Kinetic theory of instability in the interaction of an electron beam and plasma with an arbitrary anisotropic electron velocity distribution function

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Abstract

Based on the kinetic approach, this work investigates the stability of the system consisting of a fast electron beam and a dense plasma at an arbitrary (anisotropic) electron velocity distribution function. It is shown that during the interaction of a fast electron beam with a cold plasma, both the conditions for losing stability and the growth rate of disturbances do not depend on the form of the electron distribution function (EDF) of a plasma and are determined only by the ratio of the electron beam energy to the mean energy in a plasma. With an increase in the mean electron energy in the plasma, it becomes necessary to take into account the following energy moments of the EDF. It was found that the plasma anisotropy has a significant effect on both the stability loss conditions and the growth rate. The physical reason for this effect is the shift in the plasma frequency due to the Doppler effect caused by the plasma anisotropy in the coordinate system moving along with the beam. Other findings include a region of anomalous dispersion of the electron beam–plasma system and regions of negative group velocity of perturbations in such system. Physical interpretations are proposed for all the observed effects.

Introduction

Investigations of the patterns in the dynamics of electron beams in a plasma are important for the development of new devices for plasma electronics, in which a beam plasma is used. However, various types of instabilities often arise in the practical use of beam discharges, which makes their application difficult.

It has been known for about seventy years about the origin of intense oscillations of plasma parameters under conditions of propagation of a mono-directional beam of fast electrons through it, when their mean energy is much higher than that for plasma electrons. This type of plasma instability was predicted in [1, 2]. Its essence lies in the emergence and further intensification of the plasma oscillation and beam parameters (for example, density and velocity), longitudinal with respect to the direction of the initial beam velocity, to values comparable in amplitude with the values of the parameters themselves.

A large number of works were dedicated to the study of this phenomenon. Reviews of these works are given, for example, in monographs [3–6].

Previously, we solved a similar problem: we developed a kinetic theory of the stability of the electron beam–bounded plasma system for plane geometry at Knudsen numbers (Kn) of the order of 1 [7–9]. In this case, the electron distribution function (EDF) of a plasma in [9] was assumed to be isotropic, and in [7, 8], Maxwellian. It should be noted that, generally speaking, the first assumption is not entirely correct.

Indeed, consider a plasma bounded by two infinite planes. Let us introduce a coordinate system XYZ, in which the plane XY coincides with one of the planes (we will conventionally call it cathode), and the z axis is directed towards the second plane (anode). Let an electron beam move from the cathode towards the
by studying the propagation of a wave from a source of perturbations localized on a certain surface (in New J. Phys. this situation can be seen, for example, in a glow discharge near the anode, Kn of the electron beam and the electron density in the beam are of the order of those in the plasma. Indeed, \[10–12\].

Significantly exceeds that for an inelastic one. For inert gases, for example, this energy is less than 100 eV overcome the anode potential difference, there is a significant EDF anisotropy. Since the size of the anisotropy region near the completely absorbing surface is of the order of the path length, and the Knudsen number \(Kn \sim 1\), it is clear that the EDF of the plasma has an anisotropic character in a significant part of the region under consideration.

In addition, depending on the conditions, the EDF can also significantly deviate from a Maxwellian one [13]. Thus, in the strong electric fields, the drift velocity of electrons in the plasma can be an important part of the mean thermal velocity, which also leads to a significant anisotropy of the EDF.

Finally, usually when developing a kinetic theory of stability, it is assumed that the following inequality holds [7–9]:

\[
\kappa_E = \frac{E_T}{E_0} \ll 1, \tag{1}
\]

where \(E_T\) is the average plasma electron energy. At the same time, an interesting situation is when the energy of the electron beam and the electron density in the beam are of the order of those in the plasma. Indeed, this situation can be seen, for example, in a glow discharge near the anode, \(Kn \sim 1, \kappa_E < 1\), and the density of electrons moving in one direction turns out to be of the order of the plasma density. As is known, in many cases instabilities arise in gas-discharge plasmas, which are generated in a narrow near-anode region [13].

In addition, near the dielectric boundaries of the plasma it occurs that when the potential of the walls is of the order of the average electron energy, plasma electrons with higher energy, elastically scattered in the direction of the wall, can reach these surfaces. Thus, a directed flux of fast electrons is formed in direction to the surface bounding the plasma. Under the condition, \(Kn \sim 1\), it is as if there were a beam that has an energy of directed motion towards the wall above the average energy of electrons in the plasma and with a beam temperature equal to this energy. It would be interesting to study the stability of this phenomenon under fluctuations in plasma parameters.

Thus, it seems relevant to clarify the effect of the plasma EDF type and its anisotropy on the stability of the electron beam–plasma system, and also to investigate the stability of such a system when the following relations are fulfilled:

\[
\kappa_E, \kappa_b < 1, \tag{2}
\]

where \(\kappa_b = \frac{n_b}{n_0}\), and \(n_b, n_0\) are the densities of electrons in the beam and the plasma, respectively.

We consider non-relativistic electron beams when the propagating perturbations are electrostatic.

1. Basic relations

Let us designate the EDF in a plasma as \(n_0 F_0 (z, \nu, \mu)\), where \(\nu\) is the electron velocity; \(\mu = \cos \theta\) and \(\theta\) is the polar angle in the previously introduced coordinate system. Next, according to [7–9], it is necessary to write the system of kinetic equations for the electron beam and plasma electrons and the Poisson equation. Along with this, it is necessary to have a solution to the kinetic equations for the EDF without perturbation, i.e. in the region of stability of the system. Then, in this equation system, it is assumed that there is a harmonic perturbation of all parameters (including the electric field) with the frequency \(\omega\), wave number \(\gamma\), and using the resulting system of equations the relationship between these perturbation parameters is sought, which is the dispersion equation.

As known, the problem of the propagation of small-amplitude waves in a medium can be solved using two approaches:

(a) by studying the temporal dynamics of the wave, which at the moment of time \(t = 0\) has arisen at each point of a certain spatial region—this is the so-called problem with initial conditions [1, 14–16], when the frequency is assumed to be complex;

(b) by studying the propagation of a wave from a source of perturbations localized on a certain surface (in our case, the cathode plane)—this is the so-called problem with boundary conditions [17], when the wave number is assumed to be complex.

The obvious difference between these two approaches is that in the first case, the temporal dynamics of the perturbations is investigated, and in the second, the spatial ones. Depending on the properties of the environment, the results of these approaches can be either identical or radically different. An example of
such a significant difference is the problem of the propagation of sound waves in a plasma of molecular gases [18].

As a rule, when solving the problem of plasma instability, the problem with initial conditions is solved. We consider both approaches. As noted in [16], under some conditions, the growth of perturbations is possible, both within time at a fixed point in space, and along the direction of propagation of the beam at a certain time moment. We, however, investigate the second problem in more detail, as it is, in our opinion, more physically reasonable, because often in the system electron beam—self-sustained discharge the perturbations generator is the cathode (shot and thermal noise).

To obtain the dispersion equation in an explicit form, according to the previously developed theory [7–9] (see appendix A), it is necessary to calculate the integral:

\[
I_t = 8\pi^2 e \int_0^\infty \int_{-1}^1 v^2 b_t(v, \mu) \, d\mu \, dv;
\]

\[
b_t = -\frac{e}{m [i (\omega - \gamma v_z) + \nu_{ea}(v)]} \frac{\partial F_{\theta}}{\partial v_z},
\]

where \(e, m\) are the charge and mass of an electron, respectively; \(v_z = \sqrt{\frac{4k_B T_m}{m}}\) is the average thermal velocity of plasma electrons; \(\nu_{ea}(v) = \nu_{ea}[1 - \overline{\nu}_0(v)]\) is the effective frequency of elastic electron–atom collisions; \(\overline{\nu}_0\) is the mean cosine of elastic scattering of an electron by an atom; \(v_z = \mu v; \mu\) is the cosine of the angle between the electron velocity and \(z\) axis in the introduced coordinate system.

### 1.1. The case of separating variables \(\mu\) and \(v\) in the plasma EDF

For the case when the EDF variables \(\mu\) and \(v\) can be separated and it can be represented in the form (B6), the calculation of the integral (3) is given in appendix B. Using the formulas given in appendices A and B and the results [7] and acting as described above, we obtain the dispersion equation for the problem with initial conditions (when the frequency \(\omega\) is considered to be complex and the wavenumber is real and known) in the form:

\[
\frac{\kappa_b}{\epsilon^2} + P_t(\epsilon) \left[1 + i \frac{\nu_{ea}}{\gamma v_0 (1 + \epsilon)}\right]^{-1} - \frac{\gamma^2 v_0^2}{\omega_D^2} + \kappa_i \frac{1}{(2 + \epsilon)} \left[1 + \frac{\nu_{ea}(v_0)}{2 \gamma v_0} \ln \left(\frac{2 + \epsilon}{\epsilon}\right)\right]^{-1} = 0; \quad \kappa_i = \frac{\omega_D^2}{\omega^{2b}},
\]

where in general

\[
P_t(\epsilon) = \sum_{n=1}^{\infty} \left(\frac{n}{1 + \epsilon}ight)^{n+1} \left(\frac{E}{E_0}\right)^{n+1},
\]

and in the case when the plasma EDF is an even function of \(\mu\)

\[
P_t(\epsilon) = \sum_{j=1}^{\infty} \left(\frac{2j - 1}{1 + \epsilon}\right) \frac{\mu^{2j-2}}{(1 + \epsilon)^{2j}} \left(\frac{E}{E_0}\right)^{j+1}
\]

(definitions of quantities \(\mu^2\) and \(\frac{E}{E_0}\))

For the problem with boundary conditions, when the wavenumber \(\gamma\) is assumed to be complex and the frequency \(\omega\) is real and known, the dispersion equation has the form:

\[
\frac{\kappa_b}{\epsilon^2} + P_t(\epsilon) \left[1 - i \frac{\nu_{ea}}{\omega_D \kappa_{\omega}} \left(1 - i \frac{\nu_{ea}(v_0)}{\kappa_{\omega} v_0}\right)\right]^{-1} - \left(1 - i \frac{\nu_{ea}(v_0)}{\kappa_{\omega} v_0}\right)^2 + \kappa_i \frac{1}{(2 + \epsilon)} \left[1 + i \frac{\nu_{ea}(v_0)}{2 \omega_D \kappa_{\omega}} \ln \left(\frac{2 + \epsilon}{\epsilon}\right)\right]^{-1} = 0,
\]

where \(\kappa_{\omega} = \frac{\omega_D}{\omega}\).

In addition, it is satisfied:

\[
\text{Im} \left(\frac{\gamma v_0}{\omega_D}\right) = -\kappa_{\omega} \left[\frac{1 + \text{Re} (\epsilon)}{\text{Im} (\epsilon)} \left(1 \text{Re} (\epsilon) \right) \frac{\nu_{ea}(v_0)}{\kappa_{\omega} v_0} \right] + \frac{1}{(1 + \text{Re} (\epsilon))^2 + \text{Im} (\epsilon)^2};
\]

\[
\text{Re} \left(\frac{\gamma v_0}{\omega_D}\right) = \kappa_{\omega} \left[\frac{1 + \text{Re} (\epsilon) - \text{Im} (\epsilon) \frac{\nu_{ea}(v_0)}{\kappa_{\omega} v_0}}{1 + \text{Re} (\epsilon)^2 + \text{Im} (\epsilon)^2} \right].
\]
Note that when inequality (1) is satisfied, the first three terms of the function $P_t(\varepsilon)$ in general case have the form:

$$P_t(\varepsilon) = \frac{1}{(1 + \varepsilon)^2} + 2\pi \frac{E}{E_0} \frac{1}{(1 + \varepsilon)^3} + \frac{1}{(1 + \varepsilon)^4} \left[ \frac{E}{E_0} \right]^{1.5} + O \left[ \frac{E}{E_0} \right]^{3},$$  \hspace{1cm} (9)

and for an even EDF:

$$P_t(\varepsilon) = \frac{1}{(1 + \varepsilon)^2} + 3\mu E \frac{1}{(1 + \varepsilon)^3} + 5\mu^2 \frac{E}{E_0} \frac{1}{(1 + \varepsilon)^4} + O \left[ \frac{E}{E_0} \right]^{3}. \hspace{1cm} (10)$$

For an isotropic plasma, the obtained relations adequately describe the situation, including for $\kappa E < 1$. As it can be seen, in formulas (9) and (10) the first term corresponds to a ‘cold’ plasma with zero average electron energy. The following terms depend on the average energy and the angular velocity distribution of the plasma electrons. In addition, if the plasma EDF has a noticeable anisotropy, then the second term is significantly larger than for the isotropic EDF (more precisely, for an even EDF of $\mu$), since it is of order \( \left( \frac{E}{E_0} \right)^{0.5} \). The influence of this on the stability of the electron beam–plasma system and the physical reasons for this influence will be discussed below.

Let us write the functions $P_t(\varepsilon)$ for the following special cases of the angular distribution of plasma electrons, which we need when discussing the obtained results: for the isotropic EDF (which is a special case of an even EDF of $\mu$), $P_t(\varepsilon)$; for the isotropic EDF along the front hemisphere, when $\mu > 0 - P_t^i(\varepsilon)$; for the isotropic EDF along the back hemisphere, when $\mu < 0 - P_t^b(\varepsilon)$; for the sum of the isotropic EDF and a small anisotropic addition proportional to $\mu - P_t^d(\varepsilon)$. The second and third variants are characteristic for the EDFs near completely absorbing surfaces and differ only in the direction of the anisotropy relative to the beam velocity. The last variant of weak anisotropy is usually used to describe the EDFs in a gas-discharge plasma far from absorbing surfaces in the presence of the electric field. After calculations, we have for the first three EDFs:

$$P_t(\varepsilon) = \frac{1}{(1 + \varepsilon)^2} + \frac{E}{E_0} \frac{1}{(1 + \varepsilon)^3} + \frac{E}{E_0} \frac{1}{(1 + \varepsilon)^4} + O \left[ \frac{E}{E_0} \right]^{3}; \hspace{1cm} (11)$$

$$P_t^i(\varepsilon) = \frac{1}{(1 + \varepsilon)^2} + \frac{E}{E_0} \frac{1}{(1 + \varepsilon)^3} + \frac{E}{E_0} \frac{1}{(1 + \varepsilon)^4} + O \left[ \frac{E}{E_0} \right]^{3}; \hspace{1cm} (12)$$

$$P_t^b(\varepsilon) = \frac{1}{(1 + \varepsilon)^2} - \frac{4}{\pi} \frac{E}{E_0} \frac{1}{(1 + \varepsilon)^3} + \frac{E}{E_0} \frac{1}{(1 + \varepsilon)^4} + O \left[ \frac{E}{E_0} \right]^{1.5}. \hspace{1cm} (13)$$

In the case of the Maxwellian EDF, we obtain from (11):

$$P_t^i(\varepsilon) = \frac{1}{(1 + \varepsilon)^2} + \frac{4}{\pi} \frac{E}{E_0} \frac{1}{(1 + \varepsilon)^3} + \frac{E}{E_0} \frac{1}{(1 + \varepsilon)^4} + O \left[ \frac{E}{E_0} \right]^{1.5}; \hspace{1cm} (12)$$

Here, $O(x)$ is the value, for which:

$$\lim_{x \to \infty} \frac{O(x)}{x} = \text{const} \neq 0; \hspace{1cm} \text{const} < \infty.$$  

To calculate $P_t(\varepsilon)$ let us represent the corresponding model EDF in the form:

$$F_{0t}(v, \mu) = \frac{1}{2} \left[ \Phi_{0t}(v) + 3 \mu \Phi_{1t}(v) \right]; \hspace{1cm} (13)$$

where

$$\int_0^\infty F_{0t}(v, \mu) v^2 \, dv = 1.$$  

The relation (13) is the first two terms of the Legendre polynomials expansion in terms of $\mu$ of the EDF. For example, for a positive column plasma under the conditions when the EDF is determined by elastic
electron–atom collisions and the z axis in the selected coordinate system is parallel to the electric field vector, it is true that:

\[ \Phi_{1r}(v) = -u_d \frac{d\Phi_{ir}(v)}{dv}, \]

(14)

where \( u_d \) is the drift velocity of electrons. In the case of the opposite direction of the axis \( z \), the sign on the right side of (14) is reversed. Hence, in the case of the Maxwellian function \( \Phi_{ir}(v) \), we obtain:

\[ \int_0^\infty \Phi_{1r}(v) v^2 dv = \pm \sqrt{2} u_d. \]

The value \( \frac{\partial \Phi}{\partial v} \) in a gas-discharge plasma, depending on the strength of the electric field and the type of a gas, can reach values of several tenths [13]. After identical transformations, taking into account (14), we have an asymptotic series for the quantity \( P_1(r)(\varepsilon) \):

\[ P_1(r)(\varepsilon) = \frac{1}{(1+\varepsilon)^2} + \sum_{k=1}^\infty \frac{(2k-1)!!}{3^{k-1}(1+\varepsilon)^{2k+1}} \left( \frac{E_t}{E_0} \right)^k \left[ 1 \pm \frac{12k \sqrt{2} u_d}{2k+1} \frac{v}{v_t} (1+\varepsilon) \left( \frac{E_t}{E_0} \right)^{0.5} \right]. \]

(15)

The first three terms of this expansion are:

\[ P_1(r)(\varepsilon) = \frac{1}{(1+\varepsilon)^2} \left[ \frac{6}{\pi} \frac{u_d}{v_t} \frac{1}{(1+\varepsilon)^1} \left( \frac{E_t}{E_0} \right)^{0.5} + \frac{1}{(1+\varepsilon)^2} \left( \frac{E_t}{E_0} \right) + O \left( \frac{u_d}{v_t} \left( \frac{E_t}{E_0} \right)^{1.5} \right) + O \left( \frac{E_t}{E_0} \right)^2 \right]. \]

(16)

Recall that the upper and lower signs in the formulas (15) and (16) correspond to the parallel and antiparallel directions of the electric field vector in the plasma with respect to the z axis.

1.2. General case of an arbitrary EDF

Let us consider the general case of an arbitrary dependence of the plasma EDF on variables \( v \) and \( \mu \) (including when these variables cannot be separated). In this case, it is more convenient to consider the dependence of the EDF not on variables \( v, \mu \), but on variables \( v_x, v_y, v_z \)—the velocity \( v \) components in the chosen Cartesian coordinate system. Thus, in accordance with the above, it is necessary to calculate the value

\[ I_t(\varepsilon) = 4\pi e \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b_t(v_x, v_y, v_z) dv_x dv_y dv_z, \]

where

\[ b_t(v_x, v_y, v_z) = -\frac{e}{m} \frac{\partial F_{0t}(v_x, v_y, v_z)}{\partial v_z}, \]

\[ F_{0t}(v_x, v_y, v_z) dv_x dv_y dv_z = v^2 F_{0t}(v, \mu, \varphi) dv d\mu d\varphi, \]

and \( \varphi \) is the polar angle with \( z \) axis in the cylindrical coordinate system.

We assume that the plasma EDF at energies higher than the average thermal energy has a monotonic character and the number of plasma electrons with velocities of the order of \( v_0 \) is negligible. Then, acting in the same way as it was done in appendix B, and neglecting the Landau damping (for the same reasons as before), we can obtain that, at \( \frac{\omega_0}{\gamma v_0} \ll 1 \), the following asymptotic expansion is valid for \( I_t(\varepsilon) \):

\[ I_t(\varepsilon) = -i \frac{\omega_0}{\gamma v_0} \sum_{n=1}^{N} \frac{n}{(1+\varepsilon)^{n+1}} \left( \frac{v_z}{v_0} \right)^{n-1} + O \left[ \left( \frac{v_z}{v_0} \right)^N \right] \]

(17)

Here, \( E_z \) is the electron energy in the \( z \) direction.

From (17) we obtain for the function \( P_t(\varepsilon) \) an analogue of the relation (5) also in the form of an asymptotic series:

\[ P_t(\varepsilon) = -i \frac{\omega_0}{\gamma v_0} \sum_{n=1}^{N} \frac{n}{(1+\varepsilon)^{n+1}} \left( \frac{E_z}{E_0} \right)^{n-1} + O \left[ \left( \frac{E_z}{E_0} \right)^{N} \right]. \]

(18)

The formula (18) is valid for an arbitrary plasma EDF.
It is not difficult to verify that if the variables \( \nu \) and \( \mu \) can be separated, then directly from (18) we have the relation (5).

Thus, it turns out that the role of a plasma in the propagation of harmonic disturbances in a certain direction is determined by the energy moments (or velocity moments) of the unperturbed EDF in this direction: drift velocity, energy, energy flow, etc.

Leaving the first four terms of (18), we have:

\[
P_1(\varepsilon) = \frac{1}{(1+\varepsilon)^2} + \frac{2}{(1+\varepsilon)^3} \left( \frac{v_b}{v_0} \right) + \frac{3}{(1+\varepsilon)^4} \left( \frac{E_z}{E_0} \right) + \frac{5}{(1+\varepsilon)^5} \left( \frac{E_z}{E_0} \right)^{1.5} + O \left( \left( \frac{E_z}{E_0} \right)^2 \right).
\]

In order to analyze the influence of the plasma EDF anisotropy on the parameters of the propagating perturbation and to compare them with the case of the isotropic EDF, we obtain the analytical expression for the solution to the dispersion equation (4). For simplicity, let us consider the situation where \( \kappa_1 = \nu \varepsilon = 0 \) and \( \frac{\varepsilon}{\omega} = 1 \), and leave the two leading terms in the expansion (19).

Thus, the equation (4) takes the simple form:

\[
\frac{\kappa_b}{\varepsilon^2} + \frac{1}{(1+\varepsilon)^2} + \frac{2}{(1+\varepsilon)^3} \left( \frac{v_z}{v_0} \right) - 1 = 0,
\]

and in addition, in the particular case under consideration, the following is satisfied:

\[
\varepsilon = \frac{\omega - \omega_D}{\omega_D}.
\]

For the situation when \( \kappa_b, \left( \frac{\varepsilon}{\omega} \right) \leq 10^{-3} \), leaving in (20) the members \( O \left( \varepsilon^3 \right) \), one can get a solution with \( \text{Im} (\varepsilon) \leq 0 \):

\[
1 + \text{Re} (\varepsilon) = -\frac{\kappa_b^{1/2} v_b^{1/3}}{2^{4/3}} \left[ 1 + \frac{14}{3} \kappa_b - \frac{4}{9} \frac{v_z}{v_0} \right] + \frac{\kappa_b^{1/3}}{3} \cdot \frac{\kappa_b^{1/3}}{2^{3/2}} + \frac{\kappa_b^{2/3}}{3} \cdot \frac{2^{1/2}}{2^{3/2}} + O \left( \kappa_b^{1/3} \right) + O \left( \frac{\kappa_b^{2/3} \left( \frac{v_z}{v_0} \right)}{\omega_D} \right).
\]

Note that the first term in the second formula of (21) identically coincides with the corresponding formulas in [15, 19], where the problem of the stability of the electron beam–isotropic plasma system was solved within the framework of the hydrodynamic approach. Similarly, formulas (21) for \( \left( \frac{\varepsilon}{\omega} \right) = 0 \) up to values \( O \left( \kappa_b^{2/3} \right) \) coincide with the relation (42) in [7], where a kinetic theory for a low-voltage beam discharge (LVBD) in inert gases was developed.

Let us show the relationship between the parameters of the electron beam–plasma system, that correspond to the region of stability loss. For the problems with initial and boundary conditions, respectively, we have:

\[
\frac{\nu_0 (v_0)}{\gamma v_0} + \text{Im} (\varepsilon) \leq 0; \quad \frac{\nu_0 (v_0)}{\omega_D} + \frac{\text{Im} (\varepsilon)}{\text{Re} (\varepsilon)} \leq 0.
\]

The second relation directly follows from the formula (8). The equal sign corresponds to a zero-perturbation growth rate.

Phase and group velocities, respectively, for the problem with initial conditions are calculated by the formulas:

\[
\frac{v_x}{v_0} = \left[ 1 + \text{Re} (\varepsilon) \right];
\]

\[
\frac{v_x}{v_0} = \frac{\text{dRe}(\omega)}{\text{d} \gamma v_0} = 1 + \text{Re} (\varepsilon) + 2a \frac{\text{dRe} (\varepsilon)}{\text{d}a},
\]

where \( a = \frac{\kappa_b^{2}}{\omega_D} \).
Similarly, for the problem with boundary conditions, we have:

\[
\frac{v_b}{v_0} = \frac{\kappa \omega}{\kappa_0 [1 + \text{Re}(\epsilon)] - \text{Im}(\epsilon) \frac{\sigma_{v_b}(\nu)}{\omega}}
\]

\[
\frac{v_b}{v_0} = \left\{ \frac{\kappa_0 [1 + \text{Re}(\epsilon)] - \text{Im}(\epsilon) \frac{\sigma_{v_b}(\nu)}{\omega}}{[1 + \text{Re}(\epsilon)] + \text{Im}(\epsilon)^2} \right\}^{-1}
\]

The relations (23) and (24) follow from the definitions of phase and group velocities and from the formulas obtained for the terms in the right-hand sides of these formulas.

The relationship between the phase and group velocities, which we need when discussing the results, can be obtained in the following form from the definition of these velocities:

\[
\frac{d^2 v_f}{d \omega^2} = \frac{1}{\omega} \left( 1 - \frac{v_f}{v_g} \right) \left[ \left( 1 - \frac{v_f}{v_g} \right) \frac{d v_f}{d \omega} + \left( \frac{v_f}{v_g} \right)^2 \frac{d v_g}{d \omega} \right] ; \quad \frac{d v_f}{d \omega} = \frac{v_f}{\omega} \left( 1 - \frac{v_f}{v_g} \right).
\]

2. Discussion

First, we note that in the known special cases, when \( \kappa_i = 0, \frac{v_e}{v_0} = 1 + \text{Re}(\epsilon) \leq 10^{-3} \), considered in [15, 19, 20], as well as for LVBDs, and under the assumption that the plasma EDF is isotropic and has the Maxwellian velocity distribution [7, 8] our results coincide with the data obtained in other studies.

It follows from these results that, in the case of the isotropic EDF and when the inequality (1) is satisfied at the same average electron energy in the plasma, the form of the velocity EDF weakly affects the parameters of the propagating perturbation in the plasma.

Let us consider the results obtained for the anisotropic EDF in plasma. Of particular interest is the case of strong violation of isotropy, for example, for a plasma in strong electric fields or near absorbing surfaces. The calculations were carried out for the case of an inert He gas. The range of small parameters \( \kappa_b, \kappa_E \leq 10^{-3} \) for an interaction of a fast electron beam with an isotropic plasma has been extensively studied (except for the case \( \kappa_i \neq 0 \) [1, 15, 16, 19, 20]. In our opinion, the range of greater interest is when these parameters are of the order of \( 10^{-2} \) and greater. Such values are typical for various beam discharges, for example, the LVBD in rare gases [21].

We will briefly discuss the instability problem with initial conditions and in more detail the problem with boundary conditions, since, as a rule, the second formulation of the problem is often realized in plasmas, when the source of perturbations is an electron beam.

In the calculations, the mean cosine of the scattering angle for elastic electron–atom collisions was assumed to be 0.3, which for the case of the He atom corresponds to the beam energy equal to \( E_0 = 30 \text{ eV} \) (LVBD in He). From the equations (4) and (7) it follows that, for both problems (with initial and boundary conditions) the mean cosine of an angle of elastic scattering is considered, the perturbation parameters on the value \( P_0 (v_b) \) is included in the dispersion equation only for the group of electrons that have an isotropic distribution with the beam energy (i.e. at \( v = v_0 \)) since for plasma electrons having the average energy of the order of several eV, the value \( P_0 (v_b) \) is close to 0 [10, 11], at least for inert gases. The first term of the dispersion equations (4) and (7), corresponding to the beam electrons, also does not include \( P_0 (v_b) \), since any elastic scattering, regardless of the scattering angle, leads to the escape of the electron from this group. In view of the above, a weak dependence of the perturbation parameters on the value \( P_0 (v_b) \) is observed.

Figures 1 and 2 show the results of the calculations of the dependence of the relative growth rate and the relative frequency shift of the perturbation propagating in the electron beam–plasma system on the relative wave number for the problem with initial conditions at different values of the drift velocity projection \( \nu_z = \nu_0 \) on the z axis. Calculations were performed for the equation (4) where \( P_0 ^{\nu} (v) \) from the relation (16) was used. As can be seen from these data, in the case when the drift velocity (which in here is the cause of the plasma EDF anisotropy) is directed along the beam propagation, the module of the growth rate and the frequency shift decrease relative to the case of the isotropic EDF (i.e. at \( \nu_0 = 0 \)) remaining negative. On the contrary, if this velocity is in the direction opposite electron beam, then the module of these parameters increases. That is, at \( \nu_0 < 0 \), the system becomes less stable, and at \( \nu_0 > 0 \) more stable. We explain this below when we analyze the corresponding dependencies for the problem with boundary conditions.

At the same time, it should be noted that, with an increase in the wave number, the instability of the system grows. This is in agreement with the well-known fact that the growth rate grows with an increase in the perturbation frequency [7, 19].
Figure 1. Dependence of the relative growth rates $\text{Im}(\omega)/\omega_b$ on the relative wavenumber of the harmonic perturbation for the problem with initial conditions calculated by the formulas (4) and (16) at $\kappa_i = 0.01; \kappa_b = 0.06; \kappa_E = 0.05; \nu_{ea} = 0$; (1) — $\frac{\nu}{v_i} = 0$; (2) — $\frac{\nu}{v_i} = 0.2$; (3) — $\frac{\nu}{v_i} = -0.2$.

Figure 2. Dependence of the relative frequency shifts on the relative wavenumber of the harmonic perturbation for the problem with initial conditions calculated by the formulas (4) and (16) at $\kappa_i = 0.01; \kappa_b = 0.06; \kappa_E = 0.05; \nu_{ea} = 0$; (1) — $\frac{\nu}{v_i} = 0$; (2) — $\frac{\nu}{v_i} = 0.2$; (3) — $\frac{\nu}{v_i} = -0.2$.

Figures 3 and 4 show the dependences of the phase and group velocities for the problem with initial conditions (formulas (23) and (24)), respectively, on the relative wavenumber of the perturbation and for the conditions of figures 1 and 2. It is seen that, first, both the phase and group velocities decrease with the increasing of the wavenumber. This is explained as follows. At relatively low frequencies (large wavelengths and small wavenumbers), perturbations do not propagate in the plasma and, therefore, are carried by the beam with a relative velocity close to 1. As the frequency of the perturbation increases (increase in the wavenumber) and it approaches to the Langmuir frequency, oscillations in the plasma become excited, which leads to a decrease in the perturbation propagation velocity. We examine this effect in more detail when we discuss the calculation results for the problem with boundary conditions.
Figure 3. Dependence of the relative phase velocities $v_f/v_0$ on the relative wavenumber of the harmonic perturbation for the problem with initial conditions calculated by the formulas (4) and (16) at $\kappa_i = 0.01; \kappa_b = 0.06; \kappa_E = 0.05; \nu_{ea} = 0$; (1) — $\frac{\nu_f}{\nu_i} = 0$; (2) — $\frac{\nu_f}{\nu_i} = 0.2$; (3) — $\frac{\nu_f}{\nu_i} = -0.2$.

Figure 4. Dependence of the relative group velocities $v_f/v_0$ on the relative wavenumber of the harmonic perturbation for the problem with initial conditions calculated by the formulas (4) and (16) at $\kappa_i = 0.01; \kappa_b = 0.06; \kappa_E = 0.05; \nu_{ea} = 0$; (1) — $\frac{\nu_f}{\nu_i} = 0$; (2) — $\frac{\nu_f}{\nu_i} = 0.2$; (3) — $\frac{\nu_f}{\nu_i} = -0.2$.

Figure 5 shows the results of calculating the dependencies of the relative growth rate $\frac{\text{Im} \gamma_{v_0}}{\nu_D}$ in the problem with boundary conditions for the case of the anisotropic plasma, when the anisotropy is caused by the electric field (for example, a positive column plasma). The calculations were carried out for various
velocity of propagation of perturbations in the system in question at all frequencies is of the order of neglected the Landau damping. In our case this can be done for two reasons. First, because the phase frequency, the growth rate rapidly decreases regardless of the sign of the drift velocity. It can also be seen that, regardless of the value of the drift velocity, the growth rate has a pronounced maximum at a frequency close to the plasma frequency. As the frequency increases above the Langmuir frequency, the growth rate rapidly decreases regardless of the sign of the drift velocity, the growth rate has a pronounced maximum at a frequency close to the plasma frequency. As the frequency increases above the Langmuir frequency, the growth rate rapidly decreases regardless of the sign of the drift velocity.

It can also be seen that, regardless of the value of the drift velocity, the growth rate has a pronounced maximum at a frequency close to the plasma frequency. As the frequency increases above the Langmuir frequency, the growth rate rapidly decreases regardless of the sign of the drift velocity.

We recall that when deriving the dispersion equation for the problem with boundary conditions, we neglected the Landau damping. In our case this can be done for two reasons. First, because the phase velocity of propagation of perturbations in the system in question at all frequencies is of the order of \( v_0 \) and significantly exceeds the average thermal velocity of plasma electrons. Due to this, for example, for the Maxwellian EDF and the considered electron beam–plasma system the correction for the Landau damping is proportional to the factor \( \exp \left[ -\kappa_E^{-1}(1 + \varepsilon)^2 \right] \), which is exponentially small for the parameter \( \kappa_E \). Second, we take into account the damping due to elastic electron–atom collisions, which under our conditions significantly exceeds the Landau damping.

As can be seen from the data in figure 5, the frequency values at which the growth rates have maxima are different for different values of the parameter \( \frac{\omega_0}{\nu} \). At \( \frac{\omega_0}{\nu} = 0 \) the maximum of the growth rate is shifted to the right, and at \( \frac{\omega_0}{\nu} = -0.05 \) the left relative to the curve for \( \frac{\omega_0}{\nu} = 0 \).

Let us consider the reasons for this phenomenon. Oscillations of the beam parameters by generating oscillations of the electric field cause oscillations in the plasma. At \( \frac{\omega_0}{\nu} \neq 0 \) the frequency is determined in the coordinate system moving with the plasma. Let us denote the frequency of the maximum growth rate at \( \frac{\omega_0}{\nu} = 0 \) as \( \omega_{LD} \). At a positive value of the drift velocity, due to the Doppler effect, the frequency of forced plasma oscillations in the coordinate system moving with the drift velocity in the direction of the beam becomes less than the frequency for the case \( \frac{\omega_0}{\nu} = 0 \). Therefore, the extremum is reached at a certain oscillation frequency of the beam parameters in the laboratory coordinate system \( \omega_{+D} > \omega_{LD} \), such that the frequency in the coordinate system moving with the plasma is equal to \( \omega_{LD} \). A similar situation occurs when the sign of the drift velocity changes, but the frequency in the coordinate system moving with the drift
Figure 6. Dependence of the relative growth rate \( \frac{\text{Im}(\gamma v_0)}{\omega_D} \) on the parameter \( \kappa \omega_D \) for the problem with boundary conditions calculated by the formulas (7) and (16) for different negative parameters \( u_d \) and \( v_t \) at \( \kappa_i = 0.01; \kappa_b = 0.05; \kappa_E = 0.02; \omega_D = 0.06; \) (1)—\( \frac{\kappa}{\kappa_i} = 0 \); (2)—\( \frac{\kappa}{\kappa_i} = -0.01 \); (3)—\( \frac{\kappa}{\kappa_i} = -0.03 \); (4)—\( \frac{\kappa}{\kappa_i} = -0.05 \); (5)—\( \frac{\kappa}{\kappa_i} = -0.07 \); (6)—\( \frac{\kappa}{\kappa_i} = -0.1 \).

velocity towards the beam becomes higher. Therefore, the oscillation frequency of the beam parameters in the laboratory coordinate system \( \omega - D \), at which the extremum is reached, is \( \omega_D < \omega_D^0 \). Thus, the physical reason for the shift in the frequency of the growth rate maximum when the sign of the drift velocity changes is the Doppler effect. It is quite understandable that the drift velocity directed along the normal to the velocity of the beam electrons does not affect the parameters of the perturbations propagating in the electron beam–plasma system. This follows directly from the formula (19).

In figure 5 one can see another interesting feature of the curves. Namely, the growth rate for the case \( u_d < 0 \) significantly exceeds the value for \( u_d > 0 \). At the same time, as can be seen from figure 6, as the negative value of the modulus of \( u_d \) increases, the growth rate grows. Let us consider this effect in more detail.

For simplicity, let us assume that the EDF is Maxwellian with a temperature \( T \). Let \( \omega \) be the frequency of the wave propagating along the beam with the phase velocity \( v_f \) in the laboratory coordinate system. This wave excites a perturbation with a wave number \( \gamma_p \) in the plasma with a drift velocity projection \( u_d \) on the \( z \) axis. In accordance with the Doppler effect, the frequency of this perturbation \( \omega_p \) is equal to:

\[
\omega_p = \omega \left(1 - \frac{u_d}{v_f}\right). \tag{26}
\]

Let us assume that the oscillations that are excited in the plasma obey the well-known dispersion law [24]:

\[
\omega_p = \omega_D \sqrt{1 + \frac{3}{2} \gamma_p^2 r_D^2}. \tag{27}
\]

This relationship only qualitatively characterizes the plasma oscillations in the system under consideration in the plasma frequency region, since it does not take into account the interaction of these waves with the electron beam.

Using the known relationship between the Debye radius and the Langmuir frequency, from the relations (26) and (27) we obtain:

\[
\gamma_p = \left[\left(\frac{\omega}{\omega_D}\right)^2 \left(1 - \frac{u_d}{v_f}\right)^2 - 1\right]^{0.5} \frac{\omega_D}{\sqrt{\frac{3kT}{2m}}} \tag{28}
\]
In this case, the phase velocity \( v_{pf} \) of the perturbation propagation in the coordinate system associated with the plasma is equal to:

\[
v_{pf} = \frac{\frac{\omega_d}{\omega_p} \left( 1 - \frac{\omega_d}{\omega_p} \right)}{\left[ \left( \frac{\omega_p}{\omega_d} \right)^2 \left( 1 - \frac{\omega_d}{\omega_p} \right)^2 - 1 \right]} \sqrt{\frac{3kT}{2m}}.
\]  

(29)

In order for the wave excited by the perturbation of the beam to be amplified in the plasma, it is necessary that their wave numbers and velocities in the laboratory coordinate system coincide. It is easy to show that if the wave numbers coincide, the propagation velocities of the perturbations in the beam and plasma in the laboratory coordinate system also coincide. That is, if the equality

\[
\gamma = \frac{\omega}{\nu_f} = \frac{\omega_p}{v_{pf}} = \gamma_p
\]

is satisfied (where \( v_{pf} \) is the phase velocity of propagation of the perturbation in the plasma in the coordinate system associated with the plasma), then it is also satisfied that:

\[
\nu_f = v_{pf} + u_d.
\]

Using the relation (28), when the wave numbers are equal (\( \gamma = \gamma_p \)) we can obtain:

\[
\frac{\omega_p - \omega_d}{\omega_d} = \frac{\nu_p}{\nu_f} \left( \frac{\nu_0}{\nu_f} \right)^2 \frac{1}{\left( 1 - \frac{\nu_0}{\nu_f} \right)^2} + O \left( \kappa E^2 \right);
\]

(30)

\[
\frac{\omega - \omega_d}{\omega_d} = \frac{\nu_p}{\nu_f} \left( \frac{\nu_0}{\nu_f} \right)^2 \frac{1}{\left( 1 - \frac{\nu_0}{\nu_f} \right)^2} + O \left( \kappa E^2 \right).
\]

(31)

From the equality (30) it follows that for \( u_d < 0 \) an increase in the modulus of this quantity leads to a decrease in the difference between the frequency of excited oscillations in the plasma and the Langmuir frequency.

In addition, \( \omega_p > \omega_d \), as it follows from (27). If \( u_d > 0 \), then an increase in the projection of the drift velocity in the plasma leads to an increase in the difference \( \omega_p - \omega_d \). These regularities explain the increase in the growth rate with an increase in the modulus of the projection of the drift velocity when the plasma moves towards the beam. Thus, it has been established that the presence of a drift velocity in the plasma with a nonzero negative projection in the direction of the beam motion leads to an increase in the instability of the fast electron beam–plasma system, and a positive projection, on the contrary, leads to an increase in the stability.

Under the conditions in question, when the average electron energy in the plasma is significantly lower than the electron beam energy, the EDF anisotropy in the plasma leads to a significant change in the frequency dependence of the parameters of harmonic perturbations. The value of the growth rate at the maximum also changes. As can be seen from the data in figures 5 and 6, the growth rate increases with a negative projection of the drift velocity, it decreases with a positive projection. This leads to the fact that the limiting frequency of electron–atomic collisions, at which the system loses stability, decreases with decreasing drift velocity, that is, the system becomes more stable in the region of negative drift velocities.

Figure 7 shows the dependences of the relative phase velocity \( \frac{\omega - \omega_d}{\omega_d} \) on the relative frequency \( \kappa \omega \) at \( \nu_0 = 0; \pm 0.05 \) for the problem with boundary conditions. The relative growth rates are also shown here. It can be seen that the phase velocity for all parameters \( \nu_0 \) has a minimum in the region near the Langmuir frequency of plasma.

As already mentioned, at frequencies that are significantly lower \( \omega_p \), oscillations in the plasma do not propagate; therefore, they are transferred due to the beam at the velocity \( \nu_0 \). When approaching to the plasma frequency, due to the excitation of plasma, the oscillation propagation velocity decreases. With a further increase in the frequency (a decrease in the wavelength), oscillations in the plasma also become damped (see the dependence of the growth rate on frequency) and their transfer along the \( z \) axis is also possible only due to the beam with a velocity \( \nu_0 \). These assumptions are confirmed by the fact that, as can be seen from the data presented in figure 7, the growth of the phase velocity after passing the minimum begins at frequencies corresponding to a sharp decrease in the growth rate. With a further increase in frequency, the phase velocity tends to \( \nu_0 \).

Thus, for any value of the projection of the drift velocity in the plasma \( \nu_0 \) on the \( z \) axis, there is a region of anomalous dispersion in the fast electron beam–plasma system, which is located for \( \nu_0 = 0 \) at \( \kappa \omega > \kappa_\omega^0 \), for \( \nu_0 = 0.05 \) at \( \kappa \omega > \kappa_\omega^+ \), and for \( \nu_0 = -0.05 \) at \( \kappa \omega > \kappa_\omega^- \) (see figure 7). The relative
frequencies $\kappa^0_{\omega \text{ min}}, \kappa^+_{\omega \text{ min}}, \kappa^-_{\omega \text{ min}}$ correspond to the minimum phase velocity at $\frac{\kappa}{\gamma} = 0; 0.05; -0.05$, respectively.

The above considerations also explain the peculiarities of the behavior of the quantity $\Re(\gamma_0)$ (see figure 8). The reasons for the difference in frequencies corresponding to the minimum of the phase velocity at different directions of the drift velocity in the plasma are similar to those for the maximum of the growth rate (see above the explanations to figures 5 and 6). In this case, the relative wave number $\frac{\kappa}{\gamma}$ for any sign of the value $\frac{\omega}{\gamma}$ has two extrema, a maximum and a minimum. It can be seen that, at a sufficiently large distance from the plasma frequency, the functions $\Re(\gamma_0)$ are close to be linear, in the region near the Langmuir frequency all dependencies have a maximum, and at frequencies slightly higher than the plasma frequency they have a minimum. In addition, depending on the sign of the parameter $\frac{\omega}{\gamma}$ there is a shift of the curves in different directions relative to the curve calculated for $\frac{\kappa}{\gamma} = 0$. This is due to the peculiarities of the phase velocity behavior due to the Doppler effect (see figure 7) and the corresponding behavior of the wavelength.

Figure 8 presents, in our opinion, interesting data on the frequency dependence of the relative group velocities at various values of $\frac{\omega}{\gamma}$. Calculations were carried out for $\frac{\omega}{\gamma} = 0; \pm 0.05$. As already mentioned, for each of the given values of the relative drift velocity of electrons in the plasma there are two values of the relative frequency, where the relative wavenumber has an extremum, and the group velocity is discontinuous: at $\frac{\omega}{\gamma} = 0 - \kappa_{\omega 1}, \kappa_{\omega 2}$; at $\frac{\omega}{\gamma} = 0.5 - \kappa^1_{\omega}, \kappa^2_{\omega}$; at $\frac{\omega}{\gamma} = -0.5 - \kappa^1_{\omega}, \kappa^2_{\omega}$, where $\kappa_{\omega 1}, \kappa^1_{\omega}, \kappa^2_{\omega}$ are the relative frequencies corresponding to the maximum, and $\kappa_{\omega 2}, \kappa^1_{\omega}, \kappa^2_{\omega}$ corresponding to the minimum of the function $\Re(\gamma_0)$ at $\frac{\omega}{\gamma} = 0, 0.5, -0.5$, respectively. In this case, all relative frequencies $\kappa_{\omega 1} = \kappa^0_{\omega \text{ min}}, \kappa^1_{\omega}, \kappa^2_{\omega}$, $\kappa_{\omega 2} = \kappa^+_{\omega \text{ min}}, \kappa^0_{\omega}, \kappa^-_{\omega \text{ min}}$ and $\kappa_{\omega 2}, \kappa^1_{\omega}, \kappa^2_{\omega}$ lie in the regions of growth of the phase velocity with an increasing frequency for the corresponding values of the relative projection $\frac{\omega}{\gamma}$ of drift velocity.

It can be seen that, similarly to the phase velocity, the group velocity, at frequencies lower than the plasma frequency, is also close to the value $\nu_0$, but begins to decrease much earlier and also reaches a minimum at frequencies close to the plasma one. Moreover, at the minimum it is of the order of $\kappa E$. So, for example, at the parameters of the electron beam–plasma system specified in figure 8, and $\kappa E = 0.01$, the relative group velocity at the minimum is $\frac{\kappa}{\gamma} \approx 0.02$, and at $\kappa E = 0.1$, $\frac{\kappa}{\gamma} \approx 0.14$. This is in accordance with the well-known dispersion law for longitudinal electrostatic oscillations in plasmas [13, 22].

Note that for $\kappa_{\omega} = \kappa_{\text{min}}$ (where $\kappa_{\text{min}}$ is the relative frequency, where the phase velocity is minimal), $\nu_{bg} = \nu_{bg}$ according to the first of the equalities (25). In the region of anomalous dispersion, the group velocity exceeds the phase velocity. Cases of 'non-physical' behavior of the group velocity in the region of
Figure 8. Dependence of the relative wave number \( \frac{\text{Re}(\gamma v_0)}{\omega_0} \) (curves 1–3) and the relative group velocity \( \frac{v_{bg}}{v_0} \) (curves 4–6) of the perturbation propagation on the parameter \( \kappa \omega \) for the problem with boundary conditions at \( \kappa_i = 0.01; \kappa_b = 0.05; \kappa_E = 0.03; \nu_ea \omega D = 0.02 \) for the cases (1) and (4)—\( \kappa_i = 0 \); (2) and (5)—\( \kappa_i = 0.05 \); (3) and (6)—\( \kappa_i = -0.05 \); the values of the relative frequencies \( \kappa_1; \kappa_2; \kappa_{p1}; \kappa_{p2}; \kappa_{n1}; \kappa_{n2} \) indicate the boundaries of the anomalous dispersion region for the cases \( \nu_ea = 0 \); \( \nu_ea = 0.05 \); \( \nu_ea = -0.05 \), respectively.

Figure 9. Dependencies of the relative phase velocity \( \frac{v_{bf}}{v_0} \) of the perturbation propagation on the parameter \( \kappa \omega \) for the problem with boundary conditions at \( \kappa_i = 0.01; \kappa_b = 0.05; \kappa_E = 0.02; \nu_ea \omega D = 0.03 \) and different values of the parameter \( \frac{\nu_ea}{\gamma} \): (1)—\( \frac{\nu_ea}{\gamma} = -0.02 \); (2)—\( \frac{\nu_ea}{\gamma} = -0.05 \); (3)—\( \frac{\nu_ea}{\gamma} = -0.07 \).
anomalous dispersion are known. Thus, in [23] it was reported a negative group velocity of short electromagnetic pulses in a medium with anomalous dispersion. Thus, we have qualitatively explained the presence of two discontinuities in the group velocity at frequencies higher than the plasma one (in the region of anomalous dispersion).

Figure 9 shows the results of the dependence of the phase velocity on frequency at different negative drift velocities of plasma electrons. It can be seen that with an increase in the modulus of the negative value $u_d$, the frequency $\kappa \omega_{\text{min}}$ decreases. Calculations show that for a positive value $u_d$, on the contrary, with an increase in the modulus of the drift velocity, this frequency increases. This behavior of curves $\frac{v_b}{v_0} (\kappa \omega)$ is consistent with earlier considerations about the influence of the Doppler effect.

Figure 10 shows the data on the dependence $\frac{v_b}{v_0} (\kappa \omega)$ for different ratios of the plasma and beam energies $\kappa_E$ and the negative drift velocity in the plasma. It can be seen that, as expected, with a decrease in the average energy of plasma electrons, their influence on the group velocity of the perturbation propagation at a fixed frequency becomes less and less pronounced. In this case, in accordance with the above, the value of the group velocity at the minimum (at a frequency close to the plasma one) decreases proportionally to $\kappa_E$.

3. Summary

In conclusion, we remark that we have solved the stability problem for a fast electron beam–plasma system for an arbitrary, including anisotropic, plasma EDF, taking into account elastically scattered electrons that have a weak EDF anisotropy in the directions of motion and that have the same energy as the beam. The effect of the drift velocity on the stability of such system has been studied for the first time. The study showed that the Doppler effect has a significant influence on the frequency dependence of the growth rate and other parameters of the propagating perturbations due to the nonzero drift velocity depending on its direction relative to the velocity of the electron beam. In addition, the form of the energy dependence of the EDF affects the parameters of the propagating perturbations only through the energy moments (average velocity, energy, energy flow, etc).

An increase in the drift velocity directed antiparallel to the beam leads to the growth of its perturbation growth rate. This means an increase in the instability of the system. On the contrary, an increase in the drift...
velocity directed parallel to the direction of the beam leads to an increase of the stability of the electron beam–plasma system.

A region of anomalous dispersion of the electron beam–plasma system was found. In this region (in addition to the natural discontinuity of the group velocity at the boundary of the regions of normal and anomalous dispersions) there is one more discontinuity in the group velocity due to the presence of a minimum in the dependence of the wavenumber on frequency. In addition, the group velocity can take negative values, which, in our opinion, in this case has no physical meaning (in contrast, for example, to the case described in [23]).

In this work, we considered the case of a nonrelativistic beam, when the perturbations of the electric field are electrostatic. For relativistic beams a separate consideration based on the developed kinetic approach in a spatially confined plasma is required. The fact is that in this case it is necessary to consider not electrostatic field oscillations, but electromagnetic ones. Therefore, even though the beam electrons are absorbed by the anode, disturbances moving towards the beam may arise due to their reflection from the plasma boundary. This fact will significantly change the results of the study.

A similar situation is realized in the Peirce problem [26, 27], in which there also exists a wave moving towards the beam. With a significant increase in the amplitude of these disturbances, their interaction is possible. However, the fundamental difference between the system under consideration (including in the case of a relativistic beam) and the Pierce diode is that there is no dense plasma in it and the disturbance frequency is determined by the plasma frequency of the beam itself, which moves against the background of stationary ions. In our case, the perturbation frequency is determined by the plasma frequency, which is at least an order of magnitude higher and waves are transported along the beam only due to their drift at the speed of beam electrons.

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### Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

### Appendix A. Derivation of the dispersion equation

For simplicity, here we consider a homogeneous plasma and a homogeneous group of electrons with the beam energy, with an isotropic angular distribution, which have experienced several elastic collisions.

As indicated, in the absence of perturbations, the EDF in the LVBD under conditions in question can be represented as the sum of functions of three groups of electrons: the initial beam, the isotropically distributed electrons with the beam energy, and plasma electrons, respectively:

\[
F_{0b} (z,v) \approx \frac{n_{0b}}{2\pi v_0^2} \delta (\mu - 1) \delta (v - v_0) \exp \left[ -\frac{z}{\lambda_{ea} (v)} \right]; \\
F_{0b} ^i (z,v) = \frac{n_{0b}}{4\pi v_0} \delta (v - v_0);
\]

where \( n_{0b} \) is the concentration of isotropically distributed electrons with the beam energy, \( E_0 \) is the energy of beam electrons, \( v \) is the electron velocity and \( \delta (x) \) is the Dirac delta function. The plasma EDF will be denoted as \( F_0 (z,v) \).

The total EDF and the electric field in the LVBD are represented as the sum of unperturbed (A1) and harmonic perturbations with amplitudes depending on the coordinate:

\[
F_b (z,t,v) = F_{0b} (z,v) + f_b (z,t,v); \quad f_b (z,t,v) = f_{0b} (v,z) \exp \left[ -\Sigma_{ea} (v) z \right] \cdot \exp \left[ i(\omega t - \gamma z) \right]; \\
F_b ^i (z,t,v) = F_{0b} ^i (z,v) + f_b ^i (z,t,v); \quad f_b ^i (z,t,v) = f_{0b} ^i (v,z) \cdot \exp \left[ i(\omega t - \gamma z) \right]; \\
F_i (z,t,v) = F_{0i} (z,v) + f_i (z,t,v); \quad f_i (z,t,v) = f_{0i} (v,z) \cdot \exp \left[ i(\omega t - \gamma z) \right],
\]

where \( \Sigma_{ea} (v) = \frac{1}{\lambda_{ea} (v)} \).
For the perturbation amplitudes we have the system [1]:

\[ f_0 + \frac{v_z}{i(\omega - \gamma v_z) + \overline{\nu}_{ea}(v)} \frac{\partial f_0}{\partial z} = \frac{a_0(v, \mu)}{2} \int_{-1}^{1} f_{0j}(v, \mu', z) \, d\mu' + b_l(\mu, z); \tag{A3} \]

\[ f_{0b}^i + \frac{v_z}{i(\omega - \gamma v_z) + \overline{\nu}_{ea}(v)} \frac{\partial f_{0b}^i}{\partial z} = \frac{a_0^i(v, \mu)}{2} \int_{-1}^{1} f_{0b}^i(v, \mu', z) \, d\mu' + b_b^i(\mu, z); \]

\[ f_{0b} = b_b(\mu, z); \]

\[ \frac{\partial E^0}{\partial z} - i\gamma E^0 = 4\pi e \left\{ \int f_0 \, d^4v + \int f_{0b}^i \, d^3v + \int f_{0b} \, d^3v \right\}, \]

where

\[ a_i = a_b^i = \frac{i\overline{\nu}_{ea}(v)}{i(\omega - \gamma v_z) + \overline{\nu}_{ea}(v)}; \quad b_i = -\frac{eE^0(z)}{m[i(\omega - \gamma v_z) + \overline{\nu}_{ea}(v)]} \frac{\partial F_{0n}}{\partial v_z}; \]

\[ b_b^i = -\frac{eE^0(z)}{m[i(\omega - \gamma v_z) + \overline{\nu}_{ea}(v)]} \frac{\partial F_{0b}^i}{\partial v_z}; \]

\[ b_b = -\frac{eE^0(z)}{m[i(\omega - \gamma v_z) + \overline{\nu}_{ea}(v)]} \frac{\partial F_{0b}}{\partial v_z}. \]

\( \overline{\nu}_{ea}(v) = \nu_{ea}(v) \left[ 1 - \overline{\nu}_0(v) \right]; \nu_{ea}(v) \) is the frequency of elastic electron–atom collisions; \( \overline{\nu}_0(v) \) is the mean cosine of the scattering angle of the elastic electron–He atom collisions. Since their frequency \( \nu_{ea} \) weakly depends on the relative velocity of the colliding particles, the dependence of the effective frequency \( \overline{\nu}_{ea}(v) \) on the velocity \( v \) is caused only by the dependence on this quantity of the mean cosine of the scattering angle. Then, in the equations for \( f_{0b}^i \) and \( f_b \), it is necessary, obviously, to put \( \overline{\nu}_{ea}(v) = \overline{\nu}_{ea}(v_0) \), and in the equation for \( f_0, \overline{\nu}_{ea}(v) = \overline{\nu}_{ea}(v_T) \), where \( v_T \) is the average velocity of plasma thermal electrons. In this case in the first approximation, it can be assumed that \( \overline{\nu}_0(v_T) = 0 \) [10] and, therefore, \( \overline{\nu}_{ea}(v_T) = \nu_{ea} \).

Since we are considering conditions when \( \lambda_{ea} \sim d \), this allows us to search for a solution to the perturbation amplitudes of the EDF and the field in the form of series in powers of \( z \), which should, to a certain level, rapidly converge. Expanding also in similar series the densities \( n_{0b}^i(z) \) and \( n_b(z) \), we get:

\[ f_{0n}(z) = \sum_{n=0}^{N} f_{0n}^n z^n; \tag{A4} \]

\[ f_{0b}(z) = \sum_{n=0}^{N} f_{0b}^n z^n; \quad f_{0b}^i(z) = \sum_{n=0}^{N} f_{0b}^{in} z^n; \]

\[ E^0(z) = \sum_{n=0}^{N} E_{0n}^n z^n; \quad n_{0b}^i(z) = \sum_{n=0}^{N} n_{0b}^{in} z^n; \quad n_b(z) = \sum_{n=0}^{N} n_b^n z^n. \]

Substituting (A4) and (3) in (A3) and equating the coefficients at the same degrees of \( z \) to zero, we obtain for the coefficients at \( n = j \leq (N - 1) \) the following system of equations:

\[ f_{0j} + \frac{v_z(j + 1)f_{0(j + 1)}^j}{i(\omega - \gamma v_z) + \overline{\nu}_{ea}(v_0)} = \frac{a_j(v, \mu)}{2} \int_{-1}^{1} f_{0j}(v, \mu', z) \, d\mu' + b_j(\mu); \tag{A5} \]

\[ f_{0b}^j + \frac{v_z(j + 1)f_{0(j + 1)}^b}{i(\omega - \gamma v_z) + \overline{\nu}_{ea}(v_0)} = \frac{a_j^b(v, \mu)}{2} \int_{-1}^{1} f_{0b}^j(v, \mu', z) \, d\mu' + b_b^j(\mu); \]

\[ f_{0b}^j = b_b^j(\mu); \]

\[ (j + 1)E_{0(j + 1)} - i\gamma E^0_{0j} = 4\pi e \left\{ \int f_{0j} \, d^4v + \int f_{0b}^j \, d^3v + \int f_{0b} \, d^3v \right\}; \]

For \( j = N \) we have:

\[ f_{0N}^j = \frac{a_N(v, \mu)}{2} \int_{-1}^{1} f_{0N}^j(v, \mu', z) \, d\mu' + b_N^j(\mu); \tag{A6} \]

\[ f_{0N}^{bN} = \frac{a_N^b(v, \mu)}{2} \int_{-1}^{1} f_{0N}^{bN}(v, \mu', z) \, d\mu' + b_b^N(\mu); \]

\[ f_{0b}^N = b_b^N; \]

\[ -i\gamma E_{fN}^N = 4\pi e \left\{ \int f_{0N}^b \, d^3v + \int f_{0b}^N \, d^3v + \int f_{0b} \, d^3v \right\}; \]
in addition, for any \( j \leq N \)
\[
\begin{align*}
\dot{b}_l^j &= -\frac{e^{i\omega_j}m}{i(\omega_+ - \gamma v_0)} \sum_{k=0}^{j} E_n^{n-k} n_0^{n-k}; \\
\dot{b}_l^0 &= -\frac{e^{i\omega_0}m}{i(\omega_+ - \gamma v_0)} \sum_{k=0}^{j} E_n^{n} n_0^{n-k}; \\
\dot{b}_b^j &= -\frac{e^{i\omega_0}m}{i(\omega_+ - \gamma v_0)} \sum_{k=0}^{j} E_n^{n} n_0^{n-k}; \\
\dot{b}_b^0 &= -m \sum_{k=0}^{j} E_n^{n} n_0^{n-k}; \\
B_l^j &= \left( \frac{\partial F_{eb}}{\partial \nu_j} \right) \sum_{\kappa=0}^{j} \left( -1 \right)^{\nu_0} \frac{1}{k!} v_0^k e^{i\omega_0} E_k^{j-k}.
\end{align*}
\]

Thus, when expanding up to \( \varepsilon^N \), we obtain a system of \( l = 4 \cdot (N + 1) \) equations, from which \( l = 3 \cdot (N + 1) \) are the Boltzmann kinetic equations for the expansion coefficients of the EDF perturbation amplitudes and \( l_b = (N + 1) \) are the Poisson equations for the expansion coefficients of the electric field perturbation amplitude. Using \( l_b \) kinetic equations, the right-hand sides of \( l_b \) Poisson equations are calculated and we obtain a linear homogeneous system of \( l_b \) equations for \( l_b \) expansion coefficients of the electric field perturbation amplitude. The system has a nonzero solution if and only if the determinant of its matrix is equal to zero. This condition is the equation for finding the growth rate of the damping (or amplification) of perturbations in the plasma.

Appendix B. Calculation of the integral \( I_1 (\varepsilon) \)

Let us calculate the expression \( I_1 \) defined by the formula (3). Considering that
\[
\frac{d}{dv} = \mu \frac{\partial}{\partial \nu} + \frac{1 - \mu^2}{\nu} \frac{\partial}{\partial \mu},
\]
we can rewrite it as:
\[
I_1 (\varepsilon) = \int_{0}^{\infty} v^2 \left[ I_1 (v, \varepsilon) + I_2 (v, \varepsilon) \right] dv;
\]
\[
I_1 (v, \varepsilon) = \int_{-1}^{1} \left[ 1 - \mu^2 \right] \frac{\partial F_{eb}}{\partial \mu} d\mu; \quad I_2 (v, \varepsilon) = \int_{-1}^{1} \left[ 1 - \mu^2 \right] \frac{\partial F_{eb}}{\partial \mu} d\mu,
\]
where \( \varepsilon = \frac{\omega - \gamma v_0}{v_0} \).

Since in our case the phase velocity of perturbations is of the order of \( v_0 \), which is several times higher than the average thermal velocity of electrons in plasma \( v_1 \), the Landau damping can be neglected at frequencies \( \omega \sim \omega_0 \) or less [22]. Indeed, in the case, for example, of a Maxwellian EDF in plasma, the correction for the Landau damping for the considered electron beam–plasma system is of order \( \exp \left[ -n_e \gamma (1 + \varepsilon) \right] \). Then the integrals \( I_1 (v, \varepsilon) \) and \( I_2 (v, \varepsilon) \) and can be calculated in the sense of principal value (see, for example, [24]).

Bearing in mind the aforesaid about the Landau damping, let us consider the quantity \( I_1 (v, \varepsilon) \). As shown in [7], the inequalities \( 1 > \operatorname{Re} (\varepsilon) > -1 \) hold. The convergence radius of the series \((1 - x)^{-1}\) is equal to 1, that is \(|x| < 1\) [25]. Then it is easy to show that the series
\[
\frac{\mu}{1 - \mu^2 v_0 (1 + \varepsilon)} = \sum_{n=0}^{\infty} \left( \frac{v}{v_0} \right)^n \frac{\mu^{n+1}}{1 + \varepsilon}.
\]
converges at
\[
\left| \frac{\mu}{v} \frac{v}{(1 + \varepsilon) v_0} \right| < 1,
\]
or
\[
\frac{v}{v_0} < \sqrt{\left[1 + \operatorname{Re} (\varepsilon) \right]^2 + \left| \operatorname{Im} (\varepsilon) \right|^2} \sim 1.
\]

We consider the situation when \( \frac{v}{v_0} \ll 1 \), that is \( \left( \frac{v}{v_0} \right)^{0.5} \approx \frac{v}{v_0} < 0.3 \), where \( v_1 \) is the average thermal velocity of electrons in the plasma. Hence, we can conclude that for, to a certain level, a rapidly decreasing EDF, the series (B2) converges for the majority of electrons. For example, in the case of the Maxwellian distribution with an average energy \( E_0 \), this is satisfied for velocities \( v \ll 3v_1 \) (almost 100% of electrons).
Substituting (B2) into the expression for $I_1 (v, \varepsilon)$ and $I_2 (v, \varepsilon)$ and integrating by parts, we obtain:

$$I_1 (v, \varepsilon) = \sum_{n=0}^{\infty} \frac{\partial}{\partial v} \left[ \left( \frac{v}{v_0} \right)^n \frac{1}{(1 + \varepsilon)^n} \int_{-1}^{1} \mu^{n+1} F_0 (v, \mu) \, d\mu \right].$$

$$I_2 (v, \varepsilon) = \frac{1}{v} \sum_{n=0}^{\infty} \left( \frac{v}{v_0} \right)^n \frac{1}{(1 + \varepsilon)^n} \left[ (n + 2) \int_{-1}^{1} \mu^{n+1} F_0 (v, \mu) \, d\mu \right].$$

In the case of a flat cathode, when the plasma electrons are reflected from the cathode according to the mirror law, then $F_0 (v, \mu)$ at the location of the cathode (where the primary loss of stability of the LVBD occurs) is an even function of $\mu$. Then the formulas (B4) change as follows:

$$I_1 (v, \varepsilon) = -\frac{1}{v} \sum_{j=0}^{\infty} \frac{\partial}{\partial v} \left[ \left( \frac{v}{v_0} \right)^{2j+1} \frac{1}{(1 + \varepsilon)^{2j+1}} \int_{-1}^{1} \mu^{2j+1} F_0 (v, \mu) \, d\mu \right].$$

$$I_2 (v, \varepsilon) = \frac{1}{v} \sum_{j=0}^{\infty} \left( \frac{v}{v_0} \right)^{2j+1} \frac{1}{(1 + \varepsilon)^{2j+1}} \left( (2j + 3) \int_{-1}^{1} \mu^{2j+1} F_0 (v, \mu) \, d\mu \right).$$

Let us assume that the plasma EDF can be represented as:

$$F_0 (v, \mu) = \frac{m_0}{2\pi} \Phi_0 (v) M_0 (\mu),$$

$$\int_{-1}^{1} M_0 (\mu) \, d\mu = \int_{0}^{\infty} \nu^2 \Phi_0 (v) \, dv = 1.$$

In this case, using (B5), one can get:

$$\int_{0}^{\infty} v^2 I_1 (v, \varepsilon) \, dv = -\frac{1}{v_0} \sum_{j=0}^{\infty} \frac{(2j + 1) \mu^{2j}}{(1 + \varepsilon)^{2j+1}} \left( \frac{E}{E_0} \right)^{j-1} ;$$

$$\int_{0}^{\infty} v^2 I_2 (v, \varepsilon) \, dv = \frac{1}{v_0} \sum_{j=0}^{\infty} \frac{(2j + 1) \mu^{2j}}{(1 + \varepsilon)^{2j+1}} \left( \frac{E}{E_0} \right)^{j-1} - \frac{1}{v_0} \sum_{j=0}^{\infty} \frac{(2j - 1) \mu^{2j-2}}{(1 + \varepsilon)^{2j+1}} \left( \frac{E}{E_0} \right)^{j-1} ,$$

where $\mu^{2j}$ and $\left( \frac{E}{E_0} \right)^{j-1}$ are the mean values of $\mu^{2j}$ and $\left( \frac{E}{E_0} \right)^{j-1}$ calculated for the plasma EDF $F_0 (v, \mu)$. Hence from (B1) we have for $I_1 (\varepsilon)$:

$$I_1 (\varepsilon) = -\frac{\omega_D}{\gamma v_0^2} \sum_{j=0}^{\infty} \frac{(2j - 1) \mu^{2j-2}}{(1 + \varepsilon)^{2j}} \left( \frac{E}{E_0} \right)^{j-1} .$$

In the case where $F_0 (v, \mu)$ is not even, taking into account (B4), it is satisfied that:

$$I_1 (\varepsilon) = -\frac{\omega_D}{\gamma v_0^2} \sum_{n=1}^{\infty} \frac{n \mu^{n-1}}{(1 + \varepsilon)^{n+1}} \left( \frac{E}{E_0} \right)^{\frac{n}{2} - 1} .$$

For an isotropic EDF, when $M_0 (\mu) = 0.5$, it is easy to obtain:

$$\mu^{2j} = \frac{1}{2j - 1} .$$

Then (B8) and (B9) can be rewritten as:

$$I_1 (\varepsilon) = -\frac{\omega_D}{\gamma v_0^2} \sum_{j=0}^{\infty} \frac{1}{(1 + \varepsilon)^{2j}} \left( \frac{E}{E_0} \right)^{j-1} .$$

Let us calculate the value $\left( \frac{E}{E_0} \right)^{k-1}$ for the Maxwellian function

$$F_0^m (v, \mu) = A \exp \left( -\frac{mv^2}{2kT_i} \right) ; \quad A = \left( \frac{m}{2\pi kT_i} \right)^{3/2} .$$

(B11)
where $T_i$ and $k$ are the electron temperature in the plasma and the Boltzmann constant, respectively. Omitting intermediate calculations, we get:

$$
\left( \frac{E}{E_0} \right)^j = \left( \frac{2}{3} \right)^j \Gamma \left( \frac{j + 3/2}{3/2} \right) \left( \frac{E_1}{E_0} \right)^j; \quad E_i = \frac{3}{2} kT_i,
$$

(B12)

where $\Gamma (x)$ is the gamma function of the argument $x$ [25].

Using the well-known expression for $\Gamma \left( \frac{n + 1/2}{2} \right)$ (where $n$ is a positive integer) through $n$ [25] in the case of the Maxwellian EDF in the plasma, we have the formal equality:

$$
I_1 \left( \varepsilon \right) = \left( \frac{\omega_D}{\gamma v_0^2} \right) \sum_{j=1}^{\infty} a_j \left( \frac{E_i}{E_0} \right) \varepsilon \; ; \quad a_j \left( \frac{E_i}{E_0} \right) = \frac{(2j - 1)!!}{3^{j-1}(1 + \varepsilon)^{2j}} \left( \frac{E_1}{E_0} \right)^{j-1}.
$$

(B13)

It should be noted that, strictly speaking, this series diverges for any value of the parameter $\frac{E_1}{E_0}$ and $|\varepsilon| > 1$. Indeed, using the Stirling formula for the quantity $j!$ when $j \to \infty$ and noting that

$$
(2j - 1)!! = \frac{(2j - 1)!}{2^{j-1} \cdot (j - 1)!},
$$

we get

$$
\lim_{j \to \infty} \left( 2j - 1 \right)!! \left( \frac{E_1}{E_0} \right)^{j-1} = \lim_{j \to \infty} \left[ \text{const} \left( \frac{2}{3} \varepsilon \right)^{j-1} \right] = \infty
$$

for any, arbitrarily small values of $\frac{E_1}{E_0}$. However, it can be shown that the series (B13) for $\frac{E_1}{E_0} \ll 1$ is an asymptotic expansion for the quantity $I_1 \left( \varepsilon \right)$ and, therefore, describes it quite accurately. Indeed, according to the definition, an asymptotic series of a function $f \left( x \right)$ is a series $\sum_{n=0}^{\infty} b_n \left( x \right)$, if it holds that [25]:

$$
f \left( x \right) = \sum_{n=1}^{N} b_n \left( x \right) + o \left( b_N \left( x \right) \right); \quad b_{n+1} \left( x \right) = o \left( b_n \left( x \right) \right),
$$

(B14)

were $\lim_{x \to \infty} \frac{b_i}{x} = 0$.

It is easy to show that both of these relations are true for

$$
x = \frac{E_i}{E_0}; \quad f \left( \frac{E_i}{E_0} \varepsilon \right) = I_1 \left( \varepsilon \right) \; ; \quad b_n \left( \frac{E_i}{E_0} \varepsilon \right) = \frac{\omega_D}{\gamma v_0^2} a_n \left( \frac{E_1}{E_0} \varepsilon \right).
$$

Thus, for the Maxwellian EDF, one can write:

$$
I_1 \left( \varepsilon \right) \approx \frac{\omega_D}{\gamma v_0^2} \sum_{j=1}^{N_f} a_j \left( \frac{E_i}{E_0} \varepsilon \right),
$$

(B15)

where $N_f > 0$ is some integer.

The accuracy of formula (B15) depends on the number $N_f$, which, in turn, is determined by the parameter $\frac{E_1}{E_0}$. This fact is discussed in more detail in the main body of the text.

The physical reason for the divergence of the series (B13) is that, strictly speaking, for an arbitrarily small parameter $\frac{E_1}{E_0}$ there are electrons in the plasma with velocities for which the convergence condition (B3) is not satisfied. However, when the inequality $\frac{E_1}{E_0} \ll 1$ holds, the number of these electrons is negligible. The quantitative expression of this fact is the existence of an asymptotic series (B15) for the expression $I_1 \left( \varepsilon \right)$.

It is quite obvious that for any other physically well-founded plasma EDF (and not only for Maxwell’s) there are asymptotic series of the type (B15). It is clear that, the faster the EDF falls off in the region of high velocities, the wider the range of applicability of these series, which can be obtained using the above algorithm. It should be also borne in mind that in the calculations it is necessary to separately study the accuracy of the approximation of the function $I_1 \left( \varepsilon \right)$ by such series.

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