Euler characteristics and actions of automorphism groups of free groups

Shengkui Ye
Xi’an Jiaotong-Liverpool University

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Abstract

Let $M^r$ be a connected orientable manifold with the Euler characteristic $\chi(M) \not\equiv 0 \mod 6$. Denote by $\text{SAut}(F_n)$ the unique subgroup of index two in the automorphism group of a free group. Then any group action of $\text{SAut}(F_n)$ (and thus the special linear group $\text{SL}_n(\mathbb{Z})$) $(n \geq r + 2)$ on $M^r$ by homeomorphisms is trivial. This confirms a conjecture related to Zimmer’s program for these manifolds.

1 Introduction

Let $\text{SL}_n(\mathbb{Z})$ be the special linear group over integers. There is an action of $\text{SL}_n(\mathbb{Z})$ on the sphere $S^{n-1}$ induced by the linear action on the Euclidean space $\mathbb{R}^n$. It is believed that this action is minimal in the following sense.

Conjecture 1.1 Any action of $\text{SL}_n(\mathbb{Z})$ $(n \geq 3)$ on a compact connected $r$-manifold by homeomorphisms factors through a finite group action if $r < n - 1$.

The smooth version of this conjecture was formulated by Farb and Shalen [8], and is related to the Zimmer program concerning group actions of lattices in Lie groups on manifolds (see the survey articles [9] [19] for more details). When $r = 1$, Conjecture 1.1 is already proved by Witte [15]. Weinberger [16] confirms the conjecture when $M = T^r$ is a torus. Bridson and Vogtmann [5] confirm the conjecture when $M = S^r$ is a sphere. Ye [18] confirms the conjecture for all flat manifolds. For $C^{1+\beta}$ group actions of finite-index subgroup in $\text{SL}_n(\mathbb{Z})$, one of the results proved by Brown, Rodriguez-Hertz and Wang [7] confirms Conjecture 1.1 for surfaces. For $C^2$ group actions of
cocompact lattices, Brown-Fisher-Hurtado [6] confirms Conjecture 1.1. Note that the $C^0$ actions could be very different from smooth actions. It seems that very few other cases have been confirmed (for group actions preserving extra structures, many results have been obtained, cf. [9, 19]).

Let $\text{SAut}(F_n)$ denote the unique subgroup of index two in the automorphism group $\text{Aut}(F_n)$ of the free group $F_n$. Note that there is a surjection $\phi : \text{SAut}(F_n) \to \text{SL}_n(\mathbb{Z})$ given by the abelianization of $F_n$. In this note, we obtain the following general result on topological actions.

**Theorem 1.2** Let $M^r$ be a connected (resp. orientable) manifold with the Euler characteristic $\chi(M) \not\equiv 0 \mod 3$ (resp. $\chi(M) \not\equiv 0 \mod 6$). Then any group action of $\text{SAut}(F_n)$ ($n > r + 1$) on $M^r$ by homeomorphisms is trivial.

Since any group action of $\text{SL}_n(\mathbb{Z})$ could be lifted to an action of $\text{SAut}(F_n)$, Theorem 1.2 confirms Conjecture 1.1 for orientable manifolds with nonvanishing Euler characteristic modulo 6.

**Remark 1.3** (i) The bound of $n$ cannot be improved, since $\text{SAut}(F_n)$ acts through $\text{SL}_n(\mathbb{Z})$ non-trivially on $S^{n-1}$.

(ii) Belolipetsky and Lubotzky [6] prove that for any finite group $G$ and any dimension $r \geq 2$, there exists a hyperbolic manifold $M^r$ such that $\text{Isom}(M) \cong G$. Therefore, $\text{SL}_n(\mathbb{Z})$ and thus $\text{SAut}(F_n)$ could act nontrivially through a finite quotient group on such a hyperbolic manifold. This implies that the condition of Euler characteristic could not be dropped.

(iii) To satisfy the assumption on the Euler characteristic, the dimension $r$ has to be even. There are however no further restrictions on $r$, as the following example shows. Let $\{g_i\}$ be a sequence of nonnegative integers with $g_i \not\equiv 1 \mod 3$ and $\Sigma_{g_i}$ an orientable surface of genus $g_i$. For any even number $r$,

$$M^r = \Sigma_{g_1} \times \Sigma_{g_2} \times \cdots \times \Sigma_{g_r}$$

has nonzero (modulo 6) Euler characteristic and thus satisfies the condition of Theorem 1.2.

Our proof of Theorem 1.2 rely on torsion and so will not be applicable to finite-index subgroups. Actually, Theorem 1.2 does not hold for general finite-index subgroups. For example, let $q < \mathbb{Z}$ be a non-trivial ideal and $C$ a non-trivial cyclic subgroup of $\text{SL}_n(\mathbb{Z}/q)$. Let $f : \text{SAut}(F_n) \to \text{SL}_n(\mathbb{Z}) \to \text{SL}_n(\mathbb{Z}/q)$ be the group homomorphism induced by quotient ring homomorphism. The group $f^{-1}(C)$ could act non-trivially on $S^2$ by rotations.
through \( C \). By a profound result of Grunewald and Lubotzky \cite{11} (Corollary 1.2), there is a group homomorphism \( \rho \) from a finite-index subgroup \( G \) of \( \text{SAut}(F_n) \) \((n \geq 3)\) to \( \text{SL}_{n-1}(\mathbb{Z}) \) such that \( \text{Im} \ f \) is of finite index. Therefore, the group \( G \) could act through \( \text{SL}_{n-1}(\mathbb{Z}) \) on \( S^{n-2} \), which is an infinite-group action.

2 Proofs

The cohomology \( n \)-manifold mod \( p \) (a prime) considered in this article will be as in Borel \cite{2}. Roughly speaking, a cohomology \( n \)-manifold mod \( p \) is a locally compact Hausdorff space which has a local cohomology structure (with coefficient group \( \mathbb{Z}/p \)) resembling that of Euclidean \( n \)-space. Let \( L = \mathbb{Z} \) or \( \mathbb{Z}/p \). All homology groups in this section are Borel-Moore homology groups with compact supports and coefficients in a sheaf \( A \) of modules over \( L \). The homology groups of \( X \) are denoted by \( H_c^*(X; L) \) and the Alexander-Spanier cohomology groups (with coefficients in \( L \) and compact supports) are denoted by \( H^*_c(X; L) \). We define the cohomology dimension \( \dim_L X = \min\{ n \mid H_{c}^{n+1}(U; L) = 0 \text{ for all open } U \subset X \} \). If \( L = \mathbb{Z}/p \), we write \( \dim_p X \) for \( \dim_L X \).

In order to prove Theorem 1.2 we need several lemmas.

**Lemma 2.1** (Borel \cite{2}, Theorem 4.3, p.182) Let \( G \) be an elementary \( p \)-group operating on a first countable cohomology \( n \)-manifold \( X \) mod \( p \). Let \( x \in X \) be a fixed point of \( G \) on \( X \) and let \( n(H) \) be the cohomology dimension mod \( p \) of the component of \( x \) in the fixed point set of a subgroup \( H \) of \( G \). If \( r = n(G) \), we have

\[
 n - r = \sum_H (n(H) - r)
\]

where \( H \) runs through the subgroups of \( G \) of index \( p \).

**Lemma 2.2** (Mann and Su \cite{13}, Theorem 2.2) Let \( G \) be an elementary \( p \)-group of rank \( k \) operating effectively on a first countable connected cohomology \( r \)-manifold \( X \) mod \( p \). Suppose \( \dim_p F(G) = r_0 \geq 0 \) where \( F(G) \)
is the fixed point set of \( G \) on \( X \). Then \( k \leq \frac{r-r_0}{2} \) if \( p \neq 2 \) and \( k \leq r - r_0 \) if \( p = 2 \).

Let \( X \) be an oriented cohomology \( r \)-manifold over \( \mathbb{Z} \) (in the sense of Bredon \cite{1}). A homeomorphism \( f : X \to X \) is orientation-preserving if the orientation is preserved. In the following lemma, we consider the case of orientation-preserving actions.

**Lemma 2.3** Let \( G \) be a non-trivial elementary \( 2 \)-group of rank \( k \) operating effectively on a first countable connected oriented cohomology \( r \)-manifold \( X \) over \( \mathbb{Z} \) by orientation-preserving homeomorphisms. Suppose \( \dim_2 F(G) = r_0 \geq 0 \) where \( F(G) \) is the fixed point set of \( G \) on \( X \). Then \( k \leq r - 1 - r_0 \).

**Proof.** Note that the manifold \( X \) is also a cohomology \( r \)-manifold over \( \mathbb{Z}/2 \) and the fixed point set \( \text{Fix}(g) \) is a cohomology manifold over \( \mathbb{Z}/2 \) by Smith theory (see \cite{2}, Theorem 2.2 and the bottom of p.78). If there is a non-trivial element \( g \in G \) such that the dimension of the fixed point set \( \text{Fix}(g) \) is \( r \), the element \( g \) acts trivially by invariance of domain (see Bredon \cite{3}, Cor. 16.19, p.383). This is a contradiction to the assumption that \( G \) acts effectively. Therefore, we could assume that \( \text{Fix}(g) \) is of non-trivial even codimension by Bredon \cite{2} (Theorem 2.5, p.79. We use the assumption that \( X \) is a cohomology manifold over \( \mathbb{Z} \)). Now the lemma becomes obvious for \( r = 1 \). When \( r = 2 \), the dimension of \( \text{Fix}(H) \) is zero for any nontrivial subgroup \( H < G \). If \( r = 2 \) and \( k = 1 \), the fixed point set \( \text{Fix}(G) \) is of dimension 0 and the statement holds. If \( r = 2 \) and \( k \geq 2 \), this would be impossible by Borel’s formula in Lemma 2.1.

Choose a nontrivial element \( g \in G \) such that the fixed point set \( \text{Fix}(g) \) is of the maximal dimension among all nontrivial elements in \( G \). Fix a connected component \( M \) of \( \text{Fix}(g) \) containing a connected component of \( F(G) \) with the largest dimension. Choose a decomposition \( G = \langle g \rangle \bigoplus G_0 \) for some subgroup \( G_0 < G \). The action of the complement \( G_0 \) leaves \( M \) invariant. If some nontrivial element \( h \in G_0 \) acts trivially on \( M \), let \( H = \langle g, h \rangle \cong (\mathbb{Z}/2)^2 \).

By the assumption that the fixed point set \( \text{Fix}(g) \) is of the maximal dimension, each nontrivial element in \( H \) has its fixed point set of dimension \( \dim_2 \text{Fix}(g) \). This is impossible by Borel’s formula in Lemma 2.1. Therefore, the action of \( G_0 \) on \( M \) is effective. Note that \( \text{Fix}(g) \) is a cohomology manifold over \( \mathbb{Z}/2 \) (by Smith theory) of dimension at most \( r - 2 \). Thus the rank of \( G_0 \) is at most \( r - 2 - r_0 \) by Lemma 2.2. Therefore,

\[
k = \text{rank}(G_0) + 1 \leq r - 1 - r_0.
\]
The inequality in Lemma 2.3 is sharp, by considering the linear action of the diagonal subgroup \((\mathbb{Z}/2)^n \cong \text{SL}_n(\mathbb{Z})\) on \(\mathbb{R}^n\).

Let \(X\) be a locally compact Hausdorff space and a finite group \(G = (\mathbb{Z}/p)^n\) acting on \(X\) by homeomorphisms. In the remaining part of this article, we suppose that the Euler characteristic \(\sum (-1)^i \dim H^i(X; \mathbb{Z}/p) =: \chi(X; \mathbb{Z}/p)\) is defined. The following results are well-known from Smith theory (cf. [2], Theorem 3.2 on page 40 and Theorem 4.4 on page 42-43).

**Lemma 2.4** We have the following.

(i) Suppose that the cyclic group \(G = \mathbb{Z}/p\) operates freely on \(X\), whose \(\dim \mathbb{Z}X < \infty\) and \(H^*(X; \mathbb{Z}/p)\) is finite dimensional. Then \(H^*(X/G; \mathbb{Z}/p)\) is finite-dimensional and

\[ \chi(X; \mathbb{Z}/p) = p \chi(X/G; \mathbb{Z}/p). \]

(ii) Suppose that the cyclic group \(\mathbb{Z}/p\) operates on \(X\), whose \(\dim_{\mathbb{Z}/p} X < \infty\) and \(H^*(X; \mathbb{Z}/p)\) is finite dimensional. Let \(F\) be the fixed point set. Then \(H^*(F; \mathbb{Z}/p), H^*(X - F; \mathbb{Z}/p)\) are finite dimensional and

\[ \chi(X; \mathbb{Z}/p) = \chi(X - F; \mathbb{Z}/p) + \chi(F; \mathbb{Z}/p). \]

Denote by \(G_x\) the stabilizer of \(x \in X\). Suppose that \(X = \bigcup_{i=0}^n X_i\) is the union of subspaces \(X_i = \{x \in X : \text{order}(G_x) = p^i\}\). It is clear that each \(X_i\) is \(G\)-invariant and \(X_n = \text{Fix}(G)\).

**Theorem 2.5** Suppose that \(G\) is a (not necessarily abelian) \(p\)-group of order \(p^n\) acting on \(X\). Then

\[ \chi(X; \mathbb{Z}/p) = \sum_{i=0}^n \chi(X_i; \mathbb{Z}/p) = \sum_{i=0}^n p^{n-i} a_i, \]

for some integers \(a_i\). Actually, we have \(\chi(X_i; \mathbb{Z}/p) = p^{n-i} a_i\).

**Proof.** We prove the theorem by induction on \(n\). When \(n = 0\), the statement is trivial by the assumption that the Euler characteristic \(\chi(X; \mathbb{Z}/p)\) is defined. When \(n = 1\), this is Lemma 2.4 by noting that \(F = X_1\) and \(X_0 = X - F\). Choose \(a\) to be an order-\(p\) element in the center of \(G\). Let \(F = \text{Fix}(a)\) and \(X_0 = X - F\). The quotient group \(G/\langle a \rangle\) acts on the quotient space \(X_0/\langle a \rangle\) and \(F\). Denote by

\[ Y_i = \{x \in (X - F)/\langle a \rangle : |(G/\langle a \rangle)_x| = p^i\} \]
and 
\[ Z_i = \{ x \in F : |(G/\langle a \rangle)_x| = p^i \}. \]

We will denote \( \chi(X; \mathbb{Z}_p) \) by \( \chi(X) \) for short. By the induction step, we have that 
\[ \chi((X - F)/\langle a \rangle) = \sum_{i=0}^{n-1} \chi(Y_i) = \sum_{i=0}^{n-1} p^{n-1-i}a_i' \]
and 
\[ \chi(F) = \sum_{i=0}^{n-1} \chi(Z_i) = \sum_{i=0}^{n-1} p^{n-1-i}b_i. \]

The first equality in the statement of the theorem is proved by noting that 
\[ X_i = q^{-1}(Y_i) \cup Z_{i-1} \]
with the convention that \( Z_{-1} = \emptyset \), where \( q : (X - F) \to (X - F)/\langle a \rangle \) is the projection. Therefore, we have 
\[
\begin{align*}
\chi(X) &= \chi(X - F) + \chi(F) \\
&= p\chi((X - F)/\langle a \rangle) + \chi(F) \\
&= p^n a_0' + \sum_{i=1}^{n-1} p^{n-i}(a_i' + b_{i-1}) + b_{n-1}.
\end{align*}
\]

The proof is finished by choosing \( a_0 = a_0', a_i = a_i' + b_{i-1} \) for \( 1 \leq i \leq n - 1 \) and \( a_n = b_{n-1} \). The last statement that \( \chi(X_i; \mathbb{Z}/p) = p^{n-i}a_i \) could be proved by noting \( X_i = q^{-1}(Y_i) \cup Z_{i-1} \) and a similar induction argument. \( \blacksquare \)

For a group \( G \) and a prime \( p \), let the \( p \)-rank be \( \text{rk}_p(G) = \sup \{ k : (\mathbb{Z}/p)^k \to G \} \). It is possible that \( \text{rk}_p(G) = +\infty \).

**Theorem 2.6** Let \( M^r \) be a first countable connected cohomology \( r \)-manifold over \( \mathbb{Z}/p \) and Homeo(\( M \)) the group of self-homeomorphisms. We adapt the convention that \( p^n = 1 \) when \( n < 0 \). Then the \( p \)-rank satisfies 
\[
p^{\text{rk}_p(\text{Homeo}(M)) - \left\lfloor \frac{r}{2} \right\rfloor} | \chi(M; \mathbb{Z}/p)
\]
when \( p \) is odd and 
\[
2^{\text{rk}_2(\text{Homeo}(M)) - r} | \chi(M; \mathbb{Z}/2)
\]
when \( p = 2 \). If \( M^r \) (\( r \geq 1 \)) is an oriented connected cohomology \( r \)-manifold over \( \mathbb{Z} \) and Homeo_{+}(M) is the group of orientation-preserving self-homeomorphisms, we have 
\[
2^{\text{rk}_2(\text{Homeo}(M)) - r + 1} | \chi(M; \mathbb{Z}/2).
\]
Proof. Suppose that an elementary $p$-group $G = (\mathbb{Z}/p)^{n_p}$ acts effectively on $M$ for $n_p = \text{rk}_p(\text{Homeo}(M))$. If the group action is free, we have $p^{n_p} | \chi(M)$ by Theorem 2.5 and the statements are obvious. In the following, we suppose that the group action is not free. We let $X = M$ and $X_i$ as in Theorem 2.5 for the sake of sticking to the notation of Theorem 2.5. Denote by $G_x$ the stabilizer of $x \in X_i$ for nonempty $X_i$. By Lemma 2.2 we have $i := \text{rank}(G_x) \leq r^2$ if $p \neq 2$ and $\text{rank}(G_x) \leq r$ if $p = 2$. Therefore, we have $p^{n_p-i} \geq p^{n_p-r^2}$ when $p \neq 2$ (or $p^{n_p-i} \geq p^{n_p-r}$ when $p = 2$). This implies that $p^{n_p-i\chi(M;\mathbb{Z}/p)}$ (or $2^{n_p-r\chi(M;\mathbb{Z}/2)}$ when $p = 2$) considering Theorem 2.5. A similar argument proves the orientation-preserving case using Lemma 2.3, by noting that the subgroup $G_x$ acts on $M$ orientation-preservingly if so does $G$. □

Remark 2.7 When $M$ is a surface, Theorem 2.6 was already known to Kulkarni [12].

Fixing a basis $\{a_1, \ldots, a_n\}$ for the free group $F_n$, we define several elements in $\text{Aut}(F_n)$ as the following. The inversions are defined as $e_i : a_i \mapsto a_i^{-1}, a_j \mapsto a_j (j \neq i)$; while the permutations are $(ij) : a_i \mapsto a_j, a_j \mapsto a_i, a_k \mapsto a_k (k \neq i, j)$. The subgroup $N < \text{Aut}(F_n)$ generated by all $e_i (i = 1, \ldots, n)$ is isomorphic to $(\mathbb{Z}/2)^n$. The subgroup $W_n < \text{Aut}(F_n)$ is generated by $N$ and all $(ij)$ $(1 \leq i \neq j \leq n)$. Denote $SW_n = W_n \cap \text{SAut}(F_n)$ and $SN = N \cap \text{SAut}(F_n)$. The element $\Delta = e_1 e_2 \cdots e_n$ is central in $W_n$ and lies in $\text{SAut}(F_n)$ precisely when $n$ is even.

The following result is Proposition 3.1 of [5].

Lemma 2.8 Suppose $n \geq 3$ and let $f$ be a homomorphism from $\text{SAut}(F_n)$ to a group $G$. If $f|_{SW_n}$ has non-trivial kernel $K$, then one of the following holds:

1. $n$ is even, $K = \langle \Delta \rangle$ and $f$ factors through $\text{PSL}(n, \mathbb{Z})$,
2. $K = SN$ and the image of $f$ is isomorphic to $\text{SL}(n, \mathbb{Z}/2)$, or
3. $f$ is the trivial map.

When $n = 2m$ is even, for each $1 \leq i \leq m$ define $R_i : F_n \to F_n$ as $a_{2i-1} \mapsto a_{2i-1}^{-1}, a_{2i} \mapsto a_{2i}^{-1}, a_{2i-1} \mapsto a_j (j \neq 2i, 2i-1)$. Let $T < \text{SAut}(F_n)$.
be the subgroup generated by all $R_i$, $i = 1, \ldots, m$. By Lemma 3.2 of Bridson-Vogtmann [5], $T$ is isomorphic to $(\mathbb{Z}/3)^m$. The following result is Proposition 3.4 of [5].

**Lemma 2.9** For $m \geq 2$ and any group $G$, let $\phi : \text{SAut}(F_{2m}) \to G$ be a homomorphism. If $\phi|_T$ is not injective, then $\phi$ is trivial.

**Proof of Theorem 1.2** Let $f : \text{SAut}(F_n) \to \text{Homeo}(M)$ be a group homomorphism. Since the Euler characteristics $\chi(M; \mathbb{Z}/2) = \chi(M; \mathbb{Z}/3)$ (cf. [4], Theorem 5.2 and Corollary 5.7), they will be simply denoted by $\chi(M)$. Since any action of $\text{SAut}(F_n)$ on a non-orientable manifold $M$ can be uniquely lifted to be an action on the orientable double covering $\bar{M}$ (cf. [3], Cor. 9.4, p.67), we may assume that $M$ is oriented and the group action is orientation-preserving by noting that $\text{SAut}(F_n)$ is perfect (cf. [10]).

When $M$ is non-orientable and $\chi(M) \not\equiv 0 \mod 3$, we would still have $\chi(\bar{M}) \not\equiv 0 \mod 3$.

When $n = 3$, the manifold $M$ is of dimension one. This case is already proved by Bridson and Vogtmann [5]. Suppose that $n \geq 4$. Choose $m = \lceil \frac{n}{2} \rceil$ (the integer part) and $T \cong (\mathbb{Z}/3)^m$. Let $\text{SAut}(F_{2m})$ be the subgroup of $\text{SAut}(F_n)$ fixing $a_n$ if $n$ is odd. Note that $\text{SAut}(F_n)$ is normally generated by a Nielsen automorphism in $\text{SAut}(F_{2m})$ (cf. [10]). If $f$ is not trivial, the restriction $f|_{\text{SAut}(F_{2m})}$ is not trivial and thus the map $f|_T$ is injective by Lemma 2.6 Theorem 2.6 implies that $3 \mid \chi(M)$, by noting that $n-r \geq 2$. This is a contradiction in the non-orientable case. If $\text{Im} f$ contains a copy of $(\mathbb{Z}/2)^{n-2}$, Theorem 2.6 would imply that $2 \mid \chi(M)$. This would be a contradiction to the assumption that $\chi(M) \not\equiv 0 \mod 6$ for the orientable manifold $M$. Therefore, the restriction $f|_{SN}$ is not injective and Case 1 in Lemma 2.8 cannot happen, since $SN \cong (\mathbb{Z}/2)^{n-1}$. If case 2 happens, the image satisfies $\text{Im} f = \text{SL}(n, \mathbb{Z}/2)$. Let $x_{1i}(1)$ denote the matrix with 1s along the diagonal, 1 in the $(1, i)$-th position and zeros elsewhere. Since the subgroup $(x_{12}(1), x_{13}(1), \ldots, x_{1n}(1)) \cong (\mathbb{Z}/2)^{n-1}$, we still have $2 \mid \chi(M)$. This is a contradiction, which implies that $f$ has to be trivial. 

From the above proof, we see that Theorem 1.2 also holds for cohomology manifolds over $\mathbb{Z}$.

**Remark 2.10** For a specific $n$, the conditions of Theorem 1.2 may be improved. For example, when $n$ is odd, Case 1 in Lemma 2.8 cannot happen. A similar proof as that of Theorem 1.2 shows that any action of $\text{SAut}(F_{2k+1})$ ($k \geq 1$) on an orientable manifold $M^r$ with $\chi(M) \not\equiv 0 \mod 12$
(resp. $\chi(M) \not\equiv 0 \mod 2$) by homeomorphisms is trivial when $2k > r$ (resp. $2k \geq r$).

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Department of Mathematical Sciences, Xi’an Jiaotong-Liverpool University, 111 Ren Ai Road, Suzhou, Jiangsu 215123, China.
E-mail: Shengkui.Ye@xjtlu.edu.cn