Modular group representations in combinatorial quantization with non-semisimple Hopf algebras

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Abstract. The aim of this paper is to establish the main properties of the loop algebra and of the handle algebra, which are the building blocks of the graph algebras introduced by Alekseev-Grosse-Schomerus and Buffenoir-Roche in the combinatorial quantization of Chern-Simons theory. Here we assume that the gauge Hopf algebra $H$ is finite-dimensional, factorizable and ribbon, but not necessarily semi-simple. We show how the handle algebra gives rise to a projective representation of the mapping class group of the torus, namely $SL_2(\mathbb{Z})$, on $SLF(H)$ (the space of symmetric linear forms on $H$). This projective representation is shown to be equivalent to that found by Lyubaschenko and Majid. We finally apply these results to the case of the restricted quantum group $H = U_q(\mathfrak{sl}(2))$. We give explicit formulas for the action of $SL_2(\mathbb{Z})$ on a suitable basis of $SLF(U_q(\mathfrak{sl}(2)))$ and determine the structure of the projective representation in this case.

This is the first step towards a general theory for surfaces of arbitrary genus.

1 Introduction

Let $\Sigma_{g,n}$ be a compact oriented surface of genus $g$ with $n$ boundary components, let $\Gamma$ be a ribbon graph embedded on the surface, and let $H$ be a ribbon Hopf algebra. The lattice algebra of $(\Sigma_{g,n}, \Gamma, H)$ is an associative algebra, which is a quantum analogue of the algebra of functions associated to lattice gauge theory on $(\Sigma_{g,n}, \Gamma, G)$, where the role of the gauge group $G$ is now played by the Hopf algebra $H$. The lattice algebra is endowed with an $O(H)$-coaction (where $O(H)$ is the restricted dual of $H$ with its canonical Hopf algebra structure), which is the quantum analogue of gauge transformations, and which turns the lattice algebra into an $O(H)$-comodule-algebra.

These algebras were introduced and studied by Alekseev-Grosse-Schomerus (see [AGS95], [AGS96], [Ale94], [AS96a]) and Buffenoir-Roche (see [BuR95], [BuR96]) in the combinatorial quantization of the Hamiltonian Chern-Simons theory. Also see the more recent works [MW15], [BJ17].

We can choose a canonical graph on $\Sigma_{g,n}$ which has just one vertex and has one ribbon edge corresponding to each homotopy class of curves. The corresponding lattice algebra is called the graph algebra, which will be denoted here $L_{g,n}(H)$. The representation theory of $L_{g,n}(H)$ and of their algebras of $H$-invariants is investigated in [AS96a] when $H$ is finite-dimensional and semi-simple and in [Ale94] when $H$ is the quantum group $U_q(\mathfrak{g})$ for $q$ generic. Moreover, in [Ale94], it is asserted in the case $H = U_q(\mathfrak{g})$ (q generic) that there are isomorphisms:

\[
L_{g,n}(H) \cong L_{0,1}(H)^{\otimes n} \otimes L_{1,0}(H)^{\otimes g},
\]

\[
L_{0,1}(U_q(\mathfrak{g})) \cong U_q(\mathfrak{g}), \quad L_{1,0}(U_q(\mathfrak{g})) \cong \text{Fun}(T^*G_q)
\]
where $\text{Fun}(T^*G_q)$ is a quantized algebra of functions associated to the cotangent bundle $T^*G$ ($G$ being the Lie group associated to $\mathfrak{g}$). Note that in our terminology (definition [AGS95]), $\text{Fun}(T^*G_q)$ is the Heisenberg double of the Hopf algebra $\mathcal{O}(U_q(\mathfrak{g}))$ (restricted dual of $U_q(\mathfrak{g})$). These isomorphisms indicate that $\mathcal{L}_{0,1}(H)$ and $\mathcal{L}_{1,0}(H)$ are building blocks of $\mathcal{L}_{g,n}(H)$ and thus deserve particular interest. Following the terminology of [AS96a], $\mathcal{L}_{0,1}(H)$ is called the loop algebra and $\mathcal{L}_{1,0}(H)$ is called the handle algebra. A summary about the representation theory of $\mathcal{L}_{g,n}(H)$ and $\mathcal{L}_{g,n}^{\text{inv}}(H)$ and about these isomorphisms is also available in [BNR02].

An important feature of $\mathcal{L}_{g,n}(H)$ is that it gives rise to a projective representation of the mapping class group of $\Sigma_{g,n}$ (see [AS96a], [AS96b]).

In the previously cited original papers on combinatorial quantization, it is always assumed that $H$ is either $U_q(\mathfrak{g})$ with $q$ generic or a particular semi-simplified specialization of $U_q(\mathfrak{g})$ with $q$ root of unity. This latter specialization is well-defined in the context of weak quasi-Hopf algebras and the whole construction only involve simple $U_q(\mathfrak{g})$-modules having non-zero quantum dimensions (see [AGS95], [AGS96], [AS96a]).

In our setting, we assume that $H$ is a finite-dimensional, factorizable, ribbon Hopf algebra which is not necessarily semi-simple, the guiding example being $H = \overline{U}_q(\mathfrak{sl}(2))$. We do not semi-simplify $H$, and it turns that the non-semi-simple modules (especially the principal indecomposable modules) play an important role. Moreover, the combinatorial quantization of [AGS95], [AGS96] for $U_q(\mathfrak{sl}(2))$ ($q$ being a root of unity) and the one provided here for $\overline{U}_q(\mathfrak{sl}(2))$ describe different topological field theories. The first is related to topological datas associated to Wess-Zumino-Witten conformal field theory, whereas the second is related to logarithmic conformal field theory.

We now outline the work presented here. Let $H$ be a finite-dimensional, factorizable, ribbon Hopf algebra. The aim of the paper is to make a complete and careful study of the two building blocks $\mathcal{L}_{0,1}(H)$ and $\mathcal{L}_{1,0}(H)$ in this setting, to show how $\mathcal{L}_{1,0}(H)$ gives rise to a projective representation of the mapping class group of the torus on $\text{SLF}(H)$ (the space of symmetric linear forms on $H$), and to explicitly describe the theory for the important example $H = \overline{U}_q(\mathfrak{sl}(2))$. In our constructions we do not use the Clebsch-Gordan maps nor the $S$-matrix, since this objects have nice properties in the semi-simple case only.

Some well-known facts about braided Hopf algebras and the matrix coefficients of their finite-dimensional representations are recalled in section 2. In sections 3 and 4 the definition of the two building blocks $\mathcal{L}_{0,1}(H)$ and $\mathcal{L}_{1,0}(H)$ as well as the $\mathcal{O}(H)$-coaction on them are provided. We carefully prove the isomorphisms of $\mathcal{L}_{0,1}(H)$ with $H$ (Theorem 3.6) and of $\mathcal{L}_{1,0}(H)$ with the Heisenberg double of $\mathcal{O}(H)$ (Theorem 4.7), which implies that $\mathcal{L}_{1,0}(H)$ is (isomorphic to) a matrix algebra. In section 4.3 we define the representation of the algebra of coinvariants, $\mathcal{L}_{1,0}^{\text{inv}}(H)$, on $\text{SLF}(H)$ (Theorem 4.8) which is a key-point for us, and we provide useful technical formulas about it.

In section 5 the Dehn twist presentation of the mapping class group is recalled. Then we explain how to define a projective representation $\rho_{\text{SLF}}$ of $\text{SL}_2(\mathbb{Z})$ on $\text{SLF}(H)$ (Theorem 5.8). As in [AS96a], [Sch98], we will be led to associate a copy in $\mathcal{L}_{1,0}(H)$ of the ribbon element to each Dehn twist. We then show that the relations involved in the presentation of the mapping class group of the torus hold. Note that we clearly distinguish the relations which hold in $\mathcal{L}_{1,0}(H)$ itself (see Proposition 5.3) and those which hold only when they are represented on $\text{SLF}(H)$. The computations rely on several technical results where integrals on $H$ play an important role. We finally recall the Lyubaschenko-Majid projective representation $\rho_{\text{LM}}$ on $\mathcal{Z}(H)$ and show in Theorem 5.12 that $\rho_{\text{SLF}}$ and $\rho_{\text{LM}}$ are equivalent.

The last section is devoted to the example of $\overline{U}_q = \overline{U}_q(\mathfrak{sl}(2))$. All the preliminary facts about $\overline{U}_q$ and the GTA basis, which is a suitable basis of $\text{SLF}(\overline{U}_q)$, are available in [Fai18]. We recall the braided extension of $\overline{U}_q$ to which the $R$-matrix belongs and explain how to define $\mathcal{L}_{0,1}(\overline{U}_q)$ and $\mathcal{L}_{1,0}(\overline{U}_q)$. Then in Theorem 6.1 we give the explicit formulas for the action of $\text{SL}_2(\mathbb{Z})$ on the GTA basis of $\text{SLF}(\overline{U}_q)$. The multiplication formulas in this basis (see [Fai18 Section 5], [GT09]), are the crucial tool to obtain the result. The structure of $\text{SLF}(\overline{U}_q)$ under the action of $\text{SL}_2(\mathbb{Z})$ is determined.
Finally, we will write the dimensional representations. We use the abbreviation PIM for Principal Indecomposable Module. Here we consider only finite-

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Notations. If $A$ is an algebra, $V$ is a finite-dimensional $A$-module and $x \in A$, we denote by $V_x \in \text{End}_C(V)$ the representation of $x$ on the module $V$. More generally, if $X \in A^\otimes n$ and $V_1, \ldots, V_n$ are $A$-modules, we denote by $X_{V_1 \otimes \cdots \otimes V_n}$ the representation of $X$ on $V_1 \otimes \cdots \otimes V_n$. As in \[CR62\], we will use the abbreviation PIM for Principal Indecomposable Module. Here we consider only finite-dimensional representations.

We use integral indices to describe embeddings $A^{\otimes m} \hookrightarrow A^{\otimes n}$ with $m \leq n$. For instance, if we embed $A^{\otimes 3}$ in $A^{\otimes 4}$, then for $w = a \otimes b \otimes c \in A^{\otimes 3}$, we have $w_{123} = a \otimes b \otimes c \otimes 1$, $w_{312} = b \otimes c \otimes a \otimes 1$, $w_{142} = a \otimes c \otimes 1 \otimes b$ and so on. See \[Kas95\, VIII.2] for the precise definition. Note that this notation does not take into account the number of tensorands of the target space. For instance, $w_{123} = a \otimes b \otimes c \otimes 1$ if $A^{\otimes 3} \hookrightarrow A^{\otimes 4}$ but $w_{123} = a \otimes b \otimes c$ if $A^{\otimes 3} \hookrightarrow A^{\otimes 3}$. However, the target space is always clear from the context. We generalize in the obvious way this notation to embeddings $A_1 \otimes \cdots \otimes A_m \hookrightarrow B_1 \otimes \cdots \otimes B_n$ where $m \leq n$ and each $A_i$ is some $B_j$.

We denote by Mat$_n(A) = \text{Mat}_n(C) \otimes A$, the algebra of matrices of size $n$ with coefficients in the algebra $A$. Every $M \in \text{Mat}_n(A)$ is uniquely written as $M = \sum_{i,j} E_{ij} \otimes a_{ij}$, where $E_{ij}$ is the elementary matrix with $1$ at the intersection of the $i$-th row and the $j$-th column and $0$ elsewhere. We define $M'_j = a_{ij}$. If $f : A \rightarrow A$ is a morphism, then we define $f(M) = \sum_{i,j} E_{ij} \otimes f(a_{ij})$, which can also be written $f(M)'_j = f(M'_j)$. Let moreover $N = \sum_{i,j} E_{ij} \otimes b_{ij} \in \text{Mat}_n(C) \otimes A$, and let as above $M_1$ (resp. $N_2$) be the embedding of $M$ (resp. $N$) in $\text{Mat}_n(C) \otimes \text{Mat}_n(C) \otimes A = \text{Mat}_{n^2}(A)$. Then we see that $M_1N_2$ (resp. $N_2M_1$) contains all the possible products of coefficients of $M$ (resp. of $N$) by coefficients of $N$ (resp. of $M$):

$$M_1N_2 = \sum_{i,j,k,\ell} E_{ij} \otimes E_{k\ell} \otimes a_{ij}b_{k\ell}, \quad N_2M_1 = \sum_{i,j,k,\ell} E_{ij} \otimes E_{k\ell} \otimes b_{k\ell}a_{ij}$$

or equivalently:

$$(M_1N_2)_{ij}^{ik} = M'_jN'_k, \quad (N_2M_1)_{ij}^{ik} = N'_kM'_j.$$ 

In particular, $M_1N_2 = N_2M_1$ if and only if the coefficients of $M$ commute with those of $N$. More generally, if the coefficients of a tensor $M$ commute with those of a tensor $N$ and if they are embedded on different tensorands, then their embeddings commute. For instance $M_{135}N_{24} = N_{24}M_{135}$, $M_{145}N_{23} = N_{23}M_{145}$ and so on.

In order to simplify notations, we will use implicit summations. First, we use Einstein’s notation for the computations involving indices: when an index variable appears twice, one time in upper position and one time in lower position, it implicitly means summation over all the values of the index. For instance if $L, M \in \text{Mat}_n(C)^{\otimes 2} \otimes A$ and $N \in \text{Mat}_n(C)^{\otimes 3} \otimes A$, then $\langle L_{32}M_{13}N_{312} \rangle_{bd}^{ac} = L_{bd}^{ac} M_{ki} N_{ikj}^{bd}$.

Second, we use Sweedler’s notation (see \[Kas95\, Not. III.1.6]) without summation sign for the coproducts, that is we write

$$\Delta(x) = x' \otimes x'', \quad (\Delta \otimes \text{id}) \circ \Delta(x) = (\text{id} \otimes \Delta) \circ \Delta(x) = x' \otimes x'' \otimes x'''$$

and so on.

Finally, we will write the $R$-matrix as $R = a \otimes b$, with implicit summation on $i$.

For $q \in \mathbb{C} \setminus \{-1, 0, 1\}$, we define the $q$-integer $[n]$ (with $n \in \mathbb{Z}$) by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$
We will denote $\hat{q} = q - q^{-1}$ to shorten formulas. Observe that if $q$ is a $2p$-root of unity, then $[p] = 0$ and $[p - n] = [n]$.

As usual $I_n$ will denote the identity matrix of size $n$ and $\delta_{s,t}$ is the Kronecker symbol.

## 2 Some basic facts

We refer to [CRC62, Chap. IV and VIII] for background material about representation theory.

### 2.1 Dual of a finite-dimensional algebra

Let $A$ be a finite-dimensional $\mathbb{C}$-algebra. Denote by $A^*$ the dual of $A$, that is the vector space $\text{Hom}_\mathbb{C}(A, \mathbb{C})$. Let $V$ be a $n$-dimensional $A$-module. We define:

$$
V^T : A \to \text{End}_\mathbb{C}(V),
$$

$$
x \mapsto V^x.
$$

If we choose a basis in $V$, then we can express $V^x \in \text{End}_\mathbb{C}(V)$ in this basis, and hence $V^T$ becomes a matrix $V^T \in \text{Mat}_n(A^*)$. An element $V^T_{ij}$ with $1 \leq i, j \leq n$ is then called a matrix coefficient (associated to the representation $V$). This is the point of view used throughout this paper: we think about $V^T$ as a matrix.

Since $A^*$ is finite-dimensional, it is generated as a vector space by the matrix coefficients of the PIMs. Indeed, let $(x_1, \ldots, x_n)$ be a basis of $A$ with $x_1 = 1$, let $(x_1^*, \ldots, x_n^*) \subset A^*$ be the dual basis and let $A^*A$ be the regular representation. It is readily seen that $A^T_{ij}(x_j) = \delta_{i,j}$ and thus $A^T_{1j} = x^j$.

The claim is proved since the PIMs are the direct summands of $A^*A$. Note however that the matrix coefficients of the PIMs do not form a basis of $A^*$ in general. Indeed, even if we fix a family $(P_\alpha)$ representing each isomorphism class of PIMs, it is possible for $P_\alpha$ and $P_\beta$ to have a composition factor $S$ in common. In this case, both $P_\alpha^T$ and $P_\beta^T$ contain $T$ as submatrix. This is what happens for $H = U_q(\mathfrak{sl}(2))$, see e.g. [Fai18, Section 3]. In the semi-simple case this phenomenon does not occur.

An obvious but important relation is functoriality: if $\phi : V \to W$ is an $A$-linear map, then we have

$$
\phi V^T = W^T \phi.
$$

Note in particular that if $z \in A$ is central, then $V^z T = T^V z$.

### 2.2 Braided Hopf algebras, factorizability, ribbon element

For all the definitions and basic results about braided Hopf algebras, we refer to [Kas95, Chap. VIII]. Let $H$ be a braided Hopf algebra with universal $R$-matrix $R$. We will often write $R = a_i \otimes b_i$. Let us recall the main properties of $R$:

$$
(\Delta \otimes \text{id})(R) = R_{13}R_{23}, \quad (\text{id} \otimes \Delta)(R) = R_{12}R_{13}.
$$

(2)

$$
(S \otimes \text{id})(R) = (\text{id} \otimes S^{-1})(R) = R^{-1}, \quad (S \otimes S)(R) = R.
$$

(3)

$$
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
$$

(4)

The relation (2) is called the (quantum) Yang-Baxter equation.

Consider

$$
\Psi : H^* \to H
$$

$$
\beta \mapsto (\beta \otimes \text{id})(RR')
$$

$$
(\Psi(\beta))^x = \delta_{i,j} T_{ij}.
$$
where \( R' = P(R) \), with \( P \) the flip map defined by \( P(x \otimes y) = y \otimes x \). \( \Psi \) is often called the Reshetikhin-Semenov map, or the Drinfeld map. Here we will encounter several variants of the map \( \Psi \), and we reserve the name Reshetikhin-Semenov-Drinfeld for the morphism introduced in Proposition 3.4. We say that \( H \) is factorizable if \( \Psi \) is an isomorphism of vector spaces. By the remarks above, we can restrict \( \beta \) to be a matrix coefficient of some \( T \).

Define \( R^{(+)} = R \), \( R^{(-)} = (R')^{-1} \), and let
\[
\hat{I} L^{(\pm)} = (T \otimes \text{id})(R^{(\pm)}) \in \text{Mat}_{\dim(\mathcal{L})}(H).
\] (5)

We will use the letters \( I, J, \ldots \) for modules over Hopf algebras. Observe that
\[
(T \otimes \text{id})(RR') = \hat{I} \hat{L}^{(-1)}.
\]

Hence, if \( H \) is factorizable, the coefficients of the matrices \( \hat{I} \hat{L}^{(\pm)} \) generate \( H \) as an algebra. These matrices satisfy nice relations which are consequences of (2) and (4):
\[
\begin{align*}
\hat{I}^{(e)} \hat{I}^{(e)} &= \hat{I}^{(e)} \\
\Delta(\hat{I}^{(e)}) &= \hat{I}^{(e)} b \otimes \hat{I}^{(e)} a \\
R^{(e)} \hat{I}^{(e)} &= \hat{L}^{(e)} \hat{L}^{(e)} \\
R^{(e)} \hat{L}^{(e)} &= \hat{L}^{(e)} \hat{L}^{(e)} R^{(e)} \\
R^{(e)} \hat{L}^{(e)} &= \hat{L}^{(e)} \hat{L}^{(e)} R^{(e)}
\end{align*}
\] (6)

Be aware that the expression of the coproduct is not the same as in an usual matrix product, since the order of indices is inverted (compare with (11) below). If the representations \( I \) and \( J \) are fixed and arbitrary, we will simply write these relations as:
\[
L_{12}^{(e)} = L_{21}^{(e)}, \quad R_{12}^{(e)} L_{12}^{(e)} = L_{12}^{(e)} R_{12}^{(e)}, \quad R_{12}^{(e)} L_{12}^{(e)} = L_{12}^{(e)} R_{12}^{(e)}
\]

to alleviate notations (the space 1 (resp. 2) corresponds to the representation \( I \) (resp. \( J \))).

Recall that the Drinfeld element \( u \) and its inverse are:
\[
u = S(b_i) a_i = b_i S^{-1}(a_i) \quad \text{and} \quad u^{-1} = S^{-2}(b_i) a_i = S^{-1}(b_i) S(a_i) = b_i S^2(a_i).
\] (7)

We say that \( v \in H \) is a ribbon element if it is central and it satisfies:
\[
v^2 = u S(u), \quad \Delta(v) = (R'R)^{-1} v \otimes v, \quad S(v) = v.
\] (8)

We also have \( \varepsilon(v) = 1 \). A ribbon element is in general not unique. A ribbon Hopf algebra \((H, R, v)\) is a braided Hopf algebra \((H, R)\) together with a ribbon element \( v \).

We say that \( g \in H \) is a pivotal element if:
\[
\Delta(g) = g \otimes g \quad \text{and} \quad \forall x \in H, \quad S^2(x) = gxg^{-1}.
\] (9)

A pivotal element is in general not unique. But in a ribbon Hopf algebra \((H, R, v)\) there is a canonical choice:
\[
g = uv^{-1}.
\] (10)

We will always take this canonical pivotal element \( g \) in the sequel.
2.3 Dual Hopf algebra $\mathcal{O}(H)$

Let $(H, \cdot, \eta, \Delta, \varepsilon, S)$ be a finite-dimensional Hopf algebra. There exists a canonical Hopf algebra structure on $H^*$, defined for $\psi, \varphi \in H^*$ by:

$$(\psi \cdot \varphi)(x) = (\psi \otimes \varphi)(\Delta(x)), \quad \eta(1) = \varepsilon, \quad \Delta(\psi)(x \otimes y) = \psi(xy), \quad \varepsilon(\psi) = \psi(1), \quad S(\psi) = \psi \circ S.$$  

with $x, y \in H$. When it is endowed with this structure, $H^*$ is called dual Hopf algebra, and denoted $\mathcal{O}(H)$ in the sequel. In terms of matrix coefficients, we have:

$$(T_{12})^{I J} = (\otimes)_{I J}^{C}, \quad \Delta(T_0^I) = I_{\dim(I)}^I \otimes I_{\dim(I)}^I, \quad \varepsilon(T) = I_{\dim(I)}^I, \quad S(T) = I_{\dim(I)}^{-I}.$$  

where $C$ is the trivial representation. By definition of the action on the dual $I^*$, it holds:

$$S(T) = I^T.$$  

Finally, let us mention that if $g$ is a pivotal element in $H$ then by (9):

$$S^2(T) = g^I T g^{-I}.$$  

3 The loop algebra $\mathcal{L}_{0,1}(H)$

Let $H$ be a finite-dimensional, braided and factorizable Hopf algebra. We will not need the ribbon assumption in this section.

3.1 Definition of $\mathcal{L}_{0,1}(H)$ and $\mathcal{O}(H)$-comodule-algebra structure

If $V$ is a vector space, we denote by $T(V)$ the tensor algebra associated to $V$, which by definition is linearly spanned by all the formal products $v_1 \cdots v_n$ (with $n \in \mathbb{N} \setminus \{0\}$ and $v_i \in V$) satisfying the obvious multilinear relations. There is a canonical injection $j : V \to T(V)$. In the case of $V = H^*$, we denote $J = j(T)$.

**Definition 3.1.** The loop algebra $\mathcal{L}_{0,1}(H)$ is the quotient of $T(H^*)$ by the following relations:

$$I^I J_{12} J_{2 I} = M_{12}^I (R^I)_{12} M_2 (R^{-I})_{12}$$

for all finite-dimensional $H$-modules $I, J$.  

6
These relations are called fusion relations. If the two representations $I$ and $J$ are fixed and arbitrary, we will simply write
\[ M_{12} = M_1 R_{21} M_2 R_{21}^{-1}. \] (15)

**Remark 1.** By coassociativity, we have $(I \otimes J) \otimes K = I \otimes (J \otimes K)$. Hence, there are two decompositions of $M$ in this case, namely:
\[
\begin{align*}
\frac{(I \otimes J) \otimes K}{M} &= M_1(R')_{12} M_2(R^{-1})_{12} (R')_{23} (R^{-1})_{13} M_3(R^{-1})_{13} (R^{-1})_{23} \\
\frac{I \otimes J \otimes K}{M} &= M_1(R')_{12} M_2(R')_{13} M_3(R^{-1})_{13} (R^{-1})_{12}
\end{align*}
\]
where we used (2). Applying the Yang-Baxter equation twice, the reader may check that these two expressions are equal. Also recall that we have commutation relations like $M_1(R')_{12} M_2(R')_{13} = M_2(R')_{13} M_1(R')_{12}$ (see the section about notations).

We have a useful analogue of the FRT relations:

**Proposition 3.2.** The following exchange relations hold in $L_{0,1}(H)$:
\[
\frac{I}{R_{12}} M_1(R')_{12} M_2 = M_2 R_{12} M_1(R')_{12}.
\]

Such a relation is called a reflection equation. It can be written in a shortened way if the representations $I$ and $J$ are fixed and arbitrary:
\[
R_{12} M_1 R_{21} M_2 = M_2 R_{12} M_1 R_{21}. \quad \text{(16)}
\]

**Proof:** We have the isomorphism $P_{IJ} : I \otimes J \rightarrow J \otimes I$, so by functoriality: $P_{IJ} M = M P_{IJ}$. This gives:
\[
(P_{IJ})_{12} R_{12} M_1(R')_{12} M_2 = M_2 R_{12} M_1(R')_{12} = (P_{IJ})_{12} M_2 R_{21} M_1(R')_{21} R_{12} = (P_{IJ})_{12} M_2 R_{12} M_1
\]
and we have the result. \(\square\)

An important fact is that $L_{0,1}(H)$ is endowed with a $O(H)$-comodule-algebra structure.

**Proposition 3.3.** Let $i_1 : O(H) \rightarrow O(H) \otimes L_{0,1}(H)$ and $i_2 : L_{0,1}(H) \rightarrow O(H) \otimes L_{0,1}(H)$ be the canonical embeddings. Then the following map defines a structure of (left) $O(H)$-comodule-algebra on $L_{0,1}(H)$
\[
\Omega : L_{0,1}(H) \rightarrow O(H) \otimes L_{0,1}(H), \quad M \mapsto i_1(M) i_2(S(T))^\perp.
\]

If $H$ has a pivotal element $g$, then for every $\Phi \in \text{End}_H(I)$, the element $\text{tr}(g \Phi M)$ is coinvariant.

Before giving the proof, let us precise that:
\[
\Omega(M^i_b) = T^a_i S(T^i_b) \otimes M^i_j.
\]

If we identify $O(H)$ with $i_1(O(H))$ and $L_{0,1}(H)$ with $i_2(L_{0,1}(H))$, then the coaction is simply $\Omega(M) = \frac{I}{T} M S(T)$. We will always use these identifications in the sequel to shorten notations.
It is easy to check the axioms of a coaction. Let us check that $\Omega$ is compatible with the fusion relation. We use the shortened notation explained before:

$$
\Omega(M)_{12} = T_{12} M_{12} S(T_{12}) = T_{12} M_{12} R_{21} M_{2} R_{21}^{-1} S(T)_{2} S(T)_{1} \quad \text{(definition)}
$$

$$
= T_{1} M_{1} R_{21} M_{2} R_{21}^{-1} S(T)_{2} S(T)_{1} \quad \text{(commuting elements in tensor product algebra)}
$$

$$
= T_{1} M_{1} R_{21} M_{2} S(T)_{1} S(T)_{2} R_{21}^{-1} \quad \text{(eq. (13))}
$$

$$
= T_{1} M_{1} S(T)_{1} R_{21} M_{2} S(T)_{2} R_{21}^{-1} \quad \text{(eq. (13))}
$$

$$
= \Omega(M)_{1} R_{21} \Omega(M)_{2} R_{21}^{-1} \quad \text{(definition)}.
$$

The proof of the last claim is just matrix computation using (11), (14) and (13):

$$
\Omega \left( \text{tr}_{(g\Phi M)} \right) = \left( S^{2}(T) g\Phi M S(T) \right)_{a} = S^{2}(T) g\Phi M = S(T) g\Phi M = \text{tr}(g\Phi M).
$$

We denote by $L_{0,1}^{\text{inv}}(H)$ the subalgebra of coinvariants of $L_{0,1}(H)$.

### 3.2 Isomorphism $L_{0,1}(H) \cong H$

Recall that if $V$ is a left $O(H)$-comodule, then we can dualize the coaction in order to turn $V$ into a right $H$-module. Indeed, let $\Omega : V \to O(H) \otimes V$ be the coaction. If $\Omega(v) = \sum_{i} \psi_{i} \otimes v_{i}$, then we define the right action by

$$
\forall h \in H, \quad v \cdot h = \sum_{i} \psi_{i}(h)v_{i}.
$$

It is easy to check that the space of invariants is the space of coinvariants and that if moreover $V$ is a $O(H)$-comodule-algebra, then this formula endows $V$ with a structure of (right) $H$-module-algebra.

In the case of $L_{0,1}(H)$, Proposition 3.3 shows that the right action of $H$ is:

$$
\forall h \in H, \quad \cdot h = h' M S(h'').
$$

Also recall the right adjoint action of $H$ on itself defined by $a \cdot h = S(h')ah''$ with $a, h \in H$, whose invariants are the central elements of $H$.

**Proposition 3.4.** The following map is a morphism of algebras:

$$
\Psi_{0,1} : \quad L_{0,1}(H) \to H
$$

$$
\hat{M} \to \left( \hat{\Phi} \otimes \text{id} \right)(RR') = \hat{L}(\hat{\Phi}) \hat{L}(-1).
$$

*If we endow $H$ with the right adjoint action, then $\Psi_{0,1}$ is a morphism of (right) $H$-modules. Hence, $\Psi_{0,1}$ brings coinvariants to central elements.*

We will call $\Psi_{0,1}$ the Reshetikhin-Semenov-Drinfeld morphism (RSD morphism for short). The difference with the morphism $\Psi$ of section 2.2 is that the source spaces are different.

**Proof:** Using the relations of (6), we check that $\Psi_{0,1}$ preserves the relation of Definition 3.1

$$
\Psi_{0,1}(M)_{1} R_{21} \Psi_{0,1}(M)_{2} R_{21}^{-1} = L_{1}^{(+)} L_{1}^{-1} R_{21} L_{2}^{(+)} L_{2}^{-1} R_{21}^{-1} = L_{1}^{(+)} L_{2}^{(+)} R_{21} L_{1}^{-1} L_{2}^{-1} R_{21}^{-1}
$$

$$
= L_{1}^{(+)} L_{2}^{(+)} L_{2}^{-1} L_{1}^{-1} = L_{12}^{(+)} L_{12}^{-1} = \Psi_{0,1}(M)_{12}.
$$
For the \( H \)-linearity:

\[
\Psi_{0,1}(h'M S(h'')) = (T \otimes \text{id})(h' \otimes 1 \text{RR'} S(h'') \otimes 1)
\]

\[
= (T \otimes \text{id})(h' \otimes 1 \text{RR'} S(h'') \otimes S(h') h''')
\]

\[
= (T \otimes \text{id})(h'S(h'') \otimes S(h') \text{RR'} 1 \otimes h''')
\]

\[
= (T \otimes \text{id})(1 \otimes S(h') \text{RR'} 1 \otimes h'') = S(h') \Psi_{0,1}(M) h''.
\]

We used the basic properties of \( S \) and the fact that \( \Delta^{op} R = R \Delta \), with \( \Delta^{op} = P \circ \Delta \). \( \square \)

Write \( T(H^*) = \bigoplus_{n \in \mathbb{N}} T_n(H^*) \), where \( T_n(H^*) \) is the subspace generated by all the products \( \psi_1 \cdots \psi_n \), with \( \psi_i \in H^* \) for each \( i \).

**Lemma 3.5.** Each element of \( T(H^*) \) is equivalent modulo the fusion relation of \( L_{0,1}(H) \) to an element of \( T_1(H^*) \). It follows that \( \dim(L_{0,1}(H)) \leq \dim(H^*) \).

**Proof:** It suffices to show that the product of two elements of \( T_1(H^*) \) is equivalent to a linear combination of elements of \( T_1(H^*) \), and the result follows by induction. We can restrict to matrix coefficients since they linearly span \( H^* \). The idea is to invert the fusion relation. If we write \( R = a_i \otimes b_i \), then the fusion relation is rewritten as:

\[
\begin{align*}
I \otimes J \quad \gamma_{12}(R^i)_{12} &= \left((a_i)_{12} M_1 M_2 (b_i)_{12}\right) \\
&= \left((a_i)_{12} S_{12}(b_i)_{12}\right)
\end{align*}
\]

Let \( x_j \otimes y_j \in H \otimes H \), then:

\[
x_j a_i \otimes b_i y_j = 1 \otimes 1 \iff S(a_i) S(x_j) \otimes b_i y_j = 1 \otimes 1 \\
\iff x_j \otimes y_j = S^{-1}(a_i) \otimes b_j
\]

It follows that:

\[
\begin{align*}
I \otimes J \quad M_2 &= \gamma_{12} M_1 \left((a_i)_{12} M_1 M_2 (b_i)_{12}\right) \\
&= \left((a_i)_{12} S_{12}(b_i)_{12}\right)
\end{align*}
\]

and this gives the result since \( M_1 M_2 \) contains all the possible products between the coefficients of \( M \) and those of \( M \). \( \square \)

**Theorem 3.6.** Recall we assume that \( H \) is a finite-dimensional factorizable Hopf algebra. Then the RSD morphism \( \Psi_{0,1} \) gives an isomorphism of \( H \)-module-algebras \( L_{0,1}(H) \cong H \). It follows that \( \mathcal{L}_{0,1}(H) \cong \mathbb{Z}(H) \).

**Proof:** Since \( H \) is factorizable, \( \Psi_{0,1} \) is surjective. Hence \( \dim(L_{0,1}(H)) \geq \dim(H) \). But by Lemma 3.5 \( \dim(L_{0,1}(H)) \leq \dim(H^*) = \dim(H) \). Thus \( \dim(L_{0,1}(H)) = \dim(H) \). \( \square \)

Let us point out obvious consequences. First, by comparing the dimensions, we see that the canonical map \( H^* \rightarrow T(H^*) \rightarrow L_{0,1}(H) \) is an isomorphism of vector spaces. Second, this shows that the matrices \( M \) are invertible since \( \text{RR'} \) is invertible. More importantly, this theorem allows us to identify \( L_{0,1}(H) \) with \( H \) via \( \gamma = \frac{I}{\gamma} \gamma^{-1} \), where the matrices \( L_{\pm} \) are defined in (5). We will always work with this identification in the sequel.

We denote \( \text{SLF}(H) \) the space of symmetric linear forms on \( H \):

\[
\text{SLF}(H) = \{ \psi \in H^* \mid \forall x, y \in H, \ \psi(xy) = \psi(yx) \}.
\]
SLF($H$) is obviously a subalgebra of $O(H)$. Consider the following variant of the map $\Psi$ of section 2.2 which will be useful in what follows:

$$
\mathcal{D} : H^* \rightarrow H \\
\psi \mapsto (\psi \otimes \text{id}) (g_1(RR')_{12})
$$

(17)

where $g$ is the pivotal element (10). Since $H$ is factorizable, $\mathcal{D}$ is an isomorphism of vector spaces. A computation similar to that of the proof of Proposition 3.4 shows that $\mathcal{D}$ brings symmetric linear forms to central elements. Moreover, it is not difficult to show that it induces an isomorphism of algebras $\text{SLF}(H) \cong Z(H) = \mathcal{L}^{\text{inv}}_{0,\frac{1}{2}}(H)$.

Every $\psi \in H^*$ can be written as $\psi = \sum_{i,j,l} \lambda^l_{ij} T^j_i$ with $\lambda^l_{ij} \in \mathbb{C}$. In order to avoid the indices, define for each $I$ a matrix $A_I \in \text{Mat}_{\dim(I)}(\mathbb{C})$ by $(A_I)^i_j = \lambda^l_{ij}$. Then $\psi$ can be expressed as:

$$
\psi = \sum_I \text{tr}(A_I^I T).
$$

We write the summation sign because the set of indices is unusual. By applying $\mathcal{D}$, we get the following lemma. Although it is trivial, we will use it several times.

**Lemma 3.7.** Every $x \in \mathcal{L}_{0,1}(H)$ can be expressed as:

$$
x = \sum_I \text{tr}(A_I^I g M)
$$

such that $\mathcal{D}^{-1}(x) = \sum_I \text{tr}(A_I^I T)$. Moreover, if $x \in \mathcal{L}^{\text{inv}}_{0,1}(H)$, then $\mathcal{D}^{-1}(x) \in \text{SLF}(H)$.

**Remark 2.** Let us stress that, due to non-semi-simplicity, this way of writing elements of $\mathcal{L}_{0,1}(H)$ and of $\text{SLF}(H)$ is in general not unique, see the comments in section 2.1.

### 4 The handle algebra $\mathcal{L}_{1,0}(H)$

From now on, we assume that $H$ is a finite-dimensional factorizable ribbon Hopf algebra. Note however that the ribbon assumption is not needed in sections 4.1 and 4.2.

#### 4.1 Definition of $\mathcal{L}_{1,0}(H)$ and $O(H)$-comodule-algebra structure

If $A_1$ and $A_2$ are two algebras, we denote by $A_1 \ast A_2$ their free product. $A_1$ and $A_2$ are subalgebras of $A_1 \ast A_2$, hence there exists two canonical injections $j_1 : A_1 \rightarrow A_1 \ast A_2$, $j_2 : A_2 \rightarrow A_1 \ast A_2$.

Consider the free product $\mathcal{L}_{0,1}(H) \ast \mathcal{L}_{0,1}(H)$, and let $j_1$ (resp. $j_2$) be the injection in the first (resp. second) copy of $\mathcal{L}_{0,1}(H)$. We define $I = j_1(M)$ and $J = j_2(M)$.

**Definition 4.1.** The handle algebra $\mathcal{L}_{1,0}(H)$ is the quotient of $\mathcal{L}_{0,1}(H) \ast \mathcal{L}_{0,1}(H)$ by the following exchange relations:

$$
\begin{aligned}
&I^I J^J R_{12} B_1 (R')_{12} A_2 = A_2 R_{12} B_1 (R^{-1})_{12} \\
&I^I J^J R_{12} B_1 (R')_{12} A_2 = A_2 R_{12} B_1 (R^{-1})_{12}
\end{aligned}
$$

for all finite-dimensional $H$-modules $I, J$.

Like the other relations before, the $\mathcal{L}_{1,0}$-exchange relation can be written more simply as:

$$
R_{12} B_1 R_{21} A_2 = A_2 R_{12} B_1 R_{12}^{-1}. \\
(18)
$$

An important feature of $\mathcal{L}_{1,0}(H)$ is that it is endowed with a $O(H)$-comodule-algebra structure, defined in a similar way to that of $\mathcal{L}_{0,1}(H)$.
Proposition 4.2. Let \( i_1 : \mathcal{O}(H) \to \mathcal{O}(H) \otimes \mathcal{L}_{1,0}(H) \) and \( i_2 : \mathcal{L}_{1,0}(H) \to \mathcal{O}(H) \otimes \mathcal{L}_{1,0}(H) \) be the canonical embeddings. Then the following map defines a structure of (left) \( \mathcal{O}(H) \)-comodule-algebra on \( \mathcal{L}_{1,0}(H) \)

\[
\Omega : \mathcal{L}_{1,0}(H) \to \mathcal{O}(H) \otimes \mathcal{L}_{1,0}(H)
\]

\[
A \mapsto i_1(T)i_2(A)i_1(S(T))
\]

\[
B \mapsto i_1(T)i_2(B)i_1(S(T)).
\]

If \( H \) has a pivotal element \( g \), the elements

\[
\text{tr}_{12} \left( \left( \begin{array}{ccl}
I & J & I & \frac{1}{g_12} \Phi A_1(R)_{12} & B_2 R_{12}
\end{array} \right) \right)
\]

with \( \Phi \in \text{End}_H(I \otimes J) \) and \( \text{tr}_{12} = \text{tr} \otimes \text{tr} \), are coinvariants.

As before, we identify \( \mathcal{O}(H) \) with \( i_1(\mathcal{O}(H)) \) and \( \mathcal{L}_{1,0}(H) \) with \( i_2(\mathcal{L}_{1,0}(H)) \).

Proof: To show that \( \Omega \) is an algebra morphism, we just have to check that \( \Omega \) is compatible with the exchange relation. The proof is similar to that of Proposition 3.3 and is left to the reader. For the claim about coinvariants, we have:

\[
\Omega \left( \text{tr}_{12} \left( \left( \begin{array}{ccl}
I & J & I & \frac{1}{g_12} \Phi A_1(R)_{12} & B_2 R_{12}
\end{array} \right) \right) \right) = \text{tr}_{12} \left( \left( \begin{array}{ccl}
I & J & I & \frac{1}{g_12} T_{12} \Phi A_1(R)_{12} & B_2 R_{12} S(T)_{12}
\end{array} \right) \right) = \text{tr}_{12} \left( \left( \begin{array}{ccl}
I & J & I & \frac{1}{g_12} \Phi A_1(R)_{12} & B_2 R_{12}
\end{array} \right) \right).
\]

For the first equality, use (13), (11) and (1), and for the second take back the computation of the proof of Proposition 3.3 with \( A_1(R)_{12} B_2 R_{12} \) in place of \( M \).

Let \( \mathcal{L}_{1,0}^{\text{inv}}(H) \) be the subalgebra of coinvariants of \( \mathcal{L}_{1,0}(H) \). We now describe a wide family of maps \( \mathcal{L}_{0,1}(H) \to \mathcal{L}_{1,0}(H) \). For \( w \in \mathcal{L}_{0,1}^{\text{inv}}(H) = \mathcal{Z}(H) \) and \( m_1, n_1, \ldots, m_k, n_k \in \mathbb{Z} \), define:

\[
j_{w, A^{m_1} B^{n_1}, \ldots, A^{m_k} B^{n_k}} : \mathcal{L}_{0,1}(H) \to \mathcal{L}_{1,0}(H)
\]

\[
I \mapsto w A^{m_1} B^{n_1} \ldots A^{m_k} B^{n_k}
\]

It is clear that these maps are morphisms of \( \mathcal{O}(H) \)-comodules, but not of algebras in general. Hence the restriction satisfies \( j_{w, A^{m_1} B^{n_1}, \ldots, A^{m_k} B^{n_k}} : \mathcal{L}_{0,1}^{\text{inv}}(H) \to \mathcal{L}_{1,0}^{\text{inv}}(H) \). This gives a particular type of coinvariants in \( \mathcal{L}_{1,0}(H) \). We will more shortly write:

\[
x_{w, A^{m_1} B^{n_1}, \ldots, A^{m_k} B^{n_k}} = j_{w, A^{m_1} B^{n_1}, \ldots, A^{m_k} B^{n_k}}(x).
\]

(19)

Remark 3. Recall from remark 2 that the matrix coefficients do not form a basis of \( \mathcal{L}_{0,1}(H) \). They just linearly span this space. Thus, it is not totally obvious that the maps \( j_{w, A^{m_1} B^{n_1}, \ldots, A^{m_k} B^{n_k}} \) are well-defined since they are defined using matrix coefficients. First,

\[
j_A : \mathcal{L}_{0,1}(H) \xrightarrow{j_1} \mathcal{L}_{0,1}(H) * \mathcal{L}_{0,1}(H) \xrightarrow{\pi} \mathcal{L}_{1,0}(H)
\]

\[
j_B : \mathcal{L}_{0,1}(H) \xrightarrow{j_2} \mathcal{L}_{0,1}(H) * \mathcal{L}_{0,1}(H) \xrightarrow{\pi} \mathcal{L}_{1,0}(H)
\]

are well-defined. Let us show for instance that the map \( j_{A^{-1}, B^{-1}} \) is well-defined. Assume that \( \lambda^a_i T^b_a = 0 \). Applying the coproduct in \( \mathcal{O}(H) \) twice and tensoring with \( \text{id}_H \), we get:

\[
\lambda^a_i T^b_a \otimes \text{id}_H \otimes T^k_i \otimes \text{id}_H \otimes T^l_a \otimes \text{id}_H = 0.
\]
We evaluate this on \((RR')^{-1} \otimes (RR')^{-1} \otimes RR'\):

\[
\lambda_b^a (M^{-1})_k^b \otimes (M^{-1})_l^i \otimes M^j_a = 0.
\]

Finally, we apply the map \(j_A \otimes j_B \otimes j_A\) and multiplication in \(L_{1,0}(H)\):

\[
\lambda_b^a (A^{-1})_k^b (B^{-1})_l^i A^j_a = 0
\]
as desired. A similar proof can be used to show that all the other maps defined by means of matrix coefficients (like \(\Psi_{1,0}\) or \(\alpha, \beta\) below etc.) are well-defined.

### 4.2 Isomorphism \(L_{1,0}(H) \cong \mathcal{H}(\mathcal{O}(H))\)

Let us begin by recalling the following definition (see for instance [Mon93 4.1.10]).

**Definition 4.3.** Let \(H\) be a Hopf algebra. The Heisenberg double of \(\mathcal{O}(H), \mathcal{H}(\mathcal{O}(H))\), is the vector space \(\mathcal{O}(H) \otimes H\) endowed with the algebra structure defined by the following multiplication rules:

- The canonical injections \(H, \mathcal{O}(H) \rightarrow \mathcal{O}(H) \otimes H\) are algebra morphisms. Thus we identify \(\mathcal{O}(H)\) (resp. \(H\)) with \(\mathcal{O}(H) \otimes 1 \subset \mathcal{H}(\mathcal{O}(H))\) (resp. with \(1 \otimes H \subset \mathcal{H}(\mathcal{O}(H))\)).

- Under this identification, we have the exchange relation:

\[
\forall \psi \in \mathcal{O}(H), \forall h \in H, \ h\psi = \psi''(h')\psi' h''.
\]

There is a representation of \(\mathcal{H}(\mathcal{O}(H))\) on \(\mathcal{O}(H)\) defined by:

\[
h \triangleright \psi = \psi''(h)\psi', \quad \varphi \triangleright \psi = \varphi\psi \quad (h \in H, \ \psi, \varphi \in \mathcal{O}(H)).
\]

This representation is faithful (see [Mon93 Lem. 9.4.2]). Hence, if \(H\) is finite-dimensional, it follows that \(\mathcal{H}(\mathcal{O}(H))\) is a matrix algebra:

\[
\mathcal{H}(\mathcal{O}(H)) \cong \text{End}_C(H^*).
\]

Under our assumptions on \(H\), a natural set of generators for \(\mathcal{H}(\mathcal{O}(H))\) consists of the matrix coefficients of \(T\) and of \(L^{(\pm)}\).

**Lemma 4.4.** With these generators, the exchange relation of \(\mathcal{H}(\mathcal{O}(H))\) is:

\[
\frac{I_J^{(\pm)}}{I_T} \frac{T_2}{j} = \frac{T_2}{j} \frac{I_J^{(\pm)}}{I_T}, \quad \frac{I_T}{j} \frac{T_2}{j} = \frac{T_2}{j} \frac{I_T}{j}.
\]

**Proof:** Using (111) and (112), we have:

\[
\left( \frac{I_T}{j} \frac{T_2}{j} \right)^{ac} = \frac{I_T}{j} \frac{T_2}{j} \frac{I_J^{(\pm)}}{I_T} = \left( \frac{T_2}{j} \frac{I_J^{(\pm)}}{I_T} \right)^{ac} = \left( \frac{I_T}{j} \frac{T_2}{j} \right)^{ac}.
\]

Next, we have:

\[
\left( \frac{I_T}{j} \frac{T_2}{j} \right)^{ac} = \frac{I_T}{j} \frac{T_2}{j} \frac{I_J^{(\pm)}}{I_T} = \left( \frac{T_2}{j} \frac{I_J^{(\pm)}}{I_T} \right)^{ac} = \left( \frac{I_T}{j} \frac{T_2}{j} \right)^{ac}.
\]

The last equality is obvious. \(\Box\)
Proposition 4.5. The following map is a morphism of algebras:
\[ \Psi_{1,0} : \mathcal{L}_{1,0}(H) \rightarrow \mathcal{H}(\mathcal{O}(H)) \]
\[ A \mapsto L_{(+)}^{(+)L_{(-)}^{(-1)}} \]
\[ B \mapsto L_{(+)}^{(+)T_{L}^{(-1)}}. \]

Proof: We have to check that the fusion and exchange relations are compatible with \( \Psi_{1,0} \). Observe that the restriction of \( \Psi_{1,0} \) to the first copy of \( \mathcal{L}_{0,1}(H) \subset \mathcal{L}_{1,0}(H) \) is just the RSD morphism \( \Psi_{0,1} \), thus \( \Psi_{1,0} \) is compatible with the fusion relation over \( A \). For the fusion relation over \( B \), we have:

\[
\Psi_{1,0}(B)_{12} = L_{12}^{(+)T_{12}^{(-1)}} \\
= L_{1}^{(+)L_{2}^{(+)}T_{1}^{(-1)}T_{2}^{(-1)}L_{1}^{(-1)}} \\
= L_{1}^{(+)T_{1}^{(-1)}L_{1}^{(-1)}T_{1}^{(-1)}T_{2}^{(-1)}L_{2}^{(-1)}R_{21}^{(-1)}} \\
= L_{1}^{(+)T_{1}^{(-1)}L_{1}^{(-1)}T_{2}^{(-1)}L_{2}^{(-1)}R_{21}^{(-1)}} \\
= \Psi_{1,0}(B)_{1}R_{21}\Psi_{1,0}(B)_{2}R_{21}^{(-1)} \\
\text{(definition).}
\]

The same kind of computation allows one to show that \( \Psi_{1,0} \) is compatible with the \( \mathcal{L}_{1,0} \)-exchange relation.

We wish to show that \( \Psi_{1,0} \) is an isomorphism.

Lemma 4.6. Every element in \( \mathcal{L}_{1,0}(H) \) can be written as \( \sum_{i}(x_{i})A(y_{i})B \) with \( x_{i}, y_{i} \in \mathcal{L}_{0,1}(H) \). It follows that \( \dim(\mathcal{L}_{1,0}(H)) \leq \dim(\mathcal{L}_{0,1}(H))^{2} = \dim(H)^{2} \).

Proof: We use the same strategy as in Lemma 3.3. It suffices to show that an element like \( y_{B}x_{A} \) can be written as \( \sum_{i}(x_{i})A(y_{i})B \), and the result will follow because we can reorder all the elements by induction. We can restrict to matrix coefficients since they linearly span \( \mathcal{L}_{0,1}(H) \). The idea is to invert the exchange relation. If we write \( R = a_{i} \otimes b_{i} \), then proceeding as in the proof of Lemma 3.3 we find:

\[ I_{B_{1}A_{2}} = S^{-1}(a_{i})_{2}(R_{12}^{-1})_{12}A_{2}R_{12}B_{1}(R_{12}^{-1})_{12}(b_{i})_{1}. \]

This gives the result since \( B_{1}A_{2} \) contains all the possible products between the coefficients of \( I \) and those of \( J \).

Theorem 4.7. Recall that we assume that \( H \) is a finite-dimensional factorizable Hopf algebra. \( \Psi_{1,0} \) gives an isomorphism of algebras \( \mathcal{L}_{1,0}(H) \cong \mathcal{H}(\mathcal{O}(H)) \). It follows that \( \mathcal{L}_{1,0}(H) \) is a matrix algebra: \( \mathcal{L}_{1,0}(H) \cong \text{Mat}_{\dim(H)}(\mathbb{C}) \).

Proof: Observe that \( \Psi_{1,0} \circ j_{A} = i_{H} \circ \Psi_{0,1} \), where \( i_{H} : H \rightarrow \mathcal{H}(\mathcal{O}(H)) \) is the canonical inclusion. Since \( \Psi_{0,1} \) is an isomorphism, there exist matrices \( I(\pm) \) such that

\[ \Psi_{1,0}(A(\pm)) = I(\pm) \in \text{Mat}_{\dim(I)}(\mathcal{H}(\mathcal{O}(H))). \]

Moreover, we have:

\[ \Psi_{1,0}(A(\pm)B(-)) = T \in \text{Mat}_{\dim(I)}(\mathcal{H}(\mathcal{O}(H))). \]

Thus \( \Psi_{1,0} \) is surjective, and hence \( \dim(\mathcal{L}_{1,0}(H)) \geq \dim(\mathcal{H}(\mathcal{O}(H))) = \dim(H)^{2} \). This together with Lemma 1.6 gives \( \dim(\mathcal{L}_{1,0}(H)) = \dim(\mathcal{H}(\mathcal{O}(H))) \). The last claim is a general fact, see [21].

Remark 4. Contrarily to the case of \( \mathcal{L}_{0,1}(H) \), it is not relevant to systematically identify \( \mathcal{L}_{1,0}(H) \) with \( \mathcal{H}(\mathcal{O}(H)) \) through \( \Psi_{1,0} \). First because there is no natural \( \mathcal{O}(H) \)-comodule-algebra structure on \( \mathcal{H}(\mathcal{O}(H)) \) such that \( \Psi_{1,0} \) becomes an isomorphism of comodules. Second and more importantly, the natural setting to derive the action of the mapping class group really is \( \mathcal{L}_{1,0}(H) \), see below.
4.3 Representation of $\mathcal{L}^{\text{inv}}_{1,0}(H)$ on SLF($H$)

Recall from [20] that there is a faithful representation $\triangleright$ of $\mathcal{H}(\mathcal{O}(H))$ on $\mathcal{O}(H)$. Using the isomorphism $\Psi_{1,0}$, we get a representation of $\mathcal{L}_{1,0}(H)$ on $\mathcal{O}(H)$, still denoted $\triangleright$:

$$\forall x \in \mathcal{L}_{1,0}(H), \forall \psi \in \mathcal{O}(H), \ x \triangleright \psi = \Psi_{1,0}(x) \triangleright \psi$$

Using Lemma 4.9, it is easy to get:

$$A_1 \triangleright T_2 = T_2(RR')_{12} \quad \text{and} \quad B_1 \triangleright T_2 = (a_i)_{12} T_2(b_i)_{12}(R')_{12} = (a_{i,j})_{12} T_2(b_i)_{12}(R')_{12}$$

where as usual $R = a_i \otimes b_i$ and the last equality is obtained using [2].

As it was first pointed out in [4,5] in the case of $H = U_q(\mathfrak{g})$ ($q$ generic), the representation $\mathcal{O}(H)$ has an important submodule when we restrict to $\mathcal{L}^{\text{inv}}_{1,0}(H)$. This is also the case with our assumptions.

**Theorem 4.8.** The restriction of $\triangleright$ to $\mathcal{L}^{\text{inv}}_{1,0}(H)$ leaves the subspace SLF($H$) $\subset H^*$ stable:

$$\forall x \in \mathcal{L}^{\text{inv}}_{1,0}(H), \forall \psi \in \text{SLF}(H), \ x \triangleright \psi \in \text{SLF}(H).$$

**Hence, we have a representation of $\mathcal{L}^{\text{inv}}_{1,0}(H)$ on SLF($H$). We denote it $\rho_{\text{SLF}}$.**

**Proof:** We define matrices $T^{(\pm)} \in \text{Mat}_{\dim_{(J)}(\mathcal{H}(\mathcal{O}(H)))}$ by:

$$T^{(\pm)} = T_{12}^{-1} T_{2}.$$ 

Since the representation $\triangleright$ is faithful, this entirely defines $T^{(\pm)}$, and it is not difficult to show the following commutation rules:

$$L^{(\epsilon)} T^{(\sigma)} = L^{(\epsilon)} T^{(\sigma)}, \quad L^{(\epsilon)} T^{(\sigma)} = L^{(\sigma)} T^{(\epsilon)}, \quad R^{(\epsilon)} T^{(\sigma)} = R^{(\epsilon)} T^{(\sigma)}, \quad R^{(\epsilon)} T^{(\sigma)} = R^{(\sigma)} T^{(\epsilon)} \quad \forall \epsilon, \sigma \in \{\pm\}. $$

For instance, using the Yang-Baxter equation we have:

$$R^{(\epsilon)} T^{(\sigma)} = T^{(\epsilon)} R^{(\sigma)} \quad \forall \epsilon, \sigma \in \{\pm\}. $$

and the faithfulness of $\triangleright$ gives the equality. Now, let

$$T^{(\pm)} = \Psi_{1,0}(T^{(\pm)}) \in \text{Mat}_{\dim_{(J)}(\mathcal{L}_{1,0}(H))}. $$

**Lemma 4.9.** 1) Recall that $\mathcal{L}_{1,0}(H)$ is endowed with a structure of (right) $H$-module-algebra given by:

$$\forall h \in H, \quad A \cdot h = h' A S(h'') \quad \text{and} \quad B \cdot h = h' B S(h'').$$

Then, for the matrices $T^{(\pm)}$ of generators of $H$, we have:

$$A_2 \cdot S^{-1}(T^{(\pm)}) = T^{(\pm)} A_2 T^{(\pm)}.$$ 

It follows that $x \in \mathcal{L}_{1,0}(H)$ is invariant if, and only if, $x T^{(\pm)} = T^{(\pm)x}$. Then 2) Endow $H^*$ with the left $H$-action $\circ$ defined by $h \circ \psi = \psi(S^{-1}(h')h'').$ Then

$$T^{(\pm)} = T^{(\pm)} T^{(\pm)} T^{(\pm)}.$$ 

It follows that $\psi \in H^*$ is symmetric if, and only if, $T^{(\pm)} \psi = \psi T^{(\pm)}$ for all $J$. 

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Proof: 1) We compute each side of the equality. First, for $U = A$ or $B$, we get using (3):

$$\left( \frac{J}{U_2 \cdot S^{-1} \left( \frac{I}{L_1} \right) } \right)_{ac} = \frac{J}{U_2 \cdot S^{-1} \left( \frac{I}{L_1} \right) } = \frac{J}{T_i, S^{-1} \left( \frac{I}{L_1} \right) } = \frac{J}{U_j, T_i, J}, L_1^k} \right)_{ac}.$$

Second, we get using (3), (23) and the shortened notation:

$$\Psi_{1,0} \left( M_1^{(+)} A_2 M_1^{(-)} \right) = L_1^{(+)} L_2^{(+)} L_2^{(-)} L_1^{(-)} = L_1^{(+)} L_2^{(+)} L_2^{(-)} L_1^{(-)}$$

and we have equality since $\Psi_{1,0}$ is an isomorphism. The others cases are similar. The second part is based on analogous computations and is thus left to the reader. Note that the last claim of 2) is due to the fact that the invariants for $\diamond$ are the symmetric linear forms.

End of the proof of Theorem 2.8: Let $x \in L_{1,0}^B(H)$ and $\psi \in SLF(H)$. We apply the previous lemma:

$$\frac{J}{M^{(\pm)}} \triangleright (x \triangleright \psi) = \left( \frac{J}{M^{(\pm)}} x \triangleright \psi = \left( x, M^{(\pm)} \right) = \right( x \triangleright \psi ) I_{\dim(J)}.$$
Lemma 4.11. Let $z \in \mathcal{L}_{0,1}^{\text{inv}}(H) = Z(H)$ and let $\psi \in \text{SLF}(H)$. Then:

$$z_{B^{-1}} \triangleright \psi = \sum_{l,j} \text{tr}(\Lambda_l \psi^j)$$

with $\text{tr}_{12} = \text{tr} \otimes \text{tr}$ and $R = a_i \otimes b_i$. Note that we used that $D^{-1}(z)$ is symmetric. Now, denoting $m : H \otimes H \to H$ the product in $H$ and using the Yang-Baxter equation, we have:

$$b_j b_k S^2(a_j a_k) g \otimes b_i a_k = (m \otimes \text{id}) \circ (\text{id} \otimes S^2 \otimes \text{id})(1 \otimes a_i \otimes b_i \cdot b_j \otimes a_j \otimes 1 \cdot b_k \otimes 1 \otimes a_k) g \otimes 1$$

$$= (m \otimes \text{id}) \circ (\text{id} \otimes S^2 \otimes \text{id})(R_{23} R_{21} R_{31}) g \otimes 1$$

$$= (m \otimes \text{id}) \circ (\text{id} \otimes S^2 \otimes \text{id})(R_{12} R_{23} R_{21}) g \otimes 1$$

$$= b_j b_k S^2(a_j a_k) g \otimes a_i b_k = b_j b_k g a_j a_k \otimes a_i b_k = v^{-1} b_i a_k \otimes a_i b_k$$

$$= \Delta(v^{-1}) 1 \otimes v$$

where we used that $b_j g a_j = b_j S^2(a_j) g = u^{-1} g = v^{-1}$ by (7) and (10). Hence for $x \in H$:

$$(z_{B^{-1}} \triangleright \psi)(x) = D^{-1}(z)((v^{-1})' x') \psi(v(v^{-1})'^{-1} x') = (D^{-1}(z) \psi^v)(v^{-1} x) = (D^{-1}(z) \psi^v)^{v^{-1}}(x)$$

as desired. \hfill \square

**Lemma 4.11.** Let $z \in \mathcal{L}_{0,1}^{\text{inv}}(H) = Z(H)$ and let $\psi \in \text{SLF}(H)$. Then:

$$z_{B^{-1}} \triangleright \psi = \left( S(D^{-1}(z)) \psi^v \right)^{v^{-1}}.$$

It follows that if $S(\psi) = \psi$ for all $\psi \in \text{SLF}(H)$, then $\rho_{\text{SLF}}(z_{B^{-1}}) = \rho_{\text{SLF}}(z_B)$.

**Proof:** This proof is quite similar to that of the previous proposition. Using the fact that $\Psi_{1,0}(B^{-1}) = L(-) S(T) L(+)^{-1}$ together with Lemma 4.4 and formulas (12), (3) and (2), it is not too difficult to show that

$$\frac{I^r}{B_{-1}^t} \triangleright \frac{J}{T} = \left( I^r \otimes \text{id} \right) \frac{I^r}{a_i} \frac{I^r}{T_{12}} \frac{I^r}{a_j S^{-2}(b_j b_k)} \frac{I^r}{a_k b_i}$$

where $\otimes \text{id}$ means transpose on the first tensorand. By Lemma 3.7, we write $z_{B^{-1}} = \sum_l \text{tr}(\Lambda_l \psi^j)$ with $D^{-1}(z) = \sum_l \text{tr}(\Lambda_l \psi^j)$ in $\text{SLF}(H)$. We also write $\psi = \sum_l \text{tr}(\Theta_l \psi^j)$. Observe using (12) that:

$$S(D^{-1}(z)) = \sum_l \text{tr}(\Lambda_l S(T)) = \sum_l \text{tr}(\Lambda_l \psi)$$

Using the fact that $S(g) = g^{-1}$ and (12), we thus get:

$$z_{B^{-1}} \triangleright \psi = \sum_{l,j} \text{tr}_{12} \left( (\Lambda_l \psi)^j \frac{I^r}{a_i} \frac{I^r}{T_{12}} \frac{I^r}{a_j S^{-2}(b_j b_k)} \frac{I^r}{a_k b_i} \right)$$

$$= \sum_{l,j} \text{tr}_{12} \left( (\Lambda_l \psi)^j \frac{I^r}{a_i} \frac{I^r}{T_{12}} \frac{I^r}{a_j S^{-2}(b_j b_k) g^{-1}} \frac{I^r}{a_k b_i} \right)$$

$$= S(D^{-1}(z))(a_i a_j S^{-2}(b_j b_k) g^{-1}) \psi(a_k b_i)$$

$$= S(D^{-1}(z))(a_i a_j S^{-2}(b_j b_k) g^{-1}) \psi(a_k b_i)$$

$$= S(D^{-1}(z))(a_i a_j S^{-2}(b_j b_k) g^{-1}) \psi(a_k b_i).$$
Now, we have:
\[ a_j S^{-2}(b_j b_k) g^{-1} a_i \otimes a_k b_i = a_j g^{-1} b_j b_k a_i \otimes a_k b_i = v^{-1} R R = \Delta(v^{-1}) 1 \otimes v \]
where we used \( a_j g^{-1} b_j = a_j S^{-2}(b_j) g^{-1} = S^{-1}(g S^{-1}(b_j) S(a_j)) = S^{-1}(g u^{-1}) = S^{-1}(v^{-1}) = v^{-1} \) by (7) and (8). Hence we get as in the previous proof \( z_{B^{-1}} \triangleright \psi = (S(D^{-1}(z)) \psi') v^{-1} \).

\[ \square \]

5 Projective representation of \( \text{SL}_2(\mathbb{Z}) \)

As previously, \( H \) is a factorizable ribbon finite-dimensional Hopf algebra.

5.1 Mapping class group of the torus

Let \( \Sigma_{g,n} \) be a compact oriented two-dimensional surface of genus \( g \) with \( n \) punctures. Recall that the mapping class group \( \text{MCG}(\Sigma_{g,n}) \) is the group of all isotopy classes of orientation-preserving homeomorphisms which leave the set of punctures globally invariant, see [Mas09]. Let \( D \subset \Sigma_{g,n} \) be an embedded open disk. Then \( \text{MCG}(\Sigma_{g,n} \setminus D) \) is defined as \( \text{MCG}(\Sigma_{g,n}) \) except we restrict to homeomorphisms which fix pointwise the boundary circle \( C = \partial(\Sigma_{g,n} \setminus D) \).

Let us put the base point of \( \pi_1(\Sigma_{g,n} \setminus D) \) on the boundary circle \( C \). Since \( C \) is pointwise fixed, we can consider the action of \( \text{MCG}(\Sigma_{g,n} \setminus D) \) on \( \pi_1(\Sigma_{g,n} \setminus D) \), obviously defined by:

\[ \forall [f] \in \text{MCG}(\Sigma_{g,n}), \forall [\gamma] \in \pi_1(\Sigma_{g,n}), \; [f] \cdot [\gamma] = f_*(\gamma) = [f \circ \gamma]. \]

Until now, we identify \( f \) with its isotopy class \([f]\) and \( \gamma \) with its homotopy class \([\gamma]\).

Here we focus on the torus \( \Sigma_{1,0} = S^1 \times S^1 \). By the Dehn-Lickorish theorem, \( \text{MCG}(\Sigma_{1,0}) \) is generated by the Dehn twists \( \tau_a, \tau_b \) along the curves \( a = \{1\} \times S^1 \) and \( b = S^1 \times \{1\} \) respectively. It is well known (see e.g. [Mas09]) that

\[ \text{MCG}(\Sigma_{1,0}) = \text{SL}_2(\mathbb{Z}) = \langle \tau_a, \tau_b \mid \tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b, \; (\tau_a \tau_b)^6 = 1 \rangle. \]

This presentation is not the usual one of \( \text{SL}_2(\mathbb{Z}) \), which is:

\[ \text{SL}_2(\mathbb{Z}) = \langle s, t \mid (st)^3 = s^2, \; s^4 = 1 \rangle. \]

The link between the two presentations is \( s = \tau_a \tau_b \tau_a, \; t = \tau_a^{-1}. \)

We now remove from \( \Sigma_{1,0} \) an open disk \( D \) which does not intersect \( a \) nor \( b \). The surface \( \Sigma_{1,0} \setminus D \) and the curves \( a \) and \( b \) are represented in the figure below. We view these curves as elements of \( \pi_1(\Sigma_{1,0} \setminus D) \), that is we consider them up to homotopy and we provide them an orientation.

The map \( \tau_a \) defined above is also an element of \( \text{MCG}(\Sigma_{1,0} \setminus D) \): this is the Dehn twist along the dashed curve (which is freely homotopic to \( a \)) in the figure below. Its action on \( b \) is depicted by:
Similarly, $\tau_b$ is an element of $\text{MCG}(\Sigma_{1,0} \setminus D)$. The action of the twists $\tau_a$ and $\tau_b$ on $\pi_1(\Sigma_{1,0} \setminus D)$ is given by:
\[ (\tau_a)_*(a) = a, \quad (\tau_a)_*(b) = ba \quad \text{and} \quad (\tau_b)_*(a) = b^{-1}a, \quad (\tau_b)_*(b) = b. \quad (24) \]

### 5.2 Automorphisms $\alpha$ and $\beta$

The fundamental idea, proposed in [AS96a] and [AS96b], is to mimic the action of the Dehn twists of $\Sigma$ (viewed as elements of $\text{MCG}(\Sigma_{g,n} \setminus D)$) on $\pi_1(\Sigma_{g,n} \setminus D)$ at the level of the algebra $\mathcal{L}_{g,n}(H)$. Let us be more precise. We focus on the case $(g, n) = (1, 0)$.

In $\pi_1(\Sigma_{1,0} \setminus D)$ we have the two canonical curves $a$ and $b$, while in $\mathcal{L}_{1,0}(H)$ we have the matrices $\tau_a$ and $\tau_b$. In view of (24), let us try to define two morhisms $\tilde{\tau}_a, \tilde{\tau}_b : \mathcal{L}_{1,0}(H) \rightarrow \mathcal{L}_{1,0}(H)$ by the same formulas:
\[
\tilde{\tau}_a(A) = \frac{1}{A}, \quad \tilde{\tau}_a(B) = \frac{1}{BA}
\]
\[
\tilde{\tau}_b(A) = B^{-1}A, \quad \tilde{\tau}_b(B) = B.
\]
Let us see the behavior of these mappings under the fusion and exchange relations. For the exchange relation, no problem arises:
\[
R_{12} \tilde{\tau}_a(B)_1 R_{21} \tilde{\tau}_a(A)_2 = R_{12} B_1 A_1 R_{21} A_2 \quad \text{(definition)}
\]
\[
= R_{12} B_1 R_{21} A_2 R_{12} A_1 R_{12}^{-1} \quad \text{(eq. (18))}
\]
\[
= A_2 R_{12} B_1 A_1 R_{12}^{-1} \quad \text{(eq. (18))}
\]
\[
= \tilde{\tau}_a(A)_2 R_{12} \tilde{\tau}_a(B)_1 R_{12}^{-1} \quad \text{(definition)}
\]
and a similar computation holds for $\tilde{\tau}_b$. The fusion relation is almost satisfied:
\[
\tilde{\tau}_a(B)_{12} = B_{12} A_{12} \quad \text{(definition)}
\]
\[
= \Delta(v)_{12} B_1 B_2 v_1^{-1} v_2^{-1} R_{12} R_{12} A_{12} \quad \text{(trick + functoriality)}
\]
\[
= \Delta(v)_{12} B_1 R_{21} B_2 R_{21}^{-1} v_1^{-1} v_2^{-1} R_{12} R_{12} A_1 R_{12} A_2 R_{21}^{-1} \quad \text{(eq. (15))}
\]
\[
= \Delta(v)_{12} v_1^{-1} v_2^{-1} B_1 A_1 R_{21} B_2 A_2 R_{21}^{-1} \quad \text{(functoriality)}
\]
\[
= \Delta(v)_{12} v_1^{-1} v_2^{-1} \tilde{\tau}_a(B)_1 R_{21} \tilde{\tau}_a(B)_2 R_{21}^{-1} \quad \text{(definition)}
\]
and we get similarly:
\[
\tilde{\tau}_b(A)_{12} = B_{12}^{-1} A_{12} = \Delta(v^{-1})_{12} v_1 v_2 \tilde{\tau}_b(A)_1 R_{21} \tilde{\tau}_b(A)_2 R_{21}^{-1}.
\]
From this we conclude that the elements $\frac{1}{v^{-1}BA}$ and $\frac{1}{vB^{-1}A}$ satisfy the relation (15). Since $v$ is central, we see by functoriality that the exchange relation still holds with these elements. We thus have found the morphisms which mimic $\tau_a$ and $\tau_b$. We denote them by $\alpha$ and $\beta$ respectively.

**Proposition 5.1.** We have two automorphisms $\alpha, \beta$ of $\mathcal{L}_{1,0}(H)$ defined by:
\[
\alpha(A) = \frac{1}{A}, \quad \alpha(B) = \frac{1}{v^{-1}BA}
\]
\[
\beta(A) = \frac{1}{vB^{-1}A}, \quad \beta(B) = B.
\]

Moreover, these automorphisms are inner: there exist $\hat{\alpha}, \hat{\beta} \in \mathcal{L}_{1,0}(H)$ unique up to scalar such that
\[
\forall x \in \mathcal{L}_{1,0}(H), \quad \alpha(x) = \hat{\alpha}x\hat{\alpha}^{-1}, \quad \beta(x) = \hat{\beta}x\hat{\beta}^{-1}.
\]

**Proof:** It remains to show that $\alpha$ and $\beta$ are invertible, but it is obvious since their inverses are given by:
\[
\alpha^{-1}(A) = A, \quad \alpha^{-1}(B) = \frac{1}{vBA}
\]
\[
\beta^{-1}(A) = \frac{1}{v^{-1}BA}, \quad \beta^{-1}(B) = B.
\]
By Theorem 4.7, \( \mathcal{L}_{1,0}(H) \) is a matrix algebra. Hence, by the Skolem-Noether theorem, every automorphism of \( \mathcal{L}_{1,0}(H) \) is inner.

A natural question is then to find explicitly the elements \( \hat{\alpha}, \hat{\beta} \). The answer is amazingly simple (it has been provided in [AS96a] and [Sch98] for the semi-simple case).

**Theorem 5.2.** Up to scalar, \( \hat{\alpha} = v_A^{-1} \) and \( \hat{\beta} = v_B^{-1} \).

Recall that we have identified \( H \) with \( \mathcal{L}_{0,1}(H) \) under \( \Psi_{0,1} \), thus we regard the ribbon element \( v \) as an element of \( \mathcal{L}_{0,1}(H) \).

**Proof:** By the definition of \( \hat{\alpha} \) and \( \hat{\beta} \), we have:

\[
\begin{align*}
\hat{\alpha}A &= A\hat{\alpha}, & \hat{\alpha}B &= v^{-1}BA\hat{\alpha}, \\
\hat{\beta}A &= \hat{\beta}^{-1}A\hat{\beta}, & \hat{\beta}B &= \hat{\beta}B = B\hat{\beta}.
\end{align*}
\]

Conversely, every invertible element satisfying the first (resp. the second) line of equations is necessarily a scalar multiple of \( \hat{\alpha} \) (resp. of \( \hat{\beta} \)) by the Skolem-Noether theorem. Thus we will show that \( v_A^{-1} \) and \( v_B^{-1} \) satisfy these equations. It is obvious that \( v_A^{-1} \) (resp. \( v_B^{-1} \)) commutes with the matrices \( A \) (resp. \( B \)) since it is central in \( j_A(\mathcal{L}_{0,1}(H)) \) (resp. in \( j_B(\mathcal{L}_{0,1}(H)) \)). Let us show the other commutation relation for \( v_A^{-1} \). The idea is to make the computations in \( \mathcal{H}(O(H)) \). Recall that when we restrict \( \Psi_{1,0} \) to \( j_A(\mathcal{L}_{0,1}(H)) \), we just get the morphism \( \Psi_{0,1} : j_A(\mathcal{L}_{0,1}(H)) \to 1 \otimes H \subset \mathcal{H}(O(H)) \). It follows that \( \Psi_{1,0}(v_A) = v \in 1 \otimes H \). Then:

\[
\Psi_{1,0}(v_A^{-1}B) = v^{-1}L^{(+)}T L^{(-)}I^{-1} \quad \text{and} \quad \Psi_{1,0}(v_A^{-1}BAv_A^{-1}) = v^{-1}L^{(+)}T L^{(-)}I^{-1}L^{(+)}I^{-1}v^{-1}.
\]

Hence we must show that:

\[
v^{-1}T = v^{-1}T L^{(-)}I^{-1}L^{(+)}v^{-1}.
\]

Using the exchange relation of Definition 4.3 together with (11) and (8), we have:

\[
v^{-1}T_b = \left( (T_b^a)^{v^{-1}}(v^{-1})^{v^{-1}} \right) (T_b^a)^{v^{-1}} = \left( T_b^a, b_a, a_j v^{-1} \right) T_b^a a_i b_j v^{-1} = \left( T_b^a a_i b_j v^{-1} \right) T_b^a a_i b_j v^{-1} = \left( T_b^a a_i b_j v^{-1} \right) T_b^a a_i b_j v^{-1} = v^{-1}T_b^a a_i b_j v^{-1}.
\]

It follows that

\[
v^{-1}T = v^{-1}T b_i a_j v^{-1} = v^{-1}T b_i a_j v^{-1} = v^{-1}T L^{(-)}I^{-1}L^{(+)}v^{-1}
\]

as desired. We now apply the morphism \( \alpha^{-1} \circ \beta^{-1} \) to the equality \( v_A^{-1}B = v^{-1}BAv_A^{-1} \):

\[
\alpha^{-1} \circ \beta^{-1}(v_A^{-1}B) = v^{-1}BAv_A^{-1} = \alpha^{-1} \circ \beta^{-1}(v^{-1}BAv_A^{-1}) = B A^{-1} B v_B^{-1}.
\]

Using that \( v_B \) and \( B \) commute, we easily get the desired equality. \( \square \)

This theorem is important because it will allow us to use properties of the morphisms \( \alpha \) and \( \beta \) in order to show relations between \( v_A \) and \( v_B \).

### 5.3 Projective representation of \( SL_2(\mathbb{Z}) \) on \( SLF(H) \)

We now show that the elements \( v_A, v_B \) give rise to a projective representation of \( \text{MCG}(\Sigma_{1,0}) \) on \( SLF(H) \) via the following assignment:

\[
\tau_a \mapsto \rho_{\text{SLF}}(v_A), \quad \tau_b \mapsto \rho_{\text{SLF}}(v_B).
\]
We must then check that

$$\rho_{\text{SLF}}(v_A v_B v_A) \sim \rho_{\text{SLF}}(v_B v_A v_B), \quad \rho_{\text{SLF}}(v_A v_B)^6 \sim 1.$$  

where $\sim$ means equality up to scalar. As we will see, it turns out that the braid relation holds in the algebra $L_{1,0}(H)$ itself (the scalar being 1), while the relation $(v_A v_B)^6 \sim 1$ only holds in the representation $\text{SLF}(H)$. This is not surprising because the relation $((\tau_a)_*(\tau_b)_*)^6 = (\tau_a)_*(\tau_b)_*$ holds on $\pi_1(\Sigma_{1,0} \setminus D)$, while the relation $(\tau_a)_*(\tau_b)_*)^6 = 1$ only holds on $\pi_1(\Sigma_{1,0})$. Thus we see that, intuitively, applying $\rho_{\text{SLF}}$ amounts to gluing back the disk $D$.

Integrals on $H$ will play a prominent role. Recall that a left integral (resp. right integral) is a non-zero linear form $\mu^l$ (resp. $\mu^r$) on $H$ which satisfies:

$$\forall x \in H, \quad (\text{id} \otimes \mu^l) \circ \Delta(x) = \mu^l(x)1 \quad \text{(resp.} \quad (\mu^r \otimes \text{id}) \circ \Delta(x) = \mu^r(x)1). \quad (25)$$

Since $H$ is finite-dimensional, this is equivalent to:

$$\forall \psi \in \mathcal{O}(H), \quad \psi \mu^l = \varepsilon(\psi)\mu^l \quad \text{(resp.} \quad \mu^r \psi = \varepsilon(\psi)\mu^r). \quad (26)$$

It is well-known that left and right integrals always exist if $H$ is finite-dimensional. Moreover, they are unique up to scalar. We fix $\mu^l$. Then $\mu^l \circ S^{-1}$ is a right integral, and we choose

$$\mu^r = \mu^l \circ S^{-1}. \quad (27)$$

We can now state an important result for the sequel.

**Proposition 5.3.** Let $\varphi_v, \varphi_{v^{-1}} \in H^*$ defined by:

$$\varphi_v = \mu^l(v^{-1})^{-1}\mu^l(g^{-1}v^{-1}?), \quad \varphi_{v^{-1}} = \mu^l(v)^{-1}\mu^l(g^{-1}v^{-1}?).$$

Then:

$$D(\varphi_v) = v, \quad D(\varphi_{v^{-1}}) = v^{-1}.$$ 

It follows that:

1. $\varphi_v, \varphi_{v^{-1}} \in \text{SLF}(H)$,
2. $\mu^l(g^{-1}?), \mu^r(g?) \in \text{SLF}(H),$
3. $\forall x, y \in H, \quad \mu^l(xy) = \mu^l(yS^2(x)), \quad \mu^r(xy) = \mu^r(S^2(y)x)$.

**Proof:** Consider the following computation, where we use (8) and (25):

$$D(\mu^l(g^{-1}v^{-1}?)) = \langle \mu^l(g^{-1}v^{-1}?) \otimes \text{id}, g_1(RR')_{12} \rangle = \langle \text{id} \otimes \mu^l(g^{-1}v^{-1}?), g_2(R'R)_{12} \rangle$$

$$= \langle \text{id} \otimes \mu^l(g^{-1}v^{-1}?), v(v^{-1})' \otimes v(v^{-1})'' \rangle = \mu^r((v^{-1})')(v^{-1})'' = \mu^l(v^{-1})v.$$ 

Since $H$ is factorizable, the map $D$ is an isomorphism of vector spaces. The left integral $\mu^l$ is non-zero, so $\mu^l(g^{-1}v^{-1}?)$ is non-zero either. Since $D$ is an isomorphism, it follows that $D(\mu^l(g^{-1}v^{-1}?)$) is $\mu^l(v^{-1})v \neq 0$, and thus $\mu^l(v^{-1}) \neq 0$. Hence the formula for $\varphi_v$ is well defined. Moreover, we have the restriction $D : \text{SLF}(H) \rightarrow \mathcal{Z}(H)$, so since $v \in \mathcal{Z}(H)$, we get that $\varphi_v \in \text{SLF}(H)$. This allows us to deduce the properties stated about $\mu^l$. Using (27), we obtain the properties 1, 2 and 3 for $\mu^r$. We can now proceed with the computation for $\varphi_{v^{-1}}$:

$$D(\mu^l(g^{-1}v^{-1}?) = \langle \mu^l(g^{-1}v') \otimes \text{id}, v_1(R'R)^{-1} \rangle = \mu^r(v')v^{-1} = \mu^r(v)v^{-1} = \mu^l(v)v^{-1}.$$ 

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where we used (27), the property 3) previously shown and (8). We conclude as before. □

Since \( D \) is an isomorphism of algebras, we have \( \varphi_{\varphi^{-1}} = \varphi_{\varphi^{-1}} \), and

\[
\varphi_{\varphi^{v^2}} = \frac{\mu^l(v)}{\mu^l(v^{-1})} \epsilon.
\]  

(28)

By Proposition [1.10] the actions of \( v_A \) and \( v_B \) on SLF(\( H \)) are:

\[
\forall \psi \in \text{SLF}(H), \ v_A \triangleright \psi = \psi^u = \psi(v') \quad \text{and} \quad v_B \triangleright \psi = (\varphi_v \psi^v)^{v^{-1}}.
\]  

(29)

**Lemma 5.4.** \( \varphi_{\varphi^v} = \varphi_{\varphi^v} \).

**Proof:** For \( x \in H \):

\[
\langle \varphi_{\varphi^v}, x \rangle = \mu^l(v^{-1})^{-2} \mu^l(v^{-1} g^{-1} x') \mu^l(g^{-1} x'') = \mu^l(v^{-1})^{-2} \langle \mu^l(v^{-1}) \mu^l, g^{-1} x \rangle = \mu^l(v^{-1})^{-1} \mu^l(g^{-1} x) = \varphi_{\varphi^v}(x).
\]

We simply used (26). □

This lemma has an important consequence.

**Proposition 5.5.** The following braid relation holds in \( L_{1,0}(H) \):

\[
v_A v_B v_A = v_B v_A v_B.
\]

**Proof:** It is easy to check that the morphisms \( \alpha \) and \( \beta \) satisfy the braid relation \( \alpha \beta \alpha = \beta \alpha \beta \). Because \( L_{1,0}(H) \) is a matrix algebra, we have the existence of a scalar \( \lambda \) such that \( \hat{\alpha} \beta \hat{\alpha} = \lambda \hat{\beta} \hat{\alpha} \hat{\beta} \). Hence, by Theorem 5.2 we have: \( \lambda v_A v_B v_A = v_B v_A v_B \). Let us see the action of both sides on the counit:

\[
\lambda v_A v_B v_A \triangleright \varepsilon = \lambda v_A v_B \triangleright \varepsilon = \lambda v_A \triangleright \varphi_{\varphi^{-1}} = \lambda \varphi_{\varphi^{-1}}
\]

\[
v_B v_A v_B \triangleright \varepsilon = v_B v_A \triangleright \varepsilon = v_B \triangleright \varphi_{\varphi^{-1}} = (\varphi_v \varphi_{\varphi^v})^{v^{-1}} = (\varphi_v)^{v^{-1}} = \varphi_v.
\]

We used \( \varepsilon(v') = \varepsilon(v') \varepsilon = \varepsilon \) and Lemma 5.4. It follows that \( \lambda = 1 \). □

Observe that \( (\alpha \beta)^6 \neq \text{id} \), thus the other relation of MCG(\( \Sigma_{1,0} \)) does not hold in \( L_{1,0}(H) \). In order to show it in the representation, we begin with a technical lemma, in which we use the notation of [19].

**Lemma 5.6.** For all \( z \in L_{0,1}(H) = \mathcal{Z}(H) \), we have \( z_{v^{-1}A^{-1}B^{-1}} = z_{B^{-1}} \).

**Proof:** Using Lemma 5.7 write as usual \( z = \sum \mathbf{tr}(\Lambda g^M) \). We first show that \( z_{v^{-1}A^{-1}B^{-1}} = z_{B^{-1}A} \). The idea is to make the computations in the Heisenberg double. We have

\[
\Psi_{1,0}(v^{-1} A^{-1} B^{-1}) = v^{-1} L^+ T^{-1} L^+ \quad \text{and} \quad \Psi_{1,0}(v B^{-1} A) = v L^- T^{-1} L^-.
\]

Hence:

\[
\Psi_{1,0}(z_{v^{-1}A^{-1}B^{-1}}) = \sum \mathbf{tr}(\Lambda g v^{-1} L^+ T^{-1} L^+) = \sum \mathbf{tr}(\Lambda g v^{-1} A_i b_i \mathcal{S}(T) S(a_j) b_j).
\]

Using the defining relation of \( \mathcal{H}(\mathcal{O}(H)) \) together with (11), (2) and (7):

\[
(a_i)^2 b_i S(T)^l b_i = (a_i d_k)^2 S(T)^l b_i = (a_i d_k S(b_k))^2 S(T)^l b_i = (a_i d_k S^{-1}(b_k)) b_i = (a_i d_k S^{-1})^2 S(T)^l b_i
\]
It follows that
\[
\Psi_{1,0}(z_{v^{-1}AB^{-1}}) = \sum_I \text{tr} \left( A_I S^I(a_i) S(T) S(a_j) b_i b_j \right) = D^{-1}(z) \left( S^2(a_i) S(?) S(a_j) \right) b_i b_j \\
= D^{-1}(z) \left( S(?) S(a_j) S^2(a_i) \right) b_i b_j = (S(D^{-1}(z)) \otimes \text{id})(S(a_i) a_j \otimes b b_j) \\
= (S(D^{-1}(z)) \otimes \text{id})(? \otimes 1) = S(D^{-1}(z)) \in O(H) \otimes 1.
\]
A similar computation shows that \( \Psi_{1,0}(z_{v^{-1}B^{-1}A}) = S(D^{-1}(z)) \). Hence \( z_{v^{-1}AB^{-1}} = z_{vB^{-1}A} \). Applying the morphism \( \alpha \) to this equality, we find:
\[
\alpha(z_{v^{-1}AB^{-1}}) = z_{B^{-1}} = \alpha(z_{vB^{-1}A}) = z_{v^2A^{-1}B^{-1}A}
\]
as desired. \( \square \)

Consider \( \hat{\omega} \in L_{1,0}(H) \) defined by \( \hat{\omega}^{-1} = v_A v_B v_A \). Then \( \hat{\omega} \) implements the automorphism \( \omega = \alpha \beta \alpha \):
\[
\forall x \in L_{1,0}(H), \quad \omega(x) = \hat{\omega} x \hat{\omega}^{-1}.
\]
The key observation is the following lemma.

**Lemma 5.7.** For all \( \psi \in \text{SLF}(H) \):
\[
\hat{\omega}^2 \triangleright \psi = \frac{\mu^l(v^{-1})}{\mu^l(v)} S(\psi).
\]

**Proof:** Firstly, we show the formula for \( \psi = \varepsilon \):
\[
\hat{\omega}^{-2} \triangleright \varepsilon = (v_A v_B v_A)^2 \triangleright \varepsilon = v_A v_B v_A \triangleright \varphi^v = \varphi_v \varphi_v^2 = \frac{\mu^l(v)}{\mu^l(v^{-1})} \varepsilon
\]
where we used (29) and (28). Secondly, note that:
\[
\omega(A) = B\hat{A}^{-1}B^{-1}A, \quad \omega(B) = \hat{A}.
\]
Using Lemma 5.6, we get for all invariants \( z \in L^{iv}_{0,1}(H) \):
\[
\omega(z_A) = z_{v^2A^{-1}B^{-1}A} = z_{B^{-1}}, \quad \omega(z_B) = z_A \\
\omega^2(z_A) = z_{A^{-1}}, \quad \omega^2(z_B) = z_{B^{-1}}.
\]
This shows in particular that
\[
\hat{\omega}^2 z_B = z_{B^{-1}} \hat{\omega}^2.
\]
Thirdly, observe that by Proposition 4.10:
\[
\forall \psi \in \text{SLF}(H), \quad D(\psi^v)_B \triangleright \varepsilon = \psi.
\]
These three facts together with Lemma 4.11 yield:
\[
\hat{\omega}^2 \triangleright \psi = \hat{\omega}^2 D(\psi^v)_B \triangleright \varepsilon = D(\psi^v)_B^{-1} \hat{\omega}^2 \triangleright \varepsilon = \frac{\mu^l(v^{-1})}{\mu^l(v)} D(\psi^v)_B^{-1} \triangleright \varepsilon = \frac{\mu^l(v^{-1})}{\mu^l(v)} S(\psi^v)^{v^{-1}} = \frac{\mu^l(v^{-1})}{\mu^l(v)} S(\psi)
\]
as desired. \( \square \)

Recall that \( \text{PSL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})/\{\pm I_2\} \) admits the following presentations:
\[
\text{PSL}_2(\mathbb{Z}) = \langle \tau_a, \tau_b \mid \tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b, (\tau_a \tau_b)^3 = 1 \rangle = \langle s, t \mid (st)^3 = 1, s^2 = 1 \rangle.
\]
Theorem 5.8. Recall that we assume that $H$ is a finite-dimensional factorizable ribbon Hopf algebra. The following assignment defines a projective representation of $\text{MCG}(\Sigma_{1,0}) = \text{SL}_2(\mathbb{Z})$ on $\text{SLF}(H)$:

$$\tau_a \mapsto \rho_{\text{SLF}}(v_A), \quad \tau_b \mapsto \rho_{\text{SLF}}(v_B).$$

If moreover $S(\psi) = \psi$ for all $\psi \in \text{SLF}(H)$, then this defines actually a projective representation of $\text{PSL}_2(\mathbb{Z})$.

By a slight abuse of notation, we denote also this projective representation by $\rho_{\text{SLF}}$.

Proof: By Proposition 5.5, we know that the braid relation is satisfied in $L_{1,0}(H)$. By Lemma 5.7, we have:

$$(v_Av_B)^3 \triangleright \psi = \hat{\omega}^{-2} \triangleright \psi = \frac{\mu^l(v)}{\mu^l(v^{-1})} S^{-1}(\psi).$$

If $S(\psi) = \psi$, then

$$\rho_{\text{SLF}}(v_Av_B)^3 = \frac{\mu^l(v)}{\mu^l(v^{-1})} \text{id}.$$ 

Otherwise,

$$(v_Av_B)^6 \triangleright \psi = \frac{\mu^l(v)}{\mu^l(v^{-1})} \hat{\omega}^{-2} \triangleright S^{-1}(\psi) = \frac{\mu^l(v)^2}{\mu^l(v^{-1})^2} S^{-2}(\psi) = \frac{\mu^l(v)^2}{\mu^l(v^{-1})^2} \psi(g^{-1} ? g) = \frac{\mu^l(v)^2}{\mu^l(v^{-1})^2} \psi.$$ 

Observe that the quantity $\frac{\mu^l(v)}{\mu^l(v^{-1})}$ does not depend on the choice of $\mu^l$ since it is unique up to scalar.

5.4 Equivalence with the Lyubaschenko-Majid representation

Recall that $H$ is a finite-dimensional factorizable ribbon Hopf algebra. Under this assumption, two operators $S, T : H \to H$ are defined in [LM94]:

$$S(x) = (\text{id} \otimes \mu^l)(R^{-1}(1 \otimes x)R^{-1}), \quad T(x) = v^{-1}x.$$ 

It is shown that they are invertible and satisfy $(ST)^3 = \lambda S^2$, $S^2 = S^{-1}$, with $\lambda \in \mathbb{C}\setminus \{0\}$. We warn the reader that in [LM94], they consider the inverse of the ribbon element (see the bottom of the third page of their paper). That is why there is $v^{-1}$ in the formula for $T$.

Now we introduce two maps. The first is

$$\chi : H^* \to H \quad \beta \mapsto (\beta \otimes \text{id})(R'R)$$

while the second is

$$\gamma : H \to H^* \quad x \mapsto \mu^r(S(x)\ ?).$$

The map $\chi$ is another variant of the map $\Psi$ of section 2.2 and is called Drinfeld morphism in [FGST06]. The map $\gamma$ is denoted $\hat{\phi}^{-1}$ in [FGST06]. Under our assumptions, the inverse of $\gamma$ exists (see the proof of Lemma 5.9), but we do not use it.

It is not too difficult to show (see e.g. [Iba15, Remark IV.1.2]) that

$$S = \chi \circ \gamma.$$ 

Consider the space of left $q$-characters:

$$\text{Ch}^l(H) = \{ \beta \in H^* \mid \forall x, y \in H, \ \beta(xy) = \beta(S^2(y)x) \}.$$
These maps satisfy the following restrictions:

\[ \chi : \text{Ch}^l(H) \rightarrow \mathcal{Z}(H), \quad \gamma : \mathcal{Z}(H) \rightarrow \text{Ch}^l(H). \]

This is due to the fact that they intertwine the adjoint and the coadjoint actions (for the first the computation is analogous to that of the proof of Proposition 3.3.1 while the second is immediate by Proposition 5.3.1. It follows that \( \mathcal{Z}(H) \) is stable under \( S \) and \( T \). But since \( S^2 \) is inner, we have \( S^4(z) = z \) for each \( z \in \mathcal{Z}(H) \). Thus there exists a projective representation \( \rho_{LM} \) of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathcal{Z}(H) \), defined by:

\[ \rho_{LM}(s) = S_{\mathcal{Z}(H)}, \quad \rho_{LM}(t) = T_{\mathcal{Z}(H)}. \]

As a corollary of these remarks, we have the following lemma.

**Lemma 5.9.** 1) \( H \) is unimodular, which means that there exists \( c \in \mathcal{Z}(H) \), called two-sided cointegral, such that \( xc = \varepsilon(x)c \) for all \( x \in H \).

2) \( H \) is unbalanced, which means that \( \mu^t = \mu^r(g^2?) \).

**Proof:** 1) Since \( H \) is factorizable, \( \chi \) is invertible. Thus \( \gamma = \chi^{-1} \circ S \) is also invertible. Let \( c = \gamma^{-1}(\varepsilon) \). Then \( c \in \mathcal{Z}(H) \) and

\[ \gamma(xc)(h) = \mu^r(S(xc)h) = \mu^r(S(c)S(x)h) = \varepsilon(S(x)h) = \varepsilon(x)\mu^r(S(c)h) = \varepsilon(x)\gamma(c)(h). \]

Since \( \gamma \) is invertible, it follows that \( xc = \varepsilon(x)c \).

2) The terminology “unbalanced” is picked from [BBG18], where some facts about integrals and cointegrals are recalled. Let \( a \in H \) be the comodulus of \( \mu^r \): \( \psi \mu^r = \psi(a)\mu^r \) for all \( \psi \in \mathcal{O}(H) \) (see e.g. [BBG18 eq. 4.9]). By a result of Drinfeld (see [Mon93 Prop. 10.1.14]), but be aware that in this book the notations and conventions for \( \alpha \) and \( g \) are different from those we use), we know that:

\[ uS(u)^{-1} = a(a \otimes \text{id})(R) \]

where \( a \in H^* \) is the modulus of the left cointegral \( c^l \) of \( H \). Here, since \( c = c^l \) is two-sided, we have \( a = \varepsilon \). Thus \( g^2 = u^2v^{-2} = uS(u)^{-1} = a \) by [8] and [10]. We deduce that

\[ \mu^l = \mu^r \circ S = \mu^r(a?) = \mu^r(g^2?) \]

where the second equality is [BBG18 Prop. 4.7]. \( \square \)

The left \( q \)-characters are nothing more than shifted symmetric linear forms. More precisely, we have an isomorphism of algebras:

\[ (g^{-1})^* : \text{SLF}(H) \rightarrow \text{Ch}^l(H), \quad \psi \mapsto \psi(g^{-1})^?. \]

Let us define shifted versions of \( \chi \) and of \( \gamma \):

\[ \chi_{g^{-1}} = \chi \circ (g^{-1})^* : \text{SLF}(H) \rightarrow \mathcal{Z}(H), \quad \gamma_g = g^* \circ \gamma : \mathcal{Z}(H) \rightarrow \text{SLF}(H). \]

The equality \( S = \chi_{g^{-1}} \circ \gamma_g \) still holds, but we have now \( \text{SLF}(H) \) instead of \( \text{Ch}^l(H) \).

In order to show the equivalence of \( \rho_{SLF} \) and \( \rho_{LM} \), we begin with two technical lemmas.

**Lemma 5.10.** 1. \( (v^{-1})^t \otimes S^{-1}((v^{-1})^t) = S((v^{-1})^t) \otimes (v^{-1})^t) \).

2. \( \forall \psi \in \mathcal{O}(H), \forall h \in H \), \( \mu^r(h?)\psi = \mu^r(h^l?)\psi(S^{-1}(h^l?)). \)

**Proof:** The first equality is easy to show with formulas [8] and [13]. For the second one:

\[ \langle \mu^r(h?)\psi, x \rangle = \mu^r(hx)\psi(x) = \mu^r(h'x)\psi(S^{-1}(h')h''x'') = \psi(S^{-1}(h''\mu^r(h'x)h''x''')) = \psi(S^{-1}(h''\mu^r((h'x')(h'x)x')) = \mu^r(h'x)\psi(S^{-1}(h'')) \]

where we simply used the defining property [25] of \( \mu^r \). \( \square \)

We will employ an immediate consequence of Lemma 5.9

\[ \varphi_v = \mu^l(v^{-1})^{-1} \mu^r(gv^{-1}) \]

(30)
Lemma 5.11. It holds:
\[ \rho_{\text{SLF}}(v_A^2 v_B) = \mu^1(v^{-1})^{-1} \gamma_g \circ \chi_{g^{-1}}. \]

Proof: We compute each side of the equality. On the one hand:
\[ \gamma_g \circ \chi_{g^{-1}}(\psi) = \gamma_g((\psi \otimes \text{id})(g^{-1}(v^{-1})' v \otimes (v^{-1})' v)) = \psi(g^{-1}(v^{-1})' v) \mu^r(gS((v^{-1})') v) \]
whereas on the other hand:
\[ v_A^2 v_B \triangleright \psi = (\varphi_v \psi v')^v = \mu^l(v^{-1})^{-1} (\mu^r(gv^{-1} ?) \psi v')^v = \mu^l(v^{-1})^{-1} [\mu^r(g(v^{-1})') \psi(vg^{-1}S^{-1}((v^{-1})'))]^v = \mu^l(v^{-1})^{-1} \mu^r(gvS((v^{-1})') \psi(vg^{-1}(v^{-1})) \]
as desired. We used (30) and Lemma 5.10. \hfill \Box

The link between the two presentations of SL\(_2(\mathbb{Z})\) is \( s = \tau_a \tau_b \tau_a, t = \tau_a^{-1} \). Hence we define two operators \( S', T': \text{SLF}(H) \to \text{SLF}(H) \) by:
\[ S' = \rho_{\text{SLF}}(v_A v_B v_A), \quad T' = \rho_{\text{SLF}}(v_A^{-1}). \]

Theorem 5.12. Recall that we assume that \( H \) is a finite-dimensional factorizable ribbon Hopf algebra. Then the projective representation \( \rho_{\text{SLF}} \) of Theorem 5.11 is equivalent to \( \rho_{\text{LM}} \).

Proof: Consider the following isomorphism of vector spaces:
\[ f = \rho_{\text{SLF}}(v_A^{-1}) \circ \gamma_g : \ Z(H) \to \text{SLF}(H), \quad z \mapsto \gamma_g(z)^{v^{-1}} = \mu^r(gv^{-1}S(z)) . \]

By Lemma 5.11
\[ S' = \mu^l(v^{-1})^{-1} \rho_{\text{SLF}}(v_A^{-1}) \circ \gamma_g \circ \chi_{g^{-1}} \circ \rho_{\text{SLF}}(v_A). \]

Thus:
\[ f \circ S = \rho_{\text{SLF}}(v_A^{-1}) \circ \gamma_g \circ \chi_{g^{-1}} \circ \gamma_g = \mu^l(v^{-1}) S' \circ f. \]

Next,
\[ f \circ T(z) = f(v^{-1} z) = \gamma_g(z)^{v^{-2}} = \rho_{\text{SLF}}(v_A^{-1}) (\gamma_g(z)^{v^{-1}}) = T' \circ f(z) \]

Then \( f \) is an intertwiner. \hfill \Box

6 The example of \( H = \overline{U}_q(\mathfrak{sl}(2)) \)

Let \( q \) be a primitive root of unity of order \( 2p \), with \( p > 2 \). We now work in some detail the case of \( H = \overline{U}_q(\mathfrak{sl}(2)) \), the restricted quantum group associated to \( \mathfrak{sl}(2) \), which will be denoted \( \overline{U}_q \) in the sequel. The definitions, notations, conventions and main properties about \( \overline{U}_q, \ Z(\overline{U}_q), \text{SLF}(\overline{U}_q) \), their canonical bases, and about the \( \overline{U}_q \)-modules used here are summarized in the first pages of [Fai18], to which we refer in order to keep this text compact.

To explicitly describe the representation of \( \text{SL}(2, \mathbb{Z}) \), we use the GTA basis of \( \text{SLF}(\overline{U}_q) \) which is studied in detail in [Fai18], and which has been defined in [GT09] and [Ari10].

In principle, since \( \overline{U}_q \) is not braided (see below), it is not clear that the previous results still hold. In practice, the universal \( R \)-matrix belongs to an extension of \( \overline{U}_q \) by a square root of \( K \), and although some computations occur in the extension, the final result always belongs to \( \overline{U}_q \). This is explained in what follows.
6.1 The braided extension of $\overline{U}_q$

Recall that $\overline{U}_q$ is not braided itself. But its extension by a square root of $K$ is braided, as shown in [FGST06]. Let $\overline{U}_q^{1/2}$ be this extension and $R \in \overline{U}_q^{1/2} \otimes \overline{U}_q^{1/2}$ be the universal $R$-matrix, given by

$$R = q^{H \otimes H/2} \sum_{m=0}^{p-1} \frac{q^m}{[m]!} q^{m(n-1)/2} E^m \otimes F^m,$$

with $q^{H \otimes H/2} = \frac{1}{4p} \sum_{n,j=0}^{4p-1} q^{-nj/2} K^{n/2} \otimes K^{j/2}$.

where $q^{1/2}$ is a fixed square root of $q$. We use the notation $q^{H \otimes H/2}$ because $q^{H \otimes H/2} v \otimes w = q^{h/2}$ if $K^{1/2} v = q^{a/2} v$ and $K^{1/2} w = q^{b/2} w$. Recall that $\tilde{q} = q - q^{-1}$.

Then $RR' \in \overline{U}_q \otimes \overline{U}_q$ and $v \in \overline{U}_q$ (we choose $g = K^{p+1}$ as pivotal element, and by (10) this fixes the choice of $v$). Moreover, even if it is not braided, $\overline{U}_q$ is factorizable, in the sense that the morphism $\Psi$ of section 2.2 is an isomorphism of vector spaces.

Let $I$ be a $\overline{U}_q^{1/2}$-module. Since $\overline{U}_q \subset \overline{U}_q^{1/2}$, $I$ determines a $\overline{U}_q$-module, which we denote $I_{\overline{U}_q}$. We say that a $\overline{U}_q$-module $J$ is liftable if there exists a $\overline{U}_q^{1/2}$-module $\tilde{J}$ such that $J_{\overline{U}_q} = J$. Not every $\overline{U}_q$-module is liftable, see [KST11]. But the simple modules and the PIMs are liftable, which is enough for us. Indeed, it suffices to define the action of $K^{1/2}$ on these modules. Take back the notations of [Fai18] for the canonical basis of modules. For the simple module $X^\epsilon(s)$ ($\epsilon \in \{\pm\}$), there are two choices for $\epsilon^{1/2}$, and so the two possible liftings are defined by

$$K^{1/2} v_j = \epsilon^{1/2} q^{(s-1-2j)/2} v_j$$

and the action of $E$ and $F$ is unchanged. The two possible liftings of the PIM $P^\epsilon(s)$ are defined by

$$K^{1/2} b_0 = \epsilon^{1/2} q^{(s-1)/2} b_0, \quad K^{1/2} x_0 = \epsilon^{1/2} q^{(p-s-1)/2} x_0,$$

$$K^{1/2} y_0 = (-\epsilon^{1/2}) q^{(p-s-1)/2} y_0, \quad K^{1/2} a_0 = \epsilon^{1/2} q^{(s-1)/2} a_0$$

and the action of $E$ and $F$ is unchanged.

Let $X^{-}(1)$ be the 1-dimensional $\overline{U}_q^{1/2}$-module with basis $v$ defined by $Ev = Fv = 0$, $K^{1/2} v = -v$. If $\tilde{I}$ is a lifting of a simple module or a PIM $I$, then we have seen that the only possible liftings of $I$ are $\tilde{I}$ and $\tilde{I} \otimes X^{-}(1)$. Moreover, using (2), we get equalities which will be used in the next section:

$$\begin{align*}
(\tilde{I} \otimes X^{-}(1)) \tilde{J} & = (\tilde{I} \otimes X^{-}(1)) \tilde{J} \\
R_{\tilde{I} \tilde{J}} & = R_{\tilde{I} \tilde{J}}(K^p)_2, \\
(\tilde{I} \otimes X^{-}(1)) \tilde{J} & = (\tilde{I} \otimes X^{-}(1)) \tilde{J} \\
(R')_{\tilde{I} \tilde{J}} & = (R')_{\tilde{I} \tilde{J}}(K^p)_2.
\end{align*}$$

(31)

6.2 $\mathcal{L}_{0,1}(\overline{U}_q)$ and $\mathcal{L}_{1,0}(\overline{U}_q)$

We define $\mathcal{L}_{0,1}(\overline{U}_q)$ as the quotient of $T(\overline{U}_q^*)$ by the fusion relation

$$I \otimes J = M_{12} = M_{1}(R')_{12} M_{2}(R^{-1})_{12},$$

where $I, J$ are simple modules or PIMs and $\tilde{I}, \tilde{J}$ are liftings of $I$ and $J$. From (31) and the fact that $K^p$ is central, we see that this does not depend on the choice of $\tilde{I}$ and $\tilde{J}$. As we saw in section 2.1, the matrix coefficients of the PIMs linearly span $\mathcal{L}_{0,1}(H)$, thus we can restrict to them in the definition. However, the simple modules are added for convenience. All the results of section 3 remain true for $\mathcal{L}_{0,1}(\overline{U}_q)$. In particular, $\Psi_{0,1}$ is an isomorphism since $\overline{U}_q$ is factorizable.

We now describe $\mathcal{L}_{0,1}(\overline{U}_q)$ by generators and relations. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \tilde{R} = \begin{pmatrix} \tilde{x}(2) & \tilde{y}(2) \\ \tilde{y}^{(2)} & \tilde{x}(2) \end{pmatrix} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \tilde{q} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \tilde{q} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

where $q^{1/2}$ is a fixed square root of $q$. We use the notation $q^{H \otimes H/2}$ because $q^{H \otimes H/2} v \otimes w = q^{h/2}$ if $K^{1/2} v = q^{a/2} v$ and $K^{1/2} w = q^{b/2} w$. Recall that $\tilde{q} = q - q^{-1}$.
where \( \tilde{X}^+(2) \) is the lifting of \( X^+(2) \) defined by \( K^{1/2}v_0 = q^{1/2}v_0 \). By the decomposition rules of tensor products (see \[\text{Sim94}\], and also \[\text{KS11}, \text{Iba15}\]), every PIM (and every simple module) is a direct summand of some tensor power \( X^+ (2)^{\otimes n} \). Thus every matrix coefficient of a PIM is a matrix coefficient of some \( X^+ (2)^{\otimes n} \) (with \( n \geq p \)). It follows from the fusion relation that \( a, b, c, d \) generate \( \mathcal{L}_{0,1}(U_q) \). Let us seek the relations. We emphasize that each relation corresponds to a particular morphism, as we explain now. First, we have seen in Proposition 3.2 that the braiding morphism \( P\tilde{R} : X^+ (2)^{\otimes 2} \rightarrow X^+ (2)^{\otimes 2} \) provides the reflection equation. Proceeding as in the proofs of lemmas 3.5 and 4.6, we invert the reflection equation:

\[
\tilde{X}^+(2)
\]

\[
M_1M_2 = S^{-1}(a_i)\tilde{R}^{-1}_{12}M_2\tilde{R}_{12}M_1\tilde{R}_{21} (b_i).
\]

A calculation gives the following exchange relations:

\[
da = ad, \quad db = q^2bd, \quad dc = q^{-2}cd.
ba = ab + q^{-1}qbd, \quad cb = bc + q^{-1}q(da - d^2), \quad ca = ac - q^{-1}qdc.
\]

Second, since \( X^+ (2)^{\otimes 2} \cong X^+ (1) \oplus X^+ (3) \), up to scalar there exists a unique morphism \( \Phi : X^+ (1) \rightarrow X^+ (2)^{\otimes 2} \). It is easily computed:

\[
\Phi(1) = qv_0 \otimes v_1 - v_1 \otimes v_0.
\]

By functoriality and fusion, we have

\[
X^+ (2)^{\otimes 2} \quad M_{12} \Phi = M_1 R_{21} M_2 R_{21}^{-1} \Phi = \Phi.
\]

This gives just one new relation, which is the analogue of the quantum determinant:

\[
ad - q^2 bc = 1.
\]

Let us compute the RSD isomorphism on \( M \):

\[
\Psi_{0,1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{X^+ (2)}{X^+ (2)} \frac{L^+}{L^+}^{-1} = \begin{pmatrix} K^{1/2} & \hat{q}K^{1/2} F \\ 0 & K^{-1/2} \end{pmatrix} \begin{pmatrix} K^{-1/2} & 0 \\ \hat{q} K^{-1/2} E & K^{-1/2} \end{pmatrix} = \begin{pmatrix} K + q^{-1} \hat{q}^2 F E & q^{-1} \hat{q} F \\ \hat{q} K^{-1/2} E & K^{-1} \end{pmatrix}.
\]

We deduce the relations \( b^p = c^p = 0 \) and \( d^{2p} = 1 \) from the defining relations of \( U_q \). Let us mention that it is possible to find two morphisms \( f_1, f_2 \) defined by

\[
\begin{align*}
f_1 : & \quad \mathcal{P}^+(p-1) \rightarrow X^+ (2)^{\otimes p} \\
& \quad b_0^+(p-1) \mapsto -qv_1^+(2) \otimes v_1^+(2)^{\otimes p-1}, \\
& \quad b_{p-2}^+(p-1) \mapsto v_0^+(2) \otimes v_1^+(2)^{\otimes 2p-1}.
\end{align*}
\]

One can show using functoriality that the relations \( b^p = c^p = 0 \) are consequences of the existence of \( f_1 \) and that the relation \( d^{2p} = 1 \) is a consequence of the existence of \( f_2 \). The proof rely on matrix computations; the details will be provided in [Equa].

**Theorem 6.1.** \( \mathcal{L}_{0,1}(U_q) \) admits the following presentation:

\[
\begin{pmatrix}
da = ad, & db = q^2bd, & dc = q^{-2}cd \\
ba = ab + q^{-1}qbd, & cb = bc + q^{-1}q(da - d^2), & ca = ac - q^{-1}qdc \\
ad - q^2bc = 1, & b^p = c^p = 0, & d^{2p} = 1
\end{pmatrix}.
\]

A basis is given by the monomials \( b^i c^j d^k \) with \( 0 \leq i, j, k \leq p - 1, 0 \leq k \leq 2p - 1 \).

**Proof:** Let \( A \) be the algebra defined by this presentation. It is readily seen that \( a = d^{-1} + q^2bcd^{-1} \) and that the monomials \( b^i c^j d^k \) with \( 0 \leq i, j, k \leq p - 1, 0 \leq k \leq 2p - 1 \) linearly span \( A \). Thus \( \dim(A) \leq 2p^3 \). But we know that \( 2p^3 = \dim(U_q) = \dim(\mathcal{L}_{0,1}(U_q)) \) since the monomials \( E^iF^jK^k \) with \( 0 \leq i, j, k \leq p - 1, 0 \leq k \leq 2p - 1 \) form the PBW basis of \( U_q \). It follows that \( \dim(A) \leq \dim(\mathcal{L}_{0,1}(U_q)) \).

Since these relations are satisfied in \( \mathcal{L}_{0,1}(U_q) \), there exists a surjection \( p : A \rightarrow \mathcal{L}_{0,1}(U_q) \). Thus \( \dim(A) \geq \dim(\mathcal{L}_{0,1}(U_q)) \), and the theorem is proved. \( \square \)
Remark 6. A consequence of this theorem is that $L_{0,1}(U_q)$ is a restricted version (i.e., a finite-dimensional quotient by monomial central elements) of $L_{0,1}(U_q)^{spe}$, the specialization at our root of unity $q$ of the algebra $L_{0,1}(U_q)$. A complete study of the algebra $L_{0,1}(U_q)^{spe}$ will appear in [BaR]. Let us also mention that, specializing the RSD morphism of $L_{0,1}(U_q)$, we get a new morphism:

$$
\Psi_{0,1} : L_{0,1}(U_q)^{spe} \xrightarrow{\Psi_{0,1}^{spe}} U_q \xrightarrow{\pi} U_q
$$

where $\pi$ is the canonical projection. It is easy to see that $\ker(\Psi_{0,1}) = \langle b^p, c^p, d^{2p} - 1 \rangle$ and we obtain $L_{0,1}(U_q) \cong L_{0,1}(U_q)^{spe} / \ker(\Psi_{0,1})$.

Applying the isomorphism of algebras $D$ defined in (17) to the GTA basis of $SLF(U_q)$, we get a new basis of $Z(U_q)$. We introduce notations for these basis elements:

$$
\chi^s \quad \text{with } 1 \leq s \leq p \text{ and } \epsilon \in \{\pm\} \text{ and } 1 \leq s' \leq p - 1. \text{ They satisfy the same multiplication rules than the elements of the GTA basis, see [Fai18 Section 5] or [GT09] (the elements $\chi(s)$ defined in [GT09] correspond to $[s]H^s$ here).}
$$

Let us mention that under the identification $L_{0,1}(U_q) = U_q$ via $\Psi_{0,1}$, it holds by definition $W = \text{tr}(K^{p+1} M^\gamma)$, since we choose $K^{p+1}$ as pivotal element. In particular,

$$
\chi^{+(2)} \quad W = -qa - q^{-1}d = -q^2 F E - qK - q^{-1}K^{-1} = -q^2C
$$

where $C$ is the standard Casimir element of $U_q$.

Similarly, we define $L_{1,0}(U_q)$ as the quotient of $L_{0,1}(U_q) \ast L_{0,1}(U_q)$ by the exchange relations:

$$
\frac{\tilde{I} \tilde{J}}{R_{12}B_{12}(R')_{12}A_{2}} = A_{2}R_{12}B_{12}(R^{-1})_{12}
$$

where $I, J$ are simple modules or PIMs and $\tilde{I}, \tilde{J}$ are liftings of $I$ and $J$. From (31), we see again that this does not depend on the choice of $\tilde{I}$ and $\tilde{J}$. The coefficients of $A$ and of $B$ generate $L_{1,0}(U_q)$. Using the commutation relations of the Heisenberg double, it is easy to show that $\Psi_{1,0}$ indeed takes values in $H(\mathcal{O}(U_q))$ (the square root of $K$ does not appear). In order to obtain a presentation of $L_{1,0}(U_q)$, one can again restrict to $I = J = \chi^{+(2)}$ and write down the corresponding exchange relations. We do not give this presentation of $L_{1,0}(U_q)$ since we will not use it in this work.

6.3 Explicit description of the $SL_2(\mathbb{Z})$-projective representation

Note that it can be shown directly that $U_q$ is unimodular and unibalanced, see for instance [Iba15 Cor. II.2.8] (also note that in [RBGTS] it is shown that all the simply laced restricted quantum groups at roots of unity are unibalanced). In this way, we recover that $\gamma$ (and thus $\gamma_9$) is invertible. Indeed, one can check that $\gamma^{-1}(\psi) = \psi(c')c''$, where $c$ is the two-sided integral.

Proposition 6.2. For all $z \in Z(U_q)$, $S(z) = z$ and for all $\psi \in SLF(U_q)$, $S(\psi) = \psi$. It follows that in the case of $U_q$, $\rho_{SLF}$ is in fact a projective representation of $PSL_2(\mathbb{Z})$.

Proof: By [FGST06 Appendix D], the canonical central elements are expressed as $e_s = P_s(C)$, $w_s^\pm = \pi_s^\pm Q_s(C)$ where $P_s$ and $Q_s$ are polynomials, $C$ is the Casimir element and $\pi_s^\pm$ are Fourier transforms of $(K^\gamma)^{0 \leq j \leq 2p-1}$. It is easy to check that $S(C) = C$ and that $S(\pi_s^\pm) = \pi_s^\mp$, thus $S(e_s) = e_s$ and $S(w_s^\pm) = w_s^\mp$. Next, let $\psi \in SLF(U_q)$. Since $\gamma_9$ is an isomorphism, we can write $\psi = \gamma_9(z)$ as follows:

$$
S(\psi) = S(\mu^r(gz?)) = \mu^r \circ S(?zg^{-1}) = \mu^l(g^{-1}z?) = \mu^l(gz?) = \psi.
$$
We used that $S(z) = z$, the Proposition\textsuperscript{[5.3]} and the fact that $\overline{U}_q$ is unbalanced. \hfill $\Box$

We want to determine the action of $\hat{\alpha} = v_A^{-1}$ and $\hat{\beta} = v_B^{-1}$ on the GTA basis. Some preliminaries are in order. First recall (see \cite{FGST06} or \cite{Iba15}) that the expression of the ribbon element $v$ in the canonical basis $(e_s, w_s^\pm)$ of $Z(\overline{U}_q)$ is:

$$v = \sum_{s=0}^{p} v_{\chi^+(s)} e_s + \hat{q} \sum_{s=1}^{p-1} v_{\chi^+(s)} \left( \frac{p-s}{s} w_s^+ - \frac{s}{s} w_s^- \right)$$

(33)

with $\hat{q} = q - q^{-1}$ and

$$v_{\chi^+(s)} = v_{\chi^-(p-s)} = (-1)^{s-1} q^{\frac{(s^2-1)}{2}}.$$

Note that $v_{\chi^+(s)} = v_{\chi^+(0)}$ and $v_{\chi^+(0)}$ is just a notation used to unify the formula. Expressing $v$ in this basis is obvious. Second, it is easy to see that the action of $Z(\overline{U}_q)$ on $\text{SLF}(\overline{U}_q)$ is:

$$\left( \chi^+_\delta \right)^{\delta \epsilon} = \delta_{s,t} \chi^+_{\delta \epsilon}, \quad \left( \chi^-_\delta \right)^{\delta \epsilon} = \delta_{p-s,t} \chi^-_{\delta \epsilon}, \quad G^{\epsilon_1}_\delta = \delta_{s,t} G_s \quad G^{\epsilon_2}_s = \delta_{s,t} \chi^+_s, \quad G^{\epsilon_3}_s = \delta_{s,t} \chi^-_{p-s}. \quad (34)$$

Finally, we have the following lemma.

**Lemma 6.3.** Let $z \in \mathcal{L}_{0,r}^\mu(H) = Z(H)$ and let $\psi \in \text{SLF}(H)$. Then:

$$z_{B^{-1}A} \triangleright \psi = S(D^{-1}(z))^{-1} \psi.$$

**Proof:** The proof is analogous to those of the two similar results in section \textsuperscript{[4.3]} and is thus left to the reader. Note that this lemma is not specific to the case of $\overline{U}_q$. \hfill $\Box$

**Theorem 6.4.** The actions of $\hat{\alpha}$ and $\hat{\beta}$ on the GTA basis are given by:

$$\hat{\alpha} \triangleright \chi^+ = v_{\chi^+(s)} \chi^+ \quad \hat{\alpha} \triangleright G_{s'} = v_{\chi^+(s')} G_{s'} - v_{\chi^+(s)} \hat{q} \left( \frac{p-s}{s} \chi^+_s - \frac{s'}{s} \chi^-_{p-s} \right)$$

and

$$\hat{\beta} \triangleright \chi^+ = \xi(-\epsilon)^{p-1} q^{-s(\epsilon^2-1)} \left( \sum_{\ell=1}^{p-1} (-1)^{s-1} \ell (-\epsilon)^{p-\ell} (q^\ell + q^{-\ell}) \left( \chi^\ell_p + \chi^-_{p-\ell} \right) + \chi^+_{p} + (-\epsilon)^{p-1} \chi^-_{p} \right)$$

$$\hat{\beta} \triangleright G_{s'} = \xi(-\epsilon)^{s'} q^{-s(\epsilon^2-1)} \hat{q} \left( \sum_{j=1}^{p-1} (-\epsilon)^{p+j} \left[ j \right] G_j \right).$$

with $\epsilon \in \{\pm\}, \ 0 \leq s \leq p, \ 1 \leq s' \leq p-1$ and $\xi \in \mathbb{C}\{0\}$ is a scalar depending only on the normalization of $\mu^l$.

**Proof:** The formulas for $\hat{\alpha}$ are easily deduced from Proposition\textsuperscript{[4.10]} (33) and (34). Computing the action of $\hat{\beta}$ is more challenging. It turns out that it is not a good idea to use directly Proposition\textsuperscript{[4.10]} Instead, we will use the commutation relations of $\hat{\beta}$ with the $A, B$-matrices, namely

$$\hat{\beta} A = B^{-1} A \hat{\beta}, \quad \hat{\beta} B = B \hat{\beta} \quad (35)$$

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to compute the action of $\hat{\beta}$ by induction. The multiplication rules of the GTA basis (see [Fai18, Section 5]) will be used several times. Let us denote

$$\hat{\beta} \triangleright \chi^e_s = \sum_{\sigma(\pm)} \sum_{\ell=1}^{p} \lambda^\sigma_{\ell}(\epsilon, s) \chi^\sigma_\ell + \sum_{j=1}^{p-1} \delta_j(\epsilon, s) G_j.$$  

Relation (35) provides $\hat{\beta} \ W_A = \chi^{\mp(2)} A \ W_{B^{-1} A} \hat{\beta}$. On the one hand, we obtain by (32):

$$\hat{\beta} \ W_A \triangleright \chi^e_s = \hat{\beta} \triangleright \chi^e_s(-q^2 C') = \sum_{\sigma(\pm)} \sum_{\ell=1}^{p} -\epsilon(q^s + q^{-s}) \lambda^\sigma_{\ell}(\epsilon, s) \chi^\sigma_\ell + \sum_{j=1}^{p-1} -\epsilon(q^s + q^{-s}) \delta_j(\epsilon, s) G_j.$$  

On the other hand, we use Lemma 6.3 and the multiplication rules:

$$\chi^{\mp(2)} A \ W_{B^{-1} A} \hat{\beta} \triangleright \chi^e_s = \sum_{\sigma(\pm)} \sum_{\ell=1}^{p} \lambda^\sigma_{\ell}(\epsilon, s) \chi^\mp_{\ell} + \sum_{j=1}^{p-1} \delta_j(\epsilon, s) \chi^\mp_j G_j$$

$$= \sum_{\sigma(\pm)} (\lambda^\sigma_{\ell}(\epsilon, s) + 2 \lambda^\sigma_{\ell}(\epsilon, s)) \chi^\sigma_\ell + \sum_{\ell=1}^{p} \chi^\ell_{\ell-1} \chi^\ell_{\ell} + \chi^\ell_{\ell+1} \chi^\ell_{\ell} + \chi^\ell_{p-2} \chi^\ell_{p-1} + \chi^\ell_{p-1} \chi^\ell_{p-1}$$

$$+ \lambda^\sigma_{p-1}(\epsilon, s) \chi^\sigma_{p-1} + \delta_2(\epsilon, s) \frac{[2]}{[2]} G_1 + \sum_{j=2}^{p-2} [j] \left( \frac{\delta_j(\epsilon, s)}{[j]} + \frac{\delta_{j+1}(\epsilon, s)}{[j+1]} \right) G_j + \delta_{p-2}(\epsilon, s) \frac{G_{p-1}}{[2]}.$$  

This gives recurrence equations between the coefficients which are easily solved:

$$\hat{\beta} \triangleright \chi^e_s = \lambda(\epsilon, s) \left( \sum_{\ell=1}^{p-1} (-1)^s (-\epsilon) \lambda^\ell_{\ell}(q^s + q^{-s}) (\chi^\ell_{\ell} + \chi_{p-\ell}) + \chi^\ell_{p-1} + (-\epsilon) \lambda^\ell_{p-1} \chi^\ell_{p-1} \right)$$

$$+ \delta(\epsilon, s) \sum_{j=1}^{p-1} (-\epsilon)^{j+1} \frac{[j][j]}{[s]} G_j.$$  

The coefficients $\lambda(\epsilon, s) = \lambda^\ell_{p}(\epsilon, s)$ and $\delta(\epsilon, s) = \delta_1(\epsilon, s)$ are still unknown. In order to compute them by induction, we use the relation $\chi^{\mp(2)} A \ W_{B^{-1} A} \hat{\beta}$, which is another consequence of (35). Before, note that

$$\chi^{\mp(2)} A \ W_{B^{-1} A} \hat{\beta} \triangleright \chi^e_s = \left( \lambda^\ell_{\ell}(\epsilon, s) \right) \chi^\ell_{\ell-1} + \frac{\lambda^\ell_{\ell}(\epsilon, s)}{\lambda^\ell_{\ell}(s+1)} \chi^\ell_{\ell+1} = -\epsilon q^s q^{-\ell} \chi_{s-1} - \epsilon q^s q^{-\ell} \chi_{s+1}.$$  

with $1 \leq s \leq p - 1$ and the convention that $\chi^0_{s} = 0$. It follows that

$$\hat{\beta} \triangleright \chi^e_{s+1} = -\epsilon q^{-s} \frac{\chi^{\mp(2)} A \ W_{B^{-1} A} \hat{\beta}}{2} \triangleright \chi^e_{s-1}.$$  

(36)

Due to (33), (34) and the multiplication rules, we have

$$\chi^{\mp(2)} A \ W_{B^{-1} A} \hat{\beta} \triangleright \chi^e_s = \left( \lambda^\ell_{\ell}(\epsilon, s) \right) \chi^\ell_{\ell-1} + \frac{\lambda^\ell_{\ell}(\epsilon, s)}{\lambda^\ell_{\ell}(s+1)} \chi^\ell_{\ell+1} + \delta_{p-1}(\epsilon, s) \chi^\ell_{p-1} + \frac{\lambda^\ell_{p-1}(\epsilon, s)}{[2]} G_1 + \ldots$$  

where the dots (\ldots) mean the remaining of the linear combination in the GTA basis. After replacing by the values found previously and inserting in relation (36), this yields

$$\lambda(\epsilon, s) \chi^\ell_{s+1} + \delta(\epsilon, s) \chi^\ell_{s-1},$$

$$= (\epsilon q^{s+1})(q^s + q^{-s}) \lambda(\epsilon, s) + (\epsilon)^{s+1}(q^s + q^{-s}) \delta(\epsilon, s) \chi^\ell_{p-1}$$

$$+ (\epsilon q^{s+1})(q^s + q^{-s}) \delta(\epsilon, s) - \epsilon q^{-s} \delta(\epsilon, s - 1)) G_1 + \ldots.$$  

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These are recurrence equations. It just remains to determine the first values \(\lambda(\epsilon, 1), \delta(\epsilon, 1).\) Observe that, since \(\U_q\) is unbalanced:

\[
\hat{\beta} \triangleright \chi^+_1 = (\varphi_{v^{-1}}(\chi^+_1)v)^{v^{-1}} = \mu^l(v)^{-1}\mu^l(K^{p-1}v)^{v^{-1}} = \mu^l(v)^{-1}\mu^r(K^{p+1}?).
\]

In [Fai18, Section 4.3] the decomposition of \(\mu^r(K^{p+1}?\) in the GTA basis has been found (when \(\mu^r\) is suitably normalized). Thanks to this, we obtain

\[
\hat{\beta} \triangleright \chi^+_1 = \lambda(+, 1)\chi^+_p + \delta(+, 1)G_1 + \ldots = \xi(-1)^{p-1}\chi^+_p - \xi G_1 + \ldots
\]

and

\[
\hat{\beta} \triangleright \chi^-_1 = v_{\chi^-(1)} \chi^-_1 v_{\chi^-(1)} = \chi^-_1 \left(\hat{\beta} \triangleright \chi^+_1\right)^{v^{-1}}
\]

\[
= -\xi\chi^+_p + \xi G_1 + \ldots = \lambda(-, 1)\chi^+_p + \delta(-, 1)G_1 + \ldots
\]

where \(\xi\) depends only on the normalization of \(\mu^r\) (and thus of \(\mu^l\)). We are now in position to solve the recurrence equations. It is easy to check that the solutions are

\[
\delta(\epsilon, s) = \xi(1)^sq^{(s^2-1)}[s], \quad \lambda(\epsilon, s) = \xi(1)^spq^{(s^2-1)}.
\]

We now proceed with the proof of the formula for \(G_\delta.\) Relation (35) implies \(\hat{\beta} H_B^1 = H_B^1 \hat{\beta}.\) By (33), (34) and the multiplication rules, we have on the one hand:

\[
\hat{\beta} H_B^1 \triangleright \chi^+_1 = \hat{\beta} \triangleright (G_1(\chi^+_s)^v)^{v^{-1}} = [s] \hat{\beta} \triangleright G_s - \hat{q}(p-s)\hat{\beta} \triangleright \chi^+_s + \hat{q}s\hat{\beta} \triangleright \chi^-_s
\]

whereas on the other hand:

\[
H_B^1 \hat{\beta} \triangleright \chi^+_s = (\hat{\beta} \triangleright (\chi^+_s)^v)^{v^{-1}} = \hat{q}\delta \sum_{j=1}^{p-1} \delta_j(1, +, s) \left(G_j - \hat{q}_j \hat{p}^{\frac{j}{[j]}} \chi_j^+ + \hat{q}_j \hat{p}^{\frac{j}{[j]}} \chi_j^-\right).
\]

Equalizing both sides and inserting the previously found values, we obtain the desired formula.

\[\Box\]

**Remark 7.** The guiding principle of the previous computations was that the multiplication of two symmetric linear forms in the GTA basis is easy when one of them is \(\chi^+_2, \chi^-_1\) or \(G_1\) (see [Fai18, Section 5]), and that all the formulas can be derived from \(\hat{\beta} \triangleright \chi^+_1\) using only such products.

Recall that the standard representation \(\mathbb{C}^2\) of \(\text{SL}_2(\mathbb{Z})\) is \(\text{MCG}(\Sigma_{1,0})\) is defined by

\[
\tau_a \mapsto \left(\begin{array}{cc}1 & 0 \\ -1 & 1\end{array}\right), \quad \tau_b \mapsto \left(\begin{array}{cc}1 & 1 \\ 0 & 1\end{array}\right).
\]

**Lemma 6.5.** Let \(V\) be a (projective) representation of \(\text{SL}_2(\mathbb{Z})\) which admits a basis \((x_s, y_s)\) such that

\[
\tau_a x_s = \sum_{\ell} a_\ell(s)x_\ell, \quad \tau_b x_s = \sum_{\ell} b_\ell(s)(x_\ell + y_\ell)
\]

\[
\tau_a y_s = \sum_{\ell} a_\ell(s)(y_\ell - x_\ell), \quad \tau_b y_s = \sum_{\ell} b_\ell(s)y_\ell.
\]

Then there exists a (projective) representation \(W\) of \(\text{SL}_2(\mathbb{Z})\) such that \(V \cong \mathbb{C}^2 \otimes W\). More precisely, \(W\) admits a basis \((w_s)\) such that

\[
\tau_a w_s = \sum_{\ell} a_\ell(s)w_\ell, \quad \tau_b w_s = \sum_{\ell} b_\ell(s)w_\ell.
\]

**Proof:** It is easy to check that the formulas for \(\tau_a w_s\) and \(\tau_b w_s\) indeed define a \(\text{SL}_2(\mathbb{Z})\)-representation on \(W\). Let \((e_1, e_2)\) be the canonical basis of \(\mathbb{C}^2\). Then

\[
e_1 \otimes w_s \mapsto y_s, \quad e_2 \otimes w_s \mapsto x_s
\]

is an isomorphism which intertwines the \(\text{SL}_2(\mathbb{Z})\)-action.

\[\Box\]

The structure of the Lyubaschenko-Majid representation on \(\mathcal{Z}(\mathbb{U}_q)\) is described in [FGST06]. Here, we recover this result on the \(\text{SLF}(\mathbb{U}_q)\) side (recall from Theorem 5.12 that these representations are equivalent).
Theorem 6.6. The \((p+1)\)-dimensional subspace \(\mathcal{V} = \text{vect}(\chi^+_p + \chi_{p-s}, \chi^+_p)_{1 \leq s \leq p-1}\) is stable under the \(\text{SL}_2(\mathbb{Z})\)-action of Theorem 6.4 Moreover, there exists a \((p-1)\)-dimensional projective representation \(\mathcal{W}\) of \(\text{SL}_2(\mathbb{Z})\) such that

\[
\text{SLF}(\overline{U}_q) = \mathcal{V} \oplus (\mathbb{C}^2 \otimes \mathcal{W}).
\]

Proof: By [Fai18, Cor. 5.1], \(\mathcal{V}\) is an ideal of \(\text{SLF}(\overline{U}_q)\). It is easy to see that \(\mathcal{V}\) is moreover stable under the action \((\overline{34})\) of \(\mathcal{Z}(\overline{U}_q)\). Thus we deduce without any computation that \(\mathcal{V}\) is \(\text{SL}_2(\mathbb{Z})\)-stable. Next, in view of the formulas in Theorem 6.4 it is natural to define

\[
x_s = \frac{q^p - s}{2} \chi^+_s - \frac{q^p}{2} \chi_{p-s}, \quad y_s = G_s - x_s.
\]

Then:

\[
\alpha \triangleright x_s = v^{-1} \chi^+_s x_s, \quad \hat{\beta} \triangleright x_s = \xi(-1)^s q^{-(s^2 - 1)} \frac{\hat{q}^p}{2} \sum_{j=1}^{p-1} (-1)^j [j] [s] (x_j + y_j)
\]

\[
\alpha \triangleright y_s = v^{-1} \chi^+_s (y_s - x_s), \quad \hat{\beta} \triangleright y_s = \xi(-1)^s q^{-(s^2 - 1)} \frac{\hat{q}^p}{2} \sum_{j=1}^{p-1} (-1)^j [j] [s] y_j.
\]

The result follows from Lemma 6.5.

We precise that, explicitly, the projective representation \(\mathcal{W}\) has a basis \((w_s)_{1 \leq s \leq p-1}\) such that

\[
\tau_a w_s = v^{-1} \chi^+_s w_s, \quad \tau_b w_s = \xi(-1)^s q^{-(s^2 - 1)} \frac{\hat{q}^p}{2} \sum_{j=1}^{p-1} (-1)^j [j] [s] w_j.
\]

6.4 A conjecture about the representation of \(\mathcal{L}^{\text{inv}}_{1,0}(\overline{U}_q)\) on \(\text{SLF}(\overline{U}_q)\)

Another natural (but harder) question is to determine the structure of \(\text{SLF}(\overline{U}_q)\) under the action of \(\mathcal{L}^{\text{inv}}_{1,0}(\overline{U}_q)\). As mentioned in the proof of Theorem 6.6, the subspace \(\mathcal{V} = \text{vect}(\chi^+_p + \chi_{p-s}, \chi^+_p)_{1 \leq s \leq p-1}\) is quite “robust”. We have several reasons to think that the following conjecture is true.

Conjecture. \(\mathcal{V}\) is a \(\mathcal{L}^{\text{inv}}_{1,0}(\overline{U}_q)\)-submodule of \(\text{SLF}(\overline{U}_q)\).

In order to prove this conjecture one needs to find a basis or a generating set of \(\mathcal{L}^{\text{inv}}_{1,0}(\overline{U}_q)\), and then to show that \(\mathcal{V}\) is stable under the action of the basis elements (or of the generating elements). Both tasks are difficult.

We quickly summarize the reasons for which we believe in this conjecture. Let us mention that since \(\mathcal{V}\) is an ideal of \(\text{SLF}(\overline{U}_q)\), it is stable under the action of \(z_A, z_B\) and \(z_{B^{-1}} A\) for all \(z \in \mathcal{Z}(\overline{U}_q) = \mathcal{L}^{\text{inv}}_{1,0}(\overline{U}_q)\) (see Proposition 4.10 and Lemma 6.3). Also recall the wide family of invariants given by Proposition 4.2. A long computation (which is not specific to \(\overline{U}_q\)) shows that

\[
\text{tr}_{12} \left( I_{23} \otimes \hat{g}_2 \phi_{12} A_1 (R)_{12} B_2 R_{12} \right) \triangleright \chi^K = v J \text{tr}_{13} \left( I_{12} \otimes K \hat{v}_{13}^{-1} s_{IJ,K}(\phi)_{13} \right)
\]

where \(\chi^K\) is the character of \(K\), \(v = v J \text{id}\) (note that we may assume that \(I, J, K\) are simple modules) and

\[
s_{IJ,K}(\phi) = \text{tr}_2 \left( J K \hat{g}_2 R_{23} \phi_{12} (R)_{23} \right).
\]

Proving that \(\mathcal{V}\) is stable under the action of these invariants amounts to show symmetry properties between \(s_{IJ,\chi^+(s)}\) and \(s_{IJ,\chi^-(p-s)}\) for all simple \(\overline{U}_q\)-modules \(I, J\). We have checked that it is true if \(\phi = \text{id}_{I \otimes J}\) (in this case \(s_{IJ,K}(\text{id}_{I \otimes J}) = s_{I,K} \text{id}_{I \otimes K}\), where \(s_{I,K}\) is the usual \(S\)-matrix) for all simple modules \(I, J\), and also that it holds for \(I = J = \chi^+(2)\) with every \(\phi\).

Let us point out an important consequence of the Conjecture.
Proposition 6.7. Assume that the Conjecture holds. Then the $\mathcal{L}^\text{inv}_{1,p}(\mathcal{U}_q)$-modules $\mathcal{V}$ and $\text{SLF}(\mathcal{U}_q)/\mathcal{V}$ are simple. It follows that $\text{SLF}(\mathcal{U}_q)$ has length $2$ as $\mathcal{L}^\text{inv}_{1,0}(\mathcal{U}_q)$-module.

Proof: This is basically a consequence of (33) and of the multiplication rules in the GTA basis. To avoid particular cases, define $\chi^0 = 0$, $\chi^p_{p+1} = \chi_1^p$, $\chi^p_{-1} = \chi^p_{p-1}$ and $e_{-1} = e_{p+1} = 0$. Let $0 \neq \mathcal{V} \subset \mathcal{V}$ be a submodule, and let $v = \sum_{j=0}^p \lambda_j (\chi^+_j + \chi^-_{p-j}) \in \mathcal{V}$ be non-zero. Assume that $\lambda_s$ is non-zero. Then using Proposition 4.10, we get $(e_s)_A \triangleright v = \lambda_s (\chi^+_s + \chi^-_{p-s})$, and thus $\chi^+_s + \chi^-_{p-s} \in \mathcal{V}$. Apply $W_B$:

$$\chi^+_s + \chi^-_{p-s} = (\chi^+_s + \chi^-_{p-s})^v = -q^{-s+\frac{1}{2}}(\chi^+_s + \chi^-_{p-s+1}) - q^{s+\frac{1}{2}}(\chi^+_s + \chi^-_{p-s-1}).$$

Hence:

$$\chi^+_s + \chi^-_{p-s} = (e_{s-1})_A \ W_B \triangleright (\chi^+_s + \chi^-_{p-s}) = -q^{-s+\frac{1}{2}}(\chi^+_s + \chi^-_{p-s+1})$$

$$\chi^+_s + \chi^-_{p-s} = (e_{s+1})_A \ W_B \triangleright (\chi^+_s + \chi^-_{p-s}) = -q^{s+\frac{1}{2}}(\chi^+_s + \chi^-_{p-s-1}).$$

It follows that $\chi^+_s + \chi^-_{p-s+1}$, $\chi^+_s + \chi^-_{p-s-1} \in \mathcal{V}$. Continuing like this, one gets step by step that all the basis vectors belong to $\mathcal{V}$, hence $\mathcal{V} = \mathcal{V}$.

Next, let $\overline{G}_s$ and $\overline{\chi}^+_s$ be the classes of $G_s$ and $\chi^+_s$ modulo $\mathcal{V}$ (with $\overline{\chi}^+_0 = \overline{\chi}^+_p = 0$). Let $0 \neq \mathcal{W} \subset \text{SLF}(\mathcal{U}_q)/\mathcal{V}$ be a submodule and $w = \sum_{j=1}^{p-1} \nu_j \overline{G}_j + \sigma_j \overline{\chi}^+_j \in \mathcal{W}$ be non-zero. If all the $\nu_j$ are 0, then there exists $\sigma_j \neq 0$ and $(e_s)_A \triangleright w = \sigma_j \overline{\chi}^+_j \in \mathcal{W}$. If one of the $\nu_j$, say $\nu_s$, is non-zero, then $(w^+_s)_A \triangleright w = \nu_s \overline{\chi}^+_s \in \mathcal{W}$. In both cases we get $\overline{\chi}^+_s \in \mathcal{W}$. Now we proceed as previously:

$$\overline{\chi}^+_s = (e_{s-1})_A \ W_B \triangleright \overline{\chi}^+_s = -q^{-s+\frac{1}{2}}\overline{\chi}^+_s,$$

$$\overline{\chi}^+_s = (e_{s+1})_A \ W_B \triangleright \overline{\chi}^+_s = -q^{s+\frac{1}{2}}\overline{\chi}^+_s.$$

Thus we get step by step that $\overline{\chi}^+_j \in \mathcal{W}$ for all $j$. Apply $H^+_B$:

$$\overline{\chi}^+ (G_1 (\chi^+_j)) + \mathcal{V} = \mathcal{V}$$

It follows that $\overline{G}_j \in \mathcal{W}$ for all $j$, and thus $\mathcal{W} = \text{SLF}(\mathcal{U}_q)/\mathcal{V}$ as desired. □

In order to determine the structure of $\text{SLF}(\mathcal{U}_q)$ if the Conjecture is true, it will remain to determine whether the exact sequence

$$0 \rightarrow \mathcal{V} \rightarrow \text{SLF}(\mathcal{U}_q) \rightarrow \text{SLF}(\mathcal{U}_q)/\mathcal{V} \rightarrow 0$$

is split or not, i.e. whether $\mathcal{V}$ is a direct summand of $\text{SLF}(\mathcal{U}_q)$ or not.

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