On Topological Susceptibility, Vacuum Energy 
and Theta Dependence in Gluodynamics

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Abstract:

We suggest that the topological susceptibility in gluodynamics can be found in terms of the gluon condensate using renormalizability and heavy fermion representation of the anomaly. Analogous relations can be also obtained for other zero momentum correlation functions involving the topological density operator. Using these relations, we find the $\theta$ dependence of the condensates $\langle GG \rangle$, $\langle G\tilde{G} \rangle$ and of the partition function for small $\theta$ and an arbitrary number of colors.
1 Introduction

The importance of exact results in QCD or pure Yang-Mills (YM) theory is hard to over-
estimate. There is as yet no thorough understanding of their nonperturbative aspects
which are of crucial importance in most instances where strong interactions are involved.
In these circumstances, any exact model-independent statements (theorems) about non-
perturbative properties of QCD or YM theory become especially valuable. It turns out
that some exact results can be obtained even in the absence of detailed knowledge of the
confinement and chiral symmetry breaking mechanisms. These theorems will survive any
further development of the theory. On the other hand, they serve as a test case for any
nonperturbative models and confinement scenarios, which should respect them.

Inspired by a recent impressive breakthrough [1] in supersymmetric (SUSY) theories
(see e.g. [2] for a review), one may wish to address the issue of available exact results
for ordinary, non-supersymmetric QCD or YM theory which is not considered as a softly
broken SUSY model, but is rather taken on its own. Only a few of them are known so far.
One well known example is provided by the so-called t’Hooft consistency condition [3]
which ensures the anomaly matching in terms of composite and building block particles.
Another series of theorems is based on strict inequalities which yield a number of qualiti-
tative, but rigorous statements for QCD [4]. Two more classes of exact results are given
by anomalous Ward identities [5] and low energy theorems based on a combination of the
Ward identities technique with renormalizability arguments [6]. The power of the latter
method is that it allows for exact evaluation of some nonperturbative observables in QCD,
see e.g. [7] for a review. Among the variety of results, obtained within this approach,
exact calculations of zero momentum correlation functions of scalar gluon currents (to be
discussed below) are of special interest for our purposes.

In this paper we address zero momentum correlation functions of the topological den-
sity operator $\alpha_s/(8\pi)G\tilde{G}$ with itself and the scalar gluon current $\alpha_s/(8\pi)GG$, in pure YM
theory (gluodynamics) [1]. We will make an attempt to find these correlation functions in
terms of the gluon condensates $\langle GG \rangle$, $\langle G\tilde{G} \rangle$ without resorting to any particular approxi-
mation, but rather appealing to the general properties of the theory such as renormaliz-
ability and dimensional transmutation. In particular, we will argue that the topological
susceptibility in gluodynamics can be found in terms of the gluon condensate within a
particular regularization scheme. Although our arguments will be somewhat heuristic,
there is a hope that our final Eq. (21) is correct. This hope is supported by the fact that
the same relation can be also obtained along with a very different line of reasoning [8]
which implies the same regularization scheme.

Our interest in the correlation functions of the above type is mostly related to the
problem of the $\theta$ dependence [9], which has been long discussed in the literature practically
since the discovery of instantons [10]. We note that little is known about the $\theta$ dependence
in YM theory. Prior to a recent renewed interest in these issues in supersymmetric theories
[11, 1, 2], essentially all what was known about the $\theta$ dependence in YM theory was the
fact that the physics should depend on $\theta$ through the combination $\theta/N_c$. This conclusion
follows from the analysis of the famous Witten-Veneziano mass relation for the $\eta'$ [12]

\footnote{1 We would like to recall that in QCD all zero momentum correlation functions of $G\tilde{G}$ are suppressed in the chiral limit by the light quark masses.}
obtained within the large $N_c$ approach. After the works of Witten and Veneziano, the discussion of the $\theta$ dependence in the literature has mostly switched to field theoretical models other than gluodynamics. One of the purposes of this letter is to find the $\theta$ dependence of lowest dimensional condensates and the partition function in YM theory for arbitrary number of colors $N_c$ and any value of $\theta \ll \pi$.

There exist many different reasons to study the correlation function of the above type and the $\theta$ dependence of the lowest dimensional condensates. Here we would like to mention a few of them. First, such relations are needed for the construction of a low energy effective Lagrangian [13] which can be used to study the vacuum structure of the theory. Such an effective Lagrangian allows one to find the $\theta$ dependence for all (and not only for small) values of $\theta$. Secondly, the latter results for the $\theta$ dependence in YM theory and QCD can be used for the axion physics [14]. Finally, these results may be of interest for the lattice studies [15] and nonperturbative models of the YM vacuum including, in particular, instanton liquid models [16].

The starting point of our method is the well known relation

$$\frac{\alpha_s}{8\pi} C_\mu^a \tilde{C}_\mu^a = - \lim_{M \to \infty} \langle M \bar{\Psi} i\gamma_5 \Psi \rangle_A ,$$

where $\Psi$ stands for a fermion field of mass $M$. The meaning of this formula is that the topological density operator is equivalent to the fermion bilinear (1) in the external gluon field in the infinite mass limit. This observation determines our whole approach to the problem. For our purposes, we consider gluodynamics as the low energy limit of YM theory coupled to a heavy fermion field. The fermion mass is assumed to be much larger than the inverse confinement radius, which ensures that this heavy fermion does not affect infrared properties of the theory. As its mass can be made arbitrarily large, all calculations are under theoretical control, and no problem can arise with this definition. On the other hand, the inclusion of a heavy fermion requires a regularization on a much higher ultraviolet scale. For the latter, we will assume the Pauli-Villars regulators. We will therefore work in the double limit $\Lambda_{YM} \ll M \ll M_R$, where $M$ is the mass of the auxiliary fermion, $M_R$ is the Pauli-Villars fermion mass, and $\Lambda_{YM}$ stands for the dimensional transmutation parameter in gluodynamics. We will argue that a combination of the Ward identities technique with the fermion representation of the anomaly, Eq.(1), and its analog for the scalar case together with renormalizability arguments allow one to find zero-momentum correlation functions of the topological density in gluodynamics in terms of the nonperturbative gluon condensates $\langle GG \rangle, \langle G\tilde{G} \rangle$, within a particular regularization scheme.

Our presentation is organized as follows. In Sect.2 we propose a method which relates the topological susceptibility in gluodynamics with the gluon condensate. In Sect.3 we combine this result with the Witten-Veneziano [12] approach to the U(1) problem at large $N_c$. The $\eta'$ meson mass in the limit $m_q \to 0, N_c \to \infty$ will be expressed in terms of the gluon condensate of pure YM theory. Sect.4 deals with the problem of the $\theta$ dependence in YM theory for small values $\theta \ll \pi$. Sect.5 contains our conclusions.

Note that a different relation of the topological density with light flavour fermion bilinears has been used starting from [17] in the study of the $\theta$-dependence in QCD by current algebra methods.

In what follows, we use the terms gluodynamics and YM theory interchangeably.
Correlation functions of topological density

In this section we study the zero momentum two-point function of the topological density (topological susceptibility) in pure YM theory. In what follows we will use the short notation for zero momentum correlation functions

\[
\langle A, B \rangle \equiv \lim_{p \to 0} i \int dxe^{ipx} \langle 0 | T \{ A(x) B(0) \} | 0 \rangle
\]

of arbitrary operators \( A(x) \) and \( B(x) \). In this notation, we want to evaluate the two-point function :

\[
P \equiv \langle \alpha_s \frac{8}{\pi} G_{\mu \nu} \tilde{G}_{\mu \nu}, \alpha_s \frac{8}{\pi} G_{\mu \nu} \tilde{G}_{\mu \nu} \rangle .
\]

As is well known, this correlation function vanishes to all order of perturbation theory since the topological density operator is a total derivative. Therefore, it seems that explicitly nonperturbative methods (for example, the instanton approximation) are needed to evaluate the two-point function (3). What will be argued below is that this correlation function can be analysed using general arguments appealing to renormalizability of YM theory, together with the fermion representation of the anomaly, Eq.(1).

To proceed with our arguments, we would like to remind the reader how a similar problem of the scalar two-point function

\[
S \equiv \langle \alpha_s \frac{12}{\pi} G_{\mu \nu} G_{\mu \nu}, \alpha_s \frac{12}{\pi} G_{\mu \nu} G_{\mu \nu} \rangle
\]

was addressed by Novikov et. al. (NSVZ) [3]. These authors have derived a fundamental low energy theorem fixing zero momentum correlation functions of scalar gluon currents :

\[
i \int dx \langle 0 | T \{ O(x) \alpha_s \frac{12}{\pi} G^2(0) \} | 0 \rangle_{YM} = \frac{2d}{3b} \langle O \rangle
\]

Here \( O(x) \) is an arbitrary color singlet local operator of canonical dimension \( d \) and \( b = 11/3N_c \) stands for the first coefficient of the \( \beta \)-function in pure Yang-Mills theory\(^4\). As a derivation of NSVZ theorem (5) is rather simple, for the sake of completeness we would like to recall it here. One starts with a redefinition of the gluon field

\[
\tilde{G}_{\mu \nu} \equiv g_0 G_{\mu \nu} ,
\]

where \( g_0 \) is the bare coupling constant defined at the cut-off scale \( M_R \). Then the path integral representation immediately yields the relation

\[
i \int dx \langle 0 | T \{ O(x) \tilde{G}^2(0) \} | 0 \rangle = -\frac{d}{d(1/4g_0^2)} \langle O \rangle .
\]

On the other hand, the renormalizability and dimensional transmutation phenomenon in a massless theory (either QCD with massless quarks or gluodynamics) ensure that

\[
\langle O \rangle = \text{const} \left[ M_R \exp \left( \frac{8\pi^2}{bg_0^2} \right) \right]^d
\]

\(^{4}\) The same formula holds in QCD in the chiral limit with the substitution \( b = 11/3N_c - 2/3n_f \) where \( n_f \) is a number of light flavors.
with the choice \( b = \frac{11}{3}N_c - \frac{2}{3}n_f \) or \( b = \frac{11}{3}N_c \), respectively. Finally, performing the differentiation yields the final result (9). Using the NSVZ theorem (4) for the diagonal case \( O = (\alpha_s/(12\pi))G^2 \), which is proportional to the conformal anomaly \( -b\alpha_s/(8\pi)G^2 \), we obtain

\[
S = \left\langle \frac{\alpha_s}{12\pi}G^2 , \frac{\alpha_s}{12\pi}G^2 \right\rangle = \frac{2}{9b} \left\langle \frac{\alpha_s}{\pi}G^2 \right\rangle .
\]

One should note that relation (8) implies the Wick type of the T-product in the correlation function (8). Moreover, Eq. (8) means that perturbative contributions are subtracted in both sides of Eq. (9). In this sense, the correlation function (9) can be obtained by the differentiation of \( \log(Z/Z_{PT}) \) in respect to \( 1/g_0^2 \) (here \( Z \) and \( Z_{PT} \) stand for the full and perturbatively defined partition functions, respectively, see Eq. (31) below).

We would now like to interpret Eq. (9) in a different way which makes use of the view of gluodynamics as a low energy limit of YM theory coupled to a heavy fermion \( \Psi \) of mass \( M \). An idea similar to what follows was first implemented by K"uhn and Zakharov (KZ) to relate the proton matrix element of the topological density \( \langle N|G\tilde{G}|N \rangle \) to the known quantity \( \langle N|G^2|N \rangle \) fixed (in the chiral limit) by the conformal anomaly. A technical trick, which will be used below, was suggested in [19] where, in particular, we reproduced by this method the KZ relation for the matrix element \( \langle N|G\tilde{G}|N \rangle \). In the present work, a similar idea is applied to correlation functions of the topological density.

Let us introduce chiral projections

\[
\Psi_L = \frac{1}{2}(1 - \gamma_5)\Psi , \quad \Psi_R = \frac{1}{2}(1 + \gamma_5)\Psi
\]

of the fermion field. We can now use the well known expansion of the scalar density

\[
-M \left( \bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L \right) \rightarrow \frac{\alpha_s}{12\pi}G_{\mu\nu}G^\mu\nu .
\]

Using (11), we put Eq. (9) in the form

\[
\left\langle \frac{\alpha_s}{12\pi}G^2 , -M \left( \bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L \right) \right\rangle = \frac{8}{3b} \left( -M \left( \bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L \right) \right).
\]

The following observation is crucial for what follows. Generally, Eq. (12) is just equivalent to (9), and one might therefore think that we would not obtain anything new in the course of this identical re-writing. However, one can note that, as long as the heavy fermion decouples, a “rotated” fermion should also do, because the physical content of the effective low energy theory can not be changed by a redefinition of a heavy fermion field (see the end of this section for more detail). In particular, one can consider the chiral rotations (change of variables)

\[
\Psi_L \Psi_R \rightarrow \exp(i\alpha)\bar{\Psi}_L \Psi_R , \quad \Psi_R \Psi_L \rightarrow \exp(-i\alpha)\bar{\Psi}_R \Psi_L .
\]

5 The same result (9) was obtained in [16] using canonical methods with Pauli-Villars regularization. To one loop order in regulator fields, it was found that perturbative contributions add the identity to both sides of Eq. (9).

6 One should note that both the \( \beta \)-function and the dimensional parameter \( \Lambda_{YM} \) become different when YM theory is coupled to a heavy fermion. However, the decoupling ensures that the relevant parameters of a corresponding low energy effective theory (i.e. pure gluodynamics) are its own \( \beta \)-function and \( \Lambda_{YM} \).
Different transformation properties of the two terms in (12) under rotations (13) imply that Eq.(12) is equivalent to two relations

$$\langle \frac{\alpha_s}{12\pi} G^2, M\bar{\Psi}_L\Psi_R \rangle = \frac{8}{3b} \langle M\bar{\Psi}_L\Psi_R \rangle ,$$  \hspace{1cm} (14)

$$\langle \frac{\alpha_s}{12\pi} G^2, M\bar{\Psi}_R\Psi_L \rangle = \frac{8}{3b} \langle M\bar{\Psi}_R\Psi_L \rangle .$$  \hspace{1cm} (15)

Now we can take the difference of these equations to extract the combination of interest. One can see that in this case the chiral fermion bilinears will combine into the pseudoscalar gluon density $\tilde{G}$. The relevant expansion reads (cf. Eq.(1))

$$-iM \left( \bar{\Psi}_L\Psi_R - \bar{\Psi}_R\Psi_L \right) \rightarrow \frac{\alpha_s}{8\pi} G_{\mu\nu} \tilde{G}^{\mu\nu} .$$  \hspace{1cm} (16)

Using (16), we transform (14,15) into

$$\langle \frac{\alpha_s}{12\pi} G^2, \frac{\alpha_s}{8\pi} G_{\mu\nu} \tilde{G}^{\mu\nu} \rangle = \frac{8}{3b} \langle \frac{\alpha_s}{8\pi} G_{\mu\nu} \tilde{G}^{\mu\nu} \rangle .$$  \hspace{1cm} (17)

Normally, both sides of this equation vanish because of the wrong CP-parity. However, we may think of a “regularized” theory with a $\theta$-term for $\theta \ll 1$, where Eq.(17) would be perfectly sensible. In any case, such a “regularization” by the $\theta$-terms is only needed at the intermediate step (17) of our derivation. The final answer will be valid for any value of $\theta$, including $\theta = 0$. As a double-check of our procedure, one can note that Eq.(17) coincides with the general NSVZ formula (5) for the particular case $O = (\alpha_s/8\pi)G\tilde{G}$.

Now we proceed similarly with Eq.(17). We substitute the $G^2$ term in the l.h.s. and $G\tilde{G}$ term in the r.h.s. of (17) by the heavy quark expansions (11) and (16), respectively. Then (17) becomes

$$\langle \frac{\alpha_s}{8\pi} G_{\mu\nu} \tilde{G}^{\mu\nu} , M \left( \bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L \right) \rangle = i \frac{8}{3b} \langle M \left( \bar{\Psi}_L\Psi_R - \bar{\Psi}_R\Psi_L \right) \rangle$$  \hspace{1cm} (18)

Using the identity

$$\bar{\Psi}_L\Psi_R - \bar{\Psi}_R\Psi_L = \bar{\Psi}_L\gamma_5\Psi_R + \bar{\Psi}_R\gamma_5\Psi_L$$

and transformation properties under chiral rotations (13), we see that (18) is again equivalent to two equations

$$\langle \frac{\alpha_s}{8\pi} G_{\mu\nu} \tilde{G}^{\mu\nu} , M\bar{\Psi}_L\Psi_R \rangle = i \frac{8}{3b} \langle M\bar{\Psi}_L\gamma_5\Psi_R \rangle ,$$  \hspace{1cm} (19)

$$\langle \frac{\alpha_s}{8\pi} G_{\mu\nu} \tilde{G}^{\mu\nu} , M\bar{\Psi}_R\Psi_L \rangle = i \frac{8}{3b} \langle M\bar{\Psi}_R\gamma_5\Psi_L \rangle .$$  \hspace{1cm} (20)

It is obvious that the scalar combination of Eqs.(13,21) exactly reproduces Eq.(17) which, as we have seen, is a particular version of the original NSVZ theorem (5). New information is contained in the pseudoscalar part of Eqs.(13,21). Taking the difference of these equations and multiplying the whole expression by $i$, we finally arrive at the relation

$$i \int dx \langle 0|T \left\{ \frac{\alpha_s}{8\pi} G\tilde{G} , \frac{\alpha_s}{8\pi} G\tilde{G} \right\} |0 \rangle = -\frac{2}{9b} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle .$$  \hspace{1cm} (21)
The relation obtained is clearly scheme dependent. In fact, we have used above the particular scheme in which all condensates and correlation functions are defined through the path integral. Thus Eq. (21) implies the Wick type of the T-product, i.e. the two-point function (21) is the second derivative of log \( Z \) in respect to \( \theta \). Therefore, Eq. (21) does not contain ultraviolet divergences which are present in the factor \( Z_{PT} \) (see Eq. (31) below) and drop out after the differentiation in \( \theta \). This ensures that the nonperturbative gluon condensate in Eqs. (9) and (21) is the same quantity. For more comments on the scheme dependence of our results, see the end of Sect. 4. We note that after Eq. (21) is established, arbitrary n-point correlation functions of operators \( G^2, \tilde{G}G \) can be found using Eqs. (9), (17) and (21) by further differentiation in respect to \( 1/g^2 \) and \( \theta \).

Furthermore, as the whole line of reasoning leading to Eq. (21) may appear rather vague, we would now like to make further comments justifying all steps made in the derivation. First, let us discuss the use of the dimensional transmutation arguments (8) in our case where we introduce an auxiliary heavy fermion field. It might seem that in this situation Eq. (8) does not hold, since now we have the mass parameter \( M \) in the theory. Let us, however, notice that for a heavy fermion mass \( M \) any dependence on \( M \) in formulas like (8), besides a perturbative redefinition of \( \Lambda_{YM} \), can only enter via corrections in powers of \( \Lambda_{YM}/M \) which die off asymptotically as \( M \to \infty \) (see the comment after Eq. (9)). This is the standard notion of decoupling of the heavy fermion. Therefore, we are justified in the use of Eq. (8) in our problem.

Secondly, one could object that the rotation (13) is not a symmetry, and thus the subsequent manipulations are not justified. Indeed, the transformation (13) is not a symmetry of the theory, and it does change the partition function. However, the non-conservation of the current corresponding to the rotation (13) shows up only at the level of \( O(1/M^2) \), and thus can be neglected in the limit \( M \to \infty \). The latter statement is simply a rephrasing of the textbook fact that a heavy quark of mass \( M \gg \Lambda_{QCD} \) does not contribute to the axial anomaly. The key difference from the case of the light quarks is the cancellation between the physical and regulator fields at the leading order in \( 1/M^2 \).

Thirdly, one could suspect that the substitution (18) is very tricky due to the “double scaling” limit \( \Lambda_{YM} \ll M \ll M_R \) which is implied in such replacement. In particular, one may worry about a region when two densities in Eqs. (14,21) are close together and essentially form a new operator of higher dimension which has different transformation properties. However, this small region \( x \sim M^{-1} \) is under theoretical control, and the operator product expansion (OPE) can be used to find that the contribution coming from this region would be of order \( M^{-n} \) and is thus negligible.

Finally, we would like to discuss the problem of higher loop contributions in Eq. (21). In the course of our transformations we have used the one-loop integration of both the auxiliary and Pauli-Villars fermions. Therefore, our result (21) is valid up to higher loop contributions in correlation functions (4) and (21). Moreover, in the “double scaling” limit \( \Lambda_{YM} \ll M \ll M_R \) which is implied in our calculations, this problem is just equivalent to the question of higher regulator loop contributions to the conformal and axial anomalies. One expects that the inclusion of higher regulator loops results in the substitution of the one-loop anomaly \( -b \alpha_s/(8\pi)G^2 \) by the full renormalization group invariant expression \( \beta(\alpha_s)/(4\alpha_s)G^2 \) in zero momentum correlation functions.
3 Witten-Veneziano formula and gluon condensate

As an application of the relation (21), we can consider the famous Witten-Veneziano scheme of resolution of the U(1) problem, which is based on a large $N_c$ approach. We should note that our result (21) is valid for arbitrary $N_c$. However, it is the large $N_c$ picture that leads to connections of physical characteristics of the $\eta'$ with zero momentum correlations functions of pseudoscalar gluon currents in gluodynamics. As will be shown in this section, the low energy relation (21) yields a new mass relation for the $\eta'$ in terms of the gluon condensate in gluodynamics.

We recall the Witten-Veneziano mass formula for the $\eta'$:

$$f_{\eta'}^2 m_{\eta'}^2 = -12 \langle \frac{\alpha_s}{8\pi} G\tilde{G}, \frac{\alpha_s}{8\pi} G\tilde{G} \rangle_{YM},$$

where $f_{\eta'}$ is the $\eta'$ residue

$$\langle 0 | \sum_{q_i=u,d,s} \bar{q}_i \gamma_\mu \gamma_5 q_i | \eta' \rangle = i\sqrt{3} f_{\eta'} q_\mu, \quad \langle 0 | n_j \frac{\alpha_s}{4\pi} G\tilde{G} | \eta' \rangle = \sqrt{3} f_{\eta'} m_{\eta'}^2$$

and the two-point function of the topological density in (22) refers to the pure YM case. It is the latter object that is now given by Eq. (21). We thus arrive at the relation which is valid in the double limit $m_q \to 0, N_c \to \infty$.

$$f_{\eta'}^2 m_{\eta'}^2 = \frac{8}{3b} \langle \frac{\alpha_s}{\pi} G^2 \rangle_{YM}.$$ 

Our result (24) suggests an explanation why the correlation function (22) is non-zero despite the fact that it vanishes to all orders of perturbation theory. In the Witten-Veneziano scheme, this fact was assigned to a ghost pole contribution to (22). Our derivation a posteriori shows that a non-vanishing value of correlation function (24) is due to a subtraction constant (gluon condensate) in the pseudoscalar channels, which is proportional to an analogous subtraction constant for the scalar channel, and is related to regulator contributions to the conformal anomaly. The ghost is simply a way to parametrize them. We note that on a qualitative level, this interpretation of the ghost was first discussed in [21].

4 The $\theta$ dependence in YM theory for $\theta \ll \pi$

In this section we will show that the $\theta$ dependence of the vacuum energy, topological density and partition function in YM theory can be found exactly for any $N_c$ and small $\theta \ll \pi$ provided the relation (21) takes place. More precisely, we will demonstrate that the Taylor expansions in the vacuum angle $\theta$ can be re-summed exactly for these objects. Thus, formally, our results hold for any $\theta$. However, it is believed that the physics is not analytic for $\theta \sim \pi$. This non-analyticity is related to the existence of distinct disconnected vacua which cross in energy at $\theta \sim \pi$. The Taylor expansion in $\theta$ refers to a state of lowest energy (at small $\theta$) out of this set, and thus can not probe other states. As a result, it can

7 Approximate mass relations for the $\eta'$, similar to (24), have been repeatedly discussed in the literature starting from [6] and [21].
be trusted only for $\theta \ll \pi$. As will be discussed below, the existence of additional vacua is crucial to establish the correct periodicity in $\theta$ with period $2\pi$. A corresponding analysis \[13\] reveals that indeed Eqs.(27),(28) (see below) correctly describe the $\theta$ dependence for small $\theta \ll \pi$.

Let us first address the $\theta$ dependence of the vacuum energy. The latter is defined by the $\theta$-vacuum expectation value of the trace of the momentum-energy tensor:

$$E_{\text{vac}} = -\frac{b}{32} \frac{\alpha_s}{\pi} G^2 \theta = -\frac{b}{32} \left( \langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle \right) + \frac{\theta^2}{2} \int dx \, dy \langle 0 | T \left\{ \frac{\alpha_s}{\pi} G^2(0) \frac{\alpha_s}{8\pi} G \tilde{G}(x) \frac{\alpha_s}{8\pi} G \tilde{G}(y) \right\} | 0 \rangle + \ldots \right). \tag{25}$$

The three-point correlation function is now fixed by the low energy relations (5) and (21). Indeed, differentiating Eq.(17) in respect to $\theta$, using Eq.(21) and setting $\theta$ to zero in a final expression, we obtain

$$i^2 \int dx \, dy \langle 0 | T \left\{ \frac{\alpha_s}{\pi} G^2(0) \frac{\alpha_s}{8\pi} G \tilde{G}(x) \frac{\alpha_s}{8\pi} G \tilde{G}(y) \right\} | 0 \rangle = -\left( \frac{8}{3b} \right)^2 \langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle. \tag{26}$$

As before, the Wick T-product is implied in Eq.(26). We thus see that the three-point function (26) is expressed in terms of the same nonperturbative gluon condensate which enters Eq.(11) and (21). Repeating the same procedure to all orders in $\theta$ in expansion (25), we arrive at the following result for $\theta \ll \pi$:

$$E_{\text{vac}} = \langle \theta | -\frac{b\alpha_s}{32\pi} G^2 | \theta \rangle = \langle 0 | -\frac{b\alpha_s}{32\pi} G^2 | 0 \rangle \cos \left( \frac{8}{3b} \theta \right). \tag{27}$$

The $\theta$ dependence of the topological density condensate can be found analogously:

$$\langle \theta | \frac{\alpha_s}{8\pi} G \tilde{G} | \theta \rangle = -\langle 0 | \frac{\alpha_s}{12\pi} G^2 | 0 \rangle \sin \left( \frac{8}{3b} \theta \right). \tag{28}$$

Similarly, we can calculate the $\theta$ dependence of the partition function $Z_\theta$ of YM theory. Since log $Z_\theta$ generates connected n-point functions of the topological density, resumming the expansion of log $Z_\theta$, we obtain

$$\log Z_\theta = \text{const} + \frac{i b}{32} V \langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle \cos \left( \frac{8}{3b} \theta \right), \tag{29}$$

where $V$ is the four-volume. A constant in Eq.(29) can be found from the relation \[22\]

$$\log Z_0 = \log Z_{PT} + \frac{i b}{32} V \langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle, \tag{30}$$

where $Z_{PT}$ stands for the perturbatively defined partition function. We note that a perturbative contribution to the gluon condensate is absorbed in the definition of log $Z_{PT}$. We therefore obtain

$$Z_\theta = Z_{PT}(g_0^2) \exp \left\{ \frac{i b}{32} V \langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle \cos \left( \frac{8}{3b} \theta \right) \right\}. \tag{31}$$
One can see that $\theta$ enters relations (27), (28) and (31) in the combination $\theta / N_c$. Thus, our results confirm the long standing conjecture, first arrived at within the large $N_c$ picture, that a $\theta$ dependence should come in the combination $\theta / N_c$. At the same time, these results seem to suggest a “wrong” periodicity in $\theta$. As was mentioned above, this interpretation of Eqs. (27), (28) and (31) would be wrong, as they are valid only for small values of $\theta \ll \pi$. It can be shown [13] that the correct $2\pi$ periodicity in $\theta$ is recovered when additional nondegenerate disconnected vacua, which cannot be seen in the naive thermodynamic limit $V \to \infty$, are taken into account. The correct periodicity in $\theta$ is the property of this set of vacua as a whole, while some two of them cross in energy at $\theta \sim \pi$. For small $\theta \ll \pi$ these additional states do not contribute the partition function in the limit $V \to \infty$, and correspondingly are not probed by the Ward identities. Their existence is rather revealed with an effective Lagrangian approach, see [13]. A corresponding analysis shows that Eqs. (27), (28) and (31), as standing for small values of $\theta \ll \pi$, are not in conflict with general principles of analyticity and $2\pi$ periodicity in $\theta$.

Finally, we would like to comment on the scheme dependence of our results. As was mentioned earlier, there is a specific scheme which is implied in all our formulas. Namely, we use the path integral definition for all correlation functions as the derivatives with respect to appropriate sources (parameters $1/g_0^2$ or $\theta$). This definition of zero momentum correlation functions implies the Wick type of the T-product symbol. It also fixes the rule of subtraction of the perturbative UV divergent contributions to the correlation functions, once such a rule is imposed for the gluon condensate. In terms of Eq.(31), this procedure means that the NSVZ low-energy theorem (9) is obtained by differentiation of the logarithm of $Z_\theta/Z_{PT}$ with respect to $1/g_0^2$, see Eq.(31). Our relation (21) and its n-point generalizations are obtained by differentiation of the same expression log($Z_\theta/Z_{PT}$) (31) with respect to $\theta$. It is clear that a change of the subtraction prescription in this scheme will generally alter the absolute value of the condensate, but it does not change the general structure of these relations, see also the appendix in [13] for more detail.

Last, but not least, formula (31) implies a “hidden” symmetry which may be thought of as the origin of the new low-energy relations similar to (21). This hidden symmetry arises due to the appearance of the new complex parameter $\tau = 1/g_0^2 + i\theta/(12\pi^2)$ in Eq.(31). Indeed, Eq.(31) implies the relation (for small $\theta$)

$$\langle -\frac{b\alpha_s}{8\pi} G^2 \rangle_\theta = \text{const} \ Re M_R^4 \exp\left( -\frac{32\pi^2}{bg_0^2} - i\frac{8}{3b} \theta \right) \equiv \text{const} \ Re M_R^4 \exp\left( -\frac{32\pi^2}{b} \tau \right).$$  \hspace{1cm} (32)$$

Thus, the nonperturbative vacuum energy depends just on this single combination of the coupling constant and $\theta$ angle. This is exactly the origin of the new relation (21) which together with the NSVZ theorem (9) can be rewritten in a holomorphic way [13]. With our definition of the nonperturbative correlators as the derivatives of the partition function with respect to the sources $\tau, \bar{\tau}$, the nonperturbative vacuum energy in Eq.(31) is the only relevant term to differentiate. A non-holomorphic dependence on $g_0^2$ of the perturbative part $Z_{PT}$ is irrelevant in our definition for calculation of the nonperturbative correlation functions. At the same time, the $\theta$ dependence appears only in the non-perturbative part of $Z_\theta$. Therefore, the origin of our new low-energy relations can be understood as the interplay between the exact NSVZ theorem (9) and assumption of separation of the perturbative and non-perturbative contributions in Eq.(31). Once this
assumption is made, a new complex structure emerges due to the pure non-perturbative origin of the $\theta$ parameter which combines with another parameter $1/g^2$ into the unique complex combination $\tau$.

5 Conclusions

Low energy theorems provide a bridge between low energy and high energy physics, including phenomena at the boundary of the ultraviolet cut-off. The fact that the axial anomaly can be described in terms of regulator fermions was emphasized by Gribov [23] (see also [7]) long ago. In this paper we suggested using the connection between the axial anomaly term and the regulating Pauli-Villars fermion bilinear, given by Eq.(1), to derive new relations for zero momentum correlation functions of the topological density. Classically, the topological density operator is a total derivative, which suggests that its effects can only be treated within infrared, explicitly nonperturbative methods such as the instanton approach. Yet, one can use another, quantum, definition of this object, which is given by Eq.(1). In contrast to the (quasi-) classical treatment, none of the extremely complicated problems of the latter (for example, a compactification of the infrared boundary, summation over different topological classes, instanton interactions, etc.) arise in the quantum approach. Instead, the problems are shifted to the analysis of the high energy behavior which is fixed by renormalizability. Therefore, we believe that a look at the “dark”, ultraviolet side of the topological density is very instructive as it can test nonperturbative infrared methods. An interesting question which could be asked in reference to Eq.(1) is whether this quantum definition of the topological density can be used in lattice studies of QCD and gluodynamics.

New low energy relations, taken together with analogous formulas for the scalar channel [8], enable us to evaluate some nonperturbative observables in gluodynamics, which seem to be out of the reach of other methods, within the regularization scheme based on the path integral. In particular, in this scheme we have found the relation between the topological susceptibility in YM theory, which is one of the most important characteristics of the YM vacuum, and the nonperturbative gluon condensate. Furthermore, within the same scheme we calculated the $\theta$ dependence for arbitrary $N_c$ and small $\theta \ll \pi$ for lowest dimensional vacuum condensates and for the partition function in gluodynamics, and found that the $\theta$ parameter always comes in the combination $\theta/N_c$, as was expected for a long time. Our results can have various applications which have been already mentioned in the introduction.
References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19; B431 (1994) 484.
   N. Seiberg, Nucl. Phys. B435 (1995) 129.
   G. Dvali and M. Shifman, hep-th/9611213; hep-th/9612128.
   A. Kovner, M. Shifman, and A. Smilga, hep-th/9706089.
   E. Witten, hep-th/9706109.
   B. Chibisov and M. Shifman, hep-th/9706141.

[2] M. Shifman, hep-th/9704114.

[3] G. t’Hooft, in: Recent Developments in Gauge Theories, eds. G. t’Hooft et. al.,
   Plenum Press, N.Y., 1980.

[4] F.T. Tomboulis, Phys. Rev. Lett. 50 (1983) 886.
   D. Weingarten, Phys. Rev. Lett. 51 (1983) 1830.
   S. Nussinov, Phys. Rev. Lett. 51 (1983) 1081; 52 (1983) 966.
   E. Witten, Phys. Rev. Lett. 51 (1983) 2351.
   C. Vafa and E. Witten, Nucl. Phys. B234 (1984) 173; Phys. Rev. Lett. 53 (1984) 535.

[5] R.J. Crewther, Phys. Lett. B70 (1977) 349.

[6] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B191
   (1981) 301.
   M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. B166 (1980) 493.

[7] M.A. Shifman, Phys. Rep. 209 (1991) 341.

[8] I. Halperin and A. Zhitnitsky, hep-th/9802093.

[9] C. Callan, R. Dashen, and D. Gross, Phys. Lett. 63B (1976) 172.
   R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976) 172.

[10] A.A. Belavin, A.M. Polyakov, A.S. Schwarz and A.S. Tyupkin, Phys. Lett. 59B
    (1975) 85.

[11] M.A. Shifman and A.I. Vainshtein, Nucl. Phys. B296 (1988) 445.

[12] E. Witten, Nucl. Phys. B156 (1979) 269.
   G. Veneziano, Nucl. Phys. B159 (1979) 213.

[13] I. Halperin and A. Zhitnitsky, hep-ph/9711398, to appear in Phys. Rev. D; hep-
    ph/9803301, submitted to Phys. Rev. Lett.

[14] I. Halperin and A. Zhitnitsky, talk given by ARZ at the Axion Meeting, Gainsville,
    FL, March 98, to be published in proceedings.
[15] B. Alles et al., Phys. Rev. D48 (1993) 2284.
    B. Alles et al., Nucl. Phys. B (Proc. Suppl.) 34 (1994) 504.
    A. Di Giacomo, hep-lat/9606001.

[16] E.V. Shuryak, Nucl. Phys. B203 (1982) 93, 116,140.
    D.I. Diakonov and V.Yu. Petrov, Nucl. Phys. B245 (1984) 259; B272 (1986) 457.
    E.V. Shuryak, Rev. Mod. Phys. 65 (1993) 1.
    T. Schäfer and E.V. Shuryak, hep-ph/9610451.

[17] V. Baluni, Phys. Rev. D19 (1979) 2227.

[18] J.H. Kühn and V.I. Zakharov, Phys. Lett. B252 (1990) 615.

[19] I. Halperin and A. Zhitnitsky, hep-ph/9706251.

[20] N. Seiberg, Phys. Rev. D49 (1994) 6857.

[21] D.I. Diakonov and M.I. Eides, Sov. Phys. JETP 54 (1981) 232.

[22] D.I. Diakonov, M.V. Polyakov, and C. Weiss, Nucl. Phys. B461 (1996) 539.

[23] V.N. Gribov, Preprint KFKI-1981-66, Budapest, 1981.