Exploration-exploitation trade-off for continuous-time episodic reinforcement learning with linear-convex models

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Abstract. We develop a probabilistic framework for analysing model-based reinforcement learning in the episodic setting. We then apply it to study finite-time horizon stochastic control problems with linear dynamics but unknown coefficients and convex, but possibly irregular, objective function. Using probabilistic representations, we study regularity of the associated cost functions and establish precise estimates for the performance gap between applying optimal feedback control derived from estimated and true model parameters. We identify conditions under which this performance gap is quadratic, improving the linear performance gap in recent work [X. Guo, A. Hu, and Y. Zhang, arXiv preprint, arXiv:2104.09311, (2021)], which matches the results obtained for stochastic linear-quadratic problems. Next, we propose a phase-based learning algorithm for which we show how to optimise exploration-exploitation trade-off and achieve sublinear regrets in high probability and expectation. When assumptions needed for the quadratic performance gap hold, the algorithm achieves an order $O(\sqrt{N \ln N})$ high probability regret, in the general case, and an order $O((\ln N)^2)$ expected regret, in self-exploration case, over $N$ episodes, matching the best possible results from the literature. The analysis requires novel concentration inequalities for correlated continuous-time observations, which we derive.

Key words. Continuous-time reinforcement learning, linear-convex, Shannon’s entropy, quadratic performance gap, Bayesian inference, conditional sub-exponential random variable

AMS subject classifications. 68Q32, 62C15, 93C73, 93E11

1 Introduction

Reinforcement learning (RL) is a core topic in machine learning and is concerned with sequential decision-making in an uncertain environment (see [38]). Two key concepts in RL are exploration, that corresponds to learning via interactions with the random environment, and exploitation, that corresponds to optimising the objective function given accumulated information. The latter can be studied using the (stochastic) control theory, while the former relies on the theory of statistical learning. When the state dynamics is not available while learning, we talk about model-free reinforcement learning, which is only concern with the search for the optimal policy. On the other hand, model-based RL assumes the model is given, or it is learned from data. Model-based RL has advantage over model-free approaches as a) it can be systematically studied using powerful and well understood stochastic control techniques that have been developed over the last half-century, see [23, 7, 6, 15]; b) it is more appropriate for high-stakes decision-making, overcoming some shortcomings of less interpretable “black-box” approaches [34].

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In this work we study finite time model-based RL when the underlying environment is modelled by a linear stochastic differential equation with unknown drift parameter and the agent is minimising (known) convex, but possibly irregular, cost function. Before introducing the learning algorithm and the main contributions of this work we briefly review the classical stochastic linear-convex (LC) control problems, with all the parameters assumed to be known, fixing the notation along the way.

1.1 Linear-convex control problem with observable parameter

Let $T > 0$ be a given terminal time, $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which supports a $d$-dimensional standard Brownian motion $W$, and $\mathbb{F}$ be the filtration generated by $W$ augmented by the $\mathbb{P}$-null sets. Let $\theta = (A, B) \in \mathbb{R}^{d \times (d + p)}$ be given parameters.\(^1\) Consider the following control problem with parameter $\theta$:

$$V^*(\theta) = \inf_{\alpha \in \mathcal{N}_2^d(\Omega; \mathbb{R}^p)} J(\alpha; \theta), \quad \text{with} \quad J(\alpha; \theta) = \mathbb{E} \left[ \int_0^T f(t, X_t^{\theta, \alpha}, \alpha_t) \, dt + g(X_T^{\theta, \alpha}) \right], \quad \text{(1.1)}$$

where $X^{\theta, \alpha} \in S^2_d(\Omega; \mathbb{R}^d)$ is the strong solution to the following dynamics:

$$dX_t = (AX_t + B\alpha_t) \, dt + dW_t, \quad t \in [0, T], \quad X_0 = x_0, \quad \text{(1.2)}$$

and the given initial state $x_0$ and functions $f$ and $g$ satisfy the following conditions as in [16]:

**H.1.** $T > 0$, $x_0 \in \mathbb{R}^d$, and $f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R} \cup \{\infty\}$ and $g : \mathbb{R}^d \to \mathbb{R}$ such that

(1) there exist measurable functions $f_0 : [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}$ and $h : \mathbb{R}^p \to \mathbb{R} \cup \{\infty\}$ such that

$$f(t, x, a) = f_0(t, x, a) + h(a), \quad \forall (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^p. \quad \text{(1.3)}$$

For all $(t, x) \in [0, T] \times \mathbb{R}^d$, $f_0(t, x, \cdot)$ is convex, $f_0(t, \cdot, \cdot)$ is differentiable with a Lipschitz continuous derivative, and $\sup_{t \in [0, T]}(|f_0(t, 0, 0)| + |\partial_{x, a} f_0(t, 0, 0)|) < \infty$. Moreover, $h$ is proper, lower semicontinuous, and convex.\(^2\)

(2) there exists $\lambda > 0$ such that for all $t \in [0, T], (x, a), (x', a') \in \mathbb{R}^d \times \mathbb{R}^p$, and $\eta \in [0, 1],$

$$\eta f(t, x, a) + (1 - \eta) f(t, x', a') \geq f(t, \eta x + (1 - \eta) x', \eta a + (1 - \eta) a') + \eta(1 - \eta) \lambda \frac{1}{2} |a - a'|^2; \quad \text{(1.4)}$$

(3) $g$ is convex and differentiable with a Lipschitz continuous derivative.

**Remark 1.1.** (H.1) not only includes the most commonly used linear-quadratic models as special cases, but also allows for practically important nonsmooth control problems in engineering and machine learning. For example, problems with control constraints correspond to $h$ being the indicator function of action sets, and sparse control problems correspond to $h$ being the $\ell^1$-norm of control variables. Moreover, in the reinforcement learning literature (see e.g., [9, 17, 37, 41, 33]), one often consider the entropy-regularised cost function:

$$f(t, x, a) := f_0(t, x)^\top a + h_{\text{ent}}(a), \quad \forall (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^p,$$  

\(^1\)With a slight abuse of notation, we identity $\mathbb{R}^{d \times d} \times \mathbb{R}^{d \times p}$ with $\mathbb{R}^{d \times (d + p)}$ throughout the paper.

\(^2\)We say a function $h : \mathbb{R}^p \to \mathbb{R} \cup \{\infty\}$ is proper if it has a nonempty domain $h := \{a \in \mathbb{R}^p \mid h(a) < \infty\}$. 

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with \( \bar{f}_0 : [0, T] \times \mathbb{R}^d \to \mathbb{R}^p \) being a sufficiently regular function, and \( h_{en} : \mathbb{R}^p \to \mathbb{R} \cup \{ \infty \} \) being the Shannon’s entropy function such that

\[
h_{en}(a) = \begin{cases} 
\sum_{i=1}^p a_i \ln(a_i), & a \in \Delta_p := \{ a \in [0, 1]^p \mid \sum_{i=1}^p a_i = 1 \}, \\
\infty, & a \in \mathbb{R}^p \setminus \Delta_p.
\end{cases}
\]

(1.5)

It has been shown that including the entropy function \( h_{en} \) in the optimisation objective leads to more robust decision-making [33] and accelerates the convergence of gradient-descent algorithms [37]. Observe that \( h_{en} \) is only differentiable on the interior of \( \Delta_p \), and its derivative blows up near the boundary. We refer the reader to [16] and the references therein for other applications of nonsmooth control problems.

In this work we focus on Lipschitz continuous feedback controls defined as follows.

**Definition 1.1.** For each \( C \geq 0 \), let \( \mathcal{V}_C \) be the following space of feedback controls:

\[
\mathcal{V}_C := \left\{ \psi : [0, T] \times \mathbb{R}^d \to \mathbb{R}^p \mid \psi \text{ is measurable and for all } (t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \psi(t, 0) \leq C \text{ and } |\psi(t, x) - \psi(t, y)| \leq C|x - y| \right\}. \tag{1.6}
\]

The following proposition shows that under (H.1), the control problem (1.1) admits an optimal feedback control \( \psi_o \), which depends continuously on the parameter \( \theta \). The proof can be found in Theorems 2.5 and 2.6 and Lemma 2.8 in [16].

**Proposition 1.1.** Suppose (H.1) holds, and let \( \Theta \) be a bounded subset of \( \mathbb{R}^{d \times (d+p)} \). Then there exists a constant \( C \geq 0 \), such that for all \( \theta \in \Theta \), there exists \( \psi_o \in \mathcal{V}_C \) such that

1. \( \psi_o \) is an optimal feedback control of (1.1), i.e., \( V^*(\theta) = J(\psi_o; \theta) \), where for all \( \theta \in \mathbb{R}^{d \times (d+p)} \) and \( \psi \in \cup_{C \geq 0} \mathcal{V}_C \),

\[
J(\psi; \theta) := E \left[ \int_0^T f(t, X_t^{\theta, \psi}, \psi(t, X_t^{\theta, \psi})) dt + g(X_T^{\theta, \psi}) \right] \in \mathbb{R} \cup \{ \infty \}, \tag{1.7}
\]

and \( X^{\theta, \psi} \in S^2_t(\Omega; \mathbb{R}^d) \) is the state process associated with \( \theta \) and \( \psi \) satisfying for \( t \in [0, T] \),

\[
dX_t^{\theta, \psi} = \theta Z_t^{\theta, \psi} dt + dW_t, \quad X_0^{\theta, \psi} = x_0, \quad \text{with} \quad Z_t^{\theta, \psi} = \left( \begin{array}{c} X_t^{\theta, \psi} \\ \psi(t, X_t^{\theta, \psi}) \end{array} \right); \tag{1.8}
\]

2. for all \( (t, x) \in [0, T] \times \mathbb{R}^d \) and \( \theta, \theta' \in \Theta \), \( |\psi_o(t, x) - \psi_{o'}(t, x)| \leq C(1 + |x|)|\theta - \theta'| \) and \( |f(t, x, \psi_o(t, x))| \leq C(1 + |x|^2) \); and

3. \( \mathbb{R}^{d \times (d+p)} \ni \theta \mapsto V^*(\theta) \in \mathbb{R} \) is continuous.

### 1.2 Phased-based learning algorithm and our contributions

In this section we provide a road map of the key ideas and contributions of this work without introducing needless technicalities. The precise assumptions and statements of the results can be found in Sections 2 and 3.
Episodic learning and observations. To encode the fact that the controller does not observe the true parameter $\theta$, we treat it as a random variable which we denote by $\theta$. The probability distribution induced by $\theta$ describes the uncertainty the controller has about unknown $\theta$. Agent learns about $\theta$ by executing a sequence of feedback policies $(\psi_m)_{m \in \mathbb{N}}$ and observing corresponding realisation of controlled dynamics (1.8), $(X^\theta_t, \psi_m)_{t \in [0, T], m \in \mathbb{N}}$. This is often referred to as the episodic framework in the RL literature (see e.g., [30, 5, 16]). At the beginning of the $m$-th episode, the information available to the agent for designing $\psi_m$ is incorporated in the $\sigma$-algebra $\mathcal{G}_{m-1} = \sigma\{X^\theta_s, \psi_s \mid s \in [0, T], n = 1, \ldots, m - 1\}$ generated by the state processes from previous episodes.

It is worth emphasising that the agent only observes the state process $(X^\theta_t, \psi_t)$, but not the Brownian motion $W$, since otherwise the problem reduces to the classical control problem. Indeed, consider (1.8). Suppose that one can choose $\psi$ such that $\int_0^t Z^\theta_s, \psi(Z^\theta_s, \psi)^\top ds$ is almost surely invertible for some $t > 0$ (see Lemma 6.1). Then, for all $t > 0$, we can write

$$\theta = \left( \left( \int_0^t Z^\theta_s, \psi(dX^\theta_s, \psi - dW_s)^\top \right) \right)^{-1} \left( \int_0^t Z^\theta_s, \psi(Z^\theta_s, \psi)^\top ds \right).$$

But this means that if we observe both $X^\theta, \psi$ and $W$, then we effectively know $\theta$.

Performance measure and incomplete learning. We denote by $\Psi_m(\cdot)$ the (random) feedback policy used in the $m$-th episode, in order to emphasise its dependence on realisations of controlled dynamics from all previous episodes; see Definition 2.1 for details. To measure the performance of an algorithm $\Psi = (\Psi_m)_{m \in \mathbb{N}}$ with the corresponding state processes $(X^\theta_t, \Psi_m)_{m \in \mathbb{N}}$, we often consider its associated cost

$$\ell_m(\Psi, \theta) := \int_0^T f(t, X^\theta_t, \Psi_m(\cdot, t, X^\theta_t, \Psi_m)) \, dt + g(X^\theta_T, \Psi_m).$$

The regret of learning up to $N \in \mathbb{N}$ episodes is then defined by

$$\mathcal{R}(N, \Psi, \theta) := \sum_{m=1}^N \ell_m(\Psi, \theta) - V^*(\theta),$$

where $V^*(\theta) = J(\psi_\theta; \theta)$ is the optimal cost that agent can achieve knowing the parameter $\theta$ (cf. (1.7)). The regret characterises the cumulative loss from taking sub-optimal policies in all episodes. Agent’s aim is to construct a learning algorithm for which the regret (either in high-probability or in expectation) $\mathcal{R}(N, \Psi, \theta)$ grows sublinearly in $N$.

To quantify the regret of an algorithm $\Psi = (\Psi_m)_{m \in \mathbb{N}}$, one can decompose the regret into

$$\mathcal{R}(N, \Psi, \theta) = \sum_{m=1}^N \left( \ell_m(\Psi, \theta) - J(\Psi_m; \theta) \right) + \sum_{m=1}^N \left( J(\Psi_m; \theta) - J(\psi_\theta; \theta) \right),$$

where $J(\Psi_m; \theta)$ is given in (1.7) and $\psi_\theta$ is given in Proposition 1.1. The first term vanishes under expectation and can be bounded with $O(\sqrt{N \ln N})$ in the high-probability sense (Propositions 4.3 and 4.4). This means that it suffices to analyse the second term which is the (expected) regret that has been analysed in [5, 16] assuming a self-exploration property of greedy policies (see Remark 3.1). However, the following examples shows these greedy policies in general do not guarantee exploration and consequently convergence to the optimal solution, which is often referred to as incomplete learning in the literature (see e.g., [21]).
Example 1.1. Consider minimising the following quadratic cost

\[ J(\alpha; \theta) = \mathbb{E} \left[ \int_0^T (\alpha_{1,t}^2 + \alpha_{2,t}^2) dt + X_t^2 \right], \]

over all admissible controls \( \alpha_1 \) and \( \alpha_2 \), subject to a one-dimensional state dynamics:

\[ dX_t = (B_1 \alpha_{1,t} + B_2 \alpha_{2,t}) dt + dW_t, \quad t \in [0, T], \quad X_0 = x_0, \quad \text{with} \ B_1, B_2 \neq 0. \]

The classical linear-quadratic (LQ) control theory shows that for any \( B = (B_1, B_2) \), the optimal feedback control is given by \( \psi_\theta(t, x) = -p_t B^\top x \), where \( (p_t) \) satisfies \( p'_t - (BB^\top) p_t^2 = 0 \) and \( p_T = 1 \). Consequently, if the agent starts with the initial estimate \( \tilde{B}^0 = (\tilde{B}_1^0, 0) \) with some \( \tilde{B}_1^0 \neq 0 \), then, assuming that \( \tilde{B}_1^m \neq 0 \) for all \( m \in \mathbb{N} \), the greedy strategy will result in a control of the form \( \alpha_t = (\alpha_{1,t}, 0) \) for all episodes and almost surely result in estimates \( \tilde{B}^m = (\tilde{B}_1^m, 0) \). In particular, the parameter \( B_2 \) and the optimal policy will never be learnt. Note that this example contradicts the full column rank condition of \( B = (B_1, B_2) \) in [16].

Separation of exploration and greedy exploitation episodes. Motivated by the example, in this work we assume that there exists a policy \( \psi^e \) that allows to explore the space (see (H.3)). We prove that such an exploration policy can be explicitly constructed under very general conditions (see Proposition 3.3), and guarantees to improve the accuracy of parameter estimation (see Remark 3.1). With the exploration policy \( \psi^e \) at hand, we separate the exploration episodes, in which \( \psi^e \) is exercised and the estimate \( \theta_m \) of \( \theta \) is updated based on available information \( \mathcal{G}_m \), from exploitation episodes in which greedy policies \( \psi_{\theta_m} \) are exercised. From Example 1.1 we see that in the latter case there is no guarantee that the agent’s knowledge of \( \theta \) increases.

Coming back to regret analysis and noting that \( V(\theta) = J(\psi_\theta; \theta) \), the second term on the right hand side of (1.11) can be decomposed as

\[
\sum_{m=1}^N (J(\Psi_m; \theta) - J(\psi_\theta; \theta)) = \sum_{m \in [1,N] \cap \mathcal{E}^\Psi} (J(\psi^e; \theta) - J(\psi_\theta; \theta)) + \sum_{m \in [1,N] \setminus \mathcal{E}^\Psi} (J(\psi_{\theta_m}; \theta) - J(\psi_\theta; \theta)),
\]

where \( \mathcal{E}^\Psi = \{ m \in \mathbb{N} | \Psi_m = \psi^e \} \). The first term of (1.12) induces loss due to exploration and increases linearly in the size of exploration episodes. The second term of (1.12), on the other hand, describes loss due to inaccuracy of our parameter estimate. Balancing exploration and exploitation episodes to control the growth of regret is precisely where the exploration-exploitation trade-off appears in this work.

To quantify the regrets of exploitation episodes, one of the main contributions of this work is to show that there exist constants \( L_\Theta, \beta > 0, r \in [1/2, 1] \) such that for all \( \theta_0 \in \Theta \),

\[
|J(\psi_\theta; \theta_0) - J(\psi_{\theta_0}; \theta_0)| \leq L_\Theta |\theta - \theta_0|^{2r}, \quad \forall \theta \in B_\beta(\theta_0),
\]

where \( \psi_\theta \) is an optimal feedback control with parameter \( \theta \). In particular,

- We prove that the performance gap (1.13) holds with \( r = 1 \) provided that the cost functions in (1.1) are Lipschitz differentiable (Theorem 3.5) or involve the (nonsmooth) entropy regularisation function (Theorem 3.6). To the best of our knowledge, this is the first paper on the (optimal) quadratic performance gap for continuous-time RL problems beyond the LQ setting.
Regret for Phased Exploration and Greedy Exploitation algorithm. While the general framework we present in this work can be used to study many popular parameter estimators from the literature, we focus the exposition on a truncated maximum a posteriori (MAP) estimate $\tilde{\theta}_m$ based on posterior distribution $\pi(\theta|G_m)$ for all $m \in \mathbb{N}$. The accuracy of the MAP estimate is evaluated by using concentration inequalities for conditional sub-exponential random variables, in order to address correlations among all observations. Based on the precise convergence rate of $|\tilde{\theta}_m - \theta|$ in terms of $m$ and the performance gap $r$, we design Phased Exploration and Greedy Exploitation algorithms (see Algorithm 1) that balance exploration and exploitation:

- In the case where the accuracy of $(\tilde{\theta}_m)_{m \in \mathbb{N}}$ improves only after the exploration episodes, we prove that the regret of the PEGE algorithm is of the magnitude $O(N^{1/1+r})$ (up to logarithmic factors) with high probability; see Theorem 3.7. In particular, our result extends the $O(\sqrt{N})$-regret bound for tabular Markov decision problems and discrete-time LQ-RL problems to continuous-time LC-RL problems with Lipschitz differentiable costs (Theorem 3.5) and entropy-regularised costs (Theorem 3.6).

- In the case where the accuracy of $(\tilde{\theta}_m)_{m \in \mathbb{N}}$ improves after both the exploration and exploitation episodes, we achieve an improved expected regret of the order $O(N^{1-r}(\ln N)^r)$ if $r < 1$ and $O((\ln N)^2)$ if $r = 1$; see Theorem 3.8. To the best of our knowledge, this is the first logarithmic regret bound beyond the linear parameterised Markov decision setting or the LQ-RL setting.

1.3 Related works

The problem that we study here falls under the umbrella of the partially observable stochastic control problems. There, one typically assumes that the controlled dynamics is not observable, and the observation process is given by another stochastic process and in addition, the coefficients of the controlled dynamics and the observation process are given to the agent. One typically also is interested in continuous learning without explicit exploration. We refer the reader to excellent monographs on this subject \[10, 6, 25\] and more recent development \[2, 3, 9\]. The problem that we study here is different. We do observe controlled dynamics but treat its parameters as random variables. In that sense, the problem is degenerate. Our main objective is to study regret in the episodic setting.

There is a vast literature on sublinear regret bounds for discrete-time RL algorithms (see e.g., \[35, 30, 31, 18\] for bandit problems and tabular Markov decision problems, and \[1, 26, 11, 36\] for discrete-time LQ-RL problems). In particular, a Phased Exploration and Greedy Exploitation algorithm has been introduced in \[35\] for linear bandit problems. Furthermore, many results in literature e.g., \[30\] and \[19\] give a regret with order depending on the size of state and action. For RL problems with continuous-time models, attentions have been mostly on algorithm development (see e.g., \[29, 27, 28\]), while regret analysis of learning algorithms is limited.

In the seminal work \[13\] authors established an asymptotically sublinear regret for regularised least-squares algorithms in an ergodic continuous-time LQ setup without an exact order of the regret bound. Recently, \[5, 16\] extend the least-squares algorithms to the finite-time horizon episodic setting and analyze their non-asymptotic regrets. They prove that if optimal controls of the true model automatically exploit the parameter space, then a greedy least-squares algorithm with suitable initialisation admits a non-asymptotically logarithmic expected regret for LQ models \[5\], and a $O(\sqrt{N})$ expected regret for LC models \[16\]. Unfortunately, as shown in Example 1.1 and in \[21\], such a self-exploration property may not hold in general, even for LQ models. Furthermore, the learning algorithm studied here works for an arbitrary initialisation.
There are various reasons for the relatively slow theoretical progress in non-asymptotic performance analysis of continuous-time RL algorithms. Analysing exploitation (i.e., the performance gap (1.13)) for continuous-time models often requires to study high-order stability (such as Lipschitz continuity or differentiability) of associated fully nonlinear HJB PDEs with respect to model parameters, which has always been one of the formidable challenges in the control theory. Furthermore, developing an effective exploration strategy for learning the environment in the nonlinear setting adds additional complexity to the problem that we study. Indeed, in the context of LQ-RL a natural exploration strategy is to add Gaussian noises with diminishing variances to greedy policies (see e.g., [13, 26, 11, 36]). This technique exploits heavily the fact that the greedy policy of a LQ control problem is affine in state variables, which clearly does not hold for general RL problems. Moreover, to ensure a finite exploration cost, one needs to construct exploration noises with range \( \mathcal{d}(h) = \{a \in \mathbb{R}^k \mid h(a) < \infty\} \), which can be computationally challenging.

**Notation:** For each \( T > 0 \), filtered probability space \( (\Omega, \mathcal{F}, \mathbb{P}) = \{\mathcal{F}_t\}_{t \in [0,T]} \), \( \mathbb{P} \) satisfying the usual condition and Euclidean space \( (E, \| \cdot \|) \), we introduce the following spaces:

- \( \mathcal{S}_q^2(\Omega \times [t, T] ; E) \), \( q \in [2, \infty) \), \( t \in [0, T] \), is the space of \( E \)-valued \( \mathbb{F} \)-progressively measurable processes \( \gamma : \Omega \times [t, T] \rightarrow E \) satisfying \( \| \gamma \|_q = E[\sup_{s \in [t, T]} |\gamma_s|^q]^{1/q} < \infty \);

- \( \mathcal{H}_p^q(\Omega \times [t, T] ; E) \), \( q \in [2, \infty) \), \( t \in [0, T] \), is the space of \( E \)-valued \( \mathbb{F} \)-progressively measurable processes \( \gamma : \Omega \times [t, T] \rightarrow E \) satisfying \( \| \gamma \|_{\mathcal{H}_p^q} = E[(\int_t^T |\gamma_s|^q ds)^{1/q}]^{1/q} < \infty \).

For notational simplicity, we denote \( \mathcal{S}_2^q(\Omega ; E) = \mathcal{S}_2^q(\Omega \times [0, T] ; E) \) and \( \mathcal{H}_p^q(\Omega ; E) = \mathcal{H}_p^q(\Omega \times [0, T] ; E) \).

We also denote by \( C \in [0, \infty) \) a generic constant, which depends only on the constants appearing in the assumptions and may take a different value at each occurrence.

## 2 A probabilistic framework for episodic RL problems

In this section, we introduce a rigorous probabilistic framework for the episodic learning procedure outlined in Section 1.2.

We start by defining admissible learning algorithms rigorously. An essential step is to describe the available information for decision-making, namely the \( \sigma \)-algebra with respect to which a learning policy is measurable at each episode. The fact that the controller does not observe the true parameter \( \theta \) indicates that the parameter is not measurable with respect to the available information \( \sigma \)-algebra. But because a deterministic quantity is measurable with respect to any \( \sigma \)-algebra on the space, we treat the parameter as a random variable \( \theta = (A, B) \). The range \( \Theta \) of the parameter \( \theta \) is assumed to be bounded and known to the controller.

**H.2.** \( \Theta \) is a nonempty bounded measurable subset of \( \mathbb{R}^{d \times (d + p)} \).

As \( \theta \) is a random parameter, it is crucial to distinguish among different sources of randomness throughout the learning process, and to ensure that all sources of randomness are supported on the same probability space. To this end, we work with a complete probability space of the following form:

\[
(\Omega, \mathcal{F}, \mathbb{P}) := \left( \Omega^\Theta \times \prod_{m=1}^{\infty} \Omega^{W_m}, \mathcal{F}^\Theta \otimes \bigotimes_{m=1}^{\infty} \mathcal{F}^{W_m}, \mathbb{P}^\Theta \otimes \bigotimes_{m=1}^{\infty} \mathbb{P}^{W_m} \right). \tag{2.1}
\]

The space \( (\Omega^\Theta, \mathcal{F}^\Theta, \mathbb{P}^\Theta) \) supports the random variable \( \theta \) taking values in \( \Theta \), and for each \( m \in \mathbb{N} \), the space \( (\Omega^{W_m}, \mathcal{F}^{W_m}, \mathbb{P}^{W_m}) \) supports a \( d \)-dimensional Brownian motion \( W^m \), which describes the randomness involved in making the new observation at the \( m \)-th episode. Without loss of
generality, we assume that $\Omega^\Theta = \Theta$, $\theta$ is the identity map on $\Omega^\Theta$, $\Omega^{W_m} = C([0, T]; \mathbb{R}^d)$ and $W^m$ is the coordinate map of $\Omega^{W_m}$. In particular, any element $\omega \in \Omega$ can be written as $\omega = (\omega^\theta, \omega^W) \in \Theta \times \prod_{m=1}^\infty \Omega^{W_m}$ with $\omega^\theta := \theta$ and $\omega^W := (\omega^W_1, \omega^W_2, \ldots)$. We extend canonically $\theta$ on $\Omega$ by setting $\theta(\omega^\theta, \omega^W) := \theta(\omega^\theta)$, and extend similarly on $\Omega$ any random variable on $\Omega^{W_m}$.

It is important to notice that one execution of the learning algorithm with all episodes corresponds to one sample $\omega = (\theta, \omega^W_1, \omega^W_2, \ldots)$ in the space $\Omega^\Theta \times \prod_{m=1}^\infty \Omega^{W_m}$. This describes the episodic learning problem precisely, where the system parameter $\theta$ is realised at the beginning of the learning process, and the agent aims to learn the unknown parameter through a sequence of episodes driven by independent noises.

Now we give the precise definition of an admissible learning algorithm. Recall the space of Lipschitz feedback controls $\mathcal{V}_C$ defined in (1.1).

**Definition 2.1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be defined in (2.1), $\mathcal{N}$ be the $\sigma$-algebra generated by $\mathbb{P}$-null sets, and $\Psi = (\Psi_m)_{m \in \mathbb{N}}$ be a sequence of functions $\Psi_m : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^p$ such that for all $m \in \mathbb{N}$, there exists a constant $C_m \geq 0$ such that $\Psi_m(\omega, \cdot) \in \mathcal{V}_{C_m}$ for all $\omega \in \Omega$.

We say that $\Psi = (\Psi_m)_{m \in \mathbb{N}}$ is an admissible learning algorithm (simply referred to as a learning algorithm) if for all $m \in \mathbb{N}$, $\Psi_m$ is $(\mathcal{G}_m^{-1} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d))/\mathcal{B}(\mathbb{R}^p)$-measurable, with the $\sigma$-algebras $(\mathcal{G}_m^{-1})_{m \in \mathbb{N} \cup \{0\}}$ defined recursively as follows: $\mathcal{G}_0^{-1} := \mathcal{N}$ and for all $m \in \mathbb{N}$, take the $(\mathcal{G}_m^{-1} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d))/\mathcal{B}(\mathbb{R}^p)$-measurable function $\Psi_m$, and define $\mathcal{G}_m := \mathcal{G}_{m-1} \lor \sigma\{X^\theta_{t, \Psi,m} | t \in [0, T]\}$, where $X^\theta_{t, \Psi,m} \in \mathcal{S}_\Theta^2(\Omega; \mathbb{R}^d)$ is the strong solution of

$$
\mathrm{d}X^\theta_{t, \Psi,m} = \theta Z^\theta_{t, \Psi,m} \mathrm{d}t + \mathrm{d}W^m_t, \quad X^\theta_{0, \Psi,m} = x_0, \quad \text{with } Z^\theta_{t, \Psi,m} = \left( \begin{array}{c} \psi^\theta_{m,t} \\ \psi^\theta_{m,m}(X^\theta_{t, \Psi,m}) \end{array} \right), \quad (2.2)
$$

with the filtration $\mathcal{F}^m$ generated by $W^m$ augmented by $\mathcal{G}_m^{-1}$. The $\sigma$-algebra $\mathcal{G}_m^{-1}$, $m \in \mathbb{N} \cup \{0\}$, describes the available information for the agent after the $m$-th episode, which is generated by the state processes from all previous episodes (up to $\mathbb{P}$-null sets).

**Remark 2.1.** For discrete-time Markov decision problems where the environment is driven by Markov chains and the agent observes the state dynamics in discrete-time, a learning algorithm is usually defined as a sequence of deterministic functions, each mapping the history of observations (i.e., a finite number of random variables) to action space (see e.g., [30, 31]). By the Doob–Dynkin lemma, this is equivalent to defining a learning algorithm as a sequence of random functions measurable with respect to the $\sigma$-algebras generated by all historical (random) observations.

Definition 2.1 extends these concepts to continuous-time RL problems, and specifies the required measurability condition of a learning algorithm. It is required for the rigorous analysis of learning algorithms in model-based RL, see Proposition 3.1. As $\theta$ and $(W^m_n)_{n=1}^m$ in general are not measurable with respect to $\mathcal{G}_{m-1}^{-1}$, Definition 2.1 restricts the choices of learning policies and prevents the agent from using the values of $\theta$ and $(W^m_n)_{n=1}^m$ in the decision-making.

We then define the regret of a learning algorithm as introduced in Section 1.2. For any given learning algorithm $\Psi = (\Psi_m)_{m \in \mathbb{N}}$, its associated cost at the $m$-th episode is given by

$$
\ell_m(\Psi, \theta) := \int_0^T f(t, X^\theta_{t, \Psi,m}, \Psi_m(\cdot, t, X^\theta_{t, \Psi,m})) \mathrm{d}t + g(X^\theta_{T, \Psi,m}), \quad (2.3)
$$

with $X^\theta_{t, \Psi,m}$ being defined as in Definition 2.1, and the regret of learning up to $N \in \mathbb{N}$ episodes is defined by

$$
\mathcal{R}(N, \Psi, \theta) := \sum_{m=1}^N \left( \ell_m(\Psi, \theta) - V^*(\theta) \right), \quad (2.4)
$$

8
where for each $\theta \in \mathbb{R}^{d \times (d+p)}$, $V^*(\theta) = J(\psi; \theta)$ as shown in Proposition 1.1.

Note that for any given $N \in \mathbb{N}$, $\mathcal{R}(N, \Psi, \theta) : \Omega \to \mathbb{R} \cup \{\infty\}$ is not deterministic since it depends on the realisations of the random parameter $\theta$ and the Brownian motions $(W^m)_{m=1}^N$. Due to (H.1) and the measurability of $\Psi_m$, $\ell_m(\Psi, \theta) : \Omega \to \mathbb{R} \cup \{\infty\}$ is $\mathcal{F} / \mathcal{B} (\mathbb{R} \cup \{\infty\})$-measurable for all $m \in \mathbb{N}$. Moreover, by the continuity of $V^* : \Theta \to \mathbb{R}$ (see Proposition 1.1), $V^*(\theta) : \Omega \to \mathbb{R}$ is $\mathcal{F} / \mathcal{B} (\mathbb{R})$-measurable, and hence $\mathcal{R}(N, \Psi, \theta)$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ for all $N \in \mathbb{N}$.

3 Phase-based learning algorithm and main results

In this section, we propose and analyse a phase-based learning algorithm for LC-RL problems under the probabilistic framework introduced in Section 2.

3.1 Bayesian inference for parameter estimation

As indicated by Definition 2.1, for the $m$-th episode, one needs to choose a (random) strategy $\Psi_m$ based on the historical observation described by the $\sigma$-algebra $\mathcal{G}_{m-1}^\Psi$. The probabilistic setup in Section 2 suggests to first infer the random parameter $\theta$ via a Bayesian algorithm based on a given prior and the samples observed so far, and then design the strategy $\Psi_m$ using the estimated parameter.

In the sequel, we derive a Bayesian estimation of $\theta$ by imposing a matrix normal prior distribution, i.e., we assume the conditional law $\pi(\theta | \mathcal{G}_0^\Psi) = \mathcal{MN}(\theta_0, I_d, V_0)$ with some $\theta_0 \in \mathbb{R}^{d \times (d+p)}$ and $V_0 \in \mathbb{S}_+^{d \times p}$. The normality of the prior distribution enables representing the posterior distribution $\pi(\theta | \mathcal{G}_m^\Psi)$ in terms of its sufficient statistic, while the separable structure of the covariance matrix effectively reduces the dimension of the covariance process (from an $\mathbb{S}_+^{d \times p}$-valued process to an $\mathbb{S}_+^{d \times p}$-valued process), which subsequently allowing for a trackable update of the sufficient statistics for large $d$ and $p$. Note that the estimated parameters will be projected into some bounded sets to recover the desired boundedness of estimates (see (3.6)). We also emphasise that the matrix normal prior distribution is only imposed to motivate the parameter estimation scheme, and will not be used in the regret analysis of our learning algorithm.

Let $\Psi = (\Psi_m)_{m \in \mathbb{N}}$ be a given learning algorithm, $(X^\theta_m)_{m \in \mathbb{N}}$ be the associated state processes, and $(\mathcal{G}_m^\Psi)_{m \in \mathbb{N}}$ be the corresponding observation filtration (cf. Definition 2.1). Then for each $m \in \mathbb{N}$, by considering (2.2), the likelihood function of $\theta$, or the Radon–Nikodym derivative of $\frac{d \mathbb{P}^{X^\theta_0,m}}{d \mathbb{P}^{W^m}}(t, X^\theta_0,m) = \exp \left( \int_0^t \left( \theta Z_{s}^\theta_0,m \right)^\top dX^\theta_0,m - \frac{1}{2} \int_0^t \left( \theta Z_{s}^\theta_0,m \right)^\top (\theta Z_{s}^\theta_0,m) ds \right)$

Hence given the initial belief that $\theta$ follows the prior distribution $\pi(\theta | \mathcal{G}_0^\Psi) = \mathcal{MN}(\theta_0, I_d, V_0)$, the

\[ \frac{d \mathbb{P}^{X^\theta_0,m}}{d \mathbb{P}^{W^m}}(t, X^\theta_0,m) = \exp \left( \int_0^t \left( \theta Z_{s}^\theta_0,m \right)^\top dX^\theta_0,m - \frac{1}{2} \int_0^t \left( \theta Z_{s}^\theta_0,m \right)^\top (\theta Z_{s}^\theta_0,m) ds \right). \]
posterior distribution of \( \pi(\theta|G_m^\Psi) \) after the \( m \)-the episode is given by

\[
\pi(\theta|G_m^\Psi) \propto \pi(\theta|G_0^\Psi) \prod_{n=1}^{m} \frac{d\mathbb{P}_{\Psi}^\theta}{d\mathbb{P}_n}(T, X_{\Psi,\theta},n)
\]

\[
\propto \exp\left( \text{tr}\left( \left( \hat{\theta}_{\Psi,m}(V_{\theta,\Psi,m})^{-1}\theta - \frac{1}{2}\theta((V_{\theta,\Psi,m})^{-1}\theta)\right) \right) \right)
\]

\[
\propto \exp\left( -\frac{1}{2}\text{tr}\left( (\theta - \hat{\theta}_{\Psi,m})(V_{\theta,\Psi,m})^{-1}(\theta - \hat{\theta}_{\Psi,m})\right) \right)
\]

where \( \propto \) stands for proportionality up to a constant independent of \( \theta \), and for all \( m \in \mathbb{N} \cup \{0\} \),

\[
V_{\theta,\Psi,m} := \left( V_0 - \sum_{n=1}^{m} \int_0^T Z_{s,\Psi,n}^{\theta}(Z_{s,\Psi,n}^{\theta})^{\top} ds \right)^{-1} \in \mathbb{S}_{++}^{d+p},
\]

\[
\hat{\theta}_{\Psi,m} := \left( \hat{\theta}_0 V_0^{-1} + \sum_{n=1}^{m} \left( \int_0^T Z_{s,\Psi,n}^{\theta}(dX_{s,\Psi,m}^{\theta}) \right)^{\top} \right) V_{\theta,\Psi,m} \in \mathbb{R}^{d \times (d+p)},
\]

with \( Z_{s,\Psi,n}^{\theta} \) defined in (2.2) for all \( n \in \mathbb{N} \). A formal derivation of this argument can be found in [25, Section 7.6.4] and is also related to the Kallianpur–Striebel formula in stochastic filtering (see [4]). In particular \( \pi(\theta|G_m^\Psi) = MN(\hat{\theta}_{\Psi,m}, I_d, V_{\theta,\Psi,m}) \), where \( \{\hat{\theta}_{\Psi,m}, V_{\theta,\Psi,m}\} \) depends only on the policies \( (\Psi_n)_{n=1}^m \).

It is clear that the random variables \( \{\hat{\theta}_{\Psi,m}, V_{\theta,\Psi,m}\} \) are sufficient statistics to represent the posterior distribution \( \pi(\theta|G_m^\Psi) = MN(\hat{\theta}_{\Psi,m}, I_d, V_{\theta,\Psi,m}) \). The following proposition iteratively constructs learning algorithms (cf. Definition 2.1) based on deterministic feedback controls and the sufficient statistics. The main step of the proof is to show that \( X_{\theta,\Psi,m} \) in (2.2) is a semimartingale with respect to the filtration generated by state processes, whose details are given in Appendix A.

**Proposition 3.1.** Suppose (H.2) holds. Let \( \hat{\theta}_0 \in \mathbb{R}^{d \times (d+p)} \), \( V_0 \in \mathbb{S}_{++}^{d+p} \), \( \hat{\theta}_{\theta,0} = \hat{\theta}_0 \) and \( V_{\theta,\Psi,0} = V_0 \).

For each \( m \in \mathbb{N} \), let \( C_m \geq 0 \), let \( \psi_m : \mathbb{R}^{d \times (d+p)} \times \mathbb{S}_{++}^{d+p} \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^p \) be a measurable function such that \( \psi_m(\theta, S, \cdot) \in \mathcal{V}_{C_m} \) for all \( (\theta, S) \in \mathbb{R}^{d \times (d+p)} \times \mathbb{S}_{++}^{d+p} \), let \( \Psi_m : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^p \) be such that for all \( (\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d \), \( \Psi_m(\omega, t, x) = \psi_m(\theta_{\Psi,m}(\omega), V_{\theta,\Psi,m}(\omega), t, x) \) for some \( n \leq m - 1 \), and let \( \{\hat{\theta}_{\Psi,m}, V_{\theta,\Psi,m}\} : \Omega \to \mathbb{R}^{d \times (d+p)} \times \mathbb{S}_{++}^{d+p} \) be defined in (3.2). Then \( \Psi = (\Psi_m)_{m \in \mathbb{N}} \) is a learning algorithm as in Definition 2.1.

We end this section by quantifying the estimation error of \( \{\hat{\theta}_{\Psi,m}\}_{m=0}^{\infty} \) in terms of the number of learning episodes. For simplicity, we focus on the following set \( \mathcal{U}_C \) of learning algorithms \( \Psi \) that are uniformly Lipschitz continuous in the \( x \)-variable: for each \( C \geq 0 \),

\[
\mathcal{U}_C := \left\{ \Psi = (\Psi_m)_{m \in \mathbb{N}} \mid \Psi \text{ is a learning algorithm as in Definition 2.1 and for all } m \in \mathbb{N} \text{ and } \omega \in \Omega, \Psi_m(\omega, \cdot) \in \mathcal{V}_C, \text{ with } \mathcal{V}_C \text{ in (1.6)} \right\}
\]

**Theorem 3.2.** Suppose (H.2) holds. Let \( x_0 \in \mathbb{R}^d \), \( T, L \geq 0 \), \( \hat{\theta}_0 \in \mathbb{R}^{d \times (d+p)} \), \( V_0 \in \mathbb{S}_{++}^{d+p} \) and for each \( \Psi \in \mathcal{U}_L \), let \( \{\hat{\theta}_{\Psi,m}, V_{\theta,\Psi,m}\}_{m=0}^{\infty} \) be defined in (3.2). Then there exists a constant \( C > 0 \) such that for all \( \Psi \in \mathcal{U}_L \), \( \delta \in (0, 1) \) and \( m \in \mathbb{N} \cap [2, \infty) \),

\[
\mathbb{P}\left( \lambda_{\min}(G_{\theta,\Psi,m})|\hat{\theta}_{\Psi,m} - \theta| \leq C (\ln m + \ln\frac{1}{\delta}) \right) \geq 1 - \delta, \quad \mathbb{P} \text{ a.s.}
\]

where \( \lambda_{\min}(S) \) is the smallest eigenvalue of a matrix \( S \in \mathbb{S}_{++}^{d+p} \), and \( G_{\theta,\Psi,m} = (V_{\theta,\Psi,m})^{-1} \).
The proof of Theorem 3.2 is given in Section 4.2, and is based on a combination of Propositions 4.6 and 4.7. Note that the Bayesian estimator \( \hat{\theta}^{\Psi,m} \) in (3.2) is defined by using all observed trajectories from previous episodes, which are not mutually independent as those for least squares estimators in [5, 16]. Hence, instead of applying concentration inequalities for independent observations, we establish Theorem 3.2 based on concentration inequalities for martingales with conditional sub-exponential differences.

Theorem 3.2 indicates that the convergence rate of \( (\hat{\theta}^{\Psi,m})_{m=0}^{\infty} \) depends on the growth rate of \( \lambda_{\min}(G^{\theta,\Psi,m})_{m\in\mathbb{N}} \) with respect to the number of episodes. In the following sections, we design a phased-based learning algorithm such that \( (\lambda_{\min}(G^{\theta,\Psi,m}))_{m\in\mathbb{N}} \) blows up to infinity sufficiently fast as \( m \to \infty \).

### 3.2 Structural assumptions for learning algorithms

As illustrated in Section 1.2, the following structural properties of the control problem (1.1)-(1.2) are essential for the design and analysis of learning algorithms.

**H.3.** There exists \( \psi^e \in \cup_{C \geq 0} \mathcal{V}_C \) and \( L_e \geq 0 \) such that

1. if \( u \in \mathbb{R}^d \) and \( v \in \mathbb{R}^p \) satisfy \( u^T x + v^T \psi(t,x) = 0 \) for almost every \((t,x) \in [0,T] \times \mathbb{R}^d\), then \( u \) and \( v \) are zero vectors,
2. for all \((t,x) \in [0,T] \times \mathbb{R}^d\), \(|f(t,x,\psi(t,x))| \leq L_e (1 + |x|^2) \) with \( f \) in (H.1).

**H.4.** There exist constants \( L, \beta > 0, r \in (0,1] \) such that for all \( \theta_0 \in \Theta \),

\[
|J(\psi; \theta_0) - J(\psi_{\theta_0}; \theta_0)| \leq L|\theta - \theta_0|^{2r}, \quad \forall \theta \in B_\beta(\theta_0) := \{ \theta \in \mathbb{R}^{d \times (d+p)} \mid |\theta - \theta_0| \leq \beta \},
\]

where \( \psi_{\theta} \) is an optimal feedback control of (1.1) and \( J(\psi; \theta_0) \) is defined in (1.7) for all \( \psi \in \cup_{C \geq 0} \mathcal{V}_C \).

**Remark 3.1.** (H.3) assumes the existence of an exploration policy \( \psi^e \), which will be exercised to explore the unknown system parameter. In particular, (H.3(1)) ensures that the state and control processes associated with \( \psi^e \) span the entire parameter space. More precisely, let \( \lambda_{\min} : \cup_{C \geq 0} \mathcal{V}_C \times \mathbb{R}^{d \times (d+p)} \to [0, \infty) \) be the function such that for all \( \psi \in \cup_{C \geq 0} \mathcal{V}_C \) and \( \theta \in \mathbb{R}^{d \times (d+p)} \),

\[
\lambda_{\min}(\psi, \theta) := \lambda_{\min}\left( \mathbb{E}^\mathbb{P} \left[ \int_0^T \begin{bmatrix} X_{t,x,a}^\theta \psi(t, X_{t,x}^\theta \psi) \\ X_{t,x,a}^\theta \psi(t, X_{t,x,a}^\theta \psi) \end{bmatrix} \right] \right),
\]

where \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) is a generic filtered probability space satisfying the usual condition, \( X^{\theta,\psi} \in \mathbb{S}_\mathbb{F}^d(\Omega; \mathbb{R}^d) \) is the state process associated with \( \theta \) and \( \psi \), and \( \lambda_{\min}(S) \) is the smallest eigenvalue of a symmetric positive semidefinite matrix \( S \). Then (H.3(1)) is equivalent to the fact that there exists \( \lambda_0 > 0 \) such that \( \lambda_{\min}(\psi^e, \theta) \geq \lambda_0 \) for all \( \theta \in \Theta \) (see Lemma 6.1). Hence, by (3.2) and Theorem 3.2, exercising \( \psi^e \) increases \( \lambda_{\min}(G^{\theta,\Psi,m}) \) and subsequently results in an improved parameter estimation based on state trajectories generated by \( \psi^e \).

(H.3(2)) along with (H.2) implies uniformly bounded exploration costs \(|J(\psi^e; \theta)| \leq C < \infty \) for all \( \theta \in \Theta \). It also enables estimating the tail behaviour of the exploration cost and quantifying the regret for the exploration phase of the learning algorithm.

Note that [5, 16] design RL algorithms by requiring the optimal feedback control associated with the unknown true parameter to satisfy (H.3). In the case with quadratic costs \( f(t,x,a) = x^T Q x + a^T R a \) and \( g(x) = 0 \) with \( Q \in \mathbb{S}_+^d \) and \( R \in \mathbb{S}_+^p \), such a requirement is equivalent to the condition that the matrix \( B \) in (1.2) has full column rank (see [5, Proposition 3.9]). Here we remove this requirement and construct \( \psi^e \) under more general conditions (see Proposition 3.3).
The remaining of this section is devoted to verifying (H.3) and (H.4) for practically important LC-RL problems. The following proposition gives an equivalent characterisation of (H.3), which allows for an explicit construction of the exploration strategy \( \psi^e \). The proof is given in Section 5.

**Proposition 3.3.** Suppose (H.1(1)) holds, and let \( \text{dom}(h) = \{ a \in \mathbb{R}^p \mid h(a) < \infty \} \). Then (H.3) holds if and only if \( \text{dom}(h) \) contains linearly independent vectors \( a_1, \ldots, a_p \). In this case, for each partition \( \{ 0 = t_0 < t_1 < \ldots < t_p = T \} \), \( \psi^e(t, x) = a_k \) for all \( (t, x) \in [t_{k-1}, t_k) \subseteq \mathbb{R}^d \) and \( k = 1, \ldots, p \) satisfies (H.3).

To verify (H.4), we first recall the following first-order performance gap under (H.1), as shown in [16, Theorem 2.7].

**Proposition 3.4.** Suppose (H.1) and (H.2) hold. Then for any \( \beta > 0 \), there exists \( L_\Theta > 0 \) such that (H.4) holds with \( r = 1/2 \) and the constants \( L_\Theta, \beta \).

The fact that the cost function \( h \) is merely lower semicontinuous restricts the model misspecification error to scale linearly in terms of the magnitude of parameter perturbations. By assuming a Lipschitz continuously differentiability of the cost function, the following theorem improves the linear dependence in Proposition 3.4 to a quadratic dependence, which generalises the well-known quadratic performance gap with quadratic costs (see e.g., [31, 5]).

**Theorem 3.5.** Suppose (H.1) and (H.2) hold, and \( h \equiv 0 \). Then for any \( \beta > 0 \), there exists \( L_\Theta > 0 \) such that (H.4) holds with \( r = 1 \) and the constants \( L_\Theta, \beta \).

The proof of Theorem 3.5 is given in Section 5. The main step is to show the functional \( J(\cdot; \theta_0) : \mathcal{H}_F^2(\Omega; \mathbb{R}^p) \to \mathbb{R} \cup \{ \infty \} \) defined in (1.1) (with \( \theta = \theta_0 \)) is convex and has a Lipschitz continuous derivative, which implies that \( J(\alpha; \theta_0) - J(\alpha^{b_0}; \theta_0) \leq C \| \alpha - \alpha^{b_0} \|^2_{\mathcal{H}_F^2} \) for all \( \alpha \in \mathcal{H}_F^2(\Omega; \mathbb{R}^p) \), with \( \alpha^{b_0} \in \mathcal{H}_F^2(\Omega; \mathbb{R}^p) \) being the minimiser of \( J(\cdot; \theta_0) \). Unfortunately, such an argument in general cannot be applied to nonsmooth cost functions. For example, as the entropy function \( \psi \) has unbounded derivatives, one can easily show that \( L^2(\Omega; \mathbb{R}^p) \ni \alpha \mapsto \mathbb{E}[\| \nabla \psi(n) \|] \in [0, \infty] \) is discontinuous at the minimiser \( \alpha^* = \arg \min_{\alpha \in L^2(\Omega; \mathbb{R}^p)} \mathbb{E}[\| h(n) \|] \).

In the following, we overcome the difficulty by establishing a local regularity of the functional \( J(\cdot; \theta_0) \) in (1.1) along optimal feedback controls \( \psi_\theta \) of estimated models, and recover the optimal quadratic performance gap for entropy-regularised cost functions. More precisely, we prove for the cost functions defined in (1.4), the map \( \mathbb{R}^{d \times (d+p)} \ni \theta \mapsto J(\psi_\theta; \theta_0) \in \mathbb{R} \) is \( C^2 \) for all \( \theta_0 \in \mathbb{R}^{d \times (d+p)} \), if \( \bar{f}_0 \) and \( g \) are sufficiently smooth.

The proof is given in Section 5. It relies on first characterising an optimal feedback control \( \psi_\theta \) in terms of solutions to a nonlinear HJB equation, and then establishing high-order regularity of the function \( \mathbb{R}^{d \times (d+p)} \times \mathbb{R}^d \ni (\theta, t, x) \mapsto \psi_\theta(t, x) \in \mathbb{R}^p \) via a probabilistic argument. Note that the convexity condition (H.1) on \( f \) and \( g \) will not be imposed in Theorem 3.6.

**Theorem 3.6.** Suppose (H.2) holds, \( x_0 \in \mathbb{R}^d \), \( T > 0 \), and \( f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R} \cup \{ \infty \} \) be of the form

\[
f(t, x, a) := \bar{f}_0(t, x)^T a + h_{en}(a), \quad \forall (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^p,
\]

with a continuous function \( \bar{f}_0 : [0, T] \times \mathbb{R}^d \to \mathbb{R}^p \), and the entropy function \( h_{en} : \mathbb{R}^p \to \mathbb{R} \cup \{ \infty \} \) defined in (1.5). Assume further that for all \( t \in [0, T] \), \( \bar{f}_0(t, \cdot) \in C^4(\mathbb{R}^d) \) and \( g \in C^4(\mathbb{R}^d) \) with bounded derivatives uniformly in \( t \). Then for all \( \theta \in \mathbb{R}^{d \times (d+p)} \), the control problem (1.1) (with parameter \( \theta \)) admits an optimal feedback control \( \psi_\theta \in \mathcal{U}_{C^2} \). Moreover, for any \( \beta > 0 \), there exists \( L_\Theta > 0 \) such that (H.4) holds with \( r = 1 \) and the constants \( L_\Theta, \beta \).
3.3 Phased Exploration and Greedy Exploitation algorithm and its regret bound

Based on (H.3) and (H.4), we propose a Phased Exploration and Greedy Exploitation (PEGE) algorithm which achieves sublinear regrets with high probability and in expectation. The algorithm alternates between exploration and exploitation phases, and extends the PEGE algorithm in [35] for discrete-time linear bandit problems to the present setting with continuous-time linear-convex models. For technical reasons, we focus on learning algorithms of the class \( \bigcup_{C \geq 0} \mathcal{U}_C \) (cf. (3.3)) by truncating the maximum a posteriori (MAP) estimates \( (\hat{\theta}^{\Psi,m})_{m \in \mathbb{N}} \) on a set compactly containing \( \Theta \).

Definition 3.1. A measurable function \( \rho : \mathbb{R}^{d \times (d+p)} \times \mathbb{S}^{d+p}_+ \to \mathbb{R}^{d \times (d+p)} \) is called a truncation to compact neighborhoods of \( \Theta \) (simply referred to as a truncation function) if there exists a bounded set \( \mathcal{K} \subset \mathbb{R}^{d \times (d+p)} \) such that range(\( \rho \)) = \( \mathcal{K} \), cl(\( \Theta \)) \( \subset \) int(\( \mathcal{K} \)), and \( \rho(\theta, V) = \theta \) for all \( \theta \in \mathcal{K} \) and \( V \in \mathbb{S}^{d+p}_+ \). For simplicity, we denote by \( \rho_{\mathcal{K}} \) a truncation function \( \rho \) with range \( \mathcal{K} \).

In practice, one can construct a truncation function by fixing a compact subset \( \mathcal{K} \subset \mathbb{R}^{d \times (d+p)} \) with cl(\( \Theta \)) \( \subset \) int(\( \mathcal{K} \)) and considering \( \rho : \mathbb{R}^{d \times (d+p)} \times \mathbb{S}^{d+p}_+ \ni (\theta, V) \mapsto \theta + \theta_0 1_{\mathcal{K}}(\theta) + \theta_0 1_{\mathcal{K}^c}(\theta) \in \mathcal{K}, \) with some \( \theta_0 \in \mathcal{K} \). Alternatively, one may consider a measurable function \( \rho \) such that

\[
\mathbb{R}^{d \times (d+p)} \times \mathbb{S}^{d+p}_+ \ni (\theta, V) \mapsto \rho(\theta, V) \in \arg \min_{\theta' \in \mathcal{K}} \text{tr}((\theta' - \theta)V^{-1}(\theta' - \theta)^\top),
\]

which corresponds to the MAP estimator truncated by \( \mathcal{K} \) (cf. (3.1)).

We now proceed to describing the PEGE algorithm based on truncated MAP estimates. The algorithm is initialised with \( \hat{\theta}_0 \in \mathbb{R}^{d \times (d+p)}, V_0 \in \mathbb{S}^{d+p}_+ \) and a truncation function \( \rho \). Then it operates in cycles, and each cycle consists of exploration and exploitation phases. In the exploration phase of the \( k \)-th cycle with \( k \in \mathbb{N} \), we exercise the exploration strategy \( \psi_e \) in (H.3) for one episode, and obtain the current estimate \( \hat{\theta}_m \) of \( \theta \) by

\[
\hat{\theta}_m := \rho(\hat{\theta}^{\Psi,m}, V^{\theta,\Psi,m}), \tag{3.6}
\]

where \( (\hat{\theta}^{\Psi,m}, V^{\theta,\Psi,m}) \) are defined in (3.2). During the exploitation phase of the \( k \)-th cycle, we exercise a sequence of greedy policies for \( m(k) \) consecutive episodes with some prescribed \( m(k) \in \mathbb{N} \). For each exploitation episode, based on the current estimate \( \hat{\theta}_m \) of \( \theta \), we execute the optimal feedback control \( \psi_{\tilde{\theta}_m} \) (cf. Proposition 1.1) for \( m(k) \) consecutive episodes with some prescribed \( m(k) \in \mathbb{N} \), and possibly update the estimate \( \hat{\theta}_{m+1} \) by using the new observations.

The PEGE algorithm is summarised as follows. In the sequel, we denote by \( \Psi^{\text{PEGE}} \) the sequence of strategies generated by Algorithm 1.

Algorithm 1: PEGE algorithm

\[
\begin{array}{l}
\text{Input: } \hat{\theta}_0 \in \mathbb{R}^{d \times (d+p)}, V_0 \in \mathbb{S}^{d+p}_+, \text{a truncation function } \rho, \text{and } m : \mathbb{N} \to \mathbb{N}.
\\
1 \text{ Initialise } m = 0, \hat{\theta}^{\Psi,0} = \hat{\theta}_0 \text{ and } V^{\theta,\Psi,0} = V_0.
\\
2 \text{ for } k = 1, 2, \ldots \text{ do}
\\
3 \quad \text{ Execute the exploration policy } \psi_e \text{ for one episode, and } m \leftarrow m + 1.
\\
4 \quad \text{ Update } (\hat{\theta}^{\Psi,m}, V^{\theta,\Psi,m}) \text{ via (3.2), } \hat{\theta}_m \text{ via (3.6), and set } \tilde{\theta} = \hat{\theta}_m.
\\
5 \quad \text{ for } l = 1, 2, \ldots, m(k) \text{ do}
\\
6 \quad \quad \text{ Execute the greedy policy } \psi_{\tilde{\theta}_l} \text{ in Proposition 1.1 for one episode, and } m \leftarrow m + 1.
\\
7 \quad \quad \text{ [Optional: update } (\hat{\theta}^{\Psi,m}, V^{\theta,\Psi,m}) \text{ via (3.2), } \hat{\theta}_m \text{ via (3.6), and set } \tilde{\theta} = \hat{\theta}_m.]}
\\
8 \end{array}
\]

9 end

13
Remark 3.2. By the regularity of \( \mathbb{R}^{d \times (d+p)} \times [0,T] \times \mathbb{R}^d \ni (\theta,t,x) \mapsto \psi_{\theta}(t,x) \in \mathbb{R}^p \) (see Proposition 1.1), the truncation function \( \rho \) and Proposition 3.1, \( \hat{\Psi}^{\text{PEGE}} \) (with or without Step 7) is a learning algorithm as in Definition 2.1.

Observe that the exploration phase of Algorithm 1 does not depend on the confidence parameter \( \delta \) in the high-probability regret bound (3.7) (cf. the least squares algorithms in [26, 11, 36]). Consequently, Algorithm 1 achieves a sublinear regret both with high probability and in expectation.

As we shall see soon, Algorithm 1 admits the same regret order regardless of whether Step 7 is employed. Hence one may omit Step 7 to save the computational costs of matrix inversion in (3.2) and of control updates, especially for high-dimensional problems with large \( d \) and \( p \). However, including Step 7 in the algorithm incorporates all available information in each exploitation episode, and hence may lead to more sample-efficient learning algorithms (see e.g., the posterior sampling algorithms in [30, 31, 1]).

Now we state the main result of this section, which shows that the regret of Algorithm 1 (without Step 7) grows sublinearly with respect to the number of episodes. For the sake of presentation, we include a sketched proof at the end of this section and present the detailed arguments in Section 6.

**Theorem 3.7.** Suppose (H.1), (H.2), (H.3) and (H.4) hold. Let \( \hat{\theta}_0 \in \mathbb{R}^{d \times (d+p)} \), \( V_0 \in \mathbb{S}^{d+p} \), \( \rho \) be a truncation function and \( m : \mathbb{N} \to \mathbb{N} \) be such that \( m(k) = \lfloor k^r \rfloor \) for all \( k \in \mathbb{N} \), with \( r \in (0,1) \) being the same as in (H.4). Then there exists a constant \( C \geq 0 \) such that for all \( \delta \in (0,1) \), the regret (2.4) of Algorithm 1 (without Step 7) satisfies with probability at least \( 1 - \delta \),

\[
R(N, \hat{\Psi}^{\text{PEGE}}, \hat{\theta}) \leq C \left( N^{1/r} \left( (\ln N)^r + \left( \ln \left( \frac{1}{\delta} \right) \right)^r \right) + \left( \ln \left( \frac{1}{\delta} \right) \right)^{1+r} \right) \quad \forall N \in \mathbb{N} \cap [2, \infty).
\]

(3.7)

Consequently, there exists a constant \( C \geq 0 \) such that \( \mathbb{E}[R(N, \hat{\Psi}^{\text{PEGE}}, \hat{\theta})] \leq CN^{1/r} (\ln N)^r \) for all \( N \in \mathbb{N} \cap [2, \infty) \).

Remark 3.3. As mentioned above, Algorithm 1 with Step 7 enjoys the same regret bounds. The proof follows essentially the steps in the arguments for Theorem 3.7. The only difference is that due to a more frequent control update, we sum over the regret of each exploitation episode in (6.8), instead of the regret for each cycle.

In the case where the greedy policies admit a self-exploration property and improve the accuracy of parameter estimation as in [5, 16], one can increase the number of exploitation episodes and obtain an improved regret bound. Note that if (H.4) holds with for \( r = 1/2 \) (cf. Proposition 3.4), then Theorem 3.8 recovers the sublinear regret \( O(\sqrt{N}) \) in [16] for general (nonsmooth) LC-RL problems, while if (H.4) holds with \( r = 1 \) (cf. Theorems 3.5 and 3.6), Theorem 3.8 generalises the logarithmic regret bound for LQ problems in [5] to smooth convex costs and entropy-regularised costs.

To simplify the presentation, we assume that Step 7 of Algorithm 1 is omitted, and estimate the regret (2.4) in expectation. However, a similar high-probability regret of expected costs can be found in the proof, whose details are given in Section 6.

**Theorem 3.8.** Suppose (H.1), (H.2), (H.3) and (H.4) hold. Let \( \hat{\theta}_0 \in \mathbb{R}^{d \times (d+p)} \), \( V_0 \in \mathbb{S}^{d+p} \) and \( \rho_{\mathcal{K}} \) be a truncation function. Assume further that there exists a constant \( \lambda_0 > 0 \) such that \( \Lambda_{\min}(\psi_{\theta'}, \hat{\theta}) \geq \lambda_0 \) for all \( \theta \in \Theta \) and \( \theta' \in \mathcal{K} \), with the function \( \Lambda_{\min} : \cup_{C \geq 0} \mathcal{V}_C \times \mathbb{R}^{d \times (d+p)} \to [0, \infty) \) defined in (3.4).
Then there exists a constant $C \geq 0$ such that if one sets $m(k) = 2^k$ for all $k \in \mathbb{N}$, then the regret (2.4) of Algorithm 1 satisfies for all $N \in \mathbb{N} \cap [2, \infty)$,

$$
\mathbb{E}[\mathcal{R}(N, \Psi^{\text{PEGE}}, \theta)] \leq \begin{cases} 
C N^{1-r} (\ln N)^r, & r \in (0, 1), \\
C (\ln N)^2, & r = 1,
\end{cases} \tag{3.8}
$$

with the constant $r \in (0, 1]$ in (H.4).

Remark 3.4. One can easily deduce from Proposition 1.1 and the stability of (1.8), $\mathbb{R}^{d \times (d+p)} \ni (\theta', \theta) \mapsto \Lambda_{\min}(\psi_{\theta}, \theta) \in [0, \infty)$ is continuous. Hence, a sufficient condition to ensure $\Lambda_{\min}(\psi_{\theta}, \theta) \geq \lambda_0$ for all $\theta \in \Theta$ and $\theta' \in \mathcal{K}$ is that for all $\theta' \in \text{cl}(\mathcal{K})$, the functions $(t, x) \mapsto \psi_{\theta'}(t, x)$ and $(t, x) \mapsto x$ are linearly independent as in (H.3(1)) (cf. Lemma 6.1). Moreover, thanks to the exploration episodes, the same regret bound holds under a weaker condition: there exist constants $\lambda_0, \eta > 0$ such that $\Lambda_{\min}(\psi_{\theta}, \theta) \geq \lambda_0$ for all $\theta \in \Theta$ and $\theta' \in \mathcal{B}_\eta(\Theta)$, with $\mathcal{B}_\eta(\Theta) := \{\theta' \in \mathbb{R}^{d \times (d+p)} \mid \exists \theta \in \Theta \text{ s.t. } |\theta' - \theta| \leq \eta \}$.

**Sketched proofs of Theorems 3.7 and 3.8.** Here we outline the key steps of the proofs of Theorems 3.7 and 3.8. For notational simplicity, we omit the dependence on “PEGE” in the superscripts. By Proposition 1.1, $\mathcal{R}(N, \Psi, \theta)$ admits the following decomposition:

$$
\mathcal{R}(N, \Psi, \theta) = \sum_{m=1}^{N} \left( \mathbb{E}[\ell_m(\Psi) - J(\Psi_m; \theta)] \right) + \sum_{m \in [1, N] \cap \mathcal{E}^\Psi} \left( J(\psi^{\hat{\theta}_m}; \theta) - J(\psi^{\theta}; \theta) \right) \tag{3.9}
$$

where $\mathcal{E}^\Psi$ is the collection of exploration episodes and $m^c$ is the last exploration episode before the $m$-th episode.

We now estimate the terms on the right-hand side of (3.9). The truncation of the MAP estimate and Proposition 1.1 ensure that $\Psi \in \mathcal{U}_C$ for some $C \geq 0$, based on which we prove the first term in (3.9) is a martingale with conditional sub-exponential differences. Hence it vanishes under expectation and can be bounded by $\mathcal{O}(\sqrt{N} \ln N)$ with high probability due to a general Bernstein’s inequality. By (H.2) and (H.3), the second term in (3.9) is of the order $\mathcal{O}(\kappa(N))$, where $\kappa(m)$ is the total number of exploration episodes up to the $m$-th episode.

The third term in (3.9) relies on the accuracy of the MAP estimate $(\hat{\theta}^{\Psi,m})_{m \in \mathbb{N}}$. For Theorem 3.7, we prove under (H.3) that there exists $\lambda_0 > 0$ such that

$$
\lambda_{\min}((V^{\theta, \Psi, m})^{-1}) \geq \lambda_{\min}\left( \sum_{n \in [1, m] \cap \mathcal{E}^\Psi} \int_0^T Z_i^{\theta, \Psi, m}(Z_i^{\theta, \Psi, n})^\top dt \right) \geq \kappa(m) \lambda_0,
$$

which along with Theorem 3.2 leads to the estimate that $|\hat{\theta}^{\Psi, m} - \theta|^2 = \mathcal{O}((\kappa(m))^{-1} \ln m)$ with high probability. Then, by exploiting properties of the truncation function $\rho$ and (H.4), we prove that $\hat{\theta}^{\Psi, m} = \theta^{\Psi, m}$ for all large $m$ and $J(\psi_{\hat{\theta}^{m^c}}; \theta) - J(\psi^{\theta}; \theta)$ admits a high probability bound $\mathcal{O}((\kappa(m^c))^{-r} (\ln m^c)^r)$. We then quantify the relation between $\kappa(m), m^c$ and $m$ based on the choice of $m$, and estimate the precise regret of Algorithm 1.

For Theorem 3.8, by exploiting the self-exploration property, we establish with high probability that $\lambda_{\min}((V^{\theta, \Psi, m})^{-1}) \geq m \lambda_0$ for sufficiently large $m$. This along with Theorem 3.2, the truncation function $\rho$ and (H.4) proves $J(\psi_{\hat{\theta}^{m^c}}; \theta) - J(\psi^{\theta}; \theta)$ is of the magnitude $\mathcal{O}((m^c)^{-r} (\ln m^c)^r)$. Consequently, we choose $m(k) = 2^k$ to emphasise exploitation and obtain improved regret bounds.
4 Proof of Theorem 3.2

This section analyses the accuracy of the MAP estimate \((\hat{\theta}^{\Psi, m})_{m=0}^{\infty}\) associated with a learning algorithm \(\Psi\) (cf. (3.2)). The definition of a learning algorithm \(\Psi\) implies that \(\hat{\theta}^{\Psi, m}\) is in general defined with correlated observations \((X^{\theta, \Psi, m})_{m=1}^{\infty}\). Hence one cannot analyse \(\hat{\theta}^{\Psi, m}\) based on concentration inequalities for independent random variables as in [5, 16]. In the subsequent analysis, we overcome this difficulty by introducing a notation of conditional sub-exponential random variables and studying its associated concentration inequalities.

4.1 Concentration inequality for conditional sub-exponential random variables

We start by defining conditional sub-Gaussian and sub-exponential random variables with a general \(\sigma\)-algebra. It is well-known that the sub-Gaussianity/sub-exponentiality of a random variable can be equivalently characterised by the finiteness of its corresponding Orlicz norms (see e.g., [39, 40]). The notion of Orlicz norms allows us to quantify the precise tail behaviour of a random variable by establishing an upper bound of its Orlicz norm, which is particularly important for continuous-time estimators defined via integrals of stochastic processes; see (3.2) and the least-squares estimators in [5, 16].

Motivated by the above applications, we first introduce the precise definition of a conditional Orlicz norm with respect to a general \(\sigma\)-algebra, as we are not aware of a standard notation in the existing literature.

Definition 4.1. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. For every \(q \in [1, 2]\) and \(\sigma\)-algebra \(\mathcal{G} \subseteq \mathcal{F}\), we define the \((q, \mathcal{G})\)-Orlicz norm \(\|\cdot\|_{q, \mathcal{G}} : \Omega \to [0, \infty]\) such that for any random variable \(X : \Omega \to \mathbb{R}\),

\[
\|X\|_{q, \mathcal{G}} := \mathcal{G}\text{-ess inf} \{Y \in L^0(\mathcal{G}; (0, \infty)) \mid \mathbb{E}[\exp(|X|^q / Y^q)|\mathcal{G}] \leq 2\},
\]

where \(L^0(\mathcal{G}; (0, \infty))\) is the set of all \(\mathcal{G}/\mathcal{B}((0, \infty))\)-measurable functions \(Y : \Omega \to (0, \infty)\).

Definition 4.1 is a natural extension of the classical Orlicz norms in [39, 40] with \(\mathcal{G} = \{\emptyset, \Omega\}\). Compared with the classical Orlicz norm, for any given random variable \(X\), \(\|X\|_{q, \mathcal{G}}\) is a \(\mathcal{G}\)-measurable function instead of a real number. The following lemma extends some basic properties of the classical Orlicz norm to the conditional Orlicz norm. The proof follows directly from Definition 4.1, the monotonicity and convexity of \([0, \infty) \ni x \mapsto \exp(x^q) \in \mathbb{R}\) for all \(q \in [1, 2]\), and hence is omitted.

Lemma 4.1. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Then for all \(q \in [1, 2]\), \(\sigma\)-algebra \(\mathcal{G} \subseteq \mathcal{F}\), random variables \(X, Y : \Omega \to \mathbb{R}\) and \(c \in \mathbb{R}\),

\[
\text{if } X \text{ is } \mathcal{G}\text{-measurable, then } \|X\|_{q, \mathcal{G}} \leq (\ln 2)^{-1/q} |X|, \quad (4.1) \\
\|cX\|_{q, \mathcal{G}} = |c| \|X\|_{q, \mathcal{G}}, \quad \|X + Y\|_{q, \mathcal{G}} \leq \|X\|_{q, \mathcal{G}} + \|Y\|_{q, \mathcal{G}}, \quad (4.2) \\
\text{if } |X| \leq |Y|, \text{ then } \|X\|_{q, \mathcal{G}} \leq \|Y\|_{q, \mathcal{G}}, \quad (4.3) \\
\|X - \mathbb{E}[X | \mathcal{G}]\|_{q, \mathcal{G}} \leq 2 \|X\|_{q, \mathcal{G}}. \quad (4.4)
\]

\(^4\)Let \((\Omega, \mathcal{G}, \mathbb{P})\) be a probability space and \(\mathcal{Y}\) be a family of \(\mathcal{G}/\mathcal{B}(\mathbb{R})\)-measurable functions. Then there exists a function \(Z : \Omega \to [-\infty, \infty]\), called the \(\mathcal{G}\)-essential infimum of \(\mathcal{Y}\) and denoted by \(Z = \mathcal{G}\text{-ess inf } \mathcal{Y}\), such that

(1) \(Z\) is \(\mathcal{G}/\mathcal{B}([-\infty, \infty])\)-measurable and \(Z \leq Y \text{ } \mathbb{P}\text{-a.s.} \text{ for all } Y \in \mathcal{Y}\),

(2) \(Z \geq Z' \text{ } \mathbb{P}\text{-a.s.} \text{ for all functions } Z' \text{ satisfying property (1)}.

If \(\mathcal{Y}\) is directed downwards, that is for \(Y, Y' \in \mathcal{Y}\), there exists \(\tilde{Y} \in \mathcal{Y}\) such that \(\tilde{Y} \leq \min(Y, Y')\), then there exists a decreasing sequence \((Y_n)_{n \in \mathbb{N}} \subseteq \mathcal{Y}\) such that \(\lim_{n \to \infty} Y_n = \mathcal{G}\text{-ess inf } \mathcal{Y} \mathbb{P}\text{-a.s.} \text{ (see e.g., [8, Theorem 1.3.40])}.

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The following proposition establishes the relation between the conditional tail behaviour and the \((q, \mathcal{G})\)-Orlicz norms of the random variable. The proof essentially follows from the lines of [39, Propositions 2.5.2 and 2.7.1], along with Definition 4.1 and Lemma 4.1, and hence is omitted.

**Proposition 4.2.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Then for all \(q \in [1, 2]\), \(\sigma\)-algebra \(\mathcal{G} \subseteq \mathcal{F}\) and integrable random variable \(X : \Omega \to \mathbb{R}\),

1. \(\|X - \mathbb{E}[X | \mathcal{G}]\|_{2, \mathcal{G}} \leq C_1\) if and only if
   \[
   \mathbb{E}[\exp(\gamma(X - \mathbb{E}[X | \mathcal{G}])) | \mathcal{G}] \leq \exp(C_1^2 \gamma^2), \quad \forall \gamma \in \mathbb{R},
   \]
2. \(\|X - \mathbb{E}[X | \mathcal{G}]\|_{1, \mathcal{G}} \leq C_1\) if and only if
   \[
   \mathbb{E}[\exp(\gamma(X - \mathbb{E}[X | \mathcal{G}])) | \mathcal{G}] \leq \exp ((C_1^2 \gamma)^2), \quad \forall |\gamma| \leq 1/C_1',
   \]

where \(C_1, C_1' \geq 0\) differ by an absolute multiplicative factor.

In the sequel, we say an integrable random variables \(X : \Omega \to \mathbb{R}\) is \(\mathcal{G}\)-sub-Gaussian (resp. \(\mathcal{G}\)-sub-exponential) if it satisfies (4.5) (resp. (4.6)) with a constant \(C_1' \geq 0\). By (4.4) and Proposition 4.2, \(X\) is \(\mathcal{G}\)-sub-Gaussian if \(\|X\|_{2, \mathcal{G}} \leq C\) \(\mathbb{P}\) a.s. for some \(C \geq 0\), and is \(\mathcal{G}\)-sub-exponential if \(\|X\|_{1, \mathcal{G}} \leq C\) \(\mathbb{P}\) a.s. for some \(C \geq 0\).

The next proposition estimates the conditional Orlicz norm of deterministic integrals of stochastic processes.

**Proposition 4.3.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Then for all measurable functions \(X : \Omega \times [0, T] \to \mathbb{R}^d\), all measurable functions \(\psi : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}\) and \(\sigma\)-algebras \(\mathcal{G} \subseteq \mathcal{F}\),

\[
\left\| \int_0^T \psi(\cdot, t, X_t) \, dt \right\|_{1, \mathcal{G}} \leq \|\psi\|_{2, \infty} \left( \frac{T}{\ln 2} + \left( \int_0^T |X_t|^2 \, dt \right)^{1/2} \right),
\]

with \(\|\psi\|_{2, \infty} = \sup_{(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d} \|\psi(\omega, t, x)\|_{1 + |x|^2}\).

**Proof.** As \(\| \int_0^T \psi(t, X_t) \, dt \| \leq \|\psi\|_{2, \infty} (T + \int_0^T |X_t|^2 \, dt)\), by (4.2) and (4.3), \(\| \int_0^T \psi(t, X_t) \, dt \|_{1, \mathcal{G}} \leq \|\psi\|_{2, \infty} (||T||_{1, \mathcal{G}} + \int_0^T |X_t|^2 \, dt \|_{1, \mathcal{G}})\). A direct computation shows that \(||T||_{1, \mathcal{G}} \leq T/\ln 2\). We then show \(\| \int_0^T |X_t|^2 \, dt \|_{1, \mathcal{G}} \leq \left( \int_0^T |X_t|^2 \, dt \right)^{1/2} \|2, \mathcal{G}\). By the definition of \(\| \cdot \|_{2, \mathcal{G}}\), there exists \((Y_n)_{n \in \mathbb{N}} \subseteq L^0(\mathcal{G} ; (0, \infty))\) such that \(\lim_{n \to \infty} Y_n = \left( \int_0^T |X_t|^2 \, dt \right)^{1/2} \|_{2, \mathcal{G}}\) and \(\mathbb{E}[\exp(\int_0^T |X_t|^2 \, dt / Y_n^2 | \mathcal{G})] \leq 2\) for all \(n\) (see [8, Theorem 1.3.40]). This implies that \(\| \int_0^T |X_t|^2 \, dt \|_{1, \mathcal{G}} \leq Y_n^2\) for all \(n\). Passing \(n\) to infinity leads to the desired estimate. \(\square\)

We end this section with a general Bernstein-type bound for a sub-exponential martingale difference sequence, whose proof follows from [40, Theorem 2.19] and (4.6).

**Proposition 4.4.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \(\mathbb{F} = \{\mathcal{F}_n\}_{n=0}^{\infty} \subseteq \mathcal{F}\) be a filtration, \(C \geq 0\), and \((D_n)_{n \in \mathbb{N}}\) be an \(\mathbb{F}\)-adapted sequence of random variables such that for all \(n \in \mathbb{N}\), \(\mathbb{E}[D_n | \mathcal{F}_{n-1}] = 0\) and \(\|D_n\|_{1, \mathcal{F}_{n-1}} \leq C\). Then there exists a constant \(C' > 0\), depending only on \(C\), such that for all \(N \in \mathbb{N}\) and \(\varepsilon > 0\), \(\mathbb{P}(\| \sum_{n=1}^{N} D_n \|_{\mathcal{G}} \geq N\varepsilon) \leq 2 \exp (-C'N \min(\varepsilon^2, \varepsilon))\).
4.2 Error bound of maximum a posterior estimate

In this section, we consider the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) defined in (2.1) and analyse the MAP estimate \((\hat{\theta}^{\Psi,m})_{m \in \mathbb{N}}\) defined in (3.2). We first establish an error bound of \((\hat{\theta}^{\Psi,m})_{m \in \mathbb{N}}\) associated with a general learning algorithm \(\Psi\) (cf. Definition 2.1).

**Lemma 4.5.** Let \(x_0 \in \mathbb{R}^d, \hat{\theta}_0 \in \mathbb{R}^{d \times (d+p)}, V_0 \in \mathbb{S}_+^{d+p}, \Psi\) be a learning algorithm as in Definition 2.1, and for each \(m \in \mathbb{N}\), let \(Z_t^{\theta,\Psi,m}\) be defined in (2.2), \(V_t^{\theta,\Psi,m}\) be defined in (3.2) and \(G_t^{\theta,\Psi,m} = (V_t^{\theta,\Psi,m})^{-1}\). Then for all \(\delta \in (0,1)\) and \(m \in \mathbb{N}\),

\[
\mathbb{P}\left(\left\|\sum_{n=1}^m \left(\int_0^T Z_{t}^{\theta,\Psi,m}(dW_t^{m})\right)\right\|_{V_t^{\theta,\Psi,m}}^2 \leq 2 \ln \left(\frac{\det G_{t}^{\theta,\Psi,m} \cdot \det V_0}{\delta}\right)^{d/2}\right) \geq 1 - \delta,
\]

where \(\|S\|_V := \text{tr}(SVS^T)\) for all \(S \in \mathbb{R}^{d \times (d+p)}\) with \(V \in \mathbb{R}^{(d+p) \times (d+p)}\).

**Proof.** We first concatenate stochastic processes from all episodes into infinite-horizon processes. Let \(W : \Omega \times [0, \infty) \to \mathbb{R}^d\) be the cumulative concatenation of the Brownian motions \((W_t^m)_{t \in [0,T],m \in \mathbb{N}}\) and let \(Z : \Omega \times [0, \infty) \to \mathbb{R}^{d \times (d+p)}\) be the concatenation of \((Z_{t}^{\theta,\Psi,m})_{t \in [0,T],m \in \mathbb{N}}\) such that

\[
W_t := W_{t + \lfloor t/T \rfloor T} + \sum_{n=1}^{\lfloor t/T \rfloor} W_n^m, \quad Z_t := Z_{t + \lfloor t/T \rfloor T}^{\theta,\Psi,1}, \quad \forall t \geq 0,
\]

Note that \(W\) is a standard Brownian motion with respect to the filtration \(\mathcal{F}_t := \sigma(\theta, W_u | u \leq t)\).

Let \(S_t := \left(\int_0^t Z_s(dW_s)^T\right), V_t := (V_0^{-1} + \int_0^t Z_sZ_s^T ds)^{-1}\) and \(G_t := V_t^{-1}\) for all \(t \geq 0\). By (3.2), \(S_{mT} = \sum_{n=1}^m \left(\int_0^T Z_{t}^{\theta,\Psi,m}(dW_t^{m})\right), V_{mT} = V_{t}^{\theta,\Psi,m}\) and \(G_{mT} = G_{t}^{\theta,\Psi,m}\) \(\mathbb{P}\text{-a.s.}\). Hence it suffices to prove the required result by considering \(W, S, G, V\).

Observe that \((G_t - V_0^{-1})_{t \geq 0}\) is the quadratic variation process \(\langle S \rangle_t\) of \((S_t)_{t \geq 0}\). Based on this observation, we express \(\exp\left(\frac{1}{2} \|S_t\|_{V_t}^2\right)\) in terms of integrals of suitable exponential martingales. To this end, for each \(U \in \mathbb{S}^{d+p}_+\), let \(c(U) := \int_{\mathbb{R}^{d \times (d+p)}} \exp\left(-\frac{1}{2} \|\theta\|_U^2\right) d\theta = \sqrt{(2\pi)^{d+d(p)}/(\det U)}\) be the normalising constant corresponding to the density of the matrix normal distribution \(\mathcal{MN}(0, I_d, U^{-1})\). By completing the square in the matrix Gaussian density, for all \(t \geq 0\),

\[
\exp\left(\frac{1}{2} \|S_t\|_{V_t}^2\right) = \frac{1}{c(G_t)} \int \exp\left(\frac{1}{2} \|S_t\|_{G_t}^2 - \frac{1}{2} \|\theta - G_t^{-1}S_t\|_{G_t}^2\right) d\theta
\]

\[
= \frac{1}{c(G_t)} \int \exp\left(\left(\text{tr}(\theta^T S_t) - \frac{1}{2} \|\theta\|_{G_t}^2\right) d\theta = \frac{1}{c(G_t)} \int \exp\left(\left(\text{tr}(\theta^T S_t) - \frac{1}{2} \|\theta\|_{V_t}^2 - \frac{1}{2} \|\theta\|_{V_0^{-1}}^2\right) d\theta
\]

\[
= \frac{1}{c(G_t)} \int M_t^\theta \exp\left(-\frac{1}{2} \|\theta\|_{V_0^{-1}}^2\right) d\theta = \frac{\det G_t \cdot \det V_0^{(d+p)}/(\det U)}{c(V_0^{-1})} \int M_t^\theta \exp\left(-\frac{1}{2} \|\theta\|_{V_0^{-1}}^2\right) d\theta,
\]

where \(\int\) denotes the integration over \(\mathbb{R}^{d \times (d+p)}\) and for each \(\theta \in \mathbb{R}^{d \times (d+p)}\), \(M_t^\theta\) is an exponential martingale defined by

\[
M_t^\theta := \exp\left(\left(\text{tr}(\theta^T S_t) - \frac{1}{2} \|\theta\|_{V_t}^2\right), \quad \forall t \geq 0.
\]

As \(W\) is a standard Brownian motion, by Itô’s formula, \(M_t^\theta\) is a non-negative local martingale with respect to \((\mathcal{F}_t)_{t \geq 0}\) and hence a supermartingale. In particular, for all \(t \geq 0\) and \(\theta \in \mathbb{R}^{d \times (d+p)}\),

\[
\mathbb{E}[M_0^\theta] \leq \mathbb{E}[M_0^\theta] = 1.
\]
Therefore, by Markov’s inequality and Fubini’s theorem,
\[
\Pr\left(\|S_t\|_{\bar{V}_t} > 2 \log \left( \frac{(\det G_t \cdot \det V_0)^{d/2}}{\delta} \right) \right) = \Pr\left( \exp \left( \frac{1}{2} \|S_t\|_{\bar{V}_t} \right) > \frac{(\det G_t \cdot \det V_0)^{d/2}}{\delta} \right)
\]
\[
\leq \delta \mathbb{E}\left[ \exp\left( \frac{1}{2} \|S_t\|_{\bar{V}_t}^2 \right) \right] = \delta \mathbb{E}\left[ \frac{1}{c(V_0^{-1})} \int \mathcal{M}_t^\theta \exp \left( -\frac{1}{2} \|\theta\|_{V_0^{-1}}^2 \right) d\theta \right]
\]
\[
= \frac{\delta}{c(V_0^{-1})} \int \mathbb{E}[\mathcal{M}_t^\theta] \exp \left( -\frac{1}{2} \|\theta\|_{V_0^{-1}}^2 \right) d\theta \leq \frac{\delta}{c(V_0^{-1})} \int \exp \left( -\frac{1}{2} \|\theta\|_{V_0^{-1}}^2 \right) d\theta = \delta.
\]
where the last inequality follows from the fact that \( \mathbb{E}[\mathcal{M}_t^\theta] \leq 1 \) for all \( t \geq 0 \). Substituting \( t = mT \) in the above inequality leads to the desired estimate.

**Proposition 4.6.** Suppose (H.2) holds. Let \( x_0 \in \mathbb{R}^d \), \( T > 0 \), \( \hat{\theta}_0 \in \mathbb{R}^{d \times (d+p)} \), \( V_0 \in \mathbb{S}_+^{d+p} \), \( \Psi \) be a learning algorithm as in Definition 2.1, and for each \( m \in \mathbb{N} \), let \( \hat{\theta}^{\Psi,m} \) and \( G^{\theta,\Psi,m} := (V^{\theta,\Psi,m})^{-1} \) where \((\hat{\theta}^{\Psi,m}, V^{\theta,\Psi,m}) \) is defined in (3.2). There exists a constant \( C \geq 0 \) such that for all \( \delta \in (0,1) \) and \( m \in \mathbb{N} \),
\[
\Pr\left( \lambda_{\min}(G^{\theta,\Psi,m})|\hat{\theta}^{\Psi,m} - \theta|^2 \leq C \left( 1 + \ln(\det G^{\theta,\Psi,m} \cdot \det V_0) + \ln\left( \frac{1}{\delta} \right) \right) \right) \geq 1 - \delta,
\]
where \( \lambda_{\min}(S) \) is the smallest eigenvalue of a matrix \( S \in \mathbb{S}_+^{d+p} \).

**Proof.** By (2.2) and (3.2),
\[
\hat{\theta}^{\Psi,m} - \theta = \left( \hat{\theta}_0 V_0^{-1} + \sum_{n=1}^m \left( \int_0^T Z_t^{\theta,\Psi,n} (dX_t^{\theta,\Psi,n})^\top \right) \right) (V^{\theta,\Psi,m} - \theta)
\]
\[
= \left( \hat{\theta}_0 V_0^{-1} + \sum_{n=1}^m \left( \int_0^T Z_t^{\theta,\Psi,n} (\theta Z_t^{\theta,\Psi,n} dt + dW_t^n)^\top \right) \right) (G^{\theta,\Psi,m})^{-1} - \theta
\]
\[
= \left( \hat{\theta}_0 - \theta \right) V_0^{-1} + \sum_{n=1}^m \left( \int_0^T Z_t^{\theta,\Psi,n} (dW_t^n)^\top \right) \right) (G^{\theta,\Psi,m})^{-1}.
\]
Right multiplying the above identity by \( G^{\theta,\Psi,m}(\hat{\theta}^{\Psi,m} - \theta)^\top \), taking the trace and using the fact that \( V^{\theta,\Psi,m} = (G^{\theta,\Psi,m})^{-1} \) give us that
\[
\|\hat{\theta}^{\Psi,m} - \theta\|_{G^{\theta,\Psi,m}}^2 = \left\| (\hat{\theta}_0 - \theta) V_0^{-1} + \sum_{n=1}^m \left( \int_0^T Z_t^{\theta,\Psi,n} (dW_t^n)^\top \right) \right\|^2_{V^{\theta,\Psi,m}}
\]
\[
\leq \left\| (\hat{\theta}_0 - \theta) V_0^{-1} \right\|_{V^{\theta,\Psi,m}} + \left\| \sum_{n=1}^m \left( \int_0^T Z_t^{\theta,\Psi,n} (dW_t^n)^\top \right) \right\|^2_{V^{\theta,\Psi,m}},
\]
where the last inequality follows from the fact that for all \( V \in \mathbb{S}_+^{d+p} \), \( S \mapsto \|S\|_V = \sqrt{\text{tr}(SVS^\top)} \) is a norm on \( \mathbb{R}^{d \times (d+p)} \). By (3.2), \( (V^{\theta,\Psi,m})^{-1} - V_0^{-1} \) is symmetric positive semidefinite, and hence \( V_0 - (V^{\theta,\Psi,m})^{-1} \) is symmetric positive semidefinite. Consequently, for all \( S \in \mathbb{R}^{d \times (d+p)} \), \( \|S\|_{V^{\theta,\Psi,m}} \leq \|S\|_{V_0} \). The desired estimate then follows from the boundedness of \( \theta \) (cf. (H.2)), Lemma 4.5 and the fact that there exists \( C \geq 0 \) such that \( \lambda_{\min}(V)|S|^2 \leq C\|S\|_V^2 \) for all \( S \in \mathbb{R}^{d \times (d+p)} \) and \( V \in \mathbb{S}_+^{d+p} \).

\[
\Box
\]
We then focus on learning algorithms of the class $\cup_{L \geq 0} \mathcal{U}_L$ and obtain an upper bound of $\det G^{\theta, \Psi, m}$. By extending [12, Corollary 4.1] to SDEs with random coefficients, we prove Lipschitz functionals of state processes $X^{\theta, \Psi, m}$ are conditional sub-Gaussian uniformly with respect to $m$.

**Proposition 4.7.** Suppose (H.2) holds. Let $x_0 \in \mathbb{R}^d$, $T, L \geq 0$, $\Psi$ be a learning algorithm in $\mathcal{U}_L$ defined as in (3.3), and for each $m \in \mathbb{N}$, let $X^{\theta, \Psi, m}$ be defined in (2.2). Then there exists a constant $C \geq 0$, depending only on $x_0, T$ and $L$, such that for all $m \in \mathbb{N}$ and $\phi : C([0, T]; \mathbb{R}^d) \to \mathbb{R}$ with $\sup_{\rho_1 \neq \rho_2} \frac{\|\phi(\rho_1) - \phi(\rho_2)\|}{\|\rho_1 - \rho_2\|} \leq 1$ and $|\phi(0)| \leq 1$, we have $\|\phi(X^{\theta, \Psi, m})\|_{2, \mathcal{F}_{m-1}} \leq C$, where $\mathcal{F}_{m-1} := \sigma\{\theta, W^n_t \mid t \in [0, T], n = 1, \ldots, m - 1\} \vee \mathcal{N}$ with $\mathcal{N}$ being the $\sigma$-algebra generated by $\mathbb{P}$-null sets.

**Proof.** Throughout this proof, let $m \in \mathbb{N}$ and the function $\phi : C([0, T]; \mathbb{R}^d) \to \mathbb{R}$ be fixed, and $C$ be a generic constant depending only on $x_0, T, L$.

By (2.2), $X^{\theta, \Psi, m}$ satisfies the dynamics

$$
dX_t = b_m(\cdot, t, X_t) \, dt + dW^m_t, \quad t \in [0, T], \quad X_0 = x_0, \tag{4.7}$$

with $b_m(\omega, t, x) := A(\omega)x + B(\omega)\Psi_m(\omega, t, x)$ for all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d$. The boundedness of $\Theta$ and $\Psi_m(\omega, \cdot) \in V_L$ give us that $|b_m(\omega, t, 0)| \leq C$ and $|b_m(\omega, t, x) - b_m(\omega, t, x')| \leq C|x - x'|$. The measurability of $\Psi_m$ implies that $b_m$ is $(\mathcal{F}_{m-1} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d))/\mathcal{B}(\mathbb{R}^d)$-measurable. By the definition of $\mathcal{F}_{m-1}$ and the Doob–Dynkin lemma, there exists a Borel measurable function $b_m : \Theta \times C([0, T]; \mathbb{R}^d) \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that $b_m(\omega, t, x) = b_m(\theta(\omega), W^1(\omega), \ldots, W^{m-1}(\omega), t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\mathbb{P}$-almost all $\omega \in \Omega$.

The definition of $(\Omega, \mathcal{F}, \mathbb{P})$ in (2.1) and $W^m$ is the canonical process on $\Omega^W = C([0, T]; \mathbb{R}^d)$ (see Section 2) imply that all $\omega \in \Omega$ can be uniquely decomposed into $\omega = (\theta, \omega, \varpi)$ with $\theta = \theta(\omega) \in \Theta$, $\omega = (W^n(\omega))_{n=1}^{m-1} \in \bigotimes_{n=1}^{m-1} \Omega^W$ and $\varpi = (\rho^n(\omega))_{n=m}^{\infty} \in \bigotimes_{n=m}^{\infty} \Omega^W$. Moreover, for each $\omega \in \bigotimes_{n=1}^{m-1} \Omega^W$, let $X^{\theta, \Psi, m, \omega}$ be the unique solution to the following dynamics

$$
dX_t = b_m(\theta, \omega, t, X_t) \, dt + dW^m_t, \quad t \in [0, T], \quad X_0 = x_0, \tag{4.8}$$

on the space $(\bigotimes_{n=1}^{\infty} \Omega^{W_n}, \bigotimes_{n=1}^{\infty} \mathcal{F}^{W_n}, \bigotimes_{n=1}^{\infty} \mathbb{P}^{W_n})$. Applying [12, Corollary 4.1] to the law of $X^{\theta, \Psi, m, \omega}$ shows that there exists $C \geq 0$ such that for all $\omega \in \bigotimes_{n=1}^{m-1} \Omega^W$ and $\gamma \in \mathbb{R}$,

$$
\mathbb{E}\bigotimes_{n=1}^{\infty} \mathbb{P}^{W_n}\left[ \exp \left( \gamma(\phi(X^{\theta, \Psi, m, \omega}) - \mathbb{E}\bigotimes_{n=1}^{\infty} \mathbb{P}^{W_n}\left[ \phi(X^{\theta, \Psi, m, \omega}) \right]) \right) \right] \leq \exp(C \gamma^2). \tag{4.9}
$$

By (4.7) and the definition of $b_m$, $X^{\theta, \Psi, m}((\theta, \omega, \varpi)) = X^{\theta, \Psi, m, \omega}(\varpi)$ $\mathbb{P}$-almost all $\omega = (\theta, \omega, \varpi)$. It then follows from the independence between $\theta$, $(W^n)_{n=1}^{m-1}$ and $(W^n)_{n=m}^{\infty}$, [22, Theorem 13, p 76] and (4.9) that

$$
\mathbb{E}\left[ \exp \left( \gamma(\phi(X^{\theta, \Psi, m}) - \mathbb{E}[\phi(X^{\theta, \Psi, m}) \mid \mathcal{F}_{m-1}]) \right) \mid \mathcal{F}_{m-1} \right] \leq \exp(C \gamma^2), \quad \forall \gamma \in \mathbb{R}.
$$

By (4.1), (4.2) and (4.5), $\|\phi(X^{\theta, \Psi, m})\|_{2, \mathcal{F}_{m-1}} \leq C(1 + \mathbb{E}[\|\phi(X^{\theta, \Psi, m})\| \mid \mathcal{F}_{m-1}])$. Standard moment estimate of (4.7) shows that $\mathbb{E}[\sup_{t \in [0, T]} X^{\theta, \Psi, m}_{t}^2 \mid \mathcal{F}_{m-1}] \leq C$, which along the growth condition of $\phi$ implies the desired sub-Gaussianity of $\phi(X^{\theta, \Psi, m})$. \hfill \Box

**Proof of Theorem 3.2.** For each $m \in \mathbb{N} \cup \{0\}$, let $\mathcal{F}_m := \sigma\{\theta, W^n_t \mid t \in [0, T], n = 1, \ldots, m\} \vee \mathcal{N}$, with $\mathcal{N}$ being the $\sigma$-algebra generated by $\mathbb{P}$-null sets, and let $(\theta^{\Psi, m}, V^{\theta^{\Psi, m}})$ be defined in (3.2), $G^{\theta^{\Psi, m}} = (V^{\theta^{\Psi, m}})^{-1}$, and let $C$ be a generic constant independent of $m$.

Applying Proposition 4.7 with $C([0, T]; \mathbb{R}^d) \ni \rho \mapsto \phi(\rho) := \frac{1}{\sqrt{m}}(\int_0^T |\rho_t|^2 \, dt)^{\frac{1}{2}} \in \mathbb{R}$ yields that $\|\int_0^T X^{\theta^{\Psi, m}}_t \, dt\|_{2, \mathcal{F}_{m-1}} \leq C$. The fact that $\Psi \in \cup_{L \geq 0} \mathcal{U}_L$ implies that there exists $L \geq 0$ such
that $|Ψ_m(ω,t,x)| \leq L(1 + |x|)$ for all $m \in \mathbb{N}$ and $(ω,t,x) ∈ Ω × [0,T] × \mathbb{R}^d$, which along with Proposition 4.3 implies that

$$\left\| \int_0^T Z_t^{θ,Ψ,m} (Z_t^{θ,Ψ,m})^\top dt \right\|_{1,F_{m-1}} = \left\| \int_0^T \left( X_t^{θ,Ψ,m} \right) \left( Ψ_m(·,t,x)^{θ,Ψ,m} \right)^\top dt \right\|_{1,F_{m-1}} \leq C,$$

with $Z_t^{θ,Ψ,m}$ defined as in (2.2). Hence, by (4.4) and Proposition 4.4, there exists a constant $C ≥ 0$ such that for all $m \in \mathbb{N}$ and $δ ∈ (0,1)$, with probability at least $1 − δ$,

$$\left| \sum_{n=1}^m \left( \int_0^T Z_t^{θ,Ψ,n} (Z_t^{θ,Ψ,n})^\top dt - E \left[ \int_0^T Z_t^{θ,Ψ,n} (Z_t^{θ,Ψ,n})^\top dt \mid F_{n-1} \right] \right) \right| \leq m \max \left( \frac{\ln(2/δ)}{C_m}, \frac{\ln(2/δ)}{C_m} \right) \leq C(m + \ln(\frac{1}{δ})).$$

As shown in the proof of Proposition 4.7, $E[\sup_{t∈[0,T]} |X_t^{θ,Ψ,m}|^2 \mid F_{m-1}] ≤ C$, which along with the fact that $Ψ_m(ω,·) ∈ \mathcal{V}_L$ for all $ω ∈ Ω$ implies that $\sum_{n=1}^m E[\sup_{t∈[0,T]} (X_t^{θ,Ψ,n} (Z_t^{θ,Ψ,n})^\top dt \mid F_{n-1}] \leq Cm$. Consequently, by (3.2) and the fact that $G^{θ,Ψ,m} = (V^{θ,Ψ,m})^{-1}$, there exists a constant $C ≥ 0$ such that for all $m \in \mathbb{N}$ and $δ > 0$, $P( |G^{θ,Ψ,m}| ≤ C(m + \ln(\frac{1}{δ})) > 1 − δ$. Then by Proposition 4.6 and the inequality $\det(G^{θ,Ψ,m}) ≤ C|G^{θ,Ψ,m}|$, we have for all $m \in \mathbb{N}$ and $δ > 0$,$$P(\lambda_{\min}(G^{θ,Ψ,m})|\dot{θ}^{Ψ,m} - θ|^2 ≤ C(1 + \ln m + \ln(\frac{1}{δ})) > 1 − 2δ.$$

The desired estimate in Theorem 3.2 with $m ≥ 2$ then follows from the fact that $\ln 2 > 0$. □

5 Proofs of Proposition 3.3 and Theorems 3.5 and 3.6

Proof of Proposition 3.3. For the if direction, let $A := \{a_1, \ldots, a_p\} ⊂ \text{dom}(h)$ be a collection of linearly independent vector in $\mathbb{R}^p$, $ψ^e$ be defined as in the statement, and $(u, v) ∈ \mathbb{R}^d × \mathbb{R}^p$ be such that $u^\top x + v^\top ψ^e(t,x) = 0$ for almost every $(t,x) ∈ [0,T] × \mathbb{R}^d$. By the boundedness of $A$ and the definition of $ψ^e$, $u = 0$ and $v ∈ \mathbb{R}^p$ is orthogonal to the space span(A) spanned by $A$. Then the linear dependence of $\{a_i\}_{i=1}^p$ implies span(A) = $\mathbb{R}^p$ and hence $v = 0$. It is clear that $ψ^e ∈ ΨC ≥ 0V_C$, and (H.3(2)) follows directly from $A ⊂ \text{dom}(h)$ and the quadratic growth of $f_0$ in (H.1(1)).

For the only if direction, observe that any $ψ^e ∈ ΨC ≥ 0V_C$ satisfying (H.3(2)) takes values in $\text{dom}(h)$ a.e. Hence, if (H.3(1)) holds, then the space span($\text{dom}(h)$) spanned by $\text{dom}(h)$ must have dimension $p$, since otherwise there exists a non-zero $v ∈ \mathbb{R}^p$ orthogonal to $\text{dom}(h)$ such that for all $ψ^e$ satisfying (H.3(2)), $v^\top ψ^e(t,x) = 0$ for almost every $(t,x) ∈ [0,T] × \mathbb{R}^d$. Reducing $\text{dom}(h)$ to a basis for span($\text{dom}(h)$) shows that $\text{dom}(h)$ contains $p$ linearly independent vectors. □

Proof of Theorem 3.5. Throughout this proof, let $θ_0 = (A_0, B_0) ∈ Θ$ and $β > 0$ be given constants, and $J(·; θ_0) : H^2_F(Ω; \mathbb{R}^p) → \mathbb{R}$ be such that

$$J(α; θ_0) := E \left[ \int_0^T f(t, X_t^{θ_0,α}, α_t) dt + g(X_T^{θ_0,α}) \right], \quad ∀α ∈ H^2_F(Ω; \mathbb{R}^p),$$

where $X_T^{θ_0,α} ∈ S^2_β(Ω; \mathbb{R}^d)$ is the strong solution to

$$dX_t = (A_0X_t + B_0α_t) dt + dW_t, \quad t ∈ [0,T]; \quad X_0 = x_0. \quad (5.1)$$
We denote by $C \geq 0$ a generic constant which depends on $\Theta, \beta$ but is independent of $\theta_0$.

By the differentiability of $f$ and $g$ and the standard variational analysis (see e.g., [42, 37]), $J(\cdot; \theta_0)$ is Fréchet differentiable and its derivative satisfies for all $\alpha \in H^2_d(\Omega; \mathbb{R}^p)$,

$$\nabla J(\alpha; \theta_0)_t := B_0^\top Y_{t,0}^{\theta_0,\alpha} + (\partial_t f)(t, X_t^{\theta_0,\alpha}, \alpha_t), \quad d\mathbb{P} \otimes dt \text{ a.e.,} \quad (5.2)$$

where $(Y^{\theta_0,\alpha}, Z^{\theta_0,\alpha}) \in S^2_d(\Omega; \mathbb{R}^d) \times H^2_d(\Omega; \mathbb{R}^{d\times d})$ is the unique solution to

$$dY_t = -(A_0 Y_t + (\partial_x f)(t, X_t^{\theta_0,\alpha}, \alpha_t)) dt + Z_t dW_t, \quad t \in [0, T]; \quad Y_T = (\nabla g)(X_T^{\theta_0,\alpha}). \quad (5.3)$$

Standard stability analysis of (5.1) shows that $\|X^{\theta_0,\alpha} - X^{\theta_0,\alpha'}\|_{S^2} \leq C\|\alpha - \alpha'\|_{H^2}$ for all $\alpha, \alpha' \in H^2_d(\Omega; \mathbb{R}^p)$, which along with the Lipschitz continuity of $\partial_x f$ and $\nabla g$ and the stability of (5.3) in [14, Proposition 2.1] gives $\|Y^{\theta_0,\alpha} - Y^{\theta_0,\alpha'}\|_{S^2} \leq C\|\alpha - \alpha'\|_{H^2}$. Hence, by (5.2) and the Lipschitz continuity of $\partial_t f, \nabla J(\cdot; \theta_0): H^2_d(\Omega; \mathbb{R}^p) \to H^2_d(\Omega; \mathbb{R}^p)$ is Lipschitz continuous. Hence, there exists $C \geq 0$ such that $\alpha, \alpha' \in H^2_d(\Omega; \mathbb{R}^p)$,

$$J(\alpha'; \theta_0) - J(\alpha; \theta_0) - \langle \nabla J(\alpha; \theta_0), \alpha' - \alpha \rangle_{H^2}$$

$$= \int_0^1 \langle \nabla J(\alpha + s(\alpha' - \alpha); \theta_0) - \nabla J(\alpha; \theta_0), \alpha' - \alpha \rangle_{H^2} ds \leq C\|\alpha' - \alpha\|_{H^2}$$

(5.4)

where $\langle \cdot, \cdot \rangle_{H^2}$ denotes the inner product on $H^2_d(\Omega; \mathbb{R}^p)$.

For each $\theta \in B(\theta_0), $ let $X_{\theta_0, \theta}^{\psi_\theta} \in S^2_d(\Omega; \mathbb{R}^d)$ be the state process associated with $\theta_0$ and $\psi_\theta$, and let $\alpha^{\theta_0, \theta} \in H^2_d(\Omega; \mathbb{R}^p)$ be such that $\alpha^{\theta_0, \theta} = \psi_\theta(t, X_{\theta_0, \theta}^{\psi_\theta})$ for $d\mathbb{P} \otimes dt$ a.s. For any given $\theta \in B(\theta_0)$, substituting $\alpha' = \alpha^{\theta_0, \theta}$ and $\alpha = \alpha^{\theta_0, \theta_0}$ into (5.4) leads to

$$J(\alpha^{\theta_0, \theta}; \theta_0) - J(\alpha^{\theta_0, \psi_\theta}; \theta_0) - \langle \nabla J(\alpha^{\theta_0, \psi_\theta}; \theta_0), \alpha^{\theta_0, \psi_\theta} - \alpha^{\theta_0, \theta} \rangle_{H^2} \leq C\|\alpha^{\theta_0, \psi_\theta} - \alpha^{\theta_0, \theta}\|_{H^2}^2.$$ 

Observe from (1.7) that $J(\psi_\theta; \theta_0) = J(\alpha^{\theta_0, \theta}; \theta_0)$. Moreover, the definition of $\psi_\theta$ implies that $\alpha^{\theta_0, \theta} \in H^2_d(\Omega; \mathbb{R}^p)$ is Lipschitz continuous. Hence, we have (5.4) holds.

By Proposition 1.1, there exists $C \geq 0$ such that for all $\theta \in B(\theta_0)$ and $(t, x) \in [0, T] \times \mathbb{R}^d$, $\psi_\theta \in \mathcal{V}_C$ and $|\psi_\theta(t, x) - \psi_\theta(t, x)| \leq C(1 + |x|)\|\theta - \theta_0\|$. Hence standard stability analysis of SDEs shows that $\|X^{\theta_0, \psi_\theta} - X^{\theta_0, \theta_0}\|_{S^2} + \|\alpha^{\theta_0, \psi_\theta} - \alpha^{\theta_0, \theta_0}\|_{H^2} \leq C\|\theta - \theta_0\|$ and finishes the proof.

Before proving Theorem 3.6, we first establish two technical lemmas for the function

$$R^{d\times(d+p)} \times [0, T] \times \mathbb{R}^d \ni (\theta, t, x) \mapsto Y^\theta_{t,x} \in \mathbb{R}, \quad (5.5)$$

where for each $\theta = (A, B)$ and $(t, x), (X^\theta, Y^\theta, Z^\theta) \in S^2_d(\Omega \times [t, T]; \mathbb{R}^d) \times S^2_d(\Omega \times [t, T]; \mathbb{R}) \times H^2_d(\Omega \times [t, T]; \mathbb{R}^{d\times d})$ be the unique solution to the following forward-backward stochastic differential equation (FBSDE): for all $s \in [t, T],

$$dX_s = AX_s ds + dW_s, \quad X_t = x, \quad (5.6a)$$

$$dY_s = f(s, X_s, B^\top Z_s) ds + Z_s dW_s, \quad Y_T = g(X_T) \quad (5.6b)$$

with some Lipschitz continuous functions $f$ and $g$ specified as follows. For notational simplicity, we shall denote by $S^q(E)$ (resp. $H^q(E)$) the space $S^q_{E}(\Omega \times [t, T]; \mathbb{R})$ (resp. $H^q_{E}(\Omega \times [t, T]; \mathbb{E})$) for any $q \geq 2, t \in [0, T]$ and Euclidean space $E$. 

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Lemma 5.1. Let \( f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R} \) and \( g : \mathbb{R}^d \to \mathbb{R} \) be continuous functions such that for all \( t \in [0, T] \), \( f(t, \cdot, \cdot) \in C^4(\mathbb{R}^d \times \mathbb{R}^p) \) and \( g \in C^4(\mathbb{R}^d) \) with bounded derivatives uniformly in \( t \). Let \( Y : \mathbb{R}^{d \times (d+p)} \times [0, T] \times \mathbb{R}^d \to \mathbb{R} \) be defined in (5.5). Then for all bounded open set \( K \subset \mathbb{R}^{d \times (d+p)} \), there exists a constant \( C > 0 \) such that for all \( \theta \in K \), the function \( [0, T] \times \mathbb{R}^d \ni (t, x) \mapsto Y_{t,x}^\theta \in \mathbb{R} \) is of the class \( C^{1,3}(\mathbb{R}^d) \), and satisfies \( |\partial_{x_j} Y_{t,x}^\theta| + |\partial_{x_j,x_l} Y_{t,x}^\theta| + |\partial_{x_j,x_l,x_i} Y_{t,x}^\theta| \leq C \) for all \( i, j, l \in \{1, \ldots, d\} \) and \( (\theta, t, x) \in K \times [0, T] \times \mathbb{R}^d \).

Proof. We shall assume \( d = p = 1 \) for notational simplicity, but the same arguments can be easily extended to a multidimensional setting. Throughout this proof, let \( K \) be a given bounded open set of \( \mathbb{R}^{d \times (d+p)} \), and \( C \) be a generic constant independent of \( (t, x, \theta) \in [0, T] \times \mathbb{R}^d \times K \).

As \( f(t, \cdot, \cdot) \in C^4 \), \( t \in [0, T] \), and \( g \in C^4 \) have bounded derivatives, by [32, Theorem 3.2], the function \( [0, T] \times \mathbb{R} \ni (t, x) \mapsto Y_{t,x}^\theta \in \mathbb{R} \) is of the class \( C^{1,2}(\mathbb{R}^d) \), while iterating the arguments there again leads to the third-order differentiability in \( x \). Hence, it suffices to establish the boundedness of the derivatives. By Itô’s formula and [32, Lemma 2.5], the gradient of \( x \mapsto Y_{t,x}^\theta \) can be identified as \( \partial_x Y_{t,x}^\theta \) for all \( (t, x) \in [0, T] \times \mathbb{R} \), where \( (\partial_x Y_{t,x}^\theta, \partial_x Z_{t,x}^\theta) \in S^2(\mathbb{R}) \times H^2(\mathbb{R}) \) satisfies the following linear BSDE:

\[
\begin{align*}
\frac{d\partial_x Y_{t,x}^\theta}{dt} &= (\partial_{x} Y_{t,x}^\theta) \partial_x Y_{t,x}^\theta + (\partial_{x} Z_{t,x}^\theta) \partial_x Z_{t,x}^\theta) ds + \partial_x Z_{t,x}^\theta dW_s, \quad t \in [0, T), \\
\partial_x Y_{T,x}^\theta &= (\nabla g)(X_{T,x}^\theta) \partial_x X_{T,x}^\theta,
\end{align*}
\]

with \( \partial_x X_{t,x}^\theta = \exp(A(s - t)) \) for all \( s \in [0, T] \), \( F_{t,x,s}^\theta = (\partial_x f)(t, X_{t,x}^\theta, BZ_{t,x}^\theta) \) and \( F_{t,x,s}^\theta = (\partial_z f)(t, X_{t,x}^\theta, BZ_{t,x}^\theta)B \) for all \( s \in [t, T] \). The a-priori estimate [43, Theorem 4.4.4] shows that for all \( q \geq 2 \), there exists \( C_q \) such that

\[
\|\partial_x Y_{t,x}^\theta\|^q_{\mathbb{H}^q} + \|\partial_x Z_{t,x}^\theta\|^q_{\mathbb{H}^q} 
\leq C_q \left( \mathbb{E} \left[ (\nabla g)(X_{T,x}^\theta) \partial_x X_{T,x}^\theta \right]^q + \left( \int_0^T |F_{t,x,s}^\theta \partial_x Y_{t,x}^\theta| dt \right)^q \right) \leq C_q,
\]

due to the boundedness of \( \nabla g \) and \( \nabla f \), which subsequently implies that \( |\partial_x Y_{t,x}^\theta| \leq C \). Taking the derivative again and using the fact that \( \partial_x X_{t,x}^\theta \) is independent of \( x \), we identify the gradient of \( x \mapsto \partial_{xx} Y_{t,x}^\theta \) with \( \partial_{xx} Y_{t,x}^\theta \) for all \( (t, x) \in [0, T] \times \mathbb{R} \), where \( (\partial_{xx} Y_{t,x}^\theta, \partial_{xx} Z_{t,x}^\theta) \in S^2(\mathbb{R}) \times H^2(\mathbb{R}) \) satisfying the following BSDE: for all \( t \in [0, T] \),

\[
\begin{align*}
\frac{d\partial_{xx} Y_{t,x}^\theta}{dt} &= (\partial_{xx} Y_{t,x}^\theta) \partial_x Y_{t,x}^\theta + 2(\partial_{xx} Z_{t,x}^\theta) \partial_x Z_{t,x}^\theta) ds + \partial_x Z_{t,x}^\theta dW_s, \\
\partial_{xx} Y_{T,x}^\theta &= (\nabla^2 g)(X_{T,x}^\theta) (\partial_x X_{T,x}^\theta)^2,
\end{align*}
\]

with \( \nabla^2 g \) being the second-order derivative of \( g \). \( F_{t,x,s}^{2,\theta} = (\partial_{xx} f)(t, X_{t,x}^{\theta}, BZ_{t,x}^{\theta}) \), \( F_{t,x,s}^{2,\theta} = (\partial_{zz} f)(t, X_{t,x}^{\theta}, BZ_{t,x}^{\theta})B \) and \( F_{t,x,s}^{0,\theta} = (\partial_{zz} f)(t, X_{t,x}^{\theta}, BZ_{t,x}^{\theta})B^2 \). Then for all \( q \geq 2 \),

\[
\begin{align*}
\|\partial_{xx} Y_{t,x}^\theta\|^q_{\mathbb{H}^q} + \|\partial_{xx} Z_{t,x}^\theta\|^q_{\mathbb{H}^q} 
\leq C_q \left( \mathbb{E} \left[ (\nabla^2 g)(X_{T,x}^\theta) (\partial_x X_{T,x}^\theta)^2 \right]^q + \left( \int_0^T |F_{t,x,s}^{2,\theta} \partial_x Y_{t,x}^\theta + 2F_{t,x,s}^{2,\theta} \partial_x Z_{t,x}^\theta + F_{t,x,s}^{0,\theta} (\partial_x Z_{t,x}^\theta)^2 | dt \right)^q \right) 
\leq C_q (1 + \|\partial_x Z_{t,x}^\theta\|^2_{\mathbb{H}^q}) \leq C_q,
\end{align*}
\]
which follows from the fact that $f, g$ have bounded derivatives and the Cauchy–Schwarz inequality. This proves that $x \mapsto Y^t_{t,x}$ has a bounded second-order derivative. Iterating the above argument again shows the boundedness of the third-order derivative of $x \mapsto Y^t_{t,x}$ and finishes the proof. □

**Lemma 5.2.** Assume the same setting as in Lemma 5.1. Then for all $t \in [0, T]$, the function $\mathbb{R}^{d \times (d+p)} \times \mathbb{R}^d \ni (\theta, x) \mapsto \partial Y^t_{t,x} \in \mathbb{R}^d$ is of the class $C^2(\mathbb{R}^{d \times (d+p)} \times \mathbb{R}^d)$. Moreover, for all bounded open set $K \subset \mathbb{R}^{d \times (d+p)}$, there exists a constant $C > 0$ such that for all $i, j \in \{1, \ldots, d\}$, $(\theta, t) \in K \times [0, T]$ and $x, x' \in \mathbb{R}^d$,

\[
(1) \quad |\partial_{\theta_i} Y^t_{t,x}| + |\partial_{x\theta_i} Y^t_{t,x}| \leq C(1 + |x|) \quad \text{and} \quad |\partial_{x\theta_i} Y^t_{t,x}| \leq C(1 + |x|^2),
\]

\[
(2) \quad |\partial_{\theta_i} Y^t_{t,x} - \partial_{x\theta_i} Y^t_{t,x'}| + |\partial_{x\theta_i} Y^t_{t,x} - \partial_{x\theta_i} Y^t_{t,x'}| \leq C(1 + |x| + |x'|)|x - x'|, \quad \text{and} \quad |\partial_{x\theta_i} Y^t_{t,x} - \partial_{x\theta_i} Y^t_{t,x'}| \leq C(1 + |x|^2 + |x'|^2)|x - x'|.
\]

**Proof.** To simplify the presentation, we focus on establishing the desired properties for $A \mapsto \partial Y^t_{t,x}$, since the regularity of $B \mapsto \partial Y^t_{t,x}$ follows from similar arguments. We also assume $d = p = 1$ and write $A \mapsto \partial_y Y^t_{t,x}$ as $A \mapsto \partial_y Y^t_{t,x}$, but the same arguments can be easily extended to a multidimensional setting. Throughout this proof, let $K$ be a given bounded open set of $\mathbb{R}$, and $C$ be a generic constant independent of $(t, x, A) \in [0, T] \times \mathbb{R}^d \times K$.

By [14, Proposition 2.4], $\mathbb{R} \ni A \mapsto (X^t_{t,x}, Y^t_{t,x}) \in S^q(\mathbb{R}) \times S^q(\mathbb{R}) \times H^q(\mathbb{R})$ is differentiable for all $q \geq 2$, where the derivative satisfies the following BSDE: for all $t \in [0, T)$

\[
d \partial A X^t_{t,x} = (X^t_{t,x} + A \partial A X^t_{t,x}) ds + \partial A Z^t_{t,x} dW_t,
\]

\[
d \partial A Y^t_{t,x} = F^t_{t,x} \partial A X^t_{t,x} + Z^t_{t,x} dW_t,
\]

\[
d \partial A Y^t_{T,x} = \partial A X^t_{T,x},
\]

with bounded coefficients $F^t_{t,x}$ and $Z^t_{t,x}$ defined as in (5.7). Then

\[
|\partial A Y^t_{t,x}|^q_{S^q} + |\partial A Z^t_{t,x}|^q_{H^q} \leq C_q \left( \mathbb{E} \left[ |(\nabla g)(X^t_{t,x})| A X^t_{t,x} |q \right] + \left( \int_0^T |F^t_{t,x} A X^t_{t,x} |d|t |^q \right) \right) \leq C_q(1 + |x|^q),
\]

(5.10)

where the last inequality follows from the fact that $|\partial A X^t_{t,x}|_{S^q} \leq C_q(1 + |x|)$.

Applying the Lipschitz estimate in [24, Theorem 2.3] to (5.6) implies that

\[
|X^t_{t,x} - X^t_{t,x'}|_{S^q} + |Y^t_{t,x} - Y^t_{t,x'}|_{S^q} + |Z^t_{t,x} - Z^t_{t,x'}|_{H^q} \leq C_q|x - x'|, \quad \forall q \geq 2, x, x' \in \mathbb{R}.
\]

Then by the Cauchy–Schwarz inequality, for all $q \geq 2$ and $x, x' \in \mathbb{R}$,

\[
|\partial (\nabla g)(X^t_{t,x})| A X^t_{t,x} - (\nabla g)(X^t_{t,x'})| A X^t_{t,x'}|_{L^q} \leq \left( |(\nabla g)(X^t_{t,x})| A X^t_{t,x} - (\nabla g)(X^t_{t,x'})| A X^t_{t,x'}|_{L^q} + |(\nabla g)(X^t_{t,x}) A X^t_{t,x} - A X^t_{t,x'}|_{L^q} \right) \leq C(1 + |x| + |x'|)|x - x'|,
\]

(5.11)

where the last inequality used $|\partial A X^t_{t,x} - \partial A X^t_{t,x'}|_{S^q} \leq C_q|x - x'|$. Similarly, we have

\[
|F^t_{t,x} A X^t_{t,x} - F^t_{t,x'} A X^t_{t,x'}|_{S^q} \leq C_q|x - x'| \quad \text{and} \quad \partial A Z^t_{t,x} |_{H^q} \leq C(1 + |x| + |x'|)|x - x'|.
\]
Consequently, applying [24, Theorem 2.3] to (5.9) shows that for all \( q \geq 2 \), \( \| \partial_A Y^{q,t,x} - \partial_A Y^{q,t,x'} \|_{S^q} + \| \partial_A Z^{q,t,x} - \partial_A Z^{q,t,x'} \|_{H^q} \leq C(1 + |x| + |x'|)|x - x'| \).

Now we study the differentiability of \( A \mapsto \partial_A Y^{q,t,x} \). Observe that \( A \mapsto \partial_A Y^{q,t,x} \in L^q(0,T) \) is differentiable for all \( q \geq 2 \), and the derivatives are bounded uniformly in \( (A,t,x) \in K \times [0,T] \times \mathbb{R} \). By [23, Theorem 9, p. 97], for all \( q \geq 2 \) and \( A \in \mathbb{R} \),

\[
\mathbb{R} \ni A' \mapsto ((\nabla g)(X_T^{q,t,x}) \partial_A X_T^{q,t,x}, F_{x,A}^{q,t,x} \partial_A X_T^{q,t,x} + F_{z,A}^{q,t,x} \partial_A Z_T^{q,t,x}) \in L^q(\Omega) \times H^q(\mathbb{R})
\]
is differentiable. Hence, applying [14, Proposition 2.4] to (5.7) shows that \( \mathbb{R} \ni A \mapsto (\partial_A Y^{q,t,x}, \partial_A Z^{q,t,x}) \in S^q(\mathbb{R}) \times H^q(\mathbb{R}) \) is differentiable, and the derivatives \((\partial_A Y^{q,t,x}, \partial_A Z^{q,t,x})\) satisfies the BSDE:

\[
\begin{align*}
d\partial_A Y^{q,t,x}_{s,A} &= (F_{x,A}^{q,t,x}(s) \partial_A X^{q,t,x}_s + F_{x,A}^{q,t,x} \partial_A X^{q,t,x}_s + F_{z,A}^{q,t,x} \partial_A Z^{q,t,x}_s + F_{z,A}^{q,t,x} \partial_A Z^{q,t,x}_s) \, ds \\
&\quad + \partial_A Z^{q,t,x}_s \, dW_s, \quad t \in [0,T),
\end{align*}
\]

(5.12)

with the coefficients

\[
\begin{align*}
F_{x,A}^{q,t,x}(s) &= (\partial_A f)(t, X^{q,t,x}_s) BZ^{q,t,x}_s \partial_A X^{q,t,x}_s + (\partial_A f)(t, X^{q,t,x}_s) B \partial_A Z^{q,t,x}_s, \\
F_{z,A}^{q,t,x}(s) &= (\partial_A f)(t, X^{q,t,x}_s) BZ^{q,t,x}_s \partial_A X^{q,t,x}_s + (\partial_A f)(t, X^{q,t,x}_s) B \partial_A Z^{q,t,x}_s B^2 \partial_A Z^{q,t,x}_s.
\end{align*}
\]

For the growth rate of \( x \mapsto \partial_A Y^{q,t,x}_t \), by the a priori estimate for (5.12), for all \( q \geq 2 \),

\[
\begin{align*}
\| \partial_A Y^{q,t,x}_t \|_{S^q} + \| \partial_A Z^{q,t,x}_t \|_{H^q} &\leq C_q \left( |(\nabla^2 g)(X_T^{q,t,x}) \partial_A X_T^{q,t,x} | + \| (\nabla g)(X_T^{q,t,x}) \partial_A X_T^{q,t,x} | \right) \\
&\quad + \left( \int_0^T |F_{x,A}^{q,t,x} \partial_A X^{q,t,x}_s + F_{x,A}^{q,t,x} \partial_A X^{q,t,x}_s + F_{z,A}^{q,t,x} \partial_A Z^{q,t,x}_s | \, dt \right)^q \\
&\leq C_q \left( 1 + |x|^q + \left( \int_0^T |\partial_A X^{q,t,x}_s| + |\partial_A Z^{q,t,x}_s| \right) (1 + |x|^{q/2}) \right) \leq C_q (1 + |x|^q),
\end{align*}
\]

which follows from the Cauchy–Schwarz inequality, (5.10) and the fact that \( \| \partial_A Z^{q,t,x}_t \|_{H^q} \leq C_q \) for all \( q \geq 2 \). To prove the local Lipschitz continuity of \( x \mapsto \partial_A Y^{q,t,x}_t \), we introduce for all \( q \geq 2 \) and \( x, x' \in \mathbb{R}^d \), the random variables

\[
\begin{align*}
G^{q,t,x} &= (\nabla^2 g)(X_T^{q,t,x}) \partial_A X_T^{q,t,x} + \nabla g(X_T^{q,t,x}) \partial_A X_T^{q,t,x}, \\
H^{q,t,x} &= F_{x,A}^{q,t,x} \partial_A X^{q,t,x}_s + F_{x,A}^{q,t,x} \partial_A X^{q,t,x}_s + F_{z,A}^{q,t,x} \partial_A Z^{q,t,x}_s + F_{z,A}^{q,t,x} \partial_A Z^{q,t,x}_s, \quad s \in [t,T],
\end{align*}
\]

By following similar arguments as in (5.11) and using the regularity of \( f \) and \( g \), we have for all \( q \geq 2 \) and \( x, x' \in \mathbb{R}^d \),

\[
\| G^{q,t,x} - G^{q,t,x'} \|_{L^q} + \| H^{q,t,x} - H^{q,t,x'} \|_{H^q} \leq C_q (1 + |x| + |x'|)|x - x'|,
\]

which along with the Lipschitz estimate for (5.12) shows

\[
\| \partial_A Y^{q,t,x} - \partial_A Y^{q,t,x'} \|_{S^q} + \| \partial_A Z^{q,t,x} - \partial_A Z^{q,t,x'} \|_{H^q} \leq C_q (1 + |x| + |x'|)|x - x'|.
\]
For the differentiability of $A \mapsto \partial_x Y^\theta,t,x_t$, applying [14, Proposition 2.4] to (5.12) implies that

$$\mathbb{R} \ni A \mapsto (\partial_x Y^\theta,t,x_t, \partial_x Z^\theta,t,x_t) \in \mathcal{S}(\mathbb{R}) \times \mathcal{H}(\mathbb{R})$$

is differentiable, and its derivative satisfies the BSDE: for all $t \in [0,T)$,

$$d\partial_x AA_s^\theta,t,x = \left( F^\theta,t,x_{s,0} A_s + 2 F^\theta,t,x_{s,2} A_s + F^\theta,t,x s, s, Z^\theta,t,x_s \right) \partial_x Z^\theta,t,x_s ds + \partial_x AA_s^\theta,t,x dW_s,$$

$$\partial_x AA_T^\theta,t,x = \left( \nabla^3 \partial_x A_T^\theta,t,x \right) \partial_x X_T^\theta,t,x + \left( \nabla^2 \partial_x A_T^\theta,t,x \right) \partial_x X_T^\theta,t,x + \left( \nabla \partial_x A_T^\theta,t,x \right) \partial_x X_T^\theta,t,x,$$

(5.13)

with the coefficients

$$F^\theta,t,x_{s,0} = \partial_x x^\theta,t,x \partial_x A_x^\theta,t,x \partial_x x^\theta,t,x \partial_x B_\theta^\theta,t,x B_\theta A^\theta,t,x,$$

$$F^\theta,t,x_{s,2} = \partial_x x^\theta,t,x \partial_x A_x^\theta,t,x \partial_x x^\theta,t,x \partial_x B_\theta^\theta,t,x B_\theta A^\theta,t,x,$$

(5.14a)

Repeating all the above estimates, one can show $\|\partial_x AA^\theta,t,x \|_\mathcal{S} + \|\partial_x AA^\theta,t,x \|_\mathcal{H} \leq C_1 (1 + |x|)^2$.

Other partial derivatives of $\theta \mapsto \partial_x Y^\theta,t,x_t$ can be estimated by using similar arguments, whose details are omitted here.

Based on Lemmas 5.1 and 5.2, we prove the entropy regularised control problem (1.1) admits an optimal feedback control, which is sufficiently regular in the $(\theta,x)$-variables.

**Proposition 5.3.** Assume the same setting as in Theorem 3.6. Then for all $\theta \in \mathbb{R}^{d \times (d+p)}$, the control problem (1.1) with parameter $\theta$ admits an optimal feedback control $\psi\theta \in \mathcal{C}_{\geq 0} \mathcal{C}$, and

1. $\psi\theta \in C^{0,2}([0,T] \times \mathbb{R}^d)$ and for all $t \in [0,T]$, the function $\mathbb{R}^{d \times (d+p)} \times \mathbb{R} \ni (\theta, x) \mapsto \psi\theta(t, x) \in \mathbb{R}^p$ is of the class $C^2(\mathbb{R}^{d \times (d+p)} \times \mathbb{R}^d)$,

2. for all bounded open set $\mathcal{K} \subset \mathbb{R}^{d \times (d+p)}$, there exists a constant $C > 0$ such that for all $(\theta, x) \in \mathcal{K}$ and $x', x'' \in \mathbb{R}^d$, $|\nabla_x \psi\theta(t, x)| + |\text{Hess}_x \psi\theta(t, x)| \leq C$, $|\nabla_{\theta} \psi\theta(t, x)| + \left| \nabla_{\theta} \left( \nabla_x \psi\theta \right)(t, x) \right| \leq C(1 + |x|)$, $|\text{Hess}_{\theta} \psi\theta(t, x)| \leq C(1 + |x|)^2$, and $|\text{Hess}_{\theta} \psi\theta(t, x)| \leq C(1 + |x|)^2 |x - x'|$.

**Proof.** Let $h^\ast_{x} : \mathbb{R}^p \rightarrow \mathbb{R}$ be the convex conjugate of $h_{x}$ such that for all $z = (z_i)_{i=1}^p \in \mathbb{R}^p$, $h^\ast_{x}(z) = \sup_{a \in \mathcal{A}} \langle a, z \rangle - h_{x}(a) = \ln \sum_{i=1}^p \exp(z_i)$. A direct computation shows that $h^\ast_{x} \in C^\infty(\mathbb{R}^p)$, $h^\ast_{x}(z) \leq C(1 + |z|)$ for all $z \in \mathbb{R}^p$, and all derivatives of $h^\ast_{x}$ are bounded. For given $\theta = (A,B) \in \mathbb{R}^{d \times (d+p)}$ and $(t, x) \in [0,T] \times \mathbb{R}^d$, consider the following FBSDE: for all $s \in [t,T]$,

$$dX_s = AX_s ds + dW_s, \qquad X_t = x,$$

$$dY_s = h^\ast_{x}(B^\top Z^\theta_t - \bar{f}_0(s, X_s)) ds + Z_s dW_s, \qquad Y_T = g(X_T).$$

By the Lipschitz continuity of $\nabla h^\ast_{x}$ and $\bar{f}_0$, (5.14) admits a unique solution $(X^\theta,t,x, Y^\theta,t,x, Z^\theta,t,x) \in \mathcal{S}^2(\mathbb{R}^d) \times \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^{1 \times d})$ (see e.g., [43, Theorems 4.3.1]). By Lemma 5.1 and the regularity of
By Proposition 5.4, for all \( \theta \in \mathbb{R}^{d\times(d+p)} \), the function \( [0,T] \times \mathbb{R}^d \ni (t,x) \mapsto V^\theta(t,x) := Y^\theta_{t,x,x} \in \mathbb{R} \) is of the class \( C^{1,3}([0,T] \times \mathbb{R}^d) \), and for all \( (t,x) \in [0,T] \times \mathbb{R}^d \), \( \mathbb{R}^{d\times(d+p)} \ni \theta \mapsto \nabla_x V^\theta(t,x) \) is of the class \( C^2(\mathbb{R}^{d\times(d+p)}) \). Moreover, for any bounded open set \( \mathcal{K} \subset \mathbb{R}^{d\times(d+p)} \), there exists \( C > 0 \) such that for all \( (\theta,t,x) \in \mathcal{K} \times [0,T] \times \mathbb{R}^d \), \( |\nabla_x \nabla_x V^\theta(t,x)| \leq C \).

By Itô’s formula, for all \( \theta \in \mathbb{R}^{d\times(d+p)} \), the function \( V^\theta(x) \) solves the following PDE: for all \( (t,x) \in [0,T] \times \mathbb{R}^d \),

\[
\frac{d}{dt}V(t,x) + \frac{1}{2} \Delta_x V(t,x) + \langle Ax, \nabla_x V(t,x) \rangle - h_{en}^*(-B^\top \nabla_x V(t,x) - \bar{f}_0(t,x)) = 0,
\]

and \( V(T,x) = g(x) \) for all \( x \in \mathbb{R}^d \) (see e.g., [43, Theorem 5.5.8]). By (3.5) and the definition of \( h_{en}^*, (5.15) \) can be equivalently written as:

\[
\frac{d}{dt}V(t,x) + \frac{1}{2} \Delta_x V(t,x) + \inf_{a \in \mathbb{R}^p} \left( \langle Ax + Ba, \nabla_x V(t,x) \rangle + f(t,x,a) \right) = 0,
\]

which is the Hamilton-Jacobi-Bellman equation for (1.1) with parameter \( \theta \). Now let \( \psi_\theta : [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) be the function such that for all \( (t,x) \in [0,T] \times \mathbb{R}^d \),

\[
\psi_\theta(t,x) = \nabla h_{en}^*(-B^\top \nabla_x V^\theta_0(t,x) - \bar{f}_0(t,x)) = \arg \min_{a \in \mathbb{R}^p} \left( (Ba, \nabla_x V^\theta(t,x) + f(t,x,a)) \right).
\]

As \( \nabla_x V^\theta \) has a bounded first-order derivative (and hence Lipschitz continuous) in \( x \), the verification theorem in [42, Theorem 5.1, p. 268] shows that \( \psi_\theta \) is the optimal feedback control of (1.1) with parameter \( \theta \). The regularity of \( h_{en}^* \) and \( \bar{f}_0 \) implies that \( \psi_\theta \) shares the same regularity of \( \nabla_x V^\theta \). Hence by Lemmas 5.1 and 5.2, \( \psi_\theta \) admits the desired properties as stated in Items (1) and (2).

**Lemma 5.4.** Assume the same setting as in Theorem 3.6. Let \( \theta_0 = (A_0,B_0) \in \mathbb{R}^{d\times(d+p)} \), and for each \( \theta \in \mathbb{R}^{d\times(d+p)} \), let \( \psi_\theta : [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R}^p \) be the feedback control in Proposition 5.3, and let \( X^{\theta_0,\psi_\theta} \in \mathcal{S}^2(\mathbb{R}^d) \) be the solution to the following dynamics:

\[
dX_t = (A_0X_t + B_0\psi_\theta(t,X_t))dt + dW_t, \quad t \in [0,T], \quad X_0 = x_0.
\]

Then for all \( q \geq 2 \), \( \mathbb{R}^{d\times(d+p)} \ni \theta \mapsto X^{\theta_0,\psi_\theta} \in \mathcal{S}^q(\mathbb{R}^d) \) is twice continuously differentiable.

**Proof.** As \( \theta_0 \) is fixed throughout the proof, we write \( X^{\theta_0,\psi_\theta} \) as \( X^{\psi_\theta} \) for notational simplicity. Consider the function

\[
\mathbb{R}^{d\times(d+p)} \times [0,T] \times \mathbb{R}^d \ni (\theta,t,x) \mapsto b_\theta(t,x) := A_0 x + B_0 \psi_\theta(t,x) \in \mathbb{R}^d.
\]

By Proposition 5.3 Item (1), for all \( t \in [0,T] \), \( b_\theta(t,x) \) is twice differentiable with respect to \( (\theta,x) \). By Proposition 5.3 Item (2), for any bounded open set \( \mathcal{K} \subset \mathbb{R}^{d\times(d+p)} \), the first- and second-order derivatives of \( b_\theta(t,x) \) with respect to \( (\theta,x) \) have at most quadratic growth in \( x \), uniformly in \( (\theta,t) \in \mathcal{K} \times [0,T] \). Hence by [23, Theorem 4, p. 105], \( \mathbb{R}^{d\times(d+p)} \ni \theta \mapsto X^{\psi_\theta} \in \mathcal{S}^q(\mathbb{R}^d) \) is twice differentiable for all \( q \geq 2 \).

It remains to establish the continuity of the second-order derivative \( \mathbb{R}^{d\times(d+p)} \ni \theta \mapsto X^{\psi_\theta} \in \mathcal{S}^q(\mathbb{R}^d) \), for which we assume without loss of generality that \( \theta \in \mathbb{R} \) and \( X^{\psi_\theta} \) is one-dimensional. By [23, Theorem 4, p. 105], the second-order derivative \( \partial_{\theta\theta}X^{\psi_\theta} \) of \( \mathbb{R} \ni \theta \mapsto X^{\psi_\theta} \) satisfies the following SDE:

\[
d\partial_{\theta\theta}X^{\psi_\theta}_t = \left( B_0(\partial_{\theta\theta}\psi_\theta)(t,X^{\psi_\theta}_t) + 2B_0(\partial_{\theta x}\psi_\theta)(t,X^{\psi_\theta}_t) \partial_{\theta\theta}X^{\psi_\theta}_t + B_0(\partial_{xx}\psi_\theta)(t,X^{\psi_\theta}_t)(\partial_{\theta\theta}X^{\psi_\theta}_t)^2 \right. \left. + (A_0 + B_0(\partial_x\psi_\theta)(t,X^{\psi_\theta}_t))\partial_{\theta\theta}X^{\psi_\theta}_t \right)dt.
\]
The twice differentiability of \( \theta \mapsto X^{\psi_0} \) implies that \( \mathbb{R} \ni \theta \mapsto (X^{\psi_0}, \partial_\theta X^{\psi_0}) \in S^{q}(\mathbb{R}) \times S^{q}(\mathbb{R}) \) is continuous for all \( q \geq 2 \). Now let \( K \subset \mathbb{R} \) be an arbitrary open subset. By Proposition 5.3 Item (2), for all \( \theta, \eta \in K \) and \( q \geq 2 \),

\[
\mathbb{E} \left[ \int_0^T |(\partial_{\theta_0} \psi_0)(t, X^0_t) - (\partial_{\theta_0} \psi_0)(t, X^0_t)|^q \, dt \right] 
\leq C_q \mathbb{E} \left[ \int_0^T \left| (\partial_{\theta_0} \psi_0)(t, X^0_t) - (\partial_{\theta_0} \psi_0)(t, X^0_t) \right|^q + \left| (\partial_{\theta_0} \psi_0)(t, X^0_t) - (\partial_{\theta_0} \psi_0)(t, X^0_t) \right|^q \, dt \right]
\leq C_q \left( \int_0^T \left( 1 + \|X^0_t\|_{S^{q}}^2 \right) \|X^0_t - X^0_t\|_{S^{q}}^q + \mathbb{E} \left[ \int_0^T \left| (\partial_{\theta_0} \psi_0)(t, X^0_t) - (\partial_{\theta_0} \psi_0)(t, X^0_t) \right|^q \, dt \right] \right).
\]

Then, as \( \eta \) tends to \( \theta \), the first term converges to zero due to the continuity of \( \theta \mapsto X^{\psi_0} \in S^{q}(\mathbb{R}) \), and the second terms also converge to zero due to the continuity of \( \partial_{\theta_0} \psi_0 \) in \( \theta \), the fact that \( |(\partial_{\theta_0} \psi_0)(t, x)| \leq C(1 + |x|^2) \) and the dominated convergence theorem. Similar arguments show that \( \mathbb{R} \ni \theta \mapsto (\partial_{\theta_2} \psi_0)(\cdot, X^{\psi_0}), (\partial_{\theta_3} \psi_0)(\cdot, X^{\psi_0}), (\partial_{\theta_4} \psi_0)(\cdot, X^{\psi_0}) \in H^q(\mathbb{R}) \) are continuous for all \( q \geq 2 \). Thus [23, Theorem 9, p. 97] shows that for all \( q \geq 2 \) and \( x \in \mathbb{R} \),

\[
\mathbb{R} \ni \theta \mapsto B_0(\partial_{\theta_0} \psi_0) \cdot (X^{\psi_0}) + 2B_0(\partial_{\theta_2} \psi_0)(\cdot, X^{\psi_0})\partial_\theta X^{\psi_0} + B_0(\partial_{\theta_3} \psi_0)(\cdot, X^{\psi_0})(\partial_\theta X^{\psi_0})^2 + (A_0 + B_0(\partial_\theta \psi_0)(\cdot, X^{\psi_0})) x \in H^q(\mathbb{R})
\]

is continuous, which along with [23, Corollary 2, p. 103] implies the continuity of \( K \ni \theta \mapsto \partial_{\theta_0} X^{\theta_0, \psi_0} \in S^{q}(\mathbb{R}) \) for all \( q \geq 2 \).

Proof of Theorem 3.6. For every \( \theta \in \mathbb{R}^{d \times (d + p)} \), the existence of optimal feedback control \( \psi_0 \) has been established in Proposition 5.3. Hence it remains to establish the quadratic performance gap. To this end, let \( \theta_0 = (A_0, B_0) \in \Theta \) and \( \beta > 0 \) be given constants, and \( C \geq 0 \) be a generic constant which depends on \( \Theta, \beta \) but is independent of \( \theta_0 \). For each \( \theta \in \mathbb{B}_\beta(\theta_0) \), consider the dynamics:

\[
dX_t = (A_0 X_t + B_0 \psi_0(t, X_t)) \, dt + dW_t, \quad t \in [0, T], \quad X_0 = x_0,
\]

which admits a strong solution \( X^{\theta_0, \psi_0} \in S^{q}(\mathbb{R}^d) \) for all \( q \geq 2 \), due to the Lipschitz continuity of \( \psi_0 \). Lemma 5.4 shows that \( \mathbb{R}^{d \times (d + p)} \ni \theta \mapsto X^{\theta_0, \psi_0} \in S^{q}(\mathbb{R}^d) \) is twice continuously differentiable.

We start by proving the twice continuous differentiability of the map \( \mathbb{R}^{d \times (d + p)} \ni \theta \mapsto J(\psi_0; \theta_0) \in \mathbb{R} \), with \( J(\psi_0; \theta_0) \) being defined as in (1.7). Applying the Fenchel-Young identity to \( h^s_{en} \) gives that \( h^s_{en}(x) + h_{en}((\nabla h^s_{en})(x)) = (x, (\nabla h^s_{en})(x)) \) for all \( x \in \mathbb{R}^p \). Hence, by (3.5) and (5.16),

\[
f(t, x, \psi_0(t, x)) = \langle \tilde{f}_0(t, x), \psi_0(t, x) \rangle + h_{en}((\nabla h^s_{en})(-B^T \nabla \psi_0(t, x) - \bar{f}_0(t, x)))
= \langle \tilde{f}_0(t, x), \psi_0(t, x) \rangle + \langle -B^T \nabla \psi_0(t, x) - \bar{f}_0(t, x), \psi_0(t, x) \rangle
- h^s_{en}(B^T \nabla \psi_0(t, x) - \bar{f}_0(t, x))
+ \langle (B^T \nabla \psi_0(t, x) - \bar{f}_0(t, x)), \psi_0(t, x) \rangle - h^s_{en}(B^T \nabla \psi_0(t, x) - \bar{f}_0(t, x)).
\]

Proposition 5.3 shows for all \( t \in [0, T] \), \( \nabla \psi_0(t, x) \) and \( \psi_0(t, x) \) is twice differentiable in \( (\theta, x) \). Moreover, the derivatives have at most quadratic growth in \( x \) and is locally Lipschitz continuous in \( x \), provided that \( \theta \in \mathbb{B}_\beta(\theta_0) \). Hence, by [23, Theorem 9, p. 97], for all \( q \geq 2 \), \( \mathbb{R}^{d \times (d + p)} \ni \theta \mapsto J(\psi_0; \theta_0) \in \mathbb{R} \) is twice continuously differentiable.
\((\nabla_\theta V^\theta(\cdot, X_{t_0}^{\theta, \psi}), \psi^\theta(\cdot, X_{t_0}^{\theta, \psi})) \in \mathcal{H}^q(\mathbb{R}^d) \times \mathcal{H}^q(\mathbb{R}^p)\) are twice continuously differentiable, which implies the twice continuous differentiability of the following function:

\[
\mathbb{R}^{d \times (d+p)} \ni \theta \mapsto \mathbb{E}\left[ \int_0^T f(t, X_t^{\theta, \psi}, \psi^\theta(t, X_t^{\theta, \psi})) \, dt \right] \in \mathbb{R}.
\]

On the other hand, as \(g \in C^4\) has bounded derivatives, [23, Theorem 9, p. 97] shows that for all \(q \geq 2, \mathbb{R}^{d \times (d+p)} \ni \theta \mapsto g(X_T^{\theta, \psi}) \in L^q(\Omega; \mathbb{R})\) is twice continuously differentiable, which implies the same regularity of \(\mathbb{R} \ni \theta \mapsto \mathbb{E}[g(X_T^{\theta, \psi})] \in \mathbb{R}\). This finishes the proof of the statement that \(\theta \mapsto J(\psi^\theta; \theta_0)\) is \(C^2\).

Observe that the function \(\mathbb{R}^{d \times (d+p)} \ni \theta \mapsto J(\psi^\theta; \theta_0) \in \mathbb{R}\) is minimised at \(\theta = \theta_0\), which implies that \(\nabla_\theta J(\psi^\theta; \theta_0)\big|_{\theta = \theta_0} = 0\). Applying Taylor’s theorem up to the second-order terms gives the desired quadratic performance gap.

\[\square\]

6 Proofs of Theorems 3.7 and 3.8

For the sake of presentation, we introduce the following notation to identify the correspondence between the number of cycles and episodes used in the PEGE algorithm (see Algorithm 1). Let \(C : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}\) be the total number of episodes after completing a given cycle satisfying \(C(K) := K + \sum_{k=1}^K m(k)\) for all \(K \in \mathbb{N} \cup \{0\}\), \(\kappa : \mathbb{N} \to \mathbb{N}\) be the corresponding cycle of a given episode satisfying \(\kappa(m) := \min\{k \in \mathbb{N} : C(k) \geq m\}\) for all \(m \in \mathbb{N}\), and let \(E := \{m_k^\epsilon \mid m_k^\epsilon = C(k-1) + 1, k \in \mathbb{N}\}\) be the collection of all exploration episodes. The definition of \(\kappa\) implies that for all \(m \in \mathbb{N}\), \(\kappa(m) - 1 < \kappa(m)\). In particular, there exist \(\epsilon, \bar{\epsilon} \geq 0\) such that for all \(m \in \mathbb{N}\), \(m \leq \kappa(m)\) and \(\epsilon m \leq \kappa(m)^{1+\epsilon} \leq \bar{\epsilon} m\) if \(m(k) = [k^\epsilon]\) for all \(k\), and \(\epsilon m \leq 2^{\kappa(m)} \leq \bar{\epsilon} m\) if \(m(k) = 2^k\) for all \(k\).

The following lemma shows that the minimum eigenvalue of \(G^{\theta, \psi, m} = (V^{\theta, \psi, m})^{-1}\), with \(V^{\theta, \psi, m}\) defined in (3.2), increases with the number of exploration episodes.

**Lemma 6.1.** \(\psi^\epsilon \in \cup_{C \geq 0} V_C\) satisfies (H.3(1)) if and only if for any bounded subset \(\Theta \subset \mathbb{R}^{d \times (d+p)}\), there exists \(\lambda_0 > 0\) such that for all \(\theta \in \Theta\), \(\Lambda_{\text{min}}(\psi^\epsilon, \theta) \geq \lambda_0\), with the function \(\Lambda_{\text{min}} : \cup_{C \geq 0} V_C \times \mathbb{R}^{d \times (d+p)} \to [0, \infty)\) defined in (3.4).

**Proof.** Without loss of generality, let us fix a feedback control \(\psi^\epsilon \in \cup_{C \geq 0} V_C\), a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) satisfying the usual condition, and a \(d\)-dimensional standard \(\mathbb{F}\)-Brownian motion \(W\) on \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\). The Lipschitz continuity of \(\psi^\epsilon\) in the \(x\)-variable and stability of SDEs (see [43, Theorem 3.2.4]) imply the continuity of \(\mathbb{R}^{d \times (d+p)} \ni \theta \mapsto \|X^{\theta, \psi^\epsilon}\|_{S^2} \in \mathbb{R}\) and \(\mathbb{R}^{d \times (d+p)} \ni \theta \mapsto \|\psi^\epsilon(\cdot, X^{\theta, \psi^\epsilon})\|_{S^2} \in \mathbb{R}\). This shows that each entry of the matrix in \(\Lambda_{\text{min}}(\psi^\epsilon, \theta)\) is continuous in \(\theta\), as it involves only the products of \(X^{\theta, \psi^\epsilon}\) and \(\psi^\epsilon(\cdot, X^{\theta, \psi^\epsilon})\). Then, by (H.2) and the continuity of the minimum eigenvalue function, it suffices to show that the statement that (H.3(1)) holds is equivalent to for any given \(\theta \in \mathbb{R}^{d \times (d+p)}\), there exists \(\lambda_0 > 0\) such that \(\Lambda_{\text{min}}(\psi^\epsilon, \theta) \geq \lambda_0\). The latter statement holds if and only if for any \(u \in \mathbb{R}^d\) and \(v \in \mathbb{R}^p\),

\[
(u \ v)^\top \mathbb{E}^\mathbb{P}\left[ \int_0^T \begin{pmatrix} X_t^{\theta, \psi^\epsilon} \\ \psi(t, X_t^{\theta, \psi^\epsilon}) \end{pmatrix} \begin{pmatrix} X_t^{\theta, \psi} \\ \psi(t, X_t^{\theta, \psi}) \end{pmatrix}^\top \, dt \right] \begin{pmatrix} u \\ v \end{pmatrix} = 0
\]

implies that \(u\) and \(v\) are zero vectors, which can be equivalently formulated as \(u^\top X_t^{\theta, \psi^\epsilon} + v^\top \psi^\epsilon(t, X_t^{\theta, \psi^\epsilon}) = 0\) for \(\mathbb{P} \otimes \mathbb{P}\)-a.e. implies \(u\) and \(v\) are zero vectors.

Based on the above observations, the if direction of the desired result holds clearly by substituting \(x = X_t^{\theta, \psi^\epsilon}\) in (H.3(1)). We now fix an arbitrary \(\theta \in \mathbb{R}^{d \times (d+p)}\) and prove the only if direction.
Let \( u \in \mathbb{R}^d \) and \( v \in \mathbb{R}^p \) such that \( u^\top X_t^{\theta,\psi} + v^\top \psi^e(t, X_t^{\theta,\psi}) = 0 \) for \( \mathbb{P}_P \otimes \mathbb{P}_T \)-a.e., and \( E = \{(t, x) \in [0, T] \times \mathbb{R}^d \mid u^\top x + v^\top \psi^e(t, x) \neq 0\} \), we aim to show \( E \) has Lebesgue measure zero. Suppose that \( E \) has non-zero measure. As \( \psi^e \) is continuous in the \( x \)-variable, \( E_t = \{ x \in \mathbb{R}^d \mid (t, x) \in E \} \) is open for all \( t \in [0, T] \). By [20, Theorem 28.7], \( E = \cup_{n \in \mathbb{N}} (\Xi_n \times O_n) \), where \( \cup_{n \in \mathbb{N}} O_n \) is an open basis for the standard topology on \( \mathbb{R}^d \), and \( \Xi_n \subset [0, T] \) is Borel for all \( n \in \mathbb{N} \). Since \( E \) has positive measure, there exists a Borel set \( \Xi_n \subset [0, T] \) and an open set \( O_n \subset \mathbb{R}^d \) such that \( \Xi_n \) and \( O_n \) have positive measures and \( \Xi_n \times O_n \subset E \). By Section 7.6.4 of [25] and \( \|X_t^{\theta,\psi}\|_{\mathcal{S}^2} < \infty \), the law \( \mathbb{P}^{X_t^{\theta,\psi}} \) of \( X_t^{\theta,\psi} \) is equivalent to the law of \( (\xi_0 + W_t)_{t \in [0, T]} \). Hence, the fact that \( O_n \) is an open set with positive measure implies \( \mathbb{P}(\{ \omega \in \Omega \mid X_t^{\theta,\psi}(\omega) \in O_n \}) > 0 \) for all \( t \in (0, T] \), which along with the fact that \( \Xi_n \) has positive measure shows that \( \{ (\omega, t) \in \Omega \times [0, T] \mid (t, X_t^{\theta,\psi}(\omega)) \in E \} \) has positive \( \mathbb{P}_P \otimes \mathbb{P}_T \) measure. This contradicts the statement that \( u^\top X_t^{\theta,\psi} + v^\top \psi^e(t, X_t^{\theta,\psi}) = 0 \) for \( \mathbb{P}_P \otimes \mathbb{P}_T \)-a.e. Consequently, \( u^\top x + v^\top \psi^e(t, x) = 0 \) for almost every \( (t, x) \in [0, T] \times \mathbb{R}^d \), which along with the assumption implies that \( u \) and \( v \) are zero vectors and proves the only if direction. \( \square \)

Based on Lemma 6.1, we quantify the accuracy of the estimators \( (\hat{\theta}_m)_{m \in \mathbb{N}} \) generated by the PEGE algorithm (without Step 7).

**Lemma 6.2.** Suppose (H.1), (H.2) and (H.3) hold. Let \( \beta > 0 \), \( \theta_0 \in \mathbb{R}^{d \times(d+p)} \), \( V_0 \in \mathbb{S}_+^{d+p} \), \( \rho \) be a truncation function (cf. Definition 3.1), \( r \in (0, 1) \), and \( m : \mathbb{N} \to \mathbb{N} \) be such that \( m(k) = \lceil k^r \rceil \) for all \( k \in \mathbb{N} \). There exists a constant \( C \geq 0 \) such that for all \( \delta > 0 \) and \( m \geq C \left(1 + \ln\left(\frac{1}{\delta}\right)\right)^{1/r} \),

\[
\mathbb{P}\left(\left|\hat{\theta}_m^{\text{PEGE}} - \theta\right|^2 \leq \min\{Cm^{-1/(1-r)}(\ln m + \ln\left(\frac{1}{\delta}\right)), \beta^2\}\right) \geq 1 - \delta,
\]

where \( \hat{\theta}_m^{\text{PEGE}} := \rho(\hat{\theta}^{\text{PEGE},m}, V_{\theta,\psi}^{\text{PEGE},m}) \) and \( (\hat{\theta}^{\text{PEGE},m}, V_{\theta,\psi}^{\text{PEGE},m}) \) is defined in (3.2).

**Proof.** For notational simplicity, we denote by \( C \) a generic constant independent of \( m, \delta \), and omit the dependence on “PEGE” and \( \Psi^{\text{PEGE}} \) in the superscripts of all random variables, if no confusion occurs.

We first analyse the accuracy of the estimator \( \hat{\theta}^{\psi,m} \) defined in (3.2). For each \( k \in \mathbb{N} \), the state process \( X_t^{\theta,\psi,m_{k}} \) for the \( k \)-th exploration episode satisfies

\[
dX_t = (AX_t + B\psi^e(t, X_t))dt + dW_{t|\mathcal{F}_t}^{m_{k}}, \quad t \in [0, T], \quad X_0 = x_0.
\]

Let \( Z_t^{k,e} := \left(X_t^{\theta,\psi,m_{k}}\right) \) for all \( t \in [0, T] \), and let \( G^{k,e} := V_0^{-1} + \sum_{n=1}^{k}0\int_{0}^{T}Z_t^{k,e}(Z_t^{k,e})^\top dt \). By the following argument as that for (4.10), there exists \( C \geq 0 \) such that for all \( k \in \mathbb{N} \) and \( \delta > 0 \),

\[
\mathbb{P}(\|G^{k,e} - \mathbb{E}[G^{k,e}|\sigma(\theta)]\|_{\text{op}} \geq k\varepsilon(k, \delta)) \leq \delta, \quad \text{with} \quad \varepsilon(k, \delta) := \max\left(\frac{\ln(2/\delta)}{Ck}, \frac{\ln(2/\delta)}{Ck}\right),
\]

where \( \| \cdot \|_{\text{op}} \) denotes the operator norm of a matrix. By Lemma 6.1 and [22, Theorem 13, p 76], there exists \( \lambda_0 > 0 \) such that \( \lambda_{\min}(\mathbb{E}[G^{k,e}|\sigma(\theta)]) \geq k\lambda_0 \mathbb{P} \)-almost surely. Moreover, observe from (6.1) that there exists \( C \geq 0 \) such that for all \( \delta > 0 \) and \( k \geq C(1 + \ln\left(\frac{1}{\delta}\right)) \), \( \varepsilon(k, \delta) \leq \lambda_0/2 \). Hence there exists \( C \geq 0 \) such that for all \( \delta > 0 \) and \( k \geq C(1 + \ln\left(\frac{1}{\delta}\right)) \), with probability at least \( 1 - \delta \),

\[
\lambda_{\min}(G^{k,e}) = \inf_{|u| = 1} (u^\top \mathcal{K}^{k,e}u) = \inf_{|u| = 1} \left(u^\top (\mathbb{E}[G^{k,e}|\sigma(\theta)])u + u^\top (G^{k,e} - \mathbb{E}[G^{k,e}|\sigma(\theta)])u\right) \\
\geq \inf_{|u| = 1} \left(u^\top (\mathbb{E}[G^{k,e}|\sigma(\theta)])u - \|G^{k,e} - \mathbb{E}[G^{k,e}|\sigma(\theta)]\|_{\text{op}} \right) \geq \lambda_0/2.
\]

(6.2)
Using the fact that $cm \leq \kappa(m)^{1+r} \leq \bar{c}m$ for all $m \in \mathbb{N}$, there exists $C \geq 0$ such that for all $\delta > 0$ and $m \geq C(1 + \ln(\frac{1}{\delta}))^{1+r}$, with probability at least $1 - \delta$,

$$\lambda_{\min}(G^{\phi,\Psi,m}) \geq \lambda_{\min}(G^{\kappa(m),e}) \geq \frac{\kappa(m) \lambda_0}{2} \geq \frac{\lambda_0(\bar{c}m)^{1/(r+1)}}{2}.$$

Observe that $\Psi^{\text{PEGE}}$ only executes the feedback control of form $\psi^e$ and $\psi_{\theta,m}$ for some $m \in \mathbb{N}$. As $\hat{\theta}_m = \rho(\hat{\theta}^m, V^{\theta,m})$ takes values in a bounded set $\mathcal{K} := \text{range}(\rho)$ (cf. Definition 3.1), by Proposition 1.1 and the fact that $\psi^e \in \cup_{C \geq 0} \mathcal{V}_C$, we have $\Psi^{\text{PEGE}} \in \cup_{C \geq 0} \mathcal{U}_C$. Hence, by Theorem 3.2 and the above estimate for $\lambda_{\min}(G^{\phi,\Psi,m})$, there exists $C \geq 0$ such that for all $\delta > 0$ and $m \geq C(1 + \ln(\frac{1}{\delta}))^{1+r}$,

$$\mathbb{P}\left( |\hat{\theta}^{\Psi,m} - \theta|^2 \leq C m^{-(1/(1+r))} \ln m + \ln(\frac{1}{\delta}) \right) \geq 1 - \delta. \quad (6.3)$$

We then prove with high probability, $\hat{\theta}_m = \hat{\theta}^{\Psi,m}$ and $|\hat{\theta}^{\Psi,m} - \theta| \leq \beta$ for all $m$ sufficiently large, where $\beta > 0$ is the constant given in the statement. By Definition 3.1, $\text{cl}(\Theta) \subseteq \text{int}(\mathcal{K})$, with $\mathcal{K} = \text{range}(\rho)$. Hence, there exists $\eta > 0$ such that $\mathbb{B}_\eta(\Theta) \subseteq \mathcal{K}$, with $\mathbb{B}_\eta(\Theta) := \{ \theta \in \mathbb{R}^{d \times (d+p)} \mid \exists \theta \in \Theta \text{ s.t. } |\theta - \theta| \leq \eta \}$. Observe that there exists $C_0 \geq 0$ such that for all $\delta > 0$, $m \geq C_0(1 + \ln(\frac{1}{\delta}))^{r+1}$, $|C m^{-(1/(1+r))} \ln m + \ln(\frac{1}{\delta})| \leq \min(\eta^2, \beta^2)$, where $C$ is the same constant as that in (6.3), and $\beta$ is the constant given in the statement. Combining this with (6.3) yields that there exists $C \geq 0$ such that for all $\delta > 0$ and $m \geq C(1 + \ln(\frac{1}{\delta}))^{1+r}$,

$$\mathbb{P}\left( |\hat{\theta}^{\Psi,m} - \theta|^2 \leq \min \left\{ C m^{-(1/(1+r))} \ln m + \ln(\frac{1}{\delta}), \eta^2, \beta^2 \right\} \right) \geq 1 - \delta, \quad (6.4)$$

which along with $\mathbb{B}_\eta(\Theta) \subseteq \mathcal{K}$ and $\hat{\theta}_m = \rho(\hat{\theta}_m^m, V^{\theta,m}) = \hat{\theta}_m^m$ provided that $\theta^m \in \mathcal{K}$ (see Definition 3.1) implies that $\hat{\theta}_m = \hat{\theta}^{\Psi,m} \in \mathcal{K}$ on the above event, and finishes the proof of the desired estimate.

The following theorem proves a sublinear bound for the expected regret of the PEGE algorithm (without Step 7).

**Theorem 6.3.** Suppose (H.1), (H.2), (H.3) and (H.4) hold. Let $\hat{\theta}_0 \in \mathbb{R}^{d \times (d+p)}$, $V_0 \in \mathbb{S}^{d+p}$, $\rho$ be a truncation function (cf. Definition 3.1), and $m: \mathbb{N} \rightarrow \mathbb{N}$ be such that $m(k) = [k^r]$ for all $k \in \mathbb{N}$ with $r \in (0, 1]$ being the same as in (H.4). Then there exists a constant $C \geq 0$ such that for all $\delta \in (0, 1)$, with probability at least $1 - \delta$,

$$\sum_{m=1}^{N} \left( J(\Psi^{\text{PEGE}}_m; \theta) - J(\psi_{\theta}; \theta) \right) \leq C \left( N^{1+r} \left( \ln N \right)^r + \left( \ln(\frac{1}{\delta}) \right)^r + \left( \ln(\frac{1}{\delta}) \right)^{1+r} \right), \quad \forall N \in \mathbb{N} \cap [2, \infty).$$

where for each $\omega \in \Omega$, $J(\Psi^{\text{PEGE}}_m; \theta)(\omega) := J(\Psi^{\text{PEGE}}_m(\omega, \cdot, \cdot); \theta(\omega))$ and $J(\psi_{\theta}; \theta)(\omega) := J(\psi_{\theta(\omega)}; \theta(\omega))$ with $J$ and $\psi_{\theta}$ given in Proposition 1.1.

**Proof.** Throughout this proof, we denote by $C$ a generic constant independent of $m, \delta$ and $\omega$, and omit the dependence on “PEGE” in the superscripts (e.g., $\Psi = \Psi^{\text{PEGE}}$).

By Lemma 6.2 and (H.4), there exists a constant $C_0 \geq 0$ such that for all $\delta > 0$ and $m \geq C_0(1 + \ln(\frac{1}{\delta}))^{1+r}$, $m \in \mathbb{N} \setminus \mathcal{E}$,

$$\mathbb{P}\left( |J(\psi_{\theta,m}; \theta) - J(\psi_{\theta}; \theta)| \leq C_0 m^{-\frac{r}{1+r}} (\ln m + \ln(\frac{1}{\delta}))^r \right) \geq 1 - \delta. \quad (6.5)$$
Observe that there exists $C \geq 0$ such that for all $\delta > 0$, $m \geq C(1 + \ln(\frac{1}{\delta}))^{1+r}$ implies that $m \geq C_0(1 + \ln(1/\delta_m))^{1+r}$ with $\delta_m := 6\delta/(\pi^2m^2)$. Hence, applying (6.5) with $\delta_m$ for all sufficiently large $m$, summing the corresponding probabilities over all $m$ and using $m \geq C\kappa(m)^{1+r}$ yield that, for all $\delta > 0$, with probability at least $1 - \delta$, we have for all $m \in \mathbb{N}$ with $\kappa(m) \geq C(1 + \ln(\frac{1}{\delta}))$,

$$|J(\psi_{\bar{m}}; \theta) - J(\psi; \theta)| \leq Cm^{-\frac{r}{1+r}}(\ln m + \ln(\frac{1}{\delta}))^r. \quad (6.6)$$

On the event considered above, for all $N \in \mathbb{N}$, we consider the following decomposition

$$\sum_{m=1}^{N} (J(\Psi_m; \theta) - J(\psi; \theta)) = \sum_{m \in \mathbb{N} \cap \mathcal{A}_{\delta,e}} (J(\Psi_m; \theta) - J(\psi; \theta)) + \sum_{m \in \mathbb{N} \cap \mathcal{A}_{\delta,e}} (J(\Psi_m; \theta) - J(\psi; \theta)), \quad (6.7)$$

where the set $\mathcal{A}_{\delta,e}$ is defined by $\mathcal{A}_{\delta,e} := \{m \in \mathbb{N} \mid \kappa(m) \geq C(1 + \ln(\frac{1}{\delta})), m \notin \mathcal{E}\}$ (with the same constant $C$ as in (6.6)).

We first estimate the first term in (6.7). It is shown in Lemma 6.2 that $\Psi \in \mathcal{U}_C$ for some constant $C \geq 0$. Standard moment estimates of the state dynamics yield that $\mathbb{E}[\sup_t |X_0^\theta, \Psi, m|^2] \leq C$. By (H.1), Proposition 1.1 Item (2) and (H.3(2)), $|f(t, x, \Psi_m(\omega, t, x))| + |g(x)| \leq C(1 + |x|^2)$ for all $m \in \mathbb{N}$, $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d$. This estimate along with (H.2) implies that $|J(\Psi_m(\omega, \cdot, \cdot), \theta(\omega))| \leq C$ for all $m \in \mathbb{N}$ and $\omega \in \Omega$. Moreover, by using the continuity of $V^* : \Theta \to \mathbb{R}$ and (H.2), $|V^*(\theta)| = |J(\psi; \theta)| \leq C$ for all $\theta \in \Theta$. Hence for all $N \geq 2$,

$$\sum_{m \in \mathbb{N} \cap \mathcal{A}_{\delta,e}} (J(\Psi_m; \theta) - J(\psi; \theta)) \leq C\left(\{m \in \mathbb{N} \mid \kappa(m) \geq C(1 + \ln(\frac{1}{\delta}))\}\right) + \kappa(N)$$

$$\leq C\left((\ln(\frac{1}{\delta}))^{1+r} + N^{\frac{1}{1+r}}\right).$$

We then estimate the second term in (6.7) for the PEGE Algorithm (without Step 7). For all $m \in \mathbb{N} \setminus \mathcal{E}$, $J(\Psi_m; \theta) = J(\psi; \theta)$ with $\theta = \theta_{m^e_{\kappa(m)}}$, where $m^e_{\kappa(m)}$ is the latest exploration episode before the $m$-th episode. By (6.6),

$$\sum_{m \in \mathbb{N} \cap \mathcal{A}_{\delta,e}} (J(\Psi_m; \theta) - J(\psi; \theta)) \leq C \sum_{m \in \mathbb{N} \cap \mathcal{A}_{\delta,e}} (m^e_{\kappa(m)})^{-\frac{r}{1+r}}(\ln(m^e_{\kappa(m)}) + \ln(\frac{1}{\delta}))^r$$

$$\leq C \sum_{k=1}^{\kappa(N)} |k|^r \left((m^e_k)^{-\frac{r}{1+r}}(\ln(m^e_k) + \ln(\frac{1}{\delta}))^r\right)$$

$$\leq C \sum_{k=1}^{\kappa(N)} |k|^r (k^{-r}(1 + \ln(k) + \ln(\frac{1}{\delta}))^r) \leq C \sum_{k=1}^{\kappa(N)} \left(1 + \ln(k) + \ln(\frac{1}{\delta})\right)^r$$

$$\leq C\kappa(N)(\ln(\kappa(N)) + (\ln(\frac{1}{\delta}))^r) \leq CN^{\frac{1}{1+r}}((\ln N)^r + (\ln(\frac{1}{\delta}))^r)$$ \quad (6.8)

where the interchange between episodes and cycles follows from the fact that $\bar{c}m \leq \kappa(m)^{1+r} \leq \bar{c}m$ for all $m$. Combining these estimates with (6.7) leads to the desired result. \hfill \Box

**Proof of Theorem 2.7.** Throughout this proof, we denote by $C$ a generic constant independent of $m, \delta$ and omit the dependence on “PEGE” in the superscripts (e.g., $\Psi = \Psi^\text{PEGE}$).

It has been shown in the proof of Theorem 6.3 that $\Psi \in \mathcal{U}_C$ and $|g(x)| + |f(t, x, \Psi_m(\omega, t, x))| \leq C(1 + |x|^2)$ for all $m \in \mathbb{N}$, $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d$. Then by (2.3), Propositions 4.3 and 4.7,
\[ \| \ell_m(\Psi, \theta) \|_{1, F_{m-1}} \leq C \] for all \( m \in \mathbb{N} \), where \( F_{m-1} \) is defined as in Proposition 4.7. Observe that 
\[ \mathbb{E}[\ell_m(\Psi, \theta) \mid F_{m-1}] = J(\Psi_0; \theta) \] for all \( m \in \mathbb{N} \). Hence, by Proposition 4.4, there exists \( C \geq 0 \) such that for all \( N \in \mathbb{N} \) and \( \delta > 0 \),
\[ \mathbb{P}\left( \left| \sum_{n=1}^N (\ell_n(\Psi, \theta) - J(\Psi_n; \theta)) \right| \geq N \varepsilon(N, \delta) \right) \leq \delta, \]
with \( \varepsilon(N, \delta) := \max\left( \sqrt{\frac{\ln(2/\delta)}{CN}}, \frac{\ln(2/\delta)}{CN} \right) \leq CN^{-1/2}(1 + \ln(\frac{1}{\delta})) \). Then for each \( N \in \mathbb{N} \) and \( \delta > 0 \), applying the above inequality with \( \delta_N = 6\delta/(\pi^2N^2) \) and summing the corresponding probabilities for all \( N \geq 2 \), there exists \( C \geq 0 \) such that for all \( \delta > 0 \),
\[ \mathbb{P}\left( \left| \sum_{n=1}^N (\ell_n(\Psi, \theta) - J(\Psi_n; \theta)) \right| \leq CN^{1/2}\left( \ln(N) + \ln(\frac{1}{\delta}) \right), \forall N \in \mathbb{N} \cap [2, \infty) \right) \geq 1 - \delta. \]
Combining the above estimate with Theorem 6.3 and the fact that \( r \in (0, 1] \) yields that there exists a constant \( C \geq 0 \) such that for all \( \delta > 0 \), with probability at least \( 1 - \delta \),
\[ R(N, \Psi, \theta) \leq C\left( N^{\frac{1}{1+r}} \left( \ln(N)^r + \left( \ln\left( \frac{1}{\delta} \right) \right)^r \right) + \left( \ln\left( \frac{1}{\delta} \right) \right)^{1+r}, \forall N \in \mathbb{N} \cap [2, \infty). \] (6.9)

It now remains to prove the expected regret bound in Theorem 3.7. For each \( \lambda > 0 \), applying (6.9) with \( \delta = \exp(-\lambda^{\frac{1}{1+r}}) \) gives that there exists a constant \( C > 0 \) such that for all \( \lambda > 0 \),
\[ \mathbb{P}\left( CN^{\frac{1}{1+r}}R(N, \Psi, \theta) - (\ln(N))^r \geq \lambda, \forall N \in \mathbb{N} \cap [2, \infty) \right) \leq \exp(-\lambda^{\frac{1}{1+r}}). \]
For each \( N \in \mathbb{N} \), let \( D_N := CN^{\frac{1}{1+r}}R(N, \Psi, \theta) - (\ln(N))^r \). Then for all \( N \in \mathbb{N} \cap [2, \infty) \) and \( \lambda > 0 \),
\[ \mathbb{P}(\max(D_N, 0) > \lambda) = \mathbb{P}(D_N \geq \lambda) \leq \exp(-\lambda^{\frac{1}{1+r}}). \] Hence for all \( N \in \mathbb{N} \cap [2, \infty) \),
\[ \mathbb{E}[D_N] \leq \mathbb{E}\left[ \max(D_N, 0) \right] = \int_0^{\infty} \mathbb{P}(\max(D_N, 0) > \lambda) \, d\lambda \leq \int_0^{\infty} \exp(-\lambda^{\frac{1}{1+r}}) \, d\lambda < \infty, \]
where the last inequality follows from \( r \in (0, 1] \). This finishes the proof of Theorem 3.7. \( \square \)

**Proof of Theorem 3.8.** The proof follows from similar arguments as that of Theorem 3.7, and we only present the main steps here. For notational simplicity, we omit the dependence on “PEGE” (e.g., \( \Psi = \Psi^{PEGE} \)), and denote by \( C \geq 0 \) a generic constant independent of \( m, \delta \) and \( N \).

By (3.2) and (4.10), there exists a constant \( C \geq 0 \) such that for all \( m \in \mathbb{N} \) and \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \),
\[ \left| C \ell_{m, \Psi, \theta} - V_0^{-1} \right| - \sum_{n=1}^m \mathbb{E}\left[ \left| \int_0^T Z_{t}^{\theta, \Psi, m}(Z_{t}^{\theta, \Psi, m})^\top dt \mid F_{n-1} \right| \mid \mathbb{F} \right] \leq m \max\left( \sqrt{\frac{\ln(\frac{1}{\delta})}{Cm}}, \frac{\ln(\frac{1}{\delta})}{Cm} \right). \]
Observe that for all \( m \in \mathcal{E}, \Psi_m = \psi^c \) and for all \( m \in \mathbb{N} \setminus \mathcal{E}, \Psi_m = \psi_{\theta_m} \), where \( \bar{\theta}_m := \theta_{m, \kappa(m)} \) with \( \theta_m \) given in (3.6) and \( m_{\kappa(m)} \) is the latest exploration episode before the \( m \)th episode; without loss of generality, we also define \( \bar{\theta}_m \) for \( m \in \mathcal{E} \). Then by the additional assumption on \( \Lambda_{\min}(\theta, \psi^c) \), Lemma 6.1, and the fact that \( \Psi_m \) depends on \( (W_n)_{n=1}^{m-1} \) and \( \theta \) only through \( \bar{\theta}_m \), there exists \( \lambda_0 > 0 \) such that for all \( m \in \mathbb{N} \) and \( u \in \mathbb{R}^{d+p} \),
\[ u^\top \mathbb{E}\left[ \int_0^T Z_{t}^{\theta, \Psi, m}(Z_{t}^{\theta, \Psi, m})^\top dt \mid \mathbb{F}_{m-1} \right] u = u^\top \mathbb{E}\left[ \int_0^T Z_{t}^{\theta, \Psi, m}(Z_{t}^{\theta, \Psi, m})^\top dt \mid \sigma(\theta, \bar{\theta}_m) \right] u \]
\[ \geq \mathbb{E}\left( \min(\Lambda_{\min}(\psi_{\theta_m}, \theta), \Lambda_{\min}(\psi^c, \theta)) \right)^2 ||u||^2 \geq \lambda_0 ||u||^2. \]
Hence a similar argument as that for (6.2) shows that there exists \( C \geq 0 \) such that for all \( \delta > 0 \) and \( m \geq C(1 + \ln(\frac{4}{\delta})) \), \( \mathbb{P} \left( \lambda \min (C^4 \Psi, m) \geq m \lambda_0/2 \right) \geq 1 - \delta \). Proceeding along the lines of the proof of Lemma 6.2, yields that there exists \( C \geq 0 \) such that for all \( \delta > 0 \) and \( m \geq C(1 + \ln(\frac{4}{\delta})) \),

\[
\mathbb{P} \left( |\hat{\theta}_m - \theta|^2 \leq \min \{C m^{-1} \left( \ln m + \ln(\frac{4}{\delta}) \right), \beta^2 \} \right) \geq 1 - \delta,
\]

(6.10)

with the constant \( \beta \) in (H.4). By considering \( \delta_m = 6 \delta / (\pi^2 m^2) \) as in Theorem 6.3 and using (H.4), there exists \( C \geq 0 \) such that for all \( \delta > 0 \), with probability at \( 1 - \delta \),

\[
|J(\bar{\psi}_{\theta_m}; \theta) - J(\psi_{\theta}; \theta)| \leq C m^{-\gamma} \left( \ln m + \ln(\frac{4}{\delta}) \right)^{\gamma}, \quad \forall m \in \mathbb{N} \text{ s.t. } \kappa(m) \geq C + \log_2 \left( 1 + \ln(\frac{4}{\delta}) \right),
\]

(6.11)

where the condition regarding \( \kappa \) follows from the inequality \( 2^{\kappa(m)} \leq \bar{c} m \) for all \( m \in \mathbb{N} \).

On the event considered above, for all \( N \in \mathbb{N} \), we consider the decomposition as in (6.7);

\[
\sum_{m=1}^{N} \left( J(\Psi_m; \theta) - J(\psi_{\theta}; \theta) \right) = \sum_{m \in \mathbb{N} \cap [1,N] \cap A_{\delta,\epsilon}} \left( J(\Psi_m; \theta) - J(\psi_{\theta}; \theta) \right) + \sum_{m \in [1,N] \setminus A_{\delta,\epsilon}} \left( J(\Psi_m; \theta) - J(\psi_{\theta}; \theta) \right),
\]

(6.12)

with the set \( A_{\delta,\epsilon} := \{ m \in \mathbb{N} \mid \kappa(m) \geq C + \log_2 \left( 1 + \ln(\frac{4}{\delta}) \right), m \notin \mathcal{E} \} \), where \( \mathcal{E} \) is the collection of explorations episodes. The first term can be estimated as in the proof of Theorem 6.3: for all \( N \in \mathbb{N} \cap [2,\infty) \),

\[
\sum_{m \in \mathbb{N} \cap [1,N] \cap A_{\delta,\epsilon}} \left( J(\Psi_m; \theta) - J(\psi_{\theta}; \theta) \right) \leq C \left( \ln(\frac{4}{\delta}) + \ln N \right),
\]

where the inequality follows from \( \bar{c} m \leq 2^{\kappa(m)} \leq \bar{c} m \) for all \( m \in \mathbb{N} \). To estimate the second term in (6.12), recall that for the \( k \)th cycle, the exploration policy \( \psi_{\theta_m^k} \) is executed for \( 2^k \) episodes. Hence, by (6.11),

\[
\sum_{m \in [1,N] \setminus A_{\delta,\epsilon}} \left( J(\Psi_m; \theta) - J(\psi_{\theta}; \theta) \right) \leq C \sum_{k=1}^{\kappa(N)} 2^k \left( (m^e_k)^{-\gamma} \left( \ln(m^e_k) + \ln(\frac{4}{\delta}) \right)^{\gamma} \right)
\]

\[
\leq C \sum_{k=1}^{\kappa(N)} 2^k \left( 2^{-r k} (k + \ln(\frac{4}{\delta}))^{r} \right) \leq C \sum_{k=1}^{\kappa(N)} 2^{(1-r)k} \left( \kappa(N) + \ln(\frac{4}{\delta}) \right)^{\gamma}
\]

\[
\leq \begin{cases} CN^{1-r} \left( \ln N \right)^r + \left( \ln(\frac{4}{\delta}) \right)^r, & r \in (0,1), \\ C(\ln N) \left( \ln N + \ln(\frac{4}{\delta}) \right), & r = 1, \end{cases}
\]

where the last inequality follows from the facts that \( \kappa(N) \leq C \log_2(N) \) and if \( r \in (0,1) \), then \( \sum_{k=1}^{m} 2^{(1-r)k} \leq C 2^{(1-r)m} \) for all \( m \in \mathbb{N} \).

Finally, recall the random variables \( (\bar{\theta}_m)_{m \in \mathbb{N}} \) defined at the beginning of the proof. Since

\[
\mathbb{E}[R(N, \Psi, \theta)] = \sum_{m=1}^{N} \mathbb{E} \left[ \mathbb{E} \left[ \ell_m(\Psi, \theta) - J(\psi_{\theta}; \theta) \mid \sigma \{ \bar{\theta}_m, \theta \} \right] \right] = \sum_{m=1}^{N} \mathbb{E} \left[ J(\Psi_m; \theta) - J(\psi_{\theta}; \theta) \right],
\]

the desired regret bound in expectation follows from the above high probability regret bound and similar arguments as those in the proof of Theorem 3.7.
A Proof of Proposition 3.1

Proof. It is clear that for all $m \in \mathbb{N}$ and $\omega \in \Omega$, $\Psi_m(\omega, \cdot) \in \mathcal{V}_{C_m}$. We now prove by induction that for all $m \in \mathbb{N} \cup \{0\}$, $(\hat{\theta}^{\Psi,n}, V^{\theta,\Psi,n})$, $0 \leq n \leq m$, are $\mathcal{G}_m^{\Psi}$-measurable and $\Psi_{m+1}$ is $(\mathcal{G}_m^{\Psi} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d) / \mathcal{B}(\mathbb{R}^d))$-measurable, with the $\sigma$-algebra $\mathcal{G}_m^{\Psi}$ defined in Definition 2.1. Note that the statement clearly holds for $m = 0$, as $(\hat{\theta}^{\Psi,0}, V^{\theta,\Psi,0}) = (\theta_0, V_0)$ are deterministic.

Suppose that the induction hypothesis holds for some index $m - 1 \in \mathbb{N} \cup \{0\}$, namely, $(\hat{\theta}^{\Psi,n}, V^{\theta,\Psi,n})$, $0 \leq n \leq m - 1$, are $\mathcal{G}_m^{\Psi}$-measurable and $\Psi_m$ is $(\mathcal{G}_m^{\Psi} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d)) / \mathcal{B}(\mathbb{R}^d))$-measurable. Then the process $Z_t^{\theta,\Psi,m} := \left( X_t^{\theta,\Psi,m} \right)$, $t \in [0, T]$, is progressively measurable with respect to the filtration $\mathcal{G}^{X,\Psi,m} = (\mathcal{G}_t^{X,\Psi,m})_{t \in [0, T]}$ defined by $\mathcal{G}_t^{X,\Psi,m} := \mathcal{G}_m^{\Psi} \vee \sigma(X_u^{\theta,\Psi,m} \mid u \leq t)$, which along with the measurability of $V^{\theta,\Psi,m}$ and (3.2) implies that $V^{\theta,\Psi,m}$ and $\mathcal{G}_m^{\Psi}$-measurable.

To prove the $\mathcal{G}_m^{\Psi}$-measurability of $\hat{\theta}^{\Psi,m}$, we first establish a semi-martingale representation of the process $X^{\theta,\Psi,m}$ with respect to the filtration $\mathcal{G}^{X,\Psi,m}$. Let $B^{\theta,\Psi,m} = (B_t^{\theta,\Psi,m})_{t \in [0, T]}$ be the process such that

$$B_t^{\theta,\Psi,m} := X_t^{\theta,\Psi,m} - x_0 - \int_0^t \mathbb{E} \left[ \theta \mid \mathcal{G}_s^{X,\Psi,m} \right] Z_s^{\theta,\Psi,m} ds, \quad \forall t \in [0, T].$$

Note that $B^{\theta,\Psi,m} \in \mathcal{G}^{X,\Psi,m}$ is progressively measurable. We now verify that $B^{\theta,\Psi,m}$ is in fact a $\mathcal{G}^{X,\Psi,m}$-Brownian motion. Since $Z_t^{\theta,\Psi,m}$ is $\mathcal{G}_t^{\theta,\Psi,m}$-measurable, it follows from (2.2), the tower property of conditional expectation and Fubini’s theorem that for all $t \geq s$,

$$\mathbb{E} \left[ B_t^{\theta,\Psi,m} - B_s^{\theta,\Psi,m} \mid \mathcal{G}_s^{X,\Psi,m} \right] = \mathbb{E} \left[ \int_s^t dX_u^{\theta,\Psi,m} - \int_s^t \mathbb{E} \left[ \theta \mid \mathcal{G}_u^{X,\Psi,m} \right] Z_u^{\theta,\Psi,m} du \right]_{\mathcal{G}_s^{X,\Psi,m}}$$

$$= \mathbb{E} \left[ \int_s^t \left( \mathbb{E} \left[ \theta Z_u^{\theta,\Psi,m} \right] - \mathbb{E} \left[ \theta \mid \mathcal{G}_u^{X,\Psi,m} \right] Z_u^{\theta,\Psi,m} \right) du \right]_{\mathcal{G}_s^{X,\Psi,m}}$$

$$= \mathbb{E} \left[ \int_s^t \mathbb{E} \left[ \theta Z_u^{\theta,\Psi,m} \mid \mathcal{G}_u^{X,\Psi,m} \right] \left( \mathbb{E} \left[ \theta \mid \mathcal{G}_u^{X,\Psi,m} \right] Z_u^{\theta,\Psi,m} \right)_{\mathcal{G}_s^{X,\Psi,m}} \right] du$$

$$= \int_s^t \mathbb{E} \left[ \mathbb{E} \left[ \theta Z_u^{\theta,\Psi,m} \mid \mathcal{G}_u^{X,\Psi,m} \right] \left( \mathbb{E} \left[ \theta \mid \mathcal{G}_u^{X,\Psi,m} \right] Z_u^{\theta,\Psi,m} \right)_{\mathcal{G}_s^{X,\Psi,m}} \right] du = 0.$$

This shows that $B^{\theta,\Psi,m}$ is a $\mathcal{G}^{X,\Psi,m}$-martingale. By (A.1), $B_0^{\theta,\Psi,m} = 0$ and $B^{\theta,\Psi,m}$ has continuous sample paths, as $X^{\theta,\Psi,m}$ has continuous sample paths. Moreover, by (2.2) and (A.1), $B^{\theta,\Psi,m}$ has the same quadratic covariance as $W^m$. Hence, Lévy’s characterisation shows that $B^{\theta,\Psi,m}$ is a $\mathcal{G}^{X,\Psi,m}$-Brownian motion, which along with (A.1) implies that $X^{\theta,\Psi,m}$ admits the following semi-martingale representation with respect to the filtration $\mathcal{G}^{X,\Psi,m}$:

$$dX_t^{\theta,\Psi,m} = \mathbb{E} \left[ \theta \mid \mathcal{G}_t^{X,\Psi,m} \right] Z_t^{\theta,\Psi,m} dt + dB_t^{\theta,\Psi,m}, \quad t \in [0, T], \quad X_0^{\theta,\Psi,m} = x_0. \quad (A.2)$$

This implies that the stochastic integral of a $\mathcal{G}^{X,\Psi,m}$-progressively measurable process with respect to $X^{\theta,\Psi,m}$ is $\mathcal{G}^{X,\Psi,m}$-progressively measurable; note that for all $m \in \mathbb{N} \cup \{0\}$, $\mathcal{G}_m^{\Psi}$ has been augmented with $\mathbb{P}$-null sets (see Definition 2.1). Consequently, $\hat{\theta}^{\Psi,m}$ in (3.2) is $\mathcal{G}_m^{\Psi}$-measurable, which along with the fact that $\Psi_{m+1}$ depends only on $(\hat{\theta}^{\Psi,n}, V^{\theta,\Psi,n})_{n=0}^{m-1}$, the $\mathcal{G}_m^{\Psi}$-measurability of $(\hat{\theta}^{\Psi,n}, V^{\theta,\Psi,n})_{n=0}^{m-1}$, and the Borel measurability of $\psi_{m+1}$ implies the desired measurability of $\Psi_{m+1}$. This completes the induction step and finishes the proof. □
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