KAWAGUCHI-SILVERMAN CONJECTURE ON AUTOMORPHISMS OF PROJECTIVE THREEFOLDS

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Dedicated to Professor De-Qi Zhang on the occasion of his 60th birthday

Abstract. Under the framework of dynamics on normal projective varieties by Kawamata, Nakayama and Zhang [18, 39, 40, 41, 51], Hu and the author [15], we may reduce Kawaguchi-Silverman conjecture for automorphisms $f$ on normal projective threefolds $X$ with either the canonical divisor $K_X$ is trivial or negative Kodaira dimension to the following two cases: (i) $f$ is a primitively automorphism of a weak Calabi-Yau threefold (ii) $X$ is a rationally connected threefold. And we prove Kawaguchi-Silverman conjecture is true for automorphisms of normal projective varieties $X$ with the irregularity $q(X) \geq \dim X - 1$. Finally, we discuss Kawaguchi-Silverman conjecture on normal projective varieties with Picard number two.

1. Introduction

1.1. Kawaguchi-Silverman conjecture. Let $X$ be a normal projective variety of dimension $n \geq 1$ over a global field $K$ of characteristic 0. Let $f : X \dashrightarrow X$ be a dominant rational map. Then the $k$-th dynamical degree of $f$ (cf. [7, 10, 47]) is defined as

$$d_k(f) := \lim_{s \to \infty} \left( (f^s)^*(H^k) \cdot H^{n-k} \right)^{1/s},$$

where $H$ is an ample Cartier divisor on $X$. A result of Dinh and Sibony [10] says that this limit exists and is independent of the choice of the ample divisor $H$.

In [20, 21], Kawaguchi and Silverman studied another dynamical invariant called the arithmetic degree, which we now recall. Assume that $X$ and $f$ are defined over the algebraic closure $\overline{Q}$ of the rational number field $\mathbb{Q}$, and write $X(\overline{Q})_f$ for the set of points $P$ whose forward $f$-orbit

$$\mathcal{O}_f(P) = \{ P, f(P), f^2(P), \ldots \}$$

is well-defined. Further, let

$$h_H : X(\overline{Q}) \to [0, \infty)$$

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be a Weil height on $X$ associated with an ample divisor $H$ (cf. [3, 17, 22]), and let $h^+_H = \max \{1, h_H\}$. For $P \in X(\overline{\mathbb{Q}})_f$, we define the lower and the upper arithmetic degree of $f$ at $x$ by

$$\underline{\alpha}_f(P) = \liminf_{n \to \infty} \frac{h^+_H(f^n(P))}{n},$$

$$\overline{\alpha}_f(P) = \limsup_{n \to \infty} \frac{h^+_H(f^n(P))}{n}.$$

Both of these quantities are independent of the choice of ample divisor $H$ and height function $h_H$ ([21, Proposition 12]). When $\alpha_f(P) = \overline{\alpha}_f(P)$, we write

$$\alpha_f(P) = \lim_{n \to +\infty} \frac{h^+_H(f^n(P))}{n}$$

and call it the arithmetic degree. When $f$ is a morphism, the above limit exists by [21].

The following Kawaguchi-Silverman conjecture (KSC) asserts that for a dominant self-map $f : X \to X$ of a projective variety $X$ over $\mathbb{Q}$, the arithmetic degree $\alpha_f(x)$ of any point $x$ with Zariski dense $f$-orbit is equal to the first dynamical degree $d_1(f)$ of $f$.

**Conjecture 1.1.** [21, Conjecture 6] Let $f : X \to X$ be a dominant rational map of a normal projective variety $X$ over $\overline{\mathbb{Q}}$, and let $x \in X(\overline{\mathbb{Q}})_f$. If $\mathcal{O}_f(x)$ is Zariski dense in $X$ and there exists the arithmetic degree $\alpha_f(x)$ at $x$, then $\alpha_f(x) = d_1(f)$.

**Remark 1.2.** It is known that any dominant rational self-map of a normal projective variety $X$ with the Kodaira dimension $\kappa(X) > 0$ does not have any Zariski dense forward orbits (cf. [40, Theorem A] and [48, Theorem 14.10]). It is known that $1 \leq \underline{\alpha}_f(x) \leq \overline{\alpha}_f(x) \leq d_1(f)$ by [21, Theorem 4] and [28, Theorem 1.4]. Then KSC is true for $f$ if $d_1(f) = 1$. So, we may assume that $d_1(f) > 1$.

1.2. **Historical note.** Let $f$ be a surjective endomorphism of a normal projective variety $X$. When $f$ is an automorphism, one can further take an $f$-equivariant resolution (cf. [5, Theorem 13.2]). Kawaguchi and Silverman showed in [19, Theorem 2(c)] that KSC holds for such $f$ when $\dim X = 2$. When $f$ is non-isomorphic and $X$ is smooth, Matsuzawa, Sano and Shibata [32, Theorem 1.3] proved that KSC holds for $f$, by reducing the problem to three precise cases: $\mathbb{P}^1$-bundles, hyperelliptic surfaces, and surfaces of Kodaira dimension one. For a singular projective surface $X$, running an $f$-equivariant minimal model program (MMP) after iterating $f$, Meng and Zhang [37, Theorem 1.3] proved that KSC holds for any surjective endomorphism of a projective surface.

Assume that $f$ is a surjective endomorphism of a normal projective variety $X$ and $d_1(f) > 1$. Kawaguchi and Silverman [21, Theorem 5] proved KSC when $f$ is polarized, i.e. there is an ample Cartier divisor $D$ and integer $q > 1$ such that $f^*D \sim qD$. Matsuzawa, Meng, Shibata, Zhang and Zhong [31, Theorem 1.9(1)] proved that there exists a nef
Cartier $\mathbb{R}$-divisor $D'$ such that $f^*D' \sim d_1(f)D'$. And Matsuzawa and the author [25, Proposition 5.2(1)] proved that KSC holds for $f$ if the Iitaka dimension $\kappa(X, D) > 0$.

Silverman [46, Theorem 1.2] proved that KSC holds for any dominant self-map on abelian varieties. Also, KSC holds for any self-morphism on semi-abelian varieties by Matsuzawa and Sano [33, Theorem 1.1].

Matsuzawa [27] proved that KSC is true when $X$ is a projective toric variety, a linear algebraic group or a variety of Fano type. Moreover, he established in [27, Theorem 4.1] that KSC is true for any surjective endomorphism $f$ on $X$ when $\text{NS}_{\mathbb{Q}}(X) \cong \text{Pic}_{\mathbb{Q}}(X)$ and the nef cone is generated by finitely many semi-ample integral divisors.

Assume that $f$ is a surjective endomorphism of a projective variety $X$. In [26, Theorem 1.2 and Proposition 1.7], Lesieutre and Satriano proved that KSC is true when $X$ is a hyper-Kähler variety, a smooth projective threefold with $\kappa(X) = 0$ and $\deg f > 1$. Moreover, KSC is true when $f$ is an automorphism of a smooth projective variety $X$ with Picard number $\rho(X) = 2$ by Shibata [45, Theorem 4.2] or [26, Theorem 2.30].

When $X$ is a smooth rationally connected variety admitting an int-amplified endomorphism, KSC holds for every surjective endomorphism of such $X$ by Meng and Zhang [37, Theorem 1.11] and Matsuzawa and Yoshihara [35, Theorem 1.1]. Moreover, let $X$ be a $\mathbb{Q}$-factorial klt projective variety with the algebraic fundamental group $\pi_1^{\text{alg}}(X_{\text{reg}})$ of the smooth locus $X_{\text{reg}}$ of $X$ being finite. Suppose $X$ admits an int-amplified endomorphism. Lesieutre, Matsuzawa, Shibata and Zhang [30, Theorem 6.3] proved that KSC holds for any surjective endomorphism of such $X$ by [37, Theorem 1.7].

Let $\pi : X \to Y$ be a surjective endomorphism of normal projective variety. Suppose $f$ (resp. $g$) is a surjective endomorphisms of $X$ (resp. $Y$) such that $\pi \circ f = g \circ \pi$. It is well known that KSC holds for $f$ if (1) $d_1(f) = d_1(g)$ and (2) KSC holds for $g$ (cf. [26, Theorem 3.4]). Now let $X$ be a projective bundle over a smooth projective variety $Y$ with the Picard number one. Lesieutre and Satriano [26] proved that KSC is true for any surjective endomorphism of $X$ when $Y = \mathbb{P}^1$. Also, Matsuzawa and the author [25] proved that KSC is true for any surjective endomorphism of $X$ when $Y$ is Fano.

Chen, Lin and Oguiso [6] showed that KSC is true for three cases: (i) birational automorphisms of a smooth projective variety $X$ with $\kappa(X) = 0$ and the irregularity $\text{q}(X) \geq \dim X - 1$ (ii) birational automorphisms of an irregular smooth threefold $X$ modulo the case that $X$ is covered by rational surfaces (iii) automorphisms of an irregular smooth threefold $X$.

A dominant rational self-map on a projective variety is called $p$-cohomological hyperbolic if the $p$-th dynamical degree is strictly larger than other dynamical degrees. Matsuzawa and Wang [34] showed that KSC is true for a $1$-cohomological hyperbolic of a
smooth projective variety, and the arithmetic degrees can be transcendental for dominant rational self-maps by using the striking result of Bell, Diller and Jonsson [2].

1.3. **Main results.** Let \( f \) be a surjective endomorphism of a normal projective variety \( X \). We say that a rational map \( \pi : X \to Y \) is \( f \)-equivariant, if there exists a surjective endomorphism \( g \) of \( Y \) such that \( \pi \circ f = g \circ \pi \). When \( f \) is an automorphism, there is a resolution of singularities \( \pi : X' \to X \) and an automorphism \( f' \) on \( X' \) such that \( \pi \circ f' = f \circ \pi \) (cf. [5, Theorem 13.2]). Therefore, we reduce KSC for \( f \) to KSC for \( f' \), which is an automorphism on a smooth projective variety. However, there is no equivariant minimal model program for automorphism groups of projective varieties in general (cf. [15, Remark 1.3(1)]). So we consider the special but important case where \( X \) is minimal, e.g. the canonical divisor \( K_X \sim 0 \) when \( \dim X = 3 \) and \( \kappa(X) = 0 \).

Due to work of Horbing and Peternell [16, Theorem 1.5], we have the Beauville-Bogomolov decomposition for minimal models with trivial canonical class as follows. Let \( X \) be a normal projective variety at most klt singularities such that \( K_X \equiv 0 \). Then there exists a finite cover, étale in codimension one \( \pi : \tilde{X} \to X \) such that

\[
\tilde{X} \cong A \times \prod Y_j \times \prod Z_k
\]

where \( A \) is an abelian variety, the \( Y_j \) are singular Calabi-Yau varieties and the \( Z_k \) are singular irreducible holomorphic symplectic varieties (see [13, Definition 1.3]). However, it is still unclear whether we can always lift the automorphisms of \( X \) to some splitting cover \( \tilde{X} \) (cf. [15, Remark 3.5]). Instead of utilizing their strong decomposition theorem, we use a weak version (cf. [15, Lemma 2.7]) due to Kawamata [18], and developed by Nakayama-Zhang [41]. For more details about automorphisms of projective varieties with trivial canonical divisor, we refer to [51, Theorem 1.1 and 2.4] and [15, Theorem 1.2]. For automorphisms on projective varieties with negative Kodaira dimension, we can use special MRC fibration due to Nakayama [39] which have the descent property. We refer to [15, Lemma 2.11] for more details about it.

The notion of a primitively birational self-map was introduced by Zhang [50] as follows.

**Definition 1.3.** [50] A birational self-map \( f : X \to X \) is *imprimitive* if there exist a variety \( B \) with \( 1 \leq \dim B < \dim X \), a birational map \( g : B \to B \), and a dominant rational map \( \pi : X \to B \) such that \( \pi \circ f = g \circ \pi \). The map \( f \) is called *primitive* if it is not imprimitive.

Below is our main result of KSC on automorphisms of normal projective threefolds.

**Theorem 1.4.** Let \( f \) be an automorphism of a normal projective threefold with only klt singularities. Suppose \( K_X \sim_{\mathbb{Q}} 0 \) or \( \kappa(X) = -\infty \). Then we reduce Conjecture 1.1 for \((X, f)\) to the following cases:
(1) \( f \) is a primitively automorphism of a weak Calabi-Yau threefold.
(2) \( X \) is a rationally connected threefold.

Notice that a birational self-map \( f \) on a minimal Calabi-Yau threefold \( X \) of Picard number \( \rho(X) \geq 2 \) is primitive if the action \( f^*|_{\text{NS}_\mathbb{Q}(X)} \) is irreducible over \( \mathbb{Q} \) (cf. [42, Corollary 1.3]). This motivates us to ask the following question.

**Question 1.5.** Let \( f \) be a birational self-map of a weak Calabi-Yau variety \( X \) with \( \rho(X) \geq 2 \). Suppose that \( f^*|_{\text{NS}_\mathbb{Q}(X)} \) is irreducible over \( \mathbb{Q} \). Then is KSC true for \( (X, f) \)?

Motivated by [6, Theorems 1.4] and without assuming that the Kodaira dimension vanishes, we may establish that KSC holds for automorphisms of normal projective varieties \( X \) with \( q(X) \geq \dim X - 1 \) as follows.

**Theorem 1.6.** KSC is true for automorphisms of normal projective varieties \( X \) with \( q(X) \geq \dim X - 1 \).

**Remark 1.7.** Let \( \pi : X \rightarrow Y \) be dominant rational map of projective varieties. Let \( f : X \rightarrow X \) and \( g : Y \rightarrow Y \) be dominant rational self-maps such that \( f \circ \pi = \pi \circ g \). Suppose that \( \pi \) is generically finite. If one could establish that KSC holds for \( f \) if and only if KSC holds for \( g \), then KSC is true for any birational automorphism of a normal projective variety \( X \) with \( q(X) \geq \dim X - 1 \) by using the same argument in the proof of Theorem 1.6.

The paper is organized as follows. In Section 2, we collect some basic facts on KSC. In Section 3, we study invariant fibrations on projective varieties and prove Theorems 1.6 and 3.3. In Section 4, we prove Theorems 1.4. Finally, we study KSC on projective varieties with Picard number two in Section 5.

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2. Preliminaries

**Notation 2.1.** Let \( X \) be a normal projective variety Let \( \text{NS}_\mathbb{R}(X) := \text{NS}(X) \otimes \mathbb{Z} \mathbb{R} \), where \( \text{NS}(X) := \text{Pic}(X)/\text{Pic}^0(X) \) is the usual Néron-Severi group of \( X \).

Let \( f : X \rightarrow X \) be a surjective endomorphism of \( X \). Denote by \( \text{Alb}(X) := \text{Pic}^0(\text{Pic}^0(X)_{\text{red}}) \)
which is an abelian variety. Then there is an albanese morphism \( \text{alb}_X : X \to \text{Alb}(X) \) such that: \( \text{alb}_X(X) \) generates \( \text{Alb}(X) \) and for every morphism \( \varphi : X \to A \) from \( A \) to an abelian variety \( A \), there exists a unique morphism \( \psi : \text{Alb}(X) \to A \) such that \( \varphi = \psi \circ \text{alb}_X \) (cf. [11, Remark 9.5.25]). By the above universal property, \( f \) descends to surjective endomorphisms on \( \text{Alb}(X) \). The irregularity \( q(X) \) of \( X \) is defined as:

\[
q(X) := \dim \text{Alb}(X).
\]

A surjective endomorphism \( f : X \to X \) is said to be int-amplified if \( f^*H - H = L \) for some ample Cartier divisors \( H \) and \( L \), or equivalently, if all eigenvalues of \( f^*|_{\text{NS}_R(X)} \) are of modulus greater than 1 (cf. [29, Theorem 1.1]).

The Iitaka dimension \( \kappa(X,D) \) of a \( \mathbb{R} \)-Cartier divisor \( D \) on \( X \) (cf. [38]) is the largest integer \( k \) such that

\[
\limsup_{m \to \infty} \frac{h^0(X, \lfloor m\ell(D)D \rfloor)}{m^k} > 0.
\]

otherwise, \( \kappa(X,D) = -\infty \). Here, \( \ell(D) \in \{ k \in \mathbb{Z}_{>0}, |kD| \neq \emptyset \} \). The Kodaira dimension of \( X \) is \( \kappa(X) = \kappa(X,K_X) \), where \( K_X \) is the canonical divisor of \( X \).

The following lemma is well-known (cf. [27, Lemma 5.6], [37, Lemma 2.5]).

**Lemma 2.2.** Let \( \pi : X \to Y \) be a dominant rational map of projective varieties. Let \( f : X \to X \) and \( g : Y \to Y \) be surjective endomorphisms such that \( g \circ \pi = \pi \circ f \). Then the following hold.

1. Suppose that \( \pi \) is generically finite. Then KSC holds for \( f \) if and only if KSC holds for \( g \).
2. Suppose \( d_1(f) = d_1(g) \) and KSC holds for \( g \). Then KSC holds for \( f \).

A normal projective variety \( X \) is said to be \( Q \)-abelian if there is a finite surjective morphism \( \pi : A \to X \) étale in codimension 1 with \( A \) being an abelian variety.

**Theorem 2.3.**

1. KSC holds for any surjective endomorphism of an abelian variety.
2. KSC holds for any surjective endomorphism of a \( Q \)-abelian variety.

**Proof.** (1) follows from [46, Theorem 2], and (2) follows from [37, Theorem 2.8].

The following theorem is due to Matsuzawa, Meng, Shibata, Zhang and Zhong [31].

**Theorem 2.4.** [31, Theorem 1.9(1)] Let \( f : X \to X \) be a surjective endomorphism of a normal projective variety over a field \( k \) of arbitrary characteristic. Assume \( d_1(f) > 1 \). Then \( f^*D \sim_R d_1(f)D \) for some nef \( \mathbb{R} \)-Cartier divisor \( D \).

**Definition 2.5.** We say such \( D \) in Theorem 2.4 is a leading real eigendivisor of \( (X,f) \).

Note that KSC holds for \( (X,f) \) if such \( D \) has \( \kappa(X,D) > 0 \) as follows.
Proposition 2.6. Let $X$ be a $\mathbb{Q}$-factorial normal projective variety and $f : X \to X$ a surjective morphism with $d_1(f) > 1$.

1. If the lead real eigendivisor $D$ has $\kappa(X, D) > 0$, then KSC holds for $f$.
2. Assume that $f \in \text{Aut}(X)$. Let $\mathbb{R}$-divisors $D_+$ and $D_-$ respectively be the lead real eigendivisor of $(X, f)$ and $(X, f^{-1})$ respectively. If $\kappa(X, D_+ + D_-) > 0$, then KSC holds for $f$.

Proof. It follows from Theorem 2.4 and [25, Proposition 5.2].

Definition 2.7. Let $f$ be an automorphism of a projective variety $X$ with $\dim X \geq 3$ and let $D_+$ and $D_-$ be some $\mathbb{R}$-Cartier divisors on $X$. We say $(X, D_+, D_-)$ is positive on $(X, f)$ if the following hold:

$$f^* D_+ \sim_R d_1(f)D_+, (f^{-1})^* D_- \sim_R d_1(f^{-1})D_-, \kappa(X, D_+ + D_-) > 0.$$ 

Remark 2.8. Lesieutre and Satriano established in [26] that KSC is true for hyper-Kähler varieties by using the pair $(X, D_+, D_-)$ which is positive on $(X, f)$ when $\kappa(X, D_+ + D_-) = \dim X$. So an interesting question is asked as follows.

Question 2.9. Does there exist an example of a projective variety $X$ with the pair $(X, D_+, D_-)$ which is positive on $(X, f)$ such that $\kappa(X, D_+ + D_-) < \dim X$?

Theorem 2.10. (cf. [26, Theorem 1.8]) Let $f$ be an automorphism of a normal non-uniruled projective threefold $X$. If the second Chern class $c_2(X)$ is strictly positive on $\text{Nef}(X)$ and $q(X) = 0$, then $f$ has finite order.

Proof. By [4, Lemma 7.1] we know that $\{D \in \text{Nef}(X) \mid c_2(X) \cdot D \leq m\}$ is compact for all $m \geq 0$. So the function $D \mapsto c_2(X) \cdot D$ achieves a minimum positive value on $\text{Nef}(X) \cap \text{Amp}(X)$ and this value is achieved by only finitely many $D_i$. Taking the sum of these finitely many $D_i$, we obtain an ample class $A$ that is fixed by $f^*$. Then by a Fujiki-Liberman type theorem (cf. [12, 24] or [23, Theorem 1.4]), some iterate $f^n$ lies in the connected component of the identity $\text{Aut}^0(X) \subseteq \text{Aut}(X)$. For a smooth model $X'$ of $X$, the birational automorphism group $\text{Bir}(X')$ contains $\text{Aut}^0(X)$ as a subgroup. By [14, Theorem 2.1], $\text{Bir}(X')$ is a disjoint union of abelian varieties of dimension equal to $q(X') = q(X) = 0$. We conclude that $f$ has finite order. □

3. Invariant fibrations on projective varieties

Invariant fibrations play an important role in the study of rational maps in higher dimension, and the product formula of Dinh-Nguyễn-Truong [9] is useful in dealing with their dynamical degrees. Let $\pi : X \to Y$ be a dominant rational map of projective
varieties and $f : X \longrightarrow X$ a dominant rational self-map. Fix an ample divisor $H$ on $X$ and an ample divisor $H'$ on $Y$. Now we give the following definition.

**Definition 3.1.** The first dynamical degree of $f$ relative to $\pi$ is defined by

$$d_1(f|_\pi) = \lim_{n \to \infty} ((f^n)^*H \cdot \pi^*((H')^{\dim Y}) \cdot H^{\dim X - \dim Y - 1})^{1/n}.$$  

The following is due to the product formula (cf. [8, Theorem 1.1]).

**Proposition 3.2.** (cf. [26, Theorem 3.4]) Let $\pi : X \longrightarrow Y$ be a dominant rational map of normal projective varieties. Suppose $f$ (resp. $g$) is a surjective endomorphism of $X$ (or $Y$) such that $g \circ \pi = \pi \circ f$. If $d_1(f|_\pi) \leq d_1(g)$ and KSC holds for $g$, then KSC also holds for $f$. The condition $d_1(f|_\pi) \leq d_1(g)$ holds in particular if $f$ is an automorphism and $\dim Y = \dim X - 1$.

**Proof.** By the product formula [7, 8, 47], we have

$$d_1(f) = \max\{d_1(g), d_1(f|_\pi)\}.$$  

Then $d_1(f) = d_1(g)$ if $d_1(f|_\pi) \leq d_1(g)$. Therefore, if $d_1(f|_\pi) \leq d_1(g)$ and KSC holds for $g$, then KSC also holds for $f$ by Lemma 2.2.

Now assume that $f$ is an automorphism and $\dim Y = \dim X - 1$. Another application of the product formula yields that $d_{\dim X}(f) = d_{\dim X - 1}(g)d_1(f|_\pi)$. Since $f$ is an automorphism, $d_{\dim X}(f) = 1$, and so both terms on the right must be 1 as well. So $d_1(f|_\pi) \leq d_1(g)$.  

**Proof of Theorem 1.6.** By [6, Proposition 5.1], we may assume that the Albanese morphism $\pi : X \to A$ is surjective and $\dim A = q(X)$. Notice that $f$ descends to a surjective endomorphism $f_A$ of $A$ by the universal property of the Albanese morphism. If $\dim A = \dim X$, then KSC holds for $(X, f)$ by Lemma 2.2 and Theorem 2.3. If $\dim A = \dim X - 1$, then KSC holds for $(X, f)$ by Theorem 2.3 and Proposition 3.2.  

The following is motivated by [26, Proof of Proposition 1.7].

**Theorem 3.3.** Let $f$ be an automorphism on a normal projective variety $X$ with non-negative Kodaira dimension. If $X$ admits a $f$-equivariant non-constant morphism $\pi : X \to \mathbb{P}^1$, then $X$ does no have any dense orbit.

**Proof.** By [5, Theorem 13.2], there exists a resolution $\varphi : \tilde{X} \to X$ and an automorphism $\tilde{f}$ of $\tilde{X}$ such that $\tilde{f} \circ \varphi = \varphi \circ f$. Then we may assume that $X$ is smooth. Now assume that $f$ descends to an automorphism $g$ of $\mathbb{P}^1$ as $\dim \mathbb{P}^1 = 1$. Let $Z \subset \mathbb{P}^1$ be the locus of points $t$, where the fiber $X_t$ is singular. Then $g(Z) = Z$. Since $Z$ is a finite set, after replacing $f$ by a further iteration, we can assume $g$ fixes $Z$ point-wise. By [49, Theorem
we know that \( Z \) contains at least three points. It follows that \( g \) is the identity since it fixes at least three points of \( \mathbb{P}^1 \). In other words, there exists a rational function on \( X \) that is invariant under some iteration of \( f \), which contradicts the fact that \( X \) has a point with a dense orbit.

**Remark 3.4.**

1. If \( f \) is a surjective endomorphism of a smooth projective variety \( X \) with \( \kappa(X) \geq 0 \) admits a surjective morphism \( \pi : X \to Y \cong \mathbb{P}^1 \), and let \( g : Y \to Y \) be an automorphism such that \( \pi \circ f = g \circ \pi \), then \( g \) is of finite order by [40, Proposition 2.3].

2. In general, we would like to ask whether that \( X \) does not have any dense orbit if \( f \) is a birational self-map of a normal projective variety \( X \) with \( \kappa(X) \geq 0 \) admits a \( f \)-equivariant map \( \pi : X/\text{axisshort/arrowaxisright} \to Z \cong \mathbb{P}^1 \).

**Proposition 3.5.** Let \( \pi : X \dashrightarrow Y \) be a dominant rational map of normal projective varieties with \( 3 = \dim X > \dim Y \geq 1 \). Let \( f : X \to X \) and \( g : Y \to Y \) be surjective endomorphisms such that \( g \circ \pi = \pi \circ f \). Suppose \( f \) is an automorphism. Then to show KSC for \((X, f)\), we only need to assume that \( Y \) is \( \mathbb{P}^1 \) or an elliptic curve. In particular, if \( \kappa(X) \geq 0 \), then we may KSC to the case that \( Y \) is an elliptic curve.

**Proof.** If \( \dim Y = 2 \), then Conjecture 1.1 holds for \((Y, g)\) by [37, Theorem 1.3]. Then Conjecture 1.1 holds for \((X, f)\) by Proposition 3.2. Then we may assume that \( \dim Y = 1 \). If \( g(Y) \geq 2 \) (i.e., \( \kappa(Y) > 0 \), then \( Y \) does no have any dense \( g \)-orbit by Remark 1.2. So Conjecture 1.1 is true for \((X, f)\). Therefore, we may assume that \( Y \) is \( \mathbb{P}^1 \) or an elliptic curve. If \( \kappa(X) \geq 0 \) and \( \pi \) is a morphism, then the proof follows from Theorem 3.3. □

### 4. Automorphisms on Projective Threefolds

Now we quote the definition of a weak Calabi-Yau variety in [15, Definition 2.4].

**Definition 4.1.** A normal projective variety \( X \) is called a *weak Calabi-Yau variety*, if

- \( X \) has only canonical singularities;
- the canonical divisor \( K_X \sim 0 \); and
- the *augmented irregularity* \( \tilde{q}(X) = 0 \).

Here, the augmented irregularity \( \tilde{q}(X) \) of \( X \) is defined as follows:

\[
\tilde{q}(X) = \sup \{ q(Y) : Y \to X \text{ is a finite surjective and étale in codimension one} \}.
\]

The following *special MRC fibration* is due to Nakayama [39].

**Definition 4.2.** [15, Definition 2.10] Given a projective variety \( X \), a dominant rational map \( \pi : X \dashrightarrow Z \) is called the special MRC fibration of \( X \), if it satisfies the following conditions:
(1) The graph $\Gamma_\pi \subseteq X \times Z$ of $\pi$ is equidimensional over $Z$.
(2) The general fibers of $\Gamma_\pi \to Z$ are rationally connected.
(3) $Z$ is a non-uniruled normal projective variety.
(4) If $\pi' : X \to Z'$ is a dominant rational map satisfying (1)-(3), then there is a birational morphism $v : Z' \to Z$ such that $\pi = v \circ \pi'$.

**Proposition 4.3.** [39, Proposition 4.14] Let $\pi : X \to Y$ be a dominant rational map from a projective variety $X$ to a normal projective variety $Y$. Then there exists a normal projective variety $T$ and a birational map $\mu : Y \to T$ satisfying the following conditions:

1. The graph $\gamma_\mu : \Gamma_\mu \to T$ of $\mu \circ \pi$ is equi-dimensional.
2. Let $\mu' : Y \to T'$ be a birational map to another normal projective variety $T'$ such that the graph $\gamma_{\mu' \circ \pi} : \Gamma_{\mu' \circ \pi} \to T'$ of $\mu' \circ \pi$ is equi-dimensional. Then there exists a birational morphism $\nu : T' \to T$ such that $\mu = \nu \circ \mu'$.

We call the composition $\mu \circ \pi : X \to T$ above satisfying Proposition 4.3 (1)-(2) the Chow reduction of $\pi : X \to Y$, which is unique up to isomorphism.

**Theorem 4.4.** Let $\pi : X \to Z$ be the special MRC fibration, and $f$ be an automorphism of $X$. Then there exists a birational morphism $p : W \to X$ and an automorphism $f_W \in \text{Aut}(W)$ and an equi-dimensional surjective morphism $q : W \to Z$ satisfying the following conditions:

1. $W$ is a normal projective variety.
2. A general fiber of $q$ is rationally connected.
3. $W$ admits $f_W$-equivariant fibration over $X$ and $Z$.

**Proof.** It follows from [39, Theorem 4.19] or [15, Lemma 2.11]. □

**Proof of Theorem 1.4.** We first suppose $K_X \sim_\Q 0$. Then by [15, Lemma 2.7], there exists a finite surjective morphism $\pi : \tilde{X} \to X$ and an automorphism $\tilde{f}$ of $\tilde{X}$ such that the following statements hold.

- $X \cong Z \times A$ for a weak Calabi-Yau variety $Z$ and an abelian variety $A$.
- $\dim A = \bar{q}(X)$.
- There are automorphisms $\tilde{f}_Z$ and $\tilde{f}_A$ of $Z$ and $A$ respectively, such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & \tilde{X} & \xrightarrow{\cong} & Z \times A \\
\downarrow{f} & & \downarrow{\tilde{f}} & & \downarrow{\tilde{f}_Z \times \tilde{f}_A} \\
X & \xrightarrow{\pi} & \tilde{X} & \xrightarrow{\cong} & Z \times A.
\end{array}
$$
If \( \tilde{q}(X) = 3 \), then it follows from Theorem 2.3 and Proposition 2.2. If \( \dim Z > 0 \), then \( \dim Z \geq 2 \) since \( \tilde{q}(Z) = 0 \) and \( K_Z \sim 0 \). If \( \dim Z = 2 \), then \( \dim A = 1 \). Consider the natural projection \( \text{pr}_1 : \tilde{X} \to Z \). Then by Propositions 2.2 and 3.5, KSC is true for \((X, f)\). Now assume that \( \dim Z = 3 \) and \( f \) is imprimitive. Then there is a rational map \( \pi : X \dashrightarrow Y \) and a birational map \( g : Y \dashrightarrow Y' \) such that \( \pi \circ f = g \circ \pi \). By Proposition 4.3, there exists a birational morphism \( \mu : Y \to Z \) such that \( \pi' = \mu \circ \pi : X \dashrightarrow Z \) is the Chow reduction of \( \pi \). Then \( f \) descents to an automorphism \( h \) of \( Z \) by Proposition 4.3.

By taking the graph of \( \pi' \), it suffices to consider the case when \( \pi' \) is morphism by Lemma 2.2. By Proposition 3.5, we may assume that \( Y \) is an elliptic curve. This completes the proof of Theorem 1.4 (1) as \( X \) has trivial Albanese.

Now we assume that \( \kappa(X) = -\infty \). Consider the special MRC fibration \( \pi : X \dashrightarrow Z \). By Theorem 4.4 and Lemma 2.2, we may assume that \( \pi : X \to Z \) is \( f \)-equivariant morphism. There are an equi-dimensional surjective morphism \( q : W \to Z \) and the general fiber of \( q \) is rationally connected. If \( \dim Z = 0 \), then \( X \) is rationally connected. Now assume that \( \dim Z > 0 \). Here \( Z \) is non-uniruled. By Proposition 3.5, \( Z \) is an elliptic curve. So \( q(X) > 0 \). Take the \( f \)-equivariant resolution for \((X, f)\), then by Proposition 2.2, we assume that \( f \) is an automorphism of a smooth projective threefold with \( q(X) > 0 \). Therefore, KSC is true for \( f \) by [6, Theorem 1.6(2)].

5. ON PROJECTIVE VARIETIES WITH PICARD NUMBER TWO

An interesting example of projective varieties with Picard number two is a projective bundle \( X \) over a normal projective variety \( Y \) with Picard number \( \rho(Y) = 1 \). Then we ask the following question.

**Question 5.1.** Is KSC true for any surjective endomorphism of a projective bundle \( X \) over a normal projective variety \( Y \) with \( \rho(Y) = 1 \)?

**Remark 5.2.** Question 5.1 is affirmative if \( Y \) is a smooth Fano variety (cf. [25, 26]).

To study KSC on projective varieties with Picard number two, we first give the following result, which is motivated by Sano’s theorem [43] on arithmetic and dynamical degrees.

**Proposition 5.3.** Let \( f \) be a surjective endomorphism of a normal projective variety \( X \) with the Picard number \( \rho(X) = 2 \). Let \( \lambda_1 \) and \( \lambda_2 \) be two eigenvalues of \( f^*|_{\text{NS}_{\mathbb{R}}(X)} \) such that \( |\lambda_1| \geq |\lambda_2| \). To prove KSC is true for \((X, f)\), then we may assume that \( f \) is int-amplified (i.e. \( d_1(f) = \lambda_1 > \lambda_2 > 1 \)) or \( \lambda_2 = 1 \). Moreover, we may assume that \( f^*|_{\text{NS}_{\mathbb{R}}(X)} \) is diagonalizable with positive eigenvalues \( p \) and \( q \) with \( p > q \geq 1 \).

**Proof.** It is known that \( 1 \leq \alpha_f(x) \leq d_1(f) \) by [21, Theorem 4] and [28, Theorem 1.4]. Now we may assume that \( d_1(f) > 1 \). Note that \( |\lambda_1| = d_1(f) \). If \( |\lambda_1| = |\lambda_2| > 1 \) or
$|\lambda_1| > 1 > |\lambda_2|$, then the arithmetic degree $\alpha_f(x) = d_1(f)$ by [43, Theorem 1.1]. Then we may assume that $|\lambda_1| > |\lambda_2| \geq 1$. After replacing $f$ by $f^2$, $f^*|_{\text{NS}_R(X)}$ is diagonalizable with positive eigenvalues $d_1(f)$ and $\lambda_2$ with $d_1(f) > \lambda_2 \geq 1$ as $\rho(X) = 2$. If $\lambda_2 > 1$, then $f$ is int-amplified by [29, Theorem 1.1]. This completes the proof of Proposition 5.3. □

**Corollary 5.4.** Let $f$ be a surjective endomorphism of a weak Calabi-Yau variety $X$ with $\rho(X) = 2$. Let $\lambda_1$ and $\lambda_2$ be two eigenvalues of $f^*|_{\text{NS}_R(X)}$ such that $|\lambda_1| \geq |\lambda_2|$. To prove KSC is true for $(X, f)$, then we may assume that $\lambda_2 = 1$. Moreover, we may assume that $f^*|_{\text{NS}_R(X)}$ is diagonalizable with positive eigenvalues $p > q \geq 1$.

**Proof.** Since $K_X$ is pseduoeffective, the proof follows from Proposition 5.3, [29, Theorem 1.9] and Theorem 2.3. □

The following is due to [37, Lemma 10.4].

**Proposition 5.5.** Let $f : X \to X$ be a surjective endomorphism of a normal projective variety $X$. Suppose $f^*|_{\text{N}_1(X)}$ is diagonalizable with positive eigenvalues $p \geq q \geq 1$, no other eigenvalues. Let $H$ be an ample Cartier divisor. Then $H = A + B$ for some nef $\mathbb{R}$-Cartier divisors $A$ and $B$ such that $f^*A \equiv pA$ and $f^*B \equiv qB$.

**Proof.** If $p = q$, then $f^*|_{\text{N}_1(X)} = p \text{id}$ and we may take $A = H$ and $B = 0$. Assume $p > q$. Let $\varphi := f^*|_{\text{N}_1(X)}$. Let $A = \lim_{i \to +\infty} \varphi^i(H)/p^i$ and $B = \lim_{i \to +\infty} q^i\varphi^{-i}(H)$. Since $\varphi$ is diagonalizable with only eigenvalues $p$ and $q$, the above limits are $\mathbb{R}$-Cartier and $H = A + B$. It is clear that $\varphi(A) = pA$ and $\varphi(B) = qB$. Note that $A$ and $B$ are limits of ample divisors. So $A$ and $B$ are nef. □

**Remark 5.6.** To prove KSC is true for projective varieties with Picard number two, we wish construct a canonical height function $\hat{h}_D$ associated with some divisors $D$ (e.g. the lead real eigendivisor of $(X, f)$)

$$\hat{h}_D(x) = \lim_{n \to \infty} \frac{h_D(f^n(x))}{d_1(f)^{\dim X}}$$

which is positive at every point $p \in X(\overline{\mathbb{Q}})$ with a Zariski dense $f$-orbit by using $A$ and $B$ in Proposition 5.5.

Now we quote KSC for the TIR case on normal projective varieties admitting an int-amplified endomorphism in [30] as follows.

**Theorem 5.7.** Let $f$ be a surjective endomorphism of a normal projective variety $X$ admitting an int-amplified endomorphism. To show KSC holds for $f$, it suffices to show KSC is true for that $(X, f)$ is the TIR case as follows:

1. $\kappa(X, -K_X) = 0$;
(2) \( f^*D_1 = d_1(f)D_1 \) for some reduced effective Weil divisor \( D_1 \sim \mathbb{Q} - K_X \); 
(3) There is an \( f \)-equivariant Fano contraction \( \pi : X \to Y \) with \( d_1(f) > d_1(f_Y)(\geq 1) \), where \( f_Y \) is induced by \( f \) on \( Y \).

**Proof.** It follows from [37, Theorem 1.7] and [30, Lemma 6.4]. □

As a start to addressing Question 5.1 when \( \dim Y > 1 \), we show the following result.

**Theorem 5.8.** Let \( f \) be a surjective endomorphism of a projective bundle \( X \) over a normal projective variety \( Y \) with \( \rho(Y) = 1, \pi : X \to Y \) the projection. Replacing \( f \) with its iterate, \( f \) descends to \( g \) on \( Y \). To prove KSC is true for \((X, f)\), we may assume that one of the following case holds:

1. \( d_1(f) = d_1(f|_\pi) > d_1(g) > 1 \), and so \( f \) is int-amplified.
2. \( d_1(f) = d_1(f|_\pi) > d_1(g) = 1 \), and the morphisms between fibers of \( \pi \) induced by \( f \) are not isomorphism.

Notice that the degree of \( f \) is greater than one.

**Proof.** Replacing \( f \) with its iterate, by [25, Lemma 2.3] we may assume that \( f \) induces an endomorphism \( g : Y \to Y \) such that \( g \circ \pi = \pi \circ f \) (cf. discussion before [1, Theorem 2]). Since \( \rho(Y) = 1 \), \( g \) is polarized and KSC is true for \( g \) by [21, Theorem 5]. Therefore, we may assume \( d_1(f) = d_1(f|_\pi) > d_1(g) \) by Lemma 2.2 and the product formula on dynamical degrees. Note that \( \text{NS}_R(X) = \mathbb{R}O_X(1) \oplus \pi^*\text{NS}_R(Y) \). If \( d_1(g) > 1 \), then the eigenvalues of \( f^* : \text{NS}_R(X) \to \text{NS}_R(X) \) are \( d_1(f) \) and \( d_1(g) \) and, then they have modulus larger than one. Thus \( f \) is int-amplified by [29, Theorem 1.1]. As a result, \( \deg f > 1 \) by [29, Lemma 3.7]. Now Suppose \( d_1(f) > d_1(g) = 1 \). Then the morphisms between fibers of \( \pi \) induced by \( f \) are not isomorphism. Indeed, let \( \pi^{-1}(y) \) be a closed fiber. Since \( \text{NS}_R(X) = \mathbb{R}O_X(1) \oplus \pi^*\text{NS}_R(Y) \cong \mathbb{R}^2 \) and \( f^* \) fixes \( \pi^*\text{NS}_R(Y) \), we see that \( f^*O_X(1) = d_1(f)O_X(1) + \pi^*D \) for some divisor \( D \) on \( Y \).

\[
(f_*[\pi^{-1}(y)] \cdot O_X(1)^{\dim X - \dim Y}) = ([\pi^{-1}(y)] \cdot f_*O_X(1)^{\dim X - \dim Y}) \\
= ([\pi^{-1}(y)] \cdot (d_1(f)O_X(1))^{\dim X - \dim Y}) \\
= d_1(f)^{\dim X - \dim Y}.
\]

This shows that the degree of the morphism \( f : \pi^{-1}(y) \to \pi^{-1}(g(y)) \) is \( d_1(f)^{\dim X - \dim Y} \) which is greater than one. So we also have \( \deg f > 1 \). □

**Remark 5.9.** The two reduced cases in Theorem 5.8 are very difficult.

1. In the first case, it suffices to show KSC for the TIR case by Theorem 5.7.
2. In the second case we need to study the relations of arithmetic degrees and relatively dynamical degrees on projective bundles, but the morphisms on fibers
induced by $f$ may not be an endomorphism. In general, let $\pi : X \to Y$ be a surjective morphism and a surjective endomorphism $f$ of $X$ descending to a surjective endomorphism $g$ on $Y$. Then a question is asked as follows.

**Question 5.10.** Let $\pi : X \to Y$ be a surjective morphism between normal projective varieties $X$ and $Y$ with $\dim X > \dim Y > 0$ and a surjective endomorphism $f$ of $X$ descents to $g$ on $Y$. Is KSC true for $(X, f)$ when $d_1(f) > d_1(g)$?

**Remark 5.11.** When $\dim Y = 1$, Question 5.10 is affirmative if $\kappa(X) \geq 0$, $Y = \mathbb{P}^1$ and $f$ is an automorphism (cf. Theorem 3.3).

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