EXTENSION-ORTHOGONAL COMPONENTS OF NILPOTENT VARIETIES

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Abstract. Let \( Q \) be a Dynkin quiver, and let \( \Lambda \) be the corresponding preprojective algebra. Let \( \mathcal{I} = \{ C_i \mid i \in I \} \) be a set of pairwise different indecomposable irreducible components of varieties of \( \Lambda \)-modules such that generically there are no extensions between \( C_i \) and \( C_j \) for all \( i, j \). We show that the number of elements in \( \mathcal{I} \) is at most the number of positive roots of \( Q \). Furthermore, we give a module theoretic interpretation of Leclerc’s counterexample to a conjecture of Berenstein and Zelevinsky.

1. Introduction

Let \( k \) be an algebraically closed field. For a finitely generated \( k \)-algebra \( A \) let \( \text{mod}_A(d) \) be the affine variety of \( A \)-modules with dimension vector \( d \). For irreducible components \( C_1 \subseteq \text{mod}_A(d_1) \) and \( C_2 \subseteq \text{mod}_A(d_2) \) define
\[
\text{ext}^1_A(C_1, C_2) = \min \{ \dim \text{Ext}^1_A(M_1, M_2) \mid (M_1, M_2) \in C_1 \times C_2 \}.
\]
An irreducible component \( C \subseteq \text{mod}_A(d) \) is indecomposable if it contains a dense subset of indecomposable \( A \)-modules. A general theory of irreducible components and their decomposition into indecomposable irreducible components was developed in [2]. Our aim is to apply this to Lusztig’s nilpotent varieties.

If not mentioned otherwise, we always assume that \( Q \) is a Dynkin quiver of type \( \mathbb{A}_n, \mathbb{D}_n \) or \( \mathbb{E}_{6,7,8} \). By \( R^+ \) we denote the set of positive roots of \( Q \), and by \( \Lambda \) we denote the preprojective algebra associated to \( Q \), see [12]. Let \( n \) be the number of vertices of \( Q \), and let \( \Lambda(d) = \text{mod}_A(d), \, d \in \mathbb{N}^n \), be the variety of \( \Lambda \)-modules with dimension vector \( d \). The varieties \( \Lambda(d) \) are called nilpotent varieties. We refer to [8, Section 12] for basic properties. Throughout, we only consider finite-dimensional modules. Our main result is the following:

**Theorem 1.1.** Assume that \( \{ C_i \subseteq \Lambda(d_i) \mid i \in I \} \) is a set of pairwise different indecomposable irreducible components such that \( \text{ext}^1_A(C_i, C_j) = 0 \) for all \( i, j \in I \). Then \( |I| \leq |R^+| \).

As a consequence we get the following result on \( \Lambda \)-modules without self-extensions:

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Corollary 1.2. Let $M$ be a $\Lambda$-module with $\text{Ext}_\Lambda^1(M, M) = 0$. Then the number of pairwise non-isomorphic indecomposable direct summands of $M$ is at most $|R^+|$.

Let $U^-_v$ be the negative part of the quantized enveloping algebra of the Lie algebra corresponding to $Q$. We regard $U^-_v$ as a $\mathbb{Q}(v)$-algebra. Let $B$ be the canonical basis and $B^*$ the dual canonical basis of $U^-_v$, see [1], [3], [8] or [10] for definitions. By [4, Section 5], the elements of $B$ (and thus of $B^*$) correspond to the irreducible components of the nilpotent varieties $\Lambda(d)$, $d \in \mathbb{N}^n$. Let $b^*(C)$ be the dual canonical basis vector corresponding to an irreducible component $C$. We denote the structure constants of $U^-_v$ with respect to the basis $B^*$ by $\lambda^E_{C, D}$, i.e.

$$b^*(C)b^*(D) = \sum_E \lambda^E_{C, D}b^*(E).$$

Following the terminology in [1], two dual canonical basis vectors $b^*(C)$ and $b^*(D)$ are called multiplicative if

$$b^*(C)b^*(D) = \lambda^E_{C, D}b^*(E)$$

for some irreducible component $E$, and they are quasi-commutative if

$$b^*(C)b^*(D) = \lambda b^*(D)b^*(C)$$

for some $\lambda \in \mathbb{Q}(v)$. The following conjecture was stated in [1, Section 1]:

**Conjecture 1.3** (Berenstein, Zelevinsky). Two dual canonical basis vectors are multiplicative if and only if they are quasi-commutative.

One direction of this conjecture was proved by Reineke [10, Corollary 4.5]. The other direction turned out to be wrong. Namely, Leclerc [8] constructed examples of quasi-commutative elements in $B^*$ which are not multiplicative. Using preprojective algebras, we give a module theoretic interpretation of one of his examples.

Marsh and Reineke [9] conjectured that the multiplicative behaviour of dual canonical basis vectors should be related to sets of irreducible components with $\text{Ext}^1$ vanishing generically between them. This was the principal motivation for our work.

The paper is organized as follows: In Section 2 we review the main results from [2]. In Section 3 we recall known results for the case that $\Lambda$ is an algebra of finite or tame representation type. The proof of Theorem 1.1 and its corollary can be found in Section 4. Finally, Section 5 is devoted to the interpretation of Leclerc’s example.

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2. Varieties of modules - definitions and known results

In this section, we work with arbitrary finite quivers.

2.1. Let $Q = (Q_0, Q_1)$ be a finite quiver, where $Q_0$ denotes the set of vertices and $Q_1$ the set of arrows of $Q$. Assume that $|Q_0| = n$. For an arrow $\alpha$ let $s\alpha$ be its starting vertex and $e\alpha$ its end vertex. An element $d = (d_i)_{i \in Q_0} \in \mathbb{N}^n$ is called a dimension vector for $Q$. A representation of $Q$ with dimension vector $d$ is a matrix tuple $M = (M_\alpha)_{\alpha \in Q_1}$ with $M_\alpha \in M_{d_{s\alpha} \times d_{e\alpha}}(k)$. A path of length $l \geq 1$ in $Q$ is a sequence $p = \alpha_1 \cdots \alpha_l$ of arrows in $Q_1$ such that $e\alpha_i = s\alpha_{i+1}$ for $1 \leq i \leq l - 1$. Define $sp = s\alpha_1$ and $ep = e\alpha_l$. For a representation $M$ and a path $p = \alpha_1 \cdots \alpha_l$ define $M_p = M_{\alpha_1} \cdots M_{\alpha_l}$ which is a matrix in $M_{d_p \times d_{ep}}(k)$. A relation for $Q$ is a $k$-linear combination $\sum_{i=1}^{t} \lambda_i p_i$ of paths $p_i$ of length at least two such that $sp_i = sp_j$ and $ep_i = ep_j$ for all $1 \leq i, j \leq t$. A representation $M$ satisfies such a relation if $\sum_{i=1}^{t} \lambda_i M_{p_i} = 0$. Given a set $\rho$ of relations for $Q$ let $\text{rep}_{(Q, \rho)}(d)$ be the affine variety of representations of $Q$ with dimension vector $d$ which satisfy all relations in $\rho$.

2.2. One can interpret this construction in a module theoretic way. Namely, let $kQ$ be the path algebra of $Q$, and let $A = kQ/\rho$, where $(\rho)$ is the ideal generated by the elements in $\rho$. Then $\text{mod}_A(d) = \text{rep}_{(Q, \rho)}(d)$ is the affine variety of $A$-modules with dimension vector $d$. If $A = kQ/(\rho)$ is finite-dimensional, then $A$ is called a basic algebra. In this case, the vertices of $Q$ correspond to the isomorphism classes of simple $A$-modules, and the entry $d_i$, $i \in Q_0$, of $d$ is the multiplicity of the simple module corresponding to $i$ in a composition series of any $M \in \text{mod}_A(d)$. The group $GL(d) = \prod_{i \in Q_0} GL_{d_i}(k)$ acts on $\text{mod}_A(d)$ by conjugation, i.e.

$$g \cdot M = (g_{s\alpha} M_{\alpha} g_{e\alpha}^{-1})_{\alpha \in Q_1}.$$ 

The orbit of $M$ under this action is denoted by $O(M)$. There is a 1-1 correspondence between the set of orbits in $\text{mod}_A(d)$ and the set of isomorphism classes of $A$-modules with dimension vector $d$.

2.3. Given irreducible components $C_i \subseteq \text{mod}_A(d_i)$, $1 \leq i \leq t$, we consider all $A$-modules with dimension vector $d = d_1 + \cdots + d_t$, which are of the form $M_1 \oplus \cdots \oplus M_t$ with the $M_i$ in $C_i$, and we denote by $C_1 \oplus \cdots \oplus C_t$ the corresponding subset of $\text{mod}_A(d)$. This is the image of the map

$$GL(d) \times C_1 \times \cdots \times C_t \longrightarrow \text{mod}_A(d)$$

$$(g, M_1, \cdots, M_t) \mapsto g \cdot \left( \bigoplus_{i=1}^{t} M_i \right).$$

We call $C_1 \oplus \cdots \oplus C_t$ the direct sum of the components $C_i$. It follows that the closure $\overline{C_1 \oplus \cdots \oplus C_t}$ is irreducible. For an irreducible component $C$ define $C^n = \bigoplus_{i=1}^{n} C$. We call $C$ indecomposable if $C$ contains a dense subset of
indecomposable $A$-modules. The following result from [2] is an analogue of
the Krull-Remak-Schmidt Theorem.

**Theorem 2.1.** If $C \subseteq \operatorname{mod}_A(d)$ is an irreducible component, then
\[ C = C_1 \oplus \cdots \oplus C_t \]
for some indecomposable irreducible components $C_i \subseteq \operatorname{mod}_A(d_i)$, $1 \leq i \leq t$, and $C_1, \ldots, C_t$ are uniquely determined by this, up to reordering. The above
direct sum is called the canonical decomposition of $C$.

However, the closure of a direct sum of irreducible components is not in
general an irreducible component. The next result is also proved in [2].

**Theorem 2.2.** If $C_i \subseteq \operatorname{mod}_A(d_i)$, $1 \leq i \leq t$, are irreducible components,
and $d = d_1 + \cdots + d_t$, then $C_1 \oplus \cdots \oplus C_t$ is an irreducible component of
$\operatorname{mod}_A(d)$ if and only if $\operatorname{Ext}_A^1(C_i, C_j) = 0$ for all $i \neq j$.

Instead of taking direct sums of the modules in two irreducible com-
ponents, one can take extensions. Let $d = d_1 + d_2$ be dimension vectors,
let $G = \operatorname{GL}(d_1) \times \operatorname{GL}(d_2)$, and let $S$ be a $G$-stable subset of $\operatorname{mod}_A(d_1) \times
\operatorname{mod}_A(d_2)$. We denote by $E(S)$ the $\operatorname{GL}(d)$-stable subset of $\operatorname{mod}_A(d)$ corre-
sponding to all modules $M$ which belong to a short exact sequence
\[ 0 \to M_2 \to M \to M_1 \to 0 \]
with $(M_1, M_2) \in S$, see [2] for more details.

For an irreducible component $C \subseteq \operatorname{mod}_A(d)$ let
\[ \mu_g(C) = \dim C - \max\{\dim O(M) \mid M \in C\} \]
be the *generic number of parameters* of $C$. Thus $\mu_g(C) = 0$ if and only if $C$
contains a dense orbit $O(M)$. For example, if $P$ is a projective $A$-module,
then $\operatorname{Ext}_A^1(P, P) = 0$. This implies that the closure of the orbit $O(P)$ is an
irreducible component, and we get $\mu_g(O(P)) = 0$. Also, if $C = C_1 \oplus \cdots \oplus C_t$
with $\operatorname{Ext}_A^1(C_i, C_j) = 0$ for all $i \neq j$, then
\[ \mu_g(C) = \sum_{i=1}^t \mu_g(C_i). \]

3. The finite and tame cases

As in the introduction let $Q$ be a Dynkin quiver. Then $\Lambda$ is of finite
representation type if and only if $Q$ is of type $A_i$, $i \leq 4$. In this case, if
$\{C_i \subseteq \Lambda(d_i) \mid i \in I\}$ is a maximal set of pairwise different indecomposable
irreducible components such that $\operatorname{Ext}_\Lambda^1(C_i, C_j) = 0$ for all $i, j$, then $|I| = |R^+|$. This follows from [3] for $i \leq 3$, and the case $i = 4$ was done by Marsh
and Reineke.

Recall that for a tame algebra $A$ one has $\mu_g(C) \leq 1$ for any indecompos-
able irreducible component $C \subseteq \operatorname{mod}_A(d)$. It is known that $\Lambda$ is of tame
representation type if and only if $Q$ is of type $A_5$ or $D_4$. In this case, a complete classification of the indecomposable irreducible components, and a necessary and sufficient condition for $\text{ext}^1_{\Lambda}(C, D) = 0$ for any two irreducible components $C$ and $D$ was obtained in [3]. In particular, this implies the following:

**Theorem 3.1.** Assume that $Q$ is of type $A_5$ or $D_4$. Then the following hold:

1. For any irreducible component $C \subseteq \Lambda(d)$ we have $\text{ext}^1_{\Lambda}(C, C) = 0$;
2. If $C \subseteq \Lambda(d)$ is an indecomposable irreducible component, then we have $\mu_g(C) = 0$ or $\mu_g(C) = 1$. For suitable $d$ there exists an indecomposable irreducible component $C \subseteq \Lambda(d)$ with $\mu_g(C) = 1$;
3. Let $\{C_i \subseteq \Lambda(d_i) \mid i \in I\}$ be a maximal set of pairwise different indecomposable irreducible components such that $\text{ext}^1_{\Lambda}(C_i, C_j) = 0$ for all $i, j$. Then there is at most one $C_i$ with $\mu_g(C_i) = 1$. In this case, we have $|I| = |R^+| - 1$, and we get $|I| = |R^+|$, otherwise.

This leads us to the following conjecture for arbitrary Dynkin quivers of type $A_n$, $D_n$ or $E_{6,7,8}$:

**Conjecture 3.2.** If $\{C_i \subseteq \Lambda(d_i) \mid i \in I\}$ is a maximal set of pairwise different indecomposable irreducible components such that $\text{ext}^1_{\Lambda}(C_i, C_j) = 0$ for all $i, j$, then

$$|I| = |R^+| - \sum_{i \in I} \mu_g(C_i).$$

In all remaining cases the algebra $\Lambda$ is of wild representation type. So one should expect irreducible components $C$ with $\text{ext}^1_{\Lambda}(C, C) \neq 0$. Thus, maybe one should study sets $\{C_i \subseteq \Lambda(d_i) \mid i \in I\}$ of irreducible components with the weaker condition $\text{ext}^1_{\Lambda}(C_i, C_j) = 0$ for all $i \neq j$. However, we do not know how to generalize Theorem 1.1 to this case.

4. **Proof of Theorem 1.1**

As before let $Q$ be a Dynkin quiver, and let $R^+ = \{a_i \mid 1 \leq i \leq N\}$ be the set of positive roots of $Q$. By Gabriel’s Theorem there is a 1-1 correspondence between the isomorphism classes of indecomposable $kQ$-modules and the elements in $R^+$. This correspondence associates to a root $a_i$ the isomorphism class $[M(a_i)]$ of an indecomposable $kQ$-module $M(a_i)$ with dimension vector $a_i$. By the Theorem of Krull-Remak-Schmidt each $kQ$-module is isomorphic to a (up to reordering) unique direct sum of the indecomposable modules $M(a_i)$. The maps

$$\alpha = (\alpha_1, \ldots, \alpha_N) \mapsto O(M_\alpha) \mapsto C_\alpha \mapsto b^*(C_\alpha)$$

define 1-1 correspondences between $\mathbb{N}^N$, the set of orbits $O(M) \subseteq \text{mod}_kQ(d)$, $d \in \mathbb{N}^n$, the set of irreducible components of $\Lambda(d)$, $d \in \mathbb{N}^n$, and the set $B^*$
of dual canonical basis vectors, where

\[ M_\alpha = \bigoplus_{i=1}^{N} M(a_i)^{\alpha_i}, \]

\[ C_\alpha = \pi^{-1}(\mathcal{O}(M_\alpha)) \]

with

\[ \pi : \Lambda(d) \to \text{mod}_{kQ}(d) \]

the canonical projection map.

Let \( \alpha, \beta \in \mathbb{N}^N \). By Theorem 2.2 the closure \( C_\alpha \oplus C_\beta \) is an irreducible component if and only if \( \text{ext}^1_\Lambda(C_\alpha, C_\beta) = \text{ext}^1_\Lambda(C_\beta, C_\alpha) = 0 \). In this case, we have \( C_\alpha \oplus C_\beta = C_{\alpha + \beta} \).

Let \( \alpha_i = (\alpha_{i1}, \ldots, \alpha_{i,N+1}) \), \( 1 \leq i \leq N + 1 \), be non-zero pairwise different elements in \( \mathbb{N}^N \) such that \( C_{\alpha_i} \) is an indecomposable irreducible component for all \( i \). To get a contradiction, we assume that \( \text{ext}^1_\Lambda(C_{\alpha_i}, C_{\alpha_j}) = 0 \) for all \( 1 \leq i, j \leq N + 1 \). For all \( \mathbf{m} = (m_1, \ldots, m_{N+1}) \in \mathbb{N}^{N+1} \) define

\[ C_{\mathbf{m}} = \sum_{i=1}^{N+1} m_i \alpha_i. \]

We get

\[ C_{\mathbf{m}} = C_{\mathbf{m}_1} \oplus \cdots \oplus C_{\alpha_{N+1}}^{m_{N+1}}. \]

Since the \( C_{\alpha_i} \) are indecomposable, the above is the canonical decomposition of the irreducible component \( C_{\mathbf{m}} \).

We claim that there exist some elements \( \mathbf{m} = (m_1, \ldots, m_{N+1}) \neq \mathbf{l} = (l_1, \ldots, l_{N+1}) \) in \( \mathbb{N}^{N+1} \) such that

\[ \sum_{i=1}^{N+1} m_i \alpha_i = \sum_{i=1}^{N+1} l_i \alpha_i. \]

This implies \( C_{\mathbf{m}} = C_{\mathbf{l}} \). Thus, we get a contradiction to the unicity of the canonical decomposition of irreducible components, see Theorem 2.1.

Let \( \Delta = (\alpha_{ij}) \) be the \( N \times (N + 1) \)-matrix where the \( j \)th column is just the vector \( \alpha_j \). Thus we have to find some \( \mathbf{d} \in \mathbb{N}^N \) and some \( \mathbf{m} \neq \mathbf{l} \) in \( \mathbb{N}^{N+1} \) such that

\[ \Delta \mathbf{m} = \Delta \mathbf{l} = \mathbf{d}. \]

Since all entries in \( \Delta \) are in \( \mathbb{N} \), this would imply that \( \mathbf{m}, \mathbf{l} \) and \( \mathbf{d} \) are all non-zero. Furthermore, there must be a non-zero element \( \mathbf{z} = (z_1, \ldots, z_{N+1}) \in \mathbb{Z}^{N+1} \) such that \( \Delta \mathbf{z} = \mathbf{0} \).

First, we consider the case \( \mathbf{z} \in \mathbb{N}^{N+1} \). Let \( \mathbf{m} = (1, \ldots, 1) \in \mathbb{N}^{N+1} \), \( \mathbf{d} = \Delta \mathbf{m} \) and \( \mathbf{l} = \mathbf{m} + \mathbf{z} \). Obviously, \( \mathbf{d} \in \mathbb{N}^N \). We get \( \Delta \mathbf{m} = \Delta \mathbf{l} = \mathbf{d} \) with \( \mathbf{m} \neq \mathbf{l} \) in \( \mathbb{N}^{N+1} \).

Next, assume that \( \mathbf{z} \notin \mathbb{N}^{N+1} \). Define \( \lambda = -\min\{z_i \mid 1 \leq i \leq N + 1\} \). Let \( \mathbf{m} = (\lambda, \ldots, \lambda) \in \mathbb{N}^{N+1} \), \( \mathbf{d} = \Delta \mathbf{m} \) and \( \mathbf{l} = \mathbf{m} + \mathbf{z} \). Again, we get \( \mathbf{d} \in \mathbb{N}^N \).
and $\Delta m = \Delta l = d$ with $m \neq l$ in $\mathbb{N}^{N+1}$. This finishes the proof of Theorem 1.1.

Corollary 1.2 follows immediately from the fact that an orbit $O(N) \subseteq \text{mod}_A(d)$ of an $A$-module $N$ is open provided $\text{Ext}^1_A(N, N) = 0$. Clearly, $O(N)$ is open if and only if the closure $\overline{O(N)}$ is an irreducible component. Then we use Theorems 1.1 and 2.2.

5. Interpretation of Leclerc’s Example

In the following, we use the notation introduced at the beginning of Section 4.

Reineke proved in [10, Lemma 4.6] that the multiplicativity of $b^*(C_\alpha)$ and $b^*(C_\beta)$ implies that

$$b^*(C_\alpha)b^*(C_\beta) = v^m b^*(C_{\alpha+\beta})$$

for some $m \in \mathbb{Z}$. He also showed that $\lambda^E_{C,D} \neq 0$ if and only if $\lambda^E_{D,C} \neq 0$. This follows from [10, Proposition 4.4]. Thus one direction of Conjecture 1.3 holds, namely if two dual canonical basis vectors are multiplicative, then they are quasi-commutative. The following related problem should be of interest:

**Problem 5.1.** Describe the elements $\alpha, \beta \in \mathbb{N}^N$ such that

$$C_{\alpha+\beta} = C_\alpha \oplus C_\beta.$$  

As mentioned in the introduction, Leclerc recently constructed in [5] counterexamples for the other direction of the Berenstein-Zelevinsky Conjecture. We give a module theoretic interpretation of one of his examples:

Let $Q$ be the quiver of type $\bar{A}_5$ with arrows $a_i : i + 1 \rightarrow i$, $1 \leq i \leq 4$. Thus $\Lambda$ is given by the quiver

1 $\alpha_1$ 2 $\alpha_2$ 3 $\alpha_3$ 4 $\alpha_4$ 5

and the following set of relations

$$\{\bar{a}_1 a_1, a_1 \bar{a}_1 - \bar{a}_2 a_2, a_2 \bar{a}_2 - \bar{a}_3 a_3, a_3 \bar{a}_3 - \bar{a}_4 a_4, a_4 \bar{a}_4\}.$$  

Now $R^+$ contains exactly 15 elements, namely for each $1 \leq i \leq 5$ there is a positive root $[i, j] = (d_l)_{1 \leq l \leq 5}$ with $d_l = 1$ for $i \leq l \leq j$, and $d_l = 0$, else. We identify $\mathbb{N}R^+$ with $\mathbb{N}^{15}$ by fixing a linear ordering on $R^+$, namely let

$$[1, 1] < [1, 2] < [1, 3] < [1, 4] < [1, 5] < [2, 2] < [2, 3] < [2, 4] < [2, 5] < [3, 3] < [3, 4] < [3, 5] < [4, 4] < [4, 5] < [5, 5].$$

Define

$$\alpha = [1, 2] + [2, 4] + [3, 3] + [4, 5], \quad \beta = [1, 2] + [1, 4] + [2, 3] + [2, 5] + [3, 4] + [4, 5].$$
Thus, regarded as elements in \( \mathbb{N}^{15} \) we have
\[
\alpha = (0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0), \\
\beta = (0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0).
\]
In [5] Leclerc showed that
\[
b^*(C_\alpha)^2 = v^{-2} \left( b^*(C_{\alpha + \alpha}) + b^*(C_\beta) \right).
\]
This is obviously a counterexample to the Berenstein-Zelevinsky Conjecture.

Now define
\[
\beta_1 = [1, 2] + [2, 3] + [3, 4] + [4, 5], \\
\beta_2 = [1, 4] + [2, 5].
\]
Thus we have \( \beta = \beta_1 + \beta_2 \).

**Proposition 5.2.** Let \( \alpha, \beta, \beta_1, \beta_2 \) be as above. Then the following hold:

1. The irreducible components \( C_\alpha, C_{\beta_1} \) and \( C_{\beta_2} \) are indecomposable with \( \mu_g(C_\alpha) = 1 \) and \( \mu_g(C_{\beta_1}) = \mu_g(C_{\beta_2}) = 0 \);
2. We have \( C_{\alpha + \alpha} = C_\alpha \oplus C_\alpha \) and \( C_\beta = C_{\beta_1} \oplus C_{\beta_2} \). Thus
\[
b^*(C_\alpha)^2 = v^{-2} \left( b^*(C_\alpha \oplus C_\alpha) + b^*(C_{\beta_1} \oplus C_{\beta_2}) \right).
\]

**Proof.** For \( \lambda \in k \setminus \{0, 1\} \) let \( M_\lambda \) be the 8-dimensional \( \Lambda \)-module where the arrows of \( \Lambda \) operate on a basis \( \{1, \cdots, 8\} \) as in the following picture:

![Diagram](image)

Thus, for example \( 2 \cdot a_1 = 1, 2 \cdot a_2 = 4 + \lambda 5, 6 \cdot a_3 = 4 + 5, 1 \cdot a_1 = 3 \), etc. Note that \( M_\lambda \) lies in \( C_\alpha \).

The modules \( M_\lambda \) are indecomposable and \( \dim \text{End}_\Lambda(M_\lambda) = 3 \). From a well-known general fact we know that each irreducible component of \( \Lambda(1, 2, 2, 2, 1) \) has dimension \( 2+4+4+2 = 12 \), see for example [5, Section 12]. The group \( \text{GL}(1, 2, 2, 2, 1) \) acts as described in Section 2 on \( \Lambda(1, 2, 2, 2, 1) \) and has dimension 14. Thus we get \( \dim \mathcal{O}(M_\lambda) = 14 - 3 = 11 \). One checks easily that \( M_\lambda \) and \( M_\mu \) are isomorphic if and only if \( \lambda = \mu \). This implies
\[
\dim \left\{ \mathcal{O}(M_\lambda) \mid \lambda \in k \setminus \{0, 1\} \right\} = 11 + 1 = 12.
\]
We get
\[
C_\alpha = \left\{ \mathcal{O}(M_\lambda) \mid \lambda \in k \setminus \{0, 1\} \right\}.
\]
Thus \( C_\alpha \) is an indecomposable irreducible component with \( \mu_g(C_\alpha) = 1 \).
Next, let $P_2$ and $P_4$ be the indecomposable projective $\Lambda$-modules corresponding to the vertices 2 and 4, respectively. These modules are both 8-dimensional and look as in the following picture:

We have $\text{Ext}^1_\Lambda(P_i, P_j) = 0$ for all $i, j \in \{2, 4\}$. This follows directly from the projectivity of both modules. From this and the above pictures we get

\[
C_{\beta_1} = \mathcal{O}(P_2), \\
C_{\beta_2} = \mathcal{O}(P_4), \\
C_{\beta} = C_{\beta_1} \oplus C_{\beta_2}.
\]

In particular, $C_{\beta_1}$ and $C_{\beta_2}$ are indecomposable irreducible components with $\mu_g(C_{\beta_1}) = \mu_g(C_{\beta_2}) = 0$. This finishes the proof.

For irreducible components $C \subseteq \Lambda(\mathbf{d})$ and $D \subseteq \Lambda(\mathbf{e})$ define

\[
\mathcal{V}(C, D) = \bigcap_{U \subseteq C, V \subseteq D} \left\{ E \subseteq \Lambda(\mathbf{d} + \mathbf{e}) \text{ irred. comp.} \mid E \subseteq \mathcal{E}(U \times V) \right\},
\]

where $U$ (resp. $V$) runs through all non-empty $\text{GL}(\mathbf{d})$-stable (resp. $\text{GL}(\mathbf{e})$-stable) open subsets of $C$ (resp. $D$), see Section 3 for the definition of $\mathcal{E}(U \times V)$.

Using the previous proposition, and some well-known results on the representation theory of the algebra $\Lambda$, see [3] and [11], one can show that

\[
\mathcal{V}(C_\alpha, C_\alpha) = \{C_{\alpha+\alpha}, C_\beta\}.
\]

Note that $\text{Ext}^1_\Lambda(C_\alpha, C_\alpha) = 0$, since $\text{Ext}^1_\Lambda(M_\lambda, M_\mu) = 0$ for all $\lambda \neq \mu$. But one can show that $\dim \text{Ext}^1_\Lambda(M_\lambda, M_\lambda) = 2$, see [3, Section 6]. For any $M_\lambda$ there is a short exact sequence

\[
0 \to M_\lambda \to M_\lambda(2) \to M_\lambda \to 0,
\]

where $M_\lambda(2)$ is the module of quasi-length two in the same Auslander-Reiten component as $M_\lambda$ (it is known that $M_\lambda$ lies in a homogeneous tube). Additionally to this ‘natural’ self-extension, there exists a short exact sequence

\[
0 \to M_\lambda \to P_2 \oplus P_4 \to M_\lambda \to 0.
\]
Motivated by our above analysis, one might ask whether the following is true:

**Question 5.3.** Are the following two statements equivalent?

(1) $\lambda_{E,C,D}^E \neq 0$;
(2) $E \in \mathcal{V}(C, D) \cup \mathcal{V}(D, C)$.

A positive answer to the above question would imply the following:

- If an irreducible component $C$ contains an open orbit, then
  \[ b^*(C)^2 = v^m b^*(C \oplus C) \]
  for some $m \in \mathbb{Z}$. Here we use that $\text{Ext}^1_{\Lambda}(M, M) = 0$ if and only if $O(M)$ is an open orbit. This is a special feature of preprojective algebras.

- If irreducible components $C$ and $D$ contain non-empty stable open subsets $U \subseteq C$ and $V \subseteq D$ such that $\text{Ext}^1_{\Lambda}(M, N) = 0$ for all $M \in U, N \in V$, then $b^*(C)$ and $b^*(D)$ are multiplicative. Here we use [2, Theorem 1.3].

- If an irreducible component $C$ contains an open orbit, and if $D$ is an irreducible component such that $\text{ext}^1(C, D) = 0$, then $b^*(C)$ and $b^*(D)$ are multiplicative. Again this uses [2, Theorem 1.3].

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EXTENSION-ORTHOGONAL COMPONENTS OF NILPOTENT VARIETIES

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