FRAMED SHEAVES ON ROOT STACKS
AND SUPERSYMMETRIC GAUGE THEORIES ON ALE SPACES

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ABSTRACT. We develop a new approach to the study of supersymmetric gauge theories on ALE spaces using the theory of framed sheaves on root toric stacks, which illuminates relations with gauge theories on \(\mathbb{R}^4\) and with two-dimensional conformal field theory. We construct a stacky compactification of the minimal resolution \(X_k\) of the \(A_{k-1}\) toric singularity \(\mathbb{C}^2/\mathbb{Z}_k\), which is a projective toric orbifold \(\mathcal{X}_k\) such that \(\mathcal{X}_k \setminus X_k\) is a \(\mathbb{Z}_k\)-gerbe. We construct moduli spaces of torsion free sheaves on \(\mathcal{X}_k\) which are framed along the compactification gerbe. We prove that this moduli space is a smooth quasi-projective variety, compute its dimension, and classify its fixed points under the natural induced toric action. We use this construction to compute the partition functions and correlators of chiral BPS operators for \(N = 2\) quiver gauge theories on \(X_k\) with nontrivial holonomies at infinity. The partition functions are computed with and without couplings to bifundamental matter hypermultiplets and expressed in terms of toric blowup formulas, which relate them to the corresponding Nekrasov partition functions on the affine toric subsets of \(X_k\). We compare our new partition functions with previous computations, explore their connections to the representation theory of affine Lie algebras, and find new constraints on fractional instanton charges in the coupling to fundamental matter. We show that the partition functions in the low energy limit are characterised by the Seiberg-Witten curves, and in some cases also by suitable blowup equations involving Riemann theta-functions on the Seiberg-Witten curve with characteristics related to the nontrivial holonomies.

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1. Introduction and Summary

1.1. Background. Gauge theories on ALE spaces of type $A_{k-1}$ have recently received renewed interest as they provide a natural extension of the AGT correspondence [5, 95, 6] to nontrivial four-manifolds. In this paper we aim to provide a new sheaf-theoretic approach to the study of supersymmetric gauge theories on $A_{k-1}$ ALE spaces which enables one to establish new relationships between moduli spaces of sheaves and infinite-dimensional Lie algebras. In this subsection we provide some physical and mathematical background to the problems we shall address in this work.

An ALE space of type $A_{k-1}$ is a four-manifold $X$ which is diffeomorphic to the minimal resolution $\varphi_k : X_k \to \mathbb{C}^2/\mathbb{Z}_k$ of the simple Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_k$, and is equipped with a Kähler metric that is asymptotically locally Euclidean (ALE), i.e., there is a compact subset $K \subset X$ and a diffeomorphism $X \setminus K \to \left( \mathbb{C}^2 \setminus B_\varepsilon(0) \right)/\mathbb{Z}_k$ under which the metric is approximated by the standard Euclidean metric on $\mathbb{C}^2/\mathbb{Z}_k$. We call $X \setminus K$ the infinity of $X$. The ALE space $X$ can be realized as a hyper-Kähler quotient [54], hence it depends on a stability parameter $\xi := (\xi_\mathbb{R}, \xi_\mathbb{C})$ which belongs to an open subset $U$ of $H^2(X_k; \mathbb{R}) \times H^2(X_k; \mathbb{C}) \simeq \mathbb{R}^k \times \mathbb{C}^k$ [51, Section 2]; roughly speaking, a representative of $\xi_\mathbb{R}$ is a Kähler form and a representative of $\xi_\mathbb{C}$ is a holomorphic volume form on $X$. We denote this $\xi$-dependence of the ALE space by $X_k(\xi)$. The space $U$ is partitioned into “chambers”, such that for parameters belonging to the same chamber the corresponding ALE spaces are isomorphic; for parameters in distinct chambers the corresponding ALE spaces are only diffeomorphic. The natural toric structure on $\mathbb{C}^2/\mathbb{Z}_k$ lifts to its resolution $X_k$. The McKay correspondence gives a bijection between the irreducible components of $\varphi_k^{-1}(0)$, which are torus-invariant smooth projective curves $D_i$ of genus zero for $i = 1, \ldots, k - 1$, and the vertices of the Dynkin diagram of type $A_{k-1}$; the intersection matrix of these curves is exactly $-C$, where $C$ is the corresponding Cartan matrix. Any ALE space inherits these properties, so the homology group $H_2(X_k(\xi); \mathbb{Z})$ of an ALE space $X_k(\xi)$ can be identified with the root lattice $\Omega$ of type $A_{k-1}$, while the cohomology group $H^2(X_k(\xi); \mathbb{R})$ is the real Cartan subalgebra of $\mathfrak{sl}(k)$ [54, Section 4].

The connection between gauge theories on ALE spaces and two-dimensional conformal field theory in physics, or between moduli spaces and infinite-dimensional Lie algebras from a mathematical perspective, goes back to pioneering works of Nakajima [65, 64, 66], who showed that one can construct highest weight representations of affine Lie algebras using quiver varieties. Nakajima’s quiver varieties arise from the ADHM construction of $U(\tau)$ gauge theory instantons on ALE spaces. A $U(\tau)$ instanton on $X_k(\xi)$ is a pair $(E, \nabla)$ consisting of a Hermitian vector bundle $E \to X_k(\xi)$ of rank $\tau$ and a connection $\nabla$ on $E$ whose curvature is anti-selfdual and square-integrable. The connection $\nabla$ is additionally characterised by its behaviour at infinity: $\nabla$ is flat but not necessarily trivial at infinity, so it is also classified by its holonomy which takes values in the fundamental group $\pi_1(X \setminus K) \simeq \pi_1(S^3/\mathbb{Z}_k) \simeq \mathbb{Z}_k$. 

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and hence corresponds to a homomorphism \( \rho: \mathbb{Z}_k \rightarrow U(r) \). As described in [42], with each irreducible representation of \( \mathbb{Z}_k \) one can associate a Hermitian line bundle on \( X_k(\xi) \), called a tautological line bundle, which is endowed with an anti-self-dual square-integrable connection whose holonomy at infinity corresponds to the representation. The tautological line bundles form an integral basis of the Picard group \( \text{Pic}(X_k(\xi)) \cong H^2(X_k(\xi); \mathbb{Z}) \); their intersection matrix is minus the inverse of the Cartan matrix \( C \) of the \( A_{k-1} \) Dynkin diagram. In [55], Kronheimer and Nakajima provided a description of \( U(r) \) instantons on \( X_k(\xi) \) in terms of ADHM type data depending on two vectors \( \vec{v} = (v_0, v_1, \ldots, v_{k-1}), \vec{w} = (w_0, w_1, \ldots, w_{k-1}) \in \mathbb{N}^k \), which parameterize the Chern character \( \text{ch}(E) \), with \( c_1(E) \) given in terms of the first Chern classes of the tautological line bundles, and the multiplicities of the \( k \) one-dimensional irreducible representations of \( \mathbb{Z}_k \) in the holonomy at infinity \( \rho \). By using the ADHM description, moduli spaces parameterizing \( U(r) \) instantons on \( X_k(\xi) \) can be realized as hyper-Kähler quotients \( M_\xi(\vec{v}, \vec{w}) \) depending on \( \xi \). By perturbing the real part of the moment map of the hyper-Kähler quotient construction one obtains larger moduli spaces, which are Nakajima quiver varieties associated with the affine extended Dynkin diagram of type \( \hat{A}_{k-1} \); they are smooth quasi-projective varieties. Nakajima proved that the cohomology of these moduli spaces is a representation of \( \widehat{\mathfrak{sl}}(k) \) acting at level \( r \); the generators of the affine Kac-Moody algebra are defined using geometric Hecke correspondences. The McKay correspondence now gives a bijective equivalence between the tautological line bundles on \( X_k(\xi) \) and integrable highest weight representations of the affine Lie algebra \( \widehat{\mathfrak{sl}}(k) \) acting at level \( r \); in the setting of two-dimensional conformal field theory, the Chern characters \( \text{c}_1(E) \) and \( \text{ch}_2(E) \) are respectively identified with the momentum \( \vec{p} \) and the energy \( L_0 \) of the highest weight module over \( \widehat{\mathfrak{sl}}(k)_r \), corresponding to the unitary representation \( \rho: \mathbb{Z}_k \rightarrow U(r) \).

A gauge theory realization of Nakajima’s results first appeared in the seminal work of Vafa and Witten [89] who showed that the partition function of a certain topologically twisted version of \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory on \( X_k(\xi) \) computes the characters of highest weight representations of \( \widehat{\mathfrak{sl}}(k)_r \). To extend this realization beyond the level of characters, one should relate the instanton partition functions of corresponding \( \mathcal{N} = 2 \) gauge theories on ALE spaces to correlators of vertex operators of \( \widehat{\mathfrak{sl}}(k)_r \). To compute the instanton partition functions, one needs some kind of smooth completion of the moduli space of instantons. In the case of instantons on \( \mathbb{R}^4 \) this completion can be obtained in three different but equivalent ways: by passing to a noncommutative deformation of \( \mathbb{R}^4 \) [74], by an algebro-geometric blowup, or by regarding the moduli space as a hyper-Kähler quotient and perturbing the set of zeroes of the moment map. In all three cases one obtains a space that parameterizes framed torsion free sheaves on the projective plane \( \mathbb{P}^2 \). In the ALE case only the third technique is currently workable and yields Nakajima’s quiver variety. Unfortunately, in general it is very difficult to use a Nakajima quiver variety to compute the instanton partition functions. A possible solution to this problem consists of realizing it as a moduli space parameterizing framed torsion free sheaves. As explained in [68] (see also [90, Section 2.3]) this is possible at least for a family of Nakajima quiver varieties. Indeed, there exists a chamber \( C_0 \) containing a parameter \( \xi_0 \) with \( \xi_0 |_{\mathbb{R}} = 0 \) such that moduli spaces of instantons \( M_{\xi_0}(\vec{v}, \vec{w}) \) are isomorphic to moduli spaces of framed \( \mathbb{Z}_k \)-equivariant locally free sheaves on \( \mathbb{P}^2 \), which are open dense subsets of the moduli spaces of framed \( \mathbb{Z}_k \)-equivariant torsion free sheaves on \( \mathbb{P}^2 \), and can also be realized as Nakajima quiver varieties with the same stability parameter. This provides a mathematical interpretation of the work [37], where the instanton partition functions of \( \mathcal{N} = 2 \) supersymmetric gauge theories on ALE spaces with parameters in the chamber \( C_0 \) were computed by considering a torus action (given by the torus \((\mathbb{C}^*)^2 \) of \( \mathbb{P}^2 \) and the maximal torus of \( GL(r, \mathbb{C}) \)) on the framed sheaves on \( \mathbb{P}^2 \) and taking into account only the fixed points that are invariant under the action of \( \mathbb{Z}_k \).

Recent developments in string theory have sparked new interest in the correspondence between four-dimensional gauge theories and two-dimensional conformal field theories in view of the observation that the \( \mathcal{N} = 2 \) gauge theory on a four-manifold \( X \) can also be studied geometrically by embedding it in
M-theory as the (2, 0) superconformal theory compactified on the corresponding Seiberg-Witten curve \( \Sigma \); this theory is the low-energy limit of the worldvolume theory of a single M5-brane filling the six-dimensional manifold \( M_6 = \Sigma \times X \). More generally, one can compactify on any punctured Riemann surface \( \Sigma' \) such that the Seiberg-Witten curve \( \Sigma \) is a branched cover of \( \Sigma' \) [60, 73]. The chiral fields of the two-dimensional conformal field theory on \( \Sigma' \) which conjecturally describes the BPS sector of the four-dimensional \( \Omega \)-deformed (i.e. equivariant) gauge theory arise as the zero modes of the six-dimensional (2, 0) tensor multiplet. By applying this machinery to the case \( X = \mathbb{R}^4 \) one obtains the correspondence conjectured by Alday, Gaiotto and Tachikawa (AGT) [5] (see also [95]), which relates the Nekrasov partition functions [71] of \( U(r) \) gauge theories on \( \mathbb{R}^4 \) to the conformal blocks of Toda conformal field theories. From a mathematical perspective, the AGT correspondence asserts a higher rank generalization of the celebrated result of Nakajima [67] and Vasserot [91] which relates the \((\mathbb{C}^*)^2\)-equivariant cohomology of the Hilbert schemes \( \text{Hilb}^n(\mathbb{C}^2) \) of \( n \) points on \( \mathbb{C}^2 \) to representations of the Heisenberg algebra over \( H^*_{(\mathbb{C}^*)^2}(\mathbb{C}^2; \mathbb{C}) \). The higher rank generalizations of \( \text{Hilb}^n(\mathbb{C}^2) \) are the moduli spaces \( \mathcal{M}(r, n) \) of framed torsion free sheaves on \( \mathbb{P}^2 \) of rank \( r \) and second Chern class \( n \). The conjecture then supports the existence of a natural geometric action of the \( \mathcal{W}(\mathfrak{gl}_r) \)-algebra on the direct sum over \( n \) of the equivariant cohomology groups of \( \mathcal{M}(r, n) \). This correspondence was proven by Schiffmann and Vasserot [83], and independently by Maulik and Okounkov [62]. The AGT correspondence allows one to write the Nekrasov partition functions, defined as integrals over \( \mathcal{M}(r, n) \), as correlators of vertex operators of the \( \mathcal{W}(\mathfrak{gl}_r) \)-algebra.

A generalization of the AGT correspondence to \( \mathcal{N} = 2 \) gauge theories on ALE spaces could provide a better gauge-theoretic explanation of Nakajima’s work than the Vafa-Witten theory. In [12, 77, 10] the relevant algebra is conjectured to be

\[
\mathcal{A}(r, k) = \mathfrak{h} \oplus \hat{\mathfrak{sl}}(k)_r \oplus \frac{\hat{\mathcal{I}}(r)_k \oplus \hat{\mathcal{I}}(r)_{\nu-k}}{\hat{\mathfrak{sl}}(r)_\nu},
\]

where \( \mathfrak{h} \) is the infinite-dimensional Heisenberg algebra and the parameter \( \nu \) is related to the equivariant parameters of the torus action. For \( r = 1 \) this algebra is simply \( \mathcal{A}(1, k) = \mathfrak{h} \oplus \hat{\mathfrak{sl}}(k)_1 \). For \( k = 2 \) the algebra \( \mathcal{A}(2, 2) \) is isomorphic to the sum of \( \hat{\mathfrak{gl}}(2)_r \) acting at level \( r \) and the Neveu-Schwarz-Ramond algebra of supersymmetric Liouville theory. For generic values of \((r, k)\), the algebra \( \mathcal{A}(r, k) \) is the sum of the affine Lie algebra \( \hat{\mathfrak{gl}}(k)_r \) and the \( \mathbb{Z}_k \)-parafermionic \( \mathcal{W}(\mathfrak{gl}_r) \)-algebra. From this perspective, the geometric representations of \( \hat{\mathfrak{sl}}(k)_r \) constructed by Nakajima are only a part of the geometric representations of \( \mathcal{A}(r, k) \) that one expects to define on the equivariant cohomology of the Nakajima quiver varieties. The conjecture that has thus far emerged from field theory calculations implies a new type of AGT correspondence which relates the equivariant cohomology groups of moduli spaces of framed \( \mathbb{Z}_k \)-equivariant torsion free sheaves on the projective plane \( \mathbb{P}^2 \) and representations of the affine algebra \( \mathcal{A}(r, k) \); see e.g. [13, 14, 96, 48, 11, 15, 7].

There are three main outstanding issues that persist. Firstly, since the Nakajima quiver varieties are all equivariantly diffeomorphic, but isomorphic only when their stability parameters lie in the same chamber, the computations of [37] provide instanton partition functions for \( \mathcal{N} = 2 \) gauge theories with only adjoint matter hypermultiplets on any ALE space \( X_k(\xi) \), but for \( \mathcal{N} = 2 \) gauge theories with fundamental matter fields only on the ALE spaces with parameters in the chamber \( C_0 \). Secondly, since the stability parameter of the surface \( X_k \) does not belong to \( C_0 \) (see e.g. [63]), for gauge theories with only adjoint matter one expects a nontrivial equivalence between the corresponding partition functions [11, 15, 7]; on the other hand, the relationship between the two partition functions should be governed by a wall-crossing formula in the case of gauge theories with fundamental matter [49]. Thirdly, one expects [15] to find a blowup formula for the instanton partition functions on ALE spaces in terms of instanton partition functions on \( \mathbb{R}^4 \) depending on the equivariant parameters of the torus action on the affine patches of \( X_k \). This
generalizes the blowup formulas of [70] by taking into account that \( X_k \) is obtained by a weighted blowup at the singular point of \( \mathbb{C}^2/\mathbb{Z}_k \). In this paper, we address some of these problems. Instead of looking for possible smooth completions of the moduli spaces of instantons, we directly construct moduli spaces of framed sheaves which, as we will show, provide a suitable setting to compute instanton partition functions for gauge theories on \( X_k \).

1.2. Overview. In this subsection we summarise the main problems we shall tackle in some detail. The goal of this paper is to investigate a new geometrical approach to the study of \( \mathcal{N} = 2 \) gauge theories on ALE spaces of type \( A_{k-1} \) by finding a suitable compactification of \( X_k \) on which to develop a theory of framed sheaves. Our approach is the first attempt to use a theory of framed sheaves in the study of gauge theories on ALE spaces with stability parameters in the same chamber as the parameter of the minimal resolution \( X_k \). In particular, we aim to provide the “correct” moduli space on which to construct natural geometric realizations of representations of the algebra \( \mathcal{A}(r, k) \); this can be interpreted as a first step in the direction of geometric realizations of representations of more complicated infinite-dimensional Lie algebras.

Note that any such compactification of a \( \text{Kähler} \) surface \( M \) is strongly constrained by a result of Bando [8]: If \( \tilde{M} \) is a compactification of \( M \) by a smooth divisor with positive normal line bundle, which is a compact \( \text{Kähler} \) surface, then holomorphic vector bundles on \( \pi \) and \( D \) structure of \( X \) can be realized as compactification divisor \( D \). Bando [8] shows that the holonomy at infinity of the instantons should correspond to a flat connection on the associated locally free sheaves restricted to the compactification divisor; hence if \( M \) is obtained from \( M \) by adding a projective line \( D \), then only instantons on \( M \) with trivial holonomy at infinity can be described in terms of framed locally free sheaves on \( M \). When \( M = X_k \), one possible way to avoid this restriction is to look for more general compactifications, that should properly allow for the contributions of instantons with nontrivial holonomies at infinity, which in gauge theory are sometimes called “fractional” instantons.

The first attempt at such a compactification of \( X_k \) is due to Nakajima [68], who suggested an orbifold compactification; the compactification divisor \( D \) carries a \( \mathbb{Z}_k \)-action such that a framed torsion free sheaf restricted to \( D \) is isomorphic to a \( \mathbb{Z}_k \)-equivariant locally free sheaf, which should encode a fixed holonomy at infinity. Another approach was pursued in [20], where torsion free sheaves on Hirzebruch surfaces \( \mathbb{F}_p \) framed along a divisor \( D_\infty \), with \( D_\infty^2 = p \), were used to compute the Vafa-Witten partition functions of \( \mathcal{N} = 4 \) gauge theory on the total spaces \( \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-p)) \) of the line bundles \( \mathcal{O}_{\mathbb{P}^1}(-p) \), which can be realized as \( \mathbb{F}_p \setminus D_\infty \). The computations of [20] also make sense for fractional Chern classes \( c_1 \in \frac{1}{p} \mathbb{Z} \) which, although heuristic, correctly incorporate the anticipated contributions to the Vafa-Witten partition function from fractional instantons (see e.g. [38, 45]); hence the paper [20] suggests that the appropriate geometric arena for these computations should involve a “stacky” compactification of \( \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-p)) \) which, as shown in [21, Appendix D], is a root toric stack over \( \mathbb{F}_p \). This reasoning was extended to compute the partition functions of \( \mathcal{N} = 2 \) gauge theories in [13, 14, 15]; other calculations of the fractional instanton contributions to supersymmetric gauge theory partition functions can be found in [38, 45, 26, 85, 27].

In the first part of this paper we address the problem of constructing a suitable compactification of \( X_k \). This is a projective toric orbifold \( \mathcal{X}_k \) over a projective normal toric compactification of \( X_k \). The stacky structure of \( \mathcal{X}_k \) is concentrated at a smooth effective Cartier divisor \( D_\infty \), such that \( \mathcal{X}_k \setminus D_\infty \simeq X_k \) and \( D_\infty \) is a \( \mathbb{Z}_k \)-gerbe over a football, i.e., an orbifold curve over \( \mathbb{P}^1 \) with two orbifold points. One has \( \pi_1(\mathcal{D}_\infty^\text{top}) \simeq \mathbb{Z}_k \), where \( \mathcal{D}_\infty^\text{top} \) is the underlying topological stack of \( \mathcal{D}_\infty \), so there exist \( k \) complex line bundles endowed with flat connections associated with the \( k \) irreducible unitary representations of \( \mathbb{Z}_k \). Thus the locally free sheaf on \( \mathcal{D}_\infty \) which should encode the holonomy at infinity is the direct sum of these line bundles; we shall call it a framing sheaf. It is associated with a homomorphism \( \rho: \mathbb{Z}_k \to U(r) \).
By using the machinery developed in [21] we construct a moduli space of torsion free sheaves on \( \mathcal{X}_k \) which are isomorphic along \( \mathcal{Z}_\infty \) to a given framing sheaf. By defining a suitable toric action and characterizing the torus-fixed point locus, we compute partition functions of gauge theories with gauge groups \( U(r) \times U(r') \) on \( X_k \), with and without matter in the bifundamental representation; the extensions of our results to general A_n-type quiver gauge theories should be possible but are left for future endeavours. By using this new geometrical approach, the factorizations (blowup formulas) are evident. In particular, we will provide rigorous derivations of the partition functions for \( N = 2 \) gauge theories on \( X_k \) which are conjecturally formulated in [15] using heuristic arguments. On the other hand, although our formulas can be shown to match with those of [15] in several nontrivial checks, our expressions are \textit{a priori} different and we expect a nontrivial equivalence between the corresponding partition functions. In this sense our partition functions may in fact yield a more transparent connection to conformal field theory in two dimensions under the AGT duality.

In addition to the partition functions, we also address the problem of computing correlators of certain gauge invariant chiral BPS operators. The topologically twisted \( N = 2 \) gauge theory on \( X_k \) has a natural set of holomorphic observables residing in the nontrivial cohomology groups \( H^p(X_k; \mathbb{C}) \) [59, 61]. They are labelled by an invariant polynomial \( \mathcal{P} \) on the Lie algebra \( u(r) \). The “0-observables” are then \( \mathcal{O}^{(0)} := \mathcal{P}(\phi) \), where \( \phi \) is the complex scalar field of the \( N = 2 \) vector multiplet, while the “2-observables” \( \mathcal{O}^{(2)} \) are obtained by canonical descent equations from the invariant polynomial \( \mathcal{P} \); there are similarly “4-observables”, but their contributions can be absorbed into the partition functions [61]. The generating functions for the correlators of nontrivial \( p \)-observables are then of the schematic form

\[
\left\langle \exp \left( \mathcal{O}^{(0)}(P) + \mathcal{O}^{(2)}(S) \right) \right\rangle_{X_k},
\]

where \( P \in H_0(X_k; \mathbb{C}) \) and \( S \in H_2(X_k; \mathbb{C}) \), while the braces denote an expectation value in the quantum gauge theory on \( X_k \). In Donaldson-Witten theory, one takes \( \mathcal{P} \) to be linear in the Casimir invariants of the gauge group; then the \( p \)-observables correspond to single-trace chiral operators \( \mathcal{P}_s(\phi) = \frac{1}{(2\pi i)^s} \text{Tr} \phi^s \), where \( \text{Tr} \) is the trace in the \( r \)-dimensional representation, and in the \( \Omega \)-deformed gauge theory they are in a bijective correspondence with characteristic classes \( \text{ch}_k(\mathcal{E}) \) of the coherent rank \( r \) universal framed sheaf \( \mathcal{E} \) associated to “families” of \( U(r) \) instantons parametrized by points of the moduli space of framed torsion free sheaves [71, 60]. In particular, at the BRST fixed points of the topologically twisted gauge theory the scalar field \( \phi \) is a certain \( u(r) \)-valued two-form on the moduli space, and the perturbation by arbitrary powers of holomorphic operators is then represented as

\[
\mathcal{P}(\phi) = \sum_{s=0}^{\infty} \frac{\tau_s}{s+1} \text{ch}_{s+1}(\mathcal{E}),
\]

where \( \tau = (\tau_0, \tau_1, \tau_2, \ldots) \) is a collection of complex parameters. In this case the generating functions for \( p \)-observables are computed rigorously in [69] for gauge theories on \( \mathbb{R}^4 \) and on the blowup of \( \mathbb{R}^4 \) at the origin.

Of particular interest is the case with \( \mathcal{P}(\phi) = \text{Tr} \phi^2 \), corresponding to observables obtained from the quadratic Casimir invariant; this corresponds to the choice \( \tau = (0, -\tau_1, 0, 0, \ldots) \) with \( \mathcal{O}^{(0)} \) related to the second Chern character of the universal sheaf \( \mathcal{E} \). Then the deformed partition functions can also be regarded as generating functions for the equivariant cohomology version of the Donaldson-Witten invariants of \( X_k \). It is then natural to investigate the structure of these correlators in the low energy limit where the equivariant deformation (\( \Omega \)-background) is removed. In this limit, we expect a relation between correlation functions on \( X_k \) and correlation functions on \( X_k(\xi_0) \) analogous to the wall-crossing formula that should relate the partition functions of the corresponding \( \Omega \)-deformed gauge theories [49]; in the low energy limit, the exceptional set \( \varphi_k^{-1}(0) \) of the minimal resolution \( \varphi_k : X_k \to \mathbb{C}^2/\mathbb{Z}_k \) can be represented by an infinite number of local operators on \( X_k(\xi_0) \). We therefore expect a blowup equation
of the schematic form
\[
\left\langle \exp \left( O^{(0)}(P) + t \left( O^{(2)}(S) \right) \right) \right\rangle_{X_k} = \left\langle \exp \left( O^{(0)}(\varphi_k(P)) + F[t] O_2, \ldots, O_{r+1} \right) \right\rangle_{X_k(\xi_0)},
\]
where \(F\) is a holomorphic function and \(O_\alpha, \alpha = 2, \ldots, r+1\) are the generators of the ring of local BRST-invariant observables corresponding to the Casimir invariants of \(U(r)\); this formula generalizes the blowup equations of \([59, 61]\) which considered blowups of smooth points. Rigorous derivations of such blowup equations for the Nakajima-Nekrasov partition functions were obtained by Nakajima and Yoshioka in \([70, 69]\) which resemble the Fintushel-Stern formulas for Donaldson invariants \([36]\); these equations were one of the ingredients in the computations of Donaldson invariants from Neveu-Schwarz formulas \([59, 61]\) which considered blowups of smooth points. Rigorous derivations of UBRST-invariant observables corresponding to the Casimir invariants of \(\mathcal{F}\),

Let us denote by \(R_i = 1\) the surface \(X_k\) one has

\[
\omega_k := -\sum_{j=1}^{k-1} \left( C^{-1} \right)^{ij} \pi_k, \quad \omega_k := -\sum_{j=1}^{k-1} \left( C^{-1} \right)^{ij} \pi_k,
\]

We prove that these classes are integral. Let us denote by \(\mathcal{R}_i := \mathcal{O}_{X_k}(\omega_i)\) their associated line bundles on \(\mathcal{X}_k\). Their restrictions to \(X_k\) are precisely the tautological line bundles of Kronheimer and Nakajima.

**Proposition** (Proposition 3.25). **The Picard group** \(\text{Pic}(\mathcal{X}_k)\) of \(\mathcal{X}_k\) **is freely generated over** \(\mathbb{Z}\) **by** \(\mathcal{O}_{X_k}(\omega_0)\) **and** \(\mathcal{R}_i\) **with** \(i = 1, \ldots, k-1\).

We further provide a characterization of the divisor \(D_\infty\) as a toric Deligne-Mumford stack with Deligne-Mumford torus \(\mathbb{C}^* \times \mathbb{R}_{\mathbb{Z}}\) and coarse moduli space \(r_k : D_\infty \to D_\infty\).
Proposition (Proposition 3.27). $\mathcal{D}_\infty$ is isomorphic as a toric Deligne-Mumford stack to the toric global quotient stack

$$\left[\mathbb{C}^2 \setminus \{0\} \right] / \mathbb{C}^* \times \mathbb{Z}_k,$$

where the group action is given in Equation (3.28).

Since line bundles on a global quotient stack $[X/G]$ are associated with characters of the group $G$, we find that the Picard group $\text{Pic}(\mathcal{D}_\infty)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_k$; it is generated by the line bundles $\mathcal{L}_1, \mathcal{L}_2$ corresponding to the characters $\chi_1, \chi_2 : \mathbb{C}^* \times \mathbb{Z}_k \to \mathbb{C}^*$ given respectively by the projections $(t, \omega) \mapsto t$ and $(t, \omega) \mapsto \omega$, where $t \in \mathbb{C}^*$ and $\omega$ is a primitive $k$-th root of unity. In particular, $\mathcal{L}_2$ is trivial. As pointed out by [34], the fundamental group of the underlying topological stack $\mathcal{D}_{\text{top}}^\infty$ of $\mathcal{D}_\infty$ is isomorphic to $\mathbb{Z}_k$; in addition, for any $i = 0, 1, \ldots, k-1$ the line bundle $\mathcal{L}_2^i$ inherits a Hermitian metric and a unitary flat connection associated with the $i$-th irreducible representation of $\mathbb{Z}_k$.

1.3.2. Moduli spaces of framed sheaves. In order to construct moduli spaces of framed sheaves on $\mathcal{X}_k$ which are needed for the formulation of supersymmetric gauge theories on $X_k$, we first have to choose a suitable framing sheaf which should encode the fixed holonomy at infinity of instantons. Since the holonomy at infinity corresponds to a representation of $\mathbb{Z}_k$, the framing sheaf should have a Hermitian metric and a unitary flat connection associated with the representation of $\mathbb{Z}_k$. Because of this, we choose as framing sheaf the locally free sheaf

$$\mathcal{F}^0_{\mathcal{D}_\infty} := \bigoplus_{i=0}^{k-1} (\mathcal{O}_{\mathcal{D}_\infty}(0, i))^{\oplus s_i},$$

where $\mathcal{O}_{\mathcal{D}_\infty}(0, i)$ is $\mathcal{L}_2^i$ for even $k$ and $\mathcal{L}_2^{i+1} = \mathcal{L}_2^i$ for odd $k$, and $\vec{w} := (w_0, w_1, \ldots, w_{k-1}) \in \mathbb{N}^k$ is a fixed vector. If we tensor $\mathcal{F}^0_{\mathcal{D}_\infty}$ by a power $\mathcal{L}_1^s$ of $\mathcal{L}_1$, we obtain a more general framing sheaf $\mathcal{F}^s_{\mathcal{D}_\infty}$; the degree of $\mathcal{F}^s_{\mathcal{D}_\infty}$ is a rational multiple of $s$.

In Section 4, we apply the general theory of framed sheaves on projective stacks developed in [21] to construct a fine moduli space $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}^0_{\mathcal{D}_\infty})$ parameterizing $(\mathcal{D}_\infty, \mathcal{F}^0_{\mathcal{D}_\infty})$-framed sheaves $(\mathcal{E}, \phi_{\mathcal{E}} : \mathcal{E}_{\mathcal{D}_\infty} \xrightarrow{\sim} \mathcal{F}^0_{\mathcal{D}_\infty})$ on $\mathcal{X}_k$ with fixed rank $r := \sum_{i=0}^{k-1} w_i$, first Chern class $c_1(\mathcal{E}) = \sum_{i=1}^{k-1} u_i \omega_i$, and discriminant $\Delta(\mathcal{E}) = \Delta$. The vector $\vec{u} = (u_1, \ldots, u_{k-1})$ satisfies the constraint

$$k v_{k-1} = \sum_{i=0}^{k-1} i w_i \mod k,$$

where $\vec{v} := C^{-1} \vec{u}$.

Theorem (Theorem 4.13). The moduli space $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}^0_{\mathcal{D}_\infty})$ is a smooth quasi-projective variety of dimension

$$\dim_{\mathbb{C}} \left( \mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}^0_{\mathcal{D}_\infty}) \right) = 2r \Delta \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} \vec{w} \cdot \vec{w}(j),$$

where for $j = 1, \ldots, k-1$ the vector $\vec{w}(j)$ is $(w_j, \ldots, w_{k-1}, w_0, w_1, \ldots, w_{j-1})$. Moreover, the Zariski tangent space of $\mathcal{M}_{r, \vec{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}^0_{\mathcal{D}_\infty})$ at a point $[(\mathcal{E}, \phi_{\mathcal{E}})]$ is $\text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$.

In the rank one case $r = 1$, we show that there is an isomorphism of fine moduli spaces

$$\iota_{1, \vec{u}, n} : \text{Hilb}^n(X_k) \xrightarrow{\sim} \mathcal{M}_{1, \vec{u}, n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}^0_{\mathcal{D}_\infty}),$$

where $\text{Hilb}^n(X_k)$ is the Hilbert scheme of $n$ points on $X_k$. 
In gauge theory, the tangent bundle $T\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}})$ describes gauge fields and matter fields in the adjoint representation of the gauge group $U(r)$. Matter in the fundamental representation of the gauge group $U(r)$ is described by the coherent sheaf

$$V := R^1p_*\left((\mathcal{E} \otimes p_{\mathcal{X}_k}^* \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))\right),$$

where $\mathcal{E}$ is the universal sheaf of $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}})$, while $p$ and $p_{\mathcal{X}_k}$ respectively denote the projections of $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}})$ onto $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}})$ and $\mathcal{X}_k$. We call $V$ the natural bundle of $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}})$.

**Proposition (Proposition 4.18).** $V$ is a locally free sheaf on $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}})$ of rank

$$\text{rk}(V) = \Delta + \frac{1}{2r} \vec{v} \cdot C \vec{v} - \frac{k-1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} w_j$$

where $\vec{v} := C^{-1} \vec{u}$.

The computation of the rank of $V$ and the dimension of the moduli spaces of framed sheaves is addressed in Appendix A, where we use the Töen-Riemann-Roch theorem and some summation identities for complex roots of unity (derived in Appendix B) to obtain the explicit formulas.

There is a natural generalization of the vector bundles $T\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}})$ and $V$ to a virtual vector bundle $E$ on the product $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}}) \times \mathcal{M}_{r',\vec{u}',\Delta'}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}'})$ with fibre over a point $((\mathcal{E}, \phi_\mathcal{E}), (\mathcal{E}', \phi_\mathcal{E}'))$ given by

$$E((\mathcal{E}, \phi_\mathcal{E}), (\mathcal{E}', \phi_\mathcal{E}')) = \text{Ext}^1(\mathcal{E} \otimes \mathcal{E}', \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty));$$

it can be regarded as a higher rank generalization of the Carlsson-Okounkov bundle [25]. In gauge theory, the bundle $E$ describes matter in the bifundamental representation of $U(r) \times U(r')$.

Let us now explain how these moduli spaces and their natural vector bundles are used to compute partition functions of $\mathcal{N} = 2$ gauge theories on $X_k$. The $\mathcal{N} = 2$ gauge theory on a four-dimensional toric manifold $X$ in the $\Omega$-background is obtained as the reduction of a six-dimensional $\mathcal{N} = 1$ gauge theory on a flat $X$-bundle $M$ over $\mathbb{T}^2$ in the limit where the torus $\mathbb{T}^2$ collapses to a point [72, Section 3.1]. The bundle $M$ can be realized as the quotient of $\mathbb{C} \times X$ by the $\mathbb{Z}^2$-action

$$(n_1, n_2) \cdot (w, x) = (w + (n_1 + \sigma n_2), g_1^{n_1} g_2^{n_2}(x)),$$

where $x \in X$, $w \in \mathbb{C}$, $(n_1, n_2) \in \mathbb{Z}^2$, $g_1, g_2$ are two commuting isometries of $X$ and $\sigma$ is the complex structure modulus of $\mathbb{T}^2$. In the collapsing limit, fields of the gauge theory which are charged under the $\mathbb{R}$-symmetry group are sections of the pullback to $M$ of a flat $T_r$-bundle over $\mathbb{T}^2$, where $T_r \subset GL(2, \mathbb{C})$ is the torus of $X$. As pointed out in [73, Section 2.2.2], the chiral observables of the $\mathcal{N} = 2$ gauge theory in the $\Omega$-background become closed forms on the moduli spaces of framed instantons which are equivariant with respect to the action of the torus $T := T_t \times T_p$, where $T_p$ is the maximal torus of the group $GL(r, \mathbb{C})$ of constant gauge transformations which consists of diagonal matrices. Thus correlation functions of chiral BPS operators become integrals of equivariant classes over the moduli spaces.

In our setting we therefore define gauge theory partition functions as generating functions for $T$-equivariant integrals of suitable characteristic classes over the moduli spaces $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}})$; the precise choice of cohomology classes depends on the matter content of the gauge theory and on the chiral observables in question. There is a natural $T$-action on $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}})$ given, on a point $((\mathcal{E}, \phi_\mathcal{E}))$, by pullback of $\mathcal{E}$ via the natural automorphism of $\mathcal{X}_k$ induced by an element of $T_t$, and by “rotation” of the framing $\phi_\mathcal{E}$ by a diagonal matrix $\rho$ of $T_p$. The $T$-fixed point locus of $\mathcal{M}_{r,\vec{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{0,\vec{u}})$ consists of a finite number of isolated points.
Proposition (Proposition 4.22). A $T$-fixed point $[(\mathcal{E}, \phi_\mathcal{E})] \in \mathcal{M}_{T, \mathcal{U}, \Delta}(\mathcal{Y}_k, \mathcal{D}_\infty, \mathcal{T}_{2\infty}^{0, \mathcal{U}})^T$ decomposes as a direct sum of rank one framed sheaves

$$((\mathcal{E}, \phi_\mathcal{E}) = \bigoplus_{\alpha=1}^{r} (\mathcal{E}_\alpha, \phi_\alpha),$$

where for $i = 0, 1, \ldots, k - 1$ and $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^{i} w_j$:

- $\mathcal{E}_\alpha$ is a tensor product $\nu^*(I_\alpha) \otimes (\bigotimes_{j=1}^{k-1} \mathcal{R}_j^{\otimes (\mathcal{U}_\alpha)})$, where $I_{\alpha}$ is an ideal sheaf of a 0-dimensional subscheme $Z_{\alpha}$ of $X_k$ with length $n_{\alpha}$ supported at the $T_1$-fixed points $p_1, \ldots, p_k$, while $\mathcal{U}_\alpha \subset \mathbb{Z}^{k-1}$ obeys $\sum_{\alpha=1}^{r} \mathcal{U}_\alpha = \mathcal{U}$ and $\mathcal{U}_{\alpha} := C^{-1}\mathcal{U}_\alpha$ satisfies

$$k(\mathcal{U}_\alpha)_{k-1} = i \mod k;$$

- $\phi_\alpha : \mathcal{E}_\alpha \xrightarrow{\sim} \mathcal{O}_{\mathcal{D}_\infty}(0, i)$ is induced by the canonical isomorphism $\bigotimes_{j=1}^{k-1} \mathcal{R}_j^{\otimes (\mathcal{U}_\alpha)} \simeq \mathcal{O}_{\mathcal{D}_\infty}(0, i);$

- $\Delta = \sum_{\alpha=1}^{r} n_{\alpha} + \frac{1}{2} \sum_{\alpha=1}^{r} \mathcal{U}_{\alpha} \cdot C\mathcal{U}_{\alpha} - \frac{1}{2r} \sum_{\alpha, \beta=1}^{r} \mathcal{U}_{\alpha} \cdot C\mathcal{U}_{\beta} \in \frac{1}{2r} k \mathbb{Z}.$

In the standard way, we can associate to the $T_1$-invariant ideal sheaf $\nu^*(I_\alpha)$ a set of Young tableaux $Y_\alpha := \{Y^n_{\alpha}\}_{i=1,\ldots,k}$. Hence to each fixed point $[(\mathcal{E}, \phi_\mathcal{E})]$ there corresponds a combinatorial datum $(\bar{Y}, \bar{u})$, where $\bar{Y} := (\bar{Y}_1, \ldots, \bar{Y}_r)$ and $\bar{u} := (\bar{u}_1, \ldots, \bar{u}_r).$

1.3.3. Gauge theory partition functions. In Section 5 we compute partition functions and correlators of chiral observables for $\mathcal{N} = 2$ gauge theories on $X_k$, with and without bifundamental matter fields. Here we summarise the pertinent results only for the gauge theory with a single adjoint hypermultiplet of mass $\mu$, i.e., the $\mathcal{N} = 2^*$ gauge theory; the pure $\mathcal{N} = 2$ theory is recovered formally in the limit $\mu \to \infty$, while the limit $\mu = 0$ has enhanced maximal supersymmetry and reduces to the $\mathcal{N} = 4$ Vafa-Witten theory. This is the case that is conjectured to compute the correlation functions of chiral observables in a dual two-dimensional conformal field theory on a torus, obtained geometrically by a twisted compactification of the $(2, 0)$ theory with defects on an elliptic curve. The more general cases are treated in the main text (see Sections 5.1 and 5.4).

Let $\varepsilon_1, \varepsilon_2, a_1, \ldots, a_r$ be the generators of $H_T^*(\text{pt}; \mathbb{Q})$; in gauge theory on $X = \mathbb{R}^4, a_1, \ldots, a_r$ are the expectation values of the complex scalar field $\phi$ of the $\mathcal{N} = 2$ vector multiplet and $\varepsilon_1, \varepsilon_2$ parameterize the holonomy of a flat connection on the $T_r$-bundle over $\mathbb{T}^2$ used to define the $\Omega$-background. For $i = 1, \ldots, k$ define

$$\varepsilon^{(i)}_1 := (k - i + 1) \varepsilon_1 - (i - 1) \varepsilon_2 \quad \text{and} \quad \varepsilon^{(i)}_2 := -(k - i) \varepsilon_1 + i \varepsilon_2,$$

and

$$\bar{a}^{(i)} := \bar{a} - \varepsilon^{(i)}_1 (\bar{v})_i - \varepsilon^{(i)}_2 (\bar{v})_{i-1},$$

where $(\bar{v})_l := ((\bar{v})_1, \ldots, (\bar{v})_l)$ for $l = 1, \ldots, k - 1$ and $(\bar{v})_0 = (\bar{v})_k = \bar{0}$. Roughly speaking, for fixed $i = 1, \ldots, k$ the parameters $\varepsilon^{(i)}_1, \varepsilon^{(i)}_2$ are related to the coordinates of the affine toric neighbourhood of the fixed point $p_i$ of $X_k$ and the shift $\bar{a}^{(i)}$ of $\bar{a}$ is due to the equivariant Chern character of the fibre of the line bundles $\mathcal{R}_j$, with $j = 1, \ldots, k - 1$, at $p_i$.

For any $T$-equivariant vector bundle $G$ of rank $d$ on $\mathcal{M}_{T, \bar{u}, \Delta}(\mathcal{Y}_k, \mathcal{D}_\infty, \mathcal{T}_{2\infty}^{0, \mathcal{U}})$, define the $T$-equivariant characteristic class

$$E_\mu(G) := \sum_{j=0}^{d} (c_j)_T(G) \mu^{d-j}.$$
Let $\bar{v} \in \frac{1}{k} \mathbb{Z}^{k-1}$ such that $k v_{k-1} = \sum_{i=0}^{k-1} i w_i \mod k$. Then the generating function for $p$-observables (with $p = 0, 2$) of the gauge theory with massive adjoint matter on $X_k$, in the topological sector labelled by $\bar{v} = C^{-1} \bar{u}$, is defined by

$$Z^*_\bar{v}(\xi_1, \xi_2, \bar{a}, \mu; q, \vec{r}, \vec{t}^{(1)}, \ldots, \vec{t}^{(k-1)})$$

$$:= \sum_{\Delta \in \frac{1}{k} \mathbb{Z}^k} q^{\Delta + \frac{1}{kd} \vec{v} \cdot C \vec{v}} \int_{\mathcal{M}_{\bar{v}, \bar{u}, \Delta}(\mathcal{D}_\infty, \mathcal{F}_\infty^{0, \bar{u}})} \mathcal{E}_\mu(T \mathcal{M}_{\bar{v}, \bar{u}, \Delta}(\mathcal{D}_\infty, \mathcal{F}_\infty^{0, \bar{u}}))$$

$$\times \exp \left( \sum_{s=0}^{\infty} \left( \sum_{i=1}^{k-1} t^{(1)}_s \left[ \text{ch}(\mathcal{E})/\mathcal{O}_{\bar{v}} \right]_s + \tau_s \left[ \text{ch}(\mathcal{E})/\mathcal{O}_{X_k} \right]_{s-1} \right) \right),$$

where $\tau_0 := \frac{1}{2\pi i} \log q$ is the bare complex gauge coupling. Here $\mathcal{E}$ is the universal sheaf of $\mathcal{M}_{\bar{v}, \bar{u}, \Delta}(\mathcal{D}_\infty, \mathcal{F}_\infty^{0, \bar{u}})$, while $\text{ch}(\mathcal{E})/\mathcal{O}_{\bar{v}}$ denotes the slant product between $\text{ch}(\mathcal{E})$ and $\mathcal{O}_{\bar{v}}$, and the notation $[\cdot]_p$ means to take the degree $p$ part of the $T$-equivariant class; since $X_k$ is noncompact, the class $\text{ch}(\mathcal{E})/\mathcal{O}_{X_k}$ is defined by localization. In particular, setting $\vec{r} = \vec{0}$ and $\vec{t}^{(1)} = \ldots = \vec{t}^{(k-1)} = \vec{0}$ we obtain the instanton partition function $Z^*_\bar{v}^{\text{inst}}$ in the topological sector labelled by $\bar{v} = C^{-1} \bar{u}$. By taking a weighted sum over all $\bar{v}$ we obtain the full generating function for $p$-observables for $\mathcal{N} = 2^*$ gauge theory on $X_k$ as

$$Z^*_{X_k}(\xi_1, \xi_2, \bar{a}, \mu; q, \vec{r}, \vec{t}^{(1)}, \ldots, \vec{t}^{(k-1)})$$

$$:= \sum_{\bar{v} \in \frac{1}{k} \mathbb{Z}^{k-1}} \begin{cases} q^{\xi_1} Z^*_{\bar{v}^{\text{inst}}}(\xi_1, \xi_2, \bar{a}, \mu; q, \vec{r}, \vec{t}^{(1)}, \ldots, \vec{t}^{(k-1)}) & \text{if } \bar{v} \cdot \bar{u} = 0 \mod k, \\ 0 & \text{otherwise} \end{cases}$$

and the full instanton partition function as

$$Z^{*\text{inst}}_{X_k}(\xi_1, \xi_2, \bar{a}, \mu; q, \vec{r}, \vec{t}^{(1)}, \ldots, \vec{t}^{(k-1)})$$

$$:= \sum_{\bar{v} \in \frac{1}{k} \mathbb{Z}^{k-1}} \begin{cases} q^{\xi_1} Z^{*\text{inst}}_{\bar{v}^{\text{inst}}}(\xi_1, \xi_2, \bar{a}, \mu; q, \vec{r}, \vec{t}^{(1)}, \ldots, \vec{t}^{(k-1)}) & \text{if } \bar{v} \cdot \bar{u} = 0 \mod k, \\ 0 & \text{otherwise} \end{cases}$$

where $\log \xi_i$ for $i = 1, \ldots, k - 1$ are chemical potentials for the fractional instantons and $\bar{v}^{\text{inst}} := \prod_{i=1}^{k-1} \xi_i^{s_i}$.

By using the localization theorem, we provide an explicit formula for the generating function $Z^*_{X_k}$ in Proposition 5.19. It factorizes with respect to the generating functions for $p$-observables of $\mathcal{N} = 2^*$ gauge theory on the affine toric subsets of $X_k$ only for $k = 2$; for $k \geq 3$ the apparent lack of a factorization property for $Z^*_{X_k}$ is due to the fact that it involves terms which depend on pairs of exceptional divisors intersecting at the fixed points of $X_k$, and such terms do not split into terms each depending on a single affine toric subset of $X_k$. This is a new and unexpected result which extends previous generating functions for $p$-observables that depend only on one exceptional divisor [69, Section 4].

For the instanton partition function we obtain the factorization formula

$$Z^{*\text{inst}}_{X_k}(\xi_1, \xi_2, \bar{a}, \mu; q, \vec{r}, \vec{t}^{(1)}, \ldots, \vec{t}^{(k-1)})$$

$$= \prod_{\bar{v} \in \frac{1}{k} \mathbb{Z}^{k-1}} q^{\bar{v} \cdot C \bar{v}} \sum_{\bar{v} \cdot \bar{u} = 0 \mod k} \prod_{\alpha=1}^{r} \prod_{\beta=1}^{k-1} \prod_{n=1}^{k-1} \prod_{\alpha, \beta=1}^{\bar{v}} \prod_{n=1}^{k-1} k^{(n)}(\xi_1^{(n)}, \xi_2^{(n)}, \bar{a}_{\beta n} + \mu)$$

$$\times \prod_{i=1}^{k} Z^{*\text{inst}}_{\bar{v}^{\text{inst}}}(\xi_1^{(i)}, \xi_2^{(i)}, \bar{a}^{(i)}, \mu; q, \vec{r}, \vec{t}^{(1)}, \ldots, \vec{t}^{(k-1)})$$,
where \( \tilde{q} = (\tilde{v}_1, \ldots, \tilde{v}_r) \), \( \tilde{v}_\beta = \tilde{v}_\beta - \tilde{v}_\alpha \) and \( a_\beta \alpha = a_\beta - a_\alpha \), while \( Z_{n}^{*, \text{inst}} \) is the Nekrasov partition function for the \( \mathcal{N} = 2^* \) gauge theory on \( \mathbb{R}^4 \) with gauge group \( U(r) \). Here \( \ell^{(n)}_{\tilde{v}_\alpha} \) are the edge contributions; whilst their explicit expressions are somewhat complicated, we point out that they depend on the Cartan matrix (see Section 4.7 for the complete formulas and Appendix C for their derivations). The analogous results in the case that fundamental hypermultiplets are included can be found in Section 5.4, while the pertinent formulas in the generic case with \( U(r) \times U(r') \) bifundamental matter is presented in Section 4.7.

Let us compare these results with those obtained in [15] where, based on a conjectural splitting of the full partition function on \( X_k \) as a product of partition functions on the open affine subsets of \( X_k \), the authors obtain an expression for the edge contributions which depends only on the combinatorial data of the fan of the toric variety \( X_k \). The two sets of expressions appear to admit drastically different structures, and a general proof of their equivalence is not immediately evident. In the \( k = 2 \) case, our instanton partition function assumes the form

\[
Z_{X_k}^*(\varepsilon_1, \varepsilon_2, \bar{a}, \mu; q, \xi) = \sum_{v \in \mathbb{Z}} \xi^v \sum_{\tilde{v}} q^{v_0} \prod_{\alpha, \beta = 1}^r \left( \ell_{\tilde{v}_\alpha}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, a_\beta \alpha + \mu) \ell_{\tilde{v}_\alpha}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, a_\beta \alpha) \right) \times Z_{C^2}^*(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \bar{a} - 2\varepsilon_1 \tilde{v}, \mu; q) Z_{C^2}^*(\varepsilon_1 - \varepsilon_2, 2\varepsilon_2, \bar{a} - 2\varepsilon_2 \tilde{v}, \mu; q),
\]

where

\[
\ell_{\tilde{v}_\alpha}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, a_\beta \alpha) = \left\{ \begin{array}{ll}
|v_{\beta \alpha}|^{-1-2i+2(v_{\beta \alpha})} \prod_{i=0}^{v_{\beta \alpha}} \prod_{j=0}^{2i+2(v_{\beta \alpha})} (a_\beta \alpha + 2i \varepsilon_1 + j (\varepsilon_2 - \varepsilon_1)) & \text{for } |v_{\beta \alpha}| > 0, \\
1 & \text{for } |v_{\beta \alpha}| = 0, \\
-[v_{\beta \alpha}]^{-1-2i-2(v_{\beta \alpha})} \prod_{i=1}^{v_{\beta \alpha}} \prod_{j=1}^{2i+2(v_{\beta \alpha})} (a_\beta \alpha + 2(2v_{\beta \alpha} - i) \varepsilon_1 - j (\varepsilon_2 - \varepsilon_1)) & \text{for } |v_{\beta \alpha}| < 0.
\end{array} \right.
\]

Since the computation of the edge contributions in this case is equivalent to that of [20, Section 4.2], our formula agrees with [13, Equation (6)] and [14, Equation (2.13)] (which are equivalent to [15, Equation (3.24)] for \( k = 2 \)). In Appendix D we further check that, for \( k = 3 \) and rank \( r = 2 \), our partition functions agree with those in [15, Appendix C] at leading orders in the \( q \)-expansion for some choices of the holonomy at infinity and the first Chern class of the framed sheaves. The dictionary between our notation and that of [15] is as follows. Their \( \tilde{k}_\alpha \) are our \( \tilde{v}_\alpha \), while their \( a_{\alpha \beta}^{(i)} \) are \( a_\alpha + \varepsilon_1 (\tilde{k}_\alpha)_{i+1} + \varepsilon_2 (\tilde{k}_\alpha)_i \). On the other hand, to fix the holonomy at infinity they use a vector \( \tilde{I} = (I_1, \ldots, I_r) \) with \( I_\alpha \in \{0, 1, \ldots, k-1\} \) and then their \( \tilde{k}_\alpha \) satisfy an equation depending on \( \tilde{I} \). For us \( \tilde{I} = (0, 1, \ldots, 0, 1, \ldots, 1, \ldots, k-1, 1, \ldots, k-1, k-1, k-1) \), where \( i \) appears with multiplicity \( w_i \) for \( i = 0, 1, \ldots, k-1 \).

In Section 5.4 we make some new observations concerning the consistency of gauge theories on ALE spaces with \( N \) fundamental hypermultiplets. In this case, the generating function for \( p \)-observables (and all other partition functions) is defined by means of the integral of an equivariant characteristic class depending on the natural bundle \( V \) and the fundamental masses; this integral makes sense only if the degree of the class is nonnegative. For \( U(r) \) gauge theories on \( \mathbb{R}^4 \) this constraint implies \( N \leq 2r \), i.e., that the gauge theory is asymptotically free\(^1\). For \( U(r') \) gauge theories on \( X_k \) the constraint is

\[
\dim_{\mathbb{C}} \left( \mathcal{M}_{r', \tilde{s}, \Delta}(\mathcal{P}_k, D_\infty, \mathcal{F}_{\infty}^{(0, \tilde{w})}) \right) \geq \text{rk}(V) N.
\]

Since we wish to factorize our partition functions in terms of partition functions of gauge theories on \( \mathbb{R}^4 \), by combining these two constraints we get an

\(^1\)For \( \mathcal{N} = 2 \) gauge theories on \( \mathbb{R}^4 \) with nonvanishing beta-function \( \beta := 2r - N \), the effective expansion parameter is \( q \Lambda^3 \) where \( \Lambda \) is an energy scale parameter. In this paper we assume for ease of notation that all quantities have been suitably rescaled and formally set \( \Lambda = 1 \) in all nonconformal gauge theory partition functions.
inequality on the first Chern class depending on the holonomy at infinity; for \( k = 2 \) this inequality reads
\[ 4v^2 \leq w^2, \]
where \( v := \frac{1}{2} u \) gives the first Chern class. In particular if the holonomy at infinity is trivial, i.e., \( \vec{w} = (r, 0) \), we get \( v = 0 \), which is the case considered in [14]. Therefore to define the partition functions for \( N = 2 \) gauge theories on \( X_k \) with \( N \leq 2r \) fundamental matter fields, we restrict to moduli spaces \( M_{r, \vec{a}, \Delta}(Z_k, D_{\infty}, \hat{\mathcal{F}}_{\infty, \vec{w}}) \) of framed sheaves whose first Chern class obeys an additional constraint (cf. Equation (5.27)); we call the corresponding gauge theory \textit{asymptotically free}. In a similar way, one can define a \textit{conformal} gauge theory on \( X_k \) with \( N = 2r \) fundamental matter fields.

In Section 5.3 we make another important check of our results: we verify that the \( \mu = 0 \) limit of the \( N = 2^* \) gauge theory partition function correctly reduces to the partition function of the Vafa-Witten topologically twisted \( \mathcal{N} = 4 \) gauge theory on \( X_k \) [89]. We show that
\[
Z_{\text{VW}}^{\text{inst}}(q, \vec{\xi}) := \lim_{\mu \to 0} Z_{\text{ALE}}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; q, \vec{\xi}) = q^{\frac{1}{24}} \prod_{j=0}^{k-1} \left( \frac{\chi_{\vec{w}}(\zeta_j, \tau_0)}{\eta(\tau_0)} \right)^{w_j},
\]
where \( \chi_{\vec{w}}(\zeta, \tau_0) \) is the character of the integrable highest weight representation of \( \hat{\mathfrak{sl}}(k) \) at level one with weight the \( i \)-th fundamental weight \( \vec{w}_i \) of type \( A_{k-i} \) and \( \eta(\tau_0) = \exp(2\pi i(2\zeta_j - \zeta_{j-1} - \zeta_{j+1})) \) (we set \( \zeta_0 = \zeta_k = 0 \)). Thus in this case we correctly reproduce the character of the representation of the affine Lie algebra \( \hat{\mathfrak{g}}(k) \), and hence confirm the modularity (S-duality) of the partition function in the limit. One should be able to reproduce the same result by computing the Poincaré polynomial of the moduli spaces \( M_{r, \vec{a}, \Delta}(Z_k, D_{\infty}, \hat{\mathcal{F}}_{\infty, \vec{w}}) \).

Another useful limit of the generating function \( Z_{X_k}^{\text{inst}}(q, \vec{\xi}) \) is obtained by setting \( \vec{\tau} = (0, -\tau_1, 0, \ldots) \) and \( \vec{\tau}^{(1)} = \ldots = \vec{\tau}^{(k-1)} = 0 \); we denote the resulting partition function by \( Z_{X_k}^{\text{cl}} \). While this deformation simply has the effect of shifting the bare coupling \( \tau_0 \to \tau_0 + \tau_1 \) in the tree-level Lagrangian of the \( N = 2 \) gauge theory, it enables one to combine the classical and instanton partition functions into a single correlation function. In this case we find
\[
Z_{X_k}^{\text{cl}}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; q, \vec{\xi}, \tau_1) = Z_{X_k}^{\text{cl}}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; q, \vec{\xi})^{1/k} \] 
where \( q_{\text{eff}} := q e^{\tau_1} \), \( Z_{X_k}^{\text{cl}}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; q_{\text{eff}}, \vec{\xi}, \tau_1 = 0) = Z_{X_k}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, m; q, \vec{\xi}) \).

The explicit expression for \( Z_{X_k}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; q_{\text{eff}}, \vec{\xi}, \tau_1) \) is given in Equation (5.16) for pure \( N = 2 \) gauge theory; analogous expressions can be written in the case of gauge theories with matter. This factorization also holds in the pure case without matter (see Section 5.1) and in the case with fundamental matter (see Section 5.4).

To obtain a partition function that enjoys modularity properties appropriate to its conjectural geometric description in the (2, 0) theory compactified on a two-torus, we should further multiply the classical and instanton parts by the purely perturbative contribution, which is independent of \( q \) and \( \vec{\xi} \). In Section 6 we give a definition of perturbative partition functions following [78, Sections 3.1 and 3.2] and [40, Section 6]. In particular we define the perturbative part of the equivariant Chern character of the bundle \( E \): At a fixed point \( ([\{E, \phi_{\vec{\xi}}\}], [\{E', \phi_{\vec{\xi}'}\}] \) it is the sum of the equivariant Euler characteristics of the fibre \( E|_{([\{E, \phi_{\vec{\xi}}\}], [\{E', \phi_{\vec{\xi}'}\}]) \) respectively over the two affine toric neighbourhoods of the two fixed points of the compactification divisor \( D_{\infty} \); in the main part of this paper these neighbourhoods are denoted by \( U_{\infty,k} \) and \( U_{\infty,0} \). We show that the computation of the perturbative part reduces to the computation of the equivariant Euler characteristics, over \( U_{\infty,k} \) and \( U_{\infty,0} \), of a Weil divisor on \( X_k \) given by \( D_{\infty}, D_0 \) and \( D_k \), which depends only on the holonomy at infinity, i.e., only on the framing vector \( \vec{w} \). By using the perturbative part of the equivariant Chern character of the bundle \( E \) we define the perturbative partition functions. The general formulas are rather complicated; in the case \( k = 2 \) the perturbative partition
function is given by

\[
Z_{\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu) = \sum \prod_{\vec{c}} \frac{\exp \left( \gamma_{\varepsilon_2-\varepsilon_1,2\varepsilon_2}(\mu + a_\beta \alpha + c_\beta \alpha (\varepsilon_2 - \varepsilon_1) - 2\varepsilon_2) \right)}{\exp \left( \gamma_{\varepsilon_2-\varepsilon_1,2\varepsilon_2}(a_\beta \alpha + c_\beta \alpha (\varepsilon_2 - \varepsilon_1) - 2\varepsilon_2) \right)}
\]

where the sum runs over all \( \vec{c} := (c_1, \ldots, c_r) \) such that \( c_\alpha = 0 \) if \( \alpha \leq w_0 \) and \( c_\alpha = 1 \) if \( w_0 < \alpha \leq w_0 + w_1 \), and for each \( \alpha, \beta = 1, \ldots, r \) with \( \alpha \neq \beta \) the quantity \( c_{\beta \alpha} \in \{0, 1\} \) is the parity of \( c_\beta - c_\alpha \). The function \( \exp \left( \gamma_{\varepsilon_1,\varepsilon_2}(x) \right) \) is the double zeta-function regularization of the infinite product

\[
\prod_{i,j=0}^{\infty} \left( x - i \varepsilon_1 - j \varepsilon_2 \right).
\]

This formula resembles the one-loop partition function computed in [15, Section 3.2]; however, the definition of the perturbative partition function on a noncompact space requires a choice and so a direct comparison is meaningless without further input. In [14] the choice is made to match with the DOZZ (Dorn-Otto-Zamolodchikov-Zamolodchikov) three-point correlation functions of supersymmetric Liouville theory. The analogous results in the case that fundamental hypermultiplets are included can be found in Section 6.4, while the pertinent formulas in the generic case with \( U(r') \times U(r') \) bifundamental matter is presented in Section 6.1.

1.3.4. Seiberg-Witten geometry. In Section 7 we study relations between the supersymmetric gauge theory we have developed and Seiberg-Witten theory. The low energy limit of \( \mathcal{N} = 2 \) gauge theories on \( \mathbb{R}^4 \) is completely characterised by the (punctured) Seiberg-Witten curve \( \Sigma \) of genus \( r \) [84]. The curve \( \Sigma \) is equipped with a meromorphic differential \( \lambda_{\text{SW}} \), called the Seiberg-Witten differential, and its periods determine the Seiberg-Witten prepotential \( F_{\Sigma}(\vec{a}, \mu; q) \) which is a holomorphic function of all parameters. In a symplectic basis \( \{A_\alpha, B_\alpha\}_{\alpha=1, \ldots, r} \cup \{S\} \) for the homology group \( H_1(\Sigma; \mathbb{Z}) \), the periods of the Seiberg-Witten differential determine the quantities

\[
a_\alpha = \oint_{A_\alpha} \lambda_{\text{SW}}, \quad \frac{\partial F_{\Sigma^2}}{\partial a_\alpha}(\vec{a}, \mu; q) = -2\pi i \oint_{B_\alpha} \lambda_{\text{SW}} \quad \text{and} \quad \mu = \oint_S \lambda_{\text{SW}}.
\]

It follows that the period matrix \( \tau = (\tau_{\alpha\beta}) \) of the Seiberg-Witten curve \( \Sigma \) is related to the prepotential by

\[
\tau_{\alpha\beta} = -\frac{1}{2\pi i} \frac{\partial^2 F_{\Sigma^2}}{\partial a_\alpha \partial a_\beta}(\vec{a}, \mu; q),
\]

and it determines the infrared effective gauge couplings.

The Seiberg-Witten prepotential can be recovered from the partition function for the \( \Omega \)-deformed \( \mathcal{N} = 2^* \) gauge theory on \( \mathbb{R}^4 \) in the low energy limit in which the equivariant parameters \( \varepsilon_1, \varepsilon_2 \) vanish; this result was originally conjectured by Nekrasov [71] and subsequently proven in [70, 73]. In Sections 7.1 and 7.2 we prove analogous results for gauge theory on \( X_k \). Let \( \tilde{k} = k/2 \) for even \( k \) and \( \tilde{k} = k \) for odd \( k \).

**Theorem (Theorem 7.9).** \( F_{X_k}^{\varepsilon_1,\varepsilon_2}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; q, \vec{\xi}) := -\tilde{k} \varepsilon_1 \varepsilon_2 \log Z_{X_k}^{\varepsilon_1,\varepsilon_2}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; q, \vec{\xi}) \) is analytic in \( \varepsilon_1, \varepsilon_2 \) near \( \varepsilon_1 = \varepsilon_2 = 0 \) and

\[
\lim_{\varepsilon_1, \varepsilon_2 \to 0} F_{X_k}^{\varepsilon_1,\varepsilon_2}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; q, \vec{\xi}) = \frac{1}{k} F_{C^2}^{\varepsilon_1,\varepsilon_2}(\vec{a}, \mu; q),
\]

where \( F_{C^2}^{\varepsilon_1,\varepsilon_2}(\vec{a}, \mu; q) \) is the instanton part of the Seiberg-Witten prepotential of \( \mathcal{N} = 2^* \) gauge theory on \( \mathbb{R}^4 \).
Corollary. $F^*_{X_k}(\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu; q, \xi, \tau_1) := -\tilde{k} \varepsilon_1 \varepsilon_2 \log Z_{X_k}^* (\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu; q, \xi, \tau_1)$ is analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$ and
\[
\lim_{\varepsilon_1, \varepsilon_2 \to 0} F^*_{X_k}(\varepsilon_1, \varepsilon_2, \bar{\alpha}; q, \xi, \tau_1) = \frac{1}{k} \left( \frac{\tilde{k} \tau_1}{2} \sum_{\alpha=1}^{r} a^2_\alpha + F^*_{C^2}(\bar{\alpha}, \mu; \text{pert}) \right).
\]

Theorem (Theorem 7.11). For any fixed holonomy vector at infinity $\bar{\varepsilon} = (\bar{c}_1, \ldots, \bar{c}_r) \in \{0, 1, \ldots, k-1\}$ we have
\[
\lim_{\varepsilon_1, \varepsilon_2 \to 0} \varepsilon_1 \varepsilon_2 F^*_{X_k}(\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu; \bar{\varepsilon}) = \frac{1}{k} F^*_{C^2}(\bar{\alpha}, \mu),
\]
where $Z^*_{X_k}(\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu) := \sum e^{\alpha} \exp \left( - F^*_{X_k}(\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu; \bar{\varepsilon}) \right)$ and $F^*_{C^2}(\bar{\alpha}, \mu)$ is the perturbative part of the Seiberg-Witten prepotential of $\mathcal{N} = 2^*$ gauge theory on $\mathbb{R}^4$.

These results agree with the analogous derivations in [15, Section 2.1]. They confirm that $\mathcal{N} = 2^*$ gauge theories on the ALE space $X_k$, like their counterparts on $\mathbb{R}^4$, are some sort of quantization of a Hitchin system with spectral curve $\Sigma$.

In Section 7.3 we derive blowup equations which relate the generating functions for correlators of quadratic 2-observables to the instanton partition functions in the low energy limit. We focus on the case $k = 2$ in which the generating function $Z^*_{X_2}$ factorizes; in this case, the computations are manageable because of the factorization of the generating function for correlators of 2-observables into two copies of instanton partition functions. For $k \geq 3$, due to this apparent absence of factorization we are unable to derive analogous blowup equations in the case of weighted blowups.

Let $Z^*_{X_2}(\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu; q, \xi, t)$ be the generating function $Z^*_{X_2}(\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu; q, \xi, \bar{\tau}, \tilde{t})$ specialized at $\bar{\tau} = 0$ and $\tilde{t} := (0, -t, 0, \ldots)$. Let $\Theta([\bar{\mu}]|\xi)\tau)$ be the Riemann theta-function with characteristic $[\bar{\mu}]$ on the Seiberg-Witten curve $\Sigma$ for $\mathcal{N} = 2^*$ gauge theory on $\mathbb{R}^4$.

Theorem (Theorem 7.18). $Z^*_{X_2}(\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu; q, \xi, t)/Z^*_{X_2}(\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu; q, \xi)$ is analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$, and
\[
\lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{Z^*_{X_2}(\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu; q, \xi, t)}{Z^*_{X_2}(\varepsilon_1, \varepsilon_2, \bar{\alpha}, \mu; q, \xi)} = \exp \left( \left( \frac{\partial}{\partial q} \right)^2 F^*_{C^2}(\bar{\alpha}, \mu; q, \xi) t^2 + 2\pi i \sum_{\alpha=\nu_0+1}^{r} \zeta_\alpha \Theta \left[ \frac{\partial}{\partial \zeta} \right] (\zeta + \bar{\nu}) | C \tau \right),
\]
where
\[
\zeta_\alpha := -\frac{t}{2\pi i} \left( a_\alpha + q \frac{\partial^2 F^*_{C^2} \partial \alpha}{\partial q} (\bar{\alpha}, \mu; q) \right),
\]
while $\kappa_\alpha := \frac{1}{4\pi i} \log(\xi)$ for $\alpha = 1, \ldots, r$ and
\[
\nu_\alpha := \begin{cases} 
\sum_{\beta=\nu_0+1}^{\nu_0+1} \log \left( (a_\beta - a_\alpha)^2 - \mu^2 \right) - \frac{2\pi i w_1}{r} \tau_0 
+ \sum_{\beta=\nu_0+1}^{r} \frac{\partial^2 F^*_{C^2} \partial \alpha \beta}{\partial q} (\bar{\alpha}, \mu; q) & \text{for } \alpha = 1, \ldots, \nu_0, \\
- \sum_{\beta=1}^{\nu_0} \log \left( (a_\beta - a_\alpha)^2 - \mu^2 \right) + \frac{4\pi i w_0}{r} \tau_0 & \text{for } \alpha = \nu_0 + 1, \ldots, r.
\end{cases}
\]

This blowup equation underlies the modularity properties of the partition function and correlators of quadratic 2-observables on the Seiberg-Witten curve for $\mathcal{N} = 2^*$ gauge theory on $X_2$ with period matrix $\tau$ twisted by the $A_1$ Cartan matrix $C$, and it generalizes the representation of the Vafa-Witten
partition function at $\mu = 0$ in terms of modular forms. If the fixed holonomy at infinity is trivial, i.e., $\overline{w} = (w_0, w_1) = (r, 0)$, the characteristic vector $\overline{v} \in \mathbb{C}^r$ vanishes and our result resembles [70, Theorem 8.1] and [15, Equation (2.25)]. In general, the nontrivial holonomy at infinity is encoded in $\overline{v}$.

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2. Preliminaries on stacks

2.1. Deligne-Mumford stacks. In this subsection we summarise the conventions about stacks that will be used throughout this paper. Our main reference for the theory of stacks is the book [57]. In this paper all schemes are defined over $\mathbb{C}$ and are Noetherian, unless otherwise stated. A variety is an irreducible reduced separated scheme of finite type over $\mathbb{C}$. The smooth locus of a variety $X$ is denoted by $X_{\text{sm}}$.

By a Deligne-Mumford stack we mean a separated Noetherian stack $\mathscr{X}$ of finite type over $\mathbb{C}$. An orbifold is a smooth Deligne-Mumford stack with generically trivial stabilizer.

The inertia stack $\mathcal{I}(\mathscr{X})$ of a Deligne-Mumford stack $\mathscr{X}$ is the fibre product $\mathscr{X} \times_{\mathscr{X} \times \mathscr{X}} \mathscr{X}$ with respect to the diagonal morphism $\Delta: \mathscr{X} \to \mathscr{X} \times \mathscr{X}$. For a scheme $T$, an object in $\mathcal{I}(\mathscr{X})(T)$ is a pair $(x, g)$ where $x$ is an object of $\mathscr{X}(T)$ and $g: x \xrightarrow{\sim} x$ is an automorphism. A morphism $(x, g) \to (x', g')$ is a morphism $f: x \to x'$ in $\mathscr{X}(T)$ such that $f \circ g = g' \circ f$. Let $\varpi: \mathcal{I}(\mathscr{X}) \to \mathscr{X}$ be the forgetful morphism which for any scheme $T$ sends a pair $(x, g)$ to $x$.

An étale presentation of a Deligne-Mumford stack $\mathscr{X}$ is a pair $(U, u)$, where $U$ is a scheme and $u: U \to \mathscr{X}$ is a representable étale surjective morphism [57, Definition 4.1]. A morphism between two étale presentations $(U, u)$ and $(V, v)$ of $\mathscr{X}$ is a pair $(\varphi, \alpha)$, where $\varphi: U \to V$ is a morphism and $\alpha: u \sim \varpi \circ \varphi$ is a 2-isomorphism. The étale groupoid associated with the étale presentation $u: U \to \mathscr{X}$ is the groupoid

$$U \times_{\mathscr{X}} U \xrightarrow{\alpha} U.$$ 

If $P$ is a property of schemes which is local in the étale topology (for example regular, normal, reduced, Cohen-Macaulay, etc.), then $\mathscr{X}$ has the property $P$ if for one (and hence every) étale presentation $u: U \to \mathscr{X}$ the scheme $U$ has the property $P$.

A gerbe over a Deligne-Mumford stack $\mathscr{X}$ is a stack $\mathcal{Y}$ over $\mathscr{X}$ which étale locally admits a section and for which any two local sections are locally 2-isomorphic. For any integer $k \geq 2$, let $\mu_k$ denote the group of complex $k$-th roots of unity. A gerbe $\mathcal{Y} \to \mathscr{X}$ is a $\mu_k$-banded gerbe, or simply a $\mu_k$-gerbe, if for every étale presentation $U$ of $\mathscr{X}$ and every object $x \in \mathcal{Y}(U)$ there is an isomorphism $\alpha_x: \mu_k \to \text{Aut}_U(x)$ of sheaves of groups satisfying natural compatibility conditions [35, Section 6.1].

A (quasi-)coherent sheaf $\mathcal{E}$ on $\mathscr{X}$ is a collection of pairs $(\mathcal{E}_{U,\alpha}, \theta_{\varphi, \alpha})$, where for any étale presentation $u: U \to \mathscr{X}$, $\mathcal{E}_{U,\alpha}$ is a (quasi-)coherent sheaf on $U$, and for any morphism $(\varphi, \alpha): (U, u) \to (V, v)$ between two étale presentations of $\mathscr{X}$, $\theta_{\varphi, \alpha}: \mathcal{E}_{U,\alpha} \to \varphi^* \mathcal{E}_{V,\alpha}$ is an isomorphism which satisfies a cocycle condition with respect to three étale presentations (see [57, Lemma 12.2.1] and [93, Definition 7.18]). A torsion free (resp. locally free) sheaf on $\mathscr{X}$ is a coherent sheaf $\mathcal{E}$ such that all $\mathcal{E}_{U,\alpha}$ are torsion free (resp. locally free).

If $\mathscr{X}$ is a Deligne-Mumford stack, by [53, Corollary 1.3-(1)] there exists a coarse moduli space $(X, \pi)$ (or simply $X$) where $X$ is a separated algebraic space; amongst other properties, the morphism
\pi : \mathcal{X} \rightarrow X \) is proper and quasi-finite. We shall always assume that \( X \) is a scheme. We recall some properties of Deligne-Mumford stacks that we shall use in this paper:

- The functor \( \pi_* : \text{Qcoh}(\mathcal{X}) \rightarrow \text{Qcoh}(X) \) is exact and maps coherent sheaves to coherent sheaves \([4, \text{Lemma 2.3.4}]\). In particular, \( \mathcal{X} \) is tame \([3, \text{Definition 3.1}]\).
- \( H^\bullet(\mathcal{X}, \mathcal{E}) \simeq H^\bullet(X, \pi_* \mathcal{E}) \) for any quasi-coherent sheaf \( \mathcal{E} \) on \( \mathcal{X} \) \([76, \text{Lemma 1.10}]\).

**Notation.** We use the symbols \( \mathcal{E}, \mathcal{G}, \mathcal{F}, \ldots \) for sheaves on a Deligne-Mumford stack, and the symbols \( E, F, G, \ldots \) for sheaves on a scheme. For any coherent sheaf \( \mathcal{F} \) on a Deligne-Mumford stack \( \mathcal{X} \) we denote by \( \mathcal{F}^\vee \) its dual \( \text{Hom}(\mathcal{F}, \mathcal{O}_\mathcal{X}) \). We denote in the same way the dual of a coherent sheaf on a scheme. By \( \mathbb{A}^n \) we denote the \( n \)-dimensional affine space over \( \mathbb{C} \), and by \( \mathbb{G}_m \) the multiplicative group \( \mathbb{C}^* \). The projection morphism \( T \times Y \rightarrow Y \) is written as \( p_Y \) or \( p_{T \times Y, Y} \).

### 2.2. Root stacks.

In this subsection, we give a brief survey of the theory of root stacks as presented in [24] (see also [2, Appendix B]).

Let \( \mathcal{X} \) be an algebraic stack. We use the standard fact that there is an equivalence between the category of line bundles on \( \mathcal{X} \) and the category of morphisms \( \mathcal{X} \rightarrow \mathbb{G}_m \), where the morphisms in the former category are taken to be isomorphisms of line bundles. There is also an equivalence between the category of pairs \( (\mathcal{L}, s) \), with \( \mathcal{L} \) a line bundle on \( \mathcal{X} \) and \( s \in \Gamma(\mathcal{X}, \mathcal{L}) \), and the category of morphisms \( \mathcal{X} \rightarrow [\mathbb{A}^1/\mathbb{G}_m] \), where \( \mathbb{G}_m \) acts on \( \mathbb{A}^1 \) by multiplication \([79, \text{Example 5.13}]\).

Throughout this subsection, \( \mathcal{X} \) will be an algebraic stack, \( \mathcal{L} \) a line bundle on \( \mathcal{X} \), \( s \in \Gamma(\mathcal{X}, \mathcal{L}) \) a global section, and \( k \) a positive integer. The pair \((\mathcal{L}, s)\) defines a morphism \( \mathcal{X} \rightarrow [\mathbb{A}^1/\mathbb{G}_m] \) as above. Let \( \theta_k : [\mathbb{A}^1/\mathbb{G}_m] \rightarrow [\mathbb{A}^1/\mathbb{G}_m] \) be the morphism induced by the morphisms

\[
x \in \mathbb{A}^1 \mapsto x^k \in \mathbb{A}^1 \quad \quad \text{and} \quad \quad t \in \mathbb{G}_m \mapsto t^k \in \mathbb{G}_m,
\]

which sends a pair \((\mathcal{L}, s)\) to its \( k \)-th tensor power \((\mathcal{L}^\otimes k, s^\otimes k)\).

**Definition 2.1.** Let \( \sqrt[k]{(\mathcal{L}, s)}/\mathcal{X} \) be the algebraic stack obtained as the fibre product

\[
\begin{array}{ccc}
\sqrt[k]{(\mathcal{L}, s)}/\mathcal{X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\
\downarrow & & \downarrow \theta_k \\
\mathcal{X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]
\end{array}
\]

We say that \( \sqrt[k]{(\mathcal{L}, s)}/\mathcal{X} \) is the root stack obtained from \( \mathcal{X} \) by the \( k \)-th root construction. \( \Box \)

**Remark 2.2.** By [24, Example 2.4.2], if \( s \) is a nowhere vanishing section then \( \sqrt[k]{(\mathcal{L}, s)}/\mathcal{X} \simeq \mathcal{X} \). This shows that all of the stacky structure in \( \sqrt[k]{(\mathcal{L}, s)}/\mathcal{X} \) is concentrated at the zero locus of \( s \). \( \triangle \)

**Definition 2.3.** Let \( \sqrt[\mathcal{L}]/\mathcal{X} \) be the algebraic stack obtained as the fibre product

\[
\begin{array}{ccc}
\sqrt[\mathcal{L}]/\mathcal{X} & \longrightarrow & \mathbb{G}_m \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathbb{G}_m
\end{array}
\]

where \( \mathcal{X} \rightarrow \mathbb{G}_m \) is determined by \( \mathcal{L} \) and \( \mathbb{G}_m \rightarrow \mathbb{G}_m \) is given by the map \( t \in \mathbb{G}_m \mapsto t^k \in \mathbb{G}_m \). \( \Box \)
As described in [24, Example 2.4.3], $\sqrt[s]{\mathcal{L}}/\mathcal{X}$ is a closed substack of $\sqrt[s]{(\mathcal{L}, 0)}/\mathcal{X}$. In general, let $\mathcal{D}$ be the vanishing locus of $s \in \Gamma(\mathcal{X}, \mathcal{L})$. One has a chain of inclusions of closed substacks

$$\sqrt[s]{\mathcal{L}|_{\mathcal{D}}}/\mathcal{D} \subset \sqrt[s]{(\mathcal{L}|_{\mathcal{D}}, 0)}/\mathcal{D} \subset \sqrt[s]{(\mathcal{L}, s)}/\mathcal{X}.$$ 

In addition, $\sqrt[s]{\mathcal{L}|_{\mathcal{D}}}/\mathcal{D}$ is isomorphic to the reduced stack $\left(\sqrt[s]{(\mathcal{L}|_{\mathcal{D}}, 0)}/\mathcal{D}\right)_{\text{red}}$. Locally, $\sqrt[s]{\mathcal{L}}/\mathcal{X}$ is a quotient of $\mathcal{X}$ by a trivial action of $\mu_k$, though this is not true globally. In general, $\sqrt[s]{\mathcal{L}}/\mathcal{X}$ is a $\mu_k$-gerbe over $\mathcal{X}$. Its cohomology class in the étale cohomology group $H^2(\mathcal{X}; \mu_k)$ is obtained from the class $[\mathcal{L}] \in H^1(\mathcal{X}; \mathbb{G}_m)$ via the boundary homomorphism $\delta: H^1(\mathcal{X}; \mathbb{G}_m) \to H^2(\mathcal{X}; \mu_k)$ given by the Kummer exact sequence

$$1 \to \mu_k \to \mathbb{G}_m \xrightarrow{(-k)} \mathbb{G}_m \to 1.$$ 

Since the class $\delta([\mathcal{L}])$ has trivial image in $H^2(\mathcal{X}; \mathbb{G}_m)$, the gerbe $\sqrt[s]{\mathcal{L}}/\mathcal{X}$ is called essentially trivial [58, Definition 2.3.4.1 and Lemma 2.3.4.2].

By [24, Corollaries 2.3.6 and 2.3.7] we get the following result.

**Proposition 2.4.** The projection $\sqrt[s]{(\mathcal{L}, s)}/\mathcal{X} \to \mathcal{X}$ is faithfully flat and quasi-compact. If $X$ is a scheme and $L$ is a line bundle on $X$ with global section $s$, then $X$ is a coarse moduli space for both $\sqrt[s]{(\mathcal{L}, s)}/\mathcal{X}$ and $\sqrt[s]{L}/X$ under the projections to $X$.

**Theorem 2.5** ([24, Theorem 2.3.3]). If $\mathcal{X}$ is a Deligne-Mumford stack, then $\sqrt[s]{(\mathcal{L}, s)}/\mathcal{X}$ is also a Deligne-Mumford stack.

2.2.1. Roots of an effective Cartier divisor on a smooth separated algebraic stack.

**Definition 2.6.** Let $\mathcal{X}$ be a smooth algebraic stack, $\mathcal{D} \subset \mathcal{X}$ an effective Cartier divisor and $k$ a positive integer. We denote by $\sqrt[k]{\mathcal{D}}/\mathcal{X}$ the root stack $\sqrt[k]{(\mathcal{O}_\mathcal{X}(\mathcal{D}), s_\mathcal{D})}/\mathcal{X}$, where $s_\mathcal{D}$ is the tautological section of $\mathcal{O}_\mathcal{X}(\mathcal{D})$ which vanishes along $\mathcal{D}$. Let now $\mathcal{D}_i \subset \mathcal{X}$ be effective Cartier divisors and $k_i$ positive integers for $i = 1, \ldots, n$. We denote by $\sqrt[k]{\mathcal{D}_1/\mathcal{X}} \times \cdots \times \sqrt[k]{\mathcal{D}_n/\mathcal{X}}$.

There is an equivalent definition of $\sqrt[k]{\mathcal{D}}/\mathcal{X}$ in [24, Definition 2.2.4], which gives rise to a morphism $\sqrt[k]{\mathcal{D}}/\mathcal{X} \to [\mathbb{A}^1/\mathbb{G}_m]^n$. Each of the components $\sqrt[k]{\mathcal{D}}/\mathcal{X} \to [\mathbb{A}^1/\mathbb{G}_m]$ corresponds to an effective divisor $\mathcal{D}_i$, i.e., the reduced closed substack $\pi^{-1}(\mathcal{D}_i)_{\text{red}}$, where $\pi: \sqrt[k]{\mathcal{D}}/\mathcal{X} \to \mathcal{X}$ is the natural projection morphism. Moreover

$$\pi^* \mathcal{O}_\mathcal{X}(\mathcal{D}_i) \simeq \mathcal{O}_{\sqrt[k]{\mathcal{D}}/\mathcal{X}}(k_i \mathcal{D}_i).$$

As explained in [9, Section 2.1], since $\mathcal{X}$ is smooth, each $\mathcal{D}_i$ is smooth and has simple normal crossing. Moreover, $\sqrt[k]{\mathcal{D}}/\mathcal{X}$ is smooth and each $\mathcal{D}_i$ has simple normal crossing; each $\mathcal{D}_i$ is the root stack $\sqrt[k_i]{\mathcal{O}_\mathcal{X}(\mathcal{D}_i)|_{\mathcal{D}_i}/\mathcal{D}_i}$ and hence is a $\mu_{k_i}$-gerbe over $\mathcal{D}_i$.

In closing this subsection we provide a useful characterization of the Picard group of $\sqrt[k]{\mathcal{D}}/\mathcal{X}$. There is an exact sequence of groups [24, Corollary 3.1.2]

$$0 \longrightarrow \text{Pic}(\mathcal{X}) \xrightarrow{\pi} \text{Pic}\left(\sqrt[k]{\mathcal{D}}/\mathcal{X}\right) \xrightarrow{q} \prod_{i=1}^n \mu_{k_i} \longrightarrow 0.$$
Every line bundle $L \in \text{Pic} \left( \tilde{\mathcal{O}} \big/ \mathcal{O} \right)$ can therefore be written as $L \simeq \pi^* M \otimes \bigotimes_{i=1}^{n} \mathcal{O} \left( m_i \tilde{D}_i \right)$, where $M \in \text{Pic} \left( \mathcal{X} \right)$ and $0 \leq m_i < k_i$ for $i = 1, \ldots, n$; each integer $m_i$ is unique and $M$ is unique up to isomorphism. The morphism $q$ maps $L$ to $(m_1, \ldots, m_n)$.

In the following we shall denote by $[x] \in \mathbb{Z}$ the integer part (floor function) and by $\{x\} := x - [x] \in [0, 1)$ the fractional part of a rational number $x$.

**Lemma 2.7** ([24, Theorem 3.1.1]). Let $\mathcal{X}$ be a smooth algebraic stack, $M$ a coherent sheaf on $\mathcal{X}$ and $m_1, \ldots, m_n \in \mathbb{Z}$. Then

$$\pi_* \left( \pi^* M \otimes \bigotimes_{i=1}^{n} \mathcal{O} \left( m_i \tilde{D}_i \right) \right) \simeq M \otimes \bigotimes_{i=1}^{n} \mathcal{O}_\mathcal{X} \left( \left\lfloor \frac{m_i}{k_i} \right\rfloor \tilde{D}_i \right).$$

### 2.3. Toric stacks

In this subsection we shall collect some results about toric stacks. We describe two equivalent approaches to the theory. The first is due to Fantechi, Mann and Nironi [35], and is based on a generalization of the notions of tori and toric varieties. The second is of a combinatorial nature and is based on the notion of stacky fan due to Borisov, Chen and Smith [16].

Our reference for the geometry of toric varieties is the book [29]. Let $T$ be a torus. Denote by $M := \text{Hom}(T, \mathbb{C}^*)$ the lattice of characters of $T$ and by $N := \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ the dual lattice of one-parameter subgroups; for an element $m \in M$ we denote the corresponding character by $e^m$. A **toric variety** is a variety $X$ containing a torus $T$ as an open subset, such that the action of $T$ on itself extends to an algebraic action on the whole variety $X$. It is characterized by combinatorial data encoded by a **fan** $\Sigma$ in the vector space $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$. We say that $\Sigma$ is a **rational simplicial fan** in $N_\mathbb{Q}$ if every cone $\sigma \in \Sigma$ is generated by linearly independent vectors over $\mathbb{Q}$. We denote by $\Sigma(j)$ the set of $j$-dimensional cones of $\Sigma$ for $j = 0, 1, \ldots, \dim_{\mathbb{C}}(X)$. A **ray** is a one-dimensional cone of $\Sigma$. By the orbit-cone correspondence, rays of $\Sigma$ correspond to $T$-invariant divisors of $X$. We denote by $D_\rho$ the $T$-invariant divisor associated with the ray $\rho$. In order to avoid overly cumbersome notation, when the ray has an index, say $\rho_\alpha$, we denote the associated divisor by $D_\alpha$. We denote by $p_\sigma$ the torus-fixed point associated with a cone $\sigma \in \Sigma$ of maximal dimension, and by $U_\sigma$ the affine toric variety $\text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ which is an open subset of $X$. When the cone has an index, say $\sigma_i$, we denote the associated fixed point by $p_i$ and by $U_i$ the associated affine variety.

**Definition 2.8.** A **Deligne-Mumford torus** $\mathcal{F}$ is a product $T \times \mathcal{B}G$ where $T$ is an ordinary torus and $G$ is a finite abelian group.

Fantechi, Mann and Nironi give a definition of Deligne-Mumford tori in terms of **Picard stacks**, depending on suitable morphisms of finitely generated abelian groups [35, Definition 2.4]. They also show [35, Proposition 2.6] that their definition is equivalent to the present one. So a morphism between Deligne-Mumford tori is simply a morphism as Picard stacks.

**Definition 2.9.** A **toric Deligne-Mumford stack** is a smooth Deligne-Mumford stack $\mathcal{X}$ with coarse moduli space $\pi: \mathcal{X} \to X$, together with an open immersion of a Deligne-Mumford torus $\iota: \mathcal{F} \hookrightarrow \mathcal{X}$ with dense image such that the action of $\mathcal{F}$ on itself extends to an action $a: \mathcal{F} \times \mathcal{X} \to \mathcal{X}$. A **morphism** of toric Deligne-Mumford stacks $f: \mathcal{X} \to \mathcal{X}'$ is a morphism of stacks between $\mathcal{X}$ and $\mathcal{X}'$ which extends a morphism of the corresponding Deligne-Mumford tori $\mathcal{F} \to \mathcal{F}'$.

A toric Deligne-Mumford stack is an orbifold if and only if its Deligne-Mumford torus is an ordinary torus. By [35, Proposition 3.6] the toric structure on a toric Deligne-Mumford stack $\mathcal{X}$ with Deligne-Mumford torus $\mathcal{F}$ induces a toric structure on the coarse moduli space $X$ with torus $T$, which is a simplicial toric variety. In particular, $\mathcal{X}$ is irreducible.
Let $\mathcal{X}$ be a toric Deligne-Mumford stack of dimension $d$ with coarse moduli space $\pi: \mathcal{X} \to X$. By [35, Proposition 5.1, Theorem 5.2, Theorem 6.25 and Corollary 6.26], $\pi$ factorizes as

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{r} & X_{\text{rig}} \\
\beta & \xrightarrow{f_{\text{rig}}} & \mathcal{X}_{\text{can}} \\
\pi_{\text{rig}} & \xrightarrow{\pi} & X,
\end{array}
$$

(2.10)

with the following properties:

- $\mathcal{X}_{\text{can}}$ is the canonical toric orbifold of $X$, i.e., the unique (up to isomorphism) smooth $d$-dimensional toric Deligne-Mumford stack such that the locus where $\pi_{\text{can}}$ is not an isomorphism has dimension $\leq d - 2$.
- $\mathcal{X}_{\text{rig}}$ is the rigidification of $\mathcal{X}$ with respect to the generic stabilizer$^2$.
- Let $D_1, \ldots, D_m$ be torus-invariant divisors in $X$, and define the smooth integral closed substacks $\mathcal{G}_i := \langle (\pi_{\text{can}})^{-1}(D_i)\rangle_{\text{red}}$ of codimension one in $\mathcal{X}_{\text{can}}$. Then there exist integers $k_1, \ldots, k_m \in \mathbb{N}$ such that

$$
\mathcal{X}_{\text{rig}} \simeq \mathcal{G}_1 \times \cdots \times \mathcal{G}_m.
$$

- There exist $\ell$ line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_\ell \in \text{Pic}(\mathcal{X}_{\text{rig}})$ and integers $b_i \in \mathbb{N}$, $i = 1, \ldots, \ell$ such that

$$
\mathcal{X} \simeq \left( \mathcal{L}_1 / \mathcal{X}_{\text{rig}} \times \cdots \times \mathcal{L}_\ell / \mathcal{X}_{\text{rig}} \right)^{b_1 \cdots b_\ell}.
$$

Thus $r: \mathcal{X} \to \mathcal{X}_{\text{rig}}$ is an essentially trivial $\prod_{i=1}^{\ell} \mu_{b_i}$-gerbe.

2.3.1. Gale duality with torsion. Here we follow the presentation of generalized Gale duality in [16, Section 2]; this paper extends the classical Gale duality construction (see e.g. [29, Section 14.3]) to a larger class of maps. Let $N$ be a finitely generated abelian group and $\beta: \mathbb{Z}^n \to N$ a group homomorphism. Define the Gale dual map $\beta^\vee: (\mathbb{Z}^n)^* \to \text{DG}(\beta)$ as follows (here we denote $(-)^* := \text{Hom}(-, \mathbb{Z})$). Take projective resolutions $E^\bullet$ and $F^\bullet$ for $\mathbb{Z}^n$ and $N$ respectively. By [94, Theorem 2.2.6], $\beta$ lifts to a morphism $E^\bullet \to F^\bullet$, and by [94, Subsection 1.5.8] there is a short exact sequence of cochain complexes $0 \to F^\bullet \to \text{Cone}(\beta) \to E^\bullet[1] \to 0$, where $\text{Cone}(\beta)$ is the mapping cone of $\beta$. Since $E^\bullet$ is projective, it gives an exact sequence of cochain complexes

$$
0 \longrightarrow E^\bullet[1]^* \longrightarrow \text{Cone}(\beta)^* \longrightarrow (F^\bullet)^* \longrightarrow 0
$$

which induces a long exact sequence in cohomology containing the segment

$$
N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \longrightarrow H^1(\text{Cone}(\beta)^*) \longrightarrow \text{Ext}_{\mathbb{Z}}^1(N, \mathbb{Z}) \longrightarrow 0.
$$

(2.11)

Define $\text{DG}(\beta) := H^1(\text{Cone}(\beta)^*)$ and $\beta^\vee: (\mathbb{Z}^n)^* \to \text{DG}(\beta)$ to be the second map in (2.11). By this definition, it is evident that the construction is natural.

There is also an explicit description of the map $\beta^\vee$. If $d$ is the rank of $N$, one can choose a projective resolution of $N$ of the form $0 \to \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^{d+r} \to N \to 0$, where $Q$ is a matrix of integers. Then $\beta: \mathbb{Z}^n \to N$ lifts to a map $\mathbb{Z}^n \xrightarrow{B} \mathbb{Z}^{d+r}$ where $B$ is a matrix of integers. Thus the mapping cone $\text{Cone}(\beta)$ is the complex $0 \to \mathbb{Z}^{d+r} \xrightarrow{[B|Q]} \mathbb{Z}^{d+r} \to 0$, hence $\text{DG}(\beta) = (\mathbb{Z}^{d+r})^*/\text{Im}([B|Q]^*)$ and $\beta^\vee$ is the composition of the inclusion map $(\mathbb{Z}^n)^* \to (\mathbb{Z}^{d+r})^*$ with the quotient map $(\mathbb{Z}^{d+r})^* \to \text{DG}(\beta)$.

---

$^2$The generic stabilizer $G$ of $\mathcal{X}$ is defined as the union, inside the inertia stack $\mathcal{I}^{\mathcal{I}}(\mathcal{X})$, of all the components of maximal dimension and it is a subsheaf of groups of $\mathcal{I}^{\mathcal{I}}(\mathcal{X})$. Intuitively, the rigidification of $\mathcal{X}$ by its generic stabilizer $G$ is the stack with the same objects as $\mathcal{X}$ and the automorphism group of an object $x$ of $\mathcal{X}$ is the quotient $\text{Aut}_{\mathcal{X}}(x)/G$. Rigidifications can also be defined for any central subgroup of the generic stabilizer. For the general construction we refer to [3, Appendix A] (see also [1, Section 5.1]).
If \( N \) is free, then \( Q = 0 \) and \( DG(\beta) = (\mathbb{Z}^n)^*/\text{Im}(B^*) \), and moreover the kernel of \( \beta'^r \) is \( N^* \) [16, Proposition 2.2].

We give a property of the generalized Gale dual that will be useful in the following.

**Lemma 2.12** ([16, Lemma 2.3]). A morphism of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{n_1} & \longrightarrow & \mathbb{Z}^{n_2} & \longrightarrow & \mathbb{Z}^{n_3} & \longrightarrow & 0 \\
\downarrow \beta_1 & & \downarrow \beta_2 & & \downarrow \beta_3 & & \downarrow & & \\
0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & 0
\end{array}
\]

in which the columns have finite cokernels, induces a morphism of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{Z}^{n_1})^* & \longrightarrow & (\mathbb{Z}^{n_2})^* & \longrightarrow & (\mathbb{Z}^{n_3})^* & \longrightarrow & 0 \\
\downarrow \beta'_1 & & \downarrow \beta'_2 & & \downarrow \beta'_3 & & \downarrow & & \\
0 & \longrightarrow & DG(\beta_1) & \longrightarrow & DG(\beta_2) & \longrightarrow & DG(\beta_3) & \longrightarrow & 0
\end{array}
\]

2.3.2. **Stacky fans.** Let \( N \) be a finitely generated abelian group of rank \( d \). We write \( \overline{N} \) for the lattice generated by \( N \) in the \( d \)-dimensional \( \mathbb{Q} \)-vector space \( N_{\mathbb{Q}} \). The natural map \( N \to \overline{N} \) is denoted by \( b \mapsto \overline{b} \).

Let \( \Sigma \) be a rational simplicial fan in \( N_{\mathbb{Q}} \). We assume that the rays \( \rho_1, \ldots, \rho_n \) of \( \Sigma \) span \( N_{\mathbb{Q}} \) and fix an element \( b_i \in N \) such that \( b_i \) generates the cone \( \rho_i \) for \( i = 1, \ldots, n \). The set \( \{b_1, \ldots, b_n\} \) defines a homomorphism of groups \( \beta: \mathbb{Z}^n \to N \) with finite cokernel. We call the triple \( \Sigma := (N, \Sigma, \beta) \) a stacky fan.

A stacky fan \( \Sigma \) encodes a group action on a quasi-affine variety \( Z_{\Sigma} \). To describe this action, let \( \mathbb{C}[z_1, \ldots, z_n] \) be the coordinate ring of \( \mathbb{A}^n \). The quasi-affine variety \( Z_{\Sigma} \) is \( \mathbb{A}^n \setminus \mathcal{V}(J_{\Sigma}) \), where \( J_{\Sigma} \) is the ideal generated by the monomials \( \prod_{\rho_i \in \sigma} z_i \) for \( \sigma \in \Sigma \). The \( \mathbb{C} \)-valued points of \( Z_{\Sigma} \) are the points \( z \in \mathbb{A}^n \) such that the cone generated by the set \( \{\rho_i \mid z_i = 0\} \) belongs to \( \Sigma \). We equip \( Z_{\Sigma} \) with an action of the group \( G_{\Sigma} := \text{Hom}_{\mathbb{Z}}(DG(\beta), \mathbb{C}^*) \) as follows. By applying the functor \( \text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*) \) to the map \( \beta': (\mathbb{Z}^n)^* \to DG(\beta) \) we obtain a homomorphism \( \alpha: G_{\Sigma} \to (\mathbb{C}^*)^n \). Then the natural action of \((\mathbb{C}^*)^n\) on \( \mathbb{A}^n \) induces an action of \( G_{\Sigma} \) on \( \mathbb{A}^n \). Since \( \mathcal{V}(J_{\Sigma}) \) is a union of coordinate subspaces, the variety \( Z_{\Sigma} \) is \( G_{\Sigma} \)-invariant.

Let us define the global quotient stack \( X(\Sigma) := [Z_{\Sigma}/G_{\Sigma}] \). By [35, Lemma 7.15 and Theorem 7.24] one has the following result.

**Proposition 2.13.** \( X(\Sigma) \) is a toric Deligne-Mumford stack of dimension \( d \) with coarse moduli space the simplicial toric variety \( X(\Sigma) \) associated with \( \Sigma \). Conversely, let \( X \) be a toric Deligne-Mumford stack with coarse moduli space \( X \). Denote by \( \Sigma \) a fan of \( X \) in \( N_{\mathbb{Q}} \), and assume that the rays of \( \Sigma \) span \( N_{\mathbb{Q}} \). Then there exists a finitely generated abelian group \( N \) and a group homomorphism \( \beta: \mathbb{Z}^n \to N \) such that the stack \( X(\Sigma) \) associated with \( \Sigma = (N, \Sigma, \beta) \) is isomorphic to \( X \) as toric Deligne-Mumford stacks.

**Remark 2.14.** Let \( \Sigma = (N, \Sigma, \beta) \) be a stacky fan. By [35, Remark 7.19-(1)], the Picard group \( \text{Pic}(X(\Sigma)) \) is isomorphic to \( DG(\beta) \).

We will now reformulate the factorization (2.10) in terms of stacky fans. Let \( \Sigma = (N, \Sigma, \beta) \) be a stacky fan. Denote by \( \beta'^{\mathbb{G}} \) the composition of \( \beta \) with the quotient map \( N \to N/N_{\text{tor}} \), where \( N_{\text{tor}} \) is the torsion subgroup of \( N \). For \( i = 1, \ldots, n \), denote by \( v_i \) the unique generator of \( \rho_i \cap (N/N_{\text{tor}}) \). Then there exist unique integers \( a_i \) such that \( \beta'^{\mathbb{G}}(e_i) = a_i v_i \), where \( e_i \) with \( i = 1, \ldots, n \) is the standard basis of
\( \mathbb{Z}^n \). Therefore there exists a unique group homomorphism \( \beta^{\text{can}} : \mathbb{Z}^n \to N/N_{\text{tor}} \) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathbb{Z}^n & \overset{\beta}{\longrightarrow} & N \\
\downarrow{\text{diag}(a_1, \ldots, a_n)} & & \downarrow{\beta^{\text{rig}}} \\
\mathbb{Z}^n & \overset{\beta^{\text{can}}}{\longrightarrow} & N/N_{\text{tor}}
\end{array}
\]

Set \( \Sigma^{\text{can}} := (N/N_{\text{tor}}, \Sigma, \beta^{\text{can}}) \) and \( \Sigma^{\text{rig}} := (N/N_{\text{tor}}, \Sigma, \beta^{\text{rig}}) \). By [35, Lemma 7.15, Theorem 7.17 and Theorem 7.24] we have the following result.

**Proposition 2.15.** Let \( \Sigma = (N, \Sigma, \beta) \) be a stacky fan. Assume that the rays of \( \Sigma \) span \( N_{\mathbb{Q}} \). Then:

- \( \mathcal{X}(\Sigma)^{\text{rig}} \simeq \mathcal{X}(\Sigma^{\text{rig}}) \) and \( \mathcal{X}(\Sigma) \) is a toric orbifold if and only if \( N \) is free, or equivalently, if and only if \( \Sigma = \Sigma^{\text{rig}} \).
- \( \mathcal{X}(\Sigma)^{\text{can}} \simeq \mathcal{X}(\Sigma^{\text{can}}) \) and \( \mathcal{X}(\Sigma) \) is canonical if and only if \( \Sigma = \Sigma^{\text{can}} \).

### 2.3.3. Closed and open substacks

Fix a cone \( \sigma \) in the fan \( \Sigma \). Let \( N_{\sigma} \) be the subgroup of \( N \) generated by the set \( \{ b_i | \rho_i \subseteq \sigma \} \) and let \( N(\sigma) \) be the quotient group \( N/N_{\sigma} \). By extending scalars, the quotient map \( N \to N(\sigma) \) becomes a surjection \( N_{\mathbb{Q}} \to N(\sigma)_{\mathbb{Q}} \). The quotient fan \( \Sigma/\sigma \) is the set \( \{ \tau = \tau + N(\sigma)_{\mathbb{Q}} | \sigma \subseteq \tau \text{ and } \tau \in \Sigma \} \), and the link of \( \sigma \) is the set \( \text{link}(\sigma) := \{ \tau | \tau + \sigma \in \Sigma, \tau \cap \sigma = 0 \} \).

For each ray \( \rho_i \) in \( \text{link}(\sigma) \), write \( \tilde{\rho}_i \) for the ray in \( \Sigma/\sigma \) and \( b_i \) for the image of \( b_i \) in \( N(\sigma) \). Let \( \ell \) be the number of rays in \( \text{link}(\sigma) \) and let \( \beta(\sigma) : \mathbb{Z}^\ell \to N(\sigma) \) be the map determined by the list \( \{ b_i | \rho_i \in \text{link}(\sigma) \} \). The quotient stacky fan \( \Sigma/\sigma \) is the triple \( (N(\sigma), \Sigma/\sigma, \beta(\sigma)) \). By using [16, Proposition 4.2] and the characterization of the invariant divisors of a toric variety as universal geometric quotients [28] one can prove the following result.

**Proposition 2.16.** Let \( \sigma \) be a cone in the stacky fan \( \Sigma \). Then \( \mathcal{X}(\Sigma/\sigma) \) defines a closed substack of \( \mathcal{X}(\Sigma) \). In particular, if \( \rho \) is a ray in \( \Sigma \), the closed substack \( \mathcal{D}_\rho = \mathcal{X}(\Sigma/\rho) \) is the effective Cartier divisor \( \pi^{-1}(D_\rho)_{\text{red}} \) of \( \mathcal{X}(\Sigma) \) with coarse moduli space \( D_\rho \), where \( \pi : \mathcal{X}(\Sigma) \to X(\Sigma) \) is the coarse moduli space morphism.

In the following we shall call \( \mathcal{D}_\rho \) the torus-invariant divisor \( \mathcal{X}(\Sigma/\rho) \) associated with the ray \( \rho \).

**Remark 2.17.** One can explicitly construct a map \( \text{DG}(\beta) \to \text{DG}(\beta(\sigma)) \) that gives both the restriction morphism \( \text{Pic}(\mathcal{X}(\Sigma)) \to \text{Pic}(\mathcal{X}(\Sigma/\sigma)) \) and the group homomorphism \( G_{\Sigma/\sigma} \to G_\Sigma \). The method was first described in the proof of [16, Proposition 4.2] but it had a gap, so we refer here to [50, Section 5.1] where the gap was fixed. Let \( \ell \) be the number of rays in \( \text{link}(\sigma) \) and \( d \) the dimension of \( \sigma \). Consider the morphisms of short exact sequences

\[
\begin{array}{cccc}
0 & \overset{0}{\longrightarrow} & \mathbb{Z}^{\ell + d} & \overset{\beta^{\ast}}{\longrightarrow} & \mathbb{Z}^n & \overset{\beta}{\longrightarrow} & \mathbb{Z}^{n-\ell-d} & \overset{0}{\longrightarrow} & 0 \\
0 & \overset{0}{\longrightarrow} & N & \overset{\cong}{\sim} & N & \overset{\beta}{\longrightarrow} & 0 & \overset{0}{\longrightarrow} & 0
\end{array}
\]

and

\[
\begin{array}{cccc}
0 & \overset{0}{\longrightarrow} & \mathbb{Z}^d & \overset{\beta^{\ast}}{\longrightarrow} & \mathbb{Z}^{\ell + d} & \overset{\beta^{\ast}}{\longrightarrow} & \mathbb{Z}^\ell & \overset{0}{\longrightarrow} & 0 \\
0 & \overset{0}{\longrightarrow} & N_{\sigma} & \overset{\beta^{\ast}}{\longrightarrow} & N & \overset{\beta(\sigma)}{\longrightarrow} & N(\sigma) & \overset{0}{\longrightarrow} & 0
\end{array}
\]
where $\beta_\sigma : \mathbb{Z}^d \to \mathbb{N}$ is the group homomorphism determined by the set $\{b_i \mid \rho_i \subseteq \sigma\}$. By applying Lemma 2.12 to these diagrams, one obtains from the first diagram a morphism $DG(\beta) \to DG(\beta)$ and from the second an isomorphism $DG(\beta(\sigma)) \to DG(\beta)$. Composing the inverse of the latter with the former, one has a morphism $DG(\beta) \to DG(\beta(\sigma))$. $\square$

One can give a characterization of the inertia stack $\mathcal{I}(\mathcal{X}(\Sigma))$ in terms of these closed substacks of $\mathcal{X}(\Sigma)$. Recall that if $\mathcal{X}$ is a global quotient stack of the form $[Z/G]$, then $\mathcal{I}(\mathcal{X}) = \bigsqcup_{q \in G} [Z^q/G]$ where $Z^q$ is the fixed point locus of $Z$ with respect to the element $q \in G$ (see e.g. [16, Section 4]). After fixing a stacky fan $\Sigma = (N, \Sigma, \beta)$, for every maximal cone $\sigma \in \Sigma(d)$ define the set

$$\text{Box}(\sigma) := \left\{ v \in N \mid \bar{v} = \sum_{\mu_i \subseteq \sigma} q_i \bar{b}_i \quad \text{for} \quad 0 \leq q_i < 1 \right\}.$$ 

The set $\text{Box}(\sigma)$ is in a one-to-one correspondence with the finite group $N(\sigma)$. Define $\text{Box}(\Sigma) = \bigcup_{\sigma \in \Sigma(d)} \text{Box}(\sigma)$, and for every $v \in N$ call $\sigma(v)$ the unique maximal cone containing $\bar{v}$. By [16, Lemma 4.6 and Theorem 4.7] we get the following result.

**Theorem 2.18.** If $\Sigma$ is a complete fan, the elements $v \in \text{Box}(\Sigma)$ are in one-to-one correspondence with the elements $g \in G_{\Sigma}$ which fix a point in $Z_{\Sigma}$ and

$$\mathcal{X}(\Sigma(\sigma(v))) \simeq \left[ Z_{\Sigma}^q/G_{\Sigma} \right].$$

Moreover, the inertia stack can be characterized as

$$\mathcal{I}(\mathcal{X}(\Sigma)) = \bigsqcup_{v \in \text{Box}(\Sigma)} \mathcal{X}(\Sigma(\sigma(v))).$$

Viewing a $d$-dimensional cone $\sigma \in \Sigma$ as the fan consisting of the cone $\sigma$ and all of its faces, we can identify $\sigma$ with an open substack of $\mathcal{X}(\Sigma)$. The induced stacky fan $\sigma$ is the triple $(N, \sigma, \beta_\sigma)$, where $\beta_\sigma : \mathbb{Z}^d \to \mathbb{N}$ is the group homomorphism introduced in Remark 2.17.

**Proposition 2.19** ([16, Proposition 4.3]). Let $\sigma$ be a $d$-dimensional cone in the fan $\Sigma$. Then $\mathcal{X}(\sigma)$ is an open substack of $\mathcal{X}(\Sigma)$ of the form $[V(\sigma)/N(\sigma)]$, where $V(\sigma) \simeq \mathbb{C}^d$ and $N(\sigma)$ is a finite abelian group acting on it, whose coarse moduli space is the open affine toric subset $U_\sigma$ of $X(\Sigma)$.

**Remark 2.20.** By varying the $d$-dimensional cones $\sigma$ of $\Sigma$, the open substacks $\mathcal{X}(\sigma)$ form an open cover of $\mathcal{X}(\Sigma)$.

$\square$

3. Stacky compactification of the ALE space $X_k$

3.1. Minimal resolution of $\mathbb{C}^2/\mathbb{Z}_k$. Let $k \geq 2$ be an integer and denote by $\mu_k$ the group of $k$-th roots of unity in $\mathbb{C}$. A choice of a primitive $k$-th root of unity $\omega$ defines an isomorphism of groups $\mu_k \simeq \mathbb{Z}_k$. We define an action of $\mu_k \simeq \mathbb{Z}_k$ on $\mathbb{C}^2$ as

$$\omega \triangleright (z_1, z_2) := (\omega z_1, \omega^{-1} z_2).$$

The quotient $\mathbb{C}^2/\mathbb{Z}_k$ is a normal affine toric surface. To describe its fan we need to introduce some notation. Let $N \simeq \mathbb{Z}^2$ be the lattice of one-parameter subgroups of the torus $T_k := \mathbb{C}^* \times \mathbb{C}^*$. Fix a $\mathbb{Z}$-basis $\{\tilde{e}_1, \tilde{e}_2\}$ of $N$ and define the vectors $\tilde{v}_i := i \tilde{e}_1 - (i - 1) \tilde{e}_2 \in N$ for any integer $i \geq 0$. Then the fan of $\mathbb{C}^2/\mathbb{Z}_k$ consists of the two-dimensional cone $\sigma := \text{Cone}(\tilde{v}_0, \tilde{v}_k) \subseteq N_Q$ and its subcones. The origin is the unique singular point of $\mathbb{C}^2/\mathbb{Z}_k$, and is a particular case of a rational double point or Du Val singularity [29, Definition 10.4.10].
By [29, Example 10.1.9 and Corollary 10.4.9], the minimal resolution of singularities of $\mathbb{C}^2/\mathbb{Z}_k$ is the smooth toric surface $\varphi_k: X_k \to \mathbb{C}^2/\mathbb{Z}_k$ defined by the fan $\Sigma_k \subset N_\mathbb{Q}$ where

$$\Sigma_k(0) := \{\{0\}\} ,$$

$$\Sigma_k(1) := \{\rho_i := \text{Cone}(\vec{v}_i) \mid i = 0, 1, 2, \ldots, k\} ,$$

$$\Sigma_k(2) := \{\sigma_i := \text{Cone}(\vec{v}_{i-1}, \vec{v}_i) \mid i = 1, 2, \ldots, k\} .$$

The vectors $v_i$ are the minimal generators of the rays $\rho_i$ for $i = 0, 1, \ldots, k$.

**Remark 3.1.** Recall the McKay correspondence: There is a one-to-one correspondence between the irreducible representations of $\mu_k$ and the irreducible components of the exceptional divisor $\varphi_k^{-1}(0)$ of the minimal resolution $\varphi_k: X_k \to \mathbb{C}^2/\mathbb{Z}_k$, which are $T_1$-invariant rational curves $D_i$ for $i = 1, \ldots, k-1$ [29, Corollary 10.3.11]. By [29, Equation (10.4.3)], the intersection matrix $(D_i \cdot D_j)_{1 \leq i, j \leq k-1}$ is given by minus the symmetric Cartan matrix $C$ of the root system of type $A_{k-1}$, i.e., one has

$$(D_i \cdot D_j)_{1 \leq i, j \leq k-1} = -C = \begin{pmatrix} -2 & 1 & \cdots & 0 \\ 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -2 \end{pmatrix} .$$

The coordinate ring of $\mathbb{C}^2/\mathbb{Z}_k$ is $\mathbb{C}[\mathbb{C}^2/\mathbb{Z}_k] := \mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[T_1, T_1^{-1} T_2^k]$. On the other hand, $\mathbb{C}[\mathbb{C}^2/\mathbb{Z}_k] = \mathbb{C}[t_1, t_2, Z_k] = \mathbb{C}[t_1, t_2, t_1 t_2]$. These two rings are isomorphic by imposing

$$T_1 = t_1^k \quad \text{and} \quad T_2 = t_2 t_1^{1-k} . \quad (3.2)$$

Let $U_i$ be the torus-invariant affine open subset of $X_k$ corresponding to the two-dimensional cone $\sigma_i$ for $i = 1, \ldots, k$. Then $\mathbb{C}[U_i] := \mathbb{C}[\sigma_i^\vee \cap M] = \mathbb{C}[T_1^{-1} T_2^{-i}, T_1^{-i} T_2]$ for $i = 1, \ldots, k$. By the relations (3.2), $\mathbb{C}[U_i] = \mathbb{C}[t_1^{k-i+1} t_2^{-i} t_1^{-i+k} t_2^i]$.

After identifying the characters of $T_i$ with the one-dimensional $T_i$-modules, we denote by $\varepsilon_1$ and $\varepsilon_2$ the equivariant first Chern class of $t_1$ and $t_2$, respectively. Define

$$\chi_i^1(t_1, t_2) = t_1^{k-i+1} t_2^{1-i} \quad \text{and} \quad \chi_i^2(t_1, t_2) = t_1^{-i} t_2 .$$

Let $\varepsilon_j^{(i)}$ be the equivariant first Chern class of $\chi_j^i$ for $i = 1, \ldots, k$ and $j = 1, 2$. Then

$$\varepsilon_1^{(i)}(\varepsilon_1, \varepsilon_2) = (k-i+1) \varepsilon_1 - (i-1) \varepsilon_2 \quad \text{and} \quad \varepsilon_2^{(i)}(\varepsilon_1, \varepsilon_2) = -(k-i) \varepsilon_1 + i \varepsilon_2 .$$

**Lemma 3.3.** Let $i \in \{1, 2, \ldots, k\}$. The character of the tangent space to $X_k$ at the torus-invariant point $p_i$ is given by

$$\text{ch}_{T_i}(T_{p_i} X_k) = \chi_1^i + \chi_2^i .$$

By [29, Theorem 4.2.8] one gets the following result.

**Lemma 3.4.** Let $i \in \{0, 1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, k\}$. The character of the line bundle $O_{X_k}(D_i)$ at the point $p_j$ is

$$\text{ch}_{T_j}(O_{X_k}(D_i)_{p_j}) = \begin{cases} \chi_1^i, & j = i, \\ \chi_2^{i+1}, & j = i+1, \\ 0, & \text{otherwise} . \end{cases}$$
3.2. Normal compactification. Let us define the vector \( \vec{b}_\infty := -\vec{v}_0 - \vec{v}_k = -k \vec{e}_1 + (k - 2) \vec{e}_2 \) in \( N \). Denote by \( \rho_\infty \) the ray \( \text{Cone}(\vec{b}_\infty) \subset N_Q \) and by \( \vec{v}_\infty \) its minimal generator. For even \( k \) we have \( \vec{v}_\infty = \frac{1}{2} \vec{b}_\infty \), while for odd \( k \) we have \( \vec{v}_\infty = \vec{b}_\infty \). Let \( \sigma_{\infty,k} \) and \( \sigma_{\infty,0} \) be the two-dimensional cones \( \text{Cone}(\vec{v}_k, \vec{v}_\infty) \subset N_Q \) and \( \text{Cone}(\vec{v}_0, \vec{v}_\infty) \subset N_Q \), respectively.

Let \( \bar{X}_k \) be the normal projective toric surface\(^3\) defined by the complete fan \( \bar{\Sigma}_k \subset N_Q \) with
\[
\bar{\Sigma}_k(0) := \{ \{0\} \} = \Sigma_k(0),
\]
\[
\bar{\Sigma}_k(1) := \{ \rho_i \mid i = 0, 1, 2, \ldots, k \} \cup \{ \rho_\infty \} = \Sigma_k(1) \cup \{ \rho_\infty \},
\]
\[
\bar{\Sigma}_k(2) := \{ \sigma_i \mid i = 1, 2, \ldots, k \} \cup \{ \sigma_{\infty,k}, \sigma_{\infty,0} \} = \Sigma_k(2) \cup \{ \sigma_{\infty,k}, \sigma_{\infty,0} \}.
\]
The surface \( X_k \) is an open dense subset of \( \bar{X}_k \).

Henceforth we will denote by \( \tilde{k} \in \mathbb{Z} \) the integer \( k/2 \) if \( k \) is even, \( k \) if \( k \) is odd. By [29, Example 1.3.20], the affine toric open subsets \( U_{\sigma_{\infty,k}} \) and \( U_{\sigma_{\infty,0}} \) are isomorphic to \( \mathbb{C}^2/\mathbb{Z}_{\tilde{k}} \). In particular, for \( k = 2 \) one has \( \tilde{k} = 1 \), and hence the toric surface \( \bar{X}_2 \) is smooth; indeed, \( \bar{X}_2 \) is the second Hirzebruch surface \( \mathbb{F}_2 \).

**Proposition 3.5.** The intersection matrix \( (D_i \cdot D_j)_{i,j=0,1,\ldots,k,\infty} \) is given by
\[
(D_i \cdot D_j)_{i,j=0,1,\ldots,k,\infty} = \begin{pmatrix}
\frac{2 - k}{k} & 1 & 0 & \ldots & 0 & 0 & \frac{1}{k} \\
1 & -2 & 1 & \ldots & 0 & 0 & 0 \\
0 & 1 & -2 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -2 & 1 & 0 \\
0 & 0 & 0 & \ldots & 1 & \frac{2 - k}{k} & \frac{1}{k} \\
\frac{1}{k} & 0 & 0 & \ldots & 0 & \frac{1}{k} & \frac{k}{k^2}
\end{pmatrix}.
\]

**Proof.** By [29, Proposition 6.4.4-(a)] we have directly \( D_\infty \cdot D_i = 0 \) for \( i = 1, \ldots, k - 1 \). On the other hand by [29, Lemma 6.4.2] we get
\[
D_\infty \cdot D_0 = \frac{\text{mult}(\rho_0)}{\text{mult}(\sigma_{\infty,0})},
\]
where \( \text{mult}(\rho_0) \) is the index of the sublattice \( \mathbb{Z}\vec{v}_0 \) in \( \mathbb{Q}\vec{v}_0 \cap N \), so that \( \text{mult}(\rho_0) = 1 \); \( \text{mult}(\sigma_{\infty,0}) \) is the index of \( \mathbb{Z}\vec{v}_0 + \mathbb{Z}\vec{v}_\infty \) in \( (\mathbb{Q}\vec{v}_0 + \mathbb{Q}\vec{v}_\infty) \cap N \). Since \( \mathbb{Z}\vec{v}_0 + \mathbb{Z}\vec{v}_\infty = \tilde{k}\mathbb{Z}\vec{e}_1 + \mathbb{Z}\vec{e}_2 \), we have \( \text{mult}(\sigma_{\infty,0}) = \tilde{k} \).

In the same way one computes \( D_\infty \cdot D_k \).

Suppose that \( k \) is odd. We have \( \vec{v}_0 + \vec{v}_\infty + \vec{v}_k = 0 \), hence by using [29, Proposition 6.4.4] we get
\[
D_\infty \cdot D_\infty = \frac{\text{mult}(\rho_\infty)}{\text{mult}(\sigma_{\infty,k})} = \frac{1}{\tilde{k}} = \frac{k}{k^2}.
\]
In the same way, using the relation \( k \vec{v}_1 - (k - 2) \vec{v}_0 + \vec{v}_\infty = 0 \) we get \( D_0 \cdot D_0 = -\frac{k - 2}{k} \), and analogously \( k \vec{v}_{k-1} - (k - 2) \vec{v}_0 + \vec{v}_\infty = 0 \) so we have \( D_k \cdot D_k = -\frac{k - 2}{k} \). For even \( k \) one uses analogous relations. By [29, Corollary 6.4.3], one has \( D_i \cdot D_j = 1 \) for \( i = 0, j = 1 \) and \( i = k - 1, j = k \). The result now follows by Remark 3.1. \( \square \)

\(^3\)The completeness of the fan is equivalent to the completeness of the surface [29, Theorem 3.4.6]. In two dimensions, the completeness of a surface is equivalent to its projectivity [29, Proposition 6.3.25].
3.3. Canonical stack. Let $\pi_k^{\text{can}} : \mathcal{X}_k \to \tilde{X}_k$ be the two-dimensional canonical projective toric orbifold\(^4\) with Deligne-Mumford torus $T_i$ and with coarse moduli space $\tilde{X}_k$. Since $\mathcal{X}_k$ is canonical, the locus where $\pi_k^{\text{can}}$ is not an isomorphism has a nonpositive dimension. In particular, $\pi_k^{\text{can}}$ is an isomorphism precisely over the smooth locus $(\tilde{X}_k)_{\text{sm}}$ of $\tilde{X}_k$.

The stacky fan of $\mathcal{X}_k$ is $\Sigma_k := (N, \Sigma_k, \beta^{\text{can}})$, where $\beta^{\text{can}} : \mathbb{Z}^{k+2} \to N \simeq \mathbb{Z}^2$ is the map sending the $i$-th coordinate vector of $\mathbb{Z}^{k+2}$ to $\vec{v}_i$ for $i = 0, 1, \ldots, k$. As explained in Section 2.3.2, the stacky fan determines the structure of $\mathcal{X}_k$ as a global quotient stack, and $\mathcal{X}_k$ is the quotient stack $[Z_{\Sigma_k}/G_{\Sigma_k}]$, where $Z_{\Sigma_k}$ is the union over all cones $\sigma \in \Sigma_k$ of the open subsets

$$Z_{\sigma} := \{ z \in \mathbb{C}^{k+2} \mid z_i \neq 0 \text{ if } \rho_i \notin \sigma \} \subset \mathbb{C}^{k+2}.$$  

The group $G_{\Sigma_k}$ is given by

$$G_{\Sigma_k} = \text{Hom}_{\mathbb{Z}} \left( \text{DG}(\beta^{\text{can}}), \mathbb{C}^\times \right),$$  

where $\text{DG}(\beta^{\text{can}})$ is the cokernel of the map $(\beta^{\text{can}})^* : \mathbb{Z}^2 \to \mathbb{Z}^{k+2}$ dual to the map $\beta^{\text{can}}$. Thus $\text{DG}(\beta^{\text{can}}) \simeq \mathbb{Z}^k$ and $G_{\Sigma_k} \simeq (\mathbb{C}^\times)^k$. The action of $G_{\Sigma_k}$ on $Z_{\Sigma_k} \subset \mathbb{C}^{k+2}$ is given by applying the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to the quotient map $(\beta^{\text{can}})^* : \mathbb{Z}^{k+2} \to \text{DG}(\beta^{\text{can}}) \simeq \mathbb{Z}^k$ to obtain an injective group morphism

$$tG_{\Sigma_k} : G_{\Sigma_k} = \text{Hom}_{\mathbb{Z}} \left( \text{DG}(\beta^{\text{can}}), \mathbb{C}^\times \right) \longrightarrow \text{Hom}_{\mathbb{Z}} \left( \mathbb{Z}^{k+2}, \mathbb{C}^\times \right) \simeq (\mathbb{C}^\times)^{k+2}.$$

By restricting the standard action of $(\mathbb{C}^\times)^{k+2}$ on $\mathbb{C}^{k+2}$ with respect to this morphism, we find that the action of $G_{\Sigma_k}$ on $Z_{\Sigma_k}$ is given by

$$(t_1, \ldots, t_k) \cdot (z_1, \ldots, z_{k+2}) = \left\{ \begin{array}{ll}
\left( \prod_{i=1}^{k-1} t_i^{i-2-k} z_1, \prod_{i=1}^{k-1} t_i^{-(i+1)} t_k^{k-2}, t_1 z_3, \ldots, t_k z_{k+2} \right), & \text{if } k \text{ odd} \\
\left( \prod_{i=1}^{k-1} t_i^{i-1-k} z_1, \prod_{i=1}^{k-1} t_i^{-(i+1)} t_k^{k-1}, t_1 z_3, \ldots, t_k z_{k+2} \right), & \text{if } k \text{ even}
\end{array} \right. $$

for $(t_1, \ldots, t_k) \in G_{\Sigma_k}$ and $(z_1, \ldots, z_{k+2}) \in Z_{\Sigma_k}$.

The boundary divisor $\mathcal{X}_k \setminus T_i$ is a simple normal crossing divisor with $k+2$ irreducible components which we denote by $\tilde{D}_0, \tilde{D}_1, \ldots, \tilde{D}_k, \tilde{D}_\infty$. For $i = 0, 1, \ldots, k, \infty$, the effective divisor $\tilde{D}_i$ is Cartier and is $T_i$-invariant, hence it corresponds to the line bundle $\mathcal{O}_{\mathcal{X}_k}^{\tilde{D}_i}$, and the latter has a canonical section $s_i$. Since $\mathcal{X}_k$ is a global quotient stack, the line bundles on $\mathcal{X}_k$ are in one-to-one correspondence with characters of $G_{\Sigma_k}$ [35, Remark 1.1]. In particular, $\mathcal{O}_{\mathcal{X}_k}^{\tilde{D}_i}$ is associated with the character

$$G_{\Sigma_k} \xrightarrow{tG_{\Sigma_k}} (\mathbb{C}^\times)^{k+2} \xrightarrow{p_i} \mathbb{C}^\times,$$

where $p_i$ is the $i$-th projection. The canonical section $s_i$ is the $i$-th coordinate of $Z_{\Sigma_k}$.

**Remark 3.7.** For $i = 0, 1, \ldots, k, \infty$ the divisor $\tilde{D}_i$ is a projective toric orbifold with Deligne-Mumford torus $\mathbb{C}^\times$ and with coarse moduli space $D_i$. It is the torus-invariant divisor associated with the ray $\rho_i$. We shall describe in detail later on the stacky fan of $\mathcal{X}_k$.\(^{\text{\(\triangle\)}}\)

---

\(^4\)A Deligne-Mumford stack is projective if it is a global quotient stack and it has a projective scheme as coarse moduli space [76, Definition 2.20].
We can now characterize the Picard group $\text{Pic}(\mathcal{X}^\text{can}_k)$ of $\mathcal{X}^\text{can}_k$. Firstly, by [35, Remark 4.14-(b)], the group $\text{Pic}(\mathcal{X}^\text{can}_k)$ fits into the short exact sequence

$$0 \rightarrow M \rightarrow \text{Div}_{T_k}(\bar{X}_k) \rightarrow \text{Pic}(\mathcal{X}^\text{can}_k) \rightarrow 0,$$

(3.8)

where the map $\text{Div}_{T_k}(\bar{X}_k) \rightarrow \text{Pic}(\mathcal{X}^\text{can}_k)$ is given by the composition of the natural map $\text{Div}_{T_k}(\bar{X}_k) \rightarrow A^1(\bar{X}_k)$ and the pullback morphism $(\pi_k^\text{can})^*: A^1(\bar{X}_k) \rightarrow \text{Pic}(\mathcal{X}^\text{can}_k)$ which sends $[D] \in A^1(\bar{X}_k)$ to the line bundle $O_{\mathcal{X}^\text{can}_k}(\pi_k^\text{can})^{-1}(D \cap (\bar{X}_k)_{\text{sm}})$. By Remark 2.14, one has $\text{Pic}(\mathcal{X}^\text{can}_k) \simeq DG(\beta^\text{can})$ and so the morphism $\text{Div}_{T_k}(\bar{X}_k) \rightarrow \text{Pic}(\mathcal{X}^\text{can}_k)$ is the quotient map $(\beta^\text{can})^\vee$.

Characterization of $\tilde{\mathcal{G}}_\infty$. The Cartier divisor $\tilde{\mathcal{G}}_\infty$ is a one-dimensional projective toric orbifold with Deligne-Mumford torus $\mathbb{C}^*$ and with coarse moduli space $D_\infty$. Its stacky fan is $\Sigma^\text{can}_k/\rho_\infty := (N(\rho_\infty), \tilde{\Sigma}_k/\rho_\infty, \beta^\text{can}(\rho_\infty))$ where the quotient group $N(\rho_\infty) := N/\mathbb{Z}\tilde{v}_\infty$ is isomorphic to $\mathbb{Z}$ and the quotient fan $\Sigma_k/\rho_\infty \subset N(\rho_\infty)\otimes\mathbb{Q} \simeq \mathbb{Q}$ is given by

$$\tilde{\mathcal{G}}_\infty \simeq \frac{(k, \tilde{\mathcal{G}}_\infty)}{(0, \infty)/\mathbb{P}^1} \xrightarrow{\pi_k} D_\infty \simeq \mathbb{P}^1,$$

where $\pi_k := (\beta^\text{can}(\rho_\infty))_{\tilde{\mathcal{G}}_\infty}$; indeed, by [35, Remark 7.19-(2)] the orders of the root stack construction over $D_\infty$, relative to the points 0 and $\infty$, are given by the absolute values of the components of the matrix of $\beta^\text{can}(\rho_\infty)$, so they are both equal to $k$.

**Proposition 3.10.** The toric orbifold $\tilde{\mathcal{G}}_\infty$ is isomorphic as a global quotient stack to

$$\frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^* \times \mu_k},$$

where the action of $\mathbb{C}^* \times \mu_k$ on $\mathbb{C}^2 \setminus \{0\}$ is given by

$$(t, \omega) \cdot (z_1, z_2) = (t \omega z_1, t z_2)$$

(3.11)

for $(t, \omega) \in \mathbb{C}^* \times \mu_k$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$.

**Proof.** By Proposition 2.16 we have $\tilde{\mathcal{G}}_\infty \simeq [Z_{\Sigma^k/\rho_\infty}/G_{\Sigma^\text{can}_k/\rho_\infty}]$, where $Z_{\Sigma^k/\rho_\infty} := \mathbb{C}^2 \setminus \{0\}$ and $G_{\Sigma^\text{can}_k/\rho_\infty} := \text{Hom}_{\mathbb{Z}}(DG(\beta^\text{can}(\rho_\infty)), \mathbb{C}^*)$. It remains to prove that $G_{\Sigma^\text{can}_k/\rho_\infty}$ is isomorphic to $\mathbb{C}^* \times \mu_k$ and the action of $G_{\Sigma^\text{can}_k/\rho_\infty}$ on $Z_{\Sigma^k/\rho_\infty}$ is given by (3.11).

As described in Section 2.3.1, the abelian group $DG(\beta^\text{can}(\rho_\infty))$ can be realized as the cokernel of the map

$$\beta^\text{can}(\rho_\infty)^*: \mathbb{Z} \rightarrow \mathbb{Z}^2, \quad m \mapsto m \tilde{e}_1 - m \tilde{e}_2.$$ 

Hence $DG(\beta^\text{can}(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z} \tilde{e}_1$ and therefore $G_{\Sigma^\text{can}_k/\rho_\infty} \simeq \mathbb{C}^* \times \mu_k$. The quotient map $\beta^\text{can}(\rho_\infty)^\vee: \mathbb{Z}^2 \rightarrow DG(\beta^\text{can}(\rho_\infty))$ is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

(3.12)

The action of $G_{\Sigma^\text{can}_k/\rho_\infty}$ on $\mathbb{C}^2 \setminus \{0\}$ is the restriction of the standard action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^2 \setminus \{0\}$ via the immersion

$$tG_{\Sigma^\text{can}_k/\rho_\infty} : \mathbb{C}^* \times \mu_k \simeq G_{\Sigma^\text{can}_k/\rho_\infty} \simeq \text{Hom}_{\mathbb{Z}}(DG(\beta^\text{can}(\rho_\infty)), \mathbb{C}^*) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{C}^*) \simeq (\mathbb{C}^*)^2$$
obtained by applying the functor \( \text{Hom}_\mathbb{Z}(-, \mathbb{C}^*) \) to the quotient map \( \beta^{\text{can}}(\rho_\infty)^\vee \). Therefore the action of \( G_{\mathbb{C}^*}^{\text{can}}/\rho_\infty \) on \( Z_{\mathbb{C}^*}/\rho_\infty \) is exactly (3.11).

This characterization of \( \mathcal{D}_\infty \) as a global quotient stack yields the following result.

**Corollary 3.13.** The Picard group \( \text{Pic}(\mathcal{D}_\infty) \) of \( \mathcal{D}_\infty \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}} \). It is generated by the line bundles \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) corresponding, respectively, to the characters

\[
\tilde{\chi}_1 : (t, \omega) \in \mathbb{C}^* \times \mu_{\tilde{k}} \mapsto t \in \mathbb{C}^* \quad \text{and} \quad \tilde{\chi}_2 : (t, \omega) \in \mathbb{C}^* \times \mu_{\tilde{k}} \mapsto \omega \in \mathbb{C}^* .
\]

We give a geometrical interpretation of the line bundles \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). Recall that the boundary divisor \( \partial \mathcal{D}_\infty \setminus \mathbb{C}^* \) is a simple normal crossing divisor with two irreducible components \( \bar{p}_0 \) and \( \bar{p}_\infty \). These are effective Cartier divisors associated with the rays \( \rho_0 \) and \( \rho_\infty \), respectively, and they coincide with the closed substacks \( \bar{\pi}_{-1}(0)_{\text{red}} \) and \( \bar{\pi}_{-1}(\infty)_{\text{red}} \). Define the Cartier divisor \( \bar{p} := \bar{p}_0 - \bar{p}_\infty \).

**Proposition 3.14.** The line bundle \( \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}_0) \) is isomorphic to \( \mathcal{L}_1 \) and the line bundle \( \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}) \) is isomorphic to \( \mathcal{L}_2 \).

**Proof.** Recall that the quotient map \( \beta^{\text{can}}(\rho_\infty)^\vee : \mathbb{Z}^2 \to \text{DG}(\beta^{\text{can}}(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}} \) sends the coordinate vector \( \bar{e}_1 \) of \( \mathbb{Z}^2 \) to \( \bar{f}_1 + \bar{f}_2 \) and the coordinate vector \( \bar{e}_2 \) of \( \mathbb{Z}^2 \) to \( \bar{f}_1 \), where \( \{\bar{f}_1, \bar{f}_2\} \) is the standard basis of \( \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}} \). By Remark 2.14, \( \text{Pic}(\mathcal{D}_\infty) \) is isomorphic to \( \text{DG}(\beta^{\text{can}}(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}} \), hence the vector \( \bar{f}_i \) corresponds to \( \mathcal{L}_i \) for \( i = 1, 2 \) by Corollary 3.13. So the quotient map \( \beta^{\text{can}}(\rho_\infty)^\vee : \mathbb{Z}^2 \to \text{DG}(\beta^{\text{can}}(\rho_\infty)) \) can be interpreted as a map from \( \text{Div}(\mathcal{D}_\infty) \simeq \mathbb{Z} \rho_0 + \mathbb{Z} \rho_\infty \) to \( \text{Pic}(\mathcal{D}_\infty) \), hence we have \( \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}_0) \simeq \mathcal{L}_1 \otimes \mathcal{L}_2 \) and \( \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}) \simeq \mathcal{L}_1 \otimes \mathcal{L}_2 \). Then \( \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}) \simeq \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}_0) \otimes \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}_\infty) \simeq \mathcal{L}_2 \). □

The following result will be useful later on.

**Lemma 3.15.** The line bundle \( \mathcal{O}_{\mathcal{D}_\infty}^{\text{can}}(\bar{p}_\infty) \) assumes the following form with respect to the generators \( \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}_\infty) \) and \( \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}) \) of \( \text{Pic}(\mathcal{D}_\infty) \):

\[
\mathcal{O}_{\mathcal{D}_\infty}^{\text{can}}(\mathcal{D}_\infty) \simeq \left\{ \begin{array}{ll}
\mathcal{O}_{\mathcal{D}_\infty}(2\bar{p}_\infty) \otimes \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}) & \text{for even } k , \\
\mathcal{O}_{\mathcal{D}_\infty}(\bar{p}_0) \otimes \mathcal{O}_{\mathcal{D}_\infty}(k+1/2) \bar{p} & \text{for odd } k .
\end{array} \right.
\]

Moreover

\[
\mathcal{O}_{\mathcal{D}_\infty}^{\text{can}}(\bar{p}_0) \simeq \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}_0) \quad \text{and} \quad \mathcal{O}_{\mathcal{D}_\infty}^{\text{can}}(\bar{p}_\infty) \simeq \mathcal{O}_{\mathcal{D}_\infty}(\bar{p}_\infty) .
\]

**Proof.** We need to compute the restriction map \( \text{Pic}(\mathcal{D}_\infty) \to \text{Pic}(\mathcal{D}_\infty) \). Since by Remark 2.14, \( \text{Pic}(\mathcal{D}_\infty) \simeq \text{DG}(\beta^{\text{can}}) \) and \( \text{Pic}(\mathcal{D}_\infty) \simeq \text{DG}(\beta^{\text{can}}(\rho_\infty)) \), we need only to determine a restriction map from \( \text{DG}(\beta^{\text{can}}) \) to \( \text{DG}(\beta^{\text{can}}(\rho_\infty)) \). By applying the procedure described in Remark 2.17 we obtain morphisms of short exact sequences

\[
0 \to Z^{k-1} \to Z^{k+2} \to Z^3 \to 0
\]

and

\[
0 \to Z^2 \to Z^3 \to Z \to 0
\]
where $\tilde{\beta}^{can}: \mathbb{Z}^3 \to N$ is the restriction of $\beta^{can}: \mathbb{Z}^{k+2} \to N$ to the subgroup $\mathbb{Z}^3 \subset \mathbb{Z}^{k+2}$ generated by the rays $\rho_0, \rho_k, \rho_\infty$ (here we identify $\mathbb{Z}^{k+2}$ with the free abelian group generated by the rays of $\Sigma_k$). Since $N_{\rho_\infty}$ is the subgroup of $N$ generated by $\tilde{v}_\infty$, the map $\beta^{can}_{\rho_\infty}: \mathbb{Z} \to N_{\rho_\infty}$ sends 1 to $\tilde{v}_\infty$.

Let us denote by $\phi$ the isomorphism $DG(\beta^{can}_{\rho_\infty}) \cong DG(\tilde{\beta}^{can})$. One can explicitly compute the map $(\tilde{\beta}^{can})^\vee: \mathbb{Z}^3 \to DG(\tilde{\beta}^{can}) \cong \mathbb{Z} \oplus \mathbb{Z}_k$ in the commutative diagram (3.16), obtaining the matrix

$$
\begin{pmatrix}
1 & 1 & 2 \\
-1 & 0 & -1
\end{pmatrix}
$$

for even $k$ and

$$
\begin{pmatrix}
1 & 1 & 1 \\
-1 & 1 & 0
\end{pmatrix}
$$

for odd $k$.

Since $\beta^{can}(\rho_\infty)^\vee$ is given by the matrix (3.12), the map $\phi$ in the commutative diagram (3.17) is represented by the matrix

$$
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & -2
\end{pmatrix}
$$

for even $k$ and

$$
\begin{pmatrix}
1 & 0 & 0 \\
1 & -1 & 2
\end{pmatrix}
$$

for odd $k$.

Then its inverse is represented by

$$
\begin{pmatrix}
1 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}
$$

Thus the restriction map $Pic(\mathcal{X}_k) \to Pic(\mathcal{X}_k)$ is given by the composition of the map $DG(\beta^{can}) \to DG(\tilde{\beta}^{can})$ in the commutative diagram (3.16) with $\phi^{-1}$. Since the line bundle $O_{\mathcal{X}_k^{can}}(\tilde{\mathcal{D}}_\infty)$ is the image of $(0, \ldots, 0, 1) \in \mathbb{Z}^{k+2}$ via the map $(\beta^{can})^\vee$, the first statement follows.

The results for the restrictions of the line bundles $O_{\mathcal{X}_k^{can}}(\mathcal{D}_0)$ and $O_{\mathcal{X}_k^{can}}(\mathcal{D}_k)$ follow in the same way.

\begin{remark}
By this Lemma, it is easy to see that the line bundle $O_{\mathcal{X}_k^{can}}(\tilde{\mathcal{D}}_\infty)$ is $\pi_k^{can}$-ample, i.e., the representation of its fibre at any geometric point of $\mathcal{X}_k^{can}$ is faithful.
\end{remark}

### 3.4. Root toric stack

Let $\mathcal{X}_k := \sqrt[k]{\tilde{\mathcal{D}}_\infty/\mathcal{X}_k^{can}}$ be the stack obtained from $\mathcal{X}_k^{can}$ by performing a $k$-th root construction along the divisor $\tilde{\mathcal{D}}_\infty$. It is a two-dimensional projective toric orbifold with Deligne-Mumford torus $T_k$ and with coarse moduli space $\pi_k := \pi_k^{can} \circ \phi_k: \mathcal{X}_k \to \bar{X}_k$ (cf. Section 2.3). Its stacky fan is $\bar{\Sigma}_k := (N, \Sigma_k, \beta)$, where $\beta: \mathbb{Z}^{k+2} \to N$ is given by $\{\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_k, k \tilde{v}_\infty\}$.

As a global quotient stack, $\mathcal{X}_k$ is isomorphic to $[Z_{\Sigma_k}/G_{\Sigma_k}]$ where $Z_{\Sigma_k}$ is the same as for $\mathcal{X}_k^{can}$, since $\bar{\Sigma}_k$ is the fan in both stacky fans. The group $G_{\Sigma_k}$ is $Hom_{\mathbb{Z}}(DG(\beta), \mathbb{C}^*)$, where $DG(\beta)$ is the cokernel of the group homomorphism $\beta^*: \mathbb{Z}^2 \to \mathbb{Z}^{k+2}$. Then we obtain $DG(\beta) \simeq \mathbb{Z}^k$ and therefore $G_{\Sigma_k} \simeq (\mathbb{C}^*)^k$.

By applying the functor $Hom_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the quotient map $\beta^*: \mathbb{Z}^{k+2} \to DG(\beta)$ we obtain an injective morphism $G_{\Sigma_k} \to (\mathbb{C}^*)^{k+2}$ which is given by

$$(t_1, \ldots, t_k) \mapsto \begin{cases}
(\prod_{i=1}^{k-1} t_i t_k^{-k+2}, \prod_{i=1}^{k-1} t_i^{-i+1} t_k^{k-2}, t_1, \ldots, t_k) & \text{for odd } k \\
(\prod_{i=1}^{k-1} t_i t_k^{-k+2}, \prod_{i=1}^{k-1} t_i^{-i+1} t_k^{k-2}, t_1, \ldots, t_k) & \text{for even } k
\end{cases}
$$

By restricting the standard action of $(\mathbb{C}^*)^{k+2}$ on $Z_{\Sigma_k} \subset \mathbb{C}^{k+2}$ we obtain an action of $G_{\Sigma_k}$ on $Z_{\Sigma_k}$.

As before, the boundary divisor $\mathcal{X}_k \setminus T_k$ is a simple normal crossing divisor with $k + 2$ irreducible components $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_k, \mathcal{D}_\infty$, which are the effective Cartier divisors corresponding to the rays $\rho_0, \rho_1, \ldots, \rho_k, \rho_\infty$. By construction (cf. Section 2.2), we have

$$
\phi_k^*(O_{\mathcal{X}_k^{can}}(\mathcal{D}_i)) \simeq \begin{cases}
O_{\mathcal{D}_i}(\mathcal{D}_i) & \text{for } i = 0, 1, \ldots, k \\
O_{\mathcal{D}_\infty}(k \mathcal{D}_\infty) & \text{for } i = \infty
\end{cases}
$$
Remark 3.19. Since \( \mathcal{X}_k \) is a quotient stack, there is a well defined integral intersection theory [32]. As \( \mathcal{X}_k \) is smooth, its rational Chow groups are isomorphic to the rational Chow groups of \( \mathcal{X}_k \) via \( \pi_{ks} \) by [93, Proposition 6.1]. In particular, \( \pi_{ks}(\mathcal{D}_i) = D_i \) for \( i = 0, 1, \ldots, k \) and \( \pi_{ks}(\mathcal{D}_\infty) = \frac{1}{k} D_\infty \).

Recall that by Remark 3.1 the intersection matrix \((D_i \cdot D_j)_{1 \leq i, j \leq k-1}\) is minus the Cartan matrix \( C \) of the Dynkin diagram of type \( A_{k-1} \). The matrix \( C \) is not unimodular and its inverse has matrix elements

\[
(C^{-1})^{ij} = \min(i, j) - \frac{i j}{k}.
\]

In \( \text{Pic}(\mathcal{X}_k)_\mathbb{Q} \) we can define the classes

\[
\omega_i := - \sum_{j=1}^{k-1} (C^{-1})^{ij} \mathcal{D}_j
\]

for \( i = 1, \ldots, k-1 \).

Lemma 3.21. The classes \( \omega_i \) are integral combinations of \( \mathcal{D}_i \) for \( i = 0, 1, \ldots, k \) and \( \mathcal{D}_\infty \) in \( \text{Pic}(\mathcal{X}_k) \).

Proof. We argue as in [26, Section 5.2]. Let \( \bar{v}_\infty = \bar{k} \bar{v}_1 + \alpha \bar{v}_2 \) be the minimal generator of the ray \( \rho_\infty \), where \( \alpha := \bar{k} - 1 \in \mathbb{Z} \) if \( k \) is even and \( \alpha := \bar{k} - 2 \) if \( k \) is odd. Let us consider the following relations in \( \text{Pic}(\mathcal{X}_k) \):

\[
0 = \text{div}(\chi^{(1,0)}) = \mathcal{D}_1 + 2 \mathcal{D}_2 + \cdots + k \mathcal{D}_k - \bar{k} k \mathcal{D}_\infty, \\
0 = \text{div}(\chi^{(0,1)}) = \mathcal{D}_0 - \mathcal{D}_2 - \cdots - (k-1) \mathcal{D}_k + a k \mathcal{D}_\infty,
\]

where \( \chi^{(1,0)}, \chi^{(0,1)} \) are the characters of \( T_k \) associated with \((1,0), (0,1) \in M\), respectively. Since by definition

\[
\omega_1 = - \sum_{j=1}^{k-1} \frac{(k-j)}{k} \mathcal{D}_j \quad \text{and} \quad \omega_{k-1} = - \sum_{j=1}^{k-1} \frac{j}{k} \mathcal{D}_j,
\]

we get \( \omega_1 = \mathcal{D}_0 + (-\bar{k} (k-1) + a k) \mathcal{D}_\infty \) and \( \omega_{k-1} = \mathcal{D}_k - \bar{k} \mathcal{D}_\infty \). For \( i = 2, \ldots, k-2 \) we have \( \omega_i = \omega_{i-1} - \omega_{k-1} - \sum_{j=1}^{k-1} \mathcal{D}_j \), and the assertion follows.

Definition 3.22. For \( i = 1, \ldots, k-1 \), the \( i \)-th tautological line bundle \( \mathcal{R}_i \) on \( \mathcal{X}_k \) is the line bundle associated to the Cartier divisor \( \omega_i \).

By the proof of Lemma 3.21, the tautological line bundles \( \mathcal{R}_i \) for \( i = 1, \ldots, k-1 \) can be written as

\[
\mathcal{R}_i = \begin{cases} \\
\mathcal{O}_{\mathcal{X}_k} \left( \mathcal{D}_0 - \sum_{j=1}^{i-1} (j-1) \mathcal{D}_j - (i-1) \sum_{j=i}^{k} \mathcal{D}_j + (i-2) \bar{k} \mathcal{D}_\infty \right) \quad & \text{for } i = 1, \ldots, k-2, \\
\mathcal{O}_{\mathcal{X}_k} \left( \mathcal{D}_k - \bar{k} \mathcal{D}_\infty \right) \quad & \text{for } i = k-1.
\end{cases}
\]

Remark 3.24. We call these line bundles “tautological” as they play the same role as the tautological line bundles considered by Kronheimer and Nakajima [55]. Indeed, their restrictions to \( X_k \subset \mathcal{X}_k \) yield exactly their tautological line bundles. Moreover, note that

\[
\int_{\mathcal{X}_k} c_1(\mathcal{R}_i) \cdot c_1(\mathcal{R}_j) = \int_{\mathcal{X}_k} \omega_i \cdot \omega_j = -(C^{-1})^{ij} \quad \text{for } i, j = 1, \ldots, k-1,
\]

which is the same result as in [55, Theorem A.7].

Proposition 3.25. The Picard group \( \text{Pic}(\mathcal{X}_k) \) of \( \mathcal{X}_k \) is freely generated over \( \mathbb{Z} \) by \( \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty) \) and \( \mathcal{R}_i \) with \( i = 1, \ldots, k-1 \).
Proposition 3.27. It is enough to use the explicit expressions for the real Cartan subalgebra of $\mathfrak{g}$ associated with the Dynkin diagram of type $A_{k-1}$. In this picture, $H_2(X_k; \mathbb{Z})$ can be identified with the root lattice $\Omega$ of type $A_{k-1}$. Under this correspondence, the classes $[D_1], \ldots, [D_{k-1}]$ are the simple roots. Since $Pic(\mathcal{X}_k)$ has no torsion, the map $\pi_0 : \text{Pic}(\mathcal{X}_k) \to \text{Pic}(\mathcal{X}_k) \otimes \mathbb{Z}$ is injective. Consider the restriction map $\pi^* : \text{Pic}(\mathcal{X}_k) \otimes \mathbb{Z} \to \text{Pic}(X_k) \otimes \mathbb{Z}$ with respect to the inclusion morphism $\iota : X_k \to \mathcal{X}_k$. The map $\pi^*$ is surjective because of Equation (3.20).

For $j = 1, \ldots, k-1$, let $\gamma_j$ be a simple root in $\Omega$, and $\gamma := \sum_{i=1}^{k-1} y_i \gamma_i$ an element of the root lattice. Under the correspondence described above, this gives a linear combination of the divisors $\sum_{i=1}^{k-1} y_i [D_i]$ and singles out a unique line bundle $O_{X_k}(\sum_{i=1}^{k-1} y_i D_i)$. After fixing an integer $u_\infty \in \mathbb{Z}$ and setting $\bar{u} = -C\bar{y}$, where $\bar{y} := (y_1, \ldots, y_{k-1})$, we can define a line bundle $\bigotimes_{i=1}^{k-1} R_i^{\otimes u_i} \otimes O_{\mathcal{X}_k}(\mathcal{D}_\infty)^{\otimes u_\infty}$ such that

$$\bigotimes_{i=1}^{k-1} R_i^{\otimes u_i} \otimes O_{\mathcal{X}_k}(u_\infty \mathcal{D}_\infty)|_{X_k} \simeq O_{X_k}\left(\sum_{i=1}^{k-1} y_i D_i\right).$$

\[\square\]

Characterization of $\mathcal{D}_\infty$. All results presented below hold for every $k \geq 2$, but the proofs and computations are given only for the cases $k > 2$. For $k = 2$ there are some differences; since $X_2$ is the second Hirzebruch surface, for that case one can use the calculations done in [21, Appendix D] for the stacky Hirzebruch surfaces.

By construction (cf. Section 2.2), $\mathcal{D}_\infty$ is isomorphic to the root stack $\sqrt[\mathbb{Z}][\mathcal{X}_k^{\text{can}}]|_{\mathcal{X}_k}/\mathcal{D}_\infty$. By [35, Theorem 6.25-(1)] the divisor $\mathcal{D}_\infty$ is a projective toric Deligne-Mumford stack with Deligne-Mumford torus $\mathcal{T} \simeq T_k \times \mathbb{R} \mu_k$. Its stacky fan is the quotient stacky fan $\Sigma_{k/\mu_\infty} := (N(\mu_\infty), \Sigma_k/\mu_\infty, \beta(\mu_\infty))$, where $N(\mu_\infty) := N/k \mathbb{Z} \otimes \mathbb{Z} \simeq \mathbb{Z} \oplus \mathbb{Z} k$ and the quotient fan $\Sigma_{k/\mu_\infty} \subset N(\mu_\infty) \otimes \mathbb{Q} \simeq \mathbb{Q}$ is the same as the quotient fan (3.9) of $\mathcal{D}_\infty$. The quotient map $N \to N(\mu_\infty) \simeq \mathbb{Z} \oplus \mathbb{Z} k$ is given by

$$\left(\begin{array}{cc} 1 - k & -k \\ -1 & -1 \end{array}\right) \text{ for even } k \quad \text{ and } \quad \left(\begin{array}{cc} k - 2 & k \\ -k - 1 & -k + 1 \end{array}\right) \text{ for odd } k.$$ 

On the other hand, the map $\beta(\mu_\infty) : \mathbb{Z}^2 \to N(\mu_\infty) \simeq \mathbb{Z} \oplus \mathbb{Z} k$ is given by the matrix

$$M(\beta(\mu_\infty)) = \left(\begin{array}{cc} 0 & -k \\ -1 & 1 \end{array}\right) \text{ for even } k \quad \text{ and } \quad M(\beta(\mu_\infty)) = \left(\begin{array}{cc} k & -k \\ -k - 1 & k - 1 \end{array}\right) \text{ for odd } k.$$ 


By Section 2.3, the toric stack $\mathcal{D}_\infty$ is an essentially trivial gerbe with banding group $\text{Hom}_{\mathbb{Z}}(N(\mu_\infty)_{\text{tor}}, \mathbb{C}^*) \simeq \mu_k$ over its rigidification $\mathcal{D}_{\mathcal{L}}$, which is $\mathcal{D}_\infty$. Let $\tilde{\phi}_k := (\phi_k)|_{\mathcal{D}_\infty} : \mathcal{D}_\infty \to \mathcal{D}_\infty$ be the $\mu_k$-gerbe structure morphism. Then $r_k := \tilde{\pi}_k \circ \tilde{\phi}_k : \mathcal{D}_\infty \to D_\infty \simeq \mathbb{P}^1$ is the coarse moduli space of $\mathcal{D}_\infty$.

Proposition 3.27. The toric stack $\mathcal{D}_\infty$ is isomorphic to the global quotient stack

$$\left[\mathbb{C}^2 \setminus \{0\} \right] / \left[ \mathbb{C}^* \times \mu_k \right].$$
where the action is given by
\[
(t, \omega) \triangleright (z_1, z_2) = \begin{cases} 
(t^k \omega z_1, t^k \omega^{-1} z_2) & \text{for even } k, \\
(t^k \omega^{k-1} z_1, t^k \omega^{k-1} z_2) & \text{for odd } k,
\end{cases}
\]
for \((t, \omega) \in \mathbb{C}^* \times \mu_k\) and \((z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}.

**Proof.** By arguing along the lines of the analogous characterization for \(\mathcal{D}_\infty\) (cf. Proposition 3.10), we get that \(\mathcal{D}_\infty\) is isomorphic to \([Z_{\Sigma_k} / \rho_\infty] / G_{\Sigma_k / \rho_\infty}\), where \(Z_{\Sigma_k} / \rho_\infty = \mathbb{C}^2 \setminus \{0\}\) is the same as for \(\mathcal{D}_\infty\). The group \(G_{\Sigma_k / \rho_\infty}\) is \(\text{Hom}_{\mathbb{Z}}(DG(\beta(\rho_\infty)), \mathbb{C}^*)\). Since \(N(\rho_\infty)\) has torsion, \(DG(\beta(\rho_\infty))\) is obtained in the following way. Consider a free resolution of \(N(\rho_\infty)\)
\[
0 \longrightarrow \mathbb{Z} \xrightarrow{Q} \mathbb{Z}^2 \longrightarrow N(\rho_\infty) \simeq \mathbb{Z} \oplus \mathbb{Z}_k \longrightarrow 0
\]
where \(Q\) is the map which sends 1 to \(k^2 \in \mathbb{Z}\). Consider also a lifting \(B: \mathbb{Z}^2 \to \mathbb{Z}^2\) of \(\beta(\rho_\infty)\), so that \(B\) can be represented by the matrix \(M(\beta(\rho_\infty))\). Define the map \([B\, Q]\): \(\mathbb{Z}^3 \to \mathbb{Z}^2\) by adding the column \(Q\) to the matrix of \(B\). Then \(DG(\beta(\rho_\infty)) = \text{Coker}([B\, Q]^*)\) and \([B\, Q]^*\) is given by the matrix
\[
H = \begin{pmatrix}
\frac{k}{k^2} & -1 \\
-k & -1 \\
0 & k
\end{pmatrix}
\]
for even \(k\) and
\[
H = \begin{pmatrix}
k & \frac{k-1}{k^2} \\
-k & \frac{k-1}{k^2} \\
0 & k
\end{pmatrix}
\]
for odd \(k\).

In both cases, \(H\) is equivalent to the matrix
\[
K = \begin{pmatrix}
1 & 0 \\
0 & k \\
0 & 0
\end{pmatrix},
\]
in the sense that there exist two unimodular matrices \(T \in GL(3, \mathbb{Z})\) and \(P \in GL(2, \mathbb{Z})\) such that \(H = TKP\). Hence we have \(DG(\beta(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z}_k\) and \(G_{\Sigma_k / \rho_\infty} \simeq \mathbb{C}^* \times \mu_k\). The action of \(\mathbb{C}^* \times \mu_k\) on \(\mathbb{C}^2 \setminus \{0\}\) is given by composition of the standard \((\mathbb{C}^*)^2\)-action with the map \(\mathbb{C}^* \times \mu_k \to (\mathbb{C}^*)^2\) obtained by applying the functor \(\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)\) to the composition \(\mathbb{Z}^2 \to \mathbb{Z}^3 \to DG(\beta(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z}_k\), where the second map is the quotient map. This gives the assertion. 

Recall that \(\text{Pic}(\mathcal{D}_\infty) \simeq DG(\beta(\rho_\infty))\) (cf. Remark 2.14). The explicit characterization of the global quotient structure of \(\mathcal{D}_\infty\) then yields the following result.

**Corollary 3.29.** The Picard group \(\text{Pic}(\mathcal{D}_\infty)\) of \(\mathcal{D}_\infty\) is isomorphic to \(\mathbb{Z} \oplus \mathbb{Z}_k\). It is generated by the line bundles \(\mathcal{L}_1\) and \(\mathcal{L}_2\) corresponding respectively to the two characters of \(G_{\Sigma_k / \rho_\infty} \simeq \mathbb{C}^* \times \mu_k\) given by
\[
\chi_1: (t, \omega) \in \mathbb{C}^* \times \mu_k \mapsto t \in \mathbb{C}^* \quad \text{and} \quad \chi_2: (t, \omega) \in \mathbb{C}^* \times \mu_k \mapsto \omega \in \mathbb{C}^*.
\]
In particular, \(\mathcal{L}_2^{\otimes k}\) is trivial.

By the commutative diagram of [35, Equation (6.28)] we also know that \(\text{Pic}(\mathcal{D}_\infty)\) fits into a commutative diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \\
\downarrow f & & \downarrow f \\
0 & \longrightarrow & \text{Pic}(\mathcal{D}_\infty)
\end{array}
\]
\[
\begin{array}{ccc}
\phi_k^* & \longrightarrow & \text{Pic}(\mathcal{D}_\infty) \\
\downarrow & & \downarrow \text{id} \\
\phi_k^* & \longrightarrow & \text{Pic}(\mathcal{D}_\infty)
\end{array}
\]
where the morphisms \(f\) and \(\phi\) send 1 \(\mapsto \mathcal{O}\) to \(\mathcal{O}_{\mathcal{D}_\infty}(\mathcal{D}_\infty)\) and 1 \(\mapsto \mathcal{O}_{\mathcal{D}_\infty}(\mathcal{D}_\infty)\) respectively. From this commutative diagram, one can argue that every line bundle \(\mathcal{L}\) on \(\mathcal{D}_\infty\) can be written as a tensor product \(\phi_k^*(\mathcal{N}) \otimes \mathcal{O}_{\mathcal{D}_\infty}(\ell \mathcal{D}_\infty)\) for a line bundle \(\mathcal{N}\) on \(\mathcal{D}_\infty\) and \(0 \leq \ell < k\) an integer.

Now we characterize the restrictions of line bundles from \(\mathcal{D}_k\) to \(\mathcal{D}_\infty\).
Lemma 3.31. The restriction of $\mathcal{O}_{\mathfrak{X}_k}(\mathcal{D}_\infty)$ to $\mathcal{D}_\infty$ is isomorphic to $\mathcal{L}_1$. For even $k$ one has

$$\mathcal{O}_{\mathfrak{X}_k}(\mathcal{D}_0)|_{\mathcal{D}_\infty} \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2 \quad \text{and} \quad \mathcal{O}_{\mathfrak{X}_k}(\mathcal{D}_k)|_{\mathcal{D}_\infty} \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes -1},$$

while for odd $k$ one has

$$\mathcal{O}_{\mathfrak{X}_k}(\mathcal{D}_0)|_{\mathcal{D}_\infty} \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k+1}{2}} \quad \text{and} \quad \mathcal{O}_{\mathfrak{X}_k}(\mathcal{D}_k)|_{\mathcal{D}_\infty} \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k-1}{2}}.$$

Proof. The proof is analogous to that of Lemma 3.15. We need only to point out that the analogue of the map $\phi$ in this case is not uniquely determined by just imposing the commutativity of the diagram analogous to the diagram (3.17). To compute that map, one has to follow the proof of Lemma 2.12. In particular, in our case we find that it is the identity. \hfill $\square$

If one works out some of the details of the proof of this Lemma and in particular writes explicitly some of the maps in the required commutative diagrams, one can for example obtain that $\mathcal{O}_{\mathfrak{X}_k}(\mathcal{D}_i)|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}$ for $i = 1, \ldots, k - 1$. Moreover, if we denote by $p_0$ and $p_\infty$ respectively the divisors $\tilde{\phi}_k^{-1}(\tilde{p}_0)_{\text{red}}$ and $\tilde{\phi}_k^{-1}(\tilde{p}_\infty)_{\text{red}}$ in $\mathcal{D}_\infty$ corresponding to $p_0$ and $p_\infty$, from the explicit form of the Gale dual of $\beta(p_\infty)$ one has the following result.

Corollary 3.32. For even $k$ we have

$$\mathcal{O}_{\mathcal{D}_\infty}(p_0) \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2 \quad \text{and} \quad \mathcal{O}_{\mathcal{D}_\infty}(p_\infty) \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes -1}.$$ 

For odd $k$ we have

$$\mathcal{O}_{\mathcal{D}_\infty}(p_0) \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k+1}{2}} \quad \text{and} \quad \mathcal{O}_{\mathcal{D}_\infty}(p_\infty) \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k-1}{2}}.$$ 

In particular, for any $k \geq 2$ we have $\mathcal{O}_{\mathcal{D}_\infty}(p_0) \simeq \mathcal{O}_{\mathfrak{X}_k}(\mathcal{D}_0)|_{\mathcal{D}_\infty}$ and $\mathcal{O}_{\mathcal{D}_\infty}(p_\infty) \simeq \mathcal{O}_{\mathfrak{X}_k}(\mathcal{D}_k)|_{\mathcal{D}_\infty}$.

Remark 3.33. This corollary makes it clear that the line bundles associated with the torus-invariant divisors are not sufficient to generate the whole Picard group of the gerbe $\mathcal{D}_\infty$. This is evident if we consider the exact sequence (2.11), which in our case becomes

$\mathbb{Z} \xrightarrow{\beta(p_\infty)^*} \mathbb{Z}^2 \xrightarrow{\beta(p_\infty)^\vee} \text{Pic}(\mathcal{D}_\infty) \rightarrow \text{Ext}_1^2 \left( N(p_\infty), \mathbb{Z} \right) \simeq \mathbb{Z}_k \rightarrow 0,$

and our statement is equivalent to the fact that $\beta(p_\infty)^\vee$ is not surjective. \hfill $\triangle$

Corollary 3.34. The restrictions of the tautological line bundles $\mathcal{R}_i$ on $\mathfrak{X}_k$ to $\mathcal{D}_\infty$ are given by

$$\mathcal{R}_i|_{\mathcal{D}_\infty} \simeq \mathcal{L}_2^{\otimes i} \quad \text{for even } k \quad \text{and} \quad \mathcal{R}_i|_{\mathcal{D}_\infty} \simeq \mathcal{L}_2^{\otimes i \frac{k+1}{2}} \quad \text{for odd } k.$$ 

Proof. By using Equation (3.23) we get

$$\mathcal{R}_i|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathfrak{X}_k}(\mathcal{D}_0 - (i - 1) \mathcal{D}_k + (i - 2) \mathcal{D}_k)|_{\mathcal{D}_\infty} \quad \text{for } i = 1, \ldots, k - 2,$$

$$\mathcal{R}_{k-1}|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathfrak{X}_k}(\mathcal{D}_k - \mathcal{D}_k)|_{\mathcal{D}_\infty}.$$ 

The result now follows from Lemma 3.31. \hfill $\square$

Finally, we need to relate the Picard groups of $\mathfrak{D}_\infty$ and $\mathcal{D}_\infty$, in particular making the map $\tilde{\phi}_k$ in the diagram (3.30) explicit. At this stage, from the commutativity of the diagram we only know that $\tilde{\phi}_k^* \mathcal{O}_\mathfrak{X}_k(\mathcal{D}_\infty)|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathfrak{X}_k}(k \mathcal{D}_\infty)|_{\mathcal{D}_\infty} \simeq \mathcal{L}_1^{\otimes k}$.

Proposition 3.35. One has isomorphisms

$$\tilde{\phi}_k^* \mathcal{O}_\mathfrak{X}_k(\tilde{p}_0) \simeq \mathcal{O}_\mathcal{D}_\infty(p_0) \quad \text{and} \quad \tilde{\phi}_k^* \mathcal{O}_\mathfrak{X}_k(\tilde{p}_\infty) \simeq \mathcal{O}_\mathcal{D}_\infty(p_\infty).$$

For odd $k$ we have $\tilde{\phi}_k^* \mathcal{O}_\mathfrak{X}_k(\tilde{p}) \simeq \mathcal{L}_2$ while for even $k$ we have $\tilde{\phi}_k^* \mathcal{O}_\mathfrak{X}_k(\tilde{p}) \simeq \mathcal{L}_2^{\otimes 2}$. 

The result now follows by taking the images of the vectors $(1, 0), (1, 1), (0, 1)$ in $\text{Pic}(\mathcal{D}_{\infty}) \simeq \mathbb{Z} \oplus \mathbb{Z}_k$ which correspond respectively to the line bundles $\mathcal{O}_{\mathcal{D}_\infty}(\mathcal{E}_0), \mathcal{O}_{\mathcal{D}_\infty}(\mathcal{E}_0), \mathcal{O}_{\mathcal{D}_\infty}(\mathcal{E}_0)$. □

Remark 3.37.

- For odd $k$, since $\tilde{\phi}_k^* \mathcal{O}_{\mathcal{D}_\infty}(\mathcal{E}_0) \simeq \mathcal{L}_2$, the line bundle $\tilde{\phi}_k^* \mathcal{O}_{\mathcal{D}_\infty}(\mathcal{E}_0)$ generates the torsion part of the Picard group $\text{Pic}(\mathcal{D}_{\infty})$ of $\mathcal{D}_{\infty}$. For even $k$ this is not true since $\tilde{\phi}_k^* \mathcal{O}_{\mathcal{D}_\infty}(\mathcal{E}_0) \simeq \mathcal{L}^{1 \oplus 2}$ is not sufficient to generate the torsion part of $\text{Pic}(\mathcal{D}_{\infty})$.
- Following the proof of [35, Proposition 7.20], the last short exact sequence in the diagram (3.36) is an element of $\text{Ext}^1(N_{\mathcal{D}_\infty}, \text{Pic}(\mathcal{D}_{\infty}))$, which by [35, Proposition 6.9] induces an element $[\mathcal{O}_{\mathcal{D}_{\infty}}(\mathcal{E}_0), \mathcal{O}_{\mathcal{D}_{\infty}}] \in \text{Pic}(\mathcal{D}_{\infty})/k \text{Pic}(\mathcal{D}_{\infty})$. The last column of the diagram is a free (hence projective) resolution of $\mathbb{Z}_k$, so we can lift the identity map of $\mathbb{Z}_k$ to obtain a morphism of short exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{D}_{\infty} & \rightarrow & \mathcal{D}_{\infty} & \rightarrow & 0
\end{array}
\]

The choices of liftings $\tilde{f}$ and $f$ are not unique. In particular the choice of $\tilde{f}$ corresponds to a choice of a line bundle in the class $[\mathcal{O}_{\mathcal{D}_{\infty}}(\mathcal{E}_0)] \in \text{Pic}(\mathcal{D}_{\infty})/k \text{Pic}(\mathcal{D}_{\infty})$, while the choice of $f$ is equivalent to a choice of a line bundle in the class $[\mathcal{O}_{\mathcal{D}_{\infty}}(\mathcal{E}_0)] \in \text{Pic}(\mathcal{D}_{\infty})/k \text{Pic}(\mathcal{D}_{\infty})$. Note that by choosing $\mathcal{O}_{\mathcal{D}_{\infty}}(\mathcal{E}_0)$ and $\mathcal{O}_{\mathcal{D}_{\infty}}(\mathcal{E}_0)$, respectively, the diagram (3.38) coincides with the diagram (3.30). △

We conclude this section by computing the degree, i.e., the integral of the first Chern class, of line bundles on $\mathcal{D}_{\infty}$. This result will be used in the Section 4.
Lemma 3.39. The degree of a line bundle $L = L_1^a \otimes L_2^b$ on $D_\infty$ with $a, b \in \mathbb{Z}$ is given by

$$\int_{D_\infty} c_1(L) = \frac{a}{kk^2},$$

where by $\int_{D_\infty} : A^*(D_\infty)_Q \to Q$ we denote the pushforward $A^*(D_\infty)_Q \to A^*(\text{Spec}(\mathbb{C}))_Q \simeq \mathbb{Q}$.

Proof. First observe that for any $a, b \in \mathbb{Z}$ we have

$$L^\otimes k_k \simeq (L_1^a \otimes L_2^b)^{\otimes k} \simeq L_1^{\otimes k a} L_2^{\otimes k b} \simeq \mathcal{O}_{D_\infty}(akp_\infty).$$

Since $D_\infty$ is smooth, by [93, Proposition 6.1] the structure map $r_k : D_\infty \to D_\infty$ induces an isomorphism $r_k^* : A^*(D_\infty)_Q \to A^*(D_\infty)_Q \simeq A^*(\mathbb{P}^1)_Q$, and therefore

$$\int_{D_\infty} c_1(L^\otimes k_k) = \int_{D_\infty} c_1(\mathcal{O}_{D_\infty}(akp_\infty)) = \int_{D_\infty} r_k^* c_1(\mathcal{O}_{D_\infty}(akp_\infty)).$$

By [93, Example 6.7], we obtain $r_k[p_\infty] = \frac{1}{d_\infty} [\infty]$, where $d_\infty$ is the order of the stabilizer of the point $p_\infty$. By using the quotient presentation of $D_\infty$ in Proposition 3.27, we find $d_\infty = k \tilde{k}$ and hence

$$\int_{D_\infty} c_1(L) = \frac{1}{k \tilde{k}} \int_{\mathbb{P}^1} \frac{1}{k \tilde{k}} c_1(\mathcal{O}_{\mathbb{P}^1}(ak)) = \frac{a}{kk^2}.$$

\[\square\]

4. Moduli spaces of framed sheaves

4.1. Preliminaries on framed sheaves. We begin by introducing some notation. Given a vector $\bar{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{Z}^{k-1}$, we denote by $\mathcal{R}^{\bar{u}}$ the line bundle $\bigotimes_{i=1}^{k-1} R_i^{u_i}$, and by $\mathcal{R}_0$ the trivial line bundle $\mathcal{O}_{\mathcal{X}_k}$.

Let us fix $s \in \mathbb{Z}$. For $i = 0, 1, \ldots, k-1$ define the line bundles

$$\mathcal{O}_{D_\infty}(s, i) := \begin{cases} L_1^{\otimes s} \otimes L_2^{\otimes i} & \text{for even } k, \\ L_1^{\otimes s} \otimes L_2^{\otimes \frac{i+1}{2}} & \text{for odd } k. \end{cases}$$

In addition, let us fix $\bar{w} := (w_0, w_1, \ldots, w_{k-1}) \in \mathbb{N}^k$ and $r := \sum_{i=0}^{k-1} w_i$. Define

$$\mathcal{F}^{s, \bar{w}} := \bigoplus_{i=0}^{k-1} \mathcal{O}_{D_\infty}(s, i)^{\otimes w_i}.$$

We shall call $\mathcal{F}^{s, \bar{w}}$ a framing sheaf. It is a locally free sheaf on $\mathcal{D}_\infty$ of degree $\frac{sr}{kk^2}$ (cf. Lemma 3.39).

Remark 4.1. The line bundles $\mathcal{O}_{D_\infty}(0, j)$ are endowed with a unitary flat connection associated with the $j$-th irreducible unitary representation $\rho_j$ of $\mathbb{Z}_k$ for $j = 0, \ldots, k-1$ (cf. [34, Remark 6.5]). So the framing sheaf $\mathcal{F}^{s, \bar{w}}$ has a unitary flat connection associated with the unitary representation $\bigoplus_{j=0}^{k-1} \rho_j^{\otimes w_j}$.

Definition 4.2. A $(\mathcal{D}_\infty, \mathcal{F}^{s, \bar{w}})$-framed sheaf on $\mathcal{X}_k$ is a pair $(\mathcal{E}, \phi_{\mathcal{E}})$, where $\mathcal{E}$ is a torsion-free sheaf on $\mathcal{X}_k$, locally free in a neighbourhood of $\mathcal{D}_\infty$, and $\phi_{\mathcal{E}} : \mathcal{E}|_{\mathcal{D}_\infty} \to \mathcal{F}^{s, \bar{w}}$ is an isomorphism. We call $\phi_{\mathcal{E}}$ a framing of $\mathcal{E}$. A morphism between $(\mathcal{D}_\infty, \mathcal{F}^{s, \bar{w}})$-framed sheaves $(\mathcal{E}, \phi_{\mathcal{E}})$ and $(\mathcal{G}, \phi_{\mathcal{G}})$ is a morphism $f : \mathcal{E} \to \mathcal{G}$ such that $\phi_{\mathcal{G}} \circ f|_{\mathcal{D}_\infty} = \phi_{\mathcal{E}}$. \(\square\)
Remark 4.3. Let \( \mathcal{X} \) be a toric Deligne-Mumford stack with coarse moduli space a projective toric variety \( \pi: \mathcal{X} \to X \). The rank of a coherent sheaf \( \mathcal{E} \) on \( \mathcal{X} \) is the degree zero part of the Chern character \( \text{ch}(\mathcal{E}) \) of its class \([\mathcal{E}]\) in the K-theory group \( K(\mathcal{X}) \), which is isomorphic to the Grothendieck group generated by the locally free sheaves on \( \mathcal{X} \) because \( \mathcal{X} \) is projective and smooth. Roughly speaking\(^6\), the pushforward \( \pi_* \) preserves the rank of the invariant part of a sheaf \( \mathcal{E} \) with respect to the action of the generic stabilizer of \( \mathcal{X} \) on it. If \( \mathcal{X} = \mathcal{X}_k \), then since \( \mathcal{X}_k \) is an orbifold, its generic stabilizer is trivial and the rank of a coherent sheaf \( \mathcal{E} \) coincides with the rank of its pushforward \( \pi_k(\mathcal{E}) \). By applying the same argument to \( \mathcal{D}_\infty \), one finds that the rank of a coherent sheaf \( \mathcal{F} \) on \( \mathcal{D}_\infty \) does not coincide in general with the rank of \( \pi_k(\mathcal{F}) \), and indeed the latter is only the rank of \( \pi_k(\mathcal{F}) \) where \( \mathcal{F} = \mathcal{F}_k \) on \( \mathcal{D}_\infty \) with respect to the action of \( \mu_k \) \([76, Section 3.2]\). Since the K-theory groups \( K(\mathcal{X}_k) \) and \( K(\mathcal{D}_\infty) \) are both generated by line bundles (cf. \[17, Sections 4 and 6]\), the rank is preserved under the restriction to \( \mathcal{D}_\infty \). In particular, if \( (\mathcal{E}, \phi_{\mathcal{E}}) \) is a \( (\mathcal{D}_\infty, \mathcal{F}_k, \mathcal{D}_\infty) \)-framed sheaf on \( \mathcal{X}_k \), we get \( \text{rk}(\mathcal{E}) = \text{rk}(\mathcal{F}_k) = r \).

Moreover, the Picard group of \( \mathcal{X}_k \) is isomorphic to its second singular cohomology group with integral coefficients via the first Chern class map \([47, Section 3.1.2]\).\(^7\) Thus fixing the determinant line bundle of a coherent sheaf \( \mathcal{E} \) on \( \mathcal{X}_k \) is equivalent to fixing its first Chern class.

**Lemma 4.4.** Let \( (\mathcal{E}, \phi_{\mathcal{E}}) \) be a \( (\mathcal{D}_\infty, \mathcal{F}_k, \mathcal{D}_\infty) \)-framed sheaf on \( \mathcal{X}_k \). Then the determinant \( \det(\mathcal{E}) \) of \( \mathcal{E} \) is of the form \( R_{u}^k \otimes \mathcal{O}_{\mathcal{X}_k}(s \cdot r \cdot \mathcal{D}_\infty) \), where the vector \( u \in \mathbb{Z}^{k-1} \) satisfies the condition

\[
\sum_{j=1}^{k-1} j u_j = \sum_{i=0}^{k-1} i w_i \mod k.
\]

**(4.5)**

**Proof.** Let \( \det(\mathcal{E}) = \bigotimes_{j=1}^{k-1} R_{u_j}^j \otimes \mathcal{O}_{\mathcal{X}_k}(u \cdot \mathcal{D}_\infty) \) be the determinant line bundle of \( \mathcal{E} \) for some integer \( u \) and some vector \( \bar{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{Z}^{k-1} \). Since \( \det(\mathcal{F}_k) \simeq \det(\mathcal{E}_{\mid \mathcal{D}_\infty}) \), we get

\[
\bigotimes_{i=0}^{k-1} \mathcal{O}_{\mathcal{X}_k}(s, i) \otimes w_i \simeq \bigotimes_{j=1}^{k-1} R_{u_j}^j \otimes \mathcal{O}_{\mathcal{X}_k}(u \cdot \mathcal{D}_\infty)_{\mid \mathcal{D}_\infty}.
\]

By Corollary 3.34 we have \( \mathcal{R}_{i \cdot \mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(0, i) \) for \( i = 1, \ldots, k-1 \) and \( \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)_{\mid \mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(1, 0) \), hence we get the assertion. \( \square \)

**Remark 4.6.** Set \( \bar{u} := C^{-1} \bar{u} \). Then Equation (4.5) implies the relation

\[
k v_j = -j \sum_{i=0}^{k-1} i w_i \mod k
\]

for \( j = 1, \ldots, k-1 \). Note also that for \( i \in \{1, \ldots, k-1\} \) a component \( v_i \) is integral if and only if every component is. Let \( c \in \{0, 1, \ldots, k-1\} \) be the equivalence class modulo \( k \) of \( \sum_{i=0}^{k-1} i w_i \) and define \( \bar{v} := C^{-1} \bar{e}_c - \bar{u} \) if \( c > 0 \), otherwise \( \bar{v} := -\bar{u} \). Then \( \bar{v} \in \mathbb{Z}^{k-1} \). We identify \( \bar{v} \) with an element of the root lattice \( \Omega \) (cf. Remark 3.26). \( \square \)

Now we are ready to apply the machinery developed in \([21]\) to construct moduli spaces of framed sheaves on root toric orbifolds. In particular, note that \( k D_\infty \) is a big and nef Cartier divisor, which contains the singular points of \( \mathcal{X}_k \); the line bundle \( \mathcal{O}_{\mathcal{X}_k}(k D_\infty) \) is a \( \pi_k^{\text{can}} \)-ample sheaf (cf. Remark 3.18). Moreover, since \( \mathcal{F}_k \) is given as direct sum lines of bundles of the same degree, it is a good framing sheaf \([21, Definition 5.7]\). Then by \([21, Theorem 6.2]\) there exists a fine moduli space \( M_{r, u, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_k, \mathcal{D}_\infty) \)

\(^6\)A more precise discussion involves \( K(\mathcal{X}(\mathcal{X})) \) and the Töen-Riemann-Roch theorem \([21, Remark 2.28]\).

\(^7\)This result is a generalization of the analogous result for toric varieties \([29, Theorem 12.3.2]\).
parametrizing isomorphism classes of \((\mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{u,i})\)-framed sheaves \((\mathcal{E}, \phi_{\mathcal{E}})\) on \(\mathcal{X}_k\), where \(\mathcal{E}\) is a torsion-free sheaf of rank \(r\), determinant \(R_{\mathcal{X}_k} \otimes O_{\mathcal{X}_k}(s r \mathcal{D}_{\infty})\), and discriminant
\[
\Delta := \Delta(\mathcal{E}) = \int_{\mathcal{X}_k} \left( c_2(\mathcal{E}) - \frac{r - 1}{2r} c_1^2(\mathcal{E}) \right),
\]
and \(u \in \mathbb{Z}^{k-1}\) satisfies Equation (4.5).

**Remark 4.7.** By “fine” one means that there exists a universal framed sheaf \((\mathcal{E}, \phi_{\mathcal{E}})\), where \(\mathcal{E}\) is a coherent sheaf on \(\mathcal{X}_k \times \mathcal{M}_{r, u, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{u,i})\) which is flat over \(\mathcal{M}_{r, u, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{u,i})\), and \(\phi_{\mathcal{E}}: \mathcal{E} \to p^*_s(\mathcal{D}_{\infty}^{u,i})\) is a morphism such that \((\phi_{\mathcal{E}})|_{\mathcal{D}_{\infty} \times \mathcal{M}_{r, u, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{u,i})}\) is an isomorphism; the fibre over \([E, \phi_{\mathcal{E}}]\) is \(\mathcal{M}_{r, u, \Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{u,i})\) is itself the \((\mathcal{D}_{\infty}, \mathcal{F}_{\infty}^{u,i})\)-framed sheaf \((\mathcal{E}, \phi_{\mathcal{E}})\) on \(\mathcal{X}_k\). In the following we shall call \(\mathcal{E}\) a universal sheaf.

### 4.2. Smoothness of the moduli space.

**Proposition 4.8.** For any \(s \in \mathbb{Z}\) and \(i = 0, 1, \ldots, k - 1\), the pushforward \(r_{k*}(O_{\mathcal{D}_{\infty}}(s, i))\) of \(O_{\mathcal{D}_{\infty}}(s, i)\) is given by:

1. For even \(k\):
   \[
r_{k*}(O_{\mathcal{D}_{\infty}}(s, i)) = 0 \text{ if } s \neq 0 \mod k \quad \text{for even } k , \quad s = 0 \mod k \quad \text{for odd } k .
   \]

2. Otherwise
   \[
r_{k*}(O_{\mathcal{D}_{\infty}}(s, i)) \simeq \begin{cases} O_{\mathbb{P}^1}(\left\lfloor \frac{s + i + k}{k} \right\rfloor) + O_{\mathcal{D}_{\infty}}(\left\lfloor \frac{s - i}{k} \right\rfloor) & \text{for even } k , \\ O_{\mathbb{P}^1}(\left\lfloor \frac{1 + i + k}{k} \right\rfloor) + O_{\mathcal{D}_{\infty}}(\left\lfloor \frac{1}{k} \right\rfloor) + O_{\mathcal{D}_{\infty}}(\left\lfloor \frac{i + k}{k} \right\rfloor) & \text{for odd } k . \end{cases}
   \]

**Proof.** Let \(s \in \mathbb{Z}\) and \(i = 0, 1, \ldots, k - 1\). First recall that the banding group of the gerbe \(\tilde{\phi}_k: \mathcal{D}_{\infty} \to \mathcal{D}_{\infty}\) is \(\mu_k\), which fits into the exact sequence given by:

- For even \(k\):
  \[
  1 \longrightarrow \mu_k \overset{i_{ev}}{\longrightarrow} \mathbb{C}^* \times \mu_k \overset{q_{ev}}{\longrightarrow} \mathbb{C}^* \times \mu_k \longrightarrow 1 ,
  \]
  where \(i_{ev}: \eta \mapsto (\eta, \eta^k)\) and \(q_{ev}: (t, \omega) \mapsto (t^k \omega^{-1}, \omega^2)\);

- For odd \(k\):
  \[
  1 \longrightarrow \mu_k \overset{i_{odd}}{\longrightarrow} \mathbb{C}^* \times \mu_k \overset{q_{odd}}{\longrightarrow} \mathbb{C}^* \times \mu_k \longrightarrow 1 ,
  \]
  where \(i_{odd}: \eta \mapsto (\eta, 1)\) and \(q_{odd}: (t, \omega) \mapsto (t^k \omega^{-1}, \omega)\).

Any coherent sheaf on \(\mathcal{D}_{\infty}\) decomposes as a direct sum of eigensheaves with respect to the characters of \(\mu_k\). The pushforward of \(\tilde{\phi}_k\) preserves only the \(\mu_k\)-invariant part of a coherent sheaf on \(\mathcal{D}_{\infty}\). In view of the previous exact sequences we find that the pushforward \(\tilde{\phi}_{k*}(O_{\mathcal{D}_{\infty}}(s, i))\) is nonzero if and only if Equations (4.9)–(4.10) are satisfied. For even \(k\), and for \(s\) and \(i\) satisfying Equation (4.9), we get
\[
O_{\mathcal{D}_{\infty}}(s, i) \simeq \tilde{\phi}_{k*}\left( O_{\mathcal{D}_{\infty}}\left( \frac{s+i}{k} \tilde{p}_0 \right) \otimes O_{\mathcal{D}_{\infty}}\left( \frac{s+i-1}{k} \tilde{p}_{\infty} \right) \right) .
\]
By the projection formula, which holds for the rigidification morphism \(\tilde{\phi}_k\) [92], we thus have
\[
\tilde{\phi}_{k*}(O_{\mathcal{D}_{\infty}}(s, i)) \simeq O_{\mathcal{D}_{\infty}}\left( \frac{s+i}{k} \tilde{p}_0 \right) \otimes O_{\mathcal{D}_{\infty}}\left( \frac{s+i-1}{k} \tilde{p}_{\infty} \right) .
\]
Recall that \(\mathcal{D}_{\infty}\) is obtained from \(D_{\infty}\) by performing a \((\tilde{k}, \tilde{k})\)-root construction at the points \(0, \infty \in D_{\infty} = \mathbb{P}^1\). The asserted result for \(r_{k*}(O_{\mathcal{D}_{\infty}}(s, i))\) now follows by Lemma 2.7. In the same way, for odd \(k\) and for \(s\) satisfying Equation (4.10), we get the asserted result for \(r_{k*}(O_{\mathcal{D}_{\infty}}(s, i))\).

\(\square\)
Corollary 4.11. For $i \in \{0, 1, \ldots, k-1\}$ and any negative integer $s$ we have

$$H^0(\mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty} (s, i)) = H^0(\mathbb{P}^1, r_k(\mathcal{O}_{\mathcal{D}_\infty} (s, i))) = 0.$$ 

Thanks to this corollary, we can argue exactly as in the proof of the analogous result for framed sheaves on smooth toric surfaces [40, Proposition 2.1] and obtain easily the following result. Note that the proof involves Serre duality for stacks as stated in [21, Theorem B.7].

**Proposition 4.12.** The group $\text{Ext}^1(\mathcal{E}, \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$ vanishes for $i = 0, 2$ and for any pairs consisting of a $(\mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$-framed sheaf $(\mathcal{E}, \phi_\mathcal{E})$ and a $(\mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$-framed sheaf $(\mathcal{E}', \phi_{\mathcal{E}'})$ on $\mathcal{X}_k$. If in addition $s = 0$ then $H^i(\mathcal{X}_k, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = 0$ for $i = 0, 2$.

**Theorem 4.13.** The moduli space $\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$ is a smooth quasi-projective variety of dimension

$$\dim_{\mathbb{C}}(\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})) = 2r \Delta \cdot \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} \bar{w} \cdot \bar{w}(j),$$

where for $j = 1, \ldots, k-1$ the vector $\bar{w}(j)$ is $(w_j, \ldots, w_{k-1}, w_0, w_1, \ldots, w_{j-1})$ and $C$ is the Cartan matrix of the Dynkin diagram of type $A_{k-1}$. Moreover, the Zariski tangent space of $\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$ at a point $[(\mathcal{E}, \phi_\mathcal{E})]$ is $\text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$.

**Proof.** The moduli space $\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$ is a separated quasi-projective scheme of finite type over $\mathbb{C}$ by [21, Theorem 6.2]. By [21, Theorem 4.17-(ii)], the group $\text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$ contains the obstruction to the smoothness of $\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$ at $[(\mathcal{E}, \phi_\mathcal{E})]$. By Proposition 4.12, the group $\text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$ vanishes for all points $[(\mathcal{E}, \phi_\mathcal{E})]$ of $\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$, and so $\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$ is everywhere smooth. Thus $\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$ is a smooth quasi-projective variety over $\mathbb{C}$.

By [21, Theorem 4.17-(i)], the Zariski tangent space of $\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$ at a point $[(\mathcal{E}, \phi_\mathcal{E})]$ is $\text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$. Hence

$$\dim_{\mathbb{C}}(\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})) = \dim_{\mathbb{C}}(\text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))),$$

and the latter dimension is computed in Appendix A (see Corollary A.2). \(\square\)

By [34, Theorem 6.9] the moduli space $\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$ contains as an open subset the moduli space of $U(r)$-instantons on $X_k$ with first Chern class $\sum_{i=1}^{k-1} u_i c_1(\mathcal{R}(\mathcal{X}_k))$, discriminant $\Delta$ and holonomy at infinity associated with the unitary representation $\bigoplus_{j=0}^{k-1} \rho_j^{u_j}$. Because of this fact, we state the following conjecture:

**Conjecture 4.14.** The moduli spaces $\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_n^{\text{fr}})$ are isomorphic to Nakajima quiver varieties $\mathcal{M}_{\xi_k}(\bar{s}, \bar{w})$ for a suitable choice of $\bar{s} \in \mathbb{N}^k$, where $\xi_k$ is the stability parameter of $X_k$.

As we shall see in the following section, in the rank one case our moduli spaces are isomorphic to Hilbert schemes of points on $X_k$ and the latter moduli spaces are Nakajima quiver varieties by [56].

4.3. **Rank one.** Let $\text{Hilb}^n(X_k)$ be the Hilbert scheme of $n$ points of $X_k$, i.e., the fine moduli space parameterizing zero-dimensional subschemes of $X_k$ of length $n$. It is a smooth quasi-projective variety of dimension $2n$. If $Z$ is a point of $\text{Hilb}^n(X_k)$, the pushforward $\iota_* I_Z$ of the ideal sheaf $I_Z$ with respect to the inclusion morphism $\iota: X_k \to \mathcal{X}_k$ is a rank one torsion-free sheaf on $\mathcal{X}_k$ with $\text{det}(\iota_* I_Z) \simeq \mathcal{O}_{\mathcal{X}_k}$ and $\int_{\mathcal{X}_k} c_2(\iota_* I_Z) = n$. The morphism $\iota$ induces an isomorphism $\iota: X_k \to \mathcal{X}_k \setminus \mathcal{D}_\infty$, hence $Z \subset X_k$ is disjoint from $\mathcal{D}_\infty$ so that $\iota_* I_Z$ is locally free in a neighbourhood of $\mathcal{D}_\infty$. 

Let $\vec{u} \in \mathbb{Z}^{k-1}$ and fix $i \in \{0, 1, \ldots, k-1\}$ such that
\[
i = \sum_{j=1}^{k-1} j u_j \mod k.
\]

Let $s \in \mathbb{Z}$. Then $E := \iota_s(I_{\mathbb{Z}}) \otimes \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{X_k}(s, \mathcal{D}_{\infty})$ is a rank one torsion-free sheaf on $\mathcal{X}_k$, locally free in a neighbourhood of $\mathcal{D}_{\infty}$, with a framing $\phi_E : E_{|\mathcal{D}_{\infty}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}_k}(s, i)$ induced canonically by the isomorphism $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{X_k}(s, \mathcal{D}_{\infty})_{|\mathcal{D}_{\infty}} \cong \mathcal{O}_{\mathcal{X}_k}(s, i)$ (cf. Corollary 3.34). So we get a $(\mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{X}_k}(s, i))$-framed sheaf $(E, \phi_E)$ on $\mathcal{X}_k$ (as the line bundle $\mathcal{O}_{\mathcal{X}_k}(s, i)$ coincides with $\mathcal{F}_{s, \vec{u}}$ for the vector $\vec{w}$ such that $w_i = 1$ and $w_j = 0$ for $j \neq i$). Moreover, $\det(E) \cong \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{X_k}(s, \mathcal{D}_{\infty})$ and
\[
\int_{\mathcal{X}_k} \text{cl}_2(E) = \frac{1}{2} \int_{\mathcal{X}_k} c_1(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{X_k}(s, \mathcal{D}_{\infty}))^2 - n.
\]
This singles out a point $[\mathcal{E}, \phi_E]$ in $M_{1, \vec{u}, n}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{X}_k}(s, i))$, so that an injective morphism of fine moduli spaces
\[
i_{(1, \vec{u}, n)} : \text{Hilb}^n(X_k) \rightarrow M_{1, \vec{u}, n}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{X}_k}(s, i)) \tag{4.15}
\]

is defined. This argument extends straightforwardly to families of zero-dimensional subschemes of $X_k$ of length $n$.

**Proposition 4.16.** The inclusion morphism (4.15) is an isomorphism of fine moduli spaces.

**Proof.** We can define an inverse morphism $j_{(1, \vec{u}, n)} : M_{1, \vec{u}, n}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{X}_k}(s, i)) \rightarrow \text{Hilb}^n(X_k)$ in the following way. Let $[\mathcal{E}, \phi_E]$ be a point in $M_{1, \vec{u}, n}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{X}_k}(s, i))$. The torsion-free sheaf $E$ fits into the exact sequence
\[
0 \rightarrow E \rightarrow E^{\vee \vee} \rightarrow Q \rightarrow 0,
\]
where $E^{\vee \vee}$ is the line bundle $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{X_k}(s, \mathcal{D}_{\infty})$ and $Q$ is a zero-dimensional sheaf whose support has length $n$. Since $E$ is locally free in a neighbourhood of $\mathcal{D}_{\infty}$, the support of $Q$ is disjoint from $\mathcal{D}_{\infty}$. Hence the quotient
\[
\mathcal{O}_{\mathcal{X}_k} \cong E^{\vee \vee} \otimes \mathcal{R}^{-\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-s, \mathcal{D}_{\infty}) \rightarrow Q \otimes \mathcal{R}^{-\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-s, \mathcal{D}_{\infty}) \rightarrow 0
\]
defines a zero-dimensional subscheme $Z \subset \mathcal{X}_k$ of length $n$ which is disjoint from $\mathcal{D}_{\infty}$, and the quotient $O_{X_k} \rightarrow \iota^*(O_Z) \rightarrow 0$ defines a point $Z \in \text{Hilb}^n(X_k)$ with $E \cong \iota_s(I_{\mathbb{Z}}) \otimes \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{X_k}(s, \mathcal{D}_{\infty})$. It is easy to see that this argument can be generalized to families of framed sheaves. Moreover, one has $i_{(1, \vec{u}, n)} \circ j_{(1, \vec{u}, n)} = \text{id}$ and $j_{(1, \vec{u}, n)} \circ i_{(1, \vec{u}, n)} = \text{id}$. \hfill \QED

**Remark 4.17.** A consequence of this proposition is that after fixing $i \in \{0, 1, \ldots, k-1\}$, a vector $\vec{u} \in \mathbb{Z}^{k-1}$ such that $\sum_{j=1}^{k-1} j u_j = i \mod k$, and an integer $s \in \mathbb{Z}$, for any $(\mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{X}_k}(s, i))$-framed sheaf $(E, \phi_E)$ of rank one on $\mathcal{X}_k$ the torsion-free sheaf $E$ is isomorphic to $\iota_s(I_{\mathbb{Z}}) \otimes \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{X_k}(s, \mathcal{D}_{\infty})$, where $E$ is the ideal sheaf of some zero-dimensional subscheme of $X_k$, and $\phi_E$ is canonically induced by the isomorphism $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{X_k}(s, \mathcal{D}_{\infty})_{|\mathcal{D}_{\infty}} \cong \mathcal{O}_{\mathcal{X}_k}(s, i)$. \hfill $\triangle$

Since $i_{(1, \vec{u}, n)}$ is an isomorphism between fine moduli spaces, we also obtain an isomorphism between the corresponding universal objects. Let us denote by $Z \subset \text{Hilb}^n(X_k) \times X_k$ the universal subscheme of $\text{Hilb}^n(X_k)$, whose fibre over $Z \in \text{Hilb}^n(X_k)$ is the zero-dimensional subscheme $Z$ itself. Consider the commutative diagram
\[
\begin{array}{ccc}
\text{Hilb}^n(X_k) \times X_k & \xrightarrow{i_{(1, \vec{u}, n)} \times \text{id}} & M_{1, \vec{u}, n}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{X}_k}(s, i)) \times \mathcal{X}_k \\
\downarrow & & \downarrow \\
\text{Hilb}^n(X_k) & \xrightarrow{i_{(1, \vec{u}, n)}} & M_{1, \vec{u}, n}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{O}_{\mathcal{X}_k}(s, i))
\end{array}
\]
Then \((t(1,\bar{u},n), t)^*sE\) is the ideal sheaf of \(Z\) and \((t(1,\bar{u},n), t)^*s\phi_E = 0\), where \((E, \phi_E)\) is the universal framed sheaf on \(M_{1,\bar{u},n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{X}_k}(s, t)) \times \mathcal{X}_k\) introduced in Remark 4.7.

### 4.4. Natural bundle

In this subsection we set \(s = 0\). The natural bundle \(V\) on \(M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}})\) is defined in terms of the universal sheaf \(E\) as

\[
V := R^1p_*(E \otimes p_{\mathcal{X}_k}^*(\mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))),
\]

where \(p: M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}}) \times \mathcal{X}_k \rightarrow M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}})\) is the projection.

**Proposition 4.18.** \(V\) is a locally free sheaf on \(M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}})\) of rank

\[
\text{rk}(V) = \Delta + \frac{1}{2r} \bar{u} \cdot C\bar{u} - \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} w_j
\]

where \(\bar{u} := C^{-1}\bar{u}\).

**Proof.** First note that

\[
\text{id} \times \pi_k : M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}}) \times \mathcal{X}_k \rightarrow M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}}) \times \mathcal{X}_k
\]

is the coarse moduli space of \(M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}}) \times \mathcal{X}_k\). Moreover, the projection morphism

\[
M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}}) \times \mathcal{X}_k \rightarrow M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}})
\]

is proper. Since the stack \(M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}}) \times \mathcal{X}_k\) is tame over \(M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}})\), we can apply the cohomology and base change theorem [76, Theorem 1.7]. By using the vanishing results of Proposition 4.12 one can prove that \(V\) is a locally free sheaf of rank \(\text{dim}_{\mathbb{C}} H^1(\mathcal{X}_k, E \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))\), where \(\mathcal{E}\) is the underlying sheaf of a point \([(\mathcal{E}, \phi_E)] \in M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}})\). The computation of this dimension is given in Appendix A (see Theorem A.1).

### 4.5. Carlsson-Okounkov bundle

In this subsection we introduce the Carlsson-Okounkov bundle for the moduli spaces \(M_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}})\), generalizing the definition given in [25] for the rank one case. Let

\[
\mathcal{E}_i := p_i\beta^*(\mathcal{E}) \in K(\mathcal{M}_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}}) \times \mathcal{M}_{r',\bar{u}',\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}'}) \times \mathcal{X}_k)
\]

for \(i = 1, 2\), where \(p_{ij}\) is the projection of \(\mathcal{M}_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}}) \times \mathcal{M}_{r',\bar{u}',\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}'}) \times \mathcal{X}_k\) on the product of the \(i\)-th and \(j\)-th factors. Denote by \(p_3\) the projection of the same product onto \(\mathcal{X}_k\).

**Definition 4.19.** The Carlsson-Okounkov bundle is the element

\[
\mathcal{E} := p_{123}(-\mathcal{E}_1 \cdot \mathcal{E}_2 \cdot p_3\beta^*(\mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)))
\]

in the K-theory \(K(\mathcal{M}_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}}) \times \mathcal{M}_{r',\bar{u}',\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\mathcal{X}_k}^{0,\bar{u}'})).\)

The fibre of \(\mathcal{E}\) over a pair of points \([(\mathcal{E}, \phi_E)], [(\mathcal{E}', \phi_{E'})]\) is

\[
\mathcal{E}_{(([(\mathcal{E}, \phi_E)], [(\mathcal{E}', \phi_{E'})]))} = -\chi_{\mathcal{X}_k}(\mathcal{E}, \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)),
\]

(4.20)

where for any pair of coherent sheaves \(\mathcal{F}\) and \(\mathcal{G}\) on \(\mathcal{X}_k\) we set \(\chi_{\mathcal{X}_k}(\mathcal{F}, \mathcal{G}) := \sum_i (-1)^i \text{Ext}^i(\mathcal{F}, \mathcal{G})\). By Proposition 4.12 and the fact that all the Ext groups vanish in degree greater than two, the fibre (4.20) reduces to

\[
\mathcal{E}_{(([(\mathcal{E}, \phi_E)], [(\mathcal{E}', \phi_{E'})]))} = \text{Ext}^1(\mathcal{E}, \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)),
\]

(4.20)
Remark 4.21. Let us take $M_{r, \vec{u}, \Delta} (\mathcal{X}_k, \mathcal{D}_\infty, F^{s, \vec{a}})$ to be the fibre of $E$ at a point $[([\mathcal{E}, \phi_E]), ([\mathcal{E}, \phi_E])]$ is isomorphic to the tangent space of $M_{r, \vec{u}, \Delta} (\mathcal{X}_k, \mathcal{D}_\infty, F^{s, \vec{a}})$ at $[([\mathcal{E}, \phi_E]), ([\mathcal{E}, \phi_E])]$. On the other hand, when $s = 0$ the pushforward of $E$ with respect to the projection of $M_{r, \vec{u}, \Delta} (\mathcal{X}_k, \mathcal{D}_\infty, F^{s, \vec{a}}) \times M_{r, \vec{u}, \Delta} (\mathcal{X}_k, \mathcal{D}_\infty, F^{s, \vec{a}})$ to the second factor is isomorphic to the natural bundle $V$ of $M_{r, \vec{u}, \Delta} (\mathcal{X}_k, \mathcal{D}_\infty, F^{s, \vec{a}})$.

4.6. Torus action and fixed points. Let us begin by recalling some definitions which will be used in the combinatorial expressions below. Let $Y \subset \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ be a Young tableau, which we think of as sitting "in the first quadrant". Define the arm length and leg length of a box $s = (i, j) \in Y$ as $A(s) = A_{Y}(s) := \lambda_i - j$ and $L(s) = L_{Y}(s) := \lambda_j - i$ respectively, where $\lambda_i$ is the length of the $i$-th column of $Y$ and $\lambda_j$ is the length of the $j$-th row of $Y$. The arm colength and leg colength are given by $A'(s) = A_{Y}'(s) := j - 1$ and $L'(s) = L_{Y}'(s) := i - 1$, respectively. We also define the weight $\|Y\|$ of a Young tableau as the number of boxes $s \in Y$.

Let $T_\rho$ be the maximal torus of $GL(r, \mathbb{C})$ consisting of diagonal matrices and set $T := T_1 \times T_\rho$. We define an action of $T$ on $M_{r, \vec{u}, \Delta} (\mathcal{X}_k, \mathcal{D}_\infty, F^{s, \vec{a}})$ as follows. For any element $(t_1, t_2) \in T$, let $F_{(t_1, t_2)}$ be the automorphism of $\mathcal{X}_k$ induced by the torus action of $T_1$ on $\mathcal{X}_k$. Define an action of the torus $T = T_1 \times T_\rho$ on $M_{r, \vec{u}, \Delta} (\mathcal{X}_k, \mathcal{D}_\infty, F^{s, \vec{a}})$ by

$$(t_1, t_2, \rho) \triangleright [(\mathcal{E}, \phi_E)] := \left[\left( (F^{-1}_{(t_1, t_2)})^* (\mathcal{E}) , \phi_E \right) \right],$$

where $\rho = (\rho_1, \ldots, \rho_r) \in T_\rho$ and $\phi_E$ is the composition of isomorphisms

$$\phi_E : (F^{-1}_{(t_1, t_2)})^* \mathcal{E} \Rightarrow (F^{-1}_{(t_1, t_2)})^* F^{s, \vec{a}} \Rightarrow F^{s, \vec{a}} \Rightarrow \mathcal{F}^{s, \vec{a}};$$

here the middle arrow is given by the $T_1$-equivariant structure induced on $F^{s, \vec{a}}$ by restriction of the torus action of $\mathcal{X}_k$ to $\mathcal{D}_\infty$.

Proposition 4.22. A $T$-fixed point $[(\mathcal{E}, \phi_E)] \in M_{r, \vec{u}, \Delta} (\mathcal{X}_k, \mathcal{D}_\infty, F^{s, \vec{a}})^T$ decomposes as a direct sum of rank one framed sheaves

$$(\mathcal{E}, \phi_E) = \bigoplus_{\alpha=1}^r (\mathcal{E}_\alpha, \phi_\alpha),$$

where for $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^{i} w_j$ with $i = 0, 1, \ldots, k - 1$ we have that:

- $\mathcal{E}_\alpha$ is a tensor product $I_\alpha (I_{\alpha}) \otimes \mathcal{R}^{\vec{u}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k} (s, \mathcal{D}_\infty)$, where $I_{\alpha}$ is an ideal sheaf of a zero-dimensional subscheme $Z_\alpha$ of $\mathcal{X}_k$ supported at the $T_1$-fixed points $p_1, \ldots, p_k$ and $\vec{u}_\alpha \in \mathbb{Z}^{k-1}$ is such that

$$\sum_{j=1}^{k-1} j (\vec{u}_\alpha)_j = i \mod k;$$

(4.23)

- The framing $\phi_\alpha : \mathcal{E}_\alpha \sim \mathcal{O}_{\mathcal{D}_\infty} (s, i)$ is induced by the isomorphism $\mathcal{R}^{\vec{u}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k} (s, \mathcal{D}_\infty) \sim \mathcal{O}_{\mathcal{D}_\infty} (s, i)$.

Proof. By using the same arguments as in the proof of an analogous result for framed sheaves on smooth projective surfaces [20, Proposition 3.2], we obtain a decomposition

$$E = \bigoplus_{\alpha=1}^r \mathcal{E}_\alpha,$$
where each $E_\alpha$ is a $T$-invariant rank one torsion-free sheaf on $\mathcal{D}_k$. The restriction $\phi_{E|E_\alpha}$ gives a canonical framing to a direct summand of $F_\alpha^T$. By reordering the indices $\alpha$ if necessary, for $i = 0, 1, \ldots, k - 1$ and for each $\alpha$ such that $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^{i} w_j$ we get an induced framing on $E_\alpha$

$$\phi_\alpha := \phi_{E|E_\alpha} : E_\alpha \to \mathcal{O}_{F_\alpha}(s, i).$$

Thus $(E_\alpha, \phi_\alpha)$ is a $(\mathcal{D}_{\infty}, \mathcal{O}_{F_\alpha}(s, i))$-framed sheaf of rank one on $X_k$. As explained in Remark 4.17, the torsion-free sheaf $E_\alpha$ is the tensor product of an ideal sheaf $I_\alpha$ of a zero-dimensional subscheme $Z_\alpha$ of length $n_\alpha$ supported on $X_k$ and the line bundle $\mathcal{R}^{\alpha} \otimes \mathcal{O}_{F_\alpha}(s, \mathcal{D}_{\infty})$ for a vector $\vec{u}_\alpha \in \mathbb{Z}^{k-1}$ that satisfies Equation (4.23) in view of Lemma 4.4. Since the torsion-free sheaf $E$ is fixed by the $T_1$-action, $Z_\alpha$ is fixed as well and so it is supported at the $T_1$-fixed points $p_1, \ldots, p_k$. □

Let $[(\mathcal{E}, \phi_\mathcal{E})] = \left( \bigoplus_{\alpha=1}^{r} (E_\alpha, \phi_\alpha) \right)$ be a $T$-fixed point in $\mathcal{M}_{r, \vec{u}, \Delta}(X_k, \mathcal{D}_{\infty}, F_\alpha^T)$. Then

$$\mathcal{R}^{\alpha} \otimes \mathcal{O}_{F_\alpha}(s, \mathcal{D}_{\infty}) \simeq \det(\mathcal{E}) \simeq \bigotimes_{\alpha=1}^{r} \det(E_\alpha) \simeq \bigotimes_{\alpha=1}^{r} (\mathcal{R}^{\alpha} \otimes \mathcal{O}_{F_\alpha}(s, \mathcal{D}_{\infty})),
$$

hence $\sum_{\alpha=1}^{r} \vec{u}_\alpha = \vec{u}$. On the other hand, $I_\alpha$ is the ideal sheaf of a $T_1$-fixed zero-dimensional subscheme $Z_\alpha$ of length $n_\alpha$ for $\alpha \in \{1, \ldots, r\}$. Hence it is a disjoint union of zero-dimensional subschemes $Z_\alpha^{i}$ supported at the $T_1$-fixed points $p_i$ for $i = 1, \ldots, k$; each $Z_\alpha^{i}$ corresponds to a Young tableau $Y_\alpha^{i}$ [33], and $Z_\alpha$ corresponds to the set of Young tableaux $Y_\alpha = \{Y_\alpha^{i}\}_{i=1,\ldots,k}$ such that $\sum_{i=1}^{k} |Y_\alpha^{i}| = n_\alpha$.

Thus we can parametrize the point $[(\mathcal{E}, \phi_\mathcal{E})]$ by the pair $(\vec{Y}, \vec{u})$, where

- $\vec{Y} = (\vec{Y}_1, \ldots, \vec{Y}_r)$, where $\vec{Y}_\alpha = \{Y_\alpha^{i}\}_{i=1,\ldots,k}$ for any $\alpha = 1, \ldots, r$ is a set of Young tableaux such that $\sum_{i=1}^{k} |Y_\alpha^{i}| = n_\alpha$;
- $\vec{u} = (\vec{u}_1, \ldots, \vec{u}_r)$, where $\vec{u}_\alpha = \{(\vec{u}_\alpha)_1, \ldots, (\vec{u}_\alpha)_{k-1}\}$ for any $\alpha = 1, \ldots, r$ is an integer vector such that the relation (4.23) holds and $\sum_{\alpha=1}^{r} \vec{u}_\alpha = \vec{u}$.

If we set $\vec{v}_\alpha := C^{-1} \vec{u}_\alpha$ for $\alpha = 1, \ldots, r$, we denote the same point by $(\vec{Y}, \vec{v})$ where $\vec{v} = (\vec{v}_1, \ldots, \vec{v}_r)$. In this case any $\vec{v}_\alpha$ satisfies the relation

$$k(\vec{v}_\alpha)_{k-1} = i \mod k$$

if $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^{i} w_j$ for $i = 0, 1, \ldots, k - 1$. We shall call these pairs the **combinatorial data** of the torus-fixed point $[(\mathcal{E}, \phi_\mathcal{E})]$.

**Remark 4.24.** It is easy to see that

$$\int_{X_k} \chi_2(\mathcal{E}) = \sum_{\alpha=1}^{r} \int_{X_k} \chi_2 (\iota_\alpha I_\alpha \otimes \mathcal{R}^{\alpha} \otimes \mathcal{O}_{F_\alpha}(s, \mathcal{D}_{\infty}))$$

$$= \sum_{\alpha=1}^{r} \left( \int_{X_k} \frac{1}{2} c_1 \left( \mathcal{R}^{\alpha} \otimes \mathcal{O}_{F_\alpha}(s, \mathcal{D}_{\infty}) \right) \right)^2 - n_\alpha \right) = \sum_{\alpha=1}^{r} \left( \frac{s^2}{2k^2} - \frac{1}{2} \vec{v}_\alpha \cdot C \vec{v}_\alpha - n_\alpha \right)$$

$$= \frac{r s^2}{2k k^2} - \frac{1}{2} \sum_{\alpha=1}^{r} \vec{v}_\alpha \cdot C \vec{v}_\alpha - \frac{1}{2} \sum_{\alpha=1}^{r} n_\alpha \in \frac{1}{2k^2} \mathbb{Z}.$$

Then

$$\int_{X_k} c_2(\mathcal{E}) = \frac{r (r - 1) s^2}{2k k^2} + \sum_{\alpha=1}^{r} n_\alpha - \frac{1}{2} \sum_{\alpha \neq \beta} \vec{v}_\alpha \cdot C \vec{v}_\beta \in \frac{1}{2k^2} \mathbb{Z},$$

and therefore

$$\Delta = \sum_{\alpha=1}^{r} n_\alpha + \frac{1}{2r} \sum_{\alpha=1}^{r} \vec{v}_\alpha \cdot C \vec{v}_\alpha - \frac{1}{2r} \sum_{\alpha, \beta=1}^{r} \vec{v}_\alpha \cdot C \vec{v}_\beta \in \frac{1}{2r} \mathbb{Z}.$$
As a by-product, this computation shows that the discriminant of any \((\mathcal{D}_\infty, \mathcal{F}_{\infty, \alpha})\)-framed sheaf on \(\mathcal{X}_k\) takes values in \(\frac{1}{2\pi i} \mathbb{Z}\).

\[ \Delta \]

### 4.7 Euler classes

We introduce the equivariant parameters of the torus \(T = T_1 \times T_\rho\). For \(r = 1, \ldots, r\), let \(e_\alpha\) be the one-dimensional \(T_\rho\)-module corresponding to the projection \((\mathbb{C}^*)^r \to \mathbb{C}^*\) onto the \(\alpha\)-th factor and \(a_\alpha\) its equivariant first Chern class. The corresponding \(T_1\)-module parameters \(t_j\) and \(\varepsilon_j\) for \(j = 1, 2\) were introduced in Section 3.1. Then \(H_T(pt; \mathbb{Q}) = \mathbb{Q}[e_1, e_2, a_1, \ldots, a_r]\).

On \(\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty, \alpha}) \times \mathcal{M}_{r', \bar{u}', \Delta'}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty, \alpha'})\) there is a natural action of the extended torus \(\bar{T} = T_1 \times T_\rho \times T_\rho'\), which acts as \(T_1 \times T_\rho\) on the first factor (\(T_\rho\) acting trivially) and as \(T_1 \times T_\rho'\) on the second factor (\(T_\rho'\) acting trivially). We want to compute the character \(\text{ch}_T E([\mathcal{E}, \mathcal{F}_\alpha], [(\mathcal{E}', \mathcal{F}'_\alpha)])\) of the Carlsson-Okounkov bundle at a fixed point

\[
((\mathcal{E}, \mathcal{F}_\alpha), (\mathcal{E}', \mathcal{F}'_\alpha)) \in (\mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty, \alpha}) \times \mathcal{M}_{r', \bar{u}', \Delta'}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty, \alpha'}))^{\bar{T}}.
\]

Let \(((\bar{Y}, \bar{u}), (\bar{Y}', \bar{u}'))\) be the corresponding combinatorial data. Since the torsion-free sheaves \(\mathcal{E}\) and \(\mathcal{E}'\) decompose as

\[
\mathcal{E} = \bigoplus_{\alpha=1}^{r} t_\alpha(I_\alpha) \otimes \mathcal{R}_{\bar{u}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k}(s \mathcal{D}_\infty)
\]

and

\[
\mathcal{E}' = \bigoplus_{\beta=1}^{r'} t'_\beta(I'_\beta) \otimes \mathcal{R}_{\bar{u}'_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(s \mathcal{D}_\infty),
\]

we get

\[
\text{ch}_T E([\mathcal{E}, \mathcal{F}_\alpha], [(\mathcal{E}', \mathcal{F}'_\alpha)]) = \text{ch}_T \text{Ext}^1 (\mathcal{E}, \mathcal{E}' \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))
\]

\[= - \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} e_\beta e_\alpha^{-1} \text{ch}_T \text{Ext}^* (t_\alpha(I_\alpha) \otimes \mathcal{R}_{\bar{u}_\alpha} \otimes t'_\beta(I'_\beta) \otimes \mathcal{R}_{\bar{u}'_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))\,
\]

Let

\[
L_{\alpha, \beta}(t_1, t_2) := - \text{ch}_T \text{Ext}^* (\mathcal{R}_{\bar{u}_\alpha} \otimes \mathcal{R}_{\bar{u}'_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))
\]

\[
= - \chi_{T_1} (\mathcal{X}_k, \mathcal{R}_{\bar{u}'_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)),
\]

and

\[
M_{\alpha, \beta}(t_1, t_2) := \text{ch}_T \text{Ext}^* (\mathcal{R}_{\bar{u}_\alpha} \otimes \mathcal{R}_{\bar{u}'_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))
\]

\[- \text{ch}_T \text{Ext}^* (t_\alpha(I_\alpha) \otimes \mathcal{R}_{\bar{u}_\alpha} \otimes t'_\beta(I'_\beta) \otimes \mathcal{R}_{\bar{u}'_\beta} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)).
\]

Then

\[
\text{ch}_T E([\mathcal{E}, \mathcal{F}_\alpha], [(\mathcal{E}', \mathcal{F}'_\alpha)]) = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} e_\beta e_\alpha^{-1} (M_{\alpha, \beta}(t_1, t_2) + L_{\alpha, \beta}(t_1, t_2)).
\]

\[ (4.25) \]

Here and in the following we use an approach similar to that of [40], which computes the character of the tangent bundle at a fixed point of the moduli space of framed sheaves on a smooth projective toric surface, and so we shall borrow some of their terminology. In particular, we call \(M_{\alpha, \beta}(t_1, t_2)\) a **vertex contribution** to \(\text{ch}_T E([\mathcal{E}, \mathcal{F}_\alpha], [(\mathcal{E}', \mathcal{F}'_\alpha)])\) and \(L_{\alpha, \beta}(t_1, t_2)\) an **edge contribution** to \(\text{ch}_T E([\mathcal{E}, \mathcal{F}_\alpha], [(\mathcal{E}', \mathcal{F}'_\alpha)])\). The paper [40] uses this terminology because \(M_{\alpha, \beta}(t_1, t_2)\) will depend on the torus-fixed points \(p_i\) of \(X_k\) (which one can match to vertices of the toric graph) and \(L_{\alpha, \beta}(t_1, t_2)\) will depend on the torus-invariant divisors \(D_i\) of \(X_k\) (which one can match to edges of the toric graph such that an edge joins two vertices if and only if the corresponding fixed points lie in the divisor corresponding to the edge).
Edge contribution. By definition we have

\[ L_{\alpha\beta}(t_1, t_2) = -\chi T_\alpha \left( \mathcal{J}_k \bigotimes_{j=1}^{k-1} \mathcal{R}_j \right) - (u_\alpha) \otimes \mathcal{O}_\mathcal{X}_k (-\mathcal{D}_\infty) \right) . \]

The computation of this character is done in Appendix C.3.

Vertex contribution.

**Proposition 4.27.**

\[ M_{\alpha\beta}(t_1, t_2) = \sum_{i=1}^{k} (\chi_1)_{(i)} - (\vec{v}_\beta)_i + (\vec{v}_\alpha)_i (\chi_2)_{(i)} - (\vec{v}_\beta)_{i-1} + (\vec{v}_\alpha)_{i-1} \]  

\[ M_{Y^I, Y'}(x, y) := \sum_{s \in Y} x^{L_Y(s)} y^{A_Y(s)} + \sum_{s' \in Y'} x^{L_Y(s')} y^{A_Y(s')} . \]

Before proving this Proposition, we need some preliminary results. By Proposition 2.19 the open substack \( \mathcal{U}_i \) of \( \mathcal{X}_k \) corresponding to \( \sigma_i \) is \( [V_i/N(\sigma_i)] \simeq U_i \simeq \mathbb{C}^2 \) for \( i = 1, \ldots, k \), and the open substacks \( \mathcal{U}_{k+1}, \mathcal{U}_{k+2} \) of \( \mathcal{X}_k \) corresponding to \( \sigma_{\infty, j} \) for \( j = 0, k \) are \( [V_{k+1}/N(\sigma_{\infty, 0})], [V_{k+2}/N(\sigma_{\infty, k})] \simeq [\mathbb{C}^2/\mu_{k+2}] \). Set \( U = \bigsqcup_{i=1}^{k+2} V_i \). Since the morphisms \( U \to \bigsqcup_{i=1}^{k+2} \mathcal{U}_i \) and \( \bigsqcup_{i=1}^{k+2} \mathcal{U}_i \to \mathcal{X}_k \) are étale and surjective, the composition \( u: U \to \mathcal{X}_k \) is étale and surjective as well, hence the pair \( (U, u) \) is an étale presentation of \( \mathcal{X}_k \). Denote by \( U_\bullet \to \mathcal{X}_k \) the strictly simplicial algebraic space associated to the simplicial algebraic space which is obtained by taking the 0-coskeletal of \( (U, u) \) [80, Section 4.1]; for any \( n \geq 0 \) we have

\[ U_n = \bigsqcup_{i_0, i_1, \ldots, i_k \in \{1, \ldots, k+2\} \atop i_0 < i_1 < \cdots < i_k} V_{i_0} \times_{\mathcal{X}_k} V_{i_1} \times_{\mathcal{X}_k} \cdots \times_{\mathcal{X}_k} V_{i_k} . \]

By [80, Proposition 6.12], the category of coherent sheaves on \( \mathcal{X}_k \) is equivalent to the category of simplicial coherent sheaves on \( U_\bullet \) (see [80] for the definition of simplicial coherent sheaf on a strictly simplicial algebraic space).

**Proof of Proposition 4.27.** As explained in [80, Section 6], one has an isomorphism between the Ext groups on \( \mathcal{X}_k \) and the Ext groups of the corresponding simplicial coherent sheaves on \( U_\bullet \). Thus

\[ \text{Ext}^\bullet \left( R_\alpha^\bullet, R_\beta^\bullet \otimes \mathcal{O}_{\mathcal{X}_k} (-\mathcal{D}_\infty) \right) - \text{Ext}^\bullet \left( \iota_\ast(I_\alpha) \otimes R_\alpha^\bullet, \iota_\ast(I_\beta) \otimes R_\beta^\bullet \otimes \mathcal{O}_{\mathcal{X}_k} (-\mathcal{D}_\infty) \right) = \]

\[ \text{Ext}^\bullet \left( R_{U_\bullet}^\bullet, R_\alpha^\bullet \otimes \mathcal{O}_{\mathcal{X}_k} (-\mathcal{D}_\infty) \right) - \text{Ext}^\bullet \left( \iota_\ast(I_\alpha) \otimes R_{U_\bullet}^\bullet, \iota_\ast(I_\beta) \otimes R_\beta^\bullet \otimes \mathcal{O}_{\mathcal{X}_k} (-\mathcal{D}_\infty) \right) , \]

where for a coherent sheaf \( G \) on \( \mathcal{X}_k \) we denote by \( G_{U_\bullet} \) the corresponding simplicial coherent sheaf on \( U_\bullet \) [80, Proposition 6.12].

Recall that \( I_\alpha \) and \( I_\beta \) are ideal sheaves of zero-dimensional subschemes \( Z_\alpha \) and \( Z_\beta \) supported at the \( T_i \)-fixed points \( p_1, \ldots, p_k \) of \( \mathcal{X}_k \). Hence the restrictions of \( \iota_\ast I_\alpha \) and \( \iota_\ast I_\beta \) on \( \mathcal{U}_j \) are trivial for \( j = k + 1, k + 2 \). For the same reason, the restrictions of \( \iota_\ast I_\alpha \) and \( \iota_\ast I_\beta \) on \( \mathcal{U}_j \times_{\mathcal{X}_k} \mathcal{U}_j \) are also trivial since \( \mathcal{U}_i \times_{\mathcal{X}_k} \mathcal{U}_j \simeq U_i \cap U_j \) for \( i, j = 1, \ldots, k \). Then for pairwise distinct indices \( j_1, \ldots, j_i \in \{1, \ldots, k + 2\} \) we get

\[ \iota_\ast(I_\alpha|_{\mathcal{U}_{j_1} \times_{\mathcal{X}_k} \mathcal{U}_{j_2} \times_{\mathcal{X}_k} \cdots \times_{\mathcal{X}_k} \mathcal{U}_{j_i}} \simeq \mathcal{O}_{\mathcal{X}_k}|_{\mathcal{U}_{j_1} \times_{\mathcal{X}_k} \mathcal{U}_{j_2} \times_{\mathcal{X}_k} \cdots \times_{\mathcal{X}_k} \mathcal{U}_{j_i}} \simeq \iota_\ast(I_\beta|_{\mathcal{U}_{j_1} \times_{\mathcal{X}_k} \mathcal{U}_{j_2} \times_{\mathcal{X}_k} \cdots \times_{\mathcal{X}_k} \mathcal{U}_{j_i}} \right) \]

unless \( i = 1 \) and \( j_1 = 1, \ldots, k \). Then \( (\iota_\ast(I_\alpha|_{U_n}) \simeq \mathcal{O}_{U_n}|_{U_n} \simeq (\iota_\ast(I_\beta|_{U_n}) \mid_{U_n} \) for \( n \geq 1 \). So by using the local-to-global spectral sequence (which degenerates since \( U \) is a disjoint union of affine spaces) we find
Now we give the \( \tilde{Y} \) for two Young tableaux and define the set \( \mathcal{A}_{\alpha,\beta} := \{ \tilde{v}_{\beta \alpha} - \tilde{v}_{\beta \alpha}' \} \). Let \( a_{\beta \alpha} := a_{\beta} - a_{\alpha}, \tilde{v}_{\beta \alpha} := \tilde{v}_{\beta \alpha}' - \tilde{v}_{\alpha} \) and

\[
a_{\beta \alpha}^{(i)} := a_{\beta \alpha} - \langle \tilde{v}_{\beta \alpha} \rangle_{i} \varepsilon_{1} - \langle \tilde{v}_{\beta \alpha} \rangle_{i-1} \varepsilon_{2}^{(i)}
\]

for \( i = 1, \ldots, k \) and for \( \alpha = 1, \ldots, r, \beta = 1, \ldots, r' \) (we set \( \langle \tilde{v}_{\beta \alpha} \rangle_{0} = (\tilde{v}_{\beta \alpha})_{k} = 0 \)). Let \( c_{\beta \alpha} \) be the equivalence class of \( k \) \( (\tilde{v}_{\beta \alpha})_{k-1} \) modulo \( k \). Set \( (C^{-1})^{n_{c_{\beta \alpha}}} = 0 \) for \( n \in \{ 1, \ldots, k-1 \} \) and \( (C^{-1})^{k_{c_{\beta \alpha}}} = 0 \).

We further define

\[
m_{Y_{\alpha}, Y'_{\beta}}(\varepsilon_{1}, \varepsilon_{2}, a_{\beta \alpha}) := \prod_{s \in Y_{\alpha}} \left( a_{\beta \alpha} - L_{Y'_{\beta}}(s) \varepsilon_{1} + (A_{Y_{\alpha}}(s) + 1) \varepsilon_{2} \right) \times \prod_{s' \in Y'_{\beta}} \left( a_{\beta \alpha} + (L_{Y_{\alpha}}(s') + 1) \varepsilon_{1} - A_{Y'_{\beta}}(s') \varepsilon_{2} \right)
\]

for two Young tableaux \( Y_{\alpha} \) and \( Y'_{\beta} \), and \( \alpha = 1, \ldots, r, \beta = 1, \ldots, r' \).

The \( \tilde{T} \)-equivariant Euler class of the vertex contribution \( M_{\alpha, \beta}(t_{1}, t_{2}) \) is

\[
\prod_{i=1}^{k} m_{Y_{\alpha}, Y'_{\beta}}(\varepsilon_{1}^{(i)}, \varepsilon_{2}^{(i)}, a_{\beta \alpha}^{(i)}) .
\]

Now we give the \( \tilde{T} \)-equivariant Euler class of the edge contribution. For this, we introduce some more notation.

If \( (\tilde{v}_{\beta \alpha})_{n} - (C^{-1})^{n_{c_{\beta \alpha}}} > 0 \) for \( n \in \{ 1, \ldots, k-1 \} \), consider the equation

\[
i^{2} - i \left( \tilde{v}_{\beta \alpha} - \sum_{p=1}^{n-1} \left( (\tilde{v}_{\beta \alpha})_{p} - (C^{-1})^{p_{c_{\beta \alpha}}} \tilde{e}_{p} \right) \cdot C \tilde{e}_{n} + \frac{1}{2} \left( \tilde{v}_{\beta \alpha} - \sum_{p=1}^{n-1} \left( (\tilde{v}_{\beta \alpha})_{p} - (C^{-1})^{p_{c_{\beta \alpha}}} \tilde{e}_{p} \right) \right) \cdot C \left( \tilde{v}_{\beta \alpha} - \sum_{p=1}^{n-1} \left( (\tilde{v}_{\beta \alpha})_{p} - (C^{-1})^{p_{c_{\beta \alpha}}} \tilde{e}_{p} \right) - (C^{-1})^{c_{\beta \alpha}} \tilde{e}_{n} \right) = 0 ,
\]

and define the set

\[
S(\beta \alpha)_{n} := \{ i \in \mathbb{N} \mid i \leq (\tilde{v}_{\beta \alpha})_{n} - (C^{-1})^{n_{c_{\beta \alpha}}} \text{ is a solution of Equation (4.29)} \} .
\]

Let \( d(\beta \alpha)_{n} := \min(S(\beta \alpha)_{n}) \) if \( S(\beta \alpha)_{n} \neq \emptyset \), otherwise \( d(\beta \alpha)_{n} := (\tilde{v}_{\beta \alpha})_{n} - (C^{-1})^{n_{c_{\beta \alpha}}} \).

If \( (\tilde{v}_{\beta \alpha})_{n} - (C^{-1})^{n_{c_{\beta \alpha}}} < 0 \), consider the equation
\[ i^2 + i \left( \bar{v}_{\beta\alpha} - \sum_{p=1}^{n-1} \left( (\bar{v}_{\beta\alpha})_p - (C^{-1})^{p,c_{\beta\alpha}} \bar{v}_p \right) \cdot C \bar{v}_n \right) + \frac{1}{2} \left( \left( \bar{v}_{\beta\alpha} - \sum_{p=1}^{n-1} \left( (\bar{v}_{\beta\alpha})_p - (C^{-1})^{p,c_{\beta\alpha}} \bar{v}_p \right) \right)^2 \right) \cdot C \left( \bar{v}_{\beta\alpha} - \sum_{p=1}^{n-1} \left( (\bar{v}_{\beta\alpha})_p - (C^{-1})^{p,c_{\beta\alpha}} \bar{v}_p \right) \right) = 0, \quad (4.30) \]

and define the set

\[ S(\beta\alpha)_n^- := \{ i \in \mathbb{N} \mid i \leq -((\bar{v}_{\beta\alpha})_n + (C^{-1})^{n,c_{\beta\alpha}}) \text{ is a solution of Equation (4.30)} \}. \]

Let \( d(\beta\alpha)_n^+ = \min(S(\beta\alpha)_n^-) \) if \( S(\beta\alpha)_n^- \neq \emptyset \), otherwise \( d(\beta\alpha)_n^+ := -((\bar{v}_{\beta\alpha})_n + (C^{-1})^{n,c_{\beta\alpha}}) \). Define \( m_{\beta\alpha} \) to be the smallest index \( n \in \{1, \ldots, k-1\} \) such that \( S(\beta\alpha)_n^+ \) or \( S(\beta\alpha)_n^- \) is nonempty, otherwise \( m_{\beta\alpha} := k-1 \).

Then the \( \bar{T} \)-equivariant Euler class of the edge contribution \( L_{\alpha\beta}(t_1,t_2) \) is

\[ \prod_{n=1}^{k-1} \ell_{\alpha\beta(n)}(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}), \]

where for fixed \( n = 1, \ldots, m_{\beta\alpha} \) we set:

- If \( (\bar{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} > 0 \):
  - For \( \delta_{n,c_{\beta\alpha}} - (\bar{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}} + 2((\bar{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} - d(\beta\alpha)_n^+) \geq 0 \):
    \[ \ell_{\beta\alpha(n)}(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}) = \prod_{i=(\bar{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} - d(\beta\alpha)_n^+}^{2i+\delta_{n,c_{\beta\alpha}} - (\bar{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}} \prod_{j=0}^{d(\beta\alpha)_n^+} \left( a_{\beta\alpha} + \left( i + \frac{\delta_{n,c_{\beta\alpha}} - (\bar{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}}{2} \right) \varepsilon_1^{(n)} + j \varepsilon_2^{(n)} \right). \]
  - For \( 2 \leq \delta_{n,c_{\beta\alpha}} - (\bar{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}} + 2((\bar{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}}) < 2d(\beta\alpha)_n^+ \):
    \[ \ell_{\beta\alpha(n)}(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}) = \prod_{i=(\bar{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} - d(\beta\alpha)_n^+}^{2i+\delta_{n,c_{\beta\alpha}} - (\bar{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}-1} \prod_{j=1}^{d(\beta\alpha)_n^+} \left( a_{\beta\alpha} + \left( i - \frac{\delta_{n,c_{\beta\alpha}} - (\bar{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}}{2} \right) \varepsilon_1^{(n)} - j \varepsilon_2^{(n)} \right)^{-1} \times \]
    \[ \prod_{i=1}^{2((\bar{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}}) + \delta_{n,c_{\beta\alpha}} - (\bar{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}-2} \prod_{j=0}^{2i+\delta_{n,c_{\beta\alpha}} - (\bar{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}-1} \left( a_{\beta\alpha} + \left( i + \frac{\delta_{n,c_{\beta\alpha}} - (\bar{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}}{2} \right) \varepsilon_1^{(n)} + j \varepsilon_2^{(n)} \right). \]
  - For \( \delta_{n,c_{\beta\alpha}} - (\bar{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}} < 2 - 2((\bar{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}}) \):
    \[ \ell_{\beta\alpha(n)}(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}) = \prod_{i=(\bar{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} - d(\beta\alpha)_n^+}^{(\bar{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}}-1} \prod_{j=1}^{2i+\delta_{n,c_{\beta\alpha}} + (\bar{v}_{\beta\alpha})_{n+1} - (C^{-1})^{n+1,c_{\beta\alpha}}-1} \left( a_{\beta\alpha} + \left( i - \frac{\delta_{n,c_{\beta\alpha}} + (\bar{v}_{\beta\alpha})_{n+1} - (C^{-1})^{n+1,c_{\beta\alpha}}}{2} \right) \varepsilon_1^{(n)} + j \varepsilon_2^{(n)} \right). \]
\[
\left( a_{\beta\alpha} + \left(1 - \delta_{n,c_{\beta\alpha}} - \frac{\v_{\beta\alpha}(n) + (C^{-1})_{n+1,c_{\beta\alpha}}}{2} \right) \v_1^{(n)} - j \v_2^{(n)} \right)^{-1}.
\]

- If \((\v_{\beta\alpha})_n - (C^{-1})_{n,c_{\beta\alpha}} = 0:\)
  \[
  \v_{\beta\alpha}(\v_1^{(n)}, \v_2^{(n)}, a_{\beta\alpha}) = 1.
  \]

- If \((\v_{\beta\alpha})_n - (C^{-1})_{n,c_{\beta\alpha}} < 0:\)
  - For \(\delta_{n,c_{\beta\alpha}} - (\v_{\beta\alpha})_n + (C^{-1})_{n+1,c_{\beta\alpha}} + 2(\v_{\beta\alpha})_n - 2(C^{-1})_{n,c_{\beta\alpha}} < 2 - 2d(\beta\alpha)_n:\)
    \[
    \v_{\beta\alpha}(\v_1^{(n)}, \v_2^{(n)}, a_{\beta\alpha}) = -\frac{\v_{\beta\alpha}(n) + (C^{-1})_{n,c_{\beta\alpha}}}{2} - 2i(\delta_{n,c_{\beta\alpha}} - (\v_{\beta\alpha})_n + (C^{-1})_{n+1,c_{\beta\alpha}} - 1)
    \]
    \[
    \prod_{i=1}^{\frac{\delta_{n,c_{\beta\alpha}} - (\v_{\beta\alpha})_n + (C^{-1})_{n+1,c_{\beta\alpha}}}{2}} (a_{\beta\alpha} - \left(1 - \delta_{n,c_{\beta\alpha}} - \frac{\v_{\beta\alpha}(n) + (C^{-1})_{n+1,c_{\beta\alpha}}}{2} \right) \v_1^{(n)} - j \v_2^{(n)} \right)^{-1}.
    \]
  - For \(2 - 2d(\beta\alpha)_n \leq \delta_{n,c_{\beta\alpha}} - (\v_{\beta\alpha})_n + (C^{-1})_{n+1,c_{\beta\alpha}} + 2(\v_{\beta\alpha})_n - 2(C^{-1})_{n,c_{\beta\alpha}} < 0:\)
    \[
    \v_{\beta\alpha}(\v_1^{(n)}, \v_2^{(n)}, a_{\beta\alpha}) = -\frac{\v_{\beta\alpha}(n) + (C^{-1})_{n,c_{\beta\alpha}}}{2} - 2i(\delta_{n,c_{\beta\alpha}} - (\v_{\beta\alpha})_n + (C^{-1})_{n+1,c_{\beta\alpha}} - 1)
    \]
    \[
    \prod_{i=1}^{\frac{\delta_{n,c_{\beta\alpha}} - (\v_{\beta\alpha})_n + (C^{-1})_{n+1,c_{\beta\alpha}}}{2}} (a_{\beta\alpha} - \left(1 - \delta_{n,c_{\beta\alpha}} - \frac{\v_{\beta\alpha}(n) + (C^{-1})_{n+1,c_{\beta\alpha}}}{2} \right) \v_1^{(n)} - j \v_2^{(n)} \right)^{-1}.
    \]

For \(n = m_{\beta\alpha} + 1, \ldots, k - 1\) we set
\[
\v_{\beta\alpha}(\v_1^{(n)}, \v_2^{(n)}, a_{\beta\alpha}) = 1.
\]

**Remark 4.31.** Note that for any fixed \(n \in \{1, \ldots, k - 1\}, d(\beta\alpha)_n = 0\) implies \(\v_{\beta\alpha}(\v_1^{(n)}, \v_2^{(n)}, a_{\beta\alpha}) = 1.\)
Collecting all computations so far done, the $\tilde{T}$-equivariant Euler class of the Carlsson-Okounkov bundle $E$ at the fixed point $([(E, \phi_\Sigma)], [(E', \phi_\Sigma')])$ is given by

$$\text{Euler}_T(E_{([(E, \phi_\Sigma)], [(E', \phi_\Sigma')])}) = \prod_{\alpha=1}^{r} \prod_{\beta=1}^{r'} \prod_{i=1}^{k} m_{Y_{\alpha}, Y'_{\beta}}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha}) \prod_{n=1}^{k-1} \ell_{v_{\beta \alpha}}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha}).$$

From a physical viewpoint, this formula expresses the bifundamental weight of an $A_2$ quiver gauge theory, with a $U(r)$ gauge group associated to one node and a $U(r')$ gauge group associated to the other node.

Let $[(E, \phi_\Sigma)]$ be a $T$-fixed point of $\mathcal{M}_{r, \delta}(\mathcal{X}, D, \mathcal{F}_{r, \delta})$ and $(Y, \bar{v})$ its corresponding combinatorial data. By Remark 4.21 we can compute the $T$-equivariant Euler class of the tangent bundle $T\mathcal{M}_{r, \delta}(\mathcal{X}, D, \mathcal{F}_{r, \delta})$ at $[(E, \phi_\Sigma)]$ from this general expression. We get

$$\text{Euler}_T(T_{(Y, \bar{v})}\mathcal{M}_{r, \delta}(\mathcal{X}, D, \mathcal{F}_{r, \delta})) = \prod_{\alpha=1}^{r} \prod_{\beta=1}^{r'} \prod_{i=1}^{k} m_{Y_{\alpha}, Y'_{\beta}}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha}) \prod_{n=1}^{k-1} \ell_{v_{\beta \alpha}}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha}),$$

where $a_{\beta \alpha} = a_\beta - a_\alpha$ and $\bar{v}_{\beta \alpha} = \bar{v}_\beta - \bar{v}_\alpha$.

For $\alpha = 1, \ldots, r$ and for a Young tableau $Y_{\alpha}$ we define

$$m_{Y_{\alpha}}(\varepsilon_1, \varepsilon_2, a_\alpha) := \prod_{s \in Y_{\alpha}} (-L'_{s}(s) \varepsilon_1 - A'_{s}(s) \varepsilon_2 + a_\alpha).$$

Again, by Remark 4.21 we can compute the $T$-equivariant Euler class of the natural bundle $V$ at a fixed point $[(E, \phi_\Sigma)]$ from the general character formula. We obtain

$$\text{Euler}_T(V_{(Y, \bar{v})}) = \prod_{\alpha=1}^{r} \prod_{i=1}^{k} m_{Y_{\alpha}}(\varepsilon_1, \varepsilon_2, a_\alpha) \prod_{n=1}^{k-1} \ell_{v_{\alpha}}(\varepsilon_1, \varepsilon_2, a_\alpha),$$

where for $i = 1, \ldots, k$ we set

$$\bar{a}^{(i)} := \bar{a} - (\bar{v})_i \varepsilon_1 - (\bar{v})_{i-1} \varepsilon_2,$$

and $(\bar{v})_i := ((\bar{v})_1, \ldots, (\bar{v})_i)$ (we set $(\bar{v})_0) = (\bar{v})_k = 0$ for $\alpha = 1, \ldots, r$).

Example 4.32. Let $k = 2$. Then the Euler class formula becomes

$$\text{Euler}_T(E_{([(E, \phi_\Sigma)], [(E', \phi_\Sigma')])}) = \prod_{\alpha=1}^{r} \prod_{\beta=1}^{r'} \prod_{i=1}^{k} m_{Y_{\alpha}, Y'_{\beta}}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, a_{\beta \alpha} - 2v_{\beta \alpha} \varepsilon_1) \prod_{n=1}^{k-1} \ell_{v_{\beta \alpha}}(2\varepsilon_1, 2\varepsilon_2 - \varepsilon_1, a_{\beta \alpha}).$$

In this case $c_{\beta \alpha} \in \{0, 1\}$ for any $\alpha = 1, \ldots, r$, $\beta = 1, \ldots, r'$, while $\{v_{\beta \alpha}\} = (G^{-1})^{1, c_{\beta \alpha}}$. Since $m = 1, d(\beta \alpha) = [v_{\beta \alpha}]$ and $d(\beta \alpha) = [v_{\beta \alpha}]$, and we get

$$\ell_{v_{\beta \alpha}}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha}) = \begin{cases} \prod_{i=0}^{[v_{\beta \alpha}]-1} \prod_{j=0}^{2i+2[v_{\beta \alpha}]} (a_{\beta \alpha} + i \varepsilon_1 + j \varepsilon_2) & \text{for } [v_{\beta \alpha}] > 0, \\ 1 & \text{for } [v_{\beta \alpha}] = 0, \\ \prod_{i=1}^{-[v_{\beta \alpha}]} \prod_{j=1}^{2i-2[v_{\beta \alpha}]-1} (a_{\beta \alpha} + a_{2[v_{\beta \alpha}]} - i) \varepsilon_1 - j \varepsilon_2 & \text{for } [v_{\beta \alpha}] < 0. \end{cases}$$

The explicit formula for $k = 3$ is presented in Appendix D.
5. BPS CORRELATORS AND INSTANTON PARTITION FUNCTIONS

5.1. \( \mathcal{N} = 2 \) gauge theory.

Generating function for correlators of p-observables. For fixed rank \( r \geq 1 \) and holonomy at infinity \( \bar{\nu} \in \mathbb{N}^k \), let \( \bar{\nu} \in \frac{1}{k} \mathbb{Z}^{k-1} \) be such that \( k v_{k-1} = \sum_{i=0}^{k-1} i \bar{w}_i \) mod \( k \). Define

\[
\mathcal{Z}_{\theta}(\varepsilon_1, \varepsilon_2; \bar{a}; \bar{q}, \bar{\tau}, \bar{t}^{(1)}, \ldots, \bar{t}^{(k-1)}) := \sum_{\Delta \in \mathbb{Z}_T^k} q^{\Delta + \frac{1}{2r} \bar{\nu} \cdot C_{\bar{\nu}}}
\times \int_{\mathcal{M}_{r,s,\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty})} \exp \left( \sum_{s=0}^{\infty} \sum_{i=1}^{k-1} t_s^{(i)} \left[ \text{ch}_T(\mathcal{E})/\mathcal{D}_i \right]_s + \tau_s \left[ \text{ch}_T(\mathcal{E})/|X_k| \right]_{s-1} \right)
\]

where \( \mathcal{E} \) is the universal sheaf, \( \text{ch}_T(\mathcal{E})/\mathcal{D}_i \) is the slant product between \( \text{ch}_T(\mathcal{E}) \) and \( \mathcal{D}_i \), and the class \( \text{ch}_T(\mathcal{E})/|X_k| \) is defined by localization as

\[
\text{ch}_T(\mathcal{E})/|X_k| := \frac{1}{\text{Euler}_T(T_{p_\mathcal{D}, X_k}) \cdot \mathcal{M}_{r,s,\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty})} \text{ch}_T(\mathcal{E}) ;
\]

here \( \mathcal{M}_{r,s,\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}) \) denotes the inclusion map of \( \mathcal{D}_i \) into \( \mathcal{M}_{r,s,\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}) \) in \( X_k \times \mathcal{M}_{r,s,\Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}) \). The brackets \([\cdot]_s \) indicate the degree \( s \) part.

Remark 5.1. Let \( \mathcal{X} \) be a topological stack with an action of the Deligne-Mumford torus \( T_r \). As explained in [41, Section 5], there is a well posed notion of \( T_r \)-equivariant (co)homology theory on \( \mathcal{X} \). When \( \mathcal{X} \) is a topological space, their definition reduces to Borel’s definition of \( T_r \)-equivariant (co)homology theory on topological spaces. So the slant product is also well-defined for \( T_r \)-equivariant (co)homology theories on topological stacks.

Definition 5.2. The generating function \( \mathcal{Z}_{X_k}(\varepsilon_1, \varepsilon_2; \bar{a}; \bar{q}, \bar{\xi}, \bar{\tau}, \bar{t}^{(1)}, \ldots, \bar{t}^{(k-1)}) \) for correlators of p-observables \( (p = 0, 2) \) of pure \( \mathcal{N} = 2 \) gauge theory on \( X_k \) is

\[
\mathcal{Z}_{X_k}(\varepsilon_1, \varepsilon_2; \bar{a}; \bar{q}, \bar{\xi}, \bar{\tau}, \bar{t}^{(1)}, \ldots, \bar{t}^{(k-1)}) := \sum_{\bar{\nu} \in \frac{1}{k} \mathbb{Z}^{k-1}} \bar{\nu} \cdot \mathcal{Z}_{\theta}(\varepsilon_1, \varepsilon_2; \bar{a}; \bar{q}, \bar{\xi}, \bar{\tau}, \bar{t}^{(1)}, \ldots, \bar{t}^{(k-1)})
\]

where \( \bar{\xi} = (\xi_1, \ldots, \xi_{k-1}) \) and \( \bar{\nu} \cdot \mathcal{Z}_{\theta} := \prod_{i=1}^{k-1} \xi_i^\nu_i \).

In this subsection we compute explicitly \( \mathcal{Z}_{X_k}(\varepsilon_1, \varepsilon_2; \bar{a}; \bar{q}, \bar{\xi}, \bar{\tau}, \bar{t}^{(1)}, \ldots, \bar{t}^{(k-1)}) \). First note that by the localization formula we have

\[
\mathcal{Z}_{\theta}(\varepsilon_1, \varepsilon_2; \bar{a}; \bar{q}, \bar{\tau}, \bar{t}^{(1)}, \ldots, \bar{t}^{(k-1)}) = \sum_{(\bar{Y}, \bar{\nu})} \prod_{\alpha, \beta = 1}^{r} \prod_{i=1}^{k} \mathcal{M}_{\alpha, \beta} \mathcal{Y}_i(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, a^{(i)}_{\alpha \beta}) \prod_{n=1}^{k-1} \mathcal{F}_{\alpha \beta} \mathcal{E}_n(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a^{(n)}_{\alpha \beta})
\times \mathcal{Z}_{\theta}(\bar{Y}, \bar{\nu}) \exp \left( \sum_{s=0}^{\infty} \sum_{i=1}^{k-1} t_s^{(i)} \left[ \text{ch}_T(\mathcal{E})/\mathcal{D}_i \right]_s + \tau_s \left[ \text{ch}_T(\mathcal{E})/|X_k| \right]_{s-1} \right).
\]
Now we compute $\iota^*_{(\mathcal{Y}, \bar{v})} \operatorname{ch}_T(\mathcal{E})/\lbrack X_k \rbrack$. First note that

$$
\iota^*_{(\mathcal{Y}, \bar{v})} \operatorname{ch}_T(\mathcal{E})/\lbrack X_k \rbrack = \sum_{i=1}^{k} \frac{1}{\varepsilon_1 \varepsilon_2} \iota_{(p_i)}^* \operatorname{ch}_T(\mathcal{E})
$$

Let us fix an index $i \in \{1, \ldots, k\}$. Let $\left[ (\mathcal{E}, \phi_{\mathcal{E}}) \right] = \left[ \bigoplus_{\alpha=1}^{r} (I_\alpha \otimes \mathcal{R}^{C_i \bar{v}_\alpha}, \phi_{\alpha}) \right]$ be a $T$-fixed point and $(\mathcal{Y}, \bar{v})$ its corresponding combinatorial data. Then

$$
\iota_{(p_i)}^* \operatorname{ch}_T(\mathcal{E}) = \sum_{\alpha=1}^{r} e_\alpha \iota_{(p_i)}^* \operatorname{ch}_T (I_\alpha \otimes \mathcal{R}^{C_i \bar{v}_\alpha}) = \sum_{\alpha=1}^{r} e_\alpha \operatorname{ch}_T (I_\alpha) \operatorname{ch}_T (\mathcal{R}^{C_i \bar{v}_\alpha})
$$

where $\iota_{p_i}$ denotes the inclusion morphism of the $T_i$-fixed point $p_i$ into $X_k$. By [69, Equation (4.1)] we have

$$
\operatorname{ch}_T (I_\alpha) = 1 - (1 - (\chi^i_1)^{-1}) - (1 - (\chi^i_2)^{-1}) \sum_{s \in Y^i_2} (\chi^i_1)^{-L(s)} (\chi^i_2)^{-A'(s)}
$$

By Lemma 3.4 and Equation (3.20), we have $\operatorname{ch}_T (\mathcal{R}^{C_i \bar{v}_\alpha}) = (\chi^i_1)^{-L_{\alpha}} (\chi^i_2)^{-A_{\alpha}} - 1$. Summing up, we get

$$
\iota_{(p_i)}^* \operatorname{ch}_T(\mathcal{E})/\lbrack X_k \rbrack = \sum_{i=1}^{k} \operatorname{ch}_{\mathcal{Y}_i} (\varepsilon^i_1, \varepsilon^i_2, \bar{a}^i) \tag{5.3}
$$

where we introduced the notation

$$
\operatorname{ch}_{\mathcal{Y}_i} (\varepsilon^i_1, \varepsilon^i_2, \bar{a}^i) := \sum_{\alpha=1}^{r} \frac{e_\alpha a^i_\alpha}{\varepsilon^i_1 \varepsilon^i_2} \left( 1 - (1 - e^{-\varepsilon^i_1}) (1 - e^{-\varepsilon^i_2}) \sum_{s \in Y^i_2} e^{-\varepsilon^i_1 L'(s) - \varepsilon^i_2 A'(s)} \right)
$$

Remark 5.4. Note that

$$
\left[ \operatorname{ch}_{\mathcal{Y}_i} (\varepsilon^i_1, \varepsilon^i_2, \bar{a}^i) \right]_s = \frac{1}{\varepsilon^i_1 \varepsilon^i_2 (s + 2)!} \sum_{\alpha=1}^{r} \left( a^i_\alpha \right)^{s+2}
$$

$$
- \frac{1}{\varepsilon^i_1 \varepsilon^i_2 (s + 2)!} \sum_{\alpha=1}^{r} \sum_{s \in Y^i_2} \left( (a^i_\alpha - \varepsilon^i_1 L'(s) - \varepsilon^i_2 A'(s)) \right)^{s+2}
$$

$$
- (a^i_\alpha - \varepsilon^i_1 (L'(s) + 1) - \varepsilon^i_2 A'(s))^{s+2}
$$

$$
+ (a^i_\alpha - \varepsilon^i_1 (L'(s) + 1) - \varepsilon^i_2 (A'(s) + 1))^{s+2}
$$

In particular

$$
0 = \left[ \operatorname{ch}_{\mathcal{Y}_i} (\varepsilon^i_1, \varepsilon^i_2, \bar{a}^i) \right]_{-1} = \frac{1}{\varepsilon^i_1 \varepsilon^i_2} \sum_{\alpha=1}^{r} a^i_\alpha,
$$

$$
\left[ \operatorname{ch}_{\mathcal{Y}_i} (\varepsilon^i_1, \varepsilon^i_2, \bar{a}^i) \right]_0 = 2 \varepsilon^i_1 \varepsilon^i_2 \sum_{\alpha=1}^{r} (a^i_\alpha)^2 - \sum_{\alpha=1}^{r} |Y^i_\alpha| \tag{5.5}
$$

These formulas will be useful later on.

Now we compute $\iota_{(\mathcal{Y}, \bar{v})}^* \operatorname{ch}_T(\mathcal{E})/\lbrack \mathcal{B}_i \rbrack$. With the same conventions as above, we have

$$
\operatorname{ch}_T(\mathcal{E}) = \sum_{\alpha=1}^{r} e_\alpha \operatorname{ch}_T (\mathcal{R}^{C_i \bar{v}_\alpha}) \operatorname{ch}_T (I_\alpha)
$$
\[
\sum_{m=0}^{\infty} \frac{1}{m!} (-1)^m \int_{X_k} \left( \sum_{j=1}^{k-1} (\bar{v}_\alpha)_j [D_j] \right)^m \cdot [D_1]
\]

To compute the integral
\[
\int_{X_k} [D_1]^{m_1} \cdots [D_l]^{m_l+1} \cdots [D_{k-1}]^{m_{k-1}} \tag{5.6}
\]
by localization, we need to know the pullback of the class \([D_l] \) at the fixed points of \(X_k\) for each \(l = 1, \ldots, k - 1\). Since
\[
l^*_{\cdot p_l}[D_l] = \begin{cases} 
\varepsilon_1^{(l)} & \text{if } i = l , \\
\varepsilon_2^{(l+1)} & \text{if } i = l + 1 , \\
0 & \text{otherwise} ,
\end{cases}
\tag{5.7}
\]
the integral (5.6) is nonzero if there exists an index \(n \in \{1, \ldots, k - 1\} \) such that only the exponent of \([D_n] \) is nonzero or there exists an index \(n' \in \{2, \ldots, k - 1\} \) such that only the exponents of \([D_{n'}-1] \) and \([D_{n''}] \) are nonzero. Therefore we obtain
\[
\exp \left( - \sum_{j=1}^{k-1} (\bar{v}_\alpha)_j [D_j] \right) / [D_1]
\]

On the other hand, a similar argument shows that for \(l \in \{2, \ldots, k - 2\} \) we get
Proposition 5.9. The generating function $Z_{X_k}(\varepsilon_1, \varepsilon_2, \bar{a}; \bar{q}, \bar{\xi}, \bar{\tau}, \bar{\tau}^{(1)}, \ldots, \bar{\tau}^{(k-1)})$ assumes the form

$$Z_{X_k}(\varepsilon_1, \varepsilon_2, \bar{a}; \bar{q}, \bar{\xi}, \bar{\tau}, \bar{\tau}^{(1)}, \ldots, \bar{\tau}^{(k-1)}) = \sum_{\alpha=1}^{k} \frac{\sum_{n=0}^k n_{\alpha} + 1}{\prod_{\alpha, \beta=1}^{k} m_{\alpha} \prod_{i=1}^{k} \frac{\varepsilon_1^{(i)} \varepsilon_2^{(i)} \bar{a}^{(i)}}{\varepsilon_1^{(i)} \varepsilon_2^{(i)} \bar{a}^{(i)}}} \times \prod_{i=1}^{k} \exp \left( \sum_{s=0}^{\infty} \left( \frac{t_s^{(i)} \varepsilon_1^{(i)} + t_s^{(i-1)} \varepsilon_2^{(i)} + \tau_s}{\varepsilon_1^{(i)} \varepsilon_2^{(i)} \bar{a}^{(i)}} \right) \right) \times \exp \left( \sum_{s=0}^{\infty} \left( \frac{\sum_{l=1}^{k} \sum_{i=1}^{l-2} t_s^{(i)} + \sum_{i=1}^{k-1} t_s^{(i)} + \sum_{i=1}^{k-1} t_s^{(i)} \left[ \sum_{l=1}^{k-1} \left( \frac{\varepsilon_1^{(i)} \varepsilon_2^{(i)} \bar{a}^{(i)}}{\varepsilon_1^{(i)} \varepsilon_2^{(i)} \bar{a}^{(i)}} \right) \right] \right) \right) \right)$$

where we set $t_s^{(0)} = t_s^{(k)} = 0$ for any $s$.

Example 5.11. For $k = 2$ the generating function $Z_{X_2}(\varepsilon_1, \varepsilon_2, \bar{a}; \bar{q}, \bar{\xi}, \bar{\tau}, \bar{\tau})$ becomes

$$Z_{X_2}(\varepsilon_1, \varepsilon_2, \bar{a}; \bar{q}, \bar{\xi}, \bar{\tau}, \bar{\tau})$$
Observables) for $U$ where $Z$ follows that

Since the Nekrasov partition function we set

Definition 5.12. Instanton partition function.

Here we set

Instanton partition function.

Let us define

Definition 5.12. Instanton partition function is

Here we set $\vec{\tau} = 0$ and $\vec{\tau}^{(i)} = 0$ for $i = 1, \ldots, k - 1$.

By Equation (5.10) we get

Since the Nekrasov partition function for pure $U(r)$ gauge theory on $\mathbb{R}^4$ can be expressed as [19, Equation (3.16)]

we get

Correlators of quadratic 0-observables. Let us define

where we set $\vec{\tau} = (0, -\tau_1, 0, \ldots)$ and $\vec{\tau}^{(i)} = 0$ for $i = 1, \ldots, k - 1$. From Equations (5.10) and (5.5) it follows that

$$Z_{X_k}^\text{inst}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\xi})$$

$$= \sum_{\epsilon \in \frac{1}{2} \mathbb{Z}} \xi^\epsilon \prod_{i} \frac{\lambda_{\beta \gamma}(2\epsilon_{\gamma} + \epsilon_{\beta} - \epsilon_{\gamma} - \epsilon_{\beta})}{(2\epsilon_{\gamma} + \epsilon_{\beta} - \epsilon_{\gamma} - \epsilon_{\beta})} \times Z_{C^2}(2\epsilon_{\gamma} + \epsilon_{\beta} - \epsilon_{\gamma} - \epsilon_{\beta} + 2\epsilon_{\gamma} + \epsilon_{\beta} - 2\epsilon_{\gamma} - 2\epsilon_{\beta} + \vec{q}, \vec{\tau} + 2\epsilon_{\gamma} + \epsilon_{\beta} - \epsilon_{\gamma} - \epsilon_{\beta}),$$

where $Z_{C^2}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\tau})$ is the deformed Nekrasov partition function (the generating function for 0-observables) for $U(r)$ gauge theory on $\mathbb{R}^4$ defined in [69, Section 4.2]:

$$Z_{C^2}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\tau}) := \sum_{\vec{y}} \prod_{\alpha, \beta = 1}^{r} m_{Y_{\alpha}, Y_{\beta}}(\epsilon_1, \epsilon_2, a_{\alpha \beta}) \exp \left( \sum_{s=0}^{\infty} \tau_s \left( \text{ch}_2(\epsilon_1, \epsilon_2, \vec{a}) \right)_{s-1} \right).$$

However, for $k \geq 3$ we see no such nice factorizations of Equation (5.10).
\[
\begin{align*}
= & \sum_{\vec{v} \in \frac{1}{k} \mathbb{Z}^{k-1}} \zeta^{\vec{v}} \sum_{(Y, \vec{a})} \frac{\sum_{q=1}^{r} n_{\alpha} + \frac{1}{2} \sum_{a=1}^{r} \delta_{\alpha} \vec{v} \cdot C \vec{v}_{a}}{\prod_{\alpha, \beta = 1}^{r} m_{Y_{\alpha}^{\vec{a} \beta}}(\varepsilon_{1}^{(i)}, \varepsilon_{2}^{(i)}, a_{\beta a}) \prod_{n=1}^{k-1} \ell^{(n)}_{\varepsilon_{\beta a}}(\varepsilon_{1}^{(n)}, \varepsilon_{2}^{(n)}, a_{\beta a})} \\
& \times \prod_{l=1}^{k} \exp \left( -\tau_{1} \left[ \chi_{Y_{l}}(\varepsilon_{1}^{(l)}, \varepsilon_{2}^{(l)}, \vec{a}^{(l)}) \right]_{0} \right) \\
= & \sum_{\vec{v} \in \frac{1}{k} \mathbb{Z}^{k-1}} \zeta^{\vec{v}} \sum_{(Y, \vec{a})} \frac{\sum_{q=1}^{r} n_{\alpha} + \frac{1}{2} \sum_{a=1}^{r} \delta_{\alpha} \vec{v} \cdot C \vec{v}_{a}}{\prod_{\alpha, \beta = 1}^{r} m_{Y_{\alpha}^{\vec{a} \beta}}(\varepsilon_{1}^{(i)}, \varepsilon_{2}^{(i)}, a_{\beta a}) \prod_{n=1}^{k-1} \ell^{(n)}_{\varepsilon_{\beta a}}(\varepsilon_{1}^{(n)}, \varepsilon_{2}^{(n)}, a_{\beta a})} \\
& \times \prod_{l=1}^{k} \exp \left( -\tau_{1} \left( \frac{1}{2 \varepsilon_{1}^{(l)} \varepsilon_{2}^{(l)}} \sum_{\alpha = 1}^{r} (a_{\alpha}^{(l)})^{2} - \sum_{\alpha = 1}^{r} |Y_{l}| \right) \right) \\
= & \sum_{\vec{v} \in \frac{1}{k} \mathbb{Z}^{k-1}} \zeta^{\vec{v}} \sum_{(Y, \vec{a})} \frac{\sum_{q=1}^{r} n_{\alpha} + \frac{1}{2} \sum_{a=1}^{r} \delta_{\alpha} \vec{v} \cdot C \vec{v}_{a}}{\prod_{\alpha, \beta = 1}^{r} m_{Y_{\alpha}^{\vec{a} \beta}}(\varepsilon_{1}^{(i)}, \varepsilon_{2}^{(i)}, a_{\beta a}) \prod_{n=1}^{k-1} \ell^{(n)}_{\varepsilon_{\beta a}}(\varepsilon_{1}^{(n)}, \varepsilon_{2}^{(n)}, a_{\beta a})} \\
& \times \prod_{l=1}^{k} Z_{C_{2}}^{\text{class}}(\varepsilon_{1}^{(l)}, \varepsilon_{2}^{(l)}, \vec{a}^{(l)}; \tau_{1}) Z_{C_{2}}^{\text{inst}}(\varepsilon_{1}^{(l)}, \varepsilon_{2}^{(l)}, \vec{a}^{(l)}; q_{\text{eff}}, \vec{\xi}, \tau_{1}) ,
\end{align*}
\]
where \( q_{\text{eff}} := q \cdot e^{\tau_{1}} \) and we introduced the classical partition function for supersymmetric \( U(r) \) gauge theory on \( \mathbb{R}^{4} \) which is given by

\[
Z_{C_{2}}^{\text{class}}(\varepsilon_{1}, \varepsilon_{2}, \vec{a}; \tau_{1}) := \exp \left( -\frac{\tau_{1}}{2 \varepsilon_{1} \varepsilon_{2}} \sum_{\alpha = 1}^{r} a_{\alpha}^{2} \right) .
\]

**Proposition 5.14.** The partition function \( Z_{X_{k}}^{\text{inst}}(\varepsilon_{1}, \varepsilon_{2}, \vec{a}; q_{\text{eff}}, \vec{\xi}) \) factorizes as

\[
Z_{X_{k}}^{\text{inst}}(\varepsilon_{1}, \varepsilon_{2}, \vec{a}; q_{\text{eff}}, \vec{\xi}) = Z_{C_{2}}^{\text{class}}(\varepsilon_{1}, \varepsilon_{2}, \vec{a}; \tau_{1}) \cdot Z_{X_{k}}^{\text{inst}}(\varepsilon_{1}, \varepsilon_{2}, \vec{a}; q_{\text{eff}}, \vec{\xi}, \tau_{1}) ,
\]

where

\[
Z_{X_{k}}^{\text{inst}}(\varepsilon_{1}, \varepsilon_{2}, \vec{a}; q_{\text{eff}}, \vec{\xi}, \tau_{1}) := \sum_{\vec{v} \in \frac{1}{k} \mathbb{Z}^{k-1}} \zeta^{\vec{v}} \sum_{(Y, \vec{a})} \frac{\left( e^{\tau_{1}} \right) \sum_{\alpha = 1}^{r} \sum_{a=1}^{r} \delta_{\alpha} \vec{v} \cdot C \vec{v}_{a} \prod_{\alpha, \beta = 1}^{r} m_{Y_{\alpha}^{\vec{a} \beta}}(\varepsilon_{1}^{(i)}, \varepsilon_{2}^{(i)}, a_{\beta a}) \prod_{n=1}^{k-1} \ell^{(n)}_{\varepsilon_{\beta a}}(\varepsilon_{1}^{(n)}, \varepsilon_{2}^{(n)}, a_{\beta a}) \prod_{l=1}^{k} Z_{C_{2}}^{\text{inst}}(\varepsilon_{1}^{(l)}, \varepsilon_{2}^{(l)}, \vec{a}^{(l)}; q_{\text{eff}})}{\varepsilon_{1}^{(l)} \varepsilon_{2}^{(l)}} .
\]

**Proof.** From the identities

\[
\sum_{l=1}^{k} \frac{1}{\varepsilon_{1}^{(l)} \varepsilon_{2}^{(l)}} = \frac{1}{k \varepsilon_{1} \varepsilon_{2}} \quad \text{and} \quad \sum_{l=1}^{k} \frac{\varepsilon_{1}^{(l)} + (\vec{v}_{a})_{l-1} \varepsilon_{2}^{(l)}}{\varepsilon_{1}^{(l)} \varepsilon_{2}^{(l)}} = 0 ,
\]

it follows that

\[
\sum_{l=1}^{k} \frac{1}{2 \varepsilon_{1}^{(l)} \varepsilon_{2}^{(l)}} \sum_{\alpha = 1}^{r} (a_{\alpha}^{(l)})^{2} = \sum_{\alpha = 1}^{r} \frac{a_{\alpha}^{2}}{2k \varepsilon_{1} \varepsilon_{2}} + \sum_{\alpha, \beta = 1}^{r} \sum_{l=1}^{k} \left( \varepsilon_{1}^{(l)} \varepsilon_{2}^{(l)} \right) \left( (\vec{v}_{a})_{l-1} \varepsilon_{2}^{(l)} + (\vec{v}_{a})_{l} \varepsilon_{1}^{(l)} \right) .
\]

By the localization formula (cf. Equation (5.7)) we get
\[
\sum_{l=1}^{k} \left( (\bar{v}_\alpha)_l \varepsilon_1^{(l)} + (\bar{v}_\alpha)_l \varepsilon_2^{(l)} \right) \left( (\bar{v}_\beta)_l \varepsilon_1^{(l)} + (\bar{v}_\beta)_l \varepsilon_2^{(l)} \right)
= \left( \sum_{i=1}^{k-1} (\bar{v}_\alpha)_i [D_i] \right) \cdot \left( \sum_{i=1}^{k-1} (\bar{v}_\beta)_i [D_i] \right) = -\bar{v}_\alpha \cdot C\bar{v}_\beta.
\]
Thus
\[
\prod_{l=1}^{k} Z_{c^2}^{\text{cl}}(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \bar{a}^{(l)}; \tau_1) = \exp \left( - \sum_{l=1}^{k} \frac{\tau_1}{2\varepsilon_1^{(l)} \varepsilon_2^{(l)}} \sum_{\alpha=1}^{r} \left( a_{\alpha}^{(l)} \right)^2 \right)
= \exp \left( - \frac{\tau_1}{2k \varepsilon_1 \varepsilon_2} \sum_{\alpha=1}^{r} a_{\alpha}^2 \right) \exp \left( \frac{\tau_1}{2} \sum_{\alpha,\beta=1}^{r} \bar{v}_\alpha \cdot C\bar{v}_\beta \right)
= Z_{c^2}^{\text{cl}}(\varepsilon_1, \varepsilon_2, \bar{a}; \tau_1) \exp \left( \frac{\tau_1}{2} \sum_{\alpha,\beta=1}^{r} \bar{v}_\alpha \cdot C\bar{v}_\beta \right)
\]
and the assertion now follows straightforwardly.
\[\square\]

**Remark 5.17.** The partition function \( Z_{X_k}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \bar{a}; q_{\text{eff}}, \xi, \tau_1) \) for \( \tau_1 = 0 \) coincides with \( Z_{X_k}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \bar{a}; q_{\text{eff}}, \xi) \). \[\triangle\]

### 5.2. \( \mathcal{N} = 2^s \) gauge theory.

**Generating function for correlators of p-observables.** Let \( T_{\mu} = C^* \) and \( H_{T_{\mu}}^* (\text{pt}; \mathbb{Q}) = \mathbb{Q}[\mu] \). For a \( T \)-equivariant locally free sheaf \( G \) of rank \( n \) on the moduli space \( \mathcal{M}_{r,\bar{u},\Delta}(\mathcal{R}_k, \mathcal{D}_\infty, \mathcal{F}_0, \mathcal{F}_\infty) \) we define the class
\[
E_{\mu}(G) := \mu^n \cdot (e_1)_T(G) \mu^{n-1} + \cdots + (e_n)_T(G) \in H_{T \times T_{\mu}}^* \left( \mathcal{M}_{r,\bar{u},\Delta}(\mathcal{R}_k, \mathcal{D}_\infty, \mathcal{F}_0, \mathcal{F}_\infty) ; \mathbb{Q} \right).
\]
As previously, let \( \bar{v} \in \frac{1}{k} \mathbb{Z}^{k-1} \) be such that \( k v_{k-1} = \sum_{i=0}^{k-1} i w_i \mod k \) for fixed rank \( r \) and holonomy at infinity \( \bar{u} \). Define
\[
Z_{\bar{v}}^{\bar{u}}(\varepsilon_1, \varepsilon_2, \bar{a}; \mu; q, \bar{r}, \bar{r}^{(1)}, \ldots, \bar{r}^{(k-1)}) := \sum_{\Delta \in \frac{1}{k} \mathbb{Z}^{k-1}} q^{\Delta + \frac{1}{k} \bar{v}_{\mathcal{U}}} \|ar{v}\| \int_{\mathcal{M}_{r,\bar{u},\Delta}(\mathcal{R}_k, \mathcal{D}_\infty, \mathcal{F}_0, \mathcal{F}_\infty)} E_{\mu}(T M_{r,\bar{u},\Delta}(\mathcal{R}_k, \mathcal{D}_\infty, \mathcal{F}_0, \mathcal{F}_\infty))
\times \exp \left( \sum_{\delta=0}^{\infty} \left( \sum_{i=1}^{k-1} \left[ \text{ch}_T(\mathcal{E})/[\mathcal{D}_i] \right]_s + \tau_s \left[ \text{ch}_T(\mathcal{E})/[X_k] \right]_{s-1} \right) \right).
\]

**Definition 5.18.** The generating function for correlators of p-observables of \( \mathcal{N} = 2 \) gauge theory on \( X_k \) with an adjoint hypermultiplet of mass \( \mu \) is
\[
Z_{X_k}^{\bar{u}}(\varepsilon_1, \varepsilon_2, \bar{a}; \mu; q, \bar{r}, \bar{r}^{(1)}, \ldots, \bar{r}^{(k-1)}) := \sum_{\bar{v} \in \frac{1}{k} \mathbb{Z}^{k-1}} \xi^{\bar{v}} \hat{Z}_{\bar{v}}^{\bar{u}}(\varepsilon_1, \varepsilon_2, \bar{a}; \mu; q, \bar{r}, \bar{r}^{(1)}, \ldots, \bar{r}^{(k-1)}).
\]
Proposition 5.19. The generating function $Z^*_X(\varepsilon_1, \varepsilon_2, \tilde{\alpha}; q, \tilde{\xi}, \tilde{\tau}, \tilde{\ell}^{(1)}, \ldots, \tilde{\ell}^{(k-1)})$ assumes the form

$$Z^*_X(\varepsilon_1, \varepsilon_2, \tilde{\alpha}; q, \tilde{\xi}, \tilde{\tau}, \tilde{\ell}^{(1)}, \ldots, \tilde{\ell}^{(k-1)}) = \sum_{\tilde{v} \in \mathbb{Z}^{k-1}} \tilde{v}^{\tilde{v}} \sum_{(Y, \bar{v})} \frac{\sum_{\alpha=1}^r n_{\alpha} + \frac{1}{2} \sum_{\alpha=1}^r \bar{v}_{\alpha} - C \bar{v}_{\alpha}}{1 \prod_{\alpha, \beta=1}^r m_{Y, \bar{v}}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha} + \mu) \prod_{n=1}^k \ell_{\beta \alpha}^{(n)}(1, \varepsilon_2, a_{\beta \alpha} + \mu)}$$

$$\times \left( \prod_{l=1}^k \exp \left( \sum_{l=1}^{\infty} \left( \sum_{s=0}^{l} \left( \left( \varepsilon_s \sum_{i=1}^{l-2} t_i + \sum_{i=l+1}^{k-1} \sum_{i=1}^{l-2} t_i \right) \left[ \exp(\tilde{\tau}_s) \left( \varepsilon_1, \varepsilon_2, a^{(l)} \right) \right]_s \right) + \left( \left( \varepsilon_2 \sum_{i=1}^{l-2} t_i \right) \left[ \exp(\tilde{\tau}_s) \left( \varepsilon_1, \varepsilon_2, a^{(l)} \right) \right]_s \right) \right) \right).$$

Proof. By the localization formula we get

$$Z^*_X(\varepsilon_1, \varepsilon_2, \tilde{\alpha}; q, \tilde{\xi}, \tilde{\tau}, \tilde{\ell}^{(1)}, \ldots, \tilde{\ell}^{(k-1)}) = \sum_{\tilde{v} \in \mathbb{Z}^{k-1}} \tilde{v}^{\tilde{v}} \sum_{(Y, \bar{v})} \frac{\sum_{\alpha=1}^r n_{\alpha} + \frac{1}{2} \sum_{\alpha=1}^r \bar{v}_{\alpha} - C \bar{v}_{\alpha}}{1 \prod_{\alpha, \beta=1}^r m_{Y, \bar{v}}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha} + \mu) \prod_{n=1}^k \ell_{\beta \alpha}^{(n)}(1, \varepsilon_2, a_{\beta \alpha} + \mu)}$$

$$\times \left( \prod_{l=1}^k \exp \left( \sum_{l=1}^{\infty} \left( \sum_{s=0}^{l} \left( \left( \varepsilon_s \sum_{i=1}^{l-2} t_i + \sum_{i=l+1}^{k-1} \sum_{i=1}^{l-2} t_i \right) \left[ \exp(\tilde{\tau}_s) \left( \varepsilon_1, \varepsilon_2, a^{(l)} \right) \right]_s \right) + \left( \left( \varepsilon_2 \sum_{i=1}^{l-2} t_i \right) \left[ \exp(\tilde{\tau}_s) \left( \varepsilon_1, \varepsilon_2, a^{(l)} \right) \right]_s \right) \right) \right).$$

Note that

$$t_{(Y, \bar{v})}^* E_\mu (TM_{r, \bar{v}, \Delta} (\mathcal{X}, \mathcal{D}, \mathcal{F}_\infty))$$

where $d$ the dimension of the moduli space $\mathcal{M}_{r, \bar{v}, \Delta} (\mathcal{X}, \mathcal{D}, \mathcal{F}_\infty)$. Since the tangent space at the fixed point $(\bar{Y}, \bar{v})$ as a representation of the torus $T$ is a direct sum of one-dimensional $T$-modules (see Section 4.7), we get

$$t_{(Y, \bar{v})}^* E_\mu (TM_{r, \bar{v}, \Delta} (\mathcal{X}, \mathcal{D}, \mathcal{F}_\infty)) = \sum_{l=0}^d \mu^{d-l} (\alpha) t_{(Y, \bar{v})}^* (T (Y, \mathcal{E})) \mathcal{M}_{r, \bar{v}, \Delta} (\mathcal{X}, \mathcal{D}, \mathcal{F}_\infty).$$

By using the computations of Section 5.1 we then get the assertion. □
Instanton partition function.

**Definition 5.21.** The instanton partition function is
\[
Z^{\text{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q, \tilde{\xi}) := Z^*_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q, \tilde{\xi}, 0, \ldots, 0).
\]

By Equation (5.20) and the corresponding expression for the Nekrasov partition function of gauge theory on \(\mathbb{R}^d\) with one adjoint hypermultiplet of mass \(\mu\) [19, Equation (3.26)]
\[
Z^{\text{inst}}_{\mathbb{C}^2}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q) := \sum_{\gamma} \frac{1}{\prod_{\alpha, \beta=1}^r m_{Y_{\alpha, Y_{\beta}}}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha} + \mu)} 
\]
we obtain
\[
Z^{\text{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q, \tilde{\xi})
= \sum_{\tilde{v} \in \mathbb{Z}^k} \xi^\tilde{v} \sum_{\tilde{\theta}} q^{\frac{1}{2} \sum_{a=1}^r \bar{v}_a - C\bar{v}_a} \prod_{\alpha, \beta=1}^r \prod_{n=1}^{k-1} \ell_{\tilde{v}_{\alpha \beta}}^{(n)}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha} + \mu) 
\]
\times \prod_{i=1}^k Z^{\text{inst}}_{\mathbb{C}^2}((\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q, \tilde{\xi}, 0, \ldots, 0).}

**Correlators of quadratic 0-observables.** As in Section 5.1 we define
\[
Z^{\text{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q, \tilde{\xi}, \tau_1) := Z^*_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q, \tilde{\xi}, \tilde{\xi}, 0, \ldots, 0).
\]

From Equation (5.20) it follows that
\[
Z^{\text{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q, \tilde{\xi}, \tau_1)
= \sum_{\tilde{v} \in \mathbb{Z}^k} \xi^\tilde{v} \sum_{\tilde{\theta}} q^{\frac{1}{2} \sum_{a=1}^r \bar{v}_a - C\bar{v}_a} \prod_{l=1}^k \exp \left( - \tau_1 \left[ \text{ch}_{\tilde{Y}_l}(\varepsilon_1, \varepsilon_2, \tilde{a}) \right]_0 \right) 
\]
\times \prod_{i=1}^k \prod_{\alpha, \beta=1}^r m_{Y_{\alpha, Y_{\beta}}}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha} + \mu) 
\times \prod_{n=1}^{k-1} \ell_{\tilde{v}_{\alpha \beta}}^{(n)}(\varepsilon_1, \varepsilon_2, a_{\beta \alpha} + \mu)
\]
As in the case of pure \(\mathcal{N} = 2\) gauge theory we get
\[
Z^{\text{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q, \tilde{\xi}, \tau_1) = Z^{\text{cl}}_{\mathbb{C}^2}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q_{\text{eff}}, \tilde{\xi}, \tau_1) Z^{\text{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q_{\text{eff}}, \tilde{\xi}, \tau_1)
\]

where the partition function \(Z^{\text{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q_{\text{eff}}, \tilde{\xi}, \tau_1)\) is defined analogously to (5.16).

**5.3. \(\mathcal{N} = 4\) gauge theory.**

**Definition 5.22.** The Vafa-Witten partition function for \(\mathcal{N} = 4\) gauge theory on \(X_k\) is
\[
Z^{\text{W}}_{X_k}(q, \tilde{\xi}) := \lim_{\mu \to 0} Z^{\text{inst}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; q, \tilde{\xi})
\]
By using our previous computations we get

\[
Z_{X_k}^{\text{VW}}(q, \xi) = \sum_{\vec{v} \in \mathbb{Z}^{k-1}} \xi^{\vec{v}} \sum_{k \nu_k = 1} q^{\frac{\vec{v}}{2}} \sum_{\vec{v} \in \mathbb{Z}^{k-1}} \prod_{n_\alpha = 1} q^{\frac{\vec{n}_\alpha}{2}} \alpha \vec{v}, \tag{5.23}
\]

where

\[
\eta(q) := q^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)
\]

is the Dedekind \(\eta\) function.

Let \(c \in \{0, 1, \ldots, k - 1\}\) be the equivalence class of \(\sum_{i=0}^{k-1} i w_i\) modulo \(k\). Set \(\vec{v} := \vec{v} - C^{-1} \vec{e}_c\) if \(c > 0\) and \(\vec{v} := \vec{v}\) otherwise, where \(\vec{e}_c\) is the \(c\)-th coordinate vector of \(\mathbb{Z}^{k-1}\). Then \(n_l = v_l - (C^{-1})^l c\) for any \(l = 1, \ldots, k - 1\). Similarly, for any \(\alpha = 1, \ldots, r\) define \(\vec{n}_\alpha := \vec{v} - C^{-1} \vec{e}_{c_\alpha}\) if \(c_\alpha > 0\) and \(\vec{n}_\alpha := \vec{v}\) otherwise, where \(c_\alpha = i\) if \(\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^{i} w_j\) for \(i = 0, 1, \ldots, k - 1\). Both \(\vec{v}\) and \(\vec{n}_\alpha\) for \(\alpha = 1, \ldots, r\) can be regarded as elements in the root lattice \(\Omega\) of type \(A_{k-1}\) (cf. Remark 4.6). Note that \(\sum_{\alpha=1}^r c_\alpha = c\). Then Equation (5.23) becomes

\[
Z_{X_k}^{\text{VW}}(q, \xi) = q^{\frac{k}{2}} \eta(q)^{-r} \sum_{\vec{v} \in \mathbb{Z}^{k-1}} \sum_{\vec{n}_\alpha = \vec{v}} \prod_{i=1}^{k-1} q^{\frac{1}{2} \sum_{i=1}^{r} (\vec{n}_\alpha_i + (C^{-1})^i c_\alpha)} q^{\frac{1}{2} \sum_{i=1}^{r} \vec{n}_\alpha_i C \vec{v} + 2(\vec{n}_\alpha)_{c_\alpha} + \frac{k-c_\alpha}{k} c_\alpha}
\]

Recall from [30, Section 14.4] (see also [31, Appendix C.2] and [27, Section 5.4]) that the character of an integrable highest weight representation of \(\hat{\mathfrak{sl}}(k)\) with highest weight \(\lambda\), whose finite part is \(\lambda\), at level one is a combination of string-functions and theta-functions given by

\[
\chi^\lambda(q, \zeta) := \eta(q)^{-k+1} \sum_{\gamma^\vee \in \Omega^\vee} q^{\frac{1}{2} |\gamma^\vee + \lambda|^2} e^{2\pi i (\gamma^\vee + \lambda) \cdot \zeta},
\]

where \(\zeta := \sum_{i=0}^{k-1} z_i \gamma_i^\vee\) and \(\gamma_i^\vee\) denotes the \(i\)-th simple coroot for \(i = 1, \ldots, k - 1\). Recall that the \(d\)-th fundamental weight of type \(A_{k-1}\) is

\[
\omega_d := \left( 1 - \frac{d}{k}, \ldots, 1 - \frac{d}{k}, -\frac{d}{k}, \ldots, -\frac{d}{k} \right) \in \mathbb{Z}^{k}
\]

for \(d = 1, \ldots, k - 1\) and the fundamental weights of type \(\hat{A}_{k-1}\) are \(\hat{\omega}_0 = \omega_0 + \hat{\omega}_0\) for \(d = 1, \ldots, k - 1\). With \(\gamma_i^\vee := \sum_{i=1}^{k-1} m_i \gamma_i^\vee\) we then get

\[
\frac{1}{2} |\gamma^\vee + \omega_d|^2 = \sum_{i=1}^{k} \left( m_i - m_i m_{i+1} \right) + m_d + \frac{k - d}{2k} d,
\]

\[
e^{2\pi i (\gamma^\vee + \omega_d) \cdot \zeta} = y_{d}^{m_{d-1} - m_{d-2} - m_d} y_{d-1}^{2m_{d-1} - m_{d-2} - m_d} y_{d-2}^{2m_{d-2} - m_{d-3} - m_d} \ldots y_d^{2m_1 - m_0} y_{d-1}^{2m_{d-1} - m_{d-2} - m_d} \ldots y_{d-2}^{2m_{d-2} - m_{d-3} - m_d} \ldots y_0^{2m_1 - m_0}
\]

\[
\times \prod_{i=1}^{k-1} \xi_i^{m_i + (C^{-1})^i \cdot d},
\]
where we introduced $y_i := e^{2\pi i z_i}$ and
\[ \xi_1 := \frac{y_1^2}{y_2}, \quad \xi_2 := \frac{y_2^2}{y_1 y_3}, \quad \ldots, \quad \xi_{k-1} := \frac{y_{k-1}^2}{y_{k-2}}. \]
It follows that
\[ Z_{X_k}^{VW}(q, \zeta) = q^{\frac{r^2}{12}} \prod_{\alpha=1}^r \frac{\chi_{\alpha}(q, \zeta)}{\eta(q)} = q^{\frac{r^2}{12}} \prod_{j=0}^{k-1} \left( \frac{\chi_{\alpha}(q, \zeta)}{\eta(q)} \right)^{w_j}. \]
This expression agrees with the formula obtained in [39, Corollary 4.12] (see also [39, Remark 4.2.3]). For trivial holonomies at infinity, i.e., when $e_\alpha = 0$ for any $\alpha = 1, \ldots, r$, which is equivalent to $w_0 = r$ and $w_j = 0$ for $j = 1, \ldots, k - 1$, this expression agrees with [38, Equation (4.8)] and [45, Equation (3.23)]. It displays the Vafa-Witten partition function as a character of the affine Lie algebra $\hat{\mathfrak{gl}}(k)$, which confirms its correct modularity properties as implied by S-duality.

### 5.4. Fundamental matter.

**Generating function for correlators of $p$-observables.** Let $N \leq 2r$ be a positive integer, $T_N = (\mathbb{C}^*)^N$ and $H^*_T(pt; \mathbb{Q}) = \mathbb{Q}[\mu_1, \ldots, \mu_N]$. Let $V$ be the natural bundle on $\mathcal{M}_{r; k, \Delta} (\mathcal{D}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^0)$; recall that $V$ is a $T$-equivariantly locally free sheaf. To define the partition functions and correlators for gauge theories with fundamental matter we consider the class
\[ \prod_{s=1}^N E_{\mu_s}(V) \in H^*_{T \times T_N} (\mathcal{M}_{r; k, \Delta} (\mathcal{D}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^0) ; \mathbb{Q}). \]
It has degree
\[ \dim_{\mathbb{C}} (\mathcal{M}_{r; k, \Delta} (\mathcal{D}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^0)) - N \cdot \text{rk}(V). \]  
(5.24)
By using Theorem 4.13 and Proposition 4.18, we see that the degree (5.24) is nonnegative if and only if
\[ 2r \Delta - \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} \bar{u} \cdot \bar{u}(j) - N \left( \Delta + \frac{1}{2r} \bar{v} \cdot C \bar{v} - \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} w_j \right) \geq 0 \]  
(5.25)
for $\bar{v} := C^{-1} \bar{u}$.

**Remark 5.26.** We assume $N \leq 2r$ because this is the constraint on the number of fundamental matter fields in asymptotically free $\mathcal{N} = 2$ gauge theories on $\mathbb{R}^4$. We impose it in our case because we want to compare our gauge theories with fundamental matter on $X_k$ to those on $\mathbb{R}^4$. As an example, let us write explicitly the inequality (5.25) for $k = 2$, in which case it becomes
\[ w_1^2 - 4u^2 \geq 0. \]
If the fixed holonomy at infinity is trivial, i.e., $\bar{w} = (r, 0)$, this gives $v = 0$; this is the case considered in [14]. For arbitrary $k \geq 2$, if $\bar{w} = (r, 0, \ldots, 0)$ and $\bar{v} = 0$, the inequality (5.25) is automatically satisfied when $N \leq 2r$.

Since $N \leq 2r$, we get
\[ 2r \Delta - \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} \bar{u} \cdot \bar{u}(j) - N \left( \Delta + \frac{1}{2r} \bar{v} \cdot C \bar{v} - \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} w_j \right) \geq d_{\bar{w}}(\bar{v}) \]
where we defined
\[ d_{\bar{w}}(\bar{v}) := r \sum_{j=1}^{k-1} (C^{-1})^{jj} w_j - \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} \bar{u} \cdot \bar{u}(j) - \bar{v} \cdot C \bar{v}. \]
We define the set
\[ \Omega_{\bar{v}} := \left\{ \bar{v} \in \frac{1}{k} \mathbb{Z}^{k-1} \mid k v_{k-1} = \sum_{i=0}^{k-1} i w_i \mod k \text{ and } d_{\bar{v}}(\bar{v}) \geq 0 \right\}. \] (5.27)

Then the inequality (5.25) is satisfied for any choice of discriminant \( \Delta \) and for \( \bar{v} \in \Omega_{\bar{v}} \). Although the constraint \( d_{\bar{v}}(\bar{v}) \geq 0 \) is stronger than (5.25), it is natural in light of the discussion in Remark 5.26.

We call the \( N = 2 \) supersymmetric gauge theory on \( X_k \) with \( N \leq 2r \) fundamental hypermultiplets asymptotically free. If \( N = 2r \), we call the associated gauge theory conformal; in this case we consider the subsets
\[ \Omega_{\bar{v}}^{\text{conf}} := \left\{ \bar{v} \in \frac{1}{k} \mathbb{Z}^{k-1} \mid k v_{k-1} = \sum_{i=0}^{k-1} i w_i \mod k \text{ and } d_{\bar{v}}(\bar{v}) = 0 \right\} \subset \Omega_{\bar{v}} \]
for which the degree (5.24) is equal to 0. In the following we treat only the asymptotically free case. The conformal case is completely analogous, and simply amounts to restricting the set \( \Omega_{\bar{v}} \) to its subset \( \Omega_{\bar{v}}^{\text{conf}} \) in all sums below.

For fixed \( \bar{v} \in \Omega_{\bar{v}} \), we define
\[
Z^N_{\bar{v}}(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \bar{\tau}, \bar{\ell}^{(1)}, \ldots, \bar{\ell}^{(k-1)}) := \sum_{\Delta} q^{\Delta+\frac{1}{2} \bar{v} \cdot \bar{\tau}} \int_{\mathcal{M}_{r,\Delta}(\mathcal{X}_k, \mathcal{D}_N^0, \mathcal{R}^r_{\infty})} \prod_{s=1}^{N} E_{\mu_s}(V) \times \exp \left( \sum_{s=0}^{\infty} \left( k-1 \sum_{i=1}^{k-1} \bar{v}_s \left[ \text{ch}_{T}(\mathcal{E})/|\mathcal{R}_s| \right]_s + \bar{\tau}_s \left[ \text{ch}_{T}(\mathcal{E})/|X_k| \right]_{s-1} \right) \right).
\]

**Definition 5.28.** The generating function for correlators of p-observables of \( N = 2 \) gauge theory on \( X_k \) with \( N \) fundamental hypermultiplets of masses \( \mu_1, \ldots, \mu_N \) is
\[
Z^N_{X_k}(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \bar{\tau}, \bar{\ell}^{(1)}, \ldots, \bar{\ell}^{(k-1)}) := \sum_{\bar{v} \in \Omega_{\bar{v}}} \xi^{\bar{v}} Z^N_{\bar{v}}(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \bar{\tau}, \bar{\ell}^{(1)}, \ldots, \bar{\ell}^{(k-1)}).
\]

**Proposition 5.29.** The generating function \( Z^N_{X_k}(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \bar{\tau}, \bar{\ell}^{(1)}, \ldots, \bar{\ell}^{(k-1)}) \) assumes the form
\[
Z^N_{X_k}(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \bar{\tau}, \bar{\ell}^{(1)}, \ldots, \bar{\ell}^{(k-1)}) = \sum_{\bar{v} \in \Omega_{\bar{v}}} \xi^{\bar{v}} \sum_{(\varepsilon, \bar{a})} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\alpha=1}^{\sum \bar{v}_s} \frac{n_{\alpha}}{C_n^{\alpha}} \bar{v}_\alpha C^{\bar{v}_\alpha}
\]
\[
\times \prod_{s=1}^{N} \prod_{i=1}^{\alpha_1} m_{Y_s}(\varepsilon_1, \varepsilon_2, \bar{a}_s, \mu_s) \prod_{n=1}^{k-1} \frac{\rho(n)}{\nu_s} \frac{\rho(n)}{\nu_s} \frac{\rho(n)}{\nu_s} \frac{\rho(n)}{\nu_s} \frac{\rho(n)}{\nu_s} \frac{\rho(n)}{\nu_s} \frac{\rho(n)}{\nu_s} \frac{\rho(n)}{\nu_s} \frac{\rho(n)}{\nu_s} \frac{\rho(n)}{\nu_s}
\]
\[
\times \prod_{l=1}^{k} \exp \left( \sum_{s=0}^{\infty} \left( \left( k-1 \sum_{i=1}^{k-1} t_s^{(i)} \varepsilon_1^{(i-1)} + \bar{\tau}_s \left[ \text{ch}_{T}(\mathcal{E})/|X_k| \right]_{s-1} \right) \right) \right)
\]
\[
\times \exp \left( \sum_{s=0}^{\infty} \left( \sum_{i=1}^{k-1} t_s^{(i)} \varepsilon_1^{(i-1)} + \bar{\tau}_s \left[ \text{ch}_{T}(\mathcal{E})/|X_k| \right]_{s-1} \right) \right)
\]}
The Instanton partition function.

\[ Z_{\text{inst}}^{N}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \vec{q}, \vec{\xi}, 0, \ldots, 0) \]

By Equation (5.30) and the corresponding expression for the Nekrasov partition function of $U(r)$ gauge theory on $\mathbb{R}^4$ with $N$ fundamental hypermultiplets of masses $\mu_1, \ldots, \mu_N$ [19, Equation (3.22)]

\[ Z_{\mathbb{C}^2}^{N,\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \vec{q}, \vec{\xi}) \]

we get

\[ Z_{X_k}^{N,\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \vec{q}, \vec{\xi}, \tau_1) := Z_{X_k}^{N}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \vec{q}, \vec{\xi}, 0, \ldots, 0) \]

From Equation (5.30) it follows that

\[ Z_{X_k}^{N,\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \vec{q}, \vec{\xi}, \tau_1) = \sum_{\vec{v} \in \Omega_{\vec{v}}} \xi^{\vec{v}} \sum_{(\vec{Y}, \vec{d})} \xi^{\vec{Y}, \vec{d}} \sum_{\vec{a} \in \Omega_{\vec{a}}} \xi^{\vec{a}-\vec{c} \vec{v}_{\alpha}} \times \left( \sum_{\vec{v} \in \Omega_{\vec{v}}} \xi^{\vec{v}} \sum_{(\vec{Y}, \vec{d})} \xi^{\vec{Y}, \vec{d}} \sum_{\vec{a} \in \Omega_{\vec{a}}} \xi^{\vec{a}-\vec{c} \vec{v}_{\alpha}} \right) \]

The proof is completely analogous to the proof of Proposition 5.19 by noting that

\[ \tau_{\text{inst}}(\vec{Y}, \vec{d}) E_{\mu_s}(V) = \sum_{l=0}^{\text{rk}(V)} \mu_s^{\text{rk}(V)-l} (c_1)(V_{\vec{Y}, \vec{d}}) \]

and since $V_{\vec{Y}, \vec{d}}$ as a $T$-module is a direct sum of one-dimensional $T$-modules (see Section 4.7) we get

\[ \tau_{\text{inst}}(\vec{Y}, \vec{d}) E_{\mu_s}(V) = \prod_{a=1}^{r} \prod_{i=1}^{k} m_{Y_a}(\varepsilon_1, \varepsilon_2, a_{\alpha} + \mu_s) \prod_{n=1}^{k-1} \ell^{(n)}_{\vec{v}_{\alpha}}(\varepsilon_1, \varepsilon_2, a_{\alpha} + \mu_s). \]

\[ Z_{X_k}^{N,\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \vec{q}, \vec{\xi}) := \sum_{\vec{Y}} \xi^{\vec{Y}} \sum_{\vec{a} \in \Omega_{\vec{a}}} \xi^{\vec{a}-\vec{c} \vec{v}_{\alpha}} \times \left( \sum_{\vec{v} \in \Omega_{\vec{v}}} \xi^{\vec{v}} \sum_{(\vec{Y}, \vec{d})} \xi^{\vec{Y}, \vec{d}} \sum_{\vec{a} \in \Omega_{\vec{a}}} \xi^{\vec{a}-\vec{c} \vec{v}_{\alpha}} \right) \]

\[ = \sum_{\vec{v} \in \Omega_{\vec{v}}} \xi^{\vec{v}} \sum_{q} \frac{1}{r} \prod_{s=1}^{r} \prod_{a=1}^{k} m_{Y_s}(\varepsilon_1, \varepsilon_2, a_{\alpha} + \mu_s) \prod_{n=1}^{k-1} \ell^{(n)}_{\vec{v}_{\alpha}}(\varepsilon_1, \varepsilon_2, a_{\alpha} + \mu_s) \prod_{i=1}^{k} Z_{X_k}^{N,\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \vec{q}, \vec{\xi}). \]
As previously we get
\[ Z_{X_k}^{N,\phi}(\varepsilon_1, \varepsilon_2, \bar{\sigma}, \bar{\mu}; q, \xi, \tau_1) = Z_{C^2}^{\text{cl}}(\varepsilon_1, \varepsilon_2, \bar{\sigma}; \tau_1)^{1/2} Z_{X_k}^{N,\phi,\text{inst}}(\varepsilon_1, \varepsilon_2, \bar{\sigma}, \bar{\mu}; q_{\text{eff}}, \xi, \tau_1), \]
where the partition function \( Z_{X_k}^{N,\phi,\text{inst}}(\varepsilon_1, \varepsilon_2, \bar{\sigma}, q_{\text{eff}}, \xi, \tau_1) \) is defined analogously to (5.16).

6. Perturbative partition functions

6.1. Perturbative Euler classes. Let \([([\mathcal{E}, \phi_\varepsilon])] \) and \([([\mathcal{E}', \phi_{\varepsilon'}])] \) be \( T \)-fixed points of the respective moduli spaces \( \mathcal{M}_{r,\bar{\sigma},\Delta}(\mathcal{F}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{\bar{w}}) \) and \( \mathcal{M}_{r',\bar{\sigma}',\Delta'}(\mathcal{F}_k, \mathcal{D}_\infty, \mathcal{F}_{\infty}^{\bar{w}'}) \). By Equation (4.26) the corresponding character of the Carlsson-Okounkov bundle is given by
\[ \text{ch}_T E_{\{([\mathcal{E}, \phi_\varepsilon])] \}, ([\mathcal{E}', \phi_{\varepsilon'}])] = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} e_{\alpha} e_{\alpha}'^{-1} (M_{\alpha,\beta}(t_1, t_2) + L_{\alpha,\beta}(t_1, t_2)), \]
where the vertex contribution \( M_{\alpha,\beta}(t_1, t_2) \) is given in Equation (4.28) and the edge contribution \( L_{\alpha,\beta}(t_1, t_2) \) is defined by Equation (4.25) which can be written in the form
\[ L_{\alpha,\beta}(t_1, t_2) = -\chi_{T_1}(\bar{X}_k, \pi_k(\mathcal{R}^{\bar{a}_\beta'-\bar{a}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))). \]

Definition 6.2. The virtual \( \bar{T} \)-equivariant Chern character of the Carlsson-Okounkov bundle at the \( \bar{T} \)-fixed point \( ([([\mathcal{E}, \phi_\varepsilon])] \), ([\mathcal{E}', \phi_{\varepsilon'}]) \) is
\[ \text{ch}_{T}^{\text{vir}} E_{\{([\mathcal{E}, \phi_\varepsilon]) \}, ([\mathcal{E}', \phi_{\varepsilon'}])] := \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} e_{\alpha} e_{\alpha}'^{-1} (M_{\alpha,\beta}(t_1, t_2) + L_{\alpha,\beta}(t_1, t_2)), \]
where \( L_{\alpha,\beta}(t_1, t_2) := -\chi_{T_1}(X_k, \mathcal{R}^{\bar{a}_\beta'-\bar{a}_\alpha}). \) The perturbative part of the \( \bar{T} \)-equivariant Chern character of the Carlsson-Okounkov bundle at the \( \bar{T} \)-fixed point \( ([([\mathcal{E}, \phi_\varepsilon])] \), ([\mathcal{E}', \phi_{\varepsilon'}]) \) is
\[ \text{ch}_{T}^{\text{pert}} E_{\{([\mathcal{E}, \phi_\varepsilon]) \}, ([\mathcal{E}', \phi_{\varepsilon'}])] := \text{ch}_{T}^{\text{vir}} E_{\{([\mathcal{E}, \phi_\varepsilon]) \}, ([\mathcal{E}', \phi_{\varepsilon'}])] - \text{ch}_{T} E_{\{([\mathcal{E}, \phi_\varepsilon]) \}, ([\mathcal{E}', \phi_{\varepsilon'}])}. \]

By using arguments from the proof of [18, Theorem 4.1], one can decompose the equivariant Euler characteristic in (6.1) with respect to the torus-invariant open subsets of \( \bar{X}_k \) to get
\[ L_{\alpha,\beta}(t_1, t_2) = -\sum_{\sigma \in \Sigma_k(2)} S\left( \chi_{T_1}(U_\sigma, \pi_k(\mathcal{R}^{\bar{a}_\beta'-\bar{a}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|U_\sigma) \right), \]
where \( \chi_{T_1}^\sigma \) is the equivariant Euler characteristic over \( U_\sigma \) and \( S(f) \) is the sum of \( f \) for any \( f \in \mathbb{Z}[M] \) (see [18, Section 1.3] for the definitions of these notions). Since \( \pi_k \) is an isomorphism over \( X_k \) and similarly
\[ L_{\alpha,\beta}(t_1, t_2) = -\sum_{\sigma \in \Sigma_k(2)} S\left( \chi_{T_1}(U_\sigma, \mathcal{R}^{\bar{a}_\beta'-\bar{a}_\alpha}|U_\sigma) \right), \]
we get
\[ \text{ch}_{T}^{\text{pert}} E_{\{([\mathcal{E}, \phi_\varepsilon]) \}, ([\mathcal{E}', \phi_{\varepsilon'}])} \]
\[ = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} e_{\alpha} e_{\alpha}'^{-1} S\left( \chi_{T_1}(U_{\infty, k}, \pi_k(\mathcal{R}^{\bar{a}_\beta'-\bar{a}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|U_{\infty, k}) \right) \]
\[ + \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} e_{\alpha} e_{\alpha}'^{-1} S\left( \chi_{T_1}(U_{\infty, 0}, \pi_k(\mathcal{R}^{\bar{a}_\beta'-\bar{a}_\alpha} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))|U_{\infty, 0}) \right). \]
Set $\bar{u}_{\beta \alpha} := \bar{u}_{\beta} - \bar{u}_{\alpha}$ and define $\bar{v}_{\beta \alpha} := C^{-1} \bar{u}_{\beta \alpha}$. Let $c_{\beta \alpha} \in \{0, 1, \ldots, k - 1\}$ be the equivalence class modulo $k$ of $k (\bar{v}_{\beta \alpha})_{k-1}$. Set $\bar{v}_{\beta \alpha} := \bar{u}_{\beta \alpha} - c_{\beta \alpha} \bar{u}_{\alpha}$ for $c_{\beta \alpha} > 0$ and $\bar{v}_{\beta \alpha} := \bar{u}_{\beta \alpha}$ otherwise, where $\bar{v}_{\beta \alpha}$ is the $c_{\beta \alpha}$-th coordinate vector of $\mathbb{Z}^{k-1}$. Define $\tilde{s}_{\beta \alpha} := C^{-1} \bar{v}_{\beta \alpha} \in \mathbb{Z}^{k-1}$. Then

$$
\sum_{i=1}^{k-1} (\bar{u}_{\beta \alpha})_i \omega_i = \sum_{j=1}^{k-1} (\bar{v}_{\beta \alpha})_j \varphi_j + \omega_{c_{\beta \alpha}},
$$

where we set $\omega_{c_{\beta \alpha}} = 0$ if $c_{\beta \alpha} = 0$. Therefore

$$
\chi^{\text{pert}}_{\text{pert}} E((\mathcal{E}, \phi_{\mathcal{E}}), (\mathcal{E}', \phi_{\mathcal{E}'})|p) = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} e_{\alpha} e_{\alpha}'^{-1} S(\chi_{\sigma_{\alpha}^{\infty}}^{\infty} \left( U_{\infty, k} | \pi_k \left( \mathcal{R}_{c_{\beta \alpha}} \otimes \mathcal{O}_{\mathcal{X}_{k}} (\mathcal{O}_{\mathcal{X}_{\infty}}) \right) \right) | U_{\infty, k} )
$$

By using the relations (3.23) and the projection formula for $\pi_k$ we get

$$
\pi_k \left( \mathcal{R}_{c_{\beta \alpha}} \otimes \mathcal{O}_{\mathcal{X}_{k}} (\mathcal{O}_{\mathcal{X}_{\infty}}) \right) | U_{\infty, k} \simeq \mathcal{O}_{\mathcal{X}_{k}} (b_{\beta \alpha}^{b_{\beta \alpha}} + \frac{b_{\beta \alpha}}{k} D_{\infty}) | U_{\infty, k},
$$

where

$$
\begin{align*}
\beta_{\beta \alpha}^0 &= \begin{cases} 
0 & \text{for } c_{\beta \alpha} = 0, k - 1, \\
1 & \text{for } 1 \leq c_{\beta \alpha} \leq k - 2,
\end{cases} \\
\beta_{\beta \alpha}^k &= \begin{cases} 
\beta_{\beta \alpha} - 1 & \text{for } 1 \leq c_{\beta \alpha} \leq k - 2, \\
1 & \text{for } c_{\beta \alpha} = k - 1,
\end{cases} \\
\beta_{\beta \alpha}^{\infty} &= \begin{cases} 
-1 & \text{for } c_{\beta \alpha} = 0, \\
-\bar{k} (c_{\beta \alpha} - 2) - 1 & \text{for } 1 \leq c_{\beta \alpha} \leq k - 2, \\
-\bar{k} - 1 & \text{for } c_{\beta \alpha} = k - 1.
\end{cases}
\end{align*}
$$

By [18, Section 2.3] and the arguments in the proof of [18, Theorem 4.1] we can rewrite the perturbative character (6.3) as

$$
\chi^{\text{pert}}_{\text{pert}} E((\mathcal{E}, \phi_{\mathcal{E}}), (\mathcal{E}', \phi_{\mathcal{E}'})|p) = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} e_{\alpha} e_{\alpha}'^{-1} \sum_{m \in M \cap Q(\sigma_{\infty, k}; b_{\beta \alpha}^{k} D_{k} + [b_{\beta \alpha}^{\infty} / k] D_{\infty})} \frac{e^m}{(1 - e^{m_{\infty, k}}) (1 - e^{m_{\infty, k}})}
$$

where $m_{\infty, k}$ and $m_{\infty, k}^2$ are the facet normals to the rays of $\sigma_{\infty, k}$, i.e., the characters orthogonal to the rays of $\sigma_{\infty, k}$, and

$$
Q(\sigma_{\infty, k}; b_{\beta \alpha}^{k} D_{k} + [b_{\beta \alpha}^{\infty} / k] D_{\infty}) := \left\{ x_1 m_{\infty, k}^1 + x_2 m_{\infty, k}^2 \bigg| x_1, x_2 \in \mathbb{Q}, \ 0 \leq x_1 + \frac{b_{\beta \alpha}^{k}}{k} < 1, \ 0 \leq x_2 + \frac{[b_{\beta \alpha}^{\infty} / k]}{k} < 1 \right\}. \quad (6.4)
$$
Similarly, for the cone \( \sigma_{\infty,0} \) we denote by \( m^1_{\infty,0} \) and \( m^2_{\infty,0} \) the facet normals to its rays and

\[
Q(\sigma_{\infty,0} : \beta_{\beta_0} D_0 + [b^\infty_{\beta_0}/k] D_\infty) := \left\{ x_1 m^1_{\infty,0} + x_2 m^2_{\infty,0} \mid x_1, x_2 \in \mathbb{Q}, \quad 0 \leq x_1 + \frac{b^0_{\beta_0}}{k} < 1, \quad 0 \leq x_2 + \frac{[b^\infty_{\beta_0}/k]}{k} < 1 \right\}.
\]  

(6.5)

Explicitly, for \( \sigma_{\infty,k} \) we have

\[
m^1_{\infty,k} = \left\{ \begin{array}{ll} (1 - \tilde{k}, -\tilde{k}) & \text{for } k \text{ even}, \\ (2 - k, -k) & \text{for } k \text{ odd} \end{array} \right., \quad \text{and} \quad m^2_{\infty,k} = \left\{ \begin{array}{ll} (1 - 2\tilde{k}, -2\tilde{k}) & \text{for } k \text{ even}, \\ (1 - k, -k) & \text{for } k \text{ odd} \end{array} \right.,
\]

while for \( \sigma_{\infty,0} \) we get

\[
m^1_{\infty,0} = \left\{ \begin{array}{ll} (k - 1, \tilde{k}) & \text{for } k \text{ even}, \\ (k - 2, k) & \text{for } k \text{ odd} \end{array} \right., \quad \text{and} \quad m^2_{\infty,0} = \left\{ \begin{array}{ll} (-1, 0) & \text{for } k \text{ even}, \\ (-1, 0) & \text{for } k \text{ odd} \end{array} \right..
\]

To determine the sets (6.4) and (6.5) explicitly we need to consider separately two cases depending on the parity of \( k \); they each have cardinality \( \tilde{k} \).

For even \( k \) we get

\[
Q(\sigma_{\infty,k} : b^k_{\beta_0} D_k + [b^\infty_{\beta_0}/k] D_\infty) = \left\{ \left( i - b^k_{\beta_0} - \left\lfloor \frac{b^0_{\beta_0} + \frac{b^\infty_{\beta_0}}{k}}{k} \right\rfloor \right) (1 - 2\tilde{k}), i - b^k_{\beta_0} + 2\tilde{k}\left\lfloor \frac{b^0_{\beta_0} + \frac{b^\infty_{\beta_0}}{k}}{k} \right\rfloor \right\}_{i=0,1,...,\tilde{k}} \bigcup \left\{ \left( i - b^k_{\beta_0} + 1 - \left\lfloor \frac{b^0_{\beta_0} + \frac{b^\infty_{\beta_0}}{k}}{k} \right\rfloor \right) (1 - 2\tilde{k}), i - b^k_{\beta_0} - 2\tilde{k}\left( 1 - \left\lfloor \frac{b^0_{\beta_0} + \frac{b^\infty_{\beta_0}}{k}}{k} \right\rfloor \right) \right\}_{i=\tilde{k} + 1, \tilde{k} + 2, ..., \tilde{k}, \tilde{k} - 1}
\]

and

\[
Q(\sigma_{\infty,0} : b^0_{\beta_0} D_0 + [b^\infty_{\beta_0}/k] D_\infty) = \left\{ \left( i - b^0_{\beta_0} + \frac{b^0_{\beta_0} - i}{k} + \frac{b^\infty_{\beta_0}}{k} \right), i - b^0_{\beta_0} \right\}_{i=0,1,...,\tilde{k} - 1}.
\]

On the other hand, for odd \( k \) we have

\[
Q(\sigma_{\infty,k} : b^k_{\beta_0} D_k + [b^\infty_{\beta_0}/k] D_\infty) = \left\{ \left( i - b^k_{\beta_0} - \left\lfloor \frac{2b^0_{\beta_0} + \frac{b^\infty_{\beta_0}}{k}}{k} \right\rfloor \right) (1 - k), i - b^k_{\beta_0} + k\left\lfloor \frac{2b^0_{\beta_0} + \frac{b^\infty_{\beta_0}}{k}}{k} \right\rfloor \right\}_{i=0,1,...,\frac{k}{2}} \bigcup \left\{ \left( i - b^k_{\beta_0} + \left( 1 - \left\lfloor \frac{2b^0_{\beta_0} + \frac{b^\infty_{\beta_0}}{k}}{k} \right\rfloor \right) (1 - k) \right), i - b^k_{\beta_0} - k\left( 1 - \left\lfloor \frac{2b^0_{\beta_0} + \frac{b^\infty_{\beta_0}}{k}}{k} \right\rfloor \right) \right\}_{i=\frac{k}{2} + 1, \frac{k}{2} + 2, ..., \frac{k}{2} - 1}
\]

and

\[
Q(\sigma_{\infty,0} : b^0_{\beta_0} D_0 + [b^\infty_{\beta_0}/k] D_\infty) = \left\{ \left( i - b^0_{\beta_0} + \frac{2b^0_{\beta_0} - i}{k} + \frac{b^\infty_{\beta_0}}{k} \right), i - b^0_{\beta_0} \right\}_{i=0,1,...,\frac{k}{2} - 1}.
\]
The corresponding Euler classes involve infinite products which require regularization. Following [73, Appendix A], we denote by \( \Gamma_2(x|\varepsilon_1,\varepsilon_2) = \exp \left( \gamma_{\varepsilon_1,\varepsilon_2}(x) \right) \) the Barnes double gamma-function which is the double zeta-function regularization of the infinite product

\[
\prod_{i,j=0}^{\infty} \left( x - i \varepsilon_1 - j \varepsilon_2 \right).
\]

**Remark 6.6.** The function \( \varepsilon_1 \varepsilon_2 \gamma_{\varepsilon_1,\varepsilon_2}(x) \) is analytic near \( \varepsilon_1 = \varepsilon_2 = 0 \) with

\[
\lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \varepsilon_1 \varepsilon_2 \gamma_{\varepsilon_1,\varepsilon_2}(x) = -\frac{1}{2} \log x + \frac{3}{4} x^2.
\]

This result appears in Section 7. \( \triangle \)

### 6.2. \( \mathcal{N} = 2 \) gauge theory

Let \( \vec{w} := (w_0, w_1, \ldots, w_{k-1}) \in \mathbb{N}^k \) and \( r := \sum_{i=0}^{k} w_i \). Let us define \( \vec{c} := (c_1, \ldots, c_r) \in \{0, 1, \ldots, k-1\}^r \) such that \( c_\alpha = i \mod k \) if \( \sum_{i=0}^{\alpha} w_i < \alpha \leq \sum_{i=0}^{\alpha+1} w_i \). For \( \alpha, \beta = 1, \ldots, r, \alpha \neq \beta \), we define \( c_{\beta \alpha} \) to be the equivalence class modulo \( k \) of \( c_\beta - c_\alpha \). Set \( a_{\beta \alpha} = a_\beta - a_\alpha \) as before.

For even \( k \) define

\[
F_{\vec{X}_k}^{\text{pert}}(\varepsilon_1, \varepsilon_2; \vec{a}; \vec{c}) := \sum_{\alpha \neq \beta} \left( \sum_{i=0}^{k} \gamma_{-k \varepsilon_1 + k \varepsilon_2} 2k \varepsilon_2 \left( a_{\beta \alpha} + (i - b_{\beta \alpha}^k) \varepsilon_1 + \left( i - b_{\beta \alpha}^k + 2k \left( b_{\beta \alpha}^{\beta \alpha} + \frac{b_{\beta \alpha}^{\beta \alpha}}{k} \right) \right) \varepsilon_2 \right) \right)
\]

and for odd \( k \)

\[
F_{\vec{X}_k}^{\text{pert}}(\varepsilon_1, \varepsilon_2; \vec{a}; \vec{c}) := \sum_{\alpha \neq \beta} \left( \sum_{i=0}^{k} \gamma_{-k \varepsilon_1 + k \varepsilon_2} 2k \varepsilon_2 \left( a_{\beta \alpha} + (i - b_{\beta \alpha}^k) \varepsilon_1 + \left( i - b_{\beta \alpha}^k + k \left( \frac{b_{\beta \alpha}^{\beta \alpha} - 1}{k} + \frac{1 b_{\beta \alpha}^{\beta \alpha}}{k} \right) \right) \varepsilon_2 \right) \right)
\]
Definition 6.7. The perturbative partition function for pure $\mathcal{N} = 2$ gauge theory on $X_k$ is

$$Z^{\text{pert}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}) := \sum \exp \left( - F^{\text{pert}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}; \tilde{c}) \right).$$

Example 6.8. For $k = 2$ the perturbative partition function becomes

$$Z^{\text{pert}}_{X_2}(\varepsilon_1, \varepsilon_2, \tilde{a}) = \sum \prod_{\alpha \neq \beta} \exp \left( - \gamma_{\varepsilon_2 - \varepsilon_1, 2\varepsilon_2} (a_{\beta \alpha} + c_{\beta \alpha} (\varepsilon_2 - \varepsilon_1) - 2\varepsilon_2) - \gamma_{\varepsilon_1 - \varepsilon_2, 2\varepsilon_1} (a_{\beta \alpha} - 2\varepsilon_1) \right).$$

6.3. $\mathcal{N} = 2^*$ gauge theory. With the same conventions as in Section 6.2, define for even $k$

$$F^{\ast, \text{pert}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; \tilde{c})$$

$$:= \sum_{\alpha \neq \beta} \left( \sum \left( \gamma_{-k \varepsilon_1 + k \varepsilon_2, 2k \varepsilon_2} (a_{\beta \alpha} + (i - b^{k}_{\beta \alpha}) \varepsilon_1 + (i - b^{k}_{\beta \alpha} + 2k \left[ \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right]) \varepsilon_2) \right) \right)$$

$$+ \sum_{i=k}^{k-1} \left( \gamma_{-k \varepsilon_1 + k \varepsilon_2, 2k \varepsilon_2} (a_{\beta \alpha} + (i - b^{k}_{\beta \alpha}) \varepsilon_1 + (i - b^{k}_{\beta \alpha} - 2k \left( 1 - \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right)) \varepsilon_2) \right)$$

and for odd $k$

$$F^{\ast, \text{pert}}_{X_k}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; \tilde{c})$$

$$:= \sum_{\alpha \neq \beta} \left( \sum \left( \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} (a_{\beta \alpha} + (i - b^{k}_{\beta \alpha}) \varepsilon_1 + (i - b^{k}_{\beta \alpha} + k \left[ \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right]) \varepsilon_2) \right) \right)$$

$$+ \sum_{i=\left\lfloor \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right\rfloor}^{k-1} \left( \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} (a_{\beta \alpha} + (i - b^{k}_{\beta \alpha}) \varepsilon_1 + (i - b^{k}_{\beta \alpha} + k \left[ \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right]) \varepsilon_2) \right)$$

$$- \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} (a_{\beta \alpha} + (i - b^{k}_{\beta \alpha}) \varepsilon_1 + (i - b^{k}_{\beta \alpha} + k \left( 1 - \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right)) \varepsilon_2)$$

$$+ \sum_{i=\left\lfloor \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right\rfloor + 1}^{k-1} \left( \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} (a_{\beta \alpha} + (i - b^{k}_{\beta \alpha}) \varepsilon_1 + (i - b^{k}_{\beta \alpha} - k \left( 1 - \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right)) \varepsilon_2) \right)$$

$$- \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} (a_{\beta \alpha} + (i - b^{k}_{\beta \alpha}) \varepsilon_1 + (i - b^{k}_{\beta \alpha} - k \left( 1 - \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right)) \varepsilon_2)$$

$$- \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} (a_{\beta \alpha} + (i - b^{k}_{\beta \alpha}) \varepsilon_1 + (i - b^{k}_{\beta \alpha} - k \left( 1 - \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right)) \varepsilon_2)$$

$$+ \sum_{i=\left\lfloor \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right\rfloor + 1}^{k-1} \left( \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} (a_{\beta \alpha} + (i - b^{k}_{\beta \alpha}) \varepsilon_1 + (i - b^{k}_{\beta \alpha} - k \left( 1 - \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right)) \varepsilon_2) \right)$$

$$- \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} (a_{\beta \alpha} + (i - b^{k}_{\beta \alpha}) \varepsilon_1 + (i - b^{k}_{\beta \alpha} - k \left( 1 - \frac{b^{k}_{\beta \alpha} + \frac{b^{0}_{\beta \alpha}}{k}}{k} \right)) \varepsilon_2)$$
Example 6.10. For $k = 2$ the perturbative partition function becomes

\[
\mathcal{Z}_{X_k}^{\text{p}, \text{pert}}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu) := \sum_{\tilde{c}} \left( - F_{X_k}^{\text{p}, \text{pert}}(\varepsilon_1, \varepsilon_2, \tilde{a}, \mu; \tilde{c}) \right). 
\]

6.4. Fundamental matter. With the same notation as before, define for even $k$

\[
F_{X_k}^{N, \text{pert}}(\varepsilon_1, \varepsilon_2, \tilde{a}, \tilde{\mu}; \tilde{c}) := \sum_{\alpha \neq \beta} \left( \sum_{i=0}^{\tilde{k}-1} \gamma_{-k, \varepsilon_1, \varepsilon_2 \cdot 2k} \left( a_{\beta \alpha} + (i - b_{\beta \alpha}^k) \varepsilon_1 + \left( i - b_{\beta \alpha}^k - 2k \left[ \frac{b_{\beta \alpha}^{\gamma} + \frac{N}{k}}{2} \right] \right) \varepsilon_2 \right) \right)
\]

\[
+ \sum_{i=\tilde{k}}^{\tilde{k}-1} \gamma_{-k, \varepsilon_1, \varepsilon_2 \cdot 2k} \left( a_{\beta \alpha} + (i - b_{\beta \alpha}^k) \varepsilon_1 + \left( i - b_{\beta \alpha}^k - 2k \left( 1 - \frac{b_{\beta \alpha}^{\gamma}}{k} \right) \right) \varepsilon_2 \right) 
\]

\[
+ \sum_{i=0}^{\tilde{k}-1} \gamma_{k, \varepsilon_1, \varepsilon_2 \cdot 2k} \left( a_{\beta \alpha} + (i - b_{\beta \alpha}^k) \varepsilon_1 + \left( i - b_{\beta \alpha}^k - 2k \left( 1 - \frac{b_{\beta \alpha}^{\gamma}}{k} \right) \right) \varepsilon_2 \right) 
\]

\[
- \sum_{s=1}^{N} \sum_{\alpha=1}^{r} \left( \sum_{i=0}^{\tilde{k}} \gamma_{-k, \varepsilon_1, \varepsilon_2 \cdot 2k} \left( \mu_s + a_\alpha + (i - b_\alpha^k) \varepsilon_1 + \left( i - b_\alpha^k - 2k \left( 1 - \frac{b_\alpha^{\gamma} + \frac{N}{k}}{k} \right) \right) \varepsilon_2 \right) \right)
\]

\[
+ \sum_{i=\tilde{k}}^{\tilde{k}-1} \gamma_{-k, \varepsilon_1, \varepsilon_2 \cdot 2k} \left( \mu_s + a_\alpha + (i - b_\alpha^k) \varepsilon_1 + \left( i - b_\alpha^k - 2k \left( 1 - \frac{b_\alpha^{\gamma} + \frac{N}{k}}{k} \right) \right) \varepsilon_2 \right) 
\]

\[
+ \sum_{i=0}^{\tilde{k}-1} \gamma_{k, \varepsilon_1, \varepsilon_2 \cdot 2k} \left( \mu_s + a_\alpha + (i - b_\alpha^k) \varepsilon_1 + \left( i - b_\alpha^k + \frac{N}{k} \right) \varepsilon_2 \right) 
\]
and for odd $k$

\[
F_{X_k}^{N,\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \vec{c}) := \sum_{\alpha \neq \beta} \left( \sum_{i=0}^{k} \left\lfloor \frac{2^{\frac{N}{2}} + \frac{b_{\beta\alpha}^{0}}{k}}{b_{\beta\alpha}^{0}} \right\rfloor \right) \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} \left( a_{\beta\alpha} + (i - b_{\beta\alpha}^{k}) \varepsilon_1 + \left(i - b_{\beta\alpha}^{k} - k \left(1 - \frac{2^{\frac{N}{2}} + \frac{b_{\beta\alpha}^{0}}{k}}{k} \right)\right) \varepsilon_2 \right)
\]

\[
+ \sum_{i=0}^{k} \left( \sum_{\alpha \neq \beta} \left\lfloor \frac{2^{\frac{N}{2}} + \frac{b_{\beta\alpha}^{0}}{k}}{b_{\beta\alpha}^{0}} \right\rfloor \right) \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} \left( a_{\beta\alpha} + (i - b_{\beta\alpha}^{k}) \varepsilon_1 + \left(i - b_{\beta\alpha}^{k} - k \left(1 - \frac{2^{\frac{N}{2}} + \frac{b_{\beta\alpha}^{0}}{k}}{k} \right)\right) \varepsilon_2 \right)
\]

\[
- \sum_{s=1}^{N} \sum_{\alpha=1}^{r} \left( \sum_{i=0}^{k} \left\lfloor \frac{2^{\frac{N}{2}} + \frac{b_{\beta\alpha}^{0}}{k}}{b_{\beta\alpha}^{0}} \right\rfloor \right) \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} \left( \mu_{s} + a_{\alpha} + (i - b_{\alpha}^{k}) \varepsilon_1 + \left(i - b_{\alpha}^{k} - k \left(1 - \frac{2^{\frac{N}{2}} + \frac{b_{\alpha}^{0}}{k}}{k} \right)\right) \varepsilon_2 \right)
\]

\[
+ \sum_{i=0}^{k} \left( \sum_{\alpha=1}^{r} \left\lfloor \frac{2^{\frac{N}{2}} + \frac{b_{\alpha}^{0}}{k}}{b_{\alpha}^{0}} \right\rfloor \right) \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} \left( \mu_{s} + a_{\alpha} + (i - b_{\alpha}^{k}) \varepsilon_1 + \left(i - b_{\alpha}^{k} - k \left(1 - \frac{2^{\frac{N}{2}} + \frac{b_{\alpha}^{0}}{k}}{k} \right)\right) \varepsilon_2 \right)
\]

\[
+ \sum_{i=0}^{k-1} \left( \sum_{\alpha=1}^{r} \left\lfloor \frac{2^{\frac{N}{2}} + \frac{b_{\alpha}^{0}}{k}}{b_{\alpha}^{0}} \right\rfloor \right) \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} \left( \mu_{s} + a_{\alpha} + (i - b_{\alpha}^{k}) \varepsilon_1 + \left(i - b_{\alpha}^{k} - k \left(1 - \frac{2^{\frac{N}{2}} + \frac{b_{\alpha}^{0}}{k}}{k} \right)\right) \varepsilon_2 \right) + \sum_{i=0}^{k-1} \gamma_{-k \varepsilon_1 + k \varepsilon_2, k \varepsilon_2} \left( \mu_{s} + a_{\alpha} + (i - b_{\alpha}^{k}) \varepsilon_1 + \left(i - b_{\alpha}^{k} - k \left(1 - \frac{2^{\frac{N}{2}} + \frac{b_{\alpha}^{0}}{k}}{k} \right)\right) \varepsilon_2 \right)
\]

**Definition 6.11.** The perturbative partition function for $N = 2$ gauge theory on $X_k$ with $N$ fundamental hypermultiplets of masses $\mu_1, \ldots, \mu_N$ is

\[
Z_{X_k}^{N,\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}) := \sum_{\vec{c}} \exp \left( - F_{X_k}^{N,\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \vec{c}) \right).
\]

**Example 6.12.** For $k = 2$ the perturbative partition function becomes

\[
Z_{X_2}^{N,\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}) = \sum_{\varepsilon_2} \prod_{s=1}^{N} \prod_{\alpha=1}^{r} \exp \left( \gamma_{\varepsilon_2 - \varepsilon_1, 2\varepsilon_2} (\mu_{s} + a_{\alpha} + c_{\alpha} (\varepsilon_2 - \varepsilon_1) - 2\varepsilon_2 \right) + \gamma_{\varepsilon_1 - \varepsilon_2, 2\varepsilon_1} (\mu_{s} + a_{\alpha} - 2\varepsilon_1) \right) \prod_{\alpha \neq \beta} \exp \left( \gamma_{\varepsilon_2 - \varepsilon_1, 2\varepsilon_2} (a_{\beta\alpha} + c_{\beta\alpha} (\varepsilon_2 - \varepsilon_1) - 2\varepsilon_2 \right) + \gamma_{\varepsilon_1 - \varepsilon_2, 2\varepsilon_1} (a_{\beta\alpha} - 2\varepsilon_1) \right).
\]
7. **Seiberg-Witten geometry**

7.1. **Instanton prepotentials.** Define $F_{C_2}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q) := -\epsilon_1 \epsilon_2 \log Z_{C_2}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q)$.

**Proposition 7.1** ([70, Proposition 7.3]). $F_{C_2}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q)$ is analytic in $\epsilon_1, \epsilon_2$ near $\epsilon_1 = \epsilon_2 = 0$.

It is shown by [70, 73] that the instanton part of the Seiberg-Witten prepotential of pure $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^4$ is given by

$$F_{C_2}^{\text{inst}}(\vec{a}; q) := \lim_{\epsilon_1, \epsilon_2 \to 0} F_{C_2}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q).$$

This result gives a proof of the Nekrasov conjecture for $\mathcal{N} = 2$ gauge theories without matter fields on $\mathbb{R}^4$.

In the case of $\mathcal{N} = 2$ gauge theory on the ALE space $X_k$, define

$$F_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\xi}) := -\tilde{k} \epsilon_1 \epsilon_2 \log Z_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\xi}).$$

The following result is the proof of the analogue of the Nekrasov conjecture for gauge theory on $X_k$.

**Theorem 7.2.** $F_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\xi})$ is analytic in $\epsilon_1, \epsilon_2$ near $\epsilon_1 = \epsilon_2 = 0$ and

$$F_{X_k}^{\text{inst}}(\vec{a}; q) := \lim_{\epsilon_1, \epsilon_2 \to 0} F_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\xi}) = \frac{1}{\tilde{k}} F_{C_2}^{\text{inst}}(\vec{a}; q).$$

**Proof.** The proof of the theorem follows the arguments in [40, Section 5.5]. First note that

$$\prod_{i=1}^{k} Z_{C_2}^{\text{inst}}(\epsilon_1^{(i)}, \epsilon_2^{(i)}, \vec{a}^{(i)}; q) = \exp \left( - \sum_{i=1}^{k} \frac{F_{C_2}^{\text{inst}}(\epsilon_1^{(i)}, \epsilon_2^{(i)}, \vec{a}^{(i)}; q)}{\epsilon_1^{(i)} \epsilon_2^{(i)}} \right).$$

(7.3)

Moreover, one finds that

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \prod_{i=1}^{r} \prod_{n=1}^{k-1} \frac{1}{\tilde{\theta}^{(n)}(\epsilon_1^{(i)}, \epsilon_2^{(i), \vec{a}})} = \prod_{\alpha \neq \beta} \left( \frac{1}{a_{\beta} - a_{\alpha}} \right)^{\frac{1}{2} (\bar{c}^{\alpha} \beta, C \bar{c}^{\alpha} \beta - (C^{-1})^{\alpha \beta} \epsilon_{\alpha \beta})}. $$

(7.4)

Given $\vec{v} = (\vec{v}_1, \ldots, \vec{v}_r)$, define

$$F_{X_k, \vec{v}}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q) := \sum_{i=1}^{k} \frac{F_{C_2}^{\text{inst}}(\epsilon_1^{(i)}, \epsilon_2^{(i)}, \vec{a}^{(i)}; q)}{\epsilon_1^{(i)} \epsilon_2^{(i)}} + \frac{F_{C_2}^{\text{inst}}(w, u', \vec{a}; q)}{w u'} + \frac{F_{C_2}^{\text{inst}}(-w, u'', \vec{a}; q)}{-w u''},$$

where $u', u''$ are the normal weights at the two fixed points on $\mathcal{D}_0$, i.e., the weights of the $T_1$-action on the normal bundle $\mathcal{N}_{\mathcal{D}_0}/\mathcal{D}_0$ at the two points, while $w$ is the tangent weight at the intersection point with $\mathcal{D}_0$ (thus $-w$ is the tangent weight at the intersection point with $\mathcal{D}_k$). It follows that $u' = -\epsilon_1^{(1)} = -k \epsilon_1$, $u'' = -\epsilon_2^{(k)} = -k \epsilon_2$, while $w = \tilde{k} \epsilon_2 - \tilde{k} \epsilon_1$. Arguing along the lines of the proof of [40, Lemma 5.16], we can write $F_{X_k, \vec{v}}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q)$ as an equivariant integral over $\mathcal{D}_k$ of an analytic function; this is enough to conclude that $F_{X_k, \vec{v}}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q)$ is analytic in $\epsilon_1, \epsilon_2$ near $\epsilon_1 = \epsilon_2 = 0$. For this argument we use, in addition to the well-posed definition of the equivariant cohomology of a topological stack with an action of a Deligne-Mumford torus noted in Remark 5.1, the functoriality of this construction, in particular with respect to the coarse moduli space morphism [41, Section 5].

By using Equations (7.3) and (7.4), and arguing as in the proof of [40, Lemma 5.18], we find that

$$\log \left( Z_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\xi}) \right) Z_{C_2}^{\text{inst}}(w, u', \vec{a}; q) Z_{C_2}^{\text{inst}}(-w, u'', \vec{a}; q)$$

is also analytic in $\epsilon_1, \epsilon_2$ near $\epsilon_1 = \epsilon_2 = 0$. Thus the pole of $\log \left( Z_{X_k}^{\text{inst}}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\xi}) \right)$ at $\epsilon_1 = \epsilon_2 = 0$ is the same as the pole of

$$\frac{F_{C_2}^{\text{inst}}(w, u', \vec{a}; q)}{w u'} + \frac{F_{C_2}^{\text{inst}}(-w, u'', \vec{a}; q)}{-w u''}.$$
Now we follow the proof of [40, Theorem 5.20-(a)], obtaining that the function
\[-u'' \log Z_{X_k}^\circ \left( \varepsilon_1, \varepsilon_2, a; q, \tilde{\xi} \right) = \frac{-k^2}{k} F_{X_k}^{\text{inst}} \left( \varepsilon_1, \varepsilon_2, a; q, \tilde{\xi} \right) \]  
(7.5)
is analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$. By arguing along the lines of the proof of [40, Theorem 5.20-(b)] and using the identity $\tilde{k} u'' - \tilde{k} u'' = k w$, we find that its limit as $\varepsilon_1, \varepsilon_2 \to 0$ is
\[
\frac{k}{k} \mathcal{F}_{c^2}^{\text{inst}} (\tilde{a}; q) .
\]
Multiplying both sides of Equation (7.5) by $\tilde{k} \varepsilon_1 \varepsilon_2 / u'' = \tilde{k} / k^2$ then implies the statements. 

**Definition 7.6.** We call $F_{X_k}^{\text{inst}} (\tilde{a}; q)$ the **instanton prepotential** of pure $\mathcal{N} = 2$ gauge theory on $X_k$. 

In addition, one can define
\[
F_{X_k}^0 (\varepsilon_1, \varepsilon_2, a; q, \tau_1, \tilde{\xi}) := -\tilde{k} \varepsilon_1 \varepsilon_2 \log Z_{X_k}^0 (\varepsilon_1, \varepsilon_2, a; q, \tau_1, \tilde{\xi})
\]
and
\[
F_{X_k}^{0, \text{inst}} (\varepsilon_1, \varepsilon_2, a; q, \tau_1, \tilde{\xi}) := -\tilde{k} \varepsilon_1 \varepsilon_2 \log Z_{X_k}^{0, \text{inst}} (\varepsilon_1, \varepsilon_2, a; q, \tau_1, \tilde{\xi}) .
\]
By using Equation (5.15) one gets
\[
F_{X_k}^0 (\varepsilon_1, \varepsilon_2, a; q, \tau_1, \tilde{\xi}) = \frac{\tilde{k}}{2k} \tau_1 \sum_{a=1}^r a_a^2 + F_{X_k}^{0, \text{inst}} (\varepsilon_1, \varepsilon_2, a; q_{\text{eff}}, \tau_1, \tilde{\xi}) .
\]

**Corollary 7.7.** $F_{X_k}^{0, \text{inst}} (\varepsilon_1, \varepsilon_2, a; q_{\text{eff}}, \tau_1, \tilde{\xi})$ and $F_{X_k}^0 (\varepsilon_1, \varepsilon_2, a; q, \tau_1, \tilde{\xi})$ are analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$, and
\[
\lim_{\varepsilon_1, \varepsilon_2 \to 0} F_{X_k}^0 (\varepsilon_1, \varepsilon_2, a; q, \tau_1, \tilde{\xi}) = \frac{\tilde{k}}{k} \frac{\tau_1}{2} \sum_{a=1}^r a_a^2 + \lim_{\varepsilon_1, \varepsilon_2 \to 0} F_{X_k}^{0, \text{inst}} (\varepsilon_1, \varepsilon_2, a; q_{\text{eff}}, \tau_1, \tilde{\xi})
\]
\[
= \frac{1}{k} \left( \frac{\tilde{k} \tau_1}{2} \sum_{a=1}^r a_a^2 + \mathcal{F}_{c^2}^{\text{inst}} (a; q_{\text{eff}}) \right) .
\]

Now we deal with gauge theories involving matter fields. Define
\[
F_{C_2}^{s, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \mu; q) := -\varepsilon_1 \varepsilon_2 \log Z_{C_2}^{s, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \mu; q) ,
\]
\[
F_{C_2}^{N, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \bar{\mu}; q) := -\varepsilon_1 \varepsilon_2 \log Z_{C_2}^{N, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \bar{\mu}; q) .
\]

**Proposition 7.8 ([73]).** $F_{C_2}^{s, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \mu; q)$ and $F_{C_2}^{N, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \bar{\mu}; q)$ are analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$.

In [73] it is shown that the instanton part of the Seiberg-Witten prepotential of $\mathcal{N} = 2^\ast$ gauge theory on $\mathbb{R}^4$ is given by
\[
\mathcal{F}_{C_2}^{s, \text{inst}} (a, \mu; q) := \lim_{\varepsilon_1, \varepsilon_2 \to 0} F_{C_2}^{s, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \mu; q) ,
\]
while the instanton part of the Seiberg-Witten prepotential of $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^4$ with $N$ fundamental hypermultiplets is given by
\[
\mathcal{F}_{C_2}^{N, \text{inst}} (a, \bar{\mu}; q) := \lim_{\varepsilon_1, \varepsilon_2 \to 0} F_{C_2}^{N, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \bar{\mu}; q) .
\]

As shown in [40, Section 5.5], the techniques used for the pure gauge theory work also for gauge theories with matter. Thus we define
\[
F_{X_k}^{s, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \mu; q, \tilde{\xi}) := -\tilde{k} \varepsilon_1 \varepsilon_2 \log Z_{X_k}^{s, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \mu; q, \tilde{\xi}) ,
\]
\[
F_{X_k}^{N, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \bar{\mu}; q, \tilde{\xi}) := -\tilde{k} \varepsilon_1 \varepsilon_2 \log Z_{X_k}^{N, \text{inst}} (\varepsilon_1, \varepsilon_2, a, \bar{\mu}; q, \tilde{\xi}) .
\]
Theorem 7.9.

(1) $\mathcal{N} = 2^*$ gauge theory: $F_{X_k}^{s,\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; q, \vec{c})$ is analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$ and

$$F_{X_k}^{s,\text{inst}}(\vec{a}, \mu; q) := \lim_{\varepsilon_1, \varepsilon_2 \to 0} F_{X_k}^{s,\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; q, \vec{c}) = \frac{1}{k} F_{C^2}^{s,\text{inst}}(\vec{a}, \mu; q).$$

(2) Fundamental matter: $F_{X_k}^{N,\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{\mu}; q, \vec{c})$ is analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$ and

$$F_{X_k}^{N,\text{inst}}(\vec{a}, \vec{\mu}; q) := \lim_{\varepsilon_1, \varepsilon_2 \to 0} F_{X_k}^{N,\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; q, \vec{c}) = \frac{1}{k} F_{C^2}^{N,\text{inst}}(\vec{a}, \vec{\mu}; q).$$

Remark 7.10. Statements analogous to Corollary 7.7 hold as well in the case of gauge theories with matter fields. △

7.2. Perturbative prepotentials. For any holonomy vector at infinity $\vec{c} \in \{0, 1, \ldots, k - 1\}^r$ we have the following results.

Theorem 7.11.

(1) $\mathcal{N} = 2$ gauge theory:

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \varepsilon_1 \varepsilon_2 F_{X_k}^{s,\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{a}; \vec{c}) = \frac{1}{k} F_{C^2}^{s,\text{pert}}(\vec{a}),$$

where

$$F_{C^2}^{s,\text{pert}}(\vec{a}) := \sum_{\alpha \neq \beta} \left(-\frac{1}{2} (a_\alpha - a_\beta)^2 \log(a_\alpha - a_\beta) + \frac{3}{4} (a_\alpha - a_\beta)^2\right)$$

is the perturbative part of the Seiberg-Witten prepotential of pure $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^4$.

(2) $\mathcal{N} = 2^*$ gauge theory:

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \varepsilon_1 \varepsilon_2 F_{X_k}^{s,\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{\mu}; \vec{c}) = \frac{1}{k} F_{C^2}^{s,\text{pert}}(\vec{a}, \mu),$$

where

$$F_{C^2}^{s,\text{pert}}(\vec{a}, \mu) := \sum_{\alpha \neq \beta} \left(-\frac{1}{2} (a_\alpha - a_\beta)^2 \log(a_\alpha - a_\beta) + \frac{3}{4} (a_\alpha - a_\beta)^2\right) + \frac{1}{2} (a_\alpha - a_\beta + \mu)^2 \log(a_\alpha - a_\beta + \mu) - \frac{3}{4} (a_\alpha - a_\beta + \mu)^2$$

is the perturbative part of the Seiberg-Witten prepotential of $\mathcal{N} = 2^*$ gauge theory on $\mathbb{R}^4$.

(3) Fundamental matter:

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \varepsilon_1 \varepsilon_2 F_{X_k}^{N,\text{pert}}(\varepsilon_1, \varepsilon_2, \vec{\mu}; \vec{c}) = \frac{1}{k} F_{C^2}^{N,\text{pert}}(\vec{a}, \vec{\mu}),$$

where

$$F_{C^2}^{N,\text{pert}}(\vec{a}, \vec{\mu}) := \sum_{\alpha \neq \beta} \left(-\frac{1}{2} (a_\alpha - a_\beta)^2 \log(a_\alpha - a_\beta) + \frac{3}{4} (a_\alpha - a_\beta)^2\right) + \sum_{s=1}^{N} \sum_{a=1}^{r} \left(\frac{1}{2} (a_\alpha + \mu_a)^2 \log(a_\alpha + \mu_a) - \frac{3}{4} (a_\alpha + \mu_a)^2\right)$$

is the perturbative part of the Seiberg-Witten prepotential of $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^4$ with $N$ fundamental hypermultiplets.
Proof. We prove only (1). The proofs of (2) and (3) are analogous.

Let $k$ be even. Then

$$
\lim_{\epsilon_1, \epsilon_2 \to 0} \epsilon_1 \epsilon_2 F_{X_k}^{\text{pert}}(\epsilon_1, \epsilon_2, \vec{a}; \vec{c})
= \lim_{\epsilon_1, \epsilon_2 \to 0} \sum_{\alpha \neq \beta} \left( k \epsilon_1 \epsilon_2 \gamma - k \epsilon_1 + k \epsilon_2, 2k \epsilon_2 (a_\beta - a_\alpha) + k \epsilon_1 \epsilon_2 \gamma - k \epsilon_1 - k \epsilon_2, 2k \epsilon_1 (a_\beta - a_\alpha) \right).
$$

By arguing as in the proof of [40, Theorem 6.8] one gets the assertion. The case of odd $k$ is similar. \qed

7.3. Blowup equations. Throughout this subsection we set $k = 2$.

$\mathcal{N} = 2$ gauge theory. The generating function for correlators of quadratic 2-observables is defined by

$$
\mathcal{Z}_{X_2}^\bullet(\epsilon_1, \epsilon_2, \vec{a}; q, \xi, t) := \mathcal{Z}_{X_2}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\xi}, \vec{t}),
$$

where $\vec{\tau} = \vec{0}$ and $\vec{t} := (0, -t, 0, \ldots)$. By Example 5.11, this partition function becomes

$$
\mathcal{Z}_{X_2}^\bullet(\epsilon_1, \epsilon_2, \vec{a}; q, \xi, t) = \sum_{v \in \frac{1}{2} \mathbb{Z}} \xi^v \sum_{\vec{v} = (v_1, \ldots, v_r)} \prod_{\alpha, \beta = 1}^r \ell_{v_{\alpha \beta}}(2 \epsilon_1, \epsilon_2 - \epsilon_1, a_{\beta \alpha})
\times \mathcal{Z}_{C_2}^\bullet(\epsilon_1, \epsilon_2, \vec{a}(i); q, \epsilon_1(i) \vec{t}) \mathcal{Z}_{C_2}^{(2)}(\epsilon_2, \vec{a}(2); q, \epsilon_2(2) \vec{t}),
$$

Equation (7.12)

where $\mathcal{Z}_{C_2}(\epsilon_1, \epsilon_2, \vec{a}; q, \vec{\tau})$ is the deformed Nekrasov partition function for $U(r)$ gauge theory on $\mathbb{R}^4$. For $i = 1, 2$ we have

$$
\mathcal{Z}_{C_2}^\bullet(\epsilon_1(i), \epsilon_2(i), \vec{a}(i); q, \epsilon_1(i) \vec{t})
= \sum_{Y_{\alpha \beta}} \prod_{\alpha, \beta = 1}^{r} m_{Y_{\alpha \beta}} \left( \frac{1}{2} \epsilon_1^{(1)} \sum_{\alpha = 1}^{r} (a_{\alpha}^{(1)})^2 - 2 \sum_{\alpha = 1}^{r} |Y_{\alpha \beta}| \right).
$$

Since

$$
\frac{1}{2 \epsilon_1^{(1)} \gamma} \sum_{\alpha = 1}^{r} (a_{\alpha}^{(1)})^2 + \frac{1}{2 \epsilon_1^{(2)} \gamma} \sum_{\alpha = 1}^{r} (a_{\alpha}^{(2)})^2 = -\left( \frac{1}{2 \epsilon_1^{(1)} \gamma} + \frac{1}{2 \epsilon_1^{(2)} \gamma} \right) \sum_{\alpha = 1}^{r} v_{\alpha}^2 + 2 \sum_{\alpha = 1}^{r} v_{\alpha} a_{\alpha},
$$

Equation (7.12) becomes

$$
\mathcal{Z}_{X_2}^\bullet(\epsilon_1, \epsilon_2, \vec{a}; q, \xi, t) = \sum_{v \in \frac{1}{2} \mathbb{Z}} \xi^v \sum_{\vec{v} = (v_1, \ldots, v_r)} \prod_{\alpha, \beta = 1}^{r} \ell_{v_{\alpha \beta}}(2 \epsilon_1, \epsilon_2 - \epsilon_1, a_{\beta \alpha})
\times \mathcal{Z}_{C_2}^{\text{inst}}(\epsilon_1, \epsilon_2 - \epsilon_1, \vec{a} - 2 \epsilon_1 \vec{v}; q, e^{2 \epsilon_1 t}) \mathcal{Z}_{C_2}^{\text{inst}}(\epsilon_1 - \epsilon_2, 2 \epsilon_2, \vec{a} - 2 \epsilon_2 \vec{v}; q, e^{2 \epsilon_2 t}).
$$

We will now compare the generating function $\mathcal{Z}_{X_2}^\bullet(\epsilon_1, \epsilon_2, \vec{a}; q, \xi, t)$ with the original partition function $\mathcal{Z}_{X_2}(\epsilon_1, \epsilon_2, \vec{a}; q, \xi)$ in the low energy limit, i.e., in the limit where $\epsilon_1, \epsilon_2 \to 0$, and show that they are related through theta-functions on the genus $r$ Seiberg-Witten curve $\Sigma$ for pure $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^4$. The period matrix $\tau = (\tau_{\alpha \beta})$ of $\Sigma$ is given by

$$
\tau_{\alpha \beta} := \frac{1}{2 \pi i} \frac{\partial^2 F_{C_2}}{\partial a_\alpha \partial a_\beta}(\vec{a}; q)
= \frac{1}{2 \pi i} \frac{\partial^2 F_{C_2}^{\text{inst}}}{\partial a_\alpha \partial a_\beta}(\vec{a}; q)
- \frac{1}{2 \pi i} \sum_{\alpha' < \beta'} (\delta_{\alpha \alpha'} - \delta_{\alpha \beta'}) (\delta_{\beta \alpha'} - \delta_{\beta \beta'}) \log(a_{\beta'} - a_{\alpha'}) + \frac{\log q}{2 \pi i} \delta_{\alpha \beta}.
$$
where $\mathcal{F}_{C^2}(\vec{a}; q)$ is the total Seiberg-Witten prepotential defined by Corollary 7.7 and Theorem 7.11 (here we identify $q := e^{2\pi i n_1}$). We further define

$$\zeta_\alpha := -\frac{t}{2\pi i} \left( a_{\alpha} + q \frac{\partial^2 F^\text{inst}_{C^2}}{\partial q \partial a_\alpha}(\vec{a}; q) \right) \quad \text{for} \quad \alpha = 1, \ldots, r.$$ 

As in [69, Appendix B.1], we introduce basic $Sp(2r, \mathbb{Z})$ modular forms on the Seiberg-Witten curve $\Sigma$.

**Definition 7.13.** For $\vec{m}, \vec{n} \in \mathbb{C}^r$, the theta-function $\Theta \left[ \frac{\vec{m}}{\vec{m}} \right] (\zeta | \tau)$ with characteristic $\left[ \frac{\vec{m}}{\vec{m}} \right]$ is

$$\Theta \left[ \frac{\vec{m}}{\vec{m}} \right] (\zeta | \tau) := \sum_{n \in \mathbb{Z}^r} \exp \left( \pi i \sum_{\alpha, \beta = 1}^r (n_\alpha + \mu_\alpha) \tau_{\alpha\beta} (n_\beta + \mu_\beta) + 2\pi i \sum_{\alpha = 0}^r (\zeta_\alpha + \nu_\alpha) (n_\alpha + \mu_\alpha) \right).$$

**Theorem 7.14.** $Z^\Lambda_X(\varepsilon_1, \varepsilon_2; \vec{a}; \xi, t)$ is analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$, and

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} Z^\Lambda_X(\varepsilon_1, \varepsilon_2; \vec{a}; \xi, t) = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{Z^\Lambda_{X^2}(\varepsilon_1, \varepsilon_2; \vec{a}; \xi, t)}{Z^\Lambda_{X^2}(\varepsilon_1, \varepsilon_2; \vec{a}; \xi, t)} = \exp \left( \left( q \frac{\partial}{\partial q} \right)^2 \mathcal{F}^\text{inst}_{C^2}(\vec{a}; q) t^2 + 2\pi i \sum_{\alpha = 0}^r \zeta_\alpha \right) \Theta \left[ \begin{array}{cc} 0 & C \xi \\vec{C} \end{array} \right] \left( \begin{array}{c} C \vec{r} \\vec{r} \end{array} \right) \Theta \left[ \begin{array}{cc} 0 & C \xi \\vec{C} \end{array} \right] \left( \begin{array}{c} C \vec{r} \\vec{r} \end{array} \right) \frac{\Theta \left[ \begin{array}{c} C \xi \\vec{C} \end{array} \right] \left( \begin{array}{c} C \vec{r} \\vec{r} \end{array} \right)}{\Theta \left[ \begin{array}{c} C \xi \\vec{C} \end{array} \right] \left( \begin{array}{c} C \vec{r} \\vec{r} \end{array} \right)},$$

where $\kappa_\alpha := \frac{1}{4\pi i} \log \xi$ for $\alpha = 1, \ldots, r$ and

$$\nu_\alpha := \left\{ \begin{array}{ll} \sum_{\beta = \alpha+1}^r \left( \log(a_\beta - a_\alpha) - \frac{1}{r} \log q \right) & \text{for} \quad \alpha = 1, \ldots, w_0, \\
-2 \sum_{\beta = 1}^r \log(a_\beta - a_\alpha) + \frac{2w_0}{r} \log q & \text{for} \quad \alpha = w_0 + 1, \ldots, r. \end{array} \right.$$ 

**Proof.** First we show that the value of $Z^\Lambda_{X^2}(\varepsilon_1, \varepsilon_2; \vec{a}; \xi, t)$ at $\varepsilon_1 = \varepsilon_2 = 0$ is

$$\exp \left( \left( q \frac{\partial}{\partial q} \right)^2 \mathcal{F}^\text{inst}_{C^2}(\vec{a}; q) t^2 \right) \sum_{\vec{v} \in \mathbb{Z}^r} \xi^v \sum_{\vec{a} = (a_1, \ldots, a_r)} v_{\vec{a}} \prod_{\alpha \neq \beta} \left( \frac{1}{a_\beta - a_\alpha} \right) v_{\vec{a}}^{\frac{1}{2} \delta_1 \delta_{1, \beta}} \times \exp \left( \sum_{\alpha, \beta = 1}^r \frac{\partial^2 \mathcal{F}^\text{inst}_{C^2}(\vec{a}; q)}{\partial a_\alpha \partial a_\beta} v_{\vec{a}} v_\beta - 2 \sum_{\alpha = 1}^r \left( q \frac{\partial \mathcal{F}^\text{inst}_{C^2}(\vec{a}; q)}{\partial q \partial a_\alpha}(\vec{a}; q) + a_\alpha t \right) \right) \times \left[ \sum_{\vec{v} \in \mathbb{Z}^r} \xi^v \sum_{\vec{a} = (a_1, \ldots, a_r)} v_{\vec{a}} \prod_{\alpha \neq \beta} \left( \frac{1}{a_\beta - a_\alpha} \right) v_{\vec{a}}^{\frac{1}{2} \delta_1 \delta_{1, \beta}} \times \exp \left( \sum_{\alpha, \beta = 1}^r \frac{\partial^2 \mathcal{F}^\text{inst}_{C^2}(\vec{a}; q)}{\partial a_\alpha \partial a_\beta} (\vec{a}; q) v_{\vec{a}} v_\beta \right) \right]^{-1} \quad (7.15)$$

By [70, Lemma 6.3-(1)] we have

$$F^\text{inst}_{C^2}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}; \vec{a}; q) = F^\text{inst}_{C^2}(\varepsilon_1^{(2)}, \varepsilon_2^{(2)}; \vec{a}; q) |_{\varepsilon_1^{(1)} = \varepsilon_2^{(1)}}.$$

From this symmetry and some algebraic manipulations, one gets
\[
\frac{1}{\varepsilon_2^{(1)}} \left( F_{C^2}^{\text{inst}}(\xi_1^{(1)}, \xi_2^{(1)}; q \in \xi_1^{(1)} t) - F_{C^2}^{\text{inst}}(\xi_1^{(1)}, \xi_2^{(1)}; \vec{\alpha}; q) \right)
\]
\[
- \frac{F_{C^2}^{\text{inst}}(\xi_1^{(2)}, \xi_2^{(2)}; q \in \xi_2^{(2)} t) - F_{C^2}^{\text{inst}}(\xi_1^{(2)}, \xi_2^{(2)}; \vec{\alpha}; q)}{\varepsilon_2^{(2)}} \right)_{|\varepsilon_1, \varepsilon_2) = (0, 0)}
\]
\[
= - \left( \frac{\partial}{\partial q} \right)^2 \mathcal{Z}_{C^2}^{\text{inst}}(\vec{\alpha}; q) t^2 - \sum_{\alpha, \beta = 1}^{r} \frac{\partial^2 \mathcal{Z}_{C^2}^{\text{inst}}}{\partial a_{\alpha} \partial a_{\beta}}(\vec{\alpha}; q) v_{\alpha} v_{\beta} + 2 \sum_{\alpha = 1}^{r} q \frac{\partial^2 \mathcal{Z}_{C^2}^{\text{inst}}}{\partial q \partial a_{\alpha}}(\vec{\alpha}; q) v_{\alpha} t . \quad (7.16)
\]

One computes the numerator of Equation (7.15) by dividing \( Z_{X_2}^{\bullet}(\xi_1, \xi_2; \vec{\alpha} q, \xi, t) \) by the product of partition functions \( Z_{C^2}^{\text{inst}}(\xi_1^{(1)}, \xi_2^{(1)}; \vec{\alpha}; q) \) \( Z_{C^2}^{\text{inst}}(\xi_1^{(2)}, \xi_2^{(2)}; \vec{\alpha}; q) \) and by using Equation (7.16). Similarly, one gets the denominator of Equation (7.15).

The rewriting of Equation (7.15) in terms of theta-functions follows from straightforward but tedious algebraic manipulations, after setting
\[
v_{\alpha} = \begin{cases} 
  n_{\alpha} & \text{for } \alpha = 1, \ldots, w_0 , \\
  n_{\alpha} + \frac{1}{2} & \text{for } \alpha = w_0 + 1, \ldots, r ,
\end{cases}
\]
where \( n_{\alpha} \in \mathbb{Z} \), and using the identity
\[
\sum_{\alpha \neq \beta} (v_{\alpha} - v_{\beta})^2 = r \sum_{\alpha = 1}^{r} v_{\alpha}^2 - v^2.
\]

**Remark 7.17.** Our formula for the generating function \( Z_{X_2}^{\bullet}(\xi_1, \xi_2; \vec{\alpha}; q, \xi, t) \) resembles that of [70, Equation (6.10)]. The expression in Theorem 7.14 resembles the formula given in [70, Theorem 8.1]. In particular, note that when \( w_0 = r \) and \( w_1 = 0 \), i.e., when the holonomy at infinity is trivial, we get \( \vec{\nu} = \vec{\eta} \).

Recall that by Theorem 7.2, the value of \( F_{X_2}^{\text{inst}}(\xi_1, \xi_2; \vec{\alpha} q, \xi) \) at \( (\xi_1, \xi_2) = (0, 0) \) is
\[
\mathcal{Z}_{X_2}^{\text{inst}}(\vec{\alpha}; q) = \frac{1}{2} \mathcal{Z}_{C^2}^{\text{inst}}(\vec{\alpha}; q) .
\]
In this sense the blowup equation of Theorem 7.14 relates the generating function \( Z_{X_2}^{\bullet}(\xi_1, \xi_2; \vec{\alpha}; q, \xi, t) \) to the partition function \( Z_{C^2}^{\text{inst}}(\xi_1, \xi_2; \vec{\alpha}; q) \frac{1}{2} \) which in the low energy limit is identified as the partition function of \( \mathcal{N} = 2 \) gauge theory on the quotient \( C^2/\mathbb{Z}_2 \).

\( \mathcal{N} = 2^* \) gauge theory. The generating function \( Z_{X_2}^{\bullet}(\xi_1, \xi_2, \vec{\alpha}, \mu; q, \xi, t) \) of the 2-observables is the generating function \( Z_{X_2}^{\bullet}(\xi_1, \xi_2, \vec{\alpha} q, \xi, t) \) introduced in Definition 5.18 specialized at \( \vec{\tau} = \vec{0} \) and \( \vec{t} := (0, -t, 0, \ldots) \). By using similar arguments as above we get

\[
Z_{X_2}^{\bullet}(\xi_1, \xi_2, \vec{\alpha}, \mu; q, \xi, t) = \sum_{\varepsilon \in \mathbb{Z}} \sum_{\bar{\varepsilon} = (v_1, \ldots, v_r) \mod 2} (q e^{2it(\varepsilon_1+\varepsilon_2)}) \sum_{\alpha=1}^{r} e^{-2t \sum_{\alpha=1}^{r} v_{\alpha} \alpha} \prod_{\alpha, \beta=1}^{r} \ell_{v_{\alpha} \alpha}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, a_{\alpha} + \mu) e^{2\varepsilon_1 t} \prod_{\alpha, \beta=1}^{r} \ell_{v_{\alpha} \alpha}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, a_{\alpha})
\]
\[
\times Z_{C^2}^{\text{inst}}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{\alpha} - 2\varepsilon_1 \vec{\nu}, \mu; q e^{2\varepsilon_1 t}) Z_{C^2}^{\text{inst}}(\varepsilon_1 - \varepsilon_2, 2\varepsilon_2, \vec{\alpha} - 2\varepsilon_2 \vec{\nu}, \mu; q e^{2\varepsilon_2 t}) .
\]

The period matrix \( \vec{\tau} = (\tau_{\alpha \beta}) \) of the genus \( r \) Seiberg-Witten curve for \( \mathcal{N} = 2^* \) gauge theory on \( \mathbb{R}^4 \) is given by
\[ \tau_{\alpha\beta} := \frac{1}{2\pi i} \frac{\partial^2 F_{C^2}^{\text{inst}}}{\partial a_\alpha \partial a_\beta} (\bar{a}, \mu; q) + \frac{1}{2\pi i} \sum_{\alpha' < \beta'} (\delta_{\alpha\alpha'} - \delta_{\alpha\beta}) (\delta_{\beta\alpha'} - \delta_{\beta\beta'}) \log \left( \frac{a_{\beta'} - a_{\alpha'}}{a_{\beta'} - a_{\alpha'}} + \mu \right) + \frac{\log q}{2\pi i} \delta_{\alpha\beta}. \]

We define
\[ \zeta_\alpha := -\frac{t}{2\pi i} \left( a_\alpha + \frac{\partial^2 F_{C^2}^{\text{inst}}}{\partial q \partial a_\alpha} (\bar{a}, \mu; q) \right). \]

By using the same arguments as in the proof of Theorem 7.14 we arrive at the following result.

**Theorem 7.18.** $Z_{X_2}^{\ast,\ast}(\varepsilon_1, \varepsilon_2, \bar{a}, \mu; q, \xi, t)/Z_{X_2}^{\ast,\ast}(\varepsilon_1, \varepsilon_2, \bar{a}, \mu; q, \xi, t)$ is analytic in $\varepsilon_1, \varepsilon_2$ near $\varepsilon_1 = \varepsilon_2 = 0$, and
\[
\lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{Z_{X_2}^{\ast,\ast}(\varepsilon_1, \varepsilon_2, \bar{a}, \mu; q, \xi, t)}{Z_{X_2}^{\ast,\ast}(\varepsilon_1, \varepsilon_2, \bar{a}, \mu; q, \xi, t)} = \exp \left( (q \frac{\partial}{\partial q})^2 F_{C^2}^{\text{inst}}(\bar{a}, \mu; q) t^2 + 2\pi i \sum_{\alpha = u_0 + 1}^r \zeta_\alpha \frac{\Theta[0]}{\Theta[0]} (C \zeta + C \bar{\eta} | C \tau) \right),
\]
where $\kappa_\alpha := \frac{1}{4\pi i} \log \kappa_\alpha$ for $\alpha = 1, \ldots, r$ and
\[
\nu_\alpha := \begin{cases} \sum_{\beta = u_0 + 1}^r \left( \log \left( (a_\beta - a_\alpha)^2 - \mu^2 \right) + \frac{\partial^2 F_{C^2}^{\text{inst}}}{\partial a_\alpha \partial a_\beta} (\bar{a}, \mu; q) \right) - \frac{w_\alpha}{r} \log q & \text{for } \alpha = 1, \ldots, u_0, \\
- \sum_{\beta = 1}^{u_0} \log \left( (a_\beta - a_\alpha)^2 - \mu^2 \right) + 2 \frac{w_\alpha}{r} \log q & \text{for } \alpha = u_0 + 1, \ldots, r. 
\end{cases}
\]

**Fundamental matter.** We consider only the asymptotically free case, i.e., $N \leq 2r$. The conformal case, i.e., $N = 2r$, can be treated in a similar way.

The generating function for quadratic 2-observables $Z_{X_2}^N(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \xi, t)$ is the generating function $Z_{X_2}^N(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \xi, \bar{t})$ introduced in Definition 5.28 specialized at $\bar{t} = \bar{0}$ and $\bar{t} := (0, -t, 0, \ldots)$. By using similar arguments as above we get
\[
Z_{X_2}^N(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \xi, t) = \sum_{\bar{v} \in \Omega_{\bar{v}}} \sum_{\bar{v} = (v_1, \ldots, v_r)} (q e^{2t(\varepsilon_1 + \varepsilon_2)})^{v_2} \sum_{\alpha = 1}^r v_\alpha \prod_{s=1}^N \prod_{\alpha=1}^r \ell_{v_\alpha} (2\varepsilon_1, 2\varepsilon_2 - \varepsilon_1, a_\alpha + \mu_s) \prod_{\alpha, \beta = 1}^r \ell_{v_\alpha} (2\varepsilon_1, 2\varepsilon_2 - \varepsilon_1, a_\beta) \times Z_{C^2}^{\text{inst}}(2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \bar{a} - 2\varepsilon_2 \bar{v}; \bar{\mu}; q e^{2t_\bar{v}}) \times Z_{C^2}^{\text{inst}}(\varepsilon_1 - \varepsilon_2, 2\varepsilon_2, \bar{a}; \varepsilon_2 \bar{v}; q e^{2\varepsilon_2 t}).
\]

The period matrix $\tau = (\tau_{\alpha\beta})$ of the genus $r$ Seiberg-Witten curve for $N = 2$ gauge theory on $\mathbb{R}^4$ with $N$ fundamental hypermultiplets is given by
\[
\tau_{\alpha\beta} := \frac{1}{2\pi i} \frac{\partial^2 F_{C^2}^{\text{inst}}}{\partial a_\alpha \partial a_\beta} (\bar{a}, \mu; q) - \frac{1}{2\pi i} \sum_{\alpha' < \beta'} (\delta_{\alpha\alpha'} - \delta_{\alpha\beta'}) (\delta_{\beta\alpha'} - \delta_{\beta\beta'}) \log (a_{\beta'} - a_{\alpha'}) + \frac{1}{2\pi i} \sum_{s=1}^N \delta_{\alpha\beta} \log (a_{\beta} + \mu_s) + \frac{\log q}{2\pi i}.\]
We define

\[ \zeta_\alpha := -\frac{t}{2\pi i} \left( a_\alpha + \frac{\partial^2 \mathcal{F}_\text{inst}}{\partial q \partial a_\alpha}(\bar{a}, \bar{\mu}; q) \right). \]

Let us introduce the set

\[ \tilde{Q}_{\tilde{a}} := \left\{ (n_1, \ldots, n_r) \in \mathbb{Z}^r \left| \left( \sum_{\alpha=1}^r n_\alpha \right)^2 + w_1 \sum_{\alpha=1}^r n_\alpha - \frac{3}{4} w_1^2 \leq 0 \right. \right\}. \]

**Definition 7.19.** For \( \bar{\mu}, \bar{\nu} \in \mathbb{C}^r \) the modified theta function \( \tilde{\Theta} \left[ \begin{matrix} \bar{\mu} \\ \bar{\nu} \end{matrix} \right] (\zeta | \tau) \) with characteristic \( \left[ \begin{matrix} \bar{\mu} \\ \bar{\nu} \end{matrix} \right] \) is

\[ \tilde{\Theta} \left[ \begin{matrix} \bar{\mu} \\ \bar{\nu} \end{matrix} \right] (\zeta | \tau) := \sum_{\bar{n}=(n_1, \ldots, n_r) \in \tilde{Q}_{\bar{a}}} \exp \left( \pi i \sum_{\alpha, \beta=1}^r (n_\alpha + \mu_\alpha) \tau_{\beta\alpha} (n_\beta + \mu_\beta) + 2\pi i \sum_{\alpha=1}^r (\zeta_\alpha + \nu_\alpha) (n_\alpha + \mu_\alpha) \right). \]

By using the same arguments as in the proof of Theorem 7.14 we then arrive at the following result.

**Theorem 7.20.** \( Z_{X_2}^N(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \xi, t) / Z_{X_2}^N(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \xi) \) is analytic in \( \varepsilon_1, \varepsilon_2 \) near \( \varepsilon_1 = \varepsilon_2 = 0 \), and

\[ \lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{Z_{X_2}^N(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \xi, t)}{Z_{X_2}^N(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \xi)} = \exp \left( \left( q \frac{\partial}{\partial q} \right)^2 \mathcal{F}_\text{inst}^N(\bar{a}, \bar{\mu}; q) t^2 + 2\pi i \sum_{\alpha=w_0+1}^r \zeta_\alpha \right) \tilde{\Theta} \left[ \begin{matrix} 0 \\ C \bar{\nu} \end{matrix} \right] (C \bar{\zeta} + C \bar{\kappa} | C \tau), \]

where \( \kappa_\alpha := \frac{1}{4\pi i} \log \xi \) for \( \alpha = 1, \ldots, r \) and

\[ \nu_\alpha := \begin{cases} \sum_{\beta=w_0+1}^r \left( \log(a_\beta - a_\alpha)^2 + \frac{\partial^2 \mathcal{F}_\text{inst}^N(t)}{\partial a_\alpha \partial a_\beta}(\bar{a}, \bar{\mu}; q) \right) - \frac{w_1}{r} \log q & \text{for } \alpha = 1, \ldots, w_0, \\ -2 \sum_{\beta=1}^{w_0} \log(a_\beta - a_\alpha) + \frac{2w_0}{r} \log q + \sum_{s=1}^N \log(a_\alpha + \mu_s) & \text{for } \alpha = w_0 + 1, \ldots, r. \end{cases} \]

**Remark 7.21.** When the holonomy at infinity is trivial, i.e., \( w_1 = 0 \), our formula for the generating function \( Z_{X_2}^N(\varepsilon_1, \varepsilon_2, \bar{a}, \bar{\mu}; q, \xi, t) \) is analogous to \([44, \text{Equation (5.3)}]\), while the expression in Theorem 7.20 is analogous to \([44, \text{Equation (5.5)}]\).

Theorems 7.14, 7.18 and 7.20 are the analogues for \( \mathcal{N} = 2 \) gauge theories of the representation of the Vafa-Witten partition function in terms of modular forms. For the \( \mathcal{N} = 4 \) gauge theory considered in Section 5.3, whose prepotential is \( \frac{1}{2\pi i} \log q \sum_{\alpha=1}^3 a_\alpha^2 \), the \( SL(2, \mathbb{Z}) \) monodromies act on the coupling \( \frac{1}{2\pi i} \log q \), while for the \( \mathcal{N} = 2 \) gauge theories considered in this section the \( Sp(2r, \mathbb{Z}) \) monodromies act on the periods of the Seiberg-Witten curve, which determine the low energy effective gauge couplings, twisted by the intersection form (Cartan matrix) \( C \).
APPENDIX A. RANK AND DIMENSION FORMULAS

In this appendix we compute the rank of the natural bundle $V$ and the dimension of the moduli space $\mathcal{M}_{r,\mathbb{D}}(\mathcal{F}_k, \mathcal{D}_\infty, F_{\infty}^{s,\mathbb{D}})$ by using the Toën-Riemann-Roch theorem [87, 86]. Additional details can be found in [23, Appendix A].

**Theorem A.1.** The rank of the natural bundle $V$ of the moduli space $\mathcal{M}_{r,\mathbb{D}}(\mathcal{F}_k, \mathcal{D}_\infty, F_{\infty}^{s,\mathbb{D}})$ is given by

$$\text{rk}(V) = \Delta + \frac{1}{2r} \bar{v} \cdot C \bar{v} - \frac{k-1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} w_j.$$  

**Corollary A.2.** The dimension of the moduli space $\mathcal{M}_{r,\mathbb{D}}(\mathcal{F}_k, \mathcal{D}_\infty, F_{\infty}^{s,\mathbb{D}})$ is

$$\dim_C (\mathcal{M}_{r,\mathbb{D}}(\mathcal{F}_k, \mathcal{D}_\infty, F_{\infty}^{s,\mathbb{D}})) = 2r \Delta - \frac{k-1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} \bar{w} \cdot \bar{w}(j), \quad (A.3)$$

where for $j = 1, \ldots, k - 1$ the vector $\bar{w}(j)$ is $(w_j, \ldots, w_{k-1}, w_0, w_1, \ldots, w_{j-1})$.

Before attacking the proofs of these results, that we shall give in Section A.3, in Section A.1 we study the inertia stack of $\mathcal{F}_k$, and in Section A.2 we compute some topological invariants of $\mathcal{F}_k$ and $\mathcal{D}_\infty$.

### A.1. Inertia stack

In this subsection we will compute the inertia stack $\mathcal{I}(\mathcal{F}_k)$ of $\mathcal{F}_k$. This is a fundamental ingredient in the application of the Toën-Riemann-Roch theorem.

#### A.1.1. Characterization of the stacky points $p_0$ and $p_\infty$

We give a characterization of the stacky points $p_0$ and $p_\infty$ of $\mathcal{D}_\infty \subset \mathcal{F}_k$ as trivial gerbes over a point. We also characterize their Picard groups and the restrictions to them of the generators of the Picard group of $\mathcal{D}_\infty$.

**Lemma A.4.** Both stacks $p_0$ and $p_\infty$ are isomorphic to $\mathcal{B}_{\mu_{\tilde{k}}} = [\text{pt}/\mu_{\tilde{k}}]$. At the gerbe structure level, the maps between the banding groups $\mu_k$ of $\mathcal{D}_\infty$ and $\mu_{\tilde{k}}$ of $p_0$, $p_\infty$ are given by

$$\omega \in \mu_k \longmapsto \omega^\pm_k \in \mu_{\tilde{k}},$$

where we take the minus sign for $p_0$ and the plus sign for $p_\infty$.

**Proof.** Consider the cone $\sigma_{\infty,k+2}$. We can compute the quotient stacky fan $\Sigma_k/\sigma_{\infty,k+2}$. First note that $N(\sigma_{\infty,k+2}) \simeq \mathbb{Z}^2/ (\mathbb{Z}e_1 + k\mathbb{Z}e_2)$, and the quotient map $N \to N(\sigma_{\infty,k+2})$ sends $a e_1 + b e_2$ to $a \mod \tilde{k}$. The quotient fan $\Sigma_k/\sigma_{\infty,k+2} \subset N(\sigma_{\infty,k+2})$ is just $\{0\}$. Thus $\Sigma_k/\sigma_{\infty,k+2} = (\mathbb{Z}_{\tilde{k}}, 0, 0)$ and therefore $p_0$ is the trivial $\mu_{\tilde{k}}$-gerbe $\mathcal{B}_{\mu_{\tilde{k}}} = [\text{pt}/\mu_{\tilde{k}}]$ over a point pt.

The quotient map $N \to N(\sigma_{\infty,k+2})$ factorizes through the quotient map $N(p_\infty) \to N(\sigma_{\infty,k+2})$ which is given by

$$(c, d) \mapsto \begin{cases} c - d \tilde{k} \mod \tilde{k} & \text{for even } k, \\ c \frac{k-1}{2} - d \tilde{k} \mod \tilde{k} & \text{for odd } k. \end{cases}$$

The induced map between the torsion subgroups $\mathbb{Z}_k \to \mathbb{Z}_{\tilde{k}}$ is multiplication by $-\tilde{k}$, and the map between the banding groups of $\mathcal{D}_\infty$ and $p_0$ is given by

$$\mu_k \simeq \text{Hom}(N(p_\infty)_{\text{tor}}, \mathbb{C}^*) \xrightarrow{(-)^{-\tilde{k}}} \text{Hom}(N(\sigma_{\infty,k+2})_{\text{tor}}, \mathbb{C}^*) \simeq \mu_{\tilde{k}}.$$

For $p_\infty$ one can argue similarly. \qed

Now we give a characterization of line bundles over $p_0$ and $p_\infty$, regarded as trivial $\mu_{\tilde{k}}$-gerbes over a point.
Lemma A.5. The Picard group Pic(\(p_0\)) (resp. Pic(\(p_\infty\))) of \(p_0\) (resp. \(p_\infty\)) is generated by the line bundle \(L_{p_0}\) (resp. \(L_{p_\infty}\)) corresponding to the character \(\chi: \omega \in \mu_{k,k} \rightarrow \omega \in \mathbb{C}^*\). In particular, Pic(\(p_0\)) \(\simeq\) Pic(\(p_\infty\)) \(\simeq\) \(\mathbb{Z}_{k,k}\). The restrictions of the generators \(L_1, L_2\) of Pic(\(\mathcal{P}\)) to \(p_0, p_\infty\) are given by

\[
L_{1|p_0} \simeq L_{p_0} ; \quad L_{1|p_\infty} \simeq L_{p_\infty} ;
\]

\[
L_{2|p_0} \simeq \begin{cases} L_{p_0}^{\otimes k} & \text{for even } k , \\ L_{p_0}^{\otimes 2k} & \text{for odd } k ; \end{cases} \quad L_{2|p_\infty} \simeq \begin{cases} L_{p_\infty}^{\otimes k} & \text{for even } k , \\ L_{p_\infty}^{\otimes 2k} & \text{for odd } k . \end{cases}
\]

Proof. Consider the cone \(\sigma_{\infty,0}\). By arguing as in the proof of Lemma 3.31, one finds that the restrictions of the line bundles on \(\mathcal{X}_k\) are given by

\[
\mathcal{O}_{\mathcal{X}_k}(\mathcal{P}_0)|_{p_0} \simeq \begin{cases} L_{p_0}^{\otimes k} & \text{for even } k , \\ L_{p_0}^{\otimes 2k} & \text{for odd } k . \end{cases}
\]

Hence we obtain

\[
L_{1|p_0} \simeq L_{p_0} \quad \text{and} \quad L_{2|p_0} \simeq \begin{cases} L_{p_0}^{\otimes k} & \text{for even } k , \\ L_{p_0}^{\otimes 2k} & \text{for odd } k . \end{cases}
\]

For \(p_\infty\) one proceeds similarly. \(\square\)

A.1.2. Characterization of the inertia stack \(\mathcal{I}(\mathcal{X}_k)\). By Theorem 2.18, we have

\[
\mathcal{I}(\mathcal{X}_k) = \bigsqcup_{\bar{\mathcal{I}}} \mathcal{P}(\mathcal{\Sigma}_k/\sigma(\bar{v})) .
\]

One can show that the cardinality of \(\text{Box}(\mathcal{\Sigma}_k)\) is \(k(2k - 1)\), and its elements are classified as follows.

The element 0 belongs to \(\text{Box}(\sigma)\) for every two-dimensional cone \(\sigma \in \{\sigma_1, \ldots, \sigma_k, \sigma_{\infty,0}, \sigma_{\infty,k}\}\). Its corresponding minimal cone is \(\{0\} \in \Sigma(0)\). Thus \(\mathcal{P}(\Sigma/\{0\}) \simeq \mathcal{X}_k\). Moreover, \(\text{Box}(\mathcal{\Sigma}_k)\) contains \(k - 1\) elements of the form \(v_\infty, 2v_\infty, \ldots, (k - 1)v_\infty\) which belong to \(\rho_\infty \setminus 0\), so that their corresponding minimal cone is \(\rho_\infty\). Thus for \(g_i \in G\mathcal{\Sigma}_k\) corresponding to \(iv_\infty\) for \(i = 1, \ldots, k - 1\), we have an isomorphism \(\kappa_i: [Z_{\Sigma_k}^i / G\mathcal{\Sigma}_k] \xrightarrow{\sim} \mathcal{P}(\mathcal{\Sigma}_k/\rho_\infty) = \mathcal{X}_\infty\).

Let \(i = 1, \ldots, k - 1\). In the following we denote by \(\mathcal{P}_\infty\) the substack \([Z_{\Sigma_k}^i / G\mathcal{\Sigma}_k] \subset \mathcal{P}(\mathcal{X}_k)\). After fixing a primitive \(k\)-th root of unity \(\omega\), it is easy to see that the element \(g_i\) is \((1, \ldots, 1, \omega^i) \in G\mathcal{\Sigma}_k\). Then for a scheme \(S\), the objects of \(\mathcal{P}_\infty(S)\) are pairs of the form \((x, g_i)\), where \(x\) is an object of \(\mathcal{P}_\infty(S)\). The case \(i = 0\) is excluded because the pairs \((x, 1)\) with \(x \in \mathcal{P}_\infty\) are in \(\mathcal{X}_k \subset \mathcal{I}(\mathcal{X}_k)\). The group of automorphisms of \((x, g_i)\) is \(\mu_k\) and the inclusion of \(\mu_k\) into \(G\mathcal{\Sigma}_k\) is given by the map

\[
\gamma_k^i : \omega \in \mu_k \longmapsto g_i \in G\mathcal{\Sigma}_k = (\mathbb{C}^*)^k .
\]

The isomorphism \(\kappa_i\) implies \(v \circ \varphi^i_k = \gamma_k^i\), where the maps \(\varphi^i_k\) and \(v\) are given by

- for even \(k\):
  \[
  \varphi^i_k : \omega \in \mu_k \longmapsto (\omega^i, \omega^i) \in \mathbb{C}^* \times \mu_k ,
  \]
  \[
  v : (t, \omega) \in \mathbb{C}^* \times \mu_k \longmapsto (1, \ldots, 1, t^k \omega^{-1}, t) \in (\mathbb{C}^*)^k ;
  \]

- for odd \(k\):
  \[
  \varphi^i_k : \omega \in \mu_k \longmapsto (\omega^i, 1) \in \mathbb{C}^* \times \mu_k ,
  \]
  \[
  v : (t, \omega) \in \mathbb{C}^* \times \mu_k \longmapsto (1, \ldots, 1, t^k \omega^{-1}, t) \in (\mathbb{C}^*)^k .
  \]
The set $Box(\Sigma)$ also contains $k \tilde{k}$ elements belonging to $\sigma_{\infty,k}$. Among them, there are exactly $k$ elements which belong to $\rho_{\infty}$; their minimal cone is $\rho_{\infty}$. The minimal cone of the remaining $k \tilde{k} - k$ elements is $\sigma_{\infty,k}$. The corresponding group elements are $h_j = (1, \ldots, 1, \eta^{2j}k, \eta^j) \in G_{\Sigma_k}$ for $j = 0, \ldots, k \tilde{k} - k$, where $\eta$ is a primitive $k \tilde{k}$-th root of unity. For $\tilde{k} \not divides j$ we have $h_j = g_{j/\tilde{k}}$ and therefore $[Z_{\Sigma_k}^h / G_{\Sigma_k}] \simeq \mathcal{D}_{2/j}^{\tilde{k}}$. So from now on we consider only elements $h_j$ with $j = 1, \ldots, k \tilde{k} - 1, \tilde{k} \not divides j$. Then for any $h_j$ we have an isomorphism $\kappa^j_\infty : \left[ Z_{\Sigma_k}^h / G_{\Sigma_k} \right] \xrightarrow{\sim} \mathcal{D}_\infty$. Let $j = 1, \ldots, k \tilde{k} - 1, \tilde{k} \not divides j$. Denote by $p^\infty_j$ the substack $\left[ Z_{\Sigma_k}^h / G_{\Sigma_k} \right] \subset \mathcal{I}(\mathcal{D}_k)$. Then for a scheme $S$, the objects of $p^\infty_j(S)$ are pairs of the form $(y, h_j)$, where $y \in p^\infty(S)$. The group of automorphisms of $(y, h_j)$ is $\mu_{k \tilde{k}}$ and the inclusion of $\mu_{k \tilde{k}}$ into $G_{\Sigma_k}$ is given by the map

$$\gamma^j_{\infty,\tilde{k}} : \eta \in \mu_{k \tilde{k}} \mapsto h_j \in G_{\Sigma_k} = (\mathbb{C}^*)^k.$$

The isomorphism $\kappa^j_\infty$ implies that $i \circ j^\infty \circ \varphi^j_{\infty} = \gamma^j_{\infty}$, where the maps $\varphi^j_{\infty}$ and $j^\infty$ are given by

- for even $k$:

$$\varphi^j_{\infty} : \eta \in \mu_{k \tilde{k}} \mapsto \eta^j \in \mu_{k \tilde{k}};$$

$$j^\infty : \eta \in \mu_{k \tilde{k}} \mapsto (\eta, \eta^{-\tilde{k}}) \in \mathbb{C}^* \times \mu_k;$$

- for odd $k$:

$$\varphi^j_{\infty} : \eta \in \mu_{k \tilde{k}} \mapsto \eta^j \in \mu_{k \tilde{k}};$$

$$j^\infty : \eta \in \mu_{k \tilde{k}} \mapsto (\eta, \eta^{-2\tilde{k}}) \in \mathbb{C}^* \times \mu_k.$$

In a similar way, we obtain substacks $\bar{p}^j_0 \subset \mathcal{I}(\mathcal{D}_k)$ associated to $f_j = (1, \ldots, 1, \eta^j) \in G_{\Sigma_k}$, which are isomorphic to $p_0$, where $\eta$ is a primitive $k \tilde{k}$-th root of unity and $j = 1, \ldots, k \tilde{k} - 1, \tilde{k} \not divides j$. Therefore we obtain an isomorphism $i \circ j^0 \circ \varphi^j_{0} = \gamma^j_{0}$, where the maps $\varphi^j_{0}$ and $j^0$ are given by

- for even $k$:

$$\varphi^j_{0} : \eta \in \mu_{k \tilde{k}} \mapsto \eta^j \in \mu_{k \tilde{k}};$$

$$j^0 : \eta \in \mu_{k \tilde{k}} \mapsto (\eta, \eta^{\tilde{k}}) \in \mathbb{C}^* \times \mu_k;$$

- for odd $k$:

$$\varphi^j_{0} : \eta \in \mu_{k \tilde{k}} \mapsto \eta^j \in \mu_{k \tilde{k}};$$

$$j^0 : \eta \in \mu_{k \tilde{k}} \mapsto (\eta, \eta^{2\tilde{k}}) \in \mathbb{C}^* \times \mu_k.$$

Thus we can write the inertia stack as

$$\mathcal{I}(\mathcal{D}_k) = \mathcal{D}_k \cup \left( \bigsqcup_{i=1}^{k-1} \mathcal{D}_i \right) \cup \left( \bigsqcup_{j=1}^{k \tilde{k}-1} p^j_0 \right) \cup \left( \bigsqcup_{j=1}^{k \tilde{k}-1} p^j_{\infty} \right). \quad (A.6)$$

A.2. Topological invariants. We compute the integrals of Chern classes of the tangent bundles to $\mathcal{D}_k$ and $\mathcal{D}_\infty$. The canonical bundle of $\mathcal{D}_\infty$ is $\mathcal{K}_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(-p_0 - p_\infty)$; this can be regarded as a generalization of the analogous result for toric varieties [29, Theorem 8.2.3] (see also [52]). By Corollary 3.32 we obtain $\mathcal{K}_{\mathcal{D}_\infty} \simeq L_1^{\otimes -2\tilde{k}}$. Let $\mathcal{T}_{\mathcal{D}_\infty}$ denote the tangent sheaf to $\mathcal{D}_\infty$. By Lemma 3.39 one has

$$\int_{\mathcal{D}_\infty} c_1(\mathcal{T}_{\mathcal{D}_\infty}) = \int_{\mathcal{D}_\infty} c_1(\mathcal{O}_{\mathcal{D}_\infty}(p_0 + p_\infty)) = \frac{2}{k \tilde{k}}. \quad (A.7)$$
By applying [88, Theorem 3.4], which enables us to compute the integral of total Chern class \( c(T_{\mathcal{I}(\mathcal{X}_k)}) \), we have

\[
\int_{\mathcal{I}(\mathcal{X}_k)} c(T_{\mathcal{I}(\mathcal{X}_k)}) = \int_{\mathcal{X}_k} c^{SM}(\bar{X}_k) = e(\bar{X}_k) = |\Sigma_k(2)| = k + 2,
\]

where \( c^{SM}(\bar{X}_k) \) denotes the Chern-Schwartz-MacPherson class. The second equality comes from [81], while the third equality comes from [29, Theorem 12.3.9]. On the other hand, by the decomposition (A.6) of the inertia stack \( \mathcal{I}(\mathcal{X}_k) \) we have

\[
\int_{\mathcal{I}(\mathcal{X}_k)} c(T_{\mathcal{I}(\mathcal{X}_k)}) = \int_{\mathcal{X}_k} c_2(T_{\mathcal{X}_k}) + (k - 1) \int_{\mathcal{I}(\mathcal{X}_k)} c_1(T_{\mathcal{I}(\mathcal{X}_k)}) + k(\tilde{k} - 1) \int_{p_0} 1 + k(\tilde{k} - 1) \int_{p_{\infty}} 1.
\]

Recall that the order of the stabilizers of \( p_0 \) and \( p_{\infty} \) is \( k \tilde{k} \), so that \( \int_{p_0} 1 = \frac{1}{k \tilde{k}} \int_{p_{\infty}} 1 = \frac{1}{k \tilde{k}} \) where \( p_0 \) is understood to be the one-point scheme which we may regard as the coarse moduli space of \( p_0 \). For \( p_{\infty} \) one obtains the same result, so that

\[
\int_{\mathcal{X}_k} c_2(T_{\mathcal{X}_k}) = k + \frac{2}{k \tilde{k}}.
\]

(A.8)

### A.3. Euler characteristic

In this subsection we collect the results described so far and compute all the ingredients needed to prove Theorem A.1. As explained in the proof of Proposition 4.18, the rank of the natural bundle \( V \) is given by the Euler characteristic as \( -\chi(\mathcal{X}_k, E \otimes \mathcal{O}_{\mathcal{X}_k}(-D_{\infty})) \), where \( E \) is the underlying sheaf of a point \( [(E, \phi_\mathcal{X})] \in \mathcal{M}_{r, d, \Delta}(\mathcal{X}_k, D_{\infty}, \mathcal{F}^0) \).

By using the Toën-Riemann-Roch theorem we have

\[
\chi(\mathcal{X}_k, E \otimes \mathcal{O}_{\mathcal{X}_k}(-D_{\infty})) = \int_{\mathcal{X}_k} \frac{\text{ch}(\rho(\mathcal{N}'))}{\text{ch}(\rho(\lambda^{-1}(\mathcal{N}')))} \cdot \text{Td}(T_{\mathcal{I}(\mathcal{X}_k)}),
\]

where \( \mathcal{N} \) is the normal bundle to the local immersion \( \varpi : \mathcal{I}(\mathcal{X}_k) \to \mathcal{X}_k \).

Using the decomposition (A.6), the integral over the inertia stack becomes a sum of four terms given by

\[
A := \int_{\mathcal{X}_k} \text{ch}(E \otimes \mathcal{O}_{\mathcal{X}_k}(-D_{\infty})) \cdot \text{Td}(\mathcal{X}_k),
\]

\[
B := \sum_{i=1}^{k-1} \int_{D_{\infty}^i} \frac{\text{ch}(\rho(E \otimes \mathcal{O}_{\mathcal{X}_k}(-D_{\infty}))|_{D_{\infty}^i})}{\text{ch}(\rho(\lambda^{-1}(\mathcal{N}'))|_{D_{\infty}^i})} \cdot \text{Td}(D_{\infty}),
\]

\[
C := \sum_{i=1}^{k-1} \int_{p_0^i} \frac{\text{ch}(\rho(E \otimes \mathcal{O}_{\mathcal{X}_k}(-D_{\infty}))|_{p_0^i})}{\text{ch}(\rho(\lambda^{-1}(\mathcal{N}'))|_{p_0^i})} \cdot \text{Td}(p_0),
\]

\[
D := \sum_{i=1}^{k-1} \int_{p_{\infty}^i} \frac{\text{ch}(\rho(E \otimes \mathcal{O}_{\mathcal{X}_k}(-D_{\infty}))|_{p_{\infty}^i})}{\text{ch}(\rho(\lambda^{-1}(\mathcal{N}'))|_{p_{\infty}^i})} \cdot \text{Td}(p_{\infty}).
\]

We compute each part separately. In the ensuing calculations we make use of a number of identities involving sums of complex roots of unity, which are proven in Appendix B below.
Computation of $A$.

$$A = \int_{\mathcal{D}_k} \left( \text{ch}_2(\mathcal{E}) - c_1(\mathcal{E}) \cdot [\mathcal{D}_\infty] + c_1(\mathcal{E}) \cdot \text{Td}_1(\mathcal{D}_k) \right)$$

$$+ r \int_{\mathcal{D}_k} \left( \text{Td}_2(\mathcal{D}_k) + \frac{1}{2} [\mathcal{D}_\infty]^2 - [\mathcal{D}_\infty] \cdot \text{Td}_1(\mathcal{D}_k) \right) = \int_{\mathcal{D}_k} \text{ch}_2(\mathcal{E}) + r \frac{k^2 \tilde{k}^2 + 4\tilde{k}^2 - 6\tilde{k} + 1}{12 k^2 \tilde{k}^2}$$

since $c_1(\mathcal{E}) \cdot [\mathcal{D}_\infty] = c_1(\mathcal{E}) \cdot \text{Td}_1(\mathcal{D}_k) = 0$; the evaluation of the second integral follows from Equation (A.8), Proposition 3.5 and the adjunction formula [75, Theorem 3.8].

Computation of $B$. We first need to compute $\rho((\mathcal{E} \otimes \mathcal{O}_{\mathcal{D}_k}(-\mathcal{D}_\infty))[\mathcal{D}_\infty])$. For a fixed value of $i$, the homomorphism $\rho$ sends the K-theory class $[\mathcal{G}]$ of a vector bundle $\mathcal{G}$ on $\mathcal{D}_\infty$ to $\sum_m \omega^m [\mathcal{G}_m]$ where $\mathcal{G}_m$ is the $m$-th summand in the decomposition of $\mathcal{G}$ with respect to the action of $\omega$. Note that $(\mathcal{E} \otimes \mathcal{O}_{\mathcal{D}_k}(-\mathcal{D}_\infty))[\mathcal{D}_\infty] \simeq \mathcal{F}^0 \otimes \mathcal{L}_1^{-1} \oplus \bigoplus_{j=0}^{k-1} \mathcal{O}_{\mathcal{D}_\infty}(-1, j)^\oplus w_j$. Then

$$\rho((\mathcal{E} \otimes \mathcal{O}_{\mathcal{D}_k}(-\mathcal{D}_\infty))[\mathcal{D}_\infty]) = \sum_{j=0}^{k-1} w_j \rho(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)).$$

Lemma A.9. For fixed $i = 1, \ldots, k-1$ one has

$$\rho(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)) = \begin{cases} \omega^i(k-j-1) [\mathcal{O}_{\mathcal{D}_\infty}(-1, j)] & \text{for even } k, \\ \omega^{-i} [\mathcal{O}_{\mathcal{D}_\infty}(-1, j)] & \text{for odd } k. \end{cases}$$

Proof. Suppose $k$ is even. Recall that $\mathcal{O}_{\mathcal{D}_\infty}(-1, j) \simeq \mathbb{C}^0 \otimes \mathcal{L}_2^{-1} \mathcal{L}_1^{-1}$ corresponds to the character $\chi^{(-1,j)}: (t, \omega) \in \mathbb{C}^* \times \mu_k \mapsto t^{-1} \omega^j \in \mathbb{C}^*$. The element $\rho(\mathcal{O}_{\mathcal{D}_\infty}(-1, j))$ is computed with respect to the map $\varphi_k^j: \omega \in \mu_k \mapsto (\omega^i, \omega^j) \in \mathbb{C}^* \times \mu_k$, where $\omega$ is a primitive $k$-th root of unity. So the composition of the latter map with $\chi^{(-1,j)}$ gives $\omega \in \mu_k \mapsto \omega^{i(k-j-1)} \in \mathbb{C}^*$.

For odd $k$ one has a similar result. In that case the map $\varphi_k^j$ is given by $\omega \in \mu_k \mapsto (\omega^i, 1) \in \mathbb{C}^* \times \mu_k$, which by composition with $\chi^{(-1,j)}$ yields $\omega \in \mu_k \mapsto \omega^{-i} \in \mathbb{C}^*$. \qed

By Lemma A.9, we obtain

$$\text{ch} \left( \rho((\mathcal{E} \otimes \mathcal{O}_{\mathcal{D}_k}(-\mathcal{D}_\infty))[\mathcal{D}_\infty]) \right) = \begin{cases} \sum_{j=0}^{k-1} w_j \omega^i(k-j-1) \text{ch}(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)) & \text{for even } k, \\ \sum_{j=0}^{k-1} w_j \omega^{-i} \text{ch}(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)) & \text{for odd } k. \end{cases}$$

The normal bundle $\mathcal{N}_{\mathcal{D}_\infty/\mathcal{D}_k}$ is isomorphic to $\mathcal{O}_{\mathcal{D}_k}(\mathcal{D}_\infty)[\mathcal{D}_\infty] \simeq \mathcal{L}_1$ (cf. Lemma 3.31). Thus again by Lemma A.9 we get

$$\text{ch} \left( \rho(\lambda_{-1} \mathcal{N}_{\mathcal{D}_\infty/\mathcal{D}_k}) \right) = \text{ch}(1 - \rho(\mathcal{L}_1^{-1})) = 1 - \omega^{-i} \text{ch}(\mathcal{L}_1^{-1}) = 1 - \omega^{-i} (1 - c_1(\mathcal{L}_1)).$$

Set $s_j = \tilde{k} - j - 1$ if $k$ is even and $s_j = -1$ if $k$ is odd. We obtain

$$B = \sum_{i=1}^{k-1} \int_{\mathcal{D}_\infty} \left( \sum_{j=0}^{k-1} w_j \omega^{i s_j} \left( 1 + c_1(\mathcal{O}_{\mathcal{D}_\infty}(-1, j)) \right) \right)$$

$$\cdot \left( \frac{1}{1 - \omega^{-i}} - \frac{\omega^{-i}}{(1 - \omega^{-i})^2} c_1(\mathcal{L}_1) \right) \cdot (1 + \text{Td}_1(\mathcal{D}_\infty))$$

$$= \sum_{j=0}^{k-1} w_j \sum_{i=1}^{k-1} \frac{\omega^{i s_j}}{1 - \omega^{-i}} \left( \frac{1}{k} \frac{1}{k^2} - \frac{1}{k^2} \frac{1}{1 - \omega^{-i}} \right),$$
where the last equality follows from Equation (A.7) and Lemma 3.39. Now we use the identity [22]
\[
\frac{1}{k} \sum_{i=1}^{k-1} \frac{\omega^i s}{1 - \omega^{-i}} = \left\lfloor \frac{s}{k} \right\rfloor - \frac{s}{k} + \frac{k-1}{2k},
\] (A.10)
together with the fact that
\[
\left\lfloor \frac{s_j}{k} \right\rfloor - \frac{s_j}{k} = \begin{cases} 
\frac{1}{k} - 1 & \text{for odd } k, \text{ or even } k \text{ and even } j, \\
\frac{1}{k} - \frac{1}{2} & \text{for even } k \text{ and odd } j,
\end{cases}
\]
and \(\sum_{j=0}^{k-1} w_j = r\). Thus for odd \(k\) we easily get
\[
\sum_{j=0}^{k-1} w_j \sum_{i=1}^{k-1} \frac{\omega^i s_j}{1 - \omega^{-i}} \left( \frac{1}{k} - \frac{1}{k k^2} \right) = \frac{k-1}{2k} \sum_{j=0}^{k-1} w_j \left( \left\lfloor \frac{s_j}{k} \right\rfloor - \frac{s_j}{k} + \frac{k-1}{2k} \right) = r - \frac{(k-1)^2}{2k^3}.
\]
In the case of even \(k\), define the natural numbers \(r_e = \sum_{j \text{ even}} w_j\) and \(r_o = \sum_{j \text{ odd}} w_j\); then \(r = r_e + r_o\) and we have
\[
\frac{k-1}{2k^2} \sum_{j=0}^{k-1} w_j \left( \left\lfloor \frac{s_j}{k} \right\rfloor - \frac{s_j}{k} + \frac{k-1}{2k} \right) = r_e \left( \frac{k-1}{2k^2} (1-k) \right) + r_o \frac{k-1}{2k^2}.
\]
Thus
\[
B = \begin{cases} 
-r \frac{(k-1)^2}{2k^3} - \frac{r}{k^3} \sum_{i=1}^{k-1} \frac{\omega^{-2i}}{1 - \omega^{-1}} & \text{for odd } k, \\
r_e \left( \frac{k-1}{2k^2} (1-k) \right) + r_o \frac{k-1}{2k^2} - \frac{1}{k k^2} \sum_{j=0}^{k-1} w_j \sum_{i=1}^{k-1} \frac{\omega^i (k-j-2)}{1 - \omega^{-1}} & \text{for even } k. 
\end{cases}
\]
Using Lemma B.1, with \(s = k - 2\) for odd \(k\), and for even \(k\) and even \(j\), and with \(s = \tilde{k} - 2\) for even \(k\) and odd \(j\), we get
\[
B = \begin{cases} 
-\frac{5k^2 - 6k + 1}{12k^3} r & \text{for odd } k, \\
-\frac{2k^2 - 3k + 1}{3k^3} r + \frac{r_o}{2k} & \text{for even } k.
\end{cases}
\]

**Computation of \(C\) and \(D\).** We shall need the following characterization.

**Lemma A.11.** The conormal bundles to \(p_0\) and \(p_\infty\) in \(\mathcal{X}_k\) are given by
\[
\mathcal{N}_{p_0/\mathcal{X}_k} \simeq \mathcal{L}_{p_0}^{k-2} \oplus \mathcal{L}_{p_0}^{k-1} \quad \text{and} \quad \mathcal{N}_{p_\infty/\mathcal{X}_k} \simeq \mathcal{L}_{p_\infty}^{k-2} \oplus \mathcal{L}_{p_\infty}^{k-1}.
\]

**Proof.** Since \(p_0\) is a zero-dimensional substack in \(\mathcal{X}_k\), its tangent bundle is trivial and so \(\mathcal{N}_{p_0/\mathcal{X}_k} \simeq \mathcal{T}_{\mathcal{X}_k|p_0}^\vee\). The divisors \(\mathcal{D}_0\) and \(\mathcal{D}_\infty\), which intersect in \(p_0\), are normal crossing and so the tangent bundle splits as
\[
\mathcal{T}_{\mathcal{X}_k|p_0} \simeq \mathcal{T}_{\mathcal{D}_0|p_0} \oplus \mathcal{T}_{\mathcal{D}\infty|p_0}.
\]
By the adjunction formula [75, Theorem 3.8], we obtain
\[
\mathcal{T}_{\mathcal{D}_0|p_0} \simeq (K_{\mathcal{X}_k|p_0})_{p_0}^\vee \simeq (K_{\mathcal{X}_k|\mathcal{D}_0} \otimes \mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_0))_{p_0}^\vee \simeq \mathcal{L}_{1|p_0} \simeq \mathcal{L}_{p_0}.
\]
Since the canonical line bundle $K_{\mathcal{G}_\infty}$ is isomorphic to $\mathcal{O}_{\mathcal{G}_\infty}(-p_0 - p_\infty)$, we get $\mathcal{T}_{\mathcal{G}_\infty} \mid_{p_0} \simeq (\mathcal{O}_{\mathcal{G}_\infty}(p_0 + p_\infty)) \mid_{p_0} \simeq \mathcal{L}^{\otimes 2k}_{p_0}$ and the first isomorphism is proved. The second isomorphism is proven in the same way.

By Lemma A.11 we obtain

$$
\text{ch}_0 \left( \rho\left( \lambda_{-1}(\mathcal{N}_{p_0}^\vee/\mathcal{X}_k) \right) \right) = (1 - \eta^{-i})(1 - \eta^{-2i}) \chi_0 \left( \rho\left( \lambda_{-1}(\mathcal{N}_{p_0}^\vee/\mathcal{X}_k) \right) \right) .
$$

By Lemma A.5 we have

$$
\left( \mathcal{E} \otimes \mathcal{O}_{\mathcal{G}_k}(-\mathcal{G}_\infty) \right) \mid_{p_0} \simeq \left( \mathcal{F}^{0,df} \otimes \mathcal{L}^{\otimes -1}_{1} \right) \mid_{p_0} \simeq \left( \bigoplus_{j=0}^{k-1} \mathcal{O}_{\mathcal{G}_\infty}(-1, j)^{\oplus w_j} \right) \mid_{p_0} \simeq \bigoplus_{j=0}^{k-1} \left( \mathcal{L}_{p_0}^{\otimes -k_0 - j - 1} \right)^{\oplus w_j} ,
$$

and similarly

$$
\left( \mathcal{E} \otimes \mathcal{O}_{\mathcal{G}_k}(-\mathcal{G}_\infty) \right) \mid_{p_\infty} \simeq \bigoplus_{j=0}^{k-1} \left( \mathcal{L}_{p_\infty}^{\otimes k_0 + j - 1} \right)^{\oplus w_j} .
$$

Thus we have

$$
C + D = \frac{1}{k^2} \sum_{j=0}^{k-1} w_j \sum_{i=1}^{k-1} \frac{\omega^i j + \omega^{-i} j}{1 - \omega^{-2i}} \sum_{l=0}^{k-1} \frac{1}{\eta^l \omega^l - 1} \eta^i (k_0 + 1) + \eta^{-i} (k_0 + 1) (1 - \eta^{-i})(1 - \eta^{-2i}) . \tag{A.12}
$$

We now have to distinguish two cases.

**Odd $k$.** By Lemma B.5 we have

$$
C + D = \frac{1}{4k} \sum_{j=0}^{k-1} w_j \sum_{i=1}^{k-1} (\omega^i j + \omega^{-i} j) \left( \frac{3 - \omega^i}{(1 - \omega^{-i})^2} + \frac{\omega^{2i}}{1 + \omega^i} \right) .
$$

It is convenient to separate the contributions from $j = 0$ and $j \geq 1$ in the above sum; we denote them by $(C + D)_0$ and $(C + D)_>$, respectively. By Lemmas B.3 and B.1 with $s = k$ and $s = 1$ we find

$$(C + D)_0 = -\frac{k^2 - 1}{12k} w_0 ,$$

while by Lemmas B.4 and B.1 with $s = j$, $s = k - j$ and $s = j + 1$, $s = -j + 1$ we get

$$(C + D)_> = \sum_{j=1}^{k-1} w_j \left( \frac{j (k - j)}{2k} - \frac{k^2 - 1}{12k} \right) .$$

Thus

$$
C + D = -\frac{k^2 - 1}{12k} r + \sum_{j=1}^{k-1} \frac{j (k - j)}{2k} w_j .
$$
Even $k$. By Lemma B.6 we have
\[
C + D = \frac{2}{k^2} \sum_{j=0}^{k-1} \omega_j \sum_{i=1}^{k-1} \frac{\omega^{ij} + \omega^{-ij}}{1 - \omega^{-2i}} \sum_{l=0}^{k-1} \frac{(-1)^l j}{\eta^l \omega^l - 1}
\]
\[
= \frac{2}{k} \sum_{j \text{ even}} \omega_j \sum_{i=1}^{k-1} \frac{\omega^{ij} - \omega^{i(j-2)}(1 - \omega^{-2i})^2}{1 - \omega^{-2i}^2} + \frac{2}{k} \sum_{j \text{ odd}} \omega_j \sum_{i=1}^{k-1} \frac{\omega^{ij} - \omega^{i(j-2)}(1 - \omega^{-2i})^2}{1 - \omega^{-2i}^2}.
\]

Here we use Lemma B.1, with $\tilde{k}$ instead of $k$ and $\omega^2$ (a $\tilde{k}$-th root of unity) instead of $\omega$. For even $j$, we use $s = j/2 - 1$ and $s = \tilde{k} - j/2 - 1$, while for odd $j$ we use $s = (j-1)/2$ and $s = \tilde{k} - (j+1)/2$. This gives
\[
C + D = -\frac{k^2 - 1}{12k} r + \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} w_j,
\]
while for even $k$
\[
C + D = -\frac{k^2 - 4}{12k} r - \frac{r_o}{2k} + \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{jj} w_j.
\]

Proof of Theorem A.1. One just needs to sum the contributions $A$, $B$, $C$ and $D$ and change sign. \( \square \)

A.4. Dimension formula. The proof of Corollary A.2 requires a Riemann-Roch formula for the Euler characteristic
\[
\chi(\mathcal{E}, \mathcal{G}) = \sum_i (-1)^i \dim \mathbb{C} \operatorname{Ext}^i(\mathcal{E}, \mathcal{G}),
\]
where $\mathcal{E}$ and $\mathcal{G}$ are coherent sheaves on $\mathcal{X}_k$. One can prove (as in [46, Lemma 6.1.3]) that
\[
\chi(\mathcal{E}, \mathcal{G}) = \int_{\mathcal{I}(\mathcal{X}_k)} \frac{\operatorname{ch}^\vee(\rho(\varpi^* \mathcal{E})) \cdot \operatorname{ch}(\rho(\varpi^* \mathcal{G}))}{\operatorname{ch}(\rho(\lambda_1 \mathcal{N}^\vee))} \cdot \operatorname{Td}(\mathcal{X}_k),
\]
where for an element $x \otimes \omega \in K(\mathcal{I}(\mathcal{X}_k)) \otimes \mu_\infty$ one defines
\[
\operatorname{ch}^\vee(x \otimes \omega) := \sum_i (-1)^i \operatorname{ch}_i(x) \otimes \varpi^{-1}.
\]

Here $\mu_\infty \subset \mathbb{C}$ is the group of all roots of unity. If $\mathcal{E}$ is locally free then $\operatorname{ch}^\vee(\rho(\varpi^* \mathcal{E})) = \operatorname{ch}(\rho(\varpi^* \mathcal{E}^\vee))$.

By Proposition 4.12 and the proof of Theorem 4.13 we have
\[
\dim \mathbb{C} \mathcal{M}_{r, \bar{u}, \Delta}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{F}_\infty^{s, \bar{u}}) = -\chi(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)),
\]
where $[(\mathcal{E}, \varphi_q)]$ is a point in the moduli space. By the same argument as in Section A.3 we obtain the required formula.

Example A.13. As we saw in Section 4.3, $\mathcal{M}_{1, \bar{u}, n}(\mathcal{X}_k, \mathcal{D}_\infty, \mathcal{O}_{\mathcal{D}_\infty}(s, i))$ is isomorphic to the Hilbert scheme $\operatorname{Hilb}^n(X_k)$ of $\Delta = n$ points on $X_k$. In the rank one case one has $\bar{u} \cdot \bar{u}(j) = 0$ for all $j \geq 1$ and the dimension formula (A.3) agrees with the dimension of $\operatorname{Hilb}^n(X_k)$. Likewise, when $w_i = r$ for some $i \in \{0, 1, \ldots, k - 1\}$ and $w_j = 0$ for all $j \neq i$, the dimension of the moduli space is $2r \Delta$. 

For rank \( r \geq 2 \) we can give another formulation of the dimension formula (A.3). Let us consider a locally free sheaf on \( \mathcal{X}_k \) of the form

\[
\mathcal{E} := \bigoplus_{\alpha=1}^{r} \mathcal{M}_\alpha = \bigoplus_{\alpha=1}^{r} \bigotimes_{i=1}^{k-1} \mathcal{R}_i^{\otimes u_{\alpha,i}} \otimes \mathcal{O}_{\mathcal{X}_k}(s \mathcal{D}_{\infty})
\]

for some choice of integers \( u_{\alpha,j} \). The discriminant of \( \mathcal{E} \) is

\[
\Delta(\mathcal{E}) = \frac{r-1}{2r} \sum_{\alpha=1}^{r} \sum_{i,j=1}^{k-1} u_{\alpha,i} (C^{-1})^{ij} u_{\alpha,j} - \frac{1}{2r} \sum_{\alpha \neq \beta}^{r} \sum_{i,j=1}^{k-1} u_{\alpha,i} (C^{-1})^{ij} u_{\beta,j},
\]

and by Lemma 3.31 and Corollary 3.34 we have

\[
\mathcal{M}_{|\mathcal{D}_{\infty}} \simeq \mathcal{O}_{\mathcal{D}_{\infty}}(s, \sum_{i=1}^{k-1} i u_{\alpha,i}).
\]

Hence \( \mathcal{E} \) has an induced framing \( \phi_\mathcal{E} \) to \( \mathcal{F}_{s,\infty}^{\mathcal{D}_{\infty}} \), where \( \bar{w} = (w_0, w_1, \ldots, w_{k-1}) \) with

\[
w_l := \#\{ \alpha \mid \sum_{i=1}^{k-1} i u_{\alpha,i} = l \mod k \}.
\]

Then \([(\mathcal{E}, \phi_\mathcal{E})]\) is a point of the moduli space \( \mathcal{M}_{r,\bar{u},\Delta}(\mathcal{X}_k, \mathcal{D}_{\infty}, \mathcal{F}_{s,\infty}^{\mathcal{D}_{\infty}}) \), where

\[
\bar{u} := \left( \sum_{\alpha=1}^{r} u_{\alpha,1}, \ldots, \sum_{\alpha=1}^{r} u_{\alpha,k-1} \right).
\]

By the arguments of Section A.3, we find that the dimension of the moduli space (cf. Equation (A.3)) is

\[
(r-1) \sum_{\alpha=1}^{r} \sum_{i,j=1}^{k-1} u_{\alpha,i} (C^{-1})^{ij} u_{\alpha,j} - \sum_{\alpha \neq \beta}^{r} \sum_{i,j=1}^{k-1} u_{\alpha,i} (C^{-1})^{ij} u_{\beta,j} - \frac{1}{2} \sum_{j=1}^{k-1} (C^{-1})^{ij} \bar{w} \cdot \bar{w}(j).
\]

For \( k = 2 \) this formula reduces to

\[
\frac{r}{2} \sum_{\alpha=1}^{r} u_{\alpha}^2 - \frac{1}{2} \left( u^2 + w_0 w_1 \right),
\]

where \( u = \sum_{\alpha=1}^{r} u_{\alpha} \). By considering explicitly all possible parities of \( w_0 \) and \( w_1 \) one can check that this number is an integer.

**APPENDIX B. SUMMATION IDENTITIES FOR COMPLEX ROOTS OF UNITY**

In this appendix we collect a number of identities that were used in Appendix A. In the following \( \omega \) will denote a complex \( k \)-th root of unity, and \( \eta \) a complex \( \tilde{k} \)-th root of unity.

**Lemma B.1.** For \( s = 1, \ldots, k \), we have

\[
\sum_{i=1}^{k-1} \frac{\omega^{s i}}{(1 - \omega^{-i})^2} = -\frac{(k - 5) (k - 1)}{12} + \frac{s (k - 2 - s)}{2}.
\]

**Proof.** By the same arguments as in the proof of Equation (A.10) in [22], one can prove the identity

\[
\sum_{i=1}^{k-1} \frac{\omega^{s i}}{(1 - \omega^{-i})^2} = \sum_{l=0}^{k-1} (s - l) \left( -\frac{s - l}{k} + \frac{k - 1}{2k} \right).
\]

From this identity we get

\[
\sum_{i=1}^{k-1} \frac{\omega^{s i}}{(1 - \omega^{-i})^2} = -\sum_{l=0}^{s-1} (s - l) \frac{s - l}{k} - \sum_{l=s+1}^{k-1} (s - l) \frac{k - s - l}{k} + \frac{k - 1}{2k} \sum_{l=0}^{k-1} (s - l).
\]
\[
= - \sum_{m=1}^{k-1} \frac{m^2}{k} - \sum_{m=s+1}^{k-1} \frac{k-1}{2} \left( s - \frac{k-1}{2} \right).
\]

By standard algebraic manipulations one then gets the assertion. \(\square\)

We now prove various identities which were used for the calculation of the \(C\) and \(D\) contributions in Section A.3. We divide these identities according to the parity of \(k\). We start with odd \(k\).

**Lemma B.2.** For any fixed \(1 \leq i \leq k - 1\) and \(x \in \mathbb{C} \setminus \mu_k\), we have

\[
\prod_{\begin{subarray}{c} j=1 \\ j \neq i \end{subarray}}^{k-1} (x - \omega^j) = -\sum_{n=0}^{k-2} x^n \sum_{l=1}^{n+1} \omega^{-li} \quad \text{and} \quad \sum_{i=1}^{k-1} \frac{1}{x - \omega^i} = \frac{\sum_{n=0}^{k-2} (n+1) x^n}{\sum_{n=0}^{k-1} x^n}.
\]

**Proof.** We start by noting that, as pointed out in [22], one has \(\sum_{i=0}^{k-1} \omega^{s+i} = k\) if \(s = 0 \mod k\) and \(\sum_{i=0}^{k-1} \omega^{s+i} = 0\) otherwise; moreover,

\[
\prod_{\begin{subarray}{c} j=1 \\ j \neq i \end{subarray}}^{k-1} (x - \omega^j) = \frac{1}{x - \omega^i} \sum_{n=0}^{k-1} x^n =: \sum_{n=0}^{k-2} c_n x^n.
\]

The polynomial coefficients \(n!c_n\) can be obtained by differentiating the second expression \(n\) times with respect to \(x\) at \(x = 0\), and it is straightforward to prove by induction that

\[c_n = -\sum_{l=1}^{n+1} \omega^{-li}.
\]

Note in particular that \(c_{k-2} = -\sum_{l=1}^{k-1} \omega^{-li} = 1\) as expected.

For the second identity, we write

\[
\sum_{i=1}^{k-1} \frac{1}{x - \omega^i} = \frac{\sum_{i=1}^{k-1} \prod_{\begin{subarray}{c} j=1 \\ j \neq i \end{subarray}}^{k-1} (x - \omega^j)}{\prod_{i=1}^{k-1} (x - \omega^i)}.
\]

From above we have

\[
\prod_{i=1}^{k-1} (x - \omega^i) = \frac{x^{k-1} - 1}{x - 1} = \sum_{n=0}^{k-1} x^n
\]

and

\[
\sum_{i=1}^{k-1} \prod_{\begin{subarray}{c} j=1 \\ j \neq i \end{subarray}}^{k-1} (x - \omega^j) = -\sum_{n=0}^{k-2} x^n \sum_{l=1}^{n+1} \sum_{i=1}^{k-1} \omega^{-li} = \sum_{n=0}^{k-2} (n+1) x^n,
\]

and the result follows. \(\square\)

**Lemma B.3.** \(\sum_{i=1}^{k-1} \frac{\omega^{2i}}{1 + \omega^i} = -\frac{k+1}{2}\).
Proof. Since $k$ is odd, setting $x = -1$ in Lemma B.2 gives
\[
\prod_{i=1}^{k-1} (1 + \omega^i) = \sum_{n=0}^{k-1} (-1)^n = 1
\]
and
\[
\sum_{i=1}^{k-1} \omega^{2i} \prod_{j=1}^{k-1} (1 + \omega^j) = \sum_{n=0}^{k-2} \sum_{l=1}^{n+1} \sum_{i=1}^{k-1} \omega^{-(l-2)i} = -1 - \sum_{n=1}^{k-1} 2n + \sum_{n=1}^{k-1} (2n - 1) = -\frac{k+1}{2}.
\]

Lemma B.4. For any $1 \leq j \leq k - 1$, one has
\[
\sum_{i=1}^{k-1} \frac{\omega^i (j+2) + \omega^{-i} (j-2)}{1 + \omega^i} = -1.
\]

Proof. Putting $x = -1$ in Lemma B.2 again we find
\[
\sum_{i=1}^{k-1} \frac{\omega^i (j+2) + \omega^{-i} (j-2)}{1 + \omega^i} = \sum_{n=0}^{j} (-1)^{n+1} (n+1) + \sum_{n=j+1}^{k-1} (-1)^n (k-1) + \sum_{n=j+1}^{k-1} (-1)^{n+1} n.
\]
For odd $j$ this gives
\[
\frac{j+1}{2} + (k-1) - \frac{k-1}{2} - \frac{j+1}{2} = \frac{k-1}{2}
\]
while for even $j$ we get
\[
-\frac{j}{2} - 1 - \frac{k-1}{2} + \frac{j}{2} = -\frac{k+1}{2}.
\]
Now the sum
\[
\sum_{i=1}^{k-1} \frac{\omega^{-i} (j-2)}{1 + \omega^i} = \sum_{i=1}^{k-1} \frac{\omega^i (k-j+2)}{1 + \omega^i}
\]
is computed in an identical way by just replacing $j$ with $k-j$. Since $j$ and $k-j$ have opposite parity, we get the result.

Lemma B.5. Let $\eta$ be a $k$-th root of $\omega$ and $1 \leq i \leq k - 1$. Then
\[
\frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{\eta^j \omega^j - 1} = \frac{1}{\omega^i - 1}.
\]

Proof. Using Lemma B.2 with $x = \eta^{-i}$ we compute
\[
\frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{\eta^j \omega^j - 1} = -\frac{\eta^{-i}}{k} \left( \frac{1}{\eta^{-i} - 1} + \frac{\sum_{n=0}^{k-2} (n+1) \eta^{-in}}{\sum_{n=0}^{k-1} \eta^{-in}} \right)
\]
Using also \( \vec{s} \) and \( \vec{z} \).

As a consequence, a component \( v \) for a vector \( \vec{u} \) for \( \vec{v} \) of \( \vec{u} \) is integral if and only if every component is (cf. Remark 4.6). Define \( \omega \)-th root of \( \omega \). If \( c \) is the equivalence class of \( \sum_{j=0}^{k-1} \omega^j \omega^{-1} \) then for \( l = 1, \ldots, k - 1 \) we have \( k \omega_l = -l c \mod k \).

Now we consider the case of even \( k \).

**Lemma B.6.** Let \( \eta \) be a \( k \)-th root of \( \omega \) and \( 1 \leq i \leq k - 1 \). Then

\[
1 \sum_{j=0}^{k-1} \frac{(-1)^j}{\eta^j \omega^j - 1} = \frac{\omega^i}{\omega^{2i} - 1}
\]

**Proof.** The first identity follows exactly as in the proof of Lemma B.5, except that now \( \eta^k = \omega^2 \). For the second identity, we proceed as in the proof of Lemma B.5. Using \( \omega^k = -1 \) we first compute

\[
\sum_{j=1}^{k-1} (-1)^j \sum_{n=0}^{k-2} \frac{\eta^{-jn}}{\sum_{l=1}^{n+1} \omega^{-lj}} = -\sum_{n=0}^{k-2} \eta^{-jn} (n + 1) + \sum_{n=k-1}^{k-2} \eta^{-jn} (k - 1) - \sum_{n=k-1}^{k-2} \eta^{-jn} n
\]

Using also

\[
\sum_{n=0}^{k-1} \eta^{-jn} = \left( \sum_{n=0}^{k-1} + \sum_{n=k}^{2k-1} \right) \eta^{-jn} = (1 + \omega^{-i}) \sum_{n=0}^{k-1} \eta^{-jn},
\]

we arrive at

\[
1 \sum_{j=0}^{k-1} \frac{(-1)^j}{\eta^j \omega^j - 1} = -\frac{\eta^{-i}}{k} \left( \frac{1}{\eta^{-i} - 1} + \frac{\eta^i}{1 + \omega^{-i}} \right) \sum_{n=0}^{k-1} \frac{(n + 1) \omega^{-i} - \omega^{-i}) \eta^{-jn}}{\sum_{n=0}^{k-1} \eta^{-jn}}
\]

\[
= -\frac{\eta^{-i}}{k} \frac{-\omega^{-i} \omega^{-i} - 1)}{(\omega^{-i} - 1)(1 + \omega^{-i})} = \frac{\omega^i}{\omega^{2i} - 1}.
\]

\( \Box \)

**Appendix C. Edge contribution**

In this appendix we compute the equivariant Euler characteristic

\[
\chi_T_{\bar{s}}(\mathcal{R}_{\bar{u}} \otimes \mathcal{O}\mathcal{X}_{\bar{k}}(-\mathcal{D}_\infty)) := \chi_T_{\bar{s}}(\mathcal{X}_{\bar{k}} \mathcal{R}_{\bar{u}} \otimes \mathcal{O}\mathcal{X}_{\bar{k}}(-\mathcal{D}_\infty))
\]

for a vector \( \bar{u} \in \mathbb{Z}^{k-1} \).

Set \( \bar{v} := C^{-1} \bar{u} \). Then \( \bar{v} \in \mathbb{Z}^{k-1} \). If \( c \) is the equivalence class of \( \sum_{j=1}^{k-1} j u_j \mod k \), then for \( l = 1, \ldots, k - 1 \) we have

\[
k v_l = -l c \mod k. \tag{C.1}
\]

As a consequence, a component \( v_l \) is integral if and only if every component is (cf. Remark 4.6). Define \( \bar{z} := \bar{u} - \bar{e}_c \) if \( c > 0 \) and \( \bar{z} := \bar{u} \) otherwise, where \( \bar{e}_c \) is the \( c \)-th coordinate vector of \( \mathbb{Z}^{k-1} \). Set also \( \bar{v} := C^{-1} \bar{z} \). Then by construction \( \bar{v} \in \mathbb{Z}^{k-1} \).
C.1. Preliminaries.

Lemma C.2. Given a vector $\vec{u} \in \mathbb{Z}^{k-1}$, for every $l = 1, \ldots, k-1$ there is an exact sequence
\[
0 \to \mathcal{R}^{\vec{u} + C\vec{e}_1} \to \mathcal{R}^{\vec{u}} \to \mathcal{R}_{\mathcal{D}_l}^{\vec{u}} \to 0 \tag{C.3}
\]
where $C$ is the Cartan matrix of type $A_{k-1}$.

Proof. Fix $l = 1, \ldots, k-1$ and consider the short exact sequence
\[
0 \to \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_l) \to \mathcal{O}_{\mathcal{X}_k} \to \mathcal{O}_{\mathcal{D}_l} \to 0.
\]
We obtain the assertion simply by tensoring this sequence with $\mathcal{R}^{\vec{u}}$; we need only prove that $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_l) = \mathcal{R}^{\vec{u} + C\vec{e}_1}$. By definition $\mathcal{R}^{\vec{u}} = \mathcal{O}_{\mathcal{X}_k}(\sum_{i=1}^{k-1} u_i \omega_i)$, and we have
\[
\sum_{i=1}^{k-1} u_i \omega_i - \mathcal{D}_l = \sum_{i=1}^{k-1} u_i \omega_i + \sum_{i=1}^{k-1} C_{li} \omega_i = \sum_{i=1}^{k-1} (u_i + (C\vec{e}_1)_i) \omega_i.
\]
\qed

By Equation (3.23) we get the following result.

Lemma C.4. Let $\vec{u} \in \mathbb{Z}^{k-1}$ and $l = 1, \ldots, k-1$. Then $\mathcal{R}_{\mathcal{D}_l}^{\vec{u}} \simeq \mathcal{O}_{\mathcal{D}_l}(u_l)$, where $\mathcal{O}_{\mathcal{D}_l}(1) := \pi_{k|\mathcal{D}_l}^{\mathcal{X}_k} \mathcal{O}_{\mathcal{D}_l}(1)$.

C.2. Iterative procedure. Let us recall that by the arguments in Appendix A.3 the dimension of $H^1(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$ is $\frac{k}{2} (\vec{u} \cdot C\vec{u} - (C^{-1})^{cc})$, where we set $(C^{-1})^{cc} = 0$ if $c = 0$.

First, let us assume that $s_l \geq 0$ for every $l = 1, \ldots, k-1$. Consider the equation
\[
\frac{C_{ii}}{2} i^2 - i \left( \vec{u} - \sum_{p=1}^{l-1} s_p \vec{e}_p \right) \cdot C\vec{e}_i + \frac{1}{2} \left( \left( \vec{u} - \sum_{p=1}^{l-1} s_p \vec{e}_p \right) \cdot C \left( \vec{u} - \sum_{p=1}^{l-1} s_p \vec{e}_p \right) - (C^{-1})^{cc} \right) = 0, \quad (C.5)
\]
and define the set
\[
S_l^+ := \{i \in \mathbb{N} \mid i \leq s_l \text{ is a solution of Equation (C.5)}\}.
\]
Let $m$ be the smallest integer $l \in \{1, \ldots, k-1\}$ such that $S_l^+$ is nonempty; if all sets are empty, let $m := k-1$.

Now we can compute $\chi_{T_l}(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))$ by using Lemma C.2. Recall that by convention $\mathcal{R}_0 := \mathcal{O}_{\mathcal{X}_k}$.

Let $d_1^+ := \min(S_1^+)$ if $S_1^+ \neq \emptyset$, otherwise $d_1^+ := s_1$. By using the exact sequence (C.3) $d_1^+$ times for $l = 1$, we obtain
\[
\chi_{T_1}(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = \chi_{T_1}(\mathcal{R}^{\vec{u} - C\vec{e}_1} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) \quad \vdots \quad = \chi_{T_1}(\mathcal{R}^{\vec{u} - d_1^+ C\vec{e}_1} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) - \sum_{i=1}^{d_1^+} \chi_{T_l}(\mathcal{R}^{\vec{u} - i C\vec{e}_1} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) \tag{C.6}_{\mathcal{D}_l}.
\]
If $m = 1$, we conclude the inductive procedure. If not, set $d_2^+ := \min(S_2^+)$ if $S_2^+ \neq \emptyset$, otherwise $d_2^+ := s_2$. Now we do $d_2^+$ steps more with the sequence (C.3) for $l = 2$ and obtain
\[
\chi_{T_1}(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) = \chi_{T_1}(\mathcal{R}^{\vec{u} - d_2^+ C\vec{e}_1 - C\vec{e}_2} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) - \chi_{T_1}(\mathcal{R}^{\vec{u} - d_2^+ C\vec{e}_1 - C\vec{e}_2} \otimes \mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)) \tag{C.7}_{\mathcal{D}_l}.
\]
\[ - \sum_{i=1}^{d_1^+} \chi_{T_i}(R^{d_1^+} \otimes O_{\mathcal{X}_k}(-D_{\infty})|_{\mathcal{G}_l}) \]

\[ \vdots \]

\[ = \chi_{T_1}(R^{d_1^+} \otimes O_{\mathcal{X}_k}(-D_{\infty})) - \sum_{i=1}^{d_1^+} \chi_{T_i}(R^{d_1^+} \otimes O_{\mathcal{X}_k}(-D_{\infty})|_{\mathcal{G}_l}) \]

Iterating this procedure \( d_1^+ \) times using the sequence (C.3) for \( l = 3, \ldots, m \), where \( d_1^+ := \min(S_1^+) \) if \( S_1^+ \neq \emptyset \) and \( d_1^+ := s_l \) otherwise, we get

\[ \chi_{T_1}(R^{d_1^+} \otimes O_{\mathcal{X}_k}(-D_{\infty})) \]

Note that if \( d_{m+}^+ \in S_m^+ \), then \( \chi_{T_1}(R^{d_{m+}^+} \otimes O_{\mathcal{X}_k}(-D_{\infty})) = 0 \) by Equation (C.5) (as then the left-hand side of the equation is exactly the dimension of \( H^1(\mathcal{X}_k, R^{d_{m+}^+} \otimes O_{\mathcal{X}_k}(-D_{\infty})) \)). Otherwise

\[ \chi_{T_1}(R^{d_{m+}^+} \otimes O_{\mathcal{X}_k}(-D_{\infty})) \]

and the last characteristic is equal to zero thanks to the computations in Section A.3.

When \( s_l < 0 \) for \( l = 1, \ldots, k-1 \) consider the equation

\[ \frac{C_l}{2} i^2 + i \left( \bar{v} - \sum_{p=1}^{l-1} s_p \bar{e}_p \right) \cdot C \bar{e}_1 + \frac{1}{2} \left( \left( \bar{v} - \sum_{p=1}^{l-1} s_p \bar{e}_p \right) \cdot C \left( \bar{v} - \sum_{p=1}^{l-1} s_p \bar{e}_p \right) - (C^{-1})^{cc} \right) = 0 \]  

and define the set

\[ S_l^{-} := \{ i \in \mathbb{N} \mid i \leq -s_l \ is \ a \ solution \ of \ Equation \ (C.6) \} \]

One can follow the procedure just described using the short exact sequence (C.3) \( d_l^- \) times, where \( d_l^- := \min(S_l^-) \) if \( S_l^- \neq \emptyset \) and \( d_l^- := -s_l \) otherwise. In this case, one exchanges the roles played by the left and middle terms of the sequence.

In general, let \( m \) be the smallest integer \( l \in \{1, \ldots, k-1\} \) such that \( S_l^+ \) or \( S_l^- \) is nonempty; if all these sets are empty, let \( m := k-1 \). Define for \( l = 1, \ldots, m \) the \( L \)-factors as

\[ L^{(l)}(\chi_1^{(l)}, \chi_2^{(l)}) := \begin{cases} 
- \sum_{i=1}^{d_1^+} \chi_{T_i}(R^{d_1^+} \otimes O_{\mathcal{X}_k}(-D_{\infty})|_{\mathcal{G}_l}) & \text{for } s_l \geq 0 , \\
\sum_{i=0}^{d_1^-} \chi_{T_i}(R^{d_1^-} \otimes O_{\mathcal{X}_k}(-D_{\infty})|_{\mathcal{G}_l}) & \text{for } s_l < 0 .
\end{cases} \]

Then we obtain

\[ \chi_{T_1}(R^{d_{m+}^+} \otimes O_{\mathcal{X}_k}(-D_{\infty})) = \sum_{l=1}^{m} L^{(l)}(\chi_1^{(l)}, \chi_2^{(l)}) . \]
It remains just to compute the $L$-factors. By Lemma C.4, we have
\[ \mathcal{R} \qquad \mathcal{R}(\frac{z}{e^2} \frac{1}{e^2}) \qquad \mathcal{R}(\mathcal{O}_X(-\mathcal{E}_\infty)_{|\mathcal{Y}_l}) \simeq \mathcal{O}_{\mathcal{Y}_l}(\delta_{l,c} + z_l + s_{l-1} \mp 2i) . \]

Recalling that $z_l = (C\mathcal{S})_l = -s_{l-1} + 2s_l - s_{l+1}$ for $l = 1, \ldots, k-1$, we can rewrite the $L$-factors as
\[ L^{(l)}(\chi_1^{(l)}, \chi_2^{(l)}) = \begin{cases} -\sum_{i=s_l-d_l^+}^{s_l-1} \chi_T(\mathcal{O}_{\mathcal{Y}_l}(\delta_{l,c} - s_{l+1} + 2i)) & \text{for } s_l > 0 , \\ 0 & \text{for } s_l = 0 , \\ -\sum_{i=1-s_l-d_l^-}^{-s_l} \chi_T(\mathcal{O}_{\mathcal{Y}_l}(\delta_{l,c} - s_{l+1} - 2i)) & \text{for } s_l < 0 . \end{cases} \quad (C.7) \]

C.3. Characters of restrictions and $L$-factors. Here we choose the $T_l$-equivariant structure on the line bundles $\mathcal{O}_{\mathcal{Y}_l}(a)$ given by the isomorphism
\[ \mathcal{O}_{\mathcal{Y}_l}(a) \simeq \mathcal{O}_X(-\lfloor \frac{a}{2} \rfloor D_l + 2\lfloor \frac{a}{2} \rfloor D_{l+1}) . \quad (C.8) \]

**Theorem C.9.** Fix $l \in \{1, \ldots, k-1\}$. For $a \geq 0$ we have
\[ \chi_T(\mathcal{O}_{\mathcal{Y}_l}(a)) = (\chi_1^{(l)})^\frac{a}{2} \sum_{j=0}^a (\chi_2^{(l)})^j \quad \text{and} \quad \chi_T(\mathcal{O}_{\mathcal{Y}_l}(-a)) = -(\chi_1^{(l)})^{-\frac{a}{2}} \sum_{j=1}^{a-1} (\chi_2^{(l)})^j . \]

**Proof.** Let $a \geq 0$ and consider the short exact sequence
\[ 0 \rightarrow \mathcal{O}_X(( -\lfloor \frac{a}{2} \rfloor - 1) D_l + 2\lfloor \frac{a}{2} \rfloor D_{l+1}) \rightarrow \mathcal{O}_X(( -\lfloor \frac{a}{2} \rfloor D_l + 2\lfloor \frac{a}{2} \rfloor D_{l+1}) \rightarrow \mathcal{O}_X(( -\lfloor \frac{a}{2} \rfloor D_l + 2\lfloor \frac{a}{2} \rfloor D_{l+1})_{|\mathcal{Y}_l} \rightarrow 0 . \]

Then for the Euler characteristic we have
\[ \chi_T(\mathcal{O}_X(( -\lfloor \frac{a}{2} \rfloor D_l + 2\lfloor \frac{a}{2} \rfloor D_{l+1})_{|\mathcal{Y}_l}) = \chi_T(\mathcal{O}_X(( -\lfloor \frac{a}{2} \rfloor - 1) D_l + 2\lfloor \frac{a}{2} \rfloor D_{l+1}) - \chi_T(\mathcal{O}_X(( -\lfloor \frac{a}{2} \rfloor - 1) D_l + 2\lfloor \frac{a}{2} \rfloor D_{l+1})) \]
\[ = \chi_T(\mathcal{O}_X(( -\lfloor \frac{a}{2} \rfloor D_l + 2\lfloor \frac{a}{2} \rfloor D_{l+1})) - \chi_T(\mathcal{O}_X(( -\lfloor \frac{a}{2} \rfloor - 1) D_l + 2\lfloor \frac{a}{2} \rfloor D_{l+1})) , \]

where the last equality follows from the fact that the pushforward $\pi_{k \ast}$ preserves the equivariant decomposition of the cohomology groups. To complete the proof it is sufficient to compute for $m \geq 0$
\[ \chi_T(\mathcal{O}_X(( -m D_l)) - \chi_T(\mathcal{O}_X(( -(m + 1) D_l))) \quad (C.10) \]
which corresponds to the case $\{ \frac{a}{2} \} = 0$, and
\[ \chi_T(\mathcal{O}_X(D_{l+1} - m D_l)) - \chi_T(\mathcal{O}_X((D_{l+1} + (m + 1) D_l))) \quad (C.11) \]
which corresponds to the case $\{ \frac{a}{2} \} = \frac{1}{2}$.

By [29, Proposition 9.1.6], it is easy to verify that the zeroth and second cohomology groups that appear in Equation (C.10) vanish. To compute the first cohomology groups in (C.10) it is enough to count the integer points on the line of direction $(l-1, l)$ between the points $(-(l-2), -(l-1))$, $((l-1), l)$ and $(l, (l+1))$. We easily get
\[ \chi_T(\mathcal{O}_X(-(m D_l))) - \chi_T(\mathcal{O}_X((m + 1) D_l)) = \sum_{j=0}^{2m} T_1^{-(l-2) m+j} T_2^{-(l-1) m+j} l \qquad (\chi_1^{m})^l \sum_{j=0}^{2m} (\chi_2^{j})^l , \]
where the last equality follows from the expression (3.2) for the variables $T_1$ and $T_2$ introduced in Section 3.1.

In the same way, Equation (C.11) becomes

$$\chi T_i(\mathcal{O}_X(D_{t+1} - m D_t)) - \chi T_i(\mathcal{O}_X(D_{t+1} - (m + 1) D_t)) = \sum_{j=0}^{2m+1} T_i^{(l-2)m+j} T_2^{-(l-1)m+j} \sum_{j=0}^{2m+1} (\chi_1^l)^m (\chi_2^j)^j.$$

For $a < 0$ one argues in a similar way. \hfill $\square$

Now we use Theorem C.9 to compute the explicit expressions for the $L$-factors $L^{(l)}(\chi_1^{(l)}, \chi_2^{(l)})$ in (C.7). Fix $l \in \{1, \ldots, k - 1\}$ and set $(C^{-1})^{l,0} = 0$. We also set $(C^{-1})^{k,c} = 0$. Then we get:

- For $v_l - (C^{-1})^{l,c} > 0$:
  - For $\delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} + 2(v_l - (C^{-1})^{l,c} - d_l^+) \geq 0$:
    $$L^{(l)}(\chi_1^{(l)}, \chi_2^{(l)}) = - \sum_{i=v_l - (C^{-1})^{l,c} - d_l^+}^{v_l - (C^{-1})^{l,c} - 1} 2i + \delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} - 1 \sum_{j=0}^{\delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c}} (\chi_1^i)^i \left[ \delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} \right] (\chi_2^j)^j.$$
  - For $2 \leq \delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} + 2(v_l - (C^{-1})^{l,c}) < 2d_l^+$:
    $$L^{(l)}(\chi_1^{(l)}, \chi_2^{(l)}) = - \sum_{i=v_l - (C^{-1})^{l,c} - d_l^+}^{(C^{-1})^{l,c} + \delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c}} 2i + \delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} - 1 \sum_{j=0}^{\delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c}} (\chi_1^i)^i \left[ \delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} \right] (\chi_2^j)^j.$$
  - For $\delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} < 2 - 2(v_l - (C^{-1})^{l,c})$:
    $$L^{(l)}(\chi_1^{(l)}, \chi_2^{(l)}) = - \sum_{i=v_l - (C^{-1})^{l,c} - d_l^+}^{v_l - (C^{-1})^{l,c} - 1} -2i + \delta_{l,c} - v_{l+1} - (C^{-1})^{l+1,c} - 1 \sum_{j=0}^{\delta_{l,c} - v_{l+1} - (C^{-1})^{l+1,c}} (\chi_1^i)^i \left[ \delta_{l,c} - v_{l+1} - (C^{-1})^{l+1,c} \right] (\chi_2^j)^j.$$
- For $v_l - (C^{-1})^{l,c} = 0$: $\quad L^{(l)}(\chi_1^{(l)}, \chi_2^{(l)}) = 0$.
- For $v_l - (C^{-1})^{l,c} < 0$:
  - For $\delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} + 2v_l - 2(C^{-1})^{l,c} < 2 - 2d_l^-$:
    $$L^{(l)}(\chi_1^{(l)}, \chi_2^{(l)}) = - \sum_{i=v_l + (C^{-1})^{l,c}}^{v_l + (C^{-1})^{l,c} - d_l^-} 2i + \delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} - 1 \sum_{j=0}^{\delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c}} (\chi_1^i)^i \left[ \delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} \right] (\chi_2^j)^j.$$

• For $2 - 2d_i^t \leq \delta_{i,c} - v_{l+1} + (C^{-1})^{t+1,c} + 2v_l - 2(C^{-1})^{l,c} < 0$:

$$L^j(\chi_1^{(l)}, \chi_2^{(l)}) = \left[ \frac{\delta_{i,c} - v_{l+1} + (C^{-1})^{t+1,c}}{2} \right] - 2i + \delta_{i,c} - v_{l+1} + (C^{-1})^{t+1,c} \sum_{j=0}^{i} \left[ \frac{\delta_{i,c} - v_{l+1} + (C^{-1})^{t+1,c}}{2} \right] (\chi_1^j - i - \left[ \frac{\delta_{i,c} - v_{l+1} + (C^{-1})^{t+1,c}}{2} \right]) (\chi_2^j)^j.$$

- For $\delta_{i,c} - v_{l+1} + (C^{-1})^{t+1,c} \geq -2v_l + 2(C^{-1})^{l,c}$:

$$L^j(\chi_1^{(l)}, \chi_2^{(l)}) = -v_l + (C^{-1})^{l,c} - 2i + \delta_{i,c} - v_{l+1} + (C^{-1})^{t+1,c} \sum_{j=0}^{i} \left[ \frac{\delta_{i,c} - v_{l+1} + (C^{-1})^{t+1,c}}{2} \right] (\chi_1^j - i - \left[ \frac{\delta_{i,c} - v_{l+1} + (C^{-1})^{t+1,c}}{2} \right]) (\chi_2^j)^j.$$

**Remark C.12.** The procedure we described in this appendix involves some choices. In particular, one can consider other realizations of $O_{\beta}(a)$ as a restriction of a line bundle on $\mathcal{D}_k$ for $l = 1, \ldots, k - 1$. We chose the realization in Equation (C.8) because it resembles similar choices made in [20, Section 4.3] and in the proof of [70, Theorem 3.4]. Different realizations of the line bundles $O_{\beta}(a)$ yield equivalent results in equivariant K-theory.

**APPENDIX D. GAUGE THEORY ON $X_3$**

In this appendix we present some explicit calculations for $k = 3$.

D.1. **Euler classes.** For $k = 3$, the $\tilde{T}$-equivariant Euler class of the Carlsson-Okounkov bundle from Section 4.7 becomes

$$\text{Euler}_T \left( E([\mathcal{E}, \phi \mathcal{E}], ([\mathcal{E}', \phi \mathcal{E}'])] \right) = \prod_{\alpha=1}^{r} \prod_{\beta=1}^{r'} \prod_{i=1}^{3} m_{\mathcal{Y}_\alpha, \mathcal{Y}_\beta} (\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, a_{\beta\alpha}) \prod_{n=1}^{2} \rho_n (\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_{\beta\alpha}).$$

In this case $c_{\beta\alpha} \in \{0, 1, 2\}$ for any $\alpha = 1, \ldots, r$, $\beta = 1, \ldots, r'$. By Equation (C.1) we have $(C^{-1})^1 c_{\beta\alpha} = \{(\overline{v}_{\beta\alpha})_1\}$ and $(C^{-1})^2 c_{\beta\alpha} = \{(\overline{v}_{\beta\alpha})_2\}$. Hence the edge factor $E_1^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, a_{\beta\alpha})$ assumes the following form:

- For $[\overline{v}_{\beta\alpha}]_1 > 0$:
  - For $\delta_{1,c_{\beta\alpha}} - [\overline{v}_{\beta\alpha}]_2 + 2([\overline{v}_{\beta\alpha}]_1 - d(\beta\alpha)^+_1) > 0$:
    $$E_1^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, a_{\beta\alpha}) = \prod_{i=1}^{[\overline{v}_{\beta\alpha}]_1} \prod_{j=0}^{2i + \delta_{1,c_{\beta\alpha}} - [\overline{v}_{\beta\alpha}]_2} \left( a_{\beta\alpha} + (i + \left[ \frac{\delta_{1,c_{\beta\alpha}} - [\overline{v}_{\beta\alpha}]_2}{2} \right]) \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} \right).$$

- For $2 \leq \delta_{1,c_{\beta\alpha}} - [\overline{v}_{\beta\alpha}]_2 + 2([\overline{v}_{\beta\alpha}]_1) < 2d(\beta\alpha)^+_1$:
  $$E_1^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, a_{\beta\alpha}) = \prod_{i=1}^{[\overline{v}_{\beta\alpha}]_1} \prod_{j=1}^{2i - [\delta_{1,c_{\beta\alpha}} - [\overline{v}_{\beta\alpha}]_2]} \left( a_{\beta\alpha} + (i - \left[ \frac{\delta_{1,c_{\beta\alpha}} - [\overline{v}_{\beta\alpha}]_2}{2} \right]) \varepsilon_1^{(1)} + j \varepsilon_2^{(1)} \right)^{-1}.$$
\[
\prod_{i=-[\delta_1,\epsilon_{\beta\alpha}]-|[\bar{v}_{\beta\alpha}]_2|}^{2|\bar{v}_{\beta\alpha}|+\delta_1,\epsilon_{\beta\alpha}} \prod_{j=0}^{2i+\delta_1,\epsilon_{\beta\alpha}-|[\bar{v}_{\beta\alpha}]_2|} (a_{\beta\alpha} + \left( i + \left( -\frac{\delta_1,\epsilon_{\beta\alpha} - |\bar{v}_{\beta\alpha}|_2}{2} \right) \right) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)}) .
\]

- For \(\delta_1,\epsilon_{\beta\alpha} - |(\bar{v}_{\beta\alpha})_2| < 2 - 2 |(\bar{v}_{\beta\alpha})_1|:\n
\[
\varepsilon_2^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, a_{\beta\alpha}) = \prod_{i=[\bar{v}_{\beta\alpha}]_1}^{-[\bar{v}_{\beta\alpha}]_1-1} \prod_{j=1}^{2i-\delta_1,\epsilon_{\beta\alpha} + |(\bar{v}_{\beta\alpha})_2| - 1} (a_{\beta\alpha} + \left( i + \left( -\frac{\delta_1,\epsilon_{\beta\alpha} - |\bar{v}_{\beta\alpha}|_2}{2} \right) \right) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)}) .
\]

- For \([\bar{v}_{\beta\alpha}]_1 = 0:\n
\[
\varepsilon_2^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, a_{\beta\alpha}) = 1.
\]

- For \([\bar{v}_{\beta\alpha}]_1 < 0:\n
\[
\varepsilon_2^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, a_{\beta\alpha}) = \prod_{i=[\bar{v}_{\beta\alpha}]_1}^{-[\bar{v}_{\beta\alpha}]_1-1} \prod_{j=1}^{2i-\delta_1,\epsilon_{\beta\alpha} - |(\bar{v}_{\beta\alpha})_2| - 1} (a_{\beta\alpha} + \left( i + \left( -\frac{\delta_1,\epsilon_{\beta\alpha} - |\bar{v}_{\beta\alpha}|_2}{2} \right) \right) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)}) .
\]

- For \(2 - 2d(\beta\alpha)_1 \leq \delta_1,\epsilon_{\beta\alpha} - |(\bar{v}_{\beta\alpha})_2| + 2 |(\bar{v}_{\beta\alpha})_1| < 0:\n
\[
\varepsilon_2^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, a_{\beta\alpha}) = \prod_{i=[\bar{v}_{\beta\alpha}]_1}^{-[\bar{v}_{\beta\alpha}]_1-1} \prod_{j=1}^{2i-\delta_1,\epsilon_{\beta\alpha} - |(\bar{v}_{\beta\alpha})_2| - 1} (a_{\beta\alpha} + \left( i + \left( -\frac{\delta_1,\epsilon_{\beta\alpha} - |\bar{v}_{\beta\alpha}|_2}{2} \right) \right) \varepsilon_1^{(1)} - j \varepsilon_2^{(1)}) .
\]

- For \(\delta_1,\epsilon_{\beta\alpha} - |(\bar{v}_{\beta\alpha})_2| \geq -2 |(\bar{v}_{\beta\alpha})_1|:\n
\[
\varepsilon_2^{(1)}(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}, a_{\beta\alpha}) = \prod_{i=[\bar{v}_{\beta\alpha}]_1}^{-[\bar{v}_{\beta\alpha}]_1-1} \prod_{j=1}^{2i+\delta_2,\epsilon_{\beta\alpha} - |(\bar{v}_{\beta\alpha})_2|} (a_{\beta\alpha} + \left( i + \left( -\frac{\delta_1,\epsilon_{\beta\alpha} - |\bar{v}_{\beta\alpha}|_2}{2} \right) \right) \varepsilon_1^{(1)} + j \varepsilon_2^{(1)}) .
\]

If \(m_{\beta\alpha} = 1\), then

\[
\varepsilon_2^{(2)}(\varepsilon_1^{(2)}, \varepsilon_2^{(2)}, a_{\beta\alpha}) = 1 ,
\]

otherwise

\[
\varepsilon_2^{(2)}(\varepsilon_1^{(2)}, \varepsilon_2^{(2)}, a_{\beta\alpha}) = \left\{ \begin{array}{ll}
\prod_{i=-[\delta_1,\epsilon_{\beta\alpha}]-|\bar{v}_{\beta\alpha}|_2}^{2|\bar{v}_{\beta\alpha}|+\delta_1,\epsilon_{\beta\alpha}} \prod_{j=0}^{2i+\delta_2,\epsilon_{\beta\alpha}} (a_{\beta\alpha} + i \varepsilon_1^{(2)} + j \varepsilon_2^{(2)}) & \text{for } |(\bar{v}_{\beta\alpha})_2| > 0 , \\
1 & \text{for } |(\bar{v}_{\beta\alpha})_2| = 0 , \\
\prod_{i=1-[\bar{v}_{\beta\alpha}]_2-d(\beta\alpha)_1}^{-|\bar{v}_{\beta\alpha}|_2} \prod_{j=1}^{2i-\delta_2,\epsilon_{\beta\alpha}-1} (a_{\beta\alpha} - \left( -\frac{\delta_2,\epsilon_{\beta\alpha}}{2} \right) \varepsilon_1^{(2)} - (i \varepsilon_1^{(2)} + j \varepsilon_2^{(2)}) ) & \text{for } |(\bar{v}_{\beta\alpha})_2| < 0 .
\end{array} \right.
\]
D.2. \( U(2) \) gauge theory. In this subsection we provide some explicit computations for the instanton partition function (5.13) of pure \( N = 2 \) gauge theory on \( X_3 \) for rank \( r = 2 \). For \( k = 3 \) the \( U(2) \) partition function (5.13) becomes

\[
Z_{X_3}^{\text{inst}}(\varepsilon_1, \varepsilon_2, a_1, a_2; q, \xi_1, \xi_2) = \sum_{\vec{v}=(\xi_1, \xi_2) \in \frac{1}{3} \mathbb{Z}^2 \atop 3v_2 = v_1 + 2v_2 \mod 3} \xi_1^{v_1} \xi_2^{v_2} \sum_{\vec{v}_1, \vec{v}_2} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a_1, a_2; q),
\]

where

\[
Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a_1, a_2; q) := \frac{1}{2} \sum_{\alpha=1}^2 \sum_{\beta=1}^2 C_{\alpha \beta}^{\varepsilon_1 \varepsilon_2} \prod_{i=1}^3 \frac{1}{\prod_{\alpha, \beta=1}^{2} \prod_{i=1}^{2} \xi_1^{(i)}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, a^{(i)} \alpha \beta) 3 Z_{C_{\alpha \beta}^{\varepsilon_1 \varepsilon_2}}(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, a^{(i)} \alpha \beta; q).}
\]

We compute the lowest order of the expansion of this partition function in \( q \) for \(-1 \leq u_1, u_2 \leq 1\), where \((u_1, u_2) := C \vec{v} \). Let \( I_{\vec{v}}^{(w_0, w_1, w_2)} \) be the set consisting of pairs \((\vec{v}_1, \vec{v}_2)\) such that \( \vec{v}_1 + \vec{v}_2 = \vec{v} \)

if \( \sum_{j=0}^{l-1} w_j < 1 \leq \sum_{j=0}^{l} w_j \) for \( l = 0, 1, 2 \). We denote by \( \min (I_{\vec{v}}^{(w_0, w_1, w_2)}) \) the set consisting of the vectors which provide the smallest power of \( q \). Set \( a = \frac{a_1 - a_2}{2} \).

\( (w_0, w_1, w_2) = (2, 0, 0) \). We need to consider the cases \( \vec{v} = (0, 0) \), \((1, 1)\) and \((-1, -1)\).

- For \( \vec{v} = (0, 0) \) the set \( \min (I_{(0,0)}^{(2,0,0)}) \) is \( \{(0,0),(0,0)\} \) and

\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_{(0,0)}^{(2,0,0)}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = 1 + \cdots .
\]

- For \( \vec{v} = (1, 1) \) the set \( \min (I_{(1,1)}^{(2,0,0)}) \) is \( \{(0,0),(1,1),(1,1),(0,0)\} \) and

\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_{(1,1)}^{(2,0,0)}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q \left( - \frac{1}{2a(2a-\varepsilon_1-\varepsilon_2)} - \frac{1}{2a(2a+\varepsilon_1+\varepsilon_2)} \right) + \cdots .
\]

- For \( \vec{v} = (-1, -1) \) the set \( \min (I_{(-1,-1)}^{(2,0,0)}) \) is \( \{(0,0),(-1,-1),(-1,-1),(0,0)\} \) and

\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_{(-1,-1)}^{(2,0,0)}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q \left( - \frac{1}{2a(2a-\varepsilon_1-\varepsilon_2)} - \frac{1}{2a(2a+\varepsilon_1+\varepsilon_2)} \right) + \cdots .
\]

\( (w_0, w_1, w_2) = (0, 2, 0) \). We need to consider the cases \( \vec{v} = (\frac{1}{2}, \frac{2}{3}), (-\frac{2}{3}, -\frac{1}{3}) \) and \( (\frac{1}{3}, -\frac{1}{3}) \).

- For \( \vec{v} = (\frac{1}{2}, \frac{2}{3}) \) the set \( \min (I_{(\frac{1}{2},\frac{2}{3})}^{(0,2,0)}) \) is \( \{(\frac{2}{3}, \frac{1}{3}), (-\frac{1}{3}, \frac{1}{3}), (-\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3})\} \) and

\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_{(\frac{1}{2},\frac{2}{3})}^{(0,2,0)}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^2 \left( - \frac{1}{2a(2a-\varepsilon_1-\varepsilon_2)} - \frac{1}{2a(2a+\varepsilon_1+\varepsilon_2)} \right) + \cdots .
\]

- For \( \vec{v} = (-\frac{2}{3}, -\frac{1}{3}) \) the set \( \min (I_{(-\frac{2}{3},-\frac{1}{3})}^{(0,2,0)}) \) is \( \{(-\frac{1}{3}, \frac{1}{3}), (-\frac{1}{3}, -\frac{1}{3}), (-\frac{1}{3}, -\frac{1}{3}), (-\frac{1}{3}, \frac{1}{3})\} \) and

\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_{(-\frac{2}{3},-\frac{1}{3})}^{(0,2,0)}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^2 \left( - \frac{1}{2a(2a-\varepsilon_1-\varepsilon_2)} - \frac{1}{2a(2a+\varepsilon_1+\varepsilon_2)} \right) + \cdots .
\]
• For \( \vec{v} = (\frac{1}{3}, -\frac{1}{3}) \) the set \( I_0^{(0,2,0)}(\frac{4}{3}, \frac{1}{3}) \) is \( \{((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, -\frac{2}{3})), ((-\frac{1}{3}, -\frac{2}{3}), (\frac{2}{3}, \frac{1}{3})) \} \) and
\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_0^{(0,2,0)}(\frac{4}{3}, \frac{1}{3})} \mathcal{Z}_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^\frac{1}{2} \left(-\frac{1}{2a(2a-\varepsilon_1-\varepsilon_2)} - \frac{1}{2a(2a+\varepsilon_1+\varepsilon_2)} \right) + \cdots .
\]

\((w_0, w_1, w_2) = (0, 0, 2)\). We need to consider separately the cases \( \vec{v} = (\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, -\frac{2}{3}), (\frac{1}{3}, \frac{2}{3}) \) and \( (\frac{1}{3}, \frac{1}{3}) \).

• For \( \vec{v} = (\frac{2}{3}, \frac{1}{3}) \) the set \( I_0^{(0,2,0)}(\frac{4}{3}, \frac{1}{3}) \) is \( \{((\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, -\frac{1}{3})), ((\frac{1}{3}, -\frac{1}{3}), (\frac{1}{3}, \frac{2}{3})) \} \) and
\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_0^{(0,2,0)}(\frac{4}{3}, \frac{1}{3})} \mathcal{Z}_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^\frac{1}{2} \left(-\frac{1}{2a(2a-\varepsilon_1-\varepsilon_2)} - \frac{1}{2a(2a+\varepsilon_1+\varepsilon_2)} \right) + \cdots .
\]

• For \( \vec{v} = (-\frac{1}{3}, -\frac{2}{3}) \) the set \( I_0^{(0,2,0)}(\frac{4}{3}, \frac{1}{3}) \) is \( \{((-\frac{2}{3}, -\frac{1}{3}), (\frac{1}{3}, -\frac{1}{3})), ((\frac{1}{3}, -\frac{1}{3}), (-\frac{2}{3}, -\frac{1}{3})) \} \) and
\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_0^{(0,2,0)}(\frac{4}{3}, \frac{1}{3})} \mathcal{Z}_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^\frac{1}{2} \left(-\frac{1}{2a(2a-\varepsilon_1-\varepsilon_2)} - \frac{1}{2a(2a+\varepsilon_1+\varepsilon_2)} \right) + \cdots .
\]

• For \( \vec{v} = (-\frac{1}{3}, \frac{1}{3}) \) the set \( I_0^{(0,2,0)}(\frac{4}{3}, \frac{1}{3}) \) is \( \{((\frac{1}{3}, \frac{2}{3}), (-\frac{2}{3}, -\frac{1}{3})), ((-\frac{2}{3}, -\frac{1}{3}), (\frac{1}{3}, \frac{2}{3})) \} \) and
\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_0^{(0,2,0)}(\frac{4}{3}, \frac{1}{3})} \mathcal{Z}_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^\frac{1}{2} \left(-\frac{1}{2a(2a-\varepsilon_1-\varepsilon_2)} - \frac{1}{2a(2a+\varepsilon_1+\varepsilon_2)} \right) + \cdots .
\]

\((w_0, w_1, w_2) = (1, 1, 0)\). We need to consider separately the cases \( \vec{v} = (\frac{2}{3}, \frac{1}{3}), (-\frac{1}{3}, \frac{1}{3}), (-\frac{1}{3}, -\frac{2}{3}) \).

• For \( \vec{v} = (\frac{2}{3}, \frac{1}{3}) \) the set \( I_0^{(1,1,0)}(\frac{2}{3}, \frac{1}{3}) \) is \( \{(0, 0), (\frac{2}{3}, \frac{1}{3})\} \) and
\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_0^{(1,1,0)}(\frac{2}{3}, \frac{1}{3})} \mathcal{Z}_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^\frac{1}{2} + \cdots .
\]

• For \( \vec{v} = (-\frac{1}{3}, \frac{1}{3}) \) the set \( I_0^{(1,1,0)}(\frac{2}{3}, \frac{1}{3}) \) is \( \{(0, 0), (-\frac{1}{3}, \frac{1}{3})\} \) and
\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_0^{(1,1,0)}(\frac{2}{3}, \frac{1}{3})} \mathcal{Z}_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^\frac{1}{2} + \cdots .
\]

• For \( \vec{v} = (-\frac{1}{3}, -\frac{2}{3}) \) the set \( I_0^{(1,1,0)}(\frac{2}{3}, \frac{1}{3}) \) is \( \{(0, 0), (-\frac{1}{3}, -\frac{2}{3})\} \) and
\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_0^{(1,1,0)}(\frac{2}{3}, \frac{1}{3})} \mathcal{Z}_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^\frac{1}{2} + \cdots .
\]

\((w_0, w_1, w_2) = (1, 0, 1)\). We need to consider separately the cases \( \vec{v} = (\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, -\frac{1}{3}), (-\frac{2}{3}, -\frac{1}{3}) \).

• For \( \vec{v} = (\frac{1}{3}, \frac{2}{3}) \) the set \( I_0^{(1,0,1)}(\frac{1}{3}, \frac{2}{3}) \) is \( \{(0, 0), (\frac{1}{3}, \frac{2}{3})\} \) and
\[
\sum_{(\vec{v}_1, \vec{v}_2) \in I_0^{(1,0,1)}(\frac{1}{3}, \frac{2}{3})} \mathcal{Z}_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^\frac{1}{2} + \cdots .
\]
• For \( \vec{v} = (\frac{1}{3}, -\frac{1}{3}) \) the set \( \min(I_{(\frac{1}{3}, -\frac{1}{3})}^{1,0,1}) \) is \( \{((0, 0), (\frac{1}{3}, -\frac{1}{3}))\} \) and
  \[
  \sum_{(\vec{v}_1, \vec{v}_2) \in I_{(\frac{1}{3}, -\frac{1}{3})}^{1,0,1}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^{\frac{1}{3}} + \cdots.
  \]

• For \( \vec{v} = (-\frac{2}{3}, -\frac{1}{3}) \) the set \( \min(I_{(-\frac{2}{3}, -\frac{1}{3})}^{1,0,1}) \) is \( \{((0, 0), (-\frac{2}{3}, -\frac{1}{3}))\} \) and
  \[
  \sum_{(\vec{v}_1, \vec{v}_2) \in I_{(-\frac{2}{3}, -\frac{1}{3})}^{1,0,1}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^{\frac{1}{4}} + \cdots.
  \]

\( (w_0, w_1, w_2) = (0, 1, 1) \). We need to consider the cases \( \vec{v} = (0, 0), (1, 1) \) and \( (-1, -1) \).

• For \( \vec{v} = (0, 0) \) the set \( \min(I_{(0,0)}^{0,1,1}) \) is \( \{((\frac{2}{3}, \frac{1}{3}), (-\frac{2}{3}, -\frac{1}{3})), ((-\frac{2}{3}, -\frac{2}{3}), (\frac{1}{3}, \frac{2}{3})), ((-\frac{2}{3}, -\frac{2}{3}), (\frac{1}{3}, -\frac{1}{3}))\} \) and
  \[
  \sum_{(\vec{v}_1, \vec{v}_2) \in I_{(0,0)}^{0,1,1}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^{\frac{2}{3}} \left( -\frac{1}{2a(2a + \varepsilon_1 + \varepsilon_2)} - \frac{1}{(2a - 3\varepsilon_2)(2a - \varepsilon_1 - \varepsilon_2)} \right) + \cdots.
  \]

• For \( \vec{v} = (1, 1) \) the set \( \min(I_{(1,1)}^{0,1,1}) \) is \( \{((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))\} \) and
  \[
  \sum_{(\vec{v}_1, \vec{v}_2) \in I_{(1,1)}^{0,1,1}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^2 + \cdots.
  \]

• For \( \vec{v} = (-1, -1) \) the set \( \min(I_{(-1,-1)}^{0,1,1}) \) is \( \{((-\frac{1}{3}, -\frac{2}{3}), (-\frac{2}{3}, -\frac{1}{3}))\} \) and
  \[
  \sum_{(\vec{v}_1, \vec{v}_2) \in I_{(-1,-1)}^{0,1,1}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) = q^2 + \cdots.
  \]

These results coincide in all but one case with the computations in [15, Appendix C], which are claimed to coincide with those of [37]. In particular, the coefficient of \( q^\frac{2}{3} \) in the expansion of the function \( \sum_{(\vec{v}_1, \vec{v}_2) \in I_{(0,0)}^{0,1,1}} Z_{\vec{v}_1, \vec{v}_2}(\varepsilon_1, \varepsilon_2, a; q) \) does not coincide with the corresponding coefficient of \( Z_{k,0}^{(0,0)}(1, 2) \) (to compare these two sets of results, one needs to set \( C\vec{v} = -(c_1^{(1)}, c_1^{(2)}) \)). As pointed out in Remark C.12, our edge contributions depend on some choices; since different choices give equivalent expressions, we believe that one can find the correct choices in the procedure described in Appendix C which yields agreement with the results of [15].

Since our partition functions are defined by means of integrals over moduli spaces of framed sheaves on \( \mathcal{R}_k \), it seems more natural to focus on a comparison between our moduli spaces and the moduli spaces of \( \mathbb{Z}_k \)-equivariant framed sheaves on the projective plane \( \mathbb{P}^2 \); these moduli spaces may be regarded as a geometric foundation for the computations of the partition functions done in [37]. We will address this problem in a future work.

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