NOTES ON THE UNIVERSAL ELLIPTIC KZB EQUATION

RICHARD HAIN

CONTENTS

INTRODUCTION 2

Part 1. Background 4
1. The Universal Elliptic Curve 4
2. Unipotent Completion 7
3. Factors of Automorphy 10
4. Some Lie theory 12
5. Connections and Monodromy 14

Part 2. The Universal Elliptic KZB Connection 17
6. The Bundle $\mathcal{P}$ over $\mathcal{E}'$ 17
7. Eisenstein Series and Bernoulli Numbers 19
8. The Jacobi Form $F(\xi, \eta, \tau)$ 20
9. The Universal Elliptic KZB Connection 23

Part 3. Complements 32
10. Extending $\mathcal{P}$ to $\overline{\mathcal{M}}_{1,2}$ 32
11. Restriction to $E'_\tau$ 33
12. Restriction to the First-order Tate Curve 34
13. Restriction to $\mathcal{M}_{1,\overline{1}}$ 39
14. Rigidity 40
15. Hodge Theory 42
16. Pause for a Picture 45
17. The KZ-equation and the Drinfeld Associator 46
18. The Limit MHS on $\pi_1(E'_\partial, \partial/\partial w)^{un}$ 48

Part 4. The $\mathbb{Q}$-de Rham Structure 51
19. The $\mathbb{Q}$-DR Structure on $\mathcal{H}$ over $\overline{\mathcal{M}}_{1,\overline{1}}$ 52
20. The $\mathbb{Q}$-DR Structure on $\overline{\mathcal{P}}$ over $\overline{\mathcal{M}}_{1,\overline{1}}$ 56
21. The $\mathbb{Q}$-de Rham Structure on $E^{2n+1}H^1(\mathcal{M}_{1,1}, S^{2n}H)$ 58
Appendix A. An Identity 59
Appendix B. Polynomials 62
Appendix C. The Universal Elliptic Curve over $\text{Bl}^+_0 \mathbb{D}$ 62
References 64

Date: August 5, 2014.
Supported in part by the NSF through grants DMS-0706955 and DMS-1005675.
Introduction

These notes are a somewhat polished and expanded version of notes made during a seminar in the summer of 2007 at Duke University whose goal was to understand the paper of [16] of Levin and Racinet. The closely related paper [2] of Calaque, Enriquez and Etingof appeared slightly earlier and contains very similar results. Both papers construct an integrable connection on a vector bundle over $\mathcal{M}_{1,2}$ whose fiber over the point corresponding to the elliptic curve $E$ and a non-zero point $x \in E$ is the unipotent completion of $\pi_1(E',x)$, where $E' := E \setminus \{0\}$. This bundle with connection is, as we prove in Section 14, the de Rham manifestation of the canonical local system (i.e., locally constant sheaf) over $\mathcal{M}_{1,2}$ with the same fiber.\(^1\) This connection is a special case of the universal elliptic KZB\(^2\) connection.\(^3\) Flat sections of this bundle satisfy what Calaque et al call the universal elliptic KZB\(^2\) equation.

My interest lay in its relation to the theory of Universal Elliptic Motives developed by Makoto Matsumoto and myself in [12]. The local system of the Lie algebras of the unipotent fundamental groups over $\mathcal{M}_{1,2}$ is an example of a universal mixed elliptic motive. The elliptic KZB connection is its $Q$-de Rham realization, as explained in [12].

This exposition follows the approach of Levin and Racinet, which expresses the elliptic KZB connection in terms of Kronecker’s Jacobi form $F(\xi, \eta, \tau)$, [15] and Eisenstein series. This function $F$ was rediscovered by Zagier in [22] and can be expressed in terms of classical theta functions. It is a generating function for modular symbols of classical cusp forms. Calaque et al express the connection directly in terms of the theta functions and Eisenstein series. My hope is that this relationship between Kronecker’s Jacobi form and modular symbols of classical cusp forms will be useful in understanding the relationship between cusp forms, Pollack’s relations [17] and the relations in the depth graded quotients of the ring of motivic multiple zeta values.

During the seminar, we could not verify some of the computations in the Levin-Racinet paper without modifying their factors of automorphy. Such differences may have arisen because of differing conventions. This paper uses the modified factors of automorphy. In the interest of clarity, I have tried to make all conventions explicit and most computations complete so that they can be easily checked.

The paper is in four parts. The first provides some background material on moduli of elliptic curves, unipotent completion, and some basic facts from Lie theory. The second part is a complete exposition of the universal elliptic KZB connection following (with minor modifications) the paper of Levin-Racinet [16]. Full details are provided; little, if anything, here is new.

The third part is an exploration of the the universal elliptic KZB connection. We compute its restriction to various loci, such as the punctured first order neighbourhood of the Tate curve and a punctured formal neighbourhood of the identity

---

\(^1\)By this we mean that the sheaf of locally constant sections of the connection is the local system.

\(^2\)Named after the physicists Knizhnik, Zamoldchikov and Bernard.

\(^3\)The general universal elliptic KZB connection is the flat connection on the bundle over $\mathcal{M}_{1,n+1}$ whose fiber over $[E; 0, x_1, \ldots, x_n]$ is the unipotent fundamental group of the configuration space of $n$ points on $E'$ with base point $(x_1, \ldots, x_n)$. Calaque et al [2] write down the universal elliptic KZB-equation for all $n \geq 1$. 
section. This allows us to give some applications, and extend some results from both [2] and [16]. For example, we prove in Section 14 that the monodromy of the universal elliptic KZB connection is indeed the natural action of \( \pi_1(\mathcal{M}_{1,2}, [E'; 0, x]) \) on the unipotent fundamental group of \((E', x)\). We also explore certain Hodge theoretic aspects of the universal elliptic KZB equation and its monodromy about the nodal cubic. For example, we prove that the MHS on the unipotent completion of \( \pi_1(E', x) \) that one gets from the universal elliptic KZB connection (cf. Theorem 15.1) is its canonical MHS as constructed in [7]. In Section 18, we give an explicit computation of the limit mixed Hodge structure on the unipotent completion of \( \pi_1(E', \vec{v}) \) associated to the tangent vector \( \partial/\partial q \) of the origin of the \( q \)-disk, where \( \vec{v} \) is a suitable integrally defined tangent vector of the identity of the Tate curve. Its periods are all multiple zeta values, as one expects from the theory of universal elliptic motives and Brown’s Theorem [1]. One amusing consequence of this computation is the family of identities

\[
\sum_{a+b=n, a,b \geq 0} \frac{(2a-1)}{(2a)!} \frac{B_{2a}}{(2b)!} \frac{B_{2b}}{h_{a,b}(x, y)} = 0
\]

satisfied by Bernoulli numbers, where, for integers \( a, b \geq 0 \),

\[
h_{a,b}(x, y) = x^{2a-1}y^{2b} - x^{2b}y^{2a-1} + xy(x + y)^{2b-1}(y^{2a-2} - x^{2a-2}).
\]

It is a consequence of the fact that this limit MHS contains the unipotent completion of \( \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v}) \) and that it is invariant under monodromy. As a further check, we give a direct elementary proof of this identity in Appendix A.

The fourth part elaborates on computations [16, §5] of Levin-Racinet. We show that the restriction of the universal elliptic KZB connection to the moduli space \( \mathcal{M}_{1,2} \) of elliptic curves and a non-zero abelian differential is defined over \( \mathbb{Q} \) and compute an explicit expression for it. (Cf. Proposition 19.6 and Theorem 20.2.)

Background material on the topology of moduli spaces of elliptic curves (viewed as orbifolds) and their associated mapping class groups is not included. It can be found, for example, in [9]. The books of Serre [18] and Silverman [20] are excellent references for background material on modular forms.

Acknowledgments: I am indebted to all participants in the seminar, particularly Aaron Pollack, whose help in checking the factors of automorphy was invaluable. The other active members of the seminar were Jonathan Hanke and Leslie Saper. I am also indebted to Makoto Matsumoto for helpful discussions and to Francis Brown and Benjamin Enriquez for comments on the manuscript. Finally, I would like to thank Ma Luo who carefully read the manuscript and pointed out many typos.

0.1. Some Conventions: We use the topologist’s convention for path multiplication: if \( \alpha, \beta : [0, 1] \to X \) are paths in a topological space with \( \alpha(1) = \beta(0) \), then \( \alpha \beta : [0, 1] \to X \) is the path obtained by first traversing \( \alpha \) and then \( \beta \).

The adjoint action of an element \( u \) of the enveloping algebra of a Lie algebra \( \mathfrak{g} \) on an element \( x \) of \( \mathfrak{g} \) will often be denoted by \( u \cdot x \). This will be extended to power series \( u \) of elements of \( \mathfrak{g} \) when it makes sense. For example if \( t \in \mathfrak{g} \), then

\[
e^t \cdot x = \sum_{n=0}^{\infty} \text{ad}_t^n(x)/n!
\]
If $\delta$ is a derivation of $g$, then $\delta(f \cdot u) = \delta(f) \cdot u + f \cdot \delta(u)$.

We will be sloppy and denote the generic element of $\text{SL}_2(\mathbb{Z})$ by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

So, unless otherwise mentioned, the entries of $\gamma$ are $a$, $b$, $c$ and $d$.

Although we will say very little about the motivic aspects of this construction, we will occasionally use the notation $\mathbb{Z}(n)$ when it makes sense. For the uninitiated, this is just a copy of $\mathbb{Z}$ with extra structure. Sometimes (but not always) it is used to denote $(2\pi i)^n \mathbb{Z}$.

Part 1. Background

1. The Universal Elliptic Curve

The material in this section is standard. We will assume that the reader is familiar with the construction of $\mathcal{M}_{1,1}$ as the orbifold quotient of the upper half plane by $\text{SL}_2(\mathbb{Z})$, the construction of its Deligne-Mumford compactification $\overline{\mathcal{M}}_{1,1}$ (as an orbifold), the construction of the standard line bundle $\mathcal{L}$ over $\mathcal{M}_{1,1}$, and its extension $\mathcal{L}$ to $\overline{\mathcal{M}}_{1,1}$. This material can be found, for example, in the first four sections of [9].

The group $\text{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ by right multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (m, n) \mapsto (a \cdot m + b \cdot n, c \cdot m + d \cdot n).$$

Denote the corresponding semi-direct product $\text{SL}_2(\mathbb{Z}) \rtimes \mathbb{Z}^2$ by $\Gamma$. This the set $\text{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2$ with multiplication:

$$(\gamma_1, v_1)(\gamma_2, v_2) = (\gamma_1 \gamma_2, v_1 \gamma_2 + v_2)$$

where $\gamma_1, \gamma_2 \in \text{SL}_2(\mathbb{Z})$ and $v_1, v_2 \in \mathbb{Z}^2$.

The group $\Gamma$ acts on $X := \mathbb{C} \times \mathfrak{h}$ on the left:

$$(m, n) : (\xi, \tau) \mapsto (\xi + (m \cdot n), \tau)$$

and

$$\gamma : (\xi, \tau) \mapsto ((c \tau + d)^{-1} \xi, \gamma \tau)$$

where $\gamma \in \text{SL}_2(\mathbb{Z})$.

The quotient $\Gamma \backslash X$ is the universal elliptic curve $\mathcal{E}$; the map $\Gamma \backslash X \to \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$ induced by the projection $X \to \mathfrak{h}$ is the projection $\mathcal{E} \to \mathcal{M}_{1,1}$.

The universal elliptic curve can be compactified using the Tate curve to obtain a proper orbifold map $\mathcal{E} \to \overline{\mathcal{M}}_{1,1}$ whose fiber over $q = 0$ is the nodal cubic. Its pullback to the $q$-disk $\mathbb{D}$, with the double point removed, is the quotient of $\mathbb{C}^* \times \mathbb{D}$ by the group action $\mathbb{Z} \times \mathbb{C}^* \times \mathbb{D} \to \mathbb{C}^* \times \mathbb{D}$ defined by

$$n : (w, q) \mapsto \begin{cases} (q^n w, q) & q \neq 0, \\
(w, q) & q = 0. \end{cases}$$

Note that, although this group action is not continuous, the quotient (endowed with the quotient topology) is Hausdorff and is a complex manifold. The fiber over
\[ q = 0 \] is the group \( \mathbb{C}^* \). The zero section (aka, the identity section) passes through it at \( w = 1 \).

**Proposition 1.1.** The normal bundle of the zero section of \( E \) is \( L_{-1} \).

**Proof.** The line bundle \( L_{-1} \) is the quotient of \( \mathbb{C} \times h \) by the action
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\xi, \tau) \mapsto ((c\tau + d)^{-1}\xi, \gamma\tau).
\]
The identity \( \mathbb{C} \times h \to \mathbb{C} \times h \) is equivariant with respect to the natural inclusion \( \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2 \) and thus induces a quotient mapping \( L_{-1} \to E \) that commutes with the projections to \( \mathcal{M}_{1,1} \). This projection extends over \( q = 0 \). This is well known and from the result of Exercise 47 in [9, §5.2]. \( \square \)

**Corollary 1.2.** A neighbourhood of the zero section of \( L_{-1} \) is biholomorphic with a neighbourhood of the identity section of \( E \to \mathcal{M}_{1,1} \). \( \square \)

Denote by \( L' \) the complex manifold obtained by removing the 0-section from a holomorphic line bundle \( L \).

**Corollary 1.3.** The moduli space \( \mathcal{M}'_{1,1} \) of pairs \( (E, v) \), where \( E \) is a stable elliptic curve and \( v \) is a (possible vanishing) tangent vector at the identity is naturally isomorphic with \( L_{-1} \). In particular, the moduli space of smooth elliptic curves and a non-zero tangent vector at the identity \( \mathcal{M}_{1,1} \) is isomorphic to \( L'_{-1} \). \( \square \)

### 1.1. Fundamental Groups

A point of \( \mathcal{E}' \) is an isomorphism class \([E, x]\) of a pair \((E, x)\), where \( E \) is an elliptic curve and \( x \in E, x \neq 0 \).

**Proposition 1.4.** The fundamental group of \( \mathcal{E}' \) with respect to the base point \((E, x)\) is an extension
\[
1 \to \pi_1(E', x) \to \pi_1(\mathcal{E}', [E, x]) \to \text{SL}_2(\mathbb{Z}) \to 1.
\]
In particular, it is isomorphic to an extension of \( \text{SL}_2(\mathbb{Z}) \) by a free group of rank 2.

**Proof.** The function \( \mathbb{R}^2 \times h \to \mathbb{C} \times h \) defined by \((u, v, \tau) \mapsto (u + v\tau, \tau)\) is a homeomorphism. It induces a homeomorphism \( (\mathbb{R}/\mathbb{Z})^2 \times h \to \mathcal{E}_h \) which restricts to give a homeomorphism
\[
((\mathbb{R}/\mathbb{Z})^2 - \{0\}) \times h \to \mathcal{E}_h',
\]
where \( \mathcal{E}_h \) denotes the universal elliptic curve \( \mathbb{Z}^2(\mathbb{C} \times h) \) over \( h \) and \( \mathcal{E}_h' \) denotes \( \mathcal{E}_h \) with the 0-section removed. It follows that \( \mathcal{E}_h' \) is homotopy equivalent to each of its fibers \( E'_x \). In particular, the inclusion \((E', x) \to (\mathcal{E}_h', (E, x))\) induces an isomorphism on fundamental groups.

The result follows from covering space theory as the covering \( \mathcal{E}_h' \to \mathcal{E}' \) is Galois with Galois group \( \text{SL}_2(\mathbb{Z}) \). \( \square \)

**Corollary 1.5.** For each point \([E, x]\) of \( \mathcal{E}' \), there is a natural action of \( \pi_1(\mathcal{E}', [E, x]) \) on \( \pi_1(E', x) \).

**Proof.** Since \( \pi_1(E', x) \) is a normal subgroup of \( \pi_1(\mathcal{E}', [E, x]) \), one has the conjugation action \( g : \gamma \mapsto g\gamma g^{-1} \) of \( \pi_1(\mathcal{E}', [E, x]) \) on \( \pi_1(E', x) \). \( \square \)
Denote the $\mathbb{C}^*$ bundle obtained from $L_k$ by removing the 0-section by $L'_k$. When $k \neq 0$, $\text{SL}_2(\mathbb{Z})$ acts fixed point freely on $\mathbb{C}^* \times \mathfrak{h}$. In this case its fundamental group is a central extension

$$0 \to \mathbb{Z} \to \pi_1(L'_k,\ast) \to \text{SL}_2(\mathbb{Z}) \to 1.$$ 

**Remark 1.6.** It is well-known that $\pi_1(L'_{-1})$ is naturally isomorphic to each of the following groups:

(i) the braid group $B_3$ on 3-strings;
(ii) the fundamental group of $\mathbb{C}^2$ with the cusp $x^2 = y^3$ removed;
(iii) the fundamental group of the complement of the trefoil knot;
(iv) the inverse image $\text{SL}_2(\mathbb{Z})$ of $\text{SL}_2(\mathbb{Z})$ in the universal covering group $\widetilde{\text{SL}}_2(\mathbb{R})$ of $\text{SL}_2(\mathbb{R})$.

Details can be found, for example, in [9].

Proposition 1.1 implies that if $E$ is an elliptic curve and $\vec{v}$ is a non-zero tangent vector at $0 \in E$, there is a natural homomorphism

$$\pi_1(L'_{-1}, [E, \vec{v}]) \to \pi_1(E', [E, \vec{v}]).$$

Composing this with the action above we obtain an action

$$\pi_1(L'_{-1}, [E, \vec{v}]) \to \text{Aut} \pi_1(E', \vec{v}).$$

Denote the element of $\pi_1(E', \vec{v})$ that corresponds to moving once around the identity in the positive direction by $c_o$. Denote by $z_o$ the image in $\pi_1(L'_{-1}, [E, \vec{v}]) \cong \widetilde{\text{SL}}_2(\mathbb{Z})$ of the positive generator of the fundamental group of the fiber $L'_{-1,E} \cong \mathbb{C}^*$ over $[E]$ of the projection $L'_{-1} \to \mathcal{M}_{1,1}$.

**Proposition 1.7.** This action of $\pi_1(L'_{-1}, [E, \vec{v}])$ on $\pi_1(E', \vec{v})$ fixes $c_o$.

**Proof.** Observe that $c_o$ is the image of $z_o$ under the continuous mapping $L'_{-1} \to E'$. The result follows as $z_o$ is central in $\widetilde{\text{SL}}_2(\mathbb{Z})$. \qed

Since $\pi_1(L'_{-1}, [E, \vec{v}])$ acts on $\pi_1(E', x)$, we can form the semi-direct product

$$\pi_1(L'_{-1}, [E, \vec{v}]) \ltimes \pi_1(E', \vec{v}).$$

**Lemma 1.8.** The element $c_o^{-1}z_o$ is central in $\pi_1(L'_{-1}, [E, \vec{v}]) \ltimes \pi_1(E', \vec{v})$.

**Proof.** Note that $z_o$ acts on $\pi_1(E', \vec{v})$ by conjugation by $c_o$. Since $z_o$ is central in $\pi_1(L'_{-1}, [E, \vec{v}])$ and since each element of $\pi_1(L'_{-1}, [E, \vec{v}])$ fixes $c_o$, we see that $c_o^{-1}z_o$ commutes with each element of $\pi_1(L'_{-1}, [E, \vec{v}])$.

If $g \in \pi_1(E', \vec{v})$, then

$$gc_o^{-1}z_o g^{-1} = gc_o^{-1}(z_o g^{-1} z_o^{-1}) z_o = gc_o^{-1} c_o^{-1} z_o = c_o^{-1} z_o.$$ \qed

This semi-direct product can be realized as the fundamental group of the pullback $\mathcal{E}'_L$ of $\mathcal{E}'$ to $L'_{-1}$. This has a (continuous) section. Since $\mathcal{E}'_L$ is a $\mathbb{C}^*$ covering of $\mathcal{E}'$, we obtain:

**Proposition 1.9.** The kernel of the natural homomorphism

$$\pi_1(\mathcal{E}'_L, [E, \vec{v}]) \cong \pi_1(L'_{-1}, [E, \vec{v}]) \ltimes \pi_1(E', \vec{v}) \to \pi_1(\mathcal{E}', [E, \vec{v}])$$
is the infinite cyclic subgroup generated by $c^{-1}_o z_o$. This homomorphism induces an isomorphism
\[
(\pi_1(L'_{-1}, [E, \vec{v}]) \ltimes \pi_1(E', \vec{v}))/\langle c^{-1}_o z_o \rangle \to \pi_1(E', [E, \vec{v}]).
\]

In mapping class group notation, this result says that there is a natural isomorphism
\[
\Gamma_{1,2} \cong (\Gamma_{1,1} \ltimes \pi_1(E', \vec{v})) / \langle Z \rangle.
\]

In the Hodge and Galois worlds, the copy of $Z$ is a copy of $Z(1)$.

1.2. The local system $\mathbb{H}$. This is the local system (i.e., locally constant sheaf) over $M_{1,1}$ whose fiber over $[E] \in M_{1,1}$ is $H^1(E; \mathbb{C})$. We identify it, via Poincaré duality $H^1(E) \to H^1(E)$, with the local system $R^1\pi_*\mathbb{C}$ over $M_{1,1}$ associated to the universal elliptic curve $\pi : E \to M_{1,1}$. This has fiber $H^1(E; \mathbb{C})$ over $[E] \in M_{1,1}$.

We consider two ways of framing (i.e., trivializing) the pullback of $\mathbb{H}$ to $\mathfrak{h}$. Denote the universal elliptic curve over $\mathfrak{h}$ by $E_\mathfrak{h} \to \mathfrak{h}$. It is the quotient of $\mathbb{C} \times \mathfrak{h}$ by the standard action of $Z^2$ given above. The first homology of $E_\tau := \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z})$ is naturally isomorphic to $\Lambda_\tau := \mathbb{Z} \oplus \tau \mathbb{Z}$. Let $a, b$ be the basis of $H_1(E_\tau; \mathbb{Z})$ that corresponds to the basis $1, \tau$ of $\Lambda_\tau$.

Denote the dual basis of $H_1(E_\tau; \mathbb{C}) \cong \text{Hom}(H_1(E_\tau; \mathbb{Z}))$ by $\check{a}, \check{b}$. Then, under Poincaré duality,
\[
\check{a} = -b \quad \text{and} \quad \check{b} = a.
\]

Denote the element $d\xi$ of $H^1(E_\tau; \mathbb{C})$ by $w_\tau$. Then
\[
w_\tau = \check{a} + \tau \check{b} = \tau a - b.
\]

The two framings $a, b$ and $2\pi i b, \omega_\tau$ of $\mathbb{H}$ over $\mathfrak{h}$ are related by
\[
(2\pi i b \quad w_\tau) = (\check{b} \quad \check{a}) \begin{pmatrix} 2\pi i & \tau \\ 0 & 1 \end{pmatrix} = (a \quad b) \begin{pmatrix} 2\pi i & \tau \\ 0 & -1 \end{pmatrix}.
\]

2. Unipotent Completion

Suppose that $\pi$ is a discrete group and that $R$ is a commutative ring. Denote the group algebra of $\pi$ over $R$ by $R\pi$. This is an $R$-algebra. The augmentation is the homomorphism
\[
\epsilon : R\pi \to R
\]
that takes each $\gamma \in \pi$ to 1. Its kernel, denoted $J$, is called the augmentation ideal. The powers of $J$ define a topology on $R\pi$. A base of neighbourhoods of 0 consist of the powers of $J$:
\[R\pi \supset J \supset J^2 \supset J^3 \supset \cdots\]

The completion of $R\pi$ in this topology is called the $J$-adic completion of $\pi$ and is denoted by $R\pi^\wedge$. In concrete terms:
\[
R\pi^\wedge = \lim_{\rightarrow} R\pi / J^n.
\]

Denote its augmentation ideal by $J^\wedge$.

The group algebra also has a “coproduct”
\[
\Delta : R\pi \to R\pi \otimes R\pi.
\]
This is an augmentation preserving algebra homomorphism, which is continuous in the $J$-adic topology. It thus induces a ring homomorphism
\[ \Delta : R\pi^\wedge \to R\pi^\wedge \otimes R\pi^\wedge. \]

Now suppose that $R$ is a field $F$ of characteristic zero. Note that each element of $1 + J^\wedge$ is a unit. Define
\[ \mathcal{P}(F) = \{ x \in F\pi^\wedge : \epsilon(x) = 1 \text{ and } \Delta x = x \otimes x \} \]
and
\[ \mathfrak{p} = \{ x \in F\pi^\wedge : \Delta x = x \otimes 1 + 1 \otimes x \}. \]
Elements of $\mathfrak{p}$ are said to be primitive; elements of $\mathcal{P}$ are said to be group-like.

**Proposition 2.1.**

(i) $\mathcal{P}(F)$ is a subgroup of the group of the group $1 + J^\wedge$;

(ii) $\mathfrak{p}$ is a Lie algebra, with bracket $[u, v] = uv - vu$, which lies in $J^\wedge$;

(iii) The logarithm and exponential mappings
\[ J^\wedge \xrightarrow{\log} 1 + J^\wedge \xrightarrow{\exp} \mathcal{P}(F) \]
are continuous bijections, which induce continuous bijections
\[ \mathfrak{p} \xrightarrow{\log} \mathcal{P}(F) \xrightarrow{\exp} \mathfrak{p}. \]

The third part implies that the exponential map
\[ \exp : (\mathfrak{p}, \text{BCH}) \to \mathcal{P} \]
is a group isomorphism, where the multiplication on $\mathfrak{p}$ is defined using the Baker-Campbell-Hausdorff formula [19]:
\[ \text{BCH}(u, v) := \log(e^u e^v) = u + v + \frac{1}{2}[u, v] + \cdots \]

**Proof.** The first two assertions are easily verified, as is the first part of the third assertion. To prove the last assertion, note that since $\exp$ is continuous, $\exp \Delta(x) = \Delta \exp(x)$ for all $x \in J^\wedge$. Now, $x \in J^\wedge$ is primitive if and only if
\[ \Delta x = x \otimes 1 + 1 \otimes x. \]
Since $x \otimes 1$ and $1 \otimes x$ commute, this holds if and only if
\[ \Delta \exp(x) = \exp(\Delta(x)) = \exp(x \otimes 1) \exp(1 \otimes x) = \exp(x) \otimes \exp(x). \]
That is, $x \in J^\wedge$ is primitive if and only if $\exp x$ is group-like. $\square$

Since $\epsilon(\gamma) = 1$ for all $\gamma \in \pi$, there is a homomorphism $\pi \to 1 + J^\wedge$. By the definition of the coproduct $\Delta$, the image of this homomorphism lands in $\mathcal{P}(F)$. Thus, the inclusion $\pi \to F\pi$ induces a natural homomorphism $\pi \to \mathcal{P}(F)$

**Definition 2.2.** Suppose that $H_1(\pi; F)$ is finite dimensional (e.g., $\pi$ is finitely generated). The homomorphism $\pi \to \mathcal{P}(F)$ is called the unipotent (or Malcev) completion of $\pi$ over $F$. The pronunipotent group $\mathcal{P}$ is denoted $\pi^{un}$. The Lie algebra of the unipotent completion is the Lie algebra $\mathfrak{p}$. It is also called the Malcev Lie algebra associated to $\pi$. 
Unipotent completion can be viewed as a functor from the category of groups to the category of pro-unipotent groups over $F$:
\[ \pi \Rightarrow \mathcal{P}(F) \]

There is also the functor
\[ \pi \Rightarrow p \]
that assigns to a group, the Lie algebra of its unipotent completion over $F$.

There are therefore natural homomorphisms
\[ \text{Aut} \pi \to \text{Aut} \mathcal{P} \quad \text{and} \quad \text{Aut} \pi \to \text{Aut} p. \]

**Remark 2.3.** Replacing $\pi$ by $\mathcal{P}$ and its Lie algebra $p$ allows us to use the methods of formal Lie theory to study $\pi$. When $\pi$ is the fundamental group of an algebraic variety, $p$ supports additional structure: If $\pi$ is the fundamental group of a complex algebraic variety and $F = \mathbb{Q}$, then $p$ has a natural mixed Hodge structure; if $\pi$ is the fundamental group of a smooth algebraic variety defined over $\mathbb{Q}$ with $\mathbb{Q}$-rational base point, then the absolute Galois group $G_{\mathbb{Q}}$ acts on $p \otimes \mathbb{Q}_\ell$ and $\mathcal{P}(\mathbb{Q}_\ell)$.

### 2.1. The unipotent completion of a free group

Suppose that $\pi$ is the free group $\langle x_1, \ldots, x_n \rangle$ generated by the set $\{x_1, \ldots, x_n\}$.

Consider the ring
\[ F\langle\langle X_1, \ldots, X_n \rangle\rangle \]
of formal power series in the non-commuting indeterminants $X_j$. Define an augmentation
\[ \epsilon : F\langle\langle X_1, \ldots, X_n \rangle\rangle \to F \]
by sending a power series to its constant term. The augmentation ideal $\ker \epsilon$ is the maximal ideal $I = (X_1, \ldots, X_n)$.

Define a coproduct
\[ \Delta : F\langle\langle X_1, \ldots, X_n \rangle\rangle \to F\langle\langle X_1, \ldots, X_n \rangle\rangle \hat{\otimes} F\langle\langle X_1, \ldots, X_n \rangle\rangle \]
by defining each $X_j$ to be primitive:
\[ \Delta X_j := X_j \otimes 1 + 1 \otimes X_j. \]

There is a unique group homomorphism
\[ \pi \to F\langle\langle X_1, \ldots, X_n \rangle\rangle \]
that takes $x_j$ to $\exp(X_j)$. This extends to a ring homomorphism
\[ \theta : F\pi \to F\langle\langle X_1, \ldots, X_n \rangle\rangle. \]

Since $\epsilon(x_j) = 1 = \epsilon(\exp(X_j))$, $\theta$ is augmentation preserving, and therefore extends to a continuous homomorphism
\[ \hat{\theta} : F\pi \to F\langle\langle X_1, \ldots, X_n \rangle\rangle \]

As in the case of completed group algebras, one can define primitive and group-like elements of $F\langle\langle X_1, \ldots, X_n \rangle\rangle$. As there, an element of $1 + I$ is group-like if and only if it is the exponential of a primitive element. Since $\exp(X_j)$ is group-like, it is easy to check that $\hat{\theta}$ preserves both the product and the coproduct. (One says that it is a homomorphism of complete Hopf algebras.) It is easy to use universal mapping properties to prove:
Proposition 2.4. The homomorphism $\hat{\theta}$ is an isomorphism of complete Hopf algebras. □

Corollary 2.5. The restriction of $\hat{\theta}$ induces a natural isomorphism

$$d\theta : \mathfrak{p} \to \mathbb{L}(X_1, \ldots, X_n)^\wedge$$

of topological Lie algebras.

Proof. This follows immediately from the fact that $\hat{\theta}$ induces an isomorphism on primitive elements and the well-known fact that the set of primitive elements of the power series algebra $F\langle\langle X_1, \ldots, X_n \rangle\rangle$ is the completed free Lie algebra $\mathbb{L}(X_1, \ldots, X_n)^\wedge$. □

There is a weaker version of the construction of the unipotent completion of a free group that is relevant. Suppose that $\theta : \pi \to F\langle\langle X_1, \ldots, X_n \rangle\rangle$ is a homomorphism that satisfies

$$\theta(x_j) = \exp(U_j)$$

where $U_j \in J^\wedge$ and $U_j \equiv X_j \mod (J^\wedge)^2$. Then it is not difficult to show that $\theta$ induces a continuous isomorphism

$$\hat{\theta} : F\pi \to F\langle\langle X_1, \ldots, X_n \rangle\rangle$$

and, by restriction, a Lie algebra isomorphism

$$d\theta : \mathfrak{p} \to \mathbb{L}(X_1, \ldots, X_n)^\wedge$$

and a group isomorphism

$$P \to \exp \mathbb{L}(X_1, \ldots, X_n)^\wedge.$$
Note that the fibered product $\mathcal{E} \times_{\mathcal{M}_{1,1}} \mathcal{E} \to \mathcal{M}_{1,1}$ of the universal elliptic curve is the quotient of $\mathbb{C} \times \mathbb{C} \times \mathfrak{h}$ by the $\text{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2 \oplus \mathbb{Z}^2)$-action
\[(m,n), (r,s) : (\xi, \eta, \tau) = (\xi + m\tau + n, \eta + r\tau + s, \tau) \]
and
\[\gamma : (\xi, \tau) \mapsto ((c\tau + d)^{-1}\xi, (c\tau + d)^{-1}\eta, \gamma\tau)\]
where $\gamma \in \text{SL}_2(\mathbb{Z})$.

**Example 3.2.** Suppose that $G = \text{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2 \oplus \mathbb{Z}^2)$ and that $X = \mathbb{C} \times \mathbb{C} \times \mathfrak{h}$, where the $G$-action is the one defined above. Let $V = \mathbb{C}$. Define
\[A_\gamma(\xi, \eta, \tau) = \begin{cases} (c\tau + d)e(e\xi/(c\tau + d)) & \gamma \in \text{SL}_2(\mathbb{Z}), \\
e^{-m\tau}e(\xi)\gamma & \gamma = ((m,n),(r,s)). \end{cases} \]
This is a well-defined factor of automorphy. The quotient
\[G \setminus (\mathbb{C} \times X) \to G \setminus X \]
is a line bundle
\[\mathcal{N} \to \mathcal{E} \times_{\mathcal{M}_{1,1}} \mathcal{E} \]
over the self product over $\mathcal{M}_{1,1}$ of the universal elliptic curve. The restriction of $\mathcal{N}$ to the zero section $\mathcal{M}_{1,1}$ is the line bundle $\mathcal{L}$. (Just look at the factor of automorphy when $\xi = \eta = 0$.)

**Remark 3.3.** Later (Prop. 8.1) we will see that the restriction of $\mathcal{N}$ to the fiber $E^2$ over $[E]$ is the pullback of the Poincaré line bundle over $E \times E$ to $E \times E$ along the map $(\xi, \eta) \mapsto (\xi, -\eta)$.

The next example gives an alternative (and useful) description of the local system $\mathbb{H}$.

**Example 3.4.** Let $G = \text{SL}_2(\mathbb{Z})$, $X = \mathfrak{h}$ and $V = \mathbb{C}^2$. Then
\[M_\gamma(\tau) = \begin{pmatrix} (c\tau + d)^{-1} & 0 \\ 2\pi ic & c\tau + d \end{pmatrix} \]
is a factor of automorphy. The resulting bundle is the vector bundle associated to the local system $\mathbb{H} \to \mathcal{M}_{1,1}$ defined in Section 1.2. To see this, we set
\[t = \omega_\tau/2\pi i \in H^1(E_\tau, \mathbb{C}). \]
Then $a$ and $t$ comprise a framing of the pullback $\mathbb{H}_{\mathfrak{h}}$ of $\mathbb{H}$ to $\mathfrak{h}$, which gives an isomorphism
\[\mathbb{C}^2 \times \mathfrak{h} \to \mathbb{H}_{\mathfrak{h}} \]
via
\[(u, v, \tau) \mapsto ((a, \tau) (t, \tau)) \begin{pmatrix} u \\ v \end{pmatrix} \]
Here, $(a, \tau)$ denotes $a$ viewed as an element of $H_1(E_\tau)$. Likewise, $(t, \tau)$ denotes the element $\omega_\tau/2\pi i$ of $H^1(E_\tau)$.

Since $\Lambda_{\gamma\tau} = (c\tau + d)\Lambda_\tau$, multiplication by $(c\tau + d)$ induces an isomorphism
\[E_\tau \to E_{\gamma\tau}.\]
This induces the identification of the fibers of $H_\gamma$ over $\tau$ and $\gamma \tau$. For convenience, set $a = (a, \tau) \in H_1(E_\tau)$ and $a' = (a, \gamma \tau) \in H_1(E_{\gamma \tau})$. Similarly with $b$ and $b'$, and with $t$ and $t'$. Then

$$2\pi i t' = \omega_{\gamma \tau} = (c\tau + d)^{-1}(c\tau + b) = 2\pi i (c\tau + d)^{-1}t$$

and

$$\begin{pmatrix} a' & \omega_{\gamma \tau} \end{pmatrix} = (c\tau + d)^{-1} \begin{pmatrix} a' & b' \end{pmatrix} \begin{pmatrix} c\tau + d & a\tau + b \\ 0 & -(c\tau + d) \end{pmatrix}$$

$$= (c\tau + d)^{-1} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} d & b \\ c & a \end{pmatrix} \begin{pmatrix} c\tau + d & a\tau + b \\ 0 & -(c\tau + d) \end{pmatrix}$$

$$= (c\tau + d)^{-1} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} (c\tau + d)d & \tau \\ (c\tau + d)c & -1 \end{pmatrix}$$

$$= (c\tau + d)^{-1} \begin{pmatrix} a & \omega_{\tau} \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & -1 \end{pmatrix} \begin{pmatrix} (c\tau + d)d & \tau \\ (c\tau + d)c & -1 \end{pmatrix}$$

from which we conclude that

$$\begin{pmatrix} a & t \end{pmatrix} = \begin{pmatrix} a' & t' \end{pmatrix} M_{\gamma}(\tau).$$

Equation (3.1) now implies that the bundle with factor of automorphy $M_{\gamma}(\tau)$ is isomorphic to $\mathbb{H}$ as the following points correspond:

$$\left( \begin{pmatrix} u \\ v \end{pmatrix}, \tau \right) \leftrightarrow \begin{pmatrix} a & t \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} a' & t' \end{pmatrix} M_{\gamma}(\tau) \begin{pmatrix} u \\ v \end{pmatrix} \leftrightarrow \begin{pmatrix} M_{\gamma}(\tau) \begin{pmatrix} u \\ v \end{pmatrix}, \gamma \tau \end{pmatrix}$$

Note that $t$ and $a$ are both invariant under $\tau \mapsto \tau + 1$. It follows that $\mathbb{H}$ is trivial over the $q$-disk.

Since the bundle $\mathbb{H}$ exists over $\mathcal{M}_{1,1}$, this computation gives a conceptual proof that $M_{\gamma}(\tau)$ is a factor of automorphy.

**Remark 3.5.** It is useful to keep in mind that

$$a \in H_1(E_\tau, \mathbb{Z}) \text{ and } \langle a, t \rangle = -(2\pi i)^{-1} \in \mathbb{Z}(-1).$$

Note that $t$ spans a line sub-bundle of $\mathcal{H} := \mathbb{H} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{M}_{1,1}}$. This line bundle is the *Hodge bundle* $F^1 \mathcal{H}$ and is isomorphic to $\mathcal{L}$. The factor of automorphy of $\mathbb{H}$ implies that the quotient of $\mathcal{H}$ by $F^1$ is isomorphic to $\mathcal{L}^{-1}$, so that we have an exact sequence

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{L}^{-1} \to 0.$$

Later we will see that this splits, even over $\overline{\mathcal{M}}_{1,1}$. (Cf. Remark 19.2 and the last paragraph of Section 19.2.)

4. **Some Lie Theory**

Let $\mathbb{C} \langle \langle t, a \rangle \rangle$ be the completion of the free associative algebra generated by the indeterminates $t$ and $a$. It is a topological algebra. Denote the closure of the free Lie algebra $L(t, a)$ in $\mathbb{C} \langle \langle t, a \rangle \rangle$ by $p$. It is a topological Lie algebra.

Define a continuous action of $\mathbb{C} \langle \langle t, a \rangle \rangle$ on $p$ by

$$\mathbb{C} \langle \langle t, a \rangle \rangle \times p \to p$$
by

\[ f(t, a) : x \mapsto f(t, a) \cdot x := f(\text{ad}_t, \text{ad}_a)(x). \]

for all \( x \in \mathfrak{p} \).

For later use, we record the following fact:

**Proposition 4.1.** Suppose that \( A : [a, b] \to \mathbb{L}(X_1, \ldots, X_n)^\wedge \) is smooth.\(^4\) If \( X : [a, b] \to \mathbb{C}((X_1, \ldots, X_n)) \) satisfies the IVP

\[ X' = AX, \quad X(0) = 1, \]

then \( X(t) \) is group-like for all \( t \in [a, b] \).

**Proof.** This follows from standard Lie theory. It can also be proved directly as follows. Since the diagonal \( \Delta \) is linear, since \( \Delta \) is an algebra homomorphism, and since \( A \) is primitive, we have

\[ (\Delta X)' = \Delta(X') = \Delta(AX) = (\Delta A)(\Delta X) = (A \otimes 1 + 1 \otimes A)\Delta X. \]

On the other hand,

\[ (X \otimes X)' = X' \otimes X + X \otimes X' = (AX) \otimes X + X \otimes (AX) = (A \otimes 1 + 1 \otimes A)(X \otimes X). \]

Thus both \( \Delta X \) and \( X \otimes X \) satisfy the IVP

\[ Y' = (A \otimes 1 + 1 \otimes A)Y, \quad Y(0) = 1 \otimes 1, \]

where \( Y : [a, b] \to \mathbb{C}((X_1, \ldots, X_n)) \otimes \mathbb{C}((X_1, \ldots, X_n)) \). It follows that \( \Delta X = X \otimes X \) for all \( t \).

### 4.1. Two identities

For later use we recall two standard identities. To avoid confusion, we shall denote composition of endomorphisms \( \phi \) and \( \psi \) of \( \mathfrak{p} \) by \( \phi \circ \psi \).

Recall that if \( V \) is a vector space and \( u, \phi \in \text{End} \mathbb{V} \), then in \( \text{End} \mathbb{V} \) we have

\[ \exp(\text{ad}(\phi))(u) = e^\phi \circ u \circ e^{-\phi}. \]

Applying this in the case where \( V = \mathfrak{p} \), we see that for all \( \delta \in \text{Der} \mathfrak{p} \) and \( \phi \in \mathbb{C}((\langle t, a \rangle)) \),

\[ \exp(\text{ad}(\phi)) \cdot \delta = e^\phi \circ \delta \circ e^{-\phi}. \]

In particular, if \( \omega \) is a 1-form on a manifold that takes values in \( \mathfrak{p} \), then

\[ e^{-mt} \circ \omega \circ e(mt) = e(-mt) \cdot \omega. \]

**Lemma 4.2.** Suppose that \( u \in \mathbb{C}((\langle t, a \rangle)) \). If \( \delta \) is a continuous derivation of \( \mathbb{C}((\langle t, a \rangle)) \), then

\[ e^{-u} \delta(e^u) = \frac{1 - \exp(-\text{ad}_u)}{\text{ad}_u} \delta(u) \quad \text{and} \quad \delta(e^u)e^{-u} = \frac{\exp(\text{ad}_u) - 1}{\text{ad}_u} \delta(u). \]

**Proof.** The functions \( e^{-su} \delta(e^{su}) \) and \( \frac{1 - \exp(-s \text{ad}_u)}{\text{ad}_u} \delta(u) \) both satisfy the differential equation

\[ X'(s) = \delta(u) - \text{ad}_u(X). \]

Since both functions vanish when \( s = 0 \), they are equal for all \( s \in \mathbb{C} \). In particular, they are equal when \( s = 1 \). This proves the first identity. The second is proved similarly using the differential equation \( Y' = \delta(u) + \text{ad}_u(Y) \).

\(^4\)That is, each coefficient of the power series \( A(t) \) is a smooth function of \( t \in [a, b] \).
5. Connections and Monodromy

Suppose that $\Gamma$ is a discrete group, $G$ is a Lie (or proalgebraic) group and that $X$ is a topological space. Suppose that $\Gamma$ acts on $X$ on the left. (Think of this action as being discontinuous and fixed point free, but it does not have to be.) Suppose that the action of $\Gamma$ lifts to the trivial right principal $G$-bundle $G \times X \to X$:

$$\gamma : (g, x) \mapsto (M_\gamma(x)g, \gamma x)$$

where $M_\gamma : X \to G$ is a factor of automorphy.

5.1. Connections. Denote the Lie algebra of $G$ by $\mathfrak{g}$. Sections of the bundle $G \times X \to X$ will be identified with functions $X \to G$ in the obvious way. A Lie algebra valued 1-form $\omega \in \Omega^1(X) \otimes \mathfrak{g}$ defines a connection on the trivial bundle $G \times X \to X$ by the formula

$$\nabla f = df + \omega f$$

where $f$ is a locally defined function $X \to G$. 

Proposition 5.1. The connection $\nabla$ is $\Gamma$-invariant if and only if for all $\gamma \in G$,

$$\gamma^* \omega = \text{Ad}(M_\gamma)\omega - dM_\gamma M_\gamma^{-1}.$$ 

□

Proposition 5.2. The connection $\nabla$ is flat if and only if $\omega$ satisfies

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$ 

Proof. Flat sections satisfy the differential equation $df = -\omega f$. If the connection is flat then,

$$0 = -d^2 f = d(\omega f) = (d\omega) f - \omega \wedge df = (d\omega + \omega \wedge \omega) f = (d\omega + \frac{1}{2}[\omega, \omega]) f$$

holds for all flat sections. Since (locally) there is a flat frame of the bundle, this implies that $d\omega + \omega \wedge \omega = 0$. The converse follows from Frobenius’ Theorem. □

Example 5.3. The sections $a$ and $b$ of the Hodge bundle $H_b$ over $\mathfrak{h}$ are flat. Since they give local framings of the associated vector bundle $\mathcal{H} := \mathbb{H} \otimes_{\mathbb{C}} \mathcal{O}$, there is a flat connection on $\mathcal{H}$, which is characterized by the property that $\nabla a = \nabla b = 0$. Since $t = \omega t/2\pi i = (\tau a - b)/2\pi i$, we have

$$2\pi i \nabla t = \nabla(\tau a - b) = ad\tau.$$ 

It follows that, in terms of the framing $a$, $t$ of $\mathcal{H}$, the connection is given by

$$\nabla = d + (2\pi i)^{-1} a \frac{\partial}{\partial t} \otimes d\tau.$$ 

5.2. Parallel transport. Every path $\alpha : [0, 1] \to X$ has a horizontal lift $\tilde{\alpha} : [0, 1] \to G$ that starts at $1 \in G$. In other words, the section

$$t \mapsto (\tilde{\alpha}(t), \alpha(t)) \in G \times X$$

is a flat section of the bundle that projects to $\alpha$ and begins at $(1, \alpha(0))$.

The function $\tilde{\alpha}$ is the unique solution of the ODE

$$d\tilde{\gamma} = -(\alpha^* \omega) \tilde{\gamma}, \quad \tilde{\gamma}(0) = 1.$$ 

Note that this does not require the connection to be flat.
Note that the uniqueness of solutions of ODEs implies that the horizontal lift of \( \alpha \) that begins at \( g \in G \) is \( t \mapsto \tilde{\gamma}(t)g \).

Denote the value of the lift \( \tilde{\alpha} \) at \( t = 1 \) by \( T(\alpha) \). The function

\[
T : \alpha \mapsto T(\alpha)
\]

is called the (parallel) transport function associated to \( \nabla \). When \( \nabla \) is flat, \( T(\alpha) \) depends only on the homotopy class of \( \alpha \) relative to its endpoints.

An immediate consequence of the uniqueness of solutions to IVPs:

**Lemma 5.4.** If \( \alpha \) and \( \beta \) are composable paths, then \( T(\alpha \beta) = T(\beta)T(\alpha) \). \( \square \)

To make the transport multiplicative, we will work with \( T(\alpha)^{-1} \). A formula for the transport and the inverse transport can be given using Chen’s iterated integrals.

**First a basic fact from ODE.**

**Lemma 5.5.** Suppose that \( R \) is a topological algebra (such as \( \mathbb{C} \langle \langle t, a \rangle \rangle \) or \( \text{gl}_n(\mathbb{C}) \)) and that \( A : [a, b] \to R \) is a smooth function. A function \( X : [a, b] \to R^X \) is a solution of the IVP

\[
X' = -AX, \quad X(0) = 1
\]

if and only if \( Y = X^{-1}(t) \) is a solution of the IVP

\[
Y' = YA, \quad X(0) = 1.
\]

**Proof.** Suppose that \( X \) satisfies

\[
0 = X^{-1}(XX^{-1})' - X^{-1}(AXX^{-1}) = (X^{-1})' - X^{-1}A.
\]

The opposite direction is proved similarly. \( \square \)

**Corollary 5.6 (Transport Formula).** The inverse transport is given by

\[
T(\alpha)^{-1} = 1 + \int_\alpha \omega + \int_\alpha \omega \omega + \int_\alpha \omega \omega \omega + \cdots
\]

**Proof.** This follows from the transport formula [5] and the previous lemma. \( \square \)

For future use, we record the following standard fact.

**Proposition 5.7.** If the connection \( \nabla \) is \( \Gamma \)-invariant, then for all paths \( \alpha : [0, 1] \to X \) and all \( \gamma \in \Gamma \)

\[
T(\gamma \circ \alpha) = M_\gamma(\alpha(1))T(\alpha)M_\gamma(\alpha(0))^{-1}.
\]

**5.3. Monodromy.** Suppose that the connection \( d + \omega \) is \( \Gamma \)-invariant and flat. Our task in this section is to explain how to compute the associated monodromy representation from the transport function \( T \) of \( \omega \) and the factor of automorphy \( M \).

By covering space theory, the choice of a point \( x_\circ \in X \) determines a surjective homomorphism

\[
\rho : \pi_1(\Gamma \setminus X, \bar{x}_\circ) \to \Gamma
\]

whose kernel is \( \pi_1(X, x_\circ) \), where \( \bar{x}_\circ \) denotes the image of \( x_\circ \) in \( \Gamma \setminus X \).

To each \( \gamma \in \pi_1(\Gamma \setminus X) \), let \( c_\gamma \) be its lift to a path in \( X \) that begins at \( x_\circ \). Note that its end point is \( \rho(\gamma) \circ x_\circ \) and that the homotopy class of \( c_\gamma \) depends only upon \( \gamma \).

**Lemma 5.8.** If \( \gamma, \mu \in \pi_1(\Gamma \setminus X, \bar{x}_\circ) \), then \( c_{\gamma \mu} = c_{\gamma} \cdot (\rho(\gamma) \circ c_{\mu}) \).
Here \cdot denotes path multiplication and \circ denotes composition. The proof is best given by the picture Figure 1.

To obtain a homomorphism (instead of an anti-homomorphism), we need to take inverses. Define

\[ \Theta_{x_o} : \pi_1(\Gamma \backslash X, \bar{x}_o) \to G \]

by

\[ \Theta_{x_o}(\gamma) = T(c_\gamma)^{-1}M_{\rho(\gamma)}(x_o). \]

Note that \( \Theta_{x_o}(\gamma)^{-1} \) is the element of the fiber \( G \) over \( x_o \) that is identified with the point \( T(c_\gamma) \) in the fiber \( G \) over \( \rho(\gamma) \cdot x_o \). It is thus the result of parallel transporting \( 1 \in G \) about the loop \( \gamma \).

**Proposition 5.9.** The monodromy representation \( \pi_1(\Gamma \backslash X, \bar{x}_o) \to G \) of the flat bundle \( \Gamma \backslash (G \times X) \to \Gamma \backslash X \) with respect to the identification above is

\[ \Theta_{x_o} : \pi_1(\Gamma \backslash X, \bar{x}_o) \to G. \]

**Proof.** Just trace through the identifications. But for reassurance, we show that \( \Theta_{x_o} \) is a group homomorphism. (We’ll drop \( \rho \) and the \( x_o \) below.) If \( \gamma, \mu \in \pi_1(\Gamma \backslash X, \bar{x}_o) \), then

\[
\Theta(\gamma \mu) = T(c_{\gamma \mu})^{-1}M_{\gamma \mu}(x_o) = T(c_{\gamma} \circ (\gamma \cdot c_{\mu}))^{-1}M_{\gamma \mu}(x_o) = T(c_{\gamma})^{-1}T(\gamma \cdot c_{\mu})^{-1}M_{\gamma}(\mu \cdot x_o)M_{\mu}(x_o) = T(c_{\gamma})^{-1}M_{\gamma}(x_o)T(c_{\mu})^{-1}M_{\gamma}(\mu \cdot x_o)M_{\mu}(x_o) = \Theta(\gamma)\Theta(\mu).
\]

\[ \square \]

Combining this with the transport formula above, we obtain a formula for the monodromy in terms of \( \omega \) and the factor of automorphy.

**Corollary 5.10** (Chen-Dyson exponential). For all \( x \in X \) and \( \gamma \in \pi_1(\Gamma \backslash X, \bar{x}) \),

\[ \Theta_{x}(\gamma) = \left(1 + \int_{c_\gamma} \omega + \int_{c_\gamma} \omega \omega + \int_{c_\gamma} \omega \omega \omega + \cdots \right)M_{\gamma}(x). \]

\[ \square \]
Part 2. The Universal Elliptic KZB Connection

6. The Bundle $\mathcal{P}$ over $\mathcal{E}'$

Before we define the universal elliptic KZB connection, we need to define the bundle $\mathcal{P}$ over $\mathcal{E}'$ on which it lives.

6.1. The bundle $\mathcal{P}^{\text{top}}$. The bundle $\mathcal{P}$ with the KZB connection will be the de Rham realization of a topological local system $\mathcal{P}^{\text{top}}$. To provide context, we first construct it.

Denote by $Y$ the universal covering space of $\mathcal{E}'$. This is also the universal covering space of $\mathcal{E}'_h = (\mathbb{C} \times h) - \Delta_h$. Choose a base point $[E_o, x_o]$ of $\mathcal{E}'_h$ and a lift $y_o$ of it to $Y$. This determines an isomorphism of $\text{Aut}(Y/\mathcal{E}')$ with $\pi_1(\mathcal{E}'_o, [E_o, x_o])$.

Denote the unipotent completion of $\pi_1(\mathcal{E}'_o, x_o)$ over $\mathbb{C}$ by $\mathcal{P}_o$. The natural action

$$\pi_1(\mathcal{E}'_o, x_o) \times \pi_1(\mathcal{E}'_o, x_o) \to \pi_1(\mathcal{E}'_o, x_o), \quad (g, \gamma) \mapsto g\gamma g^{-1}$$

determines a left action of $\pi_1(\mathcal{E}'_o, [E_o, x_o])$ on $\mathcal{P}_o$. We can therefore form the quotient

$$\pi_1(\mathcal{E}'_o, [E_o, x_o])\backslash (\mathcal{P}_o \times Y)$$

by the diagonal $\pi_1(\mathcal{E}'_o, [E_o, x_o])$-action. This is a flat right principal $\mathcal{P}$-bundle that we shall denote by $\mathcal{P}^{\text{top}} \to \mathcal{E}'$. Its fiber over $[E, x]$ is naturally isomorphic to the unipotent completion of $\pi_1(\mathcal{E}', x)$.

Since the Lie algebra $\mathfrak{p}_o$ of $\mathcal{P}_o$ can be viewed as a group with multiplication defined by the Baker-Campbell-Hausdorff formula, we can (and will) view $\mathcal{P}^{\text{top}}$ as a local system of Lie algebras. (Cf. the comment following Prop. 2.1.)

6.2. The bundle $\mathcal{P}$. Here we construct a bundle $\mathcal{P}$ over $\mathcal{E}$ on which the universal elliptic KZB connection lives. Its fiber over each point of $\mathcal{E}'$ is the Lie algebra

$$\mathfrak{p} := L(\mathfrak{t}, \mathfrak{a})^\wedge.$$

The universal elliptic KZB connection on it is constructed in Section 9. It will have regular (i.e., logarithmic) singularities along the zero section and is thus holomorphic over $\mathcal{E}'$. In Section 14 we will prove that $\mathcal{P}^{\text{top}}$ is the local system associated to the universal KZB connection on $\mathcal{P}$.

The bundle $\mathcal{P}$ will be constructed as the quotient of $\mathfrak{p} \times \mathbb{C} \times h$ by a lift of the action of $\text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ on $\mathbb{C} \times h$ to $\mathfrak{p} \times \mathbb{C} \times h$.

The (completed) universal enveloping algebra of $\mathfrak{p}$ is the power series algebra $\mathbb{C} \langle \langle \mathfrak{t}, \mathfrak{a} \rangle \rangle$. The adjoint action defines the ring homomorphism

$$\mathbb{C} \langle \langle \mathfrak{t}, \mathfrak{a} \rangle \rangle \to \text{End} \mathfrak{p}$$

that takes $f(\mathfrak{t}, \mathfrak{a})$ to $f(\text{ad}_t, \text{ad}_a) \in \text{End} \mathfrak{p}$. This restricts to a homomorphism $\mathbb{C} \langle \langle \mathfrak{t}, \mathfrak{a} \rangle \rangle^\times \to \text{Aut} \mathfrak{p}$.

We use the notation of Section 3. Take $G = \Gamma := \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, $X = \mathbb{C} \times h$, and $V = \mathfrak{p}$. Note that $\Gamma$ acts on $\mathfrak{p} \times X$ via the projection $\Gamma \to \text{SL}_2(\mathbb{Z})$ using the factor of automorphy $M_\gamma(\tau)$ defined in (6.1).

The bundle $\mathcal{P}$ is defined using factors of automorphy $\tilde{M}_\gamma(\xi, \tau)$ which live in the group $\text{SL}_2(\mathbb{R}) \ltimes \mathbb{C} \langle \langle \mathfrak{t}, \mathfrak{a} \rangle \rangle^\times$, where $\text{SL}_2(\mathbb{R})$ acts on $\mathbb{C} \langle \langle \mathfrak{t}, \mathfrak{a} \rangle \rangle$ via its left action on the generators via the factor of automorphy $M_\gamma(\tau)$ defined in Example 3.4. Specifically,
$M_\gamma(\tau)$ is defined by

$$M_\gamma(\tau) : \begin{cases} a \mapsto (c \tau + d)^{-1}a + 2\pi i ct \\ t \mapsto (c \tau + d)t. \end{cases}$$

The general factor of automorphy is defined by

$$\tilde{M}_\gamma(\xi, \tau) = \begin{cases} M_\gamma(\tau) \circ e \left( \frac{-ct}{c\tau + d} \right) & \gamma \in \text{SL}_2(\mathbb{Z}); \\ e(-mt) & (m,n) \in \mathbb{Z}^2. \end{cases}$$

Here $e(u) := \exp(2\pi i u)$. This is a factor of automorphy for $\Gamma$ as

$$e(ct - mt) \circ M_\gamma(\tau) = \tilde{M}_\gamma(\xi + (m,n)\gamma(\tau,1)^T) \circ e(-mt).$$

where $\gamma \in \text{SL}_2(\mathbb{Z})$ and $(m,n) \in \mathbb{Z}^2$.

**Proposition 6.1.** This is a well-defined factor of automorphy.

**Proof.** The first task is to show that $\tilde{M}$ is well-defined on $\Gamma \times \mathbb{C} \times \mathfrak{h}$ — that is, it is compatible with the relation in $\Gamma = \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. This relation is

$$(m,n) \circ \gamma = \gamma \circ (m,n)(\gamma),$$

where $\gamma \in \text{SL}_2(\mathbb{Z})$, $\circ$ denotes composition in $\Gamma$ and $\cdot$ denotes the right action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$. If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then we have to show that

$$e(-mt) \circ \tilde{M}_\gamma(\xi, \tau) = \tilde{M}_\gamma(\xi + (m,n)\gamma(\tau,1)^T, \tau) \circ e\left( -(ma + nc)t \right).$$

Since

$$M_\gamma(\tau) \circ e(\phi) = e(M_\gamma(\tau) \cdot \phi) \circ M_\gamma(\tau)$$

for all $\phi \in \mathbb{C}\langle\langle t, a \rangle\rangle$, and since

$$M_\gamma(\tau) : t \mapsto (c \tau + d)t$$

(see above), we have

$$e(-mt) \circ M_\gamma(\tau) = M_\gamma(\tau) \circ e\left( \frac{-mt}{c\tau + d} \right).$$

Thus, the left-hand side expands to

$$e(-mt) \circ \tilde{M}_\gamma(\xi, \tau) = e(-mt) \circ M_\gamma(\tau) \circ e\left( \frac{ct}{c\tau + d} \right)$$

$$= M_\gamma(\tau) \circ e\left( \frac{ct - mt}{c\tau + d} \right).$$

---

*Denote left multiplication by $\phi \in \mathbb{C}\langle\langle t, a \rangle\rangle$ by $L_\phi$. For $M \in \text{Aut} \, H$, we have $M \circ L_{M^{-1} \phi} = L_\phi \circ M$. In particular, $M_\gamma(\tau) \circ L_{t/(c\tau + d)} = L_t \circ M_\gamma(\tau)$. 
The right-hand side expands to
\[
\tilde{M}_\gamma (\xi + (ma + nc) \tau + (mb + nd), \tau) \circ e\left(\frac{c(\xi + (ma + nc) \tau + (mb + nd) \tau)}{c\tau + d}\right) \circ e\left(- (ma + nc) t\right)
\]
\[
= M_\gamma (\tau) \circ e\left(\frac{c\xi t - mt}{c\tau + d}\right),
\]
which equals the left-hand side. It follows that \(\tilde{M}\) is a well-defined function on \(\Gamma \times \mathbb{C} \times \mathfrak{h}\).

Since \(\tilde{M}\) defines a homomorphism \(\mathbb{Z}^2 \to \mathbb{Q}((t))^\times\), to complete the proof, we need only check that the restriction of \(\tilde{M}\) to \(\text{SL}_2(\mathbb{Z}) \times \mathbb{C} \times \mathfrak{h}\) is a factor of automorphy. We will use the fact that \(M_\gamma (\tau)\) is a factor of automorphy.

Let
\[
\gamma_1 = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right), \quad \gamma_2 = \left(\begin{array}{cc} p & q \\ r & s \end{array}\right) \quad \text{and} \quad \gamma_1 \gamma_2 = \left(\begin{array}{cc} e & f \\ g & h \end{array}\right).
\]

Set \((\xi', \tau') = \gamma_2 (\xi, \tau) = (\xi/(r\tau + s), \gamma_2 \tau)\). Then
\[
\tilde{M}_{\gamma_1} (\xi', \tau') \tilde{M}_{\gamma_2} (\xi, \tau) = M_{\gamma_1} (\gamma_2 \tau) \circ e\left(\frac{c\xi' t}{c\tau' + d}\right) \circ M_{\gamma_2} (\tau) \circ e\left(\frac{r\xi t}{r\tau + s}\right)
\]
\[
= M_{\gamma_1} (\gamma_2 \tau) \circ e\left(\frac{c\xi t}{(c\tau' + d)(r\tau + s)}\right) \circ e\left(\frac{r\xi t}{r\tau + s}\right)
\]
\[
= M_{\gamma_1} \gamma_2 (\tau) \circ e\left(\frac{c\xi t}{(r\tau + s)(c(p\tau + q) + d(r\tau + s)) + r\xi t}{r\tau + s}\right)
\]
\[
= M_{\gamma_1 \gamma_2} (\tau) \circ e\left(\frac{g\xi t}{(r\tau + s)(g\tau + h)}\right)
\]
\[
= \tilde{M}_{\gamma_1 \gamma_2} (\xi, \tau).
\]

\[\square\]

Remark 6.2. The bundle \(\mathcal{P}\) is a bundle of Lie algebras. Its quotient by the commutator subalgebra of each fiber is the bundle over \(\mathfrak{h}\) with framing \(t, a\) and factor of automorphy \(M_\gamma (\tau)\). So it is isomorphic to \(\mathbb{H}\) by Example 3.4, as it should be.

7. Eisenstein Series and Bernoulli Numbers

First, we define the Bernoulli numbers \(B_n\) by
\[
x = e^x - 1 = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.
\]

Note that \(B_0 = 1, B_1 = -1/2\) and that \(B_{2k+1} = 0\) when \(k > 0\).

There are several ways to normalize Eisenstein series. We will use the normalization used by Zagier [22]. This differs from those used by Serre.

For \(k \geq 1\), the function \(G_{2k} : \mathfrak{h} \to \mathbb{C}\) defined by
\[
G_{2k}(\tau) = \frac{1}{2} \frac{(2k-1)!}{(2\pi i)^{2k}} \sum_{\lambda \neq 0} \frac{1}{\lambda^{2k}}
\]
defined in Sections 1.2 and 3.2 of [2] are closely related to

When $k$ is even

$$G_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n,$$

where $q = e(\tau)$ and $\sigma_k(n) = \sum_{d|n} d^k$. In particular, when $q = 0$,

$$G_{2k}|_q = -\frac{B_{2k}}{4k} = \frac{(2k-1)!}{(2\pi i)^{2k}} \zeta(2k).$$

When $k$ is even $G_k$ is a modular form for $SL_2(\mathbb{Z})$ of weight $k$ when $k > 2$:

$$G_k(\gamma \tau) = (c\tau + d)^k G_k(\tau), \quad \gamma \in SL_2(\mathbb{Z}).$$

When $k = 2$, it satisfies

$$G_2(\gamma \tau) = (c\tau + d)^2 G_2(\tau) + ic(c\tau + d)/4\pi.\quad (Cf. \ [22, p. 457], bottom of page, and \ [22, p. 459], near bottom of page.)$$

The role of $G_2(\tau)$ in this work should be clarified by the following result, which follows from the transformation law for $G_2$ above.

**Lemma 7.1.** If $SL_2(\mathbb{Z})$ acts on $\mathbb{C} \times \mathfrak{h}$ by $\gamma : (\xi, \eta, \tau) \mapsto (\xi/(c\tau + d), \gamma \tau)$, then the form

$$\frac{d\xi}{\xi} - 2 \cdot 2\pi i G_2(\tau) d\tau$$

is $SL_2(\mathbb{Z})$-invariant.

This 1-form represents a generator of $H^1(L_{\mathfrak{h},1}, \mathbb{Z}(1)) \cong \mathbb{Z}$.

### 7.1. Some useful identities

The following well-known identities are used later in the paper. Since

$$\frac{1}{2} \coth \left(\frac{u}{2}\right) = \frac{1}{2} e^{u/2} + e^{-u/2} = \frac{1}{2} e^{u} + 1 = \frac{1}{2} + \frac{1}{u} e^{u} - 1 = \sum_{m=0}^{\infty} \frac{B_{2m}}{(2m)!} u^{2m-1},$$

we have

$$\frac{1}{u} - \frac{u/4}{\sinh^2(u/2)} = \frac{1}{u} + \frac{u}{2} \frac{d}{du} \coth \left(\frac{u}{2}\right) = \sum_{m=1}^{\infty} (2m-1) \frac{B_{2m}}{(2m)!} u^{2m-1}.\quad (7.1)$$

Rearranging gives the useful alternative form

$$\sum_{m=0}^{\infty} (2m-1) \frac{B_{2m}}{(2m)!} u^{2m-1} = -\frac{u/4}{\sinh^2(u/2)}.\quad (7.2)$$

### 8. The Jacobi Form $F(\xi, \eta, \tau)$

There are two versions of the function $F(u, v, \tau)$, one used by Levin-Racinet [16], the other by Zagier [22].\footnote{Calaque-Enriquez-Etingof \[2\] do not explicitly use the Jacobi form. Instead their connection is expressed in terms of theta functions. But since $F$ can be defined using theta functions (see below), their formulas should be closely related. In particular, their functions $k(z, x|\tau)$ and $g(z, x|\tau)$ defined in Sections 1.2 and 3.2 of \[2\] are closely related to $F(z, x, \tau)$.} Denote them by $F(\xi, \eta, \tau)$ and $F^{Zag}(u, v, \tau)$, respectively. Zagier’s function is defined by

$$F^{Zag}(u, v, \tau) := \frac{\theta'(0, \tau) \theta(u + v, \tau)}{\theta(u, \tau) \theta(v, \tau)}.$$
where $\theta$ is the classical theta function

$$\theta(u, \tau) := \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n + \frac{1}{2})^2} e^{(n + \frac{1}{2})u}, \quad q = e(\tau)$$

and $\theta'$ is its derivative with respect to $u$.

Their periodicity properties imply that $u = 2\pi i \xi$, $v = 2\pi i \eta$. Since

$$F(\xi, \eta, \tau) = \frac{1}{\xi} + \frac{1}{\eta} \mod \text{holomorphic functions}$$

and

$$F^{Zag}(u, v, \tau) = \frac{1}{u} + \frac{1}{v} \mod \text{holomorphic functions}$$

near the origin, it follows that

$$F(\xi, \eta, \tau) = 2\pi i F^{Zag}(2\pi i \xi, 2\pi i \eta, \tau).$$

It satisfies the symmetry condition

$$F(\xi, \eta, \tau) = F(\eta, \xi, \tau) = -F(-\xi, -\eta, \tau).$$

### 8.1. Expansions

We use the formulas in [22], but write them using $F$ in place of $F^{Zag}$.\(^8\)

Set $q = \exp(2\pi i \tau)$. Then

$$F(\xi, \eta, \tau) = \pi \left[ \coth(\pi i \xi) + \coth(\pi i \eta) \right] + 4\pi \sum_{n=1}^{\infty} \left( \sum_{d|n} \sin \left[ 2\pi \left( \frac{n}{d} \xi + \eta \right) \right] \right) q^n. \quad (8.1)$$

$$F(\xi, \eta, \tau) = \frac{1}{\xi} + \frac{1}{\eta} - 2 \sum_{r,s=0}^{\infty} (2\pi i)^{1+\max\{r,s\}} \left( \frac{\partial}{\partial \tau} \right)^{\min\{r,s\}} G_{\left[r-s\right]+1}(\tau) \frac{\xi^r\eta^s}{r!s!}. \quad (8.2)$$

### 8.2. Derivatives

Differentiating these with respect to $\eta$ yields:

$$\frac{1}{\eta} + \eta \frac{\partial F}{\partial \eta}(\xi, \eta, \tau) = -2 \sum_{r \geq 0 \atop s \geq 1} (2\pi i)^{1+\max\{r,s\}} \left( \frac{\partial}{\partial \tau} \right)^{\min\{r,s\}} G_{\left[r-s\right]+1}(\tau) \frac{\xi^r\eta^s}{r! (s-1)!}.$$

$$= \left( \frac{1}{\eta} - \frac{(\pi i)^2 \eta}{\sinh^2(\pi i \eta)} \right) + 8\pi^2 \eta \sum_{n=1}^{\infty} \left( \sum_{d|n} d \cos \left[ 2\pi \left( \frac{n}{d} \xi + \eta \right) \right] \right) q^n. \quad (8.3)$$

Comparing the result of differentiating this with respect to $\xi$ and (8.2) with respect to $\tau$, we obtain the heat equation:

$$2\pi i \frac{\partial F}{\partial \tau}(\xi, \eta, \tau) = \frac{\partial^2 F}{\partial \xi^2}(\xi, \eta, \tau).$$

---

\(^8\)Note the conflict in notation: Zagier sets $\xi = \exp u$ and $\eta = \exp v$, which conflicts with the variables $(\xi, \eta)$ used by Levin-Racinet: $2\pi i (\xi, \eta) = (u, v)$. See [22, p. 455].
8.3. **Elliptic and modularity properties.** Here we use the standard notation $e(x) := \exp(2\pi i x)$.

The first is the elliptic property, [22, p. 456]:

$$F(\xi + m\tau + n, \eta, \tau) = e(-m\eta)F(\xi, \eta, \tau).$$

Zagier states a more general form of this, which follows from this one using the symmetry property of $F(\xi, \eta, \tau)$.

The next is the modularity property:

$$F(\xi/(c\tau + d), \eta/(c\tau + d), \gamma\tau) = (c\tau + d) e(c\xi\eta/(c\tau + d)) F(\xi, \eta, \tau).$$

In particular $F(\xi, \eta, \tau + 1) = F(\xi, \eta, \tau)$.

**Proposition 8.1.** The function $F$ induces a meromorphic section of the line bundle $\mathcal{N} \to E \times \mathcal{M}_{1,1}$ that was constructed in Example 3.2. The divisor of the section is $[\Gamma_i] - [0_1] - [0_2]$, where $\Gamma_i$ is the graph in $E \times E$ of the involution $\iota$ that takes a point of $E$ to its inverse.

**Proof.** Since $F(\xi, \eta, \tau)$ is meromorphic on $\mathbb{C}^2$ for each $\tau \in \mathfrak{h}$, $\text{div} F$ has no vertical components over $\mathcal{M}_{1,1}$. The polar locus of $F$ over $\mathcal{M}_{1,1}$ contains $[0_1] + [0_2]$ with multiplicity one on every fiber over $\mathcal{M}_{1,1}$. The zero divisor of $F$ over $\mathcal{M}_{1,1}$ contains $[\Gamma_i]$ with multiplicity one on the generic fiber over $\mathcal{M}_{1,1}$. Since the class of $\text{div} F$ in $H^2(E_2)$ is constant and since $\text{div} F$ has no vertical components, it suffices to show that the class of $\text{div} F$ is exactly $[\Gamma_i] - [0_1] - [0_2]$ on an open set of fibers and also over $q = 0$.

The identity (8.1) implies that

$$\frac{1}{\pi i} F(\xi, \eta)_{|q=0} = \frac{w + 1}{w - 1} + \frac{u + 1}{u - 1}$$

where $w = \exp(2\pi i \xi)$ and $u = \exp(2\pi i \eta)$. The coordinates on the normalization of $E_0 \times E_0$ are $(w, u)$. The identity sections are $w = 1$ and $u = 1$. The involution is given by $w = 1/w$. It is easily checked that

$$\frac{w + 1}{w - 1} + \frac{u + 1}{u - 1} = 0$$

implies that $wu = 1$. It follows that the restriction of $\text{div} F$ to $E_0 \times E_0$ is $[\Gamma_i] - [0_1] - [0_2]$. But this implies that the divisor of $F$ is $[\Gamma_i] - [0_1] - [0_2]$ on all nearby fibers. The result follows.

This result can also be proved using the formula for $F$ in terms of theta functions. $\square$

8.4. **The Weierstrass $\wp$ function.** This is defined by

$$\wp(z, \tau) = \frac{1}{z^2} + \sum_{\lambda \in \mathbb{Z}^2, \lambda \neq 0} \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2}.$$
The identity
\[
\wp(z, \tau) = \frac{1}{z^2} + \sum_{m=2}^{\infty} (2m - 1) \left( \sum_{\lambda \in \mathbb{Z} \oplus \mathbb{Z} \tau, \lambda \neq 0} \frac{1}{\lambda^{2m}} \right) z^{2m-2}
\]
\[
= \frac{1}{z^2} + \sum_{m=2}^{\infty} 2(2\pi i)^{2m} \frac{G_{2m}(\tau)}{(2m-2)!} z^{2m-2}
\]
\[
= \frac{1}{z^2} + \sum_{m=1}^{\infty} 2(2\pi i)^{2m+2} \frac{G_{2m+2}(\tau)}{(2m)!} z^{2m},
\]
which is standard, yields:

**Lemma 8.2.** Suppose that \(x, y\) are commuting indeterminates. Then

\[
\frac{1}{2} \frac{xy}{x+y} \left( (\wp(x, \tau) - \frac{1}{x^2}) - (\wp(y, \tau) - \frac{1}{y^2}) \right)
\]
\[
= \sum_{m \geq 1} \frac{(2\pi i)^{2m+2}}{(2m)!} G_{2m+2}(\tau) \sum_{j+k=2m+1, j,k>0} (-1)^j x^j y^k
\]
in the ring \(\mathcal{O}(\mathfrak{h})[[x, y]]\) of formal power series with coefficients in \(\mathcal{O}(\mathfrak{h})\).

8.5. **The addition formula.** The following identity is used in the proof of the integrability of the elliptic KZB connection.

**Proposition 8.3** (Addition Formula).

\[
F(\xi, \eta_1, \tau) \frac{\partial F}{\partial \eta}(\xi, \eta_2, \tau) - F(\xi, \eta_2, \tau) \frac{\partial F}{\partial \eta}(\xi, \eta_1, \tau)
\]
\[
= F(\xi, \eta_1 + \eta_2, \tau) (\wp(\eta_1, \tau) - \wp(\eta_2, \tau)).
\]

9. **The Universal Elliptic KZB Connection**

This is a \(\Gamma\)-invariant flat connection constructed by Calaque, Enriquez and Etingof [2] and by Levin and Racinet [16] on the bundle

\[
p \times \mathbb{C} \times \mathfrak{h} \to \mathbb{C} \times \mathfrak{h}.
\]

So it descends to a flat connection on the bundle \(\mathcal{P} \to \mathcal{E}'\). It has regular singularities along the universal lattice:

\[
A_0 := \{(m\tau + n, \tau) \in \mathbb{C} \times \mathfrak{h}\}.
\]

It therefore descends to a meromorphic connection on the bundle \(\mathcal{P} \to \mathcal{E}\) with regular singularities along the zero-section. In the second part, we compute the connection to the Tate curve (Section 12). That computation implies that its natural extension to the \(q\)-disk has regular singularities along the nodal cubic as well.

In this section we will follow Levin-Racinet (with modifications).
9.1. Derivations. We have already explained the algebra homomorphism
\[ C \langle \langle t, a \rangle \rangle \to \text{End} p, \quad f(t, a) \mapsto \{ x \mapsto f(t, a) \cdot x \}, \]
where
\[ f(t, a) \cdot x := f(\text{ad}_t, \text{ad}_a)(x). \]
We will view \( p \) as a Lie subalgebra of \( \text{Der} p \) via the adjoint action
\[ \text{ad} : p \to \text{Der} p, \]
which is an inclusion as \( p \) has trivial center.
We also have obvious linear inclusions
\[ p \frac{\partial}{\partial t} \to \text{Der} p \quad \text{and} \quad p \frac{\partial}{\partial a} \to \text{Der} p, \]
which induce a linear isomorphism
\[ p \frac{\partial}{\partial t} \oplus p \frac{\partial}{\partial a} \cong \text{Der} p. \]

9.2. The formula. The connection is defined by a 1-form
\[ \omega \in \Omega^1(C \times h \log \Lambda) \otimes \text{End} p. \]
via the formula
\[ \nabla f = df + \omega f \]
where \( f : C \times h \to p \) is a (locally defined) section of (9.1). Specifically,
\[ \omega = \frac{1}{2\pi i} dt \otimes a \frac{\partial}{\partial t} + \psi + \nu \]
where
\[ \psi = \sum_{m \geq 1} \left( \frac{(2\pi i)^{2m+1}}{(2m)!} G_{2m+2}(\tau) d\tau \otimes \sum_{j+k=2m+1, j,k>0} (-1)^j [\text{ad}_t^j(a), \text{ad}_a^k(a)] \frac{\partial}{\partial a} \right) \]
and
\[ \nu = tF(\xi, t, \tau) \cdot a d\xi + \frac{1}{2\pi i} \left( \frac{1}{t} + t \frac{\partial F}{\partial t}(\xi, t, \tau) \right) \cdot a d\tau. \]
Note that each term takes values in \( \text{Der} p \). Later we will show that its restriction to a punctured first order neighbourhood of the identity section takes values in a smaller subalgebra.

Remark 9.1. Each term of the lower central series of \( P \) is preserved by the connection. The connection that induces a connection on the bundle of abelianizations, which is isomorphic to \( \mathbb{H} \) (cf. Remark 6.2.) Example 5.3 implies that this induced connection on \( \mathbb{H} \) is the natural connection.

9.3. Modularity. Recall that \( \Gamma = \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \). To check the \( \Gamma \)-invariance, we have to check that
\[ \gamma^* \omega = \widetilde{M}_\gamma \omega - d\widetilde{M}_\gamma \widetilde{M}_\gamma^{-1} \]
for all \( \gamma \in \Gamma \). In this section we shall prove:

Proposition 9.2. The universal elliptic KZB connection is \( \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \)-invariant. That is,
\[ \gamma^* \omega = \text{Ad}(\widetilde{M}_\gamma) \cdot \omega - d\widetilde{M}_\gamma \widetilde{M}_\gamma^{-1} \]
for all \( \gamma \in \text{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \).
It suffices to check that the connection is invariant under $\mathbb{Z}^2$ and $\text{SL}_2(\mathbb{Z})$. These are proved in the two following subsections.

9.3.1. Ellipticity: invariance under $\mathbb{Z}^2$.

**Lemma 9.3.** For all $\delta \in \text{Der} \ p$ we have

$$e(-mt) \cdot \delta = \delta + \frac{1 - e(-m \text{ad}_t)}{\text{ad}_t} \delta(t).$$

**Proof.** For all $x \in p$

$$\left(e(-mt) \cdot \delta\right)(x) = e(-mt)\delta(e(mt)(x))$$

(Equation 4.1)

$$= e(-mt)\delta(e(mt))(x) + e(-mt)e(mt)\delta(x)$$

$$= \delta(x) + \frac{1 - e(-m \text{ad}_t)}{2\pi i \text{ad}_t} \delta(2\pi imt) \cdot x$$

(Lemma 4.2)

$$= \delta(x) + \frac{1 - e(-m \text{ad}_t)}{\text{ad}_t} \delta(t) \cdot x$$

$$= \left(\delta + \frac{1 - e(-m \text{ad}_t)}{\text{ad}_t} \delta(t)\right)(x).$$

\qed

**Corollary 9.4.** If $(m, n) \in \mathbb{Z}^2$, then

$$(m, n)^*\left(\frac{1}{2\pi i} a \frac{\partial}{\partial t} dt\right) - e(-mt) \cdot \left(\frac{1}{2\pi i} a \frac{\partial}{\partial t} dt\right) = -\frac{1}{2\pi i} \frac{1 - e(-mt)}{t}(a) dt.$$

**Proof.** Apply the previous lemma with $\delta = a \frac{\partial}{\partial t}$. \qed

**Corollary 9.5.** If $a, b \in \mathbb{N}$, then

$$e(-mt) \cdot [t^a \cdot a, t^b \cdot a] \frac{\partial}{\partial a} = [t^a \cdot a, t^b \cdot a] \frac{\partial}{\partial a}.$$

**Proof.** This follows directly from the previous lemma as the derivation

$$[t^a \cdot a, t^b \cdot a] \frac{\partial}{\partial a}$$

annihilates $t$. \qed

**Corollary 9.6.** If $(m, n) \in \mathbb{Z}^2$, then $(m, n)^*\psi = e(-mt) \cdot \psi = \psi$. \qed

**Lemma 9.7.** For all $(m, n) \in \mathbb{Z}^2$

$$(m, n)^*\nu - e(-mt) \cdot \nu = \frac{1}{2\pi i} \frac{1 - e(-m \text{ad}_t)}{\text{ad}_t} (a) dt.$$

**Proof.** Write $\nu = \nu_1 + \nu_2$, where

$$\nu_1 = tF(\xi, \tau) \cdot a \, d\xi$$

and $\nu_2 = \frac{1}{2\pi i} \left(1 + t \frac{\partial}{\partial t}(\xi, \tau)\right) \cdot a \, d\tau$.

Then

$$(m, n)^*\nu_1 - e(-mt) \cdot \nu_1$$

$$= tF(\xi + m\tau + n, t) \cdot a \, d(\xi + m\tau + n) - t e(-mt)F(\xi, t) \cdot a \, d\xi$$

$$= t e(-mt)F(\xi, t) \cdot a \, d(\xi + m\tau) - t e(-mt)F(\xi, t) \cdot a \, d\xi$$

$$= m t e(-mt)F(\xi, t) \cdot a \, d\tau.$$
Note that
\[ \frac{\partial F}{\partial t}(\xi + m\tau + n, t) = \frac{\partial}{\partial t}(e(-mt)F(\xi, t)) \]
\[ = e(-mt)\frac{\partial F}{\partial t}(\xi, t) - 2\pi i e(-mt)F(\xi, t). \]

Thus
\[ 2\pi i ((m, n)^*\nu_2 - e(-mt) \cdot \nu_2) \]
\[ = \left( \frac{1}{t} + t \frac{\partial F}{\partial t}(\xi + m\tau + n, t, \tau) \right) \cdot \mathbf{a} d\tau - e(-mt)\left( \frac{1}{t} + t \frac{\partial F}{\partial t}(\xi, t, \tau) \right) \cdot \mathbf{a} d\tau \]
\[ = -2\pi i m e(-mt)F(\xi, t) \cdot \mathbf{a} d\tau + \frac{1}{t}(1 - e(-mt)) \cdot \mathbf{a} d\tau. \]

\[ \square \]

If \((m, n) \in \mathbb{Z}^2\), then the results above imply that
\[(m, n)^*\omega = e(-mt) \cdot \omega(\xi, \tau). \]
Since \(e(-mt)\) does not depend on \((\xi, \tau)\), \(de(-mt) = 0\) and \(\omega\) is invariant under \(\mathbb{Z}^2\).

9.3.2. Modularity: invariance under \(\text{SL}_2(\mathbb{Z})\). Let
\[ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}). \]
Recall that \(M_\gamma(\tau)\) is defined by
\[ a \mapsto (c\tau + d)^{-1}a + 2\pi i ct \]
\[ t \mapsto (c\tau + d)t. \]
(9.2)
Its inverse is the linear map
\[ a \mapsto (c\tau + d)a - 2\pi i ct \]
\[ t \mapsto (c\tau + d)^{-1}t. \]
(9.3)

**Lemma 9.8.** If \(\gamma \in \text{SL}_2(\mathbb{Z})\) and \(a, b \in \mathbb{N}\), then
\[ e(c\xi t/(c\tau + d)) \cdot [\text{ad}_{c\xi t}^a(a), \text{ad}_{c\xi t}^b(a)] \frac{\partial}{\partial a} = [\text{ad}_{c\xi t}^a(a), \text{ad}_{c\xi t}^b(a)] \frac{\partial}{\partial a}. \]

**Proof.** This follows from (4.1) as the derivation \(\delta = [\text{ad}_{c\xi t}^a(A), \text{ad}_{c\xi t}^b(A)] \frac{\partial}{\partial a}\) vanishes on \(t\). \(\square \)

**Lemma 9.9.** If \(\gamma \in \text{SL}_2(\mathbb{Z})\) and \(a, b \in \mathbb{N}\), then
\[ \text{Ad}(M_\gamma(\tau))[\text{ad}_{c\xi t}^a(A), \text{ad}_{c\xi t}^b(a)] \frac{\partial}{\partial a} = (c\tau + d)^{a+b-1}[\text{ad}_{c\xi t}^a(A), \text{ad}_{c\xi t}^b(A)] \frac{\partial}{\partial a}. \]

**Proof.** Set \(\delta = [\text{ad}_{c\xi t}^a(a), \text{ad}_{c\xi t}^b(a)] \frac{\partial}{\partial a}\). Since \(M_\gamma(\tau)^{-1}(t) = (c\tau + d)^{-1}t\), \(\delta \circ M_\gamma^{-1}(t) = 0\). Consequently, \(\text{Ad}(M_\gamma(\tau))\delta\) is of the form \(f(t, a) \frac{\partial}{\partial a}\). The coefficient \(f(t, a)\) is
Corollary 9.10. If $\gamma \in \text{SL}_2(\mathbb{Z})$, then $\gamma^* \psi = \text{Ad}(\tilde{M}_\gamma) \psi$.

Proof. This follows as, for each $k \geq 1$, the expression

$$G_{2k+2}(\tau) d\tau \otimes \sum_{a+b=2k+1 \atop a,b > 0} [\text{ad}_\tau^a(a), \text{ad}_\tau^b(a)] \frac{\partial}{\partial a}$$

is multiplied by $(c\tau + d)^{2k}$ by both $\gamma^*$ and $\tilde{M}_\gamma(\xi, \tau)$.

Lemma 9.11. Set $v_1 = tF(\xi, t, \tau) \cdot \text{ad}_\xi$. Then

$$\gamma^* v_1 - \tilde{M}_\gamma(\xi, \tau) v_1 = -2\pi i c t d\xi - \frac{c^2 t}{c\tau + d} e(c\xi t) F(\xi, (c\tau + d)t, \tau) \cdot a d\tau.$$

Proof. First,

$$\tilde{M}_\gamma(\xi, \tau) v_1 = \tilde{M}_\gamma(\xi, \tau) [tF(\xi, t, \tau) \cdot a] d\xi$$

$$= e(c\xi t)(c\tau + d)tF(\xi, (c\tau + d)t, \tau) \cdot ((c\tau + d)^{-1} a + 2\pi i c t) d\xi$$

$$= e(c\xi t) t F(\xi, (c\tau + d)t, \tau) \cdot a d\xi + 2\pi i c t d\xi.$$

as the value of $tF(\xi, (c\tau + d)t, \tau)$ at $t = 0$ is $(c\tau + d)^{-1}$. This and the modular property of $F(\xi, t, \tau)$ then yield:

$$\gamma^* v_1 = tF(\xi/(c\tau + d), t, \gamma \tau) \cdot a \gamma^* d\xi$$

$$= (c\tau + d) t e(c\xi t) F(\xi, (c\tau + d)t, \tau) \cdot a \left( \frac{d\xi}{c\tau + d} - \frac{c\xi d\tau}{(c\tau + d)^2} \right)$$

$$= t e(c\xi t) F(\xi, (c\tau + d)t, \tau) \cdot a \left( d\xi - \frac{c\xi d\tau}{c\tau + d} \right)$$

$$= \tilde{M}_\gamma(\xi, \tau) v_1 - 2\pi i c t d\xi - \frac{c^2 t}{c\tau + d} e(c\xi t) F(\xi, (c\tau + d)t, \tau) \cdot a d\tau.$$

As a special case of the general formula, we have:

Lemma 9.12. In $\text{Der}_p$ we have:

$$e(c\xi \text{ad}_\xi) \left( \frac{1}{2\pi i} a \frac{\partial}{\partial t} \right) = \frac{1}{2\pi i} a \frac{\partial}{\partial t} + \frac{1}{2\pi i} \frac{1 - e(c\xi t)}{t} \cdot a.$$
Lemma 9.13. Set

\[ \nu_2 = \frac{1}{2\pi i} \left( \frac{1}{t} + \frac{\partial F}{\partial t}(\xi, t, \tau) \right) \cdot a \, d\tau. \]

Then

\[ \gamma^* \nu_2 - \tilde{M}_\gamma(\xi, \tau) \nu_2 = \frac{1}{2\pi i} \left( 1 - e(c\xi t) \right) \cdot \frac{a}{t} \frac{d\tau}{(c\tau + d)^2} + \frac{c \xi t}{c\tau + d} e(c\xi t) F(\xi, (c\tau + d)t, \tau) \cdot a \, d\tau. \]

**Proof.** First note that the modularity property of \( F(\xi, t, \tau) \) implies that

\[ \frac{\partial F}{\partial t}(\xi/(c\tau + d), t, \gamma \tau) = (c\tau + d) \frac{\partial}{\partial t} \left[ e(c\xi t) F(\xi, (c\tau + d)t, \tau) \right] = (c\tau + d)e(c\xi t) \left[ c\xi F(\xi, (c\tau + d)t, \tau) + (c\tau + d) \frac{\partial F}{\partial t}(\xi, (c\tau + d)t, \tau) \right]. \]

Thus

\[ 2\pi i \gamma^* \nu_2 = \left( \frac{1}{t} + \frac{\partial F}{\partial t}(\xi/(c\tau + d), t, \gamma \tau) \right) \cdot \frac{a}{t} \frac{d\tau}{(c\tau + d)^2} - (c\tau + d)e(c\xi t) \left[ \frac{\partial F}{\partial t}(\xi, (c\tau + d)t, \tau) \right] \cdot \frac{a}{t} \frac{d\tau}{(c\tau + d)^2}. \]

Since

\[ \tilde{M}_\gamma(\xi, \tau)(t) = (c\tau + d)t \quad \text{and} \quad \tilde{M}_\gamma(\xi, \tau)(a) = e(c\xi t) \cdot \frac{a}{(c\tau + d) + 2\pi i ct}, \]

we have

\[ 2\pi i \tilde{M}_\gamma(\xi, \tau) \nu_2 = \left( \frac{1}{t} + (c\tau + d) e(c\xi t) \frac{\partial F}{\partial t}(\xi, (c\tau + d)t, \tau) \right) \cdot \frac{a}{t} \frac{d\tau}{(c\tau + d)^2} \]

as \( \frac{1}{t} + \eta \frac{\partial}{\partial \eta}(\xi, \eta, \tau) \) is holomorphic in \( \eta \) and vanishes at \( \eta = 0 \). (See formula section.)

The previous lemma implies that

\[ (e(c\xi a d) - 1) \left( \frac{1}{2\pi i} \frac{a}{t} \frac{d\tau}{(c\tau + d)^2} \right) = \frac{1}{2\pi i} \left( 1 - e(c\xi t) \right) \cdot a. \]

Now assemble the pieces to obtain the result. \( \square \)

Combining the last two computations, we obtain:

**Corollary 9.14.** For all \( \gamma \in \text{SL}_2(\mathbb{Z}) \),

\[ \gamma^* \nu - \tilde{M}_\gamma \nu = \frac{1}{2\pi i} \left( 1 - e(c\xi t) \right) \cdot \frac{a}{t} \frac{d\tau}{(c\tau + d)^2} - 2\pi i c t \, d\xi. \]
Lemma 9.15. For all $\gamma \in \text{SL}_2(\mathbb{Z})$,

$$d \tilde{M}_\gamma \tilde{M}_\gamma^{-1} = e(c\xi t) \cdot (dM_\gamma M_\gamma^{-1}) + 2\pi i c t d\xi.$$

Proof. Since $\tilde{M}_\gamma(\xi, \tau) = e(c\xi t)M_\gamma(\tau)$, we have

$$d \tilde{M}_\gamma \tilde{M}_\gamma^{-1} = d(e(c\xi t)M_\gamma)M_\gamma^{-1} e(-c\xi t)$$

$$= (e(c\xi t) dM_\gamma + 2\pi i c t e(c\xi t)M_\gamma d\xi) M_\gamma^{-1} e(-c\xi t)$$

$$= e(c\xi t) \cdot (dM_\gamma M_\gamma^{-1}) + 2\pi i c t d\xi.$$

Lemma 9.16. For all $\gamma \in \text{SL}_2(\mathbb{Z})$, we have

$$\gamma^* \left( \frac{1}{2\pi i} a \frac{\partial}{\partial t} d\tau \right) - M_\gamma \left( \frac{1}{2\pi i} a \frac{\partial}{\partial t} d\tau \right) + dM_\gamma M_\gamma^{-1} = 0.$$ 

Proof. This is best done using matrices with respect to the basis $\{a, t\}$ of $H$. These are:

$$\frac{1}{2\pi i} a \frac{\partial}{\partial t} d\tau = \frac{1}{2\pi i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} d\tau$$

and

$$M_\gamma(\tau) = \begin{pmatrix} (c\tau + d)^{-1} & 0 \\ 2\pi i c & (c\tau + d) \end{pmatrix}, \quad M_\gamma^{-1} = \begin{pmatrix} c\tau + d & 0 \\ -2\pi i c & (c\tau + d)^{-1} \end{pmatrix}.$$

Then

$$dM_\gamma M_\gamma^{-1} = \begin{pmatrix} -c(c\tau + d)^{-2} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} c\tau + d & 0 \\ -2\pi i c & (c\tau + d)^{-1} \end{pmatrix} d\tau$$

$$= \begin{pmatrix} -c & 0 \\ 0 & \frac{dt}{c\tau + d} \end{pmatrix} \begin{pmatrix} c\tau + d & 0 \\ -2\pi i c & (c\tau + d)^{-1} \end{pmatrix} d\tau$$

and

$$M_\gamma \left( \frac{1}{2\pi i} a \frac{\partial}{\partial t} d\tau \right) = \frac{1}{2\pi i} \begin{pmatrix} (c\tau + d)^{-1} & 0 \\ 2\pi i c & (c\tau + d)^{-1} \end{pmatrix} \begin{pmatrix} (c\tau + d)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c\tau + d & 0 \\ -2\pi i c & (c\tau + d)^{-1} \end{pmatrix} d\tau$$

$$= \frac{1}{2\pi i} \begin{pmatrix} -2\pi i c & (c\tau + d)^{-1} \\ 2\pi i c & c\tau + d \end{pmatrix} d\tau$$

$$= \frac{1}{2\pi i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{d\tau}{(c\tau + d)^2} + \left( \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} \frac{d\tau}{c\tau + d} \right)$$

$$= \gamma^* \left( \frac{1}{2\pi i} a \frac{\partial}{\partial t} d\tau \right) + dM_\gamma M_\gamma^{-1}.$$
Final computation: For all $\gamma \in \text{SL}_2(\mathbb{Z})$, we have:
\[
\gamma^* \omega - \tilde{M}_\gamma \omega + d\tilde{M}_\gamma \tilde{M}_\gamma^{-1} = \\
gamma^* \left( \nu + \frac{a}{2\pi i} \frac{\partial}{\partial \tau} d\tau \right) - \tilde{M}_\gamma \left( \nu + \frac{a}{2\pi i} \frac{\partial}{\partial \tau} d\tau \right) + e(c\xi t) \cdot (d\tilde{M}_\gamma \tilde{M}_\gamma^{-1} + 2\pi i d\xi) \\
= \gamma^* \left( \frac{a}{2\pi i} \frac{\partial}{\partial \tau} d\tau \right) + \frac{1 - e(c\xi t)}{2\pi i} \cdot a \frac{d\tau}{(c\tau + d)^2} - e(c\xi t) \cdot (d\tilde{M}_\gamma \tilde{M}_\gamma^{-1}) \\
= e(c\xi t) \cdot \left( \gamma^* \left( \frac{1}{2\pi i} a \frac{\partial}{\partial \tau} d\tau \right) - M_\gamma \left( \frac{1}{2\pi i} a \frac{\partial}{\partial \tau} d\tau \right) + d\tilde{M}_\gamma \tilde{M}_\gamma^{-1} \right) \\
= 0.
\]

9.4. Integrability. The connection is flat if and only if $\omega$ is integrable:
\[d\omega + \frac{1}{2} [\omega, \omega] = 0.\]
Note that $[\omega, \omega] = 2\omega \wedge \omega$, so the integrability condition can also be written
\[d\omega + \omega \wedge \omega = 0.\]
The necessity of the integrability condition follows from the fact a section $f : C \times \mathfrak{h} \to \mathfrak{p}$ is flat if and only if $df = \omega f$. Then for all flat sections $f$
\[0 = d^2 f = d(df) = (d\omega) \wedge f - \omega \wedge df = (d\omega - \omega \wedge \omega) f.\]

Proposition 9.17. The 1-form $\omega$ is closed.

Proof. It is clear that
\[d \left( \frac{1}{2\pi i} d\tau \otimes a \frac{\partial}{\partial t} + \psi \right) = 0\]
as these terms do not depend upon $\xi$. The heat equation implies that
\[2\pi i d\nu = 2\pi i \text{ad}_t dF(\xi, \text{ad}_t, \tau)(a) \wedge d\xi + d \left( \frac{1}{\text{ad}_t} + \text{ad}_t \frac{\partial F}{\partial t}(\xi, \text{ad}_t, \tau) \right)(a) \wedge d\tau \]
\[= \text{ad}_t \left( 2\pi i \frac{\partial F}{\partial \tau}(\xi, \text{ad}_t, \tau) - \frac{\partial^2 F}{\partial \xi \partial t}(\xi, \text{ad}_t, \tau) \right)(a) d\tau \wedge d\xi \]
\[= 0,\]
so that $d\omega = 0$. \hfill \Box

The proof of the vanishing of $[\omega, \omega]$ is quite involved. For this we employ the elegant calculus developed by Levin and Racinet in [16, §3.1].

9.4.1. The Levin-Racinet calculus. For $U, V \in L(\mathfrak{a}, \mathfrak{t})^\wedge$, define
\[x^r y^s \circ (U, V) = [t^r \cdot U, t^s \cdot V].\]
This extends linearly to an action $f(x, y) \circ (U, V)$ of polynomials and power series $f(x, y)$ in commuting indeterminates on ordered pairs of elements of $L(\mathfrak{t}, \mathfrak{a})$. When $U$ and $V$ are equal, one has the identity $f(x, y) \circ (U, U) = -f(y, x) \circ (U, U)$, so that
\[f(x, y) \circ (U, U) = \frac{1}{2} (f(x, y) - f(y, x)) \circ (U, U).\]
As an example of how this notation is used, note that Lemma 8.2 implies that

\begin{equation}
2\pi i \psi = \frac{1}{2} \frac{xy}{x + y} \left( \varphi(x) - \frac{1}{x^2} - \varphi(y) + \frac{1}{y^2} \right) \circ (a, a) \frac{\partial}{\partial a} \otimes dt.
\end{equation}

Two more identities will be needed in the proof of the vanishing of $[\omega, \omega]$.

**Lemma 9.18.** Suppose that $U, V \in \mathbb{L}(t, a)^\wedge$.

(i) (Jacobi identity) If $f(x, y) \in \mathbb{C}[[x, y]]$, then

\[
\text{ad}_t \left( f(x, y) \circ (U, V) \right) = (x + y) f(x, y) \circ (U, V).
\]

(ii) If $\delta$ is a continuous derivation of $\mathbb{C}\langle\langle t, a \rangle\rangle$ and $g(x) \in \mathbb{C}[[x]]$, then

\[
\delta(g(\text{ad}_t V)) = g(\text{ad}_t) \delta(V) + \left( \frac{g(x + y) - g(y)}{x} \right) \circ (\delta(t), V).
\]

**Proof.** The first identity encodes the Jacobi identity and is left as an easy exercise. To prove the second, note that both sides are linear in $g$, so that, by continuity, it suffices to prove the result when $g$ is a monomial $x^n$. This holds trivially when $n \leq 1$. The general case follows by induction using the Jacobi identity. □

9.5. Integrability. The following computation completes the proof of integrability.

**Lemma 9.19.** The 2-form $[\omega, \omega]$ vanishes, so that $\omega$ is integrable.

**Proof.** Note that

\[
\pi \omega \omega = \left[ dt \otimes a \frac{\partial}{\partial t} + 2\pi i \psi \right. \\
+ \left. dt \otimes \left( \frac{1}{\text{ad}_t} + \text{ad}_t \frac{\partial F}{\partial t}(\xi, \text{ad}_t, \tau)(a, \text{ad}_t F(\xi, \text{ad}_t, \tau)(a) \otimes d\xi \right) \right].
\]

The expression (9.5) implies that the coefficient of $dt \wedge d\xi$ is

\[
[a \frac{\partial}{\partial t}, tF(t)(a)] + \frac{1}{2} \left[ \frac{xy}{x + y} \left( \varphi(x) - \frac{1}{x^2} - \varphi(y) + \frac{1}{y^2} \right) \circ (a, a) \frac{\partial}{\partial a} \cdot tF(t)(a) \right] \\
+ \left[ tF'(t)(a), tF(t)(a) \right]
\]

where $F(z)$ denotes $F(\xi, z, \tau)$ and $F'(z)$ denotes $\partial F/\partial z(\xi, z, \tau)$. We’ll compute these three terms one at a time.

Since $[\delta, \text{ad}_v] = \text{ad}_{\delta(v)}$, Lemma 9.18 and equation (9.4) imply that the first term

\[
[a \frac{\partial}{\partial t}, tF(t)(a)] = \left( \frac{(x + y)F(x + y) - yF(y)}{x} \right) \circ (a, a) \\
= \frac{1}{2} \left( \left( \frac{y^2 - x^2}{xy} \right) F(x + y) - \frac{y}{x} F(y) + \frac{x}{y} F(x) \right) \circ (a, a).
\]
The identity \([\delta, \text{ad}_e] = \text{ad}_\partial \delta\) and the Jacobi identity (Lemma 9.18) imply that the second term is

\[
\frac{1}{2} \text{ad}_t F(t) \left( \frac{xy}{x+y} \left( \varphi(x) - \frac{1}{x^2} \varphi(y) + \frac{1}{y^2} \right) \circ (a, a) \right)
= \frac{1}{2} xy F(x + y) \left( \varphi(x) - \frac{1}{x^2} \varphi(y) + \frac{1}{y^2} \right) \circ (a, a)
= \frac{1}{2} \left( \frac{x^2 - y^2}{xy} \right) F(x + y) + xy(\varphi(x) - \varphi(y)) F(x + y) \circ (a, a)
\]

The addition formula, Proposition 8.3, implies that the third term is

\[
\left( \left( \frac{1}{x} + xF'(x) \right) yF(y) \right) \circ (a, a)
= \left( \frac{y}{x} F(y) + xy F'(x) F(y) \right) \circ (a, a)
= \frac{1}{2} \left( \frac{y}{x} F(y) - \frac{x}{y} F(x) + xy(\varphi(x) F(y) - F'(y) F(x)) \right) \circ (a, a)
= \frac{1}{2} \left( \frac{y}{x} F(y) - \frac{x}{y} F(x) - xy(\varphi(x) - \varphi(y)) F(x + y) \right) \circ (a, a).
\]

These three terms clearly sum to 0. \(\square\)

**Part 3. Complements**

This part contains an exploration of the universal elliptic KZB connection. Most of what we do is straightforward. We compute its restriction to the first order neighbourhood of the identity section and to the first order smoothing of the Tate curve, from which we see that the natural extension of \(\mathcal{P} \to \mathcal{E}'\) to \(\overline{\mathcal{E}}\) is Deligne’s canonical extension of \(\mathcal{P}\). Equivalently, we see that it has regular singularities with pronilpotent monodromy along both the identity section of \(\mathcal{E}\) and along the nodal cubic. Standard facts from the theory of ODEs with regular singular points yield formulas for the monodromy about the Tate curve. With an eye to applications to multi-zeta values and mixed Tate motives, we also study the restriction of the connection to the regular locus of the nodal cubic, which can be identified with \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\).

In Section 14 we prove that the local system associated to the universal elliptic KZB connection on \(\mathcal{P}\) is the local system \(\mathcal{P}^{\text{top}}\). This extends results in [16, §4] and [2, §4.3].

Throughout, \(p = \mathbb{L}(t, a)^\wedge\) and \(\mathcal{P}\) is the corresponding pronilpotent group. Set

\[
\text{Der}^0\mathfrak{p} = \{\delta \in \text{Der} \mathfrak{p} : \delta([t, a]) = 0\}.
\]

This is the infinitesimal analogue of \(\Gamma_{1, \mathfrak{g}}\).

**10. Extending \(\mathcal{P}\) to \(\overline{\mathcal{M}}_{1,2}\)**

The punctured universal elliptic curve \(\mathcal{E}'\) is isomorphic to \(\mathcal{M}_{1,2}\) and is the complement of a normal crossing divisor in \(\overline{\mathcal{M}}_{1,2}\). This divisor has two components: the zero section and the nodal cubic. The flat bundle \(\mathcal{P}\) over \(\mathcal{E}'\) has pronilpotent monodromy about each, and thus extends naturally to a bundle over \(\overline{\mathcal{M}}_{1,2}\) with regular singularities and (pro) nilpotent residues along each component.
This extension is easily described. First, the complement of the Tate curve in \( \mathcal{M}_{1,1} \) is the universal elliptic curve \( \mathcal{E} \), which is a quotient of \( \mathbb{C} \times \mathfrak{h} \). The bundle \( \mathcal{P} \) defined in Section 6.2 is defined over \( \mathcal{E} \), not just over \( \mathcal{E}' \). We take this to be the extension across the zero section.

To extend \( \mathcal{P} \) across the Tate curve, recall from Example 3.4 that the holomorphic vector bundle \( \mathcal{H} := \mathbb{H} \otimes_{\mathbb{C}} O_{D'} \) associated to \( \mathbb{H} \) is trivial on the punctured \( q \)-disk \( \mathbb{D}' \). The framing \( t, a \) of \( \mathcal{H} \) over \( \mathbb{D}' \) determines an extension \( \overline{\mathcal{H}} \) of \( \mathbb{H} \) to the entire \( q \)-disk \( \mathbb{D} \) that is framed by \( t \) and \( a \). The formula for the natural connection on \( \mathcal{H} \) in Example 5.3 implies it extends to a meromorphic connection on \( \overline{\mathcal{H}} \) with a regular singular point at the cusp \( q = 0 \) and with nilpotent residue. This implies that \( \overline{\mathcal{H}} \) is Deligne’s canonical extension of \( \mathcal{H} \) to \( \overline{\mathcal{M}}_{1,1} \). (Cf. [3].)

Since \( p = L(t, a)^* \), this determines an extension of \( p \times \mathbb{C} \times \mathbb{D}' \to \mathbb{C} \times \mathbb{D} \) to \( \mathbb{C} \times \mathbb{D} \); its fiber over \( q \in \mathbb{D} \) is the free Lie algebra generated by the fiber of \( \overline{\mathcal{H}} \) over \( q \), which is naturally isomorphic to \( L(t, a)^* \). The pullback of the universal elliptic curve over \( \mathcal{M}_{1,1} \) to \( \mathbb{D} \) minus the double point \( P \) of the nodal cubic is the quotient of \( \mathbb{C} \times \mathfrak{h} \) by the subgroup

\[
\Gamma := \left( \begin{array}{cc} 1 & \mathbb{Z} \\ 0 & 1 \end{array} \right) \ltimes \mathbb{Z}^2.
\]

of \( SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \).

The action of \( \Gamma \) on \( p \times \mathbb{C} \times \mathfrak{h} \) induces an action of \( \Gamma \) on \( p \times \mathbb{C} \times \mathbb{D} \). The pullback of \( \mathcal{P} \) to \( \mathcal{E}_p \) thus extends to a bundle \( \mathcal{P} \) over \( \mathcal{E}_D \) (minus \( P \)) as the quotient of this action.

In subsequent sections we check that the universal elliptic KZB connection has regular singular points (relative to the extension \( \overline{\mathcal{P}} \) with pronilpotent residues, which will imply that \( \mathcal{P} \) is Deligne’s canonical extension of \( \mathcal{P} \).

11. Restriction to \( E'_\gamma \)

The first task in proving that the universal elliptic KZB connection has the expected monodromy is to check that its restriction to each fiber of \( \mathcal{E}' \to \mathcal{M}_{1,1} \) induces an isomorphism of \( \pi_1(E'_\gamma, x) \) with \( \mathcal{P} \).

Fix \( \tau \in \mathfrak{h} \). The restriction of the universal elliptic KZB connection to \( E'_\gamma := \mathbb{C}/\Lambda_\gamma \to \mathcal{E}' \) is

\[
\nabla = d + v_1 = d + u F(\xi, t, \tau) \cdot a d\xi.
\]

Identify \( p \) with the image of the adjoint action \( \text{ad} : p \to \text{Der} p \) which is injective as \( p \) has trivial center. With this identification, \( \nabla \) takes values in \( p \). Fix \( x \in \mathbb{C} - \Lambda_\gamma \). The associated monodromy representation \( \rho_x : \pi_1(E'_\gamma, x) \to \mathcal{P} \) is given by (cf. Cor. 5.10)

\[
\rho_x(\gamma) = \left( 1 + \int_{c_1} \nu_1 + \int_{c_2} \nu_1 \nu_1 + \int_{c_3} \nu_1 \nu_1 \nu_1 + \cdots \right) e(-m(\gamma)t)
\]

where \( \rho(\gamma) = (n(\gamma), m(\gamma)) \in \mathbb{Z}^2 \). (That is, the class of \( \gamma \) in \( H_1(E'_\gamma) \) is \( n(\gamma)a + m(\gamma)b \).)

**Proposition 11.1.** If \([\gamma] = na + mb \in H_1(E'_\gamma)\), then

\[
\Theta_x(\gamma) \equiv 1 + (mt + n)a - 2\pi int \mod (t, a)^2.
\]

This can be deduced, for example, from [9, §5.2].
Proof. Observe that
\[ \nu_1 = tF(\xi, t, \tau) \cdot a \, d\xi \]
\[ = t\left( \frac{1}{t} + 1 + \text{holomorphic in } \xi \right) \cdot a \, d\xi \]
\[ \equiv a \, d\xi \mod (t, a)^2 \]
and that
\[ \text{Res}_{\xi=0} \nu_1 \equiv [t, a] \mod (t, a)^3. \]
It follows that
\[ \Theta_x(a) \equiv 1 + a \mod (t, a)^2 \]
and that
\[ \Theta_x(b) \equiv (1 + \tau a)e(-t) \equiv 1 + \tau a - 2\pi i t \mod (t, a)^2. \]
\[ \square \]

**Corollary 11.2.** The universal elliptic KZB connection induces the identification
\[ \mathbb{C}a \oplus \mathbb{C}t \to H_1(E_\tau; \mathbb{C}) \]
that takes \( a \) to \( a \) and \( 2\pi i t \) to \( \tau a - b \), the Poincaré dual of \( \omega_\tau \).

This corresponds to the framing of the bundle \( \mathbb{H} \) given in Example 3.4. (This statement can also be deduced from Remark 9.1.)

The universal connection induces an isomorphism of the unipotent completion of \( \pi_1(E'_\tau, x) \) with \( \mathcal{P} \) for all \((x, \tau) \in \mathcal{E}_b \).

**Corollary 11.3.** The monodromy of the restriction of the universal elliptic KZB connection to the fiber \( E'_\tau \) of \( \mathcal{E}' \) over \([E] \in \mathcal{M}_{1,1} \) is a homomorphism \( \pi_1(E'_\tau, x) \to \mathcal{P} \) that induces an isomorphism \( \pi_1(E'_\tau, x)^{un} \to \mathcal{P} \).

12. **Restriction to the First-order Tate Curve**

In this section we compute the restriction of the universal elliptic KZB connection to the first order Tate curve. This allows us to see that the connection has regular singularities along the Tate curve and suggests a better normalization of the generators \( t \) and \( a \) of \( \mathfrak{p} \). The important derivations \( \epsilon_{2m} \in \text{Der}^0 \mathfrak{p} \) appear when computing the residue of the connection along the Tate curve. Restricting further to the the regular locus \( \mathbb{P}^1 - \{0, 1, \infty\} \) of the nodal cubic minus its identity is the first step in computing the image of \( \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \bar{\partial})^{un} \) in the limit mixed Hodge structure on the unipotent fundamental group of the first order smoothing of the Tate curve.

The restriction of the connection to \( q = 0 \) is obtained by setting \( q = 0 \) and then replacing \( d\tau \) by
\[ d\tau = \frac{1}{2\pi i} \frac{dq}{q}. \]

Rigorously, it is obtained by noting that the connection has logarithmic poles along \( q = 0 \) and then computing the image of \( \omega \) under the restriction mapping
\[ \Omega^1_{\mathcal{E}}(\log(\mathcal{M}_{1,1} \cup \mathcal{E}_0)) \to \Omega^1_{\mathcal{E}}(\log(\mathcal{M}_{1,1} \cup \overline{\mathcal{E}_0})) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}_0}, \]
where \( E_0 \) is the fiber \( \overline{E}_0 \) of \( \mathcal{E} \) over \( q = 0 \) with the double point removed.
In concrete terms the restriction mapping is given by
\[
G(\xi, q) \frac{d\xi}{\xi} + H(\xi, q) \frac{dq}{q} \mapsto G(\xi, 0) \frac{d\xi}{\xi} + H(\xi, 0) \frac{dq}{q}
\]
where \(G\) and \(H\) are holomorphic functions of \((\xi, q)\), and then setting \(w = e(\xi)\).

Formula (8.1) implies that
\[
F(\xi, \eta)|_{q=0} = \pi i \left( \frac{e(\xi) + 1}{e(\xi) - 1} + \frac{e(\eta) + 1}{e(\eta) - 1} \right) = \pi i \left( \frac{w + 1}{w - 1} + \coth(\pi i \eta) \right)
\]
where \(w := e(\xi)\) is the parameter in the Tate curve. From this and the identity (8.3), it follows that when \(q = 0\)
\[
\frac{1}{\eta} + \eta \frac{\partial F}{\partial \eta} (\xi, \eta)|_{q=0} = \frac{1}{\eta} - \frac{(\pi i)^2 \eta}{\sinh^2(\pi i \eta)} = \frac{1}{\eta} - \frac{\pi^2 \eta}{\sin^2(\pi \eta)}
\]
which is holomorphic at \(\eta = 0\).

The restriction of the connection to a first order neighbourhood of \(E_0\) is given by the 1-form:
\[
\omega_0 = \frac{1}{(2\pi i)^2} \frac{dq}{q} \otimes a \frac{\partial}{\partial t} + \psi_0 + \nu_0,
\]
where
\[
\psi_0 = \sum_{m \geq 1} \left( \frac{(2\pi i)^2 m}{(2m)!} G_{2m+2} \frac{dq}{q} \right) \otimes \sum_{j+k=2m+1, j,k>0} (-1)^j [\text{ad}_t^j(a), \text{ad}_t^k(a)] \frac{\partial}{\partial a}
\]
\[
= - \sum_{m \geq 1} \left( \frac{(2\pi i)^2 m B_{2m+2}}{(2m)!} \frac{dq}{q} \right) \otimes \sum_{j+k=2m+1, j,k>0} (-1)^j [\text{ad}_t^j(a), \text{ad}_t^k(a)] \frac{\partial}{\partial a}
\]
and
\[
\nu_0 = \frac{t}{2} \left( \frac{w + 1}{w - 1} + \frac{e(t) + 1}{e(t) - 1} \right) (a) \frac{dw}{w} + \frac{1}{(2\pi i)^2} \left( \frac{1}{t} - \frac{(\pi i)^2 t}{\sinh^2(\pi i t)} \right) (a) \frac{dq}{q}
\]
Since
\[
\frac{w + 1}{w(w - 1)} = \frac{2}{w - 1} - \frac{1}{w},
\]
and since
\[
\frac{t}{2} \left( \frac{e(t) + 1}{e(t) - 1} \right) = \frac{t}{e(t) - 1},
\]
we have
\[
\frac{t}{2} \left( \frac{w + 1}{w - 1} + \frac{e(t) + 1}{e(t) - 1} \right) (a) \frac{dw}{w}
\]
\[
= [t, a] \frac{dw}{w} + \frac{t}{2} \left( \frac{e(t) + 1}{e(t) - 1} \right) \cdot a \frac{dw}{w}
\]
\[
= [t, a] \frac{dw}{w} + \left( \frac{t}{e(t) - 1} \right) \cdot a \frac{dw}{w}.
\]
so that
$$\nu_0 = [t, a] \frac{dw}{w - 1} + \left( \frac{t}{\epsilon(t) - 1} \right) \cdot a \frac{dw}{w} + \frac{1}{(2\pi i)^2} \left( \frac{1}{t} - \frac{(\pi i)^2 t}{\sinh^2(\pi i t)} \right) \cdot a \frac{dq}{q}.$$  

Note that the connection has regular singularities along $E_0$ as well as along the identity section.

12.1. **A better framing of $\mathbb{H}$.** To get rid of the powers of $2\pi i$ in the formulas, we replace $2\pi i \frac{d\tau}{q}$ by $\frac{dq}{q}$ and set

$$T = 2\pi i t$$ and $$A = (2\pi i)^{-1} a.$$  

In this frame
$$\frac{\partial}{\partial A} = 2\pi i \frac{\partial}{\partial a} \quad \text{and} \quad 2\pi i \frac{\partial}{\partial T} = \frac{\partial}{\partial t},$$  

so that
$$a \frac{\partial}{\partial t} = (2\pi i)^2 A \frac{\partial}{\partial T}$$ and $$[t, a] = [T, A].$$  

The expressions for $\psi$, $\psi_0$ and $\nu_0$ become:

$$\psi = \sum_{m \geq 1} \left( \frac{G_{2m+2}(\tau)}{(2m)!} \frac{dq}{q} \otimes \sum_{j+k=2m+1 \atop j,k>0} (-1)^j [\text{ad}^j_T(A), \text{ad}^k_T(A)] \frac{\partial}{\partial A} \right),$$

$$\psi_0 = - \sum_{m \geq 1} \left( \frac{B_{2m+2}}{(2m+2)(2m)!} \frac{dq}{q} \otimes \sum_{j+k=2m+1 \atop j,k>0} (-1)^j [\text{ad}^j_T(A), \text{ad}^k_T(A)] \frac{\partial}{\partial A} \right) \frac{T}{2},$$

and

$$\nu_0 = [T, A] \frac{dw}{w - 1} + \left( \frac{T}{\epsilon^2 - 1} \right) \cdot a \frac{dw}{w} + \frac{1}{T} \left( 1 - \frac{T/4}{\sinh^2(T/2)} \right) \cdot A \frac{dq}{q}. $$

Note that all of the Fourier coefficients of $\psi$ and $\nu_0$ are now rational numbers. The last term can be rewritten using the identity (7.1), as in the next section.

**Remark 12.1.** This is a continuation of Remark 3.5. The formulas for $A$ and $T$ imply that the class of $A$ in $H_1(\mathfrak{p}) \cong H_1(E_\tau)$ lies in $H_1(E_\tau, \mathbb{Z}(-1))$. Later we will see that $T, A$ is a $\mathbb{Q}$-de Rham basis of the limit mixed Hodge structure on $H_1(E_\tau)$ associated to a tangent vector of the origin of the $q$-disk that splits both the Hodge filtration and the monodromy weight filtration. The class of $T$ modulo $A$ spans a copy of $\mathbb{Z}(0)$. It is the Poincaré dual of $\omega_\tau$. This implies that $[T, A]$ spans a copy of $\mathbb{Z}(-1)$ modulo the third term of the lower central series of $\mathfrak{p}$. The limit mixed Hodge structure on $H_1(E_\tau, \bar{v})$ associated to a non-zero tangent vector $\bar{v}$ of the origin of the $q$-disk is an extension

$$0 \to \mathbb{Z}(1) \to H_1(E_\tau, \bar{v}) \to \mathbb{Z}(0) \to 0$$

The copy of $\mathbb{Z}(1)$ is spanned by $a$ and $T$ projects to a generator of $\mathbb{Z}(0)$. This extension is computed in Lemma 15.3.
12.2. **Restriction to the polydisk about** \((w, q) = (1, 0)\). For later use, we compute the restriction of this connection to the polydisk about \((w, q) = (1, 0)\). This is

\[
\nabla = d + N_w \frac{dw}{w - 1} + N_q \frac{dq}{q}
\]

where \(N_w = [T, A]\) and

\[
N_q = A \frac{\partial}{\partial T} + \left( \frac{1}{T} - \frac{T/4}{\sinh^2(T/2)} \right) \cdot A
\]

\[
- \sum_{m \geq 2} (2m - 1) \frac{B_{2m}}{(2m)!} \sum_{j + k = 2m - 1, j > k > 0} (-1)^j [\text{ad}_T^j(A), \text{ad}_T^k(A)] \frac{\partial}{\partial A}.
\]

The identity (7.1) in Section 7.1 implies that

\[
N_q = A \frac{\partial}{\partial T} + \frac{1}{12} [T, A]
\]

\[
+ \sum_{m \geq 2} (2m - 1) \frac{B_{2m}}{(2m)!} \left( \text{ad}_T^{2m-1}(A) - \sum_{j + k = 2m - 1, j > k > 0} (-1)^j [\text{ad}_T^j(A), \text{ad}_T^k(A)] \frac{\partial}{\partial A} \right)
\]

\[
= A \frac{\partial}{\partial T}
\]

\[
+ \sum_{m \geq 1} (2m - 1) \frac{B_{2m}}{(2m)!} \left( \text{ad}_T^{2m-1}(A) - \sum_{j + k = 2m - 1, j > k > 0} (-1)^j [\text{ad}_T^j(A), \text{ad}_T^k(A)] \frac{\partial}{\partial A} \right).
\]

At this stage, it is convenient to define \(\epsilon_{2m} \in \text{Der}_{2m} \mathfrak{p}\) (the derivations of \(\mathfrak{p}\) of degree \(2m\)) by\(^{10}\)

\[
(12.2) \quad \epsilon_{2m} = \begin{cases} -A \frac{\partial}{\partial T} \text{ad}_T^{2m-1}(A) - \sum_{j + k = 2m - 1, j > k > 0} (-1)^j [\text{ad}_T^j(A), \text{ad}_T^k(A)] \frac{\partial}{\partial A} & \text{if } m = 0; \\ \frac{1}{2m} \frac{B_{2m}}{(2m)!} \epsilon_{2m} & \text{if } m > 0. \end{cases}
\]

With this notation

\[
(12.3) \quad N_q = \sum_{m \geq 0} (2m - 1) \frac{B_{2m}}{(2m)!} \epsilon_{2m}.
\]

Note that \(N_w\) and \(N_q\) are pronilpotent elements of \(\text{Der} \mathfrak{p}\) and that the curvature of the restricted connection is

\[
[N_w, N_q] \frac{dw}{w - 1} \wedge \frac{dq}{q}.
\]

Since the connection is flat, \([N_w, N_q] = 0\). Since \(N_w = \text{ad}_{[T, A]}\), this implies that each \(\epsilon_{2m}\) annihilates \([T, A]\), and therefore lies in

\[
\text{Der}^0 \mathfrak{p} = \{ \delta \in \text{Der} \mathfrak{p} : \delta([t, a]) = 0 \},
\]

as it should.

\(^{10}\)These derivations occur in the work [21] of Tsunogai on the action of the absolute Galois group on the fundamental group of a once punctured elliptic curve. They also occur in the paper of Calaque et al [2, §3.1]. In [13] it is shown that \(\epsilon_{2m}\) is a highest weight vector of the unique copy of \(S^{2m}H\) in \(Gr_{2m} W\), so it is not surprising that it appears in each of these related contexts.
Proposition 12.2. For all \( m \geq 0 \), \( \epsilon_{2m} \in \text{Der}^0 p \). When \( m > 1 \), \( \epsilon_{2m} \) is a highest weight vector of weight \( 2m - 2 \) for the natural \( \text{sl}_2 \)-action; that is, it is annihilated by \( A\partial/\partial T \).

Proof. As above, the flatness of the connection implies the first assertion. For the doubting Thomases, we give a direct proof. The value of \( \epsilon_{2m} \) on the second term on \([T, A]\) is

\[
- \sum_{j+k=2m-1 \atop j,k>0} (-1)^j [\text{ad}_T^j(A), \text{ad}_T^k(A)] \frac{\partial}{\partial A} ([T, A])
\]

\[
= -\frac{1}{2} [T, \sum_{j+k=2m-1 \atop j,k>0} (-1)^j [\text{ad}_T^j(A), \text{ad}_T^k(A)]]
\]

\[
= -\frac{1}{2} \left( \sum_{j+k=2m-1 \atop j,k>0} (-1)^j [\text{ad}_T^{j+1}(A), \text{ad}_T^k(A)] + \sum_{j+k=2m-1 \atop j,k>0} (-1)^j [\text{ad}_T^j(A), \text{ad}_T^{k+1}(A)] \right)
\]

\[
= -[\text{ad}_T^{2m-1}(A), [T, A]].
\]

This cancels with the value of the first term on \([T, A]\).

The assertion that when \( m > 0 \), \( \epsilon_{2m} \) is annihilated by \( A\partial/\partial T \) is most easily verified by rewriting \( \epsilon_{2m} \) in the notation of Levine and Garoufalidis, where it is clear. □

For later use:

Proposition 12.3. The value of \( N_q \) on \( T \) is

\[
N_q(T) = \frac{1}{4} \left( \frac{T^2}{\sinh^2(T/2)} \right) \cdot A.
\]

Proof. For all \( m \geq 0 \)

\[
\epsilon_{2m}(T) = (\text{ad}_T^{2m-1} A) \cdot T = -\text{ad}_T^{2m}(A) = -T^{2m} \cdot A.
\]

So, using the identity (7.2),

\[
N_q(T) = \left( \sum_{m \geq 0} (2m - 1) B_{2m} (2m)! T^{2m} \right) \cdot A = \frac{1}{4} \left( \frac{T^2}{\sinh^2(T/2)} \right) \cdot A.
\]

□

12.3. Pullback to \( \mathbb{P}^1 - \{0, 1, \infty\} \). We can further restrict the connection to the fiber over the tangent vector \( \partial/\partial q \). Just set \( dq \) to zero to get:

\[
\omega_{E'_0} = [T, A] \frac{dw}{w-1} + \left( \frac{T}{e^T - 1} \right) \cdot A \frac{dw}{w}.
\]

Since

\[
\frac{T}{e^T - 1} + \frac{T}{e^{-T} - 1} + T = 0.
\]

It follows that the residues \( R_0, R_1 \) and \( R_\infty \) of \( \omega_{E'_0} \) at \( 0, 1, \infty \) are:

\[
R_0 = \left( \frac{T}{e^T - 1} \right) \cdot A, \quad R_1 = [T, A], \quad R_\infty = \left( \frac{T}{e^{-T} - 1} \right) \cdot A.
\]
That these sum to zero can be checked directly. Note, too, that these can be expressed in terms of Bernoulli numbers as

$$\frac{T}{e^T - 1} = 1 - \frac{T}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} T^{2k}. $$

Since the exponentials of $R_0, R_1$ and $R_\infty$ represent paths in the unipotent paths torsor of $\mathbb{P}^1 - \{0, 1, \infty\}$, and since this is invariant under monodromy about $q = 0$, it should be true that $N_q$ annihilates them. Later we prove that his is indeed the case.

13. Restriction to $\mathcal{M}_{1,1}$

In this section, we compute the restriction of the universal elliptic KZB connection to the first order neighbourhood $\mathcal{M}_{1,1}$ of the 0-section of $\mathcal{E}$. In algebraic terms, this restriction map is induced by the $\mathcal{O}_\mathcal{E}$-module homomorphism

$$\Omega^1_{\mathcal{E}}(\log \mathcal{M}_{1,1}) \to \Omega^1_{\mathcal{E}}(\log \mathcal{M}_{1,1}) \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}_{\mathcal{M}_{1,1}}$$

Here we are identifying the zero-section of $E$ with $\mathcal{M}_{1,1}$. This computation will allow us to see that the restricted connection takes values in $\text{Der}^0 \mathfrak{p}$.

In concrete terms the restriction mapping is given by

$$G(\xi, \tau) \frac{d\xi}{\xi} + H(\xi, \tau) d\tau \mapsto G(0, \tau) \frac{d\xi}{\xi} + H(0, \tau) d\tau$$

where $G$ and $H$ are holomorphic functions of $(\xi, \tau)$. The restricted connection is thus given by the 1-form

$$\omega' = \frac{1}{2\pi i} d\tau \otimes a \frac{\partial}{\partial t} + \psi + \nu' = \frac{dq}{q} \otimes A \frac{\partial}{\partial T} + \psi + \nu'$$

where

$$\nu' = [t, a] \frac{d\xi}{\xi} + \frac{1}{2\pi i} \left( \frac{1}{t} + t \frac{\partial F}{\partial t}(0, t, \tau) \right) (a) d\tau$$

$$= [t, a] \frac{d\xi}{\xi} - 2 \sum_{s \geq 0} \frac{(2\pi i)^s+1}{s!} G_{s+2}(\tau) d\tau \otimes \text{ad}_t^{s+1}(a)$$

$$= [T, A] \left( \frac{d\xi}{\xi} - 2G_2(\tau) \frac{dq}{q} \right) - \sum_{m \geq 1} \frac{2}{(2m)!} G_{2m+2}(\tau) \frac{dq}{q} \otimes \text{ad}_T^{2m+1}(A).$$

Note that both terms in this last expression are $\text{SL}_2(\mathbb{Z})$-invariant and that the term $\psi$ remains unchanged:

$$\psi = \sum_{m \geq 1} \left( \frac{2}{(2m)!} G_{2m+2}(\tau) \frac{dq}{q} \otimes \sum_{j+k=2m+1 \atop j, k \geq 0} (-1)^j [\text{ad}_T^j(A), \text{ad}_T^k(A)] \frac{\partial}{\partial A} \right).$$

Proposition 13.1. The $\text{Der} \mathfrak{p}$-valued 1-form $\nu' + \psi - [t, a] \frac{d\xi}{\xi}$ takes values in $\text{Der}^0 \mathfrak{p}$.

Proof. Since $\omega$ is integrable, $\omega'$ is also integrable. Since $\langle a \partial / \partial t, [t, a] \frac{d\xi}{\xi} \rangle = 0$,

$$[\psi + \nu', [t, a] \frac{d\xi}{\xi}] = \frac{1}{2} [w', w'] = d\omega' + \frac{1}{2} [w', w'] = 0.$$

The result follows as $\nu' + \psi - [t, a] \frac{d\xi}{\xi}$ is a multiple of $d\tau$ and as $d\tau \wedge d\xi$ is nowhere zero. \qed
Set
\[ \nu'' = - \sum_{m \geq 1} \frac{2}{(2m)!} G_{2m+2}(\tau) \frac{dq}{q} \otimes \text{ad}^{2m+1}_T(A) \]
and
\[ \omega'' = \frac{dq}{q} \otimes A \frac{\partial}{\partial T} + \psi + \nu''. \]

Then
\[ \omega'' = \frac{dq}{q} \otimes \left[ A \frac{\partial}{\partial T} + \sum_{m=1}^\infty \frac{2}{(2m)!} G_{2m+2}(\tau) \left( -\text{ad}^{2m+1}_T(A) + \sum_{j+k=2m+1 \atop j > k > 0} (-1)^j [\text{ad}^j_T(A), \text{ad}^k_T(A)] \frac{\partial}{\partial A} \right) \right]. \]

Note that this is pulled back from a connection on \( \text{Der}^0 \mathfrak{p} \times \mathfrak{h} \to \mathfrak{h} \), which is \( \text{SL}_2(\mathbb{Z}) \)-invariant by Lemma 7.1.

In terms of the notation (12.2),
\[ \omega'' = - \frac{dq}{q} \otimes \epsilon_0 - \sum_{m=2}^\infty \frac{2}{(2m-2)!} G_{2m}(\tau) \frac{dq}{q} \otimes \epsilon_{2m}. \]

Note that
\[ \omega' = \omega'' + [T, A] \left( \frac{d\xi}{\xi} - 2G_2(\tau) \frac{dq}{q} \right) \]
\[ = - \frac{dq}{q} \otimes \epsilon_0 - \left( 2G_2(\tau) \frac{dq}{q} \frac{d\xi}{\xi} \right) \otimes \epsilon_2 - \sum_{m=2}^\infty \frac{2}{(2m-2)!} G_{2m}(\tau) \frac{dq}{q} \otimes \epsilon_{2m}. \]

This gives an alternative computation of \( N_q \):
\[ N_q := \text{Res}_{q=0} \omega' = \sum_{m=0}^\infty (2m-1) \frac{B_{2m}}{(2m)!} \epsilon_{2m}. \]

14. RIGIDITY

The punctured universal elliptic curve \( \mathcal{E}' \) is the moduli space \( \mathcal{M}_{1,2} \). Consequently, the fundamental group of \( \mathcal{E}' \) is the mapping class group \( \Gamma_{1,2} \). Choose a base point \( [E_\tau, x_o] \) of \( \mathcal{M}_{1,2} \). By Cor. 11.3, the restriction of the universal elliptic KZB connection to \( E_\tau \) defines a homomorphism \( \pi_1(E'_\tau, x_o) \to \mathcal{P} \) that induces an isomorphism
\[ \pi_1(E'_\tau, x_o)^{\text{un}} \to \mathcal{P}. \]

Set \( \pi = \pi_1(E'_\tau, x_o) \). Denote the natural action of \( \Gamma_{1,2} \) on \( \mathfrak{p} \) by
\[ \rho : \Gamma_{1,2} \to \text{Aut} \mathcal{P} \cong \text{Aut} \mathfrak{p}. \]

There is a natural homomorphism \( j : \pi \to \Gamma_{1,2} \) induced by the inclusion of \( (E'_\tau, x_o) \to (\mathcal{M}_{1,2}, [E; 0, x_o]) \). The composition \( \rho \circ j \) is the action of \( \pi \) on itself by inner automorphisms. It is injective as \( \pi \) is free, which implies that \( \pi \to \mathcal{P} \) is injective and that \( \mathcal{P} \) has trivial center.
Proposition 14.1. If $\phi : \Gamma_{1,2} \to \text{Aut} \mathcal{P}$ is a homomorphism whose restriction $\pi \to \text{Aut} \mathfrak{p}$ to $\pi$ is the composite

$$
\pi \longrightarrow \mathcal{P} \xrightarrow{\text{inner}} \text{Aut} \mathcal{P},
$$

then $\phi = \rho$.

This implies the following rigidity result for connections, which implies that the universal elliptic KZB connection has the correct monodromy.

Theorem 14.2. Any flat connection on the bundle $\mathcal{P} \to \mathcal{E}'$ whose monodromy on the fiber $E'$ of $\mathcal{E}$ over $[E] \in \mathcal{M}_{1,1}$ is the composition

$$
\begin{array}{c}
\pi_1(E', x_o) \\
\xrightarrow{\text{inclusion}_*} \\
\mathcal{P} \\
\xrightarrow{\text{Ad}} \\
\xrightarrow{\rho} \text{Aut} \mathfrak{p} \cong \text{Aut} \mathcal{P}
\end{array}
$$

of the natural inclusion with the inner action of $\mathcal{P}$ on itself is gauge equivalent to the canonical flat connection on $\mathcal{P} \to \mathcal{E}'$. Consequently, the fiberwise exponential mapping

$$
\exp : \mathcal{P} \to \mathcal{P}^{\text{top}}
$$

is an isomorphism of local systems.

Identify $\pi_1(E', x_o)^{\text{un}}$ with $\mathcal{P}$ via the isomorphism given by the universal elliptic KZB connection. The the monodromy representations of $\mathcal{P}$ and $\mathcal{P}^{\text{top}}$ are homomorphisms

$$
\rho : \Gamma_{1,2} \to \text{Aut} \mathfrak{p} \text{ and } \rho^{\text{top}} : \Gamma_{1,2} \to \text{Aut} \mathcal{P}.
$$

Their restrictions to the distinguished fiber $E'_\tau \subset \mathcal{E}'$ correspond under the exponential mapping. That is, the diagram

$$
\begin{array}{ccc}
\pi_1(E'_\tau, x_o) & \xrightarrow{\rho} & \text{Aut} \mathfrak{p} \\
\| & \| & \| \\
\pi_1(E'_\tau, x_o) & \xrightarrow{\rho^{\text{top}}} & \text{Aut} \mathcal{P}
\end{array}
$$

commutes. The proposition now implies:

Theorem 14.3. The exponential mapping induces an isomorphism of the locally constant sheaf over $\mathcal{E}'$ of flat sections of the universal elliptic KZB connection on $\mathcal{P}$ with the locally constant sheaf $\mathcal{P}^{\text{top}}$ over $\mathcal{E}'$. Equivalently, the diagram

$$
\begin{array}{ccc}
\pi_1(\mathcal{E}', [E, x_o]) & \xrightarrow{\rho} & \text{Aut} \mathfrak{p} \\
\| & \| & \| \\
\pi_1(\mathcal{E}', [E, x_o]) & \xrightarrow{\rho^{\text{top}}} & \text{Aut} \mathcal{P}
\end{array}
$$

commutes.

Any other connection on the extension of $\mathcal{P}$ to $\overline{\mathcal{M}}_{1,2}$ with regular singularities differ by a holomorphic map $\overline{\mathcal{M}}_{1,2} \to \text{Aut} \mathfrak{p}$, which must be constant as $\text{Aut} \mathfrak{p}$ is a pro-affine group.

Corollary 14.4. The universal elliptic KZB connection is the unique holomorphic connection on $\mathfrak{p} \times \mathbb{C} \times \mathfrak{h}$ (up to a constant change of gauge) satisfying:
that

Lemma 14.5. Suppose that the canonical homomorphism \( N \to \mathcal{N} \) is injective and that \( \mathcal{N} \) has trivial center. If \( \phi : G \ltimes N \to \text{Aut} \mathcal{N} \) is a homomorphism whose restriction to \( N \) is the inner action of \( N \) on \( \mathcal{N} \), then \( \phi = \rho \).

Proof. Identify \( N \) with its image in \( \mathcal{N} \). Denote the inner automorphism \( x \mapsto nxn^{-1} \) of \( \mathcal{N} \) by \( \iota_n \). Since the restriction of \( \phi \) to \( N \) is \( \iota_n \), for all \( g \in G \) and \( n \in N \), we have

\[
\iota_{\phi(g)(n)} = \iota_{\phi(g)n} = \phi(g)\iota_n\phi(g)^{-1} = \iota_{\phi(g)(n)}.
\]

Since \( N \) is a subgroup of \( \mathcal{N} \) and since \( \mathcal{N} \) has trivial center, this implies that

\[
\alpha(g)(n) = gng^{-1} = \phi(g)(n)
\]

for all \( g \in G \) and \( n \in N \). Thus \( \alpha = \phi \) and the result follows. \( \square \)

Remark 14.6. This argument is easily modified to prove that the local system associated to the flat connection over \( \mathcal{M}_{1,1+n} \), constructed in [2] is the canonical local system whose fiber over \([E,0,x_1,\ldots,x_n]\) is the unipotent completion of the fundamental group of the configuration space of \( n \) points on \( E' \) with base point \((x_1,\ldots,x_n)\).

15. Hodge Theory

Denote the maximal ideal \((T,A)\) of \( \mathbb{Q}(\langle T,A \rangle) \) by \( I \). Define Hodge and weight filtrations on \( \mathbb{Q}(\langle T,A \rangle) \) by setting

\[
W_m\mathbb{Q}(\langle T,A \rangle) = I^m \quad \text{and} \quad F^p\mathbb{Q}(\langle T,A \rangle) = \{ x \in \mathbb{Q}(\langle T,A \rangle) : \deg_A(x) \leq p \}.
\]

Define a relative weight filtration \( M_\bullet \) on \( \mathbb{Q}(\langle T,A \rangle) \) by

\[
M_{-2m}\mathbb{Q}(\langle T,A \rangle) = \{ x \in \mathbb{Q}(\langle T,A \rangle) : \deg_A(x) \geq m \}.
\]

These filtrations are multiplicative. By restriction, they induce Hodge, weight and relative weight filtrations on \( \mathfrak{p} \) and thus on \( \text{Der} \mathfrak{p} \). They also induce filtrations on the bundle \( \mathfrak{p} \times \mathbb{C} \times \mathfrak{h} \) over \( \mathbb{C} \times \mathfrak{h} \).
Theorem 15.1. The Hodge and weight filtrations on $p \times \mathbb{C} \times h$ descend to Hodge and weight filtrations of the local system $\mathcal{P} \to \mathcal{E}'$. With these filtrations, the local system $\mathcal{P}$ over $\mathcal{E}' = \mathcal{M}_{1,2}$ and its restriction to $\mathcal{M}_{1,1}$ are admissible variation of MHS whose weight graded quotients are direct sums of Tate twists of $S^m H$. The MHS on the fiber over $[E, x]$ is the canonical MHS on the Lie algebra of the unipotent completion of $\pi_1(\mathcal{E}', x)$. The relative weight filtration of the limit MHS associated to the tangent vector $\vec{v} = \partial/\partial q + \partial/\partial w$ at the identity of the nodal cubic is $M_\bullet$. In addition, there is a natural homomorphism

$$\pi_1(P_1 - \{0, 1, \infty\}, \partial/\partial w)^{un} \to \pi_1(E'_\lambda, \partial/\partial w)^{un}$$

whose image is invariant under monodromy and which is a morphism of MHS for all $\lambda \in \mathbb{C}^*$

Proof. It suffices to consider the case where the base is $\mathcal{E}'$. The factor of automorphy $M_\bullet(\xi, \tau)$ (Cf. (6.2)) preserves both the Hodge and weight filtrations on $p \times \mathbb{C} \times h$. They therefore descend to filtrations of the bundle $\mathcal{P}$ over $\mathcal{E}'$. Next observe that each component of the universal elliptic KZB connection is a 1-form that takes values in $F^{-1}W_0 \text{Der}^0 p$. This implies that, over $\mathcal{E}'$, the connection satisfies Griffiths transversality and that the weight bundles are sub-local systems of $\mathcal{P}$. The weight filtration is defined over $\mathbb{Q}$ as it is defined in terms of the lower central series filtration of $p$.

As explained in Section 10, the natural extension of the universal elliptic KZB connection to the $q$-disk has regular singular points along the nodal cubic and along the identity section. The extension of the Hodge and weight bundles to the $q$-disk are the quotients of the bundles

$$(F^p p) \times \mathbb{C} \times D \text{ and } (W_m p) \times \mathbb{C} \times D$$

over $\mathbb{C} \times D$ (with coordinates $(w, q)$) by the factor of automorphy. These are well defined as $M_{(m, n)}(q) = e^{-mT}$, which lies in $F^0W_0 M_0 \text{Der}^0 p$.

The residue at each point of the identity section is $N_w := \text{ad}_{[T, A]}$. It lies in $F^{-1}W_2 M_2 \text{Der}^0 p$, which implies that the Hodge, weight and (where relevant) the relative weight filtrations extend across the identity section.

According to (12.3) the residue of the connection at each point of the nodal cubic is

$$N_q = \sum_{m \geq 0} (2m - 1) \frac{B_{2m}}{(2m)!} \epsilon_{2m} \in \text{Der}^0 p.$$

It is easy to check that for each $m \geq 0$, $\epsilon_{2m} \in F^{-1}M_2W_2 M_2 \text{Der}^0 p$ for all $m \geq 0$, so that

$$N_q \in F^{-1}M_2W_0 \text{Der}^0 p.$$

Moreover,

$$\text{Gr}^W N_q : \text{Gr}^W C\langle \langle T, A \rangle \rangle \to \text{Gr}^W C\langle \langle T, A \rangle \rangle$$

is $\epsilon_0$. Since $\text{Gr}^W_m C\langle \langle T, A \rangle \rangle = S^m H$ placed in weight $-m$ and since $\text{Gr}^W N_q = -\epsilon_0 = A \partial/\partial T \in \mathfrak{sl}(H)$, it follows easily from the representation theory of $\mathfrak{sl}(H)$ that $N_q^k$ induces an isomorphism

$$\text{Gr}^{W_{-m+k}} C\langle \langle T, A \rangle \rangle \to \text{Gr}^{W_{-m-k}} C\langle \langle T, A \rangle \rangle.$$

This implies that $M_\bullet$ is the relative weight filtration of $N_q$ and for $N_q + N_w$, which completes the proof that $\mathcal{P}$ is an admissible variation of MHS over $\mathcal{M}_{1,2}$. 


To prove that the MHS on the fiber of $\mathcal{P}$ over $[E,0] \in \mathcal{M}_{1,1}$ is its canonical MHS (as defined in [6] or [14]) we consider the restriction $\mathcal{P}_E$ of $\mathcal{P}$ to the fiber $E'$. The above discussion implies that this is an admissible variation of MHS over $E'$. In fact, it is clearly a unipotent variation of MHS. Fix a base point $x \in E'$. The above discussion implies that this is an admissible variation of MHS over $E'$. In fact, it is clearly a unipotent variation of MHS. Fix a base point $x \in E'$. Theorem 14.3 implies that the fiber $p(E,x)$ of $\mathcal{P}_E$ over $x$ is naturally isomorphic to the Lie algebra of the unipotent completion of $\pi_1(E',x)$. The monodromy representation $\theta_x : p(E,x) \to \text{End} p(E,x)$ of $\mathcal{P}_E$ is the adjoint action. Since the center of $p(E,x)$ is trivial ($p$ is free of rank 2), $\theta_x$ is injective. Denote $p(E,x)$ with its canonical MHS by $p(E,x)_{\text{can}}$ and $p(E,x)$ with the MHS given by the elliptic KZB connection via Theorem 14.3 by $p(E,x)^{\text{KZB}}$. The main theorem of [14] implies that $\theta_x : p(E,x)_{\text{can}} \to \text{End} p(E,x)^{\text{KZB}}$ is a morphism of MHS. On the other hand, since $p(E,x)^{\text{KZB}}$ is a Lie algebra in the category of pro-MHS, $\theta_x : p(E,x)^{\text{KZB}} \to \text{End} p(E,x)^{\text{KZB}}$ is also a morphism of MHS. Since $\theta_x$ is injective, this implies that the MHSs $p(E,x)_{\text{can}}$ and $p(E,x)^{\text{KZB}}$ are equal.

The final statement follows from the construction [7] of limit mixed Hodge structures on homotopy groups. It will be explained in greater detail in [10]. □

We can now prove that $N_q$ annihilates the $R_\alpha$.

**Corollary 15.2.** The derivation $N_q$ annihilates $R_0$, $R_1$ and $R_\infty$.

An elementary proof is given in the Appendix. Here we sketch a more conceptual proof.

**Sketch of Proof.** Since $N_q$ annihilates $R_1 = [T,A]$, and since $R_0 + R_1 + R_\infty = 0$, it suffices to show that $N_q(R_0) = 0$.

One has the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, \partial/\partial w)^{\wedge} & \xrightarrow{\Theta_{\text{KZ}}} & \mathbb{C}\langle\langle R_0, R_1 \rangle\rangle \\
\downarrow & & \downarrow \\
\mathbb{C}\pi_1(E', \partial/\partial q)^{\wedge} & \xrightarrow{\Theta_{\lambda}} & \mathbb{C}\langle\langle T, A \rangle\rangle
\end{array}$$

where the left hand vertical mapping is the one given by the Theorem, the right hand vertical map is given by the formulas for $R_0$ and $R_1$, and where the top horizontal map is the standard isomorphism given by the KZ-connection. The result follows as the image of $\Theta_{\text{KZ}}$ is invariant under monodromy and as the logarithm of monodromy acting on $\mathbb{C}\langle\langle T, A \rangle\rangle$ is $N_q$. □

There is a discussion of a stronger statement in the appendix.

As noted in Remark 12.1, the limit MHS on $H_1(E_\tau)$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}(1)$. The basis $T$, $A$ splits both the Hodge and monodromy weight filtrations. The following statement follows directly from Corollary 11.2.

**Lemma 15.3.** The limit MHS on $H_1(E_\tau)$ associated to the tangent vector $\lambda \partial/\partial q$ of the origin of the $q$-disk is has complex basis $A$ and $T$ and integral basis spanned by $a$ and $-b$, where $-b = T - \log \lambda A$ and $a = 2\pi i A$,
so that the corresponding period matrix is
\[
\begin{pmatrix}
1 & -\log \lambda \\
0 & 2\pi i
\end{pmatrix}
\]

This lemma implies the well known fact that the restriction of \( \mathbb{H} \) to the \( q \)-disk is a nilpotent orbit of MHS.

16. Pause for a Picture

Suppose that \( q \in \mathbb{D}^* \). The fiber of the universal elliptic curve over \( q \) is \( E_q := \mathbb{C}^*/q^2 \). For \( 0 < r < 1 \), set
\[
A_r := \{ w \in \mathbb{C}^* : r^{1/2} \leq |w| \leq r^{-1/2} \}.
\]
Denote its outer and inner boundaries by \( \partial_+ A_r \) and \( \partial_- A_r \), respectively. The elliptic curve \( E_q \) is the quotient of \( A_{|q|} \) by identifying \( w \in \partial_+ A_{|q|} \) with \( qw \in \partial_- A_{|q|} \).

The homology class \( a \) corresponds to the class of the positively oriented unit circle \( \alpha \); the homology class \( b \) corresponds to a path \( \beta \) from \( w \) to \( qw \), where \( w \in \partial_+ A_{|q|} \). Note that these have intersection number +1. We also remove a very small (infinitesimal) disk about \( w = 1 \). Its boundary \( \gamma \) corresponds to the different tangent directions at the identity of \( E_q \).

\[\text{Figure 2. } E_q \text{ as a quotient of } A_{|q|}\]

\[\text{Figure 3. The “nearby fiber” } E'_{\partial/\partial q}\]

The second figure shows the disk with the identifications almost made. From this it is evident that \( a \), the class of \( \alpha \), is the vanishing cycle. The Picard Lefschetz transformation is also evident: as \( q \) travels once around \( \mathbb{D}^* \) in the positive direction, \( a \) remains invariant, but \( b \), the class of \( \beta \) changes to \( b + a \). This can also be seen from the formula
\[
T = \tau a - b = \frac{a}{2\pi i} \log q - b
\]
Note that the free homotopy class of $\gamma$ is the conjugacy class of the commutator $\{\alpha, \beta\} := \alpha \beta \alpha^{-1} \beta^{-1}$. This is consistent with the fact that
\[
\text{Res}_{w=1} \omega_{E_0} = [T, A] = \frac{1}{2\pi i} [T, a] = \frac{1}{2\pi i} [\tau a - b, a] = \frac{1}{2\pi i} [a, b].
\]
A topological model for the pointed “first order Tate curve”
\[(e^{\sigma \partial/\partial q}, \partial/\partial w)\]
is the quotient of the real oriented blow-up of $\mathbb{P}^1$ at $\{0, 1, \infty\}$ — which is a 3-holed sphere — by the identification of the real tangent direction $v$ at $w = \infty$ with the tangent direction $(e^{i\theta})^* v$ at $w = 0$, where $(e^{i\theta})^*$ is the map on the tangent bundle of $\mathbb{P}^1$ induced by $w \mapsto e^{i\theta} w$. See Appendix C for details.

17. The KZ-equation and the Drinfeld Associator

The quotient map
\[
\mathbb{C}\langle\langle X_0, X_1, X_\infty \rangle\rangle/(X_0 + X_1 + X_\infty) \to \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle
\]
is an isomorphism. We will identify these two rings. Recall that the KZ-connection on $\mathbb{P}^1 - \{0, 1, \infty\}$ is given by
\[
\omega_{KZ} = \frac{dw}{w} X_0 + \frac{dw}{w - 1} X_1 \in H^0(\Omega^1_{\mathbb{P}^1}(\log \{0, 1, \infty\})) \otimes \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle.
\]

The form $\omega_{KZ}$ defines a flat connection on the trivial bundle
\[
\mathbb{C}\langle\langle X_0, X_1 \rangle\rangle \times \mathbb{P}^1 - \{0, 1, \infty\} \to \mathbb{P}^1 - \{0, 1, \infty\}
\]
by the formula
\[
\nabla f = df - f \omega_{KZ}.
\]
Its transport function induces a transport function
\[
\{
\text{paths in } \mathbb{P}^1 - \{0, 1, \infty\} \} \to \{
\text{group-like elements of } \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle
\}
\]
in $\mathbb{P}^1 - \{0, 1, \infty\}$. It takes the path $\gamma$ in $\mathbb{P}^1 - \{0, 1, \infty\}$ to
\[
T(\gamma) = 1 + \int_\gamma \omega_{KZ} + \int_\gamma \omega_{KZ} \omega_{KZ} + \int_\gamma \omega_{KZ} \omega_{KZ} \omega_{KZ} + \cdots
\]
Since $\omega_{KZ}$ is clearly integrable, the connection is flat and $T(\gamma)$ depends only on the homotopy class of $\gamma$ relative to its endpoints.

There are six standard tangent vectors of $\mathbb{P}^1 - \{0, 1, \infty\}$. Two are anchored at each of 0, 1, $\infty$. They lie in one orbit under the action of the symmetric group $S_3$ on $\mathbb{P}^1 - \{0, 1, \infty\}$ and are thus determined by the two vectors at $w = 0$, which are $\pm \partial/\partial w$.

These have the property that their reduction mod $p$ is non-zero for all prime numbers $p$.

The (KZ/de Rham) version of the Drinfeld associator is an invertible power series $\Phi(Y, Z) \in \mathbb{C}\langle\langle Y, Z \rangle\rangle$. It begins:
\[
\Phi(Y, Z) = 1 + \zeta(2)[Y, Z] - \zeta(3)[Y, [Y, Z]] + \zeta(1, 2)[[Y, Z], Z] - \zeta(4)[Y, [Y, [Y, Z]]] - \zeta(1, 3)[Y, [[Y, Z], Z]] + \zeta(1, 1, 2)[[[Y, Z], Z], Z] + \frac{1}{2} \zeta(2)[Y, Z]^2 + \cdots
\]
where, for positive integers $n_1, \ldots, n_r$, where $n_r > 1$,
\[ \zeta(n_1, \ldots, n_r) = \sum_{0 < k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_r^{n_r}}. \]
These are the multiple zeta numbers. They generalize the values of the Riemann zeta function at positive integers. An explicit formula for $\Phi(Y, Z)$ is given in [4].

All coefficients are rational multiplies of multiple zeta values. Several (not all!) of its basic properties are summarized in the following result:

**Theorem 17.1** (Drinfeld). The Drinfeld associator $\Phi$ satisfies:

1. $\Phi(Y, Z) \Phi(Z, Y) = 1$
2. In the ring $\mathbb{C}\langle\langle X, Y, Z \rangle\rangle/(X + Y + Z)$ we have
   \[ \Phi(X, Y)e^{i\pi Y} \Phi(Y, Z)e^{i\pi Z} \Phi(Z, X)e^{i\pi X} = 1. \]

The normalized value of $T$ on the unique real path from $w = 0$ to $w = 1$ is $\Phi(X_0, X_1)$. View the symmetric group $S_3$ as $\text{Aut}\{0, 1, \infty\}$. The action of the automorphisms of $\mathbb{P}^1$ minus $\{0, 1, \infty\}$ on the cusps determines an isomorphism $\text{Aut}(\mathbb{P}^1, \{0, 1, \infty\}) \rightarrow \text{Aut}\{0, 1, \infty\}$.

Let it act on $\{X_0, X_1, X_\infty\}$ by permuting the indices. Since the connection is invariant under the $S_3$-action on $\mathbb{P}^1$ minus $\{0, 1, \infty\}$, we have, for example, the following values of the normalized transport on the real paths:

\[ T_{\text{norm}}([1, \infty]) = \Phi(X_1, X_\infty), \quad T_{\text{norm}}([0, \infty]) = \Phi(X_0, X_\infty), \]
where $[0, \infty]$ is the path from $0$ to $\infty$ along the negative real axis.

17.1. **The fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$**. Consider the category whose objects are the 6 tangent vectors of $\mathbb{P}^1$ defined above and whose morphisms are homotopy classes from one tangent vector to another.\(^{12}\) Denote it by $\Pi(\mathbb{P}^1, V)$. It is generated by the paths shown in the diagram. As above, a good topological model is to replace $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ by the real oriented blow-up of $\mathbb{P}^1$ at $\{0, 1, \infty\}$, which is a 3-holed sphere, and represent the tangent directions by the corresponding points on the boundary of the blown up sphere.

---

\(^{11}\)To get this formula for $\Phi(Y, Z)$, one have to reverse the order of all monomials — equivalently, replace each bracket $[U, V]$ by its negative $-[U, V]$. This is because Furusho uses the opposite convention for path multiplication.

\(^{12}\)That is, the path starts with one tangent vector and ends with the negative of the second. Such a path $\gamma$ must also satisfy $\gamma(t) \notin \{0, 1, \infty\}$ when $0 < t < 1$. Composition of two such homotopy classes of paths can be defined when the second path begins at the tangent vector where the first ends.
Define a functor

$$\Theta : \Pi(\mathbb{P}^1, V) \to \{\text{group-like elements of } \mathbb{C}\langle\langle X_0, X_1, X_\infty \rangle\rangle\}$$

by taking the positively oriented semi-circle about \(a \in \{0, 1, \infty\}\) to \(e(X_a/2)\) and the real interval from \(a\) to \(b\) to \(\Phi(X_a, X_b)\). Drinfeld’s relations imply that \(\Theta\) is well-defined.

This restricts to a group homomorphism

$$\Theta_{\vec{v}} : \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v}) \to \mathcal{P}$$

for each of the 6 distinguished tangent vectors \(\vec{v}\).

18. The Limit MHS on \(\pi_1(E'_\partial/\partial q, \partial/\partial w)^{\text{un}}\)

Note that \(\omega_{E'_0}\) is obtained from \(\omega_{KZ}\) by composing with the ring homomorphism

$$\mathbb{C}(\langle\langle X_0, X_1, X_\infty \rangle\rangle)/(X_0 + X_1 + X_\infty) \hookrightarrow \mathbb{C}(\langle\langle T, A \rangle\rangle)$$

defined by

$$X_0 \mapsto R_0 = \left(\frac{T}{e^T - 1}\right) \cdot A,$$

$$X_1 \mapsto R_1 = [T, A],$$

$$X_\infty \mapsto R_\infty = \left(\frac{T}{e^{-T} - 1}\right) \cdot A,$$

(18.1)

which is well defined as \(R_0 + R_1 + R_\infty = 0\). (Cf. (12.4).)

Since the periods of \(\omega_{KZ}\) are understood (they are multiple zeta numbers), this formula will allow us to compute the periods of \(\omega_{E'_0}\) in terms of multiple zeta numbers.
18.1. **The Cylinder Relation.** To construct a well-defined homomorphism

\[ \pi_1(E'_{\partial/\partial q}, \partial/\partial w) \to \mathcal{P}, \]

we need to find all solutions \( U \in \mathcal{L}(T, A)^{\wedge} \) of the equation

\[ (18.2) \quad e^{-U} e^{\lambda R_0} e^{U} e^{\lambda R_\infty} = 1 \quad \text{in} \quad \mathbb{Q} \langle \langle T, A \rangle \rangle, \]

for all \( \lambda \in \mathbb{Q}^\times \). This relation will be called the **cylinder relation**. Note that a solution to the equation with \( \lambda = 1 \) will be a solution for all \( \lambda \).

**Lemma 18.1.** For all \( \lambda \in \mathbb{C} \), the relation

\[ e^T e^{\lambda R_0} e^{-T} e^{\lambda R_\infty} = 1 \]

holds in \( \mathcal{P} \). That is, \( U = -T \) is a solution of the cylinder equation.

**Proof.** It suffices to prove the relation

\[ e^T R_0 e^{-T} = -R_\infty. \]

Since

\[ \phi \exp(u) \phi^{-1} = \exp(\phi u \phi^{-1}), \]

we have

\[ e^T R_0 e^{-T} = e^T \left( \left[ \frac{T}{e^T - 1} \right] \cdot A \right) = \left[ \frac{T e^T}{e^T - 1} \right] \cdot A = \left[ \frac{-T}{e^{-T} - 1} \right] \cdot A = -R_\infty. \]

\[ \square \]

**Proposition 18.2.** Every solution of the cylinder relation (18.2) is of the form

\[ e^U = e^{\lambda R_0} e^{-T} \]

for some \( \lambda \in \mathbb{Q} \).

**Proof.** Suppose that \( U \) is a solution of the cylinder equation. Set \( V = \log(e^U e^T) \). Then \( V \in \mathcal{L}(T, A)^{\wedge} \) and

\[ e^U = e^V e^{-T}. \]

The cylinder relation implies that

\[ e^T e^{-V} e^{R_0} e^V e^{-T} = e^{-U} e^{R_0} e^U = e^{-R_\infty} = e^T e^{R_0} e^{-T} \]

so that

\[ e^{-V} e^{R_0} e^V = e^{R_0}. \]

The result follows as the centralizer of \( R_0 \) in \( \mathcal{L}(T, A)^{\wedge} \) is \( \mathbb{Q} R_0 \).

\[ \square \]

18.2. **The homomorphisms** \( \pi_1(E'_{\partial/\partial q}, \partial/\partial w) \to \mathcal{P} \). The positive real axis determines two points \( v_0 \) and \( v_\infty \) on the real oriented blow up of \( \mathbb{P}^1 \) at \( \{0, 1, \infty\} \) — the point \( v_0 \) lies on the circle at 0 and \( v_\infty \) lies on the circle at \( \infty \). There is a natural \( U(1) \) action on each of these circles.

Suppose that \( \lambda \in \mathbb{C}^\times \). Write it in the form \( re^{i\theta} \). View \( E'_{\lambda \partial/\partial q} \) as the quotient of the real oriented blow-up of \( \mathbb{P}^1 \) at \( \{0, 1, \infty\} \) by

\[ e^{i\theta} v_\infty \sim e^{i(\theta - \phi)} v_0. \]

Denote the image of the two identified circles in \( E'_{\lambda \partial/\partial q} \) by \( C \). One can check that as \( \lambda \) moves around the unit circle in the positive direction, the identification changes by a positive Dehn twist about \( C \). Note that for each \( \lambda \) there is a natural inclusion

\[ \iota : (\mathbb{P}^1 - \{0, 1, \infty\}, \partial/\partial w) \to (E'_{\lambda \partial/\partial q}, \partial/\partial w) \]

where in both cases \( \partial/\partial w \) is in element of the tangent space of \( 1 \in \mathbb{P}^1 \).
To define a homomorphism
\[ \Theta_\lambda : \pi_1(E'_{\partial q}, \partial/\partial w) \to P \]
such that the diagram
\[
\begin{array}{c}
\pi_1((\mathbb{P}^1 - \{0, 1, \infty\}, \partial/\partial w) \ar{r}{\iota} \ar{d}{\iota} & \exp L(X_0, X_1) \ar{d} \ar{r}{\Theta_\lambda} & \pi_1(E'_{\lambda \partial q}, \partial/\partial w) \ar{r}{\Theta_\lambda} & P
\end{array}
\]
commutes, where the right-hand vertical map is defined by (18.1). We need to give a “factor of automorphy” for the identification. This is the monodromy along a “path” from \(e^{i\phi}v_\infty\) to \(e^{\theta - \phi}v_0\). In order that \(\Theta_\lambda\) be well defined, this factor of automorphy has to satisfy the cylinder relation.

For \(\lambda \in \mathbb{C}^*\), define this factor of automorphy for \(\Theta_\lambda\) to be
\[ \lambda^{R_0}e^{-T} := e^{\log \lambda R_0}e^{-T}. \]
That is, the (inverse) monodromy in going from the tangent vector \(\lambda v_0\) at \(0 \in \mathbb{P}^1\) to the tangent vector \(v_\infty\) at \(\infty\) is \(\lambda^{R_0}e^{-T}\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{path_torsor.png}
\caption{The path torsor of \(E'_{\partial q}\)}
\end{figure}

Give \(\mathbb{C}\langle\langle A, T\rangle\rangle\) the Hodge, weight and relative weight filtrations defined in Section 15.

**Proposition 18.3.** The (complete Hopf algebra) homomorphism
\[ \Theta_\lambda : \mathbb{Q}\pi_1(E'_{\lambda \partial q}, \partial/\partial w)^\wedge \to \mathbb{C}\langle\langle T, A\rangle\rangle \]
is an isomorphism after tensoring the source with \(\mathbb{C}\). This and the Hodge and weight filtrations on \(\mathbb{C}\langle\langle A, T\rangle\rangle\) defined in Section 15 define a MHS on \(\mathbb{Q}\pi_1(E'_{\lambda \partial q}, \partial/\partial w)^\wedge\). This is the canonical limit MHS the fiber of the universal enveloping algebra of \(\mathcal{P}\) corresponding to the tangent vector \(\lambda \partial q + \partial/\partial w\) at the identity of the nodal cubic. It’s relative weight filtration is the one defined in Section 15.
Proof. First observe that \( \Theta_\lambda \) induces a homomorphism
\[
H_1(E^\lambda_{\partial/\partial q}, \mathbb{Z}) \rightarrow \mathbb{C}T \oplus \mathbb{C}A.
\]
It takes \( a \) to
\[
2\pi i R_0 \mod I^2 = 2\pi i A
\]
and \( b \) to
\[
\log(\lambda R_0 e^{-T}) \mod I^2 = \log \lambda A - T.
\]
The homomorphism \( \Theta_\lambda \) is an isomorphism after complexifying as both its source and target are free and as it induces an isomorphism on \( I/I^2 \).

The homomorphism \( \Theta_\lambda \) defines a MHS on \( \mathbb{Q}\pi_1(E^\lambda_{\partial/\partial q}, \partial/\partial w) \wedge \) by pulling back the Hodge, weight and relative weight filtrations of \( \mathbb{C}[[T,A]] \). To check that this is the limit MHS associated to \( \lambda\partial/\partial q \), it suffices to check that the induced MHS on \( H_1(E^\lambda_{\partial/\partial q}) \) by \( \Theta_\lambda \) agrees with the canonical limit MHS. This follows from the discussion above and the fact that the MHS induced on the image of \( \mathbb{Q}\pi_1(P^1 - \{0,1,\infty\}, \partial/\partial w) \wedge \) by \( \Theta_\lambda \) is its canonical MHS, which follows from the computations in Section 12.3. The point being that the limit MHS on \( H_1(E^\lambda_{\partial/\partial q}) \) corresponding to \( \lambda\partial/\partial q \) determines the factor of automorphy. The computations at the beginning of the proof imply that the MHS on \( H_1(E^\lambda_{\partial/\partial q}) \) induced by \( \Theta_\lambda \) agrees with the limit MHS that was computed in Lemma 15.3. \( \square \)

Remark 18.4. Since the Lie subalgebra of \( \mathbb{L}(T,A)\wedge \) generated by \( R_0 \) and \( R_1 \) is invariant under monodromy (which implies that \( N_q(R_0) = 0 \)), the monodromy of the “path” in \( E^\lambda_{\partial/\partial q} \) that passes from a tangent vector at 0 to a tangent vector at \( \infty \) is changed by a Dehn twist. This implies that

Note that
\[
q \frac{\partial}{\partial q} (q R_0 e^{-T}) = -N_q(q R_0 e^{-T}),
\]
as it should. Indeed, the left hand side is
\[
q \frac{\partial}{\partial q} (q R_0 e^{-T}) = q R_0 R_0 e^{-T} = -q R_0 e^{-T} R_\infty
\]
by the cylinder relation. On the other hand, we have
\[
N_q(q R_0 e^{-T}) = q R_0 N_q(e^{-T})
\]
Corollary 15.2
\[
= q R_0 e^{-T} \left( \frac{1 - e^{ad_T}}{ad_T} \right) N_q(T)
\]
Lemma 4.2
\[
= q R_0 e^{-T} \left( \frac{1 - e^T}{T} \right) \left( \frac{T^2/4}{\sinh(T/2)} \right) \cdot A
\]
Proposition 12.3
\[
= q R_0 e^{-T} \left( \frac{T}{e^{-T} - 1} \right) \cdot A
\]
\[
= q R_0 e^{-T} R_\infty.
\]

Part 4. The \( \mathbb{Q} \)-de Rham Structure

Levin and Racinet show that the bundle \( \overline{\mathcal{P}} \) over \( \mathcal{E} \) and its KZB connection are defined over \( \mathbb{Q} \). This part is an expanded exposition of their computation. In particular, we explicitly compute (Thm. 20.2) the restriction of the universal elliptic KZB connection to \( \mathcal{M}_{1,1} \) in terms of the \( \mathbb{Q} \)-algebraic coordinates on \( \mathcal{M}_{1,1} \).
19. The \( \mathbb{Q} \)-DR Structure on \( \mathcal{H} \) over \( \mathcal{M}_{1,1} \)

The first step is to compute the \( \mathbb{Q} \)-DR structure on \( \mathcal{H} \) and its canonical extension \( \mathcal{H} \). Since \( \mathcal{M}_{1,1} \) is the moduli space of elliptic curves endowed with a non-zero abelian differential, the Hodge bundle \( F^1 \mathcal{H} \) is trivialized by its tautological section. We show that the canonical extension of \( \mathcal{H} \) over \( \mathcal{M}_{1,1} \) is trivial and that it and its connection are defined over \( \mathbb{Q} \). Material in this section must surely be well known and classical (19th C).

19.1. \( \mathcal{M}_{1,1} \) as a \( \mathbb{Q} \)-scheme. As explained in Section 1 (also see [9]), \( \mathcal{M}_{1,1} \) is the quotient \( \mathcal{L}_{-1} \) of \( \mathbb{C} \times \mathfrak{h} \) by the action of \( SL_2(\mathbb{Z}) \) which acts with factor of automorphy \((c\tau + d)^{-1}\). It is also the complement in \( \mathbb{C}^2 \) of the discriminant locus \( \Delta = 0 \), where

\[
\Delta = u^3 - 27v^2.
\]

The quotient mapping \( \mathbb{C} \times \mathfrak{h} \to \mathbb{C}^2 - \Delta^{-1}(0) \) is

\[
(\xi, \tau) \mapsto (\xi^4 g_2(\tau), \xi^6 g_3(\tau)),
\]

where

\[
g_2(\tau) = 20(2\pi i)^4 G_4(\tau) \text{ and } g_3 = \frac{7}{3}(2\pi i)^6 G_6(\tau).
\]

The point \((u, v)\) corresponds to the pair

\[
(E_{u,v}, \omega_{u,v}) := (y^2 = 4x^3 - ux - v, dx/y)
\]

consisting of an elliptic curve and the non-zero abelian differential. This elliptic curve has discriminant (divided by 16) equal to

\[
\Delta := u^3 - 27v^2 = \xi^{12}(g_2^3 - 27g_3^2) = (2\pi i\xi)^{12} \Delta_0,
\]

where \( \Delta_0 = q \prod_{n \geq 1} (1 - q)^{24} \) is the Ramanujan \( \tau \)-function. We will view \( \mathcal{M}_{1,1} \) as the \( \mathbb{Q} \)-scheme \( \text{Spec} \mathbb{Q}[u, v, \Delta^{-1}] \).

19.2. Trivializing \( \mathcal{H} \) over \( \mathcal{M}_{1,1} \). To trivialize \( \mathcal{H} \), we need two linearly independent sections. The first is given by the abelian differential \( dx/y \). The second by \( xdz/y \), a differential of the second kind.

Set

\[
\eta_{\tau} = \varphi(z, \tau) dz = \left(1 + 2 \sum_{m=1}^{\infty} \frac{G_{2m+2}(\tau)}{(2m)!} (2\pi i z)^{2m+2}\right) \frac{dz}{z^2}.
\]

This is a differential of the second kind on \( E_{\tau} \).

**Proposition 19.1.** If \( \gamma \in SL_2(\mathbb{Z}) \), then \( \eta_{\gamma \tau} = (c\tau + d)\eta_{\tau} \) and

\[
\int_{E_{\tau}} \omega_{\tau} \sim \eta_{\tau} = 2\pi i.
\]

In particular, \( H^1(E_{\tau}; \mathbb{C}) = \mathbb{C}\omega_{\tau} \oplus \mathbb{C}\eta_{\tau} \) for all \( \tau \).

**Proof.** The first assertion follows easily from the definition of \( \eta_{\tau} \). The second formula follows from a routine residue computation:

Choose a closed disk \( D = \{z : |z| \leq R\} \) in \( E_{\tau} \) about the origin. Let \( F \) be a holomorphic function on \( D \) satisfying \( F'(z) = \varphi(z) \). Let \( \varphi : E_{\tau} \to \mathbb{R} \) be a smooth function that vanishes outside the annulus \( A = \{z : R/3 < |z| < R/2\} \) and is identically 1 when \( |z| < R/3 \). The 1-form

\[
\psi := \eta_{\tau} - d(\varphi F(z))
\]
is smooth and closed. Since it agrees with $\eta_\tau$ outside $A$, it has the same periods as $\eta_\tau$ and thus represents the same cohomology class. Since $\omega_\tau \wedge \psi$ is supported in $D$, we have

$$\langle \omega_\tau \wedge \eta_\tau, E_\tau \rangle = \int_{E_\tau} \omega_\tau \wedge \psi = \int_D dz \wedge \psi = \int_{\partial D} z \psi = \int_{\partial D} z \varphi(z) dz = 2\pi i.$$ 

\[\square\]

Remark 19.2. This implies that the exact sequence

$$0 \to L \to H \to L^{-1} \to 0$$

over $\mathcal{M}_{1,1}$ splits; the copy of $L^{-1}$ in $H$ is spanned locally by $\eta_\tau$. We will see below that this sequence also splits over $\mathcal{M}_{1,1}$. This splitting also follows from the vanishing of $H^1(\mathcal{M}_{1,1}, L_2)$.

Corollary 19.3. The sections $\xi^{-1} \omega_\tau$ and $\xi \eta_\tau$ of $H$ over $C \times \mathfrak{h}$ are $\text{SL}_2(\mathbb{Z})$-invariant. 

For a lattice $\Lambda$ in $C$, set

$$\varphi_\Lambda(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda, \lambda \neq 0} \left[ \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right].$$

The Weierstrass $\wp$-function $\wp(z, \tau)$ defined in Section 8.4 is $\varphi_\Lambda(z)$. One checks easily that

$$\varphi_{\xi^{-1} \Lambda}(\xi^{-1} z) = \xi^2 \varphi_\Lambda(z).$$

Multiplication by $\xi^{-1}$ induces an isomorphism $E_\tau \to \mathbb{C}/\xi^{-1} \Lambda_\tau$ under which $dz$ and $\varphi_{\xi^{-1} \Lambda_\tau} dz$ pull back to $\xi^{-1} \omega_\tau$ and $\xi \eta_\tau$, respectively.

Proposition 19.4. If $(u, v) = (\xi^4 g_2(\tau), \xi^6 g_3(\tau))$, then

(i) the map

$$z \mapsto \left[ \varphi_{\xi^{-1} \Lambda_\tau}(z), \varphi'_{\xi^{-1} \Lambda_\tau}(z), 1 \right]$$

from $\mathbb{C}/\xi^{-1} \Lambda_\tau$ to $\mathbb{P}^2$ induces an isomorphism $\mathbb{C}/\xi^{-1} \Lambda_\tau \to E_{u,v}$.

(ii) Under this isomorphism

$$dx/y = dz = \xi^{-1} \omega_\tau$$

and

$$xdx/y = \varphi_{\xi^{-1} \Lambda_\tau}(z) dz$$

(iii) Under the isomorphism $E_\tau \to \mathbb{C}/\xi^{-1} \Lambda_\tau \to E_{u,v}$, $dx/y$ and $xdx/y$ pull back to $\xi^{-1} \omega_\tau$ and $\xi \eta_\tau$, respectively. 

\[\square\]

This gives the desired trivialization of $H$ over $\mathcal{M}_{1,1}$.

Corollary 19.5. For each $(u, v) \in \mathcal{M}_{1,1}$, the elements $dx/y$ and $xdx/y$ of the fiber $H^1(E_{u,v})$ of $H$ are linearly independent.

Denote these sections of $H$ over $\mathcal{M}_{1,1}$ by $\widehat{T}$ and $\widehat{S}$, respectively. They trivialize $H$ and determine the extension

$$\widehat{H} := \mathcal{O}_{\mathcal{M}_{1,1}} \widehat{S} \oplus \mathcal{O}_{\mathcal{M}_{1,1}} \widehat{T}$$

of $H$ to $\overline{\mathcal{M}_{1,1}}$. 
Proposition 19.6. The connection on $\mathcal{F}$ with respect to this trivialization is

$$\nabla_0 = d + \left( -\frac{1}{12} \frac{d\Delta}{\Delta} \otimes \hat{T} + \frac{3}{2} \frac{\alpha}{\Delta} \otimes \hat{S} \right) \frac{\partial}{\partial T} + \left( -\frac{u}{8} \frac{\alpha}{\Delta} \otimes \hat{T} + \frac{1}{12} \frac{d\Delta}{\Delta} \otimes \hat{S} \right) \frac{\partial}{\partial S}$$

where $\alpha = 2udv - 3vdu$ and $\Delta = u^3 - 27v^2$. This form is logarithmic on $\overline{\mathcal{M}}_{1,1}$ with nilpotent residue along $\Delta = 0$ and is therefore the canonical extension of $\mathcal{H}$ over $\overline{\mathcal{M}}_{1,1}$. It is defined over $\mathbb{Q}$.

Sketch of Proof. We need to understand how the classes $dx/y$ and $xdx/y$ depend on $(u,v)$. Each of the 1-forms

$$\frac{\partial}{\partial u} \left( \frac{dx}{y} \right) = \frac{1}{2} \frac{dx}{y}, \quad \frac{\partial}{\partial v} \left( \frac{dx}{y} \right) = \frac{1}{2} \frac{dx}{y^3}, \quad \frac{\partial}{\partial u} \left( \frac{xdx}{y} \right) = \frac{1}{2} \frac{xdx}{y}, \quad \frac{\partial}{\partial v} \left( \frac{xdx}{y} \right) = \frac{1}{2} \frac{xdx}{y^3}$$

is a differential of the second kind on each $E_{u,v}$. So the cohomology class of each is a linear combination of the classes of $dx/y$ and $xdx/y$.

The differentials $d(1/y)$, $d(x/y)$ and $d(x^2/y)$, the relation $2ydy = (12x^2 - u)dx$, and some linear algebra give

$$\left( \frac{dx}{y} \frac{xdx}{y} \frac{x^2dx}{y^3} \right) \equiv \frac{3}{\Delta} \left( \frac{dx}{y} \frac{xdx}{y} \right) \left( \frac{3v}{2u} \frac{-u^2/6}{-3v} \frac{uv/4}{u^2/6} \right)$$

where $\equiv$ means congruent mod exact forms of the second kind, and thus equal in cohomology.

Now

$$\nabla_0 \left( \frac{\hat{T}}{\hat{S}} \right) = \nabla_0 \left( \frac{dx}{y} \right)$$

is a differential of the second kind on each $E_{u,v}$. So the cohomology class of each is a linear combination of the classes of $dx/y$ and $xdx/y$.

The differentials $d(1/y)$, $d(x/y)$ and $d(x^2/y)$, the relation $2ydy = (12x^2 - u)dx$, and some linear algebra give

$$\frac{\partial}{\partial u} \left( \frac{dx}{y} \right) = \frac{1}{2} \left( \frac{dx}{y} \frac{x^2dx}{y^3} \right) du + \frac{1}{2} \left( \frac{dx}{y} \frac{x^2dx}{y^3} \right) dv$$

$$= \frac{3}{2\Delta} \left( \frac{dx}{y} \frac{x^2dx}{y^3} \right) \left( -3udx + 3vdu/6 - u^6dv/6 \right)$$

$$= \left( \frac{\hat{T}}{\hat{S}} \right) \left( -\frac{1}{2} \frac{dx}{y^3} \frac{x^2dx}{y^3} \right)$$

The forms $\alpha/\Delta$ and $u\alpha/\Delta$ are logarithmic. One can prove this directly. Alternatively, we can use the fact that a meromorphic form $\varphi$ on $\mathbb{C}^2$ has logarithmic singularities along $\Delta = 0$ if and only if $\Delta \varphi$ and $\Delta d\varphi$ are both holomorphic along $\Delta = 0$. This holds in our case as

$$\Delta d(\alpha/\Delta) = -du \wedge dv \quad \text{and} \quad \Delta d(u\alpha/\Delta) = udu \wedge dv.$$  

Similarly, one checks that $w\varphi$ and $wd\varphi$ are holomorphic along the line at infinity, where $w = 0$ is a local defining equation of the line at infinity, then $\varphi$ is logarithmic along the line at infinity. This is easily checked when $\varphi$ is $\alpha/\Delta$ and $u\alpha/\Delta$. □

This implies that the sequence

$$0 \to \overline{\mathcal{C}} \to \overline{\mathcal{H}} \to \overline{\mathcal{C}}_{-1} \to 0$$

splits over $\overline{\mathcal{M}}_{1,1}$. The lift of $\overline{\mathcal{C}}_{-1}$ is $\mathcal{O}_{\overline{\mathcal{M}}_{1,1}} \hat{S}$. 

19.3. Transcendental version. We rederive the formula for the connection in terms of the coordinates \((\xi, \tau) \in \mathbb{C} \times \mathfrak{h}\). This will yield some formulas that are useful in computing the algebraic version of the universal elliptic KZB connection.

As observed above, if \((u, v) = (\xi^4 g_2(\tau), \xi^6 g_3(\tau))\), then \(dx/y = \xi^{-1} \omega_\tau\). Since \(\widehat{T}\) is the class of \(dx/y\), we have
\[
\widehat{T} = \xi^{-1} T.
\]

**Proposition 19.7.** We have
\[
\eta_\tau = (2\pi i)^2 (A - 2G_2(\tau) T)
\]
so that
\[
\widehat{S} = \xi \eta_\tau = (2\pi i)^2 \xi (A - 2G_2(\tau) T).
\]

**Proof.** Using the notation of Example 3.4, we have
\[
\begin{pmatrix} a & t \end{pmatrix} \begin{pmatrix} 1 \\ 8\pi^2 G_2(\tau) \end{pmatrix} = \begin{pmatrix} a' & t' \end{pmatrix} \begin{pmatrix} (ct + d)^{-1} & 0 \\ 2\pi i c & ct + d \end{pmatrix} \begin{pmatrix} 1 \\ 8\pi^2 G_2(\tau) \end{pmatrix} = (ct + d)^{-1} \begin{pmatrix} a' & t' \end{pmatrix} \begin{pmatrix} 1 \\ 8\pi^2 G_2(\gamma \tau) \end{pmatrix}.
\]

From this it follows that
\[
A - 2G_2(\tau) T = (ct + d)^{-1} (A' - 2G_2(\gamma \tau) T').
\]
This implies that \(A - 2G_2(\tau) T\) is a section of \(\mathcal{L}_1\) over \(\mathfrak{h}\). But \(\eta_\tau\) is another such section. It follows that \(\eta_\tau\) is a holomorphic multiple of \(A - 2G_2(\tau) T\). This multiple can be determined by pairing with \(\omega_\tau\). Since
\[
\langle T, A - 2G_2(\tau) T \rangle = \langle \omega_\tau, (2\pi i)^{-1} a \rangle = (2\pi i)^{-1} \text{and } \langle T, \eta_\tau \rangle = \int_{E_\tau} \omega_\tau \sim \eta_\tau = 2\pi i,
\]
it follows that
\[
\eta_\tau = (2\pi i)^2 (A - 2G_2(\tau) T).
\]

There are several ways to prove the second assertion. One is to observe that
\[
\langle \widehat{T}, \widehat{S} \rangle = \langle \frac{dx}{y}, \frac{xdx}{y} \rangle = 2\pi i = \langle T, S \rangle = \langle \xi^{-1} T, \xi S \rangle = \langle \widehat{T}, \xi S \rangle.
\]

We’ve already seen in Example 5.3 that the connection \(\nabla_0\) on \(\mathcal{H}\) over \(\mathfrak{h}\) with respect to the framing \(A, T\) is given by
\[
\nabla_0 = d + 2\pi i A \frac{\partial}{\partial T} \otimes d\tau.
\]

**Proposition 19.8.** With respect to the framing \(\widehat{S}\) and \(\widehat{T}\) of \(\overline{\mathcal{H}}\) over \(\mathcal{M}_{1,1}\), the connection on \(\mathcal{H}\) is
\[
\nabla_0 = d + \left(4\pi i G_2 d\tau - \frac{d\xi}{\xi} \right) \otimes \widehat{T} + \frac{2\pi i}{(2\pi i \xi)^2} d\tau \otimes \widehat{S} \frac{\partial}{\partial T}
\]
\[
- \left((2\pi i \xi)^2 (8\pi i G_2' + 2G_2') d\tau \otimes \widehat{T} + (4\pi i G_2 d\tau - \frac{d\xi}{\xi}) \otimes \widehat{S} \right) \frac{\partial}{\partial S}.
\]
Proof. Since $S/(2\pi i)^2 = A - 2G_2T$,
\[
\nabla_0 \left( T \begin{array}{c} S/(2\pi i)^2 \\ \end{array} \right) = \nabla_0 \left( T \begin{array}{c} A \\ \end{array} \right) \begin{pmatrix} 0 & -2G_2 \\ 1 & 0 \end{pmatrix} + \left( T \begin{array}{c} A \\ \end{array} \right) \begin{pmatrix} 0 & -2G_2' \\ 0 & 0 \end{pmatrix} d\tau \\
= \left( T \begin{array}{c} A \\ \end{array} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2G_2 \\ 2\pi i \end{pmatrix} + \left( T \begin{array}{c} A \\ \end{array} \right) \begin{pmatrix} -2G_2' \\ 2\pi i \end{pmatrix} d\tau \\
= \left( T \begin{array}{c} \hat{S}/(2\pi i)^2 \\ \end{array} \right) \begin{pmatrix} 4\pi iG_2 \\ 2\pi i \end{pmatrix} - \left( \pi_i \hat{G}_2 + 2G_2' \right) d\tau \\

\text{Rescaling, we have}
\[
\nabla_0 \left( T \begin{array}{c} \hat{S} \\ \end{array} \right) = \left( T \begin{array}{c} S \\ \end{array} \right) \begin{pmatrix} 4\pi iG_2 \\ (2\pi i)^{-1} \end{pmatrix} \begin{pmatrix} -2(2\pi i)^2(8\pi iG_2^2 + 2G_2') \\ -4\pi iG_2 \end{pmatrix} d\tau \\

\text{Denote the } 2 \times 2 \text{ matrix of 1-forms in this expression by } B d\tau. \text{ Then}
\[
\nabla_0 \left( \hat{T} \begin{array}{c} \hat{S} \\ \end{array} \right) = \nabla_0 \left( T \begin{array}{c} S \\ \end{array} \right) \begin{pmatrix} -\xi^{-2} & 0 \\ 0 & -\xi^{-1} \end{pmatrix} d\xi \\
= \left( \hat{T} \begin{array}{c} \hat{S} \\ \end{array} \right) \begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi \end{pmatrix} + \left( \begin{array}{c} T \begin{array}{c} S \\ \end{array} \right) \begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi \end{pmatrix} \begin{pmatrix} -\xi^{-2} & 0 \\ 0 & -\xi^{-1} \end{pmatrix} \begin{pmatrix} -\xi^{-2} & 0 \\ 0 & -\xi^{-1} \end{pmatrix} d\xi \\
= \left( \hat{T} \begin{array}{c} \hat{S} \\ \end{array} \right) \begin{pmatrix} -\frac{d\xi}{\xi} + 4\pi iG_2(\tau) d\tau \\ -\frac{d\xi}{\xi} + 4\pi iG_2(\tau) d\tau \end{pmatrix} - 2(2\pi i)^2(4\pi iG_2(\tau)^2 + G_2'(\tau)) d\tau \\

\text{Comparing this formula with that in Proposition 19.6, we conclude:}
\[
(19.1) \quad \frac{d\xi}{\xi} - 4\pi iG_2(\tau) d\tau = \frac{1}{12} d\Delta + \frac{2\pi i}{(2\pi i)^2} d\tau = \frac{3\alpha}{2\Delta} \\
\]
. Taking the quotient of the two off diagonal entries of the connection matrix, we conclude that
\[
G_4(\tau) = \frac{6}{5} \left( 2G_2(\tau)^2 + G_2'(\tau)/2\pi i \right) \\
\text{This can also be verified by observing that the RHS is a modular form of weight 4 and then computing value of both sides at } q = 0.

20. THE Q-DR STRUCTURE ON $\overline{\mathcal{P}}$ OVER $\overline{\mathcal{N}}_{1,1}$

The bundle $\overline{\mathcal{P}}$ over $\overline{\mathcal{N}}_{1,1}$ is the trivial bundle whose fiber is $\mathbb{L}(\hat{S}, \hat{T})^\wedge$. We will define a Q structure on it — that is, a Q structure on its truncations by the terms of its lower central series.

Some preliminary observations will be helpful. Since the cup product of the rational differentials $dx/y$ and $xdx/y$ is $2\pi i$, it is natural to multiply their Poincaré duals by $(2\pi i)^{-1}$ to obtain a Q-de Rham basis of the first homology. Motivated by this, we define
\[
\hat{T}_0 = \hat{T}/2\pi i \text{ and } \hat{S}_0 = \hat{S}/2\pi i.
\]
Since both basis elements of $\overline{\mathcal{N}}$ are multiplied by the same constant, the formula for the connection on $\overline{\mathcal{N}}$ given in Proposition 19.6 remains valid when we replace $\hat{S}$ by $\hat{S}_0$ and $\hat{T}$ by $\hat{T}_0$. 

Define a $\mathbb{Q}$-structure on $\mathfrak{p}$ via the isomorphism

$$\mathfrak{p} = L_{\mathcal{C}}(\hat{S}, \hat{T})^\wedge \cong L_{\mathbb{Q}}(\hat{S}_0, \hat{T}_0)^\wedge \otimes \mathbb{C}.$$  

Set $p_Q = L_{\mathbb{Q}}(\hat{S}_0, \hat{T}_0)^\wedge$. Define the $\mathbb{Q}$-structure on $\mathcal{P}$ to be $\mathcal{M}_{1,1}^{\mathbb{Q}} \times p_Q$.

Define derivations $\hat{\epsilon}_{2m}$ of $p_Q$ by

$$\hat{\epsilon}_{2m} = \left\{ \begin{array}{ll}
\hat{\epsilon}_{2m} \hat{T}_0 \cdot \hat{S}_0 & m = 0; \\
\hat{\epsilon}_{2m} \hat{T}_0 - \sum_{j+k=2m-1} (-1)^j \hat{T}_0^j \cdot \hat{S}_0 \cdot \hat{T}_0^k \cdot \hat{S}_0 & m > 0.
\end{array} \right.$$  

**Lemma 20.1.** For all $m \geq 0$, we have $(2\pi i \xi)^{2m-2} \hat{\epsilon}_{2m} = \epsilon_{2m}$ in $\text{Der} \mathfrak{p}$.

**Proof.** First observe that

$$\epsilon_{2m}(T) = -T^{2m} \cdot A$$

and

$$\hat{\epsilon}_{2m}(\hat{T}_0) = -\hat{T}_0^{2m} \cdot \hat{S}_0$$

and

$$\hat{\epsilon}_{2m}(\hat{T}_0) = \sum_{j+k=2m-1} (-1)^j [\hat{T}_0^j \cdot \hat{S}_0 \cdot \hat{T}_0^k \cdot \hat{S}_0].$$

One checks easily that, when $m \geq 1$, $\epsilon_{2m}(T) = -T^{2m} \cdot (A - 2G_2T)$ and

$$\epsilon_{2m}(A - 2G_2T) = \sum_{j+k \geq 0} (-1)^j [T^j \cdot (A - 2G_2T), T^k \cdot (A - 2G_2T)].$$

The result follows by rescaling as $\hat{T}_0 = T/(2\pi i \xi)$ and $\hat{S}_0 = 2\pi i \xi (A - 2G_2T)$. 

The connection $\nabla_0$ on $\mathcal{H}$ defines, and will be viewed as, a $\mathbb{Q}$-rational connection on each graded quotient of $\mathcal{P}$.

**Theorem 20.2.** With respect to the framing of $\mathcal{P}$ over $\mathcal{M}_{1,1}$ described above, the universal elliptic KZB-connection $\nabla$ is given by

$$\nabla = \nabla_0 + \frac{d\Delta}{12\Delta} \otimes \hat{\epsilon}_2 + \sum_{m \geq 2} \frac{3}{(2m - 2)!} \frac{p_{2m}(u, v)(3uv - 2udv)}{\Delta} \otimes \hat{\epsilon}_{2m}$$

where $\Delta = u^3 - 27v^2$ is the discriminant and where $p_{2m}(u, v) = \mathbb{Q}[u, v]$ is the polynomial characterized by $(2\pi i \xi)^{2m} G_{2m}(\tau) = p_{2m}(u, v)$. The Hodge bundles $\mathcal{F}^p \mathcal{P}$ are all defined over $\mathbb{Q}$.

The polynomial $p_{2m}(u, v)$ is weighted homogeneous of weight $2m$ in $u$ and $v$, where $u$ is given weight 4 and $v$ is given weight 6. The polynomials of weight up to 24 are listed in Appendix B.

**Proof.** With respect to the framing $A$, $T$ of $\mathcal{P}$, the connection is $\nabla = d + \omega'$ where $\omega'$ is the form (13.1):

$$\omega' = -\frac{dq}{q} \otimes \epsilon_0 - \left(2G_2(\tau) \frac{dq}{q} - \frac{d\xi}{\xi}\right) \otimes \epsilon_2 - \sum_{m=2}^{\infty} \frac{2}{(2m - 2)!} G_{2m}(\tau) \frac{dq}{q} \otimes \epsilon_{2m}.$$  

Since the change of frame is homogeneous, the transformed connection is of the form

$$\nabla = \nabla_0 + \omega'.$$
We just need to express \( \omega' \) in the frame given by Lie words in \( \hat{S}_0, \hat{T}_0 \). Using the identities (19.1) and the preceding lemma we have
\[
\omega' = - \left( 2 G_2(\tau) \frac{dq}{q} - \frac{d\xi}{\xi} \right) \otimes \epsilon_2 - \sum_{m=2}^{\infty} \frac{2}{(2m-2)!} G_{2m}(\tau) \frac{dq}{q} \otimes \epsilon_{2m}
\]
\[
= \frac{1}{12} \frac{d\Delta}{\Delta} \otimes \epsilon_2 - \sum_{m \geq 2} \frac{2}{(2m-2)!} (2\pi \xi)^{2m} G_{2m}(\tau) \frac{2\pi i}{(2\pi \xi)^2} d\tau \otimes \epsilon_{2m}
\]
\[
= \frac{1}{12} \frac{d\Delta}{\Delta} \otimes \epsilon_2 - \sum_{m \geq 2} \frac{3}{(2m-2)!} p_{2m}(u,v) \alpha \Delta \otimes \epsilon_{2m}
\]
The last assertion follows from the fact that \( F^q \mathcal{P} \) is trivial and consists of those Lie words whose degree in \( \hat{T}_0 \) is \( \geq -p \). \( \square \)

21. The \( \mathbb{Q} \)-de Rham Structure on \( F^{2n+1} H^1(\mathcal{M}_{1,1}, S^{2n} \mathcal{H}) \)

Here we compute the \( \mathbb{Q} \)-structure on \( F^{2n+1} H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n} \mathcal{H}) \). The computation of the \( \mathbb{R} \)-de Rham structure on all of \( H^1(\mathcal{M}_{1,1}, S^{2n} \mathcal{H}) \) can be found in [11, §17.2].

The starting point is the isomorphisms
\[
H^0(\overline{\mathcal{M}}_{1,1}, L_{2n+2}) \rightarrow H^0(\Omega^1_{\overline{\mathcal{M}}_{1,1}}(P) \otimes F^{2n} S^{2n} \overline{\mathcal{H}}) \rightarrow F^{2n+1} H^1(\mathcal{M}_{1,1}, S^{2n} \mathcal{H}),
\]
where \( P \) denotes the cusp \( q = 0 \). The second isomorphism takes a 1-form to its cohomology class. The first follows from the isomorphisms \( L \cong F^1 \overline{\mathcal{H}} \) and \( \Omega^1_{\overline{\mathcal{M}}_{1,1}}(P) \cong L_2 \), which together induce an isomorphism
\[
L_{2n+2} \cong \Omega^1_{\overline{\mathcal{M}}_{1,1}}(P) \otimes F^{2n} S^{2n} \overline{\mathcal{H}}.
\]

We now explain the \( \mathbb{Q} \)-structure. Recall that if \( f \) is a modular form of weight \( 2n+2 \) of \( \text{SL}_2(\mathbb{Z}) \), then
\[
\omega_f := f(\tau) w^{2n} d\tau \in E^1(\mathfrak{h}) \otimes S^{2n} H
\]
is an \( \text{SL}_2(\mathbb{Z}) \)-invariant 1-form on \( \mathfrak{h} \), where \( w := 2\pi i T \) is the section of \( H \times \mathfrak{h} \rightarrow \mathfrak{h} \) that takes the value \( 2\pi i \omega_\tau \) at \( \tau \in \mathfrak{h} \) and \( \text{SL}_2(\mathbb{Z}) \) acts on \( H = \mathbb{C} A \oplus \mathbb{C} T \) via the factor of automorphy (9.2). It gives a framing of \( F^1 \mathcal{H} \) over \( \mathfrak{h} \). The section \( w \) extends to a framing of \( F^1 \overline{\mathcal{H}} \) over the \( q \)-disk.

Denote the space of modular forms of \( \text{SL}_2(\mathbb{Z}) \) of weight \( m \) whose Fourier coefficients lie in the subfield \( F \) of \( \mathbb{C} \) by \( M_{m,F} \). These form a graded ring \( M_{*,F} \) isomorphic to \( F[G_4, G_6] \).

**Proposition 21.1.** The \( \mathbb{Q} \)-structure on \( F^{2n+1} H^1_{\text{dR}}(\mathcal{M}_{1,1}, S^{2n} \mathcal{H}) \) is
\[
\{ f(q) w^{2n} \frac{dq}{q} : f(q) \in M_{2n+2,\mathbb{Q}} \}.
\]

**Proof.** Embed \( E_\tau \) into \( \mathbb{P}^2 \) via the mapping
\[
z + \Lambda_\tau \mapsto [\psi_\tau(z)/(2\pi i)^2, \psi'_\tau(z)/(2\pi i)^3, 1].
\]
The image is the plane cubic \( y^2 = 4x^3 - ux - v \) where
\[
u = g_2(\tau)/(2\pi i)^4 = 20G_4(\tau) \quad \text{and} \quad v = g_3(\tau)/(2\pi i)^6 = \frac{7}{3} G_6(\tau).
\]
\[13\text{Recall the definitions (12.1).} \]
This curve has discriminant $\Delta_0(\tau)$, where $\Delta_0$ denotes the normalized cusp form of weight 12.

With this normalization $dx/y = 2\pi i \omega_\tau = w(\tau)$.\footnote{This would have been a better normalization to use in Part 4.} We choose it because the value of the section $w$ at $q = 0$ is $dw/w$, where $w$ is the parameter on the nodal cubic that maps 0 and $\infty$ to the node and 1 to the identity. (Cf. Exercise 47 in [9].)

We regard $w$ as the section $dx/y$ of $F^1\mathcal{H}$ over $\mathcal{M}_{1,\tilde{T}} = \mathcal{H}^2_3 - \{u^3 - 27v^2 = 0\}$. It is defined over $\mathbb{Q}$. Since $x$ has weight 2 and $y$ weight 3, it has weight 1 under the $G_m$ action. If $h(u, v) \in \mathbb{Q}[u, v]$ is a polynomial of weight 2$n + 2$ (where $u$ has weight 4 and $v$ weight 6), then

$$
\frac{h(u, v)}{u^3 - 27v^2} (2uv - 3vd)w^{2n}
$$

is a $\mathbb{Q}$-rational, $G_m$-invariant section of $\Omega^1_{\mathcal{M}_{1,\tilde{T}}/\mathbb{Q}}(\log D) \otimes F^{2n}S^{2n}\mathcal{H}$ over $\mathcal{M}_{1,\tilde{T}}/\mathbb{Q}$, where $D$ denotes the discriminant locus $u^3 - 27v^2 = 0$. Since $\mathcal{M}_{1,1/\mathbb{Q}} = \mathbb{G}_m \setminus \mathcal{M}_{1,\tilde{T}}/\mathbb{Q}$ it descends to a section of $\Omega^1_{\mathcal{M}_{1,1}/\mathbb{Q}}(P) \otimes F^{2n}S^{2n}\mathcal{H}$.

The identity (19.1) implies that the pullback of this form along the map $h \to \mathcal{M}_{1,\tilde{T}}$ defined by $\tau \mapsto (20G_4(\tau), 7G_6(\tau)/3)$ is

$$
\frac{2}{3} h(20G_4(\tau), 7G_6(\tau)/3)w^{2n}\mathcal{d}q/\mathcal{q}.
$$

The result follows as $M_{2n+2,\mathbb{Q}}$ is isomorphic to the polynomials in $G_4$ and $G_6$ with rational coefficients. \hfill \Box

APPENDIX A. AN IDENTITY

In this appendix we show that the relation $N_\mathcal{Q}(R_0) = 0$ proved in Corollary 15.2 is equivalent to an identity involving Bernoulli numbers. We also give an elementary proof of this identity, which gives a proof of the vanishing of $N_\mathcal{Q}(R_0)$ that does not use limit MHSs.

For non-negative integers, define polynomials

$$
h_{a,b}(x, y) = x^{2a-1}y^{2b} - x^{2b}y^{2a-1} + xy(x + y)^{2b-1}(y^{2a-2} - x^{2a-2})
= xy(y - x)\left(2^{a-2}y^{2b-2} + (x + y)^{2b-1}\sum_{i+j=2a-3} x^iy^j\right),
$$

in commuting indeterminates $x$ and $y$. Note that

$$
h_{0,n}(x, y) = -\sum_{i+j=2n-1}^{i+j=2n-1} \left[ \binom{2n}{i+1} - \binom{2n}{j+1} \right] x^iy^j
$$

for all $n \geq 1$.

**Theorem A.1.** For all $n \geq 1$,

$$
\sum_{a+b=n, a>0}^{(2a-1)(2a)} \frac{B_{2a}B_{2b}}{B_{2n}} h_{a,b}(x, y) = \sum_{i+j=2n-1}^{i+j=2n-1} \left[ \binom{2n}{i+1} - \binom{2n}{j+1} \right] x^iy^j \in \mathbb{Z}[x, y].
$$
Equivalently,

\[ \sum_{a+b=n \atop a, b \geq 0} (2a - 1) \frac{B_{2a}}{(2a)!} \frac{B_{2b}}{(2b)!} h_{a,b}(x, y) = 0. \]

**Proof.** It suffices to show that

\[ \sum_{n \geq 0 \atop a+b=n \atop a, b \geq 0} (2a - 1) \frac{B_{2a}}{(2a)!} \frac{B_{2b}}{(2b)!} h_{a,b}(x, y) = 0. \]

(A.1)

Observe that

\[ \sum_{n \geq 0 \atop a+b=n \atop a, b \geq 0} (2a - 1) \frac{B_{2a}}{(2a)!} \frac{B_{2b}}{(2b)!} x^{2a-1} y^{2b} = \left( \sum_{a \geq 0} (2a - 1) \frac{B_{2a}}{(2a)!} x^{2a-1} \right) \left( \sum_{b \geq 0} \frac{B_{2b}}{(2b)!} y^{2b} \right) \]

\[ = \frac{4u}{\sinh^2(u/2)} \left( v e^v - 1 + \frac{v}{e^v - 1} \right) \]

\[ = \frac{2e^u}{(e^v - 1)^2} \left( v e^v - 1 + \frac{v}{e^v - 1} \right). \]

Denote this function of \((u, v)\) by \(F(u, v)\). The series (A.1) is then

\[ F(x, y) - F(y, x) + \frac{x}{x+y} F(y, x+y) - \frac{y}{x+y} F(x, x+y) \]

which is easily to vanish by elementary algebraic manipulations. \( \square \)

We finish by showing that this identity is equivalent to the vanishing of \(N_q(R_0)\).

For this we need to relate polynomials to the free Lie algebra \(\mathbb{L}(A, T)\). For this we use the Levin-Racinet calculus [16, §3.1]. Recall from Section 9.4.1 that for \(U, V \in \mathbb{L}(A, T)\),

\[ x^r y^s \circ (U, V) = [T^r \cdot U, T^s \cdot V]. \]

This extends linearly to an action \(f(x, y) \circ (U, V)\) of polynomials \(f(x, y)\) in commuting indeterminates on ordered pairs of elements of \(\mathbb{L}(T, A)\). When \(U\) and \(V\) are equal, one has the identity \(f(x, y) \circ (U, U) = -f(y, x) \circ (U, U)\), so that

\[ 2f(x, y) \circ (U, U) = (f(x, y) - f(y, x)) \circ (U, U). \]

In this case we need only consider polynomials \(f(x, y)\) satisfying \(f(x, y) + f(y, x) = 0\).

The significance of the polynomials \(h_{a,b}(x, y)\) is given by:

**Lemma A.2.** For all \(a \geq 0\) and \(b \geq 0\) with \(a + b > 0\),

\[ 2 \epsilon_{2a}(T^{2b} \cdot A) = h_{a,b}(x, y) \circ (A, A). \]

**Proof.** Observe that \(h_{a,b}(x, y) = f_{a,b}(x, y) - f_{a,b}(y, x)\) where

\[ f_{a,b}(x, y) = x^a (x + y)^{2b-1} (x^a - (-y)^a) - x^{2a-1} (x + y)^{2b} - y^{2b}. \]

The result now follows from the easily verified identity

\[ \epsilon_{2a}(T^{2b} \cdot A) = f_{a,b}(x, y) \circ (A, A). \]

\( \square \)
Corollary A.4. If \( \text{inductive step.} \) it cannot be in im ad \( T \).

So \( \langle T, A \rangle \) denotes the two sided ideal of \( \mathbb{L}(T, A) \).

\[ \langle T^i \cdot A, T^j \cdot A \rangle : i > j > 0, \ i + j = n \]

is a linearly independent subset of \( \mathbb{L}(T, A) \).

Proof. Suppose that \( n \geq 0 \). Let \( V_n \) be the subspace of \( \mathbb{L}(T, A) \) spanned by \([T^j \cdot A, T^k \cdot A]\) where \( j > k \geq 0 \) and \( j + k = n \). Then \( \dim V_n \leq \lfloor (n+1)/2 \rfloor \). To prove the result, we have to show that this is an equality. It is when \( n = 1 \). The endomorphism ad\(_T\) of \( \mathbb{L}(T, A) \) induces linear maps ad\(_T\) : \( V_n \to V_{n+1} \). This is injective for all \( n \geq 0 \) as the centralizer of \( T \) in \( \mathbb{L}(T, A) \) is \( \mathbb{Q}T \). So \( \dim V_n \leq \dim V_{n+1} \) for all \( n \geq 1 \). Assume by induction that \( \dim V_n = \lfloor (n+1)/2 \rfloor \).

If \( n \) is odd, then \( \dim V_{n+1} \leq \lfloor (n+2)/2 \rfloor = \dim V_n \). Since \( \dim V_n \leq \dim V_{n+1} \), this must be an equality.

The argument when \( n \) is even is more involved. Suppose that \( n \) is even. Then \( \dim V_n \leq \dim V_{n+1} \leq 1 + \dim V_n \).

If \( \dim V_n = \dim V_{n+1} \), then \([T^{n+1} \cdot A, A] \in \text{im ad}_T\). But since

\[ \text{ad}_T[T^j \cdot A, T^k \cdot A] = [T^{j+1} \cdot A, T^k \cdot A] + [T^j \cdot A, T^{k+1} \cdot A], \]

this would imply that

\[ [T^{n+1} \cdot A, A] \in \text{span}\{[T^j \cdot A, T^k \cdot A] : j > k > 0, \ j + k = n + 1 \} \]

A short inductive argument shows that in \( \mathbb{Q}(T, A) \)

\[ T^i \cdot A \equiv T^i A \mod (AT), \]

where \((AT)\) denotes the two sided ideal of \( \mathbb{Q}(T, A) \) generated by \( AT \). So, mod \((AT)\),

\[ [T^j \cdot A, T^k \cdot A] \equiv T^j AT^k A - T^k AT^j A \]

\[ \equiv \begin{cases} T^j A^2 & k = 0 \\ 0 & j, k > 0. \end{cases} \]

So \([T^{n+1} \cdot A, A]\) cannot be in span of \([T^j \cdot A, T^k \cdot A]\) with \( j > k > 0 \). And so it cannot be in im ad\(_T\). It follows that \( \dim V_{n+1} = 1 + \dim V_n \). This completes the inductive step.

\[ \text{Corollary A.4. If } f(x, y) \in \mathbb{C}[x, y] \text{ satisfies } f(x, y) + f(y, x) = 0, \text{ then} \]

\[ f(x, y) \circ (A, A) = 0 \text{ if and only if } f(x, y) = 0. \]

Since \( N_q(R_0) = 0 \), the degree \( n \) part of the identity

\[ N_q(R_0) = \frac{1}{2} \sum_{n \geq 0} \left( \sum_{a+b=n \atop a, b \geq 0} (2a - 1) \frac{B_{2a}}{(2a)!} \frac{B_{2b}}{(2b)!} h_{a,b}(x, y) \right) \circ (A, A) \]
yields the identities
\[ \sum_{a+b=n \atop a,b \geq 0} (2a - 1) \frac{B_{2a}}{(2a)!} B_{2b} \frac{B_{2b}}{(2b)!} h_{a,b}(x, y). \]

Thus Corollary 15.2 and Theorem A.1 are equivalent.

Appendix B. Polynomials

The polynomials \( p_{2m}(u, v) \in \mathbb{Q}[u, v] \) are defined by
\[ (2 \pi i \xi)^{2m} G_{2m}(\tau) = p_{2m}(u, v), \]
where \( (u, v) = (\xi^4 g_2(\tau), \xi^6 g_3(\tau)) \). Equivalently,
\[ G_{2m}(\tau) = p_{2m}(20G_4(\tau), 7G_6(\tau)/3). \]

Note that \( p_{2m}(u, v) \) is of weight \( 2m \) where \( u \) is given weight 4 and \( v \) is given weight 6. Below is a table up to weight 24.

\[
\begin{align*}
p_4(u,v) &= \frac{1}{20} u \\
p_6(u,v) &= \frac{3}{7} v \\
p_8(u,v) &= \frac{3}{10} u^2 \\
p_{10}(u,v) &= \frac{108}{11} uv \\
p_{12}(u,v) &= \frac{756}{65} u^3 + \frac{16200}{91} v^2 \\
p_{14}(u,v) &= 1296 uv^2 \\
p_{16}(u,v) &= \frac{174636}{85} u^4 + \frac{1166400}{17} uv^2 \\
p_{18}(u,v) &= \frac{9471168}{19} u^5 + \frac{256608000}{133} v^3 \\
p_{20}(u,v) &= \frac{25147584}{25} u^6 + \frac{678844800}{11} u^2 v^2 \\
p_{22}(u,v) &= \frac{10671720192}{23} u^7 v + \frac{103296384000}{23} u^4 v^2 \\
p_{24}(u,v) &= \frac{73581830784}{65} u^8 + \frac{141087440000}{13} u^3 v^2 + \frac{15547365504000}{91} v^4
\end{align*}
\]

Appendix C. The Universal Elliptic Curve over \( B\mathbb{P}^1_0 \mathbb{D} \)

Here we justify the claim, made at the end of Section 16, that the fiber of the universal elliptic curve over \( e^{i\theta} \frac{\partial}{\partial q} \) is obtained from the real oriented blow up of \( \mathbb{P}^1 \) at \( \{0, \infty\} \) by identifying its two boundary components with a suitable twist.

The real oriented blow up of a Riemann surface \( X \) at a finite subset \( S \) will be denoted by \( B\mathbb{P}^1_S X \). This is a bordered Riemann surface with one boundary circle for each point of \( S \). There is a continuous projection \( \pi : B\mathbb{P}^1_S X \to X \) that induces a biholomorphism
\[ B\mathbb{P}^1_S X - \partial B\mathbb{P}^1_S X \to X - S \]
The fiber of $\pi$ over $P \in S$ is the quotient of $T_P X - \{0\}$ by the multiplicative group of positive real numbers.

The map $[0,1] \times S^1 \to \mathbb{D}$ that takes $(r, \theta)$ to $e^{i\theta}$ is the real oriented blow up $\text{Bl}_0^o \mathbb{D} \to \mathbb{D}$ of the disk. More generally, the map

$$S^1 \times [0,1] \to \mathbb{P}^1$$

defined by $(\phi, t) \mapsto [te^{i\theta}, 1-t]$ is $\text{Bl}_0^{o,\infty} \mathbb{P}^1 \to \mathbb{P}^1$. With this identification, the inclusion $\mathbb{C}^* \to \text{Bl}_0^{o,\infty} \mathbb{P}^1$ takes $se^{i\phi}$ to $(\phi, s/(1+s))$.

The fiber of the universal elliptic curve over $q = re^{i\theta}$ is the quotient of

$$A := \{(w,q) \in \mathbb{C}^* \times \mathbb{D}^*: \sqrt{|q|} \leq |w| \leq 1/\sqrt{|q|} \}$$

obtained by gluing $w$ to $qw$ when $|w| = 1/\sqrt{|q|}$. Write $w = se^{i\phi}$ so that we can identify $A$ with

$$\{(s,\phi, re^{i\theta}) \in \mathbb{R} \times S^1 \times \mathbb{D}^*: \sqrt{r} \leq s \leq 1/\sqrt{r} \}.$$  

With this identification, $(1/\sqrt{r}, \phi, re^{i\theta})$ is glued to $(\sqrt{r}, \phi + \theta, re^{i\theta})$.

The function

$$h(r,s) = \left( \frac{s}{1+s} - \frac{\sqrt{r}}{1+\sqrt{r}} \right) \left( \frac{1}{1+\sqrt{r}} - \frac{\sqrt{r}}{1+\sqrt{r}} \right)^{-1}$$

induces homeomorphisms $h(r,\ ) : [\sqrt{r},1/\sqrt{r}] \to [0,1]$ for all $r \geq 0$. It has inverse $k(r,\ )$, where

$$k(r,t) = \frac{\sqrt{r}-(\sqrt{r}-1)t}{1+(\sqrt{r}-1)t}$$

Define an equivalence relation on $B := S^1 \times S^1 \times [0,1) \times [0,1)$ by

$$(\phi, \theta, 1, r) \sim (\phi + \theta, \theta, 0, r).$$

Set $B = B/\sim$. The map $(\phi, \theta, t, r) \to (r,\theta)$ defines a projection $\pi : B \to \text{Bl}_0^{o,\infty} \mathbb{D}$. This is a torus bundle over $\text{Bl}_0^{o,\infty} \mathbb{D}$. Its fiber $B_\theta$ over $(0,\theta) \in \text{Bl}_0^{o,\infty} \mathbb{D}$ is the quotient of $\text{Bl}_0^{o,\infty} \mathbb{P}^1$ obtained by identifying the two boundary components by a twist by $\theta$. The inclusion $A \hookrightarrow B$ defined by $(s,\phi, re^{i\theta}) \mapsto (\phi, \theta, h(r,s), r)$ induces a map $\mathcal{E}_\mathbb{D} \to B$ that commutes with the projections to $\mathbb{D}$ and is a homeomorphism into its image.

The map $B \to \mathbb{C}^* \times \mathbb{D}$ that takes $(\phi, \theta, t, r)$ to $(k(r,t)e^{i\theta}, re^{i\theta})$ induces a map $B \to \mathcal{E}_\mathbb{D}$ such that the diagram

$$
\begin{array}{ccc}
B & \longrightarrow & \mathcal{E}_\mathbb{D} \\
\downarrow & & \downarrow \\
\text{Bl}_0^{o,\infty} \mathbb{D} & \longrightarrow & \mathbb{D}
\end{array}
$$

commutes. For each $\theta \in S^1$, the composite $\mathbb{C}^* \hookrightarrow \text{Bl}_0^{o,\infty} \mathbb{P}^1 \to B_\theta \to E_0$ is the natural inclusion of the smooth locus of the nodal cubic $E_0$ given by the parameter $w = se^{i\phi}$. The map $\text{Bl}_0^{o,\infty} \mathbb{P}^1 \to B_\theta \to E_0$ collapses the boundary of $\text{Bl}_0^{o,\infty} \mathbb{P}^1$ to the double point of $E_0$. 


References

[1] F. Brown: Mixed Tate motives over \( \mathbb{Z} \), Ann. of Math. 175 (2012), 949–976.
[2] D. Calaque, B. Enriquez, P. Etingof: Universal KZB equations: the elliptic case, in Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, 165–266, Progr. Math., 269, Birkhäuser, Boston, 2009, [arXiv:math/0702670]
[3] P. Deligne: Équations différentielles à points singuliers réguliers, Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, 1970.
[4] H. Furusho: The multiple zeta value algebra and the stable derivation algebra, Publ. Res. Inst. Math. Sci. 39 (2003), 695–720. [arXiv:math/0011261]
[5] R. Hain: The geometry of the mixed Hodge structure on the fundamental group, Algebraic Geometry, 1985, Proc. Symp. Pure Math. 46 (1987), 247–282.
[6] R. Hain: The de Rham homotopy theory of complex algebraic varieties, I. K-Theory 1 (1987), 271–324.
[7] R. Hain: The de Rham homotopy theory of complex algebraic varieties, II. K-Theory 1 (1987), 481–497.
[8] R. Hain: Relative weight filtrations on completions of mapping class groups, in Groups of Diffeomorphisms, Advanced Studies in Pure Mathematics, vol. 52 (May, 2008), 309–368, Mathematical Society of Japan, [arXiv:0802.0814]
[9] R. Hain: Lectures on Moduli Spaces of Elliptic Curves, in Transformation Groups and Moduli Spaces of Curves, Advanced Lectures in Mathematics, edited by Lizhen Ji, S.-T. Yau no. 16 (2010), 95–166, Higher Education Press, Beijing, [arXiv:0812.1803]
[10] R. Hain: Limit Structures on Unipotent Fundamental Groups of Stable Curves, in preparation, 2013.
[11] R. Hain: The Hodge-de Rham Theory of Modular Groups, preprint, 2013. [arXiv #]
[12] R. Hain, M. Matsumoto: Universal mixed elliptic motives, in preparation.
[13] R. Hain, M. Matsumoto: A Note on Derivations of the Free Lie Algebra of Rank Two, in preparation.
[14] R. Hain, S. Zucker: Unipotent variations of mixed Hodge structure, , Invent. Math. 88 (1987), 83–124.
[15] Leopold Kronecker: Zur theorie der elliptischen Funktionen (1881). Reproduced in: Leopold Kronecker’s Werke Volume IV, pp. 313–318. Herausgegeben auf Veranlassung der Königlich Preussischen Akademie der Wissenschaften von K. Hensel, Chelsea, New York 1968.
[16] A. Levin, G. Racinet: Towards multiple elliptic polylogarithms, an unfortunately unpublished preprint, 2007, [arXiv:math/0703237]
[17] A. Pollack: Relations between derivations arising from modular forms, undergraduate thesis, Duke University, 2009. 15
[18] J.-P. Serre: A course in arithmetic, Translated from the French. Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg, 1973.
[19] J.-P. Serre: Lie algebras and Lie groups, Lectures given at Harvard University, 1964, Benjamin, 1965. (Second edition, Springer Verlag, 1992.)
[20] J. Silverman: Advanced Topics in the Arithmetic of Elliptic Curves, Graduate Texts in Mathematics, 151. Springer-Verlag, New York, 1994.
[21] H. Tsunogai: On some derivations of Lie algebras related to Galois representations, Publ. Res. Inst. Math. Sci. 31 (1995), 113–134.
[22] D. Zagier: Periods of modular forms and Jacobi theta functions, Invent. Math. 104 (1991), 449–456.

\text{Department of Mathematics, Duke University, Box 90320, Durham, NC 27708}

\text{E-mail address: hain@math.duke.edu}

\text{\footnotesize{\textsuperscript{15}}}Available at: \texttt{http://dukespace.lib.duke.edu/dspace/handle/10161/1281}