Identification of current-carrying part of a random resistor network: electrical approaches vs. graph theory algorithms

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Abstract. A set of current-carrying bonds of a random resistor network (RRN) is called the (effective) backbone. The (geometrical) backbone can be defined as a union of all self-avoiding walks between two given points on a network or between its opposite borders. These two definitions provide two different approaches for identification of backbones. On the one hand, one can treat an arbitrary network as RRN and calculate potentials and currents in this RRN. On the other hand, one can apply to the network some search algorithms on graphs. Each of these approaches are known to have both advantages and drawbacks. We have implemented several different algorithms for backbone identification. The algorithms were applied to backbone identification for different system sizes and concentrations of conducting bonds. Our analysis suggests that a universal algorithm suitable for any problem is hardly possible to offer. Most likely, each particular task needs a specific algorithm.

1. Introduction

For many decades, the physical properties of inhomogeneous media have attracted a lot of attention in the scientific community [1], particularly, due to numerous applications such as the production and use of nanocomposites [2, 3]. Theoretical prediction of the effective properties of such composites is very important for the analysis of material performance and for the design of new materials. Particular interest is paid to the electrical properties of binary materials. The theories and models relating to the electrical conductivity of mixtures of conducting and insulating species continue to attract great interest from researchers. One of the possible ways to theoretical study of binary mixtures is consideration them as random resistor networks (RRN). When a potential difference is applied between two points of such the network or between two bus bars attached to the opposite borders of the RRN, the set of current-carrying bonds is called the backbone. The remaining bonds are called dangling ends or dead ends. The backbone can be also defined as the union of all the self-avoiding walks. These two definitions provide two different approaches for identification of backbones. On the one hand, one can use Ohm’s law or Kirchhoff’s rules to calculate potentials and currents in the RRN [4, 5]. On the other hand, one can apply search algorithms on graphs [6, 7, 8, 9, 10, 11, 12]. In fact, the last algorithms of backbone identification belong to maze solving algorithms which, particularly, are applied to wire routing on chips [13]. Each of these two approaches have both advantages and drawbacks. To the best of our knowledge, a systematic comparison and analysis of the algorithms devoted to
identification of current-carrying part of the RRN is absent up to now. Particularly, the physical community is not familiar with search algorithms on graphs. In the present conference paper, we try to partially eliminate this gap. Our consideration is restricted only to two-dimensional (2D) RRN (planar graphs). The rest of the paper is constructed as follows. All necessary definitions are presented in section 2. In section 3, different method of backbone identification are briefly described and compared. Section 4 summarizes the main results.

2. Basic definitions
Consider an arbitrary 2D network of sites (nodes) connected by bonds (links). Each bond is treated a resistor with a specified electrical conductivity, $\sigma$. In the simplest case, conductivities of all resistors are equal. Suppose that this network is a subject to a potential difference. There are two natural possibilities [14], viz.,

- the “bus-bar geometry”, when two parallel conducting bars (buses) are attached to the opposite borders of the network and potentials (say, $V$ and $0$, $V > 0$) are applied to these buses [9, 10, 15] (figure 1(a)),
- the “two-point geometry”, when a potential difference is applied to two distinct sites, so that an electrical current, $I$, injected at a one site (source) and the same current withdrawn from the other site (sink) [8] (figure 1(b)).

![Figure 1. Example of a random network. The network was produced from a square lattice by removing of randomly chosen bonds. Each presented bond is associated with a given conductance. (a) bus-bar geometry; the (super)conducting buses are shown in gray. (b) two-point geometry; source and sink sites are shown as closed circles.](image)

Basically, only a certain fraction of the bonds of a particular RRN is current-carrying. This current-carrying part of the network is called the backbone or effective backbone; the remaining bonds are called dead ends or dangling ends. Some of the bonds belonging to the backbone may carry the total current. These bonds are called red bonds or singly connected bonds; when they are cut the current flow stops [16].

Alternatively, the geometrical backbone can be defined as a union of all self-avoiding walks (SAW) between two given points (case of two-point geometry) or between two opposite borders (case of bus-bar geometry) [9]. Notice, that this definition is not quite equivalent to the definition of the backbone as a set of current-carrying bonds due to presence of so-called perfectly balanced bonds. The electrical current through a perfectly balanced bond is absent because potential...
difference between its ends is equal to zero [5]. Thus, the effective backbone is defined as the set of bonds that carry a current, while the geometrical backbone is the set of bonds that either carry a current, or are perfectly balanced [17] (figure 2). A particular part of the backbone, so-called elastic backbone, is a union of all shortest paths between two given sites.

![Figure 2. Example of the random network with bus-bar geometry. Effective backbone is shown in blue, dead ends are shown in gray, difference between geometric and effective backbones is shown in magenta. The effective backbone has been extracted using a program based on the direct electrifying algorithm [5]. The geometrical backbone has been extracted using a program based on the algorithm [15].]

3. Algorithms for backbone identification

3.1. Effective backbone

The straightforward idea how to find the effective backbone is application of Ohm’s law to each bond belonging to the network [4, 18]. The potential difference between two neighboring sites \(i\) and \(j\) produces electrical current, \(I_{ij}\),

\[
I_{ij} = \sigma_{ij}(V_i - V_j).
\]

(1)

Here for generality conductivities of all bonds \(\sigma_{ij}\) are assumed to be different. Since for any site \(i\),

\[
\sum_j I_{ij} = 0,
\]

(2)

the potential of the site \(i\) is

\[
V_i = \sum_j \sigma_{ij} V_j / \sum_j \sigma_{ij}.
\]

(3)

When conductivities of all bonds are equal (say, \(\sigma_{ij} = 1\)), denominator of (3) is simply the degree (or valency) of the site \(i\)

\[
V_i = \text{deg}^{-1} i \sum_j V_j.
\]

(4)

Thus, there is a set of linear algebraic equations (SLAE), i.e., (4) and (1) for each site. This SLAE may be huge because the number of equation is of order of square of the network linear size. There may be different techniques how to build up such the SLAE [5]. When the particular RRN has several connected components, it is better to build up the SLAE for each connected component separately to avoid singular matrix [5]. In fact, it means that a kind of cluster labeling, e.g., [19], should be performed before the backbone identification. Since potentials of some sites are known (the sites at which the potentials are applied), (4) may be solved, e.g., iteratively. All bonds with non-zero currents belong to the backbone, whereas all bonds with
zero currents form the dead ends. The question is what current should be treated as zero? Any solver of SLAE involves floating-point arithmetics with inevitably round-off errors. A small current in a bond may be either ghostly (round-off error) or real. It is hardly possible to propose a strict rule on how to separate “sheep from goats”. Some parts of a geometrical backbone may carry negligible current of order of round-off errors and, hence, could be omitted. Despite a bond is not perfectly balanced, it may be almost perfectly balanced, i.e. balanced of order of computational precision. Nevertheless, when not a geometrical properties of the backbone but the effective conductivity of RRN is of primary interest, precise determination of current distribution is not so important. Hence, the methods of electrifying are suitable for calculations of effective electrical properties rather than extracting a subset of percolation cluster, i.e. its backbone.

3.2. Geometric backbone

“Burning” algorithm [8] is intended for backbone identification in two-point geometry. Its name means that the algorithm mimics a forest fire propagation. It can be divided into three steps. Let us suppose that the potential difference is applied to the sites $P$ and $Q$. At the first step, starting from $P$ and using breadth-first search (BFS) the distances between all sites and the site $P$ are determined, each site obtain a label corresponding its distance from $P$ (figure 3(a)). At the second step, all shortest paths between $P$ and $Q$ are determined. Starting from the site $Q$, each neighboring site which has a label less then the label of the current site is marked as belonging to the shortest paths. After these two steps, the elastic backbone is determined (figure 3(b)). In fact, first two steps correspond to the well-known Lee algorithm [20]. In this algorithm, the first step is called “wave expansion”, the second step is called “backtrace”. At the third step, the elastic backbone should be expanded to the geometrical backbone. This last step is most tricky. During the third step, the sites are handled starting from the sites in which the loops in the first step are closed. A site can only marked as potentially belonging to the backbone if its label is less than the labels of the sites that are lit up at time $t_i$, but in addition, the growing backbone can not burn. The question arises: if we start from a given site, an elastic backbone is reached only on one site or on several sites? In the case of multiple sites, all the sites that were handled are added to the growing backbone. This is repeated over and over until no additional part can be added to the growing backbone (figure 3(c)). Different interpretations and implementations of the third step lead to the appearance of various modifications of the algorithm [8, 21, 22, 23, 24, 25].

A recursive depth-first search (DFS) algorithm [9] which simultaneously generates a percolation cluster and finds its backbone is based on the best-known rule for traversing mazes, i.e., the wall follower or “left-hand rule”. When a pedestrian walks through a singly-connected labyrinth touching its wall by his/her left hand, an exit will be undoubtedly found if any. When a path is passed twice in opposite directions, this path is a dead end. In fact, this is a kind of well-known “Ariadne’s clew algorithm” (see, e.g., [26]). This walk through a maze should be repeated many times to find all possible paths. Each time, when a walker reaches an opposite border of a maze, a hand should be changed, i.e. when a walk from south to north corresponds to left-hand rule, then a walk from north to south corresponds right-hand rule (figure 4). Recursive implementation of the algorithm is simple but has an essential drawback connected with limited depth of recursive calls. The algorithm can be implemented without recursive calls but, in this case, a stack should be operated by hand.

A matching algorithm has been proposed [7] and applied to backbone identification for site percolation on hypercubic lattices up to 5 dimensions. The algorithm deals with a directed graph. Unfortunately, a brief sketch of the main ideas about the matching algorithm [7] is not enough for implementation of the algorithm, because several important issues have not been discussed in that paper.

4
Figure 3. Example of application of “burning” algorithm [8] to the particular random network with two-point geometry. (a) After the first step, each site has the label corresponding to the distance from lower left corner to the site. Powers correspond to number of shortest paths from the starting site to the current site. (b) After the second step, the elastic backbone, i.e., union of all shortest paths between two given sites, is found. (c) After the third step, the geometric backbone, dead ends (yellow), and red bonds (red) are marked.

Figure 4. Example of application of “wall follower” algorithm [9] to the particular random network, $L = 32$, with bus-bar geometry. (a) Percolation cluster. Different directions of walks are shown in different colors. (b) Backbone and dead ends of the same percolation cluster.

The algorithm [10] deals with both original network and its dual. Its implementation consists of three steps. During the first step, the percolation cluster should be identified. All isolated clusters which, obviously, cannot participate in the conductivity should be removed. This step
can be efficiently performed, e.g., by a single “burning” of the lattice [8]. The second step deals with the dual lattice. Notice, that the lattice of our primary interest, viz., a square lattice, is self-dual. The second step begins with identifying connected clusters in the dual lattice. Then, all bonds that are in between two neighbouring cells connected in the dual lattice should be removed. After this second step, the sites that used to be in the incipient infinite cluster after the first step are now either on the backbone or belong to disconnected clusters. It is finally straightforward to conclude the computation in a third step by extracting the incipient cluster which is now the backbone of the original RRN.

A rapid algorithm for identifying the current-carrying backbone in the 2D percolation model has been proposed by Yin and Tao [11, 12]. Its implementation has been recently presented [15]. Taking advantage of the modified Hoshen–Kopelman cluster labeling algorithm, this multistep algorithm identifies dead ends using their local properties (figure 5). The algorithm is based on Jordan’s curve theorem: a loop in a 2D graph disconnects the area inside it from the area outside. In fact, the idea of the algorithm is rather similar to [10] although a dual lattice is not utilised explicitly. The algorithm can be easily parallelized and looks more promising for most tasks.

Figure 5. Steps of backbone identification with algorithm [11, 12] implemented as in [15].
4. Conclusion

We have implemented several different algorithms for backbone identification. The algorithms were applied to backbone identification for different system sizes and concentrations of conducting bonds.

Direct calculations of electrical potentials and currents are based on floating-point arithmetic and, hence, produce round-off errors. Some ghost currents may arise due to these round-off errors which impedes the backbone extraction. Moreover, these calculations deal with huge SLAE and require a lot of computer memory. Only relative small systems can be treated in these approaches because number of equations to be solved is proportional to the square of linear size of the system.

Graph theory algorithms are sometimes difficult to understand or/and to realize. Some of them require to store not only original network but its dual [10]. Moreover, some algorithms can produce stack overflow because of recursion. In fact, application of these algorithms also is restricted to the RRN of moderate size.

Our analysis suggests that a universal algorithm suitable for any possible problem is absent. Most likely, each particular task needs a specific algorithm. Nevertheless, the algorithms which allow parallelization (e.g., [15]) look very promising in many cases.

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