The Power of Multi-step Vizing Chains

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ABSTRACT
Recent papers have addressed different variants of the \((\Delta + 1)\)-edge-colouring problem by concatenating or gluing together many Vizing chains to form what Bernshteyn coined \textit{multi-step Vizing chains}. In this paper, we consider the most general definition of this term and apply different multi-step Vizing chain constructions to prove combinatorial properties of edge-colourings that lead to (improved) algorithms for computing edge-colouring across different models of computation. This approach seems especially powerful for constructing augmenting subgraphs which respect some notion of locality.

First, we construct strictly local multi-step Vizing chains and use them to show a local version of Vizing’s Theorem thus confirming a recent conjecture of Bonamy, Delcourt, Lang and Postle. That is, we show that there exists a proper edge-colouring of a graph such that every edge \(uv\) receives a colour from the list \(\{1, 2, \ldots, \max(d(u), d(v)) + 1\}\). Our proof is constructive and also implies an \(O(n^2\Delta)\) time algorithm for computing such a colouring.

Then, we show that for any uncoloured edge there exists an augmenting subgraph of size \(O(\Delta^2\log n)\), answering an open problem of Bernshteyn. Chang, He, Li, Pettie and Uitto show a lower bound of \(\Omega(\Delta \log \frac{n}{\Delta})\) for the size of augmenting subgraphs, so the upper bound is asymptotically tight up to \(\Delta\) factors. These ideas also extend to give a faster deterministic LOCAL algorithm for \((\Delta + 1)\)-edge-colouring running in \(O(\text{poly}(\Delta)\log^6 n)\) rounds. These results improve the dependency on \(\log n\) compared to the recent breakthrough result of Bernshteyn, who showed the existence of augmenting subgraphs of size \(O(\Delta^2\log^2 n)\), and used these to give the first \((\Delta + 1)\)-edge-colouring algorithm in the LOCAL model running in \(O(\text{poly}(\Delta, \log n))\) rounds.

Finally for dynamic graphs, we show how to maintain a \((1 + \epsilon)\Delta\)-edge-colouring fully adaptive to \(\Delta\) in \(O(e^{-6}\log^9 n \log^4 \Delta)\) worst-case update time w.h.p without any restrictions on \(\Delta\). This should be compared to the edge-colouring algorithm of Duan, He and Zhang that runs in \(O(\epsilon^{-4}\log^6 n)\) amortised update time w.h.p under the condition that \(\Delta = \Omega(e^{-2}\log^2 n)\). Our algorithm avoids the use of \(O(e^{-1}\log n)\) copies of the graph, resulting in a smaller space consumption and an algorithm with provably low recourse.

1 INTRODUCTION & RELATED WORK
In the edge-colouring problem one has to assign colours to the edges of a simple graph such that no edges, sharing an endpoint, receive the same colour. More formally speaking, given a graph \(G = (V, E)\) on \(n\) vertices and \(m\) edges a (proper) \(k\)-edge-colouring of \(G\) is a function \(c : E \mapsto \{k\}\) satisfying that for any two edges \(e, e'\) such that \(e \cap e' \neq \emptyset\), we have \(c(e) \neq c(e')\). Vizing famously showed that if \(\Delta\) is the maximum degree of \(G\), then \(G\) has a \((\Delta + 1)\)-edge-colouring [31], and his proofs extends to give a polynomial time algorithm for computing such a colouring. There exist graphs for which this is tight, but there also exist graphs where \(\Delta\) colours suffice. Note that since any such graph has a vertex of degree \(\Delta\), one clearly needs at least \(\Delta\) colours to properly colour its edges. It was shown by Holyer that it is NP-complete to distinguish between when \(G\) can be \(\Delta\)-edge coloured and when it cannot [24].

Vizing actually showed something slightly stronger than the existence of a \((\Delta + 1)\)-edge-colouring: he showed that given a proper \((\Delta + 1)\)-edge-colouring of a subgraph \(G'\) of \(G\), we may extend it to a proper \((\Delta + 1)\)-edge-colouring of a larger subgraph \(G'' \supset G'\) of \(G\) by recolouring at most \(O(\Delta + n)\) edges of \(G\). These edge form what we will refer to as an \textit{augmenting subgraph} that is a subgraph such that only recolouring edges inside this subgraph allows us to extend the colouring of an edge. Small augmenting subgraphs play an important role if one wants to extend the edge-colouring by colouring multiple uncoloured edges in parallel.

Recently there has been some exciting work proving better upper bounds for the sizes of such subgraphs. As one can perhaps imagine, the existence of small augmenting subgraphs is useful for constructing algorithms, and so these results arise in different models of computation. In order to design an efficient algorithm for colouring dynamic graphs, Duan, He and Zhang [15] essentially showed that if one allows \((1 + \epsilon)\Delta\) colours, then there must exist augmenting subgraphs for every uncoloured edge of size \(O(\text{poly}(\Delta, \log n))\). With the goal of designing an efficient LOCAL algorithm, Bernshteyn [5], inspired by the approach of Grebik and Pikhurko [22], showed that in fact there exists augmenting subgraphs of size \(O(\text{poly}(\Delta, \log^2 n))\)
even if one only has \( \Delta + 1 \) colours available. All these result rely on the idea that if the augmenting subgraph constructed in Vizing’s Theorem – let us call such a subgraph a Vizing chain – is too big, then one can truncate it early and create a new one. This idea can be repeated until a short Vizing chain is found. If the Vizing chains do not overlap, the above papers show via probabilistic arguments that there has to exist a choice of chains and truncation points that terminates in a small augmenting subgraph quickly. Bernshteyn coined his construction by the name multi-step Vizing chain.

From the lower bound side, it was shown by Chang, He, Li, Pettie and Uitto [10] that augmenting subgraphs sometimes need to have size at least \( \Omega(\Delta \log \frac{n}{\Delta}) \). This lower bound immediately carries over to the round complexity of any LOCAL algorithm that works by naively extending colours through augmenting subgraphs.

In this paper, we consider the most general definition of a multi-step Vizing chain. In particular, we will allow subsequent Vizing chains to overlap in edges. We will then consider constructions of various subclasses of Vizing chains.

The different constructions from above all give rise to a subclass of Vizing chains which we will call non-overlapping. We will also consider non-overlapping multi-step Vizing chains, but in contrast to Duan, He and Zhang [15] and Bernshteyn [5] we view them through a non-probabilistic lens and show that they can be used to construct augmenting subgraphs of size \( O(\text{poly}(\Delta \log n)) \). This shows that edge-colourings of graphs with low degree are locally extendable – in the sense that they can be extended through augmenting subgraphs in the close neighbourhood of the uncoloured edges. The existence of such subgraphs also allows us to give an improved LOCAL algorithm for computing \((\Delta + 1)\) edge-colourings that falls into a long line of work where round complexity is parameterised by \( \Delta \) and \( \log n \) (see for instance [5, 10, 20, 29, 30] for a subset) which we elaborate on below.

Moreover, we define another kind of multi-step Vizing chains that we call strictly local Vizing chains and use them to prove the existence of an edge-colouring that takes the local structure around the edges into account. Namely, we will show the following conjecture of Bonamy, Delcourt, Lang and Postle [7].

**Theorem 1 (Local Vizing Theorem, Conjectured in [7]).** For any graph \( G \), there exists a proper edge-colouring of the edges of \( G \) such that all edges \( uv \) receives a colour in \( \{1, \ldots, 1 + \max\{d(u), d(v)\}\} \).

Here each edge receives a list of colour, depending on the local structure around it, and one has to find a proper edge-colouring of the graph so that every edge receives a colour from its prescribed list. This sort of local generalisation is well-studied in the context of list colouring where each element that is to be coloured, for instance the edges or the vertices of a graph, has to receive a colour from the prescribed lists. In the context of vertex colouring, Bonamy, Kelly, Nelson, and Postle recently studied list colourings in terms of local clique sizes [8], and Davies, de Joannis de Verclos, Kang, and Pirot studied the problem for triangle-free graphs [13].

In the context of edge-colourings, local generalisations of list edge-colourings was studied by Borodin, Kostochka, and Woodall [9] who showed a local generalisation of Galvin’s theorem [19] and by Bonamy, Delcourt, Lang, and Postle [7] who, under certain degree conditions, showed a local version of Kahn’s theorem [25]. Work on local generalisations of edge-colourings also appeared in the work of Erdős, Rubin, and Taylor [16].

As hinted to earlier, the constructions and ideas from above are useful when designing efficient edge-colouring algorithms in various models of computation. We will consider two such models.

**LOCAL algorithms.** In [27] Linial introduced the LOCAL model of computation. Here one is given an input graph that is viewed as a communication network. Computation is performed at each vertex in synchronous rounds. During a round, vertices are allowed to exchange messages of unbounded size with their neighbours in the communication network as well as perform unbounded computation.

Edge-colouring has been widely studied in this model of computation. Alon, Babai and Itai [2] and independently Luby [28] gave a randomised \( O(\log n) \) round algorithm for \( \Delta + 1 \) vertex colouring, which can be transformed into a \( 2\Delta - 1 \) edge-colouring algorithm by colouring the line graph. Goldberg, Plotkin and Shannon [21] improved the round complexity for graphs of small maximum degree by giving a randomised algorithm running in \( O(\Delta^2 \log^2 n) \) rounds. After a long line of work Fischer, Ghaifari and Kuhn [17] provided a deterministic algorithm for \( 2\Delta - 1 \) edge-colouring a graph in \( O(\text{poly} \log n) \) rounds. Recently, this was improved by Bariuli, Brandt, Kuhn and Olivetti [3] who gave a deterministic algorithm for \( 2\Delta - 1 \) edge-colouring a graph in \( \text{poly}(\log \Delta) + O(\log^2 n) \) rounds.

Much work has also been dedicated to going below \( 2\Delta - 1 \) colours (see for instance [10, 20] for more extensive surveys). Chang, He, Li, Pettie and Uitto [10] gave a randomized algorithm using \( \Delta + O(\sqrt{\Delta}) \) colours running in \( O(\text{poly}(\Delta, \log n)) \) rounds. Recently, Davies [14] gave a \( \log \text{poly}(\Delta, \log n) \) round randomized algorithm for computing a \( \Delta + o(\Delta) \) colouring. Su and Vu [30] designed a randomised algorithm using only \( \Delta + 2 \) colours that runs in \( O(\text{poly}(\Delta, \log n)) \) rounds. Su & Vu arrive at their \( (\Delta + 2) \)-edge-colouring algorithm by truncating Vizing chains at a randomly chosen spot, but instead of building a new one like Bernshteyn does [5], they instead use a special colour to colour the now uncoloured edge. They show that with high-probability no vertex ends up with two neighbouring edges receiving this special colour. Due to a general derandomisation technique developed by Ghaifari and Rožhoň [29] this can be turned into a deterministic algorithm. Other deterministic results include a \( O(\Delta^2) \) edge-colouring algorithm due to Ghaifari, Kuhn, Maus and Uitto [20] as well as the already mentioned result of Bernshteyn [5], who gave an algorithm using only \( \Delta + 1 \) colours. Both of these algorithms run in \( O(\text{poly}(\Delta, \log n)) \) rounds.

**Dynamic graph algorithms.** In the dynamic graph setting, one attempts to maintain solutions to graph problems as the graph is subjected to insertions and deletions of edges. The goal is that the time spent updating the solution should be significantly faster than naively re-running a static algorithm from scratch each time an edge is inserted or deleted. For many problems one aims at achieving \( O(\text{poly} \log n) \) update time.

From a static point-of-view Gabow, Nishizeki, Kariv, Leven, and Tereda [18] showed how to compute a \( (\Delta + 1) \)-edge-colouring in
We show the following local version of Vizing’s Theorem originally conjectured by Bonamy, Delcourt, Lang and Postle [7]:

**Theorem 2 (Local Vizing Theorem).** For any graph there exists a proper edge-colouring such that every edge $uv$ receives a colour from the list $L(uv) = \{1, \ldots, 1 + \max(d(u), d(v))\}$.

This Theorem immediately implies Vizing’s Theorem since $\Delta = \max_{uv} d(u)$. Then we shift our attention from local lists to extending colourings by recolouring edges locally. We show that the lower bound due to Chang, He, Li, Pettie and Uitto [10] is tight up to constant and $\Delta$ factors:

**Theorem 3.** For any graph $G$ endowed with a proper partial $(\Delta + 1)$-edge-colouring $c$, and for any edge left uncoloured by $c$, there exists an augmenting subgraph containing at most $\Omega(\Delta^2 \log n)$ edges.

Finally, we extend the techniques used to derive the two combinatorial results from above to devise efficient algorithms. First of all, the following corollary follows from our proof of Theorem 2:

**Corollary 4.** There is an algorithm that runs in $O(n^2 \Delta)$-time for computing an edge-colouring such that every edge $uv$ receives a colour from the list $L(uv) = \{1, \ldots, 1 + \max(d(u), d(v))\}$.

We improve the round complexity of the best LOCAL algorithm for computing a $(\Delta + 1)$-edge-colouring from $\tilde{O}(\max(\log \Delta, \log n))$ to $\tilde{O}(\max(\log \Delta, \log n))$.

**Theorem 5.** There is a deterministic LOCAL algorithm that computes a $\Delta + 1$ edge-colouring in $\tilde{O}(\Delta \log^6 n)$ rounds. Here $\tilde{O}(x)$ suppresses $\tilde{O}(\log \log x)$ factors.

Finally, we improve the round complexity of the best LOCAL algorithm with low recourse. More specifically, they show how to maintain a $(\Delta + 1)$-edge-colouring by recolouring edges locally. We show that the lower bound due to Bhattacarya, Grandoni and Wajc [6] can be improved by $\tilde{O}(\Delta \log \Delta)$.

**Theorem 6.** Let $G$ be a dynamic graph subject to insertions and deletions with current maximum degree $\Delta$. There exists a fully-dynamic and $\Delta$-adaptive algorithm maintaining a proper $(\Delta + 1)$-edge-colouring with $\tilde{O}(\Delta \log \Delta)$ worst-case update time with high probability.

**1.1 Results**

The dynamic case has recently received attention and seen some development. Barenboim and Maimon [4] gave a dynamic algorithm with $O(\sqrt{\Delta})$ worst-case update time using $O(\Delta)$ colours. This was subsequently improved by Bhattacarya, Chakrabarty, Henzinger and Nanongkai who gave a fully dynamic algorithm for computing a $\Delta - 1$ edge-colouring with worst case $O(\log \Delta)$ update time. Finally, Duan, He and Zhang [15] reduced the number of colours even further. They designed an algorithm using $(1 + \epsilon)\Delta$ colours when $\Delta = \Omega(\epsilon^{-2} \log^2 n)$ achieving an amortised update time of $O(\epsilon^{-4} \log^6 n)$. This approach changes between $O(\epsilon^{-1} \log n)$ copies of the graph, resulting in a $O(\epsilon^{-1} \log n)$ space overhead and a recourse which is not bounded. Bhattacarya, Grandoni, and Wajc [6] studied the problem of dynamically maintaining edge-colourings with low recourse. More specifically, they show how to maintain a $(\Delta + 1)\Delta$-edge-colouring with high probability with only $O(\Delta \log \log n)$ expected recourse in dynamic graphs of maximum degree $\Delta = \Omega(\Delta \log \log n))$.

**1.2 High-Level Overview**

In this section, we give an informal description of our techniques and approaches, before we give the precise definitions and technical proofs later. As mentioned in the introduction, the starting point is the idea of ‘gluing’ together several Vizing chains to form subgraphs with interesting properties. In the literature there are several ways of constructing Vizing chains, but we will be working with the following: a Vizing chain consists of two parts: a) a fan $F$ of edges and b) a bichromatic path $P$ consisting only of edges coloured with 2 colours, let us call them $k_1$ and $k_2$. The fan $F$ consists of a center vertex $u$ together with edges $uw_1, \ldots, uw_k$ such that the colour of $uw_{i+1}$ is available at $w_i$, meaning that no edges incident to $w_i$ has the colour $c(uw_{i+1})$. The bichromatic path $P$ has to then begin at $w_k$ and consist only of edges coloured $k_1$ and $k_2$, where $k_1$ is available at $u$ and $k_2$ is available at $w_k$. Given a Vizing chain, we can shift it by recolouring edges in $F$ and subsequently in $P$ as shown in Figure 1 to the left. This will transform one proper partial edge-colouring (the pre-shift colouring) into another proper partial edge-colouring (the post-shift colouring). In this example, we see that the shift allows us to extend the colouring to one extra edge. The idea is that if $P$ is not too long, we can shift the Vizing chain to extend our colouring, and if $P$ is too long, we can truncate it by uncolouring an edge along $P$ and only shift the first part of $P$ as shown in Figure 1 to the right.

Then we may build a new Vizing chain on top of the new uncoloured edge in the hope that this chain has a shorter length. We may iterate this to construct multi-step Vizing chains (see Figure 2). This approach is originally due to Bernshteyn [5], but where Bernshteyn only uses Vizing chains that cannot overlap in edges, we will relax this requirement and allow the concatenation of all types of Vizing chains. First we turn our attention to what we will call...
strictly local Vizing chains. In these Vizing chains, we require that the post-shift colouring is strictly local, meaning that it satisfies the condition of the Local Vizing Theorem for every coloured edge. This means that a coloured edge \( e \) needs to receive a colour from \( \{\max[d(u), d(v)] + 1\} = \{1, 2, \ldots, \max[d(u), d(v)] + 1\} \).

The key idea behind enforcing that the Vizing chains are non-overlapping is that it allows one to reason about how many Vizing chains can reach a certain point or a certain edge in the graph. The overlapping is that it allows one to reason about how many Vizing chains, then we can use them to show Theorem 2 as follows: we can reach them by truncating the Vizing chain at an edge incident to them, and shifting the uncoloured edge to be incident to such a vertex. If the bichromatic path of this chain is too long, say it has length \( > \ell \), we consider the act of truncating this Vizing chain at any of the first \( \ell \) edges and extending it to a 2-step Vizing chain through one of the points in \( R_1 \) using the construction of Grebík and Pikhurko [22]. This allows us to get a second Vizing chain build on the truncation edge such that 1) the fan from this Vizing chain does not overlap with the bichromatic path of the first Vizing chain, and 2) the new Vizing path \( P_2 \) consists of the same colours as \( P_1 \) or of two completely different colours. We then examine all of these bichromatic paths. Some of them may overlap the first Vizing chain, and this renders the respective Vizing chains, and the points that we build them on, unfit to be used as an extension point. We let \( N_1 \) contain all of the points in \( R_1 \) that are not unfit for extension. Hence, we can safely extend our multi-step Vizing chain through these extension points.

Therefore, we can consider the set of points reachable via 1-step Vizing chains of the points in \( N_1 \). These are exactly the points that we can reach via 2-step non-overlapping Vizing chain from the first point. If all of these Vizing chains have length \( > \ell \), we have not yet found a short augmenting non-overlapping Vizing chain. Therefore, we can let \( R_2 \) be the set of points that we can reach through non-overlapping 2-step Vizing chains, and similarly \( N_2 \subset R_2 \) be the set of points which we can safely extend our Vizing chain through. Similarly, if we fail to find a short, augmenting \( i \)-step Vizing chain, we can recursively define \( R_1 \) to be the set of points we can reach through \( i \)-step non-overlapping Vizing chains, and
Figure 3: The number of reachable vertices through non-overlapping Vizing chains of length \(i\) form \(R_i\). We will later show that \(R_i\) grows exponentially as a function of \(i\).

\[ N_j \subset R_i \text{ the set of points through which we can safely extend our Vizing chains.} \]

The idea is then to show that both \(R_i\) and \(N_j\) will grow exponentially in \(\Omega(\frac{\ell}{\Delta})\) for \(\ell \in \text{poly}(\Delta)\) (see Figure 3 for an illustration).

This implies that \(i \leq \log n\), and so we can conclude that there has to exist some augmenting Vizing chain that takes few steps and has a very short step-length. The idea is to show that if there is no short augmenting and non-overlapping \(j\)-step Vizing chain for any \(j \leq i\), then \(|R_j| = \Omega(\frac{\ell}{\Delta}[N_{j-1}])\) and \(|N_j| \geq \frac{|R_j|}{\Delta}\) for all \(j \leq i\). Combining these things inductively, then shows the exponential growth.

To this end, we consider the act of extending exactly one \((i-1)\)-step Vizing chain through a point in \(N_{i-1}\). We may have multiple options for which \((i-1)\)-step Vizing chain to extend, but we just pick one arbitrarily. Consider the graph \(H\) spanned by the edges of the bipartite graphs of these Vizing chains. The key observations are that \(V(H) \subset R_i \cup N_{i-1}\) and that \(\ell \cdot |N_{j-1}| = O(\Delta^3|E(H)|)\).

Indeed, each vertex in \(N_{i-1}\) contributes \(\ell\) edges to \(H\), and in this way we only count each edge of \(H\) \(O(\Delta^2)\) times. This is because a coloured edge is part of at most \(O(\Delta)\) bipartite graphs — one for each other choice of colour — and each bipartite path can be part of at most \(O(\Delta)\) Vizing chains, namely Vizing chains situated at neighbours of the endpoints of the bipartite chains. Since the density of \(H\) satisfies \(\rho(H) = \frac{|E(H)|}{|V(H)|} \leq \frac{\Delta}{2}\), we find that

\[ |R_i| = \Omega(\frac{\ell}{\Delta^3}|N_{i-1}|) \]

This leaves only the task of counting how many points in \(R_i\) that are unfit for extension. To do this, we recall that at most \(O(\Delta^3)\) Vizing chains can go through a point, and so at most

\[ O(\Delta^4 \cdot \sum_{k=1}^{i-1} |R_k| + |N_{i-1}|) \]

points in \(R_i\) can extend to overlapping Vizing chains. This follows from the fact that for a Vizing chain to be overlapping, it certainly has to go through either its own fan situated at a point in \(N_{i-1}\) or through a point in the one-hop neighbourhood of \(\frac{\sum_{k=1}^{i-1} |R_k|}{\Delta}\). By choosing \(\ell\) properly, we find that \(|N_i| \geq \frac{|R_i|}{\Delta}\).

\[ \text{Distributed algorithm. In order to turn this into a distributed algorithm, we first note that if augmenting subgraphs are vertex disjoint, then we may augment them in parallel, since changing the colours within one subgraph does not change the colour of any edges incident to vertices in the other subgraphs.} \]

The most difficult part left is to show that there actually exist many such small vertex disjoint subgraphs that we can augment in parallel. This is because of the following reduction due to Bernshteyn [5]. One can define a hypergraph on \(V(G)\) by including a hyperedge for each small augmenting subgraph containing exactly the vertices of the small augmenting subgraph. Given that we do not consider too many augmenting subgraphs and that there exists a large vertex disjoint subset of the subgraphs, one can find such a large vertex disjoint subset by computing an (approximation) of the maximum matching in the hypergraph using for instance the algorithm due to Harris [23].

Hence, we turn our attention to showing that if we include all augmenting Vizing chain with both a small number of steps and a small step-length, then we will have a large set of vertex disjoint augmenting subgraphs. The main intuition that we want to formalise is that even if we allow some adversary to remove \(O(\Delta^4)^{i+1}\) from \(N_i\) at each step above, the construction from above still goes through. This motivates the definition of what we call family-avoiding avoiding Vizing chains. Given a family of sets \(\mathcal{F} = \{F_j\}_{j=1}^{i-1}\), we say that a multi-step Vizing chain avoids \(\mathcal{F}\) if the \(j^{th}\) Vizing chain is not extended through a point belonging to \(\{F_j\}_{j=1}^{i-1}\). We also say that \(\mathcal{F}\) is \(k\)-bounded if for all \(j\) we have \(|F_j| \leq k^j\). By above, our previous construction can actually be made to avoid any \(O(\Delta^4)\)-bounded family.

This concept allows us to build a large set of vertex disjoint small augmenting subgraphs. First we pick an arbitrary such vertex disjoint subgraph. Having picked the first \(j\) vertex disjoint subgraph, we pick the \((j+1)^{th}\) as follows: for any uncoloured edge \(e\), we build a family to avoid, when we use the previous construction to build small augmenting subgraphs. In particular, we add a vertex to each subgraph containing exactly \(e\) can reach any of the vertex disjoint subgraphs that we already picked. If this is the case, we add the vertex that the \(j^{th}\) Vizing chain was extended through to \(\mathcal{F}_{j-1}(e)\). Next we note that if the total size of all subgraphs is \(T\), then, as we saw earlier, we put at most \(T \cdot O(\Delta^4)^{i+1}\) points into some \(F_j\) for all \(j\). Indeed, at most \(T \cdot O(\Delta^4)^i\) many \(i\)-step Vizing chains can reach \(T\). In the worst case, the points are added in a way that maximizes the number of families \(\mathcal{F}(e)\) that become unavoidable, meaning that they are not sufficiently bounded for our construction to be able to avoid them. However, since the number of points we can avoid grows faster than the number of points we have to avoid, the number of edges for which \(F_j(e)\) is too large falls exponentially in \(j\), and so in total at most \(O(\Delta^4 \cdot \Delta \cdot T)\) edges have unavoidable families. We pick the \((j+1)^{th}\) subgraph as a small augmenting subgraph of an edge that avoids the family constructed for it above. This means that this subgraph has to be vertex disjoint form the subgraphs we already
picked, since we are certain to never extend through a point that can reach any of these subgraphs in 1-step.

**Dynamic algorithm.** Let us first recall the approach of Duan, He and Zhang [15]. It has 3 steps: first they make an amortised reduction to the easier non-adaptive version of the problem, where we may assume \( \Delta \) is fixed. They do this by using multiple copies of the graph – each with a different upper bound on the maximum degree. The second step is to reduce the problem to one where \( \Delta = O \left( \frac{\log n}{\varepsilon} \right) \). This is done by sampling a palette \( S \subset \left( 1 + \varepsilon \right) \Delta \) such that every vertex in \( G \) has an available colour in \( S \). Note that it is crucial to have \( (1 + \varepsilon) \Delta \) colours for this step be efficient. The third and final step is to show how to insert an uncoloured edge into such a low-degree graph. To do this, Duan, He and Zhang [15] show that one can construct non-overlapping multi-step Vizing chains by using a new, disjoint palette for each step, and that short Vizing chains of this type have a good probability of being augmenting. Since they need to sample disjoint palettes for each step, they require that \( \Delta = \Omega \left( \frac{\log n}{\varepsilon} \right) \).

Our approach will circumvent the first step. Namely, we will maintain a colouring where we can control the number of edges coloured with a specific colour. In particular, for each colour \( \kappa \) we will require that the number of edges coloured \( \kappa \) is upper bounded by the number of vertices \( v \) where \( \kappa \in (1 + \varepsilon \Delta (v)) \). In fact, we will even maintain a slightly stricter invariant. To accomplish this we define a notion of \( \varepsilon \)-local Vizing chains, where we require all available colours of a vertex \( v \) to be in \( (1 + \varepsilon \Delta (v)) \). We show that shifting such Vizing chains does not violate the invariant from above, and so this allows us to reduce deletions to insertion via a very simple scheme that recolours \( O(1) \) edges in order to accommodate a deletion.

To realise this approach, we need to sample \( \varepsilon \)-local palettes. These are palettes \( S \) where every vertex \( v \) has an available colour in \( S \setminus \left( 1 + \varepsilon \Delta (v) \right) \). We show that one can sample such a palette by combining several sub-palettes sampled from intervals of length powers of \( 2 \). Finally, we use the insights from above to slightly alter an algorithm of Bernshteyn [5] so that it constructs augmenting, \( \varepsilon \)-local and non-overlapping Vizing chains. This allows us to get rid of the assumption on \( \Delta \).

### 2 PRELIMINARIES & NOTATION

For an integer \( t \), we let \( [t] = \{1, 2, \ldots, t\} \) denote the numbers in \( \mathbb{N} \) that are greater than \( 0 \) but at most \( t \). We let \( G = (V, E) \) be a graph on \( n \) vertices and \( m \) edges. A subgraph \( G' \subset G \) of \( G \) is a graph such that \( V(G') \subset V(G) \) and \( E(G') \subset E(G) \). The \( t \)-hop neighbourhood of \( G' \), \( N^t(G') \), are all of the vertices in \( G \) of distance at most \( t \) (in \( G \)) to a vertex in \( G' \). A proper (partial) \( k \)-edge-colouring \( c \) of \( G \) is a function \( c : E(G) \mapsto [k] \cup \{ \ast \} \) such that two coloured edges \( e, e' \), sharing an endpoint, does not receive the same colour. Here an edge \( e \) such that \( c(e) = \ast \) is said to be uncoloured. Two uncoloured edges may share an endpoint in a proper partial colouring. If no edges are uncoloured, we say that the colouring is a proper \( k \)-edge-colouring. Given a proper (partial) \( k \)-edge-colouring \( c \) and a set of colours \( S \subset [k] \), we let \( G[S] \) be the graph induced by the set of edges coloured with a colour from \( S \). Unless otherwise stated, we assume from now on that any edge-colouring is a \( (\Delta + 1) \) (partial) edge-colouring.

Given a proper (partial) edge-colouring of a graph \( G \), we define the set \( A(v) \) of available colours at a vertex \( v \) to consist of precisely the colours that no edge incident to \( v \) has received.

#### 2.1 Chains and Shifts

We briefly recall the chain and shift terminology as it was used by for example Bernshteyn [5] (see [5] for a more in-depth treatment).

Let \( e_1, e_2 \in E(G) \) be two adjacent edges in a graph \( G \), and let \( c \) be a proper partial colouring of \( G \). Then we can define a colouring \( \text{Shift}(c, e_1, e_2) \) by setting:

\[
\text{Shift}(c, e_1, e_2)(e) = \begin{cases} c(e_2) & \text{if } e = e_1 \\ \ast & \text{if } e = e_2 \\ c(e) & \text{if } e \notin \{e_1, e_2\} \end{cases}
\]

We say that such a pair of adjacent edges \( e_1, e_2 \) are \( \varepsilon \)-shiftable (or simply shiftable) if \( c(e_1) = \ast \) and \( c(e_2) = \ast \) and the colouring \( \text{Shift}(c, e_1, e_2) \) defined above is a proper partial colouring.

A chain \( C \) of size \( k \) is then a set of edges \( C = (e_1, \ldots, e_k) \) such that \( e_i \) and \( e_{i+1} \) are adjacent for all \( i \) and \( c(e_i) = \ast \) if \( i = 1 \). For \( 0 \leq j \leq k-1 \), we can \( j \)-shift such a chain by performing \( \text{Shift}_j(c, C) \) defined as:

\[
\text{Shift}_0(c, C) = c \\
\text{Shift}_j(c, C) = \text{Shift}(\text{Shift}_{j-1}(c, C), e_j, e_{j+1})
\]

We say that \( C \) is \( \varepsilon \)-shiftable if every pair of edges \( e_j, e_{j+1} \) is \( \varepsilon \)-shiftable. It is straightforward to check that \( j \)-shifting such a chain \( C \) yields a proper colouring, where the unique uncoloured edge in \( C \) is the edge \( e_{j+1} \).

The goal is to find a chain \( C \) such that shifting \( C \) yields an uncoloured edge \( u \) such that \( A(u) \cap A(v) \neq \emptyset \) so that we may colour \( u \) to extend our partial colouring. If this is the case, we say \( C \) is an augmenting chain, otherwise we refer to it as a truncated chain.

The initial segment of length \( s \) of a chain \( C \) is then the chain \( C_s = (e_1, \ldots, e_s) \). One can concatenate two chains \( C_1 = (e_1, \ldots, e_s) \) and \( C_2 = (e_s, f_2, \ldots, f_k) \) to get chain \( C_1 + C_2 = (e_1, \ldots, e_s, f_2, \ldots, f_k) \). Then in order to shift such a chain, we use \( \text{Shift}(c, C_1 + C_2) = \text{Shift}(\text{Shift}(c, C_1), C_2) \).

Now we can define a fan chain on \( u \) to be a chain \( F \) of the form \( F = (u w_1, \ldots, u w_k) \). Here we denote \( u \) as the center of the fan. For \( k = 1 \) this definition is ambiguous, but in any case where we do not specify the center, one can pick the center arbitrarily. Bernshteyn [5] showed that for such fans to be shiftable chains, we require \( c(u w_{i+1}) \in A(w_j) \). We will let the colour of \( c(u w_{i+1}) \) be the representative available colour at \( w_j \). We will also pick a representative available colour at \( w_k \) in \( A(w_k) \). If this representative available colour is also either available at \( u \) or if it is the representative available colour for some \( w_j \) with \( j < k \), then we say that the fan is maximal.

A \( (k_1, k_2) \)-bichromatic path is a subpath of \( G \) consisting of only edges coloured \( k_1 \) and \( k_2 \). Note that the colouring obtained by colouring all of the \( k_1 \)-coloured edges with the colour \( k_2 \), and all of the \( k_2 \)-coloured edges with the colour \( k_1 \) is also a proper colouring. We refer to this as shifting the colours of \( P \). A path chain is a chain of the form \( P = (e_1, \ldots, e_k) \), such that the edges in the chain form
a path in the graph. The length of a path chain \( P = (e_1, \ldots, e_k) \) is \( k - 1 \). A bichromatic path chain is then a chain \( P = (e_1, \ldots, e_k) \) that forms a path in \( G \) such that \( c(e_1) = u \) and the colour of \( c(e_2) = c(e_4) = \cdots = k_1 \) and \( c(e_3) = c(e_5) = \cdots = k_2 \), and furthermore such that \( e_2 \) is either the first or the last edge in some maximal \((k_1, k_2)\)-bichromatic path. In order to highlight the colours present in the chain, we will sometimes refer to such a chain as a \((k_1, k_2)\)-bichromatic path chain.

Vizing originally showed how to choose the fan and path chains in order to construct what we will refer to as an augmenting subgraph i.e. a subgraph such that by recolouring edges only in this subgraph, we can arrive at a new bigger partial edge-colouring. Note, however, that the subgraph constructed by Vizing is rather large; in order to colour a single extra edge, we might have to recolour \( \Omega(n) \) edges. We refer to this kind of construction as a Vizing chain:

**Definition 7.** Given a proper partial colouring \( c \) of \( G \), a Vizing chain of an uncoloured edge \( e = u_0 \) (or a vertex \( u \)) is a \( c \)-shiftable chain \( F + P \) consisting of a fan chain \( F = (u_0, \ldots, u_k) \) centered at \( u \) together with a \((k_1, k_2)\)-bichromatic path chain \( P = v_kx_1x_2\cdots x_ℓ \) possibly of length 0 – such that \( k_1 \) is available at \( u \) and \( k_2 \) is available at \( v_k \). If \( p \geq 1 \) we say that \( F + P \) ends at the vertex \( x_ℓ \). Finally, let \( c' \) = Shift\((c, F + P)\) be the post-shift colouring.

An important tool for proving the existence of smaller augmenting subgraphs is to glue together what we may call truncated Vizing chains to form a multi-step Vizing chain, a term coined by Bernshteyn [5], but also considered in different shapes by other papers [15, 22]. We consider a general definition of multi-step Vizing chains that encompasses all of these constructions as special types of Vizing chains. Later we will specialise to more restricted classes of multi-step Vizing chain. In particular, we note that in Section 4, we allow the Vizing chains to overlap in edges. Multi-step Vizing chains arise by combining Vizing chains:

**Definition 8.** Given a proper partial colouring \( c \) of \( G \), an \( i \)-step Vizing chain is a \( c \)-shiftable chain of the form \( F_1 + P_1 + F_2 + P_2 + \cdots + F_i + P_i \) where \( F_j + P_j \) is a Vizing chain for all \( j \).

In the case, where we do not specify \( i \), we simply refer to such chains as a multi-step Vizing chain. A (multi-step) Vizing chain is augmenting if shifting the chain leaves the final edge of the chain with a colour that is available at both endpoints of the edge. If the chain is not augmenting, it is truncated.

We let the length of an \( i \)-step Vizing chain be the sum of the lengths of the paths \( P_j \) that make up the multi-step Vizing chain, and the size of an \( i \) step Vizing chain be the total number of edges (with multiplicities) in the Vizing chain. Finally, we let \( i \) be the chain length of the multi-step Vizing chain.

### 3 Types of Multi-step Vizing Chains

The goal of these multi-step Vizing chain constructions is to glue together multiple truncated Vizing chains – all satisfying some properties – to end up with an augmenting Vizing chain that inherits some (possibly relaxed version) of these properties. We introduce these properties next.

Non-overlapping Vizing chains. This type of Vizing chain has been considered in different forms by [5, 15, 22]. We will construct these chains in similarly to Bernshteyn [5], who generalised the construction of Grebik and Pikhurko [22], but our analysis approach is completely different. First we define a non-overlapping Vizing chain.

**Definition 9.** An \( i \)-step Vizing chain is non-overlapping if every pair of Vizing chains \( F_j + P_j \) on edge \( e_j \) and \( F_k + P_k \) on edge \( e_k \) share an edge exactly when \( j = k - 1 \) and the shared edge is \( e_1 \). Furthermore, for all \( k \leq i - 1 \) no edge in the chain \( F_k + P_k = (e_1, \ldots, e_s) \) may be repeated i.e. \( e_j = e_i \) iff \( j = t \).

It is straightforward to check that the construction used in the proof of Vizing’s Theorem yields a \( 1 \)-step Vizing chain that is non-overlapping. However, it is not immediately clear how to extend it to a multi-step Vizing chain. To do so, we will apply the following lemma, due to Grebik and Pikhurko [22].

**Lemma 10** ([5, 22]). Let \( c \) be a proper partial edge-colouring and let \( e = u_0 \) be an uncoloured edge. Let \( k_1 \in [1 + A] \) be an available colour at \( u \) and let \( k_2 \in [\Delta + 1] \) be an available colour at \( v \). Then there exists a fan \( u_0w_1, \ldots, u_0w_k \) with \( w_1 = v \) such that either

1. \( k_2 \not\in A(u) \) and the fan is maximal subject to the constraint that the colours \( k_1 \) or \( k_2 \) are considered unavailable to every vertex.

2. \( w_k \neq v \), the representative available colour at \( w_k \) is \( k_2 \) and no edge \( uw_i \) is coloured \( k_2 \).

3. an available colour at \( u_0w_k \) is also available at \( u \).

We can use Lemma 10 to extend a (multi-step) Vizing chain: given an \( i \)-step Vizing chain ending at an edge \( e = u_0u_1 \) and a point \( u_1 \), we can extend this Vizing chain to an \((i + 1)\)-step Vizing chain by applying the lemma above to \( e \) such that we either avoid or reuse the colours of the bichromatic path of the last Vizing chain used to get there. Namely, in case 1) we pick an available colour at \( u \) that is not \( k_1 \) or \( k_2 \) to use in the construction of the bichromatic path. Such a colour has to exist since \( u \) is incident to at most \( A - 1 \) coloured edges, and one of these edges has to be coloured \( k_2 \). In case 2) we pick \( k_1 \) as the available colour at \( u \) so that we get a \((k_1, k_2)\)-bichromatic path. In case 3), we can construct an augmenting Vizing chain using a bichromatic path of length 0, and so we do this.

We will refer to the extension described above as extending the multi-step Vizing chain via Lemma 10. For a specific multi-step Vizing chain, we cannot guarantee that we have many choices of \( u \) such that the \((i + 1)\)-bichromatic path does not overlap with the multi-step Vizing chain used to go there. However, when we consider many choices of multi-step Vizing chains at the same time, we can provide such a guarantee.

In Section 5, we will show that for any uncoloured edge \( e \) there exists an augmenting, non-overlapping multi-step Vizing chain of size at most \( O(\text{poly}(\Delta \log n)) \). As will be evident from the proof, if the chain is long we will have many choices for how to pick the augmenting Vizing chain. This allows us to consider multiple edges at once and choose augmenting Vizing chains that do not overlap, so that we may colour multiple edges in parallel. To formalise this choice, we introduce the notion of family-avoiding Vizing chains.

**Family-avoiding Vizing chains.** The idea is that given some forbidden points \( F_i \), we would like to avoid extending our Vizing chain...
through a point in $P_i$ when constructing the $(i+1)\text{th}$ Vizing chain. In fact we will be a bit more strict. We will avoid extending our $(i+1)\text{th}$ Vizing chain through any of the sets $P_1, \ldots, P_i$. If this holds for all $i$, we say that the Vizing chain avoids the family $F = \{P_i\}_{i=1}^\infty$. Note that a Vizing chain may pass through points in $P_j$ for all $j$, but it is only allowed to be truncated at points not in $F = \{P_i\}_{i=1}^{k-1}$ at the $k\text{th}$ step. More formally, we define a family-avoiding Vizing chain as follows:

**Definition 11.** Let $F$ be a family of subsets of $V(G)$ i.e. $F = \{P_i\}_{i=1}^\infty$ for $P_i \subset V(G)$. A multi-step Vizing chain on $e$ is $F$-avoiding if the $i\text{th}$ Vizing chain is not created on a vertex in $\bigcup_{j=1}^{i-1} P_j$.

In Section 6, we use family-avoiding Vizing chains to construct many vertex disjoint augmenting Vizing chains, so that e.g. a distributed algorithm can colour many edges in parallel.

**Local Vizing chains.** For this class of Vizing chains, we want to put some restrictions on the post-shift colourings. For instance, we consider the following restricted class of Vizing chains:

**Definition 12.** We say a (multi-step) Vizing chain is strictly local if the post-shift colouring is strictly local meaning that it satisfies that every coloured edge $e = uv$ has a colour from $L(e) = \{1, 2, \ldots, \max\{d(u), d(v)\} + 1\}$.

We will show how to construct these chains in the next section.

In Section 7, we will relax this notion of locality to construct chains which are $\epsilon$-local. Before defining such chains, we define the set of $\epsilon$-available colours for a vertex $v$ as

$$A_\epsilon(v) = A(v) \cap [(1 + \epsilon)d(v)]$$

Now we can define $\epsilon$-local Vizing chains.

**Definition 13.** We say a (multi-step) Vizing chain is $\epsilon$-local if the following condition holds for all $\kappa$ in both the pre-shift colouring and the post-shift colouring $\epsilon'$ after shifting the entire (multi-step) Vizing chain:

$$\{|e : \hat{c}(e) = \kappa\| \leq \{u : \kappa \in [(1 + \epsilon)d(v)], \kappa \in A_\epsilon(v)\}$$

where $\hat{c} \in \{c, \epsilon'\}$.

### 4 APPLICATION 1: LOCAL VERSION OF VIZING’S THEOREM

We say an edge $e = uv$ is coloured strictly local if it holds that $c(e) \in \{1, 2, 3, \ldots, 1 + \max\{d(u), d(v)\}\}$. A strictly local partial colouring is then a partial edge-colouring, where all coloured edges are coloured strictly locally. The Local Vizing Conjecture says that there exists a proper colouring of any graph where all edges are coloured strictly locally. We say a fan is strictly local if the representative available colour at each point $w_i$ of the fan is in $\{1, 2, 3, \ldots, 1 + d(w_i)\}$. We observe that if the original colouring is strictly local, then so is the colouring obtained from shifting a strictly local fan. Strictly local fans gives us a way of constructing truncated Vizing chains that are strictly local, assuming that the pre-colouring is strictly local: fix a center $u$ and an uncoloured edge $uv$. The standard Vizing chain constructions can be made strictly local by choosing the available colour at a vertex $y$ in $\{1, 2, 3, \ldots, 1 + d(y)\}$ and truncating $P$ at the first problematic edge. We will need the following generic construction of strictly local Vizing chains.

**Construction:** Let $c$ be a strictly local partial colouring of a graph $G$, and let $e = uv \in E(G)$ be any edge left uncoloured by $c$. Then we consider the following strictly local Vizing chain on $e$: we construct a maximal fan $uw_1, \ldots, uw_k$ as follows. Initially, we set $w_1 = v$. Now having picked $w_j$, we pick $w_{j+1}$ or conclude our construction with $k = j$ as follows: Note first that for all vertices $y$ we have that $A(y) \cap [d(y) + 1]$ is non-empty, since at most $d(y)$ colours can be excluded from $A(y)$. We now have three cases:

**Case 1:** If there exists a colour $\kappa \in A(w_j) \cap [d(w_j) + 1]$ that is also available at $u$, pick $\kappa$ as $w_{j+1}$'s representative locally available colour and conclude the construction with $k = j$.

**Case 2:** If there exists a colour $\kappa \in A(w_j) \cap [d(w_j) + 1]$ and an index $i < j$ such that $c(A(w_i)) = \kappa$, we pick $\kappa$ as $w_i$'s representative available colour and conclude the construction with $k = j$.

**Case 3:** Otherwise, we pick an arbitrary colour $\kappa \in A(w_j) \cap [d(w_j) + 1]$ as $w_j$'s representative available colour. Note that $\kappa$ has to be incident to an edge $ux$ coloured $\kappa$ and that $\kappa \notin \{w_i\}_{i=1}^j$, since then we would be in case 1) or case 2). Therefore, we can pick $w_{j+1} = x$.

The above construction gives us a fan $F$ on the edges $uw_1, \ldots, uw_k$, such that $w_k$'s representative available colour is either available at $u$ or is present at $u$ at the edge $uw_i$ for some $i < k$. Note that $\epsilon$-shifting $F$ leaves a colouring that is also strictly local, since $uw_i$ receives a colour in $[d(w_i) + 1]$ by construction. In particular, if we stop in case 1), we have identified a strictly local Vizing chain, and we conclude the construction.

If we stop in case 2), we pick an arbitrary representative available colour $k_1 \in A(u) \cap [d(u) + 1]$ for $u$ and set $k_2$ equal to $w_k$'s representative available colour. Then we consider the $(k_1, k_2)$-bichromatic path $P' = p_1, p_2, \ldots, p_t = w_k$ ending at $w_k$. We choose $P$ to be the union of a fan edge and a subpath of $P'$ as follows: beginning from $w_k$, we traverse $P'$. We choose $P$ by truncating $P'$ at the first edge $xy$, we meet along $P'$ such that shifting the colours of $P'$ would cause the colour of $xy$ to violate strict locality of the post-shift colourings. This concludes the construction in case 2).

Finally, if no such edge exist, we can construct an augmenting strictly local Vizing chain. Currently, we have a maximal strictly local fan consisting of vertices $u, w_1, \ldots, w_k$ and the representative free colour at $w_k$ also the representative free colour at $w_i$ for some $i < k$.

Now let $k_1$ be a free colour at $u$ and $k_2$ the representative free colour at $w_k$ and $w_i$. Note that by construction all of $u, w_k, w_i$ will be endpoints of maximal $(k_1, k_2)$-bichromatic paths. Again we consider the $(k_1, k_2)$-bichromatic path $P' = p_1, p_2, \ldots, p_t = w_k$ ending at $w_k$. If it starts at $w_i$, we consider the chain

$$C = (uw_1, uw_2, \ldots, uw_i) + (uw_i, p_1, p_2, p_4, \ldots, p_{t-1}, w_k)$$

This chain is shiftable, and shifting it leaves $k_2$ as an available colour at both $p_{t-1}$ and $w_k$, and so we can extend the colouring by giving this edge the colour $k_2$. We note the following two things: 1) if the
original colouring is strictly local, then so is the one obtained by shifting the above chain, and 2) by assumption colouring the last edge like this, does not violate strict locality, since otherwise we would have truncated at this edge.

If \( P \) instead starts at \( t \), we instead consider the chain
\[
C = (u_1w_1, u_2w_2, \ldots, u_kw_k) + \left( u_{k-1}w_{k-1}, u_{k-2}w_{k-2}, \ldots, u_1w_1 \right)
\]
This chain is again shiftable, and shifting it again leaves \( k_2 \) as an available colour at both \( p_{k-1} \) and \( w_k \), and so we reach the same conclusion as before. Again we know that shifting the second part of \( C \) does not violate the strict locality of the colouring.

If \( P \) instead begins at any other vertex, we consider the chain
\[
C = (u_1w_1, u_2w_2, \ldots, u_kw_k) + \left( u_{k-1}w_{k-1}, u_{k-2}w_{k-2}, \ldots, u_1w_1 \right)
\]
It is again shiftable, and shifting it leaves either \( k_1 \) or \( k_2 \) as an available colour at both \( p_2 \) and \( p_1 \), and so we can extend the colouring to also colour this edge without violating strict locality.

Note that if we end up with a truncated Vizing chain, we may recursively perform the exact same construction at the truncated edges to construct a multi-step Vizing chain on \( e \). In particular, observe that these Vizing chains might overlap. In this section, we will only consider strictly local (multi-step) Vizing chains constructed as explained above.

Consider a strictly local Vizing chain constructed as explained above, and let \( e' \) be the last edge of \( P \) i.e. the truncated edge. A simple but key observation is that \( e' \) has to go between two vertices of (relative) low degree.

**Observation 14.** Let \( c \) be a strictly local (partial) colouring of a graph \( G \), and let \( P \) be any \((k_1, k_2)\)-bichromatic path. If the shift of \( P \) makes \( e = xy \) violate the strict locality, then \( \max\{d(x), d(y)\} + 1 < \min\{k_1, k_2\} \).

The main lemma of this section will show that if we construct a multi-step Vizing chain by repeatedly applying the construction from above on the truncated edges, then this construction cannot continue indefinitely before it ends up in an augmenting Vizing chain.

To prove this lemma, we consider the following potential function defined for any graph \( G \) and any partial colouring \( c : G \mapsto [\Delta + 1] \cup \{\} \) of \( G \):
\[
\Phi(G, c) = \sum_{\Delta+1}^{\Delta+1} \left| \{v : \kappa \in A(v) \cap [1 + d(v)]\} \right|
\]
\[
\leq n(\Delta + 1)
\]
We will show that each time one shifts one of the strictly local truncated Vizing chains constructed above, the potential of the post-shift colouring will be at least one smaller than that of the pre-shift colouring. Since for any choice of \( G, c \) we have \( \Phi(G, c) \geq 0 \), it follows that there has to exist an augmenting and strictly local multi-step Vizing chain of length at most \( O((\Delta + n) \cdot n(\Delta + 1)) \).

We defer the following two propositions to the full-version. The first shows that shifting a strictly local fan never increases the potential:

**Proposition 15.** Let \( c_1, c_2 \) be two (partial) colourings of the graph \( G \) such that \( c_2 \) is obtained from \( c_1 \) by shifting a strictly local fan. Then \( \Phi(G, c_2) \leq \Phi(G, c_1) \).

The second shows that if \( F + P \) is a strictly local 1-step Vizing chain constructed as above, then the following restrictions on \( P \) holds:

**Proposition 16.** Let \( F + P \) be a strictly local truncated Vizing chain constructed as above. Suppose \( F \) contains the edges \( u_0w_0, \ldots, u_kw_k \) and that the colour chosen as available at \( w_k \) was also chosen as available for \( w_i \). Then the last edge of \( P \) cannot have \( u, w_k \) or \( w_i \) as an endpoint.

We are now ready to prove the main lemma of this section, which says that the potential has to drop, when shifting this particular kind of strictly local truncated Vizing chain:

**Lemma 17.** Let \( F + P \) be a strictly local truncated Vizing chain constructed as before. Then shifting \( F + P \) to go from the pre-colouring \( c_1 \) to the post-colouring \( c_2 \) drops the potential by at least 1 i.e. \( \Phi(G, c_1) \geq \Phi(G, c_2) + 1 \)

**Proof.** Suppose that \( P \) is \((k_1, k_2)\)-bichromatic, and suppose that \( k_1 \) was chosen as available at \( u \). Observe first that the contribution of \( u \) and \( w_k \) will drop by 2 after the shift of \( F + P \). Indeed, after the shift \( u \) will now be incident to an edge coloured \( k_1 \in [d(u) + 1] \), and \( w_k \) will now be incident to an edge coloured \( k_2 \in [d(u) + 1] \). By Proposition 15 and Proposition 16 neither \( u \) nor \( w_k \) lose any colours incident to them, and hence their contribution to the potential has to decrease by at least 2.

Observe secondly that if \( xy \) is the last edge of \( P \), then \( c(xy) = \min\{k_1, k_2\} \) and so by Observation 14 the contribution of \( x \) is the same before and after the shift, and the contribution of \( y \) drops by at most 1.

Finally, note that any interior vertex of \( P \) sees the same colours before and after the shift, and so their contribution to the potential is unchanged. Hence, summing up the changes in contribution yields:
\[
\Phi(G, c_2) \leq \Phi(G, c_1) - 2 + 1 \leq \Phi(G, c_1) - 1
\]

Now we may prove the local version of Vizing’s Theorem:

**Proof of Theorem 2.** Suppose we are given a strictly local partial colouring of \( G \) with a minimum number of uncoloured edges. If no edge is uncoloured, we are done, so we may assume this is not the case and pick an arbitrary uncoloured edge \( e = uv \). Next, we build a strictly local multi-step Vizing chain on \( e \) as described earlier: construct a strictly local maximal fan on \( e \) with edges \( u_1w_1, \ldots, u_kw_k \). If it is augmenting, we are done, so assume it is not. Then we extend by a subpath of a bichromatic path alternating in suitably chosen colours as explained earlier. Finally, we truncate the Vizing chain if we meet an edge that would invalidate strict locality after the shift. If this happens, we repeat the same process to extend to a multi-step Vizing chain of one further step-length. The process is repeated on the post-shift colouring of the multi-step Vizing chain that we wish to extend which is not necessarily identical to the original colouring. By Lemma 17 the potential of the post-shift colouring drops by at least one everytime we perform this construction, and so we create an augmenting strictly local Vizing chain after at most \( O(n\Delta) \) steps. By augmenting this Vizing chain, we construct a larger strictly local (partial) colouring – a contradiction.
Note that this approach is constructive and also gives an algorithm for computing a local Vizing colouring. Observe that increasing the size of the colouring never increases the potential, and so we have to create and shift \(O(n + n\Delta)\) Vizing chains. Each such chain can be shifted in \(O(\Delta + n)\) time, and so we arrive at Corollary 4, which we repeat below for the readers’ convenience:

**Corollary 18 (Corollary 4).** There is an algorithm that runs in \(O(n^2 \Delta)\)-time for computing an edge-colouring such that every edge receives a colour from the list \(L(u) = \{1 + \max\{d(u), d(v)\}\}\).

5 APPLICATION 2: SMALL AUGMENTING SUBGRAPHS

The basic observation underlying our argument is the following: the vertices in \(H\) can be shifted in \(\Delta\) time, and so we arrive at Corollary 4, which we repeat below for the readers’ convenience:

\[\begin{align*}
&\text{Let } G \subset G \text{ be a subgraph of } G \text{ and let } c, c' \text{ be two (partial) } (\Delta + 1)\text{-edge-colouring of } G \text{ differing only on } H. \text{ Then, under both } c \text{ and } c', \text{ the same at most } 2(\Delta + 1)^3 \cdot |H| \text{ points in } G - N^1(H) \text{ can reach } H \text{ via a maximal bichromatic path located at a neighbour } y. \text{ Here } N^1(H) \text{ denotes the one-hop neighbourhood of } H \text{ i.e. the vertices in } G \text{ that have distance at most one from } H \text{ in } G.
\end{align*}\]

In the next section, we will consider many different \((i + 1)\)-step Vizing chains at once, and we would like to use Lemma 23 to argue that not too many of these multi-step Vizing chains can be overlapping at the same time. In order to do so, we follow a two-step approach. First we note that when we extend an \(i\)-step Vizing chain via Lemma 10 very few extension points will extend via a Vizing chain that intersects the \(i\)th Vizing chain in a vertex in the fan or in an edge of the bichromatic path. In particular, if the \((i + 1)\)th Vizing chain intersects the \(i\)th Vizing chain, then it intersects \(P_i\) in a vertex and not an edge. This is due to the way the colours of \(P_{i+1}\) is chosen in Lemma 10.

Hence, we can limit ourselves to consider extensions through points where the \((i + 1)\)th Vizing chain does not overlap with the \(i\)th Vizing chain in any edges (but possibly in some vertices). The challenge is to argue that if we consider many such extension points, not too many of them can intersect the first \((i + 1)\) steps of the chains used to reach them. In order to argue this, we would like to be able to apply Lemma 23 with \(H\) equal to the union of the first \((i + 1)\) steps of all of the multi-step Vizing chains, we consider. The idea is that any extension that intersects \(H\), has to be constructed on a point that has a neighbour \(y\) that can reach \(H\) via a maximal bichromatic path under both the \(i\)-shifted colouring \(c'\) used to reach the extension point and the original colouring \(c\). Note here that \(c'\) will differ for different extension points. This would then allow us to only reason about the original colouring \(c\) as we do in the proof of Lemma 23. However, the colourings we are considering are actually the ones obtained by shifting the first \(i\) steps of the multi-step Vizing chain, and so each of these colourings might differ from \(c\) not only on \(H\), but also along the edges in the \(i\)th step. Since we ensure that we only consider extensions that are edge-disjoint from the \(i\)th step, any Vizing chain extension that intersects \(H\) i.e. the first \((i + 1)\) steps of the chain, has to have a neighbour \(y\) that is an endpoint of a maximal bichromatic path that reaches \(H\) in the original colouring \(c\). Hence, we can apply the bound from Lemma 23 to count the number of Vizing chain extensions that might intersect the first \((i + 1)\) steps. From now on, we will be slightly imprecise and do this by saying that we apply Lemma 23, even though we technically only apply it implicitly via the above discussion.

5.1 Controlling Non-overlapping Vizing Chains

We will use the following simple upper bound on the density of a graph with maximum degree \(\Delta\):

**Observation 21.** Let \(G\) be a graph with maximum degree \(\Delta\), and let \(H\) be any subgraph of \(G\). Then \(\frac{|\Gamma(H)|}{|\Gamma(H)|} \leq \frac{\Delta}{3}\).

In order to control the expansion of \(R_i\), we will use the following lemma which shows that one cannot pack many Vizing chains into a subgraph containing few edges. We defer the proof to the full version [11].

**Lemma 22.** Let \(S\) be a set of non-overlapping Vizing chains in a graph \(G\) of maximum degree \(\Delta\) such that each Vizing chain is constructed on a different vertex. Then any coloured edge in \(G\) belongs to at most \(4\Delta^2\) of the Vizing chains.

Later we will also need to control the number of Vizing chain that intersects a vertex in a subgraph \(H\). In order to not have to worry about exactly which fans – and hence Vizing chains – that are constructible, we will upper bound the number of points that have a neighbour that is the endpoint of a maximal bichromatic path that reaches \(H\) instead. This upper bound naturally carries over to also bound the number of vertices in \(G - N^1(H)\) that have a Vizing chain reaching \(H\). An argument with a similar flavour to the one above shows that:

**Lemma 23.** Let \(H \subset G\) be a subgraph of \(G\) and let \(c, c'\) be two (partial) \((\Delta + 1)\)-edge-colouring of \(G\) differing only on \(H\). Then, under both \(c\) and \(c'\), the same at most \(2(\Delta + 1)^3 \cdot |H|\) points in \(G - N^1(H)\) can reach \(H\) via a maximal bichromatic path located at a neighbour \(y\). Here \(N^1(H)\) denotes the one-hop neighbourhood of \(H\) i.e. the vertices in \(G\) that have distance at most one from \(H\) in \(G\).

5.2 Existence of a Small Augmenting Subgraph

We are now ready to show the following main lemma, which we will use to prove Theorem 3 by induction.
Lemma 24. Suppose we are given a proper partial colouring and an uncoloured edge \(e\) in a graph of maximum degree \(\Delta\), and let \(\ell \geq 80(\Delta + 1)^7\). Suppose furthermore that \(\bigcup_{i} N_{i}(e, \ell)\) contains no vertices contained in an augmenting \(i\)-step Vizing chain with step length \(\ell\), and that for all \(1 \leq j \leq i - 1\) the following holds

\[|N_{j}(e, \ell)| \geq \frac{\ell}{8\Delta} \sum_{k=1}^{j-1} |N_{k}(e, \ell)|\]

(1) \(|N_{j}(e, \ell)| \geq |R_{j}(e, \ell)|/2\).

Then condition 1) and 2) also hold for \(j = i\).

We first show that if the conditions of Lemma 24 are satisfied then \(|R_{i}(e, \ell)|\) is big.

Lemma 25. Suppose the conditions of Lemma 24 hold. Then

\[|R_{i}| \geq \left(\frac{\ell}{2\Delta^3} - 1\right)|N_{i-1}| \geq \frac{\ell}{4\Delta^3} \sum_{k=1}^{i-1} |N_{k}|\]

Proof. By assumption, we may extend exactly one \((i - 1)\)-step Vizing chain through every vertex in \(N_{i-1}\) to get non-overlapping \((i - 1)\)-step Vizing chains, where the length of the \(i\)th Vizing chain is at least \(\ell\). Consider the subgraph \(H\) spanned by the \(\ell\) first edges of each bichromatic path in the \(i\)th Vizing chain concatenated at the vertices in \(N_{i-1}\). By definition of \(R_{i}\), we have

\[|V(H)| \leq |N_{i-1}| + |R_{i}|\]

Furthermore, since we only construct one Vizing chain from every point in \(N_{i-1}\), an edge in \(H\) belongs to at most \(4\Delta^2\) different Vizing chains by Lemma 22. Since we have assumed that each Vizing chain has length at least \(\ell\), we know that each bichromatic path contributes \(\ell\) edges to \(H\), and therefore we find:

\[4\Delta^2|E(H)| \geq \ell \cdot |N_{i-1}|\]

Hence it follows by Observation 21 that

\[\frac{|E(H)|}{|V(H)|} \leq \frac{\Delta}{2}\]

By rearranging, we conclude that \(|R_{i}| \geq \left(\frac{\ell}{2\Delta^3} - 1\right)|N_{i-1}|\). By assumption, we have \(|N_{i-1}| \geq \frac{\ell}{8\Delta} \sum_{k=1}^{i-1} |N_{k}|\), so since \(\ell \geq 80\Delta^3\) is large enough we find \(|N_{i-1}| \geq 10 \sum_{k=1}^{i-1} |N_{k}|\). Hence \(\sum_{k=1}^{i-1} |N_{k}| \leq \frac{|N_{i-1}|}{10} + |N_{i-1}|\) and it follows that \(10 \sum_{k=1}^{i-1} |N_{k}| \leq |N_{i-1}|\). Finally, we find

\[|R_{i}| \geq 10\left(\frac{\ell}{2\Delta^3} - 1\right) \sum_{k=1}^{i-1} |N_{k}| \geq \frac{\ell}{4\Delta^3} \sum_{k=1}^{i-1} |N_{k}|\]

Next we conclude that this implies that \(|N_{i}|\) also is big. We will only provide a proof-sketch, and leave the formal proof for the full-version.

Lemma 26. Let the assumptions be as in Lemma 24. Then at most

\[3(\Delta + 1)^4 \sum_{k=0}^{i-2} |R_{k}| + (3(\Delta + 1)^4 + 2\Delta) |N_{i-1}|\]

points in \(R_{i}\) has no non-overlapping Vizing chain extension.

Proof-sketch. Observe first that if we can guarantee that the Vizing chain beginning at a point in \(R_{i}\) will not intersect a vertex in the one-hop neighbourhood of \(\bigcup_{k=0}^{i-1} R_{k}\), then surely it is non-overlapping. We will show something a little weaker: namely that we can pick the Vizing chains so that most will be vertex disjoint from the one-hop neighbourhood of \(\bigcup_{k=0}^{i-1} R_{k}\), and furthermore they will not overlap the last Vizing chain, used to reach them, in any edges. Finally, we may simply remove the 1-hop neighbourhood of vertices in \(N_{i-1}\), since we do not wish to truncate our Vizing chains at these points, as we cannot guarantee that the fans constructed at these points are non-overlapping.

We can count the number of vertices that will extend their Vizing chain so that it overlaps the last Vizing chain used to reach it. This can only happen in two ways: either the Vizing chain goes through the fan situated at a vertex in \(N_{i-1}\) or it attempts to pick exactly the same augmenting path used by the last Vizing chain to reach it. There are at most \(3(\Delta + 1)^2 |N_{i-1}|\) chains of the first kind, and at most \(2\Delta |N_{i-1}|\) of the second kind.

Note secondly that at most

\[2(\Delta + 1)^4 \sum_{k=0}^{i-2} |R_{k}| + (\Delta + 1)^2 \sum_{k=0}^{i-2} |R_{k}| \leq 3(\Delta + 1)^4 \sum_{k=0}^{i-2} |R_{k}|\]

Vizing chain extensions of \(R_{i}\) can be overlapping through the one-hop neighbourhood of \(\bigcup_{k=0}^{i-1} R_{k}\). Indeed, the one-hop neighbourhood of \(R_{i}\) has size at most \((\Delta + 1) \sum_{k=0}^{i-2} |R_{k}|\), so at most \((\Delta + 1)^2 \sum_{k=0}^{i-2} |R_{k}|\) vertices either belong to the one-hop neighbourhood or can reach the one-hop neighbourhood via fans, and at most \(2(\Delta + 1)^4 \sum_{k=0}^{i-2} |R_{k}|\) vertices can reach the one-hop neighbourhood via the bichromatic path components of the Vizing chains by Lemma 23 and the proceeding discussion. Summing these contributions yields the above. Hence, in total at most

\[3(\Delta + 1)^4 \sum_{k=0}^{i-2} |R_{k}| + (3(\Delta + 1)^4 + 2\Delta) |N_{i-1}|\]

vertices of \(R_{i}\) does not belong to \(N_{i}\). □

Now we may deduce Lemma 24. We only provide a proof-sketch below and defer a formal proof to the full version [11].

Proof-sketch. By Lemma 26, we have that

\[|N_{i}| \geq |R_{i}| - 3(\Delta + 1)^4 \sum_{k=0}^{i-2} |R_{k}| + (3(\Delta + 1)^4 + 2\Delta) |N_{i-1}|\]

We will show that \(|N_{i}| \geq |R_{i}|/2\), and then the lemma will follow by Lemma 25. Observe that by assumptions 1) and 2) we have

\[3(\Delta + 1)^4 \sum_{k=0}^{i-2} |R_{k}| + (3(\Delta + 1)^4 + 2\Delta) |N_{i-1}| \leq \left(\frac{48(\Delta + 1)^7}{\ell} + 5(\Delta + 1)^4\right) |N_{i-1}|\]
Since $\ell \geq 80(\Delta + 1)^2$, we find that
\[
\left( 48(\Delta + 1)^2 + 80(\Delta + 1)^4 \right) |N_{i-1}| \leq 6(\Delta + 1)^4 |N_{i-1}|
\]
and by Lemma 25 that
\[
|R_i| \geq \left( \frac{\ell}{2\Delta^3} - 1 \right) |N_{i-1}|
\]
\[
\geq 2 \cdot \left( 3(\Delta + 1)^4 \sum_{k=0}^{i-2} |R_k| + (3\Delta + 1)^4 + 2\Delta |N_{i-1}| \right)
\]
and so we conclude the first statement by Lemma 26 and the second statement by Lemma 25.

We may now deduce Theorem 3:

**Proof of Theorem 3.** We first construct an augmenting Vizing chain on $e$ as in the proof of Vizing's Theorem. If it has length $\ell$, we are done, so we may assume this is not the case. Now we may choose to truncate the Vizing chain at any of the $\ell$ first vertices, and extend it via Lemma 10 and the discussion following the lemma. At most $2\Delta$ of these chains will overlap the edges of the path of the first Vizing chain—namely if they use the exact same bichromatic path. Furthermore, since at most $(\Delta + 1)^2$ augmenting paths can go through a vertex, at most $2(\Delta + 1)^4$ of the Vizing chains will go through the fan satiated at $e$, and so we find that $|R_i(e, \ell)| = \ell$ and that $|N(e, \ell)| \geq \ell - 2\Delta - 2(\Delta + 1)^4 \geq \ell/2$. Hence, we may apply Lemma 24 until we find a short augmenting and non-overlapping Vizing chain. If this happens at step $i$, then by construction the Vizing chain has size at most $O(\Delta^2 \cdot i)$.

**Claim 1.** For all $j$ satisfying the conditions in Lemma 24, we have $|N_j| \geq \left( \frac{\ell}{2\Delta^3} \right)^j$.

Proof. The proof is by induction on $j$. We have just handled the base case $j = 1$ above. The induction step now follows from condition 1) of Lemma 24.

and so we conclude that $i \leq \log \frac{n}{\Delta^4}$ and the theorem follows.

**6 APPLICATION 3: DISTRIBUTED $(\Delta + 1)$-EDGE-COLOURING**

We say a family $F = \{F_i\}_{i=1}^\infty$ is $k$-bounded, if for all $i$ we have $|F_i| \leq k^i$. For a $k$-bounded family with $k \geq 2$, it holds that $\left| \bigcup_{j=1}^i F_j \right| \leq 2k^i$.

Indeed, in the worst case the sets in $F$ are pairwise disjoint and hence
\[
\left| \bigcup_{j=1}^i F_j \right| \leq k^i \sum_{j=1}^{\infty} k^{-j} = \frac{k}{k-1} k^i \leq 2k^i
\]

We will show that we can construct small augmenting non-overlapping Vizing chains that furthermore also avoid any $4(\Delta + 1)^4$-bounded family $F$. The idea is simply to remove the option to extend the Vizing chains through points forbidden by $F$. The number of new starting points for Vizing chains will still grow so fast that the number of points to avoid becomes negligible. Hence, we now fix any $4(\Delta + 1)^4$-bounded family $F$—even one chosen adversarial—for the remainder of this section.

In the full-version we show how to extend Lemma 24 to show that, in fact, must exist a small augmenting $F$-avoiding Vizing chain for any uncouloured edge $e$. We may now use this to prove Theorem 5. We will outline the argument below in a proof-sketch and defer a formal proof to the full-version.

**Theorem 27.** For any partial colouring $c$ of a graph $G$, let $U$ be the set of edges left uncouloured by $c$. Then there exists a set $W \subset U$ of size $\Omega\left( \frac{|U|}{\text{poly}(\Delta \log n)} \right)$ of edges for which there exist vertex disjoint augmenting Vizing chains, each of size at most $O(\Delta^2 \log n)$.

**Proof-sketch.** Suppose we have a set $W \subset U$ of size $k$ such that there exist pairwise disjoint augmenting Vizing chains of size at most $(\Delta + 1) \cdot \ell \cdot \log n$ for each edge in $W$. Fix such a set of $k$ pairwise disjoint Vizing chains and let $G' \subseteq G$ be the subgraph containing exactly these Vizing chains.

We show that if $k < \frac{|U|}{\text{poly}(\Delta \log n)}$, then we may extend $W$ to contain $k + 1$ edges, while satisfying the same conditions.

Before we construct these families, we first consider any edge $e \in W$. If any edge in $G'$ has an endpoint in the 1-hop neighbourhood of $e$, then we will completely disregard $e$. Otherwise, we consider the points that are 1) not in the 1-hop neighbourhood in $G$ of any of the augmenting Vizing chains in $G'$ and 2) can reach a vertex in $N^4(G')$ via a Vizing chain. By Lemma 23 there are at most $2(\Delta + 1)^2 \cdot (\Delta + 1)|V(G'')| + (\Delta + 1)^2|V(G'')| \leq (3(\Delta + 1)^4)^2|V(G'')|$ such points. A similar argument may be applied to the points reaching these point via 1-step Vizing chains, which shows that at most $(3(\Delta + 1)^4)^2|V(G'')|$ points can reach the 1-hop neighbourhood of $G'$ via non-overlapping 2-step Vizing chains. By applying this argument inductively, we see that at most $(3(\Delta + 1)^4)^2|V(G'')|$ points can reach them via non-overlapping i-step Vizing chains. For any $i$, we let $P_{i-1}$ be the multi-set of points which can reach the 1-hop neighbourhood through non-overlapping i-step Vizing chains. We add a point once for each i-step Vizing chain, beginning at the point, that reaches $G''$ through edges in $G' - G'$. Then by above
\[
|P_{i-1}| \leq 3(\Delta + 1)^4 (3(\Delta + 1)^4)^{i-1} \cdot k \cdot s
\]

where $s = (\Delta + 1) \cdot \ell \cdot \log n$.

We construct a family $F(e)$ for $e$ to avoid as follows: if a non-overlapping j-step Vizing chain constructed on $e$ reaches a vertex in the 1-hop neighbourhood of $G''$ in $G$, then add the last point we extended through before reaching $w$ to $F_{j-1}(e)$. We say that the family for $e$ is *unavoidable* if it is not $4(\Delta + 1)^4$-bounded.

A vertex $w$ has to be represented more than $(4(\Delta + 1)^4)^t$ times in $P_i$ for some $i$ for the family of that vertex to be unavoidable. Hence, the maximum number of unavoidable families that we construct is upper bounded by:

\[
(\Delta + 1)k \cdot s + \sum_{i=1}^{\log \frac{n}{\text{poly}(\Delta \log n)}} \frac{|P_i|}{(4(\Delta + 1)^4)^i} \leq 3(\Delta + 1)^4 \cdot k \cdot s \sum_{i=1}^{\log \frac{n}{\text{poly}(\Delta \log n)}} \left( \frac{3}{4} \right)^i \leq 12(\Delta + 1)^4 \cdot k \cdot s
\]

hence if $k < \frac{|U|}{\text{poly}(\Delta \log n)}$, then there exists an edge $e \in U$ such that $e$ has an augmenting non-overlapping Vizing chain of size $s$ that is completely disjoint from $G'$, and so we may extend $W$ by adding this edge.
In particular, this theorem now allows us to prove Theorem 5. The approach from here on is completely symmetrical to that of Bernshteyn [5], so we leave these details for the full version [11].

7 APPLICATION 4: DYNAMIC

(1 + ε)Δ-EDGE-COLOURING

We refer the reader to Section 1.2 for a high-level overview the following approach. We first consider local palettes.

7.1 Local Palettes

We will will attempt to colour the uncoloured edge by restricting our- selves to \(G[S]\), and then use the algorithm of Bernshteyn to attempt to construct an augmenting, \(\epsilon\)-local and non-overlapping Vizing chain. To make the Vizing chains \(\epsilon\)-local, we choose only \(\epsilon\)-available colours as representative available colours, unless Lemma 10 case 2) forces us to pick a colour \(x_i\) that is not \(\epsilon\)-available at the center of the fan. However, as we show in the full version [11], the constructed Vizing chain is still \(\epsilon\)-local. It is straightforward to adapt the proofs of Bernshteyn to accommodate these changes, so we will not elaborate further on this. If the Vizing chain construction fails, we will resample \(S\) and try again. See Algorithm 1 for pseudo-code. Thus we have the following:

**Algorithm 1** An algorithm for colouring uncoloured edges.

- **function** `colour(e \leftarrow uo)`
  - Sample a local palette \(S\).
  - Construct an \(\epsilon\)-local Vizing chain \(C_1\) on \(uo\) in \(G[S]\).
  - \(i = 1\)
  - **while** `length(C_i) > \ell` **do**
    - Pick an edge \(e'\) u.a.r. from first \(\ell\) edges on \(C_i\)
    - \(i = i + 1\)
    - Truncate Vizing chain at \(e'\) and extend it via the \(\epsilon\)-local version of Lemma 10.
  - **if** Multi-step Vizing chain is overlapping or \(i > T\) **then**
    - `colour(uo)`
  - **Stop**
  - Augment \(F_1 + C_1 + \cdots + F_i + C_i\)

**Corollary 31.** If \(T = 2 \log n\) and \(\ell = 1 + 18(\Delta + 1)^6 \log n\), then Algorithm 1 is called recursively at most \((c + 1)\log n\) times with probability \(1 - n^{-c}\).

Note that the following lemma then is immediate from Corollary 31 and Lemma 29 through the use of basic data structures.

**Lemma 32.** Using `colour(uo)` an edge can be inserted in time \(O(e^{-6} \log \Delta \log n)\) with high probability.

7.2 Colouring Protocol

In this section, we briefly discuss how to replace the construction of non-overlapping augmenting Vizing chains applied by Duan, He and Zhang [15] with the construction of Bernshteyn [5] in order to get rid of the assumption that \(\Delta = \Omega(\frac{\log^2 n}{\epsilon})\).

Consider the following multi-step Vizing algorithm due to Bernshteyn: fix an uncoloured edge \(e\). Now construct an augmenting Vizing chain on \(e\) using the standard approach from Vizing’s Theorem. If the Vizing chain has length \(\ell = \Omega(\Delta^4 \log n)\), pick an edge uniformly at random among the first \(\ell\) edges of the bichromatic path and truncate the Vizing chain here. Now recursively construct a multi-step Vizing chain by applying Lemma 10 and the discussion immediately following the lemma. If the concatenated Vizing chain is both short and augmenting, we stop and augment it. Otherwise if it becomes overlapping, we stop. If the Vizing chain is still not short and augmenting after \(\Omega(\log n)\) steps, we also stop. Bernshteyn showed the following lemma:

**Lemma 30 (Bernshteyn, Lemma 6.1 p. 346 in [5]).** For any \(T \in \mathbb{N}\) and any \(\ell \in \mathbb{N}\) such that \(\lambda = \frac{\ell}{(T + 1)^{1/3}} > 1\), then the multi-step Vizing chain algorithm described above terminates with an augmenting, non-overlapping i-step Vizing chain for some \(i < T\) with probability at least

\[
1 - \frac{n}{\lambda^T} - \frac{37(\Delta + 1)^3}{\lambda - 1}
\]

We will first use Lemma 29 to sample an \(\epsilon\)-local palette \(S\). Then we will attempt to colour the uncoloured edge by restricting ourselves to \(G[S]\), and then use the algorithm of Bernshteyn to attempt to construct an augmenting, \(\epsilon\)-local and non-overlapping Vizing chain. To make the Vizing chains \(\epsilon\)-local, we choose only \(\epsilon\)-available colours as representative available colours, unless Lemma 10 case 2) forces us to pick a colour \(x_i\) that is not \(\epsilon\)-available at the center of the fan. However, as we show in the full version [11], the constructed Vizing chain is still \(\epsilon\)-local. It is straightforward to adapt the proofs of Bernshteyn to accommodate these changes, so we will not elaborate further on this. If the Vizing chain construction fails, we will resample \(S\) and try again. See Algorithm 1 for pseudo-code. Thus we have the following:

**Algorithm 1** An algorithm for colouring uncoloured edges.

- **function** `colour(e \leftarrow uo)`
  - Sample a local palette \(S\).
  - Construct an \(\epsilon\)-local Vizing chain \(C_1\) on \(uo\) in \(G[S]\).
  - \(i = 1\)
  - **while** `length(C_i) > \ell` **do**
    - Pick an edge \(e'\) u.a.r. from first \(\ell\) edges on \(C_i\)
    - \(i = i + 1\)
    - Truncate Vizing chain at \(e'\) and extend it via the \(\epsilon\)-local version of Lemma 10.
  - **if** Multi-step Vizing chain is overlapping or \(i > T\) **then**
    - `colour(uo)`
  - **Stop**
  - Augment \(F_1 + C_1 + \cdots + F_i + C_i\)

**Corollary 31.** If \(T = 2 \log n\) and \(\ell = 1 + 18(\Delta + 1)^6 \log n\), then Algorithm 1 is called recursively at most \((c + 1)\log n\) times with probability \(1 - n^{-c}\).

Note that the following lemma then is immediate from Corollary 31 and Lemma 29 through the use of basic data structures.

**Lemma 32.** Using `colour(uo)` an edge can be inserted in time \(O(e^{-6} \log \Delta \log n)\) with high probability.

7.3 Reducing Deletions to Insertions

We first note that the constructed Vizing chains are able to maintain the following invariant when they are shifted.

**Invariant 33.** For all \(\kappa\) we have \(|\{e : c(e) = \kappa\}| \leq |\{v : \kappa \in [(1 + \epsilon)d(v)], \kappa \in \mathbb{A}_v(v)\}|\).

This means that they are in fact \(\epsilon\)-local.

**Lemma 34.** Suppose that Invariant 33 holds for all \(\kappa\). Then it also holds after shifting a multi-step Vizing chain constructed as above.

Lemma 34 says that if we make sure to recolour edges in such a way that we adhere to Invariant 33 for all \(\kappa\) during deletions, then we can also colour inserted edges in such a way that we adhere to Invariant 33 for all \(\kappa\). There are only two things to consider, when maintaining the invariant under deletions. First of all, when deleting an edge \(xy\) coloured \(\kappa\), one of the following things happen: neither \(x\) nor \(y\) contribute to \(|\{v : \kappa \in [(1 + \epsilon)d(v)], \kappa \in \mathbb{A}_v(v)\}|\), exactly one of \(x\) and \(y\) contribute to \(|\{v : \kappa \in [(1 + \epsilon)d(v)], \kappa \notin \mathbb{A}_v(v)\}|\) or both \(x\) and \(y\) contribute to \(|\{v : \kappa \in [(1 + \epsilon)d(v)], \kappa \notin \mathbb{A}_v(v)\}|\). If neither \(x\) nor \(y\) or exactly one of them contribute to \(|\{v : \kappa \in [(1 + \epsilon)d(v)], \kappa \notin \mathbb{A}_v(v)\}|\), then all is fine. A problem only arises if both \(x\) and \(y\) contribute to \(|\{v : \kappa \in [(1 + \epsilon)d(v)], \kappa \notin \mathbb{A}_v(v)\}|\)
– in this case we will say the edge is a $\kappa$-heavy edge. Then if the invariant held with equality beforehand, one would now possibly have to recolour an edge coloured $\kappa$ where neither of the endpoints contribute to $\{v \in E : d(v) \leq \kappa, v \notin A_e(v)\}$ in order to restore Invariant 33. Call such an edge $\kappa$-light. Since the inequality held with equality before the deletion, it then follows that such a $\kappa$-light edge has to exist, and so we can just uncolour it in order to restore the invariant. Hence, we have reduced the problem to that of colouring this edge, and we already now how to do so while maintaining the invariant. Second of all the degree of colouring this edge, and we already now how to do so while maintaining the invariant. Therefore, we can now recolour any such edges. In order to be able to locate $\kappa$-light edges efficiently, we will with $O(\kappa)$ overhead make sure that we update a doubly-linked list $Q_\kappa$ for each colour $\kappa$ containing all $\kappa$-light edges. We have essentially shown the following lemma, and hence Theorem 6.

Lemma 35. Delete runs in $O(\varepsilon^{-6} \log^6 \Delta \log n)$ time, and if Invariant 33 holds before Delete is run, then it also holds after.

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