SEMI-TERMINAL MODIFICATIONS OF DEMI-NORMAL PAIRS

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Abstract. For a quasi-projective demi-normal pair $(X, \Delta)$, we prove that there exists a semi-canonical modification and a semi-terminal modification of $(X, \Delta)$.

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1. Introduction

It is a very classical result that for a normal algebraic surface $X$ there exists a minimal resolution $\pi: Y \to X$. The morphism $\pi$ is a projective and birational morphism such that $Y$ is smooth (i.e., $Y$ has terminal singularities) and there is no $(-1)$-curve over $X$ (i.e., $K_Y$ is nef over $X$). This result is generalized in [BCH10]; for any normal variety $X$ there exists a terminal modification $\pi: Y \to X$. More precisely, the morphism $\pi$ is a projective and birational morphism such that $Y$ has terminal singularities and $K_Y$ is nef over $X$. On the other hand, in [KSB88, §4], it is known that for any demi-normal (see Definition

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surface there exists a minimal-semi-resolution \( \pi: Y \rightarrow X \). The morphism \( \pi \) is a projective and birational morphism such that \( Y \) is semi-smooth (see Definition 2.1), \( \pi \) is an isomorphism outside a finite set over \( X \) and is an isomorphism at any generic point of the double curve \( D_Y \) of \( Y \), and there is no \((-1)\)-curve on the normalization of \( Y \) over \( X \) (this implies that \( K_Y \) is nef over \( X \)).

In this article, we consider the reducible version of terminal modifcations of normal varieties. In other words, we consider the higher-dimensional version of minimal-semi-resolutions of demi-normal surfaces. We introduce the notion of semi-terminal. This notion is a direct generalization of semi-smooth surface singularities (see Definition 2.3 (2) and Example 2.5 (3)). The following is the main result of this article (for the definitions of semi-canonical modification and semi-terminal modification, see Definition 2.6).

**Theorem 1.1** (Main Theorem). Let \((X, \Delta)\) be a demi-normal pair. (We do not assume that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier.)

1. There exists a semi-canonical modification \( f^{sc}: X^{sc} \rightarrow X \) of \((X, \Delta)\) and is unique.
2. If \( X \) is quasi-projective, then there exists a semi-terminal modification \( f^{st}: X^{st} \rightarrow X \) of \((X, \Delta)\). Moreover, the morphism \( f^{st} \) may be chosen to be projective.

**Remark 1.2.** If \((X, \Delta)\) is a normal pair, then Theorem 1.1 is obtained by [BCHM10]. If \( X \) is a demi-normal surface and \( \Delta = 0 \), then Theorem 1.1 is obtained by [KSB88, §4].

**Remark 1.3.** There are many semi-terminal modifications for a given non-normal demi-normal pair \((X, \Delta)\). For example, let \( X := (x_1x_2 = 0) \subset \mathbb{A}^3 \) and let \( \pi: \tilde{X} \rightarrow X \) be the blowing up at the origin. Then the pairs \((X, 0)\) and \((\tilde{X}, 0)\) are semi-terminal and \( K_{\tilde{X}} \) is nef over \( X \). Hence both the identity morphism and \( \pi \) are semi-terminal modiﬁcations of \((X, 0)\). Therefore, for a demi-normal surface \( X \), the notion of semi-terminal modiﬁcation of \((X, 0)\) is much weaker than the notion of minimal-semi-resolution of \( X \).

Now we organize the strategy of the proof of Theorem 1.1. For Theorem 1.1 (1), the argument is essentially same as that of [OX12], taking the normalization, taking the canonical modiﬁcation and gluing along the conductor divisors. For a demi-normal pair \((X, \Delta)\), the authors of [OX12] remark that if \( K_X + \Delta \) is not \( \mathbb{Q} \)-Cartier then the semi-log-canonical modiﬁcation of \((X, \Delta)\) does not exist in general (see [Kol13, Example 1.40] and [OX12, Example 3.1]). However, we remark that Theorem 1.1 (1) says that for every demi-normal pair \((X, \Delta)\) (without
the assumption $K_X + \Delta$ is $\mathbb{Q}$-Cartier), there exists the semi-canonical modification of $(X, \Delta)$.

The strategy of the proof of Theorem 1.1 (2) is the following. First, for a given demi-normal pair $(X, \Delta)$, we take a semi-log-resolution $Y \to X$. Then we run a MMP with scaling over $X$ for reducible varieties. It is known that the Contraction theorem for semi-log-canonical pairs was established in [Fuj12]. However, as in [Fuj12, Example 5.4], no possible minimal model program for reducible varieties has been known in general (at least in absolute setting). Our strategy is taking the semi-canonical modification instead of taking the flip for the contraction morphism. Then the program can be run in this case. Finally, we decompose each step of the program and show that the program terminates (cf. [Fuj07, §4.2]). This is the strategy of the proof of Theorem 1.1 (2).

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Throughout this paper, we will work over the complex number field $\mathbb{C}$. In this paper, a variety means a reduced algebraic (separated and of finite type) scheme over $\mathbb{C}$. For a morphism $f: Y \to X$ between equidimensional varieties, the morphism $f$ is said to be an isomorphism in codimension 1 over $X$ if there exists an open subscheme $U \subset X$ such that $\text{codim}_X(X \setminus U) \geq 2$ and $f: f^{-1}(U) \to U$ is an isomorphism. For a variety $X$, the normalization of $X$ is denoted by $\nu_X: \bar{X} \to X$. For the theory of minimal model program (MMP, for short), we refer the readers to [KM98] and [Kol13].

2. Preliminaries

In this section, we collect some basic definitions and results.

Definition 2.1. (1) Let $X$ be a variety and let $x \in X$ be a closed point. We say that $x \in X$ is a double normal crossing (dnc, for short) point if $\mathcal{O}_{X,x} \simeq \mathbb{C}[[x_0, \ldots, x_n]]/(x_0 x_1)$; a pinch point if $\mathcal{O}_{X,x} \simeq \mathbb{C}[[x_0, \ldots, x_n]]/(x_0^2 - x_1^2 x_2)$, respectively.
A variety $X$ is said to be a \textit{double normal crossing} variety (dnc variety, for short) if any closed point $x \in X$ is either smooth or dnc point; a \textit{semi-smooth} variety if any closed point $x \in X$ is one of smooth, dnc or pinch point, respectively.

**Definition 2.2** ([Kol13 §5.1]).

1. Let $X$ be an equidimensional variety. We say that $X$ is a \textit{demi-normal} variety if $X$ satisfies Serre’s $S_2$ condition and is dnc outside codimension 2.

2. For an equidimensional variety $X$, if $X$ is dnc outside codimension 2, then there exists a unique finite and birational morphism $d : X^d \to X$ such that $X^d$ is a demi-normal variety and $d$ is an isomorphism in codimension 1 over $X$. We call $d$ the \textit{demi-normalization} of $X$.

3. Let $X$ be a demi-normal variety and let $\nu_X : \bar{X} \to X$ be its normalization. Then the \textit{conductor ideal} of $X$ is defined by $\text{cond}_X := \text{Hom}_{\mathcal{O}_X}((\nu_X)_* \mathcal{O}_{\bar{X}}, \mathcal{O}_X) \subset \mathcal{O}_X$. This can be seen as an ideal sheaf $\text{cond}_X$ on $\bar{X}$. Let $D_X := \text{Spec}(\mathcal{O}_{\bar{X}}/\text{cond}_X)$ and $D_{\bar{X}} := \text{Spec}_{\bar{X}}(\mathcal{O}_{\bar{X}}/\text{cond}_{\bar{X}})$. We call $D_X$ (resp. $D_{\bar{X}}$) as the \textit{conductor divisor} of $X$ (resp. of $\bar{X}/X$). It is known that both $D_X$ and $D_{\bar{X}}$ are reduced and of pure codimension 1. Moreover, for the normalization $\nu_{D_{\bar{X}}} : \bar{D}_{\bar{X}} \to D_{\bar{X}}$, we get the Galois involution $\iota_X : \bar{D}_{\bar{X}} \to \bar{D}_{\bar{X}}$ defined by $\nu_{X}$. 

**Definition 2.3.**

1. The pair $(X, \Delta)$ is said to be a \textit{demi-normal pair} if $X$ is a demi-normal variety and $\Delta$ is a formal $\mathbb{Q}$-linear combination $\Delta = \sum_{i=1}^{k} a_i \Delta_i$ of irreducible and reduced closed subvarieties $\Delta_i$ of codimension 1 such that $\Delta_i \not\subseteq \text{Supp} D_X$ and $a_i \in [0, 1] \cap \mathbb{Q}$ for any $1 \leq i \leq k$. Furthermore, if $X$ is a normal variety, then the pair $(X, \Delta)$ is said to be a \textit{normal pair}.

2. Let $(X, \Delta)$ be a demi-normal pair and let $\nu_X : \bar{X} \to X$ be the normalization of $X$. Set $\Delta_{\bar{X}} := (\nu_X)_*^{-1} \Delta$.

   (i) [KSB88, Definition 4.17] The pair $(X, \Delta)$ is \textit{semi-canonical} if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and the pair $(\bar{X}, \Delta_{\bar{X}} + D_{\bar{X}})$ has canonical singularities.

   (ii) The pair $(X, \Delta)$ is \textit{semi-terminal} if $(X, \Delta)$ is semi-canonical and for any exceptional divisor $E$ over $\bar{X}$ the inequality $a(E, \bar{X}, \Delta_{\bar{X}} + D_{\bar{X}}) > 0$ holds unless $\text{center}_X E \subset [\Delta_{\bar{X}} + D_{\bar{X}}]$ and $\text{codim}_X (\text{center}_X E) = 2$.

**Remark 2.4.** Let $(Y, \Delta + S)$ be a normal pair with $S = [S]$.

1. If the pair $(Y, \Delta + S)$ has canonical singularities, then $\text{Diff}_S \Delta = 0$ and the pair $(S, 0)$ has canonical singularities. In particular, $S$ is normal.
(2) If the pair \((Y, \Delta + S)\) is semi-terminal, then the pair \((S, 0)\) has terminal singularities.

In particular, for a demi-normal pair \((X, \Delta)\), the following holds. (1) If \((X, \Delta)\) is semi-canonical then the pair \(\lfloor \Delta_X + D_X \rfloor, 0\) has canonical singularities. (2) If \((X, \Delta)\) is semi-terminal then the pair \(\lfloor \Delta_X + D_X \rfloor, 0\) has terminal singularities.

**Proof.** (1) Since the pair \((Y, \Delta + S)\) is plt, \(S\) is normal by \([\text{KM98}, \text{Proposition 5.51}]\). Moreover, by adjunction, \(\text{totaldiscrep}(S, \text{Diff}_S \Delta) \geq \text{discrep}(\text{center} \subset S, Y, \Delta + S) \geq 0\) holds (see \([\text{Kol13}, \text{Lemma 4.8}]\)). Thus \(\text{Diff}_S \Delta = 0\) and the pair \((S, 0)\) has canonical singularities.

(2) Assume that the pair \((S, 0)\) does not have terminal singularities. Then there exists an exceptional divisor \(E_S\) over \(S\) such that \(a(E_S, S, 0) = 0\) by (1). We note that \(\text{codim}_Y Z \geq 3\), where \(Z := \text{center}_S E_S\). By adjunction,
\[
\text{totaldiscrep}(\text{center} \subset Z, S, \text{Diff}_S \Delta) \geq \text{discrep}(\text{center} \subset Z, Y, \Delta + S)
\]
holds. Moreover, by the fact that the pair \((Y, \Delta + S)\) is semi-terminal, the right-hand of the above inequality is positive. However, the left-hand of the above inequality is less than or equal to \(a(E_S, S, 0) = 0\). This leads to a contradiction. Thus the pair \((S, 0)\) has terminal singularities. \(\square\)

**Example 2.5.**

(1) \([\text{KM98}, \text{Corollary 2.31}]\) If \((X, \Delta)\) is a normal pair such that \(X\) is a smooth variety and \(\text{Supp} \Delta \subset X\) is a smooth divisor, then the pair \((X, \Delta)\) is semi-terminal.

(2) If \((X, \Delta)\) is a demi-normal pair such that \(X\) is a semi-smooth variety, \(\text{Supp} \Delta\) is contained in the smooth locus of \(X\) and \(\text{Supp} \Delta\) is a smooth divisor, then the pair \((X, \Delta)\) is semi-terminal by (1).

(3) \([\text{KSB88}, \text{Proposition 4.12}]\) Let \(X\) be a demi-normal surface and \(x \in X\) be a closed point. The pair \((X, 0)\) is semi-canonical around \(x\) if and only if \(x \in X\) is either smooth, du Val, dnc or pinch point. The pair \((X, 0)\) is semi-terminal around \(x\) if and only if \(X\) is semi-smooth around \(x\). Thus the notion of semi-terminal singularities is a direct generalization of the notion of semi-smooth surface singularities.

**Definition 2.6.** Let \((X, \Delta)\) be a demi-normal pair and let \(f : Y \to X\) be a proper birational morphism such that \(Y\) is a demi-normal variety, \(f\) is an isomorphism in codimension 1 over \(X\) and \(f\) is an isomorphism around any generic point of \(D_Y\). Set \(\Delta_Y := f_*^{-1} \Delta\).
(1) The morphism $f$ is said to be a semi-canonical modification of $(X, \Delta)$ if $(Y, \Delta_Y)$ is semi-canonical and $K_Y + \Delta_Y$ is ample over $X$. Furthermore, if $X$ (and also $Y$) is a normal variety, then such $f$ is called a canonical modification of $(X, \Delta)$.

(2) The morphism $f$ is said to be a semi-terminal modification of $(X, \Delta)$ if $(Y, \Delta_Y)$ is semi-terminal and $K_Y + \Delta_Y$ is nef over $X$.

3. Semi-canonical modifications

The following lemma and proposition are proven essentially same as [OX12, Lemma 2.1 and Proposition 2.2].

**Lemma 3.1** (cf. [OX12, Lemma 2.1]). Let $(X, \Delta)$ be a normal pair and let $\tilde{f}: \tilde{Y} \to X$ be a projective log resolution of $(X, \Delta)$ such that $	ext{Supp} \Delta_{\tilde{Y}} \subset \tilde{Y}$ is a smooth divisor, where $\Delta_{\tilde{Y}} := \tilde{f}^{-1}_* \Delta$. If the pair $(\tilde{Y}, \Delta_{\tilde{Y}})$ has a canonical model $(Y, \Delta_Y)$ over $X$ (for the definition of canonical models of pairs, see [KM98, §3.8]), then the morphism $Y \to X$ is a canonical modification of $(X, \Delta)$.

**Proof.** It is enough to show that the pair $(Y, \Delta_Y)$ has canonical singularities. Take any exceptional divisor $E$ over $Y$. Then $E$ is either an exceptional divisor over $\tilde{Y}$ or a divisor on $\tilde{Y}$ with $E \not\subset \text{Supp} \Delta_{\tilde{Y}}$. Thus $a(E, \tilde{Y}, \Delta_{\tilde{Y}}) \geq 0$ holds since the pair $(\tilde{Y}, \Delta_{\tilde{Y}})$ has canonical singularities. By [KM98, Proposition 3.51], $a(E, Y, \Delta_Y) \geq a(E, \tilde{Y}, \Delta_{\tilde{Y}}) \geq 0$ holds. Hence the pair $(Y, \Delta_Y)$ has canonical singularities. □

**Proposition 3.2** (cf. [OX12, Proposition 2.2]). Let $(X, \Delta)$ be a normal pair. Then a canonical modification of $(X, \Delta)$ is unique, if exists.

**Proof.** Let $f: Y \to X$ be a canonical modification of the pair $(X, \Delta)$. Let $g: \tilde{Y} \to Y$ be an arbitrary log resolution of the pair $(Y, \Delta_Y)$ such that $	ext{Supp} \Delta_{\tilde{Y}} \subset \tilde{Y}$ is a smooth divisor, where $\Delta_Y := f^{-1}_* \Delta$ and $\Delta_{\tilde{Y}} := g^{-1}_* \Delta_Y$. Set $\tilde{f} := f \circ g: \tilde{Y} \to X$. Since the pair $(Y, \Delta_Y)$ has canonical singularities, we can write $K_{\tilde{Y}} + \Delta_{\tilde{Y}} = g^*(K_Y + \Delta_Y) + F$ such that $F$ is an effective exceptional divisor over $Y$. Therefore $Y$ is isomorphic to $\text{Proj}_X \bigoplus_{m \geq 0} f_* O_Y([m(K_Y + \Delta_Y)]) \simeq \text{Proj}_X \bigoplus_{m \geq 0} \tilde{f}_* O_{\tilde{Y}}([m(K_{\tilde{Y}} + \Delta_{\tilde{Y}})])$.

Therefore a canonical modification of $(X, \Delta)$ is unique. □

We recall the results in [BCHM10].

**Theorem 3.3** ([BCHM10]). Let $(X, \Delta)$ be a quasi-projective normal $\mathbb{Q}$-factorial dlt pair and let $f: X \to U$ be a projective morphism between normal quasi-projective varieties. We assume that $\Delta$ and $K_X +$
$\Delta$ are big over $U$ and $B_+(\Delta/U)$ does not contain any lc center of $(X, \Delta)$, where $B_+(\Delta/U)$ is the augmented base locus of $\Delta$ over $U$ (see [BCHM10 Definition 3.5.1]). Then any $(K_X + \Delta)$-MMP with ample scaling over $U$ induces a good minimal model $(X^m, \Delta^m)$ over $U$, that is, $K_{X^m} + \Delta^m$ is semiaffine over $U$.

Proof. By [BCHM10 Lemma 3.7.5], there exists an effective $\mathbb{Q}$-divisor $\Delta'$ such that $K_X + \Delta \sim_{\mathbb{Q}, U} K_X + \Delta'$ and the pair $(X, \Delta')$ is klt. Thus we may run $(K_X + \Delta)$-MMP with ample scaling over $U$ and terminates by [BCHM10 Corollary 1.4.2]. Since $K_X + \Delta$ is big over $U$, this MMP induces a good minimal model over $U$ by the Basepoint-free theorem [KM98 Theorem 3.24]. □

The following lemma is well-known.

Lemma 3.4. Let $(X, \Delta)$ be a quasi-projective normal pair and let $f : Y \to X$ be a projective log resolution of $(X, \Delta)$. Set $\Delta_Y := f_*^{-1} \Delta$. Then we have the following:

1. Any $\mathbb{Q}$-divisor on $Y$ is big over $X$.
2. The augmented base locus $B_+(\Delta_Y/X)$ does not contain any irreducible component of $\text{Supp} \, \Delta_Y$.

Proof. (1) is obvious. We prove (2). Take a divisor $A$ on $Y$ which is ample over $X$, take a sufficiently small rational number $0 < \epsilon \ll 1$, and take $m \in \mathbb{Z}_{>0}$ such that $m(\Delta_Y - \epsilon A)$ is Cartier. Then the sheaf $f_*O_Y (m(\Delta_Y - \epsilon A))$ is of rank one. Take a general global section of the sheaf $f_*O_Y (m(\Delta_Y - \epsilon A)) \otimes O_X (lH)$, where $H$ is ample on $X$ and $l \gg 0$. Then the pullback of the global section on $Y$ does not contain any irreducible component $S$ of $\text{Supp} \, \Delta_Y$ since $f$ is an isomorphism at the generic point of $S$. Thus $S \not\subset B_+(\Delta_Y/X)$ holds. □

As a corollary, we get the following theorem of [BCHM10]. We give a proof for the reader's convenience. We remark that this theorem is a direct consequence of [BCHM10 Corollary 1.4.2 and Lemma 3.7.5]. See also [Kol13 Theorem 1.31].

Theorem 3.5 ([BCHM10]). For any normal pair $(X, \Delta)$, there exists a canonical modification $f : Y \to X$ of $(X, \Delta)$ and is unique.

Proof. By Proposition 3.2 we can assume that $X$ is quasi-projective. Let $\tilde{f} : \tilde{Y} \to X$ be a projective log resolution of $(X, \Delta)$ such that $\text{Supp} \, \Delta_{\tilde{Y}} \subset Y$ is a smooth divisor, where $\Delta_{\tilde{Y}} := \tilde{f}_*^{-1} \Delta$. Then the pair $(\tilde{Y}, \Delta_{\tilde{Y}})$ is $\mathbb{Q}$-factorial and has canonical singularities. Hence the set of lc centers of $(\tilde{Y}, \Delta_{\tilde{Y}})$ is equal to the set of irreducible components of $[\Delta_{\tilde{Y}}]$. By Lemma 3.4 any irreducible component of $[\Delta_{\tilde{Y}}]$ is
not contained in $\mathbb{B}_+(\Delta_Y/X)$. Thus we can run $(K_Y + \Delta_Y)$-MMP with ample scaling over $X$ and induces a good minimal model over $X$ by Theorem 3.3. Hence there exists the canonical model $Y$ of $(\tilde{Y}, \Delta_{\tilde{Y}})$ over $X$. By Lemma 3.1 the morphism $Y \to X$ is the canonical modification of $(X, \Delta)$. □

**Proposition 3.6 (cf. [OX12, Corollary 2.1]).** Let $(X, \Delta + S)$ be a normal pair with $S = [S]$ and let $f: Y \to X$ be the canonical modification of $(X, \Delta + S)$. Set $S_Y := f^{-1}_* S$ and let $\nu_S: \tilde{S} \to S$ be the normalization. Then the morphism $f$ induces the birational morphism $f_S: S_Y \to \tilde{S}$ and the morphism $f_S$ is the canonical modification of $(\tilde{S}, 0)$.

**Proof.** By Remark 2.4 the pair $(S_Y, 0)$ has canonical singularities (in particular, $S_Y$ is normal) and $K_{S_Y} = (K_Y + \Delta_Y + S_Y)|_{S_Y}$ is ample over $S$. Thus the morphism $f_S$ is the canonical modification of $(\tilde{S}, 0)$. □

**Lemma 3.7 (cf. [OX12, Lemma 3.1]).** Let $(X, \Delta)$ be a demi-normal pair.

1. A semi-canonical modification of $(X, \Delta)$ is unique, if exists.
2. Let $f: Y \to X$ be the semi-canonical modification of $(X, \Delta)$, let $\nu_Y: \tilde{Y} \to Y$ and $\nu_X: \tilde{X} \to X$ be the normalizations and let $\tilde{f}: \tilde{Y} \to \tilde{X}$ be the morphism obtained by $f$. Then the morphism $\tilde{f}$ is the canonical modification of $(\tilde{X}, \Delta_{\tilde{X}} + D_{\tilde{X}})$, where $\Delta_{\tilde{X}} := (\nu_X)_*^{-1} \Delta$ and $D_{\tilde{X}}$ is the conductor divisor of $\tilde{X}/X$.

**Proof.** Let $f: Y \to X$ be a semi-canonical modification of $(X, \Delta)$. Then $K_Y + \Delta_Y + D_Y = (\nu_Y)_* (K_Y + \Delta_Y)$ is ample over $X$ and the pair $(\tilde{Y}, \Delta_{\tilde{Y}} + D_{\tilde{Y}})$ has canonical singularities. Hence the morphism $\tilde{f}$ is the canonical modification of $(\tilde{X}, \Delta_{\tilde{X}} + D_{\tilde{X}})$. Thus we get (2) and $\tilde{Y}$ is unique. On the other hand, the Galois involution $\iota_X: \tilde{D}_X \to D_{\tilde{X}}$ is extended to $\iota: D_Y \to D_{\tilde{Y}}$ uniquely, where $D_X$ is the normalization of $D_{\tilde{X}}$. Thus the quotient $\tilde{Y} \to Y$ by $\iota$ is unique by [Kol13, Proposition 5.3]. □

**Theorem 3.8** (=Theorem 111 (1)). For any demi-normal pair $(X, \Delta)$, the canonical modification of $(X, \Delta)$ exists and is unique (up to isomorphism over $X$).

**Proof.** Let $\nu_X: \tilde{X} \to X$ be the normalization and let $\Delta_{\tilde{X}} := (\nu_X)_*^{-1} \Delta$. By Theorem 3.3 there exists the canonical modification $\tilde{f}: \tilde{Y} \to \tilde{X}$ of $(\tilde{X}, \Delta_{\tilde{X}} + D_{\tilde{X}})$. Set $\Delta_{\tilde{Y}} := \tilde{f}_*^{-1} \Delta_{\tilde{X}}$ and $D_{\tilde{Y}} := \tilde{f}_*^{-1} D_{\tilde{X}}$. By Proposition 3.6 the morphism $\tilde{f}_{D_{\tilde{X}}}: D_{\tilde{Y}} \to D_{\tilde{X}}$ is the canonical modification of $(D_{\tilde{X}}, 0)$, where $D_{\tilde{X}}$ is the normalization of $D_{\tilde{X}}$. Hence the involution $\iota_X: D_{\tilde{X}} \to D_{\tilde{X}}$ can be extended to the involution $\iota: D_{\tilde{Y}} \to D_{\tilde{Y}}$. Since $K_{\tilde{Y}} + \Delta_{\tilde{Y}} + D_{\tilde{Y}}$ is ample over $X$, there exists a semi-canonical
pair \((Y, \Delta_Y)\) over \(X\) such that the normalization of \(Y\) and \(D_Y\) is exactly same as the conductor divisor of \(Y/Y\) by [Kol13, Corollary 5.37, Corollary 5.33 and Theorem 5.38]. The morphism \(Y \to X\) is exactly the semi-canonical modification of \((X, \Delta)\). \(\square\)

4. Semi-terminal modifications

In this section, we prove Theorem 1.1 (2). Let \((X, \Delta)\) be an arbitrary quasi-projective demi-normal pair. We show that there exists a projective semi-terminal modification of \((X, \Delta)\).

4.1. Semi-log-resolution. By [Kol13, Theorem 10.54], there exists a projective and birational morphism \(f: Y \to X\) such that the following properties hold:

(i) \(Y\) is semi-smooth.
(ii) \(f\) is an isomorphism in codimension 1 over \(X\) and \(f\) is an isomorphism at any generic point of \(D_Y\).
(iii) \(\text{Supp} \Delta_Y\) is contained in the smooth locus of \(Y\) and \(\text{Supp} \Delta_Y\) is a smooth divisor, where \(\Delta_Y := f_*^{-1} \Delta\).
(iv) Let \(\tilde{f}: \tilde{Y} \to \tilde{X}\) be the morphism obtained by the normalizations \(\nu_X: \tilde{X} \to X\) and \(\nu_Y: \tilde{Y} \to Y\). Then the morphism \(\tilde{f}\) is a projective log resolution of the pair \((\tilde{X}, \Delta_{\tilde{X}} + D_{\tilde{X}})\), where \(\Delta_{\tilde{X}}\) is the strict transform of \(\Delta\).

We fix a Cartier divisor \(H\) on \(Y\) which is ample over \(X\) such that \(K_Y + \Delta_Y + H\) is nef over \(X\).

4.2. Running a reducible MMP with scaling. In this section, we will construct inductively the following (for \(i \geq 0\)):

(1) Projective and birational morphisms

\[
Y_i \xrightarrow{\pi_i} W_i \xleftarrow{d_i} W_i^d \xrightarrow{\pi_i^+} Y_{i+1}
\]

over \(X\) such that all of them are isomorphisms in codimension 1 over its images and are isomorphisms at all generic points of the conductor divisors.

(2) A rational number \(\lambda_i\) such that \(0 < \lambda_i \leq 1\) and \(\lambda_0 \geq \lambda_1 \geq \cdots\).

(3) A \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(H_i\) on \(Y_i\), a positive integer \(l_i\) and an invertible sheaf \(L_i\) on \(W_i\) which is nef over \(X\).

The properties are the following:

(i) \(Y_0 = Y\) and \(H_0 = H\) holds. The pair \((Y_i, \Delta_i)\) is semi-terminal, where \(\Delta_i\) is the strict transform of \(\Delta\).
(ii) The following holds.

\[ \lambda_i = \inf \{ \lambda \in \mathbb{R}_{\geq 0} \mid K_{Y_i} + \Delta_i + \lambda H_i \text{ is nef over } X \} . \]

The morphism \( \pi_i \) is the contraction morphism associated to a \((K_{Y_i} + \Delta_i)\)-negative extremal ray \( R_i \subset \overline{\text{NE}}(Y_i/X) \) such that

\[ (K_{Y_i} + \Delta_i + \lambda_i H_i \cdot R_i) = 0. \]

(iii) The morphism \( d_i \) is the semi-normalization of \( W_i \). The morphism \( \pi_i^{\pm} \) is the semi-canonical modification of \((W_i^d, \Delta_{W_i^d})\), where \( \Delta_{W_i^d} \) is the strict transform of \( \Delta \).

(iv) The following holds:

\[
\begin{align*}
\pi_i^* L_i & \simeq \mathcal{O}_{Y_i}(l_i(K_{Y_i} + \Delta_i + \lambda_i H_i)), \\
(d_i \circ \pi_i^+)^* L_i & \simeq \mathcal{O}_{Y_{i+1}}(l_i(K_{Y_{i+1}} + \Delta_{i+1} + \lambda_i H_{i+1})).
\end{align*}
\]

**Construction.** Set \( Y_0 := Y \) and \( H_0 := H \).

Assume that we have constructed \( Y_i \) and \( H_i \) (and also \( \lambda_{i-1}, l_{i-1} \) and \( \mathcal{L}_{i-1} \), if \( i \geq 1 \)). If \( K_{Y_i} + \Delta_i \) is nef over \( X \), then we stop the program and go to Section 4.5.

We consider the case that \( K_{Y_i} + \Delta_i \) is not nef over \( X \). Set \( \lambda_i \) as in (ii). If \( i = 0 \), then \( K_{Y_i} + \Delta_i + H_i \) is nef over \( X \) by definition. If \( i \geq 1 \), then \((d_{i-1} \circ \pi_i)^* \mathcal{L}_{i-1}\) is nef over \( X \) by induction. Hence \( K_{Y_i} + \Delta_i + \lambda_{i-1} H_i \) is nef over \( X \). Thus \( 0 < \lambda_i \leq 1 \), and \( \lambda_i \leq \lambda_{i-1} \) if \( i \geq 1 \). We can find a \((K_{Y_i} + \Delta_i)\)-negative extremal ray \( R_i \subset \overline{\text{NE}}(Y_i/X) \) with \((K_{Y_i} + \Delta_i + \lambda_{i+1} H_i \cdot R_i) = 0\), and we can get the contraction morphism over \( X \) with respect to \( R_i \) by [Fuj12, Theorem 1.19]. In particular, \( \lambda_i \) is a rational number. Let \( \pi_i : Y_i \to W_i \) be the corresponding contraction morphism over \( X \). The morphism \( \pi_i \) is a projective and birational morphism. Since \((K_{Y_i} + \Delta_i + \lambda_i H_i \cdot R_i) = 0\), we can find a positive integer \( l_i \) and an invertible sheaf \( \mathcal{L}_i \) on \( W_i \) such that \( \pi_i^* \mathcal{L}_i \simeq \mathcal{O}_{Y_i}(l_i(K_{Y_i} + \Delta_i + \lambda_i H_i)) \) by the Contraction theorem [Fuj12, Theorem 1.19]. Since \( \pi_i^* \mathcal{L}_i \) is nef over \( X \), \( \mathcal{L}_i \) is also nef over \( X \). We can take the semi-normalization \( d_i : W_i^d \to W_i \) since \( W_i \) is dnc outside codimension 2. Let \( \pi_i^+ : Y_{i+1} \to W_i^d \) be the semi-canonical modification of \((W_i^d, \Delta_{W_i^d})\), where \( \Delta_{W_i^d} \) is the strict transform of \( \Delta \). (We note that \( \pi_i^+ \) exists and is unique by Theorem 4.4.) We take a \( \mathbb{Q} \)-divisor \( H_{i+1} \) on \( Y_{i+1} \) such that the following holds:

\[
(d_i \circ \pi_i^+)^* \mathcal{L}_i \simeq \mathcal{O}_{Y_{i+1}}(l_i(K_{Y_{i+1}} + \Delta_{i+1} + \lambda_i H_{i+1})).
\]

The \( \mathbb{Q} \)-divisor \( H_{i+1} \) is \( \mathbb{Q} \)-Cartier since \( K_{Y_{i+1}} + \Delta_{i+1} \) is \( \mathbb{Q} \)-Cartier.

**Claim 4.1.** The pair \((Y_{i+1}, \Delta_{i+1})\) is semi-terminal.

**Proof of Claim 4.1.** Let \( \hat{Y}_i, \hat{W}_i \) be the normalization of \( Y_i, W_i \), respectively. We note that \( \hat{W}_i \) is equal to the normalization of \( W_i^d \) by Zariski’s
Main Theorem. Let

$$\pi_i: \bar{Y}_i \to \bar{W}_i \leftarrow \bar{Y}_{i+1}$$

be the morphisms obtained by $\pi_i$ and $\pi_i^+$. Since the pair $(\bar{Y}_{i+1}, \Delta_{i+1})$ is semi-canonical, it is enough to show that the pair $(\bar{Y}_{i+1}, B_{i+1})$ is semi-terminal, where $B_i$ is the sum of $D_{\bar{Y}_i}$ and the strict transform of $\Delta_i$. We know that $-(K_{\bar{Y}_i} + B_i)$ is ample over $\bar{W}_i$ and $K_{\bar{Y}_{i+1}} + B_{i+1}$ is ample over $\bar{W}_i$. Take any exceptional divisor $E$ over $\bar{Y}_{i+1}$ such that $a(E, \bar{Y}_{i+1}, B_{i+1}) = 0$ holds. It is enough to show that $c \text{enter}_{\bar{Y}_{i+1}} E \subset [B_{i+1}]$ and $\text{codim}_{\bar{Y}_{i+1}} (\text{center}_{\bar{Y}_{i+1}} E) = 2$. Assume that either $\bar{\pi}_i$ or $\bar{\pi}_i^+$ is not an isomorphism over the generic point of $\text{center}_{\bar{W}_i} E$. Then we have

$$a(E, \bar{Y}_i, B_i) < a(E, \bar{Y}_{i+1}, B_{i+1})$$

by the negativity lemma \cite{KM98} Lemma 3.38. Since the pair $(\bar{Y}_i, B_i)$ has canonical singularities, this leads to a contradiction. (We remark that $\bar{\pi}_i$, $\bar{\pi}_i^+$ are isomorphisms over the images of the generic point of all components of $\text{Supp} B_i$, $\text{Supp} B_{i+1}$, respectively. Thus $E \not\subset \text{Supp} B_i$.) Therefore both $\bar{\pi}_i$ are $\bar{\pi}_i^+$ are isomorphisms at the generic point of $\text{center}_{\bar{W}_i} E$. In particular, $a(E, \bar{Y}_i, B_i) = 0$ holds. Since the pair $(\bar{Y}_i, B_i)$ is semi-terminal, we have $\text{center}_{\bar{Y}_i} E \subset [B_i]$ and $\text{codim}_{\bar{Y}_i} (\text{center}_{\bar{Y}_i} E) = 2$. Thus we have $\text{codim}_{\bar{Y}_{i+1}} (\text{center}_{\bar{Y}_{i+1}} E) = 2$ and $\text{center}_{\bar{Y}_{i+1}} E \subset [B_{i+1}]$. $\square$

Therefore, we can construct the objects in Section 4.2 (1), (2) and (3) inductively.

4.3. Decomposing the MMP. Let $\phi_0: Z_{0,0} \to \bar{Y}_0$ be the identity morphism and let $H_{0,0} := (\nu_{\bar{Y}_0} \circ \phi_0)^*H_0$, where $\nu_{\bar{Y}_0}: \bar{Y}_0 \to Y_0$ is the normalization.

In Section 4.3, we prove the following claim.

Claim 4.2. Let $i \geq 0$ such that $K_{\bar{Y}_i} + \Delta_i$ is not nef over $X$. Assume that there exists a projective and birational morphism $\phi_i: Z_{i,0} \to \bar{Y}_i$ (we note that $\nu_{\bar{Y}_i}: \bar{Y}_i \to Y_i$ is the normalization) and a $\mathbb{Q}$-divisor $H_{i,0}$ on $Z_{i,0}$ such that the following properties hold:

(i) The variety $Z_{i,0}$ is normal and $\mathbb{Q}$-factorial.

(ii) $K_{Z_{i,0}} + B_{i,0} = \phi^*(K_{\bar{Y}_i} + B_i)$ holds, where $B_i$ is the sum of $D_{\bar{Y}_i}$ and the strict transform of $\Delta_i$, and $B_{i,0}$ is the strict transform of $B_i$.

In particular, the pair $(Z_{i,0}, B_{i,0})$ has canonical singularities.

(iii) $H_{i,0} \sim_{\mathbb{Q}} (\nu_{\bar{Y}_i} \circ \phi_i)^*H_i$ holds.

Then we have the following results:
Thus we prove (2) for the case $j > 1$. Let $B_{i,j}$, $H_{i,j}$ be the push forward of $B_{i,0}$, $H_{i,0}$ on $Z_{i,j}$, respectively. More precisely, for $j \geq 0$, the morphism $\pi_{i,j}$ is the contraction morphism associated to a $(K_{Z_{i,j}} + B_{i,j})$-negative extremal ray $R_{i,j} \subset \mathcal{N}(Z_{i,j}/\bar{W}_i)$ and the morphism $\pi_{i,j}^+$ is the identity morphism if $\pi_{i,j}$ is divisorial and the flip if $\pi_{i,j}$ is small.

(2) Let $\mathcal{L}_{i,j}$ be the pullback of $\mathcal{L}_i$ on $Z_{i,j}$. Then we have the following:

$$\mathcal{L}_{i,j} \sim_{\mathbb{Q}} l_i(K_{Z_{i,j}} + B_{i,j} + \lambda_i H_{i,j}).$$

(3) If $K_{Z_{i,j}} + B_{i,j}$ is not nef over $\bar{W}_i$, then we have the following:

$$\lambda_i = \inf \{ \lambda \in \mathbb{R}_{\geq 0} | K_{Z_{i,j}} + B_{i,j} + \lambda H_{i,j} \text{ is nef over } X \}.$$

(4) Assume that the sequence

$$Z_{i,0} \rightarrow Z_{i,1} \rightarrow \ldots \rightarrow Z_{i,m_i}$$

terminates. In other words, $K_{Z_{i,m_i}} + B_{i,m_i}$ is nef over $\bar{W}_i$. (We will prove the termination in Section 4.4.) Then $m_i \in \mathbb{Z}_{>0}$. Set $Z_{i+1,0} := Z_{i,m_i}$ and $H_{i+1,0} := H_{i,m_i}$. Then there exists a projective and birational morphism $\phi_{i+1}: Z_{i+1,0} \rightarrow Y_{i+1}$ such that the properties [1], [11] and (11) holds.

Proof of Claim 4.2. Since the pair $(Z_{i,0}, B_{i,0})$ is $\mathbb{Q}$-factorial and has canonical singularities, we can run the MMP which described in [1]. We note that the flip $\pi_{i,j}^+$ exists if $\pi_{i,j}$ is small by [BCHM10, Corollary 1.4.1].

Now we prove (2) and (3) by induction on $j$. Let $\lambda_{i,j}$ be the right-hand of the equality in (3). We consider the case $j = 0$. By (11) in Section 4.2 $\mathcal{L}_{i,0}$ is isomorphic to $O_{Z_{i,0}}(l_i(K_{Z_{i,0}} + B_{i,0} + \lambda_i(\nu_1 \circ \phi_i)\ast H_i))$.

Thus we prove (2) for the case $j = 0$ since $H_{Z_{i,0}} \sim_{\mathbb{Q}} (\nu_1 \circ \phi_i)\ast H_i$. On the other hand,

$$\lambda_{i,0} = \inf \{ \lambda \in \mathbb{R}_{\geq 0} | \phi_i^*(K_{Y_i} + B_i + \lambda \nu_1^* H_i) \text{ is nef over } X \}$$

$$= \inf \{ \lambda \in \mathbb{R}_{\geq 0} | K_{Y_i} + \Delta_i + \lambda H_i \text{ is nef over } X \} = \lambda_i.$$

Thus we prove (3) for the case $j = 0$.

We consider the case $j \geq 1$. Since the inverse of the birational map $Z_{i,j-1} \rightarrow Z_{i,j}$ does not contract divisors, we prove (2) by induction.

We note that $\mathcal{L}_{i,j}$ is nef over $X$ since $\mathcal{L}_i$ is nef over $X$. Thus $\lambda_{i,j} \leq \lambda_i$.
Claim 4.3. 

Proof of Claim 4.3. Let \( m \) is ample over \( W \) and \( Z \) the pair \((B, W, Y)\) over \( \bar{K} \)jective and birational morphism \( \phi \) canonical model of \((B, W, Y)\) over \( \bar{K} \). On the other hand, we know that \( K_{Z,0} + B_{i,0} \) is not nef over \( \bar{W}_i \), \( K_{Z,i,j} + B_{i,j} + \lambda_i H_{i,j} \sim_Q \bar{W}_i \) and \( K_{Z,i,j} + B_{i,j} + \lambda_i H_{i,j} \) is nef over \( \bar{W}_i \). Thus \( \lambda_i \geq \lambda_j \) holds. Therefore we prove (3).

Assume that the MMP \( Z_{i,0} \rightarrow Z_{i,1} \rightarrow Z_{i,2} \rightarrow \cdots \) over \( \bar{W}_i \) terminates and induces a minimal model \((Z_{i,m_i}, B_{i,m_i})\) over \( \bar{W}_i \). We remark that \( K_{Z_i,0} + B_{i,0} \) is not nef over \( \bar{W}_i \) since \(- (K_{Y_i} + \Delta_i)\) is ample over \( \bar{W}_i \) and \( K_{Z_i,0} + B_{i,0} \) is the pullback of \( K_{Y_i} + \Delta_i \). Thus \( m_i \in \mathbb{Z}_{>0} \) holds.

Claim 4.3. \( \bar{Y}_{i+1} \) is the canonical model of the pair \((Z_{i,0}, B_{i,0})\) over \( \bar{W}_i \).

Proof of Claim 4.3. Let \( g: T \rightarrow Z_{i,0} \) be a projective log resolution of the pair \((Z_{i,0}, B_{i,0})\) such that \( \text{Supp} \ B_T \subset T \) is a smooth divisor, where \( B_T \) is the strict transform of \( B_{i,0} \). Then, by Lemmas 3.1 and 3.7 (2), \( \bar{Y}_{i+1} \) is the canonical model of the pair \((T, B_T)\) over \( \bar{W}_i \). We can write \( K_T + B_T = g^*(K_{Z_{i,0}} + B_{i,0}) + F \) with \( F \) effective and exceptional over \( Z_{i,0} \) since the pair \((Z_{i,0}, B_{i,0})\) has canonical singularities. Thus \( \bar{Y}_{i+1} \) is the canonical model of \((Z_{i,0}, B_{i,0})\) over \( \bar{W}_i \) by [KM98, Corollary 3.53].

By Claim 4.3 and [K+92] Theorem 2.22, there exists the unique projective and birational morphism \( \phi_{i+1} : Z_{i,m_i} \rightarrow \bar{Y}_{i+1} \) with \( \phi_{i+1}^*(K_{\bar{Y}_{i+1}} + B_{i+1}) = K_{Z_{i,m_i}} + B_{i,m_i} \). Set \( Z_{i+1,0} := Z_{i,m_i} \) and \( H_{i+1} := H_{i,m_i} \). We note that \( B_{i,m_i} = B_{i+1,0} \). By Claim 4.3 and Section 4.2,

\[
\mathcal{O}_{Z_{i+1,0}}(l_i((K_{Z_{i+1,0}} + B_{i+1,0} + \lambda_i H_{i+1,0})) \\
\simeq (\mathcal{L}_i)_{Z_{i+1,0}} \simeq ((\nu_{W_i} \circ \pi_i^+) \ast \mathcal{L}_i)_{Z_{i+1,0}} \\
\simeq \mathcal{O}_{Z_{i+1,0}}(l_i((K_{Z_{i+1,0}} + B_{i+1,0} + \lambda_i(\nu_{Y_{i+1}} \circ \phi_{i+1}) \ast H_{i+1,0})))
\]

holds, where \( (\mathcal{L}_i)_{Z_{i+1,0}} \), \((\nu_{W_i} \circ \pi_i^+) \ast \mathcal{L}_i)_{Z_{i+1,0}} \) is the pullback of \( \mathcal{L}_i \), \((\nu_{W_i} \circ \pi_i^+) \ast \mathcal{L}_i \) to \( Z_{i+1,0} \), respectively. Hence \( H_{i+1,0} \sim_Q (\nu_{Y_{i+1}} \circ \phi_{i+1}) \ast H_{i+1} \) holds. Thus we prove (4).

Therefore, by Claim 4.2, if we assume the termination of the sequence in (4), then we can inductively construct the diagram
for any $i$ with $K_{Y_i} + \Delta_i$ not nef over $X$.

4.4. Termination of the program. In this section, we show the following:

(a) For any $i \geq 0$ with $K_{Y_i} + \Delta_i$ not nef over $X$, the sequence $Z_{i,0} \to Z_{i,1} \to \cdots \to Z_{i+1,0}$ terminates.

(b) The sequence $Y_0 \to Y_1 \to \cdots$ terminates.

Assume either (a) does not hold for some $i$, or (a) is true and (b) does not hold. Then there exists an infinite sequence $Z_{0,0} \to Z_{0,1} \to \cdots \to Z_{1,0} \to Z_{1,1} \to \cdots$ of varieties $Z_{i,j}$ (we note that $n_i \in \mathbb{Z}_{>0}$ by Claim 4.2 (3)). For any $i$ and $j$, under the natural embedding $N_1(Z_{i,j}/\bar{W}_i) \hookrightarrow N_1(Z_{i,j}/\bar{X})$, the cone $\overline{\text{NE}}(Z_{i,j}/\bar{W}_i)$ is an extremal face in $\overline{\text{NE}}(Z_{i,j}/\bar{X})$. Hence the extremal ray $R_{i,j} \subset \overline{\text{NE}}(Z_{i,j}/\bar{W}_i)$ can be seen as a $(K_{Z_{i,j}} + B_{i,j})$-negative extremal ray $R'_{i,j} \subset \overline{\text{NE}}(Z_{i,j}/\bar{X})$. By Claim 4.2 (2), $(K_{Z_{i,j}} + B_{i,j} + \lambda_i H_{i,j}, R'_{i,j}) = 0$ holds. Moreover, $\lambda_i = \inf\{\lambda \in \mathbb{R}_{\geq 0} \mid K_{Z_{i,j}} + B_{i,j} + \lambda H_{i,j} \text{ is nef over } \bar{X}\}$ holds by Claim 4.2 (3). The contraction morphism with respect to $R'_{i,j}$ over $\bar{X}$ is equal to $\pi_{i,j}$. Thus this sequence is a $(K_{Z_{0,0}} + B_{0,0})$-MMP with scaling $H_{0,0}$ over $\bar{X}$. By Theorem 3.3, Lemma 3.4 and the facts that the pair $(Z_{0,0}, B_{0,0})$ has canonical singularities and the morphism $Z_{0,0} \to \bar{X}$ is a projective log resolution of the pair $(\bar{X}, \Delta_{\bar{X}} + D_{\bar{X}})$, this MMP over $\bar{X}$ must terminates. This leads to a contradiction. Thus both (a) and (b) are true.
4.5. **Conclusion.** By Sections 4.2 and 4.4 there exists a projective and birational morphism $f_m: Y_m \to X$ such that $K_{Y_m} + \Delta_m$ is nef over $X$. Furthermore, the pair $(Y_m, \Delta_m)$ is semi-terminal, the morphism $f_m$ is an isomorphism in codimension 1 over $X$ and is an isomorphism at any generic point of $D_{Y_m}$ by construction. Therefore the morphism $f_m$ is a semi-terminal modification of $(X, \Delta)$.

As a consequence, we have completed the proof of Theorem 1.1 (2).

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