Study on the nonlinear vibration of embedded carbon nanotube via the Hamiltonian-based method

Kang-Jia Wang and Guo-Dong Wang

Abstract
This article mainly studies the vibration of the carbon nanotubes embedded in elastic medium. A new novel method called the Hamiltonian-based method is applied to determine the frequency property of the nonlinear vibration. Finally, the effectiveness and reliability of the proposed method is verified through the numerical results. The obtained results in this work are expected to be helpful for the study of the nonlinear vibration.

Keywords
Nonlinear vibration, Hamiltonian-based method, variational principle, carbon nanotube, residual equation

Introduction
The carbon nanotube (CNT) has attracted wide attention since it is first discovered by Japanese scientist Iijima in 1991. The vibration of the CNT is usually used to directly or indirectly measure the elastic modulus or other mechanical behaviors of the CNTs. Therefore, the study of the vibration characteristics of the CNT is important. In this study, we will investigate the vibration of CNT embedded in the elastic medium, which can be governed as

\[
d\frac{d^2 \mu}{dt^2} + \left( \frac{\pi^4 E I}{P^2 \rho A} + \frac{k}{A} \right) \mu + \frac{\pi^4 E}{4P^2 \rho^3} = 0
\]

where \( l \) is the length of the CNT, \( \rho \) represents the density, \( E \) indicates Young’s modulus, and \( A \) and \( I \) mean the cross-sectional area and cross-sectional inertia moment, respectively. There are many methods available for obtaining the frequency property of equation (1.1) such as the incremental harmonic balanced method, variational iteration method, Fourier series and Stokes’ transformation, homotopy perturbation method, and He’s frequency formulation. In this study, we will use a new method called the Hamiltonian-based method to determine the frequency property. The overall structure of this study is arranged as follows. The variational principle of the problem is constructed and its Hamiltonian is obtained in

The variational principle and the Hamiltonian

The variational principle of equation (1.1) can be easily established via the semi-inverse method which reads as...
\[
J(\mu) = \int \left\{ \frac{1}{2} \left( \frac{\partial \mu}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\pi^4 E I}{16 l^4 \rho} + \frac{k}{\rho A} \right) \mu^2 + \frac{\pi^4 E}{16 l^4 \rho} \mu^4 \right\} dt.
\]

(2.1)

Where He–Weierstrass function can be obtained as\(^23\)

\[
E(t, \Xi, \Xi', \zeta) = \frac{1}{2} \zeta^2 - \frac{1}{2} \left( \frac{\pi^4 E I}{16 l^4 \rho} + \frac{k}{\rho A} \right) \mu^2 + \frac{\pi^4 E}{16 l^4 \rho} \mu^4
\]

\[
- \left\{ \frac{1}{2} \left( \frac{d\mu}{dt} \right)^2 - \frac{1}{2} \left( \frac{\pi^4 E I}{16 l^4 \rho} + \frac{k}{\rho A} \right) \mu^2 + \frac{\pi^4 E}{16 l^4 \rho} \mu^4 \right\} - \left( \zeta - \frac{d\mu}{dt} \right) \frac{d\mu}{dt}
\]

\[
= \frac{1}{2} \zeta^2 - \frac{1}{2} \left( \frac{d\mu}{dt} \right)^2 - \left( \zeta - \frac{d\mu}{dt} \right) \frac{d\mu}{dt}
\]

(2.2)

where the variable \(\zeta\) that defined as

\[
\zeta = \frac{d\mu}{dt}
\]

(2.3)

From equation (2.2), it can be proved that

\[
E(t, \mu, \mu', \zeta) = 0 \quad \text{and} \quad \frac{\partial^2 E}{\partial \zeta^2} > 0
\]

(2.4)

which shows that equation (2.1) is a minimal variational principle.

In equation (2.1), \(K = 1/2(\partial \mu / \partial t)^2\) is the kinetic energy, and \(P = 1/2(\pi^4 E I / l^4 \rho A + k / \rho A) \mu^2 + \pi^4 E / 16 l^4 \rho \mu^4\) represents the potential energy. It is well known that the total energy is conserved in the whole process of the vibration. So the Hamiltonian can be obtained as\(^24\)

\[
H = K + P = \frac{1}{2} \left( \frac{\partial \mu}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\pi^4 E I}{16 l^4 \rho} + \frac{k}{\rho A} \right) \mu^2 + \frac{\pi^4 E \mu^4}{16 l^4 \rho} = \text{constant} = H_0
\]

(2.5)

Equation (2.5) can be simplified as

\[
\frac{1}{2} \left( \frac{d\mu}{dt} \right)^2 + a \mu^2 + b \mu^4 = H_0
\]

(2.6)

where \(a = 1/2(\pi^4 E I / l^4 \rho A + k / \rho A), \beta = \pi^4 E / 16 l^4 \rho\).

**Hamiltonian-based method**

In this section, we will apply the Hamiltonian-based method to obtain the solution of equation (1.1). As is known to all, the kinetic energy and the potential energy are changed during the oscillation process, but the total energy will keep unchanged for a conservative oscillator. The Hamiltonian-based method is based on this and the variational theory, so it can present a more accurate solution compared with the VIM or HPM. Here the solution of equation (1.1) is assumed with the following form

\[
\mu(t) = M \cos(\omega t)
\]

(3.1)

With the initial conditions of equation (3.1), we can determine the Hamiltonian constant as

\[
H_0 = \alpha M^2 + \beta M^4
\]

(3.2)

Substituting equation (3.2) into equation (2.6), there is

\[
\frac{1}{2} \left( \frac{d\mu}{dt} \right)^2 + a \mu^2 + b \mu^4 - \alpha M^2 - \beta M^4 = 0
\]

(3.3)
Taking equation (3.1) into equation (3.3) leads to the residual equation, which is
\[ R = \frac{1}{2} M^2 \sin^2(\omega t) + \alpha M^2 \cos^2(\omega t) + \beta M^4 \cos^4(\omega t) - \alpha M^2 - \beta M^4 \]  
\tag{3.4}

Now we define two average residuals
\[ \tilde{R}_1 = \frac{4}{T_1} \int_0^{T_1/4} R_1 \cos(\omega_1 t) dt \]  
\tag{3.5}

\[ \tilde{R}_2 = \frac{4}{T_2} \int_0^{T_2/4} R_2 \cos(\omega_2 t) dt \]  
\tag{3.6}

So the frequency–amplitude formulation can be obtained as follows
\[ \omega^2 = \frac{\tilde{R}_2^2 - \tilde{R}_1^2}{\tilde{R}_1 - \tilde{R}_2} \omega_1 \omega_2 \]  
\tag{3.7}

Here we select two arbitrary frequencies as \( \omega_1 = 1 \) and \( \omega_2 = 2 \), and then we get the two residual equations as follows
\[ R_1 = \frac{1}{2} M^2 \sin^2(t) + \alpha M^2 \cos^2(t) + \beta M^4 \cos^4(t) - \alpha M^2 - \beta M^4 \]  
\tag{3.8}

\[ R_2 = 2M^2 \sin^2(2t) + \alpha M^2 \cos^2(2t) + \beta M^4 \cos^4(2t) - \alpha M^2 - \beta M^4 \]  
\tag{3.9}

So the two average residuals are obtained as
\[ \tilde{R}_1 = \frac{4}{T_1} \int_0^{T_1/4} R_1 \cos t dt = \frac{-M^2 (10\alpha + 14M^2 \beta - 5)}{15\pi} \]  
\tag{3.10}

\[ \tilde{R}_2 = \frac{4}{T_2} \int_0^{T_2/4} R_2 \cos(2t) dt = \frac{-2M^2 (5\alpha + 7M^2 \beta - 10)}{15\pi} \]  
\tag{3.11}

According to equation (3.7), there is
\[ \omega = \sqrt{\frac{\tilde{R}_2^2 - \tilde{R}_1^2}{\tilde{R}_1 - \tilde{R}_2}} = \sqrt{2\alpha + \frac{14}{5} \beta M^2} \]  
\tag{3.12}

With this, we get the solution of equation (1.1) as
\[ \mu(t) = M \cos \left( \sqrt{2\alpha + \frac{14}{5} \beta M^2} t \right) \]  
\tag{3.13}

That is
\[ \mu(t) = M \cos \left( \frac{\pi^4 EI}{I^2 \rho A} + \frac{k}{\rho A} + \frac{7M^2 \pi^4 E}{40I^2 \rho} t \right) \]  
\tag{3.14}

Results and discussion

Recently, He’s frequency formulation, which is first proposed by Chinese mathematician Ji-Huan He, has been widely used to solve the nonlinear vibrations arising in three-dimensional printing technology, micro-electromechanical, N/MEMS, and so on. By using He’s frequency formulation, we can get the frequency–amplitude formulation of equation (1.1) as
\[ \omega^2 = \frac{\pi^4 EI}{I^2 \rho A} + \frac{k}{\rho A} + \frac{3M^2 \pi^4 E}{16I^2 \rho} \]  
\tag{4.1}
Then the solution of equation (1.1) can be obtained as

$$\mu(t) = M \cos \left( \frac{\pi EI}{l^2 \rho A} + \frac{k}{\rho A} + \frac{3M^2 \pi^4 E}{16l^2 \rho} t \right)$$

which has a good agreement with equation (3.14).

For $E = 1, \rho = 1, l = \pi, M = 1, I = 1, A = 1$, and $k = 1$, we plot a comparison between the Hamiltonian-based method and He’s frequency formulation in Figure 1, which shows a good match between the two methods.

When we select $E = 1, \rho = 1, l = \pi, M = \frac{1}{2}, I = 1, A = 1$, and $k = 1$, the comparison of the two methods is plotted in Figure 2, which also has a good agreement.
Conclusion and future recommendation

In this study, a novel and effective method called the Hamiltonian-based method is used to study the nonlinear vibration of the CNT embedded in elastic medium. The comparison between our proposed method and He’s frequency formulation shows a good agreement, which strongly proves the correctness of the Hamiltonian-based method. The finding in this study is expected to open up new horizons for the study of the nonlinear vibration.

Recently, the fractal and fractional calculus have been widely used to model many complex problems arising in circuit, physics, filter, biomedical science, and so on. The proposed method in this work is also extended to the fractal cases.

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