Sgr A*: A Laboratory to Measure the Central Black Hole and Stellar Cluster Parameters

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ABSTRACT. Several stars orbit around a black hole candidate of mass \(3.7 \times 10^6 \, M_\odot\) in the region of the Galactic center. Looking for general relativistic (GR) periastron shifts is limited by the existence of a stellar cluster around the black hole, which modifies the cluster orbits due to classical effects that might mask the GR effect. Only if one knows the cluster parameters (its mass and core radius) it is possible to unequivocally deduce the expected GR effects and then test them. In this paper, we show that the observation of the proper motion of Sgr A* (\(v_p = 0.4 \pm 0.9 \, \text{km s}^{-1}\); Reid & Bruthaler) could help us to constrain the cluster parameters significantly, and that future measurements of the periastron shifts for at least three stars may adequately determine the cluster parameters and the mass of the black hole.

Online material: color figures

1. INTRODUCTION

GR predicts that orbits about a massive central body suffer periastron shifts yielding “rosette” shapes. However, the classical perturbing effects of other objects on inner orbits give an opposite shift. Since the periastron advance depends strongly on the compactness of the central body, the detection of such an effect may give information about the nature of the central body itself. This would apply for stars orbiting close to the Galactic center (GC), where there is a “dark object,” the black hole hypothesis being the most natural explanation of the observational data. A cluster of stars in the vicinity of the GC (at a distance of <1") has been monitored by ESO and Keck teams for several years (Genzel et al. 2003a; Schödel el et al. 2003; Ghez et al. 2003, 2004, 2005). In particular, Ghez et al. (2003) have reported on observations of several stars orbiting close to the GC’s massive black hole. Among those, the S2 star, with mass \(M_{S2} = 15 \, M_\odot\), appears to be a main-sequence star orbiting the black hole, with a Keplerian period of \(\approx 15\) yr. This yields a mass estimate (Ghez et al. 2005) of \(M_{S2} \approx 3.67 \times 10^6 \, M_\odot\) within \(4.87 \times 10^{-3}\) pc, which is the S2 semimajor axis.

Several authors have discussed the possibility of measuring the GR corrections to Newtonian orbits for Sgr A* (see, e.g., Jaroszynski 1998, 1999, 2000; Fragile & Mathews 2000; Rutilar & Eckart 2001; Weinberg et al. 2005), usually assuming that the central body is a Schwarzschild black hole. However, since black holes generally rotate, and there is no reason why they should not be rotating fast, the Kerr metric should be used instead. Not only stellar-mass black holes but also supermassive black holes are believed to spin. Indeed, X-ray observations of Seyfert galaxies, microquasars, and binary systems (Fabian et al. 1995, 2000, 2005; Tanaka et al. 1995; and references therein) show that the data could be explained by a rotating black hole model (see, e.g., Zakharov & Repin 2003a, 2003b, 2004; Zakharov et al. 2003). Furthermore, supermassive black holes at the center of QSOs, active galactic nuclei, and galaxies show beamed jet emission, implying that they have nonzero angular momentum. Hence, Kerr black holes may be fairly common in nature. The relatively short orbital period of the S2 star encourages a search for genuine GR effects, like the orbital periastron shift. Quite possibly, more suitable stars close to the GC black hole will be found in the future. Bini et al. (2005) studied the GR periastron shift around Sgr A* and estimated it for various solutions belonging to the Weyl class, including the Schwarzschild and Kerr black holes. However, they did not take into account the presence of a stellar cluster, which could in principle be sizable.

The purpose of this paper is to try to find limits for the extent
and density of the cluster about Sgr A* and to determine if those limits can yield a measurement of its spin.

Clearly, a thorough knowledge of the cluster mass and density distribution is necessary to be able to infer the mass and spin of the black hole at the GC by measuring the periastron shift and subtracting the Newtonian shift. Unfortunately, the star cluster parameters are poorly known.6 However, the measure of the Brownian motion of the central black hole due to the surrounding matter can be used to constrain the black hole mass to cluster mass ratio.5 The latest observations of the Sgr A* proper motion indicate $v_{\text{Sgr A*}} = 0.4 \pm 0.9$ km s$^{-1}$ (Reid & Bruthaler 2004), much tighter than the earlier measurement of $2\sim20$ km s$^{-1}$ (see Reid et al. 1999).

For a test particle orbiting a Schwarzschild black hole of mass $M_{\text{BH}}$, the periastron shift is given by

$$\Delta\phi_{S} = \frac{6\pi GM_{\text{BH}}}{d(1-e^2)c^2} + \frac{3(18+e^2)\pi G^2 M_{\text{BH}}^2}{2d^2(1-e^2)^3c^4}$$

(1)

(see, e.g., Weinberg 1972), where $d$ and $e$ are the semimajor axis and eccentricity of the test particle orbit, respectively. For a rotating black hole with spin parameter $a = |a| = J/GM_{\text{BH}}$, the space-time is described by the Kerr metric, and in the most favorable case of equatorial plane motion ($a \cdot v = 0$), the shift is given by

$$\Delta\phi_{K} = \Delta\phi_{S} + \frac{8a\pi GM_{\text{BH}}G^{3/2}}{d^{3/2}(1-e^2)^{3/2}c^3} + \frac{3a^2 \pi G^2}{d^2(1-e^2)^3c^4}$$

(2)

(Boyer & Price [1965], but see also Bini et al. [2005] for more details), which reduces to equation (1) for $a \rightarrow 0$. In the more general case, $a \cdot v \neq 0$, the expected periastron shift has to be evaluated numerically.

The expected periastron shifts (mas per revolution) $\Delta\phi$ (as seen from the center) and $\Delta\phi_{K}$ (as seen from Earth at the distance $R_0 \approx 8$ kpc from the GC) for the Schwarzschild and the extreme Kerr black holes turn out to be $\Delta\phi_{S} = 6.3329 \times 10^5$ and $6.4410 \times 10^5$ and $\Delta\phi_{K} = 0.661$ and 0.672, respectively, for the S2 star, and $\Delta\phi_{S} = 1.6428 \times 10^6$ and $1.6881 \times 10^6$ and $\Delta\phi_{K} = 3.307$ and 3.399, respectively, for the S16 star. Recall that

$$\Delta\phi = \frac{d(1+e)}{R_0} \Delta\phi_{S,K}.$$ 

(3)

Note that the differences between the periastron shifts for the Schwarzschild and the maximally rotating Kerr black holes are at most 0.01 mas for the S2 star and 0.009 mas for the S16 star. In order to make these measurements with the required accuracy, one needs to know the S2 orbit with a precision of at least 10 μas.

A proposal has been made to improve the angular resolution of the Very Large Telescope Interferometer (VLTI) with the Phase-Referenced Imaging and Microarcsecond Astrometry (PRIMA) facility (Röttgering et al. 2003; Delplancke et al. 2003; Quirrenbach 2003; but see also the related Web site5), which, by using a phase-referenced imaging technique, would get an angular resolution of $\sim 10$ μas. Hence, at least in principle, the effect of a maximally rotating Kerr black hole on the periastron shift of the S2 star could be distinguished from that produced by a Schwarzschild black hole with the same mass.

The outline of the paper is as follows: In the next section, we briefly discuss the effect of a central star cluster on the periastron advance. In § 3, we use the Sgr A* Brownian motion to constrain the ratio of black hole mass to star cluster mass. We then consider whether the detection of the spin of the black hole from the periastron shift of the S2 star is possible, once the cluster density and size have been adequately constrained. In § 5, we show how future measurements of the periastron shifts for at least three stars close to the GC black hole may be used to estimate the black hole mass and the star cluster mass density distribution. In the next section (§ 6), we consider what the observational requirements would be for the adequate determination of the cluster parameters in order to be able to resolve the Kerr effect. Finally, in § 7, we present some concluding remarks.

2. RETROGRADE SHIFT DUE TO A CENTRAL STELLAR CLUSTER

The star cluster surrounding the central black hole in the GC could be sizable. At least 17 members have been observed within 15 mpc up to now (Ghez et al. 2005). However, the cluster mass and density distribution, that is to say its mass and core radius, is still unknown. The presence of this cluster affects the periastron shift of stars orbiting the central black hole. The periastron advance depends strongly on the mass density profile and especially on the central density and typical length scale.

We model the stellar cluster by a Plummer model density profile

$$\rho_{\text{CS}}(r) = \rho_0 f(r), \text{ with } f(r) = \left[1 + \left(\frac{r}{r_c}\right)^3\right]^{-\alpha/2}$$

(4)

5 See http://obswww.unige.ch/Instruments/PRIMA/home/introduction.html.
(Binney & Tremaine 1987), where the cluster central density \( \rho_0 \) is given by

\[
\rho_0 = \frac{M_{\text{CL}}}{4\pi R_{\text{CL}}^2 f(r) \, dr},
\]

where \( R_{\text{CL}} \) and \( M_{\text{CL}} \) are the cluster radius and mass, respectively. According to dynamical observations toward the GC, we require that the total mass \( M(r) = M_{\text{BH}} + M_{\text{CL}}(r) \) contained within \( r \approx 5 \times 10^{-3} \) pc be \( M \approx 3.67 \times 10^6 M_\odot \). Useful information is provided by the cluster mass fraction \( \lambda_{\text{CL}} = M_{\text{CL}}/M \) and its complement, \( \lambda_{\text{BH}} = 1 - \lambda_{\text{CL}} \). As one can see, the requirement given in equation (5) implies that \( M(r) \rightarrow M_{\text{BH}} \) for \( r \rightarrow 0 \).

The total mass density profile \( \rho(r) \) is given by

\[
\rho(r) = \frac{\lambda_{\text{BH}} M_0(r) + \rho_0 f(r)}{\int_0^r 4\pi r'^2 \rho_0 f(r') \, dr'},
\]

and the mass contained within \( r \) is

\[
M(r) = \lambda_{\text{BH}} M + \int_0^r 4\pi r'^2 \rho_0 f(r') \, dr'.
\]

In Figure 1, we show the cluster mass density profile \( \rho_{\text{CL}}(r) \) as given by equation (4) for selected values of \( \lambda_{\text{BH}} \). The total mass \( M(r) \) enclosed within the radius \( r \) is also shown in Figure 2. In both figures, solid, dotted, and dashed lines correspond to \( \lambda_{\text{BH}} = 0, 0.7, \) and 0.9, respectively, and we have assumed \( r_c = 3 \) mpc (thick lines) and \( r_c = 5.8 \) mpc (thin lines).

The Newtonian gravitational potential \( \Phi_N \) at a distance \( r \) due to the mass contained within it can be evaluated as

\[
\Phi_N(r) = -G \int_r^\infty \frac{M(r')}{r'^2} \, dr'.
\]

In Figure 3, the gravitational potential \( \Phi_N(r) \) due to the mass density distribution in equation (6) is given for selected values of \( \lambda_{\text{BH}} \).

According to GR, the motion of a test particle can be fully
Fig. 6.—Expected S2 periastron shift as a function of the stellar cluster core radius. Here we have assumed a Plummer density profile for the stellar cluster. Dashed, solid, and dotted lines correspond to \( a_p = 4, 5, \) and \( 6 \), respectively. The black hole mass fraction has been fixed to \( \lambda_{BH} = 0.8 \) (top) and \( \lambda_{BH} = 0.99 \) (bottom). Note the existence of a maximum approximately corresponding to the S2 semimajor axis.

described by solving the geodesic equations. Under the assumption that the matter distribution is static and pressureless, the equation of motion of the test particle becomes

\[
\frac{d\mathbf{v}}{dt} = -\nabla(\Phi_N + 2\Phi_S^2) + 4\mathbf{v}(\mathbf{v} \cdot \nabla)\Phi_N - \mathbf{v}^2 \nabla\Phi_N, \tag{9}
\]

(see, e.g., Weinberg 1972). For a spherically symmetric mass distribution\(^6\) with a density profile given by equation (4), and for a gravitational potential given by equation (8), the previous relation becomes

\[
\frac{d\mathbf{v}}{dt} = -\frac{GM(r)}{r^3}\left(1 + \frac{4\Phi_N}{c^2} + \frac{v^2}{c^2}\right) - \frac{4v(\mathbf{v} \cdot \mathbf{r})}{c^2} \tag{10}
\]

(for details, see Rubilar & Eckart 2001), where \( \mathbf{r} \) and \( \mathbf{v} \) are the radius vector of the test particle (with respect to the center of the stellar cluster) and the velocity vector, respectively. Once the initial conditions for distance and velocity are given, the orbit of a test particle can be found by solving the set of ordinary differential equations in equation (10) numerically.

Now consider the S2 star, which is moving around the central distribution of matter on an elliptical orbit of semimajor axis \( d \) and eccentricity \( e \) in the Newtonian approximation. We take a frame with the origin in the GC, the \((x,y)\) plane on the orbital plane, and the \( x \)-axis pointing toward the periastron of the orbit. Hence, we can choose the Newtonian initial conditions to be

\[
r_0 = d(1 + e), \tag{11}
\]

\[
\mathbf{v}_0 \cdot \mathbf{r} = 0, \tag{12}
\]

\[
\mathbf{v}_0 \cdot \mathbf{v} = \sqrt{GM(r_0)^2\left(\frac{2}{d(1 + e)} - \frac{1}{d}\right)}.
\]

(see, e.g., Smart 1977).

For the S2 star, \( d \) and \( e \) given in the literature are 919 AU and 0.87, respectively. They yield the orbits of the S2 star for different values of the black hole mass fraction \( \lambda_{BH} \) shown in Figure 4. The Plummer model parameters are \( \alpha = 5 \), core radius \( r_c = 5.8 \) mpc. Note that in the case of \( \lambda_{BH} = 1 \), the expected periastron shift is that given by equation (1), while the presence of the stellar cluster leads to a retrograde periastron shift. For comparison, the expected periastron shift for the S16 star is given in Figure 5. In the latter case, the binary system orbital parameters were taken from Schödel et al. (2003), assuming also for the S16 star mass a conservative value of \( \approx 10 M_\odot \).

In Figure 6, the S2 orbital shift \( \Delta \Phi \) is given as a function of the stellar cluster core radius \( r_c \) for different power-law index values \( (\alpha = 4 \) [dashed line], \( \alpha = 5 \) [dotted line], and \( \alpha = 6 \) [solid line]). In the top panel, the black hole mass fraction is \( \lambda_{BH} = 0.8 \) in order to compare with the Rubilar & Eckart (2001) results, while the bottom panel shows the case in which \( \lambda_{BH} = 0.99 \). Note that for extremely compact clusters, \( \Delta \Phi \) is quite small. The same is true for large enough core radii, corresponding to matter density profiles that are almost constant within the S2 orbit.
Fig. 4.—Post-Newtonian orbits for different values of the black hole mass fraction $\lambda_{\text{BH}}$ for the S2 star (top two panels). Here we have assumed that the Galactic central black hole is surrounded by a stellar cluster whose density profile follows a Plummer model with $\alpha = 5$ and a core radius $r_c \approx 5.8$ mpc. The periapsis values in each panel are given in arcseconds.

Fig. 5.—Same as Fig. 4, but for the S16–Sgr A* binary system. In this case, the binary system orbital parameters were taken from Ghez et al. (2005), assuming for the S16 mass a conservative value of $\approx 10 M_{\odot}$. 
Fig. 7.—Mass enclosed within the distance $r$ for different fractions $\lambda_{\text{BH}}$ of the total mass $M$ contained within the S2 orbit. Solid, dotted, and dashed lines correspond to $\lambda_{\text{BH}} = 0.1$, 0.5, and 0.9, respectively. The stellar cluster is assumed to follow an $r^{-3/4}$ density profile.

Fig. 8.—Expected S2 periastron shift as a function of the mass ratio parameter $\lambda_{\text{BH}}$. Solid and dashed lines correspond to the S2 shift due to the black hole and the stellar cusp, respectively. We note that the shift due to the stellar cusp is independent of the $r_\ast$ value, which in this case has been assumed to be larger than the S2 semimajor axis (case b).

The shift due to the cluster is opposite in sign to the relativistic effect due to the black hole in the GC. Moreover, for each value of the cluster mass and power-law index, there exist two density profiles (corresponding to two particular core radii) that have a total shift of almost zero, implying that the periastron advance due to the cluster is equal (in magnitude) to the periastron shift due to the black hole. A numerical analysis shows that the transition from a prograde shift (due to the black hole) to retrograde shift (due to the extended mass) occurs at $\lambda_{\text{BH}} = 0.9976$, 0.9986, and 0.9990 for $\alpha = 4$, 5, and 6, respectively. This means that a small fraction of mass in the cluster drastically changes the overall shift.

We would like to note that the assumption of the Plummer model to describe the mass density distribution of the stellar cluster around the central black hole is an oversimplification. Indeed, one expects that in the presence of a central black hole, the stellar profile should follow a Bahcall-Wolf law with density distribution $\rho_\ast(r) \propto r^{-7/4}$ (Bahcall & Wolf 1977; Binney & Tremaine 1987) at least up to $r_\ast \ll r_\text{h}$, where $r_\text{h} = GM_{\text{BH}}/\sigma_\ast^2 \approx 0.5$ pc is the radius of the black hole influence sphere. In the following, we call $r_\ast$ the distance ($\ll r_\text{h}$) up to which the cluster mass density follows the Bahcall-Wolf law.

In order to study the effect of such a cusp on the expected S2 periastron shift, we consider three different scenarios: case a, the cusp is entirely contained within the S2 periastron distance $R_\ast$ (i.e., $r_\ast \leq R_\ast$); case b, the cusp extends beyond the S2 periastron distance (thus, making the S2 star move in a mass gradient); and case c, the stellar density profile follows a cusp law up to the distance $r_\ast$ from the center, and a Plummer law for $r \geq r_\ast$. In cases a and b, all stars are in a cusp density profile. In any case, we require that the total mass enclosed within $4.87 \times 10^{-3}$ pc be $M = 3.67 \times 10^6 M_\odot$.

In case a, the total S2 periastron shift is just the sum of the shift due to the black hole and the shift caused by the stellar cusp (which contributes with the same sign). Hence, the S2 shift turns out to be $\Delta \Phi \approx 0.17^\circ$ per revolution.

In case b, by requiring that the total mass enclosed within $4.87 \times 10^{-3}$ pc be $M \approx 3.67 \times 10^6 M_\odot$, we find that the dependence of the cusp mass and the induced S2 periastron shift on $r_\ast$ vanishes. Indeed, in Figure 7, we give the mass enclosed within the distance $r$ for different values of $\lambda_{\text{BH}}$. Solid, dotted, and dashed lines correspond to $\lambda_{\text{BH}} = 0.1$, 0.5, and 0.9, respectively. Figure 8 shows the expected S2 periastron shift as a function of $\lambda_{\text{BH}}$. As noted before, the shift due to the cluster is opposite in sign with respect to that due to the black hole. Moreover, for $\lambda_{\text{BH}} = 0.998$, the total shift turns out to be zero, since the contributions of the black hole and the cluster cancel out. It is worth noting that since in the case of cusp profiles the density gradient is larger than in the case of a usual ($\alpha = 5$) Plummer model, the value of the S2 periastron shift gets generally larger values. Only if the Plummer core radius is around 2 mpc are the resulting S2 periastron shifts comparable in both cases.

We then considered the superposition of a Plummer model and a Bahcall-Wolf profile (case c) extended up to $r_\ast$ such that the cusp density at $r_\ast$ equals that of the Plummer model at the same distance. Here, if $r_\ast \ll R_\text{s2}$, the S2 periastron shift will be practically equal to that caused by the Plummer model (Fig. 6, bottom), since in this case the cusp will have a minor influence. On the contrary, for an extended cusp ($r_\ast \gg R_\text{s2}$), the cusp
effect on the S2 periastron shift will dominate, reconciling with case b.

As a last point, we mention that we have also considered the effect due to an extrapolation of the observed stellar density profile—the innermost point of which is the S2 star at a distance of 0.1"—within \( R_{S2} \). Following Genzel et al. (2003b) and assuming a cusp stellar density profile, we find that the enclosed mass is in the range 30–300 \( M_\odot \) (for a constant mass density or a power law with index \( \gamma = 1.4 \)). Therefore, the cusp effect on the S2 periastron shift is negligible, since the corresponding \( \lambda_{BH} \) is always greater than 0.99992. However, we caution that the case under investigation in the present paper is different with respect to Genzel et al. (2003b), since we are assuming that a fraction of the mass contained within \( R_{S2} \) may be in a stellar cluster. Hence, the cluster mass content may be larger, thus providing a stronger effect on the S2 periastron shift.

3. Tightening mass limits of Sgr A*

We know to a high accuracy that the mass of Sgr A* within the S2 orbit is \( 3.67 \times 10^6 M_\odot \). Although there is nothing definite that is known about the mass distribution, there is strong reason to believe that there is a black hole of several solar masses, possibly surrounded by a significant cluster. In principle, the cluster mass could dominate over the black hole, be comparable to it, or be dominated by it. That there is a cluster is highly likely, on account of the large number of stars observed near Sgr A*. Although these lie outside the S2 orbit, many as yet unseen stars probably lie within the orbit as well. In this section, we use current data on the Brownian motion of Sgr A* and the evaporation time for the putative cluster to put limits on the cluster mass and hence on the black hole mass.

Chatterjee et al. (2002) have developed a simple model to describe the dynamics of a massive black hole surrounded by a dense stellar cluster. The total force acting on the black hole is separated into two independent parts, one of which is the slowly varying force due to the stellar ensemble, and the other is the rapid stochastic force due to close stellar encounters. In the case of a stellar system with a Plummer distribution, the motion of the black hole is similar to that of a Brownian particle in a harmonic potential. Thus, the one-dimensional mean-square velocity of the black hole is given by

\[
\langle v_x^2 \rangle = \frac{2GM_\odot m_\star}{9r_p M_{BH}},
\]

where it has been assumed that the cluster is composed of objects with equal mass \( m_\star \). For a Plummer (\( \alpha = 5 \)) stellar cluster, the total mass within \( R \) is

\[
M(R) = M_{BH} + \frac{M_{CL}R^3}{(R^2 + r^2_c)^{3/2}}.
\]

Since \( \langle v_x^2 \rangle \) is less than a certain maximum value \( \langle v_x^2 \rangle_{max} \), from equations (13) and (14), one obtains

\[
M_{BH} > M(R) \left\{ 1 + \frac{9}{2} \left[ \frac{\langle v_x^2 \rangle_{max} r_c R^3}{2G(R^2 + r_c^2)^{3/2} m_\star} \right] \right\}^{-1},
\]

where the right-hand side corresponds to a minimum black hole mass, as constrained by the Brownian motion of the central black hole. In Figure 9, the minimum black hole mass allowed by the Brownian motion of Sgr A* is given as a function of the stellar cluster core radius for two different proper-motion velocities of the black hole: 1.3 km s\(^{-1}\) (dashed lines) and 2 km s\(^{-1}\) (dotted lines). The total mass contained within \( R = 0.01 \) pc of Sgr A* has been taken to be \( M = 3.67 \times 10^6 M_\odot \).

Chatterjee et al. (2002) derived an evaporation time for a cluster but concentrated on the large-scale cluster \( r_c \approx 10 \) pc about Sgr A*, and hence assumed that \( M_{CL} \gg M_{BH} \). On the other hand, Rauch & Tremaine (1996) and Mouawad et al. (2005) consider only the region interior to the orbit of S2 and assume \( M_{CL} \ll M_{BH} \).

We need to allow for all possibilities while considering the cluster interior to the orbit of S2, including \( M_{CL} \approx M_{BH} \). For this purpose, we consider a cluster with a core radius \( r_c \) and mass \( M_{CL} = M - M_{BH} \). We now need to obtain the generalization of the formula from Chatterjee et al. (2002) for the median relaxation time in this more general situation. For this purpose, as usual we assume that the cluster consists of components of the same mass \( m_\star \) and evaluate the crossing time.
in the usual way to obtain the general median relaxation time

\[ T_r = \frac{0.14(1.3 r_p M_{\odot})^{3/2}}{\sqrt{GM_{\odot} m_{\ast} \log (0.4 M_{\odot}/m_{\ast})}} \quad (16) \]

It is easy to verify that in the approximation in which \( M_{\odot} \gg M_{\text{BH}} \), we recover the formula from Chatterjee et al. (2002), and in the approximation for \( M_{\odot} \ll M_{\text{BH}} \), we recover the formula from Rauch & Tremaine (1996). The evaporation time is then \( T_{\text{evap}} = 300 T \) (Binney & Tremaine 1987, p. 525).

One can assume different "reasonable" values of the time that the cluster would have been in existence and hence use the evaporation time to further limit the black hole mass in the GC. It is clear that \( 10^8 \text{ yr} = 0.1 \text{ Gyr} \) is less than the minimum value that could be regarded as reasonable, 1 Gyr is more reasonable, and 10 Gyr is likely to be a good value to assume. The results are given in Table 1 for \( m_{\ast} = 1 M_{\odot} \). Note that the tightest bound gives a very stringent upper limit of \( 9 \times 10^4 M_{\odot} \) on the cluster mass. Also note that the value decreases if the average \( m_{\ast} \) is taken to be larger.

4. THE SPIN OF THE BLACK HOLE

The periastron shift is the net contribution of the relativistic retrograde shift due to the black hole and the Newtonian prograde shift due to the surrounding cluster. Obviously, if the periastron advance due to the stellar cluster were known, the contribution of periastron advance due to the black hole could be obtained by subtracting from the measured quantity. The question arises whether the information obtained would be adequate to obtain both the black hole mass and spin parameters. Although we can put reasonably sharp bounds on the stellar cluster about the black hole, is it good enough for our purpose? If so, we could use equation (2) to obtain the spin of the black hole for different values in the possible range for the periastron shift. It is easy to see from Figure 6 that for \( \lambda_{\text{BH}} = 0.99 \), and allowing for the maximum range of unknown values of \( \alpha \) and \( r_c \), the periastron shift ranges are \( 1.8 \times 10^{-3} < -\Delta \phi < 4.7 \times 10^{-3} \) or \( 1.9 \times 10^{-3} < -\Delta \phi < 4.7 \times 10^{-3} \). For the sharpest limit obtained (\( \alpha = 5 \)) and a Brownian motion of 1.3 km s\(^{-1} \) (\( \lambda_{\text{BH}} = 0.975 \)), we find that \( \Delta \phi = -4.47 \times 10^{-3} \) or \( \Delta \phi = -4.7 \times 10^{-3} \). For Brownian a motion of 2.0 km s\(^{-1} \) (\( \lambda_{\text{BH}} = 0.964 \)), \( \Delta \phi = -5.75 \times 10^{-3} \) or \( \Delta \phi = -6.0 \times 10^{-3} \). This is a factor of 5 less than the effect of the spin. Hence, this method cannot be used to determine the spin. For this, we need the cluster parameter values, rather than upper limits for them. Alternatively, one would need to rely on the retrolensing method suggested earlier (De Paolis et al. 2005; Zakharov et al. 2005b).

5. DETERMINATION OF CLUSTER PARAMETERS

Using the stronger (1) limit of 1.3 km s\(^{-1} \) and the weaker (2) limit of 2.0 km s\(^{-1} \) to restrict the Brownian motion of Sgr A* for our calculations, in addition to evaporation times of 1 and 10 Gyr for the cluster, we obtained the minimum black hole mass. For the stronger limits on the Brownian motion and the evaporation time, the mass is \( 3.579 \times 10^6 M_{\odot} \), corresponding to a \( \lambda_{\text{BH}} = 0.975 \) for \( \alpha = 5 \). Our numerical analysis shows that the transition from a prograde shift (due to the black hole) to a retrograde shift (due to the extended mass assumed to be distributed with a Plummer density profile) occurs at \( \lambda_{\text{BH}} = 0.9976, 0.9986, \) and 0.9990 for \( \alpha = 4, 5, \) and 6, respectively. Hence, even a small cluster around the central massive black hole limits the possibility to observe and use the periastron shift of the S2 star.

Since we have modeled the star cluster density profile by using a Plummer model, the periastron shift contribution due to the stellar cluster depends on three parameters: the central density \( \rho_0 \) (or equivalently, \( \lambda_{\text{BH}} \)), the core radius \( r_c \), and the power-law index \( \alpha \). This degeneracy in the determination of the stellar cluster parameters is due to the measurement of the periastron shift of a single star. This is easily seen by inspecting Figure 10, which was obtained for illustrative purposes for the S2 star by setting \( \lambda_{\text{BH}} = 0.99 \) and varying both the core radius \( r_c \) and power-law index \( \alpha \) for the star cluster density profile. Each contour line corresponds to a given S2 periastron shift in units of degrees. To solve the parameter degeneracy and determine the stellar cluster parameters (by studying the periastron advance effect), the periastron shifts for at least three different stars have to be measured with sufficient accuracy. Consider, for example, the S16 star with an orbital period of \( \approx 36 \text{ yr} \) and eccentricity \( e = 0.97 \). Measuring its periastron shift and comparing this with the S2 result will give much better information about the stellar cluster parameters. From Figure 11, it is evident that there are regions (intersections between dashed and solid lines) in the \( \alpha-r_c \) plane in which one measures values of the periastron shift for the S2 and S16 stars. Obviously, there could be (as yet unobserved) stars with orbit apocenters comparable to S2 but with different eccentricities (for example, larger than 0.87), or stars closer to the GC black hole than the S2 or S16 stars. Monitoring their orbits and measuring their periastron shifts will be extremely helpful in reconstructing the cluster density profile. As an example, in Figure 12,
we compare the expected S2 periastron shift (solid lines, obtained for $\lambda_{\text{BH}} = 0.99$) with the periastron shift of a star whose orbit has an eccentricity of $\approx 0.87$ and semimajor axis 3 times smaller than that of S2.

As is evident from Figures 8–10, one can obtain estimates of $r_c$, $\alpha$, and $\lambda_{\text{BH}}$, provided that three stars have been observed to sufficient accuracy. Assume that we have adequate accuracy of observation to see periastron shifts of $10^{-2.5}$ mas, which is the value required to see the relativistic periastron shift. To what accuracy have we limited the cluster parameters? To determine this, we could just vary $\lambda_{\text{BH}}$ for a given $r_c$. The effect of this change would be less than the effect of changing $r_c$ and $\lambda_{\text{BH}}$. As such, if we want to know how accurately the cluster parameters are determined, we need to calculate the maximum change in $r_c$ along with the change in $\lambda_{\text{BH}}$, as allowed by the Brownian motion limit. By varying $\lambda_{\text{BH}}$ by $10^{-2}$ and $r_c$ maximally, we find that we get the required accuracy for evaporation rates from 1 to 10 Gyr and Brownian motions of 1.3–2.0 km s$^{-1}$. With this accuracy, we should also be able to separate out the classical periastron shift of the stellar cluster and the relativistic effect of a maximal (and even slightly less than maximal) Kerr black hole. With better accuracy, we should be able to get an estimate or at least an upper bound for the black hole spin as well. The question now is, what is required to achieve this observational accuracy of the periastron shift of three stars? This point is discussed in the next section.

### 6. OBSERVATIONAL REQUIREMENTS FOR DETERMINATION OF BLACK HOLE SPIN AND CLUSTER PARAMETERS

In the near-future, observations using large-diameter telescopes in combination with adaptive optics may allow us to reach the angular resolution needed to measure the periastron shift of stars close to the GC back hole. Consider, for example, an instrument with an angular resolution $\Delta \Phi$ and assume that the relative position of stars can be determined to about $1/\epsilon$ of...
the achieved angular resolution (i.e., the position accuracy is \( \Delta \Phi_p = \Delta \Phi_0 / \epsilon \)). The positional accuracy can be increased by a factor \( \sqrt{N} \) if \( N \) reference stars are used. In this case, the maximum positional accuracy is simply given by

\[
\Delta \Phi_p = \frac{\Delta \Phi_0}{\sqrt{N}}
\]  

(Rubilar & Eckart 2001). It follows that if the periastron position of a star shifts by an amount \( \Delta \Phi_0 \) (as observed from Earth), to obtain the desired accuracy, we need at least that \( \Delta \Phi_p \approx \Delta \Phi_0 \). In this case, the minimum number of reference stars can be determined once both the instrument angular and positional accuracies are known; i.e.,

\[
N_{\text{max}} = \left( \frac{\Delta \Phi_0}{\Delta \Phi_p} \right)^2.
\]  

As an example, the Large Binocular Telescope (LBT) interferometer has angular resolution of \( 0.05 \) mas, and the relative position of stars is conservatively estimated to be about 1/30th of that value (Rubilar & Eckart 2001). Therefore, to measure the periastron shift with adequate accuracy, we need at least the positional shift for the cluster parameters allowed and thereby detect the relativistic shift, we need \( N \) to be \( \approx (6.6 \times 10^{-12} \times 10^{-3})^2 \), or \( 10^5 \) reference stars, which automatically provides the accuracy required to see the maximal Kerr (spin) effect. For PRIMA,7 the relative positional accuracy is planned to be \( \approx 10 \) \( \mu \)as. As such, we would only need a single reference star.

7 See http://www.eso.org/projects/vlti/instru/prima/index_prima.html.

7. CONCLUDING REMARKS

We have used the fact that the stellar cluster close to the central black hole seems spherically symmetric to limit the Brownian motion of Sgr A* to be the observed proper motion. We have taken the stronger (1 \( \sigma \)) limit of 1.3 km s\(^{-1}\) and the weaker (2 \( \sigma \)) limit of 2.0 km s\(^{-1}\) for our calculations. We also used evaporation times of 1–10 Gyr for the cluster, appropriately modified to incorporate the gravitational well due to the black hole, to put further constraints on the cluster mass. The results of our calculations show that the stellar periastron shifts due to the cluster, even limited to the extent considered, may totally swamp not only the Kerr (spin) effect, but also the Schwarzschild effect. However, the discussion focused on the observations for a single star, S2. By modeling the star cluster density profile with a Plummer law, the periastron shift contribution due to the stellar cluster depends on three parameters: the central density \( \rho_c \) (or equivalently, \( \lambda_{\text{BH}} \)), the core radius \( r_c \), and the power-law index \( \alpha \). Consequently, with observations of three stars, we should be able to determine the cluster parameters adequately.8 We have addressed the question of what is required to obtain the desired accuracy for observing the relativistic effect. It turns out that we need about 105 reference stars with the LBT interferometer. With the accuracy expected of PRIMA, it should be enough to use only one reference star.

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