HOMOLOGICAL MIRROR SYMMETRY WITHOUT CORRECTION

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Abstract. Let $X$ be a closed symplectic manifold equipped a Lagrangian torus fibration. A construction first considered by Kontsevich and Soibelman produces from this data a rigid analytic space $Y$, which can be considered as a variant of the $T$-dual introduced by Strominger, Yau, and Zaslow. We prove that the Fukaya category of $X$ embeds fully faithfully in the derived category of coherent sheaves on $Y$, under the technical assumption that $\pi_2(X)$ vanishes (all known examples satisfy this assumption). The main new tool is the construction and computation of Floer cohomology groups of Lagrangian fibres equipped with topologised infinite rank local systems that correspond, under mirror symmetry, to the affinoid rings introduced by Tate, equipped with their natural topologies as Banach algebras.

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1. Introduction

1.1. Statement of the main result. Let $X$ be a closed symplectic manifold. One of the key tools used to understand the symplectic topology of $X$ is its Fukaya category, placing the computations of this category as a central problem in the subject. Such computations would ideally rely on geometric features of $X$. For example, the fact that cotangent bundles are fibered in Lagrangian planes ultimately accounts for the computation of their (derived) Fukaya categories as categories of modules over the chains on the based loop space of the base $\mathbb{I}$.

The case of a symplectic manifold equipped with a Lagrangian torus fibration has been the focus of much interest because of its relevance to mirror symmetry via the Strominger-Yau-Zaslow conjecture $\mathbb{I}$7. Kontsevich and Soibelman $\mathbb{I}$3 where the first to propose a mathematically precise conjecture in this context: under the assumption that the torus
fibration admits a Lagrangian section, they showed that one can associate a rigid analytic
space $Y$ (in the sense of Tate) to every such symplectic manifold, and conjectured that
the Fukaya category is equivalent to the category of (rigid analytic) coherent sheaves on
$Y$. They also took the first step in this direction by assigning line bundles on $Y$ to each
Lagrangian section of $X$, and comparing the multiplication on the Floer cohomology groups
of Lagrangian sections with the product of the sections of the corresponding bundles.

The next step was taken by Fukaya in [9], showing in complete generality that the Floer
cohomology of Lagrangian submanifolds varies \textit{analytically} with respect to the natural co-
ordinates on the space of Lagrangians coming from the flux homomorphism. Fukaya’s work
was phrased in terms of the self-Floer cohomology of Lagrangians, but a minor adaptation
shows that one can assign to an object $L$ of the Fukaya category of $X$ a sheaf $\mathcal{L}_L$ of coherent
analytic complexes on the mirror space [4]. To avoid technical difficulties, we shall from
now on restrict attention to symplectic manifolds with vanishing second homotopy group,
and consider only the subcategory of the Fukaya category consisting of \textit{tautologically unob-
structed Lagrangians}, i.e. those for which there is a choice of almost complex structure so
that all holomorphic discs are constant.

In this context, the family Floer functor was shown to be faithful in [5]. More precisely,
associated to a Lagrangian torus fibration is a gerbe $\beta$ on $Y$ classified by a class in $H^2(Y, O^*)$.
An $A_\infty$ functor from the Fukaya category of $X$ to the derived category of $\beta$-twisted coherent
sheaves on $Y$ was constructed, and the corresponding family Floer map
\begin{equation}
HF^*(L, L') \to \text{Hom}_Y(\mathcal{L}_{L'}, \mathcal{L}_L)
\end{equation}
was proved to be injective for every pair of Lagrangians. Here, and in contract to the rest
of the paper, we use the notation $\text{Hom}_Y$ to indicate morphisms in the derived category of
coherent sheaves on $Y$, without specifying a model for this category. The main result of this
paper is

\begin{theorem}
The family Floer map is surjective. In particular, there is a fully faithful
embedding of the Fukaya category of $X$ in the $\beta$-twisted derived category of coherent sheaves
on $Y$.
\end{theorem}

The main deficiencies of this result are its restrictive assumptions that (1) all Lagrangians
are tautologically unobstructed, and (2) the ambient symplectic manifold is equipped with
a \textit{non-singular} fibration. Removing the first assumption would require the use of a package
of virtual fundamental chains; the one developed by Fukaya-Oh-Ohta-Ono [10] would be
sufficient for the task at hand. The decision not to use it to prove a theorem for general
Lagrangians amounts to the desire not to add another layer of complexity to the paper. On
the other hand, admitting singular Lagrangians as fibres, e.g. immersed Lagrangians, will
require some new insights about Floer theory in families, though the first steps have been
taken by Fukaya in his announced results about mirror symmetry for K3 surfaces.

1.2. A summary of the proof.

The proof of Theorem 1.1 is based on the following idea:
the linear dual to $\mathcal{L}_L$ can also be thought of as a ($\beta$-twisted) complex of sheaves $\mathcal{R}_L$ on $Y$,
and there is a natural map, which was essentially already used in [5], from the cohomology
of the derived tensor product of these sheaves to the Floer cohomology
\begin{equation}
\mathcal{R}_{L'} \otimes_Y \mathcal{L}_L \to HF^*(L, L').
\end{equation}
Surjectivity of Equation (1.1.1) can thus be deduced from the subjectivity of the composition
\begin{equation}
\mathcal{R}_{L'} \otimes_Y \mathcal{L}_L \to HF^*(L, L') \to \text{Hom}_Y(\mathcal{L}_{L'}, \mathcal{L}_L).
\end{equation}
In fact, we shall show that this composition is an isomorphism.

The implementation of this strategy turns out to be particularly complicated; the goal of the remainder of this introduction to is indicate what the difficulties are, and how they are bypassed. We begin by specifying that as in [5] our model for the derived category of $Y$ will be of Čech nature: we fix a polyhedral cover $\{P_\sigma\}_{\sigma \in \Sigma}$ of the base $Q$ of the torus fibration, which gives rise to an affinoid cover of $Y$. A classical result of Tate [19] implies that the derived category of coherent sheaves on each of these affinoid domains is equivalent to the category of modules over the corresponding ring of functions, so that we can describe the derived category of coherent sheaves on $Y$ in terms of modules over a category $\mathcal{F}$ with objects labelled by iterated intersections of elements of the cover, in which the only morphisms are associated to inclusions. In the most basic case of a space covered by two affinoid domains, this encodes the idea that a (coherent) sheaf on the ambient space can be thought of as a pair of modules over the rings of function on the two elements of the cover, a module over the ring of functions on the intersection, together with an isomorphism between the module associated to the intersection and the restrictions of the module for each piece of the cover. This is equivalent, though less economical, than the data of the two modules together with an isomorphism between their restrictions, but it proves to be much more convenient when formulating the corresponding notion at the level of cochain complexes.

The starting point of the results of [5] is that the category $\mathcal{F}$ has a symplectic interpretation as follows: picking a basepoint $q_\sigma \in P_\sigma$, we can identify the cohomology $H_\mathcal{F}_{L}(\sigma)$ with the Floer cohomology of $L$ with the fibre $X_{q_\sigma}$ equipped with an (infinite) rank local system which depends on the polytope $P_\sigma$, and which can be expressed as a completion of the group ring of $X_{q_\sigma}$. The structure maps of the modules associated to $L$, as well as the maps in Equation (1.2.2), can then be interpreted as maps associated to families of holomorphic discs with boundary conditions on $L$ and a collection of fibres parametrised by families of paths interpolating between the basepoints chosen for each element of the cover. We can thus associate to this composition the picture on the right of Figure 1, where the label by polytopes indicates that we are considering the corresponding family of Lagrangian fibres. Each disc is labelled by the corresponding map, defined in Section 2, which gives a more precise formulation of the ideas being sketched in this introduction.

Figure 1 thus shows a picture cobordism between the moduli spaces of discs used to define the two maps we are considering and the moduli space of discs which we should try to use to prove that the composition is an isomorphism. The essential problem is to give a meaning, both in geometry and algebra, to the picture shown on the left. At the level of
geometry, the naive idea of considering families of fibres over the two polytopes runs into
a transversality problem, because two fibres intersect if and only if they are equal, and it
seems difficult to arrange for such a parametrised problem (in which the intersections
of Lagrangians change), to give moduli spaces of the correct dimension.

As is standard in Floer theory, we resolve the first problem by introducing Hamiltonian
perturbations, and associate to the node in Figure 1 with label \((P_{\tau}, P_\sigma)\) the set of intersec-
tions between a fibre over \(P_{\tau}\), and the image of a fibre over \(P_\sigma\) under a Hamiltonian isotopy.
This leads to two new problems, which are responsible for the first two real innovations of
this paper. The first problem is to ensure that the Floer cohomology groups associated to a
perturbed pair can be given appropriate meaning under mirror symmetry. At the least, we
need to know that the perturbed Floer cohomology of the pair \(P_\sigma = P_{\tau}\) is isomorphic to the
affinoid ring associated to the corresponding subset of \(Y\). The naive idea of defining such a
group as a Floer cohomology of Lagrangians equipped with local systems (see, e.g. \cite{3} for
the infinite rank case relevant to this setup) does not succeed, and produces a group that
is far larger than the desired one. This is analogous to the statement that, if we consider a
power series ring \(k[[x]]\) as a module over the polynomial ring \(k[x]\) in the natural way, then the map

\[
(1.2.3) \quad k[[x]] \rightarrow \text{Hom}_{k[x]}(k[[x]], k[[x]])
\]

is far from being an isomorphism. A solution is offered by the fact that a power series ring
acquires a natural topology from its description as an inverse limit, with respect to which
it is complete. If we consider instead \textit{continuous} morphisms, then the above map becomes
an isomorphism.

We are thus led to consider the Floer theory of Lagrangians equipped with \textit{topological}
local systems, keeping the topologies into account when defining Floer cohomology groups (see
Section 2.2). The idea of incorporating topological vector spaces in the study of Floer theory
probably goes back to Fukaya \cite{8}, who intended to use it to study Lagrangian foliations.
We do not develop the general theory, limiting ourselves to those properties required for the
proof of the main theorem. Among the indications that this approach gives the right answer
is that we succeed in proving that (i) the self-Floer cohomology of a Lagrangian with such
local systems can be computed using either a Morse-theoretic model or an appropriate
Hamiltonian perturbation (see Section 2.6.2) and (ii) the Floer cohomology associated to a
pair of polytopes in the base depends only on a neighbourhood of their intersections (see
Section 2.2.6).

This leads us to the second problem: the Floer cohomology of Lagrangians equipped
with the topological local systems that we consider is, at first sight, not invariant under
Hamiltonian isotopies, as is implicit in the statement that we can use any fibre over \(P_\sigma\) as
basepoint: distinct fibres are disjoint, hence should have trivial Floer cohomology. However,
an outcome of this paper is that, after perturbation, the Floer cohomology group for the
pair, equipped with the appropriate local systems, is isomorphic to the ring of functions
on the mirror affinoid domain. Unfortunately, the techniques used in \cite{5}, elaborating on
Fukaya’s ideas from \cite{9}, to prove the corresponding invariance statement in the construction
of the family Floer functor do not seem to be adapted to this problem.

The solution is to use the reverse isoperimetric inequality of Gromov and Solomon \cite{11}
(with a simplified proof by DuVal \cite{7}), in order to prove the desired invariance. The basic
idea is that the completions we consider can be expressed in terms of the minimal length of
the representative of any homotopy class of paths in a Lagrangian fibre, and that the reverse
isoperimetric inequality gives a bound (from above) for the lengths of the paths arising as
the boundary of a holomorphic disc of given energy. While previous work only considered the case of a single Lagrangian (rather than a pair), the method of DuVal is sufficiently flexible to immediately yield the generalisations we need, as explained in Appendix A and first implemented in Section 2.5.2.

With this at hand, we would like to interpret the picture on the right of Figure 1 as a factorisation:

\[
R_{L'} \otimes Y_L \to \text{Hom}_Y(\mathcal{L}_{L'}, \Delta) \otimes Y_L \to \text{Hom}_Y(\mathcal{L}_{L'}, \mathcal{L}_L),
\]

where \(\Delta\) corresponds to the geometric diagonal of \(Y\). At this point, we encounter the final difficulty: the model of sheaves on \(Y\) provided by modules over \(\mathcal{F}\) is not adequate for this argument. The key problem is that we have set morphisms in \(\mathcal{F}\) from \(P_{\tau}\) to \(P_{\sigma}\) to vanish whenever \(P_{\tau}\) does not contain \(P_{\sigma}\), but the corresponding Floer cohomology groups only vanish if the polytopes are disjoint. On the algebraic side, the intuition is that Floer theory recovers a model for an enlargement of the category of coherent sheaves on \(Y\), which admits as objects pushforwards of the structure sheaves of affinoid sub-domains. In algebraic geometry, this would be handled by considering the category of quasi-coherent sheaves, but the analogue in the analytic setting is poorly understood.

While it would be possible to proceed along this route, and construct a category \(P_0\) in which morphisms are not artificially set to vanish for non-inclusions, we choose instead a shortcut that allows us to study only the local analogue of this category. To this end, we introduce a bimodule over \(\mathcal{F}\), which would correspond to the pullback of the diagonal of \(P_0\) under the natural embedding \(\mathcal{F} \to P_0\), and which will play the role of the geometric diagonal of \(Y\).

Having arranged for Figure 1 to correspond to a cobordism of moduli spaces of curves, and for the left hand side to have an algebraic interpretation, we are left with the problem of showing that this alternate factorisation of the composition is an isomorphism. This is proved by showing that the corresponding maps can be computed locally in \(Q\), i.e. that the value on a polytope \(P_{\sigma}\) is determined by the restriction to the subcategory \(\mathcal{F}_\sigma\) of \(\mathcal{F}\) consisting of polytopes which intersect \(P_{\sigma}\). Having reduced the computation to a local problem, we then introduce a local category which extends \(\mathcal{F}_\sigma\) by eliminating the requirement that morphisms vanish for non-inclusions. The basic idea is then to extend \(R_L\) to a right module over this local category, and prove that it is the bimodule dual of \(\mathcal{L}_L\) (see Section 2.4.2). If \(Y_{\sigma} \subset Y\) is the corresponding analytic subspace, this essentially amounts to showing that the map

\[
R_L \otimes_{Y_{\sigma}} \mathcal{L}_{L'} \to \text{Hom}_{Y_{\sigma}}(\mathcal{L}_{L}, \Delta) \otimes_{Y_{\sigma}} \mathcal{L}_{L'},
\]

is an isomorphism. To show that the local analogue of the second map in Equation (1.2.4) is an isomorphism, we use basic algebraic arguments that amount to the statement that \(\mathcal{L}_L\) corresponds to a complex of coherent sheaves.

To complete the argument, we still have to compare the computations associated to the local category with those associated to \(\mathcal{F}_\sigma\). This ends up being a variant of the acyclicity result of Tate alluded to above, which implies that a Čech model associated to a fixed affinoid cover computes the correct (derived) space of maps between complexes of coherent sheaves, so that the coherence of \(\mathcal{L}_L\) again plays a crucial role.

1.3. Outline of the paper. Theorem 1.1 is proved in Section 4.1 which constructs the various \(A_\infty\) operations required to make precise the above outline of the proof. This in turn relies on the usual constructions of families of pseudo-holomorphic equations, constructed compatibly over various abstract moduli spaces, as explained in the preceding Section 3.
Because of the inherent combinatorial complexity of such constructions, we have chosen to precede the $A_\infty$ constructions by Section 2 which constructs all the operations at the cohomological level, and in fact proves a weak version of Theorem 1.1 for Lagrangian sections. Given the content of Section 2 and the three Appendices on reverse isoperimetric inequalities, Tate’s acyclicity results, and the computation of Floer cohomology groups with coefficients in completions of the homology of the group ring of the torus, filling in the rest of the paper is just a matter of following one’s nose.

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Notation. Let $k$ be a field, and $\Lambda$ the corresponding Novikov field consisting of series
\begin{equation}
\sum_{\lambda \in \mathbb{R}} c_\lambda T^\lambda, c_\lambda \in k
\end{equation}
with the property that the set of exponents $\lambda$ for which the coefficients $c_\lambda$ do not vanish is discrete and bounded below.

The ring $\Lambda$ admits a valuation
\begin{equation}
\text{val}(\sum_{\lambda \in \mathbb{R}} c_\lambda T^\lambda) = \lambda_0
\end{equation}
where $\lambda_0$ is the smallest exponent whose coefficient does not vanish (we set $\text{val} 0 = +\infty$).

2. Cohomological constructions

2.1. Statement of the main results. Let $X \to Q$ be a Lagrangian $n$-torus fibration. In order to bypass difficulties involving the foundations of holomorphic curve theory, we shall assume that $\pi_2(Q) = 0$. As we are interested in the Floer theory of Lagrangians in $X$, we consider a (finite) collection $A$ of Lagrangians in $X$, such that, for each $L \in A$, we have
\begin{equation}
\text{an almost complex structure } J_L \text{ with respect to which all holomorphic discs with boundary on } L \text{ are constant.}
\end{equation}
Moreover, we shall assume for simplicity that all Lagrangians in $A$ are mutually transverse.

Recall that, by the Arnol’d-Liouville theorem, we have a lattice $T^*_q X \subset T^*_q Q$, and an isomorphism
\begin{equation}
T^*_q Q/T^*_q Z \cong X_q
\end{equation}
which is canonical up to translation in the left hand side. In particular, a metric on $Q$ induces a canonical flat metric on $X_q$ for all $q \in Q$. We shall fix such a metric in the remainder of the paper.

By passing to homology, the Arnol’d-Liouville map gives rise to an isomorphism
\begin{equation}
T_q Q \cong H^1(X_q, \mathbb{R})
\end{equation}
where $X_q$ is the fibre over $q \in Q$. With respect to this map, we have a natural isomorphism of lattices
\begin{equation}
T_q^\mathbb{Z}Q \cong H^1(X_q, \mathbb{Z}),
\end{equation}
where $T_q^\mathbb{Z}Q$ is the lattice dual to $T_q^\ast \mathbb{Z}$. This lattice arises from an integral affine structure in the sense that there is an affine exponential map
\begin{equation}
H^1(X_q, \mathbb{R}) \to \mathbb{Q},
\end{equation}
which is a diffeomorphism near 0, and which induces an isomorphism of lattices at each point.

**Definition 2.1.** An integral affine polytope $P \subset \mathbb{Q}$ is the image of a polytope in $H^1(X_q, \mathbb{R})$ which is defined by inequalities $\langle \cdot, \alpha_i \rangle \geq \lambda_i$ with $\alpha_i \in H_1(X_q, \mathbb{Z})$ and $\lambda_i \in \mathbb{R}$.

As recalled in Section 2.2.1 below, we may associate to each such polytope a ring $\Gamma^P$ which is an affinoid ring in the sense of Tate [19].

We consider a finite partially ordered set $\Sigma$ such that the length of any totally ordered subset is bounded by $n + 1$, indexing a cover $\{P_\sigma\}_{\sigma \in \Sigma}$ of $Q$ by integral affine polytopes of sufficiently small diameter, and equipped with basepoints $q_\sigma \in P_\sigma$, such that $P_\sigma \subset P_\tau$ if $\tau \leq \sigma$, and
\begin{equation}
Q = \bigsqcup_{\sigma \in \Sigma} P_\sigma/\sim,
\end{equation}
where the equivalence relation identifies a point in $P_\sigma$ with its image in $P_\tau$ if $\tau \leq \sigma$. For example, starting with a finite cover of $Q$ dual to a triangulation, we can define $\Sigma$ to the set of subsets of the index set, ordered by inclusion, so that $P_\sigma$ is the intersection of the elements of the cover appearing in the label.

In this section, which can be read as an extended introduction, we construct the following structures at the cohomological level which we shall later lift to the $A_\infty$ level:

1. A category $\mathcal{H}F$ whose objects are elements $\sigma \in \Sigma$, with $\mathcal{H}F(\tau, \sigma)$ vanishing unless $\tau \leq \sigma$, and otherwise given by a Floer cohomology group $HF^\ast(P_\tau, P_\sigma)$ which is isomorphic to the affinoid ring $\Gamma^P$.
2. For each Lagrangian $L \in \mathcal{A}$ and $\sigma \in \Sigma$, Floer cohomology groups $HF^\ast(P_\sigma, L)$ and $HF^\ast(L, P_\sigma)$ which give rise to right and left modules $H\mathcal{R}_L$ and $H\mathcal{L}_L$ over $\mathcal{H}F$ (see Section 2.3).
3. For each pair of Lagrangians $(L, L')$ in $\mathcal{A}$, a map
\begin{equation}
HF^\ast(L, L') \to \text{Hom}_H(HF^\ast(L', P), HF^\ast(L, P))
\end{equation}
inducing a map into the space of left-module maps:
\begin{equation}
HF^\ast(L, L') \to \text{Hom}_{\mathcal{H}F}(H\mathcal{L}_{L'}, H\mathcal{L}_L).
\end{equation}
We define as well a map
\begin{equation}
HF^\ast(P_\sigma, L') \otimes HF^\ast(L, P_\sigma) \to HF^\ast(L, L')
\end{equation}
which is compatible with the action of morphisms in $\mathcal{H}F$, in the sense that we have an induced map
\begin{equation}
H\mathcal{R}_{L'} \otimes_{\mathcal{H}F} H\mathcal{L}_L \to HF^\ast(L, L').
\end{equation}

At the cohomological level, the structures listed above are the same as those considered in [5]. Our chain-level construction however will be different as we shall need to consider the following additional data:
(4) A \textit{geometric diagonal} bimodule $H\Delta_{H,F}$ over $H,F$, given for a pair $(\tau, \sigma)$ by a Floer cohomology group $HF^*(P_\tau, P_\sigma)$. The key point here is that these groups may be non-vanishing even if the condition $\tau \leq \sigma$ does not hold.

(5) For each $L \in A$, and for each pair $(\tau, \sigma)$ of polytopes, a map
\begin{equation}
HF^*(L, P_\tau) \otimes_{\Lambda} HF^*(P_\sigma, L) \to HF^*(P_\tau, P_\sigma),
\end{equation}
which descends to a map of bimodules:
\begin{equation}
H\mathcal{L}_L \otimes H\mathcal{R}_L \to H\Delta_{H,F}.
\end{equation}

(6) Finally, we construct a map
\begin{equation}
HF^*(P_\tau, P_\sigma) \otimes HF^*(L, P_\tau) \to HF^*(L, P_\sigma),
\end{equation}
which gives rise to a map of left modules
\begin{equation}
\Delta_{H,F} \otimes_{H,F} H\mathcal{L}_L \to H\mathcal{L}_L.
\end{equation}

For the statement of the main result of this section, it will be convenient to interpret Equation (2.1.12) as a map of right modules:
\begin{equation}
H\mathcal{R}_L \to \text{Hom}_{H,F}(H\mathcal{L}_L, \Delta_{H,F}).
\end{equation}

\textbf{Proposition 2.2.} Given pairs $(\tau, \sigma)$ in $\Sigma$ and Lagrangians $(L, L') \in A$, we have a commutative diagram
\begin{equation}
\begin{array}{ccc}
HF^*(P_\tau, L') \otimes HF^*(L, P_\tau) & \longrightarrow & HF^*(L, L') \\
\downarrow & & \downarrow \\
\text{Hom}_\Lambda(HF^*(L', P_\sigma), HF^*(P_\tau, P_\sigma)) \otimes HF^*(L, P_\tau) & \longrightarrow & \text{Hom}_\Lambda(HF^*(L', P_\sigma), HF^*(L, P_\sigma)) \\
\downarrow & & \downarrow \\
\text{Hom}_\Lambda(HF^*(L', P_\sigma), HF^*(P_\tau, P_\sigma) \otimes HF^*(L, P_\tau)). & & \end{array}
\end{equation}

Allowing arbitrary pairs in $\Sigma$, we obtain a commutative diagram
\begin{equation}
\begin{array}{ccc}
H\mathcal{R}_{L'} \otimes_{H,F} H\mathcal{L}_L & \longrightarrow & HF^*(L, L') \\
\downarrow & & \downarrow \\
\text{Hom}_{H,F}(H\mathcal{L}_{L'}, \Delta_{H,F}) \otimes_{H,F} H\mathcal{L}_L & \longrightarrow & \text{Hom}_{H,F}(H\mathcal{L}_{L'}, H\Delta_{H,F}) \\
\downarrow & & \downarrow \\
\text{Hom}_{H,F}(H\mathcal{L}_{L'}, H\Delta_{H,F} \otimes_{H,F} H\mathcal{L}_L). & & \end{array}
\end{equation}

This result is proved in Section 2.7.2 and is illustrated in Figure 1.

\textbf{Remark 2.3.} The notation that we use hides the following complication: the morphism spaces in $H,F$ will be defined using a Morse-theoretic model of Floer cohomology. On the other hand, the construction of the bimodule $\Delta_{H,F}$ will be genuinely Floer-theoretic as it is defined using Hamiltonian perturbations.

\textbf{Corollary 2.4.} If Equations (2.1.15) and (2.1.14) and the map
\begin{equation}
\text{Hom}_{H,F}(H\mathcal{L}_{L'}, H\Delta_{H,F} \otimes_{H,F} H\mathcal{L}_L) \to \text{Hom}_{H,F}(H\mathcal{L}_{L'}, H\Delta_{H,F} \otimes_{H,F} H\mathcal{L}_L)
\end{equation}
are isomorphisms, then Equation (2.1.8) is surjective.
We shall see that the conditions of Corollary 2.3 hold whenever \( L \) and \( L' \) are Lagrangian sections; we shall discuss the proofs, because they serve as models for the \( A_\infty \) analogues which hold in general.

The fact that Equation (2.1.18) is an isomorphism amounts to the statement that the module \( H_\Sigma \) behaves with respect to the functors \( \hom \) and \( \otimes \) in much the same way as a projective module over a ring, whenever \( L \) is a Lagrangian section. This is of course not surprising, because a Lagrangian section gives rise, under mirror symmetry, to a line bundle.

Next, we use the fact that the Floer cohomology groups \( HF^*(P_\tau, P_\sigma) \) vanish whenever the inputs are disjoint. This leads us to introduce, for each \( \sigma \in \Sigma \), a full subcategory \( H \mathcal{F}_\sigma \) of \( \mathcal{F} \), whose set of objects \( \Sigma_\sigma \) consists of elements of the cover which intersect \( P_\sigma \). We have the following elementary result (for the statement, we abuse notation by using the same notation for a module on \( F \) and its restriction to \( F_\sigma \)):

**Lemma 2.5.** For each left module \( \mathcal{L} \) over \( H \mathcal{F} \), the natural maps

\[
\begin{align*}
\hom_{H \mathcal{F}}(\mathcal{L}, \Sigma_{H \mathcal{F}}(\sigma, \tau)) & \to \hom_{H \mathcal{F}_\sigma}(\mathcal{L}, \Sigma_{H \mathcal{F}}(\sigma, \tau)) \\
\Sigma_{H \mathcal{F}}(\tau, \sigma) \otimes_{H \mathcal{F}_\sigma} \mathcal{L} & \to \Sigma_{H \mathcal{F}}(\tau, \sigma) \otimes_{H \mathcal{F}} \mathcal{L}
\end{align*}
\]

are isomorphisms.

**Proof.** The two arguments are entirely analogous; we explain the second. We can describe the tensor product over \( H \mathcal{F} \) as the cokernel of the map

\[
\bigoplus \Sigma_{H \mathcal{F}}(\tau-0, \sigma) \otimes_{H \mathcal{F}} (\tau-1, \tau-0) \otimes \mathcal{L}(\tau-1) \to \bigoplus \Sigma_{H \mathcal{F}}(\tau, \sigma) \otimes \mathcal{L}(\tau),
\]

with the direct sum being taken over objects of \( H \mathcal{F} \), and the arrow being given by the difference between the two compositions. The key fact is that, whenever \( H \mathcal{F}(\tau-1, \tau-0) \neq 0 \), the polytope \( P_{\tau-0} \) is contained in \( P_{\tau-1} \), so that \( \tau-0 \) being an object of \( H \mathcal{F}_\sigma \) implies the same for \( \tau-1 \). In particular, both direct sums are in fact taken over objects of \( H \mathcal{F}_\sigma \), which implies the desired isomorphism. \( \square \)

Because of the above result, whose \( A_\infty \) generalisation holds with the same proof, verifying that the analogues of Equations (2.1.15) and (2.1.14) are isomorphisms is a local computation, in the sense that the corresponding computations involve the each of the categories \( H \mathcal{F}_\sigma \) separately.

To perform these local computations, it is useful to introduce a category \( H \mathcal{P}_\sigma \), whose objects are the polytopes of \( Q \) which are contained in a polygonal neighbourhood of \( P_\sigma \) that itself contains the open star of \( \sigma \) with respect to the cover \( \Sigma \) (i.e. all \( P_\tau \) intersecting \( P_\sigma \) non-trivially). The morphisms in \( H \mathcal{P}_\sigma \) are again given by Floer cohomology groups, and we have a faithful embedding

\[
j: \ H \mathcal{F}_\sigma \to H \mathcal{P}_\sigma,
\]

which is the inclusion of a directed subcategory. We shall then prove in Section 2.7 that the pullback of the diagonal bimodule of \( H \mathcal{P}_\sigma \) is naturally isomorphic to the restriction of \( \Sigma_{H \mathcal{F}} \) to \( H \mathcal{F}_\sigma \). Moreover, for each \( L \in \mathcal{A} \) there are left and right modules \( H \mathcal{L}_{L, \sigma} \) and \( H \mathcal{R}_{L, \sigma} \) over \( H \mathcal{P}_\sigma \) whose pullbacks to \( H \mathcal{F}_\sigma \) are naturally isomorphic to the restrictions of \( H \mathcal{L}_L \) and \( H \mathcal{R}_L \).

This allows us to reduce local computations to the category \( H \mathcal{P}_\sigma \) in certain special cases, and to the corresponding \( A_\infty \) category in general. For example, the proof that Equation (2.1.14) is an isomorphism reduces to the statement that the map of the tensor product of pullbacks of left and right modules

\[
j^* \Delta_{H \mathcal{P}_\sigma}(\sigma, \tau) \otimes_{H \mathcal{F}_\sigma} j^* (H \mathcal{L}_{L, \sigma}) \to j^* H \mathcal{L}_{L, \sigma}(\sigma)
\]

...
is an isomorphism. We shall prove that the above follows from Tate acyclicity assuming
that $L_\sigma$ meets $X_{q_\sigma}$ at a point (see Section 2.3.4).

2.2. The cohomological category of polytopes.

2.2.1. Loops, paths, and local systems. We fix, for each $\sigma \in \Sigma$, a local Lagrangian section
\begin{equation}
\iota_\sigma : \nu_Q \sigma \to X
\end{equation}
of the projection $X \to Q$, defined over a contractible neighbourhood $\nu_Q \sigma$ of $P_\sigma$. It is
important for later purposes to require that $\nu_Q \sigma$ contain $P_\tau$ whenever $\tau \in \Sigma_\sigma$, i.e. whenever
$P_\sigma \cap P_\tau$ is non-empty.

The first step is to associate to each polytope $P \subset \nu_Q \sigma$ a ring $\Gamma_P$. Given a point $q \in \nu_Q \sigma$, we shall then produce a local system $U_P$ on the fibre $X_q$.

Let us write $p - q$ for the element of $H_1(X_q, \mathbb{R})$ corresponding to a point $p \in P$ under
the (affine) chart based at $q$. We have a non-archimedean valuation on the group ring $\Gamma$ of
$\pi_1(X_q, \iota_\sigma(q))$ with coefficients in the Novikov field $\Lambda$, given by
\begin{equation}
\Gamma \equiv \Lambda[\pi_1(X_q, \iota(q))] \to \mathbb{R} \cup \{+\infty\}
\end{equation}
\begin{equation}
\text{val}_p \left( \sum c_\beta z^\beta \right) = \min_{\beta} \text{val}(c_\beta) + \langle \beta, p - q \rangle.
\end{equation}

Here, $\text{val}(c_\beta)$ denotes the valuation of this element of $\Lambda$ (see Equation (1.3.2)). Taking the
minimum over all elements of $P$, we obtain the valuation
\begin{equation}
\text{val}_P \left( \sum c_\beta z^\beta \right) = \min_{p \in P} \text{val}_p \left( \sum c_\beta z^\beta \right),
\end{equation}
and define the ring $\Gamma_q^P$ to be the completion of the group ring with respect to the corre-
sponding norm (i.e. consider series such that the number of terms with valuation bounded
above by any fixed constant is finite). Elements of this ring can be written uniquely as series
\begin{equation}
\sum_{\beta \in \pi_1(X_q, \iota(q))} c_\beta z^\beta
\end{equation}
satisfying the condition
\begin{equation}
\lim_{|\beta| \to +\infty} \text{val}_P \left( c_\beta z^\beta \right) = +\infty.
\end{equation}

Given another basepoint $\iota(q)$, concatenation with a path from $\iota(q)$ to $\iota_\sigma(q)$ defines an
isomorphism between the corresponding rings, which is well defined up to conjugacy. Since
$\pi_1(X_q, \iota(q))$ is abelian, we conclude that this ring is independent of the choice of basepoint
up to canonical isomorphism.

Lemma 2.6. There are natural isomorphisms $\Gamma_q^P \to \Gamma_q^{P'}$ for each pair of points $(q, q')$ in
$\nu_Q \sigma$, with the property that for every triple $(q, q', q'')$, we have a commutative diagram:
\begin{equation}
\begin{array}{ccc}
\Gamma_q^P & \to & \Gamma_{q''}^P \\
\downarrow & & \downarrow \\
\Gamma_q^{P'} & \to & \Gamma_{q''}^{P''}
\end{array}
\end{equation}

Proof. Since $\nu_Q \sigma$ is contractible, there is a canonical identification between the fundamendal
groups of the fibres $X_q$ based at $\iota_\sigma(q)$. Writing $z_\beta^q$ for the element of $\Gamma_q^P$ associated to a
class $\beta$ in this group, the map is given by
\begin{equation}
z_\beta^q \mapsto T^{(\beta, q' - q)} z_\beta^q
\end{equation}
The appearance of $\langle \beta, q' - q \rangle$ accounts for the flux of the isotopy between $X_q$ and $X_{q'}$. □

Given these canonical isomorphisms, we shall write $\Gamma^P$ for any of these rings, discarding the choice of point in $Q$.

We now produce the local systems: let $U_\sigma$ denote the local system on $X_q$ whose value at a point $x$ is the free $\Lambda$-module on the homotopy classes of paths starting at $\iota_\sigma(q)$ and ending at $x$,

$$U_{\sigma,x} = \Lambda[\pi_0(\Omega_{\iota_\sigma(q),x}X_q)].$$

If $P$ is an integral affine polytope in $T_qQ$, the tensor product (over $\Lambda[\pi_1(X_q,\iota_\sigma(q))]$) with $\Gamma^P$ defines a local system $U^P$, with fibre

$$U^P_{\sigma,x} = \Lambda[\pi_0(\Omega_{\iota_\sigma(q),x}X_q)] \otimes \Lambda[\pi_1(X_q,\iota_\sigma(q))] \Gamma^P.$$

Note that $U_{\sigma,x}$ is a free rank-1 module over $\Lambda[\pi_1(X_q,\iota_\sigma(q))]$, with a generator corresponding to a choice of homotopy class of paths from $\iota_\sigma(q)$ to $x$. Choosing such a path, we obtain a norm on $U_{\sigma,x}$, and $U^P_{\sigma,x}$ is the completion with respect to this norm.

In order to state the analogue of Lemma 2.6, we have:

**Lemma 2.7.** If $B$ is a contractible subset of $Q$ contained in $\nu_Q\sigma$, and $x: B \to X_B$ is a Lagrangian section, there is a canonical identification

$$U^P_{\sigma,x} \cong U^P_{\sigma,x'}$$

for all points $q, q' \in B$.

**Proof.** Pick a class $\gamma$ in $\pi_0(\Omega_{\iota_\sigma(q),x}X_q) \cong \pi_0(\Omega_{\iota_\sigma(q'),x}X_{q'})$. Since $B$ is contractible, there is a unique homotopy class of maps

$$u: [0,1]^2 \to X_B$$

such that the restrictions to $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ map to $X_q$ and $X_{q'}$ and represent the class $\gamma$, and the restrictions to $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ map to the sections $\iota_\sigma$ and $x$. We define

$$F_\gamma(q,q') = \int_{[0,1]^2} u^* \omega,$$

and note that this integral is independent of the choice of representative of the given homotopy class because all boundary conditions are Lagrangian. We then define the map from $U^P_{x}$ to $U^P_{x'}$ by

$$z^\gamma \mapsto T^{F_\gamma(q,q')}z^\gamma.$$  

As the proof shows, the identification from the previous Lemma is compatible with parallel transport maps in the following sense: we can associate to a homotopy class $\gamma$ of paths with endpoints $(x_0, x_1)$ on $X_q$, and to $q' \in B$ a real number $F_\gamma(q,q')$ which is the flux of this path. Under the identification of Equation (2.2.11), the parallel transport maps associated to this homotopy class of paths in $X_q$ and the corresponding homotopy class in $X_{q'}$ differ by multiplication by $T^{F_\gamma(q,q')}$. In the setting of the above Lemma, we shall therefore omit the superscript from $x^\gamma$ unless it is required for clarity of exposition.
A key aspect of the constructions of this paper will be the need to verify the $T$-adic convergence of operators constructed using holomorphic curves. Consider points $x, y \in X_q$, a path $\gamma$ with endpoints $x$ and $y$, and the induced parallel transport map
\begin{equation}
    z^{[\gamma]}: U^P_{\sigma,x} \to U^P_{\sigma,y}
\end{equation}
for a polytope $P \subset Q$ which is in the image of the affine exponential map based at $q$. Fixing paths from $x$ to $y$ to the basepoint associated to $\sigma$, we obtain valuations on the fibres of these local systems. We can also assign a homology class in $H_1(X, \mathbb{Z})$ to $\gamma$, and the valuation of $z^{[\gamma]}$ is then bounded by the product of the norm of $[\gamma]$ with the distance from the origin to the inverse image of $P$ in $T_qQ$, where
\begin{equation}
    |[\gamma]| \equiv \min_{[\gamma']=[\gamma]} \ell(\gamma'),
\end{equation}
and $\ell$ is the length of the loop $\gamma'$, with respect to the flat metric on $X_q$ induced by the metric on $Q$.

**Lemma 2.8.** If the image of $P$ in $T_qQ$ under the inverse of the affine exponential map is contained in the ball of radius $\epsilon$, the valuation of the parallel transport map $z^{[\gamma]}$ on $U^P_{\sigma}$ is bounded below by $-\epsilon \cdot |[\gamma]|$, up to adding a constant which is independent of $\gamma$.

It is convenient to replace the condition about the inverse image of $P$ under the exponential map by a condition in $Q$. To this end, we shall assume from now on that:
\begin{equation}
    (2.2.17) \text{for all } q \in Q, \text{ the distortion of the affine exponential map in the ball of radius } 1 \text{ in } T_qQ \text{ is bounded by } 2.
\end{equation}

The following result will play a key role in the proof of convergence of various Floer theoretic constructions.

**Corollary 2.9.** Let $0 < \delta$ be a positive real number, and let $\epsilon$ be a constant which is smaller than the minimum of $1/2$ and $\delta/2$. Assume that $\{\gamma_i\}_{i=0}^\infty$ is a sequence of paths with endpoints $x$ and $y$ in $X_q$ and $\lambda_i \in \mathbb{R}$ is a sequence going to $+\infty$ such that
\begin{equation}
    (2.2.18) \quad \delta \cdot |[\gamma_i]| \leq \lambda_i + \text{ a constant independent of } i.
\end{equation}
Whenever $P$ is contained in the ball of radius $\epsilon$ about $q$ in $Q$, the map
\begin{equation}
    (2.2.19) \quad \sum_{i=1}^\infty T^{\lambda_i} z^{[\gamma_i]}: U^P_{\sigma,x} \to U^P_{\sigma,y}
\end{equation}
converges in the $T$-adic topology.

**Proof.** The valuation of $T^{\lambda_i} z^{[\gamma_i]}$ is given by
\begin{equation}
    (2.2.20) \quad \lambda_i + \text{val}_P z^{[\gamma_i]} \geq \lambda_i - 2\epsilon \cdot |[\gamma_i]| \geq (1 - 2\epsilon/\delta)\lambda_i + \text{ a constant independent of } i.
\end{equation}
The assumptions that $2\epsilon < \delta$ and $\lambda_i \to \infty$ imply that this valuation also goes to $+\infty$. □

The reader should have in mind that the constant $\delta$ which appears in the statement is obtained from applying the reverse isoperimetric inequality to a collection of holomorphic curves with boundaries $\gamma_i$, and energies $\lambda_i$. 
2.2.2. **Gradings and orientations.** In order to construct a category over a field of characteristic different from 2, we pick a Pin structure on $P_{\sigma} \subset Q$ for each $\sigma \in \Sigma$. As $P_{\sigma}$ is contractible, there is no obstruction to such a choice. Via the canonical identification, for $q \in Q$, of the tangent space of $X_q$ with the trivial vector bundle with fibre $T^*X_q$, we obtain a Pin structure on $X_q$, varying continuously over $q \in P_{\sigma}$.

**Remark 2.10.** Given a vector bundle on the 3-skeleton of $Q$, we may define a Fukaya category twisted by the corresponding class in $H^2(X, \mathbb{Z}_2)$ by picking twisted Pin structures. More generally, one can twist by general classes in $H^2(X, \mathbb{Z}_2)$, but we abstain from such generality as some of the corresponding rings would be non-commutative, and we shall later appeal to algebraic results which are not known in this generality.

To obtain a $\mathbb{Z}$-graded category we use the fact that the (canonical up to homotopy) trivialisation of the square of the top exterior power of $TQ$ induces a trivialisation of the square of the top (complex) exterior power of $TX$ with respect to any compatible almost complex structure, i.e. a complex quadratic volume form. This gives rise to a grading on all fibres $X_q$ in the sense of [15].

2.2.3. **Floer cohomology.** Given a pair $(\sigma_0, \sigma_1)$ of elements of $\Sigma$, consider a point $q \in \nu_Q \sigma_0 \cap \nu_Q \sigma_1$ together with a Morse function

$$f_{\sigma_0, \sigma_1} : X_q \to \mathbb{R},$$

whose critical locus we denote $\text{Crit}(\sigma_0, \sigma_1)$.

We denote by $\lambda_{\sigma_0, \sigma_1}$ the rank-1 free abelian group of isomorphism classes of families of Pin structures on $T_q Q$, parametrised by $[0,1]$ and twisted by the orientation line of this vector space as in [16] Equation (11.32)], which agree with the chosen structures associated to $\sigma_0$ and $\sigma_1$ at the endpoints. Given a critical point $x \in \text{Crit}(\sigma_0, \sigma_1)$, define

$$\delta_x \equiv \lambda_{\sigma_0, \sigma_1} \otimes \text{det}_x,$$

where $\text{det}_x$ is the orientation line of the stable manifold of $x$. Let $\text{deg}(x)$ denote the degree of this graded line.

The choices $\iota_{\sigma_i}$ of Lagrangian sections over $\nu_Q \sigma_i$ yield local systems $U_{\sigma_i}$ on $X_q$. Given polytopes $P_i \subset T_q Q$, we obtain by local systems $U_{P_i}$ which are constructed by completion, and define

$$CM^*(X_q, \text{Hom}^c(U_{P_0}, U_{P_1}) \otimes \delta) \equiv \bigoplus_{x \in \text{Crit}(P_0, P_1)} \text{Hom}^c(U_{P_0, x}, U_{P_1, x}) \otimes \delta_x$$

where $\text{Hom}^c(U_{P_0, x}, U_{P_1, x})$ is the space of continuous homomorphisms, i.e. those with finite valuation

$$\text{val} \phi = \inf_{f \in U_{P_0, x} \setminus \{0\}} \text{val}(\phi(f)) - \text{val}(f).$$

We now recall the construction of the differential: let $I$ denote the interval $(-\infty, +\infty)$ for which we use the parameter $t$. We define

$$T(\sigma_0, \sigma_1) = \{ \gamma : I \rightarrow X_q \bigg| \frac{d\gamma}{dt} = \nabla f_{\sigma_0, \sigma_1} \}/\mathbb{R}$$

where the gradient is taken with respect to a Riemannian metric on $X_q$, and the $\mathbb{R}$ action is by translation. We have a natural evaluation map

$$T(\sigma_0, \sigma_1) \rightarrow \text{Crit}(\sigma_0, \sigma_1) \times \text{Crit}(\sigma_0, \sigma_1)$$
given by taking the limits at \(-\infty\) and \(+\infty\), and the compactified moduli space of gradient trajectories is given by
\[
(2.2.27) \quad \mathcal{T}(\sigma_0, \sigma_1) \equiv \bigcup_k \mathcal{T}(\sigma_0, \sigma_1) \times \text{Crit}(\sigma_0, \sigma_1) \mathcal{T}(\sigma_0, \sigma_1) \times \text{Crit}(\sigma_0, \sigma_1) \cdots \times \text{Crit}(\sigma_0, \sigma_1) \mathcal{T}(\sigma_0, \sigma_1).
\]

The evaluation map \(2.2.26\) extends to the compactification
\[
(2.2.28) \quad \mathcal{T}(\sigma_0, \sigma_1) \rightarrow \text{Crit}(\sigma_0, \sigma_1) \times \text{Crit}(\sigma_0, \sigma_1),
\]
and we write \(\mathcal{T}(x_0, x_1)\) for the fibre over critical points \(x_0\) and \(x_1\).

For a generic choice of metric, \(\mathcal{T}(x_0, x_1)\) is a compact manifold with boundary of dimension
\[
(2.2.29) \quad \dim \mathcal{T}(x_0, x_1) = \deg(x_0) - \deg(x_1) - 1,
\]
which is oriented relative the tensor product \(\delta_{x_0} \otimes \delta_{x_1}\) (this requires a trivialisation of the tangent space of \(\mathcal{T}\), for which we use the standard orientation). In particular, whenever
\[
(2.2.30) \quad \deg(x_0) = \deg(x_1) + 1
\]
we can associate to each element of \(\mathcal{T}(x_0, x_1)\) a natural isomorphism
\[
(2.2.31) \quad \det_{\chi} : \det_{x_1} \rightarrow \det_{x_0}.
\]
Parallel transport also induces an isomorphism of topological vector spaces
\[
(2.2.32) \quad \Pi_1 : \text{Hom}^c(U_{\sigma_0, x_1}^{P_0}, U_{\sigma_1, x_1}^{P_1}) \rightarrow \text{Hom}^c(U_{\sigma_0, x_0}^{P_0}, U_{\sigma_1, x_0}^{P_1})
\]
\[
(2.2.33) \quad \psi \mapsto z[\gamma] \circ \psi \circ z[-\gamma].
\]
Taking the tensor product of these two maps with the identity on \(\lambda_{\sigma_0, \sigma_1}\), we define
\[
(2.2.34) \quad \mu^1 : \text{CM}^* \left( X_q, \text{Hom}^c(U_{\sigma_0}^{P_0}, U_{\sigma_1}^{P_1} \otimes \delta) \right) \rightarrow \text{CM}^* \left( X_{q', \sigma_0}^{P_0}, \text{Hom}^c(U_{\sigma_0}^{P_0}, U_{\sigma_1}^{P_1} \otimes \delta) \right)
\]
\[
(2.2.35) \quad \mu^1 \equiv \sum_{[\gamma] \in \mathcal{T}^0(\sigma_0, \sigma_1)} (-1)^{\deg(x_1)} \Pi_\gamma \otimes \text{id}_{\chi} \otimes \det_{\gamma},
\]
where \(\mathcal{T}^0(\sigma_0, \sigma_1)\) is the space of rigid gradient flow lines. Since the sum is necessarily finite, this differential is continuous.

Choosing a basepoint \(q_{\sigma_0, \sigma_1} \in \nu_Q \sigma_0 \cap \nu_Q \sigma_1\), we define
\[
(2.2.36) \quad \text{CF}^* ((\sigma_0, P_0), (\sigma_1, P_1)) \equiv \text{CM}^* \left( X_{q_{\sigma_0, \sigma_1}}^{P_0}, \text{Hom}^c(U_{\sigma_0}^{P_0}, U_{\sigma_1}^{P_1} \otimes \delta) \right).
\]
It will be useful to arrange for this complex to be independent up to isomorphism of the choice of basepoint. Recall that the choice of section \(\iota_{\sigma_0}\) induces an identification of symplectic manifolds
\[
(2.2.37) \quad X_{\nu_{Q} \sigma_0} \cong T^* \nu_{Q} \sigma_0 / T^*_\mathbb{Z} \nu_{Q} \sigma_0
\]
over the base \(\nu_{Q} \sigma_0\), hence diffeomorphisms \(X_q \cong X_{q'}\) for all pairs \((q, q') \in \nu_{Q} \sigma_0\), which are compatible for triples. We shall leave these diffeomorphisms implicit in our notation. With this in mind, the function \(f_{\sigma_0, \sigma_1}\) appearing in the right hand side of Equation \(2.2.36\) can therefore be thought of as a Morse function on any fibre over \(\nu_{Q} \sigma_0 \cap \nu_{Q} \sigma_1\), and the metric on \(X_q\) induces a metric on \(X_{q'}\). Since the base is contractible, Lemma \(2.7\) provides an isomorphism of local systems, so we obtain an isomorphism of cochain complexes
\[
(2.2.38) \quad \text{CM}^* (X_q, \text{Hom}^c(U_{\sigma_0}^{P_0}, U_{\sigma_1}^{P_1} \otimes \delta)) \rightarrow \text{CM}^* (X_{q'}, \text{Hom}^c(U_{\sigma_0}^{P_0}, U_{\sigma_1}^{P_1} \otimes \delta))
\]
which is compatible with composition for triples \((q, q', q'')\). This establishes that this Morse complex is indeed independent of the choice of basepoint in \(\nu_{Q} \sigma_0\).
Since the local systems associated to two different choices of sections are isomorphic as local systems of topological vector space, the above complex is independent up to isomorphism of the choices $\iota_{\sigma_i}$. Moreover, continuation maps in Morse theory give rise to homotopy equivalences of these Morse complexes, so that the corresponding homology group, which we denote
\begin{equation}
HF^*(P_0, P_1) \equiv HM^*(X_q, \text{Hom}^c(U_{\sigma_0}^P, U_{\sigma_1}^P \otimes \delta))
\end{equation}
is independent of all auxiliary choices, including the choice of elements $\sigma_i \in \Sigma$.

**Remark 2.11.** Note that we broke symmetry and chose $\iota_{\sigma_0}$ to identify fibres over $\nu_0 \cap \nu_1$. We could have chosen $\iota_{\sigma_1}$, or in fact any other trivialisation. In chain-level constructions, we shall need to verify compatibility between various trivialisations, by ensuring that they arise from a contractible set of choices.

2.2.4. **Morse theoretic product.** Consider a triple $\sigma = (\sigma_0, \sigma_1, \sigma_2)$ of elements of $\Sigma$ such that the intersection
\begin{equation}
\nu_0 \cap \nu_1 \cap \nu_2
\end{equation}
is non-empty. We use the above discussion to identify the functions $f_{\sigma_i, \sigma_j}$ as Morse functions on a single fibre over a point $q \in \nu_0 \sigma$. Morse theory thus induces a product on the Floer cohomology groups for pairs, whose construction, while standard, we now recall in order to set up notation for future use.

Consider the semi-infinite intervals
\begin{equation}
I_+ \equiv [0, \infty) \text{ and } I_- \equiv (-\infty, 0).
\end{equation}
We define the space of Morse data
\begin{equation}
V_{\pm}(\sigma_i, \sigma_j) \subset C^\infty(I_{\pm}, C^\infty(X_q, TX_q))
\end{equation}
to consist of families of vector fields on $X_q$, parametrised by $I_{\pm}$, which agree with $\text{grad} f_{\sigma_i, \sigma_j}$ outside a compact set. The gradient flow is taken with respect to the metric chosen in the construction of the Floer complex for the pair $(\sigma_i, \sigma_j)$.

Given $\xi_{ij}^\pm \in V_{\pm}(\sigma_i, \sigma_j)$, we then define
\begin{equation}
T_{\pm}(\sigma_i, \sigma_j) \subset C^\infty(I_{\pm}, X_q)
\end{equation}
to be the set of perturbed gradient flow lines, i.e. maps $\gamma$ from $I_{\pm}$ to $X_q$ satisfying
\begin{equation}
\frac{d\gamma}{dt} = \xi_{ij}^\pm.
\end{equation}
The limit of $\gamma$ at $\pm \infty$ yields a natural evaluation map
\begin{equation}
T_{\pm}(\sigma_i, \sigma_j) \to \text{Crit}(\sigma_i, \sigma_j).
\end{equation}
We define the spaces of broken semi-infinite perturbed gradient flow lines, to be
\begin{equation}
\overline{T}_{\pm}(\sigma_i, \sigma_j) \equiv T_{\pm}(\sigma_i, \sigma_j) \cup T_{\pm}(\sigma_i, \sigma_j) \times_{\text{Crit}(\sigma_i, \sigma_j)} \overline{T}(\sigma_i, \sigma_j).
\end{equation}
where we use the evaluation map at $\mp \infty$ for $\overline{T}(\sigma_i, \sigma_j)$. An element of this fibre product can be thought of as a broken flow line together with a semi-infinite flow line, with matching asymptotic limits.

There is a natural evaluation map
\begin{equation}
\overline{T}_{\pm}(\sigma_i, \sigma_j) \to X_q \times \text{Crit}(\sigma_i, \sigma_j),
\end{equation}
given by evaluation at 0 and of the limit of $\gamma$ at $\pm \infty$. We denote the coequaliser of the three evaluation maps from the product to $X_q$ by

\[ \overline{T}(\sigma) \equiv \text{Coeq} \left( \overline{T}_-(\sigma_0, \sigma_2) \times \overline{T}_+(\sigma_1, \sigma_2) \times \overline{T}_+(\sigma_0, \sigma_1) \to X_q \right), \]

This is the space of flow lines which map 0 to the same point, and is topologised by convergence on compact subsets together with breaking of flow lines.

Given critical points $x_i \in \text{Crit}(\sigma_i, \sigma_{i+1})$ (with index counted modulo 3), we define

\[ \overline{T}(x_0, x_2, x_1) \]

to be the inverse image of $(x_1, x_2, x_0)$ under the evaluation map

\[ \overline{T}(\sigma) \to \prod_{i=0}^2 \text{Crit}_{\sigma_i, \sigma_{i+1}}. \]

For a generic choice of triples of Morse perturbations $(\xi_{01}, \xi_{12}, \xi_{02})$, $\overline{T}(x_0, x_2, x_1)$ is a compact topological manifold with boundary, naturally oriented relative to be the inverse image of $(x_1, x_2, x_0)$ under the evaluation map

\[ \overline{T}(\sigma) \to \prod_{i=0}^2 \text{Crit}_{\sigma_i, \sigma_{i+1}}. \]

Dualising, we obtain, for each rigid element $\gamma \in \overline{T}(x_0, x_2, x_1)$, a natural map

\[ \text{det}_\gamma : \text{det}_{x_0} \otimes \text{det}_{x_2} \otimes \text{det}_{x_1}. \]

Let us now assume that we are given $P_i \subset \nu_Q \sigma_i$. Parallel transport (and composition) also induces a map

\[ \Pi_\gamma : \text{Hom}^c(U_{\sigma_1, x_2}, U_{\sigma_2, x_1}) \otimes \text{Hom}^c(U_{\sigma_0, x_1}, U_{\sigma_1, x_1}) \to \text{Hom}^c(U_{\sigma_0, x_0}, U_{\sigma_2, x_0}) \]

which can be defined as follows: let $\gamma_{ij}$ denote the path from $x_i$ to $x_j$ determined by $\gamma$. We have

\[ \Pi_\gamma(\psi_{12} \otimes \psi_{01}) = z^{[\gamma_{20}]} \circ \psi_{12} \circ z^{[\gamma_{12}]} \circ \psi_{01} \circ z^{[\gamma_{01}]}. \]

We also have a natural map

\[ \lambda_\sigma : \lambda_{\sigma_1, \sigma_2} \otimes \lambda_{\sigma_0, \sigma_1} \to \lambda_{\sigma_0, \sigma_2} \]

induced by concatenating paths.

Taking the sum, over all triples of critical points, of the tensor products of Equations (2.2.50), (2.2.51), and (2.2.52), we obtain a map

\[ CM^* \left( X_q, \text{Hom}^c(U_{\sigma_1, x_2}, U_{\sigma_2, x_1}) \otimes \text{Hom}^c(U_{\sigma_0, x_1}, U_{\sigma_1, x_1}) \right) \to CM^* \left( X_q, \text{Hom}^c(U_{\sigma_0, x_0}, U_{\sigma_2, x_0}) \right) \]

which is given by the formula

\[ \mu^2 \equiv \bigoplus_{x_i^c \in \text{Crit}(P_0, P_2)} \sum_{x_1^c \in \text{Crit}(P_0, P_1)} (-1)^{\deg(x_i)} \Pi_\gamma \otimes \lambda_\sigma \otimes \delta_\gamma. \]

Since the sum is finite, this is necessarily a continuous map. Composing the left and right hand sides with the isomorphisms of Equation (2.2.58), we obtain the product

\[ \mu^2 : CF^*((\sigma_1, P_1), (\sigma_2, P_2)) \otimes CF^*(\sigma_0, P_0, (\sigma_1, P_1)) \to CF^*(((\sigma_0, P_0), (\sigma_1, P_1)), \]

which induces a cohomological product

\[ HF^*(P_1, P_2) \otimes HF^*(P_0, P_1) \to HF^*(P_0, P_1). \]
Remark 2.12. In [5], we twisted the product by an explicit term obtained from a Čech cocycle representing the obstruction to the existence of a Lagrangian section of \( X \to Q \) which is equipped with a Pin structure. To see that our construction is equivalent, note that the obstruction to a consistent trivialisation of the local systems which is equipped with a Pin structure. To see that our construction is equivalent, note that the obstruction to a consistent trivialisation of the local systems is exactly \( \lambda_{\sigma_0, \sigma_1} \) is exactly \( \omega_2(Q) \in \mathbb{H}^2(Q, \mathbb{Z}_2) \), while resolving the ambiguity in the construction of the local systems \( U^P \) over all basepoints corresponds to the choice of a global Lagrangian section of \( X \to Q \).

2.2.5. Definition of the cohomological categories. For \( \sigma, \tau \in \Sigma \), we define

\[
\mathcal{F}(\tau, \sigma) \equiv \begin{cases} 
CF^*((\tau, P_\tau), (\sigma, P_\sigma)) & \tau \leq \sigma \\
0 & \text{otherwise.}
\end{cases}
\]

Letting \( H\mathcal{F}(\tau, \sigma) \) denote the corresponding cohomology group, we obtain a category \( H\mathcal{F} \) with compositions given as in Section 2.2.4. We omit the verification that the associativity conditions hold, which will follow from the construction of an \( A_\infty \) category in Section 4. We denote by \( H\mathcal{F}_\sigma \) the full subcategory with objects given by \( \tau \in \Sigma_\sigma \).

For each \( \sigma \in \Sigma \) and pair of polytopes \( P_0, P_1 \subseteq VQ_\sigma \), we define

\[
\text{Po}_\sigma(P_0, P_1) \equiv CF^*((\sigma, P_0), (\sigma, P_1)).
\]

Letting \( \text{HPo}_\sigma(P_0, P_1) \) denote the corresponding Floer cohomology group, we obtain a category \( \text{HPo}_\sigma \).

Remark 2.13. In Section 4.4, we shall find it useful to redefine \( \text{Po}_\sigma \) to add the choice of a basepoint \( q_i \in P_i \) to each object. The additional choice gives a category with many more objects, but it will be clear from the computations of this section that objects corresponding to different choices of basepoints on the same polytope are equivalent.

The fact that Floer cohomology groups are independent of the all auxiliary choices yields a faithful embedding \( H\mathcal{F}_\sigma \to \text{HPo}_\sigma \), which is the inclusion of a directed category. One way to make this embedding more explicit is as follows: Fix a homotopy of sections between the restrictions of \( \iota_\tau \) and \( \iota_\sigma \) to \( VQ_\sigma \cap VQ_\tau \). This induces an isomorphism of local systems:

\[
U^P_{\tau} \to U^P_{\sigma}.
\]

Taking the sum of these isomorphisms over all maxima of the Morse function \( f_{\sigma_0, \sigma_1} \), we obtain a continuation element

\[
\kappa \in CF^0((\tau, P_\tau), (\sigma, P_\sigma))
\]

which is closed and whose cohomology class is canonical. Given a pair \( (\tau_0, \tau_1) \) of objects of \( H\mathcal{F}_\sigma \), the left and right products with the corresponding continuation elements induce a map

\[
\mathcal{F}(\tau_0, \tau_1) \equiv CF^0((\tau_0, P_{\tau_0}), (\tau_1, P_{\tau_1})) \to CF^0((\sigma, P_{\tau_0}), (\sigma, P_{\tau_1})) \cong \text{Po}_\sigma(P_{\tau_0}, P_{\tau_1}).
\]

Passing to cohomology, we obtain the functor

\[
H\mathcal{F}_\sigma \to \text{HPo}_\sigma.
\]

Remark 2.14. The standard way of constructing a map of Morse complexes for different choices of Morse functions is to consider a 1-parameter family of vector fields interpolating between the two gradients, and counting solutions of the corresponding flow lines. Keeping in mind that a generic point in a manifold lies on a unique negative gradient flow line starting at a maximum, one sees that the count of pairs of gradient trees defining the product is the same count that defines a composition of continuation maps

\[
\text{CF}^0((\tau_0, P_{\tau_0}), (\tau_1, P_{\tau_1})) \to \text{CF}^0((\sigma, P_{\tau_0}), (\sigma, P_{\tau_1})).
\]
To show that this is a fully faithful embedding we construct a continuation element in $\text{CF}^0((\sigma, P_{\tau}), (\tau, P_{\tau}))$ as in Equation (2.2.63), and show that the product

$$\text{CF}^0((\tau, P_{\tau}), (\sigma, P_{\tau})) \otimes \text{CF}^0((\sigma, P_{\tau}), (\tau, P_{\tau})) \to \text{CF}^0((\sigma, P_{\tau}), (\sigma, P_{\tau}))$$

maps the tensor products of the two continuation elements to the multiplicative unit.

2.2.6. Computation of morphisms in the category. To understand the categories $H^F$ and $H_{P_{\sigma}}$, we summarise some computations established in the Appendices. The first result is a computation for inclusions, which comes in two parts, the second of which will not be used until we discuss duality in Section 2.4.

**Proposition 2.15.** If $P_1 \subseteq P_0$, the module action

$$m: \Gamma \otimes U_x \to U_x$$

extends to a quasi-isomorphism

$$\text{Hom}_A(\Gamma^{P_0}, \Gamma^{P_1}) \to \Lambda. \quad (2.2.69)$$

If $P_0$ is contained in the interior of $P_1$, there is a (natural up to sign) trace

$$\text{tr}: \text{Hom}_A(\Gamma^{P_0}, \Gamma^{P_1}) \to \Lambda. \quad (2.2.70)$$

Composing the trace with the module action, we obtain a map

$$\text{Hom}_A(\Gamma^{P_0}, \Gamma^{P_1}) \to \text{Hom}_A(\Gamma^{P_0}, \Lambda)$$

$$\theta \mapsto \text{tr} \circ \theta \circ m \quad (2.2.71)$$

which induces a quasi-isomorphism

$$\text{Hom}_A(\Gamma^{P_0}, \Lambda) \quad (2.2.73)$$

The first part of the above result, proved in Appendix C.4, implies that the morphism spaces in $H^F$ are given by

$$H^F(\tau, \sigma) \equiv \begin{cases} \Gamma^{P_{\tau}} & \tau \leq \sigma \\ 0 & \text{otherwise} \end{cases} \quad (2.2.74)$$

In particular, $H^F$ is isomorphic to the category denoted $F$ in [5], allowing us to tie the constructions of the two papers. The second part is proved in Appendix C.5.

Next, we consider a polytope $P \subset \nu_{Q_{\sigma}}$ for $\sigma \in \Sigma$, and a cover $\{P_{\alpha}\}_{\alpha \in A}$, indexed by a finite partially ordered set, in the sense that $P_{\alpha} \subseteq P_{\beta}$ if $\alpha \leq \beta$. The natural (restriction) map $\Gamma^{P_{\alpha}} \to \Gamma^{P_{\beta}}$ gives rise to a map of local systems $U^{P_{\sigma}} \to U^{P_{\alpha}}$, allowing us to form the Čech complex

$$\check{T}(P; A) \equiv \bigoplus_{\alpha_0 \in A} U^{P_{\sigma}}_{\alpha_0} \to \bigoplus_{\alpha_0 < \alpha_1 \in A} U^{P_{\sigma}}_{\alpha_0} \to \bigoplus_{\alpha_0 < \alpha_1 < \alpha_2 \in A} U^{P_{\sigma}}_{\alpha_0} \to \cdots \quad (2.2.75)$$

as a complex of (topological) local systems over $X_{Q_{\sigma}}$. Note that this is a finite direct sum of topological local systems, and thus there is no ambiguity in the construction of the topology on $\check{T}(P; A)$. Moreover, we have a canonical map of local systems

$$U^{P_{\sigma}}_{\alpha} \to \check{T}(P; A) \quad (2.2.76)$$

given by the restriction to $U^{P_{\sigma}}_{\alpha}$ for all $\alpha \in A$. 
Lemma 2.16. The map from $U^P_\sigma$ to $\tilde{T}(P; A)$ induces quasi-isomorphisms
\begin{align}
(2.2.77) & \quad P_\sigma(\tilde{T}(P; A), P') \cong CM^*(X_q, \text{Hom}^*(\tilde{T}(P; A), U_\sigma')) \otimes \delta \\
(2.2.78) & \quad P_\sigma(P', \tilde{T}(P; A)) \cong CM^*(X_q, \text{Hom}^*(U_\sigma'', \tilde{T}(P; A)) \otimes \delta)
\end{align}
equipped with the sum of the Morse differential and the internal differential of $\tilde{T}(P; A)$. We can now state an immediate consequence of Proposition B.1, which is the version of Tate acyclicity which we shall use for computations:

**Lemma 2.16.** The map from $U^P_\sigma$ to $\tilde{T}(P; A)$ induces quasi-isomorphisms
\begin{align}
(2.2.79) & \quad P_\sigma(\tilde{T}(P; A), P') \rightarrow P_\sigma(P, P') \\
(2.2.80) & \quad P_\sigma(P', P) \rightarrow P_\sigma(P', \tilde{T}(P; A))
\end{align}
for all $P' \subset \nu Q\sigma$. \qed

The above result allows us to reduce global computations to local computations. To fully make use of locality, we need the fact, proved in Appendix C.4, that the Floer complex $CF^*(([\sigma_0, P_0], [\sigma_1, P_1]))$ is acyclic whenever $P_0$ and $P_1$ are disjoint. In addition, we observe that the Floer complex $P_\sigma(\tilde{T}(P; A), P')$ is isomorphic to the complex
\begin{align}
(2.2.81) & \quad \bigoplus_{\alpha_0 \in A} P_\sigma(P, P_{\alpha_0}) \rightarrow \bigoplus_{\alpha_0 < \alpha_1 \in A} P_\sigma(P, P_{\alpha_0}) \rightarrow \bigoplus_{\alpha_0 < \alpha_1 < \alpha_2 \in A} P_\sigma(P, P_{\alpha_0}) \rightarrow \cdots,
\end{align}
and similarly for $P_\sigma(P', \tilde{T}(P; A))$.

**Corollary 2.17.** Let $(P, P', P'')$ be polytopes contained in $\nu Q\sigma$. If the intersections of $P'$ and $P''$ with an open neighbourhood of $P$ agree, there are natural isomorphisms
\begin{align}
(2.2.82) & \quad HF^*(P, P') \cong HF^*(P, P'') \\
(2.2.83) & \quad HF^*(P', P) \cong HF^*(P'', P).
\end{align}

**Proof.** By taking the intersection of $P'$ and $P''$, it suffices to prove the result under the assumption that $P'' \subset P'$. In this case, we can extend $P''$ to a cover of $P'$ with the property that all other elements of the cover intersect $P$ trivially, hence the Floer cohomology of all other elements of the cover with $P$ vanish. From Equation (2.2.81), we obtain an isomorphism
\begin{align}
(2.2.84) & \quad \text{HPO}_\sigma(\tilde{T}(P; A), P') \cong \text{HPO}_\sigma(P, P'').
\end{align}
The result then follows from Lemma 2.16. \qed

We thus see that the morphisms between objects in $\text{HPO}_\sigma$ are local in the sense that they only depend on a neighbourhood of the intersection of the corresponding polytopes.

### 2.3. Local cohomological modules

In this section, we shall assign, to each Lagrangian $L$ in $\mathcal{A}$, left and right modules over the categories $\text{HPO}_\sigma$, and use Tate acyclicity to compute these modules whenever $L$ is Hamiltonian isotopic to a Lagrangian meeting $X_{\tilde{q}}$ at a point.

We begin by imposing a condition which can be achieved by a small Hamiltonian perturbation:
\begin{align}
(2.3.1) & \quad \text{the intersection of } L \text{ with every fibre } X_q \text{ is inessential (i.e. contained in a disjoint union of closed contractible sets).}
\end{align}
For the remainder of the paper, we fix a closed neighbourhood $\nu X L$ of each element of $\mathcal{A}$ so that, for each point $q \in Q$, the intersection $\nu X L \cap X_q$ is inessential. We denote by $J_L$ the space of tame almost complex structures which agree with $J_L$ away from a fixed compact
subset of the interior of \( \nu_X L \), where \( J_L \) is the almost complex structure with respect to which we have assumed that \( L \) does not bound any non-constant holomorphic disc.

We now impose constraints on the diameter of \( \nu_X \sigma \), which will be essential in ensuring that, for each \( \sigma \in \Sigma \), we can define a Floer cohomology group for pairs \( L \in \mathcal{A} \) and \( P \in \nu_X \sigma \).

By Assumption \( (2.3.1) \), the map

\[
(2.3.2) \quad X_{\sigma} \rightarrow X_{\sigma}/\sim
\]

obtained by collapsing the components of \( X_{\sigma} \cap \nu_X L \) admits a right inverse up to homotopy, so the identity of the first homology of \( X_{\sigma} \) factors as

\[
(2.3.3) \quad H_1(X_{\sigma}; \mathbb{Z}) \rightarrow H_1(X_{\sigma}/\sim; \mathbb{Z}) \rightarrow H_1(X_{\sigma}; \mathbb{Z}).
\]

We can therefore equip \( X_{\sigma}/\sim \) with a metric so that the norm \( ||\gamma|| \) of the homology class of any loop in \( X_{\sigma} \) is bounded above by the length \( \ell(\gamma/\sim) \) of the projection.

For each \( J_L \)-holomorphic curve \( u \) from a strip to \( X \), with boundary conditions on a fixed compact subset of \( \nu_X L \) along \( t = 0 \), and \( X_{\sigma} \), at \( t = 1 \) (the coordinates on the strip \( \mathbb{R} \times [0, 1] \) are \((s, t)\)), we consider the energy \( E(u) = \int u^* \omega \) and the length \( \ell(\partial u/\sim) \) of the projection to \( X_{\sigma}/\sim \) of the boundary component of the strip labelled \( X_{\sigma} \). According to Corollary \( \ref{cor:energy-bound} \), we may choose a constant \( C \) independent of \( u \) such that this length is bounded by \( CE(u) \).

We require that

\[
(2.3.4) \quad \text{the diameter of } \nu_X \sigma \text{ is bounded by } 1/4C.
\]

Remark 2.18. This is the first of many places where we impose a condition on the diameters of the covers \( \{ \nu_X \sigma \} \) and \( \{ P_\sigma \} \). The basic idea for achieving it is to start with a given cover, and refine it until the desired constraint is satisfied. The only difficulty with this idea is that the reverse isoperimetric inequality depends on the choice of Lagrangian boundary conditions, and that a refinement of the cover entails changing which fibres are associated to elements of the cover (since we require \( q_\sigma \in \nu_X \sigma \)). The solution implemented in Section \( \ref{section:construction} \) is to establish a reverse isoperimetric inequality for choices parametrised by the space of all possible fibres (this is compact), and then pick a cover which is sufficiently fine with respect to this uniform constant.

Finally, in order to construct a module over a field of characteristic different from 2, we assume that each \( L \in \mathcal{A} \) is equipped with a Pin structure, or more generally a Pin structure relative the pull-back of a vector bundle on the 3-skeleton of \( Q \). To obtain a \( \mathbb{Z} \)-graded category, assume that the Lagrangians are graded with respect to the chosen quadratic volume form on \( X \) (in the sense of \( [13] \)).

2.3.1. Floer complexes between polytopes and Lagrangians. For each \( \sigma \in \Sigma \) and \( L \in \mathcal{A} \), we choose a Hamiltonian diffeomorphism \( \Phi_{\sigma,L} \) supported in \( \nu_X L \) such that \( L_\sigma = \Phi_{\sigma,L} L \) is transverse to \( X_{\sigma} \). Let \( \text{Crit}(\sigma, L) \) and \( \text{Crit}(L, \sigma) \) denote the set intersections of \( L_\sigma \) with \( X_{\sigma} \).

The chosen Pin structures on \( X_{\sigma} \) and \( L \) give rise to an assignment \( \delta_x \) of a 1-dimensional \( \mathbb{Z}_2 \)-graded free abelian group associated to each element \( x \) of \( \text{Crit}(\sigma, L) \) or \( \text{Crit}(L, \sigma) \). The canonical grading of the fibre with respect to the standard quadratic complex volume form on \( X \), together with a choice of grading on \( L \) determine a \( \mathbb{Z} \)-grading on \( \delta_x \).

With this in mind, we define, for each \( P \subset \nu_X \sigma \) the Floer complex

\[
(2.3.5) \quad CF^*(L, (\sigma, P)) \equiv \bigoplus_{x \in \text{Crit}(\sigma, L)} U_{\sigma,x}^P \otimes \delta_x.
\]
A choice of paths connecting the endpoints of orbits to a basepoint on $X_q$, induces a (complete) norm on these complexes, and the corresponding topology is independent of choice.

In order to define the differential, pick a family
\begin{equation}
(2.3.6) \quad J(L, \sigma): [0, 1] \to \delta_L
\end{equation}
which restricts at 0 to the pushforward of $J_L$ under $\Phi_{\sigma, L}$. We obtain a moduli space $\mathcal{R}(L, \sigma)$ of finite energy $J(L, \sigma)$ holomorphic strips with boundary conditions $L_\sigma$ along $t = 0$, and $X_{q_\sigma}$ at $t = 1$ (the coordinates on the strip $\mathbb{R} \times [0, 1]$ are $(s, t)$). To simplify the discussion later, we set
\begin{equation}
(2.3.7) \quad J(\sigma, L)(t) = J(L, \sigma)(1 - t).
\end{equation}

There is a natural evaluation
\begin{equation}
(2.3.8) \quad \mathcal{R}(L, \sigma) \to \text{Crit}(\sigma, L) \times \text{Crit}(\sigma, L)
\end{equation}
given by the asymptotic conditions at $\pm \infty$. We denote the fibre at $(x_0, x_1)$ by $\mathcal{R}(x_0, x_1)$. Choosing $J(L, \sigma)$ generically ensures that this is a topological manifold of dimension
\begin{equation}
(2.3.9) \quad \dim \mathcal{R}(x_0, x_1) = \deg(x_0) - \deg(x_2) - 1
\end{equation}
whose boundary is covered by codimension-1 strata corresponding to breaking of strips:
\begin{equation}
(2.3.10) \quad \partial \mathcal{R}(x_0, x_1) = \bigcup_{x \in \text{Crit}(\sigma, L)} \mathcal{R}(x_0, x) \times \mathcal{R}(x, x_1).
\end{equation}

The output of Floer theory is that, whenever $\mathcal{R}(x_0, x_1)$ has dimension 0, each element induces a map:
\begin{equation}
(2.3.11) \quad \delta_u: \delta_{x_1} \to \delta_{x_0}.
\end{equation}

In order to define the differential in Equation (2.3.5), we recall from Section 2.2.1 that a path from $x_0$ to $x_1$ induces a parallel transport map from $U^P_{\sigma, x_1}$ to $U^P_{\sigma, x_0}$. The boundary of an element $u \in \mathcal{R}(x_0, x_1)$ gives rise to such a path which we denote $\partial u$. For the statement of the next result, we recall that the topological energy $E(u)$ of a holomorphic curve is its area.

**Lemma 2.19.** There is a constant $A$, independent of $u$, such that whenever $P \subset \nu_Q \sigma$, we have
\begin{equation}
(2.3.12) \quad E(u) + \text{val}_P z^{[\partial u]} \geq E(u)/2 + A.
\end{equation}

**Proof.** It suffices to bound $\text{val}_q z^{[\partial u]}$ for any $q \in \nu_Q \sigma$. The condition that the distortion is bounded by 2 implies that the image of $P$ in $T_q X$ is contained in the ball of radius 2 diam $\nu_Q \sigma$. Thus
\begin{equation}
(2.3.13) \quad \text{val}_q z^{[\partial u]} \geq -2 \text{diam } \nu_Q \sigma ||[\partial u]|| \geq - \frac{||[\partial u]||}{2C},
\end{equation}
where the second inequality follows from Equation (2.3.4). At the cost of introducing an additive constant, the choice of metric fixed in the discussion preceding (2.3.4) allows us to replace $||[\partial u]||$ by the length $\ell(\partial u/\sim)$. We can then apply the reverse isoperimetric inequality: the key point is that, according to Corollary A.2 the reverse isoperimetric inequality for $J_L$-holomorphic with boundary conditions on $\nu_X L$ and $X_{q_\sigma}$ applies (with the same constant) to $J(L, \sigma)$ holomorphic curves, because the two almost complex structures agree by assumption away from a fixed compact subset of the interior of $\nu_X L$. The result thus follows. □
Corollary 2.20. For each pair \((x_0, x_1)\) of intersection points, the expression
\[
\sum_{u \in R^q(x_0, x_1)} T^E(u) z^{[\partial u]}
\]
gives a well-defined map from \(U^P_{\sigma, x_1}\) to \(U^P_{\sigma, x_0}\).

Proof. This is a direct consequence of Corollary 2.9, whose hypothesis is satisfied by the previous Lemma. □

We conclude that, for each pair \((x_0, x_1)\) of intersection points, the expression
\[
\partial_{x_0, x_1} \equiv \sum_{u \in R^q(x_0, x_1)} T^E(u) z^{[\partial u]} \otimes \delta_u
\]
gives a well-defined map from \(U^P_{\sigma, x_1} \otimes \delta_{x_1}\) to \(U^P_{\sigma, x_0} \otimes \delta_{x_0}\).

Definition 2.21. The differential on \(CF^*(L, (\sigma, P))\) is given by
\[
\bigoplus_{x_0} \sum_{x_1} (-1)^{\deg(x_1)+1} \partial_{x_0, x_1}.
\]

By the previous discussion, this differential is a continuous operator with respect to the natural topology on Floer complexes (i.e. bounded with respect to the norm induced by a choice of homotopy classes of paths to the basepoint).

Remark 2.22. Note that the sign in Equation (2.3.16) differs by one from the sign in Equation (2.2.23). The reason for this choice is that the sign conventions for modules are more intuitive if they are based on unreduced gradings.

Reversing the roles of the Lagrangian and the polytope, we consider the complex
\[
CF^*((\sigma, P), L) \equiv \bigoplus_{x \in \text{Crit}(\sigma, L)} \text{Hom}^c_A(U^P_{\sigma, x}, \Lambda) \otimes \delta_{x^q},
\]
with differential dual to the one on \(CF^*(L, (\sigma, P))\).

2.3.2. Modules over the local categories. For each \(\sigma \in \Sigma, L \in A,\) and \(P \subset \nu_Q \sigma\), we define
\[
\mathcal{L}^*_{L, \sigma}(P) \equiv CF^*(L, (\sigma, P))
\]
\[
\mathcal{R}^*_{L, \sigma}(P) \equiv CF^*((\sigma, P), L).
\]
The differentials \(\mu^{L, \sigma}_{L, \sigma}\) and \(\mu^{R, \sigma}_{L, \sigma}\) are given by Equation (2.3.16) and its dual.

We now construct the module structure on \(H\mathcal{L}_{L, \sigma}\), i.e. the left action of morphism spaces in \(H\text{Po}_\sigma\). The construction of the right module action is entirely similar, as we shall explain at the end.

Define \(\overline{\mathcal{R}}(L, \sigma, \sigma)\) to be the moduli space of strips in \(\overline{\mathcal{R}}(L, \sigma)\), with an additional marked point along the segment mapping to \(X_{q_\sigma}\). We have a natural evaluation map
\[
\overline{\mathcal{R}}(L, \sigma, \sigma) \to \text{Crit}(L, \sigma) \times X_{q_\sigma} \times \text{Crit}(L, \sigma).
\]
Let \(\overline{T}\overline{\mathcal{R}}(L, \sigma, \sigma)\) be the fibre product of \(\overline{\mathcal{R}}(L, \sigma, \sigma)\) with the space \(\overline{T}^+_{\sigma, \sigma}\) of (perturbed) positive half-gradient flow lines of the function \(f_{\sigma, \sigma}\). We shall call such moduli spaces \textit{mixed moduli spaces}, as they consist of gradient flow lines and pseudo-holomorphic discs with matching evaluation maps to Lagrangians in \(X\). Their use is standard in constructions combining Morse and Floer theory, going all the way back to \cite{14} (see for example \cite{6} in the Lagrangian setting).
We have a natural evaluation map
\[
\mathcal{TR}(L, \sigma, \sigma) \to \text{Crit}(L, \sigma) \times \text{Crit}(\sigma, \sigma) \times \text{Crit}(L, \sigma),
\]
with the ordering given counterclockwise around the boundary starting at the outgoing end. Denote by \( E \) the set of ends and marked points that we have chosen, which we decompose into \( E^{in} = \{e_{1}, e_{2}\} \) and \( E^{out} = \{e_{0}\} \), with \( E^{in} \) consisting of the incoming (positive) end and the marked point, and \( E^{out} \) consisting of the singleton output. We denote the fibre at a triple \( x = \{x_{e}\}_{e \in E} \) in the left hand side of Equation (2.3.21) by \( \mathcal{TR}(x) \). By parallel transport along the gradient flow line and the part of the boundary mapping to \( X_{q_{\sigma}} \), we obtain a map
\[
(2.3.22) \quad \text{Hom}^{c}(U_{\sigma,x_{e_{2}}}U_{\sigma,x_{e_{1}}}) \otimes U_{\sigma,x_{e_{0}}} \to U_{\sigma,x_{e_{0}}}.
\]
For generic choices of perturbations, the fibre product defining \( \mathcal{TR}(x) \) is transverse, so that it is a manifold with boundary of dimension
\[
(2.3.23) \quad \deg(x_{e_{0}}) - \sum_{e \in E^{in}} \deg(x_{e}),
\]
and which is naturally oriented relative
\[
(2.3.24) \quad \delta_{x_{e_{0}}}^\vee \otimes \bigotimes_{e \in E^{in}} \delta_{x_{e}}.
\]
A rigid element \( u \in \mathcal{TR}(\Upsilon) \) thus induces a map
\[
(2.3.25) \quad \bigotimes_{e \in E^{in}} \delta_{x_{e}} \to \delta_{x_{e_{0}}}^\vee.
\]
Given a triple \( \Upsilon = (L, P_{0}, P_{1}) \), with \( P_{0} \) and \( P_{1} \) contained in \( \nu Q_{\sigma} \), we combine Equations (2.3.22) and (2.3.25) to obtain a map
\[
(2.3.26) \quad \text{Po}_{\sigma}(P_{0}, P_{1}) \otimes L_{L,\sigma}(P_{0}) \to L_{L,\sigma}(P_{1})
\]
which, upon twisting by \( (-1)^{\deg(x_{1})+1} \), gives rise to the structure map \( \mu^{1|1}_{L,\sigma} \). Passing to cohomology, we obtain
\[
(2.3.27) \quad H\text{Po}_{\sigma}(P_{0}, P_{1}) \otimes H L_{L,\sigma}(P_{0}) \to H L_{L,\sigma}(P_{1}).
\]

The construction of the right module map proceeds as follows: we construct a moduli space \( \mathcal{R}(\sigma, \sigma, L) \) by considering strips in \( \mathcal{R}(\sigma, L) \) with an additional marked point on the boundary with label \( X_{q_{\sigma}} \), then define \( \mathcal{TR}(\sigma, \sigma, L) \) to be the fibre product over \( X_{q_{\sigma}} \) with the moduli space of semi-infinite gradient flow lines of \( f_{\sigma,\sigma} \) (see Figure 2). The count of rigid elements of these moduli spaces defines a map \( \mu^{1|1}_{L,\sigma} \) on Floer cochains
\[
(2.3.28) \quad \mathcal{R}_{L,\sigma}(P_{-0}) \otimes \text{Po}_{\sigma}(P_{-1}, P_{-0}) \to \mathcal{R}_{L,\sigma}(P_{-1})
\]
for each pair of polytopes \( (P_{-0}, P_{-1}) \) in \( \nu Q_{\sigma} \). At the level of cohomology, we obtain the desired map:
\[
(2.3.29) \quad H\mathcal{R}_{L,\sigma}(P_{-0}) \otimes H\text{Po}_{\sigma}(P_{-1}, P_{-0}) \to H\mathcal{R}_{L,\sigma}(P_{-1}).
\]
2.3.3. Computation of the module structure over $H \mathcal{F}_\sigma$. Equation (2.2.74) gives a particularly simple description of the category $H \mathcal{F}_\sigma$, with morphisms given by affinoid algebras. Identifying $\Gamma P$ with the completion of the homology of the based loop space of $X_q \sigma$, the construction of the Floer complexes yields natural maps

\begin{align*}
\Gamma P' \otimes L_{L,\sigma}(P) &\to L_{L,\sigma}(P') \\
\mathcal{R}_{L,\sigma}(P') \otimes \Gamma P' &\to \mathcal{R}_{L,\sigma}(P),
\end{align*}

whenever $P' \subset P$, arising from the map

\begin{equation}
U_{P,x}^P \otimes \Gamma P' \to U_{P,x}^P \otimes \Gamma P' \cong U_{P,x}^{P'}.
\end{equation}

Passing to cohomology, we conclude that the groups $HL_{L,\sigma}(P)$ and $HR_{L,\sigma}(P)$ give rise to modules over $H \mathcal{F}_\sigma$. This construction, which does not use any gradient trees or holomorphic discs in the module structure maps, was used in [5]. In this section, we show:

**Lemma 2.23.** The pullbacks of $HL_{L,\sigma}$ and $HR_{L,\sigma}$ under the inclusion of $H \mathcal{F}_\sigma$ in $HPo_\sigma$ are naturally isomorphic to the modules constructed from Equation (2.3.32). $\square$

The key point is the following result:

**Lemma 2.24.** All contributions to the maps

\begin{align*}
\text{Po}^0(P_0, P_1) \otimes L_{L,\sigma}^k(P_0) &\to L_{L,\sigma}^k(P_1) \\
\mathcal{R}_{L,\sigma}^k(P_{-1}) \otimes \text{Po}^0(P_{-1}, P_{-0}) &\to \mathcal{R}_{L,\sigma}^k(P_{-1})
\end{align*}

are given by configurations whose holomorphic component is a constant strip.

**Proof.** By construction, the holomorphic component of such an element is a strip with endpoints $x$ and $y$ of equal Maslov index, hence has Fredholm index 0. However, the fact that the almost complex structure on the strip is translation invariant implies that the minimal Fredholm index of a non-constant disc is 1. $\square$

In particular, if $\max x$ denotes the unique maximum of $f_{\sigma,\sigma}$ which is the endpoint of a (perturbed) gradient flow line starting at $x$, the left and right module actions are given by
the sums of the maps
\[(2.3.35) \quad \text{Po}^0(P_0, P_1) \otimes \mathcal{L}_{L_0}^k(P_0) \to \bigoplus_{x \in \text{Crit}(L, P)} \text{Hom}^c(U_{\sigma, x}^{P_0}, U_{\sigma, x}^{P_1}) \otimes U_{\sigma, x}^P \to U_{\sigma, x}^{P_1} \]
\[(2.3.36) \quad \mathcal{R}_{L_0}^k(P_0) \otimes \text{Po}^0(P_1, P_0) \to \bigoplus_{x \in \text{Crit}(L, P)} \text{Hom}^c(U_{\sigma, x}^{P_0}, U_{\sigma, x}^{P_1}) \otimes \text{Hom}(U_{\sigma, x}^{P_0}, U_{\sigma, x}^{P_1}) \to \text{Hom}^c(U_{\sigma, x}^{P_1}, \Lambda) \]
where the second map (in both cases) is induced by parallel transport along the flow line from \( x \) to max \( x \). Using the inclusion \( \Gamma^{P_1} \subset \text{Po}^0(P_0, P_1) \), we conclude:

**Corollary 2.25.** If \( P_1 \subset P_0 \) the pullback of \( \mathcal{L}_{L_0}^k(P_0) \) to \( \Gamma^{P_1} \subset \text{Po}^0(P_0, P_1) \) is a free module of rank equal to the number of elements of \( \text{Crit}(L, \sigma) \) of degree \( k \). \( \square \)

Passing to cohomology yields Lemma 2.23.

2.3.4. Computation for sections. We now prove that Equation (2.1.23) is an isomorphism whenever \( L_0 \) meets \( X_{q_x} \) transversely at a single point. The tensor product on the left of Equation (2.1.23) is the cokernel of the map
\[(2.3.37) \quad \bigoplus_{\rho_0 \leq \rho_1 \in \Sigma_{\sigma}} \text{HPO}_{\sigma}(P_{\rho_1}, P_{\sigma}) \otimes \mathcal{H}(P_{\rho_0}, P_{\rho_1}) \otimes \mathcal{H}_{L_0}(P_{\rho_0}) \to \bigoplus_{\rho_0 \in \Sigma_{\sigma}} \text{HPO}_{\sigma}(P_{\rho_0}, P_{\sigma}) \otimes \mathcal{H}_{L_0}(P_{\rho_0}). \]
If we restrict to \( \rho_0 = \rho_1 \) in the above complex, the cokernel computes the tensor product over \( \Gamma^{P_{\rho_0}} = \mathcal{H}(P_{\rho_0}, P_{\rho_0}) \) of the modules \( \text{HPO}_{\sigma}(P_{\rho_0}, P_{\sigma}) \) and \( \mathcal{H}_{L_0}(P_{\rho_0}) \). Thus the cokernel is the same as that of the map
\[(2.3.38) \quad \bigoplus_{\rho_0 < \rho_1 \in \Sigma_{\sigma}} \text{HPO}_{\sigma}(P_{\rho_1}, P_{\sigma}) \otimes \mathcal{H}(P_{\rho_0}, P_{\rho_1}) \otimes \mathcal{H}_{L_0}(P_{\rho_0}) \to \bigoplus_{\rho_0 \in \Sigma_{\sigma}} \text{HPO}_{\sigma}(P_{\rho_0}, P_{\sigma}) \otimes \Gamma^{P_{\rho_0}} \mathcal{H}_{L_0}(P_{\rho_0}). \]
The above map factors through the direct sum of the surjections
\[(2.3.39) \quad \text{HPO}_{\sigma}(P_{\rho_1}, P_{\sigma}) \otimes \mathcal{H}(P_{\rho_0}, P_{\rho_1}) \otimes \mathcal{H}_{L_0}(P_{\rho_0}) \to \text{HPO}_{\sigma}(P_{\rho_0}, P_{\sigma}) \otimes \Gamma^{P_{\rho_0}} \mathcal{H}_{L_0}(P_{\rho_0}). \]
Since \( \mathcal{H}(P_{\rho_0}, P_{\rho_1}) \) is a free rank-1 module over \( \Gamma^{P_{\rho_1}} \), the flatness of the map of rings \( \Gamma^{P_{\rho_0}} \to \Gamma^{P_{\rho_1}} \) [19, Lemma 8.6] yields an isomorphism
\[(2.3.40) \quad H(\mathcal{H}(P_{\rho_0}, P_{\rho_1})) \otimes \Gamma^{P_{\rho_0}} \mathcal{H}_{L_0}(P_{\rho_0}) \to \mathcal{H}_{L_0}(P_{\rho_0}). \]
It thus suffices to compute the cokernel of the map
\[(2.3.41) \quad \bigoplus_{\rho_0 < \rho_1 \in \Sigma_{\sigma}} \text{HPO}_{\sigma}(P_{\rho_1}, P_{\sigma}) \otimes \Gamma^{P_{\rho_0}} \mathcal{H}_{L_0}(P_{\rho_1}) \to \bigoplus_{\rho_0 \in \Sigma_{\sigma}} \text{HPO}_{\sigma}(P_{\rho_0}, P_{\sigma}) \otimes \Gamma^{P_{\rho_0}} \mathcal{H}_{L_0}(P_{\rho_0}). \]
So far, the discussion has been completely general. We now use the assumption that \( L \) meets \( X_{q_x} \) at a point: let \( x \) denote the intersection of \( L_0 \) with \( X_{q_x} \). For each \( P \subset \nu Q \sigma \), we have a canonical isomorphism
\[(2.3.42) \quad \mathcal{L}_{L_0, x}(P) = U_{\sigma, x}^P, \]
with trivial differential. Using Corollary 2.25, we conclude that each module \( \mathcal{H}_{L_0, x}(P) \) is free of rank 1 over \( \Gamma^{P_{\rho_1}} \), and we are thus reduced to prove:
Lemma 2.26. If \( L_\sigma \) meets \( X_{q_\sigma} \) transversely at a single point, the complex

\[
\bigoplus_{\rho_0 < \rho_1 \in \Sigma_\sigma} \text{HPo}_\sigma(P_{\rho_1}, P_\sigma) \rightarrow \bigoplus_{\rho_0 \in \Sigma_\sigma} \text{HPo}_\sigma(P_{\rho_0}, P_\sigma) \rightarrow \text{HPo}_\sigma(P_\sigma, P_\sigma)
\]

is right exact.

Proof. Let \( P \) be a polytope containing \( P_\sigma \) in its interior, and contained in the union of polytopes \( P_\tau \) with \( \tau \in \Sigma_\sigma \). By Corollary 2.17 the restriction

\[
\text{HPo}_\sigma(P_\tau \cap P, P_\sigma) \rightarrow \text{HPo}_\sigma(P_\tau, P_\sigma),
\]

is an isomorphism, so that the Lemma follows from the right exactness of the complex

\[
\bigoplus_{\rho_0 < \rho_1 \in \Sigma_\sigma} \text{HPo}_\sigma(P_{\rho_1} \cap P, P_\sigma) \rightarrow \bigoplus_{\rho_0 \in \Sigma_\sigma} \text{HPo}_\sigma(P_{\rho_0} \cap P, P_\sigma) \rightarrow \text{HPo}_\sigma(P_\sigma, P_\sigma).
\]

This is a consequence of Tate acyclicity: Lemma 2.16 implies that the map from the Čech complex

\[
\left( \bigoplus_{0 \leq k} \bigoplus_{\rho_0 < \cdots < \rho_k} \text{HPo}_\sigma(P_{\rho_1} \cap P, P_\sigma)[-k], \delta \right)
\]

to \( \text{HPo}_\sigma(P_\sigma, P_\sigma) \) is a quasi-isomorphism. Passing to cohomology, the quotient complex corresponding to \( 1 \leq k \) yields a map of cohomology groups which fits in a commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{\rho_0 < \rho_1 \in \Sigma_\sigma} \text{HPo}_\sigma(P_{\rho_1} \cap P, P_\sigma) & \rightarrow & H^* \left( \bigoplus_{1 \leq k} \bigoplus_{\rho_0 < \cdots < \rho_k} \text{HPo}_\sigma(P_{\rho_1} \cap P, P_\sigma)[-k], \delta \right) \\
& & \\
& & \bigoplus_{\rho_0 \in \Sigma_\sigma} \text{HPo}_\sigma(P_{\rho_0}, P_\sigma).
\end{array}
\]

Moreover, the images of these two maps agree. Applying the long exact sequence on cohomology and using the fact that \( \text{HPo}_\sigma(P_\sigma, P_\sigma) \) vanishes except in degree 0 implies the desired result. \( \square \)

Corollary 2.27. If \( L_\sigma \) meets the fibre over \( q_\sigma \) transversely at a single point, the map

\[
j^* \Delta_{\text{HPo}_\sigma}(\sigma, \cdot) \otimes H_{\mathcal{F}_\sigma} j^* (H\mathcal{L}_{L_\sigma}) \rightarrow j^* H\mathcal{L}_{L_\sigma}(\sigma)
\]

is an isomorphism. \( \square \)

2.4. Local module duality. By construction, the left and right modules associated to each Lagrangian \( L \in \mathcal{A} \) are linearly dual. In this section, we fix \( \sigma \in \Sigma \), and construct a map of bimodules

\[
H\mathcal{L}_{L_\sigma} \otimes H\mathcal{R}_{L_\sigma} \rightarrow \Delta_{\text{HPo}_\sigma}.
\]

We then prove that, if \( L_\sigma \) is a Lagrangian section, this map defines an isomorphism between the values at \( P_\sigma \) of \( H\mathcal{R}_{L_\sigma} \) and the bimodule dual of \( H\mathcal{L}_{L_\sigma} \).
2.4.1. A map to the local diagonal bimodule. Consider the triple \( \Upsilon = (\sigma, L, \sigma) \). We define \( \hat{\mathcal{R}}(\Upsilon) \) to consist of elements of \( \mathcal{R}(\sigma, L) \) equipped with a marked point along the boundary component labelled \( \sigma \), and define \( \overline{\hat{\mathcal{R}}}(\Upsilon) \) to be the fibre product of \( \hat{\mathcal{R}}(\Upsilon) \) with the space \( \overline{T}_-(\sigma, \sigma) \) of (perturbed) negative half-gradient flow lines. We have a natural evaluation map
\[
\overline{\hat{\mathcal{R}}}(\Upsilon) \to \text{Crit}(\sigma, \sigma) \times \text{Crit}(\sigma, L) \times \text{Crit}(L, \sigma).
\]
For each pair \( (P, P_-) \) of polytopes contained in \( \nu_Q \sigma \), the count of rigid elements of this moduli space thus defines a map
\[
\mathcal{L}_{L, \sigma}(P) \otimes \mathcal{R}_{L, \sigma}(P_-) \to \text{P}_\sigma(P_-, P)
\]
which, on co-homology, induces a map
\[
H\mathcal{L}_{L, \sigma}(P) \otimes H\mathcal{R}_{L, \sigma}(P_-) \to H\text{P}_\sigma(P_-, P).
\]
We omit, as usual, the straightforward verification that this arises from a map of bimodules.

\[\text{Lemma 2.28.} \quad \text{All contributions to the maps}
\]
\[
\mathcal{L}_{L, \sigma}^{n-k}(P) \otimes \mathcal{R}_{L, \sigma}^k(P_-) \to \text{Po}_\sigma^\nu(P_-, P)
\]
are given by configurations whose holomorphic component is a constant strip. \( \Box \)

Letting \( \min x \) denote the unique minimum of \( f_{\sigma, \sigma} \) which is the endpoint of a (perturbed) gradient flow line starting at \( x \), we conclude

\[\text{Corollary 2.29.} \quad \text{In degree } n, \text{ the map to the diagonal bimodule is given by the sum of the maps}
\]
\[
\mathcal{L}_{L, \sigma}^{n-k}(P) \otimes \mathcal{R}_{L, \sigma}^k(P_-) \to \bigoplus_{x \in \text{Crit}(L, \sigma)} \text{U}_{\sigma, x}^P \otimes \text{Hom}(\text{U}_{\sigma, x}^P, \Lambda) \to \text{Hom}(\text{U}_{\sigma, \min x}^P, \text{U}_{\sigma, \min x}^P)
\]
where the second map is induced by parallel transport along the flow line from \( x \) to \( \min x \). \( \Box \)

2.4.2. Computing the right module for sections. In this section, we prove:

\[\text{Lemma 2.30.} \quad \text{The restriction of Equation (2.4.5) to } P_{\sigma}
\]
\[
H\mathcal{R}_{L, \sigma}(P_{\sigma}) \to \text{Hom}_{\text{H}\text{P}_\sigma}(H\mathcal{L}_{L, \sigma}, \Delta_{\text{H}\text{P}_\sigma}(P_{\sigma}, \Upsilon))
\]
is an isomorphism whenever \( L_{\sigma} \) meets \( X_{qs} \) transversely at one point. \( \Box \)

The argument is formally dual to the one given in Section 2.3.4. We start by noting that the right hand side is the kernel of the map
\[
\prod_{\rho_0 \in \Sigma_{\sigma}} \text{Hom}(H\mathcal{L}_{L, \sigma}(P_{\rho_0}), \text{H}\text{P}_\sigma(P_{\rho_0}, P_{\rho_1})) \to \prod_{\rho_0 < \rho_1 \in \Sigma_{\sigma}} \text{Hom}(H\mathcal{J}(P_{\rho_0}, P_{\rho_1}), H\mathcal{L}_{L, \sigma}(P_{\rho_0})), \text{H}\text{P}_\sigma(P_{\rho_1}, P_{\rho_1})).
\]
We use the isomorphism \( \Gamma^{P_{\rho_0}} = H\mathcal{J}(P_{\rho_0}, P_{\rho_0}) \), the fact that \( H\mathcal{J}(P_{\rho_0}, P_{\rho_1}) \) is a free rank-1 module over \( \Gamma^{P_{\rho_1}}, \) and the flatness of the map \( \Gamma^{P_{\rho_0}} \to \Gamma^{P_{\rho_1}} \) to rewrite this as the kernel of the map
\[
\prod_{\rho_0 \in \Sigma_{\sigma}} \text{Hom}_{\Gamma^{P_{\rho_0}}}(H\mathcal{L}_{L, \sigma}(P_{\rho_0}), \text{H}\text{P}_\sigma(P_{\rho_0}, P_{\rho_1})) \to \prod_{\rho_0 < \rho_1 \in \Sigma_{\sigma}} \text{Hom}_{\Gamma^{P_{\rho_1}}}(H\mathcal{L}_{L, \sigma}(P_{\rho_1}), \text{H}\text{P}_\sigma(P_{\rho_1}, P_{\rho_1})).
\]
Assuming that $L_\sigma$ meets $X_{q_\sigma}$ at one point implies that $H\mathcal{L}_{L,\sigma}(P_\sigma)$ is free of rank one over $\Gamma^{P_\sigma}$, hence simplifying the above to:

\[(2.4.11) \prod_{\rho_0 \in \Sigma_\sigma} \text{HPo}_\sigma(P_\sigma, P_{\rho_0}) \to \prod_{\rho_0 < \rho_1 \in \Sigma_\sigma} \text{HPo}_\sigma(P_\sigma, P_{\rho_1}).\]

We now introduce a polytope $P$, containing $P_\sigma$ in its interior, and covered by its intersection with $P_\tau$. By Corollary 2.17 the restriction

\[(2.4.12) \text{HPo}_\sigma(P_\sigma, P_\tau) \to \text{HPo}_\sigma(P_\sigma, P \cap P_\tau)\]

is an isomorphism. The desired computation then follows from the following application of Tate acyclicity:

**Lemma 2.31.** The complex

\[(2.4.13) \prod_{\rho_0 \in \Sigma_\sigma} \text{HPo}_\sigma(P_\sigma, P \cap P_{\rho_0}) \to \prod_{\rho_0 < \rho_1 \in \Sigma_\sigma} \text{HPo}_\sigma(P_\sigma, P \cap P_{\rho_1})\]

is left exact.  

Since $P_\tau$ is contained in the interior of $P$, the computations of Appendix C summarised above in Section 2.2.6, together with the definition of $\mathcal{R}_{L,\sigma}(P_\sigma)$ yield an isomorphism

\[(2.4.14) \text{HPo}_\sigma(P_\sigma, P) \cong \text{Hom}^c(U_{\sigma,x}^{\mathcal{F}_\sigma}, \Lambda) \cong H\mathcal{R}_{L,\sigma}(P_\sigma)\]

As in Section 2.3.4 this isomorphism is compatible with module action, implying Lemma 2.30.

### 2.5. Global cohomological modules.

Given $L \in \mathcal{A}$ and $\sigma \in \Sigma$, we define

\[(2.5.1) \mathcal{L}^*_L(\sigma) \equiv CF^*(L, (\sigma, P_\sigma))\]
\[(2.5.2) \mathcal{R}^*_L(\sigma) \equiv CF^*((\sigma, P_\sigma), L),\]

with differentials $\mu_{\mathcal{L}}^{1|0}$ and $\mu_{\mathcal{R}}^{0|1}$ obtained from Equation (2.3.16) and its dual. In this section, we show that the corresponding cohomology groups give rise to left and right modules $H\mathcal{L}_L$ and $H\mathcal{R}_L$ over $H\mathcal{F}$. In Section 2.5.5 below, we show that the restrictions of these modules to $H\mathcal{F}_\sigma$ are isomorphic to the pullbacks of the corresponding local modules, while in Section 2.5.6 we construct the maps relating the Floer cohomology of Lagrangians in $\mathcal{A}$ with the corresponding left and right modules. All these constructions are minor variants of those appearing in [5].

#### 2.5.1. Non-Hamiltonian continuation equation.

Consider a triple

\[(2.5.3) \Upsilon = (\sigma_{-1}, \sigma_{-0}, L) \text{ or } \Upsilon = (L, \sigma_0, \sigma_1),\]

such that $P_{\sigma_{-1}} \cap P_{\sigma_0} \neq \emptyset$ (we allow $-0 = i$ for consistency with later notation). Note that the case $\sigma_i = \sigma_{i+1}$ is already covered in the previous section, so we may assume that they are different. The continuation maps between modules associated to $L$ and the two elements of the cover will be defined by a count of pseudo-holomorphic strips, so we begin by labelling the positive end $e_{in}$ of the strip $S = \mathbb{R} \times [0, 1]$ with the pair $\Upsilon_{e_{in}}$ of labels $(\sigma_0, L)$ or $(L, \sigma_{-0})$, and the negative end $e_{ou}$ with the pair $\Upsilon_{e_{ou}}$ given by $(\sigma_1, L)$ or $(L, \sigma_{-1})$. We thus obtain assignments $(\sigma_i, \Upsilon_{e_i})$ of elements of $\Sigma$ and $Q$ for each end $e$. Fix disjoint neighbourhoods $\nu_{\Upsilon_{e_i}}$ of these ends, and write $\partial S$ for the boundary component of $S$ which is labelled by elements of $\Sigma$. 

Consider the straight path \( q_{\Sigma} \) from \( q_{\sigma_e} \) to \( q_{\sigma_e} \) determined by the affine structure; we parametrise this path by the real line, with the condition that it agrees with \( q_{\sigma} \) in \( \nu_{S\sigma} \). The quotient of \( \partial S \) by the intersection with \( \nu_{S\sigma} \) is a closed interval.

Next, we fix Hamiltonians
\[
H_{\sigma,L}: [0, 1] \times X \to \mathbb{R}
\]
which generate \( \Phi_{\sigma,L} \), and choose a map
\[
H_T: \mathbb{R} \times [0, 1] \times X \to \mathbb{R}
\]
which agrees with \( H_{\tau\sigma} \) near each end \( e \) of the strip. We let
\[
\Phi_T: \mathbb{R} \to \text{Ham}(X)
\]
denote the induced path of Hamiltonian diffeomorphisms.

If we choose a family
\[
J(\Upsilon): \mathbb{R} \times [0, 1] \to \mathcal{J}
\]
of almost complex structures which agree with \( J(\Upsilon_e) \) along the end \( e \), and whose restriction to the boundary \( t = 1 \) (the coordinates are \((s, t)\) on the strip) agrees with the pushforward of \( J_L \) under the path \( \Phi_T \), we obtain a moduli space \( \mathcal{R}(\Upsilon) \) of stable \( J(\Upsilon)-\)holomorphic strips
with boundary conditions given by the path \( X_{q_{\Upsilon}} \) along the boundary \( t = 0 \), and by \( \Phi_T(L) \) along \( t = 1 \). By evaluation along the boundary \( t = 0 \), we associate to each element \( u \) of \( \mathcal{R}(\Upsilon) \) a path
\[
\partial_Q u: \mathbb{R} \to X_S,
\]
where the right hand side denotes the fibre bundle over a closed interval, obtained from the Lagrangian boundary conditions by collapsing the inverse image of the neighbourhoods \( \nu_{S\sigma} \) to the fibres \( X_{q_{\sigma_e}} \).

Consider as in Section 2.3 the quotient \( X_{S}/\sim \) by the equivalence relation which collapses the components of the intersection of \( \nu_{X}L \) with the boundaries \( X_{q_{\sigma_e}} \) and \( X_{q_{\sigma_e}} \) to points. According to Lemma A.3 there is a constant \( C \) such that, up to an additive constant, the length of \( \partial_Q u/\sim \) is bounded by \( CE_{\text{geo}}(u) \), where the geometric energy is given by
\[
E_{\text{geo}}(u) = \int |du|^2 = \int u^* \omega
\]
with the norm taken with respect to the metric induced by the almost complex structure. Using the fact that we have an isomorphism \( H_1(X_q, \mathbb{Z}) \cong H_1(X_{q_{\Upsilon}}, \mathbb{Z}) \) for any \( q \) in the path \( q_{\Upsilon} \), we arrange as before for the metric on \( X_{q_{\Upsilon}}/\sim \) to have the property that for any \( q \in q_{\Upsilon} \) the norm of the homology class in \( H_1(X_q, \mathbb{Z}) \) associated to a loop in \( X_{q_{\Upsilon}} \) is bounded by the length of the image of this loop in \( X_{q_{\Upsilon}}/\sim \). We then require that
\[
\text{diameter of } \nu_Q \sigma \text{ is bounded by } 1/8C.
\]

2.5.2. Energy of continuation maps. The moduli space \( \mathcal{R}(\Upsilon) \) is slightly unusual because the moving Lagrangian boundary conditions along \( t = 0 \) do not form a Hamiltonian family. The standard description of the Gromov-Floer bordification (in terms of breaking of strips at the ends, bubbling of discs at the boundary and of spheres at the interior) applies with unmodified proof, as does the treatment of regularity.

The main difference with the standard case of Hamiltonian families is that Gromov compactness fails in general. We now explain that it remains valid in our setting: the section
Figure 3. The boundary conditions for elements of $\overline{\mathcal{R}}(\sigma_{-1}, \sigma_{-0}, L)$ and $\overline{\mathcal{R}}(L, \sigma_0, \sigma_1)$.

$\iota_{\sigma_{\infty}}$ determines an identification of $X_{\nu_{Q_{\infty}}}$ with a neighbourhood of the 0-section in the cotangent bundle of the fibre over the basepoint $q_{\sigma_{\infty}}$, in such a way that the section corresponds to a cotangent fibre; let $\lambda_{\sigma_{\infty}}$ denote the (locally defined) Liouville 1-form obtained from this identification. We define the (topological) energy of an element $u \in \overline{\mathcal{R}}(\Upsilon)$ to be given by

$$
E(u) \equiv \int u^* \omega - \int (\partial_Q u)^* \lambda_{\sigma_{\infty}} - \int H_{\Upsilon}(s, 1, u(s, 1)) ds.
$$

Lemma 2.32. If $\{u_r\}_{r \in [0,1]}$ is a 1-parameter family of strips with boundary conditions given by the path $X_{q_{|T}}$ along the boundary $t = 0$, and by $\Phi_T(L)$ along $t = 1$, with constant asymptotic conditions at the ends, then $E(u_0) = E(u_1)$.

Proof. This is a straightforward application of Stokes’ theorem: consider the associated map

$$
[0,1] \times \mathbb{R} \times [0,1] \to X.
$$

Since $\omega$ is closed, and we have assumed that the asymptotic conditions are constant, the integral of $\omega$ over the boundary vanishes. This sum decomposes in two 4 terms; the integrals over the faces corresponding to the boundary of the first factor give rise to the terms involving $\omega$ in $E(u_0)$ and $E(u_1)$. For the integral over the face corresponding to $t = 0$, we note that the image of this face is contained in $X_{\nu_{Q_{\sigma_{\infty}}}}$, and that the choice of primitive $\lambda_{\sigma_{\infty}}$ allows us to apply Stokes’s theorem to yield the two corresponding terms in $E(u_i)$, using the fact that contributions from the end $s = \pm\infty$ vanish because the asymptotic conditions are constant. Finally, the restriction of the map to the face corresponding to $t = 1$ factors through the family

$$
L \times \mathbb{R} \to X
$$

given by applying the map $\Phi_T$ from Equation (2.5.10). Since $\Phi_T$ is generated by $H_T$, the pullback of $\omega$ to $L \times \mathbb{R}$ agrees with $-dH_T \wedge ds$. The 1-form $-H_T \wedge ds$ defines a primitive, so that a final application of Stokes’s theorem yields that the term corresponding to this stratum is the difference of the corresponding terms appearing in $E(u_0)$ and $E(u_1)$. Having accounted for all the terms in $E(u_0) - E(u_1)$, the result follows.

We now prove that Gromov compactness holds, under Condition (2.5.10):
Lemma 2.33. For $u \in R(\Upsilon)$, we have

$$E(u) \geq \frac{3}{4}E^{geo}(u) + \text{a constant independent of } u.$$  

(2.5.14)

In particular, for any positive real number $E$, the subset of $R(\Upsilon)$ consisting of curves of energy bounded by $E$ is compact.

Proof. Gromov’s argument \cite{12} shows that the geometric energy $\int u^\ast \omega$ defines a proper map $R(\Upsilon) \to [0, \infty)$, so that the first statement implies the second, and it suffices to bound the difference between the two energies.

Identifying the path $q_T$ with a path in $H^1(X_{q_{\sigma_{\epsilon_{in}}}}; \mathbb{R})$, and lifting $\partial_{\bar{q}}u$ to a path $\tilde{\partial}_{\bar{q}}u$ in $H_1(X_{q_{\sigma_{\epsilon_{in}}}}; \mathbb{R})$, we can express the integral of the 1-form $\lambda_{\sigma_{\epsilon_{in}}}$ as

$$\int (\tilde{\partial}_{\bar{q}}u)^* \lambda_{\sigma_{\epsilon_{in}}} = \int_{-\infty}^{\infty} \langle q_T(s), \partial_s \tilde{\partial}_{\bar{q}}u \rangle ds$$  

(2.5.15)

$$= \langle q_T(-\infty), \tilde{\partial}_{\bar{q}}u(-\infty) \rangle - \int_{-\infty}^{\infty} \langle \partial_s q_T(s), \tilde{\partial}_{\bar{q}}u \rangle ds$$  

(2.5.16)

$$\leq \frac{1}{8C} \cdot \ell(\tilde{\partial}_{\bar{q}}u) + \ell(\tilde{\partial}_{\bar{q}}u) \int_{-\infty}^{\infty} |\partial_s q_T(s)| ds$$  

(2.5.17)

$$\leq \frac{\ell(\tilde{\partial}_{\bar{q}}u)}{4C}$$  

(2.5.18)

Above, we have used Equation (2.5.10) to bound the length of $q_T$. The reverse isoperimetric inequality thus implies that

$$\left| \int (\tilde{\partial}_{\bar{q}}u)^* \lambda_{\sigma_{\epsilon_{in}}} \right| \leq E^{geo}(u)/4.$$  

(2.5.19)

Combining this with the fact that the last term in Equation (2.5.11) can be bounded in terms of the data $H_T$, we conclude the desired result. \qed

We now restrict to the union of components corresponding to a pair of intersection points $(x_-, x_+)$. Choosing a path, for each end $e$, from these points to the basepoint in $X_{\sigma_{\sigma_{\epsilon_{in}}}}$ obtained by intersecting with the section associated to $\sigma_{\epsilon_{in}}$, we obtain a homology class $[\partial_{\bar{q}}u]$ in $H_1(X_{q_{\sigma_{\epsilon_{in}}}}; \mathbb{Z})$ associated to $u \in R(x_-, x+)$. Using Condition (2.5.10), we find that

$$||\partial_{\bar{q}}u|| \leq C E^{geo}(u) \leq 3C/4E(u) + \text{a constant independent of } u.$$  

(2.5.20)

2.5.3. Mixed moduli spaces. We equip the strip with the marked point $e = (0, 0)$ in the case $\Upsilon = (\sigma_{-1}, \sigma_{-0}, L)$, and $e = (0, 1)$ in the case $\Upsilon = (L, \sigma_0, \sigma_1)$. We assign to this marked point the label $\Upsilon_e = (\sigma_{-1}, \sigma_{-0})$ in the first case and $\Upsilon_e = (\sigma_0, \sigma_1)$ in the second.

This choice of marked point equips the moduli space $R(\Upsilon)$ from the previous section with a natural evaluation map

$$R(\Upsilon) \to X_{q_{\Upsilon_e}}$$  

(2.5.21)

where we use the trivialisation associated to the first element of $\Upsilon_e$ to identify the two fibres. The fibre carries a Morse function $f_{\Upsilon_e}$ from Section 2.4.8 and a choice of family of vector fields on $[0, \infty)$, which agree with the gradient flow outside a compact set, yields a moduli space $\mathcal{T}_+(\Upsilon_e)$, which also admits an evaluation map to $X_{q_{\Upsilon_e}}$. We define the mixed moduli space as a fibre product (see Figure 4)

$$\mathcal{T} R(\Upsilon) \equiv R(\Upsilon) \times_{X_{q_{\Upsilon_e}}} \mathcal{T}_+(\Upsilon_e).$$  

(2.5.22)
Figure 4. An element of the moduli space $\mathcal{R}(L, \sigma_0, \sigma_1)$ defining the left module over $H$. As before, this space admits an evaluation map to the product of the intersection points associated to the triples $\{e, e_{ou}, e_{in}\}$, and we denote the fibre over a triple $x$ of generators by $\mathcal{R}(x)$.

2.5.4. The global left and right multiplications. If $\Upsilon = (L, \sigma_0, \sigma_1)$, an element of $\mathcal{R}(x)$ defines a map

$$\text{Hom}^c(U^{P_{\sigma_0}}_{\sigma_0, x_{e}}U^{P_{\sigma_1}}_{\sigma_1, x_{e}}) \otimes U^{P_{\sigma_0}}_{\sigma_0, x_{e}^\text{in}} \to U^{P_{\sigma_1}}_{\sigma_1, x_{e}^\text{in}},$$

where we have used the canonical identifications from Lemma 2.7 to omit the superscript from the intersection points of Lagrangians. The map is defined as follows: given $u = (v, \gamma)$ in this moduli space, the trivialisation $\iota_{\sigma_0, e_{in}}$, allows us to identify $x_e$ and $x_{e_{ou}}$ with points in $X_{q_e}$. Applying this trivialisation to the path along the boundary of the elements of $v$ from the positive end to $v(e)$ and the gradient flow line $\gamma$, we obtain a parallel transport map

$$U^{P_{\sigma_0}}_{\sigma_0, x_{e^\text{in}}} \to U^{P_{\sigma_0}}_{\sigma_0, x_{e^\text{in}}}.$$  

Given an element of the left hand side of Equation (2.5.23), the composition of this isomorphism with the given linear map in $\text{Hom}^c(U^{P_{\sigma_0}}_{\sigma_0, x_{e}}U^{P_{\sigma_1}}_{\sigma_1, x_{e}}) \otimes U^{P_{\sigma_0}}_{\sigma_0, x_{e}^\text{in}}$ yields an element of $U^{P_{\sigma_1}}_{\sigma_1, x_{e^\text{in}}}$ which we transport, along the image in $X_{q_{e^\text{in}}}$ of the gradient flow line and the boundary of the holomorphic strip to the fibre of $U^{P_{\sigma_1}}_{\sigma_1}$ at $x_{e_{ou}}$, which is the right hand side of Equation (2.5.23).

Remark 2.34. As explained in the discussion following Lemma 2.7, one must specify the fibre along which one performs parallel transport because the identifications of the local systems over different fibres are only compatible with parallel transport maps up to multiplication by the exponential of the flux.

Multiplying the tensor product of the isomorphism in Equations (2.5.23) with that in Equation (2.5.23) by $(-1)^{\deg(x_{e^\text{in}}) + 1}$, we obtain, for $u$ a rigid element in $\mathcal{R}(\Upsilon)$, a map

$$\mu_u: \mathcal{F}(\sigma_0, \sigma_1) \otimes \mathcal{L}_L(\sigma_0) \to \mathcal{L}_L(\sigma_1).$$

Lemma 2.35. The sum

$$\sum_{u \in \mathcal{R}(\sigma_1, \sigma_0, L)} T^{E(u)} \mu_u$$

is convergent.
Proof. The key point is to bound the valuation of $T^{E(u)}\mu_u$. The map $\mu_u$ is a composition of two parallel transport maps with an evaluation map. The evaluation map has trivial valuation and the valuation of the two parallel transport maps is bounded by the sum of the norms of the two corresponding loops obtained by concatenating with fixed paths to a basepoint. Up to an additive constant independent of $u$, the sum of norms of these loops is bounded by the length of the projection of $\partial Q u$ to $X_{q \tau}/\sim$. Applying the reverse isoperimetric inequality, we conclude that the valuation of the parallel transport maps is bounded by $CE^{n_\sigma}(u)$. By Condition (2.5.10), the polytopes $P_\tau$ are contained in the ball of radius $1/4C$ in $T_{\infty} Q$, so that applying Lemma 2.38 yields

\[
\val \mu_u \geq -E^{n_\sigma}(u)/4 + \text{ a constant independent of } u.
\]

From Lemma 2.33 we conclude that the valuation of $T^{E(u)}\mu_u$ is bounded above by $E(u)/2$, up to a constant term independent of $u$. Gromov compactness (for the geometric energy) thus implies that the sum is convergent. \qed

Remark 2.36. Note that the definitions of $E(u)$ and $\mu_u$ are asymmetric, since they rely on distinguishing the fibre $X_{\infty}$ corresponding to the incoming end. However, the product $T^{E(u)}\mu_u$ is independent of this choice.

We denote the sum in Lemma 2.35 by

\[
\mu_{\infty}^{11}: F^* (\sigma_0, \sigma_1) \otimes L^*_L (\sigma_0) \to L^*_L (\sigma_1).
\]

The standard description of the boundary of $\overline{\mathcal{M}}(Y)$ implies that this is a cochain map, hence that it induces a map on cohomology

\[
H^* (\sigma_0, \sigma_1) \otimes H L^*_L (\sigma_0) \to H L^*_L (\sigma_1),
\]

which makes the groups $H L^*_L$ into a left module over the category $H F$. The last statement requires the construction of a homotopy corresponding, in the usual language of $A_\infty$ modules to the operation $\mu_{\infty}^{11}$, and hence will be subsumed by later constructions.

In the case $Y = (\sigma_{-1}, \sigma_0, L)$, elements of the moduli space $\overline{\mathcal{M}}(x_{\infty}, x_{\infty}, x_c)$ give rise to a map

\[
\Hom^c (U_{P_0, \infty, x_{\infty}}, Y) \otimes \Hom^c (U_{P_{-1}, x_c, \infty}, U_{P_{-1}, 0, x_{\infty}}) \to \Hom^c (U_{P_{-1}, x_{\infty}, x_c}, Y).
\]

The sum of maps associated to this moduli space defines the chain-level product

\[
\mu_{\infty}^{11}: R^*_L (\sigma_0) \otimes F^* (\sigma_{-1}, \sigma_0) \to R^*_L (\sigma_{-1})
\]

which induces the structure map of the right module $H R^*_L$ at the cohomological level.

2.5.5. Comparison of local and global modules. Recall that we denote by $H F_\sigma$ the subcategory of $H F$ with objects the elements $\tau \in \Sigma$ such that $P_\tau \cap P_\sigma \neq \emptyset$. We have a faithful embedding $H F_\tau \to H F_\sigma$. In this section, we prove that the pullback of $H L^*_L, \sigma$ under this functor is naturally isomorphic to the restriction of $H L^*_L$.

We could define such a map using previously constructed moduli spaces, but it is convenient for later purposes to have a variant, in which we replace the boundary marked point by an interior marked point: let $\overline{\mathcal{M}}_{2,1}$ denote the one element subset of the moduli space of discs with one interior marked point and two boundary punctures, corresponding to the unit disc with punctures at $\pm 1$ and marked point at the origin. Consider a basepoint $\sigma \in \Sigma$, and a pair $Y = (L, \tau)$, with $\tau \in \Sigma_\sigma$. Pick a parametrisation of the path $q_{\sigma, \tau}$ by the boundary component $R \times \{1\}$ of the strip, which agrees with $q_{\sigma}$ near the positive end and with $q_{\tau}$ near the other end. We equip the strip with boundary conditions given by the corresponding
path of fibres, and by the Lagrangian $L$ along the boundary $\mathbb{R} \times \{0\}$, perturbed near the ends as in Section 2.5.1 to achieve transversality with $X_{\sigma}$ near the positive end, and $X_{\tau}$ near the negative end. Choosing a family of almost complex structures as before, we obtain a moduli space $\overline{\mathcal{R}(Y)}$. Choosing a homotopy between the sections $i_{\sigma}$ and $i_{\tau}$, the count of rigid elements of this space defines a map

$$L_{L,\sigma}(P_{\tau}) \equiv CF^*(L, (\sigma, P_{\tau})) \rightarrow CF^*(L, (\tau, P_{\tau})) \equiv L_{L}(\tau),$$

which is a quasi-isomorphism. Swapping the roles of the boundary conditions at 0 and 1, and dualising, we obtain

$$R_{L,\sigma}(P_{\tau}) \equiv CF^*((\sigma, P_{\tau}), L) \rightarrow CF^*((\tau, P_{\tau}), L) \equiv R_{L}(\tau).$$

These constructions give rise to maps of cohomological modules by the use of mixed moduli spaces as in previous sections.

2.5.6. Floer cohomology of Lagrangians, and global modules. We now proceed to construct the top horizontal and right vertical maps in Diagram (2.1.17); this is a minor variant of the maps considered in [5]. Given $L \neq L' \in \mathcal{A}$, and $\sigma \in \Sigma$, we consider the triple $Y = (L, L', \sigma)$ or $(L, \sigma, L')$. Let $S$ be a disc with 3 punctures. The triple $Y$ induces a unique labelling of the boundary components of the complement of the punctures which respects the ordering counterclockwise around the boundary; we write $Y_e$ for the pair associated to each end $e$. We pick a strip-like end on each puncture, which is positive if the corresponding edge is incoming, and negative otherwise.

![Figure 5](image-url)  
Figure 5. The boundary conditions for elements of the moduli spaces $\overline{R}(L, L', \sigma)$ and $\overline{R}(L, \sigma, L')$.

Pick Hamiltonian paths connecting $\Phi_{\sigma, L}$ and $\Phi_{\sigma, L'}$ to the identity, giving rise to moving Lagrangian boundary conditions along the boundary segments labelled $L$ and $L'$, such that the end labelled by $(L, L')$ has constant conditions given by $L$ and $L'$ (see Figure 5). As in the previous section, the pushforward of $J_L$ or $J_{L'}$ by the corresponding path of symplectomorphisms determines an almost complex structure $J(z)$ along these boundary components. We define

$$\mathcal{J}(Y) \subset C^\infty(S, \mathcal{J})$$

(2.5.34)

to be the space of almost complex structures parametrised by $S$ which extend the above map, and whose restriction to each end $e$ agrees, under a choice of strip-like ends, with the families $J(Y_e)$ assigned earlier to pairs of labels containing an element of $\Sigma$, and given by a fixed regular choice of path from $J_L$ to $J_{L'}$ for $Y_e = (L, L')$. 


If we impose constant Lagrangian conditions $X_q$, along the remaining boundary component, the choice of an element $J(\Upsilon)$ of $J(\Upsilon)$ determines a moduli space $R(\Upsilon)$ of holomorphic triangles equipped with an evaluation map

\[ R(\Upsilon) \rightarrow \prod_e \text{Crit}(\Upsilon_e), \]

where the product is taken over all the ends. Assuming that the family of almost complex structure is generic, and after imposing a further constraint on the diameter of $\nu Q\sigma$ in terms of the corresponding reverse isoperimetric inequality, the count of rigid elements of $R(\Upsilon)$ (together with parallel transport) induces maps

\[
\begin{align*}
\mathcal{L}_L(\sigma) \otimes CF^*(L, L') &\rightarrow \mathcal{L}_L(\sigma) \\
\mathcal{R}_L(\sigma) \otimes \mathcal{L}_L(\sigma) &\rightarrow CF^*(L, L'),
\end{align*}
\]

depending on whether $\Upsilon = (L, L', \sigma)$ or $(L, \sigma, L')$.

Dualising Equation (2.5.36) gives rise to a map

\[ CF^*(L, L') \rightarrow \text{Hom}_\Lambda^c(\mathcal{L}_{L'}^*(\sigma), \mathcal{L}_L^*(\sigma)). \]

Using a mixed moduli space as in Section 2.5.3, and a family of moving Lagrangian boundary conditions, we find that this map commutes with the action of morphisms groups in $H\mathcal{F}$, and obtain the map from Equation (2.4.18).

We also obtain a commutative diagram

\[
\begin{array}{ccc}
H\mathcal{R}_L^*(\sigma) \otimes H\mathcal{F}^*(\sigma_-, \sigma) \otimes H\mathcal{L}_L^*(\sigma_-) & \longrightarrow & H\mathcal{R}_L^*(\sigma) \otimes H\mathcal{L}_L^*(\sigma) \\
\downarrow & & \downarrow \\
H\mathcal{R}_L^*(\sigma_-) \otimes H\mathcal{L}_L^*(\sigma_-) & \longrightarrow & H\mathcal{F}^*(L, L'),
\end{array}
\]

which proves that we have defined a map from the tensor product of left and right modules over $H\mathcal{F}$ to the Floer cohomology of the pair $(L, L')$.

2.6. The pertubed diagonal. The constructions of Section 2.5 generalise those of Section 2.3 to the global category. In this section, we extend the results of Section 2.4: the key point is to construct a bimodule over $H\mathcal{F}$ whose restriction to $H\mathcal{F}_\sigma$, for each $\sigma \in \Sigma$, agrees with the pullback of the diagonal bimodule of $H\mathcal{P}_\sigma$.

The construction will be implemented via a perturbed Lagrangian Floer cohomology group of fibres. To this end, let $\phi: X \rightarrow X$ be a Hamiltonian diffeomorphism which is generic in the sense that

\[ \text{(2.6.1) for all pairs } (q_-, q) \in Q^2, X_{q_-} \cap \phi X_q \text{ is discrete}. \]

Fix a neighbourhood $\nu(X \cap \phi X)$ in $Q^2 \times X$ of the set

\[ \prod_{(q_-, q) \in Q^2} X_{q_-} \cap \phi X_q \subset Q^2 \times X. \]

We denote by $\nu_X(X_{q_-} \cap \phi X_q)$ the inverse image over $(q_-, q) \in Q^2$, and require that this subset be sufficiently small that

\[ \text{(2.6.3) for all pairs } (q_-, q), \text{ the intersections of } \nu_X(X_{q_-} \cap \phi X_q) \text{ with } X_{q_-} \text{ and } \phi X_q \text{ are inessential}.} \]
Let $\Sigma^\phi$ denote the set $\Sigma \times \{\phi\}$, with elements $\sigma^\phi = (\sigma, \phi)$. Given a pair $\Upsilon = (\sigma_-, \sigma^\phi) \in \Sigma \times \Sigma^\phi$, we introduce the notation
\[(2.6.4) \quad \nu_X \Upsilon \equiv \nu_X (X_{q\sigma_-} \cap \phi X_{q\sigma}).\]

Given a path $J(\Upsilon)$ of almost complex structures on $X$ which are constant away from a fixed compact subset of the interior of $\nu_X \Upsilon$, Lemma \[\text{[A1]}\] provides a reverse isoperimetric constant $C$ for $J(\Upsilon)$ holomorphic strips with boundary conditions $X_{q\sigma_-}$ and $\phi X_{q\sigma}$, with the length measured in the quotient of these Lagrangians by the equivalence relation which identifies two points in the same component of the intersection with $\nu_X \Upsilon$. We require that
\[(2.6.5) \quad \text{diam} \nu_{Q\sigma} < 1/4C.\]

As before, the reverse isoperimetric inequality is independent of the restriction of $J(\Upsilon)$ to the chosen compact subset of the interior of $\nu_X \Upsilon$, and of the intersection of the Lagrangians with this inessential region. This will allow us to pick perturbations in order to achieve transversality.

2.6.1. Perturbed Floer cohomology for pairs of polytopes. Consider, for all pairs
\[(2.6.6) \quad \Upsilon = (\sigma_-, \sigma^\phi) \in \Sigma \times \Sigma^\phi\]
a Hamiltonian function
\[(2.6.7) \quad H_{\Upsilon}: X \times [0, 1] \to \mathbb{R},\]
supported in $\nu_X \Upsilon$. We write $\Phi_{\Upsilon}$ for the composition of $\phi$ with the Hamiltonian diffeomorphism generated by $H_{\Upsilon}$, and assume that
\[(2.6.8) \quad \text{Crit}(\Upsilon) \equiv X_{q\sigma_-} \cap \Phi_{\Upsilon} X_{q\sigma}\]
consists only of transverse intersection points.

By assumption, this is a discrete space, and, given the choice of Pin structures from Section 2.2.2, we obtain a 1-dimensional free abelian group $\delta_x$ associated to each element $x$ of $\text{Crit}(\Upsilon)$. Consider the moduli space
\[(2.6.9) \quad \mathcal{R}(\Upsilon) \to \text{Crit}(\Upsilon)^2\]
of stable finite energy $J(\Upsilon)$-holomorphic strips with boundary condition $X_{q\sigma_-}$ along $t = 0$ and $\Phi_{\Upsilon} X_{q\sigma}$ along $t = 1$, equipped with its natural evaluation map at the ends to intersection points of the boundary Lagrangians. Choosing the almost complex structure generically, the fibre $\mathcal{R}(x_0, x_1)$ over a pair $(x_0, x_1)$ has the expected dimension, and whenever the moduli space is rigid, we obtain a map $\delta_u$ of orientation lines for each $u \in \mathcal{R}(x_0, x_1)$.

Given a curve $u \in \mathcal{R}(x_0, x_1)$, let $\partial_{\Sigma^-} u$ and $\partial_{\Sigma^\phi} u$ denote the boundary components labelled by $\sigma_-$ and $\sigma^\phi$. Parallel transport along these boundaries defines maps
\[(2.6.10) \quad z[\partial_{\Sigma^-} u]; U_{\sigma_-, x_1}^P \to U_{\sigma_-, x_0}^P\]
\[(2.6.11) \quad z[\partial_{\Sigma^\phi} u]; U_{\sigma^\phi, x_0}^P \to U_{\sigma^\phi, x_1}^P.\]

We define the Floer complex
\[(2.6.12) \quad CF^* (\Upsilon) \equiv \bigoplus_{x \in \text{Crit}(\Upsilon)} \text{Hom}^c (U_{\sigma^\phi, x_0}^P, U_{\sigma_-, x}^P) \otimes \delta_x,\]
with differential $\mu^{[0]1[0]}_\phi$ given by
\begin{equation}
\mu^{[0]1[0]}_\phi | \psi \otimes \delta_{x_0,x_1} \equiv \sum_{u \in R(x_0,x_1)} (-1)^{\deg(x_1) + 1} T^E(u) \cdot [\partial_u u] \cdot \psi \cdot \delta_{[\partial_u u] \otimes \delta_u}.
\end{equation}

Condition (2.6.3) implies that $P_\sigma$ and $P_{\sigma_-}$ are contained in the ball of radius $1/4C$ about $q_\sigma$ and $q_{\sigma_-}$. As in Corollary 2.9 we conclude that this differential is well-defined and continuous with respect to the topology on these Floer complexes.

To define a bimodule over $\mathcal{H} \mathcal{F}$, we set
\begin{equation}
\underline{\Sigma}(\sigma_-) \equiv CF^* \circ ((\sigma-, P_{\sigma_-}), (\sigma^\phi, P_\sigma))
\end{equation}
and denote the cohomology by $H\underline{\Sigma}(\sigma-, \sigma)$.

As it is completely analogous to the discussion from Section 2.5.4, we omit the details of the construction of the bimodule structure maps (they will appear in greater generality when we discuss the $A_\infty$ refinement).

2.6.2. Map from the diagonal bimodule. The main result of this section is the existence of a map from the diagonal bimodule on $\mathcal{H} \mathcal{F}$ to the perturbed diagonal:

**Lemma 2.37.** If $\Sigma$ labels a sufficiently fine cover, then for each triple $(\sigma, \tau, \tau_-)$ in $\Sigma$ such that $\tau$ and $\tau_-$ both lie in $\Sigma_\sigma$, there is a natural quasi-isomorphism
\begin{equation}
P_{\sigma}(P_{\tau_-}, P_\tau) \to \underline{\Sigma}(\tau_-, \tau).
\end{equation}
In particular, the restriction of $H\underline{\Sigma}$ to $\mathcal{H} \mathcal{F}_\sigma$ is isomorphic to the pullback of the diagonal bimodule of $H\mathcal{P}_{\sigma^\phi}$.

Fix a basepoint $\sigma \in \Sigma$, and consider elements $\tau$ and $\tau_-$ of $\Sigma_\sigma$. Let $\Upsilon = \{\tau_-, \tau^\phi\}$, and denote by $\overline{\Sigma}$ a copy of $\overline{\Sigma}_\Upsilon$, which we think of as parametrising a disc $S$ with a puncture at $-1$ (we denote the corresponding end by $e$), an interior marked point at $0$, and a boundary marked point at $1$. The basic idea is that the puncture corresponds to the perturbed bimodule, the marked point at $1$ to the diagonal of $H\mathcal{P}_{\sigma^\phi}$, and the interior marked point to the fact that we shall interpolate between the data defining these two bimodules.

We begin by strengthening the conditions imposed at the beginning of Section 2.6.3 by imposing the relevant isoperimetric constraint: let $\partial_S^S S$ be the part of the boundary of $S$ corresponding to the upper semi-circle, and $\partial_S^\phi S$ the lower semi-circle. We pick parametrisations of the paths $q_{\tau_-, \sigma}$ and $q_{\tau, \sigma}$
\begin{align}
q_\Upsilon: \partial_S^S S & \to Q \\
q^\phi_\Upsilon: \partial_S^\phi S & \to Q
\end{align}
which agree with $q_\sigma$ near the marked point $-1$, and have value $(q_{\tau_-}, q_\tau)$ near the end. We also pick a map
\begin{equation}
\Phi_\Upsilon: \partial_S^\phi S \to \Ham(X),
\end{equation}
which is the identity near $-1$, and agrees with $\Phi_{\tau_\Upsilon}$ near the end (recall that $\Upsilon_\sigma = (\tau_-, \tau^\phi)$).

Given this data, we form Lagrangian boundary conditions given by the path $X_{\rho_\Upsilon}$ on $\partial_S^S S$ and $\Phi_\Upsilon X_{q_\Upsilon}$ on $\partial_S^\phi S$ (see Figure 6); by construction, these boundary conditions agree with $X_{q_\sigma}$ at the marked point. Let $X_S$ to be the fibre bundle of Lagrangian boundary conditions over the quotient of the boundary of $S$ by the two components of its intersection with a fixed neighbourhood $\nu SE$ of the negative end over which these paths are locally constant. Define $X_S/\sim$ to be the quotient by the equivalence relation that collapses the intersections.
of $\nu_X Y_e$ with the fibres $X_{q_{r-}}$ and $\Phi_{Y_e} X_q$, over the endpoints. We can associate to each map $u$ from $S$ to $X$ with such boundary conditions a map

$$\partial u: \partial S \to X_{S/\sim}.$$  

We equip this quotient with a metric so that, for a loop in $X_S$, the norm of the homology class in $H_1(X_{q_{r}}, Z)$ is bounded by the length in the quotient.

Let $J(\Upsilon)$ be a family of almost complex structures on the strip which, under a choice of strip-like ends, agrees with $J(\Upsilon_e)$ near the end $e$. According to Lemma A.3 there is a constant $C$ independent of $u$, such that, whenever $u$ is $J(\Upsilon)$-holomorphic, the length of the path $\partial u/\sim$ is bounded by the product of the energy with $C$, up to an additive constant which is also independent of $u$. We require

$$\text{diam} \nu Q \sigma < 1/8 C.$$  

Since $\Sigma$ is finite, we can assume that such an isoperimetric estimate holds independently of the choice of element $\sigma \in \Sigma$. Moreover, by Lemma A.3, the estimate holds uniformly for Floer data (i.e. choices of moving Lagrangian boundary conditions and families of almost complex structures) which are arbitrary in $\nu S e \times \nu_X Y_e$, but are otherwise given by the above fixed choice.

Let $\mathcal{R}(\Upsilon)$ denote the moduli space of finite energy stable $J(\Upsilon)$ holomorphic strips, with moving Lagrangian boundary conditions given by the paths $X_{q_{r-}}$ and $\Phi_{X_{q_{r-}}}$. We have a natural evaluation map at $\pm 1$:

$$\mathcal{R}(\Upsilon) \to \text{Crit}(\tau- \tau_+, \tau_\phi).$$

By construction, the reverse isoperimetric inequality from Equation (2.6.20) applies to this moduli space.

We now consider the fibre product

$$\mathcal{R}(\Upsilon) \equiv \mathcal{R}(\Upsilon) \times_{X_{q_{r-}}} \mathcal{T}(\sigma, \sigma).$$

where $\mathcal{T}(\sigma, \sigma)$ is the moduli space of perturbed half-gradient flow lines introduced in Section 2.2.4. We have a natural evaluation map

$$\mathcal{T}(\Upsilon) \to \text{Crit}(\tau- \tau_+, \tau_\phi) \times \text{Crit}(\sigma, \sigma),$$

whose fibre at a pair $(x_0, x_1)$ we denote

$$\mathcal{T}(x_0, x_1).$$
Choosing paths connecting the intersections of $X_{q\sigma}$ with the sections associated to $\tau$, $\tau\prec$, and $\sigma$ yields isomorphisms of local systems $U_{\tau\prec}^{P_{\tau\prec}} \cong U_{\tau}^{P_{\tau\prec}}$ and $U_{\tau\prec}^{P_{\tau\prec}} \cong U_{\tau}^{P_{\tau\prec}}$. Using this together with parallel transport along the boundary of an element $u$ of this moduli space yields a map
\begin{equation}
\Hom^c(U_{\tau\sigma,\tau\prec}, U_{\tau\sigma,\tau\prec}) \to \Hom^c(U_{\tau\sigma,\tau\prec}, U_{\tau\sigma,\tau\prec}),
\end{equation}
\begin{equation}
\phi \mapsto z[\partial\nu u].\psi \cdot z[\partial\nu u],
\end{equation}
where $\partial\nu u$ and $\partial\nu u$ are the restrictions of $u$ to the boundary components $\partial\Sigma$ and $\partial\Sigma$.

For generic choices of the almost complex structure and the Morse perturbation, $\mathcal{TR}(x_0, x_1)$ is a manifold with boundary of dimension $\text{deg}(x_0) - \text{deg}(x_1)$, and rigid elements of this moduli space induce an isomorphism $\delta_u$ of orientation lines. Tensoring these with the parallel transport maps, we define
\begin{equation}
\kappa: \mathcal{PO}_\sigma(P_{\tau\prec}, P_\tau) \to \mathcal{TR}(\tau, \tau)
\end{equation}
to be given by the sum
\begin{equation}
\kappa_\phi \otimes \delta_{x_1} = - \sum_{u \in \mathcal{TR}^i(x_0, x_1)} T^E(u) \bigotimes \delta_u.
\end{equation}
The reverse isoperimetric inequality for moduli problems with moving Lagrangian boundary conditions, together with Condition (2.6.20), implies that this map is well-defined and continuous.

**Remark 2.38.** The minus sign above accounts for the fact that we use reduced gradings to define the differential on $\mathcal{PO}_\sigma(P_{\tau\prec}, P_\tau)$, but unreduced gradings for $\mathcal{TR}(\tau, \tau)$.

To prove that this map is a chain equivalence, we construct a two-sided homotopy inverse: Given a sequence $\Upsilon = (\tau, \tau\phi)$, we denote by $\mathcal{TR}_\Upsilon$ a copy of $\mathcal{TR}^i$ which we now think of as equipped with a marked point at $-1$ and a puncture at $1$. By reflecting the data chosen above, we obtain moving Lagrangian boundary conditions which agree with $X_{q\sigma}$ near $-1$ and with the pair $(X_{q\sigma}, \Phi_\Upsilon, X_{q\sigma})$, near the puncture. We obtain a moduli space $\mathcal{TR}(\Upsilon)$ with an evaluation map to $X_{q\sigma} \times \text{Crit}(\Upsilon)$. Taking the fibre product over $X_{q\sigma}$ with the moduli space $\mathcal{T}_{\lambda}(\sigma, \sigma)$ of perturbed flow lines, we obtain the moduli space $\mathcal{TR}(\Upsilon)$, equipped with an evaluation map to the product of $\text{Crit}(\sigma, \sigma)$ at the negative end, and $\text{Crit}(\tau, \tau\phi)$ at the positive end. Counts of rigid elements of this moduli space define a map
\begin{equation}
\mathcal{TR}(\tau, \tau) \to \mathcal{PO}_\sigma(P_{\tau\prec}, P_\tau).
\end{equation}

We omit the proof that this is a left and right homotopy inverse to Equation (2.6.27), noting only that this a priori requires a further reverse isoperimetric constraint on the cover. Passing to cohomology proves Lemma 2.37.

2.6.3. **Computing the bimodule: distant polytopes.** In this section, we prove the following result:

**Lemma 2.39.** If the cover $\Sigma$ is sufficiently fine, and $\text{diam} P_\sigma \ll \text{diam} \nu Q \sigma$ for all elements $\sigma \in \Sigma$, then the group $H^\Sigma(\tau, \tau)$ vanishes whenever $P_\tau$ and $P_{\tau\prec}$ are not contained in a common chart $\nu Q \sigma$.

The difference in scale between $P_\sigma$ and $\nu Q \sigma$ will be set by a reverse isoperimetric condition. Consider the space of discs with punctures at $\pm 1$, and two interior marked points, which lie on the real axis and are equidistant from the origin. This moduli space has two
boundaries, corresponding to the two marked points lying at the origin, and to the breaking of the domain into two components.

Fix a Hamiltonian isotopy from the identity to \( \phi \). For each pair \( \Upsilon = (\tau_-, \tau_\phi) \), concatenation with the Hamiltonian isotopy from \( \phi \) to \( \Phi_\Upsilon \) yield a Hamiltonian isotopy from the identity to \( \Phi_\Upsilon \), and in the inverse direction by reversing the direction of the path. As shown in Figure 7 we equip the curves over the moduli space described in the previous paragraph with Lagrangian boundary conditions which are obtained from the constant boundary conditions \( (X_{q_-}, X_{q_\phi}) \) by applying the induced homotopy of paths from \( \Phi_\Upsilon \) to itself, interpolating between the constant path and the concatenation of the path from \( \Phi_\Upsilon \) to the identity with its inverse. The constant path corresponds to the case where the two marked points agree, and the concatenation to the broken curve.

![Figure 7](image_url)

**Figure 7.** The boundary conditions, near from the ends, of the moduli space which induces a null-homotopy of Floer complexes.

We now pick almost complex structures for this family of Lagrangian boundary conditions: for the constant boundary conditions, we use the translation-invariant almost complex structure \( J(\tau_-, \tau_\phi) \) chosen in Section 2.6.1, while near the broken curve, we assume that the almost complex structure is obtained by gluing, and agrees with \( J(\tau_-, \tau_\phi) \) at the ends. Applying Lemma A.4 we obtain a reverse isoperimetric constant for holomorphic curves in this family.

Using the fact that \( X_{q_-} \) and \( X_{q_\phi} \) are disjoint, we find that the composition of maps associated to the broken curve vanishes, since it corresponds to a map factoring through a Floer complex that is the 0 group by definition. Choosing the diameter of the elements of the cover \( \{P_\sigma \}_{\sigma \in \Sigma} \) to be much smaller than that of the cover \( \{\nu Q_\sigma \}_{\sigma \in \Sigma} \), we obtain a null homotopy for the identity map on \( \Sigma(\tau_-, \tau) \), implying Lemma 2.39.

2.7. Global bimodule maps. Given a Lagrangian \( L \in \mathcal{A} \) we have defined left and right modules over \( \mathcal{H} \mathcal{F} \). In Section 2.7.1 we construct a map of bimodules

\[
(2.7.1) \quad \mathcal{H} \mathcal{L}_L \otimes \mathcal{H} \mathcal{R}_L \rightarrow \Delta
\]

by using a moduli space of holomorphic triangles, with one moving Lagrangian boundary condition. We also construct a map of left modules

\[
(2.7.2) \quad \Delta \otimes_{\mathcal{H} \mathcal{F}} \mathcal{H} \mathcal{L}_L \rightarrow \mathcal{H} \mathcal{L}_L.
\]
These are the missing ingredients in the statement of Proposition 2.2 which we proceed to prove in Section 2.7.

In order to compare these constructions with the local ones, we recall that Lemma 2.5 shows that the computation of tensor products with $\Delta$ and morphisms of left module to $\Delta$ are local. The main results of this section are summarised by the following:

**Proposition 2.40.** For each $\sigma \in \Sigma$, there are commutative diagrams of bimodules over $H\mathcal{F}_\sigma$

\[
j^*H\mathcal{L}_{L,\sigma} \otimes j^*H\mathcal{R}_{L,\sigma} \longrightarrow j^*\Delta_{H\mathcal{P}_0}\sigma
\]

and of left $H\mathcal{F}_\sigma$-modules:

\[
j^*H\Delta_{H\mathcal{P}_0}\sigma \otimes_{H\mathcal{F}_{\sigma}} j^*H\mathcal{L}_{L,\sigma} \longrightarrow j^*H\mathcal{L}_{L,\sigma}
\]

Note that the first commutative diagram in the above Proposition is equivalent to

\[
j^*H\mathcal{R}_{L,\sigma} \longrightarrow \text{Hom}_{H\mathcal{F}_\sigma}(j^*H\mathcal{L}_{L,\sigma}, j^*\Delta_{H\mathcal{P}_0}\sigma)
\]

Our previous results establish that all the vertical arrows in Diagrams (2.7.4) and (2.7.5) are isomorphisms if we restrict to the object $\sigma \in H\mathcal{F}_\sigma$. Together with Lemma 2.5 we conclude

**Lemma 2.41.** If $L_\sigma$ meets $X_{q_\sigma}$ at one point, there are natural isomorphisms

\[
\Delta(\sigma, \cdot) \otimes_{H\mathcal{F}} H\mathcal{L}_L \rightarrow H\mathcal{L}_L(\sigma)
\]

\[
H\mathcal{R}_L(\sigma) \rightarrow \text{Hom}_{H\mathcal{F}}(H\mathcal{L}_L, \Delta(\cdot, \sigma))
\]

**Corollary 2.42.** If $L, L' \in \mathcal{L}$ are Lagrangian sections, the composition

\[
H\mathcal{R}_{L'} \otimes_{H\mathcal{F}} H\mathcal{L}_L \longrightarrow \text{Hom}_{H\mathcal{F}}(H\mathcal{L}_{L'}, H\mathcal{L}_L)
\]

\[
\text{Hom}_{H\mathcal{F}}(H\mathcal{L}_{L'}, H\Delta) \otimes_{H\mathcal{F}} H\mathcal{L}_L \longrightarrow \text{Hom}_{H\mathcal{F}}(H\mathcal{L}_{L'}, H\Delta \otimes_{H\mathcal{F}} H\mathcal{L}_L).
\]

is an isomorphism.

**Proof.** It suffices to show that the bottom horizontal map is an isomorphism; this essentially follows from the techniques developed in Section 2.3.4 choosing a generator for $H\mathcal{F}(P_\tau, P_\rho)$
and $H\mathcal{L}_L(\rho)$ as rank-1 modules over $\Gamma^\rho$, we can write morphisms from $H\mathcal{L}_L$ to any module $M$ as the kernel of the map

$$\bigoplus_{\sigma_0 \in \Sigma} M(\sigma_0) \to \bigoplus_{\sigma_0 < \sigma_1 \in \Sigma} M(\sigma_1).$$

(2.7.9)

This implies that $\text{Hom}_{H\mathcal{L}}(H\mathcal{L}_L, \_)$ is right exact. Identifying $\Delta H\mathcal{L} \otimes H\mathcal{L}_L$, as a left module, with the cokernel of

$$\bigoplus_{\rho_0 < \rho_1 \in \Sigma} \Delta H\mathcal{L}(\rho_0, \_) \otimes \Gamma^{\rho_1} \Gamma^{\rho_0} \to \bigoplus_{\rho_0 \in \Sigma} \Delta H\mathcal{L}(\rho_0, \_)$$

(2.7.10)

we obtain the desired result.

**Remark 2.43.** In Section 4.3.1, we provide a less computational proof of the $A_\infty$ refinement of this statement. The reader is invited to adapt that proof to the cohomological level, or alternatively adapt this computational proof to the $A_\infty$ refinement.

\[\square\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{The boundary conditions for elements of the moduli spaces $\overline{\mathcal{R}}(\sigma_-, \sigma^\phi, L)$ and $\overline{\mathcal{R}}(\sigma_-, L, \sigma^\phi)$.}
\end{figure}

### 2.7.1. Construction of the maps.

Let $S$ be a thrice-punctured disc, equipped with a fixed neighbourhood $\nu e$ of each end $e$. Let $\Upsilon$ be a set of labels for the boundary components which is either (i) $(\sigma_-, L, \sigma^\phi)$ or (ii) $(\sigma_-, \sigma^\phi, L)$, with $L \in \mathcal{A}$, and $\sigma, \sigma_- \in \Sigma$. Pick a map

$$\Phi : \partial S \to \text{Ham}(X)$$

(2.7.11)

which agrees with the identity on the component labelled $\sigma_-$, interpolates between $\Phi_{L, \sigma}$ and $\Phi_{L, \sigma_-}$ on the component labelled $L$, and interpolates between the identity and $\Phi_{\sigma_-, \sigma^\phi}$ on the component labelled $\sigma^\phi$. Applying these Hamiltonians to the Lagrangians $L$, $X_{q^\phi}$, and $X_{q^-}$ yields moving Lagrangian boundary conditions on $S$ (see Figure 8).

In order to straightforwardly appeal to the reverse isoperimetric inequality, we require that the restriction of the path $\Phi$ to each end $\nu se$ agree away from a fixed compact subset of the interior of $\nu S \Upsilon e$ with (i) $\phi$ along the boundary component labelled by $\sigma^\phi$ and (ii) the identity on every other component. This restriction is exactly the same as the one made in the choice of perturbations used to define the Floer cohomology groups of fibres with $L$, or in Section 2.6 for pairs for fibres. We choose a family of almost complex structures on $X$ parametrised by $S$, whose restriction to $\nu se$ is obtained from $J(\Upsilon e)$ by a choice of strip-like ends. We denote by $\overline{\mathcal{R}}(\Upsilon)$ the corresponding moduli space of stable holomorphic discs.
We write $X_S$ for total space of the boundary conditions labelled $\sigma$ and $\sigma^\phi$, modulo the relation which identifies each component of the inverse image of $\nu e$ to the corresponding fibre; this is a fibre bundle over a pair of closed intervals. We consider the equivalence relation $\sim$, which is the identity in the interior, and which is given, at a boundary Lagrangian labelled by an end $e$, by collapsing the components of the intersection with $\nu_X e$ to a point. We equip this quotient with a metric such that norm of the homology class of a loop is bounded by the length of the projection.

Lemma A.3 provides a reverse isoperimetric constant $C$ for each element of $\mathcal{R}(\Upsilon)$. We require that
\begin{equation}
\text{diam} \nu Q_\sigma, \text{diam} \nu Q_\sigma^- < \frac{1}{8C}.
\end{equation}

This condition ensures the convergence of the count of rigid elements of the moduli space $\mathcal{R}(\Upsilon)$, yielding maps
\begin{align}
\Sigma(\sigma^-, \sigma) \otimes \mathcal{L}_L(\sigma^-) & \rightarrow \mathcal{L}_L(\sigma) \\
\mathcal{L}_L(\sigma) \otimes \mathcal{R}_L(\sigma^-) & \rightarrow \Sigma(\sigma^-, \sigma).
\end{align}

Passing to cohomology, we obtain the maps
\begin{align}
H\Sigma(\sigma^-, \sigma) \otimes H\mathcal{L}_L(\sigma^-) & \rightarrow H\mathcal{L}_L(\sigma) \\
H\mathcal{L}_L(\sigma) \otimes H\mathcal{R}_L(\sigma^-) & \rightarrow H\Sigma(\sigma^-, \sigma).
\end{align}

2.7.2. Proof of Proposition 2.2 Consider the moduli space $\mathcal{R}_4$ of holomorphic discs with 4 boundary punctures, one of which is distinguished as outgoing. Given a quadruple $\Upsilon = (L, \sigma^-, L', \sigma^\phi)$, with $\sigma, \sigma^-$ in $Q$ and $L, L' \in A$, we denote by $\mathcal{R}_\Upsilon$ a copy of $\mathcal{R}_4$ with the corresponding boundary labels, and by
\begin{equation}
\mathcal{S}_\Upsilon \rightarrow \mathcal{R}_\Upsilon
\end{equation}
the universal curve over this moduli space. Recall that $\mathcal{R}_\Upsilon$ is homeomorphic to a closed interval, with boundary given by the two possible configurations of stable discs consisting of two components each of which is disc with 3 punctures. These configurations are distinguished by the fact that the node is labelled by the pair $(L, L')$ in one case and by $(\sigma^-, \sigma^\phi)$ in the second.

We choose families of strip-like ends for all surfaces over $\mathcal{R}_\Upsilon$, which we assume are compatible with those made in Sections 2.5.6, 2.6.1, and 2.7.1 over the boundary strata. The choices made in these sections also determine moving Lagrangian boundary conditions on the fibres over the boundary. We extend these choices to arbitrary surfaces representing elements of $\mathcal{R}_\Upsilon$, compatibly with gluing near the boundary of the moduli space, in such a way that the moving Lagrangian boundary conditions are (i) constant (with value $X_q\sigma$) along the boundary with label $\sigma^-$, (ii) give a family of paths interpolating between $\Phi_{\sigma, L}(L)$ and $\Phi_{\sigma^-, L}(L')$, or $\Phi_{\sigma, L'}(L')$, along the boundary components with label $L$ and $L'$, and (iii) give a family of paths with endpoints on $X_q\sigma$, interpolating between the constant path and the concatenation of a path from $X_q\sigma$ to $\Phi_{\sigma^-, \sigma^\phi}(X_{q^-})$ and its inverse along the remaining boundary. Our previous choices also determine families of almost complex structures on $X$ parametrised by the fibres over the endpoints of the moduli space; we extend them to a family $J_S$ over each curve $S$ in $\mathcal{R}_\Upsilon$ in such a way that the almost complex structure along the boundary components labelled $L$ and $L'$ are the pushforwards of $J_L$ and $J_{L'}$ by the corresponding Hamiltonian diffeomorphisms, and the restrictions to the ends agree with the choices made in the construction of the Floer cohomology groups.
Each such surface is equipped with a decomposition which we call the thick-thin decomposition; for a surface far from the boundary, the thin parts are neighbourhoods of the ends where the boundary conditions are locally constant and the almost complex structure is obtained by pull-back from an interval by a choice of strip-like end. For a surface near the boundary, there is an additional component coming from the gluing region. Each component $\Theta$ of the thin part has a corresponding subset $\nu_X \Theta$ of $X$ equipped with a compact subset of the interior away from which the restrictions to $\Theta$ of the boundary conditions and the almost complex structures are constant.

For each surface $S \in \mathcal{S}$, let $X_S$ denote the fibre bundle over the pair of closed intervals obtained from the subset of $\partial S$ labelled by $\sigma^\phi$ and $\sigma^-$ by collapsing the components of the intersection with the thin part. We denote by $X_S/\sim$ the quotient of $X_S$ by the equivalence relation which, in each fibre corresponding to a component $\Theta$ of the thin part, identifies the components of the intersection with $\nu_X \Theta$ to points. As before, we pick a metric on this quotient so that the lengths of loops in the quotient provide a bound for the norm of the homology class in a fibre.

By the results of Appendix A, we can find a constant $C$ which uniformly controls the length of the projection to $X_S/\sim$ of the boundary of $J_S$-holomorphic curves in $X$ with the above boundary conditions. We then require that

\begin{equation}
\text{diam } \nu_X \sigma < \frac{1}{8C}.
\end{equation}

By construction, we have an evaluation map

\begin{equation}
\mathcal{R}(Y) \to \text{Crit}(L, \sigma^-) \times \text{Crit}(\sigma^-, L') \times \text{Crit}(L', \sigma) \times \text{Crit}(L, \sigma).
\end{equation}

Choosing the data generically, the count of rigid elements of these moduli spaces thus induces a map

\begin{equation}
\mathcal{R}_{L'}(\sigma_-) \otimes \mathcal{L}_L(\sigma_-) \rightarrow \text{Hom}_A(\mathcal{L}_{L'}(\sigma), \mathcal{L}_L(\sigma))
\end{equation}
which gives a homotopy for the following diagram

$$
\begin{array}{c}
\mathcal{R}_{L'}(\sigma_-) \otimes \mathcal{L}_L(\sigma_-) \\
\downarrow \\
\text{Hom}_\Lambda(\mathcal{L}_{L'}(\sigma_-), \mathcal{L}(\sigma_-)) \otimes \mathcal{L}_L(\sigma_-) \\
\downarrow \\
\text{Hom}_\Lambda(\mathcal{L}_{L'}(\sigma), \mathcal{L}_L(\sigma)),
\end{array}
$$

because the boundary strata of $\mathcal{R}(\Upsilon)$ over the boundary of $\mathcal{R}_\Upsilon$ correspond to the two compositions. Passing to cohomology, we obtain the commutativity of Diagram (2.1.1).

2.7.3. Compatibility of the maps of left modules. We now prove the commutativity of Diagram (2.7.4). Let $\mathcal{R}_{3,2}$ denote the moduli space of discs with 2 boundary punctures, one of which is distinguished as outgoing and the other as incoming, together with one boundary marked point, and 2 interior marked points. Consider the embedding

$$
\mathcal{R}_{3} \times (\mathcal{R}_{2,1})^2 \subset \mathcal{R}_{3,2}
$$

shown on the right of Figure 10 and corresponding to the configuration where one interior marked point escapes at each of the boundary marked point and incoming puncture. For the appropriate choices of Lagrangian boundary conditions and families of almost complex structures, this embedding corresponds to the composition of the bottom and left arrows in Diagram (2.7.4). We also have an embedding

$$
\mathcal{R}_{2,1} \times \mathcal{R}_3 \subset \mathcal{R}_{3,2}
$$

corresponding to the configuration where the two interior marked points agree, and escape to the outgoing end. Note that this is a slight abuse of notation, as we should add a factor on the left corresponding to the sphere with 3 marked points that bubbles off when two interior points collide, but we shall equip this stable disc with the Floer data obtained by forgetting the bubble, so that we can ignore it. This stratum corresponds to the other composition in Diagram (2.7.4). To construct a homotopy, we fix a path

$$
\mathcal{R}_{3,2} \subset \mathcal{R}_{3,2}
$$

interpolating between these two strata.

Figure 10. The boundary of the moduli space giving rise to the homotopy between the two compositions in Diagram (2.7.4).
Let $\sigma$ be a basepoint in $\Sigma$, $L$ a Lagrangian in $\mathcal{A}$, and $\tau, \tau^-$ elements of $\Sigma_{\sigma}$. Let $\Upsilon$ denote the sequence $(L, \tau^-, \tau \phi)$, and denote by $\mathcal{R}_{\Upsilon}$ a copy of the space $\mathcal{R}_{2,2} \Sigma_{\sigma}$, with corresponding labels on the complement in the boundary of the marked point. Let $\mathcal{S}_{\Upsilon}$ denote the universal curve.

Over the boundary boundary stratum of $\mathcal{R}_{\Upsilon}$ labelled by a configuration with two disc components, the boundary labels are given by $(L, \tau \phi)$ on the disc carrying an interior marked point, and $(L, \tau^-, \tau \phi)$ on the other. The first case was already considered in Section 2.5.5, where we chose a path of Lagrangians between the pairs $(L_\sigma, X_{q_\sigma})$ and $(L_\tau, X_{q_\tau})$. In the second case, we use the triple of constant Lagrangians $(L_\sigma, X_{q_\sigma}, X_{q_\sigma})$.

Over the other boundary stratum of $\mathcal{R}_{\Upsilon}$ the choice of moving Lagrangian boundary conditions is immediately obtained from the choices made in Sections 2.5.5 for the disc with labels $(L, \tau \phi)$, in Section 2.6.2 for the disc with interior marked point with labels $(\tau^-, \tau \phi)$, and in Section 2.7.1 for the disc with labels $(L, \tau^-, \tau \phi)$.

We now pick families of moving Lagrangian boundary conditions, which are Hamiltonian along the boundary labelled $L$, and almost complex structures on the universal curve over $\mathcal{R}_{\Upsilon}$, which near the endpoints of this interval, are obtained by gluing, up to a perturbation term supported in the thick part away from the boundaries in which we apply the reverse isoperimetric inequality. In this way, we achieve transversality using standard methods for the corresponding moduli space $\mathcal{R}(\Upsilon)$, while still being able to appeal to Lemma A.4 to ensure the existence of a uniform reverse isoperimetric constant for elements of this moduli space. To avoid bubbling problems, we assume that the family of almost complex structures at each point $z$ along the boundary condition labelled $L$ is obtained from $J_L$ by applying the Hamiltonian diffeomorphism mapping $L$ to the boundary condition over $z$. We also assume that the boundary condition at the marked point with label $(\tau^-, \tau \phi)$ agrees with $X_{q_\sigma}$, so that we have an evaluation map

$$\mathcal{R}(\Upsilon) \rightarrow X_{q_\sigma}.$$  (2.7.25)

Assuming that the diameter of $\nu_Q \sigma$ is sufficiently small relative to this constant, the count of elements of the fibre product

$$\mathcal{T}\mathcal{R}(\Upsilon) \equiv \mathcal{R}(\Upsilon) \times_{X_{q_\sigma}} \mathcal{T}_+(\sigma, \sigma)$$  (2.7.26)

space defines a homotopy for the diagram

$$\text{Po}_\sigma (P_{\tau^-}, P_\tau) \otimes \mathcal{L}_{L_\sigma} (P_{\tau^-}) \rightarrow \mathcal{L}_{L_\sigma} (P_\tau)$$

$$\downarrow$$

$$\mathcal{N}(\tau^-, \tau) \otimes \mathcal{L}_L (\tau) \rightarrow \mathcal{L}_L (\tau).$$  (2.7.27)

Passing to cohomology yields the commutativity of Diagram (2.7.4).

2.7.4. **Comparison with the map to the perturbed diagonal.** In this section, we prove the commutativity of Diagram (2.7.1); most of the arguments are essentially the same as in the previous section, except that we shall encounter a moduli space with a boundary facet containing a node for which the boundary label is given by a pair of Lagrangians which agree. This requires us to add to the moduli space an additional component consisting of pairs of discs connected by a gradient flow line of varying length. This type of additional cobordism appears throughout the literature when combining Morse-theoretic and Floer-theoretic moduli spaces.
Given a triple $\Upsilon = (\tau_-, L, \tau)$ with $L \in A$ and $\tau, \tau_- \subset \nu Q \sigma$ as before, we shall construct a moduli space interpolating between the two moduli spaces defining the compositions being compared. To this end, we let $R\Upsilon$ denote a copy of the space $R_{3,2}$, which we now think of as parametrising discs with 3 boundary punctures and 2 interior marked points whose position is constrained, as illustrated in the middle and right of Figure 11.

Over the boundary stratum of $R\Upsilon$ labelled by a configuration consisting of three discs, the moving Lagrangian boundary conditions are given by the choices made in Section 2.5.5 for the two discs carrying marked points, and by those in Section 2.7.1 for the thrice-punctured disc.

Over the other boundary stratum of $R\Upsilon$, the choice of basepoint $\sigma \in \Sigma$ yields moving Lagrangian conditions on the boundary of the disc carrying the interior marked point, which were fixed in Section 2.6.2. We forget the labels on the second component, and consider the constant boundary conditions $(X_{q_1}, L, X_{q_2})$. Note that these two conditions are compatible because, near the marked point, the boundary conditions considered in Section 2.6.2 are constant with value $X_{q_2}$; this is why we label the middle of Figure 11 with a node connecting the two components (instead of a pair of ends with matching labels).

As in the previous section we interpolate between these two boundary conditions, and the corresponding choices of almost complex structures, to obtain a moduli space $R(\Upsilon)$, equipped with an evaluation map:

$$R(\Upsilon) \to \text{Crit}(L, \sigma) \times \text{Crit}(\sigma, L) \times \text{Crit}(\tau_-, \tau^\phi).$$

There is a reverse-isoperimetric constant for these moduli space, with respect to which we impose the condition that the cover be sufficiently small.

The stratum of the boundary of this moduli space corresponding to discs with 3 components evidently gives rise to the composition

$$j^* H\mathcal{R}_{L, \sigma} \to H\mathcal{R}_L \to \text{Hom}_{H\mathcal{R}_L, \sigma} (j^* H\mathcal{L}_{L, \sigma}, \Delta).$$
around the left and bottom of Diagram (2.7.5). The other boundary does not, however, correspond to the other composition, because it should involve a Floer cohomology group of local systems over $X_{q_\sigma}$, which has been defined using Morse theory.

We therefore introduce a space $\mathcal{T}_{[0, \infty]}(\sigma, \sigma) \to X_{q_\sigma}$ corresponding to the two ends of the flow line. The boundary is covered by a copy of $X_{q_\sigma}$ on which the evaluation map is the diagonal (corresponding to the gradient flow line of length 0), and the closure of the codimension 1 stratum

\begin{equation}
\mathcal{T}_+(\sigma, \sigma) \times_{\text{Crit}(\sigma, \sigma)} \mathcal{T}_-(\sigma, \sigma)
\end{equation}

which corresponds to gradient lines of infinite length. The compactification is obtained by allowing broken flow lines with multiple components, i.e. by adding the strata:

\begin{equation}
\mathcal{T}_+(\sigma, \sigma) \times_{\text{Crit}(\sigma, \sigma)} \mathcal{T}_-(\sigma, \sigma)
\end{equation}

We now define $\mathcal{R}(\Upsilon)$ to be the union of $\mathcal{R}(\Upsilon)$ with the fibre product

\begin{equation}
\mathcal{R}(\Upsilon_{\text{cyc}}) \times_{X_{q_\sigma}} \mathcal{T}_{[0, \infty]}(\sigma, \sigma) \times_{X_{q_\sigma}} \mathcal{R}(\sigma, L, \sigma),
\end{equation}

where $\Upsilon_{\text{cyc}} = (\tau_-, \tau^b)$. By construction, we have an evaluation map

\begin{equation}
\mathcal{T}(\Upsilon) \to \text{Crit}(L, \sigma) \times \text{Crit}(\sigma, L) \times \text{Crit}(\tau_-, \tau^b).
\end{equation}

For generic choices of Floer and Morse data, this moduli space gives rise to a homotopy for the diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{L}_{L_\sigma}(P_\tau) \otimes \mathcal{R}_{L_\sigma}(\tau_-) & \longrightarrow & P_\sigma(\tau_-, \tau) \\
\downarrow & & \downarrow \\
\mathcal{L}_L(\tau) \otimes \mathcal{R}(\tau_-) & \longrightarrow & \mathcal{A}(\tau_-, \tau).
\end{array}
\end{equation}

Passing to cohomology, and using adjunction, we obtain the commutativity of Diagram (2.7.4).

3. Higher moduli spaces

In this section, we consider the families of holomorphic curves which will be used in the $\mathbb{A}_\infty$ refinement of the constructions of Section 2. We separate considerations of convergence and transversality by proceeding in two steps: (i) we first write families of equations with moving Lagrangian boundary conditions for which we state reverse isoperimetric inequalities that are robust under a certain class of perturbations, then (ii) we perturb these equations to achieve transversality among the Lagrangian boundary conditions and regularity for the moduli spaces of holomorphic curves, within the allowable class.

3.1. Abstract moduli spaces of discs. A stable tree is a finite tree $T$ with edges $E(T)$ and vertices $V(T)$ such that the valency of every vertex is larger than or equal to 3. A stable planar tree is a tree as above, equipped with a cyclic ordering of the edges $E(T_v)$ of the tree $T_v$ consisting of the edges adjacent to any vertex $v$. We write $F(T)$ for the set of flags. We allow edges which are adjacent to a single vertex, and which are called external and form a set denoted $E^{\text{ext}}(T)$, and fix a distinguished such edge (the root) which we call outgoing, forming the singleton $E^{\text{out}}(T)$. The remaining external edges are called incoming and denoted $E^{\text{in}}(T)$. The path from a vertex $v$ to the root determines a unique outgoing
edge $e_v^\text{in}$ of $T_v$, we call the remaining edges adjacent to $v$ incoming, and denote the sets of incoming and outgoing edges at a vertex $v$ by $E_v^\text{in}(T)$ and $E_v^\text{out}(T)$. The cyclic ordering and the choice of outgoing edge give rise to a unique ordering of $E_v(T)$ compatible with the cyclic structure with the property that $e_v^\text{out}$ is the last element.

The cyclic ordering determines an isotopy class of proper embeddings of the corresponding topological tree in the plane, and the choice of outgoing edge determines an ordering of the components of $R^2 \setminus T$ given counterclockwise starting with the component which is to the left of the outgoing edge when directed outwards. Given a set $A$, each ordered subset $\Upsilon$ of $A$ consisting of $E^\text{ext}(T)$ elements induces a map

$$\pi_0(R^2 \setminus T) \to A.$$  

We assign to each edge $e$ of $T$ the ordered pair $\Upsilon_e$ consisting of the labels of the two regions adjacent to $e$, with the convention that, if $e$ is oriented towards the outgoing edge, the label which is to the left appears first. We denote by $\Upsilon_v$ the induced label on the tree $T_v$ associated to each vertex $v$.

In this section, we shall consider as labels ordered subsets $\Upsilon$ of $Q \amalg Q^\phi \amalg A$ (where $Q^\phi = Q \times \{\phi\}$) such that

(i) there are at most $n+1$ distinct elements of $Q$ (respectively $Q^\phi$), and 2 elements of $A$ appearing in $\Upsilon$ (ii) the elements of $Q$ (respectively $Q^\phi$) are consecutive in $\Upsilon$ and are contained in a ball of radius 1, (iii) all elements of $Q$ precede those of $Q^\phi$, and (iv) any elements of $Q^\phi$ appear last.

In other words, the sequence is of the form

$$(L_1, \ldots, L_j, q_0, q_0^\phi, L_1', \ldots, L_k', q_1^\phi, \ldots, q_\ell^\phi)$$

where $j + k \leq 2$.

Given such a label $\Upsilon$, let $\mathcal{K}_T$ denote the moduli space of stable holomorphic discs with (i) a marked point for each cyclically successive pair of elements of $Q$ or $Q^\phi$ in $\Upsilon$ and (ii) a puncture for each other cyclically successive pair of elements of $\Upsilon$. We assume that the cyclic ordering induced by the ordering of $\Upsilon$ corresponds to the counterclockwise ordering around the boundary of the disc. There are thus $|T|$ punctures and marked points in total, and the boundary components of the complement of the marked points are labelled by the elements of $\Upsilon$. In addition, there is a distinguished puncture corresponding to the first and last element of $\Upsilon$, which we call outgoing. For each tree $T$ labelled by $\Upsilon$, we then define

$$\mathcal{K}_T^\Upsilon = \prod_{v \in V(T)} \mathcal{K}_{\Upsilon_v}.$$  

We distinguish the subset $V^\text{deg}(T) \subset V(T)$ consisting of vertices with degenerate labels, i.e. such that the label is contained in $Q$ or $Q^\phi$.

Each map $T \to T'$ which collapses internal edges induces an inclusion

$$\mathcal{K}_T^\Upsilon \to \mathcal{K}_{T'}^\Upsilon'$$

and the tree with a unique vertex corresponds to the moduli space $\mathcal{K}_T$. Moreover, if $\Upsilon \to \Upsilon'$ is a map of ordered sets which collapses successive elements that are equal, we obtain a (forgetful) map of moduli spaces

$$\mathcal{K}_T \to \mathcal{K}_{\Upsilon'}. $$

Let $\mathcal{S}_T$ denote the universal curve over $\mathcal{K}_T$. For $L \in A$, $q \in Q$, or $q^\phi \in Q^\phi$, we denote by $\partial_L \mathcal{S}_T$, $\partial_q \mathcal{S}_T$, and $\partial_{q^\phi} \mathcal{S}_T$ the corresponding boundary segment of the complement of the
marked points. We write $\partial Q S_T$ and $\partial Q S_T^*$ for the union of boundary components labelled by elements of $Q$ or $Q^\phi$.

For each tree $T$, we write $S_T^T$ for the fibre over the stratum $R_T^T$. This space decomposes as a disjoint union of components labelled by the vertices of $T$:

$$S_T^T \equiv \bigcup_{v \in V(T)} S_T^v$$

$$S_T^T \equiv S_{T,v} \times_{R_T} R_T^T.$$  

We shall need to consider certain moduli spaces of discs with conformal constraints. Fix a basepoint $q_*$ on $Q$, and consider ordered subsets $\Upsilon$ of $Q \sqcup Q^\phi \sqcup A$ of the form

$$(L, q_0, \ldots, q_e)$$

$$(q^{-r}, \ldots, q-0, L)$$

$$(q^{-r}, \ldots, q-0, q_0^\phi, q_1^\phi, \ldots, q_\ell^\phi)$$

$$(L, q^{-r}, \ldots, q-0, q_0^\phi, q_1^\phi, \ldots, q_\ell^\phi)$$

$$(q^{-r}, \ldots, q-0, L, q_0^\phi, q_1^\phi, \ldots, q_\ell^\phi),$$

where all elements of $Q$ and $Q^\phi$ are within distance 1 of $q_*$. In the first three cases, we define $R_T^\Upsilon$ to be the inverse image of the 0-dimensional submanifold $R_{2,1} \subset R_{2,1}$ fixed in Section 2.5.5 while in the last two cases, we define it to be the inverse image of the 1-dimensional submanifold $R_{3,2} \subset R_{3,2}$ fixed in Section 2.7.3. The maps to $R_{2,1}$ and $R_{3,2}$ are obtained by forgetting all marked points labelled by pairs of elements of $Q$ or $Q^\phi$.

**Remark 3.1.** There is a minor difference between the moduli spaces $R_T^\Upsilon$ for $\Upsilon$ given by Equations (3.1.12) and (3.1.13). In the first case, we consider $R_{3,2}$ as parametrising discs with one boundary marked point, and two punctures, while in the second, we have three boundary punctures. The difference will be apparent when we discuss the boundary conditions that will be imposed on these moduli spaces.

The boundary strata of $R_T^\Upsilon$ are also labelled by trees. To describe them, we begin by associating to each stratum of $R_{2,1}$ or $R_{3,2}$ a rooted planar tree $T$ equipped with a distinguished subset of vertices denoted $N(T)$ (for nodes) consisting of those vertices which correspond to disc components which carry an interior marked point. For $R_{2,1}$, we thus obtain a tree with a unique bivalent node, and no other vertices. In the same way, each boundary stratum of $R_T^\Upsilon$ corresponds to a planar tree with a distinguished set of vertices $N(T) \subset V(T)$ which are allowed to be bivalent, and such that the tree obtained by forgetting all incoming edges with labels pairs of elements in $Q$ or $Q^\phi$ corresponds to a tree labelling a boundary stratum of $R_{2,1}$ or $R_{3,2}$.

We distinguish the set of vertices $V^*(T) \subset V(T)$ with the property that they separate an incoming Floer edge from all nodes. Writing $\Upsilon_v^*$ for the sequence obtained from $\Upsilon_v$ by replacing all points in $Q$ or $Q^\phi$ by $q_*$, we have natural product decomposition

$$R_T^\Upsilon \equiv \prod_{v \in N(T)} R_T^v \times \prod_{v \in V^*(T)} R_{T_v} \times \prod_{v \notin N(T) \cup V^*(T)} R_{\Upsilon_v}.$$  

**Remark 3.2.** The distinction between $R_{T_v}$ and $R_{\Upsilon_v}$ is entirely formal, as the two spaces are naturally homeomorphic. As indicated by the labelling of Figure 12 we shall use these moduli spaces to extend the construction of Section 2.6.2 which should indicate to the reader why we introduce this notation.
As before, we also define the set $V^{\deg}(T) \subset V(T) \setminus N(T)$ to consist of those vertices with label $\Upsilon_v$ contained in $Q$ or $Q^\phi$.

By construction, elements of $\overline{R}_\Upsilon$ correspond to curves with at most two incoming ends. We set the labels on these incoming ends to be

$$\Upsilon_e = (L, q_\ast) \text{ or } \Upsilon_e = (q_\ast, L),$$

while for all other ends we use the planar embedding from Equation (3.1.1) to set the label.

Let $\overline{S}_\Upsilon$ denote the universal curves over $\overline{R}_\Upsilon$, obtained by restriction of the universal curve over $\overline{R}_\Upsilon$. As before, given a tree $T$, the fibre $\overline{S}_T$ over $\overline{R}_\Upsilon$ can be written as a union
of components labelled by vertices of $T$, and which we denote $\mathcal{S}_T$. We have

$$
\mathcal{S}_T \equiv \begin{cases} 
\mathcal{S}_T \times \tau^T_T \times R_T & v \in N(T) \\
\mathcal{S}_T \times \tau^V_T \times R_T & v \in V^*(T) \\
\mathcal{S}_T \times \tau^T_T \times R_T & \text{otherwise.}
\end{cases}
$$

3.1.1. Strip-like ends, gluing charts, and thin-thick decompositions. Recall that a positive
(respectively negative) strip-like end on a Riemann surface $S$ is a holomorphic map

$$
[0, \infty) \times [0, 1] \to S \text{ or } (-\infty, 0] \times [0, 1] \to S
$$

which is a biholomorphism in a neighbourhood of a puncture. We shall equip the outgoing
puncture of a curve in $\mathcal{R}_T$ with a negative strip-like end, and all other punctures with positive
strip-like ends. As in [16, Section (9f)], we make this choice consistently for all curves in $\mathcal{R}_T$ and
for all labels $\Upsilon$. The key consistency condition is that, near the boundary strata of $\mathcal{R}_T$, the
strip-like ends are compatible with the gluing maps which are constructed as follows:
for each tree $T$ labelling a stratum of $\mathcal{R}_T$, the choices of ends induces an open embedding
(the gluing map)

$$
\mathcal{R}_T \times (0, \infty)^{E(T)} \to \mathcal{R}_T.
$$

Fixing the identification of $(0, \infty]$ with $(-1, 0]$ which takes $R$ to $-e^{-R}$, we obtain charts

$$
\mathcal{R}_T \times (-1, 0)^{E(T)} \to \mathcal{R}_T
$$

which equip the moduli space with the structure of a smooth manifold with corners.

The gluing map lifts at the level of universal curves to a map

$$
\mathcal{S}_T \times (-1, 0)^{E(T)} \to \mathcal{S}_T,
$$

which is surjective over the image of the gluing map at the level of moduli spaces. The same
discussion applies to the moduli spaces $\mathcal{R}_T$: choices of strip-like ends induce gluing charts
near the boundary strata, which lift to the universal curve.

The gluing charts equip each curve $S$ in $\mathcal{R}_T$ or $\mathcal{R}_T$ with a thick-thin decomposition:
the thin part will be the union of the image under the gluing maps of (i) the strip-like
ends, (ii) the curves corresponding to degenerate vertices, and (iii) curves which become
unstable upon omitting repeated elements of $Q$ or $Q^\phi$. The thick part is the complement.
We associate to each component $\Theta$ of the thin part, the subset $\nu_X \Theta$ of $X$ given by (i) the
empty set if all labels appearing on the intersection of the boundary with $e$ are contained in
$Q$ or $Q^\phi$, (ii) the set $\nu_X \Upsilon_e$ if $\Theta$ contains the image of an end $e$ under gluing. Here, we set
$\nu_X \Upsilon$ to be (i) $X$ if $\Upsilon$ is degenerate, (ii) the set $\nu_X L$ if $\Upsilon$ consists of an element of $Q \cup Q^\phi$
and $L \in \mathcal{A}$, and (iii) the set $\nu_X (X_q \cup \phi X_q)$ if $\Upsilon = (q, q^\phi)$.

By construction, any two ends with images in $\Theta$ have the same label, so this definition
is consistent (it is important here to distinguish ends from marked points).

Finally, for each $S$ in $\mathcal{S}_T$, we fix neighbourhoods $\nu \partial Q S$ and $\nu \partial^\phi Q S$ in $S$ of the corresponding
boundary components of $S$, which are compatible with gluing, and which only intersect each
other in the thin part. If $S$ contains a component with degenerate labels, we assume that such
a component is contained in $\nu \partial Q S$ or $\nu \partial^\phi Q S$ as long as there is a part of the boundary with
the corresponding label. We write $\nu \partial Q \mathcal{S}_T$ and $\nu \partial^\phi Q \mathcal{S}_T$ for the union of such neighbourhoods
over all points $S \in \mathcal{R}_T$. 
We also choose such neighbourhoods of the boundary components of $S \in \mathcal{S}_\Upsilon$; the only difference is that the neighbourhood $\nu \partial Q S$ and $\nu \partial \phi Q S$ should intersect in the union of the thin part with a neighbourhood of any marked point with label in $Q \sqcup Q^\phi$.

3.2. Unperturbed families of equations. Fix a metric on the base $Q$, and assume the contractibility of the spaces of paths of length bounded by any constant smaller than $2n+4$, with endpoints fixed to be any two points of distance bounded by 2.

3.2.1. Paths of fibres. For each non-degenerate sequence $\Upsilon$, we choose maps

\begin{align}
q_{\Upsilon}: \partial Q \mathcal{S}_{\Upsilon} &\to Q \\
q^\phi_{\Upsilon}: \partial Q^\phi \mathcal{S}_{\Upsilon} &\to Q,
\end{align}

of lengths respectively bounded by twice the number of elements of $\Upsilon$ lying in $Q$ and $Q^\phi$, and varying smoothly in the choice of labels. We require these maps to satisfy the following additional properties:

1. (Values along the ends) The restrictions of $q_{\Upsilon}$ and $q^\phi_{\Upsilon}$ to the intersection of each end with $\partial Q \mathcal{S}_{\Upsilon}$ and $\partial Q^\phi \mathcal{S}_{\Upsilon}$ are constant with value $q$.

2. (Compatibility with gluing) For each tree $T$ labelling a boundary stratum of $\overline{\mathcal{R}}_{\Upsilon}$, the restrictions of $q_{\Upsilon}$ and $q^\phi_{\Upsilon}$ to a neighbourhood of $\overline{\mathcal{R}}_{\Upsilon}$ are obtained by gluing in the sense that the following diagram commutes in a neighbourhood of the origin:

\[
\left(\partial Q \mathcal{S}^T_{\Upsilon} \sqcup \partial Q^\phi \mathcal{S}^T_{\Upsilon}\right) \times (-1,0]^{E(T)} \xrightarrow{q_{\Upsilon} \sqcup q^\phi_{\Upsilon}} \partial Q \mathcal{S}_{\Upsilon} \sqcup \partial Q^\phi \mathcal{S}_{\Upsilon} \xrightarrow{q_{\Upsilon} \sqcup q^\phi_{\Upsilon}} Q.
\]

3. (Compatibility with boundary) For each tree $T$ labelling a boundary stratum of $\overline{\mathcal{R}}_{\Upsilon}$ and vertex $v \in V^{\text{deg}}(T)$, the restriction of $q_{\Upsilon}$ and $q^\phi_{\Upsilon}$ to $\mathcal{S}_{\Upsilon}$ is constant. Otherwise, these data are compatible with $(q_{\Upsilon}, q^\phi_{\Upsilon})$ in the sense that we have a commutative diagram

\[
\partial Q \mathcal{S}^v_{\Upsilon} \sqcup \partial Q^\phi \mathcal{S}^v_{\Upsilon} \xrightarrow{q_{\Upsilon} \sqcup q^\phi_{\Upsilon}} \partial Q \mathcal{S}_{\Upsilon} \sqcup \partial Q^\phi \mathcal{S}_{\Upsilon} \xrightarrow{q_{\Upsilon} \sqcup q^\phi_{\Upsilon}} Q.
\]

4. (Forgetful maps) If $\Upsilon$ is obtained from a sequence $\Upsilon'$ by repeating elements of $Q$ or $Q^\phi$, the choices of paths are compatible with the induced forgetful maps in the sense that the following diagrams commute:

\[
\partial Q \mathcal{S}_{\Upsilon} \xrightarrow{q_{\Upsilon}} Q \quad \partial Q^\phi \mathcal{S}_{\Upsilon} \xrightarrow{q^\phi_{\Upsilon}} Q.
\]
(5) (Forgetting $\phi$) If $\Upsilon$ is an ordered non-degenerate subset of $Q^\phi \amalg A$, and $\pi \Upsilon$ the corresponding ordered subset of $Q \amalg A$, we have a commutative diagram

$$
\partial^\phi_Q \mathcal{T} \longrightarrow \partial_Q \mathcal{T}_{\pi \Upsilon} \\
\downarrow \\
Q.
$$

(3.2.6)

The construction of such paths proceeds by induction on the number of element of $\Upsilon$, and the bound on the lengths of the paths ensures the contractibility of the corresponding space of choices, ensuring that there is no obstruction to extending the choices from the boundary strata to the interior of the moduli space.

For each $q_* \in Q$, and sequence $\Upsilon$ of the form given in Equations (3.1.9)–(3.1.13), we choose maps

$$
q_\Upsilon: \partial_Q \mathcal{S}_\Upsilon \to Q
$$

(3.2.7)

$$
q_\phi^\phi_\Upsilon: \partial^\phi_Q \mathcal{S}_\Upsilon \to Q
$$

(3.2.8)

varying smoothly in the choice of labels, and of length bounded by 1 plus twice the number of elements of $\Upsilon$ lying in $Q$ or $Q^\phi$. The key condition is that:

1. The maps have value $q_*$ near any marked point or node with label given by one element in $Q$ and one in $Q^\phi$.

We require these maps to satisfy the following additional properties, which are entirely analogous to those listed above:

2. (Values along the ends) The restrictions of $q_\Upsilon$ and $q_\phi^\phi_\Upsilon$ to the intersection of each end $e$ with $\partial_q \mathcal{S}_\Upsilon$ and $\partial^\phi_q \mathcal{S}_\Upsilon$ are constant with value $q$ if $e$ is an outgoing end, and $q_*$ if $e$ an incoming end.

3. (Compatibility with gluing) For each tree $T$ labelling a boundary stratum of $\mathcal{R}_\Upsilon$, the restrictions of $q_\Upsilon$ and $q_\phi^\phi_\Upsilon$ to a neighbourhood of $\mathcal{S}^*_T$ are obtained by gluing.

4. (Compatibility with boundary) For each tree $T$ labelling a boundary stratum of $\mathcal{R}_\Upsilon$ and vertex $v \in V(T)$, the restrictions of the maps $q_\Upsilon$ and $q_\phi^\phi_\Upsilon$ to $\mathcal{S}^*_T$ agree with (i) constant functions if $v$ is degenerate, (ii) the maps $q_\Upsilon_\v$ and $q_\phi^\phi_\Upsilon_\v$ if $v$ is a node, (iii) the constant function $q_*$ if $v \in V^*(T)$, and (iv) the functions $q_\Upsilon_\v$ and $q_\phi^\phi_\Upsilon_\v$ in the remaining cases.

5. (Forgetful maps) If $\Upsilon \to \Upsilon'$ is a map of labels, then $q_\Upsilon$ and $q_\phi^\phi_\Upsilon$ are obtained from $q_{\Upsilon'}$ and $q_{\phi^\phi_{\Upsilon'}}$, by composition with the forgetful map.

Note that Condition 2 above is compatible with Equation (3.1.15), which sets the labels for incoming ends to consist of an element of $A$ and the basepoint $q_*$.

3.2.2. Deformation of the diagonal. Recall that we chose a Hamiltonian diffeomorphism $\phi$ in Section 2.6. We now pick a map

$$
\phi_\Upsilon: \partial_Q^\phi \mathcal{S}_\Upsilon \to \text{Ham}(X)
$$

(3.2.9)

such that the following properties hold:

1. (Values along the ends) the restriction to an end $e$ agrees with $\phi$ if $\Upsilon_e = (\sigma_-, \sigma^\phi)$ and with the identity otherwise.
(2) (Compatibility with gluing) For each tree $T$ labelling a boundary stratum of $\overline{\mathcal{R}}_\Gamma$, the restriction of $\phi_\Gamma$ to a neighbourhood of $\overline{\mathcal{R}}_\Gamma^T$ is obtained by gluing in the sense that the following diagram commutes near the origin:

$$
\begin{array}{ccc}
\partial_0^e\overline{\mathcal{S}}_\Gamma^T & \cong & \partial_0^e\overline{\mathcal{S}}_\Gamma \\
\phi_\Gamma \downarrow & & \phi_\Gamma \\
\downarrow & & \phi_\Gamma \downarrow \\
\text{Ham}(X). & & \text{Ham}(X).
\end{array}
$$

(3.2.10)

(3) (Compatibility with boundary) For each tree $T$ labelling a boundary stratum of $\overline{\mathcal{R}}_\Gamma$ and vertex $v \in V(T)$ the map $\phi_\Gamma$ is (i) constant if $v$ is degenerate, and (ii) otherwise agrees with $\phi_{\Gamma_v}$ in the sense that we have a commutative diagram

$$
\begin{array}{ccc}
\partial_0^v\overline{\mathcal{S}}_\Gamma^v & \cong & \partial_0^v\overline{\mathcal{S}}_{\Gamma_v} \\
\phi_\Gamma \downarrow & & \phi_{\Gamma_v} \\
\downarrow & & \downarrow \\
\text{Ham}(X). & & \text{Ham}(X).
\end{array}
$$

(3.2.11)

(4) (Forgetful maps) For each forgetful map $\Upsilon \rightarrow \Upsilon'$, the following diagrams commute:

$$
\begin{array}{ccc}
\partial_0^e\overline{\mathcal{S}}_\Upsilon & \cong & \partial_0^e\overline{\mathcal{S}}_{\Upsilon'} \\
\downarrow & & \downarrow \\
\text{Ham}(X). & & \text{Ham}(X).
\end{array}
$$

(3.2.12)

(5) (Forgetting $\phi$) If $\Upsilon$ is an ordered subset of $Q^\phi \sqcup A$, $\phi_\Gamma$ constant with value the identity map.

Again, the construction proceeds by induction on the number of elements of $\Upsilon$; while the space of Hamiltonian diffeomorphisms may not be contractible, it is easy to lift all choices to the space of paths based at the identity in $\text{Ham}(X)$. In this way, all obstructions to extending choices from boundary strata to the interior vanish.

Similarly we consider a family

$$
\phi_\Upsilon: \partial_0^e\overline{\mathcal{S}}_\Upsilon \rightarrow \text{Ham}(X),
$$

(3.2.13)

such that the following properties hold:

1. The restriction to a neighbourhood of a node or marked point with label in $Q \times Q^\phi$ is constant with value the identity.
2. (Values along the ends) the restriction to each end $e$ agrees with $\phi$ if $\Upsilon_e$ contains an element of $Q$, and with the identity otherwise.
3. (Compatibility with gluing) For each tree $T$ labelling a boundary stratum of $\overline{\mathcal{R}}_\Upsilon$ the restriction of $\phi_\Gamma$ to a neighbourhood of $\overline{\mathcal{S}}_\Upsilon^T$ is obtained by gluing.
4. (Compatibility with boundary) For each tree $\overline{T}$ labelling a boundary stratum of $\overline{\mathcal{R}}_\Upsilon$ and vertex $v \in V(T)$, the restriction of $\phi_\Gamma$ to $\overline{\mathcal{S}}_\Upsilon^v$ agrees with (i) a constant function if $v$ is degenerate, (ii) the map $\phi_{\Gamma_v}$ if $v$ is a node, (iii) the identity if $v \in V^*(T)$, and (iv) the map $\phi_{\Gamma_v}$ otherwise.
5. (Forgetful maps) Whenever $\Upsilon$ is obtained from $\Upsilon'$ by repeating elements of $Q$ or $Q^\phi$, $\phi_\Upsilon$ is obtained from $\phi_{\Upsilon'}$ by composition with the forgetful map.
3.2.3. Choices of almost complex structures. Let $\mathcal{J}$ denote the space of $\omega$-tame almost complex structures on $X$. For simplicity, fix an almost complex structure $J_\omega$ on $X$, and recall that we have chosen an almost complex structure $J_L$ associated to each $L \in \mathcal{A}$, and subsets $\nu_X Y$ of $X$ for each pair $Y$ of labels. For each sequence $Y$ as in the preceding sections, we consider maps

\begin{align}
J_\nu^\prime: & \nu \partial Q S_Y \cup \nu \partial Q \overline{S}_Y \to \mathcal{J} \\
J_\nu^\prime': & \nu \partial Q S_Y \cup \nu \partial Q \overline{S}_Y \to \mathcal{J}
\end{align}

satisfying the following conditions:

1. (Values along the ends) Away from a fixed compact subset of the interior of $\nu_X Y$, the restriction to each end $e$ is constant, with value $J_L$ if $L \in Y_e$, and $J_\omega$ otherwise.
2. (Compatibility with gluing) For each tree $T$ labelling a boundary stratum of $\mathcal{R}_T$ or $\overline{\mathcal{R}}_T$, the restrictions to a neighbourhood of $\mathcal{R}_T$ or $\overline{\mathcal{R}}_T$ are obtained by gluing.
3. (Compatibility with boundary) For each tree $T$ labelling a boundary stratum of $\mathcal{R}_T$ or $\overline{\mathcal{R}}_T$ and non-degenerate vertex $v \in V(T)$, the restriction of $J_\nu^\prime$ or $J_\nu^\prime$ to $\mathcal{S}_T$ or $\overline{\mathcal{S}}_T$ agrees with the almost complex structure associated to $Y_v$.
4. (Forgetful maps) The choice of almost complex structure is compatible with forgetful maps associated to repeating elements of $Q^\omega$ or $Q$.
5. (Forgetting $\phi$) The choice of almost complex structure is compatible with the projection $Q^\omega \to Q$ if $Y$ contains no element of $Q$.

3.2.4. Reverse isoperimetric constant. For each curve $S$ in $\mathcal{R}_T$ or $\overline{\mathcal{R}}_T$, the paths $q_T$ and $q_T^\phi$ or $q_T^\omega$ determine a closed manifold with boundary $X_S$, homeomorphic to two copies of the product of a fibre with $[0, 1]$ in the first case, and one such copy in the second case, obtained from the boundary conditions over $\partial Q S$ and $\partial Q^\phi S$ by identifying the fibres over each component $\Theta \subset S$ of the thin part. We define the equivalence relation $\sim$ on $X_S$ to collapse the intersection of the fibre over $\Theta$ with each component of $\nu_X \Theta$ to a point. The assumption that this intersection is inessential allows us to fix a metric on the quotient so that, for each loop in $X_S$, the length of the projection to $X_S/\sim$ bounds the norm of the corresponding homology class in $H_1(X_q, \mathbb{Z})$ for any $q$ on the path $q_T$ or $q_T^\phi$.

Given a curve $S$ in $\mathcal{R}_T$ or $\overline{\mathcal{R}}_T$, let $u: S \to X$ be a map such that $u(z)$ lies in (i) $\nu_X L$ if $z \in \partial_Q S$, (ii) in $X_{q_T(z)}$ or $X_{q_T^\phi(z)}$ if $z \in \partial Q S$, and (iii) in $\phi_T(z)X_{q_T(z)}$ or $\phi_T(z)X_{q_T^\phi(z)}$ if $z \in \partial Q^\phi S$. Each such curve has an evaluation map

\begin{align}
\partial u/\sim: & \partial Q S \to X_S/\sim.
\end{align}

Let $\ell(\partial u/\sim)$ denote the length of this curve.

**Lemma 3.3.** There is a universal constant $C$ such that, for each choice of labels $Y$, we have

\begin{align}
\ell(\partial u/\sim) \leq CE^{\rho_Q}(u) + \text{ a constant independent of } u
\end{align}

for each curve $u$ with boundary conditions as above, whose restriction to $\nu \partial Q S$ and $\nu \partial Q^\phi S$ are holomorphic with respect to the complex structures fixed in Section 3.2.3. Moreover, for each components $\Theta$ of the thin part of $S$, this constant is independent of the restriction of the complex structure and the Lagrangian boundary conditions to $\Theta \times \nu_X \Theta$.

**Proof.** The existence of such a constant for each curve $S$ follows from Lemma 3.3. The assumption that the Floer data are compatible with gluing and boundary strata implies
that we can use Lemma A.4 to conclude that the constant can be chosen uniformly for \( S \) in \( \overline{\mathcal{R}_\Sigma} \) or \( \overline{\mathcal{R}_\Sigma} \) for any finite collection of labels. The fact that \( Q \) is compact implies that the constant may be chosen uniformly if the number of labels on the boundary is bounded. The assumption that the data are compatible with forgetful maps, and the constraint that there be at most \( n+1 \) distinct boundary labels corresponding to elements of \( Q \) or \( Q^\phi \) implies that the constant may be chosen uniformly.

3.3. Perturbed equations. We now return to the context of Section 2 and consider a partially ordered set \( \Sigma \) labelling a pair of nested covers \( \{P_\sigma \subset \nu_\sigma\} \) of \( Q \), such that the cover \( \{P_\sigma\} \) has dimension \( n \) (i.e. the maximal length of any totally ordered subset of \( \Sigma \) is \( n+1 \)). Moreover, we assume that

\[
\text{diam } \nu_\sigma \leq \max(1, \frac{1}{8C}),
\]

where the constant \( C \) is the one from Lemma 3.3. For each \( \sigma \in \Sigma \), we choose a basepoint \( q_\sigma \in P_\sigma \), which we think of as a map

\[
(3.3.2) \quad \Sigma \to Q.
\]

As before, we write \( \Sigma^\phi \) for \( \Sigma \times \{\phi\} \), whose elements are equipped with the same choices of basepoints.

Let \( \Upsilon \) denote an ordered subset of \( \Sigma \Pi \Sigma^\phi \Pi A \) such that

(i) all elements of \( \Sigma \) (respectively \( \Sigma^\phi \)) are consecutive and are increasing with respect to the partial order on \( \Sigma \), and (ii) all elements of \( \Sigma \) precede those of \( \Sigma^\phi \), (iii) any element of \( \Sigma^\phi \) appear last, and (iv) all elements of \( A \) are distinct.

In other words, the sequence is of the form

\[
(L_1, \ldots, L_j, \sigma_{-r}, \ldots, \sigma_0, L'_1, \ldots, L'_k, \sigma^\phi_0, \sigma^\phi_1, \ldots, \sigma^\phi_l)
\]

where \( \sigma_i \leq \sigma_{i+1} \). In this paper, we shall only consider the cases \( j+k \leq 2 \).

We have a map of labelling sets

\[
(3.3.5) \quad \Sigma \Pi \Sigma^\phi \Pi A \to Q \Pi Q^\phi \Pi A,
\]

so that any curve \( S \in \overline{\mathcal{R}_\Upsilon} \) is equipped with moving Lagrangians boundary conditions along the boundary components \( \partial_\Sigma S \) and \( \partial^\phi_\Sigma S \) by the construction of Section 3.2. By abuse of notation, we write

\[
(3.3.6) \quad q_T : \partial_\Sigma \overline{\mathcal{R}_T} \to Q
\]

\[
(3.3.7) \quad q^\phi_T : \partial^\phi_\Sigma \overline{\mathcal{R}_T} \to Q
\]

for these paths.

Next, we fix an element \( \sigma \in \Sigma \), and consider an ordered subset \( \Upsilon \) of \( \Sigma \Pi \Sigma^\phi \Pi A \), whose image \( \pi_\Upsilon \) under the map \( \Sigma \to Q \) is one of the sequences in Equations (3.1.10)–(3.1.13), and such that the corresponding ordered subsets of \( \Sigma \) and \( \Sigma^\phi \) respect the partial ordering.

We write \( \overline{\mathcal{R}_\Upsilon} \) for \( \overline{\mathcal{R}_{\pi_\Upsilon}} \), and shall set \( q_\sigma = q_\sigma \) in all future constructions. We have the same product decomposition as in Equation (3.1.14), replacing \( \Upsilon^\phi \) by \( \Upsilon^\phi_\sigma \), i.e. the sequence obtained by replacing all elements of \( \Sigma \) and \( \Sigma^\phi \) by \( \sigma \). We also impose the analogue of Equation (3.1.15), and set the labels of the incoming ends to be either \( \Upsilon_e = (L, \sigma) \) or \( \Upsilon_e = (\sigma, L) \).
3.3.1. Families of Hamiltonians. We shall perturb the Lagrangian boundary conditions from Section 3.2 to achieve transversality. To this end, we choose maps

\[ H_\Upsilon : \partial S_\Upsilon \rightarrow C^\infty (X \times [0, 1], \mathbb{R}) \]

\[ H_\Upsilon^0 : \partial S_\Upsilon^0 \rightarrow C^\infty (X \times [0, 1], \mathbb{R}) \]

which are subject to the following constraints:

1. (Restriction to the thick part) Away from the thin part of each curve \( S \), the restrictions of \( H_\Upsilon \) and \( H_\Upsilon^0 \) to the boundary components labelled by elements of \( \Sigma \) or \( \Sigma^0 \) vanish.

2. (Restriction to the thin part) In each component \( \Theta \) of the thin part, \( H_\Upsilon \) and \( H_\Upsilon^0 \) are supported in \( \nu_X \Theta \).

3. (Restriction to the ends) Sufficiently far along each end \( e \), \( H_\Upsilon \) and \( H_\Upsilon^0 \) agree with the Hamiltonian \( H_{\Upsilon_e} \) fixed in Sections 2.3 and 2.6 (keeping Equation (3.1.15) into account).

4. (Compatibility with gluing) For each tree \( T \) labelling a boundary stratum of \( R_\Upsilon \) or \( R_\Upsilon^0 \), the restriction of \( H_\Upsilon \) to a neighbourhood of \( R_T \) is obtained by gluing.

5. (Compatibility with boundary) For each tree \( T \) labelling a boundary stratum of \( R_\Upsilon \) or \( R_\Upsilon^0 \) and vertex \( v \in V(T) \) the restriction of \( H_\Upsilon \) or \( H_\Upsilon^0 \) to \( \Sigma^v \) or \( \overline{\Sigma}^v \) agrees with (i) a constant if \( v \) is degenerate, (ii) the family \( H_\Upsilon^v \) if \( v \) is a node, (iii) the family \( H_{\Upsilon^v} \) if \( v \in V^\sigma (T) \), and (iv) the family \( H_{\Upsilon_e} \) in the remaining case.

6. (Forgetful maps) \( H_\Upsilon \) and \( H_\Upsilon^0 \) are compatible with the forgetful maps associated to repeating elements of \( \Sigma \) or \( \Sigma^0 \).

7. (Forgetting \( \phi \)) If \( \Upsilon \) is an ordered subset of \( \Sigma^0 \Pi A \), and \( \pi \Upsilon \) the corresponding ordered subset of \( \Sigma \Pi A \), then \( H_{\Upsilon} \) agrees with \( H_{\pi \Upsilon} \) under the identification of \( \overline{\Sigma}_y \) with \( \overline{\Sigma}_{\pi \Upsilon} \).

The construction proceeds by induction, and the only non-trivial part is to ensure that the conditions imposed on the thin parts and on the ends are consistent with the requirement with compatibility of gluing. This is indeed the case because \( H_{\Upsilon_e} \) was assumed to be supported in \( \nu_X \Upsilon_e \).

Let \( \Phi_\Upsilon \) denote the map

\[ \partial S \rightarrow \text{Ham}(X), \]

which on the components labelled by elements of \( Q \Pi A \) agrees with the Hamiltonian diffeomorphisms generated by \( H_\Upsilon \), and on the components labelled by \( Q^0 \) agrees with the result of applying these diffeomorphisms to \( \phi_\Upsilon \).

3.3.2. Families of almost complex structures. For each non-degenerate pair \( \Upsilon \) in \( \Sigma \Pi \Sigma^0 \Pi A \), we fix a map

\[ J(\Upsilon) : [0, 1] \rightarrow \mathcal{J} \]

which is constant outside of \( \nu_X \Upsilon \) with value \( J_\Upsilon \), and agrees with the pushforward of \( J_L \) by \( \Phi_\Upsilon \) on an endpoint corresponding to \( L \in A \). Recall that we set \( \nu_X \Upsilon = X \) if \( \Upsilon \) is a subset of \( A \), so that there is no obstruction to choosing such a family for a pair \( (L, L') \).

We pick families of almost complex structures

\[ J_\Upsilon : \overline{\Sigma}_T \rightarrow \mathcal{J} \]

\[ J_\Upsilon^0 : \overline{\Sigma}_T^0 \rightarrow \mathcal{J} \]

subject to the following constraints:
(1) (Restriction to the boundary) In the thick part, the restriction to the neighbourhoods of the boundary components labelled by elements of $\Sigma \Pi \Sigma^\phi$ agree with those fixed in Equations (3.2.14)-(3.2.15). Moreover, the restriction to each boundary component labelled by $L \in A$ agrees with the pushforward of $J_L$ by $\Phi_T$.

(2) (Restriction to the thin parts) Over each component $\Theta$ of the thin part, $J(\Upsilon)$ or $J(\Upsilon^0)$ are constant away from $\nu_X \Theta$.

(3) (Restriction to the ends) Sufficiently far along each end $e$, $J(\Upsilon)$ and $J(\Upsilon^0)$ agree with $J_{\Upsilon_e}$ under a choice of strip-like end (it is important here to keep Equation (3.1.15) into account).

(4) (Compatibility with gluing) For each tree $T$ labelling a boundary stratum of $R(\Upsilon)$ or $R(\Upsilon^0)$, the restriction of $J(\Upsilon)$ to a neighbourhood of the corresponding boundary stratum asymptotically agrees with the map obtained by gluing. In the case of $R(\Upsilon)$, this means that the diagram

$$
\partial \mathcal{S}_T \times (-1,0]^{E(T)} \xrightarrow{J_T} \partial \mathcal{S}_\Upsilon \\
\downarrow J_T \downarrow J
$$

(3.3.14)

commutes up to a map $\mathcal{S}_T \times (-1,0]^{E(T)} \to \mathcal{J}$ which is supported in a compact subset, and which vanishes to infinite order at the origin.

(5) (Compatibility with boundary) For each tree $T$ labelling a boundary stratum of $\overline{R}_T$ or $\overline{R}_T$, the restriction of $J_T$ to a neighbourhood of the corresponding boundary stratum asymptotically agrees with the map obtained by gluing. In the case of $\overline{R}_T$, this means that the diagram

$$
\partial \mathcal{S}_T' \times (-1,0]^{E(T)} \xrightarrow{J_T'} \partial \mathcal{S}_{\Upsilon'} \\
\downarrow J_T' \downarrow J
$$

(3.3.15)

commutes up to a map $\mathcal{S}_T' \times (-1,0]^{E(T)} \to \mathcal{J}$ which is supported in a compact subset, and which vanishes to infinite order at the origin.

(6) (Forgetful maps) $J(\overline{\Upsilon})$ and $J(\Upsilon^0)$ are compatible with the forgetful maps associated to repeating elements of $\Sigma$ or $\Sigma^\phi$.

(7) (Forgetting $\phi$) For each sequence $\Upsilon$ in $\Sigma^\phi \Pi A$, $J(\Upsilon)$ agrees with $J(\pi \Upsilon)$.

3.3.3. Moduli spaces of pseudoholomorphic discs. Given a choice of Hamiltonian paths $H_T$ and almost complex structure $J_T$, we define the moduli space $\overline{R}(\Upsilon)$ to be the space of finite energy $J_T$-holomorphic maps for a curve $S \in \overline{R}_T$ to $X$, with (moving) Lagrangian boundary conditions given by (i) $\Phi_T(L)$ along $\partial_L S$, (ii) $\Phi_T(X_q \Upsilon)$ along $\partial_{X_q} S$, and (iii) $\Phi_T(X_q \phi \Upsilon)$ along $\partial_{X_q \phi} S$.

Returning to the context of Section 3.2.4, we now consider the topological energy $E(u)$ defined as in Section 2.5.2. The following result follows immediately from Lemma 3.3 and the proof of Lemma 2.33.

**Corollary 3.4.** Every element $u$ in $\overline{R}(\Upsilon)$ or $\overline{R}(\Upsilon^0)$ satisfies the reverse isoperimetric inequality

$$
\ell(u) \leq \frac{3C}{4} E(u) + \text{a constant independent of } u.
$$

The finite energy condition implies that we have a natural evaluation map

$$
\overline{R}(\Upsilon) \to \prod \text{Crit}(\Upsilon_e)
$$

where the product is taken over all ends of elements of $\overline{R}_T$, i.e. over all external edges $e$ of the underlying tree $T$ whose label $\Upsilon_e$ is non-degenerate (i.e. not contained in $\Sigma$ or $\Sigma^\phi$). We have a similar map for $\overline{R}(\Upsilon^0)$, and denote the fibre over a collection $x_e \in \text{Crit}(\Upsilon_e)$ of intersection
points by $\overline{R}(\{ x_e \})$ in either case. The following result is then a standard consequence of regularity theory for holomorphic curves; for its statement, we fix the constant codim which vanishes for the moduli spaces $\overline{R}(\Upsilon)$, and equals the codimension of $\overline{R}_T \subset \overline{R}_\Upsilon$ for the moduli spaces $\overline{R}(\Upsilon)$.

**Lemma 3.5.** For generic choices of Floer data $(H_\Upsilon, J(\Upsilon))$ and $(H_T, J(\Upsilon))$, the moduli spaces $\overline{R}(\Upsilon)$ and $\overline{R}(\Upsilon)$ are regular. In particular, $\overline{R}(\{ x_e \})$ is a manifold with boundary of dimension

$$|\Upsilon| - 3 - \text{codim} + \deg(x_{e_{\text{ou}}}) - \sum_{e \neq e_{\text{ou}}} \deg(x_e).$$

This space is naturally oriented relative the tensor product of the orientation lines of $\overline{R}_\Upsilon$ or $\overline{R}_T$ with $\delta_{x_{e_{\text{ou}}}} \otimes \bigotimes_{e \in E_{\text{int}}(T_v)} \delta_{x_e}$. The boundary is covered by the inverse image of the boundary strata of $\overline{R}_\Upsilon$ or $\overline{R}_T$, together with the union, over all ends $e$ of elements of $\overline{R}_\Upsilon$ or $\overline{R}_T$, of the images of the fibre products

$$\overline{R}(\Upsilon) \times_{\text{Crit}(\Upsilon_e)} \overline{R}(\Upsilon_e) \text{ or } \overline{R}(\Upsilon) \times_{\text{Crit}(\Upsilon_e)} \overline{R}(\Upsilon_e).$$

□

3.4. **Parametrised Morse moduli spaces.** In Section 2, we constructed various moduli spaces as fibre products of moduli spaces of discs and gradient flow lines, along common evaluation maps to Lagrangian fibres. This construction is not sufficiently flexible in general, as iterated fibre products may not be transverse. The standard solution is to take perturbations of the Floer equations and the gradient flow equations which depend on the abstract configuration being considered. For our applications, it suffices to perturb the gradient flow lines, so we adopt this simplified context.

3.4.1. **Morse and Floer edges.** Let $T$ be a rooted planar tree with a label $\Upsilon$ by elements of $\Sigma \cup \Sigma^\phi \cup \Lambda$ as in the discussion following Corollary 3.4. We define a decomposition

$$E(T) = E^{\text{Fl}}(T) \amalg E^{\text{Mo}}(T)$$

into Floer and Morse edges as follows:

$$\text{any edge labelled by a pair in } \Sigma \text{ or in } \Sigma^\phi \text{ lies in } E^{\text{Mo}}(T). \text{ All other edges lie in } E^{\text{Fl}}(T).$$

We have a corresponding decomposition of the set of flags of $T$ into Floer and Morse flags:

$$F(T) = F^{\text{Fl}}(T) \amalg F^{\text{Mo}}(T).$$

To edges in $E^{\text{Mo}}(T)$, we shall associate (perturbed) Morse flow lines in a Lagrangian fibre of $X \to Q$. To this end, we set

$$\mathcal{T}_e = \begin{cases} [0, \infty] & e \in E^{\text{int}}(T) \\ \{ \infty \} & e \in E^{\text{ext}}(T) \end{cases}.$$

For each vertex $v$, recall that $\Upsilon_v$ denotes the ordered subset of $\Upsilon$ obtained from the labels of the components adjacent to $v$, starting again with the component to the left of the outgoing edge $e_{\text{ou}}^v$. Note that we have a canonical identification between the punctures of
an element of $\mathcal{R}_{T_v}$ and the Floer edges of $T_v$, and between the boundary marked points and the Morse edges. We define

$$ TR_{T,\Upsilon} \equiv \prod_{v \in V(T)} \mathcal{R}_{T_v} \times \prod_{e \in E^{Mo}(T)} \mathcal{T}_e $$

and

$$ TB_{T,\Upsilon} \equiv \prod_{v \in N(T)} \mathcal{R}_{\Upsilon v} \times \prod_{e \in E^{Mo}(T) \cup N(T)} \mathcal{R}_{T_e} \times \prod_{v \in V(T)} \mathcal{R}_{\Upsilon v} \times \prod_{e \in E^{Mo}(T) \cup N(T)} \mathcal{T}_e. $$

Identifying $T_e$ with the interval $[-1,0]$ via the map $R \mapsto -R$, we obtain the structure of a manifold with corners on this space. The corner strata are products of the corner strata of $\mathcal{R}_{\Upsilon v}$, $\mathcal{R}_{\Upsilon v}$, and $\mathcal{R}_{T_e}$ according to the topological type of the corresponding stable disc, and the stratification of $T_e$ according to whether the length is 0, non-zero and finite, or infinite. To have a better description of the boundary strata of $TR_{T,\Upsilon}$, we write

$$ TR_{T,\Upsilon} \equiv \prod_{v \in V(T)} \mathcal{R}_{\Upsilon v} \times \prod_{e \in E^{Mo}(T)} \mathcal{T}_e, $$

for an isomorphic copy of $TR_{T,\Upsilon}$ whenever $\Upsilon$ is a sequence of the form

$$ (L,\sigma_0,\ldots,\sigma_{\ell}) $$

$$ (\sigma_{-r},\ldots,\sigma_{-0},L) $$

$$ (\sigma_{-r},\ldots,\sigma_{-0},\phi_0,\ldots,\phi_{\ell}), $$

and all elements of $\Sigma$ above lie in $\Sigma_\sigma$.

**Remark 3.6.** We stress the difference between $TR_{T,\Upsilon}$ and $TR_{T,\Upsilon}$. In the first case, the labels for Morse edges are given by pairs of elements of $\Upsilon$, and hence can include elements of $\Sigma$ which differ from the basepoint $\sigma$. In the second case, all Morse edges have label $(\sigma,\sigma)$. For each map $T' \rightarrow T$ which collapses internal edges, and $v$ a vertex of $T$, let $T'_v$ denote the subtree of $T'$ whose vertices are those mapping $v$, and whose edges all are edges adjacent to such vertices. We have a natural identification of strata

$$ TR_{T',T,\Upsilon} \cong TR_{T',T,\Upsilon} \subseteq TR_{T,\Upsilon} $$

given on one side by the locus where the lengths of all collapsed edges is 0 and on the other by the inclusion $TR_{T',T,\Upsilon} \subseteq TR_{T,\Upsilon}$. We obtain the space

$$ TR_{\Upsilon} \equiv \bigcup_{|E^{ext}(T)|=|\Upsilon|} TR_{T,\Upsilon}/\sim $$

by gluing the spaces $TR_{T,\Upsilon}$ along their common strata. Restricting to trees labelling the boundary strata of $TR_{\Upsilon}$, we similarly obtain:

$$ TR_{\Upsilon} \equiv \bigcup TR_{T,\Upsilon}/\sim. $$

We can stratify the boundary of $TR_{\Upsilon}$ in such a way that the strata are indexed by trees $T$ labelled by $\Upsilon$:

$$ TR_{\Upsilon}^T \equiv \prod_{v \in V(T)} TR_{T_v,\Upsilon_v}. $$

Such a stratum corresponds to collections where each Floer edge of $T$ can be identified with the node of a broken disc, and where the Morse edges of $T$ have infinite length. In particular, the codimension 1 boundary strata are given by a choice of tree $T$ with $|\Upsilon|$ external edges,
together with a distinguished internal edge \( e \in E \) which is either (i) the unique internal Floer edge or (ii) the unique internal Morse edge whose length is specified to equal \( \infty \).

We have a similar description for the moduli spaces \( \mathcal{TR}_{T, \Upsilon} \) with constraints, with boundary strata given by

\[
\mathcal{TR}_{T, \Upsilon}^T \equiv \prod_{v \in N(T)} \mathcal{TR}_{T_v, \Upsilon_v} \times \prod_{v \in V^s(T)} \mathcal{TR}_{T_v, \Upsilon_v} \times \prod_{v \notin V^s(T) \cup N(T)} \mathcal{TR}_{T_v, \Upsilon_v},
\]

where \( T \) is required to label a boundary stratum of \( \mathcal{R}_{\Upsilon} \). With the exception of strata corresponding to the boundary of \( \mathcal{R}_{3,2} \) in the cases corresponding to Equations (3.1.12) and (3.1.13), the boundary strata again have either a unique Floer edge, or a unique Morse edge of infinite length.

We associate to each Morse edge of \( T \) a universal interval over \( \mathcal{TR}_{T, \Upsilon} \): first, we define \( I_r \) for \( r \in [0, \infty) \) to be the interval \([0, r]\). We extend this to \( r = \infty \), by setting

\[
I_\infty \equiv I_+ \sqcup I_-.
\]

We equip the space

\[
I_{[0,\infty]} \equiv \prod_{r \in [0,\infty]} I_r
\]

with the natural topology away from \( r = \infty \), extended to the latter by the requirement that the sets

\[
\prod_{R < r \leq \infty} [0, R) \quad \text{and} \quad \prod_{R < r \leq \infty} (-R, 0]
\]

be open. We then define

\[
\mathcal{T}_e \equiv \begin{cases} I_{[0,\infty]} & e \in E^{\text{int}}(T) \\ [0, \infty) & e \in E^{\text{in}}(T) \\ (-\infty, 0] & e \in E^{\text{out}}(T) \end{cases}
\]

and note that we have a natural projection map

\[
\mathcal{T}_e \to \mathcal{T}_e
\]

for all Morse edges of \( T \).

Pulling back the universal interval over \( \mathcal{T}_e \) yields the universal interval associated to the edge \( e \):

\[
\mathcal{I}_{T, \Upsilon} \to \mathcal{TR}_{T, \Upsilon}
\]

and the union over all Morse edges is the \textit{universal interval over} \( \mathcal{TR}_{T, \Upsilon} \)

\[
\mathcal{T}_{T, \Upsilon} \to \mathcal{TR}_{T, \Upsilon},
\]

which is again equipped with a natural smooth structure. The two universal intervals over the codimension 1 strata of \( \mathcal{R}_{T, \Upsilon} \) are naturally isomorphic. We have an entirely analogous construction of a universal interval

\[
\mathcal{I}_{T, \Upsilon} \to \mathcal{TR}_{T, \Upsilon},
\]
3.4.2. Morse data and moduli spaces for pairs. Given a pair of elements $\Upsilon = (\Upsilon_0, \Upsilon_1)$ of $\Sigma$ or $\Sigma_\phi$, let $\nu Q\Upsilon$ denote the intersection $\nu Q\Upsilon_0 \cap \nu Q\Upsilon_1$. Recall $X_{\nu Q\Upsilon}$ denotes the restriction of the torus bundle $X \to Q$ to $\nu Q\Upsilon$; we pick a distinguished fibre $X_{q\Upsilon}$ which we equip with a metric, and with a Morse-Smale function

$$f_\Upsilon : X_{q\Upsilon} \to \mathbb{R}.$$ 

(3.4.24)

Pick in addition a Lagrangian section over $\nu Q\Upsilon$, which induces a trivialisation of $X_{\nu Q\Upsilon}$ by parallel transport. We write

$$\psi^{q,q'}_{\Upsilon} : X_q \to X_{q'}$$

(3.4.25)

for the induced map of fibres over points $q,q' \in \nu_q \Upsilon$. For simplicity of notation, we may sometimes omit the subscript from the above maps.

We extend the definition of Morse data given in Equation (2.2.42) as follows: for a non-negative real number $r$, we define

$$V_r(\Upsilon) = C^\infty(I_r, C^\infty(X_{q\Upsilon}, TX_{q\Upsilon})).$$

(3.4.26)

For the case $r = \infty$, we define

$$V_\infty(\Upsilon) = V_+(\Upsilon) \times V_-(\Upsilon),$$

(3.4.27)

and recall that we require that elements of $V_\pm(\Upsilon)$ correspond to a family of vector fields parametrised by $I_\pm$, which agree with the gradient vector field outside a compact set. We then define

$$V_{[0,\infty]}(\Upsilon) = \prod_{r \in [0,\infty]} V_r(\Upsilon).$$

(3.4.28)

Exactly as in Section 2.2.4, we define $T_\bullet(\Upsilon, V)$, for $\bullet \in \{\pm\} \cup [0,\infty)$, to be the set of pairs $(\gamma, \xi)$, with $\gamma$ a path in $X_{q\Upsilon}$ with domain $I_\bullet$ and $\xi \in V_r(\Upsilon)$ satisfying the perturbed gradient flow equation $\frac{d\gamma}{dt} = \xi$. We also define

$$T_\infty(\Upsilon, V) = T_-(\Upsilon, V) \times_{\text{Crit}_-} T_+(\Upsilon, V),$$

(3.4.29)

and obtain the union

$$T_{[0,\infty]}(\Upsilon, V) = \prod_{r \in [0,\infty]} V_r(\Upsilon).$$

(3.4.30)

The spaces corresponding to $\bullet \in \{\pm\} \cup [0,\infty)$ fibrewise compactifications over $V_\bullet$

$$T_\bullet(\Upsilon, V) \subset \overline{F}_\bullet(\Upsilon, V)$$

(3.4.31)

by adding to $T_\bullet(\Upsilon, V)$ the fibre products at the ends with the space $\overline{F}(\Upsilon)$ of gradient flow lines. In the case $\bullet = \infty$, the boundary stratum is given by

$$T_-(\Upsilon, V) \times_{\text{Crit}_-} \overline{F}(\Upsilon) \times_{\text{Crit}_+} T_+(\Upsilon, V).$$

(3.4.32)

Taking the union over $r \in [0,\infty]$, we obtain the space

$$T_{[0,\infty]}(\Upsilon, V) \subset \overline{F}_{[0,\infty]}(\Upsilon, V)$$

(3.4.33)

which maps to $V_{[0,\infty]}(\Upsilon)$ with compact fibres.
3.4.3. Morse moduli spaces for labelled trees. Let $T$ be a tree labelled as in Section 3.1. For each edge in $E^{Mo}(T)$, we define

\[(3.4.34)\]

$V_e(\Upsilon) \equiv \begin{cases} V_\pm(\Upsilon_e) \times \mathcal{TR}_{T,\Upsilon} & \text{if $e$ is infinite} \\ V_{[0,\infty]}(\Upsilon_e) \times \mathcal{TR}_{T,\Upsilon} & \text{otherwise.} \end{cases}$

In particular, for any edge $e$, there is thus a natural projection map

\[(3.4.35)\]

$V_e(\Upsilon) \to \mathcal{TR}_{T,\Upsilon}.$

The fibre product of these spaces over $\mathcal{TR}_{T,\Upsilon}$ for all $e \in E^{Mo}(T)$ defines the space of Morse data

\[(3.4.36)\]

$V_T(\Upsilon) \to \mathcal{TR}_{T,\Upsilon}$.

We can define spaces $V^\sigma_T(\Upsilon) \to \mathcal{TR}_{T,\Upsilon}$ and $V_T(\Upsilon) \to \mathcal{TR}_{T,\Upsilon}$ in exactly the same way.

A section of $V_T(\Upsilon)$ thus consists of a choice of perturbed gradient flow equation $f$ for each edge $e \in E^{Mo}(T)$, metrised according to the image of this point in $\mathcal{T}_T$, and hence corresponds to a map

\[(3.4.37)\]

$\mathcal{T}_e \to C^\infty(X_{q_{T,e}}, TX_{q_{T,e}}).$

for each Morse edge $e$. We shall assume that such a map is smooth, and moreover agrees to infinite order, along the boundary of the moduli space, with the map obtained in gluing.

We shall moreover pick the data consistently in the following sense: identifying the elements of $E(T) \setminus \{e\}$ and $E(T/e)$, we assume that the following diagram commutes for each Morse edge $e'$ of $T$:

\[(3.4.38)\]

In addition, for each Morse edge $e$ of $T$, and for each edge $e' \neq e_{\pm}$ of $T_{e_{\pm}}$ (with corresponding edges $e' \in E(T)$), we require the commutativity of the diagram:

\[(3.4.39)\]

We impose the analogous condition that the restrictions of the Morse data associated to $e$ agree, upon restriction to this boundary stratum, with the pullbacks of the data associated to $e_{\pm}$ on the respective components.

Such sections determine moduli spaces

\[(3.4.40)\]

$\mathcal{T}_e(\Upsilon) \to \mathcal{TR}_{T,\Upsilon},$

for each Morse edge $e$, which compactify the parametrised moduli space of (perturbed) gradient solutions corresponding to $e$. The same construction yields moduli spaces

\[(3.4.41)\]

$\mathcal{T}_e(\Upsilon) \to \mathcal{TR}_{T,\Upsilon}$

\[(3.4.42)\]

whenever $e$ is a Morse edge of a tree labelling a boundary stratum of $\mathcal{R}_T$ or $\mathcal{R}_\Upsilon$. 
Remark 3.7. Note that, while we have an identification $\mathcal{T}R_{T,\Upsilon} \cong \mathcal{T}R_{T,\Upsilon^*}$, the spaces $\mathcal{T}e_\Upsilon(Y)$ and $\mathcal{T}e(Y^*)$ are spaces of perturbed gradient flow lines for Morse functions which are a priori different, over fibres which themselves may be different.

For each flag $f$ containing $e$, we have an evaluation map
\begin{equation}
\mathcal{T}e_\Upsilon(Y) \to X^f_{q\Upsilon^e},
\end{equation}
while for each infinite end of $e$, we have an evaluation map
\begin{equation}
\mathcal{T}e_\Upsilon(Y) \to \text{Crit}_\Upsilon^e,
\end{equation}
so we obtain a decomposition of $\mathcal{T}e_\Upsilon(Y)$ into components $\mathcal{T}e(x; \Upsilon)$ labelled by such critical points.

Lemma 3.8. For generic choices of admissible Morse data, the moduli spaces $\mathcal{T}e_\Upsilon(Y)$, $\mathcal{T}e_\Upsilon(Y)$, and $\mathcal{T}e_\Upsilon(Y)$ associated to an internal Morse edge $e$ are manifolds with boundary of dimension equal to
\begin{equation}
n + |\Upsilon| - 3 - \text{codim}
\end{equation}
where codim is as in Lemma 3.5. This manifold is naturally oriented relative the tensor product of $|X_{q\Upsilon^e}|$ with the underlying abstract moduli spaces, and has boundary given by the inverse image of their boundaries under the evaluation map.

If $e$ is a external edge, then $\mathcal{T}e(x; \Upsilon)$, $\mathcal{T}e(x; \Upsilon)$, and $\mathcal{T}e(x; \Upsilon)$ are manifolds with boundary of dimension
\begin{equation}
\begin{cases}
n - \deg(x) + |\Upsilon| - 3 - \text{codim} & e \in E^{in}(T) \\
\deg(x) + |\Upsilon| - 3 - \text{codim} & e \in E^{an}(T)
\end{cases}
\end{equation}
which are naturally oriented relative the tensor product of the orientation line of the underlying moduli space with $|X_{q\Upsilon^e}| \otimes \delta_x$ in the first case, and $\delta_x$ in the second case. The inverse image of each stratum of the underlying abstract moduli space is again naturally a submanifold, of the same codimension. There is an additional boundary stratum of codimension 1, given for $\mathcal{T}(\Upsilon)$ by
\begin{equation}
\mathcal{T}(\Upsilon^e) \times_{\text{Crit}(\Upsilon^e)} \mathcal{T}e_\Upsilon(Y)
\end{equation}
and similarly for $\mathcal{T}e_\Upsilon(Y)$ and $\mathcal{T}e_\Upsilon(Y)$.

Consistency of the choices of data implies that the two moduli spaces over each codimension 1 boundary stratum of the boundary are naturally identified: in particular, if $e$ is an internal edge, then over the stratum $\ell(e) = \infty$ we have a homeomorphism
\begin{equation}
\mathcal{T}e_\Upsilon(Y) |_{\mathcal{T}R_{T^e,\Upsilon^e}} \times_{\mathcal{T}R_{T^e,\Upsilon^e}} \mathcal{T}e_\Upsilon(Y) \cong \mathcal{T}e_\Upsilon(Y) \times_{\text{Crit}(\Upsilon^e)} \mathcal{T}e_\Upsilon(Y),
\end{equation}
with the same type of decomposition for edges of $\mathcal{T}e_\Upsilon(Y)$ and $\mathcal{T}e_\Upsilon(Y)$.

3.5. Mixed moduli spaces. Let $T$ be a rooted planar tree labelled by $\Upsilon$. Given a choice of Floer data as in Section 3.3 and Morse data as in Section 3.4, we obtain moduli spaces of holomorphic discs for all vertices and of gradient flow lines for all Morse edges. We shall define mixed moduli spaces as fibre products with respect to evaluation maps to the fibres. In order to specify these evaluation maps, we need to make some additional choices:
For any Morse flag \( f \) of a tree \( T \) labelling a boundary stratum of \( \mathcal{R}_{\Gamma}, \mathcal{R}'_{\Gamma} \) or \( \mathcal{R}_{\Gamma} \), we pick maps
\begin{align}
q^f_{\Gamma}: \mathcal{R}_{T,\Gamma} &\to \nu_Q \mathcal{Y}_f \\
q^f_{\Gamma}^\sigma: \mathcal{R}'_{T,\Gamma} &\to \nu_Q \mathcal{Y}_f \\
q^f_{\Lambda}: \mathcal{R}_{T,\Lambda} &\to \nu_Q \mathcal{Y}_f
\end{align}
subject to the following constraints:
(1) (Value at marked points) For each vertex \( v \) such that \( \Upsilon_v \) is degenerate, the values of \( q^f_{\Gamma}, q^f_{\Gamma}^\sigma \), or \( q^f_{\Lambda} \) for flags containing \( v \) agree. If \( \Upsilon_v \) is non-degenerate, \( q^f_{\Gamma}^\sigma \) is constant with value \( q_v \), while \( q^f_{\Gamma} \) (or \( q^f_{\Gamma} \)) agrees with the value of the path \( q_{T_v} \) or \( q^\phi_{T_v} \) (resp. \( q_{\Lambda} \) or \( q^\phi_{\Lambda} \)) at the corresponding marked point of the surface in \( \mathcal{S}_{\Upsilon_v} \) (resp. \( \mathcal{S}_{\Upsilon} \)).

(2) (Compatibility with boundary) The two values of the maps \( q^f_{\Gamma} \) and \( q^f_{\Lambda} \) over each boundary stratum of \( \mathcal{R}_{T,\Gamma}, \mathcal{R}'_{T,\Gamma}, \) or \( \mathcal{R}_{T,\Lambda} \) agree.

To clarify the last condition, recall that we have given a description of each boundary stratum as either a product of lower dimensional moduli spaces, corresponding to the breaking of a disc giving rise to a Floer edge or of a Morse edge having infinite length, or as a boundary stratum common to two different moduli spaces, corresponding to a Morse edge having length 0, or the breaking of a disc giving rise to a Morse edge.

Let \( X^f_{\Gamma} \) denote the pullback of \( X \) to a bundle over \( \mathcal{R}_{\Gamma} \) under the map \( q^f_{\Gamma} \). For each Morse flag \( f = (v, e) \), we have a natural evaluation map
\begin{equation}
\mathcal{T}_e(\Upsilon) \to X^f_{\Gamma}
\end{equation}
obtained by applying \( \psi_\Gamma \) in order to identify the fibres of \( X^f_{\Gamma} \) with \( X_{q_{T_v}} \). We also have a map from \( \mathcal{R}(\Upsilon_v) \times_{\mathcal{S}_{\Upsilon_v}} \mathcal{R}_{T,\Gamma} \) to the same space, obtained by evaluation at the marked point associated to \( e \). Let \( X^f_{\Lambda} \) define the fibre product over \( \mathcal{R}_{\Gamma} \) of the spaces \( X^f_{\Gamma} \) for all Morse flags over \( \mathcal{R}_{\Gamma} \). We then define the mixed moduli space as the fibre product
\begin{equation}
\begin{aligned}
\prod_{v \in \mathcal{Y}_{\Delta_0}(T)} \mathcal{R}(\Upsilon_v) \times_{\mathcal{S}_{\Upsilon_v}} \mathcal{R}_{T,\Gamma} \longrightarrow X^f_{\Gamma} \prod_{f \in \mathcal{E}(T)} \text{Crit}(\Upsilon_f).
\end{aligned}
\end{equation}

Unwinding the definition of the fibre product, we find that an element of \( \mathcal{R}_{\Gamma}(\Upsilon) \) consists of (i) a point in \( \mathcal{R}_{T,\Gamma} \), (ii) a gradient flow line in \( \mathcal{F}_e \) over this point in \( \mathcal{R}_{T,\Gamma} \) for each Morse edge \( e \) in \( T \), (iii) an intersection point of the corresponding Lagrangians at each Floer edge \( e \) of \( T \), and (iv) a holomorphic map in \( \mathcal{R} \) over the projection to \( \mathcal{R}_{\Gamma} \). At each Morse flag \( v \in e \), we require that the evaluations of the gradient flow line in \( \mathcal{F}_e(\Upsilon) \) (at the end of \( e \)) and of the curve in \( \mathcal{R}(\Upsilon_v) \) (at the marked point corresponding to \( e \)) agree as points in \( X^f_{\Gamma} \), while at each Floer flag \( v \in e \) we require the asymptotic conditions for the elements of \( \mathcal{R}(\Upsilon_v) \) to be given by the intersection point of Lagrangians chosen for \( e \).

Taking the same fibre product construction yield moduli spaces \( \mathcal{R}'_{\Gamma}(\Upsilon) \to \mathcal{R}'_{T,\Gamma}, \) as well as moduli spaces with constraints
\begin{equation}
\mathcal{R}_{\Gamma}(\Upsilon) \to \mathcal{R}_{T,\Gamma}.
\end{equation}
Taking the evaluation map at all external edges yields the map
\[(3.5.7) \quad \overline{\mathcal{R}}_T(\Upsilon) \to \prod_{e \in E^{\text{ext}}(T)} \text{Crit}(\Upsilon_e),\]
whose fibre at a collection \(x = \{x_e\}\) of critical and intersection points we denote \(\overline{\mathcal{R}}_T(x)\), and similarly for the other two moduli spaces. Regularity and gluing theory imply that, for generic Morse and Floer data, the fibre product defining each stratum is transverse. The key point is that, in each fibre product over \(X_T\), one of the factors is a Morse gradient flow line, and that the inhomogeneous data for flow lines is allowed to vary with respect to the parameter in the abstract moduli space of discs and metric trees.

**Lemma 3.9.** For generic Floer and Morse data, the moduli space \(\overline{\mathcal{R}}_T(x)\) is a manifold with boundary of dimension
\[(3.5.8) \quad |\Upsilon| - 3 - \text{codim} + \deg(x_{\text{crit}}) - \sum_{e \in E^{\text{in}}(T)} \deg(x_e),\]
where \(\text{codim}\) is as in Lemma 3.5. This space is naturally oriented relative the tensor product of the tangent space of the underlying moduli space with
\[(3.5.9) \quad \delta_{x_{\text{crit}}} \otimes \bigotimes_{e \in E^{\text{in}}(T)} \delta_{x_e}.\]

The codimension 1 boundary strata of \(\overline{\mathcal{R}}_T(\Upsilon)\), \(\overline{\mathcal{R}}_T(\Upsilon_T)\), and \(\overline{\mathcal{R}}_T(\Upsilon)\) are given by the inverse images of the boundary strata of the corresponding abstract moduli spaces, together with the images of the fibre products
\[(3.5.10) \quad \begin{cases} \overline{\mathcal{R}}_T(\Upsilon) \times_{\text{Crit}(\Upsilon_e)} \overline{\mathcal{R}}(\Upsilon_e) & e \in E^{\text{Mo}}(T) \cap E^{\text{ext}}(T) \\ \overline{\mathcal{R}}_T(\Upsilon) \times_{\text{Crit}(\Upsilon_e)} \overline{\mathcal{R}}(\Upsilon_e) & e \in E^{\text{Fl}}(T) \cap E^{\text{ext}}(T), \end{cases}\]
and the analogous fibre products for \(\overline{\mathcal{R}}_T(\Upsilon)\) and \(\overline{\mathcal{R}}_T(\Upsilon)\). \(\Box\)

Since the data are chosen consistently for different trees \(T\), there are codimension 1 strata which appear in pairs: given a Morse edge \(e\), we have a map
\[(3.5.11) \quad \overline{\mathcal{R}}_T(\Upsilon)|_{\overline{\mathcal{R}}_T^{T/e}} \to \overline{\mathcal{R}}_T^{T/e}(\Upsilon)|_{\overline{\mathcal{R}}_T^{T/e}}\]
corresponding on the left to locus where this Morse edge has length 0, and on the right to the locus where it appears from a breaking of holomorphic discs.

Taking the union along the identification induced by Equation (3.5.11) over all \(T\) with \(d + 1\) external edges, we obtain a moduli space \(\overline{\mathcal{R}}(\Upsilon)\) which admits an evaluation map to \(\text{Crit}(\Upsilon_e)\) for each external edge as before. The same construction yields moduli spaces \(\overline{\mathcal{R}}(\Upsilon)\) and \(\overline{\mathcal{R}}_T(\Upsilon)\). We denote by \(\overline{\mathcal{R}}(x_0, x_d, \ldots, x_1)\) the fibre over points in \(\prod_{e \in E^{\text{ext}}(T)} \text{Crit}(\Upsilon_e)\), with \(x_0\) corresponding to the output.

**Lemma 3.10.** If the asymptotic conditions \((x_0, x_d, \ldots, x_1)\) satisfy
\[(3.5.12) \quad d - 2 - \text{codim} + \deg(x_{\text{crit}}) - \sum_{i=1}^{d} \deg(x_i^+) = 1,\]
then \(\overline{\mathcal{R}}(x_0, x_d, \ldots, x_1)\) is a 1-dimensional manifold with boundary. The boundary decomposes into strata labelled by the codimension 1 boundary strata of \(\mathcal{R}_T, \mathcal{R}_T^-,\) or \(\mathcal{R}_{\text{crit}}^+,\) together with the boundary strata described by Equation (3.5.11). \(\Box\)
4. Chain level constructions

In this section, we first refine the global structures from Section 2 to the $A_\infty$ level. We then proceed to the local case, and establish the equivalence between local and global invariants. One minor difference with the cohomological construction is that the local category will have objects labelled by additional choices of basepoints. This leads to a much larger category, which is easier to make into the target of our functor, but the quasi-isomorphism classification of objects will be the same as in Section 2.

4.1. Statement of results. Given the cover $\Sigma$ fixed in the previous section, we begin by constructing the $A_\infty$ category $F$. To each $L \in A$ we associate left and right modules $L_L$ and $R_L$ over $F$, and construct, for each pair $L$ and $L'$ of elements of $A$, maps

\[
\begin{align*}
CF^*(L, L') &\to \text{Hom}_\mathcal{F}(\mathcal{L}_{L'}, \mathcal{L}_L) \\
R_{L'} \otimes_\mathcal{F} L_L &\to CF^*(L, L').
\end{align*}
\]

Next, we construct the bimodule $\Delta$, together with module maps

\[
\begin{align*}
\text{R}_{L'} &\to \text{Hom}_\mathcal{F}(L_L, \Delta) \\
\Delta \otimes_\mathcal{F} L_L &\to L_L
\end{align*}
\]

The analogue of Proposition 2.2 is the following result, from Section 4.2 below:

Proposition 4.1. Given pairs $(L, L') \in A$, there is a homotopy commutative diagram

\[
\begin{array}{ccc}
\text{R}_{L'} \otimes_\mathcal{F} L_L & \to & CF^*(L, L') \\
\downarrow & & \downarrow \\
\text{Hom}_\mathcal{F}(\mathcal{L}_{L'}, \Delta) \otimes_\mathcal{F} L_L & \to & \text{Hom}_\mathcal{F}(\mathcal{L}_{L'}, \mathcal{L}_L) \\
\text{Hom}_\mathcal{F}(\mathcal{L}_L, \Delta) \otimes_\mathcal{F} L_L & \to & \text{Hom}_\mathcal{F}(\mathcal{L}_L, \Delta \otimes_\mathcal{F} L_L)
\end{array}
\]

In Section 4.3.1, we show the following result concerning the left modules associated to Lagrangians:

Lemma 4.2. For each $L \in A$, the left module $L_L$ lies in the triangulated closure of left Yoneda modules over $\mathcal{F}$. In particular, it is a perfect left module.

Corollary 4.3. If Equations (4.1.3) and (4.1.4) are isomorphisms, then the map

\[
HF^*(L, L') \to H \text{Hom}_\mathcal{F}(\mathcal{L}_L, \mathcal{L}_L)
\]

is surjective.

Proof. The fact that $L_{L'}$ is perfect implies that the natural map

\[
\text{Hom}_\mathcal{F}(\mathcal{L}_L, \Delta) \otimes_\mathcal{F} L_L \to \text{Hom}_\mathcal{F}(\mathcal{L}_L, \Delta \otimes_\mathcal{F} L_L)
\]

is a quasi-isomorphism; indeed, the analogous statement for Yoneda modules is true by the Yoneda Lemma, so one can apply a filtration argument. The result then follows from Diagram (4.1.5) \qed

In order to verify the assumption of the above Corollary, we introduce the subcategory $\mathcal{F}_\sigma$ of $\mathcal{F}$ with objects $\tau \in \Sigma_\sigma$. We have the following variant of Lemma 2.5, whose proof is given in Section 4.3.2 below:
Lemma 4.4. For each left module $L$ over $F$, the natural maps
\begin{align}
\text{Hom}_{F}(L, \Delta_{\sigma}(\sigma, \omega)) & \to \text{Hom}_{F}(L, \Delta_{\sigma}(\omega, \omega)) \\
\Delta_{\sigma}(\omega, \omega) \otimes_{F} L & \to \Delta_{\sigma}(\omega, \omega) \otimes_{F} L
\end{align}
are quasi-isomorphisms.

Having reduced the main result to a local computation, we now proceed, as suggested by the results of Section 2, by removing the various Hamiltonian perturbations that appear in the restriction of $L$ to $F_{\sigma}$. To this end, we introduce a category $Po_{\sigma}$ corresponding to the polytopes contained in $\nu Q_{\sigma}$. We construct this category in such a way that we have a strict $A_{\infty}$ embedding:
\begin{align}
j : F_{\sigma} \to Po_{\sigma}.
\end{align}
We then construct modules $L_{L, \sigma}$ and $R_{L, \sigma}$, as well as a map of right modules
\begin{align}
R_{L, \sigma} & \to \text{Hom}_{F_{\sigma}}(L_{L, \sigma}, \Delta_{Po_{\sigma}}).
\end{align}
The comparison between local and global constructions is summarised by the following result, from Section 4.4 below:

Proposition 4.5. There are quasi isomorphisms of modules over $F_{\sigma}$
\begin{align}
j^{*}\Delta_{Po_{\sigma}} & \to \Delta \\
j^{*}L_{L, \sigma} & \to L_{L} \\
j^{*}R_{L, \sigma} & \to R_{L}
\end{align}
which fit in homotopy commutative diagrams of right $F_{\sigma}$ modules
\begin{align}
j^{*}R_{L, \sigma} & \longrightarrow \text{Hom}_{F_{\sigma}}(j^{*}L_{L, \sigma}, j^{*}\Delta_{Po_{\sigma}}) \\
R_{L} & \longrightarrow \text{Hom}_{F_{\sigma}}(L_{L}, \Delta) \longrightarrow \text{Hom}_{F_{\sigma}}(j^{*}L_{L, \sigma}, \Delta)
\end{align}
and of left $F_{\sigma}$ modules
\begin{align}
j^{*}\Delta_{Po_{\sigma}} \otimes_{F_{\sigma}} j^{*}L_{L, \sigma} & \longrightarrow j^{*}L_{L, \sigma} \\
\Delta \otimes_{F_{\sigma}} L_{L} & \longrightarrow L_{L}.
\end{align}

This reduces computations of the global modules over the local category, to computations of the local modules. In Section 4.5, we prove the $A_{\infty}$ analogues of Corollary 2.27 and Lemma 2.30.

Lemma 4.6. The natural maps
\begin{align}
j^{*}R_{L, \sigma}(\sigma) & \to \text{Hom}_{F_{\sigma}}(j^{*}L_{L, \sigma}, j^{*}\Delta_{Po_{\sigma}}(\sigma, \omega)) \\
j^{*}\Delta_{Po_{\sigma}}(\omega, \omega) \otimes_{F_{\sigma}} j^{*}L_{L, \sigma} & \to j^{*}L_{L, \sigma}(\sigma)
\end{align}
are quasi-isomorphisms.

This is the final ingredient which we need in order to prove the main result of this paper:
Proof of Theorem 4.7. The results of [5] imply that the functor is faithful, so that it suffices to show that Equation (4.1.1) is surjective on cohomology. Corollary 4.3 reduces this to showing that the bi-module $\Delta$ behaves like the diagonal bimodule when tensored with $L_L$, and that $R_L$ is the space of left module maps from $L_L$ to $\Delta$. Both statements are reduced by Lemma 4.3 to each local category $\mathcal{F}_\sigma$, and reduced further by Proposition 4.5 to the corresponding statement for the local modules $L_{L,\sigma}$ and $R_{L,\sigma}$, and the pullback of the diagonal of $P_{L,\sigma}$. Lemma 4.6 thus completes the argument. □

4.2. Global constructions. The objects of $\mathcal{F}$ are elements $\sigma \in \Sigma$, with morphisms given by Equation (2.2.60). Associated to an element $L$ of $\mathcal{A}$, the left and right $A_\infty$ modules $L_L$ and $R_L$ over $\mathcal{F}$ have underlying cochain complexes that were introduced in Section 3.2. The bimodule $\Delta$ is given by Equation (2.6.14). We now explain the construction of the complexes and maps appearing in Diagram (4.1.5), and the homotopy for this diagram.

Given a sequence of objects $\Upsilon = \{\sigma_i\}_{i=0}^d$, the $A_\infty$ operation
\begin{equation}
\mu^d: \mathcal{F}(\sigma_{d-1}, \sigma_d) \otimes \cdots \otimes \mathcal{F}(\sigma_0, \sigma_1) \to \mathcal{F}(\sigma_0, \sigma_d)
\end{equation}
is defined by counting rigid elements of $\mathcal{TR}(\Upsilon)$ as follows: given a collection $x = (x_0; x_1, \ldots, x_d) \in \text{Crit}(\sigma_0, \ldots, \sigma_d)$, we note that every element of $\mathcal{TR}(x)$ induces a map
\begin{equation}
\text{Hom}^c(U_{\sigma_{d-1},x_d}^{\sigma_{d-1}}, U_{\sigma_d,x_d}^{\sigma_d}) \otimes \cdots \otimes \text{Hom}^c(U_{\sigma_0,x_0}^{\sigma_0}, U_{\sigma_1,x_1}^{\sigma_1}) \to \text{Hom}^c(U_{\sigma_0,x_0}^{\sigma_0}, U_{\sigma_d,x_d}^{\sigma_d})
\end{equation}
obtained by parallel transport along all gradient flow line components (the boundary of all discs are constant in this case). Tensoring with the map on orientation lines induced by Equation (3.5.9), we obtain the map
\begin{equation}
\mu_u: \mathcal{F}(\sigma_{d-1}, \sigma_d) \otimes \cdots \otimes \mathcal{F}(\sigma_0, \sigma_1) \to \mathcal{F}(\sigma_0, \sigma_d).
\end{equation}

We define the higher products as the finite sum
\begin{equation}
\mu^k \equiv \sum_{u \in \mathcal{TR}(x)} (-1)^{\star} \mu_u,
\end{equation}
where $\star = 2 - d + \sum \deg x_i$, as in [16]. The proof that these operations satisfy the $A_\infty$ relation is standard (see, e.g. [2]).

Remark 4.7. We recall that the sign conventions in [16] are associated to assigning to each generator is reduced grading.

Given a sequence $\Upsilon = (L, \sigma_1, \ldots, \sigma_\ell)$, with $L \in \mathcal{A}$ and $\sigma_i \in \Sigma$, the module structure on $L_L$ is defined by counting elements of $\mathcal{TR}(\Upsilon)$: for each $x = (x_0; x_1, \ldots, x_\ell) \in \text{Crit}(\Upsilon)$ and $u \in \mathcal{TR}(x)$, we obtain a map
\begin{equation}
\text{Hom}^c(U_{\sigma_{\ell-1},x_\ell}^{\sigma_{\ell-1}}, U_{\sigma_\ell,x_\ell}^{\sigma_\ell}) \otimes \cdots \otimes \text{Hom}^c(U_{\sigma_1,x_1}^{\sigma_1}, U_{\sigma_2,x_2}^{\sigma_2}) \otimes U_{\sigma_\ell,x_\ell}^{\sigma_\ell} \to U_{\sigma_0,x_0}^{\sigma_0}
\end{equation}
by parallel transport along the boundary of the disc components as well as along all constituent gradient flow lines of $u$. Together with the map on orientation lines given by Equation (3.5.9), a rigid element thus induces a map
\begin{equation}
\mu_u: \mathcal{F}(\sigma_{\ell-1}, \sigma_\ell) \otimes \cdots \otimes \mathcal{F}(\sigma_1, \sigma_2) \otimes L_L(\sigma_1) \to L_L(\sigma_\ell).
\end{equation}
We define the higher left module structure maps as the sum
\begin{equation}
\mu^{(-1)^1}_L \equiv \sum_{y \in \mathcal{TR}(x)} (-1)^{\star + 1} T^{E(u)} \mu_u
\end{equation}
where $E(u)$ is the sum of the topological energies of the underlying strip, and the sign $\varPhi + 1$ accounts for the fact that $x_0$ and $x_1$ are assigned their usual gradings, and all other generators their reduced grading.

**Lemma 4.8.** The sum in Equation (4.2.7) converges.

**Proof.** Fix a path connecting a basepoint on $X_{\sigma_1}$ with the intersection point of this Lagrangian with each section $\iota_{\sigma_i}$. This induces a norm on each Floer complex appearing in Equation (4.2.6), and we shall prove that there are only finitely many elements $u$ of the moduli space such that valuation of $\mu_u$ is uniformly bounded by any given constant. As $\mu_u$ is a composition of maps associated to gradient flow lines and to a holomorphic strip, and the valuation of the maps associated to gradient flow lines is bounded, the valuation of (4.2.12) is uniformly bounded by any given constant. As $\mu_u$ is thus bounded, up to a constant independent of $u$, by the product of the diameter of elements of the cover $\{P_\sigma\}_{\sigma \in \Sigma}$ with the length of the boundary of the strip. The latter is bounded by the reverse isoperimetric inequality as in Lemma 3.3, and the result thus follows from Gromov compactness. \qed

We now briefly outline how the remaining operations are constructed: if we consider instead a sequence $T = (L, L', \sigma_1, \ldots, \sigma_\ell)$, with $L, L' \in \mathcal{A}$ and $\sigma_i \in \Sigma$, then the count of elements of $\mathcal{R}(\mathcal{Y})$ induces a map

$$\text{CF}^*(L, L') \to \text{Hom}(\mathcal{F}(\sigma_{\ell-1}, \sigma_\ell) \otimes \cdots \otimes \mathcal{F}(\sigma_1, \sigma_2) \otimes \mathcal{L}_L(\sigma_1), \mathcal{L}_L(\sigma_\ell)).$$

Regarding signs, we use the convention that the generators of $\text{CF}^*(L, L')$ and $\mathcal{L}_L(\sigma_i)$ are equipped with unreduced gradings; this corresponds to the fact that the moduli space $\mathcal{R}(x_0; x_{-r+1}, \ldots, x_{-1})$ contributes with sign

$$\varPhi + 1 + \sum_{i=-r+1}^{-1} \deg(x_i)$$

in the definition of $\mu_{R_L}^{1,r-1}$. Considering the case $\mathcal{Y} = (L, \sigma_1, \ldots, \sigma_r, L')$ yields the map

$$\mathcal{R}_L(\sigma_r) \otimes \mathcal{F}(\sigma_{r-1}, \sigma_r) \otimes \cdots \otimes \mathcal{F}(\sigma_1, \sigma_2) \otimes \mathcal{L}_L(\sigma_1) \to \text{CF}^*(L', L).$$

The structure map $\mu_{\Sigma}^{0,1,r}$

$$\mathcal{F}(\sigma_{\ell-1}, \sigma_\ell) \otimes \cdots \otimes \mathcal{F}(\sigma_0, \sigma_1) \otimes \Delta(\sigma_{-r}, \sigma_0) \otimes \mathcal{F}(\sigma_{-1}, \sigma_0) \otimes \cdots \otimes \mathcal{F}(\sigma_{-r}, \sigma_{-r+1}) \to \Delta(\sigma_{-r}, \sigma_\ell).$$

of the bimodule $\Delta$ is obtained by the count of rigid elements of $\mathcal{R}(\mathcal{Y})$, with $\mathcal{Y} = (\sigma_{-r}, \ldots, \sigma_{-1}, \sigma_{-r}, \sigma_0^\phi, \sigma_1^\phi, \ldots, \sigma_\ell^\phi)$. If we consider instead a sequence $\mathcal{Y} = (\sigma_{-r}, \ldots, \sigma_{-1}, L, \sigma_0^\phi, \sigma_1^\phi, \ldots, \sigma_\ell^\phi)$, we obtain a map

$$\mathcal{F}(\sigma_{\ell-1}, \sigma_\ell) \otimes \cdots \otimes \mathcal{F}(\sigma_0, \sigma_1) \otimes \mathcal{L}_L(\sigma_0) \otimes \mathcal{R}_L(\sigma_{-r}) \otimes \mathcal{F}(\sigma_{-1}, \sigma_{-r}) \otimes \cdots \otimes \mathcal{F}(\sigma_{-r}, \sigma_{-r+1}) \to \Delta(\sigma_{-r}, \sigma_\ell).$$
The maps associated to all such sequences for fixed \( L \) yield the map of bimodules:

\[
\mathcal{L}_L \otimes \mathcal{R}_L \to \Xi.
\]

Finally, the sequence \( \Upsilon = (L, \sigma_-, \ldots, \sigma_{-1}, \sigma_0, \sigma_0^\phi, \sigma_1^\phi, \ldots, \sigma_{\ell}^\phi) \) yields the map

\[
\mathcal{F}(\sigma_{-1}, \sigma_0) \otimes \cdots \otimes \mathcal{F}(\sigma_0, \sigma_1) \otimes \Xi(\sigma_0, \sigma_0) \otimes \mathcal{F}(\sigma_1, \sigma_0) \otimes \cdots \otimes \mathcal{F}(\sigma_{r-1}, \sigma_{r+1}) \otimes \mathcal{L}_L(\sigma_{r-1}) \to \mathcal{L}_L(\sigma_{\ell}).
\]

Fixing \( L \), we obtain the map of left modules:

\[
\mathcal{L}_L \otimes \mathcal{F} \Xi \to \mathcal{L}_L.
\]

We now prove the commutativity of the main diagram:

**Proof of Proposition 4.1.** Given a sequence \( \Upsilon = (L, \sigma_-, \ldots, \sigma_{-1}, \sigma_0, \sigma_0^\phi, \sigma_1^\phi, \ldots, \sigma_{\ell}^\phi) \),

the count of rigid elements of \( \mathcal{F}(\Upsilon) \) defines a map

\[
\mathcal{F}(\sigma_{-1}, \sigma_0) \otimes \cdots \otimes \mathcal{F}(\sigma_0, \sigma_1) \otimes \mathcal{L}_{L'}(\sigma_0) \otimes \mathcal{R}_{L'}(\sigma_0) \otimes \mathcal{F}(\sigma_1, \sigma_0) \otimes \cdots \otimes \mathcal{F}(\sigma_{r-1}, \sigma_{r-1}) \otimes \mathcal{L}_L(\sigma_{r-1}) \to \mathcal{L}_L(\sigma_{\ell}),
\]

which we rewrite by adjunction as a map

\[
\mathcal{R}_{L'}(\sigma_0) \otimes \mathcal{F}(\sigma_0, \sigma_1) \otimes \mathcal{L}_{L'}(\sigma_0) \otimes \mathcal{R}_{L'}(\sigma_0) \otimes \mathcal{F}(\sigma_1, \sigma_0) \otimes \cdots \otimes \mathcal{F}(\sigma_{r-1}, \sigma_{r-1}) \otimes \mathcal{L}_L(\sigma_{r-1}) \to \mathcal{L}_L(\sigma_{\ell}).
\]

Fixing \( L \) and \( L' \), and letting the sequences of elements in \( \Sigma \) vary, we obtain a map

\[
\mathcal{R}_{L'} \otimes \mathcal{F} \mathcal{L}_L \to \mathcal{Hom}_{\mathcal{F}} (\mathcal{L}_{L'}, \mathcal{L}_L)
\]

which is a homotopy between the two compositions in Diagram 4.3.11. \( \square \)

### 4.3. Global computations

To prove our main result, we shall need some additional algebraic computations:

#### 4.3.1. Perfectness of left modules

Recall that the left module \( \mathcal{Y}_\sigma \) associated to each element \( \sigma \) of \( \Sigma \) is given by

\[
\mathcal{Y}_\sigma(\tau) = \mathcal{F}(\sigma, \tau).
\]

Because the category \( \mathcal{F} \) is directed, we can associate to each of its objects a different left module \( \mathcal{L}_\sigma \) given by

\[
\mathcal{L}_\sigma(\tau) = \begin{cases} 
\mathcal{F}(\sigma, \sigma) & \tau = \sigma \\
0 & \text{otherwise,} 
\end{cases}
\]

where the module structure of \( \mathcal{F}(\sigma, \sigma) \) is the obvious one. The next result asserts that these modules can be built from Yoneda modules. For the proof, we introduce the notion of a module being *supported at* \( \sigma \in \Sigma \) if its cohomology vanishes at every other object of \( \mathcal{F} \). By construction, the module \( \mathcal{L}_\sigma \) is supported at \( \sigma \).

**Lemma 4.9.** There is an iterated extension of Yoneda modules which is quasi-isomorphic to \( \mathcal{L}_\sigma \).

**Proof.** The proof is by induction on the number of elements larger than \( \sigma \) in the partial ordering of \( \Sigma \). By definition, the maximal elements correspond to objects \( \sigma \) of \( \mathcal{F} \) such that \( \mathcal{F}(\sigma, \tau) \) vanishes whenever \( \tau \neq \sigma \). This establishes the Lemma in the base case of maximal elements. By induction, we therefore assume that the Lemma holds for all \( \tau \) larger than a fixed element \( \sigma \).
The fact that $\mathcal{F}$ is directed provides a filtration of every left module with subquotient direct sums of modules supported on elements of $\Sigma$. Applying this to the Yoneda module $Y_\sigma$, the quasi isomorphism between $\mathcal{F}(\sigma, \tau)$ and $\mathcal{F}(\tau, \tau)$ implies that $Y_\sigma$ is filtered by modules quasi-isomorphic to $L_\tau$ for $\sigma \leq \tau$. Applying the induction hypothesis, we conclude that the Yoneda module $Y_\sigma$ admits a filtration so that one subquotient is quasi-isomorphic to $L_\sigma$, and all others are iterated extensions of Yoneda modules. This implies that $L_\sigma$ is itself obtained as an iterated extension of Yoneda modules, proving the result. □

From the above, we conclude:

**Proof of Lemma 4.2.** Consider the filtration of $L_L$ associated to the fact that $\mathcal{F}$ is directed. The subquotient associated to each object $\sigma \in \Sigma$ is the complex $L_L(\sigma)$ as a module over $\mathcal{F}(\sigma, \sigma)$. Consider the filtration of $L_L(\sigma)$ by degree of the corresponding intersection point between $L_\sigma$ and $X_{q_\sigma}$. The subquotients are, by the computations of Section 2.3.4, quasi-isomorphic to free modules of rank-1 over $\mathcal{F}(\sigma, \sigma)$. The conclusion is immediate. □

**4.3.2. Reduction to the local directed categories.**

**Proof of Lemma 4.4.** Since the result holds far more generally, and in order to simplify the notation, let $P$ and $Q$ be left and right modules over $\mathcal{F}$ such

$$(4.3.3) \quad P(\tau) \text{ and } Q(\tau) \text{ are acyclic whenever } \tau \notin \Sigma_\sigma.$$ 

By the computations of Sections 2.6.2 and 2.6.3, the left and right modules $\Delta(\sigma, \_)$ and $\Delta(\_, \sigma)$ satisfy this assumption. We shall prove that, for each left module $L$ over $\mathcal{F}$, the natural maps

$$(4.3.4) \quad \text{Hom}_\mathcal{F}(L, P) \to \text{Hom}_{\mathcal{F}_\sigma}(L, P)$$

$$(4.3.5) \quad Q \otimes_{\mathcal{F}_\sigma} L \to Q \otimes_{\mathcal{F}} L$$

are quasi-isomorphisms. To this end, consider the number filtration of the bar complex, with associated graded complex given by the direct sum

$$(4.3.6) \quad Q(\sigma_d) \otimes \mathcal{F}(\sigma_d, \sigma_d) \otimes \cdots \otimes \mathcal{F}(\sigma_0, \sigma_1) \otimes L(\sigma_0).$$

By assumption, the only non-trivial contribution occurs if $\sigma_d \in \Sigma_\sigma$. On the other hand, if $\tau \in \Sigma_\sigma$, and there is a non-trivial morphism $\rho \to \tau$ in $\mathcal{F}$, then $\rho \in \Sigma_\sigma$. Thus the fact that $\sigma_d$ lies in $\Sigma_\sigma$ implies that all other elements of the sequence above also lie in $\Sigma_\sigma$, so that the inclusion of bar complexes induces an isomorphism on the cohomology of associated graded groups. The argument for $\text{Hom}_\mathcal{F}(L, P)$ is entirely analogous. □

**4.4. Local constructions.** In this section, we revisit the local constructions from Section 2 and lift them to the $A_\infty$ level. We start in Section 4.4.1 by constructing the category $\mathcal{P}_\sigma$ associated to an element of $\Sigma$, which admits $\mathcal{F}_\sigma$ as an embedded subcategory. We then briefly indicate how the constructions of Section 4.2 can be adapted to produce left and right modules over $\mathcal{P}_\sigma$ associated to each Lagrangian $L \in \mathcal{A}$, and the local duality map from Equation (4.1.11). We then use the moduli spaces constructed in Section 3.5 to prove Proposition 4.1.
4.4.1. The local category of (based) polytopes. We shall give a relatively convoluted definition of the category \( \text{Po}_\sigma \), which ensures that there is a strict \( A_\infty \) embedding \( \mathcal{F}_\sigma \subset \text{Po}_\sigma \). The basic idea is that we can think of the structure maps of \( \text{Po}_\sigma \) as counting either (perturbed) gradient flow segments on \( X_{q_v} \), glued together along choices of diffeomorphisms with specified isotopies to the identity, or as collections of (perturbed) gradient flow segments on fibres \( X_q \) over different points in \( \nu Q \sigma \), glued together along identifications of these fibres. The diffeomorphisms \( \psi_{\sigma}^q \) associated to the chosen section of \( X \) over \( \nu Q \sigma \) will be essential in comparing these two points of view. In order to achieve an embedding of \( \mathcal{F}_\sigma \), we also choose a homotopy between the Lagrangian sections associated to each \( \tau \in \Sigma_\sigma \), and that associated to \( \sigma \) itself.

The objects of the category \( \text{Po}_\sigma \) are pairs \( q \in P \), where \( P \subset \nu Q \sigma \). The choice of basepoint \( q \) is of no real consequence; the objects corresponding to all such choices will be quasi-isomorphic, and the choice is only included to give us enough flexibility to produce the desired strict \( A_\infty \) embedding.

For each pair \( \Upsilon = ((q_0, P_0), (q_1, P_1)) \), we choose a fibre \( q_\tau \) in \( \nu Q \sigma \), a metric \( g_\tau \) on \( X_{\tau^q} \), and a Morse-Smale function \( f_\tau \) on \( X_{\tau^q} \). We assume that these choices are subject to the following constraint:

\[
(4.4.1) \quad \text{If } (q_0, q_1) = (g_{\sigma_0}, g_{\sigma_1}), \text{ the data } (g_\tau, g_\tau, f_\tau) \text{ agree with the data used to define the Floer complex } CF^*((\sigma_0, P_0), (\sigma_1, P_1)) \text{ in Equation (2.2.36).}
\]

Remark 4.10. We implicitly assume that, whenever \( \tau \neq \rho \), the basepoints \( q_\rho \) and \( q_\sigma \) are distinct. Of course, this condition can be easily achieved by generic choices, but more importantly, we can always enlarge the category \( \text{Po}_\sigma \) so that we can associate to each element of \( \Sigma_\sigma \) a fixed object of \( \text{Po}_\sigma \).

The morphisms are then given by the Morse complexes

\[
(4.4.2) \quad \text{Po}_\sigma(\Upsilon) \equiv \text{CM}^* \left( X_{\tau^q}, \text{Hom}^*((U^r_{\sigma_0}, U^r_{\sigma_1}) \otimes \delta) \right).
\]

To define the compositions, we pick, for each sequence \( \Upsilon \) of objects of \( \text{Po}_\sigma \), metric tree \( T \) labelled by \( \Upsilon \), and disc with marked points \( S_v \in \overline{R}_{\tau^q} \) for each vertex, the following data:

(i) perturbed gradient equations on \( X_{q_v} \equiv X_{\tau^q} \) for each edge of \( T \), (ii) a point \( q_v \in \nu Q \sigma \) for each vertex of \( T \), and (iii) a diffeomorphism

\[
(4.4.3) \quad \psi_f : X_{q_v} \to X_{q_v},
\]

for each flag \( f = (v \in e) \), with a specified homotopy to \( \psi_{\sigma_0}^q \).

Remark 4.11. The choice of discs associated to vertices is completely unnecessary, and only appears here to make the construction exactly the same as that for the construction of the \( A_\infty \) structure on \( \mathcal{F} \).

Because the path space is contractible, there is no obstruction to making such choices in families, compatibly with degenerations of discs and breaking of edges. In addition, to ensure compatibility with the construction of the category \( \mathcal{F}_\sigma \), we require that:

\[
(4.4.4) \quad \text{If } \Upsilon = ((q_{\sigma_0}, P_0), \ldots, (q_{\sigma_n}, P_n)), \text{ with } \sigma_1 \in \Sigma_\sigma \text{ such that } \sigma_0 \leq \cdots \leq \sigma_n, \text{ the choice for any tree labelled by } \Upsilon \text{ agrees with that in Sections 3.3.3 and 3.5.}
\]

The specified homotopy for sequences coming from objects of \( \Sigma \) arises from the isotopy, fixed above, between the Lagrangian sections associated to elements of \( \Sigma_\sigma \), and the section for \( \sigma \) itself.

It is now entirely straightforward to see that the counts of such rigid configurations equip \( \text{Po}_\sigma \) with the structure of an \( A_\infty \) category. We obtain an embedding of \( \mathcal{F}_\sigma \) as a subcategory
of $P_{0\sigma}$, corresponding to the objects $(q_{\sigma}, P_{\sigma})$, by using for each pair $\Upsilon = (\tau_0, \tau_1)$ of objects of $\mathcal{T}_\sigma$, the isomorphism

$$CM^* \left( X_{\Upsilon}, \text{Hom}^c(U_{\tau_0}^{P_{\tau_0}}, U_{\tau_1}^{P_{\tau_1}} \otimes \delta) \right) \cong CM^* \left( X_{\Upsilon'}, \text{Hom}^c(U_{\tau_0}^{P_{\tau_0}}, U_{\tau_1}^{P_{\tau_1}} \otimes \delta) \right)$$

induced by the choice of isotopy of sections associated to $\sigma$ and $\tau_1$.

4.4.2. Modules and structure maps. Given $L \in \mathcal{A}$, we have already fixed a Hamiltonian isotopic Lagrangian $L_\sigma$ which is transverse to $X_{q_\sigma}$. Given an element $(q, P)$ of $P_{0\sigma}$, we define

$$\mathcal{L}_{L_\sigma}(q, P) \equiv CF^*(L, (\sigma, P))$$

$$\mathcal{R}_{L_\sigma}(q, P) \equiv CF^*((\sigma, P), L)$$

as in Section 2.3.2. In other words, all the groups are defined using holomorphic strips with Lagrangian boundary conditions $(L_\sigma, X_{q_\sigma})$.

Let $\Upsilon$ be a sequence of the form

$$(L, (q_0, P_0), \ldots, (q_t, P_t))$$

as in Section 3.4.1, we consider a moduli space $\mathcal{M}_{\Upsilon}$ of discs and metric trees, which has a top-dimensional stratum of the form $\mathcal{M}_{\Upsilon}^T \times \prod_{T \in \mathcal{E}(M_\sigma(T))} \mathcal{T}_\sigma$ for each tree $T$ with label $\Upsilon$. We then proceed to inductively choose Morse data on all edges of the tree $T$, compatibly with the choice of Morse data on the trees which define the $A_\infty$ structure. As in Equation (4.4.5), we obtain a mixed moduli space $\mathcal{M}_{\Upsilon}(\Upsilon')$ of holomorphic discs boundary conditions $L_\sigma$ and $X_{q_\sigma}$, with two ends (adjacent to the segment labelled $L$), equipped with a collection of boundary marked points.

As in Section 3.3.3 let $\Upsilon^\sigma$ denote the sequence obtained from $\Upsilon$ by replacing all objects of $P_{0\sigma}$ by the element $\sigma$ of $\Sigma$. We have a corresponding moduli space $\mathcal{M}_{\Upsilon}(\Upsilon')$ of holomorphic discs boundary conditions $L_\sigma$ and $X_{q_\sigma}$, with two ends (adjacent to the segment labelled $L$), equipped with a collection of boundary marked points.

The count of rigid elements of the moduli space associated to Equation (4.4.8) defines a map

$$P_{0\sigma}((q_{-1}, P_{-1}), (q, P)) \otimes \cdots \otimes P_{0\sigma}((q_0, P_0), (q_1, P_1)) \otimes \mathcal{L}_{L_\sigma}(q_0, P_0) \rightarrow \mathcal{L}_{L_\sigma}(q_\ell, P_\ell),$$

which makes $\mathcal{L}_{L_\sigma}$ into a left module over $P_{0\sigma}$. We similarly obtain the structure of a right module on $\mathcal{R}_{L_\sigma}$ by considering sequences as in Equation (4.4.9).

Finally, the sequence in Equation (4.4.10) gives rise to a map

$$P_{0\sigma}((q_{-1}, P_{-1}), (q_\ell, P_\ell)) \otimes \cdots \otimes P_{0\sigma}((q_0, P_0), (q_1, P_1)) \otimes \mathcal{L}_{L_\sigma}(q_0, P_0) \otimes \mathcal{R}_{L_\sigma}(q_{-0}, P_{-0})$$

$$\otimes P_{0\sigma}((q_{-1}, P_{-1}), (q_{-0}, P_{-0}) \cdots \otimes P_{0\sigma}((q_{-r}, P_{-r}), (q_{-r+1}, P_{-r+1})) \rightarrow P_{0\sigma}((q_{-r}, P_{-r}), (q_{\ell}, P_\ell)).$$
Fixing $L$, and letting $r$ and $\ell$ vary, we obtain the map of bimodules
\[(4.4.16) \quad \mathcal{L}_{L,\sigma} \otimes \mathcal{R}_{L,\sigma} \to \Delta_{P_{\sigma}}.\]

4.4.3. The global-to-local comparison. The purpose of this section is to complete the reduction of the global computations to local ones. The argument is an entirely straightforward use of the moduli spaces $\mathcal{TR}(\Sigma)$ from Section 3.5.

**Proof of Proposition 4.5.** Fix an element $\sigma \in \Sigma$. We first begin by considering sequences $\Upsilon$ of the form $(L, \sigma_0, \ldots, \sigma_\ell)$ or $(\sigma_r, \ldots, \sigma_0, L)$, where $\sigma_i \in \Sigma$. The counts of rigid elements of the corresponding moduli spaces define maps
\[(4.4.17) \quad \mathcal{F}(\sigma_{\ell-1}, \sigma_\ell) \otimes \cdots \otimes \mathcal{F}(\sigma_0, \sigma_1) \otimes \mathcal{L}_{L,\sigma}(q_{\sigma_0}, P_{\sigma_0}) \to \mathcal{L}_{L}(\sigma_\ell)\]
\[(4.4.18) \quad \mathcal{R}_{L,\sigma}(q_{\sigma_0}, P_{\sigma_0}) \otimes \mathcal{F}(\sigma_{-1}, \sigma_0) \otimes \cdots \otimes \mathcal{F}(\sigma_{-r}, \sigma_{-r+1}) \to \mathcal{R}_{L}(\sigma_{-r})\]
which are the structures maps of left and right module homomorphisms $j^* \mathcal{L}_{L,\sigma} \to \mathcal{L}_{L}$ and $j^* \mathcal{R}_{L,\sigma} \to \mathcal{R}_{L}$. Here, we use the identification
\[(4.4.19) \quad \mathcal{F}(\sigma_i, \sigma_j) \cong P_{\sigma}(\{q_{\sigma_i}, P_{\sigma_i}\}, \{q_{\sigma_j}, P_{\sigma_j}\})\]
giving rise to the $A_{\infty}$ embedding noted at the end of Section 4.4.1.

Considering instead a sequence $\Upsilon = (\sigma_r, \ldots, \sigma_{-1}, \sigma_{-0}, \sigma_0^0, \sigma_0^1, \ldots, \sigma_0^\ell)$, we obtain a map
\[(4.4.20) \quad \mathcal{F}(\sigma_{\ell-1}, \sigma_\ell) \otimes \cdots \otimes \mathcal{F}(\sigma_0, \sigma_1) \otimes P_\sigma((q_{\sigma_{-0}}, P_{\sigma_{-0}}), (q_{\sigma_0}, P_{\sigma_0}))\]
\[\otimes \mathcal{F}(\sigma_{-1}, \sigma_{-0}) \otimes \cdots \otimes \mathcal{F}(\sigma_{-r}, \sigma_{-r+1}) \to \Xi(\sigma_{-r}, \sigma_\ell),\]
which defines the map of bimodules $j^* \Delta_{P_{\sigma}} \to \Xi$.

The homotopies in Diagrams (4.1.15) and (4.1.16) follow in exactly the same way by counting rigid elements of $\mathcal{TR}(\Sigma)$ for sequences:
\[(4.4.21) \quad (L, \sigma_{-r}, \ldots, \sigma_{-0}, \sigma_0^0, \sigma_0^1, \ldots, \sigma_\ell^\ell)\]
\[(4.4.22) \quad (\sigma_{-r}, \ldots, \sigma_{-0}, L, \sigma_0^0, \sigma_1^0, \ldots, \sigma_\ell^\ell).\]

\[\square\]

4.5. Local computations. We conclude the main part of the paper by performing to the necessary local computations:

**Proof of Lemma 4.6.** The two arguments are entirely analogous; we explain the proof of the duality isomorphism. The decomposition of the intersection points of $L_\sigma$ with $X_{\sigma}$ by degree provides a filtration of $j^* \mathcal{R}_{L,\sigma}(\sigma)$ as a cochain complex, and of $j^* \mathcal{L}_{L,\sigma}$ as a left module, hence of $\text{Hom}_{\mathcal{F}}(j^* \mathcal{L}_{L,\sigma}, j^* \Delta_{P_{\sigma}}(\sigma, \omega))$ as a cochain complex. It suffices to prove that the map at the level of each associated graded group is a quasi-isomorphism. Regularity implies that all holomorphic strips whose inputs and outputs agree are constant, so that the module structure on each subquotient of this filtration on $j^* \mathcal{L}_{L,\sigma}$ is the same as the module structure for a Lagrangian section, whose restriction to each object $\tau$ is quasi-isomorphic to a free rank-1 module. The complex we are trying to compute is thus quasi-isomorphic to the Čech complex
\[(4.5.1) \quad \bigoplus_{\tau_0 < \cdots < \tau_k} P_{\sigma}(\{q_{\sigma}, P_{\sigma}\}, \{q_{\sigma}, P_{\tau_k}\})[-k, \delta]\]
by the argument of Section 4.3.2 using the quasi-isomorphism between $(q_{\sigma}, P_{\tau_k})$ and $(q_{\tau_k}, P_{\tau_k})$. As in Section 4.4.2 we may replace $P_{\tau_k}$ in the above expression by $P_{\tau_k} \cap P$ for a polytope $P$. 

containing \( P \) in its interior, and covered by the elements of \( \Sigma \). By Tate acyclicity, we conclude that each associated graded group of \( \text{Hom}_{\Sigma \sigma} \left( j^* L_{L, \sigma}, j^* \Delta P_{\sigma} (\sigma, \omega) \right) \) is quasi-isomorphic to \( P_{\sigma} \) whose cohomology is quasi-isomorphic to the linear dual of \( \Gamma^\sigma \) by the computation of Appendix C. The associated graded group of \( j^* R_{L, \sigma} (\sigma) \) is isomorphic to this same group, and the fact that the map between them is an isomorphism then follows from Proposition 2.15. \( \square \)

**Appendix A. Reverse isoperimetric inequalities**

**A.1. The basic inequality.** Let \( X \) be a closed symplectic manifold, \( S \) a Riemann surface obtained from a closed Riemann surface with boundary by removing boundary punctures. Let \( K \subset X \) be a closed codimension 0 submanifold with boundary, and assume that we have a labelling

\[
\Upsilon_S: \pi_0(\partial S) \to \{K\} \cup \mathcal{L}
\]

of \( \partial S \) by \( \{K\} \cup \mathcal{L} \), where \( \mathcal{L} \) is a collection of Lagrangians in \( X \). Given \( i \in \pi_0(\partial S) \), we write \( \partial_i S \) for the corresponding component of the boundary. We write \( \partial_K S \) for the union of components labelled by \( K \), and \( L_i \) for the Lagrangian labelled by a component \( i \). We assume that all intersections among Lagrangians appearing in \( \mathcal{L} \) are contained in the interior of \( K \), and choose another closed subset \( K' \) containing this intersection, and contained in the interior of \( K \):

\[
K' \Subset K \subset X.
\]

In addition, we fix a Riemannian metric \( g_X \) with respect to which we shall compute all norms.

Given an almost complex structure \( J \), consider a family

\[
J_S: S \to \mathcal{J}
\]

of almost complex structures which agree with \( J \) away from \( K' \). There is an associated moduli space

\[
M(\Upsilon_S, J_S)
\]

of \( J_S \)-holomorphic curves with boundary conditions given by \( \Upsilon_S \) in the sense that a point \( z \in \partial_K S \) maps to \( K' \), while a point \( z \in \partial L_i \) on a component labelled by a Lagrangian maps to \( L_i \).

Recall that the geometric energy \( E_{geo}(u) \) of any element \( u \) of the moduli space is the area

\[
E(u) = \int ||du||^2.
\]

Moreover, given a component \( i \) of \( \partial S \), we define

\[
\ell_{L_i, K} (\partial u)
\]

the length of the part of \( \partial u \) mapping along \( L_i \) to the complement of \( K \), with respect to \( g_X |_{L_i} \).

**Lemma A.1.** For each choice of labels \( \Upsilon_S \), there exists a constant \( C \), depending on \( J \) but not on the family \( J_S \), such that for each element \( u \in M(\Upsilon_S, J_S) \) and each component \( i \in \pi_0(\partial S) \) which is labelled by a Lagrangian, we have

\[
\ell_{L_i, K} (\partial u) \leq C E(u).
\]

This constant is independent of the restriction of \( L_i \) to \( K \).
Proof. This is a minor modification of the results of Groman-Solomon and Duval. We briefly indicate how to adapt Duval’s proof: Since all intersection points between Lagrangians are contained in \( K' \), we may choose a tubular neighbourhood \( \nu X L_i \) of \( L_i \), which is equipped with a non-negative weakly plurisubharmonic function \( \rho_i \) (with respect to \( J \)) that vanishes on the intersection with \( K' \cup L_i \), and does not vanish away from the intersection with \( K \cup L_i \). Moreover, we require that \( \rho_i \) be (strictly) plurisubharmonic away from \( K' \), with weakly plurisubharmonic square root. Near \( \partial K' \cap L_i \), such a function can be locally modelled after the function

\[
\chi(x_1)\left(|y_1|^2 + \cdots + |y_n|^2\right).
\]

for the Lagrangian \([0, \infty) \times \mathbb{R}^{n-1} \subset \mathbb{C}^n\), where \( \chi \) is a smooth function vanishing for \( x_1 \leq 0 \).

For simplicity, we assume that \( g_X \) everywhere dominates the (semi)-metric induced by \( \rho_i \) and \( J \).

Let \( \ell_{\rho_i}(\partial u) \) denote the length of \( \partial u \) with respect to the (semi)-metric induced by \( dd^c \rho_i \), and \( E_{\rho_i} \) the integral of \( dd^c \rho_i \) over the part of \( u \) with image in \( \nu X L_i \). There is a constant \( C_0 \) such that

\[
(A.9) \quad \ell_{\rho_i}(\partial u) \leq C_0E_{\rho_i}(u).
\]

The basic idea of is that the function \( E_{\rho_i}(u) \) which measures the area of \( u \) in the domain \( \rho_i \leq r \) is monotonic, and by Fubini’s theorem, has limit bounded above by a constant multiple \( \ell_{\rho_i}(\partial u) \) (see [2] for details); the fact that \( \rho_i \) vanishes to order greater than 1 along \( K' \) implies that this part of boundary does not contribute to \( \ell_{\rho_i}(\partial u) \).

Away from \( K' \), the metric induced by \( dd^c \rho_i \) is uniformly comparable to \( g_X \), so we may assume the existence of a constant \( C_1 \) such that

\[
(A.10) \quad \ell_{L_i,K}(\partial u) \leq C_1 \ell_{\rho_i}(\partial u)
\]

On the other hand, the assumption that \( g_X \) dominates \( dd^c \rho_i \) implies that

\[
(A.11) \quad E_{\rho_i}(u) \leq E^{geo}(u).
\]

The result follows from combining these inequalities. \( \square \)

For application, it will be convenient to state the estimate in a different way: define \( L_i/\sim \) to be the quotient of \( L_i \) by the relation which identifies points in the same component of \( K \cap L_i \). Given a metric on \( L_i/\sim \), we have:

**Corollary A.2.** There exists a constant \( C \) such that, for each element \( u \in M(\Upsilon_S, J_S) \), we have

\[
(A.12) \quad \ell(\partial u/\sim) \leq CE(u).
\]

\( \square \)

A.2. Moving Lagrangian boundary conditions. It will be necessary to have a generalisation for moving boundary conditions, and families of almost complex structures. Let \( S \) be a Riemann surface equipped with a thick-thin decomposition: in our setting, this will simply mean a collection of subsets \( \Theta \subset S \), the components of the thin part, which include neighbourhoods of all punctures.

Let \( \Upsilon_S \) be moving boundary conditions on \( S \), i.e. a smooth assignment of a subset of \( X \) to each point on \( \partial S \), which we assume is locally constant on the thin-part. We shall only consider the situation in which the restriction to each component is either a moving family of Lagrangians, or is a constant family given by a compact subset \( K \subset X \) as in the previous section.
For each component \( i \in \pi_0(\partial S) \) labelled by a moving family of Lagrangians, we denote the graph by \( \tilde{L}_i \subset \partial_i S \times X \). By construction, this map is independent of the first factor in the thin part, so that we obtain a Lagrangian \( L_{i, \Theta} \) in \( X \) for each component \( \Theta \) of the thin part.

Assume that we are given, for each such component, an almost complex structure \( J_{\Theta} \) on \( X \) as well as a pair of codimension 0 manifolds with boundary

\[
\nu' \Theta \subset \nu_X \Theta \subset X
\]

which contain the intersection of the labels of all boundary components which intersect \( \Theta \). We pick a family \( J_S \) of almost complex structures on \( X \) parametrised by \( S \), such that

\[
\text{the restriction of } J_S \text{ to the product of a component } \Theta \text{ of the thin part with } X \setminus \nu' \Theta \text{ agrees with } J_{\Theta}.
\]

Given this data, we shall consider the moduli space

\[
M(\Upsilon_S, J_S)
\]

of \( J_S \) holomorphic curves with boundary conditions \( \Upsilon_S \): it is convenient to introduce the almost complex structure \( \tilde{J}_S \) on \( S \times X \) induced by \( J_S \), and describe this moduli space as the space of \( \tilde{J}_S \) holomorphic sections with boundary conditions given by \( \tilde{L}_i \) over \( \partial_i S \).

Our goal is to prove a reverse isoperimetric inequality for these moduli spaces. The basic idea is that Lemma A.1 provides an estimate for the moduli spaces holomorphic strips corresponding to each end; we shall extend this estimate to \( S \), at the cost of a possibly weaker proportionality constant, as well as the addition of a constant term. The main point is to provide a proof that extends to families of (broken) holomorphic curves.

We shall compare the geometric energy of each curve \( u \)

\[
E^{geo}(u) = \int |du|^2 = \int u^* \omega
\]

to the length of the boundary, which we formulate as follows: we denote by \( \tilde{L}_i/\sim \) the quotient of \( \tilde{L}_i \) by the equivalence relation which (i) collapses each component of the inverse image of \( \Theta \) in \( \tilde{L}_i \) to a single fibre, and (ii) identifies points in the same component of the intersection of \( \tilde{L}_i \) with \( \nu_X \Theta \). In particular, \( \tilde{L}_i/\sim \) maps to the quotient of \( \partial_i S \) by the components of the intersection with the thin part, agrees with \( \tilde{L}_i \) away from the inverse image of these components, and agrees with the quotient considered in the previous section over each component of the thin part.

**Lemma A.3.** There exists a constant \( C \) such that, for each \( u \in M(\Upsilon_S, J_S) \), we have

\[
\ell(\partial_i u/\sim) \leq C E^{geo}(u) + \text{a constant independent of } u.
\]

The constant depends only on the restriction of the Lagrangian boundary conditions and the almost complex structures to a neighbourhood of \( \partial_i S \), but is independent of their restriction to \( \Theta \times \nu_X' \Theta \) for each component \( \Theta \) of the thin part.

**Proof.** The graph \( \tilde{L}_i \) is a totally real submanifold with respect to the almost complex structure \( \tilde{J}_S \) which at every point in \( z \in S \) is the product of the complex structure on \( S \) with the value of the family \( J_S \) at \( x \). This almost complex structure is compatible with the symplectic form

\[
\omega_X + \omega_S,
\]
where $\omega_S$ is an area form on $S$ which we assume has area 1. If we define $E(\tilde{u})$ to be the area with respect to such a symplectic form, we have

$$E(\tilde{u}) = E(u) + 1,$$

which will be one origin for the constant term in the statement of the Lemma. The remainder of the proof proceeds in essentially the same way as that of Lemma A.1.

For each component $\Theta$ of the thin part which is adjacent to $i$, equipped with the family of almost complex structures $J_\Theta$, fix the neighbourhood $\nu_{\Theta, \tilde{L}_i}$ and the function $\rho_{i, \Theta} : \nu_{\Theta} \to \mathbb{R}$ considered in the proof of Lemma A.1.

By projection to the second factor, we obtain a function on $\Theta \times X$ which we denote $\rho_{i, \Theta}$.

We consider a neighbourhood $\nu_{S \times X} \tilde{L}_i$ of $\tilde{L}_i$ which contains the product $\Theta \times L_i, \Theta$ for all components $\Theta$ of the thin part which meet $\tilde{L}_i$, and is equipped with a weakly plurisubharmonic function

$$\rho_i : \nu_{S \times X} \tilde{L}_i \to [0, \infty)$$

such that the following properties hold

1. Over $\Theta \subset S$, $\rho_i$ agrees with $\rho_{i, \Theta}$.
2. Away from $\Theta \times X \Theta$, the function $\rho_i$ is strictly plurisubharmonic, and only vanishes on $\tilde{L}_i$.
3. The square root of $\rho_i$ is everywhere weakly plurisubharmonic.

The existence of such a function follows from the usual patching argument for plurisubharmonic functions as in [7]. For simplicity, we also assume that there is a finite diameter metric on $S$ whose product with $g_X$ everywhere dominates the metric induced by $\rho_i$.

Given a section $\tilde{u}$ corresponding to an element of $M(\Upsilon_S, J_S)$, let $\ell_{\rho_i}(\partial_i \tilde{u})$ denote the length of $\partial_i \tilde{u}$ with respect to the (semi)-metric induced by $dd^c \rho_i$, and $E_{\rho_i}$ the integral of $dd^c \rho_i$ over the part of $\tilde{u}$ with image in $\nu_{S \times X} \tilde{L}_i$. As before, there is a constant $C_0$ such that

$$\ell_{\rho_i}(\partial_i \tilde{u}) \leq C_0 E_{\rho_i}(u).$$

Away from a small neighbourhood of the inverse image of $\Theta$, the metric induced by $dd^c \rho_i$ is uniformly comparable to the product of $g_X$ with a metric on $S$. Because $\rho_i$ is non-degenerate in the fibre direction over $\Theta$, we therefore obtain a constant $C_1$ such that

$$\ell(\partial_i u / \sim) \leq C_1 \ell_{\rho_i}(\partial_i \tilde{u}) + \text{a constant independent of } u,$$

where the constant term accounts for the fact that the derivatives of $\rho_i$ in the base direction vanish identically on $\Theta_i$.

On the other hand, the assumption that $g_X$ and the metric on $S$ dominate $dd^c \rho_i$ implies that

$$E_{\rho_i}(u) \leq E(\tilde{u}).$$

The result follows from combining this inequality with Equations (A.7), (A.9), and (A.10). □

A.3. Compatibility with gluing. We shall require a uniform estimate for families of Riemann surfaces. Let us therefore consider a rooted planar tree $T$, and a collection of Riemann surfaces with punctures $S_v$ for each vertex of $T$, with an identification of the ends of $S_v$ with the edges adjacent to $v$. We write $E^{\text{int}}(T)$ for the edges of $T$ which are adjacent to two vertices. Given gluing parameters

$$R : E^{\text{int}}(T) \to [0, \infty]$$

for the family of almost complex structures $J_\Theta$.
and a choice of disjoint strip-like ends for each end of $S_v$, we obtain a Riemann surface $S_{T,R}$ by gluing. Assuming that each $S_v$ is equipped with a thick-thin decomposition, we obtain a thick-thin decomposition of $S_{T,R}$. We allow the possibility that the decomposition of a component $S_v$ be degenerate, in the sense that the thin part consists of the entire surface.

The planar structure determines an embedding $T \subset \mathbb{R}^2$ up to isotopy; assume that we have moving boundary conditions $\Upsilon_{S_v}$, which are locally constant on all components of the thin part, and are consistent for adjacent vertices, in the sense that the subsets of $X$ assigned to the two ends corresponding to each interior edge of $T$ agree. We write $\Upsilon_{S_{T,R}}$ for the moving boundary conditions on $S_{T,R}$ obtained by gluing. As in the previous section, we pick, for each component $\Theta$ of the thin part of a curve $S_v$, closed subsets $\nu'_\Theta \subset \nu \Theta$ of $X$, which contains the intersections of all labels, and an almost complex structure $J_\Theta$ on $X$; we assume that these choices are the same for the thin parts meeting the two punctures corresponding to a given edge in $T$.

Finally, we pick families of almost complex structures $J_{S_v}$ which agree with $J_\Theta$ on $\Theta \times X \setminus \nu'_\Theta$, and are also compatible across the edges of $T$. Let $J_{S_{T,R}}$ be a family of almost complex structure on $S_{T,R}$ which agrees with the family obtained by gluing (i) in the thick part near each boundary stratum, and (ii) in the thin part away from $\Theta \times \nu'_\Theta$ for each component $\Theta$ of the thin part.

Given this data, we shall consider the moduli space

$$M(\Upsilon_{S_{T,R}}, J_{S_{T,R}})$$

which is given as in the previous section if all gluing parameters are finite, and is given for infinite gluing parameters by the fibre products, along the evaluation maps for the edges, of the moduli spaces $M(\Upsilon_{S_v}, H_{S_v}, J_{S_v})$ corresponding to the vertices.

For each $i \in \pi_0(\mathbb{R}^2 \setminus T)$ with Lagrangian label, and finite gluing parameter $R$, consider the quotient $L_i/R/\sim$ of the moving Lagrangian boundary condition over the component of the boundary corresponding to $i$, by the relation considered in the previous section: collapse each component of the inverse image of the thin part, then identify points in the same component of $\nu'_\Theta$ in the fibres over a component $\Theta$ of the thin part. By construction, the spaces we obtain for different choices of gluing parameters are naturally homeomorphic; we write $L_i/\sim$ for any of these spaces, and note that we have an evaluation map

$$\partial_i u/\sim: \partial_i S_{T,R} \to L_i/\sim$$

for any choice of gluing parameter.

**Lemma A.4.** There exists a constant $C$ such that, for each $R \in [0, \infty) \mathcal{E}^{int}(T)$ and $u \in M(\Upsilon_{S_{T,R}}, J_{S_{T,R}})$, we have

$$l(\partial_i u/\sim) \leq CE^{geo}(u) + \text{ a constant independent of } u \text{ and } R.$$  

This constant depends on the restriction of $J_{S_{T,R}}$ to a neighbourhood of the corresponding boundary component, and is independent of the restriction of $(L_i,R, J_{S_{T,R}})$ to the product of each component $\Theta$ of the thin part with $\nu'_\Theta$.

**Proof.** It suffices to check that the constants in the proof of Lemma A.3 can be chosen in terms of the corresponding constants for the moduli spaces $M(\Upsilon_{S_v}, J_{S_v})$ and independently of $R$. For Equation (A.7), this follows from the fact that the choice of symplectic form on $X \times S_T$ induces a symplectic form on $X \times S_{T,R}$. For Equation (A.9), we note that a choice of function $\rho_{S_v}$ for all $v$ induces, by gluing, a function $\rho_{S_{T,R}}$, for which we have the same estimate. The same argument applies to Equation (A.10), which completes the argument. \(\square\)
Appendix B. Tate’s acyclicity theorem

Given an affinoid covering of an affinoid domain, Tate showed in [19] that the augmented Čech complex of the rings of functions is acyclic, and more generally for the Čech complex with coefficients in a complex of coherent sheaves. Tate’s argument starts by constructing a null-homotopy for Laurent coverings (see Section B.2 below), then proceeds to use standard tools of homological algebra to conclude the general case.

In this section, we imitate the strategy of Tate’s proof in order to be able to use Čech methods to compute morphism spaces in the analytic Fukaya category. Since the construction is completely local, we consider a torus $\mathbb{T}^n$ equipped with a basepoint and Morse function. Let $\mathcal{F}$ denote the $A_\infty$-category with objects integral affine polytopes $P \subset H_1(\mathbb{T}^n, \mathbb{R})$, morphisms for a pair $(P_0, P_1)$ given by the complex

$$ CF^*(P_0, P_1) \equiv CM^*(X_Q, \text{Hom}_c(U^{P_0}, U^{P_1}) \otimes \delta) $$

and $A_\infty$ operations as in Section 4.4.1.

Let $A$ be an ordered set indexing a cover $\{P_a\}_{a \in A}$ of a polytope $P$. Given a totally ordered subset $\tau \subset A$, we denote by $P_\tau$ the intersection of the polytopes $P_a$ for $a \in \tau$. If $\tau$ is a subset of $\sigma$, the inclusion $P_\sigma \subset P_\tau$ gives rise to a canonical element

$$ \delta^\tau_\sigma \in CF^0(P_\tau, P_\sigma) $$

whose construction is recalled in Section B.1 below.

We obtain a twisted complex

$$ \check{T}(P, A) \equiv \left( \bigoplus_{\sigma \in A} P_\sigma[-|\sigma|], \delta \right), $$

where $\delta$ is the Čech differential. Explicitly, if $\sigma$ equals $(a_1, \ldots, a_m)$ as an ordered set, then the restriction of the differential to $P_\sigma$ is given by

$$ \sum_{a \in A \setminus \sigma} (-1)^i \delta^{\sigma \cup \{a\}}_\sigma. $$

Proposition B.1. The natural map

$$ P \rightarrow \check{T}(P, \Sigma) $$

induced by $\bigoplus_{a \in A} \delta^\tau_\sigma$ is a quasi-isomorphism in the category of twisted complexes over $\mathcal{F}$.

The proof of this proposition is given in Section B.3 below.

B.1. Restriction maps on Floer cochains. We begin by defining the map in Equation (B.2) more precisely: if $P_1 \subset P_0$, the restriction map $\Gamma^{P_0} \rightarrow \Gamma^{P_1}$ induces a map of local systems

$$ U^{P_0} \rightarrow U^{P_1}, $$

which is a continuous inclusion (we point out again that, unlike in Section 2.2.1, we do not decorate local systems by elements of an underlying cover of $Q$). These maps are natural in sense that given a triple $P_2 \subset P_1 \subset P_0$ the map $U^{P_0} \rightarrow U^{P_2}$ is given by composition.

Taking the sum of these elements over all maxima of the Morse function on $\mathbb{T}^n$, we obtain

$$ \delta^{P_0}_{P_1} \in CF^0(P_0, P_1). $$

To tie back to Equation (B.2), we define

$$ \delta^\tau_{P_1} \equiv \delta^{P_0}_{P_1}. $$
B.2. Tate’s null-homotopy. Following Tate, we begin by consider an integral affine function \( u \) on \( H_1(\mathbb{T}^n, \mathbb{R}) \). If \( P \) is an integral affine polytope, let \( P_+ \) and \( P_- \) denote the subsets of \( P \) where \( u \) is non-negative, respectively non-positive, and let \( P_{\pm} \) denote their intersection. In the language of rigid geometry, this corresponds to a Laurent cover with two terms (Tate calls these two term special affine coverings, see [19, Lemma 8.3]).

Consider the two-term Čech complex \( \check{C}^*(\Gamma;\{+,-\}) \)

\[
\Gamma^P_+ \oplus \Gamma^P_- \xrightarrow{\check{d}} \Gamma^P_.
\]

This complex is natural in the sense that every inclusion \( P \subset P' \) induces a natural map of complexes

\[
\check{C}^*(\Gamma^{P'};\{+,-\}) \xrightarrow{\check{h}} \check{C}^*(\Gamma^P;\{+,-\})
\]

**Lemma B.2.** There is a continuous null-homotopy for the augmented Čech complex complex

\[
\Gamma_p \rightarrow \check{C}^*(\Gamma^P;\{+,-\})
\]

which is natural in \( P \).

**Proof.** Applying a change of coordinates, we may assume that \( u \) is a coordinate function on \( H_1(\mathbb{T}^n, \mathbb{R}) \), corresponding to a monomial \( z \) in the group ring \( \Gamma \). We formally write every element of the ring of functions on \( P_{\pm}, P_+, P_- \), and \( P \) as \( F(z, w) = \sum_{i=1}^{+\infty} z^i f_i(w) \) where \( w = (w_2, \ldots, w_n) \) are the other coordinates, and define

\[
F_+(z, w) \equiv \sum_{i=1}^{+\infty} z^i f_i(w)
\]

\[
F_-(z, w) \equiv \sum_{i=-\infty}^{0} z^i f_i(w).
\]

Evidently, \( F_- + F_+ = F \). On \( \Gamma^P_\pm \), the null homotopy is provided by

\[
\Gamma^P_+ \oplus \Gamma^P_- \xrightarrow{(F_+, -F_-)} \Gamma^P_-
\]

The valuation of this map is non-negative because the minimal valuation of \( F_\pm \) on \( P_\pm \) is achieved on \( P_\pm \), and \( \check{d} \circ \check{h} \mid \Gamma^P_\pm \) is the identity. On \( \Gamma^P_+ \oplus \Gamma^P_- \), the null homotopy is

\[
\Gamma^P \xrightarrow{\Gamma^P_+ \oplus \Gamma^P_-} \Gamma^P_+ \oplus \Gamma^P_-
\]

\[
F_- + G_+ \xrightarrow{(F, G)} (F_+ + G_+ + (F - G)_+, F_- + G_+ - (F - G)_-),
\]

from which the equation for a null-homotopy follows.

Naturaly with respect to restriction maps is automatic from the fact that we did not appeal to any property of \( P \) in constructing \( \check{h} \). □
Given a topological vector space $V$, we consider the complex

$$(B.11) \quad \text{Hom}^c(\check{\mathcal{C}}^\ast(\Gamma_P;\{+,-\}),V) \equiv \text{Hom}^c(\Gamma^P_+,V) \oplus \text{Hom}^c(\Gamma^P_-,V),$$

where $\text{Hom}^c$ is the space of continuous maps. Composing with $\tilde{h}$, we obtain a null-homotopy for the augmented complex

$$(B.12) \quad \text{Hom}^c(\check{\mathcal{C}}^\ast(\Gamma_P;\{+,-\}),V) \rightarrow \text{Hom}^c(\Gamma_P,V)$$

which is natural in $P$, $V$.

B.3. Acyclicity of the augmented complex. We next consider a general Laurent cover: let $\{u_m\}_{m \in M}$ be a collection of integral affine functions on $H_1(T^n,\mathbb{R})$ indexed by a finite set $M$. Given a polytope $P$, we associate to each element of $M \times \{+,-\}$ the polytope $P_{m,\pm}$ given by the subset where $u_m$ is non-negative (or non-positive). We say that the elements $P_{m,\pm}$ are the Laurent cover associated to $M \times \{+,-\}$, so that we obtain a twisted complex $\check{T}(P,M \times \{+,-\})$ on $\mathcal{F}$ given by Equation (B.3).

Let $(P,P')$ be a pair of polytopes. By definition, the space of morphisms in the category of twisted complexes on $\mathcal{F}$ from $\check{T}(P,M \times \{+,-\})$ to $P'$ is given by

$$\left( \bigoplus_{\sigma \subset M \times \{+,-\}} CF^\ast(P_\sigma,P')[-n],\delta \right)$$

with the differential induced by restriction.

**Lemma B.3.** For each pair of polytopes $(P,P')$, the natural map

$$(B.2) \quad \left( \bigoplus_{\sigma \subset M \times \{+,-\}} CF^\ast(P_\sigma,P')[-n],\delta \right) \rightarrow CF^\ast(P,P')$$

is a quasi-isomorphism.

**Proof.** Filtering by the degree of critical points of the Morse function, it suffices to prove that the augmented complex

$$\left( \bigoplus_{\sigma \subset M \times \{+,-\}} \text{Hom}^c(\Gamma^P_\sigma,\Gamma^P_\sigma') \rightarrow \text{Hom}^c(\Gamma^P,P') \right)$$

is a quasi-isomorphism. As in [19, Lemma 8.4], induction on the number of elements of $M$ reduces this to the case of a singleton, which follows immediately from Lemma B.2. □

**Corollary B.4.** If $M \times \{+,-\}$ indexes a Laurent cover of a polytope $P$, there is a natural quasi-isomorphism

$$(B.4) \quad P \rightarrow \check{T}(P,M \times \{+,-\}).$$

We now prove the main result of this section:

**Proof of Proposition B.1.** We essentially follow the method introduced by Tate in [19, Section 8]: every cover $\Sigma$ admits a refinement by a Laurent cover $M \times \{+,-\}$ obtained by considering the functions defining all boundary facets of polytopes appearing in the cover. The naturality of the construction of the complexes $\check{T}$ implies that we can write the map from $P$ to the Čech twisted complex associated to $M \times \{+,-\}$ as a composition:

$$P \rightarrow \check{T}(P,\Sigma) \rightarrow \check{T}(P,M \times \{+,-\}).$$
where the first arrow is the map which we would like to show is a quasi-isomorphism. By Corollary B.4 it suffices to show that the second map is a quasi-isomorphism. Filtering by the number of elements of a subset \( \sigma \in \Sigma \), this follows by applying Corollary B.4 to (B.6)
\[
P_\sigma \to \hat{T}(P_\sigma, M \times \{+,-\}).
\]

\[\square\]

**Appendix C. Computations of Floer cohomology groups**

As in Appendix B, we consider the local situation, by studying the category of polyotopes in \( H_1(T^n, \mathbb{R}) \). The main goal is to prove Proposition 2.15. Along the way, we shall prove that morphisms between disjoint polytopes vanish.

**Remark C.1.** A version of the results of this section hold for the completions of the homology of the based loops space of any topological space having the homotopy type of a finite CW complex, where the polygons are integral affine subsets of first cohomology. The proof would take us too far afield, so we give a computational and explicit proof in the case of tori.

**C.1. The 1-dimensional case.** Consider the circle equipped with the standard Morse function with a unique minimum and maximum, and let \( U \) denote the local system corresponding to the regular representation of the fundamental group. Identifying the space of paths from the minimum to the maximum with the space of based loops at the maximum via the choice of one segment, and the homology of the latter with the Laurent polynomial ring \( \Gamma = \mathbb{Z}[z, z^{-1}] \), the differential in the Morse complex with coefficients in \( \text{Hom}(U, U) \) can be expressed as the map
\[
\phi \mapsto \phi - z \cdot \phi \cdot z^{-1},
\]
and the kernel of this differential is naturally isomorphic to
\[
\Gamma \equiv \text{Hom}_\mathbb{R}(\Gamma, \Gamma) \subset \text{Hom}_\mathbb{Z}(\Gamma, \Gamma).
\]

Consider the map
\[
\text{Hom}_\mathbb{Z}(\Gamma, \Gamma) \leftarrow \text{Hom}_\mathbb{Z}(\Gamma, \Gamma)
\]
\[
\psi(z^i) + zh(\psi)(z^{i-1}) = h(\psi)(z^i) \leftrightarrow \psi,
\]
which we normalise by setting \( h_\psi(1) = 0 \).

**Lemma C.2.** The map \( h \) defines a homotopy from the identity of \( \text{Hom}(U, U) \) to the projection
\[
\text{Hom}_\mathbb{Z}(\Gamma, \Gamma) \to \text{Hom}_\mathbb{R}(\Gamma, \Gamma) \equiv \Gamma
\]
\[
\phi \mapsto \phi(1)
\]
from the degree 0 summand to \( \Gamma \).

For later purposes, it is convenient to derive this complex, and the corresponding null-homotopy, from a version of the Koszul complex: namely, consider
\[
\Gamma \otimes \Gamma \to \Gamma \otimes \Gamma
\]
with differential
\[
d(f \otimes g) = f \otimes g - z^{-1} \cdot f \otimes g \cdot z.
\]
The complex \( \text{Hom}(U, U) \) is naturally isomorphic to the complex of \( \Gamma \)-module maps from Equation (C.7) to \( \Gamma \), and the null-homotopy \( h \) arises from a null homotopy of this complex.
Remark C.3. Note that the choice of Koszul complex depends on a choice of decomposition of Laurent polynomials into positive and negative powers. In particular, the differential $d_-$ associated to swapping the roles of $z$ and $z^{-1}$ is related to the above differential by the equation
\[(C.9) \quad d_- = -z \cdot d \cdot z^{-1}.\]
The minus sign above accounts for the sign ambiguity we shall encounter later when we consider the dual Floer cohomology group.

C.2. The standard Morse complex on the torus. We consider the torus $T^n \equiv S^1 \times \cdots \times S^1$, whose group ring we denote $\Gamma$. The product decomposition, and an orientation of each factor, induce an isomorphism
\[(C.1) \quad \Gamma \equiv \mathbb{Z}[z^\pm_1, \cdots, z^\pm_n].\]
It is convenient to switch back and forth between this notation, and the notation wherein we write elements of $\Gamma$ as $z^\alpha$ for $\alpha \in \mathbb{Z}^n$.

Consider the standard Morse function, i.e. the sum of the standard Morse functions on each factor, having a unique minimum and maximum, and pick on each factor a path from the minimum to the maximum. Consider the universal local system whose fibre at a point is the space of paths to a basepoint, which we choose to be the maximum. Our choice of paths identifies the Morse complex with coefficients in the endomorphisms of this local system with
\[(C.2) \quad \text{Hom}(\Gamma, \Gamma) \otimes H^*(T^n, \mathbb{Z}),\]
with a differential given by
\[(C.3) \quad \partial(\phi \otimes \alpha) = \sum_i \partial_i(\phi \otimes \alpha) = \sum_i \left(\phi - z_j \cdot \phi \cdot z^{-1}_j\right) \otimes b_j \wedge \alpha,\]
where $\{b_j\}_{j=1}^n$ is the standard basis of $H^1(T^n)$.

There is a natural subcomplex of Equation (C.2) given by the inclusion of
\[(C.4) \quad \Gamma \rightarrow \text{Hom}(\Gamma, \Gamma) \otimes H^*(T^n; \mathbb{Z})\]
whose image lies is $\text{Hom}_\Gamma(\Gamma, \Gamma) \otimes H^0(T^n)$. We shall construct an explicit retraction from the right to the left hand side.

To this end, we define a map
\[(C.5) \quad h_j : \text{Hom}(\Gamma, \Gamma) \rightarrow \text{Hom}(\Gamma, \Gamma)\]
\[(C.6) \quad h_j \psi(z^\alpha) = \begin{cases} 0 & \text{if } \alpha_j = 0 \\ \psi(z^\alpha) + z_j \cdot \psi(z^{\alpha-e_j}) & \text{otherwise} \end{cases},\]
where $\alpha \in \mathbb{Z}^n$, and $e_j$ is the $j^{\text{th}}$ basis element. Note that this is a recursive definition of $h_j \psi$, and that the explicit formula is
\[(C.7) \quad h_j \psi(z^\alpha) = \begin{cases} \sum_{i=0}^{\alpha_j-1} z^{ie_j} \psi(z^{\alpha-ie_j}) & 0 \leq \alpha_j \\ -\sum_{i=\alpha_j}^{\alpha_j-1} z^{ie_j} \psi(z^{\alpha-ie_j}) & \alpha_j < 0. \end{cases}\]

We then define
\[(C.8) \quad h : \text{Hom}(\Gamma, \Gamma) \otimes H^*(T^n) \rightarrow \text{Hom}(\Gamma, \Gamma) \otimes H^*(T^n)\]
\[(C.9) \quad h = \sum_{j=1}^n h_j \otimes \iota_j,\]
where \( \iota_j \) is the slant product
\[
(C.10) \quad H^*(T^n; \mathbb{Z}) \to H^{* - 1}(T^n; \mathbb{Z})
\]
with the basis element of \( e_j \in H_1(T^n; \mathbb{Z}) \).

**Lemma C.4.** The map \( h \) is a homotopy between the identity and the projection
\[
(C.11) \quad \text{Hom}(\Gamma, \Gamma) \otimes H^*(T^n; \mathbb{Z}) \to \Gamma
\]
\[
(C.12) \quad \phi \otimes 1 \mapsto \phi(1).
\]

**Proof.** The complex \( \text{Hom}(\Gamma, \Gamma) \otimes H^*(T^n; \mathbb{Z}) \) is naturally isomorphic to the complex of \( \Gamma \)-homomorphisms from the \( n \)-fold tensor product of Equation (C.7) to \( \Gamma \), and the homotopy to the projection is induced from the corresponding homotopy in Equation (C.3). \( \square \)

**C.3. Construction of the homotopy for inclusions: I.** Our goal is to extract from Lemma C.4 a bounded homotopy for the completions. To this end, we now use \( \Gamma \) to denote the ring of Laurent polynomials over the Novikov field \( \Lambda \).

Let \( P_0 \) and \( P_1 \) be integral affine polytopes in \( H^1(T^n; \mathbb{R}) \). The Morse complex computing morphisms between the corresponding local systems is given by
\[
(C.1) \quad \text{CF}^*(P_0, P_1) \cong \text{Hom}^c(\Gamma^{P_0}, \Gamma^{P_1}) \otimes H^*(T^n; \mathbb{Z}),
\]
with differential given by Equation (C.3). Moreover, we have the inclusion of \( \text{Hom}^c(\Gamma^{P_0}, \Gamma^{P_1}) \) in degree 0.

**Lemma C.5.** If \( P_1 \subset P_0 \), the homotopy \( h \) is continuous, hence induces a retraction
\[
(C.2) \quad \text{Hom}^c(\Gamma^{P_0}, \Gamma^{P_1}) \otimes H^*(T^n; \mathbb{Z}) \to \text{Hom}^c(\Gamma^{P_0}, \Gamma^{P_1}) = \Gamma^{P_1}
\]

**Proof.** It suffices to prove that each map \( h_j \) is continuous, in which case is suffices to bound
\[
(C.3) \quad \text{val}(h_j \psi) - \text{val} \psi
\]
for any (continuous) map \( \psi \) from \( \Gamma^{P_0} \) to \( \Gamma^{P_1} \). A straightforward computation reduces this to proving that
\[
(C.4) \quad \text{val}_{P_0} z_j \leq \text{val}_{P_1} z_j
\]
which follows immediately from the inclusion \( P_1 \subset P_0 \). \( \square \)

From this, we conclude the first part of Proposition 2.15 exhibiting the isomorphism between \( HF^*(P_0, P_1) \) and \( \Gamma^{P_1} \) if \( P_1 \subset P_0 \).

**C.4. Construction of the homotopy for disjoint sets.** Let us now consider the case where \( P_0 \) and \( P_1 \) are disjoint integral affine polytopes. By a change of coordinates, we may assume that the first lies in the region where the first coordinate is strictly positive, and the second in the region where it is strictly negative. We then define
\[
(C.1) \quad h: \text{Hom}^c(\Gamma^{P_0}, \Gamma^{P_1}) \otimes H^*(T^n) \to \text{Hom}^c(\Gamma^{P_0}, \Gamma^{P_1}) \otimes H^*(T^n)
\]
\[
(C.2) \quad \psi \otimes v \mapsto \sum_{i=1}^{\infty} z_i^1 \cdot \psi \cdot z_i^{-1} \otimes \iota_1 v.
\]
We note that the infinite series on the right hand side is convergent because
\[
(C.3) \quad \text{val}(z_i^1 \cdot \psi \cdot z_i^{-1}) = \text{val}(\psi) + i (\text{val}_{P_0}(z) + \text{val}_{P_1}(z^{-1}))
\]
and the assumptions on \( P \) and \( P_1 \) respectively imply that
\[
(C.4) \quad 0 < \text{val}_{P_0} z \text{ and } 0 < \text{val}_{P_1} z^{-1}.
\]
Lemma C.6. The map $h$ defines a null-homotopy of $\text{Hom}^c(\Gamma P_0, \Gamma P_1) \otimes H^*(T^n)$.

Proof. Let us write $H^*(T^n)$ as the direct sum $\bigoplus_{i=0}^{n} H^i(T^n)$. The compositions $h \circ \partial_1: \text{Hom}^c(\Gamma P_0, \Gamma P_1) \otimes H^0(T^n) \rightarrow \text{Hom}^c(\Gamma P_0, \Gamma P_1)$ and $\partial_1 \circ h: \text{Hom}^c(\Gamma P_0, \Gamma P_1) \otimes H^1(T^n) \rightarrow \text{Hom}^c(\Gamma P_0, \Gamma P_1)$ both agree with the identity by an explicit computation. On the other hand, $h$ commutes with each differential $\partial_i$ for $i \neq 1$. The result follows.

Corollary C.7. If $P_0 \cap P_1 = \emptyset$, the cohomology group $HF^*(P_0, P_1)$ vanishes.

C.5. Computation of morphisms for inclusions: II. Consider the 1-dimensional case, with intervals $P_1 \subset P_0$. Recall that

(C.1) $\text{val}_{P_0} z < \text{val}_{P_1} z$ and $\text{val}_{P_0} z^{-1} < \text{val}_{P_1} z^{-1},$

thus, there is a constant $c$ such that

(C.2) $\text{val}_{P_1} z^i - \text{val}_{P_0} z^i > c|\i|.$

Lemma C.8. The natural inclusions

(C.3) $\text{Hom}^c(\Gamma P_0, \Gamma P_1) \leftarrow \text{Hom}^c(\Gamma P_1, \Gamma P_0) \rightarrow \text{Hom}^c(\Gamma P_0, \Gamma P_1)$

factor through the inclusions

(C.4) $\Gamma P_1 \otimes \text{Hom}^c(\Gamma P_1, \Lambda) \subset \text{Hom}^c(\Gamma P_0, \Gamma P_1).$

Proof. We can formally write any element $\phi \in \text{Hom}^c(\Gamma P_1, \Gamma P_0)$ as

(C.5) $\sum_{i,j} \phi_i(z^j) z^i \otimes \rho_j,$

where $\phi_i(z^j)$ is the coefficient of $z^j$ in $\phi(z^j)$, and $\rho_j$ is the homomorphism which assigns 1 to $z^j$ and 0 to every other monomial basis element of $\Gamma$. The assumption that $\phi$ is bounded implies that there exists a constant $K$ such that

(C.6) $\min_{i,j} (\text{val}_{P_0} (z^j) + \text{val}_{P_0} z^i - \text{val}_{P_1} z^j) \geq K.$

The result now follows from Equation (C.2), since replacing $\text{val}_{P_0} z^i$ by $\text{val}_{P_1} z^i$ allows us to add $c|\i|$ to the right hand side, so that, when considered as an element of $\text{Hom}^c(\Gamma P_1, \Gamma P_0)$, there are only finitely many terms with valuation bounded above by any given number. Replacing $\text{val}_{P_1} z^j$ by $\text{val}_{P_0} z^j$ implies the same for $\text{Hom}^c(\Gamma P_0, \Gamma P_0)$.

We conclude that there is a natural trace

(C.7) $\text{tr}: \text{Hom}^c(\Gamma P_1, \Gamma P_0) \rightarrow \Lambda,$

given by the composition of the inclusions into $\Gamma P_1 \otimes \text{Hom}^c(\Gamma P_1, \Lambda)$ with the evaluation map

(C.8) $\Gamma P_1 \otimes \text{Hom}^c(\Gamma P_1, \Lambda) \rightarrow \Lambda$

(C.9) $f \otimes \rho \mapsto \rho(f).$

Explicitly, the trace can be written as

(C.10) $\psi \mapsto \sum_{i=-\infty}^{+\infty} \psi(z^i)_i,$
where the subscript records the coefficient of $z^i$. This in particular shows that the traces defined via endomorphisms of $\Gamma^{P_0}$ and $\Gamma^{P_1}$ agree.

We now consider the map
\begin{align}
(C.11) & \quad \epsilon : \Hom^c(\Gamma^{P_1}, \Gamma^{P_0}) \to \Hom^c(\Gamma^{P_1}, \Lambda) \\
(C.12) & \quad \epsilon \psi(f) = \tr(\psi \circ f).
\end{align}

It is easy to see that $\epsilon$ composes trivially with the differential on $CF^*(P_1, P_0)$ hence defines a chain map to $\Hom^c(\Gamma^{P_1}, \Lambda)$. Consider the map
\begin{align}
(C.13) & \quad \Hom^c(\Gamma^{P_1}, \Gamma^{P_0}) \leftarrow \Hom^c(\Gamma^{P_1}, \Lambda) : \delta \\
(C.14) & \quad \rho(f) = \delta(\rho)(f).
\end{align}

**Lemma C.9.** The composition $\epsilon \circ \delta$ is the identity on $\Hom^c(\Gamma^{P_1}, \Gamma)$.

**Proof.** Consider the map $\delta(\rho) \cdot z^i$. We compute that
\begin{equation}
(C.15) \quad \delta(\rho)(z^i \cdot z^j) = \rho(z^{i+j}).
\end{equation}
Thus the trace of this map is given by setting $j = 0$, and is equal to $\rho(z^i)$. \qed

We now define a map $h$ which will serve as a homotopy between the identity and the composition $\delta \circ \epsilon$, and which is determined by the expression
\begin{align}
(C.16) & \quad \Hom^c(\Gamma^{P_1}, \Gamma^{P_0}) \leftarrow \Hom^c(\Gamma^{P_1}, \Gamma^{P_0}) \\
(C.17) & \quad \psi + h(z^{-1} \circ \psi \circ z) = h(\psi) \leftarrow \psi,
\end{align}
which we normalise by requiring that $h(\psi)$ vanish if the image of $\psi$ is contained in the image of $\delta$. We obtain an explicit expression for this map as follows: formally write
\begin{equation}
(C.18) \quad \psi = \sum_{i \in \mathbb{Z}} \psi_i
\end{equation}
with $\psi_i$ having image in the line spanned by $z^i$. We have
\begin{equation}
(C.19) \quad h\psi_i = \begin{cases} 
\sum_{j=1}^{i-1} z^{-j} \circ \psi_i \circ z^j & 1 \leq i \\
0 & i = 0 \\
\sum_{j=1}^{-1} z^{-j} \circ \psi_i \circ z^j & i \leq -1,
\end{cases}
\end{equation}
and we (formally) define
\begin{align}
(C.20) & \quad h\psi = \sum_{i \in \mathbb{Z}} h\psi_i \\
(C.21) & \quad = \sum_{j \leq -1} z^{-j} \circ \psi_{\leq j} \circ z^j + \sum_{0 \leq j} z^{-j} \circ \psi_{j+1 \leq} \circ z^j,
\end{align}
where $\psi_{\leq j}$ is the sum of all components $\psi_i$ with $i \leq j$, and $\psi_{j+1 \leq}$ is the sum of all components with $j + 1 \leq i$.

**Lemma C.10.** The expression in Equation (C.21) is convergent, and the map $h$ is continuous.
Proof. We compute that, for $j$ strictly negative, the valuation of $z^{-j} \cdot \psi_{\leq j} \cdot z^j$ is given by the infimum over all Laurent polynomials $f$ of
\begin{align*}
\text{(C.22)} & \quad \text{val}_{P_0} z^{-j} \cdot \psi_{\leq j} \cdot z^j f - \text{val}_{P_1} f = j \text{val}_{P_0} z^{-1} + \text{val}_{P_0} \psi_{\leq j} \cdot z^j f - \text{val}_{P_1} z^j f \\
\text{(C.23)} & \quad + \text{val}_{P_1} z^j f - \text{val}_{P_1} f \\
\text{(C.24)} & \quad \geq j \text{val}_{P_0} z^{-1} + \text{val}_{\psi_{\leq j}} + \text{val}_{P_1} z^j \\
\text{(C.25)} & \quad \geq -j \left(\text{val}_{P_1} z^{-1} - \text{val}_{P_0} z^{-1}\right) + \text{val}_{\psi}.
\end{align*}

The desired bound in this case thus follows from Equation \text{(C.22)}. The case of positive monomials is similar, and the result immediately follows. \hfill \square

Lemma C.11. The map $h$ defines a homotopy between the identity on $CF^*(P_1, P_0)$, and the projection $\delta \circ \epsilon$.

Proof. The fact that $h \circ d$ is the identity follows trivially from Equation \text{(C.17)}. The identity $d \circ h$ follows from computation using the formal decomposition used above. In particular, for $\phi$ with image in the line spanned by $z^i$ with $i$ positive, we have
\begin{align*}
\text{(C.26)} & \quad d h \phi = d (\phi + z^{-1} \phi z + \cdots + z^{-i+1} \phi z^{i-1}) \\
\text{(C.27)} & \quad = \phi - z^{-1} \phi z + z^{-1} \phi z - z^{-2} \phi z^2 + \cdots - z^{-i} \phi z^i \\
\text{(C.28)} & \quad = \phi - z^{-i} \phi z^i.
\end{align*}

The result thus follows from the equality $\delta \circ \epsilon (\phi) = z^{-i} \phi z^i$. \hfill \square

We now consider the higher dimensional situation: let $P_1 \subset P_0$ be nested integral affine polytopes in $\mathbb{R}^n$. As before, we have a map
\begin{align*}
\text{(C.29)} & \quad \epsilon : \text{Hom}^c(\Gamma^{P_1}, \Gamma^{P_0}) \to \text{Hom}^c(\Gamma^{P_1}, \Lambda) \\
\text{(C.30)} & \quad \epsilon \psi(f) = \text{tr} (\psi \circ f),
\end{align*}
with one-sided inverse given by the inclusion
\begin{align*}
\text{(C.31)} & \quad \delta : \text{Hom}^c(\Gamma^{P_1}, \Lambda) \to \text{Hom}^c(\Gamma^{P_1}, \Gamma^{P_0})
\end{align*}
whose image consists of maps factoring through $\Lambda \cdot 1 \subset \Gamma^{P_0}$.

Consider the maps $h_j$ given by
\begin{align*}
\text{(C.32)} & \quad \text{Hom}_Z(\Gamma^{P_1}, \Gamma^{P_0}) \leftarrow \text{Hom}_Z(\Gamma^{P_1}, \Gamma^{P_0}) \\
\text{(C.33)} & \quad \psi + h_j (z_j^{-1} \circ \psi \circ z_j) = h_j (\psi) \leftarrow \psi,
\end{align*}
which we normalise by requiring that $h_j (\psi)$ vanish if the image of $\psi$ is contained in the space spanned by monomials with trivial power of $z_j$ (i.e. convergent Laurent series in the variables $z_i$ for $i \neq j$). As in Section \text{[C.4]} we set
\begin{align*}
\text{(C.34)} & \quad h = \sum_{j=1}^{n} h_j \otimes t_j.
\end{align*}

This map provides a homotopy between the identity on $CF^*(P_1, P_0)$ and the projection to $\text{Hom}^c(\Gamma^{P_1}, \Lambda)$ given by the composition
\begin{align*}
\text{(C.35)} & \quad CF^*(P_1, P_0) \xrightarrow{\epsilon} \text{Hom}^c(\Gamma^{P_1}, \Lambda) \xrightarrow{\delta} CF^*(P_1, P_0).
\end{align*}
The proof of Proposition \text{[2.15]} is now complete.
Remark C.12. The constructions of this section are formally dual to those of Section C.4, in the sense that the formulae we use can be derived from those of that section by dualising with respect to the pairing

$$\Gamma \otimes \Gamma \to \Lambda$$

$$f \otimes g \mapsto \text{Res}(f gdz/z)$$

where the symbol Res(hdz) assigns 1 to the monomial $h = z^{-1}$, and 0 to every non-trivial monomial. One can thus link the discussion of this section with the theory of residues via Tate’s approach [18].

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