‘Baryonic’ bound-state instability in trapped fermionic atoms

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Abstract

We consider a homogeneous gas of spin-$S$ fermionic atoms, as might occur near the center of an optical trap. In the case where all scattering lengths are negative and of the same magnitude we demonstrate the instability of the Fermi sea to the condensation of bound ‘baryonic’ composites containing $2S + 1$ atoms. The gap in the excitation spectrum is calculated.

I. INTRODUCTION

Sympathetic cooling of trapped fermionic gases \[1\] is likely to lead to the creation of degenerate Fermi gases in the near future. In magnetic traps, the fermion’s spin is locked to the field direction. However the advent \[2,3\] of optical traps holds the promise of degenerate Fermi gases with unquenched spins and hence a variety of nonideal Fermi gases. Already there has been some discussion of the possibility of pairing \[4–6\] in the presence of attractive interactions, which may be of an exotic type \[7,8\]. In part the exotic possibilities occur because alkali fermions exist which have larger total spins than spin $1/2$: $^{22}$Na, $^{86}$Rb, $^{132}$Cs, $^{134}$Cs, $^{136}$Cs have $S = 5/2$, $5/2$, $3/2$, $7/2$, $9/2$ respectively \[9\]. In this paper we will point out that in addition to pairing there are other possible ground states, in particular when the atoms experience spin-independent attractive interactions.

One-dimensional models of interacting fermions with spins greater than one half have been considered for a substantial time, both for repulsive \[10\] and attractive \[11–13\] inter-
actions. In the attractive case, the ground state has been found to contain bound states of fermions with up to $2S + 1$ constituents (with equality in the absence of an applied ‘magnetic’ field which would distinguish the different spin states). One may rationalise these results by noting that the Pauli principle does not militate against binding extra fermions until a spin state must be doubly occupied, and hence extra nodes in the spatial wave function occur. In the light of these results we will examine whether the ground state of the three-dimensional homogeneous $S > 1/2$ weakly interacting attractive Fermi gas is characterised by a condensation of composites of $2S + 1$ fermions. The possible condensation of alpha particles as four-particle composites in nuclear matter and at the surface of nuclei has a long history whose relation to the current work we will discuss at the end of this paper.

II. BARYONIC GROUND STATE ENERGY

In three dimensions we use a variational approach similar to that used in the original BCS paper [14]. We will assume that the composites (‘baryons’) have zero centre of mass momentum (as in BCS) and are also total spin singlets, as in the one-dimensional results. (The latter can be understood physically as providing the lowest kinetic energy associated with relative motion in the bound state.) The condensed state, with all baryons having centre of mass momentum zero, is of the form (here $n = 2S + 1$):

$$|B\rangle = \left[ \sum_{\{k_i\}} \varphi(k_1, \ldots, k_n) c_{k_1, \sigma_1}^\dagger \cdots c_{k_n, \sigma_n}^\dagger \right]^N$$

where all the spin states in the baryon, $\sigma_i$, (with $i = 1, \ldots, n$) are different. $\varphi(k_1, \ldots, k_n)$ is the Fourier transform of the completely symmetric relative wavefunction $\varphi(r_1, \ldots, r_n)$. Unlike the BCS case, there are several grand canonical states, $|\text{BCS}\rangle$, corresponding to $|B\rangle$. We will construct the simplest one.

This family of variational states allows us to use $1/n$ as a small parameter. (Indeed if the coupling between the atoms were strong, there would be similarities with work [15] on the structure of baryons in QCD using a $1/N$ expansion.)
Consider the following (non-normalized) ground state:

\[ |\tilde{\psi}_b\rangle = \exp \left[ \sum_{k_1,\ldots,k_n} \varphi(k_1,\ldots,k_n)c_1^\dagger(k_1)\ldots c_n^\dagger(k_n) \right] |0\rangle \]

Below we show that the corresponding normalized state has the form

\[ |\psi_b\rangle = e^{(n-2)N_b/2} \prod_k u_n(k) \cdot \exp \left[ \sum_{k_1,\ldots,k_n} \varphi_\alpha(k_1,\ldots,k_n)c_1^\dagger(k_1)\ldots c_n^\dagger(k_n) \right] |0\rangle \]

\[ = e^{(n-2)N_b/2} \prod_{k_1+\ldots+k_n=0} \left[ \prod_{i=1}^n u(k_i) + v(k_1,\ldots,k_n)c_1^\dagger(k_1)\ldots c_n^\dagger(k_n) \right] |0\rangle \]

where \( c_\alpha^\dagger(k), c_\alpha(k), \alpha = 1,\ldots,n = 2S+1 \) the creation and annihilation operators of a fermion with \( \alpha \)-th projection of the spin and momentum \( k \), \( N_b \equiv N/n \) is the number of baryons (with \( N \) being the number of atoms) and

\[ \varphi_\alpha(k_1,\ldots,k_n) \equiv \frac{v(k_1,\ldots,k_n)}{u(k_1)\ldots u(k_n)} \alpha \delta_{k_1+\ldots+k_n,0} \]

The Kronecker symbol on the r.h.s. of (2) means that we consider baryonic states with total momentum equal to zero. The expression \( k_1 + \ldots + k_n = 0 \) as a subscript to the product in (1) implies that the product is taken over all sets \( \{k_1,\ldots,k_n\} \) with the total momentum equal to zero. We will apply a ‘normalization’ condition similar to that used in BCS states:

\[ |u(k)|^2 + |v(k)|^2 = 1, \quad |v(k)|^2 \equiv \sum_{k_2,\ldots,k_n} |v(k_2,\ldots,k_n)|^2 \]

This, unlike the BCS one, does not imply that the total state is normalised. The vector conjugate to \( |\psi_b\rangle \) is

\[ \langle \psi_b | = e^{(n-2)N_b/2} \prod_k \bar{u}_n(k) \cdot \langle 0 | \exp \left[ \sum_{k_1,\ldots,k_n} \bar{\varphi}_\alpha(k_1,\ldots,k_n)c_n(k_n)\ldots c_1(k_1) \right] \]

To calculate the normalization \( \langle \psi_b |\psi_b \rangle \) we make use of the following identity

\[ \exp \left[ \sum_{k_1,\ldots,k_n} \varphi_\alpha(k_1,\ldots,k_n)c_1^\dagger(k_1)\ldots c_n^\dagger(k_n) \right] \]

\[ = \int D\xi D\bar{\xi} \exp \left[ \sum_{i=1}^{n-1} \sum_k \left\{ -\bar{\xi}_i(k)\xi_i(k) + c_i^\dagger(k)\xi_i(k) \right\} - \sum_{k_1,\ldots,k_n} \varphi_\alpha(k_1,\ldots,k_n)c_n^\dagger(k_n)\bar{\xi}_1(k_1)\ldots\bar{\xi}_{n-1}(k_{n-1}) \right] \]
where $\bar{\xi}_i(k), \xi_i(k)$ are Grassmann variables and the measure of integration is denoted by

$$D\bar{\xi}\xi \equiv \prod_{i=1}^{n-1} \prod_k d\bar{\xi}_i(k) d\xi_i(k)$$

As $c^i$ enters linearly in the exponentials in (5) then the operator averaging in $\langle \psi_b|\psi_b \rangle$ can be easily fulfilled and we find

$$\langle \psi_b|\psi_b \rangle = C \int D\bar{\xi}\xi \exp \left[ -\sum_{i=1}^{n-1} \sum_k \bar{\xi}_i(k) \xi_i(k) - \sum_{k_1, \ldots, k_n} \bar{\varphi}_n(k_1, \ldots, k_n) \varphi_n(p_1, \ldots, p_n) \bar{\xi}_1(p_1) \xi_1(k_1) \cdots \bar{\xi}_{n-1}(p_{n-1}) \xi_{n-1}(k_{n-1}) \delta_{k_n,p_n} \right]$$

where $C$ denotes the factor

$$C = e^{(n-2)N_b} \prod_k |u(k)|^{n-1}$$

For the second term in the exponential we make use of the following identity:

$$\exp \left[ -\sum_{i=1}^{n-1} \sum_{k,p} \bar{\sigma}_i(k,p) \sigma_i(k,p) - \sum_{i=1}^{n-1} \sum_k \bar{\sigma}_i(k) \xi_i(k) \xi_i(p) \right]$$

$$= \int D\bar{\sigma}\sigma \exp \left[ -\sum_{i=1}^{n-1} \sum_{k,p} \bar{\sigma}_i(k,p) \sigma_i(k,p) - \sum_{i=1}^{n-1} \sum_k \bar{\sigma}_i(k) \xi_i(k) \xi_i(p) \right] + \sum_{k_1, \ldots, k_n} \bar{\varphi}_n(k_1, \ldots, k_n) \varphi_n(p_1, \ldots, p_n) \prod_{i=1}^{n-1} \sigma_i(p_i, k_i) \delta_{k_n,p_n}$$

where

$$D\bar{\sigma}\sigma \equiv \prod_{i=1}^{n-1} \prod_{k,p} d\bar{\sigma}_i(k,p) d\sigma_i(k,p)$$

The meaning of $\sigma$-fields is not clear at this point, but we will discuss this presently.

Integrating over the Grassmann fields we obtain

$$\langle \psi_b|\psi_b \rangle = C \int D\bar{\sigma}\sigma \exp \left[ -\sum_{i=1}^{n-1} \sum_{k,p} \bar{\sigma}_i(k,p) \sigma_i(k,p) + \sum_{i=1}^{n-1} \text{tr} \log[1 + \bar{\sigma}_i] + K \right]$$

where $K$ denotes the following expression

$$K = \sum_{k_1, \ldots, k_n} \bar{\varphi}_n(k_1, \ldots, k_n) \varphi_n(p_1, \ldots, p_n) \prod_{i=1}^{n-1} \sigma_i(p_i, k_i) \delta_{k_n,p_n}$$
We calculate the integral over \( \bar{\sigma}, \sigma \) using the saddle point method, where \( n \) is the large parameter. We assume that the symmetry of the exponential over \( \sigma_i \)-fields is not broken and put \( \sigma_i = \sigma, \bar{\sigma}_i = \bar{\sigma} \), so the exponent becomes

\[-(n - 1)\bar{\sigma}\sigma + (n - 1)\text{tr} \log[1 + \bar{\sigma}] + K\]

The saddle point equations are

\[
\sigma(k, p) = (1 + \bar{\sigma})^{-1}(k, p) \hspace{1cm} (7)
\]

\[
\bar{\sigma}(k, p) = \sum_{k_2, \ldots, k_n, p_2, \ldots, p_n} \bar{\varphi}_n(k, k_2, \ldots, k_n) \varphi_n(p, p_2, \ldots, p_n) \prod_{i=2}^{n-1} \sigma(p_i, k_i) \delta_{k_n, p_n} \hspace{1cm} (8)
\]

In the limit \( n \to \infty \) we expect a mean field approximation to be valid for the description of the composites:

\[v(k_1, \ldots, k_n) = B \cdot \delta_{k_1 + \ldots + k_n, 0} \prod_{i=1}^{n} v(k_i)\]

where \( B \) is chosen to satisfy the normalization condition (3). With this assumed form for \( v \), it can be shown that the following anzatz can serve as a quite general solution of the saddle point equations:

\[
\sigma(k, p) = \delta_{k, p} s_1(k) + c_2 \bar{s}_2(k) s_2(p) , \hspace{1cm} \bar{\sigma}(k, p) = \delta_{k, p} s_3(k) + c_4 \bar{s}_4(k) s_4(p) . \hspace{1cm} (10)
\]

with functions \( s_i, i = 1, 2, 3, 4 \) to be determined. If we further restrict ourselves by considering weak coupling case we obtain that the solution is (see Appendix for details)

\[
\sigma(k, p) = \delta_{k, p} |u(k)|^2 , \hspace{1cm} \bar{\sigma}(k, p) = \delta_{k, p} \left| \frac{v(k)}{u(k)} \right|^2 \hspace{1cm} (11)
\]

Hence,

\[
\langle \psi_b | \psi_b \rangle = C \exp \left[ -(n - 1) \sum_k |v(k)|^2 - (n - 1) \sum_k \log |u(k)|^2 + \sum_k \sum_{k_2, \ldots, k_n} |v(k, k_2, \ldots, k_n)|^2 \right]
\]

Because of the normalization condition (3) and expression for \( C \) we finally obtain

\[
\langle \psi_b | \psi_b \rangle = 1 + O(1/n) \hspace{1cm} (12)
\]
Let us consider the following average:

\[ \langle \psi_b | c_\alpha(k) c_\alpha^\dagger(k') | \psi_b \rangle \]

Repeating all the steps we have made before we can obtain the following result:

\[
\langle \psi_b | c_\alpha(k) c_\alpha^\dagger(k') | \psi_b \rangle = C \int \mathcal{D}\sigma \mathcal{D}\bar{\sigma} (1 + \bar{\sigma}_\alpha)^{-1}(k, k')
\times \exp \left[ - \sum_{i=1}^{n-1} \sum_{k,p} \bar{\sigma}_i(k,p) \sigma_i(k,p) + \sum_{i=1}^{n-1} \text{tr} \log[1 + \bar{\sigma}_i] + K \right]
\]

Calculating again the integral over \( \sigma \)-fields using the saddle point method and noting that the contribution of the first multiplier in the integrand to the saddle point equations can be neglected we obtain from (7):

\[
\langle \psi_b | c_\alpha(k) c_\alpha^\dagger(k') | \psi_b \rangle = \sigma(k, k')
\]

So the \( \sigma \)-field may be interpreted as momentum distribution of the holes.

In an analogous way we can obtain the following averages:

\[
\langle \psi_b | c_\alpha^\dagger(k) c_\alpha(k) | \psi_b \rangle = |v(k)|^2 + O(1/n) \quad (13)
\]

\[
\sum_q \langle \psi_b | c_\alpha^\dagger(k) c_\beta^\dagger(-k + q) c_\beta(-k' + q) c_\alpha(k') | \psi_b \rangle
= \delta_{k,k'} |v(k)|^2 \sum_q |v(q)|^2 + \bar{v}(k) u(k) \bar{u}(k') v(k') f(\hat{k}, \hat{k}') + O(1/n) \quad (14)
\]

where \( \hat{k} \equiv \frac{k}{|k|} \) and the function \( f \) has the form

\[
f(\hat{k}, \hat{k}') \equiv f(x) = 1 - \frac{3x}{2} + \frac{x^3}{2} \quad , \quad x \equiv \sin \frac{\theta}{2}
\]

with \( \theta \) being the angle between vectors \( k, k' \).

We are now in a position to consider a general Hamiltonian of the form:

\[
\hat{H} = \sum_{k,\alpha} \epsilon(k) c_\alpha^\dagger(k) c_\alpha(k) + \frac{1}{2\Omega} \sum_{k, k', q_{\alpha,\beta}} V(k, k') c_\alpha^\dagger(k) c_\beta^\dagger(-k + q) c_\beta(-k' + q) c_\alpha(k') \quad (16)
\]

where
\[
\epsilon(k) = \frac{k^2}{2m} - \mu
\]  
(17)

and \(\mu\) is the chemical potential. Then, using the results above for expectation values, we find the following expression for the ground state energy

\[
E_b = \langle \psi_b | \hat{H} | \psi_b \rangle = n \sum_k \epsilon(k) |v(k)|^2 + \frac{V(0)N^2}{2\Omega} + \frac{n^2}{2\Omega} \sum_{k,k'} V(k,k') \bar{v}(k) u(k) \bar{u}(k') v(k') f(\hat{k}, \hat{k}')
\]

(18)

with the condition

\[
\langle \psi_b | \hat{N} | \psi_b \rangle = n \sum_k |v(k)|^2 = N
\]

(19)

Note that if we consider an interaction potential depending only on the modulus of momentum: \(V(k,k') = V(|k|, |k'|)\), and look for a solution for \(u, v\) also depending only on the modulus of momentum (that is in absence of a spontaneous breaking of the rotational symmetry) then the function \(f\) in (18) can be replaced by its average value:

\[
\int_0^1 dx \ f(x) = \frac{3}{8}
\]

and we have

\[
E_b = \langle \psi_b | \hat{H} | \psi_b \rangle = n \sum_k \epsilon(k) |v(k)|^2 + \frac{V(0)N^2}{2\Omega} + \frac{n^2}{2\Omega} \sum_{k,k'} \tilde{V}(k,k') \bar{v}(k) u(k) \bar{u}(k') v(k')
\]

(20)

where \(\tilde{V} = \frac{2}{8} V\).

The normalization condition, (3), allows the introduction of the following parameterization for \(u, v\), by analogy with the usual procedure for the BCS case:

\[
v(k) = \cos \theta(k) , \quad u(k) = \sin \theta(k)
\]

(21)

and we obtain for the ground state energy

\[
E_b = n \sum_k \epsilon(k) \cos^2 \theta(k) + \frac{V(0)N^2}{2\Omega} + \frac{n^2}{2\Omega} \sum_{k,k'} V(k,k') \cos \theta(k) \sin \theta(k) \cos \theta(k') \sin \theta(k')
\]

(22)

We now minimize \(E_b\) with respect to \(\theta(k)\) to find
\[ \tan[2\theta(k)] = \frac{n}{2\epsilon(k)} \frac{n}{(2\pi)^3} \int dk' \tilde{V}(k, k') \sin[2\theta(k')] \quad (23) \]

Introducing the following notation

\[ \Delta(k) = -\frac{n}{2(2\pi)^3} \int dk' \tilde{V}(k, k') \sin[2\theta(k')] \quad , \quad E(k) = \left[ \epsilon^2(k) + \Delta^2(k) \right]^{1/2} \quad (24) \]

such that

\[ \tan[2\theta(k)] = -\frac{\Delta(k)}{\epsilon(k)} \quad , \quad \sin[2\theta(k)] = \frac{\Delta(k)}{E(k)} \quad , \quad \cos[2\theta(k)] = -\frac{\epsilon(k)}{E(k)} \quad (25) \]

we obtain the following equation for the gap \( \Delta(k) \):

\[ \Delta(k) = -\frac{n}{2(2\pi)^3} \int dk' \tilde{V}(k, k') \Delta(k') \frac{\Delta(k')}{E(k')} \quad (26) \]

and the chemical potential can be defined from

\[ \frac{n}{(2\pi)^3} \int dk' \cos^2\theta(k) = \rho \quad (27) \]

Following Anderson and Morel [16], Eq.(24) can be rewritten in terms of an effective potential, \( U_\xi(k, k') \):

\[ \Delta(k) = -\frac{1}{(2\pi)^3} \int_{|\epsilon(k')|<\xi} dk' U_\xi(k, k') \frac{\Delta(k')}{2E(k')} \quad , \quad (28) \]

where \( \xi \) is some cut-off such that \( \xi \ll \mu \equiv \frac{k_F^2}{2m} \) and \( U_\xi(k, k') \) satisfies the equation:

\[ U_\xi(k, k') = n\tilde{V}(k, k') - \frac{1}{(2\pi)^3} \int_{|\epsilon(q)|>\xi} dq n\tilde{V}(k, q)U_\xi(q, k') \frac{1}{2|\epsilon(q)|} \quad . \quad (29) \]

We take \( V(k, k') \) to have the following separable (energy dependent) form:

\[ V(k, k') = V\Theta(\mu - |\epsilon(k)|)\Theta(\mu - |\epsilon(k')|) \quad , \quad V = \frac{4\pi a}{m} \quad (30) \]

which, as we shall see, is consistent with low-energy approximation to the \( T \)-matrix. Here \( a \) is the scattering length (we put \( \hbar = 1 \)), \( \Theta(x) \) the step function. We assume that the interaction is weak (which is a good approximation experimentally) which means

\[ k_F a \ll 1 \quad (31) \]
To check that the form of interaction (30) is a consistent low-energy approximation consider the equation for the $T$-matrix $T(k, k', z)$ of zero energy, $z = 0$:

$$T(k, k', 0) = V(k, k') - \frac{1}{(2\pi)^3} \int_{| \epsilon(q) | > \xi} dq \ V(k, q) T(q, k', 0) \frac{m}{q^2}$$  \hspace{1cm} (32)

Making the anzatz

$$T(k, k', 0) = T_0 \Theta(\mu - |\epsilon(k)|) \Theta(\mu - |\epsilon(k')|)$$

we obtain the following relation for $T_0, V$:

$$\frac{1}{\rho_0 T_0} = \frac{1}{\rho_0 V} + \sqrt{2} \ , \ \rho_0 = \frac{mk_F}{2\pi^2}$$

As

$$\frac{1}{\rho_0 V} = \frac{\pi}{2ak_F} \gg 1$$

then

$$T_0 \approx V$$  \hspace{1cm} (33)

Let us define $U_\xi(k, k')$. Assuming the form

$$U_\xi(k, k') = U_\xi \Theta(\mu - |\epsilon(k)|) \Theta(\mu - |\epsilon(k')|)$$

we get the following equation for $U_\xi$:

$$U_\xi = n\bar{V} - \frac{n\bar{V} U_\xi}{4\pi^2} \int_{\xi<|\epsilon(q)|<\mu} dq \ \frac{q^2}{|\epsilon(q)|}$$

After some algebra we arrive at the relation

$$\frac{1}{\rho_0 U_\xi} = \frac{1}{\rho_0 n\bar{V}} + \log(\sqrt{2} + 1) - \log \frac{\xi}{4\mu}$$  \hspace{1cm} (34)

If we assume again that we can drop the second term in the right hand side (because the coupling is weak) and noting that $\bar{V}, U_\xi < 0$ we finally obtain

$$\frac{1}{\rho_0 |U_\xi|} = \frac{1}{\rho_0 n|V|} + \log \frac{\xi}{4\mu}$$  \hspace{1cm} (35)

Note that the energy can be expressed in terms of (24):
From (28) then we obtain
\[
\Delta = 2 \xi \exp \left[ -\frac{1}{\rho_0 |U_\xi|} \right] = 8 \mu \exp \left[ -\frac{1}{\rho_0 n |V|} \right]
\] (37)

The energy of the normal state per unit volume can be written as
\[
\frac{E_n}{\Omega} = \frac{n}{(2\pi)^3} \int_{|k| < k_F} dk \epsilon(k)
\] (38)

Hence, from (36, 37) we find that the difference of the baryonic and the normal ground state energies is:
\[
\frac{\Delta E_b}{\Omega} \equiv \frac{E_b - E_n}{\Omega} = -48 \rho \mu \exp \left[ -\frac{1}{\rho_0 n |V|} \right], \quad \rho \equiv \frac{N}{\Omega}, \quad \rho_0 = \frac{mk_F^2}{2\pi^2}
\] (39)

with \(N\) the number of atoms in the system.

We conclude this section with a note about weakening the assumption of equality of scattering lengths in all channels. If the scattering lengths are different then the interaction will take the form [17]:
\[
V = \sum_{n=0}^{[f]} V_n (S_1 \cdot S_2)^n
\]
where \([f]\) means the integer part of \(f\). In the case where the scattering lengths in all channels are equal, \(a_F = a, \forall F\), we have \(V_0 = \frac{4\pi a}{m}, V_n = 0, n \neq 0\). If the scattering lengths are slightly different then the terms in the interaction with \(V_n, n \neq 0\) are small and can be neglected but \(V_0\) is equal to some weighted average over the scattering lengths and the treatment in this section will still be approximately valid.

**III. BCS GROUND STATE ENERGY**

We will now show that the conventional BCS ground state has an energy that is higher than the baryonic one, at least if we restrict ourselves to \(s\)-wave pairing. The BCS ground state has the form [18]:

\[
E_b = \frac{n \Omega}{2(2\pi)^3} \int dk \left[ \epsilon(k) - \frac{\epsilon^2(k)}{E(k)} - \frac{\Delta^2(k)}{2E(k)} \right]
\] (36)
\[ |\psi_{\text{BCS}}\rangle = \prod_{k>0} u^n(k) \exp \left[ \sum_{k>0,\alpha,\beta} \varphi(k) c^\dagger_\alpha(k) \tilde{P} c_\beta^\dagger(-k) \right] |0\rangle \quad (40) \]

and the conjugate state is

\[ \langle \psi_{\text{BCS}}| = \prod_{k>0} \bar{u}^n(k) \langle 0 | \exp \left[ - \sum_{k>0,\alpha,\beta} \bar{\varphi}(k) c_\alpha(k) \bar{P} c_\beta^\dagger(-k) \right] \quad (41) \]

where

\[ \varphi(k) \equiv \frac{v(k)}{u(k)} , \quad |v(k)|^2 + |u(k)|^2 = 1 \quad (42) \]

\( k > 0 \) is an arbitrary ordering on momentum space which divides it into two halves (it can be defined, for example, as follows: \( k > 0 \) if \( k_z > 0 \) or \( k_z = 0, k_y > 0 \) or \( k_z = 0, k_y = 0, k_x > 0 \)).

The matrix \( \tilde{P} \) has the form

\[ \tilde{P}^T = -\tilde{P} \quad (43) \]

\( T \) means matrix transposition.

It is well known that any antisymmetric matrix \( A : A^T = -A \) can be orthogonally transformed to an antisymmetric matrix \( B \) in the canonical form:

\[ A = OBO^T , \quad O^T = O^{-1} \]

matrix \( B \) has the following quasi-diagonal (block-diagonal) form

\[ B = \text{diag} \begin{pmatrix} 0 & \lambda_1 & \ldots & 0 & \lambda_{n/2} \\ -\lambda_1 & 0 & \ldots & -\lambda_{n/2} & 0 \end{pmatrix} \]

This transformation is equivalent to

\[ c(k) \to Oc(k) , \quad c^\dagger(k) \to c^\dagger(k)O^T \quad (44) \]

As the Hamiltonian in (16) is invariant under the transformation (44) and because of the assumption of the spin symmetry of the ground state, there exists some orthogonal matrix \( O \) such that the ground state (40,41), after transformation (44), takes the following form:
$$|\psi_{\text{BCS}}\rangle = \prod_{k>0} u^n(k) \exp \left[ \sum_{k>0} \sum_{\alpha,\beta} \varphi(k) c_\alpha^\dagger(k) P_{\alpha\beta} c_\beta^\dagger(-k) \right] |0\rangle$$  \hspace{1cm} (45)

where

$$P = \text{diag} \left\{ 0, e^{i\chi_1}, 0, e^{i\chi_2}, \ldots, 0, e^{i\chi_{n/2}} \right\}$$  \hspace{1cm} (46)

The form of the matrix $P$ implies that the spin states can be enumerated after the transformation in such a way that there are $n/2$ “positively” directed spins and each of them has an oppositely directed partner. So the spin indices can be thought as taking values $\pm 1, \ldots, \pm n/2$. The phases $\chi_\alpha$ represent possible phase differences between Cooper pairs in different spin states (we consider only rotationally invariant states, so the modulus of all elements in the matrix $P$ are equal).

To calculate the norm and the energy of the ground state we consider the following generating functional:

$$G(\bar{\eta}, \eta) = \langle \psi_{\text{BCS}} | \exp \left[ \sum_{k,\alpha} c_\alpha^\dagger(k) \eta_\alpha(k) \right] \exp \left[ \sum_{k,\alpha} \bar{\eta}_\alpha(k) c_\alpha(k) \right] |\psi_{\text{BCS}}\rangle$$  \hspace{1cm} (47)

Making use of the following identity

$$\exp \left[ \sum_{k>0} \sum_{\alpha,\beta} \varphi(k) c_\alpha^\dagger(k) P_{\alpha\beta} c_\beta^\dagger(-k) \right] = \int \text{D}\bar{\xi} \text{D}\xi \exp \left[ - \sum_{k>0,\alpha} \bar{\xi}_\alpha(k) \xi_\alpha(k) \right] + \sum_{k>0,\alpha} c_\alpha^\dagger(k) \xi_\alpha(k) + \sum_{k>0,\alpha,\beta} \bar{\xi}_\alpha(k) P_{\alpha\beta} c_\beta^\dagger(-k)$$

we get the following result for the generating functional

$$G(\bar{\eta}, \eta) = \exp \left[ \sum_{k} \sum_{\alpha=1}^{n/2} \left\{ \bar{u}(k)v(k) e^{i\chi_\alpha} \bar{\eta}_\alpha(k) \eta_{-\alpha}(-k) - u(k)v(k) e^{-i\chi_\alpha} \eta_\alpha(k) \eta_{-\alpha}(-k) \right\} \right. \hspace{1cm} (48)$$

Putting $\bar{\eta}, \eta = 0$ in (48) we obtain $\langle \psi_{\text{BCS}} | \psi_{\text{BCS}} \rangle = 1$. The energy of the ground state is

$$E_{\text{BCS}} = \langle \psi_{\text{BCS}} | \hat{H} | \psi_{\text{BCS}} \rangle = \sum_{k} \epsilon(k) |v(k)|^2 + \frac{V(0)N^2}{2\Omega} + \sum_{k,k'} V(k,k')u(k)v(k')u(k')v(k)$$  \hspace{1cm} (49)
Introducing as usual the trigonometric parameterization

\[ u(k) = \sin \theta(k), \quad v(k) = \cos \theta(k) \]

and applying the variational principle we find the following equation

\[ \tan[2\theta(k)] = \frac{1}{2\epsilon(k)} \frac{1}{(2\pi)^3} \int dk' \ V(k, k') \sin[2\theta(k')] \quad (50) \]

Defining the gap as

\[ \Delta(k) = -\frac{1}{2(2\pi)^3} \int dk' \ V(k, k') \sin[2\theta(k')] \quad (51) \]

and repeating calculations of the preceding section we arrive at the following expression for the gap

\[ \Delta = 8\mu \exp \left[ -\frac{1}{\rho_0|V|} \right] \quad (52) \]

and the difference between BCS and the normal ground state energies is

\[ \frac{\Delta E_{\text{BCS}}}{\Omega} = \frac{E_{\text{BCS}} - E_n}{\Omega} = -48\rho\mu \exp \left[ -\frac{1}{\rho_0|V|} \right], \quad \rho \equiv \frac{N}{\Omega}, \quad \rho_0 = \frac{mk_F}{2\pi^2} \quad (53) \]

From eqs.(39,53) we conclude that the energy of the baryonic ground state is lower than for BCS state (for \( n \geq 4 \) or, equivalently, \( S \geq 3/2 \)). Indeed, one usually assumes that a coupling constant is proportional to \( 1/n \) in the framework of large \( n \) expansion, so we can put \( V \equiv V_0/n \). Then the ratio of the baryonic and BCS binding energies can be expressed as follows:

\[ \frac{\Delta E_{\text{BCS}}}{\Delta E_{\text{b}}} = \exp \left[ -\frac{1}{\rho_0|V|} + \frac{1}{\rho_0n|V|} \right] = \exp \left[ -\frac{3n/8 - 1}{\rho_0|V_0|} \right] < 1 \quad \text{for} \quad n \geq 4 \]

The last relation demonstrates the statement.

**IV. CONCLUSION**

In this paper we have shown that, assuming that all scattering lengths are approximately equal, the true ground state of a dilute gas of fermions with a high hyperfine spin is in fact
of baryonic nature. The energy of the state and the gap in the spectrum (interpreting $\Delta$ by analogy with the BCS case) have been calculated. For comparison we calculated the energy of BCS s-wave ground states and showed that it is higher than the energy of the baryonic ground state.

The possibility of forming a baryon-like bound state has been discussed in paper [20]. The authors investigated the formation of a four-fermion ($\alpha$-particle) condensate. While the results of [20] are very interesting they cannot be directly related to ours. Firstly, in the case of $\alpha$-particles $n = 4$ and $1/n = 1/4$ can be hardly regarded as a good expansion parameter to apply our results. Secondly, $n = 4$ is the dimension of the joint spin-isospin space and the Hamiltonian is not invariant under rotations in that space (exemplified by the absence of a di-neutron bound state, as against the existence of the deuteron).

We note that if the interaction were strong then the quasi-baryons considered in the paper would become ‘real-space’ composite bosons where the ground state might be more appropriately described as being Bose-condensed. So it would interesting to study the evolution from weak to strong coupling in the manner that Nozieres et. al. derived for an attractive fermion gas [21].

The method developed in this paper may be of interest in other fields such as the low-energy behavior of QCD [22,23], the quark-gluon plasma [24] and neutron stars [25–27].

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APPENDIX:

In this appendix we sketch the main steps of calculation of the norm of the baryonic ground state vector and averages (13,14).

We consider the following generating functional
\[ G(\bar{\eta}, \eta) = \langle \psi_b | \exp \left[ \sum_{i=1}^{n} \sum_{k} c_i^*(k) \eta_i(k) \right] \exp \left[ \sum_{i=1}^{n} \sum_{k} \bar{\eta}_i(k) c_i(k) \right] | \psi_b \rangle \]  

(A1)

from which we can obtain the norm and the required averages in an obvious way. Using the identity (3) we can transform \[(A1)\] to the form

\[ G(\bar{\eta}, \eta) = C \int D\xi D\bar{\xi} \exp \left[ \sum_{i=1}^{n-1} \sum_{k} \left\{ -\bar{\xi}_i(k) \xi_i(k) + \bar{\eta}_i(k) \xi_i(k) + \bar{\xi}_i(k) \eta_i(k) - \bar{\eta}_i(k) \eta_i(k) \right\} \right] 
+ \sum_{k_1, \ldots, k_n} \left\{ \varphi_n(k_1, \ldots, k_n) \eta_n(k_n) \xi_{n-1}(k_{n-1}) \ldots \xi_1(k_1) 
+ \varphi_n(k_1, \ldots, k_n) \xi_1(k_1) \ldots \xi_{n-1}(k_{n-1}) \bar{\eta}_n(k_n) \right\} 
+ \sum_{k_1, \ldots, k_n, p_1, \ldots, p_n} \varphi_n(k_1, \ldots, k_n) \varphi_n(p_1, \ldots, p_n) \xi_1(k_1) \ldots \xi_{n-1}(p_{n-1}) \xi_n(k_n) \delta_{k_n, p_n} \right] \]  

(A2)

Here we put \( \alpha = n \) while it can be any number from 1 to \( n \).

Consider first the norm of the ground state. Then after some transformations (see the main text) we obtain saddle point equations (7,8). If we accept anzatz (10) then from (8) we have

\[ s_3(k) = K_0 \left| \frac{v(k)}{u(k)} \right|^2, \quad s_4(k) = \frac{v(k)}{u(k)} \]

\[ K_0 \equiv K_0(k_1) = |B|^2 \sum_{k_2, \ldots, k_n} \delta_{k_1 + \ldots + k_n, 0} \prod_{i=2}^{n-1} \frac{|v(k_i)|^2 s_1(k_i)}{|u(k_i)|^2} \cdot |v(k_n)|^2 \]

\[ c_4 \equiv c_4(k_1, p_1) = |B|^2 \sum_{k_2, \ldots, k_n, p_2, \ldots, p_n} \delta_{k_1 + \ldots + k_n, 0} \delta_{p_1 + \ldots + p_{n-1} + k_n, 0} \prod_{i=2}^{n-1} \frac{\bar{v}(k_i) v(p_i)}{u(k_i) u(p_i)} \cdot |v(k_n)|^2 \]

\[ \times \left[ \prod_{i=2}^{n-1} \{ \delta_{k_i, p_i} s_1(k_i) + c_2 \bar{s}_2(p_i) s_2(k_i) \} - \prod_{i=2}^{n-1} \delta_{k_i, p_i} s_1(k_i) \right] \]  

(A3)

\( K_0(k), c_4(k, p) \) are weakly dependent on their arguments and can be approximated by constants. On other hand, from (7,10) we obtain

\[ s_1(k) = \frac{1}{1 + s_3(k)}, \quad s_2(k) = \frac{s_4(k)}{1 + s_3(k)} \]

\[ c_2 = -c_4 \left[ 1 + c_4 \sum_k \frac{|s_4(k)|^2}{1 + s_3(k)} \right]^{-1} \]  

(A4)

Choosing \( B \) such that \( K_0 = 1 \) we obtain the following solution

\[ s_1(k) = |u(k)|^2, \quad s_3(k) = \frac{|v(k)|^2}{|u(k)|^2} \]
$$s_2(k) = v(k)\bar{u}(k), \quad s_4(k) = \frac{v(k)}{u(k)}$$

It can be easily seen then that the condition $K_0 = 1$ is equivalent to the normalization condition (3). From (A3,A4) a relation for determining of the constant $c_4$ can be obtained.

In the weak coupling approximation it takes the form:

$$c_4 = \frac{\text{const}}{n^3N_b} \left[ \frac{1}{(1 + c_4N_b)^n} - 1 \right], \quad N_b = \sum_k |v(k)|^2$$

An obvious solution of this equation is $c_4 = 0$ and we obtain (11).

Consider now average (13). Using the symmetry over spin we obtain from (A2) after integrating over the Grassmann fields:

$$\langle \psi_b|c_i^\dagger(k)c_i(k)|\psi_b \rangle = \langle \psi_b|c_n^\dagger(k)c_n(k)|\psi_b \rangle = - \left. \frac{\partial}{\partial \eta_n(k)} \frac{\partial}{\partial \bar{\eta}_n(k)} G(\bar{\eta}, \eta) \right|_{\eta = \bar{\eta} = 0}$$

$$= C \int \mathcal{D}\sigma \mathcal{D}\bar{\sigma} \sum_{k_1, \ldots, k_n, p_1, \ldots, p_n} \varphi_n(k_1, \ldots, k_n, k)\varphi_n(p_1, \ldots, p_n, k) \prod_{i=1}^{n-1} [1 + \sigma_i]^{-1}(k_i, p_i)$$

$$\times \exp \left[ -\sum_{i=1}^{n-1} \sum_{k,p} \bar{\sigma}_i(k, p)\sigma_i(k, p) + \sum_{i=1}^{n-1} \text{tr} \log[1 + \sigma_i] + K \right]$$

A contribution of the first multiplier in the integrand to the saddle point equations is of the factorizable form and it can be shown that this contribution can be neglected. So we can use again solution (14). Hence

$$\langle \psi_b|c_i^\dagger(k)c_i(k)|\psi_b \rangle = |v(k)|^2|B|^2 \sum_{k_1, \ldots, k_n, -k} \delta_{k_1 + \ldots + k_n + k, 0} \prod_{i=1}^{n-1} |v(k_i)|^2 = |v(k)|^2$$

The last equality is due to the normalization condition (3).

Finally we consider the average (14). We have

$$\langle \psi_b|\sum_q c_i^\dagger(k)c_j^\dagger(-k + q)c_j(-k' + q)c_i(k')|\psi_b \rangle = \langle \psi_b|c_i^\dagger(k)c_n^\dagger(-k + q)c_n(-k' + q)c_1(k')|\psi_b \rangle$$

$$= \sum_q \left. \frac{\partial}{\partial \eta_1(k)} \frac{\partial}{\partial \eta_n(-k + q)} \frac{\partial}{\partial \bar{\eta}_n(-k' + q)} \frac{\partial}{\partial \bar{\eta}_1(k')} G(\bar{\eta}, \eta) \right|_{\eta = \bar{\eta} = 0} = \delta_{k, k'}|v(k)|^2 \sum_q |v(q)|^2$$

$$+ C \int \mathcal{D}\sigma \mathcal{D}\bar{\sigma} \sum_{k_1, \ldots, k_n, q, p_1, \ldots, p_n} \varphi_n(k_1, \ldots, k_n, -k + q)\varphi_n(p_1, \ldots, p_n, -k' + q) \prod_{i=1}^{n-1} [1 + \sigma_i]^{-1}(k_i, p_i)$$

$$\times [1 + \sigma_i]^{-1}(k_1, k)[1 + \sigma_1]^{-1}(p_1, k') \exp \left[ -\sum_{i=1}^{n-1} \sum_{k,p} \bar{\sigma}_i(k, p)\sigma_i(k, p) + \sum_{i=1}^{n-1} \text{tr} \log[1 + \sigma_i] + K \right]$$
Again it can be shown as before that we can use saddle solution (11), so we obtain
\[ \langle \psi_b | \sum_q c^\dagger_i(k)c_j(-k + q)c_j(-k' + q)c_i(k') | \psi_b \rangle = \delta_{k,k'} |v(k)|^2 \sum_q |v(q)|^2 \]
\[ + \bar{v}(k)u(k)v(k')\bar{u}(k') \sum_q \bar{v}(-k + q)v(-k' + q)|B|^2 \sum_{k_2,...,k_{n-1}} \delta_{k_2+...+k_{n-1}+q,0} \prod_{i=2}^{n-1} |v(k_i)|^2 \]
\[ \approx \delta_{k,k'} |v(k)|^2 \sum_q |v(q)|^2 + \bar{v}(k)u(k)v(k')\bar{u}(k') \frac{1}{N_b} \sum_q \bar{v}(-k + q)v(-k' + q) \]

In the weak coupling limit \(|v(k)u(k)|\) is non-zero only in a narrow region around the Fermi surface, so we can put \(|k| = |k'| = k_F\). Introducing the following notation
\[ f(\hat{k},\hat{k}') \equiv \sum_q \bar{v}(-k + q)v(-k' + q) \bigg|_{|k| = |k'| = k_F} \]
we obtain (14).
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