Deformed Quantum Relativistic Phase Spaces – an Overview

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We describe three ways of modifying the relativistic Heisenberg algebra - first one not linked with quantum symmetries, second and third related with the formalism of quantum groups. The third way is based on the identification of generalized deformed phase space with the semidirect product of two dual Hopf algebras describing quantum group of motions and the corresponding quantum Lie algebra. As an example the \(\kappa\)-deformation of relativistic Heisenberg algebra is given, determined by \(\kappa\)-deformed \(D=4\) Poincaré symmetries.

1 Introduction

The “naive” canonical quantization scheme, based on the Heisenberg relations \((i = 1, \ldots N)\)

\[
[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}
\]  

(1.1a)

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\[ [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \quad (1.1b) \]

has been applied to quantum mechanics \((N = 3; \text{in general } N = 3n \text{ finite})\) as well as to quantum field theory \((N = \infty)\). The modification of canonical quantization scheme in the presence of interactions in quantum field theory is well–known, and the difficulty with existence of canonical ET limits causes the appearance of infinite wave function renormalization. It appear however, that also quantum–mechanical system in the presence of quantized gravitational forces can not be quantized canonically. The quantized gravitational field modifies the classical space–time by replacing classical Riemannian geometry by at present not yet well understood quantum Riemannian geometry. One of expected properties of such a quantized geometry is the noncommutativity of space coordinates \([x_i, x_j] \neq 0\) \([1-4]\). Further if we supplement the fourth Minkowski coordinate \(x_0 = ct\) and add \(p_0 = \frac{E}{c}\), one gets the following relativistic extension of the relations \((1.1a-b)\)

\[ [\hat{x}_\mu, \hat{p}_\nu] = i\hbar \eta_{\mu\nu} \quad (1.2a) \]
\[ [\hat{x}_\mu, \hat{x}_\nu] = [\hat{p}_\mu, \hat{p}_\nu] = 0 \quad (1.2b) \]

Again the relations \((1.2a-b)\) will not be valid for arbitrary small distances, because taking into consideration the quantum gravity effects \([x_\mu, x_\nu] \neq 0\) \([1-4]\)

In this lecture I would like to review briefly different ways of obtaining deformations of relativistic phase space algebra \((1.2a-b)\). The distinction between different ways depends on the role played by quantum deformations of respective symmetry groups. We shall distinguish the following three classes of deformations of the Heisenberg algebra \((1.2a-b)\):

i) The deformed relations remain covariant under classical symmetry group. Such a deformation we shall call as described in the framework of classical symmetries (see Sect. 2).

ii) The quantum phase space coordinates are the corepresentations of quantum symmetry group and the deformed relations are covariant under the coaction of quantum symmetries. Such formalism is based on differential calculus on noncommutative spaces, with the derivatives representing momenta (see Sect 3). In particular the deformations of the

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\(^\dagger\)This conditions is weaker than \([\hat{x}_i, \hat{x}_j] \neq 0\) because one can have in relativistic case \([\hat{x}_i, \hat{x}_j] = 0\) but \([\hat{x}_0, \hat{x}_i^-] \neq 0\) (see also Sect. 4).
relations (1.2a-b) is obtained if we consider quantum phase space as described by the vector corepresentation of quantum Lorentz group (deformed Minkowski space) supplemented with noncommutative vector fields (deformed fourmomenta).

iii) The generalized quantum relativistic phase space is described by the semidirect product of two dual Hopf algebras, describing respectively the quantum Poincaré group and quantum Poincaré algebra. It appears that if we perform the contraction by putting trivial Lorentz group generators ($\Lambda^\mu_\nu = \delta^\mu_\nu$) one obtains the eight-dimensional deformed relativistic phase space subalgebra, span by the translation generators of quantum Poincaré group and fourmomentum generators of quantum Poincaré algebra. In particular we shall present in Sect. 4 such an example derived from deformed $\kappa$–Poincaré symmetries. In Sect. 5 we shall present some final remarks.

2 Deformations of Relativistic Heisenberg Algebra in the Framework of Classical Relativistic Symmetries

The first well-known proposal of noncommutativity for four relativistic space-time coordinates is due to Snyder ([5]; see also [6]). He assumed that (we put $\hbar = 1$):

$$[\hat{x}_\mu, \hat{x}_\nu] = i \ell^2 M_{\mu\nu},$$

(2.1)

where $M_{\mu\nu}$ are Lorentz algebra generators, and

$$[M_{\mu\nu}, \hat{x}_\rho] = i (\eta_{\nu\rho} \hat{x}_\mu - \eta_{\mu\rho} \hat{x}_\nu)$$

(2.2)

In order to describe the deformation of relativistic phase space (1.2a-b) we employ the commuting fourmomenta ($[p_\mu, p_\nu] = 0$). We get

$$M_{\mu\nu} = i \left( p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu} \right)$$

(2.3)

$$\hat{x}_\mu = i \left( 1 + \ell^2 p^2 \right)^{\frac{1}{2}} \frac{\partial}{\partial p^\mu},$$
\[ [\hat{x}_\mu, \hat{p}_\nu] = i \left( 1 + \ell^2 p^2 \right)^{\frac{1}{2}} \eta_{\mu\nu}. \] (2.4)

It is clear from the relations (2.1-4) that the deformed formalism is covariant under the classical Lorentz symmetries. Moreover, the relations (2.1-2) indicate that in the momentum space one can introduce fifth momentum coordinate \( p_5 \), satisfying the five-dimensional mass-shell condition \( p_5^2 - p^2 = \frac{1}{\ell^2} \) or \( p_5 = \frac{1}{\ell}(1 + \ell^2 p^2)^{\frac{1}{2}} \). It appears that the formalism has build in classical \( O(3,2) \) anti-de-Sitter symmetry with the deformation parameter \( \ell \) equal to the inverse de-Sitter radius in five-dimensional momentum space. This property further was developed by Kadyshevsky [6] and his collaborators [7,8] as a way of introducing regularization in local relativistic field theory.

Second model which we would like to quote here has been recently proposed by Dopplicher, Fredenhagen and Roberts (DFR) [1]. Starting from the analysis of the uncertainty relations in the presence of quantum gravity effects (see also [2-4]) they proposed the following model of noncommutative space-time coordinate:

\[ [\hat{x}_\mu, \hat{x}_\nu] = i \ell^2 \Sigma_{\mu\nu}, \] (2.5)

where \( \Sigma_{\mu\nu} = -\Sigma_{\nu\mu} \) is the two-tensor, commuting with the coordinates \( \hat{x}_\mu \). It appears that the relation (2.5) can be supplemented by the classical relations \([p_\mu, p_\nu] = 0, [\hat{x}_\mu, p_\nu] = i\eta_{\mu\nu}\). The DFR phase space relations are covariant under the classical Lorentz symmetries, provided that the Lorentz generators \( M_{\mu\nu} \) contain additional term rotating the tensor components \( \Sigma_{\mu\nu} \)

\[ [M_{\mu\nu}, \Sigma_{\rho\tau}] = i \left( \eta_{\mu\rho} \Sigma_{\nu\tau} - \eta_{\mu\tau} \Sigma_{\nu\rho} + \eta_{\nu\rho} \Sigma_{\mu\tau} - \eta_{\nu\tau} \Sigma_{\mu\rho} \right). \] (2.6)

Moreover, because \( \Sigma_{\mu\nu} \) are \( x_\mu \)-independent translation-invariant operator, we see that the DFR formalism is invariant under the classical \( D=4 \) Poincaré transformations.

The relation (2.5) can be written for any \( D \), but only in \( D=2 \) one can use standard Poincaré generators, because the central two-tensor \( \Sigma_{\mu\nu} \), will be described by a scalar:

\[ \Sigma_{\mu\nu}^{D=2} = \eta_{\mu\nu} \cdot \Sigma. \] (2.7)

Interestingly, if we consider \( D=2 \) Euclidean space, and corresponding \( (2+1) \)-dimensional Galilei group, the relation (2.5) describes the second central extension of classical \( (2+1) \)-dimensional algebra [9,10], with the nonrelativistic coordinates described by boost generators.
The DFR deformations is very mild, and its dynamical consequences can be calculated explicitly (for D=4 see [1,11]; for Euclidean D=2 case see [12]).

It should be pointed out that both deformations presented in this section introduce besides the phase space generators $Y_A = (\hat{x}_\mu, \hat{p}_\mu)$ also additional generators $U_r$ (Lorentz generators $M_{\mu\nu}$ or tensorial central charges $\Sigma_{\mu\nu}$), which form together consistent associative algebra:

\[
[Y_A, Y_B] = \Omega_{AB}(Y, U),
\]

\[
[Y_A, U_r] = \Omega_{Ar}(Y, U),
\]

\[
[U_r, U_s] = \Omega_{rs}(Y, U).
\]

(2.8)

The generalized quantum symplectic structure ($\Omega_{AB} = -\Omega_{BA}, \Omega_{Ar} = -\Omega_{ra}$) satisfy Jacobi identities. In Snyder case the algebra (2.8) is the anti-de-Sitter algebra in five-momentum space constrained by the mass-shell condition $p_5^2 - p^2 = \frac{1}{\ell^2}$.

In general case discussing the geometric and algebraic nature of deformed phase space algebra (2.8) one should ask three questions:

i) Under which symmetries the relations (2.8) are covariant? In the following two paragraphs we shall consider the examples of deformations covariant under quantum symmetries, described by noncommutative and noncocommutative Hopf algebra playing the role of symplectomorphism group.

ii) which reality conditions for the generators $Y_A, U_r$ can be consistently imposed?

iii) which are the Hilbert space (or C*-algebra) realizations of the deformed quantum phase space algebra (2.8)?

In this brief review we shall discuss the deformations in purely algebraic framework, and we shall be mainly concerned with the answer to the first question.
3 Deformations of Heisenberg Algebra Covariant under the Quantum Group Transformations

For canonical case, describing relativistic quantum mechanics (see (1.2a-b) the basic set of relations (2.8) is reduced to the first one with \( \Omega_{AB} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix} \), and the symplectomorphism group is described by Sp(8;R). The symplectomorphism algebra sp(8;R) contains the diagonal subalgebra O(3,1) describing the classical Lorentz transformations in space-time and four-momentum sectors.

First step in considering deformed Heisenberg algebra (1.2) is to consider deformed Lorentz symmetry. At present there are well-known all possible quantum Lorentz symmetries, in the form of real \( \ast \)-Hopf algebra [13,14]. The problem which has not been yet solved is the embeddings of all these quantum Lorentz groups into the possible quantum symplectomorphism groups described by quantum deformations of Sp(8;R).

At this point we would like to observe, that Heisenberg algebra (1.1) can be also presented as an algebra of creation and annihilation operators

\[ a_i = \sqrt{2}(\hat{x}_i + i\hat{p}_i), \quad a_i^+ = \sqrt{2}(\hat{x}_i - i\hat{p}_i) \]

satisfying well-known relation (we put \( \hbar = 1 \))

\[ [a_i^+, a_j] = \delta_{ij} \]

\[ [a_i, a_j] = [a_i^+, a_j^+] = 0 \]

If \( i = 1, \ldots, N \) the symplectomorphism groups is Sp(2N), but there is a subgroups \( U(N) \subset Sp(2N) \) of holomorphic transformations \( a_i' = U_i^j a_j \) where \( U_i^j \in U(N) \). The problem of deformation of relations (3.1) consistent with well-known Drinfeld-Jumbo deformation \( U_q(N) \) has been solved by Woronowicz and Pusz ([15]; see also [16]). The set of q-deformed commutation relations looks as follows ([A, B]_q \equiv AB - qBA):

\[ [a_i, a_j^+]_q = 0 \quad i > j \]

\[ [a_i', a_j^+]_q^2 = 1 + (q^2 - 1) \sum_{k=0}^{i-1} a_k^+ a_k \quad i = j \]

\[ [a_i, a_j]_q = [a_i^+, a_j^+]_{q^{-1}} = 0 \quad i < j \]
It appears that the n-dimensional modules \((a_1, \ldots, a_N)\) and \((a_1^+, \ldots, a_N^+)\) form the corepresentation of quantum holomorphic symplectomorphism group \(U_q(N)\). This construction has been further generalized to the supersymmetric set of bosonic and fermionic creation and annihilation operators [19] and to the case of quantum covariance group described by a quantum R-matrix [16,20].

In order to describe the quantum deformation of the relativistic phase space (1.2a-b) one should perform the following steps:

i) Specify the quantum deformation \(O(3,1)^{(q)}\) of Lorentz group \(O(3,1)\), which only in particular cases can be described in terms of quantum R-matrices for the spinorial Lorentz group \(SL(2;c)\).

ii) Determine the algebra of quantum Minkowski space \(M_4^{(q)}\), with four generators of algebra describing the fundamental representation of \(O^{(q)}(3,1)\).

(iii) Define the quantum fourmomenta \(p_\mu\) by noncommutative derivatives (noncommutative vector fields) acting on the functions on \(M_4^{(q)}\).

It appears that depending on the type of deformations, we need to consider besides the generators \(Y_A = (\hat{x}_\mu, \hat{p}_\mu)\) also additional generators \(U_r\) (see (2.8)). We shall mention here the following two cases:

i) Drinfeld-Jimbo deformation of Lorentz group.
   
   Such a case was elaborated in detail by Munich group [21-23]. The relations (2.8) are realized if we supplement 7 additional generators \(U_r = (M_{\mu\nu}, \Lambda)\), where \(M_{\mu\nu}\) are the Lorentz generators, and \(\Lambda\) describe the scaling generator. An interesting property of this scheme is the classical nature of time coordinate (i.e. \(\hat{x}_0\) can be chosen an central element of \(M_4^{(q)}\)).

ii) Podleś class of deformations of Poincaré group, providing 8-dimensional quantum phase space.
   
   It is known [24,25] that the Drinfeld-Jimbo deformation of Lorentz group can not be extended to quantum Poincaré group. Only the deformations of Lorentz group with quantum R-matrix satisfying \(R^2 = 1\)

\[\text{§ For the definition of } U_q(N) \text{ see [17,18].}\]
permits the extension to quantum Poincaré group $\mathcal{P}_4^{(q)}$ [26]. The translation sector of $\mathcal{P}_4^{(q)}$ describes the generators $\hat{x}_\mu$ of $\mathcal{M}_4^{(q)}$: 

\begin{equation}
(R - 1)_{\rho\tau}^{\mu\nu} (\hat{x}_\rho \hat{x}_\tau - Z_{\mu\nu}^{\rho\tau} \hat{x}_\nu + T_{\rho\tau}^\nu) = 0 \tag{3.3}
\end{equation}

where $R$ is quantum O(3,1) matrix satisfying $R^2 = 1$ and $Z_{\mu\nu}^{\rho\tau}, C_{\mu\nu}$ are numerical dimensionless coefficients, satisfying suitably set of constraints (see [26]). The fourdimensional differential calculus, with four independent basic one-forms and corresponding four dual derivatives, are obtained if the parameters in eq. (3.3) satisfy additional conditions [27]. In such a case the space-time noncommutativity (3.4) can be supplemented by the following remaining relations of deformed algebra (1.2a-b)

\begin{equation}
p_{\mu} \hat{x}_\nu - R_{\mu\nu}^{\rho\tau} \hat{x}_\tau p_\rho = \delta^{\nu}_{\mu} + Z_{\mu\nu}^{\rho\tau} p_\rho \tag{3.4a}
\end{equation}

\begin{equation}
p_{\mu} p_\nu - R_{\mu\nu}^{\rho\tau} p_\rho p_\tau = 0 \tag{3.4b}
\end{equation}

where $p_\nu$ can be identified with the noncommutative derivatives.

\section{4 Deformed Phase Space as Heisenberg Double of Quantum Space–Time Symmetries}

In previous paragraphs we described deformed relativistic phase space as a deformed Heisenberg algebra covariant under classical (Sect. 2) or quantum (Sect. 3) relativistic symmetries. In such an approach to “deformed physics” the basic object is a quantum phase algebra, and the quantum symmetries in form of respective Hopf algebras describe its covariance properties. It should be stressed that the assumptions of covariance of deformed phase space algebra (deformed Heisenberg algebra) under classical or quantum symmetries restricts substantially all possible class of deformations.

In this Section we shall consider the scheme in which primary object is the quantum symmetry. The deformed relativistic phase space will be obtained directly from a dual pair of Hopf algebras, describing quantum Poincaré group $\mathcal{P}_4^{(q)}$ and quantum Poincaré algebra $P_4^{(q)}$. The realizations describing the noncommutativity of coordinates and momenta will be obtained by introducing the semidirect product $\mathcal{P}_4^{(q)} \ltimes P_4^{(q)}$, or so–called Heisenberg double [28,29].
If we have two dual Hopf algebras $H$, $\tilde{H}$, with duality generated by inner nondegenerate product $\langle \tilde{a}, a \rangle$ ($a \in H; \tilde{a} \in \tilde{H}$), the Heisenberg double $H \bowtie \tilde{H}$ is the algebra $H \otimes \tilde{H}$ with the cross multiplication given by the relations:

$$\tilde{a} \circ a = a_{(1)} \langle \tilde{a}_{(1)}, a_{(2)} \rangle \tilde{a}_{(2)}$$

(4.1)

where we use the notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ etc. In particular one can apply this definition to the multiplication of two Abelian Hopf algebras describing translation sector of classical Poincaré group

$$\left(\tilde{H} = \{x^\mu\}; [x^\mu, x^\nu] = 0, \Delta(x^k) = x^k \otimes 1 + 1 \otimes x^k\right)$$

and fourmomentum sector of classical Poincaré algebra ($H = \{p_k; [p^k, p^\nu] = 0, \Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k\}$). Introducing the Planck constant $\hbar$ into the duality relation

$$\langle x^\mu, p^\nu \rangle = i\hbar \delta^\mu_\nu$$

(4.2a)

one obtains from (4.1)

$$x^\mu \cdot p^\nu = p^\nu \langle 1, 1 \rangle x^\mu + 1 \cdot \langle x^\mu, p^\nu \rangle \cdot 1$$

(4.2b)

Because $\langle 1, 1 \rangle = 1$, substituting (4.2a) we obtain from (4.2a) the classical relativistic Heisenberg algebra (1.2a-b). Such a scheme can be generalized to the case of any quantum - deformed relativistic symmetries. In this lecture I shall provide the example of so-called $\kappa$-deformations of D=4 relativistic symmetries.

We choose the following realization of the $\kappa$-Poincaré algebra in bi-crossproduct basis [2,8] ($\mu, \nu, \lambda, \rho = 0, 1, 2, 3; i, j = 1, 2, 3$ and $g^{\mu\nu} = g_{\mu\nu} = (-1, 1, 1, 1)$)

-algebra sector

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, M_{\lambda\sigma}] = i\hbar(g_{\mu\sigma}M_{\nu\lambda} + g_{\nu\lambda}M_{\mu\sigma} - g_{\mu\lambda}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\lambda})$$

$$[M_{ij}, P_\mu] = -i\hbar(g_{i\mu}P_j - g_{j\mu}P_i)$$

$$[M_{i0}, P_0] = i\hbar P_i$$

$$[M_{i0}, P_j] = i\delta_{ij} \left( \hbar^2 \kappa \sinh \left( \frac{P_0}{\hbar \kappa} \right) e^{-\frac{P_0}{\hbar \kappa}} + \frac{1}{2\kappa} \vec{P}^2 \right) - \frac{i}{\kappa} P_i P_j$$

(4.3)
where $\hbar$ denotes Planck's constant and $\kappa$ deformation parameter. The formulas for the antipode and counit are omitted because they are not essential for this construction.

Using the following duality relations, extending (4.2)

$$< x^\mu, P_\nu > = i\hbar \delta_\nu^\mu \quad < \Lambda^\nu_\mu, M_{\alpha\beta} > = i\hbar \left( \delta_\alpha^\mu g_{\nu\beta} - \delta_\beta^\mu g_{\nu\alpha} \right)$$  \hspace{1cm} (4.5)

we obtain the commutation relations defining $\kappa$–Poincaré group [3,8] in the form

- algebra sector

$$[x^\mu, x^\nu] = \frac{i}{\kappa} (\delta^\mu_0 x^\nu - \delta^\nu_0 x^\mu)$$

$$[\Lambda^\nu_\mu, x^\lambda] = \frac{-i}{\kappa} \left( (\Lambda^\nu_0 - \delta^\nu_0) \Lambda^\lambda_\mu + (\Lambda^\mu_0 - \delta^\mu_0) g^{\mu\lambda} \right)$$  \hspace{1cm} (4.6)

$$[\Lambda^\nu_\mu, \Lambda^0_\mu] = 0$$

- coalgebra sector

$$\Delta(x^\mu) = \Lambda^\mu_\alpha \otimes x^\alpha + x^\mu \otimes I$$  \hspace{1cm} (4.7)

$$\Delta(\Lambda^\mu_\nu) = \Lambda^\mu_\alpha \otimes \Lambda^\alpha_\nu$$

The commutation relations (4.3) and (4.6) one can supplemented by the following relations obtained from (4.1), (4.4) and (4.7)

- cross relations:
\[
[P_k, x_l] = i\hbar\delta_{kl} \quad [P_0, x_0] = i\hbar
\]

\[
[P_k, x_0] = -\frac{i}{\kappa}P_k \quad [P_0, x_l] = 0
\]

\[
[P_\mu, \Lambda_\beta^\alpha] = 0 \quad (4.8)
\]

\[
[M_{\alpha\beta}, \Lambda_\nu^\alpha] = i\hbar (\delta_\beta^\mu \Lambda_{\alpha\nu} - \delta_\alpha^\mu \Lambda_{\beta\nu})
\]

\[
[M_{\alpha\beta}, x^\mu] = i\hbar (\delta_\beta^\mu x_\alpha - \delta_\alpha^\mu x_\beta) + \frac{i}{\kappa} (\delta_\beta^0 M_{\alpha}^\mu - \delta_\alpha^0 M_{\beta}^\mu)
\]

where

\[
M_{\alpha}^\mu = g^{\mu\rho}M_{\alpha\rho}, \quad M_{\alpha}^\mu = g^{\mu\rho}M_{\rho\alpha}
\]

The relations (4.3), (4.6) and (4.8) give us the Heisenberg double \( \mathcal{H}(\mathcal{P}_k) \) of \( \kappa \)-Poincaré algebra firstly presented in [32]. In order to define the \( \kappa \)-deformed phase space we should consider subalgebra of \( \mathcal{H}(\mathcal{P}_k) \) given by the following commutation relations:

\[
[x_k, x_l] = 0 \quad [P_\mu, P_\nu] = 0
\]

\[
[x_0, x_k] = \frac{i}{\kappa}x_k
\]

\[
[x_k, P_l] = i\hbar\delta_{kl} \quad [x_k, P_0] = 0
\]

\[
[x_0, P_l] = \frac{i}{\kappa}P_l \quad [x_0, P_0] = -i\hbar
\]

(4.9)

For \( \kappa \rightarrow \infty \) we get the standard nondeformed phase space satisfying the Heisenberg commutation relations (1.2a-b).

It should be stressed that the \( \kappa \)-deformed phase space (4.9) is not a Hopf algebra, however, separately in the translation sector (generators \( x_\mu \)) and fourmomentum sector (generators \( P_\mu \)) one can read-off from the relations (4.4) and (4.7) (in eq. (4.7) after contraction \( \Lambda_\beta^\mu \rightarrow \delta_\beta^\mu \)) the coalgebraic structure. We obtain that in \( \kappa \)-deformed phase space one can add two coordinates and two momenta as follows:

\[
x_\mu^{(1+2)} = x_\mu^{(1)} + x_\mu^{(2)} \quad (4.10a)
\]

\[
P_0^{(1+2)} = P_0^{(1)} + P_0^{(2)} \quad (4.10b)
\]
We see that due to the $\kappa$–deformation there is modified the addition law of two three–momenta.

Analogous Heisenberg double construction can be performed for any quantum Poincaré group listed by Podleś and Woronowicz [26]. It should be mentioned however, that before solving such a task we should obtain the complete list of quantum Poincaré algebras, dual to quantum groups from [26] - the list which at present is yet not known.

5 Final Remarks

The considerations of deformed quantum phase spaces goes along the following three lines:

• i) Mathematical consequences of deforming space–time symmetries. In such a way one obtains the set of possible algebraic schemes, which we outlined in this talk.

• ii) Investigation of dynamical consequences of modifying Einstein gravity scheme (e.g. superstring theory) on Heisenberg uncertainty relations (see e.g. [32,33])

• iii) Considering different “gedenken-experiments” for the measurement of distances and time intervals in quantum theory, which include into analysis the quantum property of the observer (reference system) (see e.g. [34-36]). It appears that the “classical” measuring devices, with infinite mass as well as definite position and velocity are the idealizations in the description of measuring process which is not realistic. It appears that introducing “realistic” measurement theory in quantum mechanics we are bound to modify the standard Heisenberg uncertainty relations, with Planck length $l_p = \frac{\hbar}{m_p c} \simeq 10^{-33}$ cm playing a crucial role in the modifications.

We see that the future modified models of quantum mechanics should have on one side solid mathematical and dynamical bases, on the other should be consistent with “realistic” measurement theory. A modest effort in such a direction has been presented in [36], where there are given arguments based on “realistic” measurement theory for the $\kappa$–deformation of relativistic phase
space, presented in Sect. 4. At present, however, the choice of the deformation which is the most physical one is an open question.

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