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COLORING GRAPHS WITH NO INDUCED SUBDIVISION OF $K_4^+$

LOUIS ESPERET AND NICOLAS TROTIGNON

Abstract. Let $K_4^+$ be the 5-vertex graph obtained from $K_4$, the complete graph on four vertices, by subdividing one edge precisely once (i.e. by replacing one edge by a path on three vertices). We prove that if the chromatic number of some graph $G$ is much larger than its clique number, then $G$ contains a subdivision of $K_4^+$ as an induced subgraph.

Given a graph $H$, a subdivision of $H$ is a graph obtained from $H$ by replacing some edges of $H$ (possibly none) by paths. We say that a graph $G$ contains an induced subdivision of $H$ if $G$ contains a subdivision of $H$ as an induced subgraph.

A class of graphs $F$ is said to be $\chi$-bounded if there is a function $f$ such that for any graph $G \in F$, $\chi(G) \leq f(\omega(G))$, where $\chi(G)$ and $\omega(G)$ stand for the chromatic number and the clique number of $G$, respectively.

Scott [7] conjectured that for any graph $H$, the class of graphs without induced subdivisions of $H$ is $\chi$-bounded, and proved it when $H$ is a tree. But Scott’s conjecture was disproved in [1] that every graph $H$ obtained from the complete graph $K_4$ by subdividing at least 4 of the 6 edges once (in such a way that the non-subdivided edges, if any, are non-incident), is a counterexample to Scott’s conjecture. On the other hand, Scott proved that the class of graphs with no induced subdivision of $K_4$ has bounded chromatic number (see [5]). Le [4] proved that every graph in this class has chromatic number at most 24. If triangles are also excluded, Chudnovsky et al. [2] proved that the chromatic number is at most 3.

In this paper, we extend the list of graphs known to satisfy Scott’s conjecture. Let $K_4^+$ be the 5-vertex graph obtained from $K_4$ by subdividing one edge precisely once.

Theorem 1. The family of graphs with no induced subdivision of $K_4^+$ is $\chi$-bounded.

We will need the following result of Kühn and Osthus [3].

Theorem 2 ([3]). For any graph $H$ and every integer $s$ there is an integer $d = d(H, s)$ such that every graph of average degree at least $d$ contains the complete bipartite graph $K_{s,s}$ as a subgraph, or an induced subdivision of $H$.

Proof of Theorem 7. Let $k$ be an integer, let $d(\cdot, \cdot)$ be the function defined in Theorem 2 and let $R(s, t)$ be the Ramsey number of $(s, t)$, i.e. the smallest $n$ such that every graph on $n$ vertices has a stable set of size $s$ or a clique of size $t$.

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We will prove that every graph \( G \) with no induced subdivision of \( K_4^+ \), and with clique number at most \( k \), is \( d \)-colorable, with \( d = \max(k, d(K_4^+, R(4, k))) \). The proof proceeds by induction on the number of vertices of \( G \) (the result being trivial if \( G \) has at most \( k \) vertices). Observe that all induced subgraphs of \( G \) have clique number at most \( k \) and do not contain any induced subdivision of \( K_4^+ \). Therefore, by the induction, we can assume that all induced subgraphs of \( G \) are \( d \)-colorable. In particular, we can assume that \( G \) is connected.

Assume first that \( G \) does not contain \( K_{4,s} \) as a subgraph, where \( s = R(4, k) \). Then by Theorem 2, \( G \) has average degree less than \( d \), and hence contains a vertex of degree at most \( d - 1 \). By the induction, \( G - v \) has a \( d \)-coloring and this coloring can be extended to a \( d \)-coloring of \( G \), as desired.

We can thus assume that \( G \) contains \( K_{4,s} \) as a subgraph. Since \( G \) has clique number at most \( k \), it follows from the definition of \( R(4, k) \) that \( G \) contains \( K_{4,4} \) as an induced subgraph. Let \( M \) be a set of vertices of \( G \) inducing a complete multipartite graph with at least two partite sets containing at least 4 vertices. Assume that among all such sets of vertices of \( G \), \( M \) is chosen with maximum cardinality. Let \( V_1, V_2, \ldots, V_t \) be the partite sets of \( M \).

Let \( v \) be a vertex of \( G \), and \( S \) be a set of vertices not containing \( v \). The vertex \( v \) is complete to \( S \) if \( v \) is adjacent to all the vertices of \( S \), anticomplete to \( S \) if \( v \) is not adjacent to any of the vertices of \( S \), and mixed to \( S \) otherwise. Let \( R \) be the vertices of \( G \) not in \( M \). We can assume that \( R \) is non-empty, since otherwise \( G \) is clearly \( k \)-colorable and \( k \leq d \). We claim that:

If a vertex \( v \) of \( R \) has at least two neighbors in some set \( V_i \), then it is not mixed to any set \( V_j \) with \( j \neq i \). 

Assume for the sake of contradiction that \( v \) has two neighbors \( a, b \) in \( V_i \) and a neighbor \( c \) and a non-neighbor \( d \) in \( V_j \), with \( j \neq i \). Then \( v, a, b, c, d \) induce a copy of \( K_4^+ \), a contradiction. This proves (1).

Each vertex \( v \) of \( R \) has at most one neighbor in each set \( V_i \). 

Assume for the sake of contradiction that some vertex \( v \in R \) has two neighbors \( a, b \) in some set \( V_i \). Then by (1), \( v \) is complete or anticomplete to each set \( V_j \) with \( j \neq i \). Let \( \mathcal{A} \) be the family of sets \( V_j \) to which \( v \) is anticomplete, and let \( \mathcal{C} \) be the family of sets \( V_j \) to which \( v \) is complete. If \( \mathcal{A} \) contains at least two elements, i.e. if \( v \) is anticomplete to two sets \( V_j \) and \( V_j' \), then by taking \( u \in V_j \) and \( u' \in V_j' \), we observe that \( v, a, b, u, u' \) induces a copy of \( K_4^+ \), a contradiction. It follows that \( \mathcal{A} \) contains at most one element.

Next, we prove that \( v \) is complete to \( V_i \). Assume instead that \( v \) is mixed to \( V_i \). If \( v \) is complete to some set \( V_j \) containing at least two vertices, then we obtain a contradiction with (1). It follows that all the elements of \( \mathcal{C} \) are singleton. By the definition of \( M \), this implies that \( \mathcal{A} \) contains exactly one set \( V_j \), which has size at least 4. Let \( c \) be a non-neighbor of \( v \) in \( V_i \), and let \( d, d' \) be two vertices in \( V_j \). Then \( v, a, b, c, d, d' \) is an induced subdivision of \( K_4^+ \), a contradiction. We proved that \( v \) is complete to \( V_i \). Hence, every set
Each connected component of $G - M$ has at most one neighbor in each set $V_i$.  

Assume for the sake of contradiction that some connected component of $G - M$ has at least two neighbors in some set $V_i$. Then there is a path $P$ whose endpoints $u, v$ are in $V_i$, and whose internal vertices are in $R$. Choose $P, u, v, V_i$ such that $P$ contains the least number of edges. Note that by (2), $P$ contains at least 3 edges. Observe also that by the minimality of $P$, the only edges in $G$ between $V_i$ and the internal vertices of $P$ are the first and last edge of $P$. Let $V_j$ be a partite set of $M$ with at least 4 elements, with $j \neq i$ (this set exists, by the definition of $M$). By (2) and the minimality of $P$, at most two vertices of $V_j$ are adjacent to some internal vertex of $P$. Since $V_j$ contains at least four vertices, there exist $a, b \in V_j$ that are not adjacent to any internal vertex of $P$. If $V_i$ has at least three elements then it contains a vertex $w$ distinct from $u, v$. As $w$ is not adjacent to any vertex of $P$, the vertices $w, a, b$ together with $P$ induce a subdivision of $K_{4}^{+}$, a contradiction. If $V_i$ has at most two elements, then there must be an integer $\ell$ distinct from $i$ and $j$ such that $V_\ell$ has at least four elements. In particular, $V_\ell$ contains a vertex $c$ that is not adjacent to any internal vertex of $P$. As a consequence, the vertices $a, c$ together with $P$ induce a subdivision of $K_{4}^{+}$, which is again a contradiction. This proves (3).

Recall that we can assume that $R$ is non-empty. An immediate consequence of (3) is that the neighborhood of each connected component of $R$ is a clique. Since $G$ is connected, it follows that it contains a clique cutset $K$ (a clique whose deletion disconnects the graph). Let $C$ be a connected component of $G - K$, let $G_1 = G - C$, and let $G_2$ be the subgraph of $G$ induced by $C \cup K$. It follows from the induction that there exist $d$-colorings of $G_1$ and $G_2$. Furthermore, since $K$ is a clique, we can assume that the colorings coincide on $K$. This implies that $G$ is $d$-colorable and concludes the proof of Theorem 1. $\square$

We remark that we could have used $K_{3,3}$ instead of $K_{4,4}$ in the proof, at the expense of a slightly more detailed analysis. The resulting bound on the chromatic number would have been $\max(k, d(K_{4}^{+}, R(3, k)))$ instead of $\max(k, d(K_{4}^{+}, R(4, k)))$.

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