On the stress tensor for asymptotically flat gravity

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Abstract

The recent introduction of a boundary stress tensor for asymptotically flat spacetimes enabled a new construction of energy, momentum and Lorentz charges. These charges are known to generate the asymptotic symmetries of the theory, but their explicit formulae are not identical to previous constructions in the literature. This paper corrects an earlier comparison with other approaches, including terms in the definition of the stress tensor charges that were previously overlooked. We show that these terms either vanish identically (for $d > 4$) or take a form that does not contribute to the conserved charges (for $d = 4$). This verifies the earlier claim that boundary stress tensor methods for asymptotically flat spacetimes yield the same conserved charges as other approaches. We also derive some additional connections between the boundary stress tensor and the electric part of the Weyl tensor.

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1. Introduction

Conserved quantities play an important role in our description of physical systems. Particularly interesting cases arise in gauge theories, where conserved charges may be expressed as surface integrals at infinity. For a theory that is diffeomorphism invariant, like general relativity, the construction of conserved quantities depends crucially on the asymptotic structure of spacetime. In the case of asymptotically flat spacetimes in $d \geq 4$ dimensions, one is most interested in conserved charges associated with the generators of asymptotic Poincaré transformations.

There are many different methods for constructing conserved charges in asymptotically flat gravity \cite{1-10}. The various approaches offer different perspectives, but ultimately lead to equivalent results. For example, Arnowitt, Deser and Misner (ADM) \cite{1, 2} based their
construction on the initial value formulation of general relativity. This introduced a notion of asymptotic flatness at spatial infinity by demanding that the Cauchy data on some initial surface (rather than the geometry of the full spacetime) should asymptotically approach that of a corresponding surface in Minkowski space. In contrast, a fully covariant approach to asymptotically flat conserved quantities was introduced by Ashtekar and Hansen in the late 1970s. Several important results were established within this framework, including the introduction of a precise notion of asymptotic flatness at spatial infinity [3, 4], and showing that the ADM 4-momentum is obtained from the past limit of Bondi 4-momentum [11]. More recently, a definition of conserved charges was introduced that makes use of a boundary stress tensor [10]. Boundary stress tensor methods for asymptotically flat space were motivated by the success of such methods in asymptotically anti-de Sitter space (see [12–14] as well as [9] and many succeeding works), which were, in turn, inspired by the methods of [15]. Earlier steps toward using such methods in asymptotically flat space were taken in [9, 16–18].

The starting point of the boundary stress tensor method is a valid variational principle for asymptotically flat gravity. The standard form of the action, comprising the Einstein–Hilbert and Gibbons–Hawking–York terms, must be modified because it is not stationary with respect to arbitrary field variations that preserve the boundary conditions. This is directly related to the problem investigated by Regge and Teitelboim in [2], who found that the Hamiltonian formulation of general relativity with asymptotically flat boundary conditions is only well defined if it includes the additional surface term first proposed in [1]. The same conclusion applies to the Lagrangian formulation of the theory, but the necessary boundary terms were not investigated until recently; see [10] for a discussion in the second-order formalism or [19–21] for a discussion in the first-order formalism. Given a valid variational principle, the boundary stress tensor $T_{ab}$ is defined by varying the on-shell action with respect to the boundary metric, as in [12, 13, 15]. Any asymptotic Killing field $\xi^a$ can then be used to define a current $T_{ab}\xi^b$, and the flux of this current across a cut $C$ of spatial infinity is a conserved charge. According to the results of [22] these charges necessarily generate the asymptotic symmetries of the theory. In [10], it was argued—on general grounds—that the conserved quantities defined in this way must agree with other definitions. To illustrate this point the boundary stress tensor definition of energy and momentum at spatial infinity was shown to agree with the charges defined by Ashtekar and Hansen in [3].

The comparison between the boundary stress tensor approach of [10] and other methods in the literature was extended in [23], where two important results were established. First, it was shown that the canonical (space + time) reduction of the covariant action yields the ADM result [1] with the appropriate boundary terms. Second, using the form

$$T_{ab} = \frac{1}{8\pi G} (\pi_{ab} - \hat{\pi}_{ab}) \quad (1.1)$$

of the boundary stress tensor presented in [10], explicit formulae for boundary stress tensor charges associated with asymptotic translations and Lorentz transformations were shown to agree with those given by Ashtekar and Hansen in [3]. In (1.1), $\pi_{ab}$ is the conjugate momentum of the gravitational field, and $\hat{\pi}_{ab}$ is an analogous contribution from the counterterms. However, it turns out that there are several additional terms (beyond those in (1.1)) which potentially contribute to $T_{ab}$ but were overlooked in [10].

The primary purpose of this paper is to address the terms in the stress tensor that were overlooked in [10, 23]. Specifically, we want to resolve the tension between these ‘extra terms’ and the observation in [23] that the stress tensor (1.1) leads to charges that agree with the definition of Ashtekar and Hansen [3]. To this end, we explicitly calculate the extra terms.

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4 We thank Julian Le Witt and Simon Ross for bringing this fact to our attention.
and show that they do not contribute to the conserved charges. In $d > 4$ spacetime dimensions this is because the extra terms are identically zero. However, for $d = 4$ the situation is more subtle; the extra terms are not zero, but they do not contribute to the conserved charges. We also establish several additional results that summarize here in simple physical terms.

- For $d > 4$ the boundary stress tensor agrees with the electric part of the Weyl tensor at both first and second orders in an asymptotic expansion.

- For $d = 4$ the first-order term in the asymptotic expansions of the boundary stress tensor and the electric part of the Weyl tensor are the same. This result also applies to spacetimes carrying non-zero NUT charge. At second order, the boundary stress tensor is given by the second-order term in the electric part of the Weyl tensor plus contributions from the extra terms described above. The extra terms vanish for some spacetimes and are non-zero for others. However, in no case do they contribute to the conserved quantities.

- For $d = 4$, with zero NUT charge, there is a non-trivial identity relating terms in the asymptotic expansions of the electric and magnetic parts of the Weyl tensor:

$$E^{(2)}_{ab} - \sigma E^{(1)}_{ab} = \epsilon_{cde(ab}D^e\beta_{d)}.$$  

This identity helps one see more clearly the equivalence between the boundary stress tensor and Ashtekar–Hansen charges. It can also be used to simplify the analysis of [23].

The rest of the paper is organized as follows. We begin with various definitions and a brief review of asymptotic flatness in section 2. We then present the covariant counter-term action of [10] in section 3, and vary it with respect to the boundary metric to obtain the full boundary stress tensor. In sections 4.1 and 4.2, we discuss the asymptotic expansion of the stress tensor in $d = 4$ and $d > 4$ dimensions, respectively. The stress tensors are then used to construct conserved quantities at spatial infinity that agree with the usual definitions [1–5] of energy, momentum, angular momentum and boosts. We establish this by explicitly showing that the extra terms overlooked in [10, 23] do not contribute to the conserved charges. Finally, we consider some examples in section 5 and close with a brief discussion in section 6. Throughout, we use the notation of [10, 23, 24].

2. Preliminaries

This section sets the stage for our later work by providing relevant definitions and review. In the first part, we state our definition of asymptotic flatness in four and higher dimensions, followed by a discussion of the asymptotic field equations. The calculations in this part follow the approach of Beig and Schmidt [25, 26]. The second part contains definitions and a few useful results concerning the electric part of the Weyl tensor.

2.1. Asymptotic flatness

We begin by reviewing the coordinate-based definition of asymptotic flatness that was used in [10]. This approach is inspired by the covariant phase space treatment of [27] and is modeled on the definitions given in [25, 26]. Our definition is particularly close to that of [5], which treats spatial infinity as the unit timelike hyperboloid.

A spacetime of dimension $d \geq 4$ is asymptotically flat at spatial infinity if the line element admits an expansion of the form

$$ds^2 = \left(1 + \frac{2\alpha}{\rho^{d-3}} + \mathcal{O}(\rho^{-2})\right)\rho^2 + \rho^2 \left(h^{(0)}_{ab} + \frac{h^{(1)}_{ab}}{\rho^{d-3}} + \mathcal{O}(\rho^{-2})\right)\eta_a\eta_b + \rho \left(\mathcal{O}(\rho^{-2})\right)\eta_a\eta_a,$$

where

$$h^{(0)}_{ab} = \frac{1}{2\pi^2} \int \frac{d^4k}{(2\pi)^4} e^{i\vec{k}\cdot\vec{x}} \left[\epsilon_{abc}k_c + \delta_{ab}\right] D^e \beta_e.$$
for large positive $\rho$, where $h_{ab}^{0\rho}$ and $\eta^a$ are a metric and the associated coordinates on the unit $(d-2, 1)$ hyperboloid $\mathcal{H}$, and $\sigma$, $h_{ab}^{0\rho}$ are respectively a smooth function and a smooth tensor field on $\mathcal{H}$. The coordinate $\rho$ is the hyperbolic ‘radial’ function associated with some asymptotically Minkowski coordinates $x^a$ through $\rho^2 = \eta_{ab}x^ax^b$. In (2.1), the symbols $O(\rho^{-(d-2)})$ refer to terms that fall off, at least, as fast as $\rho^{-(d-2)}$ as one approaches spacelike infinity, i.e., $\rho \to +\infty$ with the coordinates $\eta^a$ fixed.

Following [25, 26] we partially fix the gauge and bring the metric into the form

$$ds^2 = N^2 d\rho^2 + h_{ab} \eta^a d\eta^b$$

where again $h_{ab}^{0\rho}$ is the metric on the unit hyperboloid. In four dimensions, there are two additional conditions that must be imposed [3, 26, 27]. First, we require $h_{ab}^{1\rho} = -2\sigma h_{ab}^{0\rho}$. This rules out the possibility of the so-called ‘spi super-translations’ [3]. Second, the function $\sigma$ must be even under the natural inversion mapping induced by the Minkowskian transformation $x^a \to -x^a$, ruling out the so-called ‘logarithmic super-translations’. These conditions ensure that the asymptotic symmetry group at spatial infinity is precisely the Poincaré group; see [3, 4, 5, 25, 28, 27] for further details.

Given the form of the metric (2.2), it is natural to decompose the Einstein equations using the (outward-pointing) unit vector $n^a = N^{-1} \delta^a_\rho$ normal to the constant $\rho$ surface $\mathcal{H}_\rho$ with induced metric $h_{ab} = g_{ab} - n_an_b$. Denoting projection onto $\mathcal{H}_\rho$ by $\perp$, this yields

$$\perp(R_{ab}) = R_{ab} + D_aD_b - \frac{\sigma}{\rho^d-3} \eta_{ab} - \frac{1}{2} \eta^{c}K_{cb} + 2K_{ab}^cK_{cb},$$

$$\perp(R_{abc}n^c) = \partial_\rho K_{ab} - D_bK = -\partial^b\pi_{ab},$$

$$R_{ab}n^an^b = -\partial_aK - K_{ab}^cK_{cb} + (D_ba^b - a^b n_b),$$

where $\mathcal{R}_{ab}$ is the intrinsic Ricci tensor on $\mathcal{H}_\rho$, $D_a$ is the (torsion-free) covariant derivative compatible with the induced metric, and $\partial_a$ denotes the Lie derivative along a vector field $v^a$.

The ‘acceleration’ $a^b$ and extrinsic curvature $K_{ab}$ are

$$a^b = n^a\nabla_ar^b, \quad K_{ab} = \frac{1}{2} \partial_a h_{ab}.$$  

One may take the fundamental variables to be the metric $h_{ab}$ on $\mathcal{H}_\rho$ and its conjugate momentum $\pi_{ab} = h_{ab}K - K_{ab}^cK_{cb}$.

We now discuss the asymptotic expansions of equations (2.4)–(2.6), taking care to distinguish between the $d = 4$ and $d > 4$ cases.

**Asymptotic expansion: $d = 4$.** The leading term in the asymptotic expansion of the field equation (2.4) implies that the three-dimensional metric $h_{ab}^{0\rho}$ is a solution of

$$\mathcal{R}_{ab}^{(0)} = 2h_{ab}^{(0)}.$$  

By inserting the expansion (2.3) into the field equations, Beig [26] showed that the first-order Einstein equations are identically satisfied, if

$$D^2 \sigma + 3\sigma = 0.$$  

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Here, we have introduced the (torsion-free) covariant derivative $D_a$ on the hyperboloid compatible with the metric $h^{(0)}_{ab}$. Beig [26] also showed that the second-order equations may be written in the form

$$h^{(2a)}_a = 12 \sigma^2 + \sigma_a \sigma^a,$$  \hspace{1cm} (2.10)  

$$D_b h^{(2b)}_a = 16 \sigma \sigma_{ab} + 2 \sigma_a \sigma_a,$$  \hspace{1cm} (2.11)  

$$D^2 h^{(2)}_{ab} - 2 h^{(2)}_{ab} = 6 \sigma_a \sigma^a h^{(0)}_{ab} + 8 \sigma_a \sigma_b + 14 \sigma \sigma_{ab} - 18 \sigma^2 h^{(0)}_{ab} + 2 \sigma_{ab} \sigma^b + 2 \sigma_{abc} \sigma^c,$$  \hspace{1cm} (2.12)  

where $\sigma_a = D_a \sigma$, $\sigma_{ab} = D_b D_a \sigma$, and $\sigma_{abc} = D_c D_b D_a \sigma$. For further details the reader is referred to [23, 25, 26].

**Asymptotic expansion: $d > 4$.** It is straightforward to carry out calculations analogous to those of [26] for $d > 4$ (see, e.g., [17]). The equations obtained from the asymptotic expansion of (2.4) are

$$\mathcal{R}^{(0)}_{ab} = (d - 2) h^{(0)}_{ab},$$  \hspace{1cm} (2.13)  

$$\mathcal{R}^{(1)}_{ab} = D_a D_b \sigma - (d - 1) \sigma h^{(0)}_{ab} - \frac{d - 3}{2} h^{(1)}_{ab} h^{(0)}_{ab} + \frac{(d - 1)}{2} h^{(1)}_{ab},$$  \hspace{1cm} (2.14)  

$$\mathcal{R}^{(2)}_{ab} = (d - 2) \left( h^{(2)}_{ab} - \frac{1}{2} h^{(2)} h^{(0)}_{ab} \right),$$  \hspace{1cm} (2.15)  

and the expansion of (2.5) yields

$$D^b h^{(1)}_{ab} - D_a h^{(1)} = 2 \left( \frac{d - 2}{d - 3} \right) D_a \sigma,$$  \hspace{1cm} (2.16)  

$$D^b h^{(2)}_{ab} - D_a h^{(2)} = 0.$$  \hspace{1cm} (2.17)  

The function $\sigma$ satisfies an equation similar to (2.9), which is obtained by taking the trace of (2.14) with $h^{(0)}_{ab}$ and using the covariant divergence of (2.16) to simplify the result. This gives

$$D^2 \sigma + (d - 3) (d - 1) \sigma + \frac{(d - 3) (d - 4)}{2} h^{(1)} = 0.$$  \hspace{1cm} (2.18)  

In four dimensions, the equations for the second-order term $h^{(2)}_{ab}$ receive contributions that are quadratic in $h^{(0)}_{ab}$ and $\sigma$, because these terms all appear at the same order in the asymptotic expansions. This is no longer the case in $d > 4$, and as a result the analog of (2.10) is

$$h^{(2)} = 0.$$  \hspace{1cm} (2.19)  

Note that in the asymptotic expansion of Minkowski space the metric $h^{(0)}_{ab}$ is a metric for the maximally symmetric hyperboloid, i.e., for de Sitter space. We will only require $h^{(0)}_{ab}$ to be a general Einstein metric satisfying (2.13). However, we will continue to refer to $H$ as the ‘hyperboloid’, for convenience.

**2.2. Electric part of the Weyl tensor**

The electric part of the Weyl tensor figures prominently in several of our results. It is defined as

$$E_{ab} := C_{abcd} n^c n^d.$$  \hspace{1cm} (2.20)  

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On a constant $\rho$ surface $\mathcal{H}_\rho$, this can be given a purely geometric expression in terms of the intrinsic and extrinsic curvatures, the acceleration $a^b$ and the covariant derivative $D_a$:

$$E_{ab} = \left( \frac{d-3}{d-2} \right) \left( \frac{1}{d-1} h_{ab} K - \xi_a K_{ab} \right) + \frac{d-4}{d-2} K_{ab} K_{cd} K_{cd} + \frac{1}{d-2} \left( \frac{1}{d-1} h_{ab} K \right) - \frac{1}{d-2} \left( R_{ab} - \frac{1}{d-1} h_{ab} R \right) + \frac{d-3}{d-2} \left( D_a a_b - \frac{1}{d-1} h_{ab} D_c a^c - a_a a_b + \frac{1}{d-1} h_{ab} a^c a^c \right).$$

(2.21)

It is straightforward to verify that $E_{ab}$ is traceless and divergenceless. In $d \geq 4$ dimensions, the asymptotic expansion takes the form

$$E_{ab} = \rho^{3-d} E^{(1)}_{ab} + \rho^{2-d} E^{(2)}_{ab} + \mathcal{O}(\rho^{1-d}).$$

(2.22)

However, there are differences between the terms in this expansion in the $d = 4$ and $d > 4$ cases. In four dimensions they are given by

$$E^{(1)}_{ab} = -\sigma_{ab} - h^{(0)}_{ab} \sigma,$$

(2.23)

$$E^{(2)}_{ab} = -h^{(2)}_{ab} + \sigma_{ab} + 5h^{(0)}_{ab} \sigma^2 - 2\sigma_{a} \sigma_{b} + h^{(0)}_{ab} \sigma^2 \sigma^c,$$

(2.24)

while in $d > 4$ dimensions the terms in the expansion are

$$E^{(1)}_{ab} = -\frac{(d-3)(d-4)}{2} h^{(1)}_{ab} - \sigma_{ab} - (d-3)h^{(0)}_{ab} \sigma,$$

(2.25)

$$E^{(2)}_{ab} = -\frac{(d-3)(d-2)}{2} h^{(2)}_{ab}.$$

(2.26)

The structure of these terms and the role they play in the construction of conserved charges are discussed in detail in section 4.

In four dimensions the second-order term (2.24) satisfies a non-trivial identity:

$$E^{(2)}_{ab} - \sigma E^{(1)}_{ab} = \epsilon_{cd} a^{d} h^{(d)}_{b},$$

(2.27)

where $\beta_{ab}$ is the second-order magnetic part of the Weyl tensor [3, 4, 23, 26]. This identity is useful in establishing the equivalence between our expressions for the Lorentz charges and the results of Ashtekar and Hanson [3].

3. A variational principle and the boundary stress tensor

It was shown in [10] that a good variational principle for gravity with asymptotically flat boundary conditions in $d \geq 4$ dimensions is given by the action

$$S = \frac{1}{16\pi G} \int_{M} \sqrt{-g} R + \frac{1}{8\pi G} \int_{\partial M} \sqrt{-h} (K - \hat{K}),$$

(3.1)

where $\hat{K} := h^{ab} \hat{K}_{ab}$, and $\hat{K}_{ab}$ is defined to satisfy

$$\hat{R}_{ab} = \hat{K}_{ab} \hat{K} - \hat{K}^{e}_{ab} \hat{K}_{eb}.$$ 

(3.2)

Equation (3.2) admits more than one solution for $\hat{K}_{ab}$—we choose the solution that asymptotes to the extrinsic curvature of the boundary of Minkowski space as $\partial M$ is taken to infinity. As described in [10], the boundary term can be motivated from the heuristic idea that one should subtract off a ‘background’ divergence. The defining equation (3.2), arrived at via the Gauss–Codazzi equations, identifies $\hat{K}_{ab}$ as the part of the extrinsic curvature that is fixed by the
kinematics of the theory. A similar boundary term has been used to construct variational principles for asymptotically linear dilaton spacetimes [29, 30] and asymptotically plane wave spacetimes [31].

Varying the action and imposing the equations of motion yields

$$\delta S = \frac{1}{16\pi G} \int_{\partial M} \sqrt{-h} (\pi^{ab} - \hat{\pi}^{ab} + \Delta^{ab}) \delta h_{ab},$$

(3.3)

where \( \hat{\pi}^{ab} := h^{ab} \hat{K} - \hat{K}^{ab} \), in analogy with \( \pi^{ab} \), and \( \Delta^{ab} \) represents a number of ‘extra’ terms. An expression for \( \Delta^{ab} \) is given in appendix A. The boundary stress tensor is defined as the functional derivative of the on-shell action with respect to \( h_{ab} \):

$$T_{ab} := \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h_{ab}} = \frac{1}{8\pi G} (\pi^{ab} - \hat{\pi}^{ab} + \Delta^{ab}).$$

(3.4)

The first two terms in (3.4) were considered in [10, 23]:

$$T_{ab}^{\Delta \pi} = \frac{1}{8\pi G} (\pi_{ab} - \hat{\pi}_{ab}).$$

(3.5)

These terms are conserved in the sense that \( D_b T_{ab}^{\Delta \pi} = 0 \), and the generators of asymptotic symmetries constructed using \( T_{ab}^{\Delta \pi} \) agree with the Ashtekar–Hansen charges. However, the last term in (3.4) was overlooked in previous investigations of the stress tensor. In the following section, we show that this does not alter the conclusions of [10, 23].

4. Conserved charges

By the general arguments given in [10, 15, 22], the boundary stress tensor leads to conserved charges that generate the expected asymptotic symmetries. The generator of diffeomorphisms along an asymptotic Killing field \( \xi^a \) is

$$Q[\xi] := \lim_{\rho \to \infty} \int_{C_{\rho}} \sqrt{-h_{C_{\rho}}} u_{C_{\rho}}^a T_{ab} \xi^b,$$

(4.1)

where each \( C_{\rho} \) is a Cauchy surface within a constant \( \rho \) hypersurface \( H_{\rho} \) such that \( C = \lim_{\rho \to \infty} C_{\rho} \) is a Cauchy surface in the boundary hyperboloid \( H \), and \( u_{C_{\rho}}^a \) is the unit normal to \( C_{\rho} \) in \( H_{\rho} \). Explicit computations in [10, 23] showed that for \( d = 4 \) the quantity

$$Q[\xi] := \lim_{\rho \to \infty} \int_{C_{\rho}} \sqrt{-h_{C_{\rho}}} u_{C_{\rho}}^a T_{ab}^{\Delta \pi} \xi^b,$$

(4.2)

computed without the term \( \Delta_{ab} \), agrees precisely with the Ashtekar–Hansen conserved quantities [3]. In section 4.1, we show that the extra terms in \( T_{ab} \) do not change this result. The conserved charges do not receive contributions from \( \Delta_{ab} \), despite the fact that in \( d = 4 \) these terms are non-zero at the relevant order in the asymptotic expansion. The proof relies on a detailed calculation of \( \Delta_{ab} \). The \( d > 4 \) case is treated in section 4.2. A calculation of \( \Delta_{ab} \) shows that the extra terms are identically zero for \( d > 4 \), up to terms that vanish too fast in the limit \( \rho \to \infty \) to contribute to conserved charges.

Asymptotic Killing fields. Recall that Minkowski space Killing fields naturally divide into two classes: translations and Lorentz transformations. In an asymptotically flat spacetime there is a one-to-one correspondence between these classes and the first two terms in the asymptotic expansion of a general asymptotic Killing field \( \xi^a \). In dimension \( d \geq 4 \) this expansion takes the form

$$\xi^a = \xi^a_{(0)} + \rho^{-1} \xi^a_{(1)} + O(\rho^{-2}).$$

(4.3)
Spatial rotations and boosts are associated with the leading order term in the expansion. They correspond to Killing vectors $\xi^a_{\text{m}}$ of $\mathcal{H}$ and therefore satisfy $D_a \xi^a_{\text{m}} = 0$. Translations appear at next-to-leading order in (4.3), and are associated with conformal Killing vectors $\xi^a_{\text{m}}$ that satisfy $D_a \xi^a_{\text{m}} = f(\eta) h^{a}_{\text{m}}$ for some non-zero function $f$ on $\mathcal{H}$. Terms of order $p^{-2}$ and higher are not relevant for our purposes—the asymptotic expansion of the stress tensor implies that they do not contribute to the conserved charges.

In $d = 4$, the surface $\mathcal{H}$ at spatial infinity is the unit hyperboloid, so the meaning of asymptotic translations and Lorentz transformations is clear. However, in dimension $d > 4$ we only require $h^{a}_{\text{m}}$ to be a general Einstein metric satisfying (2.13). In that case, the properties described above, which depend on the isometries and conformal isometries of the metric at spatial infinity, define what we mean by ‘translations’ and ‘Lorentz transformations’.

**Potentials for the stress tensor.** We now pause to establish a few useful results concerning potentials for transverse, symmetric tensors. The existence of these potentials plays an important role in the construction of the conserved charges. Much of our treatment follows that of [3, 4, 26].

A tensor $\theta_{ab}$ is said to admit a scalar potential $\alpha$, if

$$\theta_{ab} = D_a D_b \alpha - h^{(0)}_{ab} D^2 \alpha - (d - 2) \alpha h^{(0)}_{ab},$$

(4.4)

where the metric $h^{(0)}_{ab}$ satisfies (2.13). Taking the divergence of $\theta_{ab}$ and commuting covariant derivatives shows that it is conserved: $D^a \theta_{ab} = 0$. If $\xi^a_{\text{m}}$ is a Killing vector of $h^{(0)}_{ab}$, then the current $\theta_{ab} \xi^b_{a(0)}$ can be expressed as the divergence of an anti-symmetric tensor:

$$\theta_{ab} \xi^b_{a(0)} = D^b \left( 2 \xi^a_{b(0)} D_0 a + \alpha D_b \xi^a_{(0) a} \right).$$

(4.5)

Now consider a Cauchy surface $C \subset \mathcal{H}$, with timelike unit vector $u^a_{(0)}$. The component of the current along $u^a_{(0)}$ becomes a total derivative on $C$:

$$u^a_{(0)} \theta_{ab} \xi^b_{a(0)} = u^a_{(0)} D^b \left( 2 \xi^a_{b(0)} D_0 a + \alpha D_b \xi^a_{(0) a} \right) = D^a \left[ \left( 2 \xi^a_{b(0)} D_0 b + \alpha D_b \xi^a_{(0) b} \right) u^b_{(0)} \right].$$

(4.6)

This vanishes when integrated over $C$, so that currents of the form $\theta_{ab} \xi^b_{a(0)}$ do not contribute to the conserved charge associated with $\xi^a_{\text{m}}$.

Similarly, a tensor $t_{ab}$ is said to admit a symmetric, transverse tensor potential, if

$$t_{ab} = D^a \gamma_{ab} + 2 R^{(0)}_{acbd} \gamma^d$$

(4.7)

for some symmetric $\gamma_{ab}$ with $D^a \gamma_{ab} = 0$. The tensor $t_{ab}$ is conserved, and for $\xi^a_{\text{m}}$, a Killing field of $h^{(0)}_{ab}$ the current $t_{ab} \xi^b_{a(0)}$ is the divergence of an anti-symmetric tensor:

$$t_{ab} \xi^b_{a(0)} = 2 D^c \left( \xi^a_{c(0)} D_0 b + \gamma_{ca} D_b \xi^a_{(0) c} \right).$$

(4.8)

A current of this form does not contribute to conserved charges, because the component along $u^a_{(0)}$ is a total divergence on $C$. The proofs of these statements, which are more involved than in the case of the scalar potential, allow for metrics with non-vanishing Weyl curvature.

**4.1. $d = 4$**

In this section, we show that $\Delta_{ab}$ does not contribute to the conserved charges in four dimensions. That is,

$$\Delta Q[\xi] := \lim_{\rho \to \infty} \frac{1}{8 \pi G} \int_{C_\rho} \sqrt{-h_C} u^a_{C_\rho} \Delta_{ab} \xi^b_{a(0)} = 0$$

(4.9)
for any asymptotic Killing field $\xi^a$. We show that $\Delta Q[\xi]$ is zero using the asymptotic expansion of $\Delta_{ab}$. The main result of this calculation is

$$\Delta_{ab} := \rho \Delta_{ab}^{(0)} + \Delta_{ab}^{(1)} + \rho^{-1} \Delta_{ab}^{(2)} + \mathcal{O}(\rho^{-2})$$

$$= \frac{1}{4\rho} \left[ 9\sigma^a \sigma^b h^{(0)}_{ab} - 29\sigma_a \sigma_b + 63\sigma_{ab} + 24\sigma_{ap} \sigma^p - 5\sigma_{cd} \sigma^{cd} h^{(0)}_{ab} + 45\sigma^2 h^{(0)}_{ab} ight.$$

$$- 3\sigma_{mn} \sigma^{mp} h^{(0)}_{ab} + 9\sigma_{pq(\alpha} \sigma^{pq)\beta} - 3\sigma^{pq} \sigma_{p(q\beta)\alpha} - 2\sigma^2 \sigma_{(\alpha(\beta)} \right] + \mathcal{O}(\rho^{-2}).$$

(4.10)

Three remarks are in order here. First, both $\Delta_{ab}^{(0)}$ and $\Delta_{ab}^{(1)}$ are identically zero. Second, all of the terms in $\Delta_{ab}^{(2)}$ involving $h^{(2)}_{ab}$ cancel, so that it depends only on $\sigma$ and its derivatives. Third, $\Delta_{ab}^{(2)}$ is traceless; $h^{(0)ab} \Delta_{ab}^{(2)} = 0$.

In [23], it was shown that $\Delta_{ab} := \pi_{ab} - \hat{\pi}_{ab}$ is conserved. From general arguments given in [10, 22] we expect the full $T_{ab}$ to be conserved as well. It follows that $\Delta_{ab}$ should be independently conserved. Using various identities from appendix B one can verify that the divergence of $\Delta_{ab}$ is indeed zero: $D^a \Delta_{ab} = D^a \Delta_{ab}^{(2)} = 0$. Thus, any contributions to the charges from (4.9) would also be conserved. Since $\Delta_{ab} = 0$, there are no contributions to the charges associated with asymptotic translations. There may, however, be a contribution to the Lorentz charges from the non-vanishing term $\Delta_{ab}^{(2)}$:

$$\Delta Q[\xi] = \frac{1}{8\pi G} \int C \sqrt{-h} \frac{m}{a(0)} h^{(2)ab} \xi_{(0)},$$

(4.11)

where the integral is performed over a cut $C$ of the unit hyperboloid $\mathcal{H}$. In the rest of this section, we will only be concerned with Lorentz charges constructed from asymptotic Killing fields with $\xi_{(0)}^b \neq 0$.

Most of the terms in $\Delta_{ab}^{(2)}$ can be expressed in terms of the scalar and tensor potentials described above. An explicit calculation shows that

$$4\Delta_{ab}^{(2)} = \theta_{ab} + t_{ab} + \frac{1}{3} \gamma_{ab} + \frac{11}{2} \mathring{j}_{ab} - 4k_{ab},$$

(4.12)

where $\theta_{ab}$ and $t_{ab}$ were defined in (4.4) and (4.7), respectively, and

$$\alpha = -\frac{3}{2} \sigma_{mn} a^{mn} - a^m \sigma_m, \quad \gamma_{ab} = 6(2\sigma_{ab} + \frac{5}{2} \sigma^2 h^{(0)}_{ab} + \sigma^c \sigma_{cb} - \frac{1}{2} \sigma_{mn} a^{mn} h^{(0)}_{ab}),$$

(4.13)

$$k_{ab} = -\frac{1}{2} \sigma^c \sigma_{cb} \xi^{(0)}_{ab} + \sigma_a \sigma_b + \frac{3}{2} \sigma^2 h^{(0)}_{ab},$$

(4.14)

$$j_{ab} = 2(\sigma_{ab} + 2\sigma^2 h^{(0)}_{ab} + \sigma_a \sigma_b - \sigma \sigma^2 h^{(0)}_{ab}).$$

(4.15)

Note that $k_{ab}$, $j_{ab}$ and $\gamma_{ab}$ are all divergence free. Furthermore, $\gamma_{ab}$ and $\mathring{j}_{ab}$ themselves admit similar potentials [23]:

$$\frac{1}{6} \gamma_{ab} \xi^a_{(0)} = D^a \left( \xi^c_{(0)} D_{(a} \xi^{b)}_{bc} + k_{(a} D_{b)} \xi^a_{(0)} \right) + j_{ab} \xi^a_{(0)},$$

(4.17)

$$j_{ab} \xi^a_{(0)} = D^a (\sigma^2 D_{(a} \xi^b_{(0)}) - 4\sigma \sigma_{(a} \xi^b_{(0)})).$$

(4.18)

Thus, with the possible exception of $k_{ab}$, the terms in (4.12) cannot contribute to the conserved charges.

We will now show that the $k_{ab}$ term does not contribute to the conserved charges. First, consider the case of spatial rotations. It is convenient to introduce coordinates $(\tau, \theta, \phi)$ on the unit hyperboloid $\mathcal{H}$ so that the metric takes the standard form

$$h^{(0)}_{ab} \; dx^a \; dx^b = -d\tau^2 + \cosh^2 \tau (d\theta^2 + \sin^2 \theta \; d\phi^2).$$

(4.19)
Without loss of generality the cut \( C \) is taken to be the surface \( \tau = 0 \), or ‘neck’, of the hyperboloid, which is just the unit 2-sphere. In this case, the timelike vector \( u^a_{(0)} \) points in the \( \tau \)-direction, and for any rotational Killing field \( \xi^a_{(0)} \) we have

\[
u^a_{(0)} u^b_{ab} \xi^b_{(0)} = 0. \tag{4.20}\]

Thus, the contribution to the conserved charge from the \( k_{ab} \) term in (4.12) is given by the integral of \( \sigma_a \sigma_a \xi^a_{(0)} \) over the neck of the hyperboloid:

\[
Q_k = \int_{S^2} \sigma_a \sigma_a \xi^a_{(0)}. \tag{4.21}\]

Since \( \sigma \) has even parity on \( \mathcal{H} \) it is convenient to decompose it as

\[
\sigma = T_+ Y_+ + T_- Y_-, \tag{4.22}\]

where \( Y_\pm \) schematically denotes even (odd) parity spherical harmonics on \( S^2 \), and \( T_\pm \) is an even (odd) parity function of the coordinate \( \tau \) alone. Note that \( T_- \) is identically zero at \( \tau = 0 \).

Using this decomposition we have

\[
\sigma_a \xi^a = T_+ Y_+ + T_- Y_- \tag{4.23}
\]

and

\[
\sigma_a \xi^a = T_+ Y_+ + T_- Y_, \tag{4.24}\]

where \( Y_{\pm \xi} = \xi^a_{(0)} D_a Y_{\pm \xi} \). Thus,

\[
Q_k = (T_+ T_-) \int_{S^2} Y_{\xi} Y_. \tag{4.25}\]

But \( Y_{\xi} \) has even parity when \( \xi^a_{(0)} \) is a spatial rotation, so the integrand in (4.25) is odd and \( Q_k \) vanishes.

A similar argument holds for boosts. On the \( \tau = 0 \) slice of the hyperboloid, a boost Killing field \( \xi^a_{(0)} \) can be expressed as

\[
\xi^a_{(0)} = g u^a_{(0)}, \tag{4.26}\]

where \( g \) is a function on \( S^2 \) with odd parity [32]. Thus,

\[
Q_k = \int_{S^2} u^a_{(0)} k_{ab} \xi^b_{(0)} = \int_{S^2} u^a_{(0)} k_{ab} u^b_{(0) g} = \int_{S^2} k_{\tau \tau} g. \tag{4.27}\]

From the definition (4.15) we see that \( k_{\tau \tau} \) is a sum of squares, and, therefore, has even parity on \( S^2 \). But since \( g \) has odd parity on \( S^2 \), the integrand in (4.27) is again odd and \( Q_k \) vanishes. This completes the proof that the extra term \( \Delta_{ab} \) in the stress tensor does not contribute to the conserved charges in four dimensions.

**Asymptotic expansion of the full stress tensor.** We close this section by summarizing our results for the boundary stress tensor in four dimensions. The asymptotic expansion of the full stress tensor is

\[
T_{ab} = \frac{1}{8\pi G} \left[ \Delta \pi_{ab}^{(1)} + \frac{\Delta \pi_{ab}^{(2)} + \Delta \pi_{ab}^{(2)}}{\rho} + \ldots \right]. \tag{4.28}\]

\[
^5\text{On a different cut the vector will have other components.}
\]
The large $\rho$ expansion of $\Delta\pi_{ab}$ was performed in detail in appendix B of [23], where it was shown that

$$\Delta\pi_{ab}^{(1)} = \pi_{ab} + \pi h_{ab} = -E_{ab}^{(1)},$$  \hspace{1cm} (4.29)

$$\Delta\pi_{ab}^{(2)} = -\left(\frac{3}{2}\sigma^2 + \sigma\sigma^c + \frac{1}{2}\sigma_{cd}\sigma^{cd}\right)h_{ab}^{(0)} + h_{ab}^{(2)} + 2\sigma_a\sigma_b + \sigma\sigma_{ab} + \sigma_a\sigma_b$$

$$= -E_{ab}^{(2)} + \frac{1}{\rho}\gamma_{ab}.$$  \hspace{1cm} (4.30)

This expansion can be used to extract expressions for the generators of asymptotic symmetries from (4.1). The term $\Delta_{ab}$ does not contribute, so the relevant terms are

$$Q[\xi] = -\frac{1}{8\pi G} \lim_{\rho \to \infty} \int_\rho \frac{\partial S}{\partial \rho} [\pi_{ab}^{(1)}\xi^b + u^a_{(0)}(E_{ab}^{(1)}\xi^b_{(0)} + u^a_{(0)}(E_{ab}^{(1)} - \sigma E_{ab}^{(1)} - \frac{1}{\rho}\gamma_{ab}))\xi^b_{(0)}].$$  \hspace{1cm} (4.31)

Two of the terms in this expression vanish when integrated over $C$. The integral of the first term vanishes, because $\xi^b_{(0)}$ is a Killing vector on the hyperboloid and $E_{ab}^{(1)}$ takes the form (4.4), with scalar potential $\sigma$. The integral of the term involving $\gamma_{ab}$ also vanishes by virtue of (4.17). The remaining terms are

$$Q[\xi] = -\frac{1}{8\pi G} \int d^2x \sqrt{-h} \left[\pi_{ab}^{(1)}\xi^b + u^a_{(0)}(E_{ab}^{(1)}\xi^b_{(0)} + u^a_{(0)}(E_{ab}^{(1)} - \sigma E_{ab}^{(1)} - \frac{1}{\rho}\gamma_{ab}))\xi^b_{(0)}\right].$$  \hspace{1cm} (4.32)

where the $\sigma E_{ab}^{(1)}$ term is due to sub-leading terms in the asymptotic expansions of $\sqrt{-h}$ and $u^a$. The charges for asymptotic translations come from the first term in the integrand, and the Lorentz charges come from the second term. The equivalence between the stress tensor and Ashtekar–Hanson definition of the Lorentz charges follows from the identity (2.27), applied to the second term in (4.32) [23].

4.2. $d > 4$

The asymptotic expansion of $\Delta_{ab}$ in dimension $d > 4$ takes the form

$$\Delta_{ab} = \rho \Delta_{ab}^{(0)} + \rho^{-d+4} \Delta_{ab}^{(4)} + \rho^{-d+3} \Delta_{ab}^{(3)} + \mathcal{O}(\rho^{-d+2}).$$  \hspace{1cm} (4.33)

By a tedious calculation one can show that $\Delta_{ab}^{(0)} = \Delta_{ab}^{(1)} = \Delta_{ab}^{(2)} = 0$, so the stress tensor takes the form

$$T_{ab} = -\frac{1}{8\pi G} (\sigma_{ab} - \pi_{ab})$$

$$= \frac{1}{8\pi G} \rho^{d-4} \left(\sigma h_{ab}^{(0)} + \frac{d-4}{2} h_{ab}^{(1)} + \frac{1}{d-3} D_a D_b \sigma\right) + \frac{1}{8\pi G} \rho^{d-3} \frac{d-2}{2} h_{ab}^{(2)} + \mathcal{O}(\rho^{-d}).$$  \hspace{1cm} (4.34)

This is a pleasing result because it is proportional to $E_{ab}$, without the additional terms that appeared in four dimensions. The relation between the stress tensor and the electric part of the Weyl tensor is

$$T_{ab} = -\frac{1}{8\pi G} \frac{\rho}{d-3} E_{ab} + \mathcal{O}(\rho^{-d}).$$  \hspace{1cm} (4.35)

Expression (4.1) for the conserved charges becomes

$$Q[\xi] = -\frac{1}{8\pi G} \frac{1}{(d-3)} \lim_{\rho \to \infty} \int_\rho \frac{\partial S}{\partial \rho} [\pi_{ab}^{(1)}\xi^b + u^a_{(0)}(E_{ab}^{(1)}\xi^b_{(0)} + u^a_{(0)}(E_{ab}^{(1)} - \sigma E_{ab}^{(1)} - \frac{1}{\rho}\gamma_{ab}))\xi^b_{(0)}].$$  \hspace{1cm} (4.36)
The first term is proportional to \( \rho \), and must vanish for the \( \rho \to \infty \) limit to exist. In \( d = 4 \), the analogous term vanishes, because \( E^{(i)}_{ab} \) admits a simple scalar potential. The proof is more complicated for \( d > 4 \), but proceeds in essentially the same way. The term \( E^{(i)}_{ab} \) can be expressed in terms of one tensor and two scalar potentials as

\[
E^{(1)}_{ab} = -\theta_{ab}[\sigma] - \frac{d-4}{2} \left( l_{ab} - \theta_{ab}[h^{(1)}] - 2 \left( \frac{d-1}{d-3} \right) \theta_{ab}[\sigma] \right),
\]

with the potential for \( l_{ab} \) given by

\[
\gamma_{ab} = h^{(1)}_{ab} - h^{(0)}_{ab} h^{(1)} - 2 \left( \frac{d-2}{d-3} \right) h^{(0)}_{ab} \sigma.
\]

It follows that the flux across \( C \) of the current \( E^{(i)}_{ab} \) vanishes. Taking the \( \rho \to \infty \) limit gives the final expression for the conserved charges in dimension greater than four:

\[
Q[\xi] = -\frac{1}{8\pi G} \int_C d^{d-2}x \sqrt{-h^{(0)}_C} \left[ u^{(1)}_{ab} E^{(1)}_{ab} s_{(1)} + u^{(2)}_{ab} E^{(2)}_{ab} s_{(2)} \right].
\]

We remind the reader that the above expressions hold when \( h^{(0)}_{ab} \) is any Einstein metric satisfying (2.13). As in the case of \( d = 4 \), the charges for asymptotic translations come from the first term and the Lorentz charges come from the second term. However, for \( d > 4 \) the sub-leading terms in the expansions of \( \sqrt{-h^{C}_C} \) and \( u^{(i)}_{ab} \) do not contribute to (4.39) as they begin at too high of an order in \( \rho^{-1} \).

5. Examples

It is useful to illustrate the results of the previous sections with a few concrete examples. In the first example, we determine the stress tensor and conserved charges for the Kerr solution in four dimensions. Then, in the second example, we explicitly construct four-dimensional spacetimes with \( \Delta_{ab}^{(2)} \neq 0 \).

5.1. The Kerr black hole

The line element for the Kerr solution in four dimensions can be written as

\[
ds^2 = \frac{\Xi}{\Delta} dr^2 + \Xi d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 - dr^2 + \frac{2MGr}{\Xi} (a \sin^2 \theta d\phi - dr)^2.
\]

The parameters \( a \) and \( M \) are related to the mass and angular momentum, and the functions \( \Xi \) and \( \Delta \) are given by

\[
\Xi = r^2 + a^2 \cos^2(\theta), \quad \Delta = r^2 - 2MG + a^2.
\]

The line element must be expressed in the Beig–Schmidt form (2.2) before the results of the previous sections can be applied. This is accomplished by a series of coordinate redefinitions, as described in [25]. In the Beig–Schmidt coordinates, \( h^{(0)}_{ab} \) is the standard metric on the hyperboloid:

\[
h^{(0)}_{ab} dx^a dx^b = -d\tau^2 + \cosh^2 \tau (d\theta^2 + \sin^2 \theta d\phi^2),
\]

and the function \( \sigma \) is given by

\[
\sigma = MG \cosh 2\tau \sech \tau.
\]

Note that this is the only admissible solution of (2.9) that depends only on \( \tau \). The first-order terms in the expansion of the metric are \( h^{(1)}_{ab} = -2\sigma h^{(0)}_{ab} \), as required by the asymptotically flat

\[\text{A second solution is not even under } \tau \to -\tau, \text{ as required by the boundary conditions.}\]
boundary conditions. The second-order terms $h_{ab}^{(2)}$ are a bit more complicated. Their particular form is not too illuminating; it is sufficient to note that $h_{ab}^{(2)}$ is the first term in the asymptotic expansion that depends on the rotation parameter $a$.

The Kerr solution has two non-zero conserved charges: the mass associated with translations $\partial_t$ and the angular momentum associated with rotations $\partial_\phi$. Calculating the charges from (4.32) requires the $\tau-\tau$ and $\tau-\phi$ components of the electric part of the Weyl tensor. Specifically, the terms we need are

$$E_{\tau\tau}^{(1)} = -2MG \operatorname{sech}^3 \tau, \quad E_{\tau\phi}^{(1)} = 0, \quad E_{\tau\phi}^{(2)} = 3aMG \sin^2 \theta \sech^2 \tau. \quad (5.5)$$

The mass is obtained by evaluating the first term in (4.32) for the vector field $\xi_a = \rho^{-1} \cosh(\tau) \delta_a^\tau + \cdots$. (5.6)

The cut $C$ is taken to be the $\tau = 0$ surface of the unit hyperboloid, with timelike normal $u^\mu_\tau = \delta_\mu^\tau$. Then, the conserved charge is

$$Q_{\partial\tau} = -\frac{1}{8\pi G} \int_C d^3 x \sqrt{-h}^{(0)} \sin \theta u^\tau_{(0)} E^{(1)}_{\tau\tau} \xi^\tau = \frac{1}{8\pi G} \int_0^{2\pi} d\phi \int_0^\pi d\theta \cosh \tau \sin \theta \cdot 2MG \cosh \tau \cdot \cosh \tau = M. \quad (5.7)$$

The vector field for rotations is just $\xi^\mu = \delta_\mu^\phi$. Using this in the second term of (4.32) gives

$$Q_{\partial\phi} = -\frac{1}{8\pi G} \int_C d^3 x \sqrt{-h}^{(0)} (E_{\tau\phi}^{(2)} - \sigma E_{\tau\phi}^{(1)}) \xi^\phi = -\frac{1}{8\pi G} \int_0^{2\pi} d\phi \int_0^\pi d\theta \cosh^2 \tau \sin \theta \cdot 3aMG \sech^2 \tau \sin^2 \theta = -aM. \quad (5.8)$$

These are the expected results for the mass and angular momentum of the Kerr solution.

It is interesting to calculate the extra terms $\Delta_{ab}^{(2)}$ for the Kerr solution. Using the expressions for $h_{ab}^{(m)}$ and $\sigma$, all of the components are found to vanish:

$$\Delta_{ab}^{(2)} = 0. \quad (5.9)$$

This result also applies for the Schwarzschild and Reissner–Nordstrom solutions. In these cases, the results of section 4.1 are superfluous, since the boundary stress tensor reduces to the form considered in [23].

### 5.2. Spacetimes with $\Delta_{ab}^{(2)} \neq 0$

In the previous section, we saw that the extra terms vanish for a number of familiar four-dimensional spacetimes. However, it is easy to demonstrate the existence of asymptotically flat spacetimes for which the extra terms do not vanish. The condition (2.9) is a linear equation, so new solutions may be obtained by superposition. The result (4.12), on the other hand, is nonlinear, which means that these new solutions will generally have non-vanishing $\Delta_{ab}^{(2)}$.

An explicit example is a spacetime consisting of two black holes related by a boost. For simplicity we will work with the Schwarzschild solution. Since this is recovered from the $a \to 0$ limit of (5.1), it follows that $\sigma$ for the Schwarzschild metric is also given by (5.4). First, consider a single black hole that is boosted along the $z$-axis by an amount $\beta$. The function $\sigma$ becomes

$$\sigma_b = \frac{1 + 2q^2}{\sqrt{1 + q^2}} MG. \quad (5.10)$$
with \( q \) given by
\[
q = \cosh \beta \sinh \tau + \sinh \beta \cosh \tau \cos \theta.
\] (5.11)

The boost simply maps one solution of (2.9) to another, and does not change the fact that the extra terms vanish. Now, consider a spacetime containing two black holes related by the boost described above. The full metric for this spacetime may be complicated. But the first-order term in the asymptotic expansion satisfies a linear equation, so it is given by the sum of (5.4) and (5.10):
\[
\tilde{\sigma} = \sigma + \sigma_b.
\] (5.12)

As expected, the extra terms do not vanish for this solution. Assuming that the boost parameter is small, the first non-zero terms appear at second order in \( \beta \):
\[
\begin{align*}
\Delta^{(2)}_{\tau\tau} &= \beta^2 F_1(\tau, \theta), \\
\Delta^{(2)}_{\tau\theta} &= \beta^2 F_2(\tau, \theta) \sin 2\theta \tanh \tau, \\
\Delta^{(2)}_{\theta\theta} &= \beta^2 F_3(\tau, \theta), \\
\Delta^{(2)}_{\phi\phi} &= \beta^2 F_4(\tau, \theta), \\
\Delta^{(2)}_{\tau\phi} &= \Delta^{(2)}_{\theta\phi} = 0.
\end{align*}
\] (5.13-5.17)

where the \( F_n(\tau, \theta) \) are polynomials in \( \cos 2\theta \) and \( \cosh 2\tau \). The result is traceless and divergenceless, and the \( \theta \) and \( \tau \) dependence of the first two terms are such that they do not contribute to the Lorentz charges.

The functions (5.10) and (5.12) contain all the information needed to determine the energy and momentum of the solution described above. The energy and momentum for the Schwarzschild solution are the same as for Kerr, and for the boosted black hole we have
\[
E = M \cosh \beta, \quad p_z = M \sinh \beta, \quad p_x = p_y = 0.
\] (5.18)

These charges are obtained from the first term in (4.32), which is linear in \( \sigma \). Therefore, the energy and momentum for solutions like (5.12) are given by the sum of the energy and momentum of the constituent solutions.

6. Conclusion

The boundary stress tensor (3.4) obtained from the action (3.1) can be used to define a conserved charge for any asymptotic Killing field \( \xi^a \):
\[
Q[\xi] = \frac{1}{8\pi G} \int_C d^{d-2}x \sqrt{-h_C} u^a T_{ab} \xi^b,
\] (6.1)

where \((C, h_C)\) is a Cauchy surface in \((\mathcal{H}, h)\), and \(u^a\) is a timelike unit vector in \(\mathcal{H}\) normal to \(C\). Such charges necessarily generate the asymptotic symmetries of the theory.

In [23], it was shown that a slightly different expression, in which \( T^{\Delta} \) in (6.1) is replaced by
\[
T^{\Delta}_{ab} = \frac{1}{8\pi G} (\pi_{ab} - \tilde{\pi}_{ab}),
\] (6.2)
agrees with other definitions of conserved charges in the literature. However, the difference between $\Delta_{ab}^1 = T_{ab} - T_{ab}^{\infty}$ consists of a number of ‘extra’ terms whose asymptotic expansion takes the form

$$\Delta_{ab} = \rho^{d-2} \Delta_{ab}^{(1)} + \rho^{d-3} \Delta_{ab}^{(2)} + \mathcal{O}(\rho^{2-d}).$$  \hfill (6.3)

The first two terms in this expansion appear at just the right orders to contribute to the conserved charges associated with asymptotic translations and Lorentz transformations, respectively. This observation creates a certain tension with the results of [10, 23], which did not take these extra terms into account.

Our main result was to resolve this issue by proving that the extra terms do not contribute to the conserved charges. In dimension $d > 4$ the first two terms in (6.3) simply vanish, as a result of a remarkable series of cancelations. The proof in four dimensions is more subtle. In that case, the first-order term vanishes, but the second-order term may be non-zero. However, in section 4.1 we showed that $\Delta_{ab}^{(2)}$ takes the form (4.12), which implies that it cannot contribute to the conserved charges. The remaining terms in $T_{ab}$ are of the form (6.2), so that the analysis of [10, 23] applies without modification. We find that (6.1) can be expressed in terms of the electric part of the Weyl tensor. In four dimensions the charges are given by

$$Q[\xi] = -\frac{1}{8\pi G} \int_C d^2 x \sqrt{-h} \left[ u^a_{(0)} E^{(1)}_{ab} \xi^b_{(1)} + u^a_{(0)} (E^{(2)}_{ab} - \sigma E^{(1)}_{cd}) \xi^b_{(0)} \right],$$  \hfill (6.4)

while in dimension greater than four they are

$$Q[\xi] = -\frac{1}{8\pi G} \frac{1}{d - 3} \int_C d^{d-2} x \sqrt{-h} \left[ u^a_{(0)} E^{(1)}_{ab} \xi^b_{(1)} + u^a_{(0)} (E^{(2)}_{ab} - \sigma E^{(1)}_{cd}) \xi^b_{(0)} \right].$$  \hfill (6.5)

The explicit comparisons made in [23] show that these charges are equivalent to both the ADM construction [1] and (in $d = 4$) the covariant charges defined by Ashtekar and Hanson [3].

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Appendix A. Explicit form of $\Delta_{ab}$

In this appendix, we derive an expression for the extra terms $\Delta_{ab}$ that appear in the stress tensor. Varying the action (3.1) and imposing the equations of motion yields

$$\delta S = \frac{1}{16\pi G} \int_{\partial M} \sqrt{-h} \left[ (\pi^{ab} - h^{ab} \hat{K}) \delta h_{ab} - 2h^{ab} \delta \hat{K}_{ab} \right].$$  \hfill (A.1)

To simplify this we must compute $\delta \hat{K}_{ab}$. From the defining equation for $\hat{K}$ (3.2), we have

$$L_{cd}^{\hat{K}_{cd}} \delta \hat{K}_{cd} = \delta \mathcal{R}_{ab} + (\hat{K}_{ab} \hat{K}_{cd} - \hat{K}_a^d \hat{K}_b^c) \delta h_{cd}.$$  \hfill (A.2)
where following [10] we define
\[ L_{cd}^{ab} = h_{cd}^{ab} \hat{K}_{ab} + \delta_{a}^{c} \delta_{b}^{d} \hat{K} - \delta_{a}^{c} \hat{K}_{b}^{d} - \delta_{b}^{c} \hat{K}_{a}^{d}. \]  

(A.3)

Inserting equation (A.2) into (A.1) we obtain
\[ \delta S = \frac{1}{16 \pi G} \int_{\partial M} \sqrt{-h} \left[ (\pi_{ab} - \hat{\pi}_{ab} + \hat{K}^{ab} - 2 \hat{L}_{ab}^{cd} (\hat{K}_{cd}^{ef} - \hat{K}_{c}^{a} \hat{K}_{d}^{b})) \delta h_{ab} - 2 \hat{L}_{ab} \delta R_{ab} \right], \]  

(A.4)

where we define \( \hat{\pi}_{ab} = h_{ab}^{cd} \hat{K} - \hat{K}^{ab} \) and where
\[ \delta \hat{K}_{ab} = (L^{-1})_{ab}^{cd} \left( \delta R_{cd} - (\hat{K}_{cd}^{ef} - \hat{K}_{c}^{e} \hat{K}_{d}^{f}) \delta h_{ef} \right). \]  

(A.6)

We can use the fact that
\[ \delta R_{ab} = -\frac{1}{2} h^{cd} D_{a} D_{b} \delta h_{cd} - \frac{1}{2} h^{cd} D_{c} D_{d} \delta h_{ab} + h^{cd} D_{c} D_{d} \delta h_{bd}, \]  

(A.7)

where \( D_{a} \) is the torsion-free covariant derivative compatible with \( h_{ab} \). We now perform integrations by parts to write
\[ \delta S = \frac{1}{16 \pi G} \int_{\partial M} \sqrt{-h} \left[ (\pi_{ab} - \hat{\pi}_{ab} + \hat{K}^{ab} - 2 \hat{L}_{ab}^{cd} (\hat{K}_{cd}^{ef} - \hat{K}_{c}^{a} \hat{K}_{d}^{b})) \delta h_{ab} + \hat{L}_{ab}^{cd} \right. \]
\[ \left. + D^{2} \hat{L}_{ab} + h_{ab}^{cd} D_{c} D_{d} \hat{L}_{ab}^{cd} - D_{e} (D^{a} \hat{L}_{eb} + D^{b} \hat{L}_{ea}) \right] \delta h_{ab}. \]  

(A.8)

The boundary stress tensor for pure gravity can be readily obtained from this equation:
\[ T^{ab} = \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h_{ab}} = \frac{1}{8 \pi G} (\pi^{ab} - \hat{\pi}^{ab} + \Delta^{ab}), \]  

(A.9)

where \( \Delta_{ab} \) is defined to be
\[ \Delta_{ab} = \hat{K}_{ab} - 2 \hat{L}_{ab}^{cd} (\hat{K}_{cd}^{ef} - \hat{K}_{c}^{a} \hat{K}_{d}^{b}) + D^{2} \hat{L}_{ab} + h_{ab}^{cd} D_{c} D_{d} \hat{L}_{ab}^{cd} - 2 D_{e} (D_{a} \hat{L}_{eb} + D_{b} \hat{L}_{ea}). \]  

(A.10)

Some explicit results were given in [33].

Appendix B. A collection of useful identities

By commuting derivatives several times, one can prove a number of identities involving \( \sigma \) and its derivatives. Below, we present a collection of such identities in \( d = 4 \). All of these identities, in one form or another, are used in the main text. For ease of reference we state our conventions again: \( \sigma_{abc} := D_{a} D_{b} D_{c} \sigma, \sigma_{abcd} := D_{a} D_{b} D_{c} D_{d} \sigma \), etc. Also, recall that the commutator of two covariant derivatives acting on a tensor \( t_{ab} \) is
\[ [D_{a}, D_{b}] t_{c}^{d} = R_{abc}^{(0)} t_{c}^{d} + R_{abd}^{(0)} t_{e}^{d}, \]  

(B.1)

where
\[ R_{abc}^{(0)} = h_{ac}^{(0)} h_{bd}^{(0)} - h_{bc}^{(0)} h_{ad}^{(0)}. \]  

(B.2)
One can show using the equation of motion for $\sigma$ that

\[\sigma^c_c = -3\sigma,\]  \hspace{1cm} (B.3)

\[\sigma^c_{ca} = -3\sigma_a,\]  \hspace{1cm} (B.4)

\[\sigma^{ac}_c = -\sigma_a,\]  \hspace{1cm} (B.5)

\[\sigma_e\sigma^{mn} = \sigma_e\sigma^{mn} - \sigma^m\sigma^n + \sigma^e\sigma^i h^{(0)}_{mn},\]  \hspace{1cm} (B.6)

\[\sigma_{pmn}\sigma^{pmn} = 2\sigma_{pmn}\sigma^{pmn} - 3\sigma_{mn}\sigma^m,\]  \hspace{1cm} (B.7)

\[\sigma^p\sigma_{pq} = \sigma^p\sigma_{pq} - 3\sigma^p\sigma_q,\]  \hspace{1cm} (B.8)

\[\sigma_{pq(a}\sigma_{b)}pq = \sigma_{pq(a}\sigma_{b)}pq - 3\sigma_{a}\sigma_b - \sigma_{q(a}h_{ab)},\]  \hspace{1cm} (B.9)

\[\sigma_{(a\ mn}\sigma_{b)mn} = 2\sigma_{mn}\sigma_{mn} - 3\sigma_{a}\sigma_b + h_{ab}\sigma^m\sigma^m,\]  \hspace{1cm} (B.10)

\[\alpha_{ab}e = 6\sigma h_{ab} + 3\sigma_{ab},\]  \hspace{1cm} (B.11)

\[\alpha_{(a}\ b)\sigma_{(b)mn} = 2\sigma_{mn}\sigma_{mn} - 9\sigma^2,\]  \hspace{1cm} (B.12)

\[\alpha_{p(ab)\sigma_{pq}} = \alpha_{pq(ab)}\sigma_{pq} - 6\sigma_{ab} = \sigma_{pq}\sigma_{pq} - \sigma_{ab}\sigma_{pq},\]  \hspace{1cm} (B.13)

\[\alpha_{pq\sigma_{pqab}} = 30\sigma_{a\ pq} + 9\sigma_{pq}\sigma_{pq},\]  \hspace{1cm} (B.14)

\[\alpha_{pq\sigma_{pqba}} = 24\sigma_{a\ pq} + 7\sigma_{pq}\sigma_{pq}.\]  \hspace{1cm} (B.15)

Finally, recall that for any tensor $\rho_{ab}$

\[\mathcal{D}_{a}t^e = D_{a}t^e + \frac{1}{\rho} C^{(1)}_{a}t^e = -\frac{1}{\rho} C^{(1)}_{a}t^e + \frac{1}{\rho^2} C^{(2)}_{a}t^e = \frac{1}{\rho^2} C^{(2)}_{a}t^e + \cdots,\]  \hspace{1cm} (B.17)

where expressions for $C^{(1)}_{a}$ and $C^{(2)}_{a}$ are

\[C^{(1)}_{a} = -h^{(0)}_{b} \sigma_a + h^{(0)c} \sigma_b + \sigma^e h_{ab},\]  \hspace{1cm} (B.18)

\[C^{(2)}_{a} = -2\sigma (h^{(0)c} \sigma_b + h^{(0)c} \sigma_a - h^{(0)}_{ab} \sigma^e) + \frac{1}{2}(D_A h^{(2)c}_{b} + D_A h^{(2)c}_{a} - D^c h^{(2)}_{ab}).\]  \hspace{1cm} (B.19)

These expressions were also given in equations (D.12) and (D.13) of [23].

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