Greedoids from flames

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Funding information
Hungarian Scientific Research Fund, Grant/Award Number: 129211; Alexander von Humboldt-Stiftung

Abstract
A digraph $D$ with $r \in V(D)$ is an $r$-flame if for every $v \in V(D) - r$, the in-degree of $v$ is equal to the local edge-connectivity $\lambda_D(r, v)$. We show that for every digraph $D$ and $r \in V(D)$, the edge sets of the $r$-flame subgraphs of $D$ form a greedoid. Our method yields a new proof of Lovász’s theorem stating: for every digraph $D$ and $r \in V(D)$, there is an $r$-flame subdigraph $F$ of $D$ such that $\lambda_F(r, v) = \lambda_D(r, v)$ for $v \in V(D) - r$.

We also give a strongly polynomial algorithm to find such an $F$ working with a fractional generalization of Lovász’s theorem.

KEYWORDS
edge-connectivity, greedoid, rooted digraph, strongly polynomial algorithm

2020 MATHEMATICS SUBJECT CLASSIFICATION
05C20, 05B35, 05C40, 05C35, 05C85

1 INTRODUCTION

Subgraphs preserving some connectivity properties while having as few edges as possible have been a subject of interest since the beginning of graph theory. Suppose that $D$ is a digraph with $r \in V(D)$ and let us denote the local edge-connectivity$^1$ from $r$ to some $v \in V(D) - r$ by $\lambda_D(r, v)$. We are looking for a spanning subgraph $H$ of $D$ with the smallest possible number of edges in which all the local edge-connectivities outwards from the root $r$ are the same as in $D$, that is, $\lambda_H(r, v) = \lambda_D(r, v)$ for all $v \in V(D) - r$. In order to have $\lambda_D(r, v)$ many pairwise edge-disjoint paths from $r$ to $v$ in $H$, it is obviously necessary that the

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$^1$The local edge-connectivity from $r$ to $v$ is the maximal number of pairwise edge-disjoint $r \to v$ paths.

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* J Graph Theory. 2021;98:49–56. wileyonlinelibrary.com/journal/jgt | 49
in-degree $\varphi_H(v)$ of $v$ in $H$ is at least $\lambda_D(r, v)$. This leads to the estimation $|E(H)| \geq \sum_{v \in V(D) - r} \lambda_D(r, v)$. It was shown by Lovász that, maybe surprisingly, this trivial lower bound is always sharp.

**Theorem 1.1** (Lovász, Theorem 2 of [1]). For every digraph $D$ and $r \in V(D)$, there is a spanning subdigraph $H$ of $D$ such that for every $v \in V(D) - r$

$$\lambda_D(r, v) = \lambda_H(r, v) = \varphi_H(v).$$

Calvillo-Vives rediscovered Theorem 1.1 independently in [2] and named the rooted digraphs $F$ with $\lambda_F(r, v) = \varphi_F(v)$ for all $v \in V(F) - r$ “r-flames”.

We establish a direct connection between the extremal problem above and the theory of greedoids. The latter were introduced by Korte and Lovász as a generalization of matroids to capture greedy solvability in problems where the matroid concept turned out to be too restrictive. The field is actively investigated since the 1980s, for a survey we refer to [3].

We show that the subflames of a rooted digraph always form a greedoid whose bases are exactly the subdigraphs described in Theorem 1.1.

**Theorem 1.2.** Let $D = (V, E)$ be a digraph and $r \in V$. Then

$$\mathcal{F}_{D,r} := \{E(F) | F \subseteq D \text{ is an } r \text{- flame} \}$$

is a greedoid on $E$. Furthermore, for each $\subseteq$-maximal element $E(F^*)$ of $\mathcal{F}_{D,r}$ we have $\lambda_{F^*}(r, v) = \lambda_D(r, v)$ for all $v \in V - r$.

The proof of Theorem 1.1 by Lovász is algorithmic but only for simple digraphs polynomial. We prove a fractional generalization of Lovász’s theorem considering digraphs with non-negative edge-capacities and replacing “edge-connectivity” by “flow-connectivity”. Our proof provides a simple strongly polynomial algorithm to find an $H$ with properties given in Theorem 1.1.

It is worth to mention that one can formulate a structural infinite generalization of Theorem 1.1 in the same manner as Erdős conjectured such an extension of Menger’s theorem (see [4]). As in the case of Menger’s theorem, the problem is getting much harder in the infinite setting. The “vertex-variant” of this generalization was proved for countably infinite digraphs in [5] which was then further developed in [6].

### 2 | NOTATION

In this article we deal only with finite combinatorial structures. An $\mathcal{F} \subseteq 2^E$ is a greedoid on $E$ if $\emptyset \in \mathcal{F}$ and $\mathcal{F}$ has the Augmentation property, that is, whenever $F, F' \in \mathcal{F}$ with $|F| < |F'|$, there is some $e \in F' \setminus F$ such that $F + e \in \mathcal{F}$. A digraph $D$ is an ordered pair $(V, E)$ where $E$ is a set of directed edges with their endpoints in $V$ where parallel edges are allowed but loops are not. Let us fix throughout this article a vertex set $V$ and a “root vertex” $r \in V$. For $U \subseteq V$, $\text{in}_D(U)$ and $\text{out}_D(U)$ stand for the set of ingoing and outgoing edges of $U$ respectively, furthermore, let $\varphi_D(U) := |\text{in}_D(U)|$ and $\delta_D(U) := |\text{out}_D(U)|$. For simplicity we always assume that $\text{in}_D(r) = \emptyset$. We write shortly $\lambda_D(v)$ for $\lambda_D(r, v)$ wherever $v \in V - r$. Recall, this is the local edge-connectivity (i.e., the maximal number of pairwise disjoint paths) from $r$ to $v$. We define $\mathcal{G}_D(v)$ to be the set of those $I \subseteq \text{in}_D(v)$ for which there exists a system $\mathcal{P}$ of edge-disjoint $r \to v$
paths where the set of the last edges of the paths in \( P \) is \( I \). It is known that set \( G_D(v) \) is the family of independent sets of a matroid. Matroids representable this way were discovered by Perfect \cite{perfect} and Pym \cite{pym} independently (using an equivalent definition based on vertex-disjoint paths between vertex sets) and are called gammoids. A digraph \( F \) is a flame if \( G_F(v) \) is a free matroid\(^2\) for every \( v \in V - r \), equivalently \( \lambda_F(v) = \varrho_F(v) \) for every \( v \neq r \).

3 \ THE FLAME GREEDOID OF A ROOTED DIGRAPH

The core of the proof of Theorem 1.2 is the following lemma.

**Lemma 3.1.** Let \( H \) and \( D \) be digraphs and assume that \( \lambda_H(u) < \lambda_D(u) \) for some \( u \in V - r \). Then there is an \( e \in E(D) \setminus E(H) \) with head, say \( v \), such that \( e \) is a coloop\(^3\) of \( G_{H+e}(v) \), that is,

\[ G_{H+e}(v) \supseteq \{ I + e : I \in G_H(v) \} . \]

**Proof.** Let \( U := \{ U \subseteq V - r : u \in U \} \) and \( \varrho_H(U) = \lambda_H(u) \). By Menger’s theorem \( U \neq \emptyset \) and the submodularity of the map \( X \mapsto \varrho_H(X) \) ensures that \( U \) is closed under union and intersection. Let \( U \) be the \( \subseteq \)-largest element of \( U \). Since \( \lambda_H(u) < \lambda_D(u) \), there exists some edge \( e \in \text{in}_D(U) \setminus \text{in}_H(U) \). Note that in \( H + e \) every \( X \subseteq V - r \) with \( X \supseteq U \) has at least \( \lambda_H(u) + 1 = \varrho_{H+e}(U) \) many ingoing edges because of the maximality of \( U \). By applying Menger’s theorem in \( H + e \) with \( r \) and \( U \), we find a system \( P \) of edge-disjoint \( r \rightarrow U \) paths of size \( \lambda_H(u) + 1 \) (see Figure 1). The set of the last edges of the paths in \( P \) is necessarily the whole \( \text{in}_{H+e}(U) \). Let the head of \( e \) be \( v \) and let \( I \in G_H(v) \) witnessed by the path-system \( Q \). Clearly each \( Q \in Q \) enters \( U \) at least once. For \( Q \in Q \), we define \( f_Q \) as the last meeting of \( Q \) with \( \text{in}_H(U) \). Finally, we build a path-system \( R \) witnessing \( I + e \in G_{H+e}(v) \) as follows. For \( Q \in Q \), we consider the unique \( f_Q \in P \) with last edge \( f_Q \) and concatenate it with the terminal segment of \( Q \) from \( f_Q \) to obtain \( R_Q \). Moreover, let \( R_e \) be the unique path in \( P \) with last edge \( e \). Then \( \mathcal{R} := \{ R_Q : Q \in Q \} \cup \{ R_e \} \) witnesses \( I + e \in G_{H+e}(v) \) as desired. \qed

**Proof of Theorem 1.2.** Suppose that \( F_0, F_1 \subseteq D \) are flames with \( \lambda_{F_0} < \lambda_{F_1} \). Then there must be some \( u \in V - r \) for which \( \varrho_{F_0}(u) < \varrho_{F_1}(u) \). Since \( F_0 \) and \( F_1 \) are flames

\[ \lambda_{F_0}(u) = \varrho_{F_0}(u) < \varrho_{F_1}(u) = \lambda_{F_1}(u) . \]

By applying Lemma 3.1 with \( F_0, F_1 \) and \( u \), we find an \( e \in E(F_1) \setminus (F_0) \) with head \( v \) where \( e \) is a coloop of \( G_{F_0+e}(v) \). On the one hand, \( G_{F_0}(v) \) is a free matroid and the previous sentence ensures that \( G_{F_0+e}(v) \) is free as well. On the other hand, for \( w \in V \setminus \{ r, v \} \) any path-system witnessing that \( G_{F_0}(w) \) is a free matroid shows the same for \( G_{F_0+e}(w) \). By combining these we may conclude that \( F_0 + e \) is a flame.

In order to prove the last sentence of Theorem 1.2, let \( F^* \) be a maximal flame in \( D \) and suppose for a contradiction that \( \lambda_{F^*}(u) < \lambda_D(u) \) for some \( u \in V - r \). Applying

\(^2\)A free matroid is a matroid where all sets are independent.

\(^3\)A coloop is an edge of a matroid which can be added to any independent set without ruin independence.
Lemma 3.1 gives again some \( e \in E \setminus E(F^*) \) for which \( F^* + e \) is a flame contradicting the maximality of \( F^* \).

The Max flow min cut theorem (see [10]) guarantees that \( \lambda_c(v) \) is well-defined and equals to

\[
\min\{q_c(W) : W \subseteq V - r \quad \text{with} \quad v \in W\}
\]

For \( v \in V - r \) and \( c \in \mathbb{R}^E_+ \), we write \( \mathcal{G}_c(v) \) for the set of those vectors in \( \mathbb{R}^{\text{in}_D(v)}_+ \) that can be obtained as a restriction of an \( r \to v \) flow \( x \leq c \) to \( \text{in}_D(v) \) that we denote by \( x|\text{in}_D(v) \). It is not too hard to prove that \( \mathcal{G}_c(v) \) is a polymatroid and it is natural to call it a polygammoid. An \( f \in \mathbb{R}^E_+ \) is a fractional flame if \( f|\text{in}_D(v) \in \mathcal{G}_f(v) \) (equivalently \( \lambda_f(v) = \varrho_f(v) \)) for all \( v \in V - r \). For \( e \in E \),
let $\chi_e \in \mathbb{R}_+^E$ be the vector where $\chi_e(e')$ is 1 if $e = e'$ and 0 otherwise. We call a vector integral if all of its coordinates are integers.

The fractional version of Lovász’s theorem can be formulated in the following way.

**Theorem 4.1.** Let $D = (V, E)$ be a digraph and $r \in V$. Then for every $c \in \mathbb{R}_+^E$ there is an $f \leq c$ such that for every $v \in V - r$

$$\lambda_c(v) = \lambda_f(v) = \varphi_f(v),$$

moreover, if $c$ is integral then $f$ can be chosen to be integral. Such an $f$ can be found in strongly polynomial time.

**Proof.** In the contrast of Theorem 1.1, the following fractional analogue of Lemma 3.1 is not sufficient itself to provide the existence part of Theorem 4.1 but will be an important tool later. \hfill \Box

**Lemma 4.2.** Let $x, y \in \mathbb{R}_+^E$ such that $\lambda_y(u) < \lambda_x(u)$ for some $u \in V - r$. Then there is an $e \in E$ with head, say $v$, and an $\epsilon > 0$ such that $x(e) - y(e) \geq \epsilon$ and

$$G_{y+\epsilon\chi_e}(v) = \{s + \delta\chi_e : s \in G_y(v) \land 0 \leq \delta \leq \epsilon\}.$$

**Proof.** The proof goes similarly as for Lemma 3.1. By applying the Max flow min cut theorem and the submodularity of the function $X \mapsto \varphi_y(X)$, we take the maximal $U \subseteq V - r$ with $u \in U$ and $\varphi_y(U) = \lambda_y(u)$. We pick some $e \in \text{in}_D(U)$ with $x(e) > y(e)$ and let

$$\epsilon := \min\{x(e) - y(e), \varphi_y(W) - \varphi_y(U) : U \subseteq W \subseteq V - r\}.$$ 

Let $p$ be an $r \to u$ flow of maximal amount with respect to the capacity $y + \epsilon\chi_e$ in the auxiliary digraph we obtain by contracting $U$ to $u$ while deleting the arising loops. By defining $p$ on the edges with both ends in $U$ to be 0, we ensure $p \in \mathbb{R}_+^E$. The Max flow min cut theorem and the choice of $\epsilon$ guarantee that

$$p|\text{in}_D(U) = (y + \epsilon\chi_e)|\text{in}_D(U).$$

We may assume that $p$ is a nonnegative combination of $r \to U$ paths. Let $s \in G_y(u)$ witnessed by the $r \to u$ flow $q$ which is a non-negative combination of $r \to u$ paths. Take the sum of the terminal segments of these weighted paths from the last common edge with $\text{in}_D(U)$ together with the trivial path $e$ with a given weight $\delta$ with $0 \leq \delta \leq \epsilon$ to obtain a vector $q'$. Starting with $p$ one can construct a $p' \leq p$ which is a non-negative combination of $r \to U$ paths and for which $p'|\text{in}_D(U) = q'|\text{in}_D(U)$. It is easy to see that the coordinate-wise maximum of $p'$ and $q'$ witnessing $s + \delta\chi_e \in G_{y+\epsilon\chi_e}(v)$. \hfill \Box

Now we turn to the description of the algorithm. Let $V = \{v_0, \ldots, v_n\}$ where $v_0 = r$. The algorithm starts with $f_0 := c$. If $f_k \in \mathbb{R}_+^E$ is already constructed and $k < n$, then we take an $r \to v_{k+1}$ flow $z_{k+1} \leq f_k$ of amount $\lambda_{f_k}(v_{k+1})$, which we choose to be integral if $f_k$ is integral, and define

$$f_{k+1}(e) := \begin{cases} z_{k+1}(e) & \text{if } e \in \text{in}_D(v_{k+1}) \\ f_k(e) & \text{otherwise.} \end{cases}$$

The algorithm concludes with $f_n$.
Since the flow problem can be solved in strongly polynomial time, the algorithm described above is strongly polynomial with a suitable flow subroutine. We claim that $f_n$ satisfies the demands of Theorem 4.1. Since we start with $c$ and lower some values in each step, $f_n \leq c$ holds. If $c \in \mathbb{Z}_+^E$, then a straightforward induction shows that $f_n \in \mathbb{Z}_+^E$.

**Lemma 4.3.** If $z \leq x$ is an $r \rightarrow v$ flow of amount $\lambda_x(v)$ and $y(e) := \begin{cases} z(e) & \text{if } e \in \text{in}_D(v) \\ x(e) & \text{otherwise} \end{cases}$ then $\lambda_y(u) = \lambda_x(u)$ for every $u \in V - r$.

**Proof.** Suppose for a contradiction that there exists a $u \in V - r$ with $\lambda_y(u) < \lambda_x(u)$. Note that $u \neq v$ because $\lambda_x(v) = \lambda_y(v)$ is witnessed by $z$. By Lemma 4.2, there is an $e \in E$ and an $\varepsilon$ such that $x(e) - y(e) > \varepsilon > 0$ (which implies that the head of $e$ must be $v$) and $G_{y+\varepsilon}(v) = [s + \delta x : s \in G_y(v) \land 0 \leq \delta \leq \varepsilon]$. Let $s_0 := z|\text{in}_D(v)$.

$$\lambda_x(v) \geq \lambda_{y+\varepsilon}(v) \geq ||s_0||_1 + \varepsilon = \lambda_x(v) + \varepsilon$$

which is a contradiction. \[\square\]

By applying Lemma 4.3 with $x = f_k$, $y = f_{k+1}$ and $z = z_{k+1}$ we obtain the following.

**Corollary 4.4.** $\lambda_{f_k}(v) = \lambda_{f_{k+1}}(v)$ for every $k < n$ and $v \in V - r$.

It follows by induction on $k$ that $\lambda_{f_k}(v) = \lambda_{c}(v)$ for every $v \in V - r$ and $k \leq n$. In particular $\lambda_{f_n}(v) = \lambda_{c}(v)$ for all $v \in V - r$. Let $1 \leq k \leq n$ be arbitrary. Then $\varphi_{f_k}(v_k) = \lambda_{f_k}(v_k)$ follows directly from the algorithm (the common value is $\varphi_{\lambda_{c}}(v_k)$). On the one hand, the left side is equal to $\varphi_{f_k}(v_k)$ since in moving from $f_k$ to $f_n$ the algorithm no longer changes the values on the elements of $\text{in}_D(v_k)$. On the other hand, we have seen that $\lambda_{f_k}(v_k) = \lambda_{f_n}(v_k) = \lambda_{c}(v_k)$.

Finally, let us point out a special case of Lemma 4.3.

**Corollary 4.5.** Let $D$ be a directed graph and let $\mathcal{P}$ be a maximal sized family of pairwise edge-disjoint $r \rightarrow v$ paths in $D$. Then the deletion of those ingoing edges of $v$ that are unused by the path-family $\mathcal{P}$ does not reduce any local edge-connectivities of the form $\lambda_D(r, u)$ with $u \in V(D) - r$.

### 5 | OUTLOOK

By Theorem 4.1, finding a spanning subdigraph of a given digraph $D$ that preserves all the local edge-connectivities from a prescribed root vertex $r$ and has the fewest possible edges with respect to this property can be done in polynomial time. It is natural to ask the complexity of the weighted version:

**Question 5.1.** What is the complexity of the following combinatorial optimization problem?

- Input: digraph $D$, $r \in V(D)$ and cost function $c: E(D) \rightarrow \mathbb{R}_+$
Output: spanning subdigraph \( F \) of \( D \) with \( \lambda_F(r,v) = \lambda_D(r,v) \) for every \( v \in V(D) - r \) for which \( \sum_{e \in E(F)} c(e) \) is minimal with respect to this property.

The special case where \( \lambda_D(r,v) \) is the same for every \( v \in V(D) - r \) can be solved in polynomial time by using weighted matroid intersection (see [11]).

There are more general flow models involving polymatroidal bounding functions (see e.g., [12], [13] and [14]). The Max flow min cut theorem is preserved under these models.

**Question 5.2.** Is it possible to generalize Theorem 4.1 by using the polymatroidal flow model introduced by Hassin in [12] (and rediscovered later by Lawler and Martel in [13] independently)?

The relation between matroids and polymatroids motivates the following concept of polygreedoids: a polygreedoid is a compact \( \mathcal{P} \subseteq \mathbb{R}_+^E \) such that

**PG1** \( \emptyset \in \mathcal{P} \),
**PG2** whenever \( x, y \in \mathcal{P} \) with \( ||x||_1 < ||y||_1 \), there is some \( e \in E \) with \( y(e) > x(e) \) such that \( x + \varepsilon \chi_e \in \mathcal{P} \) for all small enough \( \varepsilon > 0 \).

It follows directly from Lemma 4.2 that fractional flames under a given bounding vector form a polygreedoid. Greedoids have a property called accessibility which can be considered as a weakening of the downward closedness of matroids. It tells that every \( F \in \mathcal{F} \) can be enumerated in such a way that each initial segment belongs to \( \mathcal{F} \), that is, \( F = \{ e_1, ..., e_k \} \) such that \( \{ e_1, ..., e_k \} \in \mathcal{F} \) for every \( k \leq n \). Accessibility tends to be a part of the axiomatization of greedoids via the restriction the Augmentation axiom for pairs with \( |F'| = |F| + 1 \). It is not too hard to prove that polygreedoids satisfy the following analogous property: for every \( x \in \mathcal{P} \) there is a continues strictly increasing\(^4\) function \( g: [0, 1] \to \mathcal{P} \) with \( g(0) = 0 \) and \( g(1) = x \). Finally, let us end the article with the following general question.

**Question 5.3.** How much of the theory of greedoids is preserved for polygreedoids?

**ACKNOWLEDGEMENT**
The author would like to thank the generous support of the Alexander von Humboldt Foundation and NKFIH OTKA-129211.

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\(^4\)Strictly increasing is meant with respect to the coordinate-wise partial ordering of \( \mathbb{R}_+^E \).
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**How to cite this article:** A. Joó, *Greedoids from flames*, J. Graph Theory 2021;98: 49–56. https://doi.org/10.1002/jgt.22681