The influence of friction on gas parameters in the minimum section of nozzles

V S Okhotin and E V Dzhuraeva
National Research University “Moscow Power Engineering Institute”
E-mail: Dzhuraevaev@mpei.ru

Abstract. Processes of gas flow in nozzles, accompanied by the release of frictional heat, are presented in the form of polytropic processes. The polytropic process index $n$ determines the degree of irreversibility of the gas flow process caused by the release of frictional heating. Relations are obtained to calculate the flow rate and thermodynamic properties of gas in the minimum section of the Laval nozzle and in the outlet section of the convergent nozzle at a pressure behind the nozzle less than the critical pressure. The gas calculated parameters (pressure, temperature, specific volume, velocity, cross-sectional area) in the minimum cross-section differ from the recommended values in the reference literature [1]. In particular, the gas pressure in the minimum cross section turns out to be higher than the critical pressure recommended in [1].

The proposed publication is devoted to the study of the effect of friction on gas parameters, namely, temperature, pressure, specific volume, etc., during its flow in channels of variable cross-section – nozzles that are widely used in power engineering. The problem solved in this work is to determine, on the basis of the adopted model of the flow, the parameters of the gas in the outlet section of the convergent nozzle, operating in the off-design mode, and in the minimum cross section of the Laval nozzle, operating in the design mode, when the pressure behind the nozzles is less than the critical pressure.

To solve this problem, a one-dimensional stationary flow of an ideal gas in converging nozzles, as well as in the converging part of a Laval nozzle, is considered. The heat capacity of the gas is assumed to be a constant value, independent of temperature. In a frictionless flow at a medium pressure behind the nozzle $p_{ave} \leq p_{cr}$, a critical pressure $p_{cr}$ is established in the outlet section of the converging nozzle (Fig. 1), the flow velocity $w_{cr}$ becomes equal to the speed of sound, and $d_f = 0$ (f is the cross-sectional area of the nozzle). The critical pressure is determined by the critical pressure ratio [1,2]

$$
\pi_{cr} = \frac{p_{cr}}{p_0} = \left(\frac{2}{k+1}\right)^{k-1},
$$

where $p_0$ is the pressure of the inhibited gas flow, and $k$ is the indicator of the adiabatic process.

In the minimum cross section of the Laval nozzle $d_f = 0$: in a frictionless flow in the design mode, the critical pressure (1) and the velocity equal to the local speed of sound are established. It is obvious that, within the framework of a one-dimensional stationary flow model, the processes occurring in converging nozzles at $p_{ave} < p_{cr}$ and in the converging part of the Laval nozzle operating in the design mode ($p_{ave} < p_{cr}$) are similar. In a frictionless flow, these are 0-cr processes in Figure 1.
In a frictional flow, to determine the parameters of the gas flow state in the minimum section of the Laval nozzle or in the outlet section of a converging nozzle at the pressure of the medium behind the nozzle \( p_{ave} < p_{cr} \), it is necessary to know the nature of the release of frictional heat along the length of the nozzle or to take as a model an equation describing the actual process, for example, 0-a (Fig.1), in which the entropy \( s \) increases. The used velocity coefficient \( \phi \), which allows estimating the effect of friction by comparing the flow velocities in the presence of friction \( w_a \) (in the process 0-a) and without friction \( w_{cr} \) (in the process 0-cr) at the same initial and final pressures \( p_0 \) and \( p_{cr} \), is

\[
\phi = \frac{w_a}{w_{cr}},
\]

and is equal to the speeds \( w_a \) and \( w_{cr} \):

\[
w_a = \sqrt{2(h_0 - h_a)}; \quad w_{cr} = \sqrt{2(h_0 - h_{cr})},
\]

where \( h \) is the enthalpy of the gas in the states corresponding to Fig. 1.

The velocity coefficient \( \phi \) allows calculating the real (taking friction into account) gas outflow velocity \( w_a \) and determining its enthalpy

\[
h_a = h_{cr} + (1 - \phi^2)(h_0 - h_{cr}),
\]

and its temperature for an ideal gas with constant heat capacity

\[
T_a = T_{cr} + (1 - \phi^2)(T_0 - T_{cr}).
\]

To determine the minimum cross-sectional area, it is necessary to know pressure in addition to temperature. It is generally accepted [1] that in a flow with friction, the gas pressure in the minimum cross section is the same as in the flow without friction, therefore, \( p_a = p_{cr} \), as is shown in Fig. 1.

In this publication, it is proposed to describe the flow process to assess the effect of friction by the equation of the polytropic process \( p \nu^n = \text{const} \), in which \( n \) is the index of the polytropic process - a constant value for a given process at a fixed stagnation pressure \( p_0 \) at the nozzle inlet and \( p \) is the pressure in an arbitrary section. The relationship between the rate factor \( \phi \) and the index of the polytropic process \( n \) at pressures \( p_0 \) and \( p_{cr} \) may be obtained if in equation (4) we use the relations for isentropic and polytropic processes 0-cr and 0-a:

\[
T_{cr} = T_0 (\pi_{cr})^{\frac{n-1}{n}}; \quad T_a = T_0 (\pi_{cr})^{\frac{n-1}{n}}.
\]

The relations between \( \phi \) and \( n \) obtained in this way therefore look like:

\[
\phi^2 = \frac{1-(\pi_{cr})^{\frac{n-1}{n}}}{1-(\pi_{cr})^{\frac{k-1}{k}}},
\]

\[
n = \frac{\ln(\pi_{cr})}{\ln \left( \frac{\pi_{cr}}{1-\phi^2+\phi^2(\pi_{cr})^{\frac{k-1}{k}}} \right)}.
\]
Obviously, relations (6) and (7) will be valid if, instead of $\pi_{cr}$, we use $\pi = p/p_0$, where $p_0$ is the stagnation pressure, and $p$ is the current coordinate in the arbitrary section of the nozzle. The index of the polytropic process $n$ of the considered processes of type 0-a ($dT < 0; ds > 0$) can vary in the range from 1 to $k$ ($1 < n < k$). In a reversible flow, when $q_{tr} = 0$, $\varphi = 1$; $n = k$. In irreversible processes, the greater the friction is, the lower the velocity coefficient $\varphi$ and the lower the index of the polytropic process $n$ are.

The relationship between the velocity coefficient $\varphi$ and the index of the polytropic process $n$, determined by relations (6) and (7), is shown in Figure 2 for one, two, and polyatomic gases.

![Figure 2](image1)

**Figure 2.** The relationship between the rate factor $\varphi$ and the index of the polytropic process $n$ for monatomic ($k = 1.667$), diatomic ($k = 1.4$) and polyatomic ($k = 1.3$) gases.

It should be noted that the use of the equation of the polytropic process $pv^n = \text{const}$ to describe the adiabatic irreversible (with friction) process 0-a is conditional, since in the adiabatic process $q = 0$, and in the polytropic process (for $n \neq k$) $q \neq 0$. The equation of the polytropic process is used here, first of all, to determine the parameters and functions of the gas state in the real process 0-a (Fig. 1). The heat of such a polytropic process will be equal to the heat of friction $q_{fr}$ of the adiabatic irreversible process 0-a, rather than to the heat supplied to the flow from the outside, which is equal to zero in the adiabatic process.

To ensure the maximum mass flow rate of gas through the nozzle in the minimum cross section, the ratio of velocity to specific volume ($w / v$) should be maximum:

$$m_f = \frac{w}{v} \rightarrow \text{max.} \quad (8)$$

The flow rate of an ideal gas with constant heat capacity, performing a polytropic process 0-a, has the form

$$w = \left[ 2 \frac{k}{k-1} RT_0 \left( 1 - \pi^{n-1} \right) \right]^{1/2}, \quad (9)$$

specific volume $v$ in an arbitrary section for a polytropic process $\frac{1}{v} = \frac{1}{v_0} \pi^{1/n}$, and their relation to each other is as follows

$$\frac{w}{v} = \left[ 2 \frac{k}{k-1} \frac{p_0}{v_0} \left( \pi^n - \pi^{n+1} \right) \right]^{1/2}, \quad (10)$$

where $\pi = p/p_0$, $p$ is the pressure in an arbitrary section of the nozzle, and $p_0, v_0$ are the braking parameters.

By solving the equation $\frac{d(w/v)}{dx} = 0$, the optimal pressure ratio $\pi_2$ is determined, at which the required gas flow rate $m$ will be provided in the minimum section:

$$\pi_2 = \left( \frac{2}{n+1} \right)^{n-1}, \quad (11)$$

the gas pressure in the minimum section.
and temperature

\[ p_2 = p_0 \pi_2, \]  

(12)

\[ T_2 = \frac{2p_0}{\pi_0 (n+1)}. \]  

(13)

Since the indicator of the polytropic process \( n < k \), then, in accordance with (12) and (13), the state of the gas in the minimum section 2 in the \( T, s \)-diagram (Figure 1) is shown on the polytropic \( n = \text{const} \) above the point a.

Let us determine how much the gas parameters in the minimum cross section (state 2), obtained in this work, differ from the gas parameters at point a (Figure 1) and those recommended in the reference literature [1]. Equations for \( \pi_2 \) (11) and for \( \pi_{cr} \) (1) have the same form, therefore, for a numerical example, one can use the adiabatic exponent \( k \) calculated according to the molecular kinetic theory and the \( \pi_{cr} \) value according to formula (1): for a monatomic gas \( k = 5/3 = 1.667 \) and \( \pi_{cr} = 0.487 \); for a diatomic gas \( k = 1.4 \) and \( \pi_{cr} = 0.528 \); for a polyatomic gas \( k = 1.3 \) and \( \pi_{cr} = 0.546 \).

So, for example, in the flow of an ideal monatomic gas, for which \( k = 1.667 \) with a reversible (without friction) flow in the minimum cross section \( \pi_2 = \pi_{cr} = 0.487 \). In the flow of the same monatomic ideal gas, but with friction (at \( n = 1.4 \)), the pressure ratio in the minimum section is \( \pi_2 = 0.528 \); with greater friction, when \( n = 1.3 \), \( \pi_2 = 0.546 \). Thus, the pressure \( p_2 \) in the minimum section of the nozzle in the presence of friction is greater than in the case of a reversible flow, when there is no friction (Figure 1). In addition, as it follows from the above example, the greater the friction is, the greater the pressure in the minimum section \( p_2 \) is, as shown in Figure 3.

To compare the pressure in the minimum cross section \( p_2 \), calculated according to (12) in this work, with the pressure \( p_{cr} \) obtained from (1) and recommended in [1], Figure 4 shows the dependence of the ratio \( p_2/p_{cr} \) on the polytropic process index \( n \), which was calculated as \( p_2/p_{cr} = \pi_2/\pi_{cr} \). It also shows the values of the velocity factor \( \varphi \), calculated by formula (6) for the pressure ratio \( p_2/p_{cr} \), corresponding to the gas velocity in the minimum section \( w_2 \) (in the process 0-2) and the velocity \( w_2 \) (in the reversible process 0-2, Figure 1).

For example, for a monatomic gas (Figure 4a, \( k = 1.667 \)) in a flow with friction, if it is characterized by a polytropic exponent \( n = 1.4 \), then \( p_2/p_{cr} = 1.085 \), i.e., the pressure in the minimum section \( p_2 \) is 8.5% higher than the critical pressure \( p_{cr} = 0.487p_0 \). With the same value \( n = 1.4 \) and the found pressure \( p_2 \), the velocity coefficient \( \varphi = 0.865 \). At \( n = 1.3 \), the discrepancy between \( p_2 \) and \( p_{cr} \) reaches 12.5%.

In a similar way, for a diatomic gas (\( k = 1.4 \)), Fig. 4b shows: for \( n = 1.3 \), the pressure in the minimum section \( p_2 \) is 3.3% higher than the critical pressure \( p_{cr} = 0.528p_0 \), and the velocity coefficient \( \varphi = 0.90 \); at \( n = 1.2 \), the pressure \( p_2 \) is 6.8% higher than the critical pressure \( p_{cr} \), and the velocity coefficient is \( \varphi = 0.78 \).

For a polyatomic gas (Figure 4c, \( k = 1.3 \)) at \( n = 1.2 \), the pressure \( p_2 \) is 3.5% higher than the critical pressure \( p_{cr} = 0.546p_0 \), and the velocity coefficient is \( \varphi = 0.86 \); at \( n = 1.1 \), the pressure \( p_2 \) is 6.8% higher than the critical pressure \( p_{cr} \), and the velocity coefficient \( \varphi = 0.78 \).

\[ \begin{align*}
\text{Figure 4.} & \quad \text{Dependence of the pressure ratio } p_2/p_{cr} (- - -) \text{ and the rate coefficient } \varphi (---) \text{ on the index of the polytropic process } n \text{ for monatomic (a), diatomic (b) and polyatomic (c) gases.}
\end{align*} \]
It is of interest to compare how friction affects the flow of monatomic, diatomic, and polyatomic gases at the same velocity coefficient. To do this, it is necessary to solve the inverse problem - having given the velocity coefficient \( \varphi \), determine the polytrope index \( n \) and the pressure ratio \( p_2/p_\varphi \) for various gases (for various adiabatic indices \( k \)). Let \( \varphi = 0.9 \), then for monatomic gases (\( k = 1.667 \)) the polytrope index is \( n = 1.46 \) and \( p_2 / p_\varphi = 1.065 \); for diatomic gases (\( k = 1.4 \)) the polytrope index is \( n = 1.295 \) and \( p_2 / p_\varphi = 1.035 \); for polyatomic gases (\( k = 1.3 \)), the polytrope index is \( n = 1.24 \) and \( p_2 / p_\varphi = 1.02 \).

Thus, at \( \varphi = 0.9 \), the difference between \( p_2 \) and \( p_\varphi \) for monatomic gases is 6.5%, for diatomic gases it is 3.5% and for polyatomic gases it is 2.0%. With a greater release of frictional heat, for example, at \( \varphi = 0.8 \), the difference between \( p_2 \) and \( p_\varphi \) increases: it is 11.0% for monatomic gases, 6.2% for diatomic gases, and 4.5% for polyatomic gases.

To calculate the specific volume of gas in state 2 and state \( a \) (Fig. 1), the equation of the polytropic process \( p v^n = \text{const} \), and formulas (1) and (11) are used. The calculated relations for \( v_2 \) and \( v_a \), obtained in this way have the form

\[
v_2 = v_0 / \left( \frac{2}{n+1} \right)^{\frac{1}{n-1}},
\]

\[
v_a = v_0 / \left( \frac{2}{k+1} \right)^{\frac{k}{k-1}}.
\]

To assess the difference between \( v_2 \) and \( v_a \), the ratio of specific volumes \( v_2/v_a \) is calculated; it is shown in Fig. 5 as a function of the polytropic indices \( n \) and adiabatic \( k \). Here, as in Fig. 4, a comparison with the velocity coefficient \( \varphi \) is given.

![Figure 5](image.png)

**Figure 5.** Dependence of the ratio of the specific volumes \( v_2/v_a \) (---) and the rate coefficient \( \varphi \) ( - - - ) on the index of the polytropic process \( n \) for monatomic (a), diatomic (b) and polyatomic (c) gases.

The dependences shown in Fig. 5 allow one to estimate the differences between \( v_2 \) and \( v_a \) for different gases at the same velocity coefficient \( \varphi \). Let \( \varphi = 0.9 \). Then, for monatomic gases (\( k = 1.667 \)), the specific volume \( v_2 \), calculated according to the method proposed in this work from the pressure \( p_2 \), exceeds the specific volume \( v_a \), calculated according to the existing method, at the pressure \( p_a = p_\varphi \) by 4.0%, for diatomic gases (\( k = 1.4 \)) - by 2.5% and for polyatomic gases (\( k = 1.3 \)) - by 2.0%. With a greater release of frictional heat, for example, at \( \varphi = 0.8 \), the excess of the specific volume \( v_2 \) over \( v_a \) is 7.5% for monatomic gases, 5.0% for diatomic gases, and 3.8% for polyatomic gases.

The gas flow rate in the minimum cross section is determined from (9) using the relation (11)

\[
w_2 = \left[ 2 \frac{k}{k-1} R T_0 \frac{n-1}{n+1} \right]^{1/2}.
\]

Comparison of the flow rate \( w_2 \) with the rate calculated for process 0-\( a \) is given in the form of the dependence of the ratio \( w_2/w_a \) on the index of the polytropic process \( n \) in Fig. 6 for one-, two-, and polyatomic ideal gases. It also shows the dependence of the velocity coefficient \( \varphi \) on the indicator of the polytropic process \( n \). The velocity \( w_a \) was determined by formula (9) using (1):
\[ w_a = \left[ 2 \frac{k}{k-1} RT_0 \left( 1 - \pi \frac{n-1}{n} \right) \right]^{1/2}. \] 

(17)

**Figure 6.** Dependence of the ratio of the velocities \( w_2/w_a \) (---) and the velocity coefficient \( \varphi \) (- - -) on the index of the polytropic process \( n \) for monatomic (a), diatomic (b) and polyatomic (c) gases.

The obtained dependences, shown in Figure 6, allow one to estimate the differences between the velocities \( v_2 \) and \( v_a \) for different gases at the same velocity coefficient \( \varphi \). Let's take \( \varphi = 0.9 \). Then, for monoatomic gases (\( k = 1.667 \)), the velocity \( w_2 \), calculated by the method proposed in this work from the pressure \( p_2 \), is less than the velocity \( w_a \) calculated by the existing method [1] from the pressure \( p_a = p_0 \), by 4.0%, for diatomic gases (\( k = 1.4 \)) - by 2.5% and for polyatomic gases (\( k = 1.3 \)) - by 2.0%. With a greater release of frictional heat, for example, at \( \varphi = 0.8 \), the differences in the velocities \( w_2 \) and \( w_a \) increase and amount to 7.0% for monatomic gases, 4.5% for diatomic gases, and 3.8% for polyatomic gases.

Continuity equation (8) allows calculating the minimum cross-sectional area, which for the gas state \( 2 \) and state \( a \) (Figure 1) will be different: \( f_2 \) and \( f_a \). The ratio of these areas is calculated using the specific volumes \( v_2 \) (14) and \( v_a \) (15), as well as the gas velocities \( w_2 \) (16) and \( w_a \) (17) in the minimum cross section:

\[ \frac{f_2}{f_a} = \frac{v_2 w_2}{v_a w_2}. \]

(18)

The area ratios \( f_2/f_a \) and the velocity coefficient \( \varphi \) are calculated according to (18). In contrast to the gas flow rate, its pressure and specific volume in the minimum section, the differences in the areas \( f_2 \) and \( f_a \) at \( \varphi = 0.9 \) and \( \varphi = 0.8 \) are insignificant and do not exceed 1%.

**Conclusions**

The effect of friction on the gas parameters in the minimum section of the Laval nozzle, operating in the design mode, and in the outlet section of the converging nozzle, operating in the off-design mode, at a pressure behind the nozzle \( p_{ave} \leq p_{cr} \) has been investigated. The thermal parameters of the gas in the flow have been described by the equation of a polytropic process, in which the polytropic exponent served as an indicator of irreversibility caused by the release of frictional heat. Formulas have been obtained to calculate the gas parameters (pressure, temperature, specific volume), flow rate and minimum cross-sectional area in the minimum section. The results of numerical calculations of the gas parameters have shown that the pressure in the minimum cross section of the considered nozzles is higher than the critical pressure recommended in the reference literature. As a consequence, the temperature and specific volume of the gas, the velocity in the minimum section, and the area of the minimum section differ. The comparison results are presented in the graphs.

**References**

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