On the support $t$-designs of extremal Type III and IV codes

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Abstract

Let $C$ be an extremal Type III or IV code and $D_w$ be the support design of $C$ for weight $w$. We introduce the numbers, $\delta(C)$ and $s(C)$, as follows: $\delta(C)$ is the largest integer $t$ such that, for all weights, $D_w$ is a $t$-design; $s(C)$ denotes the largest integer $t$ such that $w$ exists and $D_w$ is a $t$-design. Herein, we consider the possible values of $\delta(C)$ and $s(C)$.

Keywords: self-dual codes, $t$-designs, Assmus–Mattson theorem, harmonic weight enumerators.

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1 Introduction

Let $D_w$ be the support design of code $C$ for weight $w$. From the Assmus–Mattson theorem, if $C$ is an extremal Type III (resp. Type IV) code, then for all $w$, $D_w$ is a 5-, 3-, and 1-design for $n = 12m$ (resp. $n = 6m$), $12m + 4$ (resp. $n = 6m + 2$), and $12m + 8$ (resp. $n = 6m + 4$), respectively.

Let

$$\delta(C) := \max\{t \in \mathbb{N} \mid \forall w, D_w \text{ is a } t\text{-design}\},$$

$$s(C) := \max\{t \in \mathbb{N} \mid \exists w \text{ s.t. } D_w \text{ is a } t\text{-design}\}.$$
It is noteworthy that $\delta(C) \leq s(C)$. In our previous papers [15, 23, 24, 22], we considered the following problems.

**Problem 1.1.** Find the upper bound of $s(C)$.

**Problem 1.2.** Does the case where $\delta(C) < s(C)$ occur?

Next, we explain our motivation for this study. The first motivation is as follows. For Problem 1.1, many examples of 5-designs can be obtained from the Assmus–Mattson theorem; however, an example of a 6-design is not known. Therefore, we aim to obtain a $t$-design for $t \geq 6$ using the Assmus–Mattson theorem. For Problem 1.2, if $C$ is an extremal Type II code, an example of $\delta(C) < s(C)$ [23] does not exist. In [24], we discovered the first nontrivial examples of $\delta(C) < s(C)$ in triply even binary codes of length 48, an example of which is the moonshine code [27]. Using this result, we provide a new characterization of the moonshine code [27].

The second motivation for this study is that the Assmus–Mattson theorem is one of the most important theorems in design and coding theory. Assmus–Mattson-type theorems in lattice and vertex operator algebra theories are known as the Venkov and Höhn theorems, respectively [28, 14]. For example, the $E_8$-lattice and moonshine vertex operator algebra $V^\natural$ provide spherical 7-designs for all $(E_8)_{2m}$ and conformal 11-designs for all $(V^\natural)_m$, $m > 0$. It is noteworthy that the $(E_8)_{2m}$ and $(V^\natural)_{m+1}$ are a spherical 8-design and a conformal 12-design, respectively, if and only if $\tau(m) = 0$, where

$$q \prod_{m=1}^\infty (1 - q^m)^{24} = \sum_{m=0}^\infty \tau(m)q^m$$

Furthermore, D.H. Lehmer conjectured in [16] that

$$\tau(m) \neq 0$$

for all $m$ [20, 28, 29]. Therefore, it is interesting to determine the lattice $L$ (resp. vertex operator algebra $V$) such that $L_m$ (resp. $V_m$) are spherical (resp. conformal) $t$-designs for all $m$ by the Venkov theorem (resp. Höhn theorem) and $L'_m$ (resp. $V'_m$) are spherical (resp. conformal) $t'$-designs for some $m'$ with some $t' > t$. This study is inspired by these possibilities. For related results, see [4, 5, 6, 7, 8, 15, 21, 22, 23, 25].
Next, we explain our main results. Herein, we present Problems 1.1 and 1.2 for extremal Type III and IV codes. Let $C$ be an extremal Type III or IV code of length $n$. In 1999, Zhang [30] showed that $C$ does not exist if

$$n = \begin{cases} 
12m \ (m \geq 70), \\
12m + 4 \ (m \geq 75), \\
12m + 8 \ (m \geq 78), 
\end{cases}$$

for Type III, and

$$n = \begin{cases} 
6m \ (m \geq 17), \\
6m + 2 \ (m \geq 20), \\
6m + 4 \ (m \geq 22), 
\end{cases}$$

for Type IV. The proof of this fact is to show that the coefficient of $x^{n-d}y^d$ ($d = \min(C)$) of the extremal weight enumerators is negative. In [30], Zhang remarked that the bounds for Type III may be improved if one considers the coefficients of the highest and next-to-highest powers of $y$ of the extremal weight enumerators. We remark that using all the coefficients of the extremal weight enumerators, we obtain more strict bounds for Type III codes:

**Theorem 1.3.** Let $C$ be an extremal Type III code of length $n$. Then $C$ does not exist if

$$n = \begin{cases} 
12m \ (m \in \{6, 8, 10, 12, 14, 16, 18, 20\} \cup \{m \in \mathbb{Z} \mid m \geq 22\}), \\
12m + 4 \ (m \in \{14, 16, 18, 20, 22, 24, 26, 28\} \cup \{m \in \mathbb{Z} \mid m \geq 30\}), \\
12m + 8 \ (m \in \{22, 24, 26, 28, 30, 32, 34, 36\} \cup \{m \in \mathbb{Z} \mid m \geq 38\}). 
\end{cases}$$

This means that if there exists an extremal ternary code of length $n = 12m + r \ (r \in \{0, 4, 8\})$, then $m$ must be in the following set:

$$m \in \begin{cases} 
\{i \in \mathbb{Z} \mid 1 \leq i \leq 5\} \cup \{7, 9, 11, 13, 15, 17, 19, 21\} \text{ if } r = 0, \\
\{i \in \mathbb{Z} \mid 1 \leq i \leq 13\} \cup \{15, 17, 19, 21, 23, 25, 27, 29\} \text{ if } r = 4, \\
\{i \in \mathbb{Z} \mid 1 \leq i \leq 21\} \cup \{23, 25, 27, 29, 31, 33, 35, 37\} \text{ if } r = 8. 
\end{cases}$$

The extremal weight enumerators for Type III codes of length $n \leq 12 \times 78 + 8 = 944$) are listed in one of the author’s homepage [19].

The main results of the present study are the following theorems.

**Theorem 1.4.** Let $C$ be an extremal Type III code of length $n$.

(1) Assume that $n = 12m$. 

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(a) If \( m \neq 15 \), \( \delta(C) = s(C) = 5 \).
(b) If \( m = 15 \), \( \delta(C) = s(C) = 5 \) or 7.

(2) Assume that \( n = 12m + 4 \).

(a) If \( m \notin \{11, 21, 25\} \), \( \delta(C) = s(C) = 3 \).
(b) If \( m \in \{11, 21, 25\} \), \( \delta(C) = s(C) = 3 \) or 5.

(3) Assume that \( n = 12m + 8 \).

(a) If \( m \neq 14 \), \( \delta(C) = s(C) = 1 \).
(b) If \( m = 14 \), \( \delta(C) = s(C) = 1 \) or 3.

**Theorem 1.5.** Let \( C \) be an extremal Type IV code of length \( n \).

(1) Assume that \( n = 6m \) (\( m \neq 1, 2 \)).

(a) If \( m \notin \{10, 15\} \), \( \delta(C) = s(C) = 5 \).
(b) If \( m \in \{10, 15\} \), \( \delta(C) = s(C) = 5 \) or 7.

(2) Assume that \( n = 6m + 2 \).

(a) If \( m \neq 11 \), \( \delta(C) = s(C) = 3 \).
(b) If \( m = 11 \), \( \delta(C) = s(C) = 3, 5, 6 \) or 7.

(3) Assume that \( n = 6m + 4 \).

(a) If \( m \in \{1, 2, 4, 13\} \), \( \delta(C) = s(C) = 1 \).
(b) If \( m \in \{3, 5, 6, 7, 8, 10, 11, 12, 15, 16, 17, 18, 20, 21\} \), \( \delta(C) = s(C) = 1 \) or 3.
(c) If \( m = 9 \), \( \delta(C) = s(C) = 1, 3 \) or 4.
(d) If \( m \in \{14, 19\} \), \( \delta(C) = s(C) = 1, 3, 4 \) or 5.

Summarizing the above, we have the following theorem.

**Theorem 1.6.** Let \( C \) be an extremal Type III or IV code.

1. [An answer to the Problem \[1.1\]]

   We have \( s(C) \leq 7 \).
2. [An answer to the Problem 1.2]

The case $\delta(C) < s(C)$ does not occur.

This paper is organized as follows. In Section 2, we provide the definitions and some basic properties of self-dual codes and $t$-designs as well as review the concept of harmonic weight enumerators and some lemmas that are used in the proof of the main results. In Sections 3 and 4 we provide the proofs of Theorems 1.4 and 1.5, respectively.

All computer calculations were performed using Mathematica [26].

2 Preliminaries

2.1 Codes and support $t$-designs

Let $\mathbb{F}_q$ be a finite field of $q$ elements. A linear code $C$ of length $n$ is a linear subspace of $\mathbb{F}_q^n$. For $q = 3$, an inner product $(x, y)$ on $\mathbb{F}_q^n$ is expressed as

$$(x, y) = \sum_{i=1}^{n} x_i y_i,$$

where $x, y \in \mathbb{F}_q^n$ with $x = (x_1, x_2, \ldots, x_n)$, and $y = (y_1, y_2, \ldots, y_n)$. The Hermitian inner product $(x, y)_H$ on $\mathbb{F}_q^n$ is expressed as

$$(x, y)_H = \sum_{i=1}^{n} x_i y_i^2,$$

where $x, y \in \mathbb{F}_q^n$ with $x = (x_1, x_2, \ldots, x_n)$, and $y = (y_1, y_2, \ldots, y_n)$. The dual of a linear code $C$ is defined as follows: for $q = 3$,

$$C^\perp = \{ y \in \mathbb{F}_q^n \mid (x, y) = 0 \text{ for all } x \in C \},$$

for $q = 4$,

$$C_{q=4}^\perp,H = \{ y \in \mathbb{F}_q^n \mid (x, y)_H = 0 \text{ for all } x \in C \}.$$

A linear code $C$ is called self-dual if $C = C^\perp$ for $q = 3$, and if $C = C_{q=4}^\perp,H$ for $q = 4$. For $x \in \mathbb{F}_q^n$, the weight $\text{wt}(x)$ is the number of its nonzero components. The minimum distance of a code $C$ is $\min\{ \text{wt}(x) \mid x \in C, x \neq 0 \}$. A linear code of length $n$, dimension $k$, and minimum distance $d$ is called an $[n, k, d]$ code.

Herein, we consider the following self-dual codes [12]:
Type III: A code is defined over $\mathbb{F}_3^n$ with all weights divisible by 3.
Type IV: A code is defined over $\mathbb{F}_4^n$ with all weights divisible by 2.

Let $C$ be a Type III or Type IV code of length $n$. Then we have the following bound on the minimum weight of $C$ [17, 18]:

$$\min(C) \leq \begin{cases} 3 \left\lfloor \frac{n}{12} \right\rfloor + 3 & \text{if } C \text{ is ternary,} \\ 2 \left\lfloor \frac{n}{6} \right\rfloor + 2 & \text{if } C \text{ is quaternary.} \end{cases}$$

We say that $C$ meeting the bound (2.1) with equality is extremal.

A $t$-$(v, k, \lambda)$ design (or $t$-design for short) is a pair $D = (X, B)$, where $X$ is a set of points of cardinality $v$, and $B$ a collection of $k$-element subsets of $X$ called blocks, with the property that any $t$ points are contained in precisely $\lambda$ blocks.

The support of a nonzero vector $x := (x_1, \ldots, x_n), x_i \in \mathbb{F}_q = \{0, 1, \ldots, q-1\}$ is the set of indices of its nonzero coordinates: $\text{supp}(x) = \{i \mid x_i \neq 0\}$. The support design of a code of length $n$ for a nonzero weight $w$ is a design with $n$ points of coordinate indices; it blocks the supports of all codewords of weight $w$. The following lemma can be observed easily.

**Lemma 2.1** ([11, Page 3, Proposition 1.4]). Let $\lambda(S)$ be the number of blocks containing a set $S$ of $s$ points in a $t$-$(v, k, \lambda)$ design, where $0 \leq s \leq t$. Therefore,

$$\lambda(S) \binom{k-s}{t-s} = \lambda \binom{v-s}{t-s}.$$ 

In particular, the number of blocks is

$$\frac{v(v-1) \cdots (v-t+1)}{k(k-1) \cdots (k-t+1)} \lambda.$$

### 2.2 Harmonic weight enumerators

In this section, we extend the harmonic weight enumerator method used by Bachoc [2] and Bannai et al. [4]. For convenience, we quote (from [2, 13]) the definitions and properties of discrete harmonic functions (for more information, the reader is referred to [2, 13]).

Let $\Omega = \{1, 2, \ldots, n\}$ be a finite set (which will be the set of coordinates of the code), and let $X$ be the set of its subsets; for all $k = 0, 1, \ldots, n$, $X_k$ is the set of its $k$-subsets. We denote the free real vector spaces spanned by
the elements of $X$ and $X_k$ by $\mathbb{R}X$, $\mathbb{R}X_k$, respectively. The element of $\mathbb{R}X_k$ is denoted by

$$f = \sum_{z \in X_k} f(z)z$$

and is identified with the real-valued function on $X_k$ expressed as $z \mapsto f(z)$.

Such an element $f \in \mathbb{R}X_k$ can be extended to an element $\tilde{f} \in \mathbb{R}X$ by setting, for all $u \in X$,

$$\tilde{f}(u) = \sum_{z \in X_k, z \subset u} f(z).$$

If an element $g \in \mathbb{R}X$ is equal to some $\tilde{f}$, for $f \in \mathbb{R}X_k$, we say that $g$ has a degree of $k$. The differentiation $\gamma$ is the operator defined by linearity from

$$\gamma(z) = \sum_{y \in X_{k-1}, y \subset z} y$$

for all $z \in X_k$ and for all $k = 0, 1, \ldots, n$, and $\text{Harm}_k$ is the kernel of $\gamma$, i.e.,

$$\text{Harm}_k = \ker(\gamma|_{\mathbb{R}X_k}).$$

**Theorem 2.2** ([13, Theorem 7]). A set $B \subset X_m$ (where $m \leq n$) of blocks is a $t$-design if and only if $\sum_{b \in B} \tilde{f}(b) = 0$ for all $f \in \text{Harm}_k$, $1 \leq k \leq t$.

In [2], the harmonic weight enumerator associated with a linear code $C$ is defined as follows:

**Definition 2.3** ([2, Definition 2.1],[3, Definition 4.1]). Let $C$ be a linear code of length $n$, and let $f \in \text{Harm}_k$. The harmonic weight enumerator associated with $C$ and $f$ is

$$W_{C,f}(x, y) = \sum_{c \in C} \tilde{f}(c)x^{n-\text{wt}(c)}y^{\text{wt}(c)}.$$

Subsequently, the structure of these invariant rings is described as follows:

**Theorem 2.4** ([3, Lemma 6.1 and 6.2]). (1) Let $C$ be a Type III code of length $n$, and let $f \in \text{Harm}_k$. Let $u \in \{0, 1\}$ be such that $u \equiv k \pmod{2}$ and $v \in \{0, 1, 2\}$ be such that $v \equiv -k \pmod{3}$. Therefore, we
have $W_{C,f}(x,y) = (xy)^k Z_{C,f}(x,y)$. Moreover, the polynomial $Z_{C,f}(x,y)$ is of degree $n - 2k$ and is in $I_{G_{3,\chi_u,v}}$, where

$$I_{G_{3,\chi_u,v}} = \begin{cases} 
\langle g_4, g_{12} \rangle & \text{if } (u,v) = (0,0), \\
p_4 \langle g_4, g_{12} \rangle & \text{if } (u,v) = (0,1), \\
p_2^2 \langle g_4, g_{12} \rangle & \text{if } (u,v) = (0,2), \\
p_6 \langle g_4, g_{12} \rangle & \text{if } (u,v) = (1,0), \\
p_4p_6 \langle g_4, g_{12} \rangle & \text{if } (u,v) = (1,1), \\
p_2^2p_6 \langle g_4, g_{12} \rangle & \text{if } (u,v) = (1,2), 
\end{cases}$$

and

$$\begin{cases} 
p_4 = y(x^3 - y^3), \\
p_6 = x^6 - 20x^3y^3 - 8y^6, \\
g_4 = x^4 + 8xy^3, \\
g_{12} = y^3(x^3 - y^3)^3. 
\end{cases}$$

(2) Let $C$ be a Type IV code of length $n$, and let $f \in \text{Harm}_k$. Let $u \in \{0,1\}$ be such that $u \equiv k \pmod{2}$ and $v \in \{0,1\}$ be such that $v \equiv k \pmod{2}$. Therefore, we have $W_{C,f}(x,y) = (xy)^k Z_{C,f}(x,y)$. Moreover, the polynomial $Z_{C,f}(x,y)$ is of degree $n - 2k$ and is in $I_{G_{4,\chi_u,v}}$, where

$$I_{G_{4,\chi_u,v}} = \begin{cases} 
\langle h_2, h_6 \rangle & \text{if } (u,v) = (0,0), \\
q_3r_3 \langle h_2, h_6 \rangle & \text{if } (u,v) = (1,1), 
\end{cases}$$

and

$$\begin{cases} 
h_2 = x^2 + 3y^2, \\
h_6 = y^2(x^2 - y^2)^2, \\
q_3 = y(x^2 - y^2), \\
r_3 = x^3 - 9xy^2. 
\end{cases}$$

We recall the slightly more general definition of the notion of a $T$-design for a subset $T$ of $\{1,2,\ldots,n\}$, as follows: a set $B$ of blocks is called a $T$-design if and only if $\sum_{b \in B} \hat{f}(b) = 0$ for all $f \in \text{Harm}_k$ and for all $k \in T$. By Theorem 2.2, a $t$-design is a $T = \{1,\ldots,t\}$-design. Let $W_{C,f} = \sum_{i=0}^{n} c_f(i) x^{n-i}y^i$. Subsequently, $D_w$ is a $T$-design if and only if $c_f(w) = 0$ for all $f \in \text{Harm}_j$ with $j \in T$.

**Theorem 2.5** ([10]). (1) Let $D_w$ be the support design of weight $w$ of an extremal Type III code of length $n$ ($n \geq 12$).
If \( n \equiv 0 \pmod{12} \), then \( D_w \) is a \( \{1, 2, 3, 4, 5, 7\} \)-design.

If \( n \equiv 4 \pmod{12} \), then \( D_w \) is a \( \{1, 2, 3, 5\} \)-design.

If \( n \equiv 8 \pmod{12} \), then \( D_w \) is a \( \{1, 3\} \)-design.

Let \( D_w \) be the support design of weight \( w \) of an extremal Type IV code of length \( n \).

If \( n \equiv 0 \pmod{6} \) (\( n \geq 18 \)), then \( D_w \) is a \( \{1, 2, 3, 4, 5, 7\} \)-design.

If \( n \equiv 2 \pmod{6} \), then \( D_w \) is a \( \{1, 2, 3, 5\} \)-design.

If \( n \equiv 4 \pmod{6} \), then \( D_w \) is a \( \{1, 3\} \)-design.

2.3 Coefficients of harmonic weight enumerators of extremal Type III and IV codes

As mentioned in Section 2.2, the support designs of a code \( C \) are affected by whether the coefficients of \( W_{C,f}(x, y) \) are zero. Therefore, we performed an investigation and show the following lemmas, where the binomial coefficient is defined by

\[
\binom{n}{k} = 0
\]

if \( n < k \).

**Lemma 2.6.** Let \( Q_1 = (x^4 + 8xy^3)(x^3 - y^3)^\alpha \). If the coefficients of \( x^{3\alpha + 4 - 3i}y^{3i} \) in \( Q_1 \) are equal to 0 for \( 0 \leq i \leq \alpha + 1 \), then \( \alpha = 9i - 1 \).

**Proof.** We have

\[
Q_1 = (x^4 + 8xy^3)(x^3 - y^3)^\alpha
\]

\[
= \sum_{i=0}^{\alpha+1} (-1)^i \left( \binom{\alpha}{i} - 8 \binom{\alpha}{i-1} \right) x^{3\alpha+4-3i}y^{3i}.
\]

If the coefficients of \( x^{3\alpha+4-3i}y^{3i} \) in \( Q_1 \) are equal to 0, i.e.,

\[
\binom{\alpha}{i} - 8 \binom{\alpha}{i-1} = 0,
\]
we then have
\[
\frac{\alpha!}{i!(\alpha - i)!} - \frac{\alpha!}{(i - 1)!(\alpha - i + 1)!} = 0
\]
⇔ \(\alpha - i + 1 - 8i = 0\)
⇔ \(\alpha = 9i - 1\).

Lemma 2.7. (1) Let \(R_1 = (x^2 + 3y^2)(x^2 - y^2)^\alpha\). If the coefficients of \(x^{2\alpha+2-2i}y^{2i}\) in \(R_1\) are equal to 0 for \(0 \leq i \leq \alpha + 1\), then \(\alpha = 4i - 1\).

(2) Let \(R_2 = (x^3 - 9xy^2)(x^2 - y^2)^\alpha\). If the coefficients of \(x^{2\alpha+3-2i}y^{2i}\) in \(R_2\) are not equal to 0.

(3) Let \(R_3 = (x^2 + 3y^2)^2(x^2 - y^2)^\alpha\). If the coefficients of \(x^{2\alpha+4-2i}y^{2i}\) in \(R_3\) are equal to 0 for \(0 \leq i \leq \alpha + 2\), then \(48\alpha + 112\) is a square number.

Proof. (1) We have
\[
R_1 = (x^2 + 3y^2)(x^2 - y^2)^\alpha
= \sum_{i=0}^{\alpha+1} (-1)^i \left( \binom{\alpha}{i} - 3 \binom{\alpha}{i-1} \right) x^{2\alpha+2-2i} y^{2i}.
\]
If the coefficients of \(x^{2\alpha+2-2i}y^{2i}\) in \(R_1\) are equal to 0, i.e.,
\[
\binom{\alpha}{i} - 3 \binom{\alpha}{i-1} = 0,
\]
we then have
\[
\frac{\alpha!}{i!(\alpha - i)!} - \frac{3\alpha!}{(i - 1)!(\alpha - i + 1)!} = 0
\]
⇔ \(\alpha - i + 1 - 3i = 0\)
⇔ \(\alpha = 4i - 1\).

(2) We have
\[
R_2 = (x^3 - 9xy^2)(x^2 - y^2)^\alpha
= \sum_{i=0}^{\alpha+1} (-1)^i \left( \binom{\alpha}{i} + 9 \binom{\alpha}{i-1} \right) x^{2\alpha+3-2i} y^{2i}.
\]
If the coefficients of \(x^{2\alpha+3-2i}y^{2i}\) in \(R_2\) are equal to 0, i.e.,

\[
\binom{\alpha}{i} + 9\binom{\alpha}{i-1} = 0,
\]

we then have

\[
\frac{\alpha!}{i!(\alpha - i)!} + \frac{\alpha!}{(i - 1)!(\alpha - i + 1)!} = 0
\]

\[\Leftrightarrow \alpha - i + 1 + 9i = 0\]

\[\Leftrightarrow \alpha = -8i - 1 < 0.\]

Hence, the coefficients of \(x^{2\alpha+3-2i}y^{2i}\) in \(R_2\) are not equal to 0.

(3) We have

\[
R_3 = (x^2 + 3y^2)^2(x^2 - y^2)^\alpha
\]

\[
= \sum_{i=0}^{\alpha+1} (-1)^i \binom{\alpha}{i} - 6\binom{\alpha}{i - 1} + 9\binom{\alpha}{i - 2}\right) x^{2\alpha+4-2i}y^{2i}.
\]

If the coefficients of \(x^{2\alpha+4-2i}y^{2i}\) in \(R_3\) are equal to 0, i.e.,

\[
\binom{\alpha}{i} - 6\binom{\alpha}{i - 1} + 9\binom{\alpha}{i - 2} = 0,
\]

we then have

\[
\frac{\alpha!}{i!(\alpha - i)!} - 6\frac{\alpha!}{(i - 1)!(\alpha - i + 1)!} + 9\frac{\alpha!}{(i - 2)!(\alpha - i + 2)!} = 0
\]

\[\Leftrightarrow (\alpha - i + 2)(\alpha - i + 1) - 6i(\alpha - i + 2) + 9i(i - 1) = 0\]

\[\Leftrightarrow 16i^2 - (8\alpha + 24)i + \alpha^2 + 3\alpha + 2 = 0.\]

We have

\[
i = \frac{4\alpha + 12 \pm \sqrt{48\alpha + 112}}{16}.
\]

Because \(i\) is an integer, \(48\alpha + 112\) is a square number.
3 Proof of Theorem 1.4

3.1 Case for $n = 12m$

In this section, we consider the case of extremal Type III $[12m, 6m, 3m + 3]$ codes satisfying (1.1). Let $C$ be an extremal Type III $[12m, 6m, 3m + 3]$ code and $D_{3m+3}^{12m}$ be the support (with duplicates omitted) design of the minimum weight of $C$. By [13, Theorem 2], the number of codewords of minimum nonzero weight of $C$ is equal to

$$2 \binom{12m}{5} \binom{4m-2}{m-1} / \binom{3m+3}{5}. $$

Therefore, by the Assmus–Mattson theorem, $D_{3m+3}^{12m}$ is a 5-design with parameters

$$\left(12m, 3m + 3, \frac{4m-2}{m-1}\right).$$

Proposition 3.1. (1) If $t \geq 6$, then $D_{3m+3}^{12m}$ is a 7-design and $m = 15$.

(2) $D_{3m+3}^{12m}$ is never an 8-design.

Proof. (1) By Theorem 2.5 (1), $D_{3m+3}^{12m}$ is a 7-design if $t \geq 6$. If $D_{3m+3}^{12m}$ is a 7-design, then by Lemma 2.1

$$\lambda_6 = \frac{3m-2}{12m-5} \binom{4m-2}{m-1} \text{ and } \lambda_7 = \frac{(3m-2)(3m-3)}{(12m-5)(12m-6)} \binom{4m-2}{m-1}$$

are positive integers. By computing $m$ satisfying (1.1), if $\lambda_6$ and $\lambda_7$ are positive integers, then we have $m = 15$.

(2) We have verified that

$$\lambda_8 = \frac{(3m-2)(3m-3)(3m-4)}{(12m-5)(12m-6)(12m-7)} \binom{4m-2}{m-1}$$

is not a positive integer for $m = 15$. Therefore, by Lemma 2.1 $D_{3m+3}^{12m}$ is never an 8-design. □

For $t = 8$, we present the following proposition.

Proposition 3.2. Let $D_w^{12m}$ be the support $t$-design of weight $w$ of an extremal Type III code of length $n = 12m$. Therefore, all $D_w^{12m}$ are 8-designs simultaneously, or none of the $D_w^{12m}$ is an 8-design.
Proof. Let us assume that \( t = 8 \), and \( C \) is an extremal Type III \([12m, 6m, 3m + 3]\) code. Therefore, by Theorem 2.4 (1), we have \( W_{C,f}(x, y) = c(f)(xy)^8Z_{C,f}(x, y) \), where \( c(f) \) is a linear function from \( \text{Harm}_t \) to \( \mathbb{R} \), and \( Z_{C,f}(x, y) \in I_{G_{3},x_{0},1} \).

By Theorem 2.4 (1), \( Z_{C,f}(x, y) \) can be written in the following form:

\[
Z_{C,f}(x, y) = p_4 \sum_{i=0}^{m} a_i g_4^{3(m-i)-5} g_{12}^i.
\]

Because the minimum weight of \( C \) is \( 3m + 3 \), we have \( a_i = 0 \) for \( i \neq m - 2 \). Therefore, \( W_{C,f}(x, y) \) can be written in the following form:

\[
W_{C,f}(x, y) = c(f)(xy)^8p_4g_4g_{12}^{m-2}
= c(f)(xy)^8y^{3m-5}(x^4 + 8xy^3)(x^3 - y^3)^{3m-5}.
\]

By Lemma 2.6, the coefficients of \( x^{9m-11-3i}y^{3i} \) in \( (x^4 + 8xy^3)(x^3 - y^3)^{3m-5} \) are not equal to 0 for \( 0 \leq i \leq 3m - 4 \) because \( 3m - 5 \neq 9i - 1 \). Therefore, all \( D_{12m}^w \) are 8-designs simultaneously, or none of the \( D_{12m}^w \) is an 8-design. 

By Propositions 3.1 and 3.2 we obtained the following theorem.

Theorem 3.3.

(1) If \( D_{12m}^w \) becomes a 7-design for any \( w \), then \( m = 15 \).

(2) \( D_{12m}^w \) is never an 8-design for any \( w \).

Hence, the proof of Theorem 1.4 (1) is completed.

3.2 Case for 12m + 4

In this section, we consider the case of extremal Type III \([12m + 4, 6m + 2, 3m + 3]\) codes satisfying (1.1). Let \( C \) be an extremal Type III \([12m + 4, 6m + 2, 3m + 3]\) code and \( D_{12m+4}^w \) be the support (with duplicates omitted) design of the minimum weight of \( C \). By [18 Theorem 2], the number of codewords of the minimum nonzero weight of \( C \) is equal to

\[
2(12m + 4)(12m + 3)(12m + 2)\frac{(4m)!}{m!(3m + 3)!}.
\]

Therefore, by the Assmus–Mattson theorem, \( D_{3m+3}^{12m+4} \) is a 3-design with parameters

\[
\left( 12m + 4, 3m + 3, \binom{4m}{m} \right).
\]

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Proposition 3.4. Let $D_{3m+3}^{12m+4}$ be the support $t$-design of the minimum weight of an extremal Type III code of length $n = 12m + 4$.

(1) If $t \geq 4$, then $D_{3m+3}^{12m+4}$ is a 5-design and $m$ must be in the set $\{11, 21, 25\}$.

(2) $D_{3m+3}^{12m+4}$ is never a 6-design.

Proof. (1) By Theorem 2.5 (1), $D_{3m+3}^{12m+4}$ is a 5-design if $t \geq 4$. If $D_{3m+3}^{12m+4}$ is a 5-design, then by Lemma 2.1,

$$\lambda_4 = \frac{3m}{12m+1} \left( \frac{4m}{m} \right) \text{ and } \lambda_5 = \frac{3m(3m-1)}{(12m+1)12m} \left( \frac{4m}{m} \right)$$

are positive integers. By computing $m$ satisfying (1.1), if $\lambda_4$ and $\lambda_5$ are positive integers, then we have

$$m \in \{11, 21, 25\}.$$  

(2) If $D_{3m+3}^{12m+4}$ is a 6-design, then by Lemma 2.1,

$$\lambda_6 = \frac{3m(3m-1)(3m-2)}{(12m+1)12m(12m-1)} \left( \frac{4m}{m} \right)$$

is a positive integer. Then we do not obtain $m$ satisfying (1.1).

For $t \geq 6$, we present the following proposition.

Proposition 3.5. Let $D_w^{12m+4}$ be the support $t$-design of weight $w$ of an extremal Type III code of length $n = 12m + 4$.

(1) All $D_w^{12m+4}$ are 6-designs simultaneously, or none of the $D_w^{12m+4}$ is a 6-design.

Proof. Let $C$ be an extremal Type III $[12m + 4, 6m + 2, 3m + 3]$ code.

(1) Let us assume that $t = 6$. Therefore, by Theorem 2.4 (1), we have $W_{C,f}(x, y) = c(f)(xy)^6 Z_{C,f}(x, y)$, where $c(f)$ is a linear function from Harm to $\mathbb{R}$, and $Z_{C,f}(x, y) \in I_{G_5, \chi_0, 0}$. By Theorem 2.4 (1), $Z_{C,f}(x, y)$ can be written in the following form:

$$Z_{C,f}(x, y) = \sum_{i=0}^{m} a_i g_4^{3(3m-i)-2} g_{12}^{i}.$$
Because the minimum weight of $C$ is $3m + 3$, we have $a_i = 0$ for $i \neq m - 1$. Therefore, $W_{C,f}(x,y)$ can be written in the following form:

\[
W_{C,f}(x,y) = c(f)(xy)^6 g_{12}^{m-1} \\
= c(f)(xy)^6 y^{3m-3}(x^4 + 8xy^3)(x^3 - y^3)^{3m-3}.
\]

By Lemma 2.6 the coefficients of $x^{9m-5-3i}y^{3i}$ in $(x^4 + 8xy^3)(x^3 - y^3)^{3m-3}$ are not equal to 0 for $0 \leq i \leq 3m - 2$ since $3m - 3 \neq 9i - 1$. Therefore, all $D_{12m+4}^w$ are 6-designs simultaneously, or none of the $D_{12m+4}^w$ is a 6-design.

By Proposition 3.4 and 3.5 we obtained the following theorem.

**Theorem 3.6.** Let $D_{12m+4}^w$ be the support $t$-design of weight $w$ of an extremal Type III code of length $n = 12m + 4$ satisfying (1.1).

1. If $D_{12m+4}^w$ becomes a 5-design for any $w$, then $m$ must be in the set \{11, 21, 25\}.

2. $D_{12m+4}^w$ is never a 6-design for any $w$.

Hence, the proof of Theorem 1.4 (2) is completed.

### 3.3 Case for $12m + 8$

In this section, we consider the case of extremal Type III $[12m + 8, 6m + 4, 3m + 3]$ codes satisfying (1.1). Let $C$ be an extremal Type III $[12m + 8, 6m + 4, 3m + 3]$ code and $D_{12m+8}^w$ be the support (with duplicates omitted) design of the minimum weight of $C$. By [18, Theorem 2], the number of codewords of the minimum nonzero weight of $C$ is equal to

\[
6(12m + 8)\frac{(4m + 2)!}{m!(3m + 3)!}.
\]

Therefore, by the Assmus–Mattson theorem, $D_{3m+3}^{12m+8}$ is a 1-design with parameters

\[
\left(12m + 8, 3m + 3, 3 \binom{4m + 2}{m}\right).
\]

**Proposition 3.7.** Let $D_{3m+3}^{12m+8}$ be the support $t$-design of the minimum weight of an extremal Type III code of length $n = 12m + 8$. 

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(1) If \( t \geq 2 \), then \( D_{3m+3}^{12m+8} \) is a 3-design and \( m \) must be 14.

(2) \( D_{3m+3}^{12m+8} \) is never a 4-design.

Proof. (1) By Theorem 2.5 (1), \( D_{3m+3}^{12m+8} \) is a 3-design if \( t \geq 2 \). If \( D_{3m+3}^{12m+8} \) is a 3-design, then by Lemma 2.1

\[
\lambda_2 = \frac{3m + 2}{12m + 7} \binom{4m + 2}{m} \quad \text{and} \quad \lambda_3 = \frac{(3m + 2)(3m + 1)}{(12m + 7)(12m + 6)} \binom{4m + 2}{m}
\]

are positive integers. By computing \( m \) satisfying (1.1), if \( \lambda_2 \) and \( \lambda_3 \) are positive integers, then we have \( m = 14 \).

(2) We have verified that \( \lambda_4 \) is not a positive integer for \( m = 14 \). Therefore, by Lemma 2.1 \( D_{3m+3}^{12m+8} \) is never a 4-design.

Next, we present the following proposition.

Proposition 3.8. Let \( D_w^{12m+8} \) be the support \( t \)-design of weight \( w \) of an extremal Type III code of length \( n = 12m + 8 \). If \( m \not\equiv 0 \pmod{3} \), all \( D_w^{12m+8} \) are 4-designs simultaneously, or none of the \( D_w^{12m+8} \) is a 4-design.

Proof. Let \( C \) be an extremal Type III \([12m + 8, 6m + 4, 3m + 3]\) code. Let us assume that \( t = 4 \). Therefore, by Theorem 2.4 (1), we have \( W_{C,f}(x, y) = c(f)(xy)^4Z_{C,f}(x, y) \), where \( c(f) \) is a linear function from \( \text{Harm}_t \) to \( \mathbb{R} \), and \( Z_{C,f}(x, y) \in I_{G_3,x_0,2}^* \). By Theorem 2.4 (1), \( Z_{C,f}(x, y) \) can be written in the following form:

\[
Z_{C,f}(x, y) = p_{12}^2 \sum_{i=0}^{m} a_i g_4^{3(m-i)-2} g_{12}^i.
\]

Because the minimum weight of \( C \) is \( 3m + 3 \), we have \( a_i = 0 \) for \( i \neq m - 1 \).

Therefore, \( W_{C,f}(x, y) \) can be written in the following form:

\[
W_{C,f}(x, y) = c(f)(xy)^4p_{12}^2 g_4 g_{12}^{m-1}
= c(f)(xy)^4y^{3m-1}(x^4 + 8xy^3)(x^3 - y^3)^{3m-1}.
\]

By Lemma 2.6 if \( m \not\equiv 0 \pmod{3} \), then the coefficients of \( x^{9m+1-3i}y^{3i} \) in \((x^4 + 8xy^3)(x^3 - y^3)^{3m-1}\) are not equal to 0 for \( 0 \leq i \leq 3m \) because
3m − 3 ≠ 9i − 1. Therefore, all $D_{w}^{12m+8}$ are 4-designs simultaneously, or none of the $D_{w}^{12m+8}$ is a 4-design for $m \not\equiv 0 \pmod{3}$.

By Propositions 3.7 and 3.8 we obtained the following theorem.

**Theorem 3.9.** Let $D_{w}^{12m+8}$ be the support $t$-design of weight $w$ of an extremal Type III code of length $n = 12m + 8$ satisfying (1.1).

1. If $D_{w}^{12m+8}$ becomes a 3-design for any $w$, then $m = 14$.
2. In the case where $m = 14$, $D_{w}^{12m+8}$ is a 1 or 3-design for any $w$.
3. $D_{w}^{12m+8}$ is never a 4-design for any $w$.

Hence, the proof of Theorem 1.4 (3) is completed.

### 4 Proof of Theorem 1.5

#### 4.1 Case for $n = 6m$

In this section, we consider the case of extremal Type IV $[6m, 3m, 2m + 2]$ codes ($3 \leq m \leq 16$). Let $C$ be an extremal Type IV $[6m, 3m, 2m + 2]$ code and $D_{2m+2}^{6m}$ be the support (with duplicates omitted) design of the minimum weight of $C$. By [17, Theorem 18], $D_{2m+2}^{6m}$ is a

$$
\text{5-}\left(6m, 2m + 2, \binom{3m - 3}{m - 2}\right)
$$

**design.**

**Proposition 4.1.** Let $D_{2m+2}^{6m}$ be the support $t$-design of the minimum weight of an extremal Type IV code of length $n = 6m$.

1. If $t \geq 6$, then $D_{2m+2}^{6m}$ is a 7-design and $m$ must be in the set $\{10, 15\}$.
2. $D_{2m+2}^{6m}$ is never an 8-design.

**Proof.** (1) If $D_{2m+2}^{6m}$ is a 7-design, then by Lemma 2.1

$$
\lambda_6 = \frac{2m - 3}{6m - 5} \binom{3m - 3}{m - 2} \quad \text{and} \quad \lambda_7 = \frac{(2m - 3)(2m - 4)}{(6m - 5)(6m - 6)} \binom{3m - 3}{m - 2}
$$

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are positive integers. By computing \( m \leq 16 \), if \( \lambda_6 \) and \( \lambda_7 \) are positive integers, then we have \( m \in \{10, 15\} \).

(2) We have verified that

\[
\lambda_8 = \frac{(2m - 3)(2m - 4)(2m - 5)}{(6m - 5)(6m - 6)(6m - 7)} \left( \frac{3m - 3}{m - 2} \right)
\]

is not a positive integer for \( m \in \{10, 15\} \). Therefore, by Lemma 2.1, \( D_{2m+2}^{6m} \) is never an 8-design.

For \( t \geq 8 \), we present the following proposition.

**Proposition 4.2.** Let \( D_{w}^{6m} \) be the support \( t \)-design of weight \( w \) of an extremal Type IV code of length \( n = 6m \). Therefore, all \( D_{w}^{6m} \) are 8-designs simultaneously, or none of the \( D_{w}^{6m} \) is an 8-design.

**Proof.** Let \( C \) be an extremal Type IV \([6m, 3m, 2m + 2]\) code. Let us assume that \( t = 8 \). Therefore, by Theorem 2.4 (2), we have

\[
W_{C,f}(x, y) = c(f)(xy)^8 Z_{C,f}(x, y),
\]

where \( c(f) \) is a linear function from \( \text{Harm}_t \) to \( \mathbb{R} \), and \( Z_{C,f}(x, y) \in I_{G_{4,0,0}} \). By Theorem 2.4 (2), \( Z_{C,f}(x, y) \) can be written in the following form:

\[
Z_{C,f}(x, y) = \sum_{i=0}^{m} a_i h_2^{3(m-i)-8} h_6^i.
\]

Because the minimum weight of \( C \) is \( 2m + 2 \), we have \( a_i = 0 \) for \( i \neq m - 3 \). Therefore, \( W_{C,f}(x, y) \) can be written in the following form:

\[
W_{C,f}(x, y) = c(f)(xy)^8 h_2^{m-3} h_6^{2m-6} (x^2 + 3y^2)(x^2 - y^2)^{2m-6}.
\]

By Lemma 2.1 (1), the coefficients of \( x^{4m-10-2i} y^{2i} \) in \( (x^2 + 3y^2)(x^2 - y^2)^{2m-6} \) are not equal to 0 for \( 0 \leq i \leq 2m - 5 \) because \( 2m - 6 \neq 4i - 1 \). Therefore, all \( D_{w}^{6m} \) are 8-designs simultaneously, or none of the \( D_{w}^{6m} \) is an 8-design.

By Propositions 4.1 and 4.2, we obtained the following theorem.

**Theorem 4.3.**

(1) If \( D_{w}^{6m} \) becomes a 7-design for any \( w \), then \( m \) must be in the set \( \{10, 15\} \).

(2) \( D_{w}^{6m} \) is never an 8-design for any \( w \).

Hence, the proof of Theorem 1.5 (1) is completed.
4.2 Case for \( n = 6m + 2 \)

In this section, we consider the case of extremal Type IV \([6m+2, 3m+1, 2m+2]\) codes (\( m \leq 19 \)). Let \( C \) be an extremal Type IV \([6m+2, 3m+1, 2m+2]\) code and \( D_{2m+2}^{6m+2} \) be the support (with duplicates omitted) design of the minimum weight of \( C \). By [17, Theorem 14], the number of codewords of the minimum nonzero weight of \( C \) is equal to

\[
\frac{3(6m+1)}{m+1} \binom{3m+1}{m}.
\]

Therefore, by the Assmus–Mattson theorem, \( D_{2m+2}^{6m+2} \) is a 3-design with parameters

\[
\left( 6m+2, 2m+2, \frac{1}{3} \binom{3m}{m} \right).
\]

**Proposition 4.4.** Let \( D_{2m+2}^{6m+2} \) be the support \( t\)-design of the minimum weight of an extremal Type IV code of length \( n = 6m+2 \).

1. If \( t \geq 4 \), then \( D_{2m+2}^{6m+2} \) is a 5-design and \( m \) must be 11.

2. \( D_{2m+2}^{6m+2} \) is never an 8-design.

**Proof.** (1) By Theorem 2.5 (2), \( D_{2m+2}^{6m+2} \) is a 5-design if \( t \geq 4 \). If \( D_{2m+2}^{6m+2} \) is a 5-design, then by Lemma 2.1,

\[
\lambda_4 = \frac{2m-1}{6m-1} \cdot \frac{11}{3} \binom{3m}{m} \quad \text{and} \quad \lambda_5 = \frac{(2m-1)(2m-2)}{(6m-1)(6m-2)} \cdot \frac{1}{3} \binom{3m}{m},
\]

are positive integers. By computing \( m \leq 19 \), if \( \lambda_4 \) and \( \lambda_5 \) are positive integers, then we have \( m = 11 \).

(2) For \( m = 11 \), we have verified that

\[
\lambda_6 = \frac{(2m-1)(2m-2)(2m-3)}{(6m-1)(6m-2)(6m-3)} \cdot \frac{1}{3} \binom{3m}{m}, \quad \text{and} \quad \lambda_7 = \frac{(2m-1)(2m-2)(2m-3)(2m-4)}{(6m-1)(6m-2)(6m-3)(6m-4)} \cdot \frac{1}{3} \binom{3m}{m}
\]

are positive integers, and

\[
\lambda_8 = \frac{(2m-1)(2m-2)(2m-3)(2m-4)(2m-5)}{(6m-1)(6m-2)(6m-3)(6m-4)(6m-5)} \cdot \frac{1}{3} \binom{3m}{m}
\]

is not a positive integer. Therefore, by Lemma 2.1, \( D_{2m+2}^{6m+2} \) is never an 8-design. 

\[\square\]
For $t \geq 6$, we present the following proposition.

**Proposition 4.5.** Let $D_{w}^{6m+2}$ be the support $t$-design of weight $w$ of an extremal Type IV code of length $n = 6m + 2$.

1. All $D_{w}^{6m+2}$ are 6-designs simultaneously, or none of the $D_{w}^{6m+2}$ is a 6-design.

2. All $D_{w}^{6m+2}$ are 7-designs simultaneously, or none of the $D_{w}^{6m+2}$ is a 7-design.

3. For the case $m = 11$. All $D_{w}^{68}$ are 8-designs simultaneously, or none of the $D_{w}^{68}$ is an 8-design.

**Proof.** Let $C$ be an extremal Type IV $[6m + 2, 3m + 1, 2m + 2]$ code.

(1) Let us assume that $t = 6$. Therefore, by Theorem 2.4 (2), we have $W_{C,f}(x, y) = c(f)(xy)^{6}Z_{C,f}(x, y)$, where $c(f)$ is a linear function from Harm$_{t}$ to $\mathbb{R}$, and $Z_{C,f}(x, y) \in I_{G_{4},x_{0},0}$. By Theorem 2.4 (2), $Z_{C,f}(x, y)$ can be written in the following form:

$$Z_{C,f}(x, y) = \sum_{i=0}^{m} a_{i}h_{2}^{3(m-i)-5}h_{6}^{i}.$$ 

Because the minimum weight of $C$ is $2m + 2$, we have $a_{i} = 0$ for $i \neq m - 2$. Therefore, $W_{C,f}(x, y)$ can be written in the following form:

$$W_{C,f}(x, y) = c(f)(xy)^{6}h_{2}^{5}h_{6}^{m-2} = c(f)(xy)^{6}y^{2m-4}(x^{2} + 3y^{2})(x^{2} - y^{2})^{2m-4}.$$ 

By Lemma 2.7 (1), the coefficients of $x^{4m-6-2i}y^{2i}$ in $(x^{2} + 3y^{2})(x^{2} - y^{2})^{2m-4}$ are not equal to 0 for $0 \leq i \leq 2m - 3$ because $2m - 4 \neq 4i - 1$. Therefore, all $D_{w}^{6m+2}$ are 6-designs simultaneously, or none of the $D_{w}^{6m+2}$ is a 6-design.

(2) Let us assume that $t = 7$. Therefore, by Theorem 2.4 (2), we have $W_{C,f}(x, y) = c(f)(xy)^{7}Z_{C,f}(x, y)$, where $c(f)$ is a linear function from Harm$_{t}$ to $\mathbb{R}$, and $Z_{C,f}(x, y) \in I_{G_{4},x_{1},1}$. By Theorem 2.4 (2), $Z_{C,f}(x, y)$ can be written in the following form:

$$Z_{C,f}(x, y) = q_{3}r_{3} \sum_{i=0}^{m} a_{i}h_{2}^{3(m-i)-9}h_{6}^{i}.$$ 

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Because the minimum weight of $C$ is $2m + 2$, we have $a_i = 0$ for $i \neq m - 3$. Therefore, $W_{C,f}(x,y)$ can be written in the following form:

$$W_{C,f}(x,y) = c(f)(xy)^{7}q_{3}r_{3}h_{6}^{m-3}$$

$$= c(f)(xy)^{7}y^{2m-5}(x^{3} - 9xy^{2})(x^{2} - y^{2})^{2m-5}.$$

By Lemma 2.7 (2), the coefficients of $x^{4m-7-2i}y^{2i}$ in $(x^{3} - 9xy^{2})(x^{2} - y^{2})^{2m-5}$ are not equal to 0 for $0 \leq i \leq 2m - 4$. Therefore, all $D_{w}^{6m+2}$ are 7-designs simultaneously, or none of the $D_{w}^{6m+2}$ is a 7-design.

(3) Let us assume that $t = 8$. Therefore, by Theorem 2.4 (2), we have $W_{C,f}(x,y) = c(f)(xy)^{8}Z_{C,f}(x,y)$, where $c(f)$ is a linear function from Harm$_{t}$ to $\mathbb{R}$, and $Z_{C,f}(x,y) \in I_{G_{4},\chi_{0},0}$. By Theorem 2.4 (2), $Z_{C,f}(x,y)$ can be written in the following form:

$$Z_{C,f}(x,y) = \sum_{i=0}^{m} a_{i}h_{2}^{3(m-i)-7}h_{6}^{i}.$$  

Because the minimum weight of $C$ is $2m + 2$, we have $a_i = 0$ for $i \neq m - 3$. Therefore, $W_{C,f}(x,y)$ can be written in the following form:

$$W_{C,f}(x,y) = c(f)(xy)^{8}h_{2}^{2}h_{6}^{m-3}$$

$$= c(f)(xy)^{8}y^{2m-6}(x^{2} + 3y^{2})^{2}(x^{2} - y^{2})^{2m-6}.$$  

By Lemma 2.7 (3), if the coefficients of $x^{4m-8-2i}y^{2i}$ in $(x^{2} + 3y^{2})^{2}(x^{2} - y^{2})^{2m-6}$ are equal to 0, then $48(2m - 6) + 112$ is a square number. In the case where $m = 11$, $48(2m - 6) + 112 = 880$ is not a square number. Therefore, all $D_{w}^{68}$ are 8-designs simultaneously, or none of the $D_{w}^{68}$ is an 8-design. 

\[\square\]

By Propositions 4.4 and 1.5, we obtained the following theorem.

**Theorem 4.6.** Let $D_{w}^{6m+2}$ be the support $t$-design of weight $w$ of an extremal Type IV code of length $n = 6m + 2$ ($m \leq 19$).

(1) If $D_{w}^{6m+2}$ becomes a 5-design for any $w$, then $m = 11$.

(2) In the case where $m = 11$, $D_{w}^{68}$ is a 3-, 5-, 6-, or 7-design for any $w$.

(3) $D_{w}^{6m+2}$ is never an 8-design for any $w$.

Hence, the proof of Theorem 1.5 (2) is completed.
4.3 Case for $n = 6m + 4$

In this section, we consider the case of extremal Type IV $[6m+4, 3m+2, 2m+2]$ codes ($m \leq 21$). Let $C$ be an extremal Type IV $[6m+4, 3m+2, 2m+2]$ code and $D_{2m+2}^{6m+4}$ be the support (with duplicates omitted) design of the minimum weight of $C$. By [17, Theorem 14], the number of codewords of the minimum nonzero weight of $C$ is equal to

$$3\binom{3m+2}{m+1}.$$ 

Therefore, by the Assmus–Mattson theorem, $D_{2m+2}^{6m+4}$ is a 1-design with parameters

$$\left(6m+4, 2m+2, \binom{3m+1}{m}\right).$$

**Proposition 4.7.** Let $D_{2m+2}^{6m+4}$ be the support $t$-design of the minimum weight of an extremal Type IV code of length $n = 6m + 4$.

1. If $D_{2m+2}^{6m+4}$ is a 3-design, then $m$ must be in the set
   \{3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21\}.

2. If $D_{2m+2}^{6m+4}$ is a 4-design, then $m$ must be in the set \{9, 14, 19\}.

3. If $D_{2m+2}^{6m+4}$ is a 5-design, then $m$ must be in the set \{14, 19\}.

4. $D_{2m+2}^{6m+4}$ is never a 6-design.

**Proof.** (1) By Theorem 2.5, if $D_{2m+2}^{6m+4}$ is a 3-design if $t \geq 2$. If $m \in \{1, 2, 4, 13\}$, then $D_{2m+2}^{6m+4}$ is not a 3-design.

   For $m \in \{1, 4, 13\}$, we have verified that both

   $$\lambda_2 = \frac{2m+1}{6m+3}\binom{3m+1}{m} \quad \text{or} \quad \lambda_3 = \frac{(2m+1)2m}{(6m+3)(6m+2)}\binom{3m+1}{m}$$

   are not positive integers.

   For $m = 2$, it is known that no 3-(16, 6, 2) design exists [9].

   (2) If $D_{2m+2}^{6m+4}$ is a 4-design, then

   $$\lambda_4 = \frac{(2m+1)2m(2m-1)}{(6m+3)(6m+2)(6m+1)}\binom{3m+1}{m}$$
is a positive integer. By computing \( m \leq 21 \), if \( \lambda_4 \) is a positive integer, then we have \( m \in \{9, 14, 19\} \).

(3) For \( m \in \{9, 14, 19\} \), if

\[
\lambda_5 = \frac{(2m + 1)2m(2m - 1)(2m - 2) (3m + 1)}{(6m + 3)(6m + 2)(6m + 1)6m} \binom{m}{m}
\]

is a positive integer, then \( m \in \{14, 19\} \).

(4) We have verified that

\[
\lambda_6 = \frac{(2m + 1)2m(2m - 1)(2m - 2)(2m - 3) (3m + 1)}{(6m + 3)(6m + 2)(6m + 1)6m(6m - 1)} \binom{m}{m}
\]

is not a positive integer for \( m \in \{14, 19\} \).

For \( t \geq 4 \), we present the following proposition.

**Proposition 4.8.** Let \( D_{6m+4}^w \) be the support \( t \)-design of weight \( w \) of an extremal Type IV code of length \( n = 6m + 4 \).

1. All \( D_{6m+4}^w \) are 4-designs simultaneously, or none of the \( D_{6m+4}^w \) is a 4-design.

2. All \( D_{6m+4}^w \) are 5-designs simultaneously, or none of the \( D_{6m+4}^w \) is a 5-design.

3. If \( m \in \{14, 19\} \), all \( D_{6m+4}^w \) are 6-designs simultaneously, or none of the \( D_{6m+4}^w \) is a 6-design.

**Proof.** Let \( C \) be an extremal Type IV \([6m+4, 3m+2, 2m+2]\) code.

(1) We assume that \( t = 4 \). Therefore, by Theorem 2.4 (2), we have

\[
W_{C,f}(x, y) = c(f)(xy)^4 Z_{C,f}(x, y),
\]

where \( c(f) \) is a linear function from \( \text{Harm}_t \) to \( \mathbb{R} \), and \( Z_{C,f}(x, y) \in I_{G_4,\chi_0,0} \). By Theorem 2.4 (2), \( Z_{C,f}(x, y) \) can be written in the following form:

\[
Z_{C,f}(x, y) = \sum_{i=0}^{m} a_i h_2^{3(m-i)-2} h_6^i.
\]

Because the minimum weight of \( C \) is \( 2m + 2 \), we have \( a_i = 0 \) for \( i \neq m - 1 \). Therefore, \( W_{C,f}(x, y) \) can be written in the following form:

\[
W_{C,f}(x, y) = c(f)(xy)^4 h_2 h_6^{m-1} = c(f)(xy)^4 y^{2m-2}(x^2 + 3y^2)(x^2 - y^2)^{2m-2}.
\]
By Lemma 2.7 (1), the coefficients of $x^{4m-2-2i}y^{2i}$ in $(x^2+3y^2)(x^2-y^2)^{2m-2}$ are not equal to 0 for $0 \leq i \leq 2m-1$ because $2m-2 \neq 4i-1$. Therefore, all $D_{w}^{6m+4}$ are 4-designs simultaneously, or none of the $D_{w}^{6m+4}$ is a 4-design.

(2) Let us assume that $t = 5$. Therefore, by Theorem 2.3 (2), we have $W_{C,f}(x,y) = c(f)(xy)^5 Z_{C,f}(x,y)$, where $c(f)$ is a linear function from $\text{Harm}_t$ to $\mathbb{R}$, and $Z_{C,f}(x,y) \in I_{G_4,\chi_{1,1}}$. By Theorem 2.4 (2), $Z_{C,f}(x,y)$ can be written in the following form:

$$Z_{C,f}(x,y) = q_3 r_3 \sum_{i=0}^{m} a_i h_2^{3(m-i)-6} h_6^i.$$  

Because the minimum weight of $C$ is $2m+2$, we have $a_i = 0$ for $i \neq m-2$. Therefore, $W_{C,f}(x,y)$ can be written in the following form:

$$W_{C,f}(x,y) = c(f)(xy)^5 q_3 r_3 h_6^{m-2} = c(f)(xy)^5 y^{2m-3} (x^3 - 9xy^2) (x^2 - y^2)^{2m-3}.$$  

By Lemma 2.7 (2), the coefficients of $x^{4m-3-2i}y^{2i}$ in $(x^3 - 9xy^2)(x^2 - y^2)^{2m-3}$ are not equal to 0 for $0 \leq i \leq 2m-2$. Therefore, all $D_{w}^{6m+4}$ are 5-designs simultaneously, or none of the $D_{w}^{6m+4}$ is a 5-design.

(3) Let us assume that $t = 6$. Therefore, by Theorem 2.3 (2), we have $W_{C,f}(x,y) = c(f)(xy)^6 Z_{C,f}(x,y)$, where $c(f)$ is a linear function from $\text{Harm}_t$ to $\mathbb{R}$, and $Z_{C,f}(x,y) \in I_{G_4,\chi_{0,0}}$. By Theorem 2.4 (2), $Z_{C,f}(x,y)$ can be written in the following form:

$$Z_{C,f}(x,y) = \sum_{i=0}^{m} a_i h_2^{3(m-i)-4} h_6^i.$$  

Because the minimum weight of $C$ is $2m+2$, we have $a_i = 0$ for $i \neq m-2$. Therefore, $W_{C,f}(x,y)$ can be written in the following form:

$$W_{C,f}(x,y) = c(f)(xy)^6 h_2^2 h_6^{m-2} = c(f)(xy)^6 y^{2m-4} (x^2 + 3y^2)^2 (x^2 - y^2)^{2m-4}.$$  

By Lemma 2.7 (3), if the coefficients of $x^{4m-4-2i}y^{2i}$ in $(x^2 + 3y^2)^2(x^2 - y^2)^{2m-4}$ are equal to 0, then $48(2m-4) + 112$ is a square number. If $m = 14$ or 19, then $48(2m-4) + 112 = 1264$ or $48(2m-4) + 112 = 1744$ is not a square number. Therefore, if $m \in \{14, 19\}$, all $D_{w}^{6m+4}$ are 6-designs simultaneously, or none of the $D_{w}^{6m+4}$ is a 6-design. 

$\square$
By Propositions 4.7 and 4.8 we obtained the following theorem.

**Theorem 4.9.** Let $D_w^{6m+4}$ be the support $t$-design of weight $w$ of an extremal Type IV code of length $n = 6m + 4$ $(m \leq 21)$.

1. If $D_w^{6m+4}$ becomes a 3-design for any $w$, then $m$ must be in the set 
   \{3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21\}.

2. If $D_w^{6m+4}$ becomes a 4-design for any $w$, then $m$ must be in the set 
   \{9, 14, 19\}.

3. In the case where $m = 9$, $D_w^{58}$ is a 1, 3 or 4-design for any $w$. If 
   $m \in \{14, 19\}$, $D_w^{6m+4}$ is a 1-, 3-, 4- or 5-design for any $w$.

4. $D_w^{6m+4}$ is never a 6-design for any $w$.

Hence, the proof of Theorem 1.5 (3) is completed.

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