CONTINUOUS ANALOGS OF POLYNOMIALS . . .

Dedicated to the centenary of Mark Krein

Preface

In the recent years, the theory of orthogonal polynomials on the real line (OPRL) and on the unit circle (OPUC) enjoyed the considerable development. In these lecture notes, we will explain how to construct the continuous analogs of polynomials orthogonal on the unit circle. It is possible to build a theory which is as rigorous and complete as the theory for OPUC. Spectral theories of one-dimensional Dirac and Schrödinger operators can then be viewed in the framework of this theory which establishes a solid link between an approximation theory and quantum mechanics.

The theory is based on the ideas suggested by M.G. Krein. They were developed later by various authors, especially from Krein’s school. In the meantime, new results were obtained for OPUC and OPRL and that was a motivation for us to try to understand their continuous analogs. We also try to give systematic exposition of the theory but have to refer to the literature once in a while. Also, these notes do not cover some aspects of the theory (e.g. continuation problems for $G_r$ classes, regularity of coefficients, etc.) but we give necessary references. In general, our objective is to give only basics of the theory by presenting complete proofs and filling various gaps present in the current literature. As a prerequisite for reading these notes, we assume that a reader is familiar with main facts from the OPUC theory (see, e.g. [72, 26, 66]). The knowledge of spectral theory for Schrödinger and Dirac operators might also be very helpful.

What is not covered and what is new?

We didn’t include the following subjects that are related to our topic: solution to the continuous analogs of Schur and Caratheodory-Toeplitz problems [45, 46]. We also do not discuss matrix-valued version of the theory. For the recent progress on more specific questions (such as continuous analog of Szegő case, Rakhmanov’s Theorem, etc.) we suggest the reader to consult the journal publications, e.g. [74, 71, 14, 15, 16]. Also, we will deal with rather regular classes of coefficients (not worse than $L^2_{loc}(\mathbb{R}^+)$) but the general case can also be treated in the framework of different differential operators (see, e.g., [52, 53, 6]).

In these notes, we present quite a few new results. That includes: approximation of continuous orthogonal system by the sequence of the discrete ones (Section 8), distribution of zeroes (Section 9), new criteria for $A(r) \in L^2(\mathbb{R}^+)$ and more on that case (Section 10), the continuous analog of the Strong Szegő Theorem – sharp conditions (Section 14). We also gave complete proofs for results that were present in the literature without any proofs and gave alternative (hopefully, more transparent) proofs for several other statements (e.g. the continuous analog of Baxter’s Theorem, scattering theory for Krein systems and Dirac operators).

Acknowledgements

These notes are based on the graduate course given at Caltech in Fall 2001. We are indebted to B. Simon and P. Deift for their support and encouragement to prepare these lectures. Thanks are due to A. Teplyaev and L. Sakhnovich for their help and
insightful remarks. This work was supported by NSF grant DMS-0500177, Alfred
P. Sloan Research Fellowship, and Oswald Veblen Fund during the stay at the
Institute for Advanced Study, Princeton, NJ. Finally, it is my pleasure to dedicate
these notes to the centenary of Mark Grigorievich Krein (April 2007) who was the
founder of this theory.

**Notations**

- $\mathbb{D}$ – open unit disc in $\mathbb{C}$
- $\mathbb{T}$ – unit circle in $\mathbb{C}$
- $H^p(\Omega)$ – Hardy space in the domain $\Omega$, $0 < p \leq \infty$
- $N(\Omega)$ – Nevanlinna class of analytic functions in $\Omega$
- $B(\Omega)$ – closed unit ball in $H^\infty(\Omega)$
- $P_\Delta$ – the following projection:
  \[
  P_\Delta \left[ \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx \right] = \int_{\Delta} e^{i\lambda x} f(x) dx
  \]
- $H^2_{[0,R]}$ = Ran$P_{[0,R]}$
- $P_\pm$ – denote $P_{\mathbb{R} \pm}$ respectively
- $|O|$ = $\sqrt{\text{tr}(O^*O)}$ – absolute value of operator $O$
- $\delta(x)$ – the delta-function at zero
- $W(\mathbb{R})$ – Wiener’s Banach algebra of functions
  \[
  \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx, f(x) \in L^1(\mathbb{R})
  \]
- $W_+(\mathbb{C}^+)$ – Banach algebra of functions
  \[
  \hat{f}(\lambda) = \int_{0}^{\infty} f(x) e^{i\lambda x} dx, f(x) \in L^1(\mathbb{R}^+)
  \]
- $H^{1/2}(\mathbb{R})$ – fractional Sobolev space of functions $f$ whose Fourier transform satisfies
  \[
  \int_{-\infty}^{\infty} <t> |\hat{f}(t)|^2 dt < \infty, <t> = \sqrt{t^2 + 1}
  \]
- $C_0(\Omega)$ – continuous on $\overline{\Omega}$ functions vanishing at the boundary of $\Omega$
- $\chi_\Delta(x)$ – the characteristic function of the set $\Delta$
- $\Pi_\Delta$ – the orthogonal projection in $L^2(\mathbb{R}^+)$ onto $L^2(\Delta), \Delta \subset \mathbb{R}$
- $S^p$ – Schatten-Von Neumann class of compact operators
- $\ln^+ x = \ln x$ if $x \geq 1$ and $= 0$ for $0 < x < 1$
- $\ln^- x = \ln x$ if $0 < x < 1$ and $= 0$ for $x > 1$
- $f * g$ – means the convolution of $f$ and $g$

We usually use calligraphic letters to distinguish between operators and functions. For example, $\mathcal{O}$ stands for an operator and $\mathcal{O}(x,y)$ denotes the function of two variables.
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1. SOME CLASSES OF FUNCTIONS ON THE REAL LINE

In this section, we recall some basic facts on positive definite functions on the real line. Then, we introduce certain class of functions that we will use later on.

Let $0 < r \leq \infty$.

**Definition 1.1.** The Lebesgue-measurable function $\phi(x)$ defined on the interval $(-r,r)$ is called Hermitian if $\phi(-x) = \overline{\phi(x)}$ for a.e. $x$.

**Definition 1.2.** The Lebesgue-measurable function of two variables $K(x,y)$ is called Hermitian if

$$K(x,y) = K(y,x)$$

for a.e. $0 < x, y < r$.

**Definition 1.3.** The integral kernel $K(x,y)$ is called positive definite on $[0,r]$ if for any $N$, $\{x_j\}_{j=1}^N, \{x_j \in [0,r]\}$, $\{c_j\}_{j=1}^N, \{c_j \in \mathbb{C}\}$, we have inequality

$$\sum_{n,m=1}^N c_n \overline{c_m} K(x_n,x_m) \geq 0.$$  \hspace{1cm} (2)

Consider the integral operator $\mathcal{K}$ in $L^2[0,r]$ with kernel $K(x,y) \in C([0,r]^2)$, i.e.

$$(\mathcal{K}f)(x) = \int_0^r K(x,y)f(y)dy.$$ 

Clearly, (1) means $\mathcal{K}^* = \mathcal{K}$. It is an easy exercise to see that if the continuous kernel is Hermitian, then (2) is equivalent to $\mathcal{K} \geq 0$, where inequality is understood in the operator sense.

**Definition 1.4.** A function $\phi(x)$ is called positive definite if the integral kernel $\phi(x-y)$ is positive definite on $\mathbb{R}^+$. This is equivalent to $\phi(x-y)$ being positive on the whole line $\mathbb{R}$.

**Notation 1.** The class of continuous positive definite functions on the whole line is denoted by $P_\infty$.

If $d\mu$ is finite positive measure on $\mathbb{R}$, then

$$\phi(x) = \int_{-\infty}^{\infty} \exp(itx)d\mu(t)$$  \hspace{1cm} (3)

is positive definite. The classical result of Bochner says that the converse statement is also true. That, in a sense, is the continuous analog of the solution to trigonometric moment problem.

**Theorem 1.1.** (Bochner, [3]) A function $\phi$ belongs to the class $P_\infty$ if and only if it admits the representation (3) with finite positive measure $\mu$. The measure $\mu$ in this representation is unique.

Notice that Bochner’s theorem implies that all $P_\infty$ functions are necessarily bounded on $\mathbb{R}$. 
Notation 2. Let \( G_\infty \) denote the class of continuous Hermitian functions \( g(x) \) defined on the whole line such that \( g(0) = 0 \) and the integral kernel

\[
K(x, y) = g(x) + g(-y) - g(x - y)
\]
is positive definite on \( \mathbb{R}^+ \), i.e. on any interval \([0, r], r > 0\).

Next, we obtain some rather crude estimates on \( g \in G_\infty \). Later, these bounds will be used to prove the integral representation for functions of class \( G_\infty \).

The following inequality holds

\[
|g(2x)| \leq 8|g(x)|
\]
for any \( x \). Indeed, since the kernel \( K(x, y) \) is positive definite, estimate (2) is true.

Take \( N = 2, x_1 = x, x_2 = 2x \). If \( c_1 = \xi \in \mathbb{R}, c_2 = 1 \), we have

\[
\text{Re}(g(x))\xi^2 + \text{Re}(g(2x))\xi + \text{Re}(g(2x)) \geq 0
\]

Since \( \xi \) is arbitrary real,

\[
|\text{Re}(g(2x))| \leq 4 \text{Re}(g(x))
\]
(5)

For the same choice of \( x_{1(2)} \), we let \( c_1 = i\xi \in i\mathbb{R}, c_2 = 1 \). Then,

\[
\text{Re}(g(x))\xi^2 - [2 \text{Im}(g(x)) - \text{Im}(g(2x))]\xi + \text{Re}(g(2x)) \geq 0
\]

That yields

\[
[\text{Im}(g(2x)) - 2 \text{Im}(g(x))]^2 \leq 4 \text{Re}(g(x))\text{Re}(g(2x))
\]
Using (5), we have

\[
|\text{Im}(g(2x))| \leq 2|\text{Im}(g(x))| + 4 \text{Re}(g(x))
\]

Combining estimates for the real and imaginary parts, we obtain (4).

The following estimate holds true

\[
|g(x)| < C(1 + |x|^3)
\]
indeed, if \( \max_{x \in [-1,1]} |g(x)| = M, \) then \( \max_{-2^n,2^n} |g(x)| \leq 8^n M \) by (1). Therefore, \( |g(x)| \leq 8^{[\log_2 |x|]+1} M \leq C|x|^3 \), where \([\cdot]\) means the integer part of a number. As we will see later, the estimate (6) is very far from optimal.

The following integral representation of \( G_\infty \) functions is an analog of Bochner’s theorem for class \( G_\infty \).

Theorem 1.2. Function \( g(x) \in G_\infty \) if and only if

\[
g(x) = i\beta x + \int_{-\infty}^{\infty} \left( 1 + \frac{i\lambda x}{1 + \lambda^2} - \exp(i\lambda x) \right) \frac{d\sigma(\lambda)}{\lambda^2}
\]
where \( \beta \in \mathbb{R} \) and positive measure \( \sigma \) satisfies the estimate

\[
\int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{1 + \lambda^2} < \infty.
\]
(8)

Constant \( \beta \) and measure \( \sigma \) are uniquely defined.
Proof. Any functions of the form (7) belongs to $G_\infty$. Indeed, notice that the integral in (7) converges if (8) holds and defines the continuous function that vanishes at zero. Then, we have the following representation

$$g(x) + g(-y) - g(x - y) = \int_{-\infty}^{\infty} \frac{(1 - \exp(i\lambda x))(1 - \exp(i\lambda y))}{\lambda^2} d\sigma(\lambda)$$

which ensures the positivity of the operator with the corresponding kernel for any $r > 0$.

Conversely, due to (10), any $G_\infty$ function allows Laplace transform. Consider

$$L(z) = z^2 \int_{0}^{\infty} g(x) \exp(ixz) dx$$

This function is analytic in $\mathbb{C}^+$. Notice that

$$\frac{L(z) + L(z^{-1})}{2\text{Im } z} = |z|^2 \int_{0}^{\infty} \int_{0}^{\infty} [g(x - y) - g(x) - g(-y)] \exp(ixz - iy\zeta) dxdy \leq 0$$

Consequently, $-iL(z)$ is Herglotz function and has well-known integral representation ([4], chapter 6) which gives

$$L(z) = i\alpha z - i\beta - i \int_{-\infty}^{\infty} \frac{1 - \lambda x}{(\lambda + z)(1 + \lambda^2)} d\sigma(\lambda)$$

where

$$\int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{1 + \lambda^2} < \infty$$

and $\alpha \geq 0, \beta \in \mathbb{R}$. Let us take the inverse Laplace transform. Notice that

$$\frac{\lambda^2(1 - \lambda z)}{i(\lambda + z)(1 + \lambda^2)} = z^2 \int_{0}^{\infty} \left[ 1 + \frac{i\lambda x}{1 + \lambda^2} - \exp(i\lambda x) \right] \exp(ixz) dx, z \in \mathbb{C}^+, \lambda \in \mathbb{R}$$

Therefore,

$$L(z) = i\alpha z - i\beta + z^2 \int_{0}^{\infty} \exp(ixz) \int_{-\infty}^{\infty} \left[ 1 + \frac{i\lambda x}{1 + \lambda^2} - \exp(i\lambda x) \right] \frac{d\sigma(\lambda)}{\lambda^2} dx$$

Since

$$1 = -z^2 \int_{0}^{\infty} x \exp(ixz) dx, \quad z = -iz^2 \int_{0}^{\infty} \exp(ixz) dx, \quad z \in \mathbb{C}^+$$

we have the formula

$$g(x) = \alpha + i\beta x + \int_{-\infty}^{\infty} \left[ 1 + \frac{i\lambda x}{1 + \lambda^2} - \exp(i\lambda x) \right] \frac{d\sigma(\lambda)}{\lambda^2}$$

Due to normalization $g(0) = 0, \alpha = 0$. Uniqueness follows from the uniqueness of the Herglotz function representation. \qed
As a simple corollary one gets the following improvement of (6): \(|g(x)| < C(1 + x^2)\). If \(\sigma\) in (7) is Heaviside function and \(\beta = 0\), then \(g(x) = x^2/2\). So, the quadratic growth is possible.

If the support of \(\sigma\) is a compact, then the second derivative of \(g\) exists and is positive definite. That follows from the Bochner’s theorem. In general, the derivatives of \(g(x)\) at zero can have singularities. In what follows, the reference measure is \(\sigma(\lambda) = \lambda/(2\pi)\), and \(\beta = 0\). It is then easy to check that \(g(x) = |x|/2\). So, in this case, the second derivative in the distributional sense is delta function.

The relation between \(G_\infty\) and \(P_\infty\) can be established by

**Lemma 1.1.** If \(f(x) \in P_\infty\), then \(f(0) - f(x) \in G_\infty\).

**Proof.** From Bochner’s theorem, we have

\[
  f(x) = \int_{-\infty}^{\infty} \exp(ikt) d\mu(t) \quad \int_{-\infty}^{\infty} \exp(ikt) d\mu(t)
\]

Take \(\sigma(\lambda) = \int_{0}^{\infty} t^2 d\mu(t), \quad \beta = -\int_{-\infty}^{\infty} t(1 + t^2)^{-1} d\mu(t)\) and use Theorem 2. \(\square\)

The converse is wrong. Take \(f(x) = ix\). The corresponding kernel \(K(x, y) = 0\). At the same time, \(f(x) \notin P_\infty\) because \(f(x)\) is not bounded. The class \(G_\infty\) is convenient for description of measures generating one very important class of canonical differential systems, the Krein systems, which we plan to study in the next sections.

**Remarks and Historical Notes.**

Classes \(P_r, G_r\) can be introduced for finite \(r > 0\) (see [3], p. 190 and references there). Then, measures \(\mu\) and \(\sigma\) are not uniquely defined in general. The class of nonuniqueness for \(\sigma\) corresponds to the continuation problems [46]. The case when quadratic form (2) has not more than \(\kappa\) negative squares is more difficult. It was studied in the framework of Pontryagin \(\Pi_{\kappa}\)-spaces [46].
2. Factorization of integral operators

To understand better the algebraic aspects of the theory, we will need some rather simple results on the factorization of integral operators. As we know from the linear algebra, given any matrix $A = \{a_{ij}\}_{i,j=1}^d$ with nonzero leading principal minors, we can always find a lower-triangular matrix $X_1$ and an upper-triangular matrix $X_2$, such that $A = X_1DX_2$, $D$ is a diagonal matrix. The proof is simple. Since $a_{11} \neq 0$, using the first step of Gauss algorithm, we can find the lower-triangular matrix $L_1$ of elementary transforms, such that $L_1A$ has the first column collinear to $[1, 0, \ldots, 0]^t$. Then, find an upper-triangular $U_1$ such that the matrix $L_1AU_1$ has the first raw collinear to $[1, 0, \ldots, 0]$. Notice that the first column stays the same. The leading principle minors of $L_1AU_1$ are the same as those of $A$. Thus, we can continue this process. In the end, we get a lower-triangular $L = L_d \ldots L_1$ and an upper-triangular $U = U_1 \ldots U_d$ such that $LAU = D$. Denoting $X_1 = L^{-1}$, $X_2 = U^{-1}$, we get the desired statement. Notice that matrices $X_1$ and $X_2$ have 1 on the diagonal. Taking inverse of $A$, we get the factorization in the reverse order.

Now, what happens in the continuous case? First, we need to establish one general result about the resolvent kernels of integral operators. Fix some $R > 0$. For any $0 < r \leq R$, consider integral operators

\[ \mathcal{K}_r f(x) = \int_0^r K(x, y)f(y)dy, \]

with a kernel $K(x, y)$, continuous on $[0, R]^2$, and acting in the Hilbert space $L^2[0, r]$. Assume $-1 \notin Spec(\mathcal{K}_r)$ for any $0 < r \leq R$. Then, the resolvent kernel $\Gamma_r(s, t)$ exists and

\[ \Gamma_r(s, t) + \int_0^r K(s, u)\Gamma_r(u, t)du = K(s, t), 0 < s, t < r \]

(11)

**Lemma 2.1.** Let $K(x, y) \in C([0, R]^2)$ and $-1 \notin Spec(\mathcal{K}_r)$ for any $0 < r \leq R$. Then,

\[ \partial \Gamma_r(s, t)/\partial r = -\Gamma_r(s, r)\Gamma_r(r, t), 0 \leq s, t \leq r \]

Function $\Gamma_r(s, t)$ is jointly continuous in $s, t, r$ and is continuously differentiable in $r$.

**Proof.** Notice that Fredholm’s formula for resolvent kernel

\[ \Gamma_r(s, t) = \frac{\delta_r(s, t)}{\delta_r} \]

(12)

ensures that $\Gamma_r(s, t)$ is jointly continuous in $s, t, r$ and is continuously differentiable in $r, r > 0$ (Appendix, Lemma 17.1).

Then, differentiate (11) in $r$. We have

\[ \frac{\partial \Gamma_r(s, t)}{\partial r} + \int_0^r K(s, u)\frac{\partial \Gamma_r(u, t)}{\partial r}du = -K(s, r)\Gamma_r(r, t) \]

(13)
On the other hand, multiplying both sides of

$$\Gamma_r(s, r) + \int_0^r K(s, u) \Gamma_r(u, r) du = K(s, r) \quad (14)$$

by $-\Gamma_r(r, t)$, we obtain an equation

$$-\Gamma_r(s, r) \Gamma_r(r, t) - \int_0^r K(s, u) \Gamma_r(u, r) \Gamma_r(r, t) du = -K(s, r) \Gamma_r(r, t) \quad (15)$$

Now we see that both $\partial \Gamma_r(s, t)/\partial r$ and $-\Gamma_r(s, r) \Gamma_r(r, t)$ satisfy the same integral equation. Therefore, they are equal. \qed

Although the continuous kernels are very natural, we will also work with $K(x, y)$ from the following class.

**Definition 2.1.** Function $K(x, y)$ belongs to the class $\hat{C}([0, R]^2)$, if it is continuous in each of the triangles: $\Delta_+ = \{0 \leq x \leq y \leq R\}$ and $\Delta_- = \{0 \leq y \leq x \leq R\}$ but might have discontinuity on the diagonal $x = y$ if considered as the function on $[0, R]^2$ (i.e. the limits $K_+(x, x)$ and $K_-(x, x)$ might be different).

For this case, we have an analogous statement

**Lemma 2.2.** Let $K(x, y) \in \hat{C}([0, R]^2)$ and $-1 \notin \text{Spec}(\mathcal{K}_r)$ for any $0 < r \leq R$. Then,

$$\partial \Gamma_r(s, t)/\partial r = -\Gamma_r(s, r) \Gamma_r(r, t), \ s, t \in \Delta_\pm \quad (16)$$

Function $\Gamma_r(s, t) \in \hat{C}([0, r]^2)$. It is continuously differentiable in $r$ for $s, t \in \Delta_\pm$ and the derivative $\partial \Gamma_r(s, t)/\partial r$ is also from $\hat{C}([0, r]^2)$.

**Proof.** The proof is similar to the proof of Lemma 2.1. Instead of the usual Fredholm formula for resolvent kernel we need to use a modified one given in Lemma 17.2. Indeed, analysis of $\delta_r$ shows that it is continuously differentiable. The Carleman-Hilbert determinant $\delta_r(s, t)$ has derivative in $r$ for $s, t$ fixed in each of $\Delta_\pm$. This derivative is also from the class $\hat{C}([0, r]^2)$. \qed

Notice that $\partial \Gamma_r(s, t)/\partial r$ can be regarded as a continuous function on the whole $[0, r]^2$ because the right-hand side of (16) is continuous on $[0, r]^2$.

The natural analog of the lower-triangular matrix is Volterra integral operator

$$\mathcal{L}f(x) = \int_0^x L(x, y) f(y) dy$$

acting on the Hilbert space $L^2[0, R]$. We also assume that $L(x, y) \in C(\Delta_-)$. An operator $\mathcal{U}$ is upper-triangular, if $\mathcal{U}^*$ is lower-triangular. The product and the sum of two lower(upper)-triangular operators are lower(upper)-triangular as well. Operators $I + \mathcal{L}$ and $I + \mathcal{U}$ are both invertible and $(I + \mathcal{L})^{-1} - I$ is lower-triangular, $(I + \mathcal{U})^{-1} - I$ is upper-triangular. In fact, there is the Banach algebra of lower(upper)-triangular operators $[32, 30]$. Assuming that we have factorization

$$I + \mathcal{K}_R = (I + \mathcal{L})(I + \mathcal{U}) \quad (17)$$

where $\mathcal{L}$ is lower-triangular and $\mathcal{U}$ is upper-triangular, we immediately get that $I + \mathcal{K}_R$ is invertible and $(I + \mathcal{K}_R)^{-1} = I - \mathcal{S}_R = (I + \mathcal{V}_+)(I + \mathcal{V}_-)$, where $\mathcal{V}_\pm$
is upper(lower)-triangular with kernels $V_+(x, y), 0 < x, y < R$ and we have the following formula for the resolvent kernel

$$\Gamma_R(x, y) = \begin{cases} 
-V_+(x, y) - \int_y^R V_+(x, u)V_-(u, y)du, & x < y \\
-V_-(x, y) - \int_x^R V_+(x, u)V_-(u, y)du, & x > y 
\end{cases}$$

(18)

It should also be mentioned that $\mathcal{K}_r, (0 < r < R)$ can be factorized just by using truncations of $\mathcal{L}$ and $\mathcal{U}$.

**Theorem 2.1.** The integral operator $\mathcal{K}_R$ with kernel $K(x, y) \in \hat{C}([0, R]^2)$ admits factorization $\mathcal{T}$ if and only if $I + \mathcal{K}_r$ is invertible in $L^2[0, r]$ for any $0 < r < R$. In this case,

$$V_+(x, y) = -\Gamma_y(x, y), \quad V_-(x, y) = -\Gamma_x(x, y)$$

(19)

where $\Gamma, (x, y)$ denotes the resolvent kernel of $I + \mathcal{K}_r$.

**Proof.** Indeed, assume that $I + \mathcal{K}_r$ is invertible for any $0 < r < R$ and $\Gamma, (x, y)$ is the resolvent kernel. Define $V_\pm$ by (19). Now, let us check (18). Indeed, from Lemma 2.2,

$$\Gamma_y(x, y) - \int_y^R \Gamma_u(x, u)\Gamma_u(u, y)du = \Gamma_y(x, y) + \int_y^R \frac{\partial}{\partial u}\Gamma_u(x, y)du = \Gamma_R(x, y), 0 < x < y < R$$

The case $x > y$ can be checked in the same way. Thus, we have (18), which means that $I - \mathcal{G}_R = (I + V_+)(I + V_-)$. Now, (17) is straightforward. Conversely, assume that the factorization (17) exists. Then

$$I + \mathcal{K}_r = \Pi_{[0, r]}(1 + \mathcal{L})(1 + \mathcal{U})\Pi_{[0, r]} = \Pi_{[0, r]}(1 + \mathcal{L})\Pi_{[0, r]}(1 + \mathcal{U})\Pi_{[0, r]}$$

and it is clearly invertible. □

Another important class of factorizations is the following one. Instead of integral operator on $L^2[0, R]$ we consider an integral operator on $L^2[\mathbb{R}, \mathbb{R}]$ and define the lower-triangular operator as

$$\hat{\mathcal{L}} f(x) = \int_{-|x|}^{|x|} \hat{\mathcal{L}}(x, y)f(y)dy$$

where $\hat{\mathcal{L}}(x, y)$ is continuous in $\{0 \leq x \leq R, |y| \leq |x|\}$ and in $\{-R \leq x \leq 0, |y| \leq |x|\}$. Introduce $\hat{\Omega}_- = \{|y| \leq |x| \leq R\}, \hat{\Omega}_+ = \{|x| \leq |y| \leq R\}$ and redefine $\Delta_- = \{-R \leq y \leq x \leq R\}, \Delta_+ = \{-R \leq x \leq y \leq R\}$.

Similarly, we say that $\mathcal{U}$ is upper-triangular if $\hat{\mathcal{U}}^*$ is lower-triangular. These newly defined lower-triangular operators possess the same algebraic properties: sum and product of two lower-triangular operators is lower-triangular, $(I + \hat{\mathcal{L}})^{-1} - I$ exists and is lower-triangular. The same is true about the upper-triangular operators. In
The function resolvent kernel is upper(lower)-triangular with kernels  \( \hat{\Delta} \) and the derivative \( \partial_r \hat{\Delta} \) is also from \( \tilde{C}([-r, r]^2) \).

**Proof.** We will only check the formula for derivative. The other properties can be checked just like in Lemma 2.2. We have

\[
\hat{\Gamma}_r(s, t) + \int_{-r}^r \hat{K}(s, u)\hat{\Gamma}_r(u, t)du = \hat{K}(s, t)
\]

(21)

Differentiating in \( r \), we get \( s \neq t \)

\[
\frac{\partial \hat{\Gamma}_r(s, t)}{\partial r} + \int_{-r}^r \hat{K}(s, u)\frac{\partial \hat{\Gamma}_r(u, t)}{\partial r}du = -(\hat{K}(s, r)\hat{\Gamma}_r(r, t) + \hat{K}(s, -r)\hat{\Gamma}_r(-r, t))
\]

(22)

On the other hand, take (21) with \( t = \pm r \), multiply by \( -\hat{\Gamma}_r(\pm r, t) \) and add:

\[
-\left[-\left(\hat{\Gamma}_r(s, r)\hat{\Gamma}_r(r, t) + \hat{\Gamma}_r(s, -r)\hat{\Gamma}_r(-r, t)\right) + \int_{-r}^r \hat{K}(s, u)\left[-\left(\hat{\Gamma}_r(u, r)\hat{\Gamma}_r(r, t) + \hat{\Gamma}_r(u, -r)\hat{\Gamma}_r(-r, t)\right)\right]du =
\]

\[
-(\hat{K}(s, r)\hat{\Gamma}_r(r, t) + \hat{K}(s, -r)\hat{\Gamma}_r(-r, t))
\]

Comparison with (22) finishes the proof.

Now, if we have (21), then \((I + \hat{\mathcal{K}}_R)^{-1} = I - \hat{\mathcal{G}}_R = (I + \hat{\mathcal{V}}_+)(I + \hat{\mathcal{V}}_-),\) where \( \hat{\mathcal{V}}_\pm \) is upper(lower)-triangular with kernels \( \hat{\mathcal{V}}_\pm(x, y), -R \leq x, y \leq R \). Moreover, for the resolvent kernel

\[
\hat{\Gamma}_R(x, y) = \left\{
\begin{array}{ll}
-\hat{\mathcal{V}}_+(x, y) - \int_{|y| < |u| < R} \hat{\mathcal{V}}_+(x, u)\hat{\mathcal{V}}_-(u, y)du, & (x, y) \in \hat{\Omega}_+
\end{array}
\right.
\]

(23)

The next Theorem provides the needed factorization.
Theorem 2.2. The integral operator \( \hat{K}_R \) with kernel \( \hat{K}(x, y) \in \hat{C}([-R, R]^2) \) admits factorization \( \hat{K}_R \) if and only if \( I + \hat{K}_r \) is invertible in \( L^2[-r, r] \) for any \( 0 < r \leq R \). In this case,
\[
\hat{V}_+(x, y) = -\hat{\Gamma}_|y|(x, y), \quad (x, y) \in \hat{\Omega}_+
\]
\[
\hat{V}_-(x, y) = -\hat{\Gamma}_|x|(x, y), \quad (x, y) \in \hat{\Omega}_-
\]
(24)

Proof. Calculating the right-hand side in (23) and using Lemma 2.3 we get (for \((x, y) \in \hat{\Omega}_-\)):
\[
-\hat{V}_-(x, y) - \int_{|x| < |u| < R} \hat{V}_+(x, u)\hat{V}_-(u, y)du =
\]
\[
= \hat{\Gamma}_{|x|}(x, y) - \int_{|x|}^R \hat{\Gamma}_u(x, u)\hat{\Gamma}_u(u, y)du - \int_{-r}^{-|x|} \hat{\Gamma}_{|u|}(x, u)\hat{\Gamma}_{|u|}(u, y)du
\]
\[
= \hat{\Gamma}_{|x|}(x, y) - \int_{|x|}^R \left[\hat{\Gamma}_u(x, u)\hat{\Gamma}_u(u, y)du + \hat{\Gamma}_u(x, -u)\hat{\Gamma}_u(-u, y)\right]du
\]
\[
= \hat{\Gamma}_R(x, y)
\]
The case \((x, y) \in \Omega_+\) can be checked similarly. So, (23) is true. The other statements of the Theorem can be verified following the proof of Theorem 2.1. □

Notice that \( \hat{V}_-(x, -x) = \hat{V}_+(x, -x), x \neq 0 \). The results of last Lemma and a Theorem can be easily generalized to the case when the kernel \( K(x, y) \) is allowed to have a discontinuity of the first kind on \( \{(x, y) : y = -x\} \). We do not do that since the class \( \hat{C}([-R, R]^2) \) is exactly the one we will need later on.

Remarks and Historical Notes.
The proofs of the results in this section are partially taken from [30]. In [30], the general case of factorization along the chain is considered. Recently, the factorization problem for integral operators with less regular kernels was studied in [54, 55]. Later on, we will need to use the factorization of Fredholm integral operators along with regularity properties of the kernels.
3. Continuous analogs of polynomials orthogonal on the unit circle

In this section we start building the theory of continuous analogs of polynomials orthogonal on the unit circle (OPUC). For the OPUC basics, we refer the reader to [72, 26, 66, 40]. Let $H(x)$ be Hermitian function defined on $\mathbb{R}$ and $H(x) \in L^1(0,r)$ for any $r > 0$. In the Hilbert space $L^2[0,r]$, consider the following integral operator

$$
\mathcal{H}_r f(x) = \int_0^r H(x-t)f(t)dt, \ 0 < x < r
$$

(25)

This operator is called “truncated Toeplitz” operator or operator with the “displacement kernel” [65, 46]. It is obvious that for any $r > 0$, this operator is self-adjoint, compact, and its lower (upper) bound decreases (increases) in $r$.

Definition 3.1. Function $g(x) \in G_\infty$ has an accelerant if there exists Hermitian function $H(x)$ (accelerant) defined on $\mathbb{R}$ such that

$$
g(x) = \frac{|x|}{2} + \int_0^x (x-s)H(s)ds
$$

(26)

for all $x \in \mathbb{R}$.

Theorem 3.1. The function $H(x)$ generates $g(x) \in G_\infty$ by formula (26) if and only if

$$
I + \mathcal{H}_r \geq 0
$$

for any $r > 0$ and inequality is understood in the operator sense.

Proof. It is obvious that $g(x)$ is Hermitian, continuous, and $g(0) = 0$. Consider any $\varphi(x) \in C^\infty[0, \infty)$ with compact support on $[0,r]$. For the kernel $K(x,y) = g(x) + g(-y) - g(x-y)$, we have

$$
\int_0^\infty \int_0^\infty K(x,y)\varphi'(y)\varphi'(x)dxdy = I_1 + I_2 - I_3
$$

For $I_1$ and $I_2$,

$$
I_1 = I_2 = -\varphi(0) \int_0^\infty g(x)\varphi'(x)dx
$$

$$
I_3 = \int_0^\varphi'(x) \int_0^{\varphi(x)g(x)dydx} - \varphi(0) \int_0^\varphi'(x) \varphi'(x)dx + \int_0^\varphi'(x) \int_0^{\varphi'(x)} g'(x-y)\varphi(y)dydx
$$

Using (26) and integrating by parts, we have

$$
\int_0^\infty K(x,y)\varphi'(y)\varphi'(x)dxdy = \int_0^\varphi'(x) H(x-y)\varphi'(x)dxdy + \int_0^\varphi'(x) \int_0^{\varphi(x)} |\varphi(x)|^2 dx = ((1 + \mathcal{H}_r) \varphi, \varphi)
$$

Thus, if $g \in G_\infty$, then $I_1 + I_2 - I_3 \geq 0$ and $1 + \mathcal{H}_r \geq 0$ for any $r > 0$. Conversely, assume that $1 + \mathcal{H}_r \geq 0$ for any $r > 0$. Take any $\psi(x) \in C^\infty[0, \infty)$ with compact support in $[0,r]$. It can be written as

$$
\psi(x) = \varphi'(x)
$$
where
\[ \phi(x) = -\int_{x}^{\infty} \psi(s)ds \]
and \( \phi(x) \in C^\infty[0, \infty) \), \( \phi(x) \) is supported on \([0, r]\). Therefore,
\[ \int_0^\infty \int_0^\infty K(x, y)\psi(y)\psi(x)dxdy = ((1 + \mathcal{H}_r)\phi, \phi) \geq 0 \]
and \( g(x) \in G_\infty \). \( \square \)

As a corollary from Theorem 1.2 and Theorem 3.1, we get the following formula for an accelerant
\[ \frac{|x|}{2} + \int_0^x (x - s)H(s)ds = i\beta x + \int_{-\infty}^{\infty} \left( 1 + \frac{i\lambda x}{1 + \lambda^2} - \exp(i\lambda x) \right) \frac{d\sigma(\lambda)}{\lambda^2} \] (27)
where \( \beta \in \mathbb{R} \), and
\[ \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{1 + \lambda^2} < \infty \]

The straightforward calculation shows that the trivial case \( H(x) = 0 \) corresponds to \( \beta = 0 \) and \( \sigma_0(\lambda) = \lambda/(2\pi) \).

Essentially, (27) means that
\[ \int_{-\infty}^{\infty} \exp(i\lambda x)d\sigma(\lambda) \quad \text{“} = \quad \delta(x) + H(x) \]
or
\[ \int_{-\infty}^{\infty} \exp(i\lambda x)d(\sigma(\lambda) - \sigma_0(\lambda)) \quad \text{“} = \quad H(x) \]
In other words, \( H(x) \) are “moments” of \( \sigma - \sigma_0 \). Clearly, the integrals in the last formulas do not have to converge in the usual sense.

**Lemma 3.1.** Assume that \( \sigma \) from (7) is known and \( g \) has an accelerant, then the constant \( \beta \) is defined uniquely by the formula
\[ \beta = -i\Psi'(0) \] (28)
where
\[ \Psi(x) = -\int_{-\infty}^{\infty} \left( 1 + \frac{i\lambda x}{1 + \lambda^2} - \exp(i\lambda x) \right) \frac{d[\sigma(\lambda) - \sigma_0(\lambda)]}{\lambda^2} \in C^1(\mathbb{R}) \] (29)

**Proof.** The proof follows from the formula (27) by taking the derivative. \( \square \)

It is up to us to choose the regularity class for \( H(x) \). In these notes, we will consider two important cases:

\[ H(x) \in C[0, \infty), \] (30)

and
\[ H(x) \in L^2_{loc}(\mathbb{R}) \] (31)
Other classes of regularity (e.g., $H(x) \in L^p_{\text{loc}}(\mathbb{R}), p \geq 1$) can also be treated. Although, the case $p = 1$ needs special consideration (see discussion in [46, 45]).

We will start our construction with the continuous accelerants. Then the $L^2_{\text{loc}}$ case will be treated by an approximation argument in the separate section. For $(30)$, $H(x)$ might have discontinuity at 0 but the left and the right limits must exist and $H(-0) = H(+0)$ due to Hermite property. Notice that the operator $\mathcal{H}_r$ has a kernel from the class $\hat{C}([0, r]^2), r > 0$.

Assume that we have the strict inequality

$$1 + \mathcal{H}_r > 0$$

(32)

for any $r > 0$. Then, there is the resolvent kernel $\Gamma_r(t, s)$ with nice properties (see Lemma 2.2) such that

$$\Gamma_r(s, t) = \overline{\Gamma_r(t, s)}$$

(33)

$$\Gamma_r(t, s) + \int_0^r H(t-u)\Gamma_r(u, s)du = H(t-s),$$

(34)

$$\Gamma_r(t, s) + \int_0^r \Gamma_r(t, u)H(u-s)du = H(t-s), \quad 0 \leq s, t \leq r$$

(35)

We emphasize that the last two identities should be understood as equalities for functions from $\hat{C}([0, r]^2)$ class. In the meantime, if $H(\pm 0) \in \mathbb{R}$, then $H(x)$ is actually continuous at 0 and by Lemma 2.1 the kernel $\Gamma_r(x,y)$ is continuous on the diagonal as well.

Let us introduce the following “continuous polynomials”

$$P(r, \lambda) = \exp(i\lambda r) - \int_0^r \Gamma_r(r, s)\exp(i\lambda s)ds$$

(36)

$$P_*(r, \lambda) = 1 - \int_0^r \Gamma_r(s, r)\exp(i\lambda(r-s))ds$$

(37)

Notice that function $P(r, \lambda)$ is of exponential type exactly $r$, and $P_*(r, \lambda)$ is of exponential type not greater than $r$.

Formulas (36) and (37) can be easily explained. They are quite natural and have analogs in the OPUC theory. Let us consider positive finite measure $\mu(\theta)$ on the unit circle. We will denote the inner product of two functions $f$ and $g$ in $L^2(d\mu)$ by $(f,g)_\mu$. Let the sequence of moments be

$$c_n = \int_{-\pi}^{\pi} \exp(-in\theta)d\mu(\theta), \quad n \in \mathbb{Z}^+$$

Let $\{e_j\}, e_j = [0, \ldots, 0, 1, 0, \ldots, 0]^t, (j = 0, \ldots)$ denotes the standard orthonormal basis. Consider the following Toeplitz matrix

$$\mathcal{T}_n = \begin{bmatrix}
  c_0 & c_1 & \cdots & c_n \\
  c_1 & c_0 & \cdots & c_{n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_n & c_{n-1} & \cdots & c_0 
\end{bmatrix}$$

(38)
If $D_n = \det T_n$, then one can easily show that

\[
P_n(z) = \frac{1}{D_{n-1}} \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} & 1 \\ c_1 & c_0 & \cdots & c_{n-1} & z \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_0 & z^{n-1} \\ \overline{c}_n & \overline{c}_{n-1} & \cdots & \overline{c}_1 & z^n \end{pmatrix}
\]

are monic polynomials of degree $n$ (i.e., the coefficient in front of $z^n$ is 1) orthogonal with respect to $d\mu$. In formula \[37\], exponents $\exp(i\lambda r)$, $r \geq 0$, $\lambda \in \mathbb{C}$ play the role of $z^n$, $n \in \mathbb{Z}^+$, $z \in \mathbb{C}$. The exponential type $r$ is an analog of the integer index $n$. If one introduces the truncated discrete Toeplitz operator $T_n$ given by the matrix \[38\], then $P_n(z)$ is the last component of the following vector $D_nD_{n-1}^{-1}T_n^{-1}[1, z, \ldots, z^n]^t$. That follows from Kramer’s rule and \[39\]. Besides Kramer’s rule, there is the following algebraic explanation of orthogonality relation. Consider the last component of the vector $T_n^{-1}[1, z, \ldots, z^n]^t$, i.e. $(T_n^{-1}[1, z, \ldots, z^n]^t, e_n)$. Assume that we have two polynomials of degree not greater than $n$: $A(z) = a_n z^n + \ldots + a_0$, $B(z) = b_n z^n + \ldots + b_0$. Then

\[
(A, B)_{\mu} = (\overline{T_n a}, b)
\]

where $\overline{T_n}$ is obtained from $T_n$ by conjugating all elements, $a = [a_0, \ldots, a_n]^t$, $b = [b_0, \ldots, b_n]^t$. Therefore,

\[
(z^j, (T_n^{-1}[1, z, \ldots, z^n]^t, e_n))_{\mu} = (z^j, ([1, z, \ldots, z^n]^t, T_n^{-1} e_n))_{\mu} = (\overline{T_n e_j}, \overline{T_n^{-1} e_n}) = \delta_{jn}
\]

Now, consider the function $f_r(x) = \exp(i\lambda x)$ on an interval $x \in [0, r]$. Then, $P(r, \lambda)$ is the value of the function $(1 + H_r)^{-1}f_r$ at the point $x = r$. Because the spectrum of the truncated continuous Toeplitz operator $H_r$, always contains 0, we need to take $(1 + H_r)^{-1}$, rather than $T_n^{-1}$ in the discrete case. This normalization makes the function $P(r, \lambda)$ “monic”-- it has $\exp(i\lambda r)$ term. In the theory of orthogonal polynomials on the unit circle, there is a natural procedure that maps any polynomial of degree $n$ to its “reciprocal”. It is defined in the following way

\[
[*](a_n z^n + \ldots + a_0) = \overline{a_0} z^n + \ldots + \overline{a_n}
\]

or

\[
[*](A_n(z)) = z^n \overline{A(\overline{z})^{-1}}
\]

Notice that the degree of the polynomial might decrease under this operation. For example, in the space of polynomials of degree not greater than $n$, $[*](z^n) = 1$. The natural analog of $[*]$ operation for the function $f_r(\lambda)$ of the exponential type $r$ is given by the following formula

\[
[*](f_r(\lambda)) = \exp(i\lambda r) \overline{f_r(\lambda)}
\]

Function $P_r(r, \lambda)$ is a continuous analog of $[*](P_n(z))$, i.e.

\[
P_r(r, \lambda) = [*](P(r, \lambda))
\]

It is clear from the formula \[39\] and identity $\Gamma_r(s, t) = \overline{\Gamma_r(t, s)}$. In the discrete case, polynomials $P_n(z)$ are orthogonal with respect to $d\mu$. In the continuous setting, $\{P(r, \lambda)\}$ turns out to be orthonormal family in $L^2(\sigma \overline{\sigma})$ with $\sigma$ from the integral representation \[27\]. Orthogonality is understood in the usual sense-- just like for the Fourier transform. Thus, rather than in the discrete case, continuous monic polynomials are already normalized.
Recall the factorization results from the previous section. Notice that since $I + H_r > 0$ for any $r > 0$, the Theorem 2.1 is applicable. Let us fix any $R > 0$ and consider the factorization (17): $I + H_R = (I + L)(I + U)$, where the lower-diagonal $L$ has kernel $L(x, y)$ and the upper-diagonal $U$ has kernel $U(x, y)$, $0 < x, y < R$. Since $H$ is Hermitian, $L = U^*$. If $I + L = (I + V - I)^{-1}$ and $I + U = (I + V + I)^{-1}$, then

**Lemma 3.2.** The following formula is true

$$\exp(i\lambda r) = P(r, \lambda) + \int_0^r L(r, s)P(s, \lambda)ds, 0 < r \leq R$$  \hspace{1cm} (45)

**Proof.** Take any $R > 0$. By (19), equation (36) can be written as

$$P(r, \lambda) = (I + V - I)^{-1}\exp(i\lambda r), 0 < r < R$$

and the both sides are regarded as functions in $L^2[0, R]$. Since $I + L = (I + V - I)^{-1}$, we have the statement of the Lemma 1. \hfill □

In the discrete case, the following formula is true for any $n \in \mathbb{Z}^+$

$$z^n = P_n(z) + \sum_{j=0}^{n-1} l_{n,j} P_j(z), l_{j,n} \in \mathbb{C}$$

The Lemma 3.2 is the continuous analog of that representation. Now, we are ready to prove the following

**Theorem 3.2.** The following map is an isometry from $L^2(\mathbb{R}^+)$ into $L^2(\mathbb{R}, d\sigma)$

$$\mathcal{O} : f(r) \mapsto (\mathcal{O} f)(\lambda) = \int_0^\infty f(r)P(r, \lambda)dr$$  \hspace{1cm} (46)

In other words,

$$\int_{-\infty}^{\infty} ||(\mathcal{O} f)(\lambda)||^2d\sigma(\lambda) = \int_0^{\infty} |f(r)|^2dr$$  \hspace{1cm} (47)

Integral in (46), is understood in the $L^2$- sense.

**Proof.** The following is true for any $t, r \in \mathbb{R}$

$$\frac{|r - t|}{2} + \int_0^{r-t} (r-t-s)H(s)ds = i\beta(r-t) + \int_{-\infty}^{\infty} \left(1 + \frac{i\lambda(r-t)}{1 + \lambda^2} - \exp(i\lambda(r-t))\right) \frac{d\sigma(\lambda)}{\lambda^2}$$

Multiply this equality by $f''(t) f(t) \in C_0^\infty(\mathbb{R})$ and integrate by parts. We have the following formula

$$f(r) + \int_{-\infty}^{\infty} H(r-t) f(t)dt = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(t) \exp(i\lambda(r-t))dt \right] d\sigma(\lambda)$$  \hspace{1cm} (48)

1 Kernel $L(x, y)$ should be regarded as a transformation kernel from one basis to another [51].
For \( f(r) \),
\[
\overline{f(r)} + \int_{-\infty}^{\infty} H(r-t)f(t)dt = \int_{-\infty}^{\infty} \exp(i\lambda r) \int_{-\infty}^{\infty} f(t) \exp(i\lambda t)dt d\sigma(\lambda)
\]
(49)

Consequently, we have the following analog of (40):
\[
\int_{-\infty}^{\infty} \left| f(r) \right|^2 dr + \int_{-\infty}^{\infty} f(r) \int_{-\infty}^{\infty} f(t)H(r-t)dtdr = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \exp(i\lambda t)dt d\sigma(\lambda)
\]
(50)

If \( f \in L^2[0, r] \), then (50) implies
\[
\langle (1 + \mathbb{H}r)f, f \rangle = \int_{-\infty}^{\infty} \int_{0}^{r} f(t) \exp(i\lambda t)dt d\sigma(\lambda)
\]
(51)
by simple approximation argument. Now, let us use Lemma 3.2 for the interval [0, r]:
\[
\int_{0}^{r} f(t) \exp(i\lambda t)dt = ((I + \mathcal{L})P(t, \lambda), f(t))_{L^2[0, r]} = (P(t, \lambda), (I + \mathcal{L}^*)f(t))_{L^2[0, r]}
\]

Therefore, (51) reads
\[
\|g\|^2 = \langle (I + \mathbb{L})^{-1} (I + \mathbb{H}r)(I + \mathbb{U})^{-1} g, g \rangle = \int_{-\infty}^{\infty} \int_{0}^{r} \overline{g(t)}P(t, \lambda)dt d\sigma(\lambda)
\]
(52)
where \( g = (I + \mathbb{U})f \). Since \( I + \mathbb{U} \) is invertible in \( L^2(0, r) \), (52) holds for any \( g \in L^2[0, r] \). Since \( \|g\| = \|\overline{g}\| \) and a number \( r \) was chosen arbitrarily, \( \mathcal{O} \) is isometry on \( L^2(\mathbb{R}^+) \).

As a simple corollary of the Theorem 3.2 and the polarization identity we get the following Lemma.

**Lemma 3.3.** For any \( f(r) \in L^2(\mathbb{R}^+) \)
\[
f(r) = \int_{-\infty}^{\infty} \overline{P(r, \lambda)}(\mathcal{O}f)(\lambda)d\sigma(\lambda)
\]
where the equality is understood in the \( L^2(\mathbb{R}^+) \) sense.

**Remark 3.1.** Notice, that \( \mathcal{O} \) is not necessarily a unitary map. In the simplest case \( H(r) = 0, P(r, \lambda) = \exp(i\lambda r), \sigma(\lambda) = \lambda/(2\pi) \) and the range of \( \mathcal{O} \) is \( H^2(\mathbb{R}) \subset L^2(\mathbb{R}, d\sigma) \).

Now, let us obtain the differential system for \( P(r, \lambda) \) and \( P^*(r, \lambda) \). In the discrete case, we have
\[
\begin{align*}
P_{n+1}(z) &= zP_n(z) - \bar{a}_nP_n^*(z), \quad P_0(z) = 1 \\
P_n^*(z) &= P_n^*(z) - a_nzP_n(z), \quad P_0^*(z) = 1
\end{align*}
\]
(53)
where $P_\ast^r(z) = [\ast]P_n(z)$, $a_n$ are the so-called Verblunsky coefficients (Geronimus coefficients, Schur coefficients, circle parameters, or reflection parameters). If one starts with arbitrary positive finite measure with infinite number of growth points, then the corresponding $a_n \in \mathbb{D}$. Conversely, any sequence $a_n \in \mathbb{D}$ yields the unique probability measure with infinite number of growth points.

Let us prove one property of the resolvent kernel which would yield differential equations for $P(r, \lambda)$ and $P_\ast(r, \lambda)$. It holds only for the integral operators with the “displacement” kernel

**Lemma 3.4.** If $\Gamma_r(s, t)$ is the resolvent kernel for $H_r$, then

$$\Gamma_r(s, t) = \Gamma_r(r - t, r - s) \quad (54)$$

**Proof.** The following relation holds

$$H_r \mathcal{F}_r = \mathcal{F}_r \overline{H}_r$$

where the “flip” operator $\mathcal{F}_r$ is defined: $\mathcal{F}_r f(x) = f(r - x)$ and $\overline{H}_r$ is an integral operator with the displacement kernel $H(x)$. Then,

$$(1 + \mathcal{H}_r)^{-1} \mathcal{F}_r = \mathcal{F}_r (1 + \overline{H}_r)^{-1}$$

Writing this down in terms of the resolvent kernel gives $[54]$. □

As a simple corollary, we also get the following formulas for $P$ and $P_\ast$ which were originally used by Krein

$$P(r, \lambda) = \exp(i\lambda r) \left( 1 - \int_0^r \Gamma_r(s, 0) \exp(-i\lambda s) ds \right) \quad (55)$$

$$P_\ast(r, \lambda) = 1 - \int_0^r \Gamma_r(0, s) \exp(i\lambda s) ds \quad (56)$$

An analog of the relations $[53]$ is given by the following statement.

**Theorem 3.3.** The following equations hold

$$\begin{cases}
    P' = i\lambda P - \overline{A}P_\ast, & P(0, \lambda) = 1, \\
    P_\ast' = -AP, & P_\ast(0, \lambda) = 1
\end{cases} \quad (57)$$

where

$$A(r) = \Gamma_r(0, r) \quad (58)$$

**Proof.** Differentiate $[56]$ and use Lemma $[2.2]$ to get

$$P_\ast'(r, \lambda) = -\Gamma_r(0, r) \exp(i\lambda r) + \int_0^r \Gamma_r(0, r) \Gamma_r(r, s) \exp(i\lambda s) ds = -A(r)P(r, \lambda)$$

where we used the definition of $P(r, \lambda)$. Equation for $P(r, \lambda)$ can be obtained from the equation for $P_\ast(r, \lambda)$ and $[11]$. □

**Definition 3.2.** System $[57]$ is called the Krein system.

Obviously, coefficient $A(r)$ in the Krein system is an analog of Verblunsky coefficients.
Lemma 3.5. Under regularity conditions (30), we have $A(r) \in C[0,\infty)$. We also have $A(0) = H(-0) = H(+0)$.

Proof. The continuity of $A(r)$ follows from Lemma 2.2. The equality $A(0) = H(-0)$ can be obtained from (55) where $r \to 0$.

Notice that $H(x) \in C(\mathbb{R})$ iff $H(0) - \text{real}$ $A(0)$ is real as well.

Let us consider the model (“free”) case. If $\sigma(\lambda) = \lambda/(2\pi)$ in (27), then $\beta = 0$, $g(t) = |t|/2$, $H(t) = 0$, $\Gamma_r(s,t) = 0$, $A(r) = 0$, $P(\rho,\lambda) = \exp(i\lambda r)$, $P_\rho(\rho,\lambda) = 1$.

Lemma 3.6. The following is true
1) Christoffel-Darboux formula:

$$P_\rho(r,\lambda)P_\rho(r,\mu) = P(r,\lambda)\overline{P(r,\mu)} - i(\lambda - \mu) \int_0^r P(s,\lambda)\overline{P(s,\mu)} ds, \quad \lambda, \mu \in \mathbb{C} \quad (59)$$

2) For $\lambda \in \mathbb{C}$,

$$P(r,\lambda) = \exp(i\lambda r)\overline{P_\rho(r,\lambda)} \quad (60)$$

3) $P_\rho(r,\lambda)$ does not have any zeroes in $\mathbb{C}^+$

4) $P(r,\lambda)$ has zeroes in $\mathbb{C}^+$ only

Proof. To prove Christoffel-Darboux formula, multiply the first equation in (57) by $\overline{P(r,\mu)}$. Multiply both sides of

$$P_\rho^\prime(r,\mu) = -\overline{A(r)} P(r,\mu)$$

by $P_\rho(r,\lambda)$ and subtract two identities. One has

$$P_\rho^\prime(r,\lambda)\overline{P(r,\mu)} - P_\rho^\prime(r,\mu)P_\rho(r,\lambda) = i\lambda P(r,\lambda)\overline{P(r,\mu)} \quad (61)$$

Write the same equation with $\lambda$ and $\mu$ interchanged. Take conjugate and add to (61). One gets (59). 2) repeats (44).

Take $\lambda \in \mathbb{C}^+$ and $\mu = \lambda$. Then, (59) guarantees that $|P_\rho(r,\lambda)| > 0$, so $P_\rho(r,\lambda)$ has no zeroes in $\mathbb{C}^+$. Assume that $P_\rho(r,\lambda) = 0$ for some $\lambda \in \mathbb{R}$. Then, by 2), $P(r,\lambda) = 0$. However, functions $P(\rho,\lambda), P_\rho(\rho,\lambda)$ solve the problem (57) and then must vanish for all $\rho \geq 0$. In the meantime $P(0,\lambda) = P_\rho(0,\lambda) = 1$. This contradiction shows that $P_\rho(r,\lambda)$ has no zeroes in $\mathbb{C}^+$.

4) follows from 3) and (44).

Assume that for fixed $r$ we know $P(r,\lambda)$ as the function in $\lambda$. The natural question is whether we can find $P(\rho,\lambda)$ and $A(\rho)$ for all $0 < \rho < r$? The answer happens to be positive. Notice that since $P(r,\lambda)$ is given, we know the values of $\Gamma_r(s,0)$ for all $s \in [0,r]$. It easily follows from (55).

Consider

$$g(t,s) = \Gamma_r(0,r-t)\Gamma_r(r-s,0) - \Gamma_r(t,0)\Gamma_r(0,s), 0 < s, t < r$$

Lemma 3.7. The following formula holds true

$$\Gamma_r(t,s) = \Gamma_r(t-s,0) + \Gamma_r(s-t,0) + \int_0^{\min(s,t)} g(t-u,s-u) du, \quad 0 \leq s \neq t \leq r \quad (62)$$

where $\Gamma_r(0,s) = \Gamma_r(s,0) = 0, s < 0$ for shorthand.
Lemma 3.8. The following representation is true

\[
\frac{1}{|P_*(r, \lambda)|^2} = 1 + \int_{-\infty}^{\infty} \frac{H_r(s) \exp(i\lambda s) ds}{-\infty} \lambda \in \mathbb{R}
\]

where \( H_r(s) \) is Hermitian function, \( H_r(s) \in L^1(\mathbb{R}) \), and \( H_r(s) = H(s) \) for \( |s| < r \).

Proof. Formula (50), 3) in Lemma 3.6 and Levy-Wiener theorem yield existence of function \( H_r \in L^1(\mathbb{R}) \). Moreover, \( H_r \) is continuous on \( \mathbb{R} \) except for the points \( 0, \pm r \), where the left(right) limits exist. That is clear from the corresponding integral equation. Let us show that this \( H_r(s) \) coincides with \( H(s) \) for \( |s| < r \). We have

\[
\frac{1}{P_*(r, \lambda)} - 1 = P_*(r, \lambda) \left( 1 + \int_{-\infty}^{\infty} \frac{H_r(s) \exp(i\lambda s) ds}{-\infty} \right) - 1
\]

The left-hand side belongs \( \overline{H^2(\mathbb{R})} \). Therefore,

\[
\mathcal{P}_+ \left[ P_*(r, \lambda) \left( 1 + \int_{-\infty}^{\infty} \frac{H_r(s) \exp(i\lambda s) ds}{-\infty} \right) - 1 \right] = 0
\]

which gives

\[
\Gamma_r(s, 0) + \int_{0}^{r} H_r(s - u) \Gamma_r(u, 0) du = H_r(s), \quad 0 < s < r
\]

Formula (64) proves that \( H_r \) defines an integral operator \( \mathcal{K}_r \) on \( L^2[0, r] \) and \( I + \mathcal{K}_r > 0 \). Denote its resolvent kernel by \( \tilde{\Gamma}_r(s, t) \). The last equation shows that
\( \hat{\Gamma}_r(s,0) = \Gamma_r(s,0), 0 < s < r. \) Due to Lemma 3.7, \( \hat{\Gamma}_r(s,t) = \Gamma_r(s,t) \) for all \( 0 \leq s, t \leq r. \) So, \( H_r(s) = H(s) \) for \( 0 < s < r. \) Since \( H_r \) is Hermitian, we obtain the statement of the Lemma.

Thus, if we know \( P(r,\lambda) \), we know \( P_*(r,\lambda) \) as well and can find \( H(x) \) for \( |x| < r \) using the previous Lemma.

**Remarks and historical notes.**

Continuous analogs of polynomials orthogonal on the circle were introduced by M.G. Krein in the paper [44] but no proofs were given. We filled this gap. Lemma 3.7 is in [46], see also [24], p.100.

If one is given the function \( H(r) \in C[0,R] \), Hermitian and such that \( I + \mathfrak{H}_R > 0 \), then the Krein system can be well-defined on the interval \([0,R]\). In the meantime, the question of orthogonality with respect to some measure gives rise to certain continuation problem [46] we do not want to address here.
4. Krein systems

In the previous section we learned that any accelerant $H(r)$ gives rise to (57), the system of ODE called the Krein system. But it makes sense to study this system per se. In this section, we will show that one can start with the Krein systems and then define the accelerant $H(r)$ and measure $\sigma(\lambda)$ uniquely.

Consider the system

$$X' = VX$$

with $X(0, \lambda) = I$,

$$V = \begin{bmatrix} i\lambda & -\overline{A(r)} \\ -A(r) & 0 \end{bmatrix}$$

Matrix $V$ has very special algebraic structure and it should imply very special properties for the fundamental (transfer) matrix $X(r)$. Assume first that $A(r) \in L^1_{\text{loc}}(\mathbb{R}^+)$. The first obvious result is

**Lemma 4.1.** We have

$$\det X(r) = \exp(i\lambda r)$$

**Proof.** Indeed,

$$\det X(r) = \exp \left[ \int_0^r \text{Tr} V(t) dt \right] = \exp(i\lambda r)$$

Consider the signature matrix

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**Definition 4.1.** The matrix $M$ is said to be $J$–contraction if $M^*JM \leq J$.

**Definition 4.2.** The matrix $M$ is called $J$–unitary if $M^*JM = J$

Later, we will need the following algebraic

**Lemma 4.2.** If $M$ is $J$–unitary, then $|\det M| = 1$, and $M^{-1}, M^*$ are $J$–unitary too. If $M$ is $J$–contraction then $M^*$ is $J$–contraction also

whose proof is given in the Appendix.

The signature matrix $J$ defines the corresponding indefinite metric. For general properties of these spaces and operators acting on them, see [37, 39].

The next very important algebraic observation is

$$V^*(r)J + JV(r) = -2 \text{Im} \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

for any $r > 0$.

**Theorem 4.1.** The matrix $X$ is $J$–contraction for $\lambda \in \mathbb{C}^+$ and is $J$–unitary for $\lambda \in \mathbb{R}$. 
Proof. Consider \( Y = X^*JX \). Then, (69) yields
\[
Y' = -2 \text{Im} \lambda X^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} X, Y(0, \lambda) = J
\]
Therefore, for any \( f \in \mathbb{C}^2 \), we have
\[
(Yf, f) = (Jf, f), \lambda \in \mathbb{R} \hspace{1cm} (70)
\]
\[
(Yf, f) \leq (Jf, f), \lambda \in \mathbb{C}^+ \hspace{1cm} (71)
\]
\[
(Yf, f) \geq (Jf, f), \lambda \in \mathbb{C}^- \hspace{1cm} (72)
\]
which implies the statement of the theorem. \( \square \)

In this section, we define the functions \( P(r, \lambda), P^*(r, \lambda) \) as solutions of equation (65) corresponding the Cauchy problem \( P(0, \lambda) = P^*(0, \lambda) = 1 \). Consider also two functions \( \hat{P}(r, \lambda) \) and \( \hat{P}^*(r, \lambda) \) such that the vector \( \hat{P}(r, \lambda), -\hat{P}^*(r, \lambda) \) solves (65) and satisfies initial condition \( \hat{P}(0, \lambda) = 1, -\hat{P}^*(0, \lambda) = -1 \). The simple calculation shows that
\[
X(r) = \frac{1}{2} \begin{bmatrix} P + \hat{P} & P - \hat{P} \\ P_s - \hat{P}_s & P_s + \hat{P}_s \end{bmatrix} = \frac{1}{2} \begin{bmatrix} P & \hat{P} \\ P_s & -\hat{P}_s \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \hspace{1cm} (73)
\]
The sign “−” in the definitions of \( \hat{P}_s(r, \lambda) \) was chosen for the following reason. Notice that
\[
JV(r)J = \begin{bmatrix} i\lambda & A(r) \\ A(r) & 0 \end{bmatrix} \hspace{1cm} (74)
\]
Then,

Lemma 4.3. If \( X \) solves equation (65), then \( JXJ \) solves the same equation but with coefficient \( A(r) \) having an opposite sign.
Proof. Multiply (65) from the left by \( J \). Then use (74) and identity \( J^2 = I \). \( \square \)

Corollary 4.1. The vector \( \hat{P}(r, \lambda), \hat{P}_s(r, \lambda) \) satisfies the same initial conditions at zero as \( P(r, \lambda), P_s(r, \lambda) \) but solves system (65) with \( A(r) \) having an opposite sign. This system is called the dual Krein system.

Instead of dealing with the transfer matrix \( X(r, \lambda) \) which solves (65), we will first study functions \( P, P_s, \hat{P}, \) and \( \hat{P}_s \). Below, we list some simple properties.

Lemma 4.4.
1. For any \( \lambda \in \mathbb{C} \) and \( r \geq 0 \)
\[
P(r, \lambda)\hat{P}_s(r, \lambda) + P_s(r, \lambda)\hat{P}(r, \lambda) = 2 \exp(i\lambda r) \hspace{1cm} (75)
\]
2. All statements of Lemma 3.6 are true for both \( P(r, \lambda), P_s(r, \lambda) \) and \( \hat{P}(r, \lambda), \hat{P}_s(r, \lambda) \).
3. For \( \lambda \in \mathbb{C}^+ \), we have
\[
\text{Re} \left[ P_s^{-1}(r, \lambda)\hat{P}_s(r, \lambda) \right] \geq |P_s(r, \lambda)|^{-2} \hspace{1cm} (76)
\]
and the last inequality is equality for real \( \lambda \).

Proof. Part 1) follows from (67) and (73).
To show 2), notice that Christoffel-Darboux formula (60) is the direct consequence of the differential equations (57). Then, (60) holds because the functions \( \exp(i\lambda r)P_s(r, \lambda), \exp(i\lambda r)P(r, \lambda) \) solve the same Cauchy problem as \( P(r, \lambda), P_s(r, \lambda) \)
do. The statements about the zeroes of \( P, P_* \) can be proved in the same way as it was done in Lemma 3.6. The analogous results for \( \hat{P}, \hat{P}_* \) follow from Corollary 4.1.

To prove 3), notice that by Lemma 4.2, \( X^* \) is \( J \)-contraction for \( \lambda \in \mathbb{C}^+ \) and \( J \)-unitary for real \( \lambda \). Writing \( XJX^* \leq J \) in terms of \( P, P_* \), \( \hat{P}, \hat{P}_* \), we get

\[
\frac{1}{2} \begin{bmatrix} P\hat{P} + \hat{P}P & \hat{P}P_* - P\hat{P}_* \\ P_*\hat{P} - \hat{P}_*P & -(P_*\hat{P}_* + \hat{P}_*P) \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = J
\]

Element (2, 2) gives

\[
P_\ast\hat{P}_* + \hat{P}_*P_\ast \geq 2 \tag{77}
\]

which implies 3). For real \( \lambda \), we get equality because \( X^* \) is \( J \)-unitary.

\[\square\]

**Lemma 4.5.** If \( A(r) \) is real and \( \lambda = 0 \), the exact solution can be obtained, i.e.

\[
X(r, 0) = \begin{bmatrix} \cosh \left( -\int_0^r A(t)dt \right) & \sinh \left( -\int_0^r A(t)dt \right) \\ \sinh \left( -\int_0^r A(t)dt \right) & \cosh \left( -\int_0^r A(t)dt \right) \end{bmatrix}
\]

**Proof.** The proof is a direct calculation. \[\square\]

**Lemma 4.6.** The following estimate is true if \( \lambda \in \mathbb{R} \)

\[
\exp \left[ - \int_0^r |A(s)| ds \right] \leq |P_\ast(r, \lambda)| \leq \exp \left[ \int_0^r |A(s)| ds \right]
\]

**Proof.** The second inequality easily follows from the differential equations for \( P \) and \( P_\ast \). The first one is then immediate from (77). \[\square\]

Now, that we studied the general properties of system (65), let us show that for any \( A(r) \in C[0, \infty) \), there is the unique accelerant \( H(r) \in C[0, \infty) \) that generates it.

The following result says that solutions of Krein system are indeed continuous polynomials.

**Lemma 4.7.** For any \( r > 0 \), we have the following formulas

\[
P(r, \lambda) = \exp(i\lambda r) - \int_0^r A(r, s) \exp(i\lambda s) ds \tag{78}
\]

\[
P_\ast(r, \lambda) = 1 - \int_0^r \overline{A(r, s)} \exp(i\lambda(r - s)) ds \tag{79}
\]

where function \( A(r, s) \) is continuous in \( s \) and \( r : 0 \leq s \leq r < \infty \).

**Proof.** Consider \( Q = \exp(-i\lambda r)P \). We have the following equations for \( P \) and \( Q \)

\[
\begin{cases}
Q' = -\exp(-i\lambda r)AP_\ast, & Q(0, \lambda) = 1 \\
P_\ast' = -\exp(i\lambda r)AQ, & P_\ast(0, \lambda) = 1
\end{cases} \tag{80}
\]

The corresponding integral equations are

\[
Q(r, \lambda) = 1 - \int_0^r \exp(-i\lambda s)\overline{A(s)}P_\ast(s, \lambda) ds \tag{81}
\]
\[ P_\ast(r, \lambda) = 1 - \int_0^r \exp(i\lambda s)A(s)Q(s, \lambda)ds \]  \hspace{1cm} (82)

Let us find the solutions to (57) in the following form

\[ P(r, \lambda) = \exp(i\lambda r) - \int_0^r A(r, s) \exp(i\lambda s)ds, P_\ast(r, \lambda) = 1 - \int_0^r B(r, s) \exp(i\lambda(r-s))ds \]  \hspace{1cm} (83)

where \( A \) and \( B \) are continuous function. Then, part 2) of the Lemma 3.6 yields \( B(r, s) = A(r, s) \). Plug (83) into (81) to get the equation for \( A(r, s) \)

\[ A(r, t) = A(r-t) - \int_{r-t}^r A(s)A(s, r-t)ds \]  \hspace{1cm} (84)

Fix any positive \( R \). In the triangle \( \Delta_R = \{ 0 \leq t \leq r \leq R \} \), consider the operator

\[ [Of](r, t) = \int_{r-t}^r A(s)f(s, r-t)ds \]  \hspace{1cm} (85)

Let us shows that \( O \) is Volterra in \( C(\Delta_R) \). That would allow us to solve (84) uniquely.

The following inequalities hold true

\[ \left| \left[ O^{(2n)}f \right](r, t) \right| \leq \frac{||A||_{C[0,R]}^2||f||_{C(\Delta_R)}(r-t)^n}{n! n!} \]  \hspace{1cm} (86)

\[ \left| \left[ O^{(2n-1)}f \right](r, t) \right| \leq \frac{||A||_{C[0,R]}^{2n-1}||f||_{C(\Delta_R)}(r-t)^{n-1}t^n}{n!(n-1)!} \]  \hspace{1cm} (87)

Let us prove them by induction. For \( n = 0, 1 \), the estimates are obvious. Assume that (86) is true for \( n \). Then, we get

\[ \left| \left[ O^{(2n+1)}f \right](r, t) \right| \leq \frac{||A||_{C[0,R]}^{2n+1}||f||_{C(\Delta_R)}(r-t)^n}{(n!)^2}(r-t)^n \int_{r-t}^r (s-(r-t))^n ds \leq \frac{||A||_{C[0,R]}^{2n+1}||f||_{C(\Delta_R)}(r-t)^n}{n!(n+1)!}(r-t)^{n+1} \]

Assuming that (87) is true for \( n \) we obtain

\[ \left| \left[ O^{(2n)}f \right](r, t) \right| \leq \frac{||A||_{C[0,R]}^{2n}||f||_{C(\Delta_R)}(r-t)^n}{n!(n-1)!} \int_{r-t}^r (s-(r-t))^{n-1} ds \leq \frac{||A||_{C[0,R]}^{2n}||f||_{C(\Delta_R)}(r-t)^n}{(n!)^2}(r-t)^n \]

Therefore, \( ||O^n|| \to 0 \) as \( n \to \infty \) and \( O \) is Volterra. Notice that the estimates obtained above prove convergence of the series obtained by the iteration of (84). Therefore, the solution \( A(r, t) \) is continuous in \( 0 \leq t \leq r < \infty \). The corresponding \( Q \) and \( P_\ast \) solve integral equations (81) and (82). So, \( P \) and \( P_\ast \) from (78) and (79) solve the Krein system. \( \square \)
Remark 4.1. From (84), we have an identity
\[ A(r, 0) = A(r) \]  
(88)

Theorem 4.2. For any Krein system (57) with
\[ A(r) \in C[0, \infty), \]  
(89)
there is the unique accelerant \( H(x) \) which generates it and satisfies (30). Conversely, any accelerant satisfying (30) gives rise to the Krein system for which (89) is true.

Proof. The converse statement follows from the construction done in previous section. Now, let us find an accelerant that generates the given Krein system. The clue is given by Lemma 3.8. Consider the function \( P^{-1}_r(\lambda) \). We know that \( P_r(\lambda) \) does not have zeroes in \( \mathbb{C}^+ \). By Lemma 4.7, both \( P_r \) and \( \hat{P}_r \) are continuous polynomials, i.e.
\[ P_r(\lambda) = 1 - \int_0^r A(r, r - s) \exp(i\lambda s) \, ds, \]
\[ \hat{P}_r(\lambda) = 1 - \int_0^r \hat{A}(r, r - s) \exp(i\lambda s) \, ds \]
Therefore, Levy-Wiener theorem yields the representation
\[ P^{-1}_r(\lambda) \hat{P}_r(\lambda) = 1 + 2 \int_0^\infty H_r(s) \exp(is\lambda) \, ds, \]
(90)
with \( H_r(s) \in L^1(\mathbb{R}^+) \). \( H_r(s) \) is continuous on \([0, r]\) and \([r, \infty]\) but the right and the left limits at \( r \) are not necessarily the same. From differential equations for \( P_r \) and \( \hat{P}_r \) and Lemma 4.4 (parts (2) and (3)), we have
\[ \frac{d}{dr} \left[ P^{-1}_r(\lambda) \hat{P}_r(\lambda) \right] = \frac{2A(r) \exp(ir\lambda)}{P^2_r(\lambda)}, \lambda \in \mathbb{R} \]  
(91)
Consequently
\[ P^{-1}_r(r_2, \lambda) \hat{P}_r(r_2, \lambda) - P^{-1}_r(r_1, \lambda) \hat{P}_r(r_1, \lambda) = \int_{r_1}^{r_2} \frac{2A(s) \exp(is\lambda)}{P^2_r(s, \lambda)} \, ds, \quad 0 < r_1 < r_2 \]
and that implies
\[ H_{r_2}(s) - H_{r_1}(s) = H(s) \]  
(92)
for \( 0 < s < r_1 \). Therefore, the function \( H(s) \) is well-defined and continuous on \([0, \infty)\).

Therefore, if we let \( H_r(-s) = H_r(s), \) \( H(-s) = H(s), s > 0, \) then \( H(s) \) is Hermitian and satisfies (30). Now, let us show that it actually generates the Krein system with given coefficient \( A(r) \).

Indeed, from Lemma 4.3, part 3, we have
\[ \frac{1}{|P_r(\lambda)|^2} = \text{Re} \left[ P^{-1}_r(\lambda) \hat{P}_r(\lambda) \right] = 1 + \int_{-\infty}^{\infty} H_r(s) \exp(is\lambda) \, ds, \]  
(93)
Notice that the last identity yields
\[
\int_0^\infty |h(x)|^2 dx + \int_0^\infty H_r(x-y)\overline{h(x)}dydx \geq 0
\] (94)
for any \( h \in C_0^\infty(0, \infty) \) and the inequality is strict for any nontrivial \( h \). Indeed, one needs to rewrite (94) in terms of Fourier transform. Then, (92) shows that \( H_r + I > 0 \) for any \( r > 0 \) and \( H \) does generate the Krein system with coefficient \( A^{(1)}(r) \) that satisfies (89). Now, let us prove that \( A^{(1)}(r) = A(r) \). Denote the solutions of (57) with coefficient \( A^{(1)}(r) \) by \( P^{(1)} \) and \( P^{(1)}_* \). But \( P^{(1)}_*(r, \lambda) = P_*(r, \lambda) \) for any \( r > 0, \lambda \in \mathbb{C} \). Indeed, from Lemma 3.8 applied to \( P^{(1)}_* \) and (93), we get
\[
P^{(1)}_{[-r,r]} \left[ \frac{1}{|P^{(1)}(r, \lambda)|^2} - 1 \right] = P^{(1)}_{[-r,r]} \left[ \frac{1}{|P^{(1)}_*(r, \lambda)|^2} - 1 \right]
\]
Then, by Lemma [17.24] from Appendix, we get \( P^{(1)}_*(r, \lambda) = P_*(r, \lambda) \). Therefore, \( P^{(1)}_*(r, \lambda) = P_*(r, \lambda) \) for all \( r > 0 \) and \( A^{(1)}(r) = A(r) \).

**Remark 4.2.** Notice that the values of accelerant on \([0, R]\) depends solely on the values of \( A(r) \) on \([0, R]\) and vice versa.

The application of Levy-Wiener theorem to \( P^{(-1)}_*(r, \lambda) \tilde{P}_*(r, \lambda) \) shows that
\[
2H_r(+0) = -\hat{A}(r, r) + A(r, r)
\]
and therefore
\[
H(+0) = \lim_{r \to 0} H_r(+0) = [-\hat{A}(0, 0) + A(0, 0)]/2 = A(0)
\]
where we used (88). Therefore, if \( A(0) \in \mathbb{R} \), then \( H(x) \) is continuous at 0 and \( H(0) \in \mathbb{R} \).

Theorem 4.2 establishes a one-to-one correspondence between continuous \( A(r) \), defined on \( \mathbb{R}^+ \), and continuous accelerants for which (32) is true. But what happens to a map \( \{H(x) \to A(r)\} \) if (32) fails at a finite point? For OPUC, if the measure has only \( k \) growth points, then \( D_{k-1} \neq 0, D_k = 0 \). The corresponding \( |a_j| < 1, j = 0, \ldots, k-1, |a_k| = 1 \). For the Krein system, the situation is similar. Assume that \( 1 + \mathcal{H}_r > 0 \) for all \( r < R \) and \( \ker(I + \mathcal{H}_R) \neq 0 \). Following argument given above, one can construct \( A(r) \in C(0, R) \). Vice versa, given \( A(r) \in C(0, R) \), we can define \( H \in C(0, R) \) such that (32) holds up to \( R \). But as long as \( \ker(I + \mathcal{H}_R) \neq 0, A(r) \) blows up as \( r \) approaches \( R \) from the left. More precisely, this process is governed by a pair of simple (and clearly very crude) estimates
\[
|A(r)| \leq \|\Gamma_r(t, 0)\|_{C[0, r]} \leq \|(I + \mathcal{H}_r)^{-1}\|_{C[0, r]}\|H\|_{C[0, r]} \] (95)
and
\[
\|(I + \mathcal{H}_r)^{-1}\|_{C[0, r]} \leq 1 + Cr\|A\|_{C[0, r]} (1 + r\|A\|_{C[0, r]}) \exp[Cr\|A\|_{C[0, r]}] \] (96)
The first estimate easily follows from (94) with \( s = 0 \). It shows that \( A(r) \) can not blow up unless (32) fails at a finite point (here we also assume that \( H(x) \in C(0, R) \)). One can get (96) from the following inequalities.
\[
\|(I + \mathcal{H}_r)^{-1}\|_{C[0, r]} = \|I - \Gamma_r\|_{C[0, r]} \leq 1 + r \max_{0 \leq s, t \leq r} |\Gamma_r(s, t)|
\]
To estimate the last maximum, we follow the proof of Lemma 4.7 (estimates (86) and (87)). This gives us the following bound
\[
\max_{s \in [0,r]} |\Gamma_r(0,s)| \leq C\|A\|_{C[0,r]} \exp[Cr\|A\|_{C[0,r]}]
\]
because \(A(r,s)\) from Lemma 4.7 is actually equal to \(\Gamma_r(r,s) = \Gamma_r(r-s,0)\) by (54).

Using Lemma 3.7, we obtain an estimate
\[
\max_{0 \leq s, t \leq r} |\Gamma_r(s,t)| \leq C\|A\|_{C[0,r]}(1 + r\|A\|_{C[0,r]}) \exp[Cr\|A\|_{C[0,r]}]
\]
which yields (96). Obviously the left-hand side of (96) is non-decreasing in \(r\). If it blows up at a finite time (i.e. (32) fails at a finite time), then \(A\) blows up at the same point as well. Comparing the OPUC and Krein systems, we see that infinity (for Krein systems) plays the role of \(\mathbb{T} = \partial \mathbb{D}\) for OPUC.

**Remarks and historical notes.**

The problem of constructing accelerant from the Krein system with locally integrable coefficient \(A(r)\) was solved by Rybalko in [61]. In [61], estimates on \(A(r,t)\) from the Lemma 4.7 are a bit stronger than what we obtain. Some generalizations of Krein systems were considered by L. Sakhnovich in [65].
5. Accelerant and $A(r)$ are from $L^2_{\text{loc}}(\mathbb{R}^+)$ class

In this section, we will show that the accelerant from $L^2_{\text{loc}}(\mathbb{R})$ class generates the Krein system with $A(r) \in L^2_{\text{loc}}(\mathbb{R}^+)$ and, conversely, the Krein system with $A(r) \in L^2_{\text{loc}}(\mathbb{R}^+)$ generates the accelerant $H(x) \in L^2_{\text{loc}}(\mathbb{R})$. Moreover, this highly nonlinear map is homeomorphism in $L^2_{\text{loc}}$, i.e., in $L^2([0,R])$ for any $R > 0$. We will prove that all statements from the previous two sections find their analogs for $L^2_{\text{loc}}$-case.

First, consider an accelerant $H(x) \in L^2[0,r]$ for any $r$. Then, $I + \mathcal{H}_r > 0$ and $\Gamma_r(x,y)$ is well-defined as $L^2([0,r]^2)$-function. We also have

$$\Gamma_r(t,s) = \int_0^t H(t-u) \Gamma_r(u,s) du = H(t-s)$$

Fix any $s \in [0,r]$ in the last equation. Then, $H(t-s)$ is continuous in $s$ as $L^2[0,r]$-function in $t$. Therefore, $\Gamma_r(t,s)$ is continuous in $s$ in $L^2[0,r]$ norm with respect to the first coordinate. It is also Hermitian function so the same is true for $s$ and $t$ interchanged. Let $g_r(s) = \Gamma_r(s,0)$ for $s \in [0,r]$ and $g_r(s) = 0$ for $s \in [r,R]$. Due to the formula $\Gamma_r(s,0) = (I + \mathcal{H}_r)^{-1} H$, we have the continuity of $g_r$ in $r$ with respect to $L^2([0,R])$ norm.

The special displacement structure of the kernel of $\mathcal{H}_r$ allows to generalize Theorems [2.1] and [2.2].

**Theorem 5.1.** For Hermitian $H(x) \in L^2[-R,R]$, the operator $I + \mathcal{H}_r$ admits factorization \([17]\) if and only if $I + \mathcal{H}_r > 0$ for any $r \in (0,R]$. In this case,

$$V_+(x,y) = -\Gamma_y(x,y), \quad x < y$$

$$V_-(x,y) = -\Gamma_x(x,y), \quad x > y$$

where $\Gamma_r(x,y)$ denotes the resolvent kernel of $I + \mathcal{H}_r$.

*Proof.* Now, assume that $I + \mathcal{H}_r > 0$ for any $r \in (0,R]$. Define $V_\pm$ by \([97]\). Notice that operators $V_\pm$ are well-defined. It follows from the representation

$$[V_-f](x) = -\int_0^x \Gamma_x(x,y)f(y)dy = -\int_0^x \Gamma_x(s,0)f(x-s)ds = -\int_0^R g_x(s)f(x-s)ds$$

which shows that $V_-$ is actually bounded from $L^2[0,R]$ to $L^\infty[0,R]$. Analogous formula is true for $V_+$. These operators also have Hilbert-Schmidt and Volterra properties. Approximate $H(x)$ by $H^{(n)}(x) \in C[-R,R]$ in $L^2[-R,R]$ norm and apply Theorem [2.1]. For each $n$, formula \([18]\) is true. Moreover, $V_\pm^n(x,y) \rightarrow V_\pm(x,y), \quad \Gamma_R^n(x,y) \rightarrow \Gamma_R(x,y)$ as $n \rightarrow \infty$ and convergence is in $L^2([0,R]^2)$. Taking $n \rightarrow \infty$, we get \([18]\) for $G$ and $V_\pm$. That implies the needed factorization. The converse statement is simple and repeats the argument in Theorem [2.1].

The analog of the Theorem [2.2] can also be easily proved in the same way giving

**Theorem 5.2.** For Hermitian $H(x) \in L^2[-2R,2R]$, the operator $I + \widehat{\mathcal{H}}_r$ admits factorization \([20]\) if and only if $I + \widehat{\mathcal{H}}_r$ is invertible in $L^2[-r,r]$ for any $0 < r \leq R$. In this case,

$$\hat{V}_+(x,y) = -\hat{\Gamma}_y(x,y), \quad (x,y) \in \hat{\Omega}_+$$

$$\hat{V}_-(x,y) = -\hat{\Gamma}_y(x,y), \quad (x,y) \in \hat{\Omega}_-$$

(98)
All functions that are used in the definition of continuous polynomials are now well-defined as elements of \( L^2[0, r] \). Indeed, \( \Gamma_r(r, s) \) and \( \Gamma_\ast(0, s) \) are both from \( L^2[0, r] \) and we can consider the corresponding continuous polynomials \( P(r, \lambda), P_\ast(r, \lambda) \). Moreover, Lemma 3.2 and Theorem 3.2 are true for \( H \in L^2_{\text{loc}}(\mathbb{R}) \) as well. Indeed, their proofs were based on the factorization of Fredholm operators (the analog of integral equations) and approximate it with \( \overline{P(n)}(r) \). Notice that all arguments of \( A(r) \) and \( H(r) \) coincide. In particular, \( A(r) \in L^2[0, r] \). For any \( \lambda \in \mathbb{C} \), the polynomials \( P^{(n)}(r, \lambda), P_\ast^{(n)}(r, \lambda) \) converge to \( P(r, \lambda) \) and \( P_\ast(r, \lambda) \) uniformly in \( r \in [0, R] \). Notice that equations (57) for \( P^{(n)} \) and \( P_\ast^{(n)} \) are equivalent to the system of integral equations

\[
P^{(n)}(r, \lambda) = 1 + \int_0^r \left[ i\lambda P^{(n)}(s, \lambda) - \overline{A^{(n)}(s)} P_\ast^{(n)}(s, \lambda) \right] ds
\]

\[
P_\ast^{(n)}(r, \lambda) = 1 - \int_0^r A^{(n)}(s) P^{(n)}(s, \lambda) ds
\]

Taking the limit \( n \to \infty \), we see that \( P \) and \( P_\ast \) satisfy the corresponding equations that are equivalent to (57). The proofs of Lemma 3.6 and Lemma 3.8 work for \( L^2_{\text{loc}} \) case. Lemma 3.7 can be shown by an approximation argument.

Now, following the arguments from the Section 4, we consider \( A(r) \in L^2_{\text{loc}}(\mathbb{R}^+) \). We have

**Theorem 5.3.** For any Krein system (57) with

\[
A(r) \in L^2_{\text{loc}}(\mathbb{R}^+)
\]

there is the unique accelerant which generates it and satisfies

\[
H(x) \in L^2_{\text{loc}}(\mathbb{R})
\]

Conversely, any accelerant satisfying (100) gives rise to Krein system with \( A(r) \in L^2_{\text{loc}}(\mathbb{R}^+) \). This map is a homeomorphism.

**Proof.** As we just showed, the \( L^2_{\text{loc}} \) accelerant does generates the Krein system with \( L^2_{\text{loc}} \) coefficient. Now, let us start with \( A(r) \in L^2_{\text{loc}}(\mathbb{R}^+) \). Notice that all arguments...
from the proof of Theorem 4.2 are valid as long as we have an analog of Lemma 4.7. Therefore, we just need

**Lemma 5.1.** If $A(r) \in L^2_\text{loc}(\mathbb{R}^+)$, and $P(r, \lambda)$, $P_e(r, \lambda)$ are defined as solutions to the Krein system, then the representations (78), (79) hold with $A(r, s) \in L^2[0, r]$.  

**Proof.** We need to consider the operator $O$ given by (83) and show that it is Volterra in the space of functions $f(r, y), 0 \leq y \leq r \leq R$ such that

$$
\|f\|_{\infty, 2} = \sup_{r \in [0, R]} \left[ \int_0^r |f(r, y)|^2 dy \right]^{1/2} < \infty
$$

Let us prove by induction that

$$
\left\| \left[ O^{(n)} f \right](r, t) \right\|_{L^2[0, r]} \leq \left[ \int_0^r |A(s)| ds \right]^n \cdot \|f\|_{\infty, 2} / (n!) \tag{101}
$$

For $n = 0$, the statement is elementary. Assume that this estimate is true for $n$. Then, for $n + 1$, we have

$$
\left[ O^{(n+1)} f \right](r, t) = \int_{r-t}^r A(s) \left[ O^{(n)} f \right](s, r - t) ds
$$

Application of Minkowski inequality and induction assumption yields

$$
\left\| \left[ O^{(n+1)} f \right](r, t) \right\|_{L^2[0, r]} \leq \|f\|_{\infty, 2} (n+1)!^{-1} \left[ \int_0^r |A(s)| ds \right]^{n+1}
$$

Thus, we have (101) and

$$
\|A(r, t)\|_{L^2[0, r]} \leq \|A\|_{L^2[0, r]} \exp \left[ \|A\|_{L^1[0, r]} \right]
$$

This estimate finishes the proof of the Lemma. \hfill \square

We are left with proving

**Lemma 5.2.** The following estimates are true for any $R > 0$

$$
\|H(x)\|_{L^2[-R, R]} \leq C \|A(r)\|_{L^2[0, R]} \exp \left( C \|A\|_{L^1[0, R]} \right) \tag{102}
$$

$$
\|A(r) - \overline{H(r)}\|_{L^2[0, R]} \leq \|H\|_{L^2[0, R]} \|(I + Q_R)^{-1}\|_{2, 2} \tag{103}
$$

and the map $A(r) \rightarrow H(x)$ is homeomorphism in $L^2_\text{loc}$.  

**Proof.** From (90) and (91), we have

$$
\int_0^\infty \overline{H_R(s)} \exp(i\lambda s) ds = \int_0^R \frac{A(s) \exp(i\lambda s)}{P^2_e(s, \lambda)} ds =
$$

$$
= \frac{1}{P^2_e(R, \lambda)} \int_0^R A(s) \exp(i\lambda s) ds - \int_0^R \left[ \int_0^s A(u) \exp(i\lambda u) du \right] \frac{A(s)P(s, \lambda)}{P^3_e(s, \lambda)} ds
$$
Therefore, using $|P(s, \lambda)| = |P_*(s, \lambda)|$ and Lemma 4.6 we get
\[
\|H_R\|_2 \leq \|A\|_{L^2[0,R]} \exp(2\|A\|_{L^1[0,R]}) + \int_0^R \|A\|_{L^2[0,s]}|A(s)| \exp(2\|A\|_{L^1[0,s]})ds
\]
which yields (102) since $H(x) = H_R(x)$ for $|x| < R$. This argument also shows that $H(x)$ depends on $A(r)$ continuously in the $L^2_{loc}$.

To prove (103), we use (34) to write
\[
\Gamma_r(t,0) + \int_0^t H(t-u)\Gamma_r(u,0)du = H(t)
\]
(104)

Taking $t = r$,
\[
\Gamma_r(t,0) + \int_0^r H(r-u)\Gamma_r(u,0)du = H(r)
\]
(105)

The formula (104) implies $\Gamma_r(t,0) = (I + \mathcal{H}_r)^{-1}H$ and, therefore, $\|\Gamma_r(t,0)\|_{L^2[0,r]} \leq \|(I + \mathcal{H}_r)^{-1}\|_{L^2_{loc}} H(r)$. That yields (103). Also, the map $H(x) \rightarrow A(r)$ is continuous in $L^2_{loc}$. □

We want to mention here that slight modification of the arguments allows to prove that the map $H(x) \rightarrow A(r)$ is homeomorphism in $L^p_{loc}(\mathbb{R}) - L^p_{loc}(\mathbb{R}^+)$ for any $p > 1$. For one direction, one just have to iterate equation (104) sufficiently many times to achieve the necessary gain in regularity and then plug it in (105). The other direction is straightforward.
6. **Continuous analogs of Wall polynomials and Schur function. Bernstein-Szegő approximation**

Let us introduce the continuous analogs of the so-called Wall polynomials. Consider the functions \( \mathfrak{A}(r, \lambda), \mathfrak{B}(r, \lambda), \mathfrak{A}_s(r, \lambda), \mathfrak{B}_s(r, \lambda) \) defined by

\[
X(r, \lambda) = \begin{bmatrix} \mathfrak{A}_s(r, \lambda) & \mathfrak{B}_s(r, \lambda) \\ \mathfrak{B}(r, \lambda) & \mathfrak{A}(r, \lambda) \end{bmatrix}
\]

were \( X(r, \lambda) \) is the transfer matrix given in \((65)\). By analogy with OPUC theory, it makes sense to call \( \mathfrak{A} \) and \( \mathfrak{B} \) the continuous Wall polynomials. They can be rewritten in the following way

\[
\mathfrak{A}(r, \lambda) = \frac{P_s(r, \lambda) + \hat{P}_s(r, \lambda)}{2}, \mathfrak{A}_s(r, z) = \frac{P(r, \lambda) + \hat{P}(r, \lambda)}{2}
\]

\[
\mathfrak{B}(r, \lambda) = \frac{P_s(r, \lambda) - \hat{P}_s(r, \lambda)}{2}, \mathfrak{B}_s(r, \lambda) = \frac{P(r, \lambda) - \hat{P}(r, \lambda)}{2}
\]

**Lemma 6.1.** For continuous Wall polynomials, the following identities are true

1) For \( \lambda \in \mathbb{R} \),

\[
|\mathfrak{A}|^2 - |\mathfrak{B}|^2 = 1, |\mathfrak{A}| = |\mathfrak{A}_s|, |\mathfrak{B}| = |\mathfrak{B}_s|, \overline{\mathfrak{A}} \mathfrak{A}_s = \mathfrak{B} \mathfrak{A}
\]

2) For \( \lambda \in \mathbb{C} \),

\[
\mathfrak{A}(r, \lambda) \mathfrak{A}_s(r, \lambda) - \mathfrak{B}(r, \lambda) \mathfrak{B}_s(r, \lambda) = \exp(i \lambda r),
\]

3) For \( \lambda \in \mathbb{C}^+ \),

\[
|\mathfrak{A}|^2 - |\mathfrak{B}|^2 \geq 1, |\mathfrak{A}_s|^2 - |\mathfrak{B}_s|^2 \geq 1
\]

**Proof.** The proof follows directly from Theorem 4.1 or Lemma 4.4 \( \square \)

Notice that \((110)\) implies \( \mathfrak{B}(r, \lambda) \mathfrak{A}^{-1}(r, \lambda) \in B(\mathbb{C}^+) \).

**Theorem 6.1.** The ratio \( \mathfrak{B}(r, \lambda) \mathfrak{A}^{-1}(r, \lambda) \) converges to \( f(\lambda) \in B(\mathbb{C}^+) \). This convergence is uniform over all compacts in \( \mathbb{C}^+ \).

**Proof.** Take \( r_1 < r_2 \). Consider the Krein system on the interval \([r_1, \infty)\). Denote the transfer matrix from \( r_1 \) to \( r_2 \) by \( X(r_1, r_2, \lambda) \). Then, in our notations, \( X(0, r, \lambda) = X(r, \lambda) \). If we introduce

\[
X(r_1, r, \lambda) = \begin{bmatrix} a_s(r, \lambda) & b_s(r, \lambda) \\ b(r, \lambda) & a(r, \lambda) \end{bmatrix}, r > r_1
\]

then \( a \) and \( b \) are Wall polynomials for the same Krein system considered on the interval \([r_1, \infty)\). Obviously, Lemma 6.1 will hold for these functions as well.

The semigroup relation

\[
X(0, r_2) = X(r_1, r_2) \cdot X(0, r_1)
\]

yields

\[
\mathfrak{A}(r_2) = b(r_2) \mathfrak{B}_s(r_1) + a(r_2) \mathfrak{A}(r_1)
\]

\[
\mathfrak{B}(r_2) = b(r_2) \mathfrak{A}_s(r_1) + a(r_2) \mathfrak{B}(r_1)
\]

For the function \( \mathfrak{B} \mathfrak{A}^{-1}, \)

\[
\frac{\mathfrak{B}(r_2)}{\mathfrak{A}(r_2)} = \frac{\mathfrak{B}(r_1) + (b \ a^{-1}) \mathfrak{A}_s(r_1)}{\mathfrak{A}(r_1) + (b \ a^{-1}) \mathfrak{B}_s(r_1)}
\]
and
\[
\frac{\mathcal{B}(r_2) - \mathcal{B}(r_1)}{\mathcal{A}(r_2)} = \frac{(b a^{-1})(\mathcal{A}(r_1) \mathcal{A}_*(r_1) - \mathcal{B}(r_1) \mathcal{B}_*(r_1))}{\mathcal{A}(r_1)[\mathcal{A}(r_1) + (b a^{-1}) \mathcal{B}_*(r_1)]} = \frac{(b a^{-1}) \exp(i \lambda r_1)}{\mathcal{A}(r_1)[\mathcal{A}(r_1) + (b a^{-1}) \mathcal{B}_*(r_1)]}
\]
where we have used (109).

We have \(b a^{-1} \in B(\mathbb{C}^+)\). Therefore, from (110), we get
\[
|\mathcal{A}(r_1)| \mathcal{A}(r_1) + (b a^{-1}) \mathcal{B}_*(r_1)| \geq (|\mathcal{A}(r_1)| - |\mathcal{B}_*(r_1)|) |\mathcal{A}(r_1)| \geq \frac{|\mathcal{A}(r_1)|^2 - |\mathcal{B}_*(r_1)|^2}{2} \geq \frac{1}{2}, \lambda \in \mathbb{C}^+
\]
Then,
\[
\left| \frac{\mathcal{B}(r_2) - \mathcal{B}(r_1)}{\mathcal{A}(r_2)} \right| \leq 2 \exp(-r_1 \text{ Im } \lambda)
\]
That means \(\mathcal{B}(r, \lambda) \mathcal{A}^{-1}(r, \lambda)\) converges to a certain function \(f(\lambda)\) as \(\rho \to \infty\). This convergence is uniform in any compact in \(\mathbb{C}^+\). \(\square\)

We will call \(f(\lambda)\) the Schur function corresponding to \([0, \infty)\). Notice that \(f(\lambda) \in B(\mathbb{C}^+)\). The following formula is the consequence of (114) if one takes \(r_2 \to \infty\)
\[
f(\lambda) = \frac{\mathcal{B}(\rho, \lambda) + f_\rho(\lambda) \mathcal{A}_*(\rho, \lambda)}{\mathcal{A}(\rho, \lambda) + f_\rho(\lambda) \mathcal{B}_*(\rho, \lambda)}
\]  
(117)
where \(f_\rho(z)\) is Schur’s function for the same Krein but on the interval \([\rho, \infty)\).

Not every function from \(B(\mathbb{C}^+)\) is Schur’s function of some Krein system. The characterization of that special subclass will be given later but now we just want to mention that
\[
f(\lambda) \to 0
\]  
(118)
if \(\text{ Im } \lambda \to +\infty\). It easily follows from the formula (117) with any fixed \(\rho\) and relations
\[
f_\rho(\lambda) \in B(\mathbb{C}^+), \mathcal{A}(\rho, \lambda) \to 1, \mathcal{B}(\rho, \lambda) \to 0, \mathcal{A}_*(\rho, \lambda) \to 0, \mathcal{B}_*(\rho, \lambda) \to 0
\]
as \(\text{ Im } \lambda \to +\infty\).
In the previous section, we constructed an accelerant from the given Krein system. Then, the measure \(\sigma\) and the constant \(\beta\) can be found from the formula (27). But there is more direct way to find these data. The next Theorem develops an analog of the Weyl-Titchmarsh theory [49] for Krein systems.

**Theorem 6.2.** The ratio \(\hat{P}_*(r, \lambda) P_*^{-1}(r, \lambda)\) converges to the function \(F(\lambda)\) uniformly in any compact in \(\mathbb{C}^+\) as \(r \to \infty\). This function \(F(\lambda)\) has the positive real part in \(\mathbb{C}^+\) and allows the following representation
\[
F(\lambda)/2 = -i \beta + i \int_{-\infty}^{\infty} \frac{1 + \lambda t}{(\lambda - t)(1 + t^2)} d\sigma(t)
\]  
(119)
where \(d\sigma\) and \(\beta\) coincide with those from the formula (27). Moreover, the sequence of measures
\[
d\sigma_r(\lambda) = \frac{d\lambda}{(2\pi)|P_*(r, \lambda)|^2} \to d\sigma(\lambda)
\]  
(120)
in the weak-(\(\ast\)) sense (analog of Bernstein-Szegő approximation).
Proof. From (107),

\[ P_* = \mathcal{A} + \mathcal{B}, \quad \hat{P}_* = \mathcal{A} - \mathcal{B}, \quad P = \mathcal{A}_* + B_*, \quad \hat{P} = \mathcal{A}_* - B_* \]

Therefore,

\[ P_*^{-1}(r, \lambda) \hat{P}_*(r, \lambda) = (\mathcal{A} - \mathcal{B})(\mathcal{A} + \mathcal{B})^{-1} = \frac{1 - \mathcal{A}^{-1} \mathcal{B}}{1 + \mathcal{A}^{-1} \mathcal{B}} \]

That shows convergence of \( P_*^{-1}(r, \lambda) \hat{P}_*(r, \lambda) \) to the function

\[ F(\lambda) = (1 - f)(1 + f)^{-1} \]

Again, this convergence is uniform for compacts in \( \mathbb{C}^+ \). Function \( F(\lambda) \) has positive real part in \( \mathbb{C}^+ \) because \( f(\lambda) \in B(\mathbb{C}^+) \). Therefore, \( F(\lambda)/2 \) admits the following integral representation [4]

\[ F(\lambda)/2 = -i\beta - i\alpha \lambda + i \int_{-\infty}^{\infty} \frac{1 + \lambda t}{(\lambda - t)(1 + t^2)} d\sigma(t) \]

where \( \alpha, \beta \in \mathbb{R}, \alpha \geq 0 \) and non-decreasing function \( \sigma \) is such that

\[ \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t^2 + 1} < \infty \]

Recall the way this formula is obtained. If we map \( \lambda \in \mathbb{C}^+ \) onto \( z \in \mathbb{D} \) by conformal mapping \( z = (\lambda - i)(\lambda + i)^{-1} \), then the function \( g(z) = 2^{-1}F(i(z + 1)(1 - z)^{-1}) \) is the Herglotz function in \( \mathbb{D} \). It has a canonical representation, which can be written as follows

\[ g(z) = -i\beta + \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\tau(\xi) \]

If \( \alpha = d\sigma\{1\} \), the mass at point 1, then we have formula (122) with \((1 + t^2)^{-1} d\sigma(t) = d\tau[(t - i)(t + i)^{-1}]\).

Next, our goal is to show that \( \alpha = 0 \), and \( \beta \) and \( d\sigma \) coincide with those from (27) in section 3.

From [7], p. 630, we have

\[ \alpha = \lim_{\eta \to +\infty} iF(i\eta) \]

But (118) implies \( \lim_{\eta \to +\infty} F(i\eta) = 1 \) so \( \alpha = 0 \). In other words, measure \( d\tau \) has no mass at \( \xi = 1 \).

Notice now that each function \( P_*^{-1}(r, \lambda) \hat{P}_*(r, \lambda)/2 \) has positive real part as well and admits the same representation (122) with \( \alpha_r = 0, \beta_r \in \mathbb{R}, \) and absolutely continuous measure \( d\sigma_r \), which is the transplantation of some \( d\tau_r \). From (76), we have \( \sigma'_r(\lambda) = (2\pi)^{-1}|P_*(r, \lambda)|^{-2} \). Measures \( d\sigma_r \) are analogs of the so-called Bernstein-Szegö approximations for OPUC. They converge weakly to \( d\sigma \). Indeed, the convergence of \( P_*^{-1}(r, \lambda) \hat{P}_*(r, \lambda)/2 \) to \( F(\lambda) \) in \( \mathbb{C}^+ \) implies convergence of the corresponding functions within the unit disc \( \mathbb{D} \). By Stone-Weierstrass theorem, that yields weak convergence of measures \( d\tau_r \) to \( d\tau, \beta_r \) to \( \beta \) and so the weak convergence
of $d\sigma_r$ to $d\sigma$. Since $d\tau$ has no mass at $\xi = 1$, the family of measures $(1 + t^2)^{-1}d\sigma_r(t)$ is tight, i.e. for any $\epsilon > 0$, there is $T(\epsilon), R(\epsilon) > 0$ such that

$$ \int_{|t|>T(\epsilon)} \frac{d\sigma_r(t)}{1 + t^2} < \epsilon \quad (123)$$

if $r > R(\epsilon)$.

Let us show now that $\sigma$ coincides with the measure from (27). Using (9) and (90), we obtain

$$ \frac{1}{2} + \int_0^\infty H_r(x) \exp(i\lambda x) dx = -\frac{\lambda^2}{2} \int^\infty_0 \left[-i\beta x + \int^\infty_0 \left(1 - \frac{itx}{1 + t^2} - \exp(-itx) \right) \frac{d\sigma_r(t)}{t^2} \right] \exp(i\lambda x) dx$$

From (11), we get

$$ \frac{|t|}{2} + \int_0^t (t-s) H_r(s) ds = i\beta t + \int^\infty_0 \left(1 + \frac{its}{1 + s^2} - \exp(its) \right) \frac{d\sigma_r(s)}{s^2}$$

Fix $t$ and take $r \to \infty$ in the last equation. The formula (92) and tightness (123) yield

$$ \frac{|t|}{2} + \int_0^t (t-s) H(s) ds = i\beta t + \int^\infty_0 \left(1 + \frac{its}{1 + s^2} - \exp(its) \right) \frac{d\sigma(s)}{s^2}$$

Since the integral representation of $G_\infty$ functions is unique (Theorem 1.2), we get the statement of the Theorem.

We also have the following important

**Corollary 6.1.** For any $f(x) \in L^2[0, \rho]$, the following identity is true

$$ \int^\rho_0 \left| \int_{-\infty}^\infty f(x) \exp(i\lambda x) dx \right|^2 d\lambda = \int^\rho_0 \left| \int_{-\infty}^\infty f(x) \exp(i\lambda x) dx \right|^2 \frac{d\lambda}{2\pi|P(\rho, \lambda)|^2} \quad (124)$$

**Proof.** We have $|P(r, \lambda)| = |P_*(r, \lambda)|$ for real $\lambda$. Then, the Plancherel theorem for the Fourier integrals and (93) yield that the right hand side of (124) is equal to

$$ \int^\rho_0 |f(x)|^2 dx + \int^\rho_0 \int_0^\rho H_\rho(x-u)f(u)f(x) du dx$$

$$ = \int^\rho_0 |f(x)|^2 dx + \int^\rho_0 \int_0^\rho H(x-u)f(u)f(x) du dx \quad (125)$$

where (92) is used to get the last equality. Then, the formula (6t) shows that the l.h.s. of (124) is equal to r.h.s. of (125). \qed
The formula similar to (117) is true for Weyl-Titchmarsh function as well. From (117) and relation between \( F \) and \( f \), we get

\[
F(\lambda) = \frac{\hat{P}_s(\rho, \lambda) - \hat{P}(\rho, \lambda) + F_\rho(\lambda)(\hat{P}_s(\rho, \lambda) + \hat{P}(\rho, \lambda))}{\hat{P}_s(\rho, \lambda) + P(\rho, \lambda) + F_\rho(\lambda)(\hat{P}_s(\rho, \lambda) - P(\rho, \lambda))} \tag{126}
\]

**Remarks and historical notes.** The Weyl-Titchmarsh theory for Krein systems was developed to some extent in [61].
7. Dual system. Some simple considerations

The dual Krein system is obtained by changing the sign of the coefficient \( A(r) \) (see Corollary 4.1). Due to Corollary 4.1, functions \( \hat{P}(r, \lambda), \hat{P}_*(r, \lambda) \) are continuous orthogonal polynomials for the dual system. They are usually called the dual continuous orthogonal polynomials.

The dual Krein system can be characterized by the dual accelerant. Let us call it \( \hat{H} \). The relation between accelerant and dual accelerant is very simple.

Lemma 7.1. For the dual accelerant \( \hat{H} \), we have

\[
H(x) + \hat{H}(x) + 2 \int_0^x H(x - s) \hat{H}(s) ds = 0, \quad x \in \mathbb{R} \tag{127}
\]

Proof. We have

\[
\frac{P_*(r, \lambda)}{P_*(r, \lambda)} = 1
\]

Substitute (90) to the second factor and analogous formula to the first one gives

\[
\left(1 + 2 \int_0^x H_r(x) \exp(i\lambda x) dx\right) \left(1 + 2 \int_0^x \hat{H}_r(x) \exp(i\lambda x) dx\right) = 1
\]

which implies

\[
H_r(x) + \hat{H}_r(x) + 2 \int_0^x H_r(x - t) \hat{H}_r(t) dt = 0
\]

Then, use (92) to get (127). \( \square \)

Clearly, the last Theorem allows one to find \( \hat{H} \) from \( H \) by solving Volterra equation. The algebraic explanation to (127) is as follows. Consider two operators

\[
[\hat{A}f](x) = f(x) + 2 \int_0^x H_r(x - u)f(u) du, \quad [\hat{A}_f](x) = f(x) + 2 \int_0^x \hat{H}_r(x - u)f(u) du
\]

acting in \( L^2[0, r] \). They both have positive real parts:

\[
\text{Re} \hat{A} = I + \mathcal{H}_r, \quad \text{Re} \hat{A} = I + \mathcal{H}_r
\]

and the formula (127) is equivalent to

\[
\hat{A} = A^{-1} \tag{129}
\]

These identities arise naturally from the solution to continuous Caratheodory-Toeplitz problem [15]. The relation between \( \Gamma_r(x, y) \) and the dual resolvent kernel \( \hat{\Gamma}_r(x, y) \) is also quite simple and can be obtained from (128) and (129).

Consider the dual Weyl-Titchmarsh function \( \hat{F}(\lambda) = \lim_{r \to \infty} P_*(r, \lambda) \hat{P}_*^{-1}(r, \lambda) \). Then, by Theorem 6.2

\[
\hat{F} = F^{-1}, \quad \hat{\mathcal{F}} = - \mathcal{F}
\]

Now, let us study how the parameters of the Krein system change upon some simple transformations of the coefficient \( A(r) \).
Lemma 7.2 (Shift). Let \( A(r) \) be coefficient of \( \{ \mathcal{P}_r, \mathcal{P}_s \} \). Then \( A^{(t)}(r) = A(r) \exp(i\mu t), t \in \mathbb{R} \) corresponds to 
\[
\sigma^{(t)}(\lambda) = \sigma(\lambda + t), H^{(t)}(x) = \exp(-itx)H(x), \Gamma^{(t)}(x, y) = \Gamma_r(x, y) \exp(-it(x - y))
\]

Proof. Let pairs \( \{ P, P_s \}, \{ P^{(t)}, P_s^{(t)} \} \) be solutions of Krein system with coefficient \( A \) and \( A^{(t)} \), respectively. Introduce \( Q = \exp(-i\mu r)P, Q^{(t)} = \exp(-i\mu r)P^{(t)} \). We have (see formula (80)): 
\[
\begin{cases}
Q' = -\bar{A}\exp(-i\mu r)P, & Q(0, \lambda) = 1 \\
P_s' = -A \exp(i\mu r)Q, & P_s(0, \lambda) = 1
\end{cases}
\]
and 
\[
\begin{cases}
Q^{(t)}' = -\bar{A}\exp(-i(\mu + t)r)P^{(t)}, & Q^{(t)}(0, \lambda) = 1 \\
P_s^{(t)}' = -A \exp(i(\mu + t)r)Q^{(t)}, & P_s^{(t)}(0, \lambda) = 1
\end{cases}
\]
Therefore, \( P^{(t)}_s(r, \lambda) = P_s(r, \lambda + t) \). By Theorem 6.2, \( (2\pi)^{-1}|P^{(t)}_s(r, \lambda)|^2 d\lambda \to d\sigma^{(t)} \), we get the shift in the measure. For dual system, we have the same results. Then, by (90), 
\[
\hat{P}^{(t)}_s(r, \lambda) = 1 + 2 \int_0^\infty \frac{H^{(t)}(s) \exp(i\lambda) ds}{P^{(t)}_s(r, \lambda)} = 1 + 2 \int_0^\infty \frac{H(r) \exp(is(\lambda + t)) ds}{P^{(t)}_s(r, \lambda)},
\]
and we have the needed formula for \( H^{(t)} \). The way resolvent kernel changes is easy to obtain from (51) or (53). □

It is an easy exercise to show directly that if \( H(x) \) is an accelerant, then \( H(x) \exp(-itx) \) is an accelerant as well. Coefficient \( \beta^{(t)} \) from formula (27) changes in a more intricate way. It can be recovered by noticing that \( F^{(t)}(\lambda) = F(\lambda + t) \) and by integral representations for both Weyl-Titchmarsh functions. We then get 
\[
\beta^{(t)} = \beta + \int_{-\infty}^{\infty} \left( \frac{s - t}{1 + (s - t)^2} - \frac{s}{1 + s^2} \right) d\sigma(s)
\]

Lemma 7.2 has its analog in the OPUC theory, rather than the following Lemma

Lemma 7.3 (Dilation). For any \( \gamma > 0 \), coefficient \( \gamma A(\gamma r) \) corresponds to 
\[
\sigma(\gamma \lambda) = \gamma \sigma(\gamma^{-1} \lambda), \gamma > 0, H(\gamma)(x) = \gamma H(\gamma x), \Gamma(\gamma), r, \gamma(x, y) = \gamma \Gamma_r(\gamma x, \gamma y)
\]

Proof. Under the change of variables \( \rho = \gamma r \), system (57) changes as follows 
\[
\begin{cases}
\frac{dP(\gamma r, \lambda)}{dr} = i\gamma \lambda P(\gamma r, \lambda) - \gamma \bar{A}(\gamma r)P_s(\gamma r, \lambda), & P(0, \lambda) = 1, \\
\frac{dP_s(\gamma r, \lambda)}{dr} = -\gamma A(\gamma r)P(\gamma r, \lambda), & P_s(0, \lambda) = 1
\end{cases}
\]
which proves the Lemma following the same lines as in the proof of Lemma 7.2. □

Again, if \( H(x) \) is an accelerant, then one directly checks that \( \gamma H(\gamma x) \) is also an accelerant. Since \( F(\gamma)(\lambda) = F(\lambda/\gamma) \), we have 
\[
\beta(\gamma) = \beta + \int_{-\infty}^{\infty} \left( \frac{s \gamma^2}{1 + s^2 \gamma^2} - \frac{s}{1 + s^2} \right) d\sigma(s)
\]
Lemma 7.4 (Conjugation). The coefficient $\overline{A(r)}$ corresponds to the measure $\overline{\sigma}$: $\overline{\sigma}(I) = \sigma(-I)$ for any Borel set $I$, accelerant $\overline{H(x)}$, resolvent kernel $\overline{\Gamma_r(x,y)}$, and coefficient $-\beta$.

Proof. Take conjugation of (57). The pair $\{P(r,-\lambda), P^*(r,-\lambda)\}$ solves Krein system with parameter $\lambda$ and coefficient $\overline{A(r)}$. New Weyl-Titchmarsh function is equal to $\overline{F(-\lambda)}$. The integral representation for $F(\lambda)$ yields the value of $\beta$. □

Consider the case when $A(r)$ is real. Then, $\overline{\sigma} = \sigma$ and $H(x)$ is also real. It is continuous on the whole line provided that $A(r) \in C[0,\infty)$. In (122), constant $\beta = 0$. Later on, we will consider this case in greater details.

The next calculation will be important to understand the scattering problem for Krein system and Dirac operators. Consider Krein system on the interval $[0,R]$ with coefficient $A^{(R)}(r) = A(R-r)$ for $r \in [0,R]$. For $r > R$, we let $A^{(R)}(r) = 0$.

Lemma 7.5 (Mirror symmetry). The Schur function of Krein system with the coefficient $A^{(R)}(r)$ is equal to

\[ f^{(R)}(\lambda) = \frac{\mathfrak{B}(R,-\lambda)}{\mathfrak{A}_r(R,-\lambda)} \]

Proof. Consider the matrix $X$ that solves Krein system

\[ X' = VX, X(0,\lambda) = I \]

and $V$ is given by (66). At the same time, matrix $Y(r,\lambda) = JX(R-r,-\lambda)X^{-1}(R,-\lambda)J$ solves the following system

\[ Y' = V(R-r)Y, Y(0) = I \]

Therefore, new Wall polynomials are

\[ \mathfrak{A}^{(R)}(R,\lambda) = \exp(i\lambda R) \mathfrak{A}_r(R,\lambda) \]
\[ \mathfrak{B}^{(R)}(R,\lambda) = \exp(i\lambda R) \mathfrak{B}(R,\lambda) \]

That finishes the proof. □

Remarks and historical notes. The dual systems were studied before, e.g. [46]. They also appear in the solution to various continuation problems and in continuous Carathéodory-Toeplitz problem.
8. Szegő distance and Krein systems

As any orthogonal system, \( \{ P(r, \lambda) \} \) has the reproducing kernel. Consider the scale \( S_\rho = \mathcal{P}_{[0,\rho]}L^2(\mathbb{R}) \) of Paley-Wiener spaces. Recall that for any \( \rho > 0 \), this space consists of functions \( \hat{f}(\lambda) \) that can be represented as

\[
\hat{f}(\lambda) = \int_0^\rho f(x) \exp(i\lambda x) dx
\]

with \( f(x) \in L^2[0, \rho] \).

**Lemma 8.1.** The following function \( K_\rho(\lambda', \lambda) \):

\[
K_\rho(\lambda', \lambda) = \int_0^\rho P(x, \lambda') P(x, \lambda) dx = \frac{P_\rho(\rho, \lambda)P_\rho(\rho, \lambda') - P(\rho, \lambda)P(\rho, \lambda')}{\lambda - \lambda'}
\]

is the reproducing kernel in \( S_\rho \) space, i.e.

\[
\hat{f}(\lambda') = \langle \hat{f}(\lambda), K_\rho(\lambda', \lambda) \rangle, \ \lambda' \in \mathbb{C}
\]

where the inner product is defined as follows

\[
\langle \hat{f}_1, \hat{f}_2 \rangle = \int_{-\infty}^{\infty} \hat{f}_1(\lambda)\hat{f}_2(\lambda) d\sigma(\lambda) = \int_{-\infty}^{\infty} \hat{f}_1(\lambda)\hat{f}_2(\lambda) \frac{d\lambda}{2\pi|P(\rho, \lambda)|^2}
\]

*Proof.* The second equality in (134) is formula (59). For any fixed \( \lambda' \), \( P(x, \lambda') \in L^2[0, \rho] \), so the kernel \( K_\rho(\lambda', \lambda) \) itself is an element of \( S_\rho \). Formula (135) follows from Plancherel-type identity (47). Indeed, by (45), we have

\[
\hat{f}(\lambda') = \int_0^\rho f_1(x) P(x, \lambda) dx
\]

with some \( f_1(x) \in L^2[0, \rho] \). Now, using (47), we get

\[
\langle \hat{f}(\lambda), K_\rho(\lambda', \lambda) \rangle = \int_0^\rho f_1(x) P(x, \lambda') dx = \hat{f}(\lambda')
\]

The second equality in (136) is the contents of Corollary 6.1. □

The reproducing kernel property (135) yields

\[
K_\rho(\lambda_1, \lambda_2) = \langle K_\rho(\lambda_1, \lambda), K_\rho(\lambda_2, \lambda) \rangle
\]

Together with Cauchy inequality, that implies

\[
\left| \hat{f}(\lambda') \right|^2 \leq \| \hat{f} \|_{L^2(\mathbb{R}, d\sigma)}^2 K_\rho(\lambda', \lambda')
\]

and the equality holds if and only if \( \hat{f}(\lambda) = \gamma K_\rho(\lambda', \lambda) \), \( |\gamma| = 1 \).

**Lemma 8.2.** The following identity is true

\[
\frac{1}{K_\rho(\lambda', \lambda)} = \min_{f \in S_\rho, f(\lambda') = 1} \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\sigma(\lambda) = m_\rho^2(\lambda')
\]

for any \( \lambda' \in \mathbb{C} \). The minimizer is unique and is given by \( \hat{f}(\lambda) = K_\rho^{-1}(\lambda', \lambda) K_\rho(\lambda', \lambda) \).
Proof. Divide \( K_\rho(\lambda', \lambda) \sqrt{\hat{f}(\lambda)^2} \).

Now, the natural question to ask is what happens if \( \rho \to \infty \)? Since \( S_{\rho_1} \subset S_{\rho_2} \) for \( \rho_1 < \rho_2 \), the minimum \( m_\rho(\lambda') \) decreases. Now, can we characterize the case when it decreases to zero? To do that, we need the following classical result (see, e.g. [21], page 84). We give its proof in Appendix (Theorem 17.2).

Assume that \( d\sigma \) is a positive measure on \( \mathbb{R} \) such that
\[
\int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{1+\lambda^2} < \infty
\]

Consider the linear manifold \( X \) of functions \( \hat{f}(\lambda) \), having the following representation
\[
\hat{f}(\lambda) = \int_{r_1}^{r_2} \exp(i\lambda x)f(x)dx, \quad 0 \leq r_1 < r_2
\]
where \( f(x) \in C^1[r_1, r_2] \) and is zero outside \([r_1, r_2] \subseteq [0, \infty)\). Notice that each \( \hat{f}(\lambda) \in L^2(d\sigma) \). Denote the closure of \( X \) in \( L^2(d\sigma) \) by \( \bar{X} \).

Theorem 8.1. The linear manifold \( X \) is not dense in \( L^2(d\sigma) \) if and only if
\[
\int_{-\infty}^{\infty} \frac{\ln \sigma'(\lambda)}{1+\lambda^2} d\lambda > -\infty \quad (139)
\]
Moreover, the following formula is always true
\[
\text{Dist} \left( \frac{1}{\lambda - \lambda_0}, \bar{X} \right)_{L^2(d\sigma)} = \frac{1}{\sqrt{2 \Im \lambda_0}} \exp \left[ \frac{\Im \lambda_0}{2\pi} \int_{-\infty}^{\infty} \frac{\ln(2\pi \sigma'(\lambda))}{|\lambda - \lambda_0|^2} d\lambda \right], \quad \lambda_0 \in \mathbb{C}^+
\quad (140)
\]

Next, we will apply this Theorem to the Krein systems. Let \( d\sigma \) be the measure generated by some Krein system (the measure from (27)). Recall the definition of \( S_r \) and notice that for each finite \( r, S_r \subset L^2(d\sigma) \) (see [21]). Denote the closure of \( \cup_{r>0} S_r \) in \( L^2(d\sigma) \) by \( \bar{S} \).

Lemma 8.3. If \( d\sigma \) is generated by some Krein system, then \( \bar{X} = \bar{S} \).

Proof. It is clear that \( \bar{X} \subseteq \bar{S} \). On the other hand, any function \( \hat{f} \in S_r \) can be approximated in \( L^2(d\sigma) \) by a sequence of functions from \( X \). That easily follows from (61).

Lemma 8.4. The following formula
\[
\text{Dist} \left( \frac{2\Im \lambda_0}{\lambda - \lambda_0}, \bar{S} \right)_{L^2(d\sigma)} = \inf_{r>0} m_r(\lambda_0) = m_\infty(\lambda_0)
\quad (141)
\]
is true for any \( \lambda_0 \in \mathbb{C}^+ \).

Proof. Denote the l.h.s. by \( I_1 \) and the r.h.s. by \( I_2 \). We have
\[
\left| \frac{2\Im \lambda_0}{\lambda - \lambda_0} - \hat{f}(\lambda) \right| = \left| \frac{2i\Im \lambda_0}{\lambda - \lambda_0} - \frac{\lambda - \lambda_0}{\lambda - \lambda_0} \cdot i\hat{f}(\lambda) \right|
\]
and so
\[ I_1 = \text{Dist} \left( \frac{2i \text{Im} \lambda_0}{\lambda - \lambda_0}, \frac{\lambda - \lambda_0}{\lambda - \lambda_0} \right) \] (142)

Since
\[ 2 \text{Im} \lambda_0 \int_0^{\infty} \exp(i\lambda x) \exp(-i\lambda_0 x) dx = \frac{2i \text{Im} \lambda_0}{\lambda - \lambda_0} = 1 - \frac{2i \text{Im} \lambda_0}{\lambda - \lambda_0} \] (143)

any function
\[ \frac{2i \text{Im} \lambda_0}{\lambda - \lambda_0} - \frac{\lambda - \lambda_0}{\lambda - \lambda_0} \tilde{f}(\lambda), \tilde{f}(\lambda) \in \tilde{S} \]
can be approximated in $L^2(d\sigma)$ by the sequence of functions $\tilde{f}_n(\lambda) \in S_r$, $\tilde{f}_n(\lambda_0) = 1$. Therefore, $I_1 \geq I_2$. Assume that $\hat{f}(\lambda)$-- arbitrary function from some $S_r$ and $\hat{f}(\lambda_0) = 1$. Then
\[ \hat{f}(\lambda) = \frac{2i \text{Im} \lambda_0}{\lambda - \lambda_0} - \frac{\lambda - \lambda_0}{\lambda - \lambda_0} \hat{g}(\lambda) \] (144)

with $\hat{g}(\lambda) \in \tilde{S}$. Indeed, the function
\[ \hat{f}(\lambda) - \frac{2i \text{Im} \lambda_0}{\lambda - \lambda_0} \]
belongs to $H^2(\mathbb{C}^+)$ and has zero at $\lambda = \lambda_0$. So, by Paley-Wiener Theorem, (144) holds with $\hat{g} \in H^2(\mathbb{C}^+)$. Let us show now that $\hat{g} \in \tilde{S}$. The first formula in (143) suggests
\[ \hat{g}(\lambda) = \frac{\lambda - \lambda_0}{\lambda - \lambda_0} \left[ -\hat{f}(\lambda) - \frac{2i \text{Im} \lambda_0}{1 - \exp(in(\lambda_0 - \lambda_0))} \int_0^n \exp(ix(\lambda - \lambda_0)) dx \right] + r_n(\lambda) \]
The Paley-Wiener Theorem yields that the first term belongs to $S_{\rho}$ where $\rho = \max(r, n)$. One can easily see that $\|r_n\|_{2, \sigma} \to 0$ as $n \to \infty$. Thus, the formula (144) holds. Due to (142), $I_1 \leq I_2$ and therefore $I_1 = I_2$. \qed

The next result describes the continuous analog of the Szegő case in OPUC theory. If any of the conditions bellow is satisfied, we will say that $d\sigma \in (\text{Szegő})$.

**Theorem 8.2.** (The Szegő case) The following statements are equivalent

(a) The operator $0$ from the Theorem 3.2 is not unitary.

(b) Inequality
\[ \int_{-\infty}^{\infty} \frac{\ln \sigma'(\lambda)}{1 + \lambda^2} d\lambda > -\infty \] (145)
holds.

(c) $\sup_{r > 0, f(\lambda) \in S_r, \|f(\lambda)\|_2, \sigma = 1} |\hat{f}(\lambda_0)| = m^{-1}_\infty(\lambda_0) < \infty$
for at least one (and then for all) $\lambda_0 \in \mathbb{C}^+$.

(d) $P(r, \lambda_0) \in L^2(\mathbb{R}^+)$ for at least one (and then for all) $\lambda_0 \in \mathbb{C}^+$.

(e) $\lim_{r \to \infty} |P_r(r, \lambda_0)| < \infty$ for at least one (and then for all) $\lambda_0 \in \mathbb{C}^+$. 

Proof. (a) and (b) are equivalent. Indeed, by Lemma 8.2 the range of $\Theta$ coincides with $\bar{S}$. From Lemma 8.3 we have $\bar{X} = \bar{S}$ and then we only need to use Theorem 8.1.

The formula (59) with $\lambda = \mu = \lambda_0$ shows that if (d) holds at some $\lambda_0$ then (e) is true as well at the same point. The converse is also true.

The identity (138) and the formula for reproducing kernel yield

$$m_{\infty}^{-2}(\lambda_0) = \sup_{r>0, f(\lambda) \in \mathcal{S}_r \|f(\lambda)\|_2=1} |f(\lambda_0)|^2 = K_{\infty}(\lambda_0, \lambda_0) = \int_0^\infty |P(x, \lambda_0)|^2 dx$$

and that proves equivalence of (c) and (d) for fixed $\lambda_0$.

Let us show that (c) is satisfied with some $\lambda_0$ if and only if (a) holds. Assume that the operator $\Theta$ is unitary. That means its range is the whole $L^2(\mathcal{H})$ and $(\lambda - \lambda_0)^{-1} \in \bar{S}$, for any $\lambda_0 \in \mathbb{C}^+$. Due to Lemma 8.4 (c) fails. Conversely, assume (c) fails for some $\lambda_0 \in \mathbb{C}^+$. Then, by Lemma 8.4 $(\lambda - \lambda_0)^{-1} \in \bar{S}$. The Theorem 8.1 now implies

$$\int_{-\infty}^{\infty} \frac{\ln \sigma'(\lambda)}{|\lambda - \lambda_0|^2} d\lambda = -\infty$$

but that means (b) fails and therefore (a) fails too. Notice that both (a) and (b) do not depend on parameter $\lambda_0$. Therefore, if any of (c), (d), or (e) holds for some $\lambda_0$ then it holds for all $\lambda \in \mathbb{C}^+$. \hfill $\Box$

It is not in general true that $|P_*(r, \lambda)|, \lambda \in \mathbb{C}^+$ is even bounded as $r \to \infty$ under the conditions of the Theorem 8.2. That is due to continuous nature of the problem. Moreover, it is possible that $|P_*(r, \lambda)|$ has a limit, but $\lim_{r \to \infty} P_*(r, \lambda)$ does not exist inspite of the fact that the corresponding $A(r) \to 0$ at infinity and $A(r) \in L^p(\mathbb{R}^+)$ for any $p > 2$. This phenomena was observed for the first time by Teplyaev (see [74, 75]). That can be explained as follows: in the discrete case, the orthonormal polynomials are usually normalized such that they have the positive leading coefficient. For Krein systems, normalization is quite different: $P_*(r, \lambda)$ is normalized to be equal to 1 at infinity, the point on the boundary of $\mathbb{C}^+$. Therefore, the argument of $P_*(r, \lambda)$ is not stabilized and that leads to the ambiguity in the definition of $\lim_{r \to \infty} P_*(r, \lambda)$.

Consider some $\lambda_0 \in \mathbb{C}^+$. If conditions in Theorem 8.2 are satisfied, then there is a sequence $r_n \in [n, n+1] \to \infty$ such that $P(r_n, \lambda_0) \to 0$. Take the outer function

$$\Pi(\lambda) = \frac{1}{\sqrt{2\pi}} \exp \left[ - \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{(1 + s\lambda) \ln \sigma'(s)}{(\lambda - s)(1 + s^2)} ds \right]$$

that satisfies $|\Pi(\lambda)| = [2\pi \sigma(\lambda)]^{-1/2}$ for a.e. $\lambda \in \mathbb{R}$. Notice that $[(\lambda + i)\Pi(\lambda)]^{-1} \in H^2(\mathbb{C}^+)$ and is outer.

Lemma 8.5. If $d\sigma \in (\text{Szeg}" o)$ and $r_n \to \infty$ is such that $P(r_n, \lambda_0) \to 0$ for some $\lambda_0 \in \mathbb{C}^+$, then the following convergence $|P_*(r_n, \lambda)| \to |\Pi(\lambda)|$ takes place uniformly in $\mathbb{C}^+$.

Proof. From the Theorem 6.2 we know that the sequence $h_n(\lambda) = [(\lambda + i)P_*(r_n, \lambda)]^{-1}$ is bounded in $H^2(\mathbb{C}^+)$, i.e. $\|h_n(\lambda)\|_{L^2(\mathbb{R})}$ is bounded. Assume that $h(\lambda)$ is any
$L^2(\mathbb{R})$-weak limit point of this sequence. Then, $h(\lambda) \in H^2(\mathbb{C}^+)$ and the convergence is uniform in $\mathbb{C}^+$ over the same subsequence $n_k$. From \eqref{76},

$$\Re \left[ P_s^{-1}(r_{n_k}, \lambda) \tilde{P}_s(r_{n_k}, \lambda) \right] \geq |P_s(r_{n_k}, \lambda)|^{-2} \quad (147)$$

Taking $k \to \infty$, we get

$$|(\lambda + i)^2 |h(\lambda)|^2 \leq \Re F(\lambda)$$

for $\lambda \in \mathbb{C}^+$, where $F(\lambda)$ is the Weyl-Titchmarsh function. From \eqref{119}, we get

$$(\lambda^2 + 1)|h(\lambda)|^2 \leq 2\pi \sigma'(\lambda)$$

for a.e. $\lambda \in \mathbb{R}$. Therefore, from the multiplicative representation of $(\lambda + i)h(\lambda)$

$$|\gamma(\lambda)| \leq |\Pi^{-1}(\lambda)|, \lambda \in \mathbb{C}^+$$

At the same time, at $\lambda_0$, we have $|((\lambda_0 + i)h(\lambda_0)| = |\Pi^{-1}(\lambda_0)|$. It follows from $\eqref{134}$, $\eqref{135}$, $\eqref{140}$, $\eqref{141}$, $\eqref{142}$. Therefore, $(\lambda + i)h(\lambda)$ is an outer function different from $\Pi^{-1}(\lambda)$ only by a unimodular constant factor. Thus, for any subsequence $n_k$, $|P_s(r_{n_k}, \lambda)| \to |\Pi(\lambda)|$. That means we actually have convergence over the whole $r_n$. \hfill \Box

It is known that all outer functions $\Pi_\gamma(\lambda)$, satisfying $|\Pi_\gamma(\lambda)|^{-2} = 2\pi \sigma'(\lambda)$ a.e on $\mathbb{R}$, have the following representation

$$\Pi_\gamma(\lambda) = \frac{1}{\sqrt{2\pi}} \exp \left[ i\gamma + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(1 + s\lambda) \ln \sigma'(s)}{(\lambda - s)(1 + s^2)} ds \right], \gamma \in [0, 2\pi) \quad (148)$$

i.e. they can be parameterized by the angle $\gamma$. The function $\Pi(\lambda) = \Pi_\gamma(\lambda)$ satisfies the following normalization condition: $\Pi(i) > 0$. There are some quite interesting examples \cite{75} when $P_s(r_n, \lambda) \to \Pi_\gamma(\lambda)$ and the constant $\gamma$ depends on the choice of subsequence $r_n$. In the meantime, the following is true \cite{64}

**Lemma 8.6.** Assume that $A(r)$ is real-valued and $r_n$ is such that $P(r_n, \lambda_0) \to 0$ for at least some $\lambda_0 \in \mathbb{C}^+$. Then, $P_s(r_n, \lambda) \to \Pi(\lambda)$ uniformly in $\lambda \in \mathbb{C}^+$.

**Proof.** Following the proof of the previous Lemma, we get convergence of $P_s(r_{n_k}, \lambda)$ to some $\Pi_\gamma(\lambda)$ uniformly for $\lambda \in \mathbb{C}^+$. Taking $\lambda = i$, we get $\gamma = 0$. Indeed, $P_s(0, i) = 1$, $P_s(r, i)$ is real and has no zeroes for $r > 0$. Therefore, it must be positive for all $r > 0$. Thus $\Pi_\gamma(i) > 0$ and $\gamma = 0$. \hfill \Box

In the OPUC theory, we can not directly characterize the set of moments such that the corresponding measure belongs to Szegő class. The same is true for the Krein systems: we are not aware of the characterization of the Szegő case in terms of accelerant. In the meantime,

**Lemma 8.7.** If $d\sigma \in (\text{Szegő})$, then for the dual system we also have $d\tilde{\sigma} \in (\text{Szegő})$.

**Proof.** We know that for any Krein system,

$$\frac{\tilde{P}_s(r, \lambda)}{P_s(r, \lambda)} \to F(\lambda)$$

as $r \to \infty$ and $F(\lambda)$ has positive real part in $\mathbb{C}^+$. Therefore, if condition (e) of the Theorem\textsuperscript{[82]} is satisfied for the original Krein system, it must be satisfied for the dual one as well. \hfill \Box
Next, let us show that we have weighted $L^2$–convergence for $P_\ast(r, \lambda)$ for $\lambda \in \mathbb{R}$. First, we need the following auxiliary result

**Lemma 8.8.** Assume that the Szegő case holds. Let $r_n$ be a sequence such that $P(r_n, i) \to 0$. Then

$$
\lim_{r_n \to \infty} \int_{-\infty}^{\infty} \frac{|P(r_n, \lambda)|^2}{\lambda^2 + 1} d\sigma(\lambda) = \frac{1}{2}
$$

**Proof.** Indeed, we have

$$
P_\ast(r, \lambda) = \frac{i}{\lambda + i} \left( \frac{P(r, \lambda)}{\lambda + i} \right) \cdot \frac{P(r, i)}{\lambda + i} - \frac{P(r, \lambda)}{\lambda + i} P(r, i)
$$

or

$$
i \frac{P_\ast(r, \lambda)}{\lambda + i} = i \frac{P(r, \lambda)}{\lambda + i} . \frac{P(r, i)}{\lambda + i} + K_r(i, \lambda)
$$

(149)

From Lemma 8.5, we know that $|P_\ast(r_n, i)| \to |\Pi(i)|$. Then, since $P(r_n, i) \to 0$,

$$
\lim_{r_n \to \infty} \left\| \frac{P_\ast(r_n, \lambda)}{\lambda + i} \right\|^2_{L^2(\mathbb{R})} = |\Pi(i)|^{-2} \lim_{r_n \to \infty} |K_{r_n}(i, i)| = \frac{1}{2}
$$

☐

The following result establishes an $L^2(d\sigma, \mathbb{R})$ asymptotics of $P_\ast(r, \lambda)$. It will be used later to prove existence of wave operators for Dirac equation.

**Lemma 8.9.** Assume that $d\sigma \in (\text{Szegő})$ and $r_n \to \infty$ is such that $P(r_n, i) \to 0$ and $P_\ast(r_n, \lambda) \to \Pi_\gamma(\lambda)$ for $\lambda \in \mathbb{C}^+$, ($\gamma \in [0, 2\pi]$). Then,

$$
\int_{-\infty}^{\infty} \frac{1}{\lambda^2 + 1} \left| \frac{P_\ast(r_n, \lambda)}{\Pi_\gamma(\lambda)} - 1 \right|^2 d\lambda \to 0
$$

(150)

as $r_n \to \infty$.

**Proof.** The left-hand side of (150) is equal to

$$
\int_{-\infty}^{\infty} \frac{1}{\lambda^2 + 1} \left| \frac{P_\ast(r_n, \lambda)}{\Pi_\gamma(\lambda)} \right|^2 d\lambda + \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2 + 1} - 2 \text{Re} \int_{-\infty}^{\infty} \frac{P_\ast(r_n, \lambda)}{\lambda^2 + 1} \Pi_\gamma(\lambda) d\lambda
$$

(151)

By the Cauchy formula,

$$
\int_{-\infty}^{\infty} \frac{P_\ast(r_n, \lambda)}{\lambda^2 + 1} \Pi_\gamma(\lambda) d\lambda = \pi \frac{P_\ast(r_n, i)}{\Pi_\gamma(\lambda)} \to \pi
$$

(152)

as $r_n \to \infty$. The first term of (151) can be written as

$$
2\pi \int_{-\infty}^{\infty} \frac{1}{\lambda^2 + 1} |P_\ast(r_n, \lambda)|^2 d\sigma(\lambda) - 2\pi \int_{-\infty}^{\infty} \frac{1}{\lambda^2 + 1} |P_\ast(r_n, \lambda)|^2 d\sigma_s(\lambda)
$$

where $d\sigma_s(\lambda)$ is the singular component of $d\sigma(\lambda)$. From the Lemma 8.8 we infer

$$
\int_{-\infty}^{\infty} \frac{1}{\lambda^2 + 1} |P_\ast(r, \lambda)|^2 d\sigma(\lambda) \to \frac{1}{2}
$$

(153)
as \( r \to \infty \). Bearing in mind (151), (152), and (153), we have (150).

**Remark.** We also proved

\[
\int_{-\infty}^{\infty} \frac{1}{\lambda^2 + 1} |P_*(r_n, \lambda)|^2 d\sigma_*(\lambda) \to 0
\]

(154) as \( r_n \to \infty \).

We want to finish this section with the following observation that relates the regularity of \( |\Pi(\lambda)| \) at \(+\infty\) to some approximation problem. Consider the function

\[
f_{\lambda_0}(\lambda) = (2 \text{Im} \lambda_0)^{1/2}(\lambda - \lambda_0)^{-1}, \lambda \in \mathbb{R}.
\]

Since

\[
(2 \text{Im} \lambda_0)^{1/2}(\lambda - \lambda_0)^{-1} = i(2 \text{Im} \lambda_0)^{1/2} \int_{-\infty}^{0} e^{-i\lambda_0 x} e^{i\lambda x} dx
\]

the function \( f_{\lambda_0}(\lambda) \) has frequency concentrating near zero as \( \text{Im} \lambda_0 \to +\infty \) and the constant \( L^2 \) norm. How regular the distance \( \text{Dist}(f_{\lambda_0}(\lambda), \bar{S}) \) behaves as \( \text{Im} \lambda_0 \to +\infty \) depends on the regularity of \( \Pi(\lambda) \) at infinity. Indeed, from Lemma 8.4

\[
\text{Dist}(f_{\lambda_0}(\lambda), \bar{S}) = |\Pi(\lambda_0)|^{-1}
\]

That infinitesimal phenomena is not present in the discrete case.

**Remarks and historical notes.** The approximation results of this section can be interpreted in the framework of prediction theory for stationary Gaussian processes with continuous time [36]. The original paper by Krein contained some inaccuracies in the formulation of the Theorem 8.2 and the same mistake was made in some later papers. The correct statement was given later by Teplyaev [74, 75]. Some sufficient conditions for the Szegő case were given in the series of papers [64, 15]. There is no known criteria in terms of \( A(r) \) for the Szegő case to hold. In the meantime, if one assumes some regularity of \( A(r) \), say, \( A(r) \in L^\infty(\mathbb{R}^+) \) then \( d\sigma \in (\text{Szegő}) \) if and only if \( A(r) \in H^{-1}(\mathbb{R}^+) \) (see [15]). It is probably impossible to give reasonable characterization of the Szegő case in terms of \( A(r) \) without any apriori assumptions. For example, one can construct a sequence of compactly supported \( A^{(n)}(r) \) with growing \( H^{-1}(\mathbb{R}^+) \) norms but such that the corresponding sequence \( P_*(\infty, i) \) is bounded. This can be achieved by a simple modification of Teplyaev’s example [75].
9. Schur’s algorithm and approximation of continuous orthogonal system by discrete ones

It is well-known [66], that any function \( f(z) \in B(\mathbb{D}) \) can be expanded into the continued fraction (the so-called Schur’s algorithm). This expansion can be obtained by the iteration of

\[
f_n(z) = \frac{zf_{n+1}(z) + a_n}{1 + a_n z f_{n+1}(z)} f_0(z) = f(z), \quad a_n = f_n(0)
\]  \hspace{1cm} (155)

By doing so, we obtain the one-to-one correspondence between \( B(\mathbb{D}) \) and all sequences \( \{a_n\} \) such that \( |a_n| \leq 1 \). The Geronimus theorem asserts that these so-called Schur parameters are actually equal to Verblunsky parameters for the measure \( d\tau \) in the representation

\[
\frac{1 + zf(z)}{1 - zf(z)} = \int_\gamma^\infty \frac{\xi + z}{\xi - z} d\tau(\xi)
\]

(156)

In this section we will study Schur’s function associated to Krein system. The Schur function \( f(\lambda) \) associated to the Krein system was introduced in the Theorem [63]. Functions \( f_n(\lambda) \) in the representation \( \mathbf{117} \) are Schur’s functions that correspond to the same Krein systems but on the interval \([\rho, \infty)\). This is the same as if we would take \( A_\rho(r) = A(r + \rho) \) with \( r \in \mathbb{R}^+ \).

**Lemma 9.1.** For any fixed \( \lambda \in \mathbb{C}^+ \), the Schur functions \( f_n(\lambda) \) are continuously differentiable in \( r \) and satisfy the following equation

\[
\frac{df_n(\lambda)}{dr} = -i\lambda f_0(\lambda) + A(r) - A(0)f_n^2(\lambda)
\]

(157)

**Proof.** The smoothness of \( f_n(\lambda) \) in \( r \) follows immediately from \( \mathbf{117} \). Taking derivative of \( \mathbf{117} \) in \( r = 0 \), we get

\[
\left. \frac{df_n(\lambda)}{dr} \right|_{r=0} = -i\lambda f_0(\lambda) + A(0) - A(0)f_n^2(\lambda)
\]

(158)

Now, \( \mathbf{157} \) follows from the definition of \( f_n(\lambda) \). \( \square \)

It is very important to keep in mind that the initial condition for \( \mathbf{157} \) is \( f_0(\lambda) = f(\lambda) \) and it is **not** independent of the coefficient \( A(r) \). Now, let us compare the continuous and discrete Schur algorithms. Consider the following Möbius transform

\[
\tau_\gamma(z) = \frac{z + \gamma}{1 + \gamma z}
\]

The inverse to \( \tau_\gamma(z) \) is equal to \( \tau_{-\gamma}(z) \).

For any \( \gamma \in \mathbb{D} \), \( \tau_\gamma \) is a conformal map of \( \mathbb{D} \) onto \( \mathbb{D} \) that takes 0 to \( \gamma \). Another important property of \( \tau_\gamma \) is the preservation of the pseudohyperbolic distance on \( \mathbb{D} \):

\[
\rho(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|, \quad \rho(\tau_\gamma(z_1), \tau_\gamma(z_2)) = \rho(z_1, z_2)
\]

(159)

Given any function \( f(z) \) from \( B(\mathbb{D}) \) the Schur algorithm can also be defined as follows. The \( n \)-th Schur’s iterate \( f_n(z) \) is defined by the relation

\[
f(z) = S_{z, a_0, \ldots , a_{n-1}}(f_n) = \tau_{a_0} \circ \tau_{\tau_{a_1}} \circ \ldots \circ \tau_{\tau_{a_{n-1}}} (zf_n)
\]

(160)

and

\[
f_n(z) = S_{z, a_0, \ldots , a_{n-1}}^{-1}(f) = z^{-1} \tau_{-a_{n-1}} \circ z^{-1} \tau_{-a_{n-2}} \circ \ldots \circ z^{-1} \tau_{-a_0} \circ f
\]

(161)
Notice that for \( z \in \mathbb{T} \) the map \( S_{z,a_0,...,a_{n-1}} \) is the composition of rotations and Möbius transforms.

The differential equation (157) is a continuous analog of (155). It is Riccati equation and the Cauchy problem for solving it from the right to the left happens to be well-posed for suitable initial data:

**Lemma 9.2.** For any \( A(r) \in L^1[0, R] \) and any \( f(0) = f_0 \in \mathbb{C}, \lambda \in \mathbb{C}^+ \), there is the unique solution \( f_r(\lambda) \) to Cauchy problem for equation (157) with initial condition \( f(R) = f_0 \).

**Proof.** Consider \( Y(r, \lambda) = (y_1(r), y_2(r))^t \), solution to the following Cauchy problem
\[
\begin{aligned}
y_1' &= -i\lambda y_1 + A(r)y_2, \quad y_1(R, \lambda) = f_0, \\
y_2' &= A(r)y_1, \quad y_2(R, \lambda) = 1
\end{aligned}
\]
and solve it for \( r \in [0, R] \). Simple calculations show that
\[
Y(r, \lambda) = X_B(r)(0, R - r, \lambda)Y(R) = X^t(r, R, \lambda)Y(R)
\] (162)
where \( Y(R) = (f_0, 1)^t \) and \( X_B \) is the transfer matrix for the Krein system with coefficient \( B(r) = A(R - r) \). We have \( \lambda \in \mathbb{C}^+ \) so \( X_B \) is \( J \)-contraction by the Theorem [111]. Thus, we have \(|y_1(r, \lambda)| \leq |y_2(r, \lambda)|\). In particular, \( y_2(r, \lambda) \neq 0 \) since otherwise \( Y \equiv 0 \) on \([0, R]\). Consider \( f(r, \lambda) = y_1(r, \lambda)y_2^{-1}(r, \lambda) \). The straightforward calculation shows that \( f(r, \lambda) \) is solution to our Cauchy problem and uniqueness follows from the general theory of ODE. \( \square \)

In analogy with discrete case, we denote the solution of this Cauchy problem at zero by \( S_{\lambda,A,R}(f_0) \) and now we have \( f(\lambda) = S_{\lambda,A,R}(f_r(\lambda)) \), the direct analog of (150).

The formula (162) shows that \( S_{\lambda,A,R} \) allows the following representation
\[
S_{\lambda,A,R}(z) = \frac{\mathfrak{A}_s(r, \lambda)z + \mathfrak{B}(r, \lambda)}{\mathfrak{B}_s(r, \lambda)z + \mathfrak{A}(r, \lambda)}
\] (163)
Notice that the inverse to \( S \) in discrete case is not contraction anymore and we have the same problem in the continuous setting.

As we mentioned earlier, the class of Schur functions \( \{ \lambda \} \) in the continuous case can not be all \( B(\mathbb{C}^+) \). For example, (118) must hold. Next, we will describe the subclass of \( B(\mathbb{C}^+) \) in which \( A(r) \) have the meaning of intrinsic parameters of the function \( f(\lambda) \) just like \( \{a_n\} \) are intrinsic parameters of \( f(z) \in B(\mathbb{D}) \).

Let a function \( C(x) \in L^2_{\text{loc}}(\mathbb{R}^+) \) be given. For any \( R > 0 \), consider the operator \( \mathcal{C}_R \) acting in \( L^2[0, R] \) by the following formula
\[
\mathcal{C}_R f(x) = \int_0^x C(x - u)f(u)du
\] (164)
We start with the definition.

**Definition 9.1.** The function \( s(\lambda) \in S(\mathbb{C}^+) \) if the following is true:

1. There is a function \( C(x) \in L^2_{\text{loc}}(\mathbb{R}^+) \) such that for any \( R > 0 \) there is a function \( \Phi_R(\lambda) \in H^\infty(\mathbb{C}^+) \):
\[
s(\lambda) = \int_0^R C(x)\exp(i\lambda x)dx + \exp(i\lambda R)\Phi_R(\lambda)
\] (165)
For any $R > 0$,
\[ \|C_R\|_{L^2[0,R]} < 1 \quad (166) \]

It is an easy exercise to see that $C(x)$ and $\Phi_R(\lambda)$ are both uniquely defined for any $s(\lambda) \in S(\mathbb{C}^+)$. Conversely, if the function $C(x)$ is given, then there is at most one $s(\lambda) \in S(\mathbb{C}^+)$ having $C(x)$ as a function in the formula (165). Indeed, assume that there are two $s^{(1)}(\lambda), s^{(2)}(\lambda)$ having the same $C(x)$ in (165). Then, $(s^{(1)}(\lambda) - s^{(2)}(\lambda))/(\lambda + i) \in H^2(\mathbb{C}^+)$. At the same time,
\[
\frac{s^{(1)}(\lambda) - s^{(2)}(\lambda)}{\lambda + i} = \exp(i\lambda R) \frac{\Phi^{(1)}_R(\lambda) - \Phi^{(2)}_R(\lambda)}{\lambda + i}
\]
for any $R > 0$. Since $(\Phi^{(1)}_R(\lambda) - \Phi^{(2)}_R(\lambda))/(\lambda + i) \in H^2(\mathbb{C}^+)$ as well, we have
\[
P_{[0,R]} \left[ \frac{s^{(1)}(\lambda) - s^{(2)}(\lambda)}{\lambda + i} \right] = 0
\]
for any $R > 0$. So, $s^{(1)}(\lambda) = s^{(2)}(\lambda)$.

Consider $s(\lambda) \in S(\mathbb{C}^+)$. We have $s(\lambda) \in H^\infty(\mathbb{C}^+)$. In the space $H^2(\mathbb{C}^+)$, denote the operator of multiplication by this function by $S$. Also, consider the operator $\mathcal{C}$ acting on $L^2(0, \infty)$ and given by the formula
\[
\mathcal{C}f(x) = \int_0^x C(x - u)f(u)du
\]
(167)

Since $\Pi_{[0,R]} \mathcal{C} \Pi_{[0,R]} = \mathcal{C}_R$ and $\|\mathcal{C}_R\| < 1$, operator $\mathcal{C}$ is well-defined on $L^2(0, \infty)$ and is contraction, i.e. $\|\mathcal{C}\|_{2,2} \leq 1$.

**Lemma 9.3.** One has the following inclusion: $S(\mathbb{C}^+) \subset B(\mathbb{C}^+)$. The operators $\mathcal{C}$ and $\mathcal{S}$ are unitary equivalent.

**Proof.** Indeed, $s(\lambda) \in H^\infty(\mathbb{C}^+)$ and it is known that
\[
\|s\|_{H^\infty(\mathbb{C}^+)} = \sup_{\|g\|_{H^2(\mathbb{C}^+)} = 1} \|sg\|_{H^2(\mathbb{C}^+)} = \|S\|
\]
The condition (166) is equivalent to the estimate $\|P_{[0,R]} \mathcal{S} P_{[0,R]}\| < 1$. Then, $\|S\| \leq 1$, or $s(\lambda) \in B(\mathbb{C}^+)$. The unitary equivalence of $\mathcal{S}$ and $\mathcal{C}$ follows from the unitary equivalence of operators $\Pi_{[0,R]} \mathcal{C} \Pi_{[0,R]}$ and $P_{[0,R]} \mathcal{S} P_{[0,R]}$ via the Fourier transform. \(\square\)

Recall the definition of the accelerant: given Hermitian $H(x) \in L^2_{\text{loc}}(\mathbb{R})$, we say that it is an accelerant if the operator
\[
I + \mathcal{H}_R > 0
\]
for any $R > 0$ and $\mathcal{H}_R$ is given by (25). Given any $C(x) \in L^2_{\text{loc}}(\mathbb{R}^+)$, consider the function $H(x) \in L^2_{\text{loc}}(\mathbb{R}^+)$ which is the solution to
\[
H(x) + C(x) + \int_0^x C(x - u)H(u)du = 0 \quad (168)
\]
The direct iteration of the equation proves existence and uniqueness of this $H(x)$. Let $H(-x) = \overline{H(x)}$ for $x > 0$. 

Lemma 9.4. The function $H(x)$ is an accelerant if and only if $C(x)$ is such that \eqref{166} holds for any $R > 0$.

Proof. For any $R > 0$, consider the Caley transform of $\mathcal{C}_R$:

$$I + \mathcal{U}_R = (I - \mathcal{C}_R)(I + \mathcal{C}_R)^{-1} \tag{169}$$

Since $\mathcal{C}_R$ is a Volterra operator, the Caley transform does exist. Moreover, $\mathcal{U}_R$ is a Volterra operator with the kernel given exactly by $H$:

$$\mathcal{U}_R f(x) = 2 \int_0^x H(x - s)f(s)ds \tag{170}$$

This is an easy corollary from \eqref{168}. Clearly, Re($I + \mathcal{U}_R$) = $I + \mathcal{H}_R$. Now, the equivalence of $I + \mathcal{H}_R > 0$ and \eqref{166} is a simple algebraic fact. \hfill $\square$

Now, we can easily characterize the class of all $C(x)$ that generate $s(\lambda) \in S(\mathbb{C}^+)$.

Theorem 9.1. The Schur functions of Krein systems with $A(r) \in L^2_{\text{loc}}(\mathbb{R}^+)$ are in one-to-one correspondence with functions $s(\lambda) \in S(\mathbb{C}^+)$. For each $s(\lambda) \in S(\mathbb{C}^+)$, the coefficient $A(r)$ of the associated Krein system plays the role of the Schur parameter.

Proof. Assume $s(\lambda) \in S(\mathbb{C}^+)$ and $C(x)$ is the corresponding function. Denote by $H(x)$ the accelerant corresponding to $C(x)$, i.e.

$$H(x) + \bar{C}(x) + \int_0^x \bar{C}(x - u)H(u)du = 0 \tag{171}$$

Then, by Lemma 9.4, $H(x)$ is an accelerant that generates the Krein system with coefficient $A(x) \in L^2_{\text{loc}}(\mathbb{R}^+)$. Consider the corresponding Schur function $f(\lambda)$. Let us show that $f(\lambda) = s(\lambda)$. For each $R > 0$, we have

$$f(\lambda) = \frac{\mathcal{B}(R, \lambda) + f_R(\lambda) \mathcal{A}_s(R, \lambda)}{\mathcal{A}(R, \lambda) + f_R(\lambda) \mathcal{B}_s(R, \lambda)} \tag{172}$$

for any $R$. Clearly, by \eqref{113} with $r_1 = \infty$, $r_2 = R$

$$f(\lambda) - \frac{\mathcal{B}(R, \lambda)}{\mathcal{A}(R, \lambda)} = \frac{\exp(i\lambda R) f_R(\lambda)}{\mathcal{A}(R, \lambda) + f_R(\lambda) \mathcal{B}_s(R, \lambda)} \tag{173}$$

By \eqref{116}, the right-hand side of \eqref{173} is equal to $\exp(i\lambda R) \Phi_R(\lambda)$ with $\Phi_R(\lambda) \in H^\infty(\mathbb{C}^+)$. Consider the function $\mathcal{B}(R, \lambda) \mathcal{A}^{-1}(R, \lambda)$. Due to Levy-Wiener Theorem,

$$\mathcal{B}(R, \lambda) \mathcal{A}(R, \lambda) = \int_0^\infty C_R(x) \exp(i\lambda x)dx = \int_0^R C_R(x) \exp(i\lambda x)dx +$$

$$+ \exp(i\lambda R) \int_0^\infty C_R(x + R) \exp(i\lambda x)dx, C_R(x) \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$$

and the last term can be written as $\exp(i\lambda R) \Phi_R(\lambda)$ with $\Phi_R(\lambda) \in H^\infty(\mathbb{C}^+)$. We can also write

$$\frac{\mathcal{B}(R, \lambda)}{\mathcal{A}(R, \lambda)} = \left(1 - \frac{\hat{P}_s(R, \lambda)}{\hat{P}_s(R, \lambda)}\right) \left(1 + \frac{\hat{P}_s(R, \lambda)}{\hat{P}_s(R, \lambda)}\right)^{-1}$$
From \([\text{[91]}], \text{[92]}\) and \([\text{[171]}]\), we infer \(C_R(x) = C(x)\) for \(x \in [0, R]\). Since \(R\) is arbitrary positive, \(f(\lambda) \in S(\mathbb{C}^+)\) and \(f(\lambda) = s(\lambda)\).

Now, assume that the Krein system with the coefficient \(A(r)\) is given. Repeating the arguments above, one has \(f(\lambda) \in S(\mathbb{C}^+)\).

**Remark 9.1.** It follows from \([\text{[168]}]\) that \(C(x)\) on the interval \([0, R]\) depends only on the values of \(A(r)\) on the same interval. This is because an accelerant has analogous property.

Notice that \(s(\lambda) \in S(\mathbb{C}^+)\) implies certain regularity at infinity, a boundary point of \(\mathbb{C}^+\). For instance, \(s(iy) \to 0\) as \(y \to +\infty\). So, there are plenty of functions in \(B(\mathbb{C}^+)\) that do not belong to \(S(\mathbb{C}^+)\).

The same arguments immediately yield

**Remark 9.2.** \(A(r) \in C(0, \infty)\) iff \(H(x) \in C(0, \infty)\) iff \(C(x) \in C(0, \infty)\) in the corresponding representation for \(f(\lambda)\).

Consider this case for the rest of the section. Let \(C(x, r)\) be function associated to \(f_r(\lambda)\) by formula \([\text{[165]}]\).

**Lemma 9.5.** (Continuous analog of Geronimus theorem). The following relation holds true: \(C(0, r) = -A(r)\).

**Proof.** It is enough to prove the statement for \(r = 0\). From \([\text{[171]}]\), we have \(C(0) = -\overline{H(+0)}\). Then, by Lemma 3.3 \(C(0) = -A(0)\).

Notice that the value of \(C(r)\) at zero gives the main term of asymptotics of \(f(\lambda)\) as \(\lambda = iy, y \to +\infty\). For instance, if \(A(r) \in C^1[0, \infty)\), then \(H(x), C(x, r) \in C^1[0, \infty)\) as well (see \([\text{[53]}]\) and \([\text{[171]}]\)) and Lemma 9.5 yields \(f(r, iy) = -A(r)/y + o(y^{-1})\). In general, for \(A(r) \in C[0, \infty)\), we have asymptotics in the mean (see Lemma \([\text{[172]}]\) in Appendix):

\[
\lim_{y \to \infty} \frac{1}{y} \int_0^y s f(r, is) ds = -A(r)
\]

Anyway, the number \(C(0, r) = -A(r)\) is an intrinsic parameter of the function \(f(r, \lambda)\). Therefore, Lemma 9.5 can be regarded as the continuous analog of the celebrated Geronimus theorem which says that the Schur parameters of the function from \(B(\mathbb{D})\) coincide with the Verblunsky parameters of the associated sequence of orthogonal polynomials.

**Lemma 9.6.** Assume \(A(r) \in C^1[0, \infty)\), then \(C(x, r)\) is continuously differentiable in \(x\) and \(r\) and satisfies the following nonlinear integro-differential equation

\[
\frac{\partial C(x, r)}{\partial r} = \frac{\partial C(x, r)}{\partial x} - \frac{A(r)}{x} \int_0^x C(x - u, r) C(u, r) du \quad (174)
\]

**Proof.** Let us prove smoothness in \(\Omega_T = \{r \geq 0, x \in [0, T]\}\) for any \(T > 0\). Apply \([\text{[54]}]\) to the Krein system on \([r, \infty)\). We have the corresponding \(r\)-dependent function \(A^{(r)}(T, x) \in C^1(\Omega_T)\). Notice that \(A^{(r)}(T, x) = \Gamma^{(r)}_T(T - x, 0)\) and

\[
\Gamma^{(r)}_T(x, 0) + \int_0^T H^{(r)}(x - u) \Gamma^{(r)}_T(u, 0) du = H^{(r)}(x)
\]
To prove (174) for, say, $0 < r < R$ and $0 < x < T$, we can consider new $A_1$ equal to $A$ on $[0, R + T]$, smooth on $\mathbb{R}^+$ with compact support. Then, new $C_1(x, r) = C(x, r)$ for $0 < r < R, 0 < x < T$. Moreover, $C_1(x, r) \in L^1(\mathbb{R}^+)$ in $x$ and the formula (165) holds with $R = \infty$. Then, if one substitutes (165) to (157), equation (174) pops up.

Plug $A(r) = -C(0, r)$ into this equation and solve the first order PDE with boundary condition $C(x, 0)$ regarded as known. Then (174) becomes a nonlinear integral equation which one tries to solve by iterations. Since $A(r) = -C(0, r)$, that gives us a solution to inverse problem since $C(x, 0)$ can be read off the spectral data, say $d\sigma$.

Next, let us focus on the differential equations (157) and (57). Looking at the formula (160), one might guess that the map $S_{\lambda, A, r}$ should also be represented as a combination of Möbius transforms and certain multiplications. This is indeed the case. The following result gives approximation of continuous orthogonal polynomials by the sequence of properly scaled polynomials orthogonal on the unit circle. These discrete polynomials are given in terms of Schur parameters that depend upon the step of discretization $h$.

**Theorem 9.2.** Let $A(r) \in C[0, \infty)$. Fix any $r > 0$ and consider the sequence of Verblunsky coefficients

$$a^{(h)}_0 = hA(t_1), a^{(h)}_1 = hA(t_2), \ldots, a^{(h)}_{n-1} = hA(t_n)$$

and

$$a^{(h)}_j = 0, j \geq n, (h = r/n, t_j = jh)$$

where $h$ is chosen so small that all of these coefficients are less than one in absolute value. Consider the discrete transfer matrix generated by these coefficients

$$M(0, k, z) = W(a_k)Z \ldots W(a_0)Z$$

with

$$Z(z) = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}, W(a_j) = \begin{bmatrix} 1 & -\bar{a}_j \\ -a_j & 1 \end{bmatrix}$$

Then, we have

$$X(r, \lambda) = \lim_{h \to 0} M(0, n, \exp(i\lambda h))$$

and the convergence is uniform over $\lambda$ in compacts in $\mathbb{C}$.

**Proof.** Consider small $h > 0$. Then, from the definition and properties of the multiplicative integral [11], we have

$$X(r, \lambda) = \hat{\lambda} \int_0^\infty \exp[V(t)dt] = \lim_{h \to 0} \left[ (1 + hV(t_n)) \ldots (1 + hV(t_1)) \right] =$$

$$\lim_{h \to 0} \left[ W(a^{(h)}_{n-1})Z(w)W(a^{(h)}_{n-2})Z(w) \ldots W(a^{(h)}_0)Z(w) \right]$$

where $w = \exp(i\lambda h)$. Now, the statement of the Theorem is an elementary corollary from (178). □

The next Corollary follows directly from the Theorem.
Corollary 9.1. For continuous orthogonal polynomials,

\[
\begin{bmatrix}
P(r,\lambda) & \hat{P}(r,\lambda) \\
P^*(r,\lambda) & \hat{P}^*(r,\lambda)
\end{bmatrix}
= \lim_{h \to 0}
\begin{bmatrix}
P_n(w) & \hat{P}_n(w) \\
P^*_n(w) & \hat{P}^*_n(w)
\end{bmatrix}
\]

where \(w = \exp(i\lambda h)\), polynomials \(P_k, P^*_k\) are monic orthogonal polynomials generated by the prescribed Verblunsky parameters and \(\hat{P}_k, \hat{P}^*_k\) are dual to them. The convergence is uniform in \(\lambda\) from any compact in \(\mathbb{C}\).

Corollary 9.2. Let \(A(r) \in C[0,\infty)\). For the map \(S_{\lambda, A, r}(z)\), we have

\[
S_{\lambda, -A, r}(z) = \lim_{h \to 0} S_{w, a^{(h)}, \ldots, a^{(h)}_{n-1}}(z),
\]

\(w = \exp(i\lambda h)\) and the convergence is again uniform over the compacts in \(\mathbb{C}\).

Proof. (179) follows from the formula (163). Indeed, in discrete setting, there is a formula analogous to (163) (see [40], formula (4.19))

\[
S_{z, a_0, \ldots, a_n}(f) = \frac{A_n(z) + zB_n^*(z)f}{B_n(z) + zA_n^*(z)f}
\]

We also have

\[
\begin{bmatrix}
\mathfrak{A}_r(r,\lambda) & \mathfrak{B}_r(r,\lambda) \\
\mathfrak{B}(r,\lambda) & \mathfrak{A}(r,\lambda)
\end{bmatrix}
= \lim_{h \to 0}
\begin{bmatrix}
wB_n^*(w) & -A_n^*(w) \\
-wA_n(w) & B_n(w)
\end{bmatrix}
\]

and \(A_n, B_n\) are the standard Wall polynomials. On the other hand, by Lemma 4.3, \(JXJ\) is the transfer matrix for Krein systems with coefficient \(-A\). Comparing the corresponding formulas to (163), we get the statement of the Corollary.

Remark 9.3. One might wonder why the formula (179) contains the sign minus in front of \(A\)? The answer to this question is contained in the definition of continuous Schur function and map \(S_{\lambda, A, r}\). Indeed, we defined \(f\) as

\[
f(\lambda) = \lim_{r \to \infty} \frac{X_{21}(r,\lambda)}{X_{22}(r,\lambda)}
\]

In the discrete case, the Schur function is defined as

\[
f(z) = \lim_{n \to \infty} \frac{A_n(z)}{B_n(z)}
\]

The Schur function with opposite sign corresponds to the dual system with coefficient of opposite sign. So, having (181) in mind (see the footnote below), we see why the opposite sign was picked up.

If we view an operation \(S_{z, a_0, \ldots, a_n-1}(w)\) introduced in (160) as a map of \(w \in \mathbb{D}\) to \(\mathbb{D}\) with parameters \(z, a_0, \ldots, a_{n-1} \in \mathbb{D}\), then

- For \(z \in \mathbb{T}\), it preserves the pseudohyperbolic distance. This is simply because both multiplication by \(z\) and the Möbius transform preserve this distance.

We want to emphasize some abuse in notations in the definition of continuous Wall polynomials. If one wants to be consistent with discrete case, then the choice must be made according to (180) so that for the transfer matrix \(X\) in Krein system:

\[
X = \begin{bmatrix}
B & -A^* \\
-A & B
\end{bmatrix}
\]

In the meantime, we want to keep our notations to be consistent later on with terminology accepted in the scattering theory.
For \( z \in \mathbb{D} \), it acts as a contraction, i.e.
\[
\rho(S_{z,a_0,...,a_{n-1}}(w_1), S_{z,a_0,...,a_{n-1}}(w_2)) \leq \rho(w_1, w_2)
\]
The contractive property follows solely from the contractive property of multiplication by \( z \).

Analogous properties for the map \( S_{\lambda,A,r} \) is given in the following

**Lemma 9.7.** For any \( \lambda \in \mathbb{R} \), the map \( S_{\lambda,A,r} \) preserves the pseudohyperbolic metric and for \( \lambda \in \mathbb{C}^+ \) it is contraction.

**Proof.** For continuous \( A \), the proof follows immediately from the properties of discrete map \( S \) and Corollary 9.2. Approximating \( A \in L^1[0,r] \) by continuous functions, we get the statement of the Lemma in general case. \( \Box \)

Now, we can really regard the map \( S_{\lambda,A,r} \) as a combination of Möbius transforms and rotations. In the particular case \( \lambda = 0 \), we have the following representation
\[
S_{0,A,r}(z) = \Phi_0^r(\mathcal{A},A,z)
\]
where we use notation \( \Phi_0^r(F,G,z) \) for the continuous continued fraction invented by Puig Adam \([58]\) and later developed by Wall \([76]\). In this case, equation (157) takes the form of the Riccati-Stieltjes equation
\[
\frac{df_r(0)}{dr} = A(r) - \overline{A(r)} f_r^2(0)
\]
We do not get deeper into this subject and refer the interested reader to the original papers.

One should notice that there are many ways to approximate Krein system by the sequence of OPUC's. For example, one can take the following system of Verblunsky coefficients:
\[
hA(t_1), 0, hA(t_2), 0, \ldots, hA(t_n), 0, 0, \ldots
\]
Then, the only difference will be a different scaling, e.g.
\[
X(r, \lambda) = \lim_{h \to 0} M(0, 2n, \exp(i\lambda h/2))
\]

There are at least two other ways to approximate the Krein system with the sequence of OPUC. They are discussed below and we will make use of them later on.

Previously, we started with finite differences approximation to a system of ODE. That produced the approximation of the related analytic functions. Now we start with an accelerator and approximate it first.

**Theorem 9.3.** Assume that we are given an accelerant \( H(x) \in C[0,\infty) \). Fix any \( R > 0 \) and let \( h = R/n \). Consider the Toeplitz matrices
\[
\mathcal{T}_j = \begin{bmatrix}
1 & hH(-h) & \cdots & hH(-jh) \\
\cdots & \cdots & \cdots & \cdots \\
hH(jh) & hH((j-1)h) & \cdots & 1
\end{bmatrix}, \quad j = 1, \ldots, n
\]
For \( h \) small enough, \( \mathcal{T}_n > 0 \) and it generates the Schur coefficients \( \{a_j^{(h)}\}_{j=0}^n \) such that
\[
\lim_{h \to 0} \sup_{0 < jh < R} |A(jh) - \delta^{-1}a_j^{(h)}| \to 0
\]
where \( \delta \) is a small fixed number.
Consider the resolvent equation

$$-\alpha_{j-1}^{(h)} = \frac{\det \mathcal{T}_j}{\det \mathcal{T}_{j-1}} \mathcal{T}^{-1}_j(j, 0)$$

(185)

Proof. Indeed, we know that $I + \mathcal{H}_R > 0$, $H(-x) = \overline{H(x)}$, and $H(x) \in C[0, \infty)$. Thus $\mathcal{T}_n > 0$ for $n$ large enough and it generates the Schur parameters $\{a_j^{(h)}\}_{j=0}^{\infty}$.

Its $h$-step discretization leads to the system of linear algebraic equations with the matrix $\mathcal{T}_n$. If one takes any $\delta < r < R$, then the matrix $\mathcal{T}_{[rh^{-1}]+1}$ is the discretization of the operator $I + \mathcal{H}_r$ but with the step of discretization (relative to the length of the interval $[0, r]$) slightly bigger than that for $[0, R]$. Nevertheless, it tends to zero as $h \to 0$ and this is why we need to keep $r > \delta > 0$. It allows us to use the following argument. We have

$$\Gamma_r(x, 0) + \int_0^R H(x-u)\Gamma_r(u, 0)du = H(x),$$

The discretization with the step $h$ gives

$$\begin{bmatrix} 1 & hH(-h) & \ldots & hH(-jh) \\ hH(jh) & \ldots & \ldots & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ hH((j-1)h) & \ldots & \ldots & 1 \end{bmatrix} \begin{bmatrix} \gamma^{(r,h)}(0) \\ \gamma^{(r,h)}(j) \end{bmatrix} = \begin{bmatrix} H(+0) \\ \vdots \\ H(jh) \end{bmatrix},$$

and $j = [rh^{-1}] + 1$. Application of the standard arguments that use Hadamard’s Lemma on the determinants yields

$$\lim_{h \to 0} \sup_{k=0, \ldots, j} \left| \frac{\delta_r(kh, 0)}{\delta_r} - \gamma^{(r,h)}(k) \right| = 0$$

(186)

where $\delta_r(x, y)$ and $\delta_r$ are introduced in Lemma 17.2. From this Lemma, we also know that

$$\Gamma_r(x, 0) = \frac{\delta_r(x, 0)}{\delta_r}$$

The convergence in (186) is uniform in $r$ as long as $\delta < r < R$. Since $A(r) = \Gamma_r(0, r)$, we have

$$\sup_{j: \delta < jh < R} |A(jh) - \gamma^{(r,h)}(j)| \to 0$$

as $h \to 0$. At the same time, Kramer’s rule gives us the following

$$\gamma^{(r,h)}(j) = \frac{1}{\det \mathcal{T}_j} \det \begin{bmatrix} 1 & hH(-h) & \ldots & hH(-(j-1)h) & H(+0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ hH(jh) & hH((j-1)h) & \ldots & hH(h) & H(jh) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ hH((j-1)h) & \ldots & \ldots & hH(h) & 0 \end{bmatrix} = \frac{1}{\det \mathcal{T}_j} \det \begin{bmatrix} 1 & hH(-h) & \ldots & hH(-(j-1)h) & H(+0) - h^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ hH(jh) & hH((j-1)h) & \ldots & hH(h) & 0 \end{bmatrix} = -\frac{(H(+0) - h^{-1}) \det \mathcal{T}_{j-1}}{\det \mathcal{T}_j} \alpha_{j-1}^{(h)}$$
where the last formula follows from (185). We have $\det T_j \to \hat{\delta}_r$ uniformly in $r$ as $h \to 0$ and $\hat{\delta}_r$ is defined in Lemma 17.2. Therefore, (184) follows. □

The next Theorem is technical but we will need it later in the proof of the Strong Szegő Theorem. It will give an approximation of Krein system through its measure although we need to take $d\sigma$ of a very special kind. Assume that $d\sigma$ is purely absolutely continuous with density

$$2\pi \sigma'(\lambda) = \exp \left( \int \! l(x)e^{i\lambda x} \, dx \right)$$

with $l(x)$—Hermitian, continuous on $\mathbb{R}$ with compact support within $[-R,R]$. This measure $d\sigma$ will generate the Krein system with continuous $H(x)$ and $A(r)$. For $H(x)$, we have an expansion (see (27))

$$H(x) = l(x) + \frac{l \ast l}(x) + \frac{l \ast l \ast l}{2!} + \frac{l \ast l \ast l \ast l}{3!} + \ldots$$  (187)

Theorem 9.4. For large $n$, consider $h = Rn^{-1}$, $x_j = jh$, $j = -n, \ldots, n$ and the a.c. measure $d\mu_n$ on $\mathbb{T}$ with the density given by the formula

$$\mu_n'(\theta) = \exp \left[ \sum_{j=-n}^{n} h l(x_j)z^j \right], z = e^{i\theta}$$

Let $a_j^{(h)}$ be the associated Verblunsky parameters. Then

$$\lim_{h \to 0} \sup_{\delta < jh < R} |A(jh) - h^{-1}a_j^{(h)}| \to 0$$  (188)

where $\delta$ is any small fixed number.

Proof. The 0-th moment of the measure is equal to

$$c_n^{(0)} = 1 + h l(0) + \ldots + \frac{h^k}{k!} \sum_{j_1+\ldots+j_k=0} l(x_{j_1}) \cdot \ldots \cdot l(x_{j_k}) + \ldots$$  (189)

which can be written as

$$1 + h H(0) + \bar{o}(h)$$

as it follows from (187), approximation of the integral by the Riemann sum, and simple estimates on the tail of the series (189). The same is true about the higher moments, i.e.

$$c_n^{(k)} = h H(-kh) + \bar{o}(h), |k| < n$$

Moreover, $|h^{-1}\bar{o}(h)| \to 0$ as $h \to 0$ uniformly in $|k| < n$. Application of the same arguments that proved Theorem 9.3 completes the proof. □

Remarks and historical notes.

The continuous analogs of Schur and Caratheodory-Toeplitz problems were considered in [45]. The corresponding classes of analytic contractions were introduced in the same paper. Equation (158) is rather standard in the theory of inverse problems. For Schrödinger operators, the equation analogous to (174) was obtained and studied in [67], [28]. The explicit approximation of Krein system by sequence of scaled OPUC’s is new to the best of our knowledge. For discretization of continuous Toeplitz operators, see [23] Chapter 8. It is a very good exercise to take $A(r) = \text{const}$ on $[0, R]$ and explicitly compute polynomials that correspond to discretization with step $h = R/n$.  

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10. ZEREOES OF $P(r, \lambda)$

In this section, we study zeroes of the function $P(r, \lambda)$. It is convenient for us to write $P(r, \lambda) = \exp(i\lambda r)Q(r, \lambda)$ and consider $Q(r, \lambda)$ instead. From (55), we know that

$$Q(r, \lambda) = 1 - \int_0^r \Gamma_r(s, 0) \exp(-i\lambda s)ds$$

Assume that $A(r)$ is not identically zero. By the Hadamard Theorem, function $Q(r, \lambda)$ has the following factorization

$$Q(r, \lambda) = C\lambda^m \exp(\alpha \lambda) \prod_{n=1}^{\infty} \left(1 - \lambda/\lambda_n\right) \exp(\lambda/\lambda_n), \quad |\lambda|_1 \leq |\lambda|_2 \leq \ldots,$$

Since $P(r, \lambda)$ has zeroes in $\mathbb{C}^+$ only, $m = 0$. Also,

$$C = Q(r, 0), \quad \alpha = \frac{\partial Q(r, 0)/\partial \lambda}{Q(r, 0)}$$

Clearly, $|\lambda_1(r)| \to +\infty$ as $r \to 0$. For each $r > 0$, zeroes $\lambda_n(r)$ accumulate at infinity in a very regular way. For instance, if $A(x)$ is smooth on $x \in [0, r]$ and $A(r) \neq 0$, then $\lambda_n$ has the following trivial asymptotics at infinity (see Lemma 17.10 in Appendix)

$$\lambda_n = \lambda_n^0 + \bar{o}(1), \quad n \to \infty,$$

$$\lambda_n^0 = x_n + iy_n, \quad x_n^2 + y_n^2 = |A(r)|^2 \exp(2ry_n),$$

$$x_n = r^{-1} [\pi/2 + \pi n - \text{Arg}(A(r))], \quad n \in \mathbb{Z}$$

i.e. the zeroes are accumulating evenly near the graph of the logarithm. Moreover, as $r \to \infty$, the graph of logarithm is getting closer to the real axis and the spacing between consecutive zeroes decreases.

In the meantime, an interesting question is distribution of zeroes for finite $r$ and $\lambda$ inside the compacts in $\mathbb{C}^+$. The Fejér Theorem for polynomials $\varphi_n(z)$ orthogonal on $\mathbb{T}$ with respect to measure $d\mu$ says that all zeroes of each $\varphi_n(z)$ are inside the convex hull of $\text{supp}(d\mu)$. Let us prove similar statements for $P(r, \lambda)$. Assume that $\text{supp}(d\sigma)$ has a gap, say $(a, b)$. We want to show that $\lambda_n$ stay away from $(a, b)$. Let $M_r$ be introduced by the following formula

$$M_r = \sup_{\lambda \in \mathbb{R}} |P(r, \lambda)|$$

Lemma 4.6 and $|P(r, \lambda)| = |P_\ast(r, \lambda)|$ for $\lambda \in \mathbb{R}$ yield

$$M_r \leq \exp \left[ \int_0^r |A(s)|ds \right]$$

**Theorem 10.1.** (Continuous analog of Fejér Theorem). Let $(a, b) \cap \text{supp}(d\sigma) = \emptyset$. Then, $P(r, \lambda)$ has no zeroes in $\Omega_r$ given by

$$\Omega_r : \lambda = x + iy, \quad \int_{-\infty}^{\infty} \frac{2yd\sigma(\lambda)}{\lambda - x} < M_r^{-2}$$

---

3More on the asymptotics of $\{\lambda_n\}$ can be found in recent preprint [44].
Proof. Assume that $\lambda_0 = x + iy$ is a zero of $P(r, \lambda)$. Consider $f(\lambda) = P(r, \lambda)/(\lambda - \lambda_0) \in S_r$. By (136),
\[
\int_{-\infty}^{\infty} \frac{|f(\lambda)|^2}{2\pi|P(r, \lambda)|^2} \, d\lambda = \int_{-\infty}^{\infty} |f(\lambda)|^2 \, d\sigma(\lambda)
\]
That can be rewritten as
\[
\frac{1}{2y} = \int_{-\infty}^{\infty} \frac{|P(r, \lambda)|^2}{(\lambda - x)^2 + y^2} \, d\sigma(\lambda)
\]
Thus, a simple estimate follows
\[
1 \leq M^2 \int_{-\infty}^{\infty} \frac{2yd\sigma(\lambda)}{(\lambda - x)^2 + y^2}
\]
Clearly, $\Omega_r$ contains a domain in $\mathbb{C}^+$ contiguous to $(a, b)$.

Remark 10.1. One can modify this proof in the following way. For simplicity, assume $(a, b) = (-1, 1)$. Introduce
\[
N_r = \int_{-\infty}^{\infty} \frac{|P(r, \lambda)|^2}{\lambda^2 + 1} \, d\sigma(\lambda)
\]
Then $P(r, \lambda)$ has no zeroes in the following set
\[
\Omega'_r : \lambda = x + iy, y < \left[ 2N_r \sup_{\lambda \in \text{supp} \sigma} \frac{\lambda^2 + 1}{(\lambda - x)^2 + y^2} \right]^{-1}
\]
For a large class of coefficients $A(r)$, $N_r$ is bounded in $r$. This is because $2N_r = \text{Tr} \, \text{Im} \, G_1(r, r)$ where $G$ is the resolvent kernel for corresponding Dirac operator which will be introduced later. For example, $A \in L^\infty(\mathbb{R}^+)$ is sufficient for $N_r$ to be bounded in $r > 0$.

The next theorem yields yet another result on the distribution of $\lambda_n$.

Theorem 10.2. If $z_1$ is a zero of $P(r, \lambda)$, then there is no any other zero of $P(r, \lambda)$ in $\Omega_1$
\[
\Omega_1 : \lambda \in \mathbb{C}^+, |\lambda - z_1| < \text{Dist}(\lambda, \text{supp} \sigma)
\]
Proof. Assume $z_1$ is a zero of $P(r, \lambda)$. By the variational principle, function $f_0(\lambda) = K_r(z_1, \lambda)/K_r(z_1, z_1)$ minimizes $\|f(\lambda)\|_{2, \sigma}$ in the set of all $f(\lambda) \in S_r, |f(z_1)| = 1$. Since $P(r, z_1) = 0$,
\[
f_0(\lambda) = \frac{(z_1 - \bar{z}_1)P_*(r, \lambda)}{(\lambda - \bar{z}_1)P_*(r, z_1)}
\]
In the meantime, if $P(r, \lambda)$ has a zero $z_2 \in \Omega_1$, then $P_*(r, \bar{z}_2) = 0$ and the function
\[
f_1(\lambda) = \frac{z_1 - \bar{z}_2}{\lambda - \bar{z}_2} f_0(\lambda)
\]
belongs to $S_r, f_1(z_1) = 1$, but $\|f_1\|_{2, \sigma} < \|f_0\|_{2, \sigma}$, a contradiction. \qed
Notice that this Theorem makes no assumptions on coefficient $A(r)$. It also implies that the isosceles triangle with base $(a, b)$ and angles $\pi/6, \pi/6, 2\pi/3$ can contain only finite number of zeroes.

The next result is the continuous analog of the Widom’s theorem on the zeroes of OPUC. It says that the zeroes of $P(r, \lambda)$ can not accumulate in the compact of $\mathbb{C}^+$ provided that the support of $d\sigma$ is not the whole $\mathbb{R}$. The proof is a rather simple modification of proof for the discrete case.

For any compact $K \subset \mathbb{C}^+$, define $N_K(r)$ as the number of zeroes of $P(r, \lambda)$ in $K$. Fix any $R > 0$. We have elementary estimates

$$\max_{\lambda \in K, 0 \leq r \leq R} |P(r, \lambda)| < C(A, R, K), \quad \min_{0 \leq r \leq R} |P(r, iy)| > 1/2$$

if $y$ is large enough. Therefore, by Jensen’s formula ([60], Theorem 15.18), we know that $N_K(r)$ is bounded for $r \in [0, R]$.

**Theorem 10.3.** (Continuous analog of Widom’s theorem). Assume that the measure $d\sigma$ of the Krein system is such that $\text{supp}(d\sigma) \neq \mathbb{R}$. Then, we have

$$\sup_{r > 0} N_K(r) < \infty \quad (190)$$

**Proof.** Fix $K$ and Krein system with the measure $d\sigma$. Cover $K$ by disjoint cubes $C_j$ with side $\varepsilon$. We choose $\varepsilon$ small enough to satisfy the following conditions. For any cube $C_j$, consider $\xi \in C_j$ and a map $\phi_\xi(\lambda) = (\lambda - \xi)^{-1}$. The reflected cube $C_j^* = \{ \bar{z}, z \in C_j \}$ will be mapped to a set $D_j, \xi$ and the support of the measure $\text{supp}(d\sigma)$ to a set $F_j, \xi$, a proper subset of some circle. For each $j$, consider

$$D_j = \cup_{\xi \in C_j} D_j, \xi, F_j = \cup_{\xi \in C_j} F_j, \xi \quad (191)$$

We now require $\varepsilon$ to be so small that for each $j$ we have: $D_j$ and $F_j$ are disjoint, $\mathbb{C} \setminus F_j$ is connected. We can always satisfy these conditions because the function $\phi_\xi(\lambda)$ is jointly continuous and $\text{supp}(d\sigma)$ has a gap in it.

Fix this $\varepsilon$. Assume (190) is wrong. Clearly, among all cubes $C_j$ there will be at least one, call it $C_j^*$, such that for any $k$ we can find $r$ so that $P(r, \lambda)$ has $n$ zeroes in $C_j^*$ and $n > k$. Denote these zeroes by $\lambda_j, j = 1, \ldots, n$. Fix this cube. Let $D_j$ and $F_j$ be the corresponding sets defined by (191). Let $m < n$ be some fixed number to be specified later.

By the variational principle, the function

$$f_0(\lambda) = \frac{K_r(\lambda_1, \lambda)}{K_r(\lambda_1, \lambda_1)} = \frac{\lambda_1 - \lambda_1}{\lambda_1 - \lambda_1} \frac{P_r(r, \lambda)}{P_r(r, \lambda_1)}$$

minimizes $\|f\|_{2,\sigma}$ in the set $f \in S_r, |f(\lambda_1)| = 1$. We can write

$$f_0(\lambda) = g(\lambda)(\lambda - \lambda_2) \ldots (\lambda - \lambda_{m+1})$$

Notice that $g(\lambda_1) = 1$. We will find a polynomial $Q(\lambda)$ satisfying the following properties: $\text{deg} Q \leq m, Q(\lambda_1) = 1$, and

$$|Q(\lambda)| \leq 2^{-1} \left| \frac{\lambda_1 - \lambda_2 \ldots (\lambda - \lambda_{m+1})}{(\lambda_1 - \lambda_2) \ldots (\lambda_1 - \lambda_{m+1})} \right| \quad (192)$$
for any $\lambda \in \text{supp}(d\sigma)$. That would give us a contradiction since for the function $f(\lambda) = g(\lambda)Q(\lambda)$ we have: $f(\lambda_1) = 1,$

$$\int_{\mathbb{R}} |f(\lambda)|^2 d\sigma \leq 2^{-2} \int_{\mathbb{R}} |f_0(\lambda)|^2 d\sigma$$

and

$$f(\lambda) = f_0(\lambda)Q(\lambda) \left[ \frac{(\lambda - \bar{\lambda}_2) \ldots (\lambda - \bar{\lambda}_{m+1})}{(\lambda_1 - \bar{\lambda}_2) \ldots (\lambda_1 - \lambda_{m+1})} \right]^{-1} \in S_r$$

by Paley-Wiener Theorem. Thus we have a contradiction with the variational principle.

To find $Q(\lambda)$, we first take a map $z = (\lambda - \lambda_1)^{-1}$. It sends $\lambda_1$ to infinity, the support of $d\sigma$ will be mapped to $F_{j',\lambda_1}$, and the set $\tilde{C}_{j'}$ (reflection of $C_{j'}$ with respect to $\mathbb{R}$) will go to a compact $D_{j',\lambda_1}$.

We now use Widom’s Lemma (see Appendix, Lemma 17.7) for two compacts $D_{j'}$ and $F_{j'}$. They are disjoint and $\mathbb{C} \setminus F_{j'}$ is connected so the Lemma is applicable and the number $m$ can be chosen so that for any points $z_j \in D_{j}, j = 1, \ldots, m$, we can find a monic polynomial $\tilde{Q}(z)$ of degree $m$ such that:

$$\left| \frac{\tilde{Q}(z)}{(z - z_1) \ldots (z - z_m)} \right| \leq \frac{1}{2} \quad (193)$$

for all $z \in F$. In particular, for the points $z_j = (\bar{\lambda}_{j+1} - \lambda_1)^{-1} \in D_{j',\lambda_1} \subseteq D, j = 1, \ldots, m$ there is $Q(z)$ so that we have (193) for any $z \in F_{j',\lambda_1} \subseteq F$. Translating it back to the $\lambda$ variable, we have (192) where $Q(\lambda) = (\lambda - \lambda_1)^m \tilde{Q}((\lambda - \lambda_1)^{-1})$. Clearly, $\deg Q \leq m, Q(\lambda_1) = 1$.

**Remarks and historical notes.** The asymptotics of zeroes for the exponential functions of the special type (e.g., $P(r, \lambda)$) is a classical question. The problem here, of course, is how this asymptotics depends on the regularity of the function in representation (function $\Gamma_r(0, t)$ in our case). We do not consider this problem here, interested reader can check [34] for related results.

The results from this section are new. We addressed only some of the basic questions about the distribution of zeroes. Clearly, there are many questions left open.
11. The case $A(r) \in L^2(\mathbb{R}^+)$

Now, let us study an important class of Krein systems: $A(r) \in L^2(\mathbb{R}^+)$. 

**Theorem 11.1.** If $A(r) \in L^2[0, \infty)$, then $d\sigma \in (\text{Szegő})$. Moreover,

$$P_\star(r, \lambda) \to \Pi_\alpha(\lambda), \, \tilde{P}_\star(r, \lambda) \to \tilde{\Pi}_\beta(\lambda), \, \mathfrak{A}(r, \lambda) \to \mathfrak{A}(\lambda), \, \mathfrak{B}(r, \lambda) \to \mathfrak{B}(\lambda)$$

as $r \to \infty$ uniformly in $\text{Im} \lambda > \varepsilon, \varepsilon > 0$. Function $\mathfrak{B}(\lambda) \in N(\mathbb{C}^+)$, $f(\lambda) = \mathfrak{B}(\lambda) \mathfrak{A}^{-1}(\lambda)$ is an outer function from $B(\mathbb{C}^+)$,

$$\int_{-\infty}^{\infty} \ln |\mathfrak{A}(\lambda)| \, d\lambda = \pi \int_{0}^{\infty} |A(r)|^2 \, dr$$  \hfill (194)

**Proof.** From (82), we have

$$P_\star(r, \lambda) = 1 - \int_{0}^{r} \exp(i\lambda s)A(s) \, ds + \int_{0}^{r} \exp(i\lambda s)A(s) \int_{0}^{s} \exp(-i\lambda t)\overline{A(t)} P_\star(t, \lambda) \, dt \, ds$$

or

$$P_\star(r, \lambda) = 1 - \int_{0}^{r} \exp(i\lambda s)A(s) \, ds$$

$$+ \int_{0}^{r} P_\star(t, \lambda) \left[ \exp(-i\lambda t)\overline{A(t)} \int_{t}^{r} \exp(i\lambda s)A(s) \, ds \right] \, dt$$  \hfill (196)

Use Cauchy-Schwarz and Young inequalities to get

$$\left\| \exp(-i\lambda t)\overline{A(t)} \int_{t}^{r} \exp(i\lambda s)A(s) \, ds \right\|_{L^1[0, \infty)} \leq \frac{|A|_2^2}{\text{Im} \lambda}, \, \lambda \in \mathbb{C}^+$$

From Gronwall-Belmann inequality,

$$|P_\star(r, \lambda)| \leq C(\lambda) \exp \left( \text{Im} \lambda \right)^{-1}|A|_2^2, \, \lambda \in \mathbb{C}^+, \, C(\lambda) \leq 1 + |A|_2|\text{Im} \lambda|^{-1/2}$$

Recall the function (146) and Lemma 8.5. From (196), we have $P_\star(r, \lambda) \to \Pi_\alpha(\lambda)$ as $r \to \infty, \lambda \in \mathbb{C}^+$. Similarly, $\tilde{P}_\star(r, \lambda) \to \tilde{\Pi}_\beta(\lambda)$. Thus, $\mathfrak{A}(r, \lambda) \to \mathfrak{A}(\lambda), \, \mathfrak{B}(r, \lambda) \to \mathfrak{B}(\lambda)$ as $r \to \infty$. Moreover, from Lemma 6.1 $|\mathfrak{A}(\lambda)|^2 \geq 1 + |\mathfrak{B}(\lambda)|^2$, $\text{Im} \lambda > 0$. Consequently, $\mathfrak{A}^{-1}(\lambda) \in B(\mathbb{C}^+), \, \mathfrak{A}(\lambda) \in N(\mathbb{C}^+), \, \mathfrak{B}(\lambda) \in N(\mathbb{C}^+)$. We also have

$$\mathfrak{A}(\lambda) = (\Pi_\alpha(\lambda) + \tilde{\Pi}_\beta(\lambda))/2 = \frac{\Pi_\alpha(\lambda)}{2} \left[ 1 + \frac{\tilde{\Pi}_\beta(\lambda)}{\Pi_\alpha(\lambda)} \right]$$

Clearly, $\Pi_\alpha(\lambda)$ and $\tilde{\Pi}_\beta(\lambda)$ are outer from $N(\mathbb{C}^+)$. Due to Theorem 6.2,

$$F(\lambda) = \frac{\tilde{\Pi}_\beta(\lambda)}{\Pi_\alpha(\lambda)}$$

Since $F(\lambda)$ has positive real part, the function $1 + \tilde{\Pi}_\beta(\lambda)\Pi_\alpha^{-1}(\lambda)$ is outer from $N(\mathbb{C}^+)$. Consequently, $\mathfrak{A}(\lambda)$ is outer from $N(\mathbb{C}^+)$ and $\mathfrak{A}^{-1}(\lambda)$ is outer from $B(\mathbb{C}^+)$. 

Thus,
\[ A^{-1}(\lambda) = \exp \left[ i\gamma - \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(1 + s\lambda) \ln |A(s)|}{(s - \lambda)(1 + s^2)} ds \right], \gamma \in [0, 2\pi), \quad (197) \]

For \( A(r, \lambda) \), we have
\[ A(r, \lambda) = 1 + \int_0^r A(t, \lambda) \left[ \exp(-i\lambda t) \frac{\hat{A}(t)}{t} \int_t^r \exp(i\lambda s) A(s) ds \right] ds \quad (198) \]

Iterating this identity, estimating the lower order terms, and taking \( r \to \infty \), one has
\[ A(iy) = 1 + \int_0^\infty \left[ \exp(yt) \hat{A}(t) \int_t^\infty \exp(-ys) A(s) ds \right] + O(y^{-2}), y \to +\infty \]

Clearly,
\[ \int_0^\infty \left[ \exp(yt) \hat{A}(t) \int_t^\infty \exp(-ys) A(s) ds \right] = \int_{-\infty}^{\infty} \frac{\hat{A}(\omega)^2}{y - i\omega} d\omega \]

where \( \hat{A}(\omega) \) is the Fourier transform of the function \( A(t) \cdot \chi_{\mathbb{R}^+}(t) \) and \( \hat{A}(\omega)^2 \in L^1(\mathbb{R}) \).

Thus,
\[ A(iy) = 1 + y^{-1} \int_0^\infty |A(s)|^2 ds + o(y^{-1}), y \to +\infty \quad (199) \]

From (197), we have
\[ |A(iy)| = \exp \left[ \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\ln |A(\lambda)|}{\lambda^2 + y^2} d\lambda \right] \quad (200) \]

Since \( |A(\lambda)| \geq 1 \) for a.e. \( \lambda \in \mathbb{R} \), relations (199) and (200) imply \( \ln |A(\lambda)| \in L^1(\mathbb{R}) \) and (194).

**Corollary 11.1.** For a.e. \( \lambda \in \mathbb{R} \), we have
\[ |A(\lambda)|^2 = 1 + |B(\lambda)|^2 \quad (201) \]

**Proof.** The equation is equivalent to
\[ \frac{|\Pi_\alpha(\lambda) + \hat{\Pi}_\beta(\lambda)|^2}{4} = \frac{|\Pi_\alpha(\lambda) - \hat{\Pi}_\beta(\lambda)|^2}{4} + 1 \]

or
\[ \text{Re} F(\lambda) = \frac{1}{|\Pi_\alpha(\lambda)|^2} = 2\pi \sigma'(\lambda) \]

and the last identity is elementary and follows, e.g., from the integral representations for both functions \( F(\lambda) \) and \( \Pi_\alpha(\lambda) \). \( \Box \)

**Corollary 11.2.** If \( A(r) \in L^2(\mathbb{R}^+) \), then
\[ 2\pi \int_0^\infty |A(r)|^2 dr = - \int_{-\infty}^{\infty} \ln[1 - |f(\lambda)|^2] d\lambda \quad (202) \]
Proof. From (201), we have $1 - |f(\lambda)|^2 = |\mathfrak{A}(\lambda)|^{-2}$ for a.e. $\lambda \in \mathbb{R}$. Now, (202) follows from (194).

Notice that (202) implies
\[ f(\lambda) \in L^2(\mathbb{R}) \]  
(203)
So, $f(\lambda) \in H^2(\mathbb{C}^+) \cap B(\mathbb{C}^+)$ and (compare with (165)):
\[ f(\lambda) = \int_0^\infty C(x) \exp(i\lambda x) dx, \lambda \in \mathbb{C}^+ \]  
(204)

Corollary 11.3. If $A(r) \in L^2(\mathbb{R}^+)$, then $\ln[2\pi\sigma'(\lambda)] \in L^1(\mathbb{R}) + L^2(\mathbb{R})$.

Proof. We have for a.e. $\lambda \in \mathbb{R}$
\[ |2\pi\sigma'(\lambda)|^{-1} = |\Pi_\alpha(\lambda)|^2 = |\mathfrak{A}(\lambda) + \mathfrak{B}(\lambda)|^2 = |\mathfrak{A}(\lambda)|^2 \cdot |1 + f(\lambda)|^2 \]
and therefore
\[ -\ln[2\pi\sigma'(\lambda)] = 2 \ln |\mathfrak{A}(\lambda)| + 2 \ln |1 + f(\lambda)| \]
By (194), the first term is from $L^1(\mathbb{R})$. As about the second term,
\[ \ln |1 + f(\lambda)| \leq \ln[1 + |f(\lambda)|] \leq |f(\lambda)| \in L^2(\mathbb{R}) \]
by (203). For the negative part of the logarithm, we use an elementary estimate $|1 + f| \geq 1 - |f|$ which yields (check the definition of $\ln^-$)
\[ \ln^- |1 + f(\lambda)| \leq \ln(1 - |f(\lambda)|) = -\ln(1 - |f(\lambda)|^2) + \ln(1 + |f(\lambda)|) \]
for $|1 + f(\lambda)| < 1$. The second term is again in $L^2(\mathbb{R})$. The first one is in $L^1(\mathbb{R})$ due to Corollary 11.2.

Corollary 11.4. For the constants $\alpha$ and $\gamma$ from the multiplicative representations of $\Pi_\alpha(\lambda)$ (formula (148)) and $\mathfrak{A}(\lambda)$ (formula (197)) we have
\[ \alpha = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{s \ln[2\pi\sigma'(s)]}{1 + s^2} ds, \quad \gamma = \frac{1}{\pi} \int_{-\infty}^\infty \frac{s \ln |\mathfrak{A}(s)|}{1 + s^2} ds \]

Proof. From (197) and (199), we have
\[ \gamma = \frac{1}{\pi} \lim_{y \to -\infty} \int_{-\infty}^\infty \frac{s(y^2 - 1) \ln |\mathfrak{A}(s)|}{(s^2 + y^2)(1 + s^2)} ds = \frac{1}{\pi} \int_{-\infty}^\infty \frac{s \ln |\mathfrak{A}(s)|}{1 + s^2} ds \]
because $\ln |\mathfrak{A}(s)| \in L^1(\mathbb{R})$.

Take $y \to -\infty$ in (196). One has
\[ \Pi_\alpha(\lambda) = 1 + O(|\text{Im } \lambda|^{1/2}) \]
as $\text{Im } \lambda \to +\infty$. Then, from (148), we have
\[ \alpha = \lim_{y \to -\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{s(1 - y^2) + iy(1 + s^2)}{(s^2 + y^2)(1 + s^2)} \ln[2\pi\sigma'(s)] ds \]
Using Corollary 11.3 we get the needed formula for $\alpha$. \qed
Clearly, the Corollary implies the following integral representations

\[
\mathfrak{R}(\lambda) = \exp \left[ \frac{1}{\pi i} \int_{-\infty}^{\infty} \ln |\mathfrak{R}(s)| \frac{ds}{s - \lambda} \right], \quad (205)
\]

\[
\Pi_\alpha(\lambda) = \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \ln[2\pi \sigma'(s)] \frac{ds}{s - \lambda} \right] \quad (206)
\]

As we know from the discussion of the general Szegő case, the function \((\lambda + i)^{-1}\Pi_\alpha^{-1}(\lambda) \in H^2(\mathbb{C}^+)\). For our situation, much more is true

**Theorem 11.2.** If \(A(r) \in L^2(\mathbb{R}^+)\), then

\[
\Pi_\alpha^{-1}(\lambda) = 1 + \int_{0}^{\infty} \gamma(x)e^{ix\lambda}dx = 1 + \hat{\gamma}(\lambda), \hat{\gamma}(\lambda) \in H^2(\mathbb{C}^+) \quad (207)
\]

where

\[
\sigma(E) + \int_{0}^{\infty} |\gamma(x)|^2 dx = \int_{0}^{\infty} |A(r)|^2 dr \quad (208)
\]

where \(E\) – support of \(d\sigma\), the singular component of \(d\sigma\).

**Proof.** From Lemma 8.9, as \(r \to \infty\):

\[
P_*(r, \lambda) \to \frac{\Pi_\alpha(\lambda) \cdot \chi_{E^c}(\lambda)}{\lambda + i}
\]

in \(L^2(\mathbb{R}, d\sigma)\), where \(E^c\) – the complement of \(E\). On the other hand, for any \(r > 0, \lambda \in \mathbb{R}\), we have (by (57)):

\[
P_*(r, \lambda) = 1 - \int_{0}^{r} A(t)P(t, \lambda)dt \quad (209)
\]

That implies

\[
\frac{\Pi_\alpha(\lambda) \cdot \chi_{E^c}(\lambda)}{\lambda + i} = \frac{1}{\lambda + i} - \hat{A}(\lambda)
\]

where the generalized Fourier transform

\[
\hat{A}(\lambda) = \int_{0}^{\infty} A(r)P(r, \lambda)dr \in L^2(\mathbb{R}, d\sigma)
\]

by Theorem 3.2. Thus,

\[
\hat{A}(\lambda) = 1 - \Pi_\alpha(\lambda) \cdot \chi_{E^c}(\lambda) \quad (210)
\]

and

\[
\int_{0}^{\infty} |A(r)|^2 dr = \sigma\{E\} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \frac{1}{\Pi_\alpha(\lambda)} \right|^2 d\lambda
\]

Now, we have

\[
\int_{-\infty}^{\infty} \left| 1 - \frac{1}{\Pi_\alpha(\lambda)} \right|^2 d\lambda < \infty, \quad \frac{1}{\lambda + i} \cdot \left( 1 - \frac{1}{\Pi_\alpha(\lambda)} \right) \in H^2(\mathbb{C}^+)
\]
Therefore, by elementary Lemma 17.9 in Appendix, \( 1 - \Pi^{-1}_\alpha(\lambda) \in H^2(\mathbb{C}^+) \). The Paley-Wiener Theorem now implies (207) and (208). □

An interesting corollary from this result is that the total variation over the whole line of the singular part of the measure \( d\sigma \) is finite.

After proving the representation for \( \Pi_\alpha(\lambda) \), the following result is quite natural. By Levy-Wiener Theorem,

\[
\frac{1}{P_\ast(r, \lambda)} = 1 + \int_0^\infty \gamma_r(x)e^{i\lambda x} \, dx
\]

where \( \gamma_r(x) \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \).

**Lemma 11.1.** Assume \( A(r) \in L^2(\mathbb{R}^+) \) and \( E = \emptyset \). Then, \( \gamma_r(x) \to \gamma(x) \) in \( L^2(\mathbb{R}^+) \).

**Proof.** From (209) and Corollary 6.1 we get

\[
\int_0^r |A(x)|^2 \, dx = \int_{-\infty}^\infty |1 - P_\ast(r, \lambda)|^2 d\sigma(\lambda) = \frac{1}{2\pi} \int_{-\infty}^\infty |1 - P_\ast(r, \lambda)|^2 \frac{d\lambda}{|P_\ast(r, \lambda)|^2} \quad (211)
\]

Then, since \( P_\ast(r, \lambda) \to \Pi_\alpha(\lambda) \) for \( \lambda \in \mathbb{C}^+ \), we also have that \( \gamma_r(x) \to \gamma(x) \) weakly in \( L^2(\mathbb{R}^+) \). But since \( E = \emptyset \), we also get \( \| \gamma_r(x) \|_2 \to \| \gamma(x) \|_2 \) from (208). Therefore, \( \gamma_r(x) \to \gamma(x) \) in \( L^2(\mathbb{R}^+) \). □

If the singular part of the measure \( d\sigma \) is not trivial, then we have only the bound:

\[ \| \gamma_r \|_2 \leq \| A \|_2. \]

Now, let us characterize the class of Schur coefficients corresponding to \( A(r) \in L^2(\mathbb{R}^+) \). We start with

**Theorem 11.3.** For any \( A(r) \in L^2[0, R] \) and any \( R > 0 \), we have

\[
\int_{-\infty}^\infty \ln |A(R, \lambda)| \, d\lambda = \pi \int_0^R |A(r)|^2 \, dr \quad (212)
\]

and

\[
2\pi \int_0^R |A(r)|^2 \, dr = -\int_{-\infty}^\infty \ln(1 - |\mathfrak{B}(R, \lambda) \mathfrak{A}^{-1}(R, \lambda)|^2) \, d\lambda \quad (213)
\]

**Proof.** The proof repeats those of Theorem 11.1 and Corollary 11.2 □

Let us introduce a certain subclass of \( B(\mathbb{C}^+) \). Consider a function \( f(\lambda) \in B(\mathbb{C}^+) \) such that for its boundary value:

\[
\int_{-\infty}^\infty \ln(1 - |f(\lambda)|^2) \, d\lambda > -\infty \quad (214)
\]
Then, a simple estimate \( \ln(1 - |f(\lambda)|^2) \leq -|f(\lambda)|^2 \) yields that \( f(\lambda) \in L^2(\mathbb{R}) \). Thus \( f \in H^2(\mathbb{C}^+) \) and

\[
f(\lambda) = \int_0^{\infty} C(x)e^{i\lambda x} \, dx
\]

with \( C(x) \in L^2(\mathbb{R}^+) \).

**Definition 11.1.** We say that \( f(\lambda) \in S_0(\mathbb{C}^+) \) if \( f(\lambda) \in B(\mathbb{C}^+) \) and \( \rho_s(f, 0) = 0 \) holds.

The set \( S_0(\mathbb{C}^+) \) can be regarded as the metric space \([71]\) with the distance given by the formula

\[
\rho_s^2(f, g) = -\int_{-\infty}^{\infty} \ln \left[ 1 - \rho_s^2(f(\lambda), g(\lambda)) \right] \, d\lambda
\]

where pseudo-hyperbolic distance \( \rho_s(\cdot, \cdot) \) is defined by \([159]\). It turns out that the resulting metric space is complete (see Lemma 1.5, Theorem 1.6, Corollary 1.9 in \([71]\). The geometry of this space is studied in the same paper).

Let us write \( \rho_s(f) = \rho_s(0, f) \) for short-hand.

**Lemma 11.2.** If in Krein system \( f_*(\lambda) = S_0(\mathbb{C}^+) \) for some \( r \), then \( f(\lambda) \in S_0(\mathbb{C}^+) \) and

\[
\rho_s^2(f(\lambda)) = \rho_s^2(\mathfrak{B}(r, \lambda) \mathfrak{A}^{-1}(r, \lambda)) + \rho_s^2(f_r(\lambda))
\]

In particular, this is true for any \( A(\lambda) \in L^2(\mathbb{R}^+) \).

**Proof.** Let us prove this lemma using certain “orthogonality” argument. For \( \lambda \in \mathbb{R} \), we have

\[
\mathfrak{A}_*(r, \lambda) = e^{i\lambda r} \mathfrak{A}(r, \lambda), \quad \mathfrak{B}_*(r, \lambda) = e^{i\lambda r} \overline{\mathfrak{B}(r, \lambda)}
\]

Therefore, \([117]\) yields

\[
|f(\lambda)|^2 = \left| \frac{\mathfrak{B}(r, \lambda) \mathfrak{A}^{-1}(r, \lambda) + \exp(i\lambda r) f_r(\lambda)}{1 + \mathfrak{B}(r, \lambda) \mathfrak{A}^{-1}(r, \lambda) \exp(i\lambda r) f_r(\lambda)} \right|^2, \quad \lambda \in \mathbb{R}
\]

The formula

\[
\ln \left( 1 - \frac{z + w}{1 + \bar{z}w} \right)^2 = \ln(1 - |z|^2) + \ln(1 - |w|^2) - 2 \ln |1 + \bar{z}w|
\]

and Theorem \([113]\) give

\[
\rho_s^2(f(\lambda)) = \rho_s^2(\mathfrak{B}(r, \lambda) \mathfrak{A}^{-1}(r, \lambda)) + \rho_s^2(f_r(\lambda))
\]

\[
+ 2 \int_{-\infty}^{\infty} \ln |1 + \mathfrak{B}(r, \lambda) \mathfrak{A}^{-1}(r, \lambda) \exp(i\lambda r) f_r(\lambda)| \, d\lambda
\]

\[
= \rho_s^2(\mathfrak{B}(r, \lambda) \mathfrak{A}^{-1}(r, \lambda)) + \rho_s^2(f_r(\lambda)) + 2 \int_{-\infty}^{\infty} \ln |1 + \mathfrak{B}_*(r, \lambda) \mathfrak{A}^{-1}(r, \lambda) f_r(\lambda)| \, d\lambda
\]

The last integral is zero because

\[
\mathfrak{B}_*(r, \lambda) = \tilde{\delta}(1) \quad \text{as} \quad \lambda \in \mathbb{C}^+, \quad |\lambda| \to \infty, \quad \mathfrak{B}_*(r, iy) = \tilde{\delta}(y^{-1/2}) \quad \text{as} \quad y \to +\infty
\]

\[
f_r(\lambda) \in H^2(\mathbb{C}^+), \quad \text{so} \quad f_r(iy) = \tilde{\delta}(y^{-1/2})
\]
functions \( \mathfrak{B}_s(r, \lambda), f_r(\lambda) \in L^2(\mathbb{R}) \) and the mean-value formula (Lemma 17.8 Appendix) is applicable.

The result above is sometimes called “the layer stripping”. That is because \( \rho^2_s(f) \) is equal to the sum of the terms that correspond to different intervals of the coordinate \( r \). The formula \( \mathfrak{B}_s \) is called the non-linear Plancherel Theorem. The both results are well-known in the theory of orthogonal polynomials.

It is an important observation that any function from \( S_0(\mathbb{C}^+) \) gives rise to a certain Krein system. Indeed, we can show \( S_0(\mathbb{C}^+) \subset S(\mathbb{C}^+) \), where the class \( S(\mathbb{C}^+) \) was introduced in the Definition 9.1 and the Theorem 9.1 applies. To prove this inclusion, notice that \( C(x) \) generates operator \( \mathcal{E}_R \) (see (164)) for any \( R > 0 \). By a standard argument, \( \| \mathcal{E}_R \|_{L^2[0, R]} \leq 1 \) for all \( 0 < R < \infty \). Then, for any \( R > 0 \),

\[
f(\lambda) = \int_0^R C(x)e^{i\lambda x}dx + e^{i\lambda R}\Phi_R(\lambda), \lambda \in \mathbb{C}^+
\]

with

\[
\Phi_R(\lambda) = \int_0^\infty C(x + R)e^{i\lambda x}dx
\]

Since \( \Phi_R(\lambda) \in H^2(\mathbb{C}^+) \) by Paley-Wiener Theorem and both

\[
f(\lambda), \int_0^R C(x)e^{i\lambda x}dx \in H^\infty(\mathbb{C}^+)
\]

we get \( \Phi_R(\lambda) \in L^\infty(\mathbb{R}) \). Therefore, \( \Phi_R(\lambda) \in H^\infty(\mathbb{C}^+) \) and \( f(\lambda) \in S(\mathbb{C}^+) \).

In [71], the analog of the following Theorem was established.

**Theorem 11.4.** For Krein system, \( A(r) \in L^2(\mathbb{R}^+) \) iff the corresponding Schur function \( f(\lambda) \in S_0(\mathbb{C}^+) \).

**Proof.** If \( A(r) \in L^2(\mathbb{R}^+) \), then \( f(\lambda) \in S_0(\mathbb{C}^+) \) follows from the Corollary 11.2. Assume now that \( f(\lambda) \in S_0(\mathbb{C}^+) \). The observation made right before the Theorem says there is the corresponding Krein system with \( A(r) \in L^2_{loc}[0, \infty) \). Let us show that actually \( A(r) \in L^2(\mathbb{R}^+) \). Indeed, fix any \( R > 0 \). Then,

\[
f_R(\lambda) = \frac{f(\lambda) \mathfrak{A}(R, \lambda) - \mathfrak{B}(R, \lambda)}{\mathfrak{A}_s(R, \lambda) - f(\lambda) \mathfrak{B}_s(R, \lambda)}
\]

(217)

and

\[
|f_R(\lambda)| = \left| \frac{f(\lambda) - \mathfrak{B}(R, \lambda) \mathfrak{A}^{-1}(R, \lambda)}{1 - f(\lambda) \mathfrak{B}(R, \lambda) \mathfrak{A}^{-1}(R, \lambda)} \right|, \lambda \in \mathbb{R}
\]

Therefore,

\[
\rho_s(f_R(\lambda)) = \rho_s(f(\lambda), \mathfrak{B}(R, \lambda) \mathfrak{A}^{-1}(R, \lambda)) < \infty
\]

and \( f_R(\lambda) \in S_0(\mathbb{C}^+) \). Now, the Lemma 11.2 is applicable together with Theorem 11.3.

\[
2\pi \int_0^R |A(r)|^2dr = \rho^2_s(\mathfrak{B}(R, \lambda) \mathfrak{A}^{-1}(R, \lambda)) \leq \rho^2_s(f) < C
\]

(218)

uniformly in \( R \) which means \( A(r) \in L^2(\mathbb{R}^+) \). \( \square \)

This result has an interesting corollary.
Corollary 11.5.  
(i) If $f(\lambda)$ is the Schur function corresponding to $A_i(r) \in L^2(\mathbb{R}^+)$ and $g(\lambda) \in B(\mathbb{C}^+)$ then $g \mathcal{F}$ generates the Krein system with $A_{g \mathcal{F}} \in L^2(\mathbb{R}^+)$ and $\|A_{g \mathcal{F}}\|_2 \leq \|A_i\|_2$.

(ii) The set of measures $d\sigma$ that correspond to $A_{d\sigma}(r) \in L^2(\mathbb{R}^+)$ is convex.

Proof. The first statement is obvious due to Theorem 11.4. The second one follows from the following calculations.

$$f = \frac{1 - F}{1 + F}, 1 - |f|^2 = \frac{4 \operatorname{Re} F}{|1 + F|^2}$$

If $F_j$ and $f_j$ correspond to $d\sigma_j, j = 0, 1$ and $F_t$ and $f_t$ to $td\sigma_1 + (1 - t)d\sigma_0, t \in [0, 1]$, then

$$1 - |f_t|^2 = \frac{4(t \operatorname{Re} F_1 + (1 - t) \operatorname{Re} F_0)}{|1 + tF_1 + (1 - t)F_0|^2} = (1 - |f_1|^2) \frac{t|1 + F_t|^2}{|1 + tF_1 + (1 - t)F_0|^2}$$

$$+ (1 - |f_0|^2) \frac{(1 - t)|1 + F_0|^2}{|1 + tF_1 + (1 - t)F_0|^2} \geq \min_{j=0,1} (1 - |f_j|^2)$$

due to convexity of $|z|^2, z \in \mathbb{C}$. Therefore

$$\int_{-\infty}^{\infty} \ln(1 - |f_1(\lambda)|^2)d\lambda \geq \int_{-\infty}^{\infty} \ln(1 - |f_0(\lambda)|^2)d\lambda + \int_{-\infty}^{\infty} \ln(1 - |f_1(\lambda)|^2)d\lambda$$

and $f_t(\lambda) \in S_0(\mathbb{C}^+)$.

Remark. Clearly, the last estimate is not optimal. As about the first statement, notice that multiplication $f_A$ by any inner function does not change $L^2$ norm of the coefficient $A(r)$.

The Theorem 11.2 leads to the following natural question: what can be the singular component of $d\sigma$ if $A(r) \in L^2(\mathbb{R}^+)$? The answer is given by the following result which can be regarded as another criteria for $A(r) \in L^2(\mathbb{R}^+)$. In particular, it says that the singular component can be any singular measure finite over $\mathbb{R}^+$.

Theorem 11.5. Let $d\sigma$ be a nonnegative measure on $\mathbb{R}$ with decomposition $d\sigma = d\sigma_s + \sigma'(\lambda)d\lambda$, where $\sigma_s(\mathbb{R}) < \infty$ and $\ln[2\pi \sigma'(\lambda)] \in L^1(\mathbb{R}) + L^2(\mathbb{R})$. Assume also that

$$\exp\left[\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln[2\pi \sigma'(t)]}{t - \lambda} dt\right] - 1 = \int_{0}^{\infty} \gamma(x) \exp(i \lambda x) dx = \hat{\gamma}(\lambda) \in H^2(\mathbb{C}^+) \quad (219)$$

Then $d\sigma$ generates the Krein system with $A(r) \in L^2(\mathbb{R}^+)$ and (208) holds true.

Proof. We have

$$2\pi \sigma'(\lambda) = 1 + \hat{\gamma} + \tilde{\gamma} + |\hat{\gamma}|^2 \quad (220)$$

Take $H(x)$ Hermitian such that

$$H(x) = (2\pi)^{-1} \left[ \gamma(x) + \int_{0}^{\infty} \gamma(x + u) \gamma(u) du \right] + \int_{\mathbb{R}} \exp(i \lambda x) d\sigma_s(\lambda) \quad (221)$$

for $x > 0$ and define $\beta$ by (28). Then, it is not difficult to check that the formula (27) holds. Moreover, $H(x) \in L^2_{\text{loc}}(\mathbb{R})$ is an accelerant and generates some $A(r) \in L^2_{\text{loc}}(\mathbb{R})$. That follows from (50), (220), and the uniqueness theorem for analytic functions.
Let us show that $A(r) \in L^2(\mathbb{R}^+)$. Indeed, take $d\sigma_n$ such that it has only finite number of jumps and

$$
\int_{\mathbb{R}} \exp(i\lambda x) d\sigma_n(\lambda) \to \int_{\mathbb{R}} \exp(i\lambda x) d\sigma_s(\lambda)
$$

uniformly in $x \in [0, R]$ with any fixed $R > 0$.

Also, take purely a.c. $d\mu_\epsilon$ so that

$$
2\pi \mu'_\epsilon(\lambda) = |1 + \hat{\gamma}_\epsilon(\lambda)|^2, \hat{\gamma}_\epsilon(\lambda) = \hat{\gamma}(\lambda + i\epsilon), \epsilon > 0,
$$

Take $d\sigma_{n,\epsilon} = d\sigma_n + d\mu_\epsilon$. Then, the corresponding accelerants $H_{n,\epsilon}(x) \to H(x)$ and $A_{n,\epsilon}(r) \to A(r)$ in $L^2[0, R]$ for any fixed $R > 0$ as long as $n \to \infty, \epsilon \to 0$.

Let us show that Theorem 11.4 yields $A_{n,\epsilon} \in L^2(\mathbb{R}^+)$. Indeed, fix $n$ and $\epsilon$. Then, we can write $d\sigma_{n,\epsilon}$ as a convex combination

$$
d\sigma_{n,\epsilon} = t \left[ \frac{d\sigma_n}{t} + \frac{d\lambda}{2\pi} \right] + (1-t) \left[ \frac{\mu'_\epsilon(\lambda)}{1-t} - \frac{t}{(1-t)^2} \right] d\lambda
$$

and $t \in (0,1)$. If we can show that both measures in this convex combination generate square summable $A(r)$, then the second claim of the Corollary 11.5 finishes the argument. We will apply Theorem 11.4 to

$$
d\sigma_n = \frac{d\sigma_n}{t} + \frac{d\lambda}{2\pi}
$$

and

$$
\nu(\lambda) d\lambda, \nu(\lambda) = \frac{\mu'_\epsilon(\lambda)}{1-t} - \frac{t}{(1-t)^2}
$$ (222)

The first measure gives rise to

$$
F(\lambda) = 1 - 2it^{-1} \int_{\mathbb{R}} \frac{d\sigma_n(t)}{t - \lambda}
$$

This representation follows from the formula (122) and $F(iy) \to 1$ as $y \to +\infty$. Since

$$
1 - |f|^2 = \frac{4\Re F}{|1 + F|^2}
$$ (223)

and $d\sigma_n$ has only finite number of jumps

$$
1 - |f|^2 = 1 + O(|\lambda|^{-2})
$$

as $|\lambda| \to \infty$. The local singularities of $\ln(1 - |f|^2)$ are integrable and thus we have $\ln(1 - |f|^2) \in L^1(\mathbb{R})$ and the Theorem 11.4 can be applied.

Let us show that Theorem 11.4 can also be applied to the measure (222) as long as $t$ is chosen properly. Indeed, we can always take $t$ so small that the density $\nu(\lambda)$ of this measure is strictly positive. We have

$$
F(\lambda) = 1 + i \int_{\mathbb{R}} \frac{\hat{\gamma}(s) + \hat{\gamma}(s) + \hat{\gamma}(s) \hat{\gamma}(s)}{\pi(1-t)(\lambda - s)} ds
$$

Since $\hat{\gamma}_\epsilon$ is infinitely smooth, $F$ is continuous up to the boundary and $\ln(1 - |f|^2)$ is locally integrable by (223). Therefore, we are left with showing that $|f(\lambda)| \to 0$ if $\lambda \in \mathbb{R}, \lambda \to \infty$ and that $|f(\lambda)| \in L^2(\mathbb{R})$. Since $\hat{\gamma}(s) \in L^2(\mathbb{R})$ and it decays at infinity, the simple properties of Hilbert transform imply $F(\lambda) - 1 \in L^2(\mathbb{R})$ and $F(\lambda) - 1 \to 0$ as $\lambda \to \infty$. Since

$$
f = \frac{1 - F}{1 + F}
$$
we have $\ln(1 - |f|^2) \in L^1(\mathbb{R})$ and Theorem 11.4 applies.

The Krein system with coefficient $A_{n,\epsilon}$ has $\Pi_\epsilon(\lambda)$–function with inverse

$$\Pi_\epsilon^{-1}(\lambda) = 1 + \hat{\gamma}_\epsilon(\lambda)$$

Then, notice that (208) gives a bound $\|A_{n,\epsilon}\|_2 < C$ uniformly in $n$ and $\epsilon$ and $A(r) \in L^2(\mathbb{R}^+)$ because $A_{n,\epsilon}(r) \to A(r)$ in $L^2[0, R]$ with any $R > 0$. We get (208) and the function $\gamma(x)$ coincides with the one introduced in Theorem 11.2. \hfill \square

Later on, we will need the following bound which sharpens the Lemma 8.9.

**Lemma 11.3.** If $A(r) \in L^2(\mathbb{R}^+)$, then

$$\int_{-\infty}^\infty \frac{1}{\lambda^2 + 1} \left| \frac{P_s(r, \lambda)}{\Pi_\alpha(\lambda)} - 1 \right|^2 d\lambda + \int_{-\infty}^\infty \frac{1}{\lambda^2 + 1} |P_s(r, \lambda)|^2 d\sigma_s(\lambda) < C \int_r^\infty |A(s)|^2 ds$$

(224)

**Proof.** First, we notice that

$$\Pi_\alpha(\lambda) - P_s(r, \lambda) = - \int_r^\infty A(s)P(s, \lambda)ds, \lambda \in \mathbb{C}^+$$

Therefore,

$$|\Pi_\alpha(\lambda) - P_s(r, \lambda)|^2 \leq \left( \int_r^\infty |A(s)|^2 ds \right) \cdot \left( \int_r^\infty |P(s, \lambda)|^2 ds \right)$$

$$\leq \frac{1}{2 \Im \lambda} \left( \int_r^\infty |A(s)|^2 ds \right) \cdot \left[ \|\Pi_\alpha(\lambda)\|^2 - |P_s(r, \lambda)|^2 \right]$$

by Cauchy-Schwarz and analog of (59). For the last expression, we use

$$\|\Pi_\alpha\|^2 - |P_s|^2 \leq \|\Pi_\alpha - P_s\| \cdot (\|\Pi_\alpha\| + |P_s|)$$

and then

$$|\Pi_\alpha(\lambda) - P_s(r, \lambda)| \leq C(\lambda) \int_r^\infty |A(s)|^2 ds, \Im \lambda > 0 \quad (225)$$

The last estimate also implies

$$\int_r^\infty |P(s, \lambda)|^2 ds = (2 \Im \lambda)^{-1} \left( \|\Pi_\alpha(\lambda)\|^2 - |P_s(r, \lambda)|^2 \right)$$

$$< C(\lambda)\|\Pi_\alpha(\lambda) - P_s(r, \lambda)\| < C(\lambda) \int_r^\infty |A(s)|^2 ds \quad (226)$$

For $P(r, \lambda)$, we have

$$P^2(r, \lambda) = -2 \int_r^\infty P'(s, \lambda)P(s, \lambda)ds = -2 \int_r^\infty \left[ i\lambda P^2(s, \lambda) - \overline{A(s)P(s, \lambda)P_s(s, \lambda)} \right] ds, \lambda \in \mathbb{C}^+$$
Therefore, for \( \lambda \in \mathbb{C}^+ \)
\[
|P(r, \lambda)|^2 \leq C(\lambda) \int_\mathbb{R} |P(s, \lambda)|^2 \, ds + C(\lambda) \left[ \int_\mathbb{R} |A(s)|^2 \, ds \cdot \int_\mathbb{R} |P(s, \lambda)|^2 \, ds \right]^{1/2}
\leq C(\lambda) \int_\mathbb{R} |A(s)|^2 \, ds
\] (227)
where the last inequality follows from (226).

Now, let us improve estimates from Lemma 8.8. From (149), we have
\[
|P_*(r, i)|^2 \int_{-\infty}^{\infty} \frac{|P(r, \lambda)|^2}{\lambda^2 + 1} \, d\sigma(\lambda) = \left[ \frac{|P_*(r, i)|^2}{2} - |P(r, i)|^2 \right] + |P(r, i)|^2 \int_{-\infty}^{\infty} \frac{|P(r, \lambda)|^2}{\lambda^2 + 1} \, d\sigma(\lambda)
+ 2 \operatorname{Re} \left[ i \int_{-\infty}^{\infty} \frac{P(r, \lambda)P(r, i)}{\lambda + i} K_r(i, \lambda) \, d\sigma(\lambda) \right]
\] (229)
For real \( \lambda \), \( |P(r, \lambda)| = |P_*(r, \lambda)| \). Therefore,
\[
\int_{-\infty}^{\infty} \frac{|P_*(r, \lambda)|^2}{\lambda^2 + 1} \, d\sigma(\lambda) = \frac{1}{2} + 2 \left[ |P_*(r, i)|^2 - |P(r, i)|^2 \right]^{-1} \operatorname{Re} \left[ i \int_{-\infty}^{\infty} \frac{P(r, \lambda)K_r(i, \lambda)}{\lambda + i} \, d\sigma(\lambda) \right]
\] (230)
Using the representation
\[
K_r(i, \lambda) = K_\infty(i, \lambda) - \int_\mathbb{R} P(s, \lambda) \overline{P(s, i)} \, ds
\]
and the property of reproducing kernel, we get
\[
\int_{-\infty}^{\infty} \frac{P(r, \lambda)}{\lambda + i} K_r(i, \lambda) \, d\sigma(\lambda) = \frac{P(r, i)}{2i} - \int_{-\infty}^{\infty} \frac{P(r, \lambda)}{\lambda + i} \int_\mathbb{R} P(s, \lambda) P(s, i) \, ds \, d\sigma(\lambda)
\]
The last integral can be bounded by Cauchy-Schwarz and Theorem 3.2 as follows
\[
\left[ \int_{-\infty}^{\infty} \frac{P(r, \lambda)}{\lambda + i} \int_\mathbb{R} |P(s, \lambda) P(s, i) | \, ds \, d\sigma(\lambda) \right]^{1/2} < C(\lambda) \left[ \int_\mathbb{R} |P(r, i)|^2 \, ds \right]^{1/2} < C(\lambda) \left[ \int_\mathbb{R} |A(s)|^2 \, ds \right]^{1/2}
\]
where we used (226) for last inequality. Estimate (228) and (231) yield
\[
\int_{-\infty}^{\infty} \frac{|P_*(r, \lambda)|^2}{\lambda^2 + 1} \, d\sigma(\lambda) = \frac{1}{2} + O \left[ \int_\mathbb{R} |A(s)|^2 \, ds \right]
\] (231)
Now, repeating the proof of Lemma 8.9 with estimates (225) and (231), we obtain (224).

The estimates in the last Lemma are not sharp but good enough for us.

**Remarks and historical notes.** The results in this section are partially new. For the Helmholtz equation, analog of Lemma 11.2 was obtained in [71], where the nonlinear Fourier transform was introduced. In our case, this transform is
given by the map \( \mathcal{F} : A(r) \in L^2(\mathbb{R}^+) \rightarrow f(\lambda) \in S_0(\mathbb{C}^+) \). In [71], the space \( S_0(\mathbb{C}^+) \) is studied in detail as well as properties of the nonlinear Fourier transform. For example, its homeomorphic property is proved by means of weak convergence argument. See also [73]. For the Schrödinger operators and Jacobi matrices, the analysis is more involved [12]. The recent paper [12], contains analysis of \( L^2(\mathbb{R}) \) potentials for Schrödinger operators. In the OPUC theory, many of these results were well-known for quite a long time. The paper [10] studies the relation between the decay of the tail \( \|A\|_{L^2(\mathbb{R}^+)}^2 \) and the Hausdorff dimension of the support of \( d\sigma_s \).

In conclusion, we want to say that the case of square summable coefficient is studied pretty well by now. Perhaps, the only problem left open is the following nonlinear (non-commutative) analog of the Carleson Theorem in the Fourier analysis. Prove (or disprove) that solution of the ODE:

\[
P'(r, \lambda) = A(r) \exp(i\lambda r) P_s(r, \lambda), P_s(0, \lambda) = 1
\]

has a limit at infinity for a.e. \( \lambda \in \mathbb{R} \). We assume here, of course, that \( A(r) \in L^2(\mathbb{R}^+) \). This is a deep and difficult problem whose analog for OPUC case is also open for quite a long time. We mention the paper [10] for some recent closely related results in this direction.
12. Continuous analog of the Baxter theorem. The case
$A(r) \in L^1(\mathbb{R}^+)$

In this section, we assume that $A(r)$ and $H(x)$ are both from regularity class $L^2_{loc}(\mathbb{R}^+)$. Our goal is to prove the following analog of Baxter’s theorem in the OPUC theory. The proof is an adaptation of the one for the discrete case ([49], Chapter 5).

**Theorem 12.1.** For any Krein system, $A(r) \in L^1(\mathbb{R}^+) \cap L^2_{loc}(\mathbb{R}^+)$ if and only if the accelerant $H(r) \in L^1(\mathbb{R}) \cap L^2_{loc}(\mathbb{R}^+)$ and the Hopf-Wiener operator

$$(I + \mathcal{H}_\infty)f = f(x) + \int_0^\infty H(x - y)f(y)dy$$

is strictly positive on $L^2(\mathbb{R}^+)$. The last condition is equivalent to

$$1 + \rho(\lambda) > 0, \lambda > 0$$

where $\rho(\lambda) \in W(\mathbb{R})$ is the Fourier transform of $\overline{H(x)}$. Moreover, the measure $d\sigma$ is purely absolutely continuous, has continuous derivative and

$$\exp(-2\|A\|_1) \leq 2\pi\sigma'(\lambda) = 1 + \rho(\lambda) \leq \exp(2\|A\|_1)$$  \hspace{1cm} (232)

**Proof.** Assume that we are given $A(r) \in L^1(\mathbb{R}^+)$. We are then in the Szegö case. Indeed, an elementary application of Gronwall-Bellman inequality to ([30]) yields the uniform convergence of $P_\ast(r, \lambda)$ and $\tilde{P}_\ast(r, \lambda)$ to $\Pi_\alpha(\lambda)$ and $\tilde{\Pi}_\alpha(\lambda)$ in $\lambda \in \mathbb{C}^+$. Both $\Pi_\alpha(\lambda)$ and $\tilde{\Pi}_\alpha(\lambda)$ are continuous in $\mathbb{C}^+$. Moreover,

$$|P_\ast(r, \lambda)| \leq \exp(\|A\|_1), |\tilde{P}_\ast(r, \lambda)| \leq \exp(\|A\|_1), \lambda \in \mathbb{C}^+, r > 0$$

$$|P_\ast(r, \lambda)| \geq \exp(-\|A\|_1), |\tilde{P}_\ast(r, \lambda)| \geq \exp(-\|A\|_1), \lambda \in \mathbb{C}^+, r > 0$$

and then

$$\exp(-\|A\|_1) \leq |\Pi_\alpha(\lambda)| \leq \exp(\|A\|_1), \exp(-\|A\|_1) \leq |\tilde{\Pi}_\alpha(\lambda)| \leq \exp(\|A\|_1)$$

The Theorem 6.2 says that measures $(2\pi)^{-1}|P_\ast(r, \lambda)|^{-2}d\lambda$ converge to $d\sigma(\lambda)$ in the weak-$(\ast)$ sense. Therefore, $d\sigma(\lambda)$ is purely a.c. and its continuous density allows an estimate

$$\exp(-2\|A\|_1) \leq 2\pi\sigma'(\lambda) = |\Pi_\alpha(\lambda)|^{-2} \leq \exp(2\|A\|_1)$$  \hspace{1cm} (233)

Now, let us show that $H(x) \in L^1(\mathbb{R}^+)$. Indeed, let

$$y_1(r, \lambda) = P_\ast(r, \lambda) - 1, y_2(r, \lambda) = P(r, \lambda) - \exp(i\lambda r)$$

For each $r > 0$, $y_1(2) \in W_+(\mathbb{C}^+)$ and $\|y_1\|_{W_+} = \|y_2\|_{W_+}$ by ([35] and [37]). From ([37]), we get

$$y_1(r) = a(r) - \int_0^r A(s)y_2(s)ds$$  \hspace{1cm} (234)

where

$$a(r) = -\int_0^r A(s)\exp(is\lambda)ds \in W_+$$
and (234) is considered as an integral equation for functions with values in $W_+$. Taking the norm of the both sides in (234), we get

$$\|y_1(r)\|_{W_+} \leq \int_0^r |A(s)|ds + \int_0^r |A(s)| \cdot \|y_1(s)\|_{W_+} ds$$

The Gronwall-Bellman inequality yields convergence of $y_1(r)$ to some $y_1$ in the $W_+$ norm (as $r \to \infty$) and

$$\|y_1\|_{W_+} \leq \|A\|_1 \exp(\|A\|_1)$$

Therefore, $\Pi_{\alpha}(\lambda) = 1 + y_1$. In the same way, we have $\hat{\Pi}_{\alpha}(\lambda) = 1 + \hat{y}_1$ and $F(\lambda) = \hat{\Pi}_{\alpha}/\Pi_{\alpha} = (1 + \hat{y}_1)/(1 + y_1) = 1 + h$, where

$$h = \frac{\hat{y}_1 - y_1}{1 + y_1} \in W_+ \quad (235)$$

since the spectrum of $1 + y_1$ does not contain zero. Now, (90) implies

$$h = 2 \int_0^\infty \frac{H(x)}{\exp(i\lambda x)} dx \quad (236)$$

and so $H(x) \in L^1(\mathbb{R})$ since $h \in W_+$. We also have $2\pi\sigma'(\lambda) = 1 + \rho(\lambda)$.

Now, assume that we are given $H(x) \in L^1(\mathbb{R})$ and $I + H_\infty > 0$. Then, the equivalence of $I + H_\infty > 0$ and an estimate $1 + \rho(\lambda) > 0$ follows from the simple identity

$$((I + H_\infty)f, f) = 2\pi \int_{-\infty}^{\infty} (1 + \rho(-\lambda))|\hat{f}(\lambda)|^2 d\lambda$$

Last identity shows that $H$ generates a Krein system with $A(r) \in L^2_{\text{loc}}(\mathbb{R}^+)$. Together with formula (50) and a simple approximation argument (like in the proof of Lemma 17.10 in Appendix), it also imply $d\sigma = (2\pi)^{-1}(1 + \rho(\lambda))d\lambda$.

We need a simple

**Lemma 12.1.** If $\|H\|_1 < 1$, then $A(r) \in L^1(\mathbb{R}^+)$ and $\|A\|_1 \leq \|H\|_1/(1 - \|H\|_1)$.

**Proof.** From (34), we have

$$\Gamma_r(t, 0) + \int_0^r H(t - u)\Gamma_r(u, 0) du = H(t) \quad (237)$$

Iterating this identity, we have for

$$\Gamma_r(t, 0) = H(t) - \int_0^r H(t - u_1)H(u_1)du_1 + \int_0^r H(t - u_1) \int_0^r H(u_1 - u_2)H(u_2)du_2 du_1 - \ldots \quad (238)$$

where the series converges in $L^1[0, r]$. Then, $|\Gamma_r(t, 0)| \leq g(t)$, where

$$g = h + h * h + h * h * h + \ldots, h(t) = |H(t)|$$
Notice that \( g(t) \) does not depend on \( r \) and \( \|g\|_1 \leq \|H\|_1/(1-\|H\|_1) \). So, by taking \( t = r \) in (237), we get
\[
A(r) + \int_0^r H(r-u) \Gamma_r(u,0) du = H(r)
\]
(239)

\[
|A(r)| \leq \int_0^r |H(r-u)| \cdot |g(u)| du + |H(r)|
\]
(240)

The Young inequality finishes the proof. □

To finish the proof, we will apply this Lemma to the interval \([R, \infty)\), where \( R \) is so large that the accelerant for the Krein system considered on \([R, \infty)\) has small \( L^1 \) norm. To prove the existence of such \( R \), we need to use Baxter’s Lemma (see Appendix, Corollary 17.2) to the operator \( I + \mathcal{H}_\infty \) acting in the space \( L^1(\mathbb{R}^+) \).

Since \( H(x) \in L^1(\mathbb{R}) \) and \( 1 + \rho(\lambda) > 0 \), this Lemma is applicable and gives
\[
\|(I + \mathcal{H}_r)^{-1}\|_{L^1[0,r],L^1[0,r]} \leq C, (r > r_0); \quad \|\Gamma_r(0,x)\chi_{[0,r]}(x) - \Gamma(x)\|_1 \to 0, (r \to \infty)
\]
(241)

where
\[
\Gamma(x) = (I + \mathcal{H}_\infty)^{-1}H(x) \in L^1(\mathbb{R}^+)
\]
is solution to the Wiener-Hopf equation. Then, we are immediately in the Szegő case since
\[
P_\ast(r,\lambda) = 1 - \int_0^r \Gamma_r(0,s) \exp(i\lambda s) ds \to \Pi_\alpha(\lambda) = 1 - \int_0^\infty \Gamma(s) \exp(i\lambda s) ds
\]
and convergence is uniform in \( \mathbb{C}^+ \).

For \( F(\lambda) \), we use (119) and
\[
F(\lambda)/2 = -i\beta + i \int_{-\infty}^\infty \left[ \frac{1}{\lambda - t} + \frac{t}{t^2 + 1} \right] \frac{1 + \rho(t)}{2\pi} dt
\]

Since \( F(i\infty) = 1 \), we have
\[
F(\lambda) = 1 + \frac{i}{\pi} \int_{-\infty}^\infty \frac{1}{\lambda - t} \rho(t) dt
\]
and the integral is understood in v.p. sense. That immediately implies
\[
F(\lambda) = 1 + 2 \int_0^\infty \overline{H(x)} \exp(i\lambda x) dx
\]

From \( \tilde{F} = F^{-1} \) we have \( \tilde{H}(x) \in L^1(\mathbb{R}) \) for the dual accelerant and
\[
\|\tilde{\Gamma}_r(0,x)\chi_{[0,r]}(x) - \tilde{\Gamma}(x)\|_1 \to 0, (r \to \infty); \quad \tilde{\Gamma}(x) = (I + \tilde{\mathcal{H}}_\infty)^{-1}\tilde{H}(x) \in L^1(\mathbb{R}^+)
\]
So, we also have
\[
\|\mathfrak{B}(r,\lambda) - \mathfrak{B}(\lambda)\|_{W_+} \to 0, \|\mathfrak{A}(r,\lambda) - \mathfrak{A}(\lambda)\|_{W_+} \to 0
\]
(242)
where

\[ \mathfrak{A}(\lambda) = 1 - \int_0^\infty \frac{\Gamma(x) + \hat{\Gamma}(x)}{2} \exp(i\lambda x) dx, \quad \mathfrak{B}(\lambda) = \int_0^\infty \frac{\hat{\Gamma}(x) - \Gamma(x)}{2} \exp(i\lambda x) dx \]

From (173),

\[ \exp(i\lambda R) f_R(\lambda) = [f(\lambda) \mathfrak{A}(R, \lambda) - \mathfrak{B}(R, \lambda)] \cdot [\mathfrak{A}(R, \lambda) + f_R(\lambda) \mathfrak{B}_*(R, \lambda)] \quad (243) \]

Consider the first factor in the right-hand side. It can be written as

\[ \mathfrak{A}^{-1}(\lambda) [\mathfrak{B}(\lambda) \mathfrak{A}(R, \lambda) - \mathfrak{B}(R, \lambda) \mathfrak{A}(\lambda)] \]

Since \( \mathfrak{A}(\lambda) - 1 \in W_+ \) and \( \mathfrak{A}(\lambda) \) has no zeroes in \( \mathbb{C}^+ \), relations (242) imply this factor goes to zero in \( W_+ \) norm as \( R \to \infty \). On the other hand, this factor can be written as

\[ \mathfrak{A}(R, \lambda) \left[ f(\lambda) - \frac{\mathfrak{B}(R, \lambda)}{\mathfrak{A}(R, \lambda)} \right] \quad (244) \]

and (173) says that the Fourier coefficient of the second factor in (244) is equal to 0 on \([0, R]\). Thus,

\[ f(\lambda) \mathfrak{A}(R, \lambda) - \mathfrak{B}(R, \lambda) = \exp(i\lambda R) Q_R(\lambda) \]

where \( \|Q_R(\lambda)\|_{W_+} \to 0 \). Now,

\[ f_R(\lambda) = \frac{Q_R(\lambda) \mathfrak{A}(R, \lambda)}{1 - Q_R(\lambda) \mathfrak{B}_*(R, \lambda)} \]

and \( \|f_R(\lambda)\|_{W_+} \to 0 \) since \( \|\mathfrak{A}(R, \lambda)\|_{W_+} \leq C, \|\mathfrak{B}_*(R, \lambda)\|_{W_+} \leq C \) uniformly in \( R \). Since \( F_R = (1 - f_R)(1 + f_R)^{-1} \), we also have \( ||1 - F_R||_{W_+} \to 0 \) or \( \|H_R(x)\|_1 \to 0 \), where \( H_R(x) \) - accelerant corresponding to the interval \([R, \infty)\). The Lemma 12.1 now yields \( A(r) \in L^1(R, \infty) \) as long as \( \|H_R\| < 1 \).

Let us obtain the formula for \( \alpha \) in the representation for \( \Pi_\alpha(\lambda) \), just like we did for the square summable \( A(r) \). Since \( \Pi_\alpha(\lambda) = 1 + y_1(\lambda) \) and \( y_1(\lambda) \in W_+ \), we have a trivial asymptotics: \( \Pi_\alpha(\lambda) = 1 + O(1) \) as \( \lambda \to \infty, \lambda \in \mathbb{C}^+ \). From the multiplicative representation for \( \Pi_\alpha \), we get

\[ \alpha = \lim_{y \to -\infty} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{s(1 - y^2) + i y(1 + s^2)}{(s^2 + y^2)(1 + s^2)} \mu(s) ds, \mu(s) = \ln(1 + \rho(s)) \in W(\mathbb{R}) \]

and simple estimates yield

\[ \alpha = \frac{1}{2\pi} \text{v.p.} \int_{-\infty}^{\infty} \frac{s \mu(s)}{1 + s^2} ds \]

Thus \( \Pi_\alpha(\lambda) \) allows the same representation (206). Notice also that in contrast with square summable case, function \( \mathfrak{A}(\lambda) \) does not allow asymptotical formula (109). It should also be mentioned that the Theorem does not provide a quantitative estimate on \( \|A\|_1 \) in terms of \( \|H\|_1 \) and, say, \( \|(1 + \rho)^{-1}\|_{\infty} \). On the other hand, it is easy to bound \( \|(1 + \rho)^{-1}\|_{\infty} \) in terms of \( \|A\|_1 \).

Later on, we will need the following result

**Lemma 12.2.** Assume that conditions of Baxter’s Theorem hold. Then \( A(r) \in C_0(\mathbb{R}^+) \) iff \( H(x) \in C_0(\mathbb{R}^+) \) iff \( C(x) \in C_0(\mathbb{R}^+) \).
Proof. Since $H(x), C(x) \in L^1(\mathbb{R}^+)$, we have $H(x) \in C_0(\mathbb{R}^+)$ iff $C(x) \in C_0(\mathbb{R}^+)$ because of (168). Assume $H(x) \in C_0(\mathbb{R}^+)$. Then, (240) implies $A(r) \in C_0(\mathbb{R}^+)$. Now, let $A(r) \in L^1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$. Consider (84). Since $A(r, t) = \Gamma_r(t, r) = \Gamma_r(0, r - t)$,

$$\Gamma_r(0, t) = \overline{A(t)} - \int_0^t \overline{A(s)} \cdot \overline{\Gamma_s(0, t)} ds$$

Take $R$ large and iterate this identity for $\Delta_R = \{ R \leq t \leq r \}$. Since $A(r) \in L^1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$, we will get convergence. To be more precise, if

$$\gamma_R = \sup_{\Delta_R} |\Gamma_r(0, t)|, \alpha_R = \max_{t \geq R} |A(t)|$$

then

$$\gamma_R \leq \alpha_R + \gamma_R \int_R^\infty |A(s)| ds$$

and $\gamma_R \to 0$ as $R \to \infty$. Since $\Gamma_r(0, t) \to \Gamma(t)$ in $L^1(\mathbb{R}^+)$, we have $\|\Gamma(t)\|_{L^\infty[0, \infty)} \to 0$ as $R \to \infty$. The same is true about $\hat{\Gamma}(t)$. Application of formulas (235) and (236) finishes the proof. □

Remarks and historical notes. For an excellent exposition of the proof for the Baxter theorem in OPUC case, see [66]. In our case, some modifications were needed. Apparently, the first proof of the Baxter theorem for continuous case was given in [52]. See also [20, 47] (one has to pay attention to some inaccuracies in statements regarding the regularity of coefficients generated by summable accelerants).
13. Dirac systems

In this section, we relate Krein systems to the well-known object in mathematical physics: one-dimensional Dirac operator. Consider the Krein system, given by (57) and assume some regularity conditions, e.g. \( a(r), b(r) \in L^2_{\text{loc}}(\mathbb{R}^+) \). Let \( \lambda \in \mathbb{C} \) and

\[
\varphi(r, \lambda) = \frac{\exp(-i\lambda r)}{2} [P(2r, \lambda) + P_*(2r, \lambda)],
\psi(r, \lambda) = \frac{\exp(-i\lambda r)}{2i} [P(2r, \lambda) - P_*(2r, \lambda)]
\]

These functions are of the exponential type \( r \) and are not from \( H^2(\mathbb{C}^+) \) anymore. They should be regarded as analogs of trigonometric polynomials (or Laurent polynomials). For the free case, i.e. \( A(r) = 0 \), one has \( \varphi(r, \lambda) = \cos(r\lambda), \psi(r, \lambda) = \sin(r\lambda) \). If \( \lambda \in \mathbb{R} \),

\[
\varphi(r, \lambda) = \text{Re} \mathcal{E}(r, \lambda), \psi(r, \lambda) = \text{Im} \mathcal{E}(r, \lambda), \mathcal{E}(r, \lambda) = \exp(-i\lambda r)P'(2r, \lambda)
\]

Define \( \mathcal{E}(r, \lambda) \) for \( r < 0 \) by \( \mathcal{E}(-r, \lambda) = \overline{\mathcal{E}(r, \lambda)} \). Let \( a(r) = 2 \text{Re} A(2r), b(r) = 2 \text{Im} A(2r) \). Consider the following Dirac operator

\[
D \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -b(r) & d/dr - a(r) \\ -d/dr - a(r) & b(r) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}
\]

where the Hilbert space is \( f_1, f_2 \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+) \) and operator is made self-adjoint by imposing condition \( f_2(0) = 0 \). Another way to write \( D \) is as follows

\[
D = \mathcal{J} \frac{d}{dr} + Q(r)
\]

where potential \( Q(r) \) is

\[
Q(r) = \begin{bmatrix} -b(r) & -a(r) \\ -a(r) & b(r) \end{bmatrix}
\]

and

\[
\mathcal{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

This form of Dirac operator is called canonical. Any Dirac operator can be reduced to this form by a suitable change of variables [49], p. 48–50. Just like the Krein system, the Dirac operator in the canonical form has a lot of structure due to a special choice of potential.

Rather than, say, Schrödinger operator, \( D \) can always be defined as the closure of the naturally chosen minimal operator ([49], Theorem 7.1, p.493 or [77], p. 99). We will start with the following

**Lemma 13.1.** Functions \( \varphi \) and \( \psi \) are generalized eigenfunctions of \( D \), i.e.

\[
D \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \lambda \begin{bmatrix} \varphi \\ \psi \end{bmatrix}, \varphi(0, \lambda) = 1, \psi(0, \lambda) = 0
\]

More generally, the fundamental solution \( X_d \) for the system (249) can be expressed via the fundamental solution for Krein system in the following way

\[
X_d(r, \lambda) = \exp(-i\lambda r)U_0X(2r, \lambda)U_0^{-1}
\]
and
\[ U_0 = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \]

**Proof.** The proof is a straightforward calculation. \( \square \)

Next, we will show that the system \( \{ E(x, \lambda) \} \) is an orthogonal system in \( L^2(\mathbb{R}, d\sigma) \). There are many ways to see that but we prefer an algebraic one, based on the proper factorization of certain integral operator. Just like in Section 3, we start with the following consideration. Let \( H(x) \) be an accelerant and \( H(x) \in L_{\text{loc}}^2(0, \infty) \). For any \( r > 0 \), consider the integral operator \( \hat{G}_r \):

\[ \hat{G}_r f(x) = \int_{-r}^{r} H(x-u)f(u)du \]  \hspace{1cm} (250)

given in \( L^2[-r, r] \). Since \( \hat{G}_r \) is the translation of \( \hat{G}_{2r} \) defined on \([0, 2r] \), we have \( I + \hat{G}_r > 0 \) for any \( r > 0 \) and the conditions of the Theorem 2.2 in section 2 are satisfied. Moreover, we can express one resolvent via the other one, i.e.

\[ \hat{\Gamma}_r(x,y) = \Gamma_{2r}(r+x, r+y), \quad |x|, |y| < r \]  \hspace{1cm} (251)

Let us consider some \( R > 0 \) and the factorization \( I + \hat{G}_R = (I + \hat{L})(I + \hat{U}) \) where the lower-diagonal (in the sense of Theorems 2.2 and 5.2) operator \( \hat{L} \) has kernel \( \hat{L}(x,y), |y| < |x| < R \). Since \( \hat{G}_R \)– Hermitian, we get \( \hat{U} = \hat{L}^* \). If \( I + \hat{L} = (I + \hat{V}_-)^{-1} \) and \( I + \hat{U} = (I + \hat{V}_+)^{-1} \), then

**Lemma 13.2.** The following is true

\[ \exp(i\lambda x) = E(x, \lambda) + \int_{-|x|}^{|x|} \hat{L}(x, u)E(u, \lambda)du, \quad x \in [-R, R] \]  \hspace{1cm} (252)

**Proof.** We have the following formula for \( E(x, \lambda) \) if \( x > 0 \)

\[ E(x, \lambda) = P(2x, \lambda) \exp(-i\lambda x) = \left( \exp(2i\lambda x) - \int_0^{2x} \Gamma_{2x}(2x, s) \exp(is\lambda)ds \right) \exp(-i\lambda x) \]  \hspace{1cm} (253)

Doing the change of variables \( s - x = t \) in the integral, we obtain

\[ E(x, \lambda) = \exp(i\lambda x) - \int_{-x}^{x} \Gamma_{2x}(2x, x + t) \exp(i\lambda t)dt = \]  \hspace{1cm} (254)

\[ = \exp(i\lambda x) - \int_{-x}^{x} \Gamma_{2x}(x - t, 0) \exp(i\lambda t)dt \]  \hspace{1cm} (255)

We also have

\[ E(-x, \lambda) = \exp(-i\lambda x) - \int_{-x}^{x} \Gamma_{2x}(0, x + t) \exp(i\lambda t)dt, \quad x > 0 \]  \hspace{1cm} (256)
Therefore, from (251) and the Theorem 5.2 we get $E(x, \lambda) = (I + \hat{V}_-) \exp(i\lambda x)$. The Lemma then follow from $I + \hat{L} = (I + \hat{V}_-)^{-1}$. □

Clearly, this Lemma is an analog of Lemma 3.2 but for the different chain. Now, we are ready to relate Krein systems to Dirac operators. But first we need the following

**Definition 13.1.** We say that a non-decreasing function $\sigma_d(\lambda), \lambda \in \mathbb{R}$ is the spectral measure for Dirac operator (246), if the following is true (see [50], Chapter 8): for any $f_1 \in L^2(\mathbb{R}^+) = (I + \hat{V}_-)^{-1}$.

The next Theorem establishes a further link between the Krein systems and Dirac operators.

**Theorem 13.1.** The measure $d\sigma_d(\lambda) = 2d\sigma(\lambda)$ is the spectral measure for Dirac operator. Moreover, the mapping

$$f(x) \in L^2(\mathbb{R}) \rightarrow [Wf](\lambda) = \int_{-\infty}^{\infty} f(x)E(x, \lambda)dx$$

is unitary onto $L^2(\mathbb{R}, d\sigma)$. 

**Proof.** Let us first show that $W$ is an isometry map. Indeed, let $f(x) \in L^2(-R, R)$. From (50), we get

$$((I + \hat{H}_R)f, f) = \int_{-\infty}^{\infty} \int_{-R}^{R} \overline{f(t)} \exp(i\lambda t)dt \int_{-R}^{R} \overline{f(t)} \exp(i\lambda t)dt \sigma(\lambda)$$

In the meantime, from Lemma 13.2,

$$\int_{-R}^{R} \overline{f(t)} \exp(i\lambda t)dt = ((I + \hat{L})E(t, \lambda), f(t))_{L^2[-R,R]} = (E(t, \lambda), (I + \hat{L})^*f(t))_{L^2[-R,R]}$$

Therefore, (258) gives

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \overline{g(t)} \exp(i\lambda t)dt \right|^2 d\sigma(\lambda) = ((I + \hat{L})^{-1}(I + \hat{H}_R)(I + \hat{U})^{-1}g, g) = \|g\|^2$$

where $g = (I + \hat{L})^*f = (I + \hat{U})f$. Since $I + \hat{U}$ is invertible on $L^2[-R, R]$ and $R$ was chosen arbitrarily, we learn that $W$ is isometry. Now, let us show that $W$ is also unitary. Indeed, for any $R > 0$ and $f(x) \in L^2[-R, R]$, we get (Lemma 13.2)

$$\int_{-R}^{R} \overline{f(x)} \exp(i\lambda x)dx = \left( (I + \hat{L})E(x, \lambda), f(x) \right) = (E(x, \lambda), [(I + \hat{U})f](x))$$

Functions of that kind are dense in $L^2(\mathbb{R}, d\sigma(\lambda))$ because the span of characteristic functions of the intervals $\{[a, b]\}$ are dense and each of these characteristic functions
can be approximated by the Fourier transform of finitely supported $L^2(\mathbb{R})$ function due to the regularity condition $[\mathbb{R}]$. Therefore, the range of $\mathcal{W}$ is the whole of $L^2(\mathbb{R}, d\sigma(\lambda))$. That means $\mathcal{W}$ is unitary.

Now, we can conclude the proof of the Theorem. Take any $f_1(r) \in L^2(\mathbb{R}^+)$ and $f_2(r) \in L^2(\mathbb{R}^+)$. Let $f_1(-x) = f_1(x)$, $f_2(-x) = -f_2(x)$, $x > 0$. Consider $f(x) = f_1(x) - if_2(x)$ on $\mathbb{R}$. Function $\varphi$ is even, $\psi$ is odd. So, 

$$[\mathcal{W}f](\lambda) = \int_{-\infty}^{\infty} f(x)\mathcal{E}(x, \lambda)dx = \int_{-\infty}^{\infty} (f_1 - if_2)(\varphi + i\psi)dx = 2 \int_{0}^{\infty} (f_1\varphi + f_2\psi) dx = 2[\mathcal{F}f](\lambda)$$

That proves $\mathcal{F}$ is unitary mapping to $L^2(\mathbb{R}, 2d\sigma(\lambda))$. Since the range of $\mathcal{W}$ is the whole $L^2(\mathbb{R}, d\sigma(\lambda))$, we can write $\mathcal{W}^{-1}g = f(x) = f_1(x) - if_2(x)$, where $f_1(x) = [f(x) + f(-x)]/2$, $f_2(x) = -[f(x) - f(-x)]/(2i)$ and $g$ is arbitrary from $L^2(\mathbb{R}, d\sigma)$. Then, $2\mathcal{F}f = g$ and $\mathcal{F}$ is unitary.

It is easy to show that the spectral measure for Dirac operator is uniquely defined (see Lemma \[17.10\]) in Appendix.

As usual, the following representation can be easily obtained from the Theorem:

$$\int_{-\infty}^{\infty} \begin{bmatrix} \varphi(x, \lambda)\varphi(y, \lambda) & \varphi(x, \lambda)\psi(y, \lambda) \\
\psi(x, \lambda)\varphi(y, \lambda) & \psi(x, \lambda)\psi(y, \lambda) \end{bmatrix} d\sigma_d(\lambda) = \begin{bmatrix} \delta(x-y) & 0 \\
0 & \delta(x-y) \end{bmatrix} (259)$$

and this identity should be understood in the weak-$[L^2(\mathbb{R}^+)]^2$ sense (i.e. it is true after multiplication by $L^2$ functions and integration in $x$ and $y$).

In case $d\sigma_0(\lambda) = d\lambda/(2\pi)$ discussed above, one has $a(r) = b(r) = 0$, $\mathcal{E}(x, \lambda) = \exp(i\lambda x)$, $d\sigma_d(\lambda) = d\lambda/\pi$. The map $\mathcal{W}$ is then the standard Fourier transform.

The Dirac operator plays the role of the so-called CMV matrix for polynomials orthogonal on the unit circle. Many results about the Krein systems and functions $P(r, \lambda)$ can be viewed from that perspective.

It is also quite helpful to introduce the auxiliary dissipative operator. For any $R > 0$, consider the operator $\mathcal{D}_R$ on $L^2[0, R] \times L^2[0, R]$ given by the differential system \[246\] and the boundary conditions:

$$f_2(0) = 0, f_1(R) + if_2(R) = 0 \quad (260)$$

The domain of definition for that operator consists in functions from $W^{1,2}[0, R] \times W^{1,2}[0, R]$ satisfying \[260\]. It is an elementary calculation to show that $\mathcal{D}_R$ is dissipative since

$$\text{Im}(\mathcal{D}_R f,f) = 2|f_2(R)|^2 \geq 0$$

for all $f$ in the domain of $\mathcal{D}$. This operator has compact resolvent, an integral operator that can be written explicitly in terms of the solutions to the corresponding equation. The formula is as follows and can be easily checked

$$(\mathcal{D}_R - \lambda)^{-1}(f_1, f_2) = \begin{bmatrix} \varphi(r, \lambda) \\
\psi(r, \lambda) \end{bmatrix} \int_{r}^{R} (-Z_{12}(s, \lambda)f_1(s) + Z_{11}(s, \lambda)f_2(s))ds$$
where

\[ D = 84 \]

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\[ I \]

is understood as the regularized determinants of operator

\[ \lambda \in \mathbb{C}^+ \]

and

\[ y \in \mathbb{C} \]

Since \( (\text{kernel of resolvent has a pole at point } \lambda) \)

\[ P \in \mathbb{C} \]

half-plane. The zeroes of

\[ \mu \in \mathbb{C} \]

by the spectrum of the operator \( D \)

\[ \psi \in \mathbb{C} \]

Thus, the spectrum of this operator is discrete and coincide s with the zeroes of

\[ \mathcal{P} \]

One can say even more, infact

Lemma 13.3. The following representation is true

\[ P_2(2R, \lambda) = \exp \left[ - \int_0^R [a(s) + ib(s)] \exp(2i\lambda s)ds \right] \det_2 \left( \frac{\mathcal{D}_{R,0}^* - \lambda}{\mathcal{D}_{R,0}^* - \lambda} \right) \]

(262)

where \( \mathcal{D}_{0,R} \) denotes the operator with \( a(r) = b(r) = 0 \). The regularized determinant is understood as the regularized determinants of operator \( I + (\mathcal{D}_{0,R}^* - \lambda)^{-1}Q \) (see [68], p. 106).

Proof. A simple calculation shows that the spectrum of \( \mathcal{D}_{0,R}^* \) is empty (infact, is equal to infinity). Therefore, \( (\mathcal{D}_{0,R}^* - \lambda)^{-1} \) is always well-defined. We have

\[ (\mathcal{D}_{0,R}^* - \lambda)^{-1}(f_1, f_2)^t = \left[ \begin{array}{c} \cos(r\lambda) \\ \sin(r\lambda) \end{array} \right] \int_0^R [i \exp(i\lambda s) f_1(s) + \exp(i\lambda s) f_2(s)]ds + \left[ \begin{array}{c} \exp(i\lambda s) \\ -i \exp(i\lambda s) \end{array} \right] \int_0^R [i \cos(s\lambda) f_1(s) + i \sin(s\lambda) f_2(s)]ds \]

(263)

Simple calculations show that \( (\mathcal{D}_{0,R}^* - \lambda)^{-1}Q \in \mathbb{S}^2 \) and the regularized determinant exists.

Then, we use the following trick (see [68], p. 75). Introduce the so-called “coupling constant” \( \mu \in \mathbb{C} \) and the potentials \( Q_\mu = \mu Q \). Then, consider the corresponding functions \( P_2(2R, \lambda, \mu) \)

\[ f(\lambda, \mu) = \exp \left[ -\mu \int_0^R [a(s) + ib(s)] \exp(2i\lambda s)ds \right] \det_2(I + \mu(\mathcal{D}_{0,R}^* - \lambda)^{-1}Q) \]
Theorem 13.2.

If to get analogous result for Π

Therefore, one could have used the Carleman-Hilbert determinant instead of det

Formulas for the kernel

On the other hand, for det


c

where the constants

and zeroes

all depend on

Since

we get

Taking logarithm of both sides in

and comparing the Taylor coefficients in front of

we get


c

Comparing these two expansions, we get the statement of the Lemma (take

The determinantal representations are usually very useful in practice. They provide the natural factorization of entire functions of interest. In case potential

is small at infinity (say

one can get asymptotical expansion of any order by using the further regularization of det

Formulas for the kernel

show that it has discontinuity on the diagonal. Therefore, one could have used the Carleman-Hilbert determinant instead of det

regularization.

Since we have determinantal formula for

for finite

we might hope to get analogous result for

in case

by just taking

Theorem 13.2. If

then

\begin{equation}
\Pi_\alpha(\lambda) = \exp \left[ - \int_0^\infty [a(s) + ib(s)] \exp(2i\lambda s) ds \right] \det_2 \left( \frac{D - \lambda}{D_0 - \lambda} \right), \lambda \in \mathbb{C}^+ \tag{265} \end{equation}

Here \(D_0\) denotes the free Dirac operator, i.e. \(D\) with \(a(r) = b(r) = 0\).

Proof. The integral operator \((D_0 - \lambda)^{-1}Q\) has the following kernel

\begin{align*}
K_0(x, y, \lambda) &= \begin{bmatrix}
-e^{i\lambda y} \cos(\lambda x)[a(y) + ib(y)] & e^{i\lambda y} \cos(\lambda x)[b(y) - ia(y)] \\
-e^{i\lambda y} \sin(\lambda x)[a(y) + ib(y)] & e^{i\lambda y} \sin(\lambda x)[b(y) - ia(y)]
\end{bmatrix} \\
\text{if } y > x > 0 \text{ and}
K_0(x, y, \lambda) &= \begin{bmatrix}
-ie^{i\lambda x} \cos(\lambda y)b(y) + \sin(\lambda y)a(y) & ie^{i\lambda x} \sin(\lambda y)b(y) - \cos(\lambda y)a(y) \\
-ie^{i\lambda x} \cos(\lambda y)b(y) + \sin(\lambda y)a(y) & e^{i\lambda x} \sin(\lambda y)b(y) - a(y) \cos(\lambda y)
\end{bmatrix} \\
\text{if } 0 < y < x \text{.}
\end{align*}

Since \(A(r) \in L^2(\mathbb{R}^+)\), we have \((D_0 - \lambda)^{-1}Q \in \mathbb{S}^2\) for any \(\lambda \in \mathbb{C}^+\) and the regularized determinant exists. Now, fix \(\lambda \in \mathbb{C}^+\). The function

\[ \Pi_{\alpha}(\lambda) \text{ as } R \to \infty \text{ (see Theorem 11.1).} \] On the other hand, one can easily check that \( \det_2(I + (D^*_{0,R} - \lambda)^{-1}Q) \to \det_2(I + Q(D_0 - \lambda)^{-1}Q) \) as well. \qed 

Now, let us study the wave operators for \( \mathcal{D} \). The following result establishes a connection between the stationary and non-stationary scattering approaches.

**Theorem 13.3.** If \( a(x), b(x) \in L^2(\mathbb{R}^+) \), then the wave operators
\[ \Omega_{\pm}f = \lim_{t \to \mp \infty} e^{it\mathcal{D}}e^{-it\mathcal{D}_0}f \]
exist. The limit is understood in the strong sense, \( f = (f_1, f_2)^t \in [L^2(\mathbb{R}^+)]^2 \).

**Proof.** The free evolution of \( \mathcal{D}_0 \) is given in Lemma 17.11 from Appendix. It is actually a shift after some unitary transformations. Since each of the operators \( e^{it\mathcal{D}}, e^{-it\mathcal{D}_0} \) is unitary, it suffices to check the existence of strong limit for vectors \( f = (f_1, 0)^t \), where the scalar function \( f(x) \in C_0^\infty(\mathbb{R}^+) \). The existence of strong limit for vectors with zero as the first coordinate can be proved in the same way. Due to linearity, that is enough to conclude the convergence for all \( C_0^\infty(\mathbb{R}^+) \) vector-valued functions that give rise to subspace dense in \( L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+) \). From Lemma 17.11 we have
\[
e^{-it\mathcal{D}_0}f = \frac{1}{2} \left[ \begin{array}{c} f(x + t) + f(x - t) \\ -i(f(x - t) - f(x + t)) \end{array} \right] = \frac{1}{2} \left[ \begin{array}{c} f(|x - t|) \\ -if(|x - t|) \end{array} \right],
\]
where the last formula holds for \( t \) large enough because the support of \( f \) is finite (say, in the interval \((0, a)\)). Consider the function
\[
\vartheta(t, \lambda) = \frac{e^{it\lambda}}{2} \left[ \int_0^\infty f(|x - t|) \varphi(x, \lambda)dx - i \int_0^\infty f(|x - t|) \psi(x, \lambda)dx \right].
\]
To prove Theorem 13.3 it suffices to show that the limit of \( \vartheta(t, \lambda) \) in \( L^{2,2\sigma}(\mathbb{R}) \) exists as \( t \to \infty \). The following relations are true
\[
\vartheta(t, \lambda) = \frac{e^{it\lambda}}{2} \int_0^\infty f(|x - t|)e^{i\lambda s}P(2x, \lambda)dx = \frac{1}{2} \int_{-a}^a f(|s|)e^{-i\lambda s}P_s(2t + 2s, \lambda)ds
\]
\[
= \frac{1}{2} \int_{-a}^a \left[ \int_{-a}^s f(|\tau|)e^{-i\lambda \tau}d\tau \right]' P_s(2t + 2s, \lambda)ds = P_s(2t + 2a, \lambda) \int_0^a f(\tau) \cos(\lambda \tau)d\tau
\]
\[
+ \int_{-a}^a A(2t + 2s)P(2t + 2s, \lambda) \int_{-a}^s f(|\tau|)e^{-i\lambda \tau}d\tau ds.
\]
Since \( f(x) \) is smooth,
\[
\left| \int_{-a}^s f(|\tau|)e^{-i\tau \lambda}d\tau \right| \leq \frac{C}{\sqrt{\lambda^2 + 1}}. \tag{266}
\]
Due to Lemma 8.9 and Theorem 11.1
\[
P_s(2t + 2a, \lambda) \int_0^a f(\tau) \cos(\lambda \tau)d\tau \to \Pi_{\alpha}(\lambda + i0)\chi_{E^c}(\lambda) \int_0^a f(\tau) \cos(\lambda \tau)d\tau, \tag{267}
\]
where the possible singular component of \( d\sigma \) is supported on the Borel set \( E \), and \( \chi_{E^c} \) is the characteristic function of the complement to \( E \). The convergence is understood in \( L^2, 2\sigma (R) \) sense. Here we also used Remark after the Lemma 8.9. Generalized Minkowski inequality and (266) yield

\[
\left\| \int_{-a}^{a} A(2t + 2s)P(2t + 2s, \lambda) f(|r|) e^{-i\lambda t} d\tau ds \right\|_{2, \sigma} \leq C \left( \int_{-a}^{a} |A(2t + 2s)| |ds| \right)^{\frac{1}{2}} \left[ \sup_{x \geq 0} \int_{-\infty}^{\infty} \frac{|P(x, \lambda)|^2}{\lambda^2 + 1} d\sigma \right]^{\frac{1}{2}}.
\]

The second factor is bounded due to Lemma 8.8. Function \( A(x) \in L^2(R^+) \), therefore the first factor tends to 0 as \( t \to \infty \).

\[ \square \]

Remark. We not only proved the existence of the wave operators, but also deduced the formula for them, the right-hand side of (267). Notice that this map is isometry. Part of the arguments above are well-known in the theory of polynomials orthogonal on the unit circle [26, 66].

Very interesting effect can be observed in the case \( d\sigma \in \text{(Szeg\'o)} \), and \( A(r) \to 0 \) at infinity in some sense (say, \( A(r) \in L^p(R^+), p < \infty \)). As was discussed before (see the paragraph after Theorem 8.2), the limit \( P_s (r_n, \lambda) \) is not necessarily uniquely defined and might depend upon the choice of the subsequence \( r_n \to \infty \). One can easily show that the proof of the Theorem above can be adjusted to this situation with the exception that the limit \( \lim_{n \to \infty} e^{i t_n D} e^{-i t_n D_0} \) depends upon the choice of time sequence \( t_n \) and the limiting operators will actually differ only by the unimodular factor. However, due to Lemma 8.6 this phenomena cannot be observed for real-valued \( A(r) \) (i.e. when \( b(r) = 0 \)).

At this point, we need to mention that in OPUC theory, the free CMV matrix is unitarily equivalent to the shift in \( \ell^2(Z) \) space. In fact, the same is true about the evolution for free Dirac operator. Indeed, consider the following operator

\[
\hat{D}_0 = Z^{-1} D_0 Z, Z = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} i & -1 \\ 1 & -i \end{array} \right], \hat{D}_0 = \left[ \begin{array}{cc} -i \frac{d}{dr} & 0 \\ 0 & i \frac{d}{dr} \end{array} \right].
\]

The new domain of definition is \( [H^1(R^+)]^2 \) with additional condition \( f_1(0) = i f_2(0) \). Now, if one maps the Hilbert space \( f = (f_1, f_2)^t \in [L^2(R^+)]^2 \) to \( L^2(R) \) by

\[
f(r) \rightarrow g(x) = \begin{cases} f_1(x), & x > 0 \\ i f_2(-x), & x < 0 \end{cases}
\]

then the operator \( \hat{D} \) happens to be unitarily equivalent to the selfadjoint operator \( \mathcal{L}_0 = -i \frac{d}{dx} \) on \( L^2(R) \) with domain of definition \( H^1(R) \). Its free evolution is just the shift: \( \exp(it \mathcal{L}_0) g(x) = g(x + t), t \in R \). Since we performed only the unitary transformations, the evolution \( \exp(it \hat{D}_0) \) (see Lemma 17.11 in Appendix) is unitarily equivalent to the shift in \( L^2(R) \). Notice that potential is transformed to

\[
Z^{-1} QZ = \begin{pmatrix} 0 & 2A(2r) \\ 2A(2r) & 0 \end{pmatrix} \quad (268)
\]
The unitary transformation of $[L^2(\mathbb{R}^+)]^2$ to $\mathbb{R}$ will result in nonlocal perturbation of $L_0$: 

$$[\mathcal{L}g](x) = -i\frac{dg}{dx} + \bar{A}(x)[Sg](x)$$

where Hermitian function $\bar{A}(x)$ and operator $S$ are given by

$$\bar{A}(x) = i \begin{cases} -2A(2x), & x > 0 \\ 2A(-2x), & x < 0 \end{cases}, \quad Sg(x) = g(-x)$$

The analysis of $\mathcal{L}$ is non-trivial (rather than in the case $S = I$) and is equivalent to analysis of the original Dirac operator $\mathcal{D}$ or Krein system. Notice that in the Fourier space, this operator can be formally written as

$$-\lambda f(\lambda) + \int_{-\infty}^{\infty} V(\lambda + t)f(t)dt$$

with real-valued $V(\lambda)$ being the Fourier transform of $\bar{A}(x)$. The analysis we have done before implies the corresponding properties of this operator.

Let us consider the scattering problem for Dirac operator and relate scattering parameters to the parameters of the corresponding Krein system. Consider, for simplicity, operator $\mathcal{D}$ with finitely supported coefficients $a$ and $b$. Then, there is the so-called Jost solution $F(r, \lambda) : \mathcal{D}F = \lambda F$ defined by the asymptotics at infinity: $F(r, \lambda) = (f_1(r, \lambda), f_2(r, \lambda))^t = \exp(i\lambda r)(i, 1)^t$ for $r$ large enough. Let us introduce the scattering data for the Dirac operator:

$$A_d(\lambda) = (f_2(0, \lambda) - i f_1(0, \lambda))/2, \quad B_d(\lambda) = (f_2(0, \lambda) + i f_1(0, \lambda))/2,$$

$$T_d(\lambda) = A_d^{-1}(\lambda), \quad R_d(\lambda) = B_d(\lambda)/A_d(\lambda)$$

Coefficient $T_d(\lambda)$ is called the transmission coefficient, $R_d(\lambda)$ is the reflection coefficient, $f_2(0, \lambda)$ is Jost function. These notations are quite natural. If one extends $a$ and $b$ to the negative half-line as zero, then

$$F(r, \lambda) = A_d(\lambda) \exp(i\lambda r)(i, 1)^t + B_d(\lambda) \exp(-i\lambda r)(-i, 1)^t, \quad r < 0,$$

or

$$T_d(\lambda)F(r, \lambda) = \exp(i\lambda r)(i, 1)^t + R_d(\lambda) \exp(-i\lambda r)(-i, 1)^t, \quad r < 0.$$

Lemma 13.4. The following relations are true

$$A_d(\lambda) = \mathfrak{A}(\lambda), \quad B_d(\lambda) = \mathfrak{B}(\lambda), \quad R_d(\lambda) = f(\lambda), \quad f_2(0, \lambda) = \Pi_\alpha(\lambda),$$

$$\sigma_d'(\lambda) = \frac{1}{\pi |f_2(0, \lambda)|^2} = \frac{1}{\pi \Pi(\lambda)|^2} = 2\sigma'(\lambda) \quad (269)$$

$f(\lambda)$—Schur function of the Krein system, functions $\mathfrak{A}(\lambda), \mathfrak{B}(\lambda), \Pi_\alpha(\lambda)$ are taken from consideration of $A(r) \in L^2(\mathbb{R}^+)$ or $A(r) \in L^1(\mathbb{R}^+)$ cases.

Proof. The proof is a straightforward calculation. □

The last formula in (269) is of great importance. It gives a factorization of the spectral measure density via some function analytic in the upper half-plane. Formulas of that sort have analogs in the scattering problems for some PDE [19, 18]. The equivalence of reflection coefficient from the scattering theory of quantum mechanics and Schur function is a remarkable fact, which, perhaps, was not completely understood and used by both mathematical physicists and analysts.
The general spectral theory allows to get new natural interpretation for various quantities considered before. One example is provided by the following Lemma.

**Lemma 13.5.** For any \( \lambda_0 \in \mathbb{C}^+ \), the operator \( \text{Im}(\mathcal{D} - \lambda_0)^{-1} \) has matrix-valued kernel \( G(x, y, \lambda_0) \) and

\[
2 \text{Im} \lambda_0 \int_{-\infty}^{\infty} \frac{|P_s(2r, \lambda)|^2}{|\lambda - \lambda_0|^2} d\sigma(\lambda) = \text{Tr} G(r, r, \lambda_0)
\]

**Proof.** The spectral representation for the resolvent yields (see (259))

\[
\int_{-\infty}^{\infty} \frac{1}{\lambda - \lambda_0} \begin{bmatrix}
\varphi(x, \lambda) & \varphi(y, \lambda) \\
\psi(x, \lambda) & \psi(y, \lambda)
\end{bmatrix} d\sigma(\lambda) = (\mathcal{D} - \lambda_0)^{-1}(x, y) \quad (270)
\]

Therefore,

\[
\text{Tr} \text{Im} G(x, y, \lambda_0) = \int_{-\infty}^{\infty} \frac{\text{Im} \lambda_0}{|\lambda - \lambda_0|^2} [\varphi(x, \lambda)\varphi(y, \lambda) + \psi(x, \lambda)\psi(y, \lambda)] d\sigma(\lambda)
\]

Now, the Lemma is straightforward. \( \square \)

This Lemma allows to control the integral

\[
\int_{-\infty}^{\infty} \frac{|P_s(r, \lambda)|^2}{\lambda^2 + 1} d\sigma(\lambda)
\]

by using the standard tools of, say, perturbation theory. In particular, if \( A \to 0 \) in some sense, then this integral tends to 1/2, the value for unperturbed case.

Recall that the CMV matrix corresponding to Verblunsky coefficients \( a_n \) is

\[
\mathcal{C} = \begin{bmatrix}
* & * & * & 0 & 0 & \ldots \\
* & * & * & 0 & 0 & \ldots \\
0 & * & * & * & \ldots \\
0 & * & * & * & \ldots \\
0 & 0 & 0 & * & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} = \begin{bmatrix} A_0 & 0 & \ldots \\
0 & A_1 & \ldots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\]

\[
A_k = \begin{bmatrix}
\bar{a}_{2k}\rho_{2k-1} & -\bar{a}_{2k}\rho_{2k-1} & \bar{a}_{2k+1}\rho_{2k} & \rho_{2k+1}\rho_{2k} \\
\rho_{2k}\rho_{2k-1} & -\rho_{2k}\rho_{2k-1} & \bar{\rho}_{2k+1}\rho_{2k} & -\rho_{2k+1}\rho_{2k} \\
\rho_0 & \bar{a}_1\rho_0 & \rho_1\rho_0 \\
\rho_0 & -\bar{a}_1\rho_0 & -\rho_1\rho_0
\end{bmatrix}
\]

\[
A_0 = \begin{bmatrix}
\bar{a}_0 & \bar{a}_1\rho_0 & \rho_1\rho_0 \\
\rho_0 & \bar{a}_1\rho_0 & -\rho_1\rho_0
\end{bmatrix}
\]

and \( \rho_k = (1 - |a_k|^2)^{1/2} \). To show how CMV matrix corresponds to Dirac operators, we prefer to write it in the equivalent way (by introducing two Hilbert spaces \( \ell^2(\mathbb{Z}^+) \) corresponding to even and odd indices):

\[
\mathcal{C} = \begin{bmatrix}
\mathcal{C}_{11} & \mathcal{C}_{12} \\
\mathcal{C}_{21} & \mathcal{C}_{22}
\end{bmatrix}
\]
Consider the following formal discretization of $R$
is discretizations but that can be done.

We do not pursue the goal of making any accurate statements regarding these discretizations but that can be done.

Taking the Verblunsky coefficients

with

$$
\mathcal{C}_{11} = \begin{bmatrix}
\rho_0 & \rho_0 \rho_1 & 0 & 0 & \cdots \\
0 & -\rho_1 a_2 & \rho_2 \rho_3 & 0 & \cdots \\
0 & 0 & -\rho_3 a_4 & \rho_4 \rho_5 & \cdots \\
0 & 0 & 0 & \ast & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\mathcal{C}_{12} = \begin{bmatrix}
\rho_0 \bar{a}_1 & 0 & 0 & 0 & \cdots \\
\rho_1 \bar{a}_2 & \rho_2 \bar{a}_3 & 0 & 0 & \cdots \\
0 & \rho_3 a_4 & \rho_4 a_5 & 0 & \cdots \\
0 & 0 & \rho_5 a_6 & \ast & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\mathcal{C}_{21} = \begin{bmatrix}
\rho_0 & -\rho_1 a_0 & 0 & 0 & \cdots \\
0 & -\rho_2 a_1 & -\rho_3 a_2 & 0 & \cdots \\
0 & 0 & -\rho_4 a_3 & -\rho_5 a_4 & \cdots \\
0 & 0 & 0 & \ast & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\mathcal{C}_{22} = \begin{bmatrix}
-a_0 \bar{a}_1 & 0 & 0 & 0 & \cdots \\
\rho_1 \bar{a}_2 & -\rho_2 a_3 & 0 & 0 & \cdots \\
0 & \rho_3 \rho_4 & -\rho_4 a_5 & 0 & \cdots \\
0 & 0 & \rho_5 a_6 & \ast & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

It is well known that the formal discretization of the continuous Schrödinger operator produces a discrete Schrödinger operator, a particular case of the Jacobi matrix. For the Dirac operator, the situation is a little bit different because it is self-adjoint and the CMV matrix (an analog of Jacobi matrix in this case) is unitary. Notice that $D$ is unitarily equivalent to

$$
\hat{D} = u D u^{-1} = \begin{bmatrix}
\frac{-i}{2A(2r)} & \frac{-2iA(2r)}{i d/dr} \\
\frac{2iA(2r)}{i d/dr} & \frac{i}{2A(2r)}
\end{bmatrix},
\quad u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}
$$

with an appropriate boundary condition at zero. Conjugating $\hat{D}$ by the matrix

$$
\tau = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}
$$

we get that $\tau^{-1} \hat{D} \tau$ has the same diagonal but the off-diagonal elements have the form [208]. Consider the following formal discretization of $\hat{D}$

$$
\hat{D}_h = \begin{bmatrix}
-\frac{i h^{-1}(L - I)}{2iA(2r_n)} & -2iA(2r_n) \\
2iA(2r_n) & \frac{i h^{-1}(I - R)}{ih^{-1}(I - R)}
\end{bmatrix}
$$

where $R$ is the right shift, $L$ – the left shift. Then, consider

$$
I + \frac{i h}{2} \hat{D}_h = \begin{bmatrix}
\frac{L}{ih^{-1}(I - R)} & \frac{2hA(2r_n)}{R} \\
\frac{-2hA(2r_n)}{R} & \frac{L}{ih^{-1}(I - R)}
\end{bmatrix}
$$

Taking the Verblunsky coefficients $\{a_k^{(h)}\} : a_k^{(h)} = 0, a_{2n-1}^{(h)} = 2hA(2r_n)$, we get $I + \frac{i h}{2} \hat{D}_h = CMV + \bar{o}(h)$. Thus, formally, CMV matrices and discretization of Dirac operators are related via this very simple identity. Since $L - I$ and $I - R$ are of order $h$ when acted on smooth functions, we could also say that

$$
\exp \left( \frac{i h}{2} \hat{D}_h \right) = \begin{bmatrix}
\frac{L}{ih^{-1}(I - R)} & \frac{2hA(2r_n)}{R} \\
\frac{-2hA(2r_n)}{R} & \frac{L}{ih^{-1}(I - R)}
\end{bmatrix} + \bar{o}(h)
$$

We do not pursue the goal of making any accurate statements regarding these discretizations but that can be done.
Remarks and historical notes. The one-to-one correspondence between Krein systems and Dirac operators was discovered by M.G. Krein in his seminal paper [44]. Unfortunately, no proofs were given. The determinantal formulas obtained in this section are new to the best of our knowledge. In the meantime, analogous results for differential equations were obtained earlier (e.g. [38]). These ideas were also used quite recently [41]. If \( a(x), b(x) \in L^1(\mathbb{R}^+) \), the existence of wave operators follows from trace-class perturbation argument [59]. In the case \( a(x), b(x) \in L^p(\mathbb{R}^+), 1 < p < 2 \), the wave operators were studied by Christ and Kiselev [10]. The analysis was based on establishing the asymptotics of generalized eigenfunctions (essentially, asymptotics of \( P_r(r, \lambda) \)) for Lebesgue almost any value of \( \lambda \). If \( p > 2 \), one can use results from [43] to construct examples with no absolutely continuous spectrum. Thus in this case the wave operators might not exist at all. The proof of Theorem 13.3 is taken from [17]. Independently, Barry Simon obtained analogous results for CMV matrices in the Szegő case.
14. Schrödinger Operators

Let us consider Dirac operator (246) with \( b(r) = 0 \) and absolutely continuous \( a(r) \). For the corresponding Krein system, we have \( A(r) \in \mathbb{R} \) and from Lemma 7.3 we learn that \( H(x) \) is real and continuous on \( \mathbb{R} \), the measure \( d\sigma \) is even. Operator \( D \) takes form

\[
D = \begin{bmatrix}
0 & d/dr - a \\
-d/dr - a & 0
\end{bmatrix}
\]

It has the following domain of definition \( \{ f_1(r), f_2(r) \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+) \} \), \( f_1(2) \) are absolutely continuous, \( f_1' - af_2, f_1' + af_1 \in L^2(\mathbb{R}^+) \), \( f_2(0) = 0 \). Consider operator

\[
D^2 = \begin{bmatrix}
\mathcal{H}_1 & 0 \\
0 & \mathcal{H}_2
\end{bmatrix}
\]

where

\[
\mathcal{H}_1 = -\frac{d^2}{dr^2} + q_1, \quad f_1'(0) + a(0)f_1(0) = 0
\]

\[
\mathcal{H}_2 = -\frac{d^2}{dr^2} + q_2, \quad f_2(0) = 0
\]

and potentials are

\[
q_1 = a^2 - a', \quad q_2 = a^2 + a'
\]

Obviously, \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are different only by the order in factorization: \( \mathcal{H}_1 = \mathcal{O}^*\mathcal{O}, \mathcal{H}_2 = \mathcal{O}\mathcal{O}^* \), where \( \mathcal{O} = -d/dr - a \) is formal differential expression.

The following is true

\[
-\frac{d^2\varphi}{dr^2} + q_1\varphi = \lambda^2 \varphi, \quad \varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = -a(0)\varphi(0,\lambda)
\]

\[
-\frac{d^2\psi}{dr^2} + q_2\psi = \lambda^2 \psi, \quad \psi(0,\lambda) = 0, \quad \psi'(0,\lambda) = \lambda
\]

That means \( \varphi \) and \( \psi \) are generalized eigenfunctions for \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively.

Since \( D(\varphi(r,0),\psi(r,0))^t = 0 \), we get (compare with Lemma 4.5)

\[
\psi(r,0) = 0, \varphi(r,0) = \exp \left( - \int_0^r a(t)dt \right)
\]

Definition 1.5. The non-decreasing function \( \rho_{(h)}(E) \) is called a spectral measure for the general Schrödinger operator \( \mathcal{H}_1 = -\frac{d^2}{dr^2} + q \) with mixed boundary condition \( f'(0) = hf(0) \) if the following is true. For any \( f(r) \in L^2(\mathbb{R}^+) \), we have

\[
\int_0^\infty |f(r)|^2 dr = \int_{-\infty}^\infty \left| \int_0^\infty f(r)\varphi_{(h)}(r,E)dr \right|^2 d\rho_{(h)}(E)
\]

where \( \varphi_{(h)}(r,E) \) is the generalized eigenfunction, i.e. for any \( E \in \mathbb{R} \)

\[
-\varphi''_{(h)} + q\varphi_{(h)} = E\varphi_{(h)}, \quad \varphi_{(h)}(0,E) = 1, \varphi'_{(h)}(0,E) = h
\]

If \( h = 0 \), we get the Neumann boundary condition.
\textbf{Definition 14.1.} The non-decreasing function $\rho_{(\infty)}(E)$ is called a spectral measure for the general Schrödinger operator $\mathcal{H}_2$ with the Dirichlet boundary condition if the following is true. For any $f(r) \in L^2(\mathbb{R}^+)$, we have

$$\int_0^{\infty} |f(r)|^2 dr = \int_{-\infty}^{\infty} \left| \int_0^{\infty} f(r) \varphi_{(\infty)}(r, E) dr \right|^2 d\rho_{(\infty)}(E)$$

where $\varphi_{(\infty)}(r, E)$ is the generalized eigenfunction, i.e.

$$-\varphi_{(\infty)}' + q \varphi_{(\infty)} = E \varphi_{(\infty)}, \quad \varphi_{(\infty)}(0, E) = 0, \quad \varphi_{(\infty)}'(0, E) = 1$$

The spectral measure for the Schrödinger operator with locally integrable potential always exists. But it is not necessarily unique \cite{56, 49}.

One can use Lemma \ref{lem14.3} to prove the following Theorem. We consider the usual normalization of the measure $d\sigma$ by saying that the function $\sigma$ is odd and $\sigma(\lambda) = (\sigma(\lambda - 0) + \sigma(\lambda + 0))/2$.

\textbf{Theorem 14.1.} If $\rho_{(1,2)}$ are spectral measures for the operators $\mathcal{H}_{1,2}$, then

$$\rho_1(\lambda) = \begin{cases} 4\sigma(\sqrt{\lambda}), & \lambda \geq 0 \\ 0, & \lambda < 0 \end{cases}, \quad \rho_2(\lambda) = \begin{cases} \sqrt{\lambda}, & \lambda \geq 0 \\ 0, & \lambda < 0 \end{cases} \quad (275)$$

\textit{Proof.} Indeed, take any function $f(x) \in L^2(\mathbb{R}^+)$. Let $f(-x) = f(x), x > 0$ and consider $f(x)$ on the whole line. We have

$$[\mathcal{W}f](\lambda) = \int_{-\infty}^{\infty} E(x, \lambda)f(x) dx = 2 \int_0^{\infty} \varphi(x, \lambda)f(x) dx$$

Therefore, the first formula of (275) is straightforward due to Lemma \ref{lem13.1}. To get an expression for $\rho_2(\lambda)$, one should take the odd continuation of function $f(x)$. \hfill \Box

Notice that both $\rho_{(1,2)}(\lambda)$ are constants for $\lambda < 0$. That means $\mathcal{H}_{1,2}$ are both nonnegative operators. That is not surprising since $\mathcal{D}^2 \geq 0$ and $\mathcal{D}^2$ is decoupled into the direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$. Formula for $\rho_2(\lambda)$ shows that not only $\rho_2(\lambda)$ has no jump at 0 (no zero eigenvalue for $\mathcal{H}_2$), but also it decays at 0 in a certain way.

Consider the so-called “free” case, i.e. $a(r) = 0$. Then, $A(r) = 0$, $d\sigma = d\lambda/(2\pi)$, $q_1 = q_2 = 0$, $\varphi(x, \lambda) = \sin(\lambda x)$, $\psi(x, \lambda) = \cos(\lambda x)$. Moreover, $\rho_1(\lambda) = 2\lambda^{1/2}/\pi$ on $\mathbb{R}^+$ is the standard spectral measure for the free Schrödinger operator on $\mathbb{R}^+$ with the Neumann boundary condition and $\rho_2(\lambda) = 2\lambda^{3/2}/(3\pi)$ on $\mathbb{R}^+$ is the spectral measure for the free Schrödinger operator on $\mathbb{R}^+$ with the Dirichlet boundary condition.

Let us assume we are given nonnegative self-adjoint Schrödinger operator $\mathcal{H}_0 = -d^2/dr^2 + q$ with mixed boundary condition $f'(0) = hf(0)$ at zero and $q \in L^1_{\text{loc}}(\mathbb{R}^+)$. Denote its spectral measure by $d\rho_0(E)$. Then, there is the unique Dirac operator.
and Krein system that generate this Schrödinger operator in a way described above. Indeed, define the functions

\[ a(r) = -\frac{\psi'(r)}{\psi(r)}, \quad A(r) = a(r/2)/2 \tag{276} \]

where \( \psi(r) \) is solution to the equation \(-\psi'' + q\psi = 0, \psi'(0) = h, \psi(0) = 1\). We have \( \psi''(r) \in L_{\text{loc}}^1(\mathbb{R}^+) \) and \( \psi(r) > 0 \) by the oscillation theory for Sturm-Liouville operators ([74], p. 218). Therefore, \( a(r) \) is absolutely continuous on \( \mathbb{R}^+ \) and \( a(r) \) satisfies the Riccati equation

\[ q = a^2 - a' \tag{277} \]

Let \( A(r) = a(r/2)/2 \). Notice that \( A(r) \) is absolutely continuous. Krein system with coefficient \( A(r) \) generates \( \mathcal{K}_h \). Let us assume that there are two different \( A_1(r) \) and \( A_2(r) \) that generate the same \( \mathcal{K}_h \). Then, due to Theorem 14.1, measures \( \sigma_1 \) and \( \sigma_2 \) of these Krein systems are the same. Then, \( A_1 = A_2 \). Notice that \( A(0) = -h/2 \). Thus, we proved

**Lemma 14.1.** The operator \( \mathcal{K}_h \) is generated by the Krein systems if and only if \( \mathcal{K}_h \) is nonnegative. The Krein system is unique and \( A(r) \) is given by (270).

In general, Riccati equation (277) has many solutions. For instance, if \( q = 0 \), the general solution is given by

\[ a_h(r) = -h(1 + rh)^{-1}, h \geq 0 \tag{278} \]

But \( a_h(0) = -h \) so they can be all distinguished by the value at zero. If one considers \( q = 0 \) and boundary condition \( f'(0) = hf(0) \), then the corresponding operator is nonnegative for \( h \geq 0 \) only. In this case, \( \psi(r) = hr + 1 \) and the formula (270) gives exactly \( a_h(r) \).

The case of a Dirichlet boundary conditions is a bit subtle. Consider positive \( \mathcal{K}_\infty = -d^2/dr^2 + q \) with Dirichlet boundary condition at zero and locally summable potential \( q \). The problem here is that we don’t know \( a(0) \), the initial condition for solving Riccati equation

\[ q = a^2 + a' \]

Interestingly enough, we might have many Krein systems that correspond to the same Schrödinger operator \( \mathcal{K}_\infty \). For instance, all \( A_h(r) = -a_h(r/2)/2 \) (formula (275)) generate the same Schrödinger operator with Dirichlet boundary condition and \( q = 0 \). Different \( A_h \) have different \( \sigma_h \). Due to (275), these measures are different by the jump at zero only. Let us consider \( \sigma(\lambda) = \sigma_0(\lambda) + h\theta(\lambda)/2 \), where \( \theta(\lambda) \) is Heaviside function, \( h/2 \geq 0 \) is the jump at zero. Then, the corresponding \( H(x) = h/2 \) for all \( x \). Given any \( r > 0 \), solution of equation (35) is \( \Gamma_r(s,t) = h(2 + hr)^{-1} \). Then, \( A(r) = h(2 + hr)^{-1} = A_h(r) \).

Thus, the natural questions are when is \( \mathcal{K}_\infty \) generated by Krein system and how to describe all Krein systems that give rise to \( \mathcal{K}_\infty \)?

**Lemma 14.2.** The operator \( \mathcal{K}_\infty \) is generated by some Krein system if and only if there is some \( \psi(r) > 0 \) for \( r \geq 0 \) such that \(-\psi'' + q\psi = 0 \). Moreover, if \( A(r) \) is coefficient of Krein systems generating \( \mathcal{K}_\infty \), then

\[ A(r) = a(r/2)/2, \quad a(r) = \frac{\psi'(r)}{\psi(r)} \tag{279} \]

with some \( \psi(r) \) satisfying the properties given above.
Proof. If there is some positive $\psi(r)$ satisfying equation, then the Krein system can be easily constructed by letting

$$a(r) = \frac{\psi'(r)}{\psi(r)}, \quad A(r) = a(r/2)/2$$

Conversely, assume that there is at least one $A(r)$ generating $H_\infty$. Then, the dual system with coefficient $-A(r)$ will generate Dirac operator $D_\infty$. Let $\varphi_-(r,\lambda), \psi_-(r,\lambda)$ be the corresponding generalized eigenfunctions. Then, (273) yields

$$a(r) = \varphi'_-(r,0)/\varphi_-(r,0), A(r) = a(r/2)/2$$

and $\varphi(r,0)$ satisfies $-\varphi''(r,0) + q(r)\varphi(r,0) = 0$. □

There are several different ways to reformulate this criteria. For instance, solution $\psi(r,0)$ mentioned above exists if and only if the corresponding operator with the boundary condition $f'(0) = h_0 f(0)$ is non-negative for some $h_0$. Notice that if that is true for $h_0$, then it must be true for any $h > h_0$. One can also easily state this criteria in terms of the spectral measure $\rho_2$. If $\sigma$, obtained from the formula (275), generates a Krein systems, then this Krein system generates $H_\infty$.

In view of these two Lemmas, one can suggest the following reduction of Schrödinger operator to the Krein system. Let $H_0$ be any Schrödinger operator bounded from below. Add large positive number $\gamma$ to $H_0$ so that it becomes strictly positive. This transformation simply moves the spectrum to the right and does not change the spectral types. For $H_0 + \gamma$, the Lemma 14.1 is applicable. For $H_\infty$, the algorithm is the same but one has to apply Lemma 14.2.

Now, let us briefly discuss the solution to the inverse problem for Schrödinger operators. For $H_0$ bounded from below, the problem can be reduced to the inverse problem for Krein system which we know how to solve (just follow the construction in the first Sections). Assume that we are given the spectral measure $\rho$ of Schrödinger operator $H$ bounded from below. Assume also that the potential $q$ we want to find is continuous. Then, the asymptotics of $\rho$ at infinity is

$$\rho(\lambda) = \begin{cases} 2\lambda^{3/2}/(3\pi) + \bar{o}(\lambda^{1/2}), \text{ for the Dirichlet b.c.} \\ 2\lambda^{1/2}/\pi - h + \bar{o}(1), \text{ for mixed b.c. } f'(0) = hf(0) \end{cases}$$

From this asymptotics, we can find the corresponding boundary condition and apply one of the algorithms discussed above to find measure $\sigma$ for one of the Krein systems, generating $H$. Once $\sigma$ is known, we can find $A$, then $a$, and, finally, $q$. Notice also that this method gives a one-to-one correspondence between all Schrödinger operators, bounded from below, and spectral measures that yield accelerant: $H \in C^{m+1}(\mathbb{R}^+)$ iff the potential $q \in C^m(\mathbb{R}^+)$, $m$ is an integer. An accelerant $H$ is a.c. on $\mathbb{R}^+$ iff $q \in L_{loc}^1(\mathbb{R}^+)$.

There is a direct way of solving the inverse spectral problem for Schrödinger operators. This method is due to Gelfand and Levitan [25]. Let us discuss this method and compare it to Krein’s approach. Consider the operator (273) with a continuous potential. Assume that its spectral measure $\rho(h)$ is given. Define

$$\beta(\lambda) = \begin{cases} \rho(h)(\lambda) - 2\lambda^{1/2}/\pi, & \lambda \geq 0 \\ \rho(h)(\lambda), & \lambda < 0 \end{cases}$$
and

\[ F(x) = \lim_{n \to \infty} \int_{-\infty}^{n} \cos(\lambda^{1/2}x) d\beta(\lambda), \ x \geq 0 \]

It turns out that the limit exists and \( F(x) \) is continuously differentiable in \( x \). Consider

\[ F(x, y) = \frac{F(x + y) + F(x - y)}{2} \]

and an integral equation

\[ K(x, y) + F(x, y) + \int_{0}^{x} K(x, t) F(t, y) dt = 0, \quad 0 \leq y \leq x < \infty \quad (280) \]

One can prove that the solution \( K(x, y) \) exists and is unique. Then, the following relations solve the inverse problem.

\[ h = K(0, 0) = -F(0, 0), \quad q(x) = 2 \frac{d}{dx} K(x, x) \quad (281) \]

Now, let us make an assumption that the operator \( \mathcal{A} \) is nonnegative. Then, we can find the unique Krein system that generates \( \mathcal{A} \). Take \( \sigma(\lambda) = \rho(\lambda^2)/4 \). For an accelerant, we have the formal representation

\[ H(x) = \int_{-\infty}^{\infty} \cos(\lambda x) d(\sigma(\lambda) - \lambda/(2\pi)) = \frac{1}{2} \int_{0}^{\infty} \cos(\sqrt{\mu} x) d\left( \rho(\mu) - 2\mu^{1/2}/\pi \right) = F(x)/2 \]

One can check that

\[ -K(x, y) = \Gamma_{2x}(x + y, 0) + \Gamma_{2x}(x - y, 0) \quad (282) \]

Indeed, from (35), we have the following identities

\[ \Gamma_{2x}(x + y, 0) + \int_{0}^{x} H(y - u) \Gamma_{2x}(x + u, 0) du + \int_{0}^{x} H(y + u) \Gamma_{2x}(x - u, 0) du = H(x + y) \]
\[ \Gamma_{2x}(x - y, 0) + \int_{0}^{x} H(-y - u) \Gamma_{2x}(x + u, 0) du + \int_{0}^{x} H(u - y) \Gamma_{2x}(x - u, 0) du = H(x - y) \]

Since \( A(r) \) is real-valued, \( H(x) \) is a real-valued, even function. So, by adding the last two formulas, we get

\[ [\Gamma_{2x}(x + y, 0) + \Gamma_{2x}(x - y, 0)] + \int_{0}^{x} [H(y - u) + H(y + u)] \left[ \Gamma_{2x}(x - u, 0) + \Gamma_{2x}(x + u, 0) \right] du = \]
\[ = H(x + y) + H(x - y) \]

Therefore, we have (282), and \( K(x, x) = -[\Gamma_{2x}(2x, 0) + \Gamma_{2x}(0, 0)] \). Taking the derivative, we obtain

\[ 2 \frac{d}{dx} K(x, x) = 2(-A'(2x) + 2A^2(2x)) = a^2 - a' = q \]

and

\[ f'(0)/f(0) = -a(0) = -2A(0) = K(0, 0) \]
Thus, Krein’s approach gives the same answer and these two methods are essentially identical. The difference is that imposing condition on (280) to have the unique solution is weaker than saying that $F(x)/2$ is an accelerator. That allowed authors of [23] to deal with a more general situation. Notice that the inverse problem for $\sigma$ the Krein system is in fact the problem of the factorization for integral operators. Indeed, given measure $\sigma$, we construct the accelerator. For any $r > 0$, operator $1 + \mathcal{H}_r > 0$. Therefore, $\Gamma_r(x, y)$ exists. To find it, we need to factorize $(1 + \mathcal{H}_r)^{-1}$. Once we do that, $A(r) = \Gamma_r(0, r) = -V_r(0, r)$ by (19).

Let us consider the scattering theory for Schrödinger operator $\mathcal{H}_2$ defined by (272). We assume that $a$ has a compact support that belongs to, say, $[0, R]$. That means potential $q_2$ has a compact support too. Consider the Jost solution $F(0, \lambda) = \lambda^2 F$ defined by its asymptotics at infinity: $F(r, \lambda) = \exp(i\lambda r), r > R$. Let us introduce the scattering data

$$A_s(\lambda) = (F'(0, \lambda) + i\lambda F(0, \lambda))/(2i\lambda), B_s(\lambda) = (i\lambda F(0, \lambda) - F'(0, \lambda))/(2i\lambda),$$

$$T_s(\lambda) = A_s(\lambda)^{-1}, R_s(\lambda) = B_s(\lambda)/A_s(\lambda)$$

Function $F(0, \lambda)$ is called the Jost function for Schrödinger operator. The next Lemma relates scattering and spectral data for Schrödinger operator with Dirichlet boundary conditions to the corresponding parameters of Krein system. One should remember that we deal not with arbitrary Schrödinger but with the one generated by Krein system with compactly supported absolutely continuous coefficient.

**Lemma 14.3.** The following relations hold true

$$A_s(\lambda) = 2(\lambda) + \frac{a(0)(2(\lambda) + B(\lambda))}{2i\lambda}, B_s(\lambda) = B(\lambda) - \frac{a(0)(2(\lambda) + B(\lambda))}{2i\lambda},$$

$$R_s(\lambda) = f(\lambda) - \frac{a(0)(1 + f(\lambda))}{2i\lambda + a(0)(1 + f(\lambda))}, F(0, \lambda) = A_s(\lambda) + B_s(\lambda) = 2(\lambda) + B(\lambda) = \Pi(\lambda),$$

$$\rho^2_s(\lambda^2)/\lambda = \frac{1}{\pi|F(0, \lambda)|^2} = \frac{1}{\pi|\Pi(\lambda)|^2} = 2\sigma^2(\lambda)$$

**Proof.** The proof is a direct corollary from Lemma 13.3. Notice that if $a(0) = 0$, the data coincide with main parameters in Krein system.

Simple calculations show that $|A_s(\lambda)|^2 = 1 + |B_s(\lambda)|^2$ if $\lambda \in \mathbb{R} \setminus \{0\}$. That follows from the identity $|2(\lambda)|^2 = 1 + |B(\lambda)|^2$ which holds for any $\lambda \in \mathbb{R}$. These formulas also show that the analytical properties of $A_s(\lambda)$ and $R_s(\lambda)$ are worse than of the analogous functions for Dirac operator or Krein system. For instance, $A_s(\lambda)$ has pole at zero iff $a(0) \neq 0$. Moreover,

$$A_s(\lambda) = \frac{\Pi(\lambda)}{2} \left[ \frac{a(0)}{i\lambda} + 1 + F(\lambda) \right]$$

where $F(\lambda) = \Pi(\lambda)\Pi^{-1}(\lambda)$ is Weyl-Titchmarsh function for Krein system. Thus, we see that $A_s(\lambda)$ might also have zeroes in $\mathbb{C}^+$. These zeroes must be purely imaginary. Indeed, if $\lambda_0$ is such a zero, then $\lambda^2_0$ is eigenvalue for the Schrödinger operator considered on the whole line, a selfadjoint operator. Therefore, $\lambda^2_0 < 0$ and $\lambda_0$ is purely imaginary.

**Remarks and historical notes.**

For the first time, the inverse spectral problem for the Schrödinger operator was solved by Gelfand and Levitan [25]. Their approach is applicable to any operator,
not necessarily bounded from below. In the recent paper [67], Simon essentially introduced an “accelerant” directly for the Schrödinger operator. Different factorizations of Schrödinger operators and applications were discussed in many papers (see, for instance, [13]). These methods allow one to insert eigenvalues below the essential spectrum, for instance.
15. Scattering theory for Krein systems

In this section, we will consider the scattering theory for Krein systems and Dirac operators from slightly different perspective. The strategy is close to what is best known as approach by Marchenko and Agranovich to solution of inverse scattering problem for Sturm-Liouville operators [2]. We will try to emphasize the algebraic aspect of this argument, i.e. why Hankel operators appear and how their inversion is related to scattering data. Let us start with Krein systems. For simplicity, assume that the coefficient \( A(r) \) is finitely supported within, say, interval \([0, R]\) and is continuous on this interval. Then, clearly, there is a unique solution \( X_{sc}(r, \lambda) \) such that \( X_{sc}(r, \lambda) = X_0(r, \lambda) \), if \( r > R \) where

\[
X_0(r, \lambda) = \begin{bmatrix} e^{i\lambda r} & 0 \\ 0 & 1 \end{bmatrix}
\]

is the fundamental solution for \( A(r) = 0 \). This solution \( X_{sc}(r, \lambda) \) is normalized at infinity by its asymptotical behavior. We will study this solution and the scattering data it defines. Then, we will find its relation to spectral data and show how to solve an inverse scattering problem. This construction will be valid for more general case \( A(r) \in L^1(\mathbb{R}^+) \). In the meantime, let us assume first that \( A(r) \) is compactly supported and do some preliminary calculations. Clearly, if \( X_{sc} \) is given by its value at \( r = R \) (i.e., \( X_{sc}(R) = X_0(R) \)), we might try to study it by solving the Krein system backwards, from \( R \) to \( 0 \). That is equivalent to dealing with “mirrored” coefficient \( A^{(R)}(r) = A(R - r) \cdot X_{[0,R]}(r) \). Therefore, we can study the following problem: consider the Krein system with coefficient \( A_1(r) \), it will be later taken equal to \(-A^{(R)}(r)\) but so far it is an arbitrary function with support within \([0, R]\).

For this \( A_1(r) \), consider the following solution

\[
Y(r) = X(r) \begin{bmatrix} e^{-i\lambda R} & 0 \\ 0 & 1 \end{bmatrix}
\]

where \( X \) is the fundamental solution for \( A_1(r) \) normalized by \( X(0) = I \). Notice that

\[
X_1(r, \lambda) = Y(R-r,-\lambda)
\]

satisfies the following properties: \( X_1'(r, \lambda) = V_{\lambda,-A^{(R)}}X_1(r, \lambda) \) and \( X_1(R, \lambda) = X_0(R, \lambda) \). Thus, \( X_1(r, \lambda) \) is the scattering solution for \(-A^{(R)}_1\) and so if one wants to study the scattering problem for \( A(r) \), we just need to study \( Y(r) \) associated to \( A_1(r) = -A^{(R)}(r) \) and then use the formula \(283\).

Now, let us start with obtaining the formulas for elements of the matrix \( Y(r, \lambda) \).

To simplify the calculations, take

\[
Y_1(r, \lambda) = Y(r, \lambda) \begin{bmatrix} e^{i\lambda(R-r)} & 0 \\ 0 & 1 \end{bmatrix} = X(r) \begin{bmatrix} e^{-i\lambda r} & 0 \\ 0 & 1 \end{bmatrix}
\]

We have

\[
Y_1(r, \lambda) = \begin{bmatrix} \overline{\mathfrak{A}(r, \lambda)} e^{-i\lambda r} \mathfrak{B}(r, \lambda) \\ e^{-i\lambda r} \mathfrak{B}(r, \lambda) \mathfrak{A}(r, \lambda) \end{bmatrix}
\]

Let us write the following identity which follows from \(108\)

\[
Y_1(r, \lambda) \cdot \begin{bmatrix} \mathfrak{A}(r, \lambda) & -e^{-i\lambda r} \mathfrak{B}(r, \lambda) \\ -e^{-i\lambda r} \mathfrak{B}(r, \lambda) & \overline{\mathfrak{A}(r, \lambda)} \end{bmatrix} = I
\]

\[\text{In the calculations below we assume that } \lambda \text{ is real.}\]
or
\[
Y_1(r, \lambda) \cdot \begin{bmatrix}
1 & s_r(\lambda) \\
s_r(\lambda) & 1
\end{bmatrix} = \begin{bmatrix}
\mathcal{A}^{-1}(r, \lambda) & 0 \\
0 & \mathcal{A}(r, \lambda)^{-1}
\end{bmatrix}
\]
(285)

where \( s_r(\lambda) = -e^{-i\lambda r} \mathcal{B}(r, \lambda) \mathcal{A}^{-1}(r, \lambda) \). Let
\[
Y_1(r, \lambda) = I + \int_0^r K^{(r)}(s, 0) \begin{bmatrix}
e^{-i\lambda s} & 0 \\
0 & e^{i\lambda s}
\end{bmatrix} ds, K^{(r)}(s, 0) = \begin{bmatrix}
K_1^{(r)}(s, 0) & K_2^{(r)}(s, 0) \\
K_2^{(r)}(s, 0) & K_1^{(r)}(s, 0)
\end{bmatrix}
\]
\[
\mathcal{B}(r, \lambda) = \int_0^\infty C_r(x) e^{i\lambda x} dx, s_r(\lambda) = -\int_{-r}^\infty C_r(x+r)e^{i\lambda x} dx, J_r(x) = -C_r(-x) \chi_{[0, r]}(x)
\]

Subtract the identity matrix from both sides of (285), take adjoint of the matrices on both sides, and act by the operator
\[
\begin{bmatrix}
\mathcal{P}_+ & 0 \\
0 & \mathcal{P}_-
\end{bmatrix}
\]
from the left. We recall that \( \mathcal{P}_\pm \) are projections from \( L^2(\mathbb{R}) \) onto \( H^2(\mathbb{R}) \) and \( \overline{H^2}(\mathbb{R}) \). Then, we have the following integral equation for matrix \( K^{(r)}(x, 0), x \in [0, \infty) \)
\[
\begin{bmatrix}
I & \mathcal{G}_r^* \\
\mathcal{G}_r & I
\end{bmatrix} K^{(r)*}(., 0) + \begin{bmatrix}
0 & J_r(x) \\
J_r(x) & 0
\end{bmatrix} = 0
\]
(286)

and an operator
\[
[\mathcal{G}_r f](x) = \int_0^\infty J_r(x+u)f(u)du
\]
is acting in \( L^2(\mathbb{R}^+) \). Notice that \( K^{(r)}(x, 0) = 0 \) if \( x > r \). Consider an operator \( \mathcal{K}^{(r)} \) in \( [L^2(\mathbb{R}^+)]^2 \) such that
\[
\begin{bmatrix}
I + \mathcal{G}_r & \mathcal{G}_r^*
\end{bmatrix} (I + \mathcal{K}^{(r)*}) = I
\]
Operator \( \mathcal{G}_r \) is contractive Hankel operator since \( \mathcal{B}(r, \lambda) / \mathcal{A}(r, \lambda) \) is analytic contraction. Therefore, \( \mathcal{K}_r \) exists, is self-adjoint, and has matrix-valued kernel \( K^{(r)}(x, y) \) such that \( K^{(r)}(x, 0) \) is exactly the solution of (286). Now, let us compare these equations for different \( r \in [0, R] \). Notice that \( \mathcal{B}(R, \lambda) / \mathcal{A}(R, \lambda) \) is the Schur function for \( A_1(r) \). Denote
\[
C(x) = C_R(x), J(x) = -C(R-x) \chi_{[0, R]}(x)
\]
(287)

As it follows from (172) and (173), we have \( C_r(x) = C(x) \) for \( x \in [0, r] \). Therefore,
\[
J_r(x) = J(x + (R-r))
\]
(288)

and \( K^{(r)}(x, 0) \) can be obtained by inverting the matrix-valued Hankel operator. On the other hand, it can be expressed through kernels \( \Gamma, \hat{\Gamma} \) by formula (284). We then have
\[
K^{(r)}(x, 0) = -\frac{1}{2} \begin{bmatrix}
\Gamma_r(x, 0) + \hat{\Gamma}_r(x, 0) & \Gamma_r(x, r) - \hat{\Gamma}_r(r, x) \\
\Gamma_r(x, r) - \hat{\Gamma}_r(r, x) & \Gamma_r(0, x) + \hat{\Gamma}_r(0, x)
\end{bmatrix}
\]
Since and have the integral kernel of the operator \( A \)

Now, we are ready to translate these calculations to the original setting for coefficient \( A(r) \). Take \( A_1(r) = -A^{(R)}(r) \) and then use the formula \( 283 \). We then have

For \( Z(r, \lambda) \)

where \( L^{(r/2)}(x, y) = \mathcal{K}^{(R-r)}(x-r/2, y-r/2) \); \( x, y > r/2 \) satisfies

Now, let us obtain the formula for \( J_{R-r}(x+y-r) \). Notice that the Schur function for the coefficient \( A_1(r) = -A^{(R)}(r) \) is equal to \(-\mathfrak{B}(R, -\lambda)/\mathfrak{A}^*(R, -\lambda) \) (Lemma 7.5). If

then \( J_{R-r}(x+y-r) = D(x+y) \) by \( 287 \) and \( 288 \). Therefore, \( L^{(r/2)}(x, y) \) is an integral kernel of the operator \( \mathcal{L}^{(r/2)} \) in \( L^2([r/2, +\infty]) \) given by

and

Since \( |\mathfrak{B}(R, \lambda)/\mathfrak{A}(R, \lambda)| < 1 \), \( ||D_\rho|| < 1 \) for any \( \rho > 0 \). Therefore,

is invertible. Also, \( D_{r/2} = 0 \) if \( r > R \) so \( \mathcal{L}^{(r/2)} = 0 \) for \( r > R \).

---

The motivation to introduce \( Z(r, \lambda) \) comes from the fact that it is the scattering solution for Dirac operator, that will be made clear later.

Notice also that \( D(x) = 0 \) for \( x > R \).
Now, we want to take $R \to \infty$. For this, we need some regularity at infinity, just like we needed regularity of, say accelerant near zero in the previous constructions. The simplest and quite natural class of functions to consider is $L^1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$. 

**Theorem 15.1.** Let $A(r) \in L^1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$ be the coefficient in Krein’s system. Then, there exists the scattering solution $X_{sc}(r, \lambda)$ such that

$$X_{sc}(r, \lambda) = X_0(r, \lambda) + \tilde{\alpha}(1)$$

as $\lambda \in \mathbb{R}, r \to \infty$. This solution can be obtained as follows

$$X_{sc}(r, \lambda) = e^{i\lambda r/2}Z(r, \lambda), Z(r, \lambda) = \left[ \begin{array}{cc} e^{i\lambda r/2} & 0 \\ 0 & e^{-i\lambda r/2} \end{array} \right] + \int_{r/2}^{\infty} L^{(r/2)}(s, r/2) \left[ \begin{array}{cc} e^{i\lambda s} & 0 \\ 0 & e^{-i\lambda s} \end{array} \right] ds$$

where $L^{(r/2)}(x, y); x, y > r/2$ is the kernel of the operator $L^{(r/2)}$ :

$$[I + \mathcal{D}_{r/2}] [I + L^{(r/2)}] = I, \quad \mathcal{D}_{r/2} = \left[ \begin{array}{cc} 0 & \mathcal{D}_{r/2}^* \\ \mathcal{D}_{r/2} & 0 \end{array} \right], [\mathcal{D}_{r/2}f](x) = \int_{r/2}^{\infty} D(x+u)f(u)du$$

and $D(x)$ is given by

$$g(\lambda) = f(\lambda) \frac{\mathfrak{A}(\lambda)}{\mathfrak{A}(\lambda)} = \int_{-\infty}^{\infty} D(x)e^{i\lambda x} dx, D(x) \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$$

The coefficient $A(r)$ can be found from the following identity

$$A(r) = L_{21}^{(r/2)}(r/2, r/2)$$

*Proof.* As it follows from the section on the Baxter Theorem, Levy-Wiener Theorem, and Lemma 12.2, $D(x) \in L^1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$. Also, $\mathcal{D}_\rho$ is contraction in $L^2[0, \infty)$. By the general theory of Hankel operators, $\mathcal{D}_\rho$ is compact in any of $L^p[0, \infty), \infty > p \geq 1$ and any eigenvalue in $L^1[0, \infty)$ is also an eigenvalue in $L^2[0, \infty)$. This is because $D(x) \in L_\infty(\mathbb{R})$. In particular, means $I + \mathcal{D}_\rho$ is invertible in $L^1[0, \infty)$ and, therefore, $L^{(0)}(x, \rho) \in L^1[0, \infty) \cap C_0(\mathbb{R}^+)$. Moreover, $\|L^{(0)}(x, \rho)\|_{L^1[0, \infty)} \to 0$ as $\rho \to \infty$ which can be checked by simple iterations. Thus, $X_{sc}(r, \lambda)$ is well-defined and satisfies (289). Now, let us show that it is actually a solution.

Consider $R$– any positive number and let $A_R(r) = A(r) \cdot \chi_{[0, R]}(r)$. It has finite support and therefore the statement of the Theorem follows from the calculations given above. If $D_R(x)$ is the corresponding function, then $\|D_R(x) - D(x)\|_1 \to 0$ as $R \to \infty$. That follows from the proof of Baxter’s Theorem. Consequently, $L^R_r(x, r) \to L^R_{r}(x, r)$ and $X_{sc}(r, \lambda) \to X_{sc}(r, \lambda)$ as $R \to \infty$ and $r$ is fixed. On the other hand,

$$X_{sc}(r, \lambda) = X(r, \lambda)X^{-1}(r, \lambda)X_0(r, \lambda), r \in [0, R]$$

Then,

$$X^R_{sc}(r, \lambda) = X(r, \lambda) \left[ \begin{array}{cc} \mathfrak{A}(R, \lambda) & -\mathcal{B}(R, \lambda) \\ -\mathfrak{B}(R, \lambda) & \mathfrak{A}(R, \lambda) \end{array} \right], r \in [0, R]$$

For fixed $r$,

$$X^R_{sc}(r, \lambda) \to X(r, \lambda) \left[ \begin{array}{cc} \mathfrak{A}(\lambda) & -\mathcal{B}(\lambda) \\ -\mathfrak{B}(\lambda) & \mathfrak{A}(\lambda) \end{array} \right]$$
as $R \to \infty$. Therefore,
\[
X_{sc}(r, \lambda) = X(r, \lambda) \left[ \begin{array}{cc} \mathfrak{A}(\lambda) & -\overline{\mathfrak{B}(\lambda)} \\ -\overline{\mathfrak{B}(\lambda)} & \overline{\mathfrak{A}(\lambda)} \end{array} \right]
\]

Consequently, $X_{sc}(r, \lambda)$ is a solution. Relation (290) is true for truncated $A(r)$ and therefore holds after taking $R \to \infty$. \qed

Notice that
\[
X_{sc}(0, \lambda) = \left[ \begin{array}{cc} \mathfrak{A}(\lambda) & -\overline{\mathfrak{B}(\lambda)} \\ -\overline{\mathfrak{B}(\lambda)} & \overline{\mathfrak{A}(\lambda)} \end{array} \right]
\]

This matrix can be regarded as scattering data for Krein’s system. On the other hand, function $D(x)$ can also be regarded as scattering data. What is the relation between these functions? If $f(\lambda)$ or $\mathfrak{B}(\lambda)$ are given, then $|\mathfrak{A}(\lambda)|$ can be found from $|\mathfrak{A}(\lambda)|^{-2} = 1 - |f(\lambda)|^2$ or $|\mathfrak{A}(\lambda)|^2 = 1 + |\mathfrak{B}(\lambda)|^2$, respectively. Then, $\mathfrak{A}(\lambda)$ can be found from $|\mathfrak{A}(\lambda)|$ because it is outer. Function $\mathfrak{B}(\lambda)$ can be obtained from $\mathfrak{B}(\lambda) = f(\lambda) \mathfrak{A}(\lambda)$ and $D(x)$ may be recovered by taking inverse Fourier transform of $\mathfrak{B}(\lambda)/\mathfrak{A}(\lambda)$. The converse result can be obtained from the following calculations which is in the core of the method. Assume that $A(r) \in L^1(\mathbb{R}^+)$. Then,
\[
\left[ \begin{array}{cc} \mathfrak{A}(\lambda) & -\overline{\mathfrak{B}(\lambda)} \\ -\overline{\mathfrak{B}(\lambda)} & \overline{\mathfrak{A}(\lambda)} \end{array} \right] \left[ \begin{array}{c} 1 \\ g(\lambda) \end{array} \right] = \left[ \begin{array}{c} \mathfrak{A}^{-1}(\lambda) \\ \mathfrak{A}^{-1}(\lambda) \end{array} \right] (291)
\]

All elements in the matrices are from the Wiener algebra. Let
\[
\mathfrak{A}(\lambda) = 1 + \int_0^\infty \alpha(x)e^{i\lambda x}dx, \quad \mathfrak{B}(\lambda) = \int_0^\infty \beta(x)e^{i\lambda x}dx
\]

Writing down the integral equation for coefficients and taking suitable projections, we get
\[
\left[ \begin{array}{cc} 1 & -\mathcal{D}^* \\ -\mathcal{D} & 1 \end{array} \right] \left[ \begin{array}{c} \overline{\alpha(\cdot)} \\ \beta(\cdot) \end{array} \right] = \left[ \begin{array}{c} 0 \\ D(x) \end{array} \right] \quad (292)
\]

where $\mathcal{D}$ is the corresponding Hankel operator. Of course, this equation can be used to find $\mathfrak{B}(\lambda)$ and $\mathfrak{A}(\lambda)$. Consequently, if $D(x)$ is generated by Krein system, then it defines this system uniquely. The converse is also true.

**Theorem 15.2.** Assume that $D(x)$ is a given function from $L^1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$. If the Hankel operator $\mathcal{D}$ is contraction in $L^2(\mathbb{R}^+)$, then there is a unique Krein system that generates this $D(x)$. Moreover, the coefficient $A(r) \in L^1(\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$ generates $D(x)$ with these properties.

**Proof.** The second part follows from the arguments given above. Now, let us start with $D(x)$. Consider (291). Let $\mathfrak{A}(\lambda) = 1 + \phi(\lambda), g(\lambda) = g_1(\lambda) + g_2(\lambda), \mathfrak{A}^{-1}(\lambda) = 1 + \psi(\lambda)$, where $\phi(\lambda), g_1(\lambda), g_2(\lambda), \psi(\lambda) \in W_+(\mathbb{R})$. Then, (291) is equivalent to the algebraic system
\[
\phi = \overline{\mathfrak{B}}(g_1 + g_2), \quad \mathfrak{B} = g_1 + g_2 + \overline{\phi}(g_1 + g_2)
\]

That can be rewritten as a system for $\phi$ and $\mathfrak{B}$:
\[
\phi = \mathcal{P}_+(\mathfrak{B} g_1), \quad \mathfrak{B} = g_1 + \mathcal{P}_+(\overline{\phi} g_1) \quad (293)
\]

with two more equations
\[
g_2 = -\frac{\mathcal{P}_+(\phi \overline{g_1})}{1 + \phi}, \quad \psi = -\mathfrak{B} g_2 - \mathcal{P}_+(\mathfrak{B} \overline{g_1}) \quad (294)
\]
that determine $g_2$ and $\psi$ consecutively. The system (293) is equivalent to (292). Since $D$ is contraction, (292) has the unique solution. Now, that $\phi$ and $B$ are found, $g_2$ and $\psi$ can be found by (294). Notice carefully that we do not know yet that $\psi(\lambda) + 1$ is indeed an inverse to $1 + \phi(\lambda)$. It will be clear in a second. We have

**Lemma 15.1. Matrix**

\[
\begin{bmatrix}
\mathfrak{A}(\lambda) & \mathfrak{B}(\lambda) \\
\mathfrak{B}(\lambda) & \mathfrak{A}(\lambda)
\end{bmatrix}
\]

obtained in this way is $J$-unitary.

**Proof.** We only need to show that $|\mathfrak{A}|^2 = 1 + |B|^2$. The following is true

\[|\mathfrak{A}|^2 = 1 + |B|^2 \iff \phi + \bar{\phi} = |B|^2 - |\phi|^2 \iff \phi = P_+((|B|^2 - |\phi|^2) - \phi = P_+(B \bar{\phi} - \bar{\phi})
\]

Plug in the expressions for $g$ and $\phi$ from (293) into the last formula. Thus we just need to check that

\[\phi = P_+(B g_1) + P_+(B \bar{\phi}_+ (g_1 \bar{\phi})) - P_+(\bar{\phi} P_+ (g_1 \bar{\phi}))
\]

Due to the first identity in (293), the last equality can be written as

\[P_+(B P_+ (g_1 \bar{\phi})) = P_+(\bar{\phi} P_+ (g_1 \bar{\phi}))
\]

but this is always true since the left-hand side is

\[P_+(B P_+ (g_1 \bar{\phi})) = P_+(B g_1 \bar{\phi} - B P_- (g_1 \bar{\phi})) = P_+(g_1 \bar{\phi})
\]

and the right-hand side is

\[P_+(\bar{\phi} P_+ (g_1 \bar{\phi})) = P_+(\bar{\phi} g_1 \bar{\phi} - \bar{\phi} P_- (g_1 \bar{\phi})) = P_+(g_1 \bar{\phi})
\]

\[\square\]

As a corollary, we get $(1 + \phi) (1 + \psi) = 1$. Indeed, we have $g = B \mathfrak{A}^{-1}$ from (293) and (294). Then, $1 + \psi = B \mathfrak{A} - B \mathfrak{A} = (|\mathfrak{A}|^2 - |B|^2) \mathfrak{A}^{-1} = (1 + \phi)^{-1}$ by Lemma above.

Now, we can say that the function $f = B \mathfrak{A}^{-1}$ is analytic contraction for which the conditions of the Baxter Theorem hold true. Therefore, it generates the Krein system with $A(r) \in L^1(\mathbb{R}^+)$. Moreover, $\alpha(x), \beta(x) \in C_0(\mathbb{R}^+)$. Therefore, $C(x) \in C_0(\mathbb{R}^+)$. By Lemma 12.2 we have $A(r)$ is also from $C_0(\mathbb{R}^+)$. Clearly, the function $D(x)$ corresponds to this Krein system.

\[\square\]

The class of $D(x)$ we considered was the simplest one. In principle, the Hankel operator $\mathfrak{D}$ is bounded under much weaker conditions (e.g. BMOA space for $g_1$, see (277).

Let us make a remark regarding the Dirac operator. Notice that the matrix-function $Z(r, \lambda)$ introduced above satisfies the following properties

\[Z \equiv \frac{\lambda}{2} Z, \quad Z(r, \lambda) = \begin{bmatrix}
e^{i\lambda r/2} & 0 \\ 0 & e^{-i\lambda r/2} \end{bmatrix} + \tilde{\phi}(1)
\]

Thus, $Z(r, \lambda)$ is the scattering solution for the Dirac operator (written in a slightly different way, see (271)). All results from the section can be easily translated to this case.
Remarks and historical notes.  

The main results of this section are not new. They are essentially contained in the papers by Krein and Melik-Adamyan [45, 52, 53]. The authors were motivated by certain problem of continuation related to Hankel operators (and, essentially, coming from the complete solution of Nehari problem, see [1, 57]). Also, Melik-Adamyan considers slightly different canonical system. Using the approach of this section, one can provide another proof for continuous analog of Baxter’s Theorem. We tried to make the argument almost purely algebraic.

Also, one can use upper(lower)-triangular factorization of operator $D_r$ to represent its determinant via certain integral of $A(r)$. Analogous calculations will be done in the next section for truncated Wiener-Hopf operators. This is a way to obtain formula similar to the so-called Borodin-Okounkov identity [9].
16. Truncated Wiener-Hopf operators. The Strong Szegő Theorem

Consider the continuous accelerant \( H \). In this section, we obtain an important formula for the Fredholm determinant of \( I + \mathcal{H}_r \) in terms of the coefficient \( A(r) \) of the associated Krein system and prove continuous analog of the so-called Strong Szegő Theorem \cite{66}. We start with the following well-known result \cite{5}. We omit the proof which can be obtained, e.g., by upper(lower)-triangular factorization of \( I + \mathcal{H}_r \).

**Theorem 16.1.** If \( H(x) \) is a continuous accelerant on \( \mathbb{R} \), then

\[
\det(1 + \mathcal{H}_r) = \exp \left[ \int_0^r \Gamma_u(0,0)du \right]
\]

Now, assume that we are given a continuous accelerant \( H(x) \). We have a relation

\[
\frac{d}{du} \Gamma_u(0,0) = -|A(u)|^2
\]

and

\[
\Gamma_r(0,0) = H(0) - \int_0^r |A(s)|^2 ds
\]

\[
\det(1 + \mathcal{H}_r) = \exp[H(0)r] \exp \left[ - \int_0^r (r-s)|A(s)|^2 ds \right]
\]

(295)

and

\[
\det_2(1 + \mathcal{H}_r) = \exp \left[ - \int_0^r (r-s)|A(s)|^2 ds \right]
\]

(296)

Notice, that

\[
\frac{d}{dr} \ln \det(1 + \mathcal{H}_r) = H(0) - \int_0^r |A(s)|^2 ds, \quad \frac{d}{dr} \ln \det_2(1 + \mathcal{H}_r) = - \int_0^r |A(s)|^2 ds
\]

The fact that the both sides have limits for \( A(r) \in L^2(\mathbb{R}^+) \) can be regarded as the weak Szegő theorem. Notice that an approximation argument yields (296) without continuity assumption on \( H(x) \). The regularity \( H(x) \in L^2_{loc}(\mathbb{R}) \) is enough and the weak Szegő theorem reads: \( \ln \det_2(1 + \mathcal{H}_r) \) is nonincreasing and

\[
\lim_{r \to \infty} \frac{d}{dr} \ln \det_2(1 + \mathcal{H}_r) = - \int_0^\infty |A(s)|^2 ds
\]

The strong Szegő asymptotics is as follows

\[
\frac{\det(I + \mathcal{H}_r)}{\det_2(I + \mathcal{H}_r)} = \frac{\det_2(I + \mathcal{H}_r)}{\exp \left[ rH(0) - \int_0^r |A(s)|^2 ds \right]} = \exp \left[ \int_0^r s|A(s)|^2 ds \right]
\]
and the limit of left-hand side exists iff

$$G = \int_0^\infty r|A(r)|^2 dr < \infty$$  \hfill (297)

Notice that under this condition one has

$$r \int_0^r |A(s)|^2 ds = r \int_0^\infty |A(s)|^2 ds + o(1)$$

and assuming that $A(r) \in L^2(\mathbb{R}^+)$, we get

$$T_r = \frac{\det(I + \mathcal{H}_r)}{\exp \left[ r \left( H(0) - \int_0^\infty |A(s)|^2 ds \right) \right]} = \frac{\det_2(I + \mathcal{H}_r)}{\exp \left[ -r \int_0^\infty |A(s)|^2 ds \right]} = \exp \int_0^\infty \nu_r(s)|A(s)|^2 ds$$  \hfill (298)

where $\nu_r(x) = 1$ for $x \in [0, r]$ and $\nu_r(x) = rx^{-1}$ for $x > r$. Since $\nu_r(x)$ is non-negative, monotone in $r$, and $\nu_r(x) \leq 1$, the left-hand side is also increasing and has a finite limit iff \hfill (297) holds. The following result also gives a characterization of this case in terms of the spectral measure. That is also a continuous analog of Ibragimov’s and Golinskii-Ibragimov’s Theorems \hfill [27, 35, 66]. Some parts of the arguments below are borrowed from the discrete case \hfill [66].

**Theorem 16.2.** Assume $A(r) \in L^2(\mathbb{R}^+)$. Then, the following statements are equivalent

(i) $G < \infty$

(ii) The measure $d\sigma$ is purely a.c. and

$$\ln(2\pi \sigma'(\lambda)) = \int_{-\infty}^\infty l(x) \exp(i\lambda x) dx \in H^{1/2}(\mathbb{R})$$  \hfill (299)

(iii) $T_r$ is bounded

Moreover,

$$G = T_\infty = L = I$$  \hfill (300)

where\hfill \footnote{Recall that if $A(r) \in L^2(\mathbb{R})$, then $\ln[2\pi \sigma'(\lambda)] \in L^1(\mathbb{R}) + L^2(\mathbb{R})$ (Corollary \hfill [113]. Also, notice that condition $A(r) \in L^2(\mathbb{R})$ can be expressed purely in spectral terms, i.e. through $d\sigma$. In particular, the Theorem \hfill [113] and Lemma \hfill [177,12] from Appendix imply that conditions $d\sigma_s = 0$ and $\ln(2\pi \sigma'(\lambda)) \in H^{1/2}(\mathbb{R})$ guarantee $A(r) \in L^2(\mathbb{R}^+)$.}

$$L = \int_0^\infty x|l(x)|^2 dx, \quad I = \frac{1}{\pi} \int_{\mathbb{C}^+} \left| \frac{\partial_s \Pi_\alpha(\lambda)}{\Pi_\alpha(\lambda)} \right|^2 d^2 \lambda$$

where $\Pi_\alpha(\lambda)$ is given by \hfill (206).

**Proof.** The equivalence of (i) and (iii) as well as $G = T_\infty$ follow from the argument above (see \hfill (298)). The rest of the proof is divided into several Lemmas.

**Lemma 16.1.** If $G < \infty$, then $d\sigma_s = 0$. 

Proof. For real \( \lambda \),

\[
|P_\ast(r, \lambda)| \geq \exp\left( - \int_0^r |A(s)| ds \right) \geq \exp[-C \sqrt{\ln r}]
\]

That follows from Lemma 4.6. From (224), we get

\[
\int_{-\infty}^{\infty} \frac{|P_\ast(r, \lambda)|^2}{\lambda^2 + 1} d\sigma_\ast(\lambda) \leq C \int_{-\infty}^{\infty} |A(s)|^2 ds \leq Cr^{-1}
\]

Therefore,

\[
\int_{-\infty}^{\infty} \frac{d\sigma_\ast(\lambda)}{\lambda^2 + 1} \leq C \exp(C \sqrt{\ln r}) \rightarrow 0, \quad r \rightarrow \infty
\]

Thus \( d\sigma_\ast = 0 \). \( \square \)

Lemma 16.2. Assume \( g(\lambda) \) is outer in \( N(C^+) \), \( g(i\infty) = 1 \), and \( \ln |g(\lambda)| \in L^2(\mathbb{R}) + L^1(\mathbb{R}) \). Then, (301) below holds.

Proof. Using multiplicative representation for outer functions and normalization at \( i\infty \), we have

\[
\ln g(\lambda) = 2 \int_0^{\infty} \tilde{l}(x) \exp(i\lambda x) dx
\]

with \( \tilde{l}(x) \in L^2(\mathbb{R}^+) + W(\mathbb{R}) \cdot \chi_{\mathbb{R}^+} \). Simple calculations show that

\[
\int_0^{\infty} xe^{-2x|l(x)|^2} dx = (4\pi)^{-1} \int_{\text{Im } \lambda > \epsilon} |\partial_\lambda \ln g(\lambda)|^2 d^2 \lambda = (4\pi)^{-1} \int_{\text{Im } \lambda > \epsilon} \left| \frac{\partial_\lambda g(\lambda)}{g(\lambda)} \right|^2 d^2 \lambda
\]

Therefore, we always have

\[
\int_0^{\infty} x|\tilde{l}(x)|^2 dx = (4\pi)^{-1} \int_{C^+} \left| \frac{\partial_\lambda g(\lambda)}{g(\lambda)} \right|^2 d^2 \lambda \quad (301)
\]

even though the both quantities can be infinite. \( \square \)

For \( A(r) \in L^2(\mathbb{R}^+) \), we have \( \ln(2\pi e'(\lambda)) \in L^2(\mathbb{R}) + L^1(\mathbb{R}) \), \( \Pi_\ast(i\infty) = 1 \) and the Lemma can be applied to \( g(\lambda) = \Pi_{-\ast}^{-1}(\lambda) \) and thus we get \( L = I \) in (300).

We need the following auxiliary

Lemma 16.3. For any \( R \), we have the following inequality

\[
\int_0^R r|A(r)|^2 dr \geq \pi^{-1} \int_{C^+} \left| \frac{\partial_\lambda P_\ast(R, \lambda)}{P_\ast(R, \lambda)} \right|^2 d^2 \lambda \quad (302)
\]

Proof. For any \( T > 0 \), consider \( \Omega_T = \{ \lambda \in \mathbb{C}^+, |\lambda| \leq T \} \). Let us show that

\[
\int_0^R r|A(r)|^2 dr \geq \pi^{-1} \int_{\Omega_T} \left| \frac{\partial_\lambda P_\ast(R, \lambda)}{P_\ast(R, \lambda)} \right|^2 d^2 \lambda \quad (303)
\]
for any $T > 0$. Then the general statement follows upon taking $T \to \infty$. Fix any $T$. Then, it is sufficient to prove (303), assuming that $A(r) \in C[0, R]$. Indeed, any $A(r) \in L^2[0, R]$ can be approximated by continuous functions in $L^2[0, R]$ norm and the both sides of (303) are continuous in $A(r)$ with respect to $L^2[0, R]$ metric.

Then, we just need to use the suitable formula from the discrete case and an approximation result given by Corollary 9.1. Consider large $n$, the discretization step $h = R/n$, Verblunsky parameters given by (175), and the corresponding monic orthogonal polynomials $P_k(z)$ and $P_k^*(z)$. Then, we have the following formula (60, Theorem 2.1.4)

$$\ln \left[ \prod_{j=0}^{n} (1 - |a_j|^2)^{-j-1} \right] = \frac{1}{\pi} \int_{\mathbb{D}} \left| \frac{\partial \psi P_n^* (z)}{P_n^* (z)} \right|^2 d^2z$$  \hspace{1cm} (304)

With our choice of Verblunsky parameters,

$$\ln \left[ \prod_{j=0}^{n} (1 - |a_j|^2)^{-j-1} \right] \to \int_{0}^{R} r |A(r)|^2 dr, n \to \infty$$

by the Riemann sum approximation. Over $\Omega_T$, we have $P_n^*(e^{i\lambda h}) \to P_\lambda (R, \lambda)$ and

$$ihe^{i\lambda h} \partial_z P_n^* (e^{i\lambda h}) \to \partial_\lambda P_\lambda (R, \lambda)$$

Since $e^{i\lambda h} \to 1$ uniformly over $\Omega_T$, we also have

$$ih \partial_z P_n^* (e^{i\lambda h}) \to \partial_\lambda P_\lambda (R, \lambda)$$  \hspace{1cm} (305)

Therefore, making the change of variables and using (304) and (305), we get (303). \hfill \Box

For $A(r) \in L^2(\mathbb{R}^+)$, we have $P_\lambda (R, \lambda) \to \Pi_\alpha (\lambda)$ uniformly in $\{\text{Im } \lambda > \delta \}$ for any $\delta > 0$. Therefore, we always have an estimate $G \geq I$ and consequently (i) implies (ii) due to Lemmas 16.1, 16.2.

Now, we are left with proving that $G \leq I$. Let us assume we have purely a.c. measure with density satisfying $\ln(2\pi \sigma'(\lambda)) \in H^{1/2}(\mathbb{R})$ so that $L < \infty$ for the corresponding function $l(x)$. We need to show $G \leq L$ for the associated Krein system.

We know that $A(r) \in L^2(\mathbb{R}^+)$. Then, $2\pi \sigma'(\lambda) - 1 = \psi(x)$ with $\psi(x) \in L^2(\mathbb{R})$ by Lemma 17.12 Formulas (220) and (221) imply

$$H(x) = 2\pi \hat{\psi}(x)$$  \hspace{1cm} (306)

where $\hat{\psi}$ is Fourier transform of $\psi$.

Let us take Hermitian $l_R(x)$ such that $l_R(x)$ is continuous with compact support within $[-R, R]$ and

$$\int_{-\infty}^{\infty} (x^2 + 1)^{1/2} |l_R(x) - l(x)|^2 dx \to 0$$

as $R \to \infty$. Formula (306), Lemma 17.12 and Lemma 5.2 show that the corresponding $A_R(r) \to A(r)$ in $L^2[0, T]$ for any $T > 0$. For each $R$, we can apply the
Theorem 9.4. For the corresponding sequence of Verblunsky parameters, we have [66], Chapter 6:

$$\prod_{j=0}^{\infty} (1 - |a_j^{(n)}|^2)^{-j-1} = \exp \left[ \sum_{k=1}^{\infty} k |\hat{L}_k^{(n)}|^2 \right]$$  \hspace{1cm} (307)

where

$$\ln \mu'_n(\theta) = \sum_{k=-\infty}^{\infty} \hat{L}_k^{(n)} e^{ik\theta}$$

Clearly,

$$\sum_{k=1}^{\infty} k |\hat{L}_k^{(n)}|^2 \to \int_0^{R} x |l_R(x)|^2 dx$$

as \( n \to \infty \). On the other hand, for any \( \delta > 0 \), we have

$$\prod_{\delta < j \leq n} (1 - |a_j^{(n)}|^2)^{-j-1} \to \exp \left[ \int_{\delta}^{R} |A_R(r)|^2 dr \right]$$

as it follows from the Theorem 9.4. Therefore,

$$\int_0^{R} r |A_R(r)|^2 \leq \int_0^{R} x |l_R(x)|^2 dx$$

and then \( G \leq I \) since \( A_R(r) \to A(r) \) in \( L^2_{\text{loc}}(\mathbb{R}^+) \). So, \( G = I \) and the proof is finished. \( \square \)

The proof we used essentially utilized the strong Szegő Theorem for Toeplitz matrices and the approximation of continuous orthogonal system by the sequence of discrete ones. In the meantime, one could have adjusted the various proofs directly to continuous case (see, e.g. [9] for continuous analog of Borodin-Okounkov identity which can probably be used for this purpose). Notice that if \( d\sigma_k, k = 1, 2 \) satisfy conditions of Theorem 16.2 then \( d\sigma = [\sigma'_1]^{\gamma} [\sigma'_2]^{1-\gamma} d\lambda, \gamma \in [0, 1] \) also satisfies these conditions.

Formula (295) is very important for many applications. In the theory of random matrices, one needs to calculate asymptotics of Fredholm determinants for some specific accelerants. Assume that coefficient \( A \), corresponding to a given accelerant \( H \), tends to zero fast enough. Then, solving the inverse scattering problem by methods of the last Section, one can obtain an asymptotics of \( A(r) \) at infinity. Assume that \( A(r) \) decays at infinity fast enough such that

$$A(r) = \sum_{n=2}^{N} \frac{\gamma_n}{r^n} + \frac{C_{N+1}(r)}{r^{N+1}}$$

holds with constants \( \gamma_n \) and \( C_{N+1}(r) \in L^\infty(\mathbb{R}^+) \), \( N \) is arbitrary. Formula (295) and

$$\int_0^{r} (r-s)|A(s)|^2 ds = r \int_0^{\infty} |A(s)|^2 ds - r \int_s^{\infty} |A(s)|^2 ds - s \int_0^{\infty} |A(s)|^2 ds + \int_s^{\infty} s |A(s)|^2 ds$$
shows that as long as all $\gamma_n$, $\int_0^\infty |A(s)|^2 ds$, and $\int_0^\infty s|A(s)|^2 ds$ are known, we can compute complete asymptotic of $\det(1 + \mathcal{H}_r)$ as $r \to \infty$. But these two integrals can be explicitly expressed via the spectral data. The idea of using the inverse scattering theory to compute asymptotics of Fredholm determinants was pioneered by Dyson [22].

Remarks and historical notes.

Various generalizations of the strong Szegő formula to continuous case were obtained in [5, 39, 70]. Our version is optimal and new to our knowledge. On application of inverse scattering to random matrices see [22]. In [78], the theory of Krein systems was used to study an asymptotics of the certain Toeplitz determinants.
17. Appendix

In this Appendix, we collected the general results that we used in the main text.

17.1. For section 2. The proofs of the following two Lemmas are given in [69], p.71 and p.99.

Lemma 17.1. If $\Gamma_r(x, y)$ is the resolvent kernel for $K(x, y)$, which is continuous on $[0, r]^2$ (see Section 2), then

$$
\Gamma_r(x, y) = \frac{\delta_r(x, y)}{\delta_r}
$$

where

$$
\delta_r(x, y) = K \left( \begin{array}{c} x \\ y \end{array} \right) + \frac{1}{1!} \int_0^r K \left( \begin{array}{c} x \\ y \end{array} \right) \xi_1 \, d\xi_1 + \ldots
$$

and

$$
\delta_r = 1 + \int_0^r K \left( \begin{array}{c} \xi_1 \\ \xi_1 \end{array} \right) \, d\xi_1 + \frac{1}{2!} \int_0^r \int_0^r K \left( \begin{array}{c} \xi_1 \\ \xi_1 \end{array} \right) \xi_2 \, d\xi_1 d\xi_2 + \ldots
$$

The series converges absolutely.

The next Lemma gives Carleman-Hilbert determinantal representation for resolvent. We recommend an excellent book [31] (Theorem 2.2, p.207) for the modern presentation of that subject. It also contains the discussion of integral operators with discontinuity on the diagonal.

Lemma 17.2. If $\Gamma_r(x, y)$ is the resolvent kernel for $K(x, y) \in \hat{C}([0, r]^2)$ (see Section 2), then

$$
\Gamma_r(x, y) = \frac{\hat{\delta}_r(x, y)}{\hat{\delta}_r} \in \hat{C}([0, r]^2)
$$

where $\hat{\delta}_r(x, y)$ and $\hat{\delta}_r$ are defined as $\delta_r(x, y)$ and $\delta_r$ given above, but relative to the modified kernel $\hat{K}(x, y) = K(x, y)$ if $x \neq y$ and $\hat{K}(x, y) = 0$ on the diagonal.

17.2. For section 4. We begin with some simple facts about the linear spaces with indefinite metric. Let $[x, y] = (Jx, y)$ be indefinite inner product (in $\mathbb{C}^2$).

For any matrix $M$, introduce $M^{\circ} = JM^*J$. Then, $[Mx, y] = [x, M^{\circ}y]$. Clearly, $(AB)^{c} = B^{c}A^{c}$, $(A^{c})^{-1} = (A^{-1})^{c}$ if $A$ is invertible. The following result is well-known [37, 8]

Lemma 17.3. If $M$ is $J$–unitary, then $|\det M| = 1$, and $M^{-1}, M^{*}$ are $J$–unitary too. If $M$ is $J$–contraction then $M^{*}$ is $J$–contraction also.
Proof. Taking determinant of $M^*JM = J$, we get $|\det M| = 1$. It is straightforward that $M$ is $J$–unitary if and only if $M^{-1}$ is $J$–unitary.

Then, clearly, $M$ is $J$–unitary if and only if

$$M^*JMJ = I$$

but that means

$$M^*J = (MJ)^{-1}$$

so

$$MJM^*J = I$$

which is the same as saying that $M^*$ is $J$–unitary.

Assume that $M$ is $J$–contraction and 1 is not an eigenvalue. Then, we have a general algebraic formula

$$(M-I)^{-1}(MM^*-I)(M^*-I)^{-1} = (M-I)^{-1}(M^*M-I)(M-I)^{-1} = I+(M-I)^{-1}+(M^*-I)^{-1}$$

which can be rewritten as

$$MJM^* - J = JQ^*(M^*JM - J)QJ, Q = (M-I)^{-1}(M^*-I)$$

Since $Q$ is invertible, we have $M^*JM \leq J$ if and only if $MJM^* \leq J$. In other words, $M$ is $J$–contractive if $M^*$ is $J$–contractive. If 1 is an eigenvalue, multiply $M$ by unimodular scalar factor such that 1 is not an eigenvalue of the resulting operator and apply the argument above.

□

Lemma 17.4. Assume that

$$p(\lambda) = 1 + \int_0^r f(x)\exp(i\lambda x)dx$$

where $f(x) \in L^2[0,r]$ and $p(\lambda) \neq 0$ for $\lambda \in \mathbb{R}$. Then, $p(\lambda)$ is uniquely determined by

$$\mathcal{P}_{[-r,r]} \left[ \frac{1}{|p(\lambda)|^2} - 1 \right]$$

Proof. By Levy-Wiener theorem,

$$\frac{1}{|p(\lambda)|^2} - 1 = \int_{-\infty}^{\infty} \overline{h(x)} \exp(i\lambda x)dx$$

where $h(x) \in L^1(\mathbb{R})$ and $h(-x) = \overline{h(x)}$. We know $h(x)$ on $[-r,r]$ and need to find $f(x)$ on $[0,r]$. As in Lemma 3.8

$$\mathcal{P}_+ \left[ p(\lambda) \left( 1 + \int_{-\infty}^{\infty} \overline{h(s)} \exp(i\lambda s)ds \right) - 1 \right] = 0$$

which gives

$$f(x) + \overline{h(x)} + \int_0^r \overline{h(x-t)} f(t)dt = 0, 0 < x < r$$
This equation determines $f(x)$ on $[0, r]$ uniquely from $h(x)$ on $[-r, r]$ since $I + \mathcal{H} > 0$ where

$$\mathcal{H}g(x) = \int_0^r h(x-t)g(t)dt$$

The positivity of $I + \mathcal{H}$ follows from (308). □

17.3. For section 8.

Theorem 17.1. Let $d\mu(\lambda)$ be finite nonnegative measure defined on the whole line. Consider the linear manifold $L$ consisting of the finite linear combinations of exponents $\exp(i\lambda r)$ with $r \geq 0$. Then, $L$ is not dense in $L^2(d\mu)$ iff

$$\int_{-\infty}^{\infty} \frac{\ln \mu'(\lambda)}{1 + \lambda^2} d\lambda > -\infty$$

(309)

Proof. Denote the closure of $L$ in $L^2(d\mu)$ by $L_\mu$. Consider the function

$$f(\lambda) = \int_0^{\infty} \exp(-x + i\lambda x)dx = \frac{i}{i + \lambda}$$

We can find sequence $\{f_n\}$ in $L$ which is uniformly bounded on $\mathbb{R}$ in the $L^\infty$ norm and $f_n \rightarrow f$ in the uniform norm on any fixed compact in $\mathbb{R}$. For instance,

$$f_n(\lambda) = \sum_{j=0}^{n^2-1} \exp(i\lambda j/n) \int_{j/n-1}^{j/n+1} \exp(-x)dx$$

Therefore, $f \in L_\mu$ and $g = (\lambda - i)(\lambda + i)^{-1} = 1 - 2f \in L_\mu$. Let us show that $L_\mu$ is invariant under the multiplication by $g$. Because $g$ is an elementary Blaschke factor, $|g| = 1$ on the real line. Therefore, we only need to show that $g \psi \in L_\mu$ for any $\psi \in L$. But that clearly follows from

(i) $\psi$ is finite linear combination of exponents $\exp(i\lambda r)$ for different $r \geq 0$,
(ii) space $L_\mu$ is invariant under the multiplication by $\exp(i\lambda r)$ for any $r \geq 0$,
(iii) $g$ itself belongs to $L_\mu$.

Now, once we know that $L_\mu$ is invariant under the multiplication by $g$, we know that $g^n \in L_\mu$ for all $n \in \mathbb{Z}^+$. Let us consider the standard conformal map of $\lambda \in \mathbb{C}^+$ onto the unit disc: $w \in \mathbb{D}$, $w = (\lambda - i)(\lambda + i)^{-1}$. Under this map, measure $d\mu$ goes into a new finite measure $d\tau$ on the unit circle, which generates a new Hilbert space $L^2(d\tau)$. The subspace $L_\mu$ goes into the subspace $L_\tau$. Moreover, since all $g^n \in L_\mu$, $w^n \in L_\tau$ for any $n \in \mathbb{Z}$, $w \in \mathbb{T}$. Let us consider the subspace $Z_\tau$ obtained by closure of $Z = \text{Span}\{1, w, \dots, w^n, \dots\}$ in the $L^2(d\tau)$. Clearly $Z_\tau \subseteq L_\tau$. Let us show that actually $Z_\tau = L_\tau$. To do that, it is enough to prove that any function

$$h(w) = \exp\left[\frac{w + 1}{w - 1} r\right], r \geq 0, w \in \mathbb{T}$$

(which is an image of $\exp(i\lambda r)$ under the conformal map) can be approximated by analytic polynomials in $w$ in the $L^2(d\tau)$ metric. Consider functions

$$h_\rho(w) = \exp\left[\frac{\rho w + 1}{\rho w - 1} r\right], w \in \mathbb{T}$$
for $\rho < 1$. Since $d\mu$ is finite on $\mathbb{R}$, we have $\tau(-\varepsilon, \varepsilon) \to 0$ as $\varepsilon \to 0$. In other words, $\tau$ has no mass point at $w = 1$. Function $h(w)$ is bounded in $\mathbb{D}$ and is continuous there except for the point $w = 1$. Therefore, $h_\rho \to h$ in $L^2(d\tau)$ as $\rho \to 1$. At the same time, each $h_\rho$ is analytic in small neighborhood of $\mathbb{D}$ and therefore can be approximated by polynomials uniformly on the unit circle. Thus, $L_\tau = Z_\tau$. The Szegő theorem (66, Chapter 2) says that $Z_\tau$ is not dense in $L^2(d\tau)$ iff

$$\int_{-\pi}^{\pi} \ln \tau'(\theta) d\theta > -\infty$$

Clearly the last condition is equivalent to (309).

Notice that due to

$$\int_{-\infty}^{\infty} d\mu(\lambda) < \infty$$

we have that (309) is equivalent to

$$\int_{-\infty}^{\infty} \frac{\ln \mu'(\lambda)}{1 + \lambda^2} d\lambda > -\infty \quad (310)$$

This Theorem has interpretation in the theory of Gaussian stationary processes with continuous time and that is very useful point of view on the whole theory of Krein systems. In the meantime, there are the so-called Krein strings [21], differentiable operators more suitable to deal with stationary processes.

**Theorem 17.2.** Assume that $d\sigma$ is a measure on the real-line such that

$$\int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{1 + \lambda^2} < \infty$$

Consider the linear manifold $X$ of functions

$$\hat{f}(\lambda) = \int_{0}^{\infty} \exp(i\lambda x)f(x)dx, \quad 0 \leq r_1 < r_2$$

where $f(x) \in C^1[r_1, r_2]$ and is zero outside $[r_1, r_2] \subseteq [0, \infty)$. Then, $X$ is not dense in $L^2(d\sigma)$ iff

$$\int_{-\infty}^{\infty} \frac{\ln \sigma'(\lambda)}{1 + \lambda^2} d\lambda > -\infty \quad (311)$$

Moreover, let $\lambda_0 \in \mathbb{C}^+$. Then

$$\text{Dist} \left( \frac{1}{\lambda - \lambda_0}, \hat{X} \right)_{L^2(d\sigma)} = \frac{1}{\sqrt{2\text{Im} \lambda_0}} \exp \left[ \frac{\text{Im} \lambda_0}{2\pi} \int_{-\infty}^{\infty} \ln(2\pi \sigma'(\lambda)) \frac{d\lambda}{|\lambda - \lambda_0|^2} \right] \quad (312)$$

**Proof.** Consider a new measure $d\mu = d\sigma/(1 + \lambda^2)$ which is finite on the real line. Denote by $Y$ the linear manifold of functions of the following form $(\lambda + i)\hat{f}(\lambda), \hat{f} \in
Let $Y_\mu$ be the closure of $Y$ in $L^2(d\mu)$. We only need to show that $Y_\mu \neq L^2(d\mu)$ iff

$$\int_{-\infty}^{\infty} \frac{\ln \mu'(\lambda)}{1 + \lambda^2} d\lambda > -\infty$$  \hspace{1cm} (313)$$

Let $L_\mu$ be the space of functions from the proof of the Theorem 17.1, i.e. the closure in $L^2(d\mu)$ of finite linear combinations of exponents $\exp(i\lambda r), r \geq 0$. It is not difficult to show that $\exp(i\lambda r) \in Y_\mu$ for any $r \geq 0$. That follows from the representation

$$\exp(i\lambda r) = -i \exp(r)(\lambda + i) \int_{r_1}^{r_2} \exp(-x) \exp(i\lambda x) dx$$

So, $L_\mu \subseteq Y_\mu$. At the same time, each function

$$(\lambda + i) \int_{r_1}^{r_2} \exp(i\lambda x)f(x)dx = i(f(r_1)\exp(i\lambda r_1) - f(r_2)\exp(i\lambda r_2))$$

$$+ \int_{r_1}^{r_2} \exp(i\lambda x)(f'(x) + f(x))dx$$

can be approximated in $L^2(d\mu)$ by the finite linear combinations of exponents $\exp(i\lambda r)$. One should replace the integral by the Riemann sum and use continuity of the functions $f, f'$ to estimate the error. Thus $L_\mu = Y_\mu$ and one can use Theorem 17.1 to finish the proof of the first statement of the Theorem.

Now, let us obtain the formula for the distance. For simplicity, consider $\lambda_0 = i$. The general case can be treated in the same way. We have

$$\inf_{f \in \hat{X}} \int_{-\infty}^{\infty} \left| \frac{1}{\lambda - i} - \tilde{f}(\lambda) \right|^2 d\sigma = \inf_{f \in \hat{X}} \int_{-\infty}^{\infty} \left| 1 - \frac{\lambda - i}{\lambda + i} (\lambda + i) \tilde{f}(\lambda) \right|^2 d\sigma$$

$$= \inf_{y \in Y_\mu} \int_{-\infty}^{\infty} \left| 1 - \frac{\lambda - i}{\lambda + i} y(\lambda) \right|^2 d\mu(\lambda) = \inf_{y \in L_\mu} \int_{-\infty}^{\infty} \left| 1 - \frac{\lambda - i}{\lambda + i} y(\lambda) \right|^2 d\mu(\lambda)$$

$$= \inf_{v \in Z_\tau} \int |1 - wv(w)|^2 d\tau(w)$$

where the measure $d\tau(w)$ was obtained from $d\mu(\lambda)$ by mapping $\mathbb{C}^+$ onto $\mathbb{D}$ via $w = (\lambda - i)(\lambda + i)^{-1}$. Here we also used an approximation result from the proof of Theorem 17.1. For the last inf, we can use the Szegö formula [66], i.e.

$$\text{Dist}(1, wZ_\tau)_{L^2(d\tau)} = \exp \left[ \frac{1}{4\pi} \int_{0}^{2\pi} \ln(2\tau'(\theta))d\theta \right] = \frac{1}{\sqrt{2}} \exp \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln(2\pi \sigma'(\lambda)) d\lambda \right]$$

and the proof is finished. □
17.4. For section [9]

**Lemma 17.5.** If \( C(x) \in L^1[0, R] \), \( C(x) \) is continuous at zero and
\[
f(\lambda) = \int_0^R C(x) \exp(i\lambda x) dx
\]
then
\[
\lim_{y \to \infty} \frac{1}{y} \int_0^y s f(is) ds = C(0)
\]

**Proof.** The proof follows from the standard estimates:
\[
\lim_{y \to \infty} \frac{1}{y} \int_0^y s f(is) ds = C(0) + \lim_{y \to \infty} \frac{1}{y} \int_0^y \left[ \int_0^R [C(x) - C(0)] \exp(-sx) dx \right] ds = C(0)
\]
because the second term before the \( \lim \) can be bounded by
\[
C \left[ \omega_\delta(C) + \|C\|_1 + |C(0)| \right] \int_0^y s \exp(-s\delta) ds
\]
where \( \omega_\delta(C) = \sup_{x \in [0, \delta]} |C(x) - C(0)| \rightarrow 0 \) as \( \delta \rightarrow 0 \).

17.5. For section [10]. The following Lemma controls the zeroes of the continuous orthogonal polynomial

**Lemma 17.6.** Let
\[
p(\lambda) = 1 - \int_0^r \gamma(x)e^{-i\lambda x} dx
\]
where \( \gamma(x) \in C^2[0, r] \), \( \gamma(r) \neq 0 \). If \( \lambda_n \) are zeroes of \( p(\lambda) \) and \( |\lambda_1| \leq |\lambda_2| \leq \ldots \), then \( \lambda_n = \lambda_n^0 + o(1) \), \( n \to \infty \), where \( \lambda_n^0 = x_n + iy_n \) and
\[
x_n^2 + y_n^2 = |\gamma(r)|^2 \exp(2ry_n), \quad x_n = r^{-1} [\pi/2 + \pi n + \varphi], \quad n \in \mathbb{Z}, \quad (314)
\]
Here, \( \gamma(r) = |\gamma(r)| e^{i\varphi} \).

**Proof.** From Lebesgue-Riemann Lemma, \( \lambda_n \in \mathbb{C}^+ \) for large \( n \). Integrating by parts, we get
\[
p(\lambda) = 1 + \frac{1}{i\lambda} \left[ \gamma(r)e^{-i\lambda r} - \gamma'(0) \right] - \frac{1}{\lambda^2} \left[ \gamma'(r)e^{-i\lambda r} - \gamma'(0) \right] + \frac{1}{i\lambda} \int_0^r \gamma''(s)e^{-i\lambda s} ds
\]
Therefore, the equation \( p(\lambda) = 0 \) can be rewritten as
\[
\frac{e^{-i\lambda r}}{i\lambda} = \left[ 1 + \frac{\gamma(0)}{i\lambda} - \frac{\gamma'(0)}{\lambda^2} \right] \cdot \left[ \gamma(r) + \frac{\gamma'(r)}{i\lambda} - \frac{1}{i\lambda} \int_0^r \gamma''(s)e^{i\lambda (r-s)} ds \right]^{-1}
\]
\[
= \frac{1}{\gamma(r)} + O(|\lambda|^{-1})
\]
By Rouche’s Theorem, \( \lambda_n \) will be approaching the roots \( \lambda_n^0 \) of equation
\[
\frac{e^{-i\lambda r}}{i\lambda} = -\frac{1}{\gamma(r)}
\]
Then, the first equation in 313 easily follows upon taking the absolute value squared. The second one can be obtained by taking the real part of identity $\gamma(r)e^{-ir(x+iy)} = -i(x+iy)$, which yields $\cos(rx) = ye^{-ry|\gamma(r)|^{-1}} \to 0$, as $y \to \infty$. The last equation yields the needed quantization for $x_n$. \hfill \Box

The following result is due to Widom (see, e.g. [66], Lemma 8.1.9)

**Lemma 17.7.** (Widom’s lemma). Let $F,D$ be disjoint compact sets in $\mathbb{C}$ and $\mathbb{C}\setminus F$ connected. Then there is $m$ such that for any $z_1,z_2,\ldots,z_m \in D$ there is a monic polynomial $\tilde{Q}_m(z)$ of degree $m$, such that

$$\sup_{z \in F} \left| \frac{\tilde{Q}_m(z)}{\prod_{j=1}^m (z-z_j)} \right| \leq \frac{1}{2}$$

The following result is the mean-values formula for analytic functions of a special type.

**17.6. For section 11.**

**Lemma 17.8.** Assume that $g(\lambda) \in B(\mathbb{C}^+)$, $g(\lambda) \in L^1(\mathbb{R})$, $g(\lambda) = \tilde{\sigma}(1)$ as $|\lambda| \to \infty$ and $g(iy) = \tilde{\sigma}(y^{-1})$ as $y \to +\infty$. Then,

$$\int_{-\infty}^{\infty} \ln |1 + g(\lambda)| d\lambda = 0$$

**Proof.** Since $\Re(1 + g) > 0$ in $\mathbb{C}^+$ and $g \in B(\mathbb{C}^+)$, the function $1 + g$ is outer from $N(\mathbb{C}^+)$. Therefore,

$$\ln |1 + g(iy)| = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\ln |1 + g(\lambda)|}{\lambda^2 + y^2} d\lambda$$

Multiply the last identity by $y$ and take $y \to +\infty$. \hfill \Box

**Lemma 17.9.** If $f(\lambda)$ is such that $(\lambda+i)^{-1}f(\lambda) \in H^2(\mathbb{C}^+)$ and $f(\lambda) \in L^2(\mathbb{R})$, then $f(\lambda) \in H^2(\mathbb{C}^+)$. 

**Proof.** Since $f(\lambda) = (\lambda+i)g(\lambda)$ with $g(\lambda) \in H^2(\mathbb{C}^+)$, we have $f(\lambda) \in N(\mathbb{C}^+)$. Then, the statement of the Lemma follows, for example, from the multiplicative representation of $N(\mathbb{C}^+)$. \hfill \Box

**17.7. For section 12.** The following considerations are used in the discussion regarding the case $A(r) \in L^1(\mathbb{R}^+)$. We borrow the notations, statements, and proofs from [66], Chapter 5. For the reader’s convenience, we decided to include this material.

Let $X$ be a Banach space, $\mathcal{E}$–linear bounded operator, and $P_+$–projection (i.e. linear bounded operator such that $P_+^2 = P_+$). Notice that $P_- = I - P_+$ is also a projection.

**Definition 17.1.** The Toeplitz operator is an operator acting in $P_+(X)$ by the formula $T = P_+\mathcal{E}P_+.$

**Definition 17.2.** A linear bounded operator $\mathcal{L}$ is called upper triangular if $P_-\mathcal{L}P_+ = 0$ and $\mathcal{L}$ is lower triangular is $P_+\mathcal{L}P_- = 0.$
**Definition 17.3.** A linear bounded operator \( C \) is a Wiener-Hopf operator if \( C = L \mathcal{U} \) where \( L, \mathcal{U} \) are invertible, \( L, \mathcal{U}^{-1} \) lower triangular, \( \mathcal{U}, \mathcal{U}^{-1} \) upper triangular.

**Theorem 17.3.** (Wiener-Hopf Theorem). Let \( C \) be a Wiener-Hopf operator. Then, the corresponding Toeplitz operator \( T = P_+ CP_+ \) is invertible and

\[
T^{-1} = (P_+ \mathcal{U}^{-1} P_+)(P_+ L^{-1} P_+)
\]

Assume that \( Q, R \) are projections and

\[
QR = RQ = 0, Q + R = P_+ \tag{315}
\]

**Theorem 17.4.** (Baxter’s Lemma). Let \( C \) be a Wiener-Hopf operator so that \( C = L \mathcal{U} = \mathcal{U} L \). Consider \( Q, R \) obeying (315). Assume that

\[
RLQ = RL^{-1}Q = QU = QU^{-1}R = 0
\]

and

\[
\|P_+ L^{-1} R \mathcal{U}\| < 1/2, \|R \mathcal{U}^{-1} P_+ \| < 1/2
\]

Then, \( T_Q = Q \mathcal{E} Q \) is invertible and

\[
\|T_Q^{-1}\| < \|\mathcal{U}^{-1}\| + 2 \max(\|\mathcal{U}^{-1}\|, \|\mathcal{L}^{-1}\|)(\|P_+ \mathcal{L}^{-1}\| + \|R \mathcal{U}^{-1}\|))
\]

**Corollary 17.1.** Let \( C \) be a Wiener-Hopf operator and \( \{Q_n, R_n\} \) – sequence of projections obeying (315), with \( Q_n x \to x \) for any \( x \in P_+(X) \). If they also satisfy conditions of Baxter’s Lemma, then

\[
(Q_n C Q_n)^{-1} Q_n x - Q_n T^{-1} x \to 0, x \in P_+(X)
\]

Now, let us apply these mainly algebraic results to the concrete situation. Let \( H(x) \in L^1(\mathbb{R}) \) – Hermitian function and \( 1 + \nu(\lambda) > 0 \), where \( \nu(\lambda) \) – Fourier transform of \( H \). Let \( X \) be \( L^1(\mathbb{R}) \), \( C f = f + H * f \), \( [P_+ f] (x) = \chi_{R^+}(x) f(x) \). The function \( \nu(\lambda) \in W(\mathbb{R}) \) and \( 1 + \nu(\lambda) > 0 \). Therefore, by general result from the Wiener algebra theory, we have \( \hat{g}(\lambda) = \ln(1 + \nu(\lambda)) \in W(\mathbb{R}) \) so

\[
\hat{g} = \int_{-\infty}^{\infty} g(x) \exp(i\lambda x) dx = \int_{-\infty}^{0} g(x) \exp(i\lambda x) dx + \int_{0}^{\infty} g(x) \exp(i\lambda x) dx = \hat{g}_- + \hat{g}_+
\]

Notice that \( \hat{u} = \exp(\hat{g}_+) - 1 \in W_+ \) and \( \hat{\ell} = \exp(\hat{g}_-) - 1 \in W_- \). Therefore, \( C = \mathcal{L} \mathcal{U} \), where \( \mathcal{L} f = f + \ast f, \mathcal{U} f = f + \ast f \). Both operators \( \mathcal{U} \) and \( \mathcal{L} \) are invertible and one can easily check that \( \mathcal{U}, \mathcal{U}^{-1} \) are upper triangular, \( \mathcal{L}, \mathcal{L}^{-1} \) – lower triangular (notice that at the moment the definition of upper(lower) triangular operator is different from what we used in the section on factorization of integral operators). Therefore, the Wiener-Hopf theorem is applicable to the operator \( I + \mathcal{K}_\infty = P_+ C P_+ \). Let \( \Gamma(x) = (I + \mathcal{K}_\infty)^{-1} H(x) \).

Then, consider the following projections \( [Q_n f](x) = \chi_{[0,n]}(x) f(x) \) and \( [R_n f](x) = \chi_{[n,\infty)}(x) f(x) \). The result below is what we use in the proof of continuous analog of Baxter’s Theorem for OPUC. Recall that \( I + \mathcal{K}_\tau \) is given by (28) and can be regarded as an operator from \( L^1[0,\tau] \) to \( L^1[0,\tau] \) due to Young’s inequality.

**Corollary 17.2.** If \( n > n_0 \), then \( \|(I + \mathcal{K}_\tau)^{-1}\|_{L^1[0,\tau],L^1[0,\tau]} \to 0 \). Moreover, \( \|\Gamma_n(0,x) - \Gamma(x)\|_{L^1[0,n]} \to 0 \), where \( \Gamma_n(0,x) = (I + \mathcal{K}_\tau)^{-1} \chi_{[0,n]}(x) \cdot H(x) \).

**Proof.** Indeed, for \( n \) large enough, all conditions of Baxter’s lemma are satisfied which yields the necessary estimates on the norms. Then, in the Corollary 17.1 take \( x = H(t) \). All conditions of Corollary 17.1 are satisfied and we get the second statement on convergence. \( \square \)
The next Lemma shows that the spectral measure for Dirac operator is uniquely defined.

**Lemma 17.10.** The spectral measure $d\sigma_d$ for Dirac operator $\mathcal{D}$ is unique.

**Proof.** Indeed, if $\tau(\lambda)$ is another spectral measure, then Lemma 13.2 yields that

$$\hat{f}(\lambda) = \int_{-r}^{r} f(x) \exp(i\lambda x) dx \in L^2(\mathbb{R}, d\tau)$$

for any $f(x) \in L^2[-r, r], r > 0$. By taking $f(x) = \chi_{[0,R]}(x) \cdot \exp(-x)$ ($R$ is large), we have

$$\int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda^2 + 1} < \infty \quad (317)$$

Moreover, from the definition of the spectral measure,

$$\int_{\mathbb{R}} |\hat{f}(\lambda)|^2 d(\sigma_d(\lambda) - \tau(\lambda)) = 0$$

Using (317), we can approximate $\chi_{[a,b)}$ by functions in both $L^2(\mathbb{R}, d\sigma_d(\lambda))$ and $L^2(\mathbb{R}, d\tau(\lambda))$. We have

$$\int_{a}^{b} d(\sigma_d(\lambda) - \tau(\lambda)) = 0$$

for any $a$ and $b$. That implies $d\tau = d\sigma_d$. \hfill \Box

The following result is quite elementary. It is used in the proof of existence of wave operators for Dirac operator with square summable potential.

**Lemma 17.11.** For the unperturbed operator $\mathcal{D}_0$, the action of the group $e^{-it\mathcal{D}_0}$ is given by the formulas

$$e^{-it\mathcal{D}_0} \begin{bmatrix} f \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} f(x + t) + f(x - t) \\ -i(f(x - t) - f(x + t)) \end{bmatrix},$$

where $f(x) \in L^2(\mathbb{R}^+) \text{ is extended to the whole line as an even function.}$

$$e^{-it\mathcal{D}_0} \begin{bmatrix} 0 \\ f \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -i(f(x + t) - f(x - t)) \\ f(x - t) + f(x + t) \end{bmatrix},$$

$f(x)$ is extended to $\mathbb{R}$ as an odd function.

**Proof.** One can use the definition of $\exp(-it\mathcal{D}_0)$ to verify this statement directly. Another way to see that is to use the spectral resolution for $\mathcal{D}_0$.

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \rightarrow F(\lambda) = \int_{0}^{\infty} f_1(x) \cos(\lambda x) dx + \int_{0}^{\infty} f_2(x) \sin(\lambda x) dx,$$  

$$f_1(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\lambda) \cos(\lambda x) d\lambda, \quad f_2(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\lambda) \sin(\lambda x) d\lambda.$$
Therefore,
\[
(e^{-itD_0} f)_1(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-it\lambda} F(\lambda) \cos(\lambda x) d\lambda,
\]
\[
(e^{-itD_0} f)_2(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-it\lambda} F(\lambda) \sin(\lambda x) d\lambda.
\]

It now suffices to apply the Fourier inversion formula. □

17.9. For section [14] The following Lemma proves one simple property of the exponential map on the $H^{1/2}(\mathbb{R})$ functions.

**Lemma 17.12.** If $\gamma(x) \in H^{1/2}(\mathbb{R})$, then $e^{\gamma(x)} - 1 \in L^2(\mathbb{R})$ and this map is continuous.

**Proof.** We have
\[
e^{\gamma(x)} = 1 + \sum_{n=1}^{\infty} \frac{\gamma^n(x)}{n!}, \|\gamma^n(x)\|_2 = \|\hat{\gamma} * \ldots * \hat{\gamma}\|_2
\]
\[
\|\hat{\gamma}(\omega)\|_p \leq C \left( \frac{2 - p}{2p - 2} \right)^{(2-p)/(2p)} \|\gamma\|_{H^{1/2}(\mathbb{R})}, 1 < p < 2
\]
by Hölder’s inequality. By Young’s inequality we now have
\[
\|\hat{\gamma} * \ldots * \hat{\gamma}\|_2 \leq \|\hat{\gamma}\|_{p_n}^n, p_n = 2n(2n - 1)^{-1}
\]
So, $\|\gamma^n(x)\|_2 \leq C n^{n/2} \|\gamma\|_{H^{1/2}(\mathbb{R})}^n$. The Stirling formula for factorial yields convergence of the series and continuity of the exponential map. □
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