PLANARITY IN NONCOMMUTATIVE GAUGE THEORIES *

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Abstract

Planarity was introduced by ’t Hooft in his topological classification of diagrams in the large-N limit of $U(N)$ gauge theories. Planarity also occurs in noncommutative field theories where amplitudes possess invariance only under cyclic permutations, a feature inherited from the parent string theory. In non-commutative gauge theories both kinds of planarity merge in a context which turns out to be particularly intriguing in the two-dimensional case where gauge invariant correlators can be explicitly computed.

1. INTRODUCTION

Planarity is an important concept in usual Quantum Field Theories (QFT), especially in gauge theories (GT), after the seminal work by ’t Hooft [1], who showed that in $U(N)$ GT a topological classification of diagrams is possible in the large-N limit at a fixed value of $g^2 N$, $g$ being the coupling constant. Such a classification leads to a power expansion of amplitudes in the variable $1/N$, the planar diagrams providing the leading contribution. He also introduced a quite convenient double-line notation for diagrams, each (oriented) line being associated to a fundamental representation, in which topology turns out to be particularly transparent.

On a different side, in recent years interest has grown for QFT defined on noncommuting space-time variables, mainly triggered by the work of Seiberg and Witten [2], who related them to suitable particular limits of a string theory in the presence of a constant background. In such theories amplitudes are invariant only with respect to cyclic permutations of external momenta, a feature inherited from the parent string. In a perturbative formulation the only difference with the usual Feynman expressions is the occurrence in the vertices of a phase factor (Moyal phase), depending on the momenta and on the noncommutativity parameter $\theta$. Such a phase affects non-planar diagrams, thereby leading again to a topological classification [3]. In turn the presence of a Moyal phase in general provides a damping factor in the large-$\theta$ limit.

Noncommutative $U(N)$ GT exhibits both kinds of planarities; it is our purpose to further elaborate and elucidate this point. In so doing we follow essentially ref.[4]. The merging of space-time and “internal” symmetries does not come out unexpected in such a context; as a matter of fact, when non-commutative GT are represented in a separable Hilbert space, the gauge group embodying the mentioned symmetries turns out to be the set of all unitary operators of the kind $U = I + K$, $K$ being suitable compact operators [5] 1. To clarify this merging is one of the most fascinating and intriguing challenges in our opinion.

In Sect.2 essential definitions and notations of noncommutative GT are presented; observable quantities, in particular the open Wilson lines, which are the concrete tool of our subsequent investigation, are introduced. Qualitative consequences of planarity for open Wilson line correlators in four (space-time) dimensions are also exhibited. In Sect.3 correlators are explicitly computed in two dimensions,

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1 This does not contradict the well known Coleman-Mandula theorem, as the required hypotheses are not fulfilled in this instance. One of us (A.B.) wishes to thank T. Heinzl for calling his attention on this point.
both by a perturbative and by a non-perturbative approach, in a suitable region of the external variables (the “planar” phase of the theory), and concluding remarks are drawn.

2. Observables in noncommutative gauge theories

2.1 Notations and definitions

Noncommutativity of $D$-dimensional Minkowski space-time is encoded in a real antisymmetric matrix $\theta^{\mu\nu}$:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad \mu, \nu = 0, \ldots, D - 1,$$

and a $\star$-product of two fields $\phi_1(x)$ and $\phi_2(y)$ can be defined by means of Weyl symbols

$$\phi_1(x) \star \phi_2(y) = \int \frac{d^D p d^D q}{(2\pi)^{2D}} \exp \left[-\frac{i}{2} p_\mu \theta^{\mu\nu} q_\nu \right] \exp[i(px + qy)] \tilde{\phi}_1(p) \tilde{\phi}_2(q).$$

Then noncommutative theories are most easily formulated by replacing the usual multiplication of fields in the Lagrangian with the $\star$-product. The resulting action makes them obviously non-local.

The classical action of the $U(N)$ Yang-Mills theory in a noncommuting space-time is

$$S = -\frac{1}{2} \int d^D x Tr F_{\mu\nu} \star F^{\mu\nu},$$

where the field strength $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig(A_\mu \star A_\nu - A_\nu \star A_\mu)$$

and $A_\mu = A^a_\mu T^a$ is a $N \times N$ matrix, with $T^a$ normalized as follows: $Tr T^a T^b = \frac{1}{2} \delta^{ab}$, $a, b$ denoting $U(N)$ indices.

The action eq. (3) is invariant under infinitesimal $U(N)$ noncommutative gauge transformations

$$\delta_\lambda A_\mu = \partial_\mu \lambda - ig(A_\mu \star \lambda - \lambda \star A_\mu).$$

As noticed in [6], under this transformation the operator $Tr F^2(x)$ is not left invariant

$$Tr F^2(x) \longrightarrow Tr U(x) \star F^2(x) \star U^\dagger(x),$$

with $U(x) = \exp(x (ig \lambda(x)))$. To recover a gauge invariant operator, one has to integrate over the entire space-time, since $\star$-products inside integrals can be cyclically permuted.

As a consequence, gauge invariance in this case (star-gauge invariance) entails an integration over space-time variables and the possibility of having local probes is lost.

A Wilson line of length $l$ can be defined by means of the Moyal product as [6]

$$\Omega_\star [x, C] = P_\star \exp \left(i g \int_0^l A_\mu(x + \zeta(\sigma)) d\zeta^\mu(\sigma) \right),$$

where $C$ is the curve parameterized by $\zeta(\sigma)$, with $0 \leq \sigma \leq l$, $\zeta(0) = 0$, $\zeta(1) = l$, and $P_\star$ denotes noncommutative path ordering along $\zeta(\sigma)$ from right to left with respect to increasing $\sigma$ of $\star$-products of functions. The Wilson line is not invariant under a gauge transformation

$$\Omega_\star [x, C] \longrightarrow U(x) \star \Omega_\star [x, C] \star U^\dagger(x + l),$$

The following operator

$$W(p, C) = \int d^D x \ Tr \Omega_\star [x, C] \star e^{ipx},$$

is
turns out to be invariant provided $C$ satisfies the condition
\[ l^\nu = p_\mu \theta^{\mu\nu} \] (10)
(the Wilson line extends in the direction transverse to the momentum). The particular case $p_\mu = 0$ corresponds to a closed loop.

For simplicity in the following only straight lines will be considered. Then one can easily realize that any local operator $\mathcal{O}(x)$ in ordinary gauge theories admits a noncommutative generalization
\[ \tilde{\mathcal{O}}(p) = \text{Tr} \int d^Dx \mathcal{O}(x) \star \Omega^+ [x, C] \star e^{ipx}, \] (11)
each of the $\tilde{\mathcal{O}}(p)$’s being a genuinely different operator at different momentum.

Remarkably, owing to eq. (10), at large values of $|p|$, gauge invariance requires that the length of the Wilson line becomes large. This feature can be interpreted as a manifestation of the UV-IR mixing phenomenon.

2.2 Two open-line correlator
An interesting quantity to study is the two-point function $\langle W(p)W^+(p) \rangle$, where $W(p)$ has been defined via eqs. (7), (9). It represents the correlation function of two straight parallel Wilson lines of equal length, each carrying a transverse momentum $p$. In four dimensions such a correlator was investigated in [6], according to the perturbative expansion
\[ W(p) = \sum_{j=0}^{\infty} (ig)^j \int d^4x \int_{\zeta_j > \zeta_{j-1} > \ldots > \zeta_1} [d\zeta] \text{Tr} A(x + \zeta_1) \star \ldots \star A(x + \zeta_j) \star e^{ipx} \]
\[ W^+(p) = \sum_{j=0}^{\infty} (-ig)^j \int d^4x \int_{\zeta'_j > \zeta'_{j-1} > \ldots > \zeta'_1} [d\zeta'] \text{Tr} A(x + \zeta'_1) \star \ldots \star A(x + \zeta'_j) \star e^{-ipx}. \] (12)

By resumming ladder diagrams, the correlator was found to grow exponentially at large momenta
\[ \langle W(p)W^+(p) \rangle \propto \exp \sqrt{\frac{g^2N|p\theta||p|}{4\pi}}. \] (13)
This was correctly interpreted in [6] as a coherence effect, increasing with the length of the (parallel) lines. Ladder diagrams are leading at large $N$ and their Moyal phases cancel, so their $\theta$ dependence only occurs in the length of the line. They are planar according to the colour factor, but according to both “colour” and “geometry” criteria.

Imagine we now perform a cyclic permutation on one of the lines in a ladder diagram; the colour factor is obviously unchanged (each line entails an independent trace over colour matrices). As a consequence the diagram remains leading as far as colour is concerned. However the $\theta$-dependence is not insensitive to such a permutation: Moyal phases double instead of cancelling, producing, on an intuitive basis, a damping for large values of $\theta$. This is indeed confirmed by explicit low-order calculations. Diagrams which would be planar according to colour, do not according to geometry. Planarity in four dimensions just means “ladder”.

3. TWO-LINE CORRELATOR IN TWO DIMENSIONS
3.1 A perturbative approach
In two dimensions the situation is quite different: noncommutativity involves the time variable, but the Lorentz symmetry is not violated owing to the tensorial character of $\theta^{\mu\nu} = \theta e^{\mu\nu}$. Invariance under
area-preserving diffeomorphisms is preserved as well. If we choose the light-cone gauge, perturbative calculations are greatly simplified, thanks to the decoupling of Faddeev-Popov ghosts and to the vanishing of the vector vertices. It turns out that in all diagrams contributing to the line correlators, which are planar according to the 't Hooft’s large-$N$ limit, $\theta$-dependent phases resulting from non commutativity play no role. They are planar also according to “geometry”. This feature, which is characteristic of the theory in two dimensions, might be related to its invariance under area-preserving diffeomorphisms.

On the other hand, in two dimensions a remarkable symmetry, the Morita equivalence, allows the mapping of open Wilson lines on a noncommutative torus onto closed Wilson loops winding on a dual commutative torus [7]. In turn, in a commutative setting, Wilson loop correlations can be obtained by geometrical techniques [8]; this opens the possibility of confronting perturbative calculations with non-perturbative solutions, provided a common kinematical region of validity is found for both approaches.

We quantize the theory in the light-cone gauge $A_- = 0$ at equal times, the free propagator having the following causal expression (WML prescription)

$$D_{++}^{WML}(x) = \frac{1}{2\pi} \frac{x^-}{-x^+ + i\epsilon x^-},$$

first proposed by T.T. Wu [9]. This propagator is nothing but the restriction in two dimensions of the expression proposed by S. Mandelstam and G. Leibbrandt [10] in four dimensions and derived by means of a canonical quantization in [11]. It allows a smooth transition to an Euclidean formulation, where momentum integrals are performed by means of a “symmetric integration” [9].

We go back to eq. (12) and, with no loss of generality thanks to the persisting boost invariance, we choose the path $C$ stretching along $x^0$, so that $p$ points in the $x^1$ direction.

We then contract the $A$’s in such a way that the resulting diagram is of leading order in $N$, which yields, according to eq. (14), at a fixed perturbative order $(\alpha^2)^n$

$$(-1)^{n-k} \left( \frac{N}{4\pi} \right)^n \int [d\sigma] [d\sigma'] \int d^2x e^{ipx} \prod_{j=1}^{k} \frac{x^0 - f_j(\sigma, \sigma', \theta_p) - x^1}{-x^0 - f_j(\sigma, \sigma', \theta_p) - x^1}, \quad n \geq 1,$$

where $k$ is the number of propagators connecting the two lines and $f_j(\sigma, \sigma', \theta_p)$ is a linear function of its variables depending on the topology; the integration region for the $2n$ geometric variables is understood and the phase factors containing the noncommutativity parameter have been absorbed in the function $f_j(\sigma, \sigma', \theta_p)$.

We stress that, remarkably, factorization of propagators in coordinate variables occurs just in those diagrams which are dominant at large $N$. This feature in turn makes $\theta$-dependence trivial, as it was explicitly shown in [4], since it intervenes just through the length of the line $l$.

Surprisingly enough, it will turn out that all integrals will give the same result, no matter what the function $f_j(\sigma, \sigma', \theta_p)$ is, i.e. no matter what topology we choose in the set of planar diagrams. This finding is a direct consequence of the integration over the world volume $d^2x$, required by noncommutative gauge invariance, and of the orthogonality of the momentum with respect to the direction of the open lines. In so doing the $\theta$-dependence is washed out apart from its occurrence in $f^0$.

In order to provide a correct formulation of the theory, continuation to Euclidean variables is required: $x^0 \rightarrow i\omega_2$; we recall that, to keep the basic algebra unchanged, the noncommutativity parameter $\theta$ has also to be simultaneously continued to an imaginary value: $\theta \rightarrow i\theta$.

A symmetric integration [9] then provides the natural regularization in eq. (15)

$$\int d^2x e^{-ipx} \prod_{j=1}^{k} \frac{x_1 + i(f_j(\sigma, \sigma', \theta_p) + x_2)}{x_1 - i(f_j(\sigma, \sigma', \theta_p) + x_2)} = (-1)^k \frac{4\pi k}{p^2}.$$
Hence the integration over the geometrical variables in eq. (15) is straightforwardly carried out and yields
\[
\frac{4\pi k}{p^2} \left( \frac{Nl^2}{4\pi} \right)^n \frac{1}{n_1! n_2!},
\]  
(17)

where \( l = |p\theta| \) is the total length of the line and \( n_1, n_2 \) are the number of legs stretching out of the first and the second line, respectively (\( n_1 + n_2 = 2n \)).

Eq. (17) displays a trivial dependence on the topology of the graph, the only remnant being \( n_1! n_2! \) in the denominator. Thus, although resumming even only leading contributions in \( N \) may have seemed a formidable task when we started, it has now become feasible, provided the exact number of different configurations with fixed \( n_1, n_2 \) is known. A careful counting of all such configurations has been performed in [4]. By eventually completely resumming the perturbative series with the appropriate weight factors included, we obtain the expression

\[
\langle W(p) W^\dagger(p) \rangle = \frac{4\pi \tau^2}{p^2} \left[ I_0(2\tau) + \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz}{8\pi i} \frac{\sqrt{z^2 - 4} - 1}{\sqrt{z^2 - 4}} e^{2\tau z^2} \right]
\]  
(18)

\[
\times \left( \sqrt{1 + \left( \frac{z - \sqrt{z^2 - 4}}{2} \right)^2 - 1} - \frac{1}{2} \left( \frac{z - \sqrt{z^2 - 4}}{2} \right)^2 \right) \left( z + \sqrt{z^2 - 4} + 2\tau \right)
\]

\[
+ \int_{\nu_1-i\infty}^{\nu_1+i\infty} \int_{\nu_2-i\infty}^{\nu_2+i\infty} \frac{dz}{2\pi i} \frac{dz}{\left( 2\pi i \right)^2} e^{2(z+w)\tau^2 z^3 w^3 (1+zw) \left( 1 + \frac{2}{z^2 - 1} + \frac{4}{z^2} \right)}
\]

\[
\times \left( 1 + \frac{2}{w^2} - \sqrt{1 + \frac{4}{w^2}} \right),
\]

where \( \tau = \sqrt{\frac{g^2 N l^2}{4\pi}} \), \( \nu_1, \nu_2 > 1 \) and \( \gamma > 2 \).

One can realize that, at large \( \tau \), the term with the double integration dominates and \( \langle W(p) W^\dagger(p) \rangle \) increases like \( \exp(2\tau) = \exp(\sqrt{g^2 N l^2 / \pi}) \), disregarding a (small) power correction. As expected, the correlator depends on the 't Hooft coupling \( \sqrt{g^2 N} \); what is remarkable is that its asymptotics is an exponential linearly increasing with the line momentum \( |p| \). This is reminiscent of what was found in [6] for its four-dimensional analog.

It is tempting to argue that for a general correlator of an arbitrary number of open parallel lines in two dimensions, \( \theta \)-dependence is trivial, intervening only through the length of the lines, just in those diagrams which are dominant at large \( N \). Indeed, in ref. [12] this statement was proved at least for the correlators of three parallel Wilson lines. In the large-\( N \) planar limit, the perturbative series was resummed. Although multiple line correlators in a generic configuration are not expected to increase with the length of the lines on the basis of the estimate in [6], it was found instead they keep increasing, when the lines are parallel, at the same rate as the two-line correlator, so that the normalized three-line correlator is still increasing like a (small) power of its argument. Actually, the interference effect generated by lines with the same orientation is overwhelmed by the coherent increase due to parallelism of lines with opposite orientation.

### 3.2 An approach based on the Morita equivalence

We turn now our attention to a non-perturbative derivation of the noncommutative Wilson lines correlator, and see how it compares with eq. (18) at large \( l \). To this regard, it is worth noticing that although eq. (18) follows from a perturbative analysis, having resummed all orders, it holds also at large \( g^2 N \).

When both coordinates are compactified to form a torus, a remarkable symmetry, called Morita equivalence [7], relates different noncommutative gauge theories living on different noncommutative
tori: the duality group $SO(2, 2, \mathbb{Z})$ has an $SL(2, \mathbb{Z})$ subgroup which acts as follows

$$\begin{pmatrix} m' \\ N' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ N \end{pmatrix}, \quad \Theta' = \frac{c + d\Theta}{a + b\Theta} \tag{19}$$

$$(R')^2 = R^2(a + b\Theta)^2, \quad (g')^2 = g^2[a + b\Theta], \quad \tilde{\Phi}' = (a + b\Theta)^2\tilde{\Phi} - b(a + b\Theta), \tag{20}$$

where $\Theta \equiv \theta/(2\pi R^2)$. $\tilde{\Phi} \equiv 2\pi R^2\Phi$, $\Phi$ being a background connection and $R$ the radius of the torus, which, for simplicity, we assume to be square. The first entry $m$ denotes the magnetic flux, while $N$ characterizes the gauge group $U(N)$. It is not restrictive to consider the quantities $m$ and $\theta$ to be positive. The parameters of the transformation are integers, constrained by the condition $ad - bc = 1$.

The map in the equations above is flexible enough to allow for a commutative theory on the second torus by choosing $\Theta' = 0$. As a consequence, the parameter $d$ will be set equal to $-c/\Theta$, $\Theta$ being a suitable rational quantity. In the sequel, for notational convenience, all the primed quantities will be affected the subscript $c (N' \equiv N_c, m' \equiv m_c, ...)$.

We are eventually interested in a noncommutative theory defined on a plane $(R \to \infty)$, with a trivial first Chern class ($m = 0, \Phi = 0$) and a gauge group $U(N)$ with a large $N$, since we want to establish a comparison with the perturbative approach of the previous subsection.

We start by considering the action

$$S = \frac{1}{4g_c^2} \int d^2x \, \text{Tr} \left[ \left( F_{\mu\nu} - \frac{m_c}{2\pi R_c^2 N_c} \epsilon_{\mu\nu} \right) \left( F_{\mu\nu} - \frac{m_c}{2\pi R_c^2 N_c} \epsilon_{\mu\nu} \right) \right], \tag{21}$$

where the explicit expression for the background connection $\Phi_c = -\frac{m}{2\pi R_c^2 N_c} I$ has been introduced.

The formula for the partition function on a torus reads [8]

$$Z = \sum_{R} \exp \left[ -\frac{A}{2} C_2(R) \right], \tag{22}$$

$C_2$ being the second Casimir operator in the representation $R$ and $A = 4\pi^2(g_c R_c)^2$.

After performing a harmonic analysis, retaining only the contribution of the $m_c$-th sector and cancelling the $U(1)$ contribution against the background connection, we obtain the final expression

$$Z = \sqrt{\frac{2\pi}{AN_c N_c!}} \sum_{n, \neq n_j} \exp \left[ - \frac{4}{2} \sum_{i=1}^{N_c} n_i^2 - \frac{1}{N_c} \left( \sum_{i=1}^{N_c} n_i \right)^4 \right] \tag{23}$$

$$\times \int_{0}^{2\pi} \frac{d\alpha}{\sqrt{\pi}} \exp \left[ - \left( \alpha - \frac{2\pi}{N_c} \sum_{i=1}^{N_c} n_i \right)^2 - 2\pi i m_c \left( \frac{N_c - 1}{2} - \frac{1}{N_c} \sum_{i=1}^{N_c} n_i \right) \right].$$

Now we turn our attention to the correlation function of two straight parallel Wilson lines of equal length, lying on the noncommutative torus without winding around it, each carrying a transverse momentum $p$. The noncommutative torus will eventually be decompactified by sending its radius $R \to \infty$.

On the noncommutative torus, with the line $C$ stretching along $x_2$, we have the expression

$$W(k, C) = \frac{1}{4\pi^2 R^2} \int_{0}^{2\pi R} \int d^2x \, \text{Tr} \Omega(x, C) \ast \exp(ikx_1/R), \tag{24}$$

where $k$ is the integer associated to the transverse momentum $p = k/R$. The no-winding condition entails the constraint $\theta k < 2\pi R^2$, namely $l = p\theta < 2\pi R$, $l$ being the total length of the straight line. $W(k)$ is normalized according to $W(0) = 1$. 
Now we exploit again the Morita equivalence in order to map the open Wilson line on the noncommutative torus on a closed Polyakov loop of the ordinary Yang-Mills theory winding $k$ times around the commutative torus in the $x_2$ direction
\[ W(k) = W_r(k), \]
\[ W_r(k) = \frac{1}{4\pi^2 R_c^2} \int_0^{2\pi R_c} d^2x \frac{1}{N_c} \text{Tr} \left[ \Omega_r^{(k)}(x) \right]. \]

The trace is to be taken in the fundamental representation of $U(N_c)$ and $\Omega_r^{(k)}(x)$ is the holonomy of the closed path. This holonomy is to be computed in the flux sector $m_c$.

The correlation function of two straight parallel open Wilson lines reads
\[ \mathcal{W}_2(k) \equiv \langle W(k)W(-k) \rangle = \frac{1}{2\pi R_c} \int_0^{2\pi R_c} dx < \frac{1}{N_c} \text{Tr} \left[ \Omega_r^{(k)}(x) \right] \frac{1}{N_c} \text{Tr} \left[ \Omega_r^{(-k)}(0) \right] >. \]

By repeating the procedure we have followed in computing the partition function, keeping again the projection onto the $m_c$ sector in the decomposition $U(N_c) = U(1) \times SU(N_c)/Z_{N_c}$ and subtracting the classical background, we get [13]
\begin{equation}
\frac{1}{N_c^2} < \text{Tr} \left[ \Omega_r^{(k)}(x) \right] \text{Tr} \left[ \Omega_r^{(0)}(0) \right] > = \frac{1}{2 N_c!} \sqrt{\frac{2\pi}{AN_c}} \exp \left[ -\frac{k^2 x A}{4\pi R_c} \left( 1 - \frac{x}{2\pi R_c N_c} \right) \right] \times \sum_{n_i \neq n_j} \exp \left[ -\frac{A}{2} \left( \sum_{i=1}^{N_c} n_i^2 - \frac{1}{N_c} \sum_{i=1}^{N_c} n_i \right)^2 \right] \frac{1}{N_c} \sum_{j=1}^{N_c} \exp \left[ -\frac{x k A}{2\pi R_c} \left( n_j - \frac{1}{N_c} \sum_{i=1}^{N_c} n_i \right) \right] \times \int_0^{2\pi} \frac{d\alpha}{\sqrt{\pi}} \exp \left[ -\left( \alpha - \frac{2\pi}{N_c} \sum_{i=1}^{N_c} n_i \right)^2 - 2\pi im_c \left( \frac{N_c - 1}{2} - \frac{1}{N_c} \sum_{i=1}^{N_c} n_i \right) \right].
\end{equation}

Eqs. (23), (27) entail $N_c$ sums over the different integers $n_i$ which can take any value between $-\infty$ and $+\infty$. As a consequence of the $SU(N_c)/Z_{N_c}$ symmetry, those equations are manifestly invariant under a simultaneous shift of all the $n_i$ by an integer.

We start by considering large $N$ values in order to comply with our perturbative treatment; this forces even larger values for $N_c$ [4].

Obviously, plenty of different configurations are possible and to sum over all of them is beyond reach. We are therefore seeking for configurations which may be dominant in particular physical regimes. In a recent paper [14], three basic different regimes have been presented for a scalar noncommutative theory in two dimensions, when approximated by means of a $M \times M$ matrix model. Three different phases (disordered, planar and GMS [15]) are possible, according to the behaviour of the noncommutativity parameter $\theta$ with respect to the integer $M$ which is to be sent eventually to $\infty$ ($\theta \sim M^\nu$ with $\nu < 1$, $\nu = 1$, $\nu > 1$, respectively). This integer in turn is related to a large distance cutoff $L$ of the theory, which can be identified in our case with the length of the side of the square torus. Moreover in a $U(N)$ gauge theory, $M$ cannot be smaller than $N$.

Eqs. (23), (27) exhibit a Gaussian damping with respect to the “occupation numbers” $n_i$, which suggests that a kind of saddle-point approximation may be feasible. The most favoured configurations are those with minimal fluctuations. This happens when the integers $n_i$ assume adjacent values.

In the saddle-point approximation, for large values of $N_c$, we eventually get [4]
\begin{equation}
\mathcal{W}_2(k) \equiv \langle W(k)W(-k) \rangle \simeq \exp \left[ \frac{1}{2} \frac{1}{N_c} \left( N_c - 1 - |k| \right) \right] - 1 - \frac{1}{2} N_c |k| \left( N_c - 1 - |k| \right).
\end{equation}
Remarkably, when $|k| < N_c - 1$, we find a correlation function exponentially increasing with $|k|$, in qualitative agreement with an analogous finding in [6] and with our perturbative result. In order to reach a quantitative agreement, a fine tuning of the exponents is possible and entails a relation of the kind $\theta \sim R^\nu$ with $\nu \leq 1$ (see [4]).

If we remember that $R$ is the natural cutoff of our formulation, the condition $\theta \sim R^\nu$ with $\nu \leq 1$ is reminiscent of the analogous condition in [14] with respect to the cutoff $M$ (or the torus side length $L$), related to the dimension of their matrix model. Actually $\nu < 1$ describes the disordered phase, where quantum effects are dominant, while $\nu = 1$ is a border-line value, related to the so-called planar phase. The values $\nu > 1$ would correspond to the GMS phase which is inaccessible to our treatment.

As a final remark we notice that Eq. (28) changes dramatically when $|k| > N_c - 1$, strongly deviating from the perturbative result and thus possibly suggesting the onset of a new phase. Nevertheless, we ought to recall that in this region the saddle-point approximation we adopted no longer holds [4].

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