On the residual dependence index of elliptical distributions

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Abstract: The residual dependence index of bivariate Gaussian distributions is determined by the correlation coefficient. This tail index is of certain statistical importance when extremes and related rare events of bivariate samples with asymptotic independent components are being modeled. In this paper we calculate the partial residual dependence indices of a multivariate elliptical random vector assuming that the associated random radius is in the Gumbel max-domain of attraction. Furthermore, we discuss the estimation of these indices when the associated random radius possesses a Weibull-tail distribution.

Key words and phrases: Partial residual dependence index; Gumbel max-domain of attraction; Weibull-tail distribution; Elliptical distribution; Quadratic programming problem.

1 Introduction

Let $(X_1, X_2)$ be a bivariate elliptical random vector with stochastic representation

$$(X_1, X_2) \overset{d}{=} R(U_1, \rho U_1 + \sqrt{1-\rho^2}U_2), \quad \rho \in (-1, 1),$$

(1.1)

where the positive random radius $R$ is independent of $(U_1, U_2)$ which is uniformly distributed on the unit circle of $\mathbb{R}^2$. Here $\overset{d}{=}$ stands for equality of distribution functions. A canonical example of a bivariate elliptical random vector is when $R^2$ is Chi-square distributed, which implies that $X_1, X_2$ are standard Gaussian random variables with mean 0, variance 1 and correlation coefficient $\rho := E\{X_1 X_2\}$. It is well-known (see e.g., Reiss and Thomas (2007)) that in the Gaussian model the correlation coefficient $\rho$ does not influence the asymptotic dependence of the components. Roughly speaking this means that the sample extremes of Gaussian random vectors are asymptotically independent.

A tractable extension of the Gaussian model is the elliptical one, where $R$ is some general positive random variable with distribution function $F$. By Lemma 12.1.2 of Berman (1992)

$$X_1 \overset{d}{=} X_2 \overset{d}{=} RU_1 \overset{d}{=} RU_2$$

(1.2)

implying that distribution function of $X_1$ (denoted below by $Q$) is continuous. Clearly, the joint dependence function of $X_1$ and $X_2$ is influenced by $\rho$.

Several authors have considered elliptical distributions for modelling of rare events with specific applications in insurance and finance. Recent contributions in this directions are Peng (2008), Li and Peng (2009). In the first mentioned article the author deals with the situation that the marginal distributions are regularly varying implying that the components are asymptotically dependence. In such a model, also considered in Klüppelberg et al. (2007), novel estimation techniques are presented. In Berman (1983,1992), Hashorva (2005a,b), Abdous et al. (2005,2008) further probabilistic results are obtained when $F$ is in the Gumbel max-domain of attraction which implies that the components $X_1$ and $X_2$ are asymptotically independent.

In this paper, with motivation from above mentioned contributions, we focus on the quantification of the asymptotic dependence of elliptical random vectors.

An interesting measure of the asymptotics dependence (see e.g., Peng (1998, 2007), de Haan and Ferreira (2006), Reiss and Thomas (2007)) is the function $\chi(u)$ defined by

$$\chi(u) := \frac{P\{X_1 > u, X_2 > u\}}{P\{X_1 > u\}} \in [0, 1], \quad u > 0.$$
If for some constant $c \in [0, 1]$ we have
\[
\lim_{u \to \infty} \chi(u) = c \in (0, 1],
\]
then $X_1$ and $X_2$ are said to be asymptotically dependent. In our setup of bivariate elliptical random vectors with stochastic representation (1.1) this is the case when $R$ has distribution function $F$ in the Fréchet max-domain of attraction (or equivalently, $F$ is regularly varying with positive index $\gamma$). See Berman (1992) or Hashorva (2005a, 2006b) for further details. Important statistical applications can be found in Klüppelberg et al. (2007).

In both other cases of max-domain of attraction, i.e., $F$ is in the Gumbel or the Weibull max-domain of attraction we have (see Hashorva (2005a)) $c = 0$, which means that $X_1$ and $X_2$ are asymptotically independent.

In extreme value theory asymptotic independence is a nice property, however, $c = 0$ in (1.3) merely means that $P\{X_1 > u, X_2 > u\}$ converges faster to 0 than the marginal survival probability $P\{X_1 > u\}$ (if $u \to \infty$).

One successful approach to model the asymptotic independence is the estimation of the residual dependence index $\eta \in (0, 1]$ (see e.g., Peng (1998, 2007, 2008), de Haan and Peng (1998), or de Haan and Ferreira (2006)). We note in passing that recent ideas in testing asymptotic can be found in Hüsler and Li (2009). Now, information about $\eta$ is available if for any $x, y$ positive
\[
S_u(x, y) := \frac{\tilde{S}_u(x, y)}{S_u(1, 1)} \to S(x, y) \in (0, \infty), \quad u \to \infty, \quad (1.4)
\]
with
\[
\tilde{S}_u(x, y) := P\{Q(X_1) > 1 - x/u, Q(X_2) > 1 - y/u\}, \quad u > 0,
\]
since for any $c > 0$ and for some $\eta \in (0, 1]$ we have the important scaling relation
\[
S(cx, cy) = c^{1/\eta}S(x, y).
\]
Furthermore, the function $\tilde{S}_u(1, 1)$ is regularly varying at infinity with index $-1/\eta$. Other authors refer to $\eta$ as the coefficient of tail dependence (see e.g., Resnick (2002), or Reiss and Thomas (2007)).

In this paper we consider the problem of calculating the residual dependence index $\eta$ for the bivariate random vector $(X_1, X_2)$ with stochastic representation (1.1) assuming that the distribution function $F$ is in the Gumbel max-domain of attraction. We show that $\eta$ does not always exist. In certain instances when it exists we prove that $\eta$ is defined in terms of $\rho$ and the Weibull tail-coefficient $\theta$ (see below (2.11)).

In Section 3 we propose an estimator of the residual dependence index $\eta$. Definition, calculation and estimation of the partial residual dependence index for multivariate elliptical distributions are placed in Section 4. In the multivariate setup the partial residual dependence indexes (if they exit) are determined by the unique solution of specific quadratic programming problem, and the Weibull tail-coefficient $\theta$. Proofs of all the results are relegated to Section 5 (last one).

2 Calculation of the Residual Dependence Index

Let $(X_1, X_2)$ be an elliptical random vector with stochastic representation (1.1), and let $R$ be the positive associated random radius with distribution function $F$. We assume in the following $F(0) = 0$ and $F(x) < 1, \forall x > 0$. If $X_1$ and $X_2$ are standard Gaussian random variables, then it is well-known that (see e.g., Reiss and Thomas (2007)) $X_1$ and $X_2$ are asymptotically independent for any $\rho \in (-1, 1)$. Furthermore, the residual dependence index $\eta$ is given by
\[
\eta := (1 + \rho)/2 \in (0, 1)
\]
and (see p. 322 in Reiss and Thomas (2007))
\[
\tilde{S}_u(1, 1) = (1 + o(1)) \frac{(1 - \rho^2)^{3/2}}{(1 - \rho)^2} (4\pi)^{-\rho/(1+\rho)} (\ln u)^{-\rho/(1+\rho)} u^{-2/(1+\rho)}, \quad u \to \infty. \quad (2.5)
\]
In our notation $o(1)$ means $\lim_{u \to \infty} o(1) = 0$.

When $X_1$ and $X_2$ are independent, then $R^2 \stackrel{d}{=} X_1^2 + X_2^2$ is Chi-squared distributed with 2 degrees of freedom. The distribution function $F$ of $R$ is in this case in the max-domain of attraction of the Gumbel distribution $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$. From the extreme value theory we know (see e.g., Resnick (1987), Reiss (1989), Falk et al. (2004), Embrechts et al. (1997), or de Haan and Ferreira (2006)) that the distribution function $F$ of $R$ is in the max-domain of attraction of $\Lambda$, if for some positive scaling function $w$ we have

$$\lim_{u \to \infty} \frac{1 - F(u + x/w(u))}{1 - F(u)} = \exp(-x), \quad \forall x \in \mathbb{R}. \quad (2.6)$$

In the Gaussian case $\eta$ is strictly less than 1 and $w(u) = (1 + o(1))u, u > 0$. In the more general elliptical setup of this paper it turns out that interesting cases for calculation of $\eta$ are when

$$w(u) = u^{\theta-1}L(u), \quad \theta \in [0, \infty), \quad (2.7)$$

with $L$ a positive slowly varying function at infinity satisfying $\lim_{u \to \infty} L(cu)/L(u) = 1, \forall c > 0$. We refer to $\theta$ as the Weibull tail-coefficient index (see Girard (2004)).

We show below that the elliptical model exhibits two main two main features, namely: a) the residual dependence index $\eta$ (when it exists) depends on both $\rho$ and $\theta$, being in fact an increasing function of $\rho$ and $1/\theta$, and b) it is possible to have $\eta = 1$ when $\theta = 0$ and $\lim_{u \to \infty} L(u) = \infty$. More interestingly, both $X, Y$ are asymptotically independent even when $\eta = 1$.

The main result of this section is the following theorem.

**Theorem 2.1.** Let $(X_1, X_2), \rho \in (-1, 1)$ be a bivariate elliptical random vector with stochastic representation $\mathbf{(1.1)}$. Assume that $R$ has distribution function $F$ which satisfies (2.6) with some positive scaling function $w$.

(i) Suppose that

$$\lim_{u \to \infty} \frac{w(\alpha u)}{w(u)} = \infty, \quad \text{with } \alpha_\rho := \sqrt{2/(1 + \rho)} > 1 \quad (2.8)$$

holds, then

$$\lim_{u \to \infty} S_u(x, y) = \infty, \quad \text{if } x > 1, y > 1, \quad \lim_{u \to \infty} S_u(x, y) = 0, \quad \text{if } x, y \in (0, 1). \quad (2.9)$$

(ii) If for some $\theta \in [0, \infty)$

$$\lim_{u \to \infty} \frac{w(cu)}{w(u)} = c^{\theta-1}, \quad \forall c > 0, \quad (2.10)$$

then for any $x, y \in (0, \infty)$

$$\lim_{u \to \infty} S_u(x, y) = (x y)^{1/(2 \eta)}, \quad \eta := \left(\frac{1 + \rho}{2}\right)^{\theta/2} = \alpha_\rho^{-\theta} \in (0, 1], \quad (2.11)$$

and $S_u(1, 1)$ is regularly varying at infinity with index $-1/\eta$.

(iii) Let $Q^{-1}$ denote the inverse of the distribution function of $X_1$. As $u \to \infty$ we have the asymptotic expansion

$$\tilde{S}_u(1, 1) = (1 + o(1)) \frac{\alpha_\rho^2(1 - \rho^2)^{3/2} \left(1 - F(b_*(u))\right)}{2\pi(1 - \rho^2) b_*(u) w(b_*(u))}, \quad b_*(u) := \alpha_\rho Q^{-1}(1 - 1/u). \quad (2.12)$$

**Remarks 2.2.** 1) The scaling function $w$ in (2.6) can be defined asymptotically by (see e.g., Resnick (1987))

$$w(u) := \frac{(1 + o(1))[1 - F(u)]}{\int_u^\infty [1 - F(s)] ds}, \quad u \to \infty. \quad (2.13)$$
Further, we have 
\[
\lim_{u \to \infty} uw(u) = \infty. \tag{2.14}
\]

Hence in the model \([2.7]\) the Weibull tail-coefficient \(\theta\) is necessarily non-negative, and if \(\theta = 0\), then we need to suppose further that \(\lim_{u \to \infty} L(u) = 0\). Two interesting distributions with \(\theta = 0\) and \(L(u) = c \ln u, c \in (0, \infty), u > 0\) are Benktadler type I and Lognormal one, see Embrechts et al. (1997), pp. 149-150.

The interesting model \(\theta = 0\) is kindly suggested by the referee of the paper. A striking feature of this model is that residual dependence index \(\eta\) equals 1, thus not depending on \(\rho\) at all. Furthermore, in view of Hashorva (2005a) \(X\) and \(Y\) are asymptotically independent.

2) In view of \([1.2]\) \(X_1\) is a product of \(R\) and an independent random variable \(U_1\). Asymptotics of random products are investigated by several authors, see for recent results Tang and Tsitsiashvili (2003, 2004), Tang (2006, 2008).

Next, we present three examples.

**Example 1.** Let \((X_1, X_2), \rho\) be as in \([1.1]\) with associated random radius \(R \sim \Lambda\). Clearly, the unit Gumbel distribution \(\Lambda\) is in the Gumbel max-domain of attraction. An admissible choice for the scaling function is \(w(u) = 1, \forall u > 0\). Consequently, \([2.10]\) holds with \(\theta = 1\), implying that \(S_u(1, 1)\) is regularly varying with index 
\(- (0.5 + \rho/2)^{-1/2}\).

**Example 2.** Under the setup of Example 1 we assume further that \(R\) has distribution function \(F\) in the Gumbel max-domain of attraction with the scaling function \(w(u) = \exp(au), u > 0\), with \(a\) some positive constant. Such \(F\) exists and can easily be constructed if we assume that \(R\) possesses a density function \(f\), requiring further \(f(u)/(1 - F(u)) = w(u), \forall u > 0\). For this choice of the scaling function \(w\) \([2.8]\) holds. Hence \(\lim_{u \to \infty} S_u(x, y) = 0\) for any \(x, y \in (0, 1)\).

**Example 3.** [Kotz Type III] Again with the setup of Example 1 if for all large \(u\)
\[
P(R > u) = (1 + o(1))K u^N \exp(-ru^\theta), \quad K > 0, \theta > 0, N \in \mathbb{R}, \tag{2.15}
\]
then we refer to \((X_1, X_2)\) as a Kotz Type III elliptical random vector. Since we assume that \(\theta\) is positive, \(R\) has distribution function \(F\) in the Gumbel max-domain of attraction with the scaling function
\[
w(u) = (1 + o(1))\rho u^{\theta-1}, \quad u \to \infty.
\]
Consequently, \([2.11]\) holds and \(\eta = \alpha^{-\theta}_\rho \in (0, 1)\). Next, if we define \(b(u) := Q^{-1}(1 - 1/u), u > 0\) with \(Q\) the distribution function of \(X_1\), then Theorem \([2.11]\) implies
\[
\tilde{S}_u(1, 1) = (1 + o(1))\frac{K \alpha^{-\theta}_\rho + 2(1 - \rho^2)^{3/2}}{2\pi \rho(1 - \rho^2)^{3/2}}(b(u))^{N-\theta} \exp(-r(b(u))^{\theta})\exp(-ru^\theta), \quad u \to \infty.
\]
In view of Theorem 12.3.1 in Berman (1992) and the fact that \(Q\) is symmetric about 0 (recall \([1.1]\))
\[
1 - Q(u) = \frac{1}{2}P\{X_1^2 > u\} = \frac{1}{2}P\{R^2U_1 > u\} = (1 + o(1))\frac{K}{\sqrt{2\pi \rho \theta}} u^{N-\theta/2} \exp(-ru^\theta), \quad u \to \infty,
\]
where \(U_1^2\) is beta distributed with parameters 1/2, 1/2 being independent of \(R\). Hence we may define \(b(u)\) asymptotically as (see Embrechts et al. (1997))
\[
b(u) = (r^{-1} \ln u)^{1/\theta} \left[1 + \frac{(1 + o(1))}{\theta \ln u} [(N - \theta/2) \ln(r^{-1} \ln u)/\theta + \ln K - \frac{1}{2} \ln(2\pi \rho \theta)]\right], \quad u \to \infty.
\]
Thus we arrive at (set \(\lambda := \rho^\theta\))
\[
\tilde{S}_u(1, 1) = (1 + o(1))\frac{\alpha^{-\theta}_\rho + 2(1 - \rho^2)^{3/2}}{K(1 - \rho^2)}\frac{(K^2/2\pi \rho \theta)^{1-\lambda/2}}{(\ln u)^{(1-\lambda)N/\theta+\lambda/2-1}u^{-\lambda}}, \quad u \to \infty.
\]
In the special case

\[ K = 1, \quad r = 1/2, \quad \theta = 2, \quad N = 0, \quad \alpha = \sqrt{2/(1 + \rho)}, \]

which holds in particular if both \( X_1 \) and \( X_2 \) are standard Gaussian random variables we retrieve (2.5).

### 3 Estimation of \( \eta \) in the Weibull Model

In view of Theorem 2.1 if the scaling function \( w \) is regularly varying with index \( \theta - 1 \), then the residual dependence index \( \eta \) is defined in terms of \( \rho \) and \( \theta \). Let \((X_{k1}, X_{k2}), k = 1, \ldots, n\) be a sample of bivariate elliptical random vectors with stochastic representation (1.1) (where \( \rho \in (-1, 1) \) is assumed). Then a non-parametric estimator \( \hat{\rho}_n \) of \( \rho \) is given by (see e.g. Peng (2008), Li and Peng (2009))

\[ \hat{\rho}_n := \sin(\pi \hat{\tau}_n/2), \quad n > 1, \]  

(3.16)

where \( \hat{\tau}_n \) is the empirical estimator of the Kendall’s tau.

Good performing estimators of the so-called Weibull tail-coefficient are the Girard and Zipf estimators, see e.g., Girard (2004). Referring to the aforementioned paper we say that the random radius \( R \sim F \) possesses a Weibull-tail distribution if

\[ 1 - F(x) = \exp(-H(x)), \quad H^{-1}(x) = \inf\{ t : H(t) \geq x \} = x^{1/\theta}L_1(x) \]

(3.17)

holds with \( L_1 \) a positive slowly varying function at infinity. Gardes and Girard (2006) and Diebolt et al. (2008) give several examples of Weibull-tail distributions. Prominent instances are the Gaussian, Gamma and extended Weibull distributions.

By the properties of slowly varying functions (see e.g., de Haan and Ferreira (2006)) we may write (3.17) alternatively as

\[ 1 - F(x) = \exp(-x^\theta L_2(x)), \]

(3.18)

where \( L_2 \) is another slowly varying function which is asymptotically unique.

From the estimation point of view, a tractable class of the Weibull-tail distributions is constructed when \( F \) is in the Gumbel max-domain of attraction with the scaling function \( w \) defined asymptotically by

\[ w(u) = \frac{cu^{\theta-1}}{1 + t_1(u)}, \quad c \in (0, \infty), \]

(3.19)

where \( t_1 \) is a regularly varying function at infinity with index \( \kappa := \theta\mu, \mu \in (-\infty, 0) \) implying

\[ 1 - F(u) = \exp(-cu^\theta(1 + t_2(u))), \quad u > 0, \]

(3.20)

where \( t_2 \) is another regularly varying function at infinity with index \( \kappa \).

Under the assumption (3.19) it follows that (see Berman (1992) or Hashorva (2005a))

\[ P\{X_1 > u\} = \exp(-cu^\theta(1 + t_3(u))), \quad u \in R, \]

with \( t_3 \) again a regularly varying function at infinity with index \( \kappa \).

Next, assume that the associated random radius \( R \) defining the random sample \((X_{k1}, X_{k2}), k = 1, \ldots, n, n > 1\) possesses a Weibull-tail distribution \( F \) such that \( (3.19) \) holds. Write \( X_{1:n} \leq \cdots \leq X_{n:n} \) for the associated order statistics of \( X_{11}, \ldots, X_{n1} \). Following Gardes and Girard (2006) we might estimate \( \theta \) by

\[ \hat{\theta}_n := \frac{1}{T_n} \frac{1}{k_n} \sum_{i=1}^{n} \left( \log X_{n-i+1:n} - \log X_{n-k_n+i+1:n} \right), \]

with \( 1 \leq k_n \leq n, T_n > 0, n \geq 1 \) given constants satisfying

\[ \lim_{n \to \infty} k_n = \infty, \quad \lim_{n \to \infty} \frac{k_n}{n} = 0, \quad \lim_{n \to \infty} \log(T_n/k_n) = 1, \quad \lim_{n \to \infty} \sqrt{k_n b(\log(n/k_n))} \to \lambda \in R, \]
where the function $b$ is a regularly varying function with index $\theta$ appearing in a second order asymptotic condition imposed on $F$ (being thus related to $L_1$).

Asymptotic properties of $\hat{\theta}_n$ are discussed in Gardes and Girard (2006) and Diebolt et al. (2008). Next, based on our main result we propose an estimator for the residual dependence index $\eta$ given by

$$\hat{\eta}_n := \left((1 + \hat{\rho}_n)/2\right)^{\hat{\theta}_n/2}, \quad n > 1.$$  \hspace{1cm} (3.21)

Asymptotic properties of $\hat{\eta}_n$ follow by utilising the asymptotic properties of both $\hat{\rho}_n$ and $\hat{\theta}_n$.

We note in passing that the constant $c$ can be estimated by

$$\hat{c}_n = \frac{1}{k_n} \sum_{i=1}^{k_n} \frac{\log(n/i)}{X_{n-i+1:n}}, \quad n > 1.$$  \hspace{1cm} (3.22)

### 4 Partial Residual Dependence Index

Consider $X := (X_1, \ldots, X_k) \in \mathbb{R}^k$ an elliptical random vector in $\mathbb{R}^k$ with stochastic representation

$$X \overset{d}{=} RA^\top U,$$  \hspace{1cm} (4.23)

where $R$ is again the positive associated random radius of $X$ with distribution function $F$ independent of $U := (U_1, \ldots, U_k)^\top$ which is uniformly distributed on the unit sphere of $\mathbb{R}^k$ and $A \in \mathbb{R}^{k \times k}$ is a non-singular matrix (here $\top$ stands for the transpose sign). The distribution function of the random vector $X$ is determined by the positive definite matrix $\Sigma := A^\top A$, the distribution function $F$ and the distribution function of $U$ (which is known). See Cambanis et al. (1981) or Fang et al. (1990) for more details on elliptical distributions.

We assume in the following that $F(0) = 0, F(x) < 1, \forall x > 0$ and $\Sigma$ is a correlation matrix i.e., all the entries of the main diagonal equal 1. If the distribution function $F$ is in the Gumbel max-domain of attraction, then each pair $X_i, X_j, i \neq j, i, j \leq k$ is asymptotically independent. If further the scaling function $w$ satisfies (2.10), then by Lemma 12.1.2 in Berman (1992), Proposition 3.4 in Hashorva (2005a) and Theorem 2.4 it follows that the residual dependence index $\eta_{ij}$ (related to $(X_i, X_j)$) is

$$\eta_{ij} = \alpha_{\rho_{ij}},$$

with $\rho_{ij} \in (-1, 1)$ the $ij$-th entry of $\Sigma$.

Let $I$ be a given non-empty index subset of $\{1, \ldots, k\}$. By the assumption on $\Sigma$ we have $X_i \overset{d}{=} X_1, i = 2, \ldots, k$. Next, define $S_{u,I}(x)$ by

$$S_{u,I}(x) := P \left(Q(X_i) > 1 - \frac{x_i}{u}, \forall i \in I\right), \quad u > 0, x := (x_1, \ldots, x_k)^\top \in (0, \infty)^k,$$

with $Q$ the distribution function of $X_1$.

If $S_{u,I}(1)$ is regularly varying with index $-1/\eta_I, \eta_I \in (0, 1)$, then we refer to $\eta_I$ as the partial residual dependence index of the subvector $X_I := (X_i, i \in I)^\top$, or shortly as the partial residual dependence index.

The submatrix of $\Sigma$ obtained by retaining the rows and the columns of $\Sigma$ with indices in $J$ and $I$, respectively, (assume $I$ has less than $k$ elements) is denoted by $\Sigma_{JI}, J := \{1, \ldots, k\} \setminus I$. We define similarly $\Sigma_{II}$ and $x_I$ for $x \in \mathbb{R}^k$.

Since in our model $\Sigma = A^\top A$ is positive definite the inverse matrix $\Sigma_{II}^{-1}$ of $\Sigma_{II}$ exists. Next, we write $\alpha_I$ for the unique solution of the quadratic programming problem

$$\begin{align*}
\text{minimise the objective function} \quad & y_I^\top \Sigma_{II}^{-1} y_I, \quad y := (y_1, \ldots, y_k)^\top \in \mathbb{R}^k, \quad y_i \geq 1, \quad \forall i \in I. \quad (4.24)
\end{align*}$$

In the multivariate setup again two interesting features are observed, namely a) the partial residual index $\eta_I$ depends on both $\alpha_I$ and $\theta$ (when it exists), and b) when $\theta = 1$, then $\eta_I = 1$ for all $I \subset \{1, \ldots, k\}$.
Theorem 4.1. Let $X$ be an elliptical random vector in $\mathbb{R}^k$, $k \geq 2$, with stochastic representation \((1.23)\). Assume that the associated random radius $R$ is almost surely positive with distribution function $F$ satisfying \((2.2)\). If the scaling function $w$ satisfies \((2.10)\), then for any non-empty index set $I \subset \{1, \ldots, k\}$ with $m \leq k$ elements and any $x \in (0, \infty)^k$ we have

$$
\lim_{u \to \infty} \frac{\bar{S}_{u,t}(x)}{\bar{S}_{u,t}(1)} = \left( \prod_{j \in K} x_j^\gamma_j \right), \quad \gamma_j : = \alpha_I^{-1}(e_j^T \Sigma_{KK}^{-1} 1_K) \in (0, \infty), \quad \forall j \in K, 
$$

where $K$ is a unique subset of $I$ with $l > 0$ elements such that $\Sigma_{KK}^{-1} 1_K$ is a vector with positive elements and $e_j$ is the $j$-th unit vector in $\mathbb{R}^l$. Furthermore, if $M := I \setminus K$ is not empty, then the vector $\Sigma_{KM} \Sigma_{MM}^{-1} 1_M - 1_K$ has non-negative components and $\bar{S}_{u,t}(1)$ is regularly varying with index $-1/\eta_I$ where

$$
\eta_I := \alpha_I^{-\theta}(1^T 1_I^{-\theta} 1_I)^{-\theta}, \quad (4.26)
$$

Remarks 4.2. 1) Estimation of the partial residual dependence index $\eta_I$ related to a given index set $I$ requires estimation of the attained minimum $\alpha_I$ of the related quadratic programming problem and the Weibull tail-coefficient $\theta$. For any $i, j, i \neq j$ an estimator of $\rho_{ij}$ (the $ij$-th entry of $\Sigma$) can be defined analogously to \((3.10)\) by

$$
\hat{\rho}_{ij,n} := \sin(\pi \hat{\tau}_{ij,n}/2), \quad n > 1,
$$

with $\hat{\tau}_{ij,n}$ the corresponding empirical estimator of the Kendall’s tau.

An estimator of $\alpha_I$ can be constructed if we already have estimated the precision matrix $\Sigma_{II}^{-1}$. Estimation of $\Sigma_{II}^{-1}$ is recently discussed for Kotz distributions in Sarr and Gupta (2008).

If $\hat{\alpha}_{I,n}$ denotes an estimator of $\alpha_I$, and $\hat{\theta}_n$ an estimator of the Weibull tail-coefficient, then in view of our results we can estimate the partial residual index $\eta_I$ by

$$
\hat{\eta}_{I,n} := \hat{\alpha}_{I,n}^{-\hat{\theta}_n}. \quad (4.27)
$$

2) In the case that \((2.2)\) holds with $\alpha_I$ instead of $\alpha_\rho$, then we cannot define $\eta_I$.

3) If $\Sigma_{II}^{-1} 1_I$ has positive elements, then the subset $K$ in Theorem \((4.4)\) equals $I$. This is in particular the case if $I$ has only two elements, or when the non-diagonal elements of $\Sigma$ are all equal, say to $\rho \in (-1, 1)$. If $K \neq I$, then for estimating $\alpha_I$, we also need to identify the elements of $K$, which is not an easy task in general.

4) It is well-known that the solution of the attained minimum $\alpha_I$ of the quadratic programming problem above is related to the exact tail asymptotics of the Gaussian random vectors, see Dai and Mukherjea (2001), or Hashorva (2005b, 2007b) for more details.

5) In the particular case $w$ is given by \((2.7)\) with $\theta = 0$, then $\eta_I = 1$ for all subsets $I$ of $\{1, \ldots, k\}$. Furthermore, $\Sigma$ does not influence $\eta_I$, which is in particular the case for multivariate lognormal distributions.

Next we consider the trivariate setup in some details. The following lemma gives an explicit formula for $\alpha_I, I = \{1, 2, 3\}$, which is useful for the estimation of $\alpha_I$.

Lemma 4.3. Let $\Sigma \in \mathbb{R}^{3 \times 3}$ be a positive definite correlation matrix (with 1’s in the main diagonal) and non-diagonal entries $\rho_{ij} \in (-1, 1), i \neq j, i, j \leq 3$. Define $\rho_{\min} := \min(\rho_{12}, \rho_{13}, \rho_{23})$ and set $\alpha := \min_{x \geq 1,i=1,2,3} x^T \Sigma^{-1} x$.

(i) If $1 + 2\rho_{\min} - \rho_{12} - \rho_{13} - \rho_{23} > 0$, then we have (here $1 = (1, 1, 1)^T$)

$$
\alpha = 1^T \Sigma^{-1} 1 \quad = \frac{3 - 2(\rho_{12} + \rho_{13} + \rho_{23}) - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2(\rho_{12}\rho_{13} + \rho_{12}\rho_{23} + \rho_{13}\rho_{23})}{1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2}. \quad (4.28)
$$

(ii) If $1 + 2\rho_{\min} - \rho_{12} - \rho_{13} - \rho_{23} \leq 0$, then there exists a unique index set $\{i, j\} \subset \{1, 2, 3\}$ such that

$$
\rho_{\min} = \rho_{ij} < \min_{k \neq i, k \neq i, k \leq 3} \rho_{lk}. \quad (4.29)
$$
Moreover we have

\[ \alpha = 1_K^\top \Sigma_{KK}^{-1} 1_K = (1, 1)^\top \Sigma_{KK}^{-1}(1, 1) = \frac{2}{1 + \rho_{ij}}. \]

(4.30)

**Example 4.** [Kotz Type III, 3-dimensional Case]. Let \( X \) be an elliptical random vector in \( \mathbb{R}^3 \) with stochastic representation \( \{U, \rho_{kl}, R \} \), where the matrix \( A \) is non-singular and set \( \Sigma := A^\top A \). We denote by \( \rho_{ij} \) the \( ij \)-th entry of \( \Sigma \). Assume that \( \rho_{ii} = 1, i = 1, \ldots, k \) and \( R \) satisfies (2.15) as \( u \to \infty \). Again we refer to \( X \) as a Kotz Type III random vector. In view of Lemma 12.1.2 in Berman (1992) we have for any index set \( I = \{k, l\} \subset \{1, 23\} \) with two elements

\[ (X_k, X_l) \overset{d}{=} R(U_1, \rho_{kl} U_1 + \sqrt{1 - \rho_{kl}^2} U_2), \]

where \((U_1, U_2)\) with uniform distribution on the unit circle of \( \mathbb{R}^2 \) is independent of \( R \). Hence we can estimate \( \rho_{kl} \) as in (3.16). Let \( \hat{\rho}_{12, n}, \hat{\rho}_{13, n}, \hat{\rho}_{23, n}, n > 1 \) denote these estimators.

Next, consider the case \( I = \{1, 2, 3\} \). In view of Theorem 4.1 and Lemma 4.3 \( \eta_I = \alpha^{-\theta} \), with \( \alpha \) defined in (4.24). If

\[ 1 + 2 \min(\hat{\rho}_{12, n}, \hat{\rho}_{13, n}, \hat{\rho}_{23, n}) - \hat{\rho}_{12, n} - \hat{\rho}_{13, n} - \hat{\rho}_{22, n} > 0, \]

then the estimator \( \alpha \) is obtained by plugging in the estimators \( \hat{\rho}_{12, n}, \hat{\rho}_{13, n}, \hat{\rho}_{23, n} \). Otherwise, we estimate

\[ \hat{\rho}_{\min, n} := \min(\hat{\rho}_{12, n}, \hat{\rho}_{13, n}, \hat{\rho}_{23, n}), \quad n > 1, \]

and obtain the estimator of \( \alpha \) by plugging in \( \hat{\rho}_{\min, n} \) in (4.30). The Weibull tail-coefficient \( \theta \) can then be further estimated as previously discussed in Section 3.

5 Proofs

**Proof of Theorem 5.** Let \( Q \) be the distribution function of \( X_1 \) with inverse \( Q^{-1} \) (\( Q \) is a continuous function, see e.g., Berman (1992)). The Gumbel max-domain of attraction assumption on \( F \) implies (see e.g., Reiss (1989), or de Haan and Ferreira (2006))

\[ w(b(u))|Q^{-1}(1 - x/u) - b(u)| \to - \ln x, \quad u \to \infty \]

locally uniformly for \( x \in (0, \infty) \), with \( b(u) := Q^{-1}(1 - 1/u), u > 0 \). Next set

\[ w^*(u) := w(\alpha_u b(u)), u > 0, \quad \alpha_u := \sqrt{2/(1 + \rho)} > 1, \quad \rho \in (-1, 1). \]

For any \( u, x, y > 0 \) positive we may further write (recall \( X_1 \overset{d}{=} X_2 \))

\[ S_u(x, y) = P\left\{ Q(X_1) > 1 - \frac{x}{u}, Q(X_2) > 1 - \frac{y}{u} \right\} = P\left\{ X_1 > Q^{-1}(1 - \frac{x}{u}), X_2 > Q^{-1}(1 - \frac{y}{u}) \right\} = P\left\{ X_1 > b(u) - (1 + o(1)) \frac{\ln x}{w(b(u))}, X_2 > b(u) - (1 + o(1)) \frac{\ln y}{w(b(u))} \right\}. \]

In view of Theorem 5 in Hashorva (2007a) for any \( s, t > 0 \) positive

\[ \lim_{u \to \infty} P\left\{ w^*(u)(X_1 - b(u)) > s, w^*(u)(X_2 - b(u)) > t \big| X_1 > b(u), X_2 > b(u) \right\} = \frac{X_1 > s, X_2 > t}, \]

holds with \( X_1', X_2' \) two independent exponentially distributed random variables with mean \( \lambda_p := \sqrt{2(1 + \rho)} \). Hence if \( x, y \in (0, 1) \), then \( - \ln x, - \ln y \in (0, \infty) \), thus (2.5) follows easily. For any \( x > 1 \) and \( y > 1 \) we may write

\[ \lim_{u \to \infty} S_u(x, y) = \lim_{u \to \infty} \frac{1}{S_u(1/x, 1/y)} = \infty. \]
Next, if (2.10) holds, then
\[
\frac{w^*(u)}{\alpha_p^{-1} \ln x} \left[ G^{-1}(1 - x/u) - b(u) \right] \to -1, \quad u \to \infty
\]
locally uniformly for any \( x > 0 \). Consequently, with the same arguments as above for any \( x, y \in (0, 1] \) we obtain
\[
\lim_{u \to \infty} S_u(x, y) = \begin{cases} 
\mathbf{P}\{X'_1 > -\alpha_p^{-1} \ln x, X'_2 > -\alpha_p^{-1} \ln y\} \\
\exp\left(\frac{\alpha_p^{-1}}{\lambda_p} \ln(xy)\right) = \exp\left(\frac{\alpha_p^{-1}}{2} \ln(xy)\right) =: S(x, y).
\end{cases}
\]
The result for \( x \in (1, \infty) \) and \( y \) positive, or \( x \) positive and \( y \in (1, \infty) \) as well as the statement (iii) can be now established using directly Theorem 2 in the aforementioned paper. Since for any \( c, x, y \) positive
\[
S(cx, cy) = S(x, y)c^{1/\eta},
\]
with
\[
\eta := \alpha_p^{-\theta} = \left(\frac{1 + \rho}{2}\right)^{\theta/2} \in (0, 1]
\]
the result follows. \( \square \)

**Proof of Theorem 4.1** Let \( I \) be a non-empty subset of \( \{1, \ldots, k\} \) with \( m \leq k \) elements. The random vector \( X_I := (X_i, i \in I)^\top \) is again an elliptical random vector with stochastic representation (Cambanis et al. (1981))
\[
X_I \overset{d}{=} R_I B V,
\]
with positive associate random radius \( R_I \), square matrix \( B \) such that \( B^\top B = \Sigma_{II} \) and \( V \) uniformly distributed on the unit sphere of \( \mathbb{R}^m \) being independent of \( R_I \). As shown in Berman (1992) the associated random radius \( R_I \) has distribution function \( F_I \) in the Gumbel max-domain of attraction with the same scaling function \( w \) as \( F \) the distribution function of \( R \). By Proposition 2.1 in Hashorva (2005b) there exist a unique subset \( K \subset I \) with \( l > 0 \) elements such that
\[
\alpha_l := \min_{y \in \mathbb{R}^m, y_i \geq 1, i = 1, \ldots, m} y^\top \Sigma_{II}^{-1} y = 1_K^\top \Sigma_{KK}^{-1} 1_K > 0,
\]
\( \Sigma_{KK}^{-1} 1_K \) has non-negative components, and if \( M := I \setminus K \) is not empty, then \( \Sigma_{KM} \Sigma_{MM}^{-1} 1_M - 1_K \) has non-negative elements.

As in the proof of Theorem 2.1 for any \( x \in (0, \infty)^k \) applying further Theorem 3.4 in Hashorva (2007b) we obtain
\[
\lim_{u \to \infty} \frac{\tilde{S}_{u,I}(x)}{\tilde{S}_{u,I}(1)} = \left( \prod_{j \in K} x_j^{\mu_j \alpha_j^{\alpha_j^{-1}}} \right) =: S(x) \in (0, \infty),
\]
with \( \mu_j := e_j^\top \Sigma_{KK}^{-1} 1_K > 0 \), and \( e_j \) the \( j \)-th unit vector in \( \mathbb{R}^d \). We have further
\[
\sum_{j \in K} \mu_j = \sum_{j \in K} e_j^\top \Sigma_{KK}^{-1} 1_K = \alpha_l,
\]
hence for any \( c > 0 \) and any \( x := (x_1, \ldots, x_k)^\top \in (0, \infty)^k \) we may write
\[
\frac{S(cx_1, \ldots, cx_k)}{S(x_1, \ldots, x_k)} = \left( \prod_{j \in K} e^{\mu_j \alpha_j^{\alpha_j^{-1}}} \right) = e^{\alpha_l} \sum_{j \in K} \mu_j = e^{\alpha_l},
\]
Consequently, \( \eta_l = \alpha_l^{-\theta} \), thus the result follows. \( \square \)
The proof of the first statement is shown in Lemma 3.2 in Hasorva and Hüsler (2002). We show next the second statement. Assume therefore that

\[ 1 + 2\rho_{\min} - \rho_{12} - \rho_{13} - \rho_{23} \leq 0, \quad \text{and} \quad \rho_{\min} = \rho_{12}. \]

Since \( 1 - \max(\rho_{23}, \rho_{13}) > 0 \), then \( \rho_{12} = \rho_{23} \) or \( \rho_{12} = \rho_{13} \) is not possible, hence \( \rho_{\min} = \rho_{12} \). In view of the aforementioned lemma \( \alpha = 2/(1 + \rho_{12}) \), thus the result follows. \( \square \)

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