Revisiting the gravitomagnetic clock effect

Lorenzo Iorio

1Ministero dell’ Istruzione e del Merito. Viale Unità di Italia 68, I-70125, Bari (BA), Italy

Abstract

To the first post-Newtonian order, if two test particles revolve in opposite directions about a massive, spinning body along two circular and equatorial orbits with the same radius, they take different times to return to the reference direction relative to which their motion is measured: it is the so-called gravitomagnetic clock effect. The satellite moving in the same sense of the rotation of the primary is slower, and experiences a retardation with respect to the case when the latter does not spin, while the one circling in the opposite sense of the rotation of the source is faster, and its orbital period is shorter than it would be in the static case. The resulting time difference due to the stationary gravitomagnetic field of the central spinning body is proportional to the angular momentum per unit mass of the latter through a numerical factor which so far has been found to be $4\pi$. A numerical integration of the equations of motion of a fictitious test particle moving along a circular path lying in the equatorial plane of a hypothetical rotating object by including the gravitomagnetic acceleration to the first post-Newtonian order shows that, actually, the gravitomagnetic corrections to the orbital periods are larger by a factor of 4 in both the prograde and retrograde cases. Such an outcome, which makes the proportionality coefficient of the gravitomagnetic difference in the orbital periods of the two counter-revolving orbiters equal to $16\pi$, confirms an analytical calculation recently published in the literature by the present author. It is an important result in view of the astrophysical implications of the gravitomagnetic clock effect around Kerr black holes.

Keywords: Classical general relativity; Fundamental problems and general formalism; Experimental studies of gravity; Experimental tests of gravitational theories; Satellite orbits

1. Introduction

Within the weak-field and slow-motion approximation of the General Theory of Relativity (GTR), by gravitomagnetic clock effect it is usually meant the difference $\Delta T_{\text{gvm}}$ between the orbital periods of two counter-revolving test particles, i.e. non-spinning, moving along circular orbits of identical radius $r_0$ in the equatorial plane of a rotating body of mass $M$ and angular momentum $J$ (Mitskevich and Pulido Garcia 1970; Vladimirov et al. 1987; Cohen and Mashhoon 1993; Mashhoon 1997; You 1998; Mashhoon et al. 1999; Tartaglia 2000c; Mashhoon and Santos 2000; Mashhoon et al. 2001; Hackmann and Lämmerzahl 2014; Scheumann et al. 2023). It turns out that $\Delta T_{\text{gvm}}$ is proportional to $J/(Mc^2)$, where $c$ is the speed of light in vacuum, through a numerical factor that has been calculated in the literature to be equal to $4\pi$. Such an intriguing relativistic feature of motion was the subject of several papers investigating its possible detection as well; see Gronwald et al. (1997); Mashhoon et al. (1999); Lichtenecker et al. (2000); Tartaglia (2000a,b); Iorio (2001b,a); Iorio and Lichtenecker (2005); Lichtenecker et al. (2006). It has also relevant consequences in astrophysical contexts such as Kerr black hole spacetimes (de Felice 1995; Bonnor and Steadman 1999; Bini and Jantzen 2003; Faruque 2004; Bini et al. 2005). For other versions of the gravitomagnetic clock effect involving spinning orbiters in the Kerr spacetime, see, e.g., Bini et al. (2004); Mashhoon and Singh (2006).

The standard approach in deriving the aforementioned form of the gravitomagnetic clock effect is to calculate the time interval $T_{\text{gvm}}$ required to a test particle to come back to some fixed reference direction in the orbital plane from which it began its motion, assumed circular throughout the overall variation of the azimuthal angle $\varphi$ reckoned from such a line and spanning an interval of $2\pi$, when the general relativistic gravitomagnetic acceleration is added to the Newtonian inverse-square one. The unit vector $\hat{J}$ of the primary’s angular momentum is assumed to be known, so that one can align the reference $z$ axis with it, and the reference $(x, y)$ plane coincides with the equatorial one of the source. It is assumed the point of view of a distant observer fixed with respect

lorenzo.iorio@libero.it
to distant stars who uses the coordinate time \( t \) as own proper time; the difference of the proper times \( \tau \) of the counter-orbiting test particles is identical to \( T_{\text{gvm}} \) up to corrections of order \( O\left(c^{-4}\right) \). Indeed, it is \( d\tau = \sqrt{g_{00}} \, dt = \sqrt{1 + h_{00}} \, dt = (1 + h_{00}/2) \, dt \), where \( g_{00} \) is the “00” component of the spacetime metric tensor, and \( h_{00} \) is a small correction of post-Newtonian (pN) order.

In this paper, we will numerically check such an established result by integrating the equations of motion of a fictitious test particle orbiting a putative spinning body by removing the foregoing restrictions about \( J \). Stated differently, a circular and equatorial orbit will be considered, but its orientation in the reference frame adopted will be arbitrary. It turns out that, in such a scenario, the line of the nodes, among other things, remains fixed; then, it will be naturally assumed as reference polar axis from which the azimuthal angle \( \varphi \) is counted in the orbital plane. Furthermore, the argument of latitude \( u \), which reckons the instantaneous position of the test particle along its orbit just from the unit vector \( \hat{r} \) of the line of the nodes, will play the role of \( \varphi \). Thus, the time interval\(^1\) between two consecutive passages of the fixed line of the nodes by the test particle will be inspected by looking at the time needed to the cosine of the angle between its position vector \( r \) and \( \hat{r} \) to assume again its initial value. The numerical integration will be performed for both the senses of motion of the test particle. As a result, a discrepancy larger than the established form of the gravitomagnetic clock effect by a multiplicative factor of\(^2\) 4 will be found, in agreement with an analytical calculation recently appeared in the literature (Iorio 2016).

The paper is organized as follows. In Section 2, the gravitomagnetic acceleration experienced by a satellite orbiting a rotating primary and some of its main orbital effects are reviewed. The gravitomagnetic features of circular and equatorial orbits arbitrarily oriented in space are outlined in Section 3. Section 4 is devoted to the numerical integrations of the equations of motion in the scenario of Section 3. Section 5 deals with a tentative explanation of the resulting discrepancy with respect to the standard case. The impact of the gravitomagnetic field on the interval between two consecutive passages of the test particle at the periapsis, a scenario of Section 3. Section 4 of Lichtenegger et al. (2006) for an alternative clock effect by a multiplicative factor of\(^2\) 4 will be found, in agreement with an analytical calculation recently appeared in the literature (Iorio 2016).

The numerical integration will be performed for both the senses of motion of the test particle. As a result, a discrepancy larger than the established form of the gravitomagnetic clock effect by a multiplicative factor of\(^2\) 4 will be found, in agreement with an analytical calculation recently appeared in the literature (Iorio 2016).

2. The Lense-Thirring acceleration and some of its orbital consequences

To the first post-Newtonian (1pN) order, the mass-energy currents of an isolated, spinning body perturb the orbital motion of a test particle which experiences the gravitomagnetic Lense-Thirring (LT) acceleration (Soffel 1989; Petit and Luzum 2010; Poisson and Will 2014; Soffel and Han 2019)

\[
A_{\text{LT}} = \frac{2GJ}{c^2 a^3} \left[ 3 \left( \hat{r} \right) \hat{r} \times v + v \times \hat{J} \right].
\]

in addition to the Newtonian one.

The \( R - T - N \) components of Equation (1) turn out to be

\[
A_{R}^{\text{LT}} = \frac{2n_{\text{Kep}} GJ (1 + e \cos f)^4 \left( \hat{J} \cdot \hat{h} \right)}{c^2 a^2 (1 - e^2)^{3/2}},
\]

\[
A_{T}^{\text{LT}} = -\frac{2e n_{\text{Kep}} GJ (1 + e \cos f)^3 \sin f \left( \hat{J} \cdot \hat{h} \right)}{c^2 a^2 (1 - e \cos f)^{3/2}},
\]

\[
A_{N}^{\text{LT}} = -\frac{2n_{\text{Kep}} GJ (1 + e \cos f)^3 \left[ e \cos \omega - (2 + 3e \cos f) \cos u \right] \left( \hat{J} \cdot \hat{\bar{J}} \right) - \frac{1}{2} \left[ e \sin \omega + 4 \sin u + 3e \sin (2f + \omega) \right] \left( \hat{J} \cdot \hat{m} \right)}{c^2 a^2 (1 - e^2)^{3/2}}.
\]

Equation (1), through Equations (2)–(4), induces the LT effect (Lense and Thirring 1918; Mashhoon et al. 1984) consisting of the following secular precessions of some Keplerian orbital elements (Barker and O’Connell 1975; Damour and Schäfer 1988; Damour and Taylor 1992; Wex and Kopeikin 1999; Kraniotis 2007; Will 2008; Iorio 2017)

\[
\frac{d\hat{J}}{dt} = \frac{2GJ}{c^2 a^3 (1 - e^2)^{3/2}} \left( \hat{J} \cdot \hat{I} \right).
\]

\(^1\) In the general case, it is known as draconitic period. See Capderou (2005) or Footnote 1 of Iorio (2016) for the origin of such a name.

\(^2\) See Section 4 of Lichtenegger et al. (2006) for an alternative clock effect with the present treatment, despite less general.
\[ \frac{d\Omega}{dt} = \frac{2 G J \csc I (\hat{J} \cdot \hat{m})}{c^2 a^3 (1 - e^2)^{3/2}}, \tag{6} \]

\[ \frac{d\omega}{dt} = -\frac{2 G J \left[ 2 (\hat{J} \cdot \hat{h}) + \cot I (\hat{J} \cdot \hat{m}) \right]}{c^2 a^3 (1 - e^2)^{3/2}}. \tag{7} \]

They are currently under measurement with some Earth’s geodetic satellites (Pearlman et al. 2019) tracked with the Satellite Laser Ranging (SLR) technique (Coulot et al. 2011); see, e.g., (Iorio et al. 2011; Renzetti 2013; Lucchesi et al. 2020), and references therein.

A further gravitomagnetic effect, i.e. the spin precessions of orbiting gyroscopes (Pugh 1959; Schiff 1960), was recently measured in the Earth’s field with the Gravity Probe B (GP-B) mission to a 19% accuracy level (Everitt et al. 2011, 2015; Will 2011); the originally expected uncertainty was \(\approx 1\%\) (Everitt 1974; Everitt et al. 2001).

3. Gravitomagnetic effects for a circular and equatorial orbit

Let us consider an orbital configuration such that the satellite’s angular momentum \(\hat{h}\) is (anti-)parallel to the body’s spin axis, irrespectively of the orientation of the latter one in the reference frame adopted, i.e. such that

\[ \hat{J} \cdot \hat{h} = \pm 1, \tag{8} \]

\[ \hat{J} \cdot \hat{l} = \hat{J} \cdot \hat{m} = 0; \tag{9} \]

see Figure 1 and Figure 2.

In this case, according to Equations (5)–(6), the satellite’s orbital plane stays unchanged in space, being aligned with the equator of the primary. The same occurs also if the latter is oblate since the resulting Newtonian precessions of the node and the inclination are proportional to \(\hat{J} \cdot \hat{m}\) and \(\hat{J} \cdot \hat{l}\), respectively (Iorio 2017). In particular, the line of nodes remains fixed, and \(\hat{l}\) is constant; thus, it can be naturally assumed as a reference direction in the orbital plane. Instead, in general, the line of the apsides changes because of Equation (7). The argument of latitude \(u\), counted just from \(\hat{l}\) taken as reference polar axis, can be used as polar angle reckoning the instantaneous position of the test particle along its orbit. According to Equations (2)–(4), if Equations (8)–(9) hold, the normal component of the LT acceleration vanishes, and the gravitomagnetically perturbed motion is entirely in-plane.

If the orbit is also circular, then Equation (1) is only radial, reducing to

\[ A_{LT} = \pm \frac{2 n_{Kep} G J}{c^2 r_0^2}. \tag{10} \]

The “+” and “−” signs in Equation (10) correspond to the prograde \((\hat{J} \cdot \hat{h} = +1)\) and retrograde \((\hat{J} \cdot \hat{h} = -1)\) motion with respect to the sense of rotation of the primary, respectively; Figure 1 shows the first case, while the second one is depicted in Figure 2.

It can be noted that when the test particle orbits the central body along the same sense of the rotation of the latter, i.e. for the “+” sign, Equation (10) is radially directly outward, weakening the overall gravitational pull to which the test particle is subjected; thus, an increase in the time required to complete a full orbital revolution, which, in this case, can be naturally assumed as the time interval between two consecutive crossings of the fixed line of the nodes, is expected since the motion is slower. On the contrary, when the satellite moves in the opposite direction with respect to the sense of rotation of its primary, the overall gravitational tug experienced by the former is enhanced since Equation (10) is radially directed inward (“−” sign), the motion is faster, and the completion of an orbital revolution is expected to take less time. Mitskevich and Pulido Garcia (1970) obtained, incorrectly, the opposite result, while Vladimirov et al. (1987) agreed with the picture just outlined.

In the next Section, we will quantitatively assess such distinctive features of motion, whose non-Machian nature was discussed by Mashhoon et al. (2001), by numerically integrating the equations of motion of a fictitious test particle orbiting a hypothetical massive, spinning body.

4. Numerical integrations of the equations of motion
Figure 1. Prograde circular equatorial orbit arbitrarily oriented in space with, say, $I = 30^\circ$, $\Omega = 45^\circ$. The orbital plane is aligned with the equator of the central body, and the test particle moves along the same sense of rotation of the latter, so that $\hat{J} \cdot \hat{h} = +1$.

Table 1. Relevant physical parameters of Jupiter (Petit and Luzum 2010; Iess et al. 2018).

| Parameter | Units     | Numerical value                  |
|-----------|-----------|----------------------------------|
| $\mu$     | m$^3$ s$^{-2}$ | $1.26713 \times 10^{17}$ (Petit and Luzum 2010) |
| $\alpha_J$ | deg       | 268.057132 (Iess et al. 2018)    |
| $\delta_J$ | deg       | 64.497159 (Iess et al. 2018)     |

Let us consider a fictitious test particle orbiting a massive, spinning body with the same mass and spin axis orientation of, say, Jupiter, whose relevant physical parameters are listed in Table 1. It is assumed that the satellite moves along a circular and equatorial prograde orbit starting from the ascending node along the line of the nodes, i.e., $\mathbf{J} \cdot \mathbf{h} = +1$ and $\mathbf{r}_0 \cdot \mathbf{l} = +1$. Its equations of motion, in Cartesian coordinates, are numerically integrated with and without the LT acceleration of Equation (1) starting from the same initial conditions. The angular momentum $\mathbf{J}$ of the fictitious primary is assumed large enough to easily visualize its gravitomagnetic effects remaining, at the same time, to a level compatible with the 1pN approximation. In particular, a value for $\mathbf{J}$ is adopted such that the ratio of Equation (10) to the Newtonian monopole term is as little as 0.005. Then, a time series for the cosine of the angle between the line of the nodes and the test particle’s position vector, i.e., $\mathbf{r} \cdot \mathbf{l}$, is numerically produced in both the runs and plotted versus time in order to reckon the instant of time when $\mathbf{l}$ is crossed again after the initial instant, i.e., when $\mathbf{r} \cdot \mathbf{l} = +1$ again. Figure 3 displays both the Keplerian and the gravitomagnetic time series over one orbital revolution. It turns out that, when Equation (1) is included in the equations of motion, the passage of the test particle at the line of the nodes occurs after an amount of time equal to the Keplerian orbital period increased by the gravitomagnetic correction

$$
\delta T^+_{\text{gvm}} := +8\pi \frac{J}{M c^2},
$$

(11)
which, in the fictional example considered, amounts to the 1% of $P_{\text{Kep}}$; such a retardation is represented by the shaded area in Figure 3.

Figure 4 depicts the same scenario, but for a retrograde orbit, i.e., with $\hat{\mathbf{J}} \cdot \hat{\mathbf{h}} = -1$. In this case, the crossing of the line of the nodes occurs in advance with respect to the purely Keplerian case by the same quantity of Equation (11); such an advance is represented by the shaded area in Figure 4. Further numerical integrations showed that by varying the orbital radius does not affect the magnitude of the gravitomagnetic retardation/advance.

Figure 5 deals with the instant of time $\tilde{t}$ when the two counter-orbiting test particles meet each other for the first time after their common start. In the Keplerian case, such an event would occur just after half a revolution at $\tilde{\varphi}_{\text{Kep}} = \pi$. Instead, when the LT acceleration of Equation (1) is present, it is expected that the retrograde particle, which moves faster, reaches the prograde one, which is slower, before than in the static case. In Figure 5, the numerically produced time series of $\hat{\mathbf{r}}^\pm \cdot \hat{\mathbf{l}}$, obtained by including Equation (1) in the equations of motion of both particles, are plotted together, and the time when they cross is marked with a vertical dashed green line. It turns out that

$$\tilde{t} = \frac{P_{\text{Kep}}}{2} \left(1 - \epsilon_{\text{gvm}}^2\right),$$

(12)

with

$$\epsilon_{\text{gvm}} := \frac{|\delta T_{\text{gvm}}|}{P_{\text{Kep}}} = 8\pi \frac{J}{M c^2 P_{\text{Kep}}^2}.$$ 

(13)

5. Investigating the discrepancy with the standard scenario
Figure 3. Plot of the numerically produced time series of $\hat{r} \cdot \hat{l}$ versus time $t$ with and without the LT acceleration of Equation (1) for a circular and equatorial orbit arbitrarily oriented in space starting from the ascending node with $\hat{J} \cdot \hat{h} = +1$ (prograde motion). The relevant physical parameters of the putative primary are in Table 1. For the sake of clarity, a value of $J$ such that $A_{LT}/A_N = 0.005$ was adopted. It turns out that the time intervals between two consecutive crossings of the line of the nodes, which is fixed both in the Keplerian and LT cases, differ by $+8\pi J/(M c^2)$ which, in units of $P_{Kep}$, amounts to 0.01. The shaded area represents such a retardation with respect to the Keplerian period.

Theory predicts just what we came up with in Section 4, as shown by Eq. (74) and Eq. (77) of Iorio (2016). Such an outcome has an impact also on the gravitomagnetic clock effect which, now, is

$$\Delta T_{gvm} := T_{gvm}^+ - T_{gvm}^- = 16\pi \frac{J}{M c^2}.$$  \hspace{1cm} (14)

For the Earth, since its angular momentum per unit mass amounts to (Petit and Luzum 2010) $J_\oplus/M_\oplus \approx 9 \times 10^8 \text{ m}^2 \text{s}^{-1}$, Equation (14) returns $\Delta T_{gvm}^\oplus \approx 5 \times 10^{-7} \text{ s}$. The angular momentum of Jupiter is (Soffel et al. 2003) $J_{\text{Jup}} \approx 6.9 \times 10^{38} \text{ kg} \text{ m}^2 \text{s}^{-1}$; then, one has $\Delta T_{gvm}^{\text{Jup}} \approx 2 \times 10^{-4} \text{ s}$. Since the Sun’s angular momentum is (Pijpers 1998) $J_\odot \approx 1.90 \times 10^{41} \text{ kg} \text{ m}^2 \text{s}^{-1}$, the solar gravitomagnetic clock effect is $\Delta T_{gvm}^\odot \approx 5 \times 10^{-5} \text{ s}$. Interestingly, for a hypothetical pair of counter-revolving S-stars in the equatorial
plane of the supermassive black hole in Sgr A* at the Galactic Center, Equation (14), calculated for $J_*=\chi_*\left(M_2^*G\right)/c$ with (Daly et al. 2023) $\chi_* \approx 0.90$ and (Gillessen et al. 2009) $M_* \approx 4.1 \times 10^6 M_\odot$, yields $\Delta T_{\gamma m}^{*} \approx 9.13 \times 10^2 \text{s}$.

Also the analytical calculation of $\tilde{r}$, obtained by following the reasoning by Mashhoon et al. (2001) extended to the approach by Iorio (2016), agrees with Equation (12). One has, first, to compute with the theoretical model used the instant of times $\tilde{r}^{\pm}$ corresponding to the fixed angles $\Phi$ and $2\pi - \Phi$ for the prograde and retrograde directions, respectively. Then, by imposing the condition $\tilde{r}^+ = \tilde{r}^- = \tilde{r}$, one gets $\tilde{r}$ and $\tilde{\Phi}$. By using the calculational scheme in Iorio (2016), it can be obtained

$$\tilde{r}^+ = \left(\frac{1}{n_{\text{Kep}}} + 4 \frac{J}{M c^2}\right)\tilde{u},$$

Figure 4. Plot of the numerically produced time series of $\hat{r} \cdot \hat{l}$ versus time $t$ with and without the LT acceleration of Equation (1) for a circular and equatorial orbit arbitrarily oriented in space starting from the ascending node with $\hat{J} \cdot \hat{h} = -1$ (retrograde motion). The relevant physical parameters of the putative primary are in Table 1. For the sake of clarity, a value of $J$ such that $A_{LT}/A_N = 0.005$ was adopted. It turns out that the time intervals between two consecutive crossings of the line of the nodes, which is fixed both in the Keplerian and LT cases, differ by $-8\pi J/(Mc^2)$ which, in units of $P_{\text{Kep}}$, amounts to $-0.01$. The shaded area represents such an advance with respect to the Keplerian period.
Figure 5. Plots of the numerically produced time series of $\hat{r} \cdot \hat{I}$ and $\hat{r} \cdot \hat{l}$ versus time $t$ with the LT acceleration of Equation (1) for a circular and equatorial orbit arbitrarily oriented in space starting from the ascending node with $J \cdot \vec{n} = \pm 1$. The relevant physical parameters of the putative primary are in Table 1. For the sake of clarity, a value of $J$ such that $A_{LT}/A_{N} = 0.005$ was adopted. It turns out that the two counter-orbiting test particles meet at $\bar{\tau} = P_{Kep}/2 \left(1 - \epsilon_{gvm}^2\right)$, with $\epsilon_{gvm}$ given by Equation (13), represented by the vertical dashed green line.

$$\bar{\tau} = \left(\frac{1}{\bar{n}_{Kep}} - 4 \frac{J}{M c^2}\right) (2\pi - \bar{u})$$; 

(16)

see below for further details. By requiring that the right-hand sides of Equations (15)–(16) are equal and solving for $\bar{u}$, one gets

$$\bar{u} = \pi \left(1 - \epsilon_{gvm}\right).$$

(17)

Inserting Equation (17) in any of Equations (15)–(16) yields

$$\bar{\tau} = \frac{P_{Kep}}{2} \left(1 - \epsilon_{gvm}^2\right).$$

(18)

in agreement with Equation (12).
On the other hand, as far as $\Delta T_{\text{gvm}}$ is concerned, a discrepancy by a factor of 4 occurs with respect to the so-far accepted result$^3$ (Vladimirov et al. 1987; Cohen and Mashhoon 1993; Mashhoon 1997; You 1998; Tartaglia 2000c; Mashhoon et al. 1999; Mashhoon and Santos 2000; Mashhoon et al. 2001)

$$\delta T_{\text{gvm}}^\pm := \pm 2\pi \frac{J}{M c^2}$$

(19)

which can be obtained by equating the centripetal acceleration $\omega^2 r_0$ to the sum of the Newtonian monopole plus Equation (10). Indeed, from

$$\omega^2 r_0 = \frac{\mu}{r_0^2} + \frac{2 n_{\text{Kep}} G J}{c^2 r_0^2}$$

(20)

one gets

$$\frac{d\phi}{dr} = \pm n_{\text{Kep}} \sqrt{1 \mp \frac{2 n_{\text{Kep}} J}{M c^2}}$$

(21)

which yields

$$\frac{dr}{d\phi} = \pm \frac{1}{n_{\text{Kep}} \sqrt{1 \mp \frac{2 n_{\text{Kep}} J}{M c^2}}} \approx \pm \frac{1}{n_{\text{Kep}}} \left(1 \pm \frac{n_{\text{Kep}} J}{M c^2}\right).$$

(22)

By integrating Equation (22) with respect to $\phi$ from 0 to $+2\pi$ for the prograde motion and from 0 to $-2\pi$ for the retrograde one, the LT orbital period

$$T_{\text{gvm}}^\pm = P_{\text{Kep}} + \delta T_{\text{gvm}}^\pm := \frac{2\pi}{n_{\text{Kep}}} \pm \frac{2\pi}{M c^2}$$

(23)

is obtained. Note that $\phi$ is a polar angle counted from some fixed reference polar axis in the orbital plane aimed to instantaneously locate the test particle along its circular orbit; thus, for an equatorial orbit, it is straightforward to identify the fixed line of the nodes with the reference direction and $\phi$ with the argument of latitude $u$.

The explanation in the aforementioned discrepancy likely resides in the fact that the more general calculation in Iorio (2016), made by using the non-singular elements $q$ and $k$, accounts for the fact that, during two consecutive crossings of the line of the nodes, the orbital elements in terms of which $dt/du$ is parameterized, i.e. $p$, $q$ and $k$, do actually change instantaneously. In the general case, also the line of the nodes does not stay fixed; such a feature is captured by the calculation in Iorio (2016) as well. It turns out that such an effect does not vanish even in the limit $q, k \to 0$ corresponding to a circular orbit. Indeed, a step-by-step analysis of the calculation in Iorio (2016) made with the LT acceleration of Equation (1) shows that Equation (11) comes from the sum of

$$\int_0^{2\pi} \frac{\partial(dt/du)}{\partial q} \Delta q (u) \, du,$$

(24)

$$\int_0^{2\pi} \frac{\partial(dt/du)}{\partial k} \Delta k (u) \, du$$

(25)

which, for $q, k \to 0$, do not vanish yielding

$$4 \frac{J}{M c^2} \int_0^{2\pi} \cos u \left(\cos u - \cos u_0\right) \, du = +4\pi \frac{J}{M c^2},$$

(26)

$$4 \frac{J}{M c^2} \int_0^{2\pi} \sin u \left(\sin u - \sin u_0\right) \, du = +4\pi \frac{J}{M c^2}.$$  

(27)

Instead, it turns out that $\partial (dt/du)/\partial p \Delta p (u) = 0$ since, in the limit $q, k \to 0$, the instantaneous variation $\Delta p (u)$ of the semilatus rectum $p$ vanishes. Also the term due to the change of the line of the nodes containing $d\Omega/du$ (Iorio 2016) is zero for an equatorial orbit since, according to Equation (4) and Equations (8)–(9), $A_{N N}^T = 0$. The opposite sign in Equations (26)–(27) is obtained for the retrograde motion. It can, now, be noted that Equation (15) comes from the sum of

$$\int_0^{\pi} \frac{\partial(dt/du)}{\partial q} \Delta q (u) \, du = +2 \frac{J}{M c^2} \overline{u},$$

(28)

$^3$ Mitskevich and Pulido Garcia (1970), incorrectly, obtained the opposite signs.
\[
\int_{0}^{2\pi} \frac{\partial (dt/du)}{\partial k} \Delta k(u) \, du = +2 \frac{J}{M c^2} \bar{u},
\]  
(29)

calculated with \( \hat{J} \cdot \hat{h} = +1 \). On the other hand, Equation (16) is the sum of
\[
\int_{0}^{2\pi - \bar{u}} \frac{\partial (dt/du)}{\partial q} \Delta q(u) \, du = -2 \frac{J}{M c^2} (2\pi - \bar{u}),
\]  
(30)
\[
\int_{0}^{2\pi - \bar{u}} \frac{\partial (dt/du)}{\partial k} \Delta k(u) \, du = -2 \frac{J}{M c^2} (2\pi - \bar{u}),
\]  
(31)
calculated with \( \hat{J} \cdot \hat{h} = -1 \).

Instead, the integration based on Equation (20) is performed by considering only \( \varphi \) as variable during an orbital revolution, all the rest being kept fixed.

6. The apsidal period

Our numerical tests confirm also another result of Iorio (2016); in the case of an elliptical orbit, there is no LT correction to the apsidal period, i.e. the time interval between two consecutive crossings at the (moving) periapsis, for both the senses of motion. Figure 6 plots the numerically produced time series of the cosine of the angle between the position vector and the Laplace-Runge-Lenz vector \( A \) over an orbital revolution with and without the LT acceleration of Equation (1) for a highly elliptical and equatorial orbit arbitrarily oriented in space: both the runs share the same initial conditions corresponding to \( \hat{r}_0 \cdot \hat{A}_0 = +1 \) and \( \hat{J} \cdot \hat{h} = +1 \) (prograde motion along an equatorial orbit). It turns out that the apsidal periods are the same in both the Keplerian and LT cases. The same occurs also for \( \hat{J} \cdot \hat{h} = -1 \) (retrograde motion along an equatorial orbit), as shown by Figure 7. Further numerical integrations show that the anomalistic period is not impacted by the gravitomagnetic field of the primary also for non-equatorial orbits, in agreement with Iorio (2016).

The fact that the gravitomagnetic apsidal period is identical to the Keplerian one can be intuitively justified since, as per Equations (5)–(6), there is not net shift per orbit of the mean anomaly at epoch \( \eta \). Indeed, from the definition of the mean anomaly
\[
\mathcal{M}(t) := \eta + n_{\text{Kep}} \left( t - t_0 \right) = n_{\text{Kep}} \left( t - t_\text{p} \right),
\]  
(32)

it turns out that
\[
\eta = n_{\text{Kep}} \left( t_0 - t_\text{p} \right).
\]  
(33)

Thus, since \( n_{\text{Kep}} \) stays constant because the semimajor axis is not secularly affected by the gravitomagnetic field, the rate of change of the mean anomaly at epoch is proportional to the opposite of the pace of variation of the time of passage at pericenter according to
\[
\frac{d\eta}{dt} = -n_{\text{Kep}} \frac{dt_\text{p}}{dt}.
\]  
(34)

Should \( \eta \) increase, the crossing of the pericenter would be anticipated with respect to the Keplerian case since \( t_\text{p} \) would decrease, and vice versa. In this case, the variation of \( \eta \) would result in an orbit-by-orbit advance or delay of the passages at the pericenter, which does not occur in the present case because, in fact, \( d\eta/dt = 0 \) for Equation (1). Instead, the 1pN gravitelectric acceleration due to the mass monopole of the source affects \( \eta \) with a negative rate corresponding to an increase of \( t_\text{p} \) (Iorio 2016).

7. A gravitomagnetic clock effect for LAGEOS and LARES 2; is it possible to measure it?

Strengthened by the numerical confirmations obtained in the previous Sections of our recent analytical results for the pK orbital periods (Iorio 2016), we can trustworthily apply them to some specific situations of interest.

It turns out that a form of gravitomagnetic clock effect can be derived for the laser-ranged geodetic satellites LAGEOS (L) (Cohen and Smith 1985) and LARES 2 (LR 2) (Paolozzi et al. 2019) in view of their peculiar orbital configuration characterized by essentially identical circular orbits lying in orbital planes inclined to the Earth’s equator by
\[
I_L \approx 110^\circ,
\]  
(35)
Figure 6. Plot of the numerically produced time series of $\mathbf{r} \cdot \mathbf{A}$ versus time $t$ with and without the LT acceleration of Equation (1) for an elliptical ($e = 0.8$) and equatorial orbit arbitrarily oriented in space starting from the periapsis with $\mathbf{j} \cdot \mathbf{h} = +1$ (prograde motion). The relevant physical parameters of the putative primary are in Table 1. For the sake of clarity, a value of $J$ such that $A_{LT}/A_N = 0.005$ was adopted. It turns out that the time intervals between two consecutive crossings of the apsidal line, which varies when Equation (1) is present, are identical in both the Keplerian and LT cases.

$$I_{LR2} \approx \pi - I_L = 70^\circ,$$

respectively. Indeed, although none of them move in the Earth’s equatorial plane, their supplementary inclinations make them feasible, at least in principle, to detect the difference of the gravitomagnetic corrections to their sidereal periods meant as the time intervals between two consecutive crossings of a fixed reference direction in space.

By aligning $\mathbf{j}$ along the reference $z$ axis of a coordinate system whose fundamental plane is parallel to the Earth’s equator, Equation (76) and Equation (78) of Iorio (2016), valid to zero order in $e$, yield

$$\delta T_{gvm}^{sid} = 4\pi (-1 + 3 \cos I) \frac{J}{M c^2}.$$

(37)
Figure 7. Plot of the numerically produced time series of $\hat{r} \cdot \hat{A}$ versus time $t$ with and without the LT acceleration of Equation (1) for an elliptical ($e = 0.8$) and equatorial orbit arbitrarily oriented in space starting from the periapsis with $\hat{J} \cdot \hat{k} = -1$ (retrograde motion). The relevant physical parameters of the putative primary are in Table 1. For the sake of clarity, a value of $J$ such that $A_{LT}/A_N = 0.005$ was adopted. It turns out that the time intervals between two consecutive crossings of the apsidal line, which varies when Equation (1) is present, are identical in both the Keplerian and LT cases.

By calculating Equation (37) with Equations (35)–(36) one obtains

$$\Delta T_{gvm}^{\text{sid}} := \delta T_{gvm}^{\text{sid1}} - \delta T_{gvm}^{\text{sidLR2}} = 24 \pi \cos I L J M c^2 = -2.8 \times 10^{-7} \text{ s.} \quad (38)$$

Unfortunately, the mutual cancelation of the Keplerian orbital periods $P_{Kep}$, which should be ideally identical, is far from being perfect. Indeed, for (Ciufolini et al. 2023)

$$a_L = 12270.020705 \text{ km}, \quad (39)$$

$$a_{LR2} = 12266.1359395 \text{ km}, \quad (40)$$
it is
\[
\Delta P_{\text{Kep}} := P_{\text{Kep}}^L - P_{\text{Kep}}^{LR} = \frac{2\pi}{\sqrt{\mu_0}} \left( \sqrt{a_L^3} - \sqrt{a_{LR}^3} \right) = 6.4 \, \text{s}. \tag{41}
\]

What is worse, \(\Delta P_{\text{Kep}}\) cannot be even known with good enough accuracy to allow it to be safely subtracted from the measured difference of the satellites’ orbital periods. Indeed, since the currently claimed uncertainties in the spacecraft’s semimajor axes are (Ciufolini et al. 2023)
\[
\sigma_{a_L} = 10^{-6} \, \text{km} = 1 \, \text{mm}, \tag{42}
\]
\[
\sigma_{a_{LR}} = 10^{-7} \, \text{km} = 0.1 \, \text{mm}, \tag{43}
\]
the uncertainty in the difference of the Keplerian periods is as large as
\[
\sigma_{\Delta P_{\text{Kep}}} = 1.6 \times 10^{-6} \, \text{s}, \tag{44}
\]
which is about six times larger than the gravitomagnetic clock effect of Equation (38). Incidentally, the Earth’s gravitational parameter \(\mu_0\) would not be a limiting factor since its error (Petit and Luzum 2010) \(\sigma_{\mu_0} = 8 \times 10^5 \, \text{m}^3 \text{s}^{-2}\) contributes to \(\sigma_{\Delta P_{\text{Kep}}}\) at the \(\approx 6 \times 10^{-9} \, \text{s}\) level. Only an improvement of the already unrealistically small errors of Equations (42)–(43) by three orders of magnitude would allow to bring down \(\sigma_{\Delta P_{\text{Kep}}}\) to the \(\approx 10^{-9} \, \text{s}\) level.

8. Summary and conclusions

A recently published analytical calculation of the 1pN gravitomagnetic correction to the draconitic period of a test particle moving along a circular orbit arbitrarily oriented with respect to the equatorial plane of a massive, spinning primary can be specialized to the case in which the angular momentum \(J\) of the latter, on whose orientation in space no constraints are assumed as well, is (anti)parallel to the orbital angular momentum \(\hbar\). In such a scenario, in which the orbital plane lies in the equatorial plane from which the pole angle \(\varphi\), represented in this case by the argument of latitude \(u\), is reckoned. It turns out that the corrections \(\delta T_{\text{sym}}^L\), to be added to the Keplerian periods \(P_{\text{Kep}}\) in order to have the time intervals between two consecutive crossings of the line of the nodes, are larger than the ones present so far in the literature, equal to \(\pm 2\pi J / (M \, c^2)\) depending on the prograde or retrograde directions, respectively, by a factor of 4 in both the senses of motion. Also the amount by which the angle \(\overline{\varphi}\) marking the encounter of the two counter-rotating particles is reduced with respect to the Keplerian case is 4 times larger than the value quoted in the literature.

All such features are fully confirmed by numerical integrations of the equations of motion of a fictitious test particle orbiting a putative massive, spinning object and experiencing, to the 1pN order, the gravitomagnetic acceleration \(A_{LT}\) induced by \(J\) in addition to the Newtonian inverse-square one \(A_N\). In particular, the time required by the cosine of the angle between the satellite’s position vector \(r\) and the unit vector \(\hat{l}\) of the line of the nodes to take its initial value again, assumed equal to unity, turns out to be given by the Keplerian orbital period increased or decreased by \(8\pi J / (M \, c^2)\) for the prograde or retrograde senses of motion, respectively. Such a discrepancy with respect to the commonly known scenario may be due to the fact that the latter is based on the strict validity of the circularity condition throughout the integration over one orbital revolution, not allowing any instantaneous variations of the orbital parameters entering the analytical expression of the time derivative of the polar angle. Such a feature is, instead, accounted for in the aforementioned recent calculation which takes into account also the possible motion of the line of the nodes itself occurring, in general, for a non-equatorial orbits. As a result, the revised gravitomagnetic clock effect for identical circular and equatorial orbits traveled in opposite directions turns out to be \(16\pi J / (M \, c^2)\). Such a result has potential relevance to astrophysical observations of the environment of compact objects around which the gravitomagnetic field plays a major roles.

The numerical experiments performed in the present work confirmed also another analytically inferred distinctive feature of the temporal structure around a rotating body: there are no gravitomagnetic corrections to the anomalistic periods for both the senses of motion. It is so because the time of passage at pericenter is left unaffected by the gravitomagnetic acceleration, contrary to the 1pN gravitoelectric one.

Although they do not share the same orbital plane nor do any of them move in the Earth’s equatorial plane, the artificial satellites LAGEOS and LARES 2 may be used, at least in principle, to detect a form of clock effect involving the difference of their sidereal orbital periods which is, ideally, entirely gravitomagnetic, amounting to \(\approx 3 \times 10^{-7} \, \text{s}\). Actually, despite the close values of their semimajor axes, their Keplerian orbital periods do not cancel each other, being their difference uncertain at a \(\approx 10^{-6} \, \text{s}\) level due to the current claimed errors in the aforementioned orbital elements.
A. Notations and definitions

Here, some basic notations and definitions used throughout the text are presented (Soffel 1989; Brumberg 1991; Bertotti et al. 2003; Kopeikin et al. 2011; Poisson and Will 2014; Soffel and Han 2019).

- \( c \): speed of light in vacuum
- \( G \): Newtonian constant of gravitation
- \( M \): mass of the central body
- \( \mu := GM \): gravitational parameter of the central body
- \( J \): angular momentum of the central body
- \( J \): magnitude of the angular momentum of central body
- \( \alpha_J \): right ascension (RA) of the north pole of rotation of the central body
- \( \delta_J \): declination (DEC) of the north pole of rotation of the central body
- \( \hat{J} = \{ \cos \alpha_J \cos \delta_J, \sin \alpha_J \cos \delta_J, \sin \delta_J \} \): spin axis of the central body
- \( A \): perturbing acceleration experienced by the test particle
- \( A_N \): Newtonian inverse-square acceleration
- \( \phi \): azimuthal angle reckoning the instantaneous position of the test particle in its orbital plane
- \( \omega := d\phi/dt \): azimuthal angular speed of the test particle in its orbital plane
- \( T_{gvm} \): orbital period in presence of the LT acceleration
- \( a \): semimajor axis of the test particle
- \( n_{Kep} := \sqrt{\mu/a^3} \): Keplerian mean motion of the test particle
- \( P_{Kep} := 2\pi/n_{Kep} \): orbital period of the test particle
- \( e \): eccentricity of the test particle
- \( p := a(1 - e^2) \): semilatus rectum of the orbit of the test particle
- \( I \): inclination of the orbital plane of the test particle to the reference plane \( \{x, y\} \)
- \( \Omega \): longitude of the ascending node of the test particle
- \( \Omega \): ascending node
- \( \Omega \): descending node
- \( \omega \): argument of pericenter of the test particle
- \( q := e \cos \omega \): non-singular orbital element \( q \)
- \( k := e \sin \omega \): non-singular orbital element \( k \)
- \( f (t) \): true anomaly of the test particle
- \( u := \omega + f \) argument of latitude of the test particle
- \( \eta \): mean anomaly at epoch
- \( t_0 \): initial instant of time
\( t_p \): time of passage at pericenter

\( M(\iota) \): mean anomaly

\( r \): position vector of the test particle with respect to the central body

\( r_0 \): radius of a circular orbit

\( \hat{r} := r/r = (\cos \Omega \cos u - \cos I \sin \Omega \sin u, \sin \Omega \cos u + \cos I \cos \Omega \sin u, \sin I \sin u) \): radial unit vector

\( v \): velocity vector of the test particle

\( \mathbf{h} = r \times v \): orbital angular momentum per unit mass

\( \hat{\mathbf{h}} := (\sin I \sin \Omega, -\sin I \cos \Omega, \cos I) \): unit vector of the orbital angular momentum such that \( \hat{i} \times \hat{\mathbf{m}} = \hat{\mathbf{h}} \)

\( A := v \times \mathbf{h} - \mu \hat{r} \): Laplace-Runge-Lenz vector per unit mass

\( \hat{i} := (\cos \Omega, \sin \Omega, 0) \): unit vector directed along the line of the nodes toward the ascending node

\( \hat{\mathbf{m}} := (-\cos I \sin \Omega, \cos I \cos \Omega, \sin I) \): unit vector directed transversely to the line of the nodes in the orbital plane

\( \hat{\mathbf{s}} := \hat{\mathbf{h}} \times \hat{\mathbf{r}} = (-\sin u \cos \Omega - \cos I \sin \Omega \cos u, -\sin \Omega \sin u + \cos I \cos \Omega \cos u, \sin I \cos u) \): transverse unit vector

\( A_R := A \cdot \hat{r} \): radial component of the perturbing acceleration \( A \)

\( A_T := A \cdot \hat{s} \): transverse component of the perturbing acceleration \( A \)

\( A_N := A \cdot \hat{h} \): normal component of the perturbing acceleration \( A \)

**Data availability**

No new data were generated or analysed in support of this research.

**Conflict of interest statement**

I declare no conflicts of interest.

**References**

B. M. Barker and R. F. O’Connell. Gravitational two-body problem with arbitrary masses, spins, and quadrupole moments. *Phys. Rev. D*, 12(2):329–335, Jul 1975. [https://doi.org/10.1103/PhysRevD.12.329](https://doi.org/10.1103/PhysRevD.12.329).

B. Bertotti, P. Farinella, and D. Vokrouhlický. *Physics of the Solar System*. Kluwer, Dordrecht, Aug 2003. [https://doi.org/10.1007/978-94-010-0233-2](https://doi.org/10.1007/978-94-010-0233-2).

D. Bini and R. T. Jantzen. Gravitomagnetic Clock Effects in Black Hole Spacetimes. In G. Marmo, C. Rubano, and P. Scudellaro, editors, *General Relativity, Cosmology and Gravitational Lensing*, volume 6 of *Napoli series on Physics and Astrophysics*, pages 17–28. Bibliopolis, Napoli, Feb 2003. [https://doi.org/10.48550/arXiv.gr-qc/0102029](https://doi.org/10.48550/arXiv.gr-qc/0102029).

D. Bini, F. de Felice, and A. Geralico. Spinning test particles and clock effect in Kerr spacetime. *Class. Quantum Gravit.*, 21(23):5441–5456, Dec 2004. [https://doi.org/10.1088/0264-9381/21/23/010](https://doi.org/10.1088/0264-9381/21/23/010).

D. Bini, A. Geralico, and R. T. Jantzen. Kerr metric, static observers and Fermi coordinates. *Class. Quantum Gravit.*, 22(22):4729–4742, Nov 2005. [https://doi.org/10.1088/0264-9381/22/22/006](https://doi.org/10.1088/0264-9381/22/22/006).

W. B. Bonnor and B. R. Steadman. The gravitomagnetic clock effect. *Class. Quantum Gravit.*, 16(6):1853–1861, Jun 1999. [https://doi.org/10.1088/0264-9381/16/6/318](https://doi.org/10.1088/0264-9381/16/6/318).

V. A. Brumberg. *Essential Relativistic Celestial Mechanics*. Adam Hilger, Bristol, 1991.

M. Capderou. *Satellites. Orbits and Missions*. Springer-Verlag France, Paris, Mar 2005. [https://doi.org/10.1007/b139118](https://doi.org/10.1007/b139118).

I. Ciufolini, A. Paolozzi, E. C. Pavlis, J. C. Ries, R. Matzner, C. Paris, E. Ortero, V. Gurzadyan, and R. Penrose. The LARES 2 satellite, general relativity and fundamental physics. *Eur. Phys. J. C.*, 83(1):87, Jan 2023. [https://doi.org/10.1140/epjc/s10052-023-11230-6](https://doi.org/10.1140/epjc/s10052-023-11230-6).

J. M. Cohen and B. Mashhoon. Standard clocks, interferometry, and gravitomagnetism. *Phys. Lett. A*, 181(5):353–358, Oct 1993. [https://doi.org/10.1016/0375-9601(93)90387-F](https://doi.org/10.1016/0375-9601(93)90387-F).

S. C. Cohen and D. E. Smith. LAGEOS Scientific results: Introduction. *J. Geophys. Res.: Solid Earth*, 90(B11):9217–9220, Sep 1985. [https://doi.org/10.1029/JB090iB11p09217](https://doi.org/10.1029/JB090iB11p09217).

D. Coulot, P. Deleflie, P. Bonnefond, P. Exertier, O. Laurain, and B. de Saint-Jean. Satellite laser ranging. In H. K. Gupta, editor, *Encyclopedia of Solid Earth Geophysics*, Encyclopedia of Earth Sciences Series, pages 1049–1055. Springer, Dordrecht, 2011. [https://doi.org/10.1007/978-90-481-8702-7_98](https://doi.org/10.1007/978-90-481-8702-7_98).

R. A. Daly, M. Donahue, C. P. O’Dea, B. Sebastian, D. Haggard, and A. Lu. New black hole spin values for Sagittarius A* obtained with the outflow method. *Mon. Not. Roy. Astron. Soc.*, Oct 2023. [https://doi.org/10.1093/mnras/std3228](https://doi.org/10.1093/mnras/std3228).
B. Mashhoon, F. Gronwald, and H. I. M. Lichtenegger. Gravitomagnetism and the clock effect. In C. Lämmerzahl, C. W. F. Everitt, and F. W. Hehl, editors, *Gyros, Clocks, Interferometers ...: Testing Relativistic Gravity in Space*, volume 562 of *Lecture Notes in Physics*, pages 83–108. Springer Verlag, Berlin, Feb 2001. https://doi.org/10.1007/3-540-40988-2_5.

N. V. Mitskevich and I. Pulido Garcia. Motion of Test Masses in the Gravitational Field of a Rotating Body. *Dokl. Akad. Nauk SSSR*, 192(6):1263–1265, 1970.

A. Paolozzi, G. Sindoni, F. Felli, D. Pilone, A. Brotnu, I. Ciufolini, E. C. Pavlis, and C. Paris. Studies on the materials of LARES 2 satellite. *J. Geod.*, 93(11): 2437–2446, Nov 2019. https://doi.org/10.1007/s00190-019-01316-z.

M. Pearlman, D. Arnold, M. Davis, F. Barlier, R. Biancale, V. Vasiliev, I. Ciufolini, A. Paolozzi, E. C. Pavlis, K. Sośnica, and M. Blolffeld. Laser geodetic satellites: a high-accuracy scientific tool. *J. Geod.*, 93(11):2181–2194, Nov 2019. https://doi.org/10.1007/s00190-019-01228-y.

G. Petit and B. Luzum, editors. *IERS Conventions (2010)*, volume 36 of *IERS Technical Note*. Verlag des Bundesamts für Kartographie und Geodäsie, Frankfurt am Main, 2010.

F. P. Pijpers. Helioseismic determination of the solar gravitational quadrupole moment. *Mon. Not. Roy. Astron. Soc.*, 297(3):L76–L80, Jul 1998. https://doi.org/10.1046/j.1365-8711.1998.01801.x.

E. Poisson and C. M. Will. *Gravity*. Cambridge University Press, Cambridge, Jun 2014. https://doi.org/10.1017/CBO9781139507486.

G. E. Pugh. Proposal for a Satellite Test of the Coriolis Prediction of General Relativity. Research Memorandum 11, Weapons Systems Evaluation Group, The Pentagon, Washington D.C., Nov 1959.

G. Renzetti. History of the attempts to measure orbital frame–dragging with artificial satellites. *Centr. Eur. J. Phys.*, 11(5):531–544, Jul 2013. https://doi.org/10.2478/s11534-013-0189-1.

J. Scheumann, D. Philipp, S. Herrmann, E. Hackmann, B. Rievers, J. Ventura-Traveset, L. Mendes, and C. Lämmerzahl. Gravitomagnetic Clock Effect: Using GALILEO to explore General Relativity. *arXiv e-prints*, art. arXiv:2311.12018, Nov 2023. https://doi.org/10.48550/arXiv.2311.12018.

L. Schiffr. Possible new experimental test of general relativity theory. *Phys. Rev. Lett.*, 4(5):215–217, Mar 1960. https://doi.org/10.1103/PhysRevLett.4.215.

M. H. Soffel. *Relativity in Astronomy, Celestial Mechanics and Geodesy*. Springer, Heidelberg, 1989. https://doi.org/10.1007/978-3-642-73406-9.

M. H. Soffel and W.-B. Han. *Applied General Relativity*. Astronomy and Astrophysics Library. Springer Nature Switzerland, Cham, Oct 2019. https://doi.org/10.1007/978-3-030-19673-8.

N. Wex and S. M. Kopeikin. Frame Dragging and Other Precessional Effects in Black Hole Pulsar Binaries. *Astrophys. J.*, 514(1):388–401, Mar 1999. https://doi.org/10.1086/306933.

C. M. Will. Testing the General Relativistic “No-Hair” Theorems Using the Galactic Center Black Hole Sagittarius A*. *Astrophys. J. Lett.*, 674(1):L25, Feb 2008. https://doi.org/10.1086/528847.

C. M. Will. Finally, results from Gravity Probe B. *Physics Magazine*, 4:43, May 2011. https://doi.org/10.1103/Physics.4.43.

R. J. You. The gravitational Larmor precession of the Earth’s artificial satellite orbital motion. *Boll. Geod. Sci. Aff.*, 57(4):453–460, 1998.