Anomalous Tunneling of Spin Wave in Polar State of Spin-1 BEC

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Abstract. We investigate tunneling properties of collective spin-wave excitations in the polar state of a spin-1 spinor Bose–Einstein condensate. Within the mean-field theory at $T=0$, we show that when the condensate is in the critical supercurrent state, the spin wave mode exhibits perfect transmission through a nonmagnetic potential barrier in the low energy limit, unless the strength of a spin-independent interaction $c_0$ equals that of a spin-dependent interaction $c_1$. Such an anomalous tunneling behavior is absent in the case of a magnetic barrier. We also clarify a scaling law of the transmission probability as a function of the mode energy.

1. Introduction

The recent theoretical studies on the so-called anomalous tunneling phenomenon have revealed low energy properties of Bose-Einstein condensates in the presence of obstacles, such as impurities and barriers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In this phenomenon, the perfect transmission of Bogoliubov mode through a potential barrier is predicted in the low energy limit, which is quite different from the simple tunneling problem in quantum mechanics, where the perfect reflection is always realized in this limit.

Recently, we have extended the anomalous tunneling problem discussed in scalar BECs to the ferromagnetic and polar phases of spin-1 spinor BECs [11, 12, 13, 14]. In the polar phase [14], we showed that, not only the Bogoliubov mode, but also the spin wave exhibits the anomalous tunneling behavior. This result holds even in the presence of background superflow, except in the critical supercurrent state. In the critical current state, while the Bogoliubov excitation no longer has the anomalous tunneling property, the low-energy spin wave still tunnels through a barrier without reflection, unless the strength of a spin-independent interaction $c_0$ equals that of a spin-dependent interaction $c_1$. When $c_0 = c_1$, the spin-wave tunneling is accompanied by a finite reflection, as in the case of the Bogoliubov excitation.

In this paper, we report detailed tunneling properties of collective spin-wave excitations in the polar phase of a spin-1 spinor BEC, when the condensate is in the critical current state. Within the framework of the mean-field theory at $T=0$, we show that the perfect transmission of the spin wave occurs unless $c_0 = c_1$. When $c_0 \neq c_1$, we point out that the energy dependence of the transmission probability obeys a scaling law. We also report tunneling properties of a spin wave through a magnetic barrier, and find the perfect reflection in the low-energy limit.
2. Formulation

We consider the tunneling of a spin wave through a barrier in the polar phase of a spin-1 BEC. Assuming that the barrier potential is uniform in the y- and z-direction, one may treat this problem as a one-dimensional one. In this case, the equation of motion for the condensate wavefunction $\Phi = (\Phi_{1}, \Phi_{0}, \Phi_{-1})$ (where the subscripts represent three magnetic sub-levels in the $S = 1$ spin state) is conveniently derived from the action $I = \int dt dx \mathcal{L}$ by using the variational principle [13, 14, 16, 15]. Here, the Lagrangian density is given by

$$\mathcal{L} = i\hbar \Phi^\dagger(x, t) \partial_t \Phi(x, t) - \left[\Phi^\dagger(x, t) \left(-\frac{\hbar^2}{2m} \partial_x^2 + \frac{c_0}{2} \rho^2(x, t) + \frac{c_1}{2} \mathbf{F}^2(x, t)\right) \Phi(x, t) + V(x) \rho(x, t) + V_m(x) F_z(x, t)\right],$$

where $m$ is an atomic mass, and $\rho = \Phi^\dagger \Phi$ and $\mathbf{F} = \Phi^\dagger \mathbf{S} \Phi$ are particle and spin density, respectively (where $\mathbf{S} = (S_x, S_y, S_z)$ are the $S = 1$ spin matrices). In this paper, we take the spin quantization axis parallel to the z-axis. $c_0$ and $c_1$, respectively, describe a spin-independent and a spin-dependent interaction [16]. Since we consider the polar phase, we set $c_1 > 0$. The last two terms in Eq. (1) describe effects of potential barriers, where $V(x)$ and $V_m(x)$ are the nonmagnetic and magnetic potential, respectively.

Taking the variation of $I$ with respect to $\Phi^\dagger$, we obtain the time-dependent Gross-Pitaevskii equation. In the stationary state, the condensate wavefunction has the form $\Phi(x, t) = e^{-i\mu t/\hbar} \Phi(x)$ [13, 14], where $\Phi(x)^\dagger = (0, \Phi_0(x), 0)$ in the polar phase. $\Phi_0(x)$ obeys the time-independent GP equation, $h(x) \Phi_0(x) = 0$, where

$$h(x) = -\frac{\hbar^2}{2m} \partial_x^2 - \mu + V(x) + c_0 \rho(x).$$

In the supercurrent state with the current momentum $q$, one may set $c = c_0 \rho_0 + \hbar^2 q^2/(2m)$, where $\rho_0$ is the superfluid density at $x = \pm \infty$. In this situation, we consider spin wave excitations. For this purpose, as usual, we examine fluctuations around $\Phi_0$, by setting $\Phi(x, t) = \exp(-i\mu t/\hbar)(\Phi(x) + \phi(x, t))$. Retaining term to $O(\phi)$, we obtain the spin-wave equation as [13, 14],

$$E \begin{pmatrix} \phi_{+1}(x) \\ \phi_{-1}^*(x) \end{pmatrix} = \begin{pmatrix} (h(x) + c_1 |\Phi_0(x)|^2 - V_m(x)) \phi_{+1}(x) \\ -c_1 \Phi_0^2(x) \phi_{-1}^*(x) \end{pmatrix} - \begin{pmatrix} c_1 \Phi_0^2(x) \\ h(x) - c_1 |\Phi_0(x)|^2 - V_m(x) \end{pmatrix} \begin{pmatrix} \phi_{+1}(x) \\ \phi_{-1}^*(x) \end{pmatrix},$$

where $E$ is the mode energy, and we have set $\phi_{\pm 1}(x, t) = \phi_{\pm 1}(x) e^{\pm iEt/\hbar}$ [14, 18]. Assuming that the incident spin wave comes from $x = -\infty$, we take the asymptotic solution as

$$\begin{cases} 
\phi_{+1} \\ \phi_{-1}^* 
\end{cases} = \begin{cases} 
(\alpha_{k_1} e^{iqx}) e^{ik_1 x} + R (\alpha_{k_2} e^{iqx}) e^{ik_2 x} + A (\alpha_{k_3} e^{iqx}) e^{ik_3 x}, & (x \to \infty), \\
(\alpha_{k_1} e^{-iqx}) e^{ik_1 x} + B (\beta_{k_1} e^{-iqx}) e^{ik_4 x}, & (x \to \infty), \end{cases}$$

where $\begin{cases} 
(\alpha_{k_j}, \beta_{k_j}) = \left( c_1 \rho_0, E - c_1 \rho_0 - \hbar^2 q k_j/m - \hbar^2 k_j^2/(2m) \right) \end{cases}$ [14, 18]. $k_j (j = 1 \sim 4)$ are solutions of the equation, $\hbar^4 k_j^4/m^2 + 4(c_1 \rho_0 - \hbar^2 q^2/m)k_j^2/m + 8\hbar^2 q E/m - 4E^2 = 0$, where $(k_1, k_2)$ describe propagating waves and $(k_3, k_4)$ are damping solutions. In particular, in the low-energy region, they reduce to $(h k_1, h k_2, h k_3, h k_4) \simeq (+Em/(\sqrt{mc_1 \rho_0} + hq), -Em/(\sqrt{mc_1 \rho_0} - hq), -2i(m c_1 \rho_0 - \hbar^2 q^2)^{1/2} + 2i(m c_1 \rho_0 - \hbar^2 q^2)^{1/2}).$

In our calculations, we first determine the condensate wavefunction $\Phi_0(x)$ from the GP equation (2). Then, we solve the mode equation (3) to determine the parameter set $(T, R, A, B)$.
in the asymptotic solution in Eq. (4). The transmission probability \( \tau \), as well as the reflection probability \( r \), are then calculated from \( \tau = |T|^2 \), and \( r = |w(k_2)/w(k_1)||R|^2 \), where \( w(k) = k(|\alpha_k|^2 + |\beta_k|^2) + q(|\alpha_k|^2 - |\beta_k|^2) \). For the derivation of these equations, we refer to Ref. [14].

3. Tunneling Properties of Spin Wave in the Polar Phase

Figure 1 (a) shows the transmission probability \( \tau \) of the spin wave in the critical current state of the polar phase. In this figure, we have assumed the spatial variation of the nonmagnetic barrier as \( V(x) = 2c_0|\rho_0| \exp[-(x/\xi)^2] \) (where \( \xi \equiv \hbar/\sqrt{mc_0|\rho_0|} \) is the healing length), and have ignored the magnetic barrier \( (V_m = 0) \). Except for the case \( c_0 = c_1 \), one sees the perfect transmission of the spin wave in the low energy limit, which is quite different from the case of the Bogoliubov excitation, where a finite reflection appears in the critical current state.

Apart from the perfect transmission in the limit \( E \to 0 \), Figure 1 (a) indicates that the transmission probability \( \tau \) depends on the values of the interaction parameters \( c_0 \) and \( c_1 \). However, when we plot \( \tau \) as a function of the scaled energy \( E/E^*(c_0, c_1) \), where \( E^*(c_0, c_1) = |c_0 - c_1|/\rho_0 \), the three lines in panel (a) is well described by a universal curve, as shown in Fig. 1(b). That is, the width of the energy region where the transmission probability is enhanced in panel (a) linearly vanishes with respect to \( |c_0 - c_1| \), when \( c_1 \) approaches \( c_0 \).

Figure 2 shows the transmission probability of the spin wave in the case of the magnetic barrier \( (V, V_m) = (0, 2c_0|\rho_0| \exp[-(x/\xi)^2]) \) in the absence of supercurrent \( q = 0 \). We clearly find the absence of the anomalous tunneling phenomenon, or the perfect transmission in the low energy limit. The energy dependence of \( \tau \) is rather close to the case of the ordinary tunneling problem in quantum mechanics.

The absence of the perfect transmission in the case of magnetic barrier can be analytically confirmed, when we consider a simple \( \delta \)-functional barrier, \( V_m(x) = V_0\delta(x) \). Setting \( q = V = 0 \) to further simplify our discussion, we obtain the uniform condensate wavefunction as \( \Phi_0(x) = \sqrt{\rho_0} \), so that the upper and lower wavefunctions in Eq. (4) immediately give the expressions for the spin-wave solutions for \( x < 0 \) and \( x > 0 \), respectively. From the boundary conditions at \( x = 0 \)

\[
(\phi_{\pm 1}(0) = \phi_{\pm 1}(0), \partial_x \phi_{\pm 1}(0) - \partial_x \phi_{\pm 1}(0) = \mp(2m/\hbar^2)V_0\phi_{\pm 1}(0)), \text{we find}
\]

\[
T = A/(A - iB), \quad R = iB/(A - iB). \tag{5}
\]
Here, \( A = E[V_0 + 2\sqrt{E^2 + (c_1\rho_0)^2}] \) and \( B = V_0[E\kappa + (V_0m/h^2)\sqrt{E^2 + (c_1\rho_0)^2}] \), where \( \hbar k = (2m[\sqrt{E^2 + (c_1\rho_0)^2} - c_1\rho_0])^{1/2} \) and \( \hbar \kappa = \{2m[\sqrt{E^2 + (c_1\rho_0)^2} + c_1\rho_0]\}^{1/2} \). In the low energy limit \( E \to 0 \), one finds \( A \to 0 \) and \( B \to mc_1\rho_0V_0^2/h^2 \), leading to \( \tau = 0 \).

We note that the rotational symmetry of spin degrees of freedom is spontaneously broken in the polar phase, and the magnetic barrier directly couples with this broken symmetry. In this regard, we have shown in Ref.[13] that the perfect transmission of Bogoliubov phonon does not occur when the barrier potential couples with the \( U(1) \)-gauge symmetry, which is also spontaneously broken in the superfluid state. From these results, we expect that, for the realization of the anomalous tunneling phenomenon, it is important whether the potential barrier affects symmetries that are spontaneously broken in the superfluid phase.

4. Summary
To summarize, we have discussed tunneling properties of collective spin-wave excitations through a barrier in the polar phase of a spin-1 spinor BEC. When the barrier potential is nonmagnetic, we showed that the spin wave exhibits the anomalous tunneling behavior even in the critical current state, unless the spin-independent interaction \( c_0 \) and spin-dependent interaction \( c_1 \) take the same magnitude. When \( c_0 \neq c_1 \), the energy dependence of the transmission probability is well described by a universal curve when plotted by the scaled energy \( E/E^*(c_0,c_1) \), where \( E^*(c_0,c_1) = |c_0 - c_1|\rho_0 \). When \( c_0 = c_1 \), the perfect transmission does not occur.

When the barrier potential is magnetic, the spin wave does not exhibit the perfect transmission. Instead, the perfect reflection is realized in the low energy limit. This result is similar to the vanishing transmission probability of Bogoliubov phonon, when the barrier potential breaks the \( U(1) \)-gauge symmetry (which is spontaneously broken in the superfluid phase). Since the magnetic barrier is also related to the broken spin-rotational symmetry in the polar phase, our results indicate the importance of the relation between symmetries that are spontaneously broken in the superfluid phase and the character of potential barrier in the anomalous tunneling phenomenon.

This work was supported by Grant-in-Aid for Scientific Research (Grant No. 20500044, 21540352, 22540412, 23104723, 23500056) from MEXT and JSPS, Japan.

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[18] We omit the normalization factor used in Ref. [14]. Tunneling properties are independent of this factor.