Critical behaviour of SU(2) lattice gauge theory
A complete analysis with the $\chi^2$-method†

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ABSTRACT

We determine the critical point and the ratios $\beta/\nu$ and $\gamma/\nu$ of critical exponents of the deconfinement transition in $SU(2)$ gauge theory by applying the $\chi^2$-method to Monte Carlo data of the modulus and the square of the Polyakov loop. With the same technique we find from the Binder cumulant $g_r$ its universal value at the critical point in the thermodynamical limit to $-1.403(16)$ and for the next-to-leading exponent $\omega = 1 \pm 0.1$. From the derivatives of the Polyakov loop dependent quantities we estimate then $1/\nu$. The result from the derivative of $g_r$ is $1/\nu = 0.63 \pm 0.01$, in complete agreement with that of the 3d Ising model.

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1. Introduction

The investigation of the critical properties of any physical theory on a finite lattice is a non-trivial problem. This is so, because the singularity of the infinite volume theory at a phase transition is modified due to the finite volume into a rapid but nonsingular variation of the order parameter. The location of the critical point becomes then unclear and the infinite volume form of observables with critical behaviour like the order parameter, the susceptibility and higher cumulants cannot be used for the determination of the critical exponents. The observed finite size effects are on the other hand controlled by these exponents, as described by finite size scaling (FSS) theory. Simulations on lattices of varying size allow then to calculate the critical parameters of the theory.

Recently, a new method, the $\chi^2$-method, was proposed [1] and applied to the expectation value of the square of the magnetization. Though this quantity is not peaked at the transition, it was shown that one may deduce from its FSS behaviour the precise asymptotic critical point and the ratio of the critical exponents $\gamma$ and $\nu$ in a very straightforward manner.

In this letter we want to extend the method to the modulus and Binder’s fourth-order cumulant $g_r$ [2] of the magnetization. The intention is to find the ratio $\beta/\nu$ of exponents and to check the consistency of the results for the critical point from the different variables. As it turns out it is possible to obtain still further information on universal quantities and by examining the derivatives of the variables at the critical point one may even determine the inverse of the exponent $\nu$ of the correlation length with - at least for $SU(2)$ - unachieved precision.

2. Finite size scaling theory and the $\chi^2$-method

Let us briefly review that part of FSS theory which is relevant for our considerations. In the neighbourhood of the critical temperature $T_c$ of a second order phase transition one expects in the limit of large $N_\sigma$, the characteristic length scale of the system, for the magnetization or order parameter (denoted by $\langle L \rangle$ here) that

$$\langle L \rangle \sim (T - T_c)^{\beta}, \text{ for } T \to T_c^+. \quad (1)$$

The behaviour near to $T_c$ of the susceptibility $\chi$ and the correlation length $\xi$ in the large $N_\sigma$ limit is expected to be

$$\chi \sim |T - T_c|^{-\gamma}, \quad (2)$$
\( \xi \sim |T - T_c|^{-\nu}. \)  \hspace{1cm} (3)

A quantitative analysis of the finite lattice modifications to these formulae becomes possible by using renormalization group theory. In this framework it has been shown [3] that the singular part of the free energy density has the following form

\[
 f_s(x, h, N_\sigma) = N_\sigma^{-d} Q_f^f(g_T N^{1/\nu}_\sigma, g_h N^{(\beta+\gamma)/\nu}_\sigma, g_i N^{y_i}_\sigma).
\]  \hspace{1cm} (4)

The scaling function \( Q_f^f \) depends on the temperature \( T \) and the external field strength \( h \) in terms of a thermal and a magnetic scaling field

\[
g_T = c_T x + O(xh, x^2),
\]  \hspace{1cm} (5)

\[
g_h = c_h h + O(xh, h^2),
\]  \hspace{1cm} (6)

which are independent of \( N_\sigma \) and where \( x \) is the reduced temperature

\[
x = \frac{T - T_{c,\infty}}{T_{c,\infty}}.
\]  \hspace{1cm} (7)

Here the action contains a further symmetry breaking term \( h \cdot N^{d}_\sigma L \). Also additional irrelevant scaling fields \( g_i \) with negative exponents \( y_i \) may be present.

The order parameter \( \langle L \rangle \), the susceptibility \( \chi \) and the cumulant \( g_r \) are obtained from \( f_s \) by taking derivatives with respect to \( h \) at \( h = 0 \). The general form of the scaling relations derived in this way is

\[
 O(x, N_\sigma) = N_\sigma^{\rho/\nu} \cdot Q_O(g_T N^{1/\nu}_\sigma, g_i N^{y_i}_\sigma).
\]  \hspace{1cm} (8)

Here \( O \) is \( \langle L \rangle, \chi \) and \( g_r \) with \( (\rho = -\beta, \gamma \) and \( 0) \). Taking into account only the largest irrelevant exponent \( y_1 = -\omega \) and expanding the scaling function \( Q_O \) to first order at \( x = 0 \) results in the following equation

\[
 O(x, N_\sigma) = N_\sigma^{\rho/\nu} [c_0 + (c_1 + c_2 N^{\omega}_\sigma) x N^{1/\nu}_\sigma + c_3 N^{\omega^i}_\sigma].
\]  \hspace{1cm} (9)

Standard FSS methods are based on the evaluation of Eq. (8) and/or Eq. (9) in the neighbourhood of the infinite volume critical coupling.

The \( \chi^2 \)-method [1] utilizes Eq. (9) only at \( x = 0 \)

\[
 O(x = 0, N_\sigma) = N_\sigma^{\rho/\nu} [c_0 + c_3 N^{-\omega}_\sigma],
\]  \hspace{1cm} (10)
and determines the critical point from the leading $N_\sigma$-behaviour of Eq. (10). This is motivated by the fact, that for $x \neq 0$ the $N_\sigma$-behaviour is drastically changed due to the presence of $xN_\sigma^{1/\nu}$-terms (for $SU(2)$ e.g. $1/\nu \approx 1.6$ [4]). The critical point is then defined as that point where a fit to the leading $N_\sigma$-behaviour of Eq. (10) has the least minimal $\chi^2$.

We have two different forms of fits. If the exponent $\rho \neq 0$ the leading $N_\sigma$-behaviour is given only by the first term in Eq. (10). Taking then the logarithm we find

$$\ln O = \ln c_0 + \frac{\rho}{\nu} \ln N_\sigma ,$$

(11)
i.e. we have a linear dependence on $\ln N_\sigma$ with slope $\rho/\nu$. Linear fits to $\ln O$ as a function of $\ln N_\sigma$ give then also the exponent ratio. A different form of fit is to be used in the case $\rho = 0$, where the leading $N_\sigma$-behaviour is

$$O = c_0 + c_3 N_\sigma^{-\omega} .$$

(12)

It is interesting to consider also derivatives of the observables with respect to $x$. Since the scaling functions $Q_O$ are depending on $xN_\sigma^{1/\nu}$ we obtain

$$\frac{\partial O}{\partial x}(x,N_\sigma) = N_\sigma^{(1+\rho)/\nu} \cdot Q'_O(xN_\sigma^{1/\nu}) ,$$

(13)
where the prime denotes the derivative with respect to the argument; the dependence on the irrelevant fields is not explicitly shown, but present. Since the derivative of the scaling function is again a scaling function, derivatives of observables $O$ may be used to find the value of $(1 + \rho)/\nu$ from fits to the leading $N_\sigma$-behaviour of

$$\frac{\partial O}{\partial x}(x = 0,N_\sigma) = N_\sigma^{(1+\rho)/\nu} [c_1 + c_2 N_\sigma^{-\omega}] .$$

(14)

Derivatives may also be used to define new direct scaling functions by

$$U = x \frac{\partial \ln O}{\partial x} = U(xN_\sigma^{1/\nu}) .$$

(15)
If for any finite value of $N_\sigma$ the scaling function $Q_O$ and its derivative $Q'_O$ are finite in a neighbourhood of $x = 0$ – which is the normal case – we have immediately

$$U(0) = 0 .$$

(16)
In the thermodynamic limit, i.e. for $N_\sigma \to \infty$, we get, however a different result. From Eqs. 1–3, or in general from

$$O_\infty = c_O x^{-\rho} \quad \text{for} \quad x \to +0,$$

we find

$$U_\infty(0) = -\rho.$$

An attempt to determine $\rho$ at $x = 0$ from infinite volume formulae will therefore usually fail on finite lattices. Yet at small, but finite, positive $x$ and large $N_\sigma$, i.e. for large arguments $xN_\sigma^{1/\nu}$, $U$ will approach $-\rho$.

3. SU(2) gauge theory at finite temperature

Let us consider $SU(2)$ gauge theory on $N_\sigma^3 \times N_\tau$ lattices using the standard Wilson action

$$S(U) = \frac{4}{g^2} \sum_p (1 - \frac{1}{2} Tr U_p),$$

where $U_p$ is the product of link operators around a plaquette. The number of lattice points in the space (time) direction $N_\sigma(\tau)$ and the lattice spacing $a$ fix the volume and temperature as

$$V = (N_\sigma a)^3, \quad T = 1/(N_\tau a).$$

On an infinite volume lattice the order parameter or magnetization for the deconfinement transition is the expectation value of the Polyakov loop

$$L(x) = \frac{1}{2} Tr \prod_{\tau=1}^{N_\tau} U_{\tau,x;4},$$

or else, that of its lattice average

$$L = \frac{1}{N_\sigma^3} \sum_x L(x),$$

where $U_{x;4}$ are the $SU(2)$ link matrices at four-position $x$ in time direction.

Since, due to system flips between the two ordered states on finite lattices the expectation value $\langle L \rangle$ is always zero, the true susceptibility

$$\chi = N_\sigma^3(\langle L^2 \rangle - \langle L \rangle^2),$$
reduces there to
\[ \chi_v = N^3 \langle L^2 \rangle . \] (23)

The quantity \( \chi_v \) is monotonically rising as a function of the coupling \( 4/g^2 \) or the temperature \( T \). Below the critical point
\[ \chi_v = \chi , \] (24)
and one expects therefore \( \chi_v \) to have the same FSS behaviour as the true susceptibility for \( T < T_c \). In ref. [1] it was shown that this is indeed the case.

In the following we want to extend the analysis of ref. [1] to the remaining magnetization dependent observables and their derivatives. In particular, we want to take the expectation value of the modulus of the lattice average, \( \langle |L| \rangle \), as a kind of “finite lattice order parameter”, hoping as usual [4,5] that its FSS behaviour is controlled by the exponent \( \beta \) of the infinite volume order parameter. In addition we want to investigate the behaviour of the normalized fourth cumulant of the magnetization
\[ g_r = \frac{\langle L^4 \rangle}{\langle L^2 \rangle^2} - 3 , \] (25)
which is directly a scaling function and corresponds, up to a constant, to the renormalized coupling defined for infinite systems [6].

4. Numerical results and discussion

The \( SU(2) \) Monte Carlo data, which we want to use here were computed on \( N^3 \times N \) lattices with \( N_\sigma = 8, 12, 18, 26 \) and \( N_\tau = 4 \) and have already been reported on in refs. [5,7] and [1]. Since in the latter reference the critical point was determined to \( 4/g^2_{c,\infty} = 2.2988(1) \), we calculated in addition the still missing results at this point for \( N_\sigma = 8, 12 \) and \( 18 \) and \( N_\tau = 4 \) with high statistics. They are presented in Table 1. Proceeding in the same way as in ref. [1], we evaluated then the data in the very close vicinity (\( 2.298 \leq 4/g^2 \leq 2.300 \)) of the transition with the density of states method (DSM) [8]. The newly calculated MC data were in complete agreement with the former DSM interpolation and introduced only a tiny shift to the interpolation itself. As usual, the errors were determined with the Jackknife method.
At each $4/g^2$-value we have made linear $\chi^2$-fits to our DSM interpolation results. For $\langle|L|\rangle$ and $\chi_v$ the form of Eq. (11) was used, i.e. $\ln\langle|L|\rangle$ and $\ln\chi_v$ were fitted as a function of $\ln N_\sigma$. Correspondingly, Eq. (12) was utilized for fixed values of $\omega$ near to 1 to perform linear fits of $g_r$ as a function of $N_\sigma^{-\omega}$. In Fig. 1 we compare the obtained minimal $\chi^2/N_f$ for the three quantities. Here the number of degrees of freedom, $N_f$, is 2, and $\omega$ was fixed to 1.

We observe, that all three observables give consistent results in several respects: the positions of the minima of the $\chi^2_{\text{min}}/N_f$-curves are nearly coinciding – their differences give us an error estimate on the critical coupling; the curves have the same shape and show a very sharp increase, symmetric around the respective minimum; the size of $\chi^2_{\text{min}}/N_f$ close to the minimum is of order one, which indicates that our error estimates are reasonable. Finally, the minima themselves are very close to zero, which means that the finite size behaviours of the observables at the critical point are indeed described by the formulae, Eqs. 11 and 12. In Table 2, we list the minimum positions found.

| $N_\sigma$ | $N_\tau$ | $\langle|L|\rangle$ | $\langle L^2 \rangle$ | $g_r$ |
|-----------|----------|---------------------|--------------------|------|
| 8         | 4        | 0.14343(18)         | 0.025723(54)       | −1.3532(20) |
| 12        | 4        | 0.11591(22)         | 0.016749(48)       | −1.3695(37) |
| 18        | 4        | 0.09370(88)         | 0.01091(15)        | −1.386(16)  |
| 26        | 4        | 0.07649(75)         | 0.00733(10)        | −1.365(17)  |

Table 1

MC results at $4/g^2 = 2.2988$.

| $O$     | $4/g_{\text{min}}^2$ | $\omega$ |
|---------|----------------------|----------|
| $\langle|L|\rangle$ | 2.29895              |          |
| $\chi_v$    | 2.29890              |          |
| $g_r$     | 2.29905              | 0.9, 1.0 |
|          | 2.29900              | 1.1      |

Table 2

Minimum positions of $\chi^2_{\text{min}}/N_f$.

We conclude from their variations, that the critical point is at

$$4/g_{c,\infty}^2 = 2.29895(10).$$

(26)
Inside error bars, this value is compatible with the one of ref. [1], which was determined from the finite volume susceptibility $\chi_v$ only. The shift in $4/g_{\min}^2(\chi_v)$ due to the increased statistics is 0.0001. As can be seen from Table 2 a variation of $\omega$ in the range 0.9-1.1 has nearly no effect on the position of the minimum. Increasing $\omega$ further shifts the minimum point to smaller $4/g^2$-values, the width of the $\chi^2_{\min}/N_f$-curve becomes smaller, but at the same time the minimum value increases slightly. Decreasing $\omega$ leads to opposite effects: shift of $4/g_{\min}^2$ to higher values, widening of the $\chi^2_{\min}/N_f$-curve, decrease of the minimum. A value of $\omega = 1.0$ or slightly higher seems reasonable, because then the width of the $\chi^2_{\min}/N_f$-curve is the same as that for $\langle |L| \rangle$. Also the minimum position is then in good agreement with that of the other observables. In ref. [7], where the critical points for $N_\tau = 4$ and 6 were determined from the intersection points of the $g_r$-curves for different $N_\sigma$, a value of $\omega = 0.9$ was found for $N_\tau = 6$, whereas for $N_\tau = 4$ the contribution of irrelevant scaling fields was below visibility. With our approach we confirm the existence of these fields also for $N_\tau = 4$ with an exponent $\omega$ of comparable size. In addition, $\omega \approx 1.0$ is the value found in the 3d Ising model [4], which belongs to the same universality class as 3+1 dimensional SU(2) gauge theory [9].

Each of our three fits yields one universal quantity: $\beta/\nu$ and $\gamma/\nu$ are obtained from the $\langle |L| \rangle$ and $\chi_v$ fits; the constant $c_0$ from the $g_r$ fit is the universal value $g_r^\infty$ of $g_r$ at the critical point for $N_\sigma \to \infty$. Since all universal quantities should be equal for theories of the same universality class, we compare our results to those of the 3d Ising model [4]. This is shown in Fig. 2. Obviously we find excellent agreement between the two theories. In Table 3 we have listed the corresponding numbers at $4/g^2_{c,\infty}$. Their errors come from two sources: the uncertainty in $4/g^2_{c,\infty}$ and the fit errors at fixed $4/g^2$. The value of $g_r^\infty$ was estimated in ref. [7] from Binder’s [2] ”cumulant crossing” method to -1.38(5). Our fit leads to a lower result -1.403(16), which is in closer agreement with the Ising value -1.41. This comes about, because we include a correction to scaling term in the fit $(c_3 N_\sigma^{-\omega})$, where $c_3$ turns out to be positive.

As a final application of FSS methods we have investigated the derivatives of our observables with respect to the coupling $4/g^2$. Since the reduced temperature $x$ may be approximated by

$$x = \frac{4/g^2 - 4/g^2_{c,\infty}}{4/g^2_{c,\infty}},$$

(27)
we have proportionality between $\frac{\partial O}{\partial x}$ and $\frac{\partial O}{\partial (4/g^2)}$. As a consequence Eq. (14) may be used to find $(1 + \rho)/\nu$ from a fit of the type of Eq. (11) at the critical point.

| Source | $SU(2)$ | Ising [4] |
|--------|---------|-----------|
| $\langle |L| \rangle$ | $\beta/\nu$ | 0.525(8) | 0.518(7) |
| $D\langle |L| \rangle$ | $(1 - \beta)/\nu$ | 1.085(14) | 1.072(7) |
| | $1/\nu$ | 1.610(16) | 1.590(2) |
| | $\nu$ | 0.621(6) | 0.6289(8) |
| | $\beta$ | 0.326(8) | 0.3258(44) |
| $\chi_v$ | $\gamma/\nu$ | 1.944(13) | 1.970(11) |
| $D\chi_v$ | $(1 + \gamma)/\nu$ | 3.555(15) | 3.560(11) |
| | $1/\nu$ | 1.611(20) | 1.590(2) |
| | $\nu$ | 0.621(8) | 0.6289(8) |
| | $\gamma$ | 1.207(24) | 1.239(7) |
| | $\gamma/\nu + 2\beta/\nu$ | 2.994(21) | 3.006(18) |
| $g_r$ | $-g_r^\infty$ | 1.403(16) | 1.41 |
| $Dg_r$ | $1/\nu$ | 1.587(27) | 1.590(2) |
| $(\omega = 1)$ | $\nu$ | 0.630(11) | 0.6289(8) |

Table 3

Results from the $\chi^2$-method at $4/g_{c,\infty}^2 = 2.29895(10)$.
Here, $DO$ denotes the derivative $\partial O/\partial (4/g^2)$.

Whereas it is relatively easy to calculate the derivatives from spline interpolations to the DSM results, it seems more difficult to estimate their errors. To achieve this, we applied the Jackknife method to those spline derivatives, which were obtained from the individual Jackknife block results. In Fig. 3 we show the slope parameters from the derivative fits in the vicinity of the critical point. Again we find a surprising agreement with the 3$d$ Ising model. In Table 3 the corresponding numbers are given as well as the resulting $1/\nu(\nu)$ and exponent values. At the critical point the minimal $\chi^2/N_f$ was of the order $3 - 4$ for the derivatives of $\langle |L| \rangle$ and $\chi_v$ and about 1 for the derivative of $g_r$. We think therefore, that the result

$$\nu = 0.630(11) .$$

From the derivative of $g_r$ is the more reliable one, though all results are compatible with each other. We have checked also the hyperscaling relation

$$\gamma/\nu + 2\beta/\nu = d .$$
As can be seen from Table 3 our results are perfectly consistent with Eq. (29).

In summarizing the following comments are in order. We have shown, that also for the more complex (as compared to the Ising model) $SU(2)$ gauge theory it is possible to determine the universal quantities related to the deconfinement transition with high precision from simulations on finite lattices. The special FSS method which we applied here, the $\chi^2$-method, proved to be simple and accurate and showed consistency in the determination of the critical point from different observables. An indispensble part of the method is the DSM interpolation of the MC data. The quality of the DSM interpolation was usually superior to any single direct MC calculation apart from those with extremely high statistics.

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Figure Captions

Fig. 1 The minimal values of $\chi^2$ per degree of freedom for linear fits at fixed couplings $4/g^2$ according to Eq. (11) for $\langle |L| \rangle$ (solid line) and $\chi_v$ (short dashes). For the cumulant $g_r$ (long dashes) Eq. (12) with $\omega = 1$ was used.

Fig. 2 The slopes $\beta/\nu$ (solid line) and $\gamma/\nu$ (short dashes), and the constant $c_0 = g_r^{\infty}$ (long dashes) from the same fits as in Fig. 1. The dotted lines show the corresponding 3d Ising model values, the dashed-dotted lines the error bars of $4/g_{c,\infty}^2$.

Fig. 3 The slopes of the linear fits to the logarithms of the derivatives of $\langle |L| \rangle$ (solid line), $\chi_v$ (short dashes) and $g_r$ (long dashes). The dotted lines show the corresponding 3d Ising model values, the dashed-dotted lines the error bars of $4/g_{c,\infty}^2$. 
Figure 1
