Generalized Abel-Jacobi map on Lawson homology

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GENERALIZED ABEL-JACOBI MAP ON LAWSON HOMOLOGY

By WENCHUAN HU

Abstract. We construct an Abel-Jacobi type map on the homologically trivial part of Lawson homology groups. It generalizes the Abel-Jacobi map constructed by Griffiths. By using a result of H. Clemens, we answer affirmatively the question whether there exists a smooth projective complex variety with infinitely generated Lawson homology groups $L_p H_{2p+k}(X, \mathbb{Q})$ when $k > 0$. As a corollary, we find, for any nonnegative integer $j$, a smooth complex projective variety $X$ carrying infinitely generated semi-topological $K$-groups $K^{str}_j(X)_{\mathbb{Q}}$.

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1. Introduction. In this paper, all varieties are defined over $\mathbb{C}$. Let $X$ be a smooth projective variety with dimension $n$. Recall that the Hodge filtration

$$\cdots \subseteq F^0 H^k(X, \mathbb{C}) \subseteq F^{i-1} H^k(X, \mathbb{C}) \subseteq \cdots \subseteq F^{-i} H^k(X, \mathbb{C}) = H^k(X, \mathbb{C})$$

is defined by

$$F^q H^k(X, \mathbb{C}) := \bigoplus_{i \geq q} H^{i, k-i}(X).$$

Note that $F^q H^k(X, \mathbb{C})$ vanishes if $q > k$.

In [17], Griffiths generalized the Jacobian varieties and the Abel-Jacobi map on smooth algebraic curves to higher dimensional smooth projective varieties.
Definition 1.1. The $q$th intermediate Griffiths Jacobian of a smooth projective variety $X$ is defined by

$$J^q(X) := H^{2q-1}(X, \mathbb{C})/\{F^qH^{2q-1}(X, \mathbb{C}) + H^{2q-1}(X, \mathbb{Z})\} \cong F^{n-q+1}H^{2n-2q+1}(X, \mathbb{C})^*/H^{2q-1}(X, \mathbb{Z})^*.$$  

Let $Z_p(X)$ be the space of algebraic $p$-cycles on $X$. Set $Z^{n-p}(X) \equiv Z_p(X)$. There is a natural map $\text{cl}_q: Z^q(X) \to H^{2q}(X, \mathbb{Z})$ called the cycle class map. Set $Z^{n-q}(X)_{\text{hom}} := Z^q(X)_{\text{hom}} := \ker \text{cl}_q$

Definition 1.2. The Abel-Jacobi map

$$\Phi^q: Z^q(X)_{\text{hom}} \to J^q(X)$$

sends $\varphi \in Z^q(X)_{\text{hom}}$ to $\Phi^q_\varphi$, where $\Phi^q_\varphi$ is defined by

$$\Phi^q_\varphi(\omega) := \int_U \omega, \quad \omega \in F^{n-q+1}H^{2n-2q+1}(X, \mathbb{C}).$$

Here $\varphi = \partial U$ and $U$ is an integral current of dimension $2n - 2q + 1$.

Now let

$$J^{2q-1}(X)_{\text{alg}} \subseteq J^{2q-1}(X)$$

be the largest complex subtorus of $J^{2q-1}(X)$ whose tangent space is contained in $H^{q-1}(X)$. It can be proved that $\Phi^q(Z^q(X)_{\text{alg}})$ is a subtorus of $J^{2q-1}(X)$ contained in $J^{2q-1}(X)_{\text{alg}}$ (cf. [28], Corollary 12.19), where $Z^q(X)_{\text{alg}} \subseteq Z^q(X)$ are the subset of codimension $q$-cycles which are algebraically equivalent to zero.

The Griffiths group of codimension $q$-cycles is defined as

$$\text{Griff}^q(X) := Z^q(X)_{\text{hom}} / Z^q(X)_{\text{alg}}.$$  

Therefore we can define the transcendental part of the Abel-Jacobi map

(1.1)  

$$\Phi^q_\text{tr}: \text{Griff}^q(X) \to J^q(X)_{\text{tr}} := J^{2q-1}(X)/J^{2q-1}(X)_{\text{alg}}$$

as the factorization of $\Phi^q$.  

By using this, Griffiths showed the following:

**Theorem 1.3.** (Griffiths [17]) Let $X \subset P^4$ be a general quintic threefold, the Griffiths group $\text{Griff}^2(X)$ is nontrivial, even modulo torsion.

**Remark 1.4.** Clemens has obtained further results: Under the same assumption as those in Theorem 1.3, $\text{Griff}^2(X) \otimes \mathbb{Q}$ is an infinitely generated $\mathbb{Q}$-vector space [2].

In this paper, the Griffiths’ Abel-Jacobi map is generalized to the spaces of the homologically trivial part of Lawson homology groups.

**Definition 1.5.** The Lawson homology $L_pH_k(X)$ of $p$-cycles is defined by

$$L_pH_k(X) := \pi_{k-2p}(Z_p(X)) \quad \text{for} \quad k \geq 2p \geq 0,$$

where $Z_p(X)$ is provided with a natural topology (cf. [5], [23]). For general background, the reader is referred to [24].

In [12], Friedlander and Mazur showed that there are natural maps, called cycle class maps

$$\Phi_{p,k} : L_pH_k(X) \to H_k(X).$$

Define

$$L_pH_k(X)_{\text{hom}} := \ker\{\Phi_{p,k} : L_pH_k(X) \to H_k(X)\}$$

and

$$L_pH_k(X, \mathbb{Q}) := L_pH_k(X) \otimes \mathbb{Q}.$$

The domain of Abel-Jacobi map can be reduced to Griffiths groups as in (1.1). Similarly, our generalized Abel-Jacobi map is defined on homologically trivial part of Lawson homology groups. As an application, we show the nontriviality group.

The main result in this paper is the following:

**Theorem 1.6.** Let $X$ be a smooth projective variety. There is a well-defined map

$$\Phi : L_pH_{2p+k}(X)_{\text{hom}} \longrightarrow \left\{ \bigoplus_{r > k+1, r+s=k+1} H^{p+r,p+s}(X) \right\}^* / H_{2p+k+1}(X, \mathbb{Z})$$

which generalizes Griffiths’ Abel-Jacobi map defined in [G]. Moreover, for any $p > 0$ and $k \geq 0$, we find examples of projective manifolds $X$ for which the image of the map on $L_pH_{k+2p}(X)_{\text{hom}}$ is infinitely generated.
As the application of the main result together Clemens’ Theorem (Remark 1.4), we obtain:

**THEOREM 1.7.** For any \( k \geq 0 \), there exist a projective manifold \( X \) of dimension \( k + 3 \) such that \( L_1H_{k+2}(X)_{\text{hom}} \otimes \mathbb{Q} \) is nontrivial, in fact, infinite dimensional over \( \mathbb{Q} \).

Using the Projective Bundle Theorem proved by Friedlander and Gabber in [8], we have the following result:

**COROLLARY 1.8.** For any \( p > 0 \) and \( k \geq 0 \), there exist a smooth projective variety \( X \) such that \( L_pH_{k+2p}(X)_{\text{hom}} \otimes \mathbb{Q} \) is an infinite dimensional vector space over \( \mathbb{Q} \).

As an application to Friedlander and Walker (singular) Semi-topological \( K \)-theory (cf. [13]), we find, for each \( j \geq 0 \), a smooth projective variety \( X \) such that the semi-topological \( K \)-group \( K^\text{sst}_j(X) \) is infinitely generated.

The semi-topological \( K \)-group \( K^\text{sst}_j(X) \) is defined by

\[
K^\text{sst}_j(X) = \pi_j \left( (\mathcal{M}(\nabla(X, \text{Grass})^{an})^+ \right),
\]

where \( \text{Grass} := \bigoplus_{i,j} \text{Grass}_i(C^N) \) and \( (\mathcal{M}(\nabla(X, \text{Grass})^{an})^+ \) is the topological group given by the homotopy-theoretic completion of algebraic maps from \( X \) to \( \text{Grass} \) with the analytic topology. For general background, the reader is referred to Friedlander and Walker’ survey paper [16].

The following result is implied by Theorem 1.7 together with a result of Friedlander and Walker ([15], Theorem 4.7):

**COROLLARY 1.9.** For any \( j \geq 0 \), there is a smooth projective variety \( X \) such that \( K^\text{sst}_j(X)_{\mathbb{Q}} \) is an infinite dimensional vector space over \( \mathbb{Q} \), where \( K^\text{sst}_j(X)_{\mathbb{Q}} := K^\text{sst}_j(X) \otimes \mathbb{Q} \).

In Section 2, we will review the minimal background materials about Lawson homology and point out its relation to Griffiths groups. In Section 3, we give the definition the generalized Abel-Jacobi map. In Section 4, the nontriviality of the generalized Abel-Jacobi map is proved by using Griffiths and Clemens’ results through examples. The construction in our examples also shows this generalized Abel-Jacobi map really generalizes Griffiths’ result in [17]. In Section 5, we apply our construction to semi-topological \( K \)-theory and show that all rational semi-topological \( K \)-groups can be infinite dimensional vector space over \( \mathbb{Q} \).

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2. Lawson homology. Let $X$ be a projective variety of dimension $n$. Denote by $C_p(X)$ the space of effective algebraic $p$-cycles on $X$ and by $Z_p(X)$ the space of algebraic $p$-cycles on $X$. There is a natural, compactly generated topology on $C_p(X)$ (resp. $Z_p(X)$) and therefore $C_p(X)$ (resp. $Z_p(X)$) carries a structure of an abelian topological group ([5], [23]).

The Lawson homology $L_pH_k(X)$ of $p$-cycles is defined by

$$L_pH_k(X) := \pi_{k-2p}(Z_p(X)) \quad \text{for} \quad k \geq 2p \geq 0.$$ 

It has been proved by Friedlander in [5] that

$$L_pH_k(X) \cong \lim_{\to} \pi_{k-2p}(C_p(X)_\alpha)$$

for all $k > 0$, where the limit is taken over the connected components of $C_p(X)$ with respect to the action of $\pi_0(Z_p(X))$. For a detailed discussion of this construction and its properties we refer the reader to [12], §2 and [10], §1.

In [12], Friedlander and Mazur showed that there are natural maps, called cycle class maps

$$\Phi_{p,k}: L_pH_k(X) \rightarrow H_k(X)$$

where $H_k(X)$ is the singular homology with the integral coefficient.

Define $L_pH_k(X)_{\text{hom}}$ to be the homologically trivial part of Lawson homology group $L_pH_k(X)_{\text{hom}}$, i.e.,

$$L_pH_k(X)_{\text{hom}} := \ker\{\Phi_{p,k}: L_pH_k(X) \rightarrow H_k(X)\}.$$ 

It was proved by Friedlander [5] that $L_pH_{2p}(X) \cong Z_p(X)/Z_p(X)_{\text{alg}}$. Therefore we have

$$L_pH_{2p}(X)_{\text{hom}} \cong \text{Griff}^p_p(X),$$

where $\text{Griff}^p_p(X) := \text{Griff}^{n-p}(X)$.

For general background on Lawson homology, the reader is referred to [24].

3. The definition of generalized Abel-Jacobi map on $L_pH_{2p+k}(X)_{\text{hom}}$. In this section, $X$ denotes a smooth projective algebraic manifold with dimension $n$. Now $Z_p(X)$ is an abelian topological group with an identity element, the “null” $p$-cycle.

For $[\varphi] \in L_pH_{2p+k}(X)$, we can construct an integral $(2p+k)$-cycle $c$ in $X$ with $\Phi_{p,2p+k}([\varphi]) = [c]$, where $[c]$ is the homology class of $c$.

To see how to construct $c$ from $[\varphi]$ for the case that $p = 0$, the reader is referred to [11].
We will use this construction several times in the following. We briefly review the construction here.

A class

$$[\varphi] \in L_p H_{2p+k}(X) = \lim\pi_k(C_p(X)_{\alpha})$$

is represented by a map

$$\varphi: S^k \to C_p(X).$$

(For \(k = 0, [\varphi]\) is represented by a difference of such maps.)

We may assume \(\varphi\) to be piecewise linear (PL for short) with respect to a triangulation of \(C_p(X) \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\) respecting the smooth stratified structure ([20]). Here \(\Gamma_i\) is a subcomplex for every \(i > 0\).

Let \(\varphi\) be as above and fix \(s_0 \in S^k\) and \(x_0 \in \text{Supp}(\varphi(s_0)) \subset X\). There exist affine coordinates \((z_1, \ldots, z_p, \zeta_1, \ldots, \zeta_{a-p})\) on \(X\) with \(x_0 = 0\) such that the projection \(pr_1(z, \zeta) = z\), when restricted to \(U \times U' = \{(z, w): |z| < 1 \text{ and } |w| < 1\}\), gives a proper (hence finite) map \(pr: \text{Supp}(\varphi(s_0)) \cap (U \times U') \to U\). Slicing this cycle \(\varphi(s_0)|_{U \times U'}\) by this projection gives a PL map \(\sigma: U \to SP^d(U')\) (with respect to a triangulation of \(SP^d(U')\)) for some \(d\). Furthermore, given any such a map, we can construct a cycle in \(U \times U'\). (cf. [11].) Choose a finite number of such product neighborhood \(U_{\alpha} \times U'_{\alpha}\), \(\alpha = 1, \ldots, K\), so that the union of \(U_{\alpha} \times U'_{\alpha}(\frac{1}{2})\) covers \(\text{Supp}(\varphi(s_0))\). After shrinking each \(U'_{\alpha}\) slightly, we can find a neighborhood \(\mathcal{N}\) of \(s_0\) in \(S^k\) such that \(pr: \text{Supp}(\varphi(s)) \cap (U_{\alpha} \times U'_{\alpha}) \to U_{\alpha}\) for all \(s \in \mathcal{N}\) and for all \(\alpha\). Then \(\varphi\) is PL in \(\mathcal{N}\) if and only if \(\sigma: \mathcal{N} \times U_{\alpha} \to SP^d(U'_{\alpha})\) is PL for all \(\alpha\). One defines the cycle \(c(\varphi)\) in each neighborhood \(\mathcal{N} \times U_{\alpha} \times U'_{\alpha}\) by graphing this extended \(\sigma\). From the construction, the cycle \(c(\varphi)\) depends only on the PL map \(\varphi\). (The argument here is from [[10], page 370-371].)

**Lemma 3.1.** The homology class \(c_{\varphi} := (pr_2)_*(c(\varphi))\) is independent of the choice of PL map \(\varphi: S^k \to C_p(X)\) in \([\varphi]\), where \(pr_2: S^k \times X \to X\) is the projection onto the second factor.

**Proof.** Suppose that \(\varphi': S^k \to C_p(X)\) is another PL map in \([\varphi]\). Hence, we have a continuous map \(H: S^k \times [0, 1] \to C_p(X)\) such that \(H|_{S^k \times \{0\}} = \varphi\) and \(H|_{S^k \times \{1\}} = \varphi'\). Furthermore, this map can be chosen to be PL with respect to the triangulation of \(C_p(X)\). Therefore, by the same construction as above, we obtain that an integral current \(c_H := (pr_2)_*(c(H))\). It is clear \(\partial(c_H) = c_\varphi - c_{\varphi'}\) since the push-forward \((pr_2)_*\) commutes with the boundary map \(\partial\). \(\square\)

Alternatively, the restriction of \(\varphi\) to the interior of each top dimensional simplicies \(\Delta^k_j\) \((1 \leq j \leq N, N\) is the number of top dimensional simplices\) gives a map \(\varphi_j: \Delta^k_j \to \Gamma_{n_j}\), where \(\Delta^k_j\) is the \(j\)th \(k\)-dimensional simplex and \(n_j\) is the maximum number such that \(\Gamma_{n_j}\) contains the image of \(\varphi_j\).
The piecewise linear property of $\varphi$ with respect to the stratified structure of $C_p(X)$ have the following property:

(*) For each $s \in \Delta^k_i$, $\varphi(s) = \sum a_i V_i(s)$ with the property that $a_i(s) = a_i$ is constant in $s$ and $V_i(s)$ is irreducible.

For each $j$, $1 \leq j \leq N$, set $Z_{\Delta^k_j} := \sum a_i Z_{\Delta^k_i, j}$, where $Z^k_{\Delta^k_i, j} := \{(s, z) \in \Delta^k_i \times X | z \in V_i(s)\}$. It is clear that $Z_{\Delta^k_j}$ is an integral current. Therefore,

\begin{equation}
\phi^k_j := (pr_2)_*(Z_{\Delta^k_j})
\end{equation}

is then an integral current of real dimension $2p + k$, where $pr_2: \Delta^k_i \times X \rightarrow X$ is the projection onto the second factor. Set $Z(\varphi) := \sum^N_{j=1} \phi^k_j$.

**Lemma 3.2.** The closure of $Z(\varphi)$ is an integral cycle in $X$.

**Proof.** Since $\varphi$ is piecewise linear with respect to the triangulation of $C_p(X)$. The image of $\varphi$ on each $(k - 1)$-dimensional simplex $\Delta^k_{i-1}$ is in $\Gamma_{m_i}$, where $m_i$ is the maximum number such that $\Gamma_{m_i}$ contains the image of $\varphi|_{\Delta^k_{i-1}}$. Each $\varphi|_{\Delta^k_{i-1}}$ defines a current $\phi^k_{i-1} := (pr_2)_*(Z_{\Delta^k_{i-1}})$ as in (3.1). The sum

$$\sum_i \phi^k_{i-1}$$

is zero since, for each $\phi^k_{i-1}$, there is exactly one $\phi^k_{j-1}$ such that they have the same support but different orientation. \(\square\)

Let $c_\varphi$ be the total $(k + 2p)$-cycle in $X$ determined by $[\varphi]$. We will simply use $c$ instead of $c_\varphi$ unless it arises confusion.

**Remark 3.3.** $c_\varphi$, as current, has restricted type

$$c_\varphi = [c_\varphi]_{p+k,p} + [c_\varphi]_{p+k-1,p+1} + \cdots + [c_\varphi]_{p,p+k}.$$  

If $c$ is homologous to zero, we denote it by $c \sim_h 0$, i.e., $[\varphi] \rightarrow 0$ in $H_{2p+k}(X, \mathbb{Z})$ under the natural transformation $\Phi_{p,2p+k}: L_p H_{2p+k}(X) \rightarrow H_{2p+k}(X, \mathbb{Z})$ (see, e.g., [24], p.185). This condition translates into the fact that there exists an integral topological $(2p + k + 1)$-chain $\tilde{c}$ such that $\partial \tilde{c} = c$.

We denote by $\text{Map}(S^k, C_p(X))$ the set of piecewise linear maps with respect to a triangulation of $C_p(X)$ from the $k$-dimensional sphere to the abelian topological monoid $C_p(X)$ of $p$-cycles.

Set

$$\text{Map}(S^k, C_p(X))_{\text{hom}} \subset \text{Map}(S^k, C_p(X))$$
the subset of such maps $\varphi: S^k \to C_p(X)$ whose total cycles $c_\varphi$ is homologous to zero in $H_{2p+k}(X, \mathbb{Z})$. There is a natural induced compact open topology on the space of such maps $\text{Map}(S^k, C_p(X))$ (see, e.g., Whitehead [29]).

Now $Z_p(X)$ is the group completion of the topological monoid $C_p(X)$ (cf. [5], [23]). In the following, we will denote by $\text{Map}(S^k, Z_p(X))$ the set of piecewise linear maps with respect to a triangulation of $Z_p(X)$ from the $k$-dimensional sphere to the abelian topological group $Z_p(X)$ of $p$-cycles.

Let $\varphi: S^k \to Z_p(X)$ be a PL map which is homotopic to zero. Hence there exists a map $\tilde{\varphi}: D^{k+1} \to Z_p(X)$ such that $\tilde{\varphi}$ is PL with respect to a triangulation of $Z_p(X)$ and $\tilde{\varphi}|_{S^k} = \varphi$. Then $\tilde{\varphi}$ determines an integral current, i.e., the total $(k+1+2p)$-chain $\tilde{c}$ such that the boundary of $\tilde{c}$ is $c$, i.e., $\partial \tilde{c} = c$. From the definition, we have $\varphi \in \text{Map}(S^k, Z_p(X))_{\text{hom}}$. Denote by $\text{Map}(S^k, Z_p(X))_0$ the subspace of $\text{Map}(S^k, Z_p(X))_{\text{hom}}$ consisting of elements $\varphi$ which are homotopic to zero.

3.1. The generalized Abel-Jacobi map on $\text{Map}(S^k, Z_p(X))_{\text{hom}}$. In this subsection, suppose that

$$c \overset{\text{hom}}{\sim} 0, \text{ i.e., } [\varphi] \to 0 \in H_{2p+k}(X, \mathbb{Z})$$

under the natural transformation $\Phi_{p,2p+k}: L_pH_{2p+k}(X) \to H_{2p+k}(X, \mathbb{Z})$ (see, e.g., [24], p.185). This condition translates into the fact that there exists an integral topological $(2p+k+1)$-chain $\tilde{c}$ such that $\partial \tilde{c} = c$.

Consider

$$\omega \in \bigoplus_{r \geq k+1, r+s=k+1} \mathcal{E}^{p+r,p+s}, \quad d\omega = 0$$

and we define

$$\Phi_\varphi(\omega) = \int_{\tilde{c}} \omega.$$

We claim:

**Proposition 3.4.** $\Phi_\varphi$ is well-defined, i.e., $\Phi_\varphi(\omega)$, as an element in

$$\left\{ \bigoplus_{r,s \geq 0, r+s=k+1} H^{p+r,p+s}(X) \right\}^*/H_{2p+k+1}(X, \mathbb{Z}),$$

depends only on the cohomology class of $\omega$. Here we identify $H_{2p+k+1}(X, \mathbb{Z})$ with
the image of the composition

\[ H_{2p+k+1}(X, \mathbb{Z}) \xrightarrow{\rho} H_{2p+k+1}(X, \mathbb{C}) \cong H_{2p+k+1}(X, \mathbb{C})^* \xrightarrow{\pi} \bigoplus_{r,s \geq 0} H^{p+r,p+s}(X) \]

where \( \rho \) is the coefficient homomorphism and \( \pi \) is the projection onto the subspace.

**Proof.** We need to show:

(a) For another choice of \( \omega' \in \bigoplus_{r \geq k+1, r+s=k+1} \mathcal{E}^{p+r,p+s} \), \( \omega - \omega' = d\alpha \), we have

\[ \int_c \omega = \int_c \omega' \]

(b) If \( \tilde{c}' \) is another integral topological chain such that \( \partial \tilde{c}' = c \), then we also have \( \int_c \omega = \int_{\tilde{c}'} \omega \), where \( \tilde{c}' \) is the current determined by \( \tilde{\varphi} \).

To show the part (a), note that we can choose \( \alpha \) such that \( \omega - \omega' = d\alpha \) for some \( \alpha \) with \( r,s = 0 \) if \( r \leq k+p \) by the Hodge decomposition theorem for differential forms on \( X \). Hence

\[ \int_c \omega - \int_c \omega' = \int_c d\alpha = \int_c \alpha = 0 \]

by the Stokes Theorem and the reason of type. To see this, note that by Remark 3.3 the current \( c \) constructed above has restricted type \( c = c_{p+k,p} + c_{p+k-1,p+1} + \cdots + c_{p,p+k} \) and the differential form \( \alpha \) satisfies \( \alpha^{r,s} = 0 \) for \( r \leq k+p \). Hence \( \int_c \alpha = 0 \) follows.

This shows that the definition of \( \Phi_{\tilde{\varphi}} \) is independent of the cohomology class of \n
\[ [\omega] \in \bigoplus_{r \geq k+1, r+s=k+1} H^{p+r,p+s}(X) \]

To show the part (b), note that \( \partial(\tilde{c} - \tilde{c}') = 0 \) and hence \( \tilde{c} - \tilde{c}' = \lambda \) is an integral topological cycle and hence \( \int_\lambda \) lies in the image of the composition in (3.2). Hence \( \int_\lambda \) is well-defined independently of the choice of \( \tilde{c} \) such that \( \partial \tilde{c} = c \), as an element in

\[ \bigoplus_{r,s \geq 0, r+s=k+1} H^{p+r,p+s}(X) \]

Hence we obtain a well-defined element

\[ \Phi_{\tilde{\varphi}} \in \left\{ \bigoplus_{r \geq k+1, r+s=k+1} H^{p+r,p+s}(X) \right\}^* / H_{2p+k+1}(X, \mathbb{Z}) \]

This completes the proof of the Proposition. \( \square \)
Therefore by Proposition 3.4 we have a well-defined homomorphism

\[
\Phi: \text{Map}(S^k, \mathbb{Z}_p(X))_{\text{hom}} \rightarrow \left\{ \bigoplus_{r \geq k+1, r+s=k+1} H^{p+r, p+s}(X) \right\}^* / H_{2p+k+1}(X, \mathbb{Z})
\]

given by \( \Phi(\varphi) = \Phi_{\varphi} \).

3.2. The restriction of \( \Phi \) on \( \text{Map}(S^k, \mathbb{Z}_p(X))_0 \). In this subsection, we will study the restriction of \( \Phi \) in (3.3) to the subspace

\[
\text{Map}(S^k, \mathbb{Z}_p(X))_0 \subset \text{Map}(S^k, \mathbb{Z}_p(X))_{\text{hom}},
\]
i.e., all PL maps from \( S^k \) to \( \mathbb{Z}_p(X) \) which are homotopic to zero. Note that the image of \( \Phi \) is in

\[
\left\{ \bigoplus_{r \geq k+1, r+s=k+1} H^{p+r, p+s}(X) \right\}^* / H_{2p+k+1}(X, \mathbb{Z}).
\]

Let \( \varphi: S^k \rightarrow \mathbb{Z}_p(X) \) be an element in \( \text{Map}(S^k, \mathbb{Z}_p(X))_0 \). Denote by \( c \) the total \((2p + k)\)-cycle (maybe degenerated) determined by \( \varphi \). Hence there exists a map \( \tilde{\varphi}: D^{k+1} \rightarrow \mathbb{Z}_p(X) \) such that \( \tilde{\varphi}|_{S^k} = \varphi \) and the associated total \((2p + k + 1)\)-chain \( \tilde{c} \) such that the boundary of \( \tilde{c} \) is \( c \), i.e., \( \partial \tilde{c} = c \). The restriction of the generalized Abel-Jacobi map \( \Phi \) to the subspace of \( \text{Map}(S^k, \mathbb{Z}_p(X))_0 \) is the map

\[
\Phi_0: \text{Map}(S^k, \mathbb{Z}_p(X))_0 \rightarrow \left\{ \bigoplus_{r \geq k+1, r+s=k+1} H^{p+r, p+s}(X) \right\}^* / H_{2p+k+1}(X, \mathbb{Z}).
\]

Note that by Remark 3.3 we have

\[
c \in \left\{ \bigoplus_{r,s \geq 0, r+s=k} \mathcal{E}_{p+r, p+s} \right\}.
\]

and similarly we get

\[
\tilde{c} \in \left\{ \bigoplus_{r,s \geq 0, r+s=k+1} \mathcal{E}_{p+r, p+s} \right\}
\]

by applying the construction in Lemma 3.1 and 3.2 to the piecewise linear map \( \tilde{\varphi}: D^{k+1} \rightarrow \mathbb{Z}_p(X) \).
Hence

$$\Phi_{\varphi}(\omega) = \int_{\tilde{c}} \omega = 0$$

for

$$\omega \in \left\{ \bigoplus_{r > k+1, r+s = k+1} E^{p+r,p+s} \right\}$$

with $d\omega = 0$

by the reason of type. Therefore $\Phi_{\varphi} = 0$ on

$$\bigoplus_{r > k+1, r+s = k+1} H^{p+r,p+s}(X).$$

That is to say, the image of $\Phi$ on the subspace $\text{Map}(S^k, Z_p(X))_0$ is in

$$H^{p+k+1}(X)_* / \left\{ H^{p+k+1}(X)_* \cap \rho(H_{2p+k+1}(X, \mathbb{Z})) \right\}.$$

### 3.3. The reduction of $\Phi$ to $L_pH_{2p+k}(X)_{\text{hom}}$

We reduce the domain $\Phi$ to the quotient

$$\text{Map}(S^k, Z_p(X))_{\text{hom}} / \text{Map}(S^k, Z_p(X))_0 \cong \pi_0(\text{Map}(S^k, Z_p(X))_{\text{hom}}).$$

Now, if there are two PL maps $\varphi: S^k \to Z_p(X)$ and $\varphi': S^k \to Z_p(X)$ such that $\varphi$ is homotopic to $\varphi'$. Denote by $c$ (resp. $c'$) the total $(k + 2p)$-cycle determined by $\varphi$ (resp. $\varphi'$). For

$$\omega \in \left\{ \bigoplus_{r > k+1, r+s = k+1} E^{p+r,p+s} \right\}, \quad d\omega = 0,$$

since $c - c' \text{ hom} 0$, we have $\Phi_{\varphi - \varphi'}(\omega) = \Phi_{\varphi'}(\omega) - \Phi_{\varphi'}(\omega) = 0$ and

$$\Phi_{\varphi} = \Phi_{\varphi'} \in \left\{ \bigoplus_{r > k+1, r+s = k+1} H^{p+r,p+s}(X) \right\}^* / H_{2p+k+1}(X, \mathbb{Z})$$

by the discussion in Section 3.2.
Therefore, we have a commutative diagram

\[
\begin{array}{ccc}
\text{Map}(S^k, \mathbb{Z}_p(X))_0 & \xrightarrow{i} & \text{Map}(S^k, \mathbb{Z}_p(X))_{\text{hom}} \\
\downarrow \Phi_0 & & \downarrow \Phi \\
H^{p+k+1, p}(X)^* / H_{2p+k+1}(X, \mathbb{Z}) & \xrightarrow{i} & \left\{ \bigoplus_{r \geq k+1} H^{p+r, p+s}(X) \right\}^* / H_{2p+k+1}(X, \mathbb{Z})
\end{array}
\]

From this, we reduce \( \Phi \) to a map

\[
(3.4) \quad \Phi_{tr}: \pi_0(\text{Map}(S^k, \mathbb{Z}_p(X))_{\text{hom}}) \rightarrow \left\{ \bigoplus_{r \geq k+1} H^{p+r, p+s}(X) \right\}^* / H_{2p+k+1}(X, \mathbb{Z})
\]

given by \( \Phi_{tr}(\varphi) = \Phi_{\varphi} \). Here \( / H_{2p+k+1}(X, \mathbb{Z}) \) means modulo the image of the composed map

\[
H_{2p+k+1}(X, \mathbb{Z}) \xrightarrow{\partial} H_{2p+k+1}(X, \mathbb{C}) = \left\{ \bigoplus_{r+s=2p+k+1} H^{r,s}(X) \right\}^* \rightarrow \left\{ \bigoplus_{r \geq k+1} H^{p+r, p+s}(X) \right\}^*.
\]

We complete the construction of the generalized Abel-Jacobi map on homologically trivial part in Lawson homology

\[
L_p H_{2p+k}(X)_{\text{hom}} := \pi_0(\text{Map}(S^k, \mathbb{Z}_p(X))_{\text{hom}})
\]
i.e., the kernel of the natural transformation \( \Phi_{p,2p+k}: L_p H_{2p+k}(X) \rightarrow H_{2p+k}(X, \mathbb{Z}) \).

**Remark 3.5.** This map (3.4) defined above is exactly the usual Abel-Jacobi map on Griffiths group when \( k = 0 \) since there is a natural isomorphism \( L_p H_{2p}(X)_{\text{hom}} \cong \text{Griff}_p(X) \) (cf. [5]). This map \( \Phi \) on \( L_0 H_k(X)_{\text{hom}} \) is trivial since \( L_0 H_k(X)_{\text{hom}} = 0 \) by Dold-Thom theorem (cf. [4]).

**Remark 3.6.** Our generalized Abel-Jacobi map has been generalized to Lawson homology groups by the author [22]. The range of the more generalized Abel-Jacobi map will be certain Deligne (co)homology. The tools used there are “sparks” and “differential characters” systematically studied by Harvey, Lawson and Zweck [19] and [18].

**Remark 3.7.** Sometimes we also use \( AJ_X(c) \) to denote \( \Phi_{tr}(\varphi) \), where \( c \) is the cycle determined by \( \varphi \).

**Remark 3.8.** Prof. M. Walker told me that the Suslin Conjecture would imply the existence of such a generalized Abel-Jacobi map, at least for smooth projective 4-folds with \( p = 1 \) and \( k = 1 \) in the equation (3.4). In his recent paper, Walker has defined a morphic Abel-Jacobi map from algebraically trivial part of \( p \)-cycles to \( p \)th morphic Jacobian [27].
4. The nontriviality of the generalized Abel-Jacobi map. The natural question is the existence of smooth projective varieties such that the generalized Abel-Jacobi map $\Phi_{tr}$ on $L_pH_{2p+k}(X)_{hom}$ is nontrivial for both $p > 0$ and $k > 0$. The following example is a family of smooth 4-dimensional projective varieties $X$ with $L_1H_3(X)_{hom} \neq 0$, even modulo torsion.

**Example.** Let $E$ be a smooth elliptic curve and $Y$ be a smooth projective algebraic variety of dim $Y = 3$ such that the Griffiths group of 1-cycles of $Y$ tensored with $\mathbb{Q}$ is nontrivial. Set $X = E \times Y$. Let $[\omega] \in H^{1,0}(X)$ be a nonzero element. By the Künneth formula, we have $[\omega] = [\alpha] \wedge [\beta]$ for some $0 \neq [\alpha] \in H^{1,0}(E)$ and $0 \neq [\beta] \in H^{3,0}(Y)$.

Let $\iota: S^1 \to E$ be a homeomorphism onto its image such that $\iota(S^1) \subset E$ is not homologous to zero in $H_1(E, \mathbb{Z})$. Let $\varphi: S^1 \to Z_1(X)$ be a continuous map given by

$$\varphi(t) = (\iota(t), W) \in Z_1(X),$$

where $W \in Z_1(Y)$ a fixed element such that $W$ is homologous to zero but $W$ is not algebraic equivalent to zero, i.e., $W \notin \text{Griff}_1(Y)$. The existence of $W$ is the assumption. Then there exists an integral topological chain $U$ such that $\partial U = W$. Using the notation above, the cycle $c$ determined by $\varphi$ is $\iota(S^1) \times W$. Now $c = \iota(S^1) \times W$ is homologous to zero in $X$. Indeed,

$$\partial(\iota(S^1) \times U) = \partial(\iota(S^1)) \times U + \iota(S^1) \times \partial U = \iota(S^1) \times W = c.$$ 

Hence $\partial c = \iota(S^1) \times U + \gamma + \partial(\text{something})$, where $\partial \gamma = 0$. Therefore we have

$$\int c \omega = \int_{\iota(S^1) \times U} \omega = \left( \int_{\iota(S^1)} \alpha \right) \cdot \left( \int_U \beta \right).$$

**Proposition 4.1.** Suppose $Y$ is a smooth threefold and $W \in Z_1(Y)$ such that the image $AJ_Y(W)$ of $W$ under the Griffiths’ Abel-Jacobi map $AJ_Y$ is nontorsion in $H^{3,0}(Y)^*/\text{Im}H_3(Y, \mathbb{Z})$. The map $\varphi$ is given by (4.1) as above. Then the map $\Phi_{tr}(\varphi) \in H^{4,0}(X)/\text{Im}H_4(X, \mathbb{Z})$ is nontrivial, even modulo torsion.

**Proof.** By the Künneth formula, we have $H^{4,0}(E \times Y) \cong H^{1,0}(E) \otimes H^{3,0}(Y)$ and $H_4(E \times Y, \mathbb{Z}) \cong H_4(Y, \mathbb{Z}) \oplus \{H_1(E, \mathbb{Z}) \otimes H_3(Y, \mathbb{Z})\} \oplus \{H_2(E, \mathbb{Z}) \otimes H_2(Y, \mathbb{Z})\}$ modulo torsion. Let

$$\pi: H_4(E \times Y, \mathbb{Z}) \to \{H^{4,0}(E \times Y)^\ast\}$$

be the natural map given by $\pi(u)(\alpha \otimes \beta) = \int_u \alpha \wedge \beta$ for $u \in H_4(E \times Y, \mathbb{Z})$ and $\alpha \in H^{1,0}(E)$ and $\beta \in H^{3,0}(Y)$. Now $\pi(u) \neq 0$ only if $u \in H_1(E, \mathbb{Z}) \otimes H_3(Y, \mathbb{Z})$. 

Hence we get

\[ \{H^{4,0}(E \times Y)\}^*/\text{Im}H_4(E \times Y, \mathbb{Z}) \]
\[ \cong \{H^{1,0}(E)^* \otimes H^{3,0}(Y)^*\}/\text{Im}\{H_1(E, \mathbb{Z}) \otimes H_3(Y, \mathbb{Z})\}. \]

Therefore, by the definition of generalized Abel-Jacobi map and (4.2), we have

\[ AJ_Y(\iota(S^1) \times W)(\alpha \wedge \beta) = \Phi_{tr}(\varphi)(\alpha \wedge \beta) \]
\[ = \left( \int_{\iota(S^1)} \alpha \right) \cdot \left( \frac{1}{\mu} \int_U \beta \right) \]
\[ = \left( \int_{\iota(S^1)} \alpha \right) \cdot (AJ_Y(W)(\beta)) \]

i.e., \[ AJ_Y(\iota(S^1) \times W) = \int_{\iota(S^1)} \otimes AJ_Y(W). \]

Note that the map \[ \int_{\iota(S^1)}: H^{1,0}(E) \rightarrow \mathbb{C} \] is in the image of the embedding \[ H_1(E, \mathbb{Z}) \hookrightarrow H^{1,0}(E)^*. \] But \[ AJ_Y(W) \] is a nontorsion element in the quotient \[ H^{3,0}(Y)^*/\text{Im}H_3(Y, \mathbb{Z}). \] Now the conclusion of the proposition is from the following lemma.

**Lemma 4.2.** Let \( V_m \) and \( V_n \) be two \( \mathbb{C} \)-vector spaces of dimension of \( m \) and \( n \), respectively. Suppose that \( \Lambda_m \subset V_m, \Lambda_n \subset V_n \) be two lattices, respectively. If \( b \in V_n \) is a nontorsion element in \( V_n \), i.e., \( kb \) is not zero in \( \Lambda_n \) for any \( k \in \mathbb{Z} \), then \( a \otimes b \) is not in \( \Lambda_m \otimes \Lambda_n \) for any \( 0 \neq a \in \Lambda_m \).

**Proof.** Set \( \text{rank}(\Lambda_m) = m_0, \text{rank}(\Lambda_n) = n_0 \). Let \( \{e_i\}_{i=1}^{m_0}, \{f_j\}_{j=1}^{n_0} \) be two integral basis of \( \Lambda_m, \Lambda_n \), respectively. If the conclusion in the lemma fails, then

\[ a \otimes b = \sum_{i=1}^{m_0} \sum_{j=1}^{n_0} k_{ij} e_i \otimes f_j, \]

for some \( k_{ij} \in \mathbb{Z} \). By taking the conjugation, we can suppose that \( V_m \) and \( V_n \) are real vector spaces with lattices \( \Lambda_m \) and \( \Lambda_n \), respectively.

Suppose that \[ a = \sum_{i=1}^{m_0} k_i e_i, \] where \( k_i \in \mathbb{Z}, i = 1, \ldots, m_0 \) are not all zeros. The equation (4.3) reads as

\[ \sum_{i=1}^{m_0} k_i e_i \otimes b = \sum_{i=1}^{m_0} \sum_{j=1}^{n_0} k_{ij} e_i \otimes f_j \]

i.e.,

\[ \sum_{i=1}^{m_0} e_i \otimes \left( k_i b - \sum_{j=1}^{n_0} k_{ij} f_j \right) = 0. \]
Since \( \{e_i\}_{i=1}^{m_0} \) is a basis in \( \Lambda_m \) and hence they are linearly independent over \( \mathbb{R} \) in \( V_m \), we get

\[
(4.4) \quad k_ib - \sum_{j=1}^{m_0} k_if_j = 0
\]

for any \( i = 1, 2, \ldots, m_0 \). By assumption, at least one of \( k_i \) is nonzero since \( a \) is nonzero vector in \( V_m \). The equation (4.4) contracts to the assumption that \( kb \) is not in \( \Lambda_n \) for any \( k \in \mathbb{Z}^+ \). This completes the proof of the lemma and hence the proof of the proposition.

More generally, we have the following proposition:

**Proposition 4.3.** Suppose \( Y \) is a smooth threefold such that the image of the map \( AJ_Y \) on \( \text{Griff}_1(Y) \) under the Griffiths’ Abel-Jacobi map \( AJ_Y \) tensored by \( \mathbb{Q} \) is an infinitely dimensional \( \mathbb{Q} \)-vector space over \( \mathbb{Q} \) in \( \{ H^{3,0}(Y)/\text{Im}H_3(Y, \mathbb{Z}) \} \otimes \mathbb{Q} \).

For each \( W \in \text{Griff}_1(Y) \), the map \( \varphi_W \) is given by (4.1) as above. Then the image

\[
\left\{ \Phi_{\nu}(\varphi_W) \mid W \in \text{Griff}_1(Y) \right\} \otimes \mathbb{Q} \subset \left\{ H^{4,0}(X)/\text{Im}H_4(X, \mathbb{Z}) \right\} \otimes \mathbb{Q}
\]

is an infinite dimensional \( \mathbb{Q} \)-vector space.

**Proof.** We only need to show that:

\((\ast)\) Let \( N > 0 \) be an integer and \( W_1, \ldots, W_N \in \text{Griff}_1(Y) \) be \( N \) linearly independent elements under Griffiths Abel-Jacobi map. Then \( \varphi_{W_1}, \ldots, \varphi_{W_N} \in L_1H_3(E \times Y)_{\text{hom}} \otimes \mathbb{Q} \) are linearly independent even under the generalized Abel-Jacobi map.

The claim \((\ast)\) follows easily from Proposition 4.1 above since if \( \varphi_{W_1}, \ldots, \varphi_{W_N} \) are linearly dependent implies that \( W_1, \ldots, W_N \) are linearly dependent by Proposition 4.1. This contradicts to the assumption.

Now for a suitable choice of the 3-dimensional projective \( Y \), for example, the general quintic hypersurface in \( \mathbb{P}^4 \) (cf. [17]) or the Jacobian of a general algebraic curve with genus 3 (cf. [1]) and the 1-cycle \( W \) whose image under Abel-Jacobi map is nonzero, in fact, it is infinitely generated for general quintic hypersurface in \( \mathbb{P}^4 \) (cf. [2]). Recall the definition of Abel-Jacobi map, \( AJ_Y(W) = \int_W \) module lattice \( H^3(Y, \mathbb{Z}) \), we have \( \int_W [\beta] \neq 0 \) for this choice of \( W \) and some nonzero \( [\beta] \in H^{3,0}(Y) \).

This example also gives an affirmative answer to the following question:

**Question.** Can one show that \( L_pH_2p+j(X)_{\text{hom}} \) is nontrivial or even infinitely generated for some projective variety \( X \) where \( j > 0 \)?
Remark 4.4. From the proof of the above propositions, we see that the nontriviality of Griffiths’ Abel-Jacobi map on $Y$ implies the nontriviality of the generalized Abel-Jacobi map on homologically trivial part of certain Lawson homology groups for $X$, i.e., all the Abel-Jacobi invariants can be found by generalized Abel-Jacobi map. In [2], Clemens showed the for general quintic 3-folds, the image of the Griffiths group under the Griffiths’ Abel-Jacobi map is infinitely generated, even modulo torsion.

Remark 4.5. Friedlander proved in [7] the nontriviality of $L_{\rho}H_2^p(X)_{\text{hom}}$ for certain complete intersections by using Nori’s method in [26], which is totally different the construction here. There is no claim of any kind of infinite generated property of Lawson homology in his paper.

Remark 4.6. Nori [26] has generalized Theorem 0.1 and has shown that even the Griffiths’ Abel-Jacobi map is trivial on some Griffiths group but the Griffiths group itself is nontrivial, even non torsion. By using a total different, explicit and elementary construction, the author has constructed singular rational 4-dimensional projective varieties $X$ such that $L_1H_3(X)_{\text{hom}}$ is infinitely generated [21]. But the Able-Jacobi map is not defined on singular projective variety (at least as far as the author knows).

From the proof of Proposition 4.1, we observe that, for $Y$ as above, and $M$ is a projective manifold, if there is a map $i: S^k \rightarrow M$ such that

$$\int_{i(S^k)}: H^{k,0}(M) \rightarrow \mathbb{C}$$

is nontrivial as an element in $\{H^{k,0}(M)\}^*$, then the value of the generalized Abel-Jacobi map $\Phi_{tr}$ at $\varphi: S^k \rightarrow Z_1(X)$ defined by

$$\varphi(t) = (i(t), W) \in Z_1(M \times Y)$$

is nontrivial, even modulo torsion.

Note that if the complex Hurewicz homomorphism $\rho \otimes \mathbb{C}: \pi_k(X) \otimes \mathbb{C} \rightarrow H_k(M, \mathbb{C})$ is surjective or even a little weaker condition, i.e., the composition

$$\pi_k(X) \otimes \mathbb{C} \rightarrow H_k(M, \mathbb{C}) \rightarrow \{H^{k,0}(M)\}^*$$

is surjective, we have the nontriviality of the map $\int_{i(S^k)}: H^{k,0}(M) \rightarrow \mathbb{C}$ if $H^{k,0}(M) \neq 0$. Here the map $\pi: H_k(M, \mathbb{C}) \rightarrow \{H^{k,0}(M)\}^*$ is the Poincaré duality the projection $H^k(M, \mathbb{C}) \rightarrow H^{k,0}(M)$ in Hodge decomposition.

As a direct application to the Main Theorem in [[3], §6] and also Theorem 14 in [25], we have the following result on higher dimensional hypersurface.
PROPOSITION 4.7. Let $M$ be a smooth hypersurface in $\mathbb{P}^{n+1}$ and $n > 1$. Then the composition map

$$
\pi_k(X) \otimes \mathbb{C} \to H_k(M, \mathbb{C}) \to \{H^{k,0}(M)\}^*
$$

is surjective for any simply connected Kähler manifolds.

Therefore we obtain the following result:

**Theorem 4.8.** For any $k \geq 0$, there exist a projective manifold $X$ of dimension $k + 3$ such that $L_1H_{k+2}(X)_{\text{hom}} \otimes \mathbb{Q}$ is nontrivial or even infinite dimensional over $\mathbb{Q}$.

By using the Projective Bundle Theorem in [8], we get the following result:

**Corollary 4.9.** For any $p > 0$ and $k \geq 0$, there is a smooth projective variety $X$ such that $L_pH_{k+2p}(X)_{\text{hom}} \otimes \mathbb{Q}$ is infinite dimensional vector space over $\mathbb{Q}$.

5. An application to Friedlander-Walker Semi-topological $K$-theory. In this section, we exhibit by examples that any Semi-topological $K$-group can be infinitely generated.

Friedlander and Lawson have defined morphic cohomology groups $L^pH^k(X)$ for all $k \leq 2p$. Moreover, they have defined a duality between morphic cohomology groups $L^pH^k(X)$ with Lawson homology groups $L_{n-p}H_{2n-k}(X)$ for a projective variety $X$ ([10], [11]).

**Theorem 5.1.** [11] If $X$ is smooth projective of dim $X = n$, then the duality

$$
\mathcal{D}: L^pH^k(X) \to L_{n-p}H_{2n-k}(X)
$$

is an isomorphism for all $k \leq 2p$.

The (singular) semi-topological $K$-theory (denoted by $K^\text{sst}_n(-)$) was introduced and developed by Friedlander and Walker in a sequence of papers (cf. [13], [14], [15], [16] and references therein).

Let $K^\text{sst}(X)$ be a homotopy-theoretic group completion of a space of maps of $X$ to an infinite Grassmannian, topologized as in [13]. The semi-topological $K$-group $K^\text{sst}_j(X)$ of $X$ is defined to be the $j$th homotopy group of $K^\text{sst}(X)$. The rational $K^\text{sst}$-groups is denoted by

$$
K^\text{sst}_j(X)_{\mathbb{Q}} := K^\text{sst}_j(X) \otimes \mathbb{Q}.
$$

One of the fundamental result in semi-topological $K$-theory is that there is a natural isomorphism between rational $K^\text{sst}$-groups and certain direct sum of rational morphic groups:
THEOREM 5.2. (Friedlander-Walker [13]) There is a natural isomorphism

\[ K_j^{sst}(X)_{\mathbb{Q}} \cong \bigoplus_{q \geq 0} L^q H^{2q-j}(X, \mathbb{Q}), \quad j \geq 0 \]

for any smooth complex (quasi-)projective variety \( X \).

Combining Theorem 4.8, 5.1 and 5.2, we have the following statement on Semi-topological \( K \)-groups:

COROLLARY 5.3. For any \( j \geq 0 \), there is a smooth projective variety \( X \) such that \( K_j^{sst}(X)_{\mathbb{Q}} \) is an infinite dimensional vector space over \( \mathbb{Q} \).

Proof. Fix \( j \geq 0 \), let \( X \) be a smooth projective variety of dimension \( m = j + 3 \) in Theorem 4.8. Since

\[ K_j^{sst}(X)_{\mathbb{Q}} \cong \bigoplus_{q \geq 0} L^q H^{2q-j}(X, \mathbb{Q}), \quad \text{(by Theorem 5.2)} \]

\[ \cong \bigoplus_{q \geq 0} L_{m-q} H_{2m-2q+j}(X, \mathbb{Q}), \quad \text{(by Theorem 5.1)} \]

\[ \cong \bigoplus_{p \leq m} L_p H_{2p+j}(X, \mathbb{Q}) \]

\[ \cong \bigoplus_{p \leq j^3} L_p H_{2p+j}(X, \mathbb{Q}) \]

and the last term always contains the direct summand \( L_1 H_{2+j}(X, \mathbb{Q}) \) which is of dimension \( \dim_{\mathbb{Q}} L_1 H_{2+j}(X, \mathbb{Q}) = \infty \).

Remark 5.4. It is not difficult to find a smooth quasi-projective variety \( X \) of dimension \( j + 3 \) such that \( K_j^{sst}(X)_{\mathbb{Q}} \) is an infinite dimensional vector space over \( \mathbb{Q} \), even if the rationality of such a smooth quasi-projective variety is required. To see this, note that it has been constructed by author in [21] that there is a smooth rational quasi-projective variety \( X \) of dimension \( \dim X = j + 3(j \geq 0) \) such that \( \dim_{\mathbb{Q}} L_1 H_{2+j}(X, \mathbb{Q}) = \infty \). Since the duality between Lawson homology and morphic cohomology holds for smooth quasi-projective varieties (cf. [6], Corollary 5.3), we obtain \( \dim_{\mathbb{Q}} K_j^{sst}(X)_{\mathbb{Q}} = \infty \) by using Theorem 5.2. In particular, for \( j = 1 \), we find a smooth 4-dimensional rational quasi-projective variety \( X \) such that \( \dim_{\mathbb{Q}} K_1^{sst}(X)_{\mathbb{Q}} = \infty \) but it is impossible for a smooth 4-dimensional rational projective variety (cf. [9], Prop. 6.18).
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