ON METHOD OF STATISTICAL DIFFERENTIALS

by

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ABSTRACT: The method of statistical differentials, which approximates the mean value and variance of transformations of random variables is used in many areas of mathematics. This paper will discuss the conditions under which such an approximation will be exact, and also explore their accuracy in terms of error bounds under certain moment conditions.

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I. PRELIMINARIES:

The method of statistical differentials is a method of approximating the mean (expected value), and the variance of transformations of the random variables. The method has been presented by Johnson and Kotz [1], Elandt-Johnson and Johnson [2], and London [3]. Frye [4] has given counter examples for the approximation to be exact, and also to show that in some cases, this approximation will not hold beyond certain degree polynomials.

Throughout this paper, we will use the notations adopted by London [3]. Let $Y = g(X)$, where $X$ is a random variable such that $E[X]$ and Var$[X]$ exist and are known. Other moment conditions will be assumed, depending upon the cases, that are discussed from time to time. Let $m = E[X]$. In order to express $Y = g(X)$ as a Taylor series expanded about $X = m$, assume $g(X)$ is a function possessing derivatives of all order up to $n^{th}$ throughout the interval $a \leq X \leq b$, then there is a value $z$, with $z \in (m, X)\cup(X, m)$ such that

$$g(X) = g(m) + (X - m)g'(m) + \frac{(X - m)^2}{2!}g''(m) + \cdots$$

$$\cdots + \frac{(X - m)^{n-1}}{(n-1)!}g^{(n-1)}(m) + \frac{(X - m)^n}{n!}g^{(n)}(z).$$

The random variable $z$ lies between $m$ and $X$. Taking expected values of both sides gives,

$$E[Y] = E[g(X)] = g(m) + E[X - m]g'(m) + E\left[\frac{(X - m)^2}{2!}\right]g''(m) + \cdots$$

$$\cdots + E\left[\frac{(X - m)^{n-1}}{(n-1)!}\right]g^{(n-1)}(m) + E\left[\frac{(X - m)^n}{n!}g^{(n)}(z)\right].$$

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Since $E[X - m] = 0$ and $E[(X - m)^2] = \text{Var}[X]$, we have

\[
E[Y] = g(m) + \frac{g''(m)}{2!} \cdot \text{Var}[X] + \frac{E[(X - m)^3]}{3!} g'''(m) + \cdots
\]

\[
\cdots + E \left[ \frac{(X - m)^n}{(n-1)!} \right] g^{(n-1)}(m) + E \left[ \frac{(X - m)^n}{n!} g^{(n)}(z) \right],
\]

(1)

where $n$ is a positive integer greater than 1. It is customary to truncate this series at the second term, and to consider

\[
E[Y] \cong g(m) + \frac{g''(m)}{2!} \cdot \text{Var}[X]
\]

as statistical differential approximation. We refer to the approximation (1) as a statistical differential approximation.

II. MOTIVATION:

We first give an important order notation, which we use time to time throughout this paper. The $O$—notation (read big-oh notation) provides a special way to compare relative sizes of functions that is very useful in the analysis of error bounds. The $o$—notation (read small-oh notation) is given for completeness of the definition and will not be used in the discussion elsewhere.

**Definition 1 Landau Order Notations:**

Let $f(x)$ and $g(x)$ be given functions. Let $x_0$ be a fixed point and suppose that $g(x)$ is positive and continuous in an open interval about $x_0$, where $x_0$ may be finite or infinite.

1. If there is a constant $K$ such that

\[
|f(x)| < K g(x)
\]

in an open interval about $x_0$, then $f(x) = O\left(g(x)\right)$, $(x \to x_0)$.

2. Furthermore, if

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0,
\]

then $f(x) = o\left(g(x)\right)$, $(x \to x_0)$.

Three conditions (i), (ii) and (iii) below, each of them will essentially lead to exactness of the approximation. It is reminded that (i) and (ii) can not be true. For (ii), a random variable with a symmetric distribution around 0, $m = 0$ and $EX^j = E[(X - m)^2] = 0$ for all odd $j$ (and in particular $j = 3$), but not for even $j$.

(i). $g'''(m) = 0$ implies that all other derivatives of order greater than three evaluated at $x = m$ to be zero. In particular, $g^{(n)}(z) = 0$ for the random variable $z$ lies between $m$ and $X$. 
(ii). \( E[(X - m)^3] = 0 \), implies that all other central moments of order greater than three about mean equal zero.

(iii). Remaining terms beyond third sum up to a zero.

We use the following polynomial expansion later in the paper. The series expansion \((x + y)^n\) is symmetric with respect to the changes of variables \(x, y\), so does convergence region. The expression in parenthesis following of the series, indicates the region of convergence. If not otherwise indicated, it is to be understood that the series converges for all values of the variable.

\[
(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots, \text{ where } y^2 < x^2.
\]

**III. EXACTNESS OF THE EXPECTED VALUE:**

In this section, attention is drawn to the conditions for which an exactness of the approximation can be achieved.

**Proposition 1** If \( g(\cdot) \) is a polynomial of degree 2, then the statistical differential approximation for the expected value of the transformations of the random variable \( X \), \( E[g(X)] \) is exact.

The above proposition is extended to involve third degree polynomial in the following manner.

**Proposition 2** If \( g(\cdot) \) is a polynomial of degree 3, then the statistical differential approximation for the expected value of the transformations of the random variable \( X \), \( E[g(X)] \) is exact, provided \( E[(X - m)^3] = 0 \).

The natural question is that could this procedure be so extended to the next higher degree polynomial by requiring vanishing fourth central moment about mean of the random variable, (in addition to what have been already assumed). A more generalized version of the above proposition is the following.

Suppose \( g(\cdot) \) is a polynomial of degree \( j \), then requiring \( E[(X - m)^k] = 0 \), for all \( 3 \leq k \leq j \) will do the job! But, for \( k = 4 \), this gives that \( E[(X - m)^4] = 0 \), which immediately implies that we are dealing with a constant random variable, and there is no need to worry about anything else.

The condition of the last assertion leads to a nice relation of the 3rd moment of the random variable \( X \).

**Theorem 1** If \( X \) is a random variable such that its mean and variance exist and are known, together with the property, \( E[(X - m)^3] = 0 \), then 3rd moment of the random variable \( X \), \( E[X^3] \) exists, is finite, and satisfies the relation,

\[
E[X^3] = (E[X])^3 \left\{ 3 \cdot \frac{Var[X]}{(E[X])^2} + 1 \right\}.
\]
Proof.

From the series expansion for \((x + y)^n\) with \(y^2 < x^2\), and letting \(x = X - m\), and \(y = m > 0\), where \(E[X] = m\), we obtain,

\[
X^j = (X-m)^j + j(X-m)^{j-1}m + \frac{j(j-1)}{2!}(X-m)^{j-2}m^2 + \frac{j(j-1)(j-2)}{3!}(X-m)^{j-3}m^3 + \cdots
\]

\[
+ \frac{j(j-1)(j-2) \cdots 3}{(j-2)!}(X-m)^{j-2}m^2 + \frac{j(j-1)(j-2) \cdots 2}{(j-1)!}(X-m)^{j-1}m + j^j
\]

For \(j = 3\), taking the expected value, and using \(E[(X - m)^3] = 0\), we obtain,

\[
E[X^3] = (E[X])^3 \left\{ 3 \cdot \frac{\text{Var}[X]}{(E[X])^2} + 1 \right\},
\]

as asserted.

Our next task would be to see, are there other conditions for which this approximation is exact? One of the results in this connection is to consider, the Peano Kernel method [5];

For any \(g \in C^{n+1}[a, b]\), the Taylor expansion with integral remainder gives,

\[
g(X) = \sum_{k=0}^{n} \frac{(X - a)^k}{k!} g^{(k)}(a) + \frac{1}{n!} \int_a^X (X - \theta)^n g^{(n+1)}(\theta) d\theta,\]

where \(a \leq X \leq b\),

and \(g \in C^{n+1}[a, b]\) means \(g(\cdot)\) is \((n + 1)\) continuously differentiable function over \([a, b]\).

Based on the above formula, we have:

**Theorem 2** If \(g(\cdot)\) has derivatives of order \(j \leq 3\), \(g \in C^3[I]\), where \(I\) is some interval containing the range of \(X\), and \(E\left(\int_m^X (X - \theta)^2 g''(\theta) d\theta\right) = 0\), then the statistical differential approximations for the expected value of the transformations of the random variable \(X\), \(E[g(X)]\) is exact. Otherwise, it will be exact up to the error term \(O\left(E\left(\int_m^X (X - \theta)^2 g''(\theta) d\theta\right)\right)\).

**Proof.**

The Peano Kernel method with \(n = 2\), and \(a = m\), gives

\[
g(X) = \sum_{k=0}^{2} \frac{(X - m)^k}{k!} g^{(k)}(m) + \frac{1}{2!} \int_m^X (X - \theta)^2 g''(\theta) d\theta.
\]

Taking expected values, we have

\[
E[Y] = E[g(X)] = g(m) + \frac{g''(m)}{2!} \cdot \text{Var}[X] + \frac{1}{2!} E\left(\int_m^X (X - \theta)^2 g''(\theta) d\theta\right).
\]

Now, the assertion of this theorem follows from the last equation.

The counter examples given in [4] have this condition satisfied. The theorem also provides us to extend the statistical differential approximation beyond the third term, by requiring appropriate number of central moments about mean.
Theorem 3 If \( g(\cdot) \) has derivatives of order \( j \leq n \), \( g \in C^{n+1}[I] \), where \( I \) is some interval containing the range of \( X \), and \( E\left( \int_m^X (X - \theta)^n g^{(n+1)}(\theta) d\theta \right) = 0 \), then the statistical differential approximations for the expected value of the transformations of the random variable \( X \), \( E[g(X)] \) is exact. Otherwise, it will be exact up to the error term \( O(\int_m^X (X - \theta)^n g^{(n+1)}(\theta) d\theta) \).

One of the shortcomings of the last two theorems is that depending on the nature of the function \( g(\cdot) \), the verification of this condition may be just as difficult as finding \( E[g(X)] \) in some cases.

IV. EXACTNESS OF THE VARIANCE:

By definition,
\[
\text{Var}[Y] = E\left[ (g(X) - E[g(X)])^2 \right].
\]

Truncating the Taylor series for \( g(\cdot) \), depending on the number of terms of the approximating required, gives \( E[Y] \), which then will be used to find \( \text{Var}[g(X)] \).

Suppose for an example, the expectation of \( g(X) \),
\[
E[Y] \approx g(m) + \frac{1}{2} g''(m) \cdot \text{Var}[X]
\]
is used, then the variance of \( g(X) \) is
\[
\text{Var}[Y] = E[(Y - E[Y])^2] \approx E\left\{ g(X) - g(m) - \frac{1}{2} g''(m) \cdot \text{Var}[X] \right\}^2.
\]
Hence, the exactness of the variance formula still holds, if the function \( g(\cdot) \), and the random variable \( X \) satisfy the condition stipulated in the theorems.

Subject to first two terms of the expression for \( E[g(X)] \), we have
\[
\text{Var}[Y] \approx [g'(m)]^2 \cdot \text{Var}[X].
\]

The applicable multivariate versions, involving covariance etc. given in [3] can also be derived in a similar manner. The necessary steps and conditions in deriving the approximate expression for \( \text{Var}[Y] \) are similar to those considered in the preceding discussion.

V. ERROR BOUNDS:

Since there are only few instances, where the approximation holds to be exact, we have no alternative, but to obtain some error bounds for this approximation. Then, it would be a question of deciding how small these bounds are. The following bounds are obtained, so that the accuracy of the approximations now entirely depend on the smallness of the error bounds, so desired.

Some of the error bounds are computed for a class of functions, \( \mathcal{L} \), defined by
\[
\mathcal{L}_g \equiv \{ g : g \in C^m[m, b] & \& |g^{(j)}(\cdot)| \leq |g(\cdot)|^{(j)} \text{ for all } j \geq 1 \}.
\]
This means that the derivatives are invariant under absolute value function. Most of the functions considered in the literatures belong to this class. Note that \( g(x) = \frac{1}{x^\alpha} \) for \( x > 0 \), and \( \alpha > 0 \), does not belong to this class.
**Theorem 4** If \( g(\cdot) \) has all derivatives of order \( n \geq 1 \) such that \( |g^{(j)}(\cdot)| \leq |g(\cdot)|^{(j)} \), for all \( j \geq 3 \), and \( E\left(|X - m|^3|g'''(X)|\right) \) exists, then the statistical differential approximations for the expected value of the transformations of the random variable \( X \), \( E[g(X)] \) is exact up to the error term \( O\left(E((X - m)^3|g'''(X)|)\right) \).

**Proof.**

From the Taylor expansion about mean, we have,
\[
g(X) - g(m) - (X - m)g'(m) - \frac{(X - m)^2}{2!}g''(m) = \frac{(X - m)^3}{3!}g'''(m) + \frac{(X - m)^4}{4!}g^{(IV)}(m) + \ldots + \frac{(X - m)^n}{n!}g^{(n)}(m) + \ldots
\]
\[
= (X - m)^3 \sum_{j=0}^{n-4} \frac{X - m}{j!(j+3)!}(g'''(m))^{(j)} + \frac{(X - m)^n}{n!}g^{(n)}(m) + \ldots
\]
\[
\leq (X - m)^3 \sum_{j=0}^{n-4} \frac{j!}{j!(j+3)!}|(g'''(m))^{(j)}| + \frac{(X - m)^n}{n!}|g^{(n)}(m)| + \ldots
\]
\[
= (X - m)^3 \sum_{j=0}^{n-4} \frac{j!}{j!(j+3)!}|(g'''(m))^{(j)}| + \frac{(X - m)^n}{(n-3)!}|(g'''(m))^{(n-3)}| + \ldots
\]

Using the fact that \( |g^{(j)}(m)| \leq |g(m)|^{(j)} \), we obtain,
\[
(2) \leq (X - m)^3 \sum_{j=0}^{n-4} \frac{j!}{j!(j+3)!}|(g'''(m))^{(j)}| + \frac{(X - m)^n}{(n-3)!}|(g'''(m))^{(n-3)}| + \ldots
\]
\[
= (X - m)^3|g'''(X)|.
\]

Taking the expected values, we obtain the required error bound as in the theorem.

The following corollary follows easily.

**Corollary 1** If \( g(\cdot) \) has bounded derivatives of order \( n \geq 1 \), and \( E\left(|X - m|^3e^{X - m}\right) \) exists, then the statistical differential approximations for the expected value of the transformations of the random variable \( X \), \( E[g(X)] \) is exact up to the error term \( O\left(E((X - m)^3e^{X - m})\right) \).

**Proof.**

The proof of this corollary easily follows from the proof of the above theorem.

The following corollary follows easily.

**Corollary 2** If \( g(\cdot) \) has bounded derivatives of order \( n \geq 1 \), and \( E\left(|X - m|^3\right) \) exists, then the statistical differential approximations for the expected value of the transformations of the random variable \( X \), \( E[g(X)] \) is exact up to the error term \( O\left(E((X - m)^3)\right) \).

**Proof.**

The proof of this corollary easily follows from the proof of the above theorem.

In the following, we use the monotonicity of the norm property of random variables, in a certain fashion, so as to find an error bound for this approximation. This is in fact, so called Lyapunov Inequality. For random variable \( X \),
\[
\{E[|X|^s]\}^{1/s} \geq \{E[|X|^r]\}^{1/r}, \text{ for all } 0 < r < s.
\]
In our case, for random variable \(X - m\), and \(0 < j < n\), we have

\[
\{E[|X - m|^n]\}^{j/n} \geq \{E[|X - m|^j]\}, \text{ for all } 0 < j < n.
\]

Thus, if we assume a particular higher absolute central moment about mean equals zero, then all other lower absolute central moments about mean will be zero. This case is not much interest to us as the approximation collapses to a trivial case. One of the required conditions, namely the condition (ii); \(E[(X - m)^3] = 0\) implies \(E[(X - m)^k] = 0\), for all \(k > 3\) has no rigorous impact in this case too. However, using the monotonicity property, we can establish an easy result.

**Theorem 5** If \(g(\cdot)\) has all derivatives of order \(n \geq 1\) such that \(g^{(j)}(\cdot) \geq 0\), for all \(n \geq j \geq 3\), and \(E[|X - m|^n]\) exists for some integer \(n \geq 1\), and let \(C = \{E[|X - m|^n]\}^{1/n} < \infty\), then the statistical differential approximations for the expected value of the transformations of the random variable \(X\), \(E[g(X)]\) is exact up to the error term \(O(g''(C + m))\).

**Proof.**

From the Taylor expansion about mean, we have

\[
g(X) - g(m) = (X - m)g'(m) + \frac{(X - m)^2}{2!}g''(m) + \frac{(X - m)^3}{3!}g'''(m) + \ldots + \frac{(X - m)^n}{n!}g^{(n)}(m),
\]

Evaluating for the absolute values of the right side of the expression, by using properties of \(g^{(j)}(\cdot)\), for all \(n \geq j \geq 3\), and then taking the expected values,

\[
E[g(X)] - g(m) - \frac{1}{2}g''(m) \cdot \text{Var}[X] \leq \frac{E|X - m|^3}{3!}(g'''(m))^{(0)} + \frac{E|X - m|^4}{4!}(g''(m))^{(1)} + \ldots + \frac{E\{X - m|^n}{n!}(g''(z))^{(n-3)}\}.
\]

Now, using Lyapunov Inequality, we have

\[
\leq \frac{E|X - m|^3}{3!}(g'''(m))^{(0)} + \frac{E|X - m|^4}{4!}(g''(m))^{(1)} + \ldots + \frac{E\{X - m|^n}{n!}(g''(z))^{(n-3)}\}.
\]

In summation notation, this equals to

\[
\leq \{E|X - m|^3\}^{3/n} \sum_{j=0}^{n-3} \frac{E|X - m|^j}{(j + 3)!} (g'''(m))^{(j)} + \frac{E|X - m|^n}{n!}(g''(z))^{(n-3)}\}
\]

\[
\leq \{E|X - m|^3\}^{3/n} \sum_{j=0}^{n-3} \frac{E|X - m|^j}{j!} (g'''(m))^{(j)} + \frac{E|X - m|^n}{(n - 3)!}(g''(z))^{(n-3)}\}
\]

\[
= \{E|X - m|^3\}^{3/n} g'''(C + m)
\]

This gives the required error bound having determined an interval containing \(z\) that is independent of the range of \(X\).

Further, assuming that all derivatives of \(g(\cdot)\) are bounded, an corollary is immediate.
Corollary 2 If $g(\cdot)$ has bounded derivatives of order $n \geq 1$, and $e^{(E[|X-m|^n])^{3/n}}$ exists for some integer $n \geq 1$, and is finite, then the statistical differential approximations for the expected value of the transformations of the random variable $X$, $E[g(X)]$ is exact up to the error term $O\left(\{E[|X-m|^n]\}^{3/n} e^{(E[|X-m|^n])^{3/n}}\right)$.

Proof.
The proof of this is essentially similar to the proof of the last theorem.

All of these results derived above can be extended for the statistical differential approximation truncated beyond third term under appropriate conditions.

SUMMARY:

This topic has been presented in a course on survival methods as a prelude to the other relevant chapters to follow, but failed to discuss the conditions under which the exactness of this approximation to hold. The lack of them, students would wonder is this approximation reasonable ?, and are also eager to find out the validity and accuracy of these results. A part of this discussion enriches rather subtle, and interesting topic, thus requiring inclusion in the future additions of [3]. Relevant rates of convergence, error analysis and similar results for other series expansions can be studied, if one needs to develop this topic for further research.

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