On the second order Poincaré inequality and CLTs on Wiener-Poisson space

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Abstract

An upper bound for the Wasserstein distance is provided in the general framework of the Wiener-Poisson space. It is obtained from this bound a second order Poincaré-type inequality which is useful in terms of computations. For completeness sake, is made a survey of these results on the Wiener space, the Poisson space, and the Wiener-Poisson space, and showed several applications to central limit theorems with relevant examples: linear functionals of Gaussian subordinated fields (where the subordinated field can be processes like fractional Brownian motion or the solution of the Ornstein-Uhlenbeck SDE driven by fractional Brownian motion), Poisson functionals in the first Poisson chaos restricted to “small” jumps (particularly fractional Lévy processes) and the product of two Ornstein-Uhlenbeck processes (one in the Wiener space and the other in the Poisson space). Also, are obtained bounds for their rate of convergence to normality.

1 Introduction

In recent years many papers have looked at combining Stein’s method with Malliavin calculus in order to uncover new tools for proving various central limit theorems (CLTs). For example, I. Nourdin and G. Peccati derived an upper bound for the Wasserstein (Kantorovich) distance (and other distances) on the Wiener space using Stein’s equation [19]. Later, the same authors along with G. Reinert derived a second order Poincaré(-type) inequality which is useful (in terms of computations) for proving CLTs, and which in fact, can be seen as a quantitative extension of the Stein’s method from which upper bounds for the rate of convergence to normality can be found [20]. The first two authors with A. Réveillac extended these results to the multidimensional case [21]. In [29], G. Peccati, J. L. Solé, M. S. Taqqu and F. Utzet, were able to find an upper bound, similar to the one in [19], for the Wasserstein distance in the Poisson space. G. Peccati and C. Zheng succeeded in extending this to the multi-dimensional case in [30]. All these works are important as they give quantitative tools for computing whether a random variable converges to normality or not, and if so, its rate of convergence.

The upper bound inequality can be written as,

\[ d_W(F, N) \leq E \left| 1 - \langle DF, -DL^{-1}F \rangle_H \right| + E \left[ \langle x(DF)^2 \rangle, \langle DL^{-1}F \rangle_H \right] \]

where \( d_W \) is the Wasserstein distance, \( N \sim \mathcal{N}(0, 1) \), \( D \) is the Malliavin derivative, \( L^{-1} \) is the inverse of the infinitesimal generator of the Ornstein-Uhlenbeck (O-U) semigroup, and \( H \) is the underlying Hilbert space. In the case of the Wiener space upper bound, \( D \) is the Malliavin derivative defined in that space so this inequality holds even when the underlying Hilbert space is not \( L^2_\mu \). Also, since there are no jumps here, the second term on the right disappears (\( x \) is the size of the jump). In the Poisson space case, since we do not (as yet) have a Malliavin calculus theory developed for a general abstract Hilbert space, the underlying Hilbert space must be \( H = L^2_\mu \). A question naturally arises - can this be done for a general Lévy process; that is, is this upper bound achievable in a mixed space, the Wiener-Poisson space? The main difficulty in answering this question is that in the Wiener-Poisson space we don’t have decomposition in orthogonal polynomials (unlike with Hermite polynomials in the Wiener space, see [12] for a complete explanation), and
results like the equivalence between the Mehler semigroup and the Ornstein-Uhlenbeck semigroup. So, to overcome these shortcomings, will be necessary to deduce new formulas that will allow us to follow the ideas developed in [19] and [20], and recover their results for the Wiener-Poisson space. Will be shown that this bound still holds even when both spaces are involved.

Before getting into the details, some notation: The measure $\mu$ on the underlying Hilbert space $L^2_\mu$ is defined by the underlying Lévy process, that is, let $L_t$ be a Lévy process ($L_t$ has stationary and independent increments, is continuous in probability and $X_0 = 0$, with $E[L_t^2] < \infty$) with Lévy-triplet given by $(0, \sigma^2, \nu)$, where $\nu$ is the Lévy measure, then $\int_{\mathbb{R}^+ \times \mathbb{R}} f(z) d\mu(z) = \sigma^2 f(t,0) dt + \int_{\mathbb{R}^+ \times \mathbb{R}_0} f(t,x) x^2 d\nu(x)$, where $\mathbb{R}_0 = \mathbb{R} - \{0\}$. On the other hand, in order to define a Malliavin derivative in the Wiener-Poisson space it is sufficient to have a chaos decomposition of the space $L^2(\Omega)$. This was achieved in [14] by K. Itô, so any random variable in $L^2(\Omega)$ has a projection on the $q$th chaos given by $I_q(f_q)$, where $f_q$ is a symmetric function in $L^2_\mu$. Also, when working in the Wiener-Poisson space, the Malliavin derivative can be regarded in terms of “directions”, i.e., we can think of it as the derivative in the Wiener direction or the derivative in the Poisson direction. The fact that this can be done in this way was proven in [21] by J. L. Solé, F. Utzet and J. Vives (a quick review of the theory is given below). They showed that the Malliavin derivative with parameter $z \in \mathbb{R}^+ \times \mathbb{R}$ can be split in two cases, when $z = (t, 0)$ and when $z(t,x)$ with $x \neq 0$. The first case will be the derivative in the Wiener direction (intuitively because there are no jumps when $x = 0$), and the second will be the derivative in the Poisson direction. A distinction between the Malliavin calculus in the Wiener space or the Poisson space and in this Wiener-Poisson space is the need to define two subspaces of $L^2(\Omega)$: one where the Malliavin derivative in the Wiener direction coincides with the usual Malliavin derivative in the Wiener space and is well defined, and another where the Malliavin derivative in the Poisson direction is well defined. These subspaces are denoted by $\text{Dom} D^W$ and $\text{Dom} D^J$.

**Theorem 1.** *(Main result 1: upper bound)*

Let $N \sim \mathcal{N}(0,1)$ and let $F \in \text{Dom} D^W \cap \text{Dom} D^J$ be such that $E[F] = 0$. Then,

$$d_W(F, N) \leq E \left[ 1 - \langle DF, -DL^{-1}F \rangle_{L^2_\mu} \right] + E \left[ \langle |x(DF)|^2, |DL^{-1}F| \rangle_{L^2_\mu} \right]$$

As mentioned above, in [20], the authors used the respective Wiener space upper bound to deduce a second order Poincaré inequality. In this sense, our second main result is the second order Poincaré inequality in the Wiener-Poisson space derived from the above upper bound. This version is weaker than the one in the Wiener space because of the scarcity of results in this space, that would enable us to prove it for functionals in $\text{Dom} D^W \cap \text{Dom} D^J$; nevertheless, is possible to state it for functionals that lie in one specific Itô-chaos.

**Corollary 1.** *(Main result 2: Second order Poincaré-type inequality)*

Fix $q \in \mathbb{N}$ and let $F = I_q(f)$ with $E[F] = \mu$ and $\text{Var}[F] = \sigma^2$, and assume that $N \sim \mathcal{N}(\mu, \sigma^2)$. Then,

$$d_W(F, N) \leq \frac{\sqrt{2}}{q \sigma^2} \left( 2E \left[ \| D^2 F \|_{L^2_{\mu}}^4 \right] \frac{8}{3} E \left[ \| D^2 F \|_{L^2_{\mu}}^4 \right] \frac{8}{3} + E \left[ \langle x, (D^2 F)^2 \rangle_{L^2_{\mu}} \right] \frac{8}{3} \right) + \frac{1}{q \sigma^2} E \left[ \left( \langle |x|^2, |DF| \rangle_{L^2_{\mu}} \right)^{3/2} \right]$$

Thanks to these (Wiener, Poisson and Wiener-Poisson) upper bounds and second order Poincaré inequalities, many CLTs can be proved and generalizations made. In this paper, these bounds are reviewed for each space, showing their importance by giving applications with relevant examples. In the Wiener space case the second order Poincaré inequality is used to prove CLTs for linear functionals of Gaussian-subordinated fields when the decay rate of the covariance function of the underlying Gaussian process satisfies certain conditions. These CLTs are applied to the important cases where the underlying Gaussian process is either the fractional Brownian motion or the fractional-driven Ornstein-Uhlenbeck process, with $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

In the Poisson space case, the respective upper bound is used to prove that the small jumps process (jumps with length less than or equal to $\epsilon$) of a Poisson functional process with infinitely many jumps goes
to a normal random variable when $\epsilon$ goes to zero. Furthermore, a remarkable extension, of the known result (proved in [1]) which states that the small jumps process of a Lévy process can be approximated by Brownian motion as $\epsilon$ goes to zero, is proved. It is extended to Poisson functionals $\langle I_1(f) \rangle$ and showed that the small jumps process of this functional can be approximated by a Gaussian functional with the same kernel $f$ as $\epsilon$ goes to zero. Then is applied this result to show that in order to simulate a fractional (pure jump) Lévy process (FLP), it is sufficient to simulate a process with finitely many jumps plus an independent fractional Brownian motion (FBM).

Finally, the second order Poincaré(-type) inequality, developed in this paper, is used to prove that the time average of the product of a Wiener Ornstein-Uhlenbeck process with a Poisson Ornstein-Uhlenbeck process converges to a normal random variable as time goes to infinity. This example highlights the importance of the inequality in the Wiener-Poisson space, since it cannot be achieved by the upper bounds in the Wiener or Poisson space. An estimate of the rate of convergence to normality is obtained in the examples where the second order Poincaré inequalities is used.

The paper is organized as follows: In Section 2 is recalled the basic tools of Malliavin calculus on the Wiener space as well as the second order Poincaré inequality. Then is stated the Malliavin calculus results for the Wiener-Poisson space and explain how these tools have been extended. In Section 3 the theory developed in [20] and [19] is reproduced using Malliavin calculus theory for the Wiener-Poisson space, and with this framework is stated a “Lévy version” of the second order Poincaré inequality. Section 4 is dedicated to going over the inequalities for the Wiener, Poisson and Wiener-Poisson spaces. In the Wiener space case, a result proved in [20] concerning CLTs of linear functionals of Gaussian-subordinated fields is extended. In the Poisson space case, a result on the simulation of small jumps for processes with infinitely many jumps is given. Finally, an example of application of the second order Poincaré inequality in the Wiener-Poisson space is showed.

2 Preliminaries

As mentioned above, the most important tool for proving CLTs on the Wiener space is the upper bound developed in [20]. This requires various Malliavin calculus results on the Wiener space (Malliavin derivative, contraction of order $r$, Mehler formula, etc) which are extensively studied and explained in [25]. For the sake of the completeness, the basic tools from Malliavin calculus in the Wiener space are reviewed and then is defined the Malliavin calculus in the Wiener-Poisson setting, both needed in this paper.

2.1 Malliavin Calculus on Wiener space

Let $\mathcal{H}$ be a real separable Hilbert space. Assume there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $W := \{W(h)/ h \in \mathcal{H}\}$ is an isonormal Gaussian process, that is, $W$ is a centered Gaussian family s.t. $E[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_\mathcal{H}$. Choose $\mathcal{F}$ to be the $\sigma$-algebra generated by $W$. Let $H_q$ be the $q^\text{th}$ Hermite polynomial, $H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{\partial^q}{\partial x^q} (e^{-\frac{x^2}{2}})$, and define the $q^\text{th}$ Wiener chaos of $W$ (denoted by $\mathbb{H}_q$) as the subspace of $L^2(\Omega) := L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by $\{H_q(W(h))/ h \in \mathcal{H}, \|h\|_\mathcal{H} = 1\}$. Is important to underline that $L^2(\Omega)$ can be decomposed (Wiener chaos expansion) into an infinite orthogonal sum of the spaces $\mathbb{H}_q$: $L^2(\Omega) = \oplus_{q=0}^{\infty} \mathbb{H}_q$

**Remark 1.** In the case where $\mathcal{H} = L^2_\mu$, for any $F \in L^2(\Omega)$,

$$F = \sum_{q=0}^{\infty} I_q(f_q)$$

(1)

where $I_q$ is the $q^\text{th}$ multiple stochastic integral, $f_0 = E[F]$, $I_0$ is the identity mapping on constants and $f_q \in L^2_{\mu \otimes \mu}$ are symmetric functions uniquely determined by $F$.

Let $\mathcal{S}$ be the class of smooth random variables, i.e., if $F \in \mathcal{S}$ then there exists a function $\phi \in C^\infty(\mathbb{R}^n)$ s.t. $\frac{\partial^k \phi}{\partial x^k}(x)$ has polynomial growth for all $k = 0, 1, 2, 3, \ldots$ and $F = \phi(W(h_1), \ldots, W(h_n))$, $h_i \in \mathcal{H}$. The
Malliavin derivative of $F \in \mathcal{S}$ with respect to $W$ is the element of $L^2(\Omega, \mathcal{H})$ defined as

$$DF = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(W(h_1), \ldots, W(h_n))h_i$$

In particular $DW(h) = h$ for every $h \in \mathcal{H}$. Notice that in this particular case we have an explicit relation between the covariance of $W$ and the inner product of the Malliavin derivative, $\text{Cov}[W(h_1)W(h_2)] = E[W(h_1)W(h_2)] = \langle h_1, h_2 \rangle_{\mathcal{H}} = \langle DW(h_1), DW(h_2) \rangle_{\mathcal{H}}$.

**Remark 2.** In the case of a centered stationary Gaussian process, $X_t$, the Hilbert space can be chosen in the following way:

Consider the inner product $(1_{[0,t]}, 1_{[0,s]})_{\mathcal{H}} = \text{Cov}[X_tX_s] = C(t-s)$ and take the Hilbert space $\mathcal{H}$ as the closure of the set of step functions on $\mathbb{R}$ with respect to this inner product. With this Hilbert space one has that $X_t = W(1_{[0,t]})$ and $DX_t = 1_{[0,t]}$ where $D$ is the Malliavin derivative.

Since the Malliavin derivative verifies the chain rule, we have $Df(F) = f'(F)DF$, for any $f: \mathbb{R} \to \mathbb{R}$ of class $C^1$ with bounded derivative (is also true for functions which are only a.e. differentiable, but with the assumption that $F$ has absolutely continuous law). The second Malliavin derivative (denoted by $D^2$) can be define recursively. For $k \geq 1$ and $p \geq 1$, $\mathbb{D}^{k,p}$ denotes the closure of $S$ with respect to the norm $\|\cdot\|_{k,p}$ defined by

$$\|F\|_{k,p} = E[|F|^p] + \sum_{i=1}^{k} E[\|D^i F\|_{\mathcal{H}^\otimes i}^p]$$

Consider now an orthonormal system in $\mathcal{H}$ denoted by $\{e_k/k \geq 1\}$. Then, given elements $\phi \in \mathcal{H}^\otimes k_1$, $\psi \in \mathcal{H}^\otimes k_2$, the contraction of order $r \leq \min\{k_1, k_2\}$ is the element of $\mathcal{H}^\otimes (k_1 + k_2 - 2r)$ defined by

$$\phi \otimes_r \psi = \sum_{i_1, \ldots, i_r=1}^{\infty} \langle \phi, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H}^\otimes r} \langle \psi, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathcal{H}^\otimes r}$$

In particular, $\phi \otimes_r \psi = \langle \phi, \psi \rangle_{\mathcal{H}^\otimes r}$, when $k_1 = k_2 = r$.

**Remark 3.** Again, in the white noise framework (when $\mathcal{H} = L^2_\mu$), for symmetric functions $\phi \in L^2_\mu^\otimes k_1$, $\psi \in L^2_\mu^\otimes k_2$, the contraction is given by the integration of the first $r$ variables, i.e., $\phi \otimes_r \psi = \langle \phi, \psi \rangle_{L^2_\mu^\otimes r}$. Also, we have a formula for the product of stochastic integrals

$$I_p(f)I_q(g) = \sum_{r=0}^{\infty} r^{(p)}(r) I_{p+q-2r}(f \otimes_r g)$$

Define the divergence operator $\delta$ as the adjoint of the operator $D$, so if $F \in \text{Dom} \delta$ then $\delta(F) \in L^2(\Omega)$ and $E[\delta(F)G] = E[\langle DG, F \rangle_{\mathcal{H}}]$.

**Remark 4.** When $\mathcal{H} = L^2_\mu$, the divergence operator $\delta$ is called the Skorohod integral. It is a creation operator in the sense that for all $F \in \text{Dom} \delta \subset L^2_\mu^\otimes p(T \times \Omega)$ with chaos representation $F(t) = \sum_{q=0}^{\infty} I_q(f_q(t, \cdot))$ ($f_q \in L^2_\mu^\otimes (q+1)$ are symmetric functions in the last $q$ variables), $\delta(F) = \sum_{q=0}^{\infty} I_{q+1}(\bar{f}_q)$

For all $F \in L^2(\Omega)$ denote by $J_q F$ the projection of $F$ in the $q^{th}$ chaos. Then, the Ornstein-Uhlenbeck semigroup is the family of contraction operators $\{T_t/t \geq 0\}$ on $L^2(\Omega)$ defined by $T_t F = \sum_{q=0}^{\infty} e^{-qt} J_q F$.

Using Mehler’s formula we can find an equivalence between Mehler’s semigroup and the O-U semigroup. More formally, take $W'$ as an independent copy of $W$ defining $(W, W')$ on the product probability space $(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \times \mathbb{P})$. Each $F \in L^2(\Omega)$ can be regarded as measurable map $F(W)$ from $\mathbb{R}^H$ to $\mathbb{R}$ determined $\mathbb{P}$-$W^{-1}$-a.s. such that $T_t F = E'[F(e^{-tW} + \sqrt{1 - e^{-2tW}}')]$. The infinitesimal generator for this semigroup (denoted by $L$) is given by $LF = \sum_{q=0}^{\infty} -q J_q F$ and $\text{Dom} L = \mathbb{D}^{2,2} = \text{Dom} \delta D$. It can be proved that, for $F \in \text{Dom} L$, $\delta DF = -LF$. The pseudo-inverse of this operator (denoted by $L^{-1}$) is given by $L^{-1} F = \sum_{q=1}^{\infty} \frac{1}{q} J_q F$, and is such that $L^{-1} F \in \text{Dom} L$ and $LL^{-1} F = F - E[F]$ for any $F \in L^2(\Omega)$.

\(^1a \wedge b = \min\{a, b\} \text{ and } a \vee b = \max\{a, b\}\)

\(^2\bar{f}\) is the symmetrization of $f$, i.e., $\bar{f}(z_1, \ldots, z_q) = \frac{1}{q!} \sum_{\sigma} f(z_{\sigma(1)}, \ldots, z_{\sigma(q)})$
Lemma 1. Let \( F \) be a family of random variables. In this setting, the law of \( F \) is absolutely continuous with respect to the Lebesgue measure, then

\[
E[H_p(Z_1)H_q(Z_2)] = \begin{cases} q!(E[Z_1Z_2])^q & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}
\]

(3)

On the other hand, it is possible to expand a \( C^2 \) function \( f : \mathbb{R} \to \mathbb{R} \) in terms of Hermite polynomials, that is,

\[
f(x) = E[f(Z)] + \sum_{q=1}^{\infty} c_q H_q(x)
\]

(4)

where the real numbers \( c_q \) are given by \( c_qq! = E[f(Z)H_q(Z)] \) and \( Z \sim \mathcal{N}(0, 1) \).

Remark 5. Notice that in the white noise case we have the relationship \( H_q(W(h)) = \int_{T_q} h^{\otimes q}dW_{t_1} \cdots dW_{t_q} = I_q(h^{\otimes q}) \) so the decomposition (4) of \( f(W(h)) \) can be regarded as

\[
f(W(h)) = \sum_{q=0}^{\infty} c_q I_q(h^{\otimes q})
\]

With (3) and (4) we are able to compute the covariance of a real function \( f \) in the following way,

\[
\text{Cov}[f(Z_1)f(Z_2)] = \sum_{p,q=1}^{\infty} c_p c_q E[H_p(Z_1)H_q(Z_2)] = \sum_{q=1}^{\infty} c_q^2 q!(E[Z_1Z_2])^q
\]

(5)

With this background, is possible now to state the main tool developed in [20] and used in this paper.

Lemma 1. Let \( F \in \mathbb{D}^{2,4} \) with \( E[F] = \mu \) and \( \text{Var}[F] = \sigma^2 \). Assume that \( N \sim \mathcal{N}(\mu, \sigma^2) \), then

\[
d_W(F, N) \leq \frac{\sqrt{10}}{2\sigma^2} E[\|D^2F \otimes_1 D^2F\|_{\mathcal{H}^{\otimes 2}}]^\frac{1}{2} \times E[\|DF\|_{\mathcal{H}}] \]

(6)

where \( d_W \) is the Wasserstein distance given by \( d_W(U, N) = \sup_{\|g\|_{L_p} \leq 1} |E[g(U)] - E[g(N)]| \). If, in addition, the law of \( F \) is absolutely continuous with respect to the Lebesgue measure, then

\[
d_{TV}(F, N) \leq \frac{\sqrt{10}}{\sigma^2} E[\|D^2F \otimes_1 D^2F\|_{\mathcal{H}^{\otimes 2}}]^\frac{1}{2} \times E[\|DF\|_{\mathcal{H}}] \]

(7)

where \( d_{TV} \) is the total variation distance given by \( d_{TV}(U, N) = \sup_{A \in \mathcal{B}(\mathbb{R})} |P(U \in A) - P(N \in A)| \)

Remark 6. This lemma is quite useful since it essentially tells us that in order to obtain a CLT result for a family of random variables \( F_T \), we just need to check three conditions:

1. **Expectation of the First Derivative’s Norm:**
   \[
   E[\|DF_T\|_{\mathcal{H}}^4] = O(1) \quad \text{as } T \to \infty
   \]
   (8)

2. **Expectation of the Contraction’s Norm:**
   \[
   E[\|D^2F_T \otimes_1 D^2F_T\|_{\mathcal{H}^{\otimes 2}}] \to 0 \quad \text{as } T \to \infty
   \]
   (9)

3. **Existence of the Variance**
   \[
   \text{Var}[F_T] \to \Sigma^2 \in (0, \infty) \quad \text{exists as } T \to \infty
   \]
   (10)

Due to the Gaussian Poincaré inequality, \( \text{Var}[F_T] \leq E[\|DF_T\|_{\mathcal{H}}^4] \leq \sqrt{E[\|DF_T\|_{\mathcal{H}}^4]} \), so the variance will go to 0 if the expectation of the first Malliavin derivative’s norm goes to 0. This is why condition (3) is necessary, and the convergence to zero of the Wasserstein distance relies on condition (4).
2.2 Malliavin calculus on Wiener-Poisson space

Let $\mathcal{H} = L^2_\mu$. Assume there is a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $L_t$ is a càdlàg, centered, Lévy process: $L_t$ has stationary and independent increments, is continuous in probability and $L_0 = 0$, with $E[L_t^2] < \infty$. At the risk of causing some confusion, is denoted by $\mathcal{F}$ the filtration generated by $L_t$, completed with the null sets of the above filtration, and work on the space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume this process is represented by the triplet $(0, \sigma^2, \nu)$, where $\nu$ is the Lévy measure s.t. $d\mu(t, x) = \sigma^2 dt \delta_0(x) + x^2 dt d\nu(x)(1-\delta_0(x))$ and $\int R x^2 d\nu(x) < \infty$. This process can be represented as

$$L_t = \sigma W_t + \int xd\tilde{N}(t, x)$$

where $W_t$ is a standard Brownian motion, $\sigma \geq 0$ and $\tilde{N}$ is the compensated jump measure. A fuller exposition on Lévy processes can be found in [2] and [33]. This process is extended to a random measure $M$, which is used to construct (in an analogous way to the Itô integral construction) an integral on the step functions, and then by linearity and continuity it is extended to $L^2([0, T] \times \mathbb{R})^q, \mathcal{B}([0, T] \times \mathbb{R})^q, \mu^\otimes q)$ and denoted by $I_q$. This integral satisfies the following properties:

1. $I_q(f) = I_q(\tilde{f})$
2. $I_q(af + bg) = aI_q(f) + bI_q(g)$ (a, b $\in \mathbb{R}$)
3. $E[I_p(f)I_q(g)] = q! \int_{[0, T] \times \mathbb{R}} \tilde{f}\tilde{g} d\mu^\otimes 1_{\{q=p\}}$

These properties are stated in [33] and their proof can be found in [14]. We have a product formula in this framework which is similar to the one in remark 3 but with extra terms coming from the poisson integration part. A product formula for the pure jump framework can be found in [29]. Before stating the formula, is needed to define a general version of the contract ion. Let $\phi \in L^2_{\mu^\otimes k_1}$ and $\psi \in L^2_{\mu^\otimes k_2}$ be symmetric functions. Then the general contraction of order $r \leq \min\{k_1, k_2\}$ and $s \leq \min\{k_1, k_2\} - r$ is given by the integration of the first $r$ variables and the “sharing” of the following $s$ variables, i.e., $\phi \otimes_r^s \psi = \prod_{i=1}^{\min(k_1, k_2)} \phi_i(\cdot, z, x, \psi(\cdot, z, y))_{L^2_{\mu^\otimes k}}$, where $z \in (\mathbb{R}^2)^s$ and $(x, y) \in (\mathbb{R}^2)^{k_1-r-s} \times (\mathbb{R}^2)^{k_2-r-s}$. Now the product formula can be stated as follows. If $|f| \otimes_r^s |g| \in L^2_{\mu^\otimes r+s-k}$, for $0 \leq r \leq \min\{p, q\}$ and $0 \leq s \leq \min\{p, q\} - r$, then

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} \sum_{s=0}^{p\wedge q-r} r! s! (p \choose r) (q \choose s) (p-r \choose s) (q-s \choose s) I_{p+q-2r-s}(f \otimes_r^s g)$$

The proof of this product formula can be found in [17].

Remark 7. In the general contraction formula $z_{2i}$ is the size of the jump and $z_{2i-1}$ is the time when that jump occurs ($z = (z_1, z_2, \ldots, z_{2s-1}, z_{2s})$). If we only have the Wiener part, the factor $z_{2i}$ would be zero unless $s = 0$, and we then obtain the contraction defined in the Wiener space. Similarly, when the terms where $s \neq 0$ are zero, the formula (17) reduces to that in remark 3.

It has been also proved by Itô [14, Theorem 2] that for all $F \in L^2(\Omega) := L^2(\Omega, \mathcal{F}, \mathbb{P})$, we have,

$$F = \sum_{q=0}^{\infty} I_q(f_q), \quad f_q \in L^2_{\mu^\otimes q} := L^2(([0, T] \times \mathbb{R})^q, \mathcal{B}([0, T] \times \mathbb{R})^q, \mu^\otimes q)$$

(12)

and that this representation is unique if the $f_q$’s are symmetric function. From this chaotic representation we can define the annihilation operators and creation operators, the former will be the Malliavin derivative and the latter will be the Skorohod integral. In this way define Dom$D$ as the set of functionals $F \in L^2(\Omega)$ represented as in (12) such that $\sum_{q=1}^{\infty} q^2 \|f_q\|_{L^2_{\mu^\otimes q}}^2 < \infty$. For $F \in$Dom$D$ the Malliavin derivative of $F$ is the stochastic process given by

$$D_z F = \sum_{q=0}^{\infty} qI_{q-1}(f_q(z, \cdot)), \quad z \in [0, T] \times \mathbb{R}, \quad f_q \text{ symmetric}$$

(13)
If we define the inner product as \( \langle f, g \rangle_{L^2} = \int_{\mathbb{R}^+} f(z)g(z) d\mu(z) \), then \( \text{DomD} \) is a Hilbert space with the inner product \( \langle F, G \rangle = E[|F|^2] + E[(D_zF, D_zG)_{L^2}] \). We can embed \( \text{DomD} \) in two spaces \( \text{DomD}^0 \) and \( \text{DomD}^1 \). \( \text{DomD}^0 \) is defined as the set of all functionals \( F \in L^2(\Omega) \) with representation given as in (12) such that \( \sum_{q=1}^{\infty} q! \int_0^T \| f_q(t,0, \cdot) \|^2_{L^2} dt < \infty \), while \( \text{DomD}^1 \) is defined as the respective functionals satisfying \( \sum_{q=1}^{\infty} q! \int_{[0,T]} \| f_q(z, \cdot) \|^2_{L^2} d\mu(z) < \infty \); hence \( \text{DomD} = \text{DomD}^0 \cap \text{DomD}^1 \). We can now rewrite (due to the independency of \( W \) and \( \tilde{N} \)) \( \Omega \) as the cross product \( \Omega_W \times \Omega_N^\infty \).

**Derivative \( D_{t,0} \)**

This derivative can be interpreted as the derivative with respect to the Brownian motion part. Using the isometry \( L^2(\Omega) \cong L^2(\Omega_W; L^2(\Omega_N^\infty)) \), we can define a Malliavin derivative as we did in the Wiener case but using the \( L^2(\Omega_N^\infty) \)-valued smooth random variables \( x_N \), that is, for the functionals of the form \( F = \sum_{i=1}^n G_i H_i \), where \( G_i \in S \) and \( H_i \in L^2(\Omega_N^\infty) \). Then, this derivative will be \( D^W_0 = \sum_{i=1}^n (D^W_0 G_i) H_i \) and this \( D^W_0 \) is the derivative defined in (2). This definition is extended (see [34]) to a subspace \( \text{DomD}^W \subset \text{DomD}^0 \) and for \( F \in \text{DomD}^W \),

\[
D_{t,0} F = \frac{1}{\sigma} D^W_0 F \tag{14}
\]

Furthermore, we also have a chain rule result for functionals of the form \( F = f(G, H) \in L^2(\Omega) \) with \( G \in \text{DomD}^W, H \in L^2(\Omega_N^\infty) \) and \( f(x, y) \) continuously differentiable in the variable \( x \) with bounded partial derivative. We have that \( F \in \text{DomD}^0 \) and

\[
D_{t,0} F = \frac{1}{\sigma} \frac{\partial f}{\partial x} (G, H) D^W_0 G \tag{15}
\]

This is also true (as in the Wiener space case) for functions which are a.e. differentiable but with the restriction that \( G \) has an absolutely continuous law.

**Derivative \( D_z \) (\( z \neq (t, 0) \))**

This derivative has been shown, in [35], to be a difference operator. The idea is to introduce a jump of size \( x \) at moment \( t \). Then, the Malliavin derivative with \( z = (t, x) \) will be the translation operator given by

\[
D_z F = \Psi_{t,x} F = \frac{F(\omega_{t,x}) - F(\omega)}{x}
\]

for any \( F \in \text{DomD}^1 \) such that \( E[\int_{[0,T]} \Psi_{t,x} F^2 d\mu(z)] < \infty \). See [35] for a complete contraction on the canonical space in which this is developed, and [34] for a quick explanation on how to introduce a jump at moment \( t \) and the conditions on the \( \omega \)'s. We also have a chain rule for this Malliavin derivative but using just the difference instead of the derivative, that is, for \( F = f(G, H) \in L^2(\Omega) \) with \( G \in L^2(\Omega_W^\infty), H \in \text{DomD}^1 \) and \( f(x, y) \) continous it holds that,

\[
D_z F = \frac{f(G, H(\omega_{t,x})) - f(G, H(\omega))}{x} = \frac{f(G, xD_z H + H(\omega)) - f(G, H(\omega))}{x}
\]

Notice that if \( f \) is differentiable, then we can use the mean value theorem to obtain

\[
D_z F = \frac{\partial f}{\partial y}(G, H(\omega) + \theta_x D_z H)D_z H \tag{16}
\]

for some \( \theta_x \in (0, 1) \).

In the same way, consider the chaotic decomposition \( F(z) = \sum_{q=0}^{\infty} I_q(f_q(z, \cdot)), \) with \( f \in L^2_{\mu \otimes \delta} \), symmetric with respect to the last \( n \) variables. If \( \sum_{q=0}^{\infty} (q + 1)! \| f_q \|^2_{L^2_{\mu \otimes \delta}} < \infty \) then we say that \( F \in \text{Dom}\delta \). Now we can define the Skorohod integral of \( F \in \text{Dom}\delta \) by

\[
\delta(F) = \sum_{q=0}^{\infty} I_{q+1}(\tilde{f}_q) \in L^2(\Omega) \tag{17}
\]

7
This operator is the adjoint of the operator $D_z$, so $E[\delta(F)G] = E[(F(z), D_z G)]$ for all $G \in \text{Dom}D$. Denote by $L^{1,2}$ the set of elements $F \in L^2_{\mu \otimes P}([0, T] \times \mathbb{R} \times \Omega)$ such that $\sum_{q=0}^{\infty} q^2 \|f_q\|_{L^2_{\mu \otimes P}}^2 < \infty$. For all $F \in L^{1,2} \subset \text{Dom} \delta$ we have that $F(z) \in \text{Dom}D$, $\forall \ z \in \mu$ a.e. and that $D F(\cdot) \in L^2_{\mu \otimes P} \left(\{(0, T] \times \mathbb{R})^2 \times \Omega\right)$.

Finally, the definitions of the Ornstein-Uhlenbeck semigroup $T_t$ and its infinitesimal generator $L$ are the same as in the Wiener space case. Basically, all we need to define it is the Malliavin derivative and the Skorohod integral, that is, we can just define $L = -\delta D$. With this definition we obtain that for $F \in L^2(\Omega)$ with chaotic representation (12), $LF = \sum_{q=1}^{\infty} -q I_q(f_q)$ and $T_t F = \sum_{q=0}^{\infty} e^{-qt} I_q(f_q)$. Namely, the pseudo-inverse is given by $L^{-1} F = \sum_{q=1}^{\infty} \frac{1}{q} I_q(f_q)$ and $LL^{-1} F = F - E[F]$.

### 3 Main theorems

The first tool needed is the extension of the so-called Gaussian Poincaré inequality for the present context. But to prove this it is required to have an inequality similar to the one proved in [20, Proposition 3.1] (was proved for all $p \geq 2$ in the Wiener space case). The technique used in their proof was based on the equivalence between Mehler and Ornstein-Uhlenbeck semigroups for the Gaussian case, but in the Wiener-Poisson space we lack such an equivalence. Nevertheless, it is possible to prove it for $p = 2$ and that is, in fact, the one needed to prove the extension of the Gaussian Poincaré inequality.

**Proposition 1.** Let $F \in \text{Dom}D$ such that $E[F] = 0$. Then,

$$E\left[\|DL^{-1}F\|_{L^2_{\mu}}^2\right] \leq E\left[\|DF\|_{L^2_{\mu}}^2\right]$$

**Proof.** Assume $F$ has its chaos decomposition given by (12). By the orthogonality between chaoses we get,

$$E\left[\|DL^{-1}F\|_{L^2_{\mu}}^2\right] = E\left[\sum_{q=1}^{\infty} \frac{1}{q^2} \|DI_q(f_q)\|_{L^2_{\mu}}^2\right] \leq E\left[\sum_{q=1}^{\infty} \|DI_q(f_q)\|_{L^2_{\mu}}^2\right] = E\left[\|DF\|_{L^2_{\mu}}^2\right]$$

---

**Theorem 1.** (Extension of the Gaussian Poincaré inequality)

Let $F \in \text{Dom}D$. Then,

$$\text{Var}[F] \leq E\left[\|DF\|_{L^2_{\mu}}^2\right] \quad (18)$$

with equality if and only if $F$ is a linear combination of elements in the first and $0$th chaos.

**Proof.** Assume, without loss of generality, that $E[F] = 0$.

$$\text{Var}[F] = E[F^2] = E\left[(DF, DL^{-1}F)_{L^2_{\mu}}\right] \leq E\left[\|DF\|_{L^2_{\mu}}^2\right] \frac{1}{2} E\left[\|DL^{-1}F\|_{L^2_{\mu}}^2\right] \leq E\left[\|DF\|_{L^2_{\mu}}^2\right]$$

where Proposition 1 was used in the last step, and the fact that $F = \delta DL^{-1} F$ in the second step.

Notice is possible to combine formulas (15) and a version of (16) to write the chain rule in a unique way when $f$ is a function of just one variable which is $k$ times continuously differentiable. Similarly, it is possible to give a unified formula for the derivative of a product. Using the fact that the jump $x$ is zero for $D_{1,0}$, the following unified chain and product rules are obtained.

**Proposition 2.** (Chain rules and Product rule)

Let $F, G, H \in \text{Dom}D^W \cap \text{Dom}D^J$ such that $DF, DG, DH \in L^2_{\mu}$. Also consider $f, g \in C^{k-1}$, both with bounded
The main theorem of this paper is now stated. Recall the following bound on the Wasserstein distance for all \( z \in \mathbb{R}^+ \times \mathbb{R} \),

\[
D_z f(F) = \sum_{n=1}^{k-1} \frac{f^{(n)}(F)}{n!} x^{n-1}(D_z F)^n + \frac{f^{(k)}(F + \theta z D_z F)}{k!} x^{k-1}(D_z F)^k
\]  

(19)

for some function \( \theta \in (0, 1) \) for all \( z = (t, x) \in \mathbb{R}^+ \times \mathbb{R} \),

\[
D_z g(F) = \sum_{n=1}^{k-1} \frac{g^{(n)}(F)}{n!} x^{n-1}(D_z F)^n + \int_0^{D_z F} \frac{g^{(k)}(F + xu)}{(k-1)!} x^{k-1}(D_z F - u)^{k-1} du
\]

(20)

and

\[
D_z (GH) = D_z G \cdot H + G \cdot D_z H + x \cdot D_z G \cdot D_z H
\]

(21)

for all \( z = (t, x) \in \mathbb{R}^+ \times \mathbb{R} \).

**Proof.** If \( z = (t, 0) \), we get the chain rule formula \[15\] and the usual product rule in Wiener space. If \( z = (t, x) \) with \( x \neq 0 \), then the Malliavin derivative formula tells us that \( D_z f(F) = \frac{f(F(x, z)) - f(F(x))}{D_z x} \). Since \( f \in C^{k-1} \) and \( k \)-times differentiable, \( f(y) = f(y_0) + \sum_{n=1}^{k-1} \frac{f^{(n)}(y_0)}{n!} (y - y_0)^n + \frac{f^{(k)}(y_0 + \theta z (y - y_0))}{k!} (y - y_0)^k \) for some \( \theta \in (0, 1) \) (Taylor series of \( f \) with mean-value form for the remainder). Using this expansion with \( y = F(\omega_z) \), \( y_0 = F(\omega) \) and recalling that \( y - y_0 = F(\omega_z) - F(\omega) = x D_z F \), the chain rule result \[19\] will follows.

The second chain rule formula is obtained by using the Taylor expansion for \( g \) with integral form for the remainder, i.e. \( g(y) = g(y_0) + \sum_{n=1}^{k-1} \frac{g^{(n)}(y_0)}{n!} (y - y_0)^n + \int_{y_0}^y \frac{g^{(k)}(v)}{(k-1)!} (y - v)^{k-1} dv \). By using the values for \( y \) and \( y_0 \) as in the previous case, and applying the change of variable \( v = F + xu \) in the integral we get the chain rule formula \[20\].

For the product rule we get trivially that

\[
D_z (GH) = \frac{G(\omega_z x, \omega) - G(\omega) H(\omega)}{x} = \frac{G(\omega_z x, \omega) - G(\omega) H(\omega)}{x} + x \frac{G(\omega_z x, \omega) - G(\omega) H(\omega)}{x} H(\omega)
\]

\[
\square
\]

**3.1 Upper bound and second order Poincaré inequality**

The main theorem of this paper is now stated. Recall the following bound on the Wasserstein distance\[3\],

\[
d_W(F, N) \leq \sup_{f \in \mathcal{F}_W} |E[f'(F) - F f(F)]|
\]

where \( N \sim \mathcal{N}(0, 1) \) and \( \mathcal{F}_W := \{ f \in C^1 \mid f' \text{ is Lipschitz}, ||f'||_{\infty} \leq 1, ||f''||_{\infty} \leq 2 \}\[4\].

**Theorem 2. (Upper Bound)**

Let \( N \sim \mathcal{N}(0, 1) \) and let \( F \in \text{Dom} \mathcal{D}^W \cap \text{Dom} \mathcal{D}^J \) be such that \( E[F] = 0 \). Then,

\[
d_W(F, N) \leq E \left[ \left| F \right| - \left| DF, DL^{-1} F \right|_{L_2^2} \right] + E \left[ \left| x(DF)^2 \right|, \left| DL^{-1} F \right|_{L_2^2} \right]
\]

(22)

**Proof.** By Proposition \[2\] (with \( k = 2 \) in \[20\]) we get \( Df(F) = f'(F)DF + \int_0^{DF} f''(F + xu) x(DF - u) du \).

On the other hand, using the identity \( F = -\delta DF \) (recall \( E[F] = 0 \)) and the integration by parts formula

\[\text{see} \ [19] \text{ for further details on this bound}\]

\[\text{The class} \ \mathcal{F}_W \text{ is the collection of all continuously differentiable functions} \ f : R \rightarrow R \text{ that has derivative bounded by 1 and such that there exists a version of} \ f'' \text{ that is bounded by 2}\]
we get $E[F f(F)] = E[\langle D f(F), -D L^{-1} F \rangle_{L^2_\mu}]$. Putting them together we get

$$|E[f'(F) − F f(F)]| = |E[f'(F)] - E[\langle D f(F), -D L^{-1} F \rangle_{L^2_\mu}]|$$

$$= |E[f'(F)] - E[\langle DF, -D L^{-1} F \rangle_{L^2_\mu}] + \left( \int_0^{D F} |f''(F + xu)| x(DF - u) du, -D L^{-1} F \right)_{L^2_\mu}|$$

$$\leq E\left[|f'(F)| - \langle DF, -D L^{-1} F \rangle_{L^2_\mu}\right] + E\left[\left( \int_0^{D F} |f''(F + xu)| x(DF - u) du, -D L^{-1} F \right)_{L^2_\mu}\right]$$

$$\leq \|f''\|_{\infty} E\left[|1 - \langle DF, -D L^{-1} F \rangle_{L^2_\mu}|\right] + E\left[\left( \|f''\|_{\infty} \frac{x(DF)^2}{2}, |D L^{-1} F|\right)_{L^2_\mu}\right]$$

Finally, use the fact that $\|f''\|_{\infty} \leq 1$ and $\|f''\|_{\infty} \leq 2$ to obtain the result. □

Is desirable to use this result to obtain a nice upper bound, as in the Wiener space with Lemma [1]. The main problem faced is that equivalence between the O-U and Mehler semigroups is no longer available, so the proofs of $E[\|D^2 L^{-1} F\|_{L^2_\mu}^{\frac{q}{2}}] \leq \frac{1}{q} E[\|D^2 F\|_{L^2_\mu}^{\frac{q}{2}}]$ and $E[\|D^2 L^{-1} F\|_{L^2_\mu}^{\frac{q}{2}}] \leq \frac{1}{q} E[\|D^2 F\|_{L^2_\mu}^{\frac{q}{2}}]$ given in the Wiener space case fail for the Wiener-Poisson space case. Nevertheless, is still possible to state an equivalent version of Lemma [1] for the case when $F$ lies in one specific chaos.

**Corollary 1. (Second order Poincaré inequality)**

Fix $q \in \mathbb{N}$ and let $F = I_q(f)$ with $E[F] = \mu$ and $\text{Var}[F] = \sigma^2$. Assume that $N \sim N(\mu, \sigma^2)$, then

$$d_W(F, N) \leq \frac{\sqrt{q}}{q\sigma^2} \left( 2E[\|D^2 F\|_{L^2_\mu}^{\frac{q}{2}}] \right)^\frac{1}{q} E[\|DF\|_{L^2_\mu}^4] + E\left[\left( \langle x, (D^2 F)^2 \rangle_{L^2_\mu} \right)^{\frac{2}{q}} \right]$$

$$\leq \frac{\sqrt{q}}{q\sigma^2} \left( 2E[\|D^2 F \otimes_1 D^2 F\|_{L^2_\mu}^{\frac{q}{2}}] \right)^\frac{1}{q} E[\|DF\|_{L^2_\mu}^4] + E\left[\left( \langle x, (D^2 F)^2 \rangle_{L^2_\mu} \right)^{\frac{2}{q}} \right]$$

Proof. Assume without loss of generality that $\mu = 0$ and $\sigma = 1$. By Theorem [2] and Hölder we have that

$$d_W(F, N) \leq E[\left( 1 - \langle DF, -D L^{-1} F \rangle_{L^2_\mu} \right)^{\frac{q}{2}}] + E\left[\left( \langle x(DF)^2 \rangle_{L^2_\mu} \right)^{\frac{1}{2}} \right]$$

Also, notice that $E[\langle DF, -D L^{-1} F \rangle_{L^2_\mu}] = -\delta D^{-1} F \cdot F = E[F^2] = 1$, so if $G = \langle DF, -D L^{-1} F \rangle_{L^2_\mu}$ then $E[\left( 1 - \langle DF, -D L^{-1} F \rangle_{L^2_\mu} \right)^{\frac{q}{2}}] = \text{Var}[G]$. By Theorem [1] we have that $\text{Var}[G] \leq E[\|DG\|_{L^2_\mu}^{\frac{q}{2}}]$. Also, by the product rule [2] we have that

$$DG = \langle D^2 F, -D L^{-1} F \rangle_{L^2_\mu} + \langle DF, -D^2 L^{-1} F \rangle_{L^2_\mu} + \langle x D^2 F, -D^2 L^{-1} F \rangle_{L^2_\mu}$$

Putting all together and using the fact that $-L^{-1} F = \frac{1}{q} F$ we get,

$$E[\left( 1 - \langle DF, -D L^{-1} F \rangle_{L^2_\mu} \right)^{\frac{q}{2}}] \leq \frac{\sqrt{q}}{q} \left( 2E[\left( \langle D^2 F, DF \rangle_{L^2_\mu} \right)^{\frac{q}{2}}] + E\left[\left( \langle x, (D^2 F)^2 \rangle_{L^2_\mu} \right)^{\frac{2}{q}} \right] \right)$$

The first term in the right is bounded above in the following way

$$E[\left( \langle D^2 F, DF \rangle_{L^2_\mu} \right)^{\frac{q}{2}}] \leq E[\|D^2 F\|_{L^2_\mu}^{\frac{q}{2}}] E[\|DF\|_{L^2_\mu}^{\frac{q}{2}}]$$
since $\left\| \langle D^2 F, DF \rangle_{L^2_{\mu}} \right\|^2 \leq \left\| D^2 F \right\|_{op}^2 \left\| D F \right\|_{L^2_{\mu}}$ and by Hölder. On the other hand, and again using the fact that $F$ is in the $q^{th}$ chaos, we get

$$E \left[ \langle |x| D^2 F \rangle_{L^2_{\mu}} \right] = \frac{1}{q} E \left[ \langle |x|, |DF|^3 \rangle_{L^2_{\mu}} \right]$$

thus, obtaining the first inequality. For the second inequality, it suffices to see that if $\{\gamma_j\}_{j \geq 1}$ is the sequence of random eigenvalues of the random Hilbert-Schmidt operator $f \mapsto \langle f, D^2 F \rangle_{L^2_{\mu}}$, then

$$\|D^2 F\|_{op}^4 = \max_{j \geq 1} |\gamma_j|^4 \leq \sum_{j \geq 1} |\gamma_j|^4 = \|D^2 F \otimes_1 D^2 F\|_{L^2_{\mu}}^2$$

As in Lemma 1, this corollary basically says that if we want to show a CLT for a family of random variables $F_T$ (living in a fixed chaos) it is sufficient to check the following conditions,

1. **Expectation of the First Derivative’s Norm:**
   $$E\left[\|DF_T\|_{L^2_{\mu}}^4 \right] = O(1) \quad \text{as} \quad T \to \infty \quad (25)$$

2. **Expectation of the Cube of the First Derivative’s Norm:**
   $$E\left[\langle |x|, |DF_T|^3 \rangle_{L^2_{\mu}} \right] \to 0 \quad \text{as} \quad T \to \infty \quad (26)$$

3. **Expectation of the Contraction’s Norm:**
   $$E\left[\|D^2 F_T \otimes_1 D^2 F_T\|_{L^2_{\mu}}^2 \right] \to 0 \quad \text{as} \quad T \to \infty \quad (27)$$

4. **Expectation of the Squared Second Derivative’s Norm:**
   $$E\left[\langle x, (D^2 F_T)^2 \rangle_{L^2_{\mu}} \right] \to 0 \quad \text{as} \quad T \to \infty \quad (28)$$

5. **Existence of the Variance**
   $$\text{Var}[F_T] \to \Sigma^2 \in (0, \infty) \quad \text{exists as} \quad T \to \infty \quad (29)$$

### 4 Special cases and applications

#### 4.1 The Wiener space case:

**Linear functionals of Gaussian-subordinated fields**

When we are working in this space the jump size is always zero, so the upper bound for the Wasserstein distance becomes

$$d_W(F, N) \leq E \left| 1 - \langle DF, -DL^{-1}F \rangle_{L^2_{\mu}} \right|$$

which coincides perfectly with the bound computed in [19].
Remark 8. It is important to stress that Theorem 3 is not a direct extension of the inequality in [19], even though they appear to be similar. This is because in the pure Wiener case, the Malliavin calculus theory is developed for more abstract Hilbert spaces than $L^2_{\mu}$, and the inequality proved therein,

$$d_W(F, N) \leq E\left|1 - \langle DF, -DL^{-1}F\rangle_H\right|$$

holds for any Hilbert space $H$. However, when $H = L^2_{\mu}$, Theorem 3 is indeed an extension of Theorem 3.1 in [19].

The Wiener space is well understood and the upper bounds obtained from (30) are more powerful than Corollary 1. In fact, Lemma 1 is true for all $F \in \mathbb{D}_{+}^4$ and not just for functionals in a fixed Wiener chaos. As an application of Lemma 1 the authors of [20] proved a very useful central limit theorem for linear functionals of Gaussian-subordinated fields. Before stating it, some notation is introduced: Let $X_t$ be a centered Gaussian stationary process and define $C(t) = E[X_0X_t] = E[X_sX_{t+s}]$, its covariance function. By remark 2 is known that the Malliavin derivative of $X_t$ is well defined. Let $T > 0$, $Z \sim \mathcal{N}(0, C(0))$ and $f : \mathbb{R} \to \mathbb{R}$ be a real function of class $C^2$ not constant s.t. $E[|f(Z)|] < \infty$ and $E[|f''(Z)|^4] < \infty$. In order to simplify the notation, the following random sequence is defined,

$$F_T = T^{-\frac{1}{2}} \int_0^T (f(X_t) - E[f(Z)]) dt$$

The theorem is stated as follows,

**Lemma 2.** Suppose that $\int_\mathbb{R} |C(t)| < \infty$, and assume that $f$ is a symmetric real function. Then $\lim_{T \to \infty} \text{Var}[F_T] := \Sigma^2 \in (0, \infty)$ exists and as $T \to \infty$,

$$F_T \xrightarrow{law} N \sim \mathcal{N}(0, \Sigma^2)$$

**Remark 9.** One advantage of inequality (6) is the fact that it allows us to quantify rates of convergence to normality. Indeed, it has been proved (see [20]) that if $Z \sim \mathcal{N}(0, 1)$, then (as $T \to \infty$),

$$d_W\left(\frac{F_T}{\sqrt{\text{Var}[F_T]}}, \tilde{Z}\right) = O(T^{-\frac{1}{2}})$$

Our goal in this subsection is to extend this result to the case when $\int_\mathbb{R} |C(t)| = \infty$. This is achievable under some conditions on the decay rate of the covariance. In fact, it is very convenient that for this functional the conditions (8), (9) and (10) reduce to just one condition on the covariance of the underlying stationary Gaussian process $X_t$. Let $V(T)$ be a strictly positive continuous function with $V(T) \to 0$ as $T \to \infty$ such that either $TV(T) \to 0$ or $V \in C^1$ and $TV'(T) \to 0$ as $T \to \infty$. The following is the condition on the covariance that replace the three conditions on remark 6

**Condition:** Either $\int_\mathbb{R} |C(t)| < \infty$ or $V(T)$ (with the above characteristics) exists such that,

$$\frac{C(T)}{V(T)} \xrightarrow{T \to \infty} M \neq 0$$

$V(T)$ represents the decay rate for the covariance function. Consider the following function

$$\tilde{V}(T) = \begin{cases} T & \text{if } \int_0^\infty |C(x)| \ dx < \infty \\ \int_0^T \int_0^y V(x) \ dx \ dy & \text{if } \int_0^\infty |C(x)| \ dx = \infty \end{cases}$$

Let $\mathcal{M}_C := \{f \in C^2 / f$ is symmetric if $\int_\mathbb{R} |C(t)| < \infty$ or $E[f(Z)Z] \neq 0$ if $\int_\mathbb{R} |C(t)| = \infty \}$ and rewrite the functional $F_T$ as follows,

$$F_T = \tilde{V}(T)^{-\frac{1}{2}} \int_0^T (f(X_t) - E[f(Z)]) dt$$
Theorem 3. Suppose that condition * is verified by \( C(t) \) and that \( f \in \mathcal{M}_C \). Then \( \lim_{T \to \infty} \text{Var}[F_T] := \Sigma^2 \in (0, \infty) \) exists and as \( T \to \infty \)

\[
F_T \xrightarrow{\text{law}} N \sim \mathcal{N}(0, \Sigma^2)
\]

Furthermore, if \( \int_0^\infty |C(t)| \, dt = \infty \), then \( \Sigma^2 = 2M(E[f(Z)Z])^2 \).

Before tackling this theorem, it is necessary to verify some facts that would simplify the proof.

Proposition 3. Suppose that \( \int_0^\infty |C(t)| \, dt = \infty \). Then as \( T \to \infty \),

1. \( \left( \int_0^T V(x)dx \right)^{-1} \int_0^T |C(t)| \, dt = O(1) \)

2. \( \frac{\bar{V}(T)^{-1} \int_0^{[0,T]^2} |C(t-s)| \, dtsdt}{V(T)} = O(1) \)

3. If \( TV(T) \to 0 \):
   \[
   \bar{V}(T)^{-2}T \left( \int_0^T |C(t)| \, dt \right)^3 = O(\max\{V(T), TV(T)^2\left( \int_0^T V(x)dx \right)^{-1}\})
   \]
   
   If \( TV(T) \to 0 \) and \( TV'(T) \to 0 \),
   \[
   V(T)^{-2}T \left( \int_0^T |C(t)| \, dt \right)^3 = O(\max\{V(T), TV'(T)\})
   \]

4. For fixed \( q \geq 1 \):

\[
\bar{V}(T)^{-1} \int_0^{[0,T]^2} C(t-s)^q \, dtsdt \to 2M1_{q=1} = \begin{cases} 2M & \text{iff } q = 1 \\ 0 & \text{iff } q \neq 1 \end{cases}
\]

Proof. The proof just involves simple applications of L’Hôpital’s rule (L).

1.

\[
\lim_{T \to \infty} \frac{\int_0^T |C(t)| \, dt}{\int_0^T V(x)dx} \equiv \lim_{T \to \infty} \frac{|C(T)|}{V(T)} = |M|
\]

2. Notice first that \( \int_0^{[0,T]^2} |C(t-s)| \, dtsdt = 2 \int_0^T \int_0^T |C(x)| \, dxdt \) so

\[
\lim_{T \to \infty} \frac{\int_0^{[0,T]^2} |C(t-s)| \, dtsdt}{V(T)} = \lim_{T \to \infty} \frac{2 \int_0^T \int_0^T |C(x)| \, dxdt}{\int_0^T V(x)dx} \equiv 2 \lim_{T \to \infty} \frac{|C(T)|}{V(T)} = 2|M|
\]

3. If \( TV(T) \to 0 \):

\[
\lim_{T \to \infty} \frac{T \left( \int_0^T |C(t)| \, dt \right)^3}{V(T)^2} = \lim_{T \to \infty} \left( \int_0^T |C(t)| \, dt \right)^3 \frac{T \left( \int_0^T V(x)dx \right)^3}{V(T)^2} \equiv O(1) \lim_{T \to \infty} \left( V(T) + \frac{3TV(T)^2}{2 \int_0^T V(x)dx} \right)
\]

If \( TV(T) \to 0 \) and \( TV'(T) \to 0 \)

\[
\lim_{T \to \infty} \frac{T \left( \int_0^T |C(t)| \, dt \right)^3}{V(T)^2} = \lim_{T \to \infty} \left( \int_0^T |C(t)| \, dt \right)^3 \frac{T \left( \int_0^T V(x)dx \right)^3}{V(T)^2} \equiv O(1) \lim_{T \to \infty} \left( 4V(T) + 3TV'(T) \right)
\]

4. If for \( q > 1 \), either \( \lim_{T \to \infty} \int_0^{[0,T]^2} C(t-s)^q \, dtsdt < \infty \) or \( \lim_{T \to \infty} \int_0^T C(x)^q \, dx < \infty \), then the result will follow trivially. So let’s assume that both go to infinity as \( T \) goes to infinity.

\[
\lim_{T \to \infty} \frac{\int_0^{[0,T]^2} C(t-s)^q \, dtsdt}{V(T)} = \lim_{T \to \infty} \frac{2 \int_0^T \int_0^T C(x)^q \, dxdt}{\int_0^T \int_0^T V(x)dx} \equiv \lim_{T \to \infty} \frac{C(T)^q}{V(T)} = 2 \lim_{T \to \infty} \left( \frac{C(T)}{V(T)} \right)^q V(T)^{q-1} = 2M1_{q=1}
\]

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Now, to the proof of Theorem 3

Proof. Notice that if \( \int_{\mathbb{R}} |C(t)| < \infty \) then Theorem 3 reduces to Lemma 2 and there is nothing left to prove. Assume then, that \( \int_{\mathbb{R}} |C(t)| = \infty \). Due to remark 6, it is enough to check that condition \( * \) implies conditions (8), (9) and (10).

- **Expectation of the First Derivative’s Norm:**
  - First Malliavin Derivative:
    \[
    D F_T = \tilde{V}(T)^{-\frac{1}{2}} \int_0^T f'(X_t)[1_{[0,t]}] dt
    \]
  - Norm of the First Malliavin Derivative:
    \[
    \|DF_T\|_H^2 = \tilde{V}(T)^{-1} \int_{[0,T]^2} f''(X_t)f'(X_s)[1_{[0,t]}][1_{[0,s]}] dtds = \tilde{V}(T)^{-1} \int_{[0,T]^2} f'(X_t)f'(X_s)C(t-s) dtds
    \]
    Then,
    \[
    \|DF_T\|_H^4 = \tilde{V}(T)^{-2} \int_{[0,T]^4} f''(X_t)f'(X_s)f'(X_u)f'(X_v)C(t-s)C(u-v)dtdsdudv
    \]
  - Expectation of the First Malliavin Derivative’s Norm:
    By using Hölder (twice) on the expectation and by the stationarity of \( X_t \) we have the bound,
    \[
    |E[f'(X_t)f'(X_s)f'(X_u)f'(X_v)]| \leq E[|f'(Z)|^4]
    \]
    finally recovering the power we get,
    \[
    E[\|DF_T\|_H^4] \leq E[|f'(Z)|^4] \left( \tilde{V}(T)^{-1} \int_{[0,T]^2} |C(t-s)| dtds \right)^2 = O(1) \text{ by Proposition 3}
    \]
    All this proves that,
    \[
    E[\|DF_T\|_H^4]^{\frac{1}{4}} = O(1) \text{ as } T \to \infty
    \]

- **Expectation of the Contraction’s Norm:**
  In the same way we get,
  - Second Malliavin Derivative:
    \[
    D^2 F_T = \tilde{V}(T)^{-\frac{1}{2}} \int_0^T f''(X_t)[1_{[0,t]}^\otimes 2] dt
    \]
  - Contraction of Order 1:
    \[
    D^2 F_T \otimes_1 D^2 F_T = \tilde{V}(T)^{-1} \int_{[0,T]^2} f''(X_t)f''(X_s)[1_{[0,t]}] \otimes [1_{[0,t]}][1_{[0,s]}]\otimes [1_{[0,t]}][1_{[0,s]}] dtds = \tilde{V}(T)^{-1} \int_{[0,T]^2} f''(X_t)f''(X_s)[1_{[0,t]}] \otimes [1_{[0,s]}]C(t-s) dtds
    \]
– Norm of the Contraction:

\[ \|D^2F_T \otimes_1 D^2F_T\|_{\mathcal{H}^2}^2 = \tilde{V}(T)^{-2} \int_{[0,T]^4} f''(X_t)f''(X_s)f''(X_u)f''(X_v)C(t-s)C(u-v) \times \langle 1_{[0,t]}, 1_{[0,u]} \rangle_{\mathcal{H}} \langle 1_{[0,s]}, 1_{[0,v]} \rangle_{\mathcal{H}} \ dt ds du dv \]

\[ = \tilde{V}(T)^{-2} \int_{[0,T]^4} f''(X_t)f''(X_s)f''(X_u)f''(X_v)C(t-s)C(u-v)C(t-u)C(s-v) dt ds du dv \]

– Expectation of the Contraction’s Norm:

By using Hölder in the same way as above we get,

\[ E[\|D^2F_T \otimes_1 D^2F_T\|_{\mathcal{H}^2}^2] \leq E[|f''(Z)|^4]\tilde{V}(T)^{-2} \int_{[0,T]^4} |C(t-s)C(u-v)C(t-u)C(s-v)| dt ds du dv \]

Now, let’s make the changes of variable \( y = (t-s, u-v, t-u, v) \), and let’s denote the new region by \( \Omega \times [0,T] \). So,

\[ E[\|D^2F_T \otimes_1 D^2F_T\|_{\mathcal{H}^2}^2] \leq E[|f''(Z)|^4] \tilde{V}(T)^{-2} \int_{[0,T]^2} \int_{\Omega} |C(y_1)C(y_2)C(y_3)(y_2 + y_3 - y_1)| \ dy \]

\[ = E[|f''(Z)|^4] \tilde{V}(T)^{-2} \int_{[0,T]^2} \int_{\Omega} |C(y_1)C(y_2)C(y_3)(y_2 + y_3 - y_1)| \ dy_1 \ dy_2 \ dy_3 \]

Taking in account that by Cauchy-Schwarz, \( \forall \ t \in \mathbb{R} \),

\[ |C(t)| = \frac{|E[X_0X_t]|}{\sqrt{\text{Var}[X_0]\text{Var}[X_t]}} \sqrt{\text{Var}[X_0]\text{Var}[X_t]} \leq C(0) \]

Also, it is clear that \( \Omega \subset [-T, T]^3 \) and since the integrand is a non-negative even function, we can deduce that,

\[ E[\|D^2F_T \otimes_1 D^2F_T\|_{\mathcal{H}^2}^2] \leq E[|f''(Z)|^4]C(0)\tilde{V}(T)^{-2}T \left( 2 \int_{[0,T]} |C(y)| dy \right)^3 \]

\[ = 8E[|f''(Z)|^4]C(0) \tilde{V}(T)^{-2}T \left( \int_{[0,T]} |C(y)| dy \right)^3 \]

\[ \xrightarrow{T \rightarrow \infty} 0 \text{ by Proposition } 3 \]

All this proves that,

* If \( TV(T) \rightarrow 0 \):

\[ E[\|D^2F_T \otimes_1 D^2F_T\|_{\mathcal{H}^2}^2] = O\left( \max \left\{ V(T), TV(T)^2 \left( \int_0^T V(x) dx \right)^{-1} \right\} \right) \text{ as } T \rightarrow \infty \]

* If \( TV(T) \rightarrow 0 \) and \( TV'(T) \rightarrow 0 \)

\[ E[\|D^2F_T \otimes_1 D^2F_T\|_{\mathcal{H}^2}^2] = O\left( \max \{V(T), TV'(T)\} \right) \text{ as } T \rightarrow \infty \]

• Existence of the Variance:

Since \( f \in \mathcal{M}_C \) then \( E[f(X_0)X_0] = E[f(Z)Z] \neq 0 \). Also \( H_1(x) = x \), so the first Hermite constant in
the expansion \( q \) is not 0, i.e., \( c_1 = E[f(X_0)X_0] \neq 0 \). Using the formula \( 5 \) for the covariance of \( f \) we get,

\[
\text{Var}[F_T] = E\left[ \left( \tilde{V}(T)^{-\frac{1}{2}} \int_0^T (f(X_t) - E[f(Z)])dt \right)^2 \right] = \tilde{V}(T)^{-1} \int_{[0,T]^2} \text{Cov}[f(X_t)f(X_s)]dtds
\]

\[
= \tilde{V}(T)^{-1} \int_{[0,T]^2} \sum_{q=1}^{\infty} c_q^2 q! (E[X_tX_s])^q dtds = \sum_{q=1}^{\infty} c_q^2 q! \tilde{V}(T)^{-1} \int_{[0,T]^2} C(t-s)^qdtds
\]

\[
= c_1^2 \frac{\int_{[0,T]^2} C(t-s)dsdt}{V(T)} + \sum_{q=2}^{\infty} c_q^2 q! \frac{\int_{[0,T]^2} C(t-s)^qdtds}{V(T)} \to 2Mc_1^2
\]

All this proves that,

\[
\lim_{T \to \infty} \text{Var}[F_T] = 2M(E[f(Z)^2] \in (0, \infty) \text{ exists})
\]

Since the conditions were satisfied, Theorem \( 3 \) is proved.

\[ \square \]

**Remark 10.** Notice that during the proof of this theorem was possible to establish an estimate for the convergence rate to normality like in remark \( 2 \) i.e., if \( \tilde{Z} \sim N(0,1) \) then as \( T \to \infty \),

- If \( TV(T) \to 0 \):

\[
d_W\left( \frac{F_T}{\sqrt{\text{Var}[F_T]}}, \tilde{Z} \right) = O\left( \max \left\{ V(T), TV(T)^2 \left( \int_0^T V(x)dx \right)^{-1} \right\}^{\frac{1}{2}} \right)
\]

- If \( TV(T) \to 0 \) and \( TV'(T) \to 0 \):

\[
d_W\left( \frac{F_T}{\sqrt{\text{Var}[F_T]}}, \tilde{Z} \right) = O\left( \max \{V(T), TV'(T)\}^{\frac{1}{2}} \right)
\]

4.1.1 Examples

According to Theorem \( 2 \) the only condition we need to check in order to apply the central limit theorem to \( F_T \) is the decay rate of the covariance function for the underlying stationary Gaussian process \( X_t \) (condition \( * \)). In fact, if the decay rate is \( t^{-\alpha} \) then we can apply the CLT if \( \alpha \in (0,1) \cup (1,2) \), because in the case \( \alpha \in (1,2) \) the integral \( \int_0 T C(t) \) is finite and in the case \( \alpha \in (0,1) \) the same integral is infinite but \( V(T) = T^{-\alpha} \in C^1 \) and \( TV'(T) = -\alpha T^{-\alpha} \to 0 \) as \( T \to \infty \).

1. Fractional Brownian Motion (FBM):

In this particular case is known that the difference of FBM is a stationary process for all \( H \in (0,1) \). So, if \( B^H_t \) is FBM then \( X_t = B^H_{t+1} - B^H_t \) is a centered Gaussian stationary process. Its covariance function is,

\[
C_1(T) = E[X_TX_0] = E[(B^H_{T+1} - B^H_T)(B^H_T - B^H_0)] = \frac{|T + 1|^{2H} + |T - 1|^{2H} - 2T^{2H}}{2}
\]

Thus,

\[
\lim_{T \to \infty} T^{2-2H}C_1(T) = \lim_{T \to \infty} T^2\left(1 + \frac{1}{T}\right)^{2H} + \left(1 - \frac{1}{T}\right)^{2H} - 2 = H(2H - 1) = M \in (0, \infty)
\]

Then, the decay rate of its covariance function is \( t^{2H-2} \), so Theorem \( 2 \) is applicable to the increments of FBM for all \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \), and \( F_T \overset{\text{law}}{\to} N \sim N(0, \Sigma^2) \) as \( T \to \infty \).
2. Ornstein-Uhlenbeck Driven by FBM:
This process is given as the solution of the following SDE:

\[ Y_t^H = Y_0^H - \lambda \int_0^t Y_s^H \, ds + \tilde{B}_t^H \]

where \( \tilde{\sigma}, \lambda > 0 \) are constants, and \( B_t^H \) is a fractional Brownian motion with Hurst parameter \( H \in (0, \frac{1}{2}) \cup \left( \frac{1}{2}, 1 \right) \). This process is stationary due to the stationarity of the increments of the FBM (used in the first example). So \( X_t = Y_t^H - E[Y_0^H] \) is a centered Gaussian stationary process. In [17], the authors proved the following lemma,

**Lemma 3.** Let \( H \in (0, \frac{1}{2}) \cup \left( \frac{1}{2}, 1 \right) \) and \( N \in \mathbb{N} \). Then as \( T \to \infty \),

\[ C_2(T) = E[X_T X_0] = Cov[Y_T^H Y_0^H] = \frac{\tilde{\sigma}^2}{2} \sum_{n=1}^N \lambda^{-2n} \left( \prod_{k=0}^{2n-1} (2H - k) \right) T^{2H-2n} + O(T^{2H-2N-2}) \]

which basically tells us that for all \( H \in (0, \frac{1}{2}) \cup \left( \frac{1}{2}, 1 \right) \) the decay rate of \( C_2(T) \) is very similar to the decay rate of \( C_1(T) \) (the covariance of the FBM increments). Lemma 3 implies that,

\[ \lim_{T \to \infty} T^{2-2H}C_2(T) = \frac{H(2H - 1)}{\lambda^2} \tilde{\sigma}^2 = M \in (0, \infty) \]

As in example 1, due to this rate of decreasing, Theorem 2 is applicable to this process for all \( H \in (0, \frac{1}{2}) \cup \left( \frac{1}{2}, 1 \right) \), and \( F_T \xrightarrow{law} N \sim \mathcal{N}(0, \Sigma^2) \) as \( T \to \infty \).

According to remarks 9 and 10 we can tell that for the above examples \( F_T \) has a rate of convergence to normality of at least \( T^{(1 - \frac{1}{2H}) - \frac{1}{2}} \) for all \( H \in (0, \frac{1}{2}) \cup \left( \frac{1}{2}, 1 \right) \), that is, for \( \tilde{Z} \sim \mathcal{N}(0, 1) \),

\[ d_W \left( \frac{F_T}{\sqrt{\text{Var}[F_T]}}, \tilde{Z} \right) = O(T^{-\frac{1}{2H}} - \frac{1}{2}) \text{ as } T \to \infty \]

4.2 The Poisson space case:
Simulation of small jumps

Notice that in this space the measure \( \mu \) has support contained in \( \mathbb{R}^+ \times \mathbb{R}_0 \), and the formula obtained from the general case will be

\[ d_W(F, N) \leq E \left[ 1 - \langle DF, -DL^{-1}F \rangle_{L_p^2} \right] + E \left[ \left| \langle x(DF)^2 \rangle \right|_{L_p^2} \right] \]

(32)

**Remark 11.** This particular case was worked out by G. Peccati, J. L. Solé, M. S. Taqqu and F. Utzet in [29], with several examples (and conditions) given. They obtained the following inequality,

\[ d_W(F, N) \leq E \left[ 1 - \langle \tilde{D}F, -\tilde{D}L^{-1}F \rangle_{L_p^2} \right] + E \left[ \left| \langle (\tilde{D}F)^2 \rangle \right|_{L_p^2} \right] \]

(33)

where the factor \( x \) of the second term is missing. It is important to stress that both formulas are equivalent and that the difference lies in the definition of the Malliavin derivative and the random measure. In fact, the definition of Malliavin derivative used by them is \( \tilde{D}F = F(\omega) - \tilde{F}(\omega) \) and the random measure is \( \tilde{F} = \int_A d\tilde{N}(t, x) \) (instead of \( \int_A x d\tilde{N}(t, x) \) as in our case). So, the kernels of both chaos decomposition will differ by a factor of \( x \), that is, if \( F \) has chaos representation as in [12] for our framework with kernels \( f_q \), then

\[ F = \sum_{q=0}^{\infty} \tilde{I}_q(xf_q) \]
in their framework ($\hat{I}_q$ is the corresponding extension with their random measure). For example, the Ornstein-Uhlenbeck process (with initial value 0) is given by

$$Y_t = \int_0^t e^{-\lambda(t-s)}xd\hat{N}(s, x) = I_1(e^{-\lambda(t-s)}) = \hat{I}_1(xe^{-\lambda(t-s)})$$

and it is easy to check that $\hat{D}Y_t = xe^{-\lambda(t-s)} = xDY_t$. Also, since $d\mu(t, x) = x^2d\hat{\mu}(t, x)$, we have that

$$\left\langle \hat{D}F, -\hat{D}L^{-1}F \right\rangle_{L^2} = \left\langle DF, -DL^{-1}F \right\rangle_{L^2}$$

and

$$\left\langle (\hat{D}F)^2, |\hat{D}L^{-1}F| \right\rangle_{L^2} = \left\langle |DF|^2, |DL^{-1}F| \right\rangle_{L^2}$$

Remark 12. In [30] the authors accomplish the remarkable generalization of this bound to the multi-dimensional case. Their theorem reads as follows,

**Lemma 4.** Fix $d \geq 2$ and let $C = \{C(i, j)\}_{0 \leq i, j \leq d}$ be a $d \times d$ positive definite matrix. Suppose that $N \sim \mathcal{N}_d(0, C)$ and that $F = (F_1, \ldots, F_d)$ is a $\mathbb{R}^d$-valued random vector such that $E[F_i] = 0$ and $F_i \in \text{Dom} \hat{D}$ for all $i = 1, \ldots, d$. Then\(^5\)

$$d_2(F, N) \leq \|C^{-1}\|_{op} \|C\|_{op}^{-\frac{1}{2}} E \left[ \sum_{i,j} |C(i, j) - \left\langle \hat{D}F_i, -\hat{D}L^{-1}F_j \right\rangle_{L^2} |^2 \right]^{\frac{1}{2}}$$

$$+ \frac{\sqrt{2\pi}}{8} \|C^{-1}\|_{op} \|C\|_{op} E \left[ \left( \sum_{i=1}^d |\hat{D}F_i| \right)^2 \sum_{i=1}^d |DL^{-1}F_i| \right]_{L^2}$$

From this inequality they conclude a useful result for the first chaos case.

**Lemma 5.** For a fixed $d \geq 2$, let $N \sim \mathcal{N}_d(0, C)$, with $C$ positive definite, and let

$$F_n = (F_{n,1}, \ldots, F_{n,d}) = (\hat{I}_1(h_{n,1}), \ldots, \hat{I}_1(h_{n,d})), \ n \geq 1$$

Let $K_n$ be the covariance matrix of $F_n$, that is, $K_n(i, j) = E[\hat{I}_1(h_{n,i})\hat{I}_1(h_{n,j})] = \left\langle h_{n,i}, h_{n,j} \right\rangle_{L^2}$. Then

$$F_n \overset{law}{\rightarrow} N \text{ if } K_n(i, j) \overset{n \rightarrow \infty}{\rightarrow} C(i, j) \text{ and } \left\| h_{n,i} \right\|_{L^2}^{\frac{1}{2}} \overset{n \rightarrow \infty}{\rightarrow} 0 \text{ for all } i, j = 1 \ldots d.$$  

By remark [11], we know that it is possible to rewrite these conditions in terms of “our” framework. Indeed, denoting $h_{n,i} = \lambda x_{n,i}$, it follows that $K_n(i, j) = \left\langle h_{n,i}, h_{n,j} \right\rangle_{L^2} = (h_{n,i}, h_{n,j})_{L^2}$ and

$$\left\| h_{n,i} \right\|_{L^2}^{\frac{1}{2}} = \left\| \lambda x_{n,i} \right\|_{L^2}^{\frac{1}{2}}.$$  

In [11], the authors proved that the small jumps from a Lévy process can be approximated by Brownian motion. Before this theorem is stated, some notation needs to be introduced: Let $X_t$ be a Lévy process with triplet $(b, \sigma^2, \nu)$. To isolate the small jumps, consider the variance $\sigma^2 = \int_{|x| \leq \epsilon} x^2d\nu(x)$ and the small jumps process $X_t^\epsilon = \sigma(\epsilon)^{-1}\int_{[0,t] \times \{|x| \leq \epsilon\}} xd\hat{N}(s, x)$. So $X_t = b_t + \sigma W_t + N_t^\epsilon + \sigma(\epsilon)X_t^\epsilon$ where $N_t^\epsilon = \int_{x < t} \Delta X_x^1 \mathbb{1}_{\{|\Delta X_x| \geq \epsilon\}}$ is the part of (finitely many) jumps bigger than $\epsilon$. Their theorem reads as follows,

**Lemma 6.** $X_t^\epsilon \overset{law}{\rightarrow} \hat{W}_t$ (Brownian motion independent of $W_t$ and $N_t^\epsilon$ for all $\epsilon$) as $\epsilon \rightarrow 0$ if and only if for each $\kappa > 0$, $\sigma(\kappa \sigma(\epsilon) \land \epsilon) \sim \sigma(\epsilon)$ as $\epsilon \rightarrow 0$

---

\(^5\)see [30] for the definitions
They also proved that the condition \( \sigma(\kappa(\epsilon) \wedge \epsilon) \sim \sigma(\epsilon) \) as \( \epsilon \to 0 \) is implied by \( \lim_{\epsilon \to 0} \frac{\sigma(\epsilon)}{\epsilon} = \infty \). The importance of this lemma is that \( X_t \overset{\text{law}}{\approx} b_t + \sqrt{\sigma^2 + \sigma(\epsilon)^2} W_t + N_\epsilon^t \) (for \( \epsilon \) small enough), and the latter is quite easy to simulate.

The objective of this subsection is to extend this kind of result to functionals that are not necessarily Lévy. To focus just on the jump part, let’s assume, without loss of generality, that the triplet of the Lévy process \( X_t \) is \((0, 0, \nu)\). Then, it can be written as \( X_t = I_1(1_{[0, t]}(\epsilon)) = \int_{[0, t] \times \mathbb{R}} xd\bar{N}(s, x) \). Define \( \bar{f}_1(\epsilon)^2 = \|h_t1_{[0, t] \times \{x \leq \epsilon\}}\|_{L_2^\nu} = \int_{[0, t] \times \{x \leq \epsilon\}} h_t(s, x)^2 x^2 d\nu(x) ds \) for some \( h_t \in L_2^\nu \) and consider the processes \( \tilde{X}_t = I_1(\bar{h}_t1_{[0, t]}(\epsilon)) \). Similarly, define \( \bar{f}(\epsilon)^2 = \int_{\{x \leq \epsilon\}} x^2 d\nu(x) \) and \( \tilde{X}_t = I_1(\bar{f}(\epsilon)^{-1} h_t1_{[0, t] \times \{x \leq \epsilon\}}) \). This means that \( \tilde{X}_t = N_\epsilon^t + \bar{f}(\epsilon) \tilde{X}_t^\epsilon \) where \( N_\epsilon^t = I_1(\bar{h}_t1_{[0, t] \times \{x \geq \epsilon\}}) \) has finitely many jumps. Let \( \tilde{W} \) be an isonormal Gaussian process with covariance structure given by \( E[\tilde{W}(f)\tilde{W}(g)] = \int_R f(s)g(s)ds \).

**Theorem 4.** For a fixed \( t \), \( \tilde{X}_t^\epsilon \overset{\text{law}}{\longrightarrow} \tilde{Z} \sim \mathcal{N}(0, 1) \) as \( \epsilon \to 0 \) if

\[
\frac{\int_{[0, t] \times \{x \leq \epsilon\}} |xh_t(s, x)|^3 d\nu(x) ds}{\bar{f}_1(\epsilon)^3} \to 0
\]  

(34)

Moreover, suppose that \( h_t(s, x) = h_t(\epsilon) \). Then \( \tilde{X}_t^\epsilon \overset{\text{law}}{\longrightarrow} \tilde{W}(h_t) \) if

\[
\frac{\int_{\{x \leq \epsilon\}} |x|^3 d\nu(x)}{\bar{f}(\epsilon)^3} \to 0
\]  

(35)

**Proof.** To prove the first statement, it is enough to verify that the upper bound \( \|D\tilde{X}_t^\epsilon\|_{L_2^\nu}^2 \) goes to zero as \( \epsilon \) goes to zero. Notice that

\[
\|D\tilde{X}_t^\epsilon\|_{L_2^\nu}^2 = \|\bar{f}(\epsilon)^{-1} h_t1_{[0, t] \times \{x \leq \epsilon\}}\|_{L_2^\nu}^2 = 1
\]

so \( E \left| 1 - \langle D\tilde{X}_t^\epsilon, -DL^{-1}X_t^\epsilon \rangle_{L_2^\nu} \right| = 0 \). On the other hand,

\[
E \left[ \left| D\tilde{X}_t^\epsilon \right|^2 \left| DL^{-1}X_t^\epsilon \right|^2 \right]_{L_2^\nu} = E \left[ \left| D\tilde{X}_t^\epsilon \right|^3 \right]_{L_2^\nu} = \frac{\int_{[0, t] \times \{x \leq \epsilon\}} |xh_t(s, x)|^3 d\nu(x) ds}{\bar{f}_1(\epsilon)^3}
\]

So bound \( \|D\tilde{X}_t^\epsilon\|_{L_2^\nu}^2 \) goes to zero as \( \epsilon \) goes to zero if \( \bar{f}_1(\epsilon)^3 \) is true. To prove the second statement, it is necessary to prove that for any times \( \{t_1, \ldots, t_d\} \) the random vector \( (\tilde{X}_{t_1}^\epsilon, \ldots, \tilde{X}_{t_d}^\epsilon) \) \( \overset{\text{law}}{\longrightarrow} (\tilde{W}(h_{t_1}), \ldots, \tilde{W}(h_{t_d})) \). This is very easy to achieve thanks to Lemma 3 from remark 12 because only two conditions need to be checked. Notice that for all \( \epsilon \)

\[
K_{\epsilon}(i, j) = E \left[ \tilde{X}_{t_i}^\epsilon \tilde{X}_{t_j}^\epsilon \right] = \langle \bar{f}(\epsilon)^{-1} h_t1_{[0, t] \times \{x \leq \epsilon\}}, \bar{f}(\epsilon)^{-1} h_t1_{[0, t] \times \{x \leq \epsilon\}} \rangle_{L_2^\nu}
\]

\[
= \langle h_t1_{[0, t]}, h_t1_{[0, t]} \rangle_{L_2^\nu} = E \left[ \tilde{W}(h_t1_{[0, t]}) \right] = C(i, j)
\]

so trivially \( K_{\epsilon}(i, j) \to C(i, j) \) as \( \epsilon \to 0 \). And for the second condition we have that

\[
\left\| \left| x \right|^2 \left| \bar{f}(\epsilon)^{-1} h_t1_{[0, t] \times \{x \leq \epsilon\}} \right|^2 \right\|_{L_2^\nu}^2 = \int_0^{t_i} |h_s|^3 ds \int_{\{x \leq \epsilon\}} |x|^3 d\nu(x) \to 0 \text{ by } \text{(35)}
\]

Since both conditions were fulfilled, the conclusion of the theorem follows. \( \square \)
Remark 13. Notice that if \( h_t(s, x) = h_t(s) \) then
\[
\sigma_t(\epsilon)^2 = \int_{|x| < \epsilon} h_t(s)^2 x^2 d\nu(x) ds = \int_0^t h_t(s)^2 ds \int_{|x| < \epsilon} x^2 d\nu(x) = \int_0^t h_t(s)^2 ds \sigma(\epsilon)^2
\]
so it immediately follows that \( (34) \) is true if and only if \( (35) \) is true. Assuming then, that \( (35) \) is true, it may
true. So, for the example, assume that the measure \( \nu \)

4.2.1 Example: Fractional Lévy Process

Condition \( (34) \) is quite easy to be verified. In fact, the measure \( d\nu(x) = |x|^{-(2+\delta)} 1_{-\epsilon < x < \epsilon} dx \) for \( \delta \in (-1, 1) \)
and \( a, b > 0 \) (no jumps bigger than \( b \) or smaller than \(-a\)) is such that \( (35) \) is fulfilled. To check this, note that
\[
\int_{|x| < \epsilon} |x|^3 d\nu(x) = \frac{2 \epsilon^{2-\delta}}{(2-\delta)} \quad \text{and} \quad \sigma(\epsilon)^3 = \epsilon^{1-\frac{\delta}{1-\delta}} \left( \frac{2}{3} \right)^{\frac{1}{3}}, \quad \text{so} \quad \sigma(\epsilon)^{-3} \int_{|x| < \epsilon} |x|^3 d\nu(x) = O(\epsilon^{\frac{1+\delta}{1-\delta}}) \quad \text{and} \quad (35) \text{ is true.}
\]
So, for the example, assume that the measure \( \nu \) is such that \( (35) \) holds.

1. Fractional Lévy Process (FLP):

There are two ways to represent a fractional Brownian motion as an integral of a kernel with respect to
Brownian motion, and both deliver the same process (see \( \text{[15]} \) for a thorough explanation). One is the
so-called Mandelbrot-Van Ness representation which is an integral over the whole real line with respect to
a two sided Brownian motion. The other, is the so-called Molchan-Golosov representation which is an
integral over a compact interval. In the Lévy case, it is proved in \( \text{[37]} \) that these representations delivers
different processes with very different characteristics. Because of this “non-uniqueness”, the FLP
generated by the Mandelbrot-Van Ness representation is called FLPMvN, and the one generated by
the Molchan-Golosov representation is called FLPMG. It is known that FLP’s have the same covariance
structure as FBM. The advantage of FLPMvN over FLPMG is that the former is stationary and the
latter is not (in general), as is shown in \( \text{[37]} \). Nevertheless, since FLPMG is derived on a compact
interval, Malliavin calculus can be applied to it.

Consider \( X_t \) as an FLPMG, that is, \( X_t = I_t(K_t^H) \) where
\[
\langle K_t^H, K_s^H \rangle_{L^2} = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right)
\]
According to Theorem \( \text{[1]} \) since \( (35) \) is true and \( h_t(s, x) = K_t^H(s) \), it follows that \( X_t \overset{\text{law}}{\sim} \hat{W}(K_t^H) \)
as \( \epsilon \to \infty \). But \( \hat{W}(K_t^H) = B_t^H \) is a fractional Brownian motion. In this case, we conclude that in
order to simulate the paths of an FLPMG \( X_t \), we just need to fix \( \epsilon \) small enough, and simulate the
finite number of jumps part \( N_t^\epsilon = I_t(K_t^H 1_{\{|x| \geq \epsilon\}}) \) along with an (independent) FBM part \( B_t^H = \hat{W}(K_t^H) \),
because \( X_t \overset{\text{law}}{\sim} N_t^\epsilon + \sigma(\epsilon) B_t^H \).

4.3 The Wiener-Poisson space case:
Product of O-U processes

Finally, the second order Poincaré inequality developed in the combined space is used to obtain a CLT for
mixed processes. First, notice that if we have a double Ornstein-Uhlenbeck (O-U) process as a sum of a
Wiener O-U process \( Y_t \) plus a Poisson O-U process \( Z_t \) (independent of \( Y_t \)), it can be proved (in two different
ways) that the functional \( F_T = T^{-\frac{1}{2}} \int_0^T Y_t + Z_t dt \) converges to a normal random variable as \( T \to \infty \). The
first way is to separate \( F_T \) to the terms \( T^{-\frac{1}{2}} \int_0^T Y_t dt \) and \( T^{-\frac{1}{2}} \int_0^T Z_t dt \) and use the inequalities \( (30) \) and \( (32) \).
respectively to prove that each part goes to a normal. The other method is to use inequality \((22)\) and just do one computation. The second way is clearly faster (since the kernels are the same). This is one advantage of having the inequality in the combined space.

For our example, let's focus on a process for which we cannot use either of the inequalities, \((31)\) nor \((32)\), to prove a CLT. First, for simplicity's sake (to avoid dealing with constants), assume that the triplet for the underlying Lévy process is given by \((0, 1, \nu)\), where \(\int_{\mathbb{R}} x^2 \nu(dx) = 1\). Moreover, assume that \(\int_{\mathbb{R}} |x|^3 \nu(dx) < \infty\) and \(\int_{\mathbb{R}} x^2 \nu(dx) < \infty\). Let \(Y_t = \int_0^t \sqrt{2\lambda} e^{-\lambda(t-s)} dW_s\) and \(Z_t = \int_{[0,t] \times \mathbb{R}_0} \sqrt{2\lambda} e^{-\lambda(t-s)} x d\tilde{N}(s, x)\), so, \(Y_t\) is a Wiener O-U process and \(Z_t\) is a Poisson (pure jump) O-U process. If \(h_t(s) = \sqrt{2\lambda} e^{-\lambda(t-s)}\) then the double O-U process mentioned above is just \(Y_t + Z_t = I_t(h_t)\). Now, define \(h_t^{(0)}(s, x) = h_t(s)1_{\{x=0\}}(x)\) and \(h_t^{(1)}(s, x) = h_t(s)1_{\{x \neq 0\}}(x)\), then \(Y_t = I_t(h_t^{(0)})\) and \(Z_t = I_t(h_t^{(1)})\). Notice that due to the normalization of the Lévy triplet we have that \(C(t, s) = \langle h_t^{(0)}(0), h_s^{(0)} \rangle_{L^2_\mu} = \langle h_t^{(1)}(0), h_s^{(1)} \rangle_{L^2_\mu}\). The goal of this subsection is to prove that \(F_T = T^{-\frac{1}{2}} \int_0^T Y_t Z_t dt \xrightarrow{law} Z \sim N(0, \Sigma^2)\) as \(T \to \infty\).

Since \(h_t^{(0)}\) and \(h_t^{(1)}\) have disjoint supports (and using the product formula \((11)\)), \(Y_t Z_t = I_2(h_t^{(0)} \otimes h_t^{(1)})\), and by Fubini \(F_T = I_2(T^{-\frac{1}{2}} \int_0^T h_t^{(0)} \otimes h_t^{(1)} dt)\). Hence \(F_T\) lies in the 2\textsuperscript{nd} chaos. According to Corollary \(1\) we just need to check conditions \((25), (20), (27), (28)\) and \((29)\).

- **Expectation of the First Derivative’s Norm:**

  - First Malliavin Derivative:
  \[
  D_z F_T = T^{-\frac{1}{2}} \int_0^T I_1(h_t^{(0)}) h_t^{(1)}(z) + I_1(h_t^{(1)}) h_t^{(0)}(z) dt
  \]

  - Norm of the First Malliavin Derivative:
  \[
  \|D_z F_T\|_{L^2_\mu} = T^{-1} \int_{[0,T]^2} \langle I_1(h_t^{(0)}), h_t^{(1)}(z) + I_1(h_t^{(1)}) h_t^{(0)}(z), I_1(h_t^{(0)}), h_s^{(1)}(z) + I_1(h_s^{(1)}) h_t^{(0)}(z) \rangle_{L^2_\mu} dt ds
  \]
  \[
  = T^{-1} \int_{[0,T]^2} I_1(h_t^{(0)}) I_1(h_s^{(0)}) \langle h_t^{(1)}, h_s^{(1)} \rangle_{L^2_\mu} + I_1(h_t^{(1)}) I_1(h_s^{(1)}) \langle h_t^{(0)}, h_s^{(0)} \rangle_{L^2_\mu} ds dt
  \]
  \[
  = T^{-1} \int_{[0,T]^2} (I_1(h_t^{(0)}) I_1(h_s^{(0)}) + I_1(h_t^{(1)}) I_1(h_s^{(1)})) C(t, s) dt ds
  \]

  - Expectation of the First Derivative’s Norm:
  Notice that by the product formula \((11)\) we have that
  \[
  \prod_{i=1}^4 I_1(h_t^{(0)}) = \langle h_t^{(0)}, h_t^{(0)} \rangle_{L^2_\mu} + I_2(h_t^{(0)} \otimes h_t^{(0)}) \langle h_t^{(0)}, h_t^{(1)} \rangle_{L^2_\mu} + I_2(h_t^{(0)} \otimes h_t^{(1)}) \langle h_t^{(1)}, h_t^{(1)} \rangle_{L^2_\mu}
  \]
  and
  \[
  \prod_{i=1}^4 I_1(h_t^{(1)}) = \langle h_t^{(1)}, h_t^{(1)} \rangle_{L^2_\mu} + I_1(h_t^{(1)} \otimes h_t^{(1)}) + I_2(h_t^{(1)} \otimes h_t^{(1)}) \langle h_t^{(1)}, h_t^{(1)} \rangle_{L^2_\mu} + I_2(h_t^{(1)} \otimes h_t^{(1)}) \langle h_t^{(1)}, h_t^{(1)} \rangle_{L^2_\mu}
  \]
so,
\[ E \left[ \prod_{i=1}^{4} I_i (h_i^{(0)}) \right] = C(t_1, t_2) C(t_3, t_4) + \left\langle h_{t_1}^{(0)} \otimes h_{t_2}^{(0)}, h_{t_3}^{(0)} \otimes h_{t_4}^{(0)} \right\rangle_{L_1^2} \]

and
\[ E \left[ \prod_{i=1}^{4} I_i (h_i^{(1)}) \right] = C(t_1, t_2) C(t_3, t_4) + \left\langle h_{t_1}^{(1)} \otimes h_{t_2}^{(1)}, h_{t_3}^{(1)} \otimes h_{t_4}^{(1)} \right\rangle_{L_1^2} + \left\langle h_{t_1}^{(0)} \otimes h_{t_2}^{(1)}, h_{t_3}^{(0)} \otimes h_{t_4}^{(1)} \right\rangle_{L_1^2} \]

Also notice that,
\[ C(t, s) = \int_0^{t+s} 2\lambda e^{-\lambda(t+s-2u)} du = e^{-\lambda|t-s|} - e^{-\lambda(t+s)} \leq e^{-\lambda|t-s|} \leq 1 \]
\[ \langle h_{t_1}^{(1)} \otimes_0 h_{t_2}^{(1)}, h_{t_3}^{(1)} \otimes_0 h_{t_4}^{(1)} \rangle_{L_1^2} = \int_0^{\min\{t_1, t_2, t_3, t_4\}} \int_{\mathbb{R}_0} x^4 4\lambda^2 e^{-\lambda(t_1+t_2+t_3+t_4-4u)} d\nu(x) du \leq \lambda \int_{\mathbb{R}_0} x^4 d\nu(x) \]
\[ \langle h_{t_1}^{(0)} \otimes_0 h_{t_2}^{(0)}, h_{t_3}^{(0)} \otimes_0 h_{t_4}^{(0)} \rangle_{L_1^2} = \langle h_{t_1}^{(1)} \otimes_0 h_{t_2}^{(1)}, h_{t_3}^{(1)} \otimes_0 h_{t_4}^{(1)} \rangle_{L_1^2} = \frac{C(t_1, t_3) C(t_2, t_4) + C(t_1, t_4) C(t_2, t_3)}{2} \leq 1 \]

Putting all this together we get,
\[ E \left[ \|DF_T\|_{L_1^2}^4 \right] = 2T^{-2} \int_{[0, T]^4} \left( E \left[ \prod_{i=1}^{4} I_i (h_i^{(0)}) \right] + E \left[ \prod_{i=1}^{4} I_i (h_i^{(1)}) \right] \right) C(t_1, t_2) C(t_3, t_4) dt^1 \]
\[ \leq 2 \left( 4 + \lambda \int_{\mathbb{R}_0} x^4 d\nu(x) \right) \left( T^{-1} \int_{[0, T]^2} C(t, s) dt ds \right)^2 \]
\[ \leq 2 \left( 4 + \lambda \int_{\mathbb{R}_0} x^4 d\nu(x) \right) \left( T^{-1} \int_{[0, T]^2} e^{-\lambda|t-s|} dt ds \right)^2 \]
\[ = 2 \left( 4 + \lambda \int_{\mathbb{R}_0} x^4 d\nu(x) \right) \left( 2T^{-1} \int_{[0, T]} e^{-\lambda(t-s)} ds dt \right)^2 \]
\[ \leq 2 \left( 4 + \lambda \int_{\mathbb{R}_0} x^4 d\nu(x) \right) \left( \frac{2}{\lambda} \right)^2 \]

All this proves that
\[ E \left[ \|DF_T\|_{L_1^2}^4 \right] = O(1) \text{ as } T \to \infty \]

- **Expectation of the Cube of the First Derivative’s Norm:**
  - Cube of the First Malliavin Derivative:
    Since \( h_1^{(0)}(z) \cdot h_2^{(1)}(z) = 0 \) for all \( z \in \mathbb{R}^+ \times \mathbb{R} \) then,
    \[ |DF_T|^3 = \left| T^{-\frac{1}{2}} \int_{[0, T]^4} \left[ \prod_{i=1}^{3} I_i (h_i^{(0)}) h_i^{(1)} + \prod_{i=1}^{3} I_i (h_i^{(1)}) h_i^{(0)} \right] dt \right| \]
  - Norm of the Cube of the First Malliavin Derivative:
    Since \( x \cdot h_t^{(0)}(z) = 0 \) for all \( z = (t, x) \in \mathbb{R}^+ \times \mathbb{R} \) then,
    \[ \left\langle |x|, |DF_T|^3 \right\rangle_{L_1^2} \leq T^{-\frac{1}{2}} \int_{[0, T]^4} \prod_{i=1}^{3} \left| I_i (h_i^{(0)}) \right| \left\langle |x|, \prod_{i=1}^{3} |h_i^{(0)}| \right\rangle_{L_1^2} dt \]
– Expectation of the Cube of the First Derivative’s Norm:

Notice that by Hölder we have,

\[
E \left[ \prod_{i=1}^{3} |I_1(h_{t_t}^{(0)})|^2 \right] \leq E \left[ \prod_{i=1}^{3} \left( I_1(h_{t_t}^{(0)}) I_1(h_{t_t}^{(0)}) \right)^2 \right] \leq \left( |C(t_1,t_2)| + E \left[ \prod_{i=1}^{3} \left( I_1(h_{t_{t_2}}^{(0)}) \right)^2 \right] \right) E \left[ \prod_{i=1}^{3} \left( I_1(h_{t_{t_3}}^{(0)}) \right)^2 \right] \leq \left( |C(t_1,t_2)| + \|h_{t_t}^{(0)} \otimes h_{t_t}^{(0)} \|_{L^2_{\otimes 2}} \right) \|h_{t_t}^{(0)} \|_{L^2_t} \leq 2
\]

and

\[
\left\langle \sum_{i=1}^{3} h_{t_t}^{(1)} \right\rangle = \int_{0}^{T} \left( 2\lambda \right)^{1/2} e^{-\lambda(t_1+t_2+t_3-3u)} du \int_{R_0} |x|^3 \, d\nu(x) \leq \frac{2\sqrt{2\lambda}}{3} \left( \int_{R_0} |x|^3 \, d\nu(x) \right) e^{-\lambda(t_1+t_2+t_3-3 \min\{t_1,t_2,t_3\}}
\]

Putting all together we get,

\[
E \left[ \left( |x|, I_1(h_{t_t}^{(0)}) \right)^3 \right] \leq T^{-2} \int_{[0,T]} \left( \int_{R_0} |x|^3 \, d\nu(x) \right) e^{-\lambda(t_1+t_2+t_3-3 \min\{t_1,t_2,t_3\}} \, dt \leq \frac{4\sqrt{2\lambda}}{3} \left( \int_{R_0} |x|^3 \, d\nu(x) \right) T^{-2} \int_{[0,T]} e^{-\lambda(t_1+t_2+t_3-3 \min\{t_1,t_2,t_3\}}} \, dt \leq \frac{24\sqrt{2\lambda}}{3} \left( \int_{R_0} |x|^3 \, d\nu(x) \right) T^{-2} \int_{[0,T]} e^{-\lambda(t+s-2u)} \, dudst \leq \frac{4\sqrt{2}}{\lambda^2 \sqrt{T}} = O(T^{-1/2})
\]

All this proves that

\[
E \left[ \left( |x|, I_1(h_{t_t}^{(0)}) \right)^3 \right] \to 0 \quad \text{as} \quad T \to \infty
\]

– Second Malliavin Derivative:

\[
D^2_{x_1,x_2} F_T = T^{-1/2} \int_{0}^{T} h_t^{(1)}(z_1)h_t^{(0)}(z_2) + h_t^{(0)}(z_1)h_t^{(1)}(z_2) dt
\]

– Contraction of order 1:

\[
D^2 F_T \otimes D^2 F_T = T^{-1} \int_{[0,T]} h_t^{(1)}(z_1)h_t^{(0)}(z_1) \left\langle \left\langle h_t^{(0)}, h_t^{(0)} \right\rangle \right\rangle_{L^2_{\otimes 2}} + h_t^{(0)}(z_1)h_t^{(0)}(z_1) \left\langle \left\langle h_t^{(1)}, h_t^{(1)} \right\rangle \right\rangle_{L^2_{\otimes 2}} dt ds \leq T^{-1} \int_{[0,T]} h_t^{(1)}(z_1)h_t^{(0)}(z_1) + h_t^{(0)}(z_1)h_t^{(0)}(z_1) dt ds
\]
– Norm of the Contraction:

\[ \left\| D^2 F_T \otimes_1 D^2 F_T \right\|_{L^2_\mu}^2 \leq T^{-2} \int_{[0,T]^t} \left\langle h_{t_1}^{(0)} h_{t_2}^{(0)} h_{t_3}^{(0)} h_{t_4}^{(0)} + h_{t_1}^{(1)} h_{t_2}^{(1)} h_{t_3}^{(1)} h_{t_4}^{(1)} \right\rangle_{L^2_\mu} dt \]

– Expectation of the Contraction’s Norm:

Notice that

\[ \left\langle h_{t_1}^{(0)} h_{t_2}^{(0)} h_{t_3}^{(0)} h_{t_4}^{(0)} \right\rangle_{L^2_\mu} = \left\langle h_{t_1}^{(1)} h_{t_2}^{(1)} h_{t_3}^{(1)} h_{t_4}^{(1)} \right\rangle_{L^2_\mu} = \lambda(e^{-\lambda(t_1+t_2+t_3+t_4-4 \min(t_1,t_2,t_3,t_4)}) - e^{-\lambda(t_1+t_2+t_3+t_4)}) \]

so,

\[ \int_{[0,T]^t} \left\langle h_{t_1}^{(1)} h_{t_2}^{(1)} h_{t_3}^{(1)} h_{t_4}^{(1)} \right\rangle_{L^2_\mu} \leq 24\lambda \int_{0}^{T} \int_{0}^{t_1} \int_{0}^{t_2} \int_{0}^{t_3} \int_{0}^{t_4} e^{-\lambda(t+s+u+v)} dsdu dt dv dt \leq 4T \]

Putting all together we get,

\[ E[\left\| D^2 F_T \otimes_1 D^2 F_T \right\|_{L^2_\mu}^2] \leq 2T^{-2} \int_{[0,T]^t} \left\langle h_{t_1}^{(1)} h_{t_2}^{(1)} h_{t_3}^{(1)} h_{t_4}^{(1)} \right\rangle_{L^2_\mu} dt \leq T^{-2} \frac{8T}{\lambda^2} = \frac{8}{T\lambda^2} \]

All this proves that,

\[ E[\left\| D^2 F_T \otimes_1 D^2 F_T \right\|_{L^2_\mu}^2] \to 0 \quad \text{as} \quad T \to \infty \]

• Expectation of the Squared Second Derivative’s Norm:

– Square of the Second Malliavin Derivative:

\[ (D_{t_1}^2 F_T)^2 = T^{-1} \int_{[0,T]^2} h_t^{(0)}(z_1)h_t^{(0)}(z_1)h_{t_1}^{(1)}(z_2)h_{t_3}^{(1)}(z_2) + h_t^{(1)}(z_1)h_{t}^{(1)}(z_1)h_t^{(0)}(z_2)h_{t_2}^{(0)}(z_2) dt ds \]

– Inner Product of the Squared Second Malliavin Derivative:

Samely as above, \( x \cdot h_t^{(0)} = 0 \), so

\[ \left\langle x, (D^2 F_T)^2 \right\rangle_{L^2_\mu} = T^{-1} \int_{[0,T]^2} h_t^{(0)}(z_1) \left\langle x, h_{t_1}^{(1)}(z_1) \right\rangle_{L^2_\mu} dt ds \leq \int_{[0,T]^2} |x|^3 dv(x) T^{-1} \int_{[0,T]^2} h_t^{(0)}(z_1) dt ds \]

– Expectation of the Squared Second Derivative’s Norm:

By the above computations we have,

\[ E\left[ \left\| \left\langle x, (D^2 F_T)^2 \right\rangle \right\|_{L^2_\mu}^2 \right] \leq \left( \int_{[0,T]^2} |x|^3 dv(x) \right)^2 T^{-2} \int_{[0,T]^t} \left\langle h_{t_1}^{(0)} h_{t_2}^{(0)} h_{t_3}^{(0)} h_{t_4}^{(0)} \right\rangle_{L^2_\mu} dt \leq \frac{4}{\lambda^2 T} \]

All this proves that,

\[ E\left[ \left\| \left\langle x, (D^2 F)^2 \right\rangle \right\|_{L^2_\mu}^2 \right] \to 0 \quad \text{as} \quad T \to \infty \]

• Existence of the Variance:
Since \( F_T = I_2 (T^{-\frac{1}{2}} \int_0^T h_t^{(0)} \otimes h_t^{(1)} dt) \) then
\[
\begin{align*}
\text{Var}[F_T] &= \left\| T^{-\frac{1}{2}} \int_0^T h_t^{(0)} \otimes h_t^{(1)} dt \right\|^2 = T^{-1} \int_{[0,T]^2} \left( \int_{s_1 \vee s_2}^{T} 2\lambda e^{-\lambda (2t-s_1-s_2)} dt \right)^2 d\gamma_1 ds_2 \\
&= T^{-1} \int_{[0,T]^2} \left( e^{-\lambda |s_1-s_2|} - e^{-\lambda (2T-s_1-s_2)} \right)^2 d\gamma_1 ds_2 \\
&= T^{-1} \int_{[0,T]^2} e^{-2\lambda |s_1-s_2|} - 2e^{-2\lambda (T-s_1 \wedge s_2)} + e^{-2\lambda (2T-s_1-s_2)} d\gamma_1 ds_2 \\
&= T^{-1} \left( \frac{T}{\lambda} - 6 \frac{(1-e^{-2\lambda T})}{4\lambda^2} + 4Te^{-2\lambda T} + \frac{(1-e^{-2\lambda T})^2}{4\lambda^2} \right) = \frac{1}{\lambda} + O(T^{-1})
\end{align*}
\]

All this proves that,

\[
\text{Var}[F_T] \to \frac{1}{\lambda} \in (0, \infty) \quad \text{exists as } \quad T \to \infty
\]

Since all five conditions are met, by Corollary 11 we have that \( F_T \xrightarrow{law} N \sim \mathcal{N}(0, \frac{1}{\lambda}) \) as \( T \to \infty \). Moreover, due to the quantitative property of the inequality, is possible to estimate (from the computations above) that the rate of convergence to normality is at least \( O(T^{-\frac{1}{2}}) \), i.e., \( d_W \left( \frac{F_T}{\sqrt{\text{Var}[F_T]}}, N \right) = O(T^{-\frac{1}{2}}) \) as \( T \to \infty \). This rate is similar to the one obtained for the linear functionals of Gaussian-subordinated fields with underlying process given by the increments of FBM or the fractional-driven O-U, when \( H \in \left( 0, \frac{1}{2} \right) \). I believe that the right rate of convergence for these processes should be \( O(T^{-\frac{1}{2}}) \)!

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