Nonlocal hyperelasticity and polyconvexity in fractional spaces

José C. Bellido, Javier Cueto and Carlos Mora-Corral

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Abstract

In this paper we propose a nonlocal model of hyperelasticity obtained by substitution of the classical gradient by the Riesz fractional gradient. We show existence of solutions for those nonlocal models in Bessel fractional spaces under the main assumption of polyconvexity of the energy density. The main ingredient is the fractional Piola identity, which establishes that the fractional divergence of the cofactor matrix of the fractional gradient vanishes. This identity implies the weak convergence of the determinant of the fractional gradient, and, in turn, the existence of minimizers of the nonlocal hyperelastic energy.

1 Introduction

Elastic materials are those that deform under the action of a load, and recover their original state when the load stops acting. In those simplified situations where one can assume that the deformation of the solid is small, the mathematical model is a linear system of partial differential equations (PDE), as the Lamé system for isotropic linear elastic materials. In a more realistic situation, the deformation could be large, and, hence, hyperelastic materials and nonlinear laws are necessary for a proper modeling. Those nonlinear models, typically a fully nonlinear system of PDE, are difficult and there are well-posedness results (existence and uniqueness) only for very few particular cases. On the other hand, the system of PDE is the Euler–Lagrange system of the total potential energy of the solid, so that proving the existence of critical points for that energy would lead (under certain assumptions such as growth conditions) to the existence for the nonlinear system of elasticity. In practice, one looks for minimizers of the energy, since they are, in some sense, the most stable deformations, and the theory of existence of minimizers is relatively well understood. It was Ball [2] who first proved the existence of minimizers for hyperelastic energies with assumptions compatible in nonlinear elasticity. In that reference, the existence of solutions in Sobolev spaces is shown under certain coercivity and convexity properties on the energy density. The central convexity notion in the direct method of the calculus of variations is quasiconvexity. This concept, introduced by Morrey [26], is known to be equivalent, under natural growth conditions, to the lower semicontinuity of the energy functionals, with respect to the weak convergence in Sobolev spaces; see, e.g., [12]. Unfortunately, and despite the great deal of work over decades, quasiconvexity is not yet fully understood. For instance, given a function there is no simple way to determine whether it is quasiconvex or not. On the positive side, there is a large family of relatively simple quasiconvex functions: the polyconvex functions. Essentially, a function defined on the matrices is polyconvex if it is a convex function of its argument and its minors; see, e.g., [12]. It turns out that many density energies for classical (isotropic) hyperelastic models are polyconvex. In a general situation in hyperelasticity, the energy
functional has the form
\[ \int_{\Omega} W(x, u(x), Du(x)) \, dx \]  
with \( u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \) belonging to a prescribed Sobolev space, and \( W(x, u, \cdot) \) polyconvex for a.e. \( x \in \Omega \) and all \( u \in \mathbb{R}^n \).

Despite the huge development of the mathematical theory of hyperelasticity in the last four decades, there are still important questions to be understood \([5, 6]\). As mentioned above, in classical hyperelasticity theory one looks for minimizers of the elastic energy in a certain class of functions, typically, a Sobolev space. The election of such a class is part of the model and sometimes constitutes a controversial choice; see \([4]\). For example, Sobolev spaces \( W^{1,p} \) with \( p > n \) (or \( p = n \) with \( \det Du > 0 \)) impose deformations to be continuous, which is not realistic for many materials. For instance, there is a large amount of work for existence theories in \( W^{1,p} \) with \( p > n - 1 \) for deformations that do not present cavitation (the sudden formation of voids in the material); see \([2, 39, 28, 8]\). On the other hand, there are also existence theories for deformations allowing for cavitation \([27, 20]\): they add a surface energy term that accounts for the area of the cavity. Outside pure elasticity and within the theory of brittle materials, the deformations are not Sobolev maps but of special bounded variation: a new term is added in the energy accounting for the crack opened; see \([18, 13, 19]\).

In this paper we consider typical hyperelastic stored energy functions \( W \), as in \([1]\), but defined in spaces of fractional integrability (we remark that the term fractional Sobolev spaces usually stands for spaces of fractional order \( W^{s,p} \), which are in general different from those we consider here). In particular, and inspired by the recent works of Shieh and Spector \([34, 35]\), we explore polyconvexity as a sufficient condition for the existence of minimizers in those spaces. References \([34, 35]\) study scalar variational problems on fractional spaces defined via the so-called Riesz fractional gradient as a sufficient condition for the existence of minimizers in those spaces. References \([34, 35]\) consider integral functionals of \( W \) and all \( p > n \), their fractional Sobolev-type inequality, the compact embedding into \( \mathcal{B}_{s-1+n} \) (see \([1, 38, 31]\)).

Given a function \( u : \mathbb{R}^n \to \mathbb{R} \), and \( s \in (0, 1) \), its \( s \)-fractional gradient is given by
\[ D^s u(x) = c_{n,s} \text{pv}_{x} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \, dy, \quad x \in \mathbb{R}^n, \]  
where \( \text{pv}_{x} \) stands for the principal value centred at \( x \), and \( c_{n,s} \) is a suitable constant. In this way, the relevant functional space \( H^{s,p} = H^{s,p}(\mathbb{R}^n) \) becomes
\[ H^{s,p} = \left\{ u \in L^p(\mathbb{R}^n) : D^s u \in L^p(\mathbb{R}^n, \mathbb{R}^{n \times n}) \right\}. \]

It is also of interest the affine subspace of functions verifying a complement value condition; to be precise, given \( g \in H^{s,p} \) and a bounded domain \( \Omega \subset \mathbb{R}^n \), we consider
\[ H^{s,p}_g(\Omega) = \left\{ u \in H^{s,p} : u = g \text{ in } \Omega^c \right\}, \]
where \( \Omega^c \) stands for the complement of \( \Omega \) in \( \mathbb{R}^n \). References \([34, 35]\) consider integral functionals of the form
\[ I(u) = \int_{\mathbb{R}^n} W(x, u(x), D^s u(x)) \, dx \]  
and prove the existence of minimizers in \( H^{s,p}_g \) under the fundamental hypothesis of convexity of \( W \) in the last variable, as well as natural coercivity conditions. They also show interesting results on the functional space \( H^{s,p} \), including a fractional Sobolev-type inequality, the compact embedding into \( L^p(\mathbb{R}^n) \) and the equivalence with Bessel spaces (see \([11, 38, 31]\)).

In this paper we extend the investigations of \([34, 35]\) by considering vector variational problems on the fractional space \( H^{s,p} \). Thus, we establish the existence of minimizers under the polyconvexity assumption of the integrand. In order to obtain our results, we follow the usual steps as for classical
polyconvex variational problems, namely, we show that the determinant (or any minor) of the fractional gradient matrix \( D^s u \) is continuous with respect to weak convergence in \( H^{s,p} \). Similarly to the classical case, we need a fractional version of Piola identity, being this the key ingredient and the most remarkable contribution of this work. In this new situation we follow the work of Mengesha and Spector [25] to define a fractional divergence and establish an integration by parts formula (see also [29]). We adapt the techniques developed there for some spaces of nonlocal type to our \( H^{s,p} \) spaces. More concretely, as in [2], the Riesz \( s \)-fractional gradient of a deformation \( u : \mathbb{R}^n \to \mathbb{R}^n \) is

\[
D^s u(x) = c_{n,s} \text{pv}_x \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \otimes \frac{x - y}{|x - y|} dy, \quad x \in \mathbb{R}^n,
\]

where \( \otimes \) denotes the tensor product, and the Riesz \( s \)-fractional divergence of a vector field \( \psi : \mathbb{R}^n \to \mathbb{R}^n \) is defined as

\[
\text{div}^s \psi(x) = -c_{n,s} \text{pv}_x \int_{\mathbb{R}^n} \frac{\psi(x) + \psi(y)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} dy, \quad x \in \mathbb{R}^n.
\]

As mentioned above, we first establish the nonlocal integration by parts formula (the duality between \( \text{div}^s \) and \( D^s \)), we then prove the nonlocal Piola identity \( \text{Div}^s \text{cof} D^s u = 0 \) (where \( \text{Div}^s \) means the \( s \)-divergence by rows), we continue with the weak continuity of \( \text{det} D^s u \) and the weak lower semi-continuity of polyconvex functionals in \( H^{s,p} \), and finally we settle the existence of minimizers for \( I \).

Our results can actually be seen as a first step towards a nonlocal theory of hyperelasticity on fractional spaces \( H^{s,p} \). One primary interest of this extension is that \( H^{s,p} \) is larger than \( W^{1,p} \), and functions in \( H^{s,p} \) may exhibit singularities prohibited in \( W^{1,p} \), as we point out in Section 2. We would like to emphasize that, contrary to classical elasticity, both cavitation and fracture are compatible with the existence of optimal deformations in \( H^{s,p} \) result, Theorem 6.1. This seems to indicate that the \( L^p \) norm of \( D^s u \) not only contributes to the elastic energy, but also to a kind of surface energy, since the latter is necessary in the modelling of cavitation and fracture (see, e.g., [27, 13, 19]). On the other hand, in the last decade there has been a great deal of work on fractional PDE of elliptic type involving the fractional laplacian in some way, and our results here enlarge this theory on fractional PDE by giving existence results for vector nonlinear variational problems based on polyconvexity. The amount of references on nonlocal equations and fractional Laplacian is overwhelming, so for situations related to this work we just cite the survey [30], the paper [35] and the references therein. Moreover, we would like to remark that the fractional Piola identity has interest in itself and we hope that subsequent versions of it will be useful in several contexts, as hyperelastic nonlocal theories in bounded domains or nonlocal theories in fluid dynamics.

Related to this work, it is imperative to mention peridynamics, a nonlocal alternative model in Solid Mechanics proposed in [37]. Nonlocality is reflected in the fact that points separated by a positive distance exert a force upon each other. In this model the use of gradients is avoided by computing internal forces by integration instead of differentiation, and the elastic energy is now a double integral depending on pairs of points in the reference and deformed configurations. A main feature is that the functional space for the optimization problem is just a Lebesgue \( L^p \) space. The development of this theory in the last years has been impressive. However, most of the work until now is on linear elastic models [24]. A first attempt (to the best of the authors’ knowledge, the only one) to rigorously extend this nonlocal theory for a general nonlocal nonlinear model has been made by some of the authors of this paper in [9, 10, 11]. In those references, it is considered a general nonlocal energy of the form

\[
I(u) = \int_{\Omega} \int_{\Omega \cap B(x,\delta)} w(x - y, u(x) - u(y)) \, dy \, dx,
\]
where $\delta$ is the horizon distance of interaction of particles. Typical Calculus of Variations questions and $\Gamma$-convergence to a local model as $\delta \to 0$ were analyzed. The two main differences with the model considered here are that the nonlocality is spatially restricted by the positive horizon distance $\delta$, and that the integrand is inside the double integration rather than between the two integrals as in [3]. Although this kind of functionals is mathematically interesting and with applications in many situations, we believe it is not suitable for hyperelastic modeling, since the dependence $(x - y, u(x) - u(y))$ does not completely ‘delocalize’ $Du$, but rather it represents an average of directional derivatives of $u$. Another interesting approach related to our investigation, and going from a nonlocal peridynamic framework to a nonlocal fractional situation in the linear case, appears in the recent references [21, 32, 33]. It is also worth mentioning [14], where the well-posedness for a fractional linearly elastic equation is shown.

The outline of the paper is the following. Section 2 introduces the functional space of fractional type $H^{s,p}(\mathbb{R}^n)$ and its main properties. We also include examples of functions exhibiting singularities (namely, fracture and cavitation) belonging to these spaces. Section 3 contains several technical results necessary to prove the main theorems in the paper. In Section 4 we prove the fractional Piola identity. Section 5 shows the weak continuity of minors in $H^{s,p}$, and, finally, Section 6 proves the existence of minimizers of (3) for polyconvex integrands.

2 Functional analysis framework

This section introduces general properties of the functional space $H^{s,p}$. We start by setting the definition of principal value. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ such that $f \in L^1(B(x,r)^c)$ for every $r > 0$, we define the principal value centered at $x$ of $\int_{\mathbb{R}^n} f$, denoted by

$$\text{pv}_x \int_{\mathbb{R}^n} f$$

or $\text{pv}_x \int f$, as

$$\lim_{r \to 0} \int_{B(x,r)^c} f,$$

whenever this limit exists. We have denoted by $B(x,r)$ the open ball centered at $x$ of radius $r$, and by $B(x,r)^c$ its complement. As most integrals in this work are over $\mathbb{R}^n$, we will use the symbol $\int$ as a substitute for $\int_{\mathbb{R}^n}$.

The $s$-fractional gradient is defined as follows.

**Definition 2.1.** Let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Let $0 < s < 1$ and $x \in \mathbb{R}^n$ be such that

$$\int_{B(x,r)^c} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dy < \infty$$

(4)

for each $r > 0$. Set

$$c_{n,s} = -(n + s - 1) \frac{\Gamma \left( \frac{n+s-1}{2} \right)}{\pi \frac{1}{2} 2^{1-s} \Gamma \left( \frac{1-s}{2} \right)},$$

where $\Gamma$ is the Euler gamma function. We define $D^s u(x)$, the $s$-fractional gradient of $u$ at $x$, as

$$D^s u(x) := c_{n,s} \text{pv}_x \int \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} dy,$$

whenever the principal value exists.
We note that, due to symmetry,
\[ \text{pv}_x \int \frac{x - y}{|x - y|^{n+s+1}} \, dy = 0, \]
and consequently, the equality
\[ D^s u(x) = -c_{n,s} \text{pv}_x \int \frac{u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} \, dy \]
holds. Notice also that the constant \( c_{n,s} \) is negative.

Definition 2.1 naturally extends to vector fields. Given \( u : \mathbb{R}^n \to \mathbb{R}^m \) measurable such that (4) holds for each \( r > 0 \), its \( s \)-fractional gradient is
\[ D^s u(x) = c_{n,s} \text{pv}_x \int \frac{u(x) - u(y)}{|x - y|^{n+s}} \otimes \frac{x - y}{|x - y|} \, dy, \]
whenever it exists. Here, \( \otimes \) stands for usual tensor product of vectors. Given \( s \in (0,1) \) and \( p \in (1,\infty) \), we now define the space \( H^{s,p} \) as
\[ H^{s,p}(\mathbb{R}^n, \mathbb{R}^m) := \{ u \in L^p(\mathbb{R}^n) : D^s u \in L^p(\mathbb{R}^n, \mathbb{R}^{n \times m}) \}, \]
and we denote \( H^{s,p}(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n, \mathbb{R}) \). Since we will study complement value problems (as in, for example, [10]), we are interested in the case in which functions are prescribed in the complement of a given set. Thus, given an open set \( \Omega \subset \mathbb{R}^n \) and \( g \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^m) \), the space \( H_g^{s,p}(\Omega, \mathbb{R}^m) \) is defined as
\[ H_g^{s,p}(\Omega, \mathbb{R}^m) := \{ u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^m) : u = g \text{ in } \Omega^c \}. \]

The space \( H^{s,p} \), together with the \( s \)-fractional gradient as a mathematical object, was introduced and studied in [34, 35] (see also [29, Sect. 15.2]). The first remarkable fact is the identification of \( H^{s,p} \) with the classical Bessel potential spaces (see [1, 38, 31]) established in [34, Th. 1.7]. Thanks to this equivalence, and rewriting well-known properties for Bessel spaces in terms of \( H^{s,p} \) spaces, we obtain several basic properties that we summarize in the following proposition (see [1, Ch. 7, p. 221]). We denote by \( \hookrightarrow \) continuous inclusion.

**Proposition 2.1.** Set \( 0 < s < 1 \) and \( 1 < p < \infty \). Then:

a) \( C^\infty(\mathbb{R}^n) \) is dense in \( H^{s,p}(\mathbb{R}^n) \).

b) \( H^{s,p}(\mathbb{R}^n) \) is reflexive.

c) If \( s < t < 1 \) and \( 1 < q \leq p \leq \frac{mq}{n-(t-s)q} \), then \( H^{1,q}(\mathbb{R}^n) \hookrightarrow H^{s,p}(\mathbb{R}^n) \).

d) If \( 0 < \mu \leq s - \frac{n}{p} \), then \( H^{s,p}(\mathbb{R}^n) \hookrightarrow C^{0,\mu}(\mathbb{R}^n) \).

e) If \( p = 2 \), then \( H^{s,2}(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n) \) with equivalence of norms.

f) If \( 0 < s_1 < s < s_2 < 1 \) then \( H^{s,2}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n) \hookrightarrow H^{s_2,2}(\mathbb{R}^n) \).

We have denoted by \( W^{s,p} \) the classical fractional Sobolev space. They will not be used in this paper, but they were mentioned in Proposition 2.1 to help locate the spaces \( H^{s,p} \) in a scale of regularity. We have also denoted by \( C^{0,\mu} \) the space of Hölder continuous functions of exponent \( \mu \).

An essential tool for obtaining existence of minimizers for variational functionals is a Poincaré-type inequality. Collecting together several theorems present in the literature, we state the following result, which is not optimal but suitable for our purposes. Henceforth, given \( 1 < p < n \) and \( 0 < s < 1 \) with \( sp < n \) we define \( p^* := \frac{np}{n-sp} \).
Theorem 2.2. Set $0 < s < 1$ and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists $C = C(|\Omega|, n, p, s) > 0$ such that
$$\|u\|_{L^q(\Omega)} \leq C\|D^s u\|_{L^p(\mathbb{R}^n)}$$
for all $u \in H^{s,p}(\mathbb{R}^n)$, and any $q$ satisfying
$$\begin{cases} 
q \in [1, p^*) & \text{if } sp < n, \\
q \in [1, \infty) & \text{if } sp = n, \\
q \in [1, \infty) & \text{if } sp > n.
\end{cases}$$

The case $sp < n$ is an immediate consequence of [34, Th. 1.8], where the continuous embedding of $H^{s,p}(\mathbb{R}^n)$ in $L^{p^*}(\mathbb{R}^n)$ is shown. Case $sp = n$ is a consequence of [34, Th. 1.10], where it is proved in this context the version of Trudinger’s inequality, which implies the embedding of $H^{s,p}(\mathbb{R}^n)$ in $L^q_{\text{loc}}(\mathbb{R}^n)$ for all $q \in [1, \infty)$. Finally, the case $sp > n$ is a consequence of Proposition 2.1 d).

The following result decides which of the embeddings of Theorem 2.2 are compact. We will indicate by $\rightharpoonup$ weak convergence.

Theorem 2.3. Set $0 < s < 1$ and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $g \in H^{s,p}(\mathbb{R}^n)$. Then for any sequence $\{u_j\}_{j \in \mathbb{N}} \subset H^{s,p}(\Omega)$ such that $u_j \rightharpoonup u$ in $H^{s,p}(\mathbb{R}^n)$, for some $u \in H^{s,p}(\mathbb{R}^n)$, one has $u \in H^{s,p}(\Omega)$ and
$$u_j \to u \quad \text{in } L^q(\mathbb{R}^n),$$
for every $q$ satisfying
$$\begin{cases} 
q \in [1, p^*) & \text{if } sp < n, \\
q \in [1, \infty) & \text{if } sp = n, \\
q \in [1, \infty) & \text{if } sp > n.
\end{cases}$$

Case $sp < n$ is actually [35, Th. 2.2]. Case $sp = n$ follows from the former having in mind Proposition 2.1, part 3) or else part 7). Finally, the case $sp > n$ is a consequence of Proposition 2.1 d) and the compact embedding of $C^{0,p}(\bar{\Omega})$ into $C(\bar{\Omega})$.

2.1 Examples of functions in $H^{s,p}(\mathbb{R}^n)$

One of the motivations of this study is to propose a theory of hyperelasticity formulated in spaces wider than classical Sobolev spaces. Clearly, as a consequence of Proposition 2.1, classical Sobolev spaces are continuously embedded in $H^{s,p}$ spaces. Further, we are interested in functions that belong to $H^{s,p}$ but not to $W^{1,p}$. Necessarily, those functions must exhibit some type of singularity. We focus on two important singularities in solid mechanics: fracture and cavitation. For simplicity, we study as a model for fracture a deformation whose first component is the characteristic function $\chi_Q$ of the unit cube $Q$, while the other components are $C^\infty$ functions. As a model for cavitation, we study a radial function of compact support exhibiting one cavity at the origin. In both examples the functions have compact support: this simplifies the analysis since it avoids the issue of the integrability at infinity, and, hence, allows us to focus solely on the singularity.

We start with the case of fracture. There is an extensive literature on when the characteristic function of a set (especially, of an open bounded Lipschitz set) belongs to a functional space of fractional regularity (see, e.g., [40, 31, 36, 22, 15]). We exploit those results to give a quick proof of the following lemma.
Lemma 2.4. Set $0 < s < 1$ and $1 < p < \infty$. Let $Q = (0,1)^n$ and $\varphi_2, \ldots, \varphi_n \in C_c^\infty(\mathbb{R}^n)$. Define $u = (\chi_Q, \varphi_2, \ldots, \varphi_n)$. Then

$$u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p < \frac{1}{s}, \text{ and } u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p > \frac{1}{s}.$$

Proof. As $C_c^\infty(\mathbb{R}^n) \subset H^{s,p}(\mathbb{R}^n)$, we have that $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ if and only if $\chi_Q \in H^{s,p}(\mathbb{R}^n)$.

The fractional Sobolev space $W^{s,p}$ coincides with the Triebel–Lizorkin space $F_{p,p}^s$ and with the Besov space $B_{p,p}^s$ (see, e.g., [40, Sect. 2.3.5] or [31, Prop. 2.1.2]). This result together with [31, Lemma 4.6.3.2] shows that $\chi_Q \in W^{s,p}$ if and only if $sp < 1$. Proposition 2.1.7 concludes the proof. \hfill \Box

For the case of cavitation, the result is the following.

Lemma 2.5. Set $0 < s < 1$ and $1 < p < \infty$. Let $\varphi \in C_c^\infty([0, \infty))$ be such that $\varphi(0) > 0$, and $u(x) = \frac{x}{|x|} \varphi(|x|)$. Then

$$u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p < \frac{n}{s} \text{ and } u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \text{ if } p > \frac{n}{s}.$$

Proof. It is well known that $u \in W^{1,q}(\mathbb{R}^n, \mathbb{R}^n)$ whenever $1 < q < n$ (see, e.g., [3, Lemma 4.1]), and therefore $u \in H^{1,q}(\mathbb{R}^n, \mathbb{R}^n)$ for any $0 < t < 1$ and $1 < q < n$. Applying now Proposition 2.1.4, we have that $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ for any $s \in (0, t)$ and $p \in [q, \frac{nq}{n-(t-s)q}]$. Now we observe that the set of $(s,p) \in \mathbb{R}^2$ such that there exist $q \in (1, n)$ and $t \in (0, 1)$ for which $s \in (0, t)$ and $p \in [q, \frac{nq}{n-(t-s)q}]$ is precisely the set of $(s,p)$ such that $s \in (0, 1)$ and $p \in (1, \frac{n}{s})$. Therefore, $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ if $p < \frac{n}{s}$.

On the other hand, when $p > \frac{n}{s}$, by Proposition 2.1.4, $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ functions are continuous. Since $u$ is discontinuous, $u \notin H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ if $p > \frac{n}{s}$. \hfill \Box

3 Calculus in $H^{s,p}$

In this section we present some properties of nonlocal functionals related to $H^{s,p}$, notably, an integration by parts formula.

We start with a sufficient condition for the $s$-fractional gradient to be defined everywhere.

Lemma 3.1. Let $0 < \alpha < s < 1$ and $\varphi \in C^{0,\alpha}(\mathbb{R}^n) \cap C^{0,1}(\mathbb{R}^n)$. Then

$$\sup_{x \in \mathbb{R}^n} \int \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dy < \infty. \quad (7)$$

If, in addition, $\varphi$ has compact support then $D^s \varphi \in L^r(\mathbb{R}^n)$, for every $r \in [1, \infty)$.

Proof. Let $L$ and $C$ be, respectively, the Lipschitz and $\alpha$-Hölder constants of $\varphi$. Then, for every $x \in \mathbb{R}^n$,

$$\int \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dy \leq \int_{B(x,1)} \frac{L}{|x - y|^{n+s-1}} dy + \int_{B(x,1)^c} \frac{C}{|x - y|^{n+s-\alpha}} dy$$

$$= \int_{B(0,1)} \frac{L}{|z|^{n+s-1}} dz + \int_{B(0,1)^c} \frac{C}{|z|^{n+s-\alpha}} dz < \infty.$$

This means that $D^s \varphi \in L^\infty(\mathbb{R}^n)$. 

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Next we are going to see that $D^s \varphi \in L^1(\mathbb{R}^n)$ when $\varphi$ has compact support. Denote by $F$ the support of $\varphi$. Then
\[
\int |c_{n,s} \int \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+s}} x - y \, dy| \, dx \leq |c_{n,s}| (A + B),
\]
where
\[
A := \int \int_F \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} \, dx \, dy, \quad B := \int \int_{F^c} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} \, dy \, dx.
\]
Now, we observe that, applying Fubini’s Theorem and (7),
\[
A = \int \int_F \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} \, dy \, dx < \infty.
\]
We notice that $|\varphi(x) - \varphi(y)| = 0$ for every $(x, y) \in F^c \times F^c$. Therefore, applying again (7) we get
\[
B = \int \int_{F^c} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} \, dy \, dx \leq \int \int_{F} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} \, dy \, dx < \infty.
\]
As a consequence of (8) and (9), $D^s \varphi \in L^1(\mathbb{R}^n)$. Finally, through a standard interpolation argument, we get that $D^s \varphi \in L^r(\mathbb{R}^n)$ for all $r \in [1, \infty]$.

Lemma 3.1 implies, in particular, that $D^s \varphi$ is defined everywhere for $\varphi \in C^{0,\alpha}(\mathbb{R}^n) \cap C^{0,1}(\mathbb{R}^n)$ and $0 < \alpha < s < 1$. It also shows that $C^{1}_c(\mathbb{R}^n) \subset H^{s,p}(\mathbb{R}^n)$ for every $0 < s < 1$ and $1 < p < \infty$.

The following result defines a nonlocal operator related to the $s$-fractional gradient.

**Lemma 3.2.** Let $1 \leq q < \infty$ and $0 < \alpha < s < 1$. Let $\varphi \in C^{0,\alpha}(\mathbb{R}^n) \cap C^{0,1}(\mathbb{R}^n)$ and $k \in \{1, \ldots, n\}$. Then, the operator $K_\varphi : L^q(\mathbb{R}^n, \mathbb{R}^{k \times n}) \to L^q(\mathbb{R}^n, \mathbb{R}^k)$ defined as
\[
K_\varphi(U)(x) = c_{n,s} \int \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+s}} U(y) \frac{x - y}{|x - y|} \, dy, \quad \text{a.e. } x \in \mathbb{R}^n,
\]
is linear and bounded.

If, in addition, $\varphi$ has compact support then $K_\varphi$ is bounded from $L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$ to $L^q(\mathbb{R}^n, \mathbb{R}^n)$ for all $p \in [1, q]$.

**Proof.** Let $U \in L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$. We denote by $C$ a positive constant that does not depend on $U$ and whose value may vary along the proof. For a.e. $x \in \mathbb{R}^n$ we have
\[
|K_\varphi(U)(x)| \leq |c_{n,s}| \int \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)| \, dy,
\]
so
\[
|K_\varphi(U)(x)|^q \leq 2^{q-1} |c_{n,s}|^q \left( g(x) + h(x) \right), \quad (10)
\]
with
\[
g(x) := \left( \int_{B(x, 1)} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)| \, dy \right)^q, \quad h(x) := \left( \int_{B(x, 1)^c} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)| \, dy \right)^q.
\]
Let $L$ be a Lipschitz and an $\alpha$-Hölder constant of $\varphi$. Then, using that $\varphi$ is Lipschitz and applying Hölder’s inequality, we get
\[
g(x) \leq L^q \left( \int_{B(x, 1)} \frac{|U(y)|}{|x - y|^{n+s-1}} \, dy \right)^q = L^q \left( \int_{B(0, 1)} \frac{|U(x - z)|}{|z|^{n+s-1}} \, dz \right)^q
\]
\[
\leq L^q \int_{B(0, 1)} \frac{|U(x - z)|^q}{|z|^{n+s-1}} \, dz \left( \int_{B(0, 1)} \frac{1}{|z|^{n+s-1}} \, dz \right)^{q-1} = C \int_{B(0, 1)} \frac{|U(x - z)|^q}{|z|^{n+s-1}} \, dz.
\]
Integrating,
\[
\int g(x)\,dx \leq C \int \int_{B(0,1)} \frac{|U(x - z)|^q}{|z|^{n+s-1}}\,dz\,dx = C \int_{B(0,1)} \frac{1}{|z|^{n+s-1}} \int |U(x - z)|^q\,dx\,dz
\]
\[
= C \|U\|_{L^q(\mathbb{R}^n)} \int_{B(0,1)} \frac{1}{|z|^{n+s-1}}\,dz \leq C \|U\|_{L^q(\mathbb{R}^n)}^q.
\]
As for the term $h$, using that $\varphi$ is $\alpha$-Hölder and applying Hölder’s inequality,
\[
h(x) \leq L^q \left( \int_{B(x,1)\cap} \frac{|U(y)|}{|x - y|^{n+s-\alpha}}\,dy \right)^q \leq C \int_{B(0,1)\cap} \frac{|U(x - z)|^q}{|z|^{n+s-\alpha}}\,dz \left( \int_{B(0,1)\cap} \frac{1}{|z|^{n+s-\alpha}}\,dz \right)^{-1} \leq C \int_{B(0,1)\cap} \frac{|U(x - z)|^q}{|z|^{n+s-\alpha}}\,dz.
\]
Integrating,
\[
\int h(x)\,dx = C \int \int_{B(0,1)\cap} \frac{|U(x - z)|^q}{|z|^{n+s-\alpha}}\,dz\,dx = C \int_{B(0,1)\cap} \frac{1}{|z|^{n+s-\alpha}} \int |U(x - z)|^q\,dx\,dz
\]
\[
= C \|U\|_{L^q(\mathbb{R}^n)} \int_{B(0,1)\cap} \frac{1}{|z|^{n+s-\alpha}}\,dz \leq C \|U\|_{L^q(\mathbb{R}^n)}^q.
\]
Putting together (10), (11) and (12) we obtain
\[
\|K_\varphi(U)\|_{L^q(\mathbb{R}^n)} \leq C \|U\|_{L^q(\mathbb{R}^n)}^q,
\]
and the first part of the proof is finished.

Next we are going to see that the linear operator $K_\varphi : L^q(\mathbb{R}^n) \to L^1(\mathbb{R}^n, \mathbb{R}^n)$ is bounded in the case $\varphi$ has compact support. Denote by $F$ the support of $\varphi$. Then
\[
\int |K_\varphi(U)(x)|\,dx \leq |c_{n,s}| (A + B),
\]
where
\[
A := \int_F \int_{F'} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)|\,dy\,dx, \quad B := \int_F \int_{F'} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)|\,dy\,dx.
\]
Now, we observe that, applying Fubini’s Theorem, Hölder’s inequality and Lemma 3.1
\[
A \leq \int_F |U(y)| \int_F \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}}\,dx\,dy \leq C \left( \int_F |U(y)|^q\,dy \right)^{1/q} \leq C \|U\|_{L^q(\mathbb{R}^n)}.
\]
We notice that $|\varphi(x) - \varphi(y)| = 0$ for every $(x, y) \in F^c \times F^c$. Therefore, applying Hölder’s inequality and Lemma 3.1 we get
\[
B = \int_F \int_{F'} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)|\,dy\,dx
\]
\[
\leq \int_F \left( \int_{F'} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}}\,dy \right)^{1/\alpha'} \left( \int_{F'} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)|^q\,dy \right)^{1/q} \,dx
\]
\[
\leq C \int_F \left( \int_{F'} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)|^q\,dy \right)^{1/q} \,dx.
\]
Using again Hölder’s inequality, Lemma 3.1 and Fubini’s Theorem, we obtain
\[ B \leq C \left( \int_{F_c} \int_{F_c} |\varphi(x) - \varphi(y)| \left| U(y) \right|^q |x - y|^{n+s} \, dy \, dx \right)^{1/q} |F|^{1/q'} \]
\[ = C \left( \int_{F_c} |U(y)|^q \int_{F_c} |\varphi(x) - \varphi(y)| \, dx \, dy \right)^{1/q} \leq C \left( \int_{F_c} |U(y)|^q \, dy \right)^{1/q} \leq C \|U\|_{L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}, \]  
where $|F|$ denotes the measure of $F$. Inequalities (13), (14) and (15) lead us to
\[ \|K_\varphi(U)\|_{L^1(\mathbb{R}^n, \mathbb{R}^n)} \leq C \|U\|_{L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})}. \]
Finally, through a standard interpolation argument, we get that $K_\varphi$ is bounded from $L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$ to $L^p(\mathbb{R}^n, \mathbb{R}^n)$ for all $p \in [1, q]$. \hfill \Box

As a consequence of Lemma 3.2 and a general result, the operator $K_\varphi$ is continuous from the weak topology of $L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$ to the weak topology of $L^q(\mathbb{R}^n, \mathbb{R}^n)$ and, in the case of a $\varphi$ of compact support, from the weak topology of $L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$ to the weak topology of $L^p(\mathbb{R}^n, \mathbb{R}^n)$ for all $p \in [1, q]$.

Next we introduce a lemma about the spaces where the sequence $\{D^s u_j\}$ is convergent, provided that $\{u_j\}$ is so, in $H^{s,p}$ which will be useful later.

Lemma 3.3. Let $0 < s < 1$ and $1 < p < \infty$. Let $u \in H^{s,p}(\mathbb{R}^n)$ and let $\{u_j\}_{j \in \mathbb{N}} \subset H^{s,p}(\mathbb{R}^n)$ be a sequence converging to $u$ in $H^{s,p}(\mathbb{R}^n)$. Assume that there is a compact $K \subset \mathbb{R}^n$ such that $\bigcup_{j=1}^\infty \text{supp} \, u_j \subset K$. Then $D^s u_j \rightharpoonup D^s u$ in $L^r(\mathbb{R}^n)$ for every $r \in [1, p]$.

Proof. By linearity, we can assume that $u = 0$. Call $K_B = K + B(0, 1)$. Then
\[ \|D^s u_j\|_{L^1(K_B^{c})} \leq \|D^s u_j\|_{L^p(K_B)} |K_B|^{\frac{1}{p}} + \|D^s u_j\|_{L^1(K_B^{c})}, \]  
where $|K_B|$ denotes the Lebesgue measure of $K_B$, and $p'$ is the conjugate exponent of $p$.

On the other hand, for every $j \in \mathbb{N}$, we use Fubini’s Theorem and Hölder’s inequality to get
\[ \|D^s u_j\|_{L^1(K_B^{c})} = |c_{n,s}| \int_{K_B^{c}} \int_{K} \frac{u_j(x) - u_j(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|^{n+s}} \, dx \, dy \leq |c_{n,s}| \int_{K} \int_{K_B^{c}} \frac{|u_j(x) - u_j(y)|}{|x - y|^{n+s}} \, dx \, dy \]
\[ \leq \frac{1}{n+s} \int_{K} \left( \int_{K_B^{c}} \frac{|u_j(x) - u_j(y)|^p}{|x - y|^{n+s}} \, dx \right)^{\frac{1}{p}} \left( \int_{K_B^{c}} \frac{1}{|x - y|^{n+s}} \, dx \right)^{\frac{1}{p'}} \, dy. \]  
(17)

Now, for every $y \in K$ we have $K_B^{c} - y \subset B(0, 1)^c$, so
\[ \int_{K_B^{c}} \frac{1}{|x - y|^{n+s}} \, dx = \int_{K_B^{c} - y} \frac{1}{|z|^{n+s}} \, dz \leq \int_{B(0, 1)^c} \frac{1}{|z|^{n+s}} \, dz < \infty. \]

Now, we will use $C$ to denote a constant (depending on $n$, $s$ and $K$) which can vary through the proof. So, continuing from (17) and applying Hölder’s inequality again, we obtain
\[ \|D^s u_j\|_{L^1(K_B^{c})} \leq C \left( \int_{K} \int_{K_B^{c}} \frac{|u_j(x) - u_j(y)|^p}{|x - y|^{n+s}} \, dx \, dy \right)^{\frac{1}{p'}} \leq C \|u_j\|_{W^{s,p}(\mathbb{R}^n)} \leq C \|u_j\|_{H^{s,p}(\mathbb{R}^n)}, \]
where we have used Proposition 2.1 [11] in the last step. This inequality, together with (16), leads to
\[ \|D^s u_j\|_{L^1(\mathbb{R}^n)} \leq C \|u_j\|_{H^{s,p}(\mathbb{R}^n)} \to 0, \]
by assumption. Finally, through a standard interpolation argument, we obtain the convergence $D^s u_j \to 0$ in $L^r(\mathbb{R}^n)$ for every $r \in [1, p]$. \hfill \Box
Now we introduce a product formula for the $s$-fractional gradient. We denote by $I$ the identity matrix of dimension $n$.

**Lemma 3.4.** Let $0 < s < 1$ and $1 < p < \infty$. Let $g \in H^{s,p}(\mathbb{R}^n)$ and $\varphi \in C^1_c(\mathbb{R}^n)$. Then $\varphi g \in H^{s,p}(\mathbb{R}^n)$ and for a.e. $x \in \mathbb{R}^n$,

$$D^s(\varphi g)(x) = \varphi(x) D^s g(x) + K_\varphi(gI)(x).$$

**Proof.** Clearly $\varphi g \in L^p(\mathbb{R}^n)$. Now, for a.e. $x \in \mathbb{R}^n$ we have

$$D^s(\varphi g)(x) = c_{n,s} \text{PV}_x \int \frac{(\varphi g)(x) - (\varphi g)(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy$$

$$= c_{n,s} \text{PV}_x \int \frac{\varphi(x)g(x) - \varphi(x)g(y) + \varphi(x)g(y) - \varphi(y)g(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy$$

$$= \varphi(x) D^s g(x) + K_\varphi(gI)(x).$$

The term $\varphi D^s g$ is in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ since $\varphi \in C^1_c(\mathbb{R}^n)$, while the term $K_\varphi(gI)$ is in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ by Lemma 3.2.

Inspired by [25] (see also [29]) we introduce the $s$-fractional divergence $\text{div}^s$.

**Definition 3.1.** Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be measurable. Let $0 < s < 1$ and $x \in \mathbb{R}^n$ be such that

$$\int_{B(x,r)^c} \frac{|\phi(x) + \phi(y)|}{|x-y|^{n+s}} dy < \infty$$

for each $r > 0$. The $s$-fractional divergence of $\phi$ is defined as

$$\text{div}^s \phi(x) := -c_{n,s} \text{PV}_x \int \phi(x) + \phi(y) \frac{x-y}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy,$$

whenever the principal value exists.

Analogously, if $M : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ we denote by $\text{Div}^s M$ the column vector-function whose components are the $s$-fractional divergences of each row of $M$.

Similarly to what happened with the $s$-fractional gradient (see (5)), by symmetry, we have that

$$\text{div}^s \phi(x) = -c_{n,s} \text{PV}_x \int \phi(y) \frac{x-y}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy. \tag{18}$$

An initial property of the $s$-fractional divergence is the following, which states that it is well defined if and only if so is the $s$-fractional derivative. Its proof is analogous to that of [25, Lemma 2.3], and, hence, it will be omitted.

**Lemma 3.5.** Let $u : \mathbb{R}^n \to \mathbb{R}$ be measurable and let $x \in \mathbb{R}^n$ be such that $u(x)$ is defined and finite. Then

$$\text{PV}_x \int \frac{u(x) + u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy$$

exists and is finite if and only if

$$\text{PV}_x \int \frac{u(x) - u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy$$

exists and is finite. Moreover, in this case,

$$-c_{n,s} \text{PV}_x \int \frac{u(x) + u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy = c_{n,s} \text{PV}_x \int \frac{u(x) - u(y)}{|x-y|^{n+s}} \frac{x-y}{|x-y|} dy.$$
A consequence of Lemmas \[3.1\] and \[3.5\] is that for any \(0 < s < 1\) and \(\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)\), the number \(\text{div}^s \phi(x)\) is defined and finite for every \(x \in \mathbb{R}^n\); moreover, \(\text{div}^s \phi(x)\) is defined as a Lebesgue integral (without the need of the principal value).

The most important fact relating the \(s\)-fractional gradient and the \(s\)-fractional divergence is the integration by parts formula. The proof of this result follows the lines of [25, Th. 1.4].

**Theorem 3.6.** Let \(0 < s < 1\). Let \(u \in L^1_{\text{loc}}(\mathbb{R}^n)\) be such that

\[
\int_K \int \frac{|u(y) - u(x)|}{|x - y|^{n+s}} \, dy \, dx < \infty.
\]

for every compact \(K \subset \mathbb{R}^n\). Then \(D^s u \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)\) and for all \(\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)\),

\[
\int D^s u(x) \cdot \phi(x) \, dx = -\int u(x) \text{div}^s \phi(x) \, dx.
\]

**Proof.** Assumption (19) implies that \(D^s u\) exists a.e. as a Lebesgue integral and \(D^s u \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)\), we have

\[
\int D^s u(x) \cdot \phi(x) \, dx = c_{n,s} \int \int \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} \cdot \phi(x) \, dy \, dx.
\]

(20)

On the other hand, as \(\phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)\), by Lemma \[3.5\],

\[
-\int u(x) \text{div}^s \phi(x) \, dx = c_{n,s} \int \int u(x) \frac{\phi(x) + \phi(y)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} \, dy \, dx.
\]

(21)

Thus, it suffices to establish the equality of the right hand sides of (20) and (21); in fact, we will establish the equality of the double integrals in the domain \(D_\delta := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \geq \delta\}\) for each \(\delta > 0\). We have

\[
\int \int_{D_\delta} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} \cdot \phi(x) \, dy \, dx =
\]

\[
\int \int_{D_\delta} \frac{u(x) \phi(x)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} \, dy \, dx - \int \int_{D_\delta} \frac{u(y) \phi(y)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} \, dy \, dx.
\]

If we interchange now the roles of \(x\) and \(y\) in the second integral, using the symmetry of \(D_\delta\), we have

\[
-\int \int_{D_\delta} \frac{u(y) \phi(x)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} \, dy \, dx = \int \int_{D_\delta} \frac{u(x) \phi(y)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} \, dy \, dx,
\]

and therefore

\[
\int \int_{D_\delta} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} \cdot \phi(x) \, dy \, dx = \int \int_{D_\delta} \frac{u(x) \phi(x) + \phi(y)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} \, dy \, dx,
\]

whence the equality of the right hand sides of (20) and (21) follows.

As in Lemma \[3.4\] the following result computes the \(s\)-fractional divergence of a product.

**Lemma 3.7.** Let \(0 < s < 1\) and \(1 < p < \infty\). Let \(g \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)\) and \(\varphi \in C^1_c(\mathbb{R}^n)\). Then \(\varphi g \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)\) and for a.e. \(x \in \mathbb{R}^n\),

\[
\text{div}^s(\varphi g)(x) = \varphi(x) \text{div}^s g(x) + K_\varphi(g^T)(x).
\]
4 Fractional Piola Identity

In this section we introduce a fractional version of the Piola Identity. This is the main step in order to prove the existence of solutions for our fractional hyperelastic energy (1), since it will allow us to prove the weak continuity in $H^{s,p}$ of the determinant of the $s$-fractional gradient. Recall that the classical Piola identity asserts that, for smooth enough functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ one has $\text{Div} \, \text{cof} \, Du = 0$. Of course, cof denotes the cofactor matrix, which satisfies $\text{cof} \, AA^T = (\det A) \, I$ for every $A \in \mathbb{R}^{n \times n}$.

We start by reviewing a version of the change of variables formula for surface integrals ([27, Prop. 2.7]). Let $\Gamma$ be an oriented $(n-1)$-dimensional manifold with continuous unit normal field $\nu$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be affine and injective, with corresponding linear map $\widetilde{T}$. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be smooth. Then

$$\int_{\Gamma} g(T(x)) \cdot \text{cof} \, \widetilde{T} \nu(x) \, dS(x) = \int_{T(\Gamma)} g(x) \cdot \frac{\text{cof} \, \widetilde{T} \nu(T^{-1}x)}{|\text{cof} \, \widetilde{T} \nu(T^{-1}x)|} \, dS(x),$$

where $dS$ denotes the surface element. Now assume that $T$ is a symmetry across a hyperplane, so $T^{-1} = T$, $\det \widetilde{T} = -1$ and $T^{-1} = \widetilde{T} = -\text{cof} \, \widetilde{T}$. Therefore,

$$- \int_{\Gamma} g(T(x)) \cdot \widetilde{T} \nu(x) \, dS(x) = \int_{T(\Gamma)} g(x) \cdot \widetilde{T} \nu(Tx) \, dS(x).$$

Thus

$$\int_{\Gamma} \widetilde{T} g(T(x)) \cdot \nu(x) \, dS(x) = \int_{T(\Gamma)} \widetilde{T} g(x) \cdot \nu(Tx) \, dS(x).$$

As this is true for every $g$, we have that

$$\int_{\Gamma} g(x) \cdot \nu(x) \, dS(x) = \int_{T(\Gamma)} g(Tx) \cdot \nu(Tx) \, dS(x), \quad (22)$$

which is the formula we will use in Lemma 4.1.

In this and the next sections we will employ the following notation for the submatrices.

**Definition 4.1.** Let $k \in \mathbb{N}$ be with $1 \leq k \leq n$. Consider indices $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$.

a) We define $M = M_{i_1, \ldots, i_k; j_1, \ldots, j_k} : \mathbb{R}^{n \times n} \to \mathbb{R}^{k \times k}$ as the map such that $M(F)$ is the submatrix of $F \in \mathbb{R}^{n \times n}$ formed by the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$.

b) We define $\bar{M} = \bar{M}_{i_1, \ldots, i_k; j_1, \ldots, j_k} : \mathbb{R}^{k \times k} \to \mathbb{R}^{n \times n}$ as the map such that $\bar{M}(F)$ is the matrix whose rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$ coincide with those of $F$, whereas the rest of the entries are zero.

c) We define $N = N_{i_1, \ldots, i_k} : \mathbb{R}^n \to \mathbb{R}^k$ as the map such that $N(v)$ is the subvector of $v \in \mathbb{R}^n$ formed by the entries $i_1, \ldots, i_k$.

d) We define $\bar{N} = \bar{N}_{i_1, \ldots, i_k} : \mathbb{R}^k \to \mathbb{R}^n$ as the map such that $\bar{N}(v)$ is the vector whose entries $i_1, \ldots, i_k$ coincide with those of $v$, whereas the rest of the entries are zero.

e) We define $\tilde{N} = \tilde{N}_{i_1, \ldots, i_k} : \mathbb{R}^n \to \mathbb{R}^n$ as $\tilde{N} \circ N$.
The following formulas for the determinant will be useful. Given \( A \in \mathbb{R}^{n \times n} \), we express it as
\[
A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},
\]
where \( a_1, \ldots, a_n \in \mathbb{R}^n \) are its rows. Then \( \det A = a_i \cdot (\text{cof } A)_i \) for each \( i \in \{1, \ldots, n\} \), where \( (\text{cof } A)_i \) denotes the \( i \)-th row of \( \text{cof } A \). Now we realize that if \( b \in \mathbb{R}^n \) and
\[
A' = \begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ b \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix},
\]
then
\[
\det A' = (\text{cof } A)_i \cdot b. \tag{23}
\]

The following lemma establishes a useful bound for the principal value of an integral that will appear in the fractional Piola Identity.

**Lemma 4.1.** Let \( k \in \mathbb{N} \) be with \( 1 \leq k \leq n \). Consider indices \( 1 \leq j_1 < \cdots < j_k \leq n \) and let \( N = N_{j_1, \ldots, j_k} \) be the function of Definition 4.1. Then there exists a continuous function \( G : [0, \infty) \times (\mathbb{R}^n)^{k-1} \to \mathbb{R} \) such that for any \( a_1, \ldots, a_k \in \mathbb{R}^n \) and \( \epsilon_1, \ldots, \epsilon_k > 0 \) we have
\[
\left| \int_{(\mathbb{R}^n)^{k-1}} \det(N(x - a_1), \ldots, N(x - a_k)) \frac{dx}{|x - a_1|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \right| \leq \frac{\epsilon_1^{-s}}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} G(\epsilon_1, a_2 - a_1, \ldots, a_k - a_1).
\]

**Proof.** We can assume that the points \( a_1, \ldots, a_k \) do not lie on an affine manifold of dimension \( k - 2 \), since otherwise \( \det(N(x - a_1), \ldots, N(x - a_k)) = 0 \) for all \( x \in \mathbb{R}^n \).

Define \( h : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) as
\[
h(x) = \frac{-1}{(n + s - 1)|x|^{n+s-1}} \tag{24}
\]
and \( h_i : \mathbb{R}^n \setminus \{a_i\} \to \mathbb{R} \) as \( h_i(x) = h(x - a_i) \), for each \( i = 1, \ldots, k \). Define \( H : \mathbb{R}^n \setminus \{a_1, \ldots, a_k\} \to \mathbb{R}^k \) componentwise as \( H = (h_1, \ldots, h_k)^T \). Then
\[
DH(x) = \begin{pmatrix} \nabla h_1(x) \\ \vdots \\ \nabla h_k(x) \end{pmatrix} = \begin{pmatrix} \frac{x-a_1}{|x-a_1|^{n+s+1}} \\ \vdots \\ \frac{x-a_k}{|x-a_k|^{n+s+1}} \end{pmatrix}. \tag{25}
\]

Call \( \bar{j} = (j_1, \ldots, j_k) \) and denote by \( D_{\bar{j}}H \) the submatrix of \( DH \) formed by the columns \( j_1, \ldots, j_k \). Then, for all \( x \in \mathbb{R}^n \setminus \{a_1, \ldots, a_k\} \),
\[
\det D_{\bar{j}}H(x) = \frac{\det(N(x - a_1), \ldots, N(x - a_k))}{|x - a_1|^{n+s+1} \cdots |x - a_k|^{n+s+1}}. \tag{26}
\]
As \( DH \in L^p((\bigcup_{j=1}^{k} B(a_j, \epsilon_j))^c, \mathbb{R}^{n \times n}) \) for all \( p \in [1, \infty] \), we have \( \det D_{\bar{j}}H \in L^1((\bigcup_{j=1}^{k} B(a_j, \epsilon_j))^c) \). Therefore,
\[
\int_{(\bigcup_{j=1}^{k} B(a_j, \epsilon_j))^c} \det D_{\bar{j}}H = \lim_{R \to \infty} \int_{B(0,R) \setminus (\bigcup_{j=1}^{k} B(a_j, \epsilon_j))} \det D_{\bar{j}}H.
\]
As $H$ is smooth outside $\bigcup_{j=1}^{k} B(a_j, \epsilon_j)$, we have that
\[ \det D_jH = \text{div} \tilde{N}(h_1(\text{cof } D_jH)_1), \]
where $(\text{cof } D_jH)_1$ indicates the first row of $\text{cof } D_jH$, and $\tilde{N} = \tilde{N}_{j_1, \ldots, j_k}$ is the function of Definition 4.1. Let $R > 0$ be big enough so that $\bigcup_{j=1}^{k} \bar{B}(a_j, \epsilon_j) \subset B(0, R)$. Then, by the divergence theorem,
\[
\int_{\partial B(0,R) \setminus \bigcup_{j=1}^{k} B(a_j, \epsilon_j)} \det D_jH = -\int_{\partial \bigcup_{j=1}^{k} B(a_j, \epsilon_j)} \tilde{N}(h_1(\text{cof } D_jH)_1) \cdot \nu_j + \int_{\partial B(0,R)} \tilde{N}(h_1(\text{cof } D_jH)_1) \cdot \nu_R,
\]
where $\nu_j(x) = \frac{x - a_j}{\epsilon_j}$ in $\partial B(a_j, \epsilon_j)$ for $j = 1, \ldots, k$, and $\nu_R(x) = \frac{x}{R}$ in $\partial B(0, R)$. Having in mind the expressions (24) and (25), we find that, for some constant $C > 0$,
\[
\left| \int_{\partial B(0,R)} \tilde{N}(h_1(\text{cof } D_jH)_1) \cdot \nu_R \right| \leq \frac{C R^{(n+s)k-1}},
\]
which goes to zero as $R \to \infty$. Therefore,
\[
\int_{\left( \bigcup_{j=1}^{k} B(a_j, \epsilon_j) \right)^c} \det D_jH = -\int_{\partial \bigcup_{j=1}^{k} B(a_j, \epsilon_j)} \tilde{N}(h_1(\text{cof } D_jH)_1) \cdot \nu_j. \tag{27}
\]
For each $i = 1, \ldots, n$ we call
\[ A_i = \partial \left( \bigcup_{j=1}^{k} B(a_j, \epsilon_j) \right) \cap \partial B(a_i, \epsilon_i). \]

As a consequence of the inclusion $\partial \bigcup_{j=1}^{k} B(a_j, \epsilon_j) \subset \bigcup_{j=1}^{k} \partial B(a_j, \epsilon_j)$, we have that
\[ \partial \bigcup_{j=1}^{k} B(a_j, \epsilon_j) = \bigcup_{j=1}^{k} A_j. \]
Moreover, the $(n-1)$-dimensional area of $A_i \cap A_j$ is zero for $1 \leq i < j \leq k$. Figure 1 illustrates this situation when $k = n = 3$. Next, using (23) and (25), we have that for $j = 2, \ldots, k$ and $x \in \partial B(a_j, \epsilon_j)$,
\[
\tilde{N}(h_1(\text{cof } D_jH)_1) \cdot \nu_j(x) = \frac{\det(N(x - a_j), N(x - a_2), \ldots, N(x - a_k))}{|x - a_j| |x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} = 0.
\]
As a result, recalling (27) and the inclusion $A_j \subset \partial B(a_j, \epsilon_j)$, we have that
\begin{equation}
\int_{(\cup_{j=1}^n B(a_j, \epsilon_j))} \det D_x H \, dx = - \int_{A_1} \vec{N}(\operatorname{cof}(D_x H)_{1}) \cdot \nu_1 \, dS.
\end{equation}

Having in mind the expression (24), the multilinearity of the determinant and considering (23) and (25), we have that, for $x \in A_1$,
\begin{equation}
-\vec{N}(\operatorname{cof}(D_x H)_{1}) \cdot \nu_1(x) = \frac{1}{n+s-1} \frac{1}{\epsilon_1^{n+s}} \operatorname{cof}(D_x H)_{1} \cdot N(x - a_1)
\end{equation}
\begin{equation}
= \frac{1}{n+s-1} \frac{1}{\epsilon_1^{n+s}} \frac{\det(N(x - a_1), N(x - a_2), \ldots, N(x - a_k))}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}}
\end{equation}
\begin{equation}
= \frac{1}{n+s-1} \frac{1}{\epsilon_1^{n+s}} \frac{\det(N(x - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k))}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}}
\end{equation}
\begin{equation}
= \frac{1}{n+s-1} \frac{1}{\epsilon_1^{n+s-1}} \frac{(M(\operatorname{cof}(N(x - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k))))_{1}}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \cdot \nu_1(x),
\end{equation}
where $M = M_{i_1, \ldots, i_k; j_1, \ldots, j_k}$ is the function of Definition 4.1.

Let $\Pi_k$ be the only hyperplane in $\mathbb{R}^k$ passing through the points $N(a_1), \ldots, N(a_k)$, and consider one of the two unit normals $n \in \mathbb{R}^k$ to $\Pi_k$. Let $T_k : \mathbb{R}^k \to \mathbb{R}^k$ be the symmetry with respect to $\Pi_k$, so that for every $y \in \mathbb{R}^k$,
\begin{equation}
T_k y = y - 2(y - N(a_1)) \cdot n.
\end{equation}
Now fix a unit vector $m \in \mathbb{R}^n$ such that $N(m) = n$, and let $\Pi$ be the affine hyperplane in $\mathbb{R}^n$ with normal $m$ passing through $a_1$. Consider $T : \mathbb{R}^n \to \mathbb{R}^n$ as the symmetry across $\Pi$. Then, for all $x \in \mathbb{R}^n$,
\begin{equation}
T x = x - 2(x - a_1) \cdot m.
\end{equation}
Let $a_{k+1}, \ldots, a_n \in \Pi$ be such that the points $a_1, \ldots, a_n$ do not lie in an affine manifold of dimension $n - 2$. Define $A_1^{\pm} = \{x \in A_1 : \pm \det(x - a_1, a_1 - a_2, \ldots, a_1 - a_n) > 0\}$. Then $T(A_1^{\pm}) = A_1^{\mp}$, and $A_1^{\pm} \cup A_1^{-}$ cover $A_1$ up to a set of zero $(n - 1)$-measure; see Figure 2. Using the change of variables formula (22), we obtain
\begin{equation}
\int_{A_1^{-}} \frac{M(\operatorname{cof}(N(x - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k))))_{1}}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \cdot \nu_1(x) \, dS(x)
= \int_{A_1^{+}} \frac{M(\operatorname{cof}(N(T x - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k))))_{1}}{|T x - a_2|^{n+s+1} \cdots |T x - a_k|^{n+s+1}} \cdot \nu_1(T x) \, dS(x).
\end{equation}
Now, thanks to \([23]\), for \(x \in A_1^+\),
\[
\left(\tilde{M}(\text{cof}(N(Tx - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k)))\right)_1 \cdot \nu_1(Tx) = \frac{1}{\epsilon_1} \det(N(Tx - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k)). \quad (33)
\]

Let \(\tilde{T}_k : \mathbb{R}^k \rightarrow \mathbb{R}^k\) be the linear map corresponding to the affine map \(T_k\), and, analogously, \(\tilde{T}^i : \mathbb{R}^n \rightarrow \mathbb{R}^n\) the linear map corresponding to \(T\). We notice that \(\det(\tilde{T}_k) = -1\). Having in mind \((30)\) and \((31)\), we find that
\[
T_k y = y - 2y \cdot n, \quad y \in \mathbb{R}^k
\]
and
\[
Tx = x - 2x \cdot m, \quad x \in \mathbb{R}^n,
\]
from which we deduce that \(\tilde{T}_k \circ N = N \circ \tilde{T}^i\). Thus,
\[
\det(N(Tx - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k)) = \det(N(Tx - Ta_1), N(Ta_1 - Ta_2), \ldots, N(Ta_1 - Ta_k))
\]
\[
= \det(N(T\tilde{T}(x - a_1)), N(T\tilde{T}(a_1 - a_2)), \ldots, N(T\tilde{T}(a_1 - a_k)))
\]
\[
= \det(\tilde{T}_k(N(x - a_1)), \tilde{T}_k(N(a_1 - a_2)), \ldots, \tilde{T}_k(N(a_1 - a_k)))
\]
\[
= \det(\tilde{T}_k \circ N(x - a_1), (a_1 - a_2), \ldots, N(a_1 - a_k)) \quad (34)
\]

Putting together \((32)\), \((33)\) and \((34)\), we obtain that
\[
\int_{A_1^-} \frac{\det(N(x - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k))}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \, dS(x) = - \int_{A_1^+} \frac{\det(N(x - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k))}{|Tx - a_2|^{n+s+1} \cdots |Tx - a_k|^{n+s+1}} \, dS(x).
\]

Consequently, when we define \(f : \mathbb{R}^n \setminus \{a_2, \ldots, a_k\} \rightarrow \mathbb{R}\) as
\[
f(y) := \frac{1}{(|y - a_2| \cdots |y - a_k|)^{n+s+1}},
\]
we have that
\[
\int_{A_1^-} \frac{\det(N(x - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k))}{|x - a_2|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \, dS(x) = - \int_{A_1^+} \det(N(x - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k)) \, |f(x) - f(Tx)| \, dS(x). \quad (35)
\]

For every \(x \in A_1^+\), we join \(x\) with \(Tx\) by a curve \(\gamma_x\) inside \(A_1\), and note that the length of \(\gamma_x\) can be taken to be bounded by \(2\pi \epsilon_1\). Accordingly, let \(\gamma_x : [0, 1] \rightarrow A_1\) be of class \(C^1\) such that \(\gamma_x(0) = x, \quad \gamma_x(1) = Tx\) and \(|\gamma_x'|\) is constant with \(|\gamma_x'| \leq 2\pi \epsilon_1\). Then
\[
|f(x) - f(Tx)| = |f(\gamma_x(0)) - f(\gamma_x(1))| \leq \int_0^1 |\gamma_x'| |\nabla f(\gamma_x(t))| \, dt \leq 2\pi \epsilon_1 \int_0^1 |\nabla f(\gamma_x(t))| \, dt. \quad (36)
\]
We calculate
\[ |\nabla f(y)| = (n + s + 1) (|y - a_2| \cdots |y - a_k|)^{-n-s-2} \sum_{i=2}^{k} \prod_{j=2}^{k} |y - a_j|, \quad y \in \mathbb{R}^n \setminus \{a_2, \ldots, a_k\}. \]

Now, as \(|y - a_j| > \epsilon_j\) for every \(y \in A_1\) and \(j \in \{2, \ldots, k\},\)
\[ |\nabla f(y)| \leq \frac{n + s + 1}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \sum_{i=2}^{k} \prod_{j=2}^{k} |y - a_j| \leq \frac{n + s + 1}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \sum_{i=2}^{k} \prod_{j=2}^{k} (\epsilon_1 + |a_1 - a_j|), \]
so with (36) we obtain that
\[ |f(x) - f(Tx)| \leq 2\pi \epsilon_1 \frac{n + s + 1}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \sum_{i=2}^{k} \prod_{j=2}^{k} (\epsilon_1 + |a_1 - a_j|). \tag{37} \]

On the other hand, for all \(x \in A_1,\)
\[ |\det(N(x - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k))| \leq k! \prod_{j=2}^{k} |a_1 - a_j| = k! \epsilon_1 \prod_{j=2}^{k} |a_1 - a_j|. \tag{38} \]

Putting together (26), (28), (29), (35), (37) and (38), as well as the fact that the \((n-1)\)-dimensional area of \(A_1^\epsilon\) is bounded by a constant times \(\epsilon_1^{n-1},\) we obtain that, for a constant \(C > 0\) depending on \(n\) and \(s,\)
\[ \left| \int_{(\cup_{j=1}^{k} B(a_j, \epsilon_j)) \setminus \{x - a_1 \in \mathbb{R}^{n+s+1}, |x - a_k|^{n+s+1}\}} \frac{\det(N(x - a_1), N(a_1 - a_2), \ldots, N(a_1 - a_k))}{|x - a_1|^{n+s+1} \cdots |x - a_k|^{n+s+1}} \, dx \right| \leq \frac{C \epsilon_1^{1-s}}{(\epsilon_2 \cdots \epsilon_k)^{n+s+2}} \left( \prod_{j=2}^{k} |a_1 - a_j| \right) \sum_{i=2}^{k} \prod_{j=2}^{k} (\epsilon_1 + |a_1 - a_j|). \]

The existence of the function \(G\) of the statement follows. \(\Box\)

We are in a position to prove the fractional Piola Identity. Henceforth, supp denotes the support of a function.

**Theorem 4.2.** Let \(k \in \mathbb{N}\) be with \(1 \leq k \leq n.\) Consider indices \(1 \leq i_1 < \cdots < i_k \leq n\) and \(1 \leq j_1 < \cdots < j_k \leq n\) and the functions
\[ M = M_{i_1, \ldots, i_k, j_1, \ldots, j_k}, \quad \bar{M} = M_{i_1, \ldots, i_k, j_1, \ldots, j_k} \]
of Definition 4.1. Let \(u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)\) and \(s \in (0, 1).\) Then
\[ \text{Div}^s(\bar{M}(\text{cof} \, M(D^s u))) = 0. \]

**Proof.** Let
\[ N = N_{j_1, \ldots, j_k}, \quad \bar{N} = N_{j_1, \ldots, j_k} \]
be the maps of Definition 4.1. Naturally, \(\text{Div}^s(\bar{M}(\text{cof} \, M(D^s u))) = 0\) if and only if
\[ \text{div}^s \bar{N}(\text{cof} \, M(D^s u))_{i_\ell} = 0, \quad \ell = 1, \ldots, k. \]
We shall show $\text{div}^* \bar{N}((\text{cof } M(D^s u))_{i_1}) = 0$. The rest of the rows would proceed analogously. Using (18), we have that, for a.e. $x \in \mathbb{R}^n$,

$$\frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \text{div}^* \bar{N}((\text{cof } M(D^s u))_{i_1})(x) = \frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \text{pv}_x \int \frac{\bar{N}((\text{cof } M(D^s u))_{i_1})(x')}{|x' - x|^{n+s+1}} \cdot (x' - x) \, dx'. \quad (39)$$

Now, by (23) and (3), we have that for a.e. $x, x' \in \mathbb{R}^n$,

$$\frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \bar{N}((\text{cof } M(D^s u))_{i_1})(x') = \frac{(-1)^{k-1}}{c_{n,s}^{k-1}} \bar{N}(N(x' - x), N(D^s u_2(x')), \ldots, N(D^s u_k(x'))),$$

$$= \det \left( \frac{N(x' - x)}{|x' - x|^{n+s+1}}, \text{pv}_x \int \frac{u_{i_3}(y_2)N(x' - y_2)}{|x' - y_2|^{n+s+1}} \, dy_2, \ldots, \text{pv}_x \int \frac{u_{i_k}(y_k)N(x' - y_k)}{|x' - y_k|^{n+s+1}} \, dy_k \right),$$

$$= \lim_{\varepsilon_2 \to 0} \cdots \lim_{\varepsilon_k \to 0} f_{\varepsilon_2, \ldots, \varepsilon_k}^x(x'),$$

where for each $x \in \mathbb{R}^n$ and $\varepsilon_2, \ldots, \varepsilon_k > 0$, we have defined $f_{\varepsilon_2, \ldots, \varepsilon_k}^x : \mathbb{R}^n \to \mathbb{R}$ by

$$f_{\varepsilon_2, \ldots, \varepsilon_k}^x(x') := \det \left( \frac{N(x' - x)}{|x' - x|^{n+s+1}}, \int_{B(x', \varepsilon_2)^c} \frac{u_{i_3}(y_2)N(x' - y_2)}{|x' - y_2|^{n+s+1}} \, dy_2, \ldots, \int_{B(x', \varepsilon_k)^c} \frac{u_{i_k}(y_k)N(x' - y_k)}{|x' - y_k|^{n+s+1}} \, dy_k \right),$$

and we have used the continuity of the determinant. Let $\rho > 0$ be such that $\text{supp } u \subset B(x', \rho)$ for all $x' \in \text{supp } u$, and fix $\ell \in \{2, \ldots, k\}$. By odd symmetry, we have that

$$\int_{B(x', \varepsilon_j)^c} u_{i_\ell}(y) \frac{N(x' - y_\ell)}{|x' - y_\ell|^{n+s+1}} \, dy_\ell = \int_{B(x', \rho) \setminus B(x', \varepsilon_j)} u_{i_\ell}(y) \frac{N(x' - y_\ell)}{|x' - y_\ell|^{n+s+1}} \, dy_\ell,$n

$$= \int_{B(x', \rho) \setminus B(x', \varepsilon_j)} (u_{i_\ell}(y_\ell) - u_{i_\ell}(x'_\ell)) \frac{N(x' - y_\ell)}{|x' - y_\ell|^{n+s+1}} \, dy_\ell,$n

so, using the fact that $u$ is Lipschitz, we have, for some constant $L > 0$, that

$$\left| \int_{B(x', \varepsilon_j)^c} u_{i_\ell}(y_\ell) \frac{N(x' - y_\ell)}{|x' - y_\ell|^{n+s+1}} \, dy_\ell \right| \leq \left| \int_{B(x', \rho) \setminus B(x', \varepsilon_j)} \frac{|u_{i_\ell}(y_\ell) - u_{i_\ell}(x'_\ell)|}{|x' - y_\ell|^{n+s}} \, dy_\ell \right| \leq L \int_{B(x', \rho)} \frac{1}{|x' - y_\ell|^{n+s-1}} \, dy_\ell = L \int_{B(0, \rho)} \frac{1}{|y|^{n+s-1}} \, dy < \infty.$$n

This shows that

$$\left| f_{\varepsilon_2, \ldots, \varepsilon_k}^x(x') \right| \leq \frac{c}{|x' - x|^{n+s}},$$n

for some $c > 0$ only depending on $u$ and $n$. As

$$\int_{B(x, \varepsilon_1)^c} \frac{1}{|x' - x|^{n+s}} \, dx' < \infty,$n

for any $\varepsilon_1 > 0$, we can apply dominated convergence to conclude that

$$\int_{B(x, \varepsilon_1)^c} \lim_{\varepsilon_2 \to 0} \cdots \lim_{\varepsilon_k \to 0} f_{\varepsilon_2, \ldots, \varepsilon_k}^x(x') \, dx' = \lim_{\varepsilon_2 \to 0} \cdots \lim_{\varepsilon_k \to 0} \int_{B(x, \varepsilon_1)^c} f_{\varepsilon_2, \ldots, \varepsilon_k}^x(x') \, dx'.$$
Recalling (39) and (40), with this we obtain that
\[
\frac{(-1)^{k-1}}{\epsilon_{n,s}^k} \text{div}^s \tilde{N}((\text{cof } M(D^su))_{i_1}) (x) = \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \cdots \lim_{\varepsilon_k \to 0} \int_{B(x,\varepsilon_1)^c} f_{x_2,\ldots,x_k}(x') \, dx'.
\] (41)

Now for every \(\varepsilon_1, \ldots, \varepsilon_k > 0\) we call \(D_{\varepsilon_1, \ldots, \varepsilon_k} := B(x, \varepsilon_1) \cup \bigcup_{j=2}^k B(y_j, \varepsilon_j)\) and have that, thanks to the multilinearity of the determinant,
\[
\int_{B(x,\varepsilon_1)^c} f_{x_2,\ldots,x_k}(x') \, dx' = \int_{B(x,\varepsilon_1)^c} \int_{B(x',\varepsilon_2)^c} \cdots \int_{B(x'',\varepsilon_k)^c} \frac{\det (N(x' - x), u_{i_2}(y_2), \ldots, u_{i_k}(y_k))}{|x' - x|^n |x'' - y_2|^n |x' - y_k|^n} \, dy_2 \cdots dy_k \, dx'
\]
\[
= \int u_{i_k}(y_k) \cdots \int u_{i_2}(y_2) \int_{D_{\varepsilon_1, \ldots, \varepsilon_k}} \frac{\det (N(x' - x), N(x' - y_2), \ldots, N(x' - y_k))}{|x' - x|^n |x'' - y_2|^n |x' - y_k|^n} \, dy_2 \cdots dy_k.
\]

Call
\[
g(x, x', y_2, \ldots, y_k) := \frac{\det (N(x' - x), N(x' - y_2), \ldots, N(x' - y_k))}{|x' - x|^n |x'' - y_2|^n |x' - y_k|^n}.
\]

Then,
\[
\left| \int_{B(x,\varepsilon_1)^c} f_{x_2,\ldots,x_k}(x') \, dx' \right| \leq \left| \int_{\text{supp } u} \cdots \int_{\text{supp } u} \int_{D_{\varepsilon_1, \ldots, \varepsilon_k}} g(x, x', y_2, \ldots, y_k) \, dx' \right| dy_2 \cdots dy_k.\] (42)

Thanks to Lemma 4.1
\[
\left| \int_{D_{\varepsilon_1, \ldots, \varepsilon_k}} g(x, x', y_2, \ldots, y_k) \, dx' \right| \leq \frac{\epsilon_{k-1}^{-s}}{(\epsilon_1 \cdots \epsilon_{k-1})^{n+1}} G(\epsilon_k, x - y_k, y_2 - y_k, \ldots, y_{k-1} - y_k),\] (43)

where \(G\) is the function that appears therein. Integrating in (43), we find that
\[
\int_{\text{supp } u} \cdots \int_{\text{supp } u} \left| \int_{D_{\varepsilon_1, \ldots, \varepsilon_k}} g(x, x', y_2, \ldots, y_k) \, dx' \right| dy_2 \cdots dy_k \leq h(\varepsilon_k, x) \frac{\epsilon_{k-1}^{-s}}{(\epsilon_1 \cdots \epsilon_{k-1})^{n+1}}
\]
for some continuous function \(h : [0, \infty) \times \mathbb{R}^n \to [0, \infty)\). Consequently,
\[
\lim_{\varepsilon_k \to 0} \int_{\text{supp } u} \cdots \int_{\text{supp } u} \left| \int_{D_{\varepsilon_1, \ldots, \varepsilon_k}} g(x, x', y_2, \ldots, y_k) \, dx' \right| dy_2 \cdots dy_k = 0,
\]
and, in view of (41) and (42), we obtain that \(\text{div}^s \tilde{N}((\text{cof } M(D^su))_{i_1}) (x) = 0\). \(\square\)

5 Weak continuity of the determinant

In this section we prove that any minor (determinant of a submatrix) of \(D^su\) is a weakly continuous mapping in \(H^{s,p}\). We start by expressing a nonlocal integration by parts formula for the minors of \(D^su\) that involves the operator \(K_p\) of Lemma 3.2. Recall that for any \(F \in \mathbb{R}^{n \times n}\) and \(1 \leq i \leq n\) we denote by \(F_i\) the \(i\)-th row of \(F\).
Lemma 5.1. Let $k \in \mathbb{N}$ be with $1 \leq k \leq n$. Consider indices $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$ and the functions

\[ M = M_{i_1,j_1}, \quad \tilde{M} = \tilde{M}_{i_1,j_1}, \quad \tilde{N} = \tilde{N}_{i_1,j_1} \]

of Definition 4.1. Let $p \geq k - 1$, $q \geq \frac{p}{p-1}$ and $0 < s < 1$. Let $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ be such that $\text{cof} M(D^s u) \in \mathcal{L}^q(\mathbb{R}^n, \mathbb{R}^{k \times k})$. Then, $\det M(D^s u) \in L^1_{\text{loc}}(\mathbb{R}^n)$, and for every $\varphi \in C^\infty_c(\mathbb{R}^n)$ we have that $\tilde{N}(u) \cdot K_\varphi(\text{cof} M(D^s u)) \in L^1(\mathbb{R}^n)$ and

\[ \int \det M(D^s u)(x) \varphi(x) \, dx = -\frac{1}{k} \int \tilde{N}(u)(x) \cdot K_\varphi(\text{cof} M(D^s u))(x) \, dx. \]  

(44)

Proof. The fact $\det M(D^s u) \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a consequence of formula (23) and Hölder’s inequality, since $q \geq \frac{p}{p-1}$. Moreover, $\tilde{N}(u) \cdot K_\varphi(\text{cof} M(D^s u)) \in L^1(\mathbb{R}^n)$, since $\tilde{N}(u) \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $K_\varphi(\text{cof} M(D^s u)) \in L^r(\mathbb{R}^n, \mathbb{R}^n)$ for all $r \in [1, q]$ thanks to Lemma 3.2.

Assume first $u \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$ and let $\psi \in C^\infty_c(\mathbb{R}^n)$. Fix $x \in \mathbb{R}^n$ and $i \in \{i_1, \ldots, i_k\}$. By Lemma 3.7 and Theorem 4.2,

\[ \text{div}^s (\psi(\text{cof} M(D^s u)))(x) = K_\psi \left( (\text{cof} M(D^s u))^T \right)_i(x). \]

When we apply Theorem 3.6 to the constant function 1, we obtain from integration of the previous formula that

\[ 0 = \int \text{div}^s (\psi(\text{cof} M(D^s u)))(x) \, dx = \int K_\psi \left( (\text{cof} M(D^s u))^T \right)_i(x) \, dx. \]

By Fubini’s theorem and the definitions of $K_\psi$ and fractional gradient,

\[ \int K_\psi \left( (\text{cof} M(D^s u))^T \right)_i(x) \, dx = \int D^s \psi(y) \cdot (\text{cof} M(D^s u))_i(y) \, dy. \]

We thus have the equality

\[ \int D^s \psi(y) \cdot (\text{cof} M(D^s u))_i(y) \, dy = 0. \]

(45)

Now we assume that $u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ with $\text{cof} M(D^s u) \in \mathcal{L}^q(\mathbb{R}^n, \mathbb{R}^{k \times k})$, and, again $\psi \in C^\infty_c(\mathbb{R}^n)$. Taking into account Proposition 2.1, let $\{u_j\}_{j \in \mathbb{N}}$ be a sequence in $C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$ converging to $u$ in $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$. Then $M(D^s u_j) \rightarrow \text{cof} M(D^s u)$ in $L^p(\mathbb{R}^n, \mathbb{R}^{k \times k})$ and, hence, $\text{cof} M(D^s u_j) \rightarrow \text{cof} M(D^s u)$ in $L^{\frac{p}{p-1}}(\mathbb{R}^n, \mathbb{R}^{k \times k})$, so $M(\text{cof} M(D^s u_j)) \rightarrow \text{cof} M(\text{cof} M(D^s u))$ in $L^{\frac{p}{p-1}}(\mathbb{R}^n, \mathbb{R}^{n \times n})$. Therefore, (45) holds as well, since $D^s \psi \in L^r(\mathbb{R}^n)$ for all $r \in [1, p]$ (see Lemma 3.1). Now let $\psi \in H^{s,p}(\mathbb{R}^n)$ be of compact support, and let $\{\psi_j\}_{j \in \mathbb{N}}$ be a sequence in $C^\infty_c(\mathbb{R}^n)$ converging to $\psi$ in $H^{s,p}(\mathbb{R}^n)$ such that $\bigcup_{j \in \mathbb{N}} \text{supp} \psi_j$ is bounded. Then, by Lemma 3.3, $D^s \psi_j \rightarrow D^s \psi$ in $L^r(\mathbb{R}^n)$ for all $r \in [1, p]$. As $M(\text{cof} M(D^s u_j)) \in L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$, we have that (45) holds as well. To sum up, formula (45) is valid for any $u \in H^{s,p}(\mathbb{R}^n)$ with $\text{cof} M(D^s u) \in L^q(\mathbb{R}^n, \mathbb{R}^{k \times k})$ and any $\psi \in H^{s,p}(\mathbb{R}^n)$ of compact support.

We apply (45) to $\psi = \varphi u_i$, which is in $H^{s,p}(\mathbb{R}^n)$ thanks to Lemma 3.4, and has compact support since do $\varphi$. By the formula for $D^s \psi$ given by Lemma 3.4, we obtain that

\[ 0 = \int \varphi(y) D^s u_i(y) \cdot (\text{cof} M(D^s u))_i(y) \, dy + \int K_\varphi(u_i)(y) \cdot (\text{cof} M(D^s u))_i(y) \, dy. \]

(46)

Using formula (23), the fact $i \in \{i_1, \ldots, i_k\}$ and elementary properties of the functions of Definition 4.1, we find that for any $F \in \mathbb{R}^{n \times n}$,

\[ F_i \cdot (\text{cof} M(F))_i = \det M(F). \]
Using this and Fubini’s theorem, from (46) we arrive at
\[ 0 = \int \varphi(y) \det M(D^s u)(y) \, dy + c_{n,s} \int u_i(x) \int \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+s}} (\tilde{M}(\text{cof } M(D^s u)))_i(y) \cdot \frac{x-y}{|x-y|} \, dy \, dx. \]
We sum this equality for \( i = i_1, \ldots, i_k \) and obtain that
\[ 0 = k \int \varphi(y) \det M(D^s u)(y) \, dy + c_{n,s} \int \tilde{N}(u)(x) \cdot \int \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+s}} (\tilde{M}(\text{cof } M(D^s u))) \, \frac{x-y}{|x-y|} \, dy \, dx, \]
which is the required formula.

Now we establish the closedness and continuity properties of the minors of \( D^s u \) in the weak topology of \( H^{s,p} \). In the notation of Definition [4.1(a)], a minor of order \( k \) is a function \( \mu : \mathbb{R}^{n \times n} \to \mathbb{R} \) such that there exist \( 1 \leq i_1 < \cdots < i_k \leq n \) and \( 1 \leq j_1 < \cdots < j_k \leq n \) for which \( \mu(F) = \det M(F) \) for all \( F \in \mathbb{R}^{n \times n} \). Recall the notation \( p^* \) of Theorem 2.2 and the affine space \( H^{s,p}_g \) of (3).

**Theorem 5.2.** Let \( p \geq n - 1 \) and \( 0 < s < 1 \). Let \( g \in H^{s,p}(\mathbb{R}^n) \) and \( u \in H^{s,p}(\Omega, \mathbb{R}^n) \). Let \( \{u_j\}_{j \in \mathbb{N}} \) be a sequence in \( H^{s,p}_g(\Omega, \mathbb{R}^n) \) such that \( u_j \rightharpoonup u \) in \( H^{s,p}(\mathbb{R}^n) \). Then

**a)** If \( k \in \mathbb{N} \) with \( 1 \leq k \leq n - 2 \) and \( \mu \) is a minor of order \( k \) then \( \mu(D^s u_j) \rightharpoonup \mu(D^s u) \) in \( L^\frac{p}{p-k}(\mathbb{R}^n) \) as \( j \to \infty \).

**b)** If \( \text{cof } D^s u_j \rightharpoonup \vartheta \) in \( L^q(\mathbb{R}^n, \mathbb{R}^{n \times n}) \) for some \( q \in [1, \infty) \) and \( \vartheta \in L^q(\mathbb{R}^n, \mathbb{R}^{n \times n}) \) then \( \vartheta = \text{cof } D^s u \).

**c)** Assume \( \det D^s u_j \to \theta \) in \( L^\ell(\mathbb{R}^n) \) for some \( \ell \in [1, \infty) \) and some \( \theta \in L^\ell(\mathbb{R}^n) \). If \( sp < n \) assume, in addition, that \( \text{cof } D^s u_j \to \text{cof } D^s u \) in \( L^q(\mathbb{R}^n, \mathbb{R}^{n \times n}) \) for some \( q \in (\frac{sp}{p-k}, \infty) \). Then \( \theta = \det D^s u \).

**Proof.** We will prove by induction on \( k \). For \( k = 1 \) the result is trivial. Assume it holds for some \( k \leq n - 3 \) and let us prove it for \( k + 1 \). Let \( \mu \) be a minor of order \( k + 1 \). In the notation of Definition [4.1(a)], \( \mu(F) = \det M(F) \) for all \( F \in \mathbb{R}^{n \times n} \), where \( M = M_{i_1, \ldots, i_{k+1}; j_1, \ldots, j_{k+1}} \) for some \( 1 \leq i_1 < \cdots < i_{k+1} \leq n \) and \( 1 \leq j_1 < \cdots < j_{k+1} \leq n \). Let \( \varphi \in C^\infty_c(\mathbb{R}^n) \). By induction assumption, \( \text{cof } M(D^s u_j) \rightharpoonup \text{cof } M(D^s u) \) in \( L^\frac{p}{p-k}(\mathbb{R}^n, \mathbb{R}^{(k+1)\times (k+1)}) \) as \( j \to \infty \), so \( \tilde{M}(\text{cof } M(D^s u_j)) \rightharpoonup \tilde{M}(\text{cof } M(D^s u)) \) in \( L^\frac{p}{p-k}(\mathbb{R}^n, \mathbb{R}^{(k+1)\times (k+1)}) \) by Lemma 3.2. By Theorem 2.3, \( K_\varphi(\tilde{M}(\text{cof } M(D^s u_j))) \rightharpoonup K_\varphi(\tilde{M}(\text{cof } M(D^s u))) \) in \( L^r(\mathbb{R}^n, \mathbb{R}^n) \) for every \( r \in [1, \frac{p}{p-k}] \). Therefore, \( \tilde{N}(u_j) \to \tilde{N}(u) \) in \( L^p(\mathbb{R}^n) \), so

\[ \tilde{N}(u_j) \cdot K_\varphi(\tilde{M}(\text{cof } M(D^s u_j))) \to \tilde{N}(u) \cdot K_\varphi(\tilde{M}(\text{cof } M(D^s u))) \] in \( L^1(\mathbb{R}^n) \) \hfill (47)

since \( \frac{k}{p} + \frac{1}{p} \leq 1 \). We apply Lemma 5.1 and, in particular, formula (14) to conclude that
\[ \int \det M(D^s u_j(x)) \varphi(x) \, dx \to \int \det M(D^s u(x)) \varphi(x) \, dx. \] \hfill (48)

This shows that \( \det M(D^s u_j) \rightharpoonup \det M(D^s u) \) in the sense of distributions. As \( \{\det M(D^s u_j)\}_{j \in \mathbb{N}} \) is bounded in \( L^\frac{p}{p+1}(\mathbb{R}^n) \) and \( p > k + 1 \), we have that \( \det M(D^s u_j) \to \det M(D^s u) \) in \( L^\frac{p}{p+1}(\mathbb{R}^n) \). The proof of follows the lines of (a). Let \( \mu \) be a minor of order \( n - 1 \). In the notation of Definition [4.1(a)], \( \mu(F) = \det M(F) \) for all \( F \in \mathbb{R}^{n \times n} \), where \( M = M_{i_1, \ldots, i_{n-1}; j_1, \ldots, j_{n-1}} \) for some \( 1 \leq i_1 < \cdots < i_{n-1} \leq n \) and \( 1 \leq j_1 < \cdots < j_{n-1} \leq n \). Let \( \varphi \in C^\infty_c(\Omega) \). By part (a), \( \text{cof } M(D^s u_j) \rightharpoonup \text{cof } M(D^s u) \) in \( L^\frac{p}{p-k}(\mathbb{R}^n, \mathbb{R}^{(n-1)\times (n-1)}) \), so \( \tilde{M}(\text{cof } M(D^s u_j)) \rightharpoonup \tilde{M}(\text{cof } M(D^s u)) \) in \( L^\frac{p}{p-k}(\mathbb{R}^n, \mathbb{R}^{(n-1)\times (n-1)}) \). By Lemma 3.2, \( K_\varphi(\tilde{M}(\text{cof } M(D^s u_j))) \rightharpoonup K_\varphi(\tilde{M}(\text{cof } M(D^s u))) \) in \( L^r(\mathbb{R}^n, \mathbb{R}^n) \) for every \( r \in [1, \frac{p}{p-k-1}] \). By Theorem 2.3, \( \tilde{N}(u_j) \to \tilde{N}(u) \) in \( L^p(\mathbb{R}^n) \), so convergence (47) is also valid since \( \frac{n-2}{p} + \frac{1}{p} \leq 1 \). Thanks to (14), we conclude that convergence (48) holds. This shows that \( \mu(D^s u_j) \rightharpoonup \mu(D^s u) \) in the sense of
distributions. As this is true for every minor $\mu$ of order $n-1$, we obtain that $\text{cof } D^*u_j \rightarrow \text{cof } D^*u$ in the sense of distributions. Thanks to the assumption, $\vartheta = \text{cof } D^*u$.

We finally show part $[c]$. Let $\varphi \in C^\infty(\Omega)$. Assume first $sp < n$. By the assumption and Lemma 3.2, $K_\varphi(\text{cof } D^*u_j) \rightarrow K_\varphi(\text{cof } D^*u)$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in [1, q]$. By Theorem 2.3, $u_j \rightarrow u$ in $L^r(\mathbb{R}^n)$ for every $t \in [1, p')$, so

$$u_j \cdot K_\varphi(\text{cof } D^*u_j) \rightarrow u_j \cdot K_\varphi(\text{cof } D^*u) \quad \text{in } L^1(\mathbb{R}^n)$$

(49)

since $\frac{1}{q} + \frac{1}{p'} < 1$.

Assume now $sp \geq n$. Then $\{\text{cof } D^*u_j\}_{j \in \mathbb{N}}$ is bounded in $L^{\frac{p}{n-1}}(\mathbb{R}^n, \mathbb{R}^{n \times n})$ so, thanks to part $[b]$, $\text{cof } D^*u_j \rightarrow \text{cof } D^*u$ in $L^{\frac{p}{n-1}}(\mathbb{R}^n, \mathbb{R}^{n \times n})$. By Lemma 3.2, $K_\varphi(\text{cof } D^*u_j) \rightarrow K_\varphi(\text{cof } D^*u)$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in [1, \frac{p}{n-1}]$. By Theorem 2.3, $u_j \rightarrow u$ in $L^r(\mathbb{R}^n)$ for every $t \in [1, \infty)$, so convergence (49) holds since $p > n-1$.

In either case, we have convergence (49), so by (44) we obtain

$$\int \det D^*u_j(x) \varphi(x) \, dx \rightarrow \int \det D^*u(x) \varphi(x) \, dx.$$ 

This shows that $\det D^*u_j \rightarrow \det D^*u$ in the sense of distributions, so $\vartheta = \det D^*u$. \qed

6 Existence of minimizers

In this section we prove the existence of minimizers in $H^{s,p}$ of functionals of the form

$$I(u) := \int W(x, u(x), D^s u(x)) \, dx.$$  

(50)

under natural coercivity and polyconvexity assumptions.

We recall the concept of polyconvexity (see, e.g., [2, 12]). Let $\tau$ be the number of submatrices of an $n \times n$ matrix. We fix a function $\bar{\mu} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^\tau$ such that $\bar{\mu}(F)$ is the collection of all minors of an $F \in \mathbb{R}^{n \times n}$ in a given order. A function $W_0 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \cup \{\infty\}$ is polyconvex if there exists a convex $\Phi : \mathbb{R}^\tau \rightarrow \mathbb{R} \cup \{\infty\}$ such that $W_0(F) = \Phi(\bar{\mu}(F))$ for all $F \in \mathbb{R}^{n \times n}$.

The existence theorem of this paper is as follows. Its proof relies on a standard argument in the calculus of variations, once we have the continuity (with respect to the weak convergence) of the minors given by Theorem 5.2.

**Theorem 6.1.** Let $p \geq n-1$ satisfy $p > 1$ and $0 < s < 1$. Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:

a) $W$ is $\mathcal{L} \times \mathcal{B} \times \mathcal{B}^{n \times n}$-measurable, where $\mathcal{L}$ denotes the Lebesgue sigma-algebra in $\mathbb{R}^n$, whereas $\mathcal{B}$ and $\mathcal{B}^{n \times n}$ denote the Borel sigma-algebras in $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$, respectively.

b) $W(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^n$.

c) For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x, y, \cdot)$ is polyconvex.

d) There exist a constant $c > 0$, an $a \in L^1(\mathbb{R}^n)$ and a Borel function $h : [0, \infty) \rightarrow [0, \infty)$ such that

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} = \infty$$

23
and
\[
\begin{cases}
W(x, y, F) \geq a(x) + c|F|^p + c|\text{cof } F|^q + h(|\det F|) & \text{for some } q > \frac{p^*}{p-1}, \\
W(x, y, F) \geq a(x) + c|F|^p,
\end{cases}
\]
for a.e. \( x \in \mathbb{R}^n \), all \( y \in \mathbb{R}^n \) and all \( F \in \mathbb{R}^{n \times n} \).

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \). Let \( u_0 \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \). Define \( I \) as in (50), and assume that \( I \) is not identically infinity in \( H_{u_0}^{s,p}(\Omega, \mathbb{R}^n) \). Then there exists a minimizer of \( I \) in \( H_{u_0}^{s,p}(\Omega, \mathbb{R}^n) \).

**Proof.** Assumption (d) shows that the functional \( I \) is bounded below by \( \int a \). As \( I \) is not identically infinity in \( H_{u_0}^{s,p}(\Omega, \mathbb{R}^n) \), there exists a minimizing sequence \( \{u_j\}_{j \in \mathbb{N}} \) of \( I \) in \( H_{u_0}^{s,p}(\Omega, \mathbb{R}^n) \). Assumption (d) implies that \( \{D^s u_j\}_{j \in \mathbb{N}} \) is bounded in \( L^p(\mathbb{R}^n, \mathbb{R}^{n \times n}) \). Thanks to Theorem 2.2, \( \{u_j\}_{j \in \mathbb{N}} \) is bounded in \( L^p(\Omega, \mathbb{R}^{n \times n}) \). As \( u_j = u_0 \) in \( \Omega^c \) for all \( j \in \mathbb{N} \), we also have that \( \{u_j\}_{j \in \mathbb{N}} \) is bounded in \( L^p(\mathbb{R}^n, \mathbb{R}^n) \), and, consequently, also in \( H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \). As \( H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \) is reflexive, we can extract a weakly convergent subsequence. Using Theorem 2.3, we obtain that there exists \( u \in H^{s,p}_{u_0}(\mathbb{R}^n, \mathbb{R}^n) \) such that for a subsequence (not relabelled),
\[
u_j \rightharpoonup u \text{ in } H^{s,p}(\mathbb{R}^n, \mathbb{R}^n) \quad \text{and} \quad u_j \rightarrow u \text{ in } L^p(\mathbb{R}^n, \mathbb{R}^n).
\]

Now, by Theorem 5.2 for any minor \( \mu \) of order \( k \leq n-2 \), we have that
\[
\mu(D^s u_j) \rightharpoonup \mu(D^s u) \text{ in } L^p(\mathbb{R}^n).
\]

If \( sp < n \) then, by assumption (d), \( \{\text{cof } D^s u_j\}_{j \in \mathbb{N}} \) is bounded in \( L^q(\mathbb{R}^n, \mathbb{R}^{n \times n}) \), whereas if \( sp \geq n \) we call \( q := \frac{p^*}{n-1} \) and have that \( \{\text{cof } D^s u_j\}_{j \in \mathbb{N}} \) is bounded in \( \ell_q(\mathbb{R}^n, \mathbb{R}^{n \times n}) \). In either case we have that \( q > 1 \), so for a subsequence \( \{\text{cof } D^s u_j\}_{j \in \mathbb{N}} \) converges weakly in \( L^q(\mathbb{R}^n, \mathbb{R}^{n \times n}) \) and, by Theorem 5.2
\[
\text{cof } D^s u_j \rightharpoonup \text{cof } D^s u \text{ in } L^q(\mathbb{R}^n, \mathbb{R}^{n \times n}).
\]

If \( sp < n \) then, by assumption (d) and de la Vallée Poussin’s criterion, \( \{\det D^s u_j\}_{j \in \mathbb{N}} \) is equiintegrable, whereas if \( sp \geq n \) we have that \( \{\det D^s u_j\}_{j \in \mathbb{N}} \) is bounded in \( L^\ell(\mathbb{R}^n) \) and \( \frac{p}{n} > 1 \). In either case we have that, for a subsequence \( \{\det D^s u_j\}_{j \in \mathbb{N}} \) converges weakly in \( L^\ell(\mathbb{R}^n) \) with
\[
\left\{ \begin{array}{ll}
\ell = 1 & \text{if } sp < n, \\
\ell = \frac{p}{n} & \text{if } sp \geq n,
\end{array} \right.
\]
and, hence, by Theorem 5.2
\[
\det D^s u_j \rightharpoonup \det D^s u \text{ in } L^\ell(\mathbb{R}^n).
\]

Convergences (51)–(54) imply, thanks to a standard lower semicontinuity result for polyconvex functionals (see, e.g., [24 Th. 5.4] or [17 Th. 7.5]), that for any \( R > 0 \),
\[
\int_{B(0,R)} W(x, u(x), D^s u(x)) \, dx \leq \liminf_{j \to \infty} \int_{B(0,R)} W(x, u_j(x), D^s u_j(x)) \, dx.
\]
Therefore,
\[
\int_{B(0,R)} (W(x, u(x), D^s u(x)) - a(x)) \, dx \leq \liminf_{j \to \infty} \int_{B(0,R)} (W(x, u_j(x), D^s u_j(x)) - a(x)) \, dx \leq \liminf_{j \to \infty} \int_{B(0,R)} (W(x, u_j(x), D^s u_j(x)) - a(x)) \, dx.
\]
By monotone convergence,
\[
\int (W(x, u(x), D^s u(x)) - a(x)) \, dx \leq \liminf_{j \to \infty} \int (W(x, u_j(x), D^s u_j(x)) - a(x)) \, dx,
\]
so
\[
I(u) \leq \liminf_{j \to \infty} I(u_j).
\]
Therefore, \( u \) is a minimizer of \( I \) in \( H^{s,p}_{u_0}(\Omega, \mathbb{R}^n) \) and the proof is concluded. \( \square \)

Comparing Lemmas 2.4 and 2.5 with Theorem 6.1, we see that fracture and cavitation are compatible with the existence result of Theorem 6.1 in opposition to the case of classical elasticity (see, e.g., \cite{2, 3, 4, 5, 19, 8}). In fact, for a \( u \in H^{s,p}(\mathbb{R}^n) \) of compact support and \( p > n \), by Hölder’s inequality and Lemma 3.3, \( \text{cof} \, D^s u \in L^q(\mathbb{R}^n, \mathbb{R}^{n \times n}) \) for every \( q \in [1, \frac{p}{n-1}] \) and \( \det D^s u \in L^r(\mathbb{R}^n) \) for every \( r \in [1, \frac{p}{n}] \). Take now an \( s \in (0,1) \) such that \( sp < n \), so that this regime is compatible with cavitation (see Lemma 2.5). Considering the function \( h \) of Theorem 6.1 as \( h(t) := t^\frac{p}{n} \), we see that this map \( u \) is compatible with the assumptions of Theorem 6.1 if and only if \( \frac{p}{n-1} > \frac{p^*}{p^* - 1} \), so \( n^2 - np < sp \).

To sum up, in the regime
\[
p > n, \quad 0 < s < \frac{n}{p}
\]
a typical cavitation map is compatible with the hypothesis of Theorem 6.1. Similarly, if \( p > n \) and \( n^2 - np < sp < 1 \), i.e., in the regime
\[
p > n, \quad 0 < s < \frac{1}{p}
\]
the hypothesis of Theorem 6.1 are compatible with fracture.

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