Quasi-particle Tunneling Through a Barrier in the Fractional Quantum Hall Regime

Elad Shopen\textsuperscript{1}, Yuval Gefen\textsuperscript{2}, Yigal Meir\textsuperscript{1,3}

\textsuperscript{1}Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel
\textsuperscript{2}Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76100, Israel and
\textsuperscript{3}The Ilse Katz Center for Meso- and Nano-Scale Science and Technology, Ben-Gurion University, Beer-Sheva 84105, Israel

Tunneling of fractionally charged quasi-particles (QPs) through a barrier is considered in the context of a multiply connected geometry. In this geometry global constraints do not prohibit such a tunneling process. The tunneling amplitude is evaluated and the crossover from mesoscopic QP-dominated to electron-dominated tunneling as the system’s size is increased is found. The presence of disorder enhances both electron and QP tunneling rates.

One of the most remarkable facts about the fractional quantum Hall effect (FQHE) is the existence of fractionally charged quasi-particles (QPs). Their dynamics is manifest in a host of physical phenomena, whose observation strongly supported the veracity of Laughlin’s theory\cite{1}. It has been pointed out \cite{2, 3} that QP tunneling is distinctly different from electron tunneling. Perturbative renormalization-group analysis \cite{4} has indicated that in the weak backscattering limit inter-edge tunneling through the FQHE liquid is dominated by QP tunneling. These predictions have been confirmed by experiments \cite{5}. In the opposite limit of strong backscattering (nearly disconnected FQHE systems coupled by weak tunneling through an insulator), the same RG analysis would have predicted that tunneling should be dominated again by QP tunneling. Common wisdom, however, has it that in this limit only electron tunneling is possible. The rationale for that goes as follows: consider two FQHE puddles weakly connected through tunneling. The total number of electrons on each puddle is (nearly) a good quantum number; hence it must be an integer. QP tunneling would render this number non-integer, therefore such a process must be excluded.

Our starting point here is to note that there are setups where the above mentioned ”global constraint” (i.e. the number of electrons on each side of the barrier being an integer) does not exclude a-priori QP tunneling through a potential barrier. The common wisdom concluded to above needs then to be re-examined. Studying these setups is particularly interesting in view of recent experimental results \cite{6} which suggest the coexistence of both electron and QP tunneling under strong backscattering conditions.

Consider first the annulus depicted in Fig. 1(a). Clearly, the passage of a QP through the barrier would not violate the ’global constraint’. There are two possible trajectories (for the presumed noiseless incoming current) to traverse the system: either following the edge adiabatically, or by tunneling through the barrier. The outgoing current would then be noisy \cite{7}. By analyzing this non-equilibrium noise one may detect the effective charge involved.

Our extensive analysis, performed on a torus geometry described below, leads to three main results: (i) For our multiply-connected geometry and in the presence of a real potential barrier we confirm the existence of QP tunneling, which decreases rapidly with system size (Eq. 3). (ii) We study the amplitude of such QP tunneling processes and identify the crossover, in terms of the system’s parameters, between the electron-tunneling-dominated and the QP-dominated regimes. (iii) We show that in FQHE systems, the presence of impurities may enhance both electron and QP tunneling amplitudes (Eq. 5), in the spirit of the Shklovskii-Li-Thouless mechanism \cite{8}.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{torus.png}
\caption{(a) Proposed experimental setup for an annulus, allowing the tunneling of both QPs and electrons through a barrier. The edge states are marked. (b) The torus geometry studied in this work; x and y represent two Cartesian coordinates, the unit cell being $2\pi R \times L$; $\Phi_1$ and $\Phi_2$ are two Aharonov-Bohm fluxes threading the torus. A Hall liquid (of density $1/m$) covers the torus everywhere, except around the barrier which is initially dry (“extended-holes”). Here the barrier potential is circularly symmetric.}
\end{figure}

The torus geometry. To facilitate our analytical study and to avoid complications emanating from the system being open, we hereafter focus on a setup defined on a torus, spanned by the two periodic coordinates $0 < x < 2\pi R$ and $0 < y < L$ (Fig. 1(b)), with a uniform magnetic field perpendicular to the surface of the torus. On top of the torus surface we introduce a circular potential ridge ($V_0$). The tunneling investigated here is between the two sides of this barrier. The main steps in the analysis are: (i) We first construct modified Laughlin-Haldane-Rezayi states which corresponds to a...
bulk filling factor $1/m$ (the "wet area") and a "dry area" (made of extended holes) where the electron density is suppressed, ideally to zero. The ground state configuration is obtained by maximizing the overlap of the dry area of the many-body configuration with the barrier [9]. (ii) The Hamiltonian, hence the wave-functions, depend on two gauge fluxes, $\Phi_1$ and $\Phi_2$ (Fig. 1(b)). By adiabatically increasing $\Phi_1$ any many-body configuration will slide rigidly in the $y$-direction, giving rise to a change in its energy. As the levels of two different configurations cross, the ground state of the system changes abruptly. Below we show that such a change corresponds to a tunneling event. The set of many-body energy levels is plotted in Fig. 2(a). (iii) To enable tunneling we break the circular symmetry (in the $x$-direction), introducing an additional asymmetric potential ($V_1$). This gives rise to finite matrix elements between different configurations. Avoided-crossing gaps in the energy-flux spectrum (Fig. 2(a)) are a manifestation of tunneling: the period in flux reflects the nature of the particle that tunnels. Below we calculate these tunneling matrix elements. (iv) We demonstrate quantitatively how the presence of a multitude of $\delta$-function impurities enhances the tunneling.

Our initial Hamiltonian is $H = H_0 + U_{m} + V_0$, where $H_0$ includes the kinetic part as well as the magnetic field and fluxes and $U_{m}$ is the two-particle interaction. The barrier potential $V_0$ is assumed to be sufficiently weak to exclude mixing with higher Landau levels.

For $V_0 = 0$ and no dry area Haldane and Rezayi [10] have found a set of $m$ degenerate Laughlin wave functions. The solution is obtained for a magnetic field which is quantized according to the Dirac’s condition $RL = \ell_m^2 N_\phi$, where $N_\phi$ is the number of magnetic flux quanta perpendicular to the torus surface and $\ell_m \equiv \sqrt{\hbar/c e B}$ is the magnetic length. These wavefunctions are eigenfunctions of total quasi-momentum (TQM) [11] and, similarly to the Laughlin wave function [12], are exact (zero energy) ground states of the Hamiltonian with hard core interaction (i.e. $\nabla^2 \delta(r)$). Their ground state electron density is nearly uniform. Here $N_\phi = mN$, where $N$ is the number of electrons.

**Extended Hole Wavefunctions.** To render the barrier (and its close vicinity) "dry", we tune the magnetic field to allow for $N_h$ holes: $N_\phi = mN + N_h$. The lowest Landau level consists of $N_\phi$ single particle states $|n >$ (cf. Ref.[10]). The density profile of each single particle state is approximately a Gaussian in the $y$-direction and uniform in the $x$-direction. The distance between the guiding centers of adjacent states is $L/N_\phi$. In the subspace of the lowest Landau level (spanned by $(N_\phi$) possible Slater Determinants) the ground state is determined solely by the interaction. Diagonalizing the hard-core interaction results in a set of zero-energy ground states, each having $N_h$ holes extended in the $x$-direction. As an example, consider $N_h = 1$. When the interaction term is diagonalized one obtains $N_\phi$ Laughlin-like states of zero energy. Each of these states corresponds to a non-uniform electron density: the filling is $1/m$ almost everywhere, but the occupation of one of the single-particle quasi-momentum states $|n >$ is suppressed: the area around this guiding center is "dry" (Fig. 2(b)). We denote such a many-body state with an extended hole at $|n >$ by $\Psi_n$. It is an eigenstate of the TQM. By sliding all guiding centers rigidly by 1 we increase the TQM by $N$, shifting $\Psi_n \rightarrow \Psi_{n+1}$. The above procedure is readily generalized to $N_h > 1$. Choosing the $N_h$ holes to be contiguous leads to a dry area of a desired width [14].

**tunneling: electrons versus QPs.** The initial ground state whose dry area coincides with the barrier is denoted by $\Psi_0$. As the flux $\Phi_1$ is varied adiabatically this many-body configuration slides rigidly around the torus. The dry area of $\Psi_0$ moves and its energy $E_0(\Phi_1)$ increases. The dry area of $\Psi_1$, whose TQM differs from that of $\Psi_0$ by $N \mod (N_\phi)$, slides towards the barrier and its energy $E_1(\Phi_1)$ decreases. When $\Phi_1$ is increased by $\phi_0/2$, $E_0(\Phi_1)$ and $E_1(\Phi_1)$ intersect, and the ground state of the system switches $\Psi_0 \rightarrow \Psi_1$. This corresponds to a shift of each single electron state by 1: $|n > \rightarrow |n + 1 >$. Since the average occupation of $|n >$ is $1/m$ this process describes a shift of charge of $e/m$ from one side of the barrier to the other, i.e. QP tunneling. The level-crossing degeneracy is lifted by breaking the circular symmetry of the potential $V_0$: $H \rightarrow H + V_1$. The QP tunneling matrix element is $<\Psi_0|V_1|\Psi_1 >$. By analogy, the two many-body states $\Psi_0$ and $\Psi_n$ differing in their TQM by $mN \mod (N_\phi)$ will cross when $\Phi_1$ increases by $m\phi_0/2$, hence $<\Psi_0|V_1|\Psi_n >$ is the matrix element for electron tunneling. In order to evaluate these tunneling matrix elements we use the fact that $V_1$ is a single-particle potential. The overlap $<\Psi_0|V_1|\Psi_p > (p = 1,m$ for QP, electron tunneling respectively) consists of contributions from the respective Slater determinants components.
\[ |k_1, \ldots, k_N \rangle \in \Psi_0 \text{ and } |\ell_1, \ldots, \ell_N \rangle \in \Psi_p, \] which are identical except for a single pair \( k, \ell \). The difference in TQM is
\[ \ell - k = p \text{mod}(N_\text{p}). \] (1)

Taking \( V_i = \bar{V}_1 \delta(x) \) renders the procedure particularly simple [15]. Then \( <k|V_i|\ell> = v_p \) depends only on the difference \( \ell - k \). The tunneling amplitude can then be cast as \( T_p \equiv \langle \Psi_0|V_i|\Psi_p \rangle = f_p(L)v_p \), where \( f_p \) is a proportional factor resulting from summation over all possible pairs satisfying (1), and is calculated numerically. For \( v_p \) one readily obtains
\[ v_p = \frac{\bar{V}_1}{R} \sum_{q=-\infty}^{\infty} e^{-ip\Phi_2 \pi \left\{ \left[ (p - q)N_\Phi \right] \e^{\frac{L}{2N_\phi}} \right\}^2}. \] (2)

To reflect the overlap of two Gaussians separated by a distance \( pNL/N_\phi \) on the torus (Fig. 3(a)). For QPs the separation is of the order of \( L/\text{m} \) \((N_{\text{imp}} = \text{mN} + N_\Phi) \), so \( v_p \) scales as \( -L/2mL_\text{r}^2 \). By contrast, for electrons the separation distance (defined modulo \( L \)) is \( N_\text{h}L/N_\phi \), hence \( v_p \) scales as \( -(L/2\text{mL}_{\text{r}}^2) = \exp[-(N_\text{h}L/L_{\text{r}}^2)] \), which is \( L \)-independent (where we used the Dirac condition). Likewise we find that the factor \( f_p \) is roughly system-size independent for electron tunneling, while it rapidly decreases (Gaussian-like) for QPs.

![FIG. 3: Tunneling involves initial and final states whose TQM differ by \( P \). \( P = 1,3 \) for QPs and electrons respectively. Here \( n = 3 \). The respective matrix elements involve the overlap of two single-particle states (Gaussians), whose separation is \( \text{pNL/N}_\phi \). For QPs the latter is of the order of \( L/\text{m} \) (solid arrow, black and white Gaussians). Thus the overlap factor for QPs, \( v_{p=1} \), strongly decreases with \( L \). By contrast, for electrons (dashed arrow, black and grey) the distance (defined modulo \( L \)) is \( N_\text{h}L/N_\phi \) \( \ll L \) and the dependence on \( L \) is negligible. (b) The calculated tunneling probabilities for electrons (\( T_1 \)) and QPs (\( T_3 \)), for \( n = 3, N_\text{h} = 1 \). \( T_3 \) is practically independent of system size (squares), while for QPs \( T_1 \sim e^{-\alpha L^2/L_{\text{r}}^2} \) (circles). The solid line is a fit with \( \alpha \approx 0.07 \).

We thus summarize
\[ T_p \sim \begin{cases} e^{-\alpha L^2/L_{\text{r}}^2}, & \text{QPs (}p = 1\text{)} \vspace{1.5mm} \\ L \text{ independent}, & \text{electrons (}p = 3\text{)}. \end{cases} \] (3)

For \( N_\text{h} = 1 \) (and \( N = 2, \ldots, 6 \)) we obtain numerically \( \alpha \approx 0.07 \) (Fig. 3(b)). We expect this to approach 1/12 (= 1/4m) in the thermodynamic limit (cf. Eq. (5) for \( N_{\text{imp}} = 1 \)). Note that the factor \( f_p \) further suppresses the single-particle term \( e^{-(L/2\text{mL}_{\text{r}})^2} \rightarrow e^{-(L/2\text{mL}_{\text{r}})^2/n} \).

**Impurity-assisted tunneling.** We next introduce impurities into the system, \( \mathcal{H} = H_0 + U_{\text{int}} + V_2 \), where \( V_2 \equiv \sum_{\ell=1}^{N_{\text{imp}}} V_{\text{imp}} \delta(z - z_\ell) \) represents \( N_{\text{imp}} \) localized impurities (\( V_{\text{imp}} > 0 \)). Shklovskii and L& Thouless [8] have shown that for non-interacting electrons, impurities modify the Gaussian decay of the edge-to-edge Green function into exponential. This result is generalized here to the FQH regime. To simplify the analysis we employ the fact that QP tunneling may be interpreted as a non-local process taking place through the liquid, while electron tunneling takes place through both the potential barrier and the liquid. Thus, the torus can be effectively replaced by two cylinders whose circumference is \( 2\pi R \) and whose lengths are \( L_{\text{barrier}} = N_\text{h}L/N_\phi \) and \( L_{\text{liquid}} = L - L_{\text{barrier}} = mL/N_\Phi \). For a cylinder with impurities the ground state wavefunction \( \Psi_{(x_0)} \) contains \( N_{\text{imp}} \) localized holes at the impurity positions
\[ \Psi_{(x_0)} = \prod_{j=1}^{N} (e^{-izj/R} - e^{-iz0j/R}) \Psi_L \] (4)

\( (\Psi_L = \prod_{\ell < j} (e^{-izj/R} - e^{-izj/R}) e^{-\sum_j y_j^2/2} \) is Laughlin’s cylinder wave function.

To obtain the various tunneling matrix elements we calculate the overlap between \( \Psi_{(x_0)} \) and its shifted version \( \tilde{\Psi}_{(x_0)} = \prod_{j=1}^{N} e^{-ipz_j/R} \Psi_{(x_0)} \), \( p = 1 \) for QP and electron tunneling respectively. For \( N_{\text{imp}} = 1 \) overlap factors (up to prefactors) the perturbative result [16]: QPs tunnel more efficiently than electrons along a cylinder. We have evaluated this overlap numerically for systems with \( N \leq 6 \) (for electrons tunneling at integer filling we have considered \( N \leq 17 \) and \( N_{\text{imp}} \leq 4 \), for (i) impurities equally spaced on a line and (ii) at random positions throughout the cylinder, averaging over \( \sim 1000 \) realizations. For case (i) we find that
\[ <\Psi_{(x_0)}|\tilde{\Psi}_{(x_0)}> \sim \begin{cases} e^{-L^2/12N_{\text{imp}}L_{\text{r}}^2}, & \text{QPs} \\ e^{-L^2/4N_{\text{imp}}L_{\text{r}}^2}, & \text{electrons} \end{cases} \] (5)

agrees with the numerics. For case (ii) the decay factor of the exponent is modified, but not the parametric dependence. We find that when the longitudinal impurity density \( \lambda = L/N_{\text{imp}} \) is held constant, the typical hopping distance is kept fixed, a Gaussian-to-exponential crossover takes place. This crossover can be understood in terms of multiple impurity-assisted tunneling. For a QP, as an example, \( e^{-L^2/12N_{\text{imp}}} \rightarrow e^{-\lambda L/12} \).

**QP-electron crossover.** Studying this crossover is now experimentally feasible [17]. Here we present a framework to study it theoretically in a multiply connected geometry, e.g. the torus. As \( L_{\text{liquid}} \) is varied (compared with \( L_{\text{liquid}} \) the QP tunneling \( T_1 \) competes with
the electron tunneling $T_3$. By calculating $T_1$ and $T_p$, as explained above, varying the number of particles (modification of $L_{\text{liquid}}$) and the number of extended holes (modification of $L_{\text{barrier}}$), we obtain the ratio of the electron-tunneling amplitude to that of QP tunneling (Fig. 4).

This allows us to determine the separation between the electron-tunneling dominated and the QP-tunneling dominated regimes (dotted line). This can be compared to the solid curve obtained by taking the estimates $T_1 \sim e^{-L_{\text{liquid}}/4\ell_H^2}$, $T_3 \sim e^{-L_{\text{barrier}}/4\ell_H^2} + e^{-L_{\text{liquid}}/4\ell_H^2}$.

Discussion. Keeping the barrier size fixed and increasing the torus length we have found that the QP tunneling amplitude decreases while the electron amplitude is mostly unaffected. This supports the picture that the QP tunneling through a barrier [18], while in principle possible, is a mesoscopic effect. It may be interpreted as a QP leaping through the liquid around the barrier. In the thermodynamic limit the QP tunneling amplitude vanishes, in accordance with common wisdom. The dependence of the tunneling amplitude on length scales does not conform to the scaling resulting from the RG treatment [4]. It is strongly modified by the multiple connectedness of the system. Adding disorder enhances the tunneling amplitudes. As can be seen from Eq. (5), special arrangements of the impurities can lead to even stronger enhancement (e.g. increase the linear density of $N_{\text{imp}}$, while keeping the two-dimensional density fixed). We believe that the best candidate to test the ideas outlined here is the annulus geometry (Fig. 1(a)). The relevance of the current results to the annulus geometry will be explored in future studies.

Acknowledgments. We thank A. Altland, F.D.M. Haldane, B. Halperin, B.I. Shklovskii and D.J. Thouless for illuminating discussions on various aspects of the problem. This work was supported by the US-Israel Bina
tional Science Foundation, the Israel Academy of Science and the Minerva Foundation. Y.G. was supported by the AvH foundation. Y.M. was supported through the Einstein Minerva Center for Theoretical Physics (through the BMBF).

[1] R.B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[2] S. A. Kivelson and V. L. Pokrovsky, Phys. Rev. B 40, 1373 (1989); S. Kivelson, Phys. Rev. Lett. 65, 3369 (1990); J. A. Simmons, H.P. Wei, L.W. Engel, D.C. Tsui, and M. Shayegan, Phys. Rev. Lett. 63, 1731 (1989); J. A. Simmons, S.W. Hwang, D.C. Tsui, H.P. Wei, L.W. Engel, and M. Shayegan, Phys. Rev. B 44, 12933 (1991); V. J. Goldman and B. Su, Science 267, 1010 (1995).
[3] D.J. Thouless and Y. Gefen, Phys. Rev. Lett. 66, 806 (1991); Y. Gefen and D.J. Thouless, Phys. Rev. B 47, 10423 (1993); see also Y. Avron, R. Seiler, and B. Shapiro, Nucl. Phys. B 265, 364 (1986); D.J. Thouless, Phys. Rev. B 40, 12034 (1989).
[4] K. Moon, H. Yi, C.I. Kane, S.M. Girvin, and M.P.A. Fisher, Phys. Rev. Lett. 71, 4381 (1993); M.P.A. Fisher and L.I. Glazman, "Mesoscopic Electron Transport", edited by L. Koenen, V. Schoen, and L. Sohn, NATO ASI Series, Vol.345, p.331 (1997).
[5] R. de-Picciotto, M. Reznikov, M. Heiblum, V. Umansky, G. Bunin, and D. Mahalu, Nature 389, 162 (1997); L. Saminadayar, D.C. Glattli, Y. Jin, and B. Etienne, Phys. Rev. Lett. 79, 2526 (1997).
[6] E. Comforti, V.C. Chung, M. Heiblum, and V. Umansky, Nature 416, 5151 (2002); V.C. Chung, M. Heiblum, and V. Umansky, Phys. Rev. Lett. 91, 216804 (2003).
[7] Note that only forward scattering is involved here. This means that the transmission probability is 1, implying absence of noise in the d.c. limit. The noise alluded to above may be detected at finite frequencies.
[8] B. I. Shklovskii, Pis’ma Zh. Eksp. Teor. Fiz. 36, 43 (1982) [JETP Lett. 36, 51 (1982)]; Q. Li and D.J. Thouless, Phys. Rev. B 40, 9738 (1989).
[9] For a ridge whose extension in the y-direction is $L_{\text{barrier}}$, the number of extended holes required to render the ridge dry is $N_h = [R \times L_{\text{barrier}}/\ell_H^2] + 1$.
[10] F.D.M. Haldane and E.H. Rezayi, Phys. Rev. B 31, 2529 (1985).
[11] The total angular momentum is defined $\text{mod}(N_h)$.
[12] F.D.M. Haldane, Phys. Rev. Lett. 51, 605 (1983); S.A. Trugman and S. Kivelson, Phys. Rev. B 31, 5280 (1985).
[13] In the present paper we employ a numerical recipe for the construction of extended hole states. We note that these may also be obtained analytically, starting with Laughlin’s state with localized holes. As this requires some details, and is not crucial to understanding the present analysis, we will present it elsewhere. For $N_h = 1$: $\Psi_n = \int g_n(\zeta_0) \Psi(\zeta_0) d\zeta_0$ where $g_n$ are single particle states on the extended unit cell $2\pi R \times nL$, and $\Psi(\zeta_0)$ is a Laughlin state with a localized hole at $\zeta_0$ [10].
[14] The width of a single-particle quasi-momentum state in the y direction is $\ell_H$; when $N_h < \ell_H/\ell_H$ we have a reduced density rather than a dry area.
[15] Other forms for $V_i$ lead to similar results.
[16] A. Auerbach, Phys. Rev. Lett. 80, 817 (1998).
[17] E. Comforti, Y.C. Chung, M. Heiblum, V. Umansky, Phys. Rev. Lett. 89, 066803 (2002).
[18] The work of M. Helias and D. Pfannkuche, cond-mat/0403126, also supports the notion of QP tunneling through a barrier.