THE OVERDETERMINEDNESS OF A CLASS OF FUNCTIONAL EQUATIONS

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Abstract. We prove a uniqueness theorem for a large class of functional equations in the plane, which resembles in form a classical result of Aczél. It is also shown that functional equations in this class are overdetermined in the sense of Paneah. This means that the solutions, if they exist, are determined by the corresponding relation being fulfilled not in the original domain of validity, but only at the points of a subset of the boundary of the domain of validity.

1. Introduction

Recall the classical Cauchy functional equation:
\[ f(x + y) = f(x) + f(y). \]

To solve the functional equation usually means: given a set \( A \subseteq \mathbb{R}^2 \) and a class of functions \( \mathcal{A} \), to find the family of functions \( \mathcal{F} \subseteq \mathcal{A} \) which consists of all \( f \) such that \( f(x + y) = f(x) + f(y) \) for all \((x, y) \in A\). Following Kuczma ([3]) let us call \( A \) the domain of validity. For example, when Cauchy first treated (1), he took \( \mathcal{A} = C(\mathbb{R}) \), and showed that if the domain of validity is taken to be \( \mathbb{R}^2 \) then the set of solutions to (1) is \( \mathcal{F} = \{ f : \exists \lambda. \forall z. f(z) = \lambda z \} \). It has been shown in various works (see [5], [6], [2], [4] and the references therein) that when some additional smoothness assumptions are imposed on \( f \) then even if the domain of validity is quite small - the graph of an appropriate function, for example - the set of solutions does not grow. Thus, using the terminology of Paneah ([4]), we may say that the equation
\[ f(x + y) = f(x) + f(y), \quad (x, y) \in \mathbb{R}^2 \]
is overdetermined (for the class of functions satisfying these additional smoothness assumptions). In fact, in [7] we proved that the Cauchy

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equation in \( \mathbb{R}^n \)
\[
f(x_1 + y_1, \ldots, x_n + y_n) = f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n)
\]
is overdetermined for the class \( C^1(\mathbb{R}^n, \mathbb{R}) \).

One is led to the following questions: (a) given a class of functions \( \mathcal{A} \), what is the “smallest” domain of validity for which the solutions to (1) are only \( f(z) = \lambda z \), and (b) given a domain of validity, for what \( \mathcal{A} \) does the set of solutions to (1) remain \( f(z) = \lambda z \)?

The above questions may be asked with regards to any functional equation, and it is interesting in general to study how, given a functional equation, the set of solutions changes when the domain of validity and the class of functions considered are changed. For most classical functional equations in 2 variables, the domain of validity is usually taken to be some large, open set in \( \mathbb{R}^2 \). In [4] Paneah proved for a sample of classical functional equations that, under some smoothness assumptions, their solution is already determined by the functional equation holding on a much smaller domain of validity, e.g., a one-dimensional sub-manifold in \( \mathbb{R}^2 \), and such equations were called overdetermined. In this paper we consider a class of functional equations that turn out to be overdetermined (section 3). This class of equations contains some well known equations such as Jensen’s equation and the equation of the logarithmic mean. Our main theorem resembles in form (and in fact, was inspired by) a classical result of Aczél. Our proof depends on a minimality result in topological dynamics to which the next section is devoted.

2. SOME PRELIMINARIES IN TOPOLOGICAL DYNAMICS

A dynamical system is a pair \((X, \delta)\), where \(X\) is a metric space and \(\delta = (\delta_1, \ldots, \delta_N)\) is a set of continuous maps \(\delta_i : X \to X\). The maps in \(\delta\) generate (by composition) a semigroup of maps \(\Phi_\delta\) in the following manner:
\[
\Phi_\delta^0 = \{\text{id}_X\}, \quad \Phi_\delta^m = \{\sigma_1 \circ \cdots \circ \sigma_m | \sigma_1, \ldots, \sigma_m \in \delta\},
\]
and
\[
\Phi_\delta = \bigcup_{m=0}^{\infty} \Phi_\delta^m.
\]

**Definition 1.** For any \( x \in X \), the orbit of \( x \) is the set
\[
O(x) = \{\sigma(x) | \sigma \in \Phi_\delta\}.
\]
Definition 2. A dynamical system \((X, \delta)\) is said to be minimal if for all \(x \in X\)
\[O(x) = X.\]

Lemma 3. Let \((X, d)\) be a compact, metric space, and let \(\delta = (\delta_1, \delta_2, \ldots, \delta_N)\) be a finite family of functions \(X \to X\) satisfying
\[\delta_1(X) \cup \delta_2(X) \cup \ldots \cup \delta_N(X) = X.\]
If \(\delta\) has the property that for all \(i = 1, \ldots, N\) and all \(x, y \in X\)
\[x \neq y \Rightarrow d(\delta_i(x), \delta_i(y)) < d(x, y)\]
then the dynamical system \((X, \delta)\) is minimal.

Proof. Fix \(x_0 \in X\). We must show that for any \(y\) in \(X\) and \(\epsilon > 0\), there is a \(z \in O(x_0)\) such that \(d(z, y) \leq \epsilon\). Fix some \(y \in X\) and \(\epsilon > 0\). The set \(S \equiv \{(x_1, x_2) \in X \times X \mid d(x_1, x_2) \geq \epsilon\}\) is a compact subset of \(X \times X\), thus for every \(i = 1, 2, \ldots, N\), the continuous function \(g_i: S \to \mathbb{R}\) defined by:
\[g_i(x_1, x_2) = \frac{d(\delta_i(x_1), \delta_i(x_2))}{d(x_1, x_2)}, \quad (x_1, x_2) \in S\]
attains a maximum \(c_{\epsilon,i}\). By \(\text{(3)}\), \(c_{\epsilon,i} < 1\), for all \(i\). Set \(c_\epsilon\) to be the maximum of these constants.

Now choose some \(n\) satisfying \(c_\epsilon^n \cdot \text{diam}(X) < \epsilon\). Then for all \(\sigma \in \Phi_\delta^n\) and all \(x_1, x_2 \in X\)
\[d(\sigma(x_1), \sigma(x_2)) \leq \epsilon\]
and thus for all \(\sigma \in \Phi_\delta^n:\)
\[(4) \quad \text{diam}(\sigma(X)) \leq \epsilon.\]

But note that by virtue of \(\text{(2)}\),
\[\bigcup_{\sigma \in \Phi_\delta^n} \sigma(X) = X\]
so that there is an \(f \in \Phi_\delta^n\) s.t. \(y \in f(X)\). Now by \(\text{(4)}\) it follows that for all \(x\) it is true that \(d(f(x), y) \leq \epsilon\) so we can choose \(z = f(x_0)\) and the proof is complete. \(\square\)

3. The main result

In 1964 Aczél (see [1]) proved the following uniqueness theorem for a rather wide class of functional equations:
Theorem 4. Let $f_1, f_2 : I \to \mathbb{R}$ be continuous solutions of the equation

\[(5) \quad f(F(x, y)) = H[f(x), f(y), x, y], \quad (x, y) \in I^2\]

where $I$ is an (open, closed, half-open, finite or infinite) interval. Suppose that $F : I^2 \to I$ is continuous and internal that is,

\[\min(x, y) < F(x, y) < \max(x, y) \text{ if } x \neq y\]

and that either $u \mapsto H(u, v, x, y)$ or $v \mapsto H(u, v, x, y)$ are injections. Further, let $a, b \in I$ and

\[f_1(a) = f_2(a) \quad \text{and} \quad f_1(b) = f_2(b).\]

Then

\[\forall x \in I. f_1(x) = f_2(x).\]

This theorem motivated much work on uniqueness theorems and has been improved several times. Theorems in the same spirit were proved for different classes of $F$ and $H$ and for more general spaces ($\mathbb{R}^2, \mathbb{R}^n$, topological vector spaces, etc.). In this section we will prove a refinement of the above theorem which serves at once both as a uniqueness theorem for (5) and as a proof that all of the equations that belong to the class treated below are overdetermined.

Theorem 5. Let $I = [a, b]$, $H : \mathbb{R} \times \mathbb{R} \times I \times I \to \mathbb{R}$ any function and $F : I^2 \to I$ a continuous function that satisfies

1: $\forall x \neq y. |F(x, b) - F(y, b)|, |F(a, x) - F(a, y)| < |x - y|

2: $\exists x_0, y_0. F(a, x_0) = a$ and $F(y_0, b) = b$

For any real $A$ and $B$ there exists at most one solution $f$ to (5) that satisfies the boundary conditions

\[(6) \quad f(a) = A, \quad f(b) = B.\]

Moreover, if a function $f$ is a solution to (5) satisfying (6), then it is already determined by the functional equation

\[(7) \quad f(F(x, y)) = H[f(x), f(y), x, y], \quad (x, y) \in \Gamma\]

where $\Gamma = ([a, b] \times \{b\}) \cup (\{a\} \times [a, b])$.

\[\text{[2]} \] contains references to these developments.
Proof. Let us define two maps $\delta_1, \delta_2 : I \to I$ by the formulas

$$\delta_1(x) = F(a, x),$$

$$\delta_2(x) = F(x, b).$$

We consider the dynamical system $(I, \delta_1, \delta_2)$. By the definitions of $\delta_1, \delta_2$ and by the conditions on $F$ we have that

$$\delta_1(b) = \delta_2(a),$$

and that

$$\delta_1(x_0) = a \quad \text{and} \quad \delta_2(y_0) = b,$$

thus,

$$\delta_1(I) \cup \delta_2(I) = I.$$

In addition,

$$\forall x \neq y, |\delta_2(x) - \delta_2(y)|, |\delta_1(x) - \delta_1(y)| < |x - y|.$$

By lemma 3 it follows that the orbit of any point in $I$ is dense in $I$.

Now let $f_1$ and $f_2$ be continuous and satisfy (6) and (7). We shall show that for any $z$ in the orbit of $a$

$$f_1(z) = f_2(z).$$

For $a$ we already have by (6) that

$$f_1(a) = A = f_2(a).$$

If $z$ is a point for which we know that $f_1(z) = f_2(z)$ then

$$f_1(\delta_1(z)) = f_1(F(a, z)) = H[f_1(a), f_1(z), a, z]$$

by (7). But by our assumption on $z$ we can replace $H[f_1(a), f_1(z), a, z]$ by $H[f_2(a), f_2(z), a, z]$ and obtain

$$f_1(\delta_1(z)) = H[f_2(a), f_2(z), a, z] = f_2(\delta_1(z))$$

where the last equality follows again from (7). So we have

$$f_1(\delta_1(z)) = f_2(\delta_1(z)).$$

Arguing in just the same manner we arrive at the relation

$$f_1(\delta_2(z)) = f_2(\delta_2(z)).$$

So all the points in the orbit of $a$ inherit from $a$ the property of being given the same values by $f_1, f_2$, and so indeed for any $z \in O(a)$ we have $f_1(z) = f_2(z)$. The continuity of $f_1, f_2$ and the density of $O(a)$ imply $f_1 = f_2$ on $I$. □

Remark 6. Note that the above proof suggests an algorithm that can compute numerically a solution (when such exists) to a given functional equation on an interval with boundary data.
Remark 7. Note that it follows from the above theorem that usually (5) will not have a solution, even if (7) has a solution.

As a corollary of the above theorem we have the overdeterminedness of Jensen’s functional equation.

Corollary 8. Let $\alpha$ and $\beta$ be two positive numbers satisfying $\alpha + \beta = 1$, and let $I = [a, b]$ be some closed interval. Then all continuous solutions $f$ of the functional equation
\[
f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \quad (x, y) \in I^2
\]
are of the form
\[
f(z) = \lambda z + \mu
\]
for some constants $\lambda, \mu \in \mathbb{R}$. Moreover, these solutions are already determined by the functional equation
\[
f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \quad (x, y) \in \Gamma
\]
where $\Gamma = ([a, b] \times \{b\}) \cup (\{a\} \times [a, b])$.

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