Setting the Quantum Integrand of M-Theory

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Abstract. In anomaly-free quantum field theories the integrand in the bosonic functional integral—the exponential of the effective action after integrating out fermions—is often defined only up to a phase without an additional choice. We term this choice “setting the quantum integrand”. In the low-energy approximation to M-theory the $E_8$-model for the $C$-field allows us to set the quantum integrand using geometric index theory. We derive mathematical results of independent interest about pfaffians of Dirac operators in $8k+3$ dimensions, both on closed manifolds and manifolds with boundary. These theorems are used to set the quantum integrand of M-theory for closed manifolds and for compact manifolds with either temporal (global) or spatial (local) boundary conditions. In particular, we show that M-theory makes sense on arbitrary 11-manifolds with spatial boundary, generalizing the construction of heterotic M-theory on cylinders.

The low-energy approximation to M-theory is a refinement of classical 11-dimensional supergravity. It has a simple field content: a metric $g$, a 3-form gauge potential $C$, and a gravitino. The M-theory action contains rather subtle “Chern-Simons” terms which, on a topologically nontrivial manifold $Y$, raise delicate issues in the definition of the (exponentiated) action. Some aspects of the problem were resolved by Witten [W1]. The key ingredients are: a quantization law for $C$ and a background magnetic current induced by the fourth Stiefel-Whitney class of the underlying manifold; an expression for the exponentiated Chern-Simons terms using an $E_8$ gauge field and an associated Dirac operator in 12 dimensions; and finally a sign ambiguity in the gravitino partition function. In [DFM] the link to $E_8$ was used to construct a model for the $C$-field and define precisely the action, assuming that the metric $g$ is fixed. The present paper gives a complete treatment of the M-theory action as a function of both $C$ and $g$. Furthermore, we treat manifolds with boundary. The boundary may have several components and each component is interpreted either as a fixed time slice (temporal boundary) or a boundary in space (spatial boundary). We do not mix temporal and spatial boundary conditions. Our discussion of spatial boundaries in §4.3 generalizes the case $Y = X \times [0, 1]$, where $X$ is a closed 10-manifold, which was described in the work of Horava and

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Witten [HW1], [HW2]. Our analysis here makes it clear that the anomaly cancellation is local. (As emphasized in [BM] the locality of anomaly cancellation in the Horava-Witten model is far from obvious.) In particular, we show that there is no topological obstruction to formulating M-theory on an 11-manifold with an arbitrary number of boundary components, provided an independent $E_8$ super-Yang-Mills multiplet is present on each component.

The analysis here is more than a cancellation of anomalies in M-theory. Already in [W1] Witten showed that there is a nontrivial Green-Schwarz mechanism canceling global anomalies on closed 11-manifolds. We go further and show that the anomaly is canceled canonically. This is a crucial distinction for the following reason. The absence of anomalies is a necessary condition for a quantum theory to be well-defined, but the cancellation mechanism depends on physically measurable choices. Put differently, there are undetermined phases if the configuration space of bosonic fields is not connected. As we explain quite generally in §4.1, the exponentiated effective action after integrating out fermionic fields is naturally a section of a hermitian line bundle with covariant derivative over the space of bosonic fields. The absence of anomalies means that the line bundle is geometrically trivializable, i.e., the covariant derivative is flat with no holonomy. If there are no anomalies then global trivializations exist, and a choice of trivialization determines the integrand of the bosonic functional integral. When we make such a choice we say we have set the quantum integrand. The general uniqueness question for settings of the quantum integrand is discussed in §5.5.

Our main result is that in M-theory there is a canonical choice of trivialization, thus a canonical setting of the quantum integrand of M-theory. The procedure by which we set the quantum integrand of M-theory is, as we have mentioned, an example of the Green-Schwarz mechanism. Quite generally, by the Green-Schwarz mechanism we mean that setting the quantum integrand involves a trivialization of the tensor product of two line bundles with covariant derivative, one coming from integration over fermionic fields and the other from the simultaneous presence of electric and magnetic current; see [F2], [F3,Part 3] for a general discussion. The integral over fermionic fields is a section of a pfaffian line bundle. In this paper we use the $E_8$-model for the C-field to define the exponentiated electric coupling. This has the advantage that the associated line bundle with covariant derivative is defined by Atiyah-Patodi-Singer $\eta$-invariants associated to the $E_8$-gauge fields. With this model, then, we can analyze both line bundles in the context of standard invariants of geometric index theory and explicitly write down the trivialization which sets the quantum integrand.

The mathematical results we apply to M-theory are given in §1 for closed manifolds and in §3 for manifolds with boundary. Determinant and pfaffian line bundles are usually considered for families of Dirac operators on even dimensional manifolds, but our interest here is in the odd dimensional case. As we explain in §1.2 there is a second natural real line bundle with covariant derivative in odd dimensions, defined using the exponentiated $\eta$-invariant, and it is isomorphic to the determinant line bundle (Proposition 1.16). This isomorphism is equivalent to a trivialization of the tensor product—the trivialization needed for the physics—since the second line bundle is real. This isomorphism
induces a real structure on the determinant line bundle in odd dimensions. Also, it induces a nonflat complex trivialization of the determinant line bundle, so gives a definition of the determinant of the Dirac operator in odd dimensions as a complex number $S$; see Remark 1.20. This definition is often used in the physics literature, and is arrived at with Pauli-Villars regularization [R1], [R2], [ADM]. However, the definition as an element of the determinant line is more fundamental. There is an important refinement (Proposition 1.31) to pfaffians in dimensions 3, 11, 19, ... which includes the dimensions of interest in M-theory: 11 for the bulk and 3 for M2-branes (§5.2). This refinement is topological in a sense made precise in Appendix B (Proposition B.2).

We take up the generalization of this isomorphism to Dirac operators on manifolds with boundary in section 3. Most often considered in the geometric index theory literature are boundary conditions of global type, which in the physics correspond to a temporal boundary. The generalization of the basic theorem to this case is straightforward (§3.2). Local boundary conditions arise in the physics from spatial boundaries, but because they do not exist for every Dirac operator they are less studied. The generalization to this case is more subtle and (in general dimensions) is the subject of the forthcoming thesis of Matthew Scholl. The applications to M-theory on manifolds with boundary appear as Theorem 4.16 (temporal boundary) and Conjecture 4.35 (spatial boundary).

Our treatment falls short by not defining precisely the partition function of the Rarita-Schwinger (gravitino) field. The definition commonly used in the literature seems ill-defined due to singularities related to the zeromodes of bosonic ghosts for supersymmetry transformations. Moreover, the derivation of the standard expression in terms of pfaffians of Dirac operators assumes an off-shell formulation of supergravity, something which is lacking in the 10- and 11-dimensional cases. Nevertheless, we take the standard expression as motivation for the line bundle with covariant derivative of which the Rarita-Schwinger partition function is a section. We present a derivation of the standard formula in Appendix A, mostly to motivate the local boundary conditions for the ghost fields which are used in section 4.3. The precise definition of the Rarita-Schwinger partition function is a general issue which we leave to future work. Another issue we do not confront is the dependence of the covariant derivative on the Rarita-Schwinger line bundle on background fluxes. Nontrivial dependence can in principle arise from terms of the form $\psi G \psi$ in the supergravity action. (There are additional terms of a similar nature in heterotic $M$-theory.) We believe the above issues will not drastically alter the discussion we give, which is based on the simple assumption that the Rarita-Schwinger partition function is a section of the line bundle in equation (2.2), equipped with the standard covariant derivative.

Some general issues of independent interest arose during our investigations. One concerns the definition of anomalies and the setting of quantum integrands for manifolds with temporal boundaries. This forms part of the discussion in §4.1 and is elaborated in §5.4 where we relate it to the Hamiltonian interpretation of anomalies. There are interesting mathematical questions which underlie that discussion, but they are not treated here. Another issue concerns boundary values for fields with automorphisms, such as gauge fields. Then the boundary condition includes a choice
of isomorphism (for example, see [FQ] where gauge theories with finite gauge groups are treated carefully), and this shows up in the physics as certain phases, such as $\theta$-angles. In §5.3 we indicate how this works for the $C$-field in M-theory.

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§1 Determinants, Pfaffians, and $\eta$-Invariants

The geometry of determinant line bundles was developed in [Q], [BF]; see [F1] for a survey. In §1.1 we recall the main points. Our discussion is phrased in general terms and applies in
arbitrary dimensions. In odd dimensions Clifford multiplication by the volume form induces a real structure on the determinant line bundle, which we explain in §1.2 by introducing a manifestly real line bundle associated to the \( \eta \)-invariant [APS] and proving it is isomorphic to the determinant line bundle. In §1.3 we prove a refinement in dimensions \( 8k + 3 \) \( (k \in \mathbb{Z}^\geq 0) \) coming from the quaternionic structure. Some details about \( \zeta \)-functions are addressed in §1.4.

§1.1 Determinant line bundle

**Definition 1.1.** Let \( T \) be a smooth manifold. A geometric family of Dirac operators parametrized by \( T \) consists of:

(i) a Riemannian manifold \( \mathcal{Y} \to T \); that is, a fiber bundle \( \mathcal{Y} \to T \), a metric on the relative tangent bundle \( T(\mathcal{Y}/T) \to \mathcal{Y} \), and a horizontal distribution \( H \) on \( \mathcal{Y} \) (thus \( H \oplus T(\mathcal{Y}/T) = T\mathcal{Y} \)); and

(ii) a bundle \( M = M^0 \oplus M^1 \to \mathcal{Y} \) of complex \( \mathbb{Z}/2\mathbb{Z} \)-graded \( \text{Cliff}(\mathcal{Y}/T) \)-modules with compatible metric and covariant derivative.

The metric and horizontal distribution determine a Levi-Civita covariant derivative on \( T(\mathcal{Y}/T) \to \mathcal{Y} \). The Riemannian metrics determine a bundle \( \text{Cliff}(\mathcal{Y}/T) \to \mathcal{Y} \) of (finite dimensional) Clifford algebras: the fiber at \( y \in \mathcal{Y} \) is the real Clifford algebra of the relative cotangent space \( T^*_y(\mathcal{Y}/T) \).

The Clifford module structure on \( M \) is given as a map

\[
\gamma: T^*(\mathcal{Y}/T) \to \text{End}(M)
\]

which obeys the Clifford relation

\[
\gamma(\theta_1)\gamma(\theta_2) + \gamma(\theta_2)\gamma(\theta_1) = -2\langle \theta_1, \theta_2 \rangle, \quad \theta_1, \theta_2 \in T^*_y(\mathcal{Y}/T), \quad y \in \mathcal{Y}.
\]

We ask that the image consist of odd skew-adjoint transformations. The compatibility in the last line of Definition 1.1 also requires that (1.2) be flat. For each \( t \in T \) the Dirac operator \( D_t = \gamma \circ \nabla \) is defined on sections of \( M|_{\mathcal{Y}_t} \to \mathcal{Y}_t \). It is odd relative to the \( \mathbb{Z}/2\mathbb{Z} \)-grading.\(^1\)

To illustrate the notation let \( T \) be a point, so \( D \) is a Dirac operator on a single manifold \( Y \). Suppose first that \( \dim Y = 2m \) is even and \( Y \) is spin. Then for the standard Dirac operator \( M = S \) is the bundle of \( \mathbb{Z}/2\mathbb{Z} \)-graded spinors with homogeneous components \( S^0, S^1 \) of complex rank \( 2^{m-1} \), the bundles of chiral spinors. The Dirac operator interchanges the chirality of homogeneous spinor fields. The covariant derivative on \( S \) is induced from the Levi-Civita covariant derivative. If

\(^1\)We remark that the \( \mathbb{Z}/2\mathbb{Z} \)-grading on \( M \) is *not* the physics grading of bosonic and fermionic fields. In our exposition here sections of \( M \)—for example, spinor fields—are treated as ordinary commuting fields. When we turn to the physics applications in §2 we use the proper action for fermionic fields.
dim $Y = 2m + 1$ is odd, then we usually say that spinors are ungraded: there is no chirality. In the $\mathbb{Z}/2\mathbb{Z}$-graded setup we can take each of $S^0, S^1$ to be the ungraded spinor bundle of complex rank $2^m$. This is compatible with the observation that for any $\mathbb{Z}/2\mathbb{Z}$-graded $\text{Cliff}(Y)$-module $M \to Y$ Clifford multiplication by the volume form provides an isomorphism $M^0 \to M^1$ if dim $Y$ is odd; see the next section for consequences. For Dirac operators with coefficients in a vector bundle $E \to Y$ take $M = S \otimes E$. We occasionally denote this Dirac operator as ‘$D_M$’.

In the application to field theory the parameter space is typically an infinite dimensional space $\mathcal{B}$ of all bosonic fields; from this point of view we study the pullback by a map $T \to \mathcal{B}$.

Given a geometric family of Dirac operators parametrized by $T$ let

$$ \mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \to T $$

be the Hilbert space bundle whose fiber at $t \in T$ is the space of $L^2$ sections of $M \big|_{\mathcal{Y}_t} \to \mathcal{Y}_t$. Assume each fiber $\mathcal{Y}_t$ is closed, i.e., compact without boundary. Then the Dirac operator $D_t$ extends to an odd self-adjoint operator on $\mathcal{H}_t$, and so $D_t^2$ to an even self-adjoint operator on $\mathcal{H}_t$. The spectrum of $D_t^2$ is nonnegative, discrete, has no accumulation points, and the eigenspaces are graded finite dimensional subspaces of $\mathcal{H}_t$. Furthermore, if $\lambda^2 > 0$ is an eigenvalue of $D_t^2$, then $D_t/\lambda$ is an isometry from the even component of the $\lambda^2$-eigenspace to the odd component. Define $\text{spec}^0(D_t^2)$ to be the spectrum of $D_t^2$ restricted to $\mathcal{H}_t^0$.

There is a canonical open cover $\{U_a\}_{a \geq 0}$ of $T$:

$$ (1.4) \quad U_a = \{ t \in T : a \notin \text{spec}^0(D_t^2) \}.$$

On $U_a$ we introduce the $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle

$$ (1.5) \quad \mathcal{H}(a) = \mathcal{H}_t^0(a) \oplus \mathcal{H}_t^1(a) \to U_a $$

whose fiber $\mathcal{H}(a)_t$ at $t \in T$ is the sum of the eigenspaces of $D_t^2$ for eigenvalues less than $a$. Then $\mathcal{H}(a)$ is smooth of finite rank, with constant rank on each component of $U_a$. Furthermore, the geometric data induces a metric and covariant derivative on $\mathcal{H}(a)$. The Dirac operator $D$ restricts to an operator $D(a)$ on $\mathcal{H}(a)$. Global geometric invariants of Dirac operators are constructed by patching invariants on $U_a$.

Recall that the determinant line $\text{Det} E$ of a finite dimensional ungraded vector space $E$ is its highest exterior power. A linear map $S : E^0 \to E^1$ between vector spaces of the same dimension has a determinant

$$ (1.6) \quad \det S \in \text{Hom}(\text{Det} E^0, \text{Det} E^1) \cong \text{Det} E^1 \otimes (\text{Det} E^0)^* $
which is the induced map on the highest exterior power. The line which appears in (1.6) is the determinant line of the $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathbb{E}^0 \oplus \mathbb{E}^1$. It is natural to grade $\text{Det} \mathbb{E}$ by $\dim \mathbb{E}$. For our purposes we take the grading to lie in $\mathbb{Z}/2\mathbb{Z}$ rather than $\mathbb{Z}$.

Returning to the family of Dirac operators, define the line bundle

$$\text{Det} \mathcal{H}(a) = \text{Det} \mathcal{H}^1(a) \otimes \text{Det} \mathcal{H}^0(a)^* \rightarrow U_a.$$  

For $b > a$ we set

$$\mathcal{H}(a,b) = \mathcal{H}^0(a,b) \oplus \mathcal{H}^1(a,b) \rightarrow U_a \cap U_b,$$

whose fiber at $t$ is the sum of the eigenspaces of $D^2_t$ for eigenvalues between $a$ and $b$. There is a canonical isomorphism

$$(1.7) \quad \text{Det} \mathcal{H}(a) \otimes \text{Det} \mathcal{H}(a,b) \rightarrow \text{Det} \mathcal{H}(b) \quad \text{on } U_a \cap U_b$$

and a canonical nonzero section$^2$ $\text{det} D(a,b)$ of $\text{Det} \mathcal{H}(a,b)$, where $D(a,b)_t : \mathcal{H}^0(a,b)_t \rightarrow \mathcal{H}^1(a,b)_t$ is the restriction of the “chiral” Dirac operator $D^0_t : \mathcal{H}^0_t \rightarrow \mathcal{H}^1_t$. From (1.7) we obtain the patching isomorphism

$$(1.8) \quad \text{Det} \mathcal{H}(a) \rightarrow \text{Det} \mathcal{H}(b) \quad \text{on } U_a \cap U_b,$$

and a cocycle identity on $U_a \cap U_b \cap U_c$, whence a global smooth line bundle $\text{Det} D \rightarrow T$. Furthermore, the sections $\text{det} D(a)$ of $\text{Det} \mathcal{H}(a)$, defined analogously to $\text{det} D(a,b)$, patch to a smooth section $\text{det} D$ of $\text{Det} D \rightarrow T$.

The patching isomorphism (1.8) preserves the $\mathbb{Z}/2\mathbb{Z}$ grading of the determinant line: the parity of $\text{Det} D_t$ is the parity of index $D^0_t$.

The metric and covariant derivative on $\mathcal{H}(a)$ induce a metric and covariant derivative on $\text{Det} \mathcal{H}(a)$, but these are not preserved by (1.8). Modify the metric and covariant derivative to obtain invariance under patching: multiply the metric on $\text{Det} \mathcal{H}(a)_t$ by

$$(1.9) \quad \prod_{a < \lambda^2 \lambda^2 \in \sigma_0(D^2_t)} \lambda^2$$

and add the 1-form

$$(1.10) \quad \text{Tr}(\nabla D \circ D^{-1}|_{\mathcal{H}^0 \oplus \mathcal{H}^0(a)})$$

to the covariant derivative on $\text{Det} \mathcal{H}(a)$. We use $\zeta$-function regularization to define (1.9) and (1.10); see §1.4 for details.

$^2$For ease of notation we write ‘$\text{det} D$’ instead of the more accurate ‘$\text{det} D^0$’.  

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Proposition 1.11. Let $D$ be a geometric family of Dirac operators on closed manifolds parametrized by $T$. Then there is a functorially associated $\mathbb{Z}/2\mathbb{Z}$-graded complex line bundle

$$\text{Det } D \to T$$

with metric and covariant derivative, and a section $\text{det } D$ which vanishes at $t \in T$ for which $D_t$ has a nonzero kernel.

There are formulas for the holonomy and curvature of the determinant line bundle, but we will not need them in this paper.

§1.2 Odd dimensions

Now suppose the fibers of $\mathcal{Y} \to T$ have odd dimension and are oriented. Let $\omega$ denote Clifford multiplication by the relative volume form; it is an odd endomorphism of $M$. Multiply by a suitable power of $\sqrt{-1}$ to arrange $\omega^2 = 1$. Also, $\omega$ commutes with $\gamma(\theta)$ for any relative cotangent vector $\theta$ and is flat, so commutes with the Dirac operator $D$ on any fiber. (This commutation is not in the graded sense.) The composition $\omega D$ is a first-order even self-adjoint operator with discrete real spectrum, the spectrum has no accumulation points, and the spectrum is unbounded both positively and negatively. Observe $(\omega D)^2 = D^2$ so that eigenvalues of $\omega D$ square to eigenvalues of $D^2$. The self-adjoint operator $\omega D$ on $H^0$ is what is usually termed the Dirac operator in odd dimensions. Let $\text{spec}^0(\omega_t D_t)$ denote the spectrum of $\omega_t D_t$ on $H^0_t$ for each $t \in T$; an eigenvalue is repeated in $\text{spec}^0(\omega_t D_t)$ according to its multiplicity.

Define the open cover $\{V_\alpha\}_{\alpha \in \mathbb{R}}$ of $T$: $V_\alpha = \{t \in T : \alpha \notin \text{spec}^0(\omega_t D_t)\}$.

Note that $U_{\alpha^2} = V_\alpha \cap V_{-\alpha}$. Define

$$\eta(\alpha) : V_\alpha \to \mathbb{R}$$

to be the $\zeta$-function regularization of $\#\{\lambda \in \text{spec}^0(\omega_t D_t) : \alpha < \lambda\} - \#\{\lambda \in \text{spec}^0(\omega_t D_t) : \lambda < \alpha\}$.

Namely, for $s \in \mathbb{C}$ with Re $s >> 0$ define

$$\eta(\alpha)_t[s] = \sum_{\lambda \in \text{spec}^0(\omega_t D_t) \setminus \{0\}} \text{sign}(\lambda - \alpha)|\lambda|^{-s} - \text{sign}(\alpha) \cdot \#\{\text{spec}^0(\omega_t D_t) \cap \{0\}\}$$

and set $\eta(\alpha)_t$ to be the value of the meromorphic continuation of $\eta(\alpha)_t[s]$ at $s = 0$. For $\alpha < \beta$ we have

$$\frac{\eta(\beta)_t}{2} = \frac{\eta(\alpha)_t}{2} - \#\{\lambda \in \text{spec}^0(\omega_t D_t) : \alpha < \lambda < \beta\} \quad \text{on } V_\alpha \cap V_\beta.$$
We use (1.12) to construct two invariants. First, let $\mathbb{T}$ denote the group of unit norm complex numbers. Then

$$\tau(\alpha) = \exp \left( 2\pi i \frac{\eta(\alpha)}{2} \right) : V_\alpha \to \mathbb{T}$$

is invariant under patching, so defines a global function

$$\tau_D : T \to \mathbb{T}.$$ 

Second, we use the integers in the last term of (1.12) to patch a principal $\mathbb{Z}$-bundle on $T$: the fiber at $t \in T$ is

$$\left\{ n : \mathbb{R} \setminus \text{spec}^0(\omega_tD_t) \to \mathbb{Z} : n(\beta) = n(\alpha) - \# \left\{ \alpha < \lambda < \beta \middle| \lambda \in \text{spec}^0(\omega_tD_t) \right\} \right\}.$$

The spectral flow of $\omega D$ around a loop in $T$ is the monodromy of this principal $\mathbb{Z}$-bundle. Also, $\eta/2$ is a section of the associated real affine bundle constructed from the translation action of $\mathbb{Z}$ on $\mathbb{R}$. Topologically, each construction determines a class in $H^1(T; \mathbb{Z})$, and the classes are equal. The reduction to $H^1(T; \mathbb{Z}/2\mathbb{Z})$ is represented geometrically by a complex line bundle $L \to T$ with compatible real structure, metric, and covariant derivative. Its fiber at $t \in T$ is

$$L_t = \left\{ f : \mathbb{R} \setminus \text{spec}^0(\omega_tD_t) \to \mathbb{C} : f(\beta) = (-1)^\# \left\{ \alpha < \lambda < \beta \middle| \lambda \in \text{spec}^0(\omega_tD_t) \right\} f(\alpha) \right\}.$$

Over $V_\alpha$ the map $f \mapsto f(\alpha)$ gives an isomorphism with the trivial bundle, and we use it to define the real structure, metric, and covariant derivative. Note that the covariant derivative has order two, that is, $L^{\otimes 2}$ is canonically geometrically trivial.

On $U_\alpha \subset T$ the Clifford multiplication $\omega$ restricts to a flat isometry $\omega(a) : \mathcal{H}^0(a) \to \mathcal{H}^1(a)$, so induces a trivialization $\det \omega$ of $\text{Det} \mathcal{H}(a) \to U_\alpha$. We use it to identify the determinant line bundle with the line bundle $L$.

**Proposition 1.16.** Let $D$ be a geometric family of Dirac operators on closed odd dimensional manifolds parametrized by $T$. Then there is a functorial trivialization $1$ of $L \otimes \text{Det} D \to T$ which is geometric in the sense that

$$|1| = 1 \quad \nabla 1 = 0.$$

It induces a real structure on $\text{Det} D$ with respect to which the section $\det D$ is real.
Proof. The fiber of $L \otimes \text{Det } D$ at $t \in T$ is

$$
(L \otimes \text{Det } D)_t = \left\{ g : \mathbb{R} \setminus \text{spec}^0(\omega_t D_t) \longrightarrow \text{Det } D_t : g(\beta) = (-1)^\# \{ \lambda \in \text{spec}^0(\omega_t D_t) : \lambda < \beta < \alpha \} g(\alpha) \right\}.
$$

Let $\alpha \geq 0$. Define $\mathbb{1}$ on $U_{\alpha^2}$ by

$$
g(-\alpha) = \frac{\det \omega(\alpha^2)}{\left| \det \omega D\big|_{\mathcal{H}^0 \oplus \mathcal{H}^0(\alpha^2)} \right|} \in \text{Det } \mathcal{H}(\alpha^2).
$$

If $\beta > \alpha$ then under the patching (1.8) we have

$$
g(-\alpha) \mapsto \frac{\det \omega(\alpha^2) \det D(\alpha^2, \beta^2)}{\left| \det \omega D\big|_{\mathcal{H}^0 \oplus \mathcal{H}^0(\alpha^2)} \right|} = \frac{\det \omega(\alpha^2) \det \omega(\alpha^2, \beta^2) \det \omega D(\alpha^2, \beta^2)}{\left| \det \omega D\big|_{\mathcal{H}^0 \oplus \mathcal{H}^0(\alpha^2)} \right|} = (-1)^\# \{ \lambda \in \text{spec}^0(\omega_t D_t) : \lambda < \beta < \alpha \} \frac{\det \omega(\beta^2)}{\left| \det \omega D\big|_{\mathcal{H}^0 \oplus \mathcal{H}^0(\beta^2)} \right|} = (-1)^\# \{ \lambda \in \text{spec}^0(\omega_t D_t) : \lambda < \beta < \alpha \} g(-\beta)
$$

which matches (1.18). Thus $\mathbb{1}$ is well-defined. The definition (1.9) of the metric shows $|\mathbb{1}| = 1$. We verify $\nabla \mathbb{1} = 0$ at the end of §1.4. Finally, $\mathbb{1}$ induces an isomorphism $\text{Det } D \cong L$, in view of the isomorphism $L \cong L^{-1}$, and if $\det D_t \neq 0$ then $\det D_t$ corresponds to the function $f \in L_t$ whose value at 0 is $f(0) = |\det \omega_t D_t| (\zeta\text{-regularized})$, so is real.

Remark 1.20. The real structure on $\text{Det } D$ is a bit indirect, as opposed, say, to the real structure on the pfaffian line bundle in $8k + 3$ dimensions, which is defined in the next subsection directly in terms of the geometry. In fact, there is a natural complex trivialization of $L$, namely the square root $\tau_D^{1/2}$ of (1.14). In the notation of (1.15) it is defined by $f(\alpha) = \exp(2\pi i \eta(\alpha)/4)$. Then Proposition 1.16 renders the ratio $\det D \cdot \tau_D^{-1/2}$ a global complex function. It is sometimes defined to be the determinant in odd dimensions, both in the mathematics [S,§4] and physics literature [R1], [R2], [ADM]. Notice that the trivialization $\tau_D^{1/2}$ has unit norm but is not flat. Rather, we can modify the covariant derivative on $L$ by the imaginary 1-form $-\nabla \tau_D^{1/2}$ and then $\tau_D^{1/2}$ is flat relative to the new covariant derivative. The 1-form used to modify the covariant derivative has a local formula—up to a factor it is the 1-form component of the integral of the usual index density over the fibers of $\mathcal{Y} \rightarrow T$. 

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\[ 1.3 \] \( 8k + 3 \) dimensions

Assume the fibers of \( Y \to T \) are closed of dimension \( 8k + 3 \) for some integer \( k \geq 0 \). Then the fibers of \( \text{Cliff}(Y/T) \to Y \) have a quaternionic structure.\(^3\) Thus assume also that the fibers of \( M \to Y \) are quaternionic\(^4\) and that the geometry respects the quaternionic structure.\(^5\) For example, if the fibers of \( Y \to T \) are spin, then we can take \( M \) to be the relative spin bundle, or the relative spin bundle tensored with a real vector bundle on \( Y \). In this case the determinant and \( \eta \)-invariant have refinements due to the fact that the eigenspaces of \( D_t^2 \) are quaternionic, so have even complex dimension.

Let \( J: M \to \overline{M} \) be the quaternionic structure. Then \( J \) commutes with \( \omega \) and \( D \), hence for each \( t \in T \) with

\[
(1.21) \quad \partial_t = J \omega_t D_t : \Gamma_{Y_t}(M^0) \longrightarrow \Gamma_{Y_t}(\overline{M^0}),
\]

and the latter is formally skew-adjoint:

\[
(1.22) \quad \int_{Y_t} \left\{ \langle \partial_t \psi_1, \psi_2 \rangle + \langle \partial_t \psi_2, \psi_1 \rangle \right\} |dy| = 0
\]

for all sections \( \psi_1, \psi_2 \in \Gamma_{Y_t}(M^0) \) of \( M^0 \big|_{Y_t} \to Y_t \). Here \( \langle \cdot, \cdot \rangle \) is the hermitian metric on \( M \)—a homomorphism from \( \overline{M} \otimes M \) to the trivial bundle—and \( |dy| \) is the Riemannian measure on \( Y_t \). The operator \( (1.21) \) extends to a skew-adjoint operator \( \partial_t : \mathcal{H}_t^0 \to \overline{\mathcal{H}_t^0} \cong (\mathcal{H}_t^0)^* \).

A skew-adjoint operator \( S: E \to E^* \) on a finite dimensional complex vector space \( E \) is equivalently a 2-form \( \omega_S \in \bigwedge^2 E^* \). Suppose \( \dim E = 2n \) is even. Then the \( \text{pfaffian} \) of \( S \) is

\[
(1.23) \quad \text{pfaff } S = \frac{\omega_S^n}{n!} \in \text{Det } E^*
\]

and the graded line \( \text{Det } E^* \) has even parity. If \( \dim E \) is odd, then \( \text{pfaff } S = 0 \) and the parity of \( \text{Det } E^* \) is odd. In all cases \( (\text{pfaff } S)^{\otimes 2} = \text{det } S \) as elements of \( (\text{Det } E^*)^{\otimes 2} \). If \( J: E \to \overline{E} \) is a quaternionic structure, then of course \( \text{dim } E \) is even. Furthermore, \( \text{det } J: \text{Det } E \to \overline{\text{Det } E} \) is a real structure. If \( S \) commutes with \( J \), then \( \text{det } J(\text{pfaff } S) = \overline{\text{pfaff } S} \), i.e., \( \text{pfaff } S \) is a real element of \( \text{Det } E^* \).

\(^3\)The complex conjugate \( \overline{E} \) of a complex vector space \( E \) is defined thus: \( \overline{E} = E \) as sets, the additional laws on \( E \) and \( \overline{E} \) are equal, the scalar multiplication is conjugated. Write \( \bar{e} \in \overline{E} \) for the element which equals \( e \in E \). Then for \( \lambda \in \mathbb{C} \) we have \( \bar{\lambda} \cdot \bar{e} = \lambda \cdot e \). A quaternionic structure is a \textit{linear} map \( J: E \to \overline{E} \) such that \( \bar{J}J = -\text{id}_E \). (A linear map \( E \to \overline{E} \) is often termed an antilinear map on \( E \), but we prefer to use only linear maps.)

\(^4\)In the physics literature the quaternionic structure on spinor fields is usually written “\( J\psi = C\psi^* \)”, where \( C \) is the charge conjugation matrix. Hence \( J \) is usually regarded as anti-linear.

\(^5\)To wit, if \( J: M \to \overline{M} \) denotes the quaternionic structure, then \( J \) is unitary and \( \nabla J = 0 \). The unitarity implies

\[
(1.24) \quad (J\psi, \psi') = -(J\psi', \psi), \quad \psi, \psi' \in M.
\]

In addition \( \gamma(\theta) \in \text{End } M \) is quaternion linear for each cotangent vector \( \theta \).

\[^{11}\]
Recall from (1.4), (1.5) the open cover \( \{ U_a \}_{a \geq 0} \) of \( T \) and the finite rank complex vector bundles \( \mathcal{H}^0(a) \to U_a \). Now \( \mathcal{D} \) restricts to a skew-adjoint operator \( \mathcal{D}(a) : \mathcal{H}^0(a) \to \mathcal{H}^0(a)^* \), so defines

\[
\text{pfaff } \mathcal{D}(a) \in \text{Det } \mathcal{H}^0(a)^*.
\]

Since \( \mathcal{H}^0(a) \) is quaternionic, \( \text{Det } \mathcal{H}^0(a)^* \) has even parity, a real structure, and \( \text{pfaff } \mathcal{D}(a) \) is even. Analogous to (1.8) we patch on \( U_a \cap U_b \) using \( \text{pfaff } \mathcal{D}(a,b) \in \text{Det } \mathcal{H}^0(a,b)^* \). To patch the metric and covariant derivative we alter the correction factors (1.9) and (1.10) using the fact that every eigenvalue has even multiplicity. Namely, write \( \text{spec}^0(D_t^2) = \mathbb{H} \text{spec}^0(D_t^2) \cup \mathbb{H} \text{spec}^0(D_t^2) \) and replace (1.9) by its square root

\[
\prod_{a < \lambda^2} \lambda^2
\]

and (1.10) by

\[
\text{Tr}_{\mathbb{H}} \left( \nabla D \circ D^{-1} \big|_{\mathcal{H}^0 \oplus \mathcal{H}^0(a)} \right)
\]

Here \( \text{Tr}_{\mathbb{H}} \) is the trace of a quaternion linear operator on a quaternionic vector space.

**Proposition 1.26.** Let \( D \) be a geometric family of Dirac operators on closed \((8k+3)\)-dimensional manifolds parametrized by \( T \). Then there is a functorially associated complex line bundle

\[
\text{Pfaff } \mathcal{D} \to T
\]

of even parity with metric and covariant derivative, a compatible real structure, and a real section \( \text{pfaff } \mathcal{D} \) which vanishes at \( t \in T \) for which \( D_t \) has a nonzero kernel. There is a canonical isomorphism \( \text{Det } D \cong (\text{Pfaff } \mathcal{D})^{\otimes 2} \) which preserves the real structure, metric, and covariant derivative. Under this isomorphism \( \text{det } D = (\text{pfaff } \mathcal{D})^{\otimes 2} \).

**Remark 1.27.** Since the covariant derivative on \( \text{Pfaff } \mathcal{D} \) has order two, it follows that \( \text{Det } D \) has a canonical geometric trivialization, i.e., a section 1 which satisfies (1.17).

**Proof.** We comment only on the isomorphism. On \( U_a \) it is given as

\[
\text{det}(J \omega) \otimes \text{id}_{\text{Det } \mathcal{H}^0(a)^*} : \text{Det } \mathcal{H}^1(a) \otimes \text{Det } \mathcal{H}^0(a)^* \to (\text{Det } \mathcal{H}^0(a)^*)^{\otimes 2}.
\]

The isomorphism commutes with the patching (1.8) on \( U_a \cap U_b \) and carries \( \text{det } D \) to \( (\text{pfaff } \mathcal{D})^{\otimes 2} \). The latter fact shows that it preserves the real structures; that it preserves the metrics and covariant derivatives follows from the formulas (1.9)/(1.24) and (1.10)/(1.25).

The quaternionic structure also leads to a refinement of the \( \eta \)-invariant.

Proposition 1.28. Let $D$ be a geometric family of Dirac operators on closed $(8k + 3)$-dimensional manifolds parametrized by $T$. Then there is a functorially associated global function

\[(1.29) \quad \tau^{1/2}_D : T \to \mathbb{T}\]

whose square is (1.14).

To construct $\tau^{1/2}_D$ we simply modify (1.12) to

$$\frac{\eta(\beta)_t}{4} = \frac{\eta(\alpha)_t}{4} - \# \{ \lambda \in \mathbb{H} \operatorname{spec}^0(\omega_t D_t) : \alpha < \lambda < \beta \} \quad \text{on } V_\alpha \cap V_\beta$$

and exponentiate as in (1.13). The quaternionic structure also leads to a principal $\mathbb{Z}$-bundle over $T$ whose square (or double) is the one mentioned after (1.14), as well as a square root of the line bundle $L \to T$ constructed in (1.15):

\[(1.30) \quad L^{1/2}_t = \left\{ f : \mathbb{R} \setminus \mathbb{H} \operatorname{spec}^0(\omega_t D_t) \to \mathbb{C} : f(\beta) = (-1)^{\# \{ \lambda \in \mathbb{H} \operatorname{spec}^0(\omega_t D_t) : \alpha < \lambda < \beta \} } f(\alpha) \right\} .\]

Note that the square $L \to T$ has a canonical geometric trivialization.

Finally, we have the following refinement of Proposition 1.16.

**Proposition 1.31.** Let $D$ be a geometric family of Dirac operators on closed $(8k + 3)$-dimensional manifolds parametrized by $T$. There is a functorial geometric trivialization $1$ of $L^{1/2} \otimes \text{Pfaff } \partial$.

The trivialization $1$ is real, in addition to satisfying (1.17). Proposition 1.31 is essentially a topological statement, as is explained in Appendix B.

**Proof.** Replace (1.19) with

$$f(-\alpha) = \frac{\text{pfaff } J(\alpha^2)}{\left| \text{pfaff } \omega D \right|_{\mathcal{H}^0 \otimes \mathcal{H}^0(\alpha^2)}} \in \text{Det } \mathcal{H}^0(\alpha^2)^*,$$

where $J(\alpha^2)$ is the restriction of $J$ to $\mathcal{H}^0(\alpha^2)$ and the denominator is the $\zeta$-regularized product

$$\prod_{\alpha \in \mathcal{H} \operatorname{spec}^0(\omega_t D_t)} |\lambda|.$$ 

**Remark 1.32.** The square root $\tau^{1/4}_D$ of (1.29) is a complex section of $L^{1/2}$. In the notation of (1.30) it is defined by $f(\alpha) = \exp(2\pi i \eta(\alpha)/8)$. Its role is not analogous to that of $\tau^{1/2}_D$ in Remark 1.20,
which is used as a trivialization to define a complex determinant: the real structure on Pfaff $\mathcal{D}$ comes directly from the geometry and hence should be respected.

A direct consequence of Proposition 1.31 is that the product

$$\text{Pfaff } \mathcal{D} \cdot \tau_D^{-1/4} : T \to \mathbb{C}$$

is a well-defined global function. This product appears as part of the M-theory action; see §2.

§1.4 $\zeta$-functions

Let $D$ be a geometric family of Dirac operators on closed manifolds parametrized by $T$, and work on the open set $U_a \subset T$ of (1.4). Define the $\zeta$-functions

\begin{equation}
\zeta(a)[s] = \text{Tr} \left( (D^2)^{-s} \big|_{\mathcal{H}^0 \oplus \mathcal{H}^0(a)} \right)
\end{equation}

and

\begin{equation}
A(a)[s] = \text{Tr} \left( (D^2)^{-s} \nabla D D^{-1} \big|_{\mathcal{H}^0 \oplus \mathcal{H}^0(a)} \right).
\end{equation}

For $\text{Re}(s)$ sufficiently large $\zeta(a)[s]$ is a smooth function and $A(a)[s]$ a smooth 1-form on $U_a$. The basic analytic results are: the function $\zeta(a)[s]$ has a meromorphic continuation to $s \in \mathbb{C}$ which is holomorphic at $s = 0$; the function $A(a)[s]$ has a meromorphic continuation to $s \in \mathbb{C}$ with a simple pole at $s = 0$. The product in (1.9) is, by definition,

$$\exp \left( -\frac{d}{ds} \bigg|_{s=0} \zeta(a)[s] \right)$$

and the trace in (1.10) is

$$\frac{d}{ds} \bigg|_{s=0} \left( s A(a)[s] \right).$$

Rewrite (1.33) as

\begin{equation}
\zeta(a)[s] = \text{Tr} \left( (D^1 D^0)^{-s} \right),
\end{equation}

where $D = \begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix}$ relative to $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ and we omit from the notation the restriction to $\mathcal{H}^0 \oplus \mathcal{H}^0(a)$. Applying the differential on $T$ we find

\begin{equation}
d\zeta(a)[s] = -s \text{Tr} \left( (D^1 D^0)^{-s+1} (\nabla D^1 \circ D^0 + D^1 \circ \nabla D^0) \right).
\end{equation}

Since $D$ is self-adjoint, and using the cyclicity of the trace, we conclude

\begin{equation}
d\zeta(a)[s] = -2s \text{Re} \text{Tr} \left( (D^1 D^0)^{-s} \nabla D^1 \circ (D^1)^{-1} \right) = -2s \text{Re} A(a)[s].
\end{equation}
These manipulations are valid for Re $s >> 0$, and by analytic continuation for all $s$. The $s$-derivative at $s = 0$ is used in the proof of Proposition 1.11 to show that the metric and covariant derivative on $\text{Det} D$ are compatible.

Now assume the manifolds are odd dimensional, as in §1.2. Then $\omega D^0 = D^1 \omega$ and we have

$$\text{Tr} \left( (D^1 D^0)^{-(s+1)} D^1 \nabla D^0 \right) = \text{Tr} \left( (D^1 D^0)^{-(s+1)} D^1 \omega^2 \nabla D^0 \right)$$

$$= \text{Tr} \left( (D^1 D^0)^{-(s+1)} \omega D^0 \nabla D^1 \omega \right)$$

$$= \text{Tr} \left( \omega (D^0 D^1)^{-s} (D^1)^{-1} \nabla D^1 \omega \right)$$

$$= \text{Tr} \left( \omega D^0 (D^1 D^0)^{-(s+1)} \nabla D^1 \omega \right)$$

$$= \text{Tr} \left( (D^1 D^0)^{-(s+1)} \nabla D^1 \circ D^0 \right)$$

Comparing with (1.34) and (1.35) we see that $A(a)[s]$ is real, and so we may omit ‘Re’ in (1.35).

Now we complete the proof of Proposition 1.16 by checking that $\nabla 1 = 0$, where $1$, defined in (1.19), is in our current notation.

$$g(-\alpha) = \frac{\det \omega(\alpha^2)}{\exp \left( -\frac{1}{2} \frac{d}{ds} |_{s=0} \zeta(\alpha^2)[s] \right)} \in \text{Det} \mathcal{H}(\alpha^2).$$

The covariant derivative on $\text{Det} \mathcal{H}(\alpha^2)$ is $\nabla^0 + \frac{d}{ds} |_{s=0} (sA(\alpha^2)[s])$ for $\nabla^0$ the natural covariant derivative. Note $\nabla^0 \det \omega(\alpha^2) = 0$. A short computation shows that (1.35) (without ‘Re’) immediately implies $\nabla 1 = 0$.

§2 M-theory action on closed manifolds

M-theory is an 11-dimensional theory. Let $Y$ be an 11-manifold, which in this section we assume is closed. There are two bosonic fields: a Riemannian metric $g$ and a field $C$ which is locally a 3-form on $Y$. We review the $C$-field below. There is a single fermionic field, the gravitino or Rarita-Schwinger field $\psi$, and to accommodate it we also assume that $Y$ is spin.$^6$ Let $S^0 \oplus S^1 \rightarrow Y$ be the basic $\mathbb{Z}/2\mathbb{Z}$-graded complex $\text{Cliff}(Y)$-module. As explained in the paragraph following (1.3) we may take $S^0 = S^1$ to be the standard ungraded rank 32 complex spin bundle. Set $RS = RS^0 \oplus RS^1 = S \otimes T^*Y$. Then the Rarita-Schwinger field $\psi$ is a section of $RS^0$. Note that $S$, and so also $RS$, carries a quaternionic structure. Let $D_S$ be the Dirac operator on $S$ and $D_{RS}$ the Dirac operator on $RS$.

$^6$M-theory respects parity reversal, so exists on non-oriented and even non-orientable manifolds (which then carry a certain pin structure). In this paper, though, we assume that all manifolds are oriented.
The effective action for the gravitino is usually written as

\begin{equation}
\exp(-\Gamma_{\text{gravitino}}) = \frac{\text{pfaff } D}{(\text{pfaff } S)^3},
\end{equation}

which is a function of the metric $g$. See Appendix A for a formal derivation and discussion. Recall also from the introduction that our treatment drops certain terms from the supergravity action. While (2.1) is problematic, what seems certain is that in a definitive treatment of the gravitino the effective action is a section of the line bundle

\begin{equation}
\text{Pfaff } D \otimes (\text{Pfaff } S)^{\otimes (-3)} \rightarrow T
\end{equation}

over any family of Riemannian spin manifolds $\mathcal{Y} \rightarrow T$. In §1.3 we showed that (2.2) carries a natural real structure, metric, and covariant derivative. Furthermore, its square is canonically trivial (including its geometry).

We use the model for the $C$-field expounded in detail in [DFM], and defer to that paper for details. A $C$-field is an object in a groupoid $C_Y$. In the quantum theory one is instructed to integrate over the space of equivalence classes. The groupoid $C_Y$ depends on the Riemannian metric $g$, so the space of equivalence classes of bosonic fields in M-theory is a fiber bundle: the base is the space of equivalence classes of metrics, fiber the space of equivalence classes of $C$-fields for a fixed metric. An object in $C_Y$ is a pair $(A, c)$ consisting of a connection $A$ on a principal $E_8$-bundle $P \rightarrow Y$ and a 3-form $c \in \Omega^3(Y)$. We do not review in detail the morphisms or the space of equivalence classes; see [DFM,§3]. Our interest here is in the factor in the exponentiated action, which is usually written in terms of a local 3-form $C$ as

\begin{equation}
\text{“exp} \left(2\pi i \int_Y \left\{ \frac{1}{6} C \wedge G \wedge G - C \wedge I_8(g) \right\} \right),
\end{equation}

where $I_8(g)$ is a certain combination of Pontrjagin forms (representing $(4p_2 - p_1^2)/192$).

We now make (2.3) precise in this model for the $C$-field.

Definition 2.4. Let $T$ be a smooth manifold. A family of M-theory data on closed manifolds parametrized by $T$ consists of:

(i) a family of closed spin Riemannian 11-manifolds $\mathcal{Y} \rightarrow T$ in the sense of Definition 1.1(i);
(ii) a principal $E_8$-bundle $P \rightarrow \mathcal{Y}$ with a connection $A$; and
(iii) a 3-form $c \in \Omega^3(\mathcal{Y})$.

\footnote{In the notation ‘$A$’ is understood to signify the connection as well as the bundle $P$ which carries it.}
Let $g$ denote the metric. Associated to this data are Cliff(${\mathcal Y}/T$)-modules $S \to {\mathcal Y}$ and $RS \to {\mathcal Y}$ generalizing those described in the previous paragraph for a single manifold. We also have a Cliff(${\mathcal Y}/T$)-module $M(A) = S \otimes \text{Ad} P$ constructed from the induced connection on the real adjoint vector bundle; it also carries a compatible quaternionic structure.

**Remark 2.5.** A family of M-theory data is more information than a family of fields on 11-manifolds, all parametrized by $T$. For example, the connection $A$ has components in directions transverse to the fibers, as does the 3-form $c$. These extra components do not affect the definition of the action (2.6) below, but they do enter into the definition of the covariant derivative on the line bundles (2.2) and (2.7). In the bosonic functional integral one is meant to work once and for all with a fixed family parametrized by equivalence classes of bosonic fields.

Assume that the fibers of ${\mathcal Y} \to T$ are closed. Then (2.3) is, by definition [DFM,(4.12)],

$$
\exp(-S_{\text{gauge}}(g, (A, c))) = \tau^{1/2}_{DM(A)} \cdot \tau^{1/4}_{RS} \cdot \tau^{-3/4}_{S} \cdot \exp\left(2\pi i \int_{{\mathcal Y}/T} \Omega(g, (A, c))\right).
$$

Here $\tau^{1/2}_{DM(A)}$ is the global invariant (1.29) for the Clifford module $M(A)$; the next two factors are defined in Remark 1.32; and $\Omega(g, (A, c))$ is an 11-form whose precise formula is not of interest here. Observe that the first and last factors of (2.6) are globally defined functions on $T$, whereas the product of the two middle factors is a section of a complex line bundle, namely the bundle

$$
L^{1/2}_{RS} \otimes \left(L^{1/2}_{S}\right)^{(-3)} \to T
$$

in the notation of §1.3.

**Theorem 2.8.** Let ${\mathcal Y} \to T$ be a family of M-theory data parametrized by $T$ with closed fibers. Then the product $\exp(-\Gamma_{\text{gravitino}}) \cdot \exp(-S_{\text{gauge}})$ is a well-defined function on $T$.

**Proof.** Use the isomorphism of Proposition 1.31 to construct a trivialization of the tensor product of (2.2) and (2.7).

**Remark 2.9.** The $E_8$ model for the $C$-field includes equivalences in the groupoid $C_Y$ and it is crucial that (2.6) is invariant under those equivalences. This is shown in [DFM,§4].

**Remark 2.10.** The setup of Definition 2.4 and Theorem 2.8 includes the cancellation of gravitational anomalies. To see this, let $Y$ be a fixed closed spin 11-manifold and $\text{Met}(Y)$ the space of metrics on $Y$. Suppose $\text{Diff}'(Y)$ is a group of spin diffeomorphisms of $Y$ which acts freely on the space of metrics. Then

$$(\text{Met}(Y) \times Y)/\text{Diff}'(Y) \to \text{Met}(Y)/\text{Diff}'(Y)$$
is a family of closed spin 11-manifolds with a canonical metric along the fibers. This is part of a family of M-theory data parametrized by $\text{Met}(Y)/\text{Diff}'(Y)$. The application of Theorem 2.8 to this family is the statement that the quantum integrand is invariant under $\text{Diff}'(Y)$.

We interpret (2.6) as an “electric coupling” factor in the exponentiated action. That is more evident in the heuristic form (2.3), though the cubic self-coupling of the gauge field is a notable departure from typical electric couplings. The anomaly cancellation and setting of the quantum integrand in Theorem 2.8—of a fermionic pfaffian with an electric coupling—is an example of the Green-Schwarz anomaly cancellation mechanism.

§3 (8k + 3)-Dimensional Manifolds with Boundary

We return to the general mathematical theory of geometric invariants of Dirac operators, now on compact manifolds with boundary. There are two types of boundary conditions for Dirac operators. The global Atiyah-Patodi-Singer boundary conditions [APS] are ubiquitous in the index theory literature. Local boundary conditions do not exist for arbitrary Dirac operators in even dimensions, so have not been as widely studied. There is, however, a general class of local boundary conditions—for both even and odd dimensions—which lead to geometric invariants analogous to those in the closed case. The general story, in all dimensions, is discussed in Matthew Scholl’s thesis [Sch]. Here we state refinements of his results in the $(8k + 3)$-dimensional case, as this is what we need for M-theory.

§3.1 Generalities

Let $Y$ be an oriented odd dimensional Riemannian manifold with boundary and $M \to Y$ a complex $\mathbb{Z}/2\mathbb{Z}$-graded $\text{Cliff}(Y)$-module. Recall from §1.2 that there is an odd endomorphism $\omega$ of $M$—a multiple of Clifford multiplication by the volume form—which is flat, commutes with the Dirac operator $D$, and squares to the identity. Set $N = M|_{\partial Y}$. We decompose the restriction of $\omega$ to $\partial Y$ as

\begin{equation}
\omega = \gamma^\nu \cdot \omega^\partial \quad \text{on } \partial Y,
\end{equation}

where $\gamma^\nu$ is Clifford multiplication by the dual to the outward-pointing unit normal vector, and so $\omega^\partial$ is Clifford multiplication by a multiple of the volume form on $\partial Y$. Then $\gamma^\nu$ and $\omega^\partial$ are commuting endomorphisms of $N$, with $\gamma^\nu$ odd, $\omega^\partial$ even, and

\[(\gamma^\nu)^2 = (\omega^\partial)^2 = -\text{id}_N.\]
Decompose

\[ N = N_+ \oplus N_- \]

according to the eigenvalues \( \pm \sqrt{-1} \) of \( \omega^0 \). Since \( \omega^0 \) is even the homogeneous pieces \( N^0, N^1 \) decompose separately.

The boundary \( \partial Y \) is an oriented Riemannian manifold and \( N \to \partial Y \) restricts to a \( \text{Cliff}(\partial Y) \)-module. Let \( \gamma \) denote the Clifford action. Then \( N \) with the \( \mathbb{Z}/2\mathbb{Z} \)-grading \( (3.2) \) is a \( \text{Cliff}(\partial Y) \)-module for a modified Clifford action: \( \theta \in T^*_y(\partial Y) \) acts as \( \gamma^\nu \gamma(\theta) \). The modified Clifford action preserves \( N^0 \subset N \), whence

\[ N^0 = N^0_+ \oplus N^0_- \]

is a \( \mathbb{Z}/2\mathbb{Z} \)-graded \( \text{Cliff}(\partial Y) \)-module. The associated Dirac operator is

\[ D^0 = \begin{pmatrix} 0 & D^0_- \\ D^0_+ & 0 \end{pmatrix} \]

relative to the decomposition \( (3.2) \). By the previous remark \( (3.4) \) operates on sections of \( (3.3) \), i.e., maps sections of \( N^0 \) to sections of \( N^0 \).

Assume now that \( Y \) and \( M \) are products near the boundary. In other words, postulate for some \( \epsilon > 0 \) an isometry of the product \( (-\epsilon, 0] \times \partial Y \) onto a neighborhood \( U \) of \( \partial Y \subset Y \) and of the pullback of \( N \to \partial Y \) to \( (-\epsilon, 0] \times \partial Y \) with \( M|_U \), including the metric and covariant derivative. Then \( \gamma^\nu \) extends over \( U \) as do the decompositions \( (3.1), (3.2) \) and the Dirac operator \( (3.4) \). On \( U \) the Dirac operator \( D \) decomposes as

\[ D = \gamma^\nu(\partial_\nu - D^0), \]

where \( \partial_\nu \) is differentiation along the unit normal vector, defined using the product structure on \( M|_U \).

Specialize to \( \dim Y = 8k + 3 \) and \( M \) quaternionic, as in §1.3. The quaternionic structure \( J \) commutes with \( \omega^0 \), which has imaginary eigenvalues, and since \( J: M^0 \to \overline{M^0} \) it follows that \( J(N^0_\pm) \subset \overline{N^0_\pm} \). In particular, neither \( N^0_+ \) nor \( N^0_- \) is quaternionic. Rather, \( J \) induces a pointwise symplectic pairing

\[ \lambda_y, \check{\lambda}_y \mapsto \langle J_y \lambda_y, \check{\lambda}_y \rangle, \quad \lambda_y, \check{\lambda}_y \in N^0_y, \quad y \in \partial Y. \]
The homogeneous subspaces \((N^0_+)\) and \((N^0_-)\) are lagrangian and pair nondegenerately under (3.5). Assume \(Y\) is compact. By integration we obtain a global version of the symplectic form:

\[
\ll \lambda, \tilde{\lambda} \gg = \int_{\partial Y} \langle J\lambda, \tilde{\lambda} \rangle |dy|, \quad \lambda, \tilde{\lambda} \in \Gamma_{\partial Y}(N^0).
\]

The boundary Dirac operators

\[
D^\partial_\pm : \Gamma_{\partial Y}(N^0_\pm) \longrightarrow \Gamma_{\partial Y}(N^0_\mp)
\]

are skew-adjoint relative to (3.6).

Recall from (1.21) that on \(Y\) we set \(\mathcal{D} = J_\omega D : \Gamma_Y(M^0) \to \Gamma_Y(\overline{M}^0)\). On a closed manifold this operator is formally skew-adjoint, but on an arbitrary compact manifold the skew-adjointness equation (1.22) acquires a boundary term:

\[
\int_Y \{ \langle \mathcal{D}\psi, \tilde{\psi} \rangle + \langle \mathcal{D}\tilde{\psi}, \psi \rangle \} |dy| = \int_{\partial Y} \langle J_\omega \omega^\partial \psi^\partial, \tilde{\psi}^\partial \rangle |dy|, \quad \psi, \tilde{\psi} \in \Gamma_Y(M^0).
\]

Write \(\psi|_{\partial Y} = \psi^\partial_+ + \psi^\partial_-\) and \(\tilde{\psi}|_{\partial Y} = \tilde{\psi}^\partial_+ + \tilde{\psi}^\partial_-\). Then, up to a factor, the boundary term in (3.8) may be expressed in terms of (3.6) as

\[
\ll \psi^\partial_+, \tilde{\psi}^\partial_- \gg + \ll \psi^\partial_-, \tilde{\psi}^\partial_+ \gg.
\]

An elliptic boundary condition for \(\mathcal{D}\) is a suitable “half-dimensional” subspace of \(\Gamma_{\partial Y}(N^0)\). It determines a formally skew-adjoint operator if the subspace is lagrangian with respect to (3.9). For a local elliptic boundary condition the subspace is the space of sections of a lagrangian subbundle of \(N^0 \to \partial Y\).

§3.2 Global boundary conditions

Atiyah-Patodi-Singer [APS] introduced global boundary conditions for first-order Dirac operators in the even dimensional case. Generalizations and the odd dimensional analog were studied in many works: see, for example, [SW] and the references therein. We begin with a special class of boundary conditions [J] used to construct the global invariant (3.11) below.

Let \(\mathcal{Y} \to T\) be a family of compact \((8k + 3)\)-manifolds and \(M \to \mathcal{Y}\) a \(\mathbb{Z}/2\mathbb{Z}\)-graded \(\text{Cliff}(\mathcal{Y}/T)\)-module with compatible quaternionic structure. Set \(N = M|_{\partial \mathcal{Y}}\). On the boundary family \(\partial \mathcal{Y} \to T\) there is an induced \(\mathbb{Z}/2\mathbb{Z}\)-graded \(\text{Cliff}(\partial \mathcal{Y}/T)\)-module \(N^0 = N^0_+ \oplus N^0_-\), as in (3.3) and associated family of Dirac operators \(D^0\), as in (3.4). Let \(\mathcal{H}^0 = \mathcal{H}^0_+ \oplus \mathcal{H}^0_- \to T\) be the bundle whose fiber at \(t \in T\) is the space of \(L^2\) sections of \(N^0|_{\partial \mathcal{Y}_t} \to \partial \mathcal{Y}_t\). Write \(T = \bigcup U_a\) as in (1.4) and recall
from (1.5) the finite rank bundle $\mathcal{H}^0(a) \rightarrow U_a$. A global boundary condition for the restriction of $\mathcal{Y} \rightarrow T$ over $U_a$ is specified by a family of isometries

$$I_t: \mathcal{H}_+^0(a)_t \rightarrow \mathcal{H}_-^0(a)_t$$

which are skew-adjoint relative to the pairing (3.7):

$$\langle I_t \lambda, \bar{\lambda} \rangle + \langle I_t \bar{\lambda}, \lambda \rangle = 0, \quad \lambda, \bar{\lambda} \in \mathcal{H}_+^0(a)_t.$$ 

The corresponding boundary condition\(^8\) on $\psi \in \Gamma_{\mathcal{Y}_t}(M^0)$ is

$$\psi_+^0 + \left( I_t \oplus \frac{(D^\partial)^t}{\sqrt{(D^\partial)^2}_t} \right) \psi_-^0 = 0,$$

where $\psi^0 = \psi_+^0 + \psi_-^0 \in \Gamma_{\partial \mathcal{Y}_t}(N_0^0 \oplus N_0^0)$ is the restriction of $\psi$ to $\partial \mathcal{Y}_t$ and we decompose $(\mathcal{H}_+^0)_t$ into the direct sum of $\mathcal{H}_+^0(a)_t$ and its orthogonal complement. The Dirac operator $D_t^{(a,I)}$ is elliptic with the boundary conditions defined by $(a,I)$. Moreover, it is also formally skew-adjoint: the boundary term in (3.8) vanishes for $\psi, \bar{\psi}$ whose restriction to $\partial \mathcal{Y}_t$ satisfies (3.10), as follows immediately from (3.9) and the skew-adjointness of the operator in (3.10).

The analytic properties of $D_t^{(a,I)}$ are exactly the same as those in the closed case [DF,Appendix A]. The geometric invariants of §1.3 are defined and Proposition 1.26 holds. We do not repeat them here.

There is an important generalization of this discussion which replaces the subspace of boundary values defined by (3.10) with a subspace $W$ which is sufficiently “close” and is lagrangian with respect to (3.9). The space of admissible $W$ forms a restricted Grassmannian [Se2, Lecture 2]. We refer to [Wo] for details.

The particular invariant $\tau_{D_t^{(a,I)}}^{1/2}: U_a \rightarrow \mathbb{T}$ has a global meaning when we consider its behavior under change of global boundary condition $(a,I)$. Namely, it patches to a global section

$$\tau_{D_t^{(a,I)}}^{1/2}: T \rightarrow (\text{Pfaff } D^\partial)^{-1}$$

of the inverse pfaffian line bundle $(\text{Pfaff } D^\partial)^{-1}$ of the family of Dirac operators on $\partial \mathcal{Y} \rightarrow T$ associated to the Cliff($\partial \mathcal{Y}/T$)-module $N_0 \rightarrow \partial \mathcal{Y}$. In other words, a fiber of $(\text{Pfaff } D^\partial)^{-1}$ is the line of suitably equivariant functions $\tau$ on the space of global boundary conditions $(a,I)$, or equivalently the line of suitably equivariant functions on the restricted Grassmannian of subspaces $W$. The equivariance is defined by a cocycle $c$ which depends only on the boundary data:

$$\tau(a',I') = \tau(a,I) c((a',I'),(a,I));$$

see [DF,Theorem 1.4] for details. Note that $|\tau_{D_t}^{1/2}| = 1$. In particular, Pfaff $D^\partial$ is topologically trivial. (It is not in general geometrically trivial as $\nabla \tau_{D_t}^{1/2}$ may be nonzero.)

\(^8\)In a Hamiltonian interpretation (3.10) projects out the negative energy modes.
§3.3 Local boundary conditions

Return now to a single manifold $Y$, an $(8k + 3)$-dimensional compact Riemannian manifold; $M \to Y$ a quaternionic Cliff($Y$)-module; and $N^0 \to \partial Y$ the induced Cliff($\partial Y$)-module (3.3).

**Definition 3.13.** A local boundary condition is a flat unitary section $\epsilon$ of $\text{End} N^0 \to \partial Y$ which is even relative to (3.3), satisfies $\epsilon^2 = \text{id}_{N^0}$, anticommutes with $T^*(\partial Y) \subset \text{Cliff}(\partial Y)$, and anticommutes with $J$ in the sense that $J\epsilon = -\bar{\epsilon}J$.

The Cliff($\partial Y$)-action is defined after (3.2). It follows that the involutions $\epsilon$ and $\omega^\partial$ commute, so we can separately decompose $N^0_\pm$ and $N^0_\mp$ according to the eigenvalues $\pm 1$ of $\epsilon$:

$$N^0_\pm = N^0_\pm[+] \oplus N^0_\pm[-].$$

The domain of the Dirac operator $\partial D$ with local boundary condition $\epsilon$ is the set of all $\psi \in \Gamma_Y(M^0)$ whose restriction $\psi^\partial$ to the boundary satisfies

$$\epsilon(\psi^\partial) = \psi^\partial. \quad (3.14)$$

Because $\epsilon$ anticommutes with $J$ the boundary term in (3.8) vanishes and $\partial D$ is formally skew-adjoint. Also, these local boundary conditions are elliptic: the topological and geometric invariants of §1.3 are well-defined [Sch].

Local boundary conditions always exist in odd dimensions. For the $(8k + 3)$-dimensional case we have the special boundary conditions $\epsilon = \sqrt{-1} \omega^\partial$ and $\epsilon = -\sqrt{-1} \omega^\partial$. Furthermore, we can choose the sign independently on different components of $\partial Y$.

**Remark 3.15.** Suppose $Y$ is spin and $M = S \otimes E \to Y$, where $S$ is the $\mathbb{Z}/2\mathbb{Z}$-graded spin bundle and $E$ is a real vector bundle. Let $F = E|_{\partial Y}$. On the boundary the spin bundle splits into eigenspaces of $\omega^\partial$, so for the restriction of the even part we have $S^0|_{\partial Y} = S^0 = S^0_+ \oplus S^0_-$. A local boundary condition amounts to a splitting $F = F_+ \oplus F_-$, and the boundary condition (3.14) translates to

$$\psi^\partial \in \Gamma_{\partial Y}(S^0_+ \otimes F_+ \oplus S^0_- \otimes F_-). \quad (3.16)$$

If $\epsilon = -\sqrt{-1} \omega^\partial$ or $\epsilon = \sqrt{-1} \omega^\partial$, then $F_- = 0$ or $F_+ = 0$ and (3.16) specializes to

$$\psi^\partial \in \Gamma_{\partial Y}(S^0_+ \otimes F) \quad \text{or} \quad \psi^\partial \in \Gamma_{\partial Y}(S^0_- \otimes F).$$

Since $\epsilon$ graded commutes with Cliff($\partial Y$), it anticommutes with the Dirac operator $D^\partial$, which therefore restricts to operators

$$D^\partial[\pm]: \Gamma_{\partial Y}(N^0_+(\pm)) \longrightarrow \Gamma_{\partial Y}(N^0_+(\mp)). \quad (3.17)$$
Also, as $\epsilon$ anticommutes with $J$, the domain and codomain in (3.17) are dually paired (pointwise by (3.5)) and the operators in (3.17) are formally skew-adjoint. The operators in (3.17) are the Dirac operators $D^0[\pm]$ and $D^0[-]$ associated to the $Z/2Z$-graded $\text{Cliff}(\partial Y)$-modules

\[
N^0[\pm] = N^0_+[\pm] \oplus N^0_0\text{ and } N^0[-] = N^0_0[-] \oplus N^0_0[\pm]
\]

Now suppose $T$ parametrizes a geometric family of Dirac operators on compact $(8k+3)$-manifolds, as in the previous sections. By contrast with the global boundary conditions, a local boundary condition $\epsilon$ may be defined over the entire parameter space $T$. In particular, as remarked previously the special choices $\epsilon = \pm \sqrt{-1} \omega^{\partial}$ exist for all families.

**Theorem 3.18.** Let $D^\epsilon$ be a geometric family of Dirac operators on compact $(8k + 3)$-manifolds parametrized by $T$ with local boundary condition $\epsilon$. As in (1.21) define $\mathcal{D} = J\omega D^\epsilon$. Let $D^0[\pm]$ be the induced Dirac operators (3.17) on the boundary family. Then there is a geometric isomorphism

\[
(\text{Pfaff } \mathcal{D}) \otimes 2 \cong (\text{Pfaff } D^0[\pm]) \otimes (\text{Pfaff } D^0[-])^{-1}.
\]

The isomorphism preserves the metric and covariant derivative. There is no compatible real structure; indeed, $\text{Pfaff } \mathcal{D} \to T$ may have nontrivial curvature.

While (3.19) expresses the square of the pfaffian line bundle explicitly, in general we cannot express $\text{Pfaff } \mathcal{D}$ directly in terms of boundary data. See, for example, Proposition 1.31 for the case where the boundary is empty. But for the special boundary conditions $\epsilon = \pm \sqrt{-1} \omega^{\partial}$ we can say more. For definiteness take $\epsilon = -\sqrt{-1} \omega^{\partial}$, so that the right hand side of (3.19) is $\text{Pfaff } D^0[+] = \text{Pfaff } D^0$. Recall from (3.11) that there is a global section $\tau_D^{-1/2}$ of this line bundle, with $|\tau_D^{-1/2}| = 1$. The following simple observation defines a square root of this bundle and section.

**Lemma 3.20.** Let $K \to T$ be a smooth complex line bundle with metric and covariant derivative, and $s: T \to K$ a nonzero section. Then there is a functorially defined line bundle $K^{1/2} \to T$ with metric, covariant derivative, and nonzero section $s^{1/2}: T \to K^{1/2}$ together with an isomorphism $(K^{1/2})^{\otimes 2} \cong K$ under which $(s^{1/2})^{\otimes 2} = s$. Also, $|s^{1/2}| = |s|^{1/2}$ and $\nabla s^{1/2} = \frac{1}{2} \sum_2 s^{1/2}$.

**Proof.** The section $s$ gives an isomorphism of $K$ with the trivial bundle: the geometry is characterized by the function $|s|$ and the 1-form $\sum_2 s$. Take $K^{1/2}$ to be the trivial bundle with metric $|s|^{1/2}$ and covariant derivative $\frac{1}{2} \sum_2 s$. Under these isomorphisms both the section $s$ and its square root $s^{1/2}$ are identified with the constant function 1.
Conjecture 3.21. In the situation of Theorem 3.18 for the special boundary conditions $\epsilon = \pm \sqrt{-1} \omega^{\partial}$ there are geometric isomorphisms

\begin{align*}
Pfaff \mathcal{D} \cong (Pfaff D^{\partial})^{1/2}, & \quad \epsilon = -\sqrt{-1} \omega^{\partial}, \\
Pfaff \mathcal{D} \cong (Pfaff D^{\partial})^{-1/2}, & \quad \epsilon = \sqrt{-1} \omega^{\partial}.
\end{align*}

The square roots are defined by the sections $\tau^\pm_{1/2}$ of $(Pfaff D^{\partial})^{\pm 1}$ and Lemma 3.20. More generally, suppose $\partial Y = \sqcup (\partial Y)_i$ is written as a disjoint union of components and on each $(\partial Y)_i$ we choose either $\epsilon_i = \sqrt{-1} \omega^{\partial}$ or $\epsilon_i = -\sqrt{-1} \omega^{\partial}$. Then

\[
Pfaff \mathcal{D} \cong (Pfaff D^{\partial})^{1/2} \otimes \bigotimes_{i \text{ such that } \epsilon_i = \sqrt{-1} \omega^{\partial}} \left( Pfaff D^{(\partial Y)_i} \right)^{-1},
\]

where $D^{(\partial Y)_i}$ is the boundary Dirac operator (3.4) on the $i^{th}$ boundary component.

Notice that if $\partial Y = \emptyset$ then (3.22) reduces to Proposition 1.31.

§4 M-Theory Action on Compact Manifolds with Boundary

The two types of mathematical boundary conditions discussed in §3—global and local—correspond in physics to what we term temporal and spatial boundaries. In real time (Lorentzian signature) a temporal boundary is a spacelike hypersurface. In quantum field theory one associates a Hilbert space to a temporal boundary, and then the functional integral represents a state in the Hilbert space attached to the boundary. Global boundary conditions for fermionic fields determine a second state in the Hilbert space, and the functional integral with fixed global boundary conditions is the inner product of the two states. A spatial boundary is simply a boundary of space, so in the real time picture a boundary of each spacelike hypersurface. In quantum field theory spacetime locality requires that boundary conditions for spatial boundaries be defined pointwise, so are local. They are part of the definition of the theory.

In this section we generalize Theorem 2.8 to allow temporal or spatial boundaries. We first discuss the formal structure in general.

§4.1 Actions and Anomalies

Consider a quantum field theory defined on a closed manifold $Y$. Let $\mathcal{B}_Y$ denote the space of bosonic fields. Fields with internal symmetry, such as gauge fields and metrics, are best thought of
as objects in a groupoid, so we assume in general that \( \mathcal{B}_Y \) is a groupoid. The effective exponentiated action, after integrating out any fermionic fields in the theory, is a section

\[
\exp(-S_{\text{eff}}): \mathcal{B}_Y \rightarrow K_Y
\]

of a line bundle\(^9\) \( K_Y \rightarrow \mathcal{B}_Y \) with metric and covariant derivative \( \nabla \). The notation in (4.1) implies that morphisms (gauge transformations) in \( \mathcal{B}_Y \) are lifted to \( K_Y \) and that \( \exp(-S_{\text{eff}}) \) is invariant. The expression \( \exp(-S_{\text{eff}}) \) is meant to include everything: integrated out fermionic fields, kinetic energy factors, Chern-Simons type factors, etc. In quantum field theory one imagines that the space \( \overline{\mathcal{B}_Y} \) of equivalence classes of bosonic fields carries a measure and one defines the partition function as the integral of \( \exp(-S_{\text{eff}}) \) over \( \overline{\mathcal{B}_Y} \) with respect to that measure; correlation functions are defined similarly. Even formally this integral is not defined—one cannot add the values of \( \exp(-S_{\text{eff}}) \) at different points as they lie in distinct lines. Therefore, to define formally the quantum theory we need as well a section

\[
1: \mathcal{B}_Y \rightarrow K_Y
\]

which trivializes \( K_Y \) geometrically:

\[
\begin{align*}
|1| &= 1 \\
\nabla 1 &= 0.
\end{align*}
\]

Then

\[
\frac{\exp(-S_{\text{eff}})}{1}: \mathcal{B}_Y \rightarrow \mathbb{C}
\]

is a global function that one could integrate over \( \overline{\mathcal{B}_Y} \) if one had a measure. In some theories the effective action is naturally a function, say in a theory with only scalar fields and no fermionic fields, so a trivialization (4.2) does not enter explicitly, but in more complicated theories the effective exponentiated action is naturally in the form (4.1) and a trivialization is needed. The obstruction to the existence of a trivialization \( 1 \) which satisfies (4.3) is the \textit{anomaly}. The factors (2.1), (2.6) in the effective exponentiated M-theory action on a closed manifold each have this form. Theorem 2.8 may be interpreted as the construction of a geometric trivialization \( 1 \) for their product.

\(^9\)For our purposes a groupoid \( \mathcal{B} \) consists of a manifold \( \mathcal{B}^0 \) of objects, a manifold \( \mathcal{B}^1 \) of morphisms, and a pair of maps \( \mathcal{B}^1 \rightarrow \mathcal{B}^0 \) which define the source and target of a morphism. (Of course there is more structure: an identity map, composition laws, etc.) A line bundle with covariant derivative on \( \mathcal{B} \) is a line bundle with covariant derivative \( K \rightarrow \mathcal{B}^0 \) and an isomorphism of the two pullbacks of \( K \) to \( \mathcal{B}^1 \). (The isomorphism must satisfy compatibilities which we do not spell out here.)
As we mentioned in the introduction there can be different choices of \( 1 \), two choices differ by a locally constant function on \( B_Y \), and the different phases on the connected components are moreover constrained by locality conditions. Making a consistent choice is called setting the quantum integrand. See §5.5 for a general discussion of the uniqueness question. Part of the significance of Theorem 2.8 is that we find a canonical isomorphism, allowing us to set the quantum integrand for M-theory on closed 11-manifolds.

If \( Y \) is compact with spatial boundary then the story is much the same. Here one needs local boundary conditions on all fields, which are part of the definition of the space of fields. With \( B_Y \) understood to be the space (or groupoid) of bosonic fields satisfying the given local boundary conditions, the discussion of the previous paragraph goes through unchanged.

Now suppose \( Y \) is compact with temporal boundary. Then there is a space \( \mathcal{F}^\partial_\partial Y \) of allowed boundary conditions for all the fields; it is a fiber bundle \( \mathcal{F}^\partial_\partial Y \to B^\partial_\partial Y \) with base the space of boundary conditions \( B^\partial_\partial Y \) for the bosonic fields. The fibers are boundary conditions for the fermionic fields, which are of global type. In a well-defined theory one is meant to do the functional integral for each \( f \in \mathcal{F}^\partial_\partial Y \) over the space of fields whose boundary values equal \( f \). In that case one obtains a “function” on \( \mathcal{F}^\partial_\partial Y \), which is allowed to be rather a section of a nontrivial line bundle \( K^\partial_\partial Y \to \mathcal{F}^\partial_\partial Y \). In fact, in any theory we start with the classical action and first integrate out the fermionic fields. This yields the effective exponentiated action, which is a section

\[
(4.5) \quad \exp(-S_{\text{eff}}): B^\text{eff}_Y \to K_Y
\]

of a line bundle \( K_Y \to B^\text{eff}_Y \). Here \( B^\text{eff}_Y \) is the fiber product

\[
\begin{array}{ccc}
B^\text{eff}_Y & \xrightarrow{\pi} & \mathcal{F}^\partial_\partial Y \\
\downarrow & & \downarrow \rho \\
B_Y & \longrightarrow & B^\partial_\partial Y
\end{array}
\]

(In other words, a field in \( B^\text{eff}_Y \) is a pair \((b, f)\) consisting of a bosonic field \( b \) on \( Y \) and a boundary condition \( f \) for both bosonic and fermionic fields such that the boundary value of \( b \) is the bosonic part of \( f \).) The bosonic functional integral is an integral over the fibers of \( \pi: B^\text{eff}_Y \to \mathcal{F}^\partial_\partial Y \), nominally with integrand \((4.5)\). In other words, we integrate over the space of equivalence classes of bosonic fields with fixed boundary conditions for both bosonic and fermionic fields. But this is ill-defined as it stands, even formally, as the integrand does not take values in a fixed line. We would like the line to depend only on the boundary values of the bosonic fields on \( Y \), since these are what are held

\[\text{Footnote 10:} \text{For groupoids of fields the fiber product is in the categorical sense: a field in } B^\text{eff}_Y \text{ includes a choice of isomorphism of the boundary value of } b \text{ with the bosonic part of } f.\]
fixed in the functional integral. Therefore, to define the functional integral over the fibers of $\pi$ we must specify a generalization of the trivialization (4.2): a line bundle

$$K_{\partial Y}^\theta \to \mathcal{F}_{\partial Y}^0$$

with metric and covariant derivative together with an isomorphism

$$1: \pi^*(K_{\partial Y}^\theta) \to K_Y$$

which preserves the metric and covariant derivative. The line bundle $K_{\partial Y}^\theta \to \mathcal{F}_{\partial Y}^0$ is not derived from (4.5), but rather is constructed prior to (4.8). We explain this in §5.4. Using (4.8) we define

$$1^{-1} \circ \exp(-S_{\text{eff}}): \mathcal{B}_Y^{\text{eff}} \to \pi^*(K_{\partial Y}^\theta),$$

a section of $\pi^*(K_{\partial Y}^\theta) \to \mathcal{F}_{\partial Y}^0$. Its (formal) integral over the space of equivalence classes in fibers of $\pi$ is a section of $K_{\partial Y}^\theta \to \mathcal{F}_{\partial Y}^0$. Note (4.9) specializes to (4.4) if $\partial Y = \emptyset$.

In §5.4 we amplify this general discussion. In particular, we relate this Lagrangian (functional integral) point of view to the Hamiltonian approach to anomalies.

Remark 4.10. To clarify the notation consider a theory with a gauge field $A$ and a spinor field $\psi$ coupled to a vector bundle associated to $A$. Then $\mathcal{B}_Y$ is the groupoid of gauge fields $A$ on $Y$ and the space $\mathcal{B}_{\partial Y}^\theta$ of boundary conditions on the boson is the groupoid of gauge fields $A^\theta$ on $\partial Y$. For the fermion let $\mathcal{H}_{A^\theta}$ be the Hilbert space of $L^2$ spinor fields $\psi^\theta$ on $\partial Y$. A boundary condition for the spinor field $\psi$ on $Y$ is a “half-dimensional” subspace $W \subset \mathcal{H}_{A^\theta}$ which is roughly complementary to the boundary values of harmonic spinor fields; cf. the discussion in §3.2. So an object in $\mathcal{F}_{\partial Y}^\theta$ is a pair $(A^\theta, W)$ and an object in the groupoid $\mathcal{B}_Y^{\text{eff}}$ is a pair $(A, W)$. The partition function with boundary condition $(A^\theta, W)$ is formally written as

$$\int_{A \mid_{\partial Y} = A^\theta} dA \int_{\psi \mid_{\partial Y} \in W} d\psi \ e^{-S(A, \psi)}.$$ 

The result of the inner integral is the pfaffian of a Dirac operator, which is (4.5) in the general discussion above. The outer integral is the integral over equivalence classes in the fiber of $\pi$ in (4.6).

§4.2 Temporal Boundary Conditions

We resume our discussion of M-theory from §2, now on a compact 11-manifold $Y$ with boundary, which in this subsection is assumed temporal. The part of the effective exponentiated action which
concerns us is the product of the gravitino partition function (2.1) and an electric coupling (2.6) which depends on the $C$-field and the metric. In the notation of §4.1 the groupoid of bosonic fields has objects

$$\mathcal{B}_Y = \{(g, C)\},$$

where $g$ is a metric on $Y$ and $C = (A, c)$ is a $C$-field, in the model reviewed in §2. The groupoid of boundary conditions on both bosonic and fermionic fields has objects

$$\mathcal{F}_{\partial Y}^\partial = \{(g^\partial, C^\partial, W_{RS}, W_S)\},$$

where $g^\partial, C^\partial$ are the restrictions of $g, C$ to $\partial Y$ and $W_{RS}, W_S$ are global boundary conditions (§3.2) for the Dirac operators $D_{RS}, D_S$ which appear in (2.1); the latter are the boundary conditions on the fermionic fields. The fiber product (4.6) is

$$\mathcal{B}_Y^{\text{eff}} = \{(g, C, W_{RS}, W_S)\}.$$

The generalization of Definition 2.4 to compact manifolds with boundary is the following.

**Definition 4.11.** Let $T$ be a smooth manifold. A *family of M-theory data on compact manifolds with temporal boundary parametrized by $T$* consists of:

1. a family of compact spin Riemannian 11-manifolds $\mathcal{Y} \to T$ in the sense of Definition 1.1(i);
2. a principal $E_8$-bundle $P \to \mathcal{Y}$ with a connection $A$;
3. a 3-form $c \in \Omega^3(\mathcal{Y})$;
4. families of subspaces $W_{RS}, W_S$ which are global boundary conditions for the operators $D_{RS}^\partial, D_S^\partial$.

Here $S = S^0 \oplus S^1 \to \mathcal{Y}$ is the spinor bundle. The induced boundary $\text{Cliff}(\partial \mathcal{Y}/T)$-module is $S^\partial = (S^0|_{\partial \mathcal{Y}})_+ \oplus (S^0|_{\partial \mathcal{Y}})_-$; see (3.3). For the Rarita-Schwinger $\text{Cliff}(\mathcal{Y}/T)$-module $RS = S \otimes T^*\mathcal{Y}$ the induced $\text{Cliff}(\partial \mathcal{Y}/T)$-module is $RS^\partial \oplus S^\partial$, where

$$RS^\partial = S^0 \otimes T^*(\partial \mathcal{Y}/T).$$

Now we are in a position to define the factors (2.1), (2.6) of interest for a family of M-theory data with temporal boundary conditions. For the gravitino this is straightforward: the Dirac operators $\mathcal{D}_{RS}, \mathcal{D}_S$ with global boundary conditions $W_{RS}, W_S$ determine pfaffian line bundles (2.2) and a section

$$\exp(-\Gamma_{\text{gravitino}}): T \to \text{Pfaff } \mathcal{D}_{RS} \otimes (\text{Pfaff } \mathcal{D}_S)^{\otimes(-3)}.$$
For (2.6) we note first that the last factor is simply a function on $T$. The first factor is, according to (3.11), a global section
\[
\tau_{D_{M(A)}}^{1/2} : T \to (\text{Pfaff } D_{N(A^\theta)})^{-1}
\]
of the pfaffian line bundle of the boundary family of Dirac operators on the $\text{Cliff}(\partial \mathcal{Y}/T)$-module $N(A^\theta) = S^0 \otimes \text{Ad} P|_{\partial \mathcal{Y}}$ with connection induced from $A^\theta = A|_{\partial \mathcal{Y}}$. Now the product of the square of the middle two factors is a global section of a product $L$ of pfaffian line bundles defined from the boundary data (see (3.11)):
\[
(4.14) \quad \tau_{D_{RS}}^{1/2} \cdot \tau_{D_s}^{-3/2} : T \to L := (\text{Pfaff } D_{RS^0 \oplus S^0})^{-1} \otimes (\text{Pfaff } D_{S^0})^\otimes 3.
\]
Lemma 3.20 determines a square root
\[
(4.15) \quad \tau_{D_{RS}}^{1/4} \cdot \tau_{D_s}^{-3/4} : T \to L^{1/2}
\]
for a hermitian line bundle with covariant derivative $L^{1/2} \to T$ equipped with an isomorphism $(L^{1/2})^\otimes 2 \cong L$.

**Theorem 4.16.** Let $Y \to T$ be a family of M-theory data parametrized by $T$ with compact fibers and temporal boundary. Then there is a suitable “trivialization” 1 such that $[\exp(-\Gamma_{\text{gravitino}}) \cdot \exp(-S_{\text{gauge}})]/1$ is a section of a hermitian line bundle with connection
\[
(4.17) \quad (\text{Pfaff } D_{N(A^\theta)})^{-1} \otimes \mathcal{K} \to T
\]
which only depends on the boundary data.

The bundle $\mathcal{K} \to T$ is defined in (4.19) below. According to the discussion surrounding (4.9), the fact that (4.17) depends only on boundary data is what is needed to set up the functional integral. We remark that for fixed metric the line bundle (4.17) was discussed in [DFM,§5]. Also, Theorem 4.16 should play an important role in the extension of the computation of [DMW] to manifolds with boundary.

**Proof.** Let $\tilde{T} \to T$ be the fiber bundle whose fiber over $t \in T$ is the restricted Grassmannian of all possible boundary conditions $W_{RS}, W_S$ for the Dirac operators $\partial_{RS}, \partial_S$ at $t$; see Definition 4.11(iv). There is a section $s : T \to \tilde{T}$ which picks out the particular $W_{RS}, W_S$ in the family of M-theory data.

By the definition of the line bundle in (3.11) the product $\tau_{D_{RS}}^{1/2} \tau_{D_s}^{-3/2}$ in (4.14) lifts to an equivariant complex-valued function on $\tilde{T}$; the equivariance condition is (3.12). The square root\(^\text{11}\) of

\(^{11}\)This is a special case of Lemma 3.20 which we can make more explicit. Let $\tilde{\mathbb{C}}^\times \to \mathbb{C}^\times$ be the double cover of the nonzero complex numbers. It carries a canonical function $z^{1/2} : \tilde{\mathbb{C}}^\times \to \mathbb{C}$ which is equivariant for the $\mathbb{Z}/2\mathbb{Z}$-action and squares to the identity on $\mathbb{C}^\times$. It may be viewed as a section of a hermitian line bundle on $\mathbb{C}^\times$ with covariant derivative of order 2 whose square is isomorphic to the trivial bundle. In general we pull back from this universal case, for example here by the map $\tau_{D_{RS}}^{1/2} \tau_{D_s}^{-3/2} : \tilde{T} \to \mathbb{C}^\times$. 29
this nonzero function is also equivariant and is a section of a hermitian line bundle which we denote \( \tilde{L}^{1/2} \rightarrow \tilde{T} \). The square \((\tilde{L}^{1/2})^2\) is canonically geometrically trivial. Observe that a point in the fiber of \( L^{1/2} \) at \( t \in T \) is the space of equivariant sections of \( \tilde{L}^{1/2} \) restricted to the fiber of \( \tilde{T} \rightarrow T \) over \( t \). Now for fixed boundary conditions \((W_{RS}, W_S)\) the argument of Proposition 1.31 applies to the Dirac operators \( \mathcal{D}_{RS}, \mathcal{D}_S \) to produce a trivialization

\[
1 : T \rightarrow \left[ Pfaff \mathcal{D}_{RS} \otimes (Pfaff \mathcal{D}_S)^{\otimes (-3)} \right] \otimes s^* \tilde{L}^{1/2}.
\]

Dividing the product of (4.13) and (4.15) by (4.18) we obtain

\[
\exp(-\Gamma_{\text{gravitino}}) \cdot \tau_{\mathcal{D}_{RS}}^{1/4} \cdot \tau_{\mathcal{D}_S}^{-3/4} : T \rightarrow L^{1/2} \otimes (s^* \tilde{L}^{1/2})^{-1}.
\]

We claim that the line bundle

\[
\mathcal{K} := L^{1/2} \otimes (s^* \tilde{L}^{1/2})^{-1} \rightarrow T
\]

depends only on boundary data. This is a nontrivial claim because each factor, defined using Lemma 3.20, depends on the \( \tau^{1/2} \)-invariant of the entire manifold with boundary. Note first that \( \mathcal{K}^{\otimes 2} \cong \mathcal{L} \), since the square of \( \tilde{L}^{1/2} \) is geometrically trivial. Furthermore, as described above a section of \( \mathcal{L} \) is a suitably equivariant function on \( \tilde{T} \). Hence there is an evaluation map

\[
ev : \mathcal{L} \rightarrow s^*(\text{trivial}).
\]

This may be regarded as a nonzero section of \( s^*(\text{trivial}) \otimes \mathcal{L}^* \rightarrow T \), so by Lemma 3.20 determines a square root, which is in fact the inverse of the line bundle \( \mathcal{K} \) in (4.19). As the evaluation map (4.20) depends only on boundary data, we are done.

§4.3 Spatial Boundary Conditions

As a preliminary we state the existence of a “parity involution” \( \sigma \) on \( C \)-fields and its effect on the electric coupling (2.6). We defer the proof and discussion to §5.1.

**Proposition 4.21.** Let \( Y \) be a compact Riemannian spin 11-manifold with metric \( g \). Then there is an involution\(^{12}\) \( (A, c) \mapsto (A, c)^\sigma \) on the groupoid of \( C \)-fields such that

\[
\exp(-S_{\text{gauge}})(g, (A, c)^\sigma) = \left[ \exp(-S_{\text{gauge}})(g, (A, c)) \right]^{-1}.
\]

\(^{12}\)This is to be understood in the categorical sense: we are given an equivalence of \( \sigma^2 \) with the identity functor.
Recall that $\exp(-S_{\text{gauge}})(g, (A, c))$ is an element of a complex line, so implicit in (4.22) is a functorial isomorphism between the line for $(g, (A, c)^\sigma)$ and the inverse of the line for $(g, (A, c))$. The field strength $G$ changes sign under $\sigma$, and (4.22) is a refined version of the observation that (2.3) is an odd function of the local 3-form $C$. This involution is relevant to the parity-invariance of M-theory.

In this paper we are interested instead in the induced involution on boundary values of $C$-fields. Recall the Cliff($\partial Y$)-modules $S^0$ and $RS^0$, defined in and before (4.12). Also recall that the Rarita-Schwinger Cliff($Y$)-module $RS = S \otimes T^* Y$ induces the Cliff($\partial Y$)-module $RS^0 \oplus S^0$. Finally, the Cliff($Y$)-module $M(A) = S \otimes \text{Ad} P$ induces the Cliff($\partial Y$)-module $N(A^0) = S^0 \otimes \text{Ad} P^0$, where $A$ is a connection on the $E_8$-bundle $P \to Y$ and $P^0 \to \partial Y$ is its restriction to the boundary. Define the line

\begin{equation}
(4.23) 
L_{\text{gauge}}(g^0, A^0, c^0) = (\text{Pfaff } D_{N(A^0)})^{-1} \otimes (\text{Pfaff } D_{RS^0 \oplus S^0})^{-1/2} \otimes \left( (\text{Pfaff } D_{S^0}^{1/2}) \right)^3 \\
\cong (\text{Pfaff } D_{N(A^0)})^{-1} \otimes (\text{Pfaff } D_{RS^0})^{-1/2} \otimes (\text{Pfaff } D_{S^0}^0).
\end{equation}

The square roots in the first line of (4.23) are defined by the functions $\tau^{1/2}_{D_{RS}}$ and $\tau^{-1/2}_{D_{S}}$ which play the role of ‘s’ in Lemma 3.20; cf. (3.11). The exponentiated electric coupling (2.6) lives in the line

\begin{equation}
(4.24) 
\exp(-S_{\text{gauge}})(g, (A, c)) \in L_{\text{gauge}}(g^0, (A^0, c^0)).
\end{equation}

**Proposition 4.25.** Continuing with Proposition 4.21, let $(A^0, c^0)^\sigma$ be the parity-reversal of the boundary values of $(A, c)$. Then there is a functorial isomorphism

\begin{equation}
(4.26) 
L_{\text{gauge}}(g^0, (A^0, c^0)^\sigma) \cong L_{\text{gauge}}(g^0, (A^0, c^0)) \\
\otimes (\text{Pfaff } D_{N(A^0)})^{\otimes 2} \otimes (\text{Pfaff } D_{RS^0}) \otimes (\text{Pfaff } D_{S^0}^{1/2})^{\otimes (-2)}.
\end{equation}

Furthermore, in a family this isomorphism preserves the metrics and covariant derivatives.

Turning to M-theory on manifolds with spatial boundary, we note from the beginning that in addition to the usual fields $g, C, \psi$ there is a bosonic field $\Theta$ and a fermionic field $\chi$ which live on the boundary.

**Definition 4.27.** Let $T$ be a smooth manifold. A family of M-theory data on compact manifolds with spatial boundary parametrized by $T$ consists of:

(i) a family of compact spin Riemannian 11-manifolds $\mathcal{Y} \to T$ in the sense of Definition 1.1(i);
(ii) a principal $E_8$-bundle $P \to \mathcal{Y}$ with a connection $A$;
(iii) a 3-form $c \in \Omega^3(\mathcal{Y})$;
(iv) a principal $E_8$-bundle $Q \to \partial \mathcal{Y}$ with a connection $\Theta$;
(v) for each component $(\partial \mathcal{Y})_i$ of $\partial \mathcal{Y} = \sqcup (\partial \mathcal{Y})_i$ a choice $\epsilon_i = \sqrt{-1} \omega^0$ or $\epsilon_i = -\sqrt{-1} \omega^0$; and
(vi) a choice of boundary value isomorphism: see (4.31).
We now examine the product of (2.1) and (2.6) using local boundary conditions based on the choice of $\epsilon_i$. The restriction $\psi^\rho$ of the Rarita-Schwinger field $\psi \in \Gamma_Y(RS^0)$ to $\partial Y$ decomposes into tangential and normal components:

\begin{equation}
\psi^\rho = \psi^\rho_T + \psi^\rho_\nu.
\end{equation}

Let $(\psi^\rho)_i$ denote the restriction to $(\partial Y)_i$. Then the boundary condition for the gravitino operator $D_{RS}$ is:

\begin{equation}
\begin{aligned}
\omega^\rho(\psi^\rho_T)_i &= \mp \sqrt{-1}(\psi^\rho_T)_i \\
\omega^\rho(\psi^\rho_\nu)_i &= \mp \sqrt{-1}(\psi^\rho_\nu)_i
\end{aligned}
\end{equation}

as $\epsilon_i = \mp \sqrt{-1} \omega^\rho$.

There are three “ghost” fields; see Appendix A. The boundary conditions are given in (A.12), (A.13), and (A.14). Those in (A.12) and (A.13) lead to pfaffian line bundles which cancel, due to the opposite signs in those equations. The remaining ghost $\varrho^\rho\psi$ has boundary condition (A.14), which we repeat here as

\begin{equation}
\omega^\rho(\varrho^\rho\psi^\rho)_i = \mp \sqrt{-1}(\varrho^\rho\psi^\rho)_i
\end{equation}

as $\epsilon_i = \mp \sqrt{-1} \omega^\rho$.

Therefore, taking into account cancellations, Conjecture 3.21 implies that the effective exponentiated gravitino action $\exp(-\Gamma_{\text{gravitino}})$ is a section of

\begin{equation}
L_{\text{gravitino}} = \left[ (\text{Pfaff } D_{RS^0}^\rho)^{1/2} \otimes (\text{Pfaff } D_S^\rho)^{-1} \right] \otimes \bigotimes_{i \text{ such that } \epsilon_i = \sqrt{-1} \omega^\rho} \left[ (\text{Pfaff } D_{RS^0}^{(\partial Y)_i})^{-1} \otimes (\text{Pfaff } D_S^{(\partial Y)_i})^{\otimes 2} \right] \longrightarrow T.
\end{equation}

Note that $\psi^\rho_T$ and $\varrho^\rho\psi$ combine to give the second factor in each bracketed expression on the right hand side of (4.30).

Let $\Theta_i$ be the restriction of $\Theta$ to $(\partial Y)_i$. The boundary condition for the $C$-field $(A, c)$ on $(\partial Y)_i$ specifies an isomorphism with $\Theta$ or its parity reversal, depending on the sign of $\epsilon_i$:

\begin{equation}
(A^\rho, c^\rho)_i \cong \begin{cases} 
(\Theta_i, 0), & \text{if } \epsilon_i = -\sqrt{-1} \omega^\rho; \\
(\Theta_i, 0)^\sigma, & \text{if } \epsilon_i = \sqrt{-1} \omega^\rho.
\end{cases}
\end{equation}

The isomorphism is a morphism in the groupoid of $C$-fields; the choice of isomorphism is part of the data, and it enters (4.32) below. According to (4.24) the exponentiated electric coupling $\exp(-S_{\text{gauge}})$ takes values in the line bundle $L_{\text{gauge}}$ defined in (4.23). The boundary condition (4.31) and isomorphism (4.26) allow us to rewrite $L_{\text{gauge}}$ in terms of the field $\Theta$. Let
$N(\Theta_i) \to (\partial Y)_i$ denote the Cliff$(\partial Y)_i$-module $S^{(\partial Y)_i} \otimes \text{Ad} Q \big|_{(\partial Y)_i}$. Then the boundary isomorphism (4.31) determines an isomorphism of line bundles

$$L_{\text{gauge}} \cong \left[ (\text{Pfaff} D_{RS}^{(\partial Y)_i})^{-1/2} \otimes (\text{Pfaff} D_{S}^{(\partial Y)_i}) \right] \otimes \bigotimes_{i \text{ such that } \epsilon_i = -\sqrt{-1} \omega_0} \left[ (\text{Pfaff} D_{N(\Theta_i)}^{(\partial Y)_i}) \otimes (\text{Pfaff} D_{RS}^{(\partial Y)_i}) \otimes (\text{Pfaff} D_{S}^{(\partial Y)_i}) \otimes (-2) \right] \to T.$$  

Finally, $\chi$ is a spinor field with values in $\text{Ad} Q$; the chirality of $\chi$ depends on the sign of $\epsilon_i$. Let $\chi_i$ be the restriction of $\chi$ to $(\partial Y)_i$. Then

$$\chi_i \in \begin{cases} N(\Theta_i)_+, & \text{if } \epsilon_i = -\sqrt{-1} \omega_0; \\ N(\Theta_i)_-, & \text{if } \epsilon_i = \sqrt{-1} \omega_0. \end{cases}$$

The exponentiated effective action for $\chi_i$ is

$$\exp(-S_{\chi_i}) = \text{paff} \mathcal{L}_{N(\Theta_i)}^{(\partial Y)_i} \text{ or } (\text{paff} \mathcal{L}_{N(\Theta_i)}^{(\partial Y)_i})^{-1}$$

depending on the sign of $\epsilon_i$, so the effective exponentiated action for $\chi$ is a section of

$$L_{\chi} = \bigotimes_{i \text{ such that } \epsilon_i = -\sqrt{-1} \omega_0} (\text{Pfaff} D_{N(\Theta_i)}^{(\partial Y)_i}) \otimes \bigotimes_{i \text{ such that } \epsilon_i = \sqrt{-1} \omega_0} (\text{Pfaff} D_{N(\Theta_i)}^{(\partial Y)_i})^{-1} \to T.$$  

The tensor product of the factors of interest is therefore a section of the tensor product

$$L_{\text{gravitino}} \otimes L_{\text{gauge}} \otimes L_{\chi} \to T$$

of (4.30), (4.32), and (4.34), which manifestly has a geometric trivialization. We summarize this discussion in the following, which is rendered as a conjecture since it is based on Conjecture 3.21.

**Conjecture 4.35.** Let $\mathcal{Y} \to T$ be a family of M-theory data parametrized by $T$ with compact fibers and spatial boundary. Then the product $\exp(-\Gamma_{\text{gravitino}}) \cdot \exp(-S_{\text{gauge}}) \cdot \exp(-S_{\chi})$ is a section of a line bundle over $T$ which has a distinguished geometric trivialization 1.

Therefore, the anomalies in the individual factors cancel and the theory is well-defined with spatial boundary, at least from the point of view of anomalies. Notice that this anomaly cancellation and
Figure 1: M-theory is well-defined topologically on 11-manifolds with arbitrary numbers of spatial boundary components and arbitrary chirality projects on each component. Each component carries an independent $E_8$ super Yang-Mills multiplet.

setting of the quantum integrand work without restriction on the number of boundary components or the topology of the fibers of $\mathcal{Y} \to T$, as in Figure 1.

§5 Further discussion

In this section we record brief remarks on several aspects of the main text. We begin by outlining a construction of the involution (Proposition 4.21) which is needed in §4.3. Next, in §5.2 we remark that an analysis in 3 dimensions, which is parallel to that we did in 11 dimensions, sets the quantum integrand for an M2-brane. M-theory also admits M5-branes, and these are magnetically charged under the $C$-field. We do not know how to treat magnetic current with the $E_8$-model, however, so we do not discuss M5-branes in the present paper. The spatial boundary condition for the $C$-field involves a choice of isomorphism (4.31), and in §5.3 we explain the effect of this choice on the partition function. Finally, in §5.4 we elaborate a bit on our discussion of anomalies and setting the quantum integrand on manifolds with temporal boundary. In particular, we relate it to the Hamiltonian description of anomalies, though we leave a full development for future investigation.

§5.1 The $E_8$-model

The $E_8$-model for the gauge field $C$, as introduced in [DFM] and briefly reviewed in §2 of this paper, has many nice features. The fact that the gauge action (2.6) is expressed in terms of
geometric index theory invariants allows the direct definition of the partition function that we have given in Theorem 2.8, Theorem 4.16, and Conjecture 4.35. Also, the connection with $E_8$ is crucial for understanding the Green-Schwarz anomaly cancellation for the $E_8 \times E_8$ Type I supergravity. However, this model has a defect: the algebraic structure of $C$-fields is not at all explicit. In other words, the space of equivalence classes of $C$-fields is a torsor for an abelian group, and so the groupoid of $C$-fields must be a torsor in a categorical sense, i.e., a torsor for a Picard category. We do not need this structure, but in §4.3 we do need an involution $\sigma$ (compatible with the additive inverse on the associated abelian group). We indicate now how to construct $\sigma$.

Quite generally, we can replace a groupoid with an equivalent groupoid.

**Definition 5.1.** An equivalence of groupoids $C_1, C_2$ is a functor $F: C_1 \to C_2$ such that there exists a functor $G: C_2 \to C_1$ and natural transformations (homotopies) $G \circ F \Rightarrow id_{C_1}$ and $F \circ G \Rightarrow id_{C_2}$.

Gauge fields are modeled by groupoids, and equivalent groupoids are equally good for the purposes of setting up the functional integral. Of course, different models have different advantages as we see here. On the other hand, we can use the equivalence to transport structure between equivalent groupoids. For example, suppose $C_2$ in the definition has an involution $\sigma_2$, that is a functor $\sigma_2: C_2 \to C_2$ and a natural transformation $\sigma_2 \circ \sigma_2 \Rightarrow id_{C_2}$. Then $\sigma_1 = G \circ \sigma_2 \circ F$ is an involution on $C_1$.

We apply this discussion to $C$-fields. In [DFM,§3.4] we introduce a model for $C$-fields based on differential cocycles [HS] and prove that it is equivalent to the $E_8$-model. Let $\lambda(g) \in \Omega^4(Y)$ denote the Pontrjagin form which represents half the first Pontrjagin class. We regard it as a singular cocycle with real coefficients. Then an object in the differential cocycle model is a triple $(a,h,\omega) \in C^4(Y;\mathbb{Z}) \times C^3(Y;\mathbb{R}) \times \Omega^4(Y)$ which satisfies

\[
\begin{align*}
\delta a &= 0 \\
\delta h &= \omega - a + \frac{1}{2} \lambda(g) \\
\omega &= 0
\end{align*}
\]

The presence of $\frac{1}{2} \lambda(g)$ reflects the shift in the quantization law mentioned in the previous footnote. The field strength of $(a,h,\omega)$ is $\omega$. To define an involution on these triples we need to use a cocycle $\lambda \in C^4(Y;\mathbb{Z})$ which represents half the Pontrjagin class as an integer cohomology class; see the next paragraph for a discussion. Then the desired involution

$$
(a, h, \omega) \mapsto (\lambda - a, -h, -\omega)
$$

sends the field strength to its opposite. Transport it to the $E_8$-model to define the involution $\sigma$ of Proposition 4.21.

---

13It is not an abelian group because of the shift in the quantization condition; see [W1].

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We do not know a direct construction of the desired integer cocycle $\lambda$ on a Riemannian spin manifold $Y$, i.e., a construction which only uses the Riemannian metric. We must introduce additional data; see [BryM] for one such construction. Our choice is to pass from the category of Riemannian manifolds to an equivalent category whose objects are Riemannian spin manifolds together with a classifying map of the principal spin bundle of frames; the morphisms of the latter ignore the classifying map. Then we fix once and for all a cocycle $\lambda_{\text{univ}}$ on the classifying space and define $\lambda$ as the pullback via the classifying map.

We must still prove (4.22) and (4.26). To do so it stands to reason that the exponentiated action $\exp(-S_{\text{gauge}})$ and the line bundle $L_{\text{gauge}}$ should be defined in the model where the involution is apparent; then we can check (4.22) and (4.26) in that model. This is done in [FH]. (In fact, that paper uses an equivalent model closely related to the differential cocycle model.)

§5.2 M2-branes

For our purposes here an M2-brane is a 3-dimensional compact neat\(^a\) spin submanifold $\Sigma \subset Y$ of the 11-dimensional spin manifold of M-theory. There is a fermionic field on $\Sigma$, a partner to the position of $\Sigma$ in $Y$, namely a spinor field $\lambda$ coupled to the half spin bundle of the normal bundle of $\Sigma$ in $Y$. Assume first that $\Sigma$ is closed. We focus on two factors in the exponentiated effective action. The first is the effective action $\exp(-\Gamma_\lambda)$ of the spinor field $\lambda$, the pfaffian of a Dirac operator on $\Sigma$. This is analogous to (2.1). The second is analogous to (2.3) and may be formally written as

\[ \exp \left( 2\pi i \int_\Sigma C \right). \]

As in (2.6) we define it precisely in terms of the $E_8$-model for the $C$-field:

\[ \exp(-S_{\text{M2-gauge}})(g, (A, c)) = CS_\Sigma(A) \cdot \tau_{D_8}^{-1/4} \cdot \exp \left( 2\pi i \int_\Sigma c \right), \tag{5.2} \]

Here $CS_\Sigma$ is the exponentiated $E_8$ Chern-Simons invariant (corresponding to the generator in $H^4(BE_8; \mathbb{Z})$) and the second factor is a power of the $\tau$-invariant of the Dirac operator on $\Sigma$ acting on spinors $S$. Each of $\exp(-\Gamma_\lambda)$ and $\exp(-S_{\text{M2-gauge}})$ is a section of a line bundle with metric and covariant derivative. The analog of Theorem 2.8, which states that the product of these line bundles has a canonical geometric trivialization, is proved with the same analysis (§1.3) of Dirac operators in $8k + 3$ dimensions, specifically Proposition 1.31.

The extension to temporal boundaries for the M2-brane, in which case $\partial \Sigma \subset \partial Y$, proceeds analogously to §4. We comment only on the case of a spatial boundary, still assuming $\partial \Sigma \subset \partial Y$. There is an additional factor involving fields on $\partial \Sigma$, which plays a role analogous to (4.33). This is

\(^{14}\)‘Neat’ means that $\Sigma \cap \partial Y = \partial \Sigma$ and the intersection is transverse.
the partition function of a chiral, level 1, \(E_8\) WZW model coupled to the \(E_8\) gauge field \(\Theta\) restricted to \(\partial\Sigma\). This factor combines with the \(E_8\) Chern-Simons term in (5.2) to define the membrane amplitude as a well-defined function. These chiral degrees of freedom have a fermionic formulation when the structure group of \(\Theta\) reduces to \(SO(16)\). In this case the discussion is analogous to §4.3.

§5.3 Boundary values of \(C\)-fields

For M-theory with spatial boundary there is a specified\(^{15}\) isomorphism (4.31) of the boundary value of the \(C\)-field with the boundary \(C\)-field derived from the \(E_8\)-connection \(\Theta\). Let \(\overline{B}_Y(C)\) be the space of equivalence classes of \(C\)-fields on \(Y\) and \(\overline{B}_{\partial Y}(\Theta)\) the space of equivalence classes of \(E_8\)-connections on \(\partial Y\). There is a subset \(S \subset \overline{B}_Y(C) \times \overline{B}_{\partial Y}(\Theta)\) of the product consisting of equivalence classes which are isomorphic on \(\partial Y\) (with the boundary condition (4.31)). The space of equivalence classes of fields in the model with spatial boundary is a principal fiber bundle

\[
(5.3) \quad \overline{B}_Y^{\text{eff}}(C, \Theta) \rightarrow S
\]

with structure group

\[
(5.4) \quad \prod_i H^2((\partial Y)_i; \mathbb{R}/\mathbb{Z}).
\]

The action of the structure group changes the isomorphism (4.31) on the boundary.

The product

\[
\exp(-\Gamma_{\text{gravitino}}) \exp(-S_{\text{gauge}}) \exp(-S_{\chi}) \cdot \overline{B}_Y^{\text{eff}}(C, \Theta) \rightarrow \mathbb{C}
\]

is defined by Conjecture 4.35. This function is not constant on the fibers of (5.3), i.e., is not the pullback of a function on \(S\). Rather, \(\lambda = (\lambda_i) \in \prod_i H^2((\partial Y)_i; \mathbb{R}/\mathbb{Z})\) acts as multiplication by \(\prod_i \exp(2\pi \sqrt{-1}N_i \int_{(\partial Y)_i} \lambda_i)\) at a point of \(\overline{B}_Y^{\text{eff}}(C, \Theta)\), where \(N_i\) is the index of the Dirac operator \(D_{N_i(\Theta_i)}^{(\partial Y)}\). There is a similar formula for the effect on the partition function of an M2-brane \(\Sigma \subset Y\); then \(N_i\) is replaced by \(\pm 1\) depending on the sign of the boundary condition \(\epsilon_i = \pm \sqrt{-1} \omega^0\) in Definition 4.27(v).

Consider the special case of heterotic M-theory [HW1], [HW2]. Here \(Y = [0, 1] \times X\) for a closed spin 10-manifold \(X\). The signs in Definition 4.27(v) are \(\epsilon_0 = -\sqrt{-1} \omega^0\) and \(\epsilon_1 = \sqrt{-1} \omega^0\). One recovers the usual heterotic fields on \(X\) by integrating the M-theory fields over \([0, 1]\). Thus the

\(^{15}\) That a choice of isomorphism is needed was stated in §4.1 in the words “(4.6) is a fiber product in the categorical sense”.

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“integral” of $\Theta$—the sum over the two boundary components—is the $E_8 \times E_8$ gauge field of the heterotic theory and the integral$^{16}$ of $C$ is the $B$-field of the heterotic theory. This $B$-field is not “closed”: its differential is computed from Stokes’ theorem. (For example, the field strength $H \in \Omega^3(Y)$ of $B$ is a 3-form which satisfies

$$dH = \text{tr} F_1^2 + \text{tr} F_2^2 - \text{tr} R^2,$$

where $(F_1, F_2)$ is the curvature of the $E_8 \times E_8$ bundle, $R$ is the Riemannian$^{17}$ curvature, and the traces are suitably normalized.) Now the structure group (5.4) is the product of two copies of $H^2(X; \mathbb{R}/\mathbb{Z})$. The diagonal acts trivially in the effective heterotic theory. For the anti-diagonal we identify $H^2(X; \mathbb{R}/\mathbb{Z})$ with the group of isomorphism classes of flat gerbes.$^{18}$ Then the action of the anti-diagonal adds an equivalence class of flat gerbes to the given equivalence class of $B$-fields. Said more precisely, the $B$-field is a differential cochain of degree 3 which trivializes a fixed differential cocycle of degree 4 constructed from the metric and $E_8 \times E_8$-connection. The action in question adds a flat differential cocycle of degree 3 to the $B$-field. This can be measured through its effect on the amplitude of a worldsheet instanton, realized as an open M2-brane in heterotic M-theory.

§5.4 Temporal boundaries and the Hamiltonian anomaly

The description in §4.1 of anomalies in the case of temporal boundaries, specifically the line bundle $K_{\partial Y}^\partial \to \mathcal{F}_{\partial Y}^\partial$, is related to the Hamiltonian interpretation of anomalies. The latter was explained by Fadeev and Shatashvili [Fa], [FS] and Segal [Se3] as follows. Hamiltonian quantization of fermions on a manifold $X$ leads to a bundle of projective Hilbert spaces over the space of bosons on $X$, and to quantize the bosons one must lift this projective bundle to a vector bundle. The obstruction is measured topologically by an integral cohomology class of degree three which, more importantly, has a natural geometric realization as we now explain; see [MS], [CM], [CMM] for further discussion. Set $X = \partial Y$ to be the boundary of a spacetime to make contact with §4.1. Then this cohomology class is described by a line bundle with covariant derivative $L \to \mathcal{F}_{\partial Y}^\partial[2]$ over the fiber product

$$\begin{array}{c}
\mathcal{F}_{\partial Y}^\partial[2] \\ \downarrow p_1 \end{array} \xrightarrow{p_2} \begin{array}{c}
\mathcal{F}_{\partial Y}^\partial \\ \downarrow \rho \end{array}
$$

where $\rho$ is the map in (4.6). Recall that the fiber of $\rho$ is the space of fermionic boundary conditions for fixed bosonic boundary conditions—a restricted Grassmannian of subspaces $W$—and given two

---

$^{16}$This integral may be defined in the differential cocycle model discussed in §5.1; see [HS].

$^{17}$In a more precise treatment $R$ would be the curvature of a connection with torsion determined by $H$.

$^{18}$These are the type of “$B$-field” which appear in Type II theories. They are “closed”.

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such subspaces $W,W'$ there is a natural determinant line $L_{W,W'}$. Further, over the triple fiber product $\mathcal{F}_{\partial Y}^0[3]$ there is an isomorphism $L_{W,W''} \cong L_{W,W'} \otimes L_{W',W''}$. This is the data of a gerbe on $\mathcal{B}_{\partial Y}^0$. The lift of the projective bundle to a vector bundle mentioned above is encoded by a trivialization of this gerbe, which is a line bundle with covariant derivative $K_{\partial Y}^0 \rightarrow \mathcal{F}_{\partial Y}^0$ and an isomorphism

\[(5.5) \quad p_1^*K_{\partial Y}^0 \otimes L \rightarrow p_2^*K_{\partial Y}^0\]

with appropriate compatibility on $\mathcal{F}_{\partial Y}^0[2]$.

The line bundle $K_{\partial Y}^0 \rightarrow \mathcal{F}_{\partial Y}^0$ appears in (4.7). Thus to define a theory which includes temporal boundaries we must first resolve the Hamiltonian anomaly on $\partial Y$—i.e., produce the line bundle $K_{\partial Y}^0 \rightarrow \mathcal{F}_{\partial Y}^0$ satisfying (5.5)—and then specify the isomorphism (4.8). This must be done in a consistent way for all manifolds $Y$, constrained by physical requirements such as locality (gluing laws).

In the $8k + 3$ dimensional situation considered in this paper, the cancellation of anomalies (and setting of the quantum integrand) is Theorem 4.16. Furthermore, we expect that an analog of Proposition 1.31, which we do not work out here, may be used to construct $K_{\partial Y}^0 \rightarrow \mathcal{F}_{\partial Y}^0$ directly from the boundary data. Namely, the electric coupling term determines a gerbe over $\mathcal{B}_{\partial Y}^0$, just as the Hamiltonian quantization of fermions does, and the product of the two gerbes should have a canonical trivialization. (In this case both gerbes have a compatible real structure.) This is the Hamiltonian version of the Green-Schwarz mechanism.

It would be interesting to investigate further the many mathematical questions underlying this brief sketch.

§5.5 Topological terms

The trivialization $1$ in (4.2) which sets the quantum integrand need not be unique. The ratio of two choices is a locally constant function

\[(5.6) \quad Z : \mathcal{B}_{\partial Y}^0 \rightarrow \mathbb{C}\]

of unit norm. (We resume the notation of §4.1.) Each trivialization $1$ must satisfy gluing laws, though we have not attempted to make them precise here. Nonetheless, there is a natural gluing assumption for the ratio (5.6) of two trivializations, namely that it has the properties of the exponential of a topological term [DeF, Chapter 6] in a classical action. Thus it extends to manifolds $Y$ with boundary, it lies in a complex line which only depends on the boundary data, and it is multiplicative under gluing. Considering that (5.6) is also locally constant, we can simply say that it is the partition function of a topological quantum field theory (TQFT) with special properties.
Definition 5.7. An invertible topological quantum field theory (ITQFT) on oriented $n$-manifolds is an $n$-dimensional TQFT in which

(i) the complex vector space $L_Z$ attached to an oriented $(n-1)$-manifold $Z$ is one-dimensional;
(ii) there is a natural isomorphism $L_Z \cong L_Z^{-1}$, where $\overline{Z}$ is the oppositely oriented manifold;
(iii) the invariant $Z_Y$ of an $n$-manifold $Y$ is nonzero; and
(iv) $Z_{\overline{Y}} = Z_Y^{-1}$.

An ITQFT is unitary if in addition $L_Z$ is hermitian and $|Z_Y| = 1$.

Note that in a general TQFT there is no analog of axioms (ii) and (iv); the pairing between the vector space associated to $Z$ and that associated to $\overline{Z}$ is induced from the cylinder on $Z$; see [Qu], [Se2, Lecture 1]. Also, recall that in a general unitary TQFT the inverses in (ii) and (iv) are replaced by the complex conjugates. The category of manifolds on which the ITQFT is defined depends on the field content. For example, in M-theory it is the bordism category whose objects are spin 10-manifolds equipped with a degree four integral cohomology class. (We term this an ITQFT on 11-manifolds of the given type, as it is an 11-dimensional theory.) We remark that a related notion appears in [MZ].

Summarizing, ratios of settings of the quantum integrand are ITQFTs.

We do not have a definitive classification of ITQFTs in general, nor in the case relevant to M-theory, but will report several observations.

Proposition 5.8. Let $Z$ be the partition function of an ITQFT on compact oriented $n$-manifolds, and suppose $Z(S^n) = 1$. Then $Z$ is an oriented bordism invariant of $n$-manifolds: if $Y = \partial W$ for $W$ a compact oriented $(n+1)$-manifold, then $Z(Y) = 1$.

Proof. More generally, let $W$ be a bordism between closed oriented $n$-manifolds $Y_0$ and $Y_1$. Choose a Morse function $f : W \to [0,1]$ with $Y_0 = f^{-1}(0)$, $Y_1 = f^{-1}(1)$, and such that for each critical value $c \in (0,1)$ there is a unique critical point in $Y_c = f^{-1}(c)$. Then for $\epsilon > 0$ sufficiently small, $Y_{c+\epsilon}$ is obtained from $Y_{c-\epsilon}$ by a surgery. Namely, there is a compact $n$-manifold $Y_-$ with boundary diffeomorphic to $S^p \times S^q$, $p + q = n - 1$, so that

$$Y_{c-\epsilon} \simeq Y_- \cup (D^{p+1} \times S^q),$$
$$Y_{c+\epsilon} \simeq Y_- \cup (S^p \times D^{q+1}).$$

Here $D^{r+1}$ is the $r$-dimensional ball with boundary $S^r$. The gluing law implies that

$$Z(Y_{c-\epsilon}) = Z(Y_-)^{-1} \cdot Z(D^{p+1} \times S^q),$$
$$Z(Y_{c+\epsilon}) = Z(Y_-)^{-1} \cdot Z(S^p \times D^{q+1}),$$

where the product is the inner product in the one-dimensional Hilbert space attached to $S^p \times S^q$.

Also, the gluing law applied to the decomposition

$$S^n = (D^{p+1} \times S^q) \cup (S^p \times D^{q+1})$$

implies that

$$Z(Y_{c-\epsilon}) = Z(Y_-)^{-1} \cdot Z(D^{p+1} \times S^q),$$
$$Z(Y_{c+\epsilon}) = Z(Y_-)^{-1} \cdot Z(S^p \times D^{q+1}).$$

shows that

\[(5.10) \quad 1 = Z(S^n) = Z(D^{p+1} \times S^q)^{-1} \cdot Z(S^p \times D^{q+1}).\]

Combining (5.9) and (5.10) we conclude \(Z(Y_{c, -\varepsilon}) = Z(Y_{c, +\varepsilon})\), whence \(Z(Y_0) = Z(Y_1)\).

The converse, that the exponential of an \(\mathbb{R}/\mathbb{Z}\)-valued bordism invariant of oriented \(n\)-manifolds defines an ITQFT with \(Z(S^n) = 1\), is true. The argument, due to M. Hopkins, uses ideas from homotopy theory. We do not give it here.

Next, observe that the product of two ITQFTs is again an ITQFT. Now for each \(\lambda \in \mathbb{C}\) with \(\lambda \neq 0\) there is a simple ITQFT whose partition function \(Z_\lambda\) on a closed \(n\)-manifold \(Y\) is

\[Z_\lambda(Y) = \lambda^{\chi(Y)/2},\]

where \(\chi(Y)\) is the Euler characteristic. In even dimensions we have \(Z_\lambda(S^n) = \lambda\), which proves the following.

**Corollary 5.11.** Let \(Z\) be the partition function of an ITQFT on compact oriented \(n\)-manifolds, \(n\) even, and suppose \(Z(S^n) = \lambda\). Then \(Z = Z_\lambda \cdot Z'\), where \(Z'\) is an oriented bordism invariant.

Corollary 5.11 gives the complete story for oriented manifolds in even dimensions. We suspect that any ITQFT on compact oriented manifolds in odd dimensions has \(Z(S^n) = 1\), thus is a bordism invariant, but we do not have a proof in general. For \(n = 3\) dimensions, the first nontrivial case, there is a proof for oriented manifolds.

**Proposition 5.12.** Let \(Z\) be the partition function of an ITQFT on compact oriented \(3\)-manifolds. Then \(Z(S^3) = 1\).

*Proof.* Since the Hilbert space associated to \(S^2\) is one-dimensional, we have \(Z(S^1 \times S^2) = 1\). Choose Heegaard decompositions

\[(5.13) \quad S^1 \times S^2 \simeq Y_- \cup_{\Sigma_g} Y_+ \quad \quad S^3 \simeq Y_- \cup_{\Sigma_g} Y_+\]

for \(\Sigma_g\) a closed oriented 2-manifold of genus \(g \geq 3\). The gluing maps in (5.13) differ by an orientation-preserving diffeomorphism of \(\Sigma_g\), and the ratio of \(Z(S^3)\) to \(Z(S^1 \times S^2) = 1\) is the action of this diffeomorphism on the one-dimensional Hilbert space associated to \(\Sigma_g\). But that action is part of a one-dimensional representation of the mapping class group, and for genus \(g \geq 3\) the mapping class group is perfect [P], whence the action is trivial and \(Z(S^3) = 1\).
In higher dimensions we can show that if \( n \) is odd, then \( Z(S^n) \) is a fourth root of unity. For example, write

\[
S^5 \simeq (D^3 \times S^2) \cup_{S^2 \times S^2} (S^2 \times D^3)
\]

\[
S^3 \times S^2 \simeq (D^3 \times S^2) \cup_{S^2 \times S^2} (S^2 \times D^3)
\]

and observe that the ratio of the gluing maps, the orientation-preserving diffeomorphism \( \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \) of \( S^2 \times S^2 \), has order 2. Then

\[
S^3 \times S^2 \simeq [(D^2 \times S^1) \times S^2] \cup_{(S^1 \times S^1) \times S^2} [(S^1 \times D^2) \times S^2]
\]

\[
S^1 \times S^2 \times S^2 \simeq [(D^2 \times S^1) \times S^2] \cup_{(S^1 \times S^1) \times S^2} [(S^1 \times D^2) \times S^2]
\]

and the ratio of the gluing maps is the orientation-preserving diffeomorphism \( \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] \) of \( S^1 \times S^1 \) times the identity of \( S^2 \), which has order 4. Finally, note that \( Z(S^1 \times X) = 1 \) for any \( X \). The argument continues to all higher odd-dimensional spheres.

For the applications to quantum field theory, string theory, and M-theory we need to complete these arguments and extend them to other bordism categories of manifolds. Finally, we remark that there are ITQFTs in \( n = 3 \) dimensions which have \( Z(S^3) \neq 1 \). One such example is Chern-Simons theory for the group \( U_1 \) at the lowest level. Then \( Z(S^3) \) is a 24th root of unity. This is not a contradiction to Proposition 5.12: recall the “framing anomaly” of Chern-Simons theory [W2]. It translates to the assertion that Chern-Simons theory is defined on a different bordism category. For this particular case we also have a spin structure, so we obtain the bordism category of string manifolds (or, in the older literature, \( MO(8) \)-manifolds). The relevant bordism group \( \Omega_3^{\text{string}} \) is cyclic of order 24—it agrees with the framed bordism group in this low dimension—and we conjecture that this \( U_1 \) Chern-Simons theory is constructed from an isomorphism \( \Omega_3^{\text{string}} \rightarrow \mathbb{Z}/24\mathbb{Z} \).

## Appendix A: The Gravitino Path Integral

### §A.1 The gravitino theory

In this appendix we review the formal definition of the partition function of a gravitino as commonly used in supergravity. (See [N], [K], [vN], [FT] for a sample of the literature. For a good recent account of the boundary conditions in heterotic M-theory see [M].) As we will note this subject has not been adequately investigated in the literature, and consequently our discussion has some important gaps. Filling these gaps is beyond the scope of this paper. In this appendix we simply indicate how the standard discussion proceeds.

Let \( Y \) be a Riemannian spin manifold of any dimension and \( S \rightarrow Y \) a spin bundle. The gravitino field is a section of \( S \otimes T^*Y \). One may impose chiral and/or Majorana projections. The equations
of motion are written using an operator \( R \) defined by composition of Clifford multiplication by 3-forms with the covariant derivative:

\[
R : \Gamma(S \otimes T^*Y) \to \Gamma(S \otimes T^*Y).
\]

In local coordinates \( x^M, M = 1, \ldots, n \), we have \( \psi = \psi_M dx^M \) and we define:

\[
(R\psi)^M := \gamma^{MNP} \nabla_N \psi_P,
\]

where \( \gamma^{MNP} \) denotes Clifford multiplication by the three-form \( dx^M \wedge dx^N \wedge dx^P \). The equation of motion \( R\psi = 0 \) follows from the action

\[
(A.1) \quad \int_Y \bar{\psi} R\psi \, |dy|
\]

We would therefore like to make sense of the formal gravitino path integral

\[
(A.2) \quad Z = \int [d\psi] e^{-\int_Y \bar{\psi} R\psi \, |dy|}
\]

In general, the path integral (A.2) formally vanishes, because the action has an anticommuting gauge symmetry:

\[
(A.3) \quad \psi_M \to \psi_M + \nabla_M \epsilon
\]

One may easily check that this is a local gauge symmetry of the action (A.1) when the background Riemannian metric is Ricci-flat. In this case, one attempts to define the true partition function \( Z \) of the gravitino by “dividing by the volume of the gauge group.” Unfortunately, this volume is zero, so one must proceed somewhat formally.

When the background metric is not Ricci-flat (A.3) is not a local symmetry of (A.1). Nevertheless, the full supergravity action has a local (super) gauge invariance which involves the transformation (A.3). However, if the background is not on-shell the gauge transformations do not close into a super Lie algebra of symmetries and one must use the BV formalism to quantize the theory. This in turn leads to substantial complications in the quantization of supergravity which we will not discuss. Instead we will restrict attention in this appendix to backgrounds with vanishing \( G \)-flux and Ricci-flat metrics. This restriction is a first significant gap in our discussion.
§A.2 Local analysis of the equations of motion

Since the action (A.1) has a gauge symmetry, quantization involves ghost fields. In order to understand the nature of these ghosts let us review briefly the standard discussion of the physical degrees of freedom of the gravitino (see, for example, [vN], [F3, Appendix]). Here we study the equations of motion in flat Minkowski space. Thus \( S = S^0 \oplus S^1 \) is a \( \mathbb{Z}/2\mathbb{Z} \)-graded real representation of the Lorentz spin group. The gravitino operator fits into a complex

\[
0 \to \Omega^0(S^0) \xrightarrow{\nabla^*} \Omega^1(S^0) \xrightarrow{R} \Omega^1(S^1) \xrightarrow{\nabla} \Omega^0(S^1) \to 0
\]  

The space of gauge inequivalent solutions of the equations of motion is the degree one cohomology of (A.4). In a flat Minkowski space of \( n \) dimensions we analyze this cohomology as follows. Write the wave equation in a momentum basis

\[
\gamma^{MNP} k_N \psi_P = 0
\]

Now use

\[
\gamma_M \gamma^{MNP} = (2 - n) \gamma^{NP}
\]

Hence, for \( n - 2 \neq 0 \),

\[
\gamma^{MN} k_M \psi_N = 0
\]

Now use two more gamma matrix identities to write:

\[
\begin{align*}
\gamma^{MN} k_M \psi_N &= (\gamma \cdot k)(\gamma \cdot \psi) + k \cdot \psi \\
(\gamma^{MN} k_N)(\gamma \cdot \psi) &= \gamma^{MNP} k_N \psi_P + \gamma^M (k \cdot \psi) - (\gamma \cdot k) \psi^M
\end{align*}
\]

From (A.5), (A.6) we conclude that if we fix the gauge \( \gamma \cdot \psi = 0 \) then \( k \cdot \psi = 0 \) and \( (\gamma \cdot k) \psi^M = 0 \). We will henceforth fix this gauge. From \( (\gamma \cdot k) \psi^M = 0 \) we learn that for \( \psi^M \neq 0 \) we must have \( k^2 = 0 \).

Now, since \( k^2 = 0 \), any gravitino wave-function can be expanded

\[
\psi_M = k_M s_1 + \bar{k}_M s_2 + \epsilon^{(i)}_M s_i
\]

where \( k^2 = \bar{k}^2 = 0 \); \( k \cdot \bar{k} = 1 \); and \( \epsilon^{(i)}_i, i = 3, \ldots, n \), form a basis for the transverse space to the span of \( k, \bar{k} \). From \( k \cdot \psi = 0 \) we learn \( s_2 = 0 \). Now we still have the gauge freedom to shift \( \psi_M \to \psi_M + k_M s_1 \) preserving the gauge choice \( \gamma \cdot \psi = 0 \). Thus, we can fix the gauge completely by taking

\[
\psi_M = \epsilon^{(i)}_M s_i
\]
Note that we still have $\gamma^M \epsilon_M^{(i)} s_i = 0$ and $(\gamma \cdot k)s_i = 0$, for each $i$.

Now from $(\gamma \cdot k)s_i = 0$ we learn the following. Choose a frame so that $k = E(1, h, 0^{n-2})$ with $h = \pm 1$. Then $(\gamma \cdot k) = E\gamma^0(1 - h\gamma^0\gamma^1)$ is proportional to a projection operator, so under the decomposition

$$Spin(1, n) \supset Spin(1, 1) \times Spin(n - 2)$$

we have $s_i \in [2^{[(n-2)/2]}]_h$ is an irreducible spinor of $Spin(n - 2)$. Next from $\gamma^M \epsilon_M^{(i)} s_i = 0$ we learn that $\epsilon_M^{(i)} s_i$ is in the irreducible representation of the tensor product of the vector of $Spin(n - 2)$ with the spinor $[2^{[(n-2)/2]}]$.

Thus, the cohomology gives the expected physical solutions. An important lesson we may draw from this computation is that in the BRST quantization there will be three ghosts. Two of these will be of the same chirality as the gravitino, $k^M \psi_M$ and $k_M \epsilon$, while the third ghost, $\gamma \cdot \psi$, will be of opposite chirality.

§A.3 Partition function

We return to Euclidean field theory and so to a Riemannian manifold $Y$ with $\mathbb{Z}/2\mathbb{Z}$-graded spin bundle $S = S^0 \oplus S^1 \to Y$.

If we follow the paradigm of abelian gauge theory then the most natural definition of the gravitino partition function follows from the complex (A.4). Let $R^\perp$ be the restriction to $(\ker R)^\perp$. Then the partition function should be

$$(A.7) \quad Z = \frac{\det R^\perp \det \nabla_M}{\det'(-\nabla^* \nabla)^{-1}}$$

where $\nabla_M : \Omega^0(S^0) \to \ker R$.

Let us first formally justify (A.7) using the BRST procedure. We fix the gauge using

$$\nabla^M \psi_M = s,$$

where $s$ is an arbitrary spinor. This leaves unfixed the covariantly constant spinors, which, by our assumption of Ricci-flatness, are the same as the harmonic spinors. We will deal with gauge fixing this finite dimensional part of the gauge group below equation (A.10). Following standard procedure we now write:

$$1 = \int_{\Omega^0(S^0)^\perp} d\epsilon \delta(s - \nabla^M (\psi_M + \nabla_M \epsilon))(\det' - \nabla^* \nabla)^{-1}$$

where $\Omega^0(S^0)^\perp$ is the orthogonal complement of $\ker D = \ker(-\nabla^* \nabla)$. This expression is to be inserted in (A.2). Using gauge invariance of the action we may write:

$$Z = \left( \int_{\Omega^0(S^0)^\perp} d\epsilon \right) \left( \int d\psi \delta(s - \nabla^M \psi_M)(\det' - \nabla^* \nabla)^{-1} e^{-\int \bar{\psi} R \psi dy} \right)$$
The formal division by the gauge group removes the first factor to produce the partition function $Z$. The remaining integral may be evaluated to give (A.7), by the choice of gauge $s = 0$.

Unfortunately, (A.7) is not in a form convenient for anomaly analysis. A second form follows from the formal BRST procedure by choosing a different gauge, $s = \gamma \cdot \psi$, for an arbitrary spinor $s \in \Omega^0(S^1)$. We now write

(A.8) \[ 1 = \int_{\Omega^0(S^0)} \, d\delta (s - \gamma^M (\psi_M + \nabla_M \epsilon)) (\det 'D^+)^{-1} \]

for $D^+: \Omega^0(S^0) \rightarrow \Omega^0(S^1)$ the Dirac operator. We now insert (A.8) into (A.2), shift the field, and divide by the volume of the gauge group to obtain the gauge-fixed expression

\[ Z = \int d\psi \delta (s - \gamma \cdot \psi) (\det 'D^+)^{-1} e^{\int \bar{\psi} R \psi}. \]

The ghost fields may be introduced by writing the determinant in (A.8) in terms of commuting ghost $\epsilon$ and antighost $\beta$ fields as

\[ (\det 'D^+)^{-1} = \int d\beta d\epsilon e^{\int \bar{\beta} D^- \epsilon}. \]

At this point, rather than setting $s = 0$ we average over $s$ using the expression

(A.9) \[ 1 = \frac{1}{(\det 'D^-)^{1/2}} \int_{\Omega^0(S^1)} \, ds \int_{\Omega^0(S^0)} \, dy, \]

where $D^-: \Omega^0(S^1) \rightarrow \Omega^0(S^0)$ is the Dirac operator.

We now invoke an algebraic identity. If $\phi^M = \psi^M + \frac{1}{2} \gamma^M (\gamma \cdot \psi)$ then

\[ -\bar{\phi} D^+_{\gamma \cdot \psi} \phi = \bar{\psi} R \psi - \frac{1}{4} (n - 2)(\gamma \cdot \psi) D(\gamma \cdot \psi). \]

Thus, again formally dividing by the volume of the gauge group we obtain

(A.10) \[ Z = \frac{(\det D^+_{\gamma \cdot \psi})^{1/2}}{(\det 'D^+)(\det 'D^-)^{1/2}} \]

Since the path integral is independent of the choice of gauge we conclude that (A.10) is the same as (A.7).
There remains the gauge-fixing of the supersymmetry transformations by the covariantly constant spinors. For this finite dimensional supergauge group we will again insert

\begin{equation}
1 = \int_{\ker D} d\epsilon F(\epsilon, \psi, g)
\end{equation}

where \( F \) is a distribution concentrated on some gauge slice. (There does not appear to be any particularly natural choice for \( F \).) Now the Berezin measure \( d\epsilon \) on the odd vector space \( \ker D \) transforms as a section of the line bundle \( (\text{Det} \ker D)^{-1} \). The latter is identified with Pfaff \( \mathcal{D} \). By (A.11) \( F(\epsilon, \psi, g) \) transforms in the inverse line bundle. In the gauge fixing procedure we factor out the gauge group leaving behind \( F(0, \psi, g) \) in the path integral. The product \( F(0, \psi, g) \cdot Z \) is a section of a determinant line bundle. Specifically, for \( \text{dim} Y = 11 \) we identify \( S_0 \) and \( S_1 \) using the volume form (§1.2) and rewrite (A.10) using pfaffians (§1.3). This yields (2.1) and provides the motivation for our choice of line bundle with covariant derivative in (2.2).

§A.4 Boundary conditions for ghosts

Let us finally consider the analysis of the gravitino determinant for \( \text{dim} Y = 11 \) in the presence of a boundary, as in §4.3. For further simplicity we will assume the boundary Dirac operator \( D^\partial \) has no zeromodes. The boundary conditions on the gravitino are given in (4.29), and we make a definite choice of sign. Recall that the restriction \( \psi^\partial \) of the Rarita-Schwinger field \( \psi \) to \( \partial Y \) decomposes into its tangential and normal components (4.28). Then the boundary conditions are:

\[
\omega^\partial \psi^\partial_T = +\sqrt{-1} \psi^\partial_T
\]

\[
\omega^\partial \psi^\partial_\nu = -\sqrt{-1} \psi^\partial_\nu
\]

where \( T \) denotes the tangential component and \( \nu \) the normal component. These boundary conditions imply that the gauge group must be restricted by

\begin{equation}
\omega^\partial \nabla_T \epsilon^\partial = +\sqrt{-1} \nabla_T \epsilon^\partial
\end{equation}

\[
\omega^\partial \nabla_\nu \epsilon^\partial = -\sqrt{-1} \nabla_\nu \epsilon^\partial
\]

We then choose boundary conditions on \( \beta \) so that \( D^+ \) is skew-adjoint:

\begin{equation}
\omega^\partial \nabla_T \beta^\partial = -\sqrt{-1} \nabla_T \beta^\partial
\end{equation}

\[
\omega^\partial \nabla_\nu \beta^\partial = +\sqrt{-1} \nabla_\nu \beta^\partial
\]

The third ghost determinant comes from integrating over \( s \) (A.9), which has the same boundary condition as \( \gamma \cdot \psi \):

\begin{equation}
\omega^\partial (s^\partial) = -\sqrt{-1} s^\partial.
\end{equation}
The boundary conditions (A.12) and (A.13) do not fit the discussion on local boundary conditions for Dirac operators given in §3.3. We can relate them to the standard ones, at least topologically. Note first that if there are no covariantly constant spinors on the boundary, which we assume, then the first equation in (A.12) is equivalent to the condition \( \epsilon^0 \in \Omega^0_{\partial Y}(S^0_+) \). Similarly, the first equation in (A.13) is equivalent to the condition \( \beta^0 \in \Omega^0_{\partial Y}(S^1_-) \). Let us define

\[
\widetilde{\Omega}^0(S^0) := \{ \epsilon \in \Omega^0(S^0) | \epsilon^0 \in \Omega^0_{\partial Y}(S^0_+), (\nabla_\nu \epsilon)^0 \in \Omega^0_{\partial Y}(S^0_-) \}.
\]

We then have the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \widetilde{\Omega}^0(S^0) & \longrightarrow & \Omega^0(S^0)_{\epsilon^0 \in \Omega^0_{\partial Y}(S^0_+)} & \overset{\pi_+ \circ \nabla_\nu}{\longrightarrow} & \Omega^0(S^0, \beta) & \longrightarrow & 0 \\
D^+ & \downarrow & & & D^+ & \downarrow & \gamma_\nu & \downarrow & \\
0 & \longrightarrow & \Omega^0(S^1)_{\beta^0 \in \Omega^0_{\partial Y}(S^0_+)} & \longrightarrow & \Omega^0(S^1) & \overset{\pi_+ \circ \iota^*}{\longrightarrow} & \Omega^0(S^1, \beta) & \longrightarrow & 0
\end{array}
\]

where \( \pi_+ \) is the projection to the positive chirality spinors and \( \iota^* \) is pullback to the boundary. Each vertical arrow is a Fredholm operator. The right-most arrow is an isomorphism, and hence it follows that topologically the determinant lines (and Pfaffian lines) associated to the other two vertical arrows are isomorphic. We expect that the metrics and covariant derivatives similarly match up, but we have not checked the details.

**Appendix B: Quaternionic Fredholms and Pfaffians**

We prove a theorem about a special space of Fredholm operators which is a topological version of Proposition 1.31. We include it here to emphasize the topological nature of the latter. We remark that the similar Proposition 1.16 is not topological; there is no analog for Fredholm operators.\(^{19}\)

For determinants and pfaffians on spaces of Fredholm operators, see [Q] and [Se1].

Dirac operators on compact manifolds are special examples of Fredholm operators. Whereas Dirac operators have discrete spectrum with no accumulation points, a basic fact for the geometric constructions of §1, general Fredholm operators have continuous spectrum. Spaces of Fredholm operators are classifying spaces for \( K \)-theory [AS].

\(^{19}\)The space of self-adjoint Fredholm operators on a complex Hilbert space has only one natural real line bundle, analogous to (1.15) and the construction below; there is no nontrivial (real) determinant line bundle.
Let $\mathcal{H}$ be a separable complex Hilbert space with a unitary quaternionic structure $J: \mathcal{H} \rightarrow \overline{\mathcal{H}}$. Thus $JJ = -\text{id}_{\mathcal{H}}$ and $J$ is skew-adjoint in the sense that
\[\langle Jv, w \rangle + \langle Jw, v \rangle = 0, \quad v, w \in \mathcal{H}.\]

Define
\[\mathcal{F} = \{ T: \mathcal{H} \rightarrow \overline{\mathcal{H}} : T \text{ is Fredholm, } T \text{ is skew-adjoint, } TJ = JT \}.\]

This space of skew-adjoint quaternion Fredholm operators has two contractible components and a third component which is a classifying space for $KR^{-3}$. There are two natural real line bundles $\text{Pfaff} \rightarrow \mathcal{F}$ and $L^{1/2} \rightarrow \mathcal{F}$ which we now construct.

A subspace $A \subset \mathcal{H}$ is said to be quaternionic if $J(A) = \overline{A}$. For finite dimensional quaternionic $A \subset \mathcal{H}$ we define the open set
\[U_A = \{ T \in \mathcal{F} : T(A) \subseteq \overline{A}, T(\mathcal{H}) \oplus \overline{A} = \overline{\mathcal{H}} \}.

Let $\text{Pfaff}_A \rightarrow U_A$ be the real line bundle whose fiber at $T \in U_A$ is the pfaffian of the skewadjoint operator $T|_A: A \rightarrow \overline{A}$. The real structure is given by $\text{pfaff}J|_A$. (See the paragraph containing (1.23) for a discussion of finite-dimensional pfaffians.) If $A \subset B$ then on $U_A \cap U_B$ there is a canonical isomorphism
\[\text{Pfaff}_B \cong \text{Pfaff}_A \otimes \text{Pfaff}_{B/A},\]

where the fiber of $\text{Pfaff}_{B/A}$ at $T$ is the Pfaffian line of $T: B/A \rightarrow \overline{B/A}$. The pfaffian of that operator is nonzero and real, and this gives an isomorphism $\text{Pfaff}_A \rightarrow \text{Pfaff}_B$. For $A \subset B \subset C$ a cocycle identity is satisfied, so by patching we obtain a global real line bundle $\text{Pfaff} \rightarrow \mathcal{F}$.

To construct the bundle $L^{1/2} \rightarrow \mathcal{F}$ we observe that for $T \in \mathcal{F}$ the composition $S^{(T)} = JT: \mathcal{H} \rightarrow \mathcal{H}$ is self-adjoint Fredholm and quaternion linear in the sense that $JS^{(T)} = S^{(T)}J$. So its eigenspaces are quaternionic. For $A \subset \mathcal{H}$ finite dimensional quaternionic set $S^{(T)}_A = S^{(T)}|_A$. Let $L^{1/2}_A \rightarrow U_A$ be the real line bundle whose fiber at $T$ is
\[
\{ f_A: \mathbb{R} \setminus \mathbb{R} \text{spec}(S^{(T)}_A) \rightarrow \mathbb{R} : f_A(\beta) = (-1)^\# \{ \lambda \in \mathbb{R} \text{spec}(S^{(T)}_A) : \lambda < \alpha < \beta \} f_A(\alpha) \}.
\]

On $A \subset B$ we note that $S^{(T)}_{B/A}: B/A \rightarrow B/A$ is invertible, so the spectra of $S^{(T)}_A$ and $S^{(T)}_B$ agree in a neighborhood of $0 \in \mathbb{R}$. Define an isomorphism $L^{1/2}_B \rightarrow L^{1/2}_A$ which identifies $f_A$ and $f_B$ if $f_A = f_B$ in this neighborhood of zero. This satisfies a cocycle identity, so defines a global line bundle $L^{1/2} \rightarrow \mathcal{F}$. It carries a natural metric.
Proposition B.2. The real line bundles $\text{Pfaff} \to \mathcal{F}$ and $L^{1/2} \to \mathcal{F}$ are isomorphic.

Equivalently, the tensor product $L^{1/2} \otimes \text{Pfaff} \to \mathcal{F}$ is trivializable. For this topological theorem we make a choice (of a norm) to specify a trivialization. In the result for Dirac operators, Proposition 1.31, the trivialization is required to have unit norm, and this leads to the canonical construction in the proof of that result.

**Proof.** The space $\mathcal{F}$ is paracompact, so use a partition of unity to construct a metric on $\text{Pfaff} \to \mathcal{F}$. Set $J_A = J \big|_A$. The bundle $\text{Pfaff}_A \to U_A$ has a trivialization $\text{pfaff} J_A$. Under the patching isomorphism on $U_A \cap U_B$ the ratio of the trivializations of $\text{Pfaff}_B$ and $\text{Pfaff}_A$ is the pfaffian of the self-adjoint operator $S^{(T)}_{B/A} : B/A \to B/A$, that is, the product of its (real) eigenvalues viewing $S^{(T)}_A$ as a quaternion linear operator. Use instead the unit norm trivialization $\text{pfaff} J_A/|\text{pfaff} J_A|$ on $U_A$. Then the transition function on $U_A \cap U_B$ is

\[(B.3) \quad \frac{\text{pfaff} S^{(T)}_{B/A}}{|\text{pfaff} S^{(T)}_{B/A}|}
\]

Since $A$ is finite dimensional the spectrum of $S^{(T)}_A$ is finite, so the domain of $f_A$ in (B.1) contains an interval $(-\infty, \alpha)$ for some $\alpha \in \mathbb{R}$. Let $f_A(-\infty)$ denote the value of $f_A$ on this interval. Trivialize $L^{1/2}_A \to U_A$ by the function $f_A$ with $f_A(-\infty) = 1$. The transition functions for this trivialization are exactly (B.3).

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