On robust representation of conditional risk measures on a $L^\infty$-type module

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Abstract

The purpose of this paper is to establish a robust representation theorem for conditional risk measures by using a module-based convex analysis, where risk measures are defined on a $L^\infty$-type module. We define and study a Fatou property for this kind of risk measures, which is a generalization of the already known Fatou property for static risk measures. In order to prove this robust representation theorem we provide a modular version of Krein-Smulian theorem.

Keywords: representation; $L^0$-modules; Fatou property; $L^\infty$-type module; Krein-Smulian theorem.

Introduction

The study of risk measures was initiated by Artzner et al. [1], by defining and studying the concept of coherent risk measure. Föllmer and Schied [8] and, independently, Frittelli and Gianin [10] introduced later the more general concept of convex risk measure. This kind of risk measures are defined into a static setting, in which only two instants of time matter, today 0 and tomorrow $T$, and the analytic framework used is the classical convex analysis, which perfectly applies in this simple model cf. [4, 9].

However, when it is addressed a multiperiod setting, in which intermediate times $0 < t < T$ are considered, it appears to become quite delicate if we try to apply convex analysis, as Filipovic et al. [7] explain. This can be verified in works such as [2, 5].

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In order to overcome these difficulties Filipovic et al. [7] propose to consider a modular framework, where scalars are random variables instead of real numbers. Namely, they consider modules over $L^0(\Omega, \mathcal{F}, \mathbb{P})$ the ordered ring of (equivalence classes of) random variables. For this purpose, they establish the concept of locally $L^0$-convex module and prove randomized versions of some important theorems from convex analysis.

In this regard, it is noteworthy the extensive research done by T. Guo, who has widely researched the theorems of functional analysis under the structure of the $L^0$-modules cf. [11, 12, 13].

Likewise, the performance of this tool has been verified in works such as [6, 14]. These risk measures are defined on natural modular extensions of the classical spaces $L^p$ of measurable functions, namely, the so-called $L^p$-type modules.

The purpose of this article is to establish a robust representation theorem for conditional risk measures defined on the $L^\infty$-type module analogue to theorem 4.31 of [9] for measures on $L^\infty$. A key result to prove this theorem is the Krein-Smulian theorem, which states that in a locally convex space $E$, a convex subset $K$ of the topological conjugate $E^\ast$ is weak-* closed if, and only if, $K \cap rB^\ast$ is weak-* closed for all $r > 0$. We prove a randomized version of this theorem for modules $L^0$-normed. Guo proved [12] it is not possible to establish a general Alaoglu theorem for $L^0$-normed modules. Therefore, we provide a proof based only on completeness instead of the Alaoglu theorem.

The article is structured as follow. We give a first section of preliminaries, in which we give a review of all relevant notions within the locally $L^0$-convex analysis. The second section is devoted to provide the randomized version of Krein-Smulian. Finally, in the third section we study conditional risk measures on the $L^\infty$-type module and provide a robust representation theorem for this kind of risk measures.

1 Preliminaries and review of locally $L^0$-convex modules

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let us consider $L^0(\Omega, \mathcal{F}, \mathbb{P})$, or simply denoted by $L^0$, the set of equivalence classes of real valued $\mathcal{F}$-measurable random variables. It is known that the triple $(L^0, +, \cdot)$ endowed with the partial order of the almost sure dominance is a lattice ordered ring. We shall say “$X \geq Y$” if $\mathbb{P}(X \geq Y) = 1$, and likewise, we shall say “$X > Y$”, if $\mathbb{P}(X > Y) = 1$. And, given $A \in \mathcal{F}$, we shall say that $X > Y$ (respectively, $X \geq Y$) on $A$, if
\( \mathbb{P}(X > Y \mid A) = 1 \) (respectively, if \( \mathbb{P}(X \geq Y \mid A) = 1 \)).

We also define \( L_+^0 := \{ Y \in L^0; Y \geq 0 \} \), \( L_+^0 := \{ Y \in L^0; Y > 0 \} \) and \( \mathcal{F}^+ := \{ A \in \mathcal{F}; \mathbb{P}(A) > 0 \} \). And we shall denote by \( \tilde{L}^0 \), the set of equivalence classes of \( \mathcal{F} \)-measurable random variables taking values in \( \mathbb{R} = \mathbb{R} \cup \{ \pm \infty \} \), and the partial order of the almost sure dominance is extended to \( \tilde{L}^0 \) in a natural way.

Furthermore, given a subset \( \phi \subset L^0 \) then \( \phi \) owns both an infimum and a supremum in \( L^0 \) for the order of the almost sure dominance that will be denoted by \( \text{ess.inf} \phi \) and \( \text{ess.sup} \phi \), respectively (see A.5 of [9]).

The order of the almost sure dominance also enables us to define a topology. We define \( B_\varepsilon := \{ Y \in L^0; |Y| \leq \varepsilon \} \) the ball of radius \( \varepsilon \in L_+^0 \) centered at \( 0 \in L^0 \). Then, for all \( Y \in L^0 \), \( \mathcal{U}_Y := \{ Y + B_\varepsilon; \varepsilon \in L_+^0 \} \) is a neighborhood base of \( Y \). Thus it can be defined a topology on \( L^0 \).

In what follows we will expose more concepts that will be important throughout this paper.

**Definition 1.1.** If \( E \) is a \( L^0 \)-module and \( K \subset E \), and denote by
\[
\Pi(\Omega, \mathcal{F}) := \left\{ \{A_n\}_n \subset \mathcal{F}; \bigcup_{n \in \mathbb{N}} A_n = \Omega, A_i \cap A_j = \emptyset \text{ for all } i \neq j \right\}
\]
the set of countable partitions on \( \Omega \) to \( \mathcal{F} \). Note that we allow \( A_i = \emptyset \) for some \( i \in \mathbb{N} \).

1. Then, for a sequence \( \{X_n\}_n \subset E \) and a partition \( \{A_n\}_n \in \Pi(\Omega, \mathcal{F}) \), we define the set of countable concatenations of \( \{X_n\}_n \) and \( \{A_n\}_n \) as \( cc(\{A_n\}_n, \{X_n\}_n) := \{ X \in E; 1_{A_n}X_n = 1_{A_n}X \text{ for each } n \in \mathbb{N} \} \).

2. We say that \( K \) has the countable concatenation property, if for each sequence \( \{X_n\}_n \subset K \) and each partition \( \{A_n\}_n \in \Pi(\Omega, \mathcal{F}) \) it holds
\[
cc(\{A_n\}_n, \{X_n\}_n) \text{ is nonempty, and } cc(\{A_n\}_n, \{X_n\}_n) \subset K.
\]

3. We say that \( K \) is stable under countable concatenations, if for each sequence \( \{X_n\}_n \subset K \) and each partition \( \{A_n\}_n \in \Pi(\Omega, \mathcal{F}) \) it simply holds
\[
cc(\{A_n\}_n, \{X_n\}_n) \subset K.
\]

4. We call the countable concatenation closure of \( K \) the set defined below
\[
\overline{K}^{cc} := \bigcup cc(\{A_n\}_n, \{X_n\}_n)
\]
where \( \{ A_n \}_n \) runs through \( \Pi(\Omega, \mathcal{F}) \) and \( \{ X_n \}_n \) runs though the sequences in \( K \). Or in another way written

\[
\overline{K}^{cc} = \{ X \in E; \exists \{ A_n \}_n \in \Pi(\Omega, \mathcal{F}) \text{ with } 1_{A_n} X \in 1_{A_n} K \text{ for all } n \}
\]

Note that \( K \) is stable under countable concatenations if, and only if, \( \overline{K}^{cc} = K \).

**Definition 1.2.** A topological \( L^0 \)-module \( E[\tau] \) is a \( L^0 \)-module \( E \) endowed with a topology \( \tau \) such that

1. \( E[\tau] \times E[\tau] \rightarrow E[\tau], (X, X') \mapsto X + X' \) and
2. \( L^0[\cdot]\times E[\tau] \rightarrow E[\tau], (Y, X) \mapsto YX \)

are continuous with the corresponding product topologies.

**Definition 1.3.** A topology \( \tau \) on a \( L^0 \)-module \( E \) is a locally \( L^0 \)-convex module if there is a neighborhood base \( U \) of \( 0 \in E \) such that each \( U \in U \)

1. \( L^0 \)-convex, i.e. \( YX_1 + (1 - Y)X_2 \in U \) for all \( X_1, X_2 \in U \) and \( Y \in L^0 \) with \( 0 \leq Y \leq 1 \),
2. \( L^0 \)-absorbent, i.e. for all \( X \in E \) there is a \( Y \in L^0 \oplus \) such that \( X \in YU \),
3. \( L^0 \)-balanced, i.e. \( YX \in U \) for all \( X \in U \) and \( Y \in L^0 \) with \( |Y| \leq 1 \),
4. \( U \) is stable under countable concatenations (i.e., \( U^{cc} = U \)).

In this case, we say that \( E[\tau] \) is a locally \( L^0 \)-convex module.

An important remark is that \( \tau \) is a locally \( L^0 \)-convex topology, then \( \tau \) will be Hausdorff if, and only if, \( \bigcap_{U \in U} U = \{0\} \).

In that case, \( X, Z \in cc(\{A_n\}_n, \{X_n\}_n) \) implies \( X = Z \).

Indeed, it holds that \( X - Z \in cc(\{A_n\}_n, 0) \subset U^{cc} = U \) for all \( U \in U \), so that \( X = Z \).

Thus if \((E, \tau)\) is a Hausdorff locally \( L^0 \)-convex module, provided that \( cc(\{A_n\}_n, \{X_n\}_n) \) is nonempty it can be defined

\[
\sum_{n \in \mathbb{N}} 1_{A_n} X_n \in E
\]

the unique element belonging to \( cc(\{A_n\}_n, \{X_n\}_n) \).

**Definition 1.4.** A function \( ||\cdot|| : E \rightarrow L^0_+ \) is a \( L^0 \)-seminorm on \( E \) if:
1. \( \|YX\| = |Y| \|X\| \) for all \( Y \in L^0 \) and \( X \in E \).

2. \( \|X_1 + X_2\| \leq \|X_1\| + \|X_2\| \), for all \( X_1, X_2 \in E \).

If, moreover

3. \( \|X\| = 0 \) implies \( X = 0 \),

then \( \|\cdot\| \) is a \( L^0 \)-norm on \( E \).

**Definition 1.5.** Let \( P \) be a family of \( L^0 \)-seminorms on a \( L^0 \)-module \( E \). Given \( Q \subset P \) finite and \( \varepsilon \in L^0_{++} \), we define

\[
U_{Q,\varepsilon} := \left\{ X \in E; \sup_{\|\cdot\| \in Q} \|X\| \leq \varepsilon \right\}.
\]

Then for all \( X \in E \), \( U_{Q,X} := \{ X + \varepsilon; \varepsilon \in L^0_{++}, \ Q \subset P \) finite \} is a neighbourhood base of \( X \). Thereby, we define a topology on \( E \), which will be known as the topology induced by \( P \) and \( E \) endowed with this topology will be denoted by \( E[P] \).

It is important to note that according to the manuscript [17], a topological \( L^0 \)-module \( E[\tau] \) is a locally \( L^0 \)-convex module if, and only if, \( \tau \) is induced by a family of \( L^0 \)-seminorms.

**Definition 1.6.** Given a topological \( L^0 \)-module \( E[\tau] \), we denote by \( E[\tau]^* \), or simply by \( E^* \), the \( L^0 \)-module of continuous \( L^0 \)-linear functions \( \mu : E \to L^0 \).

We define

\[
\langle \cdot, \cdot \rangle : E \times E^* \to L^0
\]

\[
(X, X^*) := X^*(X)
\]

For each \( X^* \in E^* \) it holds that \( p_{X^*} : E \to L^0_{++}, p_{X^*}(X) := |\langle X, X^* \rangle| \) is a \( L^0 \)-seminorm.

Now, consider the topology \( \sigma(E, E^*) \) induced by the family of \( L^0 \)-seminorms \( \{p_{X^*}; X^* \in E^*\} \). Then, \( \sigma(E, E^*) \) is a locally \( L^0 \)-convex topology on \( E \), which is called the weak topology of \( E \).

Likewise, for each \( X \in E \) it holds that \( p_X : E^* \to L^0_{++}, p_X(X^*) := |\langle X, X^* \rangle| \) is a \( L^0 \)-seminorm. And we have the \( L^0 \)-convex topology \( \sigma(E^*, E) \) induced by the family of \( L^0 \)-seminorms \( \{p_X; X \in E\} \), which is called the weak-* topology of \( E \).

**Definition 1.7.** Given \( A \subset E \), we define the \( L^0 \)-convex hull

\[
co_{L^0}(A) := \left\{ \sum_{i \in I} Y_i X_i; \ I \) finite, \( X_i \in A, Y_i \in L^0_{++}, \sum_{i \in I} Y_i = 1 \right\}
\]

and we call the polar of \( A \) the subset \( A^0 \) of \( E^* \) given by

\[
A^0 := \{ Z \in E^*; |\langle X, Z \rangle| \leq 1, \text{ for all } X \in A \}.
\]
Proposition 1.1. Let $E[τ]$ be a locally $L^0$-convex module, $D, D_i \subset E$ for $i \in I$; then:

1. $D^o$ is $L^0$-convex, weak-$*$ closed and stable under countable concatenation.
2. $0 \in D^o$, $D \subset D^{oo}$. If $D_1 \subset D_2$, then $D_2^o \subset D_1^o$.
3. For $ε \in L^0_{++}$, we have that $(εD)^o = \frac{1}{ε}D^o$.
4. $(\bigcup_{i \in I} D_i)^o = \bigcap_{i \in I} D_i^o$

Proof. Let us see that $D^o$ is stable under countable concatenations.

Given $Z \in \text{cc}(\{A_n\}_n, \{Z_n\}_n)$ with $Z_n \in D^o$, it follows that for $X \in D$

$$\langle X, Z \rangle = \left(\sum_{n \in N} 1_{A_n}\right) \langle X, Z \rangle = \sum_{n \in N} 1_{A_n} \langle X, \sum_{n \in N} 1_{A_n}Z \rangle = \sum_{n \in N} 1_{A_n} \langle X, Z_n \rangle = \sum_{n \in N} 1_{A_n} \langle X, Z_n \rangle \leq 1.$$

The rest is analogue to the known proofs for locally convex spaces.

We also have a bipolar theorem for $L^0$-modules. The proof of the following result can be found in 3.26 [13].

Theorem 1.1. Let $E[τ]$ be a locally $L^0$-convex module with the countable concatenation property, $D \subset E$ then:

$$D^{oo} = \overline{coL^0cc}(D \cup \{0\})$$

Let us now turn to duality notions and results of locally $L^0$-convex modules. Given a function $f : E \rightarrow \bar{L}^0$, we define the Fenchel conjugate of $f$ as

$$f^* : E^* \rightarrow \bar{L}^0$$

$$f^*(\mu) := \text{ess. sup } (\mu(X) - f(X))$$

Likewise, we define

$$f^{**} : E \rightarrow \bar{L}^0$$

$$f^{**}(X) := \text{ess. sup } (\mu(X) - f^*(\mu))$$

Definition 1.8. Let $E[τ]$ a topological $L^0$-module. A function $f : E \rightarrow \bar{L}^0$ is called proper if $f(E) \cap L^0 \neq \emptyset$ and $f > -\infty$.

$f$ is called lower semicontinuous if the level set $V(Y_0) = \{X \in E; f(X) \leq Y_0\}$ is closed.
We have the following Fenchel-Moreau type theorem for $L^0$-modules. A version of this theorem was proved for first time in [7] but we show below the later version appeared in [13]

**Theorem 1.2.** [Fenchel-Moreau theorem] Given $E[\tau]$ a locally $L^0$-convex module with the countable concatenation property. If $f : E \to \bar{L}^0$ is a proper lower semicontinuous $L^0$-convex function. Then,

$$f = f^{**}$$

Finally, we give a short review of duality in $L^0$-normed modules.

Given a $L^0$-normed module $(E, \| \cdot \|)$ it is fulfilled that for every $\mu \in E^*$ there exists $\xi \in L^0_+$ such that $|\mu(X)| \leq \xi \|X\|$ for all $X \in E$. Then we can define

$$\|\mu\|^* := \text{ess. inf} \{ \xi \in L^0_+; |\mu(X)| \leq \xi \|X\| \text{ for all } X \in E \} = \text{ess. sup} \{ |\mu(X)|; \|X\| \leq 1 \text{ for all } X \in E \} = \text{ess. sup} \{ |\mu(X)|; \|X\| = 1_A \text{ for all } A \in \mathcal{F}^+ \}.$$

Then $(E^*, \| \cdot \|^*)$ is a $L^0$-normed module.

The following proposition is proved in the manuscript [16].

**Proposition 1.2.** Given $(E, \| \cdot \|)$ a $L^0$-normed module and $K \subset E$ (resp. $K \subset E^*$) a $L^0$-convex subset which is stable under countable concatenations, we have that the closure in $L^0$-norm coincides with the closure in the weak (resp. weak-*) topology, i. e.

$$\overline{K} = \overline{K}^{\sigma(E,E^*)} \quad \text{(resp. } \overline{K} = \overline{K}^{\sigma(E^*,E)} \text{)}).$$

Then, from now on, for any $K$ $L^0$-convex and stable under countable concatenations subset we will denote the topological closure by $\overline{K}$ without specifying whether the topology is weak or strong.

## 2 A random version of Krein-Smulian theorem

We first introduce some necessary notions that will be used in the proof of the main result of this section.

Let $(E, \| \cdot \|)$ be a $L^0$-normed module. Given $\varepsilon \in L^0_{++}$ define

$$W_\varepsilon := \{(A, B); A, B \subset E, \|X - Z\| \leq \varepsilon \text{ for all } X \in A, Z \in B \}.$$

Then $\mathcal{W} := \{W_\varepsilon; \varepsilon \in L^0_{++}\}$ is the base of an uniformity.

Let $\mathcal{G}$ be the set of all nonempty closed subsets of $E$. 
A net \( \{A_\gamma\}_{\gamma \in \Gamma} \) in \( G \) is convergent to \( A \in G \) if for each \( \varepsilon \in L^0 \) there is some \( \gamma_\varepsilon \) with \((A_\gamma, A) \in W_\varepsilon\) for all \( \gamma \geq \gamma_\varepsilon \). Note that \( A \) is unique.

We say \( \{A_\gamma\}_{\gamma \in \Gamma} \) is Cauchy if for each \( \varepsilon \in L^0 \) there is some \( \gamma_\varepsilon \) with \((A_\alpha, A_\beta) \in W_\varepsilon\) for all \( \alpha, \beta \geq \gamma_\varepsilon \).

**Proposition 2.1.** If \( (E, \| \cdot \|) \) is complete then every Cauchy net \( \{A_\gamma\}_{\gamma \in \Gamma} \) in \( G \) converges.

Furthermore, if \( \{A_\gamma\}_{\gamma \in \Gamma} \) is decreasing then it converges to \( A := \bigcap_{\gamma \in \Gamma} A_\gamma \).

**Proof.** If \( \{A_\gamma\}_{\gamma \in \Gamma} \) is Cauchy we have that for every \( \varepsilon \in L^0 \) there exists \( \gamma_\varepsilon \) such that

\[
\|X - Z\| \leq \varepsilon \text{ for all } X \in A_\alpha, Z \in A_\beta \text{ with } \alpha, \beta \geq \gamma_\varepsilon \tag{1}
\]

In particular, every net \( \{X_\gamma\}_{\gamma \in \Gamma} \) with \( X_\gamma \in A_\gamma \) is of Cauchy and therefore converges as \( E \) is complete.

Define \( A := \{\lim X_\gamma : X_\gamma \in A_\gamma\} \). Let us show that \( \{A_\gamma\}_{\gamma \in \Gamma} \) converges to \( A \).

Indeed, given \( X \in A \) there exists a net \( \{X_\gamma\}_{\gamma \in \Gamma} \) with \( X_\gamma \in A_\gamma \) which converges to \( X \). Then by (1) it holds that

\[
\|X_\gamma - Z\| \leq \varepsilon \text{ for all } Z \in A_\alpha, \text{ with } \alpha, \gamma \geq \gamma_\varepsilon.
\]

And by taking limits in \( \gamma \)

\[
\|X - Z\| \leq \varepsilon \text{ for all } Z \in A_\alpha, \text{ with } \alpha \geq \gamma_\varepsilon.
\]

Since \( X \in A \) is arbitrary \( (\overline{X}, A_\alpha) \in W_\varepsilon \) for all \( \alpha \geq \gamma_\varepsilon \). 

Finally, we will prove the Krein-Smulian theorem for \( L^0 \)-normed modules.

The classical Krein-Smulian theorem is sometime proved through the Alaoglu theorem. Nevertheless, Guo proved in [12] that the Alaoglu theorem does not hold in \( L^0 \)-normed modules, specifically, he showed that, for an arbitrary \( L^0 \)-module, the ball \( B^* \) is weak-\( * \) closed if, and only if, the probabilistic space has a concrete structure of atoms, namely being essentially purely \( \mu \)-atomic. For this reason, we provide a proof that rely only on completeness, which is inspired in the characterization of the so-known hypercomplete spaces given in 18G. chapter 5 of [15].

**Theorem 2.1.** [Randomized version of the Krein-Smulian theorem] If \( (E, \| \cdot \|) \) is a complete \( L^0 \)-normed module with the countable concatenation property and \( K \) is a \( L^0 \)-convex and stable under countable concatenations subset of \( E^* \), then the following statements are equivalent.
1. $K$ is weak-$\ast$ closed.

2. $K \cap \{Z \in E^\ast; \|Z\| \leq \varepsilon\}$ is weak-$\ast$ closed for each $\varepsilon \in L_{0+}^0$.

**Proof.** 1 $\Rightarrow$ 2: It is clear because $\{Z \in E^\ast; \|Z\| \leq \varepsilon\} = B_\varepsilon$ is weak-$\ast$ closed.

2 $\Rightarrow$ 1:

For each $\varepsilon \in L_{0+}^0$, we have that $B_\varepsilon \cap K$ is weak-$\ast$ closed. Then, the net $\{(B_\varepsilon \cap K)^o\}_{\varepsilon \in L_{0+}^0}$ is Cauchy. Indeed, for $\delta, \delta' \leq \varepsilon/2$, by using the properties of the polar and the bipolar theorem

$$(B_\delta \cap K)^o + B_\varepsilon \supset (B_\delta \cap K)^o + B_{\varepsilon/2} = (B_\delta \cap K)^o + B_{\varepsilon/2}^o \supset$$

$$\supset ((B_\delta \cap K)^oo \cap B_{\varepsilon/2}^o) \supset (B_\delta \cap K)^o.$$  

Since the net is decreasing it converges to $C := \bigcap_{\varepsilon \in L_{0+}^0} (B_\varepsilon \cap K)^o$. Then $K = C^o$.

Indeed, $C \subset (B_\varepsilon \cap K)^o$ and then $B_\varepsilon \cap K \subset C^o$ for all $\varepsilon \in L_{0+}^0$, and therefore $K \subset C^o$.

Let $\varepsilon, r \in L_{0+}^0$ be with $r > 1$. Since the net converges to $C$, there is $\delta \in L_{0+}^0$ with

$$(B_\delta \cap K)^o \subset C + (r - 1)B_\varepsilon \subset (C \cup B_\varepsilon)^oo + (r - 1)(C \cup B_\varepsilon)^oo = r(C \cup B_\varepsilon)^oo$$

and by taking polars, it follows that

$$[r(C \cup B_\varepsilon)^oo]^o = \frac{1}{r}(C \cup B_\varepsilon)^o \subset (B_\delta \cap K)^oo = B_\delta^o \cap K$$

and thus $C^o \cap B_\delta^o \subset r(B_\delta^o \cap K)$ for all $r \in L_{0+}^0, r > 1$. Therefore

$$B_\varepsilon^o \cap C^o \subset \bigcap_{r \in L_{0+}^0, r < 1} r(B_\delta^o \cap K) \subset B_\delta^o \cap K \subset K.$$

$\Box$

## 3 Robust representation of conditional risk measures on $L_\mathcal{F}^\infty(\mathcal{E})$

Firstly, we will recall the $L^p$ type modules, which are introduced in [7]

Given two $\sigma$-algebras $\mathcal{F} \subset \mathcal{E}$ and $p \in [1, +\infty]$, we define the application

$$\|\cdot | \mathcal{F}\|_p : L^0(\mathcal{E}) \to \bar{L}^0_+(\mathcal{F})$$

$$\|X | \mathcal{F}\|_p := \begin{cases} E[|X|^p | \mathcal{F}]^{1/p} & \text{if } p < \infty \\ \text{ess. inf } \{Y \in \bar{L}^0(\mathcal{F}) | Y \geq |X|\} & \text{if } p = \infty \end{cases}$$
and the set
\[ L^p_F(\mathcal{E}) := \left\{ X \in L^0(\mathcal{E}) ; \| X \|_p \in L^0(\mathcal{F}) \right\} \]

Then, \((L^p_F(\mathcal{E}), \| \cdot \|_p)\) is a \(L^0\)-normed module with the countable concatenation property and complete in the sense that every Cauchy net converges.

Moreover, the following relation holds
\[ L^p_F(\mathcal{E}) = L^0(\mathcal{F}) L^p(\mathcal{E}) = L^p(\mathcal{E})^{cc}. \]

For a proof of the last equality see proposition 3.3 [3].

And we also have the following duality result (see theorem 4.5 of [11]):

**Proposition 3.1.** Let \(1 \leq p < +\infty\) and \(1 < q \leq +\infty\) with \(1/p + 1/q = 1\). Then the application \(T : L^p_F(\mathcal{E}) \rightarrow L^p_F(\mathcal{E})^* \), \(Z \rightarrow T_Z\) defined by
\[
T_Z : L^p_F(\mathcal{E}) \rightarrow L^0(\mathcal{F}) \\
X \mapsto E[XZ \mid \mathcal{F}]
\]
is an isometric isomorphism of \(L^0\)-normed modules.

**Definition 3.1.** Given \(E\) a \(L^0\)-module, an application \(\rho : E \rightarrow L^0\) is called a conditional risk measure if \(\rho\) is:

1. monotone, i.e. if \(X \leq Z\) then \(\rho(X) \geq \rho(Z)\)
2. cash invariant, i.e. if \(Y \in L^0(\mathcal{F})\), then \(\rho(X + Y) = \rho(X) - Y\)
3. \(L^0(\mathcal{F})\)-convex, i.e. \(\rho(YX + (1 - Y)Z) \leq Y\rho(X) + (1 - Y)\rho(Z)\) for all \(Y \in L^0(\mathcal{F})\) with \(0 \leq Y \leq 1\).

And we define
\[ A_\rho := \{ X \in E ; \rho(X) \leq 0 \} \]
the acceptance set.

**Definition 3.2.**

Conditional risk measures that are defined on \(L^p\)-type modules are particularly interesting for financial applications.

We will focus our study in the special case of \(L^\infty_F(\mathcal{E})\). One of the advantages of working in this module is that, as we will show in the lemma below, whatever risk measure you consider become automatically continuous.

**Lemma 3.1.** Any \(L^0\)-convex risk measure \(\rho : L^\infty_F(\mathcal{E}) \rightarrow L^0(\mathcal{F})\) is Lipschitz continuous with respect to the \(L^0\)-norm \(\| \cdot \|_\infty\)
\[ |\rho(X) - \rho(Z)| \leq \|X - Z\|_\infty, \text{ for } X, Z \in L^\infty_F(\mathcal{E}) \]
Proof. Clearly, $X \leq Z + \|X - Z \mid F\|_\infty$, and so $\rho(Z) - \|X - Z \mid F\|_\infty \leq Z + \rho(X)$ by monotonicity and cash invariance. Reversing the roles of $X$ and $Z$ yields the assertion.

Let us show some other preliminary results.

Lemma 3.2. Given a $L^0$-convex function $\rho : E \to L^0$, then $\rho$ is local (i.e., $1_A \rho(X) = 1_A \rho(1_A X)$ for $A \in F^+$ and $X \in E$) and $A_\rho$ and $V(Y)$ (defined as in definition 1.2) are both $L^0$-convex and stable under countable concatenations.

In particular, if $E$ has the countable concatenation property then $A_\rho$ and $V(Y)$ have it as well.

Proof. $\rho$ is local due to Theorem 3.2. of [7].

Since, $A_\rho = V(0)$ it suffices to prove the result for $V(Y)$.

A trivial verification by using that $\rho$ is $L^0$-convex shows that $V(Y)$ is $L^0$-convex.

Let us show that $V(Y)$ is stable under countable concatenations. Let $X \in cc([A_n], \{X_n\})$ with $X_n \in V(Y)$, then, by using that $\rho$ is local,

$$\rho(X) = \sum_{n \in \mathbb{N}} 1_{A_n} \rho(X) = \sum_{n \in \mathbb{N}} 1_{A_n} \rho(1_{A_n} X_n) = \sum_{n \in \mathbb{N}} 1_{A_n} \rho(X_n) \leq Y.$$ 

If $E$ has the countable concatenation property, every stable under countable concatenations subset has the c.c.p. as well. So that $V(Y)$ does.

The following result provides a prior representation result on which the main theorem will be based.

Lemma 3.3. Given a $L^0$-convex risk measure $\rho : L_\infty^\mathbb{F}(\mathcal{E}) \to L^0$, it is of the form

$$\rho(X) = \text{ess. sup} \{\mu(X) - \rho^*(\mu); \mu \in L_\infty^\mathbb{F}(\mathcal{E})^*, \mu \leq 0, \mu(1) = -1\}$$

for $X \in L_\infty^\mathbb{F}(\mathcal{E})$.

Furthermore,

$$\rho^*(\mu) = \text{ess. sup}_{X \in A_\rho} \mu(X)$$

for all $\mu \in L_\infty^\mathbb{F}(\mathcal{E})^*$.

Proof. Firstly, $\rho$ is lower semicontinuous with respect to the weak topology $\sigma(L_\infty^\mathbb{F}(\mathcal{E}), L_\infty^\mathbb{F}(\mathcal{E})^*)$.

By lemma 3.2 the set $V(Y)$ is $L^0$-convex with the countable concatenation property and by lemma 3.1 it is closed as well. Then by 1.2 $V(Y)$ is weakly
closed. Thus, since \( L^\infty_F(\mathcal{E}) \) has the countable concatenation property, we can use the Fenchel-Moreau theorem \(^{1,2}\) and we have that \( \rho = \rho^{**} \). Hence,

\[
\rho(X) = \operatorname{ess. sup}_{\mu \in L^\infty_F(\mathcal{E})^*} \{ \mu(X) - \rho^*(\mu) \}.
\]

Finally, we will prove that if \( \rho^*(\mu) < \infty \), then \( \mu \leq 0 \) and \( \mu(1) = -1 \).

Indeed, if \( \rho^*(\mu) < \infty \) it holds that \( \mu(X) - \rho(X) < \infty \) for all \( X \in L^\infty_F(\mathcal{E}) \).

By way of contradiction, suppose that there exist \( X_0 \in L^\infty_F(\mathcal{E}) \), \( X_0 \geq 0 \), and \( A \in \mathcal{F}^+ \) with \( \mu(X_0) > 0 \) on \( A \).

Then, for \( \lambda \in L^0_{++} \)

\[
+\infty > \mu(\lambda X_0) - \rho(\lambda X_0) = \lambda \mu(X_0) - \rho(\lambda X_0) \geq \lambda \mu(X_0) - \rho(0), \quad \text{on } A
\]

which is impossible as \( \mu(X_0) > 0 \) on \( A \) and \( \lambda \in L^0_{++} \) arbitrary. Thus \( \mu \leq 0 \). Furthermore, for \( \alpha \in \mathbb{R} \)

\[
\mu(\alpha) - \rho(\alpha) = \alpha(\mu(1) + 1) - \rho(0) < +\infty
\]

And since \( \alpha \) is arbitrary, it holds that \( \mu(1) = -1 \).

For the second part,

\[
\rho^*(X) = \operatorname{ess. sup}_{\mu \in L^\infty_F(\mathcal{E})^*} \{ \mu(X) - \rho(X) \} \geq
\]

\[
\geq \operatorname{ess. sup}_{X \in A_\rho} \{ \mu(X) - \rho(X) \} \geq \operatorname{ess. sup}_{X \in A_\rho} \mu(X).
\]

On the other hand, for all \( X \) we have that \( X' := \rho(X) + X \in A_\rho \).

Then

\[
\operatorname{ess. sup}_{X \in A_\rho} \mu(X) \geq \mu(X') = \mu(X + \rho(X)) = \mu(X) + \rho(X) \mu(1) = \mu(X) - \rho(X).
\]

And by taking essential supremum on \( X \in L^\infty_F(\mathcal{E}) \)

\[
\operatorname{ess. sup}_{X \in A_\rho} \mu(X) \geq \rho(X)^*.
\]

It is fair to say that, unlike the version for static risk measures, we cannot say whether the supremum is attained in this case, since Alaouglu theorem is not an available tool.

Now we define below a property that generalizes the known Fatou property for static risk measures.
Definition 3.3. We say that \( \rho : L_\infty (\mathcal{E}) \to L^0 (\mathcal{F}) \) has the \( L^0 (\mathcal{F}) \)-Fatou property, if for any sequence \( \{X_n\}_n \) such that \( |X_n| \leq Y \) for some \( Y \in L^0 (\mathcal{F}) \) and \( X_n \) converges p.a.s. to some \( X \in L_\infty (\mathcal{E}) \), then

\[
\rho(X) \leq \text{ess. lim inf}_n \rho(X_n).
\]

The following proposition shows that the \( L^0 (\mathcal{F}) \)-Fatou property can be reduced to the already known Fatou property for static risk measures.

**Proposition 3.2.** A conditional risk measure \( \rho : L_\infty (\mathcal{E}) \to L^0 (\mathcal{F}) \) has the \( L^0 (\mathcal{F}) \)-Fatou property if, and only if, \( \rho |_{L_\infty (\mathcal{E})} \) has the Fatou property.

**Proof.** If \( \rho \) has the \( L^0 (\mathcal{F}) \)-Fatou property then it is clear that \( \rho |_{L_\infty (\mathcal{E})} \) has the Fatou property.

Conversely, let \( \{X_n\}_n \) be a sequence such that \( X_n \to X \) p.a.s. and \( Y \in L^0 (\mathcal{F}) \) with \( |X_n| \leq Y \). For each \( k \in \mathbb{N} \), define \( A_k := (k - 1 \leq Y < k) \), then \( \{A_k\} \in \Pi(\Omega, \mathcal{F}) \).

Thereby, \( 1_{A_k} X_n \leq 1_{A_k} Y \leq k \) for every \( k \in \mathbb{N} \).

Then since \( \rho |_{L_\infty (\mathcal{E})} \) has the Fatou property it is fulfilled that

\[
1_{A_k} \rho(1_{A_k} X) \leq 1_{A_k} \text{ess. lim inf}_n \rho(1_{A_k} X_n)
\]

By 3.2 \( \rho \) is local, thus we have the following

\[
1_{A_k} \rho(X) \leq 1_{A_k} \text{ess. lim inf}_n \rho(X_n)
\]

which yields that \( \rho(X) \leq \text{ess. lim inf}_n \rho(X_n) \).

Below we will prove the main result of this section.

**Theorem 3.1.** Suppose \( \rho : L_\infty (\mathcal{E}) \to L^0 (\mathcal{F}) \) is a \( L^0 \)-convex risk measure. Then the following conditions are equivalent

1. \( \rho \) can be represented by some penalty function \( \alpha : L_1 (\mathcal{E}) \to L^0 (\mathcal{F}) : \)

\[
\rho(X) = \text{ess. sup} \left\{ \mathbb{E} [XZ | \mathcal{F}] - \alpha(Z); \ Z \in L_1 (\mathcal{E}), Z \leq 0, \mathbb{E} [Z | \mathcal{F}] = -1 \right\},
\]

for \( X \in L_\infty (\mathcal{E}) \).

2. \( \rho \) can be represented by \( \rho^* \) restricted to \( L_1 (\mathcal{E}) : \)

\[
\rho(X) = \text{ess. sup} \left\{ \mathbb{E} [XZ | \mathcal{F}] - \rho^*(Z); \ Z \in L_1 (\mathcal{E}), Z \leq 0, \mathbb{E} [Z | \mathcal{F}] = -1 \right\},
\]

for \( X \in L_\infty (\mathcal{E}) \).

3. If \( X_n \searrow X \) p.a.s. then \( \rho(X_n) \nearrow \rho(X) \) p.a.s.
4. $\rho$ has the $L^0(\mathcal{F})$-Fatou property.

5. $\rho|_{L^\infty(\mathcal{E})}$ has the Fatou property.

6. $\rho$ is lower semicontinuous for the weak-* topology $\sigma(L^\infty(\mathcal{F}), L^1(\mathcal{F}))$.

7. The acceptance set $A_\rho := \{X; \rho(X) \leq 0\}$ is weak-* closed in $L^\infty(\mathcal{F})$ with $\sigma(L^\infty(\mathcal{F}), L^1(\mathcal{F}))$.

Proof. 2 $\Rightarrow$ 1 : It is obvious.

7 $\Rightarrow$ 2: Let $X \in L^\infty(\mathcal{F})$ and $m := \text{ess sup} \{ E[XY | \mathcal{F}] - \rho^*(Y); Y \in L^1(\mathcal{F}), Y \leq 0, E[Y | \mathcal{F}] = -1 \}$.

Since $L^1(\mathcal{F})$ is embedded into $L^\infty(\mathcal{F})^*$ via $Y \mapsto E[\cdot Y | \mathcal{F}]$, in view of the lemma [3.3] it suffices to show that $m \geq \rho(X)$ or, equivalently, that $m + X \in A_\rho$.

Suppose by way of contradiction that $m + X /\in A_\rho$.

Let $Z_0$ be such that $\rho(Z_0) > 0$ and let $B := (\rho(m + X) > 0)$.

Define $Z_1 := (m + X)1_B + Z_01_{B^c}$.

Then, $1_A Z_1 /\notin 1_A A_\rho$ for all $A \in \mathcal{F}^+$.

Indeed, if $1_A Z_1 = 1_A X_0$ for some $X_0 \in A_\rho$ and $A \in \mathcal{F}^+$, then by using that $\rho$ is local (see lemma [3.2])

$$0 \geq 1_A \rho(X_0) = 1_A \rho(1_A X_0) = 1_A \rho(1_A Z_1) =$$

$$= 1_A \rho((m + X)1_{A\cap B} + Z_01_{A\cap B^c}) =$$

$$= 1_{A\cap B} \rho(m + X) + 1_{A\cap B^c} \rho(Z_0) > 0 \text{ on } A.$$

Therefore, since $L^\infty(\mathcal{F})$ has the countable concatenation property, it holds by lemma [3.2] that $A_\rho$ is $L^0$-convex and with the c.c.p. And, by assumption, it is also a nonempty weak-* closed. Then we may apply Theorem 3.5 of [13], and we obtain a continuous linear functional $\mu$ on $(L^\infty(\mathcal{F}), \sigma(L^\infty(\mathcal{F}), L^1(\mathcal{F})))$ such that

$$\mu(Z) \geq \text{ess. inf}_{Z \in A_\rho} \mu(Z) > \mu(Z_1), \text{ for all } Z \in A_\rho. \quad (2)$$

In particular,

$$\beta := \text{ess. inf}_{Z \in A_\rho} \mu(Z) > \mu(m + X) =: \gamma, \text{ on } B. \quad (3)$$

By theorem 3.22 of [13], we have that $\mu$ must be of the form $\mu(X) = E[ZX | \mathcal{F}]$ for some $Z \in L^1(\mathcal{F})$. 
In fact, \( Z \geq 0 \).

To show this, fix \( Y \geq 0 \) and note that \( \rho(\lambda Y) \leq \rho(0) \) for \( \lambda \in L^0_+ \), by monotonicity. Hence \( \lambda Y + \rho(0) \in \mathcal{A}_\rho \) for all \( \lambda \in L^0_+ \). It follows that

\[-\infty < \mu(Z_1) < \mu(\lambda Y + \rho(0)) = \lambda \mu(Y) + \mu(\rho(0)) \]

As \( \lambda \in L^0_+ \) is arbitrary, we have that \( \mu(\lambda Y) \geq 0 \) and in turn that \( Z \geq 0 \).

(2) yields that there exists \( X_0 \) such that

\[ 0 < \mu(X_0) = E[X_0 Z | \mathcal{F}] \leq \|X_0 \mathcal{F}\| \infty E[Z | \mathcal{F}] \]

and then we have that \( E[Z | \mathcal{F}] > 0 \)

Thereby, define

\[ \tilde{Z} := \frac{Z}{E[Z | \mathcal{F}]} \]

Due to lemma (3.3)

\[ \rho^*(\tilde{Z}) = \text{ess. sup}_{Y \in \mathcal{A}_\rho} E[\tilde{Z} | \mathcal{F}] = -\frac{\beta}{E[Z | \mathcal{F}]} \]

However,

\[ E[\tilde{Z} X | \mathcal{F}] + m = \frac{\mu(X)}{E[Z | \mathcal{F}]} \frac{\mu(1)}{\mu(1)} \frac{\mu(m + X)}{E[Z | \mathcal{F}]} \]

Thus, on \( B \)

\[ E[\tilde{Z} X | \mathcal{F}] + m = \frac{\gamma}{E[Z | \mathcal{F}]} \frac{\beta}{E[Z | \mathcal{F}]} = -\rho^*(Z) \]

in contradiction to (3)

1 \( \Rightarrow \) 4:

Let \( X_n \) a sequence in \( L^\infty_{\mathcal{F}}(\mathcal{E}) \) such that \( X = \lim X_n \) pointwise almost surely and such that there exists \( Y \in L^0(\mathcal{F}) \) with \( |X_n| < Y \) for all \( n \in \mathbb{N} \).

For \( Z \in L^1_{\mathcal{F}}(\mathcal{E}) \) with \( Z \leq 0 \) and \( E[Z | \mathcal{F}] = -1 \), we have that it is defined a probability measure \( \mathbb{Q} \) such that \( -Z = \frac{\mathbb{E}[Z]}{\mathbb{P}} \) is the Radon-Nikodym derivative. Then,

\[ E[Z X_n | \mathcal{F}] = E_{\mathbb{Q}}[X_n | \mathcal{F}] E[Z | \mathcal{F}] \]

by the Dominated Convergence Theorem for conditional expectations (see Proposition 10.4.4 of [3]), the latter converges p.a.s. to

\[ E_{\mathbb{Q}}[X | \mathcal{F}] E[Z | \mathcal{F}] = E[Z X | \mathcal{F}] \]

Hence,

\[ \rho(X) = \text{ess. sup} \left\{ \lim_n E[X_n Z | \mathcal{F}] - \alpha(Z); Z \in L^1_{\mathcal{F}}(\mathcal{E}), Z \leq 0, E[Z | \mathcal{F}] = -1 \right\} \leq \]
\[
\leq \text{ess. lim inf } n \sup \{ \mathbb{E}[X_n Z] \mathbb{F} - \alpha(Z) ; \ Z \in L^1_\mathcal{F} \}, Z \leq 0, \mathbb{E}[Z \mathbb{F}] = -1 \} = \\
\quad = \text{ess. lim inf } \rho(X_n).
\]

4 \iff 5: is the proposition \ref{prop:3.2}

4 \Rightarrow 3:

Let \{X_n\}_n a sequence such that \(X_n \searrow X\) p.a.s. Then by monotonicity \(\rho(X_n) \leq \rho(X)\) for each \(n \in \mathbb{N}\).

by hypothesis \(\rho(X) \leq \text{ess. lim inf } \rho(X_n) \leq \rho(X)\).

3 \Rightarrow 6: We have to show that \(V(Y)\) is weak-* for \(Y \in L^0(\mathcal{F})\).

For this purpose, for \(\varepsilon \in L^0_+\) let

\[
W_\varepsilon := V(Y) \cap \{ X \in L^\infty_\mathcal{F}(\mathcal{E}); \|X|\mathcal{F}\|_\infty \leq \varepsilon \}.
\]

If \{\(\gamma_n\)\}_n is a net in \(W_\varepsilon\) converging with \(\|\cdot\|\mathcal{F}_1\) to \(X \in W_\varepsilon\), then for induction we can choose a increasing sequence \{\(\gamma_n\)\}_n in \(\Gamma\) such that \(\|X_\gamma - X|\mathcal{F}\|_1 \leq 1/n\).

Thus, the sequence \(X_\gamma := X_\gamma_n\) converges with \(\|\cdot\|_1\) to \(X\). Therefore, there exists a subsequence which converges p.a.s. to \(X\), and the Fatou property of \(\rho\) implies that \(X \in W_\varepsilon\). Hence \(W_\varepsilon\) is closed in \(L^1_\mathcal{F}(\mathcal{E})\).

Since \(W_\varepsilon\) is \(L^0\)-convex, stable under countable concatenations and closed in \(L^1_\mathcal{F}(\mathcal{E})\), by \ref{lem:1.2} it holds that \(W_\varepsilon\) is weak closed in \(L^1_\mathcal{F}(\mathcal{E})\).

Now, since the injection

\[
(L^\infty_\mathcal{F}(\mathcal{E}), \sigma(L^\infty_\mathcal{F}(\mathcal{E}), L^1_\mathcal{F}(\mathcal{E}))) \hookrightarrow (L^1_\mathcal{F}(\mathcal{E}), \sigma(L^1_\mathcal{F}(\mathcal{E}), L^\infty_\mathcal{F}(\mathcal{E})))
\]

is continuous, we have that \(W_\varepsilon\) is \(\sigma(L^\infty_\mathcal{F}(\mathcal{E}), L^1_\mathcal{F}(\mathcal{E}))\)-closed in \(L^\infty_\mathcal{F}(\mathcal{E})\).

Finally, due to the Krein-Smulian theorem proved \ref{thm:2.1} since \(L^\infty_\mathcal{F}(\mathcal{E})\) is complete and has the countable concatenation property and, by lemma \ref{prop:3.2}, \(V(Y)\) is \(L^0\)-convex and stable under countable concatenations, we have that \(V(Y)\) is weak-* closed.

7 \Rightarrow 8: is obvious.

A final remark is that in the article of K. Detlefsen and G. Scandolo \cite{Detlefsen2010} it addressed a kind of conditional risk measures \(\rho : L^\infty(\mathcal{E}) \to L^\infty(\mathcal{F})\) which are (i) \(L^0(\mathcal{F})\)-convex; (ii) monotone; and (iii) cash invariant, in the sense that \(\rho(X + Y) = \rho(X) - Y\) for all \(Y \in L^\infty(\mathcal{F})\). Let us call them \(L^\infty\)-conditional risk measures.

Thereby for a conditional risk measure \(\rho : L^\infty_\mathcal{F}(\mathcal{E}) \to L^0(\mathcal{F})\) such that \(\rho(L^\infty(\mathcal{E})) \subset L^\infty(\mathcal{F})\) (this happens if, and only if, \(\rho(0) \in L^\infty(\mathcal{F})\)), then the restriction

\[
\rho|_{L^\infty(\mathcal{E})} : L^\infty(\mathcal{E}) \to L^\infty(\mathcal{F})
\]
is a $L^\infty$-conditional risk measure. Furthermore, by the just proved theorem 3.1 that $\rho$ can be represented if, and only if, $\rho|_{L^\infty(\mathcal{L})}$ has the Fatou property. This means that we have theorem 1 of [5], as a consequence of 3.1.

T. Guo et al. [14] prove in a converse way that given a $L^\infty$-conditional risk measure, this can be extended in a unique way to a conditional risk measure $\tilde{\rho} : L^{\infty}(\mathcal{L}) \to L^0(\mathcal{F})$ (see theorem 4.4 [14]) and it can also be represented if, and only if, $\rho$ has the Fatou property, as it is expected from theorem 3.1.

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