The Dynamics of a Food Web System: Role of a Prey Refuge Depending on Both Species

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Abstract
This paper aims to study the role of a prey refuge that depends on both prey and predator species on the dynamics of a food web model. It is assumed that the food transfer among the web levels occurs according to Lotka-Volterra functional response. The solution properties, such as existence, uniqueness, and uniform boundedness, are discussed. The local, as well as the global, stabilities of the solution of the system are investigated. The persistence of the system is studied with the assistance of average Lyapunov function. The local bifurcation conditions that may occur near the equilibrium points are established. Finally, numerical simulation is used to confirm our obtained results. It is observed that the system has only one type of attractors that is a stable point, while periodic dynamics do not exist even on the boundary planes.

Keywords: Food web, prey refuge, Stability, bifurcation, persistence.

1. Introduction

Food webs are established in ecosystems and are significant for all living organisms. Energy needs for all activities of animals are provided through food consumption. Furthermore, food webs represent the pathways by which energy flows from a level to another. They are fundamental and inescapable in any attempt to describe how the real world life in the environment is organized or complexes of species interact. The three types of food web samples are the bases of large measure ecosystems. Therefore, to understand the dynamical behavior of an ecosystem, it is substantial to realize the dynamics of the three kinds of food web models. Hence, attention has been given to the

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dynamics with 3-species food chains and food web systems [1-8]. Later on, many researchers investigated the dynamical behavior of food webs with various types of functional responses and including many biological factors. Agarwal and Kumarg [9] proposed and studied the alternative resource effects for top predators using Holling type III model. They showed that an alternative resource has the ability to prohibit top predator annihilation. Xiao et al. [10] proposed and investigated a predator–prey model that associates time delay with a Holling type II and Allee impact in prey. They observed that if the Allee impact is high or the birth rate is low, then both species of the system are fading and hence the stability of the system is affected.

In fact, the refuge mechanism has a role in promoting the growth rate of prey and reducing both predator growth and the products lurking behavior of prey [11]. Therefore, the existence of prey refuges has substantial effects on the cohabitation of predators and their prey. There are three types of refuges; the first type is the supply of continual spatial protection for a small branch of the prey population, while the second type is the provision of temporary spatial preservation and, finally, the third type is the supply of a temporary refuge in numbers, which means the reduction of the venture of predation by rising the abundance of vulnerable prey [12]. The problem of prey-predator interactions under a prey refuge has been studied by some authors. Mukherjee [11] investigated a resource according to Holling type I and II functional responses and then showed the effect of constant prey refuge. Das et al. [13] described constant prey refuge and harvesting to both predator and prey. Ghosh et al. [14] studied the effects of extra food for predator with prey refuge. Santra et al. [15] investigated the dynamical actions with Crowley–Martin functional response-associated prey refuge. Finally, Molla et al. [16] suggested a model for Holling type-II prey–predator interactions in a prey refuge.

In this paper, a Lotka-Volterra food web model with a prey refuge that consists of prey and predator is proposed and studied. The remaining sections are organized as follows. In section (2), the mathematical model with symbols of system is introduced. Section (3) deals with the analysis of local stability. In section (4), global stability analysis is investigated. Section (5) describes the persistence of the model. In section (6), the local bifurcation of the model is studied. The numerical simulation is given in section (7), while section (8) includes some discussion of the obtained results.

2. The model formulation

In this section, a mathematical modeling approach was used to study the role of a non-constant prey refuge in the dynamical behavior of a three-species food web system. It is assumed that the food web is consisting of prey at a lower level of density at time \( T \), given by \( X(T) \). The prey grows logistically in the absence of the predators and has a non-constant refuge property that is dependent on the density of their predator. The middle predator, denoted for density at time \( T \) by \( Y(T) \), feeds on prey at the lower level and is preyed on by the predator at the higher level. It has another limited source of food in the absence of their preferred food. The top predator, denoted for density at time \( T \) by \( Z(T) \), at the higher level, feeds on both the prey at the lower level and the middle predator at the second level. Accordingly, the dynamics of such a food web system can be described by using mathematical modeling with differential equations.

\[
\begin{align*}
\frac{dx}{dt} &= rX \left(1 - \frac{X}{K_1}\right) - \alpha_1(X - X_{R_1})Y - \alpha_2(X - X_{R_2})Z, \\
\frac{dy}{dt} &= sY \left(1 - \frac{Y}{K_2}\right) + e_1\alpha_1(X - X_{R_1})Y - \alpha_3YZ - d_1Y, \tag{1}
\end{align*}
\]

\[
\frac{dz}{dt} = e_2\alpha_2(X - X_{R_2})Z + e_3\alpha_3YZ - d_2Z,
\]

with \( X_{R_1} = cXY \) and \( X_{R_2} = cXZ \), the model (1) becomes

\[
\begin{align*}
\frac{dx}{dt} &= rX \left(1 - \frac{X}{K_1}\right) - \alpha_4X(1 - c)Y - \alpha_5X(1 - c)Z, \\
\frac{dy}{dt} &= sY \left(1 - \frac{Y}{K_2}\right) + e_1\alpha_1X(1 - c)Y - \alpha_3YZ - d_1Y, \tag{2}
\end{align*}
\]

\[
\frac{dz}{dt} = e_2\alpha_2X(1 - c)Z + e_3\alpha_3YZ - d_2Z,
\]

with \( X(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0 \). The description of the parameters for the system (2) is shown in Table-1.

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Table 1: Description of Parameters

| Parameter | Description |
|-----------|-------------|
| \( x \)  | Prey density |
| \( y \)  | Middle predator density |
| \( z \)  | Top predator density |
| \( r \)  | Growth rate of prey |
| \( s \)  | Growth rate of middle predator |
| \( e_1 \) | Effect of top predator |
| \( d_1 \) | Death rate of middle predator |
| \( e_2 \) | Effect of top predator |
| \( d_2 \) | Death rate of top predator |
| \( \alpha_1 \) | Functional response |
| \( \alpha_2 \) | Functional response |
| \( \alpha_3 \) | Functional response |
| \( \alpha_4 \) | Functional response |
| \( \alpha_5 \) | Functional response |
| \( c \)  | Constant value |

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Table 1-Description of parameters in system (2)

| Parameter | Description |
|-----------|-------------|
| \( r, s \) | Intrinsic growth rates for the prey and middle predator, respectively |
| \( K_1, K_2 \) | Carrying capacity of the environment for the prey and middle predator, respectively |
| \( 1 - cY, 1 - cZ \) | Predator dependent refuge rates, where \( c \) denotes the coefficient of prey refuge in (0, 1] with \( Y \leq \frac{1}{c} \) and \( Z \leq \frac{1}{c} \), respectively. |
| \( \alpha_1, \alpha_2, \alpha_3 \) | attack rates of middle and top predator, respectively |
| \( e_1, e_2, e_3 \) | Conversion rates |
| \( d_1, d_2 \) | Natural death rates of middle and top predators, respectively |

It is assumed that all the above parameters are positive constants and hence the domain of system (2) will be given by \( \mathbb{R}_+^3 = \{ (X, Y, Z) \in \mathbb{R}^3 : X \geq 0, Y \geq 0, Z \geq 0 \} \). Now to simplify the model analysis, the following parameters and dimensionless variables are used in system (2).

\[
t = rT, x = \frac{x}{K_1}, y = cY, z = cZ,
\]

\[
w_1 = \frac{a_1}{r}, w_2 = \frac{a_1}{r}, w_3 = \frac{g}{r}, w_4 = cK_2, w_5 = \frac{\sigma_1 K_1}{r},
\]

\[
w_6 = \frac{a_2}{rc}, w_7 = \frac{d_1}{r}, w_8 = \frac{\sigma_2 K_1}{r}, w_9 = \frac{e_2 K_1}{r}, w_{10} = \frac{d_1}{r}.
\]

According to Eq. (3), the dimensionless system corresponding to system (2) will be

\[
\frac{dx}{dt} = x(1-x) - w_1x(1-y)y - w_2x(1-z)z = xf_1,
\]

\[
\frac{dy}{dt} = w_3y\left(1 - \frac{y}{w_4}\right) + w_5x(1-y)y - w_6yz - w_7y = yf_2,
\]

\[
\frac{dz}{dt} = w_8x(1-z)z + w_9yz - w_{10}z = zf_3,
\]

where \( x(0) \geq 0, y(0) \geq 0, z(0) \geq 0 \). The functions on the right hand side of system (4) are continuous and have continuous partial derivatives. Hence, they are Lipschitzian functions [17]. Therefore, the solution of system (1) exists and is unique. In addition, the following theorem shows the boundedness of the solution in \( \mathbb{R}_+^3 \).

**Theorem 1.** The region \( L_1 = \{ (x, y, z) \in \mathbb{R}_+^3 : x \leq 1; 0 < x(t) + y(t) + z(t) \leq \frac{1}{\mu_1} \left( 2 + \frac{w_3 w_4}{4} \right) \} \) is a uniformly bounded region that contains all the solutions of system (4).

**Proof.** From system (4), we have that

\[
\frac{dx}{dt} \leq x(1-x).
\]

By solving the inequality, it is simple to verify that \( x(t) \leq 1 \) as \( t \to \infty \). Let \( W(t) = x(t) + y(t) + z(t) \), then after some algebraic manipulation we have

\[
\frac{dW}{dt} \leq 2x + \frac{w_3 w_4}{4} - x - w_7y - w_{10}z.
\]

Then \( \frac{dW}{dt} + \mu_1 W \leq 2 + \frac{w_3 w_4}{4} \), where \( \mu_1 = \min\{1, w_7, w_{10}\} \). By using Gronwall inequality [18], it is easy to obtain that \( 0 < W(t) \leq \frac{1}{\mu_1} \left( 2 + \frac{w_3 w_4}{4} \right) \) as \( t \to \infty \). Hence, all solutions are uniformly bounded in \( L_1 \).

3. Local Stability Analysis

There are seven equilibrium points of system (4). Their existence conditions and local stability analyses are established.

- The vanishing equilibrium point that is denoted by \( E_0 = (0, 0, 0) \).
- The first axial equilibrium point (FAEP) that is denoted by \( E_1 = (\tilde{x}, 0, 0) = (1, 0, 0) \) and always exists.
- The second axial equilibrium point (SAEP) that is denoted by \( E_2 = (0, \tilde{y}, 0) \), where
\[ \begin{align*}
\dot{y} &= \frac{w_4(w_3-w_2)}{w_3}. \\
\text{Clearly, the SAEP exists uniquely on the } y - \text{axis if and only if it satisfies the following condition:} \\
w_3 &> w_7. \\
\text{• The prey-free equilibrium point (PFEP) that is denoted by } E_3 = (0, \bar{y}, z), \text{ where} \\
\bar{y} &= \frac{w_1}{w_2}, \quad \bar{z} = \frac{w_3w_4w_9-w_3w_{10}}{w_3w_4w_9}. \\
\text{• The middle predator-free equilibrium point (MPFEP) that is denoted by } E_4 = (\bar{x}, 0, z), \text{ where} \\
\bar{x} &= 1 - w_2\bar{z} + w_2\bar{z}^2,
\end{align*} \]

Obviously, the PFEP exists uniquely in the } yz - \text{plane if and only if the following condition holds:} \\
w_3w_4w_9 &> w_3w_{10} + w_4w_7w_9. \\
\text{• The top predator-free equilibrium point (TPFEP) that is denoted by } E_5 = (\bar{x}, \bar{y}, 0), \text{ where} \\
\bar{x} &= 1 - w_1\bar{y} + w_1\bar{y}^2, \\
\text{with } \bar{y} > 0 \text{ is the root of the equation} \\
\alpha_1\bar{y}^3 + \alpha_2\bar{y}^2 + \alpha_3\bar{y} + \alpha_4 &= 0,
\end{align*} \]

Again, by utilizing the Descartes rule of signs, Eq. (8b) has either one or three positive roots but there is no negative roots, provided the following condition holds:

\[ \begin{align*}
w_9 &> w_{10}. \\
\text{• The positive equilibrium point (PEP) that is denoted by } E_6 = (x^*, y^*, z^*), \text{ where} \\
y^* &= \frac{1}{w_9}(w_{10} - w_8x^* + w_8x^*z^*),
\end{align*} \]

where \((x^*, z^*)\) is a positive intersection point for the following two isoclines:

\[ \begin{align*}
g_1(x, z) &= 1 - x - \frac{w_1w_{10}}{w_9} + \frac{w_1w_8}{w_9}x - \frac{w_1w_8}{w_9}xz + w_1\left(\frac{w_8}{w_9}\right)^2 - 2\frac{w_1w_8w_{10}}{w_9}\frac{w_2}{w_4}x \\
&+ 2\frac{w_1w_8w_{10}}{w_9}xz + w_1\left(\frac{w_8}{w_9}\right)^2(1 - z)^2 - w_2z + w_2z^2 = 0, \\
g_2(x, z) &= w_3 - \frac{w_3w_1w_{10}}{w_4w_9} + \frac{w_3w_8}{w_4w_9}x - \frac{w_3w_8}{w_4w_9}xz + w_5x - \frac{w_3w_8w_{10}}{w_9}x + \frac{w_3w_8w_9}{w_9}x^2 \\
&- \frac{w_5w_9}{w_9}x^2z - w_6z - w_7 = 0.
\end{align*} \]

Then, as \(z \to 0\), we obtain that:

\[ \begin{align*}
g_1(x) &= w_1\left(\frac{w_8}{w_9}\right)^2x^2 - \left[1 - \frac{w_1w_8}{w_9} + 2\frac{w_1w_8w_{10}}{w_9}\frac{w_2}{w_4}\right]x + 1 - \frac{w_1w_8w_{10}}{w_9} + w_1\left(\frac{w_{10}}{w_8}\right)^2 = 0, \\
g_2(x) &= \frac{w_3w_8}{w_9}x^2 + \left[w_5 + \frac{w_3w_8w_{10}}{w_4w_9} - \frac{w_5w_{10}}{w_4w_9}\right]x + w_5 - \frac{w_3w_8w_{10}}{w_4w_9} + w_7 = 0.
\end{align*} \]

Clearly, each of Eq. (9d) and Eq. (9e) has a unique positive root on the \( x - \text{axis}, \) denoted by \( x_1 \) and \( x_2, \) respectively, provided that the next conditions hold:

\[ \begin{align*}
\text{Eq. (9d) & holds if and only if} \\
\frac{w_3w_8}{w_9}x_1^2 + \left[w_5 + \frac{w_3w_8w_{10}}{w_4w_9} - \frac{w_5w_{10}}{w_4w_9}\right]x_1 + w_5 - \frac{w_3w_8w_{10}}{w_4w_9} + w_7 &> 0, \\
\text{Eq. (9e) & holds if and only if} \\
\frac{w_3w_8}{w_9}x_2^2 + \left[w_5 + \frac{w_3w_8w_{10}}{w_4w_9} - \frac{w_5w_{10}}{w_4w_9}\right]x_2 + w_5 - \frac{w_3w_8w_{10}}{w_4w_9} + w_7 &> 0.
\end{align*} \]
Therefore, the two isoclines given by equations (9b) and (9c) have a unique positive intersection point $(x^*, z^*)$ in the interior of $xz$-plane and then the PEP exists uniquely, provided that

$$
x_1 < x_2, \quad \frac{dx}{dz} = -\left(\frac{\partial g_1(x,z)}{\partial x} + \frac{\partial g_1(x,z)}{\partial z}\right) > 0, \quad (10a)$$

$$
\frac{dx}{dz} = -\left(\frac{\partial g_2(x,z)}{\partial x} + \frac{\partial g_2(x,z)}{\partial z}\right) < 0, \quad (10b)$$

$$
w_{10} + w_9 x^* z^* > w_9 x^*. \quad (10d)$$

Now, to establish the local stability, the Jacobian matrix, that is denoted by $J(x, y, z)$ of the system (4) at the $(x, y, z)$, is determined by

$$
J(x, y, z) = \begin{bmatrix}
-x + f_1 & -w_1 x + 2 w_1 x y & -w_2 x + 2 w_2 x z \\
w_5 y - w_5 y^2 & -\frac{w_1}{w_4} y - w_5 x y + f_2 & -w_6 y \\
w_6 z - w_6 z^2 & w_9 z & -w_6 x z + f_3
\end{bmatrix}. \quad (11)
$$

It is clear that system (4) has the Jacobian at $E_0$, specified by

$$
J(E_0) = \begin{bmatrix}
1 & 0 & 0 \\
0 & w_3 - w_7 & 0 \\
0 & 0 & -w_{10}
\end{bmatrix}. \quad (12a)
$$

The eigenvalues of $J(E_0)$ are $\lambda_{0x} = 1 > 0, \lambda_{0y} = w_3 - w_7, \lambda_{0z} = -w_{10}. \quad (12b)$

Therefore, $E_0$ is a saddle point.

The Jacobian matrix at FAEP is

$$
J(E_1) = \begin{bmatrix}
-1 & -w_1 & -w_2 \\
0 & w_3 + w_5 - w_7 & 0 \\
0 & 0 & w_9 - w_{10}
\end{bmatrix}. \quad (13a)
$$

Therefore, the eigenvalues of $J(E_1)$ are given by $\lambda_{1x} = -1 < 0, \lambda_{1y} = w_3 + w_5 - w_7, \lambda_{1z} = w_9 - w_{10}. \quad (13b)$

Clearly, all the above eigenvalues are negative and $E_1$ is locally asymptotically stable in $\mathbb{R}_+^3$ if the next conditions are satisfied

$$
w_3 + w_5 < w_7, \quad w_9 < w_{10}. \quad (13c)$$

The Jacobian matrix at SAEP is

$$
J(E_2) = \begin{bmatrix}
1 - w_1 \hat{y} + w_1 \hat{y}^2 & 0 & 0 \\
-w_5 \hat{y} - w_5 \hat{y}^2 & -\frac{w_2}{w_4} \hat{y} & -w_6 \hat{y} \\
0 & 0 & w_5 \hat{y} - w_{10}
\end{bmatrix}. \quad (14a)
$$

Therefore, the eigenvalues can be written as $\lambda_{2x} = 1 - w_1 \hat{y} + w_1 \hat{y}^2, \lambda_{2y} = -\frac{w_2}{w_4} \hat{y}, \lambda_{2z} = w_5 \hat{y} - w_{10}. \quad (14b)$

The eigenvalues of $J(E_2)$ will be negative and then $E_2$ is locally asymptotically stable if the next conditions are held

$$1 + w_1 \hat{y}^2 < w_1 \hat{y}, \quad w_5 \hat{y} < w_{10}. \quad (14c)$$

The Jacobian matrix at PFEP is

$$
J(E_3) = \begin{bmatrix}
1 - w_1 \hat{y} + w_1 \hat{y}^2 - w_2 (1 - \hat{z}) \hat{z} & 0 & 0 \\
w_5 \hat{y} - w_5 \hat{y}^2 & -\frac{w_3}{w_4} \hat{y} & -w_6 \hat{y} \\
w_9 \hat{z} - w_9 \hat{z}^2 & w_9 \hat{z} & 0
\end{bmatrix} = (C_{ij}). \quad (15a)
$$

The characteristic equation of (15a) is as follows

$$
(C_{11} - \lambda) \left( \lambda^2 + \frac{w_3}{w_4} \lambda + \frac{w_6}{w_9} \hat{y} \hat{z} \right) = 0. \quad (15b)
$$

Therefore, the eigenvalues of $J(E_3)$ are $\lambda_{3x} = -w_1 \hat{y} + w_1 \hat{y}^2 - w_2 \hat{z} + w_2 \hat{z}^2 + 1. \quad (15c)$
\[ \lambda_{3y} = -\frac{w_3}{2w_4} \bar{y}^2 + \frac{\left(\frac{w_3}{w_4}\right)^2 - 4w_4w_5y^2}{2}, \]  
(15d)

\[ \lambda_{3x} = -\frac{w_3}{2w_4} \bar{y}^2 - \frac{\left(\frac{w_3}{w_4}\right)^2 - 4w_4w_5y^2}{2}. \]  
(15e)

Hence, if the next condition holds, \( E_3 \) will be locally asymptotically stable.

\[ 1 < w_3 \bar{y} - w_4 \bar{y}^2 + w_5 \bar{z} - w_6 \bar{z}^2. \]  
(15f)

The Jacobian matrix at MPFEP is presented as follows

\[
J(E_4) = \begin{bmatrix}
\bar{x} & -w_1 \bar{x} & -w_2 \bar{x} + 2w_2 \bar{z} \\
0 & w_3 + w_5 \bar{x} - w_6 \bar{z} - w_7 & 0 \\
w_5 \bar{z} - w_6 \bar{z}^2 & w_9 \bar{z} & -w_6 \bar{z} \bar{z}
\end{bmatrix} = (M_{ij}).
\]  
(16a)

The characteristic of (16a) is shown as

\[
(M_{22} - \lambda)(\lambda^2 - T_4 \lambda + D_4) = 0,
\]  
(16b)

where \( T_4 = -\bar{x}(1 + w_9 \bar{z}) < 0 \) and \( D_4 = w_9 \bar{x}^2 \bar{z} + w_2 w_8 \bar{x} \bar{z}(1 - 2\bar{z})(1 - \bar{z}) \).

Thus, the eigenvalues of \( J(E_4) \) are given by

\[ \lambda_{4y} = w_3 + w_5 \bar{x} - w_6 \bar{z} - w_7, \]  
(16c)

\[ \lambda_{4x} = \frac{T_4 + \sqrt{T_4^2 - 4D_4}}{2}, \]  
(16d)

\[ \lambda_{4z} = \frac{T_4 - \sqrt{T_4^2 - 4D_4}}{2}. \]  
(16e)

Accordingly, all the above eigenvalues have real parts of less than zero and then \( E_4 \) is locally asymptotically stable if the next conditions are satisfied

\[ w_3 + w_5 \bar{x} < w_6 \bar{z} + w_7, \]  
(16f)

\[ \bar{x} + w_2 + 2w_2 \bar{z}^2 > 3w_2 \bar{z}. \]  
(16g)

The Jacobian matrix at TPSEP is written as follows

\[
J(E_5) = \begin{bmatrix}
\bar{x} & -w_1 \bar{x} & -w_2 \bar{x} \\
w_5 \bar{y} - w_6 \bar{y}^2 & -w_3 \bar{y} - w_5 \bar{x} \bar{y} & -w_6 \bar{y} \\
0 & 0 & w_5 \bar{x} + w_9 \bar{y} - w_{10}
\end{bmatrix} = (Q_{ij}).
\]  
(17a)

Therefore, the characteristic equation of \( J(E_5) \) is given by

\[
(Q_{33} - \lambda)(\lambda^2 - T_5 \lambda + D_5) = 0,
\]  
(17b)

where \( T_5 = -\left[ \bar{x} + \frac{w_3}{w_4} \bar{y} - w_5 \bar{y} \right] < 0 \) and \( D_5 = w_9 \bar{x} \bar{y} + w_5 \bar{x} \bar{y} + w_2 w_8 \bar{x} \bar{y}(1 - 2\bar{y})(1 - \bar{y}). \)

Hence, the eigenvalues of \( J(E_5) \) are

\[ \lambda_{5x} = w_8 \bar{x} + w_9 \bar{y} - w_{10}, \]  
(17c)

\[ \lambda_{5z} = \frac{T_5 + \sqrt{T_5^2 - 4D_5}}{2}, \]  
(17d)

\[ \lambda_{5y} = \frac{T_5 - \sqrt{T_5^2 - 4D_5}}{2}. \]  
(17e)

Consequently, all the above eigenvalues have real parts of less than zero and then \( E_5 \) is locally asymptotically stable if the next conditions are satisfied

\[ w_8 \bar{x} + w_9 \bar{y} < w_{10}, \]  
(17f)

\[ \frac{w_2}{w_4} + w_5 \bar{x} + w_4 w_8 \bar{y}^2 > 3w_2 w_5 \bar{y}. \]  
(17g)

Next, the local stability conditions of the PEP are determined.

**Theorem 2:** Suppose that the PEP of the system (4) exists, then it is a locally asymptotically stable with the following sufficient conditions are satisfied

\[ y' < \frac{1}{2}, \]  
(18a)

\[ z' < \frac{1}{2}, \]  
(18b)

\[ w_9 > w_1 w_9 (1 - 2y')(1 - z'), \]  
(18c)

\[ w_9 w_2 w_5 (1 - y')(1 - 2z') < 2w_8 x' \left(\frac{w_3}{w_4} + w_5 \bar{x}' \right) + w_1 w_6 w_8 (1 - 2y')(1 - z'). \]  
(18d)

**Proof:** The Jacobian matrix at the PEP is as follows
The characteristic equation of (19a) is
\[ \lambda^3 + \delta_4 \lambda^2 + \delta_2 \lambda + \delta_3 = 0, \] (19b)
where
\[ \delta_1 = -(L_{11} + L_{12} + L_{13}), \]
\[ \delta_2 = L_{11}L_{22} - L_{12}L_{21} + L_{13}L_{23} + L_{11}L_{33} - L_{12}L_{32} - L_{13}L_{31}, \]
\[ \delta_3 = -L_{13}(L_{11}L_{22} - L_{12}L_{21}) - L_{12}(L_{11}L_{33} - L_{13}L_{31}) - L_{13}(L_{12}L_{23} - L_{13}L_{32}) + L_{11}L_{23}L_{32} + L_{12}L_{31}L_{32}, \]
and
\[ \Delta = \delta_1 \delta_2 - \delta_3 = -(L_{11} + L_{22})[L_{11}L_{22} - L_{12}L_{21}]
- (L_{11} + L_{33})[L_{11}L_{33} - L_{13}L_{31}]
- (L_{12} + L_{33})[L_{12}L_{33} - L_{13}L_{32}]
- 2L_{11}L_{22}L_{33} + L_{12}L_{23}L_{31} + L_{13}L_{21}L_{22}. \]
By using Routh-Hurwitz criterion [18], all the roots of equation (19b) have real parts of less than zero and then \( E_6 = (x^*, y^*, z^*) \) is locally asymptotically stable if \( \delta_1, \delta_3 \) and \( \Delta \) are positive. It is found by computation that all Routh-Hurwitz constraints are satisfied under the sufficient conditions given by (18a) – (18d).

4. Global stability analysis

In this section, the global dynamics of system (4) is investigated using Lyapunov functions, as demonstrated in the next theorems [18].

**Theorem 3:** Suppose that the FAEP is locally asymptotically stable. Then \( E_1 \) is a globally asymptotically stable with the following sufficient conditions are satisfied
\[ \ddot{x} \leq \min \left( \frac{w_{10}}{w_9}, \frac{w_{20} - w_5}{w_5} \right), \] (20a)
\[ \frac{w_{20} - w_5}{w_5} < \frac{w_{10}w_6 - w_{w_9}}{w_5}. \] (20b)

**Proof:** Define \( \Omega_1 \) as a real valued function that is given by
\[ \Omega_1 = q_1 \left( x - \ddot{x} - \ddot{x} \ln \frac{x}{\ddot{x}} \right) + q_2 y + q_3 z, \]
where the constants \( q_1, q_2, q_3 \) are the greater than zero constants that are to be determine. Obviously, \( \Omega_1 \) is a positive definite function that is defined for all \( x > 0, y \geq 0 \) and \( z \geq 0 \). Then, \( \frac{d\Omega_1}{dt} \) can be written as follows
\[ \frac{d\Omega_1}{dt} = -q_1(x - \ddot{x})^2 - (w_1 q_1 - w_5 q_2)xy - (-w_1 q_1 \ddot{x} - q_2 w_3 + w_7 q_2)y
\[ - (w_5 q_2 - w_1 q_1)xy^2 - \left( w_1 q_1 \ddot{x} + \frac{q_2w_3}{w_5}\right)y^2
\[ - (-w_2 q_1 + w_8 q_3)xz^2 - w_2 q_1 \ddot{x}z^2
\[ -(q_1 w_2 - q_3 w_9)xz - (q_3 w_{10} - q_1 w_2)yz - (q_2 w_6 - q_3 w_9)yz. \]
Now, by choosing the positive constants as \( q_1 = 1, q_2 = \frac{w_4}{w_5}, q_3 = \frac{w_2}{w_9} \), then we get that
\[ \frac{d\Omega_1}{dt} \leq -\left( x - \ddot{x} \right)^2 - w_1 \left( \ddot{x} + \frac{w_7 - w_3}{w_5} \right)y
\[ - w_2 \left( \frac{w_{10}}{w_9} - \ddot{x} \right)z - \left( \frac{w_1 w_6}{w_5} - \frac{w_2 w_9}{w_5} \right)y^2. \]

Then, by using the above sufficient conditions (20a)-(20b), the derivative \( \frac{d\Omega_1}{dt} \) is a negative definite function. Hence, \( E_1 \) is a globally asymptotically stable.

**Theorem 4:** Suppose that the SAEP is locally asymptotically stable. Then \( E_2 \) has a basin of attraction that satisfies the following sufficient conditions
\[ \frac{w_3 y^2}{w_2 w_9} < \frac{w_1 w_5}{w_5} \leq \frac{w_2 w_{10}}{w_9 y}. \] (21a)
\[ \frac{w_2 w_9}{w_5} < \frac{w_3 w_{10}}{w_9 y} \leq \frac{w_3 w_9}{w_9}. \] (21b)

**Proof:** Consider the following real valued function
\[ \Omega_2 = p_1 x + p_2 \left( y - \ddot{y} - \ddot{y} \ln \frac{\ddot{y}}{\ddot{y}} \right) + p_3 z. \]
where the constants $p_1, p_2, p_3$ are greater than zero and must be determined. Obviously, $\Omega_2$ is a positive definite function that is defined for all $x \geq 0, y > 0$ and $z \geq 0$. Then $\frac{d\Omega_2}{dt}$ can be written as follows

$$
\frac{d\Omega_2}{dt} = -(p_2 w_5 \dot{y} - p_1) x - p_1 x^2 - (p_1 w_1 - p_2 w_5 - p_2 w_5 \dot{y}) xy
- (p_2 w_5 - p_1 w_1) \dot{y} x y^2 - (p_1 w_2 - p_3 w_5) x z x z^2
- \frac{w_3 p_2}{w_4} (y - \dot{y})^2 - (p_2 w_6 - p_3 w_5) y z - (p_3 w_1 - p_2 w_6 \dot{y}) z.
$$

Then, by choosing the positive constants as $p_1 = 1, p_2 = \frac{w_1}{w_5}, p_3 = \frac{w_2}{w_6}$, then by substituting these constants, we obtain

$$
\frac{d\Omega_2}{dt} = -(w_1 \dot{y} - 1) x - x^2 + w_1 \dot{y} xy - \frac{w_1 w_2}{w_5 w_6} (y - \dot{y})^2
- \frac{w_1 w_6}{w_5} y z - \frac{w_2 w_10}{w_5} \dot{y} z.
$$

Using the above sufficient conditions (21a)-(21b) along with the local stability conditions, the derivative $\frac{d\Omega_2}{dt}$ is a negative definite function. Hence, $E_2$ has a basin of attraction that satisfies the given sufficient conditions.

**Theorem 5**: Assume that the PFEP of system (4) is locally asymptotically stable. Then $E_3$ has a basin of attraction that satisfies the following sufficient conditions

$$
\frac{w_6}{w_2} < \frac{w_5 w_2}{w_6} \dot{y} + w_6 \dot{z}, \quad (22a)
$$
$$
w_2 \dot{z} z < x, \quad (22b)
$$
$$\frac{w_6 w_2}{w_6} \dot{y} < \left(\frac{w_2 w_8}{w_2} - \frac{w_5 w_9}{w_6}\right)(1 - y). \quad (22c)
$$

**Proof**: Define the following real valued function

$$\Omega_3 = \beta_1 x + \beta_2 y - y \frac{\dot{y} \ln \frac{y}{\dot{y}}}{\dot{y}} + \beta_3 z - \dot{z} \ln \frac{z}{\dot{z}}$$

where the constants $\beta_1, \beta_2, \beta_3$ are the greater than zero constants that are to be determine. Obviously, $\Omega_3$ is a positive definite function that is defined for all $x \geq 0, y > 0$ and $z > 0$. Then $\frac{d\Omega_3}{dt}$ can be written as follows

$$
\frac{d\Omega_3}{dt} = -\left(\beta_2 w_5 \dot{y} + \beta_3 w_6 \dot{z} - \beta_1\right) x - \beta_1 x^2 - (\beta_2 w_5 - \beta_1 w_4) xy\dot{z}
- (\beta_1 w_2 - \beta_3 w_5) x z x z^2
- \frac{w_3 \beta_2}{w_4} (y - \dot{y})^2 - (\beta_1 w_1 - \beta_2 w_5 - w_5 \beta_3 \dot{y}) x y
- (\beta_2 w_6 - \beta_3 w_5) (y - \dot{y})(z - \dot{z}).
$$

Now, by choosing the positive constants as $\beta_1 = \frac{w_8}{w_2}, \beta_2 = \frac{w_9}{w_6}, \beta_3 = 1$, we have

$$
\frac{d\Omega_3}{dt} = -\left(\frac{w_2 w_9}{w_6} \dot{y} + \frac{w_9}{w_2} w_6 \dot{z} - \frac{w_6}{w_2} x - \frac{x}{w_6} \dot{z} x - \frac{w_9 w_2}{w_6} (y - \dot{y})^2
- \left(\frac{w_2 w_9}{w_2} - \frac{w_5 w_9}{w_6}\right)(1 - y) - \frac{w_5 w_9}{w_6} \dot{y} x y\right)
$$

Then, by using the above sufficient conditions (22a)-(22c), the derivative $\frac{d\Omega_3}{dt}$ is a negative definite function. Hence, $E_3$ has a basin of attraction that satisfies the given sufficient conditions.

**Theorem 6**: Suppose that the MPFEP is locally asymptotically stable. Then $E_4$ is a globally asymptotically stable, provided that

$$
\left[\frac{w_2 w_9}{w_1} \left(1 - z\right) - \frac{w_2 w_9}{w_1} \dot{z}\right]^2 < 4 \left(\frac{w_2}{w_1}\right) \left(\frac{w_9}{w_6} \frac{z}{\dot{z}}\right).
$$

**Proof**: Define the following real valued function

$$\Omega_4 = \beta_1 \left(x - \dot{x} - x \frac{\ln \frac{x}{\dot{x}}}{\dot{x}}\right) + \beta_2 z + \beta_3 \left(z - \dot{z} \ln \frac{z}{\dot{z}}\right),$$

where the constants $\beta_1, \beta_2, \beta_3$ are greater than zero and must be determined. Obviously, $\Omega_4$ is a positive definite function that is defined for all $x > 0, y \geq 0$ and $z > 0$. Therefore, $\frac{d\Omega_4}{dt}$ can be written as follows
\[
\frac{dn_4}{dt} = -b_1(x - \bar{x})^2 - \left[ b_1 w_2 (1 - (z + \bar{z}) - b_3 w_5 (1 - z)) \right] (x - \bar{x}) (z - \bar{z}) \\
- b_3 w_5 \bar{x} (z - \bar{z})^2 - \left[ b_1 w_1 - b_2 w_5 \right] xy \\
- b_2 \left[ w_7 - w_3 \right] + b_3 w_5 \bar{x} - b_1 w_1 \bar{x} y - \left[ b_2 w_5 - b_1 w_1 \right] xy^2 \\
- \left[ b_1 w_1 \bar{x} + b_2 \frac{w_5}{w_4} \right] y^2 - \left[ b_2 w_5 - b_3 w_5 \right] y z.
\]

Now, by choosing the positive constants as \( b_1 = \frac{w_5}{w_4}, b_2 = 1, b_3 = \frac{w_5}{w_4} \), and substituting these constants, we obtain
\[
\frac{dn_4}{dt} < - \left[ \frac{w_5}{w_4} (x - \bar{x})^2 - \left[ \frac{w_5}{w_4} \right] \left[ w_7 - w_3 \bar{x} \right] (x - \bar{x}) (z - \bar{z}) \\
- \frac{w_5}{w_4} \bar{x} (z - \bar{z})^2 - \left[ (w_7 - w_3) + w_5 \bar{x} - w_5 \bar{x} \right] y z.
\]

Accordingly, by using the condition (23) along with the local stability condition (16f), the derivative \( \frac{dn_4}{dt} \) becomes
\[
\frac{dn_4}{dt} < - \left[ \frac{w_5}{w_4} (x - \bar{x}) + \frac{w_5 w_7}{w_4} \bar{x} (z - \bar{z}) \right]^2 - \left[ (w_7 - w_3) + w_5 \bar{x} - w_5 \bar{x} \right] y z.
\]

Clearly, \( \frac{dn_4}{dt} \) is negative definite and hence \( E_4 \) is a globally asymptotically stable.

**Theorem 7:** Assume that the TPFEP of system (4) is locally asymptotically stable. Then \( E_5 \) is globally asymptotically stable, provided that
\[
\left[ \frac{w_1 w_8 - w_1 w_7}{w_7} \left( y + \bar{y} \right) \right] \left[ \frac{w_1 w_6 w_8}{w_6} \left( 1 - \bar{y} \right) \right] < 4 \left( \frac{w_5}{w_4} \left[ \frac{w_3 w_6 + w_5 x}{w_6} \right] \right).
\] (24)

**Proof:** Define the following real valued function
\[
\Omega_5 = \frac{1}{2} \left[ x - \bar{x} - \bar{x} \ln \frac{x}{\bar{x}} \right] + \frac{1}{2} \left[ y - \bar{y} - \bar{y} \ln \frac{y}{\bar{y}} \right] + \frac{1}{2} \left[ \bar{x} z \right]
\]
where the constants \( \frac{w_5}{w_4}, \frac{w_5}{w_4} \) are greater than zero and must be determined. Obviously, \( \Omega_5 \) is a positive definite function, which is defined for all \( x > 0, y > 0 \) and \( z \geq 0 \). Therefore, \( \frac{dn_5}{dt} \) can be written as follows
\[
\frac{dn_5}{dt} = -\frac{w_5}{w_4} (x - \bar{x})^2 - \left[ \frac{w_1 w_7}{w_7} \left( y + \bar{y} \right) \right] (x - \bar{x}) (y - \bar{y}) \\
- \frac{w_5}{w_4} \left[ w_5 w_3 \bar{x} \right] (y - \bar{y})^2 - \left[ \frac{w_5}{w_4} \right] \left[ w_5 w_3 \bar{x} \right] x z \\
- \left[ \frac{w_5}{w_4} \right] \left[ w_5 \bar{x} - w_5 \bar{x} \right] z - \left[ \frac{w_5}{w_4} \right] \left[ w_3 w_5 - w_1 w_2 \right] x z^2 \\
- \left[ \frac{w_5}{w_4} \right] \left[ w_5 w_3 - w_5 w_3 \right] y z.
\]

By choosing the positive constants as \( \frac{w_5}{w_4}, \frac{w_5}{w_4}, \frac{w_5}{w_4} = 1 \), and substituting these constants, we get
\[
\frac{dn_5}{dt} < - \frac{w_5}{w_4} (x - \bar{x})^2 - \left[ \frac{w_1 w_7}{w_7} \left( y + \bar{y} \right) \right] (x - \bar{x}) (y - \bar{y}) \\
- \frac{w_5}{w_4} \left[ w_5 w_3 \bar{x} + w_5 \bar{x} \right] (y - \bar{y})^2 - \left[ w_5 w_3 - w_5 w_3 \right] z.
\]

Therefore, by using the condition (24) along with the local stability condition (17f), the derivative \( \frac{dn_5}{dt} \) becomes
\[
\frac{dn_5}{dt} < - \left[ \frac{w_5}{w_4} (x - \bar{x}) + \frac{w_5}{w_4} \left[ w_3 w_5 x + w_5 \bar{x} \right] (y - \bar{y}) \right]^2 - \left[ w_5 w_3 - w_5 w_3 \right] z.
\]

Clearly, \( \frac{dn_5}{dt} \) is negative definite and hence \( E_5 \) is globally asymptotically stable.

**Theorem 8:** Suppose that the PEP is locally asymptotically stable. Then \( E_6 \) is globally asymptotically stable if the following sufficient conditions hold
\[
\begin{align*}
[w_1 - w_1 (y + y')] - w_5 + w_5 y' \right] \right] < \left[ \frac{w_3}{w_4} + w_5 x' \right]. \\
[w_2 - w_2 (z + z')] - w_9 + w_9 z' \right] \right] < w_0 x, \\
[w_6 - w_0 y] \right] \right] < \left[ \frac{w_3}{w_4} + w_5 x' \right] \left[ w_0 x \right].
\end{align*}
\] (25a, 25b, 25c)

**Proof:** Define the following real valued function
\[ \Omega_6 = x - x^* - x^* \ln \frac{x}{x^*} + y - y^* - y^* \ln \frac{y}{y^*} + z - z^* - z^* \ln \frac{z}{z^*}. \]

Clearly, \( \Omega_6 \) is a positive definite function that is defined for all \( x > 0, y > 0 \) and \( z > 0 \). Therefore, \( \frac{d\Omega_6}{dt} \) can be written as follows:

\[
\frac{d\Omega_6}{dt} = -(x - x^*)^2 - \left[ \frac{w_4}{w_4}w_4 + w_5 z^* \right](y - y^*)^2 - \left[ \frac{w_2 - w_2(z + z^*) - w_3 + w_3 z^*}{w_2} \right](x - x^*)(z - z^*) - w_6 x(z - z^*)^2 - \left[ \frac{w_6 - w_6(y - y^*)(z - z^*)}{w_6} \right].
\]

Therefore, by using the conditions (25a)-(25c), the derivative \( \frac{d\Omega_6}{dt} \) becomes:

\[
\frac{d\Omega_6}{dt} < -\frac{1}{2}(x - x^*) + \frac{\sqrt{w_2}}{w_4}w_4 + \frac{\sqrt{w_5}}{w_4}x^*(y - y^*)^2 - \frac{1}{2}\left[ \frac{\sqrt{w_2}}{w_4}w_4 + \sqrt{w_5}x^*(z - z^*) \right]^2 - \frac{1}{2}\left[ \frac{\sqrt{w_2}}{w_4}w_4 + \sqrt{w_5}x^*(y - y^*)^2 + \sqrt{w_6}x^*(z - z^*) \right]^2.
\]

Obviously, \( \frac{d\Omega_6}{dt} \) is negative definite and \( E_6 \) is globally asymptotically stable.

### 5. Persistence

In this section the persistence of system (4) is investigated using the Gard technique [19]. First, we need to check the existence of periodic dynamics in the boundary planes. Obviously, the system (4) has three possible subsystems which are obtained in the absence of \( x, y \) and \( z \) and can be written, respectively, as:

\[
\frac{dy}{dt} = y \left[ w_3 \left( 1 - \frac{y}{w_4} \right) - w_6 x - w_7 \right] = g_1(y, z),
\]

\[
\frac{dz}{dt} = z(w_3 y - w_1) = g_2(y, z).
\]

(26)

\[
\frac{dx}{dt} = x(1 - x) - w_2 (1 - z) z = g_3(x, z),
\]

\[
\frac{dz}{dt} = z(w_3 x(1 - z) - w_1) = g_4(x, z).
\]

(27)

\[
\frac{dx}{dt} = x((1 - x) - w_1 (1 - y) y) = g_5(x, y),
\]

\[
\frac{dy}{dt} = y(w_3 \left( 1 - \frac{y}{w_4} \right) + w_5 x(1 - y) - w_7) = g_6(x, y).
\]

(28)

Clearly, subsystems (26), (27) and (28) fall in the interior of the \( yz - \)plane, \( xz - \)plane and \( xy - \)plane, respectively. Now, define the function \( H_1(x, y) = \frac{1}{yz} \) which is a \( C^1 \) function in interior of \( \mathbb{R}^2_+ \) of the \( yz - \)plane. Then, determine the following quantity:

\[
\Delta(y, z) = \frac{\partial (g_1 H_1)}{\partial y} + \frac{\partial (g_2 H_1)}{\partial z} = -\frac{w_3}{w_4} \frac{1}{yz} \neq 0.
\]

According to the Dulic-Bendixon criterion [20], the quantity \( \Delta(y, z) \) does not have a changed sign and it is not identically zero, then there is no closed curve in the interior of \( \mathbb{R}^2_+ \) of the \( yz - \)plane. Further, since the subsystem (26) is bounded and has a unique positive equilibrium point in the interior of \( \mathbb{R}^2_+ \) of the \( yz - \)plane, that is given by \( (\bar{y}, \bar{z}) \), then, according to Poincare-Bendixon theorem [20], it is globally asymptotically stable, whenever it exists, and locally asymptotically stable. Similarly, by using the \( C^1 \) functions in the interior of \( \mathbb{R}^2_+ \) of the \( xz \) and \( xy \) planes, which are defined as \( H_2(x, z) = \frac{1}{xz} \) and \( H_3(x, y) = \frac{1}{xy} \) respectively, it is easy to verify that there is no periodic dynamics in the interior of \( \mathbb{R}^2_+ \) of their planes, and the positive equilibrium point of each subsystem \( (\bar{x}, \bar{z}) \) and \( (\bar{x}, \bar{y}) \) is globally asymptotically stable, whenever it exists, and locally asymptotically stable.

Consequently, the necessary and sufficient conditions for the uniform persistence of system (4) are derived in the following theorem.

**Theorem 9:** The system (4) is uniformly persistent if the following conditions hold:

\[
(w_7 < w_6 + w_5) \text{ or } (w_{10} < w_9),
\]

\[
(w_1 (1 - \bar{y}) \bar{y} < 1) \text{ or } (w_{10} < w_9),
\]

(29a)

(29b)
Proof: Define $\sigma(x, y, z) = x^\rho_1 y^\rho_2 z^\rho_3$ as an average Lyapunov function with positive constant $\rho_i; i = 1, 2, 3$.

Obviously, $\sigma(x, y, z)$ is a $C^1$ positive definite function, and if $x \to 0$ or $y \to 0$ or $z \to 0$, then $\sigma(x, y, z) \to 0$. Consequently, we get that

$$\dot{P}(x, y, z) = \frac{\sigma'(x, y, z)}{\sigma(x, y, z)} = \rho_1 [(1 - x) - w_1 (1 - y)] y - w_2 (1 - z) z$$

$$+ \rho_2 \left[ w_3 \frac{(1 - y)}{w_4} + w_5 x (1 - y) - w_6 z - w_7 \right]$$

$$+ \rho_3 \left[ w_8 x (1 - z) + w_9 y - w_{10} \right].$$

Now, according to the Gard technique, the proof is followed and the system uniformly persists if we can prove that $\dot{P}(x, y, z) > 0$ at each of the boundary equilibrium points. Since

$$\dot{P}(E_0) = \rho_1 + \rho_2 (w_3 - w_7) - \rho_3 (w_{10}).$$

Clearly, $\dot{P}(E_0) > 0$ if we chose the positive constants, so that $\rho_1$ is sufficiently large with respect to others.

$$\dot{P}(E_1) = \rho_2 (w_5 + w_6 - w_7) + \rho_3 (w_9 - w_{10}).$$

Clearly, $\dot{P}(E_1) > 0$ if any one of the conditions given by equation (29a) holds with suitable selection of positive constants $\rho_2$ and $\rho_3$.

$$\dot{P}(E_2) = \rho_1 (1 - w_1 (1 - \bar{\nu}) \bar{y}) + \rho_2 (w_9 \bar{y} - w_{10}).$$

Again, $\dot{P}(E_2) > 0$ if any one of the conditions given by equation (29b) holds with suitable selection of positive constants $\rho_1$ and $\rho_3$.

$$\dot{P}(E_3) = \rho_1 [1 - w_1 (1 - \bar{\nu}) \bar{y} - w_2 (1 - z) \bar{z}].$$

Obviously, $\dot{P}(E_3) > 0$ under condition (29c).

$$\dot{P}(E_4) = \rho_2 [w_5 + w_6 \bar{z} - w_7].$$

Obviously, $\dot{P}(E_4) > 0$ under condition (29d).

$$\dot{P}(E_5) = \rho_3 [w_9 \bar{y} + w_9 \bar{y} - w_{10}].$$

Clearly, $\dot{P}(E_5) > 0$ under the condition (29e).

Hence, all the requirements of Gard technique are satisfied.

6. Bifurcation analysis

In this section, the appearance of the local bifurcation in system (4) is investigated. In a dynamical system, a bifurcation is carried out when a small smooth variation that is made to the parameter of a model gives a sudden or topical variation in its action. In general, the local stability properties of equilibrium point, periodic orbits, or other invariant sets, change at a bifurcation. Sotomayor’s theorem [17] will be applied to study the occurrence of local bifurcation. We can write system (4) in the vector form as

$$\frac{dx}{dt} = F(X),$$

where $X = (x, y, z)^T$ and $F(X) = (x f_1, y f_2, z f_3)^T$.

Now, it is easy to verify that for any vector $V = (v_1, v_2, v_3)^T$, the second derivative of the system (30) can be written as

$$D^2 F(X, \beta) (V, V) = (c_{11})_{3 \times 1},$$

where

$$c_{11} = -2 v_1^2 - 2 w_1 v_1 v_2 + 4 w_1 y v_1 v_2 - 2 w_2 v_1 v_3$$

$$+ 4 w_2 z v_1 v_3 + 2 w_1 x v_2^2 + 2 w_2 x v_3^2,$$

$$c_{21} = -2 w_5 y v_1 v_2 + 2 w_5 v_1 v_2 - \frac{2 w_5 v_2^2}{w_6} - 2 w_5 x v_2^2 - 2 w_6 v_2 v_3,$$

$$c_{31} = -2 w_8 z v_1 v_3 + 2 w_9 v_1 v_3 + 2 w_9 v_2 v_3 - 2 w_9 x v_3^2,$$

with $\beta$ is any bifurcation parameter. Moreover, the third derivative of the system (30) is given by

$$D^3 F(X, \beta) (V, V, V) = (d_{11})_{3 \times 1},$$

where

$$d_{11} = 6 w_1 v_1 v_2^2 + 6 w_2 v_1 v_3^2; d_{21} = -6 w_5 v_1 v_2^2; d_{31} = -6 w_9 v_1 v_3^2.$$

Theorem 10. Assume that condition (13c) holds with the parameter $w_6 \equiv w_5 = w_{10}$, then the system (4) near FAEP undergoes a transcritical bifurcation provided that
However, it has a pitchfork bifurcation otherwise.

**Proof.** Clearly, under the condition (13c) with \( w_9 \equiv w_9^* \), the Jacobian matrix \( J(E_1) \) that is given by equation (13a) has the following eigenvalues: \( \lambda_{11}^* = -1, \lambda_{12}^* = w_7 + w_5 - w_7 < 0 \) and \( \lambda_{13}^* = 0 \). So, the FAEP is a non-hyperbolic point and the Jacobian matrix can be written as

\[
J_1 = J(E_1, w_9^*) = \begin{bmatrix}
-1 & -w_1 & -w_2 \\
0 & w_3 + w_5 - w_7 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Let \( V_1 = (v_{11}, v_{12}, v_{13})^T \) be the eigenvectors of \( J_1 \) corresponding to \( \lambda_{13}^* = 0 \). Then, we will get

\[
V_1 = (-w_2v_{13}, 0, v_{13})^T,
\]

where \( v_{13} \) is any nonzero real number. In the same way, \( \Phi_1 = (\Phi_{11}, \Phi_{12}, \Phi_{13})^T \) is the eigenvectors associated with the zero eigenvalue \( \lambda_{13}^* = 0 \) of \( J_1^T \). Then we obtain that

\[
\Phi_1 = (0, 0, \Phi_{13})^T,
\]

where \( \Phi_{13} \neq 0 \in \mathbb{R} \). Furthermore, it is simple to establish that

\[
F_{w_9}(X, w_9^*) = (0, 0, 0)^T,
\]

which gives \( \Phi_1^TF_{w_9}(E_1, w_9^*) = 0 \).

Thus, according to Sotomayor theorem, the saddle node bifurcation cannot occur. Moreover, since

\[
\Phi_1^T[D F_{w_9}(E_1, w_9^*) V_1] = v_{13}\Phi_{13} \neq 0,
\]

here \( DF_{w_9} \) is the derivative of \( F_{w_9} \) with respect to \( X \). Now, by substituting \( E_1, w_9^* \) and \( V_1 \) in Eq. (31a), we get that

\[
\Phi_1^T[D^2 F(E_1, w_9^*)(V_1, V_1)] = 2w_9^*(w_2 - 1)v_{13}^2\Phi_{13}.
\]

Clearly, \( \Phi_1^T[D^2 F(E_1, w_9^*)(V_1, V_1)] \neq 0 \) under the condition (32) and then system (4) undergoes a transcritical bifurcation when \( w_9 \equiv w_9^* \).

Otherwise, we have \( \Phi_1^T[D^2 F(E_1, w_9^*)(V_1, V_1)] = 0 \), when \( w_2 = 1 \). Now, by substituting \( E_1, w_9^* \) and \( V_1 \) in Eq. (31b), we get that

\[
\Phi_1^T[D^2 F(E_1, w_9^*)(V_1, V_1)] = 6w_9^*v_{13}^3\Phi_{13} \neq 0.
\]

Hence, pitchfork bifurcation takes place.

**Theorem 11.** Suppose that condition (14c) holds with the parameter \( w_{10} \equiv w_{10}^* = w_9^* \), then the system (4) near SAEP undergoes a transcritical bifurcation.

**Proof.** Clearly, under the condition (14c) with \( w_{10} \equiv w_{10}^* \), the Jacobian matrix \( J(E_2) \) that is given by equation (14a) has the following eigenvalues \( \lambda_{21}^* = 1 - w_1^* \tilde{y} + w_1^* \tilde{y}^2 < 0, \lambda_{22}^* = -\frac{w_3}{w_4} \tilde{y}^2 \) and \( \lambda_{23}^* = 0 \). So, the SAEP is a non-hyperbolic point and the Jacobian matrix can be written as

\[
J_2 = J(E_2, w_{10}^*) = \begin{bmatrix}
1 - w_1^* \tilde{y} + w_1^* \tilde{y}^2 & 0 & 0 \\
0 & w_5^* \tilde{y} - w_5^* \tilde{y}^2 & -\frac{w_2}{w_4} \tilde{y} \\
0 & 0 & 0
\end{bmatrix}.
\]

Similarly, as in theorem (10), direct computation gives that

\[
V_2 = \left( 0, -\frac{w_4 w_5}{w_3}, v_{23} \right)^T; v_{23} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{23}^* = 0 \text{ of } J_2.
\]

\[
\Phi_2 = (0, 0, \Phi_{23})^T; \quad \Phi_{23} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{23}^* = 0 \text{ of } J_2^T.
\]

\[
\Phi_2^T F_{w_{10}}(E_2, w_{10}^*) V_2 = -v_{23} \Phi_{23} \neq 0.
\]

\[
\Phi_2^T[D F(E_2, w_{10}^*)(V_2, V_2)] = -2v_{23}^2 \Phi_{23} \left( -\frac{w_4 w_5}{w_3} + w_8 \right) \neq 0.
\]

Hence, system (4) undergoes a transcritical bifurcation when \( w_{10} \equiv w_{10}^* \).

**Theorem 12.** Assume that \( w_1 \equiv w_1^* \equiv \frac{1 - w_2 (1 - 2x)}{(1 + \tilde{y}) \tilde{y}} \), then system (4) near PFEP undergoes a transcritical bifurcation, provided that

\[
1 + w_4^* \gamma_2 (1 - 2x) + w_2 \gamma_2 (1 - 2x) \neq 0.
\]

where \( \gamma_1 \) and \( \gamma_2 \) are given in the proof. However, it has a pitchfork bifurcation otherwise.

**Proof.** Clearly, when \( w_1 \equiv w_1^* \), the Jacobian matrix \( J(E_3) \) that is given by equation (15a) has the following eigenvalues

\[
\lambda_{11}^* = -1, \lambda_{12}^* = w_7 + w_5 - w_7 < 0, \lambda_{13}^* = 0.
\]

Similarly, as in theorem (10), direct computation gives that

\[
V_3 = \left( 0, -\frac{w_4 w_5}{w_3} v_{23}, v_{23} \right)^T; v_{23} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{23}^* = 0 \text{ of } J_3.
\]

\[
\Phi_3 = (0, 0, \Phi_{33})^T; \quad \Phi_{33} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{23}^* = 0 \text{ of } J_3^T.
\]

\[
\Phi_3^T F_{w_{10}}(E_3, w_{10}^*) V_3 = -v_{23} \Phi_{33} \neq 0.
\]

\[
\Phi_3^T[D F(E_3, w_{10}^*)(V_3, V_3)] = -2v_{23}^2 \Phi_{33} \left( -\frac{w_4 w_5}{w_3} + w_8 x \right) \neq 0.
\]

Hence, system (4) undergoes a transcritical bifurcation when \( w_{10} \equiv w_{10}^* \).
\[ \lambda_{31}^* = 0, \lambda_{32}^* = -\frac{w_{3,1}}{w_{3,4}} \bar{y} + \frac{(\frac{w_{3,4}}{w_{3,1}})^2 - 4w_{4,6}w_{3,1} \bar{y}^2}{2} \text{ and } \lambda_{33}^* = -\frac{w_{3,1}}{w_{3,4}} \bar{y} - \frac{(\frac{w_{3,4}}{w_{3,1}})^2 - 4w_{4,6}w_{3,1} \bar{y}^2}{2}. \]

So, the PFEP is a non-hyperbolic point and the Jacobian matrix can be written as

\[
J_3 = J(E_3, w_1) = \begin{bmatrix}
0 & 0 & 0 \\
0 & w_2 \bar{y}^3 - w_{6,2} \bar{y}^5 & w_5 \bar{y}^2 \\
-w_5 \bar{y} & w_2 \bar{y}^5 & w_5 \bar{y}^2
\end{bmatrix}.
\]

Direct computation gives that

\[
V_3 = (v_{3,1}, v_{3,2}, v_{3,3})^T; v_{3,1} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{31}^* = 0 \text{ of } J_3,
\]

where \( v_1 = -\frac{w_{3,2}(1-z)}{w_9} < 0 \) and \( v_2 = \frac{w_3w_2(1-z) + w_4w_5(1-z)}{w_4w_6w_9} > 0 \).

\[ \Phi_3 = (\Phi_{3,1}, 0, 0)^T; \Phi_{3,1} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{31}^* = 0 \text{ of } J_3^T. \]

\[ \Phi_{3,1}^T F_{w_1}(E_3, w_1) = 0. \]

\[ \Phi_{3,1}^T [D F_{w_1}(E_3, w_1^*) V_3] = -\bar{y}(1-\bar{y}) v_{3,1} \Phi_{3,1} \neq 0. \]

Clearly, \( \Phi_{3,1}^T [D^2 F(E_3, w_1^*)(V_3, V_3)] \neq 0 \) under the condition (33) and then system (4) undergoes a transcritical bifurcation when \( w_1 \equiv w_1^* \).

Otherwise, we have \( \Phi_{3,1}^T [D^2 F(E_3, w_1^*)(V_3, V_3)] = 0 \), in case of violating the condition (33). Now, by substituting \( E_3, w_1^* \) and \( V_3 \) in Eq. (31b), we get that

\[
\Phi_{3,1}^T [D^3 F(E_3, w_1^*)(V_3, V_3)] = 6(w_1^* y_1^2 + w_2 y_2^2) v_{13}^3 \Phi_{13} \neq 0.
\]

Hence, pitchfork bifurcation takes place.

**Theorem 13.** Suppose that condition (16g) holds with the parameter \( w_7 \equiv w_7^* = w_3 + w_5 \bar{x} - w_6 \bar{z} \), then the system (4) near MPFEP undergoes a transcritical bifurcation, provided that

\[ w_5y_3 - \frac{w_3}{w_4} = 0 \text{ and } w_{5,2} \neq 0, \quad (34) \]

where \( y_3 \) and \( y_4 \) are given in the proof. However, it has a pitchfork bifurcation otherwise.

**Proof.** Note that, when \( w_7 \equiv w_7^* \), the Jacobian matrix \( J(E_4) \) that is given by equation (16a) has the following eigenvalues

\[
\lambda_{41}^* = \frac{T_4}{2} + \frac{1}{2} \sqrt{T_4^2 - 4D_4}, \lambda_{42}^* = 0 \text{ and } \lambda_{43}^* = \frac{T_4}{2} - \frac{1}{2} \sqrt{T_4^2 - 4D_4},
\]

where \( T_4 < 0 \) and \( D_4 > 0 \) are given in equation (16b). So, the MPFEP is a non-hyperbolic point and the Jacobian matrix can be written as

\[
J_4 = J(E_4, w_7^*) = \begin{bmatrix}
-\bar{x} & -w_1 \bar{x} & -2w_2 \bar{x} + 2w_2 \bar{z} \\
0 & 0 & 0 \\
-w_5 \bar{z} & -w_6 \bar{z}^2 & w_6 \bar{z}
\end{bmatrix}.
\]

Direct computation gives

\[
V_4 = (v_{4,1}, v_{4,2}, v_{4,3})^T; v_{4,1} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{42}^* = 0 \text{ of } J_4,
\]

where \( y_3 = \frac{-\bar{x}w_2w_3w_4(1-z)}{w_5w_9w_1w_6} \) and \( y_4 = -\frac{-\bar{y}w_2w_3w_4(1-z)}{w_5w_9w_1w_6} \).

\[ \Phi_4 = (0, \Phi_{4,2}, 0)^T; \Phi_{4,1} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{42}^* = 0 \text{ of } J_4^T. \]

\[ \Phi_{4,2}^T F_{w_1}(E_3, w_1^*) = 0. \]

\[ \Phi_{4,2}^T [D F_{w_1}(E_4, w_7^*)] V_4 = -v_4 \Phi_{4,2} \neq 0. \]

\[ \Phi_{4,2}^T [D^2 F(E_4, w_7^*)] (V_4, V_4) = 2v_4^2 \Phi_{4,2} \left[ w_5 y_3 - \frac{w_3}{w_4} - w_5 \bar{x} - w_6 y_4 \right]. \]

Clearly, \( \Phi_{4,2}^T [D^2 F(E_4, w_7^*)] (V_4, V_4) \neq 0 \) under the condition (34) and then system (4) undergoes a transcritical bifurcation when \( w_7 \equiv w_7^* \).

Otherwise, we have \( \Phi_{4,2}^T [D^2 F(E_4, w_7^*)] (V_4, V_4) = 0 \), in case of violating the condition (34). Now, by substituting \( E_4, w_7^* \) and \( V_4 \) in Eq. (31b), we get that

\[
\Phi_{4,2}^T [D^3 F(E_4, w_7^*)] (V_4, V_4) = -6w_5 y_3 v_{4,2}^3 \Phi_{4,2} \neq 0.
\]

Hence, pitchfork bifurcation takes place.

**Theorem 14.** Suppose that condition (17g) holds with the parameter \( w_{10} \equiv w_{10}^* = w_6 \bar{x} + w_6 \bar{y} \), then the system (4) near TPFEP undergoes a transcritical bifurcation, provided that

\[
w_0 y_5 + w_0 y_6 - w_0 \bar{x} \neq 0, \quad (35)
\]

where \( y_5 \) and \( y_6 \) are given in the proof. However, it has a pitchfork bifurcation otherwise.

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Proof. Now, when $w_{10} \equiv w_{10}^*$, the Jacobian matrix $J(E_5)$ that given is by equation (17a) has the following eigenvalues

$$\lambda_{51}^* = \frac{\tau_5}{2} + \frac{1}{2} \sqrt{T_5^2 - 4D_5}, \lambda_{52}^* = \frac{\tau_5}{2} - \frac{1}{2} \sqrt{T_5^2 - 4D_5} \text{ and } \lambda_{53}^* = 0,$$

where $T_5 < 0$ and $D_5 > 0$ are given in equation (17b). So the TPFEP is a non-hyperbolic point and the Jacobian matrix can be written as

$$J_5 = J(E_5, w_{10}^*) = \begin{bmatrix} -\bar{x} & -w_1\bar{x} + 2w_2\bar{x}\bar{y} & -w_2\bar{y} \\ w_5\bar{y} - w_5\bar{y}^2 & -\frac{w_3}{w_4} - w_5\bar{x}\bar{y} & -w_6\bar{y} \\ 0 & 0 & 0 \end{bmatrix}.$$

Again, as in above theorems, by direct computation we obtain

$$V_5 = (\gamma v_{51}, \gamma v_{52}, v_{53})^T; \quad v_{53} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{53}^* = 0 \text{ of } J_5,$$

where $\gamma = -\bar{x}[w_2v_5(1-\bar{y})+w_6\bar{x}]$ and $\gamma = -\bar{x}[w_2v_5(1-\bar{y})+w_6\bar{x}] < 0$.

$$\Phi_5 = (0,0,\Phi_{53})^T; \quad \Phi_{53} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{53}^* = 0 \text{ of } J_5^T.$$

$$\Phi_5^TFw_{100}(E_5, w_{10}^*) = 0.$$

$$\Phi_5^TDw_{100}(E_5, w_{10}^*)(V_5) = -v_{53}\Phi_{53} \neq 0.$$

Clearly, $\Phi_5^T[D^2F(E_5, w_{10}^*)(V_5, V_5)] \neq 0$ under the condition (35) and then system (4) undergoes a transcritical bifurcation when $w_{10} \equiv w_{10}^*$.

Otherwise, we have $\Phi_5^T[D^2F(E_5, w_{10}^*)(V_5, V_5)] = 0$, in case of violating the condition (35). Now, by substituting $E_5, w_{10}^*$ and $V_5$ in equation (31b), we get that

$$\Phi_5^T[D^2F(E_5, w_{10}^*)(V_5, V_5)] = -6w_0v_5v_{53}^3\Phi_{53} \neq 0.$$

Hence, pitchfork bifurcation takes place.

Theorem 15. Suppose that the conditions (18a) - (18b) with the condition

$$w_6 < w_{10}[w_6(1-2\gamma')-(1-\gamma')] \quad \text{(36)}$$

are satisfied, then as the parameter

$$w_6 \equiv w_6^* = \frac{L_{12}(L_{12L_{22}}-L_{12L_{22}})+L_{13}(L_{12L_{32}}-L_{12L_{32}})}{\gamma^*(L_{12L_{32}}-L_{12L_{32}})},$$

where $L_{ij}; i,j = 1,2,3$ are given in equation (19a), then system (4) near PEP undergoes a saddle node bifurcation, provided that the following condition holds

$$y_0\Gamma_1 + y_1\Gamma_2 + \Gamma_3 \neq 0 \quad \text{(37)}$$

where all symbols are specified in the proof.

Proof. Obviously, the Jacobian matrix at the PEP with $w_6 \equiv w_6^*$ can be written as

$$J_6 = J(E_6, w_6^*) = \begin{bmatrix} -\bar{x} & -w_1\bar{x}(1-2\gamma') & -w_2\bar{x}(1-2\gamma') \\ w_5\bar{y}(1-\gamma') & -\frac{w_3}{w_4} + w_5\bar{x}\bar{y} & -w_6\bar{y} \\ w_6\bar{z}(1-\gamma') & w_6\bar{z} & -w_6\bar{z} \end{bmatrix} = (L_{ij})_{3 \times 3}.$$

Clearly, the elements of $J_6$ coincide with elements of $J(E_6)$ that are given in equation (19a), except for the element $L_{23}$, which is written in term of $w_6^*$ and denoted by $L_{23}^* = -w_6^*\bar{y}^*$. Straightforward computation shows that the determinant of $J_6$ that is denoted by $\delta_3$ in equation (19b) equals to zero ($\delta_3 = 0$) when $w_6 \equiv w_6^*$, which is positive under condition (36). Therefore, the characteristic equation that is given by equation (19b) has zero root with the other two negative real part roots, denoted by:

$$\lambda_{61}^* = 0, \lambda_{62}^* = \frac{-\delta_1 + \frac{1}{2} \sqrt{\delta_1^2 - 4\delta_2}}{2} \text{ and } \lambda_{63}^* = \frac{-\delta_1 - \frac{1}{2} \sqrt{\delta_1^2 - 4\delta_2}}{2},$$

where $\delta_1$ and $\delta_2$ are positive due conditions (18a) and (18b), and given by equation (19b). Therefore, the PEP is a non-hyperbolic equilibrium point. Now, to test the possibility of the occurrence of saddle node bifurcation, the following quantities are determined.

$$V_6 = (\gamma v_{63}, \gamma v_{63}, v_6)^T; \quad v_{63} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{61}^* = 0 \text{ of } J_6,$$

where $\gamma = \frac{L_{12L_{22}}-L_{12L_{22}}}{L_{11L_{22}}-L_{11L_{22}}}$ and $\gamma = \frac{L_{13L_{21}}-L_{11L_{21}}}{L_{11L_{22}}-L_{11L_{22}}} \neq 0 < 0$. 


\[ \Phi_6 = (y_9 \Phi_{63}, y_{10} \Phi_{63}, \Phi_{63})^T; \Phi_{63} \neq 0 \in \mathbb{R} \text{ as an eigenvector corresponding to } \lambda_{63}^* = 0 \text{ of } J_5^T, \]

where \( y_9 = \frac{L_{21}L_{32} - L_{22}L_{31}}{L_{12}L_{31} - L_{11}L_{32}} > 0 \) and \( y_{10} = \frac{L_{12}L_{31} - L_{11}L_{32}}{L_{11}L_{22} - L_{12}L_{21}} \).

\[ \Phi_6^T F_{w_6}(E_6, w_6^*) = -y_{10} y_9^* z^* \Phi_{63} \neq 0. \]

\[ \Phi_6^T [D^2 F(E_6, w_6^*) (V_6, V_6)] = -2v_{63}^2 \phi_{63} [y_9 \Gamma_1 + y_{10} \Gamma_2 + \Gamma_3]. \]

where \( \Gamma_1 = y_9^* + y_{10} y_9^*(1 - 2 y_9^*) + w_4 y_9^* (1 - 2 z^*) - w_4 x^* y_9^* - w_2 x^* y_9^*. \)

\[ \Gamma_2 = w_5 y_9^* (1 - 2 y_9^*) + w_4 x^* y_9^* y_8^* + w_6 x^* y_8^*. \]

\[ \Gamma_3 = -w_0 y_9^* (1 - 2 z^*) - w_9 y_9^* + w_9 x^*. \]

Accordingly, \( \Phi_6^T [D^2 F(E_6, w_6^*) (V_6, V_6)] \neq 0 \) under the condition (37), hence the saddle node takes place and the proof is completed.

7. Numerical Simulation

In this section, a numerical simulation of the solution of system (4) is performed to authenticate the obtained analytical results and understand the influence of the change in the parameters on the dynamical action of the system. The following hypothetical set of parameters is used in the numerical simulation, which is applied by using predictor-corrector four steps method along with the sixth orders Runge-Kutta method [21] for solving system (4). Then, MATLAB version 6.0.0.88 was used to plot the figures.

\[ w_1 = 0.5, \quad w_2 = 0.5, \quad w_3 = 1, \quad w_4 = 1, \quad w_5 = 0.2, \quad w_6 = 0.5, w_7 = 0.1, w_8 = 0.2, w_9 = 0.2, w_{10} = 0.2. \]  

(38)

It is observed that, for the data set given by equation (38) with different initial sets of values, the system (4) approaches asymptotically to a PEP, as shown in (Figure-1).

![Figure 1](attachment:image1.png)

Figure 1- The time series of the trajectories of system (4) using data set (38) with different initial points, which approach asymptotically to \( E_6 = (0.76, 0.66, 0.56). \) (a) Trajectories of \( x. \) (b) Trajectories of \( y. \) (c) Trajectories of \( z. \)

Accordingly, system (4) is globally asymptotically stable at PEP. Now, for the data set (38), with decreasing the value of the parameter \( w_8 \) up to \( w_8 = 0.01, \) it is noticed that system (4) approaches asymptotically to the TPFEP, as shown in (Figure-2). Otherwise, the system still persists at the PEP.
Similar observation was obtained for varying \( w_9 \).

Figure 2- The solution of system (4) approaches asymptotically to TPFEP that is given by \( E_5 = (0.96, 0.91, 0) \) using the data set (38) with \( w_8 = 0.01 \). (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Now, increasing the parameter \( w_{10} \), so that \( w_{10} \geq 0.38 \), with other parameters as in set (38), leads to an extinction in the top predator too, and the solution approaches asymptotically to the TPFEP as shown typically by (Figure-3). Otherwise, it is still persistent at PEP.

Figure 3- The solution of system (4) approaches asymptotically to TPFEP given by \( E_5 = (0.96, 0.91, 0) \) using the data set (38) with \( w_{10} = 0.4 \). (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Further investigation of the dynamical behavior of system (4) using data set (38) is performed with varying one parameter each time to understand their effects on the solution and persistence of the system. It is observed that all the parameters have a quantitative change on the solution of system (4). This is due to the existence of more than one source of food in each level, which makes the extinction of any species difficult through using only one parameter. Therefore, in the following, we will investigate the system under the effects of varying of multi parameters simultaneously.

For the data set (38) with \( w_1 = 1.25 \) and \( w_{10} = 0.21 \), it is noticed that the solution of system (4)
approaches asymptotically to SAEP, $E_1 = (1,0,0)$, as shown in (Figure-4).

![Figure 4](image)

**Figure 4**-The solution of system (4) approaches asymptotically to FAEP given by $E_1 = (1,0,0)$ using the data set (38) with $w_1 = 1.25, w_{10} = 0.21$. (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Note that it is simple to prove that the conditions (13c) and (13d) are held. Moreover, for the data set (38) with $w_1 = 4.5$ and $w_{10} = 0.5$, it is noticed that the solution of system (4) approaches asymptotically to SAEP, that is given by $E_2 = (0,0.5,0)$, as shown in (Figure-5).

![Figure 5](image)

**Figure 5**-The solution of system (4) approaches asymptotically to SAEP given by $E_2 = (0,0.5,0)$ using the data set (38) with $w_1 = 4.5, w_7 = 0.5$. (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Clearly, the data used in (Figure-5) satisfy the conditions (14c) and (14d). Again, for the data set (38) with $w_2 = 6.5$ and $w_4 = 2$, the system approaches asymptotically to the PFEP, that is given by $E_3 = (0,1,0.8)$, as shown in (Figure-6).
The solution of system (4) approaches asymptotically to PFEP given by \( E_3 = (0,1,0.8) \) using the data set (38) with \( w_2 = 6.5, w_4 = 2 \). (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Direct computation shows that the data used in (Figure 6) satisfy the condition (15f). Finally, the solution of system (4) approaches asymptotically to the MPFEP, that is given by \( E_4 = (0.87,0,0.54) \), using the data set (38) with \( w_3 = 0.5, w_6 = 0.5 \) and \( w_7 = 0.5 \), as shown in (Figure 7).

The solution of system (4) approaches asymptotically to MPFEP given by \( E_4 = (0.87,0,0.54) \) using the data set (38) with \( w_3 = 0.5, w_7 = 0.5, w_8 = 0.5 \). (a) The trajectory of system (4). (b) Time series of the trajectory given by (a).

Again, the data used in (Figure 7) satisfied the conditions (16f) and (16g).

1. Discussion

In this paper, a food web model incorporating a prey refuge that depends on both prey and predator species is proposed and studied. The food is consumed according to Lotka-Volterra functional response. Moreover, the intermediate predator grows logistically by the addition of favorite food at the lower level. The top predator behaves as a generalist predator and preys upon both species in the lower level and second level. All the properties of the solution of system (4) are investigated. It is observed that the system has seven nonnegative equilibrium points. The local stability conditions for each point are constructed. The global dynamics, whenever exists, is investigated too. The dynamics of all possible subsystems is also studied and it is observed that there is no periodic dynamics in the boundary planes. On the other hand, the persistence conditions of the food web system are established.
too. It is observed that the system has a wide range of persistence due to the existence of the prey refuge that is depending on both prey and predator species as well as the multisource of food for each species. The probability of occurrence of local bifurcation around the non-hyperbolic equilibrium point is also discussed. It is observed that the system has multi-types of bifurcations which may occur near the equilibrium points. On the other hand, numerical simulation is used to confirm our obtained results. It is observed that the system has only one type of attractors that is a stable point, while periodic dynamics does not exist even in the boundary planes. This indicates that the existence of the prey refuge that is depending on both prey and predator species is a stabilizing factor on the dynamics of food web model and extends the range of the parameters at which the system persists.

References
1. Krikorian, N. 1979. The Volterra model for three species predator–prey systems: boundedness and stability. Journal of mathematical biology 7(2): 117–132.
2. Hastings, A. and Powell, T. 1991. Chaos in a three-species food chain. Ecology 72(3): 896–903.
3. Huxel, G.R. and McCann, K. 1998. Food Web Stability: The Influence of Trophic Flows across Habitats. The American Naturalist 152(3): 460–469.
4. Kratina, P., LeCraw, R.M., Ingram, T. and Anholt, B.R. 2012. Stability and persistence of food webs with omnivory: are there a general pattern?. Ecosphere 3(6): art50 (18 pages).
5. Naji, R.K. 2012. Global stability and persistence of three species food web involving omnivory. Iraqi Journal of Science, 55(4): 866-876.
6. Upadhyay, R.K., Naji, R.K., Raw, S.N. and Dubey B. 2013. The role of top predator interference on the dynamics of a food chain model. Communications in Nonlinear Science and Numerical Simulation, 18: 757–768.
7. Naji, R.K. and Saady, R.R. 2014. The Dynamics of Four-Species Ecological Model. Iraqi Journal of Science, 55(2A): 506-529.
8. Hsu, S.B., Ruan, S. and Yang, T.H. 2015. Analysis of three species Lotka–Volterra food web models with omnivory. Journal of Mathematical Analysis and Application, 426: 659–687.
9. Agarwal, M. and Kumarn, A. 2017. Dynamics of Food Chain Model: Role of Alternative Resource for Top Predator. International Journal of Mathematical Modelling and Computations 07(02): 115-128.
10. Xiao, Z., Xie, X. and Xue, Y. 2018. Stability and bifurcation in a Holling type II predator–prey model with Allee effect and time delay. Advances in Difference Equations, 2018: 288 (21 pages).
11. Mukherjee, D. 2013. The effect of prey refuges on a three species food chain model. Differential Equations and Dynamical Systems, 22(4): 413–426
12. Taylor, R.J. 1984. Predation, Chapman and Hall. New York. NY. USA.
13. Das, U., Kar, T.K and Pahari, U.K. 2013. Global Dynamics of an Exploited Prey-Predator Model with Constant Prey Refuge. ISRN Biomathematics 2013, Article ID 637640, (12 pages).
14. Ghosh, J., Sahoo, B. and Poria, S. 2017. Prey-predator dynamics with prey refuge providing additional food to predator. Chaos, Solitons & Fractals, 96: 110–119.
15. Santra, P.K., Mahapatra, G.S. and Phaijoo, G.R. 2020. Bifurcation and Chaos of a Discrete Predator-Prey Model with Crowley–Martin Functional Response Incorporating Proportional Prey Refuge. Mathematical Problems in Engineering 2020, Article ID 5309814, (18 pages).
16. Molla, H., Sabir, R.M. and Sarwardi, S. 2019. Dynamics of a Predator–Prey Model with Holling Type II Functional Response Incorporating a Prey Refuge Depending on Both the Species, International Journal of Nonlinear Sciences and Numerical Simulation, 20(1): 89-104.
17. Perko L. 2001. Differential Equation and Dynamical Systems. Third Edition, New York, Springer-Verlag Inc.
18. May, R.M. 1973. Stability and complexity in model ecosystems. Princeton. New Jersey. Princeton University Press.
19. Gard, T.C. and Hallam, T.G. 1979. Persistence in food webs—I Lotka-Volterra food chains. Bulletin of Mathematical Biology, 41(6): 877–891.
20. Wiggins, S. 2003. Introduction to applied nonlinear dynamical systems and chaos. Springer’s Verlag.
21. Burden R.L. and Faires J. D. 2011. Numerical Analysis, Ninth Edition, Brooks/Cole, USA.