Isomorphism Problem Revisited: Information
Spectrum Approach

Shun Watanabe∗ and Te Sun Han†

∗Department of Computer and Information Sciences, Tokyo University of Agriculture and Technology, Japan,
E-mail:shunwata@cc.tuat.ac.jp
†Quantum ICT Advanced Development Center, National Institute of Information and Communications Technology, Japan,
E-mail:han@is.uec.ac.jp

Abstract—The isomorphism problem in the ergodic theory is revisited from the perspective of information spectrum approach, an approach that has been developed to investigate coding problems for non-ergodic random processes in information theory. It is proved that the information spectrum is invariant under isomorphisms. This result together with an analysis of information spectrum provide a conceptually simple proof of the result by Šujan, which claims that the entropy spectrum is invariant under isomorphisms. It is also discussed under what circumstances the same information spectrum implies the existence of an isomorphism.

I. INTRODUCTION

In ergodic theory, one of fundamental problems is to identify if two dynamical systems are isomorphic or not, which is known as the isomorphism problem. Inspired by the Shannon entropy in information theory, Kolmogorov and Sinai introduced the entropy of dynamical systems and showed that the entropy is an invariant under isomorphism; in other words, an isomorphism between two dynamical systems exists only if the entropies are equal. Since then, the entropy has been widely used as an invariant of the isomorphism problem. However, the entropy need not be a complete invariant, i.e., an isomorphism may not exist even if the entropies of two dynamical systems are equal. Then, an interesting question is under what circumstances the same entropy implies the existence of an isomorphism. A landmark result on this problem was provided by Ornstein in [8] (see also [9]). He proved that two i.i.d. random processes (Bernoulli shifts) are isomorphic to each other if the entropies are equal; furthermore, he also characterized the class of processes that are isomorphic to the i.i.d. random processes. See [10], [11] for other interactions between information theory and ergodic theory.

In the literature, most studies on the isomorphism problem have focused on ergodic dynamical systems with some exceptions [6], [14], [15], [13]. In [6], Kieffer and Rahe provided a sufficient condition on the existence of isomorphism between two non-ergodic mixtures of Bernoulli shifts. In [14], [15], Šujan provided a necessary condition for the existence of isomorphism in terms of “entropy spectra,” leveraging the ergodic decomposition. In [13], Takens and Verbitskiy showed that the Rényi entropy of non-ergodic dynamical system is given by the essential infimum of the spectrum of entropies of the ergodic decomposition.

On the other hand, in the 1990s, Han and Verdú developed “information spectrum” approach in information theory to investigate coding problems for general non-ergodic sources/channels [5] (see also [4]). Among other things, the key feature of the approach is that coding theorems are proved in two steps. In the first step, the performance of a coding problem is characterized by the probabilistic behavior of self-information or related quantities, which is termed the information spectrum. This step is proved without invoking probability theoretic theorems, such as the law of large number or the ergodic theorem. Then, in the second step, the probability theoretic theorems are invoked to characterize the behavior of the information spectrum in terms of information measures such as the entropy.

The main aim of this paper is to revisit the isomorphism problem from the perspective of information spectrum approach. More specifically, we prove that the information spectrum is invariant under isomorphisms between random processes.1 Then, using this result together with an analysis of information spectra, we provide an alternative proof for the aforementioned result by Šujan [14], [15], which is conceptually and technically much simpler than the argument given in [14], [15].2

Even though the information spectrum coincides with the entropy spectrum under ergodic decomposition, we are intentionally distinguishing the two concepts, “information spectrum” and “entropy spectrum.” The former is defined directly for a given random process, and we prove the invariance of information spectra without invoking the ergodic decomposition; the ergodic decomposition is only needed to prove that the information spectrum coincides with the entropy spectrum. On the other hand, the argument in [14], [15] begins with the ergodic decomposition, and the invariance of the entropy spectrum is proved via the invariance of entropy in each ergodic component.

1It should not be confused with the spectral isomorphism of linear operators induced by dynamical systems (eg. see [16]).
2An advantage of the approach in [14], [15] is that it can be applied to random processes with countably infinite alphabet, while we use the finiteness of alphabet in our proof.
II. PRELIMINARIES

In this section, we introduce our notation by reviewing some basic facts in ergodic theory. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space. A measurable map \(T : \Omega \to \Omega\) is called measure-preserving transformation if \(\mu(T^{-1}A) = \mu(A)\) for every \(A \in \mathcal{F}\). The quadruple \((\Omega, \mathcal{F}, \mu, T)\) is called a dynamical system. When \(\Omega = \mathcal{X}^\mathbb{Z}\), i.e., the set of all doubly infinite sequences

\[
x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots),
\]

where each \(x_i\) is an element of some finite set \(\mathcal{X}\), the measure-preserving transformation is given by the shift \(S^i\), i.e., \((S^i x)_i = x_{i+1}\) for \(x \in \mathcal{X}^\mathbb{Z}\); the measurable set \(\mathcal{B}_X\) is given by the \(\sigma\)-algebra generated by cylinder set

\[
[a^n_m] := \{x \in \mathcal{X}^\mathbb{Z} : x_i = a_i \ \forall m \leq i \leq n\}
\]

for \(m, n \in \mathbb{Z}\). Let us define the random process \(X = \{X_n\}_{n \in \mathbb{Z}}\) by assigning

\[
P_{X_n}(a^n_m) = \Pr\{X_i = a_i : m \leq i \leq n\} = \mu([a^n_m])
\]

for \(m, n \in \mathbb{Z}\). Owing to the measure-preserving requirement of \(S\), the random process \(X\) is stationary. When \(m = 1\), we denote \(P_{X_n}(a^n_m)\) by \(P_{X_n}(a^n)\) for \(a^n \in \mathcal{X}^n\). In this manner, the dynamical system \((\mathcal{X}^\mathbb{Z}, \mathcal{B}_X, \mu, S)\) can be identified with the random process \(X\). Throughout the rest of this paper, we mainly consider the random process \(X\) determined by \((\mathcal{X}^\mathbb{Z}, \mathcal{B}_X, \mu, S)\).

One of the most fundamental problems in ergodic theory is to determine if given two processes are “equivalent” or not. A commonly used notion of equivalence is defined as follows.

Definition 1: For two stationary random processes \(X = \{X_n\}_{n \in \mathbb{Z}}\) and \(Y = \{Y_n\}_{n \in \mathbb{Z}}\) determined by \((\mathcal{X}^\mathbb{Z}, \mathcal{B}_X, \mu, S)\) and \((\mathcal{Y}^\mathbb{Z}, \mathcal{B}_Y, \nu, S)\), respectively, we call a measurable map \(\phi : \mathcal{X}^\mathbb{Z} \to \mathcal{Y}^\mathbb{Z}\) a homomorphism if \(\nu = \phi_\# \mu\), i.e., \(\nu(B) = \mu(\phi^{-1}(B))\) for every \(B \in \mathcal{B}_Y\), and \(\phi(Sx) = S\phi(x)\) for almost sure \(x \in \mathcal{X}^\mathbb{Z}\) under \(\mu\). Furthermore, when there exists a homomorphism \(\psi : \mathcal{Y}^\mathbb{Z} \to \mathcal{X}^\mathbb{Z}\) such that \(\psi(\phi(x)) = x\) for almost sure \(x \in \mathcal{X}^\mathbb{Z}\) under \(\mu\), then \(\phi\) and \(\psi\) is called an isomorphism. When there exists an isomorphism between two stationary random processes, those processes are said to be isomorphic.

In order to determine if given two random processes are isomorphic or not, one of the most basic criterion is the ergodicity.

Definition 2: A random process \(X = \{X_n\}_{n \in \mathbb{Z}}\) determined by \((\mathcal{X}^\mathbb{Z}, \mathcal{B}_X, \mu, S)\) is called ergodic if, for every \(A \in \mathcal{B}_X\) with \(\mu(A \triangle S^{-1}A) = 0\), it holds that \(\mu(A) = 0\) or \(\mu(A) = 1\), where \(\triangle\) is the symmetric difference of sets.

From the definition, we can readily verify that ergodicity is an invariant under homomorphism (eg. see [10, Example I.2.12]).

Proposition 1: For two stationary random processes \(X = \{X_n\}_{n \in \mathbb{Z}}\) and \(Y = \{Y_n\}_{n \in \mathbb{Z}}\), suppose that there exists a homomorphism from \(X\) to \(Y\). If \(X\) is ergodic, then \(Y\) is also ergodic.

Definition 4: For a stationary random process \(X = \{X_n\}_{n \in \mathbb{Z}}\), the entropy rate is defined by

\[
H(X) := \lim_{n \to \infty} \frac{1}{n} H(X_1, \ldots, X_n),
\]

where

\[
H(X_1, \ldots, X_n) := \sum_{x^n \in \mathcal{X}^n} P_{X^n}(x^n) \log \frac{1}{P_{X^n}(x^n)}.
\]

One of the fundamental results in ergodic theory is the following.

Proposition 2 ([7], [12]): For two stationary random processes \(X = \{X_n\}_{n \in \mathbb{Z}}\) and \(Y = \{Y_n\}_{n \in \mathbb{Z}}\), if a homomorphism from \(X\) to \(Y\) exists, then it holds that

\[
H(X) \geq H(Y).
\]

Corollary 1 (Isomorphic invariance of entropy): If two stationary random processes \(X = \{X_n\}_{n \in \mathbb{Z}}\) and \(Y = \{Y_n\}_{n \in \mathbb{Z}}\) are isomorphic, then it holds that

\[
H(X) = H(Y).
\]

The entropy has been one of the most widely used invariants to determine if two random processes are isomorphic or not. In fact, when two random processes are independently identically distributed (i.i.d.) processes, i.e., Bernoulli shifts, then Ornstein proved that the entropy is the complete invariant, i.e., the two processes are isomorphic if and only if their entropies are the same [8].

III. INVARIANCE OF INFORMATION SPECTRUM

Let us introduce the information spectrum of a random process [4].

Definition 4: For a stationary random process \(X = \{X_n\}_{n \in \mathbb{Z}}\), the information spectrum is the cumulative distribution function of the normalized self-information defined by

\[
F_X(\tau) := \lim_{n \to \infty} \sup_{\gamma \to 0} \Pr\left(\frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} \leq \tau + \gamma\right)
\]

for \(\tau \in \mathbb{R}^+ := \{a \in \mathbb{R} : a \geq 0\}\).

By the definition, \(F_X(\tau)\) is right-continuous. Since

\[
\Pr\left(\frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} > \log |\mathcal{X}| + \gamma\right) = \sum_{x^n \in \mathcal{X}^n} P_{X^n}(x^n) \mathbb{1}\left[P_{X^n}(x^n) < \frac{2^{-n\gamma}}{|\mathcal{X}|^n}\right] \leq 2^{-n\gamma}
\]

for any \(\gamma > 0\), it follows that \(F_X(\tau) = 1\) for \(\tau \geq \log |\mathcal{X}|\).
When a random process $X$ is ergodic, the asymptotic equipartition property guarantees
\[
\lim_{n \to \infty} \Pr \left( \left| \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} - H(X) \right| \leq \gamma \right) = 1
\]
for any $\gamma > 0$. Thus, the information spectrum of the ergodic process is given as
\[
F_X(\tau) = 1[H(X) \leq \tau],
\]
where $1[\cdot]$ is the indicator function.

When a random process is not ergodic, the information spectrum can be computed based on the entropy spectrum of the ergodic decomposition of the process as follows.

**Theorem 1**: When the ergodic decomposition of a stationary process $X = \{X_n\}_{n \in \mathbb{Z}}$ is given as
\[
P_{X^n}(x^n) = \int_\Theta P_{X^n}^\phi(x^n)dw(\theta)
\]
for a family of ergodic processes $\{X_\theta\}_{\theta \in \Theta}$ with measure $w$ on $\Theta$, the information spectrum of the process is given as
\[
F_X(\tau) = w(\{\theta : H(X_\theta) \leq \tau\}).
\]
When all components are i.i.d., (3) was proved in [4, Lemma 1.4.4]. Exactly the same proof applies to one direction, the left hand side is less than or equal to the right hand side in (3). The opposite direction of the proof also proceeds along the line of [4, Lemma 1.4.4], but it requires more complicated argument to handle ergodic components. Because of the lack of space, the proof will be given in a full version of this paper.

Let $\overline{H}(X)$ and $\underline{H}(X)$ be defined as
\[
\overline{H}(X) := \inf \left\{ \tau : \Pr \left( \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} > \tau \right) = 0 \right\},
\]
\[
\underline{H}(X) := \sup \left\{ \tau : \Pr \left( \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} < \tau \right) = 0 \right\},
\]
which are called the spectral sup-entropy and spectral inf-entropy [4]. Then, we have $F_X(\tau) = 1$ for $\tau \geq \overline{H}(X)$ and $F_X(\tau) = 0$ for $\tau < \underline{H}(X)$.

**Remark 1**: If the information spectrum $F_X(\tau)$ is defined without the slack parameter $\gamma \downarrow 0$ in Definition 4, it may not be right-continuous in general. For instance, when $X$ is an i.i.d. process, the law of large number and the central limit theorem imply
\[
\lim_{n \to \infty} \Pr \left( \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} \leq \tau \right) = \begin{cases} 0 & \text{if } \tau < H(X) \\ 1/2 & \text{if } \tau = H(X) \\ 1 & \text{if } \tau > H(X) \end{cases},
\]
As an information spectrum counterpart of Proposition 2, we have the following theorem, which will be proved in Section IV.

**Theorem 2**: For two stationary random processes $X = \{X_n\}_{n \in \mathbb{Z}}$ and $Y = \{Y_n\}_{n \in \mathbb{Z}}$, if a homomorphism from $X$ to $Y$ exists, then it holds that
\[
F_X(\tau) \leq F_Y(\tau)
\]
for every $\tau \in \mathbb{R}^+$. Theorem 2 says that a necessary condition for the existence of a homomorphism is that the spectrum of $X$ "dominates" the spectrum of $Y$. Simpler necessary conditions are
\[
\overline{H}(Y) \leq \overline{H}(X), \quad \underline{H}(Y) \leq \underline{H}(X).
\]

**Corollary 2**: If two stationary random processes $X = \{X_n\}_{n \in \mathbb{Z}}$ and $Y = \{Y_n\}_{n \in \mathbb{Z}}$ are isomorphic, then it holds that
\[
F_X(\tau) = F_Y(\tau)
\]
for every $\tau \in \mathbb{R}^+$. By combining Corollary 2 with Theorem 1, we can recover the following result by Šujan.

**Corollary 3**: Suppose that a stationary random processes $X = \{X_n\}_{n \in \mathbb{Z}}$ is decomposed as
\[
P_{X^n}(x^n) = \int_\Theta P_{X^n}^\phi(x^n)dw(\theta)
\]
for a family of ergodic processes $\{X_\theta\}_{\theta \in \Theta}$ with measure $w$ on $\Theta$, and a stationary random processes $Y = \{Y_n\}_{n \in \mathbb{Z}}$ is decomposed as
\[
P_{Y^n}(y^n) = \int_\Xi P_{Y^n}^\psi(y^n)dv(\xi)
\]
for a family of ergodic processes $\{Y_\xi\}_{\xi \in \Xi}$ with measure $v$ on $\Xi$. If the two stationary random processes $X$ and $Y$ are isomorphic, then it holds that
\[
w(\{\theta : H(X_\theta) \leq \tau\}) = v(\{\xi : H(Y_\xi) \leq \tau\})
\]
for every $\tau \in \mathbb{R}^+$.

Theorem 2 is a generalization of Proposition 2 in the sense that the former implies the latter. In fact, by noting the ergodic decomposition of the entropy rate [3, Theorem 3.3], the identity of the expectation [1, Eq. (21.9)], and Theorem 1, we can write
\[
H(X) = \int_\Theta H(X_\theta)dw(\theta)
\]
\[
= \int_0^\infty w(\{\theta : H(X_\theta) > \tau\})d\tau
\]
\[
= \int_0^\infty (1 - F_X(\tau))d\tau;
\]
and similarly for $H(Y)$. Thus, by Theorem 2, we have
\[
H(X) = \int_0^\infty (1 - F_X(\tau))d\tau \geq \int_0^\infty (1 - F_Y(\tau))d\tau = H(Y).
\]

**IV. PROOF OF THEOREM 2**

Let $X = \{X_n\}_{n \in \mathbb{Z}}$ and $Y = \{Y_n\}_{n \in \mathbb{Z}}$ be stationary random processes determined by $(X^\mathbb{Z}, \mathcal{B}_X, \mu, S)$ and $(Y^\mathbb{Z}, \mathcal{B}_Y, \nu, S)$. For a given homomorphism $\phi : X^\mathbb{Z} \to Y^\mathbb{Z}$ and integers $m,n$, we can construct a coupling
\[
P_{X^n}(x^n, y^n) = \mu([x^m] \cap \phi^{-1}([y^m]))
\]
induced by the process $X$ and the homomorphism $\phi$. Since $\nu = \phi_*\mu$, the marginal of the joint distribution $P_{X^n}^{m, Y^n}$
coincides with the distribution induced by the process \( Y \). In the following argument, by a slight abuse of notation, we interpret that the random variable \((X^{-m}, Y^{-n})\) are distributed according to the joint distribution given by (4).

First, we approximate an arbitrary homomorphism by using a finite function (e.g., see [2, Theorem 3.1] or [10, Theorem 8.1.1]).

**Lemma 1**: For a given homomorphism \( \phi \) from \( X \) to \( Y \) and arbitrary \( \varepsilon > 0 \), there exists an integer \( \ell = \ell(\varepsilon) \) and a finite function \( f : \mathcal{X}^{2\ell+1} \to \mathcal{Y} \) such that

\[
\Pr \left( Y_0 \neq f(X^\ell) \right) \leq \varepsilon. \tag{5}
\]

Let us now fix an integer \( n \), and set \( N = 2n + 1 \). By stationarity, (5) implies

\[
\mathbb{E} \left[ \frac{1}{N} d_H(Y^n, \tilde{Y}^n) \right] = \frac{1}{N} \sum_{i = -n}^{n} \Pr \left( Y_i \neq \tilde{Y}_i \right) \leq \varepsilon, \tag{6}
\]

where \( \tilde{Y}_i = f(X_i^{\ell+i}) \) and \( d_H(\cdot, \cdot) \) is the Hamming distance. Furthermore, by the Markov inequality, (6) implies

\[
\Pr \left( \frac{1}{N} d_H(Y^n, \tilde{Y}^n) > \beta \right) \leq \frac{\varepsilon}{\beta} \tag{7}
\]

for arbitrary \( \beta > 0 \).

The next lemma is the most key part of the proof of Theorem 2.

**Lemma 2**: For a given homomorphism \( \phi \) from \( X \) to \( Y \) and arbitrary \( \varepsilon > 0 \), let \( \ell \) and \( f \) be the integer and the finite function specified by Lemma 1. For an integer \( n \), set \( m = n + \ell \) and \( N = 2n + 1 \). Then, we have

\[
\Pr \left( \frac{1}{N} \log \frac{1}{P_{X^{-m}}(x^{-m})} \leq \tau + \gamma \right)
\]

\[
\leq \Pr \left( \frac{1}{N} \log \frac{1}{P_Y(y^-n)} \leq \tau + 2\gamma \right) + \frac{\varepsilon}{\beta} + 2^{-N(\gamma - h(\beta) - \beta \log |Y|)}
\]

for any \( \tau \in \mathbb{R}^+ \), \( \gamma > 0 \), and \( \beta > 0 \), where \( h(\cdot) \) is the binary entropy function.

**Proof**: Let

\[
\mathcal{S} := \left\{ (x^{-m}, y^{-n}) \in \mathcal{X}^M \times \mathcal{Y}^N : \right. \frac{1}{N} \log \frac{1}{P_{X^{-m}}(x^{-m})} \leq \frac{1}{N} \log \frac{1}{P_Y(y^-n)} - \gamma \left. \right\}
\]

and

\[
\mathcal{C} := \left\{ (x^{-m}, y^{-n}) \in \mathcal{X}^M \times \mathcal{Y}^N : d_H(y^{-n}, f^m_n(x^{-m})) \leq N\beta \right\}
\]

where \( M = 2m + 1 \) and \( f^m_n(x^{-m}) = (f(x^{-m+1}), \ldots, f(x^\ell), \ldots, f(x^{-m}) \ldots) \). For the joint distribution given by (4), we have

\[
P_{X^{-m}Y^{-n}}(S) = P_{X^{-m}Y^{-n}}(S \cap \mathcal{C}) + P_{X^{-m}Y^{-n}}(S \cap \mathcal{C}^c)
\]

\[
\leq P_{X^{-m}Y^{-n}}(\mathcal{C}) + P_{X^{-m}Y^{-n}}(S \cap \mathcal{C})
\]

\[
\leq \frac{\varepsilon}{\beta} + P_{X^{-m}Y^{-n}}(S \cap \mathcal{C}), \tag{8}
\]

where \( \mathcal{C}^c \) is the complement of \( \mathcal{C} \) and the last inequality follows from (7).

To evaluate the second term of (8), note that \((x^{-m}, y^{-n}) \in S\) implies

\[
P_{X^{-m}Y^{-n} \mid Y^{-n}}(y^{-n}) \leq 2^{-N\gamma} P_{X^{-m}}(x^{-m}); \tag{9}
\]

also note that, for fixed \( x^{-m} \in \mathcal{X}^M \),

\[
\left\{ (y^{-n} \in \mathcal{Y}^N : (x^{-m}, y^{-n}) \in \mathcal{C}) \right\} \leq |\mathcal{Y}|^{N\beta} 2^{N\beta} \tag{10}
\]

holds for \( 0 < \beta < 1/2 \). By noting these facts, we have

\[
P_{X^{-m}Y^{-n}}(S \cap \mathcal{C})
\]

\[
= \sum_{(x^{-m}, y^{-n}) \in \mathcal{C} \cap \mathcal{S}} P_{X^{-m}Y^{-n}}(x^{-m}, y^{-n})
\]

\[
\leq \sum_{(x^{-m}, y^{-n}) \in \mathcal{C} \cap \mathcal{S}} P_{X^{-m}Y^{-n}}(y^{-n})
\]

\[
\leq 2^{-N\gamma} \sum_{(x^{-m}, y^{-n}) \in \mathcal{C} \cap \mathcal{S}} P_{X^{-m}}(x^{-m})
\]

\[
\leq 2^{-N\gamma} \sum_{x^{-m} \in \mathcal{X}^M} P_{X^{-m}}(x^{-m}) \sum_{y^{-n} \in \mathcal{Y}^N} 1[(x^{-m}, y^{-n}) \in \mathcal{C}]
\]

\[
\leq 2^{-N(\gamma - h(\beta) - \beta \log |Y|)}
\]

where the second inequality follows from (9) and the last inequality follows from (10).

Finally, note that

\[
P_{X^{-m}Y^{-n}}(S)
\]

\[
= \Pr \left( \frac{1}{N} \log \frac{1}{P_{X^{-m}}(x^{-m})} \leq \frac{1}{N} \log \frac{1}{P_Y(y^-n)} - \gamma \right)
\]

\[
\geq \Pr \left( \frac{1}{N} \log \frac{1}{P_{X^{-m}}(x^{-m})} \leq \tau + \gamma, \frac{1}{N} \log \frac{1}{P_Y(y^-n)} > \tau + 2\gamma \right)
\]

\[
\geq \Pr \left( \frac{1}{N} \log \frac{1}{P_{X^{-m}}(x^{-m})} \leq \tau + \gamma \right)
\]

\[
- \Pr \left( \frac{1}{N} \log \frac{1}{P_Y(y^-n)} \leq \tau + 2\gamma \right) \tag{12}
\]

for any \( \tau \in \mathbb{R}^+ \). By combining (8), (11), and (12), we have the claim of the lemma.

Let \( M := 2m + 1 = N + 2\ell \). In order to replace \( N \) with \( M \) in the left hand side of the bound in Lemma 2, take \( n \) sufficiently large so that \( N(\tau + \gamma)/(N + 2\ell) \geq \tau + \gamma/2 \). Then, the bound in Lemma 2 implies

\[
\Pr \left( \frac{1}{N} \log \frac{1}{P_{X^{-m}}(x^{-m})} \leq \tau + \gamma/2 \right)
\]

\[
\leq \Pr \left( \frac{1}{N} \log \frac{1}{P_Y(y^-n)} \leq \tau + 2\gamma \right) + \frac{\varepsilon}{\beta} + 2^{-N(\gamma - h(\beta) - \beta \log |Y|)}
\]

Now, take \( \beta = \beta_\gamma \) sufficiently small compared to \( \gamma \) so that the exponent of the last term becomes positive; then by taking
the limit of $n$ and by noting the stationarity of $X$ and $Y$, we have
\[
\limsup_{n \to \infty} \Pr \left( \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} \leq \tau + \gamma/2 \right) \\
\leq \limsup_{n \to \infty} \Pr \left( \frac{1}{n} \log \frac{1}{P_{Y^n}(Y^n)} \leq \tau + 2\gamma \right) + \epsilon.
\]
Since this inequality holds for arbitrary $\epsilon > 0$, by taking the limit $\epsilon \downarrow 0$, we have
\[
\limsup_{n \to \infty} \Pr \left( \frac{1}{n} \log \frac{1}{P_{X^n}(X^n)} \leq \tau + \gamma/2 \right) \\
\leq \limsup_{n \to \infty} \Pr \left( \frac{1}{n} \log \frac{1}{P_{Y^n}(Y^n)} \leq \tau + 2\gamma \right).
\]
Finally, by taking the limit $\gamma \downarrow 0$, we have the desired result.

V. SUFFICIENT CONDITION

A stationary random process $X$ is termed a B-processes if it is a stationary coding of an i.i.d. process, i.e., there exists a homomorphism from an i.i.d. process to $X$ (e.g., see [9], [10]). The theory by Ornstein says that the class of B-processes can be classified by the entropy, i.e., two B-processes having the same entropy are isomorphic. Then, it is tempting to extend this classification theory to mixtures of B-processes by using the information spectrum. However, there are some pathological cases, which will be discussed later, and not every mixtures of B-processes can be classified by the information spectrum. In the following, let us confine ourselves to the following class of processes.

Definition 5 (Countable regular mixture of B-Process): A stationary random process $X$ determined by $(\mathcal{X}^Z, \mathcal{B}_X, \mu_X, S)$ is termed a countable regular mixture of B-processes if the ergodic decomposition is given by
\[
\mu(B) = \sum_{\theta \in \mathbb{N}} w(\theta) \mu_{\theta}(B), \quad B \in \mathcal{B}_X
\]
for a family of B-processes $(X_{\theta})_{\theta \in \mathbb{N}}$ determined by $(\mathcal{X}^Z, \mathcal{B}_X, \mu_X, S)$ with a measure $w$ on the set of integers $\mathbb{N}$ such that $H(X_{\theta}) \neq H(X_{\theta'})$ for every $\theta \neq \theta'$.

This class of processes can be classified by the information spectrum.

Proposition 3: Suppose that $X$ and $Y$ are countable regular mixtures of B-processes, and $F_X(\tau) = F_Y(\tau)$ for every $\tau \in \mathbb{R}_+$. Then, there exists an isomorphism between $X$ and $Y$.

It is claimed in [15, Theorem 2] that Proposition 3 holds with neither the countability assumption nor the regularity assumption.\(^7\) Even though the countability assumption in Proposition 3 may be dispensed with but with more complicated arguments,\(^8\) the regularity assumption, i.e., $H(X_{\theta}) \neq H(X_{\theta'})$ for every $\theta \neq \theta'$, is crucial. For instance, let $X_1$ and $X_2$ be different ergodic processes having the same entropy $H(X_1) = H(X_2) = a$, and $X$ be a mixture of the two processes; let $Y$ be another ergodic process with $H(Y) = a$. Then, $X$ and $Y$ have the same information spectrum $F_X(\tau) = F_Y(\tau) = 1[a \leq \tau]$. However, these processes cannot be isomorphic since $X$ is non-ergodic while $Y$ is ergodic (cf. Proposition 1). Thus, the claim in [15, Theorem 2] has a flaw.

VI. DISCUSSION

In this paper, we proved that the information spectrum of random processes is invariant under isomorphisms. Our proof is based on the information spectrum approach developed in information theory. The proof of the invariance and the analysis of the information spectrum are conducted separately in two steps: the ergodic decomposition nor the ergodic theorem are not used in the first step (Theorem 2 and Corollary 2), and they are only used in the second step (Theorem 1). In some sense, this is a first attempt of applying the information spectrum approach to ergodic theory.

On the other hand, known constructions of isomorphisms (or homomorphisms) heavily rely on the ergodic decomposition and ergodicity of each component. Of course, since the ergodicity is preserved under isomorphisms, it is hopeless to construct isomorphisms without using ergodicity at all. However, it is worthwhile to pursue a construction that separates the use of ergodicity as much as possible. Such an approach may provide new insights into ergodic theory.

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