ON THE GEOMETRIC STRUCTURES OF CONDUCTIVE TRANSMISSION EIGENFUNCTIONS AND THEIR APPLICATION

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Abstract. This paper is concerned with the intrinsic geometric structures of conductive transmission eigenfunctions. The geometric properties of interior transmission eigenfunctions were first studied in [9]. It is shown in two scenarios that the interior transmission eigenfunction must be locally vanishing near a corner of the domain with an interior angle less than \( \pi \). We significantly extend and generalize those results in several aspects. First, we consider the conductive transmission eigenfunctions which include the interior transmission eigenfunctions as a special case. The geometric structures established for the conductive transmission eigenfunctions in this paper include the results in [9] as a special case. Second, the vanishing property of the conductive transmission eigenfunctions is established for any corner as long as its interior angle is not \( \pi \). That means, as long as the corner singularity is not degenerate, the vanishing property holds. Third, the regularity requirements on the interior transmission eigenfunctions in [9] are significantly relaxed in the present study for the conductive transmission eigenfunctions. In order to establish the geometric properties for the conductive transmission eigenfunctions, we develop technically new methods and the corresponding analysis is much more complicated than that in [9]. Finally, as an interesting and practical application of the obtained geometric results, we establish a unique recovery result for the inverse problem associated with the transverse electromagnetic scattering by a single far-field measurement in simultaneously determining a polygonal conductive obstacle and its surface conductive parameter.

Keywords: Conductive transmission eigenfunctions, corner singularity, geometric structures, vanishing, inverse scattering, uniqueness, single far-field pattern.

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1. Introduction

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), \( n = 2, 3 \), and \( V \in L^\infty(\Omega) \) and \( \eta \in L^\infty(\partial \Omega) \) be possibly complex-valued functions. Consider the following conductive transmission eigenvalue problem for \( v, w \in H^1(\Omega) \),

\[
\begin{aligned}
\Delta w + k^2(1 + V)w &= 0 \quad \text{in } \Omega, \\
\Delta v + k^2v &= 0 \quad \text{in } \Omega, \\
w &= v, \quad \partial_\nu v + \eta v = \partial_\nu w \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \nu \in S^{n-1} \) signifies the exterior unit normal vector to \( \partial \Omega \). Clearly, \( v = w \equiv 0 \) are trivial solutions to (1.1). If for a certain \( k \in \mathbb{R}_+ \), there exists a pair of nontrivial solutions \( (v, w) \in H^1(\Omega) \times H^1(\Omega) \) to (1.1), then \( k \) is called a conductive transmission eigenvalue and \( (v, w) \) is referred to as the corresponding pair of conductive transmission eigenfunctions. For a special case with \( \eta \equiv 0 \), (1.1) is known to be the interior transmission eigenvalue problem. The study of the transmission eigenvalue problems arises in the wave scattering theory and has a long and colourful history; see [10, 14, 18, 19, 22, 23, 31, 33, 35]
for the spectral study of the interior transmission eigenvalue problem, and [12, 13, 20] for the related study of the conductive transmission eigenvalue problem, and a recent survey [15] and the references therein for comprehensive discussions on the state-of-the-art developments. The problem is a type of non-elliptic and non-self-adjoint eigenvalue problem, so its study is mathematically interesting and challenging. The existing results in the literature mainly focus on the spectral properties of the transmission eigenvalues, namely their existence, discreteness, infiniteness and Weyl’s laws. Roughly speaking, the theorems for the transmission eigenvalues follow in a similar flavour to the results in the spectral theory of the Laplacian on a bounded domain. However, the transmission eigenfunctions reveal certain distinct and intriguing features. In [11, 32], it is proved that the interior transmission eigenfunctions cannot be analytically extended across the boundary \( \partial \Omega \) if it contains a corner with an interior angle less than \( \pi \). In [9], geometric structures of interior transmission eigenfunctions were discovered for the first time. It is shown that under certain regularity conditions on the interior transmission eigenfunctions, the eigenfunctions must be locally vanishing near a corner of the domain with an interior angle less than \( \pi \). With the help of numerics, it is further shown in [5, 27] that under the \( H^1 \)-regularity of the interior transmission eigenfunctions, the eigenfunctions are either vanishing or localizing at a corner with an interior angle bigger than \( \pi \). Recently, more geometric properties of the interior transmission eigenfunctions were discovered in [8, 27], which are linked with the curvature of a specific boundary point. It is noted that a corner point considered in [5, 9] can be regarded as having an infinite extrinsic curvature since the derivative of the normal vector has a jump singularity there.

In addition to the angle of the corner, we would like to emphasize the critical role played by the regularity of the transmission eigenfunctions in the existing studies of the geometric structures in the aforementioned literatures. In [9], the regularity requirements are characterized in two ways. The first one is \( H^2 \)-smoothness, and the other one is \( H^1 \)-regularity with a certain Hergoltz approximation property. The \( H^2 \)-regularity requirement can be weakened a bit to be Hölder-continuity with any Hölder index \( \alpha \in (0, 1) \).

In this paper, we establish the vanishing property of the conductive transmission eigenfunctions associated with (1.1) at a corner as long as its interior angle is not \( \pi \). That means, as long as the corner singularity is not degenerate, the vanishing property holds. In fact, in the three-dimensional case, the corner singularity is a more general edge singularity. To establish the vanishing property, we need to impose certain regularity conditions on the conductive transmission eigenfunctions which basically follow a similar manner to those considered in [9]. That is, the first regularity condition is the Hölder-continuity with any Hölder index \( \alpha \in (0, 1) \), and the second regularity condition is characterized by the Herglotz approximation. Nevertheless, for the latter case, the regularity requirement is much more relaxed in the present study compared to that in [9]. Finally, we would like to emphasize that in principle the geometric properties established for the conductive transmission eigenfunctions include the results in [9] as a special case by taking the parameter \( \eta \) to be zero. Hence, in the sense described above, the results obtained in this work significantly extend and generalize the ones in [9].

The mathematical argument in [9] is indirect which connects the vanishing property of the interior transmission eigenfunctions with the stability of a certain wave scattering problem with respect to variation of the wave field at the corner point. In [4, 8], direct mathematical arguments based on certain microlocal analysis techniques are developed for dealing with the vanishing properties of the interior transmission eigenfunctions. However, the Hölder continuity on the interior transmission eigenfunctions is an
essential assumption in [4, 8]. In this paper, in order to establish the vanishing property of the conductive transmission eigenfunctions under more general regularity conditions, we basically follow the direct approach. But we need to develop technically new ingredients for this different type of eigenvalue problem and the corresponding analysis becomes radically much more complicated.

As an interesting and practical application, we apply the obtained geometric results for the conductive transmission eigenfunctions to an inverse problem associated with the transverse electromagnetic scattering. In a certain scenario, we establish the unique recovery result by a single far-field measurement in simultaneously determining a polygonal conductive obstacle and its surface conductivity. This contributes to the well-known Schiffer’s problem in the inverse scattering theory which is concerned with recovering the shape of an unknown scatterer by a single far-field pattern; see [2, 6, 7, 16, 21, 24–26, 28, 29, 34] and the references therein for background introduction and the state-of-the-art developments on the Schiffer’s problem.

The rest of the paper is organized as follows. In Sections 2 and 3, we respectively derive the vanishing results of the conductive transmission eigenfunctions near a corner in the two-dimensional and three-dimensional cases. Section 4 is devoted to the uniqueness study in determining a polyhedral conductive obstacle as well as its surface conductivity by a single far-field pattern.

2. Vanishing near corners of conductive transmission eigenfunctions: two-dimensional case

In this section, we consider the vanishing near corners of conductive transmission eigenfunctions in the two-dimensional case. First, let us introduce some notations for the subsequent use. Let \((r, \theta)\) be the polar coordinates in \(\mathbb{R}^2\); that is, \(x = (x_1, x_2) = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2\). For \(x \in \mathbb{R}^n\), \(B_h(x)\) denotes the ball of radius \(h \in \mathbb{R}_+\) and centered at \(x\). \(B_h := B_h(0)\). Consider an open sector in \(\mathbb{R}^2\) with the boundary \(\Gamma^\pm\) as follows,

\[
W = \left\{ x \in \mathbb{R}^2 \mid x \neq 0, \quad \theta_m < \arg(x_1 + ix_2) < \theta_M \right\},
\]

(2.1)

where \(-\pi < \theta_m < \theta_M < \pi\), \(i := \sqrt{-1}\) and \(\Gamma^+\) and \(\Gamma^-\) respectively correspond to \((r, \theta_M)\) and \((r, \theta_m)\) with \(r > 0\). Henceforth, set

\[
S_h = W \cap B_h, \quad \Gamma_h^\pm = \Gamma^\pm \cap B_h, \quad S_h = \overline{W} \cap B_h, \quad \Lambda_h = S_h \cap \partial B_h, \quad \text{and} \quad \Sigma_{\Lambda_h} = S_h \setminus S_h/2.
\]

(2.2)

In Figure 1, we give a schematic illustration of the geometry considered here. For \(g_j \in L^2(\mathbb{S}^{n-1})\), we introduce

\[
v_j(x) = \int_{\mathbb{S}^{n-1}} e^{ik\xi \cdot x} g_j(\xi) d\sigma(\xi), \quad \xi \in \mathbb{S}^{n-1}, x \in \mathbb{R}^n.
\]

(2.3)

It can be easily seen that \(v_j\) is an entire solution to the Helmholtz equation \(\Delta v_j + k^2 v_j = 0\). \(v_j\) is referred to as a Herglotz wave function with kernel \(g_j\). The set of Herglotz functions is dense in the set \(\{u \in H^1(\Omega); \Delta u + k^2 u = 0\}\) in the topology induced by the \(H^1(\Omega)\)-norm. That is, for any \(v \in H^1(\Omega)\) being a solution to the Helmholtz equation in \(\Omega\), there exists a sequence of Herglotz functions which can approximate \(v\) to an arbitrary accuracy (see [36, Theorem 2.1]).

We shall also need the following lemma, which gives a particular type of planar complex geometrical optics (CGO) solution whose logarithm is a branch of the square root (cf. [4]).
Lemma 2.1. \textup{[4, Lemma 2.2]} For $x \in \mathbb{R}^2$ denote $r = |x|$, $\theta = \arg(x_1 + ix_2)$. Let

$$u_0(x) := \exp \left( \sqrt{r} \left( \cos \left( \frac{\theta}{2} + \pi \right) + i \sin \left( \frac{\theta}{2} + \pi \right) \right) \right).$$

Then $\Delta u_0 = 0$ in $\mathbb{R}^2 \setminus (\mathbb{R}^- \times \{0\} \cup \{(0,0)\})$, and $s \mapsto u_0(sx)$ decays exponentially in $\mathbb{R}_+$. Let $\alpha, s > 0$. Then

$$\int_W |u_0(sx)||x|^\alpha dx \leq \frac{2(\theta_M - \theta_m)\Gamma(2\alpha + 4)}{\delta_W^{2\alpha+4}} s^{-\alpha-2},$$

(2.5)

where $\delta_W = - \max_{\theta_m < \theta < \theta_M} \cos(\theta/2 + \pi) > 0$. Moreover

$$\int_W u_0(sx) dx = 6i(e^{-2\theta_M i} - e^{-2\theta_m i})s^{-2},$$

(2.6)

and for $h > 0$

$$\int_{W \setminus B_h} |u_0(sx)| dx \leq \frac{6(\theta_M - \theta_m)}{\delta_W^4} s^{-2}e^{-\delta_W \sqrt{h}/2}. $$

(2.7)

We are in a position to present one of the main theorems of this section.

Theorem 2.1. Let $v \in H^1(\Omega)$ and $w \in H^1(\Omega)$ be a pair of eigenfunctions to (1.1) associated with $k \in \mathbb{R}_+$. Assume that the Lipschitz domain $\Omega \subset \mathbb{R}^2$ contains a corner $\Omega \cap \mathbb{R}^2$, where $x_c$ is the vertex of $\Omega \cap \mathbb{R}^2$ and $W$ is a sector defined in (2.1). Moreover, there exits a sufficiently small neighbourhood $S_h$ (i.e. $h > 0$ is sufficiently small) of $x_c$ in $\Omega$, where $S_h$ is defined in (2.2), such that $qw \in C^\alpha(S_h)$ with $q := 1 + V$ and $\eta \in C^\alpha(\Gamma_h^+) \setminus \{0\}$ for $0 < \alpha < 1$, and $v - w \in H^2(\Sigma_{\Lambda_h})$, with $\Sigma_{\Lambda_h}$ defined in (2.2). If the following conditions are fulfilled:

(a) the transmission eigenfunction $v$ can be approximated in $H^1(S_h)$ by the Herglotz functions $v_j$, $j = 1, 2, \ldots$, with kernels $g_j$ satisfying

$$\|v - v_j\|_{H^1(S_h)} \leq j^{-1 - \gamma}, \quad \|g_j\|_{L^2(S^1)} \leq C j^{\theta},$$

(2.8)
Remark 2.1. In Theorem 2.1, we consider the case that \( v, w \) are a pair of conductive transmission eigenfunctions to (1.1) and show the vanishing property near a corner. We would like to emphasize that the result can be localized in the sense that as long as \( v, w \) satisfy all the conditions stated in Theorem 2.1 in \( \Omega \cap S_h \), then one has the vanishing property (2.11) near the corner. That is, \( v, w \) are not necessary conductive transmission eigenfunctions, and it suffices to require that \( v, w \) satisfy the equations in (1.1) in \( S_h \cap \Omega \) and the conductive transmission conditions on \( \Sigma_h \cap \partial \Omega \), then one has the same vanishing property as stated in Theorem 2.1. Indeed, the subsequent proof of Theorem 2.1 is for the aforementioned localized problem.

Remark 2.2. The condition (2.8) signifies a certain regularity condition of the transmission eigenfunction \( v \in H^1(\Omega) \). In [9], the following regularity condition was introduced,

\[
\|v - v_j\|_{L^2(\Omega)} \leq e^{-j}, \quad \|g_j\|_{L^2(\Sigma^{n-1})} \leq C(\ln j)^\beta,
\]

where the constants \( C > 0 \) and \( 0 < \beta < 1/(2n + 8), (n = 2, 3) \). Here, we allow the polynomial growth of the kernel functions. Moreover, we would like to remark that \( qw \in C^\alpha(\Sigma_h) \) is technically required in our mathematical argument of proving Theorem 2.1. It is obviously satisfied in a simple case when \( q = 0 \) in \( S_h \). We believe this condition should be able to be relaxed in the theorem, but the proof is fraught with new difficulties. Hence, we include it as a technical condition in Theorem 2.1. The interior regularity requirement \( v - w \in H^2(\Sigma_{\Lambda_h}) \) can be fulfilled in certain practical scenarios; see Theorem 4.1 in what follows on the study of an inverse scattering problem. The introduction of this interior regularity condition shall play a critical role in the proof of Theorem 4.1.

Proof of Theorem 2.1. Since the partial differential operator \( \Delta + k^2 \) is invariant under rigid motions, we assume without loss of generality that \( x_c \) is the origin. From (1.1), we have

\[
\Delta v = -k^2 v := f_1, \quad \Delta w = -k^2 qw := f_2.
\]

Subtracting the two equations of (2.13) together with the use of the boundary conditions of (1.1) we deduce that

\[
\Delta(v - w) = f_1 - f_2 \text{ in } S_h, \quad v - w = 0, \quad \partial_{\nu}(v - w) = -\eta v \text{ on } \Gamma_h^\pm.
\]

Recall that \( v \) can be approximated by the Herglotz wave function \( v_j \) given by (2.3) in the topology induced by the \( H^1 \)-norm. Since \( v \in H^1(S_h) \) is a solution to the Helmholtz equation in \( S_h \) (cf. [36, Theorem 2.1]), we can deduce that

\[
\int_{S_h} f_1(x)u_0(sx)dx = \int_{S_h} \tilde{f}_j(x)u_0(sx)dx + \delta_j(s),
\]

for some constants \( C > 0, \ Y > 0 \) and \( 0 < q < 1 \);

(b) the function \( \eta(x) \) does not vanish at the corner, i.e.,

\[
\eta(x_c) \neq 0,
\]

(c) the angles \( \theta_m \) and \( \theta_M \) of the sector \( W \) satisfy

\[
-\pi < \theta_m < \theta_M < \pi \text{ and } \theta_M - \theta_m \neq \pi;
\]

then one has

\[
\lim_{\rho \to +0} \frac{1}{m(B(x_c, \rho))} \int_{B(x_c, \rho)} |v(x)|dx = 0,
\]

where \( m(B(x_c, \rho)) \) is the area of \( B(x_c, \rho) \).
where
\[ \tilde{f}_j(x) = -k^2 v_j(x), \quad \delta_j(s) = -k^2 \int_{\mathcal{S}_h} (v(x) - v_j(x)) u_0(sx) \, dx, \] (2.14)
and \( u_0 \) is given in Lemma 2.1. Clearly \( \tilde{f}_j(x) \in H^2(S_h) \), which can be embedded into \( C^\alpha(\mathcal{S}_h) \) for \( \alpha \in (0, 1) \). Moreover by using the Cauchy-Schwarz inequality, we know that
\[ |\delta_j(s)| \leq k^2 \|v - v_j\|_{L^2(S_h)} \| u_0(sx) \|_{L^2(S_h)}. \] (2.15)
Recalling the expression of \( u_0 \) given in (2.4), using change of variables and the integral mean value theorem, we further deduce that
\[ \| u_0(sx) \|_{L^2(S_h)}^2 = \int_0^h rdr \int_{\theta_m}^{\theta_M} e^{2\sqrt{\eta} \cos(\theta/2 + \pi)} \, d\theta \leq \int_0^h rdr \int_{\theta_m}^{\theta_M} e^{-2\sqrt{\eta} \delta \sqrt{\varepsilon}} \, d\theta \]
\[ = \frac{(\theta_M - \theta_m) e^{-2\sqrt{\eta} \delta \sqrt{\varepsilon}} h^2}{2}, \] (2.16)
where \( \Theta \in [0, h] \) and \( \delta \) is defined in (2.5). Substituting (2.16) into (2.15) and using (2.8), we know that
\[ |\delta_j(s)| \leq \frac{\sqrt{\theta_M - \theta_m} k^2 e^{-2\sqrt{\eta} \delta \sqrt{\varepsilon}} h}{2} j^{-1 - \gamma}. \] (2.17)
Let \( D_\varepsilon = \mathcal{S}_h \setminus B_\varepsilon \) for \( 0 < \varepsilon < h \), it can be derived that
\[ \int_{\mathcal{S}_h} (\tilde{f}_j(x) - f_2(x)) u_0(sx) \, dx = \lim_{\varepsilon \to 0} \int_{D_\varepsilon} (\tilde{f}_j(x) - f_2(x)) u_0(sx) \, dx \] (2.18)
since \( |u_0(sx)| \leq 1 \) in \( \mathcal{S}_h \cap B_\varepsilon \) for sufficiently small \( \varepsilon \) and \( \tilde{f}_j(x) - f_2(x) \in L^2(S_h \cap B_\varepsilon) \). Denote \( \delta_{j1}(\varepsilon, s) = -k^2 \int_{B_\varepsilon} (v(x) - v_j(x)) u_0(sx) \, dx \). Since \( u_0 \notin H^2(B_\varepsilon) \) near the origin, we consider the domain \( D_\varepsilon \) in the following discussions. Using Green’s formula, we have
\[ \int_{D_\varepsilon} (\tilde{f}_j - f_2) u_0(sx) \, dx + \delta_j(s) - \delta_{j1}(\varepsilon, s) = \int_{\Lambda_h} \Delta(v - w) u_0(sx) \, dx \]
\[ = \int_{D_\varepsilon} (v - w) \Delta u_0(sx) \, dx + \int_{\partial D_\varepsilon} (u_0(sx) \partial_\nu (v - w) - (v - w) \partial_\nu u_0(sx)) \, d\sigma \]
\[ = \int_{\Lambda_h} (u_0(sx) \partial_\nu (v - w) - (v - w) \partial_\nu u_0(sx)) \, d\sigma \]
\[ + \int_{\Lambda_\varepsilon} (u_0(sx) \partial_\nu (v - w) - (v - w) \partial_\nu u_0(sx)) \, d\sigma - \int_{\Gamma_{(h, \varepsilon)}} \eta(x) u_0(sx) v(x) \, d\sigma, \] (2.19)
where \( \Lambda_h = \mathcal{S}_h \cap \partial B_h \), \( \Lambda_\varepsilon = \mathcal{S}_h \cap \partial B_\varepsilon \) and \( \Gamma_{(\varepsilon, h)}^\pm = \Gamma_{(h, \varepsilon)}^\pm \cap (B_h \setminus B_\varepsilon) \). Moreover, we have
\[ \lim_{\varepsilon \to 0} \int_{\Lambda_\varepsilon} (u_0 \partial_\nu (v - w) - (v - w) \partial_\nu u_0) \, d\sigma = 0. \] (2.20)
Since \( v \in H^1(\mathcal{S}_h \cap B_\varepsilon) \), using the trace theorem, we have \( v \in L^2(\Gamma_{(0, \varepsilon)}^\pm) \) where \( \Gamma_{(0, \varepsilon)}^\pm = \Gamma_{(h, \varepsilon)}^\pm \cap B_\varepsilon \). For sufficiently small \( \varepsilon \) and using the fact that \( |u_0(sx)| \leq 1 \) and \( \eta \in C^\alpha \left( \Gamma_{(h, \varepsilon)}^\pm \right) \) for \( 0 < \alpha < 1 \), it can be seen that
\[ \lim_{\varepsilon \to 0} \int_{\Gamma_{(0, \varepsilon)}^\pm} \eta u_0 v \, d\sigma = 0. \] (2.21)
Recall again that \(v\) can be approximated by the Herglotz wave function \(v_j\) given in (2.3) in the sense of \(H^1\)-norm. Then

\[
\int_{\Gamma_h^\pm} \eta(x)u_0(sx)v(x)\,ds = \int_{\Gamma_h^\pm} \eta(x)u_0(sx)v_j(x)\,ds + \xi_j^\pm(s),
\]

(2.22)

\[
\xi_j^\pm(s) = \int_{\Gamma_h^\pm} \eta(x)u_0(sx)(v(x) - v_j(x))\,ds.
\]

Since \(\eta \in C^\alpha(\Gamma_h^\pm)\), we have the following expansion of \(\eta(x)\) at the origin as

\[
\eta(x) = \eta(0) + \delta\eta(x), \quad |\delta\eta(x)| \leq \|\eta\|_{C^\alpha}|x|^\alpha.
\]

(2.23)

Therefore, using Cauchy-Schwarz inequality and the trace theorem, we have

\[
|\xi_j^\pm(s)| \leq |\eta(0)| \int_{\Gamma_h^\pm} |u_0(sx)||v(x) - v_j(x)|\,ds + \|\eta\|_{C^\alpha} \int_{\Gamma_h^\pm} |x|^\alpha|u_0(sx)||v(x) - v_j(x)|\,ds
\]

\[
\leq |\eta(0)||v - v_j|_{H^{1/2}(\Gamma_h^\pm)}\|u_0(sx)\|_{H^{-1/2}(\Gamma_h^\pm)}
\]

\[
+ \|\eta\|_{C^\alpha}|v - v_j|_{H^{1/2}(\Gamma_h^\pm)}\|x|^\alpha|u_0(sx)\|_{H^{-1/2}(\Gamma_h^\pm)}
\]

\[
\leq |\eta(0)||v - v_j|_{H^1(S_h)}\|u_0(sx)\|_{L^2(S_h)} + \|\eta\|_{C^\alpha}|v - v_j|_{H^1(S_h)}\|x|^\alpha|u_0(sx)\|_{L^2(S_h)}.
\]

where \(C\) is a positive constant. Hence, using polar coordinates transformation we can deduce that

\[
\|\|x|^\alpha u_0(sx)\|^2_{L^2(S_h)} = \int_0^h rdr \int_{\theta_m}^{\theta_M} r^{2\alpha}e^{2\sqrt{\pi r}\cos(\theta/2+\pi)}d\theta
\]

\[
\leq \int_0^h rdr \int_{\theta_m}^{\theta_M} r^{2\alpha}e^{-2\sqrt{\pi r}\delta W}d\theta = (\theta_M - \theta_m) \int_0^h r^{2\alpha+1}e^{-2\delta W\sqrt{\pi r}}dr \quad (t = 2\delta W\sqrt{\pi r})
\]

\[
= s^{-(2\alpha+2)} \frac{2(\theta_M - \theta_m)}{(4\delta W)^{2\alpha+2}} \int_0^{2\delta W\sqrt{\pi h}} t^{\alpha+3}e^{-t}dt \leq s^{-(2\alpha+2)} \frac{2(\theta_M - \theta_m)}{(4\delta W)^{2\alpha+2}} \Gamma(4\alpha + 4),
\]

(2.24)

where \(\delta W\) is defined in (2.5). Using (2.8), (2.16) and (2.24), we derive that

\[
|\xi_j^\pm(s)| \leq C \left( |\eta(0)| \frac{\sqrt{\theta_M - \theta_m}e^{-\sqrt{\pi \delta W h}}}{\sqrt{2}} + \|\eta\|_{C^\alpha}s^{-(\alpha+1)} \frac{2(\theta_M - \theta_m)}{(2\delta W)^{2\alpha+2}} \Gamma(4\alpha + 4) \right) j^{-1-\gamma}.
\]

(2.25)

Clearly, there holds

\[
\lim_{\varepsilon \to 0^+} \delta j_1(\varepsilon, s) = -k^2 \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon} (v(x) - v_j(x))u_0(sx)\,dx = 0,
\]

(2.26)

since \(|u_0(sx)| \leq 1\) if \(\varepsilon\) is sufficiently small and \(v - v_j \in L^2(S_h)\).

Therefore, letting \(\varepsilon \to 0^+\) in (2.19) together with (2.18), (2.20), (2.21), (2.22), and (2.26), we can derive the following integral identity:

\[
I_1 + \delta j(s) = I_3 - I_2^\pm - \xi_j^\pm(s).
\]

(2.27)
where
\[
\begin{align*}
I_1 &= \int_{S_h} u_0(sx)(\tilde{f}_{1j}(x) - f_2(x))dx, \\
I_2^\pm &= \int_{\Gamma^\pm_h} \eta(x)u_0(sx)v_j(x)dx, \\
I_3 &= \int_{\Lambda_h} (u_0(sx)\partial_\nu(v - w) - (v - w)\partial_\nu u_0(sx))dx.
\end{align*}
\] (2.28)

Clearly on $\Lambda_h$, it is easy to see that
\[
|u_0(sx)| = e^{\sqrt{s}\cos(\theta/2 + \pi)} \leq e^{-\delta_w \sqrt{s}},
\]
\[
|\partial_\nu u_0(sx)| = \left|\frac{\sqrt{s} e^{i\cos(\theta/2 + \pi)}}{2\sqrt{h}} e^{\sqrt{s}\exp((\theta/2 + \pi))}\right| \leq \frac{1}{2} \sqrt{s} e^{-\delta_w \sqrt{s}},
\]
both of which decay exponentially as $s \to \infty$. Hence we know that
\[
\|u_0(sx)\|_{L^2(\Lambda_h)} \leq e^{-\delta_w \sqrt{s}} \sqrt{\theta_M - \theta_m}, \quad \|\partial_\nu u_0(sx)\|_{L^2(\Lambda_h)} \leq \frac{1}{2} \sqrt{s} e^{-\delta_w \sqrt{s}} \sqrt{\theta_M - \theta_m}.
\]

Under the assumption $v - w \in H^2(\Sigma_{\Lambda_h})$, using Cauchy-Schwarz inequality and the trace theorem, we can prove that
\[
|I_3| \leq \|u_0(sx)\|_{L^2(\Lambda_h)} \|\partial_\nu(v - w)\|_{L^2(\Lambda_h)} + \|\partial_\nu u_0(sx)\|_{L^2(\Lambda_h)} \|v - w\|_{L^2(\Lambda_h)} \leq \frac{1}{2} \sqrt{s} e^{-\delta_w \sqrt{s}} \sqrt{\theta_M - \theta_m}.
\] (2.29)

where $c' > 0$ as $s \to \infty$.

Since $v_j$ is smooth, then $v_j$ is also $C^\alpha(S_h)$. Therefore $\tilde{f}_{1j}$ and $f_2$ are Hölder-continuous, and for $x \in S_h$ we have the splitting
\[
\tilde{f}_{1j}(x) = \tilde{f}_{1j}(0) + \delta \tilde{f}_{1j}(x), \quad |\delta \tilde{f}_{1j}(x)| \leq \|\tilde{f}_{1j}\|_{C^\alpha} |x|^\alpha,
\]
\[
f_2(x) = f_2(0) + \delta f_2(x), \quad |\delta f_2(x)| \leq \|f_2\|_{C^\alpha} |x|^\alpha.
\] (2.30)

Hence we have
\[
I_1 = (\tilde{f}_{1j}(0) - f_2(0)) \int_{S_h} u_0(sx)dx + \int_{S_h} \delta \tilde{f}_{1j}(x)u_0(sx)dx - \int_{S_h} \delta f_2(x)u_0(sx)dx.
\]

From (2.27) we can deduce the following integral equality
\[
(\tilde{f}_{1j}(0) - f_2(0)) \int_{S_h} u_0(sx)dx + \delta_j(s) = I_3 - I_2^\pm - \int_{S_h} \delta \tilde{f}_{1j}(x)u_0(sx)dx + \int_{S_h} \delta f_2(x)u_0(sx)dx - \xi_j^\pm(s).
\] (2.31)

Using the fact that
\[
\int_{S_h} u_0(sx)dx = \int_{W} u_0(sx)dx - \int_{W\setminus S_h} u_0(sx)dx,
\]
we obtain the following integral equation
\[
(\tilde{f}_{1j}(0) - f_2(0)) \int_{W} u_0(sx)dx + \delta_j(s) = I_3 - I_2^\pm - \int_{S_h} \delta \tilde{f}_{1j}(x)u_0(sx)dx + \int_{S_h} \delta f_2(x)u_0(sx)dx + (\tilde{f}_{1j}(0) - f_2(0)) \int_{W\setminus S_h} u_0(sx)dx - \xi_j^\pm(s).
\]
From (2.5) it can be derived that
\[
\left| \int_{S_h} \delta \tilde{f}_{1j}(x) u_0(sx) dx \right| \leq \int_{S_h} \left| \delta \tilde{f}_{1j}(x) \right| |u_0(sx)| dx \leq \| \tilde{f}_{1j} \|_{C^\alpha} \int_{W} |u_0(sx)||x|^{\alpha} dx \\
\leq \frac{2(\theta_M - \theta_m)\Gamma(2\alpha + 4)}{\delta W^{2\alpha + 4}} \| \tilde{f}_{1j} \|_{C^\alpha} s^{-\alpha - 2}.
\] (2.32)

Similarly we have
\[
\left| \int_{S_h} \delta f_{2}(x) u_0(sx) dx \right| \leq \frac{2(\theta_M - \theta_m)\Gamma(2\alpha + 4)}{\delta W^{2\alpha + 4}} \| f_2 \|_{C^\alpha} s^{-\alpha - 2}.
\] (2.33)

Recall that \( \tilde{f}_{1j} = -k^2 v_j(x) \) and \( v_j \) is the Herglotz wave function given by (2.3). Using the property of compact embedding of Hölder spaces, we can derive that
\[
\| \tilde{f}_{1j} \|_{C^\alpha} \leq k^2 \text{diam}(S_h)^{1-\alpha} \| v_j \|_{C^1},
\]
where \( \text{diam}(S_h) \) is the diameter of \( S_h \). After the direct computation, we have
\[
\| v_j \|_{C^1} \leq \sqrt{2\pi(1 + k)} \| g_j \|_{L^2(S^{n-1})},
\]
therefore we can deduce that
\[
\left| \int_{S_h} \delta \tilde{f}_{1j}(x) u_0(sx) dx \right| \leq \frac{2\sqrt{2\pi}(\theta_M - \theta_m)\Gamma(2\alpha + 4)}{\delta W^{2\alpha + 4}} k^2 \text{diam}(S_h)^{1-\alpha} \cdot (1 + k) \| g_j \|_{L^2(S^{n-1})} s^{-\alpha - 2}.
\] (2.34)

From (2.6) we know that
\[
(\tilde{f}_{1j}(0) - f_2(0)) \int_{W} u_0(sx) dx = 6i(\tilde{f}_{1j}(0) - f_2(0))(e^{-2\theta_M i} - e^{-2\theta_m i}) s^{-2}.
\] (2.35)

Using the Jacobi-Anger expansion (cf. [17, Page 75]), for \( v_j \) given (2.3), we have
\[
v_j(x) = v_j(0)J_0(k|x|) + 2 \sum_{p=1}^{\infty} \gamma_{pj} i^p J_p(k|x|), \quad x \in \mathbb{R}^2,
\] (2.36)

where
\[
v_j(0) = \int_{S^{n-1}} g_j(\theta) d\sigma(\theta), \quad \gamma_{pj} = \int_{S^{n-1}} g_j(\theta) \cos(p\varphi) d\sigma(\theta),
\]
and \( J_p(t) \) is the \( p \)-th Bessel function of the first kind [1]. From [1], we have the explicit expression of \( J_p(t) \) as follows:
\[
J_p(t) = \frac{\Gamma p}{2^p p!} + \frac{\Gamma p}{2^p} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell t^{2\ell}}{4^\ell (\ell!)^2}, \quad \text{for } p = 1, 2, \ldots.
\] (2.37)

Now let us investigate the boundary integral \( I_2^\pm \) defined in (2.28). In this situation, the polar coordinates \( x = (r \cos \theta, r \sin \theta) \) satisfy \( r \in (0, h) \) and \( \theta = \theta_m \) or \( \theta = \theta_M \) when \( x \in \Gamma_h^- \) or \( x \in \Gamma_h^+ \), respectively. Since \( \eta \in C^\alpha(\Gamma_h^\pm) \), recall that \( \eta \) has the expansion (2.23). Substituting (2.23) into the expression of \( I_2^- \), we have
\[
I_2^- = \eta(0) I_{21}^- + I_{22}^-,
\] (2.38)
\[ I_{21} = \int_{\Gamma_h} u_0(sx) v_j(x) \, d\sigma, \quad I_{\eta}^- = \int_{\Gamma_{\eta}} \delta \eta(x) u_0(sx) v_j(x) \, d\sigma. \]

Denote
\[ \omega(\theta) = -\cos(\theta/2 + \pi), \quad \mu(\theta) = -\cos(\theta/2 + \pi) - i \sin(\theta/2 + \pi). \] (2.39)

Then it is easy to see that \( \omega(\theta) > 0 \) for \( \theta_m \leq \theta \leq \theta_M \). From (2.23), combining with the expansion (2.36) for \( v_j \), we can derive that
\[ |I_{\eta}^-| \leq \|\eta\|_C \int_0^h r^\alpha \left( v_j(0) J_0(kr) + 2 \sum_{p=1}^{\infty} |\gamma_{pj}| J_p(kr) \right) e^{-\sqrt{sr}\omega(\theta_m)} \, dr. \]

For any \( \zeta > 0 \), using variable substitution \( t = \sqrt{sr} \), it is easy to calculate that
\[ \int_0^h r^\zeta e^{-\sqrt{sr}\omega(\theta)} \, dr = O(s^{-\zeta-1}), \] (2.40)
as \( s \to \infty \) if \( \omega(\theta) > 0 \). Here (2.40) shows that the lowest increasing term in the integral of (2.40) with respect to \( s \) as \( s \to \infty \) is \( s^{-\zeta-1} \).

Recall that
\[ J_0(t) = \sum_{p=0}^{\infty} (-1)^p \frac{t^{2p}}{4^p (p!)^2}. \] (2.41)

It is easy to see that
\[ I_1^- = \int_0^h r^\alpha J_0(kr) e^{-\sqrt{sr}\omega(\theta_m)} \, dr = I_{11}^- + I_{12}^-, \]

where
\[ I_{11}^- = \int_0^h r^\alpha e^{-\sqrt{sr}\omega(\theta_m)} \, dr, \quad I_{12}^- = \sum_{p=1}^{\infty} \frac{(-1)^p k^{2p}}{4^p (p!)^2} \int_0^h r^{\alpha+2p} e^{-\sqrt{sr}\omega(\theta_m)} \, dr. \]

From (2.40), we have
\[ I_{11}^- = O(s^{-1-\alpha}) \]
as \( s \to \infty \) since \( \omega(\theta_m) > 0 \). For \( I_{12}^- \), we have the estimation
\[ |I_{12}^-| \leq \sum_{p=1}^{\infty} \frac{h^{2p-2} k^{2p}}{4^p (p!)^2} \int_0^h r^{\alpha+2p} e^{-\sqrt{sr}\omega(\theta_m)} \, dr = O(s^{-3-\alpha}) \]
as \( s \to \infty \), where we suppose that \( kh < 1 \) for sufficiently small \( h \). Therefore, we conclude that
\[ |I_1^-| \leq O(s^{-1-\alpha}) \] (2.42)
as \( s \to \infty \). Denote
\[ I_2^- = 2 \sum_{p=1}^{\infty} \int_0^h r^\alpha \gamma_{pj} i^p J_p(kr) e^{-\sqrt{sr}\mu(\theta_m)} \, dr. \]
For sufficiently small $h > 0$, using (2.37), we have

$$|I_2^-| \leq 2\|g_j\|_{L^2(\mathbb{S}^{n-1})} \sum_{p=1}^{\infty} \left[ \frac{k^p}{2^p p!} \int_0^{h} r^{p+\alpha} e^{-\sqrt{s}\omega(\theta_m)} \, dr \right.$$  

$$+ \frac{k^p}{2^p} \sum_{\ell=1}^{\infty} \frac{k^2 \ell^2 (\ell-1)}{4^\ell (\ell!)^2} \left( \int_0^{h} r^{p+\alpha+2} e^{-\sqrt{s}\omega(\theta_m)} \, dr \right]$$  

$$\leq 2\|g_j\|_{L^2(\mathbb{S}^{n-1})} \sum_{p=1}^{\infty} \left[ \frac{k^p}{2^p p!} \int_0^{h} r^{p+\alpha} e^{-\sqrt{s}\omega(\theta_m)} \, dr \right.$$

$$+ \frac{(kh)^p}{2^p} \sum_{\ell=1}^{\infty} \frac{k^2 \ell^2 (\ell-1)}{4^\ell (\ell!)^2} \left( \int_0^{h} r^{p+\alpha+2} e^{-\sqrt{s}\omega(\theta_m)} \, dr \right]$$

$$\leq 2\|g_j\|_{L^2(\mathbb{S}^{n-1})} \sum_{p=1}^{\infty} \left[ \frac{k^p h^p}{2^p p!} \int_0^{h} r^{p+\alpha+1} e^{-\sqrt{s}\omega(\theta_m)} \, dr + O\left(s^{-\alpha-3}\right) \right],$$

where we suppose that $kh < 1$ for sufficiently small $h$. Using (2.40), we know that

$$\int_0^{h} r^{\alpha+1} e^{-\sqrt{s}\omega(\theta_m)} \, dr = O(s^{-\alpha-2})$$

as $s \to \infty$. Therefore we derive that

$$|I_2^-| \leq O(\|g_j\|_{L^2(\mathbb{S}^{n-1})} s^{-\alpha-2}), \quad (2.43)$$

as $s \to \infty$.

Now we shall investigate $I_{21}^-$. Recall that $v_j$ has the expansion (2.36). It is not difficult to see that

$$I_{21}^- = \int_0^{h} \left( v_j(0)J_0(kr) + 2 \sum_{p=1}^{\infty} \gamma_{pj} i^p J_p(kr) \right) e^{-\sqrt{s}\mu(\theta_m)} \, dr$$

$$:= v_j(0)I_{31}^- + I_{32}^-.$$

Substituting the expansion (2.41) of $J_0$ into $I_{31}$, we have

$$I_{31}^- = I_{311}^- + I_{312}^-,$$

where

$$I_{311}^- = \int_0^{h} e^{-\sqrt{s}\mu(\theta_m)} \, dr, \quad I_{312}^- = \sum_{p=1}^{\infty} \frac{(-1)^{p-2}}{4^p (p!)^2} \int_0^{h} r^{2p} e^{-\sqrt{s}\mu(\theta_m)} \, dr.$$

Using variable substitution $t = \sqrt{s}$, we can derive that

$$I_{311}^- = 2s^{-1} \left( \mu(\theta_m)^{-2} - \mu(\theta_m)^{-2} e^{-\sqrt{s}\mu(\theta_m)} - \mu(\theta_m)^{-1} \sqrt{s} e^{-\sqrt{s}\mu(\theta_m)} \right), \quad (2.44)$$

Besides, for $I_{312}^-$, we have

$$|I_{312}^-| \leq \sum_{p=1}^{\infty} \frac{k^{2p+2}}{4^p (p!)^2} \int_0^{h} r^{2p} e^{-\sqrt{s}\omega(\theta_m)} \, dr = O(s^{-3}). \quad (2.45)$$
Substituting the expansion (2.37) of $J_p$ into $I_{32}$, we can deduce that

$$
|I_{32}| \leq 2\|g_j\|_{L^2(S^{n-1})} \sum_{p=1}^{\infty} \left[ \frac{k^p}{2p!} \int_0^h r^p e^{-\sqrt{s}r^\omega(\theta_m)} \, dr 
+ \frac{(kh)^p}{2^p} \sum_{\ell=1}^{\infty} \frac{k^{2\ell}h^{2(\ell-1)}}{4^{\ell}(\ell)!^2} \int_0^h r^{2\ell} e^{-\sqrt{s}r^\omega(\theta_m)} \, dr \right]
\leq 2\|g_j\|_{L^2(S^{n-1})} \sum_{p=1}^{\infty} \left[ \frac{k^p h^{p-1}}{2^p p!} \int_0^h r^p e^{-\sqrt{s}r^\omega(\theta_m)} \, dr + O(s^{-3}) \right]
= O(\|g_j\|_{L^2(S^{n-1})} s^{-2}),
$$

(2.46)

where we suppose that $kh < 1$ for sufficiently small $h$.

Finally, substituting (2.42), (2.43), (2.44), (2.45) and (2.46) into (2.38), we have the following integral property

$$
I_2^- = 2\eta(0)v_j(0)s^{-1} \left( \mu(\theta_m)^{-2} - \mu(\theta_m) - \mu(\theta_m)^{-1}\sqrt{\eta}\mu(\theta_m) \right)
+ v_j(0)\eta(0)I_{312}^- + \eta(0)I_{32}^- + I_2^-, \\
I_{312}^- \leq O(s^{-3}), \quad I_{32}^- \leq O(\|g_j\|_{L^2(S^{n-1})} s^{-2}), \\
|I_2^-| \leq \|\eta\|_{C^\alpha} (v_j(0)I_1^- + I_2^-), \\
|I_1^-| \leq O(s^{-1-\alpha}), \quad |I_2^-| \leq O(\|g_j\|_{L^2(S^{n-1})} s^{-2-\alpha}).
$$

(2.47)

Adopting the similar argument for the integral property (2.47) of $I_2^-$, we can derive the following integral property for $I_2^+$ as follows

$$
I_2^+ = 2\eta(0)v_j(0)s^{-1} \left( \mu(\theta_M)^{-2} - \mu(\theta_M) - \mu(\theta_M)^{-1}\sqrt{\eta}\mu(\theta_M) \right)
+ v_j(0)\eta(0)I_{312}^+ + \eta(0)I_{32}^+ + I_2^+, \\
I_{312}^+ \leq O(s^{-3}), \quad I_{32}^+ \leq O(\|g_j\|_{L^2(S^{n-1})} s^{-2}), \\
|I_2^+| \leq \|\eta\|_{C^\alpha} (v_j(0)I_1^+ + I_2^+), \\
I_1^+ = \int_{\Gamma_0^+} \delta\eta(x)u_0(sx)v_j(x) \, d\sigma,
$$

(2.48)

where

$$
I_{312}^+ = \sum_{p=1}^{\infty} \frac{(-1)^p k^{2p}}{4^p p!^2} \int_0^h r^p e^{-\sqrt{s}r^\mu(\theta_M)} \, dr, \quad I_{32}^+ = 2\sum_{p=1}^{\infty} \int_0^h \gamma_{pj} r^p J_p(kr) e^{-\sqrt{s}r^\omega(\theta_M)} \, dr,
$$

$$
I_2^+ = \sum_{p=1}^{\infty} \int_0^h r^p \gamma_{pj} J_p(kr) e^{-\sqrt{s}r^\omega(\theta_M)} \, dr.
$$
Substituting (2.47) and (2.48) into (2.31), multiplying s on the both sides of (2.31), and rearranging terms, we deduce that

\[
2v_j(0)\eta(0) \left[ \left( \mu(\theta_M)^{-2} - \mu(\theta_m)^{-2} e^{-\sqrt{s\theta_M}(\theta_M)} - \mu(\theta_M)^{-1} \sqrt{s\theta_M}(\theta_M) \right) + \left( \mu(\theta_m)^{-2} - \mu(\theta_m)^{-2} e^{-\sqrt{s\theta_m}(\theta_m)} - \mu(\theta_m)^{-1} \sqrt{s\theta_m}(\theta_m) \right) \right]
\]

\[
= s \left[ I_3 - (\bar{f}_j(0) - f_2(0)) \int_{S_h} u_0(sx)dx - \delta_j(s) - v_j(0)\eta(0) (I_{312}^- + I_{312}^+) \right.
\]

\[
- \eta(0)(I_{32}^- + I_{32}^+) - I_{\eta}^+ - I_{\eta}^- - \int_{S_h} \delta f_1(x)u_0(sx)dx + \int_{S_h} \delta f_2(x)u_0(sx)dx - \xi_j^\pm(s) \right].
\]

When \(s = j\), from (2.47) and (2.48), under the assumption (2.8) we know that

\[
j|I_{32}^\pm| \leq O(j^{-1} \||g|\||L^2(S_h)) \leq O(j^{-1+\varepsilon}), \quad j|I_{32}^+| \leq O(j^{-1} \||g|\||L^2(S_h)) \leq O(j^{-1+\varepsilon}),
\]

\[
j|I_{\eta}^\pm| \leq \|\eta\|_{C^0} (v_j(0)O(j^{-\alpha}) + O(\||g|\||L^2(S_h)^{-1-\alpha}))
\]

\[
\leq \|\eta\|_{C^0} (v_j(0)O(j^{-\alpha}) + O(\||g|\||L^2(S_h)^{-1-\alpha})),
\]

\[
j|I_{312}^\pm| \leq O(j^{-2}), \quad j|I_{312}^-| \leq O(j^{-2}).
\]

Clearly, when \(s = j\), from (2.6), (2.7), (2.17), (2.25), (2.29), (2.33) and (2.34), under the assumption (2.8) it can be derived that

\[
j|I_3| \leq C |e^{-c' \sqrt{\eta}}|, \quad j \left| \int_{S_h} u_0(jx)dx \right| \leq 6|e^{-2\theta_M} - e^{-2\theta_m}j^{-1} + \frac{6(\theta_M - \theta_m)}{\delta_W^2} j^{-1} e^{-\sqrt{\eta\Delta}/2},
\]

\[
j \left| \int_{S_h} \delta f_1(x)u_0(jx)dx \right| \leq \frac{2\sqrt{2\pi}(\theta_M - \theta_m)\Gamma(2\alpha + 4)}{\delta_W^{2\alpha+1}} j^{-1} \left[ 1 + k \|g\|_{L^2(S_h)^{-1}} \right]^{-\alpha} \leq O(j^{-1+\varepsilon}),
\]

\[
j \left| \int_{S_h} \delta f_2(x)u_0(jx)dx \right| \leq \frac{2(\theta_M - \theta_m)\Gamma(2\alpha + 4)}{\delta_W^{2\alpha+1}} \|f_2\|_{C^0} j^{-\alpha-1},
\]

\[
j |\xi_j^\pm(j)| \leq C \left( \eta(0) \sqrt{\theta_M - \theta_m} e^{-\sqrt{\eta\Delta\delta_W}\delta^2} j^{-1-\gamma} + \|\eta\|_{C^0} j^{-1} \sqrt{2(\theta_M - \theta_m)\Gamma(4\alpha + 4)} \right) j^{-1+\varepsilon},
\]

\[
j |\delta_j(j)| \leq \frac{\sqrt{\theta_M - \theta_m} e^{-\sqrt{\eta\Delta\delta_W}\delta^2}}{\sqrt{2}} j^{-\gamma}, \quad \Theta \in [0, h],
\]

where \(c' > 0\) and \(\delta_W\) are defined in (2.29) and (2.5), respectively. The coefficient of \(v_j(0)\) of (2.49) with respect to the zeroth order of \(s\) is

\[
2\eta(0) \left( \mu(\theta_m)^{-2} + \mu(\theta_M)^{-2} \right).
\]
It can be calculated that
\[
\mu(\theta_m)^{-2} + \mu(\theta_M)^{-2} = \frac{(\cos \theta_m + \cos \theta_M) + i(\sin \theta_m + \sin \theta_M)}{(\cos \theta_m + i \sin \theta_m)(\cos \theta_M + i \sin \theta_M)}.
\]
Therefore under the assumption (2.10), it is not difficult to see that
\[
\cos \theta_m + \cos \theta_M \quad \text{and} \quad \sin \theta_m + \sin \theta_M
\]
can not be zero simultaneously, which implies
\[
\mu(\theta_m)^{-2} + \mu(\theta_M)^{-2} \neq 0 \tag{2.52}
\]
We take \( s = j \) in (2.49). By letting \( j \to \infty \) in (2.49), from (2.50) and (2.51), we can prove that
\[
\eta(0) \left( \mu(\theta_m)^{-2} + \mu(\theta_M)^{-2} \right) \lim_{j \to \infty} v_j(0) = 0.
\]
Since \( \eta(0) \neq 0 \) from (2.9) and (2.52), it is easy to see that
\[
\lim_{j \to \infty} v_j(0) = 0.
\]
Using the fact that
\[
\lim_{\rho \to 0} \frac{1}{m(B(x_c, \rho))} \int_{B(x_c, \rho)} |v(x)| \, dx \leq \lim_{j \to \infty} \left( \lim_{\rho \to 0} \frac{1}{m(B(x_c, \rho))} \int_{B(x_c, \rho)} |v(x) - v_j(x)| \, dx \right.
\]
\[
+ \left. \lim_{\rho \to 0} \frac{1}{m(B(x_c, \rho))} \int_{B(x_c, \rho)} |v_j(x)| \, dx \right), \tag{2.53}
\]
we readily finish the proof of this theorem. \( \square \)

We next consider the degenerate case of Theorem 2.1 with \( \eta \equiv 0 \). The conductive transmission eigenvalue problem (1.1) is reduced to the following interior transmission eigenvalue problem
\[
\begin{aligned}
\Delta w + k^2(1 + V)w &= 0 \quad \text{in} \ \Omega, \\
\Delta v + k^2 v &= 0 \quad \text{in} \ \Omega, \\
w &= v, \quad \partial_n v = \partial_n w \quad \text{on} \ \partial \Omega,
\end{aligned} \tag{2.54}
\]
By slightly modifying our proof of Theorem 2.1, we can show the following result.

**Corollary 2.1.** Suppose \( v \in H^1(\Omega) \) and \( w \in H^1(\Omega) \) are a pair of interior transmission eigenfunctions to (2.54). Let \( W \) and \( S_h \) be the same as described in Theorem 2.1. Assume that \( v - w \in H^2(\Sigma_h) \) and \( q w \in C^\alpha(S_h) \) for \( 0 < \alpha < 1 \). Under the conditions (2.10) and that the transmission eigenfunction \( v \) can be approximated in \( H^1(S_h) \) by the Herglotz functions \( v_j, \ j = 1, 2, \ldots, \) with kernels \( g_j \) satisfying
\[
\|v - v_j\|_{H^1(S_h)} \leq j^{-2-\Upsilon}, \quad \|g_j\|_{L^2(S^{n-1})} \leq C j^\rho,
\]
for some constants \( C > 0, \ \Upsilon > 0 \) and \( 0 < \rho < \alpha \), one has
\[
\lim_{\rho \to 0} \frac{1}{m(B(x_c, \rho))} \int_{B(x_c, \rho)} V(x)w(x) \, dx = 0.
\]

**Remark 2.3.** As discussed in the introduction, the vanishing near a corner of the interior transmission eigenfunctions was considered in [9]. Compared to the main result in [9], Corollary 2.1 is more general in two aspects. First, the corner in [9] must be a convex one, whereas in Corollary 2.1, the corner could be an arbitrary one as long as the corner is not degenerate, namely (2.10) is fulfilled. Second, the regularity requirement on the
Proof of Corollary 2.1. The proof follows from the one for Theorem 2.1 with some necessary modifications, and we only outline it in the following. Without loss of generality, we assume that \( x_c = 0 \). Since \( \eta(x) \equiv 0 \) near the corner, similar to (2.31), we have the following integral identity,

\[
(\tilde{f}_{ij}(0) - f_2(0)) \int_{S_h} u_0(sx)dx + \delta_j(s) = I_3 - \int_{S_h} \delta \tilde{f}_{ij}(x)u_0(sx)dx + \int_{S_h} \delta f_2(x)u_0(sx)dx,
\]

(2.56)

where \( f_2(x), \tilde{f}_{ij}(x), \delta_j(s), I_3, \delta \tilde{f}_{ij}(x) \) and \( \delta f_2(x) \) are defined in (2.13), (2.14), (2.28) and (2.30), respectively.

From (2.6), it follows that

\[
(\tilde{f}_{ij}(0) - f_2(0)) \int_{S_h} u_0(sx)dx = (\tilde{f}_{ij}(0) - f_2(0)) \int_{W} u_0(sx)dx - (\tilde{f}_{ij}(0) - f_2(0)) \int_{W \setminus S_h} u_0(sx)dx
\]

\[
= 6i(\tilde{f}_{ij}(0) - f_2(0))(e^{-2\theta_1}\eta - e^{-2\theta_2})s^2 - (\tilde{f}_{ij}(0) - f_2(0)) \int_{W \setminus S_h} u_0(sx)dx.
\]

From (2.16) and (2.55), it is not difficult to see that

\[
j^2|\delta_j(s)| \leq \frac{\sqrt{\theta_M - \theta_m}k^2e^{-\sqrt{\gamma_0}h\delta_W}}{\sqrt{2}} j^{-\gamma},
\]

(2.58)

where \( \Theta \in [0, h] \) and \( \delta_W \) is defined in (2.5). By (2.34), we can also deduce that

\[
j^2 \left| \int_{S_h} \delta \tilde{f}_{ij}(x)u_0(jx)dx \right| \leq 2\sqrt{2\pi(\theta_M - \theta_m)}\Gamma(2\alpha + 1)k^2\text{diam}(S_h)^{1-\alpha}
\]

\[
\cdot (1 + k)\|g_1\|_{L^2(\mathcal{B}^{\alpha-1})} j^{-\alpha} \leq O(j^{-(\alpha-\Theta)}),
\]

(2.59)

for \( 0 < \rho < \alpha \). After substituting (2.57) into (2.56), we take \( s = j \). Since (2.57), multiplying \( j^2 \) on both sides of (2.56), using the assumptions (2.55) and (2.10), by letting \( j \to \infty \), from (2.7), (2.29), (2.32), (2.33) and (2.58), we prove that

\[
\lim_{j \to \infty} v_j(0) = \frac{f_2(0)}{-k^2}.
\]

Since

\[
\lim_{j \to \infty} v_j(0) = \lim_{j \to \infty} \lim_{\rho \to +0} \frac{1}{m(B(0, \rho))} \int_{B(0, \rho)} v_j(x)dx = \lim_{\rho \to +0} \frac{1}{m(B(0, \rho))} \int_{B(0, \rho)} v(x)dx,
\]

and

\[
f_2(0) \frac{1}{-k^2} = \lim_{\rho \to +0} \frac{1}{m(B(0, \rho))} \int_{B(0, \rho)} qw(x)dx,
\]

together with

\[
\lim_{\rho \to +0} \frac{1}{m(B(0, \rho))} \int_{B(0, \rho)} v(x)dx = \lim_{\rho \to +0} \frac{1}{m(B(0, \rho))} \int_{B(0, \rho)} w(x)dx,
\]
we finish the proof of this corollary. □

**Remark 2.4.** If \( V(x) \) is continuous near the corner \( x_c \) and \( V(x_c) \neq 0 \), from the fact that

\[
\lim_{\rho \to 0} \frac{1}{m(B(x_c, \rho))} \int_{B(x_c, \rho)} V(x)w(x)dx = V(x_c) \lim_{\rho \to 0} \frac{1}{m(B(x_c, \rho))} \int_{B(x_c, \rho)} w(x)dx,
\]

we can prove that the vanishing property near the corner \( x_c \) of the interior transmission eigenfunctions \( v \) and \( w \) (to (1.1)), we can show that more apparent vanishing properties hold at the corner. The rest of this section is devoted to this case. In fact, we have the following theorem.

**Theorem 2.2.** Let \( v \in H^2(\Omega) \) and \( w \in H^1(\Omega) \) be eigenfunctions to (1.1). Assume that \( \Omega \subset \mathbb{R}^2 \) has a corner \( x_c \) such that \( x_c \) is the vertex of \( \Omega \cap W \) where \( W \) is the sector defined in (2.1). Moreover, there exits a sufficiently small neighbourhood \( S_h \) (i.e. \( h > 0 \) is sufficiently small) of \( x_c \) in \( \Omega \), such that \( qw \in C^\alpha(\overline{S_h}) \) and \( \eta \in C^\alpha(\overline{\Gamma_h^\pm}) \) for \( 0 < \alpha < 1 \), and \( v - w \in H^2(\Sigma_{\Lambda_h}) \). Under the following assumptions:

(a) the function \( \eta(x) \) does not vanish at the corner, i.e.,

\[
\eta(x_c) \neq 0.
\]

(b) the angles \( \theta_m \) and \( \theta_M \) of the sector \( W \) containing the corner satisfy

\[
-\pi < \theta_m < \theta_M < \pi \quad \text{and} \quad \theta_M - \theta_m \neq \pi,
\]

then we have \( v(x_c) = w(x_c) = 0 \).

**Proof.** Recall that \( f_1 \) and \( f_2 \) are defined by (2.13) and \( D_\varepsilon = S_h \setminus B_\varepsilon \), integrating by parts, we have

\[
\int_{D_\varepsilon} (f_1 - f_2)u_0(sx)dx = I_3 + \int_{\Lambda_\varepsilon} (u_0(sx)\partial_\nu(v - w) - (v - w)\partial_\nu u_0(sx))d\sigma
\]

\[
- \int_{\Gamma_\varepsilon}^\pm \eta(x)u_0(sx)v(x)d\sigma,
\]

where \( \Lambda_\varepsilon = S_h \cap \partial B_\varepsilon \), \( \Lambda_\varepsilon = S_h \cap \partial B_\varepsilon \), \( \Gamma_\varepsilon^\pm = \Gamma_\varepsilon^\pm \cap (B_\varepsilon \setminus B_\varepsilon) \) and \( I_3 \) is defined in (2.28). Using similar technique in the proof of Theorem 2.1, under the assumptions that \( v - w \in H^2(\Sigma_{\Lambda_h}) \), \( v \in H^2(S_h) \), \( qw \in C^\alpha(\overline{S_h}) \) and \( \eta \in C^\alpha(\overline{\Gamma_h^\pm}) \), we have

\[
\lim_{\varepsilon \to 0} \int_{D_\varepsilon} (f_1 - f_2)u_0(sx)dx = \int_{S_h} (f_1 - f_2)u_0(sx)dx,
\]

\[
\lim_{\varepsilon \to 0} \int_{\Lambda_\varepsilon} (u_0(sx)\partial_\nu(v - w) - (v - w)\partial_\nu u_0(sx))d\sigma = 0,
\]

\[
\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon}^\pm \eta(x)u_0(sx)v(x)d\sigma = \int_{\Gamma_\varepsilon}^\pm \eta(x)u_0(sx)v(x)d\sigma.
\]

Therefore we obtain the following integral identity:

\[
\int_{S_h} (f_1 - f_2)u_0(sx)dx = I_3 - \int_{\Gamma_h^\pm} \eta(x)u_0(sx)v(x)d\sigma.
\]
Since $f_1, f_2 \in C^\alpha(S_h)$, $\eta \in C^\alpha(\Gamma_h^\pm)$, and $v \in H^2(S_h)$ can be embedded into $C^\alpha(S_h)$, the following splitting hold

\begin{align*}
f_1(x) &= f_1(0) + \delta f_1(x), \quad |\delta f_1(x)| \leq \|f_1\|_{C^\alpha} |x|^\alpha, \\
f_2(x) &= f_2(0) + \delta f_2(x), \quad |\delta f_2(x)| \leq \|f_2\|_{C^\alpha} |x|^\alpha, \\
\eta(x) &= \eta(0) + \delta \eta(x), \quad |\delta \eta(x)| \leq \|\eta\|_{C^\alpha} |x|^\alpha, \\
v(x) &= v(0) + \delta v(x), \quad |\delta v(x)| \leq \|v\|_{C^\alpha} |x|^\alpha.
\end{align*}

(2.63)

Substituting (2.63) into (2.62), we can derive that

\begin{align*}
(f_1(0) - f_2(0)) \int_{S_h} u_0(sx) \, dx + \int_{S_h} (\delta f_1 - \delta f_2) u_0(sx) \, dx \\
= I_3 - \eta(0)v(0) \int_{\Gamma_h^+} u_0(sx) \, d\sigma - \eta(0) \int_{\Gamma_h^+} \delta v(x) u_0(sx) \, d\sigma - v(0) \int_{\Gamma_h^+} \delta \eta(x) u_0(sx) \, d\sigma \\
- \int_{\Gamma_h^+} \delta \eta(x) \delta v(x) u_0(sx) \, d\sigma.
\end{align*}

(2.64)

From (2.44), it is easy to see that

\begin{align*}
\eta(0)v(0) \int_{\Gamma_h^+} u_0(sx) \, d\sigma &= 2s^{-1}v(0) \eta(0) \left( \mu(\theta_M)^{-2} - \mu(\theta_M)^{-2} e^{-\sqrt{s}\mu(\theta_M)} \right) \\
&\quad - \mu(\theta_M)^{-1} \sqrt{s} e^{-\sqrt{s}\mu(\theta_M)} \\
\eta(0)v(0) \int_{\Gamma_h^-} u_0(sx) \, d\sigma &= 2s^{-1}v(0) \eta(0) \left( \mu(\theta_m)^{-2} - \mu(\theta_m)^{-2} e^{-\sqrt{s}\mu(\theta_m)} \right) \\
&\quad - \mu(\theta_m)^{-1} \sqrt{s} e^{-\sqrt{s}\mu(\theta_m)}
\end{align*}

(2.65)

where $\mu(\theta)$ is defined in (2.39). Besides, from (2.63), using (2.40), we can estimate

\begin{align*}
s \left| \int_{\Gamma_h^+} \delta v(x) u_0(sx) \, d\sigma \right| &\leq s \|v\|_{C^\alpha} \int_0^h r^\alpha e^{-\sqrt{s}r\omega(\theta_m)} \, dr = O(s^{-\alpha}), \\
s \left| \int_{\Gamma_h^+} \delta \eta(x) u_0(sx) \, d\sigma \right| &\leq s \|\eta\|_{C^\alpha} \int_0^h r^\alpha e^{-\sqrt{s}r\omega(\theta_m)} \, dr = O(s^{-\alpha}), \\
s \left| \int_{\Gamma_h^+} \delta \eta(x) \delta v(x) u_0(sx) \, d\sigma \right| &\leq s \|v\|_{C^\alpha} \|\eta\|_{C^\alpha} \int_0^h r^{2\alpha} e^{-\sqrt{s}r\omega(\theta_m)} \, dr = O(s^{-2\alpha}), \\
s \left| \int_{S_h} \delta f_1 u_0(sx) \, dx \right| &\leq s \cdot \|f_1\|_{C^\alpha} \int_W |u_0(sx)||x|^\alpha \, dx \\
&\leq \frac{2\|f_1\|_{C^\alpha} (\theta_M - \theta_m) \Gamma(2\alpha + 4)}{\delta_{2W}^{2\alpha+4}} s^{-\alpha-1}, \\
s \left| \int_{S_h} \delta f_2 u_0(sx) \, dx \right| &\leq s \cdot \|f_2\|_{C^\alpha} \int_W |u_0(sx)||x|^\alpha \, dx \\
&\leq \frac{2\|f_2\|_{C^\alpha} (\theta_M - \theta_m) \Gamma(2\alpha + 4)}{\delta_{2W}^{2\alpha+4}} s^{-\alpha-1}.
\end{align*}

(2.66)
Substituting (2.65) into (2.64) and multiplying s on the both sides of (2.64), after arranging terms, we obtain that

$$2v(0)\eta(0)\left(\mu(\theta_M)^{-2} + \mu(\theta_m)^{-2}\right) = 2v(0)\eta(0)\left(\mu(\theta_M)^{-2}e^{-\sqrt{\bar{\mu}_M}\theta_M} + \mu(\theta_m)^{-2}e^{-\sqrt{\bar{\mu}_m}\theta_m}\right)$$

$$+ \mu(\theta_M)^{-1}\sqrt{\bar{\mu}_M}e^{-\sqrt{\bar{\mu}_M}\theta_M} + \mu(\theta_m)^{-1}\sqrt{\bar{\mu}_m}e^{-\sqrt{\bar{\mu}_m}\theta_m}$$

$$+ s\left[I_3 - (f_1(0) - f_2(0))\int_{S_h} u_0(sx)dx - \int_{S_h} (\delta f_1 - \delta f_2)u_0(sx)dx - \eta(0)\int_{\Gamma_h^+}\delta v(x)u_0(sx)d\sigma - v(0)\int_{\Gamma_h^+}\delta\eta(x)u_0(sx)d\sigma - \int_{\Gamma_h^+}\delta\eta(x)\delta v(x)u_0(sx)d\sigma\right].$$

Since $v - w \in H^2(\Sigma_{\Lambda_h})$, (2.29) still holds. In (2.67) letting $s \to \infty$, under the assumption (2.60), from (2.7), (2.29), (2.57) and (2.66), we can show that

$$\eta(0)\left(\mu(\theta_M)^{-2} + \mu(\theta_m)^{-2}\right)v(0) = 0.$$

Under the assumption (2.61), from the proof of Theorem 2.1 we have shown that $\mu(\theta_M)^{-2} + \mu(\theta_m)^{-2} \neq 0$. Since $\eta(0) \neq 0$ from (2.60), we finish the proof of this theorem. □

**Remark 2.5.** Under the $H^2$ regularity, the interior transmission eigenfunction to (2.54) had been shown that they always vanish at a corner point if the interior angle of the corner is not $\pi$; see [4, Theorem 4.2] for more details.

### 3. Vanishing near corners of conductive transmission eigenfunctions: three-dimensional case

In this section, we study the vanishing property of the conductive transmission eigenfunctions for the 3D case. In principle, we could also consider a generic corner in the usual sense as the one for the 2D case. However, in what follows, we introduce a more general corner geometry that is described by $W \times (-M, M)$, where $W$ is a sector defined in (2.1) and $M \in \mathbb{R}_+$. It is readily seen that $W \times (-M, M)$ actually describes an edge singularity and we call it a 3D corner for notational unification. Suppose that the Lipschitz domain $\Omega \subset \mathbb{R}^3$ possesses a 3D corner. Let $x_\infty \in \mathbb{R}^3$ be the vertex of $W$ and $(x_\infty, x_n)$ is defined as the edge point of $W \times (-M, M)$. In Figure 2, we give a schematic illustration of the geometry considered in 3D. In this section, under some appropriate assumptions, we show that the conductive transmission eigenfunctions $v$ and $w$ vanish at $(x_\infty, x_n)$. Since the CGO solution constructed in Lemma 2.1 is only two dimensional, in order to make use of the similar arguments of Theorem 2.1, we introduce the following dimension reduction operator. The dimension reduction operator technique is also introduced in [4, Lemma 3.4] for studying the vanishing property of nonradiating sources and the transmission eigenfunctions at edges in three dimension. Similar to Theorem 2.1, we first assume that $v$ is only $H^1$ smooth but can be approximated by the Herglotz wave functions with some mild assumptions, where in Theorem 3.1 the interior angle of the sector $W$ cannot be $\pi$. Besides, if $v$ has $H^2$ regularity near the edge point, in Theorem 3.2 we also prove the vanishing property of $v$ and $w$ near the edge point.

**Definition 3.1.** Let $W \subset \mathbb{R}^{n-1}$ be defined in (2.1), $M > 0$. For a given function $g$ with the domain $W \times (-M, M)$. Pick up any point $x_n^\infty \in (-M, M)$. Suppose $\psi \in C_0^\infty((x_n^\infty - L, x_n^\infty + L))$ is a nonnegative function and $\psi \neq 0$, where $L$ is sufficiently small such that
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\[(x^n_c - L, x^n_c + L) \subset (-M, M), \text{ and write } x = (x', x_n) \in \mathbb{R}^n, x' \in \mathbb{R}^{n-1}. \] The dimension reduction operator \(\mathcal{R}\) is defined by

\[\mathcal{R}(g)(x') = \int_{x^n_c - L}^{x^n_c + L} \psi(x_n)g(x', x_n)dx_n\] (3.1)

where \(x' \in W\).

**Remark 3.1.** The assumption on the non-negativity of \(\psi\) plays an important role in our proof of Theorem 3.1 in what follows, where we use the integral mean value theorem to carefully investigate the asymptotic property of the parameter \(s\) appearing in the CGO solution \(u_0(sx')\) given in Lemma 2.1 as \(s \to \infty\). In order to use the two dimensional CGO solution \(u_0(sx')\) to prove the vanishing property of the conductive transmission eigenfunctions in \(\mathbb{R}^3\), we need the dimension reduction operator defined in Definition 3.1 in our proof of Theorem 3.1.

Before presenting the main results of this section, we first analyze the regularity of the functions after applying the dimension reduction operator. Using a similar argument of [4, Lemma 3.4], we can prove the following lemma, whose detailed proof is omitted.

**Lemma 3.1.** Let \(g \in H^2(W \times (-M, M)) \cap C^\alpha(W \times [-M, M]), \) where \(0 < \alpha < 1\). Then \(\mathcal{R}(g)(x') \in H^2(W) \cap C^\alpha(W)\).

**Theorem 3.1.** Let \(W \subset \mathbb{R}^{n-1}\) be defined in (2.1), \(M > 0, 0 < \alpha < 1\). For any fixed \(x^n_c \in (-M, M)\) and \(L > 0\) defined in Definition 3.1, we suppose that \(L\) is sufficiently small such that \((x^n_c - L, x^n_c + L) \subset (-M, M)\). Let \(v, w \in H^1(W \times (-M, M))\) and there exists a sufficiently small neighbourhood \(S_h\) of \(x_c \in \mathbb{R}^{n-1}\) such that \(qw \in C^\alpha(S_h \times [-M, M])\) and \(\eta \in C^\alpha(T_h^\pm \times [-M, M])\) for \(0 < \alpha < 1\), and \(v - w \in H^2(S_h \times (-M, M))\), where \(x_c\) is the vertex of \(W\) and \(S_h\) is defined in (2.2). Write \(x = (x', x_n) \in \mathbb{R}^n, x' \in \mathbb{R}^{n-1}\) and assume that

\[
\begin{align*}
\Delta v + k^2v &= 0, \quad x' \in W, -M < x_n < M, \\
\Delta w + k^2qw &= 0, \quad x' \in W, -M < x_n < M, \\
w &= v \quad \partial_n v + \eta w = \partial_n w, \quad x' \in \Gamma^\pm, -M < x_n < M.
\end{align*}
\] (3.2)

where \(\Gamma^\pm\) are the boundaries of \(W\). If the following conditions are fulfilled:
(a) the transmission eigenfunction $v$ can be approximated in $H^1(S_h \times (-M,M))$ by

the Herglotz functions $v_j$, $j = 1, 2, \ldots$, with kernels $g_j$ satisfying

$$
\|v - v_j\|_{H^1(S_h \times (-M,M))} \leq j^{-1-\gamma}, \quad \|g_j\|_{L^2(S^{n-1})} \leq Cj^{1+\epsilon},
$$

(3.3)

for some positive constant $C$, $\gamma > 0$ and $0 < \epsilon < \alpha$.

(b) the function $\eta = \eta(x')$ is independent of $x_n$ and

$$
\eta(x_c) \neq 0,
$$

(3.4)

(c) the angles $\theta_m$ and $\theta_M$ of the sector $W$ satisfy

$$
-\pi < \theta_m < \theta_M < \pi \quad \text{and} \quad \theta_M - \theta_m \neq \pi,
$$

(3.5)

then for every edge points $(x_c, x_n^c) \in \mathbb{R}^n$ of $W \times (-M,M)$ where $x_n^c \in (-M,M)$, one has

$$
\lim_{\rho \to 0} \frac{1}{m(B((x_c, x_n^c), \rho))} \int_{m(B((x_c, x_n^c), \rho))} |v(x)| dx = 0,
$$

where $m(B((x_c, x_n^c), \rho))$ is the volume of $B((x_c, x_n^c), \rho)$.

**Proof.** For the edge point $(x_c, x_n^c) \in W \times (-M,M)$, where $x_n^c \in (-M,M)$, without loss of generality, we assume that the vertex $x_c$ of the sector $W \subset \mathbb{R}^{n-1}$ is located at the origin of $\mathbb{R}^{n-1}$ and $x_n^c = 0$. Then $\Delta_{x'} v = -k^2 v - \partial_{x_n^c}^2 v$ and $\Delta_{x'} w = -k^2 qw - \partial_{x_n^c}^2 w$, by the dominate convergence theorem, integration by parts gives

$$
\Delta_{x'} R(v)(x') = \int_{-L}^{L} \psi''(x_n)v(x', x_n) dx_n - k^2 R(v)(x') := G(x'),
$$

$$
\Delta_{x'} R(w)(x') = \int_{-L}^{L} \psi''(x_n)w(x', x_n) dx_n - k^2 R(qw)(x') := \tilde{G}(x').
$$

(3.6)

Moreover, we have

$$
R(w)(x') = R(v)(x') \quad \text{on} \quad \Gamma
$$

(3.7)

in the sense of distribution, since $w(x', x_n) = v(x', x_n)$ when $x' \in \Gamma$ and $-L < x_n < L$. Similarly, using the fact that $\eta$ is independent of $x_n$, we can easily show that

$$
\partial_{x_n} R(v)(x') + \eta(x') R(v)(x') = \partial_{x_n} R(w)(x') \quad \text{on} \quad \Gamma,
$$

(3.8)

in the sense of distribution.

Recall the definition (3.6) of $G$ and $\tilde{G}$. We denote

$$
G(x') - \tilde{G}(x') = F_1(x') + F_2(x') + F_3(x'),
$$

(3.9)

where

$$
F_1(x') = \int_{-L}^{L} \psi''(x_n)(v(x', x_n) - w(x', x_n)) dx_n, \quad F_2(x') = k^2 R(qw)(x'),
$$

$$
F_3(x') = -k^2 R(v)(x').
$$

Since $v - w \in H^2(S_h \times (-L,L))$, from Lemma 3.1 we know that $F_1(x') \in H^2(S_h)$ which can be embedded into $C^0(S_h)$ for $\alpha \in (0,1)$. Also from Lemma 3.1 we have that $F_2(x') \in C^0(S_h)$, since $qw \in C^0(S_h \times [-L,L])$, $0 < \alpha < 1$.

Recall that $D_\varepsilon = S_h \setminus B_\varepsilon$ for $0 < \varepsilon < h$. It can be derived that

$$
\int_{S_h} (F_1(x') + F_2(x') + F_3(x')) u_0(sx') dx' = \lim_{\varepsilon \to 0} \int_{D_\varepsilon} (F_1(x') + F_2(x') + F_3(x')) u_0(sx') dx'
$$

$$
since $|u_0(sx)| \leq 1$ in $S_h \cap B_\varepsilon$ for sufficiently small $\varepsilon$ and $F_1(x') + F_2(x') + F_{3j}(x') \in L^2(S_h \cap B_\varepsilon)$, where

$$F_{3j}(x') = -k^2 R(v_j)(x')$$

and $v_j$ is the Herglotz wave function given by

$$v_j(x) = \int_{S^{n-1}} e^{i k d x} g_j(d) d\sigma(d), \quad d \in S^{n-1}. \quad (3.10)$$

Since $v \in H^1(S_h \times (-L,L))$ is a solution to the Helmholtz equation in $S_h \times (-L,L)$, from [36, Theorem 2.1], $v$ can be approximated by the Herglotz wave function $v_j(x)$ given in (3.10) in the topology induced by $H^1$. Therefore, we deduce that

$$\int_{S_h} -k^2 R(v)(x') u_0(sx') dx' = \int_{S_h} -k^2 R(v_j)(x') u_0(sx') dx' + \delta_j(s), \quad (3.11)$$

where

$$\delta_j(s) = -k^2 \int_{S_h} (R(v)(x') - R(v_j)(x')) u_0(sx') dx',$$

and $u_0$ is given in Lemma 2.1. Since $v_j \in H^2(S_h \times (-L,L))$, from Lemma 3.1, we have $R(v_j)(x') \in H^2(S_h)$ which can be embedded into $C^\alpha(S_h)$. Moreover by using Cauchy-Schwarz inequality, we have

$$\|R(v) - R(v_j)\|_{L^2(S_h)}^2 = \int_{S_h} \left| \int_{-L}^L \psi(x_n)(v(x',x_n) - v_j(x',x_n)) dx_n \right|^2 dx' \leq C(L,h) \|\psi\|_\infty \|v - v_j\|_{L^2(S_h \times (-L,L))}^2, \quad (3.12)$$

where $C(L,h)$ is a positive constant depending on $L$ and $h$. Recall that the $L^2$-norm of $u_0$ in $S_h$ can be estimated by (2.16), again using Cauchy-Schwarz inequality, we deduce that

$$|\delta_j(s)| \leq k^2 \|R(v) - R(v_j)\|_{L^2(S_h)} \|u_0(sx)\|_{L^2(S_h)} \leq \frac{k^2 \|\psi\|_\infty \sqrt{C(L,h)}(\theta_M - \theta_m)e^{-\sqrt{\Theta W} h} j^{1-\gamma}}{\sqrt{2} j^{-1-\gamma}}, \quad (3.13)$$

where $\Theta \in [0, h]$ and $\delta_W$ is defined in (2.5).

Let $\delta_j(\varepsilon, s) = -k^2 \int_{B_\varepsilon} (R(v)(x') - R(v_j)(x')) u_0(sx') dx'$. It can be verified that

$$\lim_{\varepsilon \to 0^+} \delta_j(\varepsilon, s) = 0,$$
Therefore, from \([4, \text{Lemma 3.2}]\), the following equation holds. Since

\[
\Gamma v
\]

in the sense of Green's formula together with (3.7) and (3.8), we have

\[
-\int_{\partial D_h} \Delta \nu(\partial \mathcal{R}(v)(x') - \mathcal{R}(w)(x'))u_0(sx')d\sigma = \int_{\partial D_h} (u_0(sx')\partial \nu(\partial \mathcal{R}(v)(x') - \mathcal{R}(w)(x'))d\sigma
\]

where

\[
\Lambda_h = S_h \cap \partial B_h, \quad \Lambda_\varepsilon = S_h \cap \partial B_\varepsilon \quad \text{and} \quad \Gamma^\pm_{\varepsilon,h} = \Gamma^\pm \cap (B_h \setminus B_\varepsilon).
\]

Since \(v - w \in H^2(S_h \times (L, L))\), from Lemma 3.1 we know that \(\mathcal{R}(v - w) \in H^2(S_h)\). Therefore, from [4, Lemma 3.2], the following equation

\[
\lim_{\varepsilon \to 0} \int_{\Lambda_\varepsilon} (u_0(sx')\partial \nu(\partial \mathcal{R}(v)(x') - \mathcal{R}(w)(x'))d\sigma = 0
\]

holds. Since \(v \in H^1((S_h \cap B_\varepsilon) \times (-L, L))\), also from Lemma 3.1 we have \(\mathcal{R}(v)(x') \in H^1(S_h \cap B_\varepsilon)\), therefore using trace theorem, we have \(\mathcal{R}(v)(x') \in L^2(\Gamma^\pm_{\varepsilon,h})\) where \(\Gamma^\pm_{\varepsilon,h} = \Gamma^\pm \cap B_\varepsilon\). For sufficiently small \(\varepsilon\), using the fact that \(|u_0(sx')| \leq 1\) and \(\eta \in C^\alpha(\Gamma^\pm_h \times [-M, M])\), it can be seen that

\[
\lim_{\varepsilon \to 0} \int_{\Gamma^\pm_{\varepsilon,h}} \eta(x')\mathcal{R}(v)(x')u_0(sx')d\sigma = 0.
\]

Recall that \(v\) can be approximated by the Herglotz wave functions \(v_j\) given in (3.10) in the sense of \(H^1\)-norm. Then

\[
\int_{\Gamma^\pm_h} \eta(x')u_0(sx')\mathcal{R}(v)(x')d\sigma = \int_{\Gamma^\pm_h} \eta(x')u_0(sx')\mathcal{R}(v_j)(x')d\sigma + \epsilon^\pm_j(s),
\]

\[
\epsilon^\pm_j(s) = \int_{\Gamma^\pm_h} \eta(x')u_0(sx')\mathcal{R}(v(x', x_n) - v_j(x', x_n))d\sigma.
\]
Using Cauchy-Schwarz inequality and the trace theorem, we have

\[
|\epsilon_j^\pm(s)| \leq |\eta(0)| \int_{\Gamma_h^\pm} |u_0(sx')||\mathcal{R}(v(x', x_n) - v_j(x', x_n))|d\sigma
+ \|\eta\|_{C^0} \int_{\Gamma_h^\pm} |x'|^\alpha|u_0(sx')||\mathcal{R}(v(x', x_n) - v_j(x', x_n))|d\sigma
\]

\[
\leq |\eta(0)||\mathcal{R}(v - v_j)||_{H^{1/2}(\Gamma_h^\pm)}|u_0(sx')||_{H^{-1/2}(\Gamma_h^\pm)}
+ \|\eta\|_{C^0}||\mathcal{R}(v - v_j)||_{H^{1}(\Sigma_s)}|u_0(sx')||_{L^2(\Sigma_s)}
\]

\[
\leq C\|\psi\|_{\infty}||v - v_j||_{H^1(\Sigma_s \times (-L, L))}(\eta(0)||u_0(sx')||_{L^2(\Sigma_s)} + \|\eta\|_{C^0}|x'|^\alpha|u_0(sx')||_{L^2(\Sigma_s)}),
\]

where \(C\) is a positive constant and the last inequality comes from Lemma 3.1. Substituting (2.16), (2.24) and (3.3) into (3.15), we obtain that

\[
|\epsilon_j^\pm(s)| \leq C\|\psi\|_{\infty} \left( |\eta(0)| \frac{\sqrt{\theta_M - \theta_m}e^{-\sqrt{\Theta}xW_h}}{\sqrt{2}} + \|\eta\|_{C^0} s^{-(\alpha+1)} \right) \int_0^1 \frac{2(\theta_M - \theta_m)\Gamma(4\alpha + 4)}{(2\delta_W)^{2\alpha+2}} j^{-2-\delta},
\]

where \(\Theta \in [0, h]\) and \(\delta_W\) is defined in (2.5).

Therefore, in (3.14), let \(\varepsilon \to 0^+\) together with (2.18), (2.20), (2.21), (2.22), and (2.26), we can derive the following integral identity:

\[
I_1 + \delta_j(s) = I_3 - I_2^\pm - \epsilon_j^\pm(s).
\]

where

\[
I_1 = \int_{S_h} u_0(sx')(F_1(x') + F_2(x') + F_3(x'))dx',
I_2^\pm = \int_{\Gamma_h^\pm} \eta(x')u_0(sx')\mathcal{R}(v_j)(x')d\sigma,
I_3 = \int_{A_h} (u_0(sx')\partial_v \mathcal{R}(v - w) - \mathcal{R}(v - w)\partial_v u_0(sx'))d\sigma.
\]

Since \(v - w \in H^2(S_h \times (-M, M))\), which implies that \(\mathcal{R}(v - w) \in H^2(\Sigma_s)\), from (2.29), we know that

\[
|I_3| \leq Ce^{-c\sqrt{x}}
\]

where \(c' > 0\) as \(s \to \infty\).

Recall that \(F_1(x') \in H^2(S_h)\), which is defined in (3.9), can be embedded into \(C^0(\Sigma_s)\). \(F_2(x') \in C^0(\Sigma_s)\) since \(qw \in C^0(\Sigma_s \times [-L, L])\). Besides, since \(v_j\) is smooth, then \(v_j \in C^0(\Sigma_s \times [-L, L])\), which means that \(\mathcal{R}(v_j)(x') \in C^0(\Sigma_s)\). Therefore for \(x' \in S_h\) we get the splitting

\[
F_1(x') = F_1(0) + \delta F_1(x'), \quad |\delta F_1(x')| \leq \|F_1\|_{C^0}|x'|^\alpha,
F_2(x') = F_2(0) + \delta F_2(x'), \quad |\delta F_2(x')| \leq \|F_2\|_{C^0}|x'|^\alpha,
F_3j(x') = F_3j(0) + \delta F_3j(x'), \quad |\delta F_3j(x')| \leq \|F_3j\|_{C^0}|x'|^\alpha.
\]
Hence we have

\[ I_1 = (F_1(0) + F_2(0) + F_{3j}(0)) \int_{S_h} u_0(sx')dx' + \int_{S_h} \delta F_1(x')u_0(sx')dx' + \int_{S_h} \delta F_2(x')u_0(sx')dx' + \int_{S_h} \delta F_{3j}(x')u_0(sx')dx'. \]

Recall that \( F_{3j}(x') = -k^2R(v_j)(x') \). Using the property of compact embedding of Hölder spaces, we can derive that for \( 0 < \alpha < 1 \),

\[ \|F_{3j}\|_{C^\alpha} \leq k^2\text{diam}(S_h)^{1-\alpha}\|R(v_j)\|_{C^1}, \]

where \( \text{diam}(S_h) \) is the diameter of \( S_h \). By the definition of the dimension reduction operator (3.1), it is easy to see that

\[ |R(v_j)(x')| \leq 4L\sqrt{\pi}\|\psi\|_{C^\infty} g_j \|L^2(\mathbb{S}^{n-1})\|, \quad |\partial_x R(v_j)(x')| \leq 4kL\sqrt{\pi}\|\psi\|_{C^\infty} g_j \|L^2(\mathbb{S}^{n-1})\|.

Thus we have

\[ \|R(v_j)\|_{C^1} \leq 4L\sqrt{\pi}\|\psi\|_{C^\infty}(1 + k)\|g_j\|_{L^2(\mathbb{S}^{n-1})}. \]

Therefore from (2.5) we have

\[ \left| \int_{S_h} \delta F_{3j}(x)u_0(sx')dx' \right| \leq \frac{8L\sqrt{\pi}\|\psi\|_{C^\infty}(\theta_M - \theta_m)\Gamma(2\alpha + 4)k^2\text{diam}(S_h)^{1-\alpha}}{\delta_2^{2\alpha+4}} (1 + k)\|g_j\|_{L^2(\mathbb{S}^{n-1})} s^{-\alpha - 2}. \]  

(3.21)

Similarly, we have

\[ \left| \int_{S_h} \delta F_1(x)u_0(sx')dx' \right| \leq \frac{2\|F_1\|_{C^\infty}(\theta_M - \theta_m)\Gamma(2\alpha + 4)}{\delta_2^{2\alpha+4}} s^{-\alpha - 2}, \]  

(3.22)

\[ \left| \int_{S_h} \delta F_2(x)u_0(sx')dx' \right| \leq \frac{2\|F_2\|_{C^\infty}(\theta_M - \theta_m)\Gamma(2\alpha + 4)}{\delta_2^{2\alpha+4}} s^{-\alpha - 2}. \]

Using the Jacobi-Anger expansion (cf. [17, Page 75]), for \( v_j \) given in (2.3), we have

\[ v_j(x) = v_j(0)j_0(k|x|) + \sum_{l=1}^{\infty} \gamma_{lj} j_\ell(2\ell + 1)j_\ell(k|x|), \quad x \in \mathbb{R}^3, \]  

(3.23)

where

\[ v_j(0) = \int_{\mathbb{S}^{n-1}} g_j(d)\sigma(d), \quad \gamma_{lj} = \int_{\mathbb{S}^{n-1}} g_j(d)P_\ell(\cos(\varphi))\sigma(d), \quad d \in \mathbb{S}^{n-1}, \]

and \( j_\ell(t) \) is the \( \ell \)-th spherical Bessel function [1] and \( \varphi \) is the angle between \( x \) and \( d \). Moreover, we have the explicit expression of \( j_\ell(t) \) as

\[ j_\ell(t) = \frac{t^\ell}{(2\ell + 1)!!} \left( 1 - \sum_{l=1}^{\infty} \frac{(-1)^l t^{2l}}{2^l l! N_{\ell,l}} \right). \]  

(3.24)
where $N_{\ell,1} = (2\ell + 3) \cdots (2\ell + 2\ell + 1)$. Therefore from the definition of the dimension reduction operator (3.1) and the integral mean value theorem, we know that

$$\mathcal{R}(j_0)(x') = \int_{-L}^{L} \psi(x_n) j_0(k|x|) \, dx_n$$

$$= \int_{-L}^{L} \psi(x_n) \, dx_n - \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l}}{2^{2l+1}(2l+1)!} \int_{-L}^{L} \psi(x_n) (|x'|^2 + x_n^2)^l \, dx_n$$

$$= C(\psi) \left[ 1 - \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l}}{2^{2l+1}(2l+1)!} (|x'|^2 + a_{0,l}^2)^l \right],$$

where $C(\psi) = \int_{-L}^{L} \psi(x_n) \, dx_n$ and $a_{0,l} \in [-L, L]$.

For $\mathcal{R}(j_\ell)(x')$, using the integral mean value theorem, we can deduce that for $\ell = 1, 2, \ldots,$

$$\int_{-L}^{L} \psi(x_n)(|x'|^2 + x_n^2)^l/2 \, dx_n = (|x'|^2 + a_{\ell}^2)^{(l-1)/2} \int_{-L}^{L} \psi(x_n)(|x'|^2 + x_n^2)^{l/2} \, dx_n$$

$$= |x'|^2 (|x'|^2 + a_{\ell}^2)^{(l-1)/2} \int_{-\arctan L/|x'|}^{\arctan L/|x'|} \psi(|x'| \tan \varpi) \sec^3 \varpi \, d\varpi$$

$$:= C'_1(\psi)|x'|^2 (|x'|^2 + a_{\ell}^2)^{(l-1)/2},$$

where $a_\ell \in [-L, L]$. Clearly, if $L < |x'|$, we know that $0 < \sec \varpi < \sqrt{\frac{L^2}{|x'|^2} + 1}$ where $\varpi \in [-\arctan L/|x'|, \arctan L/|x'|]$. Therefore we can deduce that

$$0 < C'_1(\psi) < 2^{5/2} \arctan L \|\psi\|_\infty. \tag{3.27}$$

Thus for $\ell = 1, 2, \ldots$, from (3.24) and (3.26) we have

$$\mathcal{R}(j_\ell)(x') = \int_{-L}^{L} \psi(x_n) j_\ell(k|x|) \, dx_n$$

$$= \frac{k^{\ell}}{(2\ell + 1)!} \int_{-L}^{L} \psi(x_n)(|x'|^2 + x_n^2)^{\ell/2} \left( 1 - \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l} (|x'|^2 + x_n^2)^l}{2^{2l+1}N_{\ell,l}} \right) \, dx_n$$

$$= \frac{k^{\ell} (|x'|^2 + a_{\ell}^2)^{(l-1)/2}}{(2\ell + 1)!} \left[ 1 - \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l} (|x'|^2 + a_{\ell}^2)^l}{2^{2l+1}N_{\ell,l}} \right] C'_1(\psi)|x'|^2,$$

where $a_\ell$ and $a_{\ell,l} \in [-L, L]$.

Recall that the boundary integral $I_2^\pm$ is given by (3.18). In this situation the polar coordinates $x = (r \cos \theta, r \sin \theta)$ satisfy $r \in (0, h)$ and $\theta = \theta_m$ or $\theta = \theta_M$ when $x \in \Gamma_h^\pm$ or $x \in \Gamma_M^\pm$, respectively. Since $\eta \in C^{\alpha}(\Gamma_h^\pm \times [-M, M])$, we know that

$$\eta(x') = \eta(0) + \delta \eta(x'), \quad |\delta \eta(x')| \leq ||\eta||_{C^\alpha} |x'|^\alpha.$$

Substituting the above equation into the expression of $I_2^\pm$, we have

$$I_2^\pm = \eta(0) I_{21}^\pm + I_{22}^\pm,$$  \hspace{1cm} \tag{3.29}

where

$$I_{21}^\pm = \int_{\Gamma_h^\pm} u_0(sx') \mathcal{R}(v_\ell)(x') \, d\sigma, \quad I_{22}^\pm = \int_{\Gamma_h^\pm} \delta \eta(x') u_0(sx') \mathcal{R}(v_\ell)(x') \, d\sigma.$$
Recall that $\omega(\theta) = -\cos(\theta/2 + \pi) > 0$ when $\theta_m \leq \theta \leq \theta_M$. Denote

$$I_{\eta,1}^- = \int_{\Gamma_h} \delta\eta(x')u_0(sx')\mathcal{R}(j_0)(x')d\sigma, \quad I_{\eta,2}^- = \sum_{\ell=1}^{\infty} \gamma_{\ell j} \ell^2(2\ell + 1) \int_{\Gamma_h} \delta\eta(x')u_0(sx')\mathcal{R}(j_\ell)(x')d\sigma.$$ 

Substituting (3.25) into $I_{\eta,1}^-$, we can derive that

$$|I_{\eta,1}^-| \leq |C(\psi)||\eta||c^\alpha| \int_0^h r^{\alpha} \left| 1 - \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l}}{(2l + 1)!} (r^2 + a_{0,l}^2)^l \right| e^{-\sqrt{\pi}\omega(\theta_m)} dr$$

$$= 2L||\psi||_{\infty}||\eta||_{C^\alpha} \left| 1 - \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l}}{(2l + 1)!} (\beta_{0,l}^2 + a_{0,l}^2)^l \right| \int_0^h r^{\alpha} e^{-\sqrt{\pi}\omega(\theta_m)} dr,$$

where $\beta_{0,l} \in [0, h]$ such that $k^2(\beta_{0,l}^2 + a_{0,l}^2) \leq k^2(h^2 + L^2) < 1$ for sufficiently small $h$ and $L$. From (2.40), we obtain that

$$|I_{\eta,1}^-| \leq O(s^{-\alpha - 1}) \quad (3.30)$$

as $s \to \infty$. Substituting (3.28) into $I_{\eta,2}^-$, and using (3.27), we can deduce that

$$I_{\eta,2}^- \leq C_1(\psi)\|\eta\|_{C^\alpha}$$

$$\cdot \sum_{\ell=1}^{\infty} |\gamma_{\ell j}| \int_0^h r^{\alpha} k^\ell (r^2 + a_{\ell,l}^2)^{(\ell - 1)/2} \left| 1 - \sum_{l=1}^{\infty} \frac{k^{2l}(r^2 + a_{\ell,l}^2)^l}{2l! N_{\ell,l}} \right| r^{2\alpha - \sqrt{\pi}\omega(\theta_m)} dr$$

$$\leq C_1(\psi)\|\eta\|_{C^\alpha} \|g_j\|_{L^2(S^{n-1})} \int_0^h r^{2\alpha + \beta_{\ell,l}^2} e^{-\sqrt{\pi}\omega(\theta_m)} dr$$

$$\cdot \sum_{\ell=1}^{\infty} k^\ell (\beta_{\ell,l}^2 + a_{\ell,l}^2)^{(\ell - 1)/2} \left| 1 - \sum_{l=1}^{\infty} \frac{k^{2l}(\beta_{\ell,l}^2 + a_{\ell,l}^2)^l}{2l! N_{\ell,l}} \right|$$

$$= O(||g_j||_{L^2(S^{n-1})}) s^{-\alpha - 3}, \quad (3.31)$$

where $\beta_{\ell}$ and $\beta_{\ell,l} \in [0, h]$ such that $k^2(\beta_{\ell}^2 + a_{\ell}^2) \leq k^2(h^2 + L^2) < 1$ and $k^2(\beta_{\ell,l}^2 + a_{\ell,l}^2) \leq k^2(h^2 + L^2) < 1$ for sufficiently small $h$ and $L$, by utilizing the claim that

$$|\gamma_{\ell j}| \leq ||g_j||_{L^2(S^{n-1})},$$

where we use the fact that $|P_k(t)| \leq 1$ when $|t| \leq 1$.

Substituting (3.23) and (3.28) into the expression of $I_{21}$ defined in (3.29), we can denote

$$I_{21}^- = v_j(0) \int_{\Gamma_h} u_0(sx')\mathcal{R}(j_0)(x')d\sigma + \sum_{\ell=1}^{\infty} \gamma_{\ell j} \ell^2(2\ell + 1) \int_{\Gamma_h} u_0(sx')\mathcal{R}(j_\ell)(x')d\sigma$$

$$:= v_j(0)I_{21}^- + I_{22}^-.$$
Substituting the expansion (3.25) into $I_{31}$, recalling that $\mu(\theta) = -\cos(\theta/2+\pi) - i\sin(\theta/2+\pi)$, we have

$$I_{31}^- = C(\psi) \int_0^h \left[ 1 - \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l}}{(2l + 1)!} \right] r^2 e^{\sqrt{\pi} \mu(\theta_m)} dr$$

$$= C(\psi) \left[ 1 - \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l} a_{0,l}}{(2l + 1)!} \right] \int_0^h e^{-\sqrt{\pi} \mu(\theta_m)} dr$$

$$- C(\psi) \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l}}{(2l + 1)!} \left( \sum_{i=1}^{l} C(l, i) a_{0, l}^2 \right) \int_0^h r^{2i} e^{-\sqrt{\pi} \mu(\theta_m)} dr$$

$$:= I_{311}^- + I_{312}^-,$$

where $C(l, i) = \frac{\mu_{l-1}^i}{i! (l-i)!}$ is the combinatorial number of the order $l$. Since we can choose $L$ such that $kL < 1$ and $|a_{0,l}| \leq L$, we know that

$$\left| \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l} a_{0,l}}{(2l + 1)!} \right| \leq \sum_{l=1}^{\infty} (kL)^{2l} = \frac{(kL)^2}{(1 - (kL)^2)}.$$

Moreover, from (2.44) we obtain that

$$I_{311}^- = 2s^{-1} \left( \mu(\theta_m)^{-2} - \mu(\theta_m)^{-2} e^{-\sqrt{\pi} \mu(\theta_m)} - \mu(\theta_m)^{-1} \sqrt{\pi} e^{-\sqrt{\pi} \mu(\theta_m)} \right) C(I_{311})$$

$$= C(\psi) \left[ 1 - \sum_{l=1}^{\infty} \frac{(-1)^l k^{2l}}{(2l + 1)!} \right] \left( \sum_{i=1}^{l} C(l, i) a_{0, l}^2 \right).$$

From (3.32), we know that

$$0 < \frac{C(\psi)(1 - 2(kL)^2)}{1 - (kL)^2} \leq C(I_{311}) \leq \frac{C(\psi)}{1 - (kL)^2}.$$

For $I_{312}$, we can deduce that

$$|I_{312}^-| \leq |C(\psi)| \sum_{l=1}^{\infty} \frac{k^{2l}}{(2l + 1)!} \left( \sum_{i=1}^{l} C(l, i) h^{2(i-1)} L^{2(l-i)} \right) \int_0^h r^2 e^{-\sqrt{\pi} \mu(\theta_m)} dr$$

$$= |C(\psi)| \sum_{l=1}^{\infty} \frac{k^{2l}}{(2l + 1)!} h^{2l} \left( \sum_{i=1}^{l} C(l, i) h^{2i} L^{2(l-i)} \right) \int_0^h r^2 e^{-\sqrt{\pi} \mu(\theta_m)} dr$$

$$= |C(\psi)| \sum_{l=1}^{\infty} \frac{k^{2l}}{(2l + 1)!} h^{2l} \left( (h^2 + L^2)^l - L^{2l} \right) \int_0^h r^2 e^{-\sqrt{\pi} \mu(\theta_m)} dr$$

$$\leq 2L \|\psi\| \sum_{l=1}^{\infty} \frac{k^{2l}}{(2l + 1)!} h^{2l} \left( (h^2 + L^2)^l - L^{2l} \right) \cdot O(s^{-3})$$

$$= O(s^{-3}),$$

where we choose $h$ and $L$ such that $k^2(h^2 + L^2) < 1$ and $kL < 1$. 

Substituting the expansion (3.28) of \( j \) into \( I_{32} \), we have

\[
|I_{32}| \leq C_1(\psi)\|g_j\|_{L^2(S^{n-1})} \cdot \sum_{\ell=1}^{\infty} \int_0^h r^2 e^{-\sqrt{\ell\omega}(\theta_m)} \frac{k^\ell(|r|^2 + a^2_L)^{(\ell-1)/2}}{(2\ell - 1)!!} \left| 1 - \sum_{l=1}^{\infty} \frac{(-1)^l k^l(|r|^2 + a^2_{L,l})^l}{2!l!N_{\ell,l}} \right| dr
\]

\[
= C_1(\psi)\|g_j\|_{L^2(S^{n-1})} \sum_{\ell=1}^{\infty} \frac{k^\ell(\beta_{\ell}^2 + a^2_L)^{(\ell-1)/2}}{(2\ell - 1)!!} \left| 1 - \sum_{l=1}^{\infty} \frac{(-1)^l k^l(\beta_{L,l}^2 + a^2_{L,l})^l}{2!l!N_{\ell,l}} \right|
\]

\[
= O(\|g_j\|_{L^2(S^{n-1})} s^{-3}), \quad (3.36)
\]

where \( \beta_{\ell}, \beta_{L,l} \in [0, h] \) such that \( k^2(\beta_{\ell}^2 + a^2_L) \leq k^2(h^2 + L^2) < 1 \) and \( k^2(|\beta_{L,l}|^2 + a^2_{L,l}) \leq k^2(h^2 + L^2) < 1 \) for sufficiently small \( h \) and \( L \).

Finally, substituting (3.30), (3.31), (3.33) , (3.35), and (3.36) into (3.29), we have the following integral properties

\[
I_2^- = 2\eta(0)v_j(0)s^{-1} \left( \mu(\theta_m)^{-2} - \mu(\theta_m)^{-2}e^{-\sqrt{s\mu}(\theta_m)} - \mu(\theta_m)^{-1}\sqrt{s}he^{-\sqrt{s\mu}(\theta_m)} \right) C(I_{311})
\]

\[
+ v_j(0)\eta(0)I_{312}^- + \eta(0)I_{32}^- + I_{31}^-, \quad I_{312}^- \leq O(s^{-3}), \quad I_{32}^- \leq O(\|g_j\|_{L^2(S^{n-1})} s^{-3}), \quad |I_{31}^-| \leq \|\eta\|_{C^a} \left( v_j(0)I_{\eta,1}^- + I_{\eta,2}^- \right),
\]

\[
|I_{\eta,1}^-| \leq O(s^{-1-\alpha}), \quad |I_{\eta,2}^-| \leq O(\|g_j\|_{L^2(S^{n-1})} s^{-3-\alpha}). \quad (3.37)
\]

Denote \( C(I_{311}^+) = C(\psi) \left[ 1 - \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{2(l+1)!!a_{0,l,+}} \right] \), where \( a_{0,l,+} \in [-L, L] \). For sufficiently small \( L \), similar to (3.34), we know that

\[
0 < \frac{C(\psi)(1 - 2(kL)^2)}{1 - (kL)^2} \leq C(I_{311}^+) \leq \frac{C(\psi)}{1 - (kL)^2}, \quad (3.38)
\]

where \( C(\psi) \) is defined in (3.34).

Adopting the similar arguments for the integral property (3.37) of \( I_2^- \), we can derive the following integral property for \( I_2^+ \) as follows

\[
I_2^+ = 2\eta(0)v_j(0)s^{-1} \left( \mu(\theta_M)^{-2} - \mu(\theta_M)^{-2}e^{-\sqrt{s\mu}(\theta_M)} - \mu(\theta_M)^{-1}\sqrt{s}he^{-\sqrt{s\mu}(\theta_M)} \right) C(I_{311})
\]

\[
+ v_j(0)\eta(0)I_{312}^+ + \eta(0)I_{32}^+ + I_{31}^+, \quad (3.39)
\]
where
\[
I_{312}^+ = -C(\psi) \sum_{l=1}^{\infty} \frac{(-1)^l k_{2l}^2}{(2l + 1)!!} \sum_{i_1=1}^{l} C(i, i_1) a_{0,i_1}^2 \int_{0}^{\infty} r^{2l+1} e^{-\sqrt{\sigma} \mu_0 \mu_0 \mu_0} dr, \quad |I_{312}^+| \leq O(s^{-3}),
\]
\[
I_{32}^+ = \sum_{l=1}^{\infty} \gamma_{ij}(2l + 1) \int_{\Gamma_h^+} u_0(sx') R(jx')(x') d\sigma, \quad |I_{32}^+| \leq O(\|g\|_{L^2(S_0)^{-1}}s^{-3}),
\]
\[
I_{\eta}^+ = \int_{\Gamma_h^+} \delta(\eta(x')) u_0(sx') R(jx')(x') d\sigma, \quad |I_{\eta}^+| \leq \|\eta\| C^\alpha \left( v_j(0) I_{\eta,1}^+ + I_{\eta,2}^+ \right),
\]
\[
I_{\eta,1}^+ = \int_{\Gamma_h^+} \delta(\eta(x')) u_0(sx') R(j_0)(x') d\sigma, \quad |I_{\eta,1}^+| \leq O(s^{-1-\alpha}),
\]
\[
I_{\eta,2}^+ = \sum_{l=1}^{\infty} \gamma_{ij} \delta(x') u_0(sx') R(jx')(x') d\sigma, \quad |I_{\eta,2}^+| \leq O(\|g\|_{L^2(S_0)^{-1}}s^{-3-\alpha}).
\]

We first multiply \( sj \) on the both sides of (3.17). Then substituting (3.37) and (3.39) into the resulting equation (3.17), after rearranging terms, we deduce that

\[
2v_j(0)\eta(0) \left[ \left( \mu(\theta_M)^{-2} - \mu(\theta_M)^{-2} e^{-\sqrt{\sigma} \mu_0 \mu_0 \mu_0} - \mu(\theta_M)^{-1} \sqrt{\phi} e^{-\sqrt{\sigma} \mu_0 \mu_0 \mu_0} \right) C(I_{311}^+) \right.
\]
\[
+ \left( \mu(\theta_m)^{-2} - \mu(\theta_m)^{-2} e^{-\sqrt{\sigma} \mu_0 \mu_0 \mu_0} - \mu(\theta_m)^{-1} \sqrt{\phi} e^{-\sqrt{\sigma} \mu_0 \mu_0 \mu_0} \right) C(I_{311}^-) \right]
\]
\[
= s \left[ I_3 - (F_1(0) + F_2(0) + F_3(0)) \int_{S_0} u_0(sx')dx' - \delta_j(s) \right.
\]
\[
- \eta(0)(I_{32}^+ + I_{32}^-) - I_{\eta}^+ - I_{\eta}^- - \int_{S_0} \delta F_1(x') u_0(sx') dx' - \int_{S_0} \delta F_2(x') u_0(sx') dx'
\]
\[
- \int_{S_0} \delta F_3(x') u_0(sx') dx' - v_j(0)\eta(0) \left( I_{312}^- + I_{312}^+ \right) - \epsilon_j^+(s) \right].
\]

When \( s = j \), from (3.37) and (3.39), under the assumption (3.3), we know that

\[
j|I_{32}^+| \leq O(j^{-2}||g||_{L^2(S_0)}) \leq O(j^{-1\varepsilon}), \quad j|I_{32}^-| \leq O(j^{-2}||g||_{L^2(S_0)}) \leq O(j^{-1+\varepsilon}),
\]
\[
j|I_{\eta}^+| \leq ||\eta|| C^\alpha \left( v_j(0) O(j^{-\alpha}) + O(||g||_{L^2(S_0)}j^{-2-\alpha}) \right),
\]
\[
j|I_{\eta}^-| \leq ||\eta|| C^\alpha \left( v_j(0) O(j^{-\alpha}) + O(||g||_{L^2(S_0)}j^{-2-\alpha}) \right),
\]
\[
j|I_{312}^-| \leq O(j^{-2}), \quad j|I_{312}^+| \leq O(j^{-2}).
\]
Clearly, when \( s = j \), from (2.6), (2.7), (3.13), (3.16), (3.19), (3.21) and (3.22), under the assumption (3.3), it can be derived that

\[
j \int_{S_h} \delta F_3 j(x) u_0(s'x')dx' \leq \frac{8L\sqrt{\pi}||\psi||_{C^\infty} \Gamma(\theta_M - \theta_m))}{\delta_W^{2a+4}} k^2 \delta_j(S_h)^{1-\alpha} \cdot (1 + k) ||g_j||_{L^2(S_h-1)} j^{-\alpha-1} \leq C \left( j^{-\alpha-\theta} \right),
\]

\[
j \int_{S_h} \delta F_1(x) u_0(s'x')dx' \leq \frac{2||F_1||_{C^\infty} \Gamma(\theta_M - \theta_m))}{\delta_W^{2a+4}} j^{-\alpha-1},
\]

\[
j \int_{S_h} \delta F_2(x) u_0(s'x')dx' \leq \frac{2||F_2||_{C^\infty} \Gamma(\theta_M - \theta_m))}{\delta_W^{2a+4}} j^{-\alpha-1},
\]

\[
j|\epsilon_j^+ (j)| \leq C \|\psi\|_{\infty} \left( |\eta(0)| \frac{\sqrt{\theta_M - \theta_m} e^{-j\theta\delta_W h}}{\sqrt{2}} + |\eta|_{C^\infty} j^{-\alpha} \frac{2(\theta_M - \theta_m)\Gamma(4\alpha + 4)}{(2\delta_W)^{2a+2}} \right) j^{-1-\gamma}, \quad \Theta \in [0, h],
\]

\[
j|\delta_j(j)| \leq k^2 \|\psi\|_{\infty} \sqrt{C(L,h)}(\theta_M - \theta_m) e^{-j\theta\delta_W h} j^{-\gamma}, \quad \Theta \in [0, h],
\]

where \( c' > 0 \) and \( \delta_W \) are defined in (3.19) and (2.5), respectively.

The coefficient of \( v_j(0) \) of (3.40) with respect to the zeroth order of \( s \) is

\[
2\eta(0) \left( C(I_{311}^-) \mu(\theta_m)^{-2} + C(I_{311}^+) \mu(\theta_M)^{-2} \right).
\]

It can be calculated that

\[
C(I_{311}^-) \mu(\theta_m)^{-2} + C(I_{311}^+) \mu(\theta_M)^{-2} = \frac{(C(I_{311}^+) \cos \theta_m + C(I_{311}^-) \cos \theta_M) + i(C(I_{311}^+ \sin \theta_m + C(I_{311}^- \sin \theta_M))}{(\cos \theta_m + i \sin \theta_m) (\cos \theta_M + i \sin \theta_M)}.
\]

Therefore under the assumption (2.10), we know that

\[
\cos \theta_m + \cos \theta_M \text{ and } \sin \theta_m + \sin \theta_M
\]

cannot be zero simultaneously. Without loss of generality, we assume that \( \cos \theta_m + \cos \theta_M \neq 0 \). Then we consider the following two cases:

- Case A: \( \cos \theta_m + \cos \theta_M > 0 \).
- Case B: \( \cos \theta_m + \cos \theta_M < 0 \).

For Case A, let us consider the first case that \( \cos \theta_m \) and \( \cos \theta_M \) have the same sign. From (3.34) and (3.38), it is not difficult to see that the real part of the denominator of \( C(I_{311}^-) \mu(\theta_m)^{-2} + C(I_{311}^+) \mu(\theta_M)^{-2} \) can not be zero. Therefore,

\[
C(I_{311}^-) \mu(\theta_m)^{-2} + C(I_{311}^+) \mu(\theta_M)^{-2} \neq 0.
\]
In the following, we assume that \( \cos \theta_m \) and \( \cos \theta_M \) have different signs. Then it implies that \( \cos \theta_m \leq 0 \) and \( \cos \theta_M > 0 \). From (3.34) and (3.38), we can deduce that

\[
\frac{C(\psi)}{1 - (kL)^2} (\cos \theta_m + (1 - 2(kL)^2) \cos \theta_M) \leq C(I_{311}^+) \cos \theta_m + C(I_{311}^-) \cos \theta_M \leq \frac{C(\psi)}{1 - (kL)^2} (1 - 2(kL)^2) \cos \theta_m + \cos \theta_M.
\]

Since \( L \) is flexible, for a given \( 0 < \varepsilon < 1 \), we can choose \( L \) such that \( 0 < kL < \sqrt{\varepsilon/2} \), from which we can derive the bounds as follows

\[
\frac{C(\psi)}{1 - (kL)^2} (\cos \theta_m + (1 - \varepsilon) \cos \theta_M) \leq C(I_{311}^+) \cos \theta_m + C(I_{311}^-) \cos \theta_M \leq \frac{C(\psi)}{1 - (kL)^2} ((1 - \varepsilon) \cos \theta_m + \cos \theta_M).
\]

Since \( \cos \theta_m + \cos \theta_M > 0 \), we can consider the lower bound in (3.44). Denote \( \varepsilon_0 = \min\{\cos \theta_m, \cos \theta_M\} \) and choose \( \varepsilon \in (0, \varepsilon_0) \). It can be verified that

\[
C(I_{311}^+) \cos \theta_m + C(I_{311}^-) \cos \theta_M \geq \frac{C(\psi)}{1 - (kL)^2} (\cos \theta_m + (1 - \varepsilon) \cos \theta_M) > 0,
\]

which means that (3.43) still holds.

For Case B, if \( \cos \theta_m = 0 \) or \( \cos \theta_M = 0 \) is satisfied, from the upper bound of (3.44) we can easily show that

\[
C(I_{311}^+) \cos \theta_m + C(I_{311}^-) \cos \theta_M < 0. \tag{3.45}
\]

Otherwise, if \( |\cos \theta_m| \leq |\cos \theta_M| \), from the fact that \( (1 - \varepsilon)|\cos \theta_m| \leq |\cos \theta_M| \), we know that (3.45) still holds from the upper bound of (3.44). If \( |\cos \theta_m| > |\cos \theta_M| \), we can choose \( \varepsilon \) such that \( \varepsilon > 1 - |\cos \theta_M|/|\cos \theta_m| > 0 \) to make (3.45) also be fulfilled from the upper bound of (3.44). Therefore, for Case B, we know that (3.43) is always fulfilled.

In (3.40), we take \( s = j \) and let \( j \to \infty \), using (3.41) and (3.42) under the assumption (3.3), we can prove that

\[
\lim_{j \to \infty} \eta(0) (C(I_{311}^-) \mu(\theta_m))^{-2} + C(I_{311}^+) \mu(\theta_M)^{-2} v_j(0) = 0. \tag{3.46}
\]

Under the assumption (3.5), we have shown that \( C(I_{311}^-) \mu(\theta_m)^{-2} + C(I_{311}^+) \mu(\theta_M)^{-2} \neq 0 \). Therefore, from (3.46) and (3.4), we prove that

\[
\lim_{j \to \infty} v_j(0) = 0.
\]

Using the a similar argument of (2.53), we finish the proof of this theorem. \( \square \)

**Remark 3.2.** Similar to Remark 2.1, Theorem 3.1 can be localized. Moreover, we would like to mention that in contrast to the regularity assumption on \( v - w \) near the corner in 2D of Theorem 2.1, we impose that \( v - w \in H^2(S_h \times (-M, M)) \) in Theorem 3.1, where we need to use the \( C^0 \)-continuity of \( \mathcal{R}(v - w) \) to investigate the asymptotical order of \( s \) with respect to \( s \to \infty \) for the volume integral of \( F_1(x') \) over \( S_h \) in (3.17).

Similar to Corollary 2.1, we consider the vanishing property of the interior transmission eigenfunctions \( v \in H^1(W \times (-M, M)) \) and \( w \in H^1(W \times (-M, M)) \) to (2.54) on the edge point under the assumptions (3.5) and (3.47).
Corollary 3.1. Suppose \( v \in H^1(W \times (-M,M)) \) and \( w \in H^1(W \times (-M,M)) \) is the interior transmission eigenfunctions to (2.54), where \( W \subset \mathbb{R}^{n-1} \) is defined in (2.1) and \( M > 0 \). For any fixed \( x_c \in (-M,M) \) and \( L > 0 \) defined in Definition 3.1, we suppose that \( L \) is sufficiently small such that \( (x_c - L, x_c + L) \subset (-M,M) \). Suppose that there exists a sufficiently small neighbourhood \( S_h \) of \( x_c \in \mathbb{R}^{n-1} \) such that \( qw \in C^\alpha(\overline{S_h} \times [-M,M]) \) for \( 0 < \alpha < 1 \), and \( v - w \in H^2(S_h \times (-M,M)) \), where \( x_c \) is the vertex of \( W \) and \( S_h \) is defined in (2.2). If the following conditions are fulfilled:

(a) the transmission eigenfunction \( v \) can be approximated in \( H^1(S_h \times (-M,M)) \) by the Herglotz waves \( v_j \), \( j = 1, 2, \ldots \), with kernels \( g_j \) satisfying

\[
\| v - v_j \|_{H^1(S_h \times (-M,M))} \leq j^{-2-\gamma}, \quad \| g_j \|_{L^2(\mathbb{S}^{n-1})} \leq C j^2,
\]

for some positive constant \( C, \gamma > 0 \) and \( 0 < q < \alpha \),
(b) the angles \( \theta_m \) and \( \theta_M \) of the sector \( W \) satisfy

\[
-\pi < \theta_m < \theta_M < \pi \quad \text{and} \quad \theta_M - \theta_m \neq \pi,
\]

then we have

\[
\lim_{\rho \to 0} \frac{1}{m(B(x_c, \rho))} \int_{B(x_c, \rho)} R(Vw)(x')dx' = 0,
\]

where \( q(x', x_n) = 1 + V(x', x_n) \).

Proof. Without loss of generality, we assume that \( x_c = 0 \). Since \( \eta(x) \equiv 0 \), from (3.17) we have the following integral equality

\[
(F(1) + F(2) + F_3(0)) \int_{S_h} u_0(sx')dx' + \delta_j(s) \tag{3.49}
\]

\[
= I_3 - \int_{S_h} \delta F_1(x'0)u_0(sx')dx' - \int_{S_h} \delta F_2(x'0)u_0(sx')dx' - \int_{S_h} \delta F_3(x'0)u_0(sx')dx' - \int_{S_h} \delta F_3(x'0)u_0(sx')dx'.
\]

where \( \delta_j(s) \) is defined in (3.11), \( \delta F_1(x') \), \( \delta F_2(x') \) and \( \delta F_3(x') \) are defined in (3.20), \( I_3 \) is given in (3.17). Since \( v = w \) on \( \Gamma^\pm \times (-M,M) \), it is easy to see that

\[
F(1) = \int_{-L}^L \psi'(x_n)(v(0,x_n) - w(0,x_n))dx_n = 0.
\]

Therefore, using (2.6), from (3.49), we deduce that

\[
6i(F(2) + F_3(0))(e^{-2\theta h i} - e^{-2\theta h n})s^{-2} - (F(2) + F_3(0)) \int_{W \setminus S_h} u_0(sx')dx' \tag{3.50}
\]

\[
= I_3 - \int_{S_h} \delta F_1(x'0)u_0(sx')dx' - \int_{S_h} \delta F_2(x'0)u_0(sx')dx' - \int_{S_h} \delta F_3(x'0)u_0(sx')dx' - \delta_j(s).
\]

In (3.50), we take \( s = j \) and multiply \( j^2 \) on the both sides of (3.50):

\[
6i(F(2) + F_3(0))(e^{-2\theta h i} - e^{-2\theta h n}) = j^2\left[I_3 + (F(2) + F_3(0)) \int_{W \setminus S_h} u_0(sx')dx' - \int_{S_h} \delta F_1(x'0)u_0(sx')dx' - \int_{S_h} \delta F_2(x'0)u_0(sx')dx' - \int_{S_h} \delta F_3(x'0)u_0(sx')dx' - \delta_j(s)\right].
\]

From (3.13) and (3.47), it is not difficult to see that

\[
j^2|\delta_j(s)| \leq \frac{|k|^2\|\psi\|_\infty \sqrt{C(L,h)(\theta_M - \theta_m)} e^{-\sqrt{\pi \gamma} w h}}{\sqrt{2}} j^{-\gamma},
\]

(3.52)
where \( C(L, h) \) is a positive number defined in (3.12), \( \Theta \in [0, h] \) and \( \delta_w \) is defined in (2.5).

Under the assumption (3.47), from (3.19), (3.21) and (3.22), we can obtain the following estimations

\[
j^2 \left| \int_{S_h} \delta F_{3j}(x) u_0(x') dx' \right| \leq \frac{8L \sqrt{\pi} \| \psi \|_{L^\infty} (\theta_M - \theta_m) \Gamma(2\alpha + 4)}{\delta_w^{2\alpha+4}} \cdot (1 + k) \| g_j \|_{L^2(\mathbb{S}^{n-1})} j^{-\alpha} \leq O \left( j^{-(\alpha - \varrho)} \right),
\]

\[
j^2 \left| \int_{S_h} \delta F_1(x) u_0(x') dx' \right| \leq \frac{2 \| F_1 \|_{L^\infty} (\theta_M - \theta_m) \Gamma(2\alpha + 4)}{\delta_w^{2\alpha+4}} j^{-\alpha},
\]

\[
j^2 \left| \int_{S_h} \delta F_2(x) u_0(x') dx' \right| \leq \frac{2 \| F_2 \|_{L^\infty} (\theta_M - \theta_m) \Gamma(2\alpha + 4)}{\delta_w^{2\alpha+4}} j^{-\alpha}.
\]

Under the assumption (3.48), it is easy to see that

\[
\left| e^{-2\theta_M i} - e^{-2\theta_m i} \right| = \left| 1 - e^{-2(\theta_M - \theta_m) i} \right| \neq 0
\]
since \( \theta_M - \theta_m \neq \pi \). In (3.51), by letting \( j \to \infty \), from (2.7), (3.52) and (3.53), we prove that

\[
\lim_{j \to \infty} F_{3j}(0) = -F_2(0),
\]

which implies

\[
\lim_{j \to \infty} \mathcal{R}(v_j)(0) = \mathcal{R}(qw)(0)
\]

through recalling that \( F_2 \) and \( F_{3j} \) are given in (3.9). From (3.7), we have

\[
\lim_{\rho \to +0} \frac{1}{m(B(0, \rho))} \int_{B(0, \rho)} \mathcal{R}(v)(x') dx' = \lim_{\rho \to +0} \frac{1}{m(B(0, \rho))} \int_{B(0, \rho)} \mathcal{R}(w)(x') dx'.
\]

Since

\[
\lim_{j \to \infty} \mathcal{R}(v_j)(0) = \lim_{j \to \infty} \lim_{\rho \to +0} \frac{1}{m(B(0, \rho))} \int_{B(0, \rho)} \mathcal{R}(v_j)(x') dx' = \lim_{\rho \to +0} \frac{1}{m(B(0, \rho))} \int_{B(0, \rho)} \mathcal{R}(v)(x') dx',
\]

\[
\mathcal{R}(qw)(0) = \lim_{\rho \to +0} \frac{1}{m(B(0, \rho))} \int_{B(0, \rho)} \mathcal{R}(qw)(x') dx',
\]

and from (3.54), we finish the proof of this corollary. \( \square \)

**Remark 3.3.** Corollary 3.1 states that the average value of the function \( Vw \) over the cylinder centered at the edge point \( (x_c, x_n^c) \) with the height \( L \) vanishes in the distribution sense. In addition, if \( V(x', x_n) \) is continuous near the edge point \( (x_c, x_n^c) \) where \( x_n^c \in (-M, M) \) and \( V(x_c, x_n^c) \neq 0 \), from the dominant convergent theorem and the definition the reduction operator \( \mathcal{R} \), we can prove that

\[
\lim_{\rho \to +0} \frac{1}{m(B(x_c, \rho))} \int_{B(x_c, \rho)} \int_{x_n^c - L}^{x_n^c + L} \psi(x_n) w(x', x_n) dx' dx_n = 0
\]

under the assumptions in Corollary 3.1, which also describes the vanishing property of the interior eigenfunctions \( v \) and \( w \) near the edge point in 3D. Furthermore, if \( \psi(x_n^c) \neq 0 \),
one can prove that
\[
\lim_{\rho \to +0} \frac{1}{m(B(x_c, \rho))} \int_{B(x_c, \rho)} \int_{x_n - L}^{x_n + L} w(x', x_n) dx' dx_n = 0.
\]

In the following theorem, we impose a stronger regularity requirement for the conductive transmission eigenfunction \(v\) of (3.2), i.e., \(v\) has \(H^2\)-regularity near the considering edge point. Using the dimension reduction operator given in Definition 3.1, as well as the Hölder continuity of the considering functions, we can prove the following theorem in a similar way of proving Theorem 2.2. The detailed proof of Theorem 3.2 is omitted here.

**Theorem 3.2.** Let \(v \in H^2(W \times (-M, M))\) and \(w \in H^1(W \times (-M, M))\) be the eigenfunctions to (3.2). Assume that \(W \subset \mathbb{R}^{n-1}\) is defined in (2.1), \(M > 0\), and \(x_c\) is a corner of \(W\). For any fixed \(x_n^c \in (-M, M)\) and \(L > 0\) defined in Definition 3.1, we suppose that \(L\) is sufficiently small such that \([x_n^c - L, x_n^c + L] \subset (-M, M)\). Moreover, there exists a sufficiently smaller neighbourhood \(S_h\) (i.e. \(h > 0\) is sufficiently small) of \(x_c\), such that \(qw \in C^\alpha(S_h \times [-M, M])\) and \(\eta \in C^\alpha(\Gamma^c_h \times [-M, M])\) for \(0 < \alpha < 1\) and \(v - w \in H^2(S_h \times (-M, M))\). Under the following assumptions:

\[
\begin{aligned}
(a) & \quad \text{the function } \eta = \eta(x', x_n) \text{ is independent of } x_n \text{ and does not vanish on the edge of } W \times (-M, M), \text{ i.e., } \\
& \quad \eta(x_c) \neq 0, \\
(b) & \quad \text{the angles } \theta_m \text{ and } \theta_M \text{ of the sector } W \text{ containing the corner satisfy} \\
& \quad -\pi < \theta_m < \theta_M < \pi \text{ and } \theta_M - \theta_m \neq \pi,
\end{aligned}
\]

then we have \(v\) and \(w\) vanish at the edge point \((x_c, x_n^c) \in \mathbb{R}^n\) of \(W \times (-M, M)\), where \(x_n^c \in (-M, M)\).

**Remark 3.4.** When \(\eta \equiv 0\) near the edge point, under the \(H^2\) regularity of the interior transmission eigenfunctions \(v\) and \(w\), the vanishing property of \(v\) and \(w\) is investigated in [4].

4. **Unique recovery results for the inverse scattering problem**

In this section, we apply the vanishing property of the conductive transmission eigenfunctions at a corner in 2D to investigate the unique recovery in the inverse problem associated with the following conductive scattering problem

\[
\begin{cases}
\Delta u^- + k^2 q u^- = 0 & \text{in } \Omega, \\
\Delta u^+ + k^2 u^+ = 0 & \text{in } \mathbb{R}^2 \setminus \Omega, \\
u^+ = u^-, \quad \partial_n u^+ + \eta u^+ = \partial_n u^- & \text{on } \partial \Omega, \\
u^+ = u^+ + u^s & \text{in } \mathbb{R}^2 \setminus \Omega, \\
\lim_{r \to \infty} r^{1/2} (\partial_r u^s - i k u^s) = 0, & r = |x|,
\end{cases}
\]

where \(u^s\) is an (nontrivial) entire solution to \((\Delta + k^2)u^s = 0\) signifying an incident field, and the last limit is called the Sommerfeld radiation condition which holds uniformly with respect to \(\hat{x} = x/|x| \in S^{n-1}\), and characterizes the out-radiating wave. The well-posedness
The direct problem (4.1) is known (cf. [13]), and there exists a unique solution \( u := u^- \chi_\Omega + u^+ \chi_{\mathbb{R}^n \setminus \Omega} \in H^1_{\text{loc}}(\mathbb{R}^n) \). Moreover, there holds the following asymptotic expansion
\[
 u^s(x) = \frac{e^{ik|x|}}{|x|^{(n-1)/2}} u^\infty(\hat{x}) + O\left(\frac{1}{|x|^{n/2}}\right), \quad |x| \to +\infty
\]
uniformly in all directions \( \hat{x} = x/|x| \in S^{n-1} \). The real-analytic function \( u^\infty(\hat{x}) \) is referred to as the far-field pattern or the scattering amplitude associated with \( u^i \). The inverse scattering problem is concerned with the recovery of the scatterer \( (\Omega; q, \eta) \) by knowledge of the far-field pattern \( u^\infty(\hat{x}; u^i) \); that is
\[
u^\infty(\hat{x}; u^i) \to (\Omega; q, \eta).
\]
In (4.2), if the far-field pattern is given corresponding to a single incident wave \( u^i \), then it is referred to as a single far-field measurement, otherwise it is referred to as many far-field measurements. It is known that the inverse problem (4.2) is nonlinear and ill-conditioned. For the reconstruction of the shape of the scatterer \( \Omega \) by using the factorization method for (4.2), uniqueness issue has been studied in [13]. The inverse spectral problem of gaining the information about the material properties associated to the conductive transmission eigenvalue problem has been studied in [12]. In [20], the method of uniquely recovering the conductive boundary parameter \( \eta \) from the measured scattering data as well as the convergence of the conductive transmission eigenvalues as the conductivity parameters which tend to zero has also been studied. In all of the aforementioned literatures, the unique determination results are based on the far-field patterns of all incident plane waves at a fixed frequency, which means that infinitely many far-field measurements have been used. In what follows, we show that in a rather general and practical scenario, the polyhedral shape of the scatterer, namely \( \Omega \), can be uniquely recovered by a single far-field measurement without knowing its material contents, namely \( q \) and \( \eta \). Moreover, if the surface conductive parameter \( \eta \) is constant, then it can be recovered as well.

Our main unique recovery results for the inverse scattering problem (4.2) are contained in Theorems 4.1 and 4.2. In Theorem 4.1, we establish the unique recovery results by a single far-field measurement in determining a 2D polygonal conductive scatterer without knowing its contents. In Theorem 4.2, the surface conductive parameter \( \eta \) of the scatterer can be further recovered if it is a constant. Before presenting the main results, we first show in Proposition 4.1 that the conductive parameter \( \eta \) in (4.1) has a close relationship with the wave number \( k \) from the practical point view of the TM-mode (transverse magnetic) for the time-harmonic Maxwell system [3]. This relationship helps us to show that our assumption in Theorem 4.1 can be fulfilled when the wave number \( k \) is sufficiently small.

The inverse scattering problem (4.1) is derived by the TM-mode (transverse magnetic) from the time-harmonic Maxwell system [3], where the scattering medium is covered by a thin layer with very high conductivity; see [13] for details. The conductive boundary condition has been known for a long time in the study of electromagnetic induction in the earth [37]. The full Maxwell system with the conductive boundary condition was investigated in [3], where an inhomogeneity is covered by an infinitely thin (the electric filed would not penetrate into an ideal conductor of positive thickness) and highly conducting layer. In order to illustrate the basic idea as well as to simplify the exposition, we consider the following simple model of the conductive boundary problem for the Maxwell
equation [3]:

\[
\begin{align*}
\nabla \times E - i\omega \mu_0 H &= 0 \quad \text{in } \mathbb{R}^3 \setminus \partial \Omega, \\
\nabla \times H + i\omega \varepsilon_0 E - \sigma E &= 0 \quad \text{in } \mathbb{R}^3 \setminus \partial \Omega, \\
\nu \times E\big|_+ - \nu \times E\big|_- &= 0 \quad \text{on } \partial \Omega, \\
\nu \times H\big|_+ - \nu \times H\big|_- &= \mu_0 \tau (\nu \times E) \times \nu \quad \text{on } \partial \Omega, \\
\lim_{r \to \infty} r ((H - H^i) \times \hat{x} - (E - E^i)) &= 0 \quad r = |x|, \\
\end{align*}
\]

(4.3)

where \( E, H \) are respectively the electric and magnetic fields; \( \varepsilon_0 \) and \( \mu_0 \) are two positive constants respectively signifying the electric permittivity and magnetic permeability; \( \nu(x) \) is the outer unit normal vector at \( x \in \partial \Omega \); and \( \tau \) is the positive conductive parameter define on the boundary \( \partial \Omega \) and \( \sigma \) is the conductivity satisfying

\[
\sigma = \begin{cases} 
\sigma_1 & \text{in } \Omega, \\
0 & \text{in } \mathbb{R}^3 \setminus \Omega,
\end{cases}
\]

with \( \sigma_1 \) a positive constant. In (4.3), \( E^i \) and \( H^i \) are a pair of incident waves which are entire solutions to the following homogeneous Maxwell system

\[
\begin{align*}
\nabla \times E^i - i\omega \mu_0 H^i &= 0 \quad \text{in } \mathbb{R}^3, \\
\nabla \times H^i + i\omega \varepsilon_0 E^i &= 0 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

(4.4)

where \( \omega \in \mathbb{R}_+ \) signifies the frequency of the electromagnetic waves.

**Proposition 4.1.** Under the transverse magnetic polarization, (4.3) can be reduced to a system of the form (4.1) with the boundary conductive parameter \( \eta \) satisfying

\[
\eta = i\omega \mu_0^2 \tau = \mathcal{O}(k),
\]

where \( k := \omega \sqrt{\mu_0 \varepsilon_0} \) is the wave number in (4.1).

**Proof.** As pointed in [13], (4.1) is derived from (4.3) under the transverse magnetic polarization, which means that we can suppose that

\[
E = \begin{bmatrix} 0 \\ 0 \\ u(x,y) \end{bmatrix}, \quad H = \begin{bmatrix} H_1(x,y) \\ H_2(x,y) \\ 0 \end{bmatrix},
\]

where \( E \) and \( H \) are the electromagnetic fields satisfying (4.3). Especially under the transverse magnetic polarization, we know that \( \nu = [\nu_1, \nu_2, 0]^\top \in \mathbb{R}^3 \), where \( \nu \) is the unit outward normal vector on the boundary \( \partial \Omega \). It is easy to see that

\[
\nabla \times E = \begin{bmatrix} \partial_y u \\ -\partial_x u \\ 0 \end{bmatrix}, \quad \nabla \times H = \begin{bmatrix} 0 \\ 0 \\ \partial_x H_2 - \partial_y H_1 \end{bmatrix}.
\]

Using the divergence free condition of \( E \) together with the following equality

\[
\nabla \times (\nabla \times E) = -\Delta E + \nabla (\nabla \cdot E) = -\Delta E,
\]

from (4.3) we can deduce that

\[
-\Delta E = i\omega \mu_0 \nabla \times H = i\omega \mu_0 (-i\omega \varepsilon_0 + \sigma) E.
\]
Then we have
\[ \Delta u + (\omega^2 \mu_0 \varepsilon_0 + i \omega \mu_0 \sigma) u = 0. \]
Let
\[ k = \omega \sqrt{\mu_0 \varepsilon_0} \text{ and } q = 1 + i \frac{\sigma}{\omega \varepsilon_0}. \]
It turns out that
\[ \Delta u + k^2 qu = 0. \]
Besides, it is easy to verify that
\[ \nu \times E = \begin{bmatrix} \nu_2 u \\ -\nu_1 u \\ 0 \end{bmatrix} \tag{4.5} \]
Thus substituting (4.5) into the boundary condition
\[ \nu \times E_+ - \nu \times E_- = 0, \]
we can conclude that
\[ u^+ = u^- . \]
Moreover, we can deduce that
\[ \nu \times H = \frac{1}{i \omega \mu_0} \nu \times (\nabla \times E) = \frac{1}{i \omega \mu_0} \begin{bmatrix} 0 \\ 0 \\ -\nu_1 \partial_x u - \nu_2 \partial_y u \end{bmatrix} = \frac{1}{i \omega \mu_0} \begin{bmatrix} 0 \\ 0 \\ -\partial_y u \end{bmatrix}, \]
\[ \nu \times (\nu \times H) = \frac{1}{i \omega \mu_0} \begin{vmatrix} i & j & k \\ \nu_1 & \nu_2 & 0 \\ 0 & 0 & -\partial_y u \end{vmatrix} = \frac{1}{i \omega \mu_0} \begin{bmatrix} -\nu_2 \partial_y u \\ \nu_1 \partial_y u \\ 0 \end{bmatrix}. \tag{4.6} \]
From the second boundary condition in (4.3), we have
\[ \nu \times (\nu \times H)_+ - \nu \times (\nu \times H)_- = \mu_0 \tau \nu \times (\nu \times E) \times \nu). \]
Substituting (4.6) into the above equation, we obtain that
\[ -\nu_2 \partial_y u^+ + \nu_2 \partial_y u^- = \nu_2 i \omega \mu_0^2 \tau u, \]
\[ \nu_1 \partial_y u^+ - \nu_1 \partial_y u^- = -\nu_1 i \omega \mu_0^2 \tau u, \]
which means that
\[ \partial_y u^- = \partial_y u^+ + i \omega \mu_0^2 \tau u. \]
Thus
\[ \eta = i \omega \mu_0^2 \tau = O(k). \tag{4.7} \]

The proof is complete. \qed

Proposition 4.1 basically indicates that when considering the conductive scattering problem (4.1), one may impose the low-frequency dependence behaviour (4.7) on the surface conductive parameter. As remarked earlier, Proposition 4.1 only considers the simple model (4.3) for illustration of the low-frequency behaviour (4.7). For more complex Maxwell models, one can derive the conductive scattering system (4.1) of a general form.

We are in a position to consider the inverse problem (4.2). First, we introduce the admissible class of conductive scatterers in our study.

**Definition 4.1.** Let \((\Omega; \eta, \gamma)\) be a conductive scatterer associated with the scattering problem (4.1) and \(u\) be the total wave fields therein. The scatterer is said to be admissible if it fulfils the following conditions:
(a) $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^2$, and $q \in L^\infty(\Omega)$, $\eta \in L^\infty(\partial \Omega)$.

(b) Following the notations in Theorem 2.1, if $\Omega$ possesses a corner $\Omega \cap W$, then $qu \in C^\alpha(\overline{S}_h)$, $\eta \in C^\alpha(\overline{\Gamma}_h)$.

(c) The total wave field $u$ is non-vanishing everywhere in the sense that for any $x \in \mathbb{R}^n$,
\[
\lim_{\rho \to 0^+} \frac{1}{m(B(x, \rho))} \int_{B(x, \rho)} |u(x)| \, dx \neq 0. \tag{4.8}
\]

We would like to point out that the conditions stated in Definition 4.1 can be fulfilled by the conductive scatterer $(\Omega; q, \eta)$ and the scattering problem (4.1) in certain general and practical scenarios. For example, as remarked in Remark 2.2, if $q \neq 0$ in $\overline{S}_h$, then the conditions in (b) can be easily fulfilled. If $q \neq 0$ in $\overline{S}_h$, but $\eta = 0$ on $\overline{S}_h \cap \partial \Omega$, then $u \in H^2(S_h)$. Hence, the conditions in (b) can also be easily fulfilled. There might be more cases for which the conditions in (b) are fulfilled. The condition (4.8) in (c) can also be fulfilled at least when $k$ is sufficiently small. In fact, it has been shown in Proposition 4.1 that if $\eta \neq 0$, then $\eta = O(k)$. For the scattered field $u^s$ of (4.1), from [13, Theorem 2.4], it is proved that
\[
\|u^s\|_{H^1(B)} \leq C(\|u^i\|_{H^{-1/2}(\partial \Omega)} + k^2\|qu^i\|_{L^2(\Omega)}) = O(k)\|u^i\|_{L^2(\Omega)},
\]
where $C$ is a positive number and $B$ is a large ball containing $\Omega$. Hence, if the incident field $u^i$ is non-vanishing everywhere, say $u^i = e^{ikx}d$ with $d \in S^1$ being a plane wave, and $k$ is sufficiently small, then (4.8) is obviously fulfilled. Nevertheless, by Definition 4.1, we may include more general situations into our subsequent study of the inverse problem (4.2).

**Theorem 4.1.** Consider the conductive scattering problem (4.1) associated with two conductive scatterers $(\Omega_j; q_j, \eta_j)$, $j = 1, 2$, in $\mathbb{R}^2$. Let $u^i_\infty(\hat{x}; u^i)$ be the far-field pattern associated with the scatterer $(\Omega_j; q_j, \eta_j)$ and the incident field $u^i$. Suppose that $(\Omega_j; q_j, \eta_j)$, $j = 1, 2$ are admissible and
\[
u^1_\infty(\hat{x}; u^i) = u^2_\infty(\hat{x}; u^i) \tag{4.9}
\]
for all $\hat{x} \in S^1$ and a fixed incident wave $u^i$. Then
\[
\Omega_1 \Delta \Omega_2 := (\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1) \tag{4.10}
\]
cannot possess a corner. Hence, if $\Omega_1$ and $\Omega_2$ are convex polygons in $\mathbb{R}^2$, one must have
\[
\Omega_1 = \Omega_2. \tag{4.11}
\]
Proof. By contradiction, we assume that there is a corner contained in $\Omega_1 \Delta \Omega_2$. Without loss of generality we may assume that the vertex $O$ of the corner $\Omega_2 \cap W$ is such that $O \in \partial \Omega_2$ and $O \notin \overline{\Omega_1}$.

Since $u_{\infty}^1(\hat{x}; u') = u_{\infty}^2(\hat{x}; u')$ for all $\hat{x} \in S^1$, applying Rellich’s Theorem (see [17]), we know that $u_1' = u_2'$ in $\mathbb{R}^2 \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$. Thus

$$u_1(x) = u_2(x)$$

(4.12)

for all $x \in \mathbb{R}^2 \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$. Following the notations in (2.2), we have from (4.12) that

$$u_2^\pm = u_2', \quad \partial u_2^\pm = \partial u_2' + \eta u_2' = \partial u_1^\pm + \eta u_1'$$

on $\Gamma_h^\pm$,

where the superscripts $(\cdot)^-, (\cdot)^+$ stand for the limits taken from $\Omega_2$ and $\mathbb{R}^2 \setminus \overline{\Omega_2}$ respectively. Moreover, suppose the neighbourhood $B_h(O)$ is sufficiently small such that

$$\Delta u_1^\pm + k^2 u_1^\pm = 0, \quad \Delta u_2^\pm + k^2 q_2 u_2^\pm = 0 \quad \text{in} \quad B_h(O).$$

Clearly $u_1^\pm \in H^2(S_h)$ and $u_2^\pm \in H^1(S_h)$. Now we prove that

$$u_1^\pm - u_2^\pm \in H^2(\Sigma_{\Lambda_h}),$$

where $\Sigma_{\Lambda_h}$ is defined in (2.2). We first note that on the boundary $\Gamma_h^\pm$, one has $u_2^- = u_1^+$, where $u_2^+ \in H^{3/2}(\Gamma_h^\pm)$ from the trace theorem. Since $\Gamma^+ \in C^{1,1}$, from [30, Theorem 4.18], we have the following regularity estimate for $u_2^-$ up to the boundary $\Gamma_{(h/2,h)}^+$ of $\Sigma_{\Lambda_h}$:

$$\|u_2^-\|_{H^2(\Sigma_{\Lambda_h} \cap D_1^+)} \leq C\left(\|u_2^\pm\|_{H^1(S_h)} + \|u_1^\pm\|_{H^{3/2}(\Gamma_h^+)}\right),$$

where $C > 0$ is a constant and $D_1^+$ is a open set with a boundary $\Gamma_{(h/2,h)}^+$ such that $\Sigma_{\Lambda_h} \cap D_1^+ \neq \emptyset$. Using the similar argument we can prove that $u_2^\pm$ has $H^2$-regularity up to the boundary $\Gamma_{(h/2,h)}^-$ of $\Sigma_{\Lambda_h}$. Therefore $u_2^\pm \in H^2(\Sigma_{\Lambda_h})$, which means that $u_1^\pm - u_2^\pm \in H^2(\Sigma_{\Lambda_h})$. Since $(\Omega_j; q_j, \eta_j), \ j = 1, 2$ are admissible, we know that $\eta \in C^0(\overline{\Gamma_h^\pm})$ and $q_2 u_2^- \in C^0(\overline{S_h})$. Applying Theorem 2.2 if $\eta(0) \neq 0$, and Remark 2.5 if $\eta = 0$ on $\Gamma_h^\pm$, and also using the fact that $u_1$ is continuous at the vertex $O$, we have

$$u_1(O) = 0,$$

which contradicts to the admissibility condition (c) in Definition 4.1.

The proof is complete. \qed

Based on Definition 4.1, if we further assume that the conductive parameter $\eta$ is constant, we can recover $\eta$ simultaneously once the admissible conductive scatter $\Omega$ is determined. However, in determining the conductive parameter, we need assume that $q_j = q$ for $j = 1, 2$ is known.

**Theorem 4.2.** Consider the conductive scattering problem (4.1) associated with the admissible conductive scatters $(\Omega_j; q, \eta_j)$, where $\Omega_j = \Omega$ for $j = 1, 2$ and $\eta_j \neq 0, j = 1, 2$, are two constants. Let $u_{\infty}^j(\hat{x}; u')$ be the far-field pattern associated with the scatter $(\Omega_j; q, \eta_j)$ and the incident field $u'$. Suppose that $(\Omega; q, \eta_j), \ j = 1, 2$, are admissible and

$$u_{\infty}^1(\hat{x}; u') = u_{\infty}^2(\hat{x}; u')$$

(4.13)

for all $\hat{x} \in S^1$ and a fixed incident wave $u'$. Then if $k$ is not an eigenvalue of the partial differential operator $\Delta + k^2 q$ in $H^1_0(\Omega)$, we have $\eta_1 = \eta_2$. 
Proof. Since $u^1_{\infty}(\hat{x}; u^i) = u^2_{\infty}(\hat{x}; u^i)$ for all $\hat{x} \in S^1$, we can derive that $u^1_0 = u^2_0$ for all $x \in \mathbb{R}^2 \setminus \Omega$ and thus $\partial_\nu u^1_0 = \partial_\nu u^2_0$ on $\partial \Omega$. Combining the transmission condition in the scattering problem (4.1), we deduce that

$$u^1_0 = u^2_0 = u^\pm_0 \text{ on } \partial \Omega,$$

Thus, we have

$$\partial_\nu (u^1_0 - u^2_0) = \partial_\nu (u^1_0 - u^2_0) + \eta_1 u^+_0 - \eta_2 u^-_0 = (\eta_1 - \eta_2) u^0_1 \text{ on } \partial \Omega.$$

Define $v := u^1_0 - u^2_0$. Then, $v$ fulfills

$$\begin{aligned}
\begin{cases}
(\Delta + k^2 q)v = 0 & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega, \\
\partial_\nu v = (\eta_1 - \eta_2) u^0_1 & \text{on } \partial \Omega.
\end{cases}
\end{aligned}$$

(4.14)

Since $k$ is not an eigenvalue of the operator $\Delta + k^2 q$ in $H^1_0(\Omega)$, hence one must have $v = 0$ to (4.14). Substituting this into the Neumann boundary condition of (4.14), we know that $(\eta_1 - \eta_2) u^0_1 = \partial_\nu v = 0$ on $\partial \Omega$.

Next, we prove the uniqueness of $\eta$ by contradiction. Assume that $\eta_1 \neq \eta_2$. Since $(\eta_1 - \eta_2) u^0_1 = 0$ on $\partial \Omega$ and $\eta_j$, $j = 1, 2$ are constants, we can deduce that $u^0_1 = 0$ on $\partial \Omega$. Then $u^0_1$ satisfies

$$\begin{aligned}
\begin{cases}
(\Delta + k^2 q)u^0_1 = 0 & \text{in } \Omega, \\
u^0_1 = 0 & \text{on } \partial \Omega.
\end{cases}
\end{aligned}$$

Similar to (4.14), this Dirichlet problem also only has a trivial solution $u^0_1 = 0$ in $\Omega$, since $k$ is not an eigenvalue of $\Delta + k^2 q$. Then, we can derive $u^0_1 = u^\pm_0 = 0$ and

$$\partial_\nu u^0_1 = \partial_\nu u^+_0 + \eta_1 u^+_0 = \partial_\nu u^-_0 + \eta_2 u^-_0 = 0 \text{ on } \partial \Omega,$$

which implies that $u^0_1 \equiv 0$ in $\mathbb{R}^2$ and thus $u^0_1 = -u^i$. This contradicts with the fact that $u^i_0$ satisfies the Sommerfeld radiation condition.

The proof is complete. \hfill $\Box$

Remark 4.1. In Theorem 4.2, it is required that $k$ is not an eigenvalue to $\Delta + k^2 q$ in $H^0_0(\Omega)$. Clearly, if $q$ is negative-valued in $\Omega$ or $\Im q \neq 0$ in $\Omega$, this condition is fulfilled. On the other hand, if $q$ is positive-valued in $\Omega$, then this condition can be readily fulfilled when $k \in \mathbb{R}_+$ is sufficiently small.

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