An algebraic theory of infinite classical lattices I: General theory

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Abstract

We present an algebraic theory of the states of the infinite classical lattices. The construction follows the Haag-Kastler axioms from quantum field theory. By comparison, the *-algebras of the quantum theory are replaced here with the Banach lattices ($MI$-spaces) to have real-valued measurements, and the Gelfand-Naimark-Segal construction with the structure theorem for $MI$-spaces to represent the Segal algebra as $C(X)$. The theory represents any compact convex set of states as a decomposition problem of states on an abstract Segal algebra $C(X)$, where $X$ is isomorphic with the space of extremal states of the set. Three examples are treated, the study of groups of symmetries and symmetry breakdown, the Gibbs states, and the set of all stationary states on the lattice. For relating the theory to standard problems of statistical mechanics, it is shown that every thermodynamic-limit state is uniquely identified by expectation values with an algebraic state.

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I Introduction

It is now generally recognized in statistical mechanics that in order to well-
define even such basic thermodynamic concepts as temperature and phase
transition, one must deal with systems of infinite extent \[12\]. Two approaches
to the study of infinite systems have emerged since the 1950s, Segal’s algebraic
approach in quantum field theory (QFT) (\[3\], \[8\], \[13\], \[27\]) and the theory
of thermodynamic-limit (TL) states (\[5\], \[17\], \[16\]). This paper is the first of
two papers giving an algebraic theory of measurements on infinite classical
lattices. In this paper, Part I, we give the general theory. Part II will give the
axiomatic theory of classical measurements. Construction here will be based
on a nonrelativistic variation of the Haag-Kastler axioms from QFT \[13\].

Regarding this construction, the observables of an algebraic theory are the
elements of a space satisfying the axioms of the Segal algebra. Example 2
in Segal’s original paper \[27\] is a discussion of the commutative algebras, the
setting for the classical theory. It shows in particular (Theorem 1) that any
commutative algebra satisfying the Segal axioms is representable as the space
\(C(X)\) of real-valued continuous functions on a certain compact space \(X\). By
comparison, in the quantum theory, the observable space is a \(C^*\)-algebra, and
one uses the Gelfand-Naimark-Segal (GNS) construction to represent it in a
standard form as the bounded operators on a certain Hilbert space.

Our space of observables here is a real \(MI\)-space (Banach lattice with order
unit). The structure theorem for these spaces then provides the representation
as a space \(C(X)\). We shall find that the theory focusses on the class of compact
convex sets of states on the Segal algebra. In terms of general statistical me-
chanics, this important class includes the set of Gibbs states and the compact
sets of states invariant under a group of symmetries. It also includes the set
of all stationary states.

Some of the conclusions about the structure here are new results of gen-
eral interest in statistical mechanics. In particular, we give the proof that
the unique Choquet decomposition of states into extremal states is a general
property of the state space of any infinite lattice. We show, in fact, that
any compact convex set of states may be decomposed into its extremal states.
Although much success has been had in the TL program in obtaining the de-
composability of states in large classes of lattices, the general proof of this
very basic result has not been found. We shall also show that any TL state
is uniquely identifiable by expectation values with an algebraic state. This
means that the two theories should be regarded as different approaches to a
single theory rather than as different theories.

The material in this paper is arranged as follows. Section II gives the struc-
ture of the lattices themselves and defines the spaces of local observables.
Section III introduces the theory’s axioms and applies them to obtain the al-
gebraic observables. The representation of algebraic states as threads of local
states is the object of Section IV. It is shown here that this representation enables the identification of TL states with algebraic states. In Section V, we present the theory of symmetries and symmetry breakdown, a discussion of Gibbs states, and the construction of the Segal algebra $C(X)$ for the stationary states of lattices.

II The lattice setting

The purpose of the Haag-Kastler axioms is to construct an algebraic theory as a representation of some underlying notion of local observables defined to describe measurements on a finite (laboratory-scale) system. Central to the axioms is the texture of the theory, in the classical case the assignment, to each such system, of a space of phase functions representing measurements on that system. The axioms define construction of the theory’s Segal algebra from its texture. In this section, we describe the local structure of the lattice in sufficient detail to define a texture for it.

A. The lattice.

Take the simple lattice gas first. Denote the infinite lattice by $\Gamma$, representable by $\Gamma = \mathbb{Z}^d$, where $d$ is the dimension of the lattice, and let $T$ be a fixed index set for the lattice sites in $\Gamma$. Let $\mathcal{P}$ denote the set of all finite systems (= bounded subvolumes) of $\Gamma$, and $J$ be a fixed index set for $\mathcal{P}$. $J$ is partially ordered by set inclusion, i.e., for all $s, t \in J$, write $s \leq t$ iff $\Lambda_s \subseteq \Lambda_t$, and upward directed by unions.

At any instant, each site is either empty or else it contains a particle. Denote by $\Omega_o = \{0, 1\}$ this set of single-site configurations. For all $i \in T$, set $\Omega_i \equiv \Omega_o$ and let $\Omega = \mathcal{P}_{i \in T} \Omega_i$. Then this Cartesian product $\Omega$ is the classical phase space for the problem. That is, any point $x = (x_i)_{i \in T} \in \Omega$ gives an instantaneous configuration of the whole space $\Gamma$.

Now generalize the setting. Hereafter we shall assume only that we have an abstract infinite lattice system $\Gamma$ and its phase space $\Omega$, together with the analogous set of bounded systems $\mathcal{P}$, and its index set $J$. The more complicated lattices present nothing new in these terms, although we assume throughout that the number of single-site configurations is finite. Regarding restrictions that make certain configurations impossible, these can be introduced either in $\Omega$ itself or in the distributions assigned in the theory via the Hamiltonian. The definitions here require the latter choice.

B. Local observables.

For the description of a measurement here, we will treat the infinite lattice as consisting of a system and its infinite surround—a generalized “temperature bath”—taking as possible systems of measurement the finite subvolumes of the
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lattice. We define the lattice texture to assure that the expectation values of measurements on a system are determined by the state of its surround. This requirement embodies one of the most basic facts of actual measurements. In statistical thermodynamics, the values used in the Gibbs ensembles for the intensive variables of exchange are their values in the surround. Thus, for systems that can exchange only heat, Guggenheim writes, “\( \beta \equiv 1/kT \) is determined entirely by the temperature bath and so may be regarded as a temperature scale” ([11, p.65]). A same rule obtains for the pressure and other variables, and for the same reasons.

To satisfy the requirement, we must be able to define probability distributions (=states) on the lattice configuration which limit only configurations of the surround, just as we may freely set the thermostat on a temperature bath, or pressure on a piston. Let \( \mathcal{B} \) be the Borel \( \sigma \)-algebra on \( \Omega \), and for each finite local system \( \Lambda_t \), let \( \mathfrak{A}^t \) be the sub-\( \sigma \)-algebra generated by sets of the form \( \Omega_{\Lambda_t} \times \mathcal{A}_\nu \), where \( \mathcal{A}_\nu \subseteq \Omega_{\Lambda_t} \) is any Borel subset of \( \Omega_{\Lambda_t} \). Then for any system \( \Lambda_t \), distributions on the probability space \( (\Omega, \mathfrak{A}^t) \) are of the required form.

In order to have expectation values with respect to these states, the local observables must be \( \mathfrak{A}^t \)-measurable. These are functions with preimages satisfying \( [f < a] \equiv \{ x \in \Omega : f^t(x) < a \} \subseteq \mathfrak{A}^t \forall a \in \mathbb{R} \), i.e., functions on \( \Omega \) with values depending only on configuration outside the system \( \Lambda_t \). We shall take as the local observables assigned by the texture the sets of all bounded Borel-measurable functions on the configuration space of the lattice subject to this requirement. Because of prominence of this class of functions in the theory of Gibbs states in CSM, there is a substantial literature on them ([15], [23], [25]). We adapt the term “functions from the outside” from Preston’s monograph [23] to describe them.

We define the local observable spaces of the theory as follows. For all \( t \in J \), let \( \mathfrak{W}_0(\mathfrak{A}^t) \) be the set of all real-valued, \( \mathfrak{A}^t \)-measurable simple functions on \( \Omega \), and denote by \( \mathfrak{W}(\mathfrak{A}^t) \) the uniform closure of \( \mathfrak{W}_0(\mathfrak{A}^t) \) in \( l_\infty(\Omega) \). Recall that \( l_\infty(\Omega) \) denotes the Banach space of all bounded functions on \( \Omega \), with sup norm. As constructed, \( \mathfrak{W}(\mathfrak{A}^t) \) is the smallest closed linear sublattice in \( l_\infty(\Omega) \) containing all characteristic functions. Note in particular that any bounded measurable function can be obtained as the uniform limit of a sequence of simple functions [18, page 108]. The Banach space \( \mathfrak{W}(\mathfrak{A}^t) \) with its constant function \( \chi_\Omega \) defined by \( \chi_\Omega(x) = 1 \forall x \in \Omega \) is an MI-space, i.e., a Banach lattice with order unit. Notationwise, we shall write elements as \( f^t \in \mathfrak{W}(\mathfrak{A}^t) \). Throughout the theory, we regard the elements of \( \mathfrak{W}(\mathfrak{A}^t) \) as representing measurements on the finite system \( \Lambda_t \).

There has long been available an algebraic theory for classical infinite systems [24]. In contrast with the preceding, the probability algebras are defined analogously so that local observables in this theory are functions on \( \Omega \) have
values depending only on configurations inside $\Lambda_t$. As we shall see, this leads to a much simpler construction of the Segal algebra. Obviously, our lattice requirement would not be satisfied in these algebras.

C. Local states.

We complete our discussion of the local structure by introducing the local state spaces. Denote the set of states on $\mathcal{W}(\mathcal{A}^t)$ by $E_t$, i.e., the set of all positive linear functionals on $\mathcal{W}(\mathcal{A}^t)$ with norm 1. Notationwise, write $\mu_t \in E_t$. In terms of the theory of measurement, we choose a particular system $\Lambda_t$ and fix the state of the lattice $\mu_t \in E_t$. By construction, this choice affects only configurations in the surround. Then the expectation value of any measurement $f \in \mathcal{W}(\mathcal{A}^t)$ is $\mu_t(f)$. This is the formal equivalent of determining the value of a measurement by setting the temperature of the heat bath as a reading on its thermostat.

The characterization of states is contained in the following proposition. For this result, denote by $\bigcirc^* \mathcal{W}(\mathcal{A}^t)$ the unit ball of the (topological) dual of $\mathcal{W}(\mathcal{A}^t)$ with its wk*-topology. The notation is from category theory. Unless otherwise stated, the topology on $E_t$ in the following is the wk*-topology. By compact, we shall always mean wk*-compact Hausdorff.

**Proposition II.1.** The linear functional $\mu_t$ on $\mathcal{W}(\mathcal{A}^t)$ is a state iff $\|\mu_t\| = \mu_t(\chi_\Omega) = 1$. The set of all states on $\mathcal{W}(\mathcal{A}^t)$, denoted $E_t$, is a nonempty compact subset of the unit ball $\bigcirc^* \mathcal{W}(\mathcal{A}^t)$.

**Proof.** Suppose $\mu_t$ is a state on $\mathcal{W}(\mathcal{A}^t)$. We must show $\mu_t(\chi_\Omega) = 1$. For all $f^t \in \mathcal{W}(\mathcal{A}^t)$ such that $0 \leq f^t \leq 1$, $\mu_t(\chi_\Omega - f^t) = \mu_t(\chi_\Omega) - \mu_t(f^t) \geq 0$, and therefore $1 = \|\mu_t\| = \sup_{\|f\| \leq 1} |\mu_t(f^t)| = \mu_t(\chi_\Omega)$. Conversely, suppose $\mu_t$ is a linear functional on $\mathcal{W}(\mathcal{A}^t)$ such that $\|\mu_t\| = \mu_t(\chi_\Omega) = 1$. We must show that $\mu_t \geq 0$. Fix any $f^t$ such that $0 \leq f^t \leq \chi_\Omega$. Then $0 \leq \chi_\Omega - f^t \leq \chi_\Omega$, and therefore $\|\chi_\Omega - f^t\| \leq 1$. Then $|\mu_t(\chi_\Omega - f^t)| = |\mu_t(\chi_\Omega) - \mu_t(f^t)| = |1 - \mu_t(f^t)| \leq 1$ because $\mu_t$ is contracting, and the conclusion follows.

The set $E_t$ is always nonempty $\forall t \in J$. In fact, for any $x \in \Omega$, define the point functional $\delta_x : \mathcal{W}(\mathcal{A}^t) \to \mathbb{R}$ by $\delta_x f^t = f^t(x)$. Then clearly, $\delta_x$ is a positive linear functional, and $\|\delta_x\| = \delta_x(\chi_\Omega) = 1 \forall x \in \Omega$. Hence, $\delta_x \in E_t \forall x \in \Omega$.

Now define the linear function $\mathfrak{z} : (\mathcal{W}(\mathcal{A}^t))^* \to \mathbb{R}$ by $\mathfrak{z}(\mu) = \mu(\chi_\Omega)$. Then $\mathfrak{z}$ is wk*-continuous. In fact, for any $\varepsilon > 0$, let $U = \mathcal{N}(\mu; \chi_\Omega, \varepsilon)$ be the subbasic open set $\{\nu \in (\mathcal{W}(\mathcal{A}^t))^* : |\nu(\chi_\Omega) - \mu(\chi_\Omega)| < \varepsilon\}$. Then $\mathcal{N}(\mu; \chi_\Omega, \varepsilon) = \{\nu : \mathfrak{z}(\nu) \in (\mathfrak{z}(\mu) - \varepsilon, \mathfrak{z}(\mu) + \varepsilon)\}$. Hence, for any $\mu \in (\mathcal{W}(\mathcal{A}^t))^*, \varepsilon > 0$, $\mathcal{N}(\mu; \chi_\Omega, \varepsilon) \subseteq (\mathfrak{z}(\mu) - \varepsilon, \mathfrak{z}(\mu) + \varepsilon)$. Then $\mathfrak{z}^{-1}(1)$ is closed. The unit ball $\bigcirc^* \mathcal{W}(\mathcal{A}^t)$ is wk*-compact by Alaoglu’s Theorem. Since $E_t$ is the intersection $\mathfrak{z}^{-1}(1) \cap \bigcirc^* \mathcal{W}(\mathcal{A}^t)$ of a closed hyperplane and a compact set, it is closed and compact.
III Algebraic theory

A. The Haag-Kastler frame.

1. The axioms. We are now in a position to formulate the Haag-Kastler frame for the classical case. For nonrelativistic theory, it is defined by four axioms. We state the axioms in terms of the above structure.

Axiom 1 The lattice texture is defined by the pairings

$$\Lambda_t \mapsto \mathcal{M}(A_t), \quad t \in J$$

Axiom 2 Define the order relation $\leq$ on the net $\{\mathcal{M}(A_t)\}_{t \in J}$ by $\mathcal{M}(A_s) \leq \mathcal{M}(A_t)$ iff $\mathcal{M}(A_s) \supseteq \mathcal{M}(A_t)$. Then the net $\{\mathcal{M}(A_t)\}_{t \in J}$ is an upward-directed partially ordered set, with $s \leq t \Rightarrow \mathcal{M}(A_s) \leq \mathcal{M}(A_t)$.

Axiom 3 All local observables are compatible.

Axiom 4 The theory’s Segal algebra is the completion of the inductive limit of the net of local algebras $\{\mathcal{M}(A_t)\}_{t \in J}$. It is representable as a space $\mathcal{C}(X)$ for a suitable compact space $X$.

Axiom 1 identifies each system $\Lambda_t$ with the corresponding $MI$-space of observables from outside $\Lambda_t$. Note especially that the local algebras $\mathcal{M}(A_t)$ are defined without reference to (or need for) containing walls for the systems $\Lambda_t$. Axiom 2 is an order structure imposed by the texture (3.1). Order by inclusion among the systems $\Lambda_t$ defines a partial order of the local observable algebras as well by (1.1). It follows from the definition of the $A_t$ that for all $\Lambda_s \subseteq \Lambda_t$, $\mathcal{M}(A_s) \supseteq \mathcal{M}(A_t)$. Axiom 3 has to do with the compatibility of observables on different systems. This is classical theory. The final axiom constructs the theory’s Segal algebra, the space of quasilocal observables, as the completion of an inductive limit. We shall prove that the completion of this limit is an $MI$-space. This will assure its representation as $\mathcal{C}(X)$ [23, 13.1.1.].

2. The morphisms $$(\hat{\eta}_s^t)_{s \leq t}$$. The inductive limit in Axiom 4 will be in the category $\text{Ban}_1$ of Banach spaces and linear contractions. The definition of the limit requires the upward-directed net of $MI$-spaces and, for each nested pair of systems $\Lambda_s \subset \Lambda_t$, a morphism mapping measurements on the smaller system to those measuring the same physical quantity on the larger system, i.e., a set of positive linear contractions with the following properties:

(i) $\forall f^t \in \mathcal{M}(A^t) \cap \mathcal{M}(A^s)$, $\hat{\eta}_s^t(f^t) = f^t$;
(ii) $\forall t \in J$, $\hat{\eta}_t^t f^t = f^t$; and
Hence, sup \parallel \mu = \imath mapping with a set of functions that approximate their action. For all s, t \in J, we define \eta_t^s : \mathcal{W}(\Omega) \rightarrow \mathcal{W}(\Omega) as follows. For all s \in J, \eta_t^s is the identity mapping. For s < t, denote by M_t^s = \Omega_{\Lambda_t \sim \Lambda_s}, and let

\eta_t^s f^s(x) = \frac{1}{|M_t^s|} \sum_{(M_t^s)} f^s(x), \quad x \in \Omega, f^s \in \mathcal{W}(\Omega)

(3.2)

The sum notation \((M_t^s)\) means to sum over all configurations in \(M_t^s\), holding the rest of \(x \in \Omega\) constant, and \(|M_t^s|\) is the number of such configurations. The effect of the mappings \eta_t^s is to remove the dependence on configurations in \(M_t^s\) by averaging over them. As for the approximation, note in particular that \(f^s \in \mathcal{W}(\Omega)\) and \(\eta_t^s f^s\) differ as functions on the infinite space \(\Omega\) at most at configurations in the finite region \(\Lambda_s \sim \Lambda_t\), where the average is performed.

The first two properties are immediate. For the composition rule, one has the following:

\eta_r^s f_r^s(x) = \eta_t^s f_r^s(x)

(3.3)

The similarity in properties of the function \(\eta_t^s f^s\) to a conditional expectation is apparent, although here the smoothing action is independent of state.

From the \((\eta_t^s)_{s \leq t}\), we may immediately construct a corresponding set of morphisms relating local state spaces. This is obtained as follows.

**Proposition III.1.** For each comparable pair \(s \leq t\), define the \(w^k\)-continuous mapping \(\eta_t^s : \bigodot \mathcal{W}(\Omega) \rightarrow \bigodot \mathcal{W}(\Omega)\) by \(\eta_t^s \mu_t = \mu_t \circ \eta_t^s\). The mapping \(\eta_t^s\) carries states to states and non-states to non-states. Moreover, \(\eta_t^s\) is a \(w^k\)-continuous mapping on \(E_t\) into \(E_s\). For all \(r \leq s \leq t\), \(\eta_r^s \eta_s^t = \eta_r^t\), and \(\eta_t^t\) is the identity mapping \(\iota_{E_t}\).

**Proof.** Fix any state \(\mu_t \in E_t\). By hypothesis, \(\eta_t^s \chi_{\Omega} = \chi_{\Omega}\), so that \(\eta_t^s \mu_t(\chi_{\Omega}) = \mu_t(\eta_t^s \chi_{\Omega}) = \mu_t(\chi_{\Omega}) = 1\). Moreover, \(\forall \mu_t \in E_t, \|\eta_t^s \mu_t\| = \sup_{\|f^s\| \leq 1} |\mu_t(\eta_t^s f^s)| = \sup_{\|f^s\| \leq 1} |\mu_t(\eta_t^s f^s)|\). Since \(|\eta_t^s f^s| \leq |f^s|, \sup_{\|f^s\| \leq 1} |\mu_t(\eta_t^s f^s)| \leq \sup_{\|f^s\| \leq 1} |\mu_t(f^s)| = \|\mu_t\| = 1\). But \(\sup_{\|f^s\| \leq 1} |\eta_t^s \mu_t(\chi_{\Omega})| \geq \eta_t^s \mu_t(\chi_{\Omega}) = \mu_t(\chi_{\Omega}) = 1\). Then \(\|\eta_t^s \mu_t\| = 1\). Hence, \(\eta_t^s \mu_t\) is a state by Proposition II.1. Now suppose \(\mu_t \in \bigodot \mathcal{W}(\Omega) \setminus E_t\). If
$\mu_t$ is not a positive functional, then there exists $f^t \in \mathcal{W}(\mathcal{A}^t)$, $f^t \geq 0$, such that $\mu_t(f^t) < 0$. Then $\eta^t_s\mu_t(f^t) = \mu_t(\tilde{\eta}^t_s f^t) = \mu_t(f^t) < 0$, so that $\eta^t_s\mu_t$ is not a positive functional. If $\mu_t(\chi_\Omega) < 1$, $\eta^t_s\mu_t(\chi_\Omega) = \mu_t(\chi_\Omega) < 1$. In either case, $\eta^t_s\mu_t \notin E_s$. Finally, $\forall f^t \in \mathcal{W}(\mathcal{A}^t)$, $\mu_t(f^t) = \mu_t(\tilde{\eta}^t_s f^t) = \eta^t_s\mu_t(f^t)$. The wk*-continuity is shown in [28 Proposition 6.1.8].

We have defined two systems, $\{\mathcal{W}(\mathcal{A}^t), \tilde{\eta}^t_s, J\}$ and $\{E_t, \eta^t_s, J\}$. The set $\{\mathcal{W}(\mathcal{A}^t), \tilde{\eta}^t_s, J\}$ is an inductive system in the category $\text{Ban}_1$. The set $\{E_t, \eta^t_s, J\}$ is a projective system in the category $\text{Compconv}$ of compact convex spaces and continuous affine maps. The notation follows Semadeni [28].

3. Notation. As regards notation, the transformations $\tilde{\eta}^t_s : \mathcal{W}(\mathcal{A}^s) \to \mathcal{W}(\mathcal{A}^t)$ require two indices, specifying domain and range. The form of the superscript/subscript notation follows the conventions of tensor contraction. Thus, in $\tilde{\eta}^t_s f^s$, the index $s$ cross-cancels to take a function $f^s \in \mathcal{W}(\mathcal{A}^s)$ with superscript $s$ over to a function in $\mathcal{W}(\mathcal{A}^t)$ with superscript $t$. Similarly, one encounters later $\eta^t_r\mu_t$, in which the $t$ cross-cancels to take the state $\mu_t$ with subscript $t$ to a new state with subscript $s$. In the compose $\tilde{\eta}^t_s \tilde{\eta}^r_t$, the $s$ cross-cancels to leave a transformation $\tilde{\eta}^r_t$ with superscript $t$, subscript $r$. Also encountered will be $\mu_t = \rho_t\mu_t$, taking an object $\mu$ with no index to one with subscript $t$, as well as $\sigma_s f^s = [f]$, taking an indexed function $f^t$ to $[f]$ with no index.

B. The inductive limit $\lim_{\to} \{\mathcal{W}(\mathcal{A}^t), \tilde{\eta}^t_s, J\}$.

We begin with the first part of Axiom 4, the construction of the inductive limit from the lattice texture.

**Theorem III.2.** The system $\{\mathcal{W}(\mathcal{A}^t), \tilde{\eta}^t_s, J\}$ has a unique inductive limit $\{\mathcal{W}^\infty, \sigma_t, J\} = \lim_{\to} \{\mathcal{W}(\mathcal{A}^t), \tilde{\eta}^t_s, J\}$. $\mathcal{W}^\infty$ is a Banach space, and the $\sigma_t : \mathcal{W}(\mathcal{A}^t) \to \mathcal{W}^\infty$ are linear contractions obeying the composition rule $\sigma_s = \sigma_t \tilde{\eta}^t_s$ $\forall s \leq t$.

**Proof.** The properties of the morphisms $(\tilde{\eta}^t_s)$ assure that the set $\{\mathcal{W}(\mathcal{A}^t), \tilde{\eta}^t_s, t \geq s, s \in J\}$ is a inductive system of Banach spaces, characterized by the commuting diagram:
for all $r \leq s \leq t$. A standard construction of the Banach-space limit applies to the system $\{W(A^t), \hat{\eta}_s, J\}$ as follows [28, p. 212]. Let $\bigvee_{t \in J} W(A^t)$ be the $l_1$-join of the individual algebras, i.e., the Banach space $\{(f^t)_{t \in J} \in \prod_{t \in J} W(A^t) : \sum \|f^t\|_t < \infty\}$, with the usual linear operations. The norm is $\|f\| = \sum \|f^t\|_t$, where $\|f^t\|_t$ is the norm on $W(A^t)$. The $l_1$-join is ordered by the relation $(f^t) \leq (g^t)$ iff $f^t \leq g^t \forall t \in J$. Let $\tilde{\sigma}_t : W(A^t) \to \bigvee_{t \in J} W(A^t)$ be the canonical injection, and $M$ be the closed linear subspace of $\bigvee_{t \in J} W(A^t)$ spanned by elements of the form $\tilde{\sigma}_s(f^s) - \tilde{\sigma}_t(\hat{\eta}_t^s f^s)$, $f^s \in W(A^s), s \leq t$. Then for the inductive limit $\{W^\infty, \sigma_t, J\}$, the object $W^\infty$ is the quotient space $\bigvee_{t \in J} W(A^t)/M$. Denote the elements of $W^\infty$ with square brackets. $W^\infty$ has the usual quotient norm $\|[f]\| = \inf_{h \in M} \|f + h\|$. Let $\tau : \bigvee_{t \in J} W(A^t) \to W^\infty$ be the quotient surjection. The limit homomorphism $\sigma_t : W(A^t) \to W^\infty$ is the compose $\sigma_t = \tau \circ \tilde{\sigma}_t$. All elements of $W^\infty$ are of the form $[f] = \sum \sigma_t k^t f^t$ for some countable set of functions $f^t \in W(A^t)$. The composition rule for the $(\sigma_t)_{t \in J}$ makes the following diagram commuting, for all $s \leq t$:

\[
\begin{array}{ccc}
W^\infty & \xrightarrow{\sigma_t} & W(A^t) \\
\downarrow{\sigma_s} & & \downarrow{\tilde{\eta}_s^t} \\
W(A^s) & & \\
\end{array}
\]

In introducing the morphisms $(\tilde{\eta}_s^t)$, we identified $f^s$ and its image $\tilde{\eta}_s^t f^s$ as physically equivalent local measurements. The formation of $W^\infty$ as the quotient space assures that equivalent measurements map to the same quasilocal observable, $\sigma_s f^s = \sigma_t \tilde{\eta}_s^t f^s$. We use the notation $\phi \in W^\infty$.

C. Functional representation of $W^\infty$.

Axiom 4 calls for construction of the Segal algebra as an $MI$-space formed by completion of this inductive limit. Of course, as a Banach space, $W^\infty$ is complete with respect to its norm topology, but it is not an $MI$-space. We shall show that its functional representation as an order-unit space satisfies this condition as well. We first provide the three main elements needed for this construction, namely, a partial order in $W^\infty$, an order unit in $W^\infty$, and the set of states $K(W^\infty)$ on $W^\infty$.

1. Order structure of $W^\infty$. We assign the usual quotient partial order to $W^\infty$ determined by the surjection $\tau$. Explicitly, one writes $[g] \leq [f]$ iff there
exists a finite set of pairs \((a_i, b_i) \in \bigvee_i \mathcal{M}(\mathcal{A}^i)\) such that (i) \(a_i \leq b_i \ \forall i = 1, \ldots, n\), (ii) \([g] = [a_1] \text{ and } [b_n] = [f]\), and (iii) \([h_1] = [a_2], \ldots, [b_{n-1}] = [a_n]\). \[23\] Definition 2.3.4 For example, with the choice of \(t\) fixed any \(\{\}\) respectively.

Note in particular that for \(n = 1\), \([g] \leq [f]\) iff there exists \(a \leq b\) such that \([a] = [g], \ [b] = [f]\).

The induced order relation has the following properties.

**Lemma III.3.** \([g] \leq [f]\) iff for any \(g \in \tau^\leftarrow[g], f \in \tau^\rightarrow[f]\) there exists an \(h \in \mathcal{M}\) such that \(g \leq f + h\). Moreover, the surjection \(\tau\) is order-preserving, i.e., \(\tau(C) \subseteq [C]\), where \(C\) and \([C]\) are the positive cones in \(\bigvee_i \mathcal{M}(\mathcal{A}^i)\) and \(\mathcal{M}^\infty\), respectively.

**Proof.** Suppose that the inequality \([0] \leq [f]\) is determined by \(n\) pairs \((a_i, b_i)\).

Define the set of elements \(\{h_i \in \mathcal{M}\}\) by \(h_1 = -a_1, h_{n+1} = b_n - f\), and \(h_i = b_{i-1} - a_i \ \forall i = 2, \ldots, n\). Then from \(a_i \leq b_i \ \forall i,\sum_{i=1}^n a_i \leq \sum_{i=1}^n b_i\), and thus, \(0 \leq \sum_{i=1}^n (b_i - a_i)\). Hence, \(0 \leq f + \sum_{i=1}^n (b_i - a_i)\).

Set \(h = \sum_{i=1}^{n+1} h_i \in \mathcal{M}\). For the converse, one has immediately that \(f \geq h \in \mathcal{M}\) implies (by definition) that \([f] \geq [h] = [0]\). Thus, \([0] \leq [f]\) iff there exists \(h \in \mathcal{M}\) such that \(0 \leq f + h\). Now apply this result. For any pair \([g], [f]\), \([g] \leq [f]\) or \([0] \leq [f] - [g] = [f - g]\) iff there exists \(h \in \mathcal{M}\) such that \(0 \leq (f - g) + h\) or \(g \leq f\). Finally, to show that \(\tau(C) \subseteq [C]\), fix any \([g] \in [C]\). Then for all \(f \in \tau^\leftarrow[g]\), there exists \(h \in \mathcal{M}\) such that \(f + h \in C\). But \(\tau(f + h) = [f + h] = [f] = [g]\). Conversely, \(\tau(C) \subseteq [C]\) because \(f \geq 0\) implies that \([f] \geq [0]\). Hence, \(\tau(C) = [C]\). \[\blacksquare\]

2. The order unit \(e \in \mathcal{M}^\infty\). The next result identifies an element \(e \in \mathcal{M}^\infty\) with special properties.

**Theorem III.4.** Fix any \(t \in \mathcal{J}\), and let \(e = \sigma_t(\chi_\Omega)\). Then \(e\) is independent of the choice of \(t\). The element \(e\) is an order unit for the space \(\mathcal{M}^\infty\), so that for every \([f] \in \mathcal{M}^\infty\), \(-\|f\| \leq [f] \leq \|f\|e\).

**Proof.** Recall first the definition of an order unit. \[\[1\] p.68ff\] Let \(A\) be an ordered linear space. The linear subspace \(J \subseteq A\) is called an order ideal iff for all \(a, b \in J\) and \(c \in A\), the inequality \(a \leq c \leq b\) implies that \(c \in J\). For any \(a \in A\), denote by \(J(a)\) the smallest order ideal containing \(a\). Then \(a\) is said to be an order unit of \(A\) if \(J(a) = A\).

We obtain a general form for \(J(a)\), namely, that \(\mathcal{M}^\infty = \{\sum_{i=1}^\infty \sigma_i(f^{t_i}) \in \bigvee_i \mathcal{M}(\mathcal{A}^i) : \sum_{i=1}^\infty \|f^{t_i}\|^{t_i} < \infty\}\). Indeed, by definition, everything in \(\bigvee_i \mathcal{M}(\mathcal{A}^i)\) is of the form \(\sum_{i=1}^\infty \sigma_{t_i}(f^{t_i})\) for some countable set of indices \((t_i)\). The quotient surjection \(\tau: \bigvee_i \mathcal{M}(\mathcal{A}^i) \to \mathcal{M}^\infty\) is linear and of norm \(\|\tau\| = 1\). From linearity, \(\tau(\sum_{n=1}^\infty \sigma_{t_n}f^{t_n}) = \sum_{n=1}^\infty \sigma_{t_n}f^{t_n}\) for all \(n \in \mathbb{N}\), since \(\sigma_i = \tau \circ \sigma_i\). Then from continuity, \(\tau \sum_{i=1}^\infty \sigma_{t_i}(f^{t_i}) = \sum_{i=1}^\infty \sigma_{t_i}(f^{t_i})\). But \(\tau\) is onto. Hence, everything in \(\mathcal{M}^\infty\) is attained in this way.
Fix any \( t \in J \), and define \( e = \sigma_t(\chi_\Omega) \). Then for any other \( s \in J \), there exists \( u \in J \) such that \( s, t \leq u \), because \( J \) is upward directed. Then \( \sigma_s(\chi_\Omega) - \sigma_u(\chi_\Omega) = \sigma_t(\chi_\Omega) - \sigma_u(\hat{\nu}_k^u \chi_\Omega) \in M \), so that \( \sigma_u(\chi_\Omega) = \sigma_t(\chi_\Omega) = e \) and similarly, \( \sigma_t(\chi_\Omega) = e \). Hence, the definition \( e = \sigma_t(\chi_\Omega) \) is independent of \( t \).

We construct a more general element. Fix any \( f = \sum_{k=1}^n \hat{\sigma}_t_k(f^{t_k}) \in \bigvee_1 M(A^t) \), with \( n \) finite, and consider the sum \( \sum_{k=1}^n (\| f^{t_k} \|/\| f \|) \hat{\sigma}_t_k(\chi_\Omega) \). For any \( u \geq t_k \forall k = 1, \ldots, n \),

\[
\sum_{k=1}^n (\| f^{t_k} \|/\| f \|) \hat{\sigma}_t_k(\chi_\Omega) - \hat{\sigma}_u(\chi_\Omega) = 0.
\]

But \( \hat{\sigma}_t_k(\chi_\Omega) - \hat{\sigma}_u(\hat{\nu}_k^u \chi_\Omega) \in M \) for all \( t \in J \). Since \( M \) is a linear subspace, it contains all linear combinations of its elements. Hence, \( \sum_{k=1}^n (\| f^{t_k} \|/\| f \|) \hat{\sigma}_t_k(\chi_\Omega) = e \). Then for any countable set of indices \( (t_k) \in J \), the infinite sum converges,

\[
\sum_{k=1}^\infty (\| f^{t_k} \|/\| f \|) \hat{\sigma}_t_k(\chi_\Omega) = e,
\]

because the equivalence classes are closed.

Now fix any \( f \in \bigvee_1 M(A^t) \) and any \( h \in M \). There exists a countable set of indices \( (t_k) \in J \) such that \( f + h = \sum_{k=1}^\infty \hat{\sigma}_t_k(f^{t_k} + h^{t_k}) \). Clearly, \( -a \sum f^{t_k} + h^{t_k} /\| f^{t_k} \| \hat{\sigma}_t_k(\chi_\Omega) \leq \sum \hat{\sigma}_t_k(f^{t_k} + h^{t_k}) \leq a \sum f^{t_k} + h^{t_k} /\| f^{t_k} \| \hat{\sigma}_t_k(\chi_\Omega) \) iff \( a \geq 1 \). Then \( -b(h) \sum f^{t_k} + h^{t_k} /\| f^{t_k} \| \hat{\sigma}_t_k(\chi_\Omega) \leq \sum \hat{\sigma}_t_k(f^{t_k} + h^{t_k}) \leq b(h) \sum f^{t_k} + h^{t_k} /\| f^{t_k} \| \hat{\sigma}_t_k(\chi_\Omega) \) iff \( b(h) \geq \| f + h \| \). Therefore, there exists \( h \in M \) such that \( -b(h) \sum f^{t_k} + h^{t_k} /\| f + h \| \hat{\sigma}_t_k(\chi_\Omega) \leq \sum \hat{\sigma}_t_k(f^{t_k} + h^{t_k}) \leq b(h) \sum f^{t_k} + h^{t_k} /\| f + h \| \hat{\sigma}_t_k(\chi_\Omega) \) iff \( b(h) \geq \| f + h \| \). But this is just \( -b\hat{\sigma}_{f+h}(\chi_\Omega) \leq f + h \leq b\hat{\sigma}_{f+h}(\chi_\Omega) \). Hence, \( -b \leq [f] \leq b \) for any \( b \geq \| [f] \| \). Then for all \( [f] \in M^\infty, -\| [f] \| \leq [f] \leq \| [f] \| e \). It follows that the order interval \( [-e, e] \) is absorbing, and moreover, that \( [f] \| = 1 \) implies that \( -e \leq [f] \leq e \). The conclusion that \( [-e, e] \) is an order ideal then follows immediately.

3. The states on \( M^\infty \). Since the theory’s Segal algebra is the completion of \( M^\infty \), the states on \( M^\infty \), denoted \( K M^\infty \), will be identifiable with the algebraic states. They may be characterized as follows. We give a second characterization of them in terms of an order-unit norm below (Proposition III.14).

**Proposition III.5.** Let \( \phi \in K(M^\infty) \). Then \( \| \phi \| = \phi(e) = 1 \).

**Proof.** Note that for all \( [f] \in M^\infty \), if \( \| [f] \| \leq 1 \), then \( -e \leq [f] \leq e \). Hence, if \( \phi \geq 0 \), then \( \| \phi \| = \sup_{\| [f] \| \leq 1} |\phi([f])| \leq \phi(e) = 1 \). But \( \| \hat{\sigma}_t(\chi_\Omega) \| = \| \chi_\Omega \| t = 1 \).
Then since the canonical surjection \( \tau \) is a contraction, \( \| \tau(\hat{s}_k(x_\Omega)) \| = \| e \| \leq 1 \). Hence, \( \phi(e) \leq \| \phi \| \), and therefore \( \| \phi \| = 1 \). ■

The \( \text{wk}^* \)-compactness of \( K\mathcal{M}^\infty \) is similar to that shown in Proposition II.1.

4. The Kadison representation of \( \mathcal{M}^\infty \). The functional representation is directed by the requirements of Kadison’s theorem [14]. We must begin with the most basic properties of the order on \( \mathcal{M}^\infty \). Denote by \( C \) the positive cone of \( \bigvee_1 \mathcal{M}(\mathfrak{A}) \), i.e., the set of all nonnegative elements, and by \( [C] \) the positive cone in \( \mathcal{M}^\infty \). The general properties of the order relation \( \leq \) in \( \mathcal{M}^\infty \) are given in Lemma III.3. These do not assure that the quotient order is antisymmetric [23, 2.3.4]. Since antisymmetry is needed in the functional representation of \( \mathcal{M}^\infty \), it must therefore be shown directly. The proof will depend on the following lemma.

Lemma III.6. The only element \( h \in \mathcal{M} \) comparable to 0 is \( h = 0 \) itself, i.e., \( C \cap \mathcal{M} = \{0\} \).

Proof. Fix any \( h = \sum_{k=1}^p \hat{s}_k h^{s_k} \in \mathcal{M} \), \( p \) finite. By definition, \( h \) is a linear combination of pairs of the form \( \hat{s}_k f^{s_k} - \hat{s}_k \eta_{s_k} f^{s_k} \), so we write \( \sum_{k=1}^p \hat{s}_k h^{s_k} = \sum_{k=1}^p (\hat{s}_k f^{s_k} - \hat{s}_k \eta_{s_k} f^{s_k}) \). Suppose \( h < 0 \). Let \( s \geq s_k \) for all \( k \), and denote \( \varepsilon_k = \sup_{x \in \Omega} h^{s_k}(x) \) and \( \eta_{s_k}(\varepsilon_k) = \sup_{x \in \Omega} \eta_{s_k} h^{s_k}(x) \). For some \( k \), say \( k = 1 \), \( h^{s_k} < 0 \). Then there exists \( x \in \Omega, \varepsilon > 0 \) such that \( \eta_{s_1} h^{s_1}(x) < -\varepsilon \). Of course, \( \eta_{s_k} h^{s_k}(x) \leq \eta_{s_k}(\varepsilon_k) \) for all other \( k \). Writing out each component and summing over the \( p \) inequalities yields

\[
0 < -\varepsilon + \sum_{k=2}^p \eta_{s_k}(\varepsilon_k)
\]

The 0 on the left comes from the fact that the contribution from each pair \( \hat{s}_k f^{s_k}(x) - \hat{s}_k \eta_{s_k} f^{s_k}(x) \) is just \( \eta_{s_k} f^{s_k}(x) - \eta_{s_k}(\varepsilon_k) f^{s_k}(x) = \eta_{s_k} f^{s_k}(x) - \eta_{s_k} f^{s_k}(x) = 0 \). From this equation, there exists at least one \( k \neq 1 \) such that \( \eta_{s_k}(\varepsilon_k) > 0 \). Then \( h^{s_k} \vee 0 \geq \eta_{s_k} h^{s_k} \vee 0 > 0 \). But this is impossible if \( h < 0 \). The proof for \( h > 0 \) is similar.

The countable case is simplified by the fact that the \( l_1 \)-join \( \bigvee_1 \mathcal{M}(\mathfrak{A}) \) must be a Banach space, and in particular, that the norm \( \| h \| = \sum_{k=1}^\infty \| h^{s_k} \|_k < \infty \).

With sup norm, this means that for any choice of \( \varepsilon \) in eq. (3.2), the positive contribution must come from the first \( p \) terms for \( p \) sufficiently large. The above proof therefore applies here as well. We need to show the existence of the summations. For any function \( h \in \mathcal{M} \), \( h = \sum_{k=1}^\infty \hat{s}_k h^{s_k} \), \( \sum_{k=1}^\infty |\varepsilon_k| \leq \sum_{k=1}^\infty \| h^{s_k} \|_{s_k} = \| h \| < \infty \), and since \( \eta_{s_k}(\varepsilon_k) \leq |\varepsilon_k|, \sum_{k=1}^\infty | \eta_{s_k}(\varepsilon_k) | \leq \infty \). ■

The antisymmetry of the quotient order then follows immediately. It is displayed here together with two other important (and actually equivalent) properties involving the order.
Proposition III.7. The following properties obtain:

(i) The quotient set $\mathcal{M}$ is an order ideal;
(ii) The positive cone $[C]$ is proper; and
(iii) The quotient partial order $\leq$ is antisymmetric.

Proof. For (i), $\mathcal{M}$ is an order ideal iff for any pair $h_1, h_2 \in \mathcal{M}, h_1 \leq g \leq h_2$ implies $g \in \mathcal{M}$. But $0 \leq g - h_1 \leq h_2 - h_1$ implies by the lemma that $h_2 - h_1 = 0$, or $g = h_1 \in \mathcal{M}$.

For (ii), suppose $[g] \in [C] \cap (-[C])$. The cone $[C]$ is said to be proper iff this implies that $[g] = 0$. Suppose $[g] \leq [0] \leq [g]$. The two inequalities require that there exist $h_1, h_2 \in \mathcal{M}$ such that $g \leq h_1$ and $0 \leq g + h_2$. Then $0 \leq g + h_2 \leq h_1 + h_2$. But from the lemma, $0 \leq h_1 + h_2$ implies that $h_1 + h_2 = 0$, or $0 \leq g + h_2 \leq 0$. Then by the partial order on $\bigvee_1 \mathcal{M}(\mathcal{N})$, $g + h_2 = 0$, or $g = -h_2 \in \mathcal{M}$. Hence, $[g] = [0]$. For (iii), if $[f] \leq [g] \leq [f]$, then $[0] \leq [g] - [f] \leq [0]$. Hence, from (ii), $[g - f] = [0]$, or $[g] = [f]$. ■

In the following, the term order will always imply the antisymmetric property.

Although $\mathcal{M}^\infty$ is by definition a Banach space with respect to its quotient norm, its representation in $C(X)$ will be based instead on a norm which makes direct use of its order unit $e$. Denote by $E$ the order interval $[-e, e] = \{[f] : -e \leq [f] \leq e\}$.

Proposition III.8. The Minkowski functional $p_E([f]) = \inf \{b > 0 : -be \leq [f] \leq be\}$ is a continuous seminorm on the Banach space $\mathcal{M}^\infty$, and $p_E \leq \| \cdot \|$.

Proof. Using the fact that $\mathcal{M}$ is a linear subspace, one readily shows that the order interval $E$ is a convex, balanced, and absorbing set in $\mathcal{M}^\infty$. But the Minkowski functional of any such set is a seminorm. Clearly $p_E \leq \| \cdot \|$, and hence $p_E$ is $\| \cdot \|$-continuous. ■

The seminorm $p_E$ is a norm on $\mathcal{M}^\infty$ if $\mathcal{M}^\infty$ is Archimedean [1, II.1.2]. This result is assured by the next proposition.

Proposition III.9. The Banach space $\mathcal{M}^\infty$ with its quotient partial order is Archimedean. The positive cone $[C]$ is $\| \cdot \|$-closed, and $\mathcal{M}^\infty = [C] - [C]$, i.e., $[C]$ is generating.

Proof. We show first that the order interval $[-e, e]$ is $\| \cdot \|$-closed. Fix any Cauchy sequence $(\{f_n\})_{n \in \mathbb{N}} \in [-e, e]$. Since $\mathcal{M}^\infty$ is complete, the limit $[f]$ exists in $\mathcal{M}^\infty$. We claim that $[f] \in [-e, e]$. In fact, $0 \leq |p_E([f_n]) - p_E([f])| \leq p_E([f_n] - [f]) \leq \| [f_n] - [f] \| \rightarrow 0$. Then $p_E([f]) = \lim p_E([f_n]) \leq 1$, because $[f_n] \in [-e, e]$ implies that $p_E([f_n]) \leq 1 \forall n \in \mathbb{N}$. But $p_E([f]) \leq 1$ implies that $[f] \in [-e, e]$. It follows immediately that $[-ae, ae]$ is closed for any $a > 0$. Now let $(\{f_n\})_{n \in \mathbb{N}} \in [C]$ be any $\| \cdot \|$-Cauchy sequence in the positive cone $[C]$, and let $\lim [f_n] = [f]$. We show that $[f] \in [C]$. Fix any $\varepsilon > 0$. Then
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there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0, \| f_n \| - \| f_n \| \leq \varepsilon \). Now fix any \( a \geq (1/2)(\| [f_n] \| + \varepsilon) \), so that \( [0] \leq [f_n] \leq \| f_n \| e \leq 2ae \ \forall n \geq n_0 \). The set \([0], 2ae \) is a closed neighborhood of \( ae \), so that \( [f] \in [0, 2ae] \). But \([0], 2ae \) is a completion of \( W \) and therefore \( [f] \leq [0] \) implies that \( (1/n)e - [f] = -[f] \in [C] \). To show that the cone \( [C] \) is generating, note simply that for every \( [f] \in W^\infty, [0], [f] \leq [f] \leq e \). Then \( [f] = [f] [e - [f] [e - [f] \in [C] - [C] \). \( \square \)

With the change in norm on \( W^\infty \), it is useful to introduce a new norm on the dual \((W^\infty)^* \) as well. Define \( \| \cdot \|_p \) on \((W^\infty)^* \) by \( \| \phi \|_p = \sup_{pE([f]) \leq 1} |\phi([f])| \). We may characterize the set of states \( K W^\infty \) in terms of the new norm as follows.

Proposition III.10. The linear functional \( \phi \in (W^\infty)^* \) is a state on \( W^\infty \) iff \( \| \phi \|_p = \phi(e) = 1 \).

Proof. Suppose \( \phi \in K W^\infty \). Since \( pE([f]) \leq 1 \) implies that \( -e \leq [f] \leq e, |\phi([f])| \leq \phi(e) = 1 \) from \( \phi \geq 0 \), and hence \( \| \phi \|_p = 1 \). Conversely, suppose \( \| \phi \|_p = \phi(e) = 1 \). We must show that \( \phi \geq 0 \). If \( [0] \leq [f] \leq e \), then \( [0] \leq e - [f] \leq e \), and therefore \( pE(e - [f]) \leq 1 \). Then \( |\phi(e - [f])| \leq |\phi(e - [f])| \leq \| \phi \|_p = 1 \), and hence \( 1 - \phi([f]) \leq 1 \). Then \( \phi([f]) \geq 0 \). \( \square \)

The representation of \( W^\infty \) now follows immediately from Kadison’s functional representation of an order-unit space.

Theorem III.11. Let \( W_K = A(K) \) be the Banach space of continuous affine functions on the compact convex set \( K \) of states on \( W^\infty \). The linear space \((W^\infty, e) \) with order unit \( e \) and norm \( pE \) has a functional representation \( \Delta_K : W^\infty \rightarrow W_K \) defined by \( \Delta_K([f])(\phi) = \phi([f]) \). The function \( \Delta_K \) is a \( pE \)-isometry, order-preserving in both directions. The image \( \Delta_K(W^\infty) \) is a separating uniformly dense subset of the Banach space \( W_K \) and \( \Delta_K(e) \) is the constant function \( 1_K \) on \( K \).

Proof. [1], [11] Theorem II.2.9 \( \square \)

The space \( W_K \) is the uniform closure of the subspace \( \Delta_K(W^\infty) \subset C(K) \). It is therefore a completion of \( W^\infty \) with respect to the order-unit norm \( pE \). We refer to it throughout as \textit{the} completion of \( W^\infty \). Denote its elements by \( f \in W_K \).

D. The \( MI \)-spaces of observables.

Much of the theory’s quasilocal structure depends on the fact that the space \( W_K \) is an \( MI \)-space. We now prove this fact. In particular, this will provide a representation of \( W_K \) as \( C(X) \).

Theorem III.12. The space \( W_K \) is an \( MI \)-space.
Proof. We show first that the union $\bigcup_{t \in J} \mathcal{W}(A^t)$ maps to a uniformly dense subspace of $\mathcal{W}^\infty$ under the injections $(\sigma_t)$. Fix any $[f] \in \mathcal{W}^\infty$, and any $f \in \tau^{-1}([f]) \in \bigvee_1 \mathcal{W}(A^t)$. For some countable set of indices $(t_k) \in J$, $f = \sum_{k=1}^{\infty} \sigma_{t_k} f^{s_k}$. For any given $\varepsilon > 0$, there exists $t_o \in \mathbb{N}$ such that for all $t > t_o$, $\|f - \sum_{k=1}^{t_o} \sigma_{s_k} f^{s_k}\| < \varepsilon$. From the composition rule, $\sigma_s = \sigma_t \eta_{t}$, so that $\sum_{k=1}^{t} \sigma_s f^{s_k} = \sum_{k=1}^{t_o} \sigma_t \eta_{t_k} f^{s_k} = \sigma_t \sum_{k=1}^{t_o} \eta_{t_k} f^{s_k}$. Write $g' = \sum_{k=1}^{t} \eta_{t_k} f^{s_k}$. By the definition of the quotient norm, $\|f - [g']\| < \varepsilon$.

We can readily show that the image $\tau(...)$ is a vector lattice. For any given pair $s, t \in J$, fix $f^s \in \mathcal{W}(A^s)$, $g^t \in \mathcal{W}(A^t)$. Since $J$ is upward directed, there exists a $u \in J$ such that $s, t \leq u$. From the composition rule, $\sigma_s f^s = \sigma_u \eta_u f^s$ and $\sigma_s g^t = \sigma_u \eta_u g^t$. Then $\eta_u f^s \vee \eta_u g^t \in \mathcal{W}(A^u)$ (a Banach lattice), so that $\sigma_u (\eta_u f^s \vee \eta_u g^t) = \sigma_u \eta_u f^s \vee \sigma_u \eta_u g^t$, because $\sigma_u$ is the natural injection. Since the surjection $\tau : \bigvee_1 \mathcal{W}(A^t) \to \mathcal{W}^\infty$ is order-preserving (Lemma III.3), $\sigma_u (\eta_u f^s \vee \eta_u g^t) = \sigma_u \eta_u f^s \vee \sigma_u \eta_u g^t = \sigma_u f^s \vee \sigma_u g^t$. That is, $[f^s] \vee [g^t] = [\eta_u f^s \vee \eta_u g^t] \in \mathcal{W}^\infty$. Furthermore, the subspace $\Delta_K \tau(...)$ is uniformly dense in $\mathcal{W}(K)$ because $\Delta_K(\mathcal{W}^\infty)$ is dense. Note especially that for any $f \in \bigvee_1 \mathcal{W}(A^t)$, $\|f\| = \|f\|$ from Proposition III.9 and the properties of the quotient norm on $\mathcal{W}^\infty$.

We show that the mappings $\Delta_K \circ \sigma_t : \mathcal{W}(A^t) \to \mathcal{W}(K)$ are 1:1. Recall first that for any $x \in \Omega$, the Dirac point functional $\delta(x)$ defined by $\delta(x)(f^t) \equiv f^t(x)$ is a state on $\mathcal{W}(A^t)$, i.e., $\delta(x) \in E_t$, for all $t \in J$. Furthermore, $\Delta_K \circ \sigma_t(f^t) = \phi_x(\sigma_t f^t) = \mu_x(f^t)$. Now suppose $f^t \neq g^t$. For some $x \in \Omega$, $\delta(x)(f^t) = \delta(x)(g^t)$. Then $\Delta_K \circ \sigma_t(f^t)(\delta(x)) = \Delta_K \circ \sigma_t(g^t)(\delta(x))$. Thus, $\Delta_K \circ \sigma_t(f^t) = \Delta_K \circ \sigma_t(g^t)$.

We show that for any $t \in J$, $\Delta_K \circ \sigma_t$ is order-preserving in both directions. Fix $f^t \geq g^t$. Then $\forall \mu \in E_t$, $\mu(f^t - g^t) \geq 0$, and therefore $\Delta_K \circ \sigma_t(f^t - g^t) \geq 0$. Hence, $\Delta_K \circ \sigma_t(f^t) \geq \Delta_K \circ \sigma_t(g^t)$. Conversely, suppose $\Delta_K \circ \sigma_t(f^t) \geq \Delta_K \circ \sigma_t(g^t)$. Then for all $\mu \in E_t$, $\mu(f^t) \geq \mu(g^t)$, and in particular $\delta(x)(f^t) \geq \delta(x)(g^t) \forall x \in \Omega$. Hence, $f^t(x) \geq g^t(x) \forall x \in \Omega$, and therefore $f^t \geq g^t$. It follows that $\Delta_K \circ \sigma_t$ is a lattice homomorphism.

We have thus shown that the image $\tau(...)$ is a uniformly dense linear subspace of $\mathcal{W}(K)$ and a normed vector lattice. Then its closure $\mathcal{W}(K)$ is a Banach lattice. The constant function $1_K \equiv 1 \in \mathcal{W}(K)$ is an order unit in $\mathcal{W}(K)$, i.e., $\|f\| \leq 1$ iff $\|f\| \leq 1_K$. Then $\mathcal{W}(K)$ is an M1-space (Theorem 13.2.4).

The space $\mathcal{W}(K) = \mathcal{A}(K)$, with $K = \mathcal{K}(\mathcal{W}^\infty)$, is the (essentially unique) order-unit completion of $\mathcal{W}^\infty$. We take it as the theory’s space of quasilocal observables as required by Axiom 4. We now generalize to allow a choice of $K$.

Corollary III.13. Let $K \subseteq \mathcal{K}(\mathcal{W}^\infty)$ be any nonempty compact convex set of states. Then the space $\mathcal{W}(K) = \mathcal{A}(K)$ of continuous affine functions on $K$ is
an MI-space. The Kadison function $\Delta_K : W^\infty \to W_K$ is an order-preserving mapping onto a dense subset of $W_K$, and the order unit $1_K \in W_K$.

Proof. Let $A(K; K(W^\infty))$ be the set of functions $A(K; W^\infty)$ restricted to $K$. $A(K; K(W^\infty))$ is a uniformly dense subset of $A(K)$ ([1, Corollary I.1.5], [28, 23.3.6]). For the restrictions, note that for all $\hat{f}, \hat{g} \in A(K; W^\infty)$, $\hat{f} \leq \hat{g} \Rightarrow |\hat{f}|_K \leq |\hat{g}|_K$. In particular, $|\hat{f}|_K \leq |e| \Rightarrow |\hat{f}|_K \leq |e|_K$. By the theorem, $A(K; W^\infty)$ is a Banach lattice, so that $A(K; K(W^\infty))$ is a normed vector lattice. Then $W_K$ is the completion of a normed vector lattice, and therefore, a Banach lattice [28, Proposition 3.9.5]. Since there can be no confusion in context, we shall also write $\hat{f} \in W_K$ to denote its elements.

Henceforth, $K \subseteq K(W^\infty)$ will always denote an arbitrary compact convex set of states.

E. Representation in $C(X_K)$.

Since $(W_K, 1_K)$ is a partially ordered Banach space with unit, we may characterize its states as follows.

Proposition III.14. The states on $W_K$, denoted $K(W_K)$, are a compact set consisting of the positive linear functionals on $W_K$ for which $\|\phi\| = \phi(1_K) = 1$.

Proof. The proof is similar to that in Proposition II.1. As the intersection of a wk*-closed hyperplane and the compact unit ball $\bigcap K(W_K)$ in the (topological) dual $W_K^*$ of $W_K$, $K(W_K)$ is compact.

The states on $W_K$ are related to those on $W^\infty$ by the following.

Proposition III.15. Let $K \subseteq K(W^\infty)$ be any compact convex set, and define $\alpha_K : K \to K(W_K)$ by $\alpha_K(\phi) = \hat{f}(\phi)$. Then $\alpha_K$ is an affine homeomorphism giving a parameterization or indexing of $K(W_K)$ by $x_\mu = \alpha_K(\phi_\mu)$.

Proof. [28, Theorem 23.2.3]. For the affine property, one has that for all $\mu, \nu \in K$ and $a \in (0, 1)$, $\alpha_K(a\mu + (1 - a)\nu)(\hat{f}) = \hat{f}(a\mu + (1 - a)\nu) = a\hat{f}(\mu) + (1 - a)\hat{f}(\nu) = (a\alpha_K\mu + (1 - a)\alpha_K\nu)(\hat{f})$. Note the dependence on choice of $K \subseteq K(W^\infty)$.

Corollary III.16. The set of extremal states $\partial_e K(W_K)$ is closed and therefore compact.

Proof. Clearly, $\alpha_K(\partial_e K) = \partial_e K(W_K)$. But $W_K = A(K)$ is a vector lattice. Hence, the set of states $K$ is a regular (or Bauer) simplex, i.e., a simplex for which the set of extremal points $\partial_e K$ is wk*-closed ([28, 23.7.1], [2]).

We are now able to define the algebra $C(X_K)$.
Theorem III.17. Let $X_K = \partial_e \mathcal{K}(\mathcal{W}_K)$. The mapping $\psi_K : \mathcal{W}_K \to \mathcal{C}(X_K)$ defined by $\psi_K(\hat{f})(x_\mu) = x_\mu(\hat{f})$ is an isometric vector-lattice isomorphism onto $\mathcal{C}(X_K)$ with $\psi_K(1_K) = 1_{X_K}$.

Proof. Apply the structure theorem for MI-spaces to the pair $(\mathcal{W}_K, 1_K)$ [28, Theorems 13.2.3, 13.2.4].

The MI spaces satisfy all the linear postulates of Segal algebra, but they do not have a vector multiplication needed to define powers. The isomorphism with $\mathcal{C}(X_K)$ permits us to assign the operation as follows.

Proposition III.18. Define vector multiplication on $\mathcal{W}_K$ by $\hat{f} \cdot \hat{g} = \psi_K^{-1}(\psi_K(\hat{f}) \cdot \psi_K(\hat{g}))$ for all $\hat{f}, \hat{g} \in \mathcal{W}_K$. Then $\mathcal{W}_K$ is a Segal algebra.

This completes the requirement of Axiom 4. In most of what follows, however, the representation of $\mathcal{W}_K$ as $\mathcal{C}(X_K)$ will be found to play the major role.

E. Choquet decompositions.

Compact convex sets of states play an important role in the modern theory of statistical mechanics. The theory of this class of states depends crucially on the unique decomposition of states into extremal (or pure) states. We now show that this result is assured by the fact that the set of extremal states $\partial_e \mathcal{K}(\mathcal{W}_K)$ is closed (Corollary III.16).

Let $\partial_e K$ be the set of extremal points of $K \subseteq \mathcal{K}\mathcal{W}^\infty$, and $S(\partial_e K)$ the set of Radon probability measures on $\partial_e K$ with the topology induced on it by the wk*-topology on $K$ under the Riesz representation theorem $S(\partial_e K) = \mathcal{K}\mathcal{C}(\partial_e K)$.

Theorem III.19. Let $K \subseteq \mathcal{K}\mathcal{W}^\infty$ be any compact convex set of states. Then its set of extremal states $\partial_e K$ is closed. Hence, for each state $\phi_\mu \in K$, there exists a unique Radon probability measure $\sigma'_\mu$ on $K$ with $\sigma'_\mu(\partial_e K) = 1$ such that

$$\hat{f}(\phi_\mu) = \int_{\partial_e K} \hat{f}(\phi) d\sigma'_\mu(\phi) \quad \forall \hat{f} \in \mathcal{W}_K$$

(3.5)

Let $r : K \to S(\partial_e K)$ map states to the corresponding probability measures, i.e., $r(\phi_\mu) = \sigma'_\mu$. Then $r$ is an affine homeomorphism onto $S(\partial_e K)$.

Proof. ([11, Theorem II.4.1]). The fact that $\partial_e K$ is closed (by Corollary III.16) assures the existence of the measure $\sigma'_\mu$ [28, 23.4.8]. The fact that $K$ is a simplex (by the proof of the same corollary) assures the uniqueness [28, 23.6.5].

Integrals of this form are called the Choquet decomposition ([11, [20]) of the given state $\phi_\mu \in K \subseteq \mathcal{K}(\mathcal{W}^\infty)$ into the set of pure states $\partial_e K$. A state $\phi_\mu$
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satisfying this equation is called the centroid (or resultant) of the probability measure $\sigma'_\mu$.

IV Indexing of states

Up to this point, we have seen two kinds of states. The one arises through the definition of the texture itself (Axiom 1), the mapping $\Lambda_t \mapsto \mathcal{W}(A^t)$. We immediately defined the corresponding state space $E_t = \mathcal{K}\mathcal{W}(A^t)$ (Proposition II.1) of local states $\mu_t \in E_t$ such that $\mu_t(f^t)$ is the expectation value of the observable $f^t \in \mathcal{W}(A^t)$ if the lattice is in state $\mu_t$. The other kind of state is the global state $\phi_{\mu} \in \mathcal{K}(\mathcal{W}_K)$ on the Segal algebra $\mathcal{W}_K$. For every global observable $\hat{f} \in \mathcal{W}_K$, the expectation value is $\phi_{\mu}(\hat{f})$. We know how to map any local observable $f^t$ to its global representation $\hat{f}$. In this section, we learn that every local state $\mu_t$ has a unique global representation and how to identify it.

We shall prove by traditional methods the following property of the category-theoretical limits of the states of the theory. It can be shown that the functor $\mathcal{K}$, which maps the $MI$-spaces $\mathcal{W}(A^t)$ to their sets of states $\mathcal{K}(\mathcal{W}(A^t))$ and the morphisms $\eta_{ts}^t : \mathcal{W}(A^s) \to \mathcal{W}(A^t)$ to the $\mathcal{K}(\eta_{ts}^t) = \eta_{ts}^t : E_t \to E_s$, is a directly continuous functor, i.e., it maps inductive limits to projective limits according to the rule

$$\mathcal{K}(\lim \rightarrow(\{\mathcal{W}(A^t), \eta_{ts}^t, J\})) = \lim \leftarrow(\{\mathcal{K}(\mathcal{W}(A^t)), \eta_{ts}^t, J\}) \quad (4.1)$$

The left-hand side gives the theory’s algebraic states. The right-hand side depends on the system $\{\mathcal{K}(\mathcal{W}(A^t)), \eta_{ts}^t, J\} \equiv \{E_t, \eta_{ts}^t, J\}$, which has already been introduced following Proposition III.1 (cf. [28] 11.8.6 and 23.3.2).

A. The limit $\{E_\infty, \rho_t, J\}$.

We begin by constructing the projective limit.

**Proposition IV.1.** The system $\{E_t, \eta_{ts}^t, J\}$ has a unique $\text{Compconv}$-projective limit $\{E_\infty, \rho_t, J\} = \lim \leftarrow \{E_t, \eta_{ts}^t, J\}$, with nonempty compact object set $E_\infty$ and continuous affine mappings $\rho_t : E_\infty \to E_t$ obeying the composition rule $\rho_s = \eta_{ts}^t \rho_t$ for all $t \geq s, s \in J$.

**Proof.** The transformations $(\eta_{ts}^t)_{s \leq t}$ obey the set of composition rules required to make the set $\{E_t, \eta_{ts}^t, J\}$ a projective system of compact convex spaces. The typical commuting diagram is as follows, $\forall r \leq s \leq t$: 
A proof from traditional topology then applies, as given, e.g., in [10, Theorem 3.2.10]. The construction in a category-theoretical setting is provided by [28, Proposition 11.8.2]. The limit object is the compact subspace of the Cartesian product of the \((E_t)_{s \leq t}\) defined by

\[
\lim^\leftarrow \{E_t, \eta_s^t, J\} = \{(\mu_t)_{t \in J} \in \mathcal{P}_{t \in J} E_t : \mu_s = \eta_s^t \mu_t \quad \forall s \leq t, \ s, t \in J\}
\]

Denote the limit object by \(E_\infty\). The limit morphisms \(\rho_t : E_\infty \to E_t\) are defined by \(\rho_t((\mu_s)_{s \in J}) = \mu_t\). □

The elements of \(E_\infty\) are commonly called threads.

**Proposition IV.2.** The mappings \(\rho_t : E_\infty \to E_t\) are injective, i.e., for any state \(\mu_t \in \rho_t(E_\infty)\) there exists exactly one thread \((\mu_t)_{t \in J} \in E_\infty\) such that \(\rho_t(\mu) = \mu_t\). Moreover, the transformations \(\eta_s^t : E_t \to E_s\) map the set \(\rho_t(E_\infty) \subseteq E_t\) 1:1 onto the set \(\rho_s(E_\infty) \subseteq E_s\).

**Proof.** Take the second part of the proposition first. To show that \(\eta_s^t\) is onto \(\rho_s(E_\infty)\), fix any \(\mu_s = \rho_s \mu\). Then \(\rho_t(\mu) \in E_t\), and \(\eta_s^t \rho_t \mu = \mu_s\). For \(\eta_s^t\) 1:1, fix any \(s \leq t\), and suppose that for some pair \(\mu_t, \nu_t \in \rho_t(E_\infty)\), \(\eta_s^t \mu_t = \eta_s^t \nu_t\). Then for all \(f^s \in \mathcal{W}(A^s)\), \(\mu_t(\eta_s^t f^s) = \nu_t(\eta_s^t f^s)\). In particular, for all \(f^t \in \mathcal{W}(A^t) \subseteq \mathcal{W}(A^s)\), \(\mu_t(\eta_s^t f^t) = \mu_t(f^t) = \nu_t(f^t)\), and therefore, \(\mu_t = \nu_t\). It follows that if \(\mu_t \neq \nu_t\), then \(\eta_s^t \mu_t \neq \eta_s^t \nu_t\). To show that \(\rho_s : E_\infty \to E_s\) is 1:1, fix any \(\mu, \nu \in E_\infty\), \(\mu \neq \nu\). Then there exists \(t \in J\) such that \(\mu_t \neq \nu_t\). Since the index set \(J\) is upward directed, there exists \(\mu \geq s, t\). Then \(\eta_s^u \rho_a \mu = \rho_t \mu \neq \rho_t \nu = \eta_t^u \rho_a \nu\). Hence, \(\rho_a \mu \neq \rho_a \nu\). But \(\eta_s^u\) is 1:1, so that \(\mu_s = \eta_s^u \mu_u \neq \eta_s^u \nu_u = \nu_s\). □

Let \(r_t^s : E_s \to E_t\) be the restriction operator such that \(r_t^s \mu_s = \mu_s|_{\mathcal{W}(A^t)}\). One then has from the Proposition that \(r_t^s \eta_s^t\) is the identity mapping \(\iota_{E_t}\) on \(E_t\). This may be incorporated into the following symmetrical form.

**Corollary IV.3.** The operators \(r_t^s\) and \(\eta_s^t\) are mutually inverse on the subspaces \(\rho_t(E_\infty)\) and \(\rho_s(E_\infty)\), i.e., for any comparable pair \(s \leq t\),

\[
\eta_s^t r_t^s = \iota_{\rho_t(E_\infty)} \quad \text{and} \quad \eta_s^t r_t^s = \iota_{\rho_s(E_\infty)}\]
Proof. We already have the one. For every \( \mu_r \in \rho_r(E_{\infty}) \), there exists a unique thread \( \mu \in E_{\infty} \) such that \( \mu_r = \rho_r \mu \). Then from the composition rule, \( \eta_s', \mu_r \mu = \eta_s' \eta_s \rho_r \rho_t \mu = \eta_s' \rho_g \rho_t \mu = \mu \).

Thus, higher components of a given thread are the restrictions of lower components, and the lower components extensions of the higher. Both the \( r^s \) and the \( \eta^s \) map every thread onto itself, \textit{i.e.,} there is no mixing of threads under these transformations. It follows immediately that if a state \( \mu_t \) for a particular system \( \Lambda_t \) belongs to a thread \( \mu = \rho \mu \), then it determines that thread uniquely, and hence the state on every other system in the space. Naturally, this mirrors thermodynamic equilibrium.

Now consider the defining condition for \( E_{\infty} \) (Proposition IV.1):

\[
\mu_s(f^s) = \mu_t(\hat{\eta^s}_t f^s) \quad \forall f^s \in \mathcal{M}(\mathcal{A}^t) \tag{4.2}
\]

Since \( \mu_s = \rho_s(\mu) \) and \( \mu_t = \rho_t(\mu) \) derive from the same quasilocal state \( \mu \in E_{\infty} \), they represent the same state on their respective systems \( \Lambda_s \subseteq \Lambda_t \). In introduction of the observables \( (\eta^s_t)_{t \in J} \), it was assumed that they map local observables \( f^s \in \mathcal{M}(\mathcal{A}^t) \) on the smaller space \( \Lambda^s \) to measurements \( \hat{\eta^s}_t f^s \in \mathcal{M}(\mathcal{A}^t) \) of the same physical quantity on the larger system \( \Lambda_t \supseteq \Lambda_s \). Eq. (4.2) is simply the requirement that equivalent measurements on nested systems have the same expectation value.

### B. The homeomorphism \( K \mathcal{M}_{\infty} = E_{\infty} \)

We now show \( K \mathcal{M}_{\infty} \) and \( E_{\infty} \) are homeomorphic spaces, eq. (4.1), by conventional means. This will provide the identification of the TL states to states on the Segal algebras \( \mathcal{M}_K \). The proof depends on construction of a new projective limit of compact spaces which is related to \( E_{\infty} \). For all \( t \in J \), define the mappings \( \bigcirc^s(\sigma_t) : \bigcirc^s \mathcal{M}_{\infty} \rightarrow \bigcirc^s \mathcal{M}(\mathcal{A}^t) \) by \( \bigcirc^s(\sigma_t) \phi(t^t) = \phi(\sigma_t f^t) \). (The notation is again from category theory.) Since the injection \( \sigma_t : \mathcal{M}(\mathcal{A}^t) \rightarrow \mathcal{M}_{\infty} \) is a linear contraction, the induced mapping \( \bigcirc^s(\sigma_t) \) is continuous. The mappings are in general into not onto. Fix any \( t \in J \), and since \( E_t \subseteq \bigcirc^s \mathcal{M}(\mathcal{A}^t) \), define \( F_{\infty} = \bigcirc^s(\sigma_t)^{-1}(E_t) \). The set \( F_{\infty} \) has the following properties.

**Proposition IV.4.** Let \( F_{\infty} = \bigcirc^s(\sigma_t)^{-1}(E_t) \) be the compact preimage in \( \bigcirc^s \mathcal{M}_{\infty} \) of the state space \( E_t \) for any \( t \in J \). Then for any other \( s \in J \), \( \bigcirc^s(\sigma_s)^{-1}(E_s) = F_{\infty} \), i.e., \( F_{\infty} \) is independent of choice of index. The morphisms \( \bigcirc^s(\sigma_t) \) satisfy the composition rule \( \bigcirc^s(\sigma_s) = \eta^s_t \bigcirc^s(\sigma_t) \) on \( F_{\infty} \).

**Proof.** Since \( E_t \) is compact, \( F_{\infty} \) is a closed subset of the compact set \( \bigcirc^s \mathcal{M}_{\infty} \), and therefore compact. Note that on any subspace \( \sigma_s(\mathcal{M}(\mathcal{A}^s)) \), the quotient surjection \( \tau : \sqrt{1} \mathcal{M}(\mathcal{A}^t) \rightarrow \mathcal{M}_{\infty} \) is contracting, since \( ||\tau(\hat{\sigma}_s f^s)|| = ||\sigma_s f^s|| = \inf_{h \in \mathcal{H}} ||f^s + h|| \leq ||f^s|| = ||\hat{\sigma}_s f^s||. \) Hence, \( ||f^s|| \leq 1 \) implies that \( ||\sigma_s f^s|| \leq 1. \). In fact, \( ||f^s|| \leq 1 \) iff \( ||\sigma_s f^s|| = ||f^s|| \leq 1, \) so that \( ||\sigma_s f^s|| = ||\tau(\hat{\sigma}_s f^s)|| \leq ||\hat{\sigma}_s f^s|| \leq 1. \)
1. To show that $\bigcirc^*(\sigma_s)\leftarrow (E_s) \supseteq \bigcirc^*(\sigma_t)\leftarrow (E_t)$, fix any $\phi \in \bigcirc^*(\sigma_t)\leftarrow (E_t)$. Then one has $\|\bigcirc^*(\sigma_s)\phi\| = \sup_{\|f\| \leq 1} |\bigcirc^*(\sigma_s)\phi(f^*)| \leq \sup_{\|\sigma_s f^*\| \leq 1} |\phi(\sigma_s f^*)| \leq \sup_{\|f\| \leq 1} |\phi(f)| = \|\phi\| \leq 1$, because $\phi \in \bigcirc^*(\mathcal{M}^\infty)$. But $\bigcirc^*(\sigma_t)\phi \in E_t$ implies that $\bigcirc^*(\sigma_t)\phi(\chi_{e\Omega}) = \phi(\sigma_t \chi_{e\Omega}) = \phi(e) = 1$. Then $\|\bigcirc^*(\sigma_s)\phi\| = \bigcirc^*(\sigma_s)\phi(1_{S_s}) = 1$, and therefore, $\bigcirc^*(\sigma_s)\phi$ is a state on $\mathcal{M}(\mathfrak{A}^s)$, by Proposition II.1. Show similarly that $\bigcirc^*(\sigma_t)\leftarrow (E_t) \supseteq \bigcirc^*(\sigma_s)\leftarrow (E_s)$. Now for any comparable pair $s \leq t$, one has that $\eta_t^s \bigcirc^*(\sigma_t) = \bigcirc^*(\sigma_t \eta_t^s)$, and then $\bigcirc^*(\sigma_t \eta_t^s) = \bigcirc^*(\sigma_s)$. 

We show the equivalence of $F_\infty$ and $E_\infty$.

**Proposition IV.5.** There exists a unique homeomorphic bijection $\beta : F_\infty \rightarrow E_\infty$ such that for any $t \in J$, $\bigcirc^*(\sigma_t) = \rho_t \beta$. Then $\{F_\infty, \bigcirc^*(\sigma_t), J\} = \text{lim}^{\leftarrow} \{E_t, \eta^t_s, J\}$.

**Proof.** The composition rule $\bigcirc^*(\sigma_s) = \eta^t_s \bigcirc^*(\sigma_t)$ makes the following diagram commuting:

From the uniqueness properties of the Compconv-projective limit $\{E_\infty, \rho_t, J\}$ [28 11.8.1], one therefore has that there exists a unique commuting homeomorphism $\beta : F_\infty \rightarrow E_\infty$ satisfying the composition rule $\bigcirc^*(\sigma_t) = \rho_t \beta \ \forall t \in J$:
Moreover, if \( \beta \) is onto, then \( F_\infty \) is a new (but equivalent) Compcon-projective limit of the system \( \{ E_t, \eta_t, J \} \).

Fix any \( \mu = (\mu_t)_{t \in J} \in E_\infty \), and define \( \phi_\mu : \mathcal{W}^\infty \to \mathbb{R} \) as follows. For all \( n \in \mathbb{N} \), let \( \phi_\mu(\sum_{k=1}^n \sigma_{t_k}(f_{t_k})) = \sum_{k=1}^n \mu_{t_k}(f_{t_k}) \) for any finite set of indices \( (t_k) \in J \). If \( [f] = \lim_{n \to \infty} \sum_{k=1}^n \sigma_{t_k}(f_{t_k}) \), let \( \phi_\mu([f]) = \lim_{n \to \infty} \sum_{k=1}^n \mu_{t_k}(f_{t_k}) \). The limit exits for all \( [f] \in \mathcal{W}^\infty \) because \( |\sum \mu_{t_k}(f_{t_k})| \leq \sum |\mu_{t_k}(f_{t_k})| \leq \sum \|f_{t_k}\|_1 < \infty \). (cf. Proof, Theorem III.3.) We seek to define \( \alpha : E_\infty \to F_\infty \) by \( \alpha \mu = \phi_\mu \).

Clearly, the definition of \( \phi_\mu \) assures that it is linear. In order for \( \phi_\mu \) to be well-defined on \( \mathcal{W}^\infty \), it must be independent of the choice of representation of \([f] \). Since \([f] = [g] \) implies that \([f] - [g] = [f - g] = [0] \), it suffices to show that \( \phi_\mu([0]) = 0 \). The preimage of \([0] \) under the quotient surjection is the set \( \mathcal{M} \), the closed linear span of elements of \( \bigvee I \mathcal{W}(\mathcal{A}^t_k) \) of the form \( \hat{\sigma}_s f^s - \hat{\sigma}_t \eta_t^s f^s \). But for any such pair, \( \phi_\mu(\hat{\sigma}_s f^s - \hat{\sigma}_t \eta_t^s f^s) = \mu_s(f^s) - \mu_t(\eta_t^s f^s) = (\mu_s - \eta_t^s \mu_t)(f^s) = 0 \). From the linearity of \( \phi_\mu \), any element of the linear span of such elements maps to zero, and therefore any element of the closed span, since \( \sum \mu_t(h^t) \leq \|h\|_1 \) \( \forall h = (h^t)_{t \in J} \in \mathcal{M} \).

The functional \( \phi_\mu \) is a state on \( \mathcal{W}^\infty \). For note that \( \phi_\mu(\sigma_t(\chi_\Omega)) = \mu_t(\chi_\Omega) = 1 \). Moreover, \( \phi_\mu \geq 0 \), i.e., \([f] \geq 0 \) implies that \( \phi_\mu([f]) \geq 0 \). In fact, from Lemma III.3, \([0] \leq [f] \) iff for any \( f \in [f] \), there exists \( h \in \mathcal{M} \) such that \( 0 \leq f + h \). Represent \( f + h = \sum \hat{\sigma}_t f^t + h^t \). By definition of the \( l_1 \)-join, \( 0 \leq f + h \) implies that \( 0 \leq f^t + h^t \) \( \forall t \in J \). Since \( \mu_t \geq 0 \) by definition, \( \forall t \in J, \mu_t(f^t + h^t) \geq 0 \), and hence \( 0 \leq \sum \mu_t(h^t) \leq \|h\|_1 \) \( \forall h = (h^t)_{t \in J} \in \mathcal{M} \).

Define \( \beta = \alpha^{-1} \). To show that \( \beta \) is defined on all of \( F_\infty \), fix any \( \phi \in F_\infty \), and define \( \mu_t = \bigcirc^*(\sigma_t) \phi \forall t \in J \). Then \( \mu = (\mu_t)_{t \in J} \) is a thread, i.e., \( \mu \in E_\infty \), from the composition rule in Proposition IV.4. Note especially that \( \phi(\sum \sigma_t f^t) = \sum \mu_t(f^t) \), from the linearity and (strong) continuity of \( \phi \). For the composition rule for \( \beta \), one has by construction that \( \phi_\mu \circ \sigma_t = \mu_t \forall t \in J \). Then for any \( t \in J \), \( \bigcirc^*(\sigma_t) \phi_\mu(f^t) = \phi_\mu(\sigma_t f^t) = \mu_t(f^t) \forall f^t \in \mathcal{W}(\mathcal{A}^t) \), and therefore \( \bigcirc^*(\sigma_t) \phi_\mu = \mu_t = \rho_t \beta(\phi_\mu) \). This proves the composition rule as stated. To show that \( \beta \) is injective, let \( \phi \neq \psi \) be any two elements of \( F_\infty \). Then there exists an \([f] \in \mathcal{W}^\infty \), say \([f] = [\hat{\sigma}_t f^t] \), such that \( \phi([f]) \neq \psi([f]) \). Hence, for some \( t \in J \), \( \bigcirc^*(\sigma_t) \phi \neq \bigcirc^*(\sigma_t) \psi \), so that \( \rho_t(\beta \phi) \neq \rho_t(\beta \psi) \), from the composition rule. Thus, \( \beta \phi \neq \beta \psi \).

The proof is completed by showing that \( \beta \) is continuous and therefore open, because \( F_\infty \) is compact. The subbasic neighborhoods (in the product topology) of \( \mu \in E_\infty \) are of the form \( U(\mu; f^t, \varepsilon) = \{ \nu \in E_\infty : |\mu_t f^t - \nu_t f^t| < \varepsilon \} \) for any \( t \in J, f^t \in \mathcal{W}(\mathcal{A}^t), \varepsilon > 0 \). The preimage of this set \( \beta^{-1}(U(\mu; f^t, \varepsilon)) = \{ \phi \in F_\infty : |\phi(\sigma_t f^t) - \phi_\mu(\sigma_t f^t)| < \varepsilon \} \). But this is the standard form \( N(\phi_\mu; \sigma_t f^t, \varepsilon) \) of the subbasic neighborhoods of \( \phi_\mu \) for the \( \text{wk}^* \)-topology of \( F_\infty \).
Proof. Note first that \( K^\infty \) is the intersection of a closed hyperplane and the compact unit ball \( \bigcap^*\mathfrak{M}^\infty \), and hence compact (cf. Proof, Proposition II.1). We already have that \( \beta^{-1}\mu = \phi_\mu \) is a state on \( \mathfrak{M}^\infty \), and therefore \( F_\infty \subseteq K^\infty \). For the reverse inclusion, fix any \( \phi \in K(\mathfrak{M}^\infty) \). Then \( \bigcap^*(\sigma_t)\phi(\lambda\Omega) = \phi(\sigma_t\lambda\Omega) = \phi(e) = 1 \). For any \( f^t \in \mathfrak{M}(\mathfrak{A}^t) \) with \( f^t \geq 0 \), \( \sigma_t f^t = [f^t] \geq 0 \), and hence \( \bigcap^*(\sigma_t)\phi(f^t) = \phi([f^t]) \geq 0 \) since \( \phi \geq 0 \). Therefore \( \bigcap^*(\sigma_t)\phi \in E_t \forall t \in J \), and thus, \( \phi \in \bigcap^*(\sigma_t)^\infty(E_t) = F_\infty \). Then \( F_\infty \supseteq K(\mathfrak{M}^\infty) \). ■

Combination of Proposition IV.5 and its corollary yields \( K^\infty = F_\infty = E_\infty \). We have thus proven the following.

**Theorem IV.7.** There is a 1:1 correspondence between the threads in \( E_\infty \) and the elements of \( K^\infty \). It therefore makes sense to express this correspondence as \( \phi_\mu \leftrightarrow \mu \), subscripting the elements \( \phi \in K^\infty \) with the corresponding threads \( \mu \in E_\infty \). For any system \( \Lambda_t \), the expectation value of a measurement \( f^t \in \mathfrak{M}(\mathfrak{A}^t) \) is fixed by the relation

\[
\mu_t(f^t) = \phi_\mu(\sigma_t f^t)
\]

(4.3)

i.e., by the value assigned to that property by the state \( \phi_\mu \), for any state \( \mu \in E_\infty \). The correspondence is defined by a unique homeomorphic bijection, affine in both directions, between the compact spaces \( K = K^\infty \) and \( E_\infty \).

Proof. To show that the transformation \( \mu \mapsto \phi_\mu \) is affine, i.e., that for all \( \mu, \nu \in E_\infty \) and for any \( \lambda \in (0, 1) \), \( \lambda \mu + (1 - \lambda)\nu \mapsto \lambda \phi_\mu + (1 - \lambda)\phi_\nu \), note that \( \lambda \mu + (1 - \lambda)\nu \mapsto \phi_{\lambda \mu + (1 - \lambda)\nu} \), and for any finite set \((t_k)^n \) of indices, \( \phi_{\lambda \mu + (1 - \lambda)\nu}(\sum^k \sigma_{t_k} (f^{t_k})) = \sum^k (\lambda \mu_{t_k} + (1 - \lambda)\nu_{t_k})(f^{t_k}) = (\lambda \phi_\mu + (1 - \lambda)\phi_\nu)(\sum^k \sigma_{t_k} (f^{t_k})) \). For the other direction, fix any \( t \in J \) and \( f^t \in \mathfrak{M}(\mathfrak{A}^t) \). Then \( (\lambda \phi_\mu + (1 - \lambda)\phi_\nu)(\sigma_t f^t) = \lambda \phi_\mu (f^t) + (1 - \lambda)\phi_\nu (\sigma_t f^t) \rightarrow (\lambda \mu_t + (1 - \lambda)\nu_t)(f^t) = \rho_t(\lambda \mu_t - (1 - \lambda)\nu_t)(f^t) \). ■

**C. Identification of TL states with** \( E_\infty \).

We conclude this section by showing the close relationship of the algebraic and TL approaches announced in the Introduction. It is important to be able to apply theorems from the algebraic structure such as the Choquet decomposability to the TL states. However, this relationship is also needed by the algebraic theory itself. The calculation of expectation values requires much more information about the lattice than is assumed by the algebraic theory. Since this information is embodied in TL calculations, we may solve the problem by applying the TL values themselves to the algebraic theory via the relationship of the programs.
**THEORY OF MEASUREMENT**

By a TL state $\mu$ we mean an expectation-value operator for all bounded Borel-measurable functions $\mathcal{M}(A)$ on the phase space $\Omega$ of the infinite lattice (cf. [25, p.14]). One has the following identification.

**Proposition IV.8.** Every TL state $\nu$ on $(\Omega, A)$ is related to a unique thread $(\mu_t)_{t \in J} \in E_\infty$ by its restrictions $\nu_t = \nu|_{\mathcal{M}(A^t)}$ to the individual $\mathcal{M}(A^t)$, such that for any system $\Lambda_t$, $\nu_t(f^t) = \mu_t(f^t)$ for all $f^t \in \mathcal{M}(A^t)$.

*Proof.* Consider the net of restrictions $(\nu_t)_{t \in J}$, where $\nu_t = \nu|_{\mathcal{M}(A^t)} \forall t \in J$. Homogeneity requires that for all $s, t \in J$ with $s \leq t$, the two equivalent observables $f^s \in \mathcal{M}(A^s)$ and $\hat{\nu}^t_t f^s \in \mathcal{M}(A^t)$ have the same expectation value with respect to $\nu$, i.e., that $\nu_sf^s = \nu_t \hat{\nu}^t_t f^s \forall f^s \in \mathcal{M}(A^s)$ and $\forall s \in J$ and $\forall t \geq s$. But this is eq.(4.2). Hence, $(\nu_t)_{t \in J} \in E_\infty$. The preceding proposition says that the net of projections is a unique identification of the thread.

Combining this with the fact that the set of algebraic states is homeomorphic with the set $E_\infty$, we now have that each TL state is uniquely identifiable by its expectation values with an algebraic state. We observe that the ability to form the restrictions $\nu_t = \nu|_{\mathcal{M}(A^t)}$ in this important result requires the construction to be based on functions from the outside.

TL states are commonly described in terms of a transformation $\alpha_\Lambda : \mathcal{K}\mathcal{M}(A) \to \mathcal{K}\mathcal{M}(A_\Lambda)$ that maps states on $\mathcal{M}(A)$ to states on the set of Borel-measurable functions on the configuration space $\Omega_\Lambda$ of any finite system $\Lambda$. To see that this is not something different, define $\alpha'_\Lambda : \Omega \to \Omega_\Lambda$ restricting $x \mapsto x_\Lambda$ and $\alpha_\Lambda \sigma(f_\Lambda) = \sigma(f_\Lambda \circ \alpha')$ for all $f_\Lambda \in \mathcal{M}(A_\Lambda)$ and for any state $\sigma$ on $\mathcal{M}(A)$ ([25, p.14]).

**D. Indexing of algebraic states.**

**Proposition IV.9.** The transformation $\delta_K : \mathcal{K}(\mathcal{M}_K) \to \mathcal{K}(\mathcal{C}(X_K))$ defined by $\delta_K x_\mu(f) = x_\mu(\psi_K^{-1} f)$ is an affine homeomorphism onto $\mathcal{K}(\mathcal{C}(X_K))$, and $\delta_K(X_K) = \partial K \mathcal{C}(X_K)$. Then $\delta_K$ extends the indexing of states by the definition $\zeta_\mu = \delta_K x_\mu \forall \mu \in E_\infty$.

*Proof.* The preceding proposition indexes $\mathcal{K}(\mathcal{M}_K)$ with $x_\mu(\hat{f}) \equiv \alpha_K \phi_\mu(\hat{f}) = \hat{f}(\phi_\mu)$. By Choquet’s theorem, the relation $\hat{f}(\phi_\mu) = \int_{\partial K} f(\phi) d\sigma'_\mu(\phi)$ uniquely identifies $\sigma'_\mu$ with $\phi_\mu$ and therefore $\mu$-indexes $\mathcal{S}(\partial K)$ in terms of the homeomorphism $\mathcal{S}(\partial K) = K$. By the usual integral transformation theorem [28, Proposition 18.3.3] and the fact that both $\alpha_K$ and $\psi_K$ are invertible, the mapping $\sigma'_\mu \mapsto \sigma'_\mu \circ \alpha_K^{-1} \equiv \sigma_\mu$ is a bijection from $\mathcal{S}(\partial K)$ onto $\mathcal{S}(X_K)$ defining a $\mu$-indexing on $\mathcal{S}(X_K)$. One has, for any $\mu \in E_\infty$ with representation in $K$,

$$\int_{X_K} f(x) d\sigma_\mu(x) = \int_{\partial K} f(\alpha_K(\phi)) d\sigma'_\mu(\phi) \quad (4.4)$$
For all $\phi_\nu \in \partial_s K$, $f(\alpha_K(\phi_\nu)) = \alpha_K\phi_\nu = \hat{f}(\phi_\nu)$, and hence $f(x_\nu) = x_\nu(\hat{f}) = \hat{f}(\phi_\nu)$, where $\hat{f} = \psi_K^{-1}(f)$. The integral becomes
\[
\int_{X_K} f(x)d\sigma_\mu(x) = \int_{\partial_s K} \hat{f}(\phi_\nu)d\sigma'_\mu(\phi_\nu) = \hat{f}(\phi_\mu)
\] (4.5)
where the equality on the right is from Choquet’s theorem again. That is, $\sigma_\mu \in S(X_K)$ is the unique probability measure on $X_K$ satisfying this relation. By the Riesz representation theorem, $\mathcal{KC}(X_K) = S(X_K)$, so that this result likewise $\mu$-indexes $\mathcal{KC}(X_K)$ with the definition $\zeta_\mu(f) = \hat{f}(\phi_\mu)$.

To define a mapping $\delta_K : \mathcal{K}(\mathfrak{W}_K) \to \mathcal{KC}(X_K)$, note first that for all $x_\mu \in X_K$, $f(x_\mu) = x_\mu(\hat{f}) = \hat{f}(\phi_\mu)$. Hence, $\sigma_\mu = \delta(x_\mu)$ (the Dirac point functional), so that $\zeta_\mu(f) = f(x_\mu)\forall f \in \mathcal{C}(X_K)$. This is in fact a necessary and sufficient condition for $x_\mu \in X \equiv \partial_s K(\mathfrak{W})$. Note that the condition defines a $\mu$-indexing for the extremal states $X$. Define $\delta_K : X \to \mathcal{KC}(X_K)$ by $\zeta_\mu(f) = \delta_K(x_\mu)(f) = f(x_\mu)$. Now for all convex combinations $(a_n) \in \mathbb{R}$ and sets $(x_{\mu_n})_n \in X_K$, define $\delta_K(\sum a_n x_{\mu_n}) = \sum a_n \delta_K(x_{\mu_n})$. This extends $\delta_K$ to all of $\mathcal{K}(\mathfrak{W}_K)$, because by the Krein-Milman theorem, the compact convex set $\mathcal{K}(\mathfrak{W}_K)$ is the closed convex hull of its extremal points. Clearly, $\delta_K : \mathcal{K}(\mathfrak{W}_K) \to \mathcal{KC}(X_K)$ is 1:1, because the $\mu$-indexing is unique. Since it is affine, $\delta_K$ maps extremal points to extremal points. Since $\mathcal{KC}(X_K) = S(X_K)$, and all Dirac point functionals correspond to some $x_\mu \in X$, $\delta_K$ is onto $\mathcal{KC}(X_K)$. To show that it is also continuous and open, consider the wk*-subbasic set $\mathcal{N}(x_{\mu}; \tilde{f}, \epsilon) = \{x_\nu : |x_\nu(\tilde{f}) - x_{\mu}(\hat{f})| < \epsilon\}$. One has $\delta_K(\mathcal{N}(x_{\mu}: \tilde{f}, \epsilon)) = \{\delta_K x_\nu : |\hat{f}(\phi_\nu) - \hat{f}(\phi_\mu)| < \epsilon\} = \{\delta_K x_\nu : |\zeta_\mu(f) - \zeta_\mu(f)| < \epsilon\} = \mathcal{N}(\zeta_\mu: \tilde{f}, \epsilon)$. That is, $\delta_K$ and $\delta_K^{-1}$ map subbasic sets onto subbasic sets.

Eq. (4.5) allows us to write the exp.v. in a familiar form. For any system $\Lambda_t$ and observable $f^t \in \mathfrak{W}(\xi)$, let $\hat{f}$ be the image of $f^t$ in $\mathfrak{W}_K$, so that for any $\mu_t \in E_t$, $\hat{f}(\phi_\mu) = \mu_t(f^t)$. We then have
\[
\zeta_\mu(f) = \int_{X_K} f(x)d\sigma_\mu(x) = \mu_t(f^t) \quad \forall f \in \mathcal{C}(X_K),
\] (4.6)
for all states $\zeta_\mu \in \mathcal{KC}(X_K)$.

In the algebraic QFT, the GNS construction defines a representation of the theory’s quasilocal observables as bounded linear operators on a certain abstract Hilbert space, with expectation values calculated by inner products of the form $(\psi, A\psi)$. That is, the representation brings the algebraic theory into the form of ordinary quantum mechanics. We have now seen that the representation theorem in the classical algebraic theory represents its quasilocal observables as $\mathcal{C}(X_K)$, continuous functions on a certain compact “phase
space," with expectation values calculated as integrals over that space. Thus, the representation theorem brings the algebraic theory into the form of ordinary CSM.

V Applications

Algebraic theory has to do with the abstract triple \( \{\mathcal{C}(X), \mathcal{K}\mathcal{C}(X), X\} \), where \( X \) is a compact Hausdorff space. The role of the Haag-Kastler axioms is to create a frame for interpreting mathematical conclusions about this triple in terms of a particular underlying lattice problem. Let us display the whole hierarchy of spaces defined in the algebraic construction:

\[
\begin{align*}
\mathcal{K}\mathcal{C}(X_K) &= \mathcal{G}_K \\
\mathcal{C}(X_K) &= \mathcal{K}\mathcal{W}_K \\
X_K &\subset \mathcal{K}\mathcal{W}_K \\
A(K) &\equiv \mathcal{W}_K \\
K &\subset \mathcal{K}\mathcal{W}^\infty = \mathcal{E}_\infty \\
\mathcal{W}(\mathcal{A}) &\rightarrow \mathcal{E}_\infty \rightarrow \mathcal{W}^\infty
\end{align*}
\]

We have underscored the equivalence of the threads and algebraic states in the next-to-last line. (Recall in particular that the primary identification of the TL states is with \( \mathcal{E}_\infty \).) The effect of the frame is to make everything above that line the theory of a particular choice of compact convex set \( K \subset \mathcal{K}\mathcal{W}^\infty \). In this section and the next, we study three distinct choices of \( K \). The purpose is to illustrate the importance of this class of sets in physics and the effectiveness of the theory in studying these sets provided by the freedom in the choice of \( K \).

A. Compact convex sets

The compact convex sets arise in statistical mechanics because of their connection with extremal states. These states are regarded as representing pure thermodynamic phases of a problem. These states are readily identified in the algebraic setting as the multiplicative states on \( \mathcal{C}(X) \) [28, Cor.4.5.4], the property that accounts for the zero variance of observables in these states. We may use the freedom in the choice of \( K \) to match the algebraic problem with the physical problem as follows.

**Proposition V.1.** Fix any compact convex set \( K \subset \mathcal{K}\mathcal{W}^\infty \). We may define a set \( X \) compact such that the states on \( \mathcal{C}(X) \) are isomorphic with \( K \), and \( X \) to the set \( \partial_k K \) of its extremal states. The triple \( \{X, \mathcal{C}(X), \mathcal{K}\mathcal{C}(X)\} \) so constructed is uniquely fixed by either \( \mathcal{K}\mathcal{C}(X) \) or \( X \). For all states \( \zeta \in \mathcal{K}(\mathcal{C}(X)) \), there exists a unique Radon probability measure \( \sigma \) on \( X \) such that
\[ \zeta(f) = \int_{X_K} f(x) d\sigma(x) \quad \forall f \in \mathcal{C}(X_K). \]

**Proof.** Set \( \mathcal{W}_K = \mathcal{A}(K) \), and \( X_K = \partial_e \mathcal{K}(\mathcal{W}_K) \). Then by Propositions III.15 and IV.9, \( \delta_K \alpha_K(K) = \mathcal{K}C(X_K) \), and \( X_K = \delta^{-1}_K(\partial_e \mathcal{K}C(X_K)) \). The set \( \partial_e \mathcal{K}C(X_K) \) is identified as the set of multiplicative states in \( \mathcal{K}C(X_K) \). By Proposition IV.9, the isomorphism \( \delta_X^{-1} : \mathcal{K}C(X_K) \to \mathcal{K}W_K \) maps \( \delta_e \mathcal{K}C(X_K) \) onto \( X_K \). Conversely, the set of extremal states \( \delta_K(X_K) = \partial_e \mathcal{K}C(X_K) \) determines its closed convex hull \( \mathcal{K}C(X_K) \) by the Krein-Milman Theorem. The integral result is given by the Riesz Representation Theorem.

The freedom in matching the abstract algebra to particular problems afforded by this Proposition is analogous to a flexibility in the QFT described by Emch as the essential advantage of the algebraic approach over traditional theories based on Fock space [8, p.78]. It is important to note that the choice of \( K \) in this Proposition does not restrict the number of observables. In fact, Corollary III.13 assures that the algebra \( \mathcal{C}(X_K) \) contains all the observables of the theory, for any \( K \). That is, each local observable \( f \in \mathcal{W}(A) \) maps to a unique element \( f \in \mathcal{C}(X_K) \), with its expectation value given by eq.(4.6).

Because of the identification of the extremal states with pure phases, the decomposition of states into pure states is identified with phase separation. Clearly we expect on physical grounds that the most important states are the extremal states themselves or those states that decompose into a small number of extremal states given by the Gibbs phase rule. Since the extremal property must be defined with respect to a particular compact convex set, the appearance of extremal states signifies that the physical situation itself defines a certain compact convex set of states as available to the system, especially by the equilibrium condition. The most common cases are spaces of states invariant under a particular symmetry, the equilibrium (Gibbs) states, or an intersection of these. According to the preceding Proposition, if we set \( K \subset \mathcal{K}W^\infty \) equal to the set of available states in a particular problem, then all states on \( \mathcal{C}(X_K) \) are “available,” and only these. We illustrate these principles in the following applications.

**B. Symmetry groups.**

The first application comes from the study of symmetries, following the form and notation of Ruelle [24]. A **symmetry** is an automorphism on the lattice that leaves the expectation values of the theory unchanged. A **symmetry group** is a set of symmetries with the group property. The symmetry groups are usually defined in terms of a group \( G \), and a transformation \( \tau : G \to \text{aut}(\mathcal{P}) \) mapping \( G \) to the automorphisms on the set \( \mathcal{P} \) of finite systems of the lattice. Since we are concerned with the compact convex sets of states \( K \subset \mathcal{K}W^\infty \), we
need to transform automorphisms on the lattice up to the set \( \text{aut}(\mathcal{M}^\infty) \) on \( \mathcal{M}^\infty \). Without danger of confusion, we use the same notation \( \tau_a \) to denote the corresponding transformation at each level. For simplicity, we also fix, once and for all, a particular \( a \in G \).

The local transformations are as follows. Define \( \tau_a : \mathcal{M}(\mathfrak{A}_t) \to \mathcal{M}(\mathfrak{A}_t) \) by \( \tau_a f^t(x) = f^t(\tau_a^{-1}x) \forall f^t \in \mathcal{M}(\mathfrak{A}_t) \). For the states, define \( \tau_a : E_t \to E_t \) by \( \tau_a \mu f^t = \mu_t(\tau_a f^t) \). Now for \( \tau_a : G \to \text{aut}(\mathcal{M}^\infty) \) itself, the linear subspace \( \mathcal{M} \) is generated by pairs of the form \( \tilde{\sigma}_s f^s - \tilde{\sigma}_t \hat{\eta}_s f^s = (\tilde{\sigma}_s - \tilde{\sigma}_t \hat{\eta}_s) f^s \). If we define \( \tau_a \sigma_t = \sigma_t \circ \tau_a \forall t \), then \( \tau_a (\tilde{\sigma}_s f^s - \tilde{\sigma}_t \hat{\eta}_s f^s) = (\tilde{\sigma}_s - \tilde{\sigma}_t \hat{\eta}_s) (\tau_a f^s) \in \mathcal{M} \). Hence \( \tau_a \mathcal{M} \subseteq \mathcal{M} \), i.e., the subspace \( \mathcal{M} \) is closed under \( \tau_a \). Then \( \tau_a \) does not disrupt equivalence classes, and we may define \( \tau_a \in \text{aut} (\mathcal{M}^\infty) \) by \( \tau_a[f] = [\tau_a f] \) on \( \mathcal{M}^\infty \).

Let \( \mathcal{L}_G \) be the (closed) linear subspace of \( \mathcal{M}^\infty \) of elements of the form \([g] = [f] - \tau_a[f] \) for any \( a \in G \), and define the set of states \( \mathcal{L}^1_G = \{ \phi \in \mathcal{K}\mathcal{M}^\infty : \phi[g] = 0, \, [g] \in \mathcal{L}_G, \, a \in G \} \). Clearly \( \mathcal{L}^1_G \) is \( \text{wk}^* \)-closed, \( \phi \) linear, and therefore \( \mathcal{L}^1_G \) is a compact convex subset of \( \mathcal{K}\mathcal{M}^\infty \). Then \( \mathcal{L}^1_G \) is the set of \( G \)-invariant states, i.e., states with expectation values invariant under transformations of the group \( G \). Its extremal states \( \partial_e(\mathcal{L}^1_G) \) are called the \( G \)-ergodic states. We take \( \mathcal{L}^1_G \) as the set of available states, and set \( K = \mathcal{L}^1_G \). Then the set of states on \( \mathcal{C}(X_K) \) is exactly the set of \( G \)-invariant states, and every \( G \)-invariant state admits a unique decomposition into \( G \)-ergodic states \( X_K = \partial_e \mathcal{K}\mathcal{C}(X_K) \).

The phenomenon of breakdown of symmetries gives a particularly clear picture of available states. For nested pairs of compact convex sets \( K_1 \subset K_2 \subset \mathcal{K}\mathcal{M}^\infty \), the extremal sets of \( K_1 \) are not generally extremal for \( K_2 \). Let the elements of \( K_1 \) show a certain symmetry, and suppose the state \( \phi \in K_1 \) is extremal. Then \( \phi \) is a pure thermodynamic phase with that symmetry property if the only available states are elements of \( K_1 \). But suppose instead that the set of available states is \( K_2 \), and \( K_2 \) does not possess this symmetry. We set \( K = K_2 \). If \( \phi \in \partial_e K_1 \cap (\partial_e K_2)^\prime \), then \( \phi \) is no longer extremal, but decomposes into elements of \( \partial_e K_2 \) that may not have the symmetry. We say that the symmetry has been broken. The rule is as follows: the opportunity for symmetry breakdown arises whenever the invariant set is introduced into a larger set of available states that are not all invariant.

Now suppose the group \( G \) contains a subgroup \( H \) which is energetically favored, so that only \( H \)-invariant states are available. We define as above \( \mathcal{L}_G \) and \( \mathcal{L}_H \). Clearly, \( \mathcal{L}_G \supset \mathcal{L}_H \). Since it is a stronger condition to be invariant on the larger set, \( \mathcal{L}^1_G \subset \mathcal{L}^1_H \). Since \( \mathcal{L}^1_H \) is now the available set, we take \( K = \mathcal{L}^1_H \). Then a \( G \)-ergodic state \( \phi \in \partial_e \mathcal{L}^1_G \cap (\partial_e \mathcal{L}^1_H)^\prime \) will not be represented in \( X_K \), i.e., \( \alpha_K \phi \notin X_K \). Hence, the state \( \phi \) is not extremal, but is instead decomposed into \( H \)-ergodic states in \( X_K \). We say that the \( G \)-symmetry is broken.

C. Gibbs states.
The Gibbs states of the theory are identified as those threads \( \mu \in E_\infty \) with components \( \mu_t \in E_t \) compatible with assignment of a traditional Gibbs distribution as a conditional distribution to each finite system in the space, as assured by the DLR equations. One has the result from the TL program that a translation-invariant state is an equilibrium state if, and only if, it is a Gibbs state \[25, \text{Thm.4.2}\]. Denote the invariant states on \( W^\infty \) by \( I \), and the set of all Gibbs states by \( G \). Both are compact convex sets. The invariant equilibrium states are the intersection \( I \cap G \). With \( K = I \cap G \), the states in \( \partial_e(I \cap G) \) are thermodynamic pure phases. But if all Gibbs states are energetically available, then we set \( K = G \subseteq K W^\infty \). Since \( I \cap G \subseteq G \), the above rule applies. Any invariant state in the intersection \((\partial_e(I \cap G)) \cap (\partial_e G)\)' decomposes into extremal Gibbs states that are not invariant. One says that the translational invariance of the theory is broken \[25, 4.3\].

D. Stationary states.

We conclude with the construction of the most basic set of states in classical statistical mechanics, the stationary states. Let \( E \subseteq W^\infty \) be the set of all microcanonical (MC) states on the lattice, and let \( K \) be the closed convex hull \( \text{co}(E) \) of \( E \). By MC states, we mean those states in \( W^\infty \) identified with TL states \( \mu = (\mu_t)_{t \in J} \in E_\infty \) whose components are the projections of a given MC state.

**Proposition V.2.** The set \( K \) is a compact convex set of states, and \( E = \partial_e K \).

**Proof.** The closed convex set \( \text{co}(E) \) is the same as the closure of \( \text{co}(E) \) \[7, \text{Theorem V.2.4}\]. But the closure of a convex set is convex \[7, \text{Theorem V.2.1}\]. Hence, \( K \) is a compact convex set, and we may use it to define the triple \( \{C(X_K), K C(X_K), X_K\} \). Now \( \phi_\mu \in K \) is an extremal state iff \( \zeta_\mu \in K C(X_K) \) is extremal, for given \( \mu \in E_\infty \). The extremal states of \( K C(X_K) \) are multiplicative, so that in particular, the energy density has 0 variance on \( X_K \). But this is true iff \( \mu \in E_\infty \) is a MC state. \( \blacksquare \)

The MC states are specified by pairs of values of the energy and particle-number densities, related to the two constants of the motion. Since all stationary distributions are written as Borel functions of these two constants, they may be regarded as distributions over the set of MC states. Since \( K \) is a compact convex set, we may choose it to define \( C(X_K) \). Then by the Riesz Representation Theorem, the set \( K C(X_K) \) consists of all distributions on \( X_K \), and hence all stationary states on the lattice, including in particular the traditional Gibbs equilibrium distributions.

The set \( X_K \) has the following remarkable structure.
Theorem V.3. The compact set $X_K \subset \mathcal{K}\mathcal{W}_K$ is a finite set with the discrete topology. All open sets $F \in \mathcal{B}(X_K)$ are clopen, and $X_K$ is extremely disconnected.

Proof. The set of transformations

$$\mathcal{W}(\mathcal{A}^t) \cong \mathcal{W}_\infty \overset{\Delta_f}{\rightarrow} \mathcal{W} \overset{\gamma_f}{\rightarrow} \mathcal{C}(X_K)$$

represents local measurements on the system $\Lambda_t$ as the corresponding quasilocal observables. Define $\gamma_t = \psi_K \circ \Delta_K \circ \sigma_t : \mathcal{W}(\mathcal{A}^t) \rightarrow \mathcal{C}(X_K)$ for any $t \in J$. Let $g : X_K \rightarrow \mathbb{R}^2$ be defined by $g(x) = (H, N)$, where $H$ and $N$ are the energy and particle densities, respectively, of the state $x$, and let $M = g(X_K)$. Fix once and for all a finite system $\Lambda_t$. Let $g^t \in \mathcal{W}(\mathcal{A}^t)$ and $n^t : \Omega \rightarrow \mathbb{R}$ be the energy and number densities, respectively, for $\Gamma^\infty$. Let $F \subset X_K$ be any open set, and define $A_F = g(F) \subset M$. The component $\mu_t \in E_t$ of thread $(\mu_t)_{t \in J} = \mu$ corresponding to a particular state $x_\mu \in F$ is a MC ensemble on the algebra of local observables $\mathcal{W}(\mathcal{A}^t)$ corresponding to an energy and particle density in $A_F$. For any Borel set $B \subset \mathbb{R}^2$, denote as usual $[g^t \in B] = (g^t)^+(B)$, and let $\chi_{[g^t\in B]} : \Omega \rightarrow \{0, 1\}$ be the characteristic function of $[g^t \in B]$ on the configuration space $\Omega$. Then $\chi_{[g^t\in A_F]}(a) = 1$ if $g^t(a) \in A_F$, and 0 otherwise. Clearly, $\mu_t(\chi_{[g^t\in A_F]}) = 1$ if $x_\mu \in F$, and 0 otherwise, because $[g^t \in A_F]$ is the support of the component $\mu_t$ of any thread $(\mu_t)_{t \in J} = \mu$ for which $x_\mu \in F$. But $\gamma_t(f^t)(x_\mu) = \mu_t(f^t)\forall f^t \in \mathcal{W}(\mathcal{A}^t), \mu \in \mathcal{E}_\infty$, so that $\gamma_t(\chi_{[g^t\in A_F]}(x_\mu) = 1$ if $x_\mu \in F$, and 0 otherwise. Hence, $\gamma_t(\chi_{[g^t\in A_F]} = \chi_{F}^{(X)}$. Thus, $\chi_{F}^{(X)} \in \mathcal{C}(X_K)$. But the characteristic function $\chi_{F}^{(X)}$ is continuous iff $F$ is clopen. Since $X_K$ is Hausdorff, the complement of any singleton $x \in X_K$ is open and therefore clopen. Hence, all singletons are open, and $X_K$ is discrete. But the only discrete compact spaces are finite. 

The compact extremely disconnected spaces are frequently called Stonean spaces. Note especially that this theorem results from the algebraic structure itself, without any assumptions about the topology of the lattice configuration space $\Omega$.

The Stonean topology for $X_K$ has the following consequence. Let $\mathfrak{P} \subset \mathcal{C}(X_K)$ be the lattice of idempotents in $\mathcal{C}(X_K)$. These are exactly the characteristic functions of Borel sets in $X$, i.e., functions of the form $\chi_B^{(X)}(x) = 1, x \in B$, and 0 otherwise, where $B \subset X_K$ is a Borel set. The Stonean topology on $X_K$ is equivalent to the condition that $\mathfrak{P}$ be a complete lattice \[22\] Theorem 6.2d].
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References

[1] Alfsen, E. M.: Compact Convex Sets and Boundary Integrals. Ergeb. Math. Grenzgeb., Bd.57, New York: Springer 1971.
[2] Bauer, H.: Schilowsscher Rand und Dirichletsches Problem. Ann. Inst. Fourier 11, 89-136 (1961).
[3] Bratteli, O., Robinson, D. W.: Operator Algebras and Quantum Statistical Mechanics. I. C*- and W*-algebras, Symmetry Groups, Decomposition of States. 2nd Edition. New York: Springer 1987.
[4] Choquet, G.: Unicité des representations intégrales au moyen de points extremaux dans les cônes convex réticulés. C. R. Paris 243, 555-557 (1966).
[5] Dobruschin, R. L.: Description of a random field by means of conditional probabilities and the conditions governing its regularity. Theory of Prob. and its Appl. 13, 197-224 (1968).
[6] Dubin, D. A. Solvable Models in Algebraic Statistical Mechanics. London: Oxford (1974).
[7] Dunford, N., and Schwartz, J. T. Linear Operators Part I: General Theory. New York: Interscience (1957).
[8] Emch, G. G.: Algebraic Methods in Statistical Mechanics and Quantum Field Theory. New York: Wiley 1972.
[9] Emch, G. G. Mathematical and Conceptual Foundations of 20th-Century Physics. New York: Noth-Holland, 1984.
[10] Engelking, R.: Outline of General Topology. Amsterdam: North-Holland 1968.
[11] Guggenheim, E. A. Thermodynamics. Amsterdam: North-Holland 1957.
[12] Haag, R.: Local Quantum Physics. Fields, Particles, Algebras. New York: Springer 1996.
[13] Haag, R., Kastler, D.: An algebraic approach to quantum field theory. J. Math. Phys. 5, 848-861 (1964).
[14] Kadison, R. V.: A representation theorem for commutative topological algebras. Mem. Amer. Math. Soc., 7. 1951.
[15] Ledrappier, F.: Mesures d’équilibre sur un réseau. Comm.Math.Phys. 33, 119-128 (1973).
[16] Lanford, O. E.: Entropy and Equilibrium States in Classical Statistical Mechanics. In Lecture Notes in Physics 20: Statistical Mechanics and Mathematical Problems, Ed. A Lenard. New York: Springer 1973.
[17] Lanford, O. E., Ruelle, D.: Observables at infinity and states with short-range correlations in statistical mechanics. Commun. Math. Phys. 13, 194-215 (1969)
[18] Loève, M.: Probability Theory. 3rd Ed. Princeton: Van Nostrand 1963
[19] Mackey, G. W.: Mathematical Foundations of Quantum Mechanics. New York: Benjamin 1963.
[20] Phelps, R.: Lectures on Choquet’s Theorem. Princeton: Van Nostrand 1966.
[21] Piron, C. Axiomatique quantique: Helv. Phys. Acta 37, 439-468 (1964)
[22] Porter, J. R., Wood, R. G.: Extensions and Absolutes of Hausdorff Spaces. New York: Springer 1988.
[23] Preston, C.: Random Fields. Lecture Notes in Mathematics, Bd. 534. New York: Springer 1976
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[24] Ruelle, D.: Statistical Mechanics. Rigorous Results. Reading: Benjamin 1969.
[25] Ruelle, D.: Thermodynamic Formalism. Reading: Addison-Wesley 1978.
[26] Sakai, S. $C^\ast$-algebras and $W^\ast$-algebras. New York: Springer 1971.
[27] Segal, I. E.: Postulates for general quantum mechanics. Ann. Math. 48, 930-948 (1947)
[28] Semadeni, Z.: Banach Spaces of Continuous Functions. Warszawa: PWN—Polish Scientific Publishers 1971