Method of group foliation, hodograph transformation and non-invariant solutions of the Boyer-Finley equation

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Abstract

We present the method of group foliation for constructing non-invariant solutions of partial differential equations on an important example of the Boyer-Finley equation from the theory of gravitational instantons. We show that the commutativity constraint for a pair of invariant differential operators leads to a set of non-invariant solutions of this equation. In the second part of the paper we demonstrate how the hodograph transformation of the ultra-hyperbolic version of Boyer-Finley equation in an obvious way leads to its non-invariant solution obtained recently by Mañas and Alonso. Due to extra symmetries, this solution is conditionally invariant, unlike non-invariant solutions obtained previously. We make the hodograph transformation of the group foliation structure and derive three invariant relations valid for the hodograph solution, additional to resolving equations, in an attempt to obtain the orbit of this solution.

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1 Introduction

The standard method for obtaining exact solutions of partial differential equations (PDEs) by the symmetry group analysis is the symmetry reduction which gives only \textit{invariant solutions}, \textit{i.e.} the solutions which are invariant with respect to some subgroup of symmetry group of the PDEs \textsuperscript{123}. The drawback of the standard symmetry analysis is that with this method one
loses all non-invariant and many invariant solutions, the latter being non-invariant solutions of the equations obtained by symmetry reduction.

Meanwhile, non-invariant solutions of PDEs might be very important for applications in physics. The classical example is the so-called $K3$ instanton in the Einstein’s theory of gravitation which corresponds to the Kummer surface in differential geometry. The corresponding Kähler metric has no Killing vectors and it could be constructed once one finds non-invariant solutions of the complex Monge-Ampère equation, the problem which has not been solved for over 150 years.

Recently we discovered that the method of group foliation can be made an appropriate tool for obtaining non-invariant solutions of non-linear PDEs that admit an infinite-dimensional symmetry group. The method involves the study of compatibility of given equations with a differential constraint, which is automorphic under an infinite dimensional symmetry subgroup, the latter acting transitively on the submanifold of the common solutions. By studying compatibility conditions of this automorphic system, i.e. the resolving equations, one can provide an explicit foliation of the entire solution manifold of the considered equations into separate orbits. Unlike the standard symmetry method we do not loose any solutions in the process and we can select orbits of non-invariant solutions by obtaining appropriate solutions of the resolving system.

The idea of the method, belonging to Lie and Vessiot \[4,5\], is more than a hundred years old being resurrected in a modern form by Ovsiannikov more than 30 years ago (see \[1\] and references therein). We have modified the method by introducing three important new ideas \[6,7,8,9,10\]:
1. The use of invariant cross-differentiation, involving operators of invariant differentiation and their commutator algebra for derivation of resolving equations and for obtaining their particular solutions.
2. Commutator representation of resolving system in terms of the commutators of operators of invariant differentiation.
3. Method of invariant integration for solving automorphic system.

We illustrate all this on a physically important example of the Boyer-Finley equation \[11\] which is known also as the heavenly equation

$$u_{z\bar{z}} = \kappa (e^u)_{tt} \iff u_{xx} + u_{yy} = \kappa (e^u)_{tt}, \quad \kappa = \pm 1$$  \quad (1.1)

where $u = u(t, z, \bar{z})$. This equation is a continuous version of the Toda lattice or $SU(\infty)$ Toda field \[12\]. It appears in the theory of gravitational instantons \[13,14,15\] where it describes self-dual Einstein spaces with Euclidean
signature having one rotational Killing vector. It also appears in the general theory of relativity \([16]\) and other physical theories.

In the first part of this paper, on the example of the Boyer-Finley equation, we clarify main concepts of the method including these three ideas and consider in detail the main steps which should be performed for obtaining non-invariant solutions. This approach enables us to present explicitly a family of non-invariant solutions of the Boyer-Finley equation depending on two functional parameters.

In the second part of the paper we discuss a new non-invariant solution of the ultra-hyperbolic version of the Boyer-Finley equation \((1.1)\) with \(\kappa = 1\) obtained recently by Mañas and Alonso by the hodograph transformation of associated systems of the hydrodynamic type \([17]\) (see also \([18]\) for a generalization of the approach of \([17]\)). We point out that if one makes the hodograph transformation of the Boyer-Finley equation itself, then it becomes obvious how to impose a differential constraint immediately leading to the solution of Mañas and Alonso without any use of hydrodynamical systems or the general theorem presented in \([17]\). We show that this solution possesses extra symmetries depending on eight arbitrary functions of \(u\) which are not the symmetries of the Boyer-Finley equation. Hence, this solution is conditionally invariant \([19]\) though not an invariant solution. A similar argument shows that symmetries of the solutions obtained by Calderbank and Tod \([14]\) and by Martina, Sheftel and Winternitz \([8]\) coincide with the symmetries of the Boyer-Finley equation itself and hence these solutions, having no extra symmetries, are not conditionally invariant. Finally, we make the hodograph transformation of the whole group foliation structure and obtain the relations between differential invariants valid for the solution of \([17]\) which together with the resolving equations partially describe the orbit of this solution. We have not yet succeeded to obtain a solution of the resolving system which corresponds to the orbit of the hodograph solution of \([17]\). Since the additional relations which we have found do not form a complete set of invariant differential constraints which uniquely select the hodograph solution, the corresponding solution of the resolving system would give us the orbits of more general solutions of \((1.1)\) with \(\kappa = 1\) and also solutions with \(\kappa = -1\). This work is still in progress.
2 Symmetry algebra, differential invariants and automorphic system

We start with the symmetry algebra of the generators of point transformations for the Boyer-Finley equation (1.1) [8]

\[ T = \partial_t, \quad G = t \partial_t + 2 \partial_u \]
\[ X_a = a(z) \partial_z + \bar{a}(\bar{z}) \partial_{\bar{z}} - (a'(z) + \bar{a}'(\bar{z})) \partial_u \] (2.1)

where \( a(z) \) is an arbitrary holomorphic functions of \( z \) and prime denotes derivative with respect to argument.

The Lie algebra of symmetry generators (2.1) is determined by the commutation relations

\[ [T, G] = T, \quad [T, X_a] = 0, \quad [G, X_a] = 0, \quad [X_a, X_b] = X_{ab'} - ba' \] (2.2)

They show that the generators \( X_a \) of conformal transformations form a subalgebra of Lie algebra (2.2). This subalgebra is infinite dimensional since the generators \( X_a \) depend on \( a(z) \). The corresponding finite transformations form an infinite dimensional symmetry subgroup of the equation (1.1) since instead of a group parameter they also involve an arbitrary holomorphic function of \( z \). We choose this infinite dimensional conformal group for the group foliation.

Next we find differential invariants of the symmetry subgroup of conformal transformations. They are invariants of all the generators \( X_a \) in the prolongation spaces, so that they can depend on independent variables, unknowns and also on partial derivatives of unknowns allowed by the prolongation order. The order of the differential invariant is defined as the order of the highest derivative which this invariant depends on. The highest order \( N \) of differential invariants should be larger or equal to the order of the equation \( (N \geq 2) \) and there should be \( n \) functionally independent invariants with \( n > p + q \) where \( p \) and \( q \) are the number of independent and dependent variables, respectively. Thus, we should have \( p = 3, \ q = 1, \ n > 4, \ N \geq 2 \).

The routine calculation for \( N = 2 \) gives \( n = 5 \) functionally independent differential invariants up to the second order inclusively

\[ t, \quad u_t, \quad u_{tt}, \quad \rho = e^{-u}u_{z\bar{z}}, \quad \eta = e^{-u}u_{z\bar{z}}u_{\bar{z}t} \] (2.3)

and all of them are real. This allows us to express the Boyer-Finley equation (1.1) solely in terms of the differential invariants \( u_{tt} = \kappa \rho - u_t^2 \).
The next step is to choose the general form of the automorphic system. We choose the invariants \( t, u_t, \rho \) as new invariant independent variables, the same number \( p = 3 \) as in the original equation (1.1), and require that the remaining invariants should be functions of the chosen ones. This provides us with the general form of the automorphic system that also contains the studied equation expressed in terms of invariants (2.3)

\[
\begin{align*}
  u_{tt} &= \kappa \rho - u_t^2, \\
  \eta &= F(t, u_t, \rho)
\end{align*}
\]

The real function \( F \) in the right-hand side should be determined from the resolving equations which are compatibility conditions of the system (2.4). Then the system (2.4) will have automorphic property, i.e. any of its solutions can be obtained from any other solution by an appropriate transformation of the conformal group.

### 3 Operators of invariant differentiation and the basis of differential invariants

Our next task is to find operators of invariant differentiation. They are linear combinations of total derivatives \( D_t, D_z, D_{\bar{z}} \) with respect to independent variables \( t, z, \bar{z} \) with the coefficients depending on local coordinates of the prolongation space which are defined by the special property that, acting on any (differential) invariant, they map it again into a differential invariant. As a consequence, these differential operators commute with any infinitely prolonged generator \( X_\alpha \) of the conformal symmetry group. Being first order differential operators, they raise the order of a differential invariant by one. The total number of independent operators of invariant differentiation is obviously equal to 3, the number of independent variables.

We obtain the basis for the operators of invariant differentiation [8,9,10]

\[
\delta = D_t, \quad \Delta = e^{-u}u_{zt}D_z, \quad \bar{\Delta} = e^{-u}u_{\bar{z}t}D_{\bar{z}}
\]

The next step is to find the basis of differential invariants which is defined as a minimal finite set of (differential) invariants of a symmetry group from which any other differential invariant of this group can be obtained by a finite number of invariant differentiations and operations of taking composite functions. The proof of existence and finiteness of the basis was given by Tresse [20] and in a modern form by Ovsiannikov [1].
In our example the basis of differential invariants is formed by the set of three invariants \( t, u_t, \rho \), while two other invariants \( u_{tt} \) and \( \eta \) defined in (2.3) are given by the relations
\[
u_{tt} = \delta(u_t), \quad \eta \equiv e^{-u}u_{zt}u_{zt} = \Delta(u_t) = \bar{\Delta}(u_t) \quad (3.2)
\]
All other functionally independent invariants of higher order can be obtained by acting with operators of invariant differentiation on the basis \( \{t, u_t, \rho\} \).

In particular, the following third order invariants generated from the second order invariant \( \rho \) by invariant differentiations will be used in our construction
\[
s = \Delta(\rho), \quad \bar{s} = \bar{\Delta}(\rho), \quad \tau = \delta(\rho) \equiv D_t(\rho) \quad (3.3)
\]

### 4 Commutator algebra of invariant differential operators

The operators \( \delta, \Delta \) and \( \bar{\Delta} \) defined by (3.1) form the commutator algebra which is a Lie algebra over the field of invariants of the conformal group \([1]\).

This algebra is simplified by introducing two new operators of invariant differentiation \( Y \) and \( \bar{Y} \) instead of \( \Delta \) and \( \bar{\Delta} \) and two new variables \( \lambda \) and \( \bar{\lambda} \) instead of \( \sigma \) and \( \bar{\sigma} \), defined by
\[
\Delta = \eta Y, \quad \bar{\Delta} = \eta \bar{Y}, \quad \sigma = \eta \lambda, \quad \bar{\sigma} = \eta \bar{\lambda} \quad (4.1)
\]
The resulting algebra has the form
\[
[\delta, Y] = \left( \kappa \bar{\lambda} - 3u_t - \frac{\delta(\eta)}{\eta} \right) Y, \quad [\delta, \bar{Y}] = \left( \kappa \lambda - 3u_t - \frac{\delta(\eta)}{\eta} \right) \bar{Y}
\]
\[
[Y, \bar{Y}] = \frac{(\tau + u_t \rho)}{\eta} (Y - \bar{Y}) \quad (4.2)
\]

With the use of operators \( \delta, Y \) and \( \bar{Y} \) the general form (2.4) of the automorphic system becomes
\[
\delta(u_t) = \kappa \rho - u_t^2, \quad Y(u_t) = 1 \quad (\bar{Y}(u_t) = 1) \quad (4.3)
\]
where the first equation is the Boyer-Finley equation and the second equation follows from the second relation (3.2). Here we put \( \eta = F \) in the equations (4.1) and in the commutation relations (4.2) according to the second equation in (2.4). Then we obtain
\[
Y = (1/F)\Delta, \quad Y = (1/F)\bar{\Delta}, \quad Y(\rho) = \lambda, \quad \bar{Y}(\rho) = \bar{\lambda} \quad (4.4)
\]
5 Resolving equations

The system of resolving equations is a set of compatibility conditions between the studied equation and equations added to it in order to form the automorphic system. In our case we require the compatibility between two equations \([4.3]\) which gives restrictions on the function \(F(t, u_t, \rho)\) in the right-hand side of the second equation in \([2.4]\). A new feature in our modification of the method is that we do this in an explicitly invariant manner by using the invariant cross-differentiation \([6, 7, 8, 9, 10]\) involving the operators \(\delta, Y\) and \(\bar{Y}\). The resulting five resolving equations have the form \([8, 9, 10]\)

\[
\delta(F) = \left[\kappa(\lambda + \bar{\lambda}) - 5u_t\right]F \quad (5.1)
\]

\[
F\left(Y(\bar{\lambda}) - \bar{Y}(\lambda)\right) = (\tau + u_t\rho)(\lambda - \bar{\lambda}) \quad (5.2)
\]

\[
\delta(\lambda) = Y(\tau) + 2u_t\lambda - \kappa\lambda^2 \quad (5.3)
\]

\[
\delta(\bar{\lambda}) = \bar{Y}(\tau) + 2u_t\bar{\lambda} - \kappa\bar{\lambda}^2 \quad (5.4)
\]

\[
F\left(Y(\lambda) + \bar{Y}(\bar{\lambda})\right) = -(\tau + u_t\rho)(\lambda + \bar{\lambda})
+ 2\kappa\left[\delta(\tau) + 4u_t\tau + 2F + \kappa\rho^2 + 2u_t^2\rho\right] \quad (5.5)
\]

The definitions of \(\lambda, \bar{\lambda}, \tau\) used here are given by equations \([4.4], [3.3]\).

The resolving equations \([5.1], [5.2], [5.3], [5.4], [5.5]\) form a closed resolving system where the second order differential invariant \(\eta = F\) and the third order differential invariants \(\lambda, \bar{\lambda}\) and \(\tau\) are unknown functions of the independent variables \(t, u_t, \rho\).

An explicit form of the equations \([5.1] - [5.5]\) is obtained if we use the projected operators of invariant differentiation (see \([3.1], [4.3], [4.4]\))

\[
\delta = \partial_t + (\kappa\rho - u_t^2)\partial_{u_t} + \tau\partial_\rho, \quad Y = \partial_{u_t} + \lambda\partial_\rho, \quad \bar{Y} = \partial_{u_t} + \bar{\lambda}\partial_\rho \quad (5.6)
\]

The commutator relations \([4.2]\) were satisfied identically by the operators of invariant differentiation. On the contrary, for the projected operators \([5.6]\) these commutation relations and even the Jacobi identity are satisfied only on account of the resolving equations. More than that, the theorem proved in \([8]\) claims that commutator algebra \([4.2]\) of the operators of invariant differentiation \(\delta, Y, \bar{Y}\) together with the Jacobi identity gives a commutator representation for the resolving system.
6 Solutions of the resolving system and of the Boyer-Finley equation

To find particular solutions of the resolving system, we impose various simplifying constraints. The most obvious ones, like $\bar{Y} = Y$ or $F = 0$, lead to invariant solutions. These we already know, or can obtain by much simpler standard methods. The weaker assumption that leads to non-invariant solutions is that the operators $Y$ and $\bar{Y}$ commute

$$[Y, \bar{Y}] = 0 \iff \tau = -u_t \rho$$  \hspace{1cm} (6.1)

but $\bar{Y} \neq Y$, i.e. $\bar{\lambda} \neq \lambda$ and also $F \neq 0$. With this Ansatz we find the particular solution of the resolving system [5] (assuming $2\kappa \rho - u_t^2 \geq 0$)

$$\tau = -u_t \rho, \quad \lambda = \kappa u_t + i\sqrt{2\kappa \rho - u_t^2}, \quad \bar{\lambda} = \kappa u_t - i\sqrt{2\kappa \rho - u_t^2}$$  \hspace{1cm} (6.2)

and the expression for $F$ [5] is not needed for obtaining solutions of (1.1).

To reconstruct solutions of the Boyer-Finley equation starting from the particular solution (6.2) of the resolving system we use the procedure of invariant integration which amounts to the transformation of equations to the form of exact invariant derivative [5]. Then we drop the operator of invariant differentiation in such an equation adding the term that is an arbitrary element of the kernel of this operator. This term plays the role of the integration constant. This procedure is described in detail in [5,9,10].

Here we present only the final result for solutions of the Boyer-Finley equation [5] (see also [14] in relation to (6.4))

$$u(t, z, \bar{z}) = \ln \left| \frac{(t + b(z))c'(z)}{c(z) + c(\bar{z})} \right|^2 \text{ for } \kappa = 1$$  \hspace{1cm} (6.3)

$$u(t, z, \bar{z}) = \ln \left| \frac{(t + b(z))c'(z)}{1 + |c(z)|^2} \right|^2 \text{ for } \kappa = -1$$  \hspace{1cm} (6.4)

Here $b(z)$ and $c(z)$ are arbitrary holomorphic functions. One of them is fundamental and labels a particular orbit of solutions. The other one is induced by a conformal symmetry transformation and can be transformed away. These solutions for generic $b(z)$ and $c(z)$ are non-invariant [5].
7 Hodograph transformation of the Boyer-Finley equation as a shortcut to its special solution

Hodograph transformation is the interchange of roles of the unknown $u$ and one of the independent variables $t$: $u = u(t, z, \bar{z}) \mapsto t = t(u, z, \bar{z})$. The transformed equation (7.1) has the form

$$(t_z t_{\bar{z}} - \kappa e^u) t_{uu} = t_u (t_z t_{u\bar{z}} + t_z t_{u\bar{z}} - t_u t_{z\bar{z}} - \kappa e^u)$$

(7.1)

There is an obvious simplifying differential constraint

$$t_z t_{\bar{z}} - \kappa e^u = 0$$

(7.2)

valid only for $\kappa = 1$, since $t_z t_{\bar{z}} > 0$. Indeed, together with (7.1) it implies linear dependence of $t$ on $z$ and $\bar{z}$ and hence $t_z = e^{u/2+i\alpha(u)}$, $t_{\bar{z}} = e^{u/2-i\alpha(u)}$ with the final result for the solution of (7.1)

$$t = e^{u/2+i\alpha(u)} z + e^{u/2-i\alpha(u)} \bar{z} + h(u)$$

(7.3)

where $\alpha(u), h(u)$ are arbitrary functions. This is the solution given in [17].

8 Symmetries of non-invariant solutions

Point symmetries of the equation (7.1) coincide with the symmetries (2.1) of the original equation (1.1) because the hodograph transformation belongs also to a class of point transformations. Since the solution (7.3) is the general solution of the system of differential equations (7.1), (7.2), the symmetries of this solution coincide with the point symmetries of that system. A computer package ‘Liepde’ used together with the PDE solver ‘Crack’ by Thomas Wolf, that runs under REDUCE 3.6 or 3.7, gives the symmetry generator of the solution (7.3) depending on eight arbitrary functions of $u$

$$X = a(u) \partial_u + [2zg(u) + 2\bar{z}e^u b(u) + t e(u) + h(u)] \partial_t$$

$$+ [tb(u) + zc(u) + d(u)] \partial_z + \{\bar{z}[2e(u) - a(u) - c(u)] + f(u) + tg(u)e^{-u}\} \partial_{\bar{z}}$$

(8.1)

Since the solution itself depends only on two arbitrary functions $\alpha(u), h(u)$, it is clear that in the set (8.1) there exist such symmetries with respect to which
the solution (7.3) is invariant for any fixed $\alpha(u), h(u)$. Those symmetries are not symmetries of the Boyer-Finley equation, therefore (7.3) determines conditionally invariant (though non-invariant) solution by the definition of Levi and Winternitz [19].

In a similar way we study symmetries of the solutions (6.3) and (6.4) of the Boyer-Finley equation (1.1) obtained by Calderbank and Tod [14] and in our paper [8]. They are general solutions of the system of PDEs consisting of the original equation (1.1) and the differential equations

\[
\begin{align*}
    u_{zzt} &= 0, \\
    u_{zzz} &= u_z u_{zz} + e^u u_{zt} \left[ \kappa u_t + i \sqrt{2} \kappa e^{-u} u_z \bar{z} - u_t^2 \right]
\end{align*}
\]

plus complex conjugate to the last equation, that follow from (6.2) by using the definitions (3.3), (2.3) of $\tau, \lambda, \bar{\lambda}, \rho$. The symmetries of solutions (6.3) and (6.4) with arbitrary (not fixed!) functions $b(z), c(z)$ coincide with the symmetries of this system of four PDEs which turn out to be exactly the same as symmetries (2.1) of the original equation (1.1). Hence, our solutions with arbitrary fixed generic $b(z), c(z)$ have no symmetries [8]. Therefore they are not conditionally invariant and thus the method of group foliation seems to be the only regular method which could give such solutions.

To prevent misunderstanding, we warn against mixing up the invariance of a particular solution with a fixed choice of arbitrary functions with the invariance of its orbit when arbitrary functions are allowed to transform under a symmetry transformation since the orbit is always an invariant manifold.

9 Transformations of differential invariants and invariant differential operators

Our main problem is to understand how we could arrive at the solution (7.3) by the method of group foliation, i.e. to determine the orbit of this solution.

The first step is to make the hodograph transformation of the whole group foliation structure. In this section we make the hodograph transformation of differential invariants and of invariant differential operators and in the next section we present the hodograph transformation of the resolving equations.

Since the hodograph transformation is a point transformation, it conserves point symmetries, differential invariants and operators of invariant differentiation. We change the definitions of differential invariants $\rho, \eta$ and
invariant differential operators after the hodograph transformation and therefore from now on we label the old invariants and operators with the tilde.

The set of hodograph-transformed differential invariants of up to the second order inclusive consists of the new unknown $t = t(u, z, \bar{z})$ and its derivatives $t_u = 1/u_t$, $t_{uu} = -u_{tt}/u_t^3$ and two more invariants

$$
\rho = e^{-u[t_z t_{z\bar{z}} + t_u t_{u\bar{z}} - t_u(t_z t_{u\bar{z}} + t_z t_{u\bar{z}})]}, \quad \tilde{\rho} = e^{-u} u_{zz} = -\rho/t_u^3
$$

$$
\eta = e^{-u}(t_z t_{uu} - t_u t_{u\bar{z}})(t_z t_{uu} - t_u t_{u\bar{z}}), \quad \tilde{\eta} = e^{-u} u_{zt} u_{\bar{z}t} = \eta/t_u^6
$$

The explicit invariant form of the transformed Boyer-Finley equation (7.1) becomes

$$
t_{uu} = \kappa \rho + t_u \tag{9.1}
$$

Let us denote by $D_u$ the total derivative with respect to $u$ taken for the constant values of $z, \bar{z}$. Since the invariant differential operator $\delta = D_t = (1/t_u) D_u$ is a total derivative with respect to invariant $t$, its definition is not changed: $\tilde{\delta} = \delta$. The definitions of $\Delta, \tilde{\Delta}$ become ($\tilde{\Delta} = -(1/t_u^4) \Delta, \tilde{\Delta} = -(1/t_u^4) \tilde{\Delta}$)

$$
\Delta = e^{-u}(t_z t_{uu} - t_u t_{u\bar{z}})(t_z D_u - t_u D_z), \quad \tilde{\Delta} = e^{-u}(t_z t_{uu} - t_u t_{u\bar{z}})(t_z D_u - t_u D_z)
$$

We choose $t, t_u, \rho$ as the independent invariant variables in accordance with our previous choice. The action of new operators of invariant differentiation on the independent variables becomes

$$
\delta(t) = 1, \quad \delta(t_u) = \kappa \frac{\rho}{t_u} + 1, \quad \Delta(t) = \tilde{\Delta}(t) = 0, \quad \Delta(t_u) = \tilde{\Delta}(t_u) = \eta
$$

$$
\delta(\rho) = \tau, \quad \Delta(\rho) = \sigma, \quad \tilde{\Delta}(\rho) = \tilde{\sigma} \tag{9.2}
$$

where for $\delta(t_u)$ we have used the equation (9.1).

New invariant differential operators $Y = (1/\eta) \Delta$ and $\tilde{Y} = (1/\eta) \tilde{\Delta}$ have the properties

$$
Y(t) = 0, \quad \tilde{Y}(t) = 0, \quad Y(t_u) = \tilde{Y}(t_u) = 1, \quad Y(\rho) = \lambda, \quad \tilde{Y}(\rho) = \tilde{\lambda} \tag{9.3}
$$

# 10 Hodograph transformation of resolving equations

We consider the automorphic equation

$$
\eta = \Delta(t_u) = F(t, t_u, \rho) \tag{10.1}
$$
together with the transformed Boyer-Finley equation (9.1) and derive the resolving equations as compatibility conditions of these two equations. The resulting five equations coincide with the hodograph transform of the old resolving equations with the only change caused by the difference in definitions of differential invariants and invariant differential operators

\[
\delta(F) = \frac{1}{t_u} \left[ \kappa (\lambda + \bar{\lambda}) + 1 \right] F
\]

\[
F(Y(\bar{\lambda}) - \bar{Y}(\lambda)) = (\lambda - \bar{\lambda}) \left( t_u \tau - 3\kappa \frac{\rho^2}{t_u} + \frac{2}{t_u} F - 2\rho \right)
\]

\[
\delta(\lambda) = Y(\tau) + \kappa \frac{\lambda}{t_u} \left( \frac{\rho}{t_u} - \lambda \right)
\]

\[
\delta(\bar{\lambda}) = \bar{Y}(\tau) + \kappa \frac{\bar{\lambda}}{t_u} \left( \frac{\rho}{t_u} - \bar{\lambda} \right)
\]

\[
F(Y(\bar{\lambda}) + \bar{Y}(\lambda)) = (\lambda + \bar{\lambda}) \left( -t_u \tau + 4 \frac{F}{t_u} + 3\kappa \frac{\rho^2}{t_u} + 2\rho \right) + 2\kappa t_u^2 \delta(\tau) - (12\rho + 4\kappa t_u) \left( \tau + \frac{F}{t_u^2} - \kappa \frac{\rho^2}{t_u^2} - \frac{\rho}{t_u} \right)
\]

Thus, we have 4 unknowns \( F, \lambda, \bar{\lambda} \) and \( \tau \) and 3 independent variables \( t, u_t, \rho \). Here \( \delta, Y, \bar{Y} \) denote the projected operators (see (9.2), (9.3))

\[
\delta = \partial_t + \left( \kappa \frac{\rho}{t_u} + 1 \right) \partial_{t_u} + \tau \partial_\rho, \quad Y = \partial_{t_u} + \lambda \partial_\rho, \quad \bar{Y} = \partial_{t_u} + \bar{\lambda} \partial_\rho
\] (10.3)

11 Towards the invariant description of the hodograph solution

Our final goal is to obtain a complete set of independent additional relations between differential invariants which are satisfied on the hodograph solution of \([17]\). Combining them with the resolving equations we would arrive at the particular solution of the resolving system which selects the orbit of this solution and makes it possible to reconstruct the solution \([17]\). Using these invariant relations with the opposite sign of \( \kappa \), we hope to obtain new non-invariant solutions of the Euclidean Boyer-Finley equation \([1, 1]\) with \( \kappa = -1 \) which are more physically interesting.
Up to now we have succeeded to discover only three additional relations between differential invariants satisfied identically on the solution (7.3)

\[ t_u \tau + \frac{\Phi}{t_u} = \frac{1}{4} \left[ (2\rho + t_u)(\lambda + \bar{\lambda}) - i\sqrt{4\Phi - t_u^2(\lambda - \bar{\lambda})} \right], \quad (11.1) \]

\[ \Delta(\Phi) = \frac{i}{4} \sqrt{4\Phi - t_u^2} \left\{ 2\left[ \Phi(\lambda + \bar{\lambda}) + (\rho^2 + t_u\rho)(\lambda - \bar{\lambda}) \right] - \left[ t_u^2 + i(2\rho + t_u)\sqrt{4\Phi - t_u^2} \right] \lambda \right\} + \frac{2}{t_u} \Phi F \quad (11.2) \]

together with the complex conjugate to (11.2) where \( \Phi = F - \rho^2 - t_u\rho \). We use the first one of these equations to exclude \( \tau \) from the remaining equations. It turns out that these additional equations are not sufficient for solving the resolving equations (10.2). We are now in the process of searching for missing relations.

12 Conclusions and outlook

In the first part of this paper we have given a short exposition of our development of the method of group foliation as a method for obtaining non-invariant solutions of PDEs which admit an infinite dimensional symmetry group. The method was illustrated by the physically interesting Boyer-Finley (or heavenly) equation. It was shown how the constraint of commutativity of two operators of invariant differentiation led to non-invariant solutions of the Boyer-Finley equation.

In the second part of the paper we discussed the hodograph solution of Mañas and Alonso. We have shown how it naturally arises after making the hodograph transformation of the Boyer-Finley equation without any reference to the hydrodynamic-type systems. It turns out that this solution has extra symmetries as compared to the equation itself and that, as a consequence, it is conditionally invariant, unlike non-invariant solutions obtained previously.

Our final goal in this part of the paper was to obtain a complete set of independent additional relations between differential invariants which are valid for the hodograph solution of [17] and using them to obtain a particular solution of the resolving system which selects the orbit of the hodograph solution. In this way we hoped to obtain more general solutions of resolving
equations if we would skip one of the additional relations between differential invariants and, as a consequence, more general solutions of the Boyer-Finley equation in a hodograph form. More than that, those extra relations should not be sensitive to the sign of $\kappa$ as it was in the case of solutions (6.3) and (6.4) and hence we could reconstruct solutions of the equation (7.1) with the opposite sign $\kappa = -1$ which was impossible using the method of the paper [17]. Until now we have not yet succeeded to fulfil this program and present here only partial results.

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