Sharp well-posedness for a coupled system of mKdV-type equations

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Abstract. We consider the initial value problem associated with a system consisting modified Korteweg–de Vries-type equations
\[
\begin{align*}
\partial_t v + \partial_x^3 v + \partial_x (vw^2) &= 0, & u(x, 0) &= \phi(x), \\
\partial_t w + \alpha \partial_x^3 w + \partial_x (v^2 w) &= 0, & v(x, 0) &= \psi(x),
\end{align*}
\]
and prove the local well-posedness results for given data in low regularity Sobolev spaces \(H^s(\mathbb{R}) \times H^s(\mathbb{R})\), \(s > -\frac{1}{2}\), for \(0 < \alpha < 1\). Our result covers the whole scaling subcritical range of Sobolev regularity contrary to the case \(\alpha = 1\), where the local well-posedness holds only for \(s \geq \frac{1}{4}\). We also prove that the local well-posedness result is sharp in two different ways; namely, for \(s < -\frac{1}{2}\) the key trilinear estimates used in the proof of the local well-posedness theorem fail to hold, and the flow map that takes initial data to the solution fails to be \(C^3\) at the origin. These results hold for \(\alpha > 1\) as well.

1. Introduction

In this work, we consider the initial value problem (IVP) associated with the following system of the modified Korteweg–de Vries (mKdV)-type equations
\[
\begin{align*}
\partial_t v + \partial_x^3 v + \partial_x (vw^2) &= 0, & u(x, 0) &= \phi(x), \\
\partial_t w + \alpha \partial_x^3 w + \partial_x (v^2 w) &= 0, & v(x, 0) &= \psi(x),
\end{align*}
\]
where \((x, t) \in \mathbb{R} \times \mathbb{R}; v = v(x, t)\) and \(w = w(x, t)\) are real-valued functions; and \(0 < \alpha < 1\) is a constant.

For \(\alpha = 1\), the system (1.1) reduces to a special case of a broad class of nonlinear evolution equations considered by Ablowiz et al. [1] in the inverse scattering context. In this case, the well-posedness issues along with existence and stability of solitary waves for this system are widely studied in the literature. Using the technique developed by Kenig et al. [11], Montenegro [13] proved that the IVP (1.1) with \(\alpha = 1\) is locally well posed for given data \((\phi, \psi)\) in \(H^s(\mathbb{R}) \times H^s(\mathbb{R}), s \geq \frac{1}{4}\). In this approach, one uses the smoothing property of the linear group combined with the \(L^6_t L^4_x\) Strichartz estimates and maximal function estimates. Tao [18] showed that this local result can also be

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proved by using the Fourier transform restriction norm space $X_{s,b}$ [see definition (2.1)] introduced by Bourgain [3]. In this method, the trilinear estimate
\[ \| \partial_x (uvw) \|_{X_{s,b}} \lesssim \| u \|_{X_{s,b}} \| v \|_{X_{s,b}} \| w \|_{X_{s,b}} \] (1.2)
that is valid for $s \geq \frac{1}{4}$ plays a central role to apply contraction mapping principle. The author in [13] also proved global well-posedness for given data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1$, using the conservation laws
\[ I_1(v, w) := \int_{\mathbb{R}} (v^2 + w^2) \, dx \]
and
\[ I_2(v, w) := \int_{\mathbb{R}} (v_x^2 + w_x^2 - v^2 w^2) \, dx, \]
satisfied by the flow of (1.1). This global result is further improved in [7] by proving for data with regularity $s > \frac{1}{4}$. For existence and stability of solitary waves to the system (1.1), we refer to works in [2,13]. It is worth noting that the local well-posedness result for the system (1.1) with $\alpha = 1$ is sharp as it can be justified in two different ways. First, the key trilinear estimate (1.2) fails whenever $s < \frac{1}{4}$; see [9]. Second, the associated IVP for $s < \frac{1}{4}$ is ill-posed in the sense that the mapping data solution is not uniformly continuous; see [10]. This notion of ill-posedness is a bit strong. For further works in this direction, we refer to [5].

For $0 < \alpha < 1$, very less is known regarding well-posedness issues for the IVP (1.1). In this work, we are interested to deal with these issues for given data in low regularity Sobolev spaces considering $0 < \alpha < 1$. We note that the approach of Kenig et al. [11] yields local well-posedness for $s \geq \frac{1}{4}$ for $\alpha \neq 1$ too. However, if one uses the Fourier transform restriction norm space the situation is quite different. For motivation, we recall the work of Oh [15,16] for the KdV and the Majda–Biello system introduced in [12]
\begin{equation}
\begin{aligned}
\partial_t v + \partial_x^3 v + \partial_x (w^2) &= 0, & v(x, 0) &= \phi(x), \\
\partial_t w + \alpha \partial_x^3 w + \partial_x (wv) &= 0, & w(x, 0) &= \psi(x),
\end{aligned}
\tag{1.3}
\end{equation}
where $0 < \alpha < 1$. The author in [15] used the Fourier transform restriction norm method and proved that the IVP (1.3) is locally well posed for data with regularity $s \geq 0$. He also showed that the well-posedness result is sharp in the sense that if one demands $C^2$ regularity for the flow map, the condition $s \geq 0$ is necessary. The main tool in the proof was the validity of the bilinear estimate for the interacting nonlinearity
\[ \| \partial_x (vw) \|_{X_{s,b}^{\alpha}} \lesssim \| v \|_{X_{s,b}} \| w \|_{X_{s,b}^{\alpha}} \] (1.4)
whenever $s \geq 0$. [For definition of $X_{s,b}$ and $X_{s,b}^{\alpha}$ spaces, see (2.1) and (2.2).] Recall that the bilinear estimate (1.4) in the case $\alpha = 1$ holds for $s > -\frac{3}{4}$ (see [9]). Observe that
the regularity requirement for the validity of the bilinear estimates for the interacting nonlinearity in the case $0 < \alpha < 1$ is higher than that in the case $\alpha = 1$. In the Fourier transform restriction norm method, one needs to restrict the Fourier transform in the cubics $\tau = \xi^3$ and $\tau = \alpha \xi^3$. When $\alpha = 1$, both the cubics are the same and one can obtain the similar bilinear and trilinear estimates as in the individual KdV and mKdV equations. However, if $\alpha \neq 1$ the cubics are different and the frequency interactions behave differently. As observed in [15] while studying a coupled system of the KdV equations and the Majda–Bielo system, no cancelation occurs and consequently needs higher regularity in the data to get required bilinear estimates. Now the natural question is: How about the trilinear estimate if one considers $\alpha \neq 1$?

The main focus of this work is to answer the question posed above. In fact, considering $0 < \alpha < 1$ we prove that the trilinear estimate for the interacting nonlinear terms holds true whenever $s > -\frac{1}{2}$. As a consequence, we obtain the local well-posedness result for the IVP (1.1) for given data $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -\frac{1}{2}$. More precisely, we prove the following result.

**Theorem 1.1.** Let $0 < \alpha < 1$ and $s > -\frac{1}{2}$, then for any $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, there exist $\delta = \delta(\| (\phi, \psi) \|_{H^s \times H^s})$ (with $\delta(\rho) \to \infty$ as $\rho \to 0$) and a unique solution $(v, w) \in X^{1, \delta}_{s,b} \times X^{\alpha, \delta}_{s,b}$ to the IVP (1.1) in the time interval $[0, \delta]$. Moreover, the solution satisfies the estimate

$$\| (v, w) \|_{X^{1, \delta}_{s,b} \times X^{\alpha, \delta}_{s,b}} \lesssim \| (\phi, \psi) \|_{H^s \times H^s},$$

where the norm $\| \cdot \|_{X^{1, \delta}_{s,b}}$ and $\| \cdot \|_{X^{\alpha, \delta}_{s,b}}$ are as defined in (2.3).

The local well-posedness result obtained in Theorem 1.1 improves the result obtained by smoothing effect combined with the $L^p_t L^q_x$ Strichartz estimates and maximal function estimates. The main ingredients in the proof of Theorem 1.1 are the new trilinear estimates (for exact statement see Proposition 2.3)

$$\| (v w^2)_x \|_{X^{1}_s b'} \lesssim \| v \|_{X^{1}_s b} \| w \|_{X^{\alpha}_s b}^2,$$

which hold for $s > -\frac{1}{2}$ when $0 < \alpha < 1$.

It is quite surprising to note that, in contrast to the bilinear estimate, for the validity of the above trilinear estimates with $0 < \alpha < 1$ the regularity requirement is quite lower than the one required for case with $\alpha = 1$. In fact, if one considers $\alpha = 1$ the above trilinear estimates fail to hold whenever $s < \frac{1}{4}$ (see [9]). As can be seen in the proof of Lemma 3.2, when $0 < \alpha < 1$, no cancelation occurs in the resonance relation and this can be exploited in a positive way to lower the regularity requirement for the validity of the trilinear estimate. However, as discussed above, in the case of
the corresponding bilinear estimates the regularity requirement is higher when two different cubics are considered; see [15].

We emphasize that the trilinear estimates for $0 < \alpha < 1$ are valid in the whole scaling subcritical range, i.e., $s > -\frac{1}{2}$. This is in quite contrast to the case $\alpha = 1$ where it holds only for $s > \frac{1}{4}$. At this point, we recall the work in [14] where the authors produce a counter example to prove failure of bilinear estimate for $s = -\frac{3}{4}$. So, it is natural to ask whether a similar example can be constructed for the trilinear estimates (1.5) and (1.6) for $s = -\frac{1}{2}$. Also, we mention the work in [6] where the authors prove existence of global solution to the mKdV equation for given data with negative Sobolev regularity. Although the mapping data solution fails to be uniformly continuous for $s < \frac{1}{4}$, they got such global result by obtaining an appropriate global in time $H^s$-a priori estimate for $-\frac{1}{8} < s < \frac{1}{4}$. So, the another natural question is whether global solution to the system (1.1) can be found in line of [6]. We are working on these questions and will be answered elsewhere.

We also prove that the local well-posedness result obtained in Theorem 1.1 is sharp in two different ways. First, we prove that the crucial trilinear estimates (1.5) and (1.6) fail to hold if $s < -\frac{1}{2}$. Next, we show that the application that takes the initial data to the solution fails to be $C^3$ whenever $s < -\frac{1}{2}$. More precisely, we prove the following results.

**Proposition 1.2.** Let $0 < \alpha < 1$. The trilinear estimates (1.5) and (1.6) fail to hold for any $b \in \mathbb{R}$ whenever $s < -1/2$.

**Theorem 1.3.** Let $0 < \alpha < 1$. For any $s < -\frac{1}{2}$ and for given $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, there exist no time $T = T(\|\phi, \psi\|_{H^s \times H^s})$ such that the application that takes initial data $(\phi, \psi)$ to the solution $(v, w) \in C([0, T]; H^s) \times C([0, T]; H^s)$ to the IVP (1.1) is $C^3$ at the origin.

This negative result makes sense, because if one uses contraction mapping principle to prove local well-posedness, the flow map turns out to be smooth.

**Remark 1.4.** All the results stated above hold for $\alpha > 1$ as well. This can be justified by using symmetry of the system and scaling $\tilde{v}(x, t) = v(\alpha^{-\frac{1}{3}} x, t)$ and $\tilde{w}(x, t) = w(\alpha^{-\frac{1}{3}} x, t)$, so that $(\tilde{u}, \tilde{v})$ satisfy

\[
\begin{align*}
\partial_t \tilde{v} + \frac{1}{\alpha} \partial_x^3 \tilde{v} + \partial_x(\tilde{v} \tilde{w}^2) &= 0, \\
\partial_t \tilde{w} + \partial_x^3 \tilde{w} + \partial_x(\tilde{v}^2 \tilde{w}) &= 0,
\end{align*}
\]

with $0 < \frac{1}{\alpha} < 1$.

Before finalizing this section, we outline the structure of this paper. In Sect. 2, we introduce function spaces and other notations used in this work and record some preliminary estimates. In Sect. 3, we derive the main trilinear estimate and use it to prove the local well-posedness result stated in Theorem 1.1. Finally, in Sect. 4, we prove the failure of the trilinear estimates and the ill-posedness result stated, respectively, in Proposition 1.2 and Theorem 1.3.
2. Function spaces and preliminary estimates

In this section, we introduce the function spaces and notations that are used throughout this paper. Also, we will record some preliminary estimates that are essential to prove the well-posedness result stated in Theorem 1.1. As described in the previous section, we will use the Fourier transform restriction norm space, the so-called Bourgain’s space.

For $f: \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$, we define the mixed Lebesgue space $L^p_x L^q_t$ with norm defined by

$$\|f\|_{L^p_x L^q_t} = \left( \int_{\mathbb{R}} \left( \int_0^T |f(x, t)|^q \, dt \right)^{p/q} \, dx \right)^{1/p},$$

with usual modifications when $p = \infty$. We replace $T$ by $t$ if $[0, T]$ is the whole real line $\mathbb{R}$.

We use $\hat{f}(\xi)$ to denote the Fourier transform of $f(x)$ defined by

$$\hat{f}(\xi) = c \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx$$

and $\tilde{f}(\xi, \tau)$ to denote the Fourier transform of $f(x, t)$ defined by

$$\tilde{f}(\xi, \tau) = c \int_{\mathbb{R}^2} e^{-i(x\xi + t\tau)} f(x, t) \, dx \, dt.$$

We use $H^s$ to denote the $L^2$-based Sobolev space of order $s \in \mathbb{R}$ with norm

$$\|f\|_{H^s(\mathbb{R})} = \|\langle \xi \rangle^s \hat{f}(\xi)\|_{L^2_\xi},$$

where $\langle \xi \rangle = 1 + |\xi|$.

Next, we introduce the Fourier transform restriction norm spaces, more commonly known as Bourgain’s space in the literature.

For $s, b \in \mathbb{R}$, we define the Fourier transform restriction norm spaces $X_{s,b}(\mathbb{R} \times \mathbb{R})$ and $X_{s,b}^\alpha(\mathbb{R} \times \mathbb{R})$ as completion of a space of Schwartz class functions with respective norms

$$\|f\|_{X_{s,b}} = \|(1 + D_t)^b U(t) f\|_{L^2_t(H^s_{\xi})} = \|\langle \tau - \xi^3 \rangle^b \langle \xi \rangle^s \tilde{f}(\xi, \tau)\|_{L^2_{\xi,\tau}},$$

(2.1) and

$$\|f\|_{X_{s,b}^\alpha} = \|(1 + D_t)^b U^\alpha(t) f\|_{L^2_t(H^s_{\xi})} = \|\langle \tau - \alpha\xi^3 \rangle^b \langle \xi \rangle^s \tilde{f}(\xi, \tau)\|_{L^2_{\xi,\tau}},$$

(2.2)

where $U(t) = e^{-t\partial_x^3}$ and $U^\alpha(t) = e^{-t\alpha\partial_x^3}$ are the unitary groups associated with the linear problems and the operator $(1 + D_t)^b$ is defined via the Fourier transform as $[U^\alpha(t) f]^\alpha(\xi) = e^{it\alpha\xi^3} \hat{f}(\xi)$ and $[(1 + D_t)^b f]^\alpha(\xi) = \langle \tau \rangle^b \hat{f}(\tau)$.
If \( b > \frac{1}{2} \), the Sobolev lemma implies that \( X_{s,b} \subset C(\mathbb{R}; H_x^s(\mathbb{R})) \). For any interval \( I \), we define the localized spaces \( X^I_{s,b} := X_{s,b}(\mathbb{R} \times I) \) with norm
\[
\| f \|_{X_{s,b}(\mathbb{R} \times I)} = \inf \{ \| g \|_{X_{s,b}}, \ g|_{\mathbb{R} \times I} = f \}. \tag{2.3}
\]
Sometimes we use the definition \( X^1_{s,b} := \| f \|_{X_{s,b}(\mathbb{R} \times [0,\delta])} \) and similar for \( X^\alpha_{s,b} \).

We use \( c \) to denote various constants whose exact values are immaterial and may vary from one line to the next. We use \( A \lesssim B \) to denote an estimate of the form \( A \leq cB \) and \( A \sim B \) if \( A \leq cB \) and \( B \leq cA \). Also, we use the notation \( a+ \) to denote \( a + \epsilon \) for \( 0 < \epsilon \ll 1 \).

In sequel, we record some linear and nonlinear estimates satisfied by the solution in \( X_{s,b} \) and \( X^\alpha_{s,b} \) spaces.

We define a cutoff function \( \psi_1 \in C^\infty(\mathbb{R}; \mathbb{R}^+) \) which is even, such that \( 0 \leq \psi_1 \leq 1 \) and
\[
\psi_1(t) = \begin{cases} 
1, & |t| \leq 1, \\
0, & |t| \geq 2.
\end{cases}
\]
We also define \( \psi_T(t) = \psi_1(t/T) \), for \( 0 < T \leq 1 \).

In what follows, we list some estimates that are crucial in the proof of the local well-posedness result.

**Lemma 2.1.** For any \( s, b \in \mathbb{R} \), we have
\[
\| \psi_1 U(t) \phi \|_{X_{s,b}} \leq C \| \phi \|_{H^s}, \quad \| \psi_1 U^\alpha(t) \phi \|_{X^\alpha_{s,b}} \leq C \| \phi \|_{H^s}. \tag{2.4}
\]
Further, if \( -\frac{1}{2} < b' \leq 0 \leq b < b'+1 \) and \( 0 \leq \delta \leq 1 \), then
\[
\| \psi_\delta \int_0^t U(t-t') f(u(t')) dt' \|_{X_{s,b}} \lesssim \delta^{1-b+b'} \| f(u) \|_{X_{s,b}} \tag{2.5}
\]
and
\[
\| \psi_\delta \int_0^t U^\alpha(t-t') f(u(t')) dt' \|_{X^\alpha_{s,b}} \lesssim \delta^{1-b+b'} \| f(u) \|_{X^\alpha_{s,b}}. \tag{2.6}
\]

**Proof.** For the proof of this lemma, we refer to [8]. \( \square 

**Remark 2.2.** In the proof of the local well-posedness results, we will take \( b' = -\frac{1}{2} + 2\epsilon \) and \( b = \frac{1}{2} + \epsilon \) so that \( 1 - b + b' \) is strictly positive.

Now, we state the trilinear estimate which is central in our argument.

**Proposition 2.3.** Let \( 0 < \alpha < 1 \), \( b > \frac{1}{2} \) and \( b' \) be as in Lemma 2.1. Then the following trilinear estimates
\[
\| (vw^2)_x \|_{X_{s,b'}} \lesssim \| v \|_{X_{s,b}} \| w \|_{X^\alpha_{s,b}}^2 \tag{2.7}
\]
and
\[
\| (v^2w)_x \|_{X^\alpha_{s,b'}} \lesssim \| v \|_{X_{s,b}}^2 \| w \|_{X^\alpha_{s,b}} \tag{2.8}
\]
hold for any \( s > -\frac{1}{2} \).

Proof of this proposition will be provided below in a separate section.
3. Proofs of the key trilinear estimates and local well-posedness result

In this section, we prove the crucial trilinear estimates stated in Proposition 2.3 and the local well-posedness result stated in Theorem 1.1.

3.1. Proof of the trilinear estimate

We start by defining $n$-multiplier and $n$-linear functional; see [18]. Let $n \geq 2$ be an even integer. An $n$-multiplier $M_n(\xi_1, \ldots, \xi_n)$ is a function defined on the hyper-plane $\Gamma_n := \{ (\xi_1, \ldots, \xi_n); \sum \xi_i = 0 \}$ with Dirac delta $\delta(\sum \xi_i)$ as a measure.

We define an $[n; \mathbb{R}^{n+1}]$-multiplier to be any function $m : \Gamma_n(\mathbb{R}^{n+1}) \to \mathbb{C}$ and its norm $\| m \|_{[n; \mathbb{R}^{n+1}]}$ to be the best constant such that

$$\left| \int_{\Gamma_n} m(\xi) \prod_{j=1}^n f_j(\xi) \right| \leq \| m \|_{[n; \mathbb{R}^{n+1}]} \| f_j \|_{L^2(\mathbb{R}^{n+1})}$$

(3.1)

holds true for all the test functions in $\mathbb{R}^{n+1}$.

If $M_n$ is an $n$-multiplier and $f_1, \ldots, f_n$ are functions on $\mathbb{R}$, we define an $n$-linear functional, as

$$\Lambda_n(M_n, f_1, \ldots, f_n) := \int_{\Gamma_n} M_n(\xi_1, \ldots, \xi_n) \prod_{j=1}^n \hat{f}_j(\xi_j).$$

(3.2)

We write $\Lambda_n(M_n) := \Lambda_n(M_n, f, f, \ldots, f)$ in the case when $\Lambda_n$ is applied to the $n$ copies of the same function $f$.

The following results will be useful in our argument.

Lemma 3.1. (i) If $a, b > 0$ and $a + b > 1$, we have

$$\int_{\mathbb{R}} \frac{dx}{(x - \alpha)^a (x - \beta)^b} \lesssim \frac{1}{(|\alpha - \beta|)^c}, \quad c = \min\{a, b, a + b - 1\}. \quad (3.3)$$

(ii) Let $a, \eta \in \mathbb{R}, a, \eta \neq 0, b > 1$, then

$$\int_{\mathbb{R}} \frac{dx}{a(x^2 - \eta^2)^b} \lesssim \frac{1}{|a\eta|}, \quad (3.4)$$

(iii) Let $a, \eta \in \mathbb{R}, a, \eta \neq 0, b > 1$, then

$$\int_{\mathbb{R}} \frac{|x \pm \eta| dx}{a(x^2 \pm \eta^2)^b} \lesssim \frac{1}{|a|}, \quad (3.5)$$

(iv) For $l > 1/3$,

$$\int_{\mathbb{R}} \frac{dx}{(x^3 + ax^2 + ax + a_0)^l} \lesssim 1. \quad (3.6)$$

Proof. Proof of (3.3) can be found in [17], (3.4) and (3.5) in [4] and (3.6) in [9].
Now we prove some bilinear estimates that play central role in the proof of the trilinear estimates.

**Lemma 3.2.** Let $s > -\frac{1}{2}$, $0 < \epsilon < \frac{2s+1}{15}$ and $0 < \alpha < 1$, then the following bilinear estimates

$$\|uv\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|_{X_{-\frac{1}{2}, \frac{1}{2}+2\epsilon}} \|v\|_{X_{\frac{1}{2}, \frac{1}{2}+2\epsilon}}$$  \hspace{1cm} (3.7)

and

$$\|uv\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|_{X_{\alpha, \frac{1}{2}+\epsilon}} \|v\|_{X_{\frac{1}{2}, \frac{1}{2}+\epsilon}}$$  \hspace{1cm} (3.8)

hold true.

**Proof.** We provide proofs for (3.7), and the proof of the bilinear estimate (3.8) is similar. Using Plancherel’s identity, the estimate (3.7) is equivalent to showing that

$$\|B_s(f, g)\|_{L^2(\mathbb{R}^2)^2} \leq C \|f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)},$$  \hspace{1cm} (3.9)

where

$$B_s(f, g) = \int_{\mathbb{R}^2} \frac{\langle \xi \rangle^{1/2} \tilde{f}(\xi_2, \tau_2) \tilde{g}(\xi_1, \tau_1)}{\langle \xi_1 \rangle^s (\tau_1 - \alpha \xi_1^3)^{1+\epsilon} (\tau_2 - \xi_2^3)^{1-2\epsilon}} d\xi_1 d\tau_1,$$  \hspace{1cm} (3.10)

with $\tilde{f}(\xi, \tau) = \langle \xi \rangle^{-\frac{1}{2}} \langle \tau - \xi^3 \rangle^{1-2\epsilon} e^{i\xi \tau} \tilde{u}(\xi, \tau)$, $\tilde{g}(\xi, \tau) = \langle \xi \rangle^s \langle \tau - \alpha \xi^3 \rangle^{\frac{1}{2}+\epsilon} e^{i\xi \tau}$, $\xi_2 = \xi - \xi_1$ and $\tau_2 = \tau - \tau_1$. Using Cauchy–Schwarz inequality, we obtain (3.9) if

$$L_1 := \sup_{\xi, \tau} \int_{\mathbb{R}^2} \frac{\langle \xi \rangle^{2s}}{\langle \xi_1 \rangle^{2s} (\tau_1 - \alpha \xi_1^3)^{1+2\epsilon} (\tau_2 - \xi_2^3)^{1-4\epsilon}} d\xi_1 d\tau_1 \lesssim 1.$$  \hspace{1cm} (3.11)

Applying the estimate (3.3) for $\tau_1$ integral, we obtain from (3.11) that

$$L_1 \lesssim \sup_{\xi, \tau} \int_{\mathbb{R}} \frac{\langle \xi \rangle^{2s}}{\langle \xi_1 \rangle^{2s} (\tau - \xi_2^3 - \alpha \xi_1^3)^{1-4\epsilon}} d\xi_1 := \sup_{\xi, \tau} L_1,$$  \hspace{1cm} (3.12)

for $0 < \epsilon < \frac{1}{2}$.

Now, we move to estimate the integral in (3.12) dividing in four different cases:

$$|\xi| \leq 1, \quad 1 < |\xi| \leq c_1|\xi_1|, \quad c_1|\xi_1| < |\xi| < \frac{1}{c_2}|\xi_1|, \quad \frac{1}{c_2}|\xi_1| \leq |\xi|,$$

where $c_1 = -1 + \frac{\sqrt{3} + (3 - \lambda)(1 - \alpha)}{\sqrt{3}}$ with $0 < \lambda < 3$ and $c_2 = \frac{c_1}{1 - \alpha}$.

**Case a)** $|\xi| \leq 1$. Considering $c_0 := \frac{9}{1 - \alpha}$, we further divide in two subcases:

$$|\xi_1| \geq c_0 \quad \text{and} \quad |\xi_1| < c_0.$$
Subcase a1 \(|\xi_1| \geq c_0\). In this case, we have
\[
\langle \xi_2 \rangle \langle \xi_1 \rangle \lesssim \langle \xi_1 \rangle^2 \lesssim 1 + \xi_1^2.
\] (3.13)

For fixed \(\xi\) and \(\tau\), we define
\[
H(\xi_1) = \tau - \xi_2^3 - \alpha \xi_1^3.
\] (3.14)

We have
\[
H'(\xi_1) = 3(1 - \alpha)\xi_1^2 + 3\xi_1^2 - 6\xi_1 \geq 3(1 - \alpha)\xi_1^2 - 3 - 6|\xi_1| \geq 2(1 - \alpha)\xi_1^2.
\] (3.15)

The function \(H'\) has two roots namely \(\xi_1 = r_1 \xi, r_2 \xi\) where
\[
r_1 = \frac{1 + \sqrt{\alpha}}{1 - \alpha}, \quad r_2 = \frac{1 - \sqrt{\alpha}}{1 - \alpha}.
\]

Thus the function \(y = H(\xi_1)\) is monotone on each of the intervals:
\[
(-\infty, r_2 \xi), \quad [r_2 \xi, r_1 \xi), \quad \text{and} \quad (r_1 \xi, \infty).
\] (3.16)

Thus in this case, using that \(2s + 1 > 0\), we get
\[
L_1 = \int_{|\xi_1| \geq c_0} \frac{\langle \xi_2 \rangle \langle \xi_1 \rangle}{\langle \xi_1 \rangle^2 + 1 (\tau - \xi_2^3 - \alpha \xi_1^3)^{1 - 4\epsilon}} \, d\xi_1
\lesssim \int_{|\xi_1| \geq c_0} \frac{1 + |\xi_1|^2}{\langle \xi_1 \rangle^2 + 1 (\tau - \xi_2^3 - \alpha \xi_1^3)^{1 - 4\epsilon}} \, d\xi_1
\lesssim \int_{|\xi_1| \geq c_0} \frac{1}{\langle \tau - \xi_2^3 - \alpha \xi_1^3 \rangle^{1 - 4\epsilon}} \, d\xi_1 + \int_{|\xi_1| \geq c_0} \frac{|\xi_1|^2}{\langle \xi_1 \rangle^2 (\tau - \xi_2^3 - \alpha \xi_1^3)^{1 - 4\epsilon}} \, d\xi_1
\lesssim J_1 + J_2.
\] (3.17)

Using (3.6), we have \(J_1 \lesssim 1\) provided \(0 < \epsilon < \frac{1}{6}\). In what follows, we estimate \(J_2\) considering two different cases.

Case a1.1 \(\langle H(\xi_1) \rangle \lesssim |\xi_1|^3\). In this case, considering \(2s + 1 > 0\), we get
\[
J_2 \lesssim \int_{|\xi_1| \geq c_0} \frac{H'(\xi_1)}{|\xi_1|^2 + 1 (H(\xi_1))^{1 - 4\epsilon}} \, d\xi_1
\lesssim \int_{|\xi_1| \geq c_0} \frac{H'(\xi_1) \langle H(\xi_1) \rangle^{5\epsilon}}{|\xi_1|^2 (H(\xi_1))^{1 + \epsilon}} \, d\xi_1
\lesssim \int_{|\xi_1| \geq c_0} \frac{H'(\xi_1) |\xi_1|^{15\epsilon}}{|\xi_1|^2 (H(\xi_1))^{1 + \epsilon}} \, d\xi_1.
\] (3.18)

Making a change of variables of \(x = H(\xi_1)\) on each interval of monotonicity of \(H(\xi_1)\), for \(0 < \epsilon < \frac{2s + 1}{15}\), we obtain from (3.18) that
\[
J_2 \lesssim \int_{\mathbb{R}} \frac{dx}{(x)^{1 + \epsilon}} \lesssim 1.
\]
**Case a1.2)** $\langle H(\xi_1) \rangle \gtrsim |\xi_1|^3$. In this case, one can obtain

$$J_2 \lesssim \int_{|\xi_1| \geq c_0} \frac{|\xi_1|^2}{|\xi_1|^{2s+1} |\xi_1|^{3-12\epsilon}} d\xi_1 \lesssim \int_{|\xi_1| \geq c_0} \frac{1}{|\xi_1|^{2s+2-12\epsilon}} d\xi_1 \lesssim 1,$$

where the last inequality we used that $2s + 2 - 12\epsilon > 1$, which holds for $0 < \epsilon < \frac{2s+1}{12}$. This completes the proof of the **Subcase a1**.

**Subcase a2)** $|\xi_1| \leq c_0$ Using that $2s + 1 > 0$ and (3.6), in this case we get

$$L_1 = \int_{|\xi_1| \leq c_0} \frac{\langle \xi_2 \rangle \langle \xi_1 \rangle}{\langle \xi_1 \rangle^{2s+1} (\tau - \xi_2^3 - \alpha \xi_1^3)^{1-4\epsilon}} d\xi_1 \lesssim \int_{\mathbb{R}} \frac{1}{(\tau - \xi_2^3 - \alpha \xi_1^3)^{1-4\epsilon}} d\xi_1 \lesssim 1,$$

(3.19)

provided $0 < \epsilon < \frac{1}{6}$.

**Case b)** $1 < |\xi| \leq c_1 |\xi_1|$. In this case, we have

$$H'(\xi_1) = 3(1-\alpha)\xi_1^2 + 3\xi_2^2 - 6\xi_1 \geq 3(1-\alpha)\xi_1^2 - 3\xi_2^2 - 6|\xi_1| \geq \lambda (1-\alpha)\xi_1^2.$$

(3.20)

In the last inequality, we considered the value of $c_1$ as the positive root of $3c_1^2 + 6c_1 = (3-\lambda)(1-\alpha)$, i.e, $c_1 = -1 + \sqrt{3 + (3-\lambda)(1-\alpha)} / \sqrt{3}, 0 < \lambda < 3$. Again, we divide in two subcases: $|\xi_1| \geq 1$ and $|\xi_1| \leq 1$.

**Subcase b1)** $|\xi_1| \geq 1$. In this case, we get

$$\langle \xi_2 \rangle \langle \xi_1 \rangle \lesssim \langle \xi_1 \rangle^2 \lesssim \xi_1^2.$$

(3.21)

Using the inequalities (3.20), (3.21) and similar arguments as in **Subcase a1**, we conclude that $L_1 \lesssim 1$.

**Subcase b2)** $|\xi_1| \leq 1$. In this case also $|\xi| \lesssim 1$, and similarly as in **Subcase a2**, we have $L_1 \lesssim 1$.

**Case c)** $\frac{1}{c_2} |\xi_1| \leq |\xi|, c_2 = \frac{c_1}{1-\alpha}$. In this case, we have

$$\langle \xi_2 \rangle \langle \xi_1 \rangle \lesssim \langle \xi \rangle \langle \xi_1 \rangle.$$

(3.22)

Also

$$H'(\xi_1) = 3(1-\alpha)\xi_1^2 + 3\xi_2^2 - 6\xi_1 \geq 3\xi_2^2 - 3(1-\alpha)\xi_1^2 - 6|\xi_1|,$$

(3.23)
where we considered $c_2$ as the positive root of $3(1 - \alpha)c_2^2 + 6c_2 = (3 - \lambda)$ in the last inequality. In view of Case a), we can consider $|\xi| \geq 1$. Here also, considering $b = \frac{1}{1 - 4\epsilon}$, $a + b = 3$, we divide in two subcases.

**Subcase c1** \( \langle H(\xi_1) \rangle \lesssim \langle \xi_1 \rangle^a \langle \xi \rangle^b \)  
In this case, considering $2s + 1 > 0$, we get

\[
L_1 = \int_{\mathbb{R}} \langle \xi \rangle \langle \xi_1 \rangle^{2s} \langle 1 - \xi_2^3 - \alpha \xi_1^3 \rangle^{1-4\epsilon} d\xi_1
\]

\[
\lesssim \int_{\mathbb{R}} \langle \xi \rangle H' \langle \xi_1 \rangle d\xi_1
\]

\[
\lesssim \int_{\mathbb{R}} \langle \xi \rangle^2 \langle \xi_1 \rangle^{2s} \langle H(\xi_1) \rangle^{1-4\epsilon} d\xi_1
\]

\[
\lesssim \int_{\mathbb{R}} \frac{H'(\xi)}{\langle \xi_1 \rangle^{2s}} \langle \xi \rangle \langle H(\xi_1) \rangle d\xi_1
\]

\[
\lesssim \int_{\mathbb{R}} \frac{H' \langle \xi_1 \rangle}{\langle \xi_1 \rangle^{2s} - 5a \epsilon \langle \xi \rangle} \langle H(\xi_1) \rangle^{1+\epsilon} d\xi_1
\]

Recall that $a + b = 3$, taking $0 < \epsilon < \frac{2s + 1}{15}$ and making a change of variables of $x = H(\xi_1)$ on each interval of monotonicity of $H(\xi_1)$, we obtain from (3.24) that

\[
L_1 \lesssim \int_{\mathbb{R}} \frac{dx}{\langle x \rangle^{1+\epsilon}} \lesssim 1.
\]

**Subcase c2** \( \langle H(\xi_1) \rangle \gtrsim \langle \xi_1 \rangle^a \langle \xi \rangle^b \)  
In this case, one can obtain

\[
L_1 \gtrsim \int_{\mathbb{R}} \langle \xi \rangle \langle \xi_1 \rangle^{2s+1} \langle \xi \rangle^{a(1-4\epsilon)} \langle \xi \rangle^{b(1-4\epsilon)} d\xi_1
\]

\[
= \int_{\mathbb{R}} \langle \xi \rangle \langle \xi_1 \rangle^{2s+1} \langle \xi \rangle^{2-12\epsilon} d\xi_1
\]

\[
= \int_{\mathbb{R}} \frac{1}{\langle \xi \rangle^{2s+2-12\epsilon}} d\xi_1 \lesssim 1,
\]

where in the last inequality we used that $2s + 2 - 12\epsilon > 1$ which holds for $0 < \epsilon < \frac{2s+1}{12}$. This completes the proof of the Case c).

**Case d** \( c_1|\xi_1| < |\xi| < \frac{1}{c_2}|\xi_1| \)  
In view of Case a), we can suppose $|\xi| > 1$, and consequently, $|\xi_1| \sim |\xi| > 1$. Let $\mathcal{R} = \{ \xi_1 : c_2|\xi| < |\xi_1| < \frac{1}{c_1}|\xi| \}$. We will consider two different cases: $|H(\xi_1)| \gtrsim |\xi|^3$ and $|H(\xi_1)| \lesssim |\xi|^3$. 
Subcase d1) $|H(\xi_1)| \lesssim |\xi|^3$. Since $|\xi_1| \sim |\xi| > 1$, one has $\langle \xi_2 \rangle \lesssim \langle \xi_1 \rangle$, and therefore,

$$L_1 \lesssim \int_{\mathcal{R}} \frac{\langle \xi_2 \rangle}{\langle \xi_1 \rangle^{2x+1} \langle H(\xi_1) \rangle^{1+\epsilon}} d\xi_1 \lesssim \int_{\mathcal{R}} \frac{1}{\langle \xi_1 \rangle^{2x+2-12\epsilon}} d\xi_1 \lesssim 1,$$

where the last inequality holds for $0 < \epsilon < \frac{2s+1}{12}$.

Subcase d2) $|H(\xi_1)| \lesssim |\xi|^3$. In this case, we get

$$L_1 = \int_{\mathcal{R}} \frac{\langle \xi_2 \rangle \langle \xi_1 \rangle \langle H(\xi_1) \rangle^{5\epsilon}}{\langle \xi_1 \rangle^{2x+1} \langle H(\xi_1) \rangle^{1+\epsilon}} d\xi_1 \lesssim \int_{\mathcal{R}} \frac{\langle \xi \rangle^{2x+1} \langle H(\xi_1) \rangle^{1+\epsilon}}{\langle \xi \rangle^{2x+1-15\epsilon}} d\xi_1 \lesssim \int_{\mathcal{R}} \frac{1}{\langle H(\xi_1) \rangle^{1+\epsilon}} d\xi_1. \quad (3.25)$$

Note that

$$H(\xi_1) = \tau - \xi_2^3 - \alpha \xi_1^3 = \tau - \xi^3 + (1 - \alpha)\xi_1^3 - 3\xi\xi_1^2 + 3\xi_1^2\xi_1. \quad (3.26)$$

Making a change of variables $\xi_1 = \xi x$, we obtain that

$$\chi' := \int_{\mathcal{R}} \frac{1}{\langle H(\xi_1) \rangle^{1+\epsilon}} d\xi_1 = \int_{\widetilde{\mathcal{R}}} \frac{|\xi|}{\langle \tau - \xi^3 + \xi^3 [(1 - \alpha)x^3 - 3x^2 + 3x] \rangle^{1+\epsilon}} dx \leq \int_{\widetilde{\mathcal{R}}} \frac{|\xi|}{\langle \xi^3 \kappa^{-1} [c_0 + \kappa^3 + (x - \kappa)^3 - 3x\kappa^2 \alpha] \rangle^{1+\epsilon}} dx, \quad (3.27)$$

where $c_0 = \frac{\tau - \xi^3}{\xi^3} \kappa$ and $\kappa = \frac{1}{1 - \alpha}$. $\widetilde{\mathcal{R}} = \{x; c_2 < |x| < \frac{1}{c_1}\}$. Again, we make a change of variables $x = \kappa(\eta + 1)$, next the change of variables $\eta = \omega + \frac{\alpha}{\omega}$ and finally the change of variables $\omega^3 = t$, to arrive at

$$\chi' \sim |\xi| \int_{\mathcal{A}_\alpha} \frac{|t^{2/3} - \alpha|}{|t|^{1/3-\epsilon} (|t| + |\xi_1^3| t^2 + \kappa_1 t + \alpha^3))^{1+\epsilon}} dt,$$

where $\kappa_1 = c_0(1 - \alpha)^3 + 1 - 3\alpha$ and

$$\mathcal{A}_\alpha = \{t; \left| t^{1/3} + \frac{\alpha}{t^{1/3}} + 1 - \sqrt{2 - \alpha} \right| < 1 \}.$$

It is not difficult to see that

$$0 < s_1(\alpha) < |t| < s_2(\alpha). \quad (3.28)$$
Thus $|t| \sim \alpha$, and hence,
\[ \mathcal{X} \sim \|\xi\| \int_{A_{\alpha}} \frac{|t^{2/3} - \alpha|}{(1 + |\xi|^3|t|^2 + \kappa_1 t + \alpha^3)|^{1+\epsilon}} \, dt. \quad (3.29) \]

Let the discriminant be $\Delta = \kappa_1^2 - 4\alpha^3$. We consider three cases: $\Delta = 0$, $\Delta > 0$ and $\Delta < 0$.

**Subcase d2.1) $\Delta = 0$** In this case, $\kappa_1 = \pm 2\alpha^{3/2}$ and
\[ t^2 + \kappa_1 t + \alpha^3 = \left( t + \frac{\kappa_1}{2} \right)^2 = (t \pm \alpha^{3/2})^2. \quad (3.30) \]

We will consider the positive sign in (3.30), and the estimate with negative sign is similar. From (3.29), (3.28) and (3.30), we obtain
\[ \mathcal{X} \lesssim \|\xi\| \int_{A_{\alpha}} \frac{|t^{2/3} - \alpha|}{(1 + |\xi|^3|t|^2 + \kappa_1 t + \alpha^3)|^{1+\epsilon}} \, dt \]
\[ \lesssim \|\xi\| \int_{A_{\alpha}} \frac{|(t^{1/3} - \sqrt{\alpha})(t^{1/3} + \sqrt{\alpha})|}{(1 + |\xi|^3|t| + (\sqrt{\alpha})^3|t|^{2/3} - t^{1/3}\sqrt{\alpha} + \alpha^2)|^{1+\epsilon}} \, dt \]
\[ \lesssim \|\xi\| \int_{A_{\alpha}} \frac{|(t^{1/3} + \sqrt{\alpha})|}{(1 + |\xi|^3|t^{1/3} + \alpha^2|^{2})^{1+\epsilon}} \, dt, \quad (3.31) \]
where we used that
\[ x^2 \pm xc + c^2 \geq \frac{3c^2}{4}, \quad \forall x, c \in \mathbb{R}. \quad (3.32) \]

Finally we make a change of variable $u = |\xi|^{3/2}(t^{1/3} + \sqrt{\alpha})$ in the last integral. Considering $t \in A_{\alpha}$, we have $\mathcal{X} \lesssim \frac{1}{|\xi|^2}$. Therefore, in this case, from (3.25), we have
\[ L_1 \lesssim \frac{\langle \xi \rangle^2}{|\xi|^{2s+1-15\epsilon}} \mathcal{X} \lesssim \frac{\langle \xi \rangle^2}{|\xi|^{2s+1-15\epsilon}} \frac{1}{|\xi|^2} \lesssim 1; \quad (3.33) \]
in the last inequality, $0 < \epsilon < \frac{2s+1}{15}$ was used.

For the sake of simplicity, without loss of generality, from here onwards we consider $\alpha = \frac{1}{2}$ (the general case $\alpha \in (0, 1)$ follows analogously). Observe that
\[ A_{1/2} = \left\{ t : \frac{1}{4}(-4 - \sqrt{6}) - \frac{1}{4}\sqrt{14 + 8\sqrt{6}} < t^{1/3} < \frac{1}{4}(-4 - \sqrt{6}) \right\} \]
\[ + \frac{1}{4}\sqrt{14 + 8\sqrt{6}} \]
\[ = (l_1, l_2), \quad (3.34) \]
where $l_1 = -28.6526 \cdots < l_2 = -0.004356 \cdots < 0$. 

Subcase d2.2) $\Delta > 0$: In this case, $|\kappa_1| > 2\alpha^{3/2}$ and

$$r^2 + \kappa_1 t + \alpha^3 = \left(t + \frac{\kappa_1}{2}\right)^2 + \alpha^3 - \frac{\kappa_1^2}{4} = (t + \beta)^2 - \gamma^2,$$  \hspace{1cm} (3.35)

where $\beta = \frac{\kappa_1}{2}$ and $\gamma^2 = \beta^2 - \alpha^3$; consequently, $\beta = \pm(\gamma^2 + \alpha^3)^{1/2}$. We consider $\beta = (\gamma^2 + \alpha^3)^{1/2}$ similar argument works if $\beta = -(\gamma^2 + \alpha^3)^{1/2}$. Considering a change of variable $\tau = t + \beta$ in (3.29), we get

$$\mathcal{X} \lesssim |\xi| \int_{l_1 + \beta}^{l_2 + \beta} \frac{|(\tau - \beta)^{2/3} - \alpha|}{(1 + |\xi|^2|\tau^2 - \gamma^2|)^{1+\epsilon}} \, d\tau \leq |\xi| \int_{l_1 + \beta}^{l_2 + \beta} \frac{|(\tau - (\gamma^2 + \alpha^3)^{1/2})^{2/3} - \alpha|}{(1 + |\xi|^2|\tau^2 - \gamma^2|)^{1+\epsilon}} \, d\tau.$$ \hspace{1cm} (3.36)

As $l_1 \leq \tau - \beta = \tau - (\gamma^2 + \alpha^3)^{1/2} \leq l_2$, it is not difficult to see

$$\frac{|(\tau - (\gamma^2 + \alpha^3)^{1/2})^{2/3} - \alpha|}{(1 + |\xi|^2|\tau^2 - \gamma^2|)^{1+\epsilon}} \leq \frac{|(\tau - \gamma^2 + \alpha^3)^{1/2})^{1/3} + \sqrt{\alpha}|}{(1 + |\xi|^2|\tau^2 - \gamma^2|)^{1+\epsilon}},$$ \hspace{1cm} (3.37)

where we used the identity $M^3 + N^3 = (M + N)(M^2 - MN + N^2)$, and the inequality (3.32) with $M = (\tau - (\gamma^2 + \alpha^3)^{1/2})^{1/3}$ and $N = \sqrt{\alpha}$.

Thus using (3.5), (3.36) and (3.37), we obtain

$$\mathcal{X} \lesssim |\xi| \int_{l_1 + \beta}^{l_2 + \beta} \frac{|\tau - \gamma|}{(1 + |\xi|^2|\tau^2 - \gamma^2|)^{1+\epsilon}} \, d\tau + |\xi| \int_{l_1 + \beta}^{l_2 + \beta} \frac{|\gamma - (\gamma^2 + \alpha^3)^{1/2} + \alpha^{3/2}|}{(1 + |\xi|^2|\tau^2 - \gamma^2|)^{1+\epsilon}} \, d\tau \leq \frac{1}{|\xi|^2} + |\xi| \int_{l_1 + \beta}^{l_2 + \beta} \frac{|\gamma - (\gamma^2 + \alpha^3)^{1/2} + \alpha^{3/2}|}{(1 + |\xi|^2|\tau^2 - \gamma^2|)^{1+\epsilon}} \, d\tau.$$ \hspace{1cm} (3.38)
In order to estimate the last integral, note that

\[
\frac{|\gamma - (\gamma^2 + \alpha^3)^{1/2} + \alpha^3/2|}{(1 + |\xi|^{3}|\tau^2 - \gamma^2|)^{1+\epsilon}} = \frac{|(\gamma + \alpha^3/2)^2 - (\gamma^2 + \alpha^3)|}{(1 + |\xi|^{3}|\tau^2 - \gamma^2|)^{1+\epsilon}} \frac{1}{|\gamma + \alpha^3/2 + (\gamma^2 + \alpha^3)^{1/2}|}
\]

\[
= \frac{2|\gamma|^{\alpha^3/2}}{(1 + |\xi|^{3}|\tau^2 - \gamma^2|)^{1+\epsilon}} \frac{1}{|\gamma + \alpha^3/2 + (\gamma^2 + \alpha^3)^{1/2}|}.
\]

For \(\alpha \in (0, 1)\) one has \((\gamma^2 + \alpha^3)^{1/2} \geq |\gamma|\). It follows that

\[
\gamma + \alpha^3/2 + (\gamma^2 + \alpha^3)^{1/2} \geq \gamma + \alpha^3/2 + |\gamma| \geq \alpha^3/2.
\]

Finally using (3.4), (3.38), (3.39) and (3.40), we deduce that

\[
X \lesssim \frac{1}{|\xi|^2} + |\xi| \int_{l_1+\beta}^{l_2+\beta} \frac{|\gamma|}{(1 + |\xi|^{3}|\tau^2 - \gamma^2|)^{1+\epsilon}} d\tau \lesssim \frac{1}{|\xi|^2}.
\]

Analogously as in the above case, we obtain \(L_1 \lesssim 1\), provided \(0 < \epsilon < \frac{2s+1}{15}\).

**Subcase d2.3** \([\Delta < 0]\): This case is similar to the case \(\Delta > 0\); for the sake of completeness, we give all details here. In this case, \(|\kappa_1| < 2\alpha^{3/2}\) and

\[
t^2 + \kappa_1 t + \alpha^3 = (t + \frac{\kappa_1}{2})^2 + \alpha^3 - \frac{\kappa_1^2}{4} = (t + \beta)^2 + \gamma^2,
\]

where \(\beta = \frac{\kappa_1}{2}\) and \(\gamma^2 = \alpha^3 - \beta^2\); consequently, \(\beta = \pm (\alpha^3 - \gamma^2)^{1/2}\). We consider \(\beta = (\alpha^3 - \gamma^2)^{1/2}\), similar argument works if \(\beta = -(\alpha^3 - \gamma^2)^{1/2}\). Considering a change of variable \(\tau = t + \beta\) in (3.29), we get

\[
X \lesssim |\xi| \int_{l_1+\beta}^{l_2+\beta} \frac{|(\tau - \beta)^{2/3} - \alpha|}{(1 + |\xi|^{3}|\tau^2 + \gamma^2|)^{1+\epsilon}} d\tau \lesssim |\xi| \int_{l_1+\beta}^{l_2+\beta} \frac{|(\tau - (\alpha^3 - \gamma^2)^{1/2})^{2/3} - \alpha|}{(1 + |\xi|^{3}|\tau^2 + \gamma^2|)^{1+\epsilon}} d\tau.
\]

As \(l_1 \leq \tau - \beta = \tau - (\alpha^3 - \gamma^2)^{1/2} \leq l_2\), analogously as above it is not difficult to see
\[
\frac{|(\tau - (\alpha^3 - \gamma^2)^{1/2})^{2/3} - \alpha|}{(1 + |\xi|^3|\tau^2 + \gamma^2|)^{1+\epsilon}} \\
= \frac{|(\tau - (\alpha^3 - \gamma^2)^{1/2})^{1/3} - \sqrt{\alpha}| (\tau - (\alpha^3 - \gamma^2)^{1/2})^{1/3} + \sqrt{\alpha}}{(1 + |\xi|^3|\tau^2 + \gamma^2|)^{1+\epsilon}} \\
\lesssim \frac{|(\tau - (\alpha^3 - \gamma^2)^{1/2})^{1/3} + \sqrt{\alpha}|}{(1 + |\xi|^3|\tau^2 + \gamma^2|)^{1+\epsilon}} \\
\lesssim \frac{|(\tau - (\alpha^3 - \gamma^2)^{1/2})^{1/2} + \alpha^{3/2}|}{(1 + |\xi|^3|\tau^2 - \gamma^2|)^{1+\epsilon}} |M^2 - MN + N^2| \\
\lesssim \frac{|(\tau + \gamma| + |\gamma - (\alpha^3 - \gamma^2)^{1/2} + \alpha^{3/2}|}{(1 + |\xi|^3|\tau^2 + \gamma^2|)^{1+\epsilon}},
\]

where we used the identity \(M^3 + N^3 = (M + N)(M^2 - MN + N^2)\) and the inequality (3.32) with \(M = (\tau - (\alpha^3 - \gamma^2)^{1/2})^{1/3}\) and \(N = \sqrt{\alpha}\).

Thus combining (3.5), (3.43) and (3.44), we obtain

\[
\mathcal{X} \lesssim |\xi| \int_{1+\beta}^{l_2+\beta} \frac{|\tau + \gamma|}{(1 + |\xi|^3|\tau^2 + \gamma^2|)^{1+\epsilon}} \, dt + |\xi| \int_{1+\beta}^{l_2+\beta} \frac{|\gamma|}{(1 + |\xi|^3|\tau^2 + \gamma^2|)^{1+\epsilon}} \, dt \\
+ |\xi| \int_{1+\beta}^{l_2+\beta} \frac{|(\alpha^3 - \gamma^2)^{1/2} - \alpha^{3/2}|}{(1 + |\xi|^3|\tau^2 + \gamma^2|)^{1+\epsilon}} \, dt \\
\lesssim \frac{1}{|\xi|^2} + |\xi| \int_{1+\beta}^{l_2+\beta} \frac{|(\alpha^3 - \gamma^2)^{1/2} - \alpha^{3/2}|}{(1 + |\xi|^3|\tau^2 + \gamma^2|)^{1+\epsilon}} \, dt. \tag{3.45}
\]

In order to estimate the last integral, note that

\[
\frac{|(\alpha^3 - \gamma^2)^{1/2} - \alpha^{3/2}|}{(1 + |\xi|^3|\tau^2 - \gamma^2|)^{1+\epsilon}} = \frac{|(\alpha^3 - \gamma^2) - \alpha^3|}{(1 + |\xi|^3|\tau^2 - \gamma^2|)^{1+\epsilon}} \frac{1}{|(\alpha^3 - \gamma^2)^{1/2} + \alpha^{3/2}|} \lesssim \frac{\gamma^2}{(1 + |\xi|^3|\tau^2 - \gamma^2|)^{1+\epsilon}} \alpha^{3/2}. \tag{3.46}
\]

For \(\alpha \in (0, 1)\), from definition of \(\gamma\) one has \(|\gamma| \leq \alpha^{3/2} \leq 1\). Using (3.4), (3.45) and (3.46), it follows that

\[
\mathcal{X} \lesssim \frac{1}{|\xi|^2} + |\xi| \int_{1+\beta}^{l_2+\beta} \frac{|\gamma|}{(1 + |\xi|^3|\tau^2 - \gamma^2|)^{1+\epsilon}} \, dt \\
\lesssim \frac{1}{|\xi|^2}. \tag{3.47}
\]

The rest follows as in **Case d2.2**. \(\square\)

The following result shows that the bilinear estimate proved in Lemma 3.2 is sharp.
Proposition 3.3. The bilinear estimates (3.7) and (3.8) fail to hold for any $b \in \mathbb{R}$ whenever $s < -1/2$.

Now we move to prove the trilinear estimates.

Proof of Proposition 2.3. We provide detailed proof of the estimate (2.7) only, the proof of the estimate (2.8) follows analogously.

Let $b = \frac{1}{2} + \varepsilon$ and $b' = -\frac{1}{2} + 2\varepsilon$. Then using duality, the estimate (2.7) is equivalent to

$$\| (uvw)_x \|_{X_{s, \frac{1}{2} + 2\varepsilon}} \leq \sup_{h \in X_{s, \frac{1}{2} - 2\varepsilon}} \left| \int_{\mathbb{R}^2} (uvw)_x \, dx \right| \lesssim \| u \|_{X_{s, \frac{1}{2} + \varepsilon}} \| v \|_{X_{s, \frac{1}{2} + \varepsilon}} \| w \|_{X_{s, \frac{1}{2} + \varepsilon}} \| h \|_{X_{s, \frac{1}{2} - 2\varepsilon}}.$$  

(3.48)

Using the relation $\int f g = \int f \overline{g} = \int \hat{f} \bar{\hat{g}} = \int \hat{f} \hat{g} = \int \hat{f}(\xi) \overline{\hat{g}}(-\xi)$, one has

$$\left| \int_{\mathbb{R}^2} (uvw)_x \, dx \right| = \left| \int_{\mathbb{R}^2} i \xi \int_{\xi_1 + \xi_2 + \xi_3 = \xi} \tilde{u}(\xi_1, \tau_1) \tilde{v}(\xi_2, \tau_2) \tilde{w}(\xi_3, \tau_3) \tilde{h}(\xi, -\tau) \right|$$

$$= \left| \int_{\xi_1 + \cdots + \xi_4 = 0} \xi_4 \tilde{u}(\xi_1, \tau_1) \tilde{v}(\xi_2, \tau_2) \tilde{w}(\xi_3, \tau_3) \tilde{h}(\xi_4, \tau_4) \right|.$$  

(3.49)

Using (3.49) in (3.48), we need to prove

$$\left| \int_{\xi_1 + \cdots + \xi_4 = 0} \xi_4 \tilde{u}(\xi_1, \tau_1) \tilde{v}(\xi_2, \tau_2) \tilde{w}(\xi_3, \tau_3) \tilde{h}(\xi_4, \tau_4) \right| \lesssim \| u \|_{X_{s, \frac{1}{2} + \varepsilon}} \| v \|_{X_{s, \frac{1}{2} + \varepsilon}} \| w \|_{X_{s, \frac{1}{2} + \varepsilon}} \| h \|_{X_{s, \frac{1}{2} - 2\varepsilon}}.$$  

(3.50)

Let

$$\| u \|_{X_{s, \frac{1}{2} + \varepsilon}} = \| \langle \xi \rangle^s (\tau - \xi^3)^{\frac{1}{2} + \varepsilon} \tilde{u}(\xi, \tau) \|_{L^2_{\xi \tau}} =: \| f_1 \|_{L^2_{\xi \tau}}$$

$$\| v \|_{X_{s, \frac{1}{2} + \varepsilon}} = \| \langle \xi \rangle^s (\tau - \alpha \xi^3)^{\frac{1}{2} + \varepsilon} \tilde{v}(\xi, \tau) \|_{L^2_{\xi \tau}} =: \| f_2 \|_{L^2_{\xi \tau}}$$

$$\| w \|_{X_{s, \frac{1}{2} + \varepsilon}} = \| \langle \xi \rangle^s (\tau - \alpha \xi^3)^{\frac{1}{2} + \varepsilon} \tilde{w}(\xi, \tau) \|_{L^2_{\xi \tau}} =: \| f_3 \|_{L^2_{\xi \tau}}$$

$$\| h \|_{X_{s, \frac{1}{2} - 2\varepsilon}} = \| \langle \xi \rangle^{-s} (\tau - \xi^3)^{\frac{1}{2} - 2\varepsilon} \tilde{h}(\xi, \tau) \|_{L^2_{\xi \tau}} =: \| f_4 \|_{L^2_{\xi \tau}}.$$  

(3.51)
From (3.51) and (3.50), the matter reduces to proving
\[
\left| \int_{\xi_1+\cdots+\xi_4=0 \atop \tau_1+\cdots+\tau_4=0} \xi_4 \tilde{f}_1(\xi_1, \tau_1) \tilde{f}_2(\xi_2, \tau_2) \tilde{f}_3(\xi_3, \tau_3) \tilde{f}_4(\xi_4, \tau_4) \right| \leq \Pi_{j=1}^4 \| f_j \|_{L^2_{\xi'}}. 
\] (3.52)

So, we need to prove
\[
\left| \int_{\xi_1+\cdots+\xi_4=0 \atop \tau_1+\cdots+\tau_4=0} m(\xi_1, \tau_1, \cdots, \xi_4, \tau_4) \Pi_{j=1}^4 \tilde{f}_j(\xi_j, \tau_j) \right| \leq \Pi_{j=1}^4 \| f_j \|_{L^2_{\xi'}}, 
\] (3.53)
where
\[
m(\xi_1, \tau_1, \cdots, \xi_4, \tau_4) := \frac{\xi_4 (\xi_4)^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \langle \xi_4 \rangle^s (\tau_1 - \xi_1^3)^{\frac{1}{2}+\epsilon} (\tau_2 - \alpha \xi_2^3)^{\frac{1}{2}+\epsilon} (\tau_3 - \alpha \xi_3^3)^{\frac{1}{2}+\epsilon} (\tau_4 - \xi_4^3)^{\frac{1}{2}-2\epsilon}}. 
\] (3.54)

In this way, recalling the definition of the norm \( \| m \|_{[4; \mathbb{R}^2]} \) of the multiplier \( m \), the whole matter reduces to showing that
\[
\| m \|_{[4; \mathbb{R}^2]} \lesssim 1. 
\] (3.55)

Note that \(-\xi_4 = \xi_1 + \xi_2 + \xi_3 \implies \langle \xi_4 \rangle \leq \langle \xi_1 \rangle + \langle \xi_2 \rangle + \langle \xi_3 \rangle \). Therefore, considering \( s + 1 = s_0 + s_1 \) with \( s_1 \geq 0 \), one can obtain
\[
\xi_4 \langle \xi_4 \rangle^s \leq \langle \xi_4 \rangle^{s+1} \lesssim \langle \xi_4 \rangle^{s_0} (\langle \xi_1 \rangle^{s_1} + \langle \xi_2 \rangle^{s_1} + \langle \xi_3 \rangle^{s_1}). 
\] (3.56)

From (3.56) and (3.54), we get
\[
m \leq \frac{\langle \xi_4 \rangle^{s_0} \langle \xi_1 \rangle^{s_1}}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \langle \xi_4 \rangle^s (\tau_1 - \xi_1^3)^{\frac{1}{2}+\epsilon} (\tau_2 - \alpha \xi_2^3)^{\frac{1}{2}+\epsilon} (\tau_3 - \alpha \xi_3^3)^{\frac{1}{2}+\epsilon} (\tau_4 - \xi_4^3)^{\frac{1}{2}-2\epsilon}} + \frac{\langle \xi_4 \rangle^{s_0} \langle \xi_2 \rangle^{s_1}}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \langle \xi_4 \rangle^s (\tau_1 - \xi_1^3)^{\frac{1}{2}+\epsilon} (\tau_2 - \alpha \xi_2^3)^{\frac{1}{2}+\epsilon} (\tau_3 - \alpha \xi_3^3)^{\frac{1}{2}+\epsilon} (\tau_4 - \xi_4^3)^{\frac{1}{2}-2\epsilon}} + \frac{\langle \xi_4 \rangle^{s_0} \langle \xi_3 \rangle^{s_1}}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s \langle \xi_4 \rangle^s (\tau_1 - \xi_1^3)^{\frac{1}{2}+\epsilon} (\tau_2 - \alpha \xi_2^3)^{\frac{1}{2}+\epsilon} (\tau_3 - \alpha \xi_3^3)^{\frac{1}{2}+\epsilon} (\tau_4 - \xi_4^3)^{\frac{1}{2}-2\epsilon}} =: J_1 + J_2 + J_3. 
\] (3.57)
If we take $s_0 = \frac{1}{2}$, then $s_1 = s + \frac{1}{2}$, so $J_i$, $i = 1, 2, 3$ can be written as

$$J_1 = \frac{\langle \xi_4 \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{\frac{1}{2}}}{\langle \xi_2 \rangle^s \langle \xi_3 \rangle^s (\tau_1 - \xi_1^3)^{1/2+e} (\tau_2 - \alpha \xi_2^3)^{1/2+e} (\tau_3 - \alpha \xi_3^3)^{1/2+e} (\tau_4 - \xi_4^3)^{1/2-2e}} = \frac{\langle \xi_1 \rangle^{\frac{1}{2}}}{\langle \xi_2 \rangle^s \langle \xi_3 \rangle^s (\tau_1 - \xi_1^3)^{1/2+e} (\tau_2 - \alpha \xi_2^3)^{1/2+e} (\tau_3 - \alpha \xi_3^3)^{1/2+e} (\tau_4 - \xi_4^3)^{1/2-2e}} \leq m_1(\xi_1, \tau_1, \xi_2, \tau_2) m_1(\xi_4, \tau_4, \xi_3, \tau_3),$$

(3.58)

$$J_2 = \frac{\langle \xi_4 \rangle^{\frac{1}{2}} \langle \xi_2 \rangle^{\frac{1}{2}}}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s (\tau_1 - \xi_1^3)^{1/2+e} (\tau_2 - \alpha \xi_2^3)^{1/2+e} (\tau_3 - \alpha \xi_3^3)^{1/2+e} (\tau_4 - \xi_4^3)^{1/2-2e}} = \frac{\langle \xi_2 \rangle^{\frac{1}{2}}}{\langle \xi_1 \rangle^s \langle \xi_3 \rangle^s (\tau_1 - \xi_1^3)^{1/2+e} (\tau_2 - \alpha \xi_2^3)^{1/2+e} (\tau_3 - \alpha \xi_3^3)^{1/2+e} (\tau_4 - \xi_4^3)^{1/2-2e}} \leq m_1(\xi_4, \tau_4, \xi_3, \tau_3) m_2(\xi_2, \tau_2, \xi_1, \tau_1),$$

(3.59)

$$J_3 = \frac{\langle \xi_4 \rangle^{\frac{1}{2}} \langle \xi_3 \rangle^{\frac{1}{2}}}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s (\tau_1 - \xi_1^3)^{1/2+e} (\tau_2 - \alpha \xi_2^3)^{1/2+e} (\tau_3 - \alpha \xi_3^3)^{1/2+e} (\tau_4 - \xi_4^3)^{1/2-2e}} = \frac{\langle \xi_3 \rangle^{\frac{1}{2}}}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s (\tau_1 - \xi_1^3)^{1/2+e} (\tau_2 - \alpha \xi_2^3)^{1/2+e} (\tau_3 - \alpha \xi_3^3)^{1/2+e} (\tau_4 - \xi_4^3)^{1/2-2e}} \leq m_1(\xi_4, \tau_4, \xi_2, \tau_2) m_2(\xi_3, \tau_3, \xi_1, \tau_1).$$

(3.60)

With all what we did above, using comparison principle, permutation and composition properties (see, respectively, Lemmas 3.1, 3.3 and 3.7 in [18]), it is enough to bound $\| m_j \|_{[3; \mathbb{R}^2]}$, $j = 1, 2$, where

$$m_1(\xi_1, \tau_1, \xi_2, \tau_2) = \frac{\langle \xi_1 \rangle^{\frac{1}{2}}}{(\tau_1 - \xi_1^3)^{1/2-2e} \langle \xi_2 \rangle^s (\tau_2 - \alpha \xi_2^3)^{1/2+e}},$$

(3.61)

$$m_2(\xi_1, \tau_1, \xi_2, \tau_2) = \frac{\langle \xi_1 \rangle^{\frac{1}{2}}}{(\tau_1 - \alpha \xi_1^3)^{1/2-2e} \langle \xi_2 \rangle^s (\tau_2 - \xi_2^3)^{1/2+e}}.$$

(3.62)

With the argument presented in [18], proving $\| m_j \|_{[3; \mathbb{R}^2]} \lesssim 1$, $j = 1, 2$ is equivalent, respectively, to showing the following bilinear estimates

$$\| uv \|_{L^2(\mathbb{R}^2)} \lesssim \| u \|_{X_{-\frac{1}{2}+\frac{1}{2}-2e}} \| v \|_{X_{\frac{1}{2}+e}},$$

(3.63)

and

$$\| uv \|_{L^2(\mathbb{R}^2)} \lesssim \| u \|_{X_{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-2e}} \| v \|_{X_{\frac{1}{2}+e}}.$$  

(3.64)

This equivalence can be proved using duality with similar calculations as the ones used to get (3.53).

Recall that the estimates (3.63) and (3.64) for $s > -\frac{1}{2}$ are already proved in Lemma 3.2, and this completes the proof of (2.7). \qed
Remark 3.4. For \( \alpha = 1 \), one has \( m_1 = m_2 \) and in this case the bilinear estimate (3.64), and thus, the norm of \( m_2 \) is proved to be bounded in [18] for all \( s \geq \frac{1}{4} \). Our interest here is to consider \( 0 < \alpha < 1 \).

3.2. Proof of the local well-posedness result

In this subsection, we use the linear estimates recorded in Sect. 2 and the trilinear estimates from Proposition 2.3 to sketch proof of Theorem 1.1.

Proof of Theorem 1.1. We define spaces \( Z_s, b := X_{s, b} \times X_{s, b}^\alpha \) and \( Y_s := H_s \times H_s \) with norms \( \| (v, w) \|_{X_{s, b} \times X_{s, b}^\alpha} := \max \{ \| v \|_{X_{s, b}}, \| w \|_{X_{s, b}^\alpha} \} \) and similar for \( Y_s \). Let \( a > 0 \) and consider a ball in \( Z_{s, b} \) given by

\[
X_a := \{ (v, w) \in Z_{s, b}; \| (v, w) \|_{Z_{s, b}} < a \}.
\]

(3.65)

As we are interested in finding local solution to the IVP (1.1), we define the following application with the use of cutoff functions

\[
\begin{align*}
\Phi_\phi[v, w](t) & := \psi_1(t)U(t)\phi - \psi_3(t) \int_0^t U(t - t') \partial_x (vw^2)(t') \, dt', \\
\Psi_\psi[v, w](t) & := \psi_1(t)U^\alpha(t)\psi - \psi_3(t) \int_0^t U^\alpha(t - t') \partial_x (v^2w)(t') \, dt'.
\end{align*}
\]

(3.66)

We will show that there exist \( a > 0 \) and \( \delta > 0 \) such that the application \( \Phi \times \Psi \) maps \( X_s^\alpha \) into \( Y_s^\alpha \) and is a contraction.

We start estimating the first component \( \Phi \). Using estimates (2.4), (2.5) from Lemma 2.1, we have for \( s > -\frac{1}{2} \) and \( \theta := 1 - b + b' > 0 \),

\[
\| \Phi \|_{X_{s, b}} \leq C \| \phi \|_{H^s} + C \delta^\theta \| \partial_x (vw^2) \|_{X_{s, b}^\alpha}.
\]

(3.67)

Now, using the trilinear estimates from Proposition 2.3, one obtains

\[
\| \partial_x (vw^2) \|_{X_{s, b}^\alpha} \leq C \| v \|_{X_{s, b}} \| w \|_{X_{s, b}^\alpha}^2.
\]

(3.68)

Inserting (3.68) in (3.67), one gets

\[
\| \Phi \|_{X_{s, b}} \leq C \| \phi \|_{H^s} + C \delta^\theta \| v \|_{X_{s, b}} \| w \|_{X_{s, b}^\alpha}^2 \\
\leq C \| (\phi, \psi) \|_{Y_s} + C \delta^\theta \| (v, w) \|_{Z_{s, b}}^3.
\]

(3.69)

In an analogous manner, it is not difficult to obtain

\[
\| \Psi \|_{X_{s, b}^\alpha} \leq C \| (\phi, \psi) \|_{Y_s} + C \delta^\theta \| (v, w) \|_{Z_{s, b}}^3.
\]

(3.70)

Therefore, from (3.69) and (3.70), we obtain

\[
\| (\Phi, \Psi) \|_{Z_{s, b}} \leq C \| (\phi, \psi) \|_{Y_s} + C \delta^\theta \| (v, w) \|_{Z_{s, b}}^3.
\]

(3.71)
Let us choose \( a = 2C \| (\phi, \psi) \|_{Y^s} \), then from (3.71), we get
\[
\| (\Phi, \Psi) \|_{Z_{a,b}} \leq \frac{a}{2} + C \delta^\theta a^3. \tag{3.72}
\]
Now, if we take \( \delta > 0 \) such that \( C \delta^\theta a^2 < \frac{1}{2} \), then (3.72) yields
\[
\| (\Phi, \Psi) \|_{Z_{a,b}} \leq \frac{a}{2} + \frac{a}{2} = a. \tag{3.73}
\]
Therefore, the application \( \Phi \times \Psi \) maps \( X^a_s \) into \( X^a_s \). With the similar technique, one can easily show that \( \Phi \times \Psi \) is a contraction. Hence by a standard argument one can prove that the IVP (1.1) is locally well posed for initial data \( (\phi, \psi) \in Y^s \) for any \( s > -\frac{1}{2} \). Moreover, from (3.73) and the choice of \( a \), one has
\[
\| (\Phi, \Psi) \|_{Z_{a,b}} \leq C \| (\phi, \psi) \|_{Y^s}. \tag{3.74}
\]
The rest of the proof follows standard argument, so we omit the details. \( \square \)

4. Failure of bilinear and trilinear estimates, and ill-posedness

In this section, we prove the failure of the trilinear estimates that are crucial in the argument we used to obtain local well-posedness result using Bourgain’s spaces. Also we will prove the failure of the bilinear estimate without derivative that plays a fundamental role in the proof of the trilinear estimates. Finally, we prove ill-posedness result by showing that the application of data solution is not \( C^3 \) at the origin for initial data with Sobolev regularity below \( -\frac{1}{2} \). To obtain these negative results, we construct counter examples exploiting the lack of cancelation in the resonance relation when \( 0 < \alpha < 1 \).

4.1. Failure of trilinear estimates

We start by recording two elementary results about convolution that we use to prove the failure of trilinear estimate.

**Lemma 4.1.** Let \( R_j := [a_j, b_j], j = 1, \ldots, n \) be intervals of size \( |R_j| = b_j - a_j = N \). Then
\[
\| \chi_{R_1} \ast \chi_{R_2} \ast \cdots \ast \chi_{R_n} \|_{L^2} \sim N^{n-\frac{1}{2}} = |R_j|^{n-\frac{1}{2}}. \tag{4.1}
\]

**Proof.** First note that
\[
\hat{\chi}_{R_j}(\xi) \sim \int_{a_j}^{b_j} e^{-i\xi x} dx = \frac{2}{\xi} e^{-i\xi \frac{b_j+a_j}{2}} \sin \left( \frac{b_j - a_j}{2} \xi \right). \tag{4.2}
\]
Now, using Plancherel’s identity
\[
\| \chi_{R_1} \ast \chi_{R_2} \ast \cdots \ast \chi_{R_n} \|_{L^2} \sim \left\| \sin^n \left( \frac{N \xi}{2} \right) \right\|_{L^2} \sim N^{n-\frac{1}{2}}. \tag{4.3}
\]
\( \square \)
Lemma 4.2. Let \( R_j := [a_j, b_j] \times [c_j, d_j] \subset \mathbb{R}^2 \), \( j = 1, \ldots, n \) be rectangles such that \( b_j - a_j = N \) and \( d_j - c_j = M \). Then

\[
\| \chi_{R_1} \ast \chi_{R_2} \ast \cdots \ast \chi_{R_n} \|_{L^2(\mathbb{R}^2)} \sim (NM)^{n-\frac{1}{2}} = |R_j|^{n-\frac{1}{2}}. \tag{4.4}
\]

Proof. The proof follows by using Fubini’s theorem

\[
\widehat{\chi_{R_j}}(\xi, \tau) = \widehat{\chi_{[a_j,b_j]}(\xi)}\widehat{\chi_{[c_j,d_j]}(\tau)},
\]

followed by Lemma 4.1. \( \square \)

Now prove the result on the failure of the trilinear estimates stated in Proposition 1.2.

Proof of Proposition 1.2. We give details of the proof for the failure of (1.5) whenever \( s < -\frac{1}{2} \) and \( 0 < \alpha < 1 \). The proof of (1.6) follows analogously.

In fact, we will prove a general result showing that

\[
\| \partial_s (u_1 u_2 u_3) \|_{X_{s,b}'} \lesssim \| u_1 \|_{X_{s,b}} \| u_2 \|_{X_{s,b}} \| u_3 \|_{X_{s,b}} \tag{4.5}
\]

fails to hold whenever \( s < -\frac{1}{2} \) by constructing a counter example.

Using definition of the \( X_{s,b} \)-norm and Plancherel’s identity, the estimate (4.5) is equivalent to proving

\[
\| T_s(f, g, h) \|_{L^2_{\xi}L^2_\tau} \leq \| f \|_{L^2(\mathbb{R}^2)} \| g \|_{L^2(\mathbb{R}^2)} \| h \|_{L^2(\mathbb{R}^2)}, \tag{4.6}
\]

where

\[
T_s(f, g, h) := \left\| \langle \xi \rangle^s (\tau - \xi^3)^b \xi \right\| \int_{\mathbb{R}^4} \frac{f(\xi_1, \tau_1)g(\xi_2, \tau_2)h(\xi_3, \tau_3)}{\langle \xi_1 \rangle^s \langle \tau_1 - \xi_1^3 \rangle^b \langle \xi_2 \rangle^s \langle \tau_2 - \alpha \xi_2^3 \rangle^b \langle \xi_3 \rangle^s \langle \tau_3 - \alpha \xi_3^3 \rangle^b} \right\|_{L^2_{\xi\tau}} \tag{4.7}
\]

with \( \xi_3 = \xi - \xi_1 - \xi_2 \), \( \tau_3 = \tau - \tau_1 - \tau_2 \), and

\[
\tilde{f}(\xi, \tau) := \langle \xi \rangle^s (\tau - \xi^3)^b \tilde{u}_1(\xi, \tau) \\
\tilde{g}(\xi, \tau) := \langle \xi \rangle^s (\tau - \alpha \xi^3)^b \tilde{u}_2(\xi, \tau) \\
\tilde{h}(\xi, \tau) := \langle \xi \rangle^s (\tau - \alpha \xi^3)^b \tilde{u}_3(\xi, \tau).
\]

Let \( c_1, c_2 \) and \( c_3 \) be three constants satisfying

\[
c_1 + c_2 + c_3 = 1 \\
c_1^3 + c_2^3 + c_3^3 = 1. \tag{4.8}
\]

Consider three rectangles \( R_1, R_2 \) and \( R_3 \) with centers at \((c_1 N, (c_1 N)^3), (c_2 N, \alpha (c_2 N)^3)\) and \((c_3 N, \alpha (c_3 N)^3)\), respectively, and each with dimension \( N^{-2} \times 1 \). Now, consider

\[
\tilde{f} = \chi_{R_1}, \quad \tilde{g} = \chi_{R_2}, \quad \tilde{h} = \chi_{R_3}.
\]
It is easy to show that
\[
\| f \|_{L^2(\mathbb{R}^2)} = \| g \|_{L^2(\mathbb{R}^2)} = \| h \|_{L^2(\mathbb{R}^2)} = N^{-1}.
\] (4.9)

Also,
\[
|\xi_1 - c_1 N| \leq \frac{N^{-2}}{2}, \quad |\tau_1 - (c_1 N)^3| \leq \frac{1}{2}
\] (4.10)

and
\[
|\xi_j - c_j N| \leq \frac{N^{-2}}{2}, \quad |\tau_j - \alpha(c_j N)^3| \leq \frac{1}{2}, \quad j = 2, 3.
\] (4.11)

For \((\xi, \tau) \in R_1 + R_2 + R_3\), using (4.8), (4.10) and (4.11), it is easy to prove that
\[
|\xi - N| \lesssim N^{-2}, \quad \text{and} \quad |\tau - \xi^3| \lesssim 1.
\] (4.12)

In fact, the first estimate in (4.12) is a consequence of
\[
|\xi_j - N c_j| \sim N,
\]
and the second estimate is a consequence of
\[
|\tau - \xi^3| = |\tau - N^3 + N^3 - \xi^3| = |\tau_1 - (N c_1)^3 + \tau_2 - \alpha(N c_2)^3 + \tau_3 - \alpha(N c_3)^3 + (N - \xi)(N^2 + \xi^2 + \xi^2)|.
\]

As a result, for \((\xi, \tau) \in R_1 + R_2 + R_3\) we have \(|\xi| \sim N\) and \(\langle \tau - \xi^3 \rangle \sim 1\). Also, it is not difficult to show that \(|\xi_j| \sim N, j = 1, 2, 3\), and \(\langle \tau_1 - \xi_1^3 \rangle \sim 1\) and \(\langle \tau_j - \alpha \xi_j^3 \rangle \sim 1, j = 2, 3\).

Finally, using above considerations in (4.6), we obtain
\[
\left\| N^{s+1} \int_{\mathbb{R}^4} \chi_{R_1}(\xi_1, \tau_1) \chi_{R_2}(\xi_2, \tau_2) \chi_{R_3}(\xi_3, \tau_3) \right\|_{L^2_{\xi\tau}} \lesssim N^{-3}
\]
which is equivalent to
\[
N^{-2s+1} \| \chi_{R_1} \ast \chi_{R_2} \ast \chi_{R_3} \|_{L^2_{\xi\tau}} \lesssim N^{-3}.
\] (4.13)

Now, using estimate (4.4) from Lemma 4.2 with \(n = 3\) and \(|R_j| = N^{-2}\), the estimate (4.13) yields
\[
N^{-2s+1} N^{-5} \lesssim N^{-3} \iff N^{-2s-1} \lesssim 1.
\] (4.14)

If we choose \(N\) large, the estimate (4.14) fails to hold whenever \(s < -\frac{1}{2}\), and this completes the proof of the theorem. \(\square\)
4.2. Failure of bilinear estimates

Now we prove the failure of bilinear estimates (3.7) and (3.8) stated in Lemma 3.2 whenever \( s < -\frac{1}{2} \). The following proposition shows that the bilinear estimate (3.7) fails for \( s < -\frac{1}{2} \). The proof of the failure of (3.8) is analogous.

**Proposition 4.3.** Let \( 0 < \alpha < 1 \) and \( s < -\frac{1}{2} \), then the following bilinear estimate

\[
\|uv\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|_{\mathcal{X}_{s,\frac{1}{2}+\epsilon}} \|v\|_{\mathcal{X}_{s,\frac{1}{2}-\epsilon}} \tag{4.15}
\]

fails to hold.

**Proof.** Using Plancherel’s identity, the estimate (4.15) is equivalent to showing that

\[
B_s(f, g) := \left\| \int_{\mathbb{R}^2} \frac{\langle \xi_2 \rangle^{1/2} f(\xi_2, \tau_2) g(\xi_1, \tau_1)}{\langle \xi_1 \rangle^s \langle \tau_1 - \alpha \xi_3 \rangle^{\frac{1}{2}+\epsilon} \langle \tau_2 - \alpha \xi_3 \rangle^{\frac{1}{2}-2\epsilon}} d\xi_1 d\tau_1 \right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)} \leq \|f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)}, \tag{4.16}
\]

where \( \tilde{f}(\xi, \tau) = \langle \xi \rangle^{\frac{1}{2}} (\tau - \alpha \xi_3)^{\frac{1}{2}-2\epsilon} \tilde{u}(\xi, \tau) \), \( \tilde{g}(\xi, \tau) = \langle \xi \rangle^s (\tau - \alpha \xi_3)^{\frac{1}{2}+\epsilon} \tilde{v}(\xi, \tau) \), \( \xi_2 = \xi - \xi_1 \) and \( \tau_2 = \tau - \tau_1 \).

We will construct functions \( f \) and \( g \) for which the estimate (4.16) fails to hold when \( s < -\frac{1}{2} \). For this, let \( c_1, c_2 \) be two numbers such that

\[
c_1 + c_2 = 1, \quad \alpha c_1^3 + c_2^3 = 1. \tag{4.17}
\]

Consider two rectangles \( R_1 \) and \( R_2 \) with centers, respectively, at \( (c_1 N, \alpha(c_1 N)^3) \) and \( (c_2 N, (c_2 N)^3) \), each with dimension \( N^{-2} \times 1 \). Now, consider \( f \) and \( g \) defined via their Fourier transform

\[
\tilde{g} = \chi_{R_1}, \quad \tilde{f} = \chi_{R_2}.
\]

It is easy to see that

\[
\|f\|_{L^2(\mathbb{R}^2)} = \|g\|_{L^2(\mathbb{R}^2)} = N^{-1}. \tag{4.18}
\]

Also,

\[
|\xi_1 - c_1 N| \leq \frac{N^{-2}}{2}, \quad |\tau_1 - \alpha(c_1 N)^3| \leq \frac{1}{2} \tag{4.19}
\]

and

\[
|\xi_2 - c_2 N| \leq \frac{N^{-2}}{2}, \quad |\tau_2 - (c_2 N)^3| \leq \frac{1}{2}. \tag{4.20}
\]

Using (4.17), (4.19) and (4.20), it is easy to prove that

\[
(\xi, \tau) \in R_1 + R_2 \implies |\xi - N| \lesssim N^{-2} \text{ and } |\tau - \xi^3| \lesssim 1.
\]
Consequently, we have $|\xi| \sim N$ and $\langle \tau - \xi^3 \rangle \sim 1$ for $(\xi, \tau) \in R_1 + R_2$. Also, one can prove that $|\xi_j| \sim N$ and $\langle \tau_1 - \alpha \xi^3_1 \rangle \sim 1$ and $\langle \tau_2 - \xi^2_2 \rangle \sim 1$.

With these considerations, we get from (4.16)

$$B_s(f, g) \sim \left\| N^{\frac{1}{2} - s} \int_{\mathbb{R}^2} \chi_{R_1}(\xi_1, \tau_1) \chi_{R_2}(\xi_2, \tau_2) d\xi_1 d\tau_1 \right\|_{L^2_{\xi_1}(\mathbb{R}^2)}$$

$$\sim N^{\frac{1}{2} - s} \left\| \chi_{R_1} \ast \chi_{R_2} \right\|_{L^2(\mathbb{R}^2)}$$

$$= N^{\frac{1}{2} - s} |R_j|^{2 - \frac{1}{2}} = N^{-\frac{5}{2} - s}.$$  \hspace{1cm} (4.21)

Now, using (4.18) and (4.21) in (4.16),

$$N^{-\frac{5}{2} - s} \lesssim N^{-2} \iff N^{-\frac{1}{2} - s} \lesssim 1.$$  \hspace{1cm} (4.22)

If we choose $N$ large, the estimate (4.22) fails to hold whenever $s < -\frac{1}{2}$, and this completes the proof of the theorem.  \hfill \Box

4.3. Ill-posedness result

In this subsection, we consider the ill-posedness issue for the IVP (1.1). We start with the following result which is the main ingredient to prove the ill-posedness result stated in Theorem 1.3.

**Proposition 4.4.** Let $s < -\frac{1}{2}$, $0 < \alpha < 1$ and $T > 0$. Then there does not exist a space $X^s_T \times X^s_T$ continuously embedded in $C([0, T]; H^s(\mathbb{R})) \times C([0, T]; H^s(\mathbb{R}))$ such that the estimates

$$\| (U(t)\phi, U^\alpha(t)\psi) \|_{X^s_T \times X^s_T} \lesssim \| (\phi, \psi) \|_{H^s \times H^s},$$  \hspace{1cm} (4.23)

and

$$\| (\Phi_3, \Psi_3) \|_{X^s_T \times X^s_T} \lesssim \| (v, w) \|_{X^s_T \times X^s_T},$$  \hspace{1cm} (4.24)

hold true, where

$$\Phi_3 := -6 \int_0^t U(t - t') \partial_x [v w^2](t') dt'$$

$$\Psi_3 := -6 \int_0^t U^\alpha(t - t') \partial_x [v^2 w](t') dt'.$$

**Proof.** The proof follows a contradiction argument. If possible, suppose that there exists a space $X^s_T \times X^s_T$ that is continuously embedded in $C([0, T]; H^s(\mathbb{R})) \times C([0, T]; H^s(\mathbb{R}))$ such that the estimates (4.23) and (4.24) hold true. If we consider $v = U(t)\phi$ and $w = U^\alpha(t)\psi$, then from (4.23) and (4.24), we get

$$\| \Phi_3 \|_{H^s} \leq \| (\Phi_3, \Psi_3) \|_{H^s \times H^s} \lesssim \| (\phi, \psi) \|_{H^s \times H^s}^3,$$  \hspace{1cm} (4.25)
where
\[ \Phi_3 := -6 \int_0^t U(t-t') \partial_x \left[ U(t') \phi(U^\alpha(t') \psi)^2 \right] dt' \]
\[ \Psi_3 := -6 \int_0^t U^\alpha(t-t') \partial_x \left[ (U(t') \phi)^2 U^\alpha(t') \psi \right] dt'. \]

(4.26)

The main idea to complete the proof is to find an appropriate initial data \((\phi, \psi)\) for which the estimate (4.25) fails to hold whenever \(s < -\frac{1}{2}\).

Let \(N \gg 1, \gamma = \gamma(N)\) to be chosen later such that \(0 < \gamma \ll 1\). Let us consider \(\phi, \psi\) defined via Fourier transform
\[ \hat{\phi}(\xi) = \gamma^{-\frac{1}{2}} N^{-s} \chi_{R_1}(\xi) \]
and
\[ \hat{\psi}(\xi) = \gamma^{-\frac{1}{2}} N^{-s} \left[ \chi_{R_2}(\xi) + \chi_{R_3}(\xi) \right] = \gamma^{-\frac{1}{2}} N^{-s} P(\xi), \]

(4.27) (4.28)

where
\[ R_j(\xi) = \{ \xi : |\xi - c_j N| \leq \gamma \}, \quad j = 1, 2, 3 \]
(4.29)

with constants \(c_j\) satisfying
\[ c_1 + c_2 + c_3 = 1, \quad c_1^3 + \alpha c_2^3 + \alpha c_3^3 = 1. \]
(4.30)

In view of the definition of unitary groups \(U\) and \(U^\alpha\), and choices of \(\phi\) and \(\psi\), respectively, in (4.27) and (4.28), we have
\[ U^\alpha(t)\psi(\xi) = e^{i\alpha t \xi^3} \hat{\psi}(\xi) = \gamma^{-\frac{1}{2}} N^{-s} e^{i\alpha t \xi^3} P(\xi), \]
(4.31)

and
\[ U(t)\phi(\xi) = e^{i t \xi^3} \hat{\phi}(\xi) = \gamma^{-\frac{1}{2}} N^{-s} e^{i t \xi^3} \chi_{R_1}(\xi). \]
(4.32)

Let \(\Phi_1 := U(t)\phi\) and \(\Psi_1 := U^\alpha(t)\psi\). Taking Fourier transform in (4.26), we obtain
\[ \hat{\Phi}_3(\xi, t) = -6 \int_0^t e^{i(t-t')\xi^3} i\xi \int_{\mathbb{R}^2} \hat{\Psi}_1(\xi' - \xi_1 - \xi_2) \hat{\phi}_1(\xi_1) d\xi_1 d\xi_2 dt' \]
\[ \hat{\Psi}_3(\xi, t) = -6 \int_0^t e^{i(t-t')\alpha\xi^3} i\xi \int_{\mathbb{R}^2} \hat{\Phi}_1(\xi' - \xi_1 - \xi_2) \hat{\psi}_1(\xi_1) d\xi_1 d\xi_2 dt'. \]
(4.33)
Let $\xi_3 = \xi - \xi_1 - \xi_2$. For the choices of $\phi$ and $\psi$, using the relations (4.31) and (4.32) in (4.33), we get

$$\widehat{\Phi}_3(\xi, t) = -6e^{it\xi_3^2}i\xi \int_0^t e^{-it'\xi_3^2} \int_{\mathbb{R}^2} \gamma^{-\frac{3}{2}} N^{-3s} e^{i\xi_1^2 + \alpha e^{i\xi_2^3} + \alpha e^{i\xi_3^3}} \chi_{R_1}(\xi_1) P(\xi_2) d\xi_1 d\xi_2 dt'$$

$$= -6e^{it\xi_3^2}i\xi \gamma^{-\frac{3}{2}} N^{-3s} \int_{\mathbb{R}^2} \chi_{R_1}(\xi_1) P(\xi_2) P(\xi_3) \int_0^t e^{i\xi_1^2 + \alpha e^{i\xi_2^3} + \alpha e^{i\xi_3^3} - \xi^3} dt' d\xi_1 d\xi_2$$

$$= -6e^{it\xi_3^2}i\xi \gamma^{-\frac{3}{2}} N^{-3s} \int_{\mathbb{R}^2} \chi_{R_1}(\xi_1) P(\xi_2) P(\xi_3) e^{itA} - \frac{1}{itA} d\xi_1 d\xi_2,$$

where $A := \xi_1^3 + \alpha \xi_2^3 + \alpha \xi_3^3 - \xi^3$.

To continue with the proof, we need the following lemmas.

**Lemma 4.5.** Let $N \gg 1$, $0 < \gamma \ll 1$, $|\xi - N| \leq \gamma$, $\xi_1 \in R_1$ and $R_j$, $j = 1, 2, 3$ as defined in (4.29), then

$$(\xi_1, \xi_2, \xi_3) \in R_1 \times R_2 \times R_3,$$

or

$$(\xi_1, \xi_2, \xi_3) \in R_1 \times R_3 \times R_2.$$

**Proof.** We prove it using contradiction argument. If possible, suppose that $(\xi_1, \xi_2, \xi_3) \notin R_1 \times R_2 \times R_3$ and $(\xi_1, \xi_2, \xi_3) \notin R_1 \times R_3 \times R_2$. It means one can get two members of $\{\xi_2, \xi_3\}$ belonging to the same rectangle. For simplicity of exposition, let us consider the case when $\xi_1 \in R_1$ and $\xi_2, \xi_3 \in R_2$. (The other case can be handled analogously.) Therefore, by definition

$$|\xi_1 - c_1 N| \leq \gamma, \quad |\xi_2 - c_2 N| \leq \gamma \quad \text{and} \quad |\xi_3 - c_2 N| \leq \gamma.$$

Noting that $\xi = \xi_1 + \xi_2 + \xi_3$ and using the first relation in (4.30), we get

$$|\xi - N| = |\xi_1 + \xi_2 + \xi_3 - (c_1 N + c_2 N + c_3 N)|$$

$$= |\xi_1 - c_1 N + \xi_2 - c_2 N + \xi_3 - c_2 N + (c_1 - c_2) N|$$

$$\geq |c_1 - c_2| N - 3\gamma$$

$$\geq \frac{1}{2} |c_1 - c_2| N,$$

if we choose $\gamma \leq \frac{1}{2} |c_1 - c_2| N$. From last expression, we obtain that $|\xi - N| \gtrsim N$ which contradicts the fact that $|\xi - N| \leq \gamma$. \qed
Lemma 4.6. Let $N \gg 1$, $0 < \gamma \ll 1$ and $A$ be as defined in (4.34), $|\xi - N| \leq \gamma$ and 

$$(\xi_1, \xi_2, \xi_3) \in R_1 \times R_2 \times R_3$$

or

$$(\xi_1, \xi_2, \xi_3) \in R_1 \times R_3 \times R_2.$$ 

Then one has

$$|\xi^3 - N^3| \lesssim \gamma N^2, \quad \text{and} \quad |A| \leq \gamma N^2. \quad (4.35)$$

Proof. As $|\xi| \sim N$, one has $|\xi^3 - N^3| = |\xi - N||\xi^2 + \xi N + N^2| \leq 3\gamma N^2$, which implies the first inequality in (4.35).

Now we move to prove the second inequality in (4.35). We have

$$|\xi_1 - c_1 N| \leq \gamma, \quad |\xi_2 - c_2 N| \leq \gamma \quad \text{and} \quad |\xi_3 - c_3 N| \leq \gamma \quad (4.36)$$

or

$$|\xi_1 - c_1 N| \leq \gamma, \quad |\xi_2 - c_3 N| \leq \gamma \quad \text{and} \quad |\xi_3 - c_2 N| \leq \gamma \quad (4.37)$$

and consequently $|\xi_j| \sim N$ for all $j = 1, 2, 3$.

Now, using the second condition in (4.30) and (4.36)

$$|A| = |\xi_1^3 + \alpha \xi_2^3 + \alpha \xi_3^3 - N^3 + N^3 - \xi^3|$$

$$\leq |\xi_1^3 + \alpha \xi_2^3 + \alpha \xi_3^3 - (c_1 N)^3 - \alpha (c_2 N)^3 - \alpha (c_3 N)^3| + |N^3 - \xi^3|$$

$$\leq |\xi_1^3 - (c_1 N)^3| + \alpha |\xi_2^3 - (c_2 N)^3| + \alpha |\xi_3^3 - (c_3 N)^3| + \gamma N^2$$

$$\lesssim \gamma N^2. \quad (4.38)$$

Similarly using (4.37), we also have $|A| \lesssim \gamma N^2$. \hfill \Box

The following lemma was proved in [15].

Lemma 4.7. Let $R$ and $\tilde{R}$ be rectangles centered at $a$ and $b$ whose dimensions are $2\alpha$. Let $R_0$ be the translate of $R$ centered at $a + b$. Then we have

$$\chi_R \ast \chi_{\tilde{R}}(\xi) \geq \alpha \chi_{R_0}(\xi) = \frac{1}{2} |R| \chi_{R_0}(\xi).$$

Lemma 4.8. Let $R_j$, $j = 1, 2, 3$ be rectangles centered at $a_j$, $j = 1, 2, 3$ each with dimension $2\alpha$. Let $R_0$ be the translate of $R_1$ centered at $a_1 + a_2 + a_3$. Then we have

$$\chi_{R_1} \ast \chi_{R_2} \ast \chi_{R_3}(\xi) \geq \alpha^2 \chi_{R_0}(\xi) = \frac{|R_1|^2}{4} \chi_{R_0}(\xi).$$
Proof. Using the definition of convolution and Lemma 4.7, we obtain

\[
\chi_{R_1} \ast \chi_{R_2} \ast \chi_{R_3}(\xi) = \int_{R_1} \chi_{R_2} \ast \chi_{R_3}(\xi_1 - \xi)d\xi_1 \\
\geq \int_{R_1} \alpha \chi_{R_0}(\xi_1 - \xi)d\xi_1 = \alpha \chi_{R_1} \ast \chi_{R_0}(\xi) \\
\geq \alpha^2 \chi_{R_0}(\xi),
\]

(4.39)

where \( \tilde{R}_0 \) is the translate of \( R_2 \) centered at \( a_2 + a_3 \) and \( R_0 \) is the translate of \( R_1 \) centered at \( a_1 + a_2 + a_3 \) and \( (\xi_1, \xi_2, \xi_3) \in R_1 \times R_2 \times R_3 \).

Now, we retake the proof of Proposition 4.4. We choose \( N \gg 1 \) and \( \gamma = \gamma(N) = \epsilon N^{-2} \), for \( \epsilon > \), so that \( \|A\| \lesssim \epsilon \). With this choice, we have, for any \( t > 0 \)

\[
|e^{itA} - 1| = \left| \cos \frac{tA}{A} + i \frac{\sin \frac{tA}{A}}{A} \right| \geq \left| \frac{\sin \frac{tA}{A}}{A} \right| \geq \frac{t}{2}.
\]

Therefore, if \( |\xi - N| \leq \gamma \), from (4.34) and Lemma 4.5, we obtain

\[
|\hat{\Phi}_3(\xi, t)| \gtrsim |\xi|^{-2} N^{-3s} \left| \int_{\mathbb{R}^2} \chi(\xi_1) P(\xi_2) P(\xi_3) \frac{e^{itA} - 1}{iA} d\xi_1 d\xi_2 \right| \\
\gtrsim |\xi|^{-3} N^{-3s} t \left| \chi_{R_1} \ast \chi_{R_2} \ast \chi_{R_3}(\xi) \right|.
\]

(4.40)

Now, we move to calculate the \( H^s \)-norm of \( \Phi_3 \); using Lemma 4.8, we obtain

\[
\|\Phi_3\|_{H^s} = \|(\xi) \hat{\Phi}_3(\xi, t)\|_{L^2(\mathbb{R})} \geq \|(\xi) \hat{\Phi}_3(\xi, t)\|_{L^2(|\xi - N| \leq \gamma)} \\
\gtrsim N^s N^{-3s} t \left\| \chi_{R_1} \ast \chi_{R_2} \ast \chi_{R_3}(\xi) \right\|_{L^2(|\xi - N| \leq \gamma)} \\
\gtrsim N^{1-2s} \frac{1}{2} t \|\chi_{|\xi - N| \leq \gamma}(\xi)\|_{L^2(|\xi - N| \leq \gamma)} \\
\gtrsim N^{1-2s} \frac{1}{2} t \left| \left\{ |\xi - N| \leq \gamma \right\} \right|^{1/2} \\
\gtrsim N^{1-2s} \frac{1}{2} t,
\]

(4.41)

where in the last step of above inequality used \( \left| \left\{ |\xi - N| \leq \gamma \right\} \right| \sim \gamma \).

Therefore, for the choice of \( \gamma = \gamma(N) = \epsilon N^{-2} \) we get from (4.41) that

\[
\|\Phi_3\|_{H^s} \gtrsim N^{-1-2s} t \epsilon.
\]

(4.42)

On the other hand, we have

\[
\|(\phi, \psi)\|_{H^s \times H^s} = \|\psi\|_{H^s} \sim 1.
\]

(4.43)

In view of (4.42) and (4.43), we can conclude that the estimate (4.25) fails to hold when \( s < -\frac{1}{2} \).

Now, we prove the ill-posedness result stated in Theorem 1.3.
Proof of Theorem 1.3. For \((\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})\), consider the Cauchy problem

\[
\begin{aligned}
\partial_t v + \partial_x^3 v + \partial_x (v w^2) &= 0, \\
\partial_t w + \alpha \partial_x^3 w + \partial_x (v^2 w) &= 0, \\
v(x, 0) &= \delta \phi, \quad w(x, 0) = \delta \psi,
\end{aligned}
\tag{4.44}
\]

where \(\delta > 0\) is a parameter. The solution \((v(x, t), w(x, t)) := (v^\delta(x, t), w^\delta(x, t))\) of the IVP (4.44) depends on the parameter \(\delta\). The equivalent integral equation can be written as

\[
\begin{aligned}
v(t, \delta) = \delta U(t)\phi - \int_0^t U(t - t')\partial_x (v w^2)(t')dt', \\
w(t, \delta) = \delta U^\alpha(t)\phi - \int_0^t U^\alpha(t - t')\partial_x (v^2 w)(t')dt',
\end{aligned}
\tag{4.45}
\]

where \(U(t)\) and \(U^\alpha(t)\) are the unitary groups describing the solution of the linear part of the IVP (4.44). Note that by uniqueness of solution we have \(v^0(x, t) = 0 = w^0(x, t)\).

Differentiating (4.45) with respect to \(\delta\) and evaluating at \(\delta = 0\), we get

\[
\begin{aligned}
\partial_\delta v(t, 0) &= U(t)\phi =: \Phi_1, \\
\partial_\delta w(t, 0) &= U^\alpha(t)\psi =: \Psi_1.
\end{aligned}
\tag{4.46}
\]

\[
\begin{aligned}
\partial_\delta^2 v(t, 0) &= 0 = \partial_\delta^2 w(t, 0), \\
\partial_\delta^3 v(t, 0) &= -6 \int_0^t U(t - t')\partial_x [\Phi_1(\Psi_1)^2](t')dt' =: \Phi_3, \\
\partial_\delta^3 w(t, 0) &= -6 \int_0^t U^\alpha(t - t')\partial_x [(\Phi_1)^2\Psi_1](t')dt' =: \Psi_3.
\end{aligned}
\tag{4.47}
\]

If the flow map is \(C^3\) at the origin from \(H^s(\mathbb{R}) \times H^s(\mathbb{R})\) to \(C([-T, T]; H^s(\mathbb{R})) \times C([-T, T]; H^s(\mathbb{R}))\), we must have

\[
\| (\Phi_3, \Psi_3) \|_{L^\infty_t H^s(\mathbb{R}) \times H^s(\mathbb{R})} \lesssim \| (\phi, \psi) \|_{H^s(\mathbb{R})}^3.
\tag{4.49}
\]

But from Proposition 4.4, we have seen that the estimate (4.49) fails to hold for \(s < -\frac{1}{2}\) if we consider \(\phi\) and \(\psi\) given, respectively, by (4.27) and (4.28), and this completes the proof of the theorem. \(\square\)

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