On Countable Stationary Towers

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Abstract. In this paper, we investigate properties of countable stationary towers. We derive the regularity properties of sets of reals in $L(\mathbb{R})$ from some properties of countable stationary towers without explicit use of strong large cardinals such as Woodin cardinals. We also introduce the notion of semiprecipitousness and investigate its relation to precipitousness and presaturation of countable stationary towers. We show that precipitousness of countable stationary towers of weakly compact height implies the regularity properties of sets of reals in $L(\mathbb{R})$.

Keywords: stationary tower, regularity properties of sets of reals

1 Introduction

In his expository article (written in Japanese) \cite{8}, Gaisi Takeuti called the development of a series of results concerning $L(\mathbb{R})$ and large cardinals, by Martin, Steel, and Woodin during the second half of the 1980’s, a revolutionary leap in modern set theory. Long before the 1980’s, Takeuti and Solovay pointed out that $L(\mathbb{R})$, the smallest inner model containing every real, is a natural candidate for an inner model of AD (axiom of determinacy).\cite{3} What Takeuti called a revolutionary leap has confirmed that their intuition was correct.

The first result in this revolutionary leap was proving various regularity properties, such as Lebesgue measurability, the Baire property, and the perfect set property, hold for every set of reals in $L(\mathbb{R})$ from the existence of a supercompact cardinal. Soon after, the large cardinal hypothesis was weakened and the conclusion was strengthened drastically. For convenience, throughout this paper, by regularity properties, we mean Lebesgue measurability, the Baire property, and the perfect set property.

Woodin realized the existence of a certain generic elementary embedding would imply regularity properties of sets of reals in $L(\mathbb{R})$.\cite{9}

Theorem 1 (Woodin). Let $\lambda$ be an inaccessible cardinal. Suppose there is a partial order $\mathbb{P}$ such that $\mathbb{P}$ forces the following:

There is a generic elementary embedding $j : V \to M$ with

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1. \( \text{crit}(j) = \aleph_1^V \) and \( j(\aleph_1^V) = \lambda \)
2. \( M \) is closed under taking \( \omega \) sequences in \( V^P \)
3. every real is obtained by a small (size less than \( \lambda \)) forcing, i.e. in \( V[G] \), where \( G \) is \( P \)-generic, for each real \( x \) there are a poset \( P_x \) in the ground model \( V \) with cardinality less than \( \lambda \) and a \( P_x \)-generic \( H \in V[G] \) such that \( x \in V[H] \).

Then every set of reals in \( L(\mathbb{R}) \) satisfies the regularity properties. \( \Box \)

Woodin developed the notion of stationary tower forcing to produce a generic elementary embedding satisfying the hypothesis of the above theorem. Although Woodin’s theory of stationary towers is more general, we will concentrate our attention on countable stationary towers. We present a brief sketch of the notion of countable stationary tower forcing. We refer the reader to Larson’s book [5] for a more thorough treatment of stationary tower forcing.

**Definition 1.** Let \( \gamma \) be an uncountable limit ordinal. We denote the set of all \( p \in V_\gamma \) such that \( p \) is a stationary subset of \( \bigcup p \) by \( Q_\gamma \). For \( p, q \in Q_\gamma \), we say \( p \leq Q_\gamma q \) if \( \bigcup p \supseteq \bigcup q \) and \( \forall x \in p(x \cap \bigcup q \in q) \).

The partial order \( \langle Q_\gamma, \leq Q_\gamma \rangle \) is known as a countable stationary tower of height \( \gamma \). Throughout this paper, whenever we mention a stationary tower, we are assuming its height is an uncountable limit ordinal.

If \( G \) is a \( Q_\gamma \)-generic filter over a ground model \( V \), then we can take the generic ultrapower \( \text{Ult}(V; G) \) of \( V \). We refer a reader to Larson’s book [5] for the definitions and basic facts concerning generic ultrapowers induced by stationary tower forcing.

In general, there is no guarantee that the generic ultrapower \( \text{Ult}(V; G) \) is well-founded.

**Definition 2.** \( Q_\gamma \) is said to be precipitous if \( Q_\gamma \) forces that \( \text{Ult}(V; G) \) is well-founded.

**Remark 1.** It turns out that the precipitousness of \( Q_\gamma \) is equivalent to “\( Q_\gamma \) forces that \( \text{Ult}(V_\delta; G) \) is well-founded where \( \delta = (|V_\gamma|^+)^V \)”.

This motivated the next definition:

**Definition 3.** \( Q_\gamma \) is said to be semiprecipitous if \( Q_\gamma \) forces that \( \text{Ult}(V_\gamma; G) \) is well-founded.

If \( \text{Ult}(V; G) \) is well-founded, then, by taking the transitive collapse of \( \text{Ult}(V; G) \), we can define a generic elementary embedding \( j : V \rightarrow M \) by \( j(x) = [c_x] \), where \( c_x \) is the constant function whose domain is \( [\aleph_1]^\omega \) and that takes the constant value \( x \).

Woodin proved that if \( \lambda \) is a large cardinal, such as a supercompact cardinal, then \( Q_\lambda \) is precipitous. Actually, he derived a somewhat stronger property than precipitousness from the large cardinal in his proof.

**Definition 4.** \( Q_\gamma \) is said to be presaturated if \( Q_\gamma \) forces that \( \text{Ult}(V; G) \) is closed under taking \( \omega \) sequences in generic extensions.
Remark 2. It is easy to see that presaturation of $Q_\gamma$ implies precipitousness of $Q_\gamma$. It turns out that the presaturation of $Q_\gamma$ is equivalent to “$Q_\gamma$ forces $\text{cf}(\gamma) > \omega$”. Furthermore, the presaturation of $Q_\gamma$ implies that $Q_\gamma$ forces $j(N_1^V) = \gamma$, where $j$ is the corresponding generic elementary embedding.

Woodin proved that if $\lambda$ is a supercompact cardinal, then $Q_\lambda$ is presaturated and satisfies the hypotheses of Theorem 1. Therefore, the existence of a supercompact cardinal implies that every set of reals in $L(R)$ satisfies the regularity properties.

Woodin later proved that if $\lambda$ is a Woodin cardinal which is substantially weaker than supercompact, then $Q_\lambda$ is presaturated. A generic elementary embedding generated by $Q_\lambda$ with Woodin $\lambda$ always satisfies properties 1 and 2 of the hypotheses of Theorem 1. But property 3 can fail. It is known that the existence of one Woodin cardinal, in fact even $\omega$ many Woodin cardinals, could not imply that every set of reals in $L(R)$ satisfies the regularity properties. In particular J. Steel [7] proved that if the existence of infinitely many Woodin cardinals is consistent, then it is consistent with the axiom of choice holding in $L(R)$. It turns out that if there are infinitely many Woodin cardinals and a measurable cardinal above them, then every set of reals in $L(R)$ satisfies the regularity properties.

In this paper, we investigate the following questions:

1. Can we derive the regularity properties of sets of reals in $L(R)$ from properties of countable stationary towers, such as precipitousness or presaturation, without explicit use of strong large cardinals such as supercompact cardinals or Woodin cardinals?
2. Are there any relations between presaturation, precipitousness, and semiprecipitousness other than obvious implications?

2 Results

Burke used the following game to characterize the precipitousness of a stationary tower. [1]

For an uncountable limit ordinal $\gamma$, let $\Gamma_\gamma$ denote the following two player game of length $\omega$: At each inning, players I and II choose conditions $p_n, q_n \in Q_\gamma$ alternately with $p_0 \geq q_0 \geq p_1 \geq q_1 \geq \cdots$. For the play $\langle p_n, q_n \mid n < \omega \rangle$, player II wins if there is some $x \in [V_\gamma]^\omega$ such that $x \cap \bigcup p_n \in p_n$ for every $n < \omega$. Otherwise I wins. The following theorem is due to Burke [1]:

Theorem 2. Let $\gamma$ be an uncountable limit ordinal. Then the following are equivalent:

1. $Q_\gamma$ is precipitous.
2. Player I does not have a winning strategy in the game $\Gamma_\gamma$.
For a certain $\gamma$, semiprecipitousness of $\mathbb{Q}_\gamma$ is equivalent to precipitousness of $\mathbb{Q}_\gamma$.

**Proposition 1.** Let $\lambda$ be a singular cardinal such that $|V_\lambda| = \lambda$ and $\omega < \text{cf}(\lambda) < \lambda$. If $\mathbb{Q}_\lambda$ is semiprecipitous, i.e. $\models_{\mathbb{Q}_\lambda} \text{"Ult}(V_\lambda; G)$ is well-founded”, then $\mathbb{Q}_\lambda$ is precipitous.

**Proof.** Suppose $\mathbb{Q}_\lambda$ is not precipitous. Then player I has a winning strategy $\sigma$ in $\mathcal{G}_\lambda$.

Let $T$ be the set of all finite sequences $\langle \langle q_i, x_i \rangle \mid i \leq n \rangle$ such that:

1. $q_i \in \mathbb{Q}_\lambda$ and $x_i \in q_i$.
2. $x_{i+1} \cap \bigcup q_i = x_i$ for every $i < n$.
3. $\sigma(\emptyset) \geq q_0 \geq \sigma(q_0) \geq q_1 \geq \ldots \geq \sigma(q_0, \ldots, q_{n-1}) \geq q_n$.

For $s, t \in T$, we define $s \preceq_T s' \preceq_T t$ if $s$ is an extension of $t$. Since $\sigma$ is a winning strategy for player I, $T$ is well-founded, i.e., there is no infinite descending sequence in $T$.

We note that $T \subseteq V_\lambda$. Consider the structure $(V_\lambda, \in, T)$, where we identify $T$ as a unary predicate. Since $|V_\lambda| = \lambda$, we can check the following:

1. $(V_\lambda, \in, T) \models \text{"T is well-founded"}$.
2. Since $T \cap V_\alpha \subseteq V_\lambda$ for every $\alpha < \lambda$, we have $(V_\lambda, \in, T) \models \text{"$\forall \alpha \exists x (x = T \cap V_\alpha)$"}$.
3. $(V_\lambda, \in, T) \models \text{"$\forall \alpha, T \cap V_\alpha$ is well-founded, and the rank function $f : T \cap V_\alpha \rightarrow \beta$ exists for some ordinal $\beta$"}$.

Take a $\mathbb{Q}_\lambda$-generic $G$ with $\sigma(\emptyset) \in G$. In $V[G]$, we can take the ultrapower of the structure $(V_\lambda, \in, T)$ by $G$. Since $\text{Ult}(V_\lambda; G)$ is well-founded, the ultrapower of $(V_\lambda, \in, T)$ is also well-founded. Let $M = \langle M, \in, j(T) \rangle$ be the transitive collapse of the ultrapower, and $j : V_\lambda \rightarrow M$ be the elementary embedding. Then, because $\omega < \text{cf}(\lambda) < \lambda$, we have that $j^{\omega} \lambda$ is bounded in $M \cap \text{ON}$. Let $\gamma = \sup(j^{\omega} \lambda) \in M \cap \text{ON}$. Consider $T^* = j(T) \cap M_\gamma$. By (2) above, we have $T^* \in M$, and by (3), there exists the rank function $f : T^* \rightarrow \beta$ in $M$. Hence, $T^*$ is also well-founded in $V[G]$, and there is no infinite descending sequence of $T^*$ in $V[G]$.

By induction on $n < \omega$, we define a descending sequence $\langle p_n, q_n \mid n < \omega \rangle$ as follows: First, let $p_0 = j(\sigma(\emptyset)) \in G$. Then $\{\sigma(q) \mid q \leq p_0\}$ is dense below $p_0$, so we can take $q_0 \leq p_0$ and $p_1 = \sigma(q_0) \leq q_0$ with $p_1 \in G$. We have $q_0 \in G$. Again, since $\{\sigma(q_0, q) \mid q \leq p_1\}$ is dense below $p_1$, we can choose $q_1 \leq p_1$ and $p_2 = \sigma(q_0, q_1) \in G$, and so on. We know $j(p_n) = j(\sigma)(j(q_0), \ldots, j(q_{n-1})) \geq j(q_n)$. For $n < \omega$, let $x_n = \bigcup q_n$. We know $j^{\omega} x_n \in j(q_n), \text{ and } j^{\omega} x_{n+1} \cap j(q_n) = j^{\omega} x_n$. Because $j(q_n), j^{\omega} x_n \in M_\gamma$, we have that $\langle j(q_i), j^{\omega} x_i \mid i \leq n \rangle \in T^*$ for every $n < \omega$. Therefore, the sequence $\langle j(q_n), j^{\omega} x_n \mid n < \omega \rangle$ induces an infinite descending sequence in $T^*$. This is a contradiction.

**Remark 3.** The reader might wonder if the hypothesis of the last proposition is vacuous. In our next paper [4], we prove the consistency of precipitousness of $\mathbb{Q}_\lambda$ where $\lambda$ satisfies the conditions of Proposition 1 assuming a sufficiently strong large cardinal hypothesis.
Later in this paper, we will show that, in general, semiprecipitousness of $\mathcal{Q}_\lambda$ does not imply precipitousness.

We can also use Burke’s game to prove the following proposition:

**Proposition 2.** Suppose $\alpha$ and $\beta$ are uncountable limit ordinals with $\alpha < \beta$. If $\mathcal{Q}_\alpha$ is precipitous, then, for each $p \in \mathcal{Q}_\alpha$, there is some $p^* \in \mathcal{Q}_\beta$ such that $p^* \leq p$ and $p^* \Vdash_{\mathcal{Q}_\alpha} \langle \mathcal{G} \cap \mathcal{Q}_\alpha \rangle$ is $\mathcal{Q}_\alpha$-generic” where $\mathcal{G}$ is the canonical $\mathcal{Q}_\beta$-name for a generic filter.

This proposition is an immediate consequence of the next lemma. The referee informed us that this lemma is a “tower” analogue of Proposition 2.4 of [4].

**Lemma 1.** Let $\alpha$ be an uncountable limit ordinal, and suppose $\mathcal{Q}_\alpha$ is precipitous. Then, for every $p \in \mathcal{Q}_\alpha$, the set \{ $x \in [V_{\alpha+1}]^\omega$ | $x \cap \bigcup p \in p, \forall D \in x$ (if $D \subseteq \mathcal{Q}_\alpha$ is dense, then $\exists r \in D \cap x \ (x \cap \bigcup r \in r)$ \} is stationary in $[V_{\alpha+1}]^\omega$.

**Proof.** Take $p \in \mathcal{Q}_\alpha$. Fix a well-ordering $\Delta$ on $V_{\alpha+1}$ and a function $f : [V_{\alpha+1}]^\omega \to V_{\alpha+1}$. For a set $x \subseteq V_{\alpha+1}$, let $Sk(x)$ denote the Skolem hull of $x$ under the structure $\langle V_{\alpha+1}, \in, \Delta, f \rangle$. We will find a set $x$ such that $Sk(x)$ belongs to the required set.

Now we describe a strategy for player I in $\Gamma_\alpha$. Fix a surjection $\pi : \omega \to \omega \times \omega$ such that if $\pi(n) = (i, j)$ then $i \leq n$. For a countable $x \subseteq V_{\alpha+1}$, we also fix an enumeration $\langle D^x_j | j < \omega \rangle$ of all dense subsets of $\mathcal{Q}_\alpha$ that belong to $Sk(x)$.

First, let $p_0^0 = p$. Take $j < \omega$ with $\pi(0) = (0, j)$. Since $p_0^0$ is stationary in $[\bigcup p_0^\alpha]^\omega$, there is some dense set $D_0 \subseteq \mathcal{Q}_\alpha$ with $\{ x \in p_0^0 | D_0 = D^x_j \}$ stationary in $[\bigcup p_0^\alpha]^\omega$, so $\{ x \in p_0^0 | D_0 = D^x_j \}$ is a condition of $\mathcal{Q}_\alpha$. Take $p_0^0 \in D_0$ with $p_0^0 \leq \{ x \in p_0^0 | D_0 = D^x_j \}$. Then take $p_0 \leq p_0^0$ such that $p_0 \in x$ and $Sk(x) \cap \bigcup p_0 = x$ for every $x \in p_0$. This $p_0$ is the first move of player I.

Let $n > \omega$, and suppose $p \geq p_0^0 \geq p_0 \geq q_0 \geq \cdots \geq q_{n-1}$ were chosen so that, for every $k < n$:

1. If $\pi(k) = (i, j)$, then, for every $x \in p_k^\alpha$, we have $D^x_j \cap \bigcup p_i = D_k$.
2. $p_k^0 \in D_k$.
3. $Sk(x) \cap \bigcup p_k = x$ and $p_k^0 \in x$ for every $x \in p_k$.

As in the $n = 0$ case, we can choose $p_n \leq p_n^0 \leq p_n \leq q_n \leq \cdots \geq q_{n-1}$ as required, and player I chooses $p_n$ as his move. This completes the description of our strategy for player I.

Since $\mathcal{Q}_\alpha$ is precipitous, the strategy above is not a winning strategy for player I. Thus, we can find a descending sequence $p \geq p_0^0 \geq p_0^0 \geq p_0 \geq q_0 \geq \cdots \geq q_{n-1}$ such that each $p_n, p_n^0, p_n^\alpha$ is chosen according to the above strategy, but there is some $x \subseteq V_\alpha$ such that $x \cap \bigcup p_n \in p_n$ for every $n < \omega$. We may assume that $x = \bigcup_{n<\omega} (x \cap \bigcup p_n)$. Since $p_n \geq p_n^0 \geq p_0 \geq p_n^\alpha$, we also have $x \cap \bigcup p_n^\alpha \in p_n^\alpha$, $x \cap \bigcup p_n^\alpha \subseteq p_n^\alpha$, and $x = \bigcup_{n<\omega} (x \cap \bigcup p_n^\alpha) = \bigcup_{n<\omega} (x \cap \bigcup p_n^\alpha)$.

Hence $Sk(x) = \bigcup_{n<\omega} Sk(x \cap \bigcup p_n^\alpha)$. For $n < m$, we have $Sk(x \cap \bigcup p_m^\alpha) \cap \bigcup p_n^\alpha = (x \cap \bigcup p_m^\alpha) \cap \bigcup p_n^\alpha = x \cap \bigcup p_n^\alpha$, thus $Sk(x) \cap \bigcup p_n^\alpha = x \cap \bigcup p_n^\alpha$ for every $n < \omega$. We show that, if $D \in Sk(x)$ is dense in $\mathcal{Q}_\alpha$, then there is $r \in D \cap Sk(x)$ with
Proof. Let $\text{dom}(M) = n < \omega$ and, for every $\gamma < \lambda$, such that for each $\gamma < \lambda$, either $q \leq p$ or $q = p$, and, if $q \leq p$, then there is some function $f_n^q$ with $q \Vdash_{\lambda} f_n = f_n^q$. Since the set $\{ \alpha < \lambda \mid Q_\alpha \text{ is precipitous} \}$ is stationary in $\lambda$, there is $\alpha < \lambda$ such that $Q_\alpha$ is precipitous, and, for every $n < \omega$, $A_n \cap Q_\alpha$ is pre dense in $Q_\alpha$, and dom$(f_n^q) \in V_\alpha$ for every $q \in A_n \cap V_\alpha$. Let $D_n = \{ r \in Q_\alpha \mid \exists q \in A_n \cap \alpha (r \leq q) \}$. $D_n$ is dense in $Q_\alpha$.

By the previous lemma, the set $p' = \{ x \in [V_{\alpha+1}]^\omega \mid x \cap \bigcup p \in p, \forall n < \omega \exists r \in D_n \cap x (x \cap \bigcup r \in r) \}$ is stationary in $[V_{\alpha+1}]^\omega$. We know $p' \leq p$. For $x \in p'$ and $n < \omega$, take $r_n^x \in D_n \cap x$ with $x \cap \bigcup r_n^x \in r_n^x$, and we may assume that there is a unique $q_n^x \in A_n \cap x$ with $r_n^x \leq q_n^x$. Then define $g : [V_{\alpha+1}]^\omega \to V$ by $g(x) = \{ f_n^{q_n^x}(x \cap \bigcup q_n^x) \mid n < \omega \}$. It is routine to check that $p' \Vdash_{\lambda} \{x \in [V_{\alpha+1}]^\omega \mid x \cap \bigcup p \in p, \forall n < \omega \exists r \in D_n \cap x (x \cap \bigcup r \in r) \}$. □
We use the last lemma to obtain some consequences of precipitousness of $Q_\lambda$ where $\lambda$ is a weakly compact cardinal.

**Proposition 4.** Suppose $\lambda$ is a weakly compact cardinal. If $Q_\lambda$ is precipitous, then $\{\alpha < \lambda \mid Q_\alpha \text{ is precipitous}\}$ is stationary in $\lambda$. Hence, under our hypothesis, $Q_\lambda$ is presaturated.

**Proof.** The statement that “player I does not have a winning strategy for Burke’s game $\Gamma_\lambda$” can be described by some $\Pi_1^1$-statement $\varphi$ over $\langle V_\lambda, \in \rangle$. Hence, if $Q_\lambda$ is precipitous, then $V_\lambda \models \varphi$, and, since $\lambda$ is weakly compact, we have that $\{\alpha < \lambda \mid V_\alpha \models \varphi\}$ is stationary in $\lambda$. Therefore the set $\{\alpha < \lambda \mid Q_\alpha \text{ is precipitous}\}$ is stationary in $\lambda$. $\square$

By combining Proposition 3 and Proposition 4 we obtain the next result:

**Corollary 1.** If $Q_\lambda$ is precipitous for a weakly compact cardinal $\lambda$, then every set of reals in $L(R)$ satisfies the regularity properties.

We note that the least Woodin cardinal is not a weakly compact cardinal. A weaker large cardinal could also reflect semistationarity.

**Proposition 5.** If $Q_\lambda$ is semiprecipitous for a Mahlo cardinal $\lambda$, then the set $\{\alpha < \lambda \mid Q_\alpha \text{ is semiprecipitous where } \alpha \text{ is regular}\}$ is stationary in $\lambda$.

Before we present the proof of this proposition, we state its corollary:

**Corollary 2.** The semiprecipitousness of $Q_\gamma$, in general, cannot imply the precipitousness of $Q_\gamma$.

**Proof.** Suppose otherwise. Assume that $\lambda$ is a Woodin cardinal. By Woodin’s result, we know that $Q_\lambda$ is presaturated.

Since a Woodin cardinal is Mahlo, by Proposition 5 $\{\alpha < \lambda \mid Q_\alpha \text{ is semiprecipitous}\}$ is stationary. So, by our assumption, $\{\alpha < \lambda \mid Q_\alpha \text{ is precipitous}\}$ is stationary. Then, by Proposition 5 every set of reals in $L(R)$ satisfies the regularity properties. But the existence of one Woodin cardinal cannot imply this conclusion. $\square$

To prove Proposition 5 we use the following variant of Burke’s game $\Gamma_\gamma$ to characterize the precipitousness of $Q_\gamma$.

Let $\Gamma_\gamma^*$ denote the following two player game of length $\omega$: At the $n$-th inning of the game player I plays $\langle p_n, f_n \rangle$ with $p_n \in Q_\gamma$, $f_n : p_n \to \text{ON}$, and player II plays $q_n \in Q_\gamma$ alternately with the requirements: $p_0 \geq q_0 \geq p_1 \geq q_1 \geq \cdots$ and $\forall x \in p_{n+1}(f_{n+1}(x) \in f_n(x \cap \bigcup p_n))$. Player I wins if and only if the game does not stop. By the proof of Burke’s theorem, we see that $Q_\gamma$ is precipitous if and only if player I does not have a winning strategy. This variant of Burke’s game was inspired by a similar game due to Goldring characterizing precipitousness of ideals over $\mathcal{P}_\kappa \lambda$. Note that one advantage of the game $\Gamma_\gamma^*$ over $\Gamma_\gamma$ is that, if player I does not have a winning strategy in $\Gamma_\gamma^*$, then, by the theorem of Gale and Stewart, player II does have a winning strategy.
Proof (of Proposition 4). Suppose that $Q_\lambda$ is semiprecipitous for a Mahlo cardinal $\lambda$. For an uncountable limit ordinal $\gamma$, we consider the following game $\Gamma^\text{local}_\gamma$ which is a localized version of $\Gamma^*_\gamma$.

The game $\Gamma^\text{local}_\gamma$ is almost identical to $\Gamma^*_\gamma$ with the added requirement that the function $f_n$, played by player I, must take its values in $\gamma$. It is easy to see that player I does not have a winning strategy in $\Gamma^\text{local}_\gamma$ is equivalent to $Q_\gamma$, is semiprecipitous. Since player I does not have a winning strategy for the game $\Gamma^\text{local}_\lambda$, player II does have a winning strategy $\tau$ for that game. From the Mahloness of $\lambda$, by letting player II play the strategy $\tau$, it is not too difficult to see that the set $\{\gamma < \lambda \mid \tau \text{ is a winning strategy for player II in } \Gamma^\text{local}_\gamma \text{ with } \gamma \text{ regular}\}$ is stationary in $\lambda$.

It turns out that a large cardinal which is weaker than Mahlo suffices to reflect semiprecipitousness to points of cofinality $\omega$.

Proposition 6. If $Q_\lambda$ is semiprecipitous for an inaccessible cardinal $\lambda$, then there is a club $C$ in $\lambda$ such that, for every $\gamma \in C$, if $\text{cf}(\gamma) = \omega$ then $Q_\gamma$ is semiprecipitous.

Proof. Since $Q_\lambda$ is semiprecipitous, player II in $\Gamma^\text{local}_\lambda$ has a winning strategy $\tau$.

Take a sufficiently large regular cardinal, and take an elementary submodel $M \prec H_\theta$ containing all relevant objects satisfying $M \cap \lambda \in \lambda$ and $\text{cf}(M \cap \lambda) = \omega$.

Let $\gamma = M \cap \lambda$. We shall show that player II in $\Gamma^\text{local}_\gamma$ has a winning strategy. Let us describe player II’s strategy.

Suppose player I plays $\langle p_0, f_0 \rangle$. Note that $p_0 \in Q_\gamma \subseteq M$ but $f_0$ may not be an element of $M$, since $\text{range}(f_0)$ could be cofinal in $\gamma$. However, since $\text{cf}(\gamma) = \omega$, there is $\gamma' < \gamma$ such that $\{x \in p_0 \mid f(x) \leq \gamma'\}$ is stationary. Let $\gamma^*$ be the least ordinal such that $p'_0 = \{x \in p_0 \mid f(x) \leq \gamma^*\} \in Q_\gamma$. Then we have $\langle p'_0, f_0 \restriction p'_0 \rangle \in M$, and we can identify $\langle p'_0, f_0 \restriction p'_0 \rangle$ as player I’s play in $\Gamma^\text{local}_\gamma$.

Let $q_0 = \tau(\langle p'_0, f_0 \restriction p'_0 \rangle) \in M \cap Q_\lambda = Q_\gamma$ be player II’s response. Note that $q_0 \leq p'_0 \leq p_0$ in $Q_\gamma$. If $\langle p_1, f_1 \rangle$ is a response of player I in $\Gamma^\text{local}_\gamma$, by repeating the above procedure, we can find $p'_1 \subseteq p_1$ with $\text{range}(f_1 \restriction p'_1)$ bounded in $\gamma$. Since $\bigcup p_0 = \bigcup p'_0$, for every $x \in p'_1$, we have $f_1(x) = f_0(x \cap \bigcup p_0) = f_0(x \cap \bigcup p'_0)$. Hence $\langle p'_1, f_1 \restriction p'_1 \rangle, q_0, \langle p'_0, f_0 \restriction p'_0 \rangle$ satisfy the requirements of a play in $\Gamma^\text{local}_\lambda$. Since $\langle p'_0, f_0 \restriction p'_0 \rangle, \langle p'_1, f_1 \restriction p'_1 \rangle \in M$, we let player II play $q_1 = \tau(\langle p'_0, f_0 \restriction p'_0, p'_1, f_1 \restriction p'_1 \rangle) \in M$, and so on. Since $\tau$ is a winning strategy of player II in $\Gamma^\text{local}_\lambda$, it is easy to see that the strategy above is a winning strategy of player II in $\Gamma^\text{local}_\gamma$.

Remark 4. We note that, by the proof of Corollary 2, the last proposition indicates that the hypothesis “$\omega < \text{cf}(\lambda)$” of Proposition 1 cannot be dropped.

In the late 1980’s, the first author was in Los Angeles and received a letter from Gaisi Takeuti. He was inquiring about results concerning $\mathcal{L}(\mathbf{R})$ and large cardinals. The first author still remembers the letter that professed Takeuti’s passion and enthusiasm for set theory and logic. We regret that we did not have an opportunity to ask Professor Takeuti’s view on the revolutionary leap in person.
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