Optimal Berry-Ésséen bound for maximum likelihood estimation of the drift parameter in $\alpha$-Brownian bridge

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Abstract

Let $T > 0$, $\alpha > \frac{1}{2}$. In the present paper we consider the $\alpha$-Brownian bridge defined as $dX_t = -\alpha \frac{X_t}{T-t} dt + dW_t$, $0 \leq t < T$, where $W$ is a standard Brownian motion. We investigate the optimal rate of convergence to normality of the maximum likelihood estimator (MLE) for the parameter $\alpha$ based on the continuous observation $\{X_s, 0 \leq s \leq t\}$ as $t \uparrow T$. We prove that an optimal rate of Kolmogorov distance for central limit theorem on the MLE is given by $\frac{1}{\sqrt{\log(T-t)}}$, as $t \uparrow T$. First we compute an upper bound and then find a lower bound with the same speed using Corollary 1 and Corollary 2 of Kim et al. (J Multivar Anal 155:284–304, 2017b) respectively.

Keywords $\alpha$-Brownian bridge · Rate of convergence · MLE · Kolmogorov distance · Malliavin calculus

Mathematics Subject Classification 60F05 · 60H07 · 62F12

1 Introduction

Fix a time interval $[0, T)$, with $T$ is a positive real number. We consider the $\alpha$-Brownian bridge process $X := \{X_t, t \in [0, T)\}$, defined as the solution to the stochastic differential equation

$$X_0 = 0; dX_t = -\alpha \frac{X_t}{T-t} dt + dW_t, \quad 0 \leq t < T,$$  

(1)
where $W$ is a standard Brownian motion, and $\alpha > 0$ is unknown parameter to be estimated.

In recent years, the study of various problems related to the $\alpha$-Brownian bridge (1) has attracted interest. The process (1) has been first considered by Brennan et al. (1990), where it is used to describe the evolution of the simple arbitrage opportunity associated with a given futures contract in the absence of transaction costs. For more information and further references concerning the subject, we refer the reader to (Barczy et al. 2010), as well as Mansuy (2004) and Görgens et al (2014).

An example of interesting problem related to $X$ is the statistical estimation of $\alpha$ when one observes the whole trajectory of $X$. A natural candidate is the maximum likelihood estimator (MLE), which can be easily computed for this model, due to the specific form of (1): one gets

$$\tilde{\alpha}_t = -\left(\int_0^t \frac{X_u}{T-u} dX_u\right) \bigg/ \left(\int_0^t \frac{X_u^2}{(T-u)^2} du\right), \quad t < T. \quad (2)$$

In (2), the integral with respect to $X$ must of course be understood in the Itô sense. Moreover, it is easy to find

$$\alpha - \tilde{\alpha}_t = \left(\int_0^t \frac{X_u}{T-u} dW_u\right) \bigg/ \left(\int_0^t \frac{X_u^2}{(T-u)^2} du\right). \quad (3)$$

The asymptotic behavior of the MLE $\tilde{\alpha}_t$ of $\alpha$ based on the observation $\{X_s, 0 \leq s \leq t\}$ as $t \uparrow T$ has been studied in Barczy et al. (2010). Let us describe what is known about this problem: as $t \uparrow T$,

- if $\alpha > 0$, the MLE $\tilde{\alpha}_t$ is strongly consistent, that is, $\tilde{\alpha}_t \rightarrow \alpha$ almost surely, see [(Barczy et al. 2010, Theorem 16)];
- if $0 < \alpha < \frac{1}{2}$

$$\frac{(T-t)^{\frac{1}{2}\alpha}}{(T-t)^{\frac{1}{2}\alpha}} \left(\alpha - \tilde{\alpha}_t\right) \xrightarrow{\text{law}} \frac{T}{(2\alpha - 1)} \times \mathcal{C}(1)$$

with $\mathcal{C}(1)$ the standard Cauchy admitting a density function $\pi^{-1}(1 + x^2)^{-1}, x \in \mathbb{R}$, see [(Barczy et al. 2010, Theorem 7)];
- if $\alpha = \frac{1}{2}$

$$|\log (T-t)| \left(\alpha - \tilde{\alpha}_t\right) \xrightarrow{\text{law}} \int_0^T W_s dW_s$$

$$\int_0^T W_s^2 ds$$

see [(Barczy et al. 2010, Theorem 5)];
- if $\alpha > \frac{1}{2}$

$$\sqrt{|\log (T-t)|} \left(\alpha - \tilde{\alpha}_t\right) \xrightarrow{\text{law}} N(0, 2\alpha - 1),$$
see [(Barczy et al. 2010, Theorem 10)].

The study of the asymptotic distribution of an estimator is not very useful in general for practical purposes unless the rate of convergence is known. To our knowledge, no result of the Berry-Esséen type is known for the distribution of the MLE $\tilde{\alpha}_t$ of the drift parameter $\alpha$ of the $\alpha$-Brownian bridge (1). The aim of the present work is to provide, when $\alpha > \frac{1}{2}$, an optimal rate of Kolmogorov distance for central limit theorem of the MLE $\tilde{\alpha}_t$ in the following sense: There exist constants $0 < c < C < \infty$, depending only on $\alpha$ and $T$, such that for all $t$ sufficiently near $T$,

$$
\frac{c}{\sqrt{\log (T-t)}} \leq \sup_{z \in \mathbb{R}} \left| \mathbb{P}\left( \sqrt{\frac{1}{2\alpha - 1}} \log \left( \frac{T-t}{2} \right) \leq \alpha - \tilde{\alpha}_t \right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C}{\sqrt{\log (T-t)}},
$$

where $Z$ denotes a standard normal random variable.

Let us recall the case of the Ornstein–Uhlenbeck process defined as solution to the equation $dX_t = -\theta X_t dt + dW_t$, $t \geq 0$, $X_0 = 0$, with $\theta > 0$. While (Bishwal 2000) obtained the upper bound $O(1/\sqrt{T})$ in Kolmogorov distance for normal approximation of the MLE of the drift parameter $\theta$ on the basis of continuous observation of the process $X_t$ on the time interval $[0, T]$, a lower bound with the same speed has been recently obtained by Kim et al. (2017a). This means that $O(1/\sqrt{T})$ is an optimal Berry-Esséen bound for the MLE of $\theta$. Finally, we mention that (Es-Sebaiy et al. 2013) studied the parameter estimation for so-called $\alpha$-fractional bridge which is given by the Eq. (1) replacing the standard Brownian motion $W$ by a fractional Brownian motion.

We proceed as follows. In Sect. 2 we give the basic tools of Malliavin calculus needed throughout the paper. Section 3 contains our main result, concerning the optimal rate of convergence to normality of the MLE $\tilde{\alpha}_t$.

## 2 Preliminaries

In this section, we recall some elements from stochastic analysis that we will need in the paper. See Nourdin et al. (2012), and Nualart (2006) for details. Any real, separable Hilbert space $\mathcal{S}$ gives rise to an isonormal Gaussian process: a centered Gaussian family $(G(\varphi), \varphi \in \mathcal{S})$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}(G(\varphi)G(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{S}}$. In this paper, it is enough to use the classical Wiener space, where $\mathcal{S} = L^2([0, T])$, though any $\mathcal{S}$ will also work. In the case $\mathcal{S} = L^2([0, T])$, $G$ can be identified with the stochastic differential of a Wiener process $\{W_t, t \in [0, T]\}$ and one interprets $G(\varphi) := \int_0^T \varphi(s)dW(s)$.

The Wiener chaos of order $p$, denoted by $\mathcal{S}^p$, is defined as the closure in $L^2(\Omega)$ of the linear span of the random variables $H_p(G(\varphi))$, where $\varphi \in \mathcal{S}$, $\|\varphi\|_{\mathcal{S}} = 1$ and $H_p$ is the Hermite polynomial of degree $p$. The multiple Wiener stochastic integral $I_p$ with respect to $G \equiv W$, of order $p$ is an isometry between the Hilbert space $\mathcal{S}^p$ and $L^2_{sym}([0, T]^p)$ (symmetric tensor product) equipped with the scaled norm $\|u\|_{\mathcal{S}^p}^2 = \sum_{\sigma} \langle u_{\sigma} \rangle_{\mathcal{S}}^2$ where $\sigma$ runs over all $p$-element subsets of $\{1, \ldots, p\}$.
$\sqrt{p!}\| \cdot \|_{\mathcal{H}_p}$ and the Wiener chaos of order $p$ under $L^2(\Omega)$’s norm, that is, the multiple Wiener stochastic integral of order $p$:

$$I_p : (\mathcal{H}_{p}^{\otimes p}, \sqrt{p!}\| \cdot \|_{\mathcal{H}_p}) \longrightarrow (\mathcal{H}_p, L^2(\Omega))$$

is a linear isometry defined by $I_p(f^{\otimes p}) = H_p(G(f))$.

- **Multiple Wiener-Itô integral** If $f \in L^2([0,T]^p)$ is symmetric, we can also rewrite $I_p(f)$ as the following iterated adapted Itô stochastic integral:

$$I_p(f) = \int_{[0,T]^p} f(t_1, \ldots, t_p) dW_{t_1} \cdots dW_{t_p}$$

$$= p! \int_0^T dW_{t_1} \int_0^{t_1} dW_{t_2} \cdots \int_0^{t_{p-1}} dW_{t_p} f(t_1, \ldots, t_p).$$

(4)

- **The Wiener chaos expansion** For any $F \in L^2(\Omega)$, there exists a unique sequence of functions $f_p \in \mathcal{H}_p^{\otimes p}$ such that

$$F = \mathbb{E}[F] + \sum_{p=1}^{\infty} I_p(f_p),$$

where the terms are all mutually orthogonal in $L^2(\Omega)$ and

$$\mathbb{E}[I_p(f_p)^2] = p! \| f_p \|_{\mathcal{H}_p}^2.$$

- **Product formula and contractions** For any integers $p, q \geq 1$ and symmetric integrands $f \in \mathcal{H}_p^{\otimes p}$ and $g \in \mathcal{H}_q^{\otimes q}$,

$$I_p(f)I_q(g) = \sum_{r=0}^{p+q} r! \left( \begin{array}{c} p \\ r \end{array} \right) \left( \begin{array}{c} q \\ r \end{array} \right) I_{p+q-2r}(f^{\otimes r}g);$$

(5)

where $f^{\otimes r}g$ is the contraction of order $r$ of $f$ and $g$ which is an element of $\mathcal{H}_{p+q-2r}^{\otimes (p+q-2r)}$ defined by

$$(f^{\otimes r}g)(s_1, \ldots, s_{p-r}, t_1, \ldots, t_{q-r})$$

$$:= \int_{[0,T]^{p+q-2r}} f(s_1, \ldots, s_{p-r}, u_1, \ldots, u_r) g(t_1, \ldots, t_{q-r},$$

$$\times u_1, \ldots, u_r) du_1 \cdots du_r,$$

while $f^{\otimes r}g$ denotes its symmetrization. More generally the symmetrization $\hat{f}$ of a function $f$ is defined by $\hat{f}(x_1, \ldots, x_p) = \frac{1}{p!} \sum_{\sigma} f(x_{\sigma(1)}, \ldots, x_{\sigma(p)})$ where the sum runs over all permutations $\sigma$ of $\{1, \ldots, p\}$.

- **Kolmogorov distance between random variables** If $X, Y$ are two real-valued random variables, then the Kolmogorov distance between the law of $X$ and the law of $Y$ is given by

\[d_{K}(X, Y) = \sup_{t \in \mathbb{R}} |\mathbb{P}(X \leq t) - \mathbb{P}(Y \leq t)|.\]
\[ d_{kol}(X, Y) := \sup_{z \in \mathbb{R}} |\mathbb{P}(X \leq z) - \mathbb{P}(Y \leq z)|. \]

- **Optimal Berry-Esséen bound for the CLT of** \( \frac{F_t}{G_t} \)** Let** \( Z \) denote the standard normal law. Recently, using techniques relied on the combination of Malliavin calculus and Stein’s method (see, e.g., Nourdin et al. (2012)), the following observation provided lower and upper bounds of the Kolmogrov distance for the Central Limit Theorem (CLT) of \( \frac{F_t}{G_t} \), where \( F_t \) and \( G_t \) are functionals of Gaussian fields.

Fix \( T > 0 \). Let \( f_t, g_t \in \mathcal{H} \otimes^2 \) for all \( t \in [0, T) \), and let \( b_t \) be a positive function of \( t \) such that \( I_2(g_t) + b_t > 0 \) almost surely for all \( t \in [0, T) \).

(a) If \( \max_{i=1,2,3} \psi_i(t) \to 0 \) as \( t \uparrow T \), where for every \( t \in [0, T) \),

\[
\psi_1(t) := \frac{1}{b_t^2} \sqrt{\left( b_t^2 - 2 \| f_t \|_{\mathcal{H} \otimes^2}^2 \right)^2 + 8 \| f_t \otimes_1 f_t \|_{\mathcal{H} \otimes^2}^2}, \\
\psi_2(t) := \frac{2}{b_t^2} \sqrt{2 \| f_t \otimes_1 g_t \|_{\mathcal{H} \otimes^2}^2 + \langle f_t, g_t \rangle_{\mathcal{H} \otimes^2}^2}, \\
\psi_3(t) := \frac{2}{b_t^2} \sqrt{\| g_t \otimes_1 g_t \|_{\mathcal{H} \otimes^2}^4 + 2 \| g_t \otimes_1 g_t \|_{\mathcal{H} \otimes^2}^2}.
\]

then [see Corollary 1 in Kim et al. (2017b)], there exists a positive constant \( C \) such that for all \( t \) sufficiently near \( T \),

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left( \frac{I_2(f_t)}{I_2(g_t) + b_t} \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq C \max_{i=1,2,3} \psi_i(t). \tag{6}
\]

(b) If, as \( t \uparrow T \),

\[
\| f_t \otimes_1 f_t \|_{\mathcal{H} \otimes^2} \to 0, \quad \frac{2 \| f_t \|_{\mathcal{H} \otimes^2}^2 - 1}{\langle f_t, f_t \rangle_{\mathcal{H} \otimes^2}} \to 0,
\]

and \( \frac{\| f_t \otimes_1 f_t \|_{\mathcal{H} \otimes^2}}{\langle f_t, f_t \rangle_{\mathcal{H} \otimes^2}} \to \rho \neq 0 \),

then [see Corollary 2 in Kim et al. (2017b)], there exists a positive constant \( c \) such that for all \( t \) sufficiently near \( T \),

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left( \frac{I_2(f_t)}{I_2(g_t) + b_t} \leq z \right) - \mathbb{P}(Z \leq z) \right| \geq c \| f_t \otimes_1 f_t \|_{\mathcal{H} \otimes^2}. \tag{7}
\]

Throughout the paper \( Z \) denotes a standard normal random variable. Also, \( C \) denotes a generic positive constant (perhaps depending on \( \alpha \) and \( T \), but not on anything else), which may change from line to line.
3 Optimal rate of convergence of the MLE

In this section we consider the problem of optimal rate of convergence to normality of the MLE $\tilde{\alpha}_t$ given in (2). More precisely, we want to provide an optimal Berry-Esséen bound in the Kolmogorov distance for $\tilde{\alpha}_t$. In what follows we suppose that $\alpha > \frac{1}{2}$.

To proceed, let us start with useful notations needed in what follows. Because (1) is linear, it is immediate to solve it explicitly; one then gets the following formula:

$$X_t = (T - t)^\alpha \int_0^t (T - s)^{-\alpha} dW_s, \quad 0 \leq t < T. \quad (8)$$

Define for every $t \in [0, T)$,

$$\lambda_t := \frac{|\log (T - t)|}{2\alpha - 1}. \quad (9)$$

It follows from (4) and (8) that for every $t \in [0, T)$,

$$\sqrt{\frac{2\alpha - 1}{|\log (T - t)|}} \int_0^t \frac{X_s}{T - s} dW_s = \frac{1}{\sqrt{\lambda_t}} \int_0^t \frac{X_s}{T - s} dW_s$$

$$= \frac{1}{\sqrt{\lambda_t}} \int_0^t \int_0^s (T - s)^{a-1} (T - r)^{-\alpha} dW_r dW_s$$

$$= \frac{1}{2\lambda_t} \int_0^t \int_0^t (T - s \lor r)^{a-1}$$

$$\times (T - s \land r)^{-\alpha} dW_r dW_s$$

$$=: I_2(f_t), \quad (10)$$

where $f_t$ is a symmetric function defined by

$$f_t(u, v) = \frac{1}{2\sqrt{\lambda_t}} (T - u \lor v)^{a-1} (T - u \land v)^{-\alpha} \mathbb{1}_{[0,t]}(u, v). \quad (11)$$

On the other hand, using (5) and (8), we can write

$$X_s^2 = I_1 \left( (T - s)^a (T - u)^{-\alpha} \mathbb{1}_{[0,s]}(u) \right)$$

$$= I_2 \left( (T - s)^{2a} (T - u)^{-\alpha} \mathbb{1}_{[0,s]}(u, v) \right)$$

$$+ \int_0^s (T - s)^{2a} (T - u)^{-2\alpha} du.$$

Hence, we have

$$\frac{1}{\lambda_t} \int_0^t \frac{X_s^2}{(T - s)^2} ds = I_2(g_t),$$

where

$$I_2(g_t) + b_t, \quad (12)$$

where
\[ g_t(u, v) = \frac{1}{\lambda_t} \int_0^t (T-s)^{2\alpha-2}(T-u)^{-\alpha}(T-v)^{-\alpha} \mathbb{1}_{[0,T]}(u,v)ds \]
\[ = \frac{1}{\lambda_t} (T-u)^{-\alpha}(T-v)^{-\alpha} \mathbb{1}_{[0,T]}(u,v) \int_u^t (T-s)^{2\alpha-2} ds \]
\[ = \frac{(T-u)^{-\alpha}(T-v)^{-\alpha}}{|\log(T-t)|} \times [(T-u \lor v)^{2\alpha-1} - (T-t)^{2\alpha-1}] \mathbb{1}_{[0,T]}(u,v), \tag{13} \]

and
\[ b_t = \frac{1}{\lambda_t} \int_0^t \int_0^s (T-s)^{2\alpha-2}(T-u)^{-2\alpha} duds \]
\[ = \frac{1}{|\log(T-t)|} \int_0^t (T-s)^{-\alpha} - T^{1-2\alpha}(T-s)^{2\alpha-2} ds \]
\[ = 1 + \frac{\log(T)}{|\log(T-t)|} - \frac{1}{(2\alpha-1)\log(T-t)} \left(1 - \left(\frac{T-t}{T}\right)^{2\alpha-1}\right). \tag{14} \]

Therefore, combining (3), (10) and (12), we can write
\[ \sqrt{\lambda_t}(\alpha - \tilde{\alpha}_t) = \frac{I_2(f_t)}{I_2(g_t) + b_t}, \tag{15} \]

where \( \lambda_t, f_t, g_t \) and \( b_t \) are given in (9), (11), (13) and (14), respectively.

In order to prove our main result we make use of the following technical lemmas.

**Lemma 1** Suppose that \( \alpha > \frac{1}{2} \). Let \( \lambda_t, f_t \) and \( b_t \) be the functions given by (9), (11) respectively. Then we have, as \( t \uparrow T \),
\[ 2\|f_t\|_{\mathcal{S}^{\beta_1}}^2 - 1 = \frac{\log(T)}{(2\alpha - 1)\lambda_t} - \frac{1}{(2\alpha - 1)^2 \lambda_t} + o\left(\frac{1}{\lambda_t}\right), \tag{16} \]
\[ b_t^2 - 1 = \frac{2\log(T)}{(2\alpha - 1)\lambda_t} - \frac{2}{(2\alpha - 1)^2 \lambda_t} + o\left(\frac{1}{\lambda_t}\right), \tag{17} \]
\[ \langle f_t \otimes_1 f_t, f_t \rangle_{\mathcal{S}^{\beta_2}} = \frac{3}{4(2\alpha - 1)\sqrt{\lambda_t}} + o\left(\frac{1}{\sqrt{\lambda_t}}\right), \tag{18} \]
\[ \|f_t \otimes_1 f_t\|_{\mathcal{S}^{\beta_2}}^2 = \frac{5}{4(2\alpha - 1)^2 \lambda_t} + o\left(\frac{1}{\lambda_t}\right), \tag{19} \]

where the notation \( o(1/\lambda_t^\beta) \) means that \( \lambda_t^\beta o(1/\lambda_t^\beta) \rightarrow 0 \) as \( t \uparrow T \).
Proof Suppose that \(0 < T - t < 1\). So, \(- \log(T - t) = |\log(T - t)|\). We also notice that \(\lambda_i \to \infty\) as \(t \uparrow T\). Since the function \(f_t\) is symmetric, we have

\[
\|f_t\|^2_{\mathcal{D}^{\otimes 2}} = \frac{1}{4 \lambda_i} \int_{[0,t]^2} (T - x \vee y)^{2a-2} (T - x \wedge y)^{-2a} \, dx \, dy
\]

\[
= \frac{1}{2 \lambda_i} \int_0^t dy \int_0^y (T - y)^{2a-2} (T - x)^{-2a} \, dx
\]

\[
= \frac{1}{2(2a-1) \lambda_i} \int_0^t ((T - y)^{-1} - T^{-2a+1} (T - y)^{2a-2}) \, dy
\]

\[
= \frac{1}{2 |\log(T - t)|} \left( \log(T) - \log(T - t) + \frac{((T - t)/T)^{2a-1}}{2a-1} \right)
\]

\[
\times \frac{1}{(2a-1) |\log(T - t)|}
\]

\[
= \frac{1}{2} \left( 1 + \frac{\log(T)}{(2a-1) \lambda_i} - \frac{1}{(2a-1)^2 \lambda_i} + \frac{(T - t)^{2a-1}}{(2a-1)^2 T^{2a-1} \lambda_i} \right)
\]

as \(t \uparrow T\), where the latter equality comes from the fact that \(\alpha > \frac{1}{2}\).

Thus, we can deduce

\[
2\|f_t\|^2_{\mathcal{D}^{\otimes 2}} - 1 = \frac{\log(T)}{(2a-1) \lambda_i} - \frac{1}{(2a-1)^2 \lambda_i} + o \left( \frac{1}{\lambda_i} \right)
\]

which proves (16). On the other hand, the estimate (17) is a direct consequence of (14).

Let us prove (18), we have

\[
\langle f_t \otimes_1 f_t, f_t \rangle_{\mathcal{D}^{\otimes 2}} = \int_{[0,t]^3} f_t(x_1, x_2)f_t(x_2, x_3)f_t(x_3, x_1) \, dx_1 \, dx_2 \, dx_3
\]

\[
= 3! \int_0^t dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 f_t(x_1, x_2)f_t(x_2, x_3)f_t(x_3, x_1),
\]

where we used the fact that the integrand is symmetric.

Hence, using (11), we get
\[
\langle f_1 \otimes f_1, f_1 \rangle_{\mathcal{S}^{\otimes 2}} \\
= \frac{3}{4(\lambda_t)^{3/2}} \int_0^t dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 (T - x_3)^{2a-2} (T - x_2)^{-1} (T - x_1)^{-2a} \\
= \frac{3}{4(2\alpha - 1)(\lambda_t)^{3/2}} \int_0^t dx_3 \int_0^{x_3} dx_2 (T - x_3)^{2a-2} (T - x_2)^{-1} \\
\times \left[ (T - x_2)^{1-2a} - T^{1-2a} \right] \\
= \frac{3}{4(2\alpha - 1)(\lambda_t)^{3/2}} \int_0^t dx_3 (T - x_3)^{2a-2} \\
\left[ \frac{(T - x_3)^{1-2a} - T^{1-2a}}{2\alpha - 1} + T^{1-2a} \log \left( \frac{T - x_3}{T} \right) \right] \\
= \frac{3}{4(2\alpha - 1)^2(\lambda_t)^{3/2}} \left[ \log(T) - \log(T - t) + T^{1-2a} \frac{(T - t)^{2a-1} - T^{2a-1}}{2\alpha - 1} \right. \\
\left. - T^{1-2a} \log \left( \frac{T - t}{T} \right) \right](T - t)^{2a-1} + T^{1-2a} ((T - t)^{2a-1} - T^{2a-1}) \\
= \frac{3}{4(2\alpha - 1)\sqrt{\lambda_t}} + o\left( \frac{1}{\sqrt{\lambda_t}} \right),
\]
which proves (18).

Now let us prove (19). By (11), we obtain
\[
\|f_1 \otimes_1 f_1\|^2_{\mathcal{S}^{\otimes 2}} \\
= \int_{[0,t]^3} \left( \int_0^t f_1(x_1, x_2) f_1(x_3, x_2) dx_2 \right)^2 dx_1 dx_3 \\
= \int_{[0,t]^4} f_1(x_1, x_2) f_1(x_3, x_4) f_1(x_4, x_1) dx_1 dx_2 dx_3 dx_4 \\
= \frac{16}{16\lambda_t^2} \int_{\{0 < x_1 < x_2 < x_3 < x_4 \}} (T - x_4)^{2a-2} (T - x_3)^{-1} (T - x_2)^{-1} \\
\times (T - x_1)^{-2a} dx_1 dx_2 dx_3 dx_4 \\
+ \frac{8}{16\lambda_t^2} \int_{\{0 < x_1 < x_2 < x_3 < x_4 \}} (T - x_4)^{2a-2} (T - x_2)^{2a-2} (T - x_3)^{-2a} \\
\times (T - x_1)^{-2a} dx_1 dx_2 dx_3 dx_4 \\
=: A_{t,1} + A_{t,2}.
\]
For the term \(A_{t,1}\), we have
Similarly, we obtain

\[ A_{t,1} = \frac{1}{\lambda_t^2} \int_0^t dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 (T - x_4)^{2a-2} (T - x_3)^{-1} \]

\[ \times (T - x_2)^{-1} (T - x_1)^{-2a} \]

\[ = \frac{1}{(2\alpha - 1)\lambda_t^2} \int_0^t dx_4 \int_0^{x_4} dx_3 \int_0^{x_3} dx_2 (T - x_4)^{2a-2} (T - x_3)^{-1} \]

\[ \times \left[ \left( (T - x_2)^{-2a} - T^{1-2a} (T - x_2)^{-1} \right) \right] \]

\[ = \frac{1}{(2\alpha - 1)\lambda_t^2} \int_0^t dx_4 (T - x_4)^{-1} - T^{1-2a}(T - x_4)^{2a-2} \]

\[ \times \left[ \frac{(T - x_4)^{-1} - T^{1-2a}(T - x_4)^{2a-2}}{(2\alpha - 1)^2} + \frac{T^{1-2a}}{2\alpha - 1} \log \left( \frac{T - x_4}{T} \right) (T - x_4)^{2a-2} \right. \]

\[ - \frac{T^{1-2a}}{2} \log^2 \left( \frac{T - x_4}{T} \right) (T - x_4)^{2a-2} \]

\[ = \frac{1}{(2\alpha - 1)^3 \lambda_t^2} \int_0^t dx_4 (T - x_4)^{-1} + R_t \]

\[ = \frac{1}{(2\alpha - 1)^2 \lambda_t} + o\left( \frac{1}{\lambda_t} \right). \]

where, using integration by parts, straightforward calculations lead to

\[ R_t := \frac{1}{(2\alpha - 1)\lambda_t^2} \int_0^t dx_4 \left[ -T^{1-2a}(T - x_4)^{2a-2} \right. \]

\[ + \frac{T^{1-2a}}{2\alpha - 1} \log \left( \frac{T - x_4}{T} \right) (T - x_4)^{2a-2} \]

\[ - \frac{T^{1-2a}}{2} \log^2 \left( \frac{T - x_4}{T} \right) (T - x_4)^{2a-2} \]

\[ = o\left( \frac{1}{\lambda_t} \right). \]

Similarly, we obtain
Suppose that Lemma 2

Combining (20), (21) and (22), we obtain (19), and therefore, the proof is complete.

where, by integration by parts, it is easy to check that

$$A_{t,2} = \frac{1}{2(2\alpha - 1) \lambda_i^2} \int_0^t dx_4 \int_0^{x_4} dx_2 \int_0^{x_2} dx_1 (T - x_4)^{2a-2}$$

$$\times (T - x_2)^{2a-2} (T - x_3)^{-2a} (T - x_1)^{-2a}$$

$$= \frac{1}{2(2\alpha - 1) \lambda_i^2} \int_0^t dx_4 \int_0^{x_4} dx_2 (T - x_4)^{2a-2}$$

$$\times \left[ (T - x_3)^{1-4a} - T^{1-2a} (T - x_3)^{-2a} \right]$$

$$= \frac{1}{4(2\alpha - 1)^2 \lambda_i^2} \int_0^t dx_4 \int_0^{x_4} dx_2 (T - x_4)^{2a-2}$$

$$\times \left[ (T - x_2)^{-2a} - T^{-2a} (T - x_2)^{2a-2} + 2T^{1-2a} \left( (T - x_2)^{-1} - T^{-1} (T - x_2)^{2a-2} \right) \right]$$

$$= \frac{1}{4(2\alpha - 1)^3 \lambda_i^2} \int_0^t (T - x_4)^{-1} dx_4 + S_t$$

$$= \frac{1}{4(2\alpha - 1)^2 \lambda_i} + o \left( \frac{1}{\lambda_i} \right),$$

where, by integration by parts, it is easy to check that

$$S_t := \frac{-T^{1-2a}}{4(2\alpha - 1)^3 \lambda_i^2} \int_0^t \left[ (T - x_4)^{2a-2} - T^{1-2a} \right]$$

$$\times \left[ (T - x_4)^{4a-3} + T^{2a-1} (T - x_4)^{2a-2} \right] dx_4$$

$$+ \frac{T^{1-2a}}{2(2\alpha - 1)^3 \lambda_i^2} \int_0^t (T - x_4)^{2a-2} \log \left( \frac{T - x_4}{T} \right) dx_4$$

$$= o \left( \frac{1}{\lambda_i} \right).$$

Combining (20), (21) and (22), we obtain (19), and therefore, the proof is complete.

Lemma 2 Suppose that $\alpha > \frac{1}{2}$. Let $\lambda_i$, $f$, and $g_i$ be the functions given by (9), (11) and (13), respectively. Then, for all $(T - 1/e) \vee 0 < t < T$,
Proof. Note that if \((T - 1/e) \vee 0 < t < T,\) \(-\log(T-t) = |\log(T-t)|\) and \(|\log(T-t)| > 1.\) From (13) we have

\[
\|g_t\|_{\mathcal{Y}^2}^2 \leq C \frac{\lambda_t}{t
}
\] (23)

\[
\|g_t \otimes_1 g_t\|_{\mathcal{Y}^2} \leq C \frac{1}{\lambda_t^{3/2}},
\] (24)

\[
|\langle f_t, g_t \rangle_{\mathcal{Y}^2}| \leq C \frac{1}{\sqrt{\lambda_t}},
\] (25)

\[
Vert f_t \otimes_1 g_t \Vert_{\mathcal{Y}^2} \leq C \frac{1}{\lambda_t}.
\] (26)

which proves (23).

Now let us prove (24). Using (13) and the fact that \(\alpha > \frac{1}{2},\) we get

\[
0 \leq g_t(u, v) = \frac{(T - u)^{-\alpha}(T - v)^{-\alpha}}{|\log(T-t)|} \left[(T - u \vee v)^{2\alpha-1} - (T - t)^{2\alpha-1}\right] \quad \forall (u, v) \in [0, t] \times [0, t]
\]

\[
\leq \frac{\lambda_t}{\lambda_t^{3/2}} (T - u \vee v)^{2\alpha-1} - (T - t)^{2\alpha-1} \mathbb{1}_{[0,t]}(u, v)
\]

\[
= \mathcal{H}_t(u, v).
\] (27)

Further, notice that
\[ \int_{[0,t]^4} h_t(x_1, x_2) h_t(x_2, x_3) h_t(x_3, x_4) h_t(x_4, x_1) dx_1 dx_2 dx_3 dx_4 \]

\[ = \frac{16}{|\log (T-t)|^4} \int_{\{0<x_1<x_2<x_3<x_4<t\}} (T-x_4)^{2\alpha - 2} (T-x_3)^{-1} (T-x_2)^{-1} \]

\[ \times (T-x_1)^{-2\alpha} dx_1 dx_2 dx_3 dx_4 \]

\[ + \frac{8}{|\log (T-t)|^4} \int_{\{0<x_1<x_3<x_2<x_4<t\}} (T-x_4)^{2\alpha - 2} (T-x_2)^{2\alpha - 2} (T-x_3)^{-2\alpha} \]

\[ \times (T-x_1)^{-2\alpha} dx_1 dx_2 dx_3 dx_4 \]

\[ \leq \frac{16}{(2\alpha - 1)^4 \lambda_i^2} A_{t,1} + \frac{16}{(2\alpha - 1)^4 \lambda_i^2} A_{t,2} \]

\[ \leq \frac{C}{\lambda_i^3}, \tag{28} \]

where the latter inequality comes from (21) and (22).

Thus, combining (27) and (28), we obtain

\[ \| g_t \otimes_1 g_t \|^2_{\mathcal{B}^{\otimes 2}} \]

\[ = \int_{[0,t]^2} \left( \int_0^t g_t(x_1, x_2) g_t(x_3, x_2) dx_2 \right) dx_1 dx_3 \]

\[ = \int_{[0,t]^4} g_t(x_1, x_2) g_t(x_2, x_3) g_t(x_3, x_4) g_t(x_4, x_1) dx_1 dx_2 dx_3 dx_4 \]

\[ \leq \int_{[0,t]^4} h_t(x_1, x_2) h_t(x_2, x_3) h_t(x_3, x_4) h_t(x_4, x_1) dx_1 dx_2 dx_3 dx_4 \]

\[ \leq \frac{C}{\lambda_i^3}, \]

which implies (24). For (25), since \( f_t \) and \( g_t \) are symmetric, we have

\[ | \langle f_t, g_t \rangle_{\mathcal{B}^{\otimes 2}} | = \int_0^t \int_0^t f_t(u, v) g_t(u, v) dudv \]

\[ = \frac{1}{(2\alpha - 1)(\lambda_i)^{3/2}} \int_0^t dv \int_0^v du (T-v)^{-1} (T-u)^{-2\alpha} \]

\[ \times [(T-v)^{2\alpha - 1} - (T-t)^{2\alpha - 1}] \]

\[ \leq \frac{1}{(2\alpha - 1)(\lambda_i)^{3/2}} \int_0^t dv (T-v)^{-2\alpha - 2} \int_0^v du (T-u)^{-2\alpha} \]

\[ \leq \frac{1}{(2\alpha - 1)^2(\lambda_i)^{3/2}} \int_0^t dv (T-v)^{-1} \]

\[ \leq \frac{C}{\sqrt{\lambda_i}}. \]
To finish the proof it remains to prove the estimate (26). It follows from (27) that

$$\|f_t \otimes_1 g_t\|_{\hat{g}^2}^2 = \int_0^{t}\left( \int_0^t f_t(x_1, x_2) g_t(x_3, x_2) dx_2 \right)^2 dx_1 dx_3$$

$$= \int_0^{t}\int_0^t f_t(x_1, x_2) g_t(x_2, x_3) g_t(x_3, x_4) f_t(x_4, x_1) dx_1 dx_2 dx_3 dx_4$$

$$\leq \int_0^{t}\int_0^t f_t(x_1, x_2) h_t(x_2, x_3) h_t(x_3, x_4) f_t(x_4, x_1) dx_1 dx_2 dx_3 dx_4$$

Moreover, we notice that

$$\int_0^{t}\int_0^t f_t(x_1, x_2) h_t(x_2, x_3) h_t(x_3, x_4) f_t(x_4, x_1) dx_1 dx_2 dx_3 dx_4$$

$$= \frac{16}{4(2\alpha - 1)^2 \lambda_t} \int_{\{0 < t_1 < t_2 < t_3 < t_4 \}} (T - x_1)^{-2\alpha} (T - x_4)^{-1} \left((T - x_3)^{-1} - (T - x_2)^{-1}\right)$$

$$\times (T - x_4)^{2\alpha - 2} dx_1 dx_2 dx_3 dx_4$$

$$+ \frac{8}{4(2\alpha - 1)^2 \lambda_t} \int_{\{0 < t_1 < t_2 < t_3 < t_4 \}} (T - x_1)^{-2\alpha} (T - x_3)^{-2\alpha} (T - x_2)^{2\alpha - 2}$$

$$\times (T - x_4)^{2\alpha - 2} dx_1 dx_2 dx_3 dx_4$$

$$= \frac{4}{(2\alpha - 1)^2 \lambda_t^2} A_{t,1} + \frac{4}{(2\alpha - 1)^2 \lambda_t^2} A_{t,2}$$

$$\leq \frac{C}{\lambda_t^2},$$

where the latter inequality follows from (21) and (22).

Now we are ready to state and prove our main result. In the next Theorem we give an explicit optimal bound for the Kolmogorov distance, between the law of \(\sqrt{\frac{\log(T-t)}{2\alpha-1}}(\alpha - \tilde{\alpha}_t)\) and the standard normal law.

**Theorem 1** Let \(T > 0, \alpha > 1/2,\) and let \(\tilde{\alpha}_t\) be the MLE given in (2). Then there exist constants \(0 < c < C < \infty,\) depending only on \(\alpha\) and \(T,\) such that for all \(t\) sufficiently near \(T,\)

$$\frac{c}{\sqrt{\log(T-t)}} \leq \sup_{z \in \mathbb{R}} \left| \mathbb{P}\left(\sqrt{\frac{\log(T-t)}{2\alpha-1}}(\alpha - \tilde{\alpha}_t) \leq z \right) - \mathbb{P}(Z \leq z) \right| \leq \frac{C}{\sqrt{\log(T-t)}}.$$

**Proof** According to (15) we have
where \( f_t, g_t \) and \( b_t \) are given by (11), (13) and (14), respectively. Let us first show that an upper bound in Kolmogorov distance for a normal approximation of MLE \( \tilde{\alpha}_t \) is given by the rate \( \frac{1}{\sqrt{T-t}} \). Applying (6), it suffices to prove that

\[
\max_{i=1,2,3} \psi_i(t) \leq \frac{C}{\sqrt{\log (T-t)}}.
\]

Using (16), (17) and (19), we obtain

\[
\frac{1}{\sqrt{2(\alpha - 1)}} \sqrt{T-t} \leq \frac{1}{\sqrt{\log (T-t)}}.
\]

On the other hand, by combining (25) and (26), we get

\[
\frac{1}{\sqrt{\log (T-t)}} \leq \frac{C}{\sqrt{T-t}} \leq \frac{C}{\sqrt{\log (T-t)}}.
\]

Further, the estimates (23) and (24) imply that

\[
\frac{1}{\sqrt{2(\alpha - 1)}} \sqrt{T-t} \leq \frac{1}{\sqrt{\log (T-t)}}.
\]

Therefore, (29) is obtained.

For the lower bound, combining (16), (18) and (19) together with the fact that \( 2\alpha - 1 > 0 \) and \( \lambda_t \to \infty \) as \( t \to T \), we obtain, as \( t \to T \),

\[
\|f_t \otimes_1 f_t\|_{\mathcal{H}^{\otimes 2}} = \frac{\sqrt{5}}{2(\alpha - 1)} + o \left( \frac{1}{\sqrt{\lambda_t}} \right) \to 0,
\]

\[
2\|f_t\|_{\mathcal{H}^{\otimes 2}}^2 - 1 \leq \frac{\log(T)}{2(\alpha - 1)} + o \left( \frac{1}{\sqrt{\lambda_t}} \right) \to 0,
\]

\[
\frac{\|f_t \otimes_1 f_t\|_{\mathcal{H}^{\otimes 2}}}{\langle f_t \otimes_1 f_t, f_t \rangle_{\mathcal{H}^{\otimes 2}}} = \frac{\sqrt{5}}{2} + o(1) \to \frac{2\sqrt{5}}{3} \neq 0.
\]

Moreover, using (18), there is \( c > 0 \), depending only on \( \alpha \) and \( T \), such that for all \( t \) sufficiently near \( T \),

\[
|\langle f_t \otimes_1 f_t, f_t \rangle_{\mathcal{H}^{\otimes 2}}| = \frac{1}{\sqrt{\lambda_t}} \left| 4(\alpha - 1) + o(1) \right| \geq \frac{c}{\sqrt{\log (T-t)}}.
\]

Therefore, applying (7), the desired result is obtained.

\[
\square
\]

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