A Generalized Parameter Imbedding Method

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ABSTRACT

The parameter imbedding method for the Fredholm equation converts it into an initial value problem in its parameter $\lambda$. We establish the method for general operator equations of the form $[I + \hat{f}(\lambda)]\psi = \phi$. It is particularly useful for studying spontaneous symmetry breaking problems, such as contained in the nonlinear Schwinger-Dyson equation.
I. Introduction

A very useful technique in the solution of the Fredholm integral equation is the parameter imbedding method. Consider the equation

$$\psi(x) = \lambda \int K(x,y)\psi(y)dy + \phi(x). \quad (1)$$

It is well-known that for $\phi = 0$, the equation has nontrivial solution only for certain eigenvalues $\lambda = \lambda_i$, and that for $\lambda \neq \lambda_i$, the solution is given by

$$\psi(x) = \phi(x) + \lambda \int R(x,y,\lambda)\phi(y)dy, \quad (2)$$

where the resolvent function can be written in the form

$$R(x, y, \lambda) = D(x, y, \lambda)/d(\lambda). \quad (3)$$

The parameter imbedding method consists of a pair of differential equations relating $D(x, y, \lambda)$ and $d(\lambda)$,

$$d'(\lambda) = \frac{d}{d\lambda}d(\lambda) = - \int D(x, x, \lambda)dx \quad (4)$$

$$\frac{d}{d\lambda}D(x, y, \lambda) = D(x, y, \lambda)\frac{d'(\lambda)}{d(\lambda)} + \frac{1}{d(\lambda)} \int D(x, z, \lambda)D(z, y, \lambda)dz, \quad (5)$$

with the initial conditions

$$d(0) = 1, \quad (6)$$

$$D(x, y, 0) = K(x, y). \quad (7)$$

These equations can be integrated numerically with respect to $\lambda$, yielding the solution to the original integral equation. Note that the eigenvalues of the original equation are just the zeroes of $d(\lambda)$, also, the eigenfunctions are proportional to $D(x, y^*, \lambda)$, for any fixed value of $y = y^*$. The fact that $d(\lambda)$ and $D(x, y, \lambda)$ are analytic facilitate the numerical solutions greatly. Thus, the parameter imbedding method turns the original integral equation, with variable $x$ and $y$, into an initial value problem with variable $\lambda$. The latter is often much simpler and explicit solutions can be readily obtained.
For a practical application, the reader is referred to the solutions of the BS equation obtained earlier.\(^2\)

However, for a number of problems in mathematical physics\(^3\), one encounters integral equations which are not of the Fredholm type. For instance, a question of considerable current interest concerns systems which exhibit spontaneous symmetry breaking. They are often described by nonlinear, Hammerstein equations which are of the form

\[
\psi(x, \lambda) = \lambda \int K(x, y) F(y, \psi(y, \lambda)) dy,
\]

where \(F\) is a nonlinear function of \(\psi\). The dependence on the parameter \(\lambda\) is thus rather complex. In fact, it typically has bifurcation solutions which signal the onset of spontaneous symmetry breaking. One may attempt to solve these equations by linearization and then apply the parameter imbedding method. However, because of the complicated \(\lambda\) dependence, the imbedding method for Fredholm equation can not be used.

The purpose of this paper is to find a generalization of the parameter imbedding method which can be used in this situation. To this end we have studied a general class of operator equations in a Hilbert space. They are of the form

\[
[I + \hat{f}(\lambda)]\psi = \phi.
\]

where we will assume that the operators \(\hat{f}(\lambda)\) and \(\frac{d}{d\lambda}\hat{f}(\lambda)\) are linear, analytic in \(\lambda\), and of trace class\(^4,5\). (An operator \(\hat{A}\) is of trace class if \(tr|A| < \infty\)). Provided we know the inverse operator, \([I + \hat{f}(\lambda)]^{-1}\), the solution would be \(\psi = [I + \hat{f}(\lambda)]^{-1}\phi\). Let us write

\[
\hat{R}(\lambda) = [I + \hat{f}(\lambda)]^{-1} = \frac{1}{d(\lambda)} \hat{D}(\lambda),
\]

where

\[
d(\lambda) = det[I + \hat{f}(\lambda)].
\]

Then, it is found that the functions \(d(\lambda)\) and \(\hat{D}(\lambda)\) are analytic in \(\lambda\) and they obey the following pair of parameter imbedding equations

\[
\frac{d}{d\lambda}d(\lambda) = Tr[\frac{d}{d\lambda}\hat{f}(\lambda) \cdot \hat{D}(\lambda)],
\]

\[
\frac{d}{d\lambda}\hat{D}(\lambda) = \hat{D}(\lambda) \left[ \frac{d}{d\lambda}d(\lambda) - \frac{d}{d\lambda}\hat{f}(\lambda) \cdot \hat{D}(\lambda) \right].
\]
Further, we can determine the initial conditions for $d(\lambda)$ and $\hat{D}(\lambda)$ at an arbitrary point $\lambda = \lambda_0$. Thus, one can integrate the imbedding equations and arrive at the solution $R(\lambda)$ for any $\lambda$. The imbedding equations, Eqs. (12) and (13), are very simple, yet rather general. It should be applicable to a wide class of problems.

We now comment on the background materials to be used in this work. Since we need to cite results in functional analysis extensively, it is difficult to make our paper self-contained. For this reason we will follow closely the four volume treatise “Methods of Mathematical Physics”, by Reed and Simon\textsuperscript{4}, for notations, concepts, and theorems. The only exception is that, in this paper, all operators will be denoted with a “hat”, as in $\hat{f}(\lambda), \hat{D}(\lambda)$, etc.

This paper is organized as follows. The analytic Fredholm theorem in functional analysis will be presented in Sec. II. In Sec. III and Sec. IV, we derive the two parameter imbedding equations, which are the central results of this work. The initial conditions accompanying these equations are determined in Sec. V, and concluding remarks are offered in Sec. VI. To make connections between our operator formalism and that of the usual Fredholm method, we have shown, in Appendix A, a specific realization which leads to the familiar results. Finally, we present in Appendix B an explicit formula for the partial trace, which also establishes its relation with the Plemelj and Smithies formula.

II. The Fredholm Theorem

In functional analysis, the Fredholm integral equation has been abstracted so that it is applicable to the general operator theory in a Hilbert space. We now quote the analytic Fredholm theorem which will be useful for this work.\textsuperscript{6} Theorem: Let $D$ be an open connected subset of $C$. Let $\hat{f} = D \to \mathcal{L}(\mathcal{H})$ be an analytic operator-value function such that $\hat{f}(\lambda)$ is compact for each $\lambda \in D$. Then, either

A) $[I + \hat{f}(\lambda)]^{-1}$ exists for no $\lambda \in D$, or

B) $[I + \hat{f}(\lambda)]^{-1}$ exists for all $\lambda \in D \setminus S$, where $S$ is a discrete subset of $D$ (i.e., a set which has no limit point in $D$). In this case, $[I + \hat{f}(\lambda)]^{-1}$ is meromorphic in $D$, analytic in $D \setminus S$, the residues at the poles are finite rank operators, and if $\lambda \in S$ then $-\hat{f}(\lambda)\psi = \psi$ has a nonzero solution in $\mathcal{H}$.

{}From this theorem, we see that it is only necessary to ascertain that $[I + \hat{f}(\lambda)]^{-1}$ exists for some parameter value $\lambda = \lambda_0 \in D$, then the operator $\hat{f}(\lambda)$ would have the properties described in B). As we will show later, this
amounts to the condition that \( d(\lambda_o) \neq 0 \) for some \( \lambda = \lambda_o \epsilon D \). Hereafter, we assume that this condition is always satisfied.

Recall the definition,

\[
\hat{R}(\lambda) = [I + \hat{f}(\lambda)]^{-1}.
\] (14)

As long as \( \lambda \epsilon D \backslash S \), we have

\[
\hat{R}(\lambda) + \hat{f}(\lambda)\hat{R}(\lambda) = I, \tag{15}
\]

\[
\hat{R}(\lambda) + \hat{R}(\lambda)\hat{f}(\lambda) = I. \tag{16}
\]

Thus, consistency demands that \( \hat{f}(\lambda) \) and \( \hat{R}(\lambda) \) commute. Note that \( \hat{R}(\lambda) \) and \( \hat{f}(\lambda) \) are analytic in \( \lambda \epsilon D \backslash S \) and \( \lambda \epsilon D \), respectively. They and their derivatives are bounded in \( D \backslash S \). For later uses we also note that

\[
\frac{d}{d\lambda} \hat{R}(\lambda) = -\hat{R}(\lambda) \frac{d}{d\lambda} \hat{f}(\lambda)\hat{R}(\lambda), \tag{17}
\]

which follows directly from Eqs. (15) and (16).

III. The First Parameter Imbedding Equation

We now prove our first parameter imbedding equation, Eq.(12). Recall that we demand both \( \hat{f}(\lambda) \) and \( \frac{d}{d\lambda} \hat{f}(\lambda) \) to be of trace class. Since an operator which is of trace class must also be compact,\(^7\) the results of Sec. II are applicable to \( \hat{f}(\lambda) \) and \( \frac{d}{d\lambda} \hat{f}(\lambda) \).

Let’s first define the determinant

\[
d(\lambda) = det(I + \hat{f}(\lambda)) = \sum_{k=0}^{\infty} Tr[\wedge^k(\hat{f}(\lambda))], \tag{18}
\]

where we used the definition\(^4\)

\[
Tr[\wedge^k(\hat{A})] = \sum_{1 \leq i_1 < ... < \infty} (e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}, \hat{A}e_{i_1} \wedge ... \wedge \hat{A}e_{i_k}), \tag{19}
\]

e_\iota being a basis of the Hilbert space. Also, \( Tr[\wedge^0(\hat{A})] = 1 \). The series defining \( d(\lambda) \) turns out to be uniformly convergent, which we now prove.

Let us define the norm of a trace class operator, \( \hat{A} \epsilon I_1 \), by

\[
||\hat{A}||_1 = tr|\hat{A}|. \tag{20}
\]
This gives $\mathcal{I}_1$ a norm topology under which it is a Banach space satisfying the inequality
\begin{equation}
|| \wedge^k (\hat{A}) ||_1 \leq || \hat{A} ||_1^k / k!.
\end{equation}
(21)

For any $\lambda \in D$, $\hat{f}(\lambda)$ is of trace class, hence $|| \hat{f} ||_1$ is bounded. Let’s write $M = \sup || \hat{f}(\lambda) ||_1$, for $\lambda \in D$. Then
\begin{equation}
|| \wedge^k (\hat{f}(\lambda)) ||_1 \leq || \hat{f}(\lambda) ||_1^k / k! \leq M^k / k!
\end{equation}
(22)

It follows that the series for $d(\lambda)$, Eq.(18), is uniformly and absolutely convergent. We may thus differentiate term by term
\begin{equation}
\frac{d}{d\lambda} d(\lambda) = \sum_{k=o}^{\infty} \frac{d}{d\lambda} Tr[\wedge^k (\hat{f}(\lambda))].
\end{equation}
(23)

Or, using Eq.(19), and changing orders in the exterior products, we may write the result in the form
\begin{equation}
\frac{d}{d\lambda} d(\lambda) = \sum_{k=o}^{\infty} k \sum_{1 \leq i_1 < \ldots < i_k < \infty} (e_{i_1} \wedge \ldots \wedge e_{i_k}, \frac{d}{d\lambda} \hat{f}(\lambda) e_{i_1} \wedge \ldots \wedge \hat{f}(\lambda) e_{i_k})
\end{equation}
(24)

From Appendix B, Eq.(B.1), this may be written as
\begin{equation}
\frac{d}{d\lambda} d(\lambda) = \sum_{k=o}^{\infty} k Tr[\frac{d}{d\lambda} \hat{f}(\lambda) \cdot Tr_{k-1}[\wedge^k (\hat{f}(\lambda))]],
\end{equation}
(25)

where the partial trace operator is given by Eq.(B.6),
\begin{equation}
Tr_{k-1}[\wedge^k (\hat{f}(\lambda))] = \sum_{m=1}^{k} \frac{(-1)^{m+1}}{k - m + 1} \hat{f}^{m-1}(\lambda) Tr[\wedge^{k-m} (\hat{f}(\lambda))]
\end{equation}
(26)

Note that this equation for $\frac{d}{d\lambda} d(\lambda)$ is well-defined since $\frac{d}{d\lambda} \hat{f}(\lambda)$ is of trace class, $Tr_{k-1}[\wedge^k (\hat{f}(\lambda))]$ is bounded or of trace class, and the product of a bounded operator and one of trace class is also of trace class.9 In addition, the series defined in Eq.(25) is absolutely and uniformly convergent, following a similar proof for the series of $d(\lambda)$, Eq.(18). We may thus interchange $Tr$ and $\sum_k$, resulting in
\begin{equation}
\frac{d}{d\lambda} d(\lambda) = Tr[\frac{d}{d\lambda} \hat{f}(\lambda) \sum_{k=1}^{\infty} k Tr_{k-1}[\wedge^k (\hat{f}(\lambda))]].
\end{equation}
(27)
Introducing the definition

\[ \hat{D}(\lambda) = \sum_{k=1}^{\infty} k Tr_{k-1}[\wedge^k(\hat{f}(\lambda))], \quad (28) \]

we have

\[ \frac{d}{d\lambda} d(\lambda) = Tr\left[ \frac{d}{d\lambda} \hat{f}(\lambda) \cdot \hat{D}(\lambda) \right]. \quad (29) \]

This is the first parameter imbedding equation. However, it remains to establish the relation \( \hat{R}(\lambda) = [I + \hat{f}(\lambda)]^{-1} = \hat{D}(\lambda)/d(\lambda) \), which we will do in the next section.

**IV. The Relation \( \hat{R}(\lambda) = \hat{D}(\lambda)/d(\lambda) \) and the Second Parameter Imbedding Equation.**

From Eqs. (15-16), in order to verify that \( \hat{R}(\lambda) = \hat{D}(\lambda)/d(\lambda) \), it is sufficient to establish that

\[ \hat{D}(\lambda) + \hat{f}(\lambda) \hat{D}(\lambda) = d(\lambda). \quad (30) \]

\[ \hat{D}(\lambda) + \hat{D}(\lambda) \hat{f}(\lambda) = d(\lambda). \quad (31) \]

Let us first note that \( \hat{D}(\lambda) \) and \( \hat{f}(\lambda) \) commute. This follows since the partial traces in \( \hat{D}(\lambda) \) are all polynomials in \( \hat{f}(\lambda) \). Thus, we need only to examine Eq. (30). Substituting the formulae for \( d(\lambda) \) and \( \hat{D}(\lambda) \) from Eqs. (18) and (28), we have

\[ \sum_{k=1}^{\infty} k Tr_{k-1}[\wedge^k(\hat{f}(\lambda))] + \hat{f}(\lambda) \sum_{k=1}^{\infty} k Tr_{k-1}[\wedge^k(\hat{f}(\lambda))] \]

\[ = \sum_{k=1}^{\infty} Tr[\wedge^{k-1}(\hat{f}(\lambda))]. \quad (32) \]

In this equation, we have used the convention \( Tr[\wedge^0(\hat{f}(\lambda))] = 1, \hat{f}(\lambda)^0 = 1. \) Eq. (32) is valid since, for any \( k \),

\[ k Tr_{k-1}[\wedge^k(\hat{f}(\lambda))] + \hat{f}(\lambda)(k - 1) Tr_{k-2}[\wedge^{k-1}(\hat{f}(\lambda))] \]

\[ = Tr[\wedge^{k-1}(\hat{f}(\lambda))]. \quad (33) \]

This follows from the Plemelj-Smithies formula, Eq. (B.7), when we evaluate the determinant by expanding along the first row and using Eq. (B.9). Note
also that the special case, $k=1$, amounts to $Tr_0(\hat{f}(\lambda)) = 1$, which follows from Eq. (B.6).

Having thus verified the relation $\hat{R}(\lambda) = \hat{D}(\lambda)/d(\lambda)$, we can now proceed to derive the second parameter imbedding equation. This we obtain by substituting $\hat{R}(\lambda) = \hat{D}(\lambda)/d(\lambda)$ in Eq. (17), derived in Sec. II. It follows immediately that

$$\frac{d}{d\lambda} \hat{D}(\lambda) = \frac{\hat{D}(\lambda)}{d(\lambda)} \left[ \frac{d}{d\lambda} d(\lambda) - \frac{d}{d\lambda} \hat{f}(\lambda) \hat{D}(\lambda) \right],$$

(34)

which is the second imbedding equation. Further, we shall now deduce that $\hat{D}(\lambda)$ is analytic for all $\lambda \in D \setminus S$. It is already known that $d(\lambda)$ is analytic for $\lambda \in D$. Also, if $d(\lambda) \neq 0$, the zeroes of $d(\lambda)$ are isolated. From the Fredholm theorem in Sec. II, $\hat{R}(\lambda) = [I + \hat{f}(\lambda)]^{-1}$ exists for $\lambda \in D \setminus S$ and is a meromorphic function, whose residues at its poles are operators of finite rank. Since we already know that $\hat{D}(\lambda) = \hat{R}(\lambda)d(\lambda)$ is analytic in $D \setminus S$, to prove its analyticity in all of $D$, we need only show that the rank of its zero of $d(\lambda)$ is greater than or equal to that of the pole of $\hat{R}(\lambda)$. For this purpose we may introduce the operator $\hat{P}_{\lambda_o}$ which projects $\mathcal{H}$ into the vector space spanned by the eigenvectors associated with $\lambda_o \in S$. This vector space, $\{\lambda_o\}$, is finite dimensional and has dimension $\text{dim}\{\lambda_o\}$. We decompose $\hat{R}(\lambda)$ with respect to $\hat{P}_{\lambda_o}$ and $(1 - \hat{P}_{\lambda_o})$,

$$\hat{R}(\lambda) = [I + \hat{f}(\lambda)\hat{P}_{\lambda_o}]^{-1}\hat{P}_{\lambda_o} + [I + \hat{f}(\lambda)(1 - \hat{P}_{\lambda_o})]^{-1}(1 - \hat{P}_{\lambda_o}).$$

(35)

At $\lambda = \lambda_o$, the second term is nonsingular, while for the first term, $\hat{f}(\lambda)\hat{P}_{\lambda_o}$ is an operator in the finite dimensional vector space $\{\lambda_o\}$. The inverse operator $[I + \hat{f}(\lambda)\hat{P}_{\lambda_o}]^{-1}$ can thus be obtained with the usual finite-dimensional techniques and is given by the ratio of a co-factor and the determinant of $[I + \hat{f}(\lambda)\hat{P}_{\lambda_o}]$. The latter is a polynomial of degree no higher than $\text{dim}\{\lambda_o\}$, so that the rank of its zero must be $\leq \text{dim}\{\lambda_o\}$. Thus, we have proved that $\hat{D}(\lambda) = \hat{R}(\lambda)d(\lambda)$ is an analytic function in all of $D$ (including $S$).

V. The Initial Values of $d(\lambda)$ and $\hat{D}(\lambda)$.

The parameter imbedding equations, given by Eq. (12) and (13), can be integrated provided we have the initial values of $d(\lambda)$ and $\hat{D}(\lambda)$ at some point, say $\lambda = \lambda_o$. This problem becomes trivial if it happens that $\hat{f}(\lambda_o) = 0$. In this case $\hat{R}(\lambda_o) = [I + \hat{f}(\lambda_o)]^{-1} = I$ so that $d(\lambda_o) = 1$ and $\hat{D}(\lambda_o) = I$. In general, we can enforce this condition by introducing another parameter.
As long as \( d(\lambda) \neq 0 \), we can always find a \( \lambda_o \) so that \( d(\lambda_o) \neq 0 \). Let us consider a new operator equation \([I + \xi \hat{f}(\lambda_o)]\psi = \phi\), where \( \xi \) is a complex parameter. The operator-valued functions \( \xi \hat{f}(\lambda_o) \) and \( \frac{d}{d\xi}(\xi \hat{f}(\lambda_o)) = \hat{f}(\lambda_o) \) are obviously analytic in \( \xi \) and of trace class. Defining \([I + \xi \hat{f}(\lambda_o)]^{-1} = \hat{D}(\xi, \lambda_o)/d(\xi, \lambda_o)\), we have the imbedding equations with respect to \( \xi \):

\[
\frac{d}{d\xi}d(\xi, \lambda_o) = Tr[\hat{f}(\lambda_o)\hat{D}(\xi, \lambda_o)],
\]

\[
\frac{d}{d\xi}\hat{D}(\xi, \lambda_o) = \frac{\hat{D}(\xi, \lambda_o)}{d(\xi, \lambda_o)} \left[ \frac{d}{d\xi}d(\xi, \lambda_o) - \hat{f}(\lambda_o)\hat{D}(\xi, \lambda_o) \right]
\]

For these equations, the initial values at \( \xi = 0 \) are obviously

\[
d(0, \lambda_o) = 1,
\]

\[
\hat{D}(0, \lambda_o) = I.
\]

We may therefore integrate the imbedding equations to the point \( \xi = 1 \), where

\[
d(1, \lambda_o) = d(\lambda_o),
\]

\[
\hat{D}(1, \lambda_o) = \hat{D}(\lambda_o).
\]

Provided that \( d(\lambda_o) \neq 0 \), these are then the initial values we need at \( \lambda = \lambda_o \), which can be used with the original Eq.(12) and (13).

**VI. Concluding Remarks**

In this work we have obtained a general method to solve the operator equation \([I + \hat{f}(\lambda)]\psi = \phi\). If \( \hat{f}(\lambda) \) and \( \frac{d}{d\lambda}\hat{f}(\lambda) \) are linear, analytic in \( \lambda \), and of trace class, then there is a pair of simple differential equations for the analytic functions \( \hat{D}(\lambda) \) and \( d(\lambda) \), defined by \( \hat{R}(\lambda) = [I + \hat{f}(\lambda)]^{-1} = \hat{D}(\lambda)/d(\lambda) \). The integration of these equations is rather straightforward so that one can obtain explicitly the functions \( d(\lambda) \) and \( \hat{D}(\lambda) \).

Since the parameter imbedding method tackles a problem directly with respect to the parameter \( \lambda \), it is particularly useful when the solution exhibits intriguing properties in \( \lambda \). An example is the Hammerstein equation, which includes the familiar nonlinear Schwinger-Dyson (SD) equation. There are several advantages in the application of the imbedding method to this type of problems. First of all, the SD equation is known to have bifurcation solution. However, the criterion of bifurcation is contained in the function
As long as \( d(\lambda) \neq 0 \), the implicit function theorem is valid and we have a unique solution. Thus, at each step of the integration of the parameter imbedding equations, the obtained value of \( d(\lambda) \) enables us to determine whether there is a unique solution at \( \lambda \). Therefore, we can straightforwardly arrive at a solution for a wide range of the parameter \( \lambda \). Another nice feature of the imbedding method is that, at each step of the integration, the result obtained is directly the solution in its final form. It is thus very economical, computationally. Finally, the analyticity properties of \( d(\lambda) \) and \( \hat{D}(\lambda) \) are extremely useful, since we can always choose contours around possible singularities and integrate over smooth functions. All of the numerical calculations are therefore routine.

These points are well illustrated in a concrete example, which we present in a separate paper where we have found numerical solutions to the SD equation.\(^{10}\) Our imbedding method enables one to obtain explicit bifurcation solutions to the SD equation. In the process we also uncovered some of its hitherto unknown solutions.

The imbedding equations discussed in this work are very general. There seems to be little obstacle in applying them to a number of interesting problems in mathematical physics. We hope to turn to these questions in the future.

**APPENDIX A: \( \hat{R}(\lambda) \) AND THE FREDHOLM RESOLVENT**

In this appendix, we examine the relation between the operator \( \hat{R}(\lambda) \) and the usual Fredholm resolvent. They are not identical, which is why imbedding equations for the two cases also differ. The Fredholm equation (1) can be written as \([I + \hat{f}(\lambda)]\psi = \phi\) with the definition

\[
\hat{f}(\lambda) = -\lambda \int K(x, y)\psi(y)dy. \tag{A1}
\]

Its solution [Eq.(2)] may be written in the form

\[
\psi = [I + \lambda R]\phi \tag{A2}
\]

with

\[
R\phi = \int R(x, y, \lambda)\phi(y)dy. \tag{A3}
\]

Thus, the relation between \( \hat{R}(\lambda) = [I + \hat{f}(\lambda)]^{-1} \) and \( R \) is

\[
I + \lambda R = \hat{R}(\lambda) = [I + \hat{f}(\lambda)]^{-1}, \tag{A4}
\]
or
\[ \hat{f}(\lambda) + \lambda R[I + \hat{f}(\lambda)] = 0. \tag{A5} \]

Thus
\[ R = -\frac{1}{\lambda} \hat{f}(\lambda) \hat{R}(\lambda). \tag{A6} \]

We may now proceed to find a relation of the operator imbedding equations (12) and (13) for the Fredholm equations. Introducing the coordinate basis \(|x\rangle\), so that \(\langle x|\psi\rangle = \psi(x)\), and defining
\[ \langle x|\hat{f}(\lambda)|y\rangle = -\lambda K(x, y), \tag{A7} \]
\[ \langle x|\hat{D}(\lambda)|y\rangle = \hat{D}(x, y, \lambda), \tag{A8} \]
then Eq.(12) becomes
\[
\frac{d}{d\lambda}d(\lambda) = Tr[\frac{\partial}{\partial \lambda}\hat{f}(\lambda)\hat{D}(\lambda)] = \int \int \langle x|\frac{\partial}{\partial \lambda}\hat{f}(\lambda)|y\rangle \langle y|\hat{D}(\lambda)|x\rangle \, dx\, dy \\
= -\int \int K(x, y)\hat{D}(x, y, \lambda) \, dx\, dy. \tag{A9}
\]

Also, taking the matrix element of Eq.(13), we have
\[
d(\lambda)\frac{\partial}{\partial \lambda}\hat{D}(x, y, \lambda) = \hat{D}(x, y, \lambda)\frac{d}{d\lambda}d(\lambda) + \int \hat{D}(x, x', \lambda)K(x', z)\hat{D}(z, y, \lambda) \, dx' \, dz, \tag{A10}
\]
but Eq.(A6) implies that
\[ R(x, y, \lambda) = \int K(x, z)\hat{R}(z, y, \lambda) \, dz, \tag{A11} \]
so that
\[ D(x, y, \lambda) = \int K(x, z)\hat{D}(z, y, \lambda) \, dz, \tag{A12} \]

Thus, Eq.(A9) is precisely Eq.(4). Also, Eq.(A10) is equivalent to Eq.(5) by multiplying with \(K(x', x)\) and integrating with respect to \(x\).

Although the two approaches are equivalent, the use of \(\hat{R} = [I + \hat{f}(\lambda)]^{-1}\) is convenient for general formulation. This is because the operator \(\hat{f}(\lambda)\) is compact, but in an infinite dimensional space, its inverse is not bounded.\(^{11}\)

On the other hand, the identity operator is bounded, but not compact. The combination \([I + \hat{f}(\lambda)]^{-1}\) strikes a balance and has nice invertibility properties similar to those of operators in infinite dimensional spaces. As for the Fredholm resolvent, we note that from Eq.(A5), it satisfies
\[ I + \lambda R[I + \hat{f}]^{-1} = 0, \]

or
\[ R = -[I + \hat{f}^{-1}]^{-1}/\lambda. \] The appearance of the unbounded operator \( \hat{f}^{-1} \) makes \( R \) less useful compared to \( \hat{R} \).

**APPENDIX B: PARTIAL TRACE AND ITS EXPLICIT FORM**

Important properties of the partial trace will be examined in this appendix. For this, we follow Simon,\(^{12}\) where the partial trace \( \text{Tr}_{k-1}(\hat{A}) \) is defined as an operator which projects operators in \( I_1(\otimes^k\mathcal{H}) \) to \( I_1(\mathcal{H}) \). Its satisfies

\[
\text{Tr}[\hat{C}\text{Tr}_{k-1}(\hat{A})] = \sum_{1 \leq i_1 < ... < i_k < \infty} (e_{i_1} \wedge ... \wedge e_{i_k}, \hat{C}e_{i_1} \wedge \hat{A}e_{i_2} \wedge ... \wedge \hat{A}e_{i_k}) \quad (B1)
\]

for any \( \hat{C} \in I_1(\mathcal{H}) \).

We now proceed to give an explicit realization of the partial trace \( \text{Tr}_{k-1}(\hat{A}) \). Let us expand the right-hand side of Eq.(B1) with respect to the first column

\[
\sum_{1 \leq i_1 < ... < i_k < \infty} (e_{i_1} \wedge ... \wedge e_{i_k}, \hat{C}e_{i_1} \wedge \hat{A}e_{i_2} \wedge ... \wedge \hat{A}e_{i_k}) = \frac{1}{k!} \sum_{i_1=0}^{\infty} k \sum_{1 < i_2 < ... < \infty} \left[(e_{i_1}, \hat{C}e_{i_1})(e_{i_2} \wedge ... \wedge e_{i_k}, \hat{A}e_{i_2} \wedge ... \wedge \hat{A}e_{i_k}) \right.
\]

\[
+ \sum_{j=1}^{k-1} (-1)^{j}(e_{i_j}, \hat{C}e_{i_1})(e_{i_1} ... \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} ... \wedge \hat{A}e_{i_2} \wedge ...)]
\]

\[
= \frac{1}{k} \sum_{i_1=0}^{\infty} (e_{i_1}, \hat{C}e_{i_1})\text{Tr}[\hat{A}] - \frac{k-1}{k} \sum_{i_1=1}^{\infty} \sum_{1 < i_2 < \infty} (e_{i_j}, \hat{C}e_{i_1})
\]

\[
\sum_{i_{j+1}}^{\infty} (e_{i_{j-1}} \wedge e_{i_{j+1}} ... \wedge \hat{A}e_{i_{j-1}} \wedge \hat{A}e_{i_{j+1}} ...). \quad (B2)
\]

In deducing the first equality, note that the factor \( 1/k! \) in the definition of \( \wedge^k (\hat{A}) \). For the second equality, note that there is a factor \( (-1)^{j-1} \) for bringing \( \hat{A}e_{i_j} \) to the front of the exterior product. We may now expand the second term in Eq.(B2) into two terms, etc. Using the relations

\[
\sum_{i_1=1}^{\infty} (e_{i_1}, \hat{C}e_{i_1}) = \text{Tr}(\hat{C}), \quad (B3)
\]

\[
\sum_{i_1,i_j=1}^{\infty} (e_{i_j}, \hat{C}e_{i_1})(e_{i_1}, \hat{A}e_{i_j}) = \sum_{i_j=1}^{\infty} (e_{i_j}, \hat{C}\hat{A}e_{i_j}) = \text{Tr}(\hat{C}\hat{A}), \quad (B4)
\]
etc., we have

\[ Tr[\tilde{C}Tr_{k-1}(\wedge^k(\hat{A}))] = \sum_{m=1}^{k} (-1)^{m+1} \frac{(k-m)!}{k-m+1} \tilde{T}r(\wedge^{k-m}(\hat{A}))Tr[C\hat{A}^{m-1}] . \tag{B5} \]

It is now clear that the partial trace is given by the explicit formula

\[ Tr_{k-1}[\wedge^k(\hat{A})] = \sum_{m=1}^{k} \frac{(-1)^{m+1}}{k-m+1} \hat{A}^{m-1}Tr[\wedge^{k-m}(\hat{A})] . \tag{B6} \]

Note that this expression for the partial trace indeed projects \( \mathcal{L}(\otimes^k\mathcal{H}) \) into \( \mathcal{L}(\mathcal{H}) \), as required.

Finally, we show that this result for the partial trace is intimately related to the plemelj-Smithies formula,\(^{13}\) which is very important in the general Fredholm theory. The plemelj-Smithies formula is obtained through the determinant

\[ \beta_k(\hat{A}) = \begin{vmatrix} \hat{A} & k-1 & 0 & \ldots & \ldots & 0 \\ \hat{A}^2 & Tr(\hat{A}) & k-2 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \hat{A}^k & Tr(\hat{A}^{k-1}) & \ldots & \ldots & \ldots & Tr(\hat{A}) \end{vmatrix}. \tag{B7} \]

From the following expression for the trace:

\[ Tr[\wedge^k(\hat{A})] = \frac{1}{k!} \begin{vmatrix} Tr(\hat{A}) & k-1 & 0 & \ldots & \ldots & 0 \\ Tr(\hat{A}^2) & Tr(\hat{A}) & k-2 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ Tr(\hat{A}^k) & \ldots & \ldots & \ldots & \ldots & Tr(\hat{A}) \end{vmatrix}, \tag{B8} \]

we can write Eq.(B7) by expanding according to the first column

\[ \beta_k(\hat{A}) = \sum_{m=1}^{k} (-1)^{m+1} \frac{k!}{k-m+1} \hat{A}^m Tr[\wedge^{k-m}(\hat{A})] = k! \hat{A} Tr_{k-1}[\wedge^k(\hat{A})] . \tag{B9} \]

Thus, the explicit formula of the partial trace (B6) is a generalization of the plemelj-Smithies formula. This result has been crucial in establishing the parameter imbedding theory for the trace class operator-valued functions.

References

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