Stability of Positive Solution to Fractional Logistic Equations

By

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Abstract. In this paper, we show the existence of a classical solution to a class of fractional logistic equations in an open bounded subset with smooth boundary. We use the method of sub- and super-solutions with variational arguments to establish the existence of a unique positive solution. We also establish the stability and non-degeneracy of the positive solution.

Key Words and Phrases. Variational methods, Positive solutions, Stability, Integrodifferential operators, Fractional Laplacian.

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1. Introduction

In this paper, we consider the following logistic equation with fractional Laplacian

\[
\begin{cases}
(\mathcal{A})^s u = \lambda a(x)u - b(x)u^p, & p > 1 \quad \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

(1.1)

where \( \Omega \subset \mathbb{R}^n \), \( n > 2s \) \((0 < s < 1)\), be an open bounded subset with smooth boundary, \((\mathcal{A})^s\) stands for the fractional Laplacian. Throughout this paper, we assume that \( a, b \in C^\alpha(\Omega) \), \( 0 < \alpha < 1 \), \( a(x), b(x) > 0 \), \( \lambda > 0 \) be such that \( \lambda > \lambda_1(a) \), where \( \lambda_1(a) \) is the first eigenvalue of the problem

\[
\begin{cases}
(\mathcal{A})^s u = \lambda a(x)u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

(1.2)

and \( p < 2s/(n - 2s) \).

The fractional Laplacian is the infinitesimal generator of Lévy stable diffusion process, see [4] and it appears in several branches of sciences and engineering and has substantial applications such as free boundary problem [9], thin film obstacle problem [33], conformal geometry [13], cell biology [32], chemical and contaminant transport in heterogeneous aquifers [2], phase transitions [34], crystal dislocation [23] and many others. We refer to [15] and references therein for an elementary introduction on this subject. Recently,
there has been a good amount of works on the existence and qualitative questions to fractional Laplace equations. There are several papers dealing with these questions, see for instance [7, 11] and the references therein. We refer to a very recent work [41] on recent progress in the study of nonlinear diffusion equations involving nonlocal, long-range diffusion effects.

Before we begin with to the study of (1.1), let us briefly discuss the earlier research works on the local problem of (1.1):

\[
\begin{cases}
-\Delta u = \lambda a(x)u - b(x)u^p, & p > 1 \quad \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Equations of the form (1.3) has its importance in several branches of sciences and has been investigated by many researchers. For example, they arise in mathematical biology, where they describe the steady state solutions of the logistic equation with diffusion

\[
\frac{\partial u}{\partial t} = \Delta u + a(x)u - b(x)u^p,
\]

see [16] and in Riemannian geometry as the equation for the change of the scalar curvature under a conformal change of the metric, see [25] and the references therein. There are several papers which deal with the existence and qualitative questions to the equations of type (1.3). We just name a few articles which are closely related to this paper. In case \(a(x) \equiv 1 = b(x),\) (1.3) has a unique solution for \(\lambda > \lambda_1(\Omega),\) where \(\lambda_1(\Omega)\) is the first Dirichlet eigenvalue of the Laplacian in \(\Omega,\) see [3]. We recall that for \(a(x) \equiv 1\) in (1.3), and \(b(x) > 0\) in \(\Omega,\) it is well known that \(\exists\) a unique positive solution to (1.3) if and only if \(\lambda > \lambda_1(\Omega),\) see [24, 28]. Pigola et al. [29] studied the steady state solutions of

\[
\begin{cases}
-\Delta u = \lambda a(x)u - b(x)u^p, & p > 1 \\
\end{cases}
\]

on a complete Riemannian manifold. García-Melián et al. [22] established the existence, uniqueness and stability of nonnegative solution of

\[
\begin{cases}
-\Delta u = \lambda u - u^p & \text{in } \Omega, \\
\frac{\partial u}{\partial v} = u^r & \text{on } \partial \Omega,
\end{cases}
\]

where \(p, r > 0.\) Recently, García-Melián and Rossi [21] discussed the existence and uniqueness of positive solutions to the problem

\[
\begin{cases}
-(J * u) + u = \lambda u - b(x)u^p, & p > 1 \quad \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]
where $(J * u) - u$ is nonlocal diffusion operator and $J * u$ is the usual convolution. For the existence of positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, see [36] and the references cited therein. For the existence and uniqueness questions to fractional Laplace equations including logistic type of nonlinearities, we refer to the recent works [10, 12, 30].

Recently, Frank and Lenzmann [20] established uniqueness of ground states of

$$(-\Delta)^{s} u + u - u^{2s+1} = 0 \quad \text{in } \mathbb{R}$$

and nondegeneracy of the linearized operator. Due to the rapidly increasing interest on the study of fractional Laplace equations in recent years, it is natural to ask whether can we obtain the existence of a solution to (1.1) and its qualitative properties? The aim of this paper is to answer this question.

In fact, by constructing sub- and supersolution to (1.1) and then by variational arguments, we obtain the existence of a unique solution to (1.1). We also establish a stability result to the solution $u$ of (1.1) and finally show that $u$ has zero Morse index. For the stability results to quasilinear elliptic equations with logistic type of nonlinearities, we refer to [38, 39] and the references cited therein and for the similar results on Morse index of a solution to Laplace equation and biharmonic equation, we refer to [18] and [40], respectively.

Due to the appearance of fractional Laplacian in the equation, it seems difficult to apply the classical approach to establish the stability and Morse index of the solution to (1.1).

Here, we use a different approach with an application of a simple inequality to establish Theorem 1.2, given below.

More precisely, we state the following main theorems, which we will prove in this paper:

**Theorem 1.1.** Let $0 < a, b \in C^{\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, $\lambda > 0$ be such that $\lambda > \lambda_{1}(a)$. Let $\underline{u}$ and $\bar{u}$ be respectively subsolution and supersolution of (1.1) with $0 < \underline{u} \leq \bar{u}$ a.e. in $\Omega$. Let us consider the associated functional $E : H^{s}_{0}(\Omega) \to \mathbb{R}$ defined by

$$E(u) = \frac{1}{2} \int_{R^{n}} |(-\Delta)^{s/2} u|^{2} \, dx - \frac{\lambda}{2} \int_{\Omega} a(x) u^{2} \, dx + \frac{1}{p+1} \int_{\Omega} b(x) u^{p+1} \, dx.$$ 

Let $M = \{ u \in H^{s}_{0}(\Omega) : 0 < \underline{u}(x) \leq u(x) \leq \bar{u}(x) \text{ a.e. } x \in \Omega \}$. Then $E$ attains the infimum at some point $u \in M$ and $u$ is a unique solution of (1.1). Also, $u \in C^{2s-\beta}(\Omega) \cap C^{\beta}(\overline{\Omega})$ for some $0 < \beta < 1$.

**Theorem 1.2.** Let $0 < a, b \in C^{\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, $\lambda > 0$ be such that $\lambda > \lambda_{1}(a)$. Let $u$ be a solution of (1.1). Then $u$ is stable, has zero Morse index and nondegenerate.
2. Preliminaries

Let us recall the brief preliminaries on the fractional Laplacian. We will not be using all the preliminaries in the present article, since it is interesting so we reproduce it here for the interesting reader. Let $0 < s < 1$. There are various definitions to define fractional Laplacian $(\mathcal{L}_0^s)u$ of a function $u$ defined on $\mathbb{R}^n$. It is well known that $(\mathcal{L}_0^s)$ on $\mathbb{R}^n$, $0 < s < 1$ is a nonlocal operator and it can be defined through its Fourier transform. Thus, if $u$ is a function in the Schwarz class in $\mathbb{R}^n$, $n \geq 1$, denoted by $S(\mathbb{R}^n)$, then we have

$$(\mathcal{L}_0^s)u(x) = |x|^{2s} \hat{u}(x)$$

and therefore we write

$$(\mathcal{L}_0^s)^s u = f \quad \text{if} \quad \hat{f}(\xi) = |\xi|^{2s} \hat{u}(\xi),$$

where $\hat{\cdot}$ is Fourier transform. When $u$ is sufficiently regular, the fractional Laplacian of a function $u : \mathbb{R}^n \to \mathbb{R}$ is defined as follows:

$$(\mathcal{L}_0^s)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy = C_{n,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B(\epsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy,$$

where

$$C_{n,s} = s 2^{2s} \frac{\Gamma\left(s + \frac{n}{2}\right)}{\pi^{n/2} \Gamma(1 - s)},$$

which is a normalization constant.

One can also write the above singular integral as follows:

$$(2.1) \quad (\mathcal{L}_0^s)^s u(x) = -\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} \, dy,$$

$\forall x \in \mathbb{R}^n$, $u \in S(\mathbb{R}^n)$,

see [15]. When $s < 1/2$ and $f \in C^{0,\alpha}(\mathbb{R}^n)$ with $\alpha > 2s$, or if $f \in C^{1,\alpha}(\mathbb{R}^n)$, $1 + 2s > 2\alpha$, the above integral is well-defined. By the celebrated work of Caffarelli and Silvestre [8], the nonlocal operator can be expressed as a generalized Dirichlet-Neumann map for a certain elliptic boundary value problem with nonlocal operator defined on the upper half space $\mathbb{R}_+^{n+1} := \{(x, y) \mid x \in \mathbb{R}^n, y > 0\}$, i.e., for a function $u : \mathbb{R}^n \to \mathbb{R}$, we consider the extension $v(x, y) : \mathbb{R}_+^{n+1} \to \mathbb{R}$.
\( \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) that satisfy

\[
v(x, 0) = u(x),
\]

(2.2)\[ \Delta_x v + \frac{1 - 2s}{y} v_y + v_{yy} = 0. \]

(2.2) can also be written as

(2.3)\[ \div (y^{1-2s} \nabla v) = 0, \]

which is the Euler-Lagrange equation for the functional

(2.4)\[ J(v) = \int_{y > 0} |\nabla v|^2 y^{1-2s} \, dx \, dy. \]

Then it can be seen from [8], that

(2.5)\[ C(\mathcal{A}^s u) = \lim_{y \to 0^+} y^{1-2s} v_y = \frac{1}{2s} \lim_{y \to 0^+} \frac{v(x, y) - v(x, 0)}{y^{2s}}, \]

where \( C \) is some constant depending on \( n \) and \( s \). The space \( H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n) \) is defined by

\[
H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \frac{|u(x) - u(y)|}{|x - y|^{n+2s}} \in L^2(\mathbb{R}^n) \times \mathbb{R}^n) \right\}
\]

and the norm is defined as follows:

\[
\|u\|_s := \|u\|_{H^s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + \int_{\mathbb{R}^n} |u|^2 \, dx \right)^{1/2}.
\]

The term

\[
[u]_s := \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right)^{1/2}
\]

is called the Gagliardo norm of \( u \). We recall that \( H^s(\mathbb{R}^n) \) is a Hilbert space and its norm is induced by the inner product

\[
\langle u, v \rangle_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy + \int_{\mathbb{R}^n} u(x) v(x) \, dx.
\]

Let \( \{\phi_k\} \) be an orthonormal basis of \( L^2(\Omega) \) with \( \|\phi_k\|_2 = 1 \) forming a spectral decomposition of \(-\mathcal{A}\) in \( \Omega \) with zero Dirichlet boundary conditions and
\( \lambda_k \) be the corresponding eigenvalues. Let
\[
H_0^s(\Omega) = \left\{ u = \sum_{k=0}^{\infty} a_k \phi_k \in L^2(\Omega) : \| u \|_{H_0^s(\Omega)} = \left( \sum_{k=0}^{\infty} a_k^2 \lambda_k^s \right)^{1/2} < \infty \right\}.
\]

We denote \( H^{-s}(\Omega) \) the dual space of \( H_0^s(\Omega) \). For \( u \in H_0^s(\Omega) \), \( u = \sum_{k=0}^{\infty} a_k \phi_k \) with \( a_k = \int_\Omega u \phi_k \, dx \), then one can also define \( (-\Delta)^s \) subject to Dirichlet boundary condition as follows (see [15, 31]):
\[
(-\Delta)^s u = \sum_{k=0}^{\infty} a_k \lambda_k^s \phi_k \in H^{-s}(\Omega).
\]

We call \( \{ (\phi_k, \lambda_k^s) \} \) the eigenfunctions and eigenvalues of \( (-\Delta)^s \) in \( \Omega \) with zero Dirichlet boundary conditions. We recall that these two definitions of fractional Laplacian give rise to two completely different non-local operators of fractional type, i.e., the fractional Laplacian can be defined in both these ways, but the two definitions are not equivalent in bounded domains, see for instance [5, 27]. In this paper, we will be using the definition of fractional Laplacian via Fourier transform or equivalently, as an integral in the P.V. sense.

Let \( \lambda_1(a) > 0 \) be the first eigenvalue of \( (-\Delta)^s \) in \( \Omega \) and \( \phi_1 > 0 \) be the corresponding eigenfunction (first eigenfunction), i.e.,
\[
\begin{cases}
(-\Delta)^s \phi_1 = \lambda_1(a) \phi_1 & \text{in } \Omega, \\
\phi_1 > 0 & \text{in } \Omega, \\
\phi_1 = 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

The variational characterization of \( \lambda_1 \) is given by
\[
\lambda_1 = \inf \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} v|^2 \, dx : v \in H_0^s(\Omega) \text{ and } \int_{\Omega} a(x) v^2 \, dx = 1 \right\}.
\]

### 3. Positive solution, stability and nondegeneracy

In the next proposition, we prove the existence of a weak subsolution \( u \) and weak supersolution \( \tilde{u} \) of (1.1). The weak subsolution and supersolution can be defined in a standard way. We say that \( u \) is a subsolution (supersolution) of (1.1) if
\[
\int_{\mathbb{R}^n} (-\Delta)^s u \cdot \psi \, dx \leq (\geq) \int_{\Omega} (\lambda a(x) u - b(x) u^p) \cdot \psi \, dx,
\]
\( \forall \psi \in H_0^s(\Omega), \psi \geq 0 \) a.e. in \( \Omega \) and \( u \leq 0(\geq 0) \) a.e. in \( \mathbb{R}^n \setminus \Omega \).
Proposition 3.1. Let $0 < a, b \in C^\alpha(\Omega)$, $0 < \alpha < 1$, $\lambda > 0$ such that $\lambda > \lambda_1(a)$. Then there exist a subsolution $u$ and a supersolution $\bar{u}$ of (1.1) such that $0 < u \leq \bar{u}$ a.e. $x \in \Omega$.

Proof. It is easy to see that $u(x) = k\phi_1(x) \in L^\infty(\Omega)$ is a subsolution of (1.1), where $\phi_1$ is the first eigenfunction of the eigenvalue problem (1.2) and $k$ is a constant such that

\[
0 < k \leq \left[\frac{(\lambda - \lambda_1(a))a(x)}{b(x)\phi_1^{p-1}}\right]^{\frac{1}{p-1}} \quad \text{a.e. in } \Omega.
\]

Indeed, $\forall 0 \leq \psi \in H_0^s(\Omega)$, we have

\[
\int_{\Omega^*} (-A)^s(k\phi_1) \cdot \psi \, dx = \int_{\Omega^*} (-A)^{s/2}(k\phi_1) \cdot (-A)^{s/2}{\psi} \, dx
\]

\[
= \int_{\Omega} \lambda_1(a)(k\phi_1) \cdot \psi \, dx \quad \text{(by (2.6))}
\]

\[
\leq \int_{\Omega} [\lambda a(x)(k\phi_1) - b(x)(k\phi_1)^p] \cdot \psi \, dx \quad \text{(by (3.2))}.
\]

To construct a supersolution of (1.1), let us fix $x \in \Omega$, and consider the function $F : \Omega \times (0, \infty) \rightarrow \mathbb{R}$ defined by

\[
F(x, u) = \lambda a(x)u - b(x)u^p, \quad p > 1.
\]

It is easy to see that $F(x, u)$ attains its maximum at

\[
u_0 = \left(\frac{\lambda a(x)}{pb(x)}\right)^{1/(p-1)}
\]

and the maximum value is given by

\[
F(x, u_0) = (p - 1)b(x)\left(\frac{\lambda a(x)}{p}\right)^{p/(p-1)}.
\]

Now let us consider the solution $\bar{u}$ of the problem

\[
\begin{cases}
(-A)^su = (p - 1)b(x)\left(\frac{\lambda a(x)}{p}\right)^{p/(p-1)} \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

By Theorem 1.6 [31] (see also [35]), $\bar{u} \in C^\beta(\Omega)$, $0 < \beta < 1$. It is easy to check that $\bar{u}$ is a supersolution of (1.1) and since $p > 1$ and $a(x), b(x) > 0$, so the RHS of (3.4) is nonnegative and by the maximum principle [6], $\bar{u} \geq 0$ and strong
maximum principle [6] yields that \( \bar{u} > 0 \) in \( \Omega \). Since \( \phi_1 > 0 \) so by Hopf’s principle, see Proposition 4.11 [6], we can choose \( k > 0 \) sufficiently small such that \( k\phi_1(x) \leq \bar{u}(x) \), a.e. \( x \in \Omega \).

Let us recall the following embeddings:

**Theorem 3.2** [15]. The following embeddings are continuous:

1. \( H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \ 2 \leq q \leq 2n/(n-2s), \) if \( n > 2s \),
2. \( H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \ 2 \leq q \leq \infty, \) if \( n = 2s \),
3. \( H^s(\mathbb{R}^n) \hookrightarrow C^j_b(\mathbb{R}^n), \) if \( n < 2(s-j) \).

Moreover, for any \( R > 0 \) and any \( p \in [1,2, (s)] \) the embedding \( H^s(B_R) \hookrightarrow L^p(B_R) \) is compact, where

\[
C^j_b(\mathbb{R}^n) = \{ u \in C^j(\mathbb{R}^n) : D^k u \text{ is bounded on } \mathbb{R}^n \text{ for } |k| \leq j \}.
\]

### 3.1. Proof of Theorem 1.1

**Proof.** The proof is adapted from p. 17 [37] which deal with the semilinear case. By Proposition 3.1, there exist a subsolution \( \underline{u}(x) = k\phi_1(x) \) and supersolution \( \bar{u}(x) \) of (1.1). Now let us consider \( M \) with these \( \underline{u} \) and \( \bar{u} \). It is easy to see that \( E \) is coercive in \( M \). Indeed,

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \, dx - \frac{\lambda}{2} \int_{\Omega} a(x) u^2 \, dx + \frac{1}{p+1} \int_{\Omega} b(x) u^{p+1} \, dx
\]

\[
\geq \frac{1}{2} \| u \|^2_{H^s_0(\Omega)} - \lambda c_1 \| a \|_{L^\infty(\Omega)} \| u \|_{H^s_0(\Omega)}^s \quad \text{(by Theorem 3.2)}
\]

\[
\to \infty \quad \text{as} \quad \| u \|^2_{H^s_0(\Omega)} \to \infty.
\]

Using the fact that \( 1 < p < 2, (s) - 1 \) and Theorem 3.2, it is not difficult to see that \( E \) is weakly lower semicontinuous and \( M \) is weakly closed. Therefore one can easily see that the infimum of \( E \) is achieved at some \( u \in M \). Let \( \phi \in C^1_c(\Omega), \ e > 0, \) and define

\[
v_e := \min \{ \bar{u}, \max \{ \underline{u}, u + e\phi \} \} = u + e\phi - \phi_e + \phi_e,
\]

where \( \phi_e := \max \{ 0, u + e\phi - \bar{u} \} \) and \( \phi_e := -\min \{ 0, u + e\phi - \underline{u} \} \). Since \( u \) minimizes \( E \) on \( M \) and \( E \) is a \( C^1 \) functional on \( H^s_0(\Omega) \), so we have \( \langle E'(u), v_e - u \rangle \geq 0 \), which gives

\[
\langle E'(u), \phi_e \rangle \geq \frac{\langle E'(u), \phi_e \rangle - \langle E'(u), \phi_e \rangle}{e}.
\]

Since \( \bar{u} \) is an supersolution and \( (-\Delta)^{s} \) is monotone, we have
\begin{equation}
\langle E'(u), \phi \rangle \geq \langle E'(u) - E'(\bar{u}), \phi \rangle \\
\geq \varepsilon \int_{\mathbb{R}^n} \left(( -\Delta \right)^{s/2} u - ( -\Delta \right)^{s/2} \bar{u}) \cdot ( -\Delta \right)^{s/2} \phi \, dx \\
- \varepsilon \int_{\Omega_\varepsilon} |\lambda a(x)u - u^p - \lambda a(x)\bar{u} + \bar{u}^p| |\phi| \, dx,
\end{equation}

where $\Omega_\varepsilon = \{ x \in \Omega : u(x) + \varepsilon \phi x \geq \bar{u}(x) > u(x) \}$. Now $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$, the last inequality implies that

\begin{equation*}
\langle E'(u), \phi \rangle \geq 0(\varepsilon) \quad \text{as} \quad \varepsilon \to 0.
\end{equation*}

Similarly,

\begin{equation*}
\langle E'(u), \phi_\varepsilon \rangle \leq 0(\varepsilon) \quad \text{as} \quad \varepsilon \to 0
\end{equation*}

and by (3.6), we get

\begin{equation*}
\langle E'(u), \phi \rangle \geq 0.
\end{equation*}

Replacing $\phi$ by $-\phi$, one concludes that $u$ solves (1.1). It is easy to see that $u \in L^\infty(\Omega)$. This implies that the RHS of (1.1) belongs to $L^\infty(\Omega)$. Now by Theorem 1.6 [31], we get $u \in C^\gamma(\overline{\Omega})$, $0 < \gamma < 1$. Again it is easy to observe that the RHS of (1.1) belongs to some $C^\beta(\Omega)$, $0 < \beta < 1$. Now thanks to the regularity theorem [6, 11, 42] which yields that $u \in C^{2\gamma, \beta}(\Omega) \cap C^\beta(\overline{\Omega})$. Now using the sweeping arguments, we show the uniqueness of the positive solution. Indeed, if $u$ and $v$ are classical positive solutions, then by Hopf’s principle, see Proposition 4.11 [6], we obtain that $tu \leq v$ in $\Omega$ for small positive $t$. Since $tu$ is a subsolution whenever $0 < t < 1$, we have by the sweeping principle [26] that $u \leq v$. Interchanging the roles of $u$ and $v$, we see that $u = v$. We remark that the sweeping principle is valid for the operator $( -\Delta )^s$, since it is a direct consequence of the strong maximum principle. This completes the proof.

\[ \square \]

Let us consider the stability, Morse index and nondegeneracy questions to the solution of (1.1). We say that $u \in H^s_0(\Omega)$ is a weak solution to (1.1) if

\begin{equation}
\int_{\mathbb{R}^n} ( -\Delta )^{s/2} u \cdot ( -\Delta )^{s/2} \phi \, dx \\
= \int_{\Omega} \lambda a(x)u \phi \, dx - \int_{\Omega} b(x)u^p \phi \, dx, \quad \forall \phi \in H^s_0(\Omega).
\end{equation}

The linearization of (1.1) at $u$ is defined by the operator $L_+$ on $H^s_0(\Omega)$ as follows:

\begin{equation}
L_+ v = ( -\Delta )^sv - \lambda a(x)v + pb(x)u^{p-1}v.
\end{equation}
We say that \( u \) is a stable (semistable) solution of (1.1) if the first eigenvalue of \( L_+ \) is positive (nonnegative), \( L_+ v = \mu v \) and \( \mu > (\geq)0 \), where \( \mu \) is the first eigenvalue of \( L_+ \), see [16]. In other words, it is equivalent to say that \( u \) is stable (semistable) if

\[
(\text{3.10}) \quad \int_{\mathbb{R}^n} |(-\Delta)^{s/2} v|^2 dx - \lambda \int_{\Omega} a(x)v^2 dx + \int_{\Omega} pb(x)u^{p-1}v^2 dx > (\geq)0, \quad \forall v \in C^1_c(\Omega).
\]

**Definition 3.3.** We say that the Morse index of a solution \( u \) of (1.1) is the number of negative eigenvalues of the linearized operator \( L_+ \), defined by (3.9), i.e., \( \#(\mu) \) such that \( \mu < 0 \) and

\[
(\text{3.11}) \quad L_+ v = \mu v \quad \text{in} \quad \Omega; \quad v = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega
\]

has a nontrivial solution \( v \in H^s_0(\Omega) \).

**Definition 3.4.** If \( u \) is a solution to (1.1), and 0 is not an eigenvalue of (3.9), then \( u \) is a nondegenerate solution, otherwise it is called degenerate.

In order to prove next theorem, we use the following simple inequality: For any \( a, b, c, d \in \mathbb{R}, \ c > 0, \ d > 0 \), we have

\[
(\text{3.12}) \quad (c - d) \left( \frac{a^2}{c} - \frac{b^2}{d} \right) \leq (a - b)^2.
\]

Inequality (3.12) is called Picone’s identity, see Lemma 6.2 for \( p = 2 \) in [1], see also [17] and this is also used in the context of \( p \)-fractional Laplacian, see for instance, [14].

### 3.2. Proof of Theorem 1.2

**Proof.** Since \( 0 \leq a \in C^\alpha(\overline{\Omega}), \ 0 < \alpha < 1, \ \lambda > 0 \) such that \( \lambda a(x) > \lambda_1 \) so by Theorem 1.1, (1.1) has a classical solution \( u \). Now for any \( v \in C^1_c(\Omega) \), it is easy to see that \( v^2/u \in H^s_0(\Omega) \). Let us take \( \phi = v^2/u \) as a test function in (3.8). This yields that

\[
(\text{3.13}) \quad 0 = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} \left( \frac{v^2}{u} \right) dx - \int_{\Omega} \lambda a(x)v^2 dx + \int_{\Omega} b(x)u^{p-1}v^2 dx = \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|(v(x))^2 - (v(y))^2}{|x - y|^{n+2\alpha}} dx dy
\]

\[
- \int_{\Omega} \lambda a(x)v^2 dx + \int_{\Omega} b(x)u^{p-1}v^2 dx
\]
\[
\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}} \, dx dy - \int_{\Omega} \lambda a(x)v^2 \, dx \\
+ \int_{\Omega} b(x)u^{p-1}v^2 \, dx \quad \text{(by (3.12))}
\]
\[
< \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}} \, dx dy - \int_{\Omega} \lambda a(x)v^2 \, dx + p \int_{\Omega} b(x)u^{p-1}v^2 \, dx
\]
\[
= \int_{\mathbb{R}^n} |(-\Delta)^{s/2}v|^2 \, dx - \int_{\Omega} \lambda a(x)v^2 \, dx + p \int_{\Omega} b(x)u^{p-1}v^2 \, dx,
\]
which proves the first claim. Also, using the same computation as in (3.13), from (3.9) and (3.11), we get
\[
(3.14) \quad \mu \int_{\Omega} v^2 \, dx = \int_{\mathbb{R}^n} |(-\Delta)^{s/2}v|^2 \, dx - \lambda \int_{\Omega} a(x)v^2 \, dx + p \int_{\Omega} b(x)u^{p-1}v^2 \, dx
\]
and this implies that \( \mu > 0 \) and therefore \( u \) has zero Morse index. Also, \( u \) is nondegenerate and this completes the proof.

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