MONOTONE COVERING PROPERTIES DEFINED BY CLOSURE-PRESERVING OPERATORS

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Abstract. We continue Gartside, Moody, and Stares’ study of versions of monotone paracompactness. We show that the class of spaces with a monotone closure-preserving open operator is strictly larger than those with a monotone open locally-finite operator. We prove that monotonically metacompact GO-spaces have a monotone open locally-finite operator, and so do GO-spaces with a monotone (open or not) closure-preserving operator, whose underlying LOTS has a $\sigma$-closed-discrete dense subset. A GO-space with a $\sigma$-closed-discrete dense subset and a monotone closure-preserving operator is metrizable. A compact LOTS with a monotone open closure-preserving operator is metrizable.

Dedicated to our mentor and dear friend Gary Gruenhage with gratitude and well-wishes on the occasion of his 70-th birthday.

1. Introduction

Gartside and Moody [7, Theorem 1] proved that a space is protometrizable if and only if the space has a monotone star-refinement operator, and asked whether the class of protometrizable spaces coincided with the class of spaces with a monotone open locally-finite operator.

Definition 1.1. [7] A monotone open locally-finite operator is a function $r : \mathcal{C} \to \mathcal{C}$, where $\mathcal{C}$ is the set of all open covers of $X$, such that (1) for every $U \in \mathcal{C}$, $r(U)$ is a locally-finite open refinement of $U$, and (2) if $U, V \in \mathcal{C}$ and $U$ refines $V$, then $r(U)$ refines $r(V)$.

In [15], Stares showed that different characterizations of paracompact spaces, when monotonized, may give rise to different classes of spaces, and asked which monotonized characterizations coincide. The authors showed in [14] that the class of spaces with a monotone open locally-finite operator is strictly larger than the class of protometrizable spaces.

It is well-known (E. Michael, [5, 5.1.G]) that if every open cover of a regular $T_1$ space $X$ has a closure preserving refinement (of arbitrary sets), then $X$ is paracompact, i.e. every open cover has an open locally-finite (and hence open closure-preserving) refinement. Recall that a family $\mathcal{F}$ of subsets of a space $X$ is called closure-preserving if $\overline{\bigcup \mathcal{H}} = \bigcup\{\overline{H} : H \in \mathcal{H}\}$ for every subfamily $\mathcal{H} \subseteq \mathcal{F}$. Extending Gartside, Moody, and Stares’ study, we explore spaces with a monotone closure-preserving operator.

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1 A monotone star-refinement operator is a function $r : \mathcal{C} \to \mathcal{C}$ (where $\mathcal{C}$ is the set of all open covers of $X$) such that (1) for every $U \in \mathcal{C}$, $r(U)$ star-refines $U$, and (2) if $U, V \in \mathcal{C}$ and $U$ refines $V$, then $r(U)$ refines $r(V)$. 

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Definition 1.2. A space $X$ is said to have a monotone closure-preserving operator $r$ if for every open cover $\mathcal{U}$ we have that $r(\mathcal{U})$ is a closure-preserving cover (of arbitrary sets) that refines $\mathcal{U}$, and if $\mathcal{U}, \mathcal{V}$ are open covers such that $\mathcal{U}$ refines $\mathcal{V}$, then $r(\mathcal{U})$ refines $r(\mathcal{V})$. If, in addition, each $U \in r(\mathcal{U})$ is required to be open then $r$ is called a monotone open closure-preserving operator for $X$.

In a similar manner, we can distinguish between monotone (not necessarily open) locally-finite operators and monotone open locally-finite operators. We do not know if the existence of such monotone operators necessarily implies the existence of open ones. We provide partial answers to the following two questions in Theorem 2.8 and Theorem 3.10.

Question 1.3. (a) If a space $X$ has a monotone closure-preserving operator, must it have a monotone closure-preserving open operator? (b) What if $X$ is a GO-space, or (c) if $X$ is a compact LOTS?

Question 1.4. If a space has a monotone (not necessarily open) locally-finite operator, must it have a monotone open locally-finite operator?

The following proposition shows that spaces with a monotone open closure-preserving operator form a broader class than those with a monotone open locally-finite operator.

Theorem 1.5. Any space with only one non-isolated point has a monotone open closure-preserving operator.

Proof. Suppose $p$ is the only non-isolated point of a topological space $X$. If $\mathcal{U}$ is an open cover of $X$, let $U_p = \{U \in \mathcal{U} : p \in U\}$. It is easy to check that $r(\mathcal{U}) = U_p \cup \{\{x\} : x \in X \setminus \bigcup U_p\}$ is the required monotone open closure-preserving operator. □

Definition 1.6. [1, 12] A space is monotonically (countably) metacompact if there is a function $r$ that assigns to each (countable) open cover $\mathcal{U}$ of $X$ a point-finite open refinement $r(\mathcal{U})$ covering $X$ such that if $\mathcal{V}$ is a (countable) open cover of $X$ and $\mathcal{U}$ refines $\mathcal{V}$, then $r(\mathcal{U})$ refines $r(\mathcal{V})$.

Clearly a monotone open locally-finite operator is both closure-preserving and point-finite. The one-point compactification of an uncountable discrete set of size $\kappa \geq \omega_1$ and the Sequential Fan are examples of spaces that have a monotone open closure-preserving operator but no monotone locally-finite open operator nor point-finite open operator (see [3] Theorem 2.4 and [4] Theorem 3.2, respectively). This shows the metrization theorems of Chase and Gruenhage [3, 4] on compact or separable monotonically metacompact spaces do not extend to spaces with monotone open closure-preserving operators.

However, GO-spaces with a monotone closure-preserving operator do behave similarly to monotonically metacompact GO-spaces. By modifying results in [1] and [11], in Section 2 we show that GO-spaces with a monotone closure-preserving operator are monotonically metacompact when the underlying LOTS has a $\sigma$-closed-discrete dense subset. Moreover, monotonically metacompact GO-spaces have a monotone open locally-finite operator.

Section 3 is devoted to metrization results. GO-spaces with a $\sigma$-closed-discrete dense subset and a monotone closure-preserving operator are metrizable, while every compact LOTS with a monotone open closure-preserving operator is metrizable.
2. Monotone Operators in GO-Spaces

For any GO-space \((X, \tau, <)\), we use following notation found in [1]:

\[
\begin{align*}
L_r &= \{ x \in X : \{ x \} \in \tau \}; \\
R_r &= \{ x \in X \setminus L_r : \{ x, \to \} \in \tau \}; \\
L_r &= \{ x \in X \setminus L_r : (\leftarrow, x) \in \tau \}; \\
E_r &= X \setminus (I_r \cup R_r \cup L_r) = \{ x \in X : \text{neither } [x, \to) \text{ nor } (\leftarrow, x) \text{ is open} \}.
\end{align*}
\]

For a non-empty subset \(A \subseteq X\), let \(l_A = \inf(A)\) and \(u_A = \sup(A)\) which may be gaps in \(X\), and define \([A] = [l_A, u_A] = \{ x \in X : l_A \leq x \leq u_A \}\). Let \(conv(A)\) denote the convex hull of \(A\), that is, \(conv(A) = \bigcup \{ [p, q] : p, q \in A, p \leq q \}\). Clearly \(conv(A)\) is one of the sets \([l_A, u_A]\), \([l_A, u_A]\), \([l_A, u_A]\), or \((l_A, u_A)\), depending on which of \(l_A\) and \(u_A\) belong to \(A\). It is easily seen that if \(A\) is open, then so is \(conv(A)\).

It is well-known that metacompact GO-spaces (and more generally metacompact collectionwise normal spaces) are paracompact [5, Theorem 5.3.3]. A monotone version of this result holds for GO-spaces, and partially answers our Question 2.6(b) in [14].

**Theorem 2.1.** Suppose that \((X, \tau, <)\) is a GO-space. If \(X\) is monotonically metacompact then it has a monotone open locally-finite operator.

**Proof.** Let \(r\) be a monotone metacompactness operator for \(X\). For every open set \(U\) let \(I(U)\) be the family of all convex components of \(U\), where \(C \subseteq U\) is a convex component if \(C\) is convex, and maximal with respect to set inclusion (if \(C \subseteq C_1 \subseteq U\) where \(C_1\) is convex then \(C = C_1\)). For each open cover \(U\) let \(r_1(U) = \bigcup\{ I(U) : U \subseteq r(U) \}\), that is, we replace \(r(U)\) with the cover of all convex components of elements of \(r(U)\). Then \(r_1\) is also a monotone metacompactness operator, and \(r_1(U)\) consists of convex open sets. Let \(r_2(U) = \{ U \subseteq r_1(U) : U\) is maximal in \(r_1(U)\) with respect to set inclusion \} (where \(U \subseteq r_1(U)\) is maximal if \(U \subseteq V \in r_1(U)\) implies \(U = V\)). Since \(r_1(U)\) is point-finite, it contains no \(\subseteq\)-strictly increasing sequences. Hence every element of \(r_1(U)\) is contained in a maximal one, and \(r_2(U)\) covers \(X\). Clearly, \(r_2(U)\) is point-finite.

If \(r_2(U)\) were not locally-finite, then we may fix \(p \in X\) and a family \(F \subseteq r_2(U)\) that is not locally-finite at \(p\) and such that \(p \notin V\) and \(u_V = \sup V \leq p\) for all \(V \in F\) or \(l_V = \inf V \geq p\) for all \(V \in F\). Consider the former case (the other being dealt with similarly). Then \(p \in L_r \cup E_r\), and there is some \(G \in r_2(U)\) and \(h < p\) such that \([h, p] \subseteq G\). Every \(V \in F\) is convex and \(l_V = \inf V \leq h\) (for otherwise \(V \subseteq G\)). There are infinitely many \(V \in F\) with \(h < u_V\), hence \(F\) is not point-finite at \(h\), a contradiction. Therefore, \(r_2(U)\) must be locally-finite.

We do not know if the assumption that \(X\) is a GO-space could be weakened to the assumption that \(X\) is monotonically normal, or dropped altogether.

**Question 2.2.** If \(X\) is monotonically metacompact and monotonically normal, must it have an (open or not): (a) monotone locally-finite, or (b) monotone closure-preserving operator?
One cannot strengthen the conclusion of Theorem 2.1 to protometrizable spaces. The authors [14, Example 2.1] provided an example of a LOTS with a monotone open locally-finite operator that is not protometrizable.

The following characterization of monotone metacompactness is known for GO-spaces $(X, \tau, <)$ for which the underlying LOTS $(X, \lambda, <)$ has a $\sigma$-closed-discrete dense subset.

**Theorem 2.3. [11, Theorem 1.4], [11, Theorem 12].** Let $(X, \tau, <)$ be a GO-space whose underlying LOTS $(X, \lambda, <)$ has a $\sigma$-closed-discrete dense subset. Then the following are equivalent:

(a) $(X, \tau)$ is monotonically metacompact;
(b) $(X, \tau)$ is monotonically countably metacompact;
(c) the set $R_\tau \cup L_\tau$ is $\sigma$-closed-discrete in $(X, \tau)$;
(d) the set $R_\tau \cup L_\tau$ is $\sigma$-closed-discrete in $(X, \lambda)$.

The Michael line $M$ satisfies all conditions above, even though $M$ itself has no $\sigma$-closed-discrete dense subset. It is also protometrizable – equivalent to having a monotone star-refinement operator [7, Theorem 1], and has a monotone open locally-finite operator [14, Corollary 1.7]. The following proposition was used in the proof of Theorem 2.3.

**Proposition 2.4. [11, Proposition 3.8], [11, Proposition 13].** Suppose $(X, \tau, <)$ is a GO-space for which the underlying LOTS $(X, \lambda, <)$ has a $\sigma$-closed-discrete dense set. If $(X, \tau)$ is monotonically countably metacompact, then $R_\tau \cup L_\tau$ is $\sigma$-closed-discrete as a subspace of $(X, \tau)$ and as a subspace of $(X, \lambda)$.

We will prove a similar result for spaces with a monotone closure-preserving operator $r$, but first we need to modify $r$.

**Lemma 2.5.** Suppose $(X, \tau, <)$ is a GO-space with a monotone closure-preserving operator $r$. Then $X$ has a closed convex monotone closure-preserving operator $\bar{r}$ such that, for every open cover $U$:

(a) if $x \in R_\tau \cup E_\tau$ then there is $g_x > x$ and $G \in \bar{r}(U)$ with $[x, g_x] \subseteq G$, and
(b) if $x \in L_\tau \cup E_\tau$ then there is $h_x < x$ and $H \in \bar{r}(U)$ with $[h_x, x] \subseteq H$.

**Proof.** Given any open cover $U$ let $c(U) = \{C \subseteq X : C$ is an open convex subset of $X$ and $[C] \subseteq U$ for some $U \in U\}$. It is easily seen that $c(U)$ is an open cover refining $U$, and $c$ is monotone. Let $\bar{r}(U) = \{[A] : A \in r(c(U))\}$.

Clearly $\bar{r}(U)$ is a cover of $X$ with closed convex sets and $\bar{r}$ is monotone. Also, $\bar{r}(U)$ refines $U$ since if $A \in r(c(U))$ then there are $C \subseteq X$ and $U \in U$ such that $C \in c(U)$ and $A \subseteq C \subseteq [C] \subseteq U$, hence $[A] \subseteq [C] \subseteq U$.

Suppose $\bar{r}(U)$ were not closure-preserving for some $U$. Then there is a family $A \subseteq r(c(U))$ and some $p \in \bigcup\{[A] : A \in A\} \setminus \bigcup\{[A] : A \in A\}$. Since each $[A]$ is convex we have that either $p < l_A$ or $p > u_A$. We may assume without loss of generality that $p \in R_\tau \cup E_\tau$ and $p < l_A$ for all $A \in A$. It is easily seen that $p = \inf\{l_A : A \in A\}$ and $p \in \bigcup\{A : A \in A\}$, contradicting that $r(c(U))$ is closure-preserving. Thus $\bar{r}(U)$ is closure-preserving for all open covers $U$.

To prove (a), fix $x \in R_\tau \cup E_\tau$. Then $x \in (x, \rightarrow)$. For every $y > x$ there is $A_y \in r(c(U))$ with $y \in A_y$. It is enough to show that $l_{A_y} \leq x$ for some $y > x$, then

\footnote{Note that $[A] = c_\lambda(\text{conv}(A))$ where $\lambda$ is the underlying LOTS topology. The above proof would work equally well if we used $c_\tau(\text{conv}(A))$ instead of $c_\lambda(\text{conv}(A))$.}
we would have that \([x, u_{A_x}] \subseteq [A_y] \in r(\mathcal{U})\) and we may pick any \(g_x \in (x, u_{A_x})\). If \(l_{A_x} > x\) for each \(y > x\) then the family \(\{A_y : y > x\}\) is not closure-preserving at \(x\), contradicting that \(r(c(\mathcal{U}))\) is closure-preserving. The proof of (b) is similar. \(\square\)

The proof of the following lemma is similar to the proof of \([11]\) Lemma 17.

**Lemma 2.6.** Suppose \((X, \tau, <)\) is a GO-space with a monotone closure-preserving operator \(r\), \(y_n \in R_\tau\) with \(y_{n+1} < y_n\) for each \(n \in \omega\), and the \(y_n\) converge to \(y\). Let \(\mathcal{U}_n = \{(\leftarrow, y_n), [y_n, \rightarrow)\}\). If \(\bar{r}\) is the monotone operator described in the proof of Lemma \(2.4\) and \(G_n \in \bar{r}(\mathcal{U}_n)\) such that \(y_n \in G_n\), then \(\{G_n : n \in \omega\}\) is point-finite. (A similar statement holds for \(y_n \not\rightarrow y\) and \(\mathcal{U}_n = \{(\leftarrow, y_n), (y_n, \rightarrow)\}\).)

**Proof.** Suppose \(\{G_n : n \in \omega\}\) were not point-finite. Taking a subsequence of the \(y_n\) we may assume that there is some \(p \in \bigcap\{G_n : n \in \omega\}\). Then \(y_n \in G_n \subseteq [y_n, \rightarrow)\), hence \(y_n \leq p\) for each \(n\). If \(\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \omega\}\), then \(\mathcal{U}_n\) refines \(\mathcal{U}\) for each \(n\), hence \(\bar{r}(\mathcal{U}_n)\) refines \(\bar{r}(\mathcal{U})\). There are \(H_n \in \bar{r}(\mathcal{U})\) and \(m_n \geq n\) with \(y_n \in G_n \subseteq H_n \subseteq [y_{m_n}, \rightarrow)\). Since \(y_n\) converges to \(y\), the family \(\{H_n : n \in \omega\}\) is not closure-preserving at \(y\). This contradiction completes the proof. \(\square\)

The proof of the following proposition is modeled after the proof of \([11]\) Proposition 13 and \([1]\) Proposition 3.8 (stated as Proposition \(2.3\) here).

**Proposition 2.7.** Suppose \((X, \tau, <)\) is a GO-space for which the underlying LOTS \((X, \lambda, <)\) has a \(\sigma\)-closed-discrete dense set. If \((X, \tau)\) has a monotone closure-preserving operator \(r\) then \(R_\tau \cup L_\tau\) is \(\sigma\)-closed-discrete as a subspace of \((X, \tau)\) and as a subspace of \((X, \lambda)\).

**Proof.** Let \(D = \bigcup\{D_n : n \in \mathbb{N}\}\) be dense in \((X, \lambda)\) where each \(D_n\) is closed-discrete in \((X, \lambda)\) (hence also in \((X, \tau)\)). It is easily seen that \((X, \tau)\) is first-countable. By \([11]\) Lemma 2.4, \([11]\) Lemma 16 it is enough to show that \(R_\tau \cup L_\tau\) is \(\sigma\)-relatively discrete as a subspace of \((X, \tau)\).

For each \(p \in R_\tau\), let \(\mathcal{U}(p) = \{(\leftarrow, p), [p, \rightarrow)\}\). Let \(\bar{r}\) be the monotone operator described in the proof of Lemma \(2.5\) Choose \(G(p) \in \bar{r}(\mathcal{U}(p))\) and \(x_p > p\) such that \([p, x_p) \subseteq G(p)\). There is \(n(p)\) such that \((p, x_p) \cap D_{n(p)} \neq \emptyset\). Let \(R_\tau(n) = \{p \in R_\tau : n(p) = n\}\). Clearly \(R_\tau = \cup\{R_\tau(n) : n \in \omega\}\). We claim that each \(R_\tau(n)\) is relatively discrete in \((X, \tau)\). Suppose not, then there are \(n \in \omega, p \in R_\tau(n)\) and a sequence \(\{p_k : k \in \omega\}\) of \(p\) with \(p_{k+1} < p_k\) for each \(k\). We may assume that \((p_k, x_{p_k}) \cap D_n = \emptyset\). Since \([p_k, x_{p_k}) \cap D_n \neq \emptyset\), we have \(p_0 \in [p_k, x_{p_k}) \subseteq G(p_k) \subseteq [y_k, \rightarrow)\) for each \(k\), which contradicts Lemma \(2.6\).

Hence \(R_\tau(n)\) is relatively discrete for each \(n\), which shows that \(R_\tau\) is \(\sigma\)-relatively discrete. Similarly \(L_\tau\) is \(\sigma\)-relatively discrete, which completes the proof. \(\square\)

Since every monotone open locally-finite operator is both a monotone metacompactness operator and a monotone open closure-preserving operator, Theorem \(2.1\) Theorem \(2.3\) and Proposition \(2.7\) allow us to extend Theorem \(2.3\) as follows.

**Theorem 2.8.** Let \((X, \tau, <)\) be a GO-space whose underlying LOTS \((X, \lambda, <)\) has a \(\sigma\)-closed-discrete dense subset. Then the following are equivalent: (i) \(X\) has a monotone open locally-finite operator, (ii) \(X\) is monotonically metacompact, (iii) \(X\) has a monotone open closure-preserving operator, (iv) \(X\) has a monotone closure-preserving operator.
We do not know if the requirement in Theorem 2.8 that \((X, \lambda, \prec)\) has a \(\sigma\)-closed discrete dense subset is essential.

**Question 2.9.** If a GO-space \(X\) has a monotone (open or otherwise) closure-preserving operator, must it be monotonically metacompact?

If the answer to Question 2.9 is yes, then Theorems 3.3 and 3.8 in the next section would follow from results in [1], [3], and [11].

**Question 2.10.** Can one add protometrizable to the list of equivalent conditions in Theorem 2.8?

The following is a variation of our Question 2.6(d) in [14] (where “monotone locally-finite operator” meant “monotone open locally-finite operator”).

**Question 2.11.** Does every LOTS \(X\) with a monotone locally-finite (open) operator have a Nötherian locally-finite base (as defined in [14])?

### 3. METRIZATION THEOREMS

Faber’s metrization theorem for GO-spaces was the key to results in [1] and [11] on the metrization of monotonically countably metacompact GO-spaces with a \(\sigma\)-closed-discrete dense subset.

**Theorem 3.1.** [6, Theorem 3.10] Suppose \((X, \tau, \prec)\) is a GO-space and \(Y \subseteq X\). Then the subspace \((Y, \tau_Y)\) is metrizable if and only if

(a) \((Y, \tau_Y)\) has a \(\sigma\)-closed-discrete dense subset, and

(b) \(R_{\tau_Y} \cup L_{\tau_Y}\) is \(\sigma\)-closed-discrete in the subspace \((Y, \tau_Y)\).

By Faber’s metrization theorem, to prove that a GO-space \((X, \tau, \prec)\) with a \(\sigma\)-closed-discrete dense subset is metrizable it suffices to show that \(R_{\tau} \cup L_{\tau}\) is \(\sigma\)-closed-discrete.

**Proposition 3.2.** Suppose \((X, \tau, \prec)\) is a GO-space with a \(\sigma\)-closed-discrete dense subset. If \(X\) has a monotone closure-preserving operator \(r\) then \(R_r \cup L_r\) is \(\sigma\)-closed discrete.

**Proof.** Let \(D = \bigcup \{D_n : n \in \mathbb{N}\}\) be dense in \((X, \tau)\) where each \(D_n\) is closed-discrete. Then \(X\) is perfect (and first countable) [2, Proposition 3.1]. By [11] Lemma 2.1], it is enough to show that \(R_r \cup L_r\) is \(\sigma\)-relatively discrete.

The rest of the proof of Proposition 2.7 works here without modifications. \(\Box\)

The following theorem immediately follows from the above proposition and Faber’s metrization theorem.

**Theorem 3.3.** Suppose \((X, \tau, \prec)\) is a GO-space with a \(\sigma\)-closed-discrete dense subset. If \(X\) has a monotone closure-preserving operator, then \((X, \tau)\) is metrizable.

By means of a different proof, the first author [13] has shown that the Sorgenfrey line does not have a monotone closure-preserving operator. Since the Sorgenfrey line is separable and nonmetrizable, it will not have a monotone closure-preserving operator by Theorem 3.3. (One could also use Theorem 2.8.)

**Corollary 3.4.** The Sorgenfrey line has no monotone closure-preserving operator.
Every space $X$ with a (monotone or not) closure-preserving operator must be paracompact. In particular $\omega_1$ with the order topology does not have a monotone closure-preserving operator. The next theorem shows that the compact LOTS $\omega_1+1$ has no monotone closure-preserving operator either.

**Theorem 3.5.** Let $X$ be a compact LOTS with a monotone closure-preserving operator $r$. Then $X$ is first countable.

**Proof.** If not then we may assume that there is $z \in X$ such that $z \in (\langle \leftarrow, z \rangle, \otimes)$, but if $x_n < z$ for each $n \in \omega$ then $\sup_{n \in \omega} x_n < z$.

For each $x < z$ let $U(x) = \{(x, \rightarrow)\} \cup \{(\langle \leftarrow, y \rangle) : y < z\}$. If $x < t < z$ then $U(t)$ refines $U(x)$. Fix $x_0 < z$ and let $A(x_0) = \{A \in r(U(x_0)) : x_0 \in A\}$. If $A \in A(x_0)$, then $A \subseteq (\langle \leftarrow, y \rangle)$ for some $y < z$. Then $u_A = \sup(A) \leq y < z$. Also, $\sup\{u_A : A \in A(x_0)\} = u_B < z$ for some $B \in A(x_0)$. Indeed, otherwise we could take $A_k \in A(x_0)$ with $u_{A_k} < u_{A_{k+1}}$ for all $k \in \omega$ and then the family $\{A_k : k \in \omega\}$ would not be closure-preserving at $\sup\{u_{A_k} : k \in \omega\}$.

By induction pick $x_n < z$ with $x_{n+1} > \sup\{u_A : A \in A(x_n)\}$, where $A(x_n) = \{A \in r(U(x_n)) : x_n \in A\}$. Clearly $x_n < x_{n+1}$. Let $t = \sup_{n \in \omega} x_n$, then $t < z$. For each $n$ pick $C_n \in r(U(t))$ such that $x_n \in C_n$. Since $U(t)$ refines each $U(x_n)$, there is $A_n \in A(x_n)$ with $C_n \subseteq A_n$. Hence $x_n \leq \sup(C_n) \leq \sup(A_n) < x_{n+1}$. It follows that the family $\{C_n : n \in \omega\}$ is not closure-preserving at $t$, a contradiction. 

We do not know if in the above theorem we may conclude that $X$ is metrizable. We will show that if the monotone closure-preserving operator $r$ is open, then the answer is yes. Again, we modify the monotone operator.

**Lemma 3.6.** Suppose $(X, \tau, <)$ is a GO-space with a monotone closure-preserving operator $r_1$, such that if $r$ is an open operator, then $r = r_1$.

**Proof.** Let $r_1(U) = \{\text{conv}(A) : A \in r(c(U))\}$, where $c$ is defined as in the proof of Lemma 2.6. The easy verification that $r_1$ is the desired monotone operator is left to the reader.

**Lemma 3.7.** Suppose $(X, <)$ is a compact LOTS with a monotone closure-preserving operator $r$. Then $X$ has a convex monotone closure-preserving operator $r_2$ such that $r_2(U)$ is finite, for every open cover $U$. If $r$ is open, then so is $r_2$.

**Proof.** If $r_1$ is the convex monotone closure-preserving operator operator defined in Lemma 3.6 let $r_2(U) = \{U \in r_1(U) : U$ is maximal in $r_1(U)$ with respect to inclusion\}. Every element of $r_1(U)$ is contained in a $\subseteq$-maximal one, for otherwise we could find a $\subseteq$-strictly increasing chain $J = \{J_n : n \in \omega\} \subseteq r_1(U)$ with $J_n \subset J_{n+1}$ for all $n$, but then $J$ would not be closure-preserving at either $\sup\{u_{J_n} : n \in \omega\}$ or at $\inf\{l_{J_n} : n \in \omega\}$. (Since the $J_n$ are convex and $\subseteq$-increasing, there are infinitely many $n$ for which either $l_{J_{n+1}} < l_{J_n}$ or $u_{J_n} < u_{J_{n+1}}$.) Hence $r_2(U)$ covers $X$.

Given any nonempty (usually convex) $A, B \subseteq X$ define $A \ll B$ provided that either there is $a \in A$ with $a < b$ for all $b \in B$, or there is $b \in B$ with $a < b$ for all $a \in A$. Since each element of $r_2(U)$ is convex and $\subseteq$-maximal it follows that $\ll$ totally orders $r_2(U)$ (i.e. every two distinct elements of $r_2(U)$ are $\ll$-comparable). If $r_2(U)$ were infinite for some open cover $U$ then we could find a family $I = \{I_n : n \in \omega\} \subseteq r_2(U) \subseteq r_1(U)$ with either $I_n \ll I_{n+1}$ for all $n$, or $I_{n+1} \ll I_n$ for all $n$. In
the former case \( I \) is not closure-preserving at sup\( \{ u_n : n \in \omega \} \), and in the latter case at inf\( \{ l_n : n \in \omega \} \). This contradiction shows that \( r_2(\mathcal{U}) \) is finite for all open covers \( \mathcal{U} \). Clearly \( r_2 \) is monotone, and if \( r \) is open, so is \( r_2 \).

\[ \square \]

**Theorem 3.8.** Suppose that \( X \) is a compact LOTS with a monotone open closure-preserving operator \( r \). Then \( X \) is metrizable.

**Proof.** By Lemma 3.7, \( X \) is monotonically compact (i.e. it has a monotone operator \( r_2 \) such that \( r_2(\mathcal{U}) \) is a finite open refinement covering \( X \), for every open cover \( \mathcal{U} \)). Hence \( X \) is metrizable [8, Theorem 4.1] (for LOTS). More generally, see [3] [9] [12].

If Question 3.9 has a positive answer, then the answer to the following question would also be positive.

**Question 3.9.** If \( X \) is a compact LOTS with a monotone closure-preserving operator \( r \), must \( X \) be metrizable?

If \( E_r \) is empty, then we have the following partial answer to Question 3.9.

**Theorem 3.10.** Suppose \((X, \tau, <)\) is a GO-space with a monotone closure-preserving operator \( r \). Then there is a convex open monotone operator \( \hat{r} \) such that \( \hat{r}(\mathcal{U}) \) is a closure-preserving family and \( X \setminus \cup \hat{r}(\mathcal{U}) \subseteq E_r \), for every open cover \( \mathcal{U} \). If \( E_r = \emptyset \), then \( \hat{r} \) is a convex monotone open closure-preserving operator.

**Proof.** Let \( c \) and \( \bar{r} \) be the operators described in the proof of Lemma 2.5. Let \( \hat{r}(\mathcal{U}) = \{ \text{Int}[A] : A \in r(c(\mathcal{U})) \} = \{ \text{Int}(K) : K \in \hat{r}(\mathcal{U}) \} \). Clearly \( \hat{r}(\mathcal{U}) \) is an open family refining \( \bar{r}(\mathcal{U}) \) (and hence also \( \mathcal{U} \)) and \( \hat{r} \) is monotone. The proof that \( \hat{r}(\mathcal{U}) \) is a closure-preserving family is similar to the proof for \( \bar{r}(\mathcal{U}) \) and is left to the reader.

We show that \( \cup \hat{r}(\mathcal{U}) \supseteq X \setminus E_r \). If \( x \in I_r \), then \( x \in \text{Int}[A] \in \hat{r}(\mathcal{U}) \) whenever \( x \in A \in r(c(\mathcal{U})) \). If \( x \in B_r \) then by Lemma 2.5 (a), there is some \( G \in \hat{r}(\mathcal{U}) \) with \( x \in \text{Int}(G) \). The case \( x \in L_r \) is similar, which completes the proof.

In the special case when \( E_r \) is finite, Theorem 3.10 allows us to remove the requirement in Theorem 3.8 that the operator \( r \) is open.

**Theorem 3.11.** Suppose that \( X \) is a compact LOTS with a monotone closure-preserving operator \( r \). If \( E_r \) is finite, then \( X \) is metrizable.

**Proof.** Let \( \hat{r}(\mathcal{U}) \) be as described in the proof of the preceding theorem, and let \( \hat{r}(\mathcal{U}) \) be the family of \( \subseteq \)-maximal elements of \( \hat{r}(\mathcal{U}) \). It is easily seen (using the ideas in the proof of Lemma 3.7) that \( \cup \hat{r}(\mathcal{U}) = \cup \hat{r}(\mathcal{U}) \) and that \( \hat{r}(\mathcal{U}) \) is finite, for any open cover \( \mathcal{U} \). If \( E_r = \emptyset \) then we are done as \( \hat{r} \) shows that \( X \) is monotonically compact.

If \( E_r \neq \emptyset \) then (using Theorem 3.3) for each \( x \in E_r \) fix a \( \subseteq \)-decreasing local base \( B_x = \{ B_n(x) : n \in \omega \} \) (i.e. \( B_{n+1}(x) \subseteq B_n(x) \) for all \( n \)). Given any open cover \( \mathcal{U} \) let \( V_x(\mathcal{U}) \) be the \( \subseteq \)-maximal element of \( B_x \) that is contained in some open set \( U \in \mathcal{U} \) (i.e. \( V_x(\mathcal{U}) = B_n(x) \) where \( n \) is smallest such that there is \( U \in \mathcal{U} \) with \( B_n(x) \subseteq U \)). Let \( F(\mathcal{U}) = \{ V_x(\mathcal{U}) : x \in E_r \} \). Since \( E_r \) is finite, the operator \( \hat{r}(\mathcal{U}) = \hat{r}(\mathcal{U}) \cup F(\mathcal{U}) \) shows that \( X \) is monotonically compact, and hence metrizable.

**Corollary 3.12.** The Alexandroff double arrow is a compact first-countable, hereditarily Lindelöf LOTS that has no monotone closure-preserving operator.
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