Existence and uniqueness of the measure of maximal entropy for the Teichmüller flow on the moduli space of Abelian differentials

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Abstract. The main result of the paper is the statement that the ‘smooth’ measure of Masur and Veech is the unique measure of maximal entropy for the Teichmüller flow on the moduli space of Abelian differentials. The proof is based on the symbolic representation of the flow in Veech’s space of zippered rectangles.

Bibliography: 29 titles.

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§ 1. Introduction

The Teichmüller geodesic flow \( \{ g_t \} \), first studied by Masur [1] and Veech [2], acts on the moduli space of Riemann surfaces endowed with a holomorphic differential. More precisely, let \( S \) be a closed surface of genus \( g \geq 2 \). Introduce a complex structure \( \sigma \) and a holomorphic differential \( \omega \) on \( S \). The pair \( (\sigma, \omega) \) is considered to be equivalent to another pair \( (\sigma_1, \omega_1) \) if there is a diffeomorphism of \( S \) sending \( (\sigma, \omega) \) to \( (\sigma_1, \omega_1) \). The moduli space \( M(g) \) consists of the equivalence classes, and the flow \( \{ g_t \} \) on \( M(g) \) is induced by the action on the pairs \( (\sigma, \omega) \) defined by the formula \( g_t(\sigma, \omega) = (\sigma', \omega') \), where \( \omega' = e^{t \Re(\omega)} + ie^{-t \Im(\omega)} \), while the complex structure \( \sigma' \) is determined by the requirement that \( \omega' \) be holomorphic. If \( (\sigma, \omega) \) and \( (\sigma', \omega') \) are equivalent, then the differentials \( \omega \) and \( \omega' \) have the same orders of zeros and the same area. Therefore these orders and area are well-defined on \( M(g) \). Moreover, they are preserved by the Teichmüller flow \( \{ g_t \} \). Take an arbitrary unordered collection \( \kappa = (k_1, \ldots, k_r) \) with \( k_i \in \mathbb{N}, k_1 + \cdots + k_r = 2g - 2 \), and denote the subspace of \( M(g) \) corresponding to the differentials of area 1 by \( \mathcal{M}_\kappa \), that is, \( \frac{i}{2} \int \omega \wedge \overline{\omega} = 1 \), with orders of zeros \( k_i, i = 1, \ldots, r \); \( \mathcal{M}_\kappa \) is said to be a stratum in \( M(g) \). Each stratum is a \( \{ g_t \} \)-invariant set.

The space \( \mathcal{M}_\kappa \) admits a natural topological structure, under which it is in general not connected. The number of connected components is no more than 3 and depends on \( \kappa \) (see [3] for details). Each connected component \( \mathcal{H} \) of this space is \( \{ g_t \} \)-invariant.
By a theorem of Habbard and Masur [4] \( H \) admits a local identification with the space of relative homologies \( H^1(S, \{p_1, \ldots, p_m\}, \mathbb{C}) \), where \( p_1, \ldots, p_m \) are the zeros of the differential. The Lebesgue measure on \( H^1(S, \{p_1, \ldots, p_m\}, \mathbb{C}) \) normalized in such a way that the lattice \( H^1(S, \{p_1, \ldots, p_m\}, \mathbb{Z} \oplus i\mathbb{Z}) \) has covolume 1 induces, via the above identification, a globally defined \( \{g_t\} \)-invariant measure on \( H \) (see [1], [5], [6]).

We fix an arbitrary connected component \( \mathcal{H} \) and denote by \( \mu_\kappa \) the normalized restriction to \( H \) of this \( \{g_t\} \)-invariant measure.

Veech [5] showed that \( \{g_t\} \) is a Kolmogorov flow with respect to the measure \( \mu_\kappa \), with entropy given by the formula

\[
h_{\mu_\kappa}(\{g_t\}) = 2g - 1 + r. \tag{1.1}
\]

Our aim is to establish the following

**Theorem 1.1.** The measure \( \mu_\kappa \) is the unique measure of maximal entropy for the flow \( \{g_t\} \) on \( \mathcal{H} \).

The proof of this theorem is based on the representation of the flow \( \{g_t\} \) as a suspension flow over a countable alphabet topological Markov shift. The reasoning proceeds in two steps. We begin with sufficient conditions for an invariant measure of the above suspension flow to be a measure with maximal entropy. These conditions are contained in Theorem 2.2 (see § 2.1). After stating the theorem we outline its proof. This proof is close in spirit to thermodynamic formalism for countable alphabet topological Markov shifts (see [7]–[11]). In particular, we use a uniqueness theorem by Buzzi and Sarig [12] for an equilibrium measure. An application of thermodynamic formalism to another smooth dynamical system with noncompact phase space, the geodesic flow on the modular surface, can be found in [13].

Subsections 2.2–2.5 are devoted to proving Theorem 2.2. In the rest of the paper (§§ 3–5) we deduce Theorem 1.1 from Theorem 2.2, again starting with a sketch of the proof which follows.

The following observation is at the heart of our argument in this part of the proof. The Teichmüller flow induces the stable and unstable foliations with infinitely smooth fibres, with respect to which it is ‘measurably Anosov’ in the sense of Veech [5] and Forni [14]. The Masur-Veech measure \( \mu_\kappa \) induces globally defined sigma-finite measures on unstable leaves and these measures are uniformly expanded by the flow. In other words, the Masur-Veech measure has the Margulis [15] uniform expansion property on unstable leaves. Informally, Proposition 4.8 expresses the Margulis property in terms of the symbolic representation of the Teichmüller flow.

We note that in order to establish Theorem 1.1 we need only the special case of Theorem 2.2 dealing with the countable alphabet topological Bernoulli shift. But the proof in this case would only have been a little easier than the general proof. The main results of this paper were stated without proof in [16].

### § 2. Suspension flows

Let \( G \) be an Abelian group (in what follows we will only deal with \( G = \mathbb{Z} \) or \( G = \mathbb{R} \)) and let \( \{T_g, g \in G\} \) be an action of \( G \) by measurable transformations of a metrizable topological space \( X \) endowed with its Borel \( \sigma \)-algebra \( \mathcal{B} \).
If $G = \mathbb{Z}$, then the corresponding action will be denoted by $\{X, T_n\}$ (or $\{T_n\}$) and called a cascade. If $G = \mathbb{R}$, we write $\{X, T_i\}$ (or $\{T_i\}$) and refer to this action as a flow. In the first case $T_n = T^n, n \in \mathbb{Z}$, where $T$ is a bimeasurable one-to-one transformation of $(X, \mathcal{B})$ called an automorphism. We keep the same terminology when we talk about the actions together with their invariant measures.

Two cascades $\{X, T_n\}$ and $\{X', T'_n\}$, acting on $(X, \mathcal{B})$ and $(X', \mathcal{B}')$, respectively, are called isomorphic if there is a one-to-one epimorphic bimeasurable map $\Phi: X \rightarrow X'$ such that $T'_n \circ \Phi = \Phi \circ T_n$ for all $n \in \mathbb{Z}$. If $\Phi$ is not necessarily epimorphic, we say that $\{X, T_n\}$ is embedded into $\{X', T'_n\}$.

Consider a cascade $\{X, T_n\}$ together with a $\{T_n\}$-invariant Borel probability measure $\mu$ on $X$. Denote this by $\{X, T; \mu\}$. We say that the cascades $\{X, T_n; \mu\}$ and $\{X', T'_n; \mu'\}$ are isomorphic if there are sets $X_1 \in \mathcal{B}$ and $X'_1 \in \mathcal{B}'$ invariant with respect to all the $T_n$ and all the $T'_n$, respectively, such that $\mu(X_1) = \mu'(X'_1) = 1$ and the restrictions $\{X_1, T_n|_{X_1}\}$ and $\{X'_1, T'_n|_{X'_1}\}$ are isomorphic in the above sense.

If we replace $G = \mathbb{Z}$ by $G = \mathbb{R}$ and $n$ by $t$ we automatically obtain the corresponding notions for flows.

In this paper we mostly deal with flows that can be defined as follows. Let $G$, $X$, $T$ denote the set of all $\mathbb{R}$, $X$, $T$ automorphism of the space $(X, \mathcal{B})$ and $G$ isomorphic to the direct product $X = \mathbb{R}$. If $T$ is an automorphism of the space $(X, \mathcal{B})$ and $G$ is not necessarily bimeasurable, then the corresponding action will be denoted by $\{X, T; \mu\}$.

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where \( h(T; \mu) \) is the entropy of the automorphism \( T \) with respect to the measure \( \mu \).

We define the topological entropy of the flow \((T, f)\) by

\[
h_{\text{top}}(T, f) = \sup_{\mu \in \mathcal{M}_{T, f}} h(T, f; \mu). \tag{2.2}
\]

This terminology is justified by the following well-known fact: if \( X \) is a compact space, \( T \) a homeomorphism of \( X \), and \( f \) a continuous function, then the right-hand side of (2.2) is indeed the topological entropy of the suspension flow \((T, f)\).

It should be mentioned that in the noncompact case, there are different definitions of the topological entropy that can yield different values for specific systems (see, for example [17]).

Any \( \mu \in \mathcal{M}_{T, f} \) at which the supremum in (2.2) is achieved is called a measure of maximal entropy for \((T, f)\).

2.1. Suspension flows over Markov shifts. In the specific case we will deal with, \((X, T)\) is a countable alphabet topological Markov shift, that is, \( X \) is the set of doubly-infinite paths in a directed graph \( \Gamma = (V, E) \) with vertex set \( V \) and edge set \( E \subseteq V \times V \) and \( T \) is the shift transformation: \((Tx)_i = x_{i+1}\) for every \( x = (x_i, i \in \mathbb{Z}) \in X \). In other words, \( X \) consists of all sequences \( x \in V^\mathbb{Z} \) such that \( B_{x_i,x_{i+1}} = 1 \), where \( B = B(\Gamma) \) is the adjacency matrix of the graph \( \Gamma \). The vertices \( v \in V \) will also be called letters.

In what follows we assume that \( \Gamma \) is connected. If \( \Gamma \) is the complete graph, that is, \( E = V \times V \), then we have the topological Bernoulli shift with alphabet \( V \).

We introduce the discrete topology on \( V \), the product topology on \( V^\mathbb{Z} \), and the induced topology on \( X \subseteq V^\mathbb{Z} \). The map \( T \) is clearly a homeomorphism of \( X \). We shall refer to every finite path of \( \Gamma \), that is, a sequence \( w = (v_1, \ldots, v_k) \in V^k \) such that \((v_i, v_{i+1}) \in E, i = 1, \ldots, k-1, \) as a word. Denote the set of all words (including the empty one) by \( W(\Gamma) \).

Let \( w = (v_1, \ldots, v_k) \) and \( w' = (v'_1, \ldots, v'_l) \) be two words. Their concatenation \( ww' := (v_1, \ldots, v_k, v'_1, \ldots, v'_l) \) is also a word if \((v_k, v'_1) \in E \). We say that \( w' \) contains \( w' \) (or \( w' \) is a subword of \( w \)) if \( v'_i = v_i, \ldots, v'_k = v_{i+l-1} \) for some \( i, 1 \leq i \leq k-l+1 \). In the special case when \( i = 1 \) we call \( w' \) a prefix of \( w \). Let \( w = (v_1, \ldots, v_n) \), \( n \geq 2 \), be a word and \( w' = (v_1, \ldots, v_l), 1 \leq l \leq n, \) be a prefix of \( w \). We call \( w' \) a simple prefix if there is no \( k, 2 \leq k \leq l, \) such that \((v_1, \ldots, v_{n-k+1}) = (v_k, \ldots, v_{n}) \). If \( w \) is a simple prefix of itself, then \( w \) is called a simple word. If a simple word is a prefix of another word, it is clearly a simple prefix.

Remark 2.1. Since every word \( w = (v_1, \ldots, v_n) \) is a concatenation of the single-letter words \( v_i \), we will also write \( w = v_1 \ldots v_n \).

To every word \( w \) we assign the cylinder \( C_w = \{ x \in X : (x_0, \ldots, x_{|w|-1}) = w \} \), where \(|w|\) is the length of \( w \), that is, the number of symbols in \( w \).

For a function \( f : X \to \mathbb{R} \) we set

\[
\var_n(f) = \sup\{|f(x) - f(y)| : x_i = y_i \text{ when } |i| \leq n\}, \quad n \in \mathbb{N}.
\]

We say that \( f \) has summable variations if \( \sum_{n=1}^{\infty} \var_n(f) < \infty \), and that \( f \) depends only on the future if \( x_i = y_i \) for all \( i \geq 0 \) implies that \( f(x) = f(y) \).
For a suspension flow \( \{S_t\} = (T, f) \) and a set \( C \subset X \) we put
\[
\bar{\tau}(x, C) = \inf\{t > 0 : S_t(x, 0) \in C \times \{0\}\}, \quad x \in X, \tag{2.3}
\]
so that \( \bar{\tau}(x, C) \) is the first hitting time of \( C \times \{0\} \) for a point \( x \times \{0\} \in X_f \).

**Theorem 2.2.** Let \( (X, T) \) be a countable alphabet topological Markov shift corresponding to a connected graph \( \Gamma \), let \( f : X \to [c, \infty) \), \( c > 0 \), be a function with summable variations depending only on the future, and let \( \{S_t\} = (T, f) \) be the suspension flow constructed by \( T \) and \( f \). Assume that \( \mu \in \mathcal{M}_{T,f} \) is a measure positive on all cylinders in \( X \), and that for each \( l > 0 \) there exists a simple word \( w \in W(\Gamma) \) with \( |w| > l \) such that for every word \( \tilde{w} \) that does not contain \( w \), and for \( \mu \)-almost all \( x \in C_{w\tilde{w}} \),
\[
\left| \frac{\mu(C_{w\tilde{w}})}{\mu(C_w)} - e^{-s\bar{\tau}(x, C_w)} \right| \leq e^{-\alpha |w| - s\bar{\tau}(x, C_w)}, \tag{2.4}
\]
where \( \bar{\tau}(x, C_w) \) is defined in (2.3) and \( \alpha, s \) are positive constants (depending only on \( \mu \)).

Then
(i) \( s = h_{\text{top}}(T, f) \);
(ii) if \( s = h(T, f; \mu_f) \), then \( \mu_f \) is the unique measure of maximal entropy for the flow \( \{S_t\} = (T, f) \).

**Remark 2.3.** The ratio on the left-hand side of (2.4) is clearly the conditional measure of \( C_{w\tilde{w}} \) given \( C_w \).

**Remark 2.4.** The assumption that \( f \) depends only on the future is made just for convenience: Theorem 2.2 remains true without this assumption, but in the sequel we use it only in the particular form above.

We first give an outline of the proof of Theorem 2.2. At the first stage (in §2.2) we consider the particular case when \( (X, T) \) is a Bernoulli shift, while \( f(x), x \in X \), depends only on \( x_0 \) (we then say that \( f \) depends only on the zeroth coordinate). In this case the topological entropy \( h_{\text{top}}(T, f) \) can be expressed explicitly in terms of \( f \). At the next stage we come back to the general case and prove that the supremum in the definition of \( h_{\text{top}}(T, f) \) can be taken over ergodic measures that are positive on all cylinders in \( X \) (see §2.4). This enables us to show that
\[
h_{\text{top}}(T, f) = h_{\text{top}}(T_C, f_C)
\]
for the suspension flow \( (T_C, f_C) \), where \( T_C \) is the transformation induced by \( T \) on a cylinder \( C \subset X \) and \( f_C \) is determined naturally by \( f \), \( T \) and \( C \). If we chose \( C = \{x : (x_0, \ldots, x_k) = w\} \), where \( w \in W(\Gamma) \), then \( (C, T_C) \) will be isomorphic to the countable alphabet Bernoulli shift. Hence the flow \( (T_C, f_C) \) will be isomorphic to a suspension flow \( (\sigma, \varphi) \) over this Bernoulli shift. Here we use approximation and find a function \( \varphi^w \) that depends only on the zeroth coordinate and is uniformly close to \( \varphi \) (when \( w \) is long enough). The topological entropies of the suspension flows \( (\sigma, \varphi) \) and \( (\sigma, \varphi^w) \) are also close to each other. We apply the results obtained at the first stage to the flow \( (\sigma, \varphi^w) \) and rewrite inequality (2.4) for it (see §2.5). Letting the length of \( w \) tend to infinity we complete the proof of the equality \( s = h_{\text{top}}(T, f) \), which implies that the measure under consideration has maximal
entropy. We reduce proving the uniqueness of this measure to proving the uniqueness of the corresponding equilibrium measure and here use the uniqueness theorem due to Buzzi and Sarig [12].

2.2. Entropy of suspension flows over Bernoulli shifts. The proof of Theorem 2.2 is based on some properties of suspension flows constructed by a topological Markov shift (in particular, by a Bernoulli shift) and functions of one or finitely many coordinates. Some of these properties, studied first by Savchenko [8], are described in this section. We include proofs for the reader’s convenience. Our approach is close to that of [8].

We begin with two simple lemmas. Let \( \mathcal{N} = \mathbb{N} \) or \( \mathcal{N} = \{1, \ldots, n\}, \ n \geq 2 \), and let \( c = (c_i, i \in \mathcal{N}) \) be a sequence of real numbers with \( \inf_{i \in \mathcal{N}} c_i > 0 \). Denote by \( \mathcal{P} = \mathcal{P}_{\mathcal{N},c} \) the family of sequences \( p = (p_i, i \in \mathcal{N}) \) such that

\[
p_i \geq 0, \quad i \in \mathcal{N}, \quad \sum_{i \in \mathcal{N}} p_i = 1, \quad \sum_{i \in \mathcal{N}} p_i c_i < \infty.
\]

(Clearly, \( \mathcal{P}_{\mathcal{N},c} \) does not depend on \( c \) when \( |\mathcal{N}| < \infty \).) Let

\[
H(p) = H_{\mathcal{N},c}(p) := -\left( \sum_{i \in \mathcal{N}} p_i \log p_i \right) \left( \sum_{i \in \mathcal{N}} p_i c_i \right)^{-1}, \quad p \in \mathcal{P}
\]

(as usual we let \( 0 \log 0 = 0 \)).

Lemma 2.5. If \( p \in \mathcal{P} \) is such that \( p_k = 0 \) for some \( k \in \mathcal{N} \), then there exists \( p' = (p'_i, i \in \mathcal{N}) \in \mathcal{P} \) with \( p'_i > 0 \) for all \( i \) and such that \( H(p') \geq H(p) \), where the inequality is strict when \( H(p) < \infty \).

Proof. We divide \( \mathcal{N} \) into two nonempty subsets

\[
\mathcal{N}^0 = \{i \in \mathcal{N} : p_i = 0\}, \quad \mathcal{N}^1 = \mathcal{N} \setminus \mathcal{N}^0.
\]

Fix an arbitrary \( l \in \mathcal{N}^1 \), and for \( t \in [0, p_l) \) let \( p^t = (p^t_i, i \in \mathcal{N}) \), where \( p^t_k = t, \ p^t_l = p_l - t \) and \( p^t_i = p_i \) for \( i \neq k, l \). (By assumption, \( k \in \mathcal{N}^0 \).) It is clear that \( p^t \in \mathcal{P} \) and that \( H(p^t) = \infty \) when \( H(p) = \infty \). A simple calculation shows that if \( H(p) < \infty \), then the right-hand derivative \( \frac{d}{dt} H(p^t) \) at \( t = 0 \) is \( +\infty \). Hence \( H(p^t) > H(p) \) when \( t > 0 \) is small enough.

If \( \mathcal{N}^0 = \{k\} \), the proof is complete. If \( \mathcal{N}^0 \setminus \{k\} \neq \emptyset \), we first consider the case \( H(p) < \infty \). Fix an arbitrary \( t \in (0, p_l) \) for which \( H(p^t) > H(p) \). It is easy to find positive numbers \( q_i, i \in \mathcal{N}^0 \setminus \{k\} \), such that

\[
\sum_{i \in \mathcal{N}^0 \setminus \{k\}} q_i = 1, \quad \sum_{i \in \mathcal{N}^0 \setminus \{k\}} q_i (c_i - \log q_i) < \infty.
\]

For \( s \in [0, t) \) we put \( p^{t,s} = (p^{t,s}_i, i \in \mathcal{N}) \), where

\[
p^{t,s}_i = sq_i, \quad i \in \mathcal{N}^0 \setminus \{k\}; \quad p^{t,s}_i = p_i, \quad i \in \mathcal{N}^1 \setminus \{l\}; \quad p^{t,s}_k = t - s; \quad p^{t,s}_l = p_l - t.
\]
From (2.7) it follows that $p^{t,s} \in \mathcal{P}$ and $\lim_{s \to 0} H(p^{t,s}) = H(p^t)$. Therefore $H(p^{t,s}) > H(p)$ as $s > 0$ is small enough, and since $p^{t,s}_i > 0$, we can take $p^{t,s}$ with one such $s$ for $p'$.

It remains to note that if $H(p) = \infty$, then $H(p^{t,s}) = \infty$ for all $s \in [0,t)$ (see (2.7), (2.8)).

**Lemma 2.6.** Let $\mathcal{N} = \{1, \ldots, n\}$, $n \geq 2$, and let $c$, $\mathcal{P} = \mathcal{P}_{\mathcal{N}}$ and $H = H_{\mathcal{N},c}$ be as above. Then $\text{sup}_{p \in \mathcal{P}} H(p)$ is the unique solution to the equation $F_n(\beta) = 1$, where $F_n(\beta) = \sum_{i=1}^n e^{-\beta c_i}$.

**Proof.** Since $H$ is a continuous function on the compact set $\mathcal{P} \subset \mathbb{R}^n$, its supremum is attained at a point $p^0 = (p^0_i, i = 1, \ldots, n) \in \mathcal{P}$. By Lemma 2.5 $p^0_i > 0$ for all $i$.

Let

$$\mathcal{P}^+ = \{p \in \mathcal{P} : p_i > 0, i = 1, \ldots, n\}.$$ 

For $p \in \mathcal{P}^+$ we put $p_1 = 1 - \sum_{i=2}^n p_i$ and consider the equations $\frac{\partial H(p)}{\partial p_i} = 0$, $i \in \mathcal{N} \setminus \{1\}$. From this system we see that if $p^0$ is an extremum point of $H(p)$, then $p^0_i = e^{-\beta c_i}/F_n(\beta^0)$, $1 \leq i \leq n$, where $\beta^0 = \text{const} > 0$. Hence the statement we are proving is true when $c_i = c_1$ for $i = 2, \ldots, n$. Otherwise we take any $i$ for which $c_i \neq c_1$, and from the equation $\frac{\partial H(p)}{\partial p_i} \big|_{p=p^0} = 0$, where $p_1 = 1 - \sum_{i=2}^n p_i$, we obtain $\beta^0 = H(p^0)$. On the other hand, substituting $p^0$ for $p$ in $H(p)$ we see that

$$H(p^0) = \beta^0 - \frac{F_n(\beta^0)}{F_n(\beta^0)} \log F_n(\beta^0).$$

Therefore, $\log F_n(\beta^0) = 0$, that is, $\beta^0$ is a root of the equation $F_n(\beta) = 1$. This root is unique since $F_n(\beta)$ is decreasing in $\beta$. Finally,

$$H(p^0) = \max_{p \in \mathcal{P}} H(p),$$

because, as mentioned above, every point of maximum belongs to $\mathcal{P}^+$, hence the equations $\frac{\partial H(p)}{\partial p_i} = 0$, $i \in \mathcal{N} \setminus \{1\}$, $p_1 = 1 - \sum_{i=2}^n p_i$ must hold at this point. But we already know that these equations have only one solution.

Let us now consider a countable alphabet topological Bernoulli shift $(X, T)$ with $X = V^\mathbb{Z}$, and the suspension flow $\{S_t\} = (T, f)$ constructed by $T$ and a function $f$ such that $f(x) = f_0(x_0)$, $x = (x_i, i \in \mathbb{Z})$, where $f_0 : V \to [c, \infty)$, $c > 0$. Let

$$F(\beta) = \sum_{v \in V} e^{-\beta f_0(v)}, \quad \beta \geq 0.$$

The next lemma (as well as the previous one) can be derived from Proposition 3.2 in [18] (see also [11]).

**Lemma 2.7.** If there exists $\beta_0 \geq 0$ with $F(\beta_0) = 1$, then $h_{\text{top}}(T, f) = \beta_0$. Otherwise $h_{\text{top}}(T, f) = \sup\{\beta \geq 0 : F(\beta) = \infty\}$.

**Proof.** Denote the family of all Bernoulli measures in $\mathcal{M}_{T,f}$ by $B_{T,f}$. Each $\nu \in B_{T,f}$ is determined by the one-dimensional distribution $\{p^\nu(v), v \in V\}$, where

$$p^\nu(v) = \nu(C_v) \geq 0, \quad \sum_{v \in V} p^\nu(v) = 1, \quad \sum_{v \in V} p^\nu(v) f_0(v) < \infty.$$
We note that

$$\sup_{\mu \in \mathcal{M}_{T,f}} \frac{h(T; \mu)}{\mu(f)} = \sup_{\nu \in B_{T,f}} \frac{h(T; \mu)}{\mu(f)}.$$  \tag{2.9}$$

Indeed, every $\mu \in \mathcal{M}_{T,f}$ gives rise to the measure $\mu_B \in B_{T,f}$ with $p^{\mu_B}(v) = \mu_B(C_v) = \mu(C_v)$. Clearly, $\mu_B(f) = \mu(f)$, and the basic properties of measure-theoretic entropy imply that $h(T; \mu) \leq h(T; \mu_B)$.

We number the vertices $v \in V$ in an arbitrary way and put

$$B^{(n)} = \{ \nu \in B_{T,f} : p^{\nu}(v_i) = 0, \ i \geq n + 1), \quad n \in \mathbb{N}. $$

For each $\mu \in B_{T,f}$ one can easily find a sequence of measures $\nu_n \in B^{(n)}$ such that

$$\lim_{n \to \infty} \frac{h(T; \nu_n)}{\nu_n(f)} = \frac{h(T; \mu)}{\mu(f)}.$$  

Hence, by (2.9),

$$\sup_{\mu \in \mathcal{M}_{T,f}} \frac{h(T; \mu)}{\mu(f)} = \sup_{n \in \mathbb{N}, \nu \in B^{(n)}} \frac{h(T; \nu)}{\nu(f)}. \tag{2.10}$$

We now notice that the relations

$$p_i := p^{\nu}(v_i), \quad 1 \leq i \leq n, \quad \mathbf{p} = p^{\nu} := (p_1, \ldots, p_n)$$

establish a one-to-one correspondence between $B^{(n)}$ and $\mathcal{P} = \mathcal{P}_N$ with $\mathcal{N} = \{1, \ldots, n\}$, and that $h(T; \nu)/\nu(f) = H_{\mathcal{X}, \mathcal{C}}(\mathbf{p}) = H(\mathbf{p})$, where $\mathbf{c} = (c_i, i \in \mathcal{N})$, $c_i = f_0(v_i), i \in \mathcal{N}$ (see (2.5) and (2.6)).

By Lemma 2.6 the right-hand side of (2.10) is $\sup_n \beta_n$, where $\beta_n$ is determined by $F_n(\beta_n) = 1$. Note that $F_n$ is the $n$th partial sum of the series for $F$ and that both $F_n$ and $F$ are strictly decreasing functions (for $F$ this is true on the half-axis where $F$ is finite). Hence $\sup_n \beta_n = \lim_{n \to \infty} \beta_n$. We consider two possibilities the first of which is that $F(\beta) = \infty$ for all $\beta \geq 0$. It is clear that in this case $\lim_{n \to \infty} \beta_n = \infty$. Otherwise there exists a unique $\beta_\infty > 0$ such that either $F(\beta_\infty) = 1$, or $F(\beta) < 1$ for $\beta \geq \beta_\infty$ and $F(\beta) = \infty$ for $\beta < \beta_\infty$. Since $F_n(\beta) < F_{n+1}(\beta) < F(\beta)$ for all $n \geq 1$ and $\beta \geq 0$, in both cases we have $\lim_{n \to \infty} \beta_n \leq \beta_\infty$. If $\lim_{n \to \infty} \beta_n =: \beta_\infty < \beta_\infty$, then $F(\beta_\infty) > 1$ (in the second case $F(\beta_\infty) = \infty$). Therefore $F_n(\beta_\infty) > 1$ for $n$ large enough. But $\beta_\infty > \beta_n$, hence $F_n(\beta_\infty) < F_n(\beta_n) < 1$ for all $n$. From this we see that $\lim_{n \to \infty} \beta_n = \beta_\infty$. Thus we have established both statements of the lemma.

2.3. Induced automorphisms and Markov-Bernoulli reduction. For the next lemma we recall the following definition. Let $T$ be an automorphism of the space $(X, \mathcal{B})$, and let $C \in \mathcal{B}$. Set

$$X_C = \left\{ x \in X : \sum_{n < 0} 1_C(T^n x) = \sum_{n \geq 0} 1_C(T^n x) = \infty \right\}, \quad C' = C \cap X_C. \tag{2.11}$$

Thus $C'$ consists of all points in $C$ that visit $C$ infinitely often in forward and backward time. Also let

$$\tau(T, C; x) = \min\{n > 0 : T^n x \in C\}, \quad T_{C'}x = T^{\tau(T, C; x)}x, \quad x \in C'. \tag{2.12}$$
It is clear that the sets $X_C$, $C'$ are measurable and invariant with respect to $T$ and $T_{C'}$, respectively, and that $T_{C'}$ is an automorphism of the set $C'$ provided with the induced Borel $\sigma$-algebra; $T_{C'}$ is said to be the induced automorphism on $C'$.

**Lemma 2.8.** Let $(T, f)$ be the suspension flow constructed by an automorphism $T$ of $(X, B)$ and a $B$-measurable function $f: X \to [c, \infty)$, $c > 0$, and let $C \in B$. Then the suspension flow $(T|_{X_C}, f|_{X_C})$, constructed by the restrictions of $T$ and $f$ to $X_C$ is isomorphic to the suspension flow $(T_{C'}, f_{C'})$, where

$$f_{C'}(x) = \sum_{i=0}^{\tau(T, C; x) - 1} f(T^i x), \quad x \in C'.$$

Furthermore, if $\mu \in \mathcal{M}_{T, f}$ is ergodic and such that $\mu(C) > 0$, then $\mu(f) = \int_{C'} f_{C'} d\mu$ and the suspension flow $(T, f; \mu_f)$ is isomorphic to the suspension flow $(T_{C'}, f_{C'}; (\mu_{C'})_{f_{C'}})$, where $\mu_f$ is the $f$-lifting of $\mu$, $\mu_{C'}$ is the normalized restriction of $\mu$ to $C'$ and $(\mu_{C'})_{f_{C'}}$ is the $f_{C'}$-lifting of $\mu_{C'}$.

We omit the proof of this lemma since it follows immediately from standard facts in ergodic theory (see, for instance, [19]).

The following construction, which is reminiscent of Doeblin’s first return method in the theory of Markov chains, has appeared repeatedly in the literature in different terms (presumably, it appeared first in [20], see also [18] and [7]).

Let $w = (v_1, \ldots, v_l) \in W(\Gamma)$ and let $C = C_w$. Then $X_C$ defined by (2.11) can be described as follows: $x \in X$ belongs to $X_C$ if and only if there is an increasing sequence of integers $i_k = i_k(x), -\infty < k < \infty$, such that $i_k \leq 0$ for $k \leq 0$, $i_k > 0$ for $k > 0$ and $(x_{i_k}, \ldots, x_{i_k+1-1}) = w$ for every $k$, while no other segment of $x$ agrees with $w$. Furthermore, $C'$ consists of those $x$ for which $i_0(x) = 0$. It is clear that

$$i_1(x) = \tau(T, C'; x), \quad i_k(x) \geq i_1(x) + k - 1, \quad x \in C'.$$

Denote by $A_w$ the set of all words $w' = (v'_1, \ldots, v'_l) \in W(\Gamma)$ with $l' > l$ such that $(v'_1, \ldots, v'_l) = (v'_{l-1+1}, \ldots, v'_l) = w$ and no other subword of $w'$ (that is, no word of the form $(v'_m, v'_{m+1}, \ldots, v'_n)$, $1 \leq m \leq n \leq l'$) agrees with $w$. It is easy to see that if $x \in X_C$, then for each $k \in \mathbb{Z}$ the word $(x_{i_k}, x_{i_k+1}, \ldots, x_{i_k+l-1})$ belongs to $A_w$. We thus obtain a mapping $\Psi_w: X_C \to (A_w)^\mathbb{Z}$ measurable with respect to the appropriate Borel $\sigma$-algebras; its restriction to $C'$ obviously induces a one-to-one correspondence between $C'$ and $(A_w)^\mathbb{Z}$. Moreover, if $x \in C'$, then

$$\Psi_w T_{C'} x = \sigma_w \Psi_w x,$$

where $\sigma_w$ is the shift transformation on $Y_w := (A_w)^\mathbb{Z}$, that is, $\sigma_w y_i = y_{i+1}$, $y = (y_i, i \in \mathbb{Z}) \in Y_w$. Therefore $T_{C'}$ is isomorphic to the countable alphabet Bernoulli shift $(Y_w, \sigma_w)$ with alphabet $A_w$. Here and below we consider each $a \in A_w$ either as a word in the alphabet $V$ or as a letter in the new alphabet $A_w$. Which case we are in will always be clear from the context.

This construction in essence reduces the study of the topological Markov shift $(X, T)$ to that of a topological Bernoulli shift determined by $w$, and so we refer to it as the **Markov-Bernoulli reduction** applied to $(X, T)$ and $w$.

### 2.4. Positive measures

Our next aim is to show that the topological entropy of a suspension flow over a Markov shift can be computed using only ergodic measures that are positive on all cylinders.
Lemma 2.9. Let \((X, T)\) and \(f\) be as in Theorem 2.2 and let \((T, f)\) be the suspension flow constructed by \(T\) and \(f\). Then

\[
h_{\text{top}}(T; f) = \sup_{\mu \in \mathcal{E}_{T, f}^+} h(T; f; \mu),
\]

where \(\mathcal{E}_{T, f}^+\) consists of all ergodic measures in \(\mathcal{M}_{T, f}\) that are positive on all cylinders in \(X\).

Proof. Denote by \(\mathcal{E}_{T, f}\) the set of all ergodic measures in \(\mathcal{M}_{T, f}\). If \(\mu \in \mathcal{M}_{T, f} \setminus \mathcal{E}_{T, f}\), that is, if \(\mu\) is nonergodic with respect to \(T\), then \(\mu_f\), the \(f\)-lifting of \(\mu\), is nonergodic with respect to the suspension flow \((T, f)\). The flow \((T, f; \mu_f)\) can be decomposed into ergodic components (see [21]). This means the following. There exists a measurable partition \(\zeta\) of the space \((X_f, \mu_f)\) such that \(\mu_f\)-almost every element \(C_\zeta\) of \(\zeta\) is \((T, f)\)-invariant and the conditional measure \((\mu_f)^C_\zeta\) induced by \(\mu\) on \(C_\zeta\) is invariant and ergodic with respect to the restriction of \((T, f)\) to \(C_\zeta\). We may consider \((\mu_f)^C_\zeta\) as a measure on the whole of \(X_f\); it is \((T, f)\)-invariant and ergodic.

By a general formula (see [22], §9)

\[
h(T, f; \mu_f) = \int_{X_f|\zeta} h(T, f; (\mu_f)^C_\zeta) \mu_f,\zeta(dC_\zeta),
\]

where \(\mu_f,\zeta\) is the projection of \(\mu_f\) onto the quotient space \(X_f|\zeta\). Hence for every \(\varepsilon > 0\) there is an element \(C_\zeta\) with

\[
h(T, f; (\mu_f)^C_\zeta) > h(T, f; \mu_f) - \varepsilon.
\]

On the other hand, as \((\mu_f)^C_\zeta\) is a \((T, f)\)-invariant probability measure on \(X_f\), it is the \(f\)-lifting of a \(T\)-invariant probability measure \(\mu^C_\zeta\) on \(X\), that is, \((\mu_f)^C_\zeta = (\mu^C_\zeta)_f\). It is clear that \(\mu^C_\zeta(f) < \infty\) and \((T, \mu^C_\zeta)\) is ergodic. Since \(\varepsilon > 0\) was arbitrarily small, we conclude that

\[
h_{\text{top}}(T; f) = \sup_{\mu \in \mathcal{E}_{T, f}} h(T; f; \mu_f).
\]

Let \(\mathcal{E}_{T, f}^0 := \mathcal{E}_{T, f} \setminus \mathcal{E}_{T, f}^+\) and assume that, contrary to the lemma we are proving, for some \(\delta \in (0, \infty)\),

\[
\sup_{\mu \in \mathcal{E}_{T, f}^0} \frac{h(T; \mu)}{\mu(f)} > \sup_{\mu \in \mathcal{E}_{T, f}^+} \frac{h(T; \mu)}{\mu(f)} + \delta,
\]

which in particular means that

\[
\sup_{\mu \in \mathcal{E}_{T, f}^+} \frac{h(T; \mu)}{\mu(f)} < \infty.
\]

By virtue of (2.14) there is a measure \(\mu^0 \in \mathcal{E}_{T, f}^0\) such that

\[
\frac{h(T; \mu^0)}{\mu^0(f)} \geq \sup_{\mu \in \mathcal{E}_{T, f}^+} \frac{h(T; \mu)}{\mu(f)} + \frac{\delta}{2}.
\]
To show that this is impossible we first consider the case \( h(T; \mu^0) < \infty \) and let 
\[
  h^0 = h(T; \mu^0)/\mu^0(f) .
\]
Since \( f \) has summable variations, one can find \( n_\delta \in \mathbb{N} \) such that, for every \( n \geq n_\delta \), there is a function \( f_n : X \to \mathbb{R}_+ \) with the following three properties: \( f_n(x) = f_n(y) \) whenever \( x_i = y_i \) for \( |i| \leq n \); \( \inf_{x \in X} f_n(x) \geq c \), and 
\[
  \sup_{x \in X} |f(x) - f_n(x)| \leq \frac{\delta c^2}{8h^0} .
\]

One can easily check that 
\[
  \left| \frac{h(T; \mu)}{\mu(f_n)} - \frac{h(T; \mu)}{\mu(f)} \right| < \delta
\]
for every \( \mu \in \mathcal{E}_{T,f}^+ \cup \{ \mu^0 \} \). Hence (see (2.15)) 
\[
  \frac{h(T; \mu^0)}{\mu^0(f_n)} \geq \sup_{\mu \in \mathcal{E}_{T,f}^+, f_n} \frac{h(T; \mu)}{\mu(f_n)} + \frac{\delta}{4}. 
\]

Since \( |f - f_n| < \text{const} \), either both the functions \( f \) and \( f_n \) are integrable with respect to a finite measure or neither is. Hence \( \mathcal{E}_{T,f} = \mathcal{E}_{T,f}^+ \) and \( \mathcal{E}_{T,f}^+ = \mathcal{E}_{T,f}^+ \).

If \( h(T; \mu^0) = \infty \), then (2.17) also clearly holds as well.

Using the assumption that \( \mu^0 \in \mathcal{E}_{T,f}^+ \) we find a word \( w^0 \in W(\Gamma) \) with \( \mu^0(C_{w^0})=0 \). Fix an arbitrary \( n^1 \geq \max\{n_\delta, |w^0|\} \) and a word \( w^1 \in W(\Gamma) \) with \( |w^1| = n^1 \), \( \mu^0(w^1) > 0 \). Then we set \( f^1 := f_{n^1}, C := C_{w^1} \) and apply the Markov-Bernoulli reduction to \( (X,T) \) and \( w^1 \). By Lemma 2.8 the suspension flow \( (T|_{X_{C^\prime}}; f^1|_{X_{C^\prime}}) \) is isomorphic to the suspension flow \( (\sigma, \varphi) := (\sigma_{w^1}, \varphi_{f^1, w^1}) \), where 
\[
  \varphi(y) := (f^1)_{C^\prime}(\Psi_{w^1}^{-1}(y), \quad y \in Y_{w^1}. \tag{2.18}
\]

Notice that the function \( \varphi \) is constant on every one-dimensional cylinder \( \{y \in Y : y_0 = a\}, \quad a \in A_{w^1} \); the reason is that each \( a \in A_{w^1} \) when regarded as a word from \( W(\Gamma) \) is not shorter than \( w^1 \).

Let us carry the measure \( \mu^0_{C^\prime} \), the normalized restriction of \( \mu^0 \) to \( C^\prime \) (where \( C^\prime \) is defined in (2.11)), over to \( Y \) via the mapping \( \Psi_{w^1} \) to obtain a Borel probability measure \( \nu^0 \) on \( Y \). From the above properties of \( \Psi_{w^1} \) it follows that the automorphisms \( (T_{C^\prime}; (\mu^0_{C^\prime})_{C^\prime}) \) and \( (\sigma; \nu^0) \) are isomorphic and hence, by Lemma 2.8, the suspension flow \( (T, f^1; (\mu^0_{f^1})_{f^1}) \) is isomorphic to the suspension flow \( (\sigma, \varphi; (\nu^0)_{\varphi}) \), where \( (\mu^0_{f^1})_{f^1} \) and \( (\nu^0)_{\varphi} \) are the \( f^1 \)-lifting of the measure \( \mu^0 \) and the \( \varphi \)-lifting of the measure \( \nu^0 \). Therefore, by (2.1) 
\[
  \frac{h(T; \mu^0)}{\mu^0(f^1)} = \frac{h(\sigma; \nu^0)}{\nu^0(\varphi)}. \tag{2.19}
\]

If we change \( \nu^0 \) for a \( \sigma \)-invariant Borel measure \( \nu^1 \) with the same one-dimensional distribution (that is, with \( \nu^1(C_a) = \nu^0(C_a) \) for all \( a \in A_{w^1} \), where \( C_a = \{y \in Y : y_0 = a\} \)), then the numerator on the right-hand side of (2.19) can only increase, while the denominator will not change (since \( \varphi \) is constant on every cylinder \( C_a, \quad a \in A_{w^1} \)).
It follows from the definition of \( \nu^0 \) and \( \nu^1 \) that \( \nu^0(C_{a^0}) = \nu^1(C_{a^0}) = 0 \) for some \( a^0 \in A_{w^1} \). Indeed, let \( w^1 = (v_1^1, \ldots, v_{l_1}^1) \). Since the graph \( \Gamma \) is connected, there exists a word \( (v_1, \ldots, v_r) \in W(\Gamma) \) with \( (v_1, \ldots, v_{l_1}) = w^1, (v_{r-1}^1, \ldots, v_r) = w^0, \) where \( l_0 = |w^0| \). Choose an arbitrary shortest word of this type and denote it by \( w' \). Similarly, let \( w'' \) be one of the shortest words in which there are an initial subword and a terminal subword that coincide with \( w^0 \) and \( w^1 \), respectively.

From the assumptions \( \mu^0(w'') = 0, \mu^0(w^1) > 0, |w^0| \leq |w^1| \) it follows that \( w'' = w^0 \hat{w} \), where \( \hat{w} \) can have one of the following three forms:

(a) \( \hat{w} = w^1; \)
(b) \( \hat{w} = \hat{w}^1 w^1, \hat{w}^1 \in W(\Gamma); \)
(c) \( \hat{w} = (v_{l_1}^1, \ldots, v_{l_0}^1), \) where \( 1 < k \leq l_1 \), is a terminal subword of \( w^1 \).

Consider the word \( w' \hat{w} \). One easily checks that \( w' \hat{w} \in A_{w^1} \). Moreover, \( \mu^0(w' \hat{w}) = 0 \) because \( w' \hat{w} \) contains \( w^0 \) as a subword. Hence

\[
\mu_{C_{w^1}}^0(C_{w^1} \cap C_{w' \hat{w}}) = 0.
\]

We can put \( a^0 := w' \hat{w} \). Since \( \Psi_{w^1}(C' \cap C_{w' \hat{w}}) = C_{a^0} \), we have \( \nu^1(C_{a^0}) = \nu^0(C_{a^0}) = 0 \).

We now want to perturb \( \nu^1 \) within the class of Bernoulli measures on \( Y \) so as to obtain a measure for which the right-hand side of (2.19) is bigger than for \( \nu^0 \) and which is positive on all cylinders.

Since \( \varphi(y), y = (y_i; i \in \mathbb{Z}) \in Y \), depends solely on \( y_0 \), we have \( \varphi(y) = \varphi_0(y_0) \), where \( \varphi_0 \) is a function on \( A_{w^1} \).

Using Lemma 2.5 we find a \( \sigma \)-invariant Bernoulli measure \( \nu^2 \) on \( Y \) such that if \( h(\sigma; \nu^1) < \infty \), then

\[
\frac{h(\sigma; \nu^2)}{\nu^2(\varphi)} > \frac{h(\sigma; \nu^1)}{\nu^1(\varphi)} \geq \frac{h(\sigma; \nu^0)}{\nu^0(\varphi)},
\]

and if \( h(\sigma; \nu^1) = \infty \), then \( h(\sigma; \nu^2) = \infty \) as well.

We apply the mapping \( \Psi_{w^1}^{-1} \) to transfer the measure \( \nu^2 \) to \( C' \) and denote the resulting measure by \( \mu' \). Then the suspension flow \( (\sigma, \varphi; (\nu^2)(\varphi)) \) will be isomorphic to the suspension flow \( (T_{C'}, (f^1)'_{C'}; (\mu')_{f^1}_{C'}) \). Let \( C'(n) = \{ x \in C': \tau_{T,C}(x) = n \} \), \( n = 1, 2, \ldots \), and \( \mu'_n = \mu'|_{C'(n)} \) be the restriction of \( \mu' \) to \( C'(n) \) considered as a measure on \( X \). Then the measure

\[
\mu'' := \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} T^k \circ \mu'_n
\]

is concentrated on \( X_C \) and is \( T \)-invariant. By normalizing \( \mu'' \), we obtain a probability measure \( \mu''' \). By Lemma 2.8 the flows \( (T|_{X_C}, f^1|_{X_C}) \) and \( (T_{C'}, (f^1)'_{C'}; (\mu')_{f^1}_{C'}) \) are isomorphic. Then the flow \( (T, f^1; (\mu''')_{f^1}) \) is isomorphic to the flow \( (T_{C'}, (f^1)'_{C'}; (\mu')_{f^1}_{C'}) \) and hence (see above) to the flow \( (\sigma, \varphi; (\nu^2)(\varphi)) \). Therefore

\[
\frac{h(T; \mu''')}{\mu'''(f^1)} = \frac{h(\sigma; \nu^2)}{\nu^2(\varphi)} > \frac{h(T; \mu^0)}{\mu^0(f^1)}
\]

(see (2.19), (2.20)). It is clear that \( \mu''' \in \mathbb{S}_{T,f}^+ \). Moreover, \( \mu''' \in \mathbb{S}_{T,f}^+ \). Otherwise we could apply the procedure that leads us from \( \mu^0 \) to \( \nu^0 \) to \( \mu''' \). The resulting measure would coincide with \( \nu^{t,s} \) and there would be a letter \( a \in A_{w^1} \) with \( \nu^{t,s}(C_a) = 0 \).
But we know that this is impossible. Thus (2.21) contradicts (2.17) with \( f_n = f^1 \) and hence contradicts (2.14).

**Corollary 2.10.** Let \( \Gamma \), \((X, T)\) and \( f \) be as in Theorem 2.2, let \((Y_w, \sigma_w)\) be the topological Bernoulli shift obtained by applying the Markov-Bernoulli reduction to \((X, T)\) and some \( w \in W(\Gamma) \), and let \( \varphi_{f,w} \) be the function defined in (2.18). Then the suspension flows \((T, f)\) and \((\sigma_w, \varphi_{f,w})\) have the same topological entropy.

**Proof.** As before, we let \( C = C_w \) and use the notation in (2.11)–(2.13). From the definition of \( \sigma_w \) and \( \varphi_{f,w} \) it follows immediately that the suspension flows \((TC', fC')\) and \((\sigma_w, \varphi_{f,w})\) are isomorphic and hence \( h_{\text{top}}(TC', fC') = h_{\text{top}}(\sigma_w, \varphi_{f,w}) \).

Similarly, by virtue of Lemma 2.8, \( h_{\text{top}}(TC', fC') = h_{\text{top}}(T|_{XC'}, f|_{XC'}) \). But \( h_{\text{top}}(T|_{XC'}, f|_{XC'}) \leq h_{\text{top}}(T, f) \) because \( XC \) is a \( T \)-invariant subset of \( X \). Hence

\[
h_{\text{top}}(\sigma_w, \varphi_{f,w}) = (TC', fC') \leq h_{\text{top}}(T, f).
\]

On the other hand, again by Lemma 2.8, for every \( \mu \in \mathcal{E}_{T,f}^+ \) we have

\[
h(TC', fC'; \tilde{\mu}_{C'}) = h(T, f; \tilde{\mu}),
\]

where \( \tilde{\mu}_{C'} \) is the \( f_{C'} \)-lifting of the normalized restriction of \( \mu \) to \( C' \), and \( \tilde{\mu} \) is the \( f \)-lifting of \( \mu \). The supremum in \( \mu \in \mathcal{E}_{T,f}^+ \) of the left-hand side of the last equality is clearly not bigger than \( h(TC', fC') \), while by Lemma 2.9 the supremum of the right-hand side is \( h_{\text{top}}(T, f) \). Hence

\[
h_{\text{top}}(TC', fC') \geq h_{\text{top}}(T, f),
\]

which together with (2.22) yields what we are proving.

**2.5. Proof of Theorem 2.2.** Fix the notation as in Theorem 2.2. We shall also fix \( n \) and, for a while, write \( w \) and \( C \) instead of \( w_n \) and \( C_{w_n} \), respectively. Consider the sets \( X_C, C' \), the induced transformation \( T_{C'} : C' \rightarrow C' \) and the function \( f_{C'} \) (see (2.11)–(2.13)). Apply the Markov-Bernoulli reduction to \((X, T)\) and \( w \).

From (2.3) and (2.13) it is seen that \( \tau(T, C; x) = f_{C'}(x) \) for every \( x \in C' \). Then (2.4) can be rewritten in the form

\[
\frac{\overline{\mu}(C_{w\tilde{w}w})}{\overline{\mu}(C)} - e^{-sf_{C'}(x)} \leq e^{-\alpha|w| - sf_{C'}(x)},
\]

which is true for \( \overline{\mu} \)-almost all \( x \in C_{w\tilde{w}w} \).

As \( w \) is simple each word \( a \in A_w \) is of the form \( a = w\tilde{w}w \), where \( \tilde{w} \) is in \( W(\Gamma) \) (\( \tilde{w} \) may be an empty word if \( ww \in W(\Gamma) \)) and \( \tilde{w} \) does not contain \( w \) as a subword.

By assumption the measure \( \overline{\mu} \) is positive on all cylinders and \( T \)-invariant. Hence \( \overline{\mu}(C) = \overline{\mu}(C') > 0 \), and we can normalize \( \overline{\mu} \) on \( C' \) to obtain a \( T_{C'} \)-invariant probability measure \( \overline{\mu}' \). Its push forward measure \( \nu' := (\Psi_w)_*\mu'_0 \) is a probability measure on \( Y = Y_w \) invariant with respect to the shift transformation \( \sigma \). From the definition of \( \Psi_w \) it follows that for \( a := w\tilde{w}w \in A_w \),

\[
C_a := \{ y \in Y : y_0 = a \} = \Psi_w C_{w\tilde{w}w}
\]

and hence

\[
\nu'(C_a) = \frac{\overline{\mu}(C_{w\tilde{w}w})}{\overline{\mu}(C_w)}.
\]

Notice that \( \nu'(C_a) > 0 \) for all \( a \in A_w \).
Taking the relation between $f_{C'}$ and $\varphi = \varphi_{f,w}$ into account (see (2.18)) and using (2.23), (2.24) we obtain
\[ |\nu'(C_a) - e^{-s\varphi(y)}| \leq e^{-\alpha|w| - s\varphi(y)}, \quad a \in A_w, \] (2.25)
for $\nu'$-almost all $y \in C_a$.

Our next step is to approximate $\varphi$ by a function that is constant on each cylinder $C_a$, $a \in A_w$.

Since, by assumption, $w$ is simple, we have $\tau(T, C'; x) \geq |w|$ for all $x \in C'$ (see (2.12)). We say that $x$ are equivalent ($\tau \geq 0$) if $\tau(T, C'; x^1) = \tau(T, C'; x^2)$ and $x^{(1)} = x^{(2)}$ for $0 \leq i \leq \tau(T, C'; x^{(1)})$. If $x^{(1)} \sim x^{(2)}$, then (because $w$ is simple) $x_i^{(1)} = x_i^{(2)}$ for $\tau(T, C'; x^{(1)}) \leq i \leq \tau(T, C'; x^{(1)}) + |w| - 1$ as well, from which we obtain (see (2.13))
\[ |f_{C'}(x^{(1)}) - f_{C'}(x^{(2)})| \leq \sum_{i=0}^{\tau(T, C'; x) - 1} |f(T^ix^{(1)}) - f(T^ix^{(2)})| \leq \sum_{n=|w|}^{\infty} \varn(f). \] (2.26)

Let
\[ C^w(x) := \{ x' \in C' : x' \sim x \}, \quad f^w(x) = \inf_{x' \in C^w(x)} f_{C'}(x'). \] (2.27)

It is easy to see that $C^w(x)$ is a cylinder and that these cylinders constitute a partition of $C'$. Moreover, by virtue of (2.27), (2.26) the function $f^w$ is constant on each element of this partition and
\[ 0 \leq f_{C'}(x) - f^w(x) \leq \sum_{n=|w|}^{\infty} \varn(f), \quad x \in C'. \]

Therefore
\[ 0 \leq \varphi(y) - \varphi^w(y) \leq \sum_{n=|w|}^{\infty} \varn(f), \quad y \in Y, \] (2.28)

where $\varphi^w(y) := f^w(\Psi_w^{-1}y)$ is constant on each cylinder $C_a \subset Y$, $a \in A_w$ (here by $\Psi_w^{-1}y$ we mean the unique point $x \in C'$ such that $\Psi_w x = y$) and hence there is a function $\varphi^w_0$ on $A_w$ such that $\varphi^w(y) = \varphi^w_0(y_0)$.

With Lemma 2.7 in mind we will estimate the sum $\sum_{a \in A_w} \exp[-s\varphi^w_0(a)]$. Let
\[ \delta_w := \sum_{n=|w|}^{\infty} \varn(f). \] (2.29)

Since $\nu'(C_a) > 0$ for all $a \in A_w$, for every $a$ we can choose a point $y_a \in C_a$ such that (2.25) holds for $y = y_a$. Hence
\[ \nu'(C_a) - \exp[-\alpha|w| - s\varphi(y_a)] \leq e^{-s\varphi(y_a)} \leq \nu'(C_a) + \exp[-\alpha|w| - s\varphi(y_a)], \]
so that
\[
\frac{\nu'(C_a)}{1 + e^{-\alpha|w|}} \leq e^{-s\varphi(y_u)} \leq \frac{\nu'(C_a)}{1 - e^{-\alpha|w|}}, \quad a \in A_w,
\]
\[
\frac{1}{1 + e^{-\alpha|w|}} \leq \sum_{a \in A_w} e^{-s\varphi(y_u)} \leq \frac{1}{1 - e^{-\alpha|w|}}.
\]

From (2.28), (2.29) we obtain
\[
\frac{1}{1 + e^{-\alpha|w|}} \leq \sum_{a \in A_w} e^{-s\varphi(y_u)} \leq \sum_{a \in A_w} e^{-s\varphi^w_0(a)} = \sum_{a \in A_w} e^{-s\varphi^w(y_u)}
\]
\[
\leq \frac{1}{1 - e^{-\alpha|w|}} + \sum_{a \in A_w} [e^{-s\varphi^w(y_u)} - e^{-s\varphi(y_u)}]
\]
\[
= \frac{1}{1 - e^{-\alpha|w|}} + \sum_{a \in A_w} e^{-s\varphi(y_u)}[e^{s(\varphi(y_u) - \varphi^w(y_u))} - 1]
\]
\[
\leq \frac{e^{s\delta_w}}{1 - e^{-\alpha|w|}}. \tag{2.30}
\]

By assumption we can take \(w = w_n\,\) where \(|w_n| \to \infty\) as \(n \to \infty\). From (2.30) it follows that
\[
\lim_{n \to \infty} \sum_{a \in A_{w_n}} \exp[-s\varphi^w_0(a)] = 1. \tag{2.31}
\]

Let
\[
F_n(u) := \sum_{a \in A_{w_n}} \exp[-u\varphi^w_0(a)], \quad n = 1, 2, \ldots.
\]

If, for a fixed \(n\), there is a \(u \in \mathbb{R}\) such that \(F_n(u) = 1\) (there can be only one such \(u\)), then we denote it by \(u_n\). Otherwise we put \(u_n := \sup\{u : F_n(u) = \infty\}. \) Notice that \(u_n \geq 0\) (because \(F_n(0) = \infty\)) and \(u_n < \infty\) (because of (2.30)). From the definition of \(\varphi^w_n\) it follows that \(\inf_{y \in Y} \varphi^w_n(y) \to \infty\) as \(n \to \infty\) (recall that \(Y = Y_{w_n}\)). Therefore, for every \(\gamma > 0\) we have \(\lim_{n \to \infty} (dF_n(u)/du) = -\infty\) uniformly in \(u\) on the set \(D_\gamma := \{u : \gamma < F_n(u) < \infty\}\) (we mean the right-hand derivative if \(u\) is the left-hand endpoint of the interval \(D_\gamma\)). Using this fact, it is easy to deduce from (2.31) that \(u_n \to s\) as \(n \to \infty\) (it would be sufficient to know that \(dF_n(u)/du < \text{const} < 0\) on \(D_\gamma\)).

Let us now consider the isomorphic suspension flows \((\sigma, \varphi^w)\) and \((T_{C'}, f^w)\).

By Lemma 2.7
\[
t_n = h_{\text{top}}(\sigma, \varphi^w) = h_{\text{top}}(T_{C'}, f^w), \quad n = 1, 2, \ldots,
\]
where \(C' = (C_w)'\), and hence
\[
s = \lim_{n \to \infty} h_{\text{top}}(T_{C'}, f^w).
\]

From (2.28) and the bounds \(h_{\text{top}}(\sigma, \varphi^w) \leq 2s\) (for large enough \(n\), \(\inf \varphi \geq c \) and \(\inf \varphi^w_n \geq c\)) which are obvious we obtain
\[
|h_{\text{top}}(\sigma, \varphi^w) - h_{\text{top}}(\sigma, \varphi)| \leq \frac{2s\delta_n}{c},
\]
where $\delta_n = \sum_{k=|w_n|} \var_n(f)$ (see (2.16)). Therefore $s = h_{\text{top}}(\sigma, \varphi)$ and hence (see Corollary 2.10), $s = h_{\text{top}}(T, f)$. So statement (i) is proved.

To prove statement (ii) suppose that $s = h_{\text{top}}(T, f; (\overline{\mu})_f)$. Together with (i) this means that $(\overline{\mu})_f$ is a measure with maximal entropy for the suspension flow $(T, f)$. We will show that $(T, f)$ can have only one measure with maximal entropy.

Let $(\overline{\mu})_f$ be such a measure. Then by (2.2)

$$
\frac{h(T, \mu)}{\mu(f)} \leq \frac{h(T, \overline{\mu})}{\overline{\mu}(f)} = s \quad \text{for all } \mu \in \mathcal{M}_{T, f},
$$

where $s = h_{\text{top}}(T, f)$. Hence for every $\mu \in \mathcal{M}_{T, f}$ we have $h(T, \mu) + \mu(g) \leq 0$, where $g(x) := -sf(x)$, $x \in X$, while $h(T, \overline{\mu}) + \overline{\mu}(g) = 0$, so that the topological pressure for $g$ is zero and $\overline{\mu}$ is a $g$-equilibrium measure. Using the natural projection $\pi: V^Z \to V^{Z+}$ we let $X_+ = \pi X$ and $f_+(x_+) = f(x)$ for every $x_+ \in X^+$ and every $x \in \pi^{-1}x_+$ (by assumption $f$ is constant on the set $\pi^{-1}x_+$, so that $f(x)$ depends only on $x_+$). It is easy to check that $\pi T x = T_+ \pi x$, $x \in X$, where $T_+$ is the shift transformation on $X_+$, and moreover, that $\pi$ induces a one-to-one correspondence between $\mathcal{M}_{T, f}$ and $\mathcal{M}_{T_+, f_+}$, the set of $T_+$-invariant probability measures $\mu_+$ on $X_+$ with $\mu_+(f_+) < \infty$. Let $\overline{\mu}_+ \in \mathcal{M}_{T_+, f_+}$ correspond to $\overline{\mu}$. Then $\overline{\mu}_+$ is a $g_+$-equilibrium measure, where $g_+ = -sf_+$. Note that the one-sided Markov shift $T_+$ is topologically transitive (because the graph $\Gamma$ is connected), the topological pressure of $g_+$ is zero (because this is the case for $g$) and $\inf_{x_+ \in X_+} g_+(x_+) < 0$ (because $\inf_{x \in X} f(x) > 0$). Hence by Theorem 1.1 in [12] there can only be one $g_+$-equilibrium measure. So the proof of Theorem 2.2 is complete.

§ 3. The covering flow

Now we will start to deduce Theorem 1.1 from Theorem 2.2. The aim of the present section is to recall the construction of a flow that can be viewed as ‘covering’ for the Teichmüller flow $\{g_t\}$ (see § 5). We will show (see Corollary 5.2) that our problem can be reduced to a similar problem for this covering flow (denoted by $\{P_t\}$).

We first recall some constructions due to Rauzy [23], Veech [2] and Zorich [24] (see also [25]). Using these constructions, in § 3.2 we represent the covering flow as a suspension flow over a measurable transformation defined on a bounded Borel set in a finite dimensional Euclidian space. Next, in § 4, we consider a symbolic representation of the flow $\{P_t\}$ and show that, up to an isomorphism, it is a suspension flow over a countable alphabet topological Markov shift (we denote the alphabet by $\mathcal{A}$). Theorem 2.2 cannot yet be applied directly to this suspension flow, since its roof function is not bounded away from zero and has nonsummable variations. For this reason we change the base (Poincaré section) of our flow to an appropriate cylindrical subset of the base (we in fact use a family of cylinders) and thus we go over to a new suspension representation (that goes back to Veech). The new suspension flow appears to be defined over a countable alphabet Bernoulli shift whichever cylinder set we take (this is nothing but a Markov-Bernoulli reduction as defined in § 2.3). Not all cylinders can be used for this, only those corresponding to admissible ‘positive’ words $w \in \bigcup_{n=1}^{\infty} \mathcal{A}^n$ (there is a canonical way to assign a matrix with nonnegative integer entries to each $w$; if all the entries are positive, we refer to $w$ as a positive word). If we change positivity to a stronger requirement
that each word \( w \) involved in the construction has a simple positive prefix (the definition is given in § 2.1), we can prove (see §§ 4.3 and 4.4) that the roof function has summable variations (it is even Hölder continuous) and the measure \( \mu_\kappa \) (see § 1) induces an invariant measure on the base of our suspension flow which satisfies the requirements imposed on the measure \( \bar{\mu} \) in Theorem 2.2. It remains to note that for each \( \{ P^t \} \)-invariant ergodic probability measure \( \nu \) with positive entropy there is a sufficiently large collection of words \( w \) with \( \nu(w) > 0 \) each having a simple positive prefix (see Lemma 5.5). We thus have everything we need to apply Theorem 2.2.

### 3.1. Induction maps.

Let \( \pi \) be a permutation of \( m \) symbols, which will always be assumed to be irreducible in the sense that \( \pi\{1,\ldots,k\} = \{1,\ldots,k\} \) implies \( k = m \). The Rauzy operations \( a \) and \( b \) are defined by the formulae

\[
a\pi(j) = \begin{cases} 
\pi j & \text{if } j \leq \pi^{-1}m, \\
\pi m & \text{if } j = \pi^{-1}m + 1, \\
(\pi(j - 1) & \text{if } \pi^{-1}m + 1 < j \leq m,
\end{cases}
\]

\[
b\pi(j) = \begin{cases} 
\pi j & \text{if } \pi j \leq \pi m, \\
\pi j + 1 & \text{if } \pi m < \pi j < m, \\
\pi m + 1 & \text{if } \pi j = m.
\end{cases}
\]

These operations preserve irreducibility. The Rauzy class \( \mathcal{R}(\pi) \) is defined as the set of all permutations that can be obtained from \( \pi \) by application of the transformation group generated by \( a \) and \( b \). From now on we fix a Rauzy class \( \mathcal{R} \) and assume that it consists of irreducible permutations.

For \( i, j = 1,\ldots,m \) denote by \( E_{ij} \) the \( m \times m \)-matrix whose \((i, j)\)th entry is 1, while all the others are zeros. Let \( E \) be the identity \( m \times m \)-matrix. Following Veech [2], introduce the unimodular matrices

\[
A(a, \pi) = \sum_{i=1}^{\pi^{-1}m} E_{ii} + E_{m,\pi^{-1}m+1} + \sum_{i=\pi^{-1}m}^{m-1} E_{i,i+1},
\]

\[
A(b, \pi) = E + E_{m,\pi^{-1}m}.
\]

For a vector \( \lambda = (\lambda_1,\ldots,\lambda_m) \in \mathbb{R}^m \) we write

\[
|\lambda| = \sum_{i=1}^{m} |\lambda_i|.
\]

Let

\[
\Delta_{m-1} = \{ \lambda \in \mathbb{R}^m : |\lambda| = 1, \lambda_i > 0 \text{ for } i = 1,\ldots,m \}.
\]

One can identify each pair \((\lambda, \pi), \lambda \in \Delta_{m-1} \) with an interval exchange map of the interval \( I := [0,1) \) as follows. Divide \( I \) into subintervals \( I_k := [\beta_{k-1},\beta_k) \), where \( \beta_0 = 0, \beta_k = \sum_{i=1}^{k} \lambda_i, 1 \leq k \leq m \), and then place the intervals \( I_k \) in \( I \) in the following order (from left to right): \( I_{\pi^{-1}1},\ldots,I_{\pi^{-1}m} \). We obtain a piecewise linear transformation of \( I \) that preserves the Lebesgue measure.

The space \( \Delta(\mathcal{R}) \) of interval exchange maps corresponding to \( \mathcal{R} \) is defined by

\[
\Delta(\mathcal{R}) = \Delta_{m-1} \times \mathcal{R}.
\]
Set
\[
\Delta^+_\lambda = \{ \lambda \in \Delta_{m-1} \mid \lambda_{\pi^{-1} m} > \lambda_m \}, \quad \Delta^-_\lambda = \{ \lambda \in \Delta_{m-1} \mid \lambda_m > \lambda_{\pi^{-1} m} \},
\]
\[
\Delta^+(\mathcal{R}) = \bigcup_{\pi \in \mathcal{R}} \{ (\pi, \lambda) \mid \lambda \in \Delta^+_\lambda \}, \quad \Delta^-(\mathcal{R}) = \bigcup_{\pi \in \mathcal{R}} \{ (\pi, \lambda) \mid \lambda \in \Delta^-_\lambda \},
\]
\[
\Delta^{\pm}(\mathcal{R}) = \Delta^+(\mathcal{R}) \cup \Delta^-(\mathcal{R}).
\]

The Rauzy-Veech induction map \( \mathcal{T} : \Delta^{\pm}(\mathcal{R}) \to \Delta(\mathcal{R}) \) is defined as follows:
\[
\mathcal{T}(\lambda, \pi) = \begin{cases} 
\left( \frac{A(a, \pi)^{-1} \lambda}{|A(a, \pi)^{-1}|}, a\pi \right) & \text{if } \lambda \in \Delta^+_\lambda, \\
\left( \frac{A(b, \pi)^{-1} \lambda}{|A(b, \pi)^{-1}|}, b\pi \right) & \text{if } \lambda \in \Delta^-_\lambda.
\end{cases}
\]

One can check that \( \mathcal{T}(\lambda, \pi) \) is the interval exchange map induced by the pair \((\lambda, \pi)\) on the interval \([0, 1 - \gamma)\), where \(\gamma = \min(\lambda_m, \lambda_{\pi^{-1} m})\); this interval stretches to unit length.

Set
\[
\Delta^\infty(\mathcal{R}) = \bigcap_{n \geq 0} \mathcal{T}^{-n} \Delta^{\pm}(\mathcal{R}).
\]

Every \( \mathcal{T} \)-invariant probability measure is concentrated on \( \Delta^\infty(\mathcal{R}) \). On the other hand, a natural Lebesgue measure defined on \( \Delta(\mathcal{R}) \), which is finite, but not invariant, is also concentrated on \( \Delta^\infty(\mathcal{R}) \). Veech [2] showed that \( \mathcal{T} \) has an absolutely continuous ergodic invariant measure on \( \Delta(\mathcal{R}) \), which is, however, infinite.

Following Zorich [24], for \((\lambda, \pi) \in \Delta^\infty(\mathcal{R})\) we set
\[
n(\lambda, \pi) = \begin{cases} 
\min\{k > 0 : \mathcal{T}^k(\lambda, \pi) \in \Delta^-(\mathcal{R})\} & \text{if } \lambda \in \Delta^+_\lambda, \\
\min\{k > 0 : \mathcal{T}^k(\lambda, \pi) \in \Delta^+(\mathcal{R})\} & \text{if } \lambda \in \Delta^-_\lambda.
\end{cases}
\]

The Rauzy-Veech-Zorich induction map \( \mathcal{G} \) is defined by the formula
\[
\mathcal{G}(\lambda, \pi) = \mathcal{T}^{n(\lambda, \pi)}(\lambda, \pi), \quad (\lambda, \pi) \in \Delta^\infty(\mathcal{R}).
\]

**Theorem 3.1** (see [24]). The map \( \mathcal{G} \) has an ergodic invariant probability measure \( \nu \) which is absolutely continuous with respect to the Lebesgue measure on \( \Delta^\infty(\mathcal{R}) \). The density \( \rho \) of this measure is of the form
\[
\rho(\lambda, \pi) = \frac{P_\pi(\lambda)}{Q_\pi(\lambda)}, \quad \lambda = (\lambda_1, \ldots, \lambda_m),
\]
where \(P_\pi\) and \(Q_\pi\) are homogeneous polynomials with nonnegative coefficients.

### 3.2. Zippered rectangles

We briefly recall the construction of the Veech space of zippered rectangles. We use the notation from [26].

Zippered rectangles associated with the Rauzy class \( \mathcal{R} \), are triples \((\lambda, \pi, \delta)\), where \(\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m\), \(\lambda_i > 0\), \(\pi \in \mathcal{R}\), \(\delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m\) and \(\delta\) satisfies the following inequalities:
\[
\delta_1 + \cdots + \delta_i \leq 0, \quad i = 1, \ldots, m - 1, \tag{3.7}
\]
\[
\delta_{\pi^{-1} i} + \cdots + \delta_{\pi^{-1} i} \geq 0, \quad i = 1, \ldots, m - 1. \tag{3.8}
\]

The set of all vectors \(\delta\) satisfying (3.7), (3.8) is a cone in \(\mathbb{R}^m\); we denote it by \(K(\pi)\).
For a zippered rectangle \((\lambda, \pi, \delta)\) we set
\[
h_r := - \sum_{i=1}^{r-1} \delta_i + \sum_{i=1}^{\pi r-1} \delta_{\pi^{-1} i}, \quad r = 1, 2, \ldots, m, \quad (3.9)
\]
\[
\text{Area}(\lambda, \pi, \delta) := \sum_{r=1}^{m} \lambda_r h_r. \quad (3.10)
\]
(Our convention is \(\sum_{i=1}^{u} \ldots = 0\) when \(u > v\).) By (3.7), (3.8) \(h_r > 0\) for all \(r\), and if we relate the set \(Z := \bigcup_{r=1}^{m} I_r \times [0, h_r]\) (a union of rectangles in \(\mathbb{R}^2\)) to every triple \((\lambda, \pi, \delta)\), then \(\text{Area}(\lambda, \pi, \delta)\) becomes just the Lebesgue measure (area) of \(Z\). By identifying appropriate intervals in the boundaries of different rectangles \(I_r \times [0, h_r]\) we can obtain a compact Riemann surface and a 1-form on it. A detailed description of this procedure (originated by Veech [2]) can be found in the literature (see, for example, [27], [24]). We do not use it as such and so will omit the details.

Denote the space of all zippered rectangles corresponding to the Rauzy class \(\mathcal{R}\) by \(\mathcal{V}(\mathcal{R})\), that is, \(\mathcal{V}(\mathcal{R}) = \{(\lambda, \pi, \delta) : \lambda \in \mathbb{R}_+^m, \pi \in \mathcal{R}, \delta \in K(\pi)\}\).

Also let \(\mathcal{V}^+(\mathcal{R}) = \{(\lambda, \pi, \delta) \in \mathcal{V}(\mathcal{R}) : \lambda_{\pi^{-1} m} > \lambda_m\}\), \(\mathcal{V}^-(\mathcal{R}) = \{(\lambda, \pi, \delta) \in \mathcal{V}(\mathcal{R}) : \lambda_{\pi^{-1} m} < \lambda_m\}\), \(\mathcal{V}^\pm(\mathcal{R}) = \mathcal{V}^+(\mathcal{R}) \cup \mathcal{V}^-(\mathcal{R})\).

Veech [2] introduced the flow \(\{P^t\}\) acting on \(\mathcal{V}(\mathcal{R})\) by the formula
\[
P^t(\lambda, \pi, \delta) = (e^t \lambda, \pi, e^{-t} \delta),
\]
and the map \(\mathcal{U} : \mathcal{V}^\pm(\mathcal{R}) \to \mathcal{V}(\mathcal{R})\), where \(\mathcal{U}(\lambda, \pi, \delta) = \left\{ \begin{array}{ll}
(A(\pi, a)^{-1} \lambda, a \pi, A(\pi, a)^{-1} \delta) & \text{if } \lambda_{\pi^{-1} m} > \lambda_m, \\
(A(\pi, b)^{-1} \lambda, b \pi, A(\pi, b)^{-1} \delta) & \text{if } \lambda_{\pi^{-1} m} < \lambda_m \end{array} \right. \)

(the inclusion \(\mathcal{U} \mathcal{V}^\pm(\mathcal{R}) \subset \mathcal{V}(\mathcal{R})\) is not obvious and needs to be proved; this was done in [2]). The map \(\mathcal{U}\) and the flow \(\{P^t\}\) commute on \(\mathcal{V}^\pm(\mathcal{R})\) and both preserve the measure determined on \(\mathcal{V}(\mathcal{R})\) by the volume form \(\text{Vol} = d\lambda_1 \cdots d\lambda_m d\delta_1 \cdots d\delta_m\). They also preserve the area of a zippered rectangle (see (3.10)) and hence can be restricted to the set \(\mathcal{V}^{1,\pm}(\mathcal{R}) := \{(\lambda, \pi, \delta) \in \mathcal{V}^\pm(\mathcal{R}) : \text{Area}(\lambda, \pi, \delta) = 1\}\).

The restriction of the volume form \(\text{Vol}\) to \(\mathcal{V}^{1,\pm}(\mathcal{R})\) induces a measure \(\mu_{\mathcal{R}}\) on this set which is invariant under \(\mathcal{U}\) and \(\{P^t\}\).

For \((\lambda, \pi) \in \Delta(\mathcal{R})\) denote
\[
\tau^0(\lambda, \pi) := - \log(|\lambda| - \min(\lambda_m, \lambda_{\pi^{-1} m})). \quad (3.11)
\]
From (3.1), (3.2) it follows that if \( \lambda \in \Delta^+ \cup \Delta^- \), then
\[
\tau^0(\lambda, \pi) = -\log |A^{-1}(c, \pi)\lambda|,
\]
where \( c = a \) when \( \lambda \in \Delta^+ \) and \( c = b \) when \( \lambda \in \Delta^- \).

Next set
\[
\mathcal{V}_1(\mathcal{R}) := \{ x = (\lambda, \pi, \delta) \in \mathcal{V}(\mathcal{R}) : |\lambda| = 1, \text{Area}(\lambda, \pi, \delta) = 1 \},
\]
\[
\tau(x) := \tau^0(\lambda, \pi) \quad \text{for} \quad x = (\lambda, \pi, \delta) \in \mathcal{V}_1(\mathcal{R}),
\]
\[
\mathcal{V}_{1,\tau}(\mathcal{R}) := \bigcup_{x \in \mathcal{V}_1(\mathcal{R}), 0 \leq t \leq \tau(x)} P^t x.
\]

Using the map \( \mathcal{U} \) we are going to transfer the flow \( P^t \) to the set \( \mathcal{V}_{1,\tau}(\mathcal{R}) \) (or more precisely, to a proper subset of it).

It is easy to see that \( \mathcal{U} P^t x \in \mathcal{V}_1(\mathcal{R}) \) for every \( x \in \mathcal{V}_1(\mathcal{R}) \cap \mathcal{V}^\pm(\mathcal{R}) \). Identifying the points \( P^t x \) and \( \mathcal{U} P^t x \) we can continue the trajectory of \( x \) some distance. But it can happen that \( \mathcal{U} P^t x \notin \mathcal{V}^\pm(\mathcal{R}) \), so that we cannot proceed in this way.

To make \( \{P^t\} \) well defined on an invariant set we have to reduce the domain of \( \mathcal{U} \) somewhat. Let
\[
\mathcal{V}^1(\mathcal{R}) := \{ (\lambda, \pi, \delta) \in \mathcal{V}(\mathcal{R}) : \sum_{i=1}^m \delta_i = 0 \},
\]
\[
\mathcal{V}_\infty(\mathcal{R}) := \bigcap_{n \in \mathbb{Z}} \mathcal{U}^n \mathcal{V}^1(\mathcal{R}).
\]

Clearly \( \mathcal{U}^n \) is well-defined on \( \mathcal{V}_\infty(\mathcal{R}) \) for all \( n \in \mathbb{Z} \).

We now set
\[
\mathcal{V}(\mathcal{R}) := \mathcal{V}_1(\mathcal{R}) \cap \mathcal{V}_\infty(\mathcal{R}), \quad \mathcal{W}(\mathcal{R}) := \mathcal{V}_{1,\tau}(\mathcal{R}) \cap \mathcal{V}_\infty(\mathcal{R}).
\]

The above identification enables us to define a natural flow on \( \mathcal{W}(\mathcal{R}) \), for which we retain the notation \( \{P^t\} \). (Although the bounded positive function \( \tau \) is not separated from zero, the flow \( \{P^t\} \) is well-defined.)

We also note that \( \mu_\mathcal{R}(\mathcal{V}_{1,\tau}(\mathcal{R})) > 0 \). By a theorem proved independently by both Veech [2] and Masur [1], \( \mu_\mathcal{R}(\mathcal{V}_{1,\tau}(\mathcal{R})) < \infty \), and in what follows we assume that the restriction of \( \mu_\mathcal{R} \) to \( \mathcal{V}_{1,\tau}(\mathcal{R}) \) is a probability measure. Since clearly \( \mu_\mathcal{R}(\mathcal{V}_{1,\tau}(\mathcal{R}) \setminus \mathcal{W}(\mathcal{R})) = 0 \), we can assume \( \mu_\mathcal{R} \) is defined on \( \mathcal{W}(\mathcal{R}) \). This measure is ergodic with respect to the flow \( \{P^t\} \).

Remark 3.2. The presentation here differs from Veech’s work [2] by a linear change of variables: the vector parameters \( h \) and \( a \) of a zippered rectangle \( (\lambda, \pi, \delta) \) in [2] are expressed in terms of \( \pi \) and \( \delta \) by (3.9) and the equations
\[
a_r = a_r(\delta) = -\sum_{i=1}^r \delta_i, \quad r = 1, \ldots, m.
\]
Following Zorich [24] we set
\[
\mathcal{U}^+(\mathcal{R}) = \{ x = (\lambda, \pi, \delta) \in \mathcal{Y}(\mathcal{R}) : \lambda \in \Delta^+_\pi, a_m(\delta) < 0 \},
\]
\[
\mathcal{U}^- (\mathcal{R}) = \{ x = (\lambda, \pi, \delta) \in \mathcal{Y}(\mathcal{R}) : \lambda \in \Delta^-_\pi, a_m(\delta) > 0 \},
\]
\[
\mathcal{U}^\pm (\mathcal{R}) = \mathcal{U}^+(\mathcal{R}) \cup \mathcal{U}^-(\mathcal{R}),
\]
and let \( \mathcal{U}^\pm_\infty (\mathcal{R}) \) be the set of all \( x \in \mathcal{U}^\pm(\mathcal{R}) \), for which there exist infinitely many positive \( t \) and infinitely many negative \( t \) such that \( P^t x \in \mathcal{U}^\pm(\mathcal{R}) \).

For \( x = (\lambda, \pi, \delta) \in \mathcal{U}^\pm_\infty (\mathcal{R}) \) denote the first return of \( x \) to the transversal \( \mathcal{U}^\pm(\mathcal{R}) \) under the flow \( \{ P^t \} \) by \( \mathcal{F}(x) \). The map \( \mathcal{F} \) is an extension of the map \( \mathcal{G} \) to the space of zippered rectangles:

\[
\text{if } \mathcal{F}(\lambda, \pi, \delta) = (\lambda', \pi', \delta'), \quad \text{then } (\lambda', \pi') = \mathcal{G}(\lambda', \pi'). \tag{3.14}
\]

Notice that \( \mathcal{F} \) is invertible on \( \mathcal{U}^\pm_\infty (\mathcal{R}) \) and if \( x \in \mathcal{U}^+(\mathcal{R}) \) \( (x \in \mathcal{U}^-(\mathcal{R})) \), then \( \mathcal{F}(x) \in \mathcal{U}^-(\mathcal{R}) \) \( (\mathcal{F}(x) \in \mathcal{U}^+(\mathcal{R})) \), respectively. Moreover,

\[
\mathcal{U}^\pm_\infty (\mathcal{R}) = \bigcap_{n \in \mathbb{Z}} \mathcal{F}^n \mathcal{U}^\pm (\mathcal{R}).
\]

If \( x = (\lambda, \pi, \delta) \in \mathcal{U}^\pm_\infty (\mathcal{R}) \) and \( \mathcal{F}(\lambda, \pi) = \mathcal{F}^n(\lambda, \pi) \), then by (3.3) and (3.6) the first return time of \( x \) to \( \mathcal{U}^\pm(\mathcal{R}) \) under the flow \( \{ P^t \} \) is

\[
\tau(\lambda, \pi) + \cdots + \tau(\mathcal{F}^{n-1}(\lambda, \pi)) = -\log |A^{-1}(c, c^{n-1}\pi) \cdots A^{-1}(c, \pi)\lambda|, \tag{3.15}
\]

where \( c = a \) when \( \lambda \in \Delta^+_\pi \) and \( c = b \) when \( \lambda \in \Delta^-_\pi \).

We finish this section by considering a relationship between the probability measure \( \mu_\mathcal{R} \) mentioned above and the measure \( \nu \) from Theorem 3.1.

We denote the \( \mathcal{F} \)-invariant probability measure induced by \( \mu_\mathcal{R} \) on \( \mathcal{U}^\pm_\infty (\mathcal{R}) \) by \( \mu^1_\mathcal{R} \). We note that if \( (\lambda, \pi, \delta) \in \mathcal{U}^\pm_\infty (\mathcal{R}) \), then \( (\lambda, \pi) \in \Delta^\infty \).

**Lemma 3.3** (see [2], [24]). Let \( \tilde{\psi} : \mathcal{U}^\pm_\infty (\mathcal{R}) \to \Delta^\infty (\mathcal{R}) \) be the map defined by \( \tilde{\psi}(\lambda, \pi, \delta) = (\lambda, \pi) \). Then \( \tilde{\psi}_{*} \mu^1_\mathcal{R} = \nu \).

**Proof.** Note that there is a natural Lebesgue measure on each of the spaces \( \mathcal{V}(\mathcal{R}) \), \( \mathcal{U}^\pm_\infty (\mathcal{R}) \) and \( \Delta^\infty (\mathcal{R}) \). Since \( \mu_\mathcal{R} \) is proportional to the Lebesgue measure on \( \mathcal{V}(\mathcal{R}) \), it follows from the definition of \( \{ P^t \} \) that \( \mu^1_\mathcal{R} \) is absolutely continuous with respect to Lebesgue measure on \( \mathcal{U}^\pm_\infty (\mathcal{R}) \). Let \( \nu^1 := \tilde{\psi}_{*} \mu^1_\mathcal{R} \). It is clear that \( \nu^1 \) is a probability measure which is absolutely continuous with respect to \( \text{mes}_\Delta \), the Lebesgue measure on \( \Delta^\infty (\mathcal{R}) \), while by Theorem 3.1 the probability measure \( \nu \) is equivalent to \( \text{mes}_\Delta \) and ergodic with respect to \( \mathcal{G} \). Therefore \( \nu^1 = \nu \).

**§ 4. A symbolic representation of the covering flow**

In this section we construct suspension flows over symbolic Markov shifts that are of great importance in our study of the flow \( \{ P^t \} \). Using [2], [24], we begin with a brief description of a symbolic model for the map \( \mathcal{G} \).
4.1. Symbolic dynamics for the map $\mathcal{G}$. We will only deal with interval exchanges $(\lambda, \pi)$ from $\Delta^\infty(\mathcal{R})$ (see (3.4)), so that all iterations of the map $\mathcal{G}$ are defined for them. As before, our notation follows [26].

Consider the alphabet

$$\mathcal{A} := \{(c, n, \pi) \mid c = a \text{ or } b, \ n \in \mathbb{N}, \ \pi \in \mathcal{R}\}. $$

For $w_1 = (c_1, n_1, \pi_1) \in \mathcal{A}$, $w_2 = (c_2, n_2, \pi_2) \in \mathcal{A}$ we set

$$B(w_1, w_2) = \begin{cases} 1 & \text{if } (c_1)^{n_1}, \pi_1 = \pi_2, \ c_2 \neq c_1, \\ 0 & \text{otherwise}, \end{cases}$$

and thus define a function $B : \mathcal{A} \times \mathcal{A} \to \{0, 1\}$. In other words, we have a directed graph $\Gamma_{\mathcal{A}, B} = (V, E)$, where $V = \mathcal{A}$ and $(w_1, w_2) \in E$ if and only if $B(w_1, w_2) = 1$. From the definition of the Rauzy class $\mathcal{R}$ in §3.1 it follows that the graph $\Gamma_{\mathcal{A}, B}$ is connected.

Introduce the space of words

$$\mathcal{W}_{\mathcal{A}, B} = \{w = w_1 \ldots w_n \mid w_i \in \mathcal{A}, \ B(w_i, w_{i+1}) = 1, \ i = 1, \ldots, n\}. $$

It is convenient to include the empty word in $\mathcal{W}_{\mathcal{A}, B}$. As in §2.1, given a word $w \in \mathcal{W}_{\mathcal{A}, B}$, $|w|$ denotes its length, that is, the number of symbols in it; given two words $w^{(1)}, w^{(2)} \in \mathcal{W}_{\mathcal{A}, B}$, we denote their concatenation by $w^{(1)}w^{(2)}$. Note that $w^{(1)}w^{(2)}$ need not belong to $\mathcal{W}_{\mathcal{A}, B}$ unless a compatibility condition is satisfied by the last letter of $w^{(1)}$ and the first letter of $w^{(2)}$.

To each nonempty word $w \in \mathcal{W}_{\mathcal{A}, B}$ we assign a renormalization matrix $A(w)$ as follows. If $w$ is a single-letter word, $w = (c, n, \pi) \in \mathcal{A}$, we set (see (3.1), (3.2))

$$A(w) = A(c, \pi)A(c, c\pi) \cdots A(c, c^{n-1}\pi); \quad (4.1)$$

for $w \in \mathcal{W}_{\mathcal{A}, B}$, where $w = w_1 \ldots w_n$, $w_i \in \mathcal{A}$, we set

$$A(w) = A(w_1) \cdots A(w_n). \quad (4.2)$$

Consider the sequence spaces

$$\Omega_{\mathcal{A}, B} = \{\omega = (\omega_0, \omega_1, \ldots) \mid \omega_n \in \mathcal{A}, \ B(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}_+\},$$

$$\Omega_{\mathcal{A}, B}^\mathbb{Z} = \{\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \mid \omega_n \in \mathcal{A}, \ B(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\}. $$

Denote the shift one step to the left on both these spaces by $\sigma$.

We will now describe the coding map. For every letter $w = (c, n, \pi) \in \mathcal{A}$ we set

$$\Delta(w) = \begin{cases} \Delta^+(\mathcal{R}) \cap \{(\lambda, \pi) \in \Delta^\infty(\mathcal{R}) \mid n(\lambda, \pi)\} & \text{if } c = a, \\ \Delta^-(\mathcal{R}) \cap \{(\lambda, \pi) \in \Delta^\infty(\mathcal{R}) \mid n(\lambda, \pi)\} & \text{if } c = b. \end{cases} \quad (4.3)$$

In other words, when $c = a$ ($c = b$), $\Delta(w)$ consists of all points

$$(\lambda, \pi) \in \Delta^\infty(\mathcal{R}) \cap \Delta^+(\mathcal{R}) \quad ((\lambda, \pi) \in \Delta^\infty(\mathcal{R}) \cap \Delta^-(\mathcal{R}), \text{ respectively})$$

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such that \( \mathcal{T}^k(\lambda, \pi) \in \Delta^+(\mathcal{R}) \) (\( \mathcal{T}^k(\lambda, \pi) \in \Delta^-(\mathcal{R}) \)) for \( k = 0, \ldots, n - 1 \) and \( \mathcal{T}^n(\lambda, \pi) \in \Delta^-(\mathcal{R}) \) (\( \mathcal{T}^n(\lambda, \pi) \in \Delta^+(\mathcal{R}) \)). Using (3.5) we can check that if \( w = (c, n, \pi) \), then

\[
\Delta(w) = \begin{cases} 
\Delta^\infty(\mathcal{R}) \cap \left\{ (\lambda, \pi) \mid \lambda \in \Delta^\mp_a, \frac{A(w)^{-1} \lambda}{|A(w)^{-1} \lambda|} \in \Delta_a^n \pi \right\} & \text{if } c = a, \\
\Delta^\infty(\mathcal{R}) \cap \left\{ (\lambda, \pi) \mid \lambda \in \Delta^\mp_b, \frac{A(w)^{-1} \lambda}{|A(w)^{-1} \lambda|} \in \Delta_b^n \pi \right\} & \text{if } c = b.
\end{cases}
\tag{4.4}
\]

It is easy to see that all the sets \( \Delta(w), w \in \mathcal{A} \), are nonempty and constitute a partition of \( \Delta^\infty(\mathcal{R}) \). We use this partition to construct symbolic dynamics for \( \mathcal{G} \). By iterating this partition \( n \) times under the action of the transformation \( \mathcal{G} \) we obtain a partition whose elements \( \Delta(w) \) are determined by words \( w \in \mathcal{W}_{\mathcal{G},B} \) of length \( n \). Namely, for a word \( w = w_1 \ldots w_n \in \mathcal{W}_{\mathcal{G},B}, w_i \in \mathcal{A} \), we set

\[
\Delta(w) = \bigcap_{i=0}^{n-1} \mathcal{G}^{-i} \Delta(w_{i+1}).
\tag{4.5}
\]

**Remark 4.1.** From (3.3), (3.6) and (4.1)–(4.4) it follows that if \( (\lambda, \pi) \in \Delta(w) \) and \( (\lambda', \pi') \in \mathcal{G}(\lambda, \pi) \), then

\[
\lambda' = \frac{A^{-1}(w) \lambda}{|A^{-1}(w) \lambda|}, \quad \lambda = \frac{A(w) \lambda'}{|A(w) \lambda'|}.
\]

These formulae can easily be extended by induction to the case when \( w = w_1 \ldots w_n \in \mathcal{W}_{\mathcal{G},B}, (\lambda, \pi) \in \Delta(w) \) and \( (\lambda', \pi') \in \mathcal{G}(\lambda, \pi) \).

**The coding map** \( \Phi: \Delta^\infty(\mathcal{R}) \to \Omega_{\mathcal{G},B} \) is given by the formula

\[
\Phi(\lambda, \pi) = (\omega_0, \omega_1, \ldots), \quad \mathcal{G}^n(\lambda, \pi) \in \Delta(\omega_n), \quad n = 0, 1, \ldots \tag{4.6}
\]

Consider conditions under which the coding map can be inverted. For \( q \in \mathcal{W}_{\mathcal{G},B} \) we denote by \( \Omega_q \) the set of all sequences \( \omega \in \Omega_{\mathcal{G},B} \) starting from the word \( q \) and containing infinitely many occurrences of \( q \). A key role will be played below by the words \( q \) such that all entries of the renormalization matrix \( A(q) \) are positive. For brevity we will refer to these \( q \) as positive words. Observe that each word containing a positive prefix is also positive.

The next two lemmas are due to Veech [2] (see page 237, especially the discussion around equation (13.9)).

**Lemma 4.2.** Let \( q \in \mathcal{W}_{\mathcal{G},B} \) be a positive word. Then for every \( \omega \in \Omega_q \) there exists a unique interval exchange \( (\lambda, \pi) \in \Delta^\infty(\mathcal{R}) \) such that \( \Phi(\lambda, \pi) = \omega \).

**Lemma 4.3.** If the interval exchange map \( (\lambda, \pi): [0, 1] \to [0, 1] \) has a unique invariant measure, then there exists a positive word \( q \in \mathcal{W}_{\mathcal{G},B} \) such that \( (\lambda, \pi) \in \Delta(q) \).

**Remark 4.4.** Veech [2] in fact observed that for the word \( q \) mentioned in Lemma 4.3 to exist it is sufficient that \( (\lambda, \pi) \) satisfies Keane’s infinite distinct orbit condition.
4.2. Symbolic dynamics for the flow \( \{P^t\} \). We first construct symbolic dynamics for the map \( \mathcal{F} \) introduced in § 3.2. For \( (\lambda, \pi, \delta) \in \mathcal{Y}_\infty^\pm (\mathcal{R}) \) we set

\[
\tilde{\Phi}(\lambda, \pi, \delta) = (\omega_0, \omega_1, \omega_2, \ldots, \omega_i, \ldots), \quad \omega_i \in \mathcal{A},
\]

(4.7)

if \( \mathcal{F}^n(\lambda, \pi, \delta) = (\lambda_n, \pi_n, \delta_n) \) and \( (\lambda_n, \pi_n) \in \Delta(\omega_n), n \in \mathbb{Z} \) (recall that \( \mathcal{F} \) is invertible on \( \mathcal{Y}_\infty^\pm (\mathcal{R}) \)). In parallel with the coding map \( \Phi \) (see § 4.1) we have

\[
\tilde{\Phi}(\mathcal{Y}_\infty^\pm (\mathcal{R})) \subset \Omega_{\mathcal{A}, B}^\mathcal{Z},
\]

Moreover, by (3.14) and (4.6), from (4.7) it follows that

\[
\Phi(\lambda, \pi) = (\omega_0, \omega_1, \ldots).
\]

For \( \mathbf{q} \in \mathcal{W}_{\mathcal{A}, B}, |\mathbf{q}| = l \), similarly to the definition of \( \Omega_{\mathbf{q}} \) in § 4.1 we let \( \Omega_{\mathbf{q}}^\mathcal{Z} \) denote the set of all sequences \( \omega \in \Omega_{\mathcal{A}, B}^\mathcal{Z} \) satisfying \( \omega_0 \ldots \omega_{l-1} = \mathbf{q} \) and containing infinitely many occurrences of the word \( \mathbf{q} \), both in the past and in the future.

Let

\[
\mathcal{Y}_{\mathbf{q}, \infty}(\mathcal{R}) := \tilde{\Phi}^{-1}(\Omega_{\mathbf{q}}^\mathcal{Z}), \quad \mathcal{Y}_{\mathbf{q}}(\mathcal{R}) := \bigcup_{t \in \mathbb{R}} P^t \mathcal{Y}_{\mathbf{q}, \infty}(\mathcal{R})
\]

and assume that \( \mathcal{Y}_{\mathbf{q}, \infty}(\mathcal{R}) \) (and hence \( \mathcal{Y}_{\mathbf{q}}(\mathcal{R}) \)) is nonempty.

Let \( \mathcal{F}_{\mathbf{q}} \) be the first return map of \( \mathcal{F} \) to \( \mathcal{Y}_{\mathbf{q}, \infty}(\mathcal{R}) \), that is, the map induced by \( \mathcal{F} \) on \( \mathcal{Y}_{\mathbf{q}, \infty}(\mathcal{R}) \) (see § 2.3).

By definition \( \{P^t\}_{\mathcal{Y}_{\mathbf{q}}(\mathcal{R})} \), the restriction of the flow \( \{P^t\} \) to \( \mathcal{Y}_{\mathbf{q}}(\mathcal{R}) \), is Borel isomorphic to a suspension flow \( (\mathcal{F}_{\mathbf{q}}, \tau_{\mathbf{q}}) \) over the map \( \mathcal{F}_{\mathbf{q}} \). To describe the roof function \( \tau_{\mathbf{q}} \) we take \( (\lambda, \pi) \in \Delta^\infty(\mathcal{R}) \), \( (\lambda', \pi') = \Phi(\lambda, \pi) \) and, following Veech, introduce the function

\[
\tau^1 : (\lambda, \pi) \mapsto \log |A(\omega_0)\lambda'|,
\]

where \( \omega_0 \) is determined by the equation \( \Phi(\lambda, \pi) = (\omega_0, \omega_1, \ldots) \), that is, \( (\lambda, \pi) \in \Delta(\omega_0) \).

(Recall that the norm of a vector \( v \) is given by \( |v| = \sum_i |v_i| \).) From (3.1), (3.2) and Remark 4.1 we see that \( \tau^1 : (\lambda, \pi) > 0 \). Using (3.14) and (3.15) it is easy to check that if \( x = (\lambda, \pi, \delta) \in \mathcal{Y}_\mathcal{Z}(\mathcal{R}) \) and \( (\lambda, \pi) \in \Delta(\omega_0) \), where \( \omega_0 = (c, n, \pi) \), then the first return time of \( x \) to \( \mathcal{Y}_\mathcal{Z}(\mathcal{R}) \) under the action of the flow \( \{P^t\} \) is just \( \tau^1(\lambda, \pi) \).

Now let \( (\lambda, \pi) \in \Phi^{-1}(\Omega_{\mathbf{q}}), (\omega_0, \omega_1, \ldots) = \Phi(\lambda, \pi) \) and let \( s \) be the moment when the word \( \mathbf{q} \) appears for the second time in \( (\omega_0, \omega_1, \ldots) \), that is,

\[
s = s(\omega_0, \omega_1, \ldots) = \min\{k > 0 : (\omega_k, \ldots, \omega_{k+l-1}) = \mathbf{q}\}.
\]

(4.9)

Set

\[
\tau^1_{\mathbf{q}}(\lambda, \pi) = \tau^1(\lambda, \pi) + \tau^1(\mathcal{F}(\lambda, \pi)) + \cdots + \tau^1(\mathcal{F}^{s-1}(\lambda, \pi)).
\]

(4.10)

If \( x = (\lambda, \pi, \delta) \in \mathcal{Y}_{\mathbf{q}, \infty}(\mathcal{R}) \), then \( (\lambda, \pi) \in \Phi^{-1}(\Omega_{\mathbf{q}}) \) and

\[
\tau_{\mathbf{q}}(x) = \tau^1_{\mathbf{q}}(\lambda, \pi).
\]

(4.11)

Let \( \Psi_1 \) denote the map from \( \mathcal{Y}_{\mathbf{q}}(\mathcal{R}) \) to the phase space of the flow \( (\mathcal{F}_{\mathbf{q}}, \tau_{\mathbf{q}}) \) that induces the isomorphism between the flows \( \{P^t\}_{\mathcal{Y}_{\mathbf{q}}(\mathcal{R})} \) and \( (\mathcal{F}_{\mathbf{q}}, \tau_{\mathbf{q}}) \) mentioned above.
From now on we assume that \( q \) is a positive word. For these \( q \) we construct a suspension flow \((\widehat{\sigma}_q, \widehat{\tau}_q)\) closely related to \((\mathcal{F}_q, \tau_q)\). As above, we let \( \sigma \) denote the shift one-step to the left on \( \Omega_{\mathcal{A},B}^Z \), and for \( \omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in \Omega_q^Z \) let

\[
\widehat{\sigma}_q(\omega) := \sigma^s, \quad \widehat{\tau}_q(\omega) := \tau_1^q(\Phi^{-1}(\omega_0, \omega_1, \ldots)),
\]

where \( s \) is defined in (4.9). Observe that \( \omega \in \Omega_q^Z \) implies \((\omega_0, \omega_1, \ldots) \in \Omega_q \), hence by Lemma 4.2 \( \Phi^{-1}(\omega_0, \omega_1, \ldots) \) is a uniquely defined point in \( \Delta^\infty(\mathcal{R}) \).

Proposition 6 in [26] states that if \( q \in \mathcal{W}_{\mathcal{A},B} \) is positive, then for every \( \omega \in \Omega_q^Z \) there exists at most one zippered rectangle corresponding to it; in other words \( \psi \) is a natural way a measurable injective map \( \Psi \) from the phase space of the flow \((\sigma_q, \tau_q)\) to the phase space of the flow \((\widehat{\sigma}_q, \widehat{\tau}_q)\); this sends the former flow to the latter, restricted to some invariant set. Hence \((\mathcal{F}_q, \tau_q)\) is embedded into \((\widehat{\sigma}_q, \widehat{\tau}_q)\) (in the sense of \( \S \)).

Introduce a new alphabet \( \mathcal{A}_q \); it will consist of all the words \( w = (v_1, \ldots, v_n) \in \mathcal{W}_{\mathcal{A},B}, v_i \in \mathcal{A}, n > l \), such that \((v_1, \ldots, v_l) = q, (v_{n-l+1}, \ldots, v_n) = q\) and no other subword of \( w \) coincides with \( q \). Since \( |\mathcal{A}_q| = \infty \) and the graph \( \Gamma_{\mathcal{A},B} \) is connected (see \( \S \)), we have \( |\mathcal{A}_q| = \infty \). By the Markov-Bernoulli reduction introduced in \( \S \) there is a measurable one-to-one map \( \Psi_{M-B} : \Omega_q^Z \to (\mathcal{A}_q)^Z \) which sends \( \widehat{\sigma}_q \) to \( \sigma_q \), the shift one-step to the left, on \((\mathcal{A}_q)^Z\). Hence the flow \((\widehat{\sigma}_q, \widehat{\tau}_q)\) is isomorphic to the suspension flow \((\sigma_q, f_q)\), where

\[
f_q(u) = \widehat{\tau}_q(\Psi_{M-B}^{-1}(u)), \quad u \in (\mathcal{A}_q)^Z.
\]

Denote the corresponding map from the phase space of \((\widehat{\sigma}_q, \widehat{\tau}_q)\) to the phase space of \((\sigma_q, f_q)\) by \( \Psi_3 \). Summing up we can state the following.

**Lemma 4.5.** The mapping \( \Psi := \Psi_1 \circ \Psi_2 \circ \Psi_3 \) yields an embedding of the flow \( \{P^t\}|_{\tau_q(\mathcal{R})} \) into the flow \((\sigma_q, f_q)\).

Now we turn to the probability measure \( \mu_{\mathcal{R}} \) on \( \mathcal{V}(\mathcal{R}) \) and the probability measure \( \mu_{\mathcal{R}}^1 \) induced by \( \mu_{\mathcal{R}} \) on \( \mathcal{W}_\infty(\mathcal{R}) \) (see \( \S \)).

Observe that \( \mu_{\mathcal{R}} \) assigns a positive mass to every Borel set with nonempty interior. (We assume that \( \mathcal{V}(\mathcal{R}) \) and the other spaces we encounter in this paper have the natural topology.) From this fact, using Lemma 4.2 and the definitions of \( \mathcal{F}, \mathcal{G} \) and \( \mathcal{F} \) we easily derive that

\[
\mu_{\mathcal{R}}^1\{x \in \mathcal{W}_\infty(\mathcal{R}) : x = (\lambda, \pi, \delta), (\lambda, \pi) \in \Delta(q)\} > 0.
\]

Since \( \mu_{\mathcal{R}} \) is \( \{P^t\} \)-ergodic, we have

\[
\mu_{\mathcal{R}}(\mathcal{V}_q(\mathcal{R})) = 1, \quad \mu_{\mathcal{R}}^1\left(\bigcup_{n \in \mathbb{Z}} \mathcal{F}_n^{\mathcal{V}_{\infty}^{\pm}(\mathcal{R})}\right) = 1.
\]

By normalizing the restriction of \( \mu_{\mathcal{R}}^1 \) to \( \mathcal{W}_{\mathcal{A},B}^{\pm}(\mathcal{R}) \) we obtain a probability measure \( \mu_{\mathcal{A},B}^1 \).

Let \( \psi \) be the natural projection of \( \Omega_{\mathcal{A},B}^Z \) onto \( \Omega_{\mathcal{A},B} \). Recall that in \( \S \) we introduced the natural projection \( \widehat{\psi} \) of \( \mathcal{W}_{\mathcal{R}}(\mathcal{R}) \) onto \( \Delta^\infty(\mathcal{R}) \). It is clear that
\[
\psi(\tilde{\Phi}(x)) = \Phi(\tilde{\psi}(x)) \text{ for every } x \in \bigcup_{n \in \mathbb{Z}} \mathcal{F}_n \mathcal{Y}_{q,\infty}^\pm(\mathcal{R}). \]
From this fact combined with (4.14) and Lemma 3.3 we obtain
\[
\psi_*(\Phi^1_*\mu_{\mathcal{R}}) = \Phi_*\nu. \tag{4.15}
\]

Using the positivity of \(\nu\) on all nonempty open sets we arrive at the following result.

**Lemma 4.6.** The measure \(\Psi_*\mu_{\mathcal{R}}\) is positive on every open set in the phase space of the flow \((\sigma_q, f_q)\).

### 4.3. Properties of the roof function.

Recall that a word \(w' = w_1 \ldots w_l \in \mathcal{W}_q, B\) is said to be a simple prefix of a word \(w = w_1 \ldots w_l \ldots w_n\) if \(w_1 \ldots w_{n-k+1} = w_k \ldots w_n\) implies that either \(k = 1\) or \(k > l\) (see §2.1).

**Lemma 4.7.** Let \(q \in \mathcal{W}_q, B\) be a word that has a simple positive prefix. Then the function \(f_q\) introduced in (4.13) depends only on the future, is bounded away from zero, and is Hölder continuous in the following sense: there exist positive constants \(C_q > 0\) and \(\alpha_q > 0\) (depending only on \(q\)) such that if \(u = (\ldots, u_{-1}, u_0, u_1, \ldots) \in \mathcal{A}_q^\mathbb{Z}\) and \(\tilde{u} = (\ldots, \tilde{u}_{-1}, \tilde{u}_0, \tilde{u}_1, \ldots) \in \mathcal{A}_q^\mathbb{Z}\) satisfy \(u_i = \tilde{u}_i\) for \(|i| \leq n\), then
\[
|f_q(u) - f_q(\tilde{u})| \leq C_q \exp(-\alpha_q n).
\]
In particular, the function \(f_q\) has summable variations.

**Proof.** That the function \(f_q\) depends only on the future follows readily from its definition. We shall prove that it is bounded away from zero. By (4.13), (4.12) this property of \(f_q\) would follow from the same property of the function \(\tau_1^q\) defined on \(\Phi^{-1}(\Omega_q)\). But from (4.8), (4.10) and Remark 4.1 it is easy to see that for \((\omega_0, \omega_1, \ldots) \in \Omega_q,
\[
\tau_1^q(\lambda, \pi) = \log |A(\omega_0) \cdots A(\omega_{s-1})\lambda'|, \quad (\lambda, \pi) = \Phi^{-1}(\omega_0, \omega_1, \ldots), \tag{4.16}
\]
where \(s\) is defined in (4.9) and \((\lambda', \pi') = \mathcal{G}(\lambda, \pi) = \Phi^{-1}(\omega_s, \omega_{s+1}, \ldots)\). If \(\mathcal{p}\) is a simple prefix of \(q\), then \(s \geq |\mathcal{p}|\), so that \(\mathcal{p}\) is a prefix of the word \((\omega_0, \ldots, \omega_{s-1})\) and hence this word is positive. Thus all the entries of the \((m \times m)\)-matrix
\[
A(\omega_0, \ldots, \omega_{s-1}) = A(\omega_0) \cdots A(\omega_{s-1})
\]
are positive integers while \(\lambda'\) is a positive vector with \(|\lambda'| = 1\). Therefore, for every \((\lambda, \pi) \in \Phi^{-1}(\Omega_q),
\[
\tau_1^q(\lambda, \pi) = \log |A(\omega_0, \ldots, \omega_{s-1})\lambda'| \geq \log m > 0.
\]

We now turn to the Hölder continuity of \(f_q\). Let \(u, \tilde{u}\) be as in the statement of our lemma and let \(\omega = \Psi^{-1}_{M-B}(u), \tilde{\omega} = \Psi^{-1}_{M-B}(\tilde{u})\). By definition \(\omega, \tilde{\omega} \in \Omega_q^\mathbb{Z}\), hence \((\omega_0, \omega_1, \ldots), (\tilde{\omega}_0, \tilde{\omega}_1, \ldots) \in \Omega_q\). By (4.11)–(4.13)
\[
f_q(u) = \tau_1^q(\lambda, \pi) = \log |A(\omega_0, \ldots, \omega_{s-1})\lambda'|, \tag{4.17}
f_q(\tilde{u}) = \tau_1^q(\tilde{\lambda}, \tilde{\pi}) = \log |A(\tilde{\omega}_0, \ldots, \tilde{\omega}_{s-1})\tilde{\lambda}'|, \tag{4.18}
\]
where
\[ (\lambda, \pi) = \Phi^{-1}(\omega_0, \omega_1, \ldots), \quad (\tilde{\lambda}, \tilde{\pi}) = \Phi^{-1}(\tilde{\omega}_0, \tilde{\omega}_1, \ldots), \]
\[ (\lambda', \pi') = \mathcal{G}^s(\lambda, \pi), \quad (\tilde{\lambda}', \tilde{\pi}') = \mathcal{G}^s(\tilde{\lambda}, \tilde{\pi}). \]

Since \( p \) is a simple prefix of \( q \), one can find \( k \geq nl - 1 \) such that \( \omega_i = \tilde{\omega}_i \) for \( i = 0, 1, \ldots, k \). Then by Remark 4.1, for some vectors \( \lambda'', \tilde{\lambda}'' \in \Delta_{m-1} \),
\[ \lambda = \frac{A(\omega_0, \ldots, \omega_{nl-1})\lambda''}{|A(\omega_0, \ldots, \omega_{nl-1})\lambda''|}, \quad \tilde{\lambda} = \frac{A(\omega_0, \ldots, \omega_{nl-1})\tilde{\lambda}''}{|A(\omega_0, \ldots, \omega_{nl-1})\tilde{\lambda}''|}. \] (4.19)

Introduce the Hilbert metric \( d_H \) on \( \Delta_{m-1} \) by
\[ d_H(\lambda^{(1)}, \lambda^{(2)}) = \log \frac{\max_{1 \leq i \leq m}(\lambda^{(1)}_i/\lambda^{(2)}_i)}{\min_{1 \leq i \leq m}(\lambda^{(1)}_i/\lambda^{(2)}_i)}, \]
\[ \lambda^{(j)} = (\lambda^{(j)}_1, \ldots, \lambda^{(j)}_m), \quad j = 1, 2. \] (4.20)

It is known that if a non-negative \((m \times m)\)-matrix \( A = (a_{ij}) \) is such that \( \sum_{j=1}^m a_{ij} > 0 \) for all \( i \), then the mapping \( T_A: \Delta_{m-1} \to \Delta_{m-1} \) defined by \( T_A \lambda = A\lambda/|A\lambda| \) does not increase the \( d_H \)-distance between points, while if \( a_{ij} > 0 \) for all \( i, j \), then \( T_A \) is a uniform contraction (see, for example, [25]).

By definition, the word \( \omega_0 \ldots \omega_{nl-1} \) is a concatenation, namely,
\[ \omega_0 \ldots \omega_{nl-1} = p w_1 \ldots p w_n, \]
where the word \( w_i, 1 \leq i \leq n \), is such that the sum of the entries in each row of the matrix \( A(w_i) \) is positive. It follows (see (4.19)) that
\[ d_H(\lambda, \tilde{\lambda}) \leq d_H(T_A(q)\lambda'', T_A(q)\tilde{\lambda}'') \leq C_1 \alpha^n, \] (4.21)
where \( C_1 \in \mathbb{R}_+ \) and \( \alpha \in (0, 1) \) depend only on \( q \). (We have used the fact that \( T_A(q) \) takes \( \Delta_{m-1} \) to a set of finite \( d_H \)-diameter.)

Set \( A_1 = A(\omega_0, \ldots, \omega_{s-1}) \). Since all the entries of the matrix \( A_1 \) are positive, we have (see (4.20))
\[ d_H\left(\frac{A_1 \lambda}{|A_1 \lambda|}, \frac{1}{A_1 \lambda} \right) = \log \frac{\max_{i}(A_1 \lambda)_i/(A_1 \tilde{\lambda})_i}{\min_{i}(A_1 \lambda)_i/(A_1 \tilde{\lambda})_i} \leq C_2 d_H(\lambda, \tilde{\lambda}), \]
where \((A_1 \lambda)_i\) \((A_1 \tilde{\lambda})_i\) is the \( i \)th component of the vector \( A_1 \lambda \) \((A_1 \tilde{\lambda}, \) respectively) and \( C_2 \) is determined by \( q \). Hence, by (4.17) and (4.21),
\[ |f_q(u) - f_q(\tilde{u})| = \log \left| \frac{A_1 \lambda}{|A_1 \lambda|} \right| \leq C_2 d_H(\lambda, \tilde{\lambda}) \leq C_2 C_1 \alpha^n, \]
so it remains to set \( C_q = C_2 C_1 \) and \( \alpha_q = - \log \alpha \).
4.4. Transition probabilities and the uniform expansion properties. By Theorem 3.1, the map \( \mathcal{G} \) defined on \( \Delta(\mathcal{R}) \) preserves an absolutely continuous ergodic probability measure, which was denoted by \( \nu \).

Consider a word \( w = w_1 \ldots w_k \in \mathcal{W}_{\mathcal{R}, B} \), where \( w_i = (c_i, n_i, \pi_i) \in \mathcal{R} \), \( 1 \leq i \leq k \).

We say that \( w \) is compatible with a point \( (\lambda, \pi) \in \Delta^\infty(\mathcal{R}) \) (or \( (\lambda, \pi) \) is compatible with \( w \)) if either

\[
\lambda \in \Delta^-_\pi, \quad c_k = a, \quad a^{n_k} \pi_k = \pi,
\]

or

\[
\lambda \in \Delta^+_\pi, \quad c_k = b, \quad b^{n_k} \pi_k = \pi.
\]

Assuming that \( w \) is compatible with \( (\lambda, \pi) \), we set

\[
t_w(\lambda, \pi) = \left( \frac{A(w)\lambda}{|A(w)|\lambda}, \pi_1 \right).
\]  (4.22)

From the definition of \( \mathcal{G} \) (see (3.6)) it follows that

\[
\mathcal{G}^{-n}(\lambda, \pi) = \{ t_w(\lambda, \pi) : |w| = n \text{ and } w \text{ is compatible with } (\lambda, \pi) \}.
\]  (4.23)

Note that the set \( \mathcal{G}^{-n}(\lambda, \pi) \) is infinite.

In § 4.1 for each word \( w \in \mathcal{W}_{\mathcal{R}, B} \) we introduced the set \( \Delta(w) \subset \Delta^\infty(\mathcal{R}) \) (see (4.5)). One can readily check (see Remark 4.1) that

\[
\Delta(w) = \{ t_w(\lambda, \pi) : (\lambda, \pi) \in \Delta^\infty(\mathcal{R}) \text{ is compatible with } w \}.
\]  (4.24)

For every \( n \in \mathbb{N} \) we have the \( \nu \)-measurable partition \( \mathcal{G}^{-n}\varepsilon \) of \( \Delta^\infty(\mathcal{R}) \), where \( \varepsilon \) is the partition into separate points. Each element of \( \mathcal{G}^{-n}\varepsilon \) is \( \mathcal{G}^{-n}(\lambda, \pi) \) for some \( (\lambda, \pi) \in \Delta^\infty(\mathcal{R}) \), its points correspond to words \( w \in \mathcal{W}_{\mathcal{R}, B} \) of length \( n \) compatible with \( (\lambda, \pi) \) and have the form \( t_w(\lambda, \pi) \) (see (4.22), (4.23)). Given the element \( \mathcal{G}^{-n}(\lambda, \pi) \) of the partition \( \mathcal{G}^{-n}\varepsilon \), we denote the conditional measure (determined by \( \nu \)) of the point corresponding to \( w \) by \( \nu(w | (\lambda, \pi)) \).

In § 3.5 of [26] it is proved that if \( w \) is compatible with \( (\lambda, \pi) \), then

\[
\nu(w | (\lambda, \pi)) = \frac{\rho(t_w(\lambda, \pi))}{\rho(\lambda, \pi)|A(w)|^m}.
\]  (4.25)

Now consider the set \( \Delta(q) \) corresponding to a word \( q \in \mathcal{W}_{\mathcal{R}, B} \) (see (4.5)). Every point from \( \Delta(q) \) is of the form \( (\lambda, \pi_q) \), where \( \pi_q \) is a fixed permutation and \( \lambda \) belongs to a set \( \Delta'(q) \subset \Delta_{m-1} \). Let \( d(q) \) denote the diameter of \( \Delta'(q) \) with respect to the Hilbert metric on \( \Delta_{m-1} \) introduced in (4.20).

**Proposition 4.8.** There are positive constants \( \beta_1 \) and \( \beta_2 \) (depending only on \( \mathcal{R} \)) such that for every positive word \( p' \in \mathcal{W}_{\mathcal{R}, B} \) with \( d(p') \leq \beta_1 \) the following holds. Let \( p \) be a word that has \( p' \) as a simple prefix and let a word \( r \in \mathcal{W}_{\mathcal{R}, B} \) begin and end with \( p \) and contain no other occurrences of \( p \). Then for any \( (\lambda, \pi) \in \Delta(r) \cap \Phi^{-1}(\Omega_p) \) we have

\[
\left| \frac{\nu(\Delta(r))\exp[m\tau^1_p(\lambda, \pi)]}{\nu(\Delta(p))} - 1 \right| \leq \beta_2 d(p'),
\]

where \( \tau^1_p \) is defined by (4.16).
Proof. By assumption, the word \( r \) has the form \( r = p'u \mathbf{p} = p\tilde{u} \) for some \( u, \tilde{u} \in \mathcal{W}_{A,B} \). Hence
\[
\nu(\Delta(r)) = \int_{\Delta(p)} \nu(p'u|\lambda, \pi) \, d\nu(\lambda, \pi). \tag{4.26}
\]

From (4.25) it follows that
\[
\nu(p'u|\lambda, \pi) = \frac{\rho(t_{p'u}(\lambda, \pi))}{\rho(\lambda, \pi)} \frac{1}{|A(p'u)|\lambda^m}. \tag{4.27}
\]

By (4.20), taking into account that \( |\lambda| = 1 \) when \( \lambda \in \Delta_{m-1} \), for any points \( (\lambda, \pi_p), (\tilde{\lambda}, \pi_{p'}) \in \Delta(p') \) and for \( i = 1, \ldots, m \) we have
\[
e^{-d(p')} \leq \frac{\tilde{\lambda}_i}{\lambda_i} \leq e^{d(p')} \tag{4.28}
\]

We will estimate the first ratio on the right-hand side of (4.27). As \( p'u \mathbf{p'} \in \mathcal{W}_{A,B} \) it follows that, if \( (\lambda, \pi_p) \in \Delta(p') \), then \( (\lambda, \pi_p) \) is compatible with \( p'u \). Hence \( t_{p'u} \in \Delta(p'u) \subset \Delta(\mathbf{p'}) \) (see (4.24)). Set
\[
(\tilde{\lambda}, \tilde{\pi}) := t_{p'u}(\lambda, \pi_p), \quad \tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m),
\]
and observe that \( \tilde{\pi} = \pi_{p'} \). By (4.28)
\[
e^{-d(p')} \lambda_i \leq \tilde{\lambda}_i \leq e^{d(p')} \lambda_i, \quad i = 1, \ldots, m,
\]
while by Theorem 3.1 \( \rho(\tilde{\lambda}, \tilde{\pi}) = P_{\tilde{\pi}}(\tilde{\lambda})/Q_{\tilde{\pi}}(\tilde{\lambda}), \) where \( P_{\tilde{\pi}} \) and \( Q_{\tilde{\pi}} \) are homogeneous polynomials with nonnegative coefficients. Therefore,
\[
P_{\tilde{\pi}}(\tilde{\lambda}) \leq P_{\tilde{\pi}}(e^{d(p')}) \lambda \leq e^{\gamma_1 d(p')} P_{\tilde{\pi}}(\tilde{\lambda}),
\]
\[
Q_{\tilde{\pi}}(\tilde{\lambda}) \geq Q_{\tilde{\pi}}(e^{-d(p')}) \lambda \geq e^{-\gamma_2 d(p')} Q_{\tilde{\pi}}(\tilde{\lambda}),
\]
where \( \gamma_1 \) and \( \gamma_2 \) are determined by the density \( \rho \) (and eventually, by \( \mathcal{B} \)). From this we immediately obtain
\[
\exp[-(\gamma_1 + \gamma_2)d(p')] \leq \frac{\rho(t_{p'u}(\lambda, \pi))}{\rho(\lambda, \pi)} \leq \exp[(\gamma_1 + \gamma_2)d(p')]. \tag{4.29}
\]

Using (4.28) and the fact that all the entries of the matrix \( A(p'u) \) are positive, for any \( (\lambda, \pi_p), (\tilde{\lambda}, \pi_{p'}) \in \Delta(p') \) we have
\[
\exp[-d(p')] \leq \frac{|A(p'u)|\tilde{\lambda}}{|A(p'u)|\lambda} \leq \exp[d(p')]. \tag{4.30}
\]

Now fix an arbitrary point \( (\lambda^0, \pi^0) \in \Delta(r) \cap \Phi^{-1}(\Omega_p) \). By Lemma 4.2
\[
(\lambda^0, \pi^0) = \Phi^{-1}(\omega_0, \omega_1, \ldots)
\]
for some \((ω_0, ω_1, \ldots) ∈ Ω_p\). Let \(s\) be defined by (4.9) and let \((λ', π') := G_s(λ^0, π^0)\). Clearly, \(s = |p'u|\), hence \((λ', π') ∈ Δ(p) ⊂ Δ(p')\), and by (4.26), (4.27), (4.29) and (4.30),

\[
\frac{ν(Δ(ς)) |A(p'u)λ'|^m}{ν(Δ(p))} = \frac{1}{ν(Δ(p))} \int_{Δ(p)} \frac{ρ(t_{p'u}(λ, π)) |A(p'u)λ'|^m}{ρ(λ, π) |A(p'u)λ'|^m} dν(λ, π) ≤ \exp[(γ_1 + γ_2 + 1)d(p')].
\] (4.31)

Similarly,

\[
\frac{ν(Δ(ς)) |A(p'u)λ'|^m}{ν(Δ(p))} ≥ \exp[−(γ_1 + γ_2 + 1)d(p')].
\] (4.32)

From (4.31), (4.32) we obtain

\[
\left| \frac{ν(Δ(ς)) |A(p'u)λ'|^m}{ν(Δ(p))} - 1 \right| ≤ \exp[(γ_1 + γ_2 + 1)d(p)] ≤ 2(γ_1 + γ_2 + 1)d(p'),
\]

where the last inequality holds when

\((γ_1 + γ_2 + 1)d(p') ≤ \log 2)\).

It remains to recall that by (4.16)

\[
|A(p'u)λ'| = \exp(τ_1(λ, π)).
\]

§ 5. Zippered rectangles and Abelian differentials. Completion of the proof of Theorem 1.1

In §1 we fixed an arbitrary connected component \(H\) of the space \(M_{κ}\). To this component there corresponds a unique Rauzy class \(R\) for which the following is true [2], [3].

**Theorem 5.1** (Veech). There exists a finite-to-one measurable map

\[
π_R : \tilde{V}(R) → H
\]

such that \((π_R)_*μ_R = μ_{κ}\) and \(π_R ◦ P^t = g_t ◦ π_R\) for all \(t ∈ R\).

Recall that the set \(\tilde{V}(R)\) is defined in §3.2.

**Corollary 5.2.** 1. If \(η\) is a \(\{g_t\}\)-invariant ergodic probability measure on \(H\), then there exists a \(\{P^t\}\)-invariant measure \(\tilde{η}\) on \(\tilde{V}(R)\) such that \((π_R)_*\tilde{η} = η\).

2. If \(\tilde{η}\) is a \(\{P^t\}\)-invariant probability measure on \(\tilde{V}(R)\) such that the \(\{g_t\}\)-invariant measure \((π_R)_*\tilde{η}\) is ergodic, then

\[
h_{\tilde{η}}(\{P^t\}) = h_{(π_R)_*\tilde{η}}(\{g_t\}).
\] (5.1)

**Proof.** 1. By Theorem 5.1, for every point \(p ∈ H\) the preimage \(π_R^{-1}(p)\) is finite. We claim that the function \(κ(p) := \#π_R^{-1}(p)\) is Borel measurable. To prove this, we have to look at the construction of the map \(π_R\) mentioned in Theorem 5.1.

The holomorphic differential \(ω\) (see §1) has a finite number of zeros and a finite number of horizontal fibres going out of the zeros. Each of these fibres can be taken
for the base of a zippered rectangle, whose parameters will be linear functions of the relative periods of the form $\omega$ with respect to its zeros. Moreover, if two zippered rectangles coincide, this is expressed as a linear relation between the relative periods of $\omega$. This immediately implies that the number of zippered rectangles corresponding to a given Abelian differential is a Borel measurable function on the moduli space.

Now let $\eta$ be an ergodic $\{g_t\}$-invariant probability measure on $\mathcal{H}$. Due to the ergodicity of $\eta$ and the measurability of the $\{g_t\}$-invariant function $\kappa$, there is a set $\mathcal{H}' \subset \mathcal{H}$ such that $\eta(\mathcal{H}') = 1$ and $\kappa(p)$ does not depend on $p \in \mathcal{H}'$. Moreover, if $p \in \mathcal{H}'$, then the points in $\pi_{\mathcal{R}}^{-1}(p)$ can be numbered so that the number assigned to a point will be a measurable function on $\mathcal{H}'$. With this in mind we derive from (1.1) and Corollary 5.2 that

\[ \{ \text{Corollary 5.3.} \}

\begin{align*}
\text{to the measure space (} \eta, \text{) implies that}
\end{align*}

\begin{align*}
\text{with this in mind we derive from (1.1) and Corollary 5.2 that}
\end{align*}

\[ h_{\mu_{\mathcal{R}}}(\{P^t\}) = 2g - 1 + r = m. \] (5.2)
We call a point \( x \in \mathcal{V}(\mathcal{R}) \) infinitely renormalizable if its trajectory \( \{P^t x, t \in \mathbb{R}\} \) intersects the transversal \( \mathcal{V}^\perp(\mathcal{R}) \) infinitely many times both for \( t > 0 \) and \( t < 0 \).

In §3.2 we denoted the set of infinitely renormalizable points by \( \mathcal{V}_\infty^\perp(\mathcal{R}) \).

The following proposition is in essence contained in [2] and [1].

**Proposition 5.4.** There exists a Borel measurable set \( V \subset \mathcal{H} \) such that

(i) \( \mu(V) = 1 \) for every ergodic \( \{g_t\} \)-invariant probability measure \( \mu \) on \( \mathcal{H} \);

(ii) \( \pi_{\mathcal{R}}^{-1}(p) \cap \mathcal{V}_\infty^\perp \neq \emptyset \) for every point \( p \in V \).

**Proof.** For a compact set \( K \subset \mathcal{H} \), let \( K^\pm \) denote the set of points \( p \in \mathcal{H} \) for which there exist \( t_n \to +\infty \) and \( s_n \to -\infty \) such that \( g_{t_n} p \in K, g_{s_n} p \in K \) for \( n = 1, 2, \ldots \).

Take an increasing sequence of compact sets \( K_n \) such that \( \bigcup_n K_n = \mathcal{H} \) and let \( V = \bigcup_n K_n^\pm \). Then the set \( V \) is Borel measurable. It is clear that, for every probability measure \( \mu \) on \( \mathcal{H} \), there exists \( n_0 \) such that \( \mu(K_{n_0}) > 0 \). If, in addition, \( \mu \) is \( \{g_t\} \)-invariant and ergodic, then \( \mu(K_{n_0}^\pm) = 1 \) and hence \( \mu(V) = 1 \).

Let \((\sigma, \omega)\) belong to an equivalence class \( p \in V \) (see §1). Then by Masur’s theorem [1] the foliations corresponding to \( \text{Re}(\omega) \) and \( \text{Im}(\omega) \) are both uniquely ergodic. This implies, in particular, the existence of an infinitely renormalizable zippered rectangle in \( \pi_{\mathcal{R}}^{-1} p \) (see, for instance, [29]), which is all that we had to prove.

**Lemma 5.5.** Let \( \tilde{\eta} \) be an ergodic \( \{P^t\} \)-invariant probability measure on \( \mathcal{V}(\mathcal{R}) \).

Then there exists a positive word \( \mathbf{p} \in \mathcal{W}_{\mathcal{R}, \mathcal{B}} \) such that \( \tilde{\eta}(\mathcal{V}_p(\mathcal{R})) = 1 \). Moreover, if \( h_{\tilde{\eta}}(\{P^t\}) > 0 \), then for any \( \mathbf{p} \) such that \( \tilde{\eta}(\mathcal{V}_p(\mathcal{R})) = 1 \) there exists a word \( \mathbf{q} \in \mathcal{W}_{\mathcal{R}, \mathcal{B}} \) such that \( \tilde{\eta}(\mathcal{V}_q(\mathcal{R})) = 1 \) and \( \mathbf{p} \) is a simple prefix of \( \mathbf{q} \).

**Proof.** Since the measure \( \tilde{\eta} \) is ergodic with respect to \( \{P^t\} \), its projection \( \eta := (\pi_\mathcal{R})_* \tilde{\eta} \) is ergodic with respect to \( \{g_t\} \). Let \( V \) be as in Proposition 5.4 and let \( W = \pi_{\mathcal{R}}^{-1}(V) \). By Proposition 5.4 \( \eta(W) = 1 \), which implies that \( \tilde{\eta}(W) = 1 \).

For every point \( x = (\lambda, \pi, \delta) \in W \) the interval exchange map \( \langle \lambda, \pi \rangle \) has a unique invariant measure. Now Lemma 4.3 guarantees that there exists a positive word \( \mathbf{q}_x \in \mathcal{W}_{\mathcal{R}, \mathcal{B}} \) such that \( \langle \lambda, \pi \rangle \in \Delta(\mathbf{q}_x) \). Since \( \eta(W) = 1 \), while \( \mathcal{W}_{\mathcal{R}, \mathcal{B}} \) is countable, there exists \( \mathbf{q} \in \mathcal{W}_{\mathcal{R}, \mathcal{B}} \) such that \( \tilde{\eta}\{x : \mathbf{q}_x = \mathbf{q}\} > 0 \). For this \( \mathbf{q} \) we have \( \tilde{\eta}(\mathcal{V}_{\mathbf{q}_x}(\mathcal{R})) > 0 \), and since \( \tilde{\eta} \) is ergodic, \( \tilde{\eta}(\mathcal{V}_\mathbf{q}(\mathcal{R})) = 1 \).

Consider the set \( \mathcal{W}(\mathbf{p}) \subset \mathcal{W}_{\mathcal{R}, \mathcal{B}} \) of all words that have \( \mathbf{p} \) as a prefix. If \( \mathbf{q} \in \mathcal{W}(\mathbf{p}) \) and \( \mathbf{p} \) is not a simple prefix of \( \mathbf{q} \), then \( \mathbf{q} \) is a concatenation: \( \mathbf{q} = \mathbf{q}' \cdots \mathbf{q}'\mathbf{q}'' \), where \( \mathbf{q}' \) is a prefix of \( \mathbf{p} \) and \( \mathbf{q}'' \) is either a prefix of \( \mathbf{q}' \) or empty. In this situation, either \( \tilde{\eta}(\mathcal{V}_\mathbf{q}(\mathcal{R})) = 0 \) or \( \tilde{\eta}(\mathcal{V}_\mathbf{q}(\mathcal{R})) > 0 \) and \( \tilde{\eta} \) is concentrated on the set of periodic points of the flow \( \{P^t\} \). But this cannot be the case for all \( \mathbf{q} \in \mathcal{W}(\mathbf{p}) \) since \( \tilde{\eta}(\mathcal{V}_\mathbf{q}(\mathcal{R})) = 1 \) and \( h_{\tilde{\eta}}(\{P^t\}) > 0 \).

The following statement will be used below.

**Lemma 5.6.** Let \( \Gamma = (V, E) \) be a directed graph with \( |V| = \infty \) and let \( W(\Gamma) \) be the corresponding family of words (see §2.1). Then for each \( w \in W(\Gamma) \) and each \( n \in \mathbb{N} \) there exists a word \( w' \in W(\Gamma) \) of the form \( w' = w_{n} w_{n-1} \cdots w_1 w \), \( w_i \in W(\Gamma) \), containing \( n \) disjoint subwords equal to \( w \) and such that the word \( w'' := w_{n} w_{n-1} \cdots w_1 w \) is simple.

**Proof.** Denote the first and last letters of \( w \) by \( v^- \) and \( v^+ \), respectively, and construct a sequence of letters \( v_1, v_2, \ldots \) by induction as follows. For \( v_1 \) we take an arbitrary letter that is not contained in \( w \). If \( v_1, \ldots, v_k, 1 \leq k \leq n - 1 \), have already
been chosen, \( w_k^+ \) (\( w_k^- \)) denotes the shortest word with first letter \( v^+ \) and last letter \( v_k \) (\( v_k \) and \( v^- \)), respectively. If there are several words of the same minimal length, we take any of them. Then for \( v_{k+1} \) we take an arbitrary letter other than those contained in at least one of the words \( w, w_1^+, w_1^-, \ldots, w_k^+, w_k^- \).

Consider the sequence of words

\[
 w, w_1^+, w_1^-, w, w_2^+, w_2^-, w, \ldots, w, w_n^+, w_{n-1}^-, w.
\]

Delete the last letter from each word but the last one and denote by \( w' \) the concatenation of the words thus obtained. It is easy to check that \( w' \) possesses the required properties. (To define \( w_k \) we have to remove the first and last letters from \( w_k^+ \) and the last letter from \( w_k^- \), and then take the concatenation of the two words obtained.)

**Completion of the proof of Theorem 1.1.** By Corollary 5.3 it suffices to prove that \( \mu_{\mathcal{A}} \) is a unique measure with maximal entropy for the flow \( \{P_t\} \).

Let \( \mu \) be a measure on \( \mathcal{Y}(\mathcal{A}) \) with \( h_\mu(\{P_t\}) \geq h_{\mu_{\mathcal{A}}}(\{P_t\}) \). Without loss of generality we can assume that \( \mu \) is ergodic (otherwise one could pass to an ergodic component). By Lemma 5.5 there exists a word \( q \in \mathcal{W}_{\mathcal{A},B} \) that has a simple positive prefix and is such that \( \mu(\mathcal{Y}_q(\mathcal{A})) = 1 \). Recall that \( \mu_{\mathcal{A}}(\mathcal{Y}_q(\mathcal{A})) = 1 \) as well. By Lemma 4.5 the flow \( \{P_t\}|_{\mathcal{Y}_q(\mathcal{A})} \) is embedded in the suspension flow \( (\sigma_q, f_q) \) via a mapping \( \Psi \). The roof function \( f_q \) clearly depends only on the future (see §2.1). Moreover, by Lemma 4.7, \( f_q \) is bounded away from zero and has summable variations.

By Lemma 4.6 the measure \( \Psi_* \mu_{\mathcal{A}} \) is positive on all nonempty open subsets of the phase space of the flow \( (\sigma_q, f_q) \). This measure induces, in a canonical way, a probability measure \( \pi_q \) on \( (\mathcal{A})^\mathbb{Z} \), the base of the suspension flow \( (\sigma_q, f_q) \). It follows that \( \pi_q \) is positive on all cylinders in \( (\mathcal{A})^\mathbb{Z} \).

We will prove that \( \pi_q \) satisfies the other conditions imposed on the measure \( \pi \) in Theorem 2.2. We apply this theorem to the complete graph with vertex set \( \mathcal{A}_q \); in this situation every sequence of letters, that is, of vertices of the graph, is a word.

Apply Lemma 5.6 to the connected graph \( \Gamma_{\mathcal{A},B} \) (see §4.1) and to the word \( q \) taken as \( w \). From this lemma we obtain, for each \( n \in \mathbb{N} \), a word of the form \( q_0 q \ldots q_n q \in \mathcal{W}_{\mathcal{A},B} \) whose prefix \( q_0 q \ldots q_n \) is a simple word.

Each word \( a_i := q_0 q_i, i = 1, \ldots, n \), is a letter in the alphabet \( \mathcal{A}_q \) introduced in §4.2, and \( a := (a_1, \ldots, a_n) \) is clearly a simple word for all \( n \). Let \( \tilde{a} \) denote an arbitrary word in the alphabet \( \mathcal{A}_q \) that does not contain \( a \) as a subword, and consider the cylinders \( C_a, C_{a\tilde{a}} \subset (\mathcal{A})^\mathbb{Z} \). Using the definition of the measures and maps that appear below, we have

\[
\frac{\pi_q(C_{a\tilde{a}})}{\pi_q(C_a)} = \frac{(\Phi_\ast \mu_{\mathcal{A},q})(\Psi^{-1}_{M_B} C_{a\tilde{a}})}{(\Phi_\ast \mu_{\mathcal{A},q})(\Psi^{-1}_{M_B} C_a)} = \frac{(\Phi_\ast \mu_{\mathcal{A}})(\Psi^{-1}_{M_B} C_{a\tilde{a}})}{(\Phi_\ast \mu_{\mathcal{A}})(\Psi^{-1}_{M_B} C_a)}.
\]  

Observe that \( \Psi^{-1}_{M_B} C_a = C \cap \Omega^\mathbb{Z}_q \), where \( C \) is a cylinder in \( \Omega^\mathbb{Z}_{\mathcal{A},B} \) whose support is contained in \( \mathbb{Z}_+ \). It follows that

\[
\psi^{-1}(\psi(C \cap \Omega^\mathbb{Z}_q)) = C \cap \Omega^\mathbb{Z}_q^+,
\]
where $\Omega_{q^+}^\ast$ consists of all $\omega = (\omega_k, k \in \mathbb{Z}) \in \Omega_{\mathcal{A}, B}^\ast$ such that $(\omega_n, \ldots, \omega_{n+|q|-1}) = q$ for $n = 0$ and for infinitely many $n > 0$. Hence

$$(\psi_*(\Phi_\ast\mu_\mathcal{A}^1))(\psi\Psi_{\mathcal{M}-B}^{-1}Ca) = (\psi_*(\Phi_\ast\mu_\mathcal{A}^1))(\psi(C \cap \Omega_{q}^\ast)) = (\Phi_\ast\mu_\mathcal{A}^1)(C \cap \Omega_{q}^\ast).$$

Since the measure $\Phi_\ast\mu_\mathcal{A}^1$ is shift-invariant, while the word $q$ is a prefix of $a_1 \in \mathcal{A}_{\mathcal{A}, B}$, from the Poincaré Recurrence Theorem we obtain

$$(\Phi_\ast\mu_\mathcal{A}^1)(C \cap \Omega_{q}^\ast) = (\Phi_\ast\mu_\mathcal{A}^1)(C \cap \Omega_{q}^\ast) = (\Phi_\ast\mu_\mathcal{A}^1)(C),$$

so that

$$(\psi_*(\Phi_\ast\mu_\mathcal{A}^1))(\psi\Psi_{\mathcal{M}-B}^{-1}Ca) = (\Phi_\ast\mu_\mathcal{A}^1)(\Psi_{\mathcal{M}-B}^{-1}Ca). \quad (5.4)$$

In this argument we can replace $C_a$ by $C_{a\tilde{a}a}$ to obtain

$$(\psi_*(\Phi_\ast\mu_\mathcal{A}^1))(\psi\Psi_{\mathcal{M}-B}^{-1}C_{a\tilde{a}a}) = (\Phi_\ast\mu_\mathcal{A}^1)(\Psi_{\mathcal{M}-B}^{-1}C_{a\tilde{a}a}). \quad (5.5)$$

Substituting (5.4) and (5.5) into (5.3), while taking (4.15) into account yields

$$\frac{\overline{\mu}_q(C_{a\tilde{a}a})}{\overline{\mu}_q(C_a)} = \frac{(\Phi_\ast\nu)(\psi\Psi_{\mathcal{M}-B}^{-1}C_{a\tilde{a}a})}{(\Phi_\ast\nu)(\psi\Psi_{\mathcal{M}-B}^{-1}C_a)} = \frac{\nu((\Phi^{-1}(\psi\Psi_{\mathcal{M}-B}^{-1}C_{a\tilde{a}a}))}{\nu((\Phi^{-1}(\psi\Psi_{\mathcal{M}-B}^{-1}C_a))}. \quad (5.6)$$

Similarly,

$$\psi(\Psi_{\mathcal{M}-B}^{-1}Ca) = C_{a(q)} \cap \Omega_q, \quad \psi(\Psi_{\mathcal{M}-B}^{-1}C_{a\tilde{a}a}) = C_{a(q)} \cap \Omega_q,$$

where $C_{a(q)}$, $C_{a\tilde{a}(q)}$ are cylinders in $\Omega_{\mathcal{A}, B}$ corresponding to the words $a(q) := qq_1q \ldots qq_nq \in \mathcal{A}_{\mathcal{A}, B}$ and $\tilde{a}(q) := a(q)\tilde{w}a(q) \in \mathcal{A}_{\mathcal{A}, B}$ with some $\tilde{w} \in \Omega_{\mathcal{A}, B}$. Therefore (see (5.6), (4.5)),

$$\frac{\overline{\mu}_q(C_{a\tilde{a}a})}{\overline{\mu}_q(C_a)} = \frac{\nu(\Delta(\tilde{a}(q)))}{\nu(\Delta(a(q)))}. \quad (5.7)$$

We shall now apply Proposition 4.8 with $p' = qq_1q \ldots qq_nq$, $p = a(q)$, $r = \tilde{a}(q)$, where $n$ is large enough. We may do so because of the following two facts:

1) the above choice of the word $\tilde{a}$ implies that $\tilde{w}$ contains no subwords equal to $a(q)$;
2) $d(p) \leq e^{-n\alpha}$ for some $\alpha > 0$, as can be shown using the method in §4.3.

By this proposition combined with (5.3) we obtain

$$\left|\frac{\overline{\mu}_q(C_{a\tilde{a}a})}{\overline{\mu}_q(C_a)} - \exp(-m\tau_{a(q)}^1(\lambda, \pi))\right| \leq \beta_2 \exp(-n\alpha - m\tau_{a(q)}^1(\lambda, \pi)),$$

where

$$(\lambda, \pi) \in (\Phi^{-1}\psi\Psi_{\mathcal{M}-B}C_{a\tilde{a}a}) \cap \Phi^{-1}\Omega_{a(q)}.$$
We thus see that the measure \( \bar{\mu}_q \) satisfies the assumptions of Theorem 2.2 (in particular, the constant \( s \) in (2.4) equals \( m = h_{\Psi, \mu_{\mathcal{R}}} (\sigma_q, f_q) \)). This theorem implies that \( (\bar{\mu}_q)_f \) is a measure with maximal entropy for \( (\sigma_q, f_q) \) and hence \( \mu_{\mathcal{R}} \) is a measure with maximal entropy for \( \{ P^t \} \), so that \( h_{\text{top}} (\{ P^t \}) = h_{\text{top}} (\sigma_q, f_q) = m \).

As to the measure \( \mu \), it follows that \( h_{\mu} (\{ P^t \}) = h_{\mu_{\mathcal{R}}} (\{ P^t \}) \). Then \( \Psi_* \mu \) has the same entropy with respect to the suspension flow \( (\sigma_q, f_q) \) (see (5.2)) and hence is a measure with maximal entropy for this flow. But from Theorem 2.2 we know that such a measure is unique. This completes the proof of Theorem 1.1.

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