THE CONTINUITY METHOD ON FANO FIBRATIONS

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Abstract. We study finite-time collapsing limits of the continuity method. When the continuity method starting from a rational initial Kähler metric on a projective manifold encounters a finite-time volume collapsing, this projective manifold admits a Fano fibration over a lower dimensional base. In this case, we prove the continuity method converges to a singular Kähler metric on the base in the weak sense; moreover, if the base is smooth and the fibration has no singular fibers, then the convergence takes place in Gromov-Hausdorff topology.

1. Motivation and main result

In [18], La Nave and Tian introduced a new approach to the Analytic Minimal Model Program. It is a continuity method of complex Monge-Ampère equations.

Let $X$ be an $n$-dimensional projective manifold with an ample $\mathbb{Q}$-line bundle $L$. For any fixed Kähler metric $\omega_0 \in 2\pi c_1(L)$, we consider the following continuity method on $X$ introduced by La Nave and Tian in [18] (also see Rubinstein [22]):

$$
\begin{align*}
\omega(t) &= \omega_0 - t \text{Ric}(\omega(t)) \\
\omega(0) &= \omega_0.
\end{align*}
$$

It is proved in [18, Theorem 1.1] that the maximal existence time of (1.1) is

$$
T = \sup \{ t > 0 | [\omega_0] - 2\pi tc_1(X) > 0 \} = \sup \{ t > 0 | L + t K_X \text{ is ample} \}.
$$

According to the Analytic Minimal Model Program proposed in [18], there are several independent cases to consider.

1. $T = \infty$. Then $X$ is a minimal model and it is conjectured that, after a suitable normalization, the continuity method should converge in Gromov-Hausdorff topology to a generalized Kähler-Einstein metric on the canonical model of $X$, see [18] for detailed descriptions and [14, 19, 41] for some progresses.

2. $T < \infty$. By [18, Theorem 1.1, Corollary 2.2] we know $T < \infty$ means $K_X$ is not nef (or $c_1(X)$ contains some “positive part”). In this case, in general it is expected that the continuity method will contract/collapse certain “positive part” of $c_1(X)$ in Gromov-Hausdorff topology. To be precise, we recall some facts from algebraic geometry. By the rationality theorem (see e.g. [21]) we know $T \in \mathbb{Q}$. We always assume without loss of any generality that $T = 1$. By the base-point-free theorem (see e.g. [21]) we know the limiting $\mathbb{Q}$-line bundle $L + K_X$ is semi-ample. In this case, we have a holomorphic map

$$
Y \rightarrow Y \subset \mathbb{CP}^N
$$

induced by the linear system of $m(L + K_X)$ for some sufficiently large integer $m$ (see e.g. [20]). Here $Y = f(X)$ is an irreducible normal projective variety and

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Theorem 1.1. Assume $X$ is an $n$-dimensional projective manifold with an ample $\mathbb{Q}$-line bundle $L$ and $T := \sup \{ t > 0 | L + tK_X \text{ is ample} \} = 1$. Let $f : X \to Y \subset \mathbb{CP}^N$ be the map given in (1.1) with $1 \leq \dim(Y) \leq n - 1$. Define a proper subvariety $S$ of $Y$ by the critical values of $f$. For any Kähler metric $\omega_0 \in 2\pi c_1(L)$, let $\omega(t)_{t \in [0,1]}$ be the unique smooth solution of (1.1) starting from $\omega_0$ on $X$. Then there exists a positive $(1,1)$-current $\omega_Y$ on $Y$, which is a Kähler metric on $Y \setminus S$ for some proper subvariety $S$ of $Y$ and $\omega(t)$ should converge to $f^*\omega_Y$ on $f^{-1}(Y \setminus S)$ in some more regular topology, possibly, the smooth topology.

In this paper, we will focus on the above finite-time volume collapsing case (2.3) and partially confirm conjecture in this case. More precisely, we will prove the following main results.
The current \( \omega_Y \) is canonically constructed in terms of the fibration structure and the initial metric \( \omega_0 \), see Section 2 for more discussions.

A direct consequence of Theorem 1.1 is the following

**Corollary 1.2.** Assume the same as in Theorem 1.1 and, additionally, \( S = \emptyset \), i.e. \( Y \) is smooth and \( f : X \rightarrow Y \) is a holomorphic submersion. Then for any Kähler metric \( \omega_0 \in 2\pi c_1(L) \), there exists a Kähler metric \( \omega_Y \) on \( Y \) such that the unique solution \( \omega(t)_{t\in[0,1]} \) of (1.1) starting from \( \omega_0 \) on \( X \) converges to \( f^*\omega_Y \) in \( C^0(X,\omega_0) \)-topology as \( t \rightarrow 1 \). In particular, \( (X,\omega(t)) \rightarrow (Y,\omega_Y) \) in Gromov-Hausdorff topology as \( t \rightarrow 1 \).

In fact, Corollary 1.2 can be slightly generalized as follows.

**Corollary 1.3.** Let \( f : X \rightarrow Y \) be a holomorphic submersion between two compact Kähler manifolds with \( 1 \leq \dim(Y) < \dim(X) \) and assume there exist a Kähler metric \( \chi \) on \( Y \) such that \( f^*[\chi] + 2\pi c_1(X) \) is a Kähler class on \( X \). Then for any Kähler metric \( \omega_0 \in f^*[\chi] + 2\pi c_1(X) \), there exists a Kähler metric \( \omega_Y \in [\chi] \) on \( Y \) such that \( \omega(t)_{t\in[0,1]} \), the unique solution of (1.1) starting from \( \omega_0 \) on \( X \), converges to \( f^*\omega_Y \) in \( C^0(X,\omega_0) \)-topology as \( t \rightarrow 1 \). In particular, \( (X,\omega(t)) \rightarrow (Y,\omega_Y) \) in Gromov-Hausdorff topology as \( t \rightarrow 1 \).

If we check step by step, then it is clear that Corollary 1.3 can be proved by the same arguments for Corollary 1.2 (or Theorem 1.1). We remark that, in the setting of Theorem 1.1, the rationality of \( [\omega_0] \) is used to provide the Fano fibration (1.2) and the rationality of \( [\chi] \) (see (2.1) in Section 2) is used to obtain higher order regularity of \( \omega_Y \) on \( Y \setminus S \) (where one needs Kodaira lemma, which holds for nef and big \( \mathbb{Q} \)-line bundle, see [29] for details). Therefore, if we are given the setting in Corollary 1.3, then we don’t need to assume \( [\chi] \) nor \( [\omega_0] \) to be rational.

**Remark 1.4.** Very recently, Fu, Guo and Song [14] made a big progress on studying the geometry of the continuity method. They proved that the diameter of \( \omega(t) \) is uniformly bounded. Combining with our Theorem 1.1 it seems very likely that the metrics \( (X,\omega(t)) \) converge to the limiting metric constructed in section 2 below. We shall return to this problem later.

Let’s look at an example.

**Example 1.5.** Let \( \Sigma_a \), \( a \in \mathbb{Z}_{\geq 1} \), be a complete smooth fan in \( \mathbb{R}^2 \) with the minimal generators \( u_0 = (-1,a) \), \( u_1 = (0,1) \), \( u_2 = (1,0) \) and \( u_3 = (0,-1) \). Let \( \mathcal{H}_a \) be the smooth toric variety corresponding to \( \Sigma_a \), which is called the \( a \)-th Hirzebruch surface. We shall explain how to equip a \( \mathbb{CP}^1 \)-bundle structure on \( \mathcal{H}_a \) by extremal contractions. To this end, firstly note that the wall \( \tau = \mathbb{R}_{\geq 0} \cdot u_2 \) is an extremal wall and will give an extremal ray \( \mathcal{R} \) in the Mori cone of \( \mathcal{H}_a \) (see [7] Example 6.3.23]). Moreover, since the wall relation of \( \tau \) is

\[
\begin{align*}
u_1 + 0 \cdot u_2 + u_3 &= 0,
\end{align*}
\]

by [7] Proposition 15.4.5] the extremal contraction given by \( \mathcal{R} \) is a fibration (i.e., Mori fiber space) \( f : \mathcal{H}_a \rightarrow \mathbb{CP}^1 \) and the fibers are isomorphic to the toric variety of the complete fan in \( \mathbb{R} \), that is \( \mathbb{CP}^1 \). Hence all fibers are smooth and \( f \) is in fact a locally trivial fibration (see [4] p.190)). Consequently, \( f \) is a \( \mathbb{CP}^1 \)-bundle. Now the discussions in [4] Section V.4, esp. Proposition V.4.2] imply that \( f \) is exactly the projective bundle \( \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(a)) \rightarrow \mathbb{CP}^1 \), which is the desired conclusion.
Next, by [33] Lemma 2.2] we can find a suitable constant $t_0$ such that $2\pi t_0 c_1(\mathcal{H}_a) + f^*(2\pi c_1(\mathbb{C}P^1))$ is a Kähler class on $\mathcal{H}_a$. Then by Corollary 1.2 or Remark 1.3 the continuity method (1.1) starting from any Kähler metric $\omega_0 \in 2\pi t_0 c_1(\mathcal{H}_a) + f^*(2\pi c_1(\mathbb{C}P^1))$ converges to a Kähler metric $\bar{\omega} \in 2\pi c_1(\mathbb{C}P^1)$ on $\mathbb{C}P^1$ in Gromov-Hausdorff topology. By case (2.1) the continuity method restarting from $\bar{\omega}$ on $\mathbb{C}P^1$ will converge to a point in Gromov-Hausdorff topology. Moreover, if we deform $\bar{\omega}$ by the normalized version (1.3), then it will converge in smooth topology on $\mathbb{C}P^1$ to, up to a biholomorphism, $2\omega_{FS}$ as $t \to \infty$, where $\omega_{FS}$ is the Fubini-Study metric on $\mathbb{C}P^1$ with $Ric(\omega_{FS}) = 2\omega_{FS}$.

Remark 1.6. The finite time collapsing of the Kähler-Ricci flow has been studied in many papers, see [12, 13, 15, 23, 26, 27, 31, 32, 33, 38] and the references therein. For example, the collapsing of Kähler-Ricci flow on Hirzebruch surfaces is studied in [31] (also see [12] for certain generalizations), where the similar picture as in Example 1.5 is obtained by assuming certain symmetry condition on initial metrics. From the view point of Analytic Minimal Model Program with Ricci flow (in particular, see [30, Conjecture 6.6]), the result similar to Theorem 1.4 should be true for Kähler-Ricci flow.

The rest of this paper is organized as follows. We will construct the limiting metric $\omega_Y$ on $Y$ in Section 2. Then we prove a weak convergence in Section 3 and uniform convergence of metric away from singular fibers in Section 4.

2. CONSTRUCTION OF LIMITING METRICS

Assume the same as in Theorem 1.1 Then the limiting class satisfies

$$[\omega_0] - 2\pi c_1(X) = f^* [\chi],$$  \hspace{1cm} (2.1)

where $\chi = \frac{1}{m} \omega_{FS}$ for some positive integer $m$. Here $\omega_{FS} \in 2\pi c_1(O_{\mathbb{C}P^N}(1))$ is the Fubini-Study metric on $\mathbb{C}P^N$.

(2.1) in particular implies, for $y \in Y \setminus S$,

$$\omega_0|_{X_y} \in 2\pi c_1(X_y).$$  \hspace{1cm} (2.2)

By (2.1) we fix a smooth positive volume form $\Omega$ on $X$ with

$$\sqrt{-1} \partial \bar{\partial} \log \Omega = (f^* \chi - \omega_0).$$  \hspace{1cm} (2.3)

For $y \in Y \setminus S$, we denote $\omega_{0,y} := \omega_0|_{X_y}$. Then by using (2.2) and the $\partial \bar{\partial}$-lemma one can choose a $\rho_y \in C^\infty(X_y, \mathbb{R})$ with

$$\begin{cases} Ric(\omega_{0,y}) - \omega_{0,y} = \sqrt{-1} \partial \bar{\partial} \rho_y \\ \int_{X_y} e^{\rho_y}(\omega_{0,y})^{n-k} = \int_{X_y} (\omega_{0,y})^{n-k}. \end{cases}$$  \hspace{1cm} (2.4)

Here $\partial_f$ is the restriction of $\partial$ to the smooth fiber. Define a function $\rho$ on $X_{reg} := f^{-1}(Y \setminus S)$ by setting $\rho(y, \cdot) := \rho_y(\cdot)$. Then $\rho \in C^\infty(X_{reg}, \mathbb{R})$. Moreover, by Yau [40], we have a unique $u_y \in C^\infty(X_y, \mathbb{R})$ satisfying

$$\begin{cases} (\omega_{0,y} + \sqrt{-1} \partial_f \bar{\partial}_f u_y)^{n-k} = e^{\rho_y}(\omega_{0,y})^{n-k} \\ \int_{X_y} u_y(\omega_{0,y})^{n-k} = 0. \end{cases}$$  \hspace{1cm} (2.5)
Define a function $u$ on $X_{\text{reg}} := f^{-1}(Y \setminus S)$ by setting $u(y, \cdot) := u_y(\cdot)$. Then $u \in C^\infty(X_{\text{reg}}, \mathbb{R})$ and $\bar{\omega}_0 := \omega_0 + \sqrt{-1} \partial \bar{\partial} u$ is a closed real $(1,1)$-form on $X_{\text{reg}}$. Denote $\bar{\omega}_{0,y} := \bar{\omega}_0|_{X_y}$, then $\bar{\omega}_{0,y}$ is a Kähler metric on smooth fiber $X_y$ with
\begin{equation}
\text{Ric}(\bar{\omega}_{0,y}) = \omega_{0,y}.
\end{equation}

Define a function
\[ G = \frac{\Omega}{\binom{n}{k} \bar{\omega}_0^{n-k} \wedge (f^* \chi)^k} . \]
Then $G$ can be seen as a smooth positive function on $Y \setminus S$. In fact, by direct computation:
\[ \sqrt{-1} \partial f \bar{\partial} f \log G = \sqrt{-1} \partial f \bar{\partial} f \log \Omega + \text{Ric}(\bar{\omega}_{0,y}) \]
\[ = -\omega_{0,y} + \text{Ric}(\bar{\omega}_0|_{X_y}) \]
\[ = 0 \]
by the definition of $\rho$. Hence $G$ is a constant along each smooth fiber and descends to a smooth positive function on $Y \setminus S$. In fact, as in [29, Lemma 3.3], on $Y \setminus S$ we have
\[ G = f_* \Omega \binom{n}{k} V_0^k \chi, \]
where $V_0 := \int_{X_y} (\omega_{0,y})^{n-k}$ is a positive constant.

Moreover, by [29] Proposition 3.2 and its argument, we can find two positive constants $\delta$ and $\epsilon$ such that
\[ 0 < \delta \leq G \in L^{1+\epsilon}(Y, \chi^k). \]

Consider the following complex Monge-Ampère equation on $Y$
\[ (\chi + \sqrt{-1} \partial \bar{\partial} \psi)^k = e^\psi G \chi^k , \]
where we have used the same notation $\chi$ to denote the restriction of $\chi$ on $Y$.

Having (2.8), by [11] Theorem 4.1 and [29] Theorem 3.2 (building on [40, 16]) there exists a unique solution $\psi \in \text{PSH}(Y, \chi) \cap L^\infty(Y) \cap C^\infty(Y \setminus S)$ to (2.9). We denote $\omega_Y := \chi + \sqrt{-1} \partial \bar{\partial} \psi$, which is a positive $(1,1)$-current on $Y$ and a smooth Kähler metric on $Y \setminus S$. We will see that $\omega_Y$ is the limit of the continuity method (1.1).

3. Estimates and weak convergence

From now on, we will use the following reparametrization of (1.1),
\[ \begin{cases} 
\omega(t) = \omega_0 - (1 - e^{-t}) \text{Ric}(\omega(t)) \\
\omega(0) = \omega_0.
\end{cases} \]
which will be more convenient for later discussions. Obviously, (3.1) has a unique solution $\omega(t)$ for $t \in [0, \infty)$ and the finite time collapsing of (1.1) at $t = 1$ is exactly the infinite time collapsing of (3.1). A useful fact is that for $t \in [1, \infty)$ we have the following uniform lower bound for Ricci curvature along the continuity method (3.1):
\[ \text{Ric}(\omega(t)) \geq -2\omega(t). \]

First of all we reduce the continuity method (3.1) to a complex Monge-Ampère equation as follows. Define
\[ \omega_t = e^{-t} \omega_0 + (1 - e^{-t}) f^* \chi . \]
Then $\omega(t) = \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi(t)$ solves (3.1) if $\varphi(t)$ solves
\[
\begin{aligned}
(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi(t))^n &= e^{-(n-k)t}e^{\bar{\varphi}(t)}\Omega \\
\varphi(0) &= 0.
\end{aligned}
\] (3.3)

**Lemma 3.1.** [9, 10] There exists a constant \( C \geq 1 \) such that
\[
\|\varphi(t)\|_{C^0(X \times [1, \infty))} \leq C.
\]

**Proof.** Firstly note that \( \omega^n_t \leq Ce^{-(n-k)t}\Omega \) on \( X \times [1, \infty) \) for some constant \( C \geq 1 \). Then by the maximum principle, we easily see that \( \sup_{X \times [1, \infty)} \varphi(t) \) is uniformly bounded from above. Moreover, for any \( t \in [1, \infty) \),
\[
e^{2\sup_X \varphi(t)} \int_X \Omega \geq \int_X e^{\bar{\varphi}(t)}\Omega = e^{(n-k)t}\omega^n_t \geq C^{-1}
\]
for some uniform constant \( C \geq 1 \). Hence
\[
\left| \sup_{X \times [1, \infty)} \varphi(t) \right| \leq C. \tag{3.4}
\]

On the other hand, for an arbitrary fixed positive constant \( \epsilon \) and all \( t \in [1, \infty) \), we have
\[
\int_X \left( \frac{e^{\bar{\varphi}(t)}}{e^{(n-k)[\omega_t]^n}} \right)^{1+\epsilon} \leq C,
\]
where \([\omega_t]^n := \int_X \omega^n_t\) and we have used \( e^{(n-k)[\omega_t]^n} \) is uniformly bounded from below. By applying [9, Theorem 2.2] (also see [10]), we find a constant \( C \geq 1 \) such that for all \( t \in [1, \infty) \),
\[
\sup_X \varphi(t) - \inf_X \varphi(t) \leq C. \tag{3.5}
\]

Combining (3.4) and (3.5), Lemma 3.1 is proved.

An immediate consequence of Lemma 3.1 is

**Lemma 3.2.** There exists a constant \( C \geq 1 \) such that
\[
C^{-1}e^{-(n-k)t}\Omega \leq \omega(t)^n \leq Ce^{-(n-k)t}\Omega.
\]

**Lemma 3.3.** There exists a constant \( C \geq 1 \) such that on \( X \times [1, \infty) \),
\[
tr_{\omega(t)}f^*\chi \leq C
\]

**Proof.** By a Schwarz lemma argument (see e.g. [39]), we find a constant \( C \geq 1 \) such that
\[
\Delta_{\omega(t)} \log tr_{\omega(t)}f^*\chi \geq -C tr_{\omega(t)}f^*\chi - C. \tag{3.6}
\]

On the other hand, for \( t \in [1, \infty) \),
\[
\Delta_{\omega(t)} \varphi(t) = tr_{\omega(t)}(\omega(t) - e^{-t}\omega_0 - (1-e^{-t})f^*\chi) \\
\leq n - 2^{-1}tr_{\omega(t)}f^*\chi. \tag{3.7}
\]

Combining (3.6) and (3.7), we can choose a constant \( C \geq 1 \) and a sufficiently large constant \( A \) such that
\[
\Delta_{\omega(t)}(\log tr_{\omega(t)}f^*\chi - A\varphi(t)) \geq tr_{\omega(t)}f^*\chi - C.
\]

Then by the maximum principle and Lemma 3.1, Lemma 3.3 follows.
Before next step, following [35] we fix a smooth nonnegative function \( \varsigma \) on \( X \), which vanishes exactly on singular fibers and satisfies
\[
\varsigma \leq 1, \sqrt{-1}\partial\varsigma \wedge \bar{\partial}\varsigma \leq C f^* \chi, -C f^* \chi \leq \sqrt{-1}\partial\bar{\partial}\varsigma \leq C f^* \chi
\]
on \( X \) for some constant \( C \geq 1 \).

**Lemma 3.4.** Set \( \bar{\psi}_y(t) = \int_{X_y}^{1} \varphi(t)(\omega_{0,y})^{n-k} \) for \( y \in Y \setminus S \). There exists a constant \( C \geq 1 \) such that for all \( y \in Y \setminus S \),
\[
\sup_{X_y \times [2,\infty)} e^t|\varphi(t) - \bar{\psi}_y(t)| \leq C e^{C(-t)}.
\]

**Proof.** Set \( \psi(t) := e^t(\varphi(t) - \bar{\psi}_y(t)) \) and \( \omega_{t,y} := \omega(t)|_{X_y} = e^{-t} \omega_{0,y} + \sqrt{-1} \partial\bar{\partial}\varphi(t)|_{X_y} \). Then
\[
e^t \omega_{t,y} = \omega_{0,b} + \sqrt{-1} \partial\bar{\partial}\psi(t)|_{X_y}
\]
and
\[
(\omega_{0,b} + \sqrt{-1} \partial\bar{\partial}\psi(t)|_{X_y})^{n-k} = e^{(n-k)t} \omega_{t,y}^{n-k}.
\]
Note that
\[
\frac{\omega_{t,y}^{n-k}}{\omega_{0,y}^{n-k}} = \frac{\omega(t)^{n-k} \wedge f^* \chi^k}{\omega_0^{n-k} \wedge f^* \chi^k} = \frac{\omega(t)^{n-k} \wedge f^* \chi^k}{\omega(t)^n} \frac{\omega_0^{n-k} \wedge f^* \chi^k}{\omega_0^{n-k} \wedge f^* \chi^k} \leq C (tr \omega(t) f^* \chi^k) e^{-(n-k)t} \leq C e^{-(n-k)t}.
\]
Hence,
\[
(\omega_{0,b} + \sqrt{-1} \partial\bar{\partial}\psi(t)|_{X_y})^{n-k} = F_y(t) \omega_{0,b}^{n-k}.
\]
where
\[
F_y(t) \leq C \varsigma^{(-t)}
\]
for all \( y \in Y \setminus S \) and \( t \in [1, \infty) \). Now we separate discussions into two cases.

Case (1): \( n-k \geq 2 \). For this case, by applying the arguments in [35], Lemmas 3.2-3.4] we have

1. There exists a uniform constant \( C \geq 1 \) such that for any \( y \in Y \setminus S \), \( t \in [1, \infty) \) and \( u \in C^\infty(X_y) \), we have
\[
\left( \int_{X_y} |u|^2 \omega_0^{n-k} \right)^{\frac{n-k}{n-k-1}} \leq C \left( \int_{X_y} (|\nabla u|^2 + |u|^2) \omega_0^{n-k} \right).
\]

2. There exists a uniform constant \( C \geq 1 \) such that for any \( y \in Y \setminus S \), \( t \in [1, \infty) \) and \( u \in C^\infty(X_y) \) with \( \int_{X_y} \omega_0^{n-k} = 0 \), we have
\[
\int_{X_y} |u|^2 \omega_0^{n-k} \leq C e^{C(-t)} \int_{X_y} |\nabla u|^2 \omega_0^{n-k}.
\]
Now, by combining (3.10)-(3.13) and applying Yau’s $L^\infty$-estimate, we can conclude (3.8) easily.

Case (2): $n - k = 1$. In this case, since the real dimension of a smooth fiber $X_y$ is two (strictly less than three), it seems we can’t not apply the above arguments directly. We now make use of the idea in [28, Corollary 5.2] to achieve (3.8) in this case.

Let $\Delta_{\omega_{0,y}}$ be the Laplacian of $\omega_{0,y}$ on the smooth fiber $X_y$, $G_y(\cdot, \cdot)$ the Green function with respect to $\omega_{0,y}$ on $X_y$ and $A_y := \inf_{X_y \times X_y} G_y(\cdot, \cdot)$. Then by Green formula for any $x \in X_y$ we have

$$\varphi(x) - \frac{1}{V_0} \int_{X_y} \varphi \omega_{0,y} = - \int_{x \in X_y} \Delta_{\omega_{0,y}} \varphi(z)(G_y(x, z) - A_y) \omega_{0,y}(z)$$  \hspace{1cm} (3.14)

To estimate (3.14), we first note that its right hand side

$$\left| - \int_{z \in X_y} \Delta_{\omega_{0,y}} \varphi(z)(G_y(x, z) - A_y) \omega_{0,y}(z) \right| \leq \int_{x \in X_y} \left| \Delta_{\omega_{0,y}} \varphi(z)(G_y(x, z) - A_y) \omega_{0,y}(z) \right|.$$  \hspace{1cm} (3.15)

We now collect several claims.

Claim (1): there exists a positive constant $C \geq 1$ such that for all $y \in Y \setminus S$ and $t \in [1, \infty)$ we have

$$0 < e^{-t} + \Delta_{\omega_{0,y}} \varphi \leq Ce^{-t} \varsigma^{-C}.$$  \hspace{1cm} (3.16)

Claim (1) can be checked as follows:

$$0 < tr_{\omega_{0,y}} \omega_{t,y} = \frac{\omega_{t,y}}{\omega_{0,y}} \leq \frac{\omega(t)}{\omega_0} \wedge f^*\chi = \frac{\omega(t)}{\omega_0} \wedge f^*\chi \leq Ce^{-t} \omega_0 \wedge f^*\chi \leq Ce^{-t} \varsigma^{-C},$$

where we have used Lemmas [3.2] and [3.3] Claim (1) follows.

Claim (2): there exists a constant $C \geq 1$ such that for any $y \in Y \setminus S$ and $t \in [1, \infty)$ we have

$$\frac{\omega_{0,y}}{\omega_{0,y}} \leq C \varsigma^{-C}.$$  \hspace{1cm} (3.17)

In fact, we have

$$\frac{\omega_{0,y}}{\omega_{0,y}} = \frac{\omega_0 \wedge f^*\chi}{\omega_0 \wedge f^*\chi} = \frac{\omega_0 \wedge f^*\chi}{\omega_0 \wedge f^*\chi} \frac{\Omega}{\omega_0 \wedge f^*\chi} \leq C \cdot G \leq C \varsigma^{-C}.$$  

Claim (2) follows.

Claim (3): there exists a constant $C \geq 1$ such that for any $y \in Y \setminus S$ we have

$$diam(X_y, \omega_{0,y}) \leq C \varsigma^{-C}.$$  \hspace{1cm} (3.18)
To see Claim (3), we first recall that, by applying a result of Topping [34, Theorem 1.1] (also see [35, Lemma 3.3], whose argument can be applied to our case directly), we can find a constant $C \geq 1$ such that for all $y \in Y \setminus S$ there holds
\[ \text{diam}(X_y, \omega_{0,y}) \leq C. \] (3.19)

Secondly, we apply a similar argument in Claim (2) to see
\[ \omega_{0,y} \wedge f^*\chi = \omega_{0,y} \wedge \Omega \wedge f^*\chi = \frac{1}{2G} \omega_{0,y} \wedge f^*\chi \leq C, \] (3.20)
where we have used a positive lower bound of $G$ contained in (2.8). Combining (3.19) and (3.20), Claim (3) follows.

Now we can complete the proof. Since we have
\[ \text{Ric}(\omega_{0,y}) = \omega_{0,y} > 0, \]
i.e. Ricci curvature of $\omega_{0,y}$ is uniformly bounded from below by zero, and the volume of $(X_y, \omega_{0,y})$ is a positive constant $V_0$, we can apply a result in [25] Section 1.1 in Chapter 3 (note that in [25] Appendix A of Section 3] a proof for manifold with real dimension greater than or equals to three is provided; for real 2-dimensional case, this result can be checked by using $L^1$-Sobolev inequality and lower bound of isoperimetric constant contained in [8, Proposition 4 and Theorem 13]) to find a uniform positive constant $\gamma$ such that for all $y \in Y \setminus S$ we have
\[ G_y(\cdot, \cdot) \geq -\gamma \frac{\text{diam}(X_y, \omega_{0,y})^2}{V_0}. \] (3.21)

Plugging (3.18) into (3.21) gives
\[ G_y(\cdot, \cdot) \geq -C \zeta^{-C} \]
and hence
\[ A_y \geq -C \zeta^{-C}. \] (3.22)

On the other hand, combining Claims (1) and (2) we have
\[ |\Delta_{\omega_{0,y}} \varphi| = |\Delta_{\omega_{0,y}} \varphi \frac{\omega_{0,y}}{\omega_{0,y}}| \leq C e^{-t} \zeta^{-C}. \] (3.23)

Plugging (3.22) and (3.23) into (3.15) we find that
\[ |\varphi - \frac{1}{V_0} \int_{X_y} \varphi \omega_{0,y}| \leq C e^{-t} \zeta^{-C} \]
on $X_y$ and hence
\[ \sup_{X_y} \varphi - \inf_{X_y} \varphi \leq C e^{-t} \zeta^{-C}, \]
which implies
\[ \sup_{X_y \times [2, \infty)} e^t |\varphi(t) - \bar{\varphi}_y(t)| \leq C \zeta^{-C}. \]

Lemma 3.4 is proved. □

Define a smooth function $\bar{\varphi}(t)$ on $Y \setminus S \times [1, \infty)$ by $\bar{\varphi}(y, t) := \bar{\varphi}_y(t)$. 
Lemma 3.5. [28 Lemma 5.9] There exists a uniform constant $C \geq 1$ such that on $Y \times [1, \infty)$ we have

$$\Delta_{\omega(t)}(e^t (\varphi(t) - \bar{\varphi}(t))) \leq -\text{tr}_{\omega(t)}(t\omega_0) + Ce^t + C\varsigma^{-C}. \quad (3.24)$$

Proof. For a proof, see [28, Lemma 5.9]. \hfill \Box

Lemma 3.6. There exists a uniform constant $C \geq 1$ such that on $Y \times [1, \infty)$ we have

$$\text{tr}_{\omega(t)}(e^{-t}\omega_0) \leq e^{Ce^{Ct-C}}. \quad (3.25)$$

Proof. To prove this lemma, we simply modify arguments in [28 Theorem 5.2]. Firstly, by Schwarz lemma argument (see e.g. [39]) we have a constant $C \geq 1$ such that for $t \in [1, \infty),$

$$\Delta_{\omega(t)} \log \text{tr}_{\omega(t)}(e^{-t}\omega_0) \geq -C\text{tr}_{\omega(t)}\omega - C. \quad (3.26)$$

Combining (3.24) and (3.26), we know for some constants $C \geq 1$ and $A \geq 1$ and all $t \in [1, \infty),$

$$\Delta_{\omega(t)}(\log \text{tr}_{\omega(t)}(e^{-t}\omega_0) - Ae^t(\varphi(t) - \bar{\varphi}(t))) \geq \text{tr}_{\omega(t)}(t\omega_0) - Ce^t - C\varsigma^{-C}. \quad (3.27)$$

Choose a positive constant $C_1$ such that $C_1 - 2$ satisfies Lemma 3.4. Set

$$K = \log \text{tr}_{\omega(t)}(e^{-t}\omega_0) - Ae^t(\varphi(t) - \bar{\varphi}(t))$$

and $H = e^{-C_1\varsigma^{-C_1}}K$. Compute

$$\Delta_{\omega(t)}H = e^{-C_1\varsigma^{-C_1}}\Delta_{\omega(t)}K + K\Delta_{\omega(t)}(e^{-C_1\varsigma^{-C_1}}) + 2Re(\nabla(e^{-C_1\varsigma^{-C_1}}), \nabla K)_{\omega(t)}$$

$$=: I + II + III. \quad (3.28)$$

Firstly, using (3.27), we see that

$$I = e^{-C_1\varsigma^{-C_1}}\Delta_{\omega(t)}K \geq e^{-C_1\varsigma^{-C_1}}(\text{tr}_{\omega(t)}(t\omega_0) - Ce^t - C\varsigma^{-C}) \geq e^{-C_1\varsigma^{-C_1}}\text{tr}_{\omega(t)}(t\omega_0) - Ce^t. \quad (3.29)$$

Secondly, using

$$|\Delta_{\omega(t)}(e^{-C_1\varsigma^{-C_1}})| \leq \frac{C_2e^{-C_1\varsigma^{-C_1}}}{\varsigma^{C_1+2}}$$

for some constant $C_2 \geq 1$, we see that

$$II = K\Delta_{\omega(t)}(e^{-C_1\varsigma^{-C_1}}) \geq -\frac{C_2e^{-C_1\varsigma^{-C_1}}}{\varsigma^{C_1+2}}|\log \text{tr}_{\omega(t)}(t\omega_0)| - Ct$$

$$= -\frac{C_2e^{-C_1\varsigma^{-C_1}}}{\varsigma^{C_1+2}}|2\log \sqrt{\text{tr}_{\omega(t)}(t\omega_0)}| - Ct$$

$$\geq -\frac{C_2e^{-C_1\varsigma^{-C_1}}}{\varsigma^{C_1+2}}\left(\frac{\varsigma^{C_1+2}\text{tr}_{\omega(t)}(t\omega_0)}{4C_2} + 4C_2\varsigma^{-(C_1+2)}\right) - Ct$$

$$\geq -\frac{1}{4}e^{-C_1\varsigma^{-C_1}}\text{tr}_{\omega(t)}(t\omega_0) - Ct. \quad (3.30)$$

On the other hand, using

$$|\nabla \varsigma|_{\omega(t)}^2 \leq \frac{C_3e^{-2C_1\varsigma^{-C_1}}}{\varsigma^{2C_1+2}}$$
for some constant $C_3 \geq 1$, we see that

$$III = 2\text{Re}(\nabla(e^{-C_1\varsigma - C_1}), \nabla K)_{\omega(t)}$$

$$= 2\text{Re}(\nabla(e^{-C_1\varsigma - C_1}), \nabla(\frac{H}{e^{-C_1\varsigma - C_1}}))_{\omega(t)}$$

$$= 2e^{C_1C_1} \text{Re}(\nabla(e^{-C_1\varsigma - C_1}), \nabla H)_{\omega(t)} - \frac{2H}{e^{-2C_1C_1}} |\nabla(e^{-C_1\varsigma - C_1})|^2_{\omega(t)}$$

$$\geq 2e^{C_1C_1} \text{Re}(\nabla(e^{-C_1\varsigma - C_1}), \nabla H)_{\omega(t)} - 2C_3 e^{-C_1\varsigma - C_1} [| \log tr_{\omega(t)}(e^{-t\omega_0})| + |Ae^t(\varphi(t) - \bar{\varphi}(t))|] \varsigma^{-2(1+2)}$$

$$\geq 2e^{C_1C_1} \text{Re}(\nabla(e^{-C_1\varsigma - C_1}), \nabla H)_{\omega(t)} - 2C_3 e^{-C_1\varsigma - C_1} \varsigma^{-2(1+2)} |\log tr_{\omega(t)}\omega_0| - Ct$$

$$\geq 2e^{C_1C_1} \text{Re}(\nabla(e^{-C_1\varsigma - C_1}), \nabla H)_{\omega(t)} - \frac{1}{4} e^{-C_1\varsigma - C_1} tr_{\omega(t)}\omega_0 - Ct. \quad (3.31)$$

Now we plug (3.29), (3.30) and (3.31) into (3.28) and obtain

$$\Delta_{\omega(t)}H \geq \frac{1}{2} e^{-C_1\varsigma - C_1} tr_{\omega(t)}\omega_0 + 2e^{C_1C_1} \text{Re}(\nabla(e^{-C_1\varsigma - C_1}), \nabla H)_{\omega(t)} - Ce^t. \quad (3.32)$$

For any $t \in [1, \infty)$, let $x_t$ be a maximal point of $H(t)$. If $x_t \in Y \setminus Y_{\text{reg}}$, then $H(x_t, t) \leq C$ for some uniform constant $C \geq 1$; if $x_t \in Y_{\text{reg}}$, we apply the maximal principle in (3.32) to see that

$$tr_{\omega(t)}(e^{-t\omega_0}) \leq Ce^{C_1C_1}$$

at $x_t$ and hence

$$H(x_t, t) \leq C$$

for some uniform constant $C \geq 1$. In conclusion, there exists a uniform constant $C \geq 1$ such that for all $t \in [1, \infty)$,

$$H(t) \leq C,$$

from which we see that

$$tr_{\omega(t)}(e^{-t\omega_0}) \leq Ce^{C_1C_1}$$

for some uniform constant $C \geq 1$.

Lemma 3.6 is proved. \qed

**Lemma 3.7.** There exists a constant $C \geq$ such that for all $t \in [1, \infty)$,

$$e^{-Ce^{C_1C_1}} \omega_t \leq \omega(t) \leq e^{Ce^{C_1C_1}} \omega_t. \quad (3.33)$$

**Proof.** The left hand side of (3.33) follows by combining Lemmas 3.3 and 3.6. For the right hand side,

$$tr_{\omega(t)}\omega(t) \leq (n-1)! (tr_{\omega(t)}\omega_t)^{n-1} \omega(t)n \omega_t \leq e^{C(n-1)e^{C_1C_1}} \varsigma \leq e^{Cn\varsigma e^{C_1C_1}}$$

for some uniform constant $C' \geq 1$, from which the right hand side of (3.33) follows. \qed

Lemma 3.7 is proved.

We arrive at the main result in this section.

**Theorem 3.8.** As $t \to \infty$, $\varphi(t) \to f^*\psi$ in $L^1(X, \Omega)$- and $C^{1,\alpha}(X_{\text{reg}}, \omega_0)$-topology for any $\alpha \in (0, 1)$. Here $\psi$ is the unique solution to (2.30). Consequently, as $t \to \infty$, $\omega(t) \to f^*\omega_Y$ in the current sense on $X$.\qed
Proof. Using the same arguments in [11] Lemmas 2.7, 2.8, for any time sequence \( t_j \to \infty \) we can find a subsequence, still denote by \( t_j \), and a \( \psi \in PSH(Y, \chi) \cap L^\infty(Y) \) such that \( \varphi(t_j) \to f^* \psi \) in \( L^1(X, \Omega) \)-topology and \( \psi \) satisfies (2.39) on \( Y \setminus S \), i.e. for any \( K \subset \subset Y \setminus S \) and any given \( \varrho \in C^\infty_0(K) \), there holds

\[
\int_Y \varrho(\chi + \sqrt{-1} \partial \bar{\partial} \hat{\psi})^k = \int_Y \varrho \hat{G} \hat{e} \hat{\psi}^k. \tag{3.34}
\]

To see that \( \hat{\psi} \) satisfies (2.39) on \( S \), we need to recall some arguments in [11, 29] on how to solve (2.39) on \( Y \). Firstly, we choose a resolution of singularities of \( Y \):

\[
\pi : \hat{Y} \to Y,
\]

namely, \( \hat{Y} \) is nonsingular, \( \pi(\hat{Y}) = Y \) and \( \pi : \hat{Y} \setminus \pi^{-1}(S) \to Y \setminus S \) is biholomorphic. Then \( \pi^* \chi \) is a smooth semi-positive closed real (1, 1)-form on \( \hat{Y} \), which is big in the sense that \( \int_{\hat{Y}} (\pi^* \chi)^k > 0 \). We pullback the equation (2.39) to \( \hat{Y} \):

\[
(\pi^* \chi + \sqrt{-1} \partial \bar{\partial} \hat{\psi})^k = e^{\hat{\psi}} \pi^* G(\pi^* \chi)^k. \tag{3.35}
\]

Applying [11, Theorem 4.1] or [29, Theorem 3.2] gives a unique \( \hat{\psi} \in PSH(\hat{Y}, \pi^* \chi) \cap L^\infty(\hat{Y}) \) solving (3.35) on \( \hat{Y} \). Then \( \hat{\psi} \) is constant along every fiber of \( \pi \) and hence decent to the unique bounded solution \( \psi \) on \( Y \) solving (2.39) (see e.g. [11, Theorem 6.3]).

Let’s be back to our proof. Note that the pullback \( \pi^* \hat{\psi} \) on \( \hat{Y} \) of \( \hat{\psi} \) obviously satisfies (3.35) on \( \hat{Y} \setminus \pi^{-1}(S) \). Moreover, since \( \pi^* \hat{\psi} \) is a bounded function on \( \hat{Y} \) and \( \pi^{-1}(S) \) is a proper subvariety of \( \hat{Y} \), we know \( (\pi^* \chi + \sqrt{-1} \partial \bar{\partial} \pi^* \hat{\psi})^k \) takes no mass on \( \pi^{-1}(S) \) (see e.g. [17]). On the other hand, using the fact that \( \pi^* G \in L^{1+\epsilon}(\hat{Y}) \) and Hölder inequality, one easily sees that \( e^{\pi^* \hat{\psi}} \pi^* G(\pi^* \chi)^k \) also takes no mass on \( \pi^{-1}(S) \). In conclusion, \( \pi^* \hat{\psi} \) is also a bounded solution to (3.35) on \( \hat{Y} \). By uniqueness we have \( \hat{\psi} = \pi^* \psi \) and hence \( \psi = \hat{\psi} \).

Therefore, as \( t \to \infty \), \( \varphi(t) \to f^* \psi \) in \( L^1(X, \Omega) \)-topology without passing to a subsequence.

Moreover, for any \( K \subset \subset X_{\text{reg}} \), by Lemma 3.7 we have a positive constant \( C \) such that

\[
\sup_{K \times [1, \infty)} |\Delta_{\omega_t} \varphi(t)| \leq C.
\]

Therefore, given the above \( L^1 \)-convergence, we conclude \( C^{1, \alpha} \)-convergence by standard elliptic equation theory.

Theorem 3.8 is proved. \( \square \)

4. UNIFORM CONVERGENCE AWAY FROM SINGULAR FIBERS

In this section, we will give a proof for second part of Theorem 1.1, i.e. uniform convergence of metric away from singular fibers, by using the strategy developed in [36]. To this end, we need more estimates. For convenience, we will use the following notation:

\( G(t), G_i(t), i = 1, 2, \ldots, \) will always denote a positive function of \( t \) which converge to 0 as \( t \to \infty \).

Let’s begin by the following

Lemma 4.1. [36, Lemma 4.3] There exist a constant \( C \geq 1 \) and a positive function \( G(t) \) with \( G(t) \to 0 \) as \( t \to \infty \) such that

\[
\sup_{Y} e^{-C e^{C_i} G(t)} |\varphi(t) - f^* \psi| \leq G(t). \]

Proof. For a proof, see [36, Lemma 4.3]. \( \square \)
Lemma 4.2. There exist a constant $C \geq 1$ and a positive function $G(t)$ with $G(t) \to 0$ as $t \to \infty$ such that
\[
\sup_Y e^{-C e^{C t}} |\dot{\varphi}(t)| \leq G(t).
\]

Proof. We choose a sufficiently large $C$ such that $E = e^{-C e^{C t}}$ satisfies Lemma 4.4. We begin by collecting two useful identities as follows, which can be checked by direct computations. Firstly, (3.3) can be rewritten as follows:
\[
\varphi(t) = (1 - e^{-t}) \log \frac{e^{(n-k)t\omega(t)^n}}{\Omega}.
\]

Now by taking time derivative of (4.1) we have
\[
(1 - e^{-t}) \Delta_{\omega(t)}(\varphi(t) + \dot{\varphi}(t)) = \dot{\varphi}(t) - (1 - e^{-t})(tr_{\omega(t)} f^*\chi - k) - e^{-t} \log \frac{e^{(n-k)t\omega(t)^n}}{\Omega}
\]
and
\[
\Delta_{\omega(t)}((1 - e^{-t})\dot{\varphi}(t) + 2e^{-t}\dot{\varphi}(t) - (1 - 3e^{-t})\varphi(t))
= \dot{\varphi}(t) + e^{-t} \log \frac{e^{(n-k)t\omega(t)^n}}{\Omega} + (1 - 3e^{-t})tr_{\omega(t)} f^*\chi + (n + 2k)e^{-t} + (1 - e^{-t})|\omega(t)|^2_{\omega(t)} - n.
\]

By applying the maximum principle in (4.2) and using Lemmas 3.1, 3.2 and 3.3 one finds that
\[
\sup_{X \times [1, \infty)} |\dot{\varphi}(t)| \leq C.
\]

Similarly, we apply the maximum principle in (4.3) and see that
\[
\sup_{X \times [1, \infty)} \dot{\varphi}(t) \leq C.
\]

Indeed, for any $t \in [1, \infty)$, at a maximal point $x_t$ of $(1 - e^{-t})\dot{\varphi}(t) + 2e^{-t}\dot{\varphi}(t) - (1 - 3e^{-t})\varphi(t)$, by (4.3) we have
\[
\dot{\varphi}(t)(x_t) \leq C
\]
for some uniform constant $C \geq 1$. But $2e^{-t}\dot{\varphi}(t) - (1 - 3e^{-t})\varphi(t)$ is uniformly bounded by Lemma 3.1 and 4.5. Therefore,
\[
((1 - e^{-t})\dot{\varphi}(t) + 2e^{-t}\dot{\varphi}(t) - (1 - 3e^{-t})\varphi(t))(x_t) \leq C
\]
for some uniform constant $C \geq 1$ and, using again that $2e^{-t}\dot{\varphi}(t) - (1 - 3e^{-t})\varphi(t)$ is uniformly bounded, (4.4) is checked. Now we can complete the proof by an easy argument (see [36, Lemma 4.6]). Assume this lemma fails, then we can find a constant $\delta_0 > 0$, and sequences $t_j \to \infty$, $x_j \in X$ such that
\[
E(x_j)\dot{\varphi}(t_j)(x_j) \geq \delta_0,
\]
which in particular implies that $x_j \in X_{reg}$. On the other hand, by (4.5) we see
\[
\partial_t(E\dot{\varphi}(t)) = E\ddot{\varphi}(t) \leq C.
\]
So $E(x_j)\dot{\varphi}(t, x_j) \geq \frac{\delta_0}{2}$ on $t \in [t_j, t_j + \frac{\delta_0}{C}]$ and hence, by integrating in $t$,
\[
E(x_j)(\varphi - f^*\psi)(t_j + \frac{\delta_0}{C}, x_j) \geq E(x_j)(\varphi - f^*\psi)(t_j, x_j) + \frac{\delta_0^2}{2C},
\]
which implies
\[ G(t_j + \frac{\delta_j}{C}) \geq G(t_j) + \frac{\delta_j^2}{2C}, \]  
(4.6)
where \( G(t) \) is the function in Lemma 4.1. As \( t_j \to \infty \), (4.6) is impossible to hold. Therefore,
\[ \sup_X E\dot{\phi}(t) \leq G_1(t) \]
for some positive function \( G_1(t) \) with \( G_1(t) \to 0 \) as \( t \to \infty \). Similarly, we get
\[ \inf_X E\dot{\phi}(t) \geq -G_1(t). \]
Lemma 4.2 is proved. \( \square \)

Combining Lemmas 4.1 and 4.2 we conclude

**Lemma 4.3.** There exist a constant \( C \geq 1 \) and a positive function \( G(t) \) with \( G(t) \to 0 \) as \( t \to \infty \) such that
\[ \sup_Y e^{-C e^{C_C-C}} |\varphi(t) + \dot{\varphi}(t) - f^*\psi| \leq G(t). \]

**Lemma 4.4.** Let \( G(t) \) be the same function as in Lemma 4.3. There exists a constant \( C \geq 1 \) such that
\[ \sup_Y e^{-Ce^{C_C-C}} (tr_{\omega(t)} f^*\omega_Y - k) \leq C \sqrt{G(t)}. \]

**Proof.** For convenience, we present a proof by following [36, Lemma 4.7]. Choose \( E = e^{-Ce^{C_C-C}} \) satisfies Lemma 4.3. We also choose a sufficiently large constant \( C_1 \) such that \( E_1 = e^{-C_1 e^{C_C-C}} \) satisfies
\[ \frac{\|\delta E_1\|^2_{L^2(O)}}{E_1} \leq CE \]
and \( |\Delta_{\omega(t)} E_1| \leq CE \). On the one hand, we have
\[ \Delta_{\omega(t)} (E_1(\dot{\varphi}(t) + \varphi(t) - f^*\psi)) \]
\[ = E_1 \Delta_{\omega(t)} (\dot{\varphi}(t) + \varphi(t) - f^*\psi) + (\dot{\varphi}(t) + \varphi(t) - f^*\psi) \Delta_{\omega(t)} E_1 + 2Re(\partial E_1, \bar{\partial}(\dot{\varphi}(t) + \varphi(t) - f^*\psi))_{\omega(t)} \]
\[ \leq CG(t) + CE_1 (tr_{\omega(t)} f^*\omega_Y + k + \dot{\varphi} + e^{-t}) + 2Re(\partial E_1, \bar{\partial}(\dot{\varphi}(t) + \varphi(t) - f^*\psi))_{\omega(t)} \]
\[ \leq -CE_1 (tr_{\omega(t)} f^*\omega_Y - k) + CG(t) + 2Re(\partial E_1, \bar{\partial}(\dot{\varphi}(t) + \varphi(t) - f^*\psi))_{\omega(t)}, \]
where we have used (4.2) and Lemma 4.2 and assumed without loss of generality that \( e^{-t} \leq CG(t) \) on \( [1, \infty) \). Hence,
\[ \Delta_{\omega(t)} \left( \frac{E_1(\dot{\varphi}(t) + \varphi(t) - f^*\psi)}{\sqrt{G(t)}} \right) \]
\[ \leq -C \frac{E_1 (tr_{\omega(t)} f^*\omega_Y - k)}{\sqrt{G(t)}} + C \sqrt{G(t)} + 2Re(\partial E_1, \bar{\partial}(\dot{\varphi}(t) + \varphi(t) - f^*\psi))_{\omega(t)}. \]  
(4.7)
On the other hand, using [29, Section 3], we know \( \omega_Y \leq C\zeta^{-C} \chi \) on \( Y\setminus S \) and the bisectional curvature of \( \omega_Y \) on \( Y\setminus S \) has an upper bound of the form \( \zeta^{-C'} \) for some constant \( C' \geq 1 \), so by a Schwarz lemma argument we see
\[ \Delta_{\omega(t)} tr_{\omega(t)} f^*\omega_Y \geq -C' tr_{\omega(t)} f^*\omega_Y - C' \zeta^{-C'} (tr_{\omega(t)} f^*\omega_Y)^2 \geq -C \zeta^{-C} \]  
(4.8)
Therefore, we have

\[
\Delta_{\omega(t)}(E_1(tr_{\omega(t)}f^*\omega_Y - k))
= (tr_{\omega(t)}f^*\omega_Y - k)\Delta_{\omega(t)}E_1 + E_1\Delta_{\omega(t)}(tr_{\omega(t)}f^*\omega_Y - k) + 2Re(\partial E_1, \bar{\partial}(tr_{\omega(t)}f^*\omega_Y - k))
\geq -CE_1\varsigma - C + 2Re(\partial E_1, \bar{\partial}(tr_{\omega(t)}f^*\omega_Y - k))
\geq 2Re(\partial E_1, \bar{\partial}(tr_{\omega(t)}f^*\omega_Y - k)) - C.
\]

(4.9)

Now we set \( H_1 = E_1(tr_{\omega(t)}f^*\omega_Y - k) - \frac{E_1(\bar{\varphi}(t) + \varphi(t) - f^*\bar{\psi})}{\sqrt{G(t)}}. \) Note that \( H_1 \) is uniformly bounded and we want to show that it converges to 0 uniformly by applying the maximum principle. By (4.7) and (4.9) we see that

\[
\Delta_{\omega(t)}H_1 \geq CE_1(\frac{tr_{\omega(t)}f^*\omega_Y - k}{\sqrt{G(t)}} - C\sqrt{G(t)} - C
- 2Re(\partial E_1, \bar{\partial}(\bar{\varphi(t)} + \varphi(t) - f^*\bar{\psi}))\omega(t) + 2Re(\partial E_1, \bar{\partial}(tr_{\omega(t)}f^*\omega_Y - k))\omega(t)
\]

\[
= CE_1(\frac{tr_{\omega(t)}f^*\omega_Y - k}{\sqrt{G(t)}} - C + 2Re(\partial E_1, \bar{\partial}(\frac{H_1}{E_1}))\omega(t)
\]

\[
= CE_1(\frac{tr_{\omega(t)}f^*\omega_Y - k}{\sqrt{G(t)}} - C + 2\frac{E_1}{E_1} Re(\partial E_1, \bar{\partial}H_1)\omega(t) - \frac{2H_1}{E_1} |\partial E_1|_\omega(t)
\]

\[
\geq CE_1(\frac{tr_{\omega(t)}f^*\omega_Y - k}{\sqrt{G(t)}} + \frac{2}{E_1} Re(\partial E_1, \bar{\partial}H_1)\omega(t) - C
\]

(4.10)

Then by applying the maximum principle to (4.10), we easily complete the proof of this lemma.

Lemma 4.3 is proved.

□

Lemma 4.5. For any \( K \subset Y \setminus S \), there exists a constant \( C = C_K \geq 1 \) such that for all \( y \in K \) we have

\[
|e^{t}\omega(t)|_{X_y \cap C^1(\omega_0, y)} \leq C.
\]

(4.11)

Proof. We will make use of some arguments in [37] Theorem 1.1. For any given \( y \in K \) and \( x \in X_y \), we choose a local chart \((U, w^1, \ldots, w^{n-k}, y^1, \ldots, y^k)\) on \( X \) centered at \( x \) and local chart \((V, y^1, \ldots, y^k)\) on \( Y \) centered at \( y \) such that in these local charts \( f \) is given by \((w^1, \ldots, w^{n-k}, y^1, \ldots, y^k) \mapsto (y^1, \ldots, y^k)\). We also assume that \( U = \{ (w^1, \ldots, w^{n-k}, y^1, \ldots, y^k) \in \mathbb{C}^n| |w_i| < 1, |y_i| < 1, i = 1, \ldots, n-k, \alpha = 1, \ldots, k \} \) be the polydisc in \( \mathbb{C}^n \) centered at 0 and \( D \) be the unit polydisc in \( \mathbb{C}^{n-k} \).

Consider the maps \( F_t: D \times B_{\frac{1}{2}}^k \to U, F_t(w, y) = (w, ye^{\frac{-t}{2}}) \). Note that \( F_t \) is the identity when restricting to \( D \times \{ 0 \} \). On \( U \) we can write

\[
\omega_0(w, y) = \sqrt{-1} \left( \sum_{i,j=1}^{n-k} g_{ij}(w, y)dw^i \wedge d\bar{w}^j + \sum_{\alpha,\beta=1}^{k} g_{\alpha\beta}(w, y)dy^\alpha \wedge d\bar{y}^\beta \right)
+ 2Re\sqrt{-1} \sum_{i=1}^{n-k} \sum_{\alpha=1}^{k} g_{i\alpha}(w, y)dw^i \wedge dy^\alpha
\]

(4.12)
and then
\[ F_t^*\omega_0(w, b) = \sqrt{-1} \sum_{i,j=1}^{n-k} g_{ij}(w, ye^{-\frac{t}{2}})dw^i \wedge dw^j + e^{-t} \sqrt{-1} \sum_{\alpha, \beta=1}^{k} g_{\alpha\beta}(w, ye^{-\frac{t}{2}})dy^\alpha \wedge d\bar{y}^\beta + 2e^{-\frac{t}{2}}Re \sqrt{-1} \sum_{i=1}^{n-k} \sum_{a=1}^{k} g_{i\bar{a}}(w, ye^{-\frac{t}{2}})dw^i \wedge dy^a. \]

Note that $F_t^*\omega_0$ converges on any compact subsets of $D \times \mathbb{C}^k$, as $t \to \infty$, to a smooth semi-positive real $(1,1)$-form $\eta = \sqrt{-1} \sum_{i,j=1}^{n-k} g_{ij}(w, 0)dw^i \wedge dw^j$, which is strictly positive in fiber direction. Also note that
\[ e^t F_t^* f^* \chi = \sqrt{-1} \sum_{\alpha, \beta=1}^{k} \chi_{\alpha\beta}(w, ye^{-\frac{t}{2}})dy^\alpha \wedge d\bar{y}^\beta \]
converges on any compact subsets of $D \times \mathbb{C}^k$, as $t \to \infty$, to a semi-positive real $(1,1)$-form $\eta' = \sqrt{-1} \sum_{\alpha, \beta=1}^{k} \chi_{\alpha\beta}(0)dy^\alpha \wedge d\bar{y}^\beta$, which is strictly positive in base direction. In conclusion, $e^t F_t^* \omega_0 = (1-e^{-t})e^t F_t^* f^* \chi + F_t^* \omega_0$ converges on any compact subsets of $D \times \mathbb{C}^k$, as $t \to \infty$, to a smooth Kähler metric $\eta + \eta'$, which is equivalent to the standard Euclidean metric $\omega_E$ on $D \times \mathbb{C}^k$. Therefore, for any $K' \subset D \times \mathbb{C}^k$, by Lemma 5.7 one can find a constant $C = C_{K'} \geq 1$ with
\[ C^{-1}\omega_E \leq e^t F_t^* \omega(t) \leq C\omega_E \quad (4.13) \]
on $K$, where $\omega_E$ is the Euclidean metric. On the other hand, we have
\[ Ric(e^t F_t^* \omega(t)) = \frac{1}{1 - e^{-t}} F_t^* \omega_0 - \frac{e^{-t}}{1 - e^{-t}}(e^t F_t^* \omega(t)). \quad (4.14) \]

With all the above preparations, now we are able to derive a $C^1$-estimate for $e^t F_t^* \omega(t)$, so called Calabi-type estimate, by modifying [24]. For simplicity, we write $\tilde{\omega} = \omega(t) = e^t F_t^* \omega(t)$. Firstly, we fix a slightly large $K'' \subset D \times \mathbb{C}^k$ containing $K'$ in its interior, so that, for some uniform constant $C = C_{K''}$,
\[ C^{-1}\omega_E \leq \tilde{\omega}(t) \leq C\omega_E. \quad (4.15) \]

Secondly, we choose a smooth cut-off function $\varrho$ supported in $K''$ with $\varrho \equiv 1$ on $K'$. By (4.15), we can choose a $\varrho$ satisfying
\[ |\partial \varrho|_{\omega(t)} + |\Delta \varrho|_{\omega(t)} \leq C \]
for some uniform constant $C \geq 1$. Let $\nabla^E, \Gamma^E$ be the connection and Christoffel symbols of $\omega_E$ and $\tilde{\nabla}, \tilde{\Gamma}$ those of $\tilde{\omega}$. Define $T$ to be the tensor that is difference of the Christoffel symbols of $\tilde{\omega}$ and $\omega_E$ (in fact, $\Gamma^E \equiv 0$ and hence $T = \tilde{\Gamma}$) and $S = |T|_{\tilde{\omega}}^2$. Easily, we have
\[ S = |T|_{\tilde{\omega}}^2 = |\nabla^E \tilde{\omega}|_{\tilde{\omega}}^2 = \tilde{g}^{ib} \tilde{g}^{\bar{a}j} \tilde{g}^{\bar{p}q} \partial_i \tilde{g}_{\bar{j}k} \tilde{g}_{\bar{p}q}. \quad (4.16) \]

Using commutation formula, one obtains
\[ \Delta S = |\nabla T|_{\tilde{\omega}}^2 + |\tilde{\nabla} T|_{\tilde{\omega}}^2 + 2Re(\tilde{g}^{ib} \tilde{g}^{\bar{a}j} \tilde{g}_{\bar{k}l} \Delta \omega T_{ij}^k T_{jq}) + \tilde{g}^{ib} \tilde{g}^{\bar{a}j} \tilde{g}^{\bar{p}q} T_{ij}^k T_{jq} \tilde{R}_{ab} + \tilde{g}^{ib} \tilde{g}^{\bar{a}j} \tilde{g}^{\bar{p}q} T_{ij}^k T_{jq} \tilde{R}_{ab} - \tilde{g}^{ib} \tilde{g}^{\bar{a}j} \tilde{g}^{\bar{p}q} T_{ij}^k T_{jq} \tilde{R}_{kl}. \quad (4.17) \]

By plugging (4.14) into the last three terms of (4.17), and noting (4.13) and the fact that, if we write $\tilde{\omega} = F_t^* \omega_0$, then $0 \leq \tilde{\omega} \leq C\omega_E$ on $K'' \times [1, \infty)$ for some uniform constant.
Now we combine (4.21) and (4.22) to see that, for a sufficiently large constant 
plugging (4.19) into (4.18) we see that
\[ \Delta \tilde{\omega} S \geq |\tilde{\nabla} T|_{\omega}^2 + |\overline{\nabla} T|_{\tilde{\omega}}^2 + 2Re(\tilde{g}^{ib}\tilde{g}_{ip}(\Delta \tilde{\omega} T_{ip})T'_{jq}) - CS. \] (4.18)
Moreover, by the symmetry of curvature tensor and Bianchi identities we see
\[
\Delta \tilde{\omega} T_{ip}^k = \tilde{g}^{ip} \tilde{\nabla}_a \tilde{\nabla}_b T_{ip}^k = -\tilde{g}^{ik} \tilde{\nabla}_i \tilde{R}_{ipd} = -\tilde{g}^{ik} \tilde{\nabla}_i (\frac{1}{1 - e^{-t}} \tilde{g}_{pd}) = -\frac{1}{e^t - 1} \tilde{g}^{ik} (\tilde{\nabla}_i - \tilde{\nabla}_i E) \tilde{g}_{pd} - \frac{1}{e^t - 1} \tilde{g}^{ik} \tilde{\nabla}_i \tilde{g}_{pd} \]
(4.19)
Note that for some constant \( C \geq 1 \) we have \( |\tilde{\nabla}^E \tilde{\omega}|_{\omega_E} \leq C \) on \( K'' \times [1, \infty) \). Then by plugging (4.19) into (4.18) we see that
\[ \Delta \tilde{\omega} S \geq |\tilde{\nabla} T|_{\omega}^2 + |\overline{\nabla} T|_{\tilde{\omega}}^2 - CS - C, \] (4.20)
and hence
\[ \Delta \tilde{\omega} (g^2 S) \geq g^2 (|\tilde{\nabla} T|_{\omega}^2 + |\overline{\nabla} T|_{\tilde{\omega}}^2) - CS + 2Re(\tilde{g}^2, \overline{\nabla} S)_{\tilde{\omega}} \geq -CS, \] (4.21)
where we have used the following
\[ 2Re(\tilde{g}^2, \overline{\nabla} S)_{\tilde{\omega}} \geq -Cg(|\tilde{\nabla} \tilde{\omega}, \overline{\nabla} T|_{\omega}^2) \geq -Cg|\tilde{\nabla} T|_{\omega}^2 \geq -2(|\tilde{\nabla} T|_{\omega}^2 + |\overline{\nabla} T|_{\tilde{\omega}}^2) - CS \]
Also recall that, using (4.15) and the Ricci lower bound given by (4.14), we have
\[
\Delta \tilde{\omega} \operatorname{tr} \tilde{\omega} \omega_E \geq -C + \tilde{g}^{ji} \tilde{g}^{pq} g_{E}^{ba} \tilde{\nabla}_i g_{E,pb} \tilde{\nabla}_j g_{E,aq} \]
\[ = -C + \tilde{g}^{ji} \tilde{g}^{pq} g_{E}^{ba} (-T_{ip}^{a} g_{E,e,ib}) (-T_{ja}^{a} g_{E,fq}) \]
\[ \geq CS - C. \] (4.22)
Now we combine (4.21) and (4.22) to see that, for a sufficiently large constant \( A \),
\[ \Delta \tilde{\omega} (g^2 S + A \operatorname{tr} \tilde{\omega} \omega_E) \geq S - C. \] (4.23)
For any \( t \in [1, \infty) \), we choose a maximal point \( x_t \in K'' \) of \( g^2 S + A \operatorname{tr} \tilde{\omega} \omega_E \). If \( x_t \in \partial K'' \), then
\[ (g^2 S + A \operatorname{tr} \tilde{\omega} \omega_E)(x_t) \leq A \operatorname{tr} \tilde{\omega} \omega_E(x_t) \]
is uniformly bounded from above; if \( x_t \in K'' \), then by applying the maximum principle to (4.23) we also have \( (g^2 S + A \operatorname{tr} \tilde{\omega} \omega_E)(x_t) \) is uniformly bounded from above. In conclusion
\[ \sup_{K'' \times [1, \infty)} (g^2 S + A \operatorname{tr} \tilde{\omega} \omega_E) \leq C \]
for some uniform constant \( C \geq 1 \). Therefore, using the fact that \( g \equiv 1 \) on \( K' \), we see
\[ \sup_{K' \times [1, \infty)} S \leq C. \]
Now, we restrict \( S \) to fiber \( X_y \) and use the fact that \( \tilde{\omega}|_{X_y} \) is equivalent to \( \omega_{0,y} \) to conclude that
\[ |e^t \omega(t)|_{X_y}|C^1(X_y, \omega_{0,y}) \leq C. \] (4.24)
Moreover, from the arguments we easily see that the above bound \( C \) can be chosen to be uniform when \( y \) varies in a compact subset of \( Y \setminus S \).

Lemma 4.3 is proved. \( \square \)

Before next step, we recall the following two facts from [36, Section 2.3].
Lemma 4.6. [36, Lemma 2.4] For any given $K \subset Y \setminus S$, consider a smooth function $P : f^{-1}(K) \times [1, \infty) \to \mathbb{R}$ which satisfies the following conditions:

(a) There is a constant $A$ such that for all $y \in K$ and all $t \geq 1$
\[|\nabla(P|_{Y_y})|_{\omega_0, y} \leq A.\]

(b) For all $y \in K$ and all $t \geq 1$ we have
\[\int_{x \in X_y} P(x, t) \omega^n_{0, b} = 0.\]

(c) There exists a function $G : [1, \infty) \to [0, \infty)$ such that $G(t) \to 0$ as $t \to \infty$ such that
\[\sup_{y \in f^{-1}(K)} P(y, t) \leq G(t),\]
for all $t \geq 1$.

Then there is a constant $C$ such that
\[\sup_{y \in f^{-1}(K)} |P(y, t)| \leq CG(t)^{\frac{1}{n+1}}\]
for all $t$ sufficiently large.

Lemma 4.7. [36, Lemma 2.6] Let $A$ be an $N \times N$ positive definite Hermitian symmetric matrix. Assume that there exists $\epsilon \in (0, 1)$ with
\[\text{tr } A \leq N + \epsilon, \quad \det A \geq 1 - \epsilon.\]

Then there exists a constant $C_N$ depending only on $N$ such that
\[\|A - I\| \leq C_N \sqrt{\epsilon},\]
where $\| \cdot \|$ is the Hilbert-Schmidt norm, and $I$ is the $N \times N$ identity matrix.

Now we first prove the convergence of metric on smooth fibers. We point out that in [36] one is given a Calabi-Yau fibration and hence there exists a canonical metric, i.e. Ricci-flat metric, in the initial class on any smooth Calabi-Yau fiber. Moreover, a key step in [36] is to prove the restriction to a smooth Calabi-Yau fiber of a suitably normalized equation (namely, Kähler-Ricci flow or degeneration of Ricci-flat metrics) will converge to the Ricci-flat metric. In our case, the smooth fiber $X_y$ is a Fano manifold and we may not have a canonical metric on $X_y$. However, we do have defined an $\overline{\omega}_0$ in the initial class $[\omega_0]$, which is naturally associated to $\omega_0$ in the sense that $\text{Ric}(\overline{\omega}_{0, y}) = \omega_{0, y}$. We now prove in the following Lemma 4.8 that, after restricting to a smooth fiber $X_y$, a suitably normalized continuity method will converge to $\overline{\omega}_{0, y}$. This is a key observation for later discussions.

Lemma 4.8. For any given $K \subset Y \setminus S$, there exists a positive function $G_1(t)$ such that for all $y \in K$ we have
\[|e^t \omega(t)|_{X_y} - \overline{\omega}_{0, y}|_{C^0(X_y, \omega_{0, y})} \leq G_1(t).\]

Proof. Write
\[
(e^t \omega(t)|_{X_y})^{n-k} = \frac{(e^t \omega(t)|_{X_y})^{n-k}}{\overline{\omega}_{0, y}^{n-k}} \omega_0^{n-k} = e^{(n-k)t} \omega(t)^{n-k} \cdot f^* \omega_Y^{n-k} = \left( \begin{array}{c} n \\ k \end{array} \right) e^{(n-k)t} \omega(t)^{n-k} \cdot f^* \omega_Y^{n-k} \overline{\omega}_{0, y}^{n-k} = \left( \begin{array}{c} n \\ k \end{array} \right) e^{(n-k)t} \omega(t)^{n-k} \cdot f^* \omega_Y^{n-k} \overline{\omega}_{0, y}^{n-k} \\
\cdot e^{* \psi} \Omega \omega(t)^n \overline{\omega}_{0, y}^{n-k}.
\]
Then we define a function $P$ on $X \times [1, \infty)$ as follows:

$$P = \left( \binom{n}{k} e^{\frac{\omega(t)}{n-k} - f^\ast \omega(t)^{n-k} \wedge f^\ast \omega_Y} \right).$$

Easily,

$$P|_{X_y} = \frac{(e^t \omega(t))^{n-k}}{\omega_{0,y}^{n-k}}.$$

We see that $\hat{P} := P - 1$ satisfies the following properties:

1. $\int_{X_y} \hat{P} \omega_{0,y}^{n-k} = 0$ for all $y \in K$;
2. There exist a constant $C \geq 1$ such that for all $y \in K$, $\left| \nabla \hat{P} \right|_{X_y} \omega_{0,y} \leq C$.
3. As $t \to \infty$, $\sup_{f^{-1}(K)} \hat{P} \leq G_2(t)$ for some positive function $G_2(t)$ which converges to 0 as $t \to \infty$.

Indeed, item (1) is obvious; item (2) follows from Lemma 4.4 directly; for item (3), we use Lemma 4.4 to see that

$$\left( \binom{n}{k} \frac{\omega(t)^{n-k} \wedge f^\ast \omega_Y}{\omega(t)^n} \right)^{n-k} \leq 1 + G_2(t),$$

and then, combining Lemma 4.1, we have item (3).

Using above items (1)-(3), we can conclude by Lemma 4.6 that

$$\sup_{f^{-1}(K)} |\hat{P}| \leq G_4(t)$$

and hence

$$\|(e^t \omega(t)|_{X_y})^{n-k} - \omega_{0,y}^{n-k} \|_{\mathbb{C}^0(X_y, \omega_{0,y})} \leq G_5(t).$$

Next we define the following smooth function on $X_y$ for $y \in K$:

$$Q|_{X_y} = \frac{e^t \omega(t)|_{X_y} \wedge \omega_{0,y}^{n-k-1}}{\omega_{0,y}^{n-k}}.$$

Easily, $Q$ in fact smoothly depends in $y \in K$ and hence is a smooth function on $f^{-1}(K)$, which equals

$$Q = \frac{e^t \omega(t) \wedge \omega_{0}^{n-k-1} \wedge f^\ast \omega_Y^k}{\omega_{0}^{n-k} \wedge f^\ast \omega_Y^k}.$$

Set $\tilde{Q} = 1 - Q$, then $\tilde{Q}$ satisfies the following properties:

1. $\int_{X_y} \tilde{Q} \omega_{0,y}^{n-k} = 0$ for all $y \in K$;
2. There exist a constant $C \geq 1$ such that for all $y \in K$, $\left| \nabla \tilde{Q} \right|_{X_y} \omega_{0,y} \leq C$.
3. As $t \to \infty$, $\sup_{f^{-1}(K)} \tilde{Q} \leq G_6(t)$ for some positive function $G_6(t)$ which converges to 0 as $t \to \infty$.

Again, items (1') and (2') are clear to hold; for item (3'), we use the arithmetic-geometric means inequality to see that

$$(Q|_{X_y})^{n-k} \geq \frac{(e^t \omega(t)|_{X_y})^{n-k}}{\omega_{0,y}^{n-k}} \geq 1 - G_7(t),$$

where in the last inequality we have used (4.26), so item (3') follows.

Therefore, using again Lemma 4.6 we have

$$\sup_{f^{-1}(K)} |\tilde{Q}| \leq G_8(t),$$
which in particular implies that for all \( y \in K \) we have
\[
\| (e^t \omega(t)) \|_{C^0_{(f^{-1}(K),\omega)}} \leq G_9(t). \tag{4.27}
\]

Now applying Lemma 4.7 gives the desired conclusion (4.25).

Lemma 4.8 is proved.

We are ready to prove our main result in this section.

**Theorem 4.9.** For any given \( K \subset Y \setminus S \), there exists a positive function \( G_{10}(t) \) and a constant \( T \geq 1 \) such that for all \( t \in [T, \infty) \) we have
\[
\| \omega(t) - f^* \omega_Y \|_{C^0_{(f^{-1}(K),\omega)}} \leq G_{10}(t). \tag{4.28}
\]

**Proof.** We will make use of some arguments in [36]. Set \( \dot{\omega}(t) := e^{-t} \omega_0 + (1 - e^{-t}) f^* \omega_Y \) and choose a sufficiently large \( T \) such that \( \dot{\omega}(t) \) is a Kähler metric on \( f^{-1}(K) \) for all \( t \geq T \).

Obviously, as \( t \to \infty \),
\[
\| \dot{\omega}(t) - f^* \omega_Y \|_{C^0_{(f^{-1}(K),\omega)}} \to 0. \tag{4.29}
\]

Moreover, after possibly increasing \( T \), by Lemma 4.1 we also have, for all \( t \geq T \),
\[
\begin{align*}
\frac{\dot{\omega}(t)}{\omega(t)} & \geq \frac{(n) e^{-(n-k+1)} \omega_0 \wedge f^* \omega_Y}{\omega(t)} + o(e^{-(n-k+1)t}) \\
& \geq e^{1 - e^{-t} f^* \omega_Y} - Ce^{-t} \\
& \geq 1 - G_{11}(t). \tag{4.30}
\end{align*}
\]

On the other hand, for any fix \( x \in f^{-1}(K) \) and \( y = f(x) \in K \) we write
\[
tr_{\omega(t)} \dot{\omega}(t) = tr_{\omega(t)} f^* \omega_Y + tr_{\omega(t)} (e^{-t} \omega_0) + e^{-t} (tr_{\omega(t)} f^* \omega_Y + tr_{\omega(t)} (\omega_{SRF} - \omega_{SRF,b})) \\
\leq tr_{\omega(t)} f^* \omega_Y + tr_{\omega(t)} (e^{-t} \omega_0) + e^{-t} tr_{\omega(t)} (\omega_0 - \omega_0) \
\leq tr_{\omega(t)} (\omega_0 - \omega_0) \tag{4.31}
\]

To bound the last term in (4.31), we now show that
\[
|tr_{\omega(t)} (\omega_0 - \omega_0)| \leq Ce^{\frac{t}{2}}. \tag{4.32}
\]

Indeed, if we choose a local chart \( (U, z^n) \) on \( X \) centered at \( x \) and a local chart \( (V, z^n) \) on \( Y \) centered at \( y \) such that \( f \) is given by \( (z^n) \mapsto (z_{n-k+1}, \ldots, z^n) \), then, since \( \omega_0 - \omega_0 \) vanishes on fiber \( X_y \), we can write
\[
\omega_0 - \omega_0 = \sum_{\alpha=n-k+1}^{n} \sum_{i=1}^{n} \Psi_{\alpha i} dz^\alpha \wedge d\bar{z}^i
\]

for some smooth complex-valued functions \( \Psi_{\alpha i} \). Also note that by Lemma 3.7 and Cauchy-Schwarz inequality we have
\[
|\tilde{g}^{\alpha i}| \leq Ce^\frac{t}{2}
\]
whenever \( n - k + 1 \leq \alpha \leq n \). Then
\[
|tr_{\omega(t)} (\omega_0 - \omega_0)| = 2Re \sum_{\alpha=n-k+1}^{n} \sum_{i=1}^{n} g^{\alpha i} \Psi_{\alpha i} \leq Ce^\frac{t}{2}. \tag{4.33}
\]
Of course, the constant $C$ in the above inequality can be chosen to be uniform for all $x \in f^{-1}(K)$. Therefore, by plugging (4.33) into (4.31) and then using Lemmas 4.4 and 4.8 we have

$$tr_\omega(t) \hat{\omega}(t) \leq tr_\omega(t) f^*\omega_Y + tr_\omega(t) (e^{-t} \omega_{0,Y}) + Ce^{-\frac{t}{2}}$$

$$= tr_\omega(t) f^*\omega_Y + tr_\omega(t) e^{-t} \omega_{0,Y} + Ce^{-\frac{t}{2}}$$

$$\leq k + (n-k) + G_{12}(t)$$

$$= n + G_{12}(t)$$

(4.34)

Having (4.30) and (4.34), we can apply Lemma 4.7 to see that

$$\|\omega(t) - \hat{\omega}(t)\|_{C^0(f^{-1}(K), \omega(t))} \leq G_{12}(t),$$

which, combining the fact by Lemma 3.7 that $\omega(t) \leq C\omega_0$ on $f^{-1}(K)$ for some constant $C$, implies

$$\|\omega(t) - \hat{\omega}(t)\|_{C^0(f^{-1}(K), \omega_0)} \leq G_{13}(t).$$

(4.35)

Combining (4.29) and (4.35), we have proved (4.28).

Theorem 4.9 is proved. □

Proof of Theorem 1.1

Combining Theorems 3.8 and 4.9, Theorem 1.1 follows. □

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