The quantization of constrained systems: from symplectic reduction to Rieffel induction*

N. P. Landsman†
Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Silver Street, Cambridge CB3 9EW, U.K.

DAMTP-96-05
March 24, 2022

Abstract

This is an introduction to the author’s recent work on constrained systems. Firstly, a generalization of the Marsden-Weinstein reduction procedure in symplectic geometry is presented - this is a reformulation of ideas of Mikami-Weinstein and Xu. Secondly, it is shown how this procedure is quantized by Rieffel induction, a technique in operator algebra theory. The essential point is that a symplectic space with generalized moment map is quantized by a pre-(Hilbert) $C^*$-module. The connection with Dirac’s constrained quantization method is explained. Three examples with a single constraint are discussed in some detail: the reduced space is either singular, or defined by a constraint with incomplete flow, or unproblematic but still interesting. In all cases, our quantization procedure may be carried out. Finally, we re-interpret and generalize Mackey’s quantization on homogeneous spaces. This provides a double illustration of the connection between $C^*$-modules and the moment map.

*To appear in Proc. of the XIV’th Workshop on Geometric Methods in Physics, Białowieża, 1995, eds. J.P. Antoine et. al., 1996
†E.P.S.R.C. Advanced Research Fellow
1 Generalized Marsden-Weinstein reduction

Symplectic reduction is a technique to construct new symplectic manifolds from old ones. The subject is full of subtleties, and we refer to [2] for a particularly careful discussion. The main idea is to start from a symplectic manifold \((\Sigma, \omega)\), select a subspace \(C\) (the constraint hypersurface, which is embedded in \(\Sigma\) by the identity map \(i\)), and quotient \(C\) by the null foliation \(\mathcal{F}_0\) defined by the null directions of the induced symplectic form \(i^*\omega\) on \(C\), to the extent that these are tangent to \(C\). In favourable circumstances the quotient \(C/\mathcal{F}_0\) is a symplectic manifold, referred to as the reduced space of \(\Sigma\) (with respect to \(C\)). A related construction is possible when \(\Sigma\) is merely a Poisson manifold [18], and the construction below generalizes to that case; for simplicity, we restrict ourselves to the symplectic case in what follows.

In any case, classical reduction is a two-step procedure: firstly the constraints are implemented (corresponding to the choice of \(C\)); secondly, roughly speaking, gauge-equivalent points are identified. As we shall see, in quantum theory only one of these steps has to be taken. Dirac’s approach to constrained quantization [4] selects the first step, whereas the author’s singles out the second.

Marsden-Weinstein reduction (cf. [1] and refs. therein) is a special case of the above construction. Let \((S, \omega_S)\) be a symplectic manifold on which a connected Lie group \(G\) (whose Lie algebra we denote by \(\mathfrak{g}\), with dual \(\mathfrak{g}^*\)) acts from the right in strongly Hamiltonian fashion. If \((\mathfrak{g}^*)^{-}\) stands for \(\mathfrak{g}^*\) equipped with minus the Lie-Poisson structure [1, 18], one obtains an equivariant moment map \(J : S \to (\mathfrak{g}^*)^{-}\) which is a Poisson morphism. Putting \(J_X = \{J, X\}\), this means that \(\{J_X, J_Y\} = -J_{[X,Y]}\), which is equivalent to the global property \(J(xg) = \text{Ad}^*(g)J(x)\). Choose a coadjoint orbit \(O \in \mathfrak{g}^*\), and put \(\Sigma = S \times O\), with symplectic form \(\omega = \omega_S + \omega_O\), where \(\omega_O\) is the Kirillov (etc.) symplectic form on the orbit (note that this form is the one induced on \(O\) regarded as a symplectic leaf of \(\mathfrak{g}^*\) equipped with plus the Lie-Poisson structure). The inclusion map of \(O\) into \(\mathfrak{g}^*\) is called \(\rho\). The constraint hypersurface is given by the fiber product \(C = S \ast_{\mathfrak{g}^*} O\), which stands for the collection of points \(\{(x, y) \in S \times O | J(x) = \rho(y)\}\) (which is evidently diffeomorphic to \(J^{-1}(O)\)). If \(O\)
consists of regular values of the moment map, \( C \) is co-isotropically immersed in \( \Sigma \), and the reduced space \( S^\Theta = C/F_0 \) coincides with the Marsden-Weinstein quotient \( C/G \simeq J^{-1}(\mathcal{O})/G \) (cf. \([2, 26]\) for the singular case).

We now generalize Marsden-Weinstein reduction as follows \([13]\). Instead of \( g^* \) we consider an arbitrary Poisson manifold \( P \), with realization \( S \), i.e., we suppose a Poisson morphism \( J : S \to P^- \) is given (the ‘generalized moment map’), with \( S \) symplectic. Let a second realization \( \rho : S_\rho \to P \) be given. We may then repeat the above steps: we can form the fiber product \( C = S_* P S_\rho \), which, under the assumption that either \( J_* \) or \( \rho_* \) is surjective at all points of \( S \) or \( S_\rho \) that are relevant to \( C \), turns out to be co-isotropically immersed in \( \Sigma = S \times S_\rho \). The reduced space is then defined by

\[
S^\rho = (S_* P S_\rho)/F_0,
\]

as before (the null foliation is generated by the Hamiltonian vector fields of functions of the type \( J^* f - \rho^* f, f \in C^\infty(P) \)). This is essentially the reduction procedure of Mikami-Weinstein \([19]\) and Xu \([29]\), but reformulated without reference to symplectic groupoids (also cf. \([30]\)).

Let \( \mathcal{A} \) be the Poisson subalgebra of \( C^\infty(S) \) consisting of all functions \( f \) with the property that for all \( g \in C^\infty(P) \) the Poisson bracket \( \{J^* g, f\} \) vanishes at all points of \( S \) which are relevant to \( C \). We can define an ‘induced representation’ \( \pi^\rho \) of \( \mathcal{A} \) on \( S^\rho \) by \( \pi^\rho(f)([x, y]) = f(x) \), where \( [x, y] \in S^\rho \) is the image of \( (x, y) \in C \) under the canonical projection from \( C \) to \( C/F_0 \). By our definition of \( \mathcal{A} \) this is well-defined (i.e., independent of the choice of \( x \) in \([x, y]\)). In the context of constrained systems, such functions \( f \) are called weak observables; an observable is an equivalence class of such functions under the equivalence relation \( f_1 \sim f_2 \) iff \( \pi^\rho(f_1) = \pi^\rho(f_2) \).

Our thinking of symplectic reduction as an induction construction is further supported by a classical imprimitivity theorem \([29, 13]\) and by a theorem on symplectic reduction in stages \([13, 18, 26]\), both of which parallel the (much older!) corresponding theorems on induced representations in a Hilbert space and operator algebra context \([25]\).
2 Quantized Marsden-Weinstein reduction

The problem of quantizing constrained (i.e., reduced) systems was first addressed by Dirac [6]. He realized that only one of the two steps of the classical reduction procedure needs to be implemented at the quantum level. Let us illustrate his approach in the example of Marsden-Weinstein reduction by a compact connected group at level zero (that is, the classical reduced space is $S^0 = J^{-1}(0)/G$ - we assume that 0 is a regular value of the moment map so that the reduced space is a manifold). The classical phase space $S$ is supposed to be quantized by some Hilbert space $\mathcal{H}$, the state space of the unconstrained system. The strongly Hamiltonian action of $G$ on $S$ is quantized by a unitary representation $U$ of $G$ on $\mathcal{H}$ (here and in what follows, all unitary representations are assumed to be continuous). This representation should be compatible with the requirement that for each $X \in \mathfrak{g}$, the function $J_X$ on $S$ is quantized by $idU(X)$. For reduction at zero, the classical constraints are $J_{T_a} = 0$, where $\{T_a\}_a$ is a basis of $\mathfrak{g}$. Dirac then proposed that the Hilbert space of the constrained quantum system $\mathcal{H}^0_D$ be given by the subspace of $\mathcal{H}$ which satisfies the quantum constraints, i.e.,

$$\mathcal{H}^0_D = \{\psi \in \mathcal{H} | dU(T_a)\psi = 0 \forall a\}.$$

Hence (since $G$ is connected) $\mathcal{H}^0_D = \mathcal{H}_{id} = P_{id}\mathcal{H}$, where $P_{id}\mathcal{H}$ is the projector on the subspace of $\mathcal{H}$ which transforms trivially under $U(G)$. A (weak) observable is a self-adjoint operator $A$ (assumed to be bounded for simplicity) which commutes with $P_{id}$. For such operators, the restriction $P_{id}AP_{id} = AP_{id}$ to $P_{id}\mathcal{H}$ is well-defined, and they evidently act on $\mathcal{H}^0_D$ by $\pi^0_D(A) = AP_{id}$.

In the context of a specific quantization scheme, such as geometric quantization, one may ask whether $\mathcal{H}^0_D$ (in conjunction with an action of the observables on it) coincides with the quantization of $S^0$ (in other words, do reduction and quantization commute). The mathematical literature on this difficult question goes back to (at least) [9], and we will not concern ourselves with this issue here. A discussion of examples involving compact as well as non-compact groups, using the quantization scheme of [13] and the present paper, may be found in [14].
Rather, we propose to look at the above construction in a different light. Instead of implementing the constraints, we can mimick the second step of classical reduction, viz. quotienting by the null foliation of the symplectic form. The way to do this consists in modifying the inner product on $H$. We define a new sesquilinear form $(\cdot,\cdot)_0$ on $H$ by

$$(\psi,\varphi)_0 = \int_G dg \langle (U(g)\psi, \varphi) \rangle,$$

where $dg$ is the normalized Haar measure on $G$. From standard compact group theory, we infer that $(\psi,\varphi)_0 = (P_{id}\psi, P_{id}\varphi)$. Hence the new form is positive semi-definite, with null space $N_0 = H_{id}^\perp$. (Here and in what follows, the null space of a positive semi-definite sesquilinear form $(\cdot,\cdot)_0$ is defined as the collection of vectors $\psi$ such that $(\psi,\psi)_0 = 0$; by the Cauchy-Schwartz inequality - which is the same as for positive definite sesquilinear forms - it follows that $(\psi,\varphi)_0 = 0$ for all $\varphi$.) We now define the induced space $H^0$ as $H/N_0$; clearly, $H^0 \simeq H_{id} = H^0_D$ as vector spaces.

Let $V\psi$ be the image of $\psi \in H$ in $H/N_0$; the latter becomes a Hilbert space under the inner product $(V\psi, V\varphi)^0 = (\psi, \varphi)_0$. Since we have precisely eliminated the null space of $(\cdot,\cdot)_0$, this inner product is positive definite, and $H^0 \simeq H^0_D$ also as Hilbert spaces.

The condition for an operator $A$ on $H$ to have a well-defined quotient action on $H^0$ is $AN_0 \subseteq N_0$. We can then define the (trivially) induced representation $\pi^0$ by $\pi^0(A)V\psi = VA\psi$. The requirement $\pi^0(A)^* = \pi^0(A^*)$ is equivalent to $(A\psi, \varphi)_0 = (\psi, A^*\varphi)_0$, which in turn is the condition $[A, P_{id}] = 0$ we already encountered. The property $(A\psi, \varphi)_0 = (\psi, A^*\varphi)_0$ actually implies that $AN_0 \subseteq N_0$, so that we can define a (weak) observable as a self-adjoint operator $A$ which satisfies $(A\psi, \varphi)_0 = (\psi, A\varphi)_0$. Thus the present reformulation is completely equivalent to the Dirac approach.

Now consider the case where the reduced space $S^O$ is obtained by Marsden-Weinstein reduction from a nontrivial coadjoint orbit $O$. Apart from $H$, the quantization procedure then needs a second Hilbert space $H_{\rho}$, seen as the quantization of $O$, and an irreducible unitary representation $U_{\rho}$ of $G$, which quantizes the coadjoint action of $G$ on $O$. The quantization of $S \times O$ equipped with given $G$-action on
$S$ times the coadjoint action is then given by $\mathcal{H} \otimes \mathcal{H}_\rho$ equipped with the unitary representation $U \otimes U_\rho$. However, since $S^0 \simeq (S \times \mathcal{O})^0$ as symplectic spaces (where the moment map on $S \times \mathcal{O}$ is the sum of $J$ and the inclusion map), its quantization according to Dirac can be inferred from the preceding as being

$$\mathcal{H}_D^0 = P_{id}(\mathcal{H} \otimes \mathcal{H}_\rho),$$

where $P_{id}$ projects onto the subspace of $\mathcal{H} \otimes \mathcal{H}_\rho$ transforming trivially under $U \otimes U_\rho$.

For the purpose of widening the discussion to general constrained systems, it is convenient to reformulate the construction of Dirac’s Hilbert space $\mathcal{H}_D^0$ as follows. The group algebra $C^*(G)$ is defined as the $C^*$-completion of the convolution algebra $L^1(G)$, with adjoint defined by $f^*(g) = \overline{f(g^{-1})}$ [7]. This algebra can be ‘diagonalized’ by the Plancherel transform [7]: for $\gamma \in \hat{G}$ (the unitary dual of $G$, i.e., the space of equivalence classes of irreducible unitary representations of $G$; for $G$ compact this is a discrete space) we put $\hat{f}(\gamma) = \int_G dg f(g)U_\gamma(g)$ (initially defined for $f \in L^1(G)$ and extended to $C^*(G)$ by continuity), with inverse $f(g) = \sum_{\gamma \in \hat{G}} d_\gamma \text{Tr}[U_\gamma(g)^* \hat{f}(\gamma)]$ (where $d_\gamma$ is the dimension of $\mathcal{H}_\gamma$). Note that $f(\gamma)$ is an operator on $\mathcal{H}_\gamma$, and that some choice $U_\gamma$ has been made for the unitary representatives in the equivalence class $\gamma$. The algebra $C^*(G)$ then consists of those operator-valued functions $\hat{f}$ on $\hat{G}$ for which the function $\gamma \to \| \hat{f}(\gamma) \|$ is in $C_0(\hat{G})$ (i.e., vanishes at infinity). One has $\hat{f}^*(\gamma) = \hat{f}(\gamma)^*$ and $\hat{f}_1 \hat{f}_2(\gamma) = \hat{f}_1(\gamma) \hat{f}_2(\gamma)$. The norm in $C^*(G)$ is given by $\| f \| = \sup_{\gamma \in \hat{G}} \| \hat{f}(\gamma) \|$.

Given a unitary representation $U(G)$ on a Hilbert space $\mathcal{H}$, we obtain a representation $\pi$ of $C^*(G)$ by $\pi(f) = \int_G dg f(g)U(g)$. We use the decomposition $\mathcal{H} \simeq \bigoplus_{\chi \in \hat{\mathcal{H}}} \mathcal{H}(\chi)$ under $U(G)$, where $\hat{\mathcal{H}} \subset \hat{G}$, and $\mathcal{H}(\chi) = \mathcal{H}_\chi \otimes \mathcal{K}_\chi$, the second factor taking account of the multiplicity of the irreducible $U_\chi$ in $\mathcal{H}(\chi)$. This yields $\pi(f) \simeq \bigoplus_{\chi \in \hat{\mathcal{H}}} \hat{f}(\chi) \otimes \mathbb{I}_{\mathcal{K}_\chi}$. (The symbol $\mathbb{I}_{\mathcal{K}}$ denotes the identity operator on a Hilbert space $\mathcal{K}$.) Similarly, we obtain a representation $\pi_\rho(C^*(G))$ on $\mathcal{H}_\rho$.

Also, a right-representation $\pi_R$ on $\mathcal{H}$ (that is, $\pi_R(AB) = \pi_R(B)\pi_R(A)$) may be defined by $\pi_R(f) = \int_G dg f(g)U(g^{-1})$. This may be decomposed as $\pi_R(f) \simeq \bigoplus_{\chi \in \hat{\mathcal{H}}} \hat{f}(\chi)^T \otimes \mathbb{I}_{\mathcal{K}_\chi}$ By tensoring with the relevant identity operators, $\pi_R$ and $\pi_\rho$ are
defined on \( \mathcal{H} \otimes \mathcal{H}_\rho \). It then follows that \( \mathcal{H}^\rho_D \) may alternatively be characterized as the subspace of \( \mathcal{H} \otimes \mathcal{H}_\rho \) consisting of those \( \Psi \) on which the ‘quantum constraints’

\[
(\pi_R(f) - \pi_\rho(f))\Psi = 0 \quad \forall f \in C^*(G)
\]

hold. For we can decompose \( \Psi \cong \bigoplus_{\chi \in \hat{U}} \Psi(\chi) \), where \( \Psi(\chi) \in \mathcal{H}(\chi) \otimes \mathcal{H}_\rho \). The quantum constraints imply that

\[
(\hat{f}(\chi)^T \otimes I_{\mathcal{K}_\chi} \otimes I_{\mathcal{H}_\rho} - I_{\mathcal{H}} \otimes I_{\mathcal{K}_\chi} \hat{f}(\rho))\Psi(\chi) = 0 \quad \forall \chi \in \hat{U}.
\]

For \( \chi \neq \rho \) we choose \( f \) such that \( \hat{f}(\chi) = I_{\mathcal{H}_\chi} \) and \( \hat{f}(\rho) = 0 \), which shows that \( \Psi(\chi) = 0 \) for such \( \chi \). For \( \chi = \rho \) the constraints are only satisfied if \( \Psi(\rho) = \sum_i e_i \otimes \psi \otimes e_i \), where \( \{e_i\} \) is a basis in \( \mathcal{H}_\rho \) (and in \( \mathcal{H}_\rho \)), and \( \psi \in \mathcal{K}_\chi \) is arbitrary. But these vectors precisely form \( P_0^\rho \mathcal{H} \otimes \mathcal{H}_\rho \).

The above way of writing \( \mathcal{H}^\rho_D \subset \mathcal{H} \otimes \mathcal{H}_\rho \) reflects the fact that to some extent it is the quantum analogue of the constraint hypersurface \( C = S^* g^* \mathcal{O} \subset S \times \mathcal{O} \) of the classical theory.

Evidently, we may arrive at \( \mathcal{H}^\rho_D \) by the second method of modifying the inner product in putting

\[
(\Psi, \Phi)_0 = \int_G dg (U(g) \otimes U_\rho(g)\Psi, \Phi) = (P_0^\rho \Psi, P_0^\rho \Phi).
\]

The null space is then given by \( N_0 = (P_0^\rho \mathcal{H} \otimes \mathcal{H}_\rho)^\perp \), so that the induced space \( \mathcal{H}^\rho = \mathcal{H} \otimes \mathcal{H}_\rho / N_0 \) (equipped with the inner product inherited from \( (,)_0 \)) coincides with \( \mathcal{H}^\rho_D \).

Also in this case we may rewrite the construction, making use of \( C^*(G) \). Consider the function \( \langle \psi, \varphi \rangle_{C^*(G)} \) on \( G \) defined by \( g \rightarrow (U(g) \varphi, \psi) \), where \( \psi, \varphi \in \mathcal{H} \). This defines a map \( \langle \cdot, \cdot \rangle_{C^*(G)} : \mathcal{H} \otimes \mathcal{H} \rightarrow C^*(G) \), which behaves like an inner product taking values in the \( C^* \)-algebra \( C^*(G) \). Namely, one has \( \langle \psi, \varphi \rangle_{C^*(G)} = \langle \varphi, \psi \rangle_{C^*(G)} \).

Moreover, it enjoys the ‘equivariance’ property \( \langle \psi, \pi_R(f)\varphi \rangle_{C^*(G)} = \langle \psi, \varphi \rangle_{C^*(G)} f \), where the right-hand side contains the (convolution) product of two elements of \( C^*(G) \). Finally, one has \( \langle \psi, \psi \rangle_{C^*(G)} \geq 0 \) for all \( \psi \in \mathcal{H} \). To show this, expand \( \psi = \sum_{\chi \in \hat{U}} \psi(\chi)e_i(\chi) \otimes \kappa(\chi) \), where \( \{e_i\} \) is a basis in \( \mathcal{H}_\chi \), and \( \kappa(\chi) \in \mathcal{K}_\chi \) has unit
norm. The Plancherel transform of $\langle \psi, \psi \rangle_{C^* (G)}$ is then given by $\gamma \to 0$ if $\gamma \notin \hat{U}$, and $\gamma \to \psi_1 (\tau) \overline{\psi_j (\tau)}$ if $\tau \in \hat{U}$. Positivity in $C^* (G)$ simply means that $\hat{f} (\gamma)$ is a positive matrix for all $\gamma$, which is clearly the case (cf. [13] for a different proof).

The map $\langle \cdot, \cdot \rangle_{C^* (G)}$ is the quantum analogue of the moment map [13]. If $\psi \in \mathcal{H} (\chi)$ then the Plancherel transform of $\langle \psi, \psi \rangle_{C^* (G)}$ only has support at $\gamma = \chi$, so that this ‘quantum moment map’ detects which representation of $G$ is carried by a vector $\psi$; this reflects the fact in symplectic geometry that the $G$-orbit through a point of $S$, in case this orbit happens to be symplectic, is carried into a given co-adjoint orbit by the moment map.

We can now rewrite the modified inner product on $\mathcal{H} \otimes \mathcal{H}_\rho$ by linear extension of

$$\langle \psi \otimes v, \varphi \otimes w \rangle_0 = (\pi_\rho (\langle \varphi, \psi \rangle_{C^* (G)} v, w)_{\rho};$$

the inner product on the right-hand side is the one in $\mathcal{H}_\rho$. This is a sesquilinear form by the ‘inner product’ property shown 3 paragraphs ago. It is positive semi-definite by the positivity property proved afterwards. Define

$$D_0 = \{ (\pi_R (f) - \pi_\rho (f)) \Psi | f \in C^* (G), \Psi \in \mathcal{H} \otimes \mathcal{H}_\rho \}.$$

A short computation shows that $D_0 \subseteq \mathcal{N}_0$. Using the Plancherel transform in similar vein to the argument leading to the alternative identification of $\mathcal{H}_D^\rho$, one can show the opposite inclusion $\mathcal{N}_0 \subseteq D_0$. Hence $\mathcal{N}_0 = D_0$. In other words, every vector satisfies the quantum constraints up to vectors in the null space of $(\cdot, \cdot)_0$. Since these vectors project to zero in the induced space $\mathcal{H}_\rho$, we clearly see how the second method to quantize the constrained systems in questions gets rid of the states not satisfying the constraints, without actually having to impose the latter.

There are two reasons why the Dirac method was successful in this class of examples: firstly, the constraints were first-class, and secondly (because the group was compact) they had 0 in their discrete spectrum (assuming that $\mathcal{H}_{id}$ is not empty) with common eigenspace. However, few realistic systems meet these conditions. If there are second-class constraints (that is, constraints whose Poisson bracket does not vanish on $C$), Dirac and his followers have to get rid of them already at the
classical level. If the first-class constraints fail to have 0 as an eigenvalue, it is not clear what to do (see below). Fortunately, it turns out that the second approach (of modifying the inner product) continues to work even in cases where Dirac’s method faces difficulties.

3 Quantized symplectic reduction

Returning to the general reduction procedure discussed at the end of section 1, we now assume that, apart from the quantization of \( S \) by \( \mathcal{H} \), we have quantized \( S_\rho \) by a Hilbert space \( \mathcal{H}_\rho \), and the Poisson algebra \( C^\infty(P) \) by a \(*\)-algebra \( \mathfrak{B} \). For technical reasons, we assume that \( \mathfrak{B} \) is a \( C^* \)-algebra or a pre-\( C^* \)-algebra (this could correspond to the quantization of, say, the bounded functions in \( C^\infty(P) \)); it may be thought of as the abstract operator algebra generated by the quantum constraints. The pull-back \( J^* : C^\infty(P) \to C^\infty(S) \) is quantized by a right-representation \( \pi_R \) of \( \mathfrak{B} \) on \( \mathcal{H} \). Also, the Poisson morphism \( \rho : S \to P \) (or rather its pull-back) is quantized by a representation \( \pi_\rho \) of \( \mathfrak{B} \) on \( \mathcal{H}_\rho \). As before, \( \pi_R \) and \( \pi_\rho \) may be regarded as operators on \( \mathcal{H} \otimes \mathcal{H}_\rho \) by tensoring with the appropriate identity operators.

Recall that the classical constraint hypersurface \( C \) was defined as the fiber product \( C = S \ast_P S_\rho \subset S \times S_\rho \), whose points \((x, y)\) are singled out by the condition \( J(x) = \rho(y) \). Generalizing the expression for the quantum constraints in the compact group case discussed above, Dirac’s method would analogously attempt to single out the subspace \( \mathcal{H}_D^0 \) of \( \mathcal{H} \otimes \mathcal{H}_\rho \) satisfying

\[
\pi_R(B)\Psi = \pi_\rho(B)\Psi \quad \forall B \in \mathfrak{B}.
\]

However, \( \mathcal{H}_D^0 \) is empty whenever at least one operator \( \pi_R(B) - \pi_\rho(B) \) fails to have zero in its discrete spectrum. The empty space is not the correct quantization of the classical reduced space \( S^\rho \). (Even if \( \mathcal{H}_D^0 \) is nonempty, it may not be the correct quantization.) Hence we have to look for a different approach.

Generalizing the construction of the previous section, our goal is to define a sesquilinear form \((\ ,\)_0\) on \( \mathcal{H} \otimes \mathcal{H}_\rho \) with certain properties. It turns out that very often the domain of definition of this form must be taken to be a subspace \( L \otimes \mathcal{H}_\rho \),
where \( L \subset \mathcal{H} \) is dense. In such cases, \((\cdot,\cdot)_0\) may even be non-closable as a quadratic form on \( \mathcal{H} \otimes \mathcal{H}_\rho \), cf. [10]. Also, if \( L \neq \mathcal{H} \) we require that \( \pi_R(B) \) maps \( L \) into itself. In fact, for the following construction is not even necessary that \( L \) is a subspace of a Hilbert space.

Firstly, \((\cdot,\cdot)_0\) has to be positive semi-definite (i.e., \((\Psi,\Psi)_0 \geq 0\) for all \( \Psi \in L \otimes \mathcal{H}_\rho \)); as before, we denote its null space by \( N_0 \). The ‘induced’ space \( \mathcal{H}^\rho \) is now defined as the completion of \( L \otimes \mathcal{H}_\rho / N_0 \) under the (obvious) inner product defined by
\[
(V\Psi, V\Phi)^\rho = (\Psi, \Phi)_0,
\]
where \( V : L \otimes \mathcal{H}_\rho \to L \otimes \mathcal{H}_\rho / N_0 \) is the canonical projection map. In other words, the inner product on \( \mathcal{H}^\rho \) is essentially \((\cdot,\cdot)_0\) modulo the null vectors.

Secondly, all vectors of the form \((\pi_R(B) - \pi_\rho(B))\Psi\) should lie in \( N_0 \). As before, this means that every vector satisfies the quantum constraints up to terms which project to the zero vector in \( \mathcal{H}_\rho \).

In a totally different context, an inner product meeting these requirements was given by Rieffel [25]. Namely, if we can find a ‘generalized quantum moment map’ \( \langle \cdot, \cdot \rangle_\mathcal{B} : \mathcal{T} \otimes L \to \mathcal{B} \), with (i) the ‘\( \mathcal{B} \)-valued inner product’ property \( \langle \psi, \varphi \rangle_\mathcal{B} = \langle \varphi, \psi \rangle_\mathcal{B} \), (ii) the positivity property \( \pi_\rho(\langle \psi, \psi \rangle_\mathcal{B}) \geq 0\) (as an operator on \( \mathcal{H}_\rho \)) for all \( \psi \in L \), and (iii) the ‘equivariance’ property \( \langle \psi, \pi_R(B)\varphi \rangle_\mathcal{B} = \langle \psi, \varphi \rangle_\mathcal{B}B \) for all \( B \in \mathcal{B} \) and \( \psi, \varphi \in L \), then the form defined by linear extension of
\[
(\psi \otimes v, \varphi \otimes w)_0 = (\pi_\rho(\langle \varphi, \psi \rangle_\mathcal{B})v, w)_\rho
\]
satisfies the two conditions. The three properties imply the sesquilinearity, the first condition, and the second condition, respectively.

In the mathematical literature (e.g., [3]) one finds the closely related concept of a (Hilbert) \( C^* \)-module: this is given by the above data \((L, \mathcal{B}, \pi_R, \langle \cdot, \cdot \rangle_\mathcal{B})\) if in addition \( \mathcal{B} \) is a \( C^* \)-algebra, \( \langle \psi, \psi \rangle_\mathcal{B} \geq 0 \) (as an element of \( \mathcal{B} \) - this of course implies the weaker positivity property we imposed above), and \( L \) is complete under the norm \( ||\psi||^2 = ||\langle \psi, \psi \rangle_\mathcal{B}|| \). In examples relevant to quantization theory one usually does not have \( C^* \)-modules, not even when \( L \) is a Hilbert space. (For example, if \( G = U(1) \)
acts on $L = L^2(U(1))$ in the regular representation, with our choice of the quantum
moment map the norm on $L$ as a $C^*$-module is $\|\psi\| = \sup_n |\psi_n|$, where the $\psi_n$ are
the Fourier coefficients of $\psi$. The completion of $L$ in this norm is $C_0(\mathbb{Z})$, which is
strictly bigger than $L \simeq L^2(\mathbb{Z})$.

Weak quantum observables $A$ of the constrained system are defined as operators
on $L$ which are self-adjoint with respect to $(\cdot, \cdot)_0$, i.e., $(A\Psi, \Phi)_0 = (\Psi, A\Phi)_0$ for all
$\Psi, \Phi \in L \otimes \mathcal{H}_\rho$ (here $A$ is identified with $A \otimes \mathbb{I}$). For such operators, the induced
representation $\pi^\rho$, given by $\pi^\rho(A)V\Psi = V A\Psi$, is well-defined. This is because $\mathcal{N}_0$ is mapped into itself by weak observables. In addition, $\pi^\rho(A)$ is symmetric on
the domain $V(L \otimes \mathcal{H}_\rho)$ if $A$ is a weak observable. A sufficient condition for $A$ to be a weak observable is that $\langle A\phi, \psi \rangle_\mathfrak{B} = \langle \phi, A\psi \rangle_\mathfrak{B}$ for all $\psi, \varphi \in L$. Observables are equivalence classes of weak observables, where $A_1 \sim A_2$ iff $\pi^\rho(A_1) = \pi^\rho(A_2)$.

Even if $L \subset \mathcal{H}$ and $A$ is a weak observable which is bounded as an operator
on $\mathcal{H}$, the operator $\pi^\rho(A)$ is not necessarily bounded on $\mathcal{H}^\rho$; one needs the bound $(A\Psi, A\Psi)_0 \leq C_A(\Psi, \Psi)_0$ for all $\Psi \in L \otimes \mathcal{H}_\rho$ and some constant $C_A$ to prove boundedness of $\pi^\rho(A)$. The $C^*$-algebra of weak observables is defined as the completion of the collection of weak observables which are bounded in this sense, equipped with
a norm given by $\|A\| = \sqrt{C_A}$, for the smallest possible $C_A$ in the above bound (to be precise, operators whose norm is zero are to be quotiented away). A particularly favourable case occurs when $L = \mathcal{H}$, and the collection of weak observables happens to be a $C^*$-algebra under the operator norm. In that case, the semi-positivity of $(\cdot, \cdot)_0$ implies that $\pi^\rho$ of a weak observable is bounded [25]. Unfortunately, if $L$ is a proper subspace of $\mathcal{H}$ then the set of bounded operators which map $L$ into itself is never complete under the operator norm.

There is a slight reformulation of the construction of the induced space $\mathcal{H}^\rho$. Given
the quantum moment map $(\cdot, \cdot)_\mathfrak{B}$ and a state $\omega$ on $\mathfrak{B}$, we can define a modified
inner product directly on $L$ by $(\psi, \varphi)^\omega_0 = \omega(\langle \varphi, \psi \rangle_\mathfrak{B})$. The induced space $\mathcal{H}^\omega_\rho$ is then
defined as the completion of $L/\mathcal{N}^\omega_0$, where $\mathcal{N}^\omega_0$ is the null space of $(\cdot, \cdot)^\omega_0$. A map $V_\omega : L \rightarrow L/\mathcal{N}^\omega_0$ and corresponding induced representation $\pi^\omega_\rho$ of the $C^*$-algebra of
weak observables $\mathfrak{A}$ on $L$ may then be defined as before. An arbitrary unit vector
$v \in \mathcal{H}_\rho$ defines a vector state $\omega_v$ on $\mathfrak{B}$, given by $\omega_v(B) = (\pi_\rho(B)v, v)$. For each such $\omega_v$, the pair $(\mathcal{H}_\omega^\rho, \pi_\rho^\omega(\mathfrak{A}))$ is unitarily equivalent to $(\mathcal{H}_\rho, \pi_\rho(\mathfrak{A}))$. (Note that, for any $C^*$-algebra $\mathfrak{A}$, one may take $L = \mathfrak{B} = \mathfrak{A}$ and $\langle A, B \rangle_{\mathfrak{B}} = A^*B$; the induction construction is then equivalent to the GNS construction [7].)

It is possible to regard $\mathcal{H}_\rho$ as a subspace of the algebraic dual $L^*$ of $L$. This embedding is not canonical, and depends on the choice of a state $\omega_v$ on $\mathfrak{B}$. Firstly, consider vectors in $\mathcal{H}_\rho$ of the type $V\psi$. Such a vector defines a linear functional on $L$ by $< V\psi, \varphi > = (\psi, \varphi)_{\omega_v}^0$. This is well-defined, since $(\psi, \varphi)_{\omega_v}^0 = 0$ if $\psi \in \mathcal{N}_0^{\omega_v}$. Subsequently, if $V\psi_n \to \chi$ in $\mathcal{H}_\rho$ then one can define $< \chi, \varphi > = \lim_n (\psi_n, \varphi)_{\omega_v}^0$. This is well-defined, since $V\psi_n \to 0$ in $\mathcal{H}_\rho$ is equivalent to $(\psi_n, \psi_n)_{\omega_v}^0 \to 0$, which implies $< \psi_n, \varphi > \to 0$ for all $\varphi \in L$ by the Cauchy-Schwartz inequality for positive semi-definite forms. Hence we may regard $V$ as a map from $L$ into $L^*$. Vectors in $\mathcal{N}_0^{\omega_v}$ are mapped into the zero functional. This way of looking at the theory shows how the constrained quantization formalism of [3] (also cf. [20]) is a special case of our method. (Their method corresponds to taking $\mathcal{H}_\rho = \mathbb{C}$, and $L$ is taken to be a topological vector space which is continuously embedded in $\mathcal{H}$, such that $L \subset \mathcal{H} \subset L'$ forms a Gel’fand triplet; here $L' \subset L^*$ is the topological dual.)

4 The case of commuting constraints

If classically we have $n$ commuting constraints $J_i \in C^\infty(S)$, such that the constraints generate a group action, the reduced phase space is a Marsden-Weinstein quotient $S^0 = J^{-1}(0)/G$ with respect to the group $G = \Pi_{i=1}^n G_i$, where $G_i$ is $\mathbb{R}$ or $U(1)$. The case of torus actions ($G_i = U(1)$ for all $i$) has been well-studied, and many beautiful results are available (such as the convexity theorem of Atiyah and Guillemin-Sternberg, cf. [8]). The torus action on $S$ is quantized by a unitary representation $U$ on a Hilbert space $\mathcal{H}$, which decomposes as $\mathcal{H} \simeq \oplus_l \mathcal{H}_l$, where $l = (l_1, \ldots, l_n) \in \hat{G} = \mathbb{Z}^n$, and $\mathcal{H}_l$ carries the representation $U_l(z_1, \ldots, z_n) = z_1^{l_1} \ldots z_n^{l_n}$. According to the theory above, the quantization of $S^0$ is $\mathcal{H}_0$; see [3, 8] for more information on this case.

We now look at the opposite case $G_i = \mathbb{R}$ for all $i$. In the corresponding quantum
theory, the Hilbert space $\mathcal{H}$ will carry a unitary representation $U(\mathbb{R}^n)$. By the
SNAG theorem (or the complete von Neumann spectral theorem), one has the direct
integral decomposition $\mathcal{H} \cong \int_{\hat{G}} d\mu(\lambda) \mathcal{H}(\lambda)$ (cf. [7]), where $\hat{G} = \mathbb{R}^n$, and the Hilbert
space $\mathcal{H}(\lambda)$ carries the representation $a \rightarrow \exp(i\lambda a)$. If $\mu(\lambda = 0)$ is positive, and
$\lambda = 0$ is not in the essential spectrum, then the quantization of $S^0$ is $\mathcal{H}(0)$. Assume,
instead, that 0 lies in the absolutely continuous part of $\mu$; below we assume that
there is no discrete spectrum).

Let $W : \mathcal{H} \rightarrow \int_{\hat{G}} d\mu(\lambda) \mathcal{H}(\lambda)$ be a unitary transformation that diagonalizes $U$.
We choose the dense subspace $L \subset \mathcal{H}$ in such a way that it is contained in the space
of elements $\psi$ of $\mathcal{H}$ for which $W\psi$ (regarded as a cross-section of the field $\{\mathcal{H}(\lambda)\}$
[7]) is continuous on $\hat{G}$. We now define the form $(\cdot, \cdot)_0$ as in the compact group case
(but only on the domain $L$), and compute:

$$(\psi, \varphi)_0 = \int_{\hat{G}} da (U(a)\psi, \varphi) = (2\pi)^n \left| \frac{d\mu}{d\lambda} \right| (0)(W\psi)(0), (W\varphi)(0))_{\mathcal{H}(0)}.$$

Hence we can define $V : \mathcal{H} \rightarrow \mathcal{H}^0 = \mathcal{H}(0)$ by

$$V\psi = \left((2\pi)^n \left| \frac{d\mu}{d\lambda} \right| (0)\right)^{1/2} (W\psi)(0);$$

this satisfies $(V\psi, V\varphi) = (\psi, \varphi)_0$, and shows that the induced space $\mathcal{H}^0$ indeed
coincides with $\mathcal{H}(0)$.

We illustrate this procedure in three examples, each of which is of special interest
for a different reason. The examples correspond to the classical constraint

$$H_\kappa = \frac{1}{2}(p_x^2 + \kappa e^{4x} - p_y^2) = 0,$$

defined on $S = T^*\mathbb{R}^2$ with canonical symplectic structure, and described in canonical
co-ordinates, where the parameter $\kappa$ assumes the values 0, 1, $-1$. This constraint
comes from a certain finite-dimensional approximation to the universe as described
by the general theory of relativity, cf. [15] and refs. therein.

Firstly, take $\kappa = 0$. This case also emerges in the context of the representation
theory of the Poincaré group of a two-dimensional space-time, cf. [14]. The con-
straint generates an action of $\mathbb{R}$, viz. $(x, y, p_x, p_y) \rightarrow (x + p_x t, y - p_y t, p_x, p_y)$, with
moment map $J = H_0$. However, $J$ fails to be surjective at all points of the form $(x, y, 0, 0)$ (which we will refer to as ‘the singular points’), at which $J$ vanishes. Hence 0 is not a regular value of $J$, and $J^{-1}(0)$ is not a submanifold of $S$. The Marsden-Weinstein reduced space $S^0 = J^{-1}(0)/\mathbb{R}$ does not have a constant dimension: if we look at $S^0$ as fibered over the subspace $p_x = \pm p_y$ of $\mathbb{R}^2$, then the fiber above $(0, 0)$ is two-dimensional whereas at all other points it is one-dimensional. This example is of interest partly because the general theory of singular Marsden-Weinstein reduction [26], based as it is on the assumption that the group action is proper, does not directly apply here (the $\mathbb{R}$-action is not proper precisely at the singular points).

Nonetheless, $J^{-1}(0)$ is strongly co-isotropic and locally conical in the sense of [4], so that the Marsden-Weinstein quotient $S^0$ agrees with the ‘geometric’ reduction defined by $J^{-1}(0)$. As shown in [4], it is therefore possible to define a Poisson algebra $\mathcal{C}^\infty(S^0)$ as the space of strong observables equipped with the Poisson bracket inherited from $S$. Elements of $\mathcal{C}^\infty(S^0)$ are functions $f \in C^\infty(S)$, restricted to $J^{-1}(0)$, which satisfy $\{f, H\} = 0$ on $J^{-1}(0)$. It follows that such $f$ depends on $x$ and $y$ through the combination $xp_y + yp_x$. A study of the hamiltonian flow on $S$ defined by such functions, and therefore of the corresponding flow on $S^0$ obtained by projection, shows that $S^0$ may be decomposed into five ‘symplectic leaves’ (cf. [18] for this concept in the regular case, and [26] for the singular case with proper group action): $p_x = p_y > 0$, $p_x = p_y < 0$, $p_x = -p_y > 0$, $p_x = -p_y < 0$, and $p_x = p_y = 0$. Any point in a given leaf cannot leave the leaf under a Hamiltonian flow.

For proper group actions, it is shown in [26] that the symplectic leaves of a singular quotient $J^{-1}(0)/G$ are the components of $(J^{-1}(0) \cap S(H))/G$, where $S(H)$ is the stratum in $S$ of orbit type $H \subseteq G$ (that is, the stability group of any point in $S(H)$ is conjugate to $H$). In our example, the singular points form the stratum of orbit type $\mathbb{R}$, and the other four leaves correspond to the components having orbit type $\{e\}$. Hence we have the same situation as for proper group actions.

We now turn to quantization. A remarkable feature of our quantization scheme is that it applies even if the classical reduced space is singular. We quantize $S$ by
\[ \mathcal{H} = L^2(\mathbb{R}^2), \] realized in position space. The constraint is quantized by the closure of the operator \( \hat{H}_0 = \frac{1}{2}(-\partial_x^2 + \partial_y^2) \), initially defined and essentially self-adjoint on \( C^\infty_c(\mathbb{R}^2) \), which of course has absolutely continuous spectrum equal to \( \mathbb{R} \).

We choose \( L \) to be the subspace (easily shown to be dense) of \( \mathcal{H} \) consisting of those functions \( \psi \) whose Fourier transform \( \hat{\psi} \) is in \( C^\infty_c(\mathbb{R}^2) \) and satisfies \( \hat{\psi}(0) = 0 \). It is not necessary that \( L \) contain the domain of the quantum constraint (indeed, it does not), since what matters is the unitary group generated by it, which is defined on all of \( \mathcal{H} \).

One finds
\[
(\psi, \varphi)_0 = (2\pi)^{-1} \int \frac{dp}{2|p|} [\hat{\psi}(p,p)\overline{\hat{\varphi}(p,p)} + \hat{\psi}(p,-p)\overline{\hat{\varphi}(p,-p)}].
\]

We see that the induced space may be realized as \( \mathcal{H}^0 = \oplus L^2(\mathbb{R}, dp/4\pi|p|) \); the map \( V : L \rightarrow \mathcal{H}^0 \) assumes the form \( (V\psi)_\pm(p) = \hat{\psi}(p,\pm p) \). Interestingly, we may write
\[
(V\psi)_\pm(p) = (\psi, f_\pm(p; \cdot)),
\]
where \( f_\pm(p; x, y) = \exp(-ip(x \pm y)) \). For each \( p \), \( f_\pm(p; \cdot) \) is a (generalized) solution to the quantum constraint \( \hat{H}_0 f_\pm(p; \cdot) = 0 \). These solutions do not lie in \( \mathcal{H} \), yet the expression \( (\psi, f_\pm(p; \cdot)) \) is well-defined for \( \psi \in L \) (cf. the comment below).

There is a quantum analogue of four of the five strata of the classical reduced space (the stratum of orbit type \( \mathbb{R} \) is not represented in the quantum theory). Classical (weak) observables had to be smooth functions of \( p_x, p_y, \) and \( xp_y + yp_x \). On \( \mathcal{H} \) these are quantized in the Schrödinger representation, and the corresponding induced representatives on \( \mathcal{H}^0 \) are given by \( p \oplus p, p \oplus -p, \) and \(-i(pd/dp \oplus -pd/dp)\), respectively. Note that all three are essentially self-adjoint on \( \oplus \left[ C^\infty_c(\mathbb{R}^+) \oplus C^\infty_c(\mathbb{R}^-) \right] \).

Now each copy of \( L^2(\mathbb{R}, dp/4\pi|p|) \) in \( \mathcal{H}^0 \) splits as a direct sum \( \oplus L^2(\mathbb{R}^+, dp/4\pi|p|) \oplus L^2(\mathbb{R}^-, dp/4\pi|p|) \), each summand of which is irreducible under the unitary group generated by the Lie algebra spanned by the operators in question; this construction realizes the four inequivalent massless representations of the two-dimensional Poincaré group.

A similar argument might be given in terms of the flow generated by the quantum observables. By definition, the latter are induced representatives \( \pi^0(A) \) of weak
observables \( A \) (which, in particular, must map \( L \) into itself). The difficulty is that realistic Hamiltonians are unbounded operators, so that one has to answer the question whether essential self-adjointness of \( A \) on \( L \) implies the same for \( \pi^0(A) \) on \( V(L) \), and if so, whether the restriction that \( \pi^0(A) \) must map \( V(L) \) into itself is sufficient to have the desired irreducibility with respect to the unitary group generated by \( \pi^0(A) \). We shall leave this as a topic for future work.

We now look at the case \( \kappa = 1 \), that is, \( H_1 = \frac{1}{2}(p_x^2 + e^{4x} - p_y^2) \). Points on \( C \) have to satisfy \(|p_x| < |p_y|\). The flow generated by \( H_1 \) is, restricted to \( C \),

\[
(x, y, p_x, p_y) \to (x(t), y - p_y t, p_y \tanh[2p_y(t_0 - t)], p_y),
\]

where \( x(t) \) is determined by the condition \( H_1 = 0 \), and \( t_0 = (2p_y)^{-1} \arctanh(p_x/p_y) \).

This motion is complete (that is, defined for all \( t \)), so that \( H_1 = J \) is the moment map of an \( \mathbb{R} \)-action. There are no singularities, and \( S^0 \) is duly a manifold, namely \( T^*\mathbb{R} \) with the zero section removed, equipped with the canonical symplectic structure. In fact, the functions \( f_1 = p_y \) and \( f_2 = y - \frac{1}{2} \arctanh(p_x/p_y) \) Poisson-commute with the constraint on \( C \), and project to globally defined canonical co-ordinates on \( S^0 \).

As before, \( \mathcal{H} = L^2(\mathbb{R}^2) \), on which the constraint is quantized by the closure of the operator

\[
\hat{H}_1 = \frac{1}{2}(-\partial/\partial x)^2 + \exp(4x) + (\partial/\partial y)^2,
\]

initially defined and essentially self-adjoint on \( C^\infty_c(\mathbb{R}^2) \). (The essential self-adjointness immediately follows from the positivity of \(- (\partial/\partial x)^2 + \exp(4x) \), cf. [24], or may be inferred from the explicit form of the (generalized) eigenfunctions and the Weyl-Titchmarsh theory, cf. [27].)

The operator \( \hat{H}_1 \) is diagonalized by the unitary transformation \( W_1 : \mathcal{H} \to L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R}, dp/2\pi) \) given by

\[
(W_1\psi)(\sigma, p) = (\psi, f_1(\sigma, p; \cdot)); \quad (W_1^{-1}\tilde{\psi})(x, y) = (\tilde{\psi}, f_1(\cdot; x, y)),
\]

with

\[
f_1(\sigma, p; x, y) = \pi^{-1}e^{-ipy}\sqrt{2\sinh(\pi \sqrt{\sigma})}K_i\sqrt{\pi}(\frac{1}{2}e^{2x}).
\]
This is closely related to the (Kontorovich-) Lebedev transformation, cf. [21]. The expressions as given are defined on $\psi \in \mathcal{H}$ and $\tilde{\psi} \in W\mathcal{H}$ in a suitable dense subset (e.g., functions with compact support), and then extended by continuity. The point is that $(W_1H_1W_1^{-1}\tilde{\psi})(\sigma,p) = (2\sigma - \frac{1}{2}p^2)\tilde{\psi}(\sigma,p)$. For suitable $L \subset \mathcal{H}$ (as explained above), the expression $(\psi,\varphi)_0 = \int_{\mathbb{R}} dt (e^{-itH_1}\psi,\varphi)$ is well-defined, and equal to the inner product $(V_1\psi,V_1\psi)$ in $\mathcal{H}^0 = L^2(\mathbb{R},dp/2\pi)$, with $V_1$ given by $(V_1\psi)(p) = (\psi,\tilde{f}_1(p;\cdot))$, where

$$
\tilde{f}_1(p;x,y) = \sqrt{\pi}f_1(p^2/4; p;x,y) = e^{-ipy}\sqrt{\pi^{-1/2}\sinh(\pi|p|/2)}K_{|p|/2}(\frac{1}{2}e^{2x}).
$$

For each value of $p$, $\tilde{f}_1(p;\cdot)$ is a solution to the quantum constraint, i.e.,

$$
\hat{H}_1\tilde{f}_1(p;\cdot) = 0.
$$

There is, however, another linearly independent solution as well, which has been excluded by our method, in the sense that it plays no role in the construction of the induced space $\mathcal{H}^0$. Note the difference with the $\kappa = 0$ case. There, for each $p$ the quantum constraint had 2 linearly independent solutions as well (viz. $f_{\pm}$), which both ‘contributed’ to $\mathcal{H}^0$. The discrepancy between the two cases is a consequence of the fact that the spectrum of $-(\partial/\partial x)^2$ is $\mathbb{R}^+$ with multiplicity 2, whereas that of $-(\partial/\partial x)^2 + \exp(4x)$ is $\mathbb{R}^+$ with multiplicity 1.

Finally, we consider $\kappa = -1$, i.e., $H_{-1} = \frac{1}{2}(p_x^2 - e^{4x} - p_y^2)$. The peculiar feature of this case is that the flow generated by $H_{-1}$ is incomplete. Restricting ourselves to the constraint hypersurface $H_{-1} = 0$, we have the following situation. One has the condition $|p_x| > |p_y|$. For $p_y \neq 0$, the flow is

$$(x,y,p_x,p_y) \rightarrow (x(t),y - p_y t,p_y/\tanh[2p_y(t_0 - t)],p_y).$$

For $p_x > |p_y|$, this motion is defined for $t < t_0$, and describes how the $x$-co-ordinate moves from $-\infty$ at $t = -\infty$ to $\infty$ at $t = t_0$; $p_x$ moves from $|p_y|$ at $t = -\infty$ to $\infty$ at $t = t_0$. For $p_x < -|p_y|$, the motion is defined for $t > t_0$, and takes place in the opposite direction. For $p_y = 0$, one has $(x,y,p_x,0) \rightarrow (x(t),y,p_x/(1 - \frac{1}{2}p_xt),0)$. For $p_x > 0$ the motion is defined for $-\infty < t < 2/p_x$, and for $p_x < 0$ it is defined for
Thus the constraint fails to generate an action of \( \mathbb{R} \). Nonetheless, symplectic reduction from this constraint is well-defined (one follows the ‘geometric’ reduction procedure of \([2]\)). The reduced space is a manifold with two components, symplectomorphic to \( T^*\mathbb{R} \cup T^*\mathbb{R} \). To show this, note that \( f_1 = p_y \) and \( f_2 = y - \frac{1}{2} \text{arctanh} \left( \frac{p_y}{p_x} \right) \) project to globally defined canonical co-ordinates on the quotient space, the components of which are projections of the regions \( p_x > 0 \) and \( p_x < 0 \), respectively.

To quantize, we consider the operator
\[
\hat{H}_{-1} = \frac{1}{4} (-\partial/\partial x)^2 - \exp(4x) + (\partial/\partial y)^2,
\]
initially defined on \( D = C_\infty^c(\mathbb{R}^2) \subset \mathcal{H} = L^2(\mathbb{R}^2) \). As can be expected on the basis of the fact that the classical motion is incomplete \([24, 11]\), this operator is not essentially self-adjoint. The deficiency indices are \((1, 1)\), and the self-adjoint extensions are characterized by boundary conditions at \( +\infty \). For each \( \alpha \in [0, 2\pi) \), the extension \( \hat{H}_0^\alpha \) is the closure of \( \hat{H}_{-1} \) defined on \( D \) with a function added whose asymptotic behaviour as \( x \to \infty \) is \( \sim z^{-1/2}[\exp(iz) + \exp(-i(z - \alpha))] \), with \( z = \frac{1}{4} \exp(2x) \).

It is remarkable that, in general, all self-adjoint extensions of a given incomplete Hamiltonian have the same classical limit \([11]\). In the context of constrained quantization, however, we must conclude that each choice of a self-adjoint quantization of the constraint leads to a different quantum theory of the reduced system.

For simplicity, we choose \( \alpha = 0 \). Diagonalization of \( \hat{H}_{-1}^0 \) (cf. \([21]\)) is accomplished as in the \( \kappa = 1 \) case, the only difference being that \( W_{-1} : \mathcal{H} \to L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R}, dp/2\pi) \) is now defined by
\[
f_{-1}(\sigma, p; x, y) = \frac{1}{2} e^{-ipy} \sqrt{2\cosech \left( \pi \sqrt{\sigma} \right)} (J_{i\sqrt{\sigma}} + J_{-i\sqrt{\sigma}}) \left( \frac{1}{2} e^{2x} \right).
\]
Consequently, in the definition of \( V_{-1} \) one has the transformation function
\[
\tilde{f}_{-1}(p; x, y) = \frac{1}{2} e^{-ipy} \sqrt{2\pi \cosech \left( \pi |p|/2 \right)} (J_{|p|/2} + J_{-|p|/2}) \left( \frac{1}{2} e^{2x} \right).
\]
For each \( p \), this is a solution to the quantum constraint. Since the spectrum of \( -(\partial/\partial x)^2 - \exp(4x) \) is \( \mathbb{R}^+ \) without degeneracy, the same comments as in the \( \kappa = 1 \) case apply here.
Here and in all analogous examples, it is possible (though by no means necessary) to choose \( L \) as a Hilbert space \( \mathcal{H}_+ \subset \mathcal{H} \), in which case the solutions \( f(\pm)(p) \) to the constraints lie in the continuous dual \( \mathcal{H}_- \); the inclusions \( \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \) are continuous. The ‘inner product’ of \( \psi \) and \( f(p; \cdot) \) then stands for the pairing of \( \mathcal{H}_+ \) and \( \mathcal{H}_- \). See [4, 22, 23] for this approach to self-adjoint differential operators.

Also, it is possible to choose \( \mathcal{H}_+ \) as a nuclear space which is continuously injected in \( \mathcal{H} \); this leads to a similar (Gel’fand) triplet structure. It does not seem to be a good idea to us, however, to define the physical Hilbert space as the subspace of \( \mathcal{H}_- \) consisting of all generalized solutions of the constraints [10]. This space is far too big; firstly it is not possible to make it into a separable Hilbert space in any reasonable way, and secondly, in the context of our examples, it contains both linearly independent solutions of the constraints (for fixed \( p \)). While for \( \kappa = 0 \) this happens to be reasonable, for \( \kappa = \pm 1 \) we see that essential information (on the specific way the constraints are defined as self-adjoint operators on \( \mathcal{H} \)) is thrown away. This particularly affects the so-called ‘wavefunction of the universe’, cf. [15].

5 Mackey’s quantization

To close, we briefly explain how one can understand and generalize Mackey’s quantization method [17] from the point of view of symplectic reduction and Rieffel induction [12, 13]. Mackey studied particle motion on homogeneous spaces \( Q = G'/G \) (where \( G \) is a closed subgroup of \( G' \)), and found that one may define a family of inequivalent quantizations, one for each element of \( \hat{G} \) (the unitary dual of \( G \)). These are usually construed as all being quantizations of the single phase space \( S = T^*Q \). This is a misinterpretation, however. In fact, each is the quantization of a different phase space.

There is no reason to specialize the discussion to homogeneous spaces; we will consider a general principal bundle \( P \) with gauge group \( G \) and base space \( Q = P/G \). The right-action of \( G \) on \( P \) lifts (pulls back) to a strongly Hamiltonian right-action on \( S = T^*P \) with moment map \( J \) (cf. [1] for an explicit formula). Given a co-adjoint orbit \( O \), one obtains the Marsden-Weinstein quotient \( S^O \). As first discussed
in physics by Wong, and mathematically by Sternberg and others, this is the phase space of a charged particle, moving on the configuration space $Q$, which couples to a gauge field with group $G$ \[28\].

Now we quantize using our method, for simplicity assuming that $G$ is compact (which is the case relevant to physics). We equip $P$ with a $G$-invariant faithful measure and form $L = \mathcal{H} = L^2(P)$. This Hilbert space carries a unitary representation $U(G)$ defined by $(U(g)\psi)(p) = \psi(pg)$. Also, we assume we have a Hilbert space $\mathcal{H}_\rho$ and a unitary representation $U_\rho$ which quantize $O$ (cf. section 2); evidently, this is possible only of the orbit is indeed ‘quantizable’. Constructing the modified inner product $(\cdot, \cdot)_0$ on $L \otimes \mathcal{H}_\rho$ as explained in section 2 (or using the Dirac approach), one easily finds that the induced space $\mathcal{H}^\rho$ may be identified with the (closed) subspace of $L^2(P) \otimes \mathcal{H}_\rho$ consisting of equivariant functions, i.e., satisfying $\psi(pg) = U_\rho(g^{-1})\psi(p)$ for a.e. $p \in P$ and all $g \in G$. Indeed, this subspace is precisely $P^\rho_{id}(L^2(P) \otimes \mathcal{H}_\rho)$. (For non-compact $G$, the induced space is no longer a subspace of $L^2(P) \otimes \mathcal{H}_\rho$, but is still correctly identified by Rieffel induction.) Geometrically, the induced space $\mathcal{H}^\rho$ is the $L^2$-closure of the space of smooth compactly supported cross-sections $\Gamma^\infty_{\rho}(E^\rho)$ of the Hermitian vector bundle $E^\rho$ (over $Q$) associated to $P$ by the representation $U_\rho(G)$.

To sum up, the quantization of the Marsden-Weinstein quotient $SO$, where $O$ is a quantizable orbit, is the Hilbert space $\mathcal{H}^\rho$ induced from $\mathcal{H}_\rho$ (the quantization of $O$). If $P$ is a group $G'$ and $G \subset G'$ then this construction reduces to Mackey’s theory of induced representations \[17\]. This makes it particularly clear how his inequivalent quantizations of the configuration space $Q$ are to be interpreted.

A further illustration of the parallel between symplectic quotients and $C^*$-modules (see section 3) is obtained by noting that the space $L = \Gamma_0(E^\rho)$ of continuous cross-sections of $E^\rho$ vanishing at infinity can be made into a $C^*$-module \[14\]. The $C^*$-algebra $\mathfrak{B}$ is $C_0(Q)$, which acts on $L$ on the right by $(\pi_R(f)\psi)(q) = f(q)\psi(q)$. The generalized quantum moment map $\langle \cdot, \cdot \rangle_{C_0(Q)} : \Gamma \otimes L \rightarrow C_0(Q)$ is given by $\langle \psi, \varphi \rangle_{C_0(Q)} : q \rightarrow (\varphi(q), \psi(q))_\rho$. For any pure state $q \in Q$ on $C_0(Q)$, the Rieffel-induced space $\mathcal{H}^\rho$ constructed from these data is just $\mathcal{H}_\rho$. (We could have started
from the pre-$C^*$-module $L' = \Gamma^\infty_c(E^\rho)$, with the same conclusion; we used $L = \Gamma_0(E^\rho)$ because it is complete.

If our analogy between symplectic reduction and Rieffel induction is correct, there should be a corresponding construction at the classical level. Indeed, we take as ingredients (cf. section 1) $S = S^O$ (which is a locally trivial bundle over $Q$ with projection $pr$), $P = Q$ with zero Poisson structure, and $J : S^O \to Q$ given by $J = pr$. Any point $q$ (seen as a zero-dimensional symplectic manifold) in $Q$ defines a symplectic realization $\rho_q : \{q\} \to Q$ by the inclusion map. The fiber product $C = S^O \ast_Q \{q\}$ is just the fiber $pr^{-1}(q)$ above $q$. The null foliation $\mathcal{F}_0$ is generated by functions of the type $pr^*f$, $f \in C^\infty(Q)$. Locally, $S^O$ looks like $T^*Q \times \mathcal{O}$ (though not as a symplectic space), and in this local picture the leaves of $\mathcal{F}_0$ are the fibers of $T^*Q$. Hence the reduced space $S^{Oq}$ equals $\mathcal{O}$, including its symplectic structure.

Turning the argument around, we have written the orbit $\mathcal{O}$ as a generalized Marsden-Weinstein quotient $S^{Oq}$ constructed from the unreduced space $S^O$. Assuming that $\mathcal{O}$ is quantized by a Hilbert space $\mathcal{H}_\rho$, we found (from previous analysis) that $S^O$ is quantized by the linear space $L = \Gamma_0(E^\rho)$. Applying Rieffel induction, we then showed that the quantization of the reduced space $S^{Oq}$ using our method yields $\mathcal{H}_\rho$ again. Please note that this argument is non-circular.

References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, 2nd ed. (Addison Wesley, Redwood City, 1985).

[2] J.M. Arms, M. Gotay, and G. Jennings, “Geometric and algebraic reduction for singular momentum maps”, *Adv. Math.* 79 (1990) 43-103.

[3] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, and T. Thiemann, “Quantization of diffeomorphism invariant theories of connections with local degrees of freedom”, *J. Math. Phys.* 36 (1995) 6456-6493.

[4] Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators* (A.M.S., Providence, 1968).
[5] A. Connes, *Noncommutative Geometry* (Academic Press, San Diego, 1994).

[6] P.A.M. Dirac, *Lectures on quantum mechanics*, (Yeshiva University, New York, 1964).

[7] J. Dixmier, *C*-algebras (North-Holland, Amsterdam, 1977).

[8] V. Guillemin, *Moment Maps and Combinatorial Invariants of Hamiltonian T^n Spaces* (Birkhäuser, Basel, 1994).

[9] V. Guillemin and S. Sternberg, “Geometric quantization and multiplicities of group representations”, *Inv. Math.* 67 (1982) 515-538.

[10] P. Hájíček, “Quantization of systems with constraints”, pp. 113-149 in *Canonical Gravity: From Classical to Quantum*, (Lecture Notes in Physics 434), eds. J. Ehlers and H. Friedrich (Springer, Berlin, 1994).

[11] K. Hepp, “The classical limit for quantum mechanical correlation functions, *Commun. Math. Phys.* 35 (1974) 265-277.

[12] N.P. Landsman, “Strict deformation quantization of a particle in external gravitational and Yang-Mills fields”, *J. Geom. Phys.* 12 (1993) 93-132.

[13] N.P. Landsman, “Rieffel induction as generalized quantum Marsden-Weinstein reduction”, *J. Geom. Phys.* 15, 285-319 (1995); Err. *ibid.* 17 (1995) 298.

[14] N.P. Landsman, “The infinite unitary group, Howe dual pairs, and the quantization of constrained systems” submitted to *J. Funct. Anal.* (hep-th/9411171).

[15] N. P. Landsman, “Against the Wheeler-DeWitt equation”, *Class. Quant. Grav.* 12 (1995) L119-L123.

[16] N.P. Landsman and U.A. Wiedemann, “Massless particles, electromagnetism, and Rieffel induction”, *Rev. Math. Phys.* 7, 923-958 (1995).

[17] G.W. Mackey, *The theory of unitary group representations* (University of Chicago Press, Chicago, 1976).
[18] J.E. Marsden and T.E. Ratiu, *Introduction to Mechanics and Symmetry* (Springer, New York, 1994).

[19] K. Mikami and A. Weinstein, “Moments and reduction for symplectic groupoids”, *Publ. RIMS Kyoto Univ.* 24 (1988) 121-140.

[20] D. Marolf, “Refined algebraic quantization: systems with a single constraint”, *Banach Center Publ.* ?? (1996) [gr-qc/9508015].

[21] R. Picard, *Hilbert Space Approach to some Classical Transforms* (Longman, Harlow, 1989).

[22] T. Poerschke, G. Stolz, and J. Weidmann, “Expansions in generalized eigenfunctions of self-adjoint operators”, *Math. Z.* 202 (1989) 397-408.

[23] T. Poerschke and G. Stolz, “On eigenfunction expansions and scattering theory”, *Math. Z.* 212 (1993) 397-357.

[24] M. Reed and B. Simon, *Fourier Analysis, Self-adjointness* (Academic Press, New York, 1975).

[25] M.A. Rieffel, “Induced representations of $C^{*}$-algebras”, *Adv. Math.* 13 (1974) 176-257.

[26] R. Sjamaar and E. Lerman, “Stratified symplectic spaces and reduction”, *Ann. Math.* 134 (1995) 375-422.

[27] J. Weidmann, *Spectral Theory of Ordinary Differential Operators* (Lecture Notes in Mathematics 1258) (Springer, Berlin, 1987).

[28] A Weinstein, “A universal phase space for particles in Yang-Mills fields”, *Lett. Math. Phys.* 2 (1978) 417-420.

[29] P. Xu, “Morita equivalence of Poisson manifolds”, *Commun. Math. Phys.* 142 (1991) 493-509.
[30] S. Zakrzewski, “Induced representations and induced Hamiltonian actions”, *J. Geom. Phys.* 3 (1986) 211-219.