THE FOURIER-MUKAI TRANSFORM OF A UNIVERSAL FAMILY OF STABLE VECTOR BUNDLES

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Abstract. In this note we prove that the Fourier-Mukai transform $\Phi_U$ of the universal family of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ is not fully faithful.

INTRODUCTION

To every smooth projective variety $X$ one can associate its bounded derived category of coherent sheaves $\mathcal{D}^b(X)$. The derived category contains a lot of geometric information about $X$. In some cases one can even recover $X$ from $\mathcal{D}^b(X)$ but there are also examples of different varieties with equivalent derived categories, see [7] for an introduction.

To compare the derived categories of two smooth projective varieties $X$ and $Y$, one needs to study functors between them. As it turns out, most of the interesting functors are Fourier-Mukai transforms $\Phi_F : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ for some object $F \in \mathcal{D}^b(X \times Y)$.

In this note we are interested in fully faithful Fourier-Mukai transforms because they give a semi-orthogonal decomposition of the derived category $\mathcal{D}^b(Y)$ in smaller admissible subcategories. For example Krug and Sosna prove in [10] that the Fourier-Mukai transform $\Phi_{\mathcal{I}_Z} : \mathcal{D}^b(S) \to \mathcal{D}^b(S^{[n]})$ induced by the universal ideal sheaf $\mathcal{I}_Z$ of the Hilbert scheme $S^{[n]}$ is fully faithful for a surface $S$ with $p_g = q = 0$, hence $\mathcal{D}^b(S)$ is an admissible subcategory in $\mathcal{D}^b(S^{[n]})$. This result was generalized for the Hilbert square $X^{[2]}$ to smooth projective varieties $X$ with exceptional structure sheaf and arbitrary dimension $\dim(X) \geq 2$, see [11].

Another example of this behaviour is given by the moduli space $\mathcal{M}_{\mathbb{C}^2}(2, L)$ of stable rank two vector bundles with fixed determinant $L$ of degree one on a smooth projective curve $C$ of genus $g \geq 2$. This moduli space is fine and thus there is a universal family $U$ on $C \times \mathcal{M}_{\mathbb{C}^2}(2,L)$. By work of Narasimhan, see [18] and [19], as well as Fonarev and Kuznetsov, see [6], it is known that the Fourier-Mukai transform $\Phi_U : \mathcal{D}^b(C) \to \mathcal{D}^b(\mathcal{M}_{\mathbb{C}^2}(2,L))$ is fully faithful. Thus $\mathcal{D}^b(C)$ is an admissible subcategory of $\mathcal{D}^b(\mathcal{M}_{\mathbb{C}^2}(2,L))$. This also solves the so-called Fano visitor problem for smooth projective curves of genus $g \geq 2$. This result was generalized in [2] to the higher rank case $\mathcal{M}_{\mathbb{C}^2}(r, L)$ for a line bundle $L$ of degree $d$ such that $\gcd(r, d) = 1$ and curves of genus $g \geq g_0$ for some $g_0 \in \mathbb{N}$.

In light of these examples one can ask if the Fourier-Mukai transform of the universal family $U$ on a moduli space $\mathcal{M}_{\mathbb{P}^2}(r, c_1, c_2)$ of stable sheaves on $\mathbb{P}^2$ is also fully faithful. Our main result is, that this is not always the case. We prove:

Theorem. The Fourier-Mukai transform

$$\Phi_U : \mathcal{D}^b(\mathbb{P}^2) \to \mathcal{D}^b(\mathcal{M}_{\mathbb{P}^2}(4,1,3))$$

induced by the universal family $U$ of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$ is not fully faithful.

The structure of this note is as follows: in section 1 we recall some facts about the moduli space we are interested in. We construct an explicit family of stable sheaves for this moduli space in section 2. The computation of some cohomology groups for the family of stable sheaves can be found in section 3. In the final section 4 we prove the main result.

Everything in this note is defined over the field of complex numbers $\mathbb{C}$. The projective plane $\mathbb{P}^2$ is polarized by $H = \mathcal{O}_{\mathbb{P}^2}(1)$, thus $\mu$-stability means $\mu_H$-stability. A cohomology group written in lowercase characters simply denotes its dimension as a $\mathbb{C}$-vector space.

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1. The moduli space

We begin by studying the moduli space $M_{\mathbb{P}^2}(4,1,3)$ of $S$-equivalence classes of $\mu$-semistable torsion-free sheaves $E$ on the projective plane $\mathbb{P}^2$ with the following numerical data:

$$\text{rk}(E) = 4, \quad c_1(E) = 1, \quad c_2(E) = 3.$$  

(Since the first Chern class is just an integer multiple of the polarization $H$, we simply identify it with this number.)

By this choice of rank $r$ and Chern classes $c_1$ resp. $c_2$ we get:

**Lemma 1.1.** The moduli space $M_{\mathbb{P}^2}(4,1,3)$ is fine and there are no proper semistable sheaves.

**Proof.** We have

$$\gcd \left( r, c_1.H, \frac{1}{2}(c_1 - K_{\mathbb{P}^2}) - c_2 \right) = \gcd(4, 1, -1) = 1.$$  

The result now follows from [5 Corollary 4.6.7] and [5 Remark 4.6.8].

**Remark 1.2.** This lemma shows that the moduli space $M_{\mathbb{P}^2}(4,1,3)$ has a universal family, that is a sheaf $U$ on $\mathbb{P}^2 \times M_{\mathbb{P}^2}(4,1,3)$ flat over $M_{\mathbb{P}^2}(4,1,3)$ such that for every $E$ with $[E] \in M_{\mathbb{P}^2}(4,1,3)$ there is an isomorphism $U_{|[E]} \cong E$, where $U_{|[E]}$ denotes the restriction of $U$ to the fiber over $[E]$.

The following properties of the moduli space are probably well known:

**Lemma 1.3.** The moduli space $M_{\mathbb{P}^2}(4,1,3)$ is a smooth projective variety of dimension six. Furthermore all sheaves $E$ classified by this moduli space are locally free.

**Proof.** The space $M_{\mathbb{P}^2}(4,1,3)$ is projective by construction. Since every sheaf $E$ is stable, we get by Serre duality

$$\text{Ext}^2(E, E) \cong \text{Hom}(E, E(-3))^\vee = 0$$

hence $M_{\mathbb{P}^2}(4,1,3)$ is smooth and by [5] it is also irreducible. We also recall

$$\dim(M_{\mathbb{P}^2}(r,c_1,c_2)) = \Delta - (r^2 - 1)\chi(\mathbb{P}^2, O_{\mathbb{P}^2})$$

where $\Delta = 2rc_2 - (r - 1)c_1^2$ is the discriminant. So $\dim(M_{\mathbb{P}^2}(4,1,3)) = 6$.

The double dual of a $\mu$-stable torsion-free sheaf $E$ is still $\mu$-stable and defines a smooth point in $M_{\mathbb{P}^2}(4,1,3 - \ell)$ with $\ell = \text{length}(E^\vee \vee / E)$. If $E$ were not locally free we would have $\ell \geq 1$ and $M_{\mathbb{P}^2}(4,1,3 - \ell)$ would have negative dimension, which is not possible.

**Remark 1.4.** Using Lemma 1.3 together with [5 Lemma 2.1.7.] shows that the universal family $U$ is itself locally free on $\mathbb{P}^2 \times M_{\mathbb{P}^2}(4,1,3)$. This implies that the sheaves $U_p$, the restriction the fiber over $p \in \mathbb{P}^2$, are also locally free on the moduli space.

The sheaves classified by $M_{\mathbb{P}^2}(4,1,3)$ can be described more explicitly:

**Lemma 1.5.** Let $E$ be a locally free sheaf on $\mathbb{P}^2$ with $[E] \in M_{\mathbb{P}^2}(4,1,3)$, then there is a length three subscheme $Z \subset \mathbb{P}^2$ and an exact sequence

$$0 \to O_{\mathbb{P}^2}(3) \to E \to I_Z(1) \to 0.$$  

**Proof.** Hirzebruch-Riemann-Roch shows $\chi(\mathbb{P}^2,E) = 3$. The stability of $E$ implies that we have $h^2(\mathbb{P}^2, E) = 0$ and thus $h^0(\mathbb{P}^2, E) \geq 3$.

Choose a 3-dimensional subspace $U \subset H^0(\mathbb{P}^2, E)$, then by [17] Lemma 1.5] the natural evaluation map $\varphi : U \otimes O_{\mathbb{P}^2} \to E$ is injective with torsion-free quotient $Q = \text{Coker}(\varphi)$. We get the exact sequence

$$0 \to U \otimes O_{\mathbb{P}^2} \to E \to Q \to 0.$$  

By comparing Chern classes we see that we must have $Q \cong I_Z(1)$ for a length three subscheme $Z \subset \mathbb{P}^2$, that is $Z \in \mathbb{P}^{2|3}$. This gives the desired exact sequence.

**Remark 1.6.** shows that there is a close connection between $M_{\mathbb{P}^2}(4,1,3)$ and $\mathbb{P}^{2|3}$. This connection will become clearer in the next sections.
2. Construction of a family

In this section we want to construct a $\mathbb{P}^2$-family of $\mu$-stable locally free sheaves such that every member of this family is classified by $\mathcal{M}_{\mathbb{P}^2}(4,1,3)$. The construction is based on a construction of Mukai, see [15, Section 3].

The starting point of our construction is the observation that

$$\text{ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2}) = h^1(\mathbb{P}^2, I_Z(-2)) = 3$$

for every $Z \in \mathbb{P}^2$.

We define $V := \text{Ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2})$ and observe the isomorphism

$$\text{Ext}^1(I_Z(1), V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}) \cong \text{Ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2}) \otimes V^\vee \cong \text{Hom}(V, V).$$

Hence there is a distinguished extension class $e \in \text{Ext}^1(I_Z(1), V^\vee \otimes \mathcal{O}_{\mathbb{P}^2})$ corresponding to $\text{id}_V \in \text{Hom}(V, V)$, giving rise to:

$$0 \longrightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow E_Z \longrightarrow I_Z(1) \longrightarrow 0. \quad (2)$$

Remark 2.1. The sheaf $E_Z$ is called the universal extension of $I_Z(1)$ by $\mathcal{O}_{\mathbb{P}^2}$. By construction we have $\text{Hom}(E_Z, \mathcal{O}_{\mathbb{P}^2}) = 0$.

We want to study some of the properties of the sheaf $E_Z$. For example we have:

Lemma 2.2. The sheaf $E_Z$ is a locally free sheaf on $\mathbb{P}^2$.

Proof. Tensor the exact sequence (2) with $\omega_{\mathbb{P}^2}$:

$$0 \longrightarrow V^\vee \otimes \omega_{\mathbb{P}^2} \longrightarrow E_Z \otimes \omega_{\mathbb{P}^2} \longrightarrow I_Z(-2) \longrightarrow 0. \quad \square$$

Now for every subscheme $Z' \subseteq Z$ of length $0 \leq d < 3$ we have

$$h^1(\mathbb{P}^2, I_{Z'}(-2)) < h^1(\mathbb{P}^2, I_Z(-2)),$$

which by [20] Lemma 1.2. implies that $E_Z \otimes \omega_{\mathbb{P}^2}$ is locally free, hence so is $E_Z$. \square

We also have the following result concerning the stability of $E_Z$:

Lemma 2.3. The locally free sheaf $E_Z$ is $\mu$-stable.

Proof. This follows from a more general result, see [17] Lemma 1.4.]. But in this situation we can also give a direct proof:

Let $F$ be a torsion free quotient of $E_Z$ with $1 \leq \text{rk}(F) \leq 3$, then there is the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2} & \longrightarrow & E_Z & \longrightarrow & I_Z(1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 & \longrightarrow & F & \longrightarrow & F_1 & \longrightarrow & 0
\end{array}
$$

with $F_0 = \text{Im}(\mathcal{O}_{\mathbb{P}^2} \hookrightarrow E_Z \twoheadrightarrow F)$. Thus all vertical arrows are surjective. Since $F_0$ is a quotient of a free sheaf we have $c_1(F_0).H \geq 0$. Furthermore $\text{rk}(F_1) \in \{0,1\}$ as $F_1$ is a quotient of a torsion free sheaf of rank 1. We distinguish two cases.

Case $\text{rk}(F_1) = 1$:

In this case $F_1 \cong I_Z(1)$ and hence $c_1(F).H = (c_1(F_0) + c_1(F_1)).H \geq 1$. This implies

$$\mu(F) = \frac{c_1(F).H}{\text{rk}(F)} \geq \frac{1}{3} \geq 1 \geq \frac{1}{4} = \mu(E_Z).$$

Case $\text{rk}(F_1) = 0$:

In this case we have $c_1(F_1).H \geq 0$ as $F_1$ is a torsion sheaf. The only critical case is $c_1(F_0) = c_1(F_1) = 0$, since otherwise $c_1(F) = d \geq 1$ and thus $\mu(F) \geq \frac{4}{3} > \frac{1}{3} = \mu(E_Z)$.

So assume $c_1(F_0) = c_1(F_1) = 0$. Then $F_0$ is trivial itself, see for example [14] p. 302], and $F_1$ is supported in finitely many points. This implies

$$\text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}) \cong \mathbb{C}^{\text{rk}(F_0)}.$$
On the other hand $\text{Hom}(F, \mathcal{O}_{\mathbb{P}^2}) \hookrightarrow \text{Hom}(E_Z, \mathcal{O}_{\mathbb{P}^2}) = 0$ by Remark 2.1. This shows $\text{rk}(F_0) = 0$ and hence $\text{rk}(F) = 0$. So for $\text{rk}(F) \geq 1$ the case $c_1(F_0) = c_1(F_1) = 0$ cannot occur and $E_Z$ is stable. \hfill \Box \\

The last two lemmas show:

**Corollary 2.4.** For every $Z \in \mathbb{P}^{[3]}$ the sheaf $E_Z$ defines a point $[E_Z] \in \mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$.

We want to put the $\mu$-stable locally free sheaves $E_Z$ in a family classified by $\mathbb{P}^{[3]}$. To do this we need the following maps:

\[
\begin{array}{ccc}
\mathbb{P}^2 & \xrightarrow{p} & \mathbb{P}^{[3]} \\
\mathbb{P}^2 & \xrightarrow{q} & \mathbb{P}^{[3]} \\
\end{array}
\]

where $\mathcal{Z}$ is the universal family of length 3 subschemes.

**Remark 2.5.** Recall that for any coherent sheaf $F$ on $\mathbb{P}^2$ there is the associated coherent tautological sheaf $F^{[3]}$ on $\mathbb{P}^{[3]}$ defined by

$$F^{[3]} := q_*(p^*F \otimes \mathcal{O}_Z).$$

If $F$ is locally free of rank $r$ then $F^{[3]}$ is locally free of rank $3r$.

To construct the family of stable sheaves, we first put the $\text{Ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2})$ for $Z \in \mathbb{P}^{[3]}$ in a family:

**Lemma 2.6.** The first relative Ext-sheaf $\mathcal{V} := \mathcal{E}xt^1_q(\mathcal{I}_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{[3]}})$ is a locally free sheaf of rank three on $\mathbb{P}^{[3]}$. It commutes with base change and there is an isomorphism

\[
\mathcal{E}xt^1_q(\mathcal{I}_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{[3]}}) \cong \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}.
\]

**Proof.** The morphism $q : \mathbb{P}^{[3]} \to \mathbb{N}$, $Z \mapsto \text{ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2})$ is constant due to (1). So by (3) Satz 3.] the first relative Ext-sheaf is locally free of rank three on $\mathbb{P}^{[3]}$ and commutes with base change, that is for every $Z \in \mathbb{P}^{[3]}$ we have

\[
\mathcal{E}xt^1_q(\mathcal{I}_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{[3]}}) \cong \text{Ext}^1(I_Z(1), \mathcal{O}_{\mathbb{P}^2}).
\]

Using relative Serre duality, see (9) Corollary(24)], gives an isomorphism

\[
\mathcal{E}xt^1_q(\mathcal{I}_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{[3]}}) \cong \mathcal{E}xt^1_q(\mathcal{I}_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2), \omega_q) \\
\cong \text{Hom}(R^1q_*(\mathcal{I}_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2)), \mathcal{O}_{\mathbb{P}^{[3]}}).
\]

The exact sequence

\[
0 \longrightarrow \mathcal{I}_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow p^* \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow 0
\]

and standard cohomology and base change results, see (16) II.5.], show that there is an isomorphism

\[
q_*(\mathcal{O}_{\mathbb{P}^2} \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2)) \cong R^1q_*(\mathcal{I}_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2)).
\]

We see that $R^1q_*(\mathcal{I}_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(-2)) \cong \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}$ is locally free of rank three and thus we get the desired isomorphism (3). \hfill \Box \\

As the main result of this section we can now construct the desired family:

**Theorem 2.7.** There is a locally free $\mathbb{P}^{[3]}$-family $\mathcal{E}$ of $\mu$-stable locally free sheaves, given by the exact sequence

\[
0 \longrightarrow q^*\mathcal{V}^* \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0,
\]

i.e. for every $Z \in \mathbb{P}^{[3]}$ the restriction to the fiber over $Z$ defines a point $[\mathcal{E}_Z] \in \mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$. 

Proof. For every \(Z \in \mathbb{P}^2\) we have \(\text{Hom}(I_Z(1), \mathcal{O}_{\mathbb{P}^2}) = 0\), so
\[
\mathcal{E}xt^0_{q}(I_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2(3)}) = q_* \text{Hom}(I_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2(3)}) = 0.
\]
Using this fact and the projection formula for relative Ext-sheaves \([13, \text{Lemma } 4.1.]\), the five term exact sequence of the spectral sequence
\[
H^i(\mathbb{P}^2[3], \mathcal{E}xt^j_{q}(I_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), q^* \mathcal{V}^\vee)) \Rightarrow \text{Ext}^{i+j}(I_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), q^* \mathcal{V}^\vee)
\]
reduces to an isomorphism
\[
\text{Ext}^1(I_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), q^* \mathcal{V}^\vee) \cong H^0(\mathbb{P}^2[3], \mathcal{E}xt^1_{q}(I_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), q^* \mathcal{V}^\vee))
\]
\[
\cong H^0(\mathbb{P}^2[3], \mathcal{E}xt^1_{q}(I_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2(3)} \otimes \mathcal{V}^\vee))
\]
\[
\cong \text{Hom}(\mathcal{V}, \mathcal{V}).
\]
The identity \(\text{id}_\mathcal{V}\) gives rise to an extension on \(\mathbb{P}^2 \times \mathbb{P}^2[3]:(4)\)
\[
0 \longrightarrow q^* \mathcal{V}^\vee \longrightarrow \mathcal{E} \longrightarrow I_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0
\]
with \(\mathcal{E}\) flat over \(\mathbb{P}^2[3]\), since both other terms are. Restricting to the fiber over a point \(Z \in \mathbb{P}^2[3]\) defines for flatness of \(I_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1)\) a map
\[
\text{Ext}^1(I_Z \otimes p^* \mathcal{O}_{\mathbb{P}^2}(1), q^* \mathcal{V}^\vee) \rightarrow \text{Ext}^1(I_Z(1), \mathcal{V}^\vee \otimes \mathcal{O}_{\mathbb{P}^2}).
\]
By \([13, \text{Lemma } 2.1.]\) the extension defined by \(\text{id}_\mathcal{V}\) restricts to the extension given by \(\text{id}_\mathcal{V}\) on the fiber over \(Z \in \mathbb{P}^2[3]\). Thus the pullback of \((4)\) to the fiber over \(Z \in \mathbb{P}^2[3]\) is exactly the exact sequence \((2)\), hence it defines a locally free sheaf classified by \(\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)\). Using \(\mathbb{S}\) \(\text{Lemma } 1.7.\) again, we see that \(\mathcal{E}\) is itself locally free. \(\square\)

By the universal property of \(\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)\) the family \(\mathcal{E}\) comes with a classifying morphism
\[
f_\mathcal{E} : \mathbb{P}^2[3] \rightarrow \mathcal{M}_{\mathbb{P}^2}(4, 1, 3), \ Z \mapsto [\mathcal{E}_Z].
\]
Furthermore there is \(L \in \text{Pic}(\mathbb{P}^2[3])\) and an isomorphism
\[
(\text{id}_{\mathbb{P}^2} \times f_\mathcal{E})^* \mathcal{U} \otimes q^* L \cong \mathcal{E}.
\]
We need to study some properties of the morphism \(f_\mathcal{E}\). For this we need:

Lemma 2.8. Assume \(Z \in \mathbb{P}^2[3]\) is not collinear. If there is an isomorphism \(\alpha : E_{Z'} \cong E_Z\) for some \(Z' \in \mathbb{P}^2[3]\), then \(Z = Z'\).

Proof. We look at the following diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \\
\downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & E_{Z'} \\
\downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & E_{Z} \\
\end{array}
\]
Since \(Z\) is not collinear the composition \(\beta := q \circ \alpha \circ \iota\) is zero. Consequently the free submodule of \(E_{Z'}\) maps injectively to the free submodule of \(E_Z\), which then must be an isomorphism, so we get in fact the following diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & E_{Z'} \\
\downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & E_{Z} \\
\end{array}
\]
Therefore there is an induced isomorphism \(I_{Z'}(1) \cong I_Z(1)\) and so \(Z = Z'\). \(\square\)

Thus the non-collinear subschemes in \(\mathbb{P}^2[3]\) define sheaves \(E_Z\) with exactly three global sections. It makes sense to study the Brill-Noether-locus \(S\) in \(\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)\):
\[
S := \{ [E] \in \mathcal{M}_{\mathbb{P}^2}(4, 1, 3) \mid h^0(\mathbb{P}^2, E) = 4 \}.
\]
Remark 2.9. We can write down the inverse $g$ to $f_E$ on the complement of $S$:
\[ g : \mathcal{M}_{\mathbb{P}^2}(4, 1, 3) \setminus S \to \mathbb{P}^{2[3]}, \ E \mapsto \text{supp}(Q^\vee /Q) \]
where $Q$ is the cokernel of the (in this case) canonical evaluation map from Lemma 1.3.

By [20] Corollary, p.14, lines 3-5 we get for the $E_Z$ with collinear subschemes $Z$:

Lemma 2.10. Assume $Z, Z' \in \mathbb{P}^{2[3]}$ are collinear with $Z \neq Z'$ such that there is a line $\ell \subset \mathbb{P}^2$ containing both $Z$ and $Z'$, then $E_Z = E_{Z'}$.

Remark 2.11. This shows that for a sheaf $E_Z$ with a collinear subscheme $Z \subset \mathbb{P}^2$ we have
\[ f_E^{-1}([E_Z]) = \ell^{[3]} \cong \mathbb{P}^3, \]
where $\ell \subset \mathbb{P}^2$ is the line containing $Z$.

The last two lemmas suggest that $f_E$ is the blow up of $S$ in $\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$. This is indeed the case since by [21] 5.29, Example 5.3. and we have:

Lemma 2.12. The Brill-Noether-locus $S$ in $\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$ is isomorphic to $\mathbb{P}^2$ and there is an isomorphism $\mathbb{P}^{2[3]} \cong \text{Bl}_S \mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$ such that $f_E$ can be identified with the blow up of $S$ in $\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$.

Remark 2.13. This description goes back to Drezet who proved this in terms of Kronecker modules in [11] Théorème 4.]

Corollary 2.14. For every locally free sheaf $F$ on $\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$ we have isomorphisms
\[ H^i(\mathcal{M}_{\mathbb{P}^2}(4, 1, 3), F) \cong H^i(\mathbb{P}^{2[3]}, f_E^* F). \]

Proof. Since $f_E$ is birational by Lemma 2.12 the result follows from $R^i (f_E)_* \mathcal{O}_{\mathbb{P}^{2[3]}} = 0$ for $i \geq 1$, the projection formula and the Leray spectral sequence. \qed

3. Computations

We want to understand the family $\mathcal{E}$ as a $\mathbb{P}^2$-family, that is we want to understand the sheaves $\mathcal{E}_p$ on $\mathbb{P}^{2[3]}$ for $p \in \mathbb{P}^2$. For this we first note that $Z$ is not just flat over $\mathbb{P}^{2[3]}$ but also over $\mathbb{P}^2$, see [12] Theorem 2.1.], so restricting the exact sequence
\[ 0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2[3]}} \longrightarrow \mathcal{O}_Z \longrightarrow 0 \]
to the fiber over point $p \in \mathbb{P}^2$ gives the exact sequence
\[ 0 \longrightarrow I_{S_p} \longrightarrow \mathcal{O}_{\mathbb{P}^{2[3]}} \longrightarrow \mathcal{O}_{S_p} \longrightarrow 0, \]
where $S_p := \{ Z \in \mathbb{P}^{2[3]} \mid p \in \text{supp}(Z) \}$ is a codimension two subscheme in $\mathbb{P}^{2[3]}$.

Since $Z$ is flat over $\mathbb{P}^2$ we see, using [7] Examples 5.4 vi)], that $\mathcal{O}_{S_p} = k(p)^{[3]}$ is the tautological sheaf on $\mathbb{P}^{2[3]}$ associated to the skyscraper sheaf $k(p)$ of the point $p \in \mathbb{P}^2$. This implies we can use [10] Theorem 3.17.,Remark 3.20. to find the following cohomology groups:

(7) $\text{ext}^i(\mathcal{O}_{\mathbb{P}^{2[3]}}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = 0$ for all $i$ and $\text{ext}^i(\mathcal{O}_{S_p}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = \begin{cases} 1 & i = 2 \\ 0 & i \neq 2 \end{cases}$

as well as

(8) $\text{ext}^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, \mathcal{O}_{\mathbb{P}^{2[3]}}) = \begin{cases} 6 & i = 0 \\ 0 & i \geq 1 \end{cases}$ and $\text{ext}^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, \mathcal{O}_{S_p}) = \begin{cases} 7 & i = 0 \\ 0 & i \geq 1 \end{cases}$.

Using these results we can prove:

Lemma 3.1. For $p \in \mathbb{P}^2$ we have
\[ \text{ext}^i(I_{S_p}, \mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}) = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases} \text{ and } \text{ext}^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{[3]}, I_{S_p}) = \begin{cases} \geq 1 & i = 1 \\ 0 & i \geq 2. \end{cases} \]
induced long exact sequence:

since Ext by further using Ext by (8) the first part of the long exact sequence gives

\[ \text{Theorem 3.2.} \]

Let \( \square \) Again using (8) proves the second claim.

\[ \text{Proof.} \]

If we apply Hom(\( - \), \( O_{P^2}(-2) \)) to (9) we have Ext\(^6\)(\( I_{S_p}, O_{P^2}(-2) \)) = 0 as well as isomorphisms:

\[ \text{Ext}^i(\( I_{S_p}, O_{P^2}(-2) \)) \cong \text{Ext}^{i+1}(\( O_{S_p}, O_{P^2}(-2) \)) \]

for \( 1 \leq i \leq 5 \), since \( \text{Ext}^i(\( O_{P^2}, O_{P^2}(-2) \)) = 0 \) for all \( i \) by (7). We also find hom(\( I_{S_p}, O_{P^2}(-2) \)) = 0 by further using Ext\(^i\)(\( O_{S_p}, O_{P^2}(-2) \)) = 0 for \( i = 0, 1 \). This proves the first claim.

For the second claim we apply Hom(\( O_{P^2}(-2) \), -) to (9). As Ext\(^1\)(\( O_{P^2}(-2), O_{P^2} \)) = 0 by (8) the first part of the long exact sequence gives

\[ 0 \longrightarrow \text{Hom}(\( O_{P^2}(-2), I_{S_p} \)) \longrightarrow C^6 \longrightarrow C^7 \longrightarrow \text{Ext}^1(\( O_{P^2}(-2), I_{S_p} \)) \longrightarrow 0 \]

which shows that \( \text{Ext}^1(\( O_{P^2}(-2), I_{S_p} \)) \geq 1 \). We also get isomorphisms

\[ \text{Ext}^i(\( O_{P^2}(-2), I_{S_p} \)) \cong \text{Ext}^{i-1}(\( O_{P^2}(-2), O_{S_p} \)) \]

for \( 2 \leq i \leq 6 \).

Again using (8) proves the second claim. \( \square \)

To study the locally free sheaves \( \mathcal{E}_p \) on \( \mathbb{P}^2 \) we note that the exact sequence

\[ 0 \longrightarrow q^* \mathcal{V}^\vee \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_z \otimes p^* \mathcal{O}_{P^2}(1) \longrightarrow 0 \]

restricts to the fiber over \( p \in \mathbb{P}^2 \) as

\[ (9) \]

\[ 0 \longrightarrow \mathcal{O}_{P^2}(-2) \longrightarrow \mathcal{E}_p \longrightarrow I_{S_p} \longrightarrow 0 \]

by using flatness of \( \mathcal{I}_z \) over \( \mathbb{P}^2 \) and Lemma 2.10. We can now prove:

\[ \text{Theorem 3.2.} \]

Let \( \mathcal{E} \) be the \( \mathbb{P}^2 \)-family of \( \mu \)-stable locally free sheaves, then for any pair of closed points \( p, q \in \mathbb{P}^2 \) with \( p \neq q \) we have

\[ \text{ext}^1(\( \mathcal{E}_p, \mathcal{E}_q \)) \geq 1. \]

\[ \text{Proof.} \]

By [11] Theorem 1.2 the Fourier-Mukai transform

\[ \Phi_{\mathcal{I}_z} : D^b(\mathbb{P}^2) \rightarrow D^b(\mathbb{P}^2) \]

is fully faithful, that is for \( p, q \in \mathbb{P}^2 \) with \( p \neq q \) we have by flatness of \( \mathcal{I}_z \) over \( \mathbb{P}^2 \):

\[ \text{Ext}^i(\( I_{S_p}, \mathcal{E}_q \)) \cong \text{Ext}^i(\( k(p), k(q) \)) = 0 \]

for \( 0 \leq i \leq 6 \).

So applying hom(\( I_{S_q}, - \)) with \( q \neq p \) to (9) gives isomorphisms

\[ \text{Ext}^i(\( I_{S_q}, \mathcal{E}_p \)) \cong \text{Ext}^i(\( I_{S_q}, \mathcal{O}_{P^2}(-2) \)) \]

for \( 0 \leq i \leq 6 \).

If we apply hom(\( \mathcal{O}_{P^2}(-2) \), -) and use [10] Theorem 3.17] again to see

\[ \text{Ext}^i(\( \mathcal{O}_{P^2}(-2), \mathcal{O}_{P^2}(-2) \)) = \begin{cases} 1 & i = 0 \\ 0 & i \geq 1 \end{cases} \]

we get an exact sequence

\[ 0 \longrightarrow C \longrightarrow \text{Hom}(\( \mathcal{O}_{P^2}(-2), \mathcal{E}_p \)) \longrightarrow \text{Hom}(\( \mathcal{O}_{P^2}(-2), I_{S_p} \)) \longrightarrow 0 \]

and isomorphisms

\[ \text{Ext}^i(\( \mathcal{O}_{P^2}(-2), \mathcal{E}_p \)) \cong \text{Ext}^i(\( \mathcal{O}_{P^2}(-2), I_{S_p} \)) \]

for \( 1 \leq i \leq 6 \).

Finally applying hom(\( - , \mathcal{E}_q \)) with \( q \neq p \) to (9) we get the following relevant part of the induced long exact sequence:

\[ \longrightarrow \text{Ext}^1(\( \mathcal{E}_p, \mathcal{E}_q \)) \longrightarrow \text{Ext}^1(\( \mathcal{O}_{P^2}(-2), \mathcal{E}_q \)) \longrightarrow \text{Ext}^2(\( I_{S_p}, \mathcal{E}_q \)) \]

With the previous results this sequence gets:

\[ \longrightarrow \text{Ext}^1(\( \mathcal{E}_p, \mathcal{E}_q \)) \longrightarrow \text{Ext}^1(\( \mathcal{O}_{P^2}(-2), I_{S_q} \)) \longrightarrow \text{Ext}^2(\( I_{S_p}, \mathcal{O}_{P^2}(-2) \)) \]

Using Lemma 3.1 we have Ext\(^2\)(\( I_{S_p}, \mathcal{O}_{P^2}(-2) \)) = 0 and thus

\[ \text{ext}^1(\( \mathcal{E}_p, \mathcal{E}_q \)) \geq \text{ext}^1(\( \mathcal{O}_{P^2}(-2), I_{S_q} \)) \geq 1. \]
4. Non-full faithfulness of the universal family

We want to study the full faithfulness of the Fourier-Mukai transform

$$\Phi_U : \text{D}^b(\mathbb{P}^2) \to \text{D}^b(\mathcal{M}_{\mathbb{P}^2}(4, 1, 3))$$

induced by the universal family $U$ of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$.

We will use the following corollary of the Bondal-Orlov criterion for full faithfulness:

**Lemma 4.1.** ([7, Corollary 7.5]) Let $X$ and $Y$ be two smooth projective varieties and $\mathcal{P}$ a coherent sheaf on $X \times Y$, flat over $X$. Then the Fourier-Mukai transform

$$\Phi_\mathcal{P} : \text{D}^b(X) \to \text{D}^b(Y)$$

is fully faithful if and only if the following two conditions are satisfied

1) For any closed point $x \in X$ one has $\text{Ext}^i(\mathcal{P}_x, \mathcal{P}_x) = \begin{cases} C & i = 0 \\ 0 & i > \dim(X) \end{cases}$

2) For any pair of closed points $x, y \in X$ with $x \neq y$ one has $\text{Ext}^i(\mathcal{P}_x, \mathcal{P}_y) = 0$ for all $i$.

To apply this lemma to $\mathcal{P} = U$, the universal family of $\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$, we need to be able to compute $\text{Ext}^i(U_p, U_q)$. The following lemma reduces this problem to computing $\text{Ext}^i(\mathcal{E}_p, \mathcal{E}_q)$:

**Lemma 4.2.** Let $U$ be the universal family of $\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$ and $\mathcal{E}$ be the $\mathbb{P}^2[3]$-family, then for any two points $p, q \in \mathbb{P}^2$ there are the following isomorphisms for all $i$:

$$\text{Ext}^i(U_p, U_q) \cong \text{Ext}^i(\mathcal{E}_p, \mathcal{E}_q).$$

**Proof.** We have the following chain of isomorphisms:

$$\text{Ext}^i(U_p, U_q) \cong H^i(M, \text{Hom}(U_p, U_q))$$

$$\cong H^i(\mathbb{P}^2[3], f_*^* \text{Hom}(U_p, U_q))$$

$$\cong H^i(\mathbb{P}^2[3], \text{Hom}(f_*^* U_p, f_*^* U_q))$$

$$\cong \text{Ext}^i(f_*^* U_p, f_*^* U_q)$$

$$\cong \text{Ext}^i(f_*^* U_p \otimes L, f_*^* U_q \otimes L)$$

$$\cong \text{Ext}^i(\mathcal{E}_p, \mathcal{E}_q)$$

Here the first and third isomorphism use the locally freeness of $U_p$, see Remark [1.3]. The second isomorphism is Corollary [2.13]. The fourth isomorphism uses locally freeness of $f_*^* U_p$, while the sixth isomorphism follows from restricting [5] to the fiber over $p \in \mathbb{P}^2$. □

We can now prove the main theorem of this note:

**Theorem 4.3.** The Fourier-Mukai transform

$$\Phi_U : \text{D}^b(\mathbb{P}^2) \to \text{D}^b(\mathcal{M}_{\mathbb{P}^2}(4, 1, 3))$$

induced by the universal family $U$ of the moduli space $\mathcal{M}_{\mathbb{P}^2}(4, 1, 3)$ is not fully faithful.

**Proof.** For the Fourier-Mukai transform $\Phi_U$ to be fully faithful one needs

$$\text{Ext}^i(U_p, U_q) = 0$$

for any pair of points $p, q \in \mathbb{P}^2$ with $p \neq q$ and any $i$ according to Lemma [4.1].

Lemma [4.2] shows that (10) is equivalent to

$$\text{Ext}^i(\mathcal{E}_p, \mathcal{E}_q) = 0.$$ 

But we have $\text{Ext}^i(\mathcal{E}_p, \mathcal{E}_q) \neq 0$ by Lemma [5.2], so $\Phi_U$ cannot be fully faithful. □
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