QUASI-LISSE VERTEX ALGEBRAS AND MODULAR LINEAR DIFFERENTIAL EQUATIONS

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Dedicated to the great mathematician Bertram Kostant

Abstract. We introduce a notion of quasi-lisse vertex algebras, which generalizes admissible affine vertex algebras. We show that the normalized character of an ordinary module over a quasi-lisse vertex operator algebra has a modular invariance property, in the sense that it satisfies a modular linear differential equation. As an application we obtain the explicit character formulas of simple affine vertex algebras associated with the Deligne exceptional series at level $-h^\vee/6 - 1$, which express the homogeneous Schur indices of 4d SCFTs studied by Beem, Lemos, Liendo, Peelaers, Rastelli and van Rees, as quasi-modular forms.

1. Introduction

The vertex algebra $V$ is called lisse, or $C_2$-cofinite, if the dimension of the associated variety $X_V$ is zero. For instance, a simple affine vertex algebra $V$ associated with an affine Kac-Moody algebra $\widehat{g}$ is lisse if and only if $V$ is an integrable representation as a $\widehat{g}$-module. Thus, the lisse property generalizes the integrability condition to an arbitrary vertex algebra.

It is known that a lisse vertex operator algebra $V$ has nice properties, such as the modular invariance of characters $[Z, \Miy]$, and most theories of vertex operator algebras have been build under this finiteness condition (see e.g. [DLM, Hua]). However, there do exist significant vertex algebras that do not satisfy the lisse condition. For instance, admissible affine vertex algebras do not satisfy the lisse condition unless they are integrable, but nevertheless their representations are semisimple in category $\mathcal{O}$ ([AdMi, A5]) and have the modular invariance property ([KW2, AvE]). Moreover, there are a huge number of vertex algebras constructed in [BLL+] from four dimensional $N = 2$ superconformal
field theories (SCFTs), whose character coincides with the Schur limit of the superconformal index of the corresponding four dimensional theories. These vertex algebras do not satisfy the lisse property in general either.

In this paper we propose the quasi-lisse condition that generalizes the lisse condition. More precisely, we call a conformal vertex algebra $V$ quasi-lisse if its associated variety $X_V$ has finitely many symplectic leaves. For instance, a simple affine vertex algebra $V$ associated with $\hat{g}$ is quasi-lisse if and only if $X_V$ is contained in the nilpotent cone $\mathcal{N}$ of $g$. Therefore, by [FM, A3], all the admissible affine vertex algebras are quasi-lisse. Moreover, the $W$-algebras obtained by quasi-lisse affine vertex algebras by the quantized Drinfeld-Sokolov reduction ([FF, KRW]) is quasi-lisse as well. The vertex algebras constructed from 4d SCFTs are also expected to be quasi-lisse, since their associated varieties conjecturally coincide with the Higgs branches of the corresponding four dimensional theories ([R]).

We show that the normalized character of an ordinary representation of a quasi-lisse vertex operator algebra has a modular invariance property, in the sense that it satisfies a modular linear differential equation (MLDE) (cf. [MMS], [KZ], [Mas], [Mil], [KNS] and [AKNS]). This seems to be new even for an admissible affine vertex algebra. Moreover, using MLDE, we obtain the explicit character formulas of simple affine vertex algebras associated with the Deligne exceptional series $A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$ ([D]) at level $-h^\vee/6 - 1$. These vertex algebras arose in [BLL+] as 2d chiral algebras constructed from 4d SCFTs\footnote{For types $G_2$ and $F_4$ the connection with 4d SCFTs is conjectural.}. Thus [BLL+], our result expresses the homogeneous Schur indices of the corresponding 4d SCFTs as (quasi)modular forms. This result is rather surprising especially for types $D_4, E_6, E_7$ and $E_8$ (non-admissible cases), since the characters of these vertex algebras are written [KT] in terms of non-trivial Kazhdan-Lusztig polynomials as their highest weights are not regular dominant.

We note that in [CS] the authors have obtained a conjectural expression of Schur indices in terms of Kontsevich-Soibelman wall-crossing invariants, which we hope to investigate in future works.

Acknowledgments. The first named author thanks Victor Kac, Anne Moreau, Hiraku Nakajima, Takahiro Nishinaka, Leonardo Rastelli, Shu-Heng Shao, Yuji Tachikawa and Dan Xie for valuable discussion. He thanks Christopher Beem for pointing out an error in the first version of this article. Some part of this work was done while he was visiting...
Quasi-lisse vertex algebras

Let $V$ be a conformal vertex algebra, $R_V = V/C_2(V)$ the Zhu's $C_2$-algebra of $V ([Z]),$ where $C_2(V) = \langle a_{(-2)}b \mid a, b \in V \rangle C$. The space $R_V$ is a Poisson algebra by
\[
\bar{a} \cdot \bar{b} = a_{(-1)}b, \quad \{\bar{a}, \bar{b}\} = a_{(0)}b.
\]
Here $\bar{a}$ denotes the image of $a \in V$ in $R_V$, and $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in (\text{End } V)[[z, z^{-1}]]$ is the quantum field corresponding to $a \in V$. In this paper we assume $V$ is finitely strongly generated, that is, $R_V$ is finitely generated.

The associated variety $[A2]$ of a vertex algebra $V$ is the finite-dimensional algebraic variety defined by
\[
X_V = \text{Specm}(R_V).
\]
Since $R_V$ is a Poisson algebra, we have a finite partition
\[
X_V = \bigcup_{k=0}^{r} X_k,
\]
where $X_k$ are smooth analytic Poisson varieties (see e.g. [BG]). Thus, for any point $x \in X_k$ there is a well defined symplectic leaf $L_x \subset X_k$ through it.

Definition 2.1. A finitely strongly generated vertex algebra $V$ is called quasi-lisse if $X_V$ has only finitely many symplectic leaves.

Let $V$ be a quasi-lisse vertex algebra. The finiteness of the symplectic leaves implies [BG] that the symplectic leaf $L_x$ at $x \in X_V$ coincides with the regular locus of the zero variety of the maximal Poisson ideal contained in the maximal ideal corresponding to $x$. Thus, each leaf $L_x$ is a smooth connected locally-closed algebraic subvariety in $X_V$.
In particular, every irreducible component of $X_V$ is the closure of a symplectic leaf ([Gin, Corollary 3.3]).

For us, the importance of the finiteness of the symplectic leaves is in the following fact that has been established by Etingof and Schedler.

**Theorem 2.2** ([ES]). Let $R$ be a finitely generated Poisson algebra, and suppose that Specm($R$) has finitely many symplectic leaves. Then
$$\dim R/\{R, R\} < \infty.$$  

### 3. A NECESSARY CONDITION FOR THE QUASI-LISSE PROPERTY

A finitely strongly generated vertex algebra $V$ is called *conical* if it is conformal, and $L_0$ gives a $\frac{1}{m} \mathbb{Z}_{\geq 0}$-grading
$$V = \bigoplus_{\Delta \in \frac{1}{m} \mathbb{Z}_{\geq 0}} V_{\Delta}$$
on $V$ for some $m \in \mathbb{N}$, $\dim V_{\Delta} < \infty$ for all $\Delta$, and $V_0 = \mathbb{C}$, where $V_{\Delta} = \{v \in V \mid L_0 v = \Delta v\}$. Note that if $V$ is a vertex operator algebra, that is, if $V$ is integer-graded, then a conical vertex operator algebra is the same as a vertex operator algebra of CFT type.

Let $V$ be a conical vertex algebra. The $L_0$-grading induces the grading
$$R_V = \bigoplus_{\Delta \in \frac{1}{m} \mathbb{Z}_{\geq 0}} (R_V)_{\Delta}, \quad (R_V)_0 = \mathbb{C},$$
on $R_V$. In other words, the $L_0$-grading induces a contracting $\mathbb{C}^*$-action on the associated variety $X_V$.

**Remark 3.1.** The associated variety of a simple conical quasi-lisse vertex algebra is conjecturally irreducible ([AM2]). The validity of this conjecture implies that the associated variety of a quasi-lisse vertex algebra is actually *symplectic*, that is, $X_V$ is the closure of a symplectic leaf.

**Remark 3.2.** Conical lisse ($C_2$-cofinite) conformal vertex algebras are quasi-lisse, since $X_V$ is a point in this case.

**Proposition 3.3.** Let $V$ be a conical quasi-lisse vertex algebra. Then the image $[\omega]$ of the conformal vector $\omega$ of $V$ is nilpotent in the Zhu’s $C_2$-algebra $R_V$ of $V$.

**Proof.** Since $V$ is conical, the $\mathbb{C}^*$-action $\rho$ on $X_V$ induced by the conformal grading contracts to a point, say 0, that is, $\lim_{t \to 0} \rho(t)x = 0$ for all $x \in X_V$. 

Set $z = [\omega]$. It is sufficient to show that the value of $z$ at any closed point $x$ is zero. Pick an irreducible component $Y$ of $X_V$ containing $x$. Note that $Y$ is $\mathbb{C}^*$-invariant, and hence, $0 \in Y$. On the other hand, there exists a symplectic leaf $\mathcal{L} \subset X_V$ such that $Y = \overline{\mathcal{L}}$, the Zariski closure of $\mathcal{L}$. Since it belongs to the Poisson center of $R_V$, $z$ belongs to the Poisson center of $\mathcal{O}(\mathcal{L})$. Hence $z$ is constant on $\mathcal{L}$ as $\mathcal{L}$ is symplectic, and thus, so is on $Y$. Therefore the value of $z$ at $x$ is the same as the one at $0$, which is clearly zero. □

4. Finiteness of ordinary representations

Recall that a weak $V$-module $(M, Y_M)$ is called ordinary if $L_0$ acts semi-simply on $M$, any $L_0$-eigenspace $M_\Delta$ of $M$ of eigenvalue $\Delta \in \mathbb{C}$ is finite-dimensional, and for any $\Delta \in \mathbb{C}$, we have $M_{\Delta-n} = 0$ for all sufficiently large $n \in \mathbb{Z}$.

**Theorem 4.1.** Let $V$ be a quasi-lisse conformal vertex algebra. Then the number of the simple ordinary $V$-modules is finite.

**Proof.** Let $A(V)$ be the Zhu’s algebra of $V$. By the Zhu’s theorem [Z], it is sufficient to show that the number of the simple finite-dimensional $A(V)$-modules is finite.

The algebra $A(V)$ is naturally filtered: There is a natural filtration of $G_* A(V)$ induced by the filtration $\bigoplus_{\Delta \leq p} V_\Delta$ of $V$ ([Z]) that makes the associated graded $\text{gr}_G A(V)$ a Poisson algebra. Moreover, there is a surjection map

$$R_V \to \text{gr} A(V)$$

of Poisson algebras [DSK, ALY]. Therefore, $\text{Specm}(\text{gr}_G A(V))$ is a Poisson subvariety of $X_V$, and hence has a finitely many symplectic leaves. Hence, thanks to Theorem 1.4 of [ES] that follows from Theorem 2.2, we conclude that $A(V)$ has only finitely many simple finite-dimensional representations. □

5. Modular linear differential equations

Let $\vartheta_k$ denote the Serre derivation of weight $k$:

$$\vartheta_k(f) = q \frac{d}{dq} f - \frac{k}{12} E_2 f,$$

where $E_n(\tau)$ is the normalized Eisenstein series of weight $n \geq 2$. Let $\vartheta^i_k = \vartheta_{k+2(i-1)} \circ \cdots \circ \vartheta_{k+2} \circ \vartheta_k$ be the $i$-th iterated Serre derivation of
weight $k$ with $\vartheta_k^0 = 1$. Recall that a modular linear differential equation (MLDE) of weight $k$ is a linear differential equation

$$\vartheta_k^0 f + \sum_{j=0}^{n-1} P_j \vartheta_k^j f = 0$$

with a classical modular function $P_j$ of weight $2n - 2j$ for each $0 \leq j \leq n - 1$.

In this section, we prove the following theorem. Let $\mathbb{H}$ denote the complex upper-half plane.

**Theorem 5.1.** Let $V$ be a quasi-lisse vertex operator algebra, $c \in \mathbb{C}$ the central charge of $V$. Then the normalized character

$$\chi_V(\tau) = \text{tr}_V(e^{2\pi i (L_0 - c/24)}) \quad (\tau \in \mathbb{H})$$

satisfies a modular linear differential equation of weight $0$.

Let $(V, Y(\cdot, z), \omega)$ be a quasi-lisse conformal vertex operator algebra with the weight grading $V = \bigoplus_{\Delta = 0}^{\infty} V_{\Delta}$. Y. Zhu introduced a second vertex operator algebra $(V, Y(\cdot, z), \tilde{\omega})$ associated to $V ((\mathbb{Z}))$, where the vertex operator $Y(\cdot, z)$ is defined by linearly extending the assignment

$$Y[v, z] = Y(v, e^z - 1)e^{z\Delta} = \sum_{n \in \mathbb{Z}} v[n]z^{-n-1}, \quad (v \in V_{\Delta}, \Delta \geq 0),$$

and $\tilde{\omega} = \omega - c/24$. We write $Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n]z^{-n-2}$ and $V[\Delta] = \{v \in V \mid L[0]v = \Delta v\}$ for every $\Delta \in \mathbb{Z}_{\geq 0}$.

Set $A = \mathbb{C}[\tilde{G}_4(q), \tilde{G}_6(q)]$, $V_A = V \otimes A$. Here, $q$ is a formal variable and $\tilde{G}_n(q) = \sum_{j=0}^{\infty} a_{n,j}q^j$, where $\{a_{n,j}\}_{j \geq 0}$ ($n \geq 2$) are the Fourier coefficients of the Eisenstein series $G_n(\tau)$ of weight $n$, that is, $G_n(\tau) = \sum_{j=0}^{\infty} a_{n,j}e^{2\pi i j \tau}$. Let $O_q(V)$ be the $A$-submodule of $V_A$ generated by

$$a[-2]b + \sum_{k=2}^{\infty} (2k - 1)\tilde{G}_{2k}(q)a[2k - 2]b \quad (a, b \in V).$$

Let $[V_A, V_A]$ denote the $A$-span of elements $a[0]b$, $a, b \in V$, in $V_A$.

**Proposition 5.2.** The $A$-module

$$V_A/([V_A, V_A] + O_q(V))$$

is finitely generated.

**Proof.** We set

$$U := [V_A, V_A] + O_q(V).$$

Also, put $R = R_V = V/C_2(V)$, $C_2(V)_A = C_2(V) \otimes_{\mathbb{C}} A$, and

$$R_A = R \otimes_{\mathbb{C}} A = V_A/C_2(V)_A.$$
Define an increasing filtration $G_p V_A$ of the $A$-module $V_A$ by

$$G_p V_A = \bigoplus_{\Delta \leq p} V_A \otimes A.$$ 

This induces the filtration on $U$ and $V_A/U$: $G_p U = U \cap G_p V_A$, $\text{gr}_G U = \bigoplus_p G_p U / G_{p-1} U$, $G_p (V_A/U) = G_p V_A / G_p U$, and

$$\text{gr}_G (V_A/U) = \bigoplus_p G_p (V_A/U) / G_{p-1} (V_A/U) = V_A / \text{gr}_G U.$$

Since $\text{gr}_G U \supset \{ R_A, R_A \} + C_2 (V_A)$, we have a surjective map of $A$-modules

$$R_A / \{ R_A, R_A \} \twoheadrightarrow V_A / \text{gr}_G U = \text{gr}_G (V_A/U).$$

As $R_A / \{ R_A, R_A \} = (R / \{ R, R \}) \otimes \mathbb{C} A$, the assertion follows from Theorem 2.2. □

Since $A$ is a Noetherian ring, it follows from Proposition 5.2 that $V_A / ([V_A, V_A] + O_q (V))$ is a Noetherian $A$-module. Hence, we have the following lemma.

**Lemma 5.3.** For an element $a$ of $V$, there exist $s \in \mathbb{Z}_{\geq 0}$ and $g_i(q) \in A$ ($0 \leq i \leq s - 1$) such that

$$L[-2]^s a + \sum_{i=0}^{s-1} g_i(q) L[-2]^i a \in [V_A, V_A] + O_q (V).$$

Let $M$ be an ordinary $V$-module. Define the zero-mode action $o(\cdot) : V \to \text{End}(M)$ by linearly extending the assignment

$$o(a) = \text{Res}_z Y^M (a, z) z^{-1} dz : M \to M, \quad (a \in V_{\Delta}, \ \Delta \geq 0).$$

For any $a \in V$, define the formal 1-point function $\tilde{\chi}_M (a, q)$ by

$$\tilde{\chi}_M (a, q) = \text{tr} \big| M o(a) q^{L_0 - c/24}. $$

For each $q$-series $f(q)$ and $k \geq 0$, define the formal Serre derivation $\partial_k$ of weight $k$ by

$$\partial_k f = q \frac{d}{dq} f(q) + k \tilde{G}_2 (q) f(q).$$

Let $\Delta$ and $\ell$ be non-negative integers. For each $a \in V_{[\Delta]}$ and $f(q) \in \mathbb{C}[\tilde{G}_2 (q), \tilde{G}_4 (q), \tilde{G}_6 (q)]$ of weight $n$, define the formal iterated Serre derivation $\partial^i$ by

$$\partial^i (f(q) \tilde{\chi}_M (a, q)) = \partial_{\Delta + \ell + 2i - 2} (\partial^{i-1} (f(q) \tilde{\chi}_M (a, q))). \quad (i \geq 1).$$
and $\partial^0 = \text{id}$. Here, $f(q)$ is said to be of weight $n$ if $f(q)$ is a homogeneous element of weight $n$ of the graded algebra $\mathbb{C}[\tilde{G}_2(q), \tilde{G}_4(q), \tilde{G}_6(q)]$, where the weight of $\tilde{G}_k(q)$ is $k$ for each $k = 2, 4, 6$.

**Lemma 5.4** (cf. [DLM, (5.9)]). Let $v$ be an element of $V$ and $M$ an ordinary $V$-module. We have

$$\tilde{\chi}_M(L[-2]v, q) = \partial \tilde{\chi}_M(v, q) + \sum_{\ell=2}^{\infty} \tilde{G}_{2\ell}(q) \tilde{\chi}_M(L[2\ell - 2]v, q).$$

**Proof.** The assertion follows by [Z, Proposition 4.3.5] with $a = \omega$ and $b = v$. \hfill \square

Let $a$ be a primary vector of $V$ of weight $\Delta$, that is, $L[0]a = \Delta a$ and $L[n]a = 0$ for $n \geq 1$.

**Lemma 5.5** (cf. [DLM, Lemma 6.2]). For each $i \geq 1$, there exist elements $f_j(q) \in A$ ($0 \leq j \leq i - 1$) such that for any ordinary $V$-module $M$,

$$(1) \quad \tilde{\chi}_M(L[-2]^i a, q) = \partial^i \tilde{\chi}_M(a, q) + \sum_{j=0}^{i-1} f_j(\tau) \partial^j \tilde{\chi}_M(a, q).$$

**Proof.** The proof is similar to that of [DLM, Lemma 6.2]. We prove the assertion by induction on $i$. When $i = 1$, it follows from Lemma 5.4 that $\tilde{\chi}_M(L[-2]a, q) = \partial \tilde{\chi}_M(a, q)$, and therefore $(1)$ follows. Suppose that $i \geq 2$. Then by Lemma 5.4, we see that

$$\tilde{\chi}_M(L[-2]^i a, q) = \partial \tilde{\chi}_M(L[-2]^{i-1} a, q) + \sum_{k=2}^{\infty} \tilde{G}_{2k}(q) \tilde{\chi}_M(L[2k - 2]L[-2]^{i-1} a, q).$$

By using the relation of the Virasoro algebra, we have

$$L[2k - 2]L[-2]^{i-1} a = c_k \cdot L[-2]^{i-k} a$$

with a scalar $c_k$ for each $2 \leq k \leq i$ and $L[2k - 2]L[-2]^{i-1} a = 0$ if $k \geq i + 1$. Therefore,

$$\tilde{\chi}_M(L[-2]^i a, q) = \partial \tilde{\chi}_M(L[-2]^{i-1} a, q) + \sum_{k=2}^{i} c_k \tilde{G}_{2k}(q) \tilde{\chi}_M(L[-2]^{i-k} a, q).$$

By the induction hypothesis, we have $(1)$, which completes the proof. \hfill \square

Let $u$ and $v$ be elements of $V$. 
Lemma 5.6 ([Z, Proposition 4.3.6]). For every ordinary \(V\)-module \(M\),
\[
\tilde{\chi}_M(u[0]v, q) = 0,
\]
\[
\tilde{\chi}_M(u[-2]v, q) + \sum_{k=2}^{\infty} (2k - 1) G_{2k}(q) \tilde{\chi}_M(u[2k-2]v, q) = 0.
\]

Theorem 5.7. Let \(V\) be a quasi-lisse vertex operator algebra, \(a \in V\) primary with \(L[0]a = \Delta a\). For each ordinary \(V\)-module \(M\), the series \(\tilde{\chi}_M(a, q)\) converges absolutely and uniformly in every closed subset of the domain \(\{q \mid |q| < 1\}\), and the limit function \(\chi_M(a, q)\) has the form \(q^h f(q)\) with some analytic function \(f(q)\) in \(\{q \mid |q| < 1\}\). Moreover, the space spanned by \(\chi_M(a, q)\) for all ordinary \(V\)-module \(M\) is a subspace of the space of the solutions of a modular linear differential equation of weight \(\Delta\).

Proof. The proof is similar to those of [Z, Theorem 4.4.1] and [DLM, Lemma 6.3]. By Lemma 5.3, we have \(L[-s]a + \sum_{i=0}^{s-1} g_i(q) L[-2]^i a \in O_q(V)\) where \(s \in \mathbb{Z}_{\geq 0}\) and \(g_i(q) \in A\) for each \(0 \leq i \leq s - 1\). It then follows by the definition of \(O_q(V)\) and Lemma 5.6 that \(\tilde{\chi}_M(L[-2]^s a + \sum_{i=0}^{s-1} g_i(q) L[-2]^i a, q) = 0\). By Lemma 5.5, we obtain a differential equation
\[
(2) \quad \partial^s \tilde{\chi}_M(a, q) + \sum_{i=0}^{s-1} h_i(q) \partial^i \tilde{\chi}_M(a, q) = 0.
\]

for the formal series \(\tilde{\chi}_M(a, q)\) with \(h_i(q) \in A\). Since \(h_i(q)\) converges absolutely and uniformly on every closed subset of \(\{q \mid |q| < 1\}\), and (2) is regular, it follows that \(\tilde{\chi}_M(a, q)\) converges uniformly on every closed subset of \(\{q \mid |q| < 1\}\). By using (2) again, we see that the space spanned by \(\chi_M(a, q)\) for all ordinary \(V\)-module \(M\) is a subspace of the space of the solutions of the MLDE
\[
\partial^s \Delta \chi_M(a, q) + \sum_{i=0}^{s-1} p_i(q) \partial^i \chi_M(a, q) = 0,
\]

where \(p_i(q) \in \mathbb{C}[G_4(\tau), G_6(\tau)]\) is the limit function of \(h_i(q)\) with \(q = e^{2\pi i \tau}\) and \(\tau \in \mathbb{H}\). The remainder of the theorem is clear. \(\square\)

Theorem 5.1 follows from Theorem 5.7 with \(a = |0\rangle\), the vacuum vector of \(V\).

6. Examples of quasi-lisse vertex algebras

Let \(V^k(g)\) be the universal affine vertex algebra associated with a simple Lie algebra \(g\) at level \(k \in \mathbb{C}\), and let \(V_k(g)\) be the unique simple
graded quotient of $V^k(g)$. We have $X_{V^k(g)} = g^*$, where $g^*$ is equipped with the Kirillov-Kostant-Souriau Poisson structure, and $X_{V^k(g)}$ is a conic, $G$-invariant, Poisson subvariety of $X_{V^k(g)} = g^*$, where $G$ is the adjoint group of $g$ (see [A2]).

Let $N = \{x \in g \mid \text{ad} x \text{ is nilpotent}\}$, the nilpotent cone of $g$, which is identified with the zero locus of the argumentation ideal $C[g^*]_G$ of the invariant ring $C[g^*]_G$ via the identification $g = g^*$. It is well-known since Kostant [Kos63] that the number of $G$-orbits in $N$ is finite.

**Lemma 6.1.** The affine vertex algebra $V_k(g)$ is quasi-lisse if and only if the associated variety $X_{V_k(g)} \subset N$.

**Proof.** The “if” part is clear since the symplectic leaves in $g^*$ are the coadjoint orbits of $G$. Conversely, suppose that $X_{V_k(g)} \not\subset N$. Since $X_{V_k(g)}$ is closed, there exists a nonzero semisimple element $s$ in $X_{V_k(g)}$. As it is conic, $X_{V_k(g)}$ contains infinitely many orbits of the form $G.\lambda s$, $\lambda \in C^*$.

Recall that $V_k(g)$ is called **admissible** if it is an admissible representation ([KW2]) as a module over the affine Kac-Moody algebra $\hat{g}$ associated with $g$. All the admissible affine vertex algebras are quasi-lisse, since their associated varieties are contained in $N$ ([FM, A3]). In fact, the associated variety of an admissible affine vertex algebra $V_k(g)$ is irreducible, that is, $X_{V_k(g)} = \overline{O}$ for some nilpotent orbit $O$ of $g$ (see [A3] for the explicit description of the orbit $O$).

Highest weight representations of an admissible affine vertex algebra $V_k(g)$ are exactly the admissible representations $L(\lambda)$ of $\hat{g}$ of level $k$ whose integral Weyl groups are obtained from that of $V_k(g)$ by an element of the extended affine Weyl group ([AdMi, A5]). Let $\mathfrak{h}$ be the Cartan subalgebra of $g$. The modular invariance of the normalized full characters

$$e^{2\pi i t} \text{tr}_{L(\lambda)}(q^{L_0-c/24} e^{-2\pi ix}), \quad (\tau, x, t) \in Y,$$

of those representations, where $Y$ is some domain in $\mathbb{H} \times \mathfrak{h} \times \mathbb{C}$, has been known since Kac and Wakimoto [KW1, KW2], and was extended in [AvE] to that of the general (full) trace functions. Here it is essential to consider their full characters, since an admissible representation is not an ordinary representation in general, and thus, the normalized character $\text{tr}_V(e^{2\pi ir(L_0-c/24)})$ is not always well-defined.

Theorem 5.7 states the modular invariance of the normalized character (instead of the normalized full character) of an admissible representation that is ordinary. As far as the authors know, this fact is new.

Here are more examples of quasi-lisse affine vertex algebras.
Theorem 6.2 ([AM1]). Assume that $g$ belongs to the Deligne exceptional series

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$

and let $k = -h^\vee/6-1$. Then $X_{V_k(g)} \cong \mathcal{O}_{\text{min}}$, where $\mathcal{O}_{\text{min}}$ is the minimal nilpotent orbit of $g$.

In Theorem 6.2, the affine vertex algebra $V_{-h^\vee/6-1}(g)$ is admissible for types $A_1$, $A_2$, $G_2$, $F_4$, and so the statement is contained in [A3]. However, $V_{-h^\vee/6-1}(g)$ is not admissible for types $D_4$, $E_6$, $E_7$, $E_8$. These non-admissible quasi-lisse affine vertex algebras have appeared in [BLL+] as main examples of chiral algebras coming from 4d SCFTs. In fact, the labels $D_4$, $E_6$, $E_7$, $E_8$ also appear in Kodaira’s classification of isotrivial elliptic fibrations, and the corresponding 4d SCFTs are obtained by applying the $F$-theory to these isotrivial elliptic fibrations. By construction [BLL+], the character of the above non-admissible quasi-lisse affine vertex algebras are the (homogeneous) Schur indices of these 4d SCFTs obtained from elliptic fibrations. In mathematics, such a non-admissible affine vertex algebra was first extensively studied in [Per13].

In the next section we derive the explicit form of the characters of these non-admissible quasi-lisse affine vertex algebras.

Now let us give examples of quasi-lisse vertex algebras outside affine vertex algebras. Let $W^k(g, f)$ be the $W$-algebra associated with $g$ and a nilpotent element $f \in g$ at level $k$, defined by the quantized Drinfeld-Sokolov reduction

$$W^k(g, f) = H^0_{DS,f}(V^k(g)),$$

where $H^*_{DS,f}(M)$ denotes the cohomology of the BRST complex with coefficient $M$ associated with the Drinfeld-Sokolov reduction with respect to $f$. This definition was discovered by Feigin and Frenkel [FF] in the case that $f$ is principal as a generalization of Kostant’s Whittaker model of the center of $U(g)$ ([Kos78]), and was generalized to an arbitrary $f$ by Kac and Wakimoto ([KRW]).

By [A3], the natural surjection $V^k(g) \to V_k(g)$ induces a surjective homomorphism $W^k(g, f) \to H^0_{DS,f}(V_k(g))$ of vertex algebras, and moreover,

$$X_{H^0_{DS,f}(V_k(g))} \cong X_{V_k(g)} \cap S_f,$$

where $S_f$ is the Slodowy slice at $f$, that is, $S_f = f + g^e$. Here $\{e, f, h\}$ is an $\mathfrak{sl}_2$-triple and $g^e$ is the centralizer of $e$ in $g$. Therefore, we have the following assertion.
Lemma 6.3. Let $k$ be non-critical and suppose that $V_k(\mathfrak{g})$ is quasi-lisse, that is, $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$. For any $f \in X_{V_k(\mathfrak{g})}$, $H^0_{DS,f}(V_k(\mathfrak{g}))$ is quasi-lisse, and hence, so is the simple quotient $W_k(\mathfrak{g}, f)$ of $W^k(\mathfrak{g}, f)$.

We note that $H^0_{DS,f}(V_k(\mathfrak{g})) \cong W_k(\mathfrak{g}, f)$ if $G.f \subset X_{V_k(\mathfrak{g})}$ for type $A$ ([A1]) and this conjecturally holds for any $\mathfrak{g}$ ([KW3]).

Lemma 6.3 implies admissible affine vertex algebras produce many quasi-lisse $W$-algebras by applying the Drinfeld-Sokolov reduction. For instance, if $k$ is a non-degenerate admissible number (see [A4]), then $X_{V_k(\mathfrak{g})} = \mathcal{N}$, and hence,

$$X_{H^0_{DS,f}(V_k(\mathfrak{g}))} \cong \mathcal{N} \cap \mathcal{S}_f,$$

which is irreducible and therefore symplectic ([Pre]). In particular, if $f$ is a subregular nilpotent element in types $ADE$, $X_{H^0_{DS,f}(V_k(\mathfrak{g}))}$ has the simple singularity of the same type as $\mathfrak{g}$ ([Slo]). In type $A$, it has been recently shown by Genra [Gen] that the subregular $W$-algebra $W^k(\mathfrak{sl}_n, f_{\text{subreg}})$ is isomorphic to Feigin-Semikhatov’s $W_n^{(2)}$-algebra ([FS]) at level $k$.

See [AM1, AM3] for more examples of quasi-lisse vertex algebras, and see e.g. [XYY] for more examples of vertex algebras obtained from 4d SCFTs.

7. The characters of affine vertex algebras associated with the Deligne exceptional series

In this section, we give the explicit character formulas of the quasi-lisse affine vertex algebras associated with the Deligne exceptional series appeared in Theorem 6.2 by using MLDEs.

Let $\mathfrak{g}$ be a Lie algebra in the Deligne exceptional series and $V$ the simple affine vertex algebra $V_k(\mathfrak{g})$ with $k = -h^\vee/6 - 1$. The Deligne dimension formula (7) below implies that the central charge $c$ of $V$ is given by

$$c = -2h^\vee - 2.$$

Lemma 7.1. The square of the Virasoro element $\omega$ of $V$ is 0 in the Zhu’s $C_2$-algebra $R_V$.

Proof. Let $I$ be the ideal of $R_{V_k(\mathfrak{g})} = S(\mathfrak{g})$ generated by the image of the maximal submodule of $V^k(\mathfrak{g})$, so that $R_{V_k(\mathfrak{g})} = S(\mathfrak{g})/I$. We need to show that $\Omega^2 \in I$, where $\Omega$ is the Casimir element of $S(\mathfrak{g})$.

If $\mathfrak{g}$ is not of type $A$, then this result has been already stated in Lemma 2.1 of [AM1], see (the proof of) Theorem 3.1 of [AM1]. So let $\mathfrak{g}$ be of type $A$, in which case the maximal submodule of $V^k(\mathfrak{g})$
is generated by a singular vector, say \( v \) ([KW1]). For \( \mathfrak{g} = \mathfrak{sl}_2 \), the assertion follows immediately from a result in [FM], which says that the image of \( v \) in \( I \) coincides with \( \Omega e \) up to nonzero constant multiplication, see the proof of Theorem 4.2.1 of [FM]. Finally let \( \mathfrak{g} = \mathfrak{sl}_3 \). Then the vector \( v \) has degree 2, cf. [Per]. Let \( V \) be the \( \mathfrak{g} \)-submodule of \( S^2(\mathfrak{g}) \) generated by the image \([v]\) of \( v \) in \( I \). Proposition 3.3 of [GS] (which is valid for type \( A \) cases as well) says that \( \mathfrak{g} \cdot V \subset S^3(\mathfrak{g}) \) contains a submodule isomorphic to \( \mathfrak{g} \). On the other hand, Kostant’s Separation Theorem ([Kos63], cf. Proposition 3.2 of [GS]) implies that \( \mathfrak{g} \cdot \Omega \) is the unique submodule of \( S^3(\mathfrak{g}) \) isomorphic to \( \mathfrak{g} \). Thus, \( \mathfrak{g} \cdot \Omega \subset \mathfrak{g} \cdot V \), and the assertion follows. \( \Box \)

As \((V, [\cdot, \cdot])\) is isomorphic to \((V, Y(\cdot, \cdot))\), it follows that

\[
L[-2]^2|0\rangle = \sum_{i=1}^{\ell} b_i[-2]c_i
\]

for some \( \ell \geq 0 \), where \( b_i \) and \( c_i \), \( 1 \leq i \leq \ell \), are \( L[0] \)-homogeneous elements of \( V \) such that \( L[0](b_i[-2]c_i) = 4b_i[-2]c_i \). On the other hand, it follows from the definition of \( O_q(V) \) that

\[
b_i[-2]c_i \equiv -\sum_{k=2}^{\infty} f_k(q)b_i[2k - 2]c_i \quad (\text{mod } O_q(V))
\]

for \( 1 \leq i \leq \ell \). However, \( b_i[2]c_i \in \mathbb{C}[0] \) and \( b_i[2k - 2]c_i = 0 \) for \( k \geq 3 \) as the \( L[0] \)-weight of \( b_i[-2]c_i \) is 4. Therefore, we get that

\[
L[-2]^2|0\rangle + g(q)|0\rangle \in O_q(V)
\]

with \( g(q) \in A \). By using Lemma 5.5, we see that the formal characters \( \tilde{\chi}_M(|0\rangle, q) \) of all ordinary \( V \)-modules \( M \) satisfy a second order differential equation of the form \( \partial^2 \tilde{\chi}_M(|0\rangle, q) + f_1(q)\partial \tilde{\chi}_M(|0\rangle, q) + f_2(q)\tilde{\chi}_M(|0\rangle, q) = 0 \) with \( f_1(q), f_2(q) \in A \). Hence, the characters \( \chi_M(\tau) \) of the ordinary \( V \)-modules \( M \) satisfy a second order MLDE \( L(f) = 0 \) of weight 0.

The second order MLDEs (of weight 0) have the form

\[
f''(\tau) - \frac{1}{6} E_2(\tau) f'(\tau) - \frac{k(k + 2)}{144} E_4(\tau) f(\tau) = 0
\]

with \( k \in \mathbb{C} \). Here,

\[
' = q \frac{d}{dq} = \frac{1}{2\pi i} \frac{d}{d\tau}.
\]

A function \( f(\tau) \) is called of vacuum type if \( f \) has the form \( f(\tau) = q^{-\alpha/24}(1 + \sum_{n=1}^{\infty} a_n q^n) \) with \( \alpha \in \mathbb{Q} \) and \( a_n \in \mathbb{Z}_{\geq 0} \) for each \( n \geq 1 \), where \( q = e^{2\pi i \tau} \). Let \( f \) be a solution of (3) of the form \( f(\tau) = q^{-\alpha/24}(1 + O(q)) \)
with $\alpha \in \mathbb{Q}$. Then by substituting $f$ into (3), we see that $\alpha = -k/12$ or $(k + 2)/12$. If $\alpha = -k/12$, it follows that $k$ is one of the following numbers [KNS, (3.12)]:

$$k = \frac{1}{5}, \frac{1}{2}, \frac{1}{4}, \frac{7}{5}, \frac{13}{4}, \frac{7}{2}, \frac{19}{5}, 4.$$  

On the other hand, if $\alpha = (k + 2)/12$, we have [KNS, (3.16)]

$$k = \frac{1}{5}, \frac{1}{2}, 1, 2, 3, 5, 8, 11, 17, 23, 29, 53.$$  

Since $V$ is of CFT-type with the central charge $-h^V - 2$, the character $\chi_V(\tau)$ is of vacuum type and has the form $\chi_V(\tau) = q^{h^V + 1/2}(1 + O(q))$. Therefore, (4) and (5) imply that the MLDE $L(f) = 0$ must be the following one:

$$f''(\tau) - \frac{1}{6} E_2(\tau) f'(\tau) - \frac{(h^V - 1)(h^V + 1)}{144} E_4(\tau) f(\tau) = 0.$$  

The vacuum type solutions of (6) are also given in [KK] and [KNS]. As a result, we conclude that

$$\chi_{V_{-4/3}(A_1)} = \frac{\eta(3\tau)^3}{\eta(\tau)^3}, \quad \chi_{V_{-3/2}(A_2)} = \frac{\eta(2\tau)^8}{\eta(\tau)^8}, \quad \chi_{V_{-5/3}(G_2)} = \frac{E_1^{(3)}(\tau)\eta(3\tau)^6}{\eta(\tau)^8},$$

$$\chi_{V_{-2}(D_4)} = \frac{E_4'(\tau)}{240\eta(\tau)^{16}}, \quad \chi_{V_{-5/2}(F_4)} = \frac{E_2(\tau)\eta(2\tau)^24}{\eta(\tau)^{28}},$$

$$\chi_{V_{-3}(E_6)} = -\frac{1}{462} \left( \frac{E_6(\tau) E_4'(\tau)}{240\eta(\tau)^{22}} - \eta(\tau)^2 \right),$$

$$\chi_{V_{-4}(E_7)} = \frac{1}{204204} \left( \Delta(\tau) P_2 \left( \frac{E_6(\tau)}{\sqrt{\Delta(\tau)}} \right) \frac{E_4'(\tau)}{240\eta(\tau)^{34}} - \Delta(\tau) \frac{E_6(\tau)}{\eta(\tau)^{34}} \right),$$

$$\chi_{V_{-6}(E_8)} = \frac{1}{38818159380} \left( \Delta(\tau)^2 P_4 \left( \frac{E_6(\tau)}{\sqrt{\Delta(\tau)}} \right) \frac{E_4'(\tau)}{240\eta(\tau)^{58}} - \Delta(\tau)^{5/2} Q_4 \left( \frac{E_6(\tau)}{\sqrt{\Delta(\tau)}} \right) \frac{1}{\eta(\tau)^{58}} \right).$$  

Here, $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, $\Delta(\tau) = \eta(\tau)^{24}$, $E_2^{(2)}(\tau) = 2E_2(2\tau) - E_2(\tau)$,

$$E_1^{(3)}(\tau) = 1 + 6 \sum_{n=1}^{\infty} \left( \sum_{d|n} \left( \frac{d}{3} \right) (n/d)^2 \right) q^n$$

with the Legendre symbol $\left( \frac{\cdot}{3} \right)$. $P_2(x) = x^2 + 462$, $P_4(x) = x^4 + 1341x^2 + 201894$, and $Q_4(x) = x^3 + 879x$.

In particular, it follows that the characters of $V_{-4/3}(A_1)$, $V_{-3/2}(A_2)$, $V_{-5/3}(G_2)$, and $V_{-5/2}(F_4)$ are modular forms, while those of $V_{-2}(D_4)$, $V_{-3}(E_6)$, $V_{-4}(E_7)$ and $V_{-6}(E_8)$ are quasimodular forms of positive depths.
Moreover, if $h^\vee$ is the dual Coxeter number of $D_4$, $E_6$, $E_7$ or $E_8$, then MLDE (6) has a solution with a logarithmic term (see [KK, section 5] and [KNS, Remark 3.8]). Note that the above formula for $A_1$ and $A_2$ follows also from the recent remark [KW4] by Kac and Wakimoto.

**Remark 7.2.** It should be notable that the coefficient of $E_4(\tau)f(\tau)$ in (6) is a non-constant rational function in $h^\vee$, as such phenomena are often observed for the Deligne exceptional series. In fact, the dimensions of specific modules over any Lie algebra in the Deligne exceptional series satisfy the so-called *Deligne dimension formulas*, which are rational functions in $h^\vee$. For example,

$$\dim g = \frac{2(h^\vee + 1)(5h^\vee - 6)}{h^\vee + 6},$$

and $\dim L(2\theta) = 5(h^\vee)^2(2h^\vee + 3)(5h^\vee - 6)/(h^\vee + 12)(h^\vee + 6)$ ([CdM], [D] and [LM]). Here, $L(2\theta)$ is the irreducible highest weight module of weight $2\theta$ over $g$. In the vertex algebra setting, the first example of such phenomena was first observed in [T], where the coefficients of the MLDEs which the characters of the affine vertex algebras $V_1(g)$ at level 1 associated with the Deligne exceptional series satisfy are expressed as rational functions in $h^\vee$. The second example of such phenomena was found in [K2], where the minimal affine $\mathcal{W}$-algebras associated with the Deligne exceptional series at level $-h^\vee/6$ were shown to be lisse and rational.

**Remark 7.3.** It follows from a result of [KNS] that there is a vacuum type solution of (6) with $h^\vee = 24$. Although the Coxeter number $h^\vee$ of any Lie algebra in the Deligne exceptional series does not coincide with 24, the number "$h^\vee = 24$" appears in many studies of the Deligne exceptional series (see e.g. [CdM] and [K1]).
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