DECOMPOSITIONS OF STRATIFIED INSTITUTIONS

RĂZVAN DIACONESCU

ABSTRACT. The theory of stratified institutions is a general axiomatic approach to model theories where the satisfaction is parameterised by states of the models. In this paper we further develop this theory by introducing a new technique for representing stratified institutions which is based on projecting to such simpler structures. On the one hand this can be used for developing general results applicable to a wide variety of already existing model theories with states, such as those based on some form of Kripke semantics. On the other hand this may serve as a template for defining new such model theories. In this paper we emphasise on the former application of this technique by developing general results on model amalgamation and on existence diagrams for stratified institutions. These are two most useful properties to have in institution theoretic model theory.

1. INTRODUCTION

1.1. Stratified institutions. Institution theory is a general axiomatic approach to model theory that has been originally introduced in computing science by Goguen and Burstall [18]. In institution theory all three components of logical systems – namely the syntax, the semantics, and the satisfaction relation between them – are treated fully abstractly by relying heavily on category theory. This approach has impacted significantly both theoretical computing science [30] and model theory as such [7].\footnote{Both mentioned monographs rather reflect the stage of development of institution theory and its applications at the moment they were published or even before that. In the meantime a lot of additional important developments have already taken place. At this moment the literature on institution theory and around, that has been developed over the course of four decades or so is rather vast.} In computing science the concept of institution has emerged as the most fundamental mathematical structure of logic-based formal specifications, a great deal of theory being developed at the general level of abstract institutions. In model theory the institution theoretic approach meant an axiomatic driven redesign of core parts of model theory at a new level of generality – namely that of abstract institutions – independently of any concrete logical system. Moreover, there is a strong interdependency between the two lines of developments.

The institution theoretic approach to model theory has also been refined in order to address directly some important non-classical model theoretic aspects. One such direction is motivated by models with states, which appear in myriad forms in computing science and logic. A typical important class of examples is given by the Kripke semantics (of modal logics) which itself comes in a wide variety of forms. Moreover, the concept of model with states goes beyond Kripke semantics, at least in its conventional acceptations. The institution theory answer to this is given by the theory of stratified institutions introduced in [14, 2] and further developed or invoked in works such as [9, 1, 22], etc.
1.2. **New contributions.** Our work is part of a broader effort to further develop the theory of stratified institutions in several directions that are important either from a model theory as such or a computing science perspective. Our developments consist of

- a new technique for representing stratified institutions,
- on the basis of the above mentioned technique, results establishing a couple of crucial model theoretic properties for stratified institutions.

In what follows we discuss these in more detail.

1.2.1. **Decompositions of stratified institutions.** Historically there have been two major approaches to Kripke semantics within institution theory:

1. The approach introduced in [15] and then used in [25, 8, 13, 10, 21], etc., that considers Kripke semantics as a two-layered concept. A base layer consists of unspecific structures such as the interpretations of sorts (types), function, relation (predicate) symbols, etc. An upper layer consists of the structures specific to Kripke semantics and, at the syntactic level, of modalities. In that approach the base layer can be considered as a parameter and treated fully implicitly as an abstract institution while the upper layer considers explicit Kripke structures and modalities that are parameterised by the base layer.

2. The approach of the stratified institutions that is fully abstract without any explicit Kripke structures and modalities.

The drawback of the former approach is precisely its rigid commitment to a specific common concept of Kripke semantics and modal syntax. Each time one deals with different such concepts, or even goes beyond Kripke semantics, one has to reconstruct this upper level and redevelop most of the theory often by repeating the same ideas. On the other hand, due to its high abstraction level, the latter approach is free of such issues and supports a full top down development process where concepts are introduced axiomatically on a by-need basis. A typical example of this methodology comes from [9] where Kripke semantics and modalities are not assumed explicitly but are treated implicitly in a fully modular and axiomatic manner.

In order to retain some of the benefits of the two-layered approach, such as the hierarchical shape of the respective model theories that at the bottom are based on a concept of possible worlds, here we take a step further in this methodology by introducing a concept of decomposition of a stratified institution. In brief, we associate to a stratified institution a couple of abstract projections to other stratified institutions that in examples correspond to the two layers discussed above. But now the projection corresponding to the upper layer is fully abstract. Most examples / applications of stratified institutions can be presented as decomposed stratified institutions in a meaningful way.

1.2.2. **Model amalgamations and diagrams.** The institution theoretic analysis of model theory has established model amalgamation and the method of diagrams as the most pervasive properties in model theory. While in classical concrete model theory the prominent role of the latter is recognised as such, model amalgamation has a rather implicit role. In fact it is the merit of the institution theoretic approach to model theory to bring model amalgamation to surface and reveal its importance. With respect to diagrams, this concept got a fully abstract institution theoretic formulation in [6]. The literature on institution theoretic model theory
abounds of situations supporting the claims about the role of the two properties and many of these can be found in [7]. At the general level of bare abstract stratified institutions both properties have to be assumed and then established only at the level of the concrete examples. By the decomposition technique discussed above we are able to actually establish these properties at the level of abstract stratified institutions from the corresponding properties of the two “components”. The value of these results reside in the fact that, on the one hand they define classes of abstract stratified institutions that admit these properties, and on the other hand they provide an easy way to establish them in concrete situations because the problem is reduced to the two components where things are much simplified.

1.3. **Summary of contents.** This article is structured as follows:

(1) In a preliminary section we review some basic concepts from the common institution theory and also from stratified institution theory.

(2) In the next section we define the concept of decomposition of stratified institutions.

(3) In a section dedicated to model amalgamation we explore two different concept of model amalgamation that are relevant for stratified institutions. Then we develop a general result on the existence of model amalgamation for decomposed stratified institutions.

(4) In a section dedicated to diagrams we develop a construction of diagrams in decomposed stratified institutions.

2. **Preliminaries**

In this section we recall from the literature some category and institution theoretic notions that will be used in the paper. However Section 2.7 is an exception in the sense that it introduces a new concept. In order to enhance the readability of the paper by keeping it reasonably self contained we deliberately take the choice of a relatively extensive review of the needed concepts as well as of relevant examples.

The contents of this section is as follows:

(1) We fix some category theory notations and list the category theoretic concepts needed in order to study this work.

(2) We recall the definition of institution and present briefly a couple of the most common examples of model theories captured as institutions. We provide a list of ‘sub-institutions’ of first-order logic that will be used in the paper.

(3) We recall one of the two dual concepts of mappings of institutions, namely that of institution morphism, which is the relevant one for our work.

(4) We recall the definition of stratified institutions.

(5) We present the general representation of stratified institutions as ordinary institutions that has been introduced in [9]. This is a mere technical device.

(6) We discuss a representative list of examples of stratified institutions.

(7) We extend the concept of institution morphism from ordinary institution theory to stratified institutions.

(8) By following [9] we present an implicit concept of nominals in stratified institutions.
2.1. **Category theory.** The mathematical structures in institution theory are category theoretic. We usually follow the terminology and notations of [23] with some few notable exceptions. One of them is the way we write compositions. Thus we will use the diagrammatic notation for compositions of arrows in categories, i.e. if \( f : A \to B \) and \( g : B \to C \) are arrows then \( f;g \) denotes their composition. Let \( \text{Set} \) denote the category of sets, \( \text{CAT} \) denote the “quasi-category” of categories, \( |\text{CAT}| \) the collection of all categories. We use \( \Rightarrow \) rather than \( \to \) for natural transformations.

The following category theory concepts are used in our work: opposite (dual) of category, sub-category, functor, functor category, natural transformation, lax natural transformation, comma category, (direct) product, co-product, pushout, pullback, epimorphism (epi), adjunction. All these belong to the somehow elementary level of category theory. In general institution theory seldom requires category theory beyond that level. Familiarity with these concepts is a requirement for being able to follow this work.

2.2. **Institutions.** The seminal mathematical structure of institution theory is given in Definition 2.1 below from [18].

**Definition 2.1** (Institution). An institution \( \mathcal{I} = (\text{Sign}^\mathcal{I}, \text{Sen}^\mathcal{I}, \text{Mod}^\mathcal{I}, |^\mathcal{I} = \mathcal{I}) \) consists of

- a category \( \text{Sign}^\mathcal{I} \) whose objects are called signatures,
- a sentence functor \( \text{Sen}^\mathcal{I} : \text{Sign}^\mathcal{I} \to \text{Set} \) defining for each signature a set whose elements are called sentences over that signature and defining for each signature morphism a sentence translation function,
- a model functor \( \text{Mod}^\mathcal{I} : (\text{Sign}^\mathcal{I})^{\text{op}} \to \text{CAT} \) defining for each signature \( \Sigma \) the category \( \text{Mod}^\mathcal{I}(\Sigma) \) of \( \Sigma \)-models and \( \Sigma \)-model homomorphisms, and for each signature morphism \( \varphi \) the reduct functor \( \text{Mod}^\mathcal{I}(\varphi) \),
- for every signature \( \Sigma \), a binary \( \Sigma \)-satisfaction relation \( |^\mathcal{I} = \mathcal{I} \Sigma \subseteq |\text{Mod}^\mathcal{I}(\Sigma)| \times \text{Sen}^\mathcal{I}(\Sigma) \), such that for each morphism \( \varphi : \Sigma \to \Sigma' \in \text{Sign}^\mathcal{I} \), the Satisfaction Condition

\[
M' \models |^\mathcal{I} = \mathcal{I} \Sigma \varphi(\varphi) \rho \text{ if and only if } \text{Mod}^\mathcal{I}(\varphi)M' \models |^\mathcal{I} = \mathcal{I} \rho
\]

holds for each \( M' \in |\text{Mod}^\mathcal{I}(\Sigma')| \) and \( \rho \in \text{Sen}^\mathcal{I}(\Sigma) \).

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{|^\mathcal{I} = \mathcal{I} \Sigma \varphi} & \text{Sen}^\mathcal{I}(\Sigma) \\
\varphi \downarrow & \text{Mod}^\mathcal{I}(\varphi) \downarrow & \text{Sen}^\mathcal{I}(\varphi) \\
\Sigma' & \xrightarrow{|^\mathcal{I} = \mathcal{I} \Sigma'} & \text{Sen}^\mathcal{I}(\Sigma')
\end{array}
\]

**Notation 2.1.** We may omit the superscripts or subscripts from the notations of the components of institutions when there is no risk of ambiguity. For example, if the considered institution and signature are clear, we may denote \( |^\mathcal{I} = \mathcal{I} \Sigma \varphi \rho \) just by \( \models \). For \( M = \text{Mod}(\varphi)(M') \), we say that \( M \) is the \( \varphi \)-reduct of \( M' \) and that \( M' \) is a \( \varphi \)-expansion of \( M \). Moreover in order to further simplify notations we may sometimes denote \( \text{Sen}(\varphi) \rho \) by \( \varphi \rho \) and \( \text{Mod}(\varphi)M' \) by \( \varphi M' \).
The literature shows myriads of logical systems from computing or from mathematical logic captured as institutions. Many of these are collected in \cite{BM96,BM98}. In fact, an informal thesis underlying institution theory is that any ‘logic’ may be captured by the above definition. While this should be taken with a grain of salt, it certainly applies to any logical system based on satisfaction between sentences and models of any kind. The institutions introduced in the following couple of examples will be used intensively in the paper in various ways.

**Example 2.1** *(Propositional logic (PL)).* This is defined as follows. \( \text{Sign}^{PL} = \text{Set} \), for any set \( P \), \( \text{Sen}(P) \) is generated by the grammar

\[
S ::= P \mid S \land S \mid \neg S
\]

and \( \text{Mod}^{PL}(P) = \langle 2^P, \subseteq \rangle \). For any function \( \varphi : P \to P' \), \( \text{Sen}^{PL}(\varphi) \) replaces the each element \( p \in P \) that occur in a sentence \( \rho \) by \( \varphi(p) \), and \( \text{Mod}^{PL}(\varphi)(M') = \varphi; M \) for each \( M' \in 2^P' \). For any \( P \)-model \( M \subseteq P \) and \( \rho \in \text{Sen}^{PL}(P) \), \( M \models \rho \) is defined by induction on the structure of \( \rho \) by \( (M \models p) = (p \in M) \), \( (M \models \rho_1 \land \rho_2) = (M \models \rho_1) \land (M \models \rho_2) \) and \( (M \models \neg \rho) = \neg (M \models \rho) \).

**Example 2.2** *(First order logic (FOL)).* For reasons of simplicity of notation, our presentation of first order logic as institution considers only its single sorted, without equality, variant. A detailed presentation of full many sorted first order logic with equality as institution may be found in numerous works in the literature (e.g. \cite{BM96}, etc.).

The \( \text{FOL} \) signatures are pairs \( (F = (F_n)_{n \in \omega}, P = (P_n)_{n \in \omega}) \) where \( F_n \) and \( P_n \) are sets of function symbols and predicate symbols, respectively, of arity \( n \). Signature morphisms \( \varphi : (F, P) \to (F', P') \) are tuples \((\varphi^f = (\varphi^f_n)_{n \in \omega}, \varphi^p = (\varphi^p_n)_{n \in \omega})\) such that \( \varphi^f_n : F_n \to F'_n \) and \( \varphi^p_n : P_n \to P'_n \). Thus \( \text{Sign}^{FOL} = \text{Set}^\omega \times \text{Set}^\omega \).

For any \( \text{FOL} \)-signature \( (F, P) \), the set \( S \) of the \( (F, P) \)-sentences is generated by the grammar:

\[
(2) \quad S ::= \pi(t_1, \ldots, t_n) \mid S \land S \mid \neg S \mid (\exists x)S'
\]

where \( \pi(t_1, \ldots, t_n) \) are the atoms with \( \pi \in P_n \) and \( t_1, \ldots, t_n \) being terms formed with function symbols from \( F \), and where \( S' \) denotes the set of \( (F + x, P) \)-sentences with \( F + x \) denoting the family of function symbols obtained by adding the single variable \( x \) to \( F_0 \).

An \( (F, P) \)-model \( M \) is a triple

\[
M = (|M|, \{ M_\sigma : |M|^n \to |M| \mid \sigma \in F_n, n \in \omega \}, \{ M_\pi \subseteq |M|^n \mid \pi \in P_n, n \in \omega \}).
\]

where \( |M| \) is a set called the *carrier* of \( M \). An \( (F, P) \)-model homomorphism \( h : M \to N \) is a function \( |M| \to |N| \) such that \( h(M_\sigma(x_1, \ldots, x_n)) = N_\sigma(h(x_1), \ldots, h(x_n)) \) for any \( \sigma \in F_n \) and \( h(M_\pi) \subseteq N_\pi \) for each \( \pi \in P_n \).

The satisfaction relation \( M \models^FOL (F, P) \rho \) is the usual Tarskian style satisfaction defined on induction on the structure of the sentence \( \rho \).

Given a signature morphism \( \varphi : (F, P) \to (F', P') \), the induced sentence translation \( \text{Sen}^{FOL}(\varphi) \) just replaces the symbols of any \( (F, P) \)-sentence with symbols from \( (F', P') \) according \( \varphi \), and the induced model reduct \( \text{Mod}^{FOL}(\varphi)(M') \) leaves the carrier set as it is and for any \( x \) function or predicate symbol of \( (F, P) \), it interprets \( x \) as \( M'_\varphi(x) \).
In what follows we shall also consider the following parts (or ‘sub-institutions’) of FOL that are determined by restricting the FOL signatures as follows:

- **REL**: no function symbols (hence $\text{Sign}_{\text{REL}} \cong \text{Set}^\omega$);
- **REL C**: no function symbols of arity greater than 0;
- **BREL**: no function symbols and only one binary predicate symbol $\lambda$ (hence $\text{Sign}_{\text{BREL}} \cong \{\lambda\}$);
- **SETC**: no predicate symbols and no function symbols of arity greater than 0 (hence $\text{Sign}_{\text{SETC}} \cong \text{Set}$);
- **BREL C**: one binary predicate symbol and no function symbols of arity greater than 0 (hence $\text{Sign}_{\text{BREL C}} \cong \text{Set}$);

2.3. **Institution morphisms.** From the perspective of the mathematical structure, institution morphisms [18] are just ‘homomorphisms of institutions’. So they are mappings between institutions that preserve the mathematical structure of institutions.

**Definition 2.2** (Morphism of institutions). Given two institutions $\mathcal{I}_i = (\text{Sign}_i, \text{Sen}_i, \text{Mod}_i, \models_i)$, with $i \in \{1, 2\}$, an institution morphism $(\Phi, \alpha, \beta) : \mathcal{I}_2 \rightarrow \mathcal{I}_1$ consists of

- a signature functor $\Phi : \text{Sign}_2 \rightarrow \text{Sign}_1$,
- a natural transformation $\alpha : \Phi; \text{Sen}_1 \Rightarrow \text{Sen}_2$, and
- a natural transformation $\beta : \text{Mod}_2 \Rightarrow \Phi^\text{op}; \text{Mod}_1$

such that the following Satisfaction Condition holds for any $\mathcal{I}_2$-signature $\Sigma_2$, $\Sigma_2$-model $M_2$ and $\Phi(\Sigma_2)$-sentence $\rho$:

$$M_2 \models_2 \alpha_{\Sigma_2} \rho \text{ if and only if } \beta_{\Sigma_2} M_2 \models_1 \rho.$$ 

There is a dual notion of ‘homomorphism of institutions’ in which the direction of the sentence translations is reversed [26, 33, 34, 27, 19]. These are currently called comorphism and although they bear symmetry with morphisms their usage is very different. While institution morphisms have a projection feeling the comorphism have an embedding feeling. However the latter are also used for encoding ‘more complex’ institutions to ‘simpler’ institution by using the technique of institutional theories (more details on that may be found in [7]).

For examples we refer to [7]. Under a straightforward concept of composition, defined component-wise on the three components (see [7]), we get a category with the institutions as objects and the institution morphisms as arrows.

2.4. **Stratified institutions.** Informally, the main idea behind the concept of stratified institution as introduced in [2] is to enhance the concept of institution with ‘states’ for the models. Thus each model $M$ comes equipped with a set $[M]$. A typical example is given by the Kripke models, where $[M]$ is the set of the possible worlds in the Kripke structure $M$. However this is not the only possibility for models with states.

The following definition has been given in [9] and represents an important upgrade of the original definition from [2], the main reason being to make the definition of stratified institutions really usable for doing in-depth model theory. Independently another upgrade has been proposed in [1]; however there is a strong convergence between the two upgrades.
**Definition 2.3** (Stratified institution). A stratified institution $\mathcal{S}$ is a tuple 

$$(\text{Sign}^\mathcal{S}, \text{Sen}^\mathcal{S}, \text{Mod}^\mathcal{S}, \models^\mathcal{S})$$

consisting of:

– a category $\text{Sign}^\mathcal{S}$ of signatures,
– a sentence functor $\text{Sen}^\mathcal{S} : \text{Sign}^\mathcal{S} \to \text{Set}$;
– a model functor $\text{Mod}^\mathcal{S} : (\text{Sign}^\mathcal{S})^{op} \to \text{CAT}$;
– a “stratification” lax natural transformation $[\_]^\mathcal{S} : \text{Mod}^\mathcal{S} \Rightarrow \text{SET}$, where $\text{SET} : \text{Sign}^\mathcal{S} \to \text{CAT}$ is a functor mapping each signature to $\text{Set}$; and
– a satisfaction relation between models and sentences which is parameterized by model states,

$$(3) \quad \text{Mod}^\mathcal{S}(\varphi)M'^{(\models^\mathcal{S})_{\Sigma}^{M'}} \models^\mathcal{S} \varphi \text{ if and only if } \text{Mod}^\mathcal{S}(\varphi)M'^{(\models^\mathcal{S})_{\Sigma}^{M'}} \models^\mathcal{S} \varphi \text{ holds for any signature morphism } \varphi : \Sigma \to \Sigma', \Sigma'-\text{model } M', w \in [M']^\mathcal{S}_{\Sigma'} \text{, and } \Sigma-\text{sentence } \rho.$$

Like for ordinary institutions, when appropriate we shall also use simplified notations without superscripts or subscripts that are clear from the context.

The lax natural transformation property of $[\_]$ is depicted in the diagram below

$$\Sigma'' \xrightarrow{\varphi'} \Sigma' \xrightarrow{\varphi} \Sigma \xrightarrow{\text{Mod}(\Sigma)} \text{Set} \xrightarrow{[\_]} \text{Set}$$

with the following compositionality property for each $\Sigma''$-model $M''$:

$$(4) \quad [M'']_{(\varphi';\varphi)} = [M']_{\varphi'}; [\text{Mod}(\varphi')(M'')]_{\varphi}.$$

Moreover the natural transformation property of each $[\_]_\varphi$ is given by the commutativity of the following diagram:

$$(5) \quad \xymatrix{ M' \ar[d]_{\kappa'} \ar[r]^{[M']_{\Sigma}} \ar[rd]_{[\text{Mod}(\varphi')(M')]_{\Sigma}} & [M']_{\Sigma} \ar[d]_{[\text{Mod}(\varphi')]_{\Sigma}} \\
N' \ar[r]_{[\text{Mod}(\varphi')]_{\Sigma}} & [N']_{\Sigma} \ar[r]_{[\text{Mod}(\varphi')]_{\Sigma}} & [\text{Mod}(\varphi')(N')]_{\Sigma} \ar[d]_{[\text{Mod}(\varphi')(\kappa')]_{\Sigma}} \ar[ru]_{[\text{Mod}(\varphi')]_{\Sigma}} \ar[r] & \text{Set} }$$

The satisfaction relation can be presented as a natural transformation $\models : \text{Sen} \Rightarrow [\text{Mod}(\_)] : \text{Sign} \to \text{Set}$ where the functor $[\text{Mod}(\_)] : \text{Sign} \to \text{Set}$ is defined by

– for each signature $\Sigma \in |\text{Sign}|$, $[\text{Mod}(\Sigma)] \to \text{Set}$ denotes the set of all the mappings $f : [\text{Mod}(\Sigma)] \to \text{Set}$ such that $f(M) \subseteq [M]_{\Sigma}$; and
– for each signature morphism $\varphi : \Sigma \to \Sigma'$,

$$[\Mod(\varphi) \to \Set](f)(M') = [M']_{\varphi}^{-1}(f(\Mod(\varphi)M')).$$

A straightforward check reveals that the Satisfaction Condition (3) appears exactly as the naturality property of $|=_{\Sigma}$:

$$\Sigma \quad \xrightarrow{\text{Sen}(\Sigma)} \quad [\text{Mod}(\Sigma) \to \Set]$$

$$\varphi \quad \downarrow \quad \xrightarrow{\text{Sen}(\varphi)} \quad [\Mod(\varphi) \to \Set]$$

$$\Sigma' \quad \xrightarrow{\text{Sen}(\Sigma')}{|_{\Sigma'}} \quad [\text{Mod}(\Sigma') \to \Set]$$

Ordinary institutions are the stratified institutions for which $[[M]]_{\Sigma}$ is always a singleton set. In Definition 2.3 we have removed the surjectivity condition on $[M']_{\varphi}$ from the definition of the stratified institutions of [2] and will rather make it explicit when necessary. This is motivated by the fact that most of the results developed do not depend upon this condition which however holds in all examples known by us. In fact in many of the examples $[M']_{\varphi}$ are even identities, which makes $[[\_]]$ a strict rather than lax natural transformation. In such cases the stratified institution itself is called a strict stratified institution. A notable exception, when $[[\_]]$ is a proper lax natural transformation is given by Example 2.8. Also the definition of stratified institution of [2] did not introduce $[[\_]]$ as a lax natural transformation, but rather as an indexed family of mappings without much compositionality properties, which was enough for the developments in [2].

The following very expected property does not follow from the axioms of Definition 2.3, hence we impose it explicitly. It holds in all the examples discussed in this paper.

**Assumption:** In all considered stratified institutions the satisfaction is preserved by model isomorphisms, i.e. for each $\Sigma$-model isomorphism $h : M \to N$, each $w \in [[M]]_{\Sigma}$, and each $\Sigma$-sentence $\rho$,

$$M \models w \rho \text{ if and only if } N \models [h]_{w} \rho.$$

2.5. Reducing stratified institutions to ordinary institutions. The following construction from [9] will be used systematically in what follows for reducing stratified institution theoretic concepts to ordinary institution theoretic concepts, and consequently for reusing results from the latter to the former realm.

**Fact 2.1.** Each stratified institution $\mathcal{S} = (\text{Sign}, \text{Sen}, \text{Mod}, [[\_]], |=)$ determines the following ordinary institution $\mathcal{S}^{\times} = (\text{Sign}, \text{Sen}, \text{Mod}^{2}, [\_], |\_|)$ (called the local institution of $\mathcal{S}$) where

– the objects of $\text{Mod}^{2}(\Sigma)$ are the pairs $(M, w)$ such that $M \in |\text{Mod}(\Sigma)|$ and $w \in [[M]]_{\Sigma}$;

– the $\Sigma$-homomorphisms $(M, w) \to (N, v)$ are the pairs $(h, w)$ such that $h : M \to N$ and $[h]_{\Sigma}w = v$;

– for any signature morphism $\varphi : \Sigma \to \Sigma'$ and any $\Sigma'$-model $(M', w')$

$$\text{Mod}^{2}(\varphi)(M', w') = (\text{Mod}(\varphi)M', [[M']_{\varphi}w']);$$

– for each $\Sigma$-model $M$, each $w \in [[M]]_{\Sigma}$, and each $\rho \in \text{Sen}(\Sigma)$

$$(M, w) \models_{\Sigma}^{w} \rho = (M \models_{\Sigma}^{w} \rho).$$
The preservation of | - under model isomorphisms imply the preservation of \( |--^{\sharp} \) under model isomorphisms. This follows immediately by noting that \((h, w)\) is a model isomorphism in \(S^{\sharp}\) if and only if \(h\) is a model isomorphism in \(S\).

The following second interpretation of stratified institutions as ordinary institutions has been given in [2]. Note that unlike \(S^{\sharp}\) above, \(S^{\star}\) below shares with \(S\) the model functor.

**Definition 2.4.** For any stratified institution \(S = (\text{Sign}, \text{Sen}, \text{Mod}, [\_], \models)\) we say that \([\_]\) is surjective when for each signature morphism \(\varphi : \Sigma \to \Sigma'\) and each \(\Sigma'-\text{model} M'\), \([M']_{\varphi} : [\text{Mod}(\varphi)M']_{\Sigma'} \to [\text{Mod}(\varphi)M']_{\Sigma}\) is surjective.

**Fact 2.2.** Each stratified institution \(S = (\text{Sign}, \text{Sen}, \text{Mod}, [\_], \models)\) with \([\_]\) surjective determines an (ordinary) institution \(S^{\star} = (\text{Sign}, \text{Sen}, \text{Mod}, \models^{\star})\) (called the global institution of \(S\)) by defining

\[
(M \models^{\star} \rho) = \bigwedge \{M \models_{\Sigma} w \rho \mid w \in [M]_{\Sigma}\}.
\]

From now on whenever we invoke an institution \(S^{\star}\) we tacitly assume that \([\_]\) is surjective.

The institutions \(S^{\sharp}\) and \(S^{\star}\) represent generalizations of the concepts of local and global satisfaction, respectively, from modal logic (e.g. [5]). While \(S^{\star}\) “forgets” the stratification of \(S\), \(S^{\sharp}\) fully retains it (but in an implicit form). This is the reason why \(S^{\sharp}\) rather than \(S^{\star}\) can be used for reflecting concepts and results from ordinary institution theory to stratified institutions. It is important to avoid a possible confusion regarding \(S^{\sharp}\), namely that through the flattening represented by the \(\sharp\) construction stratified institution theory gets reduced to ordinary institution theory. This cannot be the case because although \(S^{\sharp}\) being an ordinary institution it has a particular character induced by the stratified structure of \(S\). This means that many general institution theoretic concepts are not refined enough to reflect properly the stratification aspects.

### 2.6. Concrete examples of stratified institutions.

Most of the examples presented below are various forms of modal logics with Kripke semantics. However a few of them go beyond the Kripke semantics. They can be found in greater detail in [9].

**Example 2.3 (Modal propositional logic \((MPL)\)).** This is the most common form of modal logic (e.g. [5], etc.).

Let \(\text{Sign}^{\text{MPL}} = \text{Set}\). For any signature \(P\), commonly referred to as ‘set of propositional variables’, the set of its sentences \(\text{Sen}^{\text{MPL}}(P)\) is the set \(S\) defined by the following grammar

\[
S ::= P \mid S \land S \mid \neg S \mid \Box S
\]

A \(P\)-model is Kripke structure \((W, M)\) where

- \(W = (|W|, W_\lambda)\) consists of set (of ‘possible worlds’) \(|W|\) and an ‘accessibility’ relation \(W_\lambda \subseteq |W| \times |W|\); and

- \(M : |W| \to 2^P\).

A homomorphism \(h : (W, M) \to (V, N)\) between Kripke structures is a homomorphism of binary relations \(h : W \to V\) (i.e. \(h : |W| \to |V|\) such that \(h(W_\lambda) \subseteq V_\lambda\)) and such that for each \(w \in |W|, M^w \subseteq N^{h(w)}\).
The satisfaction of any \( P \)-sentence \( \rho \) in a Kripke structure \((W, M)\) at \( w \in |W|\) is defined by recursion on the structure of \( \rho \):

- \( [(W, M) \models _P \pi] = (\pi \in M^w) \);
- \( [(W, M) \models _P \rho_1 \land \rho_2] = ((W, M) \models _P \rho_1) \land ((W, M) \models _P \rho_2) \);
- \( [(W, M) \models _P \neg \rho] = \neg ((W, M) \models _P \rho) \); and
- \( [(W, M) \models _P \Diamond \rho] = \bigvee_{(w, w') \in W} ((W, M) \models _P \rho) \).

For any function \( \varphi : P \to P' \) the \( \varphi \)-translation of a \( P \)-sentence just replaces each \( \pi \in P \) by \( \varphi(\pi) \) and the \( \varphi \)-reduct of a \( P' \)-structure \((W, M')\) is the \( P \)-structure \((W, M)\) where for each \( w \in |W|\), \( M^w = \varphi; M'^w \).

The stratification is defined by \([(W, M)]_P = |W|\).

Various ‘sub-institutions’ of \( MPL \) are obtained by restricting the semantics to particular classes of frames. Important examples are \( MPL_t \), \( MPL_s4 \), and \( MPL_s5 \) which are obtained by restricting the frames \( W \) to those which are respectively, reflexive, preorder, or equivalence (see e.g. [5]).

**Example 2.4 (First order modal logic (MFOL)).** First order modal logic [16] extends classical first order logic with modalities in the same way propositional modal logic extends classical propositional logic. However there are several variants that differ slightly in the approach of the quantifications. Here we present a capture of one of the most common variants of first order modal logic as a stratified institution.

\( MFOL \) has the category of signatures of \( FOL \) but for the sentences adds \( S ::= \Diamond S \) to the \( FOL \) grammar (2). The \( MFOL \) \((F, P)\)-models upgrade the \( MPL \) Kripke structures \((W, M)\) to the first order situation by letting \( M : |W| \to |Mod^FOL|(F, P)| \) such that the following sharing conditions hold: for any \( i, j \in |W| \), \(|M'| = |M|\) and also \( M^i_x = M^j_x \) for each constant \( x \in F_0 \). The concept of \( MFOL \)-model homomorphism is also an upgrading of the concept of \( FOL \)-model homomorphism as follows: \( h : (W, M) \to (V, N) \) is pair \((h_0, h_1)\) where \( h_0 : W \to V \) is a homomorphism of binary relations (like in \( MPL \)) and \( h_1 : M^w \to N^{h_0(w)} \) is an \((F, P)\)-homomorphism of \( FOL \)-models for each \( w \in |W| \).

The satisfaction \((W, M) \models _{MFOL}(F, P) \rho\) is defined by recursion on the structure of \( \rho \), like in \( MPL \) for \( \land, \neg \), and \( \Diamond \), for the atoms the \( FOL \) satisfaction relation is used, and for the quantifier case \((W, M) \models _{(F, P)} (\exists x) \rho\) if and only if there is a valuation of \( x \) into \(|M|\) such that \((W, M') \models _{(F \times P)} (\exists x) \rho\) for the corresponding expansion \((W, M')\) of \((W, M)\) to \((F \times x, P)\). (This makes sense because in any \( MFOL \) Kripke structure the interpretations of the carriers and of the constants are shared.)

The translation of sentences and the model reducts corresponding to an \( MFOL \) signature morphism are obtained by the obvious blend of the corresponding translations and reducts, respectively, in \( MPL \) and \( FOL \).

The stratification is like in \( MPL \), with \([(W, M)]_{(F, P)} = |W|\).

In the institution theory literature (e.g. [7, 15, 25, 8]) first order modal logic is often considered in a more general form in which the symbols that have shared interpretations are ‘user defined’ rather than being ‘predefined’ like here. In short this means that the signatures exhibit designated symbols (sorts, function, or predicate) that are ‘rigid’ in the sense that in a given Kripke structure they share the same interpretations across the possible worlds. For
the single reason of making the reading easier we stick here with a simpler variant that has constants and the single sort being predefined as rigid.

**Example 2.5 (Hybrid logics (HPL, HFOL)).** Hybrid logics [28, 4] refine modal logics by adding explicit syntax for the possible worlds. Our presentation of hybrid logics as stratified institutions is related to the recent institution theoretic works on hybrid logics [25, 8].

The refinement of modal logics to hybrid ones is achieved by adding a set component (Nom) to the signatures for the so-called ‘nominals’ and by adding to the respective grammars

\[(8) \quad S ::= \text{i-sen} | @i S | (\exists i) S'
\]

where \(i \in \text{Nom} \) and \(S'\) is the set of the sentences of the signature that extends \(\text{Nom}\) with the nominal variable \(i\). The models upgrade the respective concepts of Kripke structures to \((W, M)\) by adding to \(W\) interpretations of the nominals, i.e. \(W = (|W|, \{W_i \in |W| \mid i \in \text{Nom}\}, W_\lambda)\). The satisfaction relations between models (i.e. Kripke structures) and sentences extend the satisfaction relations of the corresponding non-hybrid modal institutions with

- \((w, M) \models i - \text{sen} = (w_i = w)\);
- \((w, M) \models \@i \rho = ((W, M) \models W_i \rho);\) and
- \((w, M) \models (\exists i) \rho = \bigvee \{(W', M) \models w \rho \mid W' \text{ expansion of } W \text{ to } \text{Nom}+i\} \).

Note that quantifiers over nominals allow us to simulate the binder operator \((\downarrow, \rho)\) of [20] by \((\forall i) i \Rightarrow \rho\).

The translation of sentences and model reducts corresponding to signature morphisms are canonical extensions of the corresponding concepts from MPL and MFOL.

The stratifications of HPL and HFOL are like for MPL and MFOL, i.e. \([[(W, M)](\text{Nom}, \Sigma) = |W|\).

**Example 2.6 (Polyadic modalities (MMPL, MHPL, MMFO L, MHFOL)).** Multi-modal logics (e.g. [17]) exhibit several modalities instead of only the traditional \(\Diamond\) and \(\Box\) and moreover these may have various arities. If one considers the sets of modalities to be variable then they have to be considered as part of the signatures. We may extend each of MPL, HPL, MFOL and HFOL to the multi-modal case,

- by adding an ‘\(M\)’ in front of each of these names;
- by adding a component \(\Lambda = (\Lambda_n)_{n \in \omega} \) to the respective signature concept (with \(\Lambda_n\) standing for the modalities symbols of arity \(n\)), e.g. an MHFOL signature would be a tuple of the form \((\text{Nom}, \Lambda, (F, P))\);
- by replacing in the respective grammars the rule \(S ::= \Diamond S\) by the set of rules
  \[\{S ::= (\lambda)S^n \mid \lambda \in \Lambda_{n+1}, n \in \omega\}\);
- by replacing the binary relation \(W_\lambda\) from the models \((W, M)\) with a set of interpretations
  \[\{W_\lambda \subseteq |W|^n \mid \lambda \in \Lambda_n, n \in \omega\}\].

Consequently the definition of the satisfaction relation gets upgraded with

for each \(\lambda \in \Lambda_{n+1}, ((W, M) \models w (\lambda) (\rho_1, \ldots, \rho_n)) = \bigvee_{(w, w_1, \ldots, w_n) \in W_\lambda} \bigwedge_{1 \leq i \leq n} (W, M) \models w_i \rho_i\).

The stratification is the same like in the previous examples, i.e. \([[(W, M)]_{(\text{Nom}, \Lambda, \Sigma)} = |W|\).
Example 2.7 (Modalizations of institutions; HHPL). In a series of works [15, 25, 8] modal logic and Kripke semantics are developed by abstracting away details that do not belong to modality, such as sorts, functions, predicates, etc. This is achieved by extensions of abstract institutions (in the standard situations meant in principle to encapsulate the atomic part of the logics) with the essential ingredients of modal logic and Kripke semantics. The result of this process, when instantiated to various concrete logics (or to their atomic parts only) generate uniformly a wide range of hierarchical combinations between various flavours of modal logic and various other logics. Concrete examples discussed in [15, 25, 8] include various modal logics over non-conventional structures of relevance in computing science, such as partial algebra, preordered algebra, etc. Various constraints on the respective Kripke models, many of them having to do with the underlying non-modal structures, have also been considered. All these arise as examples of stratified institutions like the examples presented above in the paper. This great multiplicity of non-conventional modal logics constitute an important range of applications for this work.

An interesting class of examples that has emerged quite smoothly out of the general works on hybridization\(^2\) of institutions is that of multi-layered hybrid logics that provide a logical base for specifying hierarchical transition systems (see [24]). As a single simple example let us present here the double layered hybridization of propositional logic, denoted HHPL.\(^3\) This amounts to a hybridization of HPL, its models thus being “Kripke structures of Kripke structures”.

The HHPL signatures are triples \((\text{Nom}^0, \text{Nom}^1, P)\) with \text{Nom}^0 and \text{Nom}^1 denoting the nominals of the first and second layer of hybridization, respectively. The \((\text{Nom}^0, \text{Nom}^1, P)\)-sentences are built over the two hybridization layers by taking the \((\text{Nom}^0, P)\)-sentences as atoms in the grammar for the HPL sentences with nominals from \text{Nom}^1. In order to prevent potential ambiguities, in general we tag the symbols of the respective layers of hybridization by the superscripts 0 (for the first layer) and 1 (for the second layer). This convention should include nominals and connectives \((\Diamond, \wedge, \text{etc.})\) as well as quantifiers. For instance, the expression \(\Diamond_p \text{Nom}^0 \wedge \text{Nom}^1 \Diamond^1 \rho\) is a sentence of HHPL where the symbols \(k\) and \(j\) represent nominals of the first and second level of hybridization and \(\rho\) a PL sentence. On the other hand, according to this tagging convention the expression \(\Diamond_j \text{Nom}^0 \wedge \text{Nom}^1 \Diamond^1 \rho\) would not parse.

Our tagging convention extends also to HHPL models. A \((\text{Nom}^0, \text{Nom}^1, P)\)-model is a pair \((W^1, M^1)\) with \(W^1\) being a \(\text{Mod}^{BREL}(\lambda)\) model and \(M^1 = ((M^1)^w)_{w \in |W^1|}\) where \((M^1)^w\) is a \((\text{Nom}^0, P)\)-model in HPL, denoted \((W^0)^w, (M^0)^w\). We also require that for all \(w, w' \in |W^1|\), we have that \(|(W^0)^w| = |(W^0)^{w'}|\) and \((W^0)^w_i = (W^0)^{w'}_i\) for each \(i \in \text{Nom}^0\).

These definitions extend in the obvious way to signature morphisms, sentence translations, model reducts and satisfaction relation. We leave these details as exercise for the reader. Then HHPL has the same stratified structure like HPL and HFOL, namely \([([W^1, M^1])_{\text{Nom}^0, \text{Nom}^1, P}] = |W^1|\).

\(^2\)I.e. Modalization including also hybrid logic features.
\(^3\)Other interesting examples that may be obtained by double or multiple hybridizations of logics would be HHFOL, HHHPL, etc., and also their polyadic multi-modalities extensions.
It is easy to see that in \(HHPL\) the semantics of the Boolean connectors and of the quantifications with nominals of the lower layer is invariant with respect to the hybridization layer; this means that in these cases the tagging is not necessary. For example if \(\rho\) is an \(HPL\) sentence then \(\langle v^0, 0 \rangle \rho\) and \(\langle v^0, 0 \rangle \rho\) are semantically equivalent, while if \(\rho\) is not an \(HPL\) sentence (which means it has some ingredients from the second layer of hybridization) then \(\langle v^0, 0 \rangle \rho\) would not parse. In both cases just using the notation \(\langle v^0 \rangle \rho\) would not carry any ambiguities.

The next series of examples include multi-modal first order logics whose semantics are given by ordinary first order rather than Kripke structures.

**Example 2.8 (Multi-modal open first order logic (**\(OFOL\), **\(MOFOL\), **\(HOFOL\), **\(HMOFOL\)).**\)

The stratified institution \(OFOL\) is the \(FOL\) instance of \(St(I)\), the ‘internal stratification’ abstract example developed in [2]. An \(OFOL\) signature is a pair \((\Sigma, X)\) consisting of \(FOL\) signature \(\Sigma\) and a finite block of variables. An \(OFOL\) signature morphism \(\varphi : (\Sigma, X) \rightarrow (\Sigma', X')\) is just a \(FOL\) signature morphism \(\varphi : \Sigma \rightarrow \Sigma'\) such that \(X \subseteq X'\).

We let \(\text{Sen}^{OFOL}((F, P), X) = \text{Sen}^{FOL}(F + X, P)\) and \(\text{Mod}^{OFOL}((F, P), X) = \text{Mod}^{FOL}(F, P)\).

For each \((F, P), X\)-model \(M\), each \(w \in |M|^{X}\), and each \((F, P), X\)-sentence \(\rho\) we define
\[
(M |_{\langle (F, P), X \rangle}^w \rho) = (M |_{\langle F + X, P \rangle}^w \rho)
\]
where \(M^w\) is the expansion of \(M\) to \((F + X, P)\) such that \(M^w_X = w\). This is a stratified institution with \([M]_{\Sigma, X} = |M|^X\) for each \((\Sigma, X)\)-model \(M\). For any signature morphism \(\varphi : (\Sigma, X) \rightarrow (\Sigma', X')\) and any \((\Sigma', X')\)-model \(M'\), \([M']_{\varphi}(a) = a|_X\) (i.e. the restriction of \(a\) to \(X\)). Note that \([M']_{\varphi}\) is surjective and that this provides an example when \([\cdot]\) is a proper lax natural transformation.

We may refine \(OFOL\) to a multi-modal logic \((MOFOL)\) by adding
\[
\{S := \langle \pi \rangle S^n \mid \pi \in P_{n+1}, n \in \omega\}
\]
to the grammar defining each \(\text{Sen}^{OFOL}((F, P), X)\) and consequently by extending the definition of the satisfaction relation with
\[
(M \models^w \langle \pi \rangle (\rho_1, \ldots, \rho_n)) = \bigvee_{(w, w_1, \ldots, w_n) \in (M^X)^n} \bigwedge_{1 \leq i \leq n} (M \models^{w_i} \rho_i) \text{ for each } \pi \in P_{n+1}, n \in \omega.
\]
(Here and elsewhere \(M^X\) denotes the \(X\)-power of \(M\) in the category of \(FOL\) \((F, P)\)-models.)

Or else we may refine \(OFOL\) with nominals \((HOFOL)\) by adding the grammar for nominals (8), for each constant \(i \in F_0\), to the grammar defining each \(\text{Sen}^{OFOL}((F, P), X)\) and consequently extending the definition of the satisfaction relation with
\[
\begin{align*}
(M \models^w \langle (F, P), X \rangle \text{i-sen}) &= ((M^X)_i = w); \\
(M \models^w \langle (F, P), X \rangle \otimes_i \rho) &= (M \models^w \langle (F, P), X \rangle \rho); \\
(M \models^w \langle (F, P), X \rangle (\exists i) \rho) &= \bigvee \{M' \models^{w} \langle (F + i, P), X \rangle \rho \mid M' \text{ expansion of } M \text{ to } (F + i, P)\}.
\end{align*}
\]
We can also have \(HMOFOL\) as the blend between \(HOFOL\) and \(MOFOL\).

**2.7. Stratified institution morphisms.** They extend the concept of institution morphism (Definition 2.2) from ordinary institutions to stratified institutions. The 2-dimensional aspect
of the stratified institutions leads to a higher complexity in the following definition of morphisms of stratified institutions. This concept will be instrumental when defining our concept of decompositions of stratified institutions.

**Definition 2.5 (Morphism of stratified institutions).** Given two stratified $S$ and $S'$ a stratified institution morphism $(\Phi, \alpha, \beta) : S' \to S$ consists of

- a functor $\Phi : \text{Sign}' \to \text{Sign}$,
- a natural transformation $\alpha : \Phi; \text{Sen} \Rightarrow \text{Sen}'$, and
- a lax natural transformation $\beta : \text{Mod}' \Rightarrow \Phi \text{op}; \text{Mod}'$ such that $\beta; \Phi \text{op} \models \models$,

and such that the following Satisfaction Condition holds for any $S'$-signature $\Sigma'$, any $\Sigma'$-model $M'$, any $w \in [M']_{\Sigma'}$ and any $\Phi(\Sigma')$-sentence $\rho$:

$$M' \models^w \alpha_{\Sigma'} \rho \text{ if and only if } \beta_{\Sigma'} M' \models^w \rho.$$

When $\beta$ is strict, $(\Phi, \alpha, \beta)$ is called strict too.

The condition on $\beta$ means the following:

- for each $S'$-signature $\Sigma$ the following diagram commutes

$$\begin{array}{ccc}
\text{Mod}'(\Sigma) & \xrightarrow{\beta_\Sigma} & \text{Mod}(\Phi \Sigma) \\
[\_\Sigma] & \searrow & \downarrow \Phi_{\Sigma} \\
\text{Set} & \xrightarrow{\_\Sigma} & \text{Set}
\end{array}$$

- for each $S'$-signature morphism $\varphi : \Sigma \to \Omega$

$$\beta_\Sigma [\_\varphi] = [\_\varphi'] \text{ which can be visualised as the commutativity of the following diagram:}$$

Morphisms of stratified institutions form a category under a composition that is defined component-wise like in the case of morphisms of ordinary institutions:

$$(\Phi', \alpha', \beta') \circ (\Phi, \alpha, \beta) = (\Phi; \Phi', \alpha \Phi'; \alpha', \beta; \beta \Phi \text{op}).$$

**Fact 2.3.** For any morphism of stratified institutions $S' \to S$, if $S$ is strict then $S'$ is strict too.

The proof of the following result consists of straightforward verifications; we will skip them.
Proposition 2.1. Any strict stratified institution morphism \((\Phi, \alpha, \beta) : S' \to S\) induces an institution morphism \((\Phi, \alpha, \beta^\sharp) : S'^\sharp \to S^\sharp\) where for each \(S'\)-signature \(\Sigma\), each \(\Sigma\)-homomorphism \(h : M \to N\) and each \(w \in [\![M]\!]_\Sigma\)

\[
\beta^\sharp h = \beta h : (\beta^\Sigma M, w) \to (\beta^\Sigma N, [\![h]\!]_\Sigma w).
\]

2.8. Nominals in stratified institutions. The definitions of this section are inherited from [9].

Definition 2.6 (Nominals extraction). Given a stratified institution \(S\), a nominals extraction is a pair \((N, Nm)\) consisting of a functor \(N : \text{Sign}_S \to \text{Sign}_{SETC}\) and a lax natural transformation \(Nm : \text{Mod}_S \Rightarrow N; \text{Mod}_{SETC}\) such that \([\![\cdot]\!] = Nm; N(\text{Mod}_{SETC} \Rightarrow \text{SET})\).

Fact 2.4. A nominals extraction \((N, Nm)\) is precisely a stratified institution morphism \((N, \emptyset, Nm) : S \to \text{SETC}\) where \(\text{SETC}\) is considered as a stratified institution with no sentences and for each \(\text{SETC}\)-model \(M\), \([\![M]\!] = |M|\) (the underlying set of \(M\)).

Example 2.9. The following table shows some nominals extractions for the stratified institutions introduced above. Note that \(HHPL\) admits two such nominals extractions.

| stratified institution | \(N\) | \(Nm\) |
|-----------------------|-------|-------|
| \(HPL, HFOL, MHPL, MHFOL\) | \(N(\text{Nom}, \Sigma) = \text{Nom}\) | \(Nm(\text{Nom}, \Sigma)(W, M) = (|W|, (W_i)_{i \in \text{Nom}})\) |
| \(HHPL\) | \(N(\text{Nom}^0, \text{Nom}^1, P) = \text{Nom}^0\) | \(Nm(W^1, M^1) = (|W^0|^1, (W_i^0)_{i \in \text{Nom}^0})\) |
| \(HOFOL, HMOFOL\) | \(N((F, P), X) = F_0\) | \(Nm(M) = (|M|^{X}, ((M^X)_i)_{i \in F_0})\) |

Definition 2.7. Let \(S\) be a stratified institution endowed with a nominals extraction \(N, Nm\). For any \(i \in \text{N}(\Sigma)\)

- a \(\Sigma\)-sentence \(i\text{-sen}\) is an \(i\)-sentence when
  \([M \models w \ i\text{-sen}] = (\text{Nom}_\Sigma(M))_i = w\);
- for any \(\Sigma\)-sentence \(\rho\), a \(\Sigma\)-sentence \(\@_i \rho\) is the satisfaction of \(\rho\) at \(i\) when
  \([M \models w \ @_i \rho] = (M \models (\text{Nom}_\Sigma(M))_i \rho)\)

for each \(\Sigma\)-model \(M\) and for each \(w \in [\![M]\!]_\Sigma\).

The stratified institution \(S\) has explicit local satisfaction when there exists a satisfaction at \(i\) for each sentence and each appropriate \(i\).

Example 2.10. The following table shows what of the properties of Definition 2.7 are satisfied by the examples of stratified institutions given above in the paper.
3. Decompositions of Stratified Institutions

An analysis of the structure of conventional Kripke semantics reveals the following situation for an individual Kripke model:

- There is a *family* of models in a “lower” logical system, usually propositional or first order logic. The indexing of the family is what is usually referred to as “worlds”.
- Then there is a certain structure imposed upon this family of models. This happens at the level of the “worlds” commonly in the form of relations.

In this section we address this general structure of Kripke semantics from an abstract axiomatic perspective. The result is a general abstract class of stratified institutions that does not necessarily consider explicitly Kripke frames nor modalised sentences, but which retains in an abstract form the essential idea of a stratified institution in which a “header” institution structures a certain multiplication of a “base” institution.

The contents of this section is as follows:

1. We introduce the concept of “base” of a stratified institution $S$ which represents the institution in which the “worlds” are interpreted as models.
2. We extend the reduction of a stratified institution $S$ to an ordinary institution $S^\#$ introduced in Section 2.5 to an adjunction between the categories of stratified institutions and of ordinary institutions.
3. Finally, on the basis of the above mentioned adjunction we introduce the main concept of this section, namely that of decomposition of a stratified institution.

3.1. Bases for stratified institutions. In a stratified institution $S$ with Kripke semantics, if $M$ is a Kripke model and $w \in [M]$ then the model $(M, w)$ of $S^\#$ represents a “localisation” in $M$ of the “world” $w$. This corresponds to a model in “lower” institution. However the construction of $S^\#$ is independent of the fact that $M$ is really a Kripke model, so this process of semantic localisation is a very general one. On the other hand we should be able to have the (syntax of the) “lower” logic available at the level of $S$. These ideas are captured by the following definition.

**Definition 3.1** (Base of stratified institution). A base for a stratified institution $S$ is an institution morphism $(\Phi, \alpha, \beta) : S^\# \to B$. A base is shared when for each signature $\Sigma$, each $\Sigma$-model $M$ of $S$, and any $w, w' \in [M]_\Sigma$ we have that $\beta_\Sigma(M, w) = \beta_\Sigma(M, w')$.

**Example 3.1** (Base for MPL). For the stratified institutions that are based on some form of Kripke semantics we may consider $B$ to be the institution that at the syntactic level removes from $S$ all syntactic entities that involve modalities, and whose models are the individual “worlds” of the respective Kripke semantics. For instance, in the case of $MPL$:

- $B = PL$ and $\Phi$ is the identity functor on $\mathbf{Set}$,
- $\alpha_P$ is the inclusion $\text{Sen}^{PL}(P) \subseteq \text{Sen}^{MPL}(P)$,
- $\beta_P(M, w) = M^w$, etc.

**Example 3.2** (Shared base for MFOL). For the stratified institutions with Kripke semantics based on first order models with some form of sharing, $B$ may remove even more structure.
from $S$ such that at the semantic level $\beta$ maps a Kripke structure to the respective shared underlying domain. For instance, if $S$ is MFOL:

- $B$ is SETC, i.e. the sub-institution of FOL induced by signatures that contain only constants, $\Phi$ removes from the signatures the predicates and the non-constant operations,
- $\alpha$ is empty (as SETC does not contain any sentences), and
- $\beta_{(F,P)}(M,w) = (|M|, (M_c)_{c \in F_0})$ where $|M|$ is the shared underlying domain of $M$, and each $M_c$ is the shared interpretation of the constant $c$.

This has been an example of a shared base.

**Example 3.3 (Non-shared base for MFOL).** When we allow more structure for $B$ we obtain another relevant base for MFOL.

- We let $B = FOL$; then $\Phi$ is identity.
- $\alpha$ consists of the canonical inclusions of sets of sentences.
- $\beta_{(F,P)}(M,w) = M^w$.

All examples of Section 2.6 admit various bases in the manner of the previous couple of examples. For instance HHPL may admit HPL as a base.

### 3.2. The adjunction between stratified and ordinary institutions.

The representation of stratified institutions as ordinary institutions given by the flattening of Fact 2.1 is part of an adjunction which is instrumental for defining the main concept introduced in this work, that of decompositions of stratified institutions.

Let INS be the category of institution morphisms and SINS be the category of strict stratified institution morphisms.

**Proposition 3.1.** Let $(\_)^\sharp : SINS \to INS$ be the canonical extension of the mapping $S \mapsto S^\sharp$ defined in Fact 2.1. Then $(\_)^\sharp$ has a right adjoint which we denoted as $(\tilde{\_}) : INS \to SINS$.

**Proof.** For any institution $B$ we define the following stratified institution $\tilde{B}$:

- $\text{Sign}^{\tilde{B}} = \text{Sign}^B$ and $\text{Sen}^{\tilde{B}} = \text{Sen}^B$,
- $|\text{Mod}^{\tilde{B}}(\Sigma)| = \{ (W,B : W \to |\text{Mod}^B(\Sigma)|) \mid W \text{ set} \}$,
- $\text{Mod}^{\tilde{B}}(\Sigma)((W,B),(V,N))$ consists of $h = (h_0 : W \to V,(h^w : B^w \to N^{h(w)})_{w \in W})$,
- for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each $\Sigma'$-model $(W',B')$:
  $$\text{Mod}^{\tilde{B}}(\varphi)(W',B') = (W',B';\text{Mod}^B(\varphi)),$$
- $[W,B]^{\tilde{B}} = W$ and $[h]_{\Sigma} = h_0$,
- $[,]_{\varphi}$ are identities, and
- $(W,B)(\models^{\tilde{B}})_{\Sigma}^w \rho$ if and only if $B^w \models^{\tilde{B}} \rho$.

The proof that $\tilde{B}$ is a stratified institution consists of straightforward verifications. Let us do only the Satisfaction Condition:

$$(W',B') \models^w \alpha \rho \iff B'^w \models \alpha \rho \quad \text{by definition}$$

$$\iff \text{Mod}^{\tilde{B}}(\varphi)(B'^w) \models \rho \quad \text{by the Satisfaction Condition in } \tilde{B}$$

$$\iff (W',B';\text{Mod}(\varphi)) \models^w \rho \quad \text{by definition}$$

$$\iff \text{Mod}^{\tilde{B}}(\varphi)(W',B') \models^w \rho \quad \text{by definition}.$$
Then \( \tilde{\mathcal{D}} \) extends canonically to a functor \( \text{INS} \to \text{SINS} \). In order to prove that this is a right adjoint to \( \mathcal{D} \) we first define the co-unit of the adjunction as follows. For each institution \( B \) we let the institution morphism \( \varepsilon_B : \tilde{B} \to B \) have identities for the signature and sentence translation functors and maps each \( \tilde{B} \Sigma \)-model \( ((W, (B^w)_{w \in W}), w) \) to \( B^w \). Then we prove the universal property of \( \tilde{B} \), namely that for each base \( (\Phi, \alpha, \beta) : S \to B \) there exists an unique strict stratified institution morphism \( (\Phi, \alpha, \tilde{\beta}) : S \to \tilde{B} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
S & \xrightarrow{(\Phi, \alpha, \beta)} & \tilde{B} \\
\downarrow & & \downarrow \\
\tilde{B} & \xleftarrow{(\Phi, \alpha, \beta)} & S
\end{array}
\]

Because the signature and the sentences translation functors of \( \varepsilon_B \) are identities there is no other choice for the signature and the sentence translation functors of \( (\Phi, \alpha, \tilde{\beta}) \). By the commutativity (9) it follows that the definition of \( \tilde{\beta} \) is constrained to

\[
\tilde{\beta}_\Sigma M = ([M]_\Sigma (\beta_\Sigma(M, w))_{w \in [M]_\Sigma}).
\]

We may skip a few straightforward things related to establishing that \( (\Phi, \alpha, \tilde{\beta}) \) is indeed a strict stratified institution morphism and only show its Satisfaction Condition:

\[
\begin{align*}
\tilde{\beta} M \models^w \rho & \iff \beta(M, w) \models \rho & \text{by definition} \\
& \iff (M, w) \models \alpha \rho & \text{by the Satisfaction Condition in } S^\sharp \\
& \iff M \models^w \alpha \rho & \text{by definition.}
\end{align*}
\]

As a matter of notation, in what follows, for any base \( (\Phi, \alpha, \beta) : S^\sharp \to B \) we will denote its correspondent through the natural isomorphism \( \text{INS}(S^\sharp, B) \cong \text{SINS}(S, \tilde{B}) \) by \((\Phi, \alpha, \tilde{\beta})\).

3.3. Decompositions of stratified institutions. In many Kripke semantics examples the models are subject to certain constraints, such as for instance the sharing constraints discussed in Example 2.4 (MFOL) or in Example 2.7 (HHPL). Such constraints are treated abstractly in the following definition as a sub-functor of the model functor of \( \tilde{B} \).

**Definition 3.2** (Decomposition of stratified institution). Let \( S \) be a stratified institution and \( (\Phi, \alpha, \beta) : S^\sharp \to B \) be a base for \( S \). Let \( \text{Mod}^C \subseteq \text{Mod}^{\tilde{B}} \) be a sub-functor such that for each signature \( \Sigma \),

\[
\tilde{\beta}_\Sigma(\text{Mod}^S(\Sigma)) \subseteq \text{Mod}^C(\phi \Sigma),
\]

referred to as the constraint model sub-functor. Let \( \tilde{B}^C \) denote the stratified sub-institution of \( \tilde{B} \) induced by \( \text{Mod}^C \). A decomposition of \( S \) consists of two stratified institution morphisms like below

\[
\begin{array}{ccc}
S^0 & \xrightarrow{(\Phi^0, \alpha^0, \beta^0)} & S \\
\downarrow & & \downarrow \\
S & \xrightarrow{(\Phi, \alpha, \tilde{\beta})} & \tilde{B}^C
\end{array}
\]
such that for each \( S \)-signature \( \Sigma \)

\[
\begin{array}{ccc}
\text{Mod}^0(\Phi^0 \Sigma) & \xrightarrow{\beta^0} & \text{Mod}^S(\Sigma) \\
\downarrow \alpha^0 & & \downarrow \beta^0 \\
\text{Set} & & \text{Mod}^C(\Phi \Sigma)
\end{array}
\]

is a pullback in \( \text{CAT} \).

The following important property follows from Fact 2.3 because \( \tilde{B} \) is strict and so is \( \tilde{B}^C \).

**Fact 3.1.** Any stratified institution that admits a decomposition is strict.

Let us note the following aspects emerging from Definition 3.2.

- The models of \( S \) can be represented as pairs of \( S^0 \)-models and families of \( B \)-models satisfying certain constraints (hence \( \tilde{B}^C \) models) such that the “worlds” of the corresponding \( \tilde{B}^C \) model constitutes the stratification of the corresponding \( S^0 \) model. This means that at the semantic level \( S \) is completely determined by the two components of the decomposition.

- The situation at the syntactic level is different. The syntax (signatures and sentences) of each of the two components is represented in the syntax of \( S \), but the latter is not completely determined by the former syntaxes. In other words \( S \) may have signatures and sentences that do not originate from either of the two components. This is what Definition 3.2 gives us. However while there are hardly any examples / applications where any sentence comes from one of the two components, in many examples the signatures of \( S \) are composed from the signatures of \( S^0 \) and those from \( \tilde{B} \). In Lemma 4.1 below we will provide a general such situation.

**Example 3.4 (Decompositions of MPL, HPL).** The sub-institutions \( REL, BREL, RELC, BREL C, SETC \) of FOL can be regarded trivially as stratified institutions by letting for each model \( W \) and sentence \( \rho \), \( \llbracket W \rrbracket = |W| \) (i.e. the underlying set of \( W \)) and \( (W \models^w \rho) = (W \models \rho) \) for each \( w \in |W| \). Under this perspective we let

\[
S^0 = \begin{cases} 
\text{BREL}, & S = MPL, \\
\text{BREL C}, & S = HPL.
\end{cases}
\]

Then

- The functor \( \Phi^0 \) forgets / erases the sets \( P \) of predicate symbols from the signature.
- \( \alpha^0 \) are empty in the case of MPL (because BREL does not have sentences) and is defined by

\[
\alpha^0_{\text{(Nom, } P)}(i, j) = \ominus_i \ominus j (= \ominus_i \neg \ominus j).
\]

for the atoms and then for any sentence by induction on the structure of the respective sentence such that \( \alpha^0_{\text{(Nom, } P)} \) commutes with the connectives (Boolean and quantifiers).

- \( \beta^0_{\text{(Nom, } P)}(W, M) = W \).
- The Satisfaction Condition of \((\Phi^0, \alpha^0, \beta^0)\) can be verified easily by induction on the structure of the sentences.
The bases of $S$ are those of Example 3.1, i.e. $B = PL$, etc. In both situations the model constraint model sub-functor $Mod^C$ is just $Mod^B$, so it is not a proper sub-functor. With respect to the $R, S4$, etc. variants of $MPL$ and $HPL$, in these cases $Mod^0(\Sigma)$ is restricted to those (sub-)categories of relations that satisfy the respective constraints.

**Example 3.5 (Decompositions of MFOL, HFOL).** The decompositions of $MFOL$ and $HFOL$ parallel the decompositions of $MPL$ and $HPL$, respectively, by replacing $PL$ with $FOL$ in Example 3.4. The only significant difference that needs a special mention is at the level of the constraint model sub-functor $Mod^C$. In both $MFOL$ and $HFOL$, a model $(W, B)$ belongs to $Mod^C(F, P)$ when $|B^w| = |B^v|$ and $B^w_c = B^v_c$ for all $w, v \in |W|$ and all $c \in F_0$. Likewise a model homomorphism $(h^0, (h^w)_{w \in |W|}) : (W, B) \to (V, N)$ is such that $h^w = h^v$ for all $w, v \in |W|$.

**Example 3.6 (Decompositions of polyadic modalities stratified institutions).** The cases of $MMPL$, $MMFOL$, $MHPL$, $MHFOL$ are similar to those of Examples 3.4 and 3.5 by taking $S^0 = REL$ when $S$ is non-hybrid and $S^0 = RELC$ when $S$ is hybrid.

Apart of the stratified institutions of Example 2.8 which are not strict all other stratified institutions given as examples in Section 2.6 admit decompositions in a similar manner as in the examples of this section. But Definition 3.2 has a theoretical potential related to the $S^0$ component that may generate situations much beyond Kripke semantics in its common acceptations. For instance we may consider $S^0$ to be an institution of algebras, which will mean algebraic operations on the “worlds” in $S$ models. To unleash the full potential of Definition 3.2 in this direction is an interesting topic of further investigation.

**Implicit nominal structures via decompositions.** In the applications the eventual nominal structures of $S$ come from $S^0$. The following fact clarifies mathematically this situation in a full generality.

**Fact 3.2.** Consider a decomposition of a stratified institution

$$
S^0 \xrightarrow{(\Phi^0, \alpha^0, \beta^0)} S \xrightarrow{(\Phi, \alpha, \beta)} \widetilde{B}^C.
$$

Then any nominals extraction of $S^0$ induces canonically a nominals extraction of $S$ by composition with $(\Phi^0, \alpha^0, \beta^0)$.

In [9] an implicit axiomatic approach to modalities is introduced in a manner similar to Definitions 2.9 and 2.7 and that is based on the concept of “frame extraction”. It is then possible to have a replica of Fact 3.2 for frame extractions.

4. **MODEL AMALGAMATION IN STRATIFIED INSTITUTIONS**

In this section we study model amalgamation in the context of stratified institutions. We do this in two parts as follows:

1. We define a concept of model amalgamation specific to stratified institutions.
2. We develop a general result that builds the model amalgamation property in a stratified institution that admits a decomposition, from the model amalgamation properties of the two components.
4.1. **Concepts of model amalgamation in stratified institutions.** The following definition extends the concept of model amalgamation [29, 31, 26, 12, 7, 11, 30], etc., from ordinary institution theory to stratified institutions. This introduces two concepts. The first one represents just the ordinary institution theoretic concept of model amalgamation formulated for stratified institutions (it does not involve the stratification structure). The second one is specific to stratified institutions.

**Definition 4.1 (Model amalgamation).** Consider a stratified institution $\mathcal{S}$ and a commutative square of signature morphisms like below:

\[
\begin{array}{c}
\Sigma \\
\downarrow \quad \varphi_2 \\
\Sigma_2 \\
\downarrow \quad \theta_2 \\
\Sigma' \\
\end{array}
\]

Then this square
- is a model amalgamation square when for each $\Sigma_k$-model $M_k$, $k = 1, 2$ such that $\varphi_1 M_1 = \varphi_2 M_2$ there exists an unique $\Sigma'$-model $M'$ such that $\theta_k M' = M_k$, $k = 1, 2$, and
- is a stratified model amalgamation square when for each $\Sigma_k$-model $M_k$ and each $w_k \in \llbracket M_k \rrbracket_{\Sigma_k}$, $k = 1, 2$, such that $\varphi_1 (M_1) = \varphi_2 (M_2)$ and $\llbracket M_1 \rrbracket_{\varphi_1} w_1 = \llbracket M_2 \rrbracket_{\varphi_2} w_2$ there exists an unique $\Sigma'$-model $M'$ and an unique $w' \in \llbracket N' \rrbracket_{\Sigma'}$ such that $\theta_k M' = M_k$ and $\llbracket M' \rrbracket_{\theta_k} w' = w_k$, $k = 1, 2$.

The model $M'$ is called the (stratified) amalgamation of $M_1$ and $M_2$.

When all pushout squares of signature morphisms are (stratified) model amalgamation squares we say that $\mathcal{S}$ is (stratified) semi-exact.

Definition 4.1 can be extended to other variants of model amalgamation in the literature.

The following straightforward fact reduces stratified model amalgamation to ordinary model amalgamation.

**Fact 4.1.** A commutative square of signature morphisms like (10) is a stratified model amalgamation square in $\mathcal{S}$ if and only if it is a model amalgamation square in $\mathcal{S}^\sharp$.

The following result provides a couple of conditions that are sufficient for stratified model amalgamation. However they fall short of being also necessary conditions. In particular this situation shows that plain model amalgamation cannot be derived from the seemingly more refined concept of stratified model amalgamation.

**Proposition 4.1.** A commutative square of signature morphisms like (10) is a stratified model amalgamation square if

- Mod($\Sigma$) $\xleftarrow{\text{Mod}(\varphi_1)}$ Mod($\Sigma_1$) $\xrightarrow{\text{Mod}(\varphi_2)}$ Mod($\Sigma_2$) $\xrightarrow{\text{Mod}(\theta_2)}$ Mod($\Sigma'$) $\xleftarrow{\text{Mod}(\theta_1)}$ Mod($\Sigma'$)
is a pullback in \(|\text{CAT}|\), and

- for each \(\Sigma\)-model \(M\)

\[
\begin{array}{ccc}
\varphi(\theta M') & \xrightarrow{[\varphi(\theta_1 M')]_{\varphi_2}} & [\theta_1 M']_{\varphi_1} \\
[\theta_2 M']_{\varphi_2} & \downarrow & [\theta_1 M']_{\varphi_1} \\
[\theta_2 M']_{\varphi_2} & \downarrow & [\theta_1 M']_{\varphi_1} \\
[M']_{\varphi_2} & \downarrow & [M']_{\varphi_1} \\
[M']_{\varphi_2} & \downarrow & [M']_{\varphi_1} \\
\end{array}
\]

is a pullback in \(\text{Set}\).

**Proof.** Note that the first condition just says that (10) is a model amalgamation square. We consider \(M_1, w_1, M_2\) and \(w_2\) like in the definition of stratified model amalgamation. Then we consider \(M'\) to be the unique amalgamation of \(M_1\) and \(M_2\) and apply the second condition for \(w_1\) and \(w_2\). \(\square\)

Note that stratified model amalgamation implies the second condition of Proposition 4.1 (by considering \(M_k = \theta_k(M')\)) but it does not technically imply the first condition.

**Example 4.1.** When the stratification is strict then the concept of stratified model amalgamation collapses to that of (ordinary) model amalgamation. For instance this is the case in \(MPL, MFOL, \) etc., where model amalgamation can be thus established from the model amalgamation in \(PL, FOL, \) etc. (see [7]).

**Example 4.2.** In \(OFOL, MOFOL, HOFO\), \(HMOFOL\) the stratification is a proper lax natural transformation. In all these examples ordinary model amalgamation and stratified model amalgamation are different concepts. Let us look in some detail into the \(OFOL\) case. Let us consider a pushout square of \(FOL\) signature morphisms

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\
\varphi_2 & \downarrow & \downarrow \theta_1 \\
\Sigma_2 & \xrightarrow{\theta_2} & \Sigma' \\
\end{array}
\]

and sets of variables \(X, X_1, X_2, X'\) such that \(X = X_1 \cap X_2\) and \(X' = X_1 \cup X_2\). Then

\[
\begin{array}{ccc}
(\Sigma, X) & \xrightarrow{\varphi_1} & (\Sigma_1, X_1) \\
\varphi_2 & \downarrow & \downarrow \theta_1 \\
(\Sigma_2, X_2) & \xrightarrow{\theta_2} & (\Sigma', X') \\
\end{array}
\]

is a stratified model amalgamation square in \(OFOL\) because

- it is an ordinary model amalgamation square since (11) is a model amalgamation square in \(FOL\) as \(FOL\) is semi-exact (according to the literature, eg. [7]), and

- for each \((\Sigma', X')\)-model \(M'\) (aka \(FOL\) \(\Sigma\)-model) and each \(a_k : X_k \to |M_k|, k = 1, 2\), such that \(a_1(x) = a_2(x)\) for each \(x \in X\), \(a' : X' \to |M'|\) defined by \(a'(x) = a_k(x)\) when \(x \in X_k\) is unique such that \([M']_{\theta_2} a = a_k, k = 1, 2\). (Note that \(|M_1| = |M_2| = |M'|\). Then we apply Proposition 4.1.
4.2. Model amalgamation by decomposition. In this part we establish model amalgamation in decomposed stratified institutions on the basis of the model amalgamation properties of the components (Proposition 4.3). Although developed in a much more general theoretical framework this result goes somehow in the same direction with a corresponding model amalgamation study from [15]. For instance both share the same practical goal of providing an easier route for establishing model amalgamation in concrete situations.

The following preliminary result shows how the model amalgamation in the “base” stratified component can be reduced to model amalgamation in the base institution.

Proposition 4.2 (Model amalgamation in $\tilde{B}$). Let $\mathcal{B}$ be any institution. Any model amalgamation square in $\mathcal{B}$ is a model amalgamation square in $\tilde{B}$ too.

Proof. Let the commutative square of signature morphisms below be a model amalgamation square in $\mathcal{B}$.

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\
\varphi_2 & \downarrow & \downarrow \theta_1 \\
\Sigma_2 & \xrightarrow{\theta_2} & \Sigma'
\end{array}
$$

Let $(W_k, B_k) \in |\text{Mod}^B(\Sigma_k)|, k = 1, 2$ such that $\varphi_1(W_1, B_1) = \varphi_2(W_2, B_2)$. This means $W_1 = W_2$ and $\varphi_1 B_1^w = \varphi_2 B_2^w$ for each $w \in W(= W_1 = W_2)$. By the model amalgamation hypothesis in $|\mathcal{B}|$, for each $w \in W$ there exists an unique $\Sigma'$-model $B'^w$ such that $\theta_k B'^w = B_k^w, k = 1, 2$. This gives $B' : W \rightarrow |\text{Mod}^B(\Sigma')|$. Then $(W, B')$ is the unique amalgamation of $(W, B_1)$ and $(W, B_2)$ in $\tilde{B}$. □

In the “base” component of a decomposition we actually need model amalgamation at the level of the constrained models. The following definition provides a general condition that allows for the model amalgamation established at the level of $\tilde{B}$ in Proposition 4.2 to be transferred to $\tilde{B}^C$. The example after Definition 4.2 illustrate how this may function in concrete situations.

Definition 4.2. Let $\mathcal{B}$ be any institution. A constraint model sub-functor $\text{Mod}^C \subseteq \text{Mod}^\tilde{B}$ preserves amalgamation when for any pushout square of signature morphisms

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\varphi_1} & \Sigma_1 \\
\varphi_2 & \downarrow & \downarrow \theta_1 \\
\Sigma_2 & \xrightarrow{\theta_2} & \Sigma'
\end{array}
$$

and for any $\tilde{B} \Sigma'$-model $(W, B'), \theta_k(W, B') \in |\text{Mod}^C(\Sigma_k)|, k = 1, 2$, implies $(W, B') \in |\text{Mod}^C(\Sigma')|$.

The following transfer of model amalgamation from $\mathcal{B}$ to $\tilde{B}^C$ is an immediate consequence of Proposition 4.2 and of Definition 4.2.
Corollary 4.1. If $\mathcal{B}$ is semi-exact and $\text{Mod}^C \subseteq \text{Mod}^{\mathcal{B}}$ preserves amalgamation then $\tilde{\mathcal{B}}^C$ is semi-exact.

Example 4.3. MPL, HPL provide trivial cases for Definition 4.2 because in both cases $\mathcal{B} = PL$ and $\text{Mod}^C = \text{Mod}^{\mathcal{B}}$. The situation is different for MFOL, HFOL, HHPL, etc. As an example let us see how is it with MFOL, this case being quite emblematic for a whole class of examples. In this case $\mathcal{B} = FOL$.

At the level of the underlying carriers the things are simple: since $W$ is invariant when taking the reducts, i.e. $\theta_k(W, B') = (W, B_k)$, and the same happens with the underlying carriers, i.e. $|B'^w| = |B^w_k| = |B^w_v|$. For the interpretations of the constants let us consider a constant $c'$ of the signature $\Sigma'$.

The following is the main result of this section.

Proposition 4.3 (Model amalgamation by decomposition). Consider a decomposition of a stratified institution $\mathcal{S}$

$$
\begin{array}{c}
\mathcal{S}^0 \xrightarrow{\Phi^0, \alpha^0, \beta^0} \mathcal{S} \xrightarrow{\Phi, \alpha, \tilde{\beta}} \tilde{\mathcal{B}}^C
\end{array}
$$

such that
(1) $\mathcal{S}^0$ is strict,
(2) $\Phi$ and $\Phi^0$ preserve pushouts,
(3) $\mathcal{B}$ and $\mathcal{S}^0$ are semi-exact, and
(4) $\text{Mod}^C$ preserves amalgamation.

Then $\mathcal{S}$ is semi-exact too.

Proof. Consider a pushout square of signature morphisms in $\mathcal{S}$

$$
\begin{array}{c}
\Sigma \xrightarrow{\varphi_1} \Sigma_1 \\
\varphi_2 \\
\Sigma_2 \xrightarrow{\theta_2} \Sigma'
\end{array}
$$

and let $M_k \in |\text{Mod}^S(\Sigma_k)|$, $k = 1, 2$, such that $\varphi_1 M_1 = \varphi_2 M_2$. By relying on the preservation pushout condition we have that the squares below
are pushout squares in $\text{Sign}^0$ and $\text{Sign}^B$, respectively (where the left-hand square above is the result of applying $\Phi^0$ to (12)). We let $M^0_k = \beta^0_{\Sigma_k} M_k$ and $(W_k, N_k) = \tilde{\beta}_{\Sigma_k} M_k$, $k = 1, 2$.

Our plan is to obtain the amalgamation of $M_1$ and $M_2$ through the pullback property on the category of models of the decomposition of $\mathcal{S}$ by joining together the amalgamations of $M^0_1$ and $M^0_2$ and of $(W_1, N_1)$ and $(W_2, N_2)$. The first step is to establish the conditions for the two amalgamations:

- In the case of $M^0_1$ and $M^0_2$ we have that
  \[
  \varphi^0_k M_k = \varphi^0_k (\beta^0_{\Sigma_k} M_k) = \beta^0_{\Sigma_k} (\varphi_k M_k) \quad (\text{by the naturality of } \beta^0)
  \]
  Since by hypothesis $\varphi_1 M_1 = \varphi_2 M_2$ it follows that $\varphi^0_1 M^0_1 = \varphi^0_2 M^0_2$.

- A similar argument applies also to $(W_1, N_1)$ and $(W_2, N_2)$, by using the naturality of $\tilde{\beta}$ and the assumption $\varphi_1 M_1 = \varphi_2 M_2$.

Now, by the semi-exactness hypotheses, let $M'^0$ be the unique amalgamation of $M^0_1$ and $M^0_2$, and $(W', N')$ be the unique amalgamation of $(W_1, N_1)$ and $(W_2, N_2)$. Note that in the case of the latter amalgamation we rely upon the result of Proposition 4.2 and on the preservation of amalgamation by $\text{Mod}^C$ hypothesis. Note also that since $\tilde{\mathcal{B}}^C$ is strict we have that $W_1 = W_2 = W'$. We have that

\[
[M'^0] = [(\Phi \theta_1) M'^0] = [M^0_1] \quad \text{S}^0 \text{ strict}
\]
\[
= [(W_1, N_1)] \quad \text{definition of } M^0_1
\]
\[
= W_1 = W'.
\]

Hence $[M'^0] = [(W', N')]$ which allows us to apply the pullback property of the model decomposition and define $M'$ to be the unique $\Sigma'$-model such that $\beta^0_{\Sigma'} M' = M'^0$ and $\tilde{\beta}_{\Sigma'} M' = (W', N')$.

We show that $M'$ is the amalgamation of $M_1$ and $M_2$. On the one hand, by the naturality of $\beta^0$ and since $M'^0$ has been defined as the amalgamation of $M^0_1$ and $M^0_2$ we have:

\[
\beta^0_{\Sigma_k} (\theta_k M') = \theta^0_k (\beta^0_{\Sigma_k} M') = \theta^0_k M'^0 = M^0_k.
\]

On the other hand, by the naturality of $\tilde{\beta}$ and since $(W', N')$ is the amalgamation of $(W_1, N_1)$ and $(W_2, N_2)$ we have:

\[
\tilde{\beta}_{\Sigma_k} (\theta_k M') = (\Phi \theta_k) (\tilde{\beta}_{\Sigma'} M') = (\Phi \theta_k) (W', N') = (W_k, N_k).
\]

From (13) and (14), by the uniqueness aspect of the pullback property of the model decomposition, it follows that $\theta_k M' = M_k$, $k = 1, 2$. The uniqueness of $M'$ follows by relying on the uniqueness of the model amalgamation in both $\mathcal{S}^0$ and $\tilde{\mathcal{B}}^C$. \qed

In many concrete examples the second condition of Proposition 4.3 is established through corresponding instances of the following lemma.

**Lemma 4.1.** If the decomposition of the stratified institution has the property that

\[
\begin{array}{ccc}
\text{Sign}^0 & \xleftarrow{\Phi^0} & \text{Sign}^S \\
\Phi & \xrightarrow{\Phi} & \text{Sign}^B
\end{array}
\]
is a product in $\text{CAT}$, then both $\Phi^0$ and $\Phi$ preserve pushouts. Moreover any pair of pushout squares of signatures, one from $S^0$ and the other one from $B$, determine canonically a pushout square of $S$ signatures.

**Proof.** By a straightforward general categorical argument. □

The following corollary provides an example of how the result of Proposition 4.3 can be applied in order to obtain model amalgamation properties in concrete stratified institutions. It is rather comprehensive with respect to the conditions of Proposition 4.3. Other such model amalgamation properties can be established in a multitude of concrete stratified institutions (such as those given in Section 2.6) in a similar manner.

**Corollary 4.2.** $\text{MMFOL}$ is semi-exact.

**Proof.** We apply Proposition 4.3 by performing a check on its conditions as follows. Step 0 consists of recalling from Example 3.6 (see also Examples 3.4 and 3.5) the decomposition of $\text{MMFOL}$. This is

$$
\begin{array}{c}
\text{REL} \\ (\Phi^0, \alpha^0, \beta^0) \ar[rr]^< & \text{MMFOL} \ar[rr]_(\Phi, \alpha, \beta) & \tilde{\text{FOL}}^C
\end{array}
$$

where

- $\text{REL}$ is considered as a stratified institution by letting $[W] = |W|$, i.e. the underlying set of the $\text{REL}$-model $W$. Note that because the $\text{REL}$-signatures have only predicate symbols there are no $\text{REL}$ sentences. This situation would be different if instead of $\text{MMFOL}$ we would consider $\text{MHFOL}$ (see Example 3.6).
- The constraint model functor $\text{Mod}^C$ that defines $\tilde{\text{FOL}}^C$ is given by the sharing of the underlying sets and of the interpretations of the constants (see Examples 2.6, 2.4, 4.3).

Now we focus on how the four conditions of Proposition 4.3 hold.

1. As stratified institution $\text{REL}$ is a strict one because the reducts in $\text{REL}$ preserve the underlying sets of the models.
2. We apply Lemma 4.1. The signatures of $\text{MMFOL}$ are indeed pairs $(\Lambda, (F, P))$ where $\Lambda$ is a $\text{REL}$ signature and $(F, P)$ is a $\text{FOL}$ signature; hence the product condition of Lemma 4.1 is fulfilled.
3. In the literature $\text{FOL}$ is a classic example of a semi-exact institution (see [12, 7], etc.) although usually $\text{FOL}$ is considered in its many-sorted form. For our single sorted variant it is just enough to note that the pushouts of single sorted $\text{FOL}$ signatures are still single sorted, and thus the semi-exactness of single sorted $\text{FOL}$ is inherited from the more general many sorted $\text{FOL}$. The same argument applies to $\text{REL}$ too, as $\text{REL}$ is a fragment (or a sub-institution) of (single sorted) $\text{FOL}$.
4. This has essentially been established in Example 4.3. □
5. Diagrams in stratified institutions

In conventional model theory the method of diagrams is one of the most important methods. The institution-independent method of diagrams plays a significant role in the development of a lot of model theoretic results at the level of abstract institutions, many of its applications being presented in [7]. These include existence of co-limits of models, free models along theory morphisms, axiomatizability results, elementary homomorphisms results, filtered power embeddings results, saturated models results (including an abstract version of Keisler-Shelah isomorphism theorem), the equivalence between initial semantics and quasi-varieties, Robinson consistency results, interpolation theory, definability theory, proof systems, predefined types, etc.

In institution theory diagrams had been introduced for the first time by Tarlecki in [31, 32] in a form different from ours. In the form presented here it has been introduced at the level of institution-independent model theory in [6] as a categorical property which formalizes the idea that

the class of model homomorphisms from a model \( M \) can be represented (by a natural isomorphism) as a class of models of a theory in a signature extending the original signature with syntactic entities determined by \( M \).

This can be seen as a coherence property between the semantic and the syntactic structures of the institution. By following the basic principle that a structure is rather defined by its homomorphisms (arrows) than by its objects, the semantic structure of an institution is given by its model homomorphisms. On the other hand the syntactic structure of a(ny concrete) institution is based upon its corresponding concept of atomic sentence.

The goal of this section is twofold. On the one hand we need to clarify the concept of diagrams in stratified institutions. This is quite straightforward:

the diagrams in a stratified institution \( S \) are the diagrams in \( S^{\#} \).

On the other hand it is useful to have a general result on the existence of diagrams at the level of abstract stratified institutions that would be applicable to a wide class of concrete situations. In this section we will develop such a result by reliance on decompositions of stratified institutions.

The structure of the section is as follows:

1. We recall the established institution theoretic concept of diagrams.
2. We introduce some preliminary technical concepts that will support the development of the main result of this section.
3. We formulate and prove a general result on the existence of diagrams in stratified institutions. This comes in two versions: for \( S^{*} \) and for \( S^{\#} \) (where \( S \) is a stratified institution).
4. By means of a (counter)example we show the necessity of the main specific technical condition underlying our result on the existence of diagrams, namely the specific infrastructure supporting nominals.

5.1. A reminder of institution-theoretic diagrams. Below we recall from [6, 7] the main concept of the institution theoretic method of diagrams.
Definition 5.1 (The method of diagrams). An institution\( \mathcal{I} \) has diagrams when for each signature\( \Sigma \) and each\( \Sigma \)-model\( M \), there exists a signature\( \Sigma_M \) and a signature morphism \( i_\Sigma(M) : \Sigma \to \Sigma_M \), functorial in \( \Sigma \) and \( M \), and a set \( E_M \) of \( \Sigma_M \)-sentences such that \( \text{Mod}(\Sigma_M, E_M) \) and the comma category \( M/\text{Mod}(\Sigma) \) are naturally isomorphic, i.e. the following diagram commutes by the isomorphism \( i_{\Sigma, M} \) that is natural in \( \Sigma \) and \( M \)

\[
\begin{array}{c}
\text{Mod}(\Sigma_M, E_M) \xrightarrow{i_{\Sigma, M}} M/\text{Mod}(\Sigma) \\
\downarrow \text{forgetful}
\end{array}
\]

The signature morphism \( i_\Sigma(M) : \Sigma \to \Sigma_M \) is called the elementary extension of \( \Sigma \) via \( M \) and the set \( E_M \) of \( \Sigma_M \)-sentences is called the diagram of the model \( M \).

In the institution theoretic literature, especially in [7], one can find a wealth of examples of systems of diagrams. Below we remind two of the most common ones.

Example 5.1 (Diagrams in PL). For any PL signature\( P \) and any \( P \)-model\( M \in 2^P \) the extension\( i_{P, M} \) is just the identity function on \( P \), while \( E_M = M \). Then, that \( N \in 2^P \) satisfied \( E_M \) means just that \( M \subseteq N \); this gives \( i_{P, M} N \).

Example 5.2 (Diagrams in FOL). For any FOL signature\( \Sigma = (F, P) \) and any \( (F, P) \)-model\( M \) the extension\( i_{(F, P), M} \) just adds the set of the elements \( |M| \) of \( M \) as new constants to \( F \). The let \( M \) be the expansion of \( M \) along \( i_{(F, P), M} \) that interprets the new constants by themselves, i.e. \( (M_M)_c = c \) for any \( c \in |M| \). \( E_M \) is defined as the set of the quantifier-free equations satisfied by \( M \). For any \( \Sigma_M \) model \( N' \) that satisfied \( E_M \), \( i_{\Sigma, M} N' \) is the \( (F, P) \)-homomorphism \( h : M \to N \) defined by \( h(x) = N'_x \), where \( N \) is the \( i_{\Sigma, M} \)-reduct of \( N' \).

In order to keep the exposition technically simpler, for the rest of this section we will ignore the properties of the functoriality of \( i \) and of the naturality of \( i \) and rather focus on the primary property of diagrams, i.e. the isomorphism property of \( i_{\Sigma, M} \) and the commutativity shown in the diagram (15). Moreover, in most applications of institution theoretic diagrams only this primary property is used. However the interested reader may develop by himself what the functoriality and the naturality properties mean in explicit form, or else he may consult them from [7].

5.2. Some supporting technical concepts. The main idea underlying our development of a general result on the existence of diagrams in stratified institutions is to consider decompositions of stratified institutions, to assume diagrams for each of the two components (which in concrete situations are already known / established), and then to combine these at the level of the stratified institution. However this process requires some technical conditions that we will spell out explicitly in what follows.

The first condition supports the lifting of diagrams from \( \mathcal{B} \) to \( \tilde{\mathcal{B}} \).

Definition 5.2. Let \( \mathcal{B} \) be an institution and \( \text{Mod}^C \) be a constraint model sub-functor for \( \tilde{\mathcal{B}} \). A system of diagrams for \( \mathcal{B} \) is coherent with respect to \( \text{Mod}^C \) when for each \( \mathcal{B} \)-signature \( \Sigma \) and each \( (W, \mathcal{B}) \in |\text{Mod}^C(\Sigma)| \) we have that
(1) For all $i, j \in W$, $\iota_{\Sigma, B^i} = \iota_{\Sigma, B^j}$; in this case all $\iota_{\Sigma, B^i}$s will be denoted by $\iota_{\Sigma, B^i} : \Sigma \to \Sigma_{B^i}$.

(2) for each $(W, B) \in \mathcal{M}^C(\Sigma)$ there exists a canonical isomorphism $i_{\Sigma,(W,B)}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Mod}(\Sigma_B,E_{(W,B)}) & \xrightarrow{i_{\Sigma,(W,B)}} & (W,B)/\text{Mod}^C(\Sigma) \\
\downarrow & & \downarrow \text{forgetful} \\
\text{Mod}^C(\Sigma_B) & \xleftarrow{\iota_{\Sigma,B}} & \text{Mod}^C(\Sigma)
\end{array}
\]

where $\text{Mod}(\Sigma_B,E_{(W,B)})$ denotes the subcategory of the comma category $[\mathcal{M}]/[\underline{\_}]_{\Sigma_B}$ (where $[\underline{\_}]_{\Sigma_B} : \text{Mod}^C(\Sigma_B) \to \text{Set}$) induced by those objects $(f : W \to V, (V,N'))$ such that $N'f(i) = E_{B^i}$ for each $i \in W$.\(^4\)

**Example 5.3 (Coherence in $\widetilde{PL}$).** If $B = PL$ then $\text{Mod}^C = \text{Mod}^{\widetilde{B}}$. The first condition is trivially satisfied because in $PL$ all elementary extensions $\iota_{p,B}$ are identities. On this basis the second condition is also trivially satisfied.

Although the following example requires a more intricate verification, this is still rather straightforward.

**Example 5.4 (Coherence in $\widetilde{FOL}$).** If $B = FOL$ then the diagrams of $FOL$ are coherent with respect to $\text{Mod}^C$ where $(W, B) \in \mathcal{M}^C(F,P)$ if and only if for all $i, j \in W$, $|B^i| = |B^j|$ and $B^i_c = B^j_c$ for each $c \in F_0$.

On the one hand, this is so because for each $FOL (F,P)$-model $M$, $\iota_{(F,P),M}$ is the extension of $(F,P)$ with new constants which are the elements of $M$.

On the other hand, the second condition of Definition 5.2 goes as follows. For any $(f : W \to V, (V,N')) \in \text{Mod}(\Sigma_B,E_{(W,B)})$ we let $i_{\Sigma,(W,B)}(f,(V,N')) = (f,h)$ where $h : |B^i| \to |N'f(i)|$ is the function that is invariant with respect to $i \in W$ and which is given by the diagrams of $B^i : h^i = i_{\Sigma,B^i}N'(f^i)$ where $N'(f^i)$ is the reduct of $Nf(i)$ along $i_{\Sigma,B}^i$ (just forgets the interpretations of the new constants corresponding to the elements of the models $B^i$, which are in fact shared by all $B^j$). We can talk about one function $h$ because as functions $h^i = h^j$ for all $i, j \in W$. This is so because for each element $c \in |B^i| = |B^j|$ we have that $h^i(c) = N'^i_c = N'^j_c$ (because $(V,N')$ being a constraint model its components share the interpretations of the constants) = $h^j_c$. This also makes $(f,h)$ a constraint model homomorphism, so $i_{\Sigma,(W,B)}N'$ belongs to $(W,B)/\text{Mod}^C(\Sigma)$ indeed.

The inverse $i_{\Sigma,(W,B)}^{-1}$ is defined as follows. Given $(f,h) : (W,B) \to (V,N)$, for each $i \in W$ we let $Nf(i)$ be $i_{\Sigma,B}^{-1}h$ where $h : B^i \to Nf(i)$. This is correctly defined because if $f(i) = f(j)$ then $Nf(i) = Nf(j)$ and $Nf(i)$, $Nf(j)$ are just the expansions of $N^f(i), N^f(j)$, respectively, with interpretations of new constants, i.e. $N^f(i) = h(c) = N^f(j)$. When $v \in V$ is outside the image of $f$, $N'$ is uniquely determined by the constraint as $N'^v$ is the expansion of $N^v$ with the interpretations of the elements of $B$ as new constants which are shared with other $N'^i$.

\(^4\)Note that unlike $E_{B^i}$, $E_{(W,B)}$ is not a set of sentences.
The following defines the workable situation when both components of a decomposition of a stratified institution admit diagrams, this being the root condition of our approach to the existence of diagrams in stratified institutions.

**Definition 5.3.** A decomposition of a stratified institution $S$

$$S^0 \xrightarrow{(\Phi^0, \alpha^0, \beta^0)} S \xrightarrow{(\Phi, \alpha, \beta)} \bar{B}$$

admits diagrams when $S^0^*$ and $B$ have diagrams such that the diagrams of $B$ are coherent with respect to $\text{Mod}^C$.

**Example 5.5 (The case of HPL).** We have to recall Example 3.4. The diagrams of $S^0$ are as follows. For any $S^0$-signature $\text{Nom}$ and any $\text{Nom}$-model $W = (|W|, W_\lambda \subseteq |W| \times |W|)$, $\iota_{\text{Nom},W}$ is the extension of $\text{Nom}$ with the elements of the set $|W|$ and

$$E_W = \{\lambda(i,j) \mid (i,j) \in W_\lambda\}.$$  

Since in this example $B = \text{PL}$ the coherence of the diagrams of $B$ with respect to $\text{Mod}^C$ is explained by Example 5.3.

**Example 5.6 (The case of HFO L).** We have to recall Example 3.5. Since $S^0$ is the same like in the HPL case, the diagrams of $S^0$ are those given in Example 5.5. In this example $B = \text{FOL}$ and therefore the coherence of the diagrams of $B$ with respect to $\text{Mod}^C$ is explained by Example 5.4.

**Notation 5.1.** For any decomposition of a stratified institution that admits diagrams (like in Definition 5.3) for any $\Sigma \in |\text{Sign}|$ and $M \in |\text{Mod}^S(\Sigma)|$, we introduce the following abbreviations:

$$\Sigma_0 = \Phi^0 \Sigma, \Sigma_1 = \Phi \Sigma, M_0 = \beta^0 \Sigma M, M_1 = \beta \Sigma M.$$  

We let $\iota_{\Sigma_0, M_0} : \Sigma_0 \to (\Sigma_0 M_0, E_{M_0})$ and (for each $i \in [M]$) $\iota_{\Sigma_1, M_1^i} : \Sigma_1 \to (\Sigma_1 M_1^i, E_{M_1^i})$ be the diagrams of $M_0$ and $M_1^i$, respectively. By the coherence hypothesis we have $\iota_{\Sigma_1, M_1^i} = \iota_{\Sigma_1, M_1^j}$ for all $i, j \in [M]$. This allows us to denote all $\iota_{\Sigma_1, M_1^i}$ by $\iota_{\Sigma_1, M_1}$.

If

$$\text{Sign}^0 \xrightarrow{\Phi^0} \text{Sign}^S \xrightarrow{\Phi} \text{Sign}^B$$

is a product in $\text{CAT}$ then we define the $\text{Sign}^S$ morphism $\iota_{\Sigma, M} : \Sigma \to \Sigma M$ by using the product property of $(\Phi^0, \Phi)$:

$$\iota_{\Sigma, M} = (\iota_{\Sigma, M_0}, \iota_{\Sigma, M_1}).$$

The last technical concept supporting the main result of this section expresses the possibility that each element of the underlying stratifications has a syntactic designation. Although it has a rather heavy technical appearance it holds naturally in the examples.

**Definition 5.4.** Consider a decomposition of a stratified institution that admits diagrams like in Definition 5.3. We say that the diagrams (of the decomposition) denote the stratification when

$$\text{Sign}^0 \xleftarrow{\Phi^0} \text{Sign}^S \xrightarrow{\Phi} \text{Sign}^B$$
is a product in CAT,

- \( S \) has a nominals extraction \((N, Nm)\),

- for each \( S \)-signature \( \Sigma \) and each \( \Sigma \)-model \( M \), there exists a function
  \[
  n_{\Sigma, M} : [M]_\Sigma \to N(\Sigma_M)
  \]
  such that \( n \) is natural in \( \Sigma \) and \( M \), and

- for each \( \Sigma_M \)-model \( N \) such that \( N \models \alpha_0 E_{M_0} \),
  \[
  n_{\Sigma, M}; Nm_{\Sigma_M}(N) = [i_{\Sigma_0, M_0} N_0]_{\Sigma_0}.
  \]

**Example 5.7 (The case of HPL).** According to Example 2.9, HPL has nominals extraction \((N, Nom)\) where for each HPL signature \((Nom, P)\), \( N(Nom, P) = Nom \) and for each \((Nom, P)\)-model \((W, M)\), \( Nm(W, M) = (|W|, (W_i)_{i \in Nom}) \). Then we define \( n_{(Nm, P), (W, M)} \) as the canonical injection \(|W| \to Nom + |W|\), where \( Nom + |W| \) denotes the disjoint union of \( Nom \) and \(|W|\).

Now let us consider any \((Nom, P)_{(W, M)}\)-model \((V', N')\) such that \( (V', N') \models \alpha_0 E_W \) which means \( V' \models E_W \). Then for each \( w \in |W| \) we have that
  \[
  (Nm(V', N'))_{n(w)} = (|V|, (V'_i)_{i \in Nom + |W|})_w = V'_w = (i_{Nom, W} V')_w = [i_{Nom, W} V']_w.
  \]

**Example 5.8 (The case of HFO L).** This is similar to Example 5.7 because the property of Definition 5.4 depends essentially on the \( S^0 \) part of the decomposition of \( S \), which is shared between HPL and HFO L.

5.3. The existence of diagrams in stratified institutions.

**Theorem 5.1.** For any decomposition of a stratified institution \( S \) that admits diagrams that denote the stratification:

- \( S^* \) has diagrams when \( S \) has explicit local satisfaction, and

- \( S^i \) has diagrams when \( S \) has explicit local satisfaction and has \( i \)-sentences too.

**Proof.** For each \( S \) signature \( \Sigma \) and each \( \Sigma \)-model \( M \) we define \( E_M \subseteq Sen^S(\Sigma_M) \) by
  \[
  E_M = \alpha_0^0 \Sigma_M E_{M_0} \cup \bigcup_{i \in |M|} @_i (\alpha_{\Sigma_M} E_{M_1})
  \]
  where \( @_i (\alpha_{\Sigma_M} E_{M_1}) \) abbreviates \( @_{\alpha_{\Sigma, M}(i) \alpha_{\Sigma_M}} \rho \mid \rho \in E_{M_1} \).

We will prove that \( i_{\Sigma, M} : \Sigma \to (\Sigma_M, E_M) \) (see Notation 5.1) is the diagram of \( M \) in \( S^* \). The coherence diagram commutes:

Let \( \gamma_{\Sigma, M} : Mod(\Sigma_M) \to Mod(\Sigma_{1, M_1}) \) be the functor defined by
  \[
  \gamma_{\Sigma, M} N' = (f : [M] \to [N'], \beta_{\Sigma, M} N')
  \]
where \( f(i) = Nm_{\Sigma_M}(N)_{\Sigma_M,i} \). Then the restriction of \( \gamma_{\Sigma,M} \) to \( \text{Mod}(\Sigma_M, E_M) \) yields a functor \( \text{Mod}(\Sigma_M, E_M) \to \text{Mod}(\Sigma_1M_1, E_1M_1) \), as follows.

Let \( N' \) be a \( \Sigma_M \)-model such that \( N' \models E_M \). Then we have that

1. \( N' \models \Sigma_M \otimes_i (\alpha_{\Sigma_M} E_{M_1}) \) for each \( i \in [M] \)
2. \( N' \models \Sigma_M \otimes_{\Sigma_M,i} (\alpha_{\Sigma_M} E_{M_1}) \) for each \( i \in [M] \)
3. \( N' \models Nm_{\Sigma_M}(N)_{\Sigma_M,i} \alpha_{\Sigma_M} E_{M_1} \) for each \( i \in [M] \)
4. \( \beta_{\Sigma_M}(N')_{\Sigma_M,i} E_{M_1} \) for each \( i \in [M] \)
5. \( (\lambda i. Nm_{\Sigma_M}(N'_{\Sigma_M,i}, \beta_{\Sigma_M}(N')) \in \text{Mod}(\Sigma_1M_1, E_1M_1) \)
6. \( \gamma_{\Sigma,M} N' \in \text{Mod}(\Sigma_1M_1, E_1M_1) \)

In the diagram below the upper left square is a pullback square. This follows by general categorical considerations on the basis of the pullback condition on model categories from the decomposition of \( S \).

\[
\begin{array}{ccc}
[M]/\text{Set} & \overset{\downarrow \varepsilon_0}{\longrightarrow} & M_0/\text{Mod}^0(\Sigma_0) \\
| | & \beta^0_{\Sigma} & | | \\
M_1/\text{Mod}(\Sigma_1) & \overset{\beta_{\Sigma}}{\longrightarrow} & M/\text{Mod}^S(\Sigma) \\
\varepsilon_1 \downarrow & & \varepsilon_\Sigma \downarrow \\
\text{Mod}(\Sigma_1M_1, E_1M_1) & \overset{\beta^0_{\Sigma_M}}{\longrightarrow} & \text{Mod}(\Sigma_M, E_M)
\end{array}
\]

If we proved that the outer hexagon of the above diagram represents a pullback too, then we obtain the isomorphism \( i_{\Sigma,M}: \text{Mod}^S(\Sigma_M, E_M) \to M/\text{Mod}^S(\Sigma) \).

We first show the commutativity of the outer hexagon. For each \( \Sigma_M \)-model \( N' \) such that \( N' \models E_M \) we have that

\[
[i_{\Sigma_1M_1}(\gamma_{\Sigma,M} N')]_{\Sigma_1} = \gamma_{\Sigma_1M_1} = \] \[\text{definition of } \gamma\]

\[
= [i_{\Sigma_1M_1}(\lambda i. Nm_{\Sigma_M}(N)_{\Sigma_M,i}, \beta_{\Sigma_M}(N'))]_{\Sigma_1} = \] \[\text{definition of } i_{\Sigma_1M_1}\]

\[
= \beta_{\Sigma_M}(N')_{\Sigma_M,i} \] \[\text{Definition 5.4.}\]

Finally, we show that the hexagon represents a pullback. We must prove that given any \((f, N_1) \in \text{Mod}(\Sigma_1M_1, E_1M_1)\) and \(N_0 \in |\text{Mod}^0(\Sigma_0M_0, E_0M_0)|\) such that \([i_{\Sigma_0M_0}N_0] = f\) there exists an unique \( N \in \text{Mod}(\Sigma_M, E_M) \) such that \( \gamma_{\Sigma,M}N = (f, N_1) \) and \( \beta^0_{\Sigma_M}N = N_0 \). It follows that \( N_1 = \beta_{\Sigma_M}N \). Note that from \([i_{\Sigma_0M_0}N_0] = f\) it also follows that \([N_0] = [N_1]\).

Hence by the pullback of the categories of models of the decomposition there exists an unique \( N \) such that \( N_1 = \beta_{\Sigma_M}N \) and \( N_0 = \beta^0_{\Sigma_M}N \). Moreover \( f \) is uniquely determined by the condition \([i_{\Sigma_0M_0}N_0] = f\).
For the second conclusion of the theorem, for each $\mathcal{S}$-signature $\Sigma$, each $\Sigma$-model $M$, and each $w \in [M]_{\Sigma}$, let us abbreviate $n_{\Sigma,M}(w)$-sen by $w$-sen. Let $i_{\Sigma,M} : \Sigma \to (\Sigma, E_M)$ be the diagram corresponding to $M$ in $\mathcal{S}$ as established above. The we prove that for each $\mathcal{S}$ $\Sigma$-model $(M, w)$, $i_{\Sigma,M} : \Sigma \to (\Sigma, E_M)$ is a diagram for $(M, w)$ where

$$E_{(M,w)} = E_M \cup \{w\text{-sen}\}.$$ 

We prove that the isomorphism $i_{\Sigma,M}$ (in $\mathcal{S}$) can be extended to an isomorphism

$$i_{\Sigma,(M,w)} : \text{Mod}^\sharp(\Sigma, E_{(M,w)}) \to (M, w)/\text{Mod}^\sharp(\Sigma).$$

For any $\mathcal{S}$ $\Sigma$-model $(N', v)$ that satisfies $E_{(M,w)}$ we have that $N' \models E_M$. Let $h = i_{\Sigma,M} : M \to N$. It remains to show that $(h, w) : (M, w) \to (N, v)$ is a homomorphism in $\mathcal{S}$ is equivalent to $(N', v) \models w$-sen:

$$(N', v) \models w \text{-sen}$$

$$\Leftrightarrow Nm_{\Sigma,M}(N')_{\Sigma,M}w = v \quad \text{definition of satisfaction of } i\text{-sentences}$$

$$\Leftrightarrow \bigl[i_{\Sigma_0,M_0}\bigl[N'\bigr]\bigr]_{\Sigma_0}w = v \quad \text{by Definition 5.4}$$

$$\Leftrightarrow \bigl[i_{\Sigma,M}\bigl[N'\bigr]\bigr]_{\Sigma}w = v \quad \text{by the commutativity of the upper right half of (16)}$$

$$\Leftrightarrow h(w) = v.$$

We can apply Theorem 5.1 for $\mathcal{S} = \text{HPL}$ and $\mathcal{S} = \text{HFOL}$ and obtain the following two corollaries, which are emblematic for applications of this general result.

**Corollary 5.1.** $\text{HPL}^\star$ and $\text{HPL}^\sharp$ have diagrams.

**Proof.** We have to recall Example 5.5. For each $(\text{Nom}, P)$-model $(W, M)$:

- $i_{\text{(Nom,P)},(W,M)}$ is the signature extension with nominals $(\text{Nom}, P) \to (\text{Nom} + |W|, P)$; and
- $E_{(\text{Nom,P}),,(W,M)} = \{\otimes_i \otimes_j \mid (i,j) \in W^2 \} \cup \{\otimes_i \pi \mid \pi \in M^i, i \in |W|\}$.

Consequently $\text{HPL}^\sharp$ has diagrams that are defined for each model $((W, M), w)$ as follows:

- the elementary extensions are the same as for the $\text{HPL}$ diagrams; and
- $E_{(\text{Nom,P}),,(W,M),w} = E_{(\text{Nom,P}),,(W,M)} \cup \{w\text{-sen}\}$. 

**Corollary 5.2.** $\text{HFOL}^\star$ and $\text{HFOL}^\sharp$ have diagrams.

**Proof.** We have to recall Example 5.6. For each $(\text{Nom}, F, P)$-model $(W, M)$:

- $i_{\text{(Nom,F,P)},(W,M)}$ is the signature extension to $(\text{Nom} + |W|, F + |M|, P)$; and
- $E_{(\text{Nom,F,P}),,(W,M)} = \{\otimes_i \otimes_j \mid (i,j) \in W^2 \} \cup \{\otimes_i \rho \mid \rho \in E_{(F,P),M}, i \in |W|\}$, where $E_{(F,P),M}$ denotes the FOL diagram of $M^i$.

$\text{HFOL}^\sharp$ has diagrams that are defined for each $((W, M), w)$ as follows:

- the elementary extensions are the same as for the $\text{HFOL}$ diagrams; and
- $E_{(\text{Nom,F,P}),,(W,M),w} = E_{(\text{Nom,F,P}),,(W,M)} \cup \{w\text{-sen}\}$. 

\[\square\]
5.4. **Non-existence of diagrams.** One of the general conclusions of our study of diagrams for stratified institutions is that they are dependent on some kind of hybrid infrastructure. While mathematically Theorem 5.1 only says that such infrastructure is sufficient, this also feels necessary because the whole idea of diagrams is related to having syntactic designations for all elements of the models, either “worlds” or elements of their interpretations. The following negative result\(^5\) provides some support for this conclusion.

**Proposition 5.1.** Neither MPL\(^*\) nor MPL\(^\sharp\) admit institution theoretic diagrams.

**Proof.** In both cases we perform a *Reductio ad Absurdum* proof by initially assuming that each of the two institutions admit diagrams.

- In the case of MPL\(^*\), let \((W, M)\) be any \(P\)-model in MPL such that \(|W| \neq \emptyset\). Since \(1_{(W, M)}\) is initial in the comma category \((W, M)/\text{Mod}^{\text{MPL}}(P)\) it follows that \(i_{P,(W,M)}^{-1}(1_{(W,M)})\) is initial in the category of the \(P_{(W,M)}\)-models satisfying \(E_{P,(W,M)}\). But it is easy to note that the empty MPL Kripke structure trivially satisfies any sentence in MPL hence \(i_{P,(W,M)}^{-1}(1_{(W,M)})\) is bound to be this trivial empty Kripke structure. This is a contradiction because, according to the axioms of diagrams, when reducing \(i_{P,(W,M)}^{-1}(1_{(W,M)})\) via \(\iota_{P,(W,M)}\) we should obtain \((W, M)\) which by our assumption is not empty.
- For the case of MPL\(^\sharp\) let us consider a singleton signature \(P = \{\pi\}\) and the MPL\(^\sharp\) \(P\)-model \(((W, M), w)\) where
  - \(|W| = \{w, v\}\) and \(W_\pi = \emptyset\); and
  - \(M^w = \emptyset\) and \(M^v = P\).

Since \(P\) is a singleton, without any loss of generality we may assume that the elementary extension \(\iota_{P,(W,M),w}\) is an inclusion \(P \subseteq P'\). Let

\[
((W, M'), w) = i_{P,(W,M),w}^{-1}(1_{(W,M),w})
\]

and let the MPL\(^\sharp\) \(P'\)-model \(((W, N'), w)\) be defined by \(N'^w = M'^w\) and \(N'^v = M'^v \setminus P\).

By induction on the structure of \(\rho\), it is easy to establish that for any \(P'\)-sentence \(\rho\) we have that

\[
((W, M'), w) \models \rho \text{ if and only if } ((W, N'), w) \models \rho.
\]

Since \(((W, M'), w)\) is a model of the diagram of \(((W, M), w)\), it follows that \(((W, N'), w)\) is a model of that diagram too. Hence \(i_{P,(W,M),w}((W, N'), w))\) is a homomorphism \(((W, M), w) \rightarrow ((W, N), w)\) where \(((W, N), w)\) is the \(P\)-reduct of \(((W, N'), w)\). Let us denote it by \(h\). Then by the homomorphism property of \(h\) we have that \(M^v \subseteq N^{h(v)}\).

But \(M^v = P\) and \(N^w = N^v = \emptyset\), hence \(M^v \not\subseteq N^{h(v)}\). This contradiction invalidates our supposition of the existence of diagrams.

\[\square\]

6. **Conclusions**

In this paper we have introduced a new technique for representing stratified institutions by a decomposition at the level of the models. Then we applied this decomposition technique

\(^5\)Developed jointly with Manuel-Antonio Martins.
for developing general results on the existence of model amalgamation and of diagrams in stratified institutions. In the latter case it has emerged that some nominals infrastructure is needed. This is hardly surprising because fundamentally diagrams reflect a fine balance between syntax and semantics (i.e. model homomorphisms represented as models of theories) and the presence of nominals restore such a balance for Kripke semantics.

**Future work.** The potential of our decomposition technique should be further explored along the following directions:

1. Quasi-varieties and initial semantics in stratified institutions. These have been studied for the particular half-abstract case of hybridised institutions in [8]. However by the decomposition technique we should be able to do those at the higher level of generality of abstract stratified institutions, one of the consequences being a wider class of concrete applications.

2. Develop some general results supporting the existence of important structures in stratified institutional model theory that in the current literature have an “assumed” status. An example is that of filtered products of models [9].

3. Generate new interesting examples of stratified institutions that break from the modal logics tradition. In this respect pragmatic motivations may come from computing science which has many areas whose foundations involve some form of models with states. In those situations there is usually an almost automatic reliance on modal logics in their more or less conventional acceptations, although those have not been developed for those specific purposes, but rather for pure logic interests. Stratified institutions and their decomposition technique has the potential to offer a powerful theoretical tool for going beyond modal logics by defining model theoretic frameworks that are finer tuned to respective concrete applications.

**Acknowledgement.** This work was supported by a grant of the Romanian Ministry of Education and Research, CNCS – UEFISCDI, project number PN-III-P4-ID-PCE-2020-0446, within PNCDI III.

**References**

[1] Marc Aiguier and Isabelle Bloch. Logical dual concepts based on mathematical morphology in stratified institutions: applications to spatial reasoning. *Journal of Applied Non-Classical Logics*, 29(4):392–429, 2019.

[2] Marc Aiguier and Răzvan Diaconescu. Stratified institutions and elementary homomorphisms. *Information Processing Letters*, 103(1):5–13, 2007.

[3] M. Arrais and José L. Fiadeiro. Unifying theories in different institutions. In Magne Haveraaen, Olaf Owe, and Ole-Johan Dahl, editors, *Recent Trends in Data Type Specification*, volume 1130 of *Lecture Notes in Computer Science*, pages 81–101. Springer, 1996.

[4] Patrick Blackburn. Representation, reasoning, and relational structures: a hybrid logic manifesto. *Logic Journal of IGPL*, 8(3):339–365, 2000.

[5] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge University Press, 2001.

[6] Răzvan Diaconescu. Elementary diagrams in institutions. *Journal of Logic and Computation*, 14(5):651–674, 2004.

[7] Răzvan Diaconescu. *Institution-independent Model Theory*. Birkhäuser, 2008.
[8] Răzvan Diaconescu. Quasi-varieties and initial semantics in hybridized institutions. *Journal of Logic and Computation*, 26(3):855–891, 2016.

[9] Răzvan Diaconescu. Implicit Kripke semantics and ultraproducts in stratified institutions. *Journal of Logic and Computation*, 27(5):1577–1606, 2017.

[10] Răzvan Diaconescu. Introducing H, an institution-based formal specification and verification language. *Logica Universalis*, 14(2):259–277, 2020.

[11] Răzvan Diaconescu and Ionuț Țuțu. On the algebra of structured specifications. *Theoretical Computer Science*, 412(28):3145–3174, 2011.

[12] Răzvan Diaconescu, Joseph Goguen, and Petros Stefaneas. Logical support for modularisation. In Gerard Huet and Gordon Plotkin, editors, *Logical Environments*, pages 83–130. Cambridge, 1993. Proceedings of a Workshop held in Edinburgh, Scotland, May 1991.

[13] Răzvan Diaconescu and Alexandre Madeira. Encoding hybridized institutions into first order logic. *Mathematical Structures in Computer Science*, 26:745–788, 2016.

[14] Răzvan Diaconescu and Petros Stefaneas. Modality in open institutions with concrete syntax. *Bulletin of the Greek Mathematical Society*, 49:91–101, 2004. Previously published as JAIST Tech Report IS-RR-97-0046, 1997.

[15] Răzvan Diaconescu and Petros Stefaneas. Ultraproducts and possible worlds semantics in institutions. *Theoretical Computer Science*, 379(1):210–230, 2007.

[16] Melvin Fitting and Richard L. Mendelsohn. *First-order Modal Logic*. Kluwer/Springer, 1998.

[17] Dov M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications*. Elsevier, 2003.

[18] Joseph Goguen and Rod Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the Association for Computing Machinery*, 39(1):95–146, 1992.

[19] Joseph Goguen and Grigore Roșu. Institution morphisms. *Formal Aspects of Computing*, 13:274–307, 2002.

[20] Valentin Goranko. Hierarchies of modal and temporal logics with reference pointers. *Journal of Logic, Language and Information*, 5(1):1–24, 1996.

[21] Daniel Găină. Foundations of logic programming in hybridised logics. In Codescu M., Diaconescu R., and Țuțu I., editors, *Recent Trends in Algebraic Development Techniques. WADT 2015*, volume 9463, pages 69–89. Springer, Cham, 2015.

[22] Daniel Găină. Forcing and calculi for hybrid logics. *Journal of ACM*, 67(4):25:1–25:55, 2020.

[23] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer, second edition, 1998.

[24] Alexandre Madeira. *Foundations and techniques for software reconfigurability*. PhD thesis, Universidades do Minho, Aveiro and Porto (Joint MAP-i Doctoral Programme), 2014.

[25] Manuel-Antonio Martins, Alexandre Madeira, Răzvan Diaconescu, and Luis Barbosa. Hybridization of institutions. In Andrea Corradini, Bartek Klin, and Corina Cîrstea, editors, *Algebra and Coalgebra in Computer Science*, volume 6859 of *Lecture Notes in Computer Science*, pages 283–297. Springer, 2011.

[26] José Meseguer. General logics. In H.-D. Ebbinghaus et al., editors, *Proceedings, Logic Colloquium, 1987*, pages 275–329. North-Holland, 1989.

[27] Till Mossakowski. Different types of arrow between logical frameworks. In F. Meyer auf der Heide and B. Monien, editors, *Proc. ICALP 96*, volume 1099 of *Lecture Notes in Computer Science*, pages 158–169. Springer Verlag, 1996.

[28] Arthur N. Prior. *Past, Present and Future*. Oxford University Press, 1967.

[29] Donald Sannella and Andrzej Tarlecki. Specifications in an arbitrary institution. *Information and Control*, 76:165–210, 1988.

[30] Donald Sannella and Andrzej Tarlecki. *Foundations of Algebraic Specifications and Formal Software Development*. Springer, 2012.

[31] Andrzej Tarlecki. On the existence of free models in abstract algebraic institutions. *Theoretical Computer Science*, 37:269–304, 1986.
[32] Andrzej Tarlecki. Quasi-varieties in abstract algebraic institutions. *Journal of Computer and System Sciences*, 33(3):333–360, 1986.

[33] Andrzej Tarlecki. Moving between logical systems. In Magne Haveraaen, Olaf Owe, and Ole-Johan Dahl, editors, *Recent Trends in Data Type Specification*, volume 1130 of *Lecture Notes in Computer Science*, pages 478–502. Springer, 1996.

[34] Andrzej Tarlecki. Towards heterogeneous specifications. In D. Gabbay and M. van Rijke, editors, *Proceedings, International Conference on Frontiers of Combining Systems (FroCoS’98)*, pages 337–360. Research Studies Press, 2000.

**Simion Stoilow Institute of Mathematics of the Romanian Academy**

*Email address*: Razvan.Diaconescu@ymail.com