Strings from Matrices*

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We identify Type IIA and IIB strings, as excitations in the matrix model of M theory.

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1. Introduction

The matrix model of M theory [1-3] purports to be a unified description of all string vacuum states, as well as a nonperturbative quantum mechanical framework for string theory. In this paper we will demonstrate the existence of some of the degrees of freedom that describe these different vacuum states. We will find freely propagating ten dimensional Type IIA and Type IIB strings. Results related to ours have recently been obtained by Sethi and Susskind [4] and by Motl [5].

We begin by recalling and refining the description of Type IIA strings [3] in the matrix model. One compactifies the ninth transverse dimension of the matrix model on a circle of radius $R_9$ by restricting attention to large $N$ matrices of the form

$$X^9 = \frac{1}{i} \frac{\partial}{\partial \sigma} - A(\sigma)$$  \hspace{1cm} (1.1)

$A$ is a $U(M)$ gauge potential, and $M$ goes to infinity. The other matrix degrees of freedom are restricted to be functions of $\sigma$ which transform in the adjoint of $U(M)$. $\sigma$ has period $1/R_9$. This ansatz [3,6,7,8] is motivated by the observation that shifts of $X^9$ by $2\pi R_9$ are gauge transformations (in the full matrix model gauge group, not just its $U(M)$ subgroup) and by extrapolating the description of zero branes in weakly coupled IIA string theory. For higher dimensional tori, one obtains, by analogous arguments, the dimensional reduction of ten dimensional SYM theory to the dual torus.

We would like to emphasize that the local dynamics of the Super Yang Mills (SYM) theory on the dual torus, which encodes the dynamics of the compactified matrix model, is not of direct physical relevance in the matrix model. Indeed, translations of the dual torus coordinate are matrix model gauge transformations by the unitary “matrix” $e^{i\alpha \mathbf{P}}$ ($\mathbf{P}$ is the torus translation generator). We will see that on the subset of matrix model degrees of freedom which represent strings, invariance under this gauge transformation becomes the Virasoro condition $L_0 = \overline{L}_0$ of light cone gauge string theory. States which do not satisfy this condition, i.e. states which carry momentum in the SYM theory, will be interpreted as strings stretched along the longitudinal direction. The true dynamics of M theory corresponds to scattering of SYM excitations in the moduli space of the SYM theory.

Another unusual feature of the SYM theory which arises from the matrix model is that its coupling constant scales as $g_{SYM}^2 \sim R_9^{-1}$, (more generally it scales like the volume of the dual torus). This is because the integral over the dual torus coordinates arises as the limit of the trace in the matrix model. The trace of the unit matrix is the total longitudinal momentum of the system, and should be independent of $R_9$. This is achieved by rescaling
the coupling. The appropriate dimensions of the coupling are made up by powers of the eleven dimensional Planck length, which we set equal to one.

The gauge potential $A$ can be gauged away in one spatial dimension, apart from its Wilson line degree of freedom. As $R_9 \to 0$, the radius of the $\sigma$ circle goes to infinity. In this limit, the quantum dynamics of the Wilson lines is frozen. The field strength variable conjugate to the Wilson line becomes a classical variable $E$, which behaves like a two dimensional $\theta$ angle \[9,10\]. Its allowed values are discrete, corresponding to Casimir operators in various representations of the gauge group (the lowest Casimir operator in each conjugacy class). For a given background electric field $E$, the energy scales like $R_9^{-2}E^2$. The first factor of $R_9^{-1}$ comes from the volume of the dual circle and the second factor of $R_9^{-1}$ from the scaling of the coupling described in the previous paragraph.

We will identify type IIA strings with the degrees of freedom in $U(1)$ subgroups of the $U(M)$ group\[1\]. If we rescale $\sigma$ to go from 0 to $2\pi$ and $X$ and $t$ so that the quadratic terms in the Hamiltonian are independent of $R_9$, then the commutator terms scale like $R_9^{-3}$ (bosonic) and $R_9^{-3/2}$ (fermionic). Thus, in the $R_9 \to 0$ limit we should restrict attention to commuting matrices. As shown in \[3\], the matrix model Lagrangian reduces to the multiple copies of the Type IIA Green-Schwarz lagrangian on this subset of matrix configurations. Our general comment about gauging of translations in the field theories which represent compactifications of the matrix model, shows that the correct Virasoro constraints of the light cone Green-Schwarz superstring follow from the gauge symmetries of the matrix model\[3\].

States of the field theory which do not satisfy the level matching constraint can be viewed as strings wrapped around the longitudinal direction. This follows from the usual Virasoro equation of light cone gauge field theory $\partial_\sigma X^- = p(\sigma)$, where $p$ is the world sheet momentum density. A similar phenomenon will arise in Section 3, when we study the emergence of Type IIB strings from SYM$_{2+1}$. There the values of world volume momenta are the charges under the NS-NS and the R-R two form gauge potentials for longitudinally wound elementary and Dirichlet strings. This may be seen by examining the supersymmetry algebra of the $2 + 1$ dimensional theory

$$
\{Q^A_\alpha, Q^B_\beta\} = 2\delta_{\alpha\beta} \gamma^A_i \gamma_i^B \p^i \tag{1.2}
$$

\[1\] It is important that we work in the light cone gauge rather than in the static gauge. In the light cone gauge the two spinors on the world sheet of the IIA string have opposite space time chirality. This is exactly as we find in the $U(1)$ gauge theory. This is to be distinguished from the IIB theory, where they have the same chirality. In the standard study of D strings in IIB theory, the static gauge is used. There the space time chiralities are opposite.

\[2\] This result was shown independently in \[3\].
where \( i = 0, 1, 2 \) and \( A, B = 1, 2 \) are indices of the \( 2 + 1 \) Lorentz symmetry (vector and spinor respectively) and \( \alpha, \beta = 1, ..., 8 \) are \( \text{spin}(8) \) indices. From the space time point of view \((1.2)\) is interpreted as part of the IIB supersymmetry algebra in the light cone frame. Here \( \alpha, \beta \) are spinors of the transverse Lorentz \( \text{spin}(8) \) symmetry and \( A, B \) label the two supercharges of the type IIB theory. The Hamiltonian \( p^0 \) in \((1.2)\) is the minus component of the space time momentum \( P^- \). \( p^{1,2} \) appear precisely as the two central charges for strings stretched along the longitudinal direction. More explicitly, as in \([11]\) we can identify them with

\[
\begin{align*}
p^1 &= \dot{X}^a [X^8, X^a] + ... = \dot{X}^a D_1 X^a + ... \\
p^2 &= \dot{X}^a [X^9, X^a] + ... = \dot{X}^a D_2 X^a + ... 
\end{align*}
\]

where \( a \) runs over the transverse directions and we used the fact that \( X^8, 9 \) become the two covariant derivatives in the spatial directions.

We wish to make one further comment on the construction presented in \([3]\). It produces a chiral two dimensional field theory as a limit of finite matrix constructions. This is not terribly surprising. From the matrix model point of view, the derivative operator arises as the limit of the matrix \( p \), which is taken to be a matrix with eigenvalues equal to the \( \frac{1}{i} \) times the logarithm of the \( N' \)th roots of unity, and to commute with the matrix \( V \) of \([3]\). Thus, our construction resembles the SLAC derivative of lattice gauge theories \([12]\).

In the matrix model, the lack of periodicity in the spectrum of \( p \) is required to describe wrapping configurations of membranes.

The strings which were exhibited in \([3]\) all have the same longitudinal momentum. Motl \([5]\) has described how strings with larger values of \( p_L \) emerge from the matrix model\([3]\). We have argued above that the stable semiclassical configurations are (in a particular gauge) diagonal \( X^i(\sigma) \) matrices. However, there is no need for these matrices to be periodic in \( \sigma \). Rather, they are periodic up to a gauge transformation, which preserves the diagonal gauge, \textit{i.e.} a permutation of eigenvalues. \( P \times P \) diagonal configurations, which have been “screwed together” in this fashion by a permutation of rank \( P \), are in one to one correspondence with strings of \( p_L = \epsilon P \), where \( \epsilon \) is the unit of longitudinal momentum carried by a \( 1 \times 1 \) matrix. Furthermore, the prescription that the light cone string coordinate measures the string’s longitudinal momentum follows immediately from this ansatz. To get arbitrary ratios of \( p_L \) we have to study the large \( M \) limit.

It remains to show that the correct string interactions emerge from this picture in the limit \( R_9 \to 0 \). Motl has argued that the correct scaling of the string coupling indeed emerges, but much work remains to be done along these lines.

\[\text{Motl’s construction was prefigured in work on black hole dynamics in string theory \([13]\).}\]
Motl’s construction has a beautiful interpretation in terms of the moduli space of SYM theory. We will show that this can be generalized to describe arbitrary toroidally compactified IIA and IIB strings. We will therefore turn in the next section to a general description of these SYM moduli spaces. In Section 3 we show how the Coulomb branch of the moduli space of toroidally compactified SYM$_{d+1}$ maps, in the limit that one radius of the SYM torus is much larger than others, into the Fock space of light cone gauge IIA string field theory compactified on the torus dual to the small SYM directions. This embedding in SYM theory provides a natural nonperturbative prescription for string interactions. We then show how a similar picture for Type IIB strings emerges in another limiting regime of the torus geometry, as first proposed by Aspinwall and Schwarz [14].

2. Some Properties of SYM Theories with Sixteen Supercharges

The properties of SYM theories with sixteen hermitian supercharges have recently been investigated in [15]. Here we will present some relevant results of the analysis of these theories, and refer the reader to [15] for more details.

2.1. The $d = 3$ Theory

We start with the study of field theories with $N = 8$ supersymmetry in $d = 3$. The super generators are in the real two dimensional representation of the Lorentz group. The automorphism of the algebra (R symmetry) is $spin(8)_R$ and the supergenerators transform as an eight dimensional representation, which we take to be the spinor $8_s$.

Since for massless particles the little group is trivial, there is only one massless representation of the superalgebra. It consists of 8 bosons in the $8_v$ of the R symmetry and 8 fermions in the $8_c$. Starting in a higher dimensional field theory with the same number of supersymmetries (e.g. $N = 4$ in $d = 4$) we find a vector field, 7 scalars and 8 fermions. The R symmetry, which is manifest in this description, is $spin(7) \subset spin(8)_R$. The vector is a singlet of $spin(7)$, the scalars are in $7$ and the fermions in $8$. After performing a duality transformation on the vector it becomes a scalar.

Interacting Lagrangians with $N = 8$ supersymmetry do not necessarily exhibit the maximal possible R symmetry. In particular, the Yang-Mills Lagrangian is invariant only under the $spin(7)$ subgroup.

The gauge coupling $g$ has dimension $\frac{1}{2}$, and therefore the theory is superrenormalizable. To analyze its long distance behavior we start by considering the moduli space of vacua. Along the flat directions the $U(M)$ gauge symmetry is broken to $U(1)^M$. The low energy degrees of freedom are in $M$ identical free $N = 8$ multiplets, each of which includes
seven scalars $X^{Ai} (i = 1, ..., 7)$ and a photon. The dual of the photons are compact scalars $\phi^A$ which live on the Cartan torus of $U(M)$ (in general, they live on the Cartan torus of the dual gauge group [15]). The Lagrangian is:

$$\frac{1}{g^2} (\partial X^{Ai})^2 + g^2 (\partial \phi^A)^2.$$  \hfill (2.1)

Because of $N = 8$ supersymmetry the Lagrangian (2.1) is not corrected in the quantum theory. Therefore, the $8M$ real dimensional moduli space of vacua $\mathcal{M}$ is flat. The $X^{Ai}$ label $\mathbb{R}^{7M}$ and $\phi^A$ labels $T^M$. The Weyl group of $U(M)$ is $S_M$. It permutes the $A$ indices and so

$$\mathcal{M} = \frac{\mathbb{R}^{7M} \times T^M}{S_M}. \hfill (2.2)$$

It has singularities whenever $X^{Ai} = X^{Bi}$ and $\phi^A = \phi^B$ for some $A$ and $B$ and all $i = 1, ..., 7$. The metric around these singular points is an orbifold metric.

Since the theory is superrenormalizable, the only dynamics which survives at energies smaller than $g^2$ is the infrared dynamics of massless modes. Note that if we define $g^2 \phi^A = X^{A8}$, to emphasize the spin$(8)$ symmetry of the lagrangian, then the radius of the $X^{A8}$ torus goes to infinity and we can focus on a neighborhood in the moduli space. At the generic point we find a free field theory. The theory at the orbifold singularities is more interesting. The moduli space around each of them looks like $\mathbb{R}^{8M}/S_M$. We believe that the theory describing these points is an interacting superconformal fixed point. A more extensive discussion is given in [15].

At long distance, the theory must flow to a scale invariant theory. It is expected that if the theory is interacting, it is also superconformal invariant. The conformal algebra in 3 dimensions is spin$(3,2)$. The eight supersymmetry generators combine with eight superconformal generators to eight spinors of spin$(3,2)$. For the closure of the algebra we must include the the spin$(8)_R$ symmetry [16].

The long distance theory is scale and superconformal invariant. As such it has a global spin$(8)$ symmetry which acts as an automorphism of the supersymmetry algebra. Along the flat directions the long distance theory is free and then the spin$(8)$ symmetry is manifest.

Below we will interpret these results in terms of the derivation of Type IIB string theory from the matrix model. The emergence of the spin$(8)$ symmetry in the infrared dynamics of the $2 + 1$ dimensional theory will imply that the string theory has an eight dimensional rotational invariance relating a dimension which arises from membrane winding to the manifest noncompact dimensions of the matrix model. The fact that the two $(2 + 1)$ Lorentz components of the eight SUSY generators transform in the same spinor representation of spin$(8)_R$ will there imply that the spacetime SUSY of the string theory is the chiral IIB algebra.
2.2. Compactification from $d = 4$

Consider now starting in a higher dimensional theory with 16 supercharges and compactifying on a torus to three dimensions. Some of the scalars in the three dimensional Lagrangian originate from components of gauge fields in the higher dimensional theory. Therefore, the corresponding directions in the moduli space of the three dimensional theory must be compact. Let us start by considering the free $U(1) \ N = 4$ theory in $d = 4$ with gauge coupling $g_4$ and compactify it on a circle of radius $R$ to three dimensions. The three dimensional gauge coupling $g_3$ satisfies

$$\frac{1}{g_3^2} = \frac{R}{g_4^2}. \tag{2.3}$$

The six scalars in the vector multiplet in four dimensions become $\phi^i$ with $i = 1,\ldots,6$. $\phi^7$ arises from a component of the four dimensional gauge field $\phi^7 = A_4$. It corresponds to a $U(1)$ Wilson line around the circle. A gauge transformation, which winds around this circle, identifies $\phi^7$ with $\phi^7 + \frac{1}{R}$. Therefore, we define the dimensionless field $\phi_e = RA_4$, whose circumference is one. When we dualize the three dimensional photon to a scalar $\phi_m$, we find the Lagrangian \[17\]

$$\frac{R}{g_4^2}(\partial \phi^i)^2 + \frac{1}{Rg_4^2}(\partial \phi_e)^2 + \frac{g_4^2}{R}(\partial \phi_m)^2. \tag{2.4}$$

The moduli space of vacua is

$$\mathbb{R}^6 \times T^2 \tag{2.5}$$

where the two circles in $T^2$ correspond to the two compact bosons $\phi_e$ and $\phi_m$. They represent a $U(1)$ Wilson line and a $U(1)$ 'tHooft line around the circle we compactified on.

In other words, these two scalars are the fourth component of the $d = 4$ photon $A_4$ and the fourth component of the magnetic photon $\tilde{A}_4$. The non-trivial duality transformation in $d = 4$ is translated to

$$\phi_e \rightarrow \phi_m$$

$$\phi_m \rightarrow -\phi_e$$

$$g_4 \rightarrow \frac{1}{g_4}. \tag{2.6}$$

It is easy to add the $\theta$ angle in four dimensions and recover the $SL(2, \mathbb{Z})$ action in four dimension as an action on the $T^2$ in the moduli space (2.5) \[17\].

As we said above, at long distance in the three dimensional theory only the local structure of the moduli space (2.3) matters. It is $\mathbb{R}^8$. The eight scalars transform as a
vector under the enhanced $\text{spin}(8)_R$ symmetry. The duality transformation (2.4) becomes part of the $\text{spin}(8)_R$ symmetry.

We can easily extend this discussion to compactified interacting theories. For example, consider the $U(M)\, N = 4$ theory in $d = 4$. Repeating the analysis of the $U(1)$ theory and modding out by the Weyl group, we find the moduli space of vacua

$$\mathbb{R}^{6M} \times T^{2M} / S_M.$$ (2.7)

The full theory is invariant only under the $\text{spin}(6)$ symmetry of the four dimensional theory. The $SL(2, \mathbb{Z})$ duality is not a symmetry of the theory. It relates theories with different values of the coupling constant. After the compactification this $SL(2, \mathbb{Z})$ acts on the $T^{2M}$ factor (it acts as the usual discrete diffeomorphism symmetry on each of the $M$ $T^2$ factors). Again, it is not a symmetry. However, at long distance its $\mathbb{Z}_2$ subgroup (2.4) becomes a symmetry. Therefore, the symmetry at long distance includes $\text{spin}(6) \times \mathbb{Z}_2$. The three dimensional Lagrangian is obtained by shrinking the compactification radius $R$ with $g_3$ fixed. Then, the $\text{spin}(6)$ R symmetry of the four dimensional theory is enhanced to $\text{spin}(7)$, which is manifest in the three dimensional Lagrangian. Since in this limit $g_4 \to 0$, the $\mathbb{Z}_2$ subgroup of $SL(2, \mathbb{Z})$ is not visible. In the long distance limit we should find a symmetry, which includes both this $\text{spin}(7)$ R symmetry and $\text{spin}(6) \times \mathbb{Z}_2$. This must be $\text{spin}(8)$. This leads to an independent derivation of the $\text{spin}(8)_R$ symmetry of the long distance theory (the other derivation was based on its superconformal invariance). This argument is similar to that of [4].

We conclude that the electric-magnetic duality of the four dimensional theory becomes a symmetry of the three dimensional theory at long distance. It is included in its $\text{spin}(8)_R$ R symmetry.

2.3. Generic Toroidal Compactifications of SYM$_{d+1}$

The final result which we will need in our discussion of the matrix model is that for the moduli space of SYM$_{d+1}$ compactified on a torus of generic dimension. The term generic means that we will omit discussion of the special consequences of duality in low dimensions. We will also restrict attention to the gauge group $U(M)$.

If we compactify SYM$_{d+1}$ to $k$ noncompact spatial dimensions, we obtain SYM$_{k+1}$, which contains $9 - k$ scalar fields in the adjoint representation of $U(M)$. $9 - d$ of these fields, $X^i$, are noncompact variables. The other $d - k$ arise from integrating the $d$ dimensional gauge potentials over one cycles of the $d - k$ torus. We call these variables $\Phi^a$. Along the generic flat direction, the gauge group is broken to $U(1)^M$. The $M(d - k)$ variables $\Phi^a$ are
now angle variables which live in \((d - k)\) copies of the Cartan torus of \(U(M)\). The moduli space is thus

\[
\mathbb{R}^{M(9-d)} \times T^{M(d-k)} \quad S_M
\]  

(2.8)

The kinetic term for the compact fields takes the form

\[
\frac{1}{g_k^2} G_{ab} \partial \Phi^a \partial \Phi^b
\]  

(2.9)

where \(g_k^2\) is the effective SYM\(_{k+1}\) coupling, including a factor of the inverse volume of the \(d - k\) torus. \(G_{ab}\) is the metric of \(T^{M(d-k)}\). Since the \(\Phi^a\) are Wilson loops, integrals of gauge field components along cycles of the original torus, the scale of this torus is the inverse of the compactification size. For example, for \(M = 1\), \(T^{d-k}\) is the dual of the compactification torus, while for general \(M\) it is the product of \(M\) copies of this dual torus. In the matrix model application below, it is this dual torus which plays the role of the spacetime on which strings propagate.

3. M Theory on Tori

3.1. Generalities

The compactified matrix model is SYM\(_{d+1}\). Compactified IIA strings should be thought of as M theory 2-branes wrapped around a one dimensional cycle of \(T^d\). In the weakly coupled type IIA theory from which the matrix model was extracted in [3], the membrane is described as a Dirichlet brane\(^4\). In the T dual prescription which leads to \(d + 1\) SYM theory, membranes wrapped around a single cycle of the torus are, from the SYM point of view, \(d - 1\) dimensional domain walls wrapped around the dual cycle of the dual torus. So, a membrane wound around the ninth direction of a rectilinear torus is, in SYM language, a domain wall wrapped around the first \(d - 1\) directions (we label the dual torus directions by \(\sigma^{a-9+d}\) for the direction dual to the \(a\)th direction in target space). Having identified these configurations in the weakly coupled Type IIA limit we now go to strong coupling via the conjectures of [3]. Namely, the theory which describes the short distance interactions of zero branes at weak coupling, is taken to be the entire theory at

\(4\) The reader should carefully distinguish the Type IIA string theory in the present paragraph from that discussed in the rest of the paper. Here, the longitudinal direction is thought of as small, while \(T^d\) is of string scale. IIA strings are membranes wrapped around the longitudinal direction. We will quickly return to a situation in which the longitudinal direction is large, where we derive another copy of perturbative IIA strings by taking a transverse dimension small.
strong coupling. What we have learned via this excursion is how to identify the degrees of freedom which will represent Type IIA strings in another weak coupling limit in which the longitudinal direction is large while one of the transverse directions is shrunk to zero. We will see that the representation of IIA strings as \(d-1\) dimensional domain walls arises naturally from SYM\(_{d+1}\) itself.

The limit of SYM\(_{d+1}\) which is supposed to describe IIA string theory compactified on a torus is one in which the radius \(R_9\) is taken very much smaller than the eleven dimensional Planck scale, while the other dimensions are taken large. Indeed, the typical size of these other directions are of order the scale set by the weakly coupled Type IIA string tension. From the SYM point of view this means that we have one large and \(d-1\) small dimensions, and it is clear that, to first approximation, we should ignore modes which carry momentum in the small directions. Thus, directly in the SYM theory, we can understand that the degrees of freedom which dominate the IIA limit are \(1+1\) dimensional fields, corresponding to integrals of the underlying degrees of freedom over \(d-1\) dimensional domain walls.

Of course, what we have done here is to dimensionally reduce \(U(M)\) SYM\(_{d+1}\) to SYM\(_{1+1}\). As we discussed in the previous section the moduli space of the dimensionally reduced theory is \(\mathbb{R}^{M(9-d)} \times T^{M(d-1)}/S_M\).

Dynamics along the moduli space is thus described by eight free \(1+1\) dimensional scalar fields and their superpartners, modded out by a discrete gauge symmetry. The boundary conditions obeyed by these scalar fields may be twisted by any element of the discrete group, which is the semidirect product of the weight lattice of \(U(M)\) and its Weyl group

\[
\mathbb{Z}^M \rtimes S_M.
\]  

(3.1)

The conjugacy classes of this group are easily worked out. Each group element is the product of a permutation and a shift. Write the permutation as a product of commuting cycles. It is obvious that we can work within the subspace corresponding to a given cycle. The permutation is then the cycle \(S\) which takes \(x_k \rightarrow x_{k+1}\). Conjugating this by a shift \(x_k \rightarrow x_k + s_k\) we get the product of \(S\) and a shift by \(s_k - s_{k+1}\). This fails to be a general shift because it is traceless. Thus the most general element is conjugate to a permutation times a shift which shifts the whole subspace acted on by each cycle of the permutation by the same lattice vector.

This is precisely what we need for the interpretation of the moduli space as compactified Type IIA string theory. The sector corresponding to a given permutation \(S\) is interpreted in string theory language as follows: Writing \(S\) as a product of commuting cycles, and arranging the matrix elements of the \(X^i\) in cyclic order, we obtain Motl’s picture of long strings. The sector of the gauge theory with \(S\) a product of \(k\) cycles of
lengths $p_k$ is in one to one correspondence with multistring states of $p_k$ units of longitudinal momentum. Note that the usual light cone correspondence between string length and longitudinal momentum follows directly from the matrix model identification of the longitudinal momentum with the rank of matrices. The shifts provide us with the winding numbers of these strings around cycles of the torus. The nontrivial conjugacy classes correspond to assigning a winding number around each cycle to each string of each value of the longitudinal momentum. In particular, we do not have separate winding numbers for the different diagonal elements of the matrix which makes up a long string.

We note that in the full $U(N)$ theory, the permutation sectors are not really topological, since permutations can be continuously deformed to the identity in $U(N)$. However, it is easy to see that as $R_9 \to 0$, the masses of the fields transforming as roots of the Lie algebra go to infinity. Combining this with SUSY nonrenormalization theorems we see that in this limit the free string picture becomes exact. The different sectors, representing strings with different values of longitudinal momenta, do not transform into each other. The challenge of deriving string interactions as corrections to this limit will be taken up elsewhere.

We have thus shown that the large $M$ SYM$_{d+1}$ theory, reproduces, in the appropriate limits both compactified and uncompactified Type IIA string theory. We have worked in the limit $R_9$ goes to zero with other radii fixed. In this limit the strings are free. We can thus do the usual T duality transformations on them. What is not clear at this juncture is how the string interactions defined by SYM$_{d+1}$ transform under those transformations. Since the string variables are only a small subset of the SYM$_{d+1}$ degrees of freedom, this is far from obvious, particularly for those values of $d$ in which SYM$_{d+1}$ is nonrenormalizable. Moreover, we do not expect the transformation rules to be simple, since T duality in a single circle is supposed to reproduce Type IIB string theory.

The nonperturbative formulation of Type IIB string theory in ten dimensions is instead supposed to derive from the theory at hand by taking a different limit of the radii. We turn to this problem in the next subsection.

### 3.2. The Matrix Model on $T^2$ and Type IIB Strings

As shown by Aspinwall and Schwarz [14], ten dimensional Type IIB string theory is supposed to emerge when M theory is compactified on $T^2$ whose area goes to zero at fixed complex structure. The eleven dimensional Planck scale is rescaled so that the $(p, q)$ string tensions $\sqrt{p^2 R_9^2 + q^2 R_8^2 J_{11}^3}$ are kept fixed. Here we see this in the framework of the matrix model.
The corresponding SYM\textsubscript{2+1} theory lives on a torus with area going to infinity. As we noted above, the SYM coupling scales as $g^2 \sim (R_8 R_9)^{-1} l_{11}$ which means that it scales to infinity like $R^{-5/3}$ (in string tension units) as we go to the Type IIB limit. We recall that $g^2$ has dimensions of mass in three dimensions. Thus, in three space time dimensions, the Yang Mills coupling is relevant. SYM theory with sixteen SUSY generators has a classical moduli space which (apart from singular points) consists of abelian field configurations. Near the singular points of the moduli space the theory is likely to be described by a nontrivial infrared fixed point. The scalings noted above suggest that apart from this extreme infrared dynamics on the moduli space, all other features of SYM\textsubscript{2+1} will decouple from the dynamics in the IIB limit. The limiting theory will be described by infrared fixed points, trivial along the flat directions in the moduli space and perhaps nontrivial near the singularities.

As we discussed above, the strong coupling limit of SYM\textsubscript{2+1} has a spin\textsubscript{(8)} global symmetry. In terms of the membranes it rotates the space time momentum component which arises as membrane winding number into the ordinary transverse space time momenta. It is interesting how the required enhanced Lorentz symmetry, which should arise in the area going to zero limit, appears in the strong coupling limit of the SYM\textsubscript{2+1} theory.

To actually see the IIB strings in SYM\textsubscript{2+1}, we must integrate the moduli fields over a one cycle of the dual torus. This follows the general prescription we have described above for finding strings in SYM\textsubscript{d+1}. The new wrinkle here is that the appropriate fields to integrate include the dual to the photon field, rather than the gauge field itself. This is because we have a strongly coupled gauge theory in the infinite coupling limit. Then the dynamics is fully described by the moduli. In order to see the proper scalings of the Lagrangian, we will redo the duality transformation.

In terms of a gauge coupling $g^2$ and a flat metric $G_{\mu\nu}$, the duality transformation of a three dimensional gauge theory is performed by doing the functional integral over $F_{\mu\nu}$ of the action,

$$S = \int \left( \frac{\sqrt{G}}{g^2} [G^{\mu\nu} G^{\lambda\kappa} F_{\mu\lambda} F_{\nu\kappa} + G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i] + \epsilon^{\mu\nu\lambda} F_{\mu\nu} \partial_\lambda \phi^8 \right),$$  

(3.2)

where $i$ runs from one to seven. The integral leads to

$$S = \int \left[ \frac{\sqrt{G}}{g^2} G^{\mu\nu} \partial_\mu \phi^8 \partial_\nu \phi^8 + \frac{\sqrt{G}}{g^2} G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i \right]$$  

(3.3)

In the present context, $\frac{\sqrt{G}}{g^2}$ is independent of the radii of the torus, and the spin\textsubscript{(8)} invariance is manifest. The nonvanishing metric components are $G_{00} = 1, G_{11} = 1/R_8^2$ and $G_{22} = 1/R_9^2$, all in eleven dimensional Planck units.
Elementary IIB strings are found by taking $R_9 \ll R_8$. In this limit, fields which vary in the $\sigma^2$ direction on the dual torus are energetically costly. The low energy excitations are functions only of $\sigma^1$. Thus, as in the case of compactified IIA strings, the $1+1$ dimensional nature of the low energy excitations is a consequence of taking one cycle of a (spacetime) torus much smaller than the others. It is easy to verify that on this set of configurations, the SYM$_{2+1}$ moduli space Lagrangian reduces to multiple copies of the Green Schwarz IIB Lagrangian:

$$\int dt d\sigma^1 [ (\partial_0 X^A)^2 + R^2_9 (\partial_{\sigma^1} X^A)^2 ] + \text{fermions} \quad (3.4)$$

Note that unlike the construction of the IIA theory, here the two spinors on the world sheet have the same space time chirality. This is exactly as it should be on the world sheet of the IIB string in the light cone gauge.

Our analysis of the correspondence between sectors in the moduli space Lagrangian and the Fock space of strings with arbitrary longitudinal momentum goes through as well. The only subtle point is that $X^8$ is a periodic variable, but its period goes to infinity in the zero area limit.

We can also describe Dirichlet strings in this formalism. We simply perform an $SL(2, \mathbb{Z})$ transformation on the elementary string. The Lagrangian is not invariant under this. The metric $G_{ij}$ transforms as $G \rightarrow M^T G M$ where $M$ is the $SL(2, \mathbb{Z})$ matrix \( \begin{pmatrix} m & n \\ p & q \end{pmatrix} \). This reproduces the correct formula for the Dirichlet string tensions \[14\]. Of course, closed D-strings in ten dimensions are not stable excitations. They interact strongly and will decay rapidly. We do not yet know how to derive these interactions from the matrix model. Despite these caveats our derivation of the Dirichlet string tensions is a correct one because we can apply it to large smooth string configurations which approach the infinite straight BPS strings.

We have given only a brief description of IIB strings here, since everything follows precisely the pattern outlined by Aspinwall and Schwarz \[14\]. Nonetheless it is rewarding to see it emerge so nicely from the matrix model formalism.

The domain wall character of the IIB string excitations of SYM$_{2+1}$ removes what might have been a paradox in the emergence of IIB strings as a zero area limit. Formally, the theory on the dual torus goes to infinite volume in this limit and we might have imagined that the rotation group in the toroidal volume is restored (and perhaps even elevated to the spin($2, 1$) Lorentz group). However, there is no such restoration of symmetry for the wrapped excitations which we are studying. These always feel the toroidal boundary conditions. The question of the significance of local excitations of the SYM theory (for which the restored symmetry might have some significance) deserves further study. We note
in addition that the discrete subgroup of rotations which preserves the toroidal boundary conditions when $R_8 = R_9$ is clearly a gauge transformation of the matrix model. It is induced by a unitary transformation of the fundamental matrices which preserves the trace in the compactified theory. In the present case it is simply the $\tau \to -\frac{1}{2}$ subgroup of the $SL(2, \mathbb{Z})$ gauge symmetry of toroidally compactified M theory.

It is of some interest to understand more completely the role of the $spin(2, 1)$ Lorentz group and its extension to the conformal group at the nontrivial fixed point. It is clear that there can be no physical symmetry between the time of the $2 + 1$ field theory, which is the same as light cone time in the ambient spacetime, and its spatial dimensions, whose corresponding translation generators are set equal to zero on physical states. Nonetheless, recalling the role of the light cone Virasoro algebra in string theory, we may anticipate that these world volume generators are crucial to the proof of ten dimensional Lorentz invariance in the nonperturbative formulation of IIB string theory. A similar conclusion is also suggested by the connection which we pointed out above, between states of nonzero two momentum and longitudinally wrapped strings.

4. Conclusions

We have shown that the prescription of compactifying the matrix model of M theory as $SYM_{d+1}$ on a dual torus, correctly reproduces the expected string theories in various limits. Our most complete results were for ten dimensional Type IIA string theory. The moduli space of large $M$ SYM$_{1+1}$ precisely reproduces the Fock space of light cone Type IIA string field theory, and the SYM theory gives a nonperturbative prescription for string interactions. We have not yet shown that the interactions it prescribes reduce to conventional perturbative string theory in the zero radius limit. The light cone level matching condition follows from gauge symmetries of the matrix model which go beyond those of SYM. We also showed that toroidally compactified IIA strings arise in the requisite manner from SYM$_{d+1}$. Here our analysis must be deemed less complete, if only because it does not really distinguish those cases where SYM$_{d+1}$ is a sensible continuum field theory from those where it isn’t.

Next we showed that the zero area limit of the compactification of the matrix model on a two torus contained excitations which propagate like free ten dimensional IIB strings with arbitrary $(p, q)$ charge (more precisely, infinitely long strings carry charge, while the finite excitations we have constructed do not). In the limit of large complex structure of the small torus, the (0, 1) string is weakly coupled and even closed strings are almost stable. The freely propagating strings have a $spin(8)$ symmetry rotating the membrane
winding number direction into the ordinary dimensions of space. We gave an argument based on 2 + 1 dimensional field theory that this is an exact symmetry of the model in the zero area limit. We may anticipate that the discussion of interactions will be more complicated in the IIB case, since it seems to involve the construction of a nontrivial fixed point theory at the origin of moduli space.

One of the most intriguing aspects of our study is the way in which the 1 + 1 dimensional character of string theory arises. Weakly coupled limits of toroidally compactified M theory arise when a single cycle of the torus is much smaller than all the others. In the SYM_{d+1} description this corresponds to 1 large and d − 1 small cycles and leads to a Kaluza Klein reduction of the degrees of freedom which accounts for the stringiness of the dynamics. We are led to the conclusion that in general, at strong coupling M theory is not stringy. Rather, the picture which appears to emerge is that the dynamics on the moduli spaces of supersymmetric field theories of higher dimension (we emphasize that it is the moduli spaces which are to be thought of as space time), is generally involved. We anticipate a particularly important role for superconformal fixed point theories, such as that which we conjecture to describe the nonperturbative interactions of IIB strings.

We cannot refrain at this point from making some remarks about the fact that for d > 3 compactification on a d torus leads to nonrenormalizable field theories. First we emphasize that as far as spacetime is concerned, this is an infrared problem. This follows from the dual relation between the world volume of SYM_{d+1} and the spacetime torus. Shenker [18] has also emphasized the way in which the ultravioletSYM_{d+1} behavior mirrors the infrared properties of the transverse Coulomb potential. Thus, resolving this problem may tell us interesting things about low energy spacetime physics.

There are a variety of possible responses to the problem of nonrenormalizability. The first is to search for a continuum definition of SYM_{d+1}. For example, strongly coupled SYM_{4+1} may be viewed as a limit of S^1 compactification of a nontrivial fixed point theory in 5 + 1 dimensions. Recent work of Rozali [19] suggests that such a limit may indeed be relevant to the physics of the matrix model compactified on a four torus. We suggest that this may be part of a larger story, and that once we understand compactifications with fewer supersymmetries, a whole range of nontrivial fixed point theories may prove to be relevant to the exploration of interesting nonperturbative physics in M theory. This would be analogous to the role that 1+1 dimensional conformal field theories play in perturbative string theory.

With the full complement of SUSYs however, there does not seem to be a possibility of nontrivial fixed points above d = 4 (note that even for d = 4 we have to appeal to toroidal compactification of a theory with chiral SUSY in a higher dimension). Perhaps this is
related to the fact that M theory only has two branes and five branes, as a consequence of which nothing interesting happens when cycles of higher dimension shrink to zero volume. To make this remark more transparent, imagine defining toroidally compactified $\text{SYM}_{d+1}$ as the limit of a cut off theory. If there are no strong coupling fixed points, the bulk dynamics of $\text{SYM}_{d+1}$ approaches that of free field theory as the cutoff is taken to infinity. However, in the toroidally compactified theory there are zero modes whose infrared dynamics exhibits the full complications of lower dimensional Yang Mills theory.

Let us now remember that the relevance of the bulk $\text{SYM}_{d+1}$ dynamics to the physics of M theory is only apparent when we take a limit in which all of the radii of the space-time torus are taken much smaller than the Planck length. In other limits of the space of compactifications, the $\text{SYM}_{d+1}$ torus has fewer large dimensions and only the lowest momentum modes around the small dual tori are included in the low energy dynamics. Thus, the *triviality* of high dimensional $\text{SYM}_{d+1}$ may be simply telling us that there are no interesting limits of the space of compactifications of the matrix model with unbroken eleven dimensional SUSY, in which cycles of dimension higher than four are shrunk to zero.

We would like to stress an assumption that we have made implicitly throughout this paper. When considering situations in which one radius of a torus was much larger than others, we have made the assumption that we could do the standard dimensional reduction of $\text{SYM}_{d+1}$ (or in the IIB limit, of its dual theory). While this is obviously correct along the flat part of the moduli space, it remains an assumption about the dynamics of whatever definition of the nonrenormalizable $\text{SYM}_{d+1}$ theory we may supply for $d > 3$. We believe that this property is necessary to the consistency of the rules for compactification of the matrix model to various dimensions. Decompactification of a spacetime circle should always lead back to the theory compactified on one fewer dimension.

Finally, we would like to stress one of the main conclusions of the present work. The matrix model system, made up only of zero branes and minimally stretched strings, is capable of reproducing the full spectrum of various string theories in appropriate limits. And it does so within the context of a nonperturbatively defined, unitary theory of string interactions. In the various weakly coupled limits it is clear that there are no low energy excitations apart from the appropriate strings. Thus the matrix model is sure to lead to

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5 The most natural cutoff is one in which the derivatives on the world volume are written as the limit of large $N$ matrices, following one of the derivations of the $\text{SYM}_{d+1}$ prescription from the matrix quantum mechanics of [3]. This cutoff preserves SUSY and gauge invariance, and may be applicable to numerical approximations of chiral and SUSY gauge theories in a more general context. This will be discussed in a future paper by one of the authors [20].
an effective theory of perturbatively interacting strings. It remains to be seen whether the interactions it prescribes are those of conventional string theory, which are the only string interactions compatible with ten dimensional Lorentz invariance. This question is under active study [21] and we hope to be able to answer it soon.

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