decays and compare them with the measured values or the theoretical predictions in QCDF approach. The summary and some discussions are included in the final section.

II. THEORETICAL FRAMEWORK

For the non-leptonic B decays, the dominant theoretical uncertainty comes from the evaluation of the hadronic matrix element $< M_1 M_2 | O_i | B >$. Now there are two popular factorization approaches being used to calculate the matrix elements: the QCDF approach [4] and the pQCD approach [19, 20, 21]. The pQCD approach has been developed earlier from the QCD hard-scattering approach [21]. Some elements of this approach are also present in the QCD factorization approach [14, 15]. The two major differences between these
two approaches are (a) the form factors are calculable perturbatively in pQCD approach, but taken as the input parameters extracted from other experimental measurements in the QCDF approach; and (b) the annihilation contributions are calculable and play an important role in producing CP violation for the considered decay modes in pQCD approach, but it could not be evaluated reliably in QCDF approach. Of course, one should remember that the assumptions behind the pQCD approach, specifically the possibility to calculate the form factors perturbatively, are still under discussion [23]. More efforts are needed to clarify these problems.

In pQCD approach, the decay amplitude is separated into soft, hard, and harder dynamics characterized by different energy scales \((t, m_b, M_W)\). It is conceptually written as the convolution,

\[
\mathcal{A}(B \to M_1 M_2) \sim \int d^4k_1 d^4k_2 d^4k_3 \text{Tr} [C(t)\Phi_B(k_1)\Phi_{M_1}(k_2)\Phi_{M_2}(k_3)H(k_1, k_2, k_3, t)],
\]

where \(k_i\)’s are momenta of light quarks included in each mesons, and \(\text{Tr}\) denotes the trace over Dirac and color indices. \(C(t)\) is the Wilson coefficient which results from the radiative corrections at short distance. In the above convolution, \(C(t)\) includes the harder dynamics at larger scale than \(M_B\) scale and describes the evolution of local 4-Fermi operators from \(m_W\) (the \(W\) boson mass) down to \(t \sim \mathcal{O}(\sqrt{\Lambda M_B})\) scale, where \(\Lambda \equiv M_B - m_b\). The function \(H(k_1, k_2, k_3, t)\) is the hard part and can be calculated perturbatively. The function \(\Phi_{M_i}\) is the wave function which describes hadronization of the quark and anti-quark to the meson \(M\). While the function \(H\) depends on the process considered, the wave function \(\Phi_{M_i}\) is independent of the specific process. Using the wave functions determined from other well measured processes, one can make quantitative predictions here.

Since the \(b\) quark is rather heavy we consider the \(B\) meson at rest for simplicity. It is convenient to use light-cone coordinate \((p^+, p^-, p_T)\) to describe the meson’s momenta: \(p^\pm = (p^0 \pm p^3)/\sqrt{2}\) and \(p_T = (p^1, p^2)\).

Using the light-cone coordinates the \(B\) meson and the two final state meson momenta can be written as

\[
P_1 = \frac{M_B}{\sqrt{2}}(1, 1, 0_T), \quad P_2 = \frac{M_B}{\sqrt{2}}(1, 0, 0_T), \quad P_3 = \frac{M_B}{\sqrt{2}}(0, 1, 0_T),
\]

respectively, here the light meson masses have been neglected. Putting the light (anti-) quark momenta in \(B, \eta'\) and \(\eta\) mesons as \(k_1, k_2,\) and \(k_3\), respectively, we can choose

\[
k_1 = (x_1 p_1^+, 0, k_{1T}), \quad k_2 = (x_2 p_2^+, 0, k_{2T}), \quad k_3 = (0, x_3 p_3^-, k_{3T}).
\]

Then, for \(B \to \eta \eta'\) decay for example, the integration over \(k_1^-, k_2^-\), and \(k_3^+\) in eq.(4) will lead to

\[
\mathcal{A}(B \to \eta \eta') \sim \int dx_1 dx_2 dx_3 b_1 b_2 b_2 b_3 \text{Tr} \left[ C(t)\Phi_B(x_1, b_1)\Phi_{\eta'}(x_2, b_2)\Phi_{\eta}(x_3, b_3)H(x_1, b_1, t)S_t(x_i) e^{-S(t)} \right],
\]

where \(b_i\) is the conjugate space coordinate of \(k_{iT}\) and \(t\) is the largest energy scale in function \(H(x_i, b_i, t)\). The large logarithms \(\ln(m_W/t)\) are included in the Wilson coefficients \(C(t)\). The large double logarithms \((\ln^2 x_i)\) on the longitudinal direction are summed by
we use the formulae as given in Ref. [8] directly. The renormalization group evolution of the Wilson coefficients from higher scale to lower scale, the next-to-leading order (NLO) results already exist in the literature [18]. This is the consistent way to cancel the explicit $\mu$ dependence effectively [25]. Thus it makes the perturbative calculation of the hard part $H$ applicable at intermediate scale, i.e., $M_B$ scale. We will calculate analytically the function $H(x_i, b_i, t)$ for the considered decays in the first order in $\alpha_s$ expansion and give the convoluted amplitudes in next section.

### A. Wilson Coefficients

For the two-body charmless B meson decays, the related weak effective Hamiltonian $H_{\text{eff}}$ can be written as [18]

$$
H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \left[ V_{ub} V_{ud}^* (C_1(\mu) O_1^u(\mu) + C_2(\mu) O_2^u(\mu)) - V_{ub} V_{td}^* \sum_{i=3}^{10} C_i(\mu) O_i(\mu) \right],
$$

where $C_i(\mu)$ are Wilson coefficients at the renormalization scale $\mu$ and $O_i$ are the four-fermion operators for the case of $b \to d$ transition,

$$
O_1^u = \bar{d}_\alpha \gamma^\mu Lb_\beta \cdot \bar{u}_\alpha \gamma_\mu Lb_\beta , \quad O_2^u = \bar{d}_\alpha \gamma^\mu Lb_\beta \cdot \bar{u}_\beta \gamma_\mu Lb_\beta ,
$$

$$
O_3 = \bar{d}_\alpha \gamma^\mu Lb_\beta \cdot \sum_{q'} q'_\beta \gamma_\mu Ld_\beta , \quad O_4 = \bar{d}_\alpha \gamma^\mu Lb_\beta \cdot \sum_{q'} q'_\beta \gamma_\mu Ld_\beta ,
$$

$$
O_5 = \bar{d}_\alpha \gamma^\mu Lb_\beta \cdot \sum_{q'} e_q q'_\beta \gamma_\mu Rq'_\beta , \quad O_6 = \bar{d}_\alpha \gamma^\mu Lb_\beta \cdot \sum_{q'} e_q q'_\beta \gamma_\mu Rq'_\beta ,
$$

$$
O_7 = \frac{3}{2} \bar{d}_\alpha \gamma^\mu Lb_\beta \cdot \sum_{q'} q'_\beta \gamma_\mu Rq'_\beta , \quad O_8 = \frac{3}{2} \bar{d}_\alpha \gamma^\mu Ld_\beta \cdot \sum_{q'} e_q q'_\beta \gamma_\mu Rq'_\beta ,
$$

$$
O_9 = \frac{3}{2} \bar{d}_\alpha \gamma^\mu Lb_\beta \cdot \sum_{q'} q'_\beta \gamma_\mu Ld_\beta , \quad O_{10} = \frac{3}{2} \bar{d}_\alpha \gamma^\mu Ld_\beta \cdot \sum_{q'} e_q q'_\beta \gamma_\mu Ld_\beta ,
$$

where $\alpha$ and $\beta$ are the $SU(3)$ color indices; $L$ and $R$ are the left- and right-handed projection operators with $L = (1 - \gamma_5), R = (1 + \gamma_5)$. The sum over $q'$ runs over the quark fields that are active at the scale $\mu = O(m_b)$, i.e., $q' = \{u, d, s, c, b\}$. For the Wilson coefficients $C_i(\mu)$ ($i = 1, \ldots, 10$), we will use the leading order (LO) expressions, although the next-to-leading order (NLO) results already exist in the literature [18]. This is the consistent way to cancel the explicit $\mu$ dependence in the theoretical formulae. For the renormalization group evolution of the Wilson coefficients from higher scale to lower scale, we use the formulae as given in Ref. [8] directly.

### B. Wave Functions

In the resummation procedures, the $B$ meson is treated as a heavy-light system. In general, the $B$ meson light-cone matrix element can be decomposed as [4, 26]

$$
\int_0^1 \frac{d^4z}{(2\pi)^4} e^{i k_1 \cdot z} \langle 0| \bar{b}_\alpha(0) d_\beta(z) |B(p_B)\rangle
$$

$$
= -\frac{i}{\sqrt{2N_c}} \left\{ (\bar{\phi}_B + m_B) \gamma_5 \left[ \phi_B(k_1) - \frac{\gamma^\mu - \gamma^\nu}{\sqrt{2}} \bar{\phi}_B(k_1) \right] \right\}_{\beta\alpha},
$$

where $n = (1, 0, 0_T)$, and $v = (0, 1, 0_T)$ are the unit vectors pointing to the plus and minus directions, respectively. From the above equation, one can see that there are
two Lorentz structures in the B meson distribution amplitude (DA). They obey to the following normalization conditions

\[
\int \frac{d^4k_1}{(2\pi)^4} \phi_B(k_1) = \frac{f_B}{2\sqrt{2N_c}}, \quad \int \frac{d^4k_1}{(2\pi)^4} \bar{\phi}_B(k_1) = 0. \quad (11)
\]

In general, one should consider these two Lorentz structures in calculations of B meson decays. However, it can be argued that the contribution of \(\bar{\phi}_B\) is numerically small \[27, 28\], thus its contribution can be numerically neglected. Therefore, we only consider the contribution of Lorentz structure

\[
\Phi_B = \frac{1}{\sqrt{2N_c}} (\bar{\psi}_B + m_B) \gamma_5 \phi_B(k_1),
\]

in our calculation. We use the same wave functions as in Refs. \[8, 9, 28\]. Throughout this paper, we use the light-cone coordinates to write the four momentum as \(k_1^+, k_1^-, k_1^\perp\).

In the next section, we will see that the hard part is always independent of one of the \(k_1^+\) and/or \(k_1^-\), if we make some approximations. The B meson wave function is then the function of variable \(k_1^-\) (or \(k_1^+\)) and \(k_1^\perp\).

\[
\phi_B(k_1^-, k_1^+) = \int dk_1^+ \phi(k_1^+, k_1^-, k_1^\perp).
\]

The wave function for \(d\bar{d}\) components of \(\eta^{(i)}\) meson are given as \[14\]

\[
\Phi_{\eta_d d}(P, x, \zeta) \equiv \frac{i\gamma_5}{\sqrt{2N_c}} \left[ \bar{\psi} \phi_{\eta_d d}^A(x) \right. + m_0 \phi_{\eta_d d}^P(x) + \left. \zeta \bar{\psi} m_0 \phi_{\eta_d d}^T(x) \right] ,
\]

where \(P\) and \(x\) are the momentum and the momentum fraction of \(\eta_{d\bar{d}}\) respectively, while \(\phi_{\eta_d d}^A\), \(\phi_{\eta_d d}^P\) and \(\phi_{\eta_d d}^T\) represent the axial vector, pseudoscalar and tensor components of the wave function respectively. Following Ref. \[14\], we here also assume that the wave function of \(\eta_{d\bar{d}}\) is same as the \(\pi\) wave function based on SU(3) flavor symmetry. The parameter \(\zeta\) is either +1 or −1 depending on the assignment of the momentum fraction \(x\).

The transverse momentum \(k_\perp\) is usually converted to the \(b\) parameter by Fourier transformation. The initial conditions of the function \(\phi_i(x)\) with \(i = (B, \eta, \eta')\) are of non-perturbative origin, satisfying the normalization

\[
\int_0^1 \phi_i(x, b=0)dx = \frac{1}{2\sqrt{6}} f_i,
\]

with \(f_i\) the meson decay constants.

III. PERTURBATIVE CALCULATIONS

In this section, we will calculate and show the decay amplitude for each diagram including wave functions. The hard part \(H(t)\) involves the four quark operators and the necessary hard gluon connecting the four quark operator and the spectator quark. We first consider \(B \to \eta\eta^{(i)}\) decay mode as an example, and then extend our study to \(B \to \eta'\eta'\) decay.
A. Decay amplitudes

Similar to the $B \to \pi \eta^{(')}$ decays in $[16]$, there are 8 type diagrams contributing to the $B \to \eta \eta^{(')}$ decays, as illustrated in Figure 1. We first calculate the usual factorizable diagrams (a) and (b). Operators $O_1$, $O_2$, $O_3$, $O_4$, $O_9$, and $O_{10}$ are $(V-A)(V-A)$ currents, the sum of their amplitudes is given as

$$F_{en} = 8 \pi C_F m_B^4 \int_0^1 dx_1 dx_3 \int_0^\infty b_1 db_1 b_3 db_3 \phi_B(x_1, b_1)$$

$$\times \left\{ \left[ (1 + x_3)\phi_\eta(x_3, b_3) + (1 - 2x_3)r_\eta\phi_\eta^P(x_3, b_3) + \phi_\eta^P(x_3, b_3) \right] \right\}$$

$$\times \alpha_s(t_e^1) h_e(x_1, x_3, b_1, b_3) \exp[-S_{ab}(t_e^1)]$$

$$+ 2 r_\eta\phi_\eta^P(x_3, b_3) \alpha_s(t_e^2) h_e(x_3, b_1, b_3) \exp[-S_{ab}(t_e^2)] \right\} . \quad (16)$$

where $r_\eta = m_\eta^2/m_B^4$, $C_F = 4/3$ is a color factor. The function $h_e$, the scales $t_e^1$ and the Sudakov factors $S_{ab}$ are displayed in Appendix A.

The form factors of $B$ to $\eta$ decay, $F_{0,1}^{B \to \eta}(0)$, can thus be extracted from the expression in Eq. (16), that is

$$F_{0,1}^{B \to \eta}(q^2 = 0) = F_{1,1}^{B \to \eta}(q^2 = 0) = F_{en}/m_B^2 , \quad (17)$$

which is identical with that defined in Ref. [27].

The operators $O_5$, $O_6$, $O_7$, and $O_8$ have a structure of $(V-A)(V+A)$. In some decay channels, some of these operators contribute to the decay amplitude in a factorizable way. Since only the axial-vector part of $(V+A)$ current contribute to the pseudo-scalar meson production, $\langle \eta|V - A|B\rangle\langle \eta'|V + A|0\rangle = -\langle \eta|V - A|B\rangle\langle \eta'|V - A|0\rangle$, that is

$$F_{en}^{P1} = -F_{en} . \quad (18)$$

For other cases, we need to do Fierz transformation for the corresponding operators to get right flavor and color structure for factorization to work. We may get $(S - P)(S + P)$ operators from $(V - A)(V + A)$ ones. For these $(S - P)(S + P)$ operators, Fig. 1(a) and 1(b) give

$$F_{en}^{P2} = 16 \pi C_F m_B^4 r_\eta^{(')} \int_0^1 dx_1 dx_3 \int_0^\infty b_1 db_1 b_3 db_3 \phi_B(x_1, b_1)$$

$$\times \left\{ \left[ \phi_\eta(x_3, b_3) + r_\eta(2 + x_3)\phi_\eta^P(x_3, b_3) - x_3\phi_\eta^P(x_3, b_3) \right] \right\}$$

$$\times \alpha_s(t_e^1) h_e(x_1, x_3, b_1, b_3) \exp[-S_{ab}(t_e^1)]$$

$$+ [x_1\phi_\eta(x_3, b_3) - 2(x_1 - 1)r_\eta\phi_\eta^P(x_3, b_3)]$$

$$\times \alpha_s(t_e^2) h_e(x_3, x_1, b_1, b_3) \exp[-S_{ab}(t_e^2)] \right\} . \quad (19)$$

For the non-factorizable diagrams 1(c) and 1(d), all three meson wave functions are involved. The integration of $b_3$ can be performed using $\delta$ function $\delta(b_3 - b_1)$, leaving only integration of $b_1$ and $b_2$. For the $(V - A)(V - A)$ operators, the result is

$$M_{en} = \frac{16\sqrt{6}}{3} \pi C_F m_B^4 \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_1 db_1 b_2 db_2 \phi_B(x_1, b_1)\phi_\eta^{(')}(x_2, b_2)$$

$$\times \left\{ \left[ 2x_3r_\eta\phi_\eta^P(x_3, b_1) - x_3\phi_\eta(x_3, b_1) \right] \right\}$$

$$\times \alpha_s(t_f) h_f(x_1, x_2, x_3, b_1, b_2) \exp[-S_{cd}(t_f)] \right\} . \quad (20)$$
FIG. 1: Typical Feynman diagrams contributing to the $B \to \eta \eta'$ decays, where diagram (a) and (b) contribute to the $B \to \eta$ form factor $F_{0,1}^{B\to \eta}$.

For the $(V - A)(V + A)$ operators the formulae are different. Here we have two kinds of contributions from $(V - A)(V + A)$ operators: $M_{\eta}^{P1}$ and $M_{\eta}^{P2}$ is for the $(V - A)(V + A)$ and $(S - P)(S + P)$ type operators respectively:

$$M_{\eta}^{P1} = 0, \quad M_{\eta}^{P2} = -M_{\eta}.$$

(21)

The factorizable annihilation diagrams 1(g) and 1(h) involve only $\eta$ and $\eta'$ wave functions. There are also three kinds of decay amplitudes for these two diagrams. $F_{\eta \eta}$ is for $(V - A)(V - A)$ type operators, $F_{\eta \eta}^{P1}$ is for $(V - A)(V + A)$ type operators, while $F_{\eta \eta}^{P2}$
is for \((S - P)(S + P)\) type operators.

\[
F_{\alpha n}^{P_1} = F_{\alpha n} = -8\pi C_F m_B^4 \int_0^1 dx_2 dx_3 \int_0^\infty b_2 db_2 b_3 db_3 \left\{ \begin{array}{c}
\left[ x_3 \phi_\eta(x_3, b_3) \phi_\eta^{(\gamma)}(x_2, b_2) \\
+ 2r_\eta r_\eta^{(\gamma)} \left[ (x_3 + 1) \phi_\eta^P(x_3, b_3) + (x_3 - 1) \phi_\eta^t(x_3, b_3) \right] \phi_\eta^{(\gamma)}(x_2, b_2) \end{array} \right]
\cdot \alpha_s(t_\gamma^3) h_a(x_2, x_3, b_2, b_3) \exp[-S_{gh}(t_\gamma^3)]
- \left[ x_2 \phi_\eta(x_3, b_3) \phi_\eta^{(\gamma)}(x_2, b_2) \\
+ 2r_\eta r_\eta^{(\gamma)} \left[ (x_2 + 1) \phi_\eta^P(x_2, b_2) + (x_2 - 1) \phi_\eta^t(x_2, b_2) \right] \phi_\eta^{(\gamma)}(x_3, b_3) \right]
\cdot \alpha_s(t_\gamma^3) h_a(x_3, x_2, b_2, b_3) \exp[-S_{gh}(t_\gamma^3)] \right\}
\]  

\[
F_{\alpha n}^{P_2} = -16\pi C_F m_B^4 \int_0^1 dx_2 dx_3 \int_0^\infty b_2 db_2 b_3 db_3 
\times \left\{ \begin{array}{c}
\left[ x_3 r_\eta \phi_\eta^P(x_3, b_3) - \phi_\eta^t(x_3, b_3) \right] \phi_\eta^{(\gamma)}(x_2, b_2) + 2r_\eta r_\eta^{(\gamma)} \phi_\eta(x_3, b_3) \phi_\eta^{(\gamma)}(x_2, b_2) \\
\cdot \alpha_s(t_\gamma^3) h_a(x_2, x_3, b_2, b_3) \exp[-S_{gh}(t_\gamma^3)] \\
+ \left[ x_2 r_\eta^{(\gamma)} \phi_\eta^P(x_2, b_2) - \phi_\eta^t(x_2, b_2) \right] \phi_\eta(x_3, b_3) + 2r_\eta r_\eta^{(\gamma)} \phi_\eta(x_2, b_2) \phi_\eta^{(\gamma)}(x_3, b_3) \\
\cdot \alpha_s(t_\gamma^3) h_a(x_3, x_2, b_2, b_3) \exp[-S_{gh}(t_\gamma^3)] \right\}.
\]  

For the non-factorizable annihilation diagrams 1(e) and 1(f), again all three wave functions are involved. Here we have three kinds of contributions. \(M_{\alpha n}, M_{\alpha n}^{P_1}\) and \(M_{\alpha n}^{P_2}\) describe the contributions from \((V - A)(V - A)\), \((V - A)(V + A)\) and \((S - P)(S + P)\) type operators respectively.

\[
M_{\alpha n} = \frac{16\sqrt{6}}{3} \pi C_F m_B^4 \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_1 db_1 b_2 db_2 \phi_B(x_1, b_1) 
\times \left\{ \begin{array}{c}
\left[ x_2 \phi_\eta(x_3, b_2) \phi_\eta^{(\gamma)}(x_2, b_2) \\
+ r_\eta r_\eta^{(\gamma)} \left[ (x_2 + x_3 + 2) \phi_\eta^P(x_2, b_2) + (x_2 - x_3) \phi_\eta^t(x_2, b_2) \right] \phi_\eta^{(\gamma)}(x_3, b_2) \end{array} \right]
\cdot \alpha_s(t_\gamma^3) h_3^3(x_1, x_2, x_3, b_1, b_2) \exp[-S_{ef}(t_\gamma^3)]
\right\}
+ \left\{ \begin{array}{c}
\left[ x_3 \phi_\eta(x_3, b_2) \phi_\eta^{(\gamma)}(x_2, b_2) \\
+ r_\eta r_\eta^{(\gamma)} \left[ (x_2 + x_3 - 2) \phi_\eta^P(x_2, b_2) + (x_2 - x_3) \phi_\eta^t(x_2, b_2) \right] \phi_\eta^{(\gamma)}(x_3, b_2) \end{array} \right]
\cdot \alpha_s(t_\gamma^4) h_3^4(x_1, x_2, x_3, b_1, b_2) \exp[-S_{ef}(t_\gamma^4)] \right\},
\]  

(24)
\[ M^{P1}_{an} = \frac{16\sqrt{6}}{3} \pi C_F m^4_B \int_0^1 dx_1dx_2 dx_3 \int_0^\infty b_1db_1b_2db_2 \phi_B(x_1, b_1) \]
\[ \times \left\{ \left[(x_3 - 2)r_\eta \phi_{\eta}(x_2, b_2)(\phi^P_\eta(x_3, b_2) + \phi^t_\eta(x_3, b_2)) - (x_2 - 2)r_\eta \phi_{\eta}(x_3, b_2) \right] \right. \]
\[ \left. \left(\phi^P_\eta(x_2, b_2) + \phi^t_\eta(x_2, b_2)\right)\right\} \cdot \alpha_s(t_f^3) h_3^f(x_1, x_2, x_3, b_1, b_2) \exp[-S_{ef}(t_f^3)] \]
\[ - \left[ x_3r_\eta \phi_{\eta}(x_2, b_2)(\phi^P_\eta(x_3, b_2) + \phi^t_\eta(x_3, b_2)) \right] \]
\[ - x_2r_\eta \phi_{\eta}(x_3, b_2)(\phi^P_\eta(x_2, b_2) + \phi^t_\eta(x_2, b_2)) \]
\[ \cdot \alpha_s(t_f^3) h_3^f(x_1, x_2, x_3, b_1, b_2) \exp[-S_{ef}(t_f^3)] \right\} , \tag{25} \]

\[ M^{P2}_{an} = \frac{16\sqrt{6}}{3} \pi C_F m^4_B \int_0^1 dx_1dx_2 dx_3 \int_0^\infty b_1db_1b_2db_2 \phi_B(x_1, b_1) \]
\[ \times \left\{ \left\{ x_3 \phi_{\eta}(x_3, b_2)\phi_{\eta}(x_2, b_2) \right\} \right. \]
\[ + r_\eta \left[ (x_2 + x_3 + 2)\phi^P_{\eta}(x_2, b_2) + (x_3 - x_2)\phi^t_{\eta}(x_2, b_2) \right] \phi^P_\eta(x_3, b_2) \]
\[ + r_\eta \left[ (x_3 - x_2)\phi^P_{\eta}(x_3, b_2) + (x_2 + x_3 - 2)\phi^t_{\eta}(x_2, b_2) \right] \phi^t_\eta(x_3, b_2) \]
\[ \left. \cdot \alpha_s(t_f^3) h_3^f(x_1, x_2, x_3, b_1, b_2) \exp[-S_{ef}(t_f^3)] \right\} \]
\[ - \left\{ x_2 \phi_{\eta}(x_3, b_2)\phi_{\eta}(x_2, b_2) \right\} \]
\[ + r_\eta \left[ (x_2 + x_3)\phi^P_{\eta}(x_2, b_2) + (x_2 - x_3)\phi^t_{\eta}(x_2, b_2) \right] \phi^P_\eta(x_3, b_2) \]
\[ + r_\eta \left[ (x_2 - x_3)\phi^P_{\eta}(x_3, b_2) + (x_2 + x_3)\phi^t_{\eta}(x_2, b_2) \right] \phi^t_\eta(x_3, b_2) \]
\[ \cdot \alpha_s(t_f^3) h_3^f(x_1, x_2, x_3, b_1, b_2) \exp[-S_{ef}(t_f^3)] \right\} . \tag{26} \]

In the above equations, we have assumed that \( x_1 \ll x_2, x_3 \). Since the light quark momentum fraction \( x_1 \) in \( B \) meson is peaked at the small region, while quark momentum fraction \( x_3 \) of \( \eta \) is peaked around 0.5, this is not a bad approximation. The numerical results also show that this approximation makes very little difference in the final result. After using this approximation, all the diagrams are functions of \( k^+_1 = x_1m_B/\sqrt{2} \) of \( B \) meson only, independent of the variable of \( k^-_1 \). Therefore the integration of eq.\tag{13} is performed safely.

For the \( B \to \eta \eta' \) decay, besides the Feynman diagrams as shown in Fig. 1 where the upper emitted meson is the \( \eta' \), the Feynman diagrams obtained by exchanging the position of \( \eta \) and \( \eta' \) also contribute to this decay mode. The corresponding expressions of amplitudes for new diagrams will be similar with those as given in Eqs.\tag{10,26}, since the \( \eta \) and \( \eta' \) are all light pseudoscalar mesons and have the similar wave functions. The expressions of amplitudes for new diagrams can be obtained by the replacements
\[ \phi_\eta \leftrightarrow \phi_{\eta'}, \quad \phi^P_\eta \leftrightarrow \phi^P_{\eta'}, \quad \phi^t_\eta \leftrightarrow \phi^t_{\eta'}, \quad r_\eta \leftrightarrow r_{\eta'}. \tag{27} \]

For example, we find that:
\[ F_{\eta \eta'} = F_{\eta \eta}, \quad F_{\eta \eta'} = -F_{\eta \eta}, \quad F_{\eta \eta'}^{P1} = -F_{\eta \eta}^{P1}, \quad F_{\eta \eta'}^{P2} = F_{\eta \eta}^{P2}. \tag{28} \]
B. Mixing of $\eta$ and $\eta'$ meson

Before we write down the complete decay amplitude for the studied decay modes, we firstly give a brief discussion about the $\eta - \eta'$ mixing and the gluonic component of the $\eta'$ meson. There exist two popular mixing basis for $\eta - \eta'$ system, the octet-singlet and the quark flavor basis, in literature. Here we use the $SU(3)_F$ octet-singlet basis with the two mixing angle scheme \[29, 30\] instead of the simple one mixing angle scheme to describe the mixing of $\eta$ and $\eta'$ mesons, since the former scheme can archive a better agreement with the relevant data, such as the decay width of $\eta^{(i)} \to \gamma \gamma$, the $\eta^{(i)} \gamma$ transition form factors, the radiative $J/\Psi$ decays\[30, 31\]. In the two mixing angle scheme, the meson $\eta$, $\eta'$ and the decay constants can be defined as

\[
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix} = \begin{pmatrix}
\cos \theta_8 & -\sin \theta_1 \\
\sin \theta_8 & \cos \theta_1
\end{pmatrix} \begin{pmatrix}
\eta_8 \\
\eta_1
\end{pmatrix},
\]

with the flavor $SU(3)$-octet and -singlet components

\[
\eta_1 = (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}, \quad \eta_8 = (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6},
\]

(29)
in the quark model, and the two mixing angles $\theta_1$ and $\theta_8$, in principle, can be determined by various related experiments \[30, 31\]. In the numerical calculations, we will use the following $\eta - \eta'$ mixing parameters \[30\]

\[
\theta_8 = -21.2^\circ, \quad \theta_1 = -2.4^\circ, \\
f_1 = 151\text{MeV}, \quad f_8 = 169\text{MeV},
\]

(30)

obtained by setting $f_q = f_\pi = 130\text{MeV}$ and $f_s = \sqrt{2f_K^2 - f_\pi^2} = 1.41f_\pi$ \[30\]. The second set of the $\eta - \eta'$ mixing parameters obtained phenomenological and used frequently in literature reads \[30\]

\[
\theta_8 = -21.2^\circ, \quad \theta_1 = -9.2^\circ, \\
f_1 = 1.17f_\pi = 152\text{MeV}, \quad f_8 = 1.26f_\pi = 164\text{MeV},
\]

(31)

for $f_\pi = 130\text{MeV}$. We usually use the first set of mixing parameters in numerical calculations, unless otherwise stated.

As shown in Eq. (30), $\eta$ and $\eta'$ are generally considered as a linear combination of light quark pairs $u\bar{u}, d\bar{d}$ and $s\bar{s}$. But it should be noted that the $\eta'$ meson may have a gluonic component in order to interpret the anomalously large branching ratios of $B \to K\eta'$ \[13, 14, 32, 33\]. Although some progress have been achieved in recent years about this problem, we currently still do not know how to calculate reliably the gluonic contributions to the B meson two body decay modes involving $\eta'$ meson. For the studied decay modes in this paper, on the other hand, currently available measurements for the branching ratios still have big uncertainties. It is therefore reasonable at present to consider only the dominant contributions from the quark contents of $\eta'$ meson, while take the subdominant contribution from the possible gluonic content of $\eta'$ meson as a source of theoretical uncertainties. Following Ref. \[33\], on the other hand, we also estimated the possible gluonic contributions to $B \to \eta^{(i)}\eta^{(j)}$ decays induced by the gluonic corrections to the $B \to \eta^{(i)}$ transition form factors \[33\] and found that these corrections to both the branching ratios and CP violating asymmetries are indeed very small.
C. Complete decay amplitudes

For $B^0 \to \eta \eta$ decay, by combining the contributions from different diagrams, the total decay amplitude can be written as

$$\mathcal{M}(\eta \eta) = \sqrt{2} \left\{ F_{e1} F_1(\theta_1, \theta_8) \left\{ \xi_u \left( C_1 + \frac{1}{3} C_2 \right) \right. \right. \\
- \xi_t \left( \frac{7}{3} C_5 + \frac{5}{3} C_4 - 2C_5 - \frac{2}{3} C_6 - \frac{1}{2} C_7 - \frac{1}{6} C_8 + \frac{1}{3} C_9 - \frac{1}{3} C_{10} \right) f^d_{\eta} \\
- \xi_t \left( C_3 + \frac{1}{3} C_4 - C_5 - \frac{1}{3} C_6 + \frac{1}{2} C_7 + \frac{1}{6} C_8 - \frac{1}{2} C_9 - \frac{1}{6} C_{10} \right) f^s_{\eta} \right\} \\
- F_{e1}^p F_1(\theta_1, \theta_8) \xi_t \left( \frac{1}{3} C_5 + C_6 - \frac{1}{6} C_7 - \frac{1}{2} C_8 \right) f^d_{\eta} \\
+ M_{e1} F_1(\theta_1, \theta_8) \left\{ \xi_u C_2 - \xi_t \left( C_3 + 2C_4 - \frac{1}{2} C_9 + \frac{1}{2} C_{10} \right) \right\} F_1(\theta_1, \theta_8) \\
- \xi_t \left( C_4 - \frac{1}{2} C_{10} \right) F_2(\theta_1, \theta_8) \left\{ \xi_u C_2 - \xi_t \left( C_3 + 2C_4 - \frac{1}{2} C_9 + \frac{1}{2} C_{10} \right) \right\} F_1(\theta_1, \theta_8)^2 \\
- M_{e1}^p \xi_t \left( C_5 - \frac{1}{2} C_7 \right) F_1(\theta_1, \theta_8)^2 \\
- M_{e1}^p \xi_t \left( \left( 2C_6 + \frac{1}{2} C_8 \right) F_1(\theta_1, \theta_8)^2 + \left( C_6 - \frac{1}{2} C_8 \right) F_2(\theta_1, \theta_8)^2 \right) \\
- F_{e1}^p \xi_t \left( \frac{1}{3} C_5 + C_6 - \frac{1}{6} C_7 - \frac{1}{2} C_8 \right) F_1(\theta_1, \theta_8)^2 \cdot f_B \right\},$$

where $\xi_u = V_{ub}^* V_{ud}$, $\xi_t = V_{tb}^* V_{td}$, while the mixing parameters and the relevant decay constants are

$$F_1(\theta_1, \theta_8) = \sqrt{\frac{1}{6}} \cos \theta_8 - \sqrt{\frac{1}{3}} \sin \theta_1, \quad F_2(\theta_1, \theta_8) = -\sqrt{\frac{2}{3}} \sin \theta_8 + \sqrt{\frac{1}{3}} \cos \theta_1,$$

$$f^d_{\eta} = \frac{f_8}{\sqrt{6}} \cos \theta_8 - \frac{f_1}{\sqrt{3}} \sin \theta_1, \quad f^s_{\eta} = -\frac{2f_8}{\sqrt{3}} \cos \theta_8 + \frac{f_1}{\sqrt{3}} \sin \theta_1.$$

It should be mentioned that the Wilson coefficients $C_i = C_i(t)$ in Eq. (33) should be calculated at the appropriate scale $t$ using equations as given in the Appendices of Ref. [8]. Here the scale $t$ in the Wilson coefficients should be taken as the same scale appeared in the expressions of decay amplitudes from Eqs. (16) to (26). This is the way in pQCD approach to eliminate the scale dependence.
Similarly, the decay amplitude for \( B^0 \to \eta \eta' \) can be written as

\[
\mathcal{M}(\eta \eta') = (F_{\eta} F_1(\theta_1, \theta_8) f_{\eta}^d + F_{\eta'} F'_1(\theta_1, \theta_8) f_{\eta'}^d) \cdot \left[ \xi_t \left( C_1 + \frac{1}{3} C_2 \right) - \xi_t \left( \frac{7}{3} C_3 + \frac{5}{3} C_4 - 2C_5 - \frac{2}{3} C_6 - \frac{1}{2} C_7 - \frac{1}{6} C_8 + \frac{1}{3} C_9 - \frac{1}{3} C_{10} \right) \right] \\
- (F_{\eta} F_1(\theta_1, \theta_8) f_{\eta}^s + F_{\eta'} F'_1(\theta_1, \theta_8) f_{\eta'}^s) \cdot \xi_t \left( C_3 + \frac{1}{3} C_4 - C_5 - \frac{1}{3} C_6 + \frac{1}{2} C_7 + \frac{1}{6} C_8 - \frac{1}{2} C_9 - \frac{1}{6} C_{10} \right) \\
- (F_{\eta}^2 F_1(\theta_1, \theta_8) f_{\eta}^d + F_{\eta'}^2 F'_1(\theta_1, \theta_8) f_{\eta'}^d) \xi_t \left( \frac{1}{3} C_5 + C_6 - \frac{1}{2} C_7 - \frac{1}{2} C_{8} \right) \\
+ (M_{\eta} + M_{\eta'}) F_1(\theta_1, \theta_8) F'_1(\theta_1, \theta_8) \left[ \xi_t C_2 - \xi_t \left( C_3 + 2C_4 - \frac{1}{2} C_9 + \frac{1}{2} C_{10} \right) \right] \\
- [M_{\eta} F_1(\theta_1, \theta_8) F'_2(\theta_1, \theta_8) + M_{\eta'} F'_1(\theta_1, \theta_8) F_2(\theta_1, \theta_8)] \xi_t \left( C_4 - \frac{1}{2} C_{10} \right) \\
- (M_{\eta} + M_{\eta'}) F_1(\theta_1, \theta_8) F'_1(\theta_1, \theta_8) \xi_t \left( 2C_6 + \frac{1}{2} C_{8} \right) \\
- (M_{\eta}^2 F_1(\theta_1, \theta_8) F'_2(\theta_1, \theta_8) + M_{\eta'}^2 F'_1(\theta_1, \theta_8) F_2(\theta_1, \theta_8)) \xi_t \left( C_6 - \frac{1}{2} C_{8} \right) \\
[\cdots] \\
- f_B \cdot (F_{\eta}^2 + F_{\eta'}^2) F_1(\theta_1, \theta_8) F'_1(\theta_1, \theta_8) \xi_t \left( \frac{1}{3} C_5 + C_6 - \frac{1}{6} C_7 - \frac{1}{2} C_{8} \right), \quad (36)
\]

where the relevant mixing parameters and decay constants are

\[
F'_1(\theta_1, \theta_8) = \sqrt{\frac{1}{6}} \sin \theta_8 + \sqrt{\frac{1}{3}} \cos \theta_1, \quad F'_2(\theta_1, \theta_8) = -\sqrt{\frac{2}{3}} \sin \theta_8 + \sqrt{\frac{1}{3}} \cos \theta_1, \quad (37)
\]

\[
f_{\eta}^d = \frac{f_s}{\sqrt{6}} \sin \theta_8 + \frac{f_1}{\sqrt{3}} \cos \theta_1, \quad f_{\eta'}^d = -\frac{2f_s}{\sqrt{3}} \sin \theta_8 + \frac{f_1}{\sqrt{3}} \cos \theta_1, \quad (38)
\]

The complete decay amplitude \( \mathcal{M}(\eta' \eta') \) for \( B \to \eta' \eta' \) decay can be obtained easily from Eq.\( (33) \) by the following replacements

\[
f_{\eta}^d, f_{\eta'}^d, f_{\eta}^s, f_{\eta'}^s \quad \rightarrow \quad f_{\eta'}^d, f_{\eta'}^s, \quad F_1(\theta_1, \theta_8) \quad \rightarrow \quad F'_1(\theta_1, \theta_8), \quad F_2(\theta_1, \theta_8) \quad \rightarrow \quad F'_2(\theta_1, \theta_8). \quad (39)
\]
Note that the contributions from the possible gluonic component of $\eta'$ meson have not been included here.

IV. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we will calculate the branching ratios and CP violating asymmetries for those considered decay modes. The input parameters and the wave functions to be used are given in Appendix B. In numerical calculations, central values of input parameters will be used implicitly unless otherwise stated.

Based on the definition of the form factor $F_{B \to \eta \to \eta', \eta'}^{\eta, \eta'}$ as given in Eq. (17) and consider the relation of $F_{B \eta} = F_{B \eta'}$ at the leading order of pQCD approach, we find the numerical values of the corresponding form factors at zero momentum transfer:

$$F_{B \to \eta \to \eta, \eta'}(q^2 = 0) = F_{B \to \eta \to \eta, \eta'}^{\eta, \eta'}(q^2 = 0) = \frac{F_{B \eta}}{m_B^2} = 0.30^{+0.05}_{-0.04}(\omega_b),$$

for $\omega_b = 0.40 \pm 0.04$ GeV, which agrees well with those as given in Refs. [34, 35].

A. Branching ratios

Using the decay amplitudes obtained in last section, it is straightforward to calculate the branching ratios. For $B^0 \to \eta \eta, \eta \eta'$ and $\eta' \eta'$ decays, the decay amplitudes as given in Eqs. (33) and (36) can be rewritten as

$$\mathcal{M} = V_{ub}V_{ud}T - V_{tb}V_{td}P = V_{ub}V_{ud}T \left[1 + ze^{i(\alpha + \delta)}\right],$$

where

$$z = \left|\frac{V_{ub}V_{ud}}{V_{tb}V_{td}}\right| \left|\frac{P}{T}\right|$$

is the ratio of penguin to tree contributions, $\alpha = \arg \left(-\frac{V_{td}V_{ub}^*}{V_{tb}V_{ud}^*}\right)$ is the weak phase (one of the three CKM angles), and $\delta$ is the relative strong phase between tree (T) and penguin (P) diagrams 1. In the pQCD approach, the ratio $z$ and the strong phase $\delta$ can be calculated perturbatively.

From Eq. (41), it is easy to write the decay amplitude for the corresponding charge conjugated decay mode

$$\overline{\mathcal{M}} = V_{ub}V_{ud}^*T - V_{tb}V_{td}^*P = V_{ub}V_{ud}T \left[1 + ze^{i(-\alpha + \delta)}\right].$$

Therefore the CP-averaged branching ratio for $B^0 \to \eta(\eta') \eta(\eta')$ decays is

$$Br = \frac{(|\mathcal{M}|^2 + |\overline{\mathcal{M}}|^2)}{2} = |V_{ub}V_{ud}T|^2 \left[1 + 2z \cos \alpha \cos \delta + z^2\right],$$

1 The “T” and “P” term refers to the part of the decay amplitude $\mathcal{M}$ in Eq. (41), which is proportional to $V_{ub}V_{ud}$ and $V_{tb}V_{td}$ respectively.
where the ratio $z$ and the strong phase $\delta$ have been defined in Eqs. (41) and (42).

By employing the two mixing angle scheme of $\eta - \eta'$ system and using the mixing parameters as given in Eq. (31), one finds the CP averaged branching ratios for the considered three decays as follows

$$Br( B^0 \to \eta \eta) = \left[ 1.5 \pm 0.3(\omega_b)^{+0.1}_{-0.2}(m_\pi^0) \pm 0.4(\alpha) \right] \times 10^{-7}, \quad (45)$$

$$Br( B^0 \to \eta \eta') = \left[ 0.60^{+0.19}_{-0.14}(\omega_b) \pm 0.05(m_\pi^0) \pm 0.04(\alpha) \right] \times 10^{-7}, \quad (46)$$

$$Br( B^0 \to \eta' \eta') = \left[ 0.68^{+0.08}_{-0.09}(\omega_b) \pm 0.05(m_\pi^0)^{+0.22}_{-0.19}(\alpha) \right] \times 10^{-7}, \quad (47)$$

where the main errors are induced by the uncertainties of $\omega_b = 0.4 \pm 0.04$ GeV, $m_\pi^0 = 1.4 \pm 0.1$ GeV and $\alpha = 100^\circ \pm 20^\circ$, respectively.

Besides the theoretical uncertainties as shown in Eqs. (45-47), the change of the mixing scheme of $\eta - \eta'$ system or changes of mixing parameters within a given scheme may also induce moderate or even significant changes to the theoretical predictions. The central values of the branching ratios for the considered three decays, for example, will be

$$Br( B^0 \to \eta \eta)\mid_{CV} = 1.9 \times 10^{-7}, \quad (48)$$

$$Br( B^0 \to \eta \eta')\mid_{CV} = 0.82 \times 10^{-7}, \quad (49)$$

$$Br( B^0 \to \eta' \eta')\mid_{CV} = 0.66 \times 10^{-7}, \quad (50)$$

if the second set of mixing parameters in the two mixing angle scheme, as given in Eq. (32), are used, and

$$Br( B^0 \to \eta \eta)\mid_{CV} = 1.9 \times 10^{-7}, \quad (51)$$

$$Br( B^0 \to \eta \eta')\mid_{CV} = 1.2 \times 10^{-7}, \quad (52)$$

$$Br( B^0 \to \eta' \eta')\mid_{CV} = 0.69 \times 10^{-7}, \quad (53)$$

if the one mixing angle scheme with $\theta_\rho = -12.3^\circ$ are employed. From the above numerical results one can see that (a) the induced variation is about 2% or 20% for $B \to \eta' \eta'$ and $B \to \eta \eta$ decay respectively, which is consistent with the general expectation; (b) for $B \to \eta \eta'$ decay, however, the resultant change of the branching ratio is roughly a factor of two. Such variation is channel-dependent and should be considered as one kind of theoretical uncertainties. The large change in $Br(B \to \eta \eta')$ is induced by destructive interference between the individual decay amplitudes from different Feynman diagrams when two mixing angle scheme are employed.

As a comparison, furthermore, we also list here the theoretical predictions for the branching ratios of the three decays in the QCDF approach[5] and the SCET approach[2]

\footnote{The theoretical predictions for the branching ratio and the direct CP violating asymmetries in the SCET approach, as shown in Eqs. (54,55) and Eqs. (70,72), are directly quoted from Table VII of Ref. [17], for the case of Theory II.}
FIG. 2: The $\alpha$ dependence of the branching ratios (in units of $10^{-6}$) of $B^0 \to \eta\eta$ decays. Here (a) is for $m_0^\pi = 1.4$ GeV, $\omega_b = 0.40 \pm 0.04$ GeV; and (b) is for $\omega_b = 0.4$ GeV, $m_0^\pi = 1.4 \pm 0.1$ GeV.

\[ Br( B^0 \to \eta\eta) = \begin{cases} (0.16 \pm 0.45) \times 10^{-6}, & \text{QCDF}, \\ (1.0 \pm 1.5) \times 10^{-6}, & \text{SCET}, \end{cases} \] (54)

\[ Br( B^0 \to \eta^\prime\eta^\prime) = \begin{cases} (0.16 \pm 0.018) \times 10^{-6}, & \text{QCDF}, \\ (2.2 \pm 5.5) \times 10^{-6}, & \text{SCET}, \end{cases} \] (55)

\[ Br( B^0 \to \eta\eta^\prime) = \begin{cases} (0.06 \pm 0.25) \times 10^{-6}, & \text{QCDF}, \\ (1.2 \pm 3.8) \times 10^{-6}, & \text{SCET}, \end{cases} \] (56)

where the individual errors as given in Refs. [5] and [17] have been added in quadrature. It is easy to see that (a) the theoretical predictions for $Br(B \to \eta\eta)$ and $Br(B \to \eta^\prime\eta^\prime)$ in both the QCDF and the pQCD approaches agree very well; (b) for $Br(B \to \eta\eta^\prime)$, the central value of the pQCD prediction is about half of the QCDF prediction, but still agree within one standard deviation; and (c) the central values of the theoretical predictions in the SCET approach are much larger than those in QCDF and pQCD approaches, but still consistent with them if one takes the very large theoretical uncertainties in SCET approach into account.

It is worth of mentioning that the FSI effects are not considered here. The smallness of FSI effects for B meson decays into two light mesons has been put forward by Bjorken [36] based on the color transparency argument [21], and also supported by further renormalization group analysis of soft gluon exchanges among initial and final state mesons [25].

At present, the pQCD predictions still have large theoretical errors (say $\sim 50\%$) induced by the large uncertainties of many input parameters. In our analysis, we considered the constraints on these parameters from analysis of other well measured decay channels. For example, the constraint $1.1 \text{ GeV} \leq m_0^\pi \leq 1.9 \text{ GeV}$ was obtained from the...
FIG. 3: The same as Fig. 2 but for $B^0 \rightarrow \eta \eta'$ decay.

FIG. 4: The same as Fig. 2 but for $B^0 \rightarrow \eta' \eta'$ decay.

phenomenological studies for $B \rightarrow \pi \pi$ decays, while the constraint of $\alpha = \left(93^{+11}_{-9}\right)^\circ$ are obtained from the direct measurements. In estimating the uncertainties, we still conservatively consider the range of $\alpha = 100^\circ \pm 20^\circ$.

In Figs. 2, 3 and 4 we show the parameter dependence of the pQCD predictions for the branching ratios of $B \rightarrow \eta \eta$, $\eta \eta'$ and $\eta' \eta'$ decays for $\omega_b = 0.4 \pm 0.04$ GeV, $m_0^\pi = 1.4 \pm 0.1$ GeV and $\alpha = [0^\circ, 180^\circ]$. From the numerical results, we observe that the pQCD predictions are sensitive to the variations of $\omega_b$, $m_0^\pi$ and $\alpha$. 
B. Branching ratios with updated wave functions

In pQCD approach, the only input matters are wave functions of the involved particles, which stand for nonperturbative contributions. Since currently available wave functions are not exactly determined, large error of the theoretical predictions will be produced from the uncertainty of the relevant $B$, $\eta$ and $\eta'$ wave functions.\footnote{This constitutes a large theoretical uncertainty common to the pQCD, QCDF and other factorization approaches where the meson wave functions are used as input.}

In last subsection, we choose the pion DAs derived from QCD sum rule as given in Ref. [35]. Although the structure of Refs. [34, 38] with a fixed decay constant $f_B = 190$ MeV (see Appendix B for more details). The resultant pQCD predictions for CP averaged branching ratios are consistent with the measured values, which may be regarded as an indication that above inputs are reasonable.

In this subsection, in order to check the theoretical uncertainty induced by the variation of wave functions, we recalculate the CP averaged branching ratios by employing the updated models of the pion DAs as given in Ref. [35]. Although the structure of $\phi^{A,P,T}_\pi$ as given in Appendix B remain unchanged, but the Gegenbauer moments $a_2^\pi$ and $a_4^\pi$ in the updated pion DAs are changed significantly, they are now

\[
(a_2^\pi, a_4^\pi) = (0.115, -0.015),
\]

instead of the old (0.44, 0.25) as shown in Eq. (B5).

Using the updated pion DAs with $(a_2^\pi, a_4^\pi) = (0.115, -0.015)$ and $f_B = 210$ MeV\footnote{Ref. [39]}, we find numerically that

(i) The values of the form factors as given in Eq. (50) will be decreased by about 10%, and we now have

\[
F_{0,1}^{B \rightarrow \eta}(q^2 = 0) = F_{0,1}^{B \rightarrow \eta'}(q^2 = 0) = 0.27_{-0.03}^{+0.05}(\omega_b),
\]

for $\omega_b = 0.40 \pm 0.04$ GeV.

(ii) The CP averaged branching ratios of the considered decays are

\[
Br(B^0 \rightarrow \eta\eta) = \left[0.67_{-0.13}^{+0.14}(\omega_b) + 0.08(m_0^\pi) + 0.12(a_2^\pi)\right] \times 10^{-7},
\]

\[
Br(B^0 \rightarrow \eta\eta') = \left[0.18_{-0.03}^{+0.05}(\omega_b) \pm 0.01(m_0^\pi) \pm 0.01(\alpha)^{+0.10}_{-0.08}(a_2^\pi)\right] \times 10^{-7},
\]

\[
Br(B^0 \rightarrow \eta'\eta') = \left[0.11_{-0.01}^{+0.04}(\omega_b) \pm 0.01(m_0^\pi) \pm 0.04(\alpha)^{+0.10}_{-0.07}(a_2^\pi)\right] \times 10^{-7},
\]

where the main errors are induced by the uncertainties of $\omega_b = 0.4 \pm 0.04$ GeV, $m_0^\pi = 1.4 \pm 0.1$ GeV, $\alpha = 100^\circ \pm 20^\circ$, $a_2^\pi = 0.115 \pm 0.115$, respectively. The errors induced by varying $a_4^\pi$ in the range of $a_4^\pi = -0.015 \pm 0.015$ are very small: only about $\pm 0.03, \pm 0.004$ and $\pm 0.01$ (in unit of $10^{-7}$) for $B \rightarrow \eta\eta, \eta\eta'$ and $\eta'\eta'$ decay, respectively. We here have assumed that the updated Gegenbauer moments $a_2^\pi$ and $a_4^\pi$ can vary by 100%.
FIG. 5: \( a_2^\pi \)- and \( a_4^\pi \)-dependence of the CP averaged branching ratio (in units of \( 10^{-7} \)) \( Br(B \to \eta\eta) \) for \( 0 \leq a_2^\pi \leq 0.50 \) (a) and \( -0.10 \leq a_4^\pi \leq 0.30 \) (b).

It is easy to see that the pQCD predictions in Eqs.(59-61) is smaller than those in Eqs.(45-47) by roughly a factor of 2 – 5. It is not difficult to understand such situation: the large decrease is induced by the great changes of the updated Gegenbauer moments in the wave functions. The updated Gegenbauer moment \( a_2^\pi = 0.115 \) is indeed much smaller than the previous one \( 0.44 \) for the leading twist-2 distribution amplitude, a factor of four decrease in magnitude. For Gegenbauer moment \( a_4^\pi \), which governs the high order contributions in pion DA, it changes from \( a_4^\pi = 0.25 \) to \( a_4^\pi = -0.015 \), even the sign is changed yet. The form factor \( F_{B \to \eta\eta} \) then reduces from 0.30 to 0.27, which leads to a smaller branching ratio. Besides the effect of a smaller form factor \( F_{B \to \eta\eta} \) which were extracted from the Fig. 1(a) and 1(b), the contributions to the branching ratios from the remaining Feynman diagrams 1(c)-1(h) also become smaller than before due to the significant variations of \( a_2^\pi \) and \( a_4^\pi \). The total decrease of the branching ratios therefore becomes significant. This fact tell us that current theoretical predictions still have a strong dependence on the form of meson wave functions.

We now take \( B \to \eta\eta \) decay as an example to show the \( a_2^\pi \)- and \( a_4^\pi \)-dependence of the theoretical predictions explicitly. Firstly, if we vary only the value of Gegenbauer moment \( a_2^\pi \) in the range of \( 0 \leq a_2^\pi \leq 0.50 \), the pQCD prediction will be

\[
0.55 \times 10^{-7} \leq Br(B \to \eta\eta) \leq 1.23 \times 10^{-7}. \tag{62}
\]

This can be seen more clearly in Fig. 5b, where the CP averaged branching ratio \( Br(B \to \eta\eta) \) shows a linear dependence on \( a_2^\pi \). For the parameter \( a_4^\pi \), we also find the similar linear dependence:

\[
0.53 \times 10^{-7} \leq Br(B \to \eta\eta) \leq 1.40 \times 10^{-7} \tag{63}
\]

for \( -0.10 \leq a_4^\pi \leq 0.30 \).
Although the pQCD predictions for the branching ratios obtained by employing the previous or updated pion DAs are all consistent with the measured values due to still large theoretical and experimental errors, the pQCD predictions in Eqs. (59–61) are more reliable, in our opinion, since the updated Gegenbauer moments as given in Ref. [35] are used to obtain these results. Of course, better wave functions of particles involved are clearly needed to reduce the errors of theoretical predictions.

In next section, we will use the updated Gegenbauer moments \((a_\pi^2, a_\pi^4) = (0.115, -0.015)\) to calculate the CP-violating asymmetries for the three considered decay modes.

C. CP-violating asymmetries

Now we turn to the evaluations of the CP-violating asymmetries of \(B \to \eta^{(t)}\eta^{(t)}\) decays in pQCD approach. We here use the wave functions as presented in Appendix B, but with the updated Gegenbauer moments \((a_\pi^2, a_\pi^4) = (0.115, -0.015)\). Because these decays are neutral B meson decays, so we should consider the effects of \(B^0 - \bar{B}^0\) mixing. For \(B^0\) meson decays into a CP eigenstate \(f\), the time-dependent CP-violating asymmetry can be defined as

\[
\frac{Br(B^0(t) \to f) - Br(\bar{B}^0(t) \to f)}{Br(B^0(t) \to f) + Br(\bar{B}^0(t) \to f)} = A_{dir}^{CP} \cos(\Delta m t) + A_{mix}^{CP} \sin(\Delta m t),
\]

where \(\Delta m\) is the mass difference between the two \(B^0\) mass eigenstates, \(t = t_{CP} - t_{tag}\) is the time difference between the tagged \(B^0\) (\(\bar{B}^0\)) and the accompanying \(\bar{B}^0\) (\(B^0\)) with opposite b flavor decaying to the final CP-eigenstate \(f_{CP}\) at the time \(t_{CP}\). The direct and mixing induced CP-violating asymmetries \(A_{dir}^{CP}\) and \(A_{mix}^{CP}\) can be written as

\[
A_{dir}^{CP} = \frac{|\lambda_{CP}|^2 - 1}{1 + |\lambda_{CP}|^2}, \quad A_{mix}^{CP} = \frac{2\text{Im}(\lambda_{CP})}{1 + |\lambda_{CP}|^2},
\]

where the CP-violating parameter \(\lambda_{CP}\) is

\[
\lambda_{CP} = \frac{V_{tb}V_{td}\langle f|H_{eff}|B^0\rangle}{V_{tb}V_{td}^*\langle f|H_{eff}|\bar{B}^0\rangle} = e^{2ia}\frac{1 + ze^{i(\delta - \alpha)}}{1 + ze^{i(\delta + \alpha)}}.
\]

Here the ratio \(z\) and the strong phase \(\delta\) have been defined previously. In pQCD approach, since both \(z\) and \(\delta\) are calculable, it is easy to find the numerical values of \(A_{dir}^{CP}\) and \(A_{mix}^{CP}\) for the considered decay processes.

By using the mixing parameters in Eq. (31) and the input parameters as given in Appendix B, one found the pQCD predictions (in units of \(10^{-2}\)) for the direct and mixing
induced CP-violating asymmetries of the considered decays

\[
A_{CP}^{\text{dir}}(B^0 \to \eta \eta) = -33^{+2.6}_{-2.8}(\alpha)^{+4.1}_{-3.8}(\omega_b)^{+3.5}_{-3.0}(m_0^\pi),
\]

\[
A_{CP}^{\text{mix}}(B^0 \to \eta \eta) = +53.5^{+0.0}_{-3.4}(\alpha)^{+3.1}_{-2.7}(\omega_b)^{+2.1}_{-0.1}(m_0^\pi),
\]

\[
A_{CP}^{\text{dir}}(B^0 \to \eta \eta') = +77.4^{+0.0}_{-6.9}(\alpha)^{+6.9}_{-11.2}(\omega_b)^{+8.0}_{-9.0}(m_0^\pi),
\]

\[
A_{CP}^{\text{mix}}(B^0 \to \eta \eta') = -13.1^{+54.7}_{-48.8}(\alpha)^{+9.0}_{-9.9}(\omega_b)^{+10.0}_{-6.2}(m_0^\pi),
\]

\[
A_{CP}^{\text{dir}}(B^0 \to \eta' \eta') = +23.7^{+10.0}_{-6.9}(\alpha)^{+18.5}_{-16.9}(\omega_b)^{+6.0}_{-8.5}(m_0^\pi),
\]

\[
A_{CP}^{\text{mix}}(B^0 \to \eta' \eta') = +93.2^{+4.9}_{-2.4}(\alpha)^{+5.2}_{-11.1}(\omega_b)^{+2.2}_{-2.1}(m_0^\pi),
\]

where the dominant errors come from the variations of \( \alpha = 100^\circ \pm 20^\circ \), \( \omega_b = 0.4 \pm 0.04 \) GeV and \( m_0^\pi = 1.4 \pm 0.1 \) GeV.

In Fig. 6 we show the \( \alpha \)-dependence of the pQCD predictions for the direct and the mixing-induced CP-violating asymmetries for \( B^0 \to \eta \eta \) (dotted curve), \( B^0 \to \eta \eta' \) (solid curve) and \( B^0 \to \eta' \eta' \) (dashed curve) decay, respectively.

As a comparison, we present the theoretical predictions for \( A_{CP}^{\text{dir}}(B^0 \to \eta^{(0)} \eta^{(0)}) \) (in units of \( 10^{-2} \)) in both the QCDF approach \[5\] \(^4\) and the SCET approach \[17\]

\[
A_{CP}^{\text{dir}}(B^0 \to \eta \eta) = \begin{cases} +63^{+32}_{-74} & \text{QCDF}, \\ +48 \pm 32 & \text{SCET}, \end{cases}
\]

\[
A_{CP}^{\text{dir}}(B^0 \to \eta \eta') = \begin{cases} +56^{+32}_{-144} & \text{QCDF}, \\ +70 \pm 24 & \text{SCET}, \end{cases}
\]

\[
A_{CP}^{\text{dir}}(B^0 \to \eta' \eta') = \begin{cases} +46^{+43}_{-147} & \text{QCDF}, \\ +60 \pm 38 & \text{SCET}, \end{cases}
\]

where the individual errors as given in Refs. \[5\] and \[17\] have been added in quadrature. From above numerical results one can see that

(i) In the considered three kinds of factorization approaches, the theoretical predictions for the CP violating asymmetries are generally large in magnitude, and consistent with each other if one takes the very large theoretical uncertainty into account.

(ii) For \( B \to \eta \eta \) decay, the pQCD prediction for \( A_{CP}^{\text{dir}} \) has an opposite sign with those given in QCDF and SCET approach, which may be tested in the future B experiments.

If we integrate the time variable \( t \), we will get the total CP asymmetry for \( B^0 \to \eta^{(0)} \eta^{(0)} \) decays,

\[
A_{CP} = \frac{1}{1 + x^2} A_{CP}^{\text{dir}} + \frac{x}{1 + x^2} A_{CP}^{\text{mix}},
\]

\(^4\) There is a sign difference between the term \( A_{CP}^{\text{dir}} \) defined here and the term \( C_f \) defined in Ref. \[5\]: i.e., \( A_{CP}^{\text{dir}} = -C_f \).
FIG. 6: The direct and mixing-induced CP asymmetry (in percentage) of $B^0 \rightarrow \eta\eta$ (dotted curve), $\eta\eta'$ (solid curve) and $\eta\eta'$ (dashed curve) decay as a function of CKM angle $\alpha$.

where $x = \Delta m / \Gamma = 0.771$ for the $B^0 - \overline{B}^0$ mixing [40]. Numerically, we found (in units of $10^{-2}$) that

$$\mathcal{A}_{CP}^{tot}(B^0 \rightarrow \eta\eta) = 5.2^{+2.3}_{-3.4}(\alpha)^{+4.0}_{-3.7}(\omega_b)^{+3.2}_{-0.0}(m_0^\pi), \quad (74)$$
$$\mathcal{A}_{CP}^{tot}(B^0 \rightarrow \eta\eta') = 42.2^{+24.0}_{-27.1}(\alpha)^{+8.7}_{-11.9}(\omega_b)^{+2.1}_{-0.8}(m_0^\pi), \quad (75)$$
$$\mathcal{A}_{CP}^{tot}(B^0 \rightarrow \eta'\eta') = 59.9^{+0.0}_{-3.5}(\alpha)^{+6.3}_{-8.0}(\omega_b)^{+2.8}_{-4.2}(m_0^\pi). \quad (76)$$

D. Effects of possible gluonic component of $\eta'$

Up to now, we have not considered the possible contributions to the branching ratios and CP-violating asymmetries of $B \rightarrow \eta^{(0)}\eta^{(0)}$ decays induced by the possible gluonic component of $\eta'$ [13, 14, 41]. When a non-zero gluonic component exist in $\eta'$ meson, an additional decay amplitude $\mathcal{M}'$ will be produced. Such decay amplitude may interfere constructively or destructively with the ones from the $q\bar{q}$ ($q = u, d, s$) components of $\eta'$, the branching ratios of the decays in question may be increased or decreased accordingly.

In Ref. [32], Beneke and Neubert computed the leading two-gluon contribution to the $B \rightarrow \eta^{(0)}$ form factors using the framework of QCD factorization. In Ref. [13], Kou examined the gluonic component of $\eta'$ and the contributions to the process $gg \rightarrow \eta'$. In his paper [13], the $\eta$ and $\eta'$ meson were written as

$$|\eta > = X_\eta|\eta_q > + Y_\eta|\eta_s >,$$
$$|\eta' > = X_{\eta'}|\eta_q > + Y_{\eta'}|\eta_s > + Z_{\eta'}|gluonium >, \quad (77)$$

where $\eta_q = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $\eta_s = s\bar{s}$. From the experimental data on the radiative light meson decays, such as $\phi \rightarrow \eta'\gamma, \eta' \rightarrow (\phi, \rho, \gamma)$ and $J/\psi \rightarrow \eta'\gamma$ decays, the author found that the gluonic component in $\eta'$ should be less than 26%.

In Ref. [33], by employing the pQCD factorization approach, Chang, Kurimoto and Li calculated the flavor-singlet contribution to the $B \rightarrow \eta^{(0)}$ transition form factors induced
by the Feynman diagrams with the two gluons emitted from the light quark of the B meson (see Fig. 1 of Ref. [33]), and they found that this gluonic contribution is negligible ($\sim 5\%$) in the $B \to \eta$ form factors, but can reach $10\% - 40\%$ in the $B \to \eta'$ ones in the whole parameter space. Such enhancement to $B \to \eta'$ transition form factor can be helpful to explain the large branching ratio $Br(B \to K\eta')$.

In order to check the gluonic effects on the decay modes under study, we here follow the same procedure as being used in Ref. [33] to include the possible gluonic contributions to the $B \to \eta^{(l)}$ transition form factors $F_{0,1}^{B\to\eta^{(l)}}$ and in turn to the branching ratios and CP violating asymmetries. We found that the gluonic contributions to $B \to \eta^{(l)}\eta^{(l)}$ decays are rather small:

- The central values of the pQCD predictions for the branching ratios remain basically unchanged (the variation is less than $2\%$) for $B \to \eta\eta$ and $B \to \eta'\eta'$ decay. For $B \to \eta\eta'$ decay, the enhancement is only about $5\%$: the central value is changed from $0.60 \times 10^{-7}$ to $0.63 \times 10^{-8}$.

- As for the CP-violating asymmetries of $B \to \eta^{(l)}\eta^{(l)}$ decays, the possible gluonic corrections are largely canceled in the ratio, and therefore negligible (less than $3\%$).

- The smallness of the gluonic corrections to the branching ratios can be understood as follows: (a) the gluonic correction to $B \to \eta$ transition form factor itself is negligibly small [33]; (b) only the first two diagrams Fig. 1(a) and 1(b) are affected by the gluonic corrections to $B \to \eta'$ form factor, while the contributions from other six diagrams remain unchanged; and (c) the total effects are consequently small.

Although much progress have been achieved in recent years, but frankly speaking, we currently still do not know how to calculate reliably the contribution of the possible gluonic component in $\eta'$ meson. From our previous works, as presented in Refs. [15, 16] where only the dominant contributions from quark contents of $\eta$ and $\eta'$ were taken into account, the pQCD predictions for the branching ratios of $B \to \rho\eta^{(l)}$ and $B \to \pi\eta^{(l)}$ decays also show a very good agreement with currently available data. It seems that large gluonic contributions are unnecessary for these decay modes. For $B \to K\eta'$ decays, on the contrary, the gluonic contribution may play an important role in explaining the so-called $K\eta'$ puzzle [32, 33].

Of course, more theoretical studies about the effects of possible gluonic component in $\eta'$ and better experimental measurements for the relevant decay modes are needed to clarify this point.

V. SUMMARY

In this paper, we calculated the branching ratios and CP-violating asymmetries of $B^0 \to \eta\eta$, $\eta\eta'$ and $\eta'\eta'$ decays at the leading order by using the pQCD factorization approach. Besides the usual factorizable diagrams, the non-factorizable and annihilation diagrams are also calculated analytically in the pQCD approach. Furthermore, the annihilation diagrams provide the necessary strong phase required by a non-zero CP-violating asymmetry for the considered decays.

From our calculations and phenomenological analysis, we found the following results:
• The pQCD predictions for the form factors of $B \to \eta$ and $\eta'$ transitions agree well with those obtained in QCD sum rule calculations [34, 35].

• Using the two mixing angle scheme and the updated Gegenbauer moments $a_2^\pi$ and $a_4^\pi$, the pQCD predictions for the CP-averaged branching ratios are

$$Br(B^0 \to \eta \eta) = (0.67^{+0.32}_{-0.25}) \times 10^{-7},$$

(78)

$$Br(B^0 \to \eta' \eta') = (0.18 \pm 0.11) \times 10^{-7},$$

(79)

$$Br(B^0 \to \eta' \eta') = (0.11^{+0.12}_{-0.09}) \times 10^{-7},$$

(80)

where the various errors as given in Eqs. [59, 61] have been added in quadrature. The leading pQCD predictions are consistent with currently available data, but both the theoretical and experimental errors are still large.

• For the CP-violating asymmetries of the considered three decay modes, the pQCD predictions are generally large in magnitude, and have large theoretical uncertainty.

Acknowledgments

We are very grateful to Cai-Dian Lü, Ying Li, Xin Liu and Huisheng Wang for helpful discussions. This work is partly supported by the National Natural Science Foundation of China under Grant No. 10275035, 10575052, and by the Specialized Research Fund for the doctoral Program of higher education (SRFDP) under Grant No. 20050319008.

APPENDIX A: RELATED FUNCTIONS

We show here the function $h_i$’s, coming from the Fourier transformations of the function $H^{(0)}$,

$$h_e(x_1, x_3, b_1, b_3) = K_0(\sqrt{x_1 x_3} m_B b_1) \left[ \theta(b_1 - b_3) K_0(\sqrt{x_3 m_B b_1}) I_0(\sqrt{x_3 m_B b_3}) \right. + \left. \theta(b_3 - b_1) K_0(\sqrt{x_3 m_B b_3}) I_0(\sqrt{x_3 m_B b_1}) \right] S_1(x_3),$$

(A1)

$$h_a(x_2, x_3, b_2, b_3) = K_0(i\sqrt{x_2 x_3} m_B b_2) \left[ \theta(b_3 - b_2) K_0(i\sqrt{x_3 m_B b_2}) I_0(\sqrt{x_3 m_B b_3}) \right. + \left. \theta(b_2 - b_3) K_0(i\sqrt{x_3 m_B b_3}) I_0(\sqrt{x_3 m_B b_2}) \right] S_1(x_3),$$

(A2)

$$h_f(x_1, x_2, x_3, b_1, b_2) = \left\{ \theta(b_2 - b_1) I_0(M_B \sqrt{x_1 x_3} b_1) K_0(M_B \sqrt{x_1 x_3} b_2) \right. + (b_1 \leftrightarrow b_2) \right\} \cdot \left( \frac{K_0(M_B F_{(1)} b_2)}{\frac{\pi}{2} H_0^{(1)}(M_B \sqrt{|F_{(1)}|^2} b_2)}, \quad \text{for } F_{(1)}^2 > 0 \right),$$

(A3)

$$h_f^2(x_1, x_2, x_3, b_1, b_2) = \left\{ \theta(b_1 - b_2) K_0(i\sqrt{x_2 x_3} b_1 M_B) I_0(i\sqrt{x_2 x_3} b_2 M_B) + (b_1 \leftrightarrow b_2) \right\}$$

$$\cdot K_0(\sqrt{x_1 + x_2 + x_3 - x_1 x_3 - x_2 x_3} b_1 M_B),$$

(A4)
\[ h_{ij}^4(x_1, x_2, x_3, b_1, b_2) = \left\{ \theta(b_1 - b_2)K_0(i\sqrt{x_2x_3b_1M_B}l_0(i\sqrt{x_2x_3b_2M_B}) \right. \\
+ (b_1 \leftrightarrow b_2) \left. \right\} \cdot \left( \frac{K_0(M_BF_2(2)b_1)}{\pi \sqrt{[F_2(2)|b_1]}} \right) \right. \\
\left. \text{for } F_2^2 > 0 \right) , (A5) \]

where \( J_0 \) is the Bessel function and \( K_0, I_0 \) are modified Bessel functions \( K_0(-ix) = -(\pi/2)Y_0(x) + i(\pi/2)J_0(x) \), and \( F_{(j)}'s \) are defined by

\[ F_{(1)}^2 = (x_1 - x_2)x_3 , \]  
\[ F_{(2)}^2 = (x_1 - x_2)x_3 . \]  

The threshold resummation form factor \( S_t(x_i) \) is adopted from Ref. [28]

\[ S_t(x) = \frac{2^{1+2c}\Gamma(3/2 + c)}{\sqrt{\pi}\Gamma(1 + c)} [x(1-x)]^c , \]  

where the parameter \( c = 0.3 \). This function is normalized to unity.

The Sudakov factors used in the text are defined as

\[ S_{\alpha\beta}(t) = s \left( x_1m_B/\sqrt{2}, b_1 \right) + s \left( x_3m_B/\sqrt{2}, b_3 \right) + s \left( (1-x_3)m_B/\sqrt{2}, b_3 \right) - \frac{1}{\beta_1} \left[ \ln \frac{\ln(t/\Lambda)}{-\ln(b_1\Lambda)} + \ln \frac{\ln(t/\Lambda)}{-\ln(b_2\Lambda)} \right] , \]  
\[ S_{\alpha\delta}(t) = s \left( x_1m_B/\sqrt{2}, b_1 \right) + s \left( x_2m_B/\sqrt{2}, b_2 \right) + s \left( (1-x_2)m_B/\sqrt{2}, b_2 \right) + s \left( (1-x_3)m_B/\sqrt{2}, b_1 \right) - \frac{1}{\beta_1} \left[ 2\ln \frac{\ln(t/\Lambda)}{-\ln(b_1\Lambda)} + \ln \frac{\ln(t/\Lambda)}{-\ln(b_2\Lambda)} \right] , \]  
\[ S_{\epsilon\gamma}(t) = s \left( x_1m_B/\sqrt{2}, b_1 \right) + s \left( x_2m_B/\sqrt{2}, b_2 \right) + s \left( (1-x_2)m_B/\sqrt{2}, b_2 \right) + s \left( (1-x_3)m_B/\sqrt{2}, b_2 \right) - \frac{1}{\beta_1} \left[ \ln \frac{\ln(t/\Lambda)}{-\ln(b_1\Lambda)} + 2\ln \frac{\ln(t/\Lambda)}{-\ln(b_2\Lambda)} \right] , \]  
\[ S_{gh}(t) = s \left( x_2m_B/\sqrt{2}, b_2 \right) + s \left( x_3m_B/\sqrt{2}, b_3 \right) + s \left( (1-x_2)m_B/\sqrt{2}, b_2 \right) + s \left( (1-x_3)m_B/\sqrt{2}, b_3 \right) - \frac{1}{\beta_1} \left[ \ln \frac{\ln(t/\Lambda)}{-\ln(b_1\Lambda)} + \ln \frac{\ln(t/\Lambda)}{-\ln(b_2\Lambda)} \right] \]  

where the function \( s(q, b) \) are defined in the Appendix A of Ref. [8]. The scale \( t_i 's \) in the
above equations are chosen as
\[
\begin{align*}
t_1^e &= \max(\sqrt{x_3} m_B, 1/b_1, 1/b_3), \\
t_2^e &= \max(\sqrt{x_1} m_B, 1/b_1, 1/b_3), \\
t_3^e &= \max(\sqrt{x_3} m_B, 1/b_2, 1/b_3), \\
t_4^e &= \max(\sqrt{x_2} m_B, 1/b_2, 1/b_3), \\
t_f^e &= \max(\sqrt{x_1 x_3} m_B, \sqrt{(x_1 - x_2) x_3} m_B, 1/b_1, 1/b_2), \\
t_3^f &= \max(\sqrt{x_1 + x_2 + x_3 - x_1 x_3 - x_2 x_3} m_B, \sqrt{x_2 x_3} m_B, 1/b_1, 1/b_2), \\
t_4^f &= \max(\sqrt{x_2 x_3} m_B, \sqrt{(x_1 - x_2) x_3} m_B, 1/b_1, 1/b_2).
\end{align*}
\] (A13)

They are chosen as the maximum energy scale appearing in each diagram to kill the large logarithmic radiative corrections.

**APPENDIX B: INPUT PARAMETERS AND WAVE FUNCTIONS**

In this Appendix we show the input parameters and the light meson wave functions to be used in the numerical calculations.

The masses, decay constants, QCD scale and $B^0$ meson lifetime are

\[
\begin{align*}
\Lambda_{\text{MS}}^{(f=4)} &= 250 \text{MeV}, \quad f_\pi = 130 \text{MeV}, \quad f_B = 190 \text{MeV}, \\
m_0^{\eta d\bar{d}} &= 1.4 \text{GeV}, \quad m_0^{\eta s\bar{s}} = 2.4 \text{GeV}, \quad m_\pi = 140 \text{MeV}, \quad f_K = 160 \text{MeV}, \\
M_B &= 5.2792 \text{GeV}, \quad M_W = 80.41 \text{GeV}, \quad \tau_{B^0} = 1.54 \times 10^{-12} \text{s}.
\end{align*}
\] (B1)

The central values of the CKM matrix elements as given in Ref. [40] are

\[
\begin{align*}
|V_{ud}| &= 0.9745, \quad |V_{ub}| = 0.0040, \\
|V_{tb}| &= 0.9990, \quad |V_{td}| = 0.0075.
\end{align*}
\] (B2)

For the $B$ meson wave function, we adopt the model

\[
\phi_B(x, b) = N_B x^2 (1 - x)^2 \exp \left[ - \frac{M_B^2}{2 \omega_b^2} - \frac{1}{2} (\omega_b b)^2 \right],
\] (B3)

where $\omega_b$ is a free parameter and we take $\omega_b = 0.4 \pm 0.04$ GeV in numerical calculations, and $N_B = 91.745$ is the normalization factor for $\omega_b = 0.4$. This is the same wave functions as being used in Refs. [8, 9, 28], which is a best fit for most of the measured hadronic B decays.

For the distribution amplitudes $\phi_{\eta d}^A, \phi_{\eta d}^P$ and $\phi_{\eta d}^T$ appeared in Eq. (14), we utilize the
result from the light-cone sum rule \[34, 42\] including twist-3 contribution:

\[
\phi_{\eta d}^A(x) = \frac{3}{\sqrt{2}N_c} f_x x(1 - x) \left\{ 1 + a_2^{\eta d} \frac{3}{2} [5(1 - 2x)^2 - 1] + a_4^{\eta d} \frac{15}{8} [21(1 - 2x)^4 - 14(1 - 2x)^2 + 1] \right\},
\]

\[
\phi_{\eta d}^P(x) = \frac{1}{2\sqrt{2}N_c} f_x \left\{ 1 + \frac{1}{2} \left( 30\eta_3 - \frac{5}{2} \rho_{\eta d}^2 \right) [3(1 - 2x)^2 - 1] + \frac{1}{8} \left( -3\eta_3 \omega_3 - \frac{27}{20} \rho_{\eta d}^2 - \frac{81}{10} \rho_{\eta d} a_2^{\eta d} \right) [35(1 - 2x)^4 - 30(1 - 2x)^2 + 3] \right\},
\]

\[
\phi_{\eta d}^T(x) = \frac{3}{\sqrt{2}N_c} f_x (1 - 2x) \cdot \left[ \frac{1}{6} + (5\eta_3 - \frac{1}{2} \eta_3 \omega_3 - \frac{7}{20} \rho_{\eta d}^2 - \frac{3}{5} \rho_{\eta d} a_2^{\eta d} ) (10x^2 - 10x + 1) \right],
\]  

\[B4\] with \[34, 38\]

\[
a_2^{\eta d} = a_2^\eta = 0.44, \quad a_4^{\eta d} = a_4^\eta = 0.25,
\]

\[
\rho_{\eta d} = m_\pi/m_0^{\eta d}, \quad \eta_3 = 0.015, \quad \omega_3 = -3.0. \quad \[B5\]

The updated Gegenbauer moments as given in Ref. \[35\] are

\[
a_2^\eta = 0.115, \quad a_4^\eta = -0.015. \quad \[B6\]

We also assume that the wave function of \(u\bar{u}\) is the same as the wave function of \(d\bar{d}\) \[14\]. For the wave function of the \(s\bar{s}\) components, we also use the same form as \(d\bar{d}\) but with \(m_{0s}\) and \(f_y\) instead of \(m_{0d}\) and \(f_x\), respectively. For \(f_x\) and \(f_y\), we use the values as given in Ref. \[38\] where isospin symmetry is assumed for \(f_x\) and \(SU(3)\) breaking effect is included for \(f_y\):

\[
f_x = f_\pi, \quad f_y = \sqrt{2f_K^2 - f_\pi^2}. \quad \[B7\]

These values are translated to the values in the two mixing angle scheme, which is often used in vacuum saturation approach as:

\[
f_1 = 151\text{MeV}, \quad f_8 = 169\text{MeV}, \quad \theta_1 = -2.4^\circ, \quad \theta_8 = -21.2^\circ. \quad \[B8\]

The parameters \(m_0^i (i = \eta d\bar{d}(u\bar{u}), \eta s\bar{s})\) are defined as:

\[
m_0^{\eta d\bar{d}(u\bar{u})} \equiv m_0^\pi \equiv \frac{m_\pi^2}{(m_u + m_d)}, \quad m_0^{\eta s\bar{s}} \equiv \frac{2M_K^2 - m_\pi^2}{2m_s}. \quad \[B9\]

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