Sylvester-Cayley vector partitions algorithm and the Gaussian polynomials

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August 2, 2023

Abstract

We extend an algorithm suggested in 1858 by Sylvester [12] and implemented in 1860 by Cayley [4] for a problem of double partitions and apply it to derivation of explicit expressions for coefficients of the Gaussian polynomials through convolution of restricted partition functions.

Keywords: double partitions, Gaussian polynomials, Bernoulli polynomials of higher order, Eulerian numbers.

2010 Mathematics Subject Classification: 11P82.

1 Partition functions

1.1 Scalar partition function

Consider a problem of finding a number of nonnegative integer solutions of a Diophantine equation

\[ \sum_{i=1}^{m} x_i d_i = x \cdot d = s, \quad d = \{d_1, d_2, \ldots, d_m\}. \] (1)

A restricted partition function \( W(s, d) \) solving the above problem is a number of partitions of an integer \( s \) into positive integers \( \{d_1, d_2, \ldots, d_m\} \), each not greater than \( s \). The generating function for \( W(s, d) \) has a form

\[ G(t, d) = \prod_{i=1}^{m} \frac{1}{1 - t^{d_i}} = \sum_{s=0}^{\infty} W(s, d) \ t^s. \] (2)

In other words the partition function equals a constant term in Taylor expansion of the following expression

\[ W(s, d) = \text{const} \left[ t^{-s} \prod_{i=1}^{m} (1 - t^{d_i})^{-1} \right]. \] (3)

Sylvester proved [13] a statement about splitting of the partition function into periodic and non-periodic parts and showed that the restricted partition function may be presented as a sum of the Sylvester waves [10]

\[ W(s, d) = \sum_{j=1}^{\infty} W_j(s, d), \] (4)
where summation runs over all distinct factors $j$ of the elements $d_i$. The wave $W_j(s,d)$ is a quasipolynomial in $s$ closely related to prime roots $\rho_j$ of unity; it is a coefficient of $t^{-1}$ in the series expansion in ascending powers of $t$ of the generator

$$F_j(s,t) = \sum_{\rho_j} \frac{\rho_j^{-s}e^{st}}{\prod_{k=1}^{m} \left(1 - \rho_j^{d_k} e^{-d_k t}\right)}.$$  

(5)

The summation is made over all prime roots of unity $\rho_j = \exp(2\pi in/j)$ for $n$ relatively prime to $j$ (including unity) and smaller than $j$. It was shown [10] that it is possible to express the Sylvester wave as a finite sum of the Bernoulli polynomials of higher order.

1.2 Vector partition function

Consider a function $W(s,D)$ counting the number of integer nonnegative solutions $x \geq 0$ to the linear system $D \cdot x = s$, where $D$ is a nonnegative integer $l \times m$ matrix. The vector partition function (VPF) $W(s,D)$ is a natural generalization of the restricted partition function to the vector argument.

The generating function for the VPF reads

$$G(t,D) = \prod_{i=1}^{m} \frac{1}{1 - t^{c_i}} = \sum_{s} W(s,D)t^{s}, \quad t^{s} = \prod_{k=1}^{l} t^{s_k}, \quad t^{c_i} = \prod_{k=1}^{l} t^{d_{ki}},$$

(6)

where $c_i$, $1 \leq i \leq m$, denotes the $i$-th column of the matrix $D$. Note that some elements $d_{ki}$ might equal zero.

The partition problems that include inequalities can be reduced to the VPF problem as follows. Assume that we have a system of $k$ linear equations and $l - k$ linear inequalities for $m > l$ integer nonnegative variables $x_i$:

$$\sum_{i=1}^{m} d_{ji}x_i = s_j, \quad (1 \leq j \leq k), \quad \sum_{i=1}^{m} d_{ji}x_i \leq n_j, \quad (k + 1 \leq j \leq l).$$

(7)

Introduce $l - k$ new integer variables $x_i$, $(k + 1 \leq i \leq l)$ that enter additively all above equations and inequalities with corresponding factors $d_{ji} = 0$ for $1 \leq j \leq k$, and $d_{ji} = 1$ for $k + 1 \leq j \leq l$, and transform all inequalities into equations

$$\sum_{i=1}^{m+l-k} d_{ji}x_i = s_j, \quad (1 \leq j \leq k), \quad \sum_{i=1}^{m+l-k} d_{ji}x_i = n_j, \quad (k + 1 \leq j \leq l).$$

(8)

It is easy to see that the number of integer nonnegative solutions of (8) equals that of (7), and thus any system of linear equations with constraints can be reduced to the standard vector partition problem.

1.3 Sylvester-Cayley method of compound partitions

The problem of vector partitions has a long history and J.J. Sylvester made a significant contribution to its solution. He put forward [12] an idea to reduce VPF into a sum of scalar partitions (2) (see also [8]). The reduction is an iterative process based on the variable elimination in the generating function (6). Sylvester considered a specific double partition problem as an illustration of his method and determined regions (chambers) of a plane $\{s_1, s_2\}$ each having a unique expression for
VPF valid in this region only. He showed that the expressions in the adjacent chambers coincide at their common boundary (see also [11]).

This approach was successfully applied by A.Cayley [4] to double partitions subject to some restrictions on the positive elements of matrix \( D \) (the columns \( c_i \) are linearly independent and the inequality \( c_{i2} < s_2 + 2 \) should hold for all \( 1 \leq i \leq m \)).

In this manuscript we present the result for double partitions obtained by Cayley and extend it to apply to the Gaussian polynomial problem.

2 Double partitions

Introduce an augmented matrix \( E \) obtained by prepending the column vector \( s \) to the matrix \( D \), so that \( e_{10} = s_1 = r \) and \( e_{20} = s_2 = \rho \). Denote the rest elements of first row of \( E \) as \( e_{1i} = d_{1i} = b_i, \ (1 \leq i \leq m) \), and of the second row by \( e_{2i} = d_{2i} = \beta_i, \ (1 \leq i \leq m) \). For simplicity assume that all \( \beta_i > 0 \), and perform a partial fraction expansion (PFE) step to present \( G(t, D) \) as sum of \( m \) fractions

\[
G(t, D) t^{-s} = t^{-s} \prod_{i=1}^{m} \frac{1}{1 - t_1^{i} t_2^{j}} = \sum_{i=1}^{m} T_i(t), \quad T_i = \frac{A_i(t) t^{-s}}{1 - t_1^{i} t_2^{j}}.
\]

Then the solution reads as a sum of the terms

\[
U_i = \text{const}_t \left[ T_i(t) \right].
\]

Cayley showed that the first term \( U_1 \) corresponding to elimination of the second column of matrix \( E \) can be written as

\[
U_1 = \text{const}_t \left[ t_1^{-(r \beta_1 - b_1 \rho)} \prod_{j \neq 1} (1 - t_1^{b_j \beta_1 - b_1 \beta_j})^{-1} \right].
\]

Comparing this to (8) we see that \( U_1 \) corresponds to \( W(r \beta_1 - b_1 \rho, d_1) \), where the elements \( d_{1j} \) of the vector \( d_1 \) are given by \( d_{1j} = b_j \beta_1 - b_1 \beta_j, \ j \neq 1 \). Effectively, this result corresponds to elimination [6] of the first unknown \( x_1 \) in the system \( D \cdot x = s \) that leads to a single Diophantine equation \( d_1 \cdot x' = r \beta_1 - b_1 \rho \), where \( x' \) is the vector of unknowns obtained from \( x \) by dropping \( x_1 \).

Similar transformations allow to find all \( m \) contributions to obtain

\[
W(s, D) = \sum_{i=1}^{m} U_i = \sum_{i=1}^{m} W(L_i, d_i), \quad L_i = r \beta_i - b_i \rho, \quad d_{ij} = b_j \beta_i - b_i \beta_j, \ j \neq i,
\]

where both \( L_i \) and \( d_{ij} \) as the determinants of \( 2 \times 2 \) matrices made of the columns \( \{c_0, c_i\} \) and \( \{c_j, c_i\} \), respectively. Cayley mentioned [4] that the contribution of each term in (12) is nonzero only when \( L_i \) is nonnegative, thus reintroducing the notion of chambers of the vector partition. He also pointed out that when some elements of the vector \( d_i \) are negative (say, \( d_{ijk} < 0 \) for \( 1 \leq k \leq K \)) one has

\[
W(L_i, d_i) = (-1)^K W(L_i - \sum_{k=1}^{K} |d_{ijk}|, |d_i|), \quad |d_i| = \{|d_{ij}|\}.
\]

Consider now a case when one or more elements in one (say, the second) row of the matrix \( D \) are zeroes. Denote the number of nonzero elements as \( m_0 \), and without loss of generality we assume that \( \beta_i \neq 0 \) for \( 1 \leq i \leq m_0 \), and \( \beta_i = 0 \) for \( m_0 + 1 \leq i \leq m \). Using the above result for the columns
with \( \beta_i = 0 \) we observe that each of them leads to a contribution \( W(L_i, d_i) = 0 \) that vanishes as its argument \( L_i = -b_i \rho < 0 \) is negative. Thus we obtain

\[
W(s, D) = \sum_{i=1}^{m_0} W(L_i, d_i), \quad L_i = d_{i0} = r \beta_i - b_i \rho, \quad d_{ij} = b_j \beta_i - b_i \beta_j, \quad j \neq i.
\] (14)

### 3 Gaussian polynomials and their coefficients

The Gaussian polynomial coefficients \( P_n^m(s) \) are defined as a number of nonnegative integer solutions of a linear Diophantine equation with a constrain

\[
\sum_{i=1}^{m} i x_i = s, \quad \sum_{i=1}^{m} x_i \leq n.
\] (15)

The coefficients \( P_n^m(s) \) of the Gaussian polynomial \( G_n^m(t) \) satisfy

\[
G_n^m(t) = \frac{G_m(t)G_n(t)}{G_{m+n}(t)} = \sum_{s=0}^{mn} P_n^m(s)t^s, \quad G_m(t) = \prod_{i=1}^{m} \frac{1}{1 - t^i}.
\] (16)

where \( G_m(t) \) denotes the generating function for the restricted partition function \( W_m(s) = W(s, d^m = \{1, 2, \ldots, m\}) \). The quasipolynomials \( P_n^m(s) \) have a finite order \( mn \) and the following properties

\[
P_n^m(0) = P_n^m(mn) = 1, \quad P_n^m(s) = 0, \quad \text{for} \quad s > mn,
\]

\[
P_n^m(s) = P_n^m(s) = P_n^m(mn - s), \quad P_n^m(mn/2 - s) = P_n^m(mn/2 + s),
\] (17)

Cayley [3] considered a problem of computation of the Gaussian polynomial coefficients and found the generating functions (16) and (19, see below). Some interesting results about explicit expression of \( P_n^m(s) \) through the regular partition functions were reported recently in [7].

#### 3.1 Sylvester-Cayley algorithm for Gaussian polynomial coefficients

The problem (15) is a particular case of (7) for \( k = 1, \quad l = 2, \) and \( d_1 = i, \quad d_2 = 1, \) thus it can be transformed into a VPF problem

\[
\sum_{i=0}^{m} ix_i = s, \quad \sum_{i=0}^{m} x_i = n,
\] (18)

and dealt with using the Sylvester-Cayley algorithm.

Write [14] for

\[
D = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 2 & \cdots & m
\end{pmatrix}, \quad S = \begin{pmatrix}
n \\
s
\end{pmatrix},
\]

that determines a generating function for \( P_n^m(s) \) with fixed integers \( m \) and \( n \)

\[
\prod_{i=0}^{m} \frac{1}{1 - at^i} = \sum_{n=0}^{\infty} G_m(t)a^n = \sum_{n=0}^{\infty} \sum_{s=0}^{mn} P_n^m(s)t^s.
\] (19)
In (14) we have \( m_0 = m, r = n, \rho = s, b_i = 1, \beta_i = i \) and obtain \( L_i = ni - s \) and \( d_{ij} = \beta_i - \beta_j = i - j \). Thus the set \( d_i \) can be written as a union of \( d^i \) and \( -d^{m-i} \). We then observe from (14)

\[
P^m_m(s) = \sum_{i=1}^{m} w_i(m, n; s), \quad w_i(m, n; s) = W(ni - s, -d^{m-i} \cup d^i), \tag{20}
\]

where the \( i \)-th term in the above sum contributes for \( 0 \leq s \leq ni \). This immediately leads to a conclusion that \( P^m_m(s) = 0 \) for \( s > nm \) as all \( m \) terms in (20) vanish. By the representation (20) the Gaussian polynomial \( P^m_m(s) \) has \( m \) chambers bounded by lines \( s = ni, 0 \leq i \leq m \). Specifically, in the \( k \)-th chamber bounded by the lines \( s = n(m - k) \) and \( s = n(m - k + 1) \) one has to retain only \( k \) terms in the sum with \( m - k + 1 \leq i \leq m \). Using (13) we arrive at

\[
w_i(m, n; s) = (-1)^{m-i}W(ni - s - s_{m-i}, \hat{d}^i \cup \hat{d}^{m-i}), \quad s_m = m(m + 1)/2, \tag{21}
\]

where the partition function \( W \) can be computed using the algorithm discussed in [10]. It is convenient to introduce \( P^m_m(r, s) \) that explicitly accounts for the summands entering \( P^m_m(s) \) in the \( r \)-th chamber

\[
P^m_m(r, s) = \sum_{i=r}^{m} w_i(m, n; s), \quad (r - 1)n \leq s \leq rn. \tag{22}
\]

### 3.2 Convolution of restricted partitions

Consider a general term \( W(s, \hat{d}^{k_1} \cup \hat{d}^{k_2}) \) in (21). Its generating function reads \( G(t) = G_{k_1}(t)G_{k_2}(t) \) and we obtain

\[
\sum_{s=0}^{\infty} W(s, \hat{d}^{k_1} \cup \hat{d}^{k_2}) t^s = \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} W_{k_1}(s_1)W_{k_2}(s_1) t^{s_1+s_2},
\]

leading to a discrete convolution (the Cauchy product)

\[
W(s, \hat{d}^{k_1} \cup \hat{d}^{k_2}) = \sum_{k=0}^{s} W_{k_1}(k)W_{k_2}(s - k),
\]

and setting here \( k_1 = i, \ k_2 = m - i \) we find

\[
W(s, \hat{d}^i \cup \hat{d}^{m-i}) = \sum_{k=0}^{s} W_i(k)W_{m-i}(s - k).
\]

Substituting it in (21) and (20) we obtain

\[
P^m_m(s) = \sum_{i=1}^{m} \sum_{k=0}^{ni-s-s_{m-i}} (-1)^{m-i}W_i(k)W_{m-i}(ni - s - s_{m-i} - k),
\]

where the chamber boundaries remain unchanged. Noticing that \( W_0(s) = \delta_{s,0} \) we rewrite the above expression as

\[
P^m_m(s) = W_m(mn - s) + \sum_{i=1}^{m-1} \sum_{k=0}^{ni-s-s_{m-i}} (-1)^{m-i}W_i(k)W_{m-i}(ni - s - s_{m-i} - k), \tag{23}
\]
where the first term in r.h.s. contributes to all chambers. The partial expression $P_m^n(r, s)$ for the $r$-th chamber reads

$$P_m^n(r, s) = W_m(mn-s) + \sum_{i=r}^{m-1} \sum_{k=0}^{ni-s-s_{m-i}} (-1)^{m-i}W_i(k)W_{m-i}(ni-s-s_{m-i}-k), \ (r-1)n \leq s \leq rn. \quad (24)$$

The relations (17) imply a symmetry $s \leftrightarrow mn - s$ leading to an equivalent representation

$$P_m^n(r, s) = W_m(s) + \sum_{i=1}^{r-1} \sum_{k=0}^{s-in-s_i} (-1)^iW_{m-i}(k)W_i(s - ni - s_i - k), \ (r-1)n \leq s \leq rn, \quad (25)$$

and we obtain a counterpart of (23)

$$P_m^n(s) = W_m(s) + \sum_{i=1}^{m-1} \sum_{k=0}^{s-in-s_i} (-1)^iW_{m-i}(k)W_i(s - ni - s_i - k). \quad (26)$$

Comparison of (24) to (25) leads to a symmetry relation

$$P_m^n(r, s) = P_m^n(m + 1 - r, mn - s), \quad (27)$$

that reduces the computation of $P_m^n(s)$ to the first $[(m + 1)/2]$ chambers.

The formulas (25, 27) present a closed form of the Gaussian polynomial coefficients as a superposition of the restricted partitions discrete convolutions. Below we illustrate the above approach by deriving explicit expressions of $P_m^n(s)$ for small $m = 3, 4$ (see also [5]). More cumbersome expressions required for derivation of $P_5^n(s)$ and $P_6^n(s)$ are presented in Appendix.

### 3.3 Explicit formula for $P_3^n(s)$

Use (23) or (26) to find

$$P_3^n(s) = W_3(s) + \sum_{i=1}^{2} \sum_{k=0}^{s-in-s_i} (-1)^iW_{3-i}(k)W_i(s - ni - s_i - k),$$

or

$$P_3^n(s) = W_3(3n - s) + \sum_{i=1}^{2} \sum_{k=0}^{ni-s-s_3-s_i} (-1)^{i+1}W_i(k)W_{3-i}(ni - s - s_{3-i} - k),$$

and we have $W_1(s) = 1$, $s_2 = 3$, $s_1 = 1$ to obtain

$$P_3^n(1, s) = W_3(s) = W_3(3n - s) + \sum_{k=0}^{n-s-3} W_2(k) - \sum_{k=0}^{2n-s-1} W_2(k), \ 0 \leq s \leq n,$$

$$P_3^n(2, s) = W_3(3n - s) - \sum_{k=0}^{2n-s-1} W_2(k), \ n \leq s \leq 2n,$$

$$P_3^n(3, s) = W_3(3n - s), \ 2n \leq s \leq 3n. \quad (28)$$

Recall that $W_2(s) = s/2 + 3/4 + (-1)^s/4$ and find

$$\Sigma_2(s) = \sum_{k=0}^{s} W_2(k) = \frac{(s + 1)(s + 3)}{4} + \frac{1 + \cos \pi s}{8},$$
to arrive at the final expression

\[ P_3^n(s) = \begin{cases} 
W_3(s), & 0 \leq s \leq n, \\
W_3(3n - s) - \Sigma_2(2n - s - 1), & n \leq s \leq 2n, \\
W_3(3n - s), & 2n \leq s \leq 3n,
\end{cases} \quad (29) \]

where

\[ W_3(s) = \frac{47}{72} + \frac{s}{2} + \frac{s^2}{12} + \frac{1}{8} \cos \pi s + \frac{2}{9} \cos \frac{2\pi s}{3}. \quad (30) \]

Note that in [5] the authors completely describe \( P_3^n(s) \) using 36 explicit expressions while our representation requires only three formulas in (29). It indicates that the approach based on Sylvester-Cayley algorithm employing the idea of chambers provides more compact and clear result.

### 3.4 Explicit formula for \( P_4^n(s) \)

Start with

\[ P_4^n(s) = W_4(s) + \sum_{i=1}^{3} \sum_{k=0}^{s-in-s_i} (-1)^i W_4-i(k) W_i(s-ni-s_i-k), \]

and use \( s_3 = 6 \) to write

\[ P_4^n(s) = W_4(s) - \sum_{k=0}^{s-n-1} W_3(k) + \sum_{k=0}^{s-2n-3} W_2(k) W_2(s-2n-3-k) - \sum_{k=0}^{s-3n-6} W_3(s-3n-6-k), \quad (31) \]

where

\[ W_4(s) = \frac{2s^3 + 30s^2 + 135s + 175}{288} + \frac{(s + 5)}{32} \cos \pi s + \frac{1}{8} \cos \frac{\pi s}{2} + \frac{2}{27} \left( \cos \frac{2\pi s}{3} - \cos \frac{2\pi (s+1)}{3} \right). \quad (32) \]

### 3.5 Maximal coefficient of Gaussian polynomial

The problem of computation of the maximal coefficient of Gaussian polynomial was addressed recently in [5] (also L. Fel, private communication).

First note that by (17) the maximal coefficient \( p_m^n \) is given by \( p_m^n = P_m^n(mn/2) \) for even \( mn \) and \( p_m^n = P_m^n((mn + 1)/2) \) for odd \( mn \). For even \( m = 2k \) we have to compute \( P_m^n(s) \) at \( s = kn \) that belongs to the \((k + 1)\)-th chamber where \( kn \leq s \leq (k + 1)n \). For odd \( m = 2k - 1 \) the argument of \( P_m^n \) belongs to the middle \( k \)-th chamber with \((k - 1)n \leq s \leq kn \). Thus we have three cases

\[ p_m^n = \begin{cases} 
P_{2k}^n(k + 1, kn), & m = 2k, \\
P_{2k-1}^n(k, 2kr - r), & m = 2k - 1, n = 2r, \\
P_{2r-1}^n(k, 2kr - k - r + 1), & m = 2k - 1, n = 2r - 1.
\end{cases} \quad (33) \]

Use here \( k = 2 \) and for \( m = 3 \) we find \( p_3^{2r} = P_3^{2r}(2, 3r) \), and \( p_3^{2r-1} = P_3^{2r-1}(2, 3r - 1) \); with \( m = 4 \) we obtain \( p_4^1 = P_4^n(3, 2n) \).

Consider first the case \( m = 3 \). For even \( n = 2r \) we obtain \( s = 3r \), so that \( 3n - s = 3r \) and \( 2n - s - 1 = r - 1 \) leading to \( P_3^{2r}(2, 3r) = W_3(3r) - \Sigma_2(r - 1) \). For odd \( n = 2r - 1 \) we use \( s = 3r - 1 \), that gives \( 3n - s = 3r - 2 \) and \( 2n - s - 1 = r - 1 \) producing \( P_3^{2r}(2, 3r-1) = W_3(3r-2) - \Sigma_2(r - 1) \). After algebraic transformations we arrive at

\[ p_3^{2r} = \frac{1}{2} \left( (r + 1)^2 + \cos \frac{\pi r}{2} \right), \quad p_3^{2r-1} = \frac{r(r + 1)}{2}. \quad (34) \]
For \( m = 4 \) we find
\[
P_4^n(3, 2n) = W_4(2n) + \sum_{k=0}^{n-1} W_3(k),
\]
that reduces to
\[
p_4^n = \frac{(2n + 5)^3}{288} + \frac{(2n + 5)}{32} + \frac{3}{16} \cos \pi n + \frac{4}{27} \cos \frac{2\pi n}{3} + \frac{4}{27} \sin \frac{\pi (8n + 1)}{6}.
\]
(35)

Turning to case \( m = 5 \) we use \( k = 3 \) and consider two separate cases – for even \( n = 2r \) we obtain \( p_5^{2r} = P_5^{2r}(3, 5r) \), while with odd \( n = 2r - 1 \) we have \( p_5^{2r-1} = P_5^{2r-1}(3, 5r - 2) \). The corresponding expressions read
\[
p_5^{2r} = \frac{(23r^2 + 69r + 49)(r + 1)(r + 2)}{288} + \frac{425}{1728} + \frac{3(2r + 3)}{64} \cos \pi r + \frac{4}{27} \cos \frac{2\pi r}{3},
\]
(36)
\[
p_5^{2r-1} = \frac{(23r^2 + 46r + 36)r(r + 2)}{288} - \frac{37}{1728} - \frac{1}{64} \cos \pi r - \frac{1}{8} \sin \frac{\pi r}{2}
\]
\[+ \frac{2}{27} \cos \frac{2\pi r}{3} - \frac{2}{27} \sin \frac{\pi (8r + 1)}{6}.
\]
(37)

Finally, for \( m = 6 \) we again take \( k = 3 \) to obtain \( p_6^n = P_6^n(4, 3n) \) leading to
\[
p_6^n = \frac{(2n + 7)(66n^4 + 924n^3 + 4606n^2 + 9604n + 12061)}{172800} + \frac{2n^2 + 14n + 55}{256} \cos \pi n
\]
\[+ \frac{1}{16} \left( \cos \frac{\pi r}{2} - \sin \frac{\pi r}{2} \right) + \frac{4}{81} \cos \frac{2\pi n}{3} + \frac{4}{81} \sin \frac{\pi (4n + 1)}{6} + \frac{8}{125} \left( \cos \frac{2\pi n}{5} + \cos \frac{4\pi n}{5} \right)
\]
\[+ \frac{4}{125} \left( - \cos \frac{\pi (2n + 1)}{5} + \cos \frac{\pi (4n + 1)}{5} + 2 \cos \frac{\pi (8n + 1)}{5} \right)
\]
\[+ \frac{4}{125} \left( \sin \frac{\pi (4n + 1)}{10} + 2 \sin \frac{\pi (8n + 1)}{10} - \sin \frac{\pi (12n + 1)}{10} \right).
\]
(38)

It is easy to see from (34-38) that the behavior of the maximal coefficient is mainly determined by the polynomial part of the corresponding expressions. This observation leads us to consideration of the polynomial part of Gaussian polynomial coefficients.

# 4 Polynomial part of Gaussian polynomial coefficients

## 4.1 Polynomial part of partition function

It is known that a leading contribution to the partition function \( W(s, d) \) for the set of generators \( d = \{d_1, d_2, \ldots, d_m\} \) is provided by its polynomial part \( w(s, d) \) that can be expressed through the Bernoulli polynomials of higher order \( B_k(s, d) \) [10]

\[
w(s, d) = \frac{B_{m-1}(s + \sigma_m, d)}{(m - 1)!\pi_m}, \quad \sigma_m = \sum_{i=1}^{m} d_i, \quad \pi_m = \prod_{i=1}^{m} d_i,
\]
(39)

where the Bernoulli polynomials of higher order can be defined in umbral calculus notation as

\[
B_k(s, d) = (s + \sum_{i=1}^{m} i d_i)^k,
\]
after the expansion the replacement \((iBd_i)^k \to B_k d_i^k\) is applied, and \(B_k\) denotes the \(k\)-th Bernoulli number. This definition implies

\[
B_k(s, d) = \sum_{l=0}^{k} \binom{k}{l} s^l B_{k-l}(d),
\]

(40)

and \(B_k(d) \equiv B_k(0, d)\) denotes the Bernoulli number of higher order.

### 4.2 Polynomial part of Gaussian polynomial coefficients

It is instructive to consider a polynomial part \(P_m^n(s)\) of quasipolynomial \(P_m^n(s)\)

\[
P_m^n(s) = \sum_{i=1}^{m} W_i(m, n; s),
\]

(41)

where \(W_i(m, n; s)\) denotes the polynomial part of \(w_i(m, n; s)\) defined in (21). For \(\hat{d}^m\) we have

\[
\sigma_m = s_m, \quad \pi_m = m!
\]

and we find that

\[
w(s, \hat{d}^i \cup \hat{d}^{m-i}) = \frac{B_{m-1}(s + s_i + s_{m-i}, \hat{d}^i \cup \hat{d}^{m-i})}{(m-1)!i!(m-i)!}.
\]

Use here the replacement \(s \to ni - s - s_{m-i}\) to obtain

\[
P_m^n(s) = \sum_{i=1}^{m} (-1)^{m-i} \frac{B_{m-1}(ni - s + s_i, \hat{d}^i \cup \hat{d}^{m-i})}{(m-1)!i!(m-i)!}.
\]

(42)

The polynomial part \(P_m^n(r, s)\) of the solution in \(r\)-th chamber reads

\[
P_m^n(r, s) = \sum_{i=r}^{m} (-1)^{m-i} \frac{B_{m-1}(ni - s + s_i, \hat{d}^i \cup \hat{d}^{m-i})}{(m-1)!i!(m-i)!}, \quad (r-1)n \leq s \leq rn.
\]

(43)

### 5 Polynomial part of maximal coefficient

The last result leads to expressions for the polynomial part of the maximal coefficient of the Gaussian polynomial. For small \(m \leq 6\) we have

\[
P_2^n(2, n) = \frac{2n + 3}{4},
\]

\[
P_3^n(2, 3r) = \frac{(r+1)^2}{2} + \frac{1}{36}, \quad P_3^n(2, 3r - 1) = \frac{(3r + 1)(3r + 2)}{18},
\]

\[
P_4^n(3, 2n) = \frac{(2n + 5)^3}{288},
\]

\[
P_5^n(3, 5r) = \frac{(r+1)(r+2)(23r^2 + 69r + 49)}{288} + \frac{571}{14400},
\]

\[
P_5^n(3, 5r - 2) = \frac{(r+2)(23r^2 + 46r + 36)}{288} + \frac{89}{1600},
\]

\[
P_6^n(4, 3n) = \frac{(2n + 7)(66n^4 + 924n^3 + 4606n^2 + 9604n + 7511)}{172800}.
\]

(44)
5.1 Polynomial part for even $m$

For even $m = 2k$ the maximal coefficient equals

$$
P_{2k}^n(k + 1, kn) = \sum_{i=k+1}^{2k} (-1)^i \frac{B_{2k-1}(ni - nk + s_i, \hat{d}_i \cup \hat{d}^{2k-i})}{(2k-1)!i!(2k-i)!}
$$

$$
= \sum_{j=0}^{k} (-1)^j \frac{B_{2k-1}(n(k-j) + s_{2k-j}, \hat{d}_j \cup \hat{d}^{2k-j})}{(2k-1)!j!(2k-j)!}
$$

$$
= \sum_{j=0}^{k} (-1)^j \frac{B_{2k-1}(n(k-j), -\hat{d}^{2k-j} \cup \hat{d}^j)}{(2k-1)!(2k)!}, \quad (45)
$$

where we employ the relation $B_n(s, -d) = B_n(s + \sigma_m, d)$. Use here (40) to write

$$
P_{2k}^n(k + 1, kn) = \frac{1}{(2k-1)!(2k)!} \sum_{p=0}^{2k-1} \binom{2k-1}{p} n^p \sum_{j=0}^{k} (-1)^j \binom{2k}{j} B_{2k-p-1}(-\hat{d}^{2k-j} \cup \hat{d}^j)(k-j)^p. \quad (46)
$$

5.2 Polynomial part for odd $m = 2k - 1$ and even $n = 2r$

In this case the maximal coefficient reads

$$
P_{2k-1}^{2r}(k, (2k-1)r) = \sum_{i=k}^{2k-1} (-1)^{i+1} \binom{2k-1}{i} \frac{B_{2k-2}(2r(i-k) + r, -\hat{d}_i \cup \hat{d}^{2k-1-i})}{(2k-2)!(2k-1)!}
$$

$$
= \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} \frac{B_{2k-2}(2r(k-1-j) + r, -\hat{d}^{2k-1-j} \cup \hat{d}^j)}{(2k-2)!(2k-1)!}, \quad (47)
$$

and we find with (40)

$$
P_{2k-1}^{2r}(k, (2k-1)r) = \frac{1}{(2k-2)!(2k-1)!} \sum_{p=0}^{2k-2} \binom{2k-2}{p} r^p
$$

$$
\times \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} B_{2k-p-2}(-\hat{d}^{2k-1-j} \cup \hat{d}^j)(2(k-1-j) + 1)^p. \quad (48)
$$

5.3 Polynomial part for odd $m = 2k - 1$ and $n = 2r - 1$

We observe

$$
P_{2k-1}^{2r-1}(k, (2k-1)r - k + 1) = \sum_{i=k}^{2k-1} (-1)^{i+1} \binom{2k-1}{i} \frac{B_{2k-2}(2r-1(i-k) + r - 1, -\hat{d}_i \cup \hat{d}^{2k-1-i})}{(2k-2)!(2k-1)!}
$$

$$
= \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} \frac{B_{2k-2}(2r-1(k-1-j) + r - 1, -\hat{d}^{2k-1-j} \cup \hat{d}^j)}{(2k-2)!(2k-1)!}
$$

$$
= \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} \frac{B_{2k-2}(2r(k-1-j) + r + (j-k), -\hat{d}^{2k-1-j} \cup \hat{d}^j)}{(2k-2)!(2k-1)!},
$$
leading to

\[
P_{2k-1}^{2r-1}(k, (2k-1)r - k + 1) = \frac{1}{(2k-2)!(2k-1)!} \sum_{p=0}^{2k-2} \binom{2k-2}{p} r^p \\
\times \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} B_{2k-p-2}(j - k, -d^{2k-1-j} \cup \hat{d}^j)(2(k - 1 - j) + 1)^p.
\] (49)

6 Leading term of maximal coefficient

Consider derivation of a general expression for the leading term of the maximal coefficient. The leading in \(n\) term in the summand in (45) evaluates to

\[
(-1)^j n^{2k-1} \binom{2k}{j} \frac{(k - j)^{2k-1}}{(2k-1)!(2k)!},
\]

so that the leading term \(L_{n,k}\) of \(P_{2k}^{n}(k+1, kn)\) reads

\[
L_{kn} = \frac{n^{2k-1}}{(2k-1)!(2k)!} \sum_{j=0}^{k} (-1)^j \binom{2k}{j} (k - j)^{2k-1},
\] (50)

where the sum evaluates to \(A_{2k-1,k}\) being a particular case of the Eulerian numbers of type A \([2]\)

\[
A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k - j)^n.
\] (51)

These numbers used in a definition of the polylogarithm function

\[
Li_{-n}(\rho) = \sum_{k=1}^{\infty} k^n \rho^k = \frac{1}{(1 - \rho)^{n+1}} \sum_{i=0}^{n} A_{n,i} \rho^{n-i}, \quad \rho \neq 1,
\]

which in its turn relates to the Eulerian polynomials \(H_n(\rho)\) defined through the generating function

\[
\frac{1 - \rho}{e^\rho - \rho} = \sum_{n=0}^{\infty} H_n(\rho) \frac{\rho^n}{n!}, \quad H_n(\rho) = (-1)^n \frac{1 - \rho}{\rho} Li_{-n}(\rho).
\]

It is worth to note that the function \(H_n(\rho)\) can be extended into the Eulerian polynomials of higher order used in the derivation \([10]\) of the partition function \(W(s, d^m)\). From \([50, 51]\) it follows that for even \(m\) the leading term \(L_{mn}\) of \(P_{m}^{n}(m/2 + 1, mn/2)\) reads

\[
L_{mn} = \frac{n^{m-1} A_{m-1,m/2}}{(m-1)!m!},
\]

producing \(L_{2n} = n/2, \ L_{4n} = n^3/36, \ L_{6n} = 11n^5/14400\) that coincide with the leading term of the corresponding expressions in \([44]\).

In case of odd \(m = 2k-1\) we observe from \([44]\) that \(L_{2k-1,2r-1} = L_{2k-1,2r}\) (it is also can be seen by comparing \([48]\) to \([49]\)), so that it is sufficient to consider the case of even \(n = 2r\) only. Using \([48]\) we obtain

\[
L_{2k-1,2r} = \frac{r^{2k-2}}{(2k-2)!(2k-1)!} \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} (2(k - 1 - j) + 1)^{2k-2}.
\] (52)
The sum in (52) evaluates to the central MacMahon numbers $B_{2k-1,k}^2$ being a particular case of the Eulerian numbers $B_{n,k}$ of type B that satisfy [1] (also see Ch.11 in [9])

$$B_{n,k} = (2(n-k) + 1)B_{n-1,k-1} + (2k+1)B_{n-1,k},$$

and we find

$$B_{n,k} = \sum_{j=0}^{k} (-1)^{k-j} \binom{n+1}{k-j}(2j+1)^n,$$

and we find

$$L_{2k-1,2r} = \frac{r^{2k-2}B_{2k-1,k}}{(2k-2)!(2k-1)!}, \quad \rightarrow \quad L_{mn} = \frac{(n/2)^{m-1}B_{m,(m+1)/2}}{(m-1)!m!},$$

producing $L_{3,2r} = r^2/2$, $L_{5,2r} = 23r^4/288$ coinciding with the leading term of the corresponding expressions in [41].

Collecting the above results we obtain a general expression for the leading term of the maximal coefficient of the Gaussian polynomial through the Eulerian numbers

$$L_{mn} = \frac{1}{(m-1)!m!} \left\{ \begin{array}{ll}
\frac{n^{m-1}A_{m-1,m/2}}{(m/2)^{m-1}B_{m,(m+1)/2}}, & \text{even } m, \\
\frac{(n/2)^{m-1}B_{m,(m+1)/2}}{((n+1)/2)^{m-1}B_{m,(m+1)/2}}, & \text{odd } m, \text{ even } n,
\end{array} \right. \quad (53)$$

**Acknowledgments**

The author thanks L.Fel for drawing attention to the Gaussian polynomials problem and numerous fruitful discussions.

**A Computation of $P_5^n(s)$ and $P_6^n(s)$**

As it follows from (20) and (21) the solution for $m = 5$ can be constructed using only three basic expressions. Namely, we have

$$W(s, \hat{d}^2 \cup \hat{d}^3) = \frac{s^4 + 18s^3 + 112s^2 + 279s + 220}{288} + \frac{37}{1728} + \frac{2s + 9}{64} (-1)^s + \frac{2}{27} \cos \frac{2\pi s}{3},$$

$$W(s, \hat{d}^1 \cup \hat{d}^4) = \frac{s^4 + 22s^3 + 166s^2 + 495s + 465}{576} + \frac{25}{3456} + \frac{2s + 11}{128} (-1)^s + \frac{2}{27} \sin \frac{\pi s}{6} + \frac{1}{16} \left( \cos \frac{\pi s}{2} + \sin \frac{\pi s}{2} \right),$$

$$W(s, \hat{d}^5) = \frac{s^4 + 30s^3 + 310s^2 + 1275s + 1687}{2880} + \frac{41}{86400} + \frac{2s + 15}{128} (-1)^s + \frac{2}{27} \cos \frac{2\pi s}{3} + \frac{1}{16} \left( \cos \frac{\pi s}{2} + \sin \frac{\pi s}{2} \right) + \frac{2}{25} \left( \cos \frac{2\pi s}{5} + \cos \frac{4\pi s}{5} \right). \quad (A1)$$

Then we use (21) to find $w_i(5,n,s)$ to insert into (20) or (22) for explicit expression of $P_5^n(s)$ or $P_6^n(r,s)$ respectively.

The solution for $m = 6$ is produced based on the four following expressions

$$W(s, \hat{d}^3 \cup \hat{d}^3) = \frac{6s^5 + 180s^4 + 2020s^3 + 10440s^2 + 24299s + 19650}{25920} + \frac{s + 6}{64} (-1)^s + \frac{6s + 34}{243} \cos \frac{2\pi s}{3} + \frac{4}{243} \sin \frac{\pi (4s + 1)}{6},$$

$$+ \frac{2s + 9}{64} (-1)^s + \frac{2}{27} \cos \frac{2\pi s}{3} + \frac{1}{16} \left( \cos \frac{\pi s}{2} + \sin \frac{\pi s}{2} \right) + \frac{2}{25} \left( \cos \frac{2\pi s}{5} + \cos \frac{4\pi s}{5} \right). \quad (A2)$$
\[
W(s, \hat{d}^2 \cup \hat{d}^4) = \frac{12s^5 + 390s^4 + 4720s^3 + 26130s^2 + 64458s + 54275}{69120} + \frac{2s^2 + 26s + 75(-1)^s}{512}
\]
\[
+ \frac{2}{81} \cos \frac{2\pi s}{3} \cos \frac{\pi(4s + 1)}{6} + \frac{1}{32} \left( \cos \frac{\pi s}{2} + \sin \frac{\pi s}{2} \right).
\]
\[
W(s, \hat{d}^1 \cup \hat{d}^5) = \frac{6s^5 + 240s^4 + 3560s^3 + 24000s^2 + 71325s + 70888}{128} + \frac{s + 8(-1)^s}{128}
\]
\[
+ \frac{2}{81} \cos \frac{2\pi s}{3} \cos \frac{\pi(4s + 1)}{6} + \frac{1}{16} \sin \frac{\pi s}{2} + \frac{2}{125} \left( \cos \frac{2\pi s}{5} + \cos \frac{4\pi s}{5} \right)
\]
\[
+ \frac{2}{125} \left( \cos \frac{\pi(2s + 1)}{5} + 3 \cos \frac{\pi(6s + 1)}{5} + 2 \cos \frac{\pi(8s + 1)}{5} \right)
\]
\[
- \frac{2}{125} \left( 2 \sin \frac{\pi(8s + 1)}{10} + \sin \frac{\pi(12s + 1)}{10} + 3 \sin \frac{\pi(16s + 1)}{10} \right),
\]
\[
W(s, \hat{d}^6) = \frac{12s^5 + 630s^4 + 12320s^3 + 110250s^2 + 439810s + 598731}{1036800} + \frac{6s^2 + 126s + 581(-1)^s}{4608}
\]
\[
+ \frac{6s + 61}{486} \cos \frac{2\pi s}{3} + \frac{2}{243} \sin \frac{\pi(4s + 1)}{6} + \frac{1}{18} \cos \frac{\pi s}{3}
\]
\[
+ \frac{1}{32} \left( \cos \frac{\pi s}{2} + \sin \frac{\pi s}{2} \right) + \frac{2}{125} \left( \cos \frac{2\pi s}{5} + \cos \frac{4\pi s}{5} \right)
\]
\[
+ \frac{2}{125} \left( \cos \frac{\pi(2s + 1)}{5} + 3 \cos \frac{\pi(6s + 1)}{5} + 2 \cos \frac{\pi(8s + 1)}{5} \right)
\]
\[
- \frac{2}{125} \left( 2 \sin \frac{\pi(8s + 1)}{10} + \sin \frac{\pi(12s + 1)}{10} + 3 \sin \frac{\pi(16s + 1)}{10} \right).
\]  

Now employ (21) to obtain \( w_i(6, n; s) \) and then use it in (20) or (22) for explicit expression of \( P^n_6(s) \) or \( P^n_6(r, s) \) respectively.

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