The Construction of the mKdV $N$-soliton Solution by the Bäcklund Transformation

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Abstract

We study group theoretical structures of the mKdV equation. The Schwarzian type mKdV equation has the global Möbius group symmetry. The Miura transformation makes a connection between the mKdV equation and the KdV equation. We find the special local Möbius transformation on the mKdV one-soliton solution which can be regarded as the commutative KdV Bäcklund transformation can generate the mKdV $N$-soliton solution. In this algebraic construction to obtain multi-soliton solutions, we could observe the addition formula.

1 Introduction

The discovery of the soliton [1–3] has given the breakthrough to exactly solve non-linear equations. There have been many interesting developments to understand soliton systems such as the AKNS formulation [4,5], the Bäcklund transformation [6–9], the Hirota equation [10–12], the Sato theory [13], the vertex construction of the soliton solution [14,15], and the Schwarzian type mKdV/KdV equation [16].

In this paper, we focus on algebraic soliton systems. The algebraic soliton system which we call here is a subclass of soliton systems in which soliton equations allow to construct $N$-soliton solution by applying algebraic addition formula to one-soliton solution without solving differential equations directly. The algebraic soliton system can be regarded as a non-linear generalization of the linear system. In order to construct solutions of linear differential equations, a linear superposition plays a key role. The algebraic soliton equation which is the special type of non-linear differential equation allows the algebraic non-linear superposition i.e. “Bäcklund transformation”.

In the AKNS formulation, the mKdV equation comes from the integrability of $2 \times 2$ matrix and Möbius group $\text{GL}(2,\mathbb{R})$ naturally appears [4]. However, the group structure behind the mKdV equation and its solutions has not yet been well studied. In this paper, we would like to reveal that the algebraic addition formula in the algebraic soliton system is nothing but the local Abelian sub-“gauge” transformation of the local non-Abelian Möbius ($\text{GL}(2,\mathbb{R})$) “gauge”
transformation. More precisely we would like to answer the following questions; From what kind of special one-soliton solution, can we algebraically construct the $N$-soliton solution? What kind of addition structure of the Möbius group appears in the algebraic $N$-soliton solution?

2 Various types of mKdV equations and algebraic construction of solutions

2.1 mKdV equations and global Möbius group symmetries

The mKdV equation with the variable $v = w_x$ is given by

$$v_t - v_{xxx} + 6v^2v_x = 0, \quad (2.1)$$
$$w_t - w_{xxx} + 2w^3 = 0, \quad (2.2)$$

where we set integration constants to be zero. In order to see symmetries of the soliton equations, we first rewrite them to the Schwarzian type mKdV equation. Introducing new variable $\varphi$ through $\varphi_x = e^{2w}$ [16] and manipulating the following expressions,

$$v_t - v_{xxx} + 6v^2v_x = \frac{1}{2} \partial_x \left( \frac{1}{\varphi_x} (\varphi_t - \varphi_x S(\varphi, x)) \right) = 0,$$
$$v = w_x = \frac{1}{2} \varphi_{xx} = \frac{1}{2} (\log \varphi_x)_x,$$

with the Schwarzian derivative $S(\varphi, x)$ defined as

$$S(\varphi, x) = \frac{\varphi_{xxx}}{\varphi_x} - \frac{3 \varphi_{xx}^2}{2 \varphi_x^3} = -2\sqrt{\varphi_x} \partial_x^2 \left( \frac{1}{\sqrt{\varphi_x}} \right), \quad (2.3)$$

we arrive at the Schwarzian type mKdV equation

$$\frac{\varphi_t}{\varphi_x} = S(\varphi, x), \quad (2.4)$$

where we impose an integration constant to be zero.

We also prepare the Hirota type mKdV equation [11][12]. If we put

$$\tanh \frac{w}{2} = \frac{g}{f} \iff e^w = \frac{f + g}{f - g}, \quad (2.5)$$

the standard mKdV equation (2.2) becomes the following Hirota form

$$\frac{(-D_t + D_x^3)f \cdot g}{D_x f \cdot g} = 3 \cdot \frac{D_x^2(f \cdot f - g \cdot g)}{(f^2 - g^2)}, \quad (2.6)$$

where $D_x$ and $D_t$ are Hirota derivatives. An example of the Hirota derivative is given by

$$D_x^3 f(x) \cdot g(x) = f(x)(\bar{\partial}_x - \partial_x)^3 g(x).$$

The bilinear form with the Hirota derivatives is called as the Hirota form. As the special case of Eq. (2.6), we consider here the following Hirota type mKdV equation

$$(-D_t + D_x^3)f \cdot g = 0, \quad (2.7)$$
$$D_x^2(f \cdot f - g \cdot g) = 0. \quad (2.8)$$
We refer Eqs. \((2.7)\) and \((2.8)\) as the dynamical equation and the structure equation, respectively. It should be noted that in the Hirota type “KdV” equation, it consists of only the time-dependent dynamical equation, and the Möbius group structure is in disguise. Thus we consider here the “mKdV” equation instead of the KdV equation to study the Möbius group structure for the soliton system.

We work with four different types of forms Eq.\((2.1)\), Eq.\((2.2)\), Eq.\((2.4)\), Eq.\((2.7)\) and Eq.\((2.8)\) for the mKdV equation. Eq.\((2.1)\) or Eq.\((2.2)\) is the standard mKdV equation. We use Eq.\((2.4)\), and Eqs.\((2.7)\) and \((2.8)\) to see global and local Möbius group structures of the mKdV solution, respectively.

The Schwarzian type equation has nice global Möbius group \((\text{GL}(2,\mathbb{R}))\) symmetry thanks to the Schwarzian derivative. One can directly show the Schwarzian type equation \((2.4)\) is invariant under the following global Möbius transformation,

\[
\varphi(x,t) \rightarrow \varphi'(x,t) = \frac{\alpha \varphi(x,t) + \beta}{\gamma \varphi(x,t) + \delta}, \quad (\alpha, \beta, \gamma, \delta = \text{const.}, \alpha \delta - \beta \gamma \neq 0),
\]

observing

\[
\left(\varphi'(x,t)\right)_x = \frac{(\alpha \delta - \beta \gamma)}{(\gamma \varphi(x,t) + \delta)^2} \varphi(x,t)_x,
\]

\[
\left(\varphi'(x,t)\right)_t = \frac{(\alpha \delta - \beta \gamma)}{(\gamma \varphi(x,t) + \delta)^2} \varphi(x,t)_t,
\]

\[
S(\varphi'(x,t),x) = S(\varphi(x,t),x).
\]

This global Möbius group \((\text{GL}(2,\mathbb{R}))\) symmetry can be decomposed by three symmetries, i.e. addition formula of \(\tanh\), the scale transformation, and the translation of \(\varphi\):

a) \(S(\tanh(x + \alpha),x) = S\left(\frac{\tanh x + \tanh \alpha}{1 + \tanh x \tanh \alpha},x\right) = -2, \quad (\alpha = \text{const.}),\) \((2.10a)\)

b) \(S(\lambda \varphi(x,t),x) = S(\varphi(x,t),x), \quad (\lambda = \text{const.}),\) \((2.10b)\)

c) \(S(\varphi(x,t) + \beta,x) = S(\varphi(x,t),x), \quad (\beta = \text{const.}).\) \((2.10c)\)

In the global Möbius group \((\text{GL}(2,\mathbb{R}))\) symmetry, the addition formula of the algebraic function “\(\tanh\)” is essential. It has been suggested that the addition formula of the algebraic function is connected with the Lie groups \([17,18]\).

### 2.2 Algebraic construction of \(N\)-soliton solution via local Möbius group structure

Through the Schwarzian type mKdV equation, it is clear that there exists the global Möbius group structure in the mKdV equation. We next try to construct \(N\)-soliton solution through the special local “gauge” transformation of the full Möbius transformation. As we explain in the next section, Bäcklund transformation can be considered as such a special local gauge transformation. In the group theoretical approach, there are two ways to construct the mKdV \(N\)-soliton solution.

a) **Hirota’s direct method:**

Though Hirota’s method uses the structure equation \((2.8)\) of the Möbius group, this method is not algebraic. First we solve the structure equation \((2.8)\) by the perturbation \(f = 1 + O(\epsilon^2) + O(\epsilon^4) + \cdots, \quad g = O(\epsilon) + O(\epsilon^3) + \cdots\), for fixed \(N\). For \(N = 2\), the addition structure is determined in the form

\[
\tanh \frac{w}{2} = \frac{g}{f} = \frac{e^{X_1} + e^{X_2}}{1 + b_{12} e^{X_1} e^{X_2}}, \quad (2.11)
\]
with \( X_i = a_i x + c_i, \) \( b_{12} = (a_1 - a_2)^2/(a_1 + a_2)^2. \) This is the addition structure of the local Möbius group for \( N = 2. \) As this addition structure is the local generalization of the global Möbius group structure, this is quite similar to the addition formula of the algebraic function \( \tanh \), because if we replace \( e^{X_i} \to \tanh(\theta_i) \) and \( b_{12} \to 1, \) we obtain the addition formula of \( \tanh \) itself. Then, by finding the one-soliton solution of Eq. (2.7), we have \( \tanh(w/2) = g/f = e^{a_1 x + a_2^2 t + c_1}. \)

As we explain later, it is not necessary to solve the dynamical equation (2.7) for solution satisfies the Hirota type mKdV equation. From Eq. (2.5), we have \( \tanh(w/2) = g/f = e^{a_1 x + a_2^2 t + c_1}. \) Of course this solution has the global Möbius group symmetry. We next examine whether the group for \( N \geq 2 \) allows \( f \) and \( g \) which reproduces the well-known one-soliton solution with \( N = 2 \) soliton solution. The structure equation (2.8) is nothing but the soliton number preserving Bäcklund transformation itself. In the Hirota type mKdV equation, the information of the Möbius group structure is build in from the beginning in the form of Eq. (2.8). We do not use this method here. See Hirota’s textbook [12].

**b) Bäcklund transformation method:**
This method is the algebraic method. The Bäcklund transformation is the special local gauge transformation of the Möbius group, which gives new soliton solution with one increased soliton number. This method might be a natural one to understand that the mKdV system is the algebraic soliton system. By such a Bäcklund transformation, we first construct the addition structure such as Eq. (2.11). In order to obtain \( N \)-soliton solution from the above addition structure, we simply replace \( X_i = a_i x + c_i \to X_i = a_i x + a_i^2 t + c_i. \) However, as we explain in the next section, the obstacle here is that all Bäcklund transformation is not always commutative. In other words, the sub-“gauge” transformation of the non-Abelian Möbius “gauge” transformation is not always Abelian. If the Bäcklund transformation is not commutative, the algebraic addition formula does not work to obtain the \( N \)-soliton solution. Then such soliton systems might fall into non-algebraic ones.

### 2.3 Global Möbius symmetric mKdV one-soliton solutions

In order to algebraically construct the \( N \)-soliton solution, we first prepare one-soliton solutions. We here give typical global Möbius symmetric one-soliton solutions.

**a) Schwarzian type mKdV solution:**
We first clarify an argument of the solution. Let us assume that \( x \) and \( t \) come in the combination of \( X(x, t) = ax + bt^n + c. \) Rewriting operators as \( \partial_x = a \partial_X \) and \( \partial_t = b t^{n-1} \partial_X, \) and applying these on functions \( v \) and \( w \) in Eqs. (2.11) and (2.12), it turns out that \( n \) should be fixed to be 1, i.e. \( X(x, t) = ax + bt + c. \)

One soliton solution, which satisfies the Schwarzian type equation (2.11), is given by \( \varphi(x, t) = \varphi(ax - 2a^3 t + c) = \tanh(ax - 2a^3 t + c), \) which reproduces the well-known one-soliton solution

\[
\varphi_x = \frac{a}{\cosh^2(ax - 2a^3 t + c)} = e^{2w} = \frac{(f + g)^2}{(f - g)^2},
\]

(2.14)

which allows \( f(x, t) = 2\sqrt{a} e^x + e^{2X} + 1, \) \( g(x, t) = 2\sqrt{a} e^x - e^{2X} - 1, \) and \( X = ax - 2a^3 t + c. \) These \( f \) and \( g \) read

\[
\frac{(-D_x + D_x^3)}{D_x f \cdot g} = 3a^2,
\]

(2.15)

\[
\frac{D_x^2 (f \cdot g)}{(f^2 - g^2)} = a^2.
\]

(2.16)
While this solution satisfies the mKdV equation \((2.6)\), it does not satisfy the original Hirota type mKdV equations \((2.7)\) and \((2.8)\), but does the generalized Hirota type mKdV equations

\[
(-D_t + D_x^3)f \cdot g = 3\lambda^2 D_x f \cdot g,
\]
\[
D_x^2(f \cdot g - g \cdot f) = \lambda^2(f^2 - g^2),
\]
with \(\lambda = a\).

b) Hirota type mKdV solution:
Another well-known one-soliton solution can be derived from

\[
\tanh \frac{w}{2} = \frac{g}{f} = e^{ax + a^3t + c}.
\]

We can easily see that this one-soliton solution satisfies the Hirota type mKdV equation, since taking \(f = 1\), Eqs.\((2.7)\) and \((2.8)\) are reduced to

\[
-g_t + g_{xxx} = 0, \quad g_{xx}g - g_x^2 = 0.
\]

Writing \(X = ax + a^3t + c\), Eq.\((2.19)\) gives

\[
e^w = \frac{1 + e^X}{1 - e^X},
\]

which reproduces the well-known solution

\[
v = w_x = -\frac{a}{\sinh X} = -\frac{a}{\sinh(ax + a^3t + c)}.
\]

Next we examine whether this solution satisfies the Schwarzian type equation or not. Having \(\varphi_x = e^{2w} = (1 + e^X)^2/(1 - e^X)^2\), and integrating this expression by choosing the integration constant properly, we could arrive at

\[
\varphi = x - 4 \frac{e^X}{a e^X - 1}.
\]

Observing

\[
\varphi_x = \frac{e^{2X} + 2 e^X + 1}{e^{2X} - 2 e^X + 1}, \quad \varphi_t = \frac{4a^2 e^X}{e^{2X} - 2 e^X + 1},
\]

we confirm \(\varphi\) given by Eq.\((2.23)\) satisfies the Schwarzian type mKdV equation \((2.4)\). We can also check Eq.\((2.23)\) reads the same \(v(x, t)\) as Eq.\((2.22)\) via \(v = \frac{1}{2}(\log \varphi_x)_x\). One-soliton solution \((2.23)\) may have the global Möbius symmetry.

It might be interesting to point out that if we choose an integration constant in such a way as

\[
\tilde{\varphi} = X - 4 \frac{e^X}{a e^X - 1},
\]

we confront the result \(\tilde{\varphi}_t - \tilde{\varphi}_x S(\tilde{\varphi}, x) = a^2\). Though \(\tilde{\varphi}\) is a solution of KdV equation, it is not that of the Schwarzian type equation \((2.4)\), so that it has no Möbius group symmetry.

We have two global Möbius symmetric one-soliton solutions i.e. \((2.12)\) and \((2.23)\). In the next section, we show that only the Hirota type mKdV one-soliton solution \((2.23)\) is connected with the algebraic \(N\)-soliton solution through the local Möbius "gauge" transformation, i.e. Bäcklund transformation.
3 Bäcklund transformation and construction of \(N\)-soliton solution

3.1 Bäcklund transformation of mKdV equation

The well-known Bäcklund transformation of the mKdV equation is given by [7, 8]

\[
\begin{align*}
    w_x' + w_x &= a \sinh(w' - w), \\
    w_t' + w_t &= -2a^2 w_x - 2aw_{xx} \cosh(w' - w) + (a^3 - 2aw_x^2) \sinh(w' - w).
\end{align*}
\]  

(3.1) (3.2)

This Bäcklund transformation can be considered as the special “gauge” transformation of the Möbius group [19]. Using the AKNS formalism [4], the spacial derivative and its “gauge” transformed equation of the \(2 \times 2\) inverse scattering transform are given by

\[
\begin{align*}
    \frac{\partial}{\partial x} \left( \begin{array}{c}
        \psi_1(x) \\
        \psi_2(x)
    \end{array} \right) &= \left( \begin{array}{cc}
        a/2 & v(x) \\
        v(x) & -a/2
    \end{array} \right) \left( \begin{array}{c}
        \psi_1(x) \\
        \psi_2(x)
    \end{array} \right), \\
    \frac{\partial}{\partial x} \left( \begin{array}{c}
        \lambda(x) \left( -\psi_2(x) \\
        \psi_1(x)
    \end{array} \right) \right) &= \lambda(x) \left( \begin{array}{cc}
        a/2 & v'(x) \\
        v'(x) & -a/2
    \end{array} \right) \left( \begin{array}{c}
        -\psi_2(x) \\
        \psi_1(x)
    \end{array} \right).
\end{align*}
\]  

(3.3) (3.4)

Defining \(\Gamma = \psi_1/\psi_2\), we have \(\Gamma_x = a\Gamma + v(1 - \Gamma^2)\) from Eq.(3.3). Using \(\Gamma' = \psi_1'/\psi_2' = -\psi_2'/\psi_1 = -1/\Gamma\) and \(\Gamma' = a\Gamma' + v'(1 - \Gamma'^2)\), we obtain Eq.(3.1) by eliminating \(\Gamma\).

Consistency of \(\partial_x\psi_1(x)\) and \(\partial_x\psi_2(x)\) in Eqs. (3.3) and (3.4) gives

\[
(w_x' + w_x)^2 = (\log \lambda)_x^2 - a^2.
\]  

(3.5)

If we compare the above with the Bäcklund transformation (3.1), we have

\[
(\log \lambda)_x = \pm a \cosh(w' - w).
\]  

(3.6)

In this way, \(\lambda(x)\) depends on both \(w(x)\) and \(w'(x)\).

We can write this “gauge” transformation in the form

\[
A' = U x U^{-1} + U A U^{-1},
\]  

(3.7)

with

\[
A = \left( \begin{array}{cc}
        a/2 & v(x) \\
        -v'(x) & -a/2
    \end{array} \right), \quad A' = \left( \begin{array}{cc}
        a/2 & v'(x) \\
        -v'(x) & -a/2
    \end{array} \right), \quad U = \left( \begin{array}{cc}
        0 & -\lambda(x) \\
        \lambda(x) & 0
    \end{array} \right).
\]

The Bäcklund transformation (3.2) is not necessary to obtain the \(N\)-soliton solution. We explain this situation by using the Hirota type equation of \(\tanh(w/2) = g/f\). The Bäcklund transformation of the Hirota type, which corresponds to Eq.(3.1), is given by [9]

\[
D_x(f \pm g) \cdot (f' \pm g') = \frac{a}{2} (f \mp g)(f' \pm g'),
\]  

(3.8)

which gives

\[
D_x^2(f \pm g) \cdot (f' \pm g') = \frac{a^2}{4} (f \pm g)(f' \pm g').
\]  

(3.9)

If we write the Bäcklund transformation in the Hirota form, the Bäcklund transformation (3.9) and the generalized structure equation (2.18) becomes strongly related in that form. Using Eqs.(2.8) and (3.8), we can show \(D_x^2(f' \cdot f' - g' \cdot g') = 0\) [9]. This means that if we construct \(f'\) and \(g'\) from \(f\) and \(g\), which satisfy Eq.(3.8), \(f'\) and \(g'\) automatically satisfy the Hirota type dynamical equation \((-D_t + D_x^2)f' \cdot g' = 0\) by using the primed bilinear Hirota form (2.6)

\[
\frac{(-D_t + D_x^2)f' \cdot g'}{D_x f' \cdot g'} = 3 \cdot \frac{D_x^2(f' \cdot f' - g' \cdot g')}{(f'^2 - g'^2)}.
\]  

(3.10)
We use the Hirota type dynamical equation (2.7) only in the case of solving the one-soliton solution.

If \( \tanh(w/2) = g/f \) is the mKdV solution, \( \tanh(-w/2) = -g/f \) is also a solution. Then the soliton number preserving Bäcklund transformation is given by \( a = 0, f' = f, g' = -g \) in Eqs. (3.8) and (3.9). Eq. (3.9) in this case is given as

\[
D_1^2(f + g) \cdot (f - g) = D_2^2(f \cdot f - g \cdot g) = 0. \tag{3.11}
\]

This is nothing but the Hirota type structure equation Eq. (2.8).

The problem of the above Bäcklund transformation is that this Bäcklund transformation is not commutative. We show this by reductio ad absurdum. Assuming the commutativity \( w_{12} = w_{21} \), we have

\[
w_{1,x} + w_{0,x} = a_1 \sinh(w_1 - w_0), \tag{3.12a}
\]

\[
w_{2,x} + w_{0,x} = a_2 \sinh(w_2 - w_0), \tag{3.12b}
\]

\[
w_{12,x} + w_{1,x} = a_2 \sinh(w_{12} - w_1), \tag{3.12c}
\]

\[
w_{12,x} + w_{2,x} = a_1 \sinh(w_{12} - w_2). \tag{3.12d}
\]

Manipulating Eq. (3.12a) − Eq. (3.12b) − Eq. (3.12c) + Eq. (3.12d), the derivative terms are canceled out, so that we have an algebraic relation

\[
\tanh\left(\frac{w_{12} - w_0}{2}\right) = -\frac{a_1 + a_2}{a_1 - a_2} \tanh\left(\frac{w_1 - w_2}{2}\right). \tag{3.13}
\]

However, it turns out that all equations (3.12a)−(3.12d) are not always satisfied by this \( w_{12} \), which gives \( w_{12} \neq w_{21} \). This shows that this Bäcklund transformation is not commutative.

We could explain the non-commutativity of this Bäcklund transformation in another way, that is, by constructing the three-soliton solution. Putting \( w_0 = 0 \), we have

\[
\tanh\left(\frac{w_{12}}{2}\right) = -a_{12} \tanh\left(\frac{w_1 - w_2}{2}\right) = -a_{12} \frac{\tanh(w_1/2) - \tanh(w_2/2)}{1 - \tanh(w_1/2) \tanh(w_2/2)}, \tag{3.14}
\]

\[
\tanh\left(\frac{w_{13}}{2}\right) = -a_{13} \tanh\left(\frac{w_1 - w_3}{2}\right) = -a_{13} \frac{\tanh(w_1/2) - \tanh(w_3/2)}{1 - \tanh(w_1/2) \tanh(w_3/2)}, \tag{3.15}
\]

with \( a_{ij} = (a_i + a_j)/(a_i - a_j) \). Next, let us construct a three-soliton solution. Assuming the commutativity \( w_{123} = w_{132} \), we have

\[
w_{12,x} + w_{1,x} = a_2 \sinh(w_{12} - w_1), \tag{3.16a}
\]

\[
w_{13,x} + w_{1,x} = a_3 \sinh(w_{13} - w_1), \tag{3.16b}
\]

\[
w_{123,x} + w_{12,x} = a_3 \sinh(w_{123} - w_{12}), \tag{3.16c}
\]

\[
w_{123,x} + w_{13,x} = a_2 \sinh(w_{123} - w_{13}). \tag{3.16d}
\]

Making Eq. (3.16a) − Eq. (3.16b) − Eq. (3.16c) + Eq. (3.16d), an algebraic relation shows up

\[
\tanh\left(\frac{w_{123} - w_1}{2}\right) = \frac{a_2 + a_3}{a_2 - a_3} \tanh\left(\frac{w_{12} - w_{13}}{2}\right) = A. \tag{3.17}
\]

We have \( \tanh(w_{123}/2) = (A + \tanh(w_1/2))/(1 + A \tanh(w_1/2)) \). Our question is, by using the addition formula Eqs. (3.13), (3.15), and (3.17), whether \( \tanh(w_{123}/2) \) becomes the cyclic symmetric expression or not. This can be the check of commutativity in this case. The answer is in the negative, since we have the following expression without cyclic symmetry for \( t_1 = \tanh(w_1/2), t_2 = \tanh(w_2/2) \), and \( t_3 = \tanh(w_3/2) \):

\[
tanh\frac{w_{123}}{2} = \frac{\tanh(w_1/2)(1 - \tanh(w_{12}/2) \tanh(w_{13}/2)) - a_{23} (\tanh(w_{12}/2) - \tanh(w_{13}/2))}{1 - \tanh(w_{12}/2) \tanh(w_{13}/2) - a_{23} \tanh(w_1/2) (\tanh(w_{12}/2) - \tanh(w_{13}/2))}, \tag{3.18}
\]

with Eqs. (3.14) and (3.15). The situation differs for the well-known Bäcklund transformation of KdV equation and this Bäcklund transformation becomes commutative. We see that in the next section.
3.2 Bäcklund transformation of KdV equation

Since we would like to construct the mKdV $N$-soliton solution through the KdV equation, we here review the KdV equation.

Using the variable $u = z_x$, the standard KdV equation is given by

$$u_t - u_{txx} + 6uu_x = 0,$$

where we put an integration constant to be zero. We use the following Hirota type KdV equation

$$(-D_t D_x + D_x^4) \tau \cdot \tau = 0,$$

where $D_x = \partial_x$. From Eq.(3.25) we obtain Eq.(3.23) by eliminating $\Gamma$.

By using $\Gamma = \psi / \psi_2$ with $\psi = \psi(x,t)$, we can write this “gauge” transformation in the form

$$D_x \psi = \frac{a^2}{2} - \frac{(\partial_x \lambda / \lambda)^2}{2}.$$

As we did for the case of mKdV equations, defining $\Gamma = \psi_1 / \psi_2$ then we have $\Gamma_x = -u + a \Gamma + \Gamma^2$ from Eq.(3.25). By using $\Gamma' = \psi_1'/\psi_2' = -(\psi_1 + a\psi_2) / \psi_2 = -\Gamma - a$ and $\Gamma_x' = -u' + a\Gamma' + \Gamma^2$, we obtain Eq.(3.23) by eliminating $\Gamma$.

Consistency of $\partial_x \psi_1(x)$ and $\partial_x \psi_2(x)$ in Eqs.(3.25) and (3.26) gives

$$z'_x + z_x = \frac{a^2}{2} + \frac{(\partial_x \lambda / \lambda)^2}{2}.$$

If we compare the above with the structure KdV Bäcklund transformation (3.24), we have

$$(\log \lambda)_x = \pm(z - z').$$

We can write this “gauge” transformation in the form

$$A' = U_x U^{-1} + U A U^{-1},$$

with

$$A = \begin{pmatrix} a/2 & -u(x) \\ -1 & -a/2 \end{pmatrix}, \quad A' = \begin{pmatrix} a/2 & -u'(x) \\ -1 & -a/2 \end{pmatrix}, \quad U = \begin{pmatrix} \lambda(x) & a\lambda(x) \\ 0 & -\lambda(x) \end{pmatrix}.$$
3.3 Connection between KdV and mKdV equations

The connection between the mKdV equation and the KdV equation is given by the Miura transformation in the form \( u = \pm v_x + v^2 \),

\[
  u_t - u_{xxx} + 6uu_x = \pm (\partial_x \pm 2v)(v_t - v_{xxx} + 6v^2v_x). \tag{3.31}
\]

First, we decompose the \( \tau \) function of KdV equation into the even part and the odd part. As an example, we consider the two-soliton solution of KdV equation

\[
  \tau(x,t) = 1 + e^{X_1} + e^{X_2} + b_{12} e^{X_1} e^{X_2}. \tag{3.32}
\]

We define the even and odd part of \( \tau \) as behavior under \( e^{X_i} \rightarrow -e^{X_i} \). We then decompose the even part by \( f_1 = 1 + b_{12} e^{X_1} e^{X_2} \) and the odd part by \( g_1 = e^{X_1} + e^{X_2} \), which gives \( \tau = f_1 + g_1 \).

In this decomposition, \( u \) is expressed by

\[
  u = -2(\log \tau)_{xx} = -2(\log(f_1 + g_1))_{xx} = -2\partial_x \left( \frac{f_{1,x} + g_{1,x}}{f_1 + g_1} \right). \tag{3.33}
\]

In terms of mKdV equation, using \( e^w = (f + g)/(f - g) \) of Eq. (2.5), \( v \) is expressed by

\[
  v = w_x = \frac{2(fg_x - f_x g)}{f^2 - g^2}. \tag{3.34}
\]

Surprisingly, if we identify \( f_1 = f \) and \( g_1 = g \), we can show that the Miura transformation \( u = -v_x + v^2 \) gives the Hirota type structure equation \( (2.8) \) itself by using the relation \( 0 = (f^2 - g^2)\left(-u - v_x + v^2\right) = D_x^2(f \cdot f - g \cdot g) \). Therefore, as Eqs. (2.7) and (2.8) are satisfied, the solution of mKdV equation \( v = w_x \) with \( \tanh(w/2) = g/f \) corresponds to the solution of KdV equation \( u = -2(\log \tau)_{xx} \) with \( \tau = f + g \) through the Miura transformation in the following form

\[
  \text{mKdV equation} \quad \xleftarrow{\text{Miura tr.}} \quad \text{KdV equation} \quad v = w_x, \quad \frac{\tanh w}{2} = \frac{g}{f} \quad \xleftarrow{\text{u = -v_x + v^2}} \quad u = -2(\log \tau)_{xx}, \quad \tau = f + g. \tag{3.35}
\]

That is, if we know the KdV solution, we can obtain the mKdV solution with the same \( f \) and \( g \) and vice versa.

For the KdV equation, if \( \tau = f + g \) is the solution with \( f \) being the even function and \( g \) being the odd function under \( e^{X_i} \rightarrow -e^{X_i}, \tau' = f - g \) is also the solution. Then the soliton number preserving Bäcklund transformation is given by putting \( a = 0, \tau = f + g, \tau' = f - g \) in Eq. (3.30), which gives

\[
  D_x^2(f + g) \cdot (f - g) = D_x^2(f \cdot f - g \cdot g) = 0. \tag{3.36}
\]

We again obtain the Hirota type structure equation. The Hirota type equation is not unique for the given standard type soliton equation. For the standard type mKdV equation \( (2.1) \), there are various different Hirota type equation by assuming \( v = g/f \) or \( w = \log(g/f) \) or \( \tanh(w/2) = g/f \) \( [11, 12] \). The Miura transformation \( u = -v_x + v^2 \) from \( v \) to \( u \) is easily performed by the differentiation, but the inverse Miura transformation from \( u \) to \( v \) is in general quite difficult because we must solve the non-linear differential equation. In order to algebraically find the \( N \)-soliton solution of the mKdV equation, this Miura transformation must also be algebraic. In order to do so, we must use the special Hirota type equation coming from \( \tanh(w/2) = g/f \). In that case, the Miura transformation connects mKdV equation and KdV equation just only by the correspondence of the same \( f \) and \( g \) with \( \tanh(w/2) = g/f \) in
mKdV and $\tau = f + g$ in KdV, which means that the Miura transformation and the inverse Miura transformation is algebraic.

Furthermore, if we use the special Hirota type mKdV equation coming from $\text{tanh}(w/2) = g/f$, the Miura transformation becomes equivalent to the soliton number preserving KdV Bäcklund transformation (3.36). Then such mKdV Hirota type structure equation has Abelian group structure. This is because 
i) Hirota type structure equation is the same as the soliton number preserving KdV Bäcklund transformation, 
ii) KdV Bäcklund transformation has Abelian group structure. The Abelian group structure of this Hirota type structure equation is the reason why the Hirota’s direct method works well for such a special Hirota type equation, where it is not necessary to connect the mKdV equation and the KdV equation.

### 3.4 Three soliton solution — The demonstration of construction of mKdV $N$-soliton solutions

We demonstrate to construct three-soliton solution, which is the nontrivial case to demonstrate the commutativity of the Bäcklund transformation. We start from 3 one-soliton solutions of the Hirota type equation,

$$\text{tanh}\left(\frac{w_i}{2}\right) = \frac{g_i}{f_i}, \quad f_i = 1, \quad g_i = e^{X_i},$$

with

$$X_i = a_i x + a_i^3 t + c_i, \quad (i = 1, 2, 3), \quad v_i = -\frac{a_i}{\sinh X_i}.$$

From the Miura transformation, we have

$$u_i = z_{i,x} = -v_{i,x} + v_i^2 = -\frac{a_i^2}{2} \frac{1}{\cosh^2(X_i/2)},$$

$$z_i = -a_i \tanh\left(\frac{X_i}{2}\right) = a_i \frac{(1 - e^{X_i})}{(1 + e^{X_i})}.$$  

Let us first construct two-soliton solution by the Bäcklund transformation (3.23). Assuming the commutativity, $z_{12} = z_{21}$, we have

$$z_{1,x} + z_{0,x} = -\frac{a_1^2}{2} + \frac{(z_1 - z_0)^2}{2},$$

$$z_{2,x} + z_{0,x} = -\frac{a_2^2}{2} + \frac{(z_2 - z_0)^2}{2},$$

$$z_{12,x} + z_{1,x} = -\frac{a_2^2}{2} + \frac{(z_{12} - z_1)^2}{2},$$

$$z_{12,x} + z_{2,x} = -\frac{a_1^2}{2} + \frac{(z_{12} - z_2)^2}{2}.$$  

As we did before, making Eq.(3.40a)−Eq.(3.40b)−Eq.(3.40c)+Eq.(3.40d), the derivative terms are canceled out and we have $z_{12} - z_0 = (a_1^2 - a_2^2)/(z_1 - z_2)$. Putting $z_0 = 0$, $z_{12}$, $z_{13}$ are given by

$$z_{12} = \frac{a_1^2 - a_2^2}{z_1 - z_2}, \quad z_{13} = \frac{a_2^2 - a_3^2}{z_1 - z_3}.$$  

We can check that Eq.(3.41) satisfies Eqs.(3.40a)−(3.40d), which means that it is commutative in this level.
Next let us construct the three-soliton solution. Assuming the commutativity, we have

\[ \begin{align*}
  z_{12,x} + z_{1,x} &= -\frac{a_2^2}{2} + \frac{(z_{12} - z_1)^2}{2}, \\
  z_{13,x} + z_{1,x} &= -\frac{a_3^2}{2} + \frac{(z_{13} - z_1)^2}{2}, \\
  z_{123,x} + z_{12,x} &= -\frac{a_2^2}{2} + \frac{(z_{123} - z_{12})^2}{2}, \\
  z_{123,x} + z_{13,x} &= -\frac{a_2^2}{2} + \frac{(z_{123} - z_{13})^2}{2}.
\end{align*} \tag{3.42a,b,c,d} \]

Making Eq. (3.42a) − Eq. (3.42b) − Eq. (3.42c) + Eq. (3.42d), we obtain

\[ z_{123} = z_1 + \frac{a_2^2 - a_3^2}{z_{12} - z_{13}}, \]

\[ = \frac{(a_1^2 - a_2^2)z_{12} + (a_2^2 - a_3^2)z_{23} + (a_3^2 - a_1^2)z_3}{(a_1^2 - a_2^2)z_3 + (a_2^2 - a_3^2)z_1 + (a_3^2 - a_1^2)z_2}. \tag{3.43} \]

We can check that Eq. (3.43) really satisfies Eqs. (3.42a + 3.42d) by the mathematical reduction, which means that it is commutative in this level. Generally, we can show the commutativity of this Bäcklund transformation by the mathematical reduction. See also Bianchi and Eisenhart [20,21].

In the above, we use only \( z_{12}, z_{13} \) and do not use \( z_{23} \). However, as the result, \( z_{123} \) is cyclic symmetric in \( z_1, z_2, z_3 \), which is another check of the commutativity of this Bäcklund transformation. Using Eq. (3.39) and redefine the constants \( c_i \) \( (i = 1, 2, 3) \) in such a way as

\[ \begin{align*}
  e^{\dot{X}_i} &= \frac{(a_1 + a_2)(a_1 + a_3)}{(a_1 - a_2)(a_1 - a_3)} e^{X_1}, \\
  e^{\dot{X}_2} &= \frac{(a_2 + a_1)(a_2 + a_3)}{(a_2 - a_1)(a_2 - a_3)} e^{X_2}, \\
  e^{\dot{X}_3} &= \frac{(a_3 + a_1)(a_3 + a_2)}{(a_3 - a_1)(a_3 - a_2)} e^{X_3},
\end{align*} \]

this gives

\[ z_{123} = (a_1 + a_2 + a_3) - \frac{\tau_x}{\tau}, \tag{3.45} \]

where

\[ \tau = 1 + e^{\dot{X}_1} + e^{\dot{X}_2} + e^{\dot{X}_3} + b_{12} e^{\dot{X}_1 + \dot{X}_2} + b_{13} e^{\dot{X}_1 + \dot{X}_3} + b_{23} e^{\dot{X}_2 + \dot{X}_3} + b_{12} b_{13} b_{23} e^{\dot{X}_1 + \dot{X}_2 + \dot{X}_3}, \tag{3.46} \]

with

\[ b_{ij} = \frac{(a_i - a_j)^2}{(a_i + a_j)^2}. \]

The constructed \( z_{123} \) differs from \( -2\tau_x/\tau \) by the constant factor \( (a_1 + a_2 + a_3) \), but this constant factor does not contribute to \( u = z_{123,x} = -2(\tau_x/\tau)_x \). This is equivalent to the constant shift of \( z_{123} \to z_{123} + (a_1 + a_2 + a_3) \). Then we have even and odd part of \( \tau \) in the form

\[ \begin{align*}
  f &= 1 + b_{12} e^{\dot{X}_1 + \dot{X}_2} + b_{13} e^{\dot{X}_1 + \dot{X}_3} + b_{23} e^{\dot{X}_2 + \dot{X}_3}, \\
  g &= e^{\dot{X}_1} + e^{\dot{X}_2} + e^{\dot{X}_3} + b_{12} b_{13} b_{23} e^{\dot{X}_1 + \dot{X}_2 + \dot{X}_3}.
\end{align*} \tag{3.47,48} \]

This gives the mKdV three-soliton solution

\[ \frac{\tanh w}{2} = \frac{g}{f} = \frac{e^{\dot{X}_1} + e^{\dot{X}_2} + e^{\dot{X}_3} + b_{12} b_{13} b_{23} e^{\dot{X}_1 + \dot{X}_2 + \dot{X}_3}}{1 + b_{12} e^{\dot{X}_1 + \dot{X}_2} + b_{13} e^{\dot{X}_1 + \dot{X}_3} + b_{23} e^{\dot{X}_2 + \dot{X}_3}}. \tag{3.49} \]

This addition structure is quite similar to the addition formula of \( \tanh \), since we obtain the addition formula of \( \tanh \) by the replacement \( e^{\dot{X}_i} \to \tanh(\theta_i) \) and \( b_{ij} \to 1 \).

The general \( N \)-soliton solution obtained by the Hirota’s direct method which is non-algebraic method is given in Hirota’s paper [11].
4 Summary and discussions

We have the dogma that it is quite surprising that the soliton equation has infinitely many exact solutions ($N$-soliton solution with $N \to \infty$) despite of its non-linearity. There must exist some nice structures and they must be some local Lie group structures behind the non-linear soliton equation. According to our dogma, we study to construct mKdV $N$-soliton solution and we elucidate the mechanism why we can algebraically construct $N$-soliton solution from the group theoretical point of view.

First of all, the Schwarzian type mKdV equation has the global Möbius group ($GL(2, \mathbb{R})$) symmetry. It is natural to expect that there might be the local Möbius group ($GL(2, \mathbb{R})$) symmetry. We then try to construct $N$-soliton solution algebraically from one-soliton solutions through the Bäcklund transformation. The Bäcklund transformation is considered as the local Möbius transformation, but the Möbius group is generally non-Abelian, so that only some special Bäcklund transformation is commutative for the addition of the transformations. While the well-known mKdV Bäcklund transformation is not commutative, the well-known KdV Bäcklund transformation is commutative. Then, in order to construct the mKdV $N$-soliton solution by the algebraic addition formula of the Bäcklund transformation, we first transform the mKdV equation to the KdV equation by the Miura transformation, and use the KdV Bäcklund transformation to obtain the KdV $N$-soliton solution from one-soliton solutions and finally come back to the mKdV $N$-soliton solution by the inverse Miura transformation. For $N = 2$ case, we use the following scheme:

\[
\begin{align*}
\tanh \frac{w_1}{2} &= e^{x_1} \quad \text{Miura tr.} \quad z_1 = a_i \frac{(1 - e^{X_1})}{(1 + e^{X_1})} \quad \text{Bäcklund tr.} \quad z_{12} = \frac{a_1^2 - a_2^2}{2} = -\frac{2\tau_x}{\tau} \\
\uparrow & \uparrow & \uparrow \\
\text{mKdV one-soliton sol.} & \quad \text{KdV one-soliton sol.} & \quad \text{KdV two-soliton sol.}
\end{align*}
\]

\[
\tau = 1 + e^{X_1} + e^{X_2} + b_{12} e^{X_1} e^{X_2} = f + g \quad \text{inverse Miura tr.} \quad \tanh \frac{w_{12}}{2} = g \frac{e^{X_1} + e^{X_2}}{1 + b_{12} e^{X_1} e^{X_2}} \quad \uparrow \\
\text{KdV two-soliton } \tau \text{ funct.} & \quad \quad \quad \quad \text{mKdV two-soliton sol.}
\]

However, if we algebraically construct $N$-soliton solution, the above Bäcklund transformation and the inverse Bäcklund transformation must be algebraic. If we use the special type of Hirota equation which comes from $\tanh(w/2) = g/f$, as it is quite surprising, this Miura transformation becomes equivalent to the structure equation of this Hirota type mKdV equation \(^{(2.8)}\) and also becomes equivalent to the soliton number preserving KdV Bäcklund transformation. Therefore, the local Möbius group structure is already build in as the structure equation of this Hirota type mKdV equation and this structure equation has the Abelian group structure. This is the reason why the Hirota’s direct method works well, where it is not necessary to connection the mKdV equation and the KdV equation.

In order that the whole construction of the $N$-soliton solution becomes algebraic, the one-soliton solution must be some special one-soliton solution, that is, it must be the one-soliton solution of the special Hirota type equation which comes from $\tanh(w/2) = g/f$. Regarding the addition structure of $\tanh(w/2) = g/f$, we have Eq.\((2.11)\) for $N = 2$ and Eq.\((3.49)\) for $N = 3$, which are quite similar to the addition formula of tanh, which reflect the addition formula of the global Möbius group symmetry of the Schwarz type equation, where the addition formula of tanh is essential. The global addition structure is given by the global transformation from one-soliton solution to another one-soliton solution in the following form:
i) \( w \rightarrow w' = w + c \), which gives another one-soliton solution
\[
\tanh(w'/2) = \frac{\tanh(w/2) + \tanh(c/2)}{1 + \tanh(w/2) \tanh(c/2)},
\]

ii) \( c_1 \rightarrow c_1 + \log \lambda \), which gives another one-soliton solution
\[
\tanh(w'/2) = \lambda e^{X_1},
\]
which is the scale transformation. Combining i) and ii), we have the global Möbius group symmetry.

We can apply our method to the \( N \)-soliton solution of the sinh-Gordon equation \( \theta_{xt} = \sinh \theta \). The Bäcklund transformation of the sinh-Gordon equation \( \theta'/2 + \theta/2 = a \sinh(\theta'/2 - \theta/2) \) has the same form as Eq. (3.1) in mKdV equation. Then, by putting \( \tanh(\theta/4) = g/f \) and \( X_i = a_i x + t/a_i + c_i \), we obtain the Hirota’s result [22] by using our method.

Our result is analogous to the result of the Galois theory to solve the algebraic equation. If the Galois group of the algebraic equation is Abelian, such algebraic equation is algebraically solvable. In our case, the soliton equation is the special non-linear equation, which means that it has the non-Abelian “gauge” symmetry, and some special soliton solution has the special Abelian sub-“gauge” symmetry. Because of that, from such special one-soliton solution, we can algebraically construct the \( N \)-soliton solution.

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