Comments on the quantum correlation and entanglement

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In recent decades it was established that the quantum measurements of physical quantities in space-time points divided by space-like intervals may be correlated. Though such correlation follows from the formulas of quantum mechanics its physics so far remains unclear and there is a number of different and rather contradictory interpretations. They concern particularly the so-called Einstein-Podolsky-Rosen paradox where the momentary action at a distance together with non-local entangled states is used for the interpretation (see, e.g. [1–9]).

We assume that the quantum theory can be formulated as local and look for the consequences of this assumption. Accordingly we try to explain the correlation phenomena in a local way looking for the origin of correlation. To exclude a presupposed correlation of participating quantum particles we consider two independent particle sources and two detectors that are independent as well. We show that the origin of the correlation is the feature that the occupation number of a particle (and other its measurable quantities) is formed by a pair of complex conjugated wave functions with in general arbitrary phases. We consider this point as crucial as it provides interpretation of the observed correlation phenomena that may otherwise look puzzling.

We briefly discuss a special type of noise that is typical for the quantum correlation phenomena.

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We consider two independent sources of quantum particles (bosons or fermions) and two independent detectors to register them. We assume that there is a source $U$ generating particles in the state $u$ and another source $V$ generating particles in the state $v$. Let the wave functions of these states be the eigenfunctions of the time-independent Hamiltonian $H$ so their space shape remains unchanged during the time evolution

$$\psi(r,t) = \exp(-iHt)\psi(r,0) = \exp(-i\omega t)\psi(r,0).$$

We assume also that there are two spatially separated detectors measuring physical quantities $A$ and $B$ and the results of measurements can be recorded and compared. For simplicity we assume that the measurements are performed at the same time $t$.

To begin with, let us suppose that the particles from source $U$ can reach detector $A$ while the particles from source $V$ can reach detector $B$. Then the observable quantities $\bar{A}$ and $\bar{B}$ are given by

$$\bar{A} = \langle u|A|u \rangle F_u, \quad \bar{B} = \langle v|B|v \rangle F_v.$$ (1)

Here $F_u$ and $F_v$ are the state occupancies. They describe the intensities of the sources. Since
the source particles do not interact their occupancies are constant

\[ F(t) = F(0). \]

Further on we will use the Dirac notation and name the wave function \(|\ldots\rangle\) as ket while the complex conjugated wave function \(<\ldots|\) will be named as bra. Let us emphasize that to present a result of measurement of a physical quantity one needs two items, i.e. a bra and a ket. Thus an observed quantum particle is always represented by a pair bra + ket. For brevity we will call such a pair as "wavicle". (This term was sometimes used in the popular literature in order to accentuate the distinction between quantum particles and usual waves). We will say that the sources emit wavicles and the detectors register them. It is important for what follows that the emitted bra and ket have arbitrary initial phases \(\varphi\) but the phase values of the bra and ket of the the same source should be the same (with the opposite signs). Therefore they cancel in the expressions for such quantities as \(\bar{A}\) and \(\bar{B}\) in Eq. (1). The additional phase factors \(e^{i\omega t}\) and \(e^{-i\omega t}\) acquired by the bra and ket during the time evolution are also cancelled.

Let us assume that the sources emit particles one by one and they reach the detectors producing data. According to the quantum mechanics, the measurements give eigenvalues of operators \(A\) and \(B\) with certain probabilities. For instance, the mean value \(\bar{A}\) can be represented as the sum

\[ \langle u|A|u \rangle F_u = F_u \sum_j c_j^*(u)c_j(u)A_j = F_u \sum_j |c_j(u)|^2 A_j \equiv F_u \sum_j A_j w_j(u). \]  

We introduce the functions \(f_j\) as the orthogonal and normalized eigenfunctions of the operator \(A\) with eigenvalues \(A_j\) so that

\[ Af_j = A_j f_j, \quad |u\rangle = \sum_j c_j(u)f_j, \quad w_j(u) = |c_j(u)|^2. \]

The aforementioned initial phases of the bra and ket of the state \(u\) do not influence the probabilities \(w_j(u)\) in Eq. (2).

As the next step we assume that the particles from the sources \(U\) and \(V\) can reach both detectors, \(A\) and \(B\). We know that for a measurement their bra and ket should "meet" in the detector. If they both come from the same source one has the situation of Eq. (2). Then we have for the mean data in each detector the sum of independent contributions from both sources:

\[ \bar{A} = \langle u|A|u \rangle F_u + \langle v|A|v \rangle F_v, \quad \bar{B} = \langle u|B|u \rangle F_u + \langle v|B|v \rangle F_v. \]  

The bra and ket of each source evolve independently according to their equations of motion but neither the time factors \(e^{\pm i\omega t}\) nor the initial phases enter into Eq. (3).

The correlation \(\bar{A}\bar{B}\) according to the detector data of Eq. (3) is:

\[ \bar{A}\bar{B} = [\langle u|A|u \rangle \langle v|B|v \rangle + \langle v|A|v \rangle \langle u|B|u \rangle] F_u F_v. \]
In this expressions a flow of weavicles emitted by the sources looks as a flow of classical point-like particles that hit both detectors with the probability $F_u F_v$. We have a U-particle and a V-particle. One of them hits one detector and another one hits the other. In this case there is no correlation between the detector data. (Of course, the sources themselves may be initially correlated and to describe this phenomenon one should have replaced the product of independent intensities $F_u F_v$ in Eq. (4) by the average product $\overline{F_u F_v}$. This type of correlation is of the classical nature and we do not consider it).

We remind that weavicles emitted by the sources are represented by pairs $bra + ket$. Thus we actually have not two but four objects, i.e. two $bra$ and two $ket$ with two values of initial phases $\pm \varphi_u$ and $\pm \varphi_v$. We understand that a pair of $bra$ and $ket$ is necessary for the performance of a detector device.

A new and interesting situation emerges where the $bra$ and $ket$ come into detectors pairwise from two different sources. In this case the wavicles that hit the detectors are not the wavicles emitted by a single source. To get these new wavicles one should exchange either two $bra$ or two $ket$ of the pairs of previous case (4). It is crucial that such new weavicles entering both detectors should be inevitably correlated since they have equal phases $\phi$ of the opposite sign. Such a phase is the difference between the initial phases and the phases acquired by the $bra$ and $ket$ during the time evolution:

$$\phi = \pm i(\omega_u t - \varphi_u - \omega_v t + \varphi_v).$$

Making the exchange procedure in Eq. (4)

$$|u\rangle \leftrightarrow |v\rangle \text{ and } \langle u| \leftrightarrow \langle v|$$

we get the following correlation contributions

$$(\overline{AB})_{cor} = \pm [\langle u|A|v\rangle \langle v|B|u\rangle + \langle u|B|v\rangle \langle v|A|u\rangle] F_u F_v.$$  

Note that phase $\phi$ does not enter into the final expression for the average $(\overline{AB})_{cor}$. The correlation contribution is negative for fermions and positive for bosons. The total average $\overline{AB}$ is the sum of the uncorrelated and correlated parts:

$$\overline{AB} = [\langle u|A|u\rangle \langle v|B|v\rangle \pm \langle u|A|v\rangle \langle v|B|u\rangle] F_u F_v + (u \leftrightarrow v).$$

Here the upper (lower) sign is for bosons (fermions). Let us note that the correlation contribution reveals itself only after joint averaging of actual measurements of both detectors. On the one hand, if one averages the data of each detector separately one gets Eq. (4) and finds no trace of correlation. On the other hand, the quantity $\overline{AB}$ is not measured directly but is obtained by the multiplication of recorded detector data that are measured independently. It follows that these data should implicitly contain the correlation contributions that vanish after the separate averaging but emerge as a result of the joint averaging.
The new correlated weavicles correspond to the mixed quantum states. Let us consider the measurements in mixed quantum states in more detail and interpret them. The pair of *bra* and *ket* taken from different sources comes into the detector with an arbitrary initial phase difference $\phi$. The contributions from such pairs are always proportional to a phase factor $e^{i\phi}$. Though this factor becomes zero after averaging over $\phi$ the vanishing itself occurs as a result of summation over many measurement events. By analogy with Eq. (2) for an observable quantity in a mixed quantum state with the initial phase difference $\phi$ we have:

$$\text{Re}\langle u|A|v \rangle = \sum_j |c_j^* (u) c_j (v)| A_j \cos[(\alpha_j + \phi)].$$

(8)

Here $\alpha_j$ is an additional phase difference depending on the coefficients $c_j$. The quantities $\langle u|A|v \rangle$ and $\langle v|A|u \rangle = \langle u|A|v \rangle^*$ participate with equal probabilities so that one can consider only their real part.

It is natural to assume that in the mixed quantum state the readings of the detector such as

$$\text{Re}\langle u|A|v \rangle = \sum_j A_j w_j (u,v) \cos \Phi_j$$

(9)

are averaged out to zero because of the random phase. In other words, the measurements in the mixed quantum state give values of $A_j \cos \Phi_j$ (that may be positive or negative) with the probabilities depending on the phase. For a random initial phase $\phi$ the phases $\Phi_j = \alpha_j + \phi$ remain random and $\text{Re}\langle u|A|v \rangle$ vanishes after averaging over series of measurements:

$$\cos \Phi_j = 0.$$

Note that in Eqs. (6) – (7) for the $AB$ correlation there is no phase difference and the pairs from the different sources always give a contribution irrespective of the phase $\Phi_j$. This cancelation of the phases looks like a cancelation of the initial phase $\varphi$ in Eqs. (1) or (2).

Thus the physical picture of the correlation of quantum particles emitted by two independent sources and registered by two detectors looks as follows. The sources produce uncorrelated quantum particles (wavicles). These wavicles produce no correlation in the measuring devices. Using the popular word "entanglement" one can say that they are not entangled. The data obtained from such particles remain the same either after separate or after joint averaging. By separate averaging we mean the independent averaging of the data of each particular detector.

The source weavicle is a pair of *bra*–*ket* with arbitrary phases of the opposite signs. As one can see above the quantum correlation occurs when the uncorrelated wavicles exchange their *bra* or *ket* and become new wavicles mutually correlated due to the same phase difference of the opposite signs. Just these correlated wavicles may be called the entangled quantum particles. They behave in a different way under separate and joint measurements. Their contributions vanish under the separate averaging thus looking as a sort of noise for a detector. However, due to their common phase these contributions can be redeemed by the
joint averaging. The common phase being the real reason of data correlation does not enter into the final expression for the quantity $\overline{AB}$. Thus the wave function phase determines the result of measurements but is absent in the final result. Therefore it may be called the actual hidden variable of correlation.

The random phase noise mentioned above can be measured and analyzed. We believe that predictions concerning properties of such a quantum noise can make a new feature as a part of the theory of quantum entanglement.

Now let us consider some correlation phenomena. For the intensity correlation of two wave sources (known as the Hanbury Brown-Twiss-effect) the operators $A$ and $B$ are the space position operators at points $r_1 - r_2 = R$ and the wave functions are the plane waves $u \Rightarrow p, v \Rightarrow p'$ ($p$ and $p'$ being the wave vectors). Then the correlation is given by

$$AB = [1 \pm \cos (p - p') R] 2 F_p F_{p'}.$$  \hfill (10)

Here the first term corresponds to uncorrelated waveicles creating the homogeneous background while the second term is the interference contribution of the waveicles created by the bra and ket of different sources. Eq. (10) describes also the bunching of bosons and anti-bunching of fermions while the flow of them falls on a detecting screen. We see that for $R \equiv (r_1 - r_2) \to 0$

the fermions "avoid" each other while the probability to find two bosons in one point becomes twice as big as compared to the background value. For the Bohm version of EPR-paradox the spin measurements are relating to two space points as well as to two time moments $t$ and $t'$. The observable spin values are time-independent with the same result as for $t = t'$. Because of importance of this case for the correlation interpretation we consider it in more details.

The operators $A$ and $B$ now represent the spin values measured at given directions. Such an operator is the scalar product $S$ of the spin vector $\sigma$ formed by the Pauli matrices and the unit vector $n$ of a certain direction. In the polar coordinates it is given by

$$S \equiv (n, \sigma) = \sin \theta \cos \varphi \sigma_x + \sin \theta \sin \varphi \sigma_y + \cos \theta \sigma_z.$$  

The source wave functions $|u\rangle \equiv |↑\rangle$ and $|v\rangle \equiv |↓\rangle$ now are the eigenfunctions of $\sigma_z$ with the eigenvalues $s_u = 1$ and $s_v = -1$.

For pure spin states only $\sigma_z$ contributes to the matrix elements of the operator $S$ and we have

$$\langle ↑ | S | ↑ \rangle = \cos \theta \langle ↑ | \sigma_z | ↑ \rangle = \cos \theta,$$

$$\langle ↓ | S | ↓ \rangle = \cos \theta \langle ↓ | \sigma_z | ↓ \rangle = - \cos \theta.$$  \hfill (11)

For the mixed states only $\sigma_x$ and $\sigma_y$ give results so we have:
\begin{align*}
\langle \uparrow | S | \downarrow \rangle &= \cos \varphi \sin \theta \langle \uparrow | \sigma_x | \downarrow \rangle + \sin \varphi \sin \theta \langle \uparrow | \sigma_y | \downarrow \rangle = e^{i\varphi} \sin \theta, \\
\langle \downarrow | S | \uparrow \rangle &= \cos \varphi \sin \theta \langle \downarrow | \sigma_x | \uparrow \rangle + \sin \varphi \sin \theta \langle \downarrow | \sigma_y | \uparrow \rangle = e^{-i\varphi} \sin \theta.
\end{align*}

Let the directions of spin measurements of the detectors \( A \) and \( B \) be given by the unit vectors \( a \) and \( b \). Then using Eqs. (11) and (12) for the matrix elements and following Eq. (3) we get:

\[ \overline{A} = \langle \uparrow | A | \uparrow \rangle F_\uparrow + \langle \downarrow | A | \downarrow \rangle F_\downarrow = \cos \theta_a (F_\uparrow - F_\downarrow), \]
\[ \overline{B} = \langle \uparrow | B | \uparrow \rangle F_\uparrow + \langle \downarrow | B | \downarrow \rangle F_\downarrow = \cos \theta_b (F_\uparrow - F_\downarrow). \]

Note that for \( F_\uparrow = F_\downarrow \) the mean values of \( \overline{A} \) and \( \overline{B} \) are zero irrespective of the measurement directions. For the quantity \( \overline{AB} \) as before we have two contributions. The first one is given by Eq. (11)

\[ (\overline{AB})_{\text{uncor}} = - \cos \theta_a \cos \theta_b 2F_\uparrow F_\downarrow. \]

We see that the product of two spin measurements of polarized spins of both sources enters into the detectors in such a way that each detector measures the spin of one of the sources. The second term corresponds the expression (6) where each detector measures spin in the mixed state created by bra+ket of both sources i.e. the spin of correlated wavicles. We have for this correlated contribution

\[ (\overline{AB})_{\text{cor}} = \mp \cos (\varphi_a - \varphi_b) \sin \theta_a \sin \theta_b 2F_\uparrow F_\downarrow. \]

Finally for the spin correlation we have

\[ \overline{AB} = - \cos \gamma 2F_u F_v \]

where \( \gamma \) is the angle between the measured spin directions \( a \) and \( b \):

\[ \cos \gamma = \cos \theta_a \cos \theta_b + \cos (\varphi_a - \varphi_b) \sin \theta_a \sin \theta_b. \]

We see that the sum of two contributions becomes independent of the direction of the spin vector \( \sigma \) being proportional to the scalar product of the unit vectors \( a \) and \( b \) which are the directions of independent spin observations in two space points in one time moment.

Now that we understand the physical cause of quantum correlation, the dependence (15) does not look puzzling and mysterious even though the measurement directions \( a \) and \( b \) are arbitrary and according to the quantum mechanics the spin values do not exist before the measurements.

For the case where \( \overline{AB} \) is the matrix element \( \langle \Psi | AB | \Psi \rangle \) with the singlet wave function \( \Psi \), namely

\[ \Psi = | \uparrow_a \rangle | \downarrow_b \rangle - | \uparrow_b \rangle | \downarrow_a \rangle \]
we have
\[ \overline{AB} = \langle \Psi | AB | \Psi \rangle, \quad \overline{A} = \langle \Psi | A | \Psi \rangle = 0, \quad \overline{B} = \langle \Psi | B | \Psi \rangle = 0 \]
that corresponds to our result of two fully independent sources under condition
\[ F_\uparrow = F_\downarrow = 1. \]  

We see that the singlet wave function (17) actually describes the flow of oppositely polarized spins of equal intensities. This equality is a unique property of the measurements described by matrix elements with the singlet wave function \( \Psi \).

One should also remember that the correlation is of statistical nature and to observe it a series of measurements is required. The law
\[ \overline{S} = \langle \uparrow | S | \uparrow \rangle F_\uparrow = \cos \theta F_\uparrow \]
for a flow of polarized spins in the \( z \) direction measured at the angle \( \theta \) can be realized as random series of positive and negative units that appear with the probabilities depending of \( \theta \):
\[ \cos \theta = (+1) \cos^2 \theta/2 + (-1) \sin^2 \theta/2 = \cos^2 \theta/2 - \sin^2 \theta/2. \]  
As we see from Eq. (13) for two spin flows of the opposite signs the mean observed value is proportional to the difference of intensities of the flows:
\[ \overline{S} = \langle \uparrow | S | \uparrow \rangle F_\uparrow + \langle \downarrow | A | \downarrow \rangle F_\downarrow = \cos \theta (F_\uparrow - F_\downarrow). \]  
An observation of spins in the singlet states gives the zero value for any measurement direction just because of the parity of intensities \( F_\uparrow = F_\downarrow \) and is not due to the special properties of a singlet state. For \( \theta = \pi/2 \) the detector always gives mean zero irrespective of the flow intensity. In this case there will be equal average number of positive and negative units fixed by the detector. (Note that the expressions (19) and (20) can be used for the probability interpretation of \( \cos \Psi \) in Eq. (9))

In conclusion we again emphasize that observed physical quantities should be connected with pairs \( bra + ket \) together with the explicit introduction of occupation numbers and their phases. We believe that the physical picture of correlation phenomena based on such an approach would not to look puzzling.
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