THE BOUNDEDNESS OF A CLASS OF FRACTIONAL TYPE
ROUGH HIGHER ORDER COMMUTATORS ON VANISHING
GENERALIZED WEIGHTED MORREY SPACES

FERİT GÜRBUZ

Abstract. This paper includes new bounds concerning the vanishing generalized weighted Morrey space. In this sense, it is outlined improved bounds about the a class of fractional type rough higher order commutators on vanishing generalized weighted Morrey spaces.

1. Introduction

Let $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$. $\Omega$ is the function defined on $\mathbb{R}^n \setminus \{0\}$ satisfying the homogeneous of degree zero condition, that is,

$$(1.1) \quad \Omega(\lambda x) = \Omega(x) \text{ for any } \lambda > 0, x \in \mathbb{R}^n \setminus \{0\}$$

and the integral zero property (=the vanishing moment condition) over the unit sphere $S^{n-1}$, that is,

$$(1.2) \quad \int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

In this paper we consider the following higher order (= $k$-th order) commutator operators of rough fractional integral and maximal operators,

$$T_{\Omega,\alpha}^{A,k}f(x) = T_{\Omega,\alpha} \left( (A(x) - A(\cdot))^k f(\cdot) \right)(x), \quad k = 0, 1, 2, \ldots,$$

$$(1.3) \quad \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} (A(x) - A(y))^k f(y)dy$$

and

$$M_{\Omega,\alpha}^{A,k}f(x) = M_{\Omega,\alpha} \left( (A(x) - A(\cdot))^k f(\cdot) \right)(x), \quad k = 0, 1, 2, \ldots,$$

$$(1.4) \quad \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|<r} |\Omega(x-y)||A(x) - A(y)|^k f(y)|dy,$$

as long as the integrals above make sense, where rough fractional integral operator $T_{\Omega,\alpha}$ and rough fractional maximal operator $M_{\Omega,\alpha}$ are defined by

$$T_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy \quad 0 < \alpha < n$$

$2000 \ Mathematics \ Subject \ Classification. \ 42B20, 42B25.$

Key words and phrases. Fractional type higher order (= $k$-th order) commutator operators; rough kernel; $A \left( \frac{2n}{p}, \frac{4}{q} \right)$ weight; vanishing generalized weighted Morrey space.
For \( k = 1 \) above, \( T_{A,k}^{\alpha} \Omega \) and \( M_{A,k}^{\alpha} \Omega \) are obviously reduced to the rough commutator operators of \( T_{\Omega}^{\alpha} \) and \( M_{\Omega}^{\alpha} \), respectively:

\[
[A, T_{\Omega}^{\alpha}] f (x) = A (x) T_{\Omega}^{\alpha} f (x) - T_{\Omega}^{\alpha} (A f) (x) = \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n - \alpha}} (A (x) - A (y)) f(y) dy
\]

and

\[
[A, M_{\Omega}^{\alpha}] f (x) = A (x) M_{\Omega}^{\alpha} f (x) - M_{\Omega}^{\alpha} (A f) (x) = \sup_{r > 0} \frac{1}{r^{n - \alpha}} \int_{|x - y| < r} |\Omega(x - y)| |A (x) - A (y)| |f(y)| dy.
\]

Moreover, \( T_{A,k}^{\alpha} \Omega \) and \( M_{A,k}^{\alpha} \Omega \) are trivial generalizations of the above commutators, respectively.

Here and henceforth, \( F \approx G \) means \( F \gtrsim G \gtrsim F \); while \( F \gtrsim G \) means \( F \geq C G \) for a constant \( C > 0 \); and also \( C \) stands for a positive constant that can change its value in each statement without explicit mention.

Now, let us list some definitions that we need in the proof of following Theorem

**Definition 1.** (Bounded Mean Oscillation (BMO)) We denote the mean value of \( f \) on \( B = B(x, r) \subset \mathbb{R}^n \) by

\[
f_B = M (f, B) = M (f, x, r) = \frac{1}{|B|} \int_B f(y) dy,
\]

and the mean oscillation of \( f \) on \( B = B(x, r) \) by

\[
MO (f, B) = MO (f, x, r) = \frac{1}{|B|} \int_B |f(y) - f_B| dy.
\]

We also define for a non-negative function \( \phi \) on \( \mathbb{R}^n \)

\[
MO_{\phi} (f, B) = MO_{\phi} (f, x, r) = \frac{1}{\phi(|B|)|B|} \int_B |f(y) - f_B| dy.
\]

Now we define

\[
BMO_{\phi} = \left\{ f \in L^1_{loc} (\mathbb{R}^n) : \sup_B MO_{\phi} (f, B) < \infty \right\}
\]

and

\[
\|f\|_{BMO_{\phi}} = \sup_B MO_{\phi} (f, B).
\]

The most important of these spaces occurs when \( \phi = 1 \), in which case \( BMO_{\phi} = BMO \).
Definition 2. (Weighted Lebesgue space) Let $1 \leq p \leq \infty$ and given a weight $w(x) \in A_p(\mathbb{R}^n)$, we shall define weighted Lebesgue spaces as

$$L_p(w) \equiv L_p(\mathbb{R}^n, w) = \left\{ f : \|f\|_{L_p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 \leq p < \infty.$$ 

For $p = \infty$, we have

$$L_\infty(w) = L_\infty(\mathbb{R}^n, w) = \left\{ f : \|f\|_{L_\infty(w)} = \text{esssup}_{x \in \mathbb{R}^n} |f(x)| w(x) \right\}.$$

Here and later, we refer to $A_p$ as the the Muckenhoupt classes. That is, $w(x) \in A_p(\mathbb{R}^n)$ for some $1 < p < \infty$ if

$$\left( \frac{1}{|B|} \int_B w(y) \, dy \right)^{\frac{1}{p}} \left( \frac{1}{|B|} \int_B w(y)^{-\frac{1}{p-1}} \, dy \right)^{\frac{1}{p-1}} \leq C$$

for all balls $B$ (see [1] for more details).

Now, let us consider the Muckenhoupt-Wheeden class $A(p, q)$ in [5]. One says that $w(x) \in A(p, q)$ for $1 < p < q < \infty$ if and only if

$$[w]_{A(p, q)} := \sup_B \left( |B|^{-\frac{1}{q}} \int_B w(x)^q \, dx \right)^{\frac{1}{q}} \left( |B|^{-\frac{1}{p}} \int_B w(x)^{-p'} \, dx \right)^{\frac{1}{p'}} < \infty,$$

where the supremum is taken over all the balls $B$. Note that, by Hölder’s inequality, for all balls $B$ we have

$$[w]_{A(p, q)} \geq [w]_{A(p, q')(B)} = |B|^{\frac{1}{q'} - \frac{1}{q} - 1} \|w\|_{L_q(B)} \|w^{-1}\|_{L_{q'}(B)} \geq 1.$$

By (1.5), we have

$$\left( \int_B w(x)^q \, dx \right)^{\frac{1}{q}} \left( \int_B w(x)^{-p'} \, dx \right)^{\frac{1}{p'}} \lesssim |B|^{\frac{1}{q'} + \frac{1}{p'}}.$$

On the other hand, let $\mu(x) = w(x)^{\tilde{q}}$, $\tilde{p} = \frac{p}{q'}$ and $\tilde{q} = \frac{q}{q'}$. If $w(x)^{\tilde{q}} \in A(\tilde{p}, \tilde{q})$, then we get $\mu(x) \in A(\tilde{p}, \tilde{q})$.

Now, we introduce some spaces which play important roles in PDE. Except the weighted Lebesgue space $L_p(w)$, the weighted Morrey space $L_{p, \kappa}(w)$, which is a natural generalization of $L_p(w)$ is another important function space. Then, the definition of generalized weighted Morrey spaces $M_{p, \varphi}(w)$ which could be viewed as extension of $L_{p, \kappa}(w)$ has been given as follows:

For $1 \leq p < \infty$, positive measurable function $\varphi(x, r)$ on $\mathbb{R}^n \times (0, \infty)$ and nonnegative measurable function $w$ on $\mathbb{R}^n$, $f \in M_{p, \varphi}(w) \equiv M_{p, \varphi}(\mathbb{R}^n, w)$ if $f \in L_{p, \kappa}(w)$ and

$$\|f\|_{M_{p, \varphi}(w)} = \mathop{\sup}_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi(x, r)} \|f\|_{L_p(B(x, r), w)} < \infty.$$

is finite. Note that for $\varphi(x, r) \equiv w(B(x, r))^{\frac{1}{\kappa}}$, $0 < \kappa < 1$ and $\varphi(x, r) \equiv 1$, we have $M_{p, \varphi}(w) = L_{p, \kappa}(w)$ and $M_{p, \varphi}(w) = L_p(w)$, respectively. Moreover, Gürbüz [2] proved that the operators $T^{A, k}_{\Omega, \alpha}$ and $M^{A, k}_{\Omega, \alpha}$ are bounded from one generalized weighted Morrey space $M_{p, \varphi_1}(w^\rho, \mathbb{R}^n)$ to another $M_{q, \varphi_2}(w^q, \mathbb{R}^n)$.

The following definition was introduced by Gürbüz [4].
Definition 3. (Vanishing generalized weighted Morrey spaces) For $1 \leq p < \infty$, $\varphi(x,r)$ is a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and nonnegative measurable function $w$ on $\mathbb{R}^n$, $f \in VM_{p,\varphi}(w) \equiv VM_{p,\varphi}(\mathbb{R}^n, w)$ if $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ and

$$
\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{1}{\varphi(x,r)} \|f\|_{L^p(B(x,r),w)} = 0.
$$

Inherently, it is appropriate to impose on $\varphi(x,t)$ with the following circumstances:

$$
\lim_{t \to 0} \sup_{x \in \mathbb{R}^n} \frac{(w(B(x,t)))^\frac{1}{p}}{\varphi(x,t)} = 0,
$$

and

$$
\inf_{t > 1} \sup_{x \in \mathbb{R}^n} \frac{(w(B(x,t)))^\frac{1}{p}}{\varphi(x,t)} > 0.
$$

From [14] and [18], we easily know that the bounded functions with compact support belong to $VM_{p,\varphi}(w)$. On the other hand, the space $VM_{p,\varphi}(w)$ is Banach space with respect to the following finite quasi-norm

$$
\|f\|_{VM_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi(x,r)} \|f\|_{L^p(B(x,r),w)},
$$

such that

$$
\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{1}{\varphi(x,r)} \|f\|_{L^p(B(x,r),w)} = 0,
$$

we omit the details. Moreover, we have the following embeddings:

$$
VM_{p,\varphi}(w) \subset M_{p,\varphi}(w), \quad \|f\|_{M_{p,\varphi}(w)} \leq \|f\|_{VM_{p,\varphi}(w)}.
$$

Henceforth, we denote by $\varphi \in B(w)$ if $\varphi(x,r)$ is a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and positive for all $(x,r) \in \mathbb{R}^n \times (0, \infty)$ and satisfies (1.7) and (1.8).

Inspired of [2], the aim of the present paper is to study the boundedness of the operators $T_{\Omega,\alpha}^{A,k}$ and $M_{\Omega,\alpha}^{A,k}$ generated by $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ with a BMO functions on vanishing generalized weighted Morrey spaces, respectively. That is, in this paper we will consider this problem.

2. Main results

Let us state our main result as follows.

Theorem 1. Suppose that $0 < \alpha < n$, $1 \leq s' < p < \frac{n}{\alpha}$, $\frac{1}{p} = \frac{1}{p} - \frac{\alpha}{n}$, $1 < q < \infty$, $\Omega \in L_s(S^{n-1})$ (s > 1) satisfies [14] such that $k \in \mathbb{N}$, $w(x)^s \in A(\frac{p}{s'}, \frac{q}{q})$, $A \in BMO(\mathbb{R}^n)$, $T_{\Omega,\alpha}^{A,k}$, $M_{\Omega,\alpha}^{A,k}$ are defined as [13], [14] and $T_{\Omega,\alpha}^{A,k}$ satisfies (13) in [2]. If $\varphi_1 \in B(w^p)$, $\varphi_2 \in B(w^q)$ and the pair $(\varphi_1, \varphi_2)$ satisfies the conditions

$$
c_\delta := \int_\delta^\infty \left(1 + \ln \frac{t}{r}\right)^k \sup_{a \in \mathbb{R}^n} \frac{\varphi_1(x,t)}{(w^q(B(x,t)))^{\frac{n}{q}}} \frac{1}{t} dt < \infty
$$

for all $\delta > 0$.
for every $\delta > 0$, and

\[
\int_r^\infty \left( 1 + \ln \frac{t}{r} \right)^k \frac{\varphi_1(x,t)}{(w^p(B(x,t)))^\frac{1}{p}} \frac{1}{t} dt \lesssim \frac{\varphi_2(x,r)}{(w^p(B(x,t)))^\frac{1}{p}}.
\]

then the operator $T_{\Omega,k}^\alpha$ is bounded from $VM_{p,q_1}(w^p)$ to $VM_{q,q_2}(w^q)$. Moreover,

\[
\begin{align*}
|T_{\Omega,k}^\alpha f|_{VM_{q,q_2}(w^q)} &\lesssim \|A\|_k \|f\|_{VM_{p,q_1}(w^p)}^p, \\
|M_{\Omega,k}^\alpha f|_{VM_{q,q_2}(w^q)} &\lesssim \|A\|_k \|f\|_{VM_{p,q_1}(w^p)}^p.
\end{align*}
\]

For $\alpha = 0$, from Theorem 1 we get the following:

**Corollary 1.** Suppose that $1 < p < \infty$, $s' < p$, $\Omega \in L_\alpha(S^{n-1})(s > 1)$ satisfies (1.1) and (1.2) such that $k \in \mathbb{N}$, $w(x)^{s'} \in A_\infty$, $A \in BMO(\mathbb{R}^n)$, $T_{\Omega,k}^\alpha$, $M_{\Omega,k}^\alpha$ are defined as

\[
T_{\Omega,k}^\alpha f(x) = T_{\Omega}((A(x) - A(\cdot)) f(\cdot))(x), \quad k = 0, 1, 2, \ldots,
\]

\[
= p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (A(x) - A(y))^k f(y) dy
\]

and the corresponding higher order (= $k$-th order) commutator operator of $M_{\Omega}$:

\[
M_{\Omega,k}^\alpha f(x) = M_{\Omega}((A(x) - A(\cdot)) f(\cdot))(x), \quad k = 0, 1, 2, \ldots,
\]

\[
= \sup_{r > 0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)| |A(x) - A(y)|^k |f(y)| dy
\]

and $T_{\Omega,k}^\alpha$ satisfies (11) in [3]. If $\varphi \in B(w)$ and the pair $(\varphi_1, \varphi_2)$ satisfies the conditions

\[
c_{s'} \overset{\delta}{:=} \int_\delta^\infty \left( 1 + \ln \frac{t}{r} \right)^k \sup_{x \in \mathbb{R}^n} \frac{\varphi_1(x,t)}{(w^p(B(x,t)))^\frac{1}{p}} \frac{1}{t} dt < \infty
\]

for every $\delta' > 0$, and

\[
\int_r^\infty \left( 1 + \ln \frac{t}{r} \right)^k \frac{\varphi_1(x,t)}{(w^p(B(x,t)))^\frac{1}{p}} \frac{1}{t} dt \lesssim \frac{\varphi_2(x,r)}{(w^p(B(x,t)))^\frac{1}{p}}.
\]

Then,

\[
\begin{align*}
|T_{\Omega,k}^\alpha f|_{VM_{p,q_2}(w^q)} &\lesssim \|A\|_k \|f\|_{VM_{p,q_1}(w^q)}^p, \\
|M_{\Omega,k}^\alpha f|_{VM_{p,q_2}(w^q)} &\lesssim \|A\|_k \|f\|_{VM_{p,q_1}(w^q)}^p.
\end{align*}
\]
Proof of Theorem 1

Proof. By Definition 3 (13) in [2] and (2.2) we get
\[
\left\| T_{\Omega, \alpha}^{A, k} f \right\|_{V_{M_p, \varphi_2(w^q, \mathbb{R}^n)}} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{\left\| T_{\Omega, \alpha}^{A, k} f \right\|_{L_q(w^q, B(x, r))}}{\varphi_2(x, r)} \\
\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi_2(x, r)} \left\| A \right\|_k^k (w^q (B(x, r)))^\frac{1}{q} \\
\times \int_r^\infty \left( 1 + \ln \frac{t}{r} \right)^{k} \| f \|_{L_p(w^p, B(x, t))} (w^q (B(x, t)))^{-\frac{1}{q}} \frac{1}{t} dt \\
\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi_2(x, r)} \left\| A \right\|_k^k (w^q (B(x, r)))^\frac{1}{q} \\
\times \int_r^\infty \left( 1 + \ln \frac{t}{r} \right)^{k} \varphi_1(x, t) \left( w^q (B(x, t)) \right)^{-\frac{1}{q}} \frac{1}{t} \| f \|_{L_p(w^p, B(x, t))} \varphi_1(x, t)^{-1} dt \\
\lesssim \left\| A \right\|_k^k \| f \|_{V_{M_p, \varphi_1(w^p, \mathbb{R}^n)}}.
\]

At last, we need to prove that
\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_2(x, r)} \left\| T_{\Omega, \alpha}^{A, k} f \right\|_{L_q(w^q, B(x, r))} \lesssim \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_2(x, r)} \| f \|_{L_p(w^p, B(x, r))} = 0.
\]

Indeed, for any \( \epsilon > 0 \), let \( 0 < r < \psi \). By (13) in [2], we have
\[
\frac{\left\| T_{\Omega, \alpha}^{A, k} f \right\|_{L_q(w^q, B(x, r))}}{\varphi_2(x, r)} \lesssim \left[ F_\psi (x, r) + G_\psi (x, r) \right],
\]
where
\[
F_\psi (x, r) := \left\| A \right\|_k^k (w^q (B(x, r)))^\frac{1}{q} \int_r^{\psi} \left( 1 + \ln \frac{t}{r} \right)^{k} \| f \|_{L_p(w^p, B(x, t))} (w^q (B(x, t)))^{-\frac{1}{q}} \frac{1}{t} dt
\]
and
\[
G_\psi (x, r) := \left\| A \right\|_k^k (w^q (B(x, r)))^\frac{1}{q} \int_r^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{k} \| f \|_{L_p(w^p, B(x, t))} (w^q (B(x, t)))^{-\frac{1}{q}} \frac{1}{t} dt.
\]
For \( \sup_{x \in \mathbb{R}^n} \sup_{0 < r < t} \frac{\| f \|_{L_p(w^p, B(x, r))}}{\varphi_2(x, r)} < \frac{\epsilon}{2} \), we can select any constant \( \psi > 0 \). This allows
to guess the first term properly from the type \( r \in (0, \psi) \) such that
\[
\sup_{x \in \mathbb{R}^n} F_\psi (x, r) < \frac{\epsilon}{2}.
\]
For the second term, in view of (2.1), we obtain
\[
G_\psi (x, r) \lesssim \left\| A \right\|_k^k \| f \|_{V_{M_p, \varphi_1(w^p, \mathbb{R}^n)}} (w^q (B(x, r)))^\frac{1}{q} \frac{1}{\varphi_2(x, r)}.
\]
Since \( \varphi_2 \in \mathcal{B}(w^n) \), it gets along to select \( r \) minor sufficient such that
\[
\sup_{x \in \mathbb{R}^n} \frac{w^n(B(x,r))}{\varphi_2(x,r)} \lesssim \left( \frac{\epsilon}{2 \| A \|_k^{1/q} \| f \|_{VM_{p,\varphi}(w^p,\mathbb{R}^n)}} \right)^q.
\]
Hence,
\[
\sup_{x \in \mathbb{R}^n} \mathcal{F}_\psi(x,r) < \frac{\epsilon}{2}.
\]
Thus,
\[
\left\| T_{\Omega, \alpha}^{A,k}f \right\|_{L_q(w^q,B(x,r))} \lesssim \epsilon.
\]
Therefore,
\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{1}{\varphi_2(x,r)} \left\| T_{\Omega, \alpha}^{A,k}f \right\|_{L_q(w^q,B(x_0,r))} = 0.
\]
As a result, (2.3) holds. On the other hand, since \( M_{\Omega, \alpha}^{A,k}f(x) \leq T_{[\Omega],[\alpha]}^{A,k}(|f|)(x) \), \( x \in \mathbb{R}^n \) (see Lemma 6 in [2]) we can also use the same method for \( M_{\Omega, \alpha}^{A,k} \), so we omit the details. As a result, we complete the proof of Theorem 1. \( \square \)

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HAKKARI UNIVERSITY, FACULTY OF EDUCATION, DEPARTMENT OF MATHEMATICS EDUCATION, HAKKARI 30000, TURKEY

E-mail address: feritgurbuz@hakkari.edu.tr