Partial matching width and its application to lower bounds for branching programs

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Abstract

We introduce a new structural graph parameter called partial matching width. In particular, for a graph $G$ and $V \subseteq V(G)$, the matching width of $V$ is such a largest $k$ such that for any permutation $SV$ of vertices with $V$, there is a prefix $SV'$ of $SV$ such that there is a matching of size $k$ and between $SV'$ (being treated as a set of vertices) and $G \setminus SV'$.

For each (sufficiently large) integer $k \geq 1$, we introduce a class $G_k$ of graphs of treewidth at most $k$ and max-degree 7 such that for each $G \in G_k$ and each (sufficiently large) $V \subseteq V(G)$, the partial matching width of $V$ is $\Omega(k \log |V|)$.

We use the above lower bound to establish a lower bound on the size of non-deterministic read-once branching programs (NROBPs). In particular, for each sufficiently large integer $k$, we introduce a class $\Phi_k$ of CNFs of (primal graph) treewidth at most $k$ such that for any $\varphi \in \Phi_k$ and any Boolean function $F \subseteq \varphi$ and such that $|\varphi| / |F| \leq 2^{\sqrt{n}}$ (here the functions are regarded as sets of assignments on which they are true), a NROBP implementing $F$ is of size $n^{\Omega(k)}$. This result significantly generalises an earlier result of the author showing a non-FPT lower bound for NROBPs representing CNFs of bounded treewidth. Intuitively, we show that not only those CNFs but also their arbitrary one side approximations with an exponential ratio still attain that lower bound.

The non-trivial aspect of this approximation is that due to a small number of satisfying assignments for $F$, it seems difficult to establish a large bottleneck: the whole function can ‘sneak’ through a single rectangle corresponding to just one vertex of the purported bottleneck. We overcome this problem by simultaneously exploring $\sqrt{n}$ bottlenecks and showing that at least one of them must be large. This approach might be useful for establishing other lower bounds for branching programs.

1 Introduction

In this paper we introduce a new structural graph parameter partial matching width defined as follows. Let $G$ be a graph, $V \subseteq V(G)$. The partial matching
width of $V$ is the largest $k$ such that for any permutation $SV$ of vertices of $V$, there is a prefix $SV'$ of $SV$ such that there is a matching of size $k$ and between $SV'$ (being treated as a set of vertices) and $G \setminus SV'$.

The partial matching width generalizes matching width of a graph [6], which is the partial matching width of $V = V(G)$. In light of a linear relationship between matching width and pathwidth [8], partial matching width can be considered a generalization of the latter.

We show that, similarly to pathwidth, the partial matching width can be much larger than the treewidth. In particular, for each (sufficiently large) integer $k \geq 1$, we introduce a class $G_k$ of graphs of treewidth at most $k$ and max-degree $7$ such that for each $G \in G_k$ and each (sufficiently large) $V \subseteq V(G)$, the partial matching width of $V$ is $\Omega(k \log |V|)$. This class is essentially the same as we used in [7] with the only difference that, for the convenience of the reasoning, instead of the underlying binary trees we use ternary ones.

Intuitively, we can say that the partial matching width serves for the above class as an expansion-like equivalent of pathwidth. A similar in spirit connection between treewidth and standard expansion has been established in [3] but in a much more general context.

We use the above lower bound on partial matching width to prove a lower bound for read-once branching programs, significantly generalizing our earlier result [7]. In particular, in [7], for each sufficiently large $k$ we introduced a class $\Phi_k$ of CNFs whose primal graph is of treewidth at most $k$ and showed that NROBPs representing this class must be of size $n^{\Omega(k)}$. Thus, we demonstrated that NROBPs are not FPT on CNFs of bounded treewidth. In this paper, we show that this lower bound is very robust because it holds for arbitrary one-side approximations of the functions of $\Phi_k$ with ratio up to $2\sqrt{n}$. Specifically, we show if we take any function $\varphi \in \Phi_k$ and consider an arbitrary function $F$ with $F \subseteq \varphi$ and $|\varphi|/|F| \leq 2\sqrt{n}$ the lower bound of $n^{\Omega(k)}$ still holds.

This result has two interesting aspects: the approach that we used and the connection between the ‘approximation’ lower bound and randomized branching programs. Let us overview both these aspects.

We overview the approach in comparison with the one we used in [7]. In particular, in [7] we considered a NROBP $Z$ representing a CNF $\varphi \in \Phi_k$ and fixed a large bottleneck of $Z$: a source-sink cut such that, for some universal constant $c$, at most $n^{-k/c}$-th path of satisfying assignments of $\varphi$ ‘passes’ through a single vertex of this cut. Then we concluded that the total number of vertices in the cut must be $n^{\Omega(k)}$. In our approximation case this approach does not work: if a function $F$ has $|\varphi|/2\sqrt{n}$ satisfying assignments, all of them can ‘sneak’ through a single vertex of the cut! Using partial matching width instead of just matching width allows us to avoid fixing a single bottleneck. Instead, we

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1In other words, we obtain $F$ by arbitrary removal of satisfying assignments of $\varphi$ so that at least $2^{-\sqrt{n}}$-th part of the initial satisfying assignment remains.

2In this paper we use slightly tweaked version of the CNFs we used in [7] replacing the underlying binary tree in the tree decomposition with ternary one. This makes reasoning by induction reasoning more elegant but the result itself remains true for both initial and tweaked classes of CNFs.
simultaneously consider $\sqrt{n}$ different bottlenecks and prove that at least one of them must be large.

This approach may be useful for other cases where the methodology of establishing a single bottleneck does not seem to work. One such notable question is the complexity of semantic NROBPs. For domains of size $r > 3$, exponential lower bounds for $r$-way semantic NROBPs have been known for long [1, 4], recently culminating with a lower bound for $r = 3$ [2]. However, to the best of our knowledge, the binary case remains open. Moreover, [2] provides an indication that fixing a single large bottleneck may not be the right technique for tackling the binary case. In light of this, it is interesting to investigate whether our approach of multiple bottlenecks would bring any new insight concerning the binary case.

Our result is related to Randomized Read-Once branching programs through a well known result of Sauerhoff [9] (see also Theorem 11.8.3. of [10]) who showed that a lower bound for a randomized (deterministic) read-once branching program for a particular function follows from a deterministic read-once branching program lower bound for an arbitrary constant approximation of this function. The approximation of [9] is different from ours: it is two sided and taken over the whole set of $2^n$ truth assignments, not just the satisfying ones. It is interesting to see whether our approach can yield a lower bound for a two-side approximation for the considered classes of CNFs (and thus a non-FPT lower bound for randomized read-once branching programs). A natural initial step is to establish a one sided approximation for deterministic read-once branching programs representing the negations of CNFs $\Phi_k$ as above.

The rest of the paper is organized as follows. Section 2 introduces the necessary background. Section 3 introduces the notion of partial matching width and proves existence of a class of graphs of small treewidth in which each sufficiently large set of vertices has large partial matching width. Section 4 proves a non-FPT lower bound for NROBPs representing functions approximating CNFs of small primal graph treewidth.

2 Preliminaries

Sets of literals and variables. In this paper when we refer to a set of literals we assume that it does not contain an occurrence of a variable and its negation. For a set $S$ of literals we denote by $Var(S)$ the set of variables whose literals occur in $S$. If $F$ is a Boolean function or its representation by a specified structure, we denote by $Var(F)$ the set of variables of $F$. A truth assignment to $Var(F)$ on which $F$ is true is called a satisfying assignment of $F$. A set $S$ of literals represents the truth assignment to $Var(S)$ where variables occurring positively in $S$ (i.e. whose literals in $S$ are positive) are assigned with true and the variables occurring negatively are assigned with false.

Projections, restrictions. For $V \subseteq Var(S)$, the projection of $S$ on $V$ denoted by $Proj(S, V)$ is the subset $S' \subseteq S$ such that $Var(S') = V$. Let $F$ be a Boolean function, $S$ be a set of literals such that $Var(S) \subseteq Var(F)$. 
The restriction $F|_S$ is a function $Var(F) \setminus Var(S)$ such that $S'$ is a satisfying assignment of $F|_S$ if and only if $S \cup S'$ is a satisfying assignment for $F$. If $S$ consists of a single literal $\ell$ then we write $F|_v$ rather than $F|_{(\ell)}$.

**Boolean functions as sets of satisfying assignments.** In this paper we regard Boolean functions and CNFs as their sets of satisfying assignments. In this context if for instance $F$ and $\varphi$ are CNFs and we write $F \subseteq \varphi$ this means that each satisfying assignment of $F$ is also a satisfying assignment of $\varphi$. We also use $|F|$ and $|\varphi|$ to denote the sizes of respective sets of satisfying assignments.

**CNFs $\varphi(G)$**. Let $G$ be a graph without isolated vertices. Then $\varphi(G)$ is a CNF with $V$ as the set of variables and $\{(u \lor v)|\{u, v\} \in E(G)\}$ as the set of clauses. This definition allows us to identify variables of $\varphi(G)$ and vertices of $G$ and to use phrases like ‘let $S$ be a set of literals of $V(G)$’ and let $V \subseteq V(G)$ be the set of all $v$ such that $\neg v \in S$.

**Definition 1 (Nondeterministic Read-once branching programs (NROBPs).)**

Let $V$ be a set of Boolean variables. Let $Z$ be a directed acyclic graph (DAG) with one source and one sink so that some of the edges are labelled with literals of $V$. We say that $v \in V$ occurs on edge $e$ if $Z$ is labelled with a literal of $v$. The occurrence can be positive or negative if the labelling literal is $v$ and $\neg v$, respectively.

We say that $Z$ is a nondeterministic read-once branching program (NROBP) implementing a function on $V$ if each variable of $V$ occurs exactly once on each source-sink path of $Z$.

For a path $P$ of $Z$ we denote by $A(P)$ the set of literals labelling $P$ and $Var(A(P))$ is denoted by $Var(P)$. Then the set of satisfying assignments of the function represented by $Z$ consists of all $A(P)$ such that $P$ is a source-sink path of $Z$.

**Proposition 1** Let $Z$ be a NROBP and let $P_1, P_2$ be two paths having the same initial and final vertices. Then $Var(P_1) = Var(P_2)$.

**Separation of sets of variables by a vertex of a NROBP.** In light of Proposition[1] for NROBP $Z$, a variable $x$, and a vertex $v$ of $Z$, we can say that $x$ is located before $v$ (that is, on each source-sink path $P$ including $v x$ occurs on the prefix of $P$ ending with $v$) or $x$ is located after $v$ (replace the prefix by the suffix). We say that two sets $X$ and $Y$ of variables are separated by $X$ if either (i) all of $X$ occur before $v$ and all of $Y$ occur after $v$ or (ii) all of $Y$ occur before $v$ and all of $X$ occur after $v$.

**Proposition 2** Let $G$ be a graph without isolated vertices, $F \subseteq \varphi(G)$ and $Z$ be a NROBP representing $F$. Let $u$ be a vertex of $Z$ and let $M = \{\{x_1, y_1\}, \ldots, \{x_q, y_q\}\}$ be a matching of $G$ such that $\{x_1, \ldots, x_q\}$ and $\{y_1, \ldots, y_q\}$ are separated by $u$.

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[1] The requirement of exactly one occurrence rather than at most one means that the NROBP is uniform. Since it is known that every NROBP can be simulated by a uniform one with only a polynomial increase of the number of nodes (see e.g. [5]), we assume the uniformity w.l.o.g.
Then there is a set $X$ of $q$ vertices of $G$ consisting of exactly one vertex of each $\{x_i, y_i\}$ such that for each source-sink path $P$ passing through $u$, $X \subseteq A(P)$.

**Proof.** If the statement does not hold then for some $\{x_i, y_i\}$, neither $x_i$ nor $y_i$ belongs to $A(P)$ for each source-sink path passing through $u$. It follows that there are two such paths $P_1$ and $P_2$ such that $\neg x_i \in A(P_1)$ and $\neg y_i \in A(P_2)$. Assume w.l.o.g. that all of $x_1, \ldots, x_q$ occur before $u$ and all of $y_1, \ldots, y_q$ occur after $u$. Let $P_1'$ be the prefix of $P_1$ ending with $u$ and let $P_2''$ be the suffix of $P_2$ beginning with $u$. Then $P = P_1' + P_2''$ is a source-sink path of $Z$ such that $\{\neg x_i, \neg y_i\} \subseteq A(P)$. As $F \subseteq \varphi(G)$ it follows that $A(P)$ is also a satisfying assignment of $\varphi(G)$. However, this is a contradiction because $A(P)$ falsifies $(x_i \lor y_i)$.

## 3 Partial matching width

**Definition 2 (Partial matching width)** Let $G$ be a graph and let $V \subseteq V(G)$. The partial matching width of $V$ (w.r.t. $G$) is the largest $k$ such that any permutation $SV$ of $V$ has a prefix $SV'$ such that there is a matching of size at least $k$ between $SV'$ and the rest of $G$.

The main theorem of this section is Theorem 2 where we prove that for each sufficiently large $k$ there is a class of graphs with treewidth at most $k$ such that for each graph $G$ of this class and each sufficiently large $V \subseteq V(G)$, the partial matching width of $V$ is $\Omega(k \log n)$.

For the purpose of proving a lower bound on partial matching width, it will be easier for us to use a related notion of witnessing matching.

**Definition 3 (Witnessing matching)** Let $V \subseteq V(G)$ and let $SV$ be a permutation of $V$. Let $\{u, v\} \in E(G)$. Then $\{u, v\}$ is supported by a partition $SV_1, SV_2$ into a prefix and a suffix if either (i) say, $u \in SV_1$ and $v \in SV_2$ or (ii) say, $u \in SV$, $v \in V(G) \setminus V$. A matching $M$ is supported by $SV_1, SV_2$ if every edge of $M$ is supported by $SV_1, SV_2$.

A matching $M$ is witnessing for $SV$ if there is a partition of $SV$ into a prefix $SV_1$ and a suffix $SV_2$ supporting $M$.

**Proposition 3** If every permutation $SV$ of $V \subseteq V(G)$ has a witnessing matching of size at least $k$ then the partial matching width of $V$ is at least $k/2$.

**Proof.** Let $SV$ be a permutation of $V$. We need to show that there is a prefix $SV'$ of $SV$ such that there is a matching of size at least $k/2$ between $SV'$ and $V(G) \setminus SV'$. Let $M'$ be a witnessing matching of size $k$ for $SV$. This means that there is a partition of $SV$ into a prefix $SV_1$ and a suffix $SV_2$ and a partition of $M'$ into $M'_1$ and $M'_2$ such that the edges of $M'_1$ connect vertices of $SV_1$ to vertices of $SV_2$ and edges of $M'_2$ connect vertices of $SV$ to vertices of $V(G) \setminus V$.

Clearly either $M'_1$ or $M'_2$ is of size $k/2$. In the former case set $SV' = SV$ and $M = M'_1$, in the latter case set $SV' = SV$ and $M = M'_2$. Clearly, in both
cases, we have a matching of size at least \(k/2\) connecting the chosen prefix to the rest of the graph.

**Graphs** \(T(H)\). Let \(T\) be a tree and \(H\) be an arbitrary graph. The graph \(T(H)\) is the union of disjoint copies \(H^v\) of \(H\) for each \(v \in V(T)\) plus additional edges defined as follows. Let \(V(H) = \{1, \ldots, p\}\). For \(i \in \{1, \ldots, p\}\), denote the copy of \(i\) in \(H^v\) by \(v^i\). Then for each \(i \in \{1, \ldots, p\}\) there is an edge between \(u^i\) and \(v^j\) whenever \(u\) and \(v\) are adjacent in \(T\).

Let \(V \subseteq T(H)\). \(u \in V(T)\) is occupied by \(V\) if \(V(H^u) \cap V \neq \emptyset\). \(OC(T, V)\) denotes the set of all vertices of \(T\) occupied by \(V\). If \(V(H^u) \subseteq V\) then we say that vertex \(u\) is complete in \(V\).

Suppose \(V \subseteq V(T(H))\), and let \(V_1, V_2 \subseteq V\) (one of \(V_1, V_2\) possibly empty) such that \(V_1 \cup V_2 = V\) and \(V_1 \cap V_2 = \emptyset\). Then the vertices of \(T(H)\) may have three different roles w.r.t. \(V_1, V_2\): belonging to \(V_1\), belonging to \(V_2\), belonging to \(V(T(H)) \setminus V\). A vertex \(u \in V(T)\) is homogenous if all the vertices of \(V(H^u)\) have the same role.

**Ternary trees and the \(tr\) function.** A complete rooted ternary tree of height \(h\) is a tree with a special designated root vertex \(rt\) (naturally determining the parent-child relation between the vertices) in which each root-leaf path has exactly \(h\) edges and each non-leaf vertex has exactly 3 children. For \(x \geq 1\), we denote by \(tr(x)\) the largest \(h\) such that \(ht\) is at least as the number of vertices of a complete rooted ternary tree \(T\) of height \(h\). It is not hard to see that \(tr\) is a logarithmic function and that \(|V(T)| \leq 3|V(T)|\).

**Lemma 1** Let \(T\) be a tree and \(H\) a graph. Let \(V \subseteq V(T(H))\), \(V_1, V_2 \subseteq V\), \(V_1 \cup V_2 = V\), \(V_1 \cap V_2 = \emptyset\). Assume that there are two vertices \(u\) and \(v\) of \(T\) and a subset \(\{1, \ldots, t\}\) of vertices of \(H\) such that for each \(i \in \{1, \ldots, t\}\), the role of \(u^i\) is not the same as the role of \(v^i\). Then \(T(H)\) has a matching \(M\) of size \(t\) such the ends of each edge \(e\) of \(M\) have different roles.

**Proof.** Let \(P = u_1, \ldots, u_q\) be the path between \(u\) and \(v\) such that \(u_1 = u\) and \(u_q = v\). Then for each \(i \in \{1, \ldots, t\}\), \(P^i = u_1^i, \ldots, u_q^i\) is a path where the first and the last vertices have different roles. It follows that \(P^i\) has an edge \(e^i\) whose ends have different roles. Each such \(e^i\) connects two copies of vertex \(i\) of \(H\), hence for \(i \neq j\), edges \(e^i\) and \(e^j\) cannot have a joint end. It follows that edges \(e^1, \ldots, e^t\) constitute a matching of size \(t\).

**Lemma 2** Let \(T\) be a tree with at least \(p\) vertices. Let \(H\) be a connected graph of at least \(2p\) vertices. Let \(V \subseteq V(T(H))\) such that \(OC(T, V) = V(T)\). Let \(V_1, V_2\) be a partition of \(V\) so that \(|V_i| \leq |V(T(H))| - p^2\) for each \(i \in \{1, 2\}\). Then \(T(H)\) has a matching of size \(p\) in which the ends of each edge have different roles w.r.t. \(V_1, V_2\).

**Proof.** Assume first that \(T\) has at least \(p\) non-homogenous vertices. Then, due to the connectedness of \(H\), for each non-homogenous vertex \(u\), there is an edge \(e_u\) of \(H^u\) whose ends have different roles. As these edges belong to different copies of \(H\), no two of them have a joint end and hence they constitute a matching of size \(p\).
If \( T \) does not have \( p \) non-homogenous vertices, it has at least one homogenous vertex \( u \). As by assumption \( V(H^u) \) intersects with \( V \), it is either that \( V(H^u) \subseteq V_1 \) or \( V(H^u) \subseteq V_2 \). W.l.o.g., assume the former. If \( T \) has another vertex \( v \) such that \( V(H^v) \subseteq V_2 \) then we are immediately done by Lemma \( \text{I} \). Thus, we conclude that for all vertices \( v \) of \( V(T) \) except at most \( p-1 \) ones, \( (H^v) \subseteq V_1 \).

Observe that there is \( w \in V(T) \) such that \( |V(H^w) \setminus V_1| \geq p \). Indeed, assume the opposite. Let \( W \) be the set of most \( p-1 \) non-homogenous vertices. It follows that each \( w \in W \) has at most \( p-1 \) vertices outside \( V_1 \) and hence, the total number of vertices that are not in \( V_1 \) is at most \( (p-1)^2 < p^2 \) in contradiction to our assumption. Thus the desired vertex \( w \) does exist.

Denote by \( \{1, \ldots, p\} \) the vertices of \( H \) such that \( w^i \notin V_1 \) for each \( i \in \{1, \ldots, p\} \). As \( V(H^w) \subseteq V_1 \), \( u^i \in V_1 \) for all \( i \in \{1, \ldots, p\} \). Then the desired matching exists by Lemma \( \text{I} \) as witnessed by vertices \( u, w, \) and \( \{1, \ldots, p\} \).

**Lemma 3** Let \( p \geq 1 \) be a natural number, \( H \) be a connected graph with at least \( 2p \) vertices and \( T \) be a complete ternary tree of at least \( p \) vertices. Let \( V \subseteq V(T(H)) \) with \( OC(T, V) = V(T) \) and let \( SV \) be a permutation of \( SV \). Then \( T(H) \) has a witnessing matching \( M \) for \( SV \) of size at least \( p(tr(|T|) - tr(p)) \). Moreover, \( M \) is supported by a partition \( SV_1, SV_2 \) of \( SV \) into a prefix and a suffix that is balanced in the following sense. For each \( i \in \{1, 2\} \) \( |V(T(H))| - |SV_i| \geq p^2 \).

**Proof.** By induction on \( tr(|T|) \). Assume first that \( tr(|T|) - tr(p) \leq 1 \). Partition \( SV \) into a prefix \( SV_1 \) and a suffix \( SV_2 \) of size different by at most 1. Note that this partition is balanced. Indeed, if \( |SV| \geq 2p^2 \), then \( |SV_i| \geq p^2 \) for each \( i \in \{1, 2\} \). Then \( |V(T(H))| - |SV_i| \geq |SV| - |SV_i| = |SV_3-i| \geq p^2 \). Otherwise, \( |SV_i| \leq p^2 \) for each \( i \in \{1, 2\} \). Note also that \( |V(T(H))| \geq 2p^2 \). Hence \( |V(T(H))| - |SV_i| \geq p^2 \). It follows from Lemma \( \text{I} \) that \( T(H) \) has a matching \( M \) of size at least \( p \) with the ends of each edge having different roles w.r.t. \( SV_1, SV_2 \) seen as sets of vertices. Clearly, \( M \) is a witnessing matching for \( SV \) supported by \( SV_1, SV_2 \). Hence, the lemma holds in the considered case.

Assume now that \( tr(|T|) - tr(p) \geq 2 \). Let \( rt \) be the root of \( T \) and let \( T_1, T_2, T_3 \) be the subtrees of \( T \) rooted by the children of \( T \). For \( i \in \{1, 2, 3\} \), let \( V_i = V \cap V(T_i(H)) \) and \( SV_i \) be the permutation of \( V_i \) where the order of elements is the same as in \( SV \). Note that as the height of \( T \) is one less than that of \( T \), \( tr(|T_i|) \geq tr(p) + 1 \) and hence \( |T_i| \geq p \), hence the lemma is correct for \( T_i, V_i, SV_i \) by the induction assumption. It follows that \( T_i(H) \) has a witnessing matching \( M_i \) for \( SV_i \) having size at least \( p(tr(|T|) - tr(p)) \) and supported by a balanced partition \( SV_i', SV_i'' \) of \( V_i \) into a prefix and a suffix.

Let \( u_1, u_2, u_3 \) be the final vertices of \( SV_1', SV_2', SV_3' \), respectively. Assume w.l.o.g. that they occur in \( SV \) in the order they are listed. Let \( SV_1 \) be the prefix of \( SV \) ending with \( u_2 \). Let \( SV_2 = SV \setminus SV_1 \).

Let \( T^* = T \setminus T_2, V^* = V \cap V(T^*(H)) \). For \( i \in \{1, 2\} \), let \( V^*_i = V^* \cap SV_i \) (the latter is treated as a set). Then for each \( i \in \{1, 2\} \) \( V(T^*_i(H)) \setminus V^*_i \geq p^2 \). Indeed, note that \( SV_1^* = V_1^* \cap V(T_1(H)) \) and \( SV_2^* = V_2^* \cap V(T_2(H)) \).
Therefore, $|V(T^*(H)) \setminus V_1^*| \geq |V(T_1(H)) \setminus V_1^*| = |V(T_1(H)) \setminus SV_1^*| \geq p^2$, the last inequality follows from the induction assumption. Symmetrically $|V(T^*(H)) \setminus V_2^*| \geq |V(T_3(H)) \setminus V_2^*| = |V(T_3(H)) \setminus SV_2^*| \geq p^2$. It follows from Lemma 2 that $T^*(H)$ has a matching $M^*$ of size $p$ where the ends of each edge have different roles w.r.t. $V_1^*$ and $V_2^*$.

Let $M = M_2 \cup M^*$. As $M_2$ and $M^*$ are matchings in vertex-disjoint subgraphs of $T(H), M$ is a matching of size $|M_2| + |M^*| \geq p(tr(|T|) - tr(p))$. We claim that $M$ is in fact a witnessing matching for $SV_1, SV_2$. Indeed, let $\{u, v\} \in M$. If $\{u, v\} \in M^*$ then either, say $u \in V^* \subseteq SV_1$ and $v^* \in V_2^* \subseteq SV_2$ or, say $u \in V^* \subseteq SV$ and $v \in V(T^*(H)) \setminus V^* = V(T^*(H)) \setminus SV \subseteq V(T(H)) \setminus SV$. That is, $\{u, v\}$ is supported by $SV_1, SV_2$ in this case. It remains to assume that $\{u, v\} \in M_2$. Then either, say $u \in SV_2 \subseteq SV_1$ and $v \in SV_2 \subseteq SV_2$ or, say $u \in SV_2 \subseteq SV$ and $v \in V(T_2(H)) \setminus SV_2 = V(T_2(H)) \setminus SV \subseteq V(T(H)) \setminus SV$, confirming again that $\{u, v\}$ is supported by $SV_1, SV_2$.

It remains to verify that $SV_1$ and $SV_2$ satisfy the balancing constraints. For that, notice that for each $i \in \{1, 2\}$, $T^*(H) \setminus V_i^* \subseteq T^*(H) \setminus SV_i \subseteq T(H) \setminus SV_i$, as we have already proved that $|T^*(H) \setminus V_i| \geq p^2$, the balancing constraints follow.

**Immediate subtrees.** Let $T$ be a complete rooted binary tree. Then $T^*$ is an immediate subtree of $T$ if $T'$ is rooted by a child of the root of $T$. Let $V \subseteq V(T(H))$. Then $T'$ is the largest immediate subtree w.r.t. $V$ if $|OC(T, V) \cap V(T')|$ is the largest among the immediate subtrees of $T$.

**Definition 4** Let $H$ be a graph. Let $T$ be a complete rooted ternary tree and let $V \subseteq V(T(H))$. Let us sequences $T_1, \ldots, T_q$ and $V_1, \ldots, V_q$ as follows.

- $T_1 = T$, $V_1 = V$.
- Assume that for $1 \leq i \leq q$, $T_i$ and $V_i$ have been defined. Then $T_{i+1}$ is an immediate largest subtree of $T_i$ w.r.t. $V_i$ and $V_{i+1} = V_i \cap V(T_{i+1}(H))$. Then $T_1, \ldots, T_q$ is called a sequence of largest subtrees of $T$ w.r.t. $V$ and $V_1, \ldots, V_q$ is the respective sequence of sets.

Assume that $|OC(T_1, V_1)| - |OC(T_2, V_2)| \geq p$, while $|OC(T_1, V_1)| - |OC(T_{q-1}, V_{q-1})| < p$. Then we say that $T_1, \ldots, T_q$ is a minimal sequence of largest subtrees of $T$ w.r.t. $V$ lacking $p$.

**Lemma 4** Assume that $OC(T, V) > p$. Then there exists a minimal sequence of largest subtrees of $T$ w.r.t. $V$ lacking $p$.

**Proof.** Let $T_1, \ldots, T_q$ be a sequence of largest subtrees of $T$ w.r.t. $V$ such that $|V(T_q)| = 1$ and $V_1, \ldots, V_q$ be the corresponding sequence of sets (such a sequence clearly exists: if the height of $T$ is $n - 1$ then a sequence of largest immediate trees of $n$ elements will be one). Then $|OC(T_1, V_1)| > p$ and $|OC(T_q, V_q)| \leq 1$. Then, take a minimal subsequence $T_1, \ldots, T_q'$ of $T_1, \ldots, T_q$ such that $|OC(T_1, V_1)| - |OC(T_{q'}, V_{q'})| \geq p$. Clearly, $T_1, \ldots, T_q'$ is a desired sequence.
Lemma 5  Let $T_1, \ldots, T_q$ be a minimal sequence of largest subetrees of $T$ w.r.t. $V$ lacking $p$ and let $V_1, \ldots, V_q$ be the corresponding sequence of sets. Then $|OC(T_q, V_q)| \geq (|OC(T_1, V_1)| - p)/3$.

**Proof.**

**Claim 1** Let $T'$ be a complete rooted ternary tree of height at least 1 and let $T'_1, T'_2, T'_3$ be immediate subtrees of $T'$. Let $V' \subseteq V(T'(H))$ and let $V_i' = V' \cap V(T_i'(H))$. Let $i \in \{1, 2, 3\}$ be such that $OC(T_i', V_i')$ is the largest. Then $|OC(T_i', V_i')| \geq (|OC(T', V')| - 1)/3$.

**Proof.** It is not hard to see that $|OC(T', V')| = |OC(T', V') \cap V(T_1)| + |OC(T', V') \cap V(T_2)| + |OC(T', V') \cap V(T_3)| + 1$ Further on, it is not hard to see that for each $j \in \{1, 2, 3\}$, $OC(T'_j, V'_j) = OC(T', V') \cap V(T'_j)$. That is, $|OC(T', V')| = |OC(T'_1, V'_1)| + |OC(T'_2, V'_2)| + |OC(T'_3, V'_3)| + 1$. Clearly, the largest of the set sizes on the right hand side is at least one third of $|OC(T', V')| - 1$. □

By minimality of $q$, $|OC(T_{q-1}, V_{q-1})| \geq |OC(T_1, V_1)| - (p - 1)$. By the above claim, $|OC(T_q, V_q)| \geq (|OC(T_{q-1}, V_{q-1})| - 1)/3 \geq (|OC(T_1, V_1)| - (p - 1) - 1)/3$, so the desired inequality follows. ■

Lemma 6  Let $T_1, \ldots, T_q$ be a minimal sequence of largest subetrees of $T$ w.r.t. $V$ lacking $p$ and let $V_1, \ldots, V_q$ be the corresponding sequence of sets. Suppose that $OC(T, V) \subseteq V(T)$. Let $T^* = T \setminus T_q$ and let $V^* = V \cap V(T^*(H))$. Then $OC(T^*, V^*) \subseteq V(T^*)$.

**Proof.** Let $T' = T \setminus T_2$ and let $V' = V \cap T'(H)$.

**Claim 2** $OC(T', V') \subseteq V(T')$.

**Proof.** Let $rt$ be the root of $T$. By definition, $T_2$ is the subtree of $T$ whose root is one of children of $rt$. If we assume that $OC(T', V') = V(T')$ then both $V(H^{rt})$ and $V(H^u)$ for each $u \in T'$ have non-empty intersections with $V$. In particular, for any other child $T'_2$ of $T$, each copy of $H$ of each vertex of $T'_2$ has a non-empty intersection with $V$. By selection, $T_2$ has the largest number of vertices whose copies of $H$ intersect with $V$. Then, for this maximality to be true, the copy of $H$ associated with each vertex of $T_2$ must have a non-empty intersection with $V$ too. But this means that the copies of $H$ of all the vertices of $T$ have a non-empty intersection with $V$ in contradiction to our assumption that $OC(T, V) \subseteq V(T)$. □

If $q = 2$ then we are done by the above claim. Otherwise, let $u \in V(T')$ such that $V(H^u) \cap V' = \emptyset$. Note that $V^*$ is obtained by adding to $V'$ the elements of $V$ intersecting $V(H^v)$ for $v \in V(T^*) \setminus V(T')$. This means that the intersection of $V^*$ with $H(T^u)$ remains empty. ■

Lemma 7  Suppose that $H$ is a connected graph of at least $p$ vertices. Assume that $|OC(T, V)| \geq p$ and that $V(T) \setminus OC(T, V) \neq \emptyset$. Then $T(H)$ has a matching of size $p$ all edges of which have one end in $V$ and the other end outside $V$. 


Proof. Assume first that all the vertices of $OC(T, V)$ are incomplete. Then, as $H$ is connected, for each $u \in OC(T, V)$, there is an edge $e_u$ connecting a vertex of $V(H^u) \cap V$ with a vertex $V(H^u) \setminus V$. Let $M$ be the set of these edges. As they belong to different copies of $H$, they cannot have joint ends and hence $M$ is a matching. As $|OC(T, V)| \geq p$ and $|M| = |OC(T, V)|$, $M$ is a matching required by the lemma.

Otherwise, let $u \in OC(T, V)$ be a complete vertex and let $v$ be a vertex such that $V(H^u) \cap V = \emptyset$. Then the statement immediately follows from lemma 1 by taking $V_1 = V$ and $V_2 = \emptyset$. ■

Theorem 1 Let $T$ be a complete rooted ternary tree. Let $H$ be a connected graph of at least $2p$ vertices. Let $V \subseteq V(T')$ such that $|OC(T, V)| \geq p$. Let $x = tr(OC(T, V))$. Finally, let $SV$ be a permutation of $V$. Then $T(H)$ has a witnessing matching for $SV$ of size at least $p \times (x - tr(p))/2$.

Proof. By induction on $|OC(T, V)|$. As a matching size is non-negative, the statement is trivially true if $x - tr(p) < 2$. Assume now that $x - tr(p) \geq 2$. Then, clearly, $|OC(T, V)| > p$. We claim that in this case there is a witnessing matching for $SV$ of size at least $p$. Indeed, if $OC(T, V) = V(T)$, this follows from Lemma 8; otherwise, this follows from Lemma 7. This establishes the statement of theorem for the case where $x - p < 4$. Assume now that $x - tr(p) \geq 4$ and that the theorem has been established for all the smaller values of $OC(T, V)$.

If $OC(T, V) = V(T)$ then the statement follows from Lemma 8, hence we assume that $OC(T, V) \subset V(T)$.

Let $T_1, \ldots, T_q$ be a minimal sequence of largest subtrees of $T$ w.r.t. $V$ lacking $p$ existing by Lemma 4. Let $V_1, \ldots, V_q$ be the corresponding sequence of sets. By Lemma 5 $|OC(T_q, V_q)| \geq (|OC(T, V)| - p)/3$. Our next step is to apply the induction assumption to $T_q$ and $V_q$. In order to do this, we must verify that (i) $|OC(T_q, V_q)| < |OC(T, V)|$ and (ii) $|OC(T_q, V_q)| \geq p$. Now, (i) follows by construction. To show (ii), let us perform the following calculation.

Let $T'$ and $T''$ be complete rooted ternary trees of height $tr(p) + 1$ and $tr(OC(T, V))$, respectively. Then $|T'| > p$ and $|T''| \leq |OC(T, V)|$. The assumption $tr(OC(T, V)) - tr(p) \geq 4$ implies that the height of $T''$ is greater than the height of $T'$ by at least 3. It follows that $|OC(T, V)| \geq |T''| \geq 27|T'| \geq 27p$ implying (ii).

Let $y = tr(OC(T_q, V_q))$. Let $SV_q$ be the permutation of $V_q$ where the order of elements is the same as in $SV$. By the induction assumption, $T_q(H)$ has a witnessing matching $M_q$ for $SV_q$ having size at least $p \times (y - tr(p))/2$.

Let $T' = T \setminus T_q$ and let $V' = V \cap V(T'(H))$. By Lemma 6 $OC(T', V') \subset V(T')$. Taking into account that $OC(T', V') \geq p$ by construction, it follows from Lemma 7 that $T'(H)$ has a matching $M'$ of size at least $p$ between $V'$ and $T'(H) \setminus V'$.

As $M'$ and $M_q$ are matching in vertex-disjoint subgraphs of $T(H)$, $M = M' \cup M_q$ is also a matching of size $|M'| + |M_q|$. We claim that $M$ is in fact a witnessing matching for $SV$ of size at least $p \times (x - tr(p))/2$ thus implying the theorem.
As \( M_q \) is a witnessing matching of \( T_q(H) \) for \( SV_q \), \( SV_q \) has a prefix \( SV'_q \) such that for any edge \( \{u, v\} \) either \( u \in SV'_q \) and \( v \in SV_q \setminus SV'_q \) or (ii) \( u \in SV_q \) and \( v \in TV_q \setminus SV_q \). Let \( w \) be the last vertex of \( SV'_q \) and let \( SV' \) be the prefix of \( SV \) ending with \( w \).

Now, let \( \{u, v\} \in M \). Assume first that \( \{u, v\} \in M_q \). If \( \{u, v\} \) satisfies condition (i) in the previous paragraph then it is not hard to see that \( u \in SV' \) and \( v \in SV \setminus SV' \) (because by construction \( SV'_q \subseteq SV' \) and \( SV_q \setminus SV'_q \subseteq SV \setminus SV' \)). If \( \{u, v\} \) is of type (ii) then, clearly \( u \in SV \) and, as \( SV \setminus SV'_q \subseteq V(T') \), \( v \) does not belong to \( SV \). Assume now that \( \{u, v\} \in M' \). Then, say, \( u \in V' \) and hence \( u \in SV \). Also, \( v \in T'(H) \setminus V' \) and hence \( v \in T(H) \setminus V \) as \( V \setminus V' \) is a subset of \( V(T_q(H)) \) disjoint with \( V(T'(H)) \). Thus we have established \( M \) is a witnessing matching for \( SV \) where \( SV' \) serves as a witnessing prefix.

It remains to verify that \( M \) is of a required size. For this, let us first observe that \( y \geq x - 2 \). Indeed, let \( T' \) and \( T'' \) be complete rooted ternary trees of height \( x \) and \( x - 2 \) respectively. Note that by definition of \( x \), \( |T'| \leq |OC(T, V)| \) and, by definition of \( y \), it is sufficient to show that \( |T''| \leq |OC(T_q, V_q)| \).

Then,

\[
|T''| = \left| \frac{|T'| - 4}{9} \right| \leq \frac{|OC(T, V)| - 4}{9} \leq \frac{3|OC(T_q, V_q)| + p - 4}{3} \leq \frac{|OC(T_q, V_q)| + p - 4}{3}
\]

where the second equality follow \( |OC(T_q, V_q)| \geq \frac{(|OC(T, V)| - p)}{3} \) proven earlier. Clearly, \((p - 4)/9 \leq 2|OC(T_q, V_q)|/3\) will immediately imply \( |T''| \leq |OC(T_q, V_q)| \). Assume the opposite. Then as \( |OC(T_q, V_q)| \geq p \) this means that \((p - 4)/9 > 2p/3 \) which is \( 5p + 4 < 0 \), a contradiction due to the non-negativity of \( p \). Thus \( y \geq x - 2 \) has been established.

Now, \( |M| = |M_q| + |M'| \geq p*|(x - 2 - tr(p))/2| + p = p*|(x - tr(p))/2 - 1| + p = p*|(x - tr(p))/2| \), as required. ■

**Theorem 2** There are constants \( c_0, c_1, c_2 \geq 1 \) such that the following is true. There is an infinite set of integer numbers \( k \geq c_0 \) such that for each \( k \) there is a class \( \mathcal{G}_k \) of treewidth at most \( k \) and such that for any \( G \in \mathcal{G}_k \) and for any \( V \subseteq V(G) \) of size at least \( k^{c_1} \), the partial matching width of \( V \) in \( G \) is at least \( (k \log |V|)/c_2 \).

**Proof.** Consider first numbers \( k > 0 \) that are multiples of 4. For each such \( k \), let \( \mathcal{G}_k \) be the set of all graphs \( T(H) \) where \( H \) is a path of \( k/2 \) vertices and \( T \) is a complete rooted ternary tree. It is not hard to see that the treewidth of each such a graph is at most \( k \): let \( T \) be the underlying tree decomposition, the bag of the root vertex include the root copy of \( H \), and the bag of each non-root vertex include its own copy of \( H \) plus that of the parent.

Put \( p = k/4 \). Then by Theorem 1 for each \( V \subseteq T(H) \) and each permutation \( SV \) of \( V \) there is a witnessing matching for \( SV \) of size \( p*|(|OC(T, V)| - tr(p))/2| \). As for a sufficiently large \( x \), \( \log x/2 \leq tr(x) \leq \log x \), it is not hard to observe that there are constants \( d_0, d_1, d_2 \) such that for each \( p \geq d_0 \) and \( |V| \geq p^{d_1}, p*|((tr(|V|)/2p) - tr(p))/2| \geq (p \log |V|)/d_2 = 11 \)
(k \log V)/4d_2$. By Proposition 3, the partial matching width of $V$ is at least $(k \log V)/4d_2$. Now, let $c_0 = 4d_0$, $c_1 = d_1$ and $c_2 = 8d_2$.

For numbers $k$ that are not necessarily multiples of $k$, set $c_2 := 2 \cdot c_2$ and $p = \lfloor k/4 \rfloor$.

4 A branching program lower bound involving partial matching width

In this section we prove the following theorem.

**Theorem 3** For each sufficiently large $k$, there is a class $\Phi_k$ of CNFs of primal treewidth at most $k$ such that for each $\varphi \in \Phi_k$ and each $F \subseteq \varphi$ such that $|\varphi|/|F| \leq 2 \sqrt{n}$, a NROBP representing $F$ is of size $n^{\log k/c}$ for some universal constant $c$.

Let $c_0, c_1$ be constants as in Theorem 2. Then $\Phi_k = \{ \varphi(G) | G \in G_k, |V(G)| \geq k^{2c_1} \}$. We introduce the lower bound on the number of vertices of $G$ so as to make sure that the matching width lower bound as specified in Theorem 2 holds for any $V \subseteq V(G)$ st. $|V| \geq \sqrt{n}$.

An important property of the CNFs $\varphi(G)$ is that, as a result of fixing many positive literals, the number of satisfying assignments decreases exponentially.

To make the above statement more precise, we need to introduce additional notation. Let $F'$ be a Boolean function, let $S'$ be a set of literals with $\text{Var}(S') \subseteq \text{Var}(F')$. Then $F' \leftarrow S'$ denotes the Boolean function with the set of satisfying assignments $\{ S | S \in F', S' \subseteq S \}$. That is, $F' \leftarrow S'$ consists of those satisfying assignments of $F'$ that include $S'$ as a subset. Alternatively, $F' \leftarrow S'$ can be obtained by adding $S'$ to each satisfying assignment of $F'|_{S'}$.

**Theorem 4** For each $d$ there is a constant $b_d > 1$ such that the following is true. Let $G$ be a graph of max-degree $d$ and let $U \subseteq V(G)$. Then $|\varphi(G) \leftarrow U| \leq |\varphi(G)|/2^{U'/b_d}$.

The proof of Theorem 4 is provided in the appendix.

The proof of Theorem 3 is based on simultaneous exploration of $\sqrt{n}$ bottlenecks. In order to highlight the need for multiple bottlenecks, we first prove Theorem 5 below using only a single bottleneck. Theorem 4 is a restricted version of Theorem 3 in which the approximation ratio is bounded by a constant. Then we show why this approach does not seem to work when the approximation ratio is bounded by $2^{\sqrt{n}}$ and provide an actual proof for Theorem 3.

**Theorem 5** Let $\varphi \in \Phi_k$ and let $F \subseteq \varphi$ be such that $|\varphi|/|F| \leq 2$ Let $Z$ be a NROBP solving $F$. Then the size of $Z$ is $n^{O(k)}$.

**Proof.** Let $P$ be a source-sink path of $Z$. The variable occurrences on $P$ form a permutation $SV$ of $V(G)$. It follows from Theorem 2 that there is a prefix
Let \( P_1 \) be the prefix of \( P \) such that \( \text{var}(P_1) = SV' \) and let \( P_2 \) be the remaining suffix of \( P \). Let \( a = a(P) \) be the final vertex of \( P_1 \) (and the initial vertex of \( P_2 \)). Clearly, the ends of each edge of \( M \) are separated by \( a \). Therefore, by Proposition 2, there is a set of vertices \( U_a \) one per edge of \( M \) such that for each path \( Q \) passing through \( a \), \( A(Q) \subseteq F \leftarrow U_a \).

Let \( X = \{a_1, \ldots, a_q\} \) be the set of vertices \( a(P) \) over all source-sink paths of \( Z \). Then we claim that \( q = n^{\Omega(k)} \) implying the lower bound. Indeed, as vertices of \( X \) form a source-sink cut, each satisfying assignment of \( X \) is carried through one of these vertices and hence belongs to some \( F \leftarrow U_a \). That is, \( F = \bigcup_{a \in X} F \leftarrow U_a \), and hence \( |F| \leq \sum_{a \in X} |F \leftarrow U_a| \). Let \( a \in X \) such that \(|F \leftarrow U_a| \) is the largest one. Then \( |F| \leq q^* |F \leftarrow U_a| \) and, since \( F \leftarrow U_a \subseteq F \leftarrow U_a \), we conclude that \(|F| \leq q^* |F \leftarrow U_a| \). Then, according to Theorem 3, \(|F| \leq q^* |\phi|/2^{|U_a|/b^7} \leq q^* |\phi|/2^{k \log n/(b^7 + c^2)} = q^* |\phi|/n^{k/(b^7 + c^2)} \), where \( b^7 \) is as in Theorem 3.

On the other hand, by our assumption, \(|F| \geq |\phi|/2\). Combining this with the previous paragraph, we observe \( q^* |\phi|/n^{k/(b^7 + c^2)} \geq |\phi|/2 \). Hence \( q^* |\phi|/n^{k/(b^7 + c^2)} \geq 1/2 \) from where the desired lower bound on \( q \) immediately follows.

The above approach works only if \(|F| \) is sufficiently large compared with \(|\phi|/n^k \); say \(|F| = |\phi|/2 \) or \(|F| = |\phi|/n^{\sqrt{k}} \). If, however, \(|F| = |\phi|/2^{\sqrt{n}} \), all the satisfying assignments of \( F \) can go through a single vertex \( a(P) \). Hence, the approach will yield only a trivial lower bound of 1.

We overcome this difficulty by associating each source-sink path of a NROBP representing \( F \) with a tuple rather than with a single vertex as specified below.

**Lemma 8** Let \( \phi \in \Phi_k \), let \( G \) be such that \( \phi = \varphi(G) \) and let \( F \subseteq \phi \).

Let \( Z \) be a NROBP representing \( F \) and let \( P \) be a source-sink path of \( Z \). Then \( P \) has vertices \( a_1, \ldots, a_q \), \( q \leq \Theta(\sqrt{n}) \) for some constant \( q = \sqrt{n} \) in case \( \sqrt{n} \) is an integer) such that there is a set \( U \subseteq V(G) \), \(|U| = \Omega(k \log n) \) such that for each source-sink path \( Q \) passing through all of \( a_1, \ldots, a_q \), \( U \subseteq A(Q) \).

**Proof.** Throughout the proof, we assume, for the sake of simplicity, that \( \sqrt{n} \) is an integer. At the end of the proof we will briefly outline a way to adjust the construction to the general case.

Let \( q = \sqrt{n} \). Partition \( P \) into subpaths \( P_1, \ldots, P_q \) (meaning that the first subpath start with the source of \( Z \), the last subpath ends with the sink and the last vertex of \( P_i \) is the first vertex of \( P_{i+1} \)) so that \(|\text{Var}(P_i)| = \sqrt{n} \) for \( 1 \leq i \leq q \). It is not hard to see that such a partition exists: \( P_i \) included the first \( \sqrt{n} \) labelled edges, \( P_2 \) includes the second \( \sqrt{n} \) labelled edges and so on.

The order in which elements of \( \text{Var}(P_i) \) occur on \( P_i \) in fact determines a permutation \( SV_i \) of \( \text{Var}(P_i) \). Any prefix \( SV' \) of \( SV_i \) clearly corresponds to a prefix of \( P_i \) where \( SV' \) is the set of variables occurring on it. We therefore apply Theorem 2 directly to \( P_i \) (rather than to \( SV_i \)) and conclude that each \( P_i \) has a prefix \( P_i' \) such that there is a matching \( M_i' \) between \( \text{Var}(P_i') \) and the rest of
Let \( \{u, v\} \in M' \). Then one of the ends of the edge, say \( u \) belongs to \( Var(P_i') \). For \( v \), there are three possible occurrences: on \( P_i \setminus P_i' \), on \( P \) before \( P_i \) and on \( P \) after \( P_i \). Clearly, there is \( M_i \subseteq M'_i \), \( |M_i| \geq |M'_i|/3 \) such that all the ‘non \( P_i' \)' ends of the edges of \( M_i \) occur in exactly one of these three locations. Let us call the location of non \( P_i' \) ends of the edges of \( M_i \) the popular location.

If the popular location is on \( P_i \) before \( P_i \) then let \( a_i \) be the last vertex of \( P_i \). If the popular location is on \( P_i \) before \( P_i \) then let \( a_i \) be the first vertex of \( P_i \). Finally, if the popular location is on \( P_i \) after \( P_i \) then let \( a_i \) be the last vertex of \( P_i \).

Clearly, in any case, \( a_i \) separates the ends of each edge of \( M_i \). By Proposition 2 there is a set \( U_i \subseteq V(G) \) including exactly one end of each edge of \( M_i \) such that for each source-sink path \( Q \) of \( Z \) passing through \( a_i \), \( U_i \subseteq A(Q) \). Let \( U = U_1 \cup \ldots U_q \). Clearly, for any source-sink path \( Q \) of \( Z \) passing through all of \( a_1, \ldots a_q \), \( U \subseteq A(Q) \).

It remains to verify that the size of \( U \) satisfies the required lower bound. To this end, let \( M = M_1 \cup \ldots \cup M_q \). Note that \( U \) covers \( M \) that is each edge of \( M \) is incident to at least one vertex of \( U \). Since \( G \) of max-degree 7, a vertex of \( U \) cannot cover more than 7 edges. It follows that \( |U| \geq |M|/7 \). Consequently, it is sufficient to verify that \( |M| = \Omega(k \log n) \).

To this end, observe that each edge \( \{u, v\} \in M \) can belong to at most two different \( M_i \). Indeed, assume that there are distinct \( i_1, i_2, i_3 \) such that \( \{u, v\} \in M_{i_1} \cap M_{i_2} \cap M_{i_3} \). By definition of \( M_i \) all of \( Var(P_{i_1}), Var(P_{i_2}), Var(P_{i_3}) \) must intersect with \( \{u, v\} \). A simple pigeonhole principle implies that one of \( \{u, v\} \) must occur in at least two of \( Var(P_{i_1}), Var(P_{i_2}), Var(P_{i_3}) \). However, this is a contradiction because by definition \( Var(P_1), \ldots, Var(P_q) \) are pairwise disjoint!

Since each edge of \( M \) contributes to at most two different \( M_i \), \( |M| \geq (\sum_{i=1}^q |M_i|)/2 \geq k \log n/12c_2 \), as required.

If \( n \) is not an integer number, we partition \( P \) into subpaths \( P_1, \ldots, P_r \) so that \( Var(P_1), \ldots, Var(P_{r-1}) \) are of size \( \lfloor \sqrt{n} \rfloor \), set \( q = r - 1 \) and define \( P_i = P_i' \) for \( 1 \leq i \leq q - 1 \) and \( P_q = P_q' \cup P_{q+1}' \). This way we ensure each \( Var(P_i) \) is of size at least \( \sqrt{n} \) (this is needed for application of Lemma 2) and \( q \leq \Theta(\sqrt{n}) \) for . Then we apply the reasoning as above. ■

**Proof of Theorem 3** As in Lemma 8 we assume that \( \sqrt{n} \) is integer and then outline a way to adjust the proof to the general case.

For each source-sink path \( P \) of \( Z \), fix vertices \( a_1, \ldots, a_q \) as per Lemma 8 such that \( q = \sqrt{n} \). Assume w.l.o.g. that these vertices occur on \( P \) in the order listed. Denote the tuple \( (a_1, \ldots, a_q) \) by \( a(P) \) and call it the characteristic tuple of \( P \). Recall that \( a_i \) is called the \( i \)-th component of \( a(P) \).

Let \( TP \) be the set of characteristic tuples of all the source-sink paths of \( Z \). For \( 1 \leq i \leq q \), let \( B_i \) be the set of all \( i \)-th components of elements of \( TP \) and let \( \mu \) be the size of the largest component. We treat the sets \( B_1, \ldots, B_q \) as the bottlenecks of \( Z \). Then \( \mu \) is the size of the largest bottleneck. In the rest of the proof we demonstrate that \( \mu \) must be large, implying the theorem. Note that this will tell us that a large bottleneck exist but will not point out to a
particular large bottleneck.

For each \( a \in TP \), let \( F_a \) be the function whose satisfying assignments are exactly those of \( F \) that are carried by paths passing through all the components of \( a \). Observe that for each satisfying assignment \( S \) of \( F \) there is \( a \in TP \) such that \( S \in F_a \). Indeed, let \( P \) be a source-sink path of \( Z \) carrying \( S \). Then, by definition, \( S \in F_a \). Therefore \( F = \bigcup_{a \in TP} F_a \) and hence \( |F| \leq \sum_{a \in TP} |F_a| \).

Let \( b \in TP \) be such that \( |F_b| \) is the largest. Then

\[
|F| \leq |F_b| * |TP| \tag{2}
\]

Let \( G \) be the graph such that \( \varphi = \varphi(G) \). Then, according to Theorem 8, there is \( U \subseteq V(G) \) of size at most \( k \log n \sqrt{n}/a_1 \) for some constant \( a_1 \) such that \( F_b \subseteq F \leftarrow U \). Clearly, \( F \leftarrow U \subseteq \varphi(G) \leftarrow U \). By Theorem 3 \( |\varphi(G) \leftarrow U| \leq |\varphi|/2^k \log n \sqrt{n}/b_7 \ast a_2 \). Therefore, denoting \( b_7 \ast a_1 \) by \( a_2 \), we obtain

\[
|F_b| \leq |\varphi|/(n^k \log n/a_2) \sqrt{n} \tag{3}
\]

Each tuple of \( TP \) is obtained by taking one element of \( B_1 \), one element of \( B_2 \), . . . , one element of \( B_q \). Therefore,

\[
|TP| \leq \prod_{1 \leq i \leq q} |B_i| = \mu^q = \mu \sqrt{n} \tag{4}
\]

Substituting (3) and (4) into (2), we obtain

\[
|F| \leq |\varphi| * (\mu/n^k \log n/a_2) \sqrt{n} \tag{5}
\]

By our assumption, \( |F| \geq |\varphi|/2 \sqrt{n} \). Combining this with (5), we obtain that \( |\varphi| * (\mu/n^k \log n/a_2) \sqrt{n} \geq |\varphi|/2 \sqrt{n} \), hence \( (\mu/n^k \log n/a_2) \sqrt{n} \geq (1/2) \sqrt{n} \), hence \( \mu/n^k \log n/a_2 \geq 1/2 \) from where the desired lower bound for \( \mu \) immediately follows.

If \( \sqrt{n} \) is not an integer then, by Lemma 8 there is a universal constant \( c \) such that for each sufficiently large \( n \), \( q \leq c \sqrt{n} \). Replace the equality \( \mu^q = \mu \sqrt{n} \) in (4) by \( \mu^q \leq \mu^c \sqrt{n} \) and update the rest of the reasoning accordingly. \( \blacksquare \)

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Definition 5 (Decision tree) Let $F$ be a function that is not constant zero. Then a decision tree $T$ for $F$ is defined as follows. Suppose that $\text{Var}(F) = \{x\}$. Then the root $rt$ of $T$ is labelled with $x$. If both $\{x\}$ and $\{-x\}$ are satisfying assignments of $F$ then $rt$ has two outgoing edges labelled with $x$ and $-x$, respectively (the leaves are not labelled with variables). If the only satisfying assignment of $F$ is $\{x\}$ then the only outgoing edge of $rt$ is labelled with $x$. Finally, if the only satisfying assignment of $rt$ is $\{-x\}$ then the only outgoing edge is labelled with $-x$.

Now, assume that $|\text{Var}(F)| > 1$. Let $x \in \text{Var}(F)$. Label $rt$ with $x$. If both $x$ and $-x$ occur in satisfying assignments of $F$ then $rt$ has two outgoing edges labelled with $x$ and $-x$. Let $u_1$ and $u_2$ be the respective heads of these edges. Then the subtree of $T$ rooted by $u_1$ is a decision tree for $F|_x$ and the subtree rooted by $u_2$ is a decision tree for $F|_{-x}$.

If there is exactly one $\ell \in \{x, -x\}$ in the satisfying assignments of $F$ then there is only one outgoing edge labelled with $\ell$ and the subtree rooted by the head of this edge is a decision tree for $F|_{\ell}$.

For a path $P$ of $T$, we denote (similarly to NROBPs) by $A(P)$ the set of literals labelling the edges of $P$.

Definition 6 (Solution counting decision tree) A solution counting decision tree (SCDT) $T$ for $F$ is a decision tree for $F$ whose edges are associated with weights defined as follows. Let $rt$ be the root of $T$ and let $e$ be an outgoing edge of $rt$ labelled with a literal $\ell$. Then the weight of $e$ is $|F|_{\ell}|/|F|$.

A Proof of Theorem 4

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Let $u$ be an internal node of $T$ and let $P$ be the unique $rt - u$ path of $P$. Let $e$ be an outgoing edge of $u$ and let $\ell$ be the literal labelling $e$. Then the weight of $e$ is $|F|_{A(P) \cup \{e\}}/|F|_{A(P)}$.

The weight of a path of $T$ is the product of weights of its edges. The weight of an edge $e$ is denoted by $\text{weight}(e)$, the weight of a path $P$ is denoted by $\text{weight}(P)$. Finally if $P$ is a set of paths then the weight of $P$, denoted by $\text{weight}(P)$, is the sum of weights of paths in $P$.

**Lemma 9** Let $T$ be a SCDT for a Boolean function $F$ and let $P$ be a root-leaf path of $T$. Then $\text{weight}(P) = 1/|F|$.

**Proof.** By induction on $|\text{Var}(F)|$. If $|\text{Var}(F)| = 1$, this can be verified by a direct inspection. Assume that $|\text{Var}(F)| > 1$. Let $P$ be a root-leaf path. Let $e$ be the first edge of $P$ and let $\ell$ be the literal labelling $e$. Let $u$ be the head of $e$ and let $T'$ be the subtree of $T$ rooted by $u$. It is not hard to observe that $T'$ is a SCDT for $F|_{\ell}$. Let $P'$ be the suffix of $P$ starting at $u$. By the induction assumption, $\text{weight}(P') = 1/|F|_{\ell}|$. Now $\text{weight}(P) = \text{weight}(P') \ast \text{weight}(e) = (1/|F|_{\ell}) \ast (|F|_{\ell}/|F|) = 1/|F|$ as required. □

In light of Lemma 9 if we need to estimate a proportion of a certain set of satisfying assignments of a Boolean function, we can calculate the weight of paths carrying these assignments in the SCDT of this Boolean function. This is exactly the approach we are taking for proving Theorem 4. From now on, fix $T$ a SCDT for $\varphi(G)$. Let us introduce additional notation related to $T$.

Let $u$ be a node of $T$. Let $P$ be the unique root-$u$ path of $T$. We denote $A(P)$ by $A_u$. Let $x \in V(G)$. We denote by $N_u(x)$ the set of neighbours $y$ of $x$ such that $y$ is neither assigned nor forced to 1 by $A_u$ (that no neighbour of $x$ occurs negatively in $A_u$). We let $F_u = \varphi(G)|_{A_u}$.

**Lemma 10** Let $x$ be a variable of $V(G)$ labelling $u$. Assume that $x$ is not forced to 1 by $A_u$. Then $|F_u|_{\neg x}/|F^u| \geq (1/2)^{|N_u(x)|+1}$. Also, $|F_u|_{|x|}/|F^u| \geq 1/2$.

**Proof.** Note first that since $A_u$ is an assigned labelling a path of $T$, it has a satisfying extension. That is $|F_u| > 0$, hence the left-hand sides of the desired inequalities are defined.

**Claim 3** Let $F$ be a Boolean function. Let $V \subseteq \text{Var}(F)$. Let $S = \text{Proj}(F, \text{Var}(F) \setminus V)$. Then $|S|/|F| \geq 1/2^{|V|}$.

**Proof.** Each $S \in S$ can be extended in $2^{|V|}$ to a set of literals of $\text{Var}(F)$. Let $S^*$ be the set of all such extensions for the elements of $S$. Clearly $|S^*| = |S| \ast 2^{|V|}$ and $F \subseteq S^*$. □

Let $V^u = \text{Var}(F^u) \setminus (N^u(x) \cup \{x\})$. In other words, $V^u$ is obtained from the set of variables not assigned by $A_u$ by removal $x$ and those neighbours of $x$ that are forced to 1 by $A_u$.

**Claim 4** $\text{Proj}(F^u, V^u) \subseteq \text{Proj}(F^u|_{\neg x}, V^u)$. 

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Proof. Let \( S \in \text{Proj}(F^u, V^u) \). Let \( A^* \) be the set of literals over \( N^u(x) \cup \{x\} \) assigning \( x \) negatively and the rest of the variables positively. Note that \( \text{Var}(S), \text{Var}(A_u) \) and \( \text{Var}(A^*) \) partition \( V(G) \).

We claim that \( S^* = S \cup A_u \cup A^* \) is a satisfying assignment of \( \varphi(G) \). To prove this let us show that each clause \( (y \cup z) \) of \( \varphi(G) \) is satisfied by \( S^* \).

Note that by definition of \( S, A_u \cup S \) has a satisfiable extension to the rest of the variables of \( V(G) \). Therefore, the claim holds if both \( y \) and \( z \) belong to \( \text{Var}(S \cup A_u) \). The same is true with \( \text{Var}(A^*) \) simply because \( A^* \) contains negative occurrence of exactly one variable. It remains to assume that one variable, say, \( y \) is contained in \( \text{Var}(A^*) \) and the other, \( z \), is contained in \( \text{Var}(S \cup A_u) \). Then \( y = x \) because otherwise \( y \) is positively assigned and the clause is satisfied. Then \( z \) is a neighbour of \( x \). If \( z \in \text{Var}(A_u) \) then \( z \) occurs positively in \( A_u \) because otherwise \( x \) is forced to 1 in contradiction to our assumption. If \( z \in \text{Var}(S) \) then, as \( z \notin N^u(x) \), we conclude that \( z \) is forced to 1 by \( A_u \) and hence also positively assigned. Now, since \( S^* \) contains \( \lnot x \), \( S^* \setminus (A_u \cup \{\lnot x\}) \) is a satisfying assignment of \( F^u\lnot x \) and hence \( S \) belongs to the projection of \( F^u\lnot x \) to \( V^u \). □

It follows from the Claim 4 that \( |\text{Proj}(F^u, V^u)| \leq |\text{Proj}(F^u\lnot x, V^u)| \). As the size of a set of literals cannot be smaller than the size of its projection, it also follows that \( |\text{Proj}(F^u, V^u)| \leq |F^u_u| \).

As \( |\text{Proj}(F^u, V^u)|/|F^u_u| \geq (1/2)^{|N^u(x)|+1} \) according to Claim 3 the second statement of the lemma follows. The first statement can be established by the same approach taking \( V^u = \text{Var}(F^u) \setminus \{x\} \).

Lemma 11 With the data as in Lemma 10, assume that \( x \) is not forced by \( A_u \). Then the weight of the outgoing edge of \( u \) labelled with \( \lnot x \) is between \( (1/2)^{|N^u(x)|+1} \) and \( 1/2 \) and the weight of the outgoing edge of \( u \) labelled with \( x \) is between \( 1/2 \) and \( 1 - (1/2)^{|N^u(x)|+1} \).

Proof. Immediately from Lemma 10. Lower bounds are given directly. The upper bound is 1 minus the lower bound for the opposite literal. □

For \( d \geq 0 \), let \( c_d = 1 - 2^{-(2d+1)} \). For \( S \subseteq V(G) \), we denote \( \prod_{x \in S} c_{|N^u(x)|} \) by \( \alpha^u(S) \).

Let \( u \) be a vertex of \( T, S \subseteq V(G) \). We denote by \( P^u(S) \) the set of paths \( P \) of \( T \) from \( u \) to a leaf such that \( S \subseteq A(P) \).

Theorem 6 Let \( u \) be a vertex of \( T \) labelled by a variable \( x \) and let \( S \subseteq V(G) \) such that none of elements of \( S \) are neighbours or having common neighbours and none of elements of \( S \) is forced to 1 by \( A_u \). Then \( \text{weight}(P^u(S)) \leq \alpha^u(S) \).

The proof of Theorem 6 is quite tedious. Therefore, we first show that how Theorem 4 follows from it.

Proof of Theorem 4. Recall that \( d \) is the max-degree of \( G \). It is not hard to see that there is a subset \( S \) of \( U \) of size at least \( |U|/(d+1) \) such that no two elements of \( S \) are adjacent.

Clearly, for each \( x \in V(G), |N^u(x)| \leq d \). Hence, it follows from Theorem 6 that for each \( u \in V(T), \text{weight}(P^u(S)) \leq c^{|S|}_d \). In particular, \( \text{weight}(P^t(S)) \leq \alpha^t(S) \).
where \( rt \) is the root of \( T \). Note that \( \{A(P)|P \in \mathbf{P}^{rt}(S)\} = \varphi(G)|_{S} \). It follows from Lemma \( 9 \) that \( \text{weight}(\mathbf{P}^{rt}(S)) = |\varphi(G)|_{S}||\varphi(G)| \).

Note that \( |\varphi(G) \leftarrow S| = |\varphi(G)|_{S} \). Then we conclude that \( |\varphi(G) \leftarrow S|/|\varphi(G)| \leq c_{d}^{\lceil S \rceil} \leq (1/2)^{(d+1)}|U| \). Then \( |\varphi(G) \leftarrow S| \leq |\varphi(G)|/2U|/db \) where \( 2^{ad} = (1/c_{d})^{1/(d+1)} \).

Lemma 12 Let \( u \) be a vertex of \( T \), let \( v \) be its child. Let \( S \subseteq V(G) \). Then \( \alpha^{v}(S) \leq \alpha^{u}(S) \).

**Proof.** We only need to verify that for each \( x \in V(G) \), \( c_{|N^{v}(x)|} \leq c_{|N^{u}(x)|} \). Indeed, any neighbour of \( x \) that is assigned or forced to 1 by \( A_{u} \) retains this status regarding \( A_{v} \). Hence \( N^{v}(x) \subseteq N^{u}(x) \) implying that \( |N^{v}(x)| \leq |N^{u}(x)| \). The lemma now follows immediately from an easy to observe fact that if \( d' < d \) then \( c_{d'} \leq c_{d} \).

Lemma 13 Let \( u \) be a vertex of \( T \) labelled by \( x \). Let \( S \subseteq V(G) \) such that \( x \in S \). Let \( v \) be the positive out-neighbour of \( u \) and let \( p \) be the weight of \( (u,v) \). Then \( p \ast \alpha^{v}(S \setminus \{x\}) \leq \alpha^{u}(S) \).

**Proof.** By definition, \( \alpha^{u}(S) = c_{|N^{u}(x)|} \ast \alpha^{u}(S \setminus \{x\}) \). By Lemma \( 12 \) \( \alpha^{v}(S \setminus \{x\}) \leq \alpha^{u}(S \setminus \{x\}) \). Hence, it remains to verify that \( p \leq c_{|N^{v}(x)|} \). By Lemma 11 \( p \leq 1 - (1/2)^{|N^{v}(x)|+1} \leq c_{|N^{v}(x)|} \).

Lemma 14 Let \( u \) be a vertex of \( T \) labelled by \( x \). Let \( S \subseteq V(G) \) contains a neighbour \( y \) of \( x \). Assume that \( x \) is not forced to 1 by \( A_{u} \) and let \( vp \) and \( vn \) be respective positive and negative out-neighbours of \( u \). Let \( p \) be the weight of \( (u,v) \). Then \( p \ast \alpha^{vp}(S) + (1-p) \ast \alpha^{vn}(S \setminus \{y\}) \leq \alpha^{u}(S) \).

**Proof.** By definition \( \alpha^{vp}(S) = c_{|N^{vp}(y)|} \ast \alpha^{vp}(S \setminus \{y\}) \) and \( \alpha^{u}(S) = c_{|N^{u}(y)|} \ast \alpha^{u}(S \setminus \{y\}) \). As both \( \alpha^{vp}(S \setminus \{y\}) \) and \( \alpha^{vn}(S \setminus \{y\}) \) do not exceed \( \alpha^{u}(S \setminus \{y\}) \) by Lemma \( 12 \) it follows that it is sufficient to verify that \( p \ast c_{|N^{vp}(y)|} + (1-p) \leq c_{|N^{vn}(y)|} \).

To this end, note that that by our assumption, \( x \in N^{u}(y) \) while \( x \notin N^{vp}(y) \). As \( N^{vp}(y) \subseteq N^{u}(y) \), \( |N^{vp}(y)| \leq |N^{u}(y)| - 1 \). As \( c_{d} \) is a non-decreasing function in \( d \), we conclude that \( c_{|N^{vp}(y)|} \leq c_{|N^{u}(y)|-1} \). Thus \( p \ast c_{|N^{vp}(y)|} + (1-p) \leq p \ast c_{|N^{u}(y)|-1} + (1-p) = p \ast (1 - (1/2)^{|N^{u}(y)|-1}) + 1 - p = 1 - p \ast (1/2)^{|N^{u}(y)|-1} \).

The rightmost part of the above derivation grows with decrease of \( p \). As \( p \geq 1/2 \) by Lemma \( 11 \) \( 1 - p \ast (1/2)^{|N^{u}(y)|-1} \leq 1 - (1/2)^{|N^{u}(y)|} \leq c_{|N^{u}(x)|} \).

Let \( v \) be a vertex of \( T \) and let \( P \) be a set of paths, all starting from \( u \). Let \( (u,v) \) be an edge of \( T \). Then \( (u,v) + P \) denotes the set of paths \( \{(u,v) + P|P \in \mathbf{P}\} \). Clearly, \( \text{weight}((u,v) + P) = \text{weight}(u,v) \ast \text{weight}(P) \).

Lemma 15 Let \( u \) be a non-leaf vertex of \( T \). Assume that \( u \) is labelled with a variable \( x \notin S \). Let \( vp \) be the positive out-neighbour of \( u \). If \( vp \) is the only out-neighbour of \( u \) then \( \mathbf{P}^{u}(S) = (u,vp) + \mathbf{P}^{vp}(S) \). Otherwise, let \( vn \) be the negative out-neighbour of \( u \). In this case, \( \mathbf{P}^{u}(S) = [(u,vp) + \mathbf{P}^{vp}(S)] \cup [(u,vn) + \mathbf{P}^{vn}(S)] \)

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Proof. Let \( P \in \mathbf{P}(S) \) and let \( P' \) be the suffix of \( P \) starting from an out-neighbour of \( u \). Then \( S \subseteq A(P') \) because otherwise, since \( x \notin S \), \( S \) is not a subset of \( A(P) \) either.

Conversely, if \( S \subseteq A(P') \) where \( P' \) is a path starting from an out-neighbour of \( u \). Then, clearly, \( S \subseteq A((u, v) + P') \) and hence \( (u, v) + P' \in \mathbf{P}(S) \). ■

Lemma 16 Let \( u \) be a non-leaf vertex of \( T \). Assume that \( u \) is labelled with a variable \( x \in S \). Let \( v_p \) be the positive out-neighbour of \( u \). \( \mathbf{P}(S) = (u, v_p) + \mathbf{P}(S \setminus \{x\}) \).

Proof. Assume that \( P \in (u, v) + \mathbf{P}(S \setminus \{x\}) \). Then, by assumption, \((u, v_p)\) contributes \( x \) to \( A(P) \) and the rest of the edges contribute \( S \setminus \{x\} \), hence \( S \subseteq A(P) \).

Conversely, assume that \( P \in \mathbf{P}(S) \) and let \( v \) be the immediate successor of \( u \) on \( P \). If \( v \neq v_p \) then \( v \) is the negative out-neighbour of \( u \). Hence \( v \notin A(P) \), that is \( x \notin A(P) \) (due to read-onceness) and hence, in particular, \( S \notin A(P) \), a contradiction. It remains to assume that \( v = v_p \). Then, as \((u, v)\) contributes \( x \) to \( A(P) \), it does not contribute to \( S \setminus \{x\} \) and hence this must be contributed by the prefix of \( P \) starting at \( v_p \). ■

Lemma 17 Let \( u \) be a non-leaf vertex of \( T \). Assume that \( u \) is labelled with a variable \( x \notin S \) and a neighbour of \( y \in S \) such that \( y \) does not occur in \( A_u \). Assume further that \( u \) has two out-neighbours. Denote the positive and negative out-neighbours of \( u \) by \( v_p \) and \( v_n \), respectively. Then \( \mathbf{P}(S) = [(u, v_p) + \mathbf{P}(S) \setminus \{x\} \cup [(u, v_n) + \mathbf{P}(S \setminus \{y\})] \).

Proof. As \( [(u, v_p) + \mathbf{P}(S) \setminus \{x\} \cup [(u, v_n) + \mathbf{P}(S \setminus \{y\})] \subseteq [(u, v_p) + \mathbf{P}(S) \setminus [(u, v_n) + \mathbf{P}(S \setminus \{y\})] \), \( \mathbf{P}(S) \subseteq [(u, v_p) + \mathbf{P}(S) \setminus [(u, v_n) + \mathbf{P}(S \setminus \{y\})] \) by Lemma 16

Assume now that \( P \in [(u, v_p) + \mathbf{P}(S) \setminus [(u, v_n) + \mathbf{P}(S \setminus \{y\})] \). If \( P \in [(u, v_p) + \mathbf{P}(S) \) then apply again Lemma 16. So, we assume that the immediate successor of \( u \) in \( P \) is \( v_n \). Let \( P' \) be the suffix of \( P \) starting at \( v_n \). We show that \( S \subseteq A(P) \). By assumption, it is enough to show that \( y \in A(P) \). By assumption \( y \) has not been assigned by \( A_v \) and is forced to \( 1 \). ■

Proof of Theorem 6 Note that if at least one of the elements of \( S \) is assigned by \( A_u \), then \( \mathbf{P}(S) = 0 \). Hence, in this case \( \text{weight}(\mathbf{P}(S)) = 0 \) and the theorem holds as \( \alpha^u(S) \) is non-negative by definition. So, we can assume that no element of \( S \) is assigned by \( A_u \).

Assume first that \( u \) is a leaf. In light of the previous paragraph, \( S = 0 \). Hence, \( \mathbf{P}(S) = \{u\} \) and \( \text{weight}(\mathbf{P}(S)) = 1 \). On the other hand, \( \alpha^u(\emptyset) = 1 \) as well. Hence the theorem holds.

Assume now that \( u \) is not a leaf and the theorem holds for all the descendants of \( u \). Consider first the case where \( u \) has exactly one out-neighbour. Then it is a positive out-neighbour, denote it \( v_p \). Let \( x \) be the variable labelling \( u \). It follows that \( x \) is forced to \( 1 \) by \( A_u \), hence \( x \notin S \).

It follows from Lemma 16 that \( \mathbf{P}(S) = (u, v_p) + \mathbf{P}(S) \). Hence \( \text{weight}(\mathbf{P}(S)) = \text{weight}((u, v_p)) \times \text{weight}(\mathbf{P}(S)) \). As \( v_p \) is the only out-neighbour of \( u \), the
weight of \((u, vp)\) is 1. Hence \(weight(P^u(S)) = weight(P^{vp}(S))\). Note that no element of \(S\) is forced to 1 by \(A_{vp}\) as \(A_{vp} = A_u \cup \{x\}\), this is true regarding \(A_u\) by assumption and regarding \(x\) due to it being a positive literal. Hence, we may apply the induction assumption, according to which \(weight(P^{vp}(S)) \leq \alpha^{vp}(S)\).

As \(\alpha^{vp}(S) \leq \alpha^u(S)\) by Lemma 12, the theorem holds in this case.

It remains to assume that \(u\) has two out-neighbours. Denote the positive and negative ones by \(vp\) and \(vn\), respectively. Let \(p\) be the weight of \((u, vp)\) (and hence the weight of \((u, vn)\) is \(1 - p\)).

Assume first that \(x \notin S\) and \(x\) is not a neighbour of any element of \(S\). By Lemma 15 \(P^u(S) = [(u, vp) + P^{vp}(S)] \cup [(u, vn) + P^{vn}(S)]\). As \([(u, vp) + P^{vp}(S)]\) is disjoint with \([(u, vn) + P^{vn}(S)]\), \(weight(P^u(S)) = weight((u, vp) + P^{vp}(S)) + weight((u, vn) + P^{vn}(S)) = p \ast weight(P^{vp}(S)) + (1 - p) \ast weight(P^{vn}(S))\).

As \(x\) is not a neighbour of any element of \(S\), none of its literals forces any element of \(S\) into 1. Hence, arguing as in the previous case, the induction assumption can be applied to both \(weight(P^{vp}(S))\) and \(weight(P^{vn}(S))\). Hence \(weight(P^u(S)) \leq p \ast \alpha^{vp}(S) + (1 - p) \ast \alpha^{vn}(S) \leq p \ast \alpha^u(S) + (1 - p) \ast \alpha^u(S) = \alpha^u(S)\), the last inequality follows from Lemma 12.

Assume now that \(x\) is a neighbour of some element \(y\) of \(S\). By our assumption about \(S\), \(x \notin S\) and \(y\) is the only neighbour of \(x\) in \(S\). As \(y\) is not assigned by \(A_u\) by our assumption, it follows from Lemma 17 that \(P^u(S) = [(u, vp) + P^{vp}(S)] \cup [(u, vn) + P^{vn}(S \setminus \{y\})]\). Arguing as in the previous case, we conclude that \(weight(P^u(S)) = p \ast weight(P^{vp}(S)) + (1 - p) \ast weight(P^{vn}(S \setminus \{y\}))\). As \(x\) is a positive literal it does not force to 1 any element of \(S\). Also, since \(x\) does not have neighbours in \(S \setminus \{y\}\), \(\neg x\) does not force any fo elements of \(S \setminus \{y\}\) to 1. It follows that the induction assumption can be applied to \(weight(P^{vp}(S))\) and \(weight(P^{vn}(S \setminus \{y\}))\). Thus we obtain \(weight(P^u(S)) \leq p \ast \alpha^{vp}(S) + (1 - p) \ast \alpha^{vn}(S \setminus \{y\}) \leq \alpha^u(S)\), the last inequality follows from Lemma 14.

It remains to assume that \(x \in S\). By Lemma 16 \(P^u(S) = (u, vp) + P^{vp}(S \setminus \{x\})\), and hence \(weight(P^u(S)) = p \ast weight(P^{vp}(S \setminus \{x\}))\). Being a positive literal, \(x\) does not force any variable of \(S \setminus \{x\}\) into 1, hence the induction assumption can be applied. It follows that \(weight(P^u(S)) \leq p \ast \alpha^{vp}(S \setminus \{x\}) \leq alpha^u(S)\), the last inequality follows from Lemma 13.