Geometric Quantization of free fields in space of motions

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Abstract

Via Kähler polarization we geometrically quantize free fields in the spaces of motions, namely the space of solutions of equations of motion. We obtain the correct results just as that given by the canonical quantization. Since we follow the method of covariant symplectic current proposed by Crnkovic, Witten and Zuckerman et al, the canonical commutator we obtained are naturally invariant under proper Lorentz transformation and the discrete parity and time transverse transformations, as well as the equations of motion.

1 Introduction

The symplectic geometrical description of classical mechanics and geometric quantization are essentially globalizations of, respectively, Hamiltonian mechanics and canonical quantization. Geometric quantization is also considered as a so far most mathematically

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thorough approach to quantization. This formalism has been shown to provide an natural way to investigate global and geometrical properties of physical systems with geometrical invariance, such as Chern-Simons theory [3], anyon system [1], and so on. The method of geometry also provide principles by itself to clarify some ambiguities in traditional canonical quantization.

It is well known, not as the path integral which can keep the classical symmetries very well, the traditional descriptions of geometrical and canonical formalism of classical theories are not manifestly covariant because one has to explicitly single out a ”time” coordinate to define the canonical conjugate momenta and the initial data of systems. In the end of eighties E.Witten [5] and G.Zuckerman [7] and C.Crnkovic [6] et al. suggested a manifestly covariant geometric descriptions, where the space of solutions of the equation of motion (called space of motion $M$) is taken as the state space[2]. This definition is independent of any special time choice so that is manifestly covariant. Then this method were used to discuss Yang-Mills theory, general relativity, string theory and theory of supersymmetry. The presymplectic form constructed by covariant symplectic current is not only invariant under Lorentz transformation, gauge transformation and diffeomorphism transformation but also has zero component along gauge and diffeomorphism orbits.

In order to use the symplectic structure to study the quantization of fields theory, it is convenient to express the symplectic form in terms of ladder fields in momenta space by the Fourier decompositions of fields. This will be discussed in this paper in detail. This step is necessary for K"ahler polarization [11] in geometric quantization. Via the covariant method we directly obtain the Poisson brackets of fields. The symplectic form is invariant under proper Lorentz transformation and the discrete parity and time transverse transformations(LPT), so naturally are the Poisson brackets and the corresponding quantum commutator.

This paper is organized as followings. In section 2 we review the Crnkovic, Zuckerman and Witten’s descriptions of space of motion. In section 3, we obtain the expressions of symplectic form in the solution space in terms of ladder fields in momentum space of free fields. Then we calculate the Poisson brackets of fields by the symplectic form. Finally we
complete the geometric quantization in space of motion via the Kähler polarization.

2 Crnkovic-Witten-Zuckerman’s Covariant Symplectic Current Description

In this section we will briefly review the description of covariant phase space developed by E.Witten [5], G.Zuckerman [7] and C.Crnkovic [6] et al. With $\phi^a$ a collection of fields which form a representation of its symmetry group of the theory the variation of a local Lagrangian density $L = L(\phi^a, \partial_\mu \phi^a)$ is

$$\delta L = \partial_\mu j^\mu + (E - L)_a \delta \phi^a, \quad (1)$$

where

$$j^\mu = \frac{\partial L}{\partial \partial_\mu \phi^a} \delta \phi^a, \quad (2)$$

and

$$(E - L)_a = \frac{\partial L}{\partial \phi^a} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi^a} \quad (3)$$

the extreme of the action leads to the Euler-Lagrange equations $(E - L)_a = 0$. Now the phase space is defined to be the space $Z$ of solutions of the Euler-Lagrange equations which is one to one correspondence to the traditional phase space with a fixed time. Before introducing the symplectic form on $Z$, we first define the vector field and the differential forms on $Z$. For simplicity, we assume field $\phi^a$ to be a real scalar field $\phi$. For a fixed $x \in M$ with $M$ the base manifold on which the field $\phi$ is defined, a mapping from $Z$ to real numbers that assign to every function $\phi$ its value at $x$ is a function on $Z$. a function $\hat{x} : Z \mapsto R$, where $\hat{x}(\phi) = \phi(x)$.

For every $\phi \in Z$, every small displacement $\delta \phi$ which is the solution of the linearized equation of motion, is defined to be the vector of the tangent space $T_\phi Z$ to $Z$ at $\phi$. The tangent vector field $\Delta$ is the section of tangent bundle $T_Z$.

1-forms, being elements of the dual vector space to the tangent vector field, will map vector fields $\Delta$ into functions on $Z$. For vector field $\Delta$ and $x \in M$, we define functions
\( \triangle_x \) on \( Z \)

\[
\triangle_x(x) \equiv \delta \phi(x)
\]

where \( \delta \phi(x) \) is a number, the value of the displacement \( \delta \phi \) at \( x \). We define the 1-form \( x^* \) on \( Z \) by demanding the contraction of \( x^* \) with \( \triangle \) gives function \( \triangle_x \)

\[
x^*(\triangle) = \triangle_x
\]

The variation \( \delta \) in (1) can be interpreted as the infinite dimensional exterior derivative operator and \( \delta \phi \) the one-form on phase space. So in the following for simplicity we just use \( \delta \phi \) as both vector in \( T_Z \) and 1-form in \( T^*Z \) and \( \delta \) as the exterior derivative. There will not be confusion. The properties of differential forms on \( T^*Z \) is similar to that of the finite dimensional case.

Now the covariant symplectic current is defined to be

\[
\delta j^\mu = \delta \frac{\partial L}{\partial \partial_{\mu} \phi^a} \wedge \delta \phi^a.
\]  

(4)

With \( \Sigma \) to be the spacelike supersurface of space-time manifold, the presymplectic form is defined to be

\[
\Omega = \int d\Sigma_\mu \delta j^\mu,
\]  

(5)

which is obviously closed in the covariant phase space. To ensure the presymplectic form is well defined on solution space \( Z \), it is needed to prove that \( \Omega \) is independent of the choice of the spacelike supersurface. Consider the presymplectic form defined on two spacelike surface \( \Sigma_1 \) and \( \Sigma_2 \), we get

\[
\Omega(\Sigma_1) - \Omega(\Sigma_2) = \int d\Sigma_1 \mu \delta j^\mu - \int d\Sigma_2 \mu \delta j^\mu = \int_V dV \partial_{\mu} \delta j^\mu - \int \Sigma_3 \delta j^\mu
\]  

(6)

where \( V \) is the four dimensional area enveloped by \( \Sigma_1, \Sigma_2 \), and \( \Sigma_3, \Sigma_3 = \partial V - \Sigma_2 - \Sigma_1 \) is a time-like surface at infinity. Using (1), the above expression becomes

\[
\Omega(\Sigma_1) - \Omega(\Sigma_2) = \int_V dV \delta \delta L - \delta((E - L)_a \delta \phi^a) - \int \Sigma_3 \delta j^\mu
\]  

(7)

Obviously the first term vanishes because \( \delta \delta = 0 \), the second term vanishes because in solution space equations of motion \( (E - L)_a \) are satisfied, and the last term vanishes.
because the fields tend to zero fast enough, so we have

$$\Omega(\Sigma_1) - \Omega(\Sigma_2) = 0,$$

(8)

which ensure $\Omega$ is well defined on solution space.

Now we give the real scalar field as a simple examples to illustrate the idea.

The Lagrangian density of scalar field theory is

$$L = \frac{1}{2} (\partial^\alpha \phi \partial_{\alpha} \phi - V(\phi)),$$

(9)

from which the equation of motion

$$\partial_\mu \partial^\mu \phi + V'(\phi) = 0.$$

(10)

be obtained. Now take a solution of the equation of motion as a point of the phase space $Z$. Thus we can define a function on $Z$ by $\phi(x) : \phi \mapsto \phi(x) \in R$. The small displacements $\delta \phi$ is determined by the linearized equation of motion.

$$\partial_\mu \partial^\mu \delta \phi + V''(\phi) \delta \phi = 0.$$

(11)

From eq. (4) and (9) the symplectic current is

$$\delta J^\mu(x) = \delta \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \wedge \delta \phi(x) = \delta (\partial^\mu \phi(x)) \wedge \delta \phi(x).$$

(12)

Then the presymplectic form is

$$\Omega = \int_\Sigma d\Sigma_\mu(x) \delta J^\mu(x) = \int_\Sigma d\Sigma^\mu \delta (\partial_\mu \phi(x)) \wedge \delta \phi(x),$$

(13)

where $\Sigma$ is a space-like hypersurface of Minkowski space-time. Being a closed and non-degenerate two form, the expression in (13) gives a symplectic structure for scalar field on $Z$.

The above formulation has been used to the Yang-Mills theory[7][8] and general relativity and the gravitational WZW-model[10] and the string theory[11]. In the next section we will use it to Palatini and Ashtekar gravity[12].
3 Geometric Quantization of Free Fields in Space of Motions

In the previous section, the classical symplectic description in space of motions has been established. In this section, we will complete geometric quantization of free fields in solution space via Kähler polarization.

From the Lagrangian density of free real scalar field given in (9), the Hamiltonian density
\[ h = \frac{1}{2} (\partial^\alpha \phi \partial_\alpha \phi + m^2 \phi^2); \]

is obtained.

The general solutions of equation of motion are:
\[ \varphi(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2\epsilon(k)}} (a(\vec{k})e^{-ikx} + a^*(\vec{k})e^{ikx}), \]

where \( \epsilon(\vec{k}) = k^0 = \sqrt{\vec{k}^2 + m^2} \) and \( kx = k_\mu x^\mu = k^0 t - \vec{k} \cdot \vec{x} \) and \( a(\vec{k}) \), complex conjugate to \( a^*(\vec{k}) \), is
\[ a(\vec{k}) = \frac{1}{\sqrt{2}} (k^0 \varphi(\vec{k}) + i\pi(\vec{k})) \]
\[ = (2\pi)^{-3/2} \int \frac{dp_x}{\sqrt{2\epsilon(k)}} [k^0 \varphi(x) + i\dot{\varphi}(x)] e^{ikx}. \]

So the solution space \( M_0 \) can be coordinated by \((a^*(\vec{k}), a(\vec{k}))\). Putting the solution (13) into (12), we derive the symplectic form in \( M_0 \) for real scalar fields
\[ \omega = i \int d^3k \delta a^*(\vec{k}) \wedge \delta a(\vec{k}), \]

where antisymmetry of the infinity exterior operator \( \delta \wedge \delta \) have been used. Eq. (17) explicitly shows that \( \omega \) on solutions space of free real scalar field is a well defined non-degenerate closed 2-form. This expression is an analogue of the finite dimensional harmonic oscillator whose symplectic form is
\[ \epsilon = dp_i \wedge dq^i = ida_i^* \wedge da_i. \]

The Hamiltonian vector field of function \( f \) is defined as
\[ i_{X_f} \omega = -\delta f, \]
where $i$ here is used to represent the contraction between vector field and differential forms in infinite space. So, the Hamiltonian vector field of $a(\vec{k})$ and $a^*(\vec{k})$ can be obtained immediately

$$X_{a(\vec{k})} = i \frac{\delta}{\delta a^*(\vec{k})},$$  \hspace{1cm} (20)

$$X_{a^*(\vec{k})} = -i \frac{\delta}{\delta a(\vec{k})}. \hspace{1cm} (21)$$

(20) and (21) implies the basic Poisson bracket

$$\{a^*(\vec{k}), a(\vec{k}')\} = -i\omega (X_{a^*(\vec{k})}, X_{a(\vec{k}')}) = X_{a^*(\vec{k})} a(\vec{k}') = -i\delta^3(\vec{k} - \vec{k}').$$  \hspace{1cm} (22)

which lead to commutator distribution [9]

$$\triangle(x, y) = \{\varphi(x), \varphi(y)\}$$

$$= i(D(x - y) - D(y - x))$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3k(\vec{k})}{k^0} \sin k(x - y),$$  \hspace{1cm} (23)

where

$$D(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k(\vec{k})}{2k^0} e^{-ikx}. \hspace{1cm} (24)$$

making use of solution of free equation (15) and density expressions (14), the Hamiltonian become

$$H = \int d^3k(\vec{k}) a^*(\vec{k}) a(\vec{k}),$$  \hspace{1cm} (25)

whose Hamiltonian vector field is obtained as

$$X_H = i \int d^3k(\vec{k}) [a^*(\vec{k}) \frac{\delta}{\delta a^*(\vec{k})} - a(\vec{k}) \frac{\delta}{\delta a(\vec{k})}].$$  \hspace{1cm} (26)

Next, we explicitly write down the symplectic structures of Maxwell field with Lagrangian density

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \hspace{1cm} (27)$$

with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The presymplectic form is then defined as

$$\Omega = \int d^3x [\delta \partial^\mu A^\nu \wedge \delta A_\nu + \partial^\nu \delta A^\mu \wedge \delta A_\mu].$$  \hspace{1cm} (28)
Using the Lorentz condition $\partial_\mu A^\mu = 0$ and, omitting the integral on boundary, (28) becomes
\[ \Omega = \int d\Sigma \delta \partial_\mu A^\nu \wedge \delta A_\nu. \] (29)
Make use of solutions of equations of motion
\[ A_\mu(x) = (2\pi)^{-3/2} \int \frac{d^3k}{2\epsilon(k)} (c_\mu(\vec{k}) e^{-ikx} + c^*_\mu(\vec{k}) e^{ikx}), \] (30)
the presymplectic form Eq. (29) can be written as
\[ \omega = i \int d^3k \delta c^*_\mu(\vec{k}) \wedge \delta c^\mu(\vec{k}). \] (31)
Similar to (23) for scalar field, the commutator distribution are
\[ \{A_\mu(x), A_\nu(y)\} = i\eta_{\mu\nu}(D(x-y) - D(y-x)), \] (32)
where $D(x-y)$ is defined as before.

As pointed out in the previous section, the presymplectic form (29) degenerates along the $U(1)$ gauge orbits. Therefore only when it restricts to module space, (29) can be defined as symplectic form. In the solution (momentum) space Lorentz constraint becomes
\[ k_\mu c^\mu(\vec{k}) = 0; \quad k_\mu c^\mu(\vec{k}) = 0. \] (33)
We choose the polarized coordinate of photon
\[ k_\mu = (k, -k, 0, 0). \] (34)
Then we obtain from Eq. (34) and (33)
\[ c^0(\vec{k}) = c^3(\vec{k}) = -c_3(\vec{k}); \quad c^*0(\vec{k}) = c^{*3}(\vec{k}) = -c^*_3(\vec{k}). \] (35)
So the symplectic form is
\[ \omega = i \int d^3k \delta c^*_i(\vec{k}) \wedge \delta c_i(\vec{k}), \quad i = 1, 2, \] (36)
which implies only the transverse degrees of freedom have contribution to $\omega$ which is consistent with the degeneracy of $\omega$ along non-physical components.
The third example is free Dirac field with Lagrangian density
\[ \mathcal{L} = \bar{\psi}(i\hat{\partial} - m)\psi \] (37)

where \( \hat{\partial} = \gamma^\mu \partial_\mu \). Here \( \psi \) and \( \bar{\psi} \) are valued in Grassmannian numbers. The general solutions of equation are
\[
\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} (e^{-ipx}a_i(\vec{p})u_i(\vec{p}) + e^{ipx}b_i^*(\vec{p})v_i(\vec{p})), \tag{38}
\]
\[
\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} (e^{ipx}a_i^*(\vec{p})\bar{u}_i(\vec{p}) + e^{-ipx}b_i(\vec{p})\bar{v}_i(\vec{p})). \tag{39}
\]

The solution manifold \( M_F \) is Grassmannian\([13]\) which is coordinated by \( (a_i^*, a_i, b_i^*, b_i) \). The functional and vector field and differential forms on Grassmannian manifold \( M_F \) can also be defined. For example, the Hamiltonian is
\[
H = \int d^3p E_p (a_i^*a_i + b_i^*b_i) \tag{40}
\]

The symplectic 2-form is then defined as
\[
\omega = \int d^3x \delta i \gamma^0 \delta \bar{\psi}. \tag{41}
\]

substituting (38) and (39) in (41) and using relation
\[
\bar{u}_i \gamma^0 u_j(\vec{p}) = u_i^+ u_j = 2E_p \delta_{ij} \tag{42}
\]
\[
\bar{v}_i \gamma^0 u_j(\vec{p}) = v_i^+ v_j = 2E_p \delta_{ij}, \tag{43}
\]
we obtain
\[
\omega = \int d^3p (\delta a_i^* \wedge \delta a_i + \delta b_i^* \wedge \delta b_i) \tag{44}
\]

where we have used the antisymmetry of both differential forms and Grassmanian numbers. The Hamiltonian vector field \( X_f \) is defined as
\[
i_{X_f} \omega = -\delta f \tag{45}
\]

Then the anticommutator distribution can be obtained
\[
\{\psi(x), \psi(y)\} = i(i \hat{\partial}_x + m)(D(x - y) - D(y - x)). \tag{46}
\]
Now we perform the quantization of free real scalar field via Kähler polarization. The positive polarization $P = F_B$ (B means Bargmann-Fock) is spanned by the frame fields $X_a(\vec{k})$ in Eq.(20). From eqs(20)(21) it is obvious that

$$F_B \cap \overline{F_B} = 0, \quad F_B \cup \overline{F_B} = \mathcal{T}^cX,$$

where $\mathcal{T}^cX$ is the tangent space of the complexized phase space $X$ coordinated by $(k^0, \varphi(\vec{k}), \pi(\vec{k}))$. So the polarization is strongly admissible and complete. Being a manifold of $R^{6\infty}$, $X$ is contractible formally. The metalinear frame fields of $\overline{F_B}$ is defined to be $\tilde{X}_a(\vec{k})$ who covers $X_a(\vec{k})$ and the half form bundle is defined to be $\delta_{-\frac{1}{2}}(\mathcal{P})$ whose section is denoted by $\nu_{\tilde{X}_a(\vec{k})}$ satisfying

$$\nu_{\tilde{X}_a(\vec{k})}^\# \cdot \tilde{X}_a(\vec{k}) = 1,$$

where $\#$ denotes the horizontal lift. Then we denote $(B_0, \nabla, <, >)$ the prequantization structure with $B_0$ the trivial line bundle and $\nabla$ the covariant derivative and $\lambda$ the trivialization section and $<, >$ the inner product in the fiber, respectively. The prequantization structure satisfies

$$\nabla \lambda_0 = -i \frac{1}{\hbar} \int d^3k \pi(\vec{k}) \wedge \delta(k^0, \varphi(\vec{k})) \lambda_0,$$

where $\lambda_0$ is unit section satisfying

$$< \lambda_0, \lambda_0 > = 1.$$

Introducing the new trivialization section $\lambda_1$ of $B_0$ by

$$\lambda_1 = \exp[-\frac{1}{4\hbar} \int d^3k ((k^0, \varphi(\vec{k}))^2 + \pi(\vec{k})^2 - 2i k^0 \varphi(\vec{k}) \pi(\vec{k}))] \lambda_0,$$

we have

$$\nabla \lambda_1 = -i \frac{1}{\hbar} \theta_1 \lambda_1,$$

where the connection $\theta_1$ is the symplectic potential by

$$\theta_1 = -i \int d^3k a(\vec{k}) \delta a^*(\vec{k}).$$
Eq. (52) suggests that \( \lambda_1 \) is constant along \( \mathcal{F}_B \), namely
\[
\nabla_{\mathcal{F}_B} \lambda_1 = 0.
\] (54)
So, symplectic potential in (53) is adapted to polarization \( P \). Clearly the section of \( B_0 \otimes \delta_{-\frac{1}{2}}(\mathcal{P}) \), which is constant along \( \mathcal{F}_B \), has the expression
\[
\sigma = \psi(a^*(\vec{k}))\lambda_1 \otimes \nu_{\bar{X}^*(\vec{k})},
\] (55)
where \( \psi(a^*(\vec{k})) \) is the holomorphic functional of \( a^*(\vec{k}) \). We define \( \mathcal{H} \) the representation space corresponding to polarization \( \mathcal{F}_B \) on which the scalar product is given by
\[
<\psi_1(a^*(\vec{k}))\lambda_1 \otimes \nu_{\bar{X}^*(\vec{k})}|\psi_2(a^*(\vec{k}))\lambda_1 \otimes \nu_{\bar{X}^*(\vec{k})}> = \int_{\mathbb{R}^6} \psi_1^*(a^*(\vec{k}))\psi_2(a^*(\vec{k})) \lambda_1, \lambda_1 > \Pi \delta\varphi(\vec{k})\delta\pi(\vec{k}).
\] (56)
From Eqs. (50) and (51)
\[
<\lambda_1, \lambda_1 >= \exp(-\frac{1}{\hbar} \int d^3ka^*(\vec{k})a(\vec{k}))
\] (57)
is obtained. Putting Eqs. (57) into (56), one gets
\[
<\psi_1(a^*(\vec{k}))\lambda_1 \otimes \nu_{\bar{X}^*(\vec{k})}|\psi_2(a^*(\vec{k}))\lambda_1 \otimes \nu_{\bar{X}^*(\vec{k})}> = \int_{\mathbb{R}^6} \psi_1^*(a^*(\vec{k}))\psi_2(a^*(\vec{k})) \exp(-\frac{1}{\hbar} \int d^3ka^*(\vec{k})a(\vec{k})) \Pi \delta\varphi(\vec{k})\delta\pi(\vec{k}).
\] (58)
This result can be also obtained by requiring \( a^*(\vec{k}) \) and \( a(\vec{k}) \) to be self-adjoint on the Hilbert space.

In the Bargmann-Fock representation, like in the finite dimensional harmonic oscillator, the observable functional \( f \) has the representation
\[
\hat{O}_f = -i\hbar \nabla_{X_f} + f - i\hbar \frac{1}{2} \int d^3k n_0(\vec{k}, \vec{k})
\] (59)
where \( n_0(\vec{k}, \vec{k}) \) is determined by
\[
\mathcal{L}_{X_f} \delta a^*(\vec{k}) = \int n_0(\vec{k}, \vec{k}^\prime)\delta a^*(\vec{k}^\prime) d^3\vec{k}^\prime.
\] (60)
Using (26), one gets the Li.e. derivative of \( \delta a^*(\vec{k}) \) along \( X_H \)
\[
\mathcal{L}_{X_H} \delta a^*(\vec{k}) = i\epsilon(\vec{k})\delta a^*(\vec{k}),
\] (61)
which means $X_H$ preserve the polarization and gives

$$n_0(\vec{k}, \vec{k}') = i\epsilon(\vec{k})\delta^3(\vec{k} - \vec{k}'), \quad (62)$$

or the $\mathcal{L}$i.e. derivative of the half form along $X_H$ is

$$\mathcal{L}_{X_H} \sqrt{\delta a^*(\vec{k})} = \frac{i}{2} \epsilon(\vec{k}) \sqrt{\delta a^*(\vec{k})}. \quad (63)$$

So, in the polarization $P = F_B$, one has

$$\hat{O}_H(\lambda_1 \otimes \nu_{\tilde{X}_{a^*(\vec{k})}}) = (-i\hbar \nabla_{X_H} + H - \frac{i}{2} \int d^3k n_0(\vec{k}, \vec{k}')\lambda_1 \otimes \nu_{\tilde{X}_{a^*(\vec{k})}}). \quad (64)$$

From Eqs. (52), (61)

$$-i\hbar \nabla_{X_H} \lambda_1 + H\lambda_1 = 0. \quad (65)$$

Then finally, using Eqs. (62), (64), one obtains

$$\hat{O}_H(\psi(a^*(\vec{k})))\lambda_1 \otimes \nu_{\tilde{X}_{a^*(\vec{k})}}) = \int d^3k \epsilon(\vec{k})\hbar(N(\vec{k}) + \frac{1}{2}\delta^3(0))(\psi(a^*(\vec{k}))\lambda_1 \otimes \nu_{\tilde{X}_{a^*(\vec{k})}}). \quad (66)$$

in which $N(\vec{k}) = a^*(\vec{k})\frac{\delta}{\delta a^*(\vec{k})}$ is the operator representation of particles number. Due to the infinite dimensions of degrees, the zero-point energy $\int d^3k \frac{1}{2}\epsilon(\vec{k})\delta^3(0)$ is divergent, which can be removed by defining normal product. Similarly, the observable momentum has the representation as follows

$$\hat{O}_{\vec{p}} = \int d^3k \hbar(N(\vec{k}) + \frac{1}{2}\delta^3(0)). \quad (67)$$

In the same way, we can obtain the representation of field operator $O(\phi(x))$ and its quantum commutator

$$[O(\phi(x)), O(\phi(y))] = \hbar(D(x - y) - D(y - x)) \quad (68)$$

which is ($-i\hbar$) times classical commutator distribution. We point out here that since in solution space we use covariant symplectic structure which preserve Poincare symmetry, and the procedure we take is in a coordinate-free form, the quantum commutator must also preserve the Poincare symmetry. In traditional canonical quantization, since the
special time is singled out, the Poincare invariance of commutators needs a proof\textsuperscript{[14]}. But here as we adopt the scheme preserving Poincare invariance from the very beginning, the Poincare invariance of commutator is a natural result.

The procedures to deal with Maxwell field and free Dirac field can be performed in a similar way as real scalar field, except considering constraint in Maxwell field and properties of Grassmannian numbers in free Dirac field. The results are the same as canonical quantization in traditional phase space, which will not be listed here.

So far, the geometric quantization of free fields in solution space gives the correct results as that obtained in traditional canonical quantization. The remarkable property is that the Poincare covariance are kept all through the procedure, just as path integral approach does.

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