Anomalous currents on closed surfaces: extended proximity, partial quantization and qubits

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Abstract
Motivated by the surfaces of topological insulators, the Dirac anomaly’s discontinuous dependence on the sign of the mass, \( m/|m| \), is investigated on closed topologies when the mass terms are weak or only partially cover the surface. It is found that, unlike the massive Dirac theory on an infinite plane, there is a smoothly decreasing current when the mass region is not infinite; also, a massive finite region fails to exhibit a Hall current edge—exerting an extended proximity effect, which can, however, be uniformly small—and oppositely orientated Hall phases are fully quantized while accompanied by diffuse chiral modes. Examples are computed using Dirac energy eigenstates on a flat torus (genus one topology) and a closed cap cylinder (genus zero topology) for various mass-term geometries. Finally, from the resulting properties of the surface spectra, a potential application for a flux-charge qubit is presented.

1. Introduction

The \( 2 + 1 \) dimensional massive Dirac theory on an infinite plane exhibits an anomalous contribution to the Hall conductance \( (h, c = 1, e = |e|) \), \( \sigma_{xy} = \frac{m^2}{c m^2} \frac{e^2}{4\pi} \) \([1–3]\), representing a half charge pumped contribution per unit of threaded flux in a Laughlin type setup \([4, 5]\). Unfortunately, this half quantization is not directly observable, since a flat two-dimensional lattice theory such as that for graphene must have an even number of Dirac modes \([6–8]\). Having effectively two or more pairs of fermion flavors, single fermion results are effectively doubled.

In contrast, the surfaces of three-dimensional topological insulators contain an odd number of massless Dirac fermion modes on closed topologies \([9–18]\). Effective Dirac mass terms can be given to the surface theory at selective regions by coupling to a ferromagnetic time-reversal breaking material \([19]\). An application of the naive infinite plane results to this system suggests several questions. If the massive region only covers a small area, will charge accumulate on its edge? If not, can a sign change of a weak local mass induce a sign flipping transition of the quantized current globally? Finally, how do magnetic fluxes induce fractional charge pumping in specific cases? In this work these issues are investigated and clarified.

Rosenberg et al \([20]\) gave a solution to the last question by concluding that fractional charge accumulation does not occur when a flux is inserted through a topological insulator. The surface theory was said to break down at \( \pi \) flux, as gapless bands induced in the bulk were responsible for discharging the ends of the flux tube. This was called the wormhole effect. Yet, it is not possible to explain a half-charge quantized transport on the basis of bands. In fact, the two-dimensional plane result implies the existence of a current in a fully gapped theory. The results to be presented will clarify these points and it is conjectured that fractional charge could indeed be observed.
In this work it is emphasized that the surface theory is sufficient to resolve all previous questions. Three results will be shown through solved examples and motivated from the effective action of topological insulators. These are (1) that mass terms do not induce perfect quantization in weak limits for closed finite surfaces (partial quantization), (2) that mass edges do not result in fractional Hall conductance edges (the ‘extended proximity’ effect) and (3) when the transition region separates oppositely orientated masses the phases are fully quantized but overlapped by a sufficiently diffuse chiral band. The existence of chiral bands is not a new result [18, 21, 22], but rather the full quantization of the separated phases. It is also suggested that the wormhole effect [20] only occurs in the localized flux limit and in that case can be thought of as an example of the extended proximity effect on a genus one surface.

Section 2 will first offer motivation for these results from the effective surface action. Sections 3 and 4 will then present specific examples and constructions showing how the effects are manifested by the surface Dirac theory. The solutions to the Dirac equation have been investigated in other contexts and geometries involving topological insulators and graphene [18, 23–25]. The anomalous Hall current is understood from the wavefunction’s spectral asymmetry (see appendix A). Finally, gaining intuition from the different mass geometries computed, a potential configuration of masses with inserted flux is proposed as an architecture for a qubit in section 5.

2. Effective action

The conclusions reached in this work can be motivated from the effective field theory for topological insulators. Starting from topological BF theory it was shown in [26] that the entire electromagnetic response comes from the surface. After integrating out the effective fields, the Hall action was obtained as

\[ S_{\text{surf}} = -\frac{e^2}{8\pi^2} \int \epsilon^{\mu\nu\rho} \partial_\mu A_\rho \partial_\nu A_\rho. \]  

(1)

Here, and throughout, Greek indices take values over the 2 + 1 space–time describing the surface. \( \theta \) is a parameter or background field giving a Hall conductance \( \sigma_{xy} = \theta e^2 / 4\pi^2 \). \( \theta \) is required to be \( \pm \pi \) to respect time-reversal invariance, and comparison with the Dirac theory on a plane allows one to identify \( \theta = \text{sign}(m) \pi \ [1–3] \).

Partial quantization. First it is noted that on the surface of the insulator there is no reason to expect \( \theta \) to be quantized if time-reversal invariance is being broken by mass terms. Importantly, for the Abelian theory on a closed surface (as in equation (1)), gauge invariance is consistent with \( \theta \) unquantized. A fixed value of \( \theta \) other than \( \pm \pi \) would lead to an unquantized Hall current.

Extended proximity effect. For a more complicated arrangement of mass terms \( \theta \rightarrow \theta(x) \) might be allowed, as an effective parameter not necessarily equivalent to the local sign of the mass. An abrupt change in the value of \( \theta \) from non-zero to zero can be thought of as a Hall edge. However, gauge invariance in general requires that \( \theta \) remain constant,

\[ S_{\text{surf}} \rightarrow -\int \epsilon^{\mu\nu\rho} \partial_\mu A_\rho \partial_\nu A_\rho = \int \epsilon^{\mu\nu\rho} \partial_\mu \theta \partial_\nu A_\rho \neq 0 \]  

(2)

(0 mod 2\( \pi \) is implied for the last term). Therefore, \( \theta \neq \theta(x) \), which is presumably responsible for the extended proximity effect and the absence of a Hall edge.

Quantization of opposite phases. The exception to equation (2) is if a region exists of zero magnetic field (which is the case for closed surfaces that do not surround monopoles). In this case \( \theta \) can transition (over a region of zero field) from one fully quantized (half-charge) Hall phase to the oppositely quantized phase accompanied by a chiral band.

To see this, working in the Coulomb gauge \( (A_0 = 0) \), zero field strength requires \( A_i = \partial_i \Gamma \) (i.e. 1, 2). Inserting this into the action where \( \theta \) varies naively gives

\[ S_{\theta \neq \theta_0} = \frac{e^2}{4\pi^2} \int \epsilon^{ij \rho} \partial_i \theta (\partial_j \Gamma \partial_k \Gamma). \]  

(3)

Normally the electromagnetic field is physically determined by a single choice of \( \Gamma \). Then there are no unique modes and no way to form propagating packets. However, on the compact surface there exist different disjoint \( \Gamma \) values distinguished by winding numbers. These different winding numbers can act as different modes forming a compact (discrete) chiral band.

If \( \theta \) varies in the \( j \) direction, the \( j \) direction is topologically equivalent to a circle separating the two phases, and with periodic boundary conditions in time, the action can be integrated to give (setting \( e = 1 \))

\[ S_{\theta \neq \theta_0} = \Delta \theta kl, \]  

(4)

where \( k, l \) are integers. This remains gauge invariant for \( \Delta \theta = 2\pi n \) \( (n \in \mathbb{Z}) \). Therefore \( \theta \) must separate two fully quantized oppositely orientated phases. The complete picture is that the chiral band can absorb or release a whole charge, half of which comes from each separate phase. The modes may be diffuse over the entire region where the field strength vanishes, however.

In the rest of this work, these suggested results will be shown in various solvable examples.

3. Flat torus

The first easily solvable case is a flat torus, which can be considered as a simplification for a crystal wafer with a hole drilled through it. Labeling coordinates \( z \) and \( \phi \) for the two orthogonal directions on the torus, a flat metric can be used. It is important to note that four inequivalent forms exist for the Dirac equation on the torus, corresponding to the inequivalent spin structures. For example, writing the torus as a square with edges identified naturally leads one to a Cartesian-like form (with constant Pauli matrices). However, this will not agree with an embedding in three-dimensional Euclidean space as used in [27, 28, 20]. While both forms are mathematically consistent, they lead to different physical results. The spectrum, for example, has a finite-size gap present in one case but not in the other. However, none of the
momentum. It remains fully gapped for any threaded flux and is shown in figure 2 as a function of the azimuthal angular position in appendix B. The corresponding spectrum is unremarkable.

\[ \sigma \rightarrow l \]

other quantities are measured in units of \( L \) is shown in figure 1. The units are set with \( e \).

\[ z \leq \frac{3}{2} \pi \]

Nevertheless, the anomalous Hall current, \( \tilde{j}(\phi) \), is non-zero and can be computed numerically using equation (A.5).

\[ e^{-i\phi} \left( -i\sigma \hat{\alpha}_z - i\sigma \hat{\alpha}_\phi + \sigma^3 m - eV(\phi) \right) \tilde{\psi} = E\tilde{\psi}, \quad (5) \]

with \( 0 \leq z < 2\pi \) and \( -L_2/2 \leq \phi < L_2/2 \), while the mass and energy are normalized to units of \( 2\pi/L_1 \). The solutions in each region are matched and the quantization condition is derived in appendix B. The corresponding spectrum is unremarkable and is shown in figure 2 as a function of the azimuthal angular momentum. It remains fully gapped for any threaded flux.

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3.1. Weak Hall current

To ascertain an anomalous Hall conductance as a function of mass strength I consider a fully massive Dirac fermion with a step potential in the \( \phi \) direction \( V(\phi) = v(\Theta(\phi + l/2) - \Theta(\phi - l/2)) \) corresponding to localized electric fields at \( \pm l/2 \), \( \tilde{E} = \frac{1}{2}(\delta(\phi + l/2) + \delta(\phi - l/2))\tilde{\psi} \). This construction is shown in figure 1. The units are set with \( L_1 \).

\[ \tilde{l} \rightarrow 2\pi l/L_1, \]

Then, the Dirac equation is \( \tilde{\psi} = \sigma^0, \tilde{\gamma} = \sigma^0 \sigma^1, \tilde{\gamma}^\phi = \sigma^0 \sigma^2 \).

According to the infinite plane result, one expects \( \frac{\tilde{j}(\phi)}{l} = \frac{\delta(\phi + l/2) - \delta(\phi - l/2)}{l} \frac{z}{2} \frac{\sign(m)}{2} \). However, the computed result turns out to have a smoother mass dependence, as shown in figure 3, illustrating partial quantization. The current smoothly approaches zero as the mass approaches zero with the onset of full quantization only with a sufficiently large mass distance \( m \sim 20 \) (in units of \( 2\pi/L_1 \)). If the current is computed along any window other than the delta function peaks, none is found, as expected. There is no further dependence on the potential strength beyond the expected (linear) scaling. Finally, note that the system exhibits a current despite remaining fully gapped, consistent with the understanding that bands are not responsible for fractional charge transport.

3.2. The extended proximity effect: first suggestions

A second case that can be easily computed suggests that, while the current depends on the mass strength, it does not respond locally to a massive region. To see this, a torus with a strip of mass \( m \) is considered, as shown in
psi ∈ k (blue) points are at the allowed azimuthal quantum numbers in appendix B and eigenvalues are shown in figure 5. The large considered in section 4.

only suggestive, much more compelling examples will be like an edge and a Hall phase is globally induced. While threading; in other words, a strip of mass does not behave like an edge and a Hall phase is globally induced. While only suggestive, much more compelling examples will be considered in section 4.

The Dirac equation and solutions in this case are given in appendix B and eigenvalues are shown in figure 5. The large (blue) points are at the allowed azimuthal quantum numbers $k \in \mathbb{Z} + 1/2$ of the wavefunction $\tilde{\psi}$ defined in appendix B as $\tilde{\psi}_\alpha = e^{i \frac{\phi}{2}} \psi_\alpha$. Because of the half-integer quantization there is a finite-size energy gap, of order $1/L_2$. When a flux $\Phi$ is inserted its effect can be undone through a transformation that amounts to shifting $\tilde{k} \rightarrow \tilde{k} + \Phi$. The smaller (red) points show the evolution of the eigenvalues under this threaded flux.

For the massless case (figure 5(a)), the bands traverse the gap. These bands are not responsible for the Hall current and the evolution of the eigenvalues under this threaded flux.

For any small strip of mass, a gap forms for all fluxes, for the more interesting case showing fractional charge accumulation and direct detection of weak quantization and the absence of a Hall edge.

4. The surface of a closed cylinder

The surface of a closed cylinder offers analytic solutions for the more interesting case showing fractional charge accumulation and direct detection of weak quantization and the absence of a Hall edge.

4.1. Closed cylinder cases: weak currents and the extended proximity effect

The relevant dimensions and coordinates for the cylinder surface are described in figure 6. The Dirac equations in the three different sections and matching conditions are derived in appendix C, they are on the top and bottom caps (with and without masses)

$$\left[(\sigma^1 \cos \phi + \sigma^2 \sin \phi) \partial_r + (\sigma^2 \cos \phi - \sigma^1 \sin \phi) \frac{\partial \phi}{r} + m_{l(III)} \sigma^3 \right] \tilde{\psi}_{l(III)} = E \tilde{\psi}_{l(III)}(r, \phi)$$

and on the side

$$e^{-i \frac{\phi}{2}} (-i \sigma^1 \partial_z - i \sigma^2 \partial_{\phi} + \sigma^3 m_{II}) e^{i \frac{\phi}{2}} \tilde{\psi}_{II} = E \tilde{\psi}_{II}(z, \phi).$$

or, defining in each region $\tilde{\psi} = \exp(i \frac{\phi}{2}) \psi$,

$$\left(-i \sigma^1 \partial_z - i \sigma^2 \partial_{\phi} - i \sigma^1 \frac{\partial \phi}{2r} + m_{l(III)} \sigma^3 \right) \tilde{\psi}_{l(III)} = E \tilde{\psi}_{l(III)},$$

and

$$(-i \sigma^1 \partial_z - i \sigma^2 \partial_{\phi} + m_{II} \sigma^3) \tilde{\psi}_{II} = E \tilde{\psi}_{II}.$$

Figure 4. Coordinates and dimensions (left) for the smooth torus (right). Note that the sides are identified in the left figure and $\Phi$ and $z$ have been exchanged relative to figure 1. However, on the right, $\phi$ is still in the vertical direction. The torus is separated into two regions, or three with edges identified, as shown on the left labeled I, II and III. A mass term, $m$, is present only in the shaded area, region II.

Figure 5. The spectrum for the configuration of figure 4. (a) Massless case, $m = 0$ with $l = 0.5$ and $L_2 = 2$. (b) With mass $m = 1.5$, and the same aspect ratios. Units are in terms of $2\pi/L_1$ as described in the text.
The matching conditions are (see appendix C)

\[ \psi_{I}^{\dagger} |_{r=R} = \sigma^{2} \psi_{II} |_{z=d/2}, \]
\[ \psi_{III}^{\dagger} |_{r=R} = \psi_{II} |_{z=-d/2}. \] (10)

I consider six cases \((m > 0 \neq m(x))\):
(a) the massless case, \(m_{I} = m_{II} = m_{III} = 0\), (b) positive mass on the top cap alone, \(m_{I} = m, m_{II} = m_{III} = 0\), (c) positive mass on the side, \(m_{II} = -m, m_{I} = m_{III} = 0\), (d) positive mass on the top and bottom, \(m_{I} = -m_{II} = m, m_{II} = m_{III} = 0\), (e) positive mass everywhere, \(m = m_{II} = m_{III} = m\), and finally (f) oppositely oriented masses on the two caps, \(m_{I} = +m_{II} = m, m_{II} = 0\). Note that the relative minus sign between the top and the rest of the cylinder in cases (b)–(e) to describe the same sign mass comes from the \(\sigma^{2}\) transformation in the top-to-side matching, stemming from the opposite orientation of the \(r\) and \(z\) directions at the top-to-side boundary.

Unlike the torus, the effect of the flux is not simply to shift the azimuthal quantum number. In particular, some boundary condition must be implemented at the origin that physically involves the flux details. Interestingly, a simple shift of \(k\) alone (the azimuthal quantum number of \(\psi\)) would allow for extra solutions, unconstrained by normalizability at the origin (see [29]). The simplest possibility is to take a profile for the flux as a localized delta function ring with a small radius \(\epsilon\), \(B(r) = \Phi \delta(r - \epsilon)/(2\pi \epsilon)\) [29], which is expected to be qualitatively similar to a diffuse flux of width \(\epsilon\).

Figure 7 shows the spectra for the different cases as a function of the azimuthal quantum number \(k\). As before, the large (blue) points represent the spectra with no flux and the smaller (red) points represent the evolution of the spectra as one unit of flux is inserted. The effect of the flux is manifested by the appearance of extra states at the end of the cycle inserting one unit of flux (marked A and B). The new states relative to the original vacuum will have half-charges associated with them (see appendix A). For all cases a simple pattern emerges: ‘anomalous’ bands create extra states near \(-m_{I}\) and \(m_{III}\) (A and B points). One can show that the states are never exactly at these values, including \(E = 0\) for the massless case. All other bands flow back to, or very near to, the original spectrum. The difference between the cases occurs in the relative distribution of the wavefunctions after one unit of flux is inserted. This is now described. The Fermi energy is assumed to be at zero or mid-gap.

**Partial quantization.** First, in case (a) (the massless case), no charge accumulation is seen. Indeed, the two states that appear near \(E = 0\) are each equally split between the top and bottom flux piercing, so that no net charge density appears to be accumulated. In cases (b)–(e), the new states are now unevenly localized, resulting in a net charge pumping. The negative energy states are localized near the bottom cap flux while the positive ones are localized near the top. However, the total charge pumped to (from) the cap depends on the mass-region strength, going to zero as this value goes to zero. Quantitative results for the pumped charge are shown in figure 8, showing incomplete quantization if the mass region is too small.

**The extended proximity effect.** While the positive and negative bands both approach the value of the local mass, the net charge pumped responds to the configuration globally. A striking example is case (c), where charge is pumped to the caps, despite the mass only being present on the side and the similarity of the spectrum with case (a). Also, in case (b), even though the mass is present on only one cap, the amount of charge pumped is equal and symmetric on the top and bottom caps. In case (b), the equality of \(|Q_{\text{top}}| = |Q_{\text{bot}}|\) is verified to high accuracy for all the masses shown in figure 8 up to long cylinders of ratio \(d/R = 8\), thereby indicating that the whole cylinder behaves in a similar fashion, as though a mass were present everywhere.

**Opposite phases and the chiral mode.** An example of the expected chiral band separating the two Hall phases is shown in case (f). As in the previous cases, two new states now appear at \(E = -|m|\). Both of these states represent a half-charge at each cap relative to the flux-less vacuum. After unit flux these states give a deficit from each cap while a chiral mode becomes occupied.

As a final note, if the solenoid could pierce only one cap (a net outward flux), then a single \(E = \pm m\) (with \(m\) the mass of that cap with flux) state would be found for \(\Phi = 1\), in agreement with index theorems (see references within [1, 30]). In the present case the pair of new states ensures that the overall system remains neutral, but the relative distribution of charge does not remain uniform.

### 4.2. The wormhole effect and surface theory fidelity

Before concluding the examples, I wish to comment on whether the closed cylinder results of section 4.1 are applicable for a real system. It was argued in [20] that surface electrons will tunnel through the bulk of a topological insulator for a very localized flux or if a hole is bored through the material. This was called the wormhole effect. Nevertheless, I will consider this question open for a general flux. Noting the results of section 3, one notices similarities.
Figure 7. The spectra corresponding to the cases (a)–(f) defined in the text. The insets reproduce cylinders for each case with the following color scheme: yellow for the massless regions, blue for positive mass and, in the case of (f), red for a negative orientated mass. The blue (larger) points at half-integer azimuthal number $\tilde{k}$ are the spectra with no flux inserted. The red (smaller) points represent the spectral flow as a localized flux is inserted, as described in the text and shown in figure 6. After one unit of flux, new states appear at the points marked A and B. For all cases $|m|R = 5, d/R = 1/2$, and the energy, $E$, is in units of $1/R$.

with the work of [20]. While I considered a smooth case, with a finite-size gap $1/L^2$ (the blue points in figure 5), it is clear that such a gap will be of order $1/R$, where $R$ is the radius of the interior of a bored cylinder. In these cases, the effect of inserting a flux tube through a bored hole or a single plaquette of the microscopic lattice can be removed by simply shifting the azimuthal number, and the spectral flow will be similar to figure 5(a).

In fact, a similar picture will result whenever a simple global transformation can remove the effect of a threaded flux, even if it goes through a hole or a single plaquette. Thus, the following ansatz is proposed: if an extremely localized flux string manages to pierce a single plaquette or a few plaquettes throughout the bulk, then the currents can be described by a ‘surface’ of genus one as in figure 9(a), since for either a lattice or continuum theory the flux simply shifts the azimuthal number. The spectrum of the surface theory of the torus contains the gap closing seen in [20] and corresponds to the ability of electrons to propagate through the interior surface of the torus. As noted in section 3, this band is actually not responsible for the anomalous current. In any case, the fact that charge will not accumulate with a mass partially covering the surface is just a case of the extended proximity effect that forces the Hall current not to exhibit an edge.

If instead the flux is smoothly varying and extended over many plaquettes, then it is still reasonable to expect the low energy theory on the surface be described faithfully by the Dirac theory, but on a surface of genus zero (this is illustrated in figure 9(b)). In this case it turns out that the physical inclusion of the flux in the manifold matters beyond simply shifting the azimuthal angle, and a fractional localized charge might be observed.
Figure 8. The total charge $\times 2 (Q$ is in units of $-e/2$) in the top cap after one unit of flux is inserted for cases (b)–(e) shown clockwise, with (b) starting at the top left. The horizontal axis plots a log scale of $m$ that is physically $mR$ since the units are normalized by $R$. The various lines are for ratios $d/R = 1/32$, $1/8$ and $2$ shown by the circles (blue), squares (purple) and diamonds (yellow) respectively. The side-mass case also has $d/R = 1$ shown by triangles (green).

Figure 9. Illustration contrasting a microscopically localized flux (a), which is qualitatively equivalent to the genus one surface discussed in section 3, and a diffuse flux (b), which remains faithfully described by a surface theory of genus zero and penetrating flux.

This potential cross-over is supported by the results of [31], which numerically solved for the case of a constant magnetic field through the two disjoint surfaces: a cube with $x$ and $y$ faces identified and open $z$ faces. It was found that the expected fractional charge smoothed out over each face was preserved.

5. Flux-charge qubit

The pattern of the results from section 4 implies that unit fluxes induce new states with energy equal to the local mass (as opposed to the Hall current, which depends on the masses globally). Taking this into account, it is possible to conceive of a configuration that will produce close to degenerate $E = 0$ states separated by an arbitrary energy gap—a desirable property for a qubit. Qubits using topological insulators have been proposed in [37] involving Majorana fermions. The current proposal would use (fractional) electric charge to distinguish between the states, while requiring magnetic flux for their stabilization. It might therefore be considered a ‘flux-charge’ qubit. The basic idea is shown in figure 10, with details described below. One imagines the cylinder...
coated with mass almost entirely (shown in blue around the topological insulator surface), except for a region of radius \( r_2 \). If \( r_2 \sim 1/m \) then any state localized in \( r_2 \) will be gapped by order \( m \) as well, thereby making the entire spectrum be gapped by \( m \). Now, if a localized flux can be inserted within \( r_2 \), with a width of say \( r_1 \), it will be localized in a different region: \( E \sim +0 \) states will be localized in the bottom flux piercing while \( E \sim -0 \) states will be on top. Therefore \( E > 0 \) and \( E < 0 \) states will not be mixed by local noise (different \( k \) states, however, can be). If, however, the flux is fractional there will in general be one state in the mid-gap region spoiling the energy separation (between \( k = -1/2 \) and \( 1/2 \) in figure 11). Therefore a limiting factor is the need for integer flux. A second issue is that for the energies to flow to a value sufficiently close to \( E = 0 \), \( r_1 \), the flux width, must be sufficiently small compared to \( r_2 \), otherwise the mass exerts a local proximity effect for the \( k = 1/2 \) state. From figure 11, at least \( r_1 < 0.1r_2 \) is required for reasonable energy scale separation.

With interactions turned on, the true ground state is the most neutral occupation, which is zero for even flux and \( \pm e/2 \) for odd flux (split on caps). A fluctuation through an even flux would destroy the charge and so for all practical purposes precision up to a single flux is needed, or specifically exactly one flux quantum.

These constraints may have a potential solution. To shield flux from regions where it is not wanted, a (type-I) superconductor can be put over the top and bottom surfaces, as shown in figure 10, with holes of radius \( r_1 \). The induced current of the superconductor would favor integer flux through \( r_1 \), which because it is small could easily be accommodated by a reasonable macroscopic magnetic field over the whole cap. To further stabilize the single flux constraint, a type-II superconductor could be used with the temperature of the system tuned so that the vortex coherence length \( \sim r_1 \) favors a single vortex. Other possibilities perhaps using SQUIDs might also be suited.

If a single flux quantum can be stabilized then the basis of the qubit is simply the occupation of the \( E \sim 0^{(+)} \) state, as shown in figure 12, \( |1\rangle = a_{0+}^\dagger (\prod_{E>0} b_E^\dagger)|0\rangle \), \( |2\rangle = a_{0-}^\dagger (\prod_{E<0} b_E^\dagger)|0\rangle \) (in the convention of appendix A). \( |1\rangle \) has a \(-e/2\) charge at the top window \((+e/2 \) at bottom) and vice versa for \( |2\rangle \). Then, in this basis, applying an electric field, or a potential \( V \) with \( V_{\text{top}} = -V_{\text{bottom}} = \nu \), generates a term in the Hamiltonian of \(-\nu \sigma_z \) in the qubit subspace. A \( \sigma^x \) or \( \sigma^z \) matrix element necessary for full unitary evolution is less trivial. Effectively, switching of the electron occupation is required. One potential route is simply to insert metallic electrodes connecting the top and bottom windows, allowing the electron to tunnel with the aid of a bias voltage. Exploration of these possibilities is left for future work.

6. Discussion

In summary, the details of how anomalous currents manifest themselves on the surfaces of topological insulators has been presented as well as a potential application in the form of a flux-charge qubit. Most striking is the extended proximity effect, which is nonetheless rendered reasonable by a lack of quantization. This effect is manifested through the non-local nature of the wavefunctions, despite the spectrum responding...
to mass terms locally. The lack of quantization is more physical than the sharp sign($m$) dependence found for the
infinite plane. The existence of a chiral band, which was
previously known, is consistent with the result that oppositely
orientated Hall phases are fully quantized, so that the chiral
band receives or releases a whole charge. An effective
unquantized current is restored in the sense that the chiral
band can be diffuse.

It was noted for the torus that $mL/2\pi \sim 20$ was necessary
for the onset of full quantization while for the closed cap
geometry, from figure 8, onset occurs at
for the closed cap

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reaches or releases a whole charge. An effective
unquantized current is restored in the sense that the chiral
band can be diffuse.

The required magnitudes for the qubit proposal are more
constraining. Using the value for $v_1$ above, for a gap of
3 K–0.3 meV, a value of $T_2 = 2.5 \mu$m is required. Therefore,$r_1 \lesssim 0.1 \mu$m, which is small. The value of $R$ and the aspect
ratio are essentially unconstrained.

Finally, whether the fractional charges remains intact
for a diffuse enough magnetic field remains an open
question, which can be investigated both numerically and
experimentally. If this is the case, then the flux-charge qubit

could be supported and offer an alternative approach to
quantum computation.

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**Appendix A. Spectral asymmetry and anomaly**

The quantum anomaly can be understood by two main routes.
For a $(2 + 1)$ dimensional Dirac theory on the infinite

plane coupled to a background gauge field $A^\mu(x)$, one can
simply compute the effective action and extract the current term [2, 1]

$$
\langle j^\mu \rangle = i \frac{\delta}{\delta A^\mu} \ln \det(iD + eA - m) \tag{A.1}
$$

$$
= \frac{e^2}{8\pi} \frac{m}{|m|} e^{\nu r B} F_{\nu\lambda} + \cdots \tag{A.2}
$$

$$
\therefore \langle Q \rangle = \int \langle j^0 \rangle = \frac{e}{2} \Phi, \tag{A.3}
$$

where the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\Phi = -e \int d^2 r B/(2\pi)$. While this gives an exact analytic result,
explicit computation of the log of the determinant is difficult
in all but the simplest cases such as the infinite plane.

Alternatively, one can diagonalize the single particle
Dirac Hamiltonian $H$, $H\psi_\lambda(x) = E_\lambda \psi_\lambda(x)$. The $\psi$s
are $n$-component classical spinors, $H$ contains $g^\mu$ matrices satisfying $[g^\mu(x), g^\nu(x)] = g^{\mu\nu}(x)$ and $g$ is the coordinate metric (see [32, 33]). Then the fermion annihilation operator
at $\vec{x}$ can be expanded in this basis [34] as

$$
\Psi(\vec{x}) = \sum_\lambda a^\dagger_\lambda \psi_\lambda + \sum_\lambda \psi_\lambda b^\dagger_\lambda, \tag{A.4}
$$

where $\psi_{\pm\lambda}$ are the positive (negative) energy eigenstates,
$\psi_0$ are possible zero energy states, and $a, b$ are fermionic
annihilation operators satisfying the usual commutation relations (here and below, $\lambda$ with no $\pm$ qualifier is taken

Then the vacuum expectation of the current, $j^\mu(\vec{x}) = \frac{-ie}{\pi} [\Psi^\dagger(\vec{x})g^{0\mu}(x) - (\psi(\vec{x}))^T g^{0\mu}(x) (\psi(\vec{x}))^T] \tag{A.5}$

$$
\frac{1}{-e} \langle j^\mu(\vec{x}) \rangle = -\frac{1}{2} \sum_{\pm\lambda} (\psi^\dagger_{+\lambda} g^{0\mu}\psi_{+\lambda} - \psi^\dagger_{-\lambda} g^{0\mu}\psi_{-\lambda}) + \frac{1}{2} \sum_o \psi^\dagger_o g^{0\mu}\psi_o - \frac{1}{2} \sum_u \psi^\dagger_u g^{0\mu}\psi_u,
$$

where the $\pm\lambda$ refer to non-zero states and $o, u$ sum over
occupied (unoccupied) zero modes. To extract the charge in
one region versus another, $\langle j^0(\vec{x}) \rangle$ may be integrated over
regions of interest.
Appendix B. Wavefunctions and the quantization condition on the torus

Although it is simple enough to write the Dirac equation and the forms of the solutions for both of the cases in the text, for clarity in notation, I will split them. For a review on vielbeins and spinors in curved space see [35, 36, 33].

B.1. Case 1: fully massive torus; step potential in the $\hat{\phi}$ direction

As discussed in the text (with unit definitions), for a fully massive torus with a potential $V_\alpha = (-v, +v, -v)$ for regions I, II, III as in figure 1 the Dirac equation is

\[( -i\sigma^1 \partial_z - i\sigma^2 \partial_{\phi} + \alpha^3 m_\alpha ) \tilde{\psi}_\alpha = ( E - V_\alpha ) \tilde{\psi}_\alpha. \tag{B.1} \]

To simplify the formulas I have put $v$ instead of $-ev/2$. In this case the $\hat{z}$ direction is trivial and the solutions are of the form $\tilde{\psi} = \exp(i k\hat{z} f_\alpha(\phi))$ with $k \in \mathbb{Z}$ (the same $k$ in all regions). If $E \neq \pm\sqrt{k^2 + m^2} \pm v$ and $E \neq \pm |k|$, $f_\alpha(\phi)$ is given by

\[ f(\phi) = A_\alpha \left( \frac{1}{i\sqrt{(E - V_\alpha)^2 - k^2 - m^2 + k}} \right) \]
\[ \times e^{i\sqrt{(E - V_\alpha)^2 - k^2 - m^2} \phi} \]
\[ + B_\alpha \left( \frac{1}{-i\sqrt{(E - V_\alpha)^2 - k^2 - m^2 + k}} \right) \]
\[ \times e^{-i\sqrt{(E - V_\alpha)^2 - k^2 - m^2} \phi}. \tag{B.2} \]

The coefficients in each region and the energy quantization are determined by normalization and matching conditions

\[ f_1(L_2/2) = \eta f_1(-L_2/2), \]
\[ f_1(l/2) = f_1(l/2) \]
\[ \text{and} \]
\[ f_1((-l)/2) = f_1((-l)/2), \]

with $\eta = -1$ for the chosen spin structure in the $\phi$ direction. The quantization condition is

\[ |a|^2 \cos(pl + q(L_2 - l)) - 1) = |b|^2 \cos(pl - q(L_2 - l)), \tag{B.3} \]

where $l$ have defined $q = \sqrt{(E + v)^2 - k^2 - m^2}$, $p = \sqrt{(E - v)^2 - k^2 - m^2}$, $a = k + i p \over E + m + k - i q \over E + m$ and $b = k + i p \over E + m - k + i q \over E + m$. This is solved numerically and shown (for a representative case) in figure 2. The special cases $E \neq \pm\sqrt{k^2 + m^2} \pm v$ and $E \neq \pm |k|$ (and special sub-cases of these such as $E = 0$) must be treated separately although in a similar manner to that above. The solutions in these cases are linear in one of the regions, and in general do not yield new solutions.

B.2. Case 2: partially massive strip

In this case the Dirac equation is

\[ (-i\sigma^1 \partial_z - i\sigma^2 \partial_{\phi} + \alpha^3 m_\alpha ) \tilde{\psi}_\alpha = ( E - V_\alpha ) \tilde{\psi}_\alpha. \tag{B.4} \]

where $m_\alpha = (0, m, 0)$ for regions I, II, III, as in figure 4. Different spin structures are encoded in whether the boundary condition on $\tilde{\psi}$ is anti-periodic or periodic in $\phi$. In each region I, II and III the solutions are of the form $\tilde{\psi} = \exp(ik\hat{z})f(z)$ with $\tilde{k} \in \mathbb{Z} + 1/2$. If $E \neq \pm\sqrt{k^2 + m^2}$ and $E \neq \pm |\tilde{k}|$, $f(z)$ is given by

\[ f(z) = A \left( \frac{1}{\sqrt{E^2 - k^2 - m^2 + i\tilde{k}}} \right) \]
\[ \times e^{i\sqrt{E^2 - k^2 - m^2} \phi} \]
\[ + B \left( \frac{1}{-\sqrt{E^2 - k^2 - m^2 + i\tilde{k}}} \right) \]
\[ \times e^{-i\sqrt{E^2 - k^2 - m^2} \phi}. \tag{B.5} \]

where the region label has been omitted and the mass $m$ is understood to be zero in regions I and III. The coefficients in each region and the energy quantization are determined by normalization and matching conditions $f_1(L_2/2) = \eta f_1(-L_2/2)$, $f_1(l/2) = f_1(l/2)$ and $f_1((-l)/2) = f_1((-l)/2)$, with $\eta = 1$ for the chosen spin structure along $\hat{\phi}$ loops. The quantization condition is as before,

\[ |a|^2 \cos(pl + q(L_2 - l)) - 1) = |b|^2 \cos(pl - q(L_2 - l)), \tag{B.6} \]

but in this case the definitions are $q = \sqrt{E^2 - k^2}$, $p = \sqrt{E^2 - k^2 - m^2}$, $a = k + i p \over E + m + k - i q \over E + m$ and $b = k + i p \over E + m - k + i q \over E + m$. This is solved numerically and is shown (for a representative case) in figure 5. The special cases $E = \pm\sqrt{k^2 + m^2}$ and $E = \pm |\tilde{k}|$ (and special sub-cases of these such as $E = 0$) must be treated separately although in a similar manner to that above. The solutions in these cases in general do not yield new solutions.

Appendix C. Derivation of the Dirac equation on a closed cylinder surface

In the notation of figure 6, the closed cylinder is divided into three regions. The local form on the caps is as the Dirac equation for the disk described in equation (6), which can be obtained by transforming the Dirac equation in Cartesian coordinates into polar coordinates. To guess the form on the side, region II, one cannot simply replace $r \to R$ as the metric suggests. In particular, the spin connection is wrong, and stems from the fact that all the curvature is at the sharp corner where the coordinate transformation is required. For a review on vielbeins and spinors in curved space see [35, 36, 33]. A real manifold must be smooth and the transition between charts should be a map $\mathbb{R}^2 \to \mathbb{R}^2$, although in practice one dimension can be made of infinitesimal width if the manifold is smooth. This suggests that a more careful procedure is to smooth the corners into a semi-circle of radius $\delta$ as in figure C.1. Replacing the relation $\delta^2 = (r - R)^2 + (z + \frac{\delta}{2})^2$ in the three-dimensional Cartesian metric, the induced metric is

\[ g = \frac{\delta^2}{\delta^2 - (r - R)^2} dr \otimes dr - r^2 d\phi \otimes d\phi. \tag{C.1} \]
and leads to the Dirac equation (using \( \phi \) dependent vielbeins) in \((r, \phi)\)
\[
e^{-i \frac{\omega}{2} \phi} \left( e^{i \frac{\omega}{2} \phi} - i \frac{\omega}{2} \frac{\partial}{\partial r} - \frac{\omega}{2r} \right) e^{i \frac{\omega}{2} \phi} = E \psi. \tag{C.2}
\]
As \( r \to R \), the equation becomes
\[
e^{-i \frac{\omega}{2} \phi} \left( -i \sigma^1 \partial_r - i \sigma^2 \frac{\partial}{\partial \phi} - \frac{\omega}{2R} \right) e^{i \frac{\omega}{2} \phi} = E \psi, \tag{C.3}
\]
just what we expect on the cap evaluated at \( r = R \). However, replacing \( r(z) \) and taking \( z \to \frac{z}{r} \) one gets
\[
e^{-i \frac{\omega}{2} \phi} \left( -i \sigma^1 \partial_z - i \sigma^2 \frac{\partial}{\partial \phi} - \frac{\omega}{2z} \right) e^{i \frac{\omega}{2} \phi} = E \psi, \tag{C.4}
\]
just what one expects from the side using an embedding, without any further transformation of \( \psi \), i.e. \( \tilde{\psi}_{1,3}(R, \phi) = \psi_{1,3}(-d/2, \phi) \). Less trivial is the boundary condition between the top and the side. The derivation proceeds in the same way as if the local \( \tilde{z} \) were facing down. Therefore, if one defines a new \( \tilde{z}_n = -z \), and calls the tentative wavefunction \( \tilde{\psi}_{1n} \) with the notation \( \tilde{\psi}_{1n} = e^{i \frac{\omega}{2} \tilde{z}} \tilde{\psi}_{1n} \), one would find \((-i \sigma^1 \tilde{z}_n - i \sigma^2 \frac{\partial}{\partial \phi}) \tilde{\psi}_{1n}(\tilde{z}_n, \phi) = E \tilde{\psi}_{1n}(\tilde{z}_n, \phi)\) (with boundary condition \( \tilde{\psi}_{1n}|_{z=d/2} = (i \sigma^1 \tilde{z}_n - i \sigma^2 \frac{\partial}{\partial \phi}) \psi_{1n}(\tilde{z}_n, \phi) = E \psi_{1n}(\tilde{z}_n, \phi)\)). Evidently, then, the new \( \tilde{\psi}_{1n} = \sigma^2 \psi_{1n} \), since \( \sigma^2 \sigma^1 \sigma^2 = -\sigma^1 \).

In summary, equations (8) and (10) are obtained.

### Appendix D. Wavefunctions and the quantization condition on the cylinder

The local solutions to equation (8) (if there is no flux) are for the caps \( \tilde{\psi}_{1,3} = \exp(ik\phi)f_{1,3}(r) \). If \( E \neq \pm \sqrt{k^2 + m^2} \) and \( E \neq \pm m_1, m_3 \), \( f(r) \) has the form
\[
f(r) = C_1 \left( \frac{J_{k-1}((E^2 - m^2)R)}{E + m} \right) \left( \frac{Y_{k+2}((E^2 - m^2)R)}{E + m} \right), \tag{D.1}
\]
where a region label = I, III should be understood for the mass \( m \), \( f(r) \) and the coefficients \( C_1 \) and \( C_2 \). \( \tilde{k} \in \mathbb{Z} + 1/2 \) is globally the same value for all regions, and different \( \tilde{k} \) are linearly independent, as required by the boundary condition along the \( \phi \) direction (they are good quantum numbers). \( J_0(r) \) and \( Y_0(r) \) are Bessel functions, the two independent solutions satisfying \( r^2 \frac{d^2 \psi}{dr^2} + r \frac{d \psi}{dr} + (r^2 - \nu^2)\psi = 0 \), \( J_0(r) \) and \( J_{1,\nu}(r) \) also work if \( n \) is not an integer. The solution on the side is \( \tilde{\psi}_{1n} = \exp(ik\phi)f_{1,3}(r) \), and if \( E \neq \pm \sqrt{k^2 + m^2} \) and \( E \neq \pm m, f(r) \) is (again omitting the region II label)
\[
f(z) = A \left( \frac{1}{\sqrt{E^2 - k^2 - m^2 + i k}} \right) e^{i \sqrt{E^2 - k^2 - m^2} z} + B \left( \frac{1}{\sqrt{E^2 - k^2 - m^2 + i k}} \right) e^{-i \sqrt{E^2 - k^2 - m^2} z}. \tag{D.2}
\]

Square integrability at the origin sets one of the coefficients of the cap to zero when \( |\tilde{k}| \geq 1/2 \). Using the boundary conditions discussed in appendix C, the remaining coefficients are determined along with the quantization condition,
\[
e^{-2d\sqrt{E^2 - k^2 - m^2}} \left( \frac{i \hbar_{11} Y^* - \hbar_{11} g_{11}}{g_{11} + i \hbar_{11} y} \right) = i \hbar_{11} y^* - \hbar_{11} \tag{D.3}
\]
with \( \hbar_{11} = \frac{\sqrt{E^2 - m^2} J_{k+\frac{1}{2}}((E^2 - m^2)R)}{E + m} \). When a localized flux is inserted through the origin, then away from origin the solutions are still of the form given by (D.1) with the replacement \( \tilde{k} \to \tilde{k} + \Phi \) for \( \Phi \) flux quantum. In this case, square integrability at the origin no longer constrains the coefficients in (D.1), instead a matching to the solutions in the B-field region must be carried out. As discussed in the text, the simplest flux profile allowing for an analytical solution is to take a delta function ring [29]. \( B(r) = \Phi \delta(r - \epsilon)/(2\pi \epsilon) \). Then, inside the ring, normalizability again constrains one coefficient to zero. Matching this at \( r = \epsilon \) gives the required relation between the coefficients in the interior of the flux and the rest of the cap. Then, the quantization condition is approximately the same as (D.3) with \( \tilde{k} \to \tilde{k} + \Phi \) for \( |\tilde{k} + \Phi| > 1/2 \). For \( |\tilde{k} + \Phi| \leq 1/2 \), the explicit form of the B-field must be used to match the wavefunctions at \( \epsilon \). The result can be summarized by making the following redefinitions in equation (D.3) (assuming \( \Phi \geq 0 \)):
\[
h_{11} = \frac{\sqrt{E^2 - m^2} d_{11} J_{k+\frac{1}{2}}((E^2 - m^2)R)}{E + m} - d_{21} J_{k-\frac{1}{2}}((E^2 - m^2)R), \quad g_{11} = d_{11} J_{k+\frac{1}{2}}((E^2 - m^2)R) + d_{21} J_{k-\frac{1}{2}}((E^2 - m^2)R),
\]
where

\[\text{Figure C.1. Side view of the smoothed out cylinder edge.}\]
\[d_1 = \frac{J_{-k-1/2} - d_j J_{-k+1/2-\Phi}}{J_{-k+1/2+\Phi}},\]
\[d_2 = \frac{J_{k-1/2+\Phi} J_{k+1/2} - J_{k+1/2+\Phi} J_{k-1/2}}{J_{k+1/2+\Phi} J_{k-1/2+\Phi} - J_{k-1/2+\Phi} J_{k+1/2+\Phi}},\]

with all unspecified J\(_s\) evaluated at \(\sqrt{E^2 - M^2 \epsilon}\) so long as \(\Phi\) is not an integer. For \(\Phi\) integer, one must further replace \(J_{-k-}\) with \(Y\) (the second Bessel function) and \(-d_2\) with \(+d_2\) in \(\hbar_a\).

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