TOPOLOGICAL, SMOOTH AND HOLOMORPHIC CLASSIFICATIONS
OF NONAUTONOMOUS LINEAR DIFFERENTIAL SYSTEMS
AND PROJECTIVE MATRIX RICCATI EQUATIONS

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Abstract

The questions of global topological, smooth and holomorphic classifications of the
differential systems, defined by covering foliations, are considered. The received results are
applied to nonautonomous linear differential systems and projective matrix Riccati equations.

Key words: covering foliation, global topological, smooth and holomorphic classifications,
nonautonomous linear differential system, projective matrix Riccati equation.

2000 Mathematics Subject Classification: 34A26.

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Introduction

The questions of global topological classification of differential systems (i.e. defined by them foliations) have been considered for the first time in the work [1]. In it the criterion of global topological equivalence for real autonomous linear ordinary differential systems of general situation, in particular, has been received. Further the given problem was considered in [2] and has definitively been solved in [3]. Similar problems for real completely solvable autonomous linear differential systems in case of two independent variables were studied in [4], and in a case, when number of independent variables on 1 less then numbers of dependent variables, were studied in [5] and [6].

In a complex case global topological classification of autonomous linear ordinary differential systems of general situation has been spent in [7 – 11] and in [12] given problem has been considered in a case of completely solvable autonomous linear differential systems. Besides, in the works [13 – 15] this question was studied for complex autonomous polynomial ordinary differential systems of the second order.

For nonautonomous differential systems the problem of global topological classification was considered only for the scalar complex linear ordinary differential equation [16; 17].

It is necessary to notice, that all criteria of global topological equivalence of corresponding differential systems received only for integrated in the quadratures cases (that has essentially facilitated reception of these criteria).

In given article we will spend global topological [12; 18; 20 – 26], smooth [21; 22; 24 – 26], \( \mathbb{R} \)-holomorphic [19; 24; 25] (in a complex case) and holomorphic [24; 26] classifications of nonautonomous linear differential systems and projective matrix Riccati equations [27], generally speaking, not integrated in quadratures.

1. Covering foliations and their classifications

Definition 1.1. Let \( A \) and \( B \) be path connected smooth varieties of dimensions \( \dim A = n \) and \( \dim B = m \). Smooth foliation \( \mathcal{F} \) of dimension \( m \) on the variety \( A \times B \), locally transversal to \( A \times \{b\} \) for all \( b \in B \), we will name a covering foliation, if the projection \( p: A \times B \to B \) on the second factor defines for each layer of it foliation covering of the variety \( B \). Thus variety \( A \) we will name a phase layer, and variety \( B \) we will name a base of covering foliation \( \mathcal{F} \).

Definition 1.2. Let \( \mathcal{F}_c \) be a layer of the covering foliation \( \mathcal{F} \), containing the point \( c \in A \times B \). The phase group \( \text{Ph}(\mathcal{F}, b_0), b_0 \in B \), of the covering foliation \( \mathcal{F} \) we will name the group of the diffeomorphisms \( \text{Diff}(A, \pi_1(B, b_0)) \) of the actions on the phase layer \( A \) by fundamental group \( \pi_1(B, b_0) \) with noted point \( b_0 \), defined under formulae

\[
\Phi^\gamma(a) = q \circ r \circ s(1) \quad \text{for all } a \in A, \quad \text{for all } \gamma \in \pi_1(B, b_0),
\]

where \( r \) is a lifting of one of ways \( s(\tau) \subset B \) for all \( \tau \in [0, 1] \), corresponding to the element \( \gamma \) of the group \( \pi_1(B, b_0) \), on the layer \( \mathcal{F}_c(a, s(0)) \) of the covering foliation \( \mathcal{F} \) in the point \( (a, s(0)) \), and \( q: A \times B \to A \) is a projection to the first factor.

It is easy to see, that owing to path connectivity and smoothness of the variety \( B \), then phase groups \( \text{Ph}(\mathcal{F}, b_1) \) and \( \text{Ph}(\mathcal{F}, b_2) \) are smoothly conjugated for any two points \( b_1 \) and \( b_2 \) of the base \( B \). Therefore further, as a rule, we will speak simply about of the phase group \( \text{Ph}(\mathcal{F}) \) of the covering foliation \( \mathcal{F} \), not connecting it with any point of the base \( B \).

Further by consideration of the questions connected with topological (smooth, \( \mathbb{R} \)-holomorphic, holomorphic) classifications of the covering foliations, we will believe everywhere, that their phase layers are homeomorphically (diffeomorphically, \( \mathbb{R} \)-holomorphically, holomorphically) equivalent each other, and in two last cases covering foliations we will consider, accordingly, holomorphic and \( \mathbb{R} \)-holomorphic. Thus under \( \mathbb{R} \)-holomorphism (holomorphism) we will understand bijective \( \mathbb{R} \)-holomorphic (holomorphic) map, having \( \mathbb{R} \)-holomorphic (holomorphic) to themselves inverse map.
Definition 1.3. We will say that the covering foliation $\mathcal{F}^1$ on the variety $A_1 \times B_1$ is topologically (smoothly, $\mathbb{R}$-holomorphically, holomorphically) equivalent to the covering foliation $\mathcal{F}^2$ on the variety $A_2 \times B_2$ if exists the homeomorphism (the diffeomorphism, the $\mathbb{R}$-holomorphism, the holomorphism) $h: A_1 \times B_1 \to A_2 \times B_2$ such that

$$q_2 \circ h(A_1 \times B_1) = A_2, \quad h(\mathcal{F}^1_{c_1}) = \mathcal{F}^2_{h(c_1)} \quad \text{for all } c_1 \in A_1 \times B_1,$$

where $q_2$ is a projection to the first factor.

Definition 1.4. We will say that the covering foliation $\mathcal{F}^1$ on the variety $A_1 \times B_1$ is embeddable (smoothly embeddable, $\mathbb{R}$-holomorphically embeddable, holomorphically embeddable) in the covering foliation $\mathcal{F}^2$ on the variety $A_2 \times B_2$ if there is such embedding (smooth embedding, $\mathbb{R}$-holomorphic embedding, holomorphic embedding)

$$h: A_1 \times B_1 \hookrightarrow A_2 \times B_2,$$

that $q_2 \circ h(A_1 \times B_1) = A_2$ and $h(\mathcal{F}^1_{c_1}) \hookrightarrow \mathcal{F}^2_{h(c_1)}$ for all $c_1 \in A_1 \times B_1$.

Definition 1.5. We will say that the covering foliation $\mathcal{F}^1$ on the variety $A_1 \times B_1$ covers (smoothly covers, $\mathbb{R}$-holomorphically covers, holomorphically covers) the covering foliation $\mathcal{F}^2$ on the variety $A_2 \times B_2$ if exists such covering (smooth covering, $\mathbb{R}$-holomorphic covering, holomorphic covering) $h: A_1 \times B_1 \to A_2 \times B_2$, that $q_2 \circ h(A_1 \times B_1) = A_2$ and $h(\mathcal{F}^1_{c_1}) \to \mathcal{F}^2_{h(c_1)}$ for all $c_1 \in A_1 \times B_1$.

Theorem 1.1. For topological (smooth, $\mathbb{R}$-holomorphic, holomorphic) equivalence of the covering foliations $\mathcal{F}^1$ and $\mathcal{F}^2$ it is necessary and enough existence of the isomorphism $\mu$ of the fundamental groups $\pi_1(B_1)$ and $\pi_1(B_2)$, generated by the homeomorphism (diffeomorphism, $\mathbb{R}$-holomorphism, holomorphism) $g_\mu: B_1 \to B_2$ of the bases, and existence of the homeomorphism (diffeomorphism, $\mathbb{R}$-holomorphism, holomorphism) $f: A_1 \to A_2$ of phase layers such that

$$f \circ \Phi_{\gamma_1} = \Phi_{\gamma_1} \circ f \quad \text{for all } \gamma_1 \in \pi_1(B_1), \quad (1.1)$$

where $\Phi_{\xi} \in \text{Ph}(\mathcal{F}_\xi)$, $\gamma_1 \in \pi_1(B_1)$, $\xi = 1, 2$.

Proof. At first we will notice, that for the smooth ($\mathbb{R}$-holomorphic, holomorphic) varieties, which are bases of the covering foliations, definitions of fundamental groups and their actions on smooth ($\mathbb{R}$-holomorphic, holomorphic) phase layers by means of continuous and by means of smooth ($\mathbb{R}$-holomorphic, holomorphic) ways are equivalent.

The necessity. Let the map $h: A_1 \times B_1 \to A_2 \times B_2$ defines topological (smooth, $\mathbb{R}$-holomorphic, holomorphic) equivalence of the covering foliations $\mathcal{F}^1$ and $\mathcal{F}^2$. Map $h$ induces the homeomorphism (the diffeomorphism, the $\mathbb{R}$-holomorphism, the holomorphism) $g_\mu = p_2 \circ h$ of bases $B_1$ and $B_2$ which, in turn, induces the isomorphism $\mu: \pi_1(B_1) \to \pi_1(B_2)$ of their fundamental groups. Let $b_1^0$ is noted point of the variety $B_1$. Then on the basis of definition 1.2 and that fact that map $h$ translates the layer $\mathcal{F}^1_{(a_1, b_1^0)}$ in the layer $\mathcal{F}^2_{h(a_1, b_1^0)}$, we come to relations (1.1), where $f = q_2 \circ h$.

The sufficiency. Let for operations of phase groups $\text{Ph}(\mathcal{F}^1)$ and $\text{Ph}(\mathcal{F}^2)$ relations (1.1) are fulfilled at the homeomorphism (the diffeomorphism, the $\mathbb{R}$-holomorphism, the holomorphism) $g_\mu$ of bases $B_1$ and $B_2$. We take the way $s_1: [0, 1] \to B_1$ such that $s_1(0) = b_1$, $s_1(1) = b_1^0$. Also we will suppose

$$h(a_1, b_1) = (l_2^{-1} \circ f \circ l_1(a_1), g_\mu(b_1)) \quad \text{for all } a_1 \in A_1, \quad \text{for all } b_1 \in B_1, \quad (1.2)$$

where $l_1(a_1) = q_1 \circ r_1(1)$, $q_1$ is a projection on the first factor, $r_1(\tau)$ is a raising of the way $s_1$ on a layer of foliation $\mathcal{F}^1$ in the point $(a_1, b_1)$, $l_2^{-1}(a_2) = q_2 \circ r_2^{-1}(1)$, $r_2^{-1}(\tau)$ is an outcome of a raising of the way $s_2^{-1} = g_\mu \circ s_1^{-1}$ on a layer of foliation $\mathcal{F}^2$ in the point $(f \circ l_1(a_1), g_\mu(b_1^0))$, $s_1^{-1}$ is a way, inverse to the way $s_1$. 

\[3\]
Now directly we come to a conclusion, that fiber bijective map (1.2) sets topological (smooth, \( \mathbb{R} \)-holomorphic, holomorphic) equivalence of covering foliations \( \mathfrak{F}^1 \) and \( \mathfrak{F}^2 \).

Similarly to given theorem it is proved following two assertions.

**Theorem 1.2.** For embedding (smooth embedding, \( \mathbb{R} \)-holomorphic embedding, holomorphic embedding) covering foliation \( \mathfrak{F}^1 \) in covering foliation \( \mathfrak{F}^2 \) it is necessary and enough existence of the homomorphism \( \mu \) of fundamental group \( \pi_1(B_1) \) in fundamental group \( \pi_1(B_2) \), generated by the embedding (the smooth embedding, the \( \mathbb{R} \)-holomorphic embedding, the holomorphic embedding) \( g_\mu: B_1 \hookrightarrow B_2 \) bases, and existence of the homeomorphism (the diffeomorphism, the \( \mathbb{R} \)-holomorphism, the holomorphism) \( f: A_1 \to A_2 \) of phase layers such that relations (1.1) are carried out.

**Theorem 1.3.** That covering foliation \( \mathfrak{F}^1 \) covered (smoothly covered, \( \mathbb{R} \)-holomorphic covered, holomorphically covered) covering foliation \( \mathfrak{F}^2 \) it is necessary and enough existence of the monomorphism \( \mu \) fundamental group \( \pi_1(B_1) \) in fundamental group \( \pi_1(B_2) \), generated by the covering (the smooth covering, the \( \mathbb{R} \)-holomorphic covering, the holomorphic covering) \( g_\mu: B_1 \hookrightarrow B_2 \) of the bases, and existence of the homeomorphism (the diffeomorphism, the \( \mathbb{R} \)-holomorphism, the holomorphism) \( f: A_1 \to A_2 \) of phase layers such that relations (1.1) are carried out.

Since theorems 1.1 – 1.3 reduce problems of topological, smooth, \( \mathbb{R} \)-holomorphic, and holomorphic classifications of covering foliations to the problems of corresponding classifications of their phase groups at morphisms (isomorphisms, homomorphisms, monomorphism) we will consider further questions of topological, smooth, \( \mathbb{R} \)-holomorphic, and holomorphic conjunctions of phase groups of covering foliations, defined by the nonautonomous linear differential systems and projective matrix Riccati equations.

## 2. Phase groups of covering foliations, defined by complex nonautonomous linear differential systems

We will consider linear differential systems

\[
dw = \sum_{j=1}^{m} A_j(z_1, \ldots, z_m) \, dw_j \quad (2.1)
\]

and

\[
dw = \sum_{j=1}^{m} B_j(z_1, \ldots, z_m) \, dw_j ,
\]

ordinary at \( m = 1 \) and completely solvable [28] at \( m > 1 \), where \( w = (w_1, \ldots, w_n) \), square matrices \( A_j(z_1, \ldots, z_m) = \|a_{ikj}(z_1, \ldots, z_m)\| \) and \( B_j(z_1, \ldots, z_m) = \|b_{ikj}(z_1, \ldots, z_m)\| \) of the size \( n \) consist from holomorphic functions \( a_{ikj}: A \to \mathbb{C} \) and \( b_{ikj}: B \to \mathbb{C} \), \( i, k = 1, \ldots, n, j = 1, \ldots, m \), multiplication of matrices we will carry out by multiplication of columns of the first matrix for corresponding lines of the second, path connected holomorphic varieties \( A \) and \( B \) are holomorphically equivalent each other, fundamental groups \( \pi_1(A) \) and \( \pi_1(B) \) have final number \( \nu \in \mathbb{N} \) of the forming.

The common solutions of linear differential systems (2.1) and (2.2) define covering foliations \( \mathfrak{L}^1 \) and \( \mathfrak{L}^2 \), accordingly, on varieties \( \mathbb{C}^n \times A \) and \( \mathbb{C}^n \times B \).

We will say that linear differential systems (2.1) and (2.2) are **topologically (smoothly, \( \mathbb{R} \)-holomorphically, holomorphically) equivalent** if exists the homeomorphism (the diffeomorphism, the \( \mathbb{R} \)-holomorphism, the holomorphism) \( h: \mathbb{C}^n \times A \to \mathbb{C}^n \times B \), translating layers of the covering foliation \( \mathfrak{L}^1 \) in layers of the covering foliation \( \mathfrak{L}^2 \).

Similarly we introduce the concepts of embedding (smooth embedding, \( \mathbb{R} \)-holomorphic embedding, holomorphic embedding) and covering (smoothly covering, \( \mathbb{R} \)-holomorphically covering, holomorphically covering) of linear differential systems.
The phase group $Ph(\mathcal{L}^1)$ of the covering foliation $\mathcal{L}^1$ is generated by the forming nondegenerate linear transformations $P_r w$ for all $w \in \mathbb{C}^n$, $P_r \in GL(n, \mathbb{C})$, $r = 1, \ldots, \nu$, and the phase group $Ph(\mathcal{L}^2)$ of the covering foliation $\mathcal{L}^2$ is generated by the forming nondegenerate linear transformations $Q_r w$ for all $w \in \mathbb{C}^n$, $Q_r \in GL(n, \mathbb{C})$, $r = 1, \ldots, \nu$.

### 3. Conjunctions of linear actions on $\mathbb{C}^n$

We will consider a problem about a finding of necessary and sufficient conditions of existence such homeomorphism (diffeomorphism, $\mathbb{R}$-holomorphism, holomorphism) $f : \mathbb{C}^n \to \mathbb{C}^n$, that identities

$$f(P_r w) = Q_r f(w) \quad \text{for all} \quad w \in \mathbb{C}^n, \quad \text{for all} \quad r \in I, \quad (3.1)$$

take place, where $f(w) = (f_1(w), \ldots, f_n(w))$, square matrices $P_r \in GL(n, \mathbb{C})$, $Q_r \in GL(n, \mathbb{C})$ for all $r \in I$, $I$ are some set of indexes. Group of linear actions on $\mathbb{C}^n$, formed by the matrices $P_r$ for all $r \in I$, we will designate by $L^1$, and through $L^2$ we will designate the similar group, formed by the matrices $Q_r$ for all $r \in I$. Besides, further everywhere a set $\{\lambda_1, \ldots, \lambda_n\}$ of nonzero complex numbers we will name simple if

$$\frac{\lambda_k}{\lambda_l} \notin s_{lk}^{k+1}, \quad s_{lk} \in \mathbb{N}, \quad l \neq k, \quad k = 1, \ldots, n, \quad l = 1, \ldots, n,$$

and a square matrix of the size $n > 1$ we will name simple if it has simple structure and a simple collection of eigenvalues.

Consider at first a topological conjunction of the Abelian linear groups $L^1$ and $L^2$. In this case we will notice that if all matrices $P_r$ (all matrices $Q_r$) have simple structure, for all $r \in I$, they are reduce to a diagonal kind by the common transformation of similarity.

**Theorem 3.1** [16; 17]. *For the topological conjunction (3.1) at $n = 1$ of linear groups $L^1$ and $L^2$ it is necessary and enough, that either*

$$q_r = p_r |p_r|^\alpha, \quad \Re \alpha > -1 \quad \text{for all} \quad r \in I, \quad (3.2)$$

*or*

$$q_r = \overline{p_r} |p_r|^\alpha, \quad \Re \alpha > -1 \quad \text{for all} \quad r \in I. \quad (3.3)$$

**Proof.** The necessity. We will assume, that conjugating homeomorphism $f$ keeps orientation (a case, when homeomorphism $f$ changes orientation, it is considered similarly).

As rotations of a complex plane around of an origin of coordinates on the angles $\varphi$ and $\psi$, where $-\pi < \varphi \leq \pi$, $-\pi < \psi \leq \pi$, are topologically conjugated, if and only if $\varphi = \psi$, that for all $r \in I$ with a condition $|p_r| = 1$, we have $q_r = p_r$, and, so, the relations (3.2) are carried out at any $\alpha$.

If $|p_{r_1}| \neq 1$ and $|p_{r_2}| \neq 1$ for some $r_1$ and $r_2$ from $I$, then

$$\left| p_{r_1} \right| = \left| p_{r_2} \right|.$$  

(3.4)

It is easy to see, that there are such sequences $\{l_s\}$ and $\{m_s\}$ of integers, that

$$\lim_{s \to +\infty} p_{r_1}^{l_s} p_{r_2}^{-m_s} = 1,$$

(3.5)

and moreover

$$\lim_{s \to +\infty} |l_s| = \lim_{s \to +\infty} |m_s| = +\infty.$$

From identity (3.1) follows that

$$f\left(p_{r_1}^{l_s} p_{r_2}^{-m_s} w\right) = q_{r_1}^{l_s} q_{r_2}^{-m_s} f(w) \quad \text{for all} \quad w \in \mathbb{C}, \quad \text{for all} \quad s \in \mathbb{N}.$$

From here taking into account (3.5) it is had, that

$$\lim_{s \to +\infty} q_{r_1}^{l_s} q_{r_2}^{-m_s} = 1.$$  

(3.6)
Therefore from (3.5) and (3.6) for some values of logarithms we receive equalities

\[
\frac{\ln |p_{r_1}|}{\ln |p_{r_2}|} = - \lim_{s \to +\infty} \frac{m_s}{l_s}
\]

(3.7)

and

\[
l_s \ln p_{r_1} + m_s \ln p_{r_2} = l_s \ln q_{r_1} + m_s \ln q_{r_2}.
\]

(3.8)

Let’s divide the left and right parts of equality (3.8) on \( l_s \) and we will pass to a limit at \( s \to +\infty \). Then taking into account (3.7) we will receive that

\[
\frac{\ln q_{r_1} - \ln p_{r_1}}{\ln |p_{r_1}|} = \frac{\ln q_{r_2} - \ln p_{r_2}}{\ln |p_{r_2}|}.
\]

(3.9)

Now, believing, that

\[
\alpha_{r_1} = \frac{\ln q_{r_1} - \ln p_{r_1}}{\ln |p_{r_1}|},
\]

we receive equality (3.2), where \( \alpha = \alpha_{r_1} \). Besides, from (3.9) follows that \( \alpha_{r_1} = \alpha_{r_2} = \alpha \) for all \( r_1 \) and \( r_2 \) from \( I \) taking into account that \( |p_{r_1}| \neq 1 \) and \( |p_{r_2}| \neq 1 \). And, at last, the inequality \( \Re \alpha > -1 \) follows from equalities \( \Re \alpha = \Re \alpha_{r_1} = \frac{\ln |q_{r_1}|}{\ln |p_{r_1}|} - 1 \) and (3.4).

The sufficiency is proved by construction of conjugating homeomorphism \( f(w) = \gamma w |w|^\alpha \) for all \( w \in \mathbb{C} \) at performance of relations (3.2); and \( f(w) = \gamma \overline{w} |w|^\alpha \) for all \( w \in \mathbb{C} \) at performance of relations (3.3).

**Theorem 3.2.** Let at \( n > 1 \) the matrices

\[
P_r = S \text{diag}\{p_{1r}, \ldots, p_{nr}\} S^{-1}, \quad Q_r = T \text{diag}\{q_{1r}, \ldots, q_{nr}\} T^{-1},
\]

and the matrices \( \ln P_r \) and \( \ln Q_r \) be simple for all \( r \in I \). Then for the topological conjunction (3.1) of linear groups \( L^1 \) and \( L^2 \) it is necessary and enough existence of such permutation \( \varphi: (1, \ldots, n) \to (1, \ldots, n) \) and complex numbers \( \alpha_k \) with \( \Re \alpha_k > -1 \), \( k = 1, \ldots, n \), that either

\[
q_{\varphi(k)r} = p_{kr} |p_{kr}|^{\alpha_k} \quad \text{for all } r \in I, \quad k = 1, \ldots, n,
\]

(3.10)

or

\[
q_{\varphi(k)r} = \overline{p}_{kr} |p_{kr}|^{\alpha_k} \quad \text{for all } r \in I, \quad k = 1, \ldots, n.
\]

(3.11)

Proof. With the help of replacement \( \xi(w) = T^{-1} f(Sw) \) for all \( w \in \mathbb{C}^n \), from identities (3.1) we pass to identities

\[
\xi(\text{diag}\{p_{1r}, \ldots, p_{nr}\} w) = \text{diag}\{q_{1r}, \ldots, q_{nr}\} \xi(w) \quad \text{for all } w \in \mathbb{C}^n, \quad \text{for all } r \in I.
\]

(3.12)

Therefore the topological conjunction of linear groups \( L^1 \) and \( L^2 \) is equivalent to performance of identities (3.12).

The necessity. Let identities (3.12) are carried out. Holomorphism \( u_r(w) = P_r w \) for all \( w \in \mathbb{C}^n \) (holomorphism \( v_r(w) = Q_r w \) for all \( w \in \mathbb{C}^n \)) defines on space \( \mathbb{C}^n \) the invariant holomorphic foliation \( \mathcal{F}_r \) (the invariant holomorphic foliation \( \mathcal{F}_r \)) of complex dimension 1, defined by the basis of nondegenerate absolute invariants \( [29] \)

\[
w_k^{\ln p_{kr}} w_l^{\ln p_{kr}}, \quad k = 1, \ldots, n-1
\]

(by the basis of nondegenerate absolute invariants \( w_k^{\ln q_{kr}} w_l^{\ln q_{kr}}, \quad k = 1, \ldots, n-1 \)) for all \( r \in I \).

We will designate through \( \mathbb{C}_k \) the coordinate complex plane \( w_l \neq 0, \ l \neq k, \ l = 1, \ldots, n \), and through \( \mathcal{C}_k \) we will designate the coordinate complex plane \( \mathbb{C}_k \) with the pricked out an origin of coordinates, \( k = 1, \ldots, n \).

As matrices \( \ln P_r \) (matrices \( \ln Q_r \)) are simple for all \( r \in I \), that at:
and its projection \( \alpha \). From here on the basis of Theorem 3.1 we come to conclusion, that there are such complex numbers \( \alpha > -1 \), that one of relations (3.10) or (3.11), \( k = 1, \ldots, n \), is carried out.

The sufficiency is proved by the construction of conjugating homeomorphism \( \xi: \mathbb{C}^n \to \mathbb{C}^n \) such that its projection \( \xi_{p(k)}(w) = \gamma_k w_k |w_k|^{\alpha_k} \) for all \( w \in \mathbb{C}^n \) if relations (3.10) take place; and its projection \( \xi_{\rho(k)}(w) = \gamma_k \overline{w_k} |w_k|^{\alpha_k} \) for all \( w \in \mathbb{C}^n \) if relations (3.11) take place; \( k = 1, \ldots, n \).

Consider now a topological conjugation of the non-Abelian linear groups \( L^1 \) and \( L^2 \).

**Theorem 3.3.** From a topological conjugation of the non-Abelian linear groups \( L^1 \) and \( L^2 \) of general situation follows them \( \mathbb{R} \)-holomorphic conjugation.

Proof of the theorem 3.3 directly follows from following two auxiliary statements (the lemmas 3.1 and 3.2).

**Lemma 3.1.** Let at \( n = 1 \) and \( I = \{1, 2\} \) linear groups \( L^1 \) and \( L^2 \) are topological conjugated, and the group of subgroup \( \mathbb{C}^* \) of nonzero complex numbers on multiplication, generated by numbers \( p_{11} \) and \( p_{12} \), is dense in the set \( \mathbb{C} \) of complex numbers. Then conjugating homeomorphism \( f \) is set either by the formula \( f(w) = \gamma w |w|^{\alpha} \) for all \( w \in \mathbb{C} \), or by the formula \( f(w) = \gamma \overline{w} |w|^{\alpha} \) for all \( w \in \mathbb{C} \), where \( \Re \alpha > -1 \).

Proof. Let conjugating homeomorphism \( f \) keeps orientation (the case when it changes orientation, it is considered similarly). Owing to the theorem 3.1 relations

\[
q_{1r} = p_{1r} |p_{1r}|^{\alpha}, \quad r = 1, 2, \quad \Re \alpha > -1,
\]

take place. On the basis of identities (3.1) we come to conclusion about justice of equalities

\[
f(p_{11}^l p_{12}^m) = f(1) p_{11}^l p_{12}^m \overline{p}_{12}^l |p_{12}^m|^{\alpha} \quad \text{for all} \quad l \in \mathbb{Z}, \quad \text{for all} \quad m \in \mathbb{Z}.
\]
Owing to density of our subgroup of group $\mathbb{C}^*$ in the set $\mathbb{C}$ of complex numbers it is had, that for any complex number $w \in \mathbb{C}$ there are such sequences $\{l_s(w)\}$ and $\{m_s(w)\}$ of integers, that $\lim_{s \to +\infty} p_{11}(w)^{l_s(w)}p_{12}(w)^{m_s(w)} = w$, $\lim_{s \to +\infty} |l_s(w)| = \lim_{s \to +\infty} |m_s(w)| = +\infty$. From here we receive the first representation of the statement of a lemma, where $\gamma = f(1)$. □

Lemma 3.2. Let the matrices

$$P_r = S_r \text{diag}\{p_{1r}, \ldots, p_{nr}\} S_r^{-1}, \quad Q_r = T_r \text{diag}\{q_{1r}, \ldots, q_{nr}\} T_r^{-1},$$

and the matrices $\ln P_r$ and $\ln Q_r$ are simple for all $r \in \mathbb{I}$. Then from a topological conjunction of non-Abelian linear groups $L^1$ and $L^2$ of general situation follows them $\mathbb{R}$-linear conjunction (i.e. homeomorphism $f : \mathbb{C}^n \to \mathbb{C}^n$ in identities (3.1) is nondegenerate linear $\mathbb{R}$-holomorphic transformation).

Proof. Let identities (3.1) take a place. By means of replacement $\xi(w) = T^{-1}_{11} f(S_1 w)$ for all $w \in \mathbb{C}^n$ from them we pass to identities

$$\xi(\text{diag}\{p_{11}, \ldots, p_{n1}\} w) = \text{diag}\{q_{11}, \ldots, q_{n1}\} \xi(w) \quad \text{for all } w \in \mathbb{C}^n. \quad (3.13)$$

Now similarly, as well as at the proof of the theorem 3.1, we come to conclusion, that an origin of coordinates of space $\mathbb{C}^n$ is a fixed point of the homeomorphism $\xi$, and its projections $\xi_k : \mathbb{C}^n \to \mathbb{C}$ are that, that relations

$$\xi_{\rho(k)}(0, \ldots, 0, p_{k1}w_k, 0, \ldots, 0) = q_{\rho(k)1} \xi_{\rho(k)}(0, \ldots, 0, w_k, 0, \ldots, 0),$$

$$\xi_l(0, \ldots, 0, w_k, 0, \ldots, 0) = 0 \quad \text{for all } w_k \in \mathbb{C}, \quad l \neq \rho(k), \quad l = 1, \ldots, n, \quad k = 1, \ldots, n,$$

take place; where $\rho : (1, \ldots, n) \to (1, \ldots, n)$ is some permutation.

From here in case of general situation on the basis of relations (3.1), the theorem 3.2 and lemma 3.1 we come to conclusion about justice of identities

$$\xi_{\rho(k)}(0, \ldots, 0, w_k, 0, \ldots, 0) = \gamma_k w_k^{|\alpha_k|} \quad \text{for all } w_k \in \mathbb{C}, \quad \Re \alpha_k > -1, \quad k = 1, \ldots, n,$$

where $w_k^* = w_k \vee \overline{w}_k$, $k = 1, \ldots, n$. From these identities further we come to relations

$$\xi_{\rho(k)}(w) = \gamma_k w_k^{|\alpha_k|} \left(1 + \varphi_{\rho(k)}(w)\right) \quad \text{for all } w \in \mathbb{C}^n, \quad \Re \alpha_k > -1,$$

$$\varphi_{\rho(k)}(0, \ldots, 0, w_k, 0, \ldots, 0) = 0 \quad \text{for all } w_k \in \mathbb{C}\{0\}, \quad k = 1, \ldots, n. \quad (3.14)$$

Now on the basis of identities (3.13) it is had, that $\varphi_k(\text{diag}\{p_{11}, \ldots, p_{n1}\} w) = \varphi_k(w)$ for all $w \in \mathbb{C}^n$, $k = 1, \ldots, n$. Considering relations (3.14) and basis of nondegenerate absolute invariants of foliation $\Sigma_1$ from the proof of the theorem 3.2, we receive representations

$$\xi_{\rho(k)}(w) = \gamma_k w_k^{|\alpha_k|} \left(1 + \psi_{\rho(k)} \left( w_1 w_k^{-\ln p_{11}/\ln p_{k1}}, \ldots, w_{k-1} w_k^{-\ln p_{{k-1},1}/\ln p_{k1}}, \right.\right.$$

$$w_{k+1} w_k^{-\ln p_{k+1,1}/\ln p_{k1}}, \ldots, w_n w_k^{-\ln p_{n1}/\ln p_{k1}} \left.\right) \quad \text{for all } w \in \mathbb{C}^n, \quad k = 1, \ldots, n,$$

(3.15)

where functions $\psi_{\rho(k)}$ are continuous on the arguments, $k = 1, \ldots, n$.

Taking into consideration noncommutativity of linear groups and identity (3.13), on the basis of relations (3.14) and (3.15) taking into account simplicity of matrixes from a lemma condition we come to its statement. □

Consider cases of smooth, $\mathbb{R}$-holomorphic and holomorphic conjunctions of linear groups $L^1$ and $L^2$.

Theorem 3.4. Linear groups $L^1$ and $L^2$ are smooth ($\mathbb{R}$-holomorphic) conjugated if and only if they are $\mathbb{R}$-linearly conjugated.

Proof. The necessity. Let linear groups $L^1$ and $L^2$ are smooth ($\mathbb{R}$-holomorphic) conjugated. Then identities (3.1) take place at the diffeomorphism (at the $\mathbb{R}$-holomorphism) $f$.
Calculating in them full differential in the point \( w = 0 \), we have
\[
\mathbf{D}_w f(0) P_r dw + \mathbf{D}_w f(0) \overline{P}_r d\overline{w} = Q_r \mathbf{D}_w f(0) dw + Q_r \mathbf{D}_w f(0) d\overline{w} \quad \text{for all } r \in I.
\]

Therefore \( \mathbb{R} \)-linear transformation \( \mathbf{D}_w f(0) w + \mathbf{D}_w f(0) \overline{w} \) for all \( w \in \mathbb{C}^n \) defines the conjunction (3.1). As map \( f \) is a diffeomorphism (an \( \mathbb{R} \)-holomorphism), it is nondegenerate.

The sufficiency is proved by direct calculations. \( \blacksquare \)

Similarly we prove the following statement.

**Theorem 3.5.** Linear groups \( L^1 \) and \( L^2 \) are holomorphic conjugated if and only if they are linearly conjugated.

### 4. Applications to the complex nonautonomous linear differential systems

Theorems 3.1, 3.2, 3.4, 3.5 and Lemma 3.2 allow on the basis of Theorems 1.1 – 1.3 to spend topological, smooth, \( \mathbb{R} \)-holomorphic and holomorphic classifications of complex nonautonomous linear differential systems of a kind (2.1). Besides, from the theorem 3.3 it is had such statement.

**Theorem 4.1.** From topological equivalence of complex nonautonomous linear differential systems with non-Abelian phase groups of general situation follows them \( \mathbb{R} \)-holomorphic equivalence.

From the theorem 4.1 follows that in the case of two and more dependent variables complex nonautonomous linear differential systems with coefficients, holomorphic on path connected holomorphic varieties with non-Abelian fundamental groups, are structurally unstable.

Let’s result now concrete examples.

Consider the following linear equations: ordinary differential
\[
\frac{dw}{dz} = \sum_{r=1}^{\nu} \sum_{l=1}^{\eta_r} \frac{\Lambda_{rl1}}{(z - z_{r1})^l} w
\]
and
\[
\frac{dw}{dz} = \sum_{r=1}^{\nu} \sum_{l=1}^{\eta_r} \frac{\Lambda_{rl2}}{(z - z_{r2})^l} w; \tag{4.1}
\]
integral
\[
w(z) = \int_0^1 \prod_{r=1}^{\nu} \left( \frac{z - z_{r1}}{z - z_{r2}} \right)^{\Lambda_{rl1}} \exp \left( \sum_{l=2}^{\eta_r} \frac{\Lambda_{rl1}}{1 - l} \left( \frac{1}{(z - z_{r1})^{l-1}} - \frac{1}{(z - z_{r2})^{l-1}} \right) \right) w(\zeta) \, d\zeta \tag{4.2}
\]
and
\[
w(z) = \int_0^1 \prod_{r=1}^{\nu} \left( \frac{z - z_{r2}}{z - z_{r1}} \right)^{\Lambda_{rl2}} \exp \left( \sum_{l=2}^{\eta_r} \frac{\Lambda_{rl2}}{1 - l} \left( \frac{1}{(z - z_{r2})^{l-1}} - \frac{1}{(z - z_{r1})^{l-1}} \right) \right) w(\zeta) \, d\zeta; \tag{4.3}
\]
integral-differential
\[
\frac{dw}{dz} = \int_0^1 \sum_{r=1}^{\nu} \sum_{l=1}^{\eta_r} \frac{\Lambda_{rl1}}{(z - z_{r1})^l} \prod_{r=1}^{\nu} \left( \frac{z - z_{r1}}{z - z_{r2}} \right)^{\Lambda_{rl1}} \exp \left( \sum_{l=2}^{\eta_r} \frac{\Lambda_{rl1}}{1 - l} \left( \frac{1}{(z - z_{r1})^{l-1}} - \frac{1}{(z - z_{r2})^{l-1}} \right) \right) w(\zeta) \, d\zeta \tag{4.4}
\]
and
\[
w(z) = \int_0^1 \prod_{r=1}^{\nu} \left( \frac{z - z_{r1}}{z - z_{r2}} \right)^{\Lambda_{rl1}} \exp \left( \sum_{l=2}^{\eta_r} \frac{\Lambda_{rl1}}{1 - l} \left( \frac{1}{(z - z_{r1})^{l-1}} - \frac{1}{(z - z_{r2})^{l-1}} \right) \right) w(\zeta) \, d\zeta; \tag{4.5}
\]
\[
\frac{dw}{dz} = \int_0^1 \sum_{r=1}^\nu \sum_{l=1}^{\eta_r} \frac{\Lambda_{rl2}}{(z-z_{r2})^l} \prod_{r=1}^\nu \left( \frac{z-z_{r2}}{\zeta-z_{r2}} \right)^{\Lambda_{rl2}} \cdot \exp \left( \sum_{l=2}^{\eta_r} \frac{\Lambda_{rl2}}{l-1} \left( \frac{1}{(z-z_{r2})^{l-1}} - \frac{1}{(\zeta-z_{r2})^{l-1}} \right) \right) w(\zeta) \, d\zeta,
\]

(4.6)

where \( \Lambda_{rl1} \) and \( \Lambda_{rl2} \) are complex numbers, the path from integrals (4.3) – (4.6), connecting points 0 and 1, are homeomorphic to segments and do not pass through points \( z_{1r} \) and \( z_{2r}, \ r = 1, \ldots, \nu. \)

The common solutions of the linear equations (4.1), (4.3), and (4.5) define the same covering foliation \( \mathcal{L}^3 \) on the variety \( \mathbb{C} \times \Gamma'_1 \), and the common solutions of the linear equations (4.2), (4.4), and (4.6) define the same covering foliation \( \mathcal{L}^4 \) on the variety \( \mathbb{C} \times \Gamma'_2 \), where \( \Gamma'_1 \) (accordingly, \( \Gamma'_2 \)) is an open complex plane with \( \nu \) eliminated points \( z_{1r}, \ r = 1, \ldots, \nu \) (with \( \nu \) eliminated points \( z_{2r}, \ r = 1, \ldots, \nu \)). The phase group \( Ph(\mathcal{L}^3) \) of covering foliation \( \mathcal{L}^3 \) is generated by nondegenerate linear transformations \( p_r \) for all \( w \in \mathbb{C}, \ r = 1, \ldots, \nu \), and the phase group \( Ph(\mathcal{L}^4) \) of covering foliation \( \mathcal{L}^4 \) is generated by nondegenerate linear transformations \( q_r w \) for all \( w \in \mathbb{C}, \ r = 1, \ldots, \nu, \) where \( p_r = \exp(2\pi i \Lambda_{rl1}), \ q_r = \exp(2\pi i \Lambda_{rl2}), \ r = 1, \ldots, \nu, \ i^2 = -1. \)

Owing to Theorems 3.1 and 1.1 it is received the following statement.

**Theorem 4.2.** For topological equivalence of covering foliations \( \mathcal{L}^3 \) and \( \mathcal{L}^4 \) it is necessary and enough existence of such permutation \( \kappa: (1 \ldots \nu) \rightarrow (1 \ldots \nu) \) and complex number \( \alpha \) with \( \text{Re} \alpha \neq -1, \) that either \( q_{\kappa(r)} = p_r |p_r|^\alpha, \ r = 1, \ldots, \nu, \) or \( q_{\kappa(k)} = \overline{p_r} |p_r|^\alpha, \ r = 1, \ldots, \nu. \)

Notice that the linear ordinary differential equations (4.1) and (4.2) in special case \( \eta_1 = \ldots = \eta_2 = 1, \ z_{1r} = z_{2r}, \ r = 1, \ldots, \nu, \) are considered in [17]. Besides, set example shows possibility of application of the device of covering foliations for the mathematical objects which are distinct from differential systems.

Consider linear ordinary differential systems

\[
\frac{dw}{dz} = A_1(z) w
\]

and

\[
\frac{dw}{dz} = B_1(z) w,
\]

(4.7)

(4.8)

where square matrices \( A_1(z) = \|a_{ik1}(z)\| \) and \( B_1(z) = \|b_{ik1}(z)\| \) of the size \( n, \ n > 1, \) consist from 1-periodic holomorphic functions \( a_{ik1}: \mathbb{C} \rightarrow \mathbb{C} \) and \( b_{ik1}: \mathbb{C} \rightarrow \mathbb{C}, \ i, k = 1, \ldots, n. \)

The common solutions of linear ordinary differential systems (4.7) and (4.8) define covering foliations \( \mathcal{L}^5 \) and \( \mathcal{L}^6 \), accordingly, on the variety \( \mathbb{C}^n \times Z, \) where \( Z \) is the cylinder \( S^1 \times \mathbb{R}, \) \( S^1 \) is an unit circle. The phase group \( Ph(\mathcal{L}^5) \) of covering foliation \( \mathcal{L}^5 \) is generated by nondegenerate linear transformation \( P_1 w \) for all \( w \in \mathbb{C}^n, \ P_1 \in GL(n, \mathbb{C}), \) and the phase group \( Ph(\mathcal{L}^6) \) covering foliation \( \mathcal{L}^6 \) is generated by nondegenerate linear transformation \( Q_1 w \) for all \( w \in \mathbb{C}^n, \ Q_1 \in GL(n, \mathbb{C}). \)

From here on the basis of Theorems 3.2 and 1.1 it is had such statement.

**Theorem 4.3.** Let at \( n > 1 \) the matrices

\[
P_1 = S \text{diag}\{p_{11}, \ldots, p_{nn}\} S^{-1}, \quad Q_1 = T \text{diag}\{q_{11}, \ldots, q_{nn}\} T^{-1},
\]

the matrices \( \ln P_1 \) and \( \ln Q_1 \) are simple. Then for topological equivalence of linear ordinary differential systems (4.7) and (4.8) it is necessary and enough existence of such permutation
\( p: (1, \ldots, n) \rightarrow (1, \ldots, n) \) and complex numbers \( \alpha_k \) with Re \( \alpha_k \neq -1 \), \( k = 1, \ldots, n \), that either \( q_{\alpha(k)} = p_{kr}|p_{kr}|^{\alpha_k}, \) or \( q_{\alpha(k)} = p_{kr}|p_{kr}|^{\alpha_k}, \) \( k = 1, \ldots, n \).

Now we will consider linear ordinary differential systems (4.7) and (4.8) in a case when square matrices \( A_1(z) \) and \( B_1(z) \) of the size \( n \) consist from holomorphic functions \( a_{ik1} : \Gamma_1 \rightarrow \mathbb{C} \) and \( b_{ik1} : \Gamma_2 \rightarrow \mathbb{C} \), \( i = 1, \ldots, n \), \( k = 1, \ldots, n \). In this case the common solutions of linear ordinary differential systems (4.7) and (4.8) define covering foliations \( \mathcal{L}^T \) and \( \mathcal{L}^S \), accordingly, on the varieties \( \mathbb{C}^n \times \Gamma_1 \) and \( \mathbb{C}^n \times \Gamma_2 \). The phase group \( Ph(\mathcal{L}^T) \) of covering foliation \( \mathcal{L}^T \) is generated by nondegenerate linear transformations \( P_r w \) for all \( w \in \mathbb{C}^n \), \( P_r \in GL(n, \mathbb{C}) \), \( r = 1, \ldots, \nu \), and the phase group \( Ph(\mathcal{L}^S) \) of covering foliation \( \mathcal{L}^S \) is generated by nondegenerate linear transformations \( Q_r w \) for all \( w \in \mathbb{C}^n \), \( Q_r \in GL(n, \mathbb{C}) \), \( r = 1, \ldots, \nu \).

In a commutative case on the basis of Theorems 3.2 and 1.1 we receive the statement.

**Theorem 4.4.** Let at \( n > 1 \) the matrices

\[
P_r = S \text{diag}\{p_{ir}, \ldots, p_{ir}\} S^{-1} \quad (Q_r = T \text{diag}\{q_{1r}, \ldots, q_{nr}\} T^{-1}),
\]

the matrices \( \ln P_r \) and \( \ln Q_r \) are simple, \( r = 1, \ldots, \nu \). Then for topological equivalence of linear ordinary differential systems (4.7) and (4.8) it is necessary and enough existence of such permutations \( \varpi: (1, \ldots, \nu) \rightarrow (1, \ldots, \nu) \), \( q: (1, \ldots, n) \rightarrow (1, \ldots, n) \) and complex numbers \( \alpha_k \) with Re \( \alpha_k \neq -1 \), \( k = 1, \ldots, n \), that either

\[
q_{\alpha(k)} = p_{kr}|p_{kr}|^{\alpha_k}, \quad k = 1, \ldots, n, \quad r = 1, \ldots, \nu,
\]

or

\[
q_{\alpha(k)} = \overline{p_{kr}}|p_{kr}|^{\alpha_k}, \quad k = 1, \ldots, n, \quad r = 1, \ldots, \nu.
\]

In a noncommutative case on the basis of Lemmas 3.1, 3.2 and Theorem 1.1 we have concrete constructive criteria of topological equivalence of differential systems (4.7) and (4.8).

5. Phase groups of covering foliations, defined by complex nonautonomous projective matrix Riccati equations

We will consider homogeneous projective matrix Riccati equations [30]

\[
dv = \sum_{j=1}^{m} \mathfrak{A}_j(z_1, \ldots, z_m) v dz_j
\]

(5.1)

and

\[
dv = \sum_{j=1}^{m} \mathfrak{B}_j(z_1, \ldots, z_m) v dz_j,
\]

(5.2)

ordinary at \( m = 1 \) and completely solvable at \( m > 1 \), where \( v = (v_1, \ldots, v_{n+1}) \) are homogeneous coordinates, square matrices \( \mathfrak{A}_j(z_1, \ldots, z_m) = \|a_{ikj}(z_1, \ldots, z_m)\| \) and \( \mathfrak{B}_j(z_1, \ldots, z_m) = \|b_{ikj}(z_1, \ldots, z_m)\| \) of the size \( n + 1 \) consist from holomorphic functions \( a_{ikj} : A \rightarrow \mathbb{C} \) and \( b_{ikj} : B \rightarrow \mathbb{C} \), \( i = 1, \ldots, n + 1 \), \( k = 1, \ldots, n + 1 \), \( j = 1, \ldots, m \), path connected holomorphic varieties \( A \) and \( B \) are holomorphically equivalent each other, fundamental groups \( \pi_1(A) \) and \( \pi_1(B) \) have final number \( \nu \in \mathbb{N} \) of the forming.

The common solutions of homogeneous projective matrix Riccati equations (5.1) and (5.2) define covering foliations \( \mathcal{P}^T \) and \( \mathcal{P}^S \), accordingly, on the varieties \( \mathbb{C}^n \times A \) and \( \mathbb{C}^n \times B \).

We will say, that homogeneous projective matrix Riccati equations (5.1) and (5.2) are topologically smoothly or holomorphically equivalent, if exists the homeomorphism, (the diffeomorphism, the \( \mathbb{R} \)-holomorphism, the holomorphism) \( h: \mathbb{C}^n \times A \rightarrow \mathbb{C}^n \times B, \)
translating the layers of the covering foliation $\mathcal{F}^1$ in the layers of the covering foliation $\mathcal{F}^2$.

Similarly we introduce the concepts of embedding (smooth embedding, $\mathbb{R}$-holomorphic embedding, holomorphic embedding) and covering (smoothly covering, $\mathbb{R}$-holomorphically covering, holomorphically covering) of homogeneous projective matrix Riccati equations. The phase group $Ph(\mathcal{F}^1)$ of the covering foliation $\mathcal{F}^1$ is generated [30] by the forming nondegenerate linear-fractional transformations $P_r v$ for all $v \in \mathbb{CP}^n$, $P_r \in GL(n+1, \mathbb{C})$, $r = 1, \ldots, \nu$, and the phase group $Ph(\mathcal{F}^2)$ of the covering foliation $\mathcal{F}^2$ is generated by the forming nondegenerate linear-fractional transformations $Q_r v$ for all $v \in \mathbb{CP}^n$, $Q_r \in GL(n+1, \mathbb{C})$, $r = 1, \ldots, \nu$.

6. Conjunctions of linear-fractional actions on $\mathbb{CP}^n$

Now we will consider a problem about a finding of necessary and sufficient conditions of existence such homeomorphism (diffeomorphism, $\mathbb{R}$-holomorphism, holomorphism) $f: \mathbb{CP}^n \to \mathbb{CP}^n$, that identities

$$f(P_r v) = Q_r f(v) \quad \text{for all } v \in \mathbb{CP}^n, \quad \text{for all } r \in I,$$

(6.1)

take place, where $f(v) = (f_1(v), \ldots, f_n(v))$, square matrices

$$P_r \in GL(n+1, \mathbb{C}), \quad Q_r \in GL(n+1, \mathbb{C}) \quad \text{for all } r \in I.$$

Thus group of linear-fractional actions on $\mathbb{CP}^n$, formed by the matrices $P_r$ for all $r \in I$ (by the matrices $Q_r$ for all $r \in I$), we will designate through $PL^1$ (through $PL^2$).

Consider at first a topological conjunction of the Abelian linear-fractional groups $PL^1$ and $PL^2$.

**Lemma 6.1.** Let at $n = 1$ linear-fractional groups $PL^1$ and $PL^2$ be topological conjugated. Then normal Jordan forms of the matrices $P_r$ and $Q_r$, defining by nonidentical is linear-fractional transformations, have identical number of blocks of Jordan, for all $r \in I$.

Proof of the given statement is spent on the basis of that fact, that the quantity of fixed points of linear-fractional transformations coincides with the number of eigenvectors of the matrices, defining by these transformations.

On the basis of Lemma 6.1 by direct calculations we come to such statement.

**Lemma 6.2.** For a topological conjugation at $n = 1$ of Abelian linear-fractional groups $PL^1$ and $PL^2$ it is necessary, that normal Jordan forms of all matrices $P_r$ and $Q_r$, defining by nonidentical linear-fractional transformations, for all $r \in I$, had identical number of blocks of Jordan.

**Theorem 6.1.** Let at $n = 1$ the matrices

$$P_r = S \text{diag}\{p_{1r}, p_{2r}\} S^{-1} \quad \text{for all } r \in I, \quad Q_r = T \text{diag}\{q_{1r}, q_{2r}\} T^{-1} \quad \text{for all } r \in I.$$

Then for a topological conjunction of linear-fractional groups $PL^1$ and $PL^2$ it is necessary and enough, that either

$$\frac{q_{1r}}{q_{2r}} = \frac{P_{1r}}{P_{2r}} \frac{|P_{1r}|^\alpha}{|P_{2r}|^\alpha}, \quad \text{Re } \alpha \neq -1, \quad \text{for all } r \in I,$$

(6.2)

or

$$\frac{q_{1r}}{q_{2r}} = \frac{P_{1r}}{P_{2r}} \frac{|P_{1r}|^\alpha}{|P_{2r}|^\alpha}, \quad \text{Re } \alpha \neq -1, \quad \text{for all } r \in I.$$

(6.3)

Proof. With the help of replacement $\xi(v) = T^{-1} f(Sv)$ for all $v \in \mathbb{CP}^1$, from identities (6.1) at $n = 1$ we pass to identities

$$\xi(\text{diag}\{p_{1r}, p_{2r}\} v) = \text{diag}\{q_{1r}, q_{2r}\} \xi(v) \quad \text{for all } v \in \mathbb{CP}^1, \quad \text{for all } r \in I.$$  

(6.4)

Therefore the topological conjunction of linear-fractional groups $PL^1$ and $PL^2$ is equivalent to performance of identities (6.4).
The necessity. Let identities (6.4) are carried out.

If all \( \frac{p_{1r}}{p_{2r}} = 1 \) for all \( r \in I \) from (6.4) it is had, as \( \frac{q_{1r}}{q_{2r}} = 1 \) for all \( r \in I \). Therefore in this case relations (6.2) are carried out at any \( \alpha \) with \( \Re \alpha \neq -1 \).

Let now \( \frac{p_{1r}}{p_{2r}} \neq 1 \), \( r \in I \). On the basis of identities (6.4) it is received, that either \( \xi(O_1) = O_1 \), or \( \xi(O_1) = O_2 \), where \( O_r \) (the origins of coordinates of the affine cards \( M_r = \{v, v_r \neq 0\} \) of the atlas \( M \) of variety \( \mathbb{CP}^1 \), \( r = 1, 2 \), are the common fixed points of linear-fractional transformations \( \text{diag}(p_{1r}, p_{2r})v \) for all \( v \in \mathbb{CP}^1 \), and \( \text{diag}(q_{1r}, q_{2r})v \) for all \( v \in \mathbb{CP}^1 \) for all \( r \in I \).

If \( \xi(O_1) = O_1 \), then on the basis of Theorem 3.1 we do a conclusion, that at \( \Re \alpha > -1 \) take place either equalities (6.2), or equalities (6.3).

If \( \xi(O_1) = O_2 \), then with the help of replacement \( \zeta(v) = (\xi_2(v), \xi_1(v)) \) for all \( v \in \mathbb{CP}^1 \), we come to the previous case and as a result it is received either equalities (6.2) at \( \Re \alpha < -1 \), or equalities (6.3) at \( \Re \alpha < -1 \).

The sufficiency is proved by construction of conjugating homeomorphism

\[
\xi(v) = (\gamma_1 v_1 | v_1 |^\alpha, \gamma_2 v_2 | v_2 |^\alpha) \quad \text{for all} \quad v \in \mathbb{CP}^1,
\]

in case of performance of relations (6.2), and by construction of conjugating homeomorphism

\[
\xi(v) = (\gamma_1 \overline{v}_1 | v_1 |^\alpha, \gamma_2 \overline{v}_2 | v_2 |^\alpha) \quad \text{for all} \quad v \in \mathbb{CP}^1,
\]

in case of performance of relations (6.3).

**Theorem 6.2.** Let at \( n > 1 \) the matrices

\[
P_r = S \text{diag}(p_{1r}, \ldots, p_{n+1, r})S^{-1}, \quad Q_r = T \text{diag}(q_{1r}, \ldots, q_{n+1, r})T^{-1},
\]

sets of numbers \( \left\{ \ln \frac{p_{1r}}{p_{n+1, r}}, \ldots, \ln \frac{p_{nr}}{p_{n+1, r}} \right\} \) and \( \left\{ \ln \frac{q_{1r}}{q_{n+1, r}}, \ldots, \ln \frac{q_{nr}}{q_{n+1, r}} \right\} \) are simple, for all \( r \in I \). Then for a topological conjunction of linear-fractional groups \( PL^1 \) and \( PL^2 \) it is necessary and enough existence of such permutation \( \varrho: (1, \ldots, n + 1) \to (1, \ldots, n + 1) \) and complex number \( \alpha \) with \( \Re \alpha > -1 \), that either

\[
\frac{q_{\varrho(k)r}}{q_{\varrho(n+1)r}} = \frac{p_{kr}}{p_{n+1, r}} \left| \frac{p_{kr}}{p_{n+1, r}} \right|^\alpha \quad \text{for all} \quad r \in I, \quad k = 1, \ldots, n, \quad (6.5)
\]

or

\[
\frac{q_{\varrho(k)r}}{q_{\varrho(n+1)r}} = \frac{\overline{p_{kr}}}{p_{n+1, r}} \left| \frac{p_{kr}}{p_{n+1, r}} \right|^\alpha \quad \text{for all} \quad r \in I, \quad k = 1, \ldots, n. \quad (6.6)
\]

**Proof.** With the help of replacement \( \xi(v) = T^{-1} f(Sv) \) for all \( v \in \mathbb{CP}^n \), from identities (6.1) we pass to identities

\[
\xi(\text{diag}(p_{1r}, \ldots, p_{n+1, r})v) = \text{diag}(q_{1r}, \ldots, q_{n+1, r}) \xi(v) \quad \text{for all} \quad v \in \mathbb{CP}^n, \quad \text{for all} \quad r \in I. \quad (6.7)
\]

So, the topological conjunction of linear-fractional groups \( PL^1 \) and \( PL^2 \) is equivalent to performance of identities (6.7).

Thus, not belittling a generality, we will consider, that conjugating homeomorphism \( \xi \) leaves invariant the common fixed points \( O_r \) (the origins of coordinates of the affine cards \( M_r = \{v, v_r \neq 0\} \) of the atlas \( M \) of variety \( \mathbb{CP}^n \), \( r = 1, \ldots, n + 1 \), of nondegenerate linear-fractional transformations

\[
\text{diag}(p_{1r}, \ldots, p_{n+1, r})v \quad \text{for all} \quad v \in \mathbb{CP}^n, \quad \text{for all} \quad r \in I,
\]

and
come to conclusion, that
\( (6.7) \) vectors
carried out. Taking into consideration last equalities, and also that fact, that in identities
linear-fractional transformations.
is carried out by either nondegenerate linear-fractional, or nondegenerate antiholomorphic
\( r \)
for all \( \alpha \), \( k = 1, \ldots, n - 1 \), for all \( r \in I \).
(basis of nondegenerate absolute invariants
\( v_k \)
\( \ln(p_{or}/p_{or+1}) \) \( v_n \)
\( \ln(p_{or}/p_{or+1}) \) \( v_{n+1} \), \( k = 1, \ldots, n - 1 \), for all \( r \in I \).
Considering, that conjugating homeomorphism \( \xi \) takes the layers of the foliations \( \mathcal{C}_r \) to
the homeomorphic to them the layers of the foliations \( \mathcal{D}_r \) for all \( r \in I \), and also taking into
consideration simplicity of sets of numbers from a theorem condition, we come to conclusion,
that homeomorphism \( \xi \) translates Riemann spheres
\( \overline{C}_{is}: v_k = 0, \ k \neq l, \ k \neq s, \ s \neq l, \ k = 1, \ldots, n + 1, \)
in Riemann spheres of a kind \( \overline{C}_{ij}, j \neq i \), and all homeomorphisms of Riemann spheres
simultaneously either keep, or change orientation.

Let homeomorphisms of Riemann spheres keep orientation (the case of homeomorphisms,
changing orientation, it is considered similarly). Then owing to a course of the proof of
Theorems 3.2 and 6.2 we come to conclusion, that in the affine card \( M_{n+1} = \{ v, v_{n+1} \neq 0 \} \)
relations
\[
\frac{q_{\xi}(k)r}{q_{\xi}(n+1)r} = \frac{P_{kr}}{P_{n+1,r}} = \frac{P_{kr}^{|\alpha_k|}}{P_{n+1,r}^{|\alpha_k|}}, \ r = 1, \ldots, \nu, \ k = 1, \ldots, n, \]
are fulfilled and at complex numbers \( \alpha_k \) the real parts \( \Re \alpha_k > -1 \), \( k = 1, \ldots, n \), are
carried out. Taking into consideration last equalities, and also that fact, that in identities
(6.7) vectors \( \{ p_{1r}, \ldots, p_{n+1,r} \} \) and \( \{ q_{1r}, \ldots, q_{n+1,r} \} \) are defined to within scalar multipliers,
for all \( r \in I \), and spending similar reasonings in other affine cards \( M_{\tau}, \tau = 1, \ldots, n \), we
come to conclusion, that \( \alpha_k = \alpha, \ k = 1, \ldots, n \). As a result we come to the relations (6.5).

The sufficiency is proved by the construction of conjugating homeomorphism \( \xi: \mathbb{C}P^n \rightarrow \mathbb{C}P^n \) such that its projection
\[
\xi_{\xi}(k)(v) = \gamma_k v_k |v_k|^{\alpha} \quad \text{for all} \quad v \in \mathbb{C}P^n, \ k = 1, \ldots, n + 1, \]
if relations (6.5) take place; and its projections
\[
\xi_{\xi}(k)(v) = \gamma_k \overline{v}_k |v_k|^{\alpha} \quad \text{for all} \quad v \in \mathbb{C}P^n, \ k = 1, \ldots, n + 1, \]
if parities (6.6) take place.

Consider now a topological conjunction of the non-Abelian linear-fractional groups \( PL^1 \)
and \( PL^2 \).

**Theorem 6.3.** From a topological conjunction at \( n = 1 \) of non-Abelian linear-fractional
groups \( PL^1 \) and \( PL^2 \) of general situation follows them \( \mathbb{R} \)-holomorphic conjunction which
is carried out by either nondegenerate linear-fractional, or nondegenerate antiholomorphic
linear-fractional transformations.

**Proof** of Theorem 6.3 directly follows from the following auxiliary statement.

**Lemma 6.3.** Let at \( n = 1 \) and \( I = \{ 1, 2 \} \) linear-fractional groups \( PL^1 \) and \( PL^2 \) are
topological conjugated, and conjugating homeomorphism \( f \) is such that:

1) \( f(O_{\tau}) = O_{\tau}, \ \tau = 1, 2; \quad (6.10) \)
2) either

\[ f(\lambda v_1, v_2) = (\lambda|\lambda|^\alpha f_1(v), f_2(v)) \quad \text{for all } v \in \mathbb{C}P^1, \quad \Re \alpha > -1, \]  

or

\[ f(\lambda v_1, v_2) = (\overline{\lambda}|\lambda|^\alpha f_1(v), f_2(v)) \quad \text{for all } v \in \mathbb{C}P^1, \quad \Re \alpha > -1; \]  

3) \( f(av_1+bv_2,cv_1+dv_2) = (Af_1(v)+Bf_2(v), Cf_1(v)+Df_2(v)) \) for all \( v \in \mathbb{C}P^1, |b|+|c|>0; \)

4) matrices \( P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = S \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} S^{-1} \) and \( Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = T \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix} T^{-1} \) are such that \( |\mu| \neq 1, |\theta| \neq 1, S = \begin{pmatrix} a_* & b_* \\ c_* & d_* \end{pmatrix}, \quad T = \begin{pmatrix} A_* & B_* \\ C_* & D_* \end{pmatrix}, \quad \frac{B_*}{D_*} = (\frac{b_*}{d_*})^{1+\beta} \), and

\[ \text{Re}(\beta-\alpha) \ln \left| \frac{b_*}{d_*} \right| - \text{Im}(\beta-\alpha) \arg \frac{b_*}{d_*} \neq 0; \]

5) the subgroup of the group \( \mathbb{C}^* \) of nonzero complex numbers on the multiplication, formed by numbers \( \lambda \) and \( \frac{b_*}{d_*} \), is dense in the set \( \mathbb{C} \) of complex numbers.

Then at (6.11) this gomeomorphism has a kind \( f(v) = (v_1|v_1|^\alpha, v_2|v_2|^\alpha) \) for all \( v \in \mathbb{C}P^1; \)
and at (6.12) this gomeomorphism has a kind \( f(v) = (\overline{v}_1|v_1|^\alpha, \overline{v}_2|v_2|^\alpha) \) for all \( v \in \mathbb{C}P^1.

Proof. Owing to relations (6.10) — (6.12) we come to conclusion, that \( \ln |\mu| \ln |\theta| > 0. \)

Let the identity (6.11) is carried out (a case, when the identity (6.12) takes place, is considered similarly). On the basis of (6.11) and conditions 3 of given lemma it is had following relations in the card \( M_2 \) of the atlas \( M \) of variety \( \mathbb{C}P^1:\)

\[ \psi(\lambda^k (P^m)^l v^2) = \lambda^k |\lambda|^\alpha (Q^m)^l \psi(v^2) \quad \text{for all } v^2 \in M_2, \quad \text{for all } k, l, m \in \mathbb{Z}. \]

Passing in them to a limit at \( m \to -\infty \) if \( |\mu| > 1 \), and passing in them to a limit at \( m \to +\infty \) if \( |\mu| < 1 \), we receive, that

\[ \psi\left(\lambda^k \left(\frac{b^*}{d^*}\right)^l\right) = \lambda^k |\lambda|^\alpha \left(\frac{B_*}{D_*}\right)^l \quad \text{for all } k \in \mathbb{Z}, \quad \text{for all } l \in \mathbb{Z}. \]  

(6.13)

From the condition 5 of lemma follows, that for any complex number \( w \in \mathbb{C} \) there are such sequences \( \{k_s(w)\} \) and \( \{l_s(w)\} \) of integers, that

\[ \lim_{s \to +\infty} \lambda^{k_s(w)} \left(\frac{b^*}{d^*}\right)^{l_s(w)} = w, \quad \lim_{s \to +\infty} |k_s(w)| = \lim_{s \to +\infty} |l_s(w)| = +\infty. \]

From here on the basis of the relations (6.13) it is had, that

\[ \psi(v^2) = v^2 |v^2|^\alpha, \quad \lim_{s \to +\infty} \left(\frac{b^*}{d^*}\right)^{(\beta-\alpha)l_s(v^2)} = 1. \]

Now from the condition 5 of given lemma follows, that \( \left| \frac{b^*}{d^*} \right| \neq 1 \), and from a condition 4 of given lemma follows, that \( \lim_{s \to +\infty} \left(\frac{b^*}{d^*}\right)^{(\beta-\alpha)l_s(v^2)} = 1. \) As a result it is received the first representation from a condition of the lemma 6.3. ■

Theorem 6.4. From a topological conjunction at \( n > 1 \) of non-Abelian linear-fractional groups \( PL^1 \) and \( PL^2 \) of general situation follows them \( \mathbb{R} \)-holomorphic conjunction.

Justice of the given theorem follows from the following auxiliary statement.

Lemma 6.4. Let at \( n > 1 \) the matrices

\[ P_r = S_r \text{ diag } \{p_{1r}, \ldots, p_{n+1,r}\} S_r^{-1}, \quad Q_r = T_r \text{ diag } \{q_{1r}, \ldots, q_{n+1,r}\} T_r^{-1}, \]
sets of numbers \( \left\{ \ln \frac{p_{1r}}{p_{n+1,r}}, \ldots, \ln \frac{p_{nr}}{p_{n+1,r}} \right\} \) and \( \left\{ \ln \frac{q_{1r}}{q_{n+1,r}}, \ldots, \ln \frac{q_{nr}}{q_{n+1,r}} \right\} \) are simple, for all \( r \in I \). Then from a topological conjunction of non-Abelian linear-fractional groups \( PL^1 \) and \( PL^2 \) of general situation follows them \( \mathbb{R} \)-holomorphic conjunction which is carried out by either nondegenerate linear-fractional, or nondegenerate antiholomorphic linear-fractional transformations.

Proof. Let identities (6.1) are carried out. With the help of replacement \( \xi(v) = T_1^{-1}f(S_1v) \) for all \( v \in \mathbb{C}P^n \), from them we pass to identities

\[
\xi(\text{diag}\{p_{11}, \ldots, p_{n+1}\})v = \text{diag}\{q_{11}, \ldots, q_{n+1}\}\xi(v) \text{ for all } v \in \mathbb{C}P^n. \tag{6.14}
\]

Similarly, as well as at the proof of the theorem 6.2, we come to conclusion, that conjugating homeomorphism \( \xi \) leaves invariant the common fixed points \( O_\tau, \tau = 1, \ldots, n+1, \) of the linear-fractional transformations (6.8) and (6.9) at \( r = 1 \).

Let homeomorphisms of all Riemann spheres \( \overline{\mathbb{C}}_{l_s}, \ s \neq l \), keep orientation (the case of homeomorphisms, changing of orientation, it is considered similarly). Then on the basis of the theorem 3.2, lemmas 3.1, 3.2 and a course of the proof of the theorem 6.2 we come to the statement of the lemma 6.4.

Consider smooth, \( \mathbb{R} \)-holomorphic and holomorphic conjunctions of Abelian linear-fractional groups \( PL^1 \) and \( PL^2 \).

**Theorem 6.5.** Let the conditions of the theorem 6.1 are satisfied. Then for a smooth (an \( \mathbb{R} \)-holomorphic) conjunction of fractional-linear groups \( PL^1 \) and \( PL^2 \) it is necessary and enough, that either

\[
\frac{q_{1r}}{q_{2r}} = \left( \frac{p_{kr}}{p_{n+1,r}} \right)^\varepsilon \text{ for all } r \in I,
\]

or

\[
\frac{q_{1r}}{q_{2r}} = \left( \frac{\overline{p}_{kr}}{p_{n+1,r}} \right)^\varepsilon \text{ for all } r \in I; \quad \varepsilon^2 = 1.
\]

**Proof** of the given statement is similar to the proof of the theorem 6.1 and is based on the theorem 3.4.

**Theorem 6.6.** Let the conditions of the theorem 6.2 are satisfied. Then for a smooth (an \( \mathbb{R} \)-holomorphic) conjunction of fractional-linear groups \( PL^1 \) and \( PL^2 \) it is necessary and enough existence of such permutation \( \rho \): \( (1, \ldots, n+1) \rightarrow (1, \ldots, n+1) \), that either

\[
\frac{q_{\rho(k)r}}{q_{\rho(n+1)r}} = \frac{p_{kr}}{p_{n+1,r}} \text{ for all } r \in I, \quad k = 1, \ldots, n,
\]

or

\[
\frac{q_{\rho(k)r}}{q_{\rho(n+1)r}} = \frac{\overline{p}_{kr}}{p_{n+1,r}} \text{ for all } r \in I, \quad k = 1, \ldots, n.
\]

**Proof** of the statements is carried out with use of a course of the proof of the theorem 6.2 by differentiation of identities (6.7).

Similarly to Theorems 6.5 and 6.6 it is received such statements.

**Theorem 6.7.** Let the conditions of the theorem 6.1 are satisfied. Then for a holomorphic conjunction of fractional-linear groups \( PL^1 \) and \( PL^2 \) it is necessary and enough, that

\[
\frac{q_{1r}}{q_{2r}} = \left( \frac{p_{kr}}{p_{n+1,r}} \right)^\varepsilon \text{ for all } r \in I; \quad \varepsilon^2 = 1.
\]

**Theorem 6.8.** Let the conditions of the theorem 6.2 are satisfied. Then for a holomorphic conjunction of fractional-linear groups \( PL^1 \) and \( PL^2 \) it is necessary and enough existence of such permutation \( \rho \): \( (1, \ldots, n+1) \rightarrow (1, \ldots, n+1) \), that
Then at (6.15) pass to identities (6.14). Further owing to the theorem 6.2 we come to conclusion, that take
formation.

This homeomorphism looks like (6.16)

PL
conjunction of non-Abelian linear-fractional groups PL1 and PL2.

**Theorem 6.9.** Let the conditions of the lemma 6.3 are satisfied. Then for a smooth conjunction of non-Abelian linear-fractional groups PL1 and PL2 of general situation it is necessary and sufficient them conjunction, which is carried out by either nondegenerate linear-fractional transformation, or nondegenerate antiholomorphic linear-fractional transformation.

Proof of the theorem 6.9 is similar to the proof of the theorem 6.3 and is based on the following auxiliary statement.

**Lemma 6.5.** Let at \( n = 1 \) and \( I = \{1\} \) linear-fractional groups PL1 and PL2 are smooth conjugated, and conjugating diffeomorphism \( f \) is such that:

1) relations (6.10) are carried out;
2) either

\[
 f(p_1v_1, v_2) = (p_1f_1(v), f_2(v)) \quad \text{for all } v \in \mathbb{C}P^1, \quad p_1 \neq 0, \quad p_1 \neq 1; \tag{6.15}
\]

or

\[
 f(p_1v_1, v_2) = (\overline{p}_1f_1(v), f_2(v)) \quad \text{for all } v \in \mathbb{C}P^1, \quad p_1 \neq 0, \quad p_1 \neq 1. \tag{6.16}
\]

Then at (6.15) this homeomorphism looks like \( f(v) = (av_1, v_2) \) for all \( v \in \mathbb{C}P^1 \); and at (6.16) this homeomorphism looks like \( f(v) = (a\overline{v}_1, \overline{v}_2) \) for all \( v \in \mathbb{C}P^1 \).

Proof. Let the relations (6.10) are carried out. Differentiating in the card \( M_2 \) of the atlas \( M \) of the variety \( \mathbb{C}P^1 \) the identity (6.1) at \( v^2 = 0 \), we receive equality

\[
 p_1 D_{v^2} \psi(0) dv^2 + \overline{p}_1 D_{\overline{v}^2} \psi(0) d\overline{v}^2 = q_1 \left( D_{v^2} \psi(0) dv^2 + D_{\overline{v}^2} \psi(0) d\overline{v}^2 \right).
\]

Owing to that \( f \) is a diffeomorphism, we have, that \( |D_{v^2} \psi(0)| + |D_{\overline{v}^2} \psi(0)| > 0 \). Therefore from last equality we come either to the relation \( q_1 = p_1 \), or to the relation \( q_1 = \overline{p}_1 \).

In the first case the identity (6.15) takes place. From it we receive identities

\[
 \psi(p_1v^2) = p_1^{l} \psi(v^2) \quad \text{for all } v^2 \in M_2, \quad \text{for all } l \in \mathbb{Z},
\]

holomorphic differentiating which on \( v^2 \), we come to identities

\[
 D_{v^2} \psi(p_1v^2) = D_{v^2} \psi(v^2) \quad \text{for all } v^2 \in M_2, \quad \text{for all } l \in \mathbb{Z}.
\]

Owing to that transformation \( \psi \) is a diffeomorphism, we receive identity

\[
 D_{v^2} \psi(v^2) = a \quad \text{for all } v^2 \in M_2.
\]

From here taking into account the relation (6.15) we come to the first representation from the given lemma.

Similarly in the second case it is received the second representation from Lemma 6.5.

**Theorem 6.10.** Let the conditions of the lemma 6.4 are satisfied. Then for a smooth conjunction of non-Abelian linear-fractional groups PL1 and PL2 of general situation it is necessary and sufficient them conjunction, which is carried out by either nondegenerate linear-fractional transformation, or nondegenerate antiholomorphic linear-fractional transformation.

Proof. The necessity. As well as at the proof of the lemma 6.4, from identities (6.1) we pass to identities (6.14). Further owing to the theorem 6.2 we come to conclusion, that take place either relations (6.5) at \( r = 1 \), or relations (6.6) at \( r = 1 \).

Let relations (6.5) are carried out at \( r = 1 \) (a case of performance of relations (6.6) at \( r = 1 \) is considered similarly). We will consider the affine card \( M_{n+1} \) of the atlas \( M \) of projective space \( \mathbb{C}P^n \). Owing to a course of the proof of the theorem 6.2 and theorems 3.4
smooth conjunction of narrowings of linear-fractional groups \( PL^1 \) and \( PL^2 \) on this card is carried out by nondegenerate linear \( \mathbb{R} \)-holomorphic transformation. By direct calculations on the basis of parities (6.5) at \( r = 1 \) we are convinced, that \( \alpha = 0 \) and that nondegenerate linear transformation is holomorphic, as keeping eigenvalues of matrices. With the help of linear-fractional functions of transition between affine cards of the atlas \( M \) on the basis of the aforesaid nondegenerate linear transformation we receive nondegenerate linear-fractional transformation. It also will be required conjugating diffeomorphism.

The sufficiency is checked by direct calculations.

Similarly to theorems 6.9 and 6.10 it is received following statements.

**Theorem 6.11.** Let the conditions of the lemma 6.3 are satisfied. Then for a holomorphic conjunction of non-Abelian linear-fractional groups \( PL^1 \) and \( PL^2 \) of general situation it is necessary and sufficient them conjunction, which is carried out by nondegenerate linear-fractional transformation.

**Theorem 6.12.** Let the conditions of the lemma 6.4 are satisfied. Then for a holomorphic conjunction of non-Abelian linear-fractional groups \( PL^1 \) and \( PL^2 \) of general situation it is necessary and sufficient them conjunction, which is carried out by nondegenerate linear-fractional transformation.

7. Applications to the complex nonautonomous projective matrix Riccati equations

Theorems 6.1 — 6.3, 6.5 — 6.12, Lemmas 6.3 and 6.4 give the chance, being based on Theorems 1.1 — 1.3 to make topological, smooth, \( \mathbb{R} \)-holomorphic, and holomorphic classifications of complex nonautonomous homogeneous projective matrix Riccati equations of a kind (5.1), and from Theorem 6.4 is received the following statement.

**Theorem 7.1.** From topological equivalence of complex nonautonomous homogeneous projective matrix Riccati equations with non-Abelian phase groups of general situation follows them \( \mathbb{R} \)-holomorphic equivalence.

As well as in point 4, from this theorem it is had, that complex nonautonomous homogeneous projective matrix Riccati equations with coefficients, holomorphic on path connected holomorphic varieties with non-Abelian fundamental groups, are structurally unstable.

Consider ordinary homogeneous projective scalar Riccati equations (i.e. ordinary homogeneous projective matrix Riccati equations at \( n = 1 \))

\[
\frac{dv}{dz} = \mathfrak{A}_1(z)v
\]  

(7.1)

and

\[
\frac{dv}{dz} = \mathfrak{B}_1(z)v,
\]  

(7.2)

where square matrices \( \mathfrak{A}_1(z) = \|a_{lk1}(z)\| \) and \( \mathfrak{B}_1(z) = \|b_{lk1}(z)\| \) of the size 2 consist from holomorphic functions \( a_{lk1}: \Gamma^v_1 \rightarrow \mathbb{C} \) and \( b_{lk1}: \Gamma^v_2 \rightarrow \mathbb{C}, l = 1, 2, k = 1, 2. \) The common solutions of ordinary homogeneous projective scalar Riccati equations (7.1) and (7.2) define covering foliation \( \mathfrak{B}^3 \) and \( \mathfrak{B}^4 \), accordingly, on the varieties \( \mathbb{C}P^1 \times \Gamma^v_1 \) and \( \mathbb{C}P^1 \times \Gamma^v_2. \)

The phase group \( Ph(\mathfrak{B}^3) \) of covering foliation \( \mathfrak{B}^3 \) is generated by nondegenerate linear-fractional transformations \( P_v \) for all \( v \in \mathbb{C}P^1, P_r \in GL(2, \mathbb{C}), r = 1, \ldots, \nu_1, \) and phase group \( Ph(\mathfrak{B}^4) \) of covering foliation \( \mathfrak{B}^4 \) is generated by nondegenerate linear-fractional transformations \( Q_v \) for all \( v \in \mathbb{C}P^1, Q_r \in GL(2, \mathbb{C}), r = 1, \ldots, \nu_2. \)

In a commutative case on the basis of Theorems 6.1 and 1.2 we receive the statement.

**Theorem 7.2.** Let matrices

\[
P_r = S \text{diag}\{p_{1r}, p_{2r}\} S^{-1}, \ r = 1, \ldots, \nu_1, \quad Q_r = T \text{diag}\{q_{1r}, q_{2r}\} T^{-1}, \ r = 1, \ldots, \nu_2, \ \nu_1 \leq \nu_2.
\]

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Then for embedding of ordinary homogeneous projective scalar Riccati equation (7.1) in ordinary homogeneous projective scalar Riccati equation (7.2) it is necessary and enough existence of such bijective map \( \varphi : (1, \ldots, \nu_2) \to \Delta \) and complex number \( \alpha \) with \( \Re \alpha \neq -1 \), that either

\[
\frac{q_{1\varphi(r)}}{q_{2\varphi(r)}} = \frac{p_{1r}}{p_{2r}} \left| \frac{p_{1r}}{p_{2r}} \right|^\alpha, \quad r = 1, \ldots, \nu_1,
\]
or

\[
\frac{q_{1\varphi(r)}}{q_{2\varphi(r)}} = \overline{\frac{p_{1r}}{p_{2r}}} \left| \frac{p_{1r}}{p_{2r}} \right|^\alpha, \quad r = 1, \ldots, \nu_1,
\]

where the set \( \Delta \) consists from \( \nu_1 \) numbers \( \{1, \ldots, \nu_2\} \).

In a noncommutative case on the basis of Lemma 6.3 and Theorem 1.2 it is had concrete constructive criteria of embedding of ordinary homogeneous projective scalar equations.

And, at last, we will consider ordinary homogeneous projective scalar Riccati equations (7.1) and (7.2) in a case when square matrices \( \mathfrak{A}_1(z) = ||a_{lk1}(z)|| \) and \( \mathfrak{B}_1(z) = ||b_{lk1}(z)|| \) of the size of 2 consist of 1-periodic holomorphic functions \( a_{lk1} : \mathbb{C} \to \mathbb{C} \) and \( b_{lk1} : \mathbb{C} \to \mathbb{C} \), and, besides, functions \( b_{lk1} \) are such that

\[
b_{lk1}(x + i) = b_{lk1}(x) \quad \text{for all} \quad x \in [0, 1], \quad l = 1, 2, \quad k = 1, 2.
\]

The common solutions of ordinary homogeneous projective scalar Riccati equations (7.1) and (7.2) define covering foliations \( \mathfrak{P}^5 \) and \( \mathfrak{P}^6 \), accordingly, on the varieties \( \mathbb{C}P^1 \times \mathbb{Z} \) and \( \mathbb{C}P^n \times T^2 \), where \( T^2 \) is a torus, defined by evolvement

\[
K = \{ z = x + iy \in \mathbb{C} : x \in [0, 1], \quad y \in [0, 1] \}.
\]

The phase group \( \text{Ph}(\mathfrak{P}^5) \) of covering foliation \( \mathfrak{P}^5 \) is generated by nondegenerate linear-fractional transformation \( P_1v \) for all \( v \in \mathbb{C}P^1 \), \( P_1 \in \text{GL}(2, \mathbb{C}) \), and the phase group \( \text{Ph}(\mathfrak{P}^6) \) of covering foliation \( \mathfrak{P}^6 \) is generated by nondegenerate linear-fractional transformations \( Q_rv \) for all \( v \in \mathbb{C}P^1 \), \( Q_r \in \text{GL}(2, \mathbb{C}) \), \( r = 1, 2 \). We will consider, that the cylinder \( \mathbb{Z} \) covers the torus \( T^2 \), under a condition, that the band of \( 0 \leq \Re z \leq 1 \) covers the square \( K \).

Owing to Theorems 6.1 and 1.3 we have the statement.

**Theorem 7.3.** Let matrices

\[
P_1 = S \text{diag} \{p_{11}, p_{21}\} S^{-1}, \quad Q_r = T \text{diag} \{q_{1r}, q_{2r}\} T^{-1}, \quad r = 1, 2.
\]

Then for covering of the projective scalar Riccati equation (7.2) by the projective scalar Riccati equation (7.1) it is necessary and enough existence of such index \( r \in \{1, 2\} \) and corresponding to it complex numbers \( \alpha_r \) with \( \Re \alpha_r \neq -1 \), that either

\[
\frac{q_{1r}}{q_{2r}} = \frac{p_{11}}{p_{21}} \left| \frac{p_{11}}{p_{21}} \right|^\alpha_r, \quad \text{or} \quad \frac{q_{1r}}{q_{2r}} = \overline{\frac{p_{11}}{p_{21}}} \left| \frac{p_{11}}{p_{21}} \right|^\alpha_r.
\]

**8. Phase groups of covering foliations, defined by real nonautonomous linear differential systems**

We will consider linear differential systems

\[
dx = \sum_{j=1}^m A_j(t_1, \ldots, t_m) x \, dt_j \tag{8.1}
\]

and

\[
dx = \sum_{j=1}^m B_j(t_1, \ldots, t_m) x \, dt_j, \tag{8.2}
\]
ordinary at \( m = 1 \) and completely solvable at \( m > 1 \), where \( x = (x_1, \ldots, x_n) \), square matrices \( A_j(t_1, \ldots, t_m) = \|a_{ikj}(t_1, \ldots, t_m)\| \) and \( B_j(t_1, \ldots, t_m) = \|b_{ikj}(t_1, \ldots, t_m)\| \) of the size \( n \) consist from holomorphic functions

\[
a_{ikj}: A \to \mathbb{R} \quad \text{and} \quad b_{ikj}: B \to \mathbb{R}, \quad i = 1, \ldots, n, \quad k = 1, \ldots, n, \quad j = 1, \ldots, m,
\]

path connected holomorphic varieties \( A \) and \( B \) are holomorphically equivalent each other, fundamental groups \( \pi_1(A) \) and \( \pi_1(B) \) have final number \( \nu \in \mathbb{N} \) of the forming.

The common solutions of linear differential systems (8.1) and (8.2) define covering foliations \( \mathcal{L}^9 \) and \( \mathcal{L}^{10} \), accordingly, on varieties \( \mathbb{R}^n \times A \) and \( \mathbb{R}^n \times B \).

We will say, that the linear differential systems (8.1) and (8.2) are \textit{topologically (smoothly, holomorphically) equivalent} if exists the homeomorphism (the diffeomorphism, the holomorphism) \( h: \mathbb{R}^n \times A \to \mathbb{R}^n \times B \), translating the layers of the covering foliation \( \mathcal{L}^9 \) in the layers of the covering foliation \( \mathcal{L}^{10} \). Similarly we introduce the concepts of \textit{embedding (smooth embedding, holomorphic embedding)} and covering (smoothly covering, holomorphically covering) of linear differential systems.

The phase group \( Ph(\mathcal{L}^9) \) of the covering foliation \( \mathcal{L}^9 \) is generated by the forming non-degenerate linear transformations \( P_r w \) for all \( x \in \mathbb{R}^n \), \( P_r \in GL(n, \mathbb{R}) \), \( r = 1, \ldots, \nu \), and the phase group \( Ph(\mathcal{L}^{10}) \) of the covering foliation \( \mathcal{L}^{10} \) is generated by the forming non-degenerate linear transformations \( Q_r x \) for all \( x \in \mathbb{R}^n \), \( Q_r \in GL(n, \mathbb{R}) \), \( r = 1, \ldots, \nu \).

\section*{9. Conjunctions of linear actions on \( \mathbb{R}^n \)}

We will consider a problem about a finding of necessary and sufficient conditions of existence such homeomorphism (diffeomorphism, holomorphism) \( f: \mathbb{R}^n \to \mathbb{R}^n \), that identities

\[
f(P_r x) = Q_r f(x) \quad \text{for all} \quad x \in \mathbb{R}^n, \quad \text{for all} \quad r \in I, \tag{9.1}
\]

take place, where \( f(x) = (f_1(x), \ldots, f_n(x)) \), square matrices \( P_r, Q_r \in GL(n, \mathbb{R}) \) for all \( r \in I \).

Group of linear actions on \( \mathbb{R}^n \), formed by matrices \( P_r \) for all \( r \in I \), we will designate through \( L^3 \), and through \( L^4 \) we will designate the similar group, formed by matrices \( Q_r \) for all \( r \in I \). Besides, further everywhere \textit{strongly hyperbolic} we will name matrices at which all own values are various among themselves and on the module are distinct from 1.

\textbf{Theorem 9.1.} For the topological conjunction (9.1) at \( n = 1 \) of linear groups \( L^3 \) and \( L^4 \) it is necessary and enough, that

\[
q_r = p_r |p_r|^\alpha \quad \text{for all} \quad r \in I, \quad \alpha > -1. \tag{9.2}
\]

\textbf{Proof.} The necessity. We will consider at first a case \( I = \{1\} \). If \( p_1 = 1 \), that from (9.1) follows, that \( q_1 = 1 \), and, hence, the relation (9.2) is carried out.

If \( p_1 = -1 \), that on the basis of (9.1) it is had, that \( f(p_1^2 x) = q_1^2 f(x) \) for all \( x \in \mathbb{R} \), and, therefore \( q_1^2 = 1 \). At \( q_1 = 1 \) the coordination of orientations of maps in different parts of identity (9.1) is broken, owing to what \( q_1 = -1 \). Hence, and in this case the relation (9.2) takes place.

Let now \( |p_1| \neq 1 \). Then from (9.1) follows, as \( |q_1| \neq 1 \). And, means, exists \( \alpha \neq -1 \), that \( q_1 = p_1 |p_1|^\alpha \) or \( q_1 = -p_1 |p_1|^\alpha \).

If \( q_1 = p_1 |p_1|^\alpha \), that of (9.1) follows, that \( f(p_1^{2k} x) = p_1^{2(k+\alpha)} f(1) \) for all \( k \in \mathbb{Z} \). We will admit, that \( \alpha < -1 \). Now we will pass in this equality to a limit: in the case \( |p_1| > 1 \) at \( k \to +\infty \), and in the case \( |p_1| < 1 \) at \( k \to -\infty \). Every time we will receive the contradiction. Therefore in the case \( q_1 = p_1 |p_1|^\alpha \) number \( \alpha > -1 \).

If \( q_1 = -p_1 |p_1|^\alpha \), that is broken a coordination of orientation of maps in different parts of identity (9.1). Uniting considered above possibility, we come to conclusion about justice of a relation (9.2) at \( I = \{1\} \).
Consider now the case \( I \neq \{1\} \). Owing to the proof of the previous part of the given statement it is had, that \( q_r = p_r |p_r|^{\alpha_r}, \alpha_r > -1 \), for all \( r \in I \), and thus \( |p_r| = 1 \) in only case when \( |q_r| = 1 \) for all \( I \).

Through \( I_1 \) we will designate set of such indexes \( r \), that \( |p_r| = |q_r| = 1 \).

We will consider 3 logic possibilities, when addition \( CI_1 \) of sets \( I_1 \) to set \( I \): 1) is empty; 2) consists of one index; 3) consists of more than one index.

If \( CI_1 = \emptyset, I = I_1 \), and \( |p_r| = |q_r| = 1 \) for all \( r \in I \), that relations (9.1) take place at any real \( \alpha \). If \( CI_1 = \{r\} \), that in relations (9.1) can be put \( \alpha = \alpha_r \).

Let \( |p_{r_1}| \neq 1, |p_{r_2}| \neq 1, r_1 \neq r_2, r_1 \in CI_1, r_2 \in CI_1 \). Then from (9.1) follows, that

\[
f(p_{r_1}^l p_{r_2}^n x) = q_{r_1}^l q_{r_2}^n f(x) \quad \text{for all} \quad x \in \mathbb{R}, \quad \text{for all} \quad l \in \mathbb{Z}, \quad \text{for all} \quad n \in \mathbb{Z}.
\]

Owing to the that the set of rational numbers is everywhere dense on set of real numbers, we conclude about existence of such sequences \( \{l_s\} \) and \( \{n_s\} \) of integers, that

\[
\lim_{s \to +\infty} p_{r_1}^{2l_s} p_{r_2}^{2n_s} = 1, \quad \lim_{s \to +\infty} |l_s| = \lim_{s \to +\infty} |n_s| = +\infty.
\]

Then

\[
f(x) = \lim_{s \to +\infty} |p_{r_1}|^{2l_s(\alpha_{r_1} - \alpha_{r_2})} f(x) \quad \text{for all} \quad x \in \mathbb{R}.
\]

From here follows, that \( \alpha_{r_1} = \alpha_{r_2} = \alpha^* > -1 \). Therefore in relations (9.2) for \( \alpha \) it is possible to take \( \alpha^* \).

The sufficiency is proved by construction of conjugating homeomorphism

\[
f: x \to x|x|^\alpha \quad \text{for all} \quad x \in \mathbb{R}, \quad \alpha > -1. \]

**Theorem 9.2.** Let at \( n > 1 \) and \( I = \{1\} \) real normal Jordan form of strongly hyperbolic matrix \( P_1 \in GL(n, \mathbb{R}) \) looks like

\[
J(P_1) = \begin{pmatrix} J_s(P_1) & 0 \\ 0 & J_u(P_1) \end{pmatrix},
\]

where all eigenvalues of the matrix \( J_s(P_1) \) on the module are less than 1, and all eigenvalues of the matrix \( J_u(P_1) \) on the module are more than 1, \( p = \dim J_s(P_1) \); real normal Jordan form of strongly hyperbolic matrix \( Q_1 \in GL(n, \mathbb{R}) \) looks like

\[
J(Q_1) = \begin{pmatrix} J_s(Q_1) & 0 \\ 0 & J_u(Q_1) \end{pmatrix},
\]

where all eigenvalues of the matrix \( J_s(Q_1) \) on the module are less than 1, and all eigenvalues of the matrix \( J_u(Q_1) \) on the module are more than 1, \( q = \dim J_s(Q_1) \). Then linear groups \( L^3 \) and \( L^4 \) are topological conjugated in only if, when

\[
p = q, \quad \det J_s(P_1) \det J_s(Q_1) > 0, \quad \det J_u(P_1) \det J_u(Q_1) > 0.
\]

**Proof.** Let the identities (9.1) take place at \( I = \{1\} \).

It is easy to see, that for linear actions the dimensions of stable both unstable invariant subspaces and orientations (positive or negative) of narrowings on these invariant subspaces are invariant at a topological conjuction. Therefore for the proof of the theorem it is enough to show a conjunction between real normal Jordan forms of the above-stated linear actions and the linear action defined by one of canonical matrices \( \text{diag}\{e^{-1}, e^{-1}, \ldots, e^{-1}, e, \ldots, e, \delta e\} \), where \( \varepsilon^2 = 1, \delta^2 = 1 \) (such linear action we will name canonical).

For proof end we will resort conjugating to canonical linear actions homeomorphisms of spaces \( \mathbb{R} \) and \( \mathbb{R}^2 \) for narrowings on stable and unstable invariant subspaces of the linear
action, corresponding to strongly hyperbolic matrix:

1) **homeomorphism**

\[ x_1 \rightarrow x_1|\ln |x_1||^{-1} \text{ for all } x_1 \in \mathbb{R} \]

conjugates the linear action \( \lambda_1 x_1 \) for all \( x_1 \in \mathbb{R} \), \( 0 < \lambda_1 < 1 \), with the linear action \( e^{-1}x_1 \) for all \( x_1 \in \mathbb{R} \);

2) **homeomorphism**

\[ x_1 \rightarrow x_1|\ln |x_1||^{-1} \text{ for all } x_1 \in \mathbb{R} \]

conjugates the linear action \( \lambda_1 x_1 \) for all \( x_1 \in \mathbb{R} \), \( -1 < \lambda_1 < 0 \), with the linear action \( -e^{-1}x_1 \) for all \( x_1 \in \mathbb{R} \);

3) **homeomorphism**

\[ x_1 \rightarrow x_1|\ln |x_1||^{-1} \text{ for all } x_1 \in \mathbb{R} \]

conjugates the linear action \( \lambda_1 x_1 \) for all \( x_1 \in \mathbb{R} \), \( \lambda_1 > 1 \), with the linear action \( ex_1 \) for all \( x_1 \in \mathbb{R} \);

4) **homeomorphism**

\[ x_1 \rightarrow x_1|\ln |x_1||^{-1} \text{ for all } x_1 \in \mathbb{R} \]

conjugates the linear action \( \lambda_1 x_1 \) for all \( x_1 \in \mathbb{R} \), \( \lambda_1 < -1 \), with the linear action \( -ex_1 \) for all \( x_1 \in \mathbb{R} \);

5) **homeomorphism**

\[ (x_1, x_2) \rightarrow \left( x_1 \cos \left( \pi \ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right) \right) - x_2 \sin \left( \pi \ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right) \right), x_1 \sin \left( \frac{\alpha \arctg \beta}{\ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right)} \right) + x_2 \cos \left( \frac{\alpha \arctg \beta}{\ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right)} \right) \right) \text{ for all } (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}, \ (0, 0) \rightarrow (0, 0) \]

conjugates the linear action \( -e^{-1}Ix \) for all \( x \in \mathbb{R}^2 \), where \( I \) there is an identity matrix of the second order, with the linear action \( e^{-1}Ix \) for all \( x \in \mathbb{R}^2 \); and also conjugates the linear action \( -eIx \) for all \( x \in \mathbb{R}^2 \) with the linear action \( eIx \) for all \( x \in \mathbb{R}^2 \);

6) **homeomorphism**

\[ (x_1, x_2) \rightarrow \left( x_1 \cos \left( \frac{\alpha \arctg \beta}{\ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right)} \right) - x_2 \sin \left( \frac{\alpha \arctg \beta}{\ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right)} \right), x_1 \sin \left( \frac{\alpha \arctg \beta}{\ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right)} \right) + x_2 \cos \left( \frac{\alpha \arctg \beta}{\ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right)} \right) \right)^{-1} \text{ for all } (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}, \ (0, 0) \rightarrow (0, 0) \]

conjugates the linear action \( \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) x \) for all \( x \in \mathbb{R}^2 \), \( \sqrt{\alpha^2 + \beta^2} < 1 \), \( \beta \neq 0 \), with the linear action \( e^{-1}Ix \) for all \( x \in \mathbb{R}^2 \);

7) **homeomorphism**

\[ (x_1, x_2) \rightarrow \left( x_1 \cos \left( \frac{\alpha \arctg \beta}{\ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right)} \right) - x_2 \sin \left( \frac{\alpha \arctg \beta}{\ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right)} \right), x_1 \sin \left( \frac{\alpha \arctg \beta}{\ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right)} \right) + x_2 \cos \left( \frac{\alpha \arctg \beta}{\ln \left( \frac{x_1^2 + x_2^2}{\alpha^2 + \beta^2} \right)} \right) \right) \]
$x_1 \sin \left( \frac{\arctg \beta}{\ln \sqrt{\alpha^2 + \beta^2}} \ln \sqrt{x_1^2 + x_2^2} \right) + x_2 \cos \left( \frac{\arctg \beta}{\ln \sqrt{\alpha^2 + \beta^2}} \ln \sqrt{x_1^2 + x_2^2} \right) \ln \sqrt{\alpha^2 + \beta^2}^{-1}$

for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $(0,0) \to (0,0)$

conjugates the linear action $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} x$ for all $x \in \mathbb{R}^2$, $\sqrt{\alpha^2 + \beta^2} > 1$, $\beta \neq 0$, with the linear action $e^{ix}$ for all $x \in \mathbb{R}^2$. ■

Let's consider now a topological associativity of Abelian linear groups $L^3$ and $L^4$ at $n > 1$ and $I \neq \{1\}$. Thus everywhere we will suppose further, that matrices $P_r$ and $Q_r$ for all $r \in I$, are strongly hyperbolic. In this case all matrices $P_r$ (matrices $Q_r$) are reduced to real normal Jordan forms $J(P_r)$ (real normal Jordan forms $J(Q_r)$), for all $r \in I$, by the general homothetic transformation and have an identical order of a disposition of real blocks of Jordan corresponding to real and complex eigenvalues. Further by means of replacement $\xi(x) = T^{-1}f(Sx)$ for all $x \in \mathbb{R}^n$, where $P_r = SJ(P_r)S^{-1}$, $Q_r = TJ(Q_r)T^{-1}$ for all $r \in I$, from identities (9.1) we will pass to identities

$$\xi(J(P_r)x) = J(Q_r)\xi(x) \quad \text{for all } x \in \mathbb{R}^n, \text{ for all } r \in I. \quad (9.3)$$

Therefore the topological associativity of Abelian linear groups $L^3$ and $L^4$ is equivalent to performance of identities (9.3).

Owing to a course of the proof of Theorem 9.2 we come to conclusion, that the space $\mathbb{R}^n$ can be divided into the direct sum of $s$ co-ordinate subspaces $\mathbb{R}^l_k$ dimensions $\dim \mathbb{R}^l_k = l_k$ such that they are invariant steady or invariant unstable for contractions $J_k(P_r)x^k$ for all $x^k \in \mathbb{R}^l_k$ and $J_k(Q_r)x^k$ for all $x^k \in \mathbb{R}^l_k$ accordingly, on these invariant subspaces, and the given contractions have identical orientation, for all $r \in I$; thus $x = (x^1, \ldots, x^s)$,

$$J(P_r) = \text{diag}\{J_1(P_r), \ldots, J_s(P_r)\}, \quad J(Q_r) = \text{diag}\{J_1(Q_r), \ldots, J_s(Q_r)\} \quad \text{for all } r \in I, \sum_{k=1}^s l_k = n.$$

Now let us consider the invariant subspace $\mathbb{R}^l_k$. For the homeomorphism-narrowing $\xi_k: \mathbb{R}^l_k \to \mathbb{R}^l_k$ of the homeomorphism $\xi: \mathbb{R}^n \to \mathbb{R}^n$ on this invariant subspace from identities (9.3) follow the identities

$$\xi_k(J_k(P_r)x^k) = J_k(Q_r)\xi_k(x^k) \quad \text{for all } x^k \in \mathbb{R}^l_k, \text{ for all } r \in I.$$

Not belittling a generality, we will consider, that the invariant subspace $\mathbb{R}^l_k$ is unstable for linear maps $J_k(P_1)x^k$ for all $x^k \in \mathbb{R}^l_k$ and $J_k(Q_1)x^k$ for all $x^k \in \mathbb{R}^l_k$ (for in case of a stability of an invariant subspace $\mathbb{R}^l_k$ it is enough to consider linear maps $J_k^{-1}(P_1)x^k$ for all $x^k \in \mathbb{R}^l_k$ and $J_k^{-1}(Q_1)x^k$ for all $x^k \in \mathbb{R}^l_k$). Using the homeomorphisms $u_k: \mathbb{R}^l_k \to \mathbb{R}^l_k$ and $v_k: \mathbb{R}^l_k \to \mathbb{R}^l_k$ of aspects 1) – 7) from the proof of the theorem 9.2, we will reduce linear maps $J_k(P_1)x^k$ for all $x^k \in \mathbb{R}^l_k$ and $J_k(Q_1)x^k$ for all $x^k \in \mathbb{R}^l_k$ accordingly, in one of two canonical (see the proof of the theorem 9.2) aspects: $ex^k$ for all $x^k \in \mathbb{R}^l_k$ (in case of positive orientation of aforementioned linear maps), or $\text{diag}\{e, \ldots, e, -e\} x^k$ for all $x^k \in \mathbb{R}^l_k$ (in case of their negative orientation). Thus, taking into account the disposition of real blocks of Jordan noted before an order, by direct evaluations we are convinced, that images of maps $J_k(P_r)x^k$ for all $x^k \in \mathbb{R}^l_k$ and $J_k(Q_r)x^k$ for all $x^k \in \mathbb{R}^l_k$ keep linearity, and matrices thus corresponding to again received linear maps are real normal Jordan forms of the same structure, as real normal Jordan forms $J_k(P_r)$ and $J_k(Q_r)$, accordingly, for all $r \in I \setminus \{1\}$. 

\[ \text{V.N. Gorbuzov, V.Yu. Tyshchenko} \]
Following auxiliary statements allow to solve a problem about a topological associativity for Abelian linear phase groups $L^3$ and $L^4$ of general situation.

**Lemma 9.1.** Let at $I = \{1, 2\}$ the matrices $P_1 = Q_1 = eI$, and the matrices $P_2$ and $Q_2$ are represented by real normal Jordan forms. Then at a topological conjunction of linear groups $L^3$ and $L^4$ the matrices $P_2$ and $Q_2$ have identical real structure.

Proof. Let linear groups $L^3$ and $L^4$ are topologically conjugated, i.e. identities

$$\xi(e \cdot x) = e \cdot \xi(x) \quad \text{for all} \quad x \in \mathbb{R}^n,$$  \hspace{1cm} (9.4)

and

$$\xi(P_2 \cdot x) = Q_2 \cdot \xi(x) \quad \text{for all} \quad x \in \mathbb{R}^n.$$  \hspace{1cm} (9.5)

take place. On the basis of (9.4) it is received, that

$$\frac{\xi_k(e \cdot x)}{\xi_n(e \cdot x)} \equiv \frac{\xi_k(x)}{\xi_n(x)}, \quad k = 1, \ldots, n - 1,$$

and, means, are fair representations

$$\frac{\xi_k(x)}{\xi_n(x)} \equiv g_k \left( \frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n} \right), \quad k = 1, \ldots, n - 1,$$  \hspace{1cm} (9.6)

as $\frac{x_k}{x_n}, \quad k = 1, \ldots, n - 1$, is a base of nondegenerate absolute invariants of map $e \cdot x$ for all $x \in \mathbb{R}^n$. Owing to relations (9.4) — (9.6) we come to conclusion, that the topological conjunction

$$h(P_2 \cdot v) = Q_2 \cdot h(v) \quad \text{for all} \quad v \in \mathbb{R}^n,$$  \hspace{1cm} (9.7)

implies from a topological conjunction of linear groups $L^3$ and $L^4$, where

$$h(v) = (h_1(v), \ldots, h_n(v)).$$

Now justice of the statement of Lemma 9.1 implies from this the fact, that the number of fixed points of linear-fractional map $P_2 \cdot v$ for all $v \in \mathbb{R}^{n-1}$ (linear-fractional map $Q_2 \cdot v$ for all $v \in \mathbb{R}^{n-1}$) coincides with number of eigenvectors of the matrix $P_2$ (the matrix $Q_2$).

**Lemma 9.2.** Let at $I = \{1, 2\}$ the matrices $P_1 = Q_1 = eI$, and the matrices

$$P_2 = \text{diag} \left\{ \begin{pmatrix} \alpha_{12} & \beta_{12} \\ -\beta_{12} & \alpha_{12} \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_{s2} & \beta_{s2} \\ -\beta_{s2} & \alpha_{s2} \end{pmatrix}, p_{2s+1,2}, \ldots, p_{n2} \right\}$$

and

$$Q_2 = \text{diag} \left\{ \begin{pmatrix} \gamma_{12} & \delta_{12} \\ -\delta_{12} & \gamma_{12} \end{pmatrix}, \ldots, \begin{pmatrix} \gamma_{s2} & \delta_{s2} \\ -\delta_{s2} & \gamma_{s2} \end{pmatrix}, q_{2s+1,2}, \ldots, q_{n2} \right\}$$

are strongly hyperbolic,

$$\pi^{-1} \arg(\alpha_{k2} + i\beta_{k2}) \notin \mathbb{Q}, \quad \pi^{-1} \arg(\gamma_{k2} + i\delta_{k2}) \notin \mathbb{Q}, \quad \frac{\ln(\alpha_{k2} + i\beta_{k2})}{\ln(\alpha_{l2} + i\beta_{l2})} \notin \mathbb{Q}, \quad \frac{\ln(\gamma_{k2} + i\delta_{k2})}{\ln(\gamma_{l2} + i\delta_{l2})} \notin \mathbb{Q},$$

$$l \neq k, \quad k = 1, \ldots, s, \quad l = 1, \ldots, s.$$

Then for a topological conjunction of linear groups $L^3$ and $L^4$ it is necessary and enough existence of such permutations $\rho: (1, \ldots, s) \to (1, \ldots, s), \chi: (2s+1, \ldots, n) \to (2s+1, \ldots, n)$ that

$$\gamma_{\rho(k)2} = \alpha_{k2}, \quad \delta_{\rho(k)2} = \beta_{k2}, \quad k = 1, \ldots, s; \quad q_{\chi(k)2} = p_{k2}, \quad k = 2s+1, \ldots, n.$$  \hspace{1cm} (9.8)

Proof. The necessity. As well as at the proof of the lemma 9.1, we receive relations (9.4) — (9.7). On their foundation we come to conclusion, that the conjugating homeomor-
phism $L_k = \{ v, v_\tau = 0, \tau = 1, \ldots, n, \tau \neq 2k - 1, \tau \neq 2k \}, k = 1, \ldots, s$, of linear-fractional maps $P_2 v$ for all $v \in \mathbb{RP}^{n-1}$ and $Q_2 v$ for all $v \in \mathbb{RP}^{n-1}$. It implies from this, that:

1) $\mathcal{O}(v) = L_k$ for all $v \in L_k, k = 1, \ldots, s$, where $O$ there is an orbit of a corresponding point at operations of the previous linear-fractional maps (it is proved on the basis of a consequence from the theorem of Kronecker [31] similarly course of the proof of Theorem 6.12);

2) for remaining possible arcwise connected one-dimensional invariant sets $I^1$, homeomorphic to projective straight lines, property $\mathcal{O}(v) = I^1$ for all $v \in I^1$ is not fulfilled (proved by reviewing of operations of the previous linear-fractional maps on bases of nondegenerate absolute invariants)

$$
\left( \frac{v_{2k}^2 + v_{2k}^2}{v_{2s-1}^2 + v_{2s}^2} \right)^{-\arctg \frac{\beta_{2s}^2}{\alpha_{2s}} - \arctg \frac{\beta_{2s}^2}{\alpha_{2s}}} \exp \left( \frac{\alpha_{2s}^2 + \beta_{2s}^2}{\alpha_{2s}^2 + \beta_{2s}^2} \arctg \frac{v_{2s}}{v_{2s-1}} \right),
$$

$$
- \arctg \frac{\beta_{2s}^2}{\alpha_{2s}} \arctg \frac{v_{2s}}{v_{2s-1}} + \arctg \frac{\beta_{2s}^2}{\alpha_{2s}} v_{2s}, \quad k = 1, \ldots, s;
$$

$$
\left( \frac{v_{2s-1}^2 + v_{2s}^2}{v_n^2} \right)^{-\arctg \frac{\beta_{2s}^2}{\alpha_{2s}}} \exp \left( \frac{\alpha_{2s}^2 + \beta_{2s}^2}{p_{n-1,2}^2} \arctg \frac{v_{2s}}{v_{2s-1}} \right);
$$

$$
\left( \frac{v_{2s-1}^2 + v_{2s}^2}{v_n^2} \right)^{-\arctg \frac{\beta_{2s}^2}{\alpha_{2s}}} \exp \left( \frac{\alpha_{2s}^2 + \beta_{2s}^2}{p_{n-1,2}^2} \arctg \frac{v_{2s}}{v_{2s-1}} \right);
$$

$$
\left| \frac{p_{n-1,2}}{|p_{n-1,2}|} \right| \left| v_{n-1} \right| \left| v_n \right|, \quad k = 2s + 1, \ldots, n - 2;
$$

and

$$
\left( \frac{v_{2k}^2 + v_{2k}^2}{v_{2s-1}^2 + v_{2s}^2} \right)^{-\arctg \frac{\beta_{2s}^2}{\alpha_{2s}}} \exp \left( \frac{\alpha_{2s}^2 + \beta_{2s}^2}{q_{n-1,2}^2} \arctg \frac{v_{2s}}{v_{2s-1}} \right),
$$

$$
- \arctg \frac{\beta_{2s}^2}{\alpha_{2s}} \arctg \frac{v_{2s}}{v_{2s-1}} + \arctg \frac{\beta_{2s}^2}{\alpha_{2s}} v_{2s}, \quad k = 1, \ldots, s;
$$

$$
\left( \frac{v_{2s-1}^2 + v_{2s}^2}{v_n^2} \right)^{-\arctg \frac{\delta_{2s}^2}{\gamma_{2s}}} \exp \left( \frac{\gamma_{2s}^2 + \delta_{2s}^2}{q_{n-1,2}^2} \arctg \frac{v_{2s}}{v_{2s-1}} \right);
$$

$$
\left( \frac{v_{2s-1}^2 + v_{2s}^2}{v_n^2} \right)^{-\arctg \frac{\delta_{2s}^2}{\gamma_{2s}}} \exp \left( \frac{\gamma_{2s}^2 + \delta_{2s}^2}{q_{n-1,2}^2} \arctg \frac{v_{2s}}{v_{2s-1}} \right);
$$

$$
\left| \frac{q_{n-1,2}}{|q_{n-1,2}|} \right| \left| q_{n-1,2} \right| \left| q_n \right|, \quad k = 2s + 1, \ldots, n - 2;
$$

of zero degree of a homogeneity corresponding to them).

Introducing auxiliary variables $z_k = v_{2k-1} + iv_{2k}, k = 1, \ldots, s$, on the basis of Theorem 3.1 we come to the first part of relations (9.8).

Besides, the homeomorphism $h$ translates each other points

$$
O_k = \{ v, v_\tau = 0, \tau = 1, \ldots, n, \tau \neq k \}, \quad k = 2s + 1, \ldots, n
$$

(being fixed for linear-fractional maps $Q_2 v$ for all $v \in \mathbb{RP}^{n-1}$). Now by similar reasonings on the basis of (9.5) — (9.7) and Theorem 9.1 it is received the second part of relations (9.8).
The sufficiency is proved by direct evaluations application of conjugating homeomorphism
\[ \xi(x) = (x_{2(1)}^{-1}, x_{2(1)}^{-1}, \ldots, x_{2(s)}^{-1}, x_{2(s)}^{-1}, x_{x(n)}^{-1}) \] for all \( x \in \mathbb{R}^n \).

Following statements it is received on the basis of two previous, Theorem 9.1 and that fact, that for linear map \( \text{diag} \{e, \ldots, e, -e\} x \) for all \( x \in \mathbb{R}^n \) dimension of a maximum invariant subspace on which the contraction of this map has positive orientation, is an invariant at a topological conjunction.

**Lemma 9.3.** Let at \( I = \{1, 2\} \) the matrixes \( P_1 = Q_1 = \text{diag} \{e, \ldots, e, -e\} \), and the matrixes \( P_2 \) and \( Q_2 \) are represented by real normal Jordan forms. Then at a topological conjunction of linear groups \( L^3 \) and \( L^4 \) the matrixes \( P_2 \) and \( Q_2 \) have identical real structure.

**Lemma 9.4.** Let at \( I = \{1, 2\} \) the matrixes \( P_1 = Q_1 = \text{diag} \{e, \ldots, e, -e\} \), and the matrixes
\[
P_2 = \text{diag} \left\{ \begin{pmatrix} \alpha_{12} & \beta_{12} \\ -\beta_{12} & \alpha_{12} \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_{s2} & \beta_{s2} \\ -\beta_{s2} & \alpha_{s2} \end{pmatrix}, p_{2s+1,2}, \ldots, p_{n2} \right\}
\]
and
\[
Q_2 = \text{diag} \left\{ \begin{pmatrix} \gamma_{12} & \delta_{12} \\ -\delta_{12} & \gamma_{12} \end{pmatrix}, \ldots, \begin{pmatrix} \gamma_{s2} & \delta_{s2} \\ -\delta_{s2} & \gamma_{s2} \end{pmatrix}, q_{2s+1,2}, \ldots, q_{n2} \right\}
\]
are strongly hyperbolic,
\[
\pi^{-1} \arg(\alpha_{k2} + i\beta_{k2}) \notin \mathbb{Q}, \quad \pi^{-1} \arg(\gamma_{k2} + i\delta_{k2}) \notin \mathbb{Q},
\]
\[
\frac{\ln(\alpha_{k2} + i\beta_{k2})}{\ln(\alpha_{l2} + i\beta_{l2})} \notin \mathbb{Q}, \quad \frac{\ln(\gamma_{k2} + i\delta_{k2})}{\ln(\gamma_{l2} + i\delta_{l2})} \notin \mathbb{Q}, \quad k = 1, \ldots, s, \ l = 1, \ldots, s, \ l \neq k.
\]

Then for a topological conjunction of linear groups \( L^3 \) and \( L^4 \) it is necessary and enough existence of such permutations \( \rho: (1, \ldots, s) \rightarrow (1, \ldots, s), \ \chi: (2s+1, \ldots, n) \rightarrow (2s+1, \ldots, n) \) that relations (9.8) are fulfilled.

In case of smooth and holomorphic conjunctions of linear groups \( L^3 \) and \( L^4 \) the following statement takes place.

**Theorem 9.3.** Linear groups \( L^3 \) and \( L^4 \) are smoothly (holomorphic) conjugated in only case when they are linearly conjugated.

**Proof. The necessity.** Let identities (9.1) take place. Calculating in them Jacobi matrix in the point \( x = 0 \), we have matrix equalities \( D_x f(0)P_r = Q_r D_x f(0) \) for all \( r \in I \).

Therefore linear map \( D_x f(0)x \) for all \( x \in \mathbb{R}^n \) satisfies to identities (9.1). As \( f \) is a diffeomorphism (holomorphism), it is nondegenerate.

The sufficiency is checked by direct evaluations.

10. Applications to real nonautonomous linear differential systems

Theorems 9.1 — 9.3, and also the algorithm based on Lemmas 9.1 — 9.4, allows on the basis of Theorems 1.1 — 1.3, to spend topological, smooth and holomorphic classifications of real nonautonomous linear differential systems of the aspect (8.1).

Besides, on the basis of the received algorithm it is possible to draw a conclusion, that real completely solvable (i.e. at two and more independent variables) linear differential systems with periodic coefficients are structurally unstable (the commutativity of phase groups of the given class of differential systems implies from a commutativity of fundamental group of a many-dimensional torus).

We will notice, that the theorem 9.2 on the basis of the theorem 1.1 gives criterion of a structural stability of real linear ordinary differential systems with periodic coefficients.
11. Phase groups of covering foliations, defined by real nonautonomous Riccati equations

We will consider Riccati equations

$$dx = \sum_{j=1}^{m} \left( a_{ij}(t_1, \ldots, t_m) x^2 + a_{0j}(t_1, \ldots, t_m) x + a_{ij}(t_1, \ldots, t_m) \right) dt_j$$

(11.1)

and

$$dx = \sum_{j=1}^{m} \left( b_{ij}(t_1, \ldots, t_m) x^2 + b_{1j}(t_1, \ldots, t_m) x + b_{0j}(t_1, \ldots, t_m) \right) dt_j$$

(11.2)

ordinary at \( m=1 \) and completely solvable at \( m > 1 \), where holomorphic functions \( a_{ij} : A \to \mathbb{R} \) and \( b_{ij} : B \to \mathbb{R} \), \( i = 0, 1, 2, j = 1, \ldots, m \), path connected holomorphic varieties \( A \) and \( B \) are holomorphic equivalent each other, fundamental groups \( \pi_1(A) \) and \( \pi_1(B) \) have final number \( \nu \in \mathbb{N} \) of the forming.

The common solutions of Riccati equations (11.1) and (11.2) define covering foliations \( \mathcal{F}^7 \) and \( \mathcal{F}^8 \), accordingly, on varieties \( \mathbb{R} \times A \) and \( \mathbb{R} \times B \), where \( \mathbb{R} \) is a real straight line \( \mathbb{R} \), supplemented by the point at infinity \( \infty \).

We will say, that real Riccati equations (11.1) and (11.2) are topologically (smoothly, holomorphically) equivalent if exists a homeomorphism (a diffeomorphism, a holomorphism) \( h : \mathbb{R} \times A \to \mathbb{R} \times B \), translating the layers of the covering foliation \( \mathcal{F}^7 \) in the layers of the covering foliation \( \mathcal{F}^8 \). Similarly we enter concepts of embedding (smooth embedding, holomorphic embedding) and covering (smoothly covering, holomorphically covering) of Riccati equations.

The phase group \( Ph(\mathcal{F}^7) \) of the covering foliation \( \mathcal{F}^7 \) is generated by the forming non-degenerate linear-fractional transformations

$$P_r(x) = \frac{p_{1r}x + p_{2r}}{p_{3r}x + p_{4r}} \quad \text{for all} \quad x \in \mathbb{R}, \quad r = 1, \ldots, \nu,$$

(11.3)

to which we will put in accordance non-degenerate matrices \( P_r = \begin{pmatrix} p_{1r} & p_{2r} \\ p_{3r} & p_{4r} \end{pmatrix} \), \( r = 1, \ldots, \nu \); phase group \( Ph(\mathcal{F}^8) \) of the covering foliation \( \mathcal{F}^8 \) is generated by the forming non-degenerate linear-fractional transformations

$$Q_r(x) = \frac{q_{1r}x + q_{2r}}{q_{3r}x + q_{4r}} \quad \text{for all} \quad x \in \mathbb{R}, \quad r = 1, \ldots, \nu,$$

(11.4)

to which we will put in accordance non-degenerate matrices \( Q_r = \begin{pmatrix} q_{1r} & q_{2r} \\ q_{3r} & q_{4r} \end{pmatrix} \), \( r = 1, \ldots, \nu \).

12. Conjunctions of linear-fractional actions on \( \mathbb{R} \)

Consider a problem about a finding of necessary and sufficient conditions of existence such homeomorphism (diffeomorphism, holomorphism) \( f : \mathbb{R} \to \mathbb{R} \) that identities

$$f(P_r(x)) = Q_r(f(x)) \quad \text{for all} \quad x \in \mathbb{R}, \quad \text{for all} \quad r \in I,$$

(12.1)

take place, where square matrices \( P_r \in GL(2, \mathbb{R}), \ Q_r \in GL(2, \mathbb{R}) \) for all \( r \in I \).

Group of linear-fractional actions on \( \mathbb{R} \), formed by matrices \( P_r \) for all \( r \in I \) we will designate through \( PL^3 \), and through \( PL^4 \) we will designate the similar group formed by matrices \( Q_r \) for all \( r \in I \).

Consider at first a case of the Abelian real linear-fractional phase groups.

Let’s prove some auxiliary statements on which basis we will receive criteria of topological, smooth and holomorphic conjunctions of Abelian real linear-fractional phase groups.

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**Lemma 12.1.** Let $f$ is a homeomorphism, conjugating linear-fractional phase groups $PL^3$ and $PL^4$. Then real normal Jordan forms of all matrices $P_r$ and $Q_r$, defining by nonidentical linear-fractional transformations, have the same structure, $r = 1, \ldots, \nu$.

The proof of the given statement is spent on the basis of that fact, that the quantity of fixed points of nonidentical linear-fractional maps (11.3) (maps (11.4)) coincides with number of eigenvectors of corresponding matrices $P_r$ (matrices $Q_r$) for all $r \in I$.

**Lemma 12.2.** For a topological conjugation of Abelian linear-fractional groups $PL^3$ and $PL^4$ it is necessary, that real normal Jordan forms of all matrices $P_r$ and $Q_r$ for all $r \in I$ defining nonidentical linear-fractional transformations, have the same structure.

The proof of the given statement is spent on the basis of Lemma 12.1 and that fact that material normal Jordan forms of all permutable among themselves the matrices of the second order defining nonidentical linear-fractional transformations, have the same structure.

**Theorem 12.1.** Let matrices $P_r = S \text{diag}(p_{1r}, p_{2r}) S^{-1}$, $Q_r = T \text{diag}(q_{1r}, q_{2r}) T^{-1}$ for all $r \in I$. Then for a topological conjugation of linear-fractional groups $PL^3$ and $PL^4$ it is necessary and enough, that

$$\frac{q_{1r}}{q_{2r}} = \frac{p_{1r}}{p_{2r}} \left| \begin{array}{c} 1 + \alpha \\ 1 - \alpha \end{array} \right|$$

for all $r \in I$, $\alpha \neq -1$. (12.2)

**Proof.** Since groups $PL^3$ and $PL^4$ are topological conjugated, then identities (12.1) take place. With the help of replacement $\xi: x \rightarrow T^{-1} \circ f \circ S(x)$ for all $x \in \mathbb{R}$, from identities (12.1) we pass to identities

$$\xi(\text{diag}(p_{1r}, p_{2r})(x)) = \text{diag}(q_{1r}, q_{2r})(\xi(x))$$

for all $x \in \mathbb{R}$, for all $r \in I$. (12.3)

Hence, the topological conjugation of groups $PL^3$ and $PL^4$ is equivalent to performance of identities (12.3).

**The necessity.** Let identities (12.3) take place.

If all $\frac{p_{1r}}{p_{2r}} = 1$ for all $r \in I$, then from (12.3) it is received, that $\frac{q_{1r}}{q_{2r}} = 1$ for all $r \in I$.

Therefore in this case relations (12.2) are carried out at $\alpha = 0$.

Let now $\frac{p_{1r}}{p_{2r}} \neq 1$, $r \in \{1, \ldots, \nu\}$. Then on the basis of identities (12.3) we come to conclusion, that either $\psi(0) = 0$, or $\psi(0) = \infty$.

Let $\xi(0) = 0$. Directly from the theorem 9.1 we have relations (12.2) at $\alpha > -1$.

Let now $\xi(0) = \infty$. Then by means of replacement $\zeta = \frac{1}{\xi}$ we come to the previous case and as a result we receive relations (12.2) at $\alpha < -1$.

The sufficiency is proved by construction of conjugating homeomorphism

$$\xi: x \rightarrow \gamma x |x|^{\alpha}$$

for all $x \in \mathbb{R}$. ■

**Theorem 12.2.** Let matrices

$$P_r = S \begin{pmatrix} \cos \alpha_r & -\sin \alpha_r \\ \sin \alpha_r & \cos \alpha_r \end{pmatrix} S^{-1}, \quad \alpha_r \in (-\pi, \pi],$$

$$Q_r = T \begin{pmatrix} \cos \beta_r & -\sin \beta_r \\ \sin \beta_r & \cos \beta_r \end{pmatrix} T^{-1}, \quad \beta_r \in (-\pi, \pi], \text{ for all } r \in I.$$

Then for topological, smooth and holomorphic conjunctions of linear-fractional groups $PL^3$ and $PL^4$ it is necessary and enough, that either

$$\beta_r = \alpha_r \text{ for all } r \in I,$$  

or

$$\beta_r = -\alpha_r \text{ for all } r \in I.$$
Proof. Similarly, as well as at the proof of the theorem 12.1, we come to conclusion, that the topological conjunctions of groups $PL^3$ and $PL^4$ is equivalent to performance of identities

$$\xi \left( x \cos \alpha_r - \sin \alpha_r \over x \sin \alpha_r + \cos \alpha_r \right) = \xi(x) \cos \beta_r - \sin \beta_r \over \xi(x) \sin \beta_r + \cos \beta_r \quad \text{for all } x \in \mathbb{R}, \text{ for all } r \in I.$$ 

We will establish a homeomorphism between expanded real line $\mathbb{R}$ and an unit circle $S^1$ by means of map

$$\zeta: x \mapsto 2 \arctg x \quad \text{for all } x \in \mathbb{R}, \quad \arctg x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).$$

Now with the help of replacement

$$\varphi: t \mapsto \zeta \circ \xi \circ \zeta^{-1}(t) \quad \text{for all } t \in (-\pi, \pi],$$

from last identities we come to equivalent identities

$$\varphi((t - 2\alpha_r)(\mod 2\pi)) = (\varphi(t) - 2\beta_r)(\mod 2\pi) \quad \text{for all } t \in (-\pi, \pi], \text{ for all } r \in I.$$

The necessity. On the basis of that fact, that circle rotations on angles $\alpha$ and $\beta$, where $-\pi < \alpha \leq \pi, -\pi < \beta \leq \pi$, are topological conjugated by:

1) positive oriented homeomorphism, if and only if $\alpha = \beta$;

2) oriented homeomorphism, if and only if $\alpha = -\beta$;

we come either to conditions (12.4), or to conditions (12.5), accordingly.

The sufficiency is established by application of conjugating identical map at performance of conditions (12.4) and by application of conjugating holomorphism

$$\varphi: t(\mod 2\pi) \mapsto (-t)(\mod 2\pi) \quad \text{for all } t \in (-\pi, \pi]$$

in case of performance of conditions (12.5). ■

**Theorem 12.3.** Let matrices $P_r = S \begin{pmatrix} 1 & p_r \ 0 & 1 \end{pmatrix} S^{-1}$, $Q_r = T \begin{pmatrix} 1 & q_r \ 0 & 1 \end{pmatrix} T^{-1}$ for all $r \in I$. Then for topological, smooth and holomorphic conjunctions of linear-fractional groups $PL^3$ and $PL^4$ it is necessary and enough, that

$$q_r = \lambda p_r \quad \text{for all } r \in I, \quad \lambda \neq 0.$$ 

(12.6)

Proof. As well as earlier, we receive, that the topological conjunction of groups $PL^3$ and $PL^4$ is equivalent to performance of identities

$$\xi(x + p_r) = \xi(x) + q_r \quad \text{for all } x \in \mathbb{R}, \text{ for all } r \in I.$$ 

(12.7)

The necessity. Let identities (12.7) take place.

If all $p_r = 0$ for all $r \in I$, from (12.7) it is received that $q_r = 0$ for all $r \in I$. Therefore in this case the relations (12.6) are carried out.

If $p_r = 0$ for all $r \in I, r \neq k$, $p_k \neq 0$, then from (12.7) it is had, that $q_r = 0$ for all $r \in I, r \neq k, q_k \neq 0$, and in quality of $\lambda$ it is possible to take number $q_k / p_k$.

Let now $p_r, p_r \neq 0, r_1 \in I, r_2 \in I$. From (12.7) it is received, as $q_{r_1}q_{r_2} \neq 0$.

Consider at first a case, when $p_{r_2} = \lambda = \frac{l}{n} \in \mathbb{Q}, l \in \mathbb{Z}, n \in \mathbb{Z}$. Then owing to identities (12.7) at $r = r_1$ it is had, that $\xi(x + lp_{r_1}) = \xi(x) + lq_{r_1}$ for all $x \in \mathbb{R}$, and at $r = r_2$ it is received, that $\xi(x + lp_{r_2}) = \xi(x + nq_{r_2}) = \xi(x) + nq_{r_2}$ for all $x \in \mathbb{R}$. Comparing the right parts of last expressions, we receive relations (12.6) at $r = r_1$ and $r = r_2$.

Let $p_{r_2} = \lambda \neq \mathbb{Q}$. Owing to density of set of rational numbers in set of real numbers
there are such sequences \( \{l_s\} \) and \( \{n_s\} \) of integers, that
\[
|l_s| \to +\infty, \quad |n_s| \to +\infty, \quad \frac{l_s}{n_s} \to \lambda \quad s \to +\infty.
\]

From identities (12.7) at \( r = 1 \) it is received, that \( \xi(x + l_s p_{r_1}) - \xi(x) = l_s q_{r_1} \) for all \( x \in \mathbb{R} \), and at \( r = 2 \) it is received, that \( \xi(x + n_s \lambda p_{r_1}) - \xi(x) = n_s q_{r_2} \) for all \( x \in \mathbb{R} \). Proceeding from two last expressions, we receive the following chain of relations:
\[
1 = \lim_{s \to +\infty} \frac{\xi(x + l_s p_{r_1}) - \xi(x)}{\xi(x + n_s \lambda p_{r_1}) - \xi(x)} = \lim_{s \to +\infty} l_s q_{r_1} n_s^{-1} q_{r_2}^{-1} = \lambda q_{r_1} q_{r_2}^{-1}.
\]

From here follows justice of relations (12.6) at \( r = 1 \) and \( r = 2 \).

Justice of relations (12.6) is similarly proved and at \( r = 3 \), \( p_{r_3} \neq 0, r_3 \neq r_2, r_3 \neq r_1 \). If \( p_{r_3} = 0 \), then from (12.7) it is had \( q_{r_3} = 0 \) and relations (12.6) at \( r = 3 \) take place.

The sufficiency is established by application of conjugating holomorphism
\[
\xi(x) = q_{r_1} p_{r_1}^{-1} x \quad \text{for all} \quad x \in \mathbb{R}
\]
in case there is such index \( r_1 \in I \), that \( p_{r_1} \neq 0 \), and identical map otherwise. ■

On the basis of the theorem 9.4 it is similar to the theorem 4.1 we receive the statement.

**Theorem 12.4.** Let the conditions of the theorem 12.1 are satisfied. Then for smooth and holomorphic conjunctions of linear-fractional groups \( PL^3 \) and \( PL^4 \) it is necessary and enough, that
\[
\frac{q_{1r}}{q_{2r}} = \left( \frac{p_{1r}}{p_{2r}} \right)^\varepsilon, \quad r = 1, \ldots, \nu, \quad \varepsilon^2 = 1.
\]

Consider now a case of non-Abelian linear-fractional groups \( PL^3 \) and \( PL^4 \).

**Theorem 12.5.** From a topological conjunction of non-Abelian linear-fractional groups \( PL^3 \) and \( PL^4 \) of general situation follows them holomorphic conjunction which is carried out by nondegenerate linear-fractional transformation.

Proof of the given theorem is spent on the basis of following auxiliary statements from which Theorems 12.6 and 12.7 define constructive criteria of topological, smooth and holomorphic conjunctions. ■

**Theorem 12.6.** Let conjugating of nondegenerate linear-fractional transformations (11.3) and (11.4) homeomorphism \( f : \mathbb{R} \to \mathbb{R} \) is such that:
1) the matrix \( P_1 \) has pair of complex conjugated roots \( \cos \alpha \pm i \sin \alpha, \quad \alpha \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \);
2) \( \frac{\alpha}{\pi} \notin \mathbb{Q} \).

Then this homeomorphism represents nondegenerate linear-fractional transformation.

Proof. Let identities (12.1) are carried out. Present the matrix \( P_1 \) in a kind
\[
P_1 = S \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} S^{-1}.
\]

Owing to Lemma 12.1 and Theorem 12.2 for the matrix \( Q_1 \) one of following two representations takes place:
\[
Q_1 = T \begin{pmatrix} \cos \alpha & \mp \sin \alpha \\ \pm \sin \alpha & \cos \alpha \end{pmatrix} T^{-1}.
\]

It is similar, as well as at the proof of the theorem 12.2, from identity (12.1) at \( r = 1 \) we pass to identity
\[
\varphi((t - 2\alpha)(\mod 2\pi)) = (\varphi(t) \mp 2\alpha)(\mod 2\pi) \quad \text{for all} \quad t \in (-\pi, \pi].
\]

On the basis of (12.8) it is had following relations
\[
\varphi((2l\alpha)(\text{mod } 2\pi)) = (\varphi(0) \pm 2\alpha)(\text{mod } 2\pi) \quad \text{for all } l \in \mathbb{Z}. \quad (12.9)
\]

Owing to a consequence from Kronecker theorem \([31, \text{p. } 314 - 315]\) and conditions 2 of these theorems it is had, that for everyone \(t \in (-\pi, \pi]\) there is such sequence \(\{l_s(t)\}\) integers, that \(\lim_{s \to +\infty} (2l_s(t) \alpha)(\text{mod } 2\pi) = t\). From here, using relations (12.9), we come to conclusion, that homeomorphism, satisfying to identity (12.8), knows
\[
\varphi(t) = (\varphi(0) \pm t)(\text{mod } 2\pi) \quad \text{for all } t \in (-\pi, \pi].
\]

Applying now in other replacement procedure, inverse already used, from last conjugating homeomorphism we come to required nondegenerate linear-fractional transformation. ■

**Lemma 12.3.** Let conjugating of nondegenerate linear-fractional transformations (11.3) and (11.4) homeomorphism \(f: \mathbb{R} \to \mathbb{R}\) is such that:

1) \(f(0) = 0, \; f(\infty) = \infty; \)
2) \(f(\lambda x) = \lambda |\lambda|^\alpha f(x) \; \text{for all} \; x \in \mathbb{R}, \; \alpha > -1; \)
3) \(f\left(\frac{ax + b}{cx + d}\right) = A f(x) + B \quad \text{for all} \; x \in \mathbb{R}, \; |b| + |c| > 0; \)
4) the matrix \(P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = S \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} S^{-1} \) is such that \(|\mu| \neq 1, \; S = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix}; \)
5) the subgroup of the group \(\mathbb{R}^*_+\) of positive real numbers on the multiplication, formed by numbers \(|\lambda|\) and \(\left|\frac{b_s}{d_s}\right|\), is dense in the set \(\mathbb{R}_+\) of positive real numbers.

Then this homeomorphism looks like
\[
f(x) = x|x|^\alpha \quad \text{for all} \; x \in \mathbb{R}. \quad (12.13)
\]

**Proof.** Let conditions (12.10) — (12.12) are satisfied. Owing to the condition 4 of lemma, relations (12.10), (12.11), Lemma 12.1 and Theorem 9.1 we come to conclusion, that the matrix \(Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = T \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix} T^{-1}\) is such that \(\ln |\mu| \ln |\theta| > 0, \; T = \begin{pmatrix} A_s & B_s \\ C_s & D_s \end{pmatrix}.\)

On the basis of (12.11) and (12.12) we receive relations
\[
f(\lambda^k(P^m)^l(x)) = \lambda^k |\lambda|^\alpha (Q^m)^l(f(x)) \quad \text{for all} \; x \in \mathbb{R}, \; \text{for all } k, l, m \in \mathbb{Z}.
\]

Passing in them to a limit at \(m \to -\infty\) if \(|\mu| > 1\), and to a limit at \(m \to +\infty\) if \(|\mu| < 1\), we have
\[
f\left(\lambda^k\left(\frac{b_s}{d_s}\right)^l\right) = \lambda^k |\lambda|^\alpha \left(\frac{B_s}{D_s}\right)^l \quad \text{for all} \; k \in \mathbb{Z}, \; \text{for all} \; l \in \mathbb{Z}. \quad (12.14)
\]

From the condition 5 of Lemma 12.3 follows, that for any positive real number \(x > 0\) exist such sequences \(\{k_s(x)\}\) and \(\{l_s(x)\}\) of even integers, that
\[
\lim_{s \to +\infty} |k_s(x)| = \lim_{s \to +\infty} |l_s(x)| = +\infty, \quad \lim_{s \to +\infty} \lambda^{k_s(x)} \left(\frac{b_s}{d_s}\right)^{l_s(x)} = x.
\]

From here on the basis of relations (12.14) it is received, that
\[
f(x) = x|x|^\alpha \lim_{s \to +\infty} \left|\frac{b_s}{d_s}\right|^{(\beta - \alpha)l_s(x)} \quad \text{for all} \; x > 0,
\]
where \(\left|\frac{B_s}{D_s}\right| = \left|\frac{b_s}{d_s}\right|^{1+\beta}\). As from the condition 5 this lemma follow, that
\[
\lim_{s \to +\infty} \left|\frac{b_s}{d_s}\right|^{(\beta - \alpha)l_s(x)} = 1.
\]
Therefore,

\[ f(x) = x|\alpha| \text{ for all } x > 0. \tag{12.15} \]

Having executed replacement \( \xi(x) = -f(-x) \) for all \( x \in \mathbb{R} \), for homeomorphism \( \xi \) by the reasonings similar resulted above, we come to relations \( \xi(x) = x|\alpha| \) for all \( x > 0 \), and from them with the help of return replacement we come to relations

\[ f(x) = x|\alpha| \text{ for all } x < 0. \tag{12.16} \]

Uniting formulas (12.10), (12.15) and (12.16), we come to representation (12.13).

Now by direct calculations on the basis of identity (12.13) we do a conclusion, that in case of general situation at performance of conditions of the lemma 12.3 conjugating homeomorphism looks like \( f(x) = x \) for all \( x \in \mathbb{R} \).

Now on the basis of the reasonings similar resulted at proof of the theorem 12.1, taking into account the theorem 12.6, we come to the statement of the theorem 12.5.

At first we will consider smooth and holomorphic conjunction of Abelian linear-fractional groups \( PL^3 \) and \( PL^4 \).

**Theorem 12.7.** Let conditions of the theorem 12.1 are satisfied. Then for smooth (holomorphic) conjunction of Abelian linear-fractional groups \( PL^3 \) and \( PL^4 \) it is necessary and enough, that

\[ \frac{q_1}{q_2} = \frac{(p_1r)^2}{(p_2r)^2} \text{ for all } r \in I, r^2 = 1. \tag{12.17} \]

Proof of the given statement is similar to the proof of the theorem 12.1 and is based on the theorem 9.3.

In case of non-Abelian linear-fractional groups \( PL^3 \) and \( PL^4 \) the following statement takes place.

**Theorem 12.8.** From a smooth conjunction of non-Abelian linear-fractional groups \( PL^3 \) and \( PL^4 \) follows them conjunction, which is carried out by nondegenerate linear-fractional transformation.

Proof of this theorem is spent similarly to the proof of the theorem 12.5 on the basis of following three auxiliary statements.

**Lemma 12.4.** Let diffeomorphism \( f: \mathbb{R} \to \mathbb{R} \) is such that:

1) relations (12.10) are carried out;

2) \( f(\lambda x) = \lambda f(x) \) for all \( x \in \mathbb{R}, \lambda \neq 0, \lambda \neq 1. \tag{12.18} \)

Then this diffeomorphism looks like

\[ f(x) = ax \text{ for all } x \in \mathbb{R}. \]

Proof. From (12.17) we come to conclusion about justice of relations

\[ f(\lambda^l x) = \lambda^l f(x) \text{ for all } x \in \mathbb{R}, \text{ for all } l \in \mathbb{Z}, \]

differentiating which on \( x \), we come to identities

\[ f'(\lambda^l x) = \lambda^l f'(x) \text{ for all } x \in \mathbb{R}, \text{ for all } l \in \mathbb{Z}. \]

Owing to that map \( f \) is a diffeomorphism, we receive identity

\[ f'(x) = a \text{ for all } x \in \mathbb{R}. \]

From here taking into account relations (12.17) we come to representation (12.18).

Similarly we prove the following statement.

**Lemma 12.5.** Let diffeomorphism \( f: \mathbb{R} \to \mathbb{R} \) is such that:

1) \( f(\infty) = \infty; \)

2) \( f(x + p) = f(x) + q \) for all \( x \in \mathbb{R}, \ pq \neq 0. \)

Then this diffeomorphism looks like \( f(x) = qp^{-1}x + a \) for all \( x \in \mathbb{R}. \)
Lemma 12.6. Let diffeomorphism \( f : \mathbb{R} \to \mathbb{R} \) is such that
\[
f\left(\frac{x \cos \alpha - \sin \alpha}{x \sin \alpha + \cos \alpha}\right) = \frac{f(x) \cos \beta \mp \sin \beta}{\pm f(x) \sin \beta + \cos \beta}
\]
for all \( x \in \mathbb{R}, \sin \alpha \neq 0 \).

Then this diffeomorphism represents nondegenerate linear-fractional transformation.

Proof. It is similar, as well as at the proof of theorems 12.2 and 12.6, from identity from a condition of the lemma 12.6 we pass to identities
\[
\varphi((t - 2l\alpha)(\text{mod } 2\pi)) = (\varphi(t) \mp 2l\alpha)(\text{mod } 2\pi)
\]
for all \( t \in (-\pi, \pi], \) for all \( l \in \mathbb{Z} \).

Differentiating last identities on \( t \), taking into account that map \( \varphi \) is a diffeomorphism, in a similar way, as well as earlier, we receive representations
\[
\varphi(t) = (\pm t + a)(\text{mod } 2\pi)
\]
for all \( t \in (-\pi, \pi] \).

Now it is similar to the proof of the theorem 12.6 we come to conclusion, that conjugating homeomorphism \( f \) is nondegenerate linear-fractional transformation.

On the basis of the theorem 12.8 and a course of proofs of theorems 12.2 — 12.4 it is had such statement.

Theorem 12.9. From a smooth conjunction of linear-fractional groups \( PL^3 \) and \( PL^4 \) follows them holomorphic conjunction which is carried out by nondegenerate linear-fractional transformation.

13. Applications to real nonautonomous Riccati equations

Theorems 12.1 — 12.9 and Lemmas 12.1 — 12.6 allow on the basis of Theorems 1.1 — 1.3 to spend topological, smooth and holomorphic classifications of real nonautonomous Riccati equations of a kind (11.1). Besides, from the theorem 12.5 it is had such statement.

Theorem 13.1. From topological equivalence of real nonautonomous Riccati equations with non-Abelian phase groups of general situation follows them holomorphic equivalence.

From the given theorem, in particular, follows, that real nonautonomous Riccati equations with coefficients, holomorphic on path connected holomorphic varieties with non-Abelian fundamental groups, are structurally unstable.
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