Evolution of Massive Black Hole Binaries in Collisionally Relaxed Nuclear Star Clusters - Impact of Mass Segregation

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ABSTRACT
Massive Black Hole (MBH) binaries are considered to be one of the most important sources of Gravitational Waves (GW) that can be detected by GW detectors like LISA. However, there are a lot of uncertainties in the dynamics of MBH binaries in the stages leading up to the GW-emission phase. It has been recently suggested that Nuclear Star Clusters (NSCs) could provide a viable route to overcome the final parsec problem for MBH binaries at the center of galaxies. NSCs are collisional systems where the dynamics would be altered by the presence of a mass spectrum. In this study, we use a suite of high-resolution N-body simulations with over 1 million particles to understand how collisional relaxation under the presence of a mass spectrum of NSC particles affects the dynamics of the MBH binary under the merger of two NSCs. We consider MBH binaries with different mass ratios and additional non-relaxed models. We find that mass-segregation driven by collisional relaxation can lead to accelerated hardening in lower mass ratio binaries but has the opposite effect in higher mass ratio binaries. Crucially, the relaxed models also demonstrate much lower eccentricities at binary formation and negligible growth during hardening stages leading to longer merger timescales. The results are robust and highlight the importance of collisional relaxation on changing the dynamics of the binary. Our models are state-of-the-art, use zero softening, and high enough particle numbers to model NSCs realistically.

Key words: galaxies: kinematics and dynamics – galaxies: nuclei – black hole physics – gravitational waves

1 INTRODUCTION
Observations in the past twenty years have demonstrated that Massive Black Holes (MBHs), despite being point sources in the center of galaxies, play a vital role in galaxy evolution and growth (e.g., Kormendy & Gebhardt 2001; Kormendy & Ho 2013). In the hierarchical growth of structures, galaxies frequently merge and form larger systems (e.g., Rodriguez-Gomez et al. 2016). Upon the merger of galaxies, two MBHs get the opportunity to come close to one another to form a bound pair. Recent observation searches have revealed the presence of well-separated accreting MBHs seen as multiple Active Galactic Nuclei (AGN) in a single galaxy, as well as circumstantial evidence for bound Keplerian binaries (e.g., Komossa et al. 2003; Komossa 2006; Bogdanović 2015).

MBH binaries are purported to be one of the strongest sources of Gravitational Waves (GW) in the universe. Observations of GW during the final inspiral phase of the binary would reveal information not only regarding the merger history of the galaxy over time but also constrain dynamical properties of galactic nuclei surrounding the MBH binary. Therefore, modeling the dynamical evolution of MBH binaries inside galactic nuclei is central to the astrophysical interpretation of galactic environment and dynamics. MBH binaries are expected to be sources of millihertz (mHz) GW that will be detectable by future space based GW detectors like LISA (Amaro-Seoane et al. 2017) or Tianqin (Luo et al. 2016).

Due to variances in galactic environments in which they are embedded in, the dynamics of MBH coalescence is up for much debate. It is theorized that the merger is a three step process before the final GW-emission step (Begelman et al. 1980; Merritt 2013, chapter 8). In the first step, the dynamical friction of stars and dark matter and that of the interstellar gas plays a role in reducing the angular momentum of the black holes which then sink towards the center of the merged galaxy. When the black holes get close enough, they form a bound binary which signals the beginning of the second stage. This stage proceeds rapidly and the separation of the binary decreases due to dynamical friction and three-body scattering events. The third stage prior to GW-driven coalescence begins as the black holes form a hard binary. Once that happens further orbital decay occurs via three-body scattering. If not enough MBH binary-star scattering occurs, the orbital decay of the binary stalls before it can reach the GW-emission phase. This is called the final parsec problem (e.g., Milosavljević & Merritt 2003).

The timescale associated with the shrinkage of the binary in the hard binary stage is unclear and heavily depends on the environment. For example, in spherical gas-poor galaxies, simulations have shown that the orbital decay of the MBH binary essentially halts due to the lack of stars in the loss-cone and the GW merger timescales often exceed the Hubble time (e.g., Vasiliev et al. 2015). Addition-
ally, in cosmological simulations MBH seeds have been found to be inefficient at sinking to the center of the nuclei leading to longer merger timescales (e.g., Ma et al. 2021). In case of merging galaxies, however, the non-sphericity of the merger product introduces global torques which can populate the loss-cone more effectively leading to a continued orbital decay of the MBH binary system. Simulations have also shown that merger timescales in such cases are less than the Hubble time (e.g., Berczik et al. 2006; Khan et al. 2013; Vasiliev et al. 2015; Vasiliev 2017).

One way to overcome the final parsec problem is by embedding the MBHs in Nuclear Star Clusters (NSCs). Nuclear Star Clusters (NSCs) represent some of the densest stellar systems in the universe. They can have mass densities of \( \rho \geq 10^9 M_\odot pc^{-3} \) (e.g., Neumayer et al. 2020). As the name suggests, NSCs are found in galactic nuclei. The masses and presence of NSCs correlate with the mass of the host galaxy. Sánchez-Janssen et al. (2019) showed that their presence in galaxies depends on the stellar mass of the host galaxies, with a peak of 90% at \( M_{stellar} \sim 10^9 M_\odot \). NSCs and MBHs can coexist in many cases. In fact, the Milky Way galaxy contains an NSC at the Galactic Center that has an MBH embedded in it (e.g., Ghez et al. 2008). Using data from Sánchez-Janssen et al. (2019), Ogiya et al. (2020) speculate that under the assumption that all NSCs contain an MBH at the center, 50% of all Milky Way sized galaxies should have both an NSC and an MBH present in their nuclei.

In Ogiya et al. (2020), the authors show that if MBHs are embedded in NSCs prior to merger, tidal effects from the merging NSCs accelerate the orbital evolution timescale before and around the time the binary is formed, thus circumventing the final parsec problem. In the presence of NSCs the formation of a hard binary occurs faster and the whole process of decay into the GW regime is accelerated. The authors found that the mergers were extremely efficient with lower mass ratio binaries merging in \( \sim 100 \) Myr while binaries with mass ratio of unity merging in 5 Gyr, which is still less than the Hubble time.

Since NSCs are collisional stellar systems they undergo collisional relaxation even under the presence of an MBH at the center. A collisionally relaxed state implies the presence of a Bahcall-Wolf cusp (Bahcall & Wolf 1976). If there is a mass spectrum present, the more-massive objects form a steeper cusp than the less-massive objects (Bahcall & Wolf 1977; Alexander & Hopman 2009). During the merger of two galaxies containing NSCs, in the absence of MBH binaries, the cusp is expected to be retained (Dehnen 2005). The presence of MBH binaries leads to a partial or complete disruption of the cusp (e.g., Dehnen 2005).

In Ogiya et al. (2020), the authors considered a one-component mass function to study the effect of NSCs on MBH binaries. Realistic NSCs, however, are comprised of a spectrum of masses (e.g., Preto & Amaro-Seoane 2010; Gualandris & Merritt 2012). The effects of a mass spectrum have been explored previously by Gualandris & Merritt (2012) and Khan et al. (2018) using a fixed binary mass ratio. However, a systematic comparison of unsegregated versus segregated models as a function of the binary mass ratio is missing. A segregated Bahcall-Wolf cusp could lead to enhanced hardening rates and could potentially accelerate the evolution to GW driven coalescence stage. In addition, the effects of the relaxed cusp on the hardening rates of the binary would be an interesting investigation. We, therefore, are motivated to understand how collisionally relaxed NSCs, under the presence of a mass spectrum, affect the dynamics of the binary in the stages leading up to GW driven coalescence.

In this work, we extend the models presented in Ogiya et al. (2020) to understand the effects of mass-segregation in NSCs on the dynamics of the MBH binary. With the usage of higher mass resolution compared to previous studies and a two-component mass function, our models are able to better represent realistic NSCs. We use the Fast Multipole Method (FMM) (Greengard & Rokhlin 1987; Cheng et al. 1999) based N-body code Taichi (Zhu 2021; Mukherjee et al. 2021) which has been shown to reproduce collisional effects as accurately as direct-summation based N-body codes while using a fraction of the computational time. To understand the effects of mass segregation we use a two-component mass function: one consisting of objects roughly a solar mass or less and the other consisting of heavier objects like stellar mass black holes. We systematically study the effect of relaxed, segregated cusps on the dynamics of MBH binaries with different mass ratios using a suite of N-body simulations and discuss the role of the relaxed cusp in the merger process.

The paper is organized as follows: in section 2 we describe the numerical methods and improvements made to Taichi to handle dense systems more accurately. In section 3, we describe the models of the mergers. In section 4 we provide the results and in section 5 we discuss the impact of stochasticity and compare our results to those of previous studies. This is followed conclusions section 6.

## 2 NUMERICAL METHODS

We perform a suite of N-body simulations using \( N \sim 1.32 \times 10^6 \) particles to study the formation and evolution of the MBHs embedded in NSCs and their evolution after the formation of the MBH binary. The simulation setup is similar to that presented in Ogiya et al. (2020) with changes to improve the resolution and the modeling of the clusters to include a two-component mass species. The detailed description of the models is provided in the next section.

To simulate the system, we utilize the FMM based code Taichi (Zhu 2021; Mukherjee et al. 2021). Mukherjee et al. (2021) showed that Taichi can simulate systems as accurately as direct-summation based collisional N-body codes while scaling as \( O(N) \). The accuracy of the force calculation in Taichi can be tuned via the usage of an input accuracy parameter \( \epsilon \) which controls the opening angle and a multipole expansion parameter \( \rho \) which controls the number of expansion terms used in the force calculation. Using Taichi we can simulate large-N systems without the usage of specialized hardware within a physically reasonable amount of time. In this work, we extend Taichi to include a fourth-order force-gradient integrator and regularization using the AR-ChaIn scheme (Mikkola & Tanikawa 1999). Additionally, we improve the accuracy of the force solver in Taichi in this work. We briefly detail the improvements below.

### 2.1 Updated Integration Scheme

Contemporary direct-summation based N-body codes use a fourth-order time integration scheme like the Hermite method (e.g., Makino & Aarseth 1992). In our previous work (Mukherjee et al. 2021) we adopted an integrator based on hierarchical Hamiltonian splitting which is only second-order accurate (Pelupessy et al. 2012). In this work, we extend Taichi to include a novel fourth-order force-gradient integrator (Rantala et al. 2021) which decomposes the system into slow and fast subsystems based on the interaction timesteps of the particles. The fast subsystem is then hierarchically split until the slow-fast split results in no particles in the fast subsystem. We refer to this scheme as the HHS-FSI scheme hereafter. Hamiltonian splitting integrators are suitable due to the large dynamical range present in our simulations. Unlike conventional composition symplectic integrators presented by Yoshida (Yoshida 1990), HHS-FSI
utilizes strictly positive sub-steps made possible by computing an additional gradient term along with the Newtonian accelerations (Chin 1997; Chin & Chen 2005) which is only possible because the potential term does not depend on momentum.

Unlike Rantala et al. (2021) where the gradient term is calculated by direct summation, we utilize an extrapolation method described in Omelyan (2006) which uses a fictitious middle step to approximate the gradient-force term. Tests by Omelyan (2006) have shown this approach is indistinguishable from the Chin & Chen (Chin & Chen 2005) method. To do this, we follow the method (see also Farr & Bertschinger 2007, section 3.1) where

\[
p \leftarrow p - \frac{h}{6} \nabla V(q) \quad (1)
\]

\[
q \leftarrow q + \frac{h}{2} \frac{p}{m} \quad (2)
\]

\[
p \leftarrow p - \frac{2h}{3} \nabla \left( q - \frac{h^2}{24m} \nabla V(q) \right) \quad (3)
\]

\[
q \leftarrow q + \frac{h}{2} \frac{p}{m} \quad (4)
\]

\[
p \leftarrow p - \frac{h}{6} \nabla V(q) \quad (5)
\]

where \( h \) is the step-size, \( q \) is the position, \( p \) is the momentum and \( \nabla V \) represents the gradient of the potential \( V \) evaluated at the position given in the parenthesis. In total, four calls to the Poisson solver are needed in one step.

The forward symplectic nature ensures more accurate and efficient integration compared to that of the Yoshida scheme (e.g., Chin 2007). The integrator is manifestly momentum conserving and includes an individual symmetrized timestepping scheme similar to that described in Mukherjee et al. (2021). Despite the loss of symplecticity due to the usage of individual timesteps, the usage of time symmetrization ensures that there is no secular drift in the energy leading to much better energy conservation (e.g., Makino et al. 2006).

### 2.2 Algorithmic regularization

Even with the inclusion of a fourth order scheme, treatment of close encounters with the MBH binary can prove to be computationally challenging. However, hierarchical Hamiltonian splitting integrators are easy to modify to include regularization as a result of clean separation of fast system from the slow system. This enables accurate handling of close binaries and/or addition of post-Newtonian terms as one can plug-in any accurate few-body solvers to evolve the Hamiltonian of the fast system. Therefore, in order to handle the dynamics of the MBH binary and its interactions with scattering particles more accurately, we include regularization in Tai\textchi. We have utilized the SpaceHub API (Wang et al. 2021) which includes multiple regularization algorithms for accurate few-body integration. Tai\textchi can be used along with any of the integration schemes present inside SpaceHub. For this work, we found that the AR\textDash Chain\textDash Sym\textsuperscript{6+} regularization scheme is the most optimal. AR\textDash Chain\textDash Sym\textsuperscript{6+} is an updated AR\textDash Chain scheme which is more accurate than Mikkola’s (Mikkola & Tanikawa 1999) original implementation.

The improvements in AR\textDash Chain\textDash Sym\textsuperscript{6+} include an updated chain coordinate transformation, which improves on the CPU time taken to perform the coordinate transformation, active round-off error compensation, and the usage of a sixth order symplectic integration scheme instead of the traditional GBS extrapolation scheme used in the original AR\textDash Chain (Mikkola & Tanikawa 1999) method. This method is extremely efficient at handling highly eccentric systems.

The usage of a fixed timestep maintains the time symmetry and as such helps achieve higher precision in round-off error dominated regime. For the same relative tolerance parameter, Wang et al. (2021) found that the AR\textDash Chain\textDash Sym\textsuperscript{6+} is at least 1-2 orders of magnitude better at conserving energy. For more information, we refer the interested reader to Wang et al. (2021). Ta\textchi can be configured to allow an arbitrary number of particles to be treated by the regularization scheme. We performed tests and found that treating up to 20 particles with the regularization scheme was optimal in terms of performance and accuracy.

### 2.3 Updates to FMM based Force Solver

The multipole-to-local (M2L) kernel plays a crucial role in FMM by translating the multipole moments to local expansions for approximated force calculations. In the previous version of Tai\textchi, the order of expansions in the M2L kernel is kept at \( p \), for both multipole moments and the derivatives of \( 1/r \). After various optimizations, it is found that M2L kernel is evidently memory-bound instead of compute-bound. To increase the efficiency of this kernel, we increase the expansion order for \( 1/r \) derivatives from \( p \) to 2\( p \). This is called the double height M2L kernel (Coulaud et al. 2008) as opposed to the single height formulation in our previous version. We found that this modifications improves the force error by a factor of \( \sim 10\times \) for the same settings compared to the single height version. As a result, we relax the force accuracy parameter \( \epsilon \) by the same factor if double-height M2L is used. For more information, we refer the interested reader to Coulaud et al. (2008).

All of the improvements presented above enhance the accuracy and capability of Tai\textchi. We tested Tai\textchi with the initial conditions from Ogiya et al. (2020) and compared the results from \#body6++GPU (Wang et al. 2015) to ensure correspondence between the two codes. The results are briefly presented in the Appendix. We found that Tai\textchi was able to simulate the systems as accurately as \#body6++GPU. For the purposes of our simulations, we found that an FMM input relative force accuracy parameter \( \epsilon = 2 \times 10^{-5} \) and a multipole expansion parameter \( p = 12 \) was most optimal. For more information on these parameters, we refer the reader to Mukherjee et al. (2021).

We ran tests with different values of \( \epsilon \) and \( p \) and found no difference in the final results. For our simulations we used a timestep parameter \( \tau_T = 0.3 \) unless more accuracy was demanded. For most simulations this results in a relative energy conservation of the order of \( \sim 0.01\% \). Under the presence of a dense segregated cusp, we found that the total relative energy conservation was \( \sim 0.1%-1\% \). We note that no softening was used in the simulations. All of the simulations presented in this study were performed using only 32 threads on an AMD Epyc 7742 node. The simulations took \( \sim 14-18 \) days for \( N \sim 1.32 \times 10^8 \) to run to completion. The simulations with highly eccentric binaries or extremely dense cusp took much longer due to the formation of stable multiple systems.

### 3 Models

As mentioned in the previous section our choice of models is motivated by the work described in Ogiya et al. (2020). We are interested in MBHs whose coalescence will be detectable by LISA and Tianqin. We set the mass of the primary \( M_1 = 10^6 M_\odot \). The masses of the secondaries are generated such that we have mass ratios \( q = 1.0, 0.1, 0.01 \) where \( q \equiv M_2 / M_1 \) and \( M_2 \) is the mass of the

\[ M_2 \]
Table 1. Summary of the initial parameters used in the generation of the $N$-body models.

| Parameter | Value |
|-----------|-------|
| $N_{\text{MS}}$ | 655360 |
| $N_{\text{BH}}$ | 3276 |
| $N$ | 1317272 |
| $M_{\text{MS}}$ | $9.525 \times 10^6 M_\odot$ |
| $M_{\text{BH}}$ | $4.75 \times 10^5 M_\odot$ |
| $M_{\text{part:MS}}$ | $14.5 M_\odot$ |
| $M_{\text{part:BH}}$ | $145 M_\odot$ |

3.1 Non-relaxed NSC Models

To generate both the non-relaxed and the relaxed models, we start off with the Dehnen density profile (Dehnen 1993) for both the MS and BH particles. The density profile for each component $i$ is given as:

$$
\rho_i(r) = \rho_0 \left( \frac{r}{r_0} \right)^{-\gamma_i} \left( 1 + \frac{r}{r_0} \right)^{-\gamma_i-4}
$$

where $\rho_0$ is the normalizing factor, $r_0$ is the scale radius and $\gamma_i$ determines the inner slope of the component $i$. We set $r_0 = 1.4$ pc following Ogiya et al. (2020). We use $\gamma_{\text{MS}} = \gamma_{\text{BH}} = 0.5$ which is the lowest value of $\gamma$ that can support an MBH at the center with an isotropic velocity distribution (Baes et al. 2005). We truncate the density profile at 1000 pc using an exponential truncation function. The distribution functions of both the MS and BH particles are generated by using the density function of each individual component and the combined potential of all components including the MBH, with an isotropic velocity distribution. This is all done under the self consistent framework of Agama.

3.2 Relaxed NSC Models

To generate the collisionally relaxed models, we have to follow a few more steps. We input the non-relaxed density profiles of both the MS and BH particles along with the mass of the MBH at the center. Then, we evolve the system using the Fokker-Planck code Phaseflow until the system has evolved to a collisionally relaxed state. For the model with a $10^6 M_\odot$ MBH at the center, the relaxed state is achieved in $\sim 0.5$ Gyr. This occurs when the inner density profile of the BH particles falls off as $r^{-2}$ and that of the MS particles falls off as $r^{-1.5}$ (Bahcall & Wolf 1977; Hopman & Alexander 2006; Alexander & Hopman 2009). The output from Phaseflow can be easily used to generate isotropic models using Agama in a fashion similar to the one described above. We present a comparison of the analytic density and mass profiles in Figure 1 for the non-relaxed and the relaxed cases when we have a $10^6 M_\odot$ SMBH at the center. In both the non-relaxed and the relaxed cases, we verified the $N$-body models accurately reproduced the analytic density and mass profiles.

3.3 Generating the Merger Models

To initialize the merger between two NSCs, we follow the steps outlined in Ogiya et al. (2020). The two NSCs and their corresponding MBHs are initially unbound and are allowed to become bound over the course of the simulation. The initial separation between the two MBHs is denoted as $d_{\text{in}}$. In our simulations we set $d_{\text{in}} = 20$ pc. We verified that this is less than the effective radius of the NSCs.

To generate the initial relative velocity of the two NSCs, we use a free parameter $\xi$ similar to the parameter $\eta$ described in Ogiya et al. (2020) equation (9). $\xi$ quantifies the initial angular momentum of the orbit. Smaller values of $\xi$ imply a more eccentric orbit. In our models, the relative velocity $v_{\text{in}}$ is defined as

$$
v_{\text{in}} = \xi \sqrt{\frac{G M_\ast(d_{\text{in}})}{d_{\text{in}}}}
$$

where $M_\ast (d_{\text{in}})$ accounts for the total mass (excluding the MBH masses) within a distance of $d_{\text{in}}$ from the center of each NSC. Once $v_{\text{in}}$ has been obtained, the NSC of the secondary along with its MBH is placed at a position centered around $r_{\text{in}} = (d_{\text{in}}, 0, 0)$ with a velocity $v_{\text{in}} = (0, v_{\text{in}}, 0)$, while the other NSC is placed at the origin with zero bulk velocity. In each simulation, we verified that the initial relative
velocity was the same to maintain consistency. We found that using $d_{in} = 20$ pc, $v_{in} \approx 55.5$ km/s. Six simulations are generated with the relaxed and the non-relaxed NSCs with $\xi = 1.0$. We label the simulations where relaxed NSCs are used as $r_\star$ simulations and the simulations where non-relaxed NSCs are used as $nr_\star$. To reduce the number of simulations performed due to computational constraints we perform mergers of relaxed NSCs with only other relaxed NSCs and non-relaxed NSCs with other non-relaxed NSCs. We expect the results of mixed simulations to lie in-between the results obtained in this study.

We expect most mergers to happen on eccentric orbits rather than circular. Thus, it is imperative to understand the effects of initial eccentricity and its evolution under the presence of relaxed and non-relaxed cusps. To understand the effects of an initially eccentric orbit, we perform four additional simulations with $q = 0.1$, $\xi = 0.5$ and $q = 0.1$, $\xi = 0.1$. The former simulations are labelled $ecc_1$ whereas the latter simulations are labelled $ecc_2$. The circular orbit models are used to understand how the evolution of the binary changes as a function of the mass-ratio while the eccentric models are used to understand how the eccentricity is affected by the different density profiles for a given mass ratio.

To demarcate the three stages of evolution, we need the influence radius and the hard binary radius. In order to determine the influence radius and the hardening radius, we consult Merritt (2013) equation (8.71). We use the following definitions which are more suitable for N-body simulations; once a bound binary is formed, an influence radius of the primary can be defined as

$$d_{infl} = r_{enc}(2M_1)$$  \hspace{1cm} (10)

where $r_{enc}$ is the radius enclosing the amount of mass in the parenthesis. The corresponding hard binary radius can be defined then as

$$a_h = \frac{q}{(1+q)^2} \frac{d_{infl}}{4}.$$  \hspace{1cm} (11)

For more definitions, we refer the reader to Merritt & Szell (2006).

All the simulations were run for a total time of 10 Myr. In most simulations, this was enough for the MBH binary to harden to $a_h/5$ which is usually sufficient to study the effects of core scouring as reported in previous studies (e.g., Merritt & Szell 2006).

4 RESULTS

To visually examine the evolution of the MBH binary over time, we present snapshots of the evolution over the first 1.5 Myr for the $r_{q,0.1}$ model in Figure 2. The orbit of the MBHs is in the $x-y$ plane, and the plot’s origin is the center of mass. We find that the NSCs merge within $\approx 1.5$ Myr leading to the formation of a hard MBH binary. To quantitatively understand the dynamics in more detail, we plot the evolution of the orbital separation, eccentricity, and inverse semi-major axis of the binary over time in Figures 3 and 6. They help us understand the differences between relaxed and non-relaxed models as a function of the mass ratio $q$ and the initial eccentricity. We analyze each step of the evolution in the sections below. We first present our analysis of the circular orbit models before moving on to the eccentric ones.

4.1 Pre-binary phase ($r > d_{infl}$)

The first stage of evolution, the pre-binary phase, lasts until a bound binary has formed. Examining the orbital elements in Figure 3, we

![Figure 1. The analytic density $\rho(r)$ and the enclosed cumulative mass $M(<r)$ a function of $r$, the distance from the center of the cluster under the presence of a $10^6 M_\odot$ MBH at the center. The analytic profiles have been computed using PhaseFlow. The differences in the relaxed and the non-relaxed cases are evident with collisional relaxation implying mass segregation. The relaxation produces a denser cusp near the $10^6 M_\odot$ MBH and stellar mass black holes dominate the total mass for all radii $<0.1$ pc. The MBH is dominant in regions with $r < 1$ pc.]

| Simulation ID   | $\gamma_{\text{MS}}$ | $\gamma_{\text{BH}}$ | $q$  | $\xi$ |
|-----------------|-----------------------|-----------------------|------|-------|
| $r_{q,0.1}$     | 1.5                   | 2.0                   | 1.0  | 1.0   |
| $r_{q,0.1}$     | 1.5                   | 2.0                   | 0.1  | 1.0   |
| $nr_{q,0.1}$    | 0.5                   | 0.5                   | 1.0  | 1.0   |
| $nr_{q,0.1}$    | 0.5                   | 0.5                   | 0.1  | 1.0   |
| $nr_{q,0.1,ecc}$| 0.5                   | 2.0                   | 0.1  | 0.5   |
| $nr_{q,0.1,ecc}$| 1.5                   | 2.0                   | 0.1  | 0.1   |
| $nr_{q,0.1,ecc}$| 0.5                   | 2.0                   | 0.1  | 0.5   |

Table 2. Summary of the model parameters used for the NSC-NSC merger simulations. All of the above simulations use the same number of particles. The first six models are on circular orbits initially and used to study the effect of relaxation as a function of $q$ and the last four models are eccentric and used to study the effect of initial eccentricity and evolution of eccentricity at a fixed $q$. $ecc_1$ models are moderately eccentric whereas $ecc_2$ models are highly eccentric.

find that this phase lasts for the first $\approx 1.5$ Myr for both relaxed and non-relaxed models across different $q$. The NSCs merge in roughly $\approx 1.5$ Myr bringing the secondary to within the influence radius of the primary ($\approx 1$ pc) leading to the formation of a bound binary.

In this phase the evolution is dominated by the dynamical friction of the stars and the drag force from the tidally stripped stars which help in reducing the angular momentum of the MBHs (Ogiya et al. 2020). The latter effect is more important in this stage as dynamical friction typically acts on longer timescales. The process involves the
Figure 2. A scatter plot of the two NSCs with MBHs projected onto the $x-y$ plane at different points in time during the merger process. The simulation being pictured here is $r_{q,0.1}$. As the simulation proceeds, the NSCs belonging to the primary (black circle) and secondary (black cross) are brought closer to each other by the combined effects of dynamical friction and tidal forces from stripped stars leading to a mixture of the MS (blue, red dots) and BH particles (brown, yellow dots) from both NSCs. The NSCs merge within ~ 1.5 Myr resulting in the formation of a hard binary at the center.

Transfer of angular momentum from the NSC cores to the stripped stars which expand their orbits. As demonstrated by Huang (1963) and later by Ogiya et al. (2020), if the two NSCs are considered to be part of a binary system, then in the event of a mass loss, the change in specific angular momentum $l$ can be written as

$$\delta l = (I_s - l) \frac{\delta m_s}{m} \quad (12)$$

where $m$ is the mass of the NSC binary system and $m_s$ and $I_s$ are the mass and specific angular momentum of the stars that have been tidally stripped from the NSC. $\delta m_s$ is large in the early phase of the dynamical evolution, while it can be negligible in the later phase. Under the assumption that the eccentricity of the stripped stars has not changed and that the mass loss through tidal disruption is negligible compared to that of the mass of the NSC, the expressions for the specific angular momentum of the NSC binary and that of the stripped stars as follows:

$$l = \sqrt{Gma(1-e^2)} \quad (13)$$

$$I_s = \sqrt{Gm(a + \delta a)(1-e^2)} \quad (14)$$

In order to satisfy the condition that $\delta m_s < 0$ and $\delta l < 0$, we find that $\delta a > 0$. Thus, as the expansion of orbit is associated with an increase in angular momentum, and the total angular momentum remains conserved, the distances between the NSC cores, and MBHs embedded in them, shrink as a result. This mechanism is especially important for lower $q$ cases as dynamical friction works inefficiently to decay the orbit of less massive BHs (Ogiya et al. 2020). The results are consistent with those presented in Ogiya et al. (2020). This is not surprising since we compared the amount of mass losing and gaining angular momentum between simulations and found that they were equivalent. This leads to negligible differences in the evolution during this period. However, we warn the readers that the effect of tidal stripping would be reduced in realistic galaxies. Interestingly, we find that the time of binary formation, which is dictated by the tidal
Figure 3. The evolution of the binary parameters as a function of time for the circular orbit models. The dashed line represents the influence radius of the binary and the dash-dotted line represents the hard-binary radius. Top: evolution of the separation $r$ between the two MBHs as a function of time. Middle: evolution of the eccentricity ($e$) as a function of time. Bottom: evolution of the inverse semi-major axis ($1/a$) as a function of time. The different evolutionary tracks between the non-relaxed and the relaxed cases highlight the imprint of the surrounding NSC on the dynamics of the MBH binary. We find that while non-relaxed models reach hard binary radius and harden faster for $q = 1.0$, the opposite happens for $q = 0.01$.

4.2 Bound binary phase ($d_{\text{infl}} < r < a_{\text{h}}$)

In the second phase of evolution, after a bound binary has formed, orbital decay occurs due to a mix of dynamical friction and three-body scattering events. When the two MBHs are sufficiently far apart, dynamical friction acts on each body independently to shrink the binary (Merritt 2013, section 8.2.2). When the binary gets closer, hardening via scattering becomes more important and the efficiency of scattering depends on the binary mass ratio $q$ (e.g., Merritt 2013, section 8.2.2).

Examining Figure 3 we find that for $q = 1.0, 0.1$, the combined phase proceeds extremely quickly leading to the formation of the hard binary immediately after the end of the first phase. The evolution in the $q = 0.01$ models is more gradual. In addition, we find some notable differences in this phase between the relaxed and the non-relaxed models that depend on $q$. We notice from the evolution of the inverse semi-major axis that in $r_q 1.0$, the binary is able to settle at a smaller separation than in $r_q 1.0$. This is quite surprising since intuitively we would expect the denser relaxed cusp to yield a faster orbital decay. As the mass ratio is lowered, the situation changes and in the $q = 0.01$ case, we find that the the relaxed model actually accelerates the transition to the hard binary stage. Since the orbital energy of the binary is given as

$$E_{\text{binary}} = -\frac{G M_1 M_2}{2a}$$

(15)

where $a$ is the semi-major axis of the binary, we deduce that for the $q = 1.0$ case, the binary is able to lose more energy in the non-relaxed model compared to that in the relaxed model. As we lower the mass ratio, the energy loss in the relaxed models increases. We seek to understand the processes in work that change the results across mass-ratios.

In order to understand the differences, we first approximately quantify the amount of energy lost by dynamical friction and scattering.
We follow Merritt (2013) equations (8.73) and (8.74) which state
\[
\frac{dE}{dt}\bigg|_{df} \approx -4.4 \frac{G^2 M^2 \rho(r) \ln(\Lambda)}{\sigma} \tag{16}
\]
and
\[
\frac{dE}{dt}\bigg|_{s} = -\frac{H(a) G^2 M^2 \rho(r)}{2q} \tag{17}
\]
where \(\frac{dE}{dt}\bigg|_{df}\) and \(\frac{dE}{dt}\bigg|_{s}\) are the energy losses from dynamical friction and scattering respectively, \(\ln(\Lambda)\) is the Coulomb logarithm, \(\sigma\) is the velocity dispersion and \(H(a)\) is the dimensionless scattering rate. As the binary hardens, the energy losses via scattering become more important as the scattering efficiency increases as \(q^{-1}\). Physically, this makes sense since the larger the mass of the secondary MBH, the more energy would have to be extracted by the intruder to harden the binary by a fixed amount.

From equations 16 and 17 we notice that \(\frac{dE}{dt} \propto M^2\). Thus, the rate of loss of energy is faster for binaries with larger secondary mass, which explains why the combined phase proceeds rapidly for \(q = 1.0, 0.1\) in contrast to the \(q = 0.01\) case. Dynamical friction is less efficient in the \(q = 0.01\) models leading to a more gradual orbital decay. However, this does not explain why the orbital decay is more efficient in the \(nr_q=0.01\) model compared to the \(r_q=0.01\) model.

To understand the differences in the evolution of the inverse semi-major axis between the non-relaxed and relaxed models we note that \(\frac{dE}{dt} \propto \rho(r)\). Therefore, we focus on the differences in the initial density profile of the non-relaxed and the relaxed models. For clarity, we compare the density profile for the MS particles in the case where the central MBH mass is \(10^6 M_\odot\) in Figure 4. Since the total mass of the MS particles is much more than that of BH particles, we expect any major discrepancies to arise out of differences in the MS density and mass profiles.

Looking at all radii < 10 pc, we find that in the relaxed models, the density of MS particles is lower than that of the non-relaxed models for all radii > 0.1 pc. Since the NSC mass across our models is fixed and the relaxed models have higher central density, the density in the outskirts decreases. This counter-intuitive result was also reported in Gualandris & Merritt (2012). To explain it, the authors accounted for the effect of the BH particles on the MS particles near the SMBH. Since the BH particles are more massive than the MS particles, they dynamically heat the MS particles leading to a lower density in the above-mentioned region. However, Gualandris & Merritt (2012) showed that at smaller radii, due to the scattering effects of the BH particles, the MS particles end up forming a denser, Bahcall-Wolf cusp. This cusp has a higher density only for very small radii, typically ~ 0.1\(d_{\text{eff}}\).

As the MBH binary hardens to \(a_{\text{BH}}\) and energy loss due to scattering becomes more dominant, differences in evolution appear between the non-relaxed and the relaxed models. We find that \(a_{\text{BH}} = 0.205\) pc for \(q = 1.0\), the largest amongst all our models. This lies in the region where the density \(\rho(r)\) for the relaxed models is lower than that of the non-relaxed models implying that the energy loss in the non-relaxed models must be higher for \(nr_q=0.1\).

The situation changes as we lower the mass-ratio and the hard-binary radius decreases. We find that the total density \(\rho\) increases after \(r \sim 0.1\) pc. Owing to the higher density, the binary is able to compensate or even more than compensate for the differences in the energy loss in the beginning of the combined stage in the \(q = 0.1, 0.01\) models. This is quite evident while comparing the evolution of the binary separation for non-relaxed and relaxed \(q = 0.01\) models. The initial inspiral of the secondary is slower in \(r_q=0.1\) compared to \(nr_q=0.01\). Once the secondary is about ~ 0.1 pc away, the inspiral accelerates and it reaches the hard binary radius faster than the non-relaxed model. The situation is quite similar to the RUR3 model in Khan et al. (2015) where the authors found that initial inspiral of the secondary for the more centrally concentrated model to be slower in the beginning. In the semi-analytic model developed in Gualandris et al. (2022), the authors found that the inspiral due to dynamical friction was slower in the models with steeper inner cusps due to lower stellar density in the outskirts.

Incidentally, for the chosen set of initial conditions, \(r_q=0.1\) and \(nr_q=0.1\) show almost identical evolution of binary parameters. For \(q \lesssim 10^{-1}\), higher central concentration drives the binary towards faster inspiral. For really low mass ratio binaries this has the potential to accelerate transition to hard-binary stages even faster but further studies with higher resolution models are required.

As a back-reaction to the shrinkage of the binary due to dynamical friction and scattering, energy is induced into the particles nearby leading to an expansion of their orbits and therefore, a disruption of the cusp. Figure 5 provides a visual description of said expansion. Comparing the Lagrange radii, i.e. the radii enclosing a particular fraction of the total mass, plots of both MS and BH particles, we find that in the combined phase, higher mass ratios inject more energy into the surrounding particles leading to a rapid expansion and disruption of the cusp, which also leads to reversal of mass-segregation. The disruption is less-violent in \(q = 0.01\) case because of the lower mass of the secondary.

Since the MBHs form a bound binary, we can also examine the evolution of eccentricity during this stage. For the larger mass-ratio models we find that the binary’s orbit is approximately circular as it reaches the hard binary radius. We see similar trends in both relaxed and non-relaxed models for \(q = 1.0, 0.1\). This is similar to the observation made by Gualandris & Merritt (2012) in their circular orbit models. For \(q = 0.01\), the story is a little different. While the binaries initially start circular, the growth of eccentricity is more stochastic in this case because of the lower mass of the secondary. However, we find that evolution of eccentricity is generally in the
direction of higher eccentricity. This was also noted in Merritt (2013) for Intermediate Mass Black Holes (IMBHs) in this mass range. Curiously, we find that for $r_q \approx 0.01$, the binary is able to reach a higher eccentricity in this phase compared to $r_q \approx 0.01$. Intuitively, we would expect the opposite because when the periapsis of the binary falls within the denser, relaxed cusp, it should circularize the binary. The increase in eccentricity happens around the time the secondary reaches the distances where the mass is dominated by BH rather than MS particles. The secondary hardens by having more encounters with the BH particles. In asymmetric MBH binaries the eccentricity growth is mostly driven by the companion-perturber mass ratio Sesana et al. (2008). A larger intruder mass in the relaxed model results in growth of eccentricity. This may not be physical because in realistic NSCs, the mass of the perturbing BH particles would be smaller.

We also noted that the evolution of eccentricity for $q = 0.01$ model is quite dependent on resolution. We found that a simulation using $N \approx 4 \times 10^6$ particles (see appendix A) resulted in the MBH binary reaching extremely large values of eccentricity (0.9) in both non-relaxed and relaxed models. It highlights the importance of using larger resolution to estimate binary evolution parameters, especially for lower mass ratio binaries. Our results for this mass range are quite similar to those presented in Arca-Sedda & Gualandris (2018) where the authors found that unless the IMBH starts off in an highly eccentric orbit, it is not able to reach large values of eccentricity. We caution against consulting single simulations to track the evolution of eccentricity with the current resolution. Unlike semi-major axis, the evolution of eccentricity is a second order effect in angular momentum and subject to more stochasticity and more simulations are needed to model the evolution more realistically.

4.3 Hard-binary phase ($r < a_h$)

The last phase before the GW emission state is the hard-binary phase where the binary hardens by three-body scattering. We investigate the differences in hardening rates between the relaxed and non-relaxed models. In the full loss-cone regime the binary should harden at a fixed rate. This would imply that,

$$\frac{d}{dt} \left( \frac{1}{a} \right) = s$$

where $s$ is some constant. To find the value of $s$, we fit straight lines to the inverse semi-major axis plots. For $q = 1.0, 0.1$ we fit the lines to the values between 5 Myr and 10 Myr. For $q = 0.01$, we do it between 7 Myr and 10 Myr since the binary reaches the hard binary radius a little before 7 Myr. We notice that there is a sharp jump in the evolution of the inverse semi-major axis in the $r_{q_0} = 0.01$ model around ~ 8 Myr. Jumps in the evolution of the inverse semi-major axis
In the hard binary phase, the non-relaxed models demonstrate a slight growth in eccentricity whereas in the relaxed models, the eccentricity remains roughly constant. This was also observed in Gualandris & Merritt (2012) where the authors found that the eccentricity evolution was roughly constant with time in the model with initial eccentricity. $nr_q \cdot 1\_ecc \_2$ is able to reach a high eccentricity of 0.8 by 10 Myr whereas its relaxed counterpart only reaches 0.3. The eccentricity evolution in this stage is consistent with the findings of Sesana (2010) where the authors report that shallower cusp models remain circular whereas the mildly eccentric orbits ($ecc \_1$), we observe that the hardening rates are lower than that in the circular case whereas in the highly eccentric orbit models ($ecc \_2$) demonstrate a larger hardening rate. This is due to the differences in the structure of the final merger product.

The presence of a relaxed cusp affects the evolution of eccentricity. The relaxed cusp leads to a lower eccentricity at binary formation. For $r_q \cdot 0\_1\_ecc \_1$ the eccentricity at binary formation is almost 0.0 even though the initial orbit was eccentric. Even for highly eccentric initial orbits like in the $r_q \cdot 0\_1\_ecc \_2$ scenario, we find that the eccentricity at formation is about 0.3. The situation for the non-relaxed models is quite different as $nr_q \cdot 0\_1\_ecc \_1$ and $nr_q \cdot 0\_1\_ecc \_2$ models demonstrate eccentricities of 0.19 and 0.55 at binary formation respectively. A This is because, in the presence of the relaxed cusp, the dynamical friction is able to circularize the binary leading to lower eccentricity as also explained in the previous section. A higher initial eccentricity results in a higher eccentricity at binary formation in our models. Gualandris et al. (2022) also report similar findings where the eccentricity at binary formation for the models with steeper cusps is systematically lower than that in the models with shallow cusps.

4.4 Eccentric Orbital Parameters

Realistic galaxy mergers are more likely to happen on eccentric orbits. The aforementioned parameter $\xi$ can be changed to change the initial amount of angular momentum. Reducing the parameter can lead to more eccentric orbits. We studied the effect of eccentric initial conditions and how they interplay with relaxed and non-relaxed models. Figure 6 shows that the evolution of the separation and the binary hardening is qualitatively identical to the initial conditions of $\xi = 1$ and $q = 0.1$. The hardening rates between the relaxed and the non-relaxed models are within 10% of each other and is not affected by the initial eccentricity of the orbit. This indicates that the hardening rate is dependent on morphology of the merger product rather than the density of the central cusp. For the mildly eccentric orbits ($ecc \_1$), we observe that the hardening rates are lower than that in the circular case whereas in the highly eccentric orbit models ($ecc \_2$) demonstrate a larger hardening rate. This is due to the differences in the structure of the final merger product.

Table 3. Summary of the slopes of the inverse semi-major axes of the various simulations. We find that the hardening rates between relaxed and non-relaxed models are within ~30% of each other contrary to the findings of Khan et al. (2018). In addition, we find that $nr_\_0$ models harden faster for $q \geq 0.1$ whereas the opposite is observed for $q = 0.01$.

| Simulation ID | $s$ [pc$^{-1}$Myr$^{-1}$] |
|---------------|----------------------------|
| $r_q \cdot 1\_0$ | 1.90                      |
| $r_q \cdot 0\_1$ | 13.9                      |
| $r_q \cdot 0\_0\_1$ | 46.6                      |
| $nr_q \cdot 1\_0$ | 2.6                       |
| $nr_q \cdot 0\_1$ | 14.6                      |
| $nr_q \cdot 0\_0\_1$ | 39.4                      |
| $r_q \cdot 0\_1\_ecc \_1$ | 8.2                      |
| $nr_q \cdot 0\_1\_ecc \_1$ | 9.2                      |
| $r_q \cdot 0\_1\_ecc \_2$ | 16.2                      |
| $nr_q \cdot 0\_1\_ecc \_2$ | 15.7                      |

In the hard binary phase, the non-relaxed models demonstrate a slight growth in eccentricity whereas in the relaxed models, the eccentricity remains roughly constant. This was also observed in Gualandris & Merritt (2012) where the authors found that the eccentricity evolution was roughly constant with time in the model with initial eccentricity. $nr_q \cdot 0\_1\_ecc \_2$ is able to reach a high eccentricity of 0.8 by 10 Myr whereas its relaxed counterpart only reaches 0.3. The eccentricity evolution in this stage is consistent with the findings of Sesana (2010) where the authors report that shallower cusps with initial eccentricity. The evolution of eccentricity in this phase has important consequences on determination of MBH merger timescales. Binaries that are able to reach high eccentricities can merge orders of magnitude faster than those on circular orbits. We plan on systematically studying the evolution of eccentricity as a function of the binary mass ratio in relaxed and non-relaxed models in a future study.

4.5 GW Emission from SMBH Binaries

We follow the evolution of the MBH binaries into the GW coalescence phase semi-analytically after the simulations have stopped at $t = 10$ Myr. To do so, we assume that the evolution of the inverse semi-major axis is constant, i.e. that the hardening due to stellar scattering takes place in the full loss cone limit. This has been the strategy adopted in previous studies (e.g., Gualandris & Merritt 2012; Ogiya et al. 2020). Under this assumption,

$$\frac{d}{dt} \left(\frac{1}{a}\right) \approx s \rightarrow \frac{da}{dt} = -a^2(t)s \tag{19}$$

where $s$ is a constant and can be figured out from the inverse semi-major axis data by fitting a straight line through it and measuring its...
Figure 6. The evolution of the binary orbital parameters for the eccentric models as a function of time $t$ in Myr. Top: $\text{ecc}_1$ models that are mildly eccentric initially. Bottom: $\text{ecc}_2$ models that are highly eccentric initially. We find similar trends as we found for the circular orbit models of $q = 0.1$ in Figure 3 for the binary hardening rates. We, however, notice that the presence of the denser relaxed cusp affects the eccentricity evolution of the binary. The relaxed cusp circularizes the binary more than the non-relaxed cusp. This is more evident in the highly eccentric scenario (bottom) where the binary in the relaxed cusp forms at much lower eccentricity and does not show any growth over time.

Figure 7. The evolution of the inverse-semi major axis as a function of time in the GW dominated phase for different mass-ratios. Left: Evolution and coalescence in models with circular orbits. Right: same but with eccentric models. The evolution is carried by taking the results from the simulations (solid lines) and evolving them semi-analytically (dashed lines) using the Peters (Peters 1964) equation. In the circular models (left), we find that the merger timescales for the non-relaxed models are smaller than their relaxed counterparts for $q = 1.0, 0.1$ and opposite for $q = 0.01$. In the eccentric models (right), we find that the binaries in the non-relaxed cusps always merge faster. All models merge within a Hubble time making NSCs a promising source of GWs.

The overall evolution of $a$ can be written as follows:

$$\frac{da}{dt} = \left. \frac{da}{dt} \right|_{GW} + \left. \frac{da}{dt} \right|_s$$

Peters (1964) provides the rate of change of the orbital elements due to the emission of GW. The rate of change of semi-major axis and eccentricity are given as a set of coupled differential equations:

$$\frac{da}{dt} = -\frac{64}{3} \frac{\beta F(e)}{a^3}$$

$$\frac{de}{dt} = -\frac{304}{15} \frac{\beta eG(e)}{a^4}$$

Peters (1964) provides the rate of change of the orbital elements due to the emission of GW. The rate of change of semi-major axis and eccentricity are given as a set of coupled differential equations:
where 
\[ \beta = \frac{G^3}{c^5} (M_1 M_2 (M_1 + M_2)) \]  
\[ G(e) = (1 - e^2)^{-5/2} \left( 1 + \frac{121}{304} e^2 \right) \]  
\[ F(e) = (1 - e^2)^{-7/2} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right). \]

We numerically solved the coupled differential equations for both \( a \) and \( e \) using the orbital elements obtained at end of the \( N \)-body integration. We do not take into account the growth of eccentricity due to stellar scattering while solving the differential equations.

From Figure 7, we find that in all cases with \( q \geq 0.1 \), the MBH binary in the non-relaxed models merge faster. We find that the \( r_{q, 1.0} \) model undergoes coalescence almost 23% faster compared to the \( r_{q, 1.0} \) model. This is due to the fact that the binary separation itself is lower in the non-relaxed case along with the fact that the scattering rate is also larger because of the reason explained in the previous section. In the \( q = 0.01 \) case, the binary in the relaxed model is at a smaller separation and hardens faster resulting in almost 15% faster coalescence compared to its non-relaxed counterpart. Since all of our models are roughly circular by the end of the integration time, we expect the timescales presented in 7 to be the upper-limit on the GW merger timescales.

Interestingly, the coalescence timescale for \( r_{q, 1.0} \) and \( r_{q, 1.0} \) simulations is comparable to the relaxation timescale of those systems. It is reasonable to expect the erosion of the non-spherical nature of the system over such timescales in addition to the occurrence of mass-segregation which can affect the hardening rate at later stages for equal mass ratio binaries. However, this is beyond the scope of this study but presents a novel area that merits further investigation in future studies.

We know that the coalescence timescale of the binary depends sensitively on the eccentricity. As such, we expect the binary in the non-relaxed eccentric models to merge faster. Figure 7 shows that this is indeed the case. The MBH binary in \( r_{q, 0.1} \_ecc_1 \) merges by 320 Myr whereas its relaxed counterpart takes around 360 Myr. The effect is stark when considering binaries on highly eccentric orbits. \( r_{q, 0.1} \_ecc_2 \) merges within 95 Myr but \( r_{q, 0.1} \_ecc_2 \) takes 2x longer. Binaries that form in very eccentric orbits thus more efficiently merge in galaxies where the central densities are lower. Our results are in line with those found by Gualandris et al. (2022) where the authors found that models with shallower slopes are more efficient at merging binaries. Our results also underscore the importance of including a mass-spectrum in \( N \)-body simulations as it can affect collisional relaxation in the NSC which in turn can affect the coalescence timescales.

### 4.6 Core Scouring

As the binary hardens, it displaces particles from the cusp. The effects of the binary can be strong enough to disrupt the cusp entirely and create a flat core. To understand the effects of the hardening of the binary on the particles, we plot the density profile of both the MS and BH particles in the merged system at different points of hardening. Due to computational limitations, we study the effects up until the time the binary hardens to a semi-major axis \( a_{h}/5 \) for \( r_{q, 1.0} \) and \( r_{q, 0.1} \) and upto \( a_{h}/2 \) for \( r_{q, 0.01} \). The latter model is extremely computationally intensive to evolve longer because of the formation of stable multiple systems in the cusp. Improving the integration scheme to handle secular systems more efficiently (Rantala et al. 2022) can alleviate this issue.

In the \( q = 1.0 \) scenario, we find that the inner cusp is completely disrupted and a large flat core is produced as the binary hardens from \( a_{h} \) to \( a_{h}/5 \). This is seen in the density profiles of both the MS and BH particles. The effective density of the core is \( \sim 10^5 M_\odot pc^{-3} \) signalling that the effects of the binary were so strong that the NSC itself was partially disrupted.

The situation is different for lower mass ratio binaries. For \( q = 0.1 \), the cusp of MS particles is partially disrupted as the binary hardens from \( a_{h} \) to \( a_{h}/3 \) but further disruption is not seen with more hardening. The original cusp is not retained. In fact, for the MS particles, the density profile of the merged system has a faint \( \gamma_{MS} = 0.5 \) inner slope. For the BH particles, we find a faint \( \gamma_{BH} = 0.7 \) slope. For \( q = 0.01 \), the effects are even more minuscule. However, since the binary could not be evolved to \( a_{h}/5 \) due to computational limitations, we exclude it from this analysis.

The partial retention of the cusp has implications on the regrowth of the Bahcall-Wolf cusp post MBH binary coalescence. Whereas the time required to achieve the collisionally relaxed state for \( q = 1.0 \) may exceed the Hubble time because of the presence of a flat core in both the MS and the BH particles (Merritt 2010), the same cannot be said for \( q = 0.1, 0.01 \) which will have faster regrowth. Crucially, we would need to understand how regrowth inter-plays with the galaxy geometry post merger. As such, \( N \)-body simulations are required to quantify the exact amount of time required for the regrowth of cusps.

Since Extreme Mass Ratio Inspiral (EMRI) rates are usually extrapolated under the assumption of a Bahcall-Wolf cusp at the center, we would expect EMRIs to arise out of galaxies that have undergone mergers with lower mass ratios. However, multiple mergers even with lower mass ratios can lead to the formation of a core and the exact number of mergers leading to the formation of a core as a function of \( q \) requires further studies. The time dependent rate of EMRIs post merger would be interesting to understand as well and we plan on exploring this in future studies.

How does the disruption of the cusp affect the velocity distribution? To analyze that, we plot the velocity anisotropy parameter for the \( r_{q, 1.0} \) model in Figure 9 during different stages of hardening. The velocity anisotropy parameter is defined as

\[ \beta = 1 - \frac{\sigma_t^2}{2\sigma_r^2} \]

where \( \sigma_r \) is the radial velocity dispersion and \( \sigma_t \) is the tangential velocity dispersion. We find that as the binary hardens, the velocity profile, which was initially isotropic, becomes tangentially biased. This is caused because the MBH binary preferentially ejects particles on radial orbits. The anisotropy parameter can act as an observational evidence for the presence of an MBH binary because of this reason.

This was also noted in previous studies like Merritt & Szell (2006).

### 5 DISCUSSION

#### 5.1 Impact of stochasticity

As there is inherent stochasticity involved while generating \( N \)-body samples, one may be curious as to whether the results mentioned in the previous sections are robust and reproducible. To understand the effect of stochasticity on the results, we generate four additional statistically independent merger models for the \( r_{q, 1.0} \) and \( r_{q, 1.0} \) simulations. To save computational resources, we only perform the simulations for the \( q = 1.0 \) models up to a termination time of 5
Figure 8. The density of particles $\rho$ presented as a function of the distance $r$ from the center of mass of the binary at different points in hardening for $r_\ast$ simulations. The initial cusp is also presented for comparison. Top: Density of MS particles. We can see that for $q = 1.0$ as the binary hardens, a core is formed. This is not observed for $q = 0.1$. Bottom: Density of BH particles. Similar observations are noted in this case.

Myr with a resolution of $N \sim 4 \times 10^5$. This should be sufficient to study any discrepancies in the three phases of evolution despite the lower resolution. The impact of resolution for higher mass ratio binaries is minimal and is also further discussed in the next section and appendix A.

In Figure 10, we plot the mean and standard deviation of the evolution of the inverse semi-major axis from the five simulations. We find that the results from the original simulations are reproducible and the stochastic scatter for the evolution of the separation and semi-major axis is low. This indicates that the differences in the results of the non-relaxed and relaxed models arise out of physical and not numerical reasons.

We caution the reader that such agreement might not be present for other orbital elements like eccentricity. Previous studies like Nasim et al. (2020) have highlighted this issue and found that it stems from insufficient numerical resolution. The stochasticity of the stellar encounters with the MBH binary affect eccentricity more leading to larger scatter among random realizations. This can potentially affect GW merger timescales since they are sensitive to the eccentricity of the binary. However, the resolution used in this work is sufficient to study the evolution of the semi-major axis and separation.

5.2 Comparison with previous studies

Gualandris & Merritt (2012) and Khan et al. (2018) used a non-uniform mass function to model the galaxy mergers. The resolution used in our simulations is lower compared to that in Gualandris & Merritt (2012) but is comparable to that used in Khan et al. (2018). Although our set up is quite different compared to Gualandris & Merritt (2012) we find qualitatively similar results. Comparing Figure 7 from Gualandris & Merritt (2012) to Figure 5 in our study, we observe that for the expansion of Lagrangian radius is quite sudden for mass ratios between $0.1 - 1.0$. Due to the energy injected into the cusp by the MBH binary there is an expansion in the orbits of stars leading to the destruction of the cusp as explained in the previous sections. Our work also suggests consistency with the evolution of the density profile. Although not directly comparable, we find that the effect of binary leading to core scouring presented for $q = 1/3$ in Gualandris & Merritt (2012) lies between results from our $q = 0.1, 1.0$ simulations.

Our results are in contrast with those presented in Khan et al. (2018) where the authors found that the hardening rates in mass segregated cases were significantly higher than those in the non-mass segregated cases, which was not found in our case. The discrepancy could result from the usage of a different initial mass function in Khan et al. (2018) or the usage of a different mass ratio of the binary. The authors admit that the effects of relaxation in their work could be enhanced because of the lower resolution used while the galaxy itself is relaxing prior to the merger. This effect would be negligible in our case since the collisionally relaxed models have been generated from Fokker-Planck simulation rather than $N$-body simulations. It could also occur from different galaxy merger shapes.
The upper limit on the coalescence timescale in our simulations is a dense cusp (relaxed), so the results are not directly comparable. Our models are characterized by an initial shallow cusp (non-relaxed) or of MBH binaries is robust as indicated by Ogiya et al. (2020). Our verification that the effect of NSCs in accelerating the orbital decline properly studied and presents a further avenue of research. We find that our results are robust. This implies that the discrepancies in the results arise out of physical rather than numerical reasons.

Figure 9. The evolution of the velocity anisotropy for $r_{q,1.0}$ model. The anisotropy parameter ($\beta$) is plotted as a function of the distance from the binary center ($r$). The shaded portions denote the standard error in calculating the anisotropy parameter. The models start initially with an isotropic distribution of velocity. As the binary hardens, it preferentially ejects particles with radial velocities producing a tangentially biased velocity structure near the MBH binary.

Figure 10. The evolution of the inverse semi-major axis $1/a$ as a function of time $t$ for five independent realizations of $r_{q,1.0}$ and $r_{q,1.0}$ simulations but with a resolution of $N \sim 4 \times 10^5$ for computational constraints. The solid lines represent the mean value of the inverse semi-major axis and the shaded region represents the standard deviation from five simulations. We find that our results are robust. This implies that the discrepancies in the results arise out of physical rather than numerical reasons.

The interplay between triaxiality and mass segregation has not been properly studied and presents a further avenue of research. We find similar results compared to Ogiya et al. (2020). This verifies that the effect of NSCs in accelerating the orbital decline of MBH binaries is robust as indicated by Ogiya et al. (2020). Our models are characterized by an initial shallow cusp (non-relaxed) or a dense cusp (relaxed), so the results are not directly comparable. The upper limit on the coalescence timescale in our simulations is $\sim 700$ Myr for the $q = 1.0$ models whereas that found in Ogiya et al. (2020) is $\sim 5$ Gyr. This could be because the presence of a cusp rather than a core leads to a more efficient hardening. However, Khan & Holley-Bockelmann (2021) found coalescence timescales similar to ours for large mass-ratio mergers and attributed the discrepancy is to better resolution of the three-body scattering process with higher mass resolution. Nevertheless, we find that the merger timescale depends on the mass ratio of the binary with $q = 0.01$ mass ratio binaries merging in $\sim 150 - 170$ Myr. Interestingly, this is in contrast to the upper limit to the timescales found in Ogiya et al. (2020).

We noted that the timescales for lower mass ratio binaries ($q \lesssim 10^{-2}$) is quite sensitive to the resolution. Lower resolution simulations with $N \sim 4 \times 10^5$ (see appendix A) resulted in merger timescales of $\sim 90 - 100$ Myr for the $q = 0.01$ merger simulations. This is almost half the merger timescale found using the higher resolution simulations with $N \sim 1.32 \times 10^6$. As the mass ratio of the binary was increased, the discrepancies between the lower and the higher resolution simulations decreased. This is in contrast to the findings of Preto et al. (2011) where the authors noted that in MBH binaries formed in galactic mergers, the triaxiality of the non-spherical merger product ensured that the hardening rate is independent of $N$. However, our results seem to be in line with that presented in Vasiliev et al. (2015) where the authors found that in triaxial galaxies the hardening rate asymptotically reaches a fixed value as the resolution is increased. According to Vasiliev et al. (2014) collisional effects account for a non-trivial portion of the hardening rate and cannot be neglected while considering the hardening rates of MBH binaries. Since NSCs are collisional systems, it highlights the importance of resolving collisional effects properly for a proper determination of LISA timescales for MBH binaries, especially for $q \approx 0.01$. This would require the usage of even higher particle numbers, even for simulations where the initial orbit is circular. To determine the minimum resolution required, we need to perform more simulations with varying $N$, which is beyond the scope of this study. We plan on investigating the asymptotic limits of hardening rates as a function of the resolution in future studies. We would like to stress, however, that the binaries merge efficiently well within the Hubble time in line with the conclusions made in Ogiya et al. (2020) about NSCs being potentially important sources for LISA detections.

6 CONCLUSIONS

MBH binaries are touted to be one of the most important sources of GW signals detectable by future generation of GW detectors like LISA. The presence of NSCs surrounding the MBHs can accelerate their evolution to the hard-binary and therefore, the GW emission phase. However, the dynamics of the binary can be quite sensitive to the composition of the NSC. The presence of a mass spectrum can lead to mass segregation since the NSC is a collisional system and can affect the evolution of the binary in non-intuitive ways.

In this work we have explored the effects of mergers of collisionally relaxed NSC on the dynamics of MBHs embedded in them. Using a suite of $N$-body simulations, we have demonstrated the non-intuitive ways in which a collisionally relaxed nuclei with a two-component initial mass function can affect the overall dynamics depending on the mass ratio of the binary. For simplicity, we considered the mass of the lighter objects to be $1M_\odot$ representing stars, white dwarfs, and neutron stars and the mass of heavier objects to be $10M_\odot$ representing stellar mass black holes.

Through the usage of a Fokker-Planck code, we evolved the NSCs to the collisionally relaxed state under the presence of an MBH at the center. We then set up the mergers with different MBH mass ratios and for comparison, also evolved mergers with non-relaxed NSCs.

During the three stages of evolution, we found that the dynamics during the pre-binary phase is similar amongst simulations, even with different mass ratios and is consistent with results from Ogiya.
et al. (2020). However, due to changes in the density profile between relaxed and non-relaxed systems, differences arise during the combined phase.

The presence of a heavier mass species leads to a decline in the density profile of the lighter species for all radii greater than $\sim 0.1d_{\text{infl}}$ within the sphere of influence of the primary. As a result, in larger mass ratio binaries, the binary is able to settle at a lower separation in the non-relaxed models after the combined phase compared to the relaxed models. However, this trend slowly changes with the mass-ratio of the binary as the scattering efficiency of the binary increases with the decrease in mass-ratio and the binary is able to lose more energy by scattering the particles that are more tightly bound to the MBHs. For $q = 0.1$, we find that the evolution in the relaxed and non-relaxed models are similar and for $q = 0.01$, we find that the presence of the denser cusp actually accelerated the evolution of the binary to the hard binary stage.

In the hard binary stage, the evolution is similar amongst non-relaxed and relaxed models. The binaries harden at a fixed rate consistent with previous studies on NSC and galaxy mergers. This indicates that even when the cusp is disrupted less, the primary mode of evolution is collisionless where the loss-cone of the binary is populated by stars on centrophilic orbits. This is driven by the shape of the merger product rather than by any relaxation effects. However, we do find that the hardening rate depends on the particle number $N$ in contrast with some previous studies. This underscores the importance of accounting for properly accounting for collisional effects as they plan a non-trivial role in the evolution of MBH binaries in collisional systems like NSCs.

Crucially, we find that the relaxed cusp plays a big role in the determination of the eccentricity at the binary binding and hardening stages. Non-relaxed cusps show higher eccentricities at binding stage and show growth in eccentricity which is absent in relaxed cusps. The eccentricity “suppression” is quite large in the relaxed models and binaries on highly eccentric initial orbits can merge almost in the non-relaxed models are similar and for $q = 0.01$, we find that the presence of the denser cusp actually accelerated the evolution of the binary to the hard binary stage.

The expected GW coalescence time for all of our models (relaxed and non-relaxed) is significantly less than the Hubble time, making merging NSCs a promising GW source. In addition, we find that the expected coalescence time for the non-relaxed models is lower than their relaxed counterparts for all models with $q > 10^{-2}$. The eccentricity evolution of the MBH binary is strongly affected by the initial density profile of the NSCs with relaxed models exhibiting lower eccentricity. This underscores the importance of properly modeling the initial conditions of the NSC including the usage of a realistic mass-spectrum. Generation of LISA waveforms will require proper modeling of the environment surrounding the binary.

While our initial conditions are idealized in the sense that only two mass species are used, we demonstrate the necessity of modeling NSCs with multiple mass species as collisional relaxation can often evolve the density and mass profiles in non-intuitive ways which can affect the dynamics of the MBH binary. Our simulations open up avenues of further exploration with regards to IMBH-MBH mergers and the impact of triaxiality on mass segregation. We also demonstrate the effectiveness of Taichi at handling problems of this resolution and scale. Taichi is among the first $N$-body codes built with a fourth-order symplectic integrator with time symmetric step solver and regularization. Nevertheless, some of the simulations from this work clearly show that directly integrating over many orbits of the hard binaries can be computationally demanding. A proper treatment of stable hierarchical systems (e.g., Wang et al. 2020; Rantala et al. 2022) is worth the investment. With further improvements, Taichi presents an effective method to simulate galaxy mergers, MBH binaries and simulations of NSCs.

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DATA AVAILABILITY

The data from the $N$-body simulations, and the version of Taichi used for this work, are available upon reasonable requests. Please contact the authors if you would be interested in using a development version.

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To understand the effect of resolution on the results, we simulate lower resolution models of mergers on circular orbits with $N = 4 \times 10^5$. For clarity, we have only presented the full evolution for the $q = 0.01$ models. Looking at Figure A1, we do not find any differences in the pre-binary phase indicating that the lower resolution simulations are able to simulate the tidal stripping process as accurately as the higher resolution simulations. The time of binary formation is consistent among the two sets of simulations as well. However, once the binary has formed, differences appear between the lower resolution and the higher resolution simulations. In simulations with $N = 4 \times 10^5$, we find that the binary is able to reach large values of eccentricity which is not observed in the simulations with $N = 1.32 \times 10^6$ particles. This was only observed in the $q = 0.01$ models but not in the larger mass-ratio models. In addition, as mentioned before, the hardening rate in the lower resolution simulation is about 2× higher than that in the higher resolution simulation. To understand how the resolution plays a role in the hardening rate, we plot the average hardening rate as a function of the mass-ratio in Figure A2. We find that the hardening rates are consistent among the lower and higher resolution simulations for $q = 1.0$. As the mass ratio is lowered, the higher resolution models demonstrate lower rates of hardening. This indicates that although the primary method for loss-cone scattering in the hard binary phase may be collisionless, driven by the non-spherical nature of the merger product, collisional effects cannot be discounted, especially in collisional systems like NSCs. The dependence of hardening rate on $N$ is more prominent for models with lower $q$. This is in line with the observations of Vasiliev et al. (2014) where the authors found that even in non-spherical galaxies, collisional loss cone refilling can play a significant part.

APPENDIX A: EFFECT OF RESOLUTION

We use the initial conditions for the $q = 1.0$ model directly obtained from Ogiya et al. (2020) to test Taichi against NBODY6++GPU. We present the evolution of the orbital elements and show that using the parameters chosen in this study, Taichi is able to simulate the system as accurately as NBODY6++GPU. Even without the usage of specialized hardware such as GPUs, Taichi is able to simulate the systems 2× faster than NBODY6++GPU using 2 GPUs. The energy conservation at the end of the simulation for Taichi was ~0.1% whereas that for NBODY6++GPU was ~1%. For these simulations, the number of particles was the same as that used in Ogiya et al. (2020), $N = 131072$. We find that there are no significant differences between the evolution of the binaries using the two codes. We find the differences in the evolution of binary separation are minuscule. Even though the eccentricity evolution is inherently stochastic, qualitatively the evolution is similar in both cases. From the plot of the inverse semi-major axis, we deduce that in both scenarios the binary is hardening at similar rates. This indicates that Taichi is able to model both collisional and collisionless processes as accurately as NBODY6++GPU and is suitable to handle the class of problems mentioned in this work.

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Evolution of MBH Binaries in Collisionally Relaxed NSCs

Figure A1. Evolution of the binary parameters for the relaxed $q = 0.01$ model presented as a function of time for lower resolution and higher resolution models. We find that although there are no differences in pre-binary phase and the time of binary formation between the lower resolution and higher resolution models, differences appear once the binary is in the bound-binary and the hard binary phases. This is quite notable for the evolution of eccentricity and the rate of hardening where the lower resolution model demonstrates a higher value compared to the high resolution model. The results are in contrast with Preto et al. (2011) as we find the hardening rate depends on $N$ indicating that the effects of collisional loss-cone refilling cannot be discounted.

Figure A2. Hardening rates of circular relaxed models with different resolutions presented as a function of the mass-ratio $q$. The hardening rates have been computed by taking the average of the hardening rates every 1 Myr after a hard binary has been formed. The error bars correspond to the standard deviation. We find that the hardening rate strongly depends on $q$ and resolution. As the mass-ratio is lowered, the hardening rate decreases as we increase $N$. Similar observations were noted for the non-relaxed scenario.
\textbf{Figure B1.} Evolution of the orbital parameters as a function of time for the \( q = 1.0 \) model presented in \textit{Ogiya et al.} (2020). Top: Evolution of binary separation. Middle: Evolution of eccentricity. Bottom: Evolution of inverse semi-major axis. Right: Evolution of the relative energy error. We find the results between \texttt{NBODY6++GPU} and \texttt{Taichi} are consistent with each other and \texttt{Taichi} is better at energy conservation by a factor of \( \sim 10 \) compared to \texttt{NBODY6++GPU}.