MATRIX VALUED COMMUTING DIFFERENTIAL OPERATORS WITH $A_2$ SYMMETRY

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ABSTRACT. We study the algebra of invariant differential operators on a certain homogeneous vector bundle over a Riemannian symmetric space of type $A_2$. We computed radial parts of its generators explicitly to obtain matrix-valued commuting differential operators with $A_2$ symmetry. Moreover, we generalize the commuting differential operators with respect to a parameter and the potential function.

INTRODUCTION

In this paper, we give matrix-valued commuting differential operators with $A_2$ symmetry that come from radial parts with respect to a certain $K$-type on a real semisimple Lie group of rank 2. Moreover, we give some generalization of commuting differential operators.

First we give a brief outline of the scalar case, which motivates our study. Let $G$ be a connected noncompact real semisimple Lie group with finite center, $K$ be a maximal compact subgroup of $G$, and $G = KAN$ be an Iwasawa decomposition. Harmonic analysis on the Riemannian symmetric space $G/K$ of the noncompact type has been extensively studied ([9]). In particular, the algebra of the left $G$-invariant differential operators on $G/K$ is commutative and $K$-invariant joint eigenfunctions, which are called zonal spherical functions are important. Heckman and Opdam constructed commuting differential operators by allowing the root multiplicities in the radial part of the Laplace-Beltrami operator on $G/K$ to be continuous parameters. Moreover, they constructed a real analytic joint eigenfunction, which is a generalization of the radial part of the zonal spherical function ([5, Part I], [16]). By a gauge transformation, the second order operator that is the radial part of the Laplace-Beltrami operator in group case becomes a Schrödinger operator without first order terms and the commuting differential operators containing it give a completely integrable system with Weyl group symmetry, which is called the Calogero-Sutherland model. The potential function is given by trigonometric function $1/\sinh^2$. It is known
that there are quantum integrable models with elliptic potential functions (\cite{17, 18}).

Vector-valued functions naturally arise in group case, if we consider $K$-types of higher dimensions. Indeed, $K$-finite matrix coefficients of representations of $G$ such as principal series representations satisfy differential equations coming from the universal enveloping algebra $U(g_{\mathbb{C}})$. Let $\tau$ be an irreducible representation of $K$ and $E_\tau$ be the associated homogeneous vector bundle over $G/K$, and $D(E_\tau)$ be the algebra of the left $G$-invariant differential operators on $E_\tau$. Deitmar \cite{2} proved that $D(E_\tau)$ is commutative if and only if $\tau|_M$ is multiplicity-free, where $M$ is the centralizer of $A$ in $K$.

Even when $D(E_\tau)$ is commutative, it seems to be hard to understand its structure and representations except the case of one-dimensional $K$-types.

In this paper, we consider the case of $G = SL(3, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) and $\tau$ is the standard representation of $K$. Then $D(E_\tau)$ is commutative and the Weyl group $S_3$ acts transitively on constituents of $\tau|_M$. In this case, $D(E_\tau)$ is easy to understand and we give radial parts of its generators explicitly. Moreover, we generalize these matrix-valued commuting differential operators by allowing the root multiplicity to be continuous parameter and also to the case of elliptic potential function.

This paper is organized as follows. In Section 1, we review on some known facts on the algebra of invariant differential operators on a homogeneous vector bundle over a symmetric space. In Section 2, we study invariant differential operators on vector bundles over symmetric spaces of type $A_2$. In Section 3, we give generalizations of matrix-valued commuting differential operators.

1. Invariant differential operators on a homogeneous vector bundle over a Riemannian symmetric space

In this section, we review on the algebra of invariant differential operators on a homogeneous vector bundle over a Riemannian symmetric space after \cite{2, 3, 14}.

1.1. Notation. Let $G$ be a connected noncompact real semisimple Lie group and $K$ be a maximal compact subgroup of $G$. Let $g$ and $\mathfrak{k}$ denote the Lie algebras of $G$ and $K$ respectively. For a Cartan involution $\theta$ of $g$ such that $g_\theta = \mathfrak{k}$, let $g = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. Let $G = KAN$ be an Iwasawa decomposition and $g = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be the corresponding decomposition. Let $\Sigma = \Sigma(g, a)$ denote the restricted root system and $\Sigma^+$ denote the positive system corresponding to $\mathfrak{n}$. For $\alpha \in \Sigma$ let $g^\alpha$ denote the root space for $\alpha$ and $m_\alpha = \dim g^\alpha$. Put $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. Let
Let \( M' \) and \( M \) be the normalizer and centralizer of \( \mathfrak{a} \) in \( K \) respectively. Then the Weyl group \( W \) of \( \Sigma \) is isomorphic to \( M'/M \).

For a real vector space \( u \) let \( u_C \) denote its complexification. Let \( U(\mathfrak{g}_C) \) denote the universal enveloping algebra of \( \mathfrak{g}_C \) and \( Z(\mathfrak{g}_C) \) its center. Let \( U(\mathfrak{g}_C)^K \) denote the set of the \( K \)-invariants in \( U(\mathfrak{g}_C) \).

1.2. The algebra of invariant differential operators. Let \((\tau, V_\tau)\) be an irreducible representation of \( K \) and \( E_\tau \) be the homogeneous vector bundle over \( G/K \) associated with \( \tau \). The space of \( C^\infty \)-sections of \( E_\tau \) is identified with a subspace of the \( V_\tau \)-valued \( C^\infty \)-functions on \( G \):

\[
C^\infty(E_\tau) \simeq \{ f \in C^\infty(G, V_\tau) : f(gk) = \tau(k^{-1})f(g) \ (g \in G, k \in K) \}.
\]

The action of \( G \) on \( C^\infty(E_\tau) \) is defined by

\[
l(g)f(x) = f(g^{-1}x) \quad (f \in C^\infty(E_\tau), \ g, x \in G).
\]

Let \( \mathbb{D}(E_\tau) \) denote the algebra of differential operators \( D : \ C^\infty(E_\tau) \to C^\infty(E_\tau) \) that satisfy \( D \circ l(g) = l(g) \circ D \) for all \( g \in G \). We call an element \( D \in \mathbb{D}(E_\tau) \) an invariant differential operator on \( E_\tau \).

Let \( \top \) denote the canonical anti-automorphism of \( U(\mathfrak{g}_C) \) defined by \( 1^\top = 1, \ X^\top = -X, \ (XY)^\top = Y^\top X^\top \ (X \in \mathfrak{g}_C) \). Let \( \Gamma_\tau \) denote the kernel of \( \tau \) in \( U(\mathfrak{t}_C) \). Then we have

\[
\mathbb{D}(E_\tau) \simeq U(\mathfrak{g}_C)^K/(U(\mathfrak{g}_C)^K \cap U(\mathfrak{g}_C)\Gamma_\tau^\top).
\]

By the Iwasawa decomposition and the Poincaré-Birkhoff-Witt theorem, we have

\[
U(\mathfrak{g}_C) = U(\mathfrak{a}_C)U(\mathfrak{t}_C) \oplus \mathfrak{n}_CU(\mathfrak{g}_C).
\]

Let \( p \) denote the projection from \( U(\mathfrak{g}_C) \) to \( U(\mathfrak{a}_C)U(\mathfrak{t}_C) \). Then \( p|_{U(\mathfrak{g}_C)^K} \) maps \( U(\mathfrak{g}_C)^K \) to \( U(\mathfrak{a}_C) \otimes U(\mathfrak{t}_C)^M \). Let \( \eta \) denote the automorphism of \( U(\mathfrak{a}_C) \) defined by \( \eta(H) = H + \rho(H) \ (H \in \mathfrak{a}_C) \). Then the homomorphism \( \gamma_\tau = (\eta \otimes (\tau \circ \top)) \circ p \) from \( U(\mathfrak{g}_C)^K \) to \( U(\mathfrak{a}_C) \otimes \text{End}_M(V_\tau)^M \) induces the following injective algebra homomorphism, which we denote by the same notation:

\[
\gamma_\tau : \mathbb{D}(E_\tau) \to (U(\mathfrak{a}_C) \otimes \text{End}_M(V_\tau))^M.
\]

The above homomorphism is not necessarily surjective.

The algebra \( \mathbb{D}(E_\tau) \) is not necessarily commutative. If \( \tau|_M \) decomposes into multiplicity free sum of irreducible representations of \( M \), then \( U(\mathfrak{a}_C) \otimes \text{End}_M(V_\tau) \) is commutative by Schur’s lemma, hence \( \mathbb{D}(E_\tau) \) is commutative. Deitmar [2] proved that \( \mathbb{D}(E_\tau) \) is commutative if and only if \( \tau|_M \) is multiplicity free.

In the next section, we will study some examples of \( E_\tau \) such that \( \mathbb{D}(E_\tau) \) are commutative and the homomorphisms \((\ref{eq:gamma})\) are surjective.
1.3. **Spherical functions.** Let \((\tau, V_\tau)\) be an irreducible representation of \(K\). We call a function \(f : G \to \text{End}(V_\tau)\) is \(\tau\)-spherical if it satisfies the condition
\[
f(k_1 x k_2) = \tau(k_2)^{-1} f(x) \tau(k_1)^{-1} \quad (x \in G, \ k_1, k_2 \in K).
\]
Alternatively, it is naturally identified with a function \(f : G \to V_\tau^* \otimes V_\tau\) that satisfies
\[
f(k_1 x k_2) = \tau^*(k_1) \otimes \tau(k_2)^{-1} f(x) \quad (x \in G, \ k_1, k_2 \in K).
\]
By the Cartan decomposition \(G = KAK\), a \(\tau\)-spherical function \(f\) is determined by its restriction to \(A\). For a differential operator \(D\) on \(E_\tau\) or an element of \(U(g_C)\), there exists a differential operator \(R_\tau(D)\) on \(C^\infty(A, (V_\tau^* \otimes V_\tau)^M)\) that satisfies
\[
D f|_A = R_\tau(D)(f|_A)
\]
for any \(\tau\)-spherical functions \(f\). We call \(R_\tau(D)\) the \(\tau\)-radial part of \(D\). We recall two well-known lemmas. Fix an element
\[
H \in a_+ = \{X \in a : \alpha(X) > 0 \ (\alpha \in \Sigma^+)\}
\]
and put \(a = \exp H \in A_+ = \exp a_+\).

**Lemma 1.1** ([21] Proof of Proposition 9.1.2.11, [13] Lemma 8.24). For \(\alpha \in \Sigma^+, \ X \in g_\alpha\), we have
\[
X - \theta X = \coth \alpha(H)(X + \theta X) - \frac{1}{\sinh \alpha(H)} \text{Ad}(a^{-1})(X + \theta X).
\]

Let \(\Omega, \Omega_a, \) and \(\Omega_m\) denote the Casimir elements in \(U(g_C), U(a_C)\), and \(U(m_C)\), respectively. For \(\alpha \in \Sigma\), let \(H_\alpha\) be the element of \(a\) such that \(\alpha(X) = B(H_\alpha, X)\) for all \(X \in a\), where \(B\) is the Killing form for \(g\). For each \(\alpha \in \Sigma^+\) we choose a basis \(\{X_{\alpha,i}\}_{1 \leq i \leq m_\alpha}\) of \(g_\alpha^C\) that is orthonormal with respect to the inner product \((X, Y) = -B(X, \theta Y)\). We write \(X_{\alpha,i} = Z_{\alpha,i} + Y_{\alpha,i}\) where \(Z_{\alpha,i} \in \xi_C\) and \(Y_{\alpha,i} \in \pi_C\).

**Lemma 1.2** ([21] Proposition 9.1.2.11).
\[
\Omega = \Omega + \Omega_m + \sum_{\alpha \in \Sigma^+} m_\alpha \coth \alpha(H) H_\alpha
\]
\[
+ 2 \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} \sinh^{-2} \alpha(H) \{\text{Ad}(a^{-1})(Z_{\alpha,i}^2) + Z_{\alpha,i}^2
\]
\[
- 2 \cosh \alpha(H) \text{Ad}(a^{-1})(Z_{\alpha,i}) Z_{\alpha,i}\}.
\]
2. Invariant differential operators on a homogeneous vector bundle over a symmetric space of type $A_2$

In this section, we consider symmetric spaces

$$G/K = SL(3, \mathbb{K})/SU(3, \mathbb{K}) \ (\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}),$$

that is,

$$G/K = SL(3, \mathbb{R})/SO(3), \ SL(3, \mathbb{C})/SU(3), \ SU^*(6)/Sp(3).$$

The restricted root system $\Sigma$ of $G/K$ is of type $A_2$ and the Weyl group $W$ is isomorphic to $S_3$. We regard $a$ as a subspace of $\mathbb{R}^3$

$$a \simeq \{(t_1, t_2, t_3) : t_1, t_2, t_3 \in \mathbb{R}, t_1 + t_2 + t_3 = 0\}.$$

We will give the above identification explicitly in each case of $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ in the following subsections. We put

$$\partial_i = \frac{\partial}{\partial t_i}, \ \partial'_i = \partial_i - \frac{1}{3} (\partial_1 + \partial_2 + \partial_3), \ t_{ij} = t_i - t_j \ (1 \leq i \neq j \leq 3).$$

In the $GL$-picture, functions on $a$ are regarded as functions on $\mathbb{R}^3$ that are killed by $\partial_1 + \partial_2 + \partial_3$.

Let $\tau$ be the standard representation of $K$ and $E_\tau \to G/K$ be the associated homogeneous vector bundle. We have the following theorem for the algebra $\mathcal{D}(E_\tau)$ of invariant differential operators on $E_\tau$.

**Theorem 2.1.** Let $G/K = SL(3, \mathbb{K})/SU(3, \mathbb{K}) \ (\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$ and $\tau$ be the standard representation of $K$.

(i) $\tau|_M$ decomposes into multiplicity free sum of three irreducible representations of $M$. These $M$-modules are in a single $W$-orbits.

(ii) $\mathcal{D}(E_\tau)$ is commutative. Moreover, $\gamma_\tau$ in (1) is surjective and gives an algebra isomorphism. The composition of $\gamma_\tau$ and the projection to the third factor gives an isomorphism $\mathcal{D}(E_\tau) \simeq U(a_{\mathbb{C}})^{W_{e_1-e_2}}$, where $W_{e_1-e_2} = \{e, s_{12}\}$. Moreover, $\mathcal{D}(E_\tau)$ is generated by two operators $D_1$ and $D_2$, which are algebraically independent and the order of $D_i$ is $i$ for $i = 1, 2$. $D_1$ is unique up to a constant multiple and a constant difference.

(iii) Let $k = 1/2, 1, 2$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. We may take $D_2$ to be the image of a constant multiple of $\Omega - \tau(\Omega_m)$. With respect to a basis of
(V_τ \otimes V_τ)_M, \, \tau\text{-radial parts of } D_1 \text{ and } D_2 \text{ have matrix expressions}

\[ R_\tau(D_1) = \begin{pmatrix} \partial_1' & 0 & 0 \\ 0 & \partial_2' & 0 \\ 0 & 0 & \partial_3' \end{pmatrix} + k \begin{pmatrix} \coth t_{12} + \coth t_{13} & 0 & 0 \\ 0 & \coth t_{12} + \coth t_{23} & 0 \\ 0 & 0 & \coth t_{12} + \coth t_{23} \end{pmatrix}, \]

\[ R_\tau(D_2) = L_2 + k \begin{pmatrix} \sinh^2 t_{12} & 0 & 0 \\ 0 & -\sinh^2 t_{12} & 0 \\ 0 & 0 & -\sinh^2 t_{12} \end{pmatrix} \begin{pmatrix} \cosh t_{12} & 0 & 0 \\ 0 & \cosh t_{12} & 0 \\ 0 & 0 & \cosh t_{12} \end{pmatrix}, \]

where

\[ L_2 = \partial_1' \partial_2' + \partial_2' \partial_3' + \partial_3' \partial_1' - k \sum_{1 \leq i < j \leq 3} (\coth t_{ij}) (\partial_i - \partial_j). \]

We will prove the above theorem in the following subsections.

By the identification \( U(a_C) \simeq S(a_C) \), we have

\[ \gamma_\tau(D_1)(\lambda) = \text{diag}(\lambda_1, \lambda_2, \lambda_3), \]

\[ \gamma_\tau(D_2)(\lambda) = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 + 4k^2 \]

for \( \lambda \in a_C^* \). Hence, the operator \( D_1 \) and \( D_2 \) in Theorem 2.1 correspond to

\[ \lambda_3 \text{ and } \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 + 4k^2 \]

respectively by the isomorphism

\[ \mathbb{D}(E_\tau) \simeq U(a_C)^{W_{e_1-e_2}} \simeq S(a_C)^{W_{e_1-e_2}}. \]

There are several choices of a second order operator such that \( D_1 \) and it generate \( \mathbb{D}(E_\tau) \). For example, there exists an invariant differential operator \( \tilde{D}_2 \) such that its \( \tau \)-radial part is of the form

\[ R_\tau(\tilde{D}_2) = \begin{pmatrix} \partial_2' \partial_3' & 0 & 0 \\ 0 & \partial_1' \partial_3' & 0 \\ 0 & 0 & \partial_1' \partial_2' \end{pmatrix} + (\text{lower order terms}). \]

Remark 2.2. The operator \( L_2 \) in the above theorem is a constant multiple of the radial part of the Laplace-Beltrami operator on \( G/K \) ([9 Proposition 3.9]).

Remark 2.3. Though the restricted root system of the exceptional symmetric pair \((e_6(-26), f_4)\) is also of type \( A_2 \), it seems that there are no \( K \)-type \( \tau \) such that results like Theorem 2.1 hold.
Remark 2.4. For $G/K = SL(2, \mathbb{K})/SU(2, \mathbb{K})$ ($\mathbb{K} = \mathbb{C}, \mathbb{H}$), that is $G/K = SL(2, \mathbb{C})/SU(2)$, $SU^*(4)/Sp(2)$, the algebra $\mathbb{D}(E_\tau)$ is commutative for the standard representation $\tau$ of $K$. $SL(2, \mathbb{C})$ and $SU^*(4)$ are isomorphic to $Spin(3, 1)$ and $Spin(5, 1)$, respectively. Moreover the standard representations of $SU(2)$ and $Sp(2)$ correspond to the spin representations of $Spin(3)$ and $Spin(5)$, respectively. For the symmetric space $G/K = Spin(2m + 1, 1)/Spin(2m + 1)$ ($m \geq 1$) and the spin representation $\tau$ of $K$, the algebra $\mathbb{D}(E_\tau)$ is commutative and generated by the Dirac operator ([7], [11]).

2.1. The case of $SL(3, \mathbb{R})/SO(3)$.

2.1.1. Notation. Let $G = SL(3, \mathbb{R})$ and $K = SO(3)$. The Lie algebras of $G$ and $K$ are

$$g = \mathfrak{sl}(n, \mathbb{R}) = \{X \in M(3, \mathbb{R}) : \text{Tr}X = 0\}$$

and $\mathfrak{k} = \mathfrak{so}(3)$. The Cartan involution of $g$ is given by $\theta X = -^tX$ for $X \in g$. We have the Cartan decomposition $g = \mathfrak{k} + \mathfrak{p}$ into $\pm 1$-eigenspaces of $\theta$. Here $\mathfrak{k}$ and $\mathfrak{p}$ consist of the real skew-symmetric and symmetric matrices respectively.

Let $\{K_1, K_2, K_3\}$ denote the basis of $\mathfrak{k}$ defined by

$$K_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 \\ 1 & -1 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$ 

Let $a$ denote the subspace of $\mathfrak{p}$ consisting of diagonal matrices

$$a = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} : t_1, t_2, t_3 \in \mathbb{R}, t_1 + t_2 + t_3 = 0 \right\}. \quad (2.2)$$

The centralizer $M$ of $a$ in $K$ is given by

$$M = \left\{ \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

2.1.2. Invariant differential operators. Let $\tau$ denote the standard representation of $K$ on $V_\tau = \mathbb{C}^3$. Equip $\mathbb{C}^3$ with the standard basis $\{e_1 = t(1, 0, 0), e_2 = t(0, 1, 0), e_3 = t(0, 0, 1)\}$.

Since $\tau|M$ decomposes into multiplicity free sum of irreducible representations, the algebra $\mathbb{D}(E_\tau)$ of invariant differential operators on the homogeneous vector bundle $E_\tau \rightarrow G/K$ associated with $\tau$ is commutative ([2]). Moreover, the irreducible constituents of $\tau|M$ are in a single $W$-orbits.
2.1.3. **First order invariant differential operator.** \((\tau, \mathbb{C}^3)\) and \((\text{Ad}, p_\mathbb{C})\) are irreducible representation of dimension 3 and 5 respectively and the tensor product \(p_\mathbb{C} \otimes \mathbb{C}^3\) contains \((\tau, \mathbb{C}^3)\) with multiplicity 1. Hence, there is a unique first order operator in \(\mathcal{D}(E_\tau)\) up to a constant multiple.

\(\tau\)-radial part of a first order invariant differential operator was computed by Sono [20, Theorem 6.4]. We will give a proof in a way parallel to our discussions for the cases of \(SL(3, \mathbb{K})/SU(3, \mathbb{K}) \ (\mathbb{K} = \mathbb{C}, \mathbb{H})\), which are given in subsequent sections.

Let \(E_{ij}\) denote the \(3 \times 3\) matrix with \((i, j)\) entry 1 and the other entries 0 and put \(E'_i = E_{ii} - \frac{1}{3}(E_{11} + E_{22} + E_{33})\). Then

\[
\begin{align*}
\{E_{ij} & (1 \leq i \neq j \leq 3), E'_k (1 \leq k \leq 3) \}
\end{align*}
\]

forms a basis of \(\mathfrak{sl}(3, \mathbb{R})\).

**Lemma 2.5** (Manabe-Ishii-Oda [10]). *The dimension of \((p_\mathbb{C} \otimes \text{End}(V_\tau))^K\) is 1 and a basis vector is represented by the following matrix*

\[
(2.3) \quad \begin{pmatrix}
E'_1 & \frac{1}{2}(E_{12} + E_{21}) & \frac{1}{2}(E_{13} + E_{31}) \\
\frac{1}{2}(E_{12} + E_{21}) & E'_2 & \frac{1}{2}(E_{23} + E_{32}) \\
\frac{1}{2}(E_{13} + E_{31}) & \frac{1}{2}(E_{23} + E_{32}) & E'_3
\end{pmatrix}
\]

*with respect to the basis \(\{e_1, e_2, e_3\}\) of \(V_\tau\).*

**Proof.** A first order invariant differential operator is given by [10, Lemma 3.3] with respect to the standard weight vectors of \(\mathfrak{so}(3) \simeq \mathfrak{su}(2)\), Applying the change of basis given in [10 §4.3], the matrix expression follows. \(\square\)

**Remark 2.6.** There exists an element \(D \in U(\mathfrak{g})^K\) such that (2.3) is the image of \(D\) under the surjective map \(U(\mathfrak{g}_\mathbb{C})^K \to (U(\mathfrak{g}_\mathbb{C}) \otimes _{\mathbb{C}} \text{End}(V_\tau))^K\). We may take \(D\) in \((p_\mathbb{C} \otimes d_\mathbb{C} \otimes d_\mathbb{C})^K\). The proof of [2] Lemma 1] with some additional computations shows that \(D\) is

\[
(2.4) \quad E'_1 K_1^2 + E'_2 K_2^2 + E'_3 K_3^2 + \frac{1}{2} (E_{12} + E_{21})(K_2 K_1 + K_1 K_2) + \frac{1}{2} (E_{13} + E_{31})(K_1 K_3 + K_3 K_1) + \frac{1}{2} (E_{23} + E_{32})(K_2 K_3 + K_3 K_2).
\]

We will give the \(\tau\)-radial part of (2.3). The space \((V^*_\tau \otimes V_\tau)^M\) is a 3-dimensional vector space with a basis

\[
(2.5) \quad \{e_1^* \otimes e_1, e_2^* \otimes e_2, e_3^* \otimes e_3\}.
\]

The following proposition gives the radial part of (2.3) with respect to this basis in the coordinates (2.2) of \(\mathfrak{a}\).
Proposition 2.7. (Sono [20, Theorem 6.4]) With respect to the basis \((V_{\tau^*} \otimes V_{\tau})^M\), the \(\tau\)-radial part of \((2.3)\) has the following matrix expression

\[
\begin{pmatrix}
\frac{\partial_1'}{2} + \coth t_{12} + \coth t_{13} & -\frac{1}{2}\sinh t_{12} & -\frac{1}{2}\sinh t_{13} \\
\frac{1}{2}\sinh t_{12} & \frac{\partial_2'}{2} + \coth t_{12} + \coth t_{23} & -\frac{1}{2}\sinh t_{23} \\
\frac{1}{2}\sinh t_{13} & -\frac{1}{2}\sinh t_{23} & \frac{\partial_3'}{2} + \coth t_{23} + \coth t_{32}
\end{pmatrix}.
\]

Proof. We consider the first row of the matrix \((2.3)\). By Lemma 1.1, we have

\[
E_1' e_1 + \frac{1}{2}(E_{12} + E_{21}) e_2 + \frac{1}{2}(E_{13} + E_{31}) e_3
\]

\[= E_1' e_1 + \left(-\frac{1}{2}\coth t_{12} K_3 + \frac{1}{2}\sinh t_{12} \text{Ad}(a^{-1}) K_3\right) e_2
\]

\[+ \left(\frac{1}{2}\coth t_{13} K_2 - \frac{1}{2}\sinh t_{13} \text{Ad}(a^{-1}) K_2\right) e_3
\]

\[= \left(E_1' + \frac{1}{2}\coth t_{12} + \frac{1}{2}\coth t_{13}\right) e_1 + \frac{1}{2}\sinh t_{12} \text{Ad}(a^{-1}) K_3 e_2
\]

\[+ \frac{1}{2}\sinh t_{13} \text{Ad}(a^{-1}) K_2 e_3.
\]

Since \(K_2 e_1^* = e_3^*\) and \(K_3 e_1^* = -e_2^*\), the action of the first row of \((2.3)\) on the coefficient of \(e_1^* \otimes e_1\) is given by

\[
\left(E_1' + \frac{1}{2}\coth t_{12} + \frac{1}{2}\coth t_{13}\right) e_1^* \otimes e_1 + \frac{1}{2}\sinh t_{12} e_2^* \otimes e_2
\]

\[+ \frac{1}{2}\sinh t_{13} e_3^* \otimes e_3.
\]

Similarly, we can compute the radial part of the second and third rows of the matrix \((2.3)\) and obtain the matrix expression. \(\square\)

2.1.4. Radial part of the Casimir operator. The following proposition gives the \(\tau\)-radial part of the Casimir operator \(\Omega\).

Proposition 2.8. The \(\tau\)-radial part \(R_\tau(\Omega)\) of the Casimir operator is given by

\[
-3R_\tau(\Omega) = \partial_1 \partial_2 + \partial_2 \partial_3 + \partial_3 \partial_1
\]

\[+ \frac{1}{2}\left(\coth t_{12}(\partial_1 - \partial_2) + \coth t_{13}(\partial_1 - \partial_3) + \coth t_{23}(\partial_2 - \partial_3)\right)
\]

\[+ \frac{1}{2}\left(\frac{1}{\sinh^2 t_{12}} + \frac{1}{\sinh^2 t_{13}} - \frac{\cosh t_{12}}{\sinh t_{12}} \frac{1}{\sinh^2 t_{13}} - \frac{\cosh t_{13}}{\sinh t_{13}} \frac{1}{\sinh^2 t_{12}} \right)
\]

\[+ \frac{1}{2}\left(\frac{1}{\sinh^2 t_{12}} + \frac{1}{\sinh^2 t_{23}} - \frac{\cosh t_{12}}{\sinh t_{12}} \frac{1}{\sinh^2 t_{23}} - \frac{\cosh t_{23}}{\sinh t_{23}} \frac{1}{\sinh^2 t_{12}} \right)
\]

\[+ \frac{1}{2}\left(\frac{1}{\sinh^2 t_{13}} + \frac{1}{\sinh^2 t_{23}} - \frac{\cosh t_{13}}{\sinh t_{13}} \frac{1}{\sinh^2 t_{23}} - \frac{\cosh t_{23}}{\sinh t_{23}} \frac{1}{\sinh^2 t_{13}} \right)
\]

with respect to the basis \((2.5)\) of \((V_{\tau^*} \otimes V_{\tau})^M\).
The Killing form of $g = \mathfrak{sl}(3, \mathbb{R})$ is given by $B(X, Y) = 6 \text{Tr} XY$ for $X, Y \in g$. \{(E_1' - E_2')/\sqrt{12}, E_3'/2\}$ forms an orthonormal basis of $\mathfrak{a}$. Moreover, $E_{ij} \in \mathfrak{g}^{i-j}$ and $B(E_{ij}/\sqrt{6}, E_{ji}/\sqrt{6}) = 1$ for $1 \leq i \neq j \leq 3$. The Casimir operator is given by

\begin{equation}
\Omega = \frac{1}{12} (E_1' - E_2')^2 + \frac{1}{4} E_3'^2
+ \frac{1}{6} (E_{12}E_{21} + E_{21}E_{12} + E_{13}E_{31} + E_{31}E_{13} + E_{23}E_{32} + E_{32}E_{23})
= \frac{1}{9} (E_1 + E_2 + E_3)^2 - \frac{1}{3} (E_1E_2 + E_2E_3 + E_3E_1)
+ \frac{1}{6} (E_{12}E_{21} + E_{21}E_{12} + E_{13}E_{31} + E_{31}E_{13} + E_{23}E_{32} + E_{32}E_{23}).
\end{equation}

By Lemma 1.2, we have

\begin{align*}
\Omega &= \frac{1}{9} (\partial_1 + \partial_2 + \partial_3)^2 + \frac{1}{3} (\partial_1\partial_2 + \partial_2\partial_3 + \partial_3\partial_1) \\
&+ \frac{1}{6} (\coth t_{12}(\partial_1 - \partial_2) + (\coth t_{13}(\partial_1 - \partial_3) + (\coth t_{23}(\partial_2 - \partial_3)) \\
&- \frac{1}{12} \sinh^{-2} t_{12} \{-\text{Ad}(a^{-1})(K_2) - K_3^2 + 2 \cosh t_{12} (\text{Ad}(a^{-1})K_3)K_3\} \\
&- \frac{1}{12} \sinh^{-2} t_{13} \{-\text{Ad}(a^{-1})(K_2) - K_2^2 + 2 \cosh t_{13} (\text{Ad}(a^{-1})K_2)K_2\} \\
&- \frac{1}{12} \sinh^{-2} t_{23} \{-\text{Ad}(a^{-1})(K_1) - K_1^2 + 2 \cosh t_{23} (\text{Ad}(a^{-1})K_1)K_1\}.
\end{align*}

Since $K_i e_i = 0$, $K_i e_j = e_k$, $K_i e_k = -e_j$ for $(i, j, k) = (1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$, and the actions on $e_i^*$'s have opposite signs, the action of $-3\Omega$ on the coefficient of $e_1^* \otimes e_1$ is

\begin{align*}
&\left(-\frac{1}{3} (\partial_1 + \partial_2 + \partial_3)^2 + \partial_1\partial_2 + \partial_2\partial_3 + \partial_3\partial_1\right) e_1^* \otimes e_1 \\
&- \frac{1}{2} \sum_{1 \leq i < j \leq 3} \coth t_{ij}(\partial_i - \partial_j) e_1^* \otimes e_1 + \frac{1}{2} \left(\frac{1}{\sinh^2 t_{12}} + \frac{1}{\sinh^2 t_{13}}\right) e_1^* \otimes e_1 \\
&- \frac{\cosh t_{12}}{2\sinh^2 t_{12}} e_2^* \otimes e_2 - \frac{\cosh t_{13}}{2\sinh^2 t_{13}} e_3^* \otimes e_3.
\end{align*}

The actions of $\Omega$ on $e_2 \otimes e_2^*$ and $e_3 \otimes e_3^*$ are given in similar way and $\partial_1 + \partial_2 + \partial_3$ acts by zero on functions on $\mathfrak{a}$. Thus we have the matrix expression \[\text{[2.6]}\].

\section{The case of $SL(3, \mathbb{C})/SU(3)$}

\subsection{Notation}
Throughout this section we use notation of \[\text{[6]}\]. Namely, the imaginary unit is denoted by $J$. Hence complex numbers are of the form
\[ \alpha + J\beta \text{ with } \alpha, \beta \in \mathbb{R}. \] For a real vector space \( I \), its complexification \( I \otimes_{\mathbb{R}} \mathbb{C} \) is denoted by \( I_{\mathbb{C}} \). Let \( \sqrt{-1} \) denote the complex structure on \( I_{\mathbb{C}} \). Namely, \( I_{\mathbb{C}} = \{ X + \sqrt{-1}Y : X, Y \in I \} \), and \( \sqrt{-1} \) acts on \( I_{\mathbb{C}} \) as a linear operator with \( (\sqrt{-1})^2 = -\text{id}_{I_{\mathbb{C}}} \), and
\[
(\alpha + J\beta)Z = \alpha Z + \beta \sqrt{-1}Z \quad (\alpha, \beta \in \mathbb{R}, Z \in I).
\]

We employ the \( GL \)-picture as in [6]. Let \( G = GL(3, \mathbb{C}) \) and \( K = U(3) \). The Lie algebras of \( G \) and \( K \) are
\[
g = \mathfrak{gl}(3, \mathbb{C}) = M(3, \mathbb{C}) = \{ 3 \times 3 \text{ complex matrices} \},
g = \mathfrak{u}(3) = \{ X \in M(3, \mathbb{C}) : X^* = -X \}
\]
respectively. Here \( X^* = tX \) for \( X \in \mathfrak{gl}(3, \mathbb{C}) \). Define \( \theta X = -X^* \) for \( X \in g \). Then \( \theta \) is the Cartan involution of \( g \) that satisfies
\[
k = g_{\theta} = \{ X \in g : \theta X = X \}.
\]
Let \( p \) denote the \(-1\)-eigenspace of \( \theta \) in \( g \):
\[
p = \{ X \in M(3, \mathbb{C}) : X^* = X \}.
\]
We have a Cartan decomposition \( g = k \oplus p \). Let \( a \) denote the set of the diagonal matrices in \( p \):
\[
(2.8) \quad a = \{ \text{diag} (t_1, t_2, t_3) : t_i \in \mathbb{R} \ (1 \leq i \leq 3) \}.
\]
Then \( a \) is a maximal abelian subspace of \( p \). Let \( M \) denote the centralizer of \( a \) in \( K \). We have
\[
M = \{ \text{diag} (u_1, u_2, u_3) : u_i \in U(1) \ (1 \leq i \leq 3) \} \simeq U(1)^3.
\]
We view \( g \) and its subalgebras as real Lie algebras. Then the Killing form of \( g \) is given by
\[
B_{\mathbb{R}}(X, Y) = 2 \text{Re} \ B_g(X, Y) = 12 \text{Re} (\text{Tr} \ XY - 4 \text{Tr} \ X \text{Tr} \ Y) \quad (X, Y \in g).
\]
The Killing form \( B_{g_{\mathbb{C}}} \) on \( g_{\mathbb{C}} \) is given by linear extension of \( B_{\mathbb{R}} \).
Let \( X, Y \in g \) and \( X + \sqrt{-1}Y \in g_{\mathbb{C}} \). The mapping
\[
X + \sqrt{-1}Y \mapsto (X + JY) \oplus (X - JY)
\]
is a Lie algebra isomorphism of \( g_{\mathbb{C}} \) onto \( g \oplus g \). The inverse mapping is given by
\[
g \oplus g \ni Z \oplus W \mapsto \frac{1}{2} \{ Z + W - \sqrt{-1}J(Z - W) \} \in g_{\mathbb{C}}.
\]
Moreover, both the mapping \( Z \mapsto \frac{1}{2}(Z - \sqrt{-1}JZ) \) and \( W \mapsto \frac{1}{2}(W + \sqrt{-1}JW) \) are isometries of \( g \) into \( g_{\mathbb{C}} \) with respect to the Killing forms \( B_g \) and \( B_{g_{\mathbb{C}}} \).
By the above isomorphism
\begin{equation}
\mathfrak{g}_C \simeq \mathfrak{g} \oplus \mathfrak{g},
\end{equation}
we have an isomorphism
\begin{equation}
U(\mathfrak{g}_C) \simeq U(\mathfrak{g}) \otimes U(\mathfrak{g}).
\end{equation}

For \(1 \leq i, j \leq 3\), let \(E_{ij}\) (resp. \(E'_{ij}\)) denote the \(3 \times 3\) matrix with \((i, j)\)-entry 1 (resp. \(J\)) and the remaining entries 0. Then \(\{E_{ij}, E'_{ij} \mid 1 \leq i, j \leq 3\}\) forms an \(\mathbb{R}\)-basis of \(\mathfrak{g}\). Define \(H_{ij} \in \mathfrak{a}, H'_{ij} \in \mathfrak{t}\) by \(H_{ij} = E_{ii} - E_{jj}, H'_{ij} = E'_{ii} - E'_{jj}\). Let \(I_3\) denote the identity matrix in \(M(3, \mathbb{C})\) and put \(I'_3 = JI_3 \in \mathfrak{t}\).

The semisimple part \(\mathfrak{g}_0\) of \(\mathfrak{g}\) is

\[\mathfrak{g}_0 = \mathfrak{sl}(3, \mathbb{C}) = \{X \in M(3, \mathbb{C}) : \text{Tr} X = 0\}.\]

Put \(\mathfrak{t}_0 = \mathfrak{t} \cap \mathfrak{g}_0 = \mathfrak{su}(3), \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0,\) and \(\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0\). Then \(\mathfrak{t} = \mathfrak{t}_0 \oplus \mathbb{R}I'_3, \mathfrak{p} = \mathfrak{p}_0 \oplus \mathbb{R}I_3,\) and \(\{H_{ij}, H'_{ij} \mid 1 \leq i < j \leq 3\}\) forms a basis of \(\mathfrak{a} \cap \mathfrak{sl}(3, \mathbb{C})\).

Let \(G_0 = SU(3), K_0 = SU(3),\) and \(M_0 = M \cap G_0\).

Define \(I^\mathfrak{t}_3, H^\mathfrak{t}_{ij}, E^\mathfrak{t}_{ij} \in \mathfrak{t}_\mathbb{C}\) and \(I^\mathfrak{p}_3, H^\mathfrak{p}_{ij}, E^\mathfrak{p}_{ij} \in \mathfrak{p}_\mathbb{C}\) by

\[I^\mathfrak{t}_3 = \sqrt{-1}I'_3, \quad H^\mathfrak{t}_{ij} = \sqrt{-1}JH_{ij}, \quad E^\mathfrak{t}_{ij} = \frac{1}{2} \left\{ (E_{ij} - E_{ji}) - \sqrt{-1}(E'_{ij} + E'_{ji}) \right\}, \]
\[I^\mathfrak{p}_3 = I_3, \quad H^\mathfrak{p}_{ij} = H_{ij}, \quad E^\mathfrak{p}_{ij} = \frac{1}{2} \left\{ (E_{ij} + E_{ji}) - \sqrt{-1}(E'_{ij} - E'_{ji}) \right\}.\]

The element \(E_{ij} \oplus 0\) and \(0 \oplus E_{ij}\) in \(\mathfrak{g} \oplus \mathfrak{g}\) correspond \(\frac{1}{2}(E^\mathfrak{p}_{ij} + E^\mathfrak{t}_{ij})\) and \(\frac{1}{2}(E^\mathfrak{p}_{ij} - E^\mathfrak{t}_{ij})\) in \(\mathfrak{g}_C\) under the isomorphism \((2.9)\), respectively. Similar identifications hold for \(I_3\) and \(H_{ij}\). Define \(E^\mathfrak{p}_i \in \mathfrak{p}_0\) (\(1 \leq i \leq 3\)) by

\[E^\mathfrak{p}_i = E_{ii} - \frac{1}{3}I_3.\]

Let \(\mathfrak{t}\) denote the set of the diagonal matrices in \(\mathfrak{t}\):

\[\mathfrak{t} = \{ \text{diag} (Jt_1, Jt_2, Jt_3) : t_i \in \mathbb{R} \mid 1 \leq i \leq 3\}.\]

Then \(\mathfrak{t}\) is a Cartan subalgebra of \(\mathfrak{t}\). Let \(\varepsilon_i\) denote the linear form on \(\mathfrak{t}\) defined by \(\varepsilon_i(\text{diag} (Jt_1, Jt_2, Jt_3)) = t_i \mid 1 \leq i \leq 3\). The set of the dominant integral weights on \(\mathfrak{t}\) is given by

\[\Lambda = \{ \mu = \mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 + \mu_3 \varepsilon_3 : \mu_1 \geq \mu_2 \geq \mu_3, \mu_i \in \mathbb{Z} \mid 1 \leq i \leq 3\}.\]

The equivalence classes of irreducible representations of \(K\) are parametrized by \(\Lambda\).
2.2.2. Invariant differential operators. Let $\tau$ denote the standard representation of $K = U(3)$ on $V_{\tau} = \mathbb{C}^3$. The restriction $\tau|_{K_0}$ is irreducible. Equip $\mathbb{C}^3$ with the standard basis $\{e_1 = t(1, 0, 0), e_2 = t(0, 1, 0), e_3 = t(0, 0, 1)\}$. The highest weight of $\tau$ is $\varepsilon_1$ and $e_1$ is a highest weight vector. Since $\tau|_{M_0}$ decomposes into multiplicity-free sum of irreducible representations, the algebra $D(E_{\tau})$ of invariant differential operators on the homogeneous vector bundle $E_{\tau} \to G_0/K_0$ associated with $\tau$ is commutative ([2]). Moreover, the irreducible constituents of $\tau|_{M_0}$ are in a single $W$-orbits.

2.2.3. First order invariant differential operator. The highest weights of the standard representation $\tau$ of $K_0$ on $V_{\tau} = \mathbb{C}^3$ and the adjoint representation of $K_0$ on $(p_0) \otimes V_{\tau}$ are $\varepsilon_1$ and $\varepsilon_1 - \varepsilon_3$, respectively. By the Littlewood-Richardson rule for $K_0 = SU(3)$, $(p_0) \otimes V_{\tau}$ decomposes into multiplicity-free sum of three irreducible representations with highest weights $\varepsilon_1, 2\varepsilon_1 - 2\varepsilon_3, 3\varepsilon_1 + \varepsilon_2$. Hence, the isotypic component of $(p_0) \otimes V_{\tau}$ with the highest weight $\varepsilon_1$ gives a first order invariant differential operator on $E_{\tau}$, which is unique up to a constant multiple.

Lemma 2.9. The dimension of $((p_0) \otimes End(V_{\tau}))^{K_0}$ is 1 and a basis vector is given by the following matrix

\[
\begin{pmatrix}
\tilde{E}_1^p & E_{12}^p & E_{13}^p \\
E_{21}^p & \tilde{E}_2^p & E_{23}^p \\
E_{31}^p & E_{32}^p & \tilde{E}_3^p
\end{pmatrix},
\]

with respect to the basis $\{e_1, e_2, e_3\}$.

Proof. We employ the $GL$-picture as in [6]. The highest weight of $\tau$ is $\varepsilon_1$ and the Gelfand-Zelevinsky basis of $V_{\tau}$ is parametrized by the set of $G$-patterns

\[
\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 \\ 0 \end{pmatrix},
\]

which correspond to $e_1, e_2, e_3$ respectively (cf. [6] Lemma 4.1]).

There is a unique $K$-homomorphism $\iota_2$ of $V_{\tau}$ into $(p_0) \otimes V_{\tau}$ and an explicit description of $\iota_2$ is given by [6] Lemma 4.3, Theorem 4.4. For a $G$-pattern $M$, let $f(M)$ denote the corresponding Gelfand-Zelevinsky basis. We put

\[
u_1 = f \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \nu_2 = f \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \nu_3 = f \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Applying [6, Theorem 4.4] to the highest weight $\varepsilon_1$ of $\tau$, we have

$$\nu_2(u_1) = f \left( \begin{array}{ccc} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \otimes u_1 - f \left( \begin{array}{ccc} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \otimes u_2 + f \left( \begin{array}{ccc} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \otimes u_3$$

$$\nu_2(u_2) = f \left( \begin{array}{ccc} 1 & 0 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 0 \end{array} \right) \otimes u_1 - f \left( \begin{array}{ccc} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \otimes u_2 + f \left( \begin{array}{ccc} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \otimes u_3$$

$$\nu_2(u_3) = f \left( \begin{array}{ccc} 1 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & 0 \end{array} \right) \otimes u_1 - f \left( \begin{array}{ccc} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right) \otimes u_2 + f \left( \begin{array}{ccc} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \otimes u_3.$$ 

By [6, Lemma 4.3], we have

$$\begin{pmatrix} \nu_2(u_1) \\ \nu_2(u_2) \\ \nu_2(u_3) \end{pmatrix} = \begin{pmatrix} E^p_{11} & E^p_{12} & E^p_{13} \\ E^p_{21} & E^p_{22} & E^p_{23} \\ E^p_{31} & E^p_{32} & E^p_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. $$

Remark 2.10. There is a minor misprint in [6]. The right hand side of the fifth line of [6, Lemma 4.2] should be

$$-\frac{1}{3}(H^p_{12} + 2H^p_{23}).$$

We will give the $\tau$-radial part of (2.11). The contragredient representation $\tau^*$ has the highest weight $-\varepsilon_3$. The space $(V_{\tau^*} \otimes V_{\tau})^M_0$ is a 3-dimensional vector space with a basis

$$\begin{pmatrix} e^*_1 \otimes e_1, e^*_2 \otimes e_2, e^*_3 \otimes e_3 \end{pmatrix}. $$

The following proposition gives the radial part of (2.11) with respect to this basis in the coordinates (2.8) of $a$.

**Proposition 2.11.** With respect to the basis (2.12) of $(V_{\tau^*} \otimes V_{\tau})^M_0$, the $\tau$-radial part of the first order invariant differential operator (2.11) has the following matrix expression.

$$\begin{pmatrix} \partial' + \coth t_{12} + \coth t_{13} \\ -\frac{1}{\sinh t_{21}} & \partial + \coth t_{21} + \coth t_{23} \\ -\frac{1}{\sinh t_{31}} & -\frac{1}{\sinh t_{32}} & \partial + \coth t_{31} + \coth t_{32} \end{pmatrix}. $$
Proof. We consider the first row of the matrix (2.11):

\[ \tilde{E}_1^p e_1 + E_{12}^p e_2 + E_{13}^p e_3. \]

Let \( j = 2 \) or \( 3 \). By Lemma 1.1, we have

\[ E_{1j}^p e_j = \coth t_{1j} e_1 - \frac{1}{\sinh t_{1j}} \text{Ad}(a^{-1}) E_{1j}^p e_j. \]

Since \( \text{Ad}(a^{-1}) E_{1j}^p \) acts on \( V_r \cong \mathbb{C}^3 \) as the multiplication by \( t_{ij} E_{ij}^p = E_{ij}^p \),
\( \text{Ad}(a^{-1}) E_{1j}^p e_j = e_j^* \) by [6, Table 2]. Thus the action of the first row of (2.11)
on the coefficient of \( e_1^* \otimes e_1 \) is given by

\[
(\tilde{E}_1^p + \coth t_{12} + \coth t_{13}) e_1^* \otimes e_1 - \frac{1}{\sinh t_{12}} e_2^* \otimes e_2 - \frac{1}{\sinh t_{13}} e_3^* \otimes e_3.
\]

We can do similar computations for the second and third rows of the matrix (2.11) and obtain the matrix expression.

\[ \square \]

2.2.4. Radial part of the Casimir operator. We put \( E_{ij}^{(1)} = \frac{1}{2}(E_{ij}^p + E_{ij}^p) \) and \( E_{ij}^{(2)} = \frac{1}{2}(E_{ij}^p - E_{ij}^p) \). For \( 1 \leq i \neq j \leq 3 \), \( \{E_{ij}^{(1)}, E_{ij}^{(2)}\} \) forms a basis of the root space \( \mathfrak{g}^{e_i - e_j} \).

The Casimir operator \( \Omega_{\mathfrak{g}_0} \) for \( \mathfrak{g}_0 = \mathfrak{sl}(3, \mathbb{C}) \) is given by (2.7). Let \( \Omega^{(1)} \) and \( \Omega^{(2)} \) denote the elements of \( U((\mathfrak{g}_0)_{\mathbb{C}}) \) that correspond to \( \Omega_{\mathfrak{g}_0} \otimes 1 \) and \( 1 \otimes \Omega_{\mathfrak{g}_0} \) by the isomorphism (2.10), respectively. That is,

\[
\Omega^{(i)} = \frac{1}{9}(E_{11}^{(i)} + E_{22}^{(i)} + E_{33}^{(i)})^2 - \frac{1}{3}(E_{11}^{(i)} E_{22}^{(i)} + E_{22}^{(i)} E_{33}^{(i)} + E_{33}^{(i)} E_{11}^{(i)}) + \frac{1}{6}(E_{12}^{(i)} E_{21}^{(i)} + E_{21}^{(i)} E_{12}^{(i)} + E_{13}^{(i)} E_{31}^{(i)} + E_{31}^{(i)} E_{13}^{(i)} + E_{23}^{(i)} E_{32}^{(i)} + E_{32}^{(i)} E_{23}^{(i)})
\]

for \( i = 1, 2 \). The Casimir operator \( \Omega \) for \( (\mathfrak{g}_0)_{\mathbb{C}} \) is given by \( \Omega = \Omega^{(1)} + \Omega^{(2)} \).

Remark 2.12. \( \Omega^{(i)} (i = 1, 2) \) belongs to the center of the universal enveloping algebra of \( \mathfrak{g}_{\mathbb{C}} \). \( -3\Omega^{(i)} + \frac{1}{3}(E_{11}^{(i)} + E_{22}^{(i)} + E_{33}^{(i)})^2 \) is denoted by \( C P_2^{(i)} \) in [6, Section 5].

The following proposition gives the radial part of the Casimir operator. We take coordinate \((t_1, t_2, t_3)\) of \( \mathfrak{a} \) and put \( t_{ij} = t_i - t_j \) as in § 2.2.1.

Proposition 2.13. The \( \tau \)-radial part of \( \Omega \) is given by

\[
-6R_\tau(\Omega) + \frac{1}{3} = \partial_1 \partial_2 + \partial_2 \partial_3 + \partial_3 \partial_1 - \{\coth t_{12}(\partial_1 - \partial_2) + \coth t_{13}(\partial_1 - \partial_3) + \coth t_{23}(\partial_2 - \partial_3)\}
\]

\[
+ \left( \begin{array}{ccc}
\frac{1}{\sinh^2 t_{12}} & \frac{1}{\sinh^2 t_{13}} & \frac{\cosh t_{12}}{\sinh^2 t_{12}} \\
\frac{\cosh t_{12}}{\sinh^2 t_{12}} & \frac{1}{\sinh^2 t_{23}} & \frac{1}{\sinh^2 t_{23}} \\
\frac{\sinh t_{13}}{\sinh^2 t_{13}} & \frac{\sinh t_{23}}{\sinh^2 t_{23}} & \frac{1}{\sinh^2 t_{23}}
\end{array} \right)
\]
with respect to the basis \( \{2.12\} \) of \((V_\tau \otimes V_\tau)^{M_0}\).

Proof. It follows from Lemma 1.2 that

\[
-6 \Omega = -\frac{1}{3}(E^p_{11} + E^p_{22} + E^p_{33})^2 + E^p_{11}E^p_{22} + E^p_{22}E^p_{33} + E^p_{33}E^p_{11}
- \frac{1}{3}(E^t_{11} + E^t_{22} + E^t_{33})^2 + E^t_{11}E^t_{22} + E^t_{22}E^t_{33} + E^t_{33}E^t_{11}
- \sum_{1 \leq i < j \leq 3} \coth(e_i - e_j) (E^p_{ii} - E^p_{jj})
+ \frac{1}{2} \sum_{1 \leq i < j \leq 3} \frac{1}{\sinh^2(e_i - e_j)} E^t_{ij}E^t_{ji} + \Ad(a^{-1})(E^t_{ij}E^t_{ji})
- 2 \cosh(e_i - e_j) (\Ad(a^{-1})E^t_{ij})E^t_{ji}.
\]

Since \( E^t_{ij} e_k = \delta_{jk} e_i \) and \( E^t_{ij} e_k^* = -\delta_{ik} e_j^* \) for \( 1 \leq i, j, k \leq 3 \), the proposition follows. \( \square \)

2.3. The case of \( SU^*(6)/Sp(3) \).

2.3.1. Notation. Let \( G/K = SL(3, \mathbb{H})/SU(3, \mathbb{H}) \simeq SU^*(6)/Sp(3) \). The Lie algebra \( \mathfrak{sl}(3, \mathbb{H}) \) is isomorphic to

\[
\mathfrak{g} = \mathfrak{su}^*(6) = \left\{ \left( \begin{array}{cc} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{array} \right) : Z_1, Z_2 \in M(3, \mathbb{C}), \Tr Z_1 + \Tr \overline{Z}_1 = 0 \right\}.
\]

Define a Cartan involution by \( \theta X = -{}^t\overline{X} \) for \( X \in \mathfrak{g} \). Then we have the corresponding Cartan decomposition \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) with

\[
\mathfrak{k} = \left\{ \left( \begin{array}{cc} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{array} \right) : Z_1, Z_2 \in M(3, \mathbb{C}), {}^t\overline{Z}_1 = -Z_1, {}^tZ_2 = Z_2 \right\} \simeq \mathfrak{sp}(3)
\]

\[
\mathfrak{p} = \left\{ \left( \begin{array}{cc} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{array} \right) : Z_1, Z_2 \in M(3, \mathbb{C}), {}^t\overline{Z}_1 = Z_1, \Tr Z_1 = 0, {}^tZ_2 = -Z_2 \right\}.
\]

We have \( \dim G/K = \dim \mathfrak{p} = 14 \).

Complexifications of \( \mathfrak{k} \) and \( \mathfrak{p} \) are

\[
\mathfrak{k}_C = \mathfrak{sp}(3, \mathbb{C}) = \left\{ \left( \begin{array}{cc} A & B \\ C & -{}^tA \end{array} \right) : A, B, C \in M(3, \mathbb{C}), {}^tB = B, {}^tC = C \right\},
\]

\[
\mathfrak{p}_C = \left\{ \left( \begin{array}{cc} A & B \\ C & {}^tA \end{array} \right) : A, B, C \in M(3, \mathbb{C}), {}^tB = -B, {}^tC = -C, \Tr A = 0 \right\}.
\]
Let $E_{ij}$ denote the $3 \times 3$ matrix with $ij$ entry 1 and all other entries 0. Define

$$Z_{ij}^{(1)} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} \quad (1 \leq i, j \leq 3),$$

$$Z_{ij}^{(2)} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix} \quad (1 \leq i \leq j \leq 3),$$

$$Z_{ij}^{(3)} = \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix} \quad (1 \leq i \leq j \leq 3).$$

Then

$$\{Z_{ij}^{(1)} (1 \leq i, j \leq 3), Z_{ij}^{(2)} (1 \leq i \leq j \leq 3), Z_{ij}^{(3)} (1 \leq i \leq j \leq 3)\}$$

forms a basis of $\mathfrak{t}_\mathbb{C} = \mathfrak{sp}(3, \mathbb{C})$. Define

$$Y_{ij}^{(1)} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} \quad (1 \leq i, j \leq 3),$$

$$Y_{ij}^{(2)} = \begin{pmatrix} 0 & E_{ij} - E_{ji} \\ 0 & 0 \end{pmatrix} \quad (1 \leq i < j \leq 3),$$

$$Y_{ij}^{(3)} = \begin{pmatrix} 0 & 0 \\ E_{ij} - E_{ji} & 0 \end{pmatrix} \quad (1 \leq i < j \leq 3).$$

Then

$$\{Y_{11}^{(1)} - Y_{22}^{(1)}, Y_{22}^{(1)} - Y_{33}^{(1)}, Y_{ij}^{(1)} (1 \leq i \neq j \leq 3), Y_{ij}^{(2)} (1 \leq i < j \leq 3), Y_{ij}^{(3)} (1 \leq i < j \leq 3)\}$$

forms a basis of $\mathfrak{p}_\mathbb{C}$. Let

$$\tilde{Y}_{ii}^{(1)} = Y_{ii}^{(1)} - \frac{1}{3}(Y_{11}^{(1)} + Y_{22}^{(1)} + Y_{33}^{(1)}) \quad (1 \leq i \leq 3).$$

and

$$a = \{t_1 \tilde{Y}_{11}^{(1)} + t_2 \tilde{Y}_{22}^{(1)} + t_3 \tilde{Y}_{33}^{(1)} : t_1, t_2, t_3 \in \mathbb{R}, t_1 + t_2 + t_3 = 0\}.$$

Then $a$ is a maximal abelian subspace of $\mathfrak{p}$. Let $e_i$ denote the linear form on $a$ defined by $e_i(Y_{jj}^{(1)}) = \delta_{ij} (1 \leq i, j \leq 3)$. Then the restricted root system for $(\mathfrak{g}, a)$ is given by

$$\Sigma = \{e_i - e_j : 1 \leq i \neq j \leq 3\}.$$

Let $\Sigma^+$ denote the positive system defined by

$$\Sigma^+ = \{e_1 - e_2, e_1 - e_3, e_2 - e_3\}.$$

The root system $\Sigma$ is of type $A_2$ and the Weyl group $W$ is $S_3$. 

For $i < j$ define

$$X_{ij}^{(1)} = \begin{pmatrix} E_{ij} & 0 \\ 0 & E_{ij} \end{pmatrix}, \quad X_{ij}^{(2)} = \begin{pmatrix} \sqrt{-1}E_{ij} & 0 \\ 0 & -\sqrt{-1}E_{ij} \end{pmatrix},$$

$$X_{ij}^{(3)} = \begin{pmatrix} 0 & -E_{ij} \\ E_{ij} & 0 \end{pmatrix}, \quad X_{ij}^{(4)} = \begin{pmatrix} 0 & \sqrt{-1}E_{ij} \\ \sqrt{-1}E_{ij} & 0 \end{pmatrix}.$$  

Then $\{X_{ij}^{(1)}, X_{ij}^{(2)}, X_{ij}^{(3)}, X_{ij}^{(4)}\}$ forms a basis of $\mathfrak{g}_{e_i - e_j}$.

Define

$$t = \bigoplus_{i=1}^{3} \mathbb{R}\sqrt{-1}Z_{ii}^{(1)}.$$  

It is a Cartan subalgebra of $\mathfrak{k}$. The root system for $(\mathfrak{k}, \mathfrak{t})$ is

$$\Delta(\mathfrak{k}, \mathfrak{t}) = \{ \pm \epsilon_i \pm \epsilon_j \ (1 < i < j < 3), \pm 2\epsilon_k \ (1 < k < 3) \},$$

where $\epsilon_i$ is a linear forms on $\mathfrak{k}$ defined by $\epsilon_i(Z_{jj}^{(1)}) = \delta_{ij} (1 < i, j < 3)$. $Z_{ij}^{(1)} (i \neq j), Z_{ij}^{(2)} (i \leq j), Z_{ij}^{(3)} (i < j)$ are root vectors for $\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, - (\epsilon_i + \epsilon_j) \in \Delta(\mathfrak{k}, \mathfrak{t})$, respectively. Define positive system $\Delta(\mathfrak{k}, \mathfrak{t})^+$ by

$$\Delta(\mathfrak{k}, \mathfrak{t})^+ = \{ \epsilon_i \pm \epsilon_j \ (1 < i < j < 3), \ 2\epsilon_k \ (1 < k < 3) \}.$$  

The adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}_C$ is an irreducible representation with the highest weight $\epsilon_1 + \epsilon_2$. The subspace $\mathfrak{a}_C \subset \mathfrak{p}_C$ consists of zero weight vectors, $Y_{ij}^{(1)} (i \neq j)$ is a weight vector with the weight $\epsilon_i - \epsilon_j$, and $Y_{ij}^{(2)}, Y_{ij}^{(3)}$ are weight vectors with the weight $\epsilon_i + \epsilon_j, -(\epsilon_i + \epsilon_j)$ respectively.

Let $m$ and $M$ denote the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and $K$ respectively. Then $m = sp(1) \oplus sp(1) \oplus sp(1), M = Sp(1) \times Sp(1) \times Sp(1)$, and $m_C$ is generated by $Z_{ii}^{(1)}, Z_{ii}^{(2)}, Z_{ii}^{(3)} (1 \leq i \leq 3)$.

2.3.2. Invariant differential operators. Let $(\tau, V_\tau)$ be the standard six dimensional representation of $K = Sp(3)$. The highest weight of $\tau$ is $\epsilon_1$ and weights of $\tau$ are $\pm \epsilon_i \ (1 \leq i \leq 3)$. Let $\rho_i$ denote the $(i+1)$-dimensional irreducible representation of $Sp(1) \simeq SU(2)$. The restriction of $\tau$ to $M \simeq Sp(1) \times Sp(1) \times Sp(1)$ decomposes into two-dimensional irreducible representations $\rho_1 \boxtimes \rho_0 \boxtimes \rho_0, \rho_0 \boxtimes \rho_1 \boxtimes \rho_0, \rho_0 \boxtimes \rho_0 \boxtimes \rho_1$ with highest weights $\epsilon_1, \epsilon_2$, and $\epsilon_3$, respectively.

Since $\tau|_M$ decomposes into multiplicity-free sum of irreducible representations, the algebra $\mathbb{D}(E_\tau)$ of invariant differential operators on the homogeneous vector bundle $E_\tau \rightarrow G/K$ associated with $\tau$ is commutative (2). Moreover, the irreducible constituents of $\tau|_M$ are in a single $W$-orbits.
2.3.3. First order invariant differential operator. We recall that representations of $K = Sp(6)$ on $p_C$ and $V_\tau$ are irreducible representations with the highest weights $\varepsilon_1 + \varepsilon_2$ and $\varepsilon_1$, respectively. By the Littlewood-Richardson rule for $Sp(6)$, $p_C \otimes V_\tau$ decomposes into multiplicity-free sum of three irreducible representations of $\mathfrak{sl}_6\mathbb{C}$ with highest weights $\varepsilon_1$, $2\varepsilon_1 + \varepsilon_2$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ ([11]). Hence, the isotypic component of $p_C \otimes V_\tau$ with the highest weight $\varepsilon_1$ gives a first order invariant differential operator on $E_\tau$, which is unique up to a constant multiple. We will give weight vectors of the isotypic component of $p_C \otimes V_\tau$ with the highest weight $\varepsilon_1$.

Let $e_i$ be the $i$-th standard unit vector in $\mathbb{R}^6$, that is, its $i$-th component is one and other components are all zero. Then $e_1, e_2, \ldots, e_6$ are weight vectors for the standard representation of $\mathfrak{sp}(3)$ on $\mathbb{C}^6$ with weights $\varepsilon_1, \varepsilon_2, \varepsilon_3, -\varepsilon_1, -\varepsilon_2, -\varepsilon_3$, respectively.

**Lemma 2.14.** The mapping $e_j \mapsto w_j$ ($1 \leq j \leq 6$) defined by the following formulae gives an $K$-equivariant injection from $V_\tau = \mathbb{C}^6$ into $p_C \otimes V_\tau$. In particular, $w_1$ is a highest weight vector with the highest weight $\varepsilon_1$.

$$w_1 = \tilde{Y}^{(1)}_{12} \otimes e_1 + \tilde{Y}^{(1)}_{13} \otimes e_2 + Y^{(1)}_{13} \otimes e_3 + Y^{(2)}_{12} \otimes e_5 + Y^{(2)}_{13} \otimes e_6,$$

$$w_2 = Y^{(1)}_{21} \otimes e_1 + Y^{(2)}_{22} \otimes e_2 + Y^{(1)}_{23} \otimes e_3 + Y^{(2)}_{21} \otimes e_4 + Y^{(2)}_{23} \otimes e_5,$$

$$w_3 = Y^{(1)}_{31} \otimes e_1 + Y^{(2)}_{32} \otimes e_2 + \tilde{Y}^{(1)}_{33} \otimes e_3 + Y^{(2)}_{31} \otimes e_4 + Y^{(2)}_{32} \otimes e_5,$$

$$w_4 = Y^{(3)}_{12} \otimes e_2 + Y^{(3)}_{13} \otimes e_3 + \tilde{Y}^{(1)}_{11} \otimes e_4 + Y^{(1)}_{21} \otimes e_5 + Y^{(1)}_{31} \otimes e_6,$$

$$w_5 = Y^{(3)}_{21} \otimes e_1 + Y^{(3)}_{23} \otimes e_3 + Y^{(1)}_{12} \otimes e_4 + \tilde{Y}^{(1)}_{22} \otimes e_5 + Y^{(1)}_{32} \otimes e_6,$$

$$w_6 = Y^{(3)}_{31} \otimes e_1 + Y^{(3)}_{32} \otimes e_2 + Y^{(1)}_{31} \otimes e_4 + Y^{(1)}_{23} \otimes e_5 + \tilde{Y}^{(1)}_{33} \otimes e_6.$$

**Proof.** The subspace of $p_C \otimes V_\tau$ with the weight $\varepsilon_1$ is spanned by

$$\tilde{Y}^{(1)}_{12} \otimes e_1, \tilde{Y}^{(1)}_{22} \otimes e_1, Y^{(1)}_{12} \otimes e_2, Y^{(1)}_{13} \otimes e_3, Y^{(2)}_{12} \otimes e_5, Y^{(2)}_{13} \otimes e_6.$$

A highest weight vector with the highest weight $\varepsilon_1$ is a linear combination of the above six weight vectors that is killed by $Z^{(1)}_{12}$, $Z^{(1)}_{23}$, and $Z^{(2)}_{33}$. Computing actions of these positive root vectors, we see that $w_1$ is a unique highest weight vector with the highest weight $\varepsilon_1$ up to a constant multiple. Other weight vectors are given by $w_2 = Z^{(1)}_{21} e_1$, $w_3 = Z^{(1)}_{32} w_2$, $w_4 = Z^{(3)}_{13} w_3$, $w_5 = -Z^{(1)}_{12} w_4$, $w_6 = -Z^{(1)}_{23} w_5$. \hfill \square

We write a $C^\infty$-section $f$ of $E_\tau$ as $f = \sum_{i=1}^6 f_i e_i$ ($f_i \in C^\infty(G)$). Then the action of $w_1$ on $f$ is given by

$$w_1 f = \tilde{Y}^{(1)}_{11} f_1 e_1 + Y^{(1)}_{12} f_2 e_2 + Y^{(1)}_{13} f_3 e_3 + Y^{(2)}_{12} f_5 e_5 + Y^{(2)}_{13} f_6 e_6.$$

Actions of $w_2, \ldots, w_6$ are given in similar ways.
By Lemma [1.1] we have

\[(2.14) \quad Y_{ij}^{(l)} = \coth t_{ij} Z_{ij}^{(l)} - \frac{1}{\sinh t_{ij}} \text{Ad}(a^{-1}) Z_{ij}^{(l)} \quad (1 \leq i \neq j \leq 3, \ 1 \leq l \leq 3).\]

Since \(Z_{ij}^{(1)} e_j = e_1\) and \(Z_{ij}^{(2)} e_{4+j} = e_1\) for \(j = 1, 2\), it follows from Lemma [2.14] that

\[w_1 = (\tilde{Y}_{11}^{(1)} + 2 \coth t_{12} + 2 \coth t_{13}) \otimes e_1\]

\[- \frac{1}{\sinh t_{12}} \text{Ad}(a^{-1}) Z_{12}^{(1)} \otimes e_2 - \frac{1}{\sinh t_{13}} \text{Ad}(a^{-1}) Z_{13}^{(1)} \otimes e_3\]

\[- \frac{1}{\sinh t_{12}} \text{Ad}(a^{-1}) Z_{12}^{(2)} \otimes e_5 - \frac{1}{\sinh t_{13}} \text{Ad}(a^{-1}) Z_{13}^{(2)} \otimes e_6.\]

In a similar way, we have

\[w_2 = \frac{1}{\sinh t_{12}} \text{Ad}(a^{-1}) Z_{21}^{(1)} \otimes e_1 + (\tilde{Y}_{22}^{(1)} - 2 \coth t_{12} + 2 \coth t_{23}) \otimes e_2\]

\[- \frac{1}{\sinh t_{23}} \text{Ad}(a^{-1}) Z_{23}^{(1)} \otimes e_3 + \frac{1}{\sinh t_{23}} \text{Ad}(a^{-1}) Z_{23}^{(2)} \otimes e_4\]

\[- \frac{1}{\sinh t_{23}} \text{Ad}(a^{-1}) Z_{23}^{(2)} \otimes e_6,\]

\[w_3 = \frac{1}{\sinh t_{13}} \text{Ad}(a^{-1}) Z_{31}^{(1)} \otimes e_1 + \frac{1}{\sinh t_{23}} \text{Ad}(a^{-1}) Z_{32}^{(1)} \otimes e_2\]

\[+ (\tilde{Y}_{33}^{(1)} - 2 \coth t_{13} - 2 \coth t_{23}) \otimes e_3\]

\[+ \frac{1}{\sinh t_{13}} \text{Ad}(a^{-1}) Z_{13}^{(2)} \otimes e_4 + \frac{1}{\sinh t_{23}} \text{Ad}(a^{-1}) Z_{23}^{(2)} \otimes e_5,\]

\[w_4 = - \frac{1}{\sinh t_{12}} \text{Ad}(a^{-1}) Z_{12}^{(1)} \otimes e_2 - \frac{1}{\sinh t_{13}} \text{Ad}(a^{-1}) Z_{13}^{(1)} \otimes e_3\]

\[+ (\tilde{Y}_{11}^{(1)} + 2 \coth t_{12} + 2 \coth t_{13}) \otimes e_4\]

\[+ \frac{1}{\sinh t_{12}} \text{Ad}(a^{-1}) Z_{12}^{(1)} \otimes e_5 + \frac{1}{\sinh t_{13}} \text{Ad}(a^{-1}) Z_{13}^{(1)} \otimes e_6,\]

\[w_5 = \frac{1}{\sinh t_{12}} \text{Ad}(a^{-1}) Z_{12}^{(3)} \otimes e_1 - \frac{1}{\sinh t_{23}} \text{Ad}(a^{-1}) Z_{23}^{(3)} \otimes e_3\]

\[+ (\tilde{Y}_{22}^{(1)} - 2 \coth t_{12} + 2 \coth t_{23}) \otimes e_5 + \frac{1}{\sinh t_{23}} \text{Ad}(a^{-1}) Z_{23}^{(1)} \otimes e_6,\]

\[w_6 = \frac{1}{\sinh t_{13}} \text{Ad}(a^{-1}) Z_{13}^{(3)} \otimes e_1 + \frac{1}{\sinh t_{23}} \text{Ad}(a^{-1}) Z_{23}^{(3)} \otimes e_2\]

\[- \frac{1}{\sinh t_{13}} \text{Ad}(a^{-1}) Z_{13}^{(1)} \otimes e_4 - \frac{1}{\sinh t_{23}} \text{Ad}(a^{-1}) Z_{23}^{(1)} \otimes e_5\]

\[+ (\tilde{Y}_{33}^{(1)} - 2 \coth t_{13} - 2 \coth t_{23}) \otimes e_6.\]
Since \( f(\alpha a) = f(\alpha m) \) for \( m \in M \) and \( \alpha \in A \), \( f \) vanishes outside \((V_\tau \otimes V_\tau)^M\). Since \( M \cong Sp(1) \times Sp(1) \times Sp(1) \) and the positive root for each \( Sp(1) \) is \( 2\varepsilon_i \) \((1 \leq i \leq 3)\),

\[
(2.15) \quad \{e_i^* \otimes e_1 - e_i^* \otimes e_4, \quad e_i^* \otimes e_2 - e_i^* \otimes e_5, \quad e_i^* \otimes e_3 - e_i^* \otimes e_6\}
\]

forms a basis of \((V_\tau \otimes V_\tau)^M\). For \( Z \in \mathfrak{t}_C, \) \( \text{Ad}(a^{-1})Z \) acts on \( f \) as \( \tau^*(Z) \).

It follows from the above expressions of \( w_1, \ldots, w_6 \), we have the following proposition.

**Proposition 2.15.** With respect to the basis \((2.15)\), the \( \tau \)-radial part of the first order invariant differential operator given in Lemma 2.14 has the following matrix expression.

\[
\begin{pmatrix}
\partial_1' & 0 & 0 \\
0 & \partial_2' & 0 \\
0 & 0 & \partial_3'
\end{pmatrix}
+ \begin{pmatrix}
2(\coth t_{12} + \coth t_{13}) & -\frac{2}{\sinh t_{12}} & -\frac{2}{\sinh t_{13}} \\
\frac{2}{\sinh t_{12}} & 2(-\coth t_{12} + \coth t_{23}) & -\frac{2}{\sinh t_{23}} \\
\frac{2}{\sinh t_{13}} & -\frac{2}{\sinh t_{23}} & -2(\coth t_{13} + \coth t_{23})
\end{pmatrix}.
\]

2.3.4. Radial part of the Casimir operator. The following proposition gives the \( \tau \)-radial part of \( \Omega \).

**Proposition 2.16.** The \( \tau \)-radial part of the Casimir operator is given by

\[
-12R_\tau(\Omega) + 3 = \partial_1' \partial_1' + \partial_2' \partial_2' + \partial_3' \partial_3' + 2\{\coth t_{12}(\partial_1 - \partial_2) + \coth t_{13}(\partial_1 - \partial_3) + \coth t_{23}(\partial_2 - \partial_3)\}
\]

\[
+ 2 \begin{pmatrix}
\frac{1}{\sinh^2 t_{12}} + \frac{1}{\sinh^2 t_{13}} & -\frac{\cosh t_{12}}{\sinh^2 t_{12}} & -\frac{\cosh t_{13}}{\sinh^2 t_{13}} \\
-\frac{\cosh t_{12}}{\sinh^2 t_{12}} & \frac{1}{\sinh^2 t_{12}} + \frac{1}{\sinh^2 t_{23}} & -\frac{\cosh t_{23}}{\sinh^2 t_{23}} \\
-\frac{\cosh t_{13}}{\sinh^2 t_{13}} & -\frac{\cosh t_{23}}{\sinh^2 t_{23}} & \frac{1}{\sinh^2 t_{13}} + \frac{1}{\sinh^2 t_{23}}
\end{pmatrix}
\]

with respect to the basis \((2.15)\) of \((V_\tau \otimes V_\tau)^M\).

**Proof.** It follows from Lemma 2.2 that

\[
-12 \Omega = -\frac{1}{3}(Y_{11}^{(1)} + Y_{22}^{(1)} + Y_{33}^{(1)})^2 + Y_{11}^{(1)} Y_{22}^{(1)} + Y_{22}^{(1)} Y_{33}^{(1)} + Y_{33}^{(1)} Y_{11}^{(1)}
\]

\[
- \frac{1}{4} \sum_{i=1}^3 \{2(Z_{ii}^{(1)})^2 + Z_{ii}^{(2)} Z_{ii}^{(3)} + Z_{ii}^{(3)} Z_{ii}^{(2)}\}
\]

\[
- 2 \sum_{1 \leq i < j \leq 3} \coth(e_i - e_j) (Y_{ii}^{(1)} - Y_{jj}^{(1)})
\]

\[
+ \frac{1}{2} \sum_{1 \leq i < j \leq 3} \frac{1}{\sinh^2(e_i - e_j)} \{Z_{ij}^{(1)} Z_{ji}^{(1)} + \text{Ad}(a^{-1})(Z_{ij}^{(1)} Z_{ji}^{(1)})\}
\]
\[ + Z_{ij}^{(1)} Z_{ij}^{(1)} + \text{Ad}(a^{-1})(Z_{ij}^{(1)} Z_{ij}^{(1)}) + Z_{ij}^{(3)} Z_{ij}^{(2)} + \text{Ad}(a^{-1})(Z_{ij}^{(3)} Z_{ij}^{(2)}) \\
+ Z_{ij}^{(2)} Z_{ij}^{(3)} + \text{Ad}(a^{-1})(Z_{ij}^{(2)} Z_{ij}^{(3)}) \}
- \sum_{1 \leq i < j \leq 3} \frac{\cosh(e_i - e_j)}{\sinh^2(e_i - e_j)} \{ Z_{ij}^{(1)} \text{Ad}(a^{-1})Z_{ji}^{(1)} + Z_{ji}^{(1)} \text{Ad}(a^{-1})Z_{ij}^{(1)} \\
+ Z_{ij}^{(3)} \text{Ad}(a^{-1})Z_{ji}^{(2)} + Z_{ji}^{(2)} \text{Ad}(a^{-1})Z_{ij}^{(3)} \}. \]

The proposition follows by computing actions of \( Z_{ij}^{(k)} \) \( (1 \leq i, j, k \leq 3) \) on weight vectors.

2.4. **Proof of Theorem 2.1.** In subsection 2.1, 2.2, and 2.3, we have proved commutativity of \( \mathbb{D}(E_\tau) \) and part (iii) of Theorem 2.1. Moreover, we have seen that \( \tau|_M \) decomposes into multiplicity free sum of three irreducible representations of \( M \).

We will show that \( W \simeq M'/M \) acts transitively on three irreducible constituents of \( \tau|_M \). We first consider the case of \( K = \mathbb{R} \). For \( 1 \leq i < j \leq 3 \), let \( s_{ij} \in M' \) denote representative of the reflection \( s_{ij} \in W \) given by

\[ s_{12}' = \exp \frac{\pi}{2} K_3, \quad s_{23}' = \exp \frac{\pi}{2} K_1, \quad s_{13}' = \exp \frac{\pi}{2} K_2. \]

The irreducible constituents of \( \tau|_M \) are \( \mathbb{R} e_1, \mathbb{R} e_2, \mathbb{R} e_3 \). Since \( s_{ij}' \mathbb{R} e_k = \mathbb{R} e_{\delta_{ij,k}} \), \( W = S_3 \) acts transitively on the irreducible constituents of \( \tau|_M \) as a permutation group. The case of \( K = \mathbb{C} \) is almost same as the case of \( K = \mathbb{R} \). In the case of \( K = \mathbb{H} \), the irreducible constituents of \( \tau|_M \) are \( \mathbb{R} e_1 \oplus \mathbb{R} e_4, \mathbb{R} e_2 \oplus \mathbb{R} e_5, \mathbb{R} e_3 \oplus \mathbb{R} e_6 \). For \( 1 \leq i < j \leq 3 \), we can take a representative of \( s_{ij} \in W = S_3 \) in \( M' \) as \( s_{ij}' = \exp \frac{\pi}{2} (Z_{ij}^{(1)} - Z_{ji}^{(1)}) \) and we can show easily that \( W = S_3 \) acts transitively on the irreducible constituents of \( \tau|_M \) as a permutation group. Thus part (i) of Theorem 2.1 is proved.

Next, we will prove part (ii) of Theorem 2.1. First we consider the case of \( K = \mathbb{R} \). By Schur’s lemma, \( \text{End}_M(V_\tau) \simeq (V_{\tau^*} \otimes V_\tau)^M \) is a three dimensional vector space with basis \( \{ e_1^* \otimes e_1, e_2^* \otimes e_2, e_3^* \otimes e_3 \} \). The Weyl group \( W = S_3 \) acts on \( (V_{\tau^*} \otimes V_\tau)^M \) as a permutation group. Hence, the map from \( (U(\mathfrak{a}_C) \otimes \text{End}_M(V_\tau))^M \) to \( U(\mathfrak{a}_C) \) defined by \( \text{diag}(u_1, u_2, u_3) = u_3 \) is an algebra isomorphism onto \( U(\mathfrak{a}_C)^{W_{\tau_1-e_2}} \).

An element \( D \in \mathbb{D}(E_\tau) \) acts on the principal series representation of \( G \) with the \( K \)-type \( \tau \) and a parameter \( \lambda \in \mathfrak{a}_C^* \) as the multiplication by \( \gamma_\tau(D)(\lambda) \). For the Casimir operator, we have

\[ \gamma_\tau(\Omega - \Omega_m)(\lambda) = \langle \lambda, \lambda \rangle - \langle \rho, \rho \rangle \]

by [10] Proposition 3.2. For the first order operator \( D_1 \), we have

\[ \gamma_\tau(D_1)(\lambda) = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \]
by [10, Proposition 3.6]. Since $\lambda_3$ and $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$ are algebraically independent generators of $U(a_C)^{W_{e_1-e_2}} \simeq S(a_C)^{W_{e_1-e_2}}$, the algebra homomorphism $[1,1]$ is surjective, hence $\mathcal{D}(E_T) \simeq U(a_C)^{W_{e_1-e_2}}$.

We can prove Theorem 2.1 (ii) for $K = \mathbb{C}, \mathbb{H}$ in similar ways.

3. Commuting differential operators with $A_2$ symmetry

3.0.1. Matrix-valued commuting differential operators. In Theorem 2.1 we obtained matrix-valued commuting differential operators with $A_2$ symmetry as radial parts of invariant differential operators. We use the notation

$$\partial_i = \frac{\partial}{\partial t_i}, \quad t_{ij} = t_i - t_j$$

as before. In the $GL$-picture, we have differential operators $P_1, \tilde{Q}_1, \text{and } \tilde{P}_2$ given by

$$P_1 = \partial_1 + \partial_2 + \partial_3,$$

$$\tilde{Q}_1 = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \end{pmatrix},$$

$$+ k \begin{pmatrix} \coth t_{12} + \coth t_{13} & \sinh t_{12} & \sinh t_{13} \\ -\coth t_{12} + \coth t_{23} & -\coth t_{13} & -\coth t_{23} \\ -\coth t_{12} & -\coth t_{13} & -\coth t_{23} \end{pmatrix},$$

$$\tilde{P}_2 = L_2 - 4k^2$$

$$+ k \begin{pmatrix} \frac{1}{\sinh^2 t_{12}} & \frac{1}{\sinh^2 t_{13}} \\ -\frac{1}{\cosh^2 t_{12} - \cosh^2 t_{13}} & -\frac{1}{\cosh^2 t_{12} - \cosh^2 t_{13}} \\ -\frac{1}{\sinh^2 t_{13}} & -\frac{1}{\sinh^2 t_{13}} \end{pmatrix},$$

where

$$L_2 = \partial_1 \partial_2 + \partial_2 \partial_3 + \partial_3 \partial_1 - k \sum_{1 \leq i < j \leq 3} (\coth t_{ij})(\partial_i - \partial_j),$$

and $P_1, \tilde{Q}_1$, and $\tilde{P}_2$ mutually commute for $k = 1/2, 1, \text{and } 2$.

Put

$$\delta_k^{1/2} = \prod_{1 \leq i < j \leq 3} (\sinh t_{ij})^k.$$ 

The function $\delta_k$ is the density function for $G/K$ up to a constant multiple. Define $Q_1 = \delta_k^{1/2} \circ Q_1 \circ \delta_k^{-1/2}$ and $P_2 = \delta_k^{1/2} \circ \tilde{P}_2 \circ \delta_k^{-1/2}$. By easy computations, we have the following corollary of Theorem 2.1.
Corollary 3.1. For $k = 1/2, 1, $ and $2$, the differential operators $P_1, Q_1, P_2$ given by

$$P_1 = \partial_1 + \partial_2 + \partial_3,$$

$$Q_1 = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \end{pmatrix} + k \begin{pmatrix} 0 & \frac{1}{\sinh t_{12}} & \frac{1}{\sinh t_{12}} \\ \frac{1}{\sinh t_{12}} & 0 & \frac{1}{\sinh t_{23}} \\ \frac{1}{\sinh t_{12}} & \frac{1}{\sinh t_{23}} & 0 \end{pmatrix},$$

$$P_2 = \partial_1 \partial_2 + \partial_2 \partial_3 + \partial_3 \partial_1 + k(k - 1) \sum_{1 \leq i < j \leq 3} \frac{1}{\sinh^2 t_{ij}}$$

mutually commute.

It is likely that the operators in Theorem 2.1 or Corollary 3.1 mutually commute for any $k \in \mathbb{C}$. Indeed, we can see by direct computations of commutators that $P_1, Q_1,$ and $P_2$ mutually commute for any $k \in \mathbb{C}$. Moreover, the form of $P_1, Q_1,$ and $P_2$ suggest the following generalization.

Let $\beta(t)$ be an odd meromorphic function with a simple pole at the origin and define $P_1, Q_1,$ and $P_2$ by

$$P_1 = \partial_1 + \partial_2 + \partial_3,$$

$$Q_1 = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \end{pmatrix} + k \begin{pmatrix} 0 & \frac{1}{\sinh t_{12}} & \frac{1}{\sinh t_{12}} \\ \frac{1}{\sinh t_{12}} & 0 & \frac{1}{\sinh t_{23}} \\ \frac{1}{\sinh t_{12}} & \frac{1}{\sinh t_{23}} & 0 \end{pmatrix},$$

$$P_2 = \partial_1 \partial_2 + \partial_2 \partial_3 + \partial_3 \partial_1 + k(k - 1) \sum_{1 \leq i < j \leq 3} \frac{1}{\sinh^2 t_{ij}}$$

By computing commutators, operators (3.1), (3.2), and (3.3) mutually commute if and only if $\beta$ satisfies the following functional equation

$$- \beta(s) \beta^2(s + t) + \beta(s) \beta^2(t) + \beta(s + t) \beta'(t) + \beta'(s + t) \beta(t) = 0.$$

We know that $\beta(t) = 1/\sinh t$ is a solution of the above functional equation. Moreover, since (3.4) does not depend on $k$, $P_1$, $P_2$, and $Q_1$ in Corollary 3.1 mutually commute for any $k \in \mathbb{C}$.

Olshanetsky-Perelomov [15, Appendix A] solved a functional equation that is essentially equivalent to (3.4) in their study of classical integrability.
of a system associated with a symmetric space. They proved that the general solution of (3.4) is given by
\[ \beta(t) = \frac{a}{\text{sn}(ax|\kappa)} \quad (a \neq 0), \]
where sn is a Jacobi’s elliptic function. Moreover, \( \beta(t) \) is a real valued function on \( \mathbb{R} \) if and only if \( |a| = 1, \kappa = 1/a^2 \) or \( a \in \mathbb{R} \cup \sqrt{-1}\mathbb{R}, \kappa \in \mathbb{R} \). Hence, we have the following theorem.

**Theorem 3.2.** The operators (3.1), (3.2), and (3.3) mutually commute for any \( k \in \mathbb{C} \) and \( \beta(t) = a/\text{sn}(ax|\kappa) \) for any \( a \neq 0 \) and \( \kappa \).

### 3.1. \( W \)-equivariance of differential operators

Differential operators in Theorem 2.1 and Corollary 3.1 are originally radial components on \( \mathfrak{a} \) of differential operators on \( G \) with certain \( K \)-equivariance, hence they have certain \( W \)-equivariance.

Let \( d \) be a \( 3 \times 3 \) matrix-valued differential operator on \( \mathfrak{a} \). For \( w \in S_3 \), let \( P_w \) denote the permutation matrix defined by \( P_w = (\delta_{iw(j)})_{1 \leq i, j \leq 3} \). Let \( d^w \) denote the matrix-valued differential operator replacing \( t \) by \( w^{-1}t \) and \( \partial_j (1 \leq j \leq 3) \) by \( \partial_{w^{-1}(j)} \) in \( d \). Then our operators in Theorem 2.1 Corollary 3.1 and Theorem 3.2 have \( S_3 \)-invariance
\[ d^w = P_w^{-1}dP_w \quad (w \in S_3). \]

**Remark 3.3.** In the case of \( \beta(t) = 1/t \), the first order operator \( Q_1 \) is essentially the Dunkl operator and \( P_2 \) is essentially the Dunkl Laplacian for \( A_2 \) type root system (II). We do not know whether there is an analogous relation with the Cherednik operator in trigonometric case.

In the forthcoming paper, we will discuss joint eigenfunctions of mutually commuting differential operators in Theorem 3.2 in trigonometric case \( \beta(t) = 1/\sinh t \). In group case (\( k = 1/2, 1, 2 \)), \( \tau \)-radial part of a matrix coefficient of a principle series representation is a joint eigenfunction that is real analytic on \( \mathfrak{a} \). In the case of \( k = 1/2 \), Sekiguchi [19] and Sono [20] computed explicitly Harish-Chandra’s \( c \)-function. We will construct a real analytic joint eigenfunction for generic \( k \), which is a vector-valued analogue of the Heckman-Opdam hypergeometric function.

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