SHARP SPECTRAL TRANSITION FOR EIGENVALUES EMBEDDED INTO THE SPECTRAL BANDS OF PERTURBED PERIODIC JACOBI OPERATORS

WENCAI LIU AND DARREN C. ONG

Abstract. We are interested in diagonal perturbations of a periodic Jacobi operator that introduce embedded eigenvalues in its essential spectrum. Embedding multiple points in the essential spectrum has been known to be difficult, given that eigenvalues are destroyed easily by small perturbations. However, given a finite or countably infinite set of points within an absolutely continuous band of the original periodic operator (subject only to a very weak non-resonance condition) we are able to construct a diagonal perturbation that preserves the essential spectrum and places eigenvalues in all of those points.

1. Introduction

We consider a Jacobi eigenvalue equation. In other words, with boundary condition \( u(-1) = 0 \) we consider the difference equation

\[
(H_0 u)(n) := a_{n+1} u(n+1) + a_n u(n-1) + b_{n+1} u(n) = Eu(n), \quad n \geq 0,
\]

where the \( \{a_j, b_j\} \) are real sequences indexed by \( j \geq 1 \) with \( a_j \) assumed to be positive. Alternatively, we can view this eigenvalue equation in terms of an operator on \( l^2(\mathbb{Z}_{\geq 0}) \). We also consider perturbations of this equation, namely,

\[
(H u)(n) = (a_{n+1} + a'_{n+1}) u(n+1) + (a_n + a'_n) u(n-1) + (b_{n+1} + b'_{n+1}) u(n) = Eu(n), \quad n \geq 0,
\]

where \( a'_j \) and \( b'_j \) are real sequences chosen so \( a_j + a'_j \) is always positive.

Let us assume in addition that the \( a_j \) and \( b_j \) sequences are periodic with period \( q \geq 1 \). Then through basic Floquet theory, we know that the essential spectrum of the operator \( H_0 \) consists of \( q \) absolutely continuous bands.

We are interested in perturbations that induce embedded eigenvalues in the absolutely continuous bands of the periodic Jacobi operator. This has been a topic that has received recent interest: see [3, 4, 9] for papers on this problem that have appeared in the past couple of years. For other operators, most prominently the Schrödinger operator, the list of papers on these eigenvalue-producing perturbations of periodic operators include [6–8, 10]. This problem is a natural progression of the classical problem (which goes back to the 1920s with [11]) of finding perturbations of free operators that induce eigenvalues in the essential spectrum.

We introduce a perturbation that induces eigenvalues in a chosen set of points within an absolutely continuous band of a given periodic Jacobi operator. This set of points can be finite or countably infinite. Indeed, the only restriction we require on this set is a weak non-resonance condition. This is a very significant improvement over previous results in the literature. Eigenvalues are in a sense very fragile, and so forcing multiple embedded eigenvalues to appear simultaneously is often challenging. Compare for instance the result in [4], which introduces a perturbation that can only produce two embedded eigenvalues.

We remark that perturbing a periodic operator is much more challenging that perturbing a free operator. In particular, this means our construction is more delicate than the construction
for the free perturbed Schrödinger operator in [1]. We use the generalized Prüfer variables in [6] rather than the standard Prüfer variables, and this makes several parts of the proof more complicated. A significant technical difficulty of our proof is in bounding decaying oscillatory terms that involve these generalized Prüfer variables. Through careful analysis, we determine that the positive parts of these decaying oscillatory terms cancel out the negative parts well enough for our purposes.

This paper is a companion paper to [10]. Indeed, our construction is very similar. However, the proof that the construction produces the desired set of eigenvalues is more difficult in the Jacobi setting compared to the continuous Schrödinger setting in [10]. The technical reason is that some key terms in our construction can be bounded by a constant in the continuous case, but in the Jacobi setting those same terms are bounded by a term that grows like $\varepsilon \ln n$ for small positive $\varepsilon$ and as $n \to \infty$. For this reason ensuring that our solutions remain $\ell^2$ requires more delicate handling compared to the continuous Schrödinger setting.

In Section 2, we will state precisely our main results. In Section 3 we will prove an auxiliary result about how sufficiently small perturbations of a periodic Jacobi operator will produce no eigenvalues in the absolutely continuous bands. Section 4 contains some important bounds on certain important terms in our construction involving the generalized Prüfer variables. It is in this section that we see why the perturbed periodic Jacobi setting is more challenging than the perturbed free setting and periodic continuous setting. Finally, in Section 5 we introduce explicitly our perturbation and prove that it produces the desired embedded eigenvalues.

2. Main results

Recalling the equation (1) we denote

$$\sigma_{ac}(H_0) = \sigma_{ess}(H_0) = \bigcup_k [c_k, d_k].$$

Let $E \in (a_k, b_k)$ and $\varphi$ be the Floquet solution of $q$-periodic operator. Suppose

$$\varphi(n, E) = p(n)e^{i k(E) q n},$$

where $p(n)$ is a real $q$-periodic function and $k(E) \in (0, \pi)$ is called the quasimomentum ($q$ is the period for $a_n, b_n$). Sometimes, we omit the dependence on $E$.

**Theorem 2.1.** Suppose $a'_n = O(1)$ and $b'_n = O(1)$. Let $H$ be given by (2). Then there exists no non-trivial $\ell^2(\mathbb{Z} \geq 0)$ solution of $Hu = Eu$ for any $E \in \bigcup_k (c_k, d_k)$.

**Theorem 2.2.** Suppose $\{E_j\}_{j=1}^N \subset \bigcup_k (c_k, d_k)$ such that quasimomenta $\{k(E_j)\}_{j=1}^N$ are different. Suppose for any $i, j \in \{1, 2, \ldots, N\}$, $k(E_i) + k(E_j) \neq \pi$. Let $a'_n = 0$. Then for any given $\{\theta_j\}_{j=1}^N \subset [0, \pi]$, there exist $b'_n$ such that

$$b'_n = \frac{O(1)}{1 + n}$$

as $n \to \infty$ and the

$$Hu = E_j u$$

has an $\ell^2(\mathbb{Z} \geq 0)$ solution with boundary condition

$$\frac{u(1)}{u(0)} = \tan \theta_j.$$

1The difficulty of [10] is to deal with the manifolds.
**Theorem 2.3.** Suppose \( \{E_j\}_{j=1}^\infty \subset \cup_k (c_k, d_k) \) such that quasimomenta \( \{k(E_j)\}_j \) are different. Suppose for any \( i, j \), \( k(E_i) + k(E_j) \neq \pi \). Let \( h(n) > 0 \) be any function on \( \mathbb{Z}_{\geq 0} \) with \( \lim_{n \to \infty} h(n) = \infty \). Let \( h'_n = 0 \) for any \( n \).

Then for any given \( \{\theta_j\}_{j=1}^\infty \subset [0, \pi] \), there exist sequence \( b'_n \) such that

\[
|b'(n)| \leq \frac{h(n)}{1+n} \text{ for } n,
\]

and

\[
Hu = E_j u
\]

has an \( \ell^2(\mathbb{Z}_{\geq 0}) \) solution with boundary condition

\[
\frac{u(1)}{u(0)} = \tan \theta_j.
\]

3. Generalized Prüfer Transformation and Proof of Theorem 2.3.

This section is a summary of the generalized Prüfer variables developed in [9]. In (1), we have a Jacobi matrix \( J \) with coefficients \( a_n > 0 \), \( b_n \in \mathbb{R} \), viewed as an operator \( H_0 \) on \( \ell^2(\mathbb{Z}_{\geq 0}) \).

We consider also its perturbation, a Jacobi matrix \( \tilde{J} \) with coefficients \( a_n + a'_n > 0 \), \( b_n + b'_n \in \mathbb{R} \), and viewed as an operator \( H \) on \( \ell^2(\mathbb{Z}_{\geq 0}) \). Consider, for \( E \in \mathbb{R} \), a solution \( \varphi \) of the eigenvalue equation \( H_0 \varphi = E \varphi \). In other words,

\[
a_{n+1} \varphi(n + 1) + b_{n+1} \varphi(n) + a_n \varphi(n-1) = E \varphi(n),
\]

We also consider an eigenfunction \( u \) for \( H \),

\[
(a_{n+1} + a'_{n+1})u(n + 1) + (b_{n+1} + b'_{n+1})u(n) + (a_n + a'_n)u(n-1) = Eu(n).
\]

We define \( \gamma(n) \) as the argument of \( \varphi(n) \). In other words,

\[
\varphi(n) = |\varphi(n)|e^{i\gamma(n)}.
\]

We can ensure uniqueness of \( \gamma \) by setting \( \gamma(0) \in [0, 2\pi) \), \( \gamma(n) - \gamma(n-1) \in [0, 2\pi) \).

Note that \( \varphi \) is complex, and is linearly independent with its complex conjugate \( \bar{\varphi} \). On the other hand, we assume that \( u \) is a real-valued eigensolution.

We now introduce \( Z(n) \). Our Prüfer variables will be define as the argument and absolute value of \( Z(n) \). It is defined as follows:

\[
a_{n+1} \varphi(n + 1) + b_{n+1} \varphi(n) + a_n \varphi(n-1) = E \varphi(n),
\]

(6)

We define \( \gamma(n) \) as the argument of \( \varphi(n) \). In other words,

\[
\varphi(n) = |\varphi(n)|e^{i\gamma(n)}.
\]

We can ensure uniqueness of \( \gamma \) by setting \( \gamma(0) \in [0, 2\pi) \), \( \gamma(n) - \gamma(n-1) \in [0, 2\pi) \).

Note that \( \varphi \) is complex, and is linearly independent with its complex conjugate \( \bar{\varphi} \). On the other hand, we assume that \( u \) is a real-valued eigensolution.

We now introduce \( Z(n) \). Our Prüfer variables will be define as the argument and absolute value of \( Z(n) \). It is defined as follows:

\[
\left( \frac{(a_n + a'_n)u(n) - u(n-1)}{u(n-1)} \right) = \frac{1}{2i} \left( \begin{array}{c} Z(n) \left( \frac{a_n \varphi(n)}{\varphi(n-1)} \right) \\ Z(n) \left( \frac{a_n \varphi(n)}{\varphi(n-1)} \right) \end{array} \right) - \frac{1}{2i} \left( \begin{array}{c} Z(n) \left( \frac{a_n \varphi(n)}{\varphi(n-1)} \right) \\ Z(n) \left( \frac{a_n \varphi(n)}{\varphi(n-1)} \right) \end{array} \right)
\]

(9)

By linear independence of \( \varphi \) and \( \bar{\varphi} \) and reality of \( u \), (9) uniquely determines \( Z(n) \). The Prüfer amplitude \( R(n) > 0 \) and Prüfer phase \( \eta(n) \in \mathbb{R} \) are defined as

\[
Z(n) = R(n)e^{i\eta(n)}.
\]

We will also need a version of the Wronskian. For two sequences \( f, g \), we have

\[
W_{0,0}(f, g) = a_{n+1} f(n) g(n+1) - a_n f(n+1) g(n),
\]

\[
W_{a,a'}(f, g) = (a_n + a'_{n+1}) f(n) g(n+1) - (a_n + a'_{n+1}) f(n+1) g(n),
\]

\[
W_{0,a'}(f, g) = (a_n + a'_{n+1}) f(n) g(n+1) - a_n f(n+1) g(n).
\]

If we assume

\[
a_{n+1} f(n + 1) + a_n f(n-1) = (x - b_{n+1}) f(n),
\]
and 
\[(a_{n+1} + a'_{n+1})g(n+1) + (a_n + a'_n)g(n-1) = (x - b_{n+1} - b'_{n+1})g(n),\]
then
\[
W_{0,a'}(f, g)(n) - W_0(a', g)(n-1) = -b'_{n+1}f(n)g(n) - a'_n(f(n)g(n-1) + f(n-1)g(n)).
\]  
(12)

Since \(\varphi, \varphi\) are linearly independent solutions of (3), by constancy of the Wronskian, we have
\[
W_{0,0}(\varphi, \varphi)(n) = 2ia_{n+1}\text{Im}(\varphi(n)\varphi(n+1)) = i\omega,
\]  
(13)
for some real nonzero constant \(\omega\). Thus,
\[
2|\varphi(n)| \cdot |\varphi(n+1)|a_{n+1} \sin(\gamma(n+1) - \gamma(n)) = \omega.
\]  
(14)

We can use Wronskians to invert (10) to get
\[
Z(n) = \frac{2}{\omega}W_{0,a'}(\varphi, u)(n-1).
\]  
(15)

**Theorem 3.1** (Theorem 5 of [3]). Prüfer variables obey the first-order recursion relation
\[
\frac{Z(n+1)}{Z(n)} = 1 - \frac{i}{\omega}a_\frac{a'_n}{a_n + a'_n} |\varphi(n)|^2 \left( e^{-2i(\eta(n)+\gamma(n))} - 1 \right)
+ \frac{i}{\omega}a'_n |\varphi(n-1)| : |\varphi(n)| e^{i(\gamma(n-1) - \gamma(n))} 
- \frac{i}{\omega}a'_n |\varphi(n-1)| : |\varphi(n)| e^{-2i\eta(n)} e^{-i(\gamma(n-1) + \gamma(n))} 
+ \frac{i}{\omega}a_n a'_n (1 - e^{-2i(\eta(n)+\gamma(n))}) |\varphi(n-1)| \cdot |\varphi(n)| e^{-i(\gamma(n-1) - \gamma(n))}.
\]

**Remark 3.2.** In this paper, we assume \(a'_n = \frac{a(1)}{n}\) and \(b'_n = \frac{a(1)}{n}\). Since \(a_n, b_n\) are periodic, \(a_n, a_n + a'_n > 0\) for all \(n\), then
\[
\frac{1}{a_n + a'_n} = O(1).
\]  
(16)

We define the Prüfer amplitude \(R\) and the Prüfer phase \(\eta\) by
\[
R(n) = |Z(n)|, \eta(n) = \text{Arg}(Z(n)).
\]  
(17)

In that case, we have
\[
\frac{R(n+1)}{R(n)} = \left| \frac{Z(n+1)}{Z(n)} \right|.
\]  
(18)

Note the following bound on \(R(n)\):

**Proposition 3.3.** For a constant \(K\),
\[
\sqrt{(a_n + a'_n)^2 u(n)^2 + u(n-1)^2} 
K \sqrt{a_n ^2 + \varphi(1)^2 + \varphi(0)^2} \leq R(n) \leq \frac{2}{\omega} \left( K a_n \sqrt{\varphi(1)^2 + \varphi(0)^2} \sqrt{u(n)^2 + u(n-1)^2} + |a'_n \varphi(n-1) u(n)| \right)
\]  
(19)

**Proof.** The left inequality simply follows from (10). The right inequality follows from (15) and the Cauchy-Schwarz inequality. \(\square\)
Let us set $a'_j = 0$ for all $j$. This changes Theorem 3.1 into a much simpler formula,

$$(20) \quad \frac{Z(n+1)}{Z(n)} = 1 - \frac{i}{\omega} b'_{n+1} |\varphi(n)|^2 (e^{-2i(\eta(n)+\gamma(n))} - 1)$$

Using (20) and (18) we have

$$\frac{R(n+1)^2}{R(n)^2} = \left(1 - \frac{i}{\omega} b'_{n+1} |\varphi(n)|^2 (\cos(2\eta(n) + 2\gamma(n)) - i \sin(2\eta(n) + 2\gamma(n)) - 1) \right) \times \left(1 + \frac{i}{\omega} b'_{n+1} |\varphi(n)|^2 (\cos(2\eta(n) + 2\gamma(n)) + i \sin(2\eta(n) + 2\gamma(n)) - 1) \right)
= 1 - \frac{b'_{n+1}^2}{\omega} \sin^2(2\eta(n) + 2\gamma(n)) |\varphi(n)|^2 + \frac{4(b'_{n+1})^2 |\varphi(n)|^4}{\omega^2} \sin^2(\eta(n) + \gamma(n)).$$

Also, starting with (20) and multiplying by $Z(n)e^{i\gamma(n)}$ we obtain

$$R(n+1) \exp(i\eta(n+1) + i\gamma(n)) = R(n) \exp(i\eta(n) + i\gamma(n)) - \frac{i}{\omega} b'_{n+1} |\varphi(n)|^2 \exp(-i\eta(n) - i\gamma(n)) R(n).$$

Dividing the real part by the imaginary part for both sides of the above equation, we get

$$\cot(\eta(n+1) + \gamma(n)) = \cot(\eta(n) + \gamma(n)) - \frac{2}{\omega} b'_{n+1} |\varphi(n)|^2$$

**Proof of Theorem 2.1** Suppose $u$ is an eigensolution with corresponding $E \in (a_k, b_k)$. By Theorem 3.1 and (18), we have

$$\frac{R(n+1)}{R(n)} = 1 - \frac{o(1)}{n}.$$ 

This implies that

$$\ln R(n+1) - \ln R(n) = \frac{o(1)}{n}.$$ 

Thus for large $n_0$, and $n > n_0$, we have

$$\ln R(n) \geq \ln R(n_0) - \frac{1}{3} \sum_{k=n_0}^{n} \frac{1}{k}.$$ 

This implies for large $n$,

$$R(n) \geq \frac{1}{Cn^n}.$$ 

This contradicts $u \in l^2(\mathbb{Z}_{\geq 0})$ by Proposition 3.3.
4. Some preparations for constructions

We always assume $a' = 0$. In this section, we indicate the dependence on $E$; thus we will write $R(n, E)$, $Z(n, E)$, $η(n, E)$ and $γ(n, E)$. Let $θ(n, E) = η(n, E) + γ(n, E)$.

By (22) and Prop.2.4, one has

$$\left(η(n + 1) + γ(n)\right) - \left(η(n) + γ(n)\right) = O(\|b'_{n+1}\|).$$

This implies

$$\theta(n + 1, E) - \theta(n, E) = γ(n + 1) - γ(n) + O(\|b'_{n+1}\|).$$

We will add another equation to complete our construction:

$$\ln R(n + 1, E) - \ln R(n, E) = -\frac{b'_{n+1}}{\omega} \sin(2\eta(n, E) + 2γ(n, E))|φ(n, E)|^2 + O(\|b'_{n+1}\|^2).$$

We will construct $b'_n$ piecewisely. Let $H_0$ be the periodic operator with Jacobi coefficient sequences $a_n, b_n$ and $H_0 + b'\text{Id}$ be the perturbation with coefficient sequences $a_n, b_n + b'_n$.

**Proposition 4.1.** Let $E$ be in $\cup_i (c_i, d_i)$ such that $k(E) \neq \frac{π}{2}$. Let $A = \{E_j\}_{j=1}^m$ be in $\cup_i (c_i, d_i)$ such that $k(E) \neq k(E_j)$ and $k(E) + k(E_j) \neq π$ for all $j = 1, 2, \cdots, m$. Suppose $θ_0 \in (0, π)$. Let $n_1 > n_0 > v$. Then there exist constants $K(E, A), C(E, A)$ (independent of $v, n_0$ and $n_1$) and perturbation $b'(n, E, A, n_0, n_1, v, θ_0)$ such that for $n_0 - v > K(E, A)$ the following holds:

**Perturbation:** for $n_0 \leq n \leq n_1$, supp$(b') \subset (n_0, n_1)$, and

$$|b'(n, E, A, n_0, n_1, v, θ_0)| \leq \frac{C(E, A)}{n - v}.$$  

**Solution for $E$:** the solution of $(H_0 + b'\text{Id})u = Eu$ with boundary condition $θ(n_0, E) = θ_0$ satisfies

$$R(n_1, E) \leq C(E, A)\left(\frac{n_1 - v}{n_0 - v}\right)^{-100} R(n_0, E)$$

and for $n_0 < n < n_1$,

$$R(n, E) \leq C(E, A)R(n_0, E).$$

In particular, for any $ε > 0$, if $\frac{n_1 - v}{n_0 - v} > K(E, A, ε)$,

$$R(n, E) \leq \left(\frac{n_1 - v}{n_0 - v}\right)^ε R(n_0, E).$$

**Solution for $E_j$:** any solution of $(H_0 + b'\text{Id})u = E_j u$ satisfies for $n_0 < n \leq n_1$ and $ε > 0$

$$R(n, E_j) \leq D(E, A, ε)\left(\frac{n_1 - v}{n_0 - v}\right)^ε R(n_0, E_j).$$

In particular, if $\frac{n_1 - v}{n_0 - v} > K(E, A, ε)$,

$$R(n, E_j) \leq \left(\frac{n_1 - v}{n_0 - v}\right)^ε R(n_0, E_j).$$

For simplicity, denote by $K = K(E, A), C = C(E, A)$ etc.. We mention that $K \gg C > 0$.

We solve the following equation for $η(n, E)$ with initial condition $η(n_0, E) = θ_0 - γ(n_0, E)$ (or in other words, $θ(n_0, E) = θ_0$):

$$\cot(η(n + 1, E) + γ(n, E)) = \cot(η(n, E) + γ(n, E)) - \frac{2}{ω} b'_{n+1}|φ(n, E)|^2$$
with

\[(34) \quad b'_{n+1} = b'_{n+1}(E, A, n_0, n_1, v, \theta_0) = \frac{C}{n - v} \sin(2\eta(n) + 2\gamma(n)).\]

We will show that this choice of \(b'_{n}\) satisfies our construction. Obviously, \((27)\) follows from \((34)\).

First, we require a technical lemma:

**Lemma 4.2.** Let \(b'_{n}\) be given in \((34)\), and let \(E\) and \(A\) satisfy the assumptions of Proposition 4.1. Let \(f(n)\) be a sequence with \(q\) period. For any \(\varepsilon > 0\), there exists \(D(E, A, \varepsilon)\) such that

\[(35) \quad \left| \sum_{t=n_0}^{n} f(t) \cos(4\theta(t, E) + \phi) \right| \leq D(E, A, \varepsilon) + \varepsilon \ln \frac{n - v}{n_0 - v},\]

and

\[(36) \quad \left| \sum_{t=n_0}^{n} f(t) \sin(2\theta(t, E_j) \sin 2\theta(t, E)) \right| \leq D(E, A, \varepsilon) + \varepsilon \ln \frac{n - v}{n_0 - v},\]

for all \(E_j \in A\).

**Proof.** We only give the proof of \((35)\). The proof of \((36)\) proceeds similarly.

**Case 1:** \(k(E) \\pi\) is rational. Since \(k(E) \notin \frac{\pi}{2}\), we can assume \(k(E) \pi = \frac{N}{N} \) for some \(N \geq 3\). Thus for any \(\phi\),

\[(37) \quad \sum_{j=0}^{N-1} \cos(4jk(E) + \phi) = 0.\]

By \((39)\), \((8)\), \((20)\) and \((27)\), one has

\[(38) \quad k(E) = \theta(n_0 + q, E) - \theta(n_0, E) + O\left(\frac{1}{n_0 - v}\right) \mod \mathbb{Z}.\]

Iterating, we obtain for any positive integer \(j \leq N - 1\),

\[(39) \quad jk(E) = \theta(n_0 + jq, E) - \theta(n_0, E) + O\left(\frac{1}{n_0 - v}\right) \mod \mathbb{Z}.\]

Thus by \((37)\) and \((39)\), we can translate \(n_0\) by \(p\) and use \(\phi = \theta(n_0 + p, E)\) to get

\[\sum_{j=0}^{N-1} \cos 4\theta(n_0 + jq + p, E) = O\left(\frac{1}{n_0 + p - v}\right),\]

for all \(p = 0, 1, \ldots, q - 1\). This implies

\[\sum_{j=0}^{N-1} f(n_0 + jq + p) \cos 4\theta(n_0 + jq + p, E) = O(1) \quad \frac{(n_0 + p - v)^2}{n_0 + jq + p},\]
for all \( p = 0, 1, \cdots, q - 1 \). Let us define an integer \( w \) so that \( n - n_0 \geq Nqw - 1 \). Then

\[
\left| \sum_{t=n_0}^{n} f(t) \frac{\cos 4\theta(t, E)}{t - v} \right| \leq \frac{|O(1)|}{n_0 - v} + \sum_{i=n_0}^{n_0+q-1+Nqw} \frac{|O(1)|}{(i - v)^2}
\]

(40)

\[
\leq \frac{|O(1)|}{n_0 - v} + \sum_{i=n_0}^{\infty} \frac{|O(1)|}{(i - v)^2}
\]

(41)

This completes the proof of (35) for rational \( \frac{k(E)}{\pi} \).

Case 2: \( \frac{k(E)}{\pi} \) is irrational. By the ergodic theorem, for any \( \epsilon > 0 \), there exists \( N > 0 \) such that

\[
\left| \sum_{j=0}^{N-1} \cos(4jk(E) + \phi) \right| \leq N\epsilon.
\]

(42)

By (42) and (38), one has

\[
\left| \sum_{j=n_0}^{n} f(t) \frac{\cos 4\theta(t, E)}{t - v} \right| \leq \frac{|O(1)|}{n_0 - v} + O(1) \ln \left( \frac{n - v}{n_0 - v} \right)
\]

for all \( p = 0, 1, \cdots, q - 1 \). This implies

\[
\sum_{j=0}^{N-1} \cos 4\theta(n_0 + jq + p, E) \leq N \left( \epsilon + O \left( \frac{1}{n_0 - v} \right) \right),
\]

for all \( p = 0, 1, \cdots, q - 1 \).

We note that

\[
\sum_{j=n_0}^{n} \frac{1}{j - v} \leq O(1) \ln \left( \frac{n - v}{n_0 - v} \right).
\]

Thus, performing an estimate analogous to (41) we obtain

\[
\left| \sum_{t=n_0}^{n} f(t) \frac{\cos 4\theta(t, E)}{t - v} \right| \leq D(E, A, \epsilon) + \epsilon \ln \left( \frac{n - v}{n_0 - v} \right).
\]

This concludes our proof of (35) for irrational \( \frac{k(E)}{\pi} \).

**Proof of Proposition 4.1.** Equation (26) becomes

\[
\ln R(n + 1, E) - \ln R(n, E) = -|\varphi(n, E)|^2 \frac{C}{n - v} \sin^2(2\eta(n) + 2\gamma(n)) + \frac{|O(1)|}{(n - v)^2}.
\]

(44)

This implies

\[
\ln R(n + 1, E) - \ln R(n, E) \leq \frac{C}{(n - v)^2}.
\]

(45)

It is easy to see that (29) follows from (45) since \( n_0 - v > K \).
Rewrite (42) as
\begin{equation}
\ln R(n+1, E) - \ln R(n, E) = -|\varphi(n, E)|^2 \frac{C}{n-v} + O(1)|\varphi(n, E)|^2 \frac{\cos 4\theta(n, E)}{n-v} + \frac{O(1)}{(n-v)^2}.
\end{equation}
Applying (35) with \( \varepsilon = 1 \) to (46), we have for \( n \geq n_0 \),
\begin{equation}
\ln R(n, E) - \ln R(n_0, E) \leq \sum_{t=n_0}^{n} \frac{C}{t-v} + O(1)|\varphi(n, E)|^2 \frac{\cos 4\theta(t, E)}{j-v} + \frac{O(1)}{(j-v)^2}
\end{equation}
\begin{equation}
\leq C - C \ln \left( \frac{n-v}{n_0-v} \right).
\end{equation}
This implies (28).

Now let us consider the solution \( u(n, E_j) \) of \( (H_0 + b')Id = E_j u \).

By (26) again, one has
\begin{equation}
\ln R(n+1, E_j) - \ln R(n, E_j) = -C|\varphi(n, E_j)|^2 \frac{\sin 2\theta(n, E) \sin 2\theta(n, E_j)}{n-b} + \frac{O(1)}{(n-b)^2}
\end{equation}
By (35) (following Lemma 4.2) and following the proof of (51), we can prove (53). We finish the proof.

\section{Construction}

We will give the construction of the perturbation \( b' \). The idea is to glue the potential \( b'(n, E, A, x_0, x_1, v, \theta_0) \) in a piecewise manner. Our construction is inspired by 2 and 10.

Let us fix a band of the absolutely continuous spectrum, and enumerate the desired embedded eigenvalues in our band spectrum as \( E_j \) (we always assume there are countably many). Let \( N : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \) be a non-decreasing function, \( N(1) = 1 \) and \( N(w) \) grows very slowly (in other words, we expect \( N(w) = N(w+1) \) to be true for “most” \( w \in \mathbb{Z}_{\geq 0} \)). Furthermore, we define \( N \) so if \( N(w + 1) > N(w) \) then \( N(w + 1) = N(w) + 1 \). Let
\begin{equation}
\varepsilon_w = \frac{1}{100N(w)}.
\end{equation}
Let \( C_w \) be a large constant that depends on the eigenvalues \( E_1 \ldots E_{N(w)} \). We write
\begin{equation}
C_w = C(E_1, E_2, \ldots, E_{N(w)}).
\end{equation}
We emphasize that the dependence of \( C_{w+1} \) on the \( E_j \) does not take into account multiplicity. Thus if \( N(w+1) = N(w+2) \) (which we expect to happen very frequently) then \( C_{w+1} = C_{w+2} \).

Let \( K_w \) be large enough such that \( K_w > K(E, \{E_j\}_{j=1}^{N(w)} \setminus E, \varepsilon_w) \) for all \( E \in \{E_j\}_{j=1}^{N(w)} \) in Proposition 4.4.

We have \( N(w) = \max_j N(j) \) for sufficiently large \( w \) in the construction of Theorem 2.2 and we instead have \( \lim_w N(w) = \infty \) in the construction of Theorem 2.3.

Define
\begin{equation}
T_{w+1} = T_wC_{w+1}
\end{equation}
and \( T_0 = C_1 \). By modifying \( C_w \), we can assume \( T_w \) is large enough so that
\begin{equation}
T_w \geq K_w
\end{equation}
for any \( E \in \{E_j\}_{j=1}^{N(w)} \) in Proposition 4.1.

Let \( E_j \) and \( \theta_j \) be given by Theorem 2.2 and Theorem 2.3. Fix \( w \). By Proposition 4.1, then there exist constants \( K_w, C_w \) (independent of \( v, n_0 \) and \( n_1 \)) and perturbation \( b'(n, E_j, A, n_0, n_1, v, \theta_0) \) such that for \( n_0 - v > K_w \) the following holds:
Potential: for \( n_0 \leq n \leq n_1 \), supp\((V) \subset (n_0, n_1)\), and
\[
|b'(n, E_j, A, n_0, n_1, v, \theta_0)| \leq \frac{C_w}{n - v}.
\]

Solution for \( E_j \): the solution of \((H_0 + b'Id)u = E_ju\) with boundary condition \( \theta(n_0, E_j) = \theta_0 \) satisfies
\[
R(n_1, E_j) \leq C_w \left( \frac{n_1 - v}{n_0 - v} \right)^{-100} R(n_0, E)
\]
and for \( n_0 < n < n_1 \),
\[
R(n, E_j) \leq \left( \frac{n_1 - v}{n_0 - v} \right)^{\varepsilon_w} R(n_0, E_j).
\]

Solution for \( E_{j'} \) with \( j' \neq j \): any solution of \((H_0 + b'Id)u = E_{j'}u\) satisfies for \( n_0 < n \leq n_1 \),
\[
R(n, E_{j'}) \leq \left( \frac{n_1 - v}{n_0 - v} \right)^{\varepsilon_w} R(n_0, E_{j'}).
\]

On the other hand, if \( N(w) \) goes to infinity arbitrarily slowly, then \( C_w \) can also go to infinity arbitrarily slowly. This doesn’t contradict our previous statement that \( T_w \) is “large enough”, since we can choose the \( C_w \) to be large but also choose it to be constant for long stretches of \( w \in \mathbb{Z}_{\geq 0} \). We do however choose \( C_w \) so that it goes to infinity faster than \( N(w) \): let us in fact choose \( C_w \) so that
\[
C_w \geq 4^{N(w+1)}.
\]

We can also assume for large \( w \),
\[
T_w \geq 1000^w.
\]
and for large \( w \),
\[
C_w \leq \ln w.
\]
Thus eventually, one has
\[
C_{w+1} \leq T_w.
\]
Let
\[
J_w = \sum_{i=1}^{w} N(i)T_i.
\]
By letting \( N(w) \) go to infinity arbitrarily slow, we assume
\[
C_w^2N(w) \leq \frac{1}{100} \min_{n \in [J_{w-1}, J_w]} h(n),
\]
where \( h(n) \) is given by Theorem 2.3.

Notice that \( J_w \) and \( T_w \) go to infinity faster than \( C_w \). More precisely, we will have \( C_w/J_w \) and \( C_w/T_w \) both tending to 0 as \( w \) tends to infinity.

We will also define potential \( b'_n \) and \( u(n, E_j), j = 1, 2, \ldots \) on \((0, J_w)\) by induction, such that
1. \( u(n, E_j) \) solves for \( n \in (0, J_w) \)
\[
Hu(n, E_j) = E_ju(n, E_j),
\]
and satisfies boundary condition
\[
\frac{u(1, E_j)}{u(0, E_j)} = \tan \theta_j,
\]
2. \( u(n, E_i) \) for \( i = 1, 2, \cdots, N(w) \) and \( w \geq 2 \), satisfies

\[
R(J_w, E_i) \leq 2^{N(w)} N(w)^{50} C_{w+1}^{-50} R(J_{w-1}, E_i).
\]

3.

\[
|b'_n| \leq 100 \frac{N(w) C_{w+1}^2}{n+1}
\]

for \( J_{w-1} \leq n \leq J_w \).

By our construction, one has

\[
\frac{J_w}{T_{w+1}} \leq 2 \sum_i N(i) T_i
\]

\[
\leq 2 \frac{N(w)}{C_{w+1}} \sum_i \frac{T_i}{T_w}
\]

\[
\leq 4 \frac{N(w)}{C_{w+1}}.
\]

The last inequality comes from (49) and (54)

Let \( u(n, E_j) \) be the solution of

\[
Hu = E_j u
\]

with boundary condition

\[
\frac{u(1, E_j)}{u(0, E_j)} = \tan \theta_j.
\]

We construct \( b'_n \) for step \( w \) in an inductive manner identical to that of the construction of \( V(x) \) in [10, Section 5]. Of course, we replace \( V(x) \) with \( b'_n \) and \( x \) with \( n \).

Now we should show that the \( b'_n \) derived from this construction satisfies the \( w + 1 \)-step conditions (59)-(62). There are small but important differences between our version and the version in [10], so for the readers' convenience we rewrite everything explicitly.

Let \( R(n, E_i) \) for \( i = 1, 2, \cdots, N(w+1) \).

\[
R(n, E_i) \text{ decreases from point } J_w + (i - 1)T_{w+1} \text{ to } J_w + iT_{w+1}, \quad i = 1, 2, \cdots, N(w+1), \text{ and may increase from any point } J_w + (m-1)T_{w+1} \text{ to } J_w + mT_{w+1}, \quad m = 1, 2, \cdots, N(w+1) \text{ and } m \neq i. \text{ That is}
\]

\[
R(J_w + iT_{w+1}, E_i) \leq N^{50}(w) C_{w+1}^{-50} R(J_w + (i - 1)T_{w+1}, E_i),
\]

and for \( m \neq i \) (see (53)),

\[
R(J_w + mT_{w+1}, E_i) \leq C_{w+1} \sigma_{w+1} R(J_w + (m - 1)T_{w+1}, E_i),
\]

by Proposition 3.1.

Thus by (47), for \( i = 1, 2, \cdots, N(w+1) \),

\[
R(J_{w+1}, E_i) \leq N^{50}(w) C_{w+1}^{-50} C_{w+1}^{N(w+1)} R(J_w, E_i) \leq N^{50}(w) C_{w+1}^{-49} R(J_w, E_i).
\]

This implies (61) for \( w + 1 \). By the construction of \( b'_n \), we calculate, in a manner identical to the derivation of [10, (67)] that

\[
|b'_n| < 100 \frac{N(w+1) C_{w+1}^2}{n+1},
\]

for \( J_w \leq n \leq J_{w+1} \). This implies (62).
5.1. **Proof of Theorems 2.2 and 2.3.**

*Proof.* In the construction of Theorem 2.2 eventually $N(w)$ and $C_w$ are bounded. In the construction of Theorem 2.3, $N(w)$ and $C_w$ grow to infinity arbitrarily slowly. By (62) and (68), (4) and (5) hold.

It suffices to show that for any $j$, $R(n, E_j) \in \ell^2$. Below we give the details. For any $N(w_0 - 1) < j \leq N(w_0)$, by the construction (see (61)), we have for $w \geq w_0$

$$R(J_w+1, E_j) \leq N(w)^{50} C^{-49} w^{-49} R(J_w, E_j)$$

$$\leq C^{-25} w^{-1} R(J_w, E_j)$$

$$\leq T_{w_0}^{-25} T_{w+1}^{-24} R(J_w, E_j)$$

(69)

where the second inequality holds by (54) and the third inequality holds by (49).

By (47), (52), (53), (54), (69), (67) and (56), for all $n \in [J_w+1, J_{w+2}]$

$$R(n, E_j) \leq C_{w+2} T_{w+2}^{-25} T_{w+1}^{-24} R(J_w, E_j)$$

(70)

Then by (70), we have

$$\sum_{n=J_{w_0}+1}^{\infty} R^2(n, E_j) = \sum_{w \geq w_0} \sum_{n=J_w+1}^{J_w+2} R^2(n, E_j)$$

$$\leq \sum_{w \geq w_0} \sum_{n=J_w+1}^{J_w+2} T_{w_0}^{50} T_{w+1}^{-48} R^2(J_w, E_j)$$

$$\leq T_{w_0}^{50} R^2(J_w, E_j) \sum_{w \geq w_0} N(w+2) T_{w+2}^{-48} T_{w+1}^{-47}$$

$$= T_{w_0}^{50} R^2(J_w, E_j) \sum_{w \geq w_0} N(w+2) C_{w+2} T_{w+1}^{-47}$$

$$\leq T_{w_0}^{50} R^2(J_w, E_j) \sum_{w \geq w_0} T_{w+1}^{-40} < \infty,$$

since $N(w)$ and $C_w$ go to infinity slowly and $T_w$ satisfies $T_5$. This completes the proof. □

**Acknowledgments**

W.L. was supported by the AMS-Simons Travel Grant 2016-2018, NSF DMS-1401204 and NSF DMS-1700314. D.O. was supported by a Xiamen University Malaysia Research Fund (Grant No: XMUMRF/2018-C1/IMAT/0001).

**References**

[1] S. Jitomirskaya and W. Liu. Noncompact complete riemannian manifolds with dense eigenvalues embedded in the essential spectrum of the Laplacian. Preprint.

[2] S. Jitomirskaya and W. Liu. Noncompact complete Riemannian manifolds with dense eigenvalues embedded in the essential spectrum of the Laplacian. Preprint.

[3] E. Judge, S. Naboko, and I. Wood. Eigenvalues for perturbed periodic Jacobi matrices by the Wigner-von Neumann approach. *Integral Equations and Operator Theory*, 85(3):427–450, 2016.

[4] E. Judge, S. Naboko, and I. Wood. Spectral results for perturbed periodic Jacobi matrices using the discrete Levinson technique. *Studia Mathematica*, in press.
A. Kiselev, Y. Last, and B. Simon. Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators. *Communications in mathematical physics*, 194(1):1–45, 1998.

A. Kiselev, C. Remling, and B. Simon. Effective perturbation methods for one-dimensional Schrödinger operators. *J. Differential Equations*, 151(2):290–312, 1999.

H. Krüger. On the existence of embedded eigenvalues. *J. Math. Anal. Appl.*, 395(2):776–787, 2012.

M. Lukic and D. C. Ong. Wigner-von Neumann type perturbations of periodic Schrödinger operators. *Trans. Amer. Math. Soc.*, 367(1):707–724, 2015.

M. Lukic and D. C. Ong. Generalized Prüfer variables for perturbations of Jacobi and CMV matrices. *J. Math. Anal. Appl.*, 444(2):1490–1514, 2016.

D. C. Ong and W. Liu. Sharp spectral transition for eigenvalues embedded into the spectral bands of perturbed periodic Schrödinger operators. Preprint.

J. von Neuman and E. Wigner. Über merkwürdige diskrete Eigenwerte. Uber das Verhalten von Eigenwerten bei adiabatischen Prozessen. *Zhurnal Physik*, 30:467–470, 1929.

(Wencai Liu) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92697-3875, USA

E-mail address: liuwencai1226@gmail.com

(Darren C. Ong) DEPARTMENT OF MATHEMATICS, XIAMEN UNIVERSITY MALAYSIA, JALAN SUNSURIA, BANDAR SUNSURIA, SEPANG, 43900, SELANGOR, MALAYSIA

URL: https://dongcl.wixsite.com/darrenong

E-mail address: darrenong@xmu.edu.my