Improved Bounds for the Approximate QFT

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Abstract
It has previously been established that the logarithmic-depth approximate quantum Fourier transform (AQFT) provides a suitable replacement for the regular QFT in many quantum algorithms. Since the AQFT is less accurate by definition, polynomially many more applications of the AQFT are required to achieve the original accuracy. However, in many quantum algorithms, the smaller size of the AQFT circuit yields a net improvement over using the QFT.

This paper presents a more thorough analysis of the AQFT circuit, which leads to the surprising conclusion that for sufficiently large input sizes, the difference between the QFT and the logarithmic-depth AQFT is negligible. In effect, the AQFT can be used as a direct replacement for the QFT, yielding improvements in any application which does not require exact quantum computation.

1 Introduction
The quantum Fourier transform is a fundamental component of many of the significant quantum algorithms known to date. The main concern with the efficiency of the QFT algorithm is its use of controlled phase-shift gates which involve very small phases. As the number of qubits in the input, \( n \), increases, the QFT algorithm requires exponentially smaller phase shifts, which may be increasingly difficult or even infeasible to implement physically.

Fortunately, for applications using the QFT which do not require an exact answer, the idea of an approximate QFT was suggested[3], in which the phase shift gates requiring the most precision are omitted, since these are also the phase shift gates which have the least effect on the output. There is clearly a trade-off between the accuracy of the AQFT and its circuit complexity. Barenco, Ekert, Suominen and Törmä established a lower bound for the probability that the AQFT returns the correct answer, and concluded that the accuracy of the QFT can be achieved by \( O(n^3/m^3) \) iterations of the AQFT, where \( n \) is the size of the input register, and \( m \) is the depth of the AQFT circuit. Usually, we have \( m = O(\log n) \).
However, using a more thorough analysis of the AQFT circuit, we derive a much better lower bound on the accuracy of the AQFT which allows us to conclude that the accuracy of the QFT can be achieved by just $1 + o(1)$ iterations of the AQFT.

## 2 The Quantum Fourier Transform

The quantum Fourier transform is a quantum circuit which performs a discrete Fourier transform on the complex-valued vector of $2^n$ probability amplitudes associated with an $n$-qubit quantum system. Specifically, given an $n$-qubit state as a superposition of basis states $|0\rangle$, $|1\rangle$, ..., $|2^n - 1\rangle$, the QFT maps each basis state $|j\rangle$ to

$$QFT(|j\rangle) = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i jk/2^n} |k\rangle.$$  

Figure 1 gives a quantum circuit which performs a QFT. The input register contains an $n$-qubit basis state $|x\rangle \equiv |x_1 x_2 \cdots x_n\rangle$. The gates labelled $H$ are Hadamard gates, and the gates labelled $R_m$ represent a series of one-qubit phase rotation gates. For each integer $m \geq 2$, the gate $R_m$ shifts the phase of the $|1\rangle$ component of the input qubit by a factor of $e^{2\pi i / 2^m}$. Note that the output state of the QFT can be factored into a tensor product as indicated in the diagram.

In practice, the QFT is usually used in reverse, where the states

$$\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i (2^{p-1} \phi)} |1\rangle)$$

are prepared for $p = 1, 2, \ldots, n$ by some other means, and used as input into the inverse QFT circuit in order to obtain an estimate of the value of $\phi$ from the state $|x\rangle = |x_1 x_2 \cdots x_n\rangle$. Specifically, $\phi$ is estimated by $\hat{\phi} = (0.x_1 x_2 \cdots x_n)$, where the parentheses denote a binary (base-2) fraction. The estimate $\hat{\phi}$ is the nearest multiple of $2^{-n}$ to $\phi$ with probability $\frac{1}{2^n}$, and one of the two nearest multiples with probability $\frac{1}{2^n}$.  

As for the generation of the required input states, since it is not important how they are generated, we may assume that they are generated by using the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ as the control qubit for a controlled-$U^{2^{p-1}}$ gate, where $U$ is a quantum gate with $\phi$ as an eigenvalue.
Griffiths and Niu observed that it is not necessary for the entire inverse QFT to be performed at once[4]. Instead, since each output qubit, \(|x_p\rangle\) is controlled only by higher-indexed qubits, and the last qubit \(|x_n\rangle\) can be obtained independently of all other qubits, they suggested a “semiclassical” version of the inverse QFT circuit in which each output qubit is obtained one at a time, starting with \(|x_n\rangle\). Proceeding one qubit at a time, the results from measuring the previous qubits are used to classically determine whether a particular phase rotation gate will be applied to the current qubit. This process does not affect the final answer, but has the advantage of replacing the controlled-rotation gates in the inverse QFT with regular rotation gates. Thus, we can analyze the entire inverse QFT circuit simply by looking at each individual “trial”.

In the circuit in Figure 2, the controlled rotation gates are replaced with a rotation by a phase of \(-\chi_p\), which is determined by the results of the previous trials, from \(|x_n\rangle\) to \(|x_{p+1}\rangle\). Specifically, \(\chi_p = (0.0x_{p+1} \ldots x_n)\).

### 3 Approximate QFT Circuits

In Coppersmith’s AQFT circuits[3], instead of performing all phase rotations \(R_p\) that are used in the QFT algorithm, we set a lower limit on the amount of phase shifted by any particular gate and ignore any phase rotation gates that do not shift by at least that amount. In other words, given a positive integer threshold, \(m\), we will define the approximate QFT circuit \(\text{AQFT}_m\) to be the circuit formed from the QFT, except that phase rotation gates \(R_p\) are ignored whenever we have \(m > p\). Note that \(\text{AQFT}_n\) is simply the regular QFT circuit. As before, the inverse AQFT can be divided into individual trials.

Suppose we are given a phase \(\phi\), and we are to use the \(\text{AQFT}_m\) circuit to find an approximation \(\hat{\phi} = (0.x_1x_2 \ldots x_n)\). The individual trial for bit \(x_p\) starts with the input state \(|0\rangle + e^{2\pi i(2^{p+1}-2\phi)}|1\rangle\), and performs a phase rotation of \(e^{-2\pi i\chi_p}\) on the probability amplitude of \(|1\rangle\). Here, \(\chi_p\) corresponds to the phase rotation amount performed by the regular QFT circuit, \((0.0x_{i+1}x_{i+2} \ldots x_n)\), but without the effect of any gates which are smaller than the given threshold \(m\). Therefore, we have \(\chi_p = (0.0x_{p+1} \ldots x_{p+m-1})\) when \(p \leq n - m\), and \(\chi_p = (0.0x_{p+1} \ldots x_n)\) otherwise.

We will now compute the probability that \(\hat{\phi}\) is the nearest integer multiple of \(1/2^n\) to \(\phi\), i.e., \(|\phi - \hat{\phi}| \leq 2^{-n-1}\). First, using the double angle formula and induction, we can easily establish the fact that for any integer \(n \geq 1\), we have

\[
\prod_{p=1}^{n} \cos^2 \frac{\theta}{2^p} = \left( \frac{\sin \theta}{2^n \sin \frac{\theta}{2^n}} \right)^2.
\]
Now observe that, in general, applying a Hadamard gate to the state $|0⟩ + e^{2\pi i((0.x_p)+\delta_p)}|1⟩$ and measuring yields $|x_p⟩$ with probability $\cos^2(\pi\delta_p)$. However, for the individual AQFT$_m$ trial for bit $x_p$, we know what $\delta_p$ has to be. We have two cases.

In the first case, where $p + m - 1 < n$ and $\chi_p = (0.0x_{p+1} \ldots x_{p+m-1})$, $\delta_p$ contains the phase rotation amount that was not applied by the AQFT$_m$ trial that would have been applied by the corresponding QFT trial, and the difference between $\phi$ and its nearest integer estimate, amplified by a factor of $2^p$ by the AQFT$_m$ circuit. In other words, we have

$$\delta_p = 2^{-m}(0.x_{p+m}x_{p+m+1} \ldots x_n) + 2^{p-1}\delta,$$

where $\delta = \phi - \hat{\phi}$. Also, if $\hat{\phi}$ is the nearest estimate of $\phi$, we have

$$-2^{p-n-1} \leq 2^{p-1}\delta \leq 2^{p-n-1}.$$

Now, since $(0.x_{p+m} \ldots x_n)$ reaches its minimum value when all the bits are 0, and its maximum value when all its bits are 1, we have

$$0 \leq 2^{-m}(0.x_{p+m} \ldots x_n) \leq 2^{-m}(1 - 2^{p+m-n-1}) = 2^{-m} - 2^{p-n-1}.$$

Adding the two inequalities together, we obtain $-2^{-m} < -2^{p-n-1} \leq \delta_p \leq 2^{-m}$. This means that final measurement will yield $|x_p⟩$ with a probability of at least $\cos^2(\pi2^{-m})$ in this case.

In the second case, where $p \geq n - m + 1$, we simply have $\delta_p = 2^{p-1}\delta$. Since $|\delta| \leq 2^{-n-1}$, we have an error probability lower bound of $\cos^2(\frac{\pi}{2}2^{-n})$. The probability $P$ of getting the correct output with the AQFT$_m$ circuit is simply the product of each individual bit trial being correct. Putting these lower bounds together, we can give a lower bound for $P$:

$$P \geq \prod_{p=n-m+1}^{n} \cos^2 \left( \frac{\pi}{2}2^{p-n-1} \right) \prod_{p=1}^{m} \cos^2(\pi2^{-m})$$

$$= \left( \frac{\sin \frac{\pi}{2}}{2^m \sin \frac{\pi}{2m}} \right)^2 \left( \cos^2(\pi2^{-m}) \right)^{n-m}$$

$$\geq \left( \frac{1}{\pi/2} \right)^2 \left( \cos^2(\pi2^{-m}) \right)^{n-m}$$

$$= \frac{4}{\pi^2} \left( \cos^2(\pi2^{-m}) \right)^{n-m}.$$
It is clear that if \( m \) grows too slowly in comparison to \( n \), then the expression will approach 0 asymptotically as \( n \) approaches \( \infty \), making this bound useless. If we take \( m \geq \log_2 n + 2 \) so that \( 2^m \geq 4n \), we would have

\[
P \geq \frac{4}{\pi^2} \left( \cos^2 \left( \frac{\pi}{4n} \right) \right)^{n-m} \geq \frac{4}{\pi^2} \left( \cos^2 \left( \frac{\pi}{4n} \right) \right)^n,
\]
giving us

\[
\lim_{n \to \infty} P \geq \lim_{n \to \infty} \frac{4}{\pi^2} \left( \cos^2 \left( \frac{\pi}{4n} \right) \right)^n = \frac{4}{\pi^2}.
\]

Since Cleve and Watrous\cite{2} establish a lower bound of \( \Omega(\log n) \) on the depth of any AQFT circuit with a constant error bound, requiring \( m \geq \log_2 n + 2 \) is a reasonable restriction on a logarithmic-depth AQFT circuit.

Asymptotically, for \( m \geq \log_2 n + 2 \), the lower bound for the accuracy of the AQFT\(_m\) circuit approaches the lower bound for the accuracy of the full QFT, meaning that for sufficiently large \( n \), the QFT may be substituted by the AQFT\(_m\) circuit with only a negligible effect. This is a significant improvement, since the full QFT on an \( n \)-qubit register requires \( O(n^2) \) gates to implement, while the AQFT\(_m\) circuit uses only \( O(nm) = O(n \log_2 n) \) gates to achieve nearly the same level of accuracy.

Finally, we will also establish a lower bound for \( P \) when we have \( m \geq \log_2 n + 2 \) for some fixed value of \( n \), since an asymptotic bound gives no information about the value of \( P \) for practical values of \( n \). We have

\[
P \geq \frac{4}{\pi^2} \left( \cos^2 \left( \frac{\pi}{4n} \right) \right)^{n-m}
= \frac{4}{\pi^2} \left( 1 - \sin^2 \left( \frac{\pi}{4n} \right) \right)^{n-m}
\geq \frac{4}{\pi^2} \left( 1 - (n - m) \sin^2 \left( \frac{\pi}{4n} \right) \right),
\]

using the Bernoulli inequality, which states that \((1 + t)^c \geq 1 + ct\) whenever \( t > -1 \) and \( c \geq 0 \). Since \( x \geq \sin x \) when \( x \geq 0 \), we have

\[
P \geq \frac{4}{\pi^2} \left( 1 - (n - m) \left( \frac{\pi}{4n} \right)^2 \right)
\geq \frac{4}{\pi^2} - \frac{1}{4n}.
\]

This indicates that even for smaller values of \( n \), the probability of the AQFT\(_m\) circuit returning the best estimate of \( \phi \) is high enough to make it a practical alternative to using the full QFT circuit.

Note that since we stipulated that \( m \geq \log_2 n + 2 \), this result is only meaningful if \( n \geq 4 \). Otherwise, the logarithmic-depth AQFT\(_m\) circuit is simply the full QFT, with lower bound \( P \geq \frac{4}{\pi^2} \). We can now establish the fixed bound

\[
P \geq \frac{4}{\pi^2} - \frac{1}{16}
\]

for AQFT\(_m\) circuits where \( m \geq \log_2 n + 2 \).
4 Conclusion

The fixed bound of \( P \geq \frac{4}{\pi^2} - \frac{1}{16} \) derived here is much stronger than the bound of

\[
P \geq \frac{8}{\pi^2} \sin^2 \left( \frac{\pi m}{4n} \right)
\]

previously established by Barenco et al. in [1]. They conclude that \( O(n^3/m^3) \) iterations are necessary which means that the AQFT\(_m\) yields an advantage only when the QFT itself is affected in the presence of decoherence, as the AQFT\(_m\) would be less susceptible.

The new analysis presented in this paper establishes the effectiveness of the AQFT\(_m\) as a direct replacement for the QFT for a suitable size of input register. Furthermore, since we have

\[
\lim_{n \to \infty} \frac{8}{\pi^2} \sin^2 \left( \frac{\pi m}{4n} \right) = 0
\]

when \( m \) is fixed at \( \log_2 n + 2 \), it was not previously known that the AQFT\(_m\) circuit would perform so well for large \( n \). This is a significant result for the feasibility of any large-scale quantum computation requiring the use of the QFT circuit.

References

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