POSITIVE SOLUTIONS FOR CHOQUARD EQUATION IN EXTERIOR DOMAINS

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Abstract. This work concerns with the following Choquard equation
\[
\begin{cases}
-\Delta u + u = \left( \int_{\Omega} \frac{u^2(y)}{|x-y|^N} \, dy \right) u \\
u \in H^1_0(\Omega),
\end{cases}
\]
where \( \Omega \subseteq \mathbb{R}^N \) is an exterior domain with smooth boundary. We prove that the equation has at least one positive solution by variational and topological methods. Moreover, we establish a nonlocal version of global compactness result in unbounded domain.

1. Introduction. In this paper we are concerned with the following equation
\[
\begin{cases}
-\Delta u + u = \left( \int_{\Omega} \frac{u^2(y)}{|x-y|^N} \, dy \right) u \\
u \in H^1_0(\Omega),
\end{cases}
\]
where \( \Omega \subseteq \mathbb{R}^N \) is an unbounded domain, \( \partial \Omega \neq \emptyset \) is bounded.

This nonlocal elliptic equation is closely related to the nonlinear Choquard equation
\[
-\Delta u + u = \left( I_\alpha * |u|^p \right) |u|^{p-2} u \quad \text{in} \ \mathbb{R}^N,
\]
where \( I_\alpha : \mathbb{R}^N \rightarrow \mathbb{R} \) is a Riesz potential defined for every \( x \in \mathbb{R}^N \setminus \{0\} \) by
\[
I_\alpha(x) = \frac{\Gamma \left( \frac{N-\alpha}{2} \right)}{\Gamma \left( \frac{\alpha}{2} \right) \pi^{N/2} 2^{\alpha} |x|^{N-\alpha}}.
\]
In the physical case \( N = 3, \alpha = 2 \) and \( p = 2 \), equation (1.2) reduces to
\[
-\Delta u + u = \left( I_2 * |u|^2 \right) u \quad \text{in} \ \mathbb{R}^3,
\]
which appeared at least in 1954, in a work by S. Pekar describing the quantum mechanics of a polaron at rest [24]. In 1967 P. Choquard used (1.3) to describing an electron trapped in its own hole, in a certain approximation to Hartree-Fock
theory of one component plasma [13]. In 1996 R. Penrose proposed (1.3) as a model self-gravitating matter, in a programme in which quantum state reduction is understood as a gravitational phenomenon [23]. Sometimes equation (1.2) was also known as the Schrödinger-Newton equation, since the convolution part might be treated as a coupling with a Newton equation.

In the pioneering work [13], Lieb proved that the ground state of (1.3) is radial and unique up to translations. Later Lions [14, 15] obtained the existence of infinitely many radially symmetric solutions (1.3). In [26], Wei and Winter showed the nondegeneracy of the ground state and studied the multi-bump solutions for (1.3). Xiang [29] proved the uniqueness and nondegeneracy of ground states for Choquard equations in three dimensions. In [22], Ma and Zhao proved that every positive solution of (1.2) is radially symmetric and monotone decreasing about some point by the moving plane method under some assumptions on $N, \alpha$ and $p$. Moroz and Schaftingen [18] considered problem (1.2). They eliminated the restriction of [22], proved the regularity, positivity, radial symmetry and decay asymptotics at infinity of the ground states for optimal range of parameters. Subsequently, Moroz and Schaftingen [19] proved the existence of ground state solutions to the nonlinear Choquard equation with a general nonlinearity satisfying almost necessary conditions in the spirit of Berestycki and Lions. Nodal solutions for Choquard equation (1.2) was studied in [5, 30]. For Choquard equation (1.2) with potential, see [1, 4, 10, 11, 16, 21, 25] references therein. For more result on Choquard equations, we refer the reader to the survey [17] for comprehensive summary.

As we cited above, Choquard equation on the whole space $\mathbb{R}^N$ have been extensively studied in past decades. Recently, some researchers pay attention to Choquard equation on other types of domain. In [9], Gao and Yang studied a Brezis-Nirenberg problem of Choquard type in bounded domain and obtained some existence results. In [7], Goel and Sreenadh concerned with a coron problem problem involving Choquard nonlinearity in an annular type domain. Goel and Sreenadh [8] also studied Choquard equation over an annular type bounded domain and proved the existence of four positive solutions by Lusternik-Schnirelmann theory and variational methods, when the inner hole of the annulus is sufficiently small. Ghimenti and Pagliardini [6] studied a slightly subcritical Choquard problem on a bounded domain and showed the number of positive solutions of nonlinear Choquard equation depends on the topology of the domain when the exponent is very close to the critical one. However, to our best knowledge, there is few results about Choquard equation in exterior domain except [20], where the authors investigated the following general Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * u^p)u^q \quad \text{in } \Omega.$$  

They established some sharp Liouville type nonexistence results for positive supersolutions and obtained optimal decay rates when supersolutions exist. One open question is the existence of positive solutions to Choquard equation in exterior domain. Motivated by the pioneering work of Benci and Cerami [2], we study Choquard equation (1.1) and give an affirmative answer to this open problem.

Our main result is the following:

**Theorem 1.1.** Assume that $N = 4, 5$ and $x_0 \in \mathbb{R}^N \setminus \Omega$. Then, there is a $\rho$ such that if $\mathbb{R}^N \setminus \Omega \subset B_\rho(x_0)$, problem (1.1) has at least one positive solution.

**Remark 1.** The assumption on $N$ in Theorem 1.1 are a consequence of the fact that the ground state solution of the limit problem (2.3) has an exponential decay
when $3 < N < 6$ (see Proposition 6.5 and Remark 6.1 in [18]) and the uniqueness of positive solutions for (2.3) is known so far to us when $N = 3, 4, 5$ (see [27]).

To prove Theorem 1.1, we follow the argument in [2]. Firstly, we study the limit problem (2.3) associated to (1.1)(see Proposition 1). With this result in hand, we can show the nonexistence of ground state to equation (1.1) (see Theorem 2.3). Hence we need to find a solution in high energy level. To this aim, then we establish a global compactness result (see Proposition 2). Some estimates are established for later use. Finally, we prove our main result by variational and topological methods.

This paper is organized as follows. In Section 2, we give some notations and preliminaries. Section 3 is devoted to establish a global compactness result for the energy functional associated to problem (1.1). In Section 4 we provide some estimates that will be used in Section 5, where the proof of Theorem 1.1 is given.

2. Notations and preliminary results. Throughout this paper we shall denote by $$\|u\| = \left(\int_{\Omega}(|\nabla u|^2 + u^2)dx\right)^{\frac{1}{2}}$$ and $\|u\|_{L^N} = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)dx\right)^{\frac{1}{2}}$ the norms in $H^1_0(\Omega)$ and $H^1(\mathbb{R}^N)$, respectively.

For the sake of brevity, let us define the operator

$$T_{\Omega} : [H^1_0(\Omega)]^4 \to \mathbb{R}$$

such that for any $(u, v, w, z) \in [H^1_0(\Omega)]^4$:

$$T_{\Omega}(u, v, w, z) = \int_{\Omega} \int_{\Omega} \frac{u(x)v(x)w(y)z(y)}{|x-y|^{N-2}} dy dx.$$

When $\Omega = \mathbb{R}^N$, we omit the subscript and denote $T(u, v, w, z)$. Moreover, we denote the operator $T_{\Omega}(u, v, w, z)$ by $T(u, w)$ for convenience if $u = v$ and $w = z$. Consequently, we have $T(u, w) = T_{\Omega}(u, u, w, w)$.

We first recall the Hardy-Littlewood-Sobolev inequality and then establish a splitting lemma for the linear operator $T_{\Omega}$.

Lemma 2.1 (Hardy-Littlewood-Sobolev inequality [12]). Let $0 < \alpha < N$, $p, q > 1$ and $1 \leq r < s < \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \quad \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}.$$

(i) For any $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, one has

$$\left|\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} dy dx\right| \leq C(N, \alpha, p) \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)};$$

(ii) For any $f \in L^r(\mathbb{R}^N)$, one has

$$\left\|\frac{1}{1+|x-y|^{\alpha}} f\right\|_{L^s(\mathbb{R}^N)} \leq C(N, \alpha, r) \|f\|_{L^r(\mathbb{R}^N)}.$$

Lemma 2.2. (i) For any $f, g \in H^1(\mathbb{R}^N)$, one has

$$T(f, g) \leq C\|f\|_{L^\infty(\mathbb{R}^N)}^2 \|g\|_{L^\infty(\mathbb{R}^N)}^2$$

and

$$T(f, f) \leq C\|f\|_{H^1(\mathbb{R}^N)}^4 \quad (3 \leq N \leq 6).$$

(ii) Assume $3 \leq N \leq 6$ and $u_m \to u$, $v_m \to v$, $w_m \to w$ in $H^1_0(\Omega)$ and $z \in H^1_0(\Omega)$, then

$$T_{\Omega}(u_m, v_m, w_m, z) \to T_{\Omega}(u, v, w, z).$$
(iii) Assume $3 \leq N \leq 6$ and $u_m \to u$ in $H_0^1(\Omega)$, then
\[ T_\Omega(u_m, u_m) - T_\Omega(u_m - u, u_m - u) \to T_\Omega(u, u). \]

Proof. (i) The first inequality follows from Lemma 2.1 (i) by setting $p = q$ and $\alpha = 2$. The Sobolev embedding yields the second inequality.

(ii) We divide the proof into several steps.

Step 1. Suppose that $v_m = v, w_m = w$ for all $m \in \mathbb{N}$. We show
\[ T_\Omega(u_m, v, w, z) \to T_\Omega(u, v, w, z). \]

In fact, using the Hölder inequality, it is easy to prove
\[ \int_\Omega \frac{w(y)z(y)}{|x - y|^{N-2}} dy \cdot v(x) \in L^2(\Omega), \quad N = 3, 4 \]
and
\[ \int_\Omega \frac{w(y)z(y)}{|x - y|^{N-2}} dy \cdot v(x) \in L^{\frac{2N}{N+2}}(\Omega), \quad N = 5, 6. \]

Note that $u_m \to u$ in $L^2(\Omega)$ ($N = 3, 4$) and $u_m \to u$ in $L^{\frac{2N}{N+2}}(\Omega)$ ($N = 5, 6$). Hence, we have
\[ T_\Omega(u_m, v, w, z) \to T_\Omega(u, v, w, z). \]

Step 2. Suppose that $w_m = w$ for all $m \in \mathbb{N}$. We claim
\[ T_\Omega(u_m, v, w, z) \to T_\Omega(u, v, w, z). \]

Note that
\[ T_\Omega(u_m, v, w, z) = T_\Omega(u_m - u, v, w, z) + T_\Omega(u, v, w, z). \]

We have $T_\Omega(u, v, w, z) \to T_\Omega(u, v, w, z)$ by step 1. To prove the claim, we just need to prove $T_\Omega(u_m - u, v, w, z) \to 0$. Indeed, using the Hölder inequality to the function $(u_m(x) - u(x))w(y)$ and $v_m(x)z(y)$, we have
\[ T_\Omega(u_m - u, v, w, z)^2 \leq T_\Omega(u_m - u, w)T_\Omega(v, z). \]

It is easy to show that $T_\Omega(v, z)$ is bounded. Next, we show that the $T_\Omega(u_m - u, w)$ converges to 0.

Set
\[ \phi_u(y) = \int_\Omega \frac{u^2(x)}{|x - y|^{N-2}} dx. \]

Then $\phi_u \in D^{1,2}(\Omega)$ is a solution of
\[ -\Delta \phi = u^2 \quad \text{in } \Omega. \]

Thus, we have
\[ T_\Omega(u_m - u, w) = \int_\Omega \phi_u(u_m - u) \to 0. \]

This concludes the proof of Step 2.

Step 3. Assume now that $u_m = u$ for all $m \in \mathbb{N}$. We claim
\[ T_\Omega(u, v_m, w_m, z) \to T_\Omega(u, v, w, z). \]

Note that
\[ T_\Omega(u, v_m, w_m, z) = T_\Omega(u, v_m - v, w_m, z) + T_\Omega(u, v, w_m, z). \]

We have $T_\Omega(u, v, w_m, z) \to T_\Omega(u, v, w, z)$ by step 1. Similar to Step 2, we have
\[ T_\Omega(u, v_m - v, w_m, z)^2 \leq T_\Omega(u, w_m)T_\Omega(v_m - v, z). \]

Observing that $T_\Omega(u, w_m)$ is bounded and $T_\Omega(v_m - v, z) \to 0$ by Step 2, we find that $T_\Omega(u, v_m - v, w_m, z) \to 0$. This concludes the proof of Step 3.
Step 4. Finally we consider the general case and prove

\[ T_\Omega(u_m, v_m, w_m, z) \to T_\Omega(u, v, w, z). \]

Note that

\[ T_\Omega(u_m, v_m, w_m, z) = T_\Omega(u_m - u, v_m, w_m, z) + T_\Omega(u, v_m, w_m, z). \]

We have \( T_\Omega(u, v_m, w_m, z) \to T_\Omega(u, v, w, z) \) by step 3. Next, we show

\[ T_\Omega(u_m - u, v_m, w_m, z) \to 0. \]

Using the Hölder inequality to the function \((u_m(x) - u(x))z(y)\) and \(v_m(x)w_m(y)\), we have

\[ T_\Omega(u_m - u, v_m, w_m, z)^2 \leq T_\Omega(u_m - u, z)T_\Omega(v_m, w_m) \to 0, \]

where we used the fact that \( T_\Omega(v_m, w_m) \) is bounded and \( T_\Omega(u_m - u, z) \to 0 \) by Step 1. This concludes the proof.

(iii) By direct calculations, we have

\[ T_\Omega(u_m - u, u_m - u) = T_\Omega(u_m, u_m) + 2T_\Omega(u_m, u) + T_\Omega(u, u) + 4T_\Omega(u_m, u, u_m, u) \]

\[ - 4T_\Omega(u_m, u, u_m, u_m) - 4T_\Omega(u_m, u, u, u). \]

Then the conclusion follows by Lemma 2.2 (ii).

Let

\[ M = \inf \{|u|^2 : u \in H_0^1(\Omega) \text{ and } T_\Omega(u, u) = 1\} \]  (2.1)

and

\[ M_\infty = \inf \{|u|^2_{L^\infty} : u \in H^1(\mathbb{R}^N) \text{ and } T(u, u) = 1\}. \]  (2.2)

Now, we recall a uniqueness result to the limit problem associated to (1.1)

\[ \begin{cases} -\Delta u + u = (\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^N} \, dy)u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \]  (2.3)

The functional corresponding to (2.3) is

\[ I_\infty(u) = \frac{1}{2} |u|^2_{L^2} - \frac{1}{4} T_{L^\infty}(u, u, u, u) = \frac{1}{2} |u|^2_{L^2} - \frac{1}{4} T(u, u). \]

**Proposition 1.** The infimum in (2.2) is achieved by some \( u \in H^1(\mathbb{R}^N) \) with \( |u|^2_{L^2} = M_\infty \) and \( T(u, u) = 1 \) corresponds to a positive regular solutions of (2.3). Furthermore, assume \( N = 4, 5 \), then any positive regular solutions \( u(x) \) of (2.3) is unique and \( u(x) = v(|x - x_0|) \) for some point \( x_0 \in \mathbb{R}^N \) and \( v : (0, \infty) \to \mathbb{R} \) a nonnegative nonincreasing function and

\[ \lim_{|x| \to \infty} u(x)|x|^{N-1}e^{-|x|} = \gamma > 0. \]  (2.4)

**Proof.** By proposition 2.2 in [18], we see that the infimum in (2.2) is achieved. The regularity, positivity and radial symmetry of solution to (2.3) are also obtained in [18]. For the decay property (2.4), see Proposition 6.5 and Remark 6.1 in [18]. The uniqueness of solutions of (2.3) is proved in [27] when \( N = 3, 4, 5 \).

**Theorem 2.3.** Problem (1.1) has no ground state solution.
Proof. We show that $M = M_{\infty}$ and $M$ is not achieved. First we observe that, since any $u \in H^1_0(\Omega)$ can be extended by zero outside $\Omega$, we may consider $H^1_0(\Omega)$ as a subspace of $H^1(\mathbb{R}^N)$, and so $M \geq M_{\infty}$.

Let $w \in H^1(\mathbb{R}^N)$ be a minimizer of (2.2) spherically symmetric about the origin and a sequence $\{y_n\} \subseteq \Omega$, $|y_n| \to \infty$. Fix $\zeta \in C^\infty(\mathbb{R}^N; [0,1])$ defined by

$$
\zeta(x) = \eta\left(\frac{|x|}{\rho}\right),
$$

where $\rho$ is the smallest positive number such that

$$(\mathbb{R}^N \setminus \Omega) \subseteq B_\rho = \{x \in \mathbb{R}^N : |x| < \rho\}
$$

and $\eta : [0, \infty) \to [0,1]$ being a $C^\infty$ non decreasing function such that $\eta(t) = 0$, $t \leq 1$, $\eta(t) = 1$, $t \geq 2$.

Consider the sequence defined by

$$
\phi_n(x) = c_n \zeta(x) w(x - y_n),
$$

where $c_n = T(\zeta(\cdot) w(\cdot - y_n), \zeta(\cdot) w(\cdot - y_n))^{1/2}$ is a normalization constant. It is easy to see $\phi_n(x) \in H^1_0(\Omega)$. We claim that

$$
\|\phi_n\|^2 \to M_{\infty}.
$$

We first show that

$$
T(\zeta(\cdot) w(\cdot - y_n), \zeta(\cdot) w(\cdot - y_n)) - T(w(\cdot - y_n), w(\cdot - y_n)) \to 0. \quad (2.5)
$$

If (2.5) is proved, then we have $c_n \to 1$.

Indeed, after the change of variables $\tilde{x} = x - y_n$ and $\tilde{y} = y - y_n$, we have

$$
T(\zeta(\cdot) w(\cdot - y_n), \zeta(\cdot) w(\cdot - y_n)) - T(w(\cdot - y_n), w(\cdot - y_n)) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi_n(x, y) dx dy
$$

where

$$
\Phi_n(x, y) = \frac{[\zeta^2(x + y_n) \zeta^2(y + y_n) - 1] w^2(x) w^2(y)}{|x - y|^{N-2}}.
$$

Recalling that $|y_n| \to \infty$, we have

$$
\Phi_n(x, y) \to 0 \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N.
$$

On the other hand,

$$
|\Phi_n(x, y)| \leq C \frac{|w(x)|^2 |w(y)|^2}{|x - y|^{N-2}} \in L^1(\mathbb{R}^N \times \mathbb{R}^N).
$$

Thus, the Lebesgue’s theorem immediately yields (2.5).

On the other hand, by (2.4),

$$
\|\zeta(\cdot) w(\cdot - y_n) - w(\cdot - y_n)\|_{L^2}^2 = \|\zeta(\cdot) w(\cdot - y_n) - w(\cdot - y_n)\|_{H^1_0(B_{2\rho})}^2
$$

$$
\leq C \int_{B_{2\rho}} |\nabla w(x - y_n)|^2 dx + C' \int_{B_{2\rho}} |w(x - y_n)|^2 dx
$$

$$
\leq C' \int_{B_{2\rho}} \left(\frac{1}{e|x-y_n| |x-y_n|^{N-1}}\right) dx = o\left(\frac{1}{|y_n|}\right).
$$

Hence,

$$
\|\zeta(\cdot) w(\cdot - y_n)\|^2_{L^2} \to \|w(\cdot - y_n)\|^2_{L^2} = M_{\infty}.
$$

This fact and $c_n \to 1$ yields

$$
\|\phi_n\|^2 = \|\phi_n\|^2_{L^2} \to M_{\infty} \text{ and } T_{\Omega}(\phi_n, \phi_n) = 1.
$$
Therefore, by the definition of $M$, we have $M \leq M_\infty$. Thus $M = M_\infty$.

Next, we prove that $M$ is not achieved. Suppose on the contrary, $M$ is achieved by some $u \in H^1_0(\Omega)$. Let us define

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in (\mathbb{R}^N \setminus \Omega), \end{cases}$$

then $\tilde{u}$ is a minimizer of (2.2) and so, a solution of problem

$$\begin{cases} -\Delta u + u = M_\infty (\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^{N-2}}dy)u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

(2.6)

It is easy to see that $v = M_\infty^{\frac{1}{2}} \tilde{u}$ solves equation (2.3). By Proposition 5.1 in [18], we known that $v > 0$ and so $\tilde{u} > 0$ in $\mathbb{R}^N$. This is a contradiction. The proof is complete.

3. A global compactness result. In this section we establish a compactness result involving the energy functional associated to problem (1.1) and given by

$$I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{4} T_\Omega(u,u,u,u) = \frac{1}{2} \|u\|^2 - \frac{1}{4} T_\Omega(u,u).$$

Proposition 2. Let $\{u_n\} \subset H^1_0(\Omega)$ be a sequence such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty.$$ 

Then, up a subsequence, there exist a weak solution $u^0 \in H^1_0(\Omega)$ of (1.1), a number $k \in \mathbb{N}$, a sequence $\{y^j_n\} \subset \mathbb{R}^N$ and $k$ functions $\{u^j_n\} \subset H^1(\mathbb{R}^N)$, $1 \leq j \leq k$ such that

$$|y^j_n| \to \infty \quad \text{for } 1 \leq j \leq k,$$

$$u^0_n \to u^0 \quad \text{in } H^1_0(\Omega),$$

$$u^j_n \to u^j \quad \text{in } H^1(\mathbb{R}^N),$$

where $u^j$ are nontrivial weak solutions of (2.3), for every $1 \leq j \leq k$. Furthermore,

$$\|u_n\|^2 \to \|u^0\|^2 + \sum_{j=1}^k \|u^j\|^2_{\mathbb{R}^N}$$

and

$$I(u_n) \to I(u^0) + \sum_{j=1}^k I(u^j).$$

Proof. We divide the proof into several steps.

Step 1. The sequence $\{u_n\}$ is bounded in $H^1_0(\Omega)$.

Indeed,

$$c_1 + \frac{c_2}{4} \geq I(u_n) - \frac{1}{4} \langle I'(u_n), u_n \rangle = \frac{1}{4} \|u_n\|^2.$$ 

The above inequality gives the boundedness of the sequence $\{u_n\}$ in $H^1_0(\Omega)$. Hence, up to a subsequence, there exists $u^0 \in H^1_0(\Omega)$ such that

$$u_n \rightharpoonup u^0 \quad \text{in } H^1_0(\Omega) \quad \text{and} \quad u_n \to u^0 \quad \text{a.e. in } \Omega.$$ (3.1)

Therefore, by Lemma 2.2 (ii), we have

$$T_\Omega(u_n, u_n) \to T_\Omega(u^0, u^0).$$ (3.2)
By standard argument, it is easy to see $u^0$ solves problem (1.1). Now, let $\psi_n^1$ be the function given by
\[
\psi_n^1(x) = \begin{cases} (u_n - u^0)(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}
\]
By (3.1), it follows that $\psi_n^1 \to 0$ in $H^1(\mathbb{R}^N)$ and $T(\psi_n^1, \psi_n^1) \to 0$. With these notations we are able to prove the following facts:

Step 1.
\[
I_{\infty}(\psi_n^1) = I(\psi_n^1) + o_n(1) = I(u_n) - I(u^0) + o_n(1). \tag{3.3}
\]

By (3.2) and Lemma 2.2 (iii), we have
\[
\|\psi_n^1\|_{\mathbb{R}^N}^2 = \|\psi_n^1\|^2 - \|u^0\|^2 + o_n(1), \tag{3.4}
\]
\[
T(\psi_n^1, \psi_n^1) = T(0, u_n - u^0, u_n - u^0) = T(u_n, u_n) - T(u^0, u^0) + o_n(1). \tag{3.5}
\]
Combining (3.4) and (3.5), we get
\[
I_{\infty}^\prime(\psi_n^1) = I(\psi_n^1) = I'(u_n) - I'(u^0) + o_n(1). \tag{3.6}
\]
From the definition of $\psi_n^1$, it is easy to see that
\[
I_{\infty}^\prime(\psi_n^1) = I(\psi_n^1).
\]

Now, we are going to show
\[
I'(\psi_n^1) = I'(u_n) - I'(u^0) + o_n(1) = o_n(1).
\]
Indeed,
\[
(I'(\psi_n^1) - I'(u_n) + I'(u^0), v) = T(\psi_n^1, v, \psi_n^1) - T(u_n, v, u_n, u_n) + T(u^0, v, u^0, u^0) = T(\psi_n^1, v, u_n - u^0, u_n - u^0) - T(u_n, v, u_n, u_n) + T(u^0, v, u^0, u^0).
\]
By Lemma 2.2 (iii), we get our conclusion.

If $\psi_n^1 \to 0$ in $H^1(\mathbb{R}^N)$, the statement of the main result are verified. Thus, we can suppose that $\psi_n^1 \to 0$ in $H^1(\mathbb{R}^N)$.

By the fact that
\[
I_{\infty}(\psi_n^1) = \frac{1}{2} \|\psi_n^1\|_{\mathbb{R}^N}^2 - \frac{1}{4} T(\psi_n^1, \psi_n^1)
\]
and $I_{\infty}^\prime(\psi_n^1) = o_n(1)$, we have
\[
(I_{\infty}^\prime(\psi_n^1), \psi_n^1) = \|\psi_n^1\|_{\mathbb{R}^N}^2 - T(\psi_n^1, \psi_n^1) = o_n(1).
\]
Then
\[
I_{\infty}(\psi_n^1) = \frac{1}{2} \|\psi_n^1\|_{\mathbb{R}^N}^2 - \frac{1}{4} \|\psi_n^1\|_{\mathbb{R}^N}^2 + o_n(1) = \frac{1}{4} \|\psi_n^1\|_{\mathbb{R}^N}^2 + o_n(1).
\]
Since $\psi_n^1 \to 0$ in $H^1(\mathbb{R}^N)$, there is $\alpha > 0$ such that
\[
I_{\infty}(\psi_n^1) \geq \alpha > 0.
\]

Now, let us decompose $\mathbb{R}^N$ into $N$-dimensional unit hypercubes $Q_i$ whose vertices have integer coordinates and put
\[
d_n = \sup_i \left( \int_{Q_i} \int_{\mathbb{R}^N} \frac{(\psi_n^1)^2(x)(\psi_n^1)^2(y)}{|x-y|^{N-2}} dy dx \right)^{\frac{1}{2}}.
\]
We claim that $d_n \geq \gamma > 0$ for some $\gamma$. 

Indeed, since $I'_{\infty}(\psi_{n}^{1}) = o_n(1)$,

$$4I_{\infty}(\psi_{n}^{1}) + o_n(1) = T(\psi_{n}^{1}, \psi_{n}^{1}) = \sum_{i} \int_{Q_i} \int_{\mathbb{R}^{N}} \frac{(\psi_{n}^{1})^{2}(x)(\psi_{n}^{1})^{2}(y)}{|x - y|^{N-2}} dy dx$$

$$\leq d_n^{2} \sum_{i} \left( \int_{Q_i} \int_{\mathbb{R}^{N}} \frac{(\psi_{n}^{1})^{2}(x)(\psi_{n}^{1})^{2}(y)}{|x - y|^{N-2}} dy dx \right)^{\frac{1}{2}}$$

$$\leq C d_n^{2} \sum_{i} \|\psi_{n}^{1}\|_{H^1(Q_i)}^{2} = C d_n^{2} \|\psi_{n}^{1}\|_{H^1(\mathbb{R}^{N})}^{2}.$$  

By the fact that $I_{\infty}(\psi_{n}^{1}) \geq \alpha > 0$, the claim is proved.

Now, let us denote by $\{y_{n}^{1}\}$ the center of a hypercube $Q_{i}$ such that

$$\left( \int_{Q_i} \int_{\mathbb{R}^{N}} \frac{(\psi_{n}^{1})^{2}(x)(\psi_{n}^{1})^{2}(y)}{|x - y|^{N-2}} dy dx \right)^{\frac{1}{2}} \geq d_n - \frac{1}{n}.$$  

We show that $\{y_{n}^{1}\}$ is unbounded in $\mathbb{R}^{N}$. Suppose on the contraction that $\{y_{n}^{1}\}$ is bounded in $\mathbb{R}^{N}$. Then there is $R > 0$ such that $Q_{i} \subset B(0, R)$. Hence, we have

$$\left( \int_{B(0, R)} \int_{\mathbb{R}^{N}} \frac{(\psi_{n}^{1})^{2}(x)(\psi_{n}^{1})^{2}(y)}{|x - y|^{N-2}} dy dx \right)^{\frac{1}{2}} \geq \left( \int_{Q_i} \int_{\mathbb{R}^{N}} \frac{(\psi_{n}^{1})^{2}(x)(\psi_{n}^{1})^{2}(y)}{|x - y|^{N-2}} dy dx \right)^{\frac{1}{2}} \geq d_n - \frac{1}{n} \geq \gamma + o(1). \quad (3.7)$$

Since $\psi_{n}^{1} \to 0$ in $H^1(\mathbb{R}^{N})$, the left hand side in $(3.7)$ converges to 0. We get a contradiction. Therefore, the sequence $\{y_{n}^{1}\}$ is unbounded in $\mathbb{R}^{N}$.

Since $\|\psi_{n}^{1}(\cdot + y_{n}^{1})\|_{\mathbb{R}^{N}} = \|\psi_{n}^{1}\|_{\mathbb{R}^{N}}$, we can assume that

$$\psi_{n}^{1}(\cdot + y_{n}^{1}) \rightharpoonup u^{1} \text{ in } H^1(\mathbb{R}^{N}).$$

Step 4. $u^{1}$ is a nontrivial weak solutions of $(2.3)$.

First, by $(3.7)$, we derive that $u^{1} \neq 0$. By straightforward computation

$$\langle I'_{\infty}(\psi_{n}^{1}(\cdot + y_{n}^{1}), v) = o_n(1), \forall v \in C_{0}^{\infty}(\mathbb{R}^{N}).$$

Then, taking the limit above, we find

$$\int_{\mathbb{R}^{N}} (\nabla u^{1} \cdot \nabla v + u^{1} \cdot v) dx = T(u^{1}, v, u^{1}), \forall v \in C_{0}^{\infty}(\mathbb{R}^{N}).$$

Then the conclusion follows by the density of $C_{0}^{\infty}(\mathbb{R}^{N})$ in $H^1(\mathbb{R}^{N})$.

Iterating this process, we obtain sequences

$$\psi_{n}^{j}(x) = \psi_{n}^{j-1}(x + y_{n}^{j-1}) - u^{j-1}(x), \quad j \geq 2,$$

with $|y_{n}^{j}| \to \infty$ and

$$\psi_{n}^{j-1}(x + y_{n}^{j-1}) \rightharpoonup u^{j-1} \text{ in } H^1(\mathbb{R}^{N}),$$

where each $u^{j}$ is a nontrivial weak solutions of $(2.3)$. Moreover, we have the following equalities

$$\|\psi_{n}^{j}\|_{\mathbb{R}^{N}}^{2} = \|\psi_{n}^{j-1}\|_{\mathbb{R}^{N}}^{2} - \|u^{j-1}\|_{\mathbb{R}^{N}}^{2} + o_n(1) = \|u_{n}\|_{\mathbb{R}^{N}}^{2} - \|u_{0}\|_{\mathbb{R}^{N}}^{2} - \sum_{i=1}^{j-1} \|u_{i}\|_{\mathbb{R}^{N}}^{2} + o_n(1) \quad (3.8)$$

and

$$I_{\infty}(\psi_{n}^{j}) = I_{\infty}(\psi_{n}^{j-1}) - I_{\infty}(u^{j-1}) + o_n(1) = I(u_{n}) - I(u_{0}) - \sum_{i=1}^{j-1} I(u_{i}) + o_n(1). \quad (3.9)$$
Since $u^j$ is a nontrivial solution of (2.3), by the definition of $M_\infty$, we can get
\[ \|u^j\|_{\mathbb{R}^N}^2 \geq M_\infty^2. \]

From the above inequality, (3.8) and (3.9), we deduce that above argument will stop at some index $k$. □

**Corollary 1.** Let $\{u_n\}$ be as in Proposition 2 and let $c < \frac{1}{4}M_\infty^2$, then $\{u_n\}$ admits a strongly convergent subsequence.

**Proof.** It is easy to see that $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Then, up to a subsequence, $u_n \rightharpoonup u_0$ in $H^1_0(\Omega)$. Suppose on the contrary that $u_n \not\rightharpoonup u_0$ in $H^1_0(\Omega)$. By Proposition 2, there exists $k \geq 1$ such that
\[ I(u_n) = I(u^0) + \sum_{j=1}^{k} I_\infty(u^j) + o_n(1). \]

In the proof of Proposition 2, we saw that
\[ \|u^j\|_{\mathbb{R}^N}^2 \geq M_\infty^2. \]
Since $u^j$ solves (2.3), we have
\[ I_\infty(u^j) = \frac{1}{4}\|u^j\|_{\mathbb{R}^N}^2 \geq \frac{1}{4}M_\infty^2. \]
Moreover, since $u^0$ solves (1.1) we have $I(u^0) \geq 0$. Hence,
\[ c = I(u^0) + \sum_{j=1}^{k} I_\infty(u^j) \geq \frac{1}{4}M_\infty^2 \]
which is a contradiction to the assumption. □

**Corollary 2.** Let $\mathcal{P}$ the set of nonnegative functions in $H^1_0(\Omega)$. Assume that there is $\{u_n\} \subseteq \mathcal{P}$ that satisfies the assumptions of Proposition 2. If
\[ \frac{1}{4}M_\infty^2 < c < \frac{1}{4}(M_\infty^2 + M^2) = \frac{1}{2}M_\infty^2, \]
then $\{u_n\}$ admits a strongly convergent subsequence.

**Proof.** It is easy to see that $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Then, up to a subsequence, $u_n \rightharpoonup u_0$ in $H^1_0(\Omega)$. Suppose on the contrary that $u_n \not\rightharpoonup u_0$ in $H^1_0(\Omega)$. Arguing as in the proof of Corollary 1, there exists $k \geq 1$ such that
\[ I(u_n) = I(u^0) + \sum_{j=1}^{k} I_\infty(u^j) + o_n(1) \geq \frac{k}{4}M_\infty^2 + o_n(1). \]
Hence, for $k \geq 2$, we have
\[ c = \lim_{n \to \infty} I(u_n) \geq \frac{1}{2}M_\infty^2 \]
which is a contradiction to the assumption. Therefore, $k$ can not be greater than 1. If $u^0 = 0$, we have that $u^1$ is the unique positive ground state solution of (2.3), thus we deduce that
\[ I_\infty(u^1) = \frac{1}{4}M_\infty^2. \]
Hence,
\[ I(u_n) = I(u^0) + I_\infty(u^1) + o_n(1) = \frac{1}{4}M_\infty^2 + o_n(1), \]
which contradicts the assumption. Therefore \( u^0 \neq 0 \) and
\[
I(u^0) \geq \frac{1}{4} M^2 \quad \text{and} \quad I_\infty(u^1) \geq \frac{1}{4} M^2_\infty.
\]
Consequently,
\[
I(u_n) \geq \frac{1}{4} M^2 + \frac{1}{4} M^2_\infty + o_n(1) = \frac{1}{2} M^2_\infty + o_n(1),
\]
which also contradicts the assumption. Thus, we can not have \( k = 1 \). Therefore we conclude that \( u_n \to u_0 \) in \( H^1_0(\Omega) \).

In the sequel, let us consider the set
\[
\mathcal{V} := \{ u \in H^1_0(\Omega) : T_\Omega(u, u) = 1 \} \quad (3.10)
\]
and the functional \( J : H^1_0(\Omega) \to \mathbb{R} \) defined by \( J(u) = \|u\|^2 \). Moreover, we consider the norm
\[
\|J'(u)\|_* = \sup_{w \in T_u \mathcal{V}, \|w\| \leq 1} |\langle J'(u), w \rangle|
\]
where
\[
T_u \mathcal{V} = \{ v \in H^1_0(\Omega) : G'(u)v = 0 \}
\]
and \( G : H^1_0(\Omega) \to \mathbb{R} \) be the functional given by
\[
G(u) = T_\Omega(u, u).
\]

**Corollary 3.** The functional \( J \) satisfies the Palais-Smale condition in
\[
\mathcal{Z} := \mathcal{P} \cap \mathcal{V} \cap \{ u \in H^1_0(\Omega) : M < J(u) < \sqrt{2} M \}.
\]

**Proof.** Let \( \{ u_n \} \subset \mathcal{Z} \) be a sequence satisfying
\[
J(u_n) \to c \in (M, \sqrt{2} M) \quad \text{and} \quad \|J'(u_n)\|_* \to 0.
\]

Setting \( v_n = \frac{c}{d^2} u_n \) and \( d = \frac{1}{4} c^2 \in \left( \frac{1}{4} M^2, \frac{1}{2} M^2 \right) \), we have
\[
I(v_n) = \frac{c}{2} \|u_n\|^2 - \frac{1}{4} c^2 \to d.
\]

Now, we claim that \( I'(v_n) \to 0 \). Indeed, by Proposition 5.12 in [28],
\[
\|J'(u_n)\|_* = \min_{\mu \in \mathbb{R}} \|J'(u_n) - \mu T_\Omega'(u_n, u_n)\| = \|J'(u_n) - \mu_n T_\Omega'(u_n, u_n)\|
\]
for some \( \mu_n \in \mathbb{R} \). Hence, we have
\[
J'(u_n) - \mu_n T_\Omega'(u_n, u_n) = o_n(1) \text{ in } (H^1_0(\Omega))^*.
\]

Therefore, by a straightforward computation,
\[
I'(v_n) = o_n(1) \text{ in } (H^1_0(\Omega))^*.
\]

Using the above fact, we can apply Corollary 2 to deduce that \( \{ v_n \} \) admits a convergent subsequence. Consequently, the functional \( J \) satisfies the (PS) condition in \( \mathcal{Z} \).
4. Some estimates. Let $\Phi_\rho$ be the operator

$$\Phi_\rho : \mathbb{R}^N \to H^1(\mathbb{R}^N)$$

$y \mapsto \phi_{y,\rho}(x) = \frac{v_{y,\rho}(x)}{T(v_{y,\rho}, v_{y,\rho})}$

where

$$v_{y,\rho}(x) = \zeta(x)w(x-y) = \eta\left(\frac{|x|}{\rho}\right)w(x-y),$$

and $\zeta, \eta, w$ are given as in the proof of Theorem 2.3. Note that $v_{y,\rho} \in H^1_0(\Omega)$ and $\||v_{y,\rho}|| = ||v_{y,\rho}||_{\mathbb{R}^N}, T_{\Omega}(v_{y,\rho}, v_{y,\rho}) = T(v_{y,\rho}, v_{y,\rho})$

and

$$\||\phi_{y,\rho}|| = ||\phi_{y,\rho}||_{\mathbb{R}^N}, T_{\Omega}(\phi_{y,\rho}, \phi_{y,\rho}) = T(\phi_{y,\rho}, \phi_{y,\rho}).$$

Lemma 4.1. (i) $\Phi_\rho(y)$ is continuous in $y$ for every $\rho$;
(ii) $\phi_{y,\rho} \to w(\cdot - y)$ in $H^1(\mathbb{R}^N)$ as $\rho \to 0$, uniformly in $y$;
(iii) $||\phi_{y,\rho}||_{\mathbb{R}^N} \to M_\infty$ as $|y| \to +\infty$, for every $\rho$.

Proof. (i) Since $\Phi_\rho(\cdot)$ is the composition of continuous functions, the conclusion is obvious.

(ii) It suffices to prove

$$T(v_{y,\rho}, v_{y,\rho}) \to 1, \text{ uniformly in } y \text{ as } \rho \to 0$$

and

$$||v_{y,\rho}(x)||^2_{\mathbb{R}^N} \to ||w(x)||^2_{\mathbb{R}^N}, \text{ uniformly in } y \text{ as } \rho \to 0.$$

Taking into account that $\eta\left(\frac{|x|}{\rho}\right) = 1$ for every $|x| \geq 2\rho$, since $\eta$ is bounded and $w$ is radially symmetric and non-increasing, we have

\[
T(v_{y,\rho}, v_{y,\rho}) - T(w(\cdot - y), w(\cdot - y)) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_{y,\rho}(x)v_{y,\rho}(z))^2 - (w(x-y)w(z-y))^2}{|x-z|^{N-2}}\,dz\,dx
\]

\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[\eta\left(\frac{|x|}{\rho}\right)]^2[\eta\left(\frac{|z|}{\rho}\right)]^2 - 1][w(x-y)]^2[w(z-y)]^2}{|x-z|^{N-2}}\,dz\,dx
\]

\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[\eta\left(\frac{|x+y|}{\rho}\right)]^2[\eta\left(\frac{|z+y|}{\rho}\right)]^2 - 1][w(x)]^2[w(z)]^2}{|x-z|^{N-2}}\,dz\,dx
\]

\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi_\rho(x, z)\,dz\,dx.
\]

Since

$$\Phi_\rho(x, z) \leq C\frac{|w(x-y)|^2|w(z-y)|^2}{|x-z|^{N-2}} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$$

and

$$\Phi_\rho(x, z) \to 0 \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N \text{ as } \rho \to 0,$$

the Lebesgue’s theorem immediately yields

$$T(v_{y,\rho}, v_{y,\rho}) \to T(w(\cdot - y), w(\cdot - y)) = 1, \text{ uniformly in } y \text{ as } \rho \to 0.$$

On the other hand, by (2.4),

$$\||v_{y,\rho}(x) - w(x-y)||^2_{\mathbb{R}^N}$$
Proof. It can be derived by Lemma 4.1 (ii).

Let $B > R$ for some $R$.

Corollary 4. There is $\bar{\rho} > 0$ such that $\forall \rho \leq \bar{\rho}$

$$\sup_{y \in \mathbb{R}^N} \|\phi_{y,\rho}\|_{\mathbb{R}^N}^2 < \sqrt{2}M.$$

Proof. It can be derived by Lemma 4.1 (ii).

Hereafter, let us fix $\rho < \bar{\rho}$, where $\rho$ is the smallest positive number such that $\mathbb{R}^N \setminus \Omega \subset B(0, \rho)$. Now we define the barycenter function

$$\beta : H^1(\mathbb{R}^N) \to \mathbb{R}^N$$

$$u \mapsto \beta(u) = \int_{\mathbb{R}^N} |u(x)|^2 \chi(|x|) \cdot x dx,$$

where $\chi \in C^\infty(\mathbb{R}^+, \mathbb{R})$ is a non-increasing real function such that

$$\chi(t) = \begin{cases} 1 & t \in (0, R] \\ R/t & t > R \end{cases}$$

for some $R > 0$ such that $\mathbb{R}^N \setminus \Omega \subset B(0, R)$. By definition of $\chi$, we see

$$\chi(|x|) |x| \leq R, \quad \forall x \in \mathbb{R}^N.$$

Let $B_0$ be the subset of $H^1_0(\Omega)$

$$B_0 = \{ u \in \mathcal{P} \cap \mathcal{V} : \beta(u) = 0 \},$$

where $\mathcal{V}$ is defined by (3.10).

Lemma 4.2. Let $c_0 = \inf_{u \in B_0} \|u\|^2$, then $c_0 > M$ and there is $R_0 > \rho$ such that

(i) if $y \in \mathbb{R}^N$ with $|y| \geq R_0$, then

$$\|\Phi_{\rho}(y)\|^2 \in (M, \frac{c_0 + M}{2});$$

(ii) if $y \in \mathbb{R}^N$ with $|y| = R_0$, then

$$(\beta(\Phi_{\rho}(y)), y) > 0.$$
Proof. Since $B_0 \subset \mathcal{V}$ and $M = \inf_{u \in \mathcal{V}} \|u\|^2$, we have $c_0 \geq M$. Now we are going to show that $c_0 \neq M$. Suppose by contradiction that $c_0 = M$. Then, there exists a minimizing sequence $\{u_n\} \subset H^1_0(\Omega)$ such that

$$\|v_n\|^2 \to M, \quad T_{\Omega}(v_n, v_n) = 1 \quad \text{and} \quad \beta(v_n) = 0.$$ 

By the Ekeland variational principle, we can suppose that

$$\|J'(v_n)\|_* \to 0.$$ 

Let $u_n = M^{\frac{1}{2}}v_n$, it is easy to see

$$I(u_n) \to \frac{1}{4}M^2 \quad \text{and} \quad I'(u_n) \to 0.$$ 

Moreover, by Proposition 2, one has

$$\|u_n\|^2 \to \|u^0\|^2 + \sum_{j=1}^{k} \|u^j\|_{\mathcal{A}^N}^2$$

and

$$I(u_n) \to I(u^0) + \sum_{j=1}^{k} I_{\infty}(u^j).$$

Therefore,

$$I(u_n) \to I(u^0) + \sum_{j=1}^{k} I_{\infty}(u^j) \geq I(u^0) + k \frac{1}{4}M^2.$$ 

Since $I(u^0) \geq 0$, then $k \leq 1$. If $k = 0$, we obtain $\|u_n\|^2 \to \|u^0\|^2$, which means

$$u_n \to u^0 \quad \text{and} \quad \|u^0\|^2 = M^2.$$ 

This is impossible, because $M$ is not achieved in $\mathcal{V}$, and so $k = 1$. We must have $u^0 = 0$. Consequently, by the fact that $u^1$ is a nontrivial weak solution of (2.3), we have

$$\|u_n\|^2 \to \|u^1\|_{\mathcal{A}^N}^2 \quad \text{and} \quad I(u^1) = \frac{1}{4}M^2.$$ 

Since $u_n \to u^0 = 0$, we get

$$\psi^1_n(x + y^1_n) = u_n(x + y^1_n) \to u^1(x)$$

and

$$\|\psi^1_n(\cdot + y^1_n)\|_{\mathcal{A}^N} = \|u_n(\cdot + y^1_n)\|_{\mathcal{A}^N} = \|u_n\|_{\mathcal{A}^N}^2 = \|u_n\|^2 \to \|u^1\|^2,$$

where $\{y^1_n\}$ be a sequence such that $|y^1_n| \to \infty$. Therefore

$$u_n(\cdot + y^1_n) \to u^1 \in H^1_0(\Omega).$$

Setting

$$u = u^1, \quad y_n = y^1_n, \quad \text{and} \quad w_n(x + y^1_n) = u_n(x + y^1_n) - u^1(x),$$

we have

$$w_n(x) = u_n(x) - u(x - y_n) \quad \text{and} \quad \|w_n\|_{\mathcal{A}^N}^2 = \|w_n(\cdot + y_n)\|_{\mathcal{A}^N}^2 = \|u_n(\cdot + y_n) - u\|_{\mathcal{A}^N}^2.$$ 

Therefore, the strong convergent of $u_n(\cdot + y_n)$ yields $w_n \to 0$ in $H^1(\mathbb{R}^N)$.

Next, we consider the following sets

$$(\mathbb{R}^N)^+_n = \{x \in \mathbb{R}^N : (x, y_n) > 0\} \quad \text{and} \quad (\mathbb{R}^N)^-_n = \mathbb{R}^N \setminus (\mathbb{R}^N)^+_n.$$ 

Using the fact that $|y_n| \to \infty$, we claim that there is a ball

$$B(y_n, \tilde{r}) = \{x \in \mathbb{R}^N : |x - y_n| < \tilde{r}\} \subset (\mathbb{R}^N)^+_n$$
such that, for \( n \) large enough,
\[
    u(x - y_n) \geq \frac{1}{2} u(0) > 0, \quad \forall x \in B(y_n, \tilde{r}). \tag{4.1}
\]
Indeed, by Proposition 1, we can assume that \( u(0) = \max_{x \in \mathbb{R}^N} u(x) \). As \( u \) be a positive radial decreasing function, then by [3]
\[
    u(z) \leq C_N \frac{|u|_{L^2(\mathbb{R}^N)}}{|z|^{N/2}},
\]
which implies that
\[
    u(z) \to 0, \text{ as } |z| \to \infty.
\]
Then by the Intermediate value theorem, there exists \( \tilde{r} > 0 \) such that
\[
    u(z) = \frac{1}{2} u(0) > 0, \quad \forall z \in \mathbb{R}^N \text{ with } |z| = \tilde{r}. \tag{4.2}
\]
Substituting \( z = x - y_n \) into (4.2), we get (4.1). On the other hand, for each \( \tilde{r} > 0 \) fixed, there is \( n_0 \) such that
\[
    (x, y_n) > \frac{|x|^2 + |y_n|^2 - \tilde{r}^2}{2} \geq \frac{|y_n|^2 - \tilde{r}^2}{2} > 0, \quad \forall n \geq n_0, \quad \forall x \in B(y_n, \tilde{r}).
\]
The above fact implies that
\[
    B(y_n, \tilde{r}) \subset (\mathbb{R}^N)_n^+, \text{ for } n \text{ large enough.}
\]
Thus, for \( n \) large enough,
\[
    |u(x - y_n)|^2 \chi(|x|), (x, y_n) > 0, \quad \forall x \in (\mathbb{R}^N)_n^+
\]
and \( |x| > R \) for every \( x \in B(y_n, \tilde{r}) \). By a straightforward computation, we have
\[
    \int_{(\mathbb{R}^N)_n^+} |u(x - y_n)|^2 \chi(|x|)(x, y_n) dx \geq \frac{R |u(0)|^2}{4} |B(y_n, \tilde{r})| \cdot |y_n|. \tag{4.3}
\]
Observing that for each \( x \in (\mathbb{R}^N)_n^+ \),
\[
    |x - y_n| \geq |x|,
\]
we have
\[
    |u(x - y_n)|^2 \chi(|x|)|x| \leq R |u(x)|^2 \in L^1(\mathbb{R}^N).
\]
Combining with the fact that \( u(x - y_n) \to 0 \) as \( |y_n| \to \infty \), we get
\[
    \int_{(\mathbb{R}^N)_n^+} |u(x - y_n)|^2 \chi(|x|)|x| dx = o_n(1). \tag{4.4}
\]
Therefore, by (4.3), (4.4) and the Cauchy-Schwartz inequality,
\[
    \left( \beta(u(x - y_n)), \frac{y_n}{|y_n|} \right)_{\mathbb{R}^N}
\]
\[
    = \int_{(\mathbb{R}^N)_n^+} |u(x - y_n)|^2 \chi(|x|)(x, \frac{y_n}{|y_n|}) dx + \int_{(\mathbb{R}^N)_n^+} |u(x - y_n)|^2 \chi(|x|)(x, \frac{y_n}{|y_n|}) dx
\]
\[
    \geq \frac{R |u(0)|^2}{4} |B(y_n, \tilde{r})| - o_n(1) > 0. \tag{4.5}
\]
Now, using the fact that \( w_n \to 0 \) in \( H^1(\mathbb{R}^N) \) and \( \beta(u_n) = 0 \), we find that
\[
    \beta(u(x - y_n)) = o_n(1),
\]
which contradicts (4.5), and so, \( c_0 > M \).
(i) Since $\Phi_p(y) = \phi_{y,\rho} \in H^1_0(\Omega)$ and $T_\Omega(\phi_{y,\rho}, \phi_{y,\rho}) = 1$, by Theorem 2.3, we have
\[
\|\phi_{y,\rho}\|_{L^2}^2 = M, \quad \forall y \in \mathbb{R}^N.
\]
By Lemma 4.1 (iii), for each fixed $\rho$, $\|\phi_{y,\rho}\|_{L^2}^2 \to M_\infty = M$ as $|y| \to +\infty$. Therefore, for a given $\epsilon \in (0, \frac{c_0 - M}{2})$, there is $R_0 > 0$ such that
\[
\left|\|\phi_{y,\rho}\|_{L^2}^2 - M\right| < \epsilon \text{ whenever } |y| \geq R_0.
\]
From the above inequality, we get the desired conclusion.

(ii) By definition of $\Phi_p(y)$ and arguing as above with $|y|$ large enough, we have
\[
\left(\beta(\Phi_p(y)), y\right)_{\mathbb{R}^N} \geq C - o(1) > 0.
\]

5. Existence of high energy solution. We define a set $\Sigma \subset \mathcal{P} \subset H^1_0(\Omega)$ as follows
\[
\Sigma := \{\Phi_p(y) : |y| \leq R_0\}.
\]
Let
\[
\mathcal{H} = \{h \in C(\mathcal{P} \cap \mathcal{V}, \mathcal{P} \cap \mathcal{V}) : h(u) = u, \forall u \in \mathcal{P} \cap \mathcal{V} \text{ with } \|u\|^2 < \frac{c_0 + M}{2}\}
\]
and
\[
\Gamma = \{A \subset \mathcal{P} \cap \mathcal{V} : A = h(\Sigma), h \in \mathcal{H}\}.
\]

Lemma 5.1. If $A \in \Gamma$, then $A \cap \mathcal{B}_0 \neq \emptyset$.

Proof. We are going to show that, for every $A \in \Gamma$, there exists $u \in A$ such that $\beta(u) = 0$. Equivalently, it suffices to prove that for every $h \in \mathcal{H}$, there exists $\tilde{y} \in \mathbb{R}^N$ with $|\tilde{y}| \leq R_0$ such that
\[
(\beta \circ h \circ \Phi_p)(y) = 0.
\]
For any $h \in \mathcal{H}$, we define
\[
\mathcal{J} = \beta \circ h \circ \Phi_p : \mathbb{R}^N \to \mathbb{R}^N
\]
and $\mathcal{F} : [0, 1] \times \bar{B}_{R_0}(0) \to \mathbb{R}^N$ given by
\[
\mathcal{F}(t, y) = t\mathcal{J}(y) + (1 - t)y.
\]
We claim that $0 \notin \mathcal{F}(t, \partial B(0, R_0))$. Indeed, for $|y| = R_0$, by Lemma 4.2 (i), we have
\[
\|\Phi_p(y)\|^2 < \frac{c_0 + M}{2}.
\]
Hence, it follows that
\[
\mathcal{F}(t, y) = t(\beta \circ \Phi_p)(y) + (1 - t)y,
\]
and
\[(F(t,y),y) = t(\beta(\Phi_\rho(y)),y) + (1-t)(y,y)\].

Now, if \(t = 0\), then \((F(t,y),y)) = |y|^2 = R_0^2 > 0\). If \(t = 1\), then by Lemma 4.2 (ii), we have \(F(1,y) = t(\beta(\Phi_\rho(y)),y) > 0\). If \(t \in (0,1)\), then \((F(t,y),y) > 0\), since the terms \(t, 1 - t, (\beta(\Phi_\rho(y)),y)\) and \(|y|^2\) are all positive.

Then, by using the invariance under homotopy of the Brouwer degree, one has
\[d(F(t,\cdot), B(0,R_0), 0) = \text{constant}, \forall t \in [0,1].\]

Since \(d(J,B(0,R_0),0) \neq 1\), there exists \(\tilde{y} \in B(0,R_0)\) such that \(J(\tilde{y}) = 0\), that is \(J(\tilde{y}) = (\beta \circ h \circ \Phi_\rho)(\tilde{y}) = 0\).

This completes the proof.

Now, let us denote
\[c := \inf_{A \in \Gamma} \sup_{u \in A} \|u\|^2,\] (5.1)
\[K_c = \{u \in P \cap V : J(u) = \|u\|^2 = c \text{ and } \nabla J|_V(u) = 0\},\]
and
\[L_\gamma = \{u \in V : J(u) \leq \gamma\},\]
for every \(\gamma \in \mathbb{R}\).

**Proof of Theorem 1.1.** We choose \(\rho = \tilde{\rho}\) as given in Corollary 4. We will show that \(c\) given by (5.1) is a critical value, that is \(K_c \neq \emptyset\). First, we claim that
\[M < c < \sqrt{2}M.\]

In fact, by Lemma 5.1, for each \(A \in \Gamma\), there is \(\hat{u} \in A \cap \mathcal{B}_0\). Then
\[c_0 = \inf_{u \in \mathcal{B}_0} \|u\|^2 \leq \inf_{u \in A \cap \mathcal{B}_0} \|u\|^2 \leq \|\hat{u}\|^2 \leq \sup_{u \in A \cap \mathcal{B}_0} \|u\|^2 \leq \sup_{u \in A} \|u\|^2.\]
Since \(M < c_0\) by Lemma 4.2, we obtain
\[M < c_0 \leq \sup_{u \in A} \|u\|^2, \forall A \in \Gamma.\]
Thus
\[M < c_0 \leq \inf_{A \in \Gamma} \sup_{u \in A} \|u\|^2 = c.\] (5.2)

By the definition of \(c\),
\[c \leq \sup_{u \in A} \|u\|^2, \forall A \in \Gamma,\]
it follows that
\[c \leq \sup_{\Phi_\rho(y) \in \Sigma} \|h(\Phi_\rho(y))\|^2, \forall h \in \mathcal{H}.\]

Now taking \(h \equiv I\), we find
\[c \leq \sup_{\Phi_\rho(y) \in \Sigma} \|\Phi_\rho(y)\|^2.\]
Hence,
\[c \leq \sup_{|y| \leq R_0} \|\Phi_\rho(y)\|^2 \leq \sup_{y \in \mathbb{R}^N} \|\Phi_\rho(y)\|^2.\]
By Corollary 4, we have
\[c \leq \sup_{y \in \mathbb{R}^N} \|\Phi_\rho(y)\|^2 < \sqrt{2}M.\] (5.3)
Combining (5.2) and (5.3), one has
\[ M < c < \sqrt{2}M. \]
This proves the claim. Suppose on the contrary \( K_c = \emptyset \). Since
\[ \frac{c_0 + M}{2} \leq \frac{c + M}{2} < c < \sqrt{2}M, \]
by the deformation Lemma [28], there exists a continuous map
\[ \eta : [0, 1] \times V \cap P \to V \cap P \]
and a positive number \( \epsilon_0 \) such that
(a) \( L_{c+\epsilon_0} \setminus L_{c-\epsilon_0} \subset \subset L_{\sqrt{2}M} \setminus L_{c+\epsilon_0}, \)
(b) \( \eta(t, u) = u, \forall u \in L_{c-\epsilon_0} \cup \{ V \cap P \setminus L_{c+\epsilon_0} \} \) and \( \forall t \in [0, 1], \)
(c) \( \eta(1, L_{c+\frac{\epsilon_0}{2}}) \subset L_{c-\frac{\epsilon_0}{2}}. \)
Fix \( \tilde{A} \in \Gamma \) such that
\[ c \leq \sup_{u \in \tilde{A}} J(u) < c + \frac{\epsilon_0}{2}. \]
Since
\[ J(u) < c + \frac{\epsilon_0}{2}, \forall u \in \tilde{A}, \]
it follows that
\[ \tilde{A} \subset L_{c+\frac{\epsilon_0}{2}}. \]
Now, by the item (c) above, one has
\[ J(u) < c - \frac{\epsilon_0}{2}, \forall u \in \eta(1, \tilde{A}), \]
that is
\[ \sup_{u \in \eta(1, \tilde{A})} J(u) < c - \frac{\epsilon_0}{2}. \tag{5.4} \]
On the other hand, we notice that \( \eta(1, \cdot) \in C(V \cap P, V \cap P) \). Moreover, since \( \tilde{A} \in \Gamma \),
there exists \( h \in \mathcal{H} \) such that \( \tilde{A} = h(\Sigma) \). Consequently,
\[ \tilde{h} = \eta(1, \cdot) \circ h \in C(V \cap P, V \cap P). \]
By the definition of \( \mathcal{H} \),
\[ h(u) = u, \forall u \in P \cap V \text{ with } \|u\|^2 < \frac{c_0 + M}{2}, \]
and
\[ \tilde{h}(u) = \eta(1, u), \forall u \in P \cap V \text{ with } \|u\|^2 < \frac{c_0 + M}{2}. \]
Taking into account that
\[ \frac{c_0 + M}{2} < c - c_0, \]
by item (b), we have
\[ \tilde{h}(u) = \eta(1, u) = u, \forall u \in P \cap V \text{ with } \|u\|^2 < \frac{c_0 + M}{2} < c - c_0. \]
This means \( \tilde{h} \in \mathcal{H} \). Moreover \( \eta(1, \tilde{A}) \in \Gamma \) since \( \eta(1, \tilde{A}) = \tilde{h}(\Sigma) \). Therefore, by the definition of \( c \), we have
\[ c \leq \sup_{u \in \eta(1, \tilde{A})} J(u), \]
which contradicts (5.4). Consequently, \( K_c \neq \emptyset \) and \( c \) is a critical value of functional \( J \) on \( P \cap V \), namely there is at least one positive solution of (1.1). \( \square \)
REFERENCES

[1] C. O. Alves, A. B. Nóbrega and M. Yang, Multi-bump solutions for Choquard equation with deepening potential well, *Calc. Var. Partial Differ. Equ.*, **55** (2016), 48.

[2] V. Benci and G. Cerami, Positive solutions of some nonlinear elliptic problems in exterior domains, *Arch. Rational Mech. Anal.*, **99** (1987), 283–300.

[3] H. Berestycki and P. L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.*, **82** (1983), 313–345.

[4] L. Battaglia and J. Van Schaftingen, Groundstates of the Choquard equations with a signchanging self-interaction potential, *Z. Angew. Math. Phys.*, **69** (2018), 16pp.

[5] M. Ghimenti and J. Van Schaftingen, Nodal solutions for the Choquard equation, *J. Funct. Anal.*, **271** (2016), 107–135.

[6] M. Ghimenti and D. Pagliardini, Multiple positive solutions for a slightly subcritical Choquard problem on bounded domains, *Calc. Var. Partial Differ. Equ.*, **58** (2019).

[7] D. Goel and K. Sreenadh, Coron problem for nonlocal equations involving Choquard nonlinearity, *Adv. Nonlinear Stud.*, **20** (2020), 141–161.

[8] D. Goel and K. Sreenadh, Critical growth elliptic problems involving Hardy-Littlewood-Sobolev critical exponent in non-contractible domains, *Adv. Nonlinear Anal.*, **9** (2020), 803–835.

[9] F. Gao and M. Yang, The Brezis-Nirenberg type critical problem for the nonlinear Choquard equation, *Sci. China Math.*, **61** (2018), 1219–1242.

[10] F. Gao, E D. da Silva, M. Yang and J. Zhou, Existence of solutions for critical Choquard equations via the concentration-compactness method, *Proc. Roy. Soc. Edinburgh Sect. A*, **150** (2020), 921–954.

[11] L. Guo, T. Hu, S. Peng and W. Shuai, Existence and uniqueness of solutions for Choquard equation involving Hardy-Littlewood-Sobolev critical exponent, *Calc. Var. Partial Differ. Equ.*, **58** (2019), 34 pp.

[12] E.H. Lieb and M. Loss, *Analysis*, American Mathematical Society, Providence, RI, second ed., 2001.

[13] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, *Stud. Appl. Math.*, **57** (1976/1977), 93–105.

[14] P. L. Lions, The Choquard equation and related questions, *Nonlinear Anal.*, **4** (1980), 1063–1072.

[15] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 109–145.

[16] X. Liu, S. Ma and X. Zhang, Infinitely many bound state solutions of Choquard equations with potentials, *Z. Angew. Math. Phys.*, **69** (2018), 118.

[17] V. Moroz and J. Van Schaftingen, A guide to the Choquard equation, *J. Fixed Point Theory Appl.*, **19** (2017), 773–813.

[18] V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, **265** (2013), 153–184.

[19] V. Moroz and J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, *Trans. Amer. Math. Soc.*, **367** (2015), 6557–6579.

[20] V. Moroz and J. Van Schaftingen, Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains, *J. Differ. Equ.*, **254** (2013), 3089–3145.

[21] V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent, *Commun. Contemp. Math.*, **17** (2015), 1550005, 12pp.

[22] L. Ma and Z. Lin, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Ration. Mech. Anal.*, **195** (2010), 455–467.

[23] I.M. Moroz, R. Penrose and P. Tod, Spherically-symmetric solutions of the Schrödinger-Newton equations, *Classical Quantum Gravity*, **15** (1998), 2733–2742.

[24] S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin, 1954.

[25] J. Van Schaftingen and J. Xia, Choquard equations under confining external potentials, *Nonlinear Differ. Equ. Appl.*, **24** (2017), 24pp.

[26] J. Wei and M. Winter, Strongly interacting bumps for the Schrödinger-Newton equations, *J. Math. Phys.*, **50** (2009), 012905, 22 pp.

[27] T. Wang and T. Yi, Uniqueness of positive solutions of the Choquard type equations, *Appl. Anal.*, **96** (2017), 409–417.
[28] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[29] C. Xiang, Uniqueness and nondegeneracy of ground states for Choquard equations in three dimensions, *Calc. Var. Partial Differ. Equ.*, 55 (2016), 25pp.

[30] J. Xia and Z. Wang, Saddle solutions for the Choquard equation, *Calc. Var. Partial Differ. Equ.*, 58 (2019), 30pp.

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