Smoothed functional average variance estimation for dimension reduction

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Abstract. We propose an estimation method that we call functional average variance estimation (FAVE), for estimating the EDR space in functional semiparametric regression model, based on kernel estimates of density and regression. Consistency results are then established for the estimator of the interest operator, and for the directions of EDR space. A simulation study that shows that the proposed approach performs as well as traditional ones is presented.

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1 Introduction

In recent years, much attention has been given to functional statistics, which can be described as the set of statistical methods for processing data having the form of curves considered as observations of functions belonging to given functional spaces. Among the references in this field, there are the books by Ramsay and Silverman\cite{16} for the applied aspects, Bosq\cite{1} for the theoretical aspects, Ferraty and Vieu\cite{6}, and Horváth and Kokoszka\cite{11} for recent developments. Many works in this field deal with problems that appear in the general framework of functional regression models which are usually used to find the best link between a real random variable $Y$ and a random curve $X$ whose values belong to $\mathcal{H} = L^2([0, 1])$, the set of square integrable functions from $[0, 1]$ to $\mathbb{R}$. An abundant literature has examined cases of parametric
functional regression models (e.g., [2], [10], [18]) described by the relation
\[ Y = f_\theta(X, \varepsilon), \]
where \( f_\theta \) belongs to a well-known family of functions parameterized by the unknown parameter \( \theta \), which is to be estimated, and \( \varepsilon \) is an error term. In contrast to this, some works deal with a nonparametric model
\[ Y = f(X) + \varepsilon \]
where \( f \) is an unknown and arbitrary function to estimate, and have introduced nonparametric estimation approaches, such as methods based on kernel estimators ([6], [7]). Alternatively, between these two different approaches, a semiparametric regression model
\[ Y = f(<\beta_1, X>_{\mathcal{H}}, <\beta_2, X>_{\mathcal{H}}, \ldots, <\beta_K, X>_{\mathcal{H}}, \varepsilon) \]
was considered ([4], [8], [9], [13]). In the model (1), \( <\cdot, \cdot>_{\mathcal{H}} \) denotes the inner product of \( \mathcal{H} \) defined for all \( g_1 \) and \( g_2 \) belonging to \( \mathcal{H} \) by
\[ <g_1, g_2>_\mathcal{H} = \int_0^1 g_1(t)g_2(t)dt, \]
and \( \beta_1, \ldots, \beta_K \) are elements of \( \mathcal{H} \) to be estimated. This model just is an extension in the functional case of the model introduced by Li[12] in the multivariate context and which has been intensively studied since then. It expresses the fact that the information in \( X \) about \( Y \) depends only on the projection of \( X \) onto the subspace spanned by \( \{\beta_1, \ldots, \beta_K\} \), called effective dimension-reduction (EDR) space. Li[12] showed that the problem of estimating the EDR space comes down, under a fairly general condition, to the spectral analysis of an operator depending on the covariance operator of the conditional expectation \( \mathbb{E}(X|Y) \) of \( X \) given \( Y \). Then, he proposed an estimation method, called sliced inverse regression (SIR), based on an estimate of an approximation of this covariance operator obtained by slicing the range of \( Y \). Alternatively, Cook[3] proposed another method, called sliced average variance estimation (SAVE), for estimating the EDR by using an estimate of an approximation of an operator depending on the conditional covariance operator \( \text{Var}(X|Y) \) of \( X \) given \( Y \). SIR and SAVE are the most popular methods for dimension reduction in the multivariate context, and smoothed estimation methods, based on kernel estimates, have been proposed for them respectively by Zhu and Fang[19] and Zhu and Zhu[20]. In the functional context, SIR has been extended to functional SIR (FSIR) by Ferré and Yao[8] who also proposed later a smoothed estimation procedure based on kernel estimates, so defining smoothed functional inverse regression (FIR). On the other hand, more recently, Lian and Li[13] extended SAVE to functional SAVE (FSAVE). To the best of our knowledge, a smoothed estimation of SAVE have not been proposed yet in the context of functional data. Taking all this into consideration, we introduce in this paper a kernel functional average variance estimation (FAVE) method for estimating the EDR space.
related to model [1]. The rest of the paper is organized as follows. In Section 2, we recall some basic facts about FAVE in the functional context, and we specify the interest operator to estimate. In Section 3, an estimator based on kernel estimates is proposed for this estimating this operator. Section 4 is devoted to an asymptotic study of the introduced estimator. A simulation study that permits to evaluate the performance of our proposal is presented in Section 5. The proofs of theorems are postponed in Section 6.

2 Functional Sliced Average Variance

Let us consider the random variables \( Y \) and \( X \) involved in the model (1); we assume, without loss of generality, that \( E(X) = 0 \), and that \( E(\|X\|^2_H) < +\infty \). Then, the covariance operator of \( X \) is defined by \( \Gamma = E(X \otimes X) \), where for any \( x,y \in H \), \( x \otimes y \) denotes the linear operator from \( H \) to itself such that \( (x \otimes y)(h) = \langle x, h \rangle_H y \) for any \( h \in H \). Throughout the paper, \( \Gamma \) will be assumed to be non-singular and positive definite. Letting \( B = (\langle \beta_1, X \rangle_H, \langle \beta_2, X \rangle_H, \cdots, \langle \beta_K, X \rangle_H) \) and denoting by \( \text{Var}(X|B) \) the conditional covariance operator of \( X \) given \( B \), we consider the following assumptions:

\[ (\mathcal{A}_1) : \text{ for all } b \in H, \text{ one has } E(\langle b, X \rangle_H | B) = \sum_{k=1}^K c_k < \beta_k, X >_H, \text{ where } c_1, \cdots, c_K \text{ are real numbers}; \]

\[ (\mathcal{A}_2) : \text{Var}(X|B) \text{ is nonrandom}. \]

Lian and Li\[13\] showed that under the assumptions \((\mathcal{A}_1)\) and \((\mathcal{A}_2)\), one has the inclusion

\[ R(\Gamma - \text{Var}(X|Y)) \subset \Gamma \exists, \quad (2) \]

where \( R(A) \) denotes the range of the operator \( A \), \( \text{Var}(X|Y) \) denotes the conditional covariance operator of \( X \) given \( Y \), and \( \exists \) is the EDR space, that is the space spanned by \( \beta_1, \cdots, \beta_K \). Therefore, \( R(\Gamma_I) \subset \exists \), where

\[ \Gamma_I := \Gamma^{-1}\mathbb{E}\left( \Gamma - 2\text{Var}(X|Y) + \text{Var}(X|Y)\Gamma^{-1}\text{Var}(X|Y) \right). \]

An important consequence is that \( \Gamma_I \) is degenerate in any direction orthonormal to the \( \beta_k \)'s (\( k = 1, 2, \cdots, K \)). Then \( \Gamma_I \) is a finite rank operator whose
range is contained into the EDR space. This space can, therefore, be approached by the subspace spanned by the eigenvectors of $\Gamma_I$ associated with the $K$ largest non-null eigenvalues of $\Gamma_I$ in the same way as in the multivariate case. In the following we suppose that $\text{rank}(\Gamma_I)=K$. We see, therefore, that the eigenvectors associated with the $K$ largest eigenvalues of $\Gamma_I$ form an base to EDR space, which make the EDR space identifiable. So $\Gamma_I$ is the interest operator of the FAVE method. Since the domain of $\Gamma_I^{-1}$ is not the whole $H$, $\Gamma_I$ may not be well-defined. Conditions under which this operator is well defined are established in [13] and recalled below.

Let

$$X = \sum_{j=1}^{\infty} \xi_j \phi_j,$$

be the well-known Karhunen-Love expansion of $X$, where $E[\xi_j^2] = \alpha_j$ are the eigenvalues and $\phi_j$ are the eigenfunctions. As usual in the functional data literature (e.g.,[13],[9]), we assume that $\alpha_1 > \alpha_2 > \cdots > 0$. We now introduce the assumptions:

$(\mathcal{A}_3)$: $E(\|X\|_H^4) < +\infty$;

$(\mathcal{A}_4)$: $E \left[ \left( \sum_{j=1}^{\infty} \alpha_j^{-2} \sum_{i=1}^{\infty} \text{Cov}^2(\xi_i, \xi_j|Y) \right)^2 \right] < +\infty$. It is known that if $(\mathcal{A}_3)$ and $(\mathcal{A}_3)$ hold, then $\Gamma_I$ is well-defined (see Proposition 1 in [13]).

3 Kernel estimator of the interest operator

For performing the FAVE method $\Gamma_I$, has to be estimated. Lian and Li[13] introduced an estimator obtained by slicing the range of $Y$. In this section, we propose another estimator of this operator based on kernel estimates of density and regression. Since $\Gamma = E[\text{Var}(X|Y)] + \text{Var}[E(X|Y)]$, we have

$$\Gamma_I = \Gamma^{-1} \left( 2\Gamma_e + \Psi - \Gamma \right)$$

(3)

where $\Gamma_e = \text{Var}[E(X|Y)]$ is the covariance operator of the conditional expectation $E(X|Y)$ and $\Psi = E(\text{Var}(X|Y)\Gamma^{-1} \text{Var}(X|Y))$. Ferré and Yao [8]
introduced a kernel estimator of $\Gamma_e$ and showed its consistency. Here, we will use this estimator, and also a kernel estimator of $\Psi$ together with the empirical counterpart of $\Gamma$ in order to define an estimator of $\Gamma_I$. Letting $f$ be the density of $Y$ and putting $m(y) = \mathbb{E}(1_{\{Y=y\}} X)$, $M(y) = \mathbb{E}(1_{\{Y=y\}} X \otimes X)$,
\[ r(Y) = \mathbb{E}(X|Y) = \frac{m(Y)}{f(Y)} \quad \text{and} \quad R(Y) = \mathbb{E}(X \otimes X|Y) = \frac{M(Y)}{f(Y)}, \]
we have $\Psi = \mathbb{E}(C(Y)\Gamma^{-1}C(Y))$ where $C(Y) = Var(X|Y) = R(Y) - r(Y) \otimes r(Y)$. As it was done in [19], in order to avoid the effect of the small values in the denominator, we consider $f_{e_n} = \max(f, e_n)$ instead of $f$, where $(e_n)_{n \in \mathbb{N}^*}$ is a sequence of real numbers which tends to zero as $n \to +\infty$. Then, we consider
\[ r_{e_n}(Y) = \frac{m(Y)}{f_{e_n}(Y)}, \quad R_{e_n}(Y) = \frac{M(Y)}{f_{e_n}(Y)} \quad \text{and} \quad C_{e_n}(Y) = R_{e_n}(Y) - r_{e_n}(Y) \otimes r_{e_n}(Y) \]
instead of $r(Y)$, $R(Y)$ and $C(Y)$. The definition of $\Gamma_I$ given in (3) requires to use the inverse of $\Gamma$. But since $\Gamma$ is an Hilbert-Schmidt operator, even though its inverse exists it is not generally bounded. To avoid this difficulty, we consider instead the finite-rank operator $\Gamma_D = \Pi_D \Gamma \Pi_D$, where $D \in \mathbb{N}^*$ and $\Pi_D$ is the projector onto the subspace $S_D$ spanned by the system $\{\phi_1, \cdots, \phi_D\}$ consisting of the $D$ first elements of an orthonormal basis of $\mathcal{H}$. This basis can, for example, be obtained either from principal component analysis (PCA) of $X$ or by using $B$-splines basis. This operator has a bounded (pseudo-)inverse defined by $\Gamma_D^{-1} = \Pi_D \Gamma^{-1} \Pi_D$.

Let $\{(X_i, Y_i)\}_{1 \leq i \leq n}$ be an i.i.d. sample of $(X, Y)$; the empirical counterpart of $\Gamma$ is given by $\hat{\Gamma}_n = n^{-1} \sum_{i=1}^n X_i \otimes X_i$. Considering the estimate $\hat{\Pi}_D$ of $\Pi_D$ defined as the projector onto an estimate $\hat{S}_D$ of $S_D$, we estimate $\Gamma_D$ by $\hat{\Gamma}_D = \hat{\Pi}_D \hat{\Gamma}_n \hat{\Pi}_D$. If we use PCA (resp. $B$-splines basis) then $\hat{S}_D$ consists of the eigenvectors associated with the $D$ largest eigenvalues of $\hat{\Gamma}_n$ (resp. $\hat{S}_D = S_D$). For a given kernel function $K : \mathbb{R} \to \mathbb{R}_+$ and a given real $h > 0$, we consider the estimates
\[ \hat{f}(y) = \frac{1}{nh} \sum_{i=1}^n K \left( \frac{Y_i - y}{h} \right), \quad \hat{m}(y) = \frac{1}{nh} \sum_{i=1}^n K \left( \frac{Y_i - y}{h} \right) X_i \]
and
\[
\hat{M}(y) = \frac{1}{nh} \sum_{i=1}^{n} K \left( \frac{Y_i - y}{h} \right) X_i \otimes X_i
\]
of \(f, m\) and \(M\) respectively. Then, putting
\[
\hat{f}_{en}(y) = \max\{e_n, \hat{f}(y)\}, \quad \hat{r}_{en}(y) = \frac{\hat{m}(Y)}{\hat{f}_{en}(y)}, \quad \hat{R}_{en}(y) = \frac{\hat{M}(y)}{\hat{f}_{en}(y)}
\]
and
\[
\hat{C}_{en}(y) = \hat{R}_{en}(y) - \hat{r}_{en}(y) \otimes \hat{r}_{en}(y)
\]
we consider
\[
\hat{\Gamma}_{e,n} = \frac{1}{n} \sum_{i=1}^{n} \hat{r}_{en}(Y_i) \otimes \hat{r}_{en}(Y_i), \quad \hat{\Psi}_{en,D} = \frac{1}{n} \sum_{i=1}^{n} \hat{C}_{en}(Y_i) \hat{\Gamma}^{-1} \hat{C}_{en}(Y_i)
\]
and we estimate \(\Gamma_I\) by the random operator
\[
\hat{\Gamma}_{I,n} = \hat{\Gamma}^{-1}_D \left( 2\hat{\Gamma}_{e,n} + \hat{\Psi}_{en,D} - \Gamma_n \right).
\]
This random operator determines our kernel FAVE approach for estimating the EDR space. This estimation procedure is achieved by considering the space spanned by the eigenvectors \(\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_K\) of \(\hat{\Gamma}_{I,n}\), associated respectively with the \(K\) largest eigenvalues \(\hat{\lambda}_1, \ldots, \hat{\lambda}_K\).

4 Asymptotics study of \(\hat{\Gamma}_{I,n}\)

In this section, we deal with asymptotics for \(\hat{\Gamma}_{I,n}\). More precisely, we first establish its consistency as an estimator of \(\Gamma_I\). Then we show the \(\hat{\beta}_k\)'s are also consistent estimators of the \(\beta_k\)'s. We need the following assumptions:

(\(\mathcal{A}_5\)): \(\Gamma\) is positive definite.

(\(\mathcal{A}_6\)): \(f, r\) and \(R\) belong to \(C^k\);

(\(\mathcal{A}_7\)): the kernel \(K\) is of order \(k > 2\), has compact support \([a, b]\), is symmetric about zero and satisfies \(K \leq 1, \int_a^b |u|^k K(u) du < +\infty\).
(\(\mathcal{A}_g\)): there exist real numbers \(d_1, d_2\) and \(d_3\) such that \(\sup_{y \in \mathbb{R}} |f^{(k)}(y)| \leq d_1,\) \(\sup_{y \in \mathbb{R}} \|m^{(k)}(y)\|_H \leq d_2\) and \(\sup_{y \in \mathbb{R}} \|m^{(k)}(y)\|_{hs} \leq d_3\), where \(\| \cdot \|_{hs}\) denotes the Hilbert-Schmidt norm of operators;

(\(\mathcal{A}_9\)): \(h \sim n^{-c_1}\) and \(e_n \sim n^{-c_2}\), where \(c_1\) and \(c_2\) are real numbers satisfying \(c_1 > 0, 0 < c_2 < \frac{k-2}{4(k+1)}\) and \(\frac{2c_2}{k} + \frac{1}{2k} < c_1 < \frac{1}{4} - c_2\);

(\(\mathcal{A}_{10}\)): \(\sqrt{n} \mathbb{E} \left[ \|R(Y)\|_{hs}^2 \chi_{\{f(y) < e_n\}} \right], \sqrt{n} \mathbb{E} \left[ \|R(Y)\|_{hs} \|r(Y)\|_{hs}^2 \chi_{\{f(y) < e_n\}} \right]\) and \(\sqrt{n} \mathbb{E} \left[ \|R(Y)\|_H^2 \chi_{\{f(y) < e_n\}} \right]\) tends to 0 as \(n \to +\infty\);

(\(\mathcal{A}_{11}\)): the function \(y \mapsto \mathbb{E} [\|X\|_{hs}^2 | Y = y] is continuous.

**Remark 1** Zhu and Fang [13] introduced \(\hat{f}_{e_n}(y) = \max(\hat{f}(y), e_n)\) to overcome technical difficulties due to small values in the denominator of \(\hat{r}(y)\). But this approach does not guarantee that we get a good estimator of \(f\). Indeed, if we take for example \(e_n = n^{-1/11}\), then until \(n = 2000\) we still have \(e_n > 1/2\) and, therefore, \(\hat{f}_{e_n}(y) = 1/2\) for all \(y \in \mathbb{R}\). To overcome this later problem, Nkou and Nkiet [14] propose to take \(e_n = \min(a; n^{-c_2})\), where \(a\) is a fixed strictly positive number. When \(a\) is sufficiently small \(\hat{f}_{e_n}(y)\) is a good estimator of \(f\), because \(\sup_{x \in \mathbb{R}} |\hat{f}_{e_n}(y) - \hat{f}(y)| \leq a\) and we still have \(e_n \sim n^{-c_2}\).

For \(D \in \mathbb{N}^*\), we consider

\[
\Psi_D = \mathbb{E} \left[ \text{Var}(X|Y) \Gamma_D^{-1} \text{Var}(X|Y) \right]
\]

and denoting by \(t_D\) the minimum positive eigenvalue of \(\Gamma_D\), we have:

**Theorem 4.1** Under assumptions \((\mathcal{A}_1)\) to \((\mathcal{A}_3)\) and \((\mathcal{A}_7)\) to \((\mathcal{A}_{11})\), if we suppose that when \(D \to +\infty\), we have \(\|\hat{\Gamma}_D - \Gamma_D\|_{\infty} = o_p(t_D)\), then

\[
\|\Psi_D - \hat{\Psi}_{e_n,D}\|_{hs} = O_p \left( \frac{1}{\sqrt{n}} \right) + O_p \left( \frac{1}{t_D \sqrt{n}} \right) + O_p \left( \frac{1}{e_n t_D} \left( h^k + \frac{\sqrt{\log(n)}}{h \sqrt{n}} \right) \right)
\]

\[
= O_p \left( \frac{1}{\sqrt{n}} \right) + O_p \left( \frac{1}{n^\gamma t_D} \right),
\]

where \(\gamma\) is a constant satisfying \(0 < \gamma < 1/4\).
Remark 2 This theorem gives an idea on the convergence rate of each component of \( \hat{\Gamma}_{I,n} \) as we know the one of \( \hat{\Gamma}_{e,n} \) from [8]. We cannot reach the \( \sqrt{n} \)-convergence, because the rate of convergence will be penalized by the one of \( t_D \). The assumption \( \| \hat{\Gamma}_D - \Gamma_D \|_\infty = o_p(t_D) \) was also used in [13] for obtaining a similar result for the case of Functional SAVE. A justification of this assumption can be found in this paper.

In the following theorem consistency of \( \hat{\Gamma}_{I,n} \) is established under some conditions.

Theorem 4.2 Under the assumptions (\( A_1 \)) to (\( A_{11} \)), if we suppose that for some \( 0 < \gamma < 1/4 \), when \( D \to +\infty \), \( \| \hat{\Gamma}_D - \Gamma_D \|_\infty = o_p(t_D) \), \( 1/(t_D \sqrt{n}) \to 0 \), \( 1/(n^{\gamma} t_D^2) \to 0 \), then \( \hat{\Gamma}_{I,n} - \Gamma_I = o_p(1) \).

Remark 3 This result only gives the convergence in probability of \( \hat{\Gamma}_{I,n} \) to \( \Gamma_I \) without specifying the rate. For the functional SAVE, Lian and Li [13] don’t show the convergence of their estimator of \( \Gamma_I \).

Now, we deal with the \( \hat{\beta}_k \)'s. For doing that, we assume that \( \beta_1, \beta_2, \ldots, \beta_K \) are the \( K \) eigenvectors of \( \Gamma_I \) associated with the \( K \) eigenvalues \( \lambda_1, \ldots, \lambda_K \) respectively, and that \( \lambda_1 > \lambda_2 > \cdots > \lambda_K > 0 \).

Theorem 4.3 Under the assumptions (\( A_1 \)) to (\( A_{11} \)), if we suppose that for some \( 0 < \gamma < 1/4 \), when \( D \to +\infty \), \( \| \hat{\Gamma}_D - \Gamma_D \|_\infty = o_p(t_D) \), \( 1/(t_D \sqrt{n}) \to 0 \), \( 1/(n^{\gamma} t_D^{5/2}) \to 0 \), then \( \| \hat{\beta}_j - \beta_j \|_H = o_p(1) \) for \( j = 1, 2, \ldots, K \).

Remark 4 This result is similar to that of FSIR obtained by Ferré and Yao [8]. It is an extension to the functional case of a property of the kernel method for sliced average variance estimation developed by Zhu and Zhu [20] in a multivariate context.

5 Simulation study

In this section, we use simulations to illustrate the kernel FAVE method and to compare it with existing methods. In all the examples, the predictor \( X \) is a standard brownian motion on \([0, 1]\), observed on a grid of \( p = 100 \) equally spaced points. Two models are considered:
Model 1: \( Y = \sin(\pi < \beta_1, X >_H / 2) + < \beta_2, X >^2_H + \varepsilon, \) where \( \beta_1(t) = (2t - 1)^3 + 1, \beta_2(t) = \cos(\pi(2t - 1)) + 1 \) and \( \varepsilon \sim N(0, 0.1^2). \)

Model 2: \( Y = 50 < \beta_1, X >^2_H + < \beta_2, X >^2_H + \varepsilon, \) where \( \beta_1(t) = 4t^2, \beta_2(t) = \sin(5\pi t/2) \) and \( \varepsilon \sim N(0, 0.1^2). \)

We set \( n = 100 \) and we consider both functional PCA and quadratic B-spline basis for computing \( \hat{\Pi}_D. \) For the B-spline basis, the knots are chosen to be equally spaced on \([0, 1]\). Various dimensions \( D \) are used, \( D = 4, 5, 6, 7, 8. \) The bandwidth \( h \) is selected by the cross-validation.

The plots of \( \hat{\beta}_1 \) and \( \hat{\beta}_2, \) obtained from kernel FAVE, together with that of \( \beta_1 \) and \( \beta_2 \) are given in Figure 1-2 for Model 1, and in Figure 3-4 for Model 2. They reveal very good estimations. In order to verify if the prior projection space is well estimates by our FAVE method, we plot in Figure 5 to 8, the index \( < \beta_j, X >_H \) versus \( < \hat{\beta}_j, X >_H \) for \( j = 1, 2 \) and for Model 1-2. These scatter plots reveal a strong correlation in both cases. All the previous plot are made using \( D = 4. \)

In order to compare our method to the FSIR and FSAVE methods, we use various dimensions \( D = 4 \) to \( D = 8. \) FSAVE is performed with number of slices \( H = 10. \) The distance between the true EDR space and its estimation is computed via \( \mathcal{E} = \| P - \hat{P} \|_{hs}, \) where \( P \) (resp. \( \hat{P} \)) denotes the projector onto the space spanned by \( \beta_1 \) and \( \beta_2 \) (resp. \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \)). We use \( m = 100 \) simulated datasets in each scenarios to get the boxplot of \( \mathcal{E}. \) In the left hand of each, figure from 9 to 14, the boxplots are built by using functional PCA basis expansion, whereas the ones in the right hand are based on B-spline basis functions. The boxplot results related to the B-spline projections are almost better than the ones from functional PCA. For Model 1 the three methods perform similarly as showed by the boxplots, but in the case of Model 2 FSIR does not work as well as FSAVE and FAVE. As a general observation the three methods are sensitive to the choice of \( D. \) Therefore, methods are needed for choosing of \( D \) and will perfect the practical use of the FSIR, FAVE and FSAVE methods.
Figure 1: Plot of $\hat{\beta}_1$ (red) and $\beta_1$ (black) for Model 1.

Figure 2: Plot of $\hat{\beta}_2$ (red) and $\beta_2$ (black) for Model 1.

Figure 3: Plot of $\hat{\beta}_1$ (red) and $\beta_1$ (black) for Model 2.

Figure 4: Plot of $\hat{\beta}_2$ (red) and $\beta_2$ (black) for Model 2.
Figure 5: Plot of $<\hat{\beta}_1, X>_H$ versus $<\beta_1, X>_H$ for Model 1.

Figure 6: Plot of $<\hat{\beta}_2, X>_H$ versus $<\beta_2, X>_H$ for Model 1.

Figure 7: Plot of $<\hat{\beta}_1, X>_H$ versus $<\beta_1, X>_H$ for Model 2.

Figure 8: Plot of $<\hat{\beta}_2, X>_H$ versus $<\beta_2, X>_H$ for Model 2.
Figure 9: Boxplots showing $\| P - \hat{P} \|_{hs}$ for Model 1, from FSAVE with $H = 10$. 
6 Proofs

6.1 Preliminary results

In this section we will give some lemmas necessary to get the proofs of the previous Theorems.

Lemma 6.1 Under assumption (A6) to (A9), we have

\[ \sup_{y \in \mathbb{R}} \| \hat{M}(y) - M(y) \|_{hs} = O_p \left( h^k + \frac{\sqrt{\log(n)}}{h \sqrt{n}} \right). \]

Proof. It is easy to check that for all \( y \in \mathbb{R} \), one has

\[ \mathbb{E} \left[ \hat{M}(y) \right] = \frac{M * K_h(y)}{h}. \]
Figure 12: Boxplots showing $\|P - \hat{P}\|_{hs}$ for Model 2, from FSAVE with $H = 10$
Figure 13: Boxplots showing $\|P - \hat{P}\|_{hs}$ for Model 2, using kernel FAVE.

Figure 14: Boxplots showing $\|P - \hat{P}\|_{hs}$ for Model 1, using kernel FSIR.
Then
\[ E \left[ \hat{M}(y) \right] - M(y) = \frac{1}{h} \int_{\mathbb{R}} R(v) K_h(v - y) f(v) dv - M(y) \]
\[ = \frac{1}{h} \int_{\mathbb{R}} [M(v) - M(y)] K_h(v - y) dv \]
\[ = \int_{\mathbb{R}} [M(y + hw) - M(y)] K(w) dw \]
\[ = \int_{\mathbb{R}} \left[ \sum_{j=1}^{k-1} \frac{(wh)^j}{j!} M^{(j)}(y) + \frac{(wh)^k}{k!} M^{(k)}(y + \theta hw) \right] K(w) dw \]
\[ = \frac{h^k}{k!} \int_{a}^{b} w^k M^{(k)}(y + \theta hw) K(w) dw. \]

Hence
\[ \| E \left[ \hat{M}(y) \right] - M(y) \|_{hs} \leq \frac{h^k}{k!} \sup_{y \in I} \| M^{(k)}(y) \|_{hs} \int_{a}^{b} |w|^k K(w) dw = Ch^k, \]

that is \( \sup_{y \in I} \left| \frac{M \ast K_h(y)}{h} - M(y) \right|_{hs} = O(h^k) \). We deduce that

\[ \sup_{y \in \mathbb{R}} \| \hat{M}(y) - M(y) \|_{hs} \leq \sup_{y \in \mathbb{R}} \| \hat{M}(y) - E \left[ \hat{M}(y) \right] \|_{hs} + \sup_{y \in \mathbb{R}} \| E \left[ \hat{M}(y) \right] - M(y) \|_{hs} \]
\[ = D_1 + D_2. \]

Let \( \varepsilon > 0 \) and \( (a_n)_{n \in \mathbb{N}} \), a sequence of non-negative reals numbers converging to +\( \infty \). We have:

\[ P(D_1 > \varepsilon) = P \left( \sup_{y \in \mathbb{R}} \| \hat{M}(y) - E \left[ \hat{M}(y) \right] \|_{hs} > \varepsilon \right) \]
\[ \leq P \left( \sup_{y \in \mathbb{R}} \| \hat{M}(y) - E \left[ \hat{M}(y) \right] \|_{hs} > \varepsilon; \| X \otimes X \|_{hs} \leq a_n \right) \]
\[ + P \left( \sup_{y \in \mathbb{R}} \| \hat{M}(y) - E \left[ \hat{M}(y) \right] \|_{hs} > \varepsilon; \| X \otimes X \|_{hs} > a_n \right) \]
\[ \leq P \left( \sup_{y \in \mathbb{R}} \| \hat{M}(y) - E \left[ \hat{M}(y) \right] \|_{hs} > \varepsilon; \| X \otimes X \|_{hs} \leq a_n \right) \]
\[ + P \left( \| X \otimes X \|_{hs} > a_n \right). \]
Since $K \leq 1$, we have for all $y \in \mathbb{R}$

\[
\| \hat{M}(y) - \mathbb{E} \left[ \hat{M}(y) \right] \|_{h, s} = \left\| \frac{1}{nh} \sum_{i=1}^{n} \left( X_i \otimes X_i K \left( \frac{Y_i - y}{h} \right) - \mathbb{E}[X_i \otimes X_i K \left( \frac{Y_i - y}{h} \right)] \right) \right\|_{h, s}
\]

\[
\leq \frac{1}{nh} \sum_{i=1}^{n} \{ \| X_i \otimes X_i \|_{h, s} + \mathbb{E}[\| X_i \otimes X_i \|_{h, s}] \}.
\]

Thus

\[
\sup_{y \in \mathbb{R}} \| \hat{M}(y) - \mathbb{E} \left[ \hat{M}(y) \right] \|_{h, s} \leq \frac{1}{nh} \sum_{i=1}^{n} \{ \| X_i \otimes X_i \|_{h, s} + \mathbb{E}[\| X_i \otimes X_i \|_{h, s}] \}
\]

and

\[
P \left( \sup_{y \in \mathbb{R}} \| \hat{M}(y) - \mathbb{E} \left[ \hat{M}(y) \right] \|_{h, s} > \varepsilon; \| X \otimes X \|_{h, s} \leq a_n \right)
\]

\[
\leq P \left( \frac{1}{nh} \sum_{i=1}^{n} \{ \| X_i \otimes X_i \|_{h, s} + \mathbb{E}[\| X_i \otimes X_i \|_{h, s}] \} \right) > \varepsilon; \| X \otimes X \|_{h, s} \leq a_n \right)
\]

\[
\leq P \left( \frac{1}{nh} \sum_{i=1}^{n} \{ \| X_i \otimes X_i \|_{h, s} + \mathbb{E}[\| X_i \otimes X_i \|_{h, s}] \} \mathbf{1}_{\{ \| X \otimes X \|_{h, s} \leq a_n \}} > \varepsilon \right).
\]

However, for any $i \in \{1, \ldots, n\}$, one has

\[
\frac{1}{h} \left( \| X_i \otimes X_i \|_{h, s} + \mathbb{E}[\| X_i \otimes X_i \|_{h, s}] \right) \mathbf{1}_{\{ \| X \otimes X \|_{h, s} \leq a_n \}} \leq \frac{2a_n}{h}.
\]

Then using Bernstein inequality, we obtain

\[
P \left( \sup_{y \in \mathbb{R}} \| \hat{M}(y) - \mathbb{E} \left[ \hat{M}(y) \right] \|_{h, s} > \varepsilon; \| X \otimes X \|_{h, s} \leq a_n \right) \leq 2 \exp \left( - \frac{n \varepsilon^2 h^2}{16a_n^2} \right)
\]

from what we deduce

\[
P(D_2 > \varepsilon) \leq P(\| X \otimes X \|_{h, s} > a_n) + 2 \exp \left( - \frac{n \varepsilon^2 h^2}{16a_n^2} \right)
\]

\[
\leq \frac{\mathbb{E}(\| X \otimes X \|^2_{h, s})}{a_n^2} + 2 \exp \left( - \frac{n \varepsilon^2 h^2}{16a_n^2} \right)
\]

\[
= \frac{\mathbb{E}(\| X \|^4_{H})}{a_n^2} + 2 \exp \left( - \frac{n \varepsilon^2 h^2}{16a_n^2} \right).
\]
Taking $\varepsilon = \frac{\varepsilon_0}{h} \sqrt{\frac{\log(n)}{n}}$, where $\varepsilon > 0$, and $\alpha_n = (\log(n))^{1/4}$, we have

$$P \left(D_2 > \frac{\varepsilon_0}{h} \sqrt{\frac{\log(n)}{n}}\right) \leq \mathbb{E} \left(\|X\|_{H}^{4}\right) \left(\log(n)\right)^{1/2} + 2 \exp\left(-\frac{\varepsilon_0 (\log(n))^{1/2}}{16}\right)$$

and since

$$\lim_{n \to +\infty} \left(\mathbb{E} \left(\|X\|_{H}^{4}\right) \left(\log(n)\right)^{1/2} + 2 \exp\left(-\frac{\varepsilon_0 (\log(n))^{1/2}}{16}\right)\right) = 0$$

we conclude that $D_2 = O_p \left(h^{-1}n^{-1/2}(\log(n))^{1/2}\right)$ and, consequently, that

$$\sup_{y \in \mathbb{R}} \|\hat{M}(y) - M(y)\|_{hs} = O_p \left(h^k + \frac{\sqrt{\log(n)}}{h\sqrt{n}}\right).$$

\[
\text{Lemma 6.2} \quad \text{We have:} \quad \frac{f_n(Y_j) - \hat{f}(Y_j)}{f_n(Y_j)} \leq 2 \left[1_{\{f(Y_j) < 2e_n\}} + \frac{\left(\sup_{y \in \mathbb{R}} |\hat{f}(y) - f(y)|\right)^2}{e_n^2}\right].
\]

\text{Proof.} \quad \text{Since}

$$\left|\frac{f_n(Y_j) - \hat{f}(Y_j)}{f_n(Y_j)}\right| = \left|e_n 1_{\{f(Y_j) < e_n\}} + \hat{f}(Y_j) 1_{\{f(Y_j) \geq e_n\}} - \hat{f}(Y_j)\right|

= \left|e_n 1_{\{f(Y_j) < e_n\}} - \hat{f}(Y_j) 1_{\{f(Y_j) < e_n\}}\right|

\leq \left(e_n + \hat{f}(Y_j)\right) 1_{\{f(Y_j) < e_n\}}$$

we obtain

$$\left|\frac{f_n(Y_j) - \hat{f}(Y_j)}{f_n(Y_j)}\right| \leq \left(\frac{e_n}{f_n(Y_j)} + \frac{\hat{f}(Y_j)}{f_n(Y_j)}\right) 1_{\{f(Y_j) < e_n\}} \leq 2 1_{\{f(Y_j) < e_n\}}.$$

It is easy to check that

$$1_{\{f_n(Y_j) < e_n\}} \leq 1_{\{f(Y_j) < 2e_n\}} + \frac{\left(\sup_{y \in \mathbb{R}} |\hat{f}(y) - f(y)|\right)^2}{e_n^2}.$$
Thus
\[
\left| \frac{\hat{f}_n(Y_j) - \hat{f}(Y_j)}{f_n(Y_j)} \right| \leq 2 \left[ 1_{\{f(Y_j) < 2\epsilon_n\}} + \frac{\left( \sup_{y \in \mathbb{R}} |\hat{f}(y) - f(y)| \right)^2}{\epsilon_n^2} \right].
\]

\[\square\]

**Lemma 6.3** Under the assumption ($\mathcal{A}_3$), if we suppose $\|\hat{\Gamma}_D - \Gamma_D\|_\infty = o_p(t_D)$, we have
\[
A_{2n} = \frac{1}{n} \sum_{j=1}^n C(Y_j)\Gamma_D^{-1} C(Y_j) - \frac{1}{n} \sum_{j=1}^n C(Y_j)\hat{\Gamma}_D^{-1} C(Y_j) = O_p\left( \frac{1}{t_D \sqrt{n}} \right).
\]

**Proof.**

\[
\|A_{2n}\|_{hs} = \frac{1}{n} \sum_{j=1}^n C(Y_j) \left\| \left[ \Gamma_D^{-1} - \hat{\Gamma}_D^{-1} \right] C(Y_j) \right\|_{hs}
= \frac{1}{n} \sum_{j=1}^n C(Y_j) \left\| \hat{\Gamma}_D^{-1} (\hat{\Gamma}_D - \Gamma_D) \Gamma_D^{-1} \right\|_{hs}
\leq \frac{1}{n} \sum_{j=1}^n C(Y_j) \left\| \hat{\Gamma}_D^{-1} (\hat{\Gamma}_D - \Gamma_D) \Gamma_D^{-1} \right\|_{hs}
\leq \left\| \hat{\Gamma}_D^{-1} (\hat{\Gamma}_D - \Gamma_D) \right\|_\infty \frac{1}{n} \sum_{j=1}^n \|C(Y_j)\|_{hs} \|\Gamma_D^{-1} C(Y_j)\|_{hs}.
\]

Since $\|\hat{\Gamma}_D - \Gamma_D\|_\infty = o_p(t_D)$ and $\|\hat{\Gamma}_D^{-1}\|_\infty = O_p(\frac{1}{t_D})$, we deduce from the preceding inequality that $\|A_{2n}\|_{hs} = o_p(\frac{1}{t_D \sqrt{n}})$. \[\square\]

**Lemma 6.4** Under assumptions ($\mathcal{A}_3$) and ($\mathcal{A}_{10}$), if we suppose $\|\hat{\Gamma}_D - \Gamma_D\|_\infty = o_p(t_D)$, we have
\[
A_{3n} = \frac{1}{n} \sum_{j=1}^n C(Y_j)\hat{\Gamma}_D^{-1} C(Y_j) - \frac{1}{n} \sum_{j=1}^n C_{e_n}(Y_j)\hat{\Gamma}_D^{-1} C_{e_n}(Y_j) = O_p\left( \frac{1}{t_D \sqrt{n}} \right).
\]
Proof. We can write

\[ A_{3n} = \frac{1}{n} \sum_{j=1}^{n} [C(Y_j) - C_{e_n}(Y_j)] \hat{\Gamma}_D^{-1} C(Y_j) - \frac{1}{n} \sum_{j=1}^{n} C_{e_n}(Y_j) \hat{\Gamma}_D^{-1} [C_{e_n}(Y_j) - C(Y_j)] \]

\[ = A_{31n} - A_{32n}. \]

First, we deal with \( A_{31n} \). We have

\[ A_{31n} = \frac{1}{n} \sum_{j=1}^{n} [R(Y_j) - R_{e_n}(Y_j)] \hat{\Gamma}_D^{-1} C(Y_j) \]

\[ + \frac{1}{n} \sum_{j=1}^{n} [r_{e_n}(Y_j) \otimes r_{e_n}(Y_j) - r(Y_j) \otimes r(Y_j)] \hat{\Gamma}_D^{-1} C(Y_j) \]

\[ = A_{311n} + A_{312n}. \]

Further,

\[ \sqrt{n} \|A_{311n}\|_{hs} = \sqrt{n} \left\| \frac{1}{n} \sum_{j=1}^{n} [R(Y_j) - R_{e_n}(Y_j)] \hat{\Gamma}_D^{-1} C(Y_j) \right\|_{hs} \]

\[ \leq \frac{\|\hat{\Gamma}_D^{-1}\|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \|M(Y_j)\|_{hs} \|C(Y_j)\|_{hs} \left| \frac{1}{f(Y_j)} - \frac{1}{f_{e_n}(Y_j)} \right| \]

and since

\[ \left| \frac{1}{f(Y_j)} - \frac{1}{f_{e_n}(Y_j)} \right| \leq \frac{1}{f(Y_j)} 1_{\{f(Y_j) < e_n\}} \]

it follows

\[ \sqrt{n} \|A_{311n}\|_{hs} \leq \frac{\|\hat{\Gamma}_D^{-1}\|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \|M(Y_j)\|_{hs} \|C(Y_j)\|_{hs} \frac{1}{f(Y_j)} 1_{\{f(Y_j) < e_n\}} \]

\[ = \frac{\|\hat{\Gamma}_D^{-1}\|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \|R(Y_j)\|_{hs} \|C(Y_j)\|_{hs} 1_{\{f(Y_j) < e_n\}} \]

\[ \leq \frac{\|\hat{\Gamma}_D^{-1}\|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \|R(Y_j)\|_{hs} \|R(Y_j) - r(Y_j) \otimes r(Y_j)\|_{hs} 1_{\{f(Y_j) < e_n\}} \]

\[ \leq \frac{\|\hat{\Gamma}_D^{-1}\|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \left[ \|R(Y_j)\|_{hs}^2 + \|R(Y_j)\|_{hs} \|r(Y_j)\|_{hs}^2 \right] 1_{\{f(Y_j) < e_n\}}. \]
Thus
\[
\mathbb{E} \left[ \frac{\sqrt{n} \| A_{311n} \|_{hs}}{\| \Gamma_D^{-1} \|_\infty} \right] \leq \sqrt{n} \mathbb{E} \left[ \| R(Y) \|_{hs} 1_{\{f(Y) < e_n\}} \right] + \sqrt{n} \mathbb{E} \left[ \| R(Y) \|_{hs} \| r(Y) \|_{hs}^2 1_{\{f(Y) < e_n\}} \right]
\]
and since \( \| \Gamma_D^{-1} - \Gamma_D^{-1} \|_\infty = o_p(t_D), \| \Gamma_D^{-1} \|_\infty = O_p(\frac{1}{t_D}) \), we deduce from the preceding inequality, assumption \((\mathcal{A}10)\) and Markov inequality that \( A_{311n} = o_p(\frac{1}{t_D \sqrt{n}}) \). On the other hand,

\[
\sqrt{n} \| A_{312n} \|_{hs} \leq \frac{\| \Gamma_D^{-1} \|_\infty \sqrt{n}}{n} \sum_{j=1}^{n} \| m(Y_j) \otimes m(Y_j) \left[ \frac{1}{f^2(Y_j)} - \frac{1}{f_x^2(Y_j)} \right] \|_{hs} \| C(Y_j) \|_{hs}
\]

\[
\leq \frac{\| \Gamma_D^{-1} \|_\infty \sqrt{n}}{n} \sum_{j=1}^{n} \| m(Y_j) \otimes m(Y_j) \|_{hs} \| C(Y_j) \|_{hs} \left[ \frac{1}{f^2(Y_j)} - \frac{1}{f_x^2(Y_j)} \right] \| 1_{\{f(Y_j) < e_n\}}
\]

\[
\leq \frac{\| \Gamma_D^{-1} \|_\infty \sqrt{n}}{n} \sum_{j=1}^{n} \| r(Y_j) \otimes r(Y_j) \|_{hs} \| C(Y_j) \|_{hs} \| 1_{\{f(Y_j) < e_n\}}
\]

\[
\leq \frac{\| \Gamma_D^{-1} \|_\infty \sqrt{n}}{n} \sum_{j=1}^{n} \left[ \| r(Y_j) \|^2_H \| R(Y_j) \|_{hs} + \| r(Y_j) \|^4_H \right] \| 1_{\{f(Y_j) < e_n\}}
\]

Thus
\[
\mathbb{E} \left[ \frac{\sqrt{n} \| A_{312n} \|_{hs}}{\| \Gamma_D^{-1} \|_\infty} \right] \leq \sqrt{n} \mathbb{E} \left[ \| r(Y) \|^2_H \| R(Y) \|_{hs} 1_{\{f(Y) < e_n\}} \right] + \sqrt{n} \mathbb{E} \left[ \| r(Y) \|^4_H 1_{\{f(Y) < e_n\}} \right]
\]

and since \( \| \Gamma_D^{-1} \|_\infty = O_p(\frac{1}{t_D}) \), we deduce from the preceding inequality, assumption \((\mathcal{A}10)\) and Markov inequality that \( A_{312n} = o_p(\frac{1}{t_D \sqrt{n}}) \). This permits to conclude that \( A_{31n} = A_{311n} + A_{312n} = o_p(\frac{1}{t_D \sqrt{n}}) \). Now, we deal with \( A_{32n} \); we have

\[
A_{32n} = \frac{1}{n} \sum_{j=1}^{n} C_{e_n}(Y_j) \Gamma_D^{-1} \left[ r_{e_n}(Y_j) - R(Y_j) \right] - \frac{1}{n} \sum_{j=1}^{n} C_{e_n}(Y_j) \Gamma_D^{-1} \left[ r_{e_n}(Y_j) \otimes r_{e_n}(Y_j) - r(Y_j) \otimes r(Y_j) \right]
\]

\[
= A_{321n} - A_{322n}
\]
and

\[ A_{321n} = \frac{1}{n} \sum_{j=1}^{n} C_{en}(Y_j) \hat{\Gamma}^{-1}_D \left[ R_{en}(Y_j) - R(Y_j) \right] \]

\[ = \frac{1}{n} \sum_{j=1}^{n} R_{en}(Y_j) \hat{\Gamma}^{-1}_D \left[ R_{en}(Y_j) - R(Y_j) \right] - \frac{1}{n} \sum_{j=1}^{n} \Gamma_{en}(Y_j) \otimes \Gamma_{en}(Y_j) \hat{\Gamma}^{-1}_D \left[ R_{en}(Y_j) - R(Y_j) \right] \]

\[ = A_{3211n} - A_{3212n}. \]

Moreover

\[ \sqrt{n} \| A_{3211n} \|_{hs} \leq \frac{\| \hat{\Gamma}^{-1}_D \|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \| R_{en}(Y_j) \|_{hs} \cdot \left[ \frac{1}{f_{en}(Y_j)} - \frac{1}{f(Y_j)} \right] \]

\[ \leq \frac{\| \hat{\Gamma}^{-1}_D \|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \| R_{en}(Y_j) \|_{hs} \cdot \| R(Y_j) \|_{hs} 1 \{ f(Y_j) < \epsilon_n \} \]

\[ \leq \frac{\| \hat{\Gamma}^{-1}_D \|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \| R(Y_j) \|_{hs}^2 1 \{ f(Y_j) < \epsilon_n \}. \]

Hence

\[ \mathbb{E} \left[ \sqrt{n} \| A_{3211n} \|_{hs} \right] \leq \sqrt{n} \mathbb{E} \left[ \| R(Y) \|_{hs}^2 1 \{ f(Y) < \epsilon_n \} \right] \]

the using \( \| \hat{\Gamma}^{-1}_D \|_{\infty} = O_p(\frac{1}{t_D}) \), assumption (A_{10}) and Markov inequality, we conclude that \( A_{3211n} = o_p(\frac{1}{t_D \sqrt{n}}) \). Furthermore, we have

\[ \sqrt{n} \| A_{3212n} \|_{hs} \leq \frac{\| \hat{\Gamma}^{-1}_D \|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \| r(Y_j) \otimes r(Y_j) \|_{hs} \cdot M(Y_j) \|_{hs} \cdot \left[ \frac{1}{f_{en}(Y_j)} - \frac{1}{f(Y_j)} \right] \]

\[ \leq \frac{\| \hat{\Gamma}^{-1}_D \|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \| r(Y_j) \|_{H}^2 \| R(Y_j) \|_{hs} 1 \{ f(Y_j) < \epsilon_n \}. \]
what implies

$$\sqrt{n}E \left[\frac{\|A_{3212n}\|_{hs}}{\|\hat{\Gamma}^{-1}_D\|_{\infty}}\right] \leq \sqrt{n}E \left[\|r(Y)\|^2_H \|R(Y)\|_{hs}1_{\{f(Y) < e_n\}}\right] = o_p(1).$$

Thus, $A_{3212n} = o_p(\frac{1}{t_D\sqrt{n}})$ and we can then conclude that $A_{321n} = o_p(\frac{1}{t_D\sqrt{n}})$.

It remains to treat $A_{322n}$. We have:

$$A_{322n} = \frac{1}{n} \sum_{j=1}^{n} R_{e_n}(Y_j)\hat{\Gamma}^{-1}_D [r_{e_n}(Y_j) \otimes r_{e_n}(Y_j) - r(Y_j) \otimes r(Y_j)]$$

$$- \frac{1}{n} \sum_{j=1}^{n} r_{e_n}(Y_j) \otimes r_{e_n}(Y_j)\hat{\Gamma}^{-1}_D [r_{e_n}(Y_j) \otimes r_{e_n}(Y_j) - r(Y_j) \otimes r(Y_j)]$$

$$= A_{3221n} - A_{3222n}$$

and since

$$\sqrt{n}E \left[\|A_{3221n}\|_{hs}\right] \leq \|\hat{\Gamma}^{-1}_D\|_{\infty} \sqrt{n}E \left[\|r(Y)\|^2_H \|R(Y)\|_{hs}1_{\{f(Y) < e_n\}}\right]$$

we obtain from assumption $(\mathcal{A}_{10})$ that $A_{3221n} = o_p(\frac{1}{t_D\sqrt{n}})$. Further,

$$\sqrt{n}E \left[\|A_{3222n}\|_{hs}\right] \leq \|\hat{\Gamma}^{-1}_D\|_{\infty} \sqrt{n}E \left[\|r(Y)\|^2_H 1_{\{f(Y) < e_n\}}\right]$$

and from assumption $(A_8)$ and Markov inequality we deduce that $A_{3222n} = o_p(\frac{1}{t_D\sqrt{n}})$. Consequently, $A_{322n} = o_p(\frac{1}{t_D\sqrt{n}})$ and $A_{32n} = o_p(\frac{1}{t_D\sqrt{n}})$. All of the above permit to conclude that $A_{3n} = o_p(\frac{1}{t_D\sqrt{n}}).$ \hfill \qed

**Lemma 6.5** Under assumptions $(\mathcal{A}_3)$, $(\mathcal{A}_6)$ to $(\mathcal{A}_9)$ if we suppose that $\|\hat{\Gamma}_D - \Gamma_D\|_{\infty} = o_p(t_D)$, we have:

$$A_{4n} = \frac{1}{n} \sum_{j=1}^{n} C_{e_n}(Y_j)\hat{\Gamma}^{-1}_D C_{e_n}(Y_j) - \frac{1}{n} \sum_{j=1}^{n} \hat{C}_{e_n}(Y_j)\hat{\Gamma}^{-1}_D \hat{C}_{e_n}(Y_j)$$

$$= O_p \left( \frac{1}{t_D e_n} \left( h^k + \frac{\sqrt{\log(n)}}{h\sqrt{n}} \right) \right) + O_p \left( \frac{1}{t_D n^{\gamma}} \right)$$

where $\gamma$ is a real constant satisfying $0 < \gamma < 1/4$. 

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Proof.

\[ A_{4n} = \frac{1}{n} \sum_{j=1}^{n} \left[ C_e(n) - \tilde{C}_e(n) \right] \hat{\Gamma}_D^{-1} C_e(n) - \frac{1}{n} \sum_{j=1}^{n} \left[ \tilde{C}_e(n) - C_e(n) \right] \hat{\Gamma}_D^{-1} \left[ \tilde{C}_e(n) - C_e(n) \right] \]

\[ + \frac{1}{n} \sum_{j=1}^{n} C_e(n) \hat{\Gamma}_D^{-1} \left[ C_e(n) - \tilde{C}_e(n) \right] = A_{41n} - A_{42n} + A_{43n}. \]

First, we deal with \( A_{41n} \); we have:

\[ A_{41n} = \frac{1}{n} \sum_{j=1}^{n} \left[ R_e(n) - \hat{R}_e(n) \right] \hat{\Gamma}_D^{-1} C_e(n) \]

\[ + \frac{1}{n} \sum_{j=1}^{n} \left[ \hat{r}_e(n) \otimes \hat{r}_e(n) - r_e(n) \otimes r_e(n) \right] \hat{\Gamma}_D^{-1} C_e(n) \]

\[ = A_{411n} + A_{412n} + A_{413n} + A_{414n}. \]
and
\[
\| A_{411n} \|_{hs} \leq \| \hat{\Gamma}_D^{-1} \|_{\infty} \frac{1}{n} \sum_{j=1}^{n} \| R_{e_n}(Y_j) - \hat{R}_{e_n}(Y_j) \|_{hs} \| C(Y_j) \|_{hs}
\]
\[
\leq \| \hat{\Gamma}_D^{-1} \|_{\infty} \frac{1}{n} \sum_{j=1}^{n} \| R_{e_n}(Y_j) - \hat{R}_{e_n}(Y_j) \|_{hs} \| C(Y_j) \|_{hs}
\]
\[
\leq \| \hat{\Gamma}_D^{-1} \|_{\infty} \frac{1}{n} \sum_{j=1}^{n} \| R_{e_n}(Y_j) \|_{hs} \| f(Y_j) - \hat{f}(Y_j) \| + \frac{1}{\hat{f}_{e_n}(Y_j)} \left[ \hat{M}(Y_j) - M(Y_j) \right] \|_{hs} \| C(Y_j) \|_{hs}
\]
\[
\leq \| \hat{\Gamma}_D^{-1} \|_{\infty} \frac{1}{n} \sum_{j=1}^{n} \| R(Y_j) \|_{hs} \| C(Y_j) \|_{hs} \sup_{y \in \mathbb{R}} | f(y) - \hat{f}(y) |
\]
\[
+ \| \hat{\Gamma}_D^{-1} \|_{\infty} \frac{1}{n} \sum_{j=1}^{n} \| C(Y_j) \|_{hs} \sup_{y \in \mathbb{R}} \| M(y) - \hat{M}(y) \|_{hs}.
\]

It is known from [15] that
\[
\sup_{y \in \mathbb{R}} | \hat{f}(y) - f(y) | = O_p(h^k + \frac{\sqrt{\log(n)}}{h\sqrt{n}}); \quad (4)
\]
then, this property together with Lemma 6.1 assumption (A9) and the preceding inequality imply
\[
A_{411n} = O_p \left( \frac{1}{tD e_n} \left( h^k + \frac{\sqrt{\log(n)}}{h\sqrt{n}} \right) \right) = O_p \left( \frac{1}{tD n^\gamma} \right).
\]

A similar reasoning, but by using instead of Lemma 6.1 the following result from [18]:
\[
\sup_{y \in \mathbb{R}} \| \hat{m}(y) - m(y) \|_{H} = O_p \left( h^k + \frac{\sqrt{\log(n)}}{h\sqrt{n}} \right)
\]
permits to obtain
\[
A_{413n} = O_p \left( \frac{1}{tD e_n} \left( h^k + \frac{\sqrt{\log(n)}}{h\sqrt{n}} \right) \right) = O_p \left( \frac{1}{tD n^\gamma} \right)
\]
and
\[ A_{414n} = O_p \left( \frac{1}{t_D c_n} \left( h^k + \frac{\sqrt{\log(n)}}{h \sqrt{n}} \right) \right) = O_p \left( \frac{1}{t_D n^{\gamma}} \right). \]

On the other hand,
\[
\|A_{412n}\|_{h^s} = \frac{1}{n} \sum_{j=1}^{n} \| (\hat{r}_{en}(Y_j) - r_{en}(Y_j)) \otimes (\hat{r}_{en}(Y_j) - r_{en}(Y_j)) \hat{\Gamma}_D^{-1} C_{en}(Y_j) \|_{h^s} \\
\leq \|\hat{\Gamma}_D^{-1}\|_{\infty} \frac{1}{n} \sum_{j=1}^{n} \|\hat{r}_{en}(Y_j) - r_{en}(Y_j)\|_{h^s}^2 \|C(Y_j)\|_{h^s}.
\]

Similar developments as previously done for \(\|A_{411n}\|_{h^s}\) permit to obtain
\[
\frac{1}{n} \sum_{j=1}^{n} \|\hat{r}_{en}(Y_j) - r_{en}(Y_j)\|_{h^s}^2 \|C(Y_j)\|_{h^s} = O_p \left( \frac{1}{\sqrt{n}} \right), \text{ and since } \|\hat{\Gamma}_D^{-1}\|_{\infty} = O_p \left( \frac{1}{t_D} \right), \text{ we conclude that } A_{412n} = O_p \left( \frac{1}{t_D \sqrt{n}} \right).
\]

Therefore,
\[ A_{41n} = O_p \left( \frac{1}{t_D c_n} \left( h^k + \frac{\sqrt{\log(n)}}{h \sqrt{n}} \right) \right) + O_p \left( \frac{1}{t_D \sqrt{n}} \right) = O_p \left( \frac{1}{t_D n^{\gamma}} \right). \]

Further,
\[
A_{42n} = \frac{1}{n} \sum_{j=1}^{n} \left[ \hat{R}_{en}(Y_j) - R_{en}(Y_j) \right] \hat{\Gamma}_D^{-1} \left[ \hat{R}_{en}(Y_j) - R_{en}(Y_j) \right] \\
- \frac{1}{n} \sum_{j=1}^{n} \left[ \hat{R}_{en}(Y_j) - R_{en}(Y_j) \right] \hat{\Gamma}_D^{-1} \left[ \hat{r}_{en}(Y_j) \otimes \hat{r}_{en}(Y_j) - r_{en}(Y_j) \otimes r_{en}(Y_j) \right] \\
- \frac{1}{n} \sum_{j=1}^{n} \left[ \hat{r}_{en}(Y_j) \otimes \hat{r}_{en}(Y_j) - r_{en}(Y_j) \otimes r_{en}(Y_j) \right] \hat{\Gamma}_D^{-1} \left[ \hat{R}_{en}(Y_j) - R_{en}(Y_j) \right] \\
+ \frac{1}{n} \sum_{j=1}^{n} \left[ \hat{r}_{en}(Y_j) \otimes \hat{r}_{en}(Y_j) - r_{en}(Y_j) \otimes r_{en}(Y_j) \right] \hat{\Gamma}_D^{-1} \left[ \hat{r}_{en}(Y_j) \otimes \hat{r}_{en}(Y_j) - r_{en}(Y_j) \otimes r_{en}(Y_j) \right]
\]
\[ = A_{421n} - A_{422n} - A_{423n} + A_{424n}. \]
and

\[ \sqrt{n} \| A_{421n} \|_{hs} \leq \frac{\| \hat{\Gamma}_D^{-1} \|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \| R_{e_n}(Y_j) - R_{e_n}(Y_j) \|_{h,s}^2 \]

\[ \leq \frac{\| \hat{\Gamma}_D^{-1} \|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \left\| \frac{R_{e_n}(Y_j)}{f_{e_n}(Y_j)} \left[ f_{e_n}(Y_j) - \hat{f}_{e_n}(Y_j) \right] - \frac{1}{\hat{f}_{e_n}(Y_j)} \left[ \hat{M}(Y_j) - M(Y_j) \right] \right\|_{h,s}^2 \]

\[ = \frac{\| \hat{\Gamma}_D^{-1} \|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \left\| \frac{R_{e_n}(Y_j)}{f_{e_n}(Y_j)} \left[ f_{e_n}(Y_j) - \hat{f}_{e_n}(Y_j) \right] \right\|_{h,s}^2 \]

\[ + \frac{\| \hat{\Gamma}_D^{-1} \|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \left\| \frac{1}{\hat{f}_{e_n}(Y_j)} \left[ \hat{M}(Y_j) - M(Y_j) \right] \right\|_{h,s}^2 \]

\[ - \frac{2\| \hat{\Gamma}_D^{-1} \|_{\infty} \sqrt{n}}{n} \sum_{j=1}^{n} \left\langle \frac{R_{e_n}(Y_j)}{f_{e_n}(Y_j)} \left[ f_{e_n}(Y_j) - \hat{f}_{e_n}(Y_j) \right] , \frac{1}{\hat{f}_{e_n}(Y_j)} \left[ \hat{M}(Y_j) - M(Y_j) \right] \right\rangle_{h,s} \]

\[ \leq \frac{\| \hat{\Gamma}_D^{-1} \|_{\infty} \sqrt{n}}{n e_n^2} \sum_{j=1}^{n} \| R_{e_n}(Y_j) \|_{h,s}^2 \left\| f_{e_n}(Y_j) - \hat{f}_{e_n}(Y_j) \right\| \]

\[ + \frac{\| \hat{\Gamma}_D^{-1} \|_{\infty} \sqrt{n}}{n e_n^2} \sum_{j=1}^{n} \left\| \hat{M}(Y_j) - M(Y_j) \right\|_{h,s} \left\| f_{e_n}(Y_j) - \hat{f}_{e_n}(Y_j) \right\| \]

\[ \leq \frac{\| \hat{\Gamma}_D^{-1} \|_{\infty} \sqrt{n}}{n e_n^2} \sum_{j=1}^{n} \| R(Y_j) \|_{h,s}^2 \left( \sup_{y \in \mathbb{R}} \left| \hat{f}(y) - f(y) \right| \right)^2 \]

\[ + \frac{\| \hat{\Gamma}_D^{-1} \|_{\infty} \sqrt{n}}{n e_n^2} \sum_{j=1}^{n} \left( \sup_{y \in \mathbb{R}} \left| \hat{M}(y) - M(y) \right| \right)^2 \]

\[ + \frac{2\| \hat{\Gamma}_D^{-1} \|_{\infty} \sqrt{n}}{n e_n^2} \sum_{j=1}^{n} \| R(Y_j) \|_{h,s} \sup_{y \in \mathbb{R}} \left| \hat{M}(y) - M(y) \right| \sup_{y \in \mathbb{R}} \left| \hat{f}(y) - f(y) \right|. \]

From the weak law of large numbers we obtain \( \frac{1}{n} \sum_{j=1}^{n} \| R(Y_j) \|_{h,s} = O_p(1) \)
and \( \frac{1}{n} \sum_{j=1}^{n} \| R(Y_j) \|_{h,s}^2 = O_p(1) \). Then using [4], Lemma 6.1, the assumption
and the fact that $\|\hat{\Gamma}_D^{-1}\|_\infty = O_p(\frac{1}{t_D})$, we obtain

$$\sqrt{n} \|A_{421n}\|_{hs} = O_p \left[ \frac{1}{t_D} n^{1/2+2c_2} \left( h^k + \frac{1}{h} \sqrt{\frac{\log(n)}{n}} \right)^2 \right]$$

$$= O_p \left[ \frac{1}{t_D} \left( n^{c_2-k+c_1+1/4} + n^{c_1+c_2-1/4} \sqrt{\log(n)} \right)^2 \right]$$

$$= O_p(\frac{1}{t_D})$$

from what we deduce that $A_{421n} = O_p(\frac{1}{t_D \sqrt{n}})$. In the same way, we show that $A_{422n} = O_p(\frac{1}{t_D \sqrt{n}})$, $A_{423n} = O_p(\frac{1}{t_D \sqrt{n}})$, $A_{424n} = O_p(\frac{1}{t_D \sqrt{n}})$. Thus, $A_{42n} = O_p(\frac{1}{t_D \sqrt{n}})$. Now, we deal with $A_{43n}$; since $A_{43n} = (A_{41n})^*$, we also have

$$A_{43n} = O_p \left( \frac{1}{t_D n} \left( h^k + \sqrt{\frac{\log(n)}{n}} \right) \right) + O_p \left( \frac{1}{t_D \sqrt{n}} \right) = O_p \left( \frac{1}{t_D n^\gamma} \right)$$

Finally, we obtain

$$A_{4n} = \frac{1}{n} \sum_{j=1}^{n} C_{e_n}(Y_j) \hat{\Gamma}_D^{-1} C_{e_n}(Y_j) - \frac{1}{n} \sum_{j=1}^{n} \hat{C}_{e_n}(Y_j) \hat{\Gamma}_D^{-1} \hat{C}_{e_n}(Y_j)$$

$$= O_p \left( \frac{1}{t_D \sqrt{n}} \left( h^k + \frac{\sqrt{\log(n)}}{h \sqrt{n}} \right) \right) + O_p \left( \frac{1}{t_D \sqrt{n}} \right)$$

$$= O_p \left( \frac{1}{t_D n^\gamma} \right)$$

\(\square\)
6.2 Proof of the Theorems

6.2.1 Proof of Theorem 4.1

Since

\[ \mathbb{E} \left[ \text{Var}(X|Y) \Gamma_D^{-1} \text{Var}(X|Y) \right] - \frac{1}{n} \sum_{j=1}^{n} \hat{C}_{e_n}(Y_j) \hat{\Gamma}_D^{-1} \hat{C}_{e_n}(Y_j) \]

\[ = \left( \mathbb{E} \left[ \text{Var}(X|Y) \Gamma_D^{-1} \text{Var}(X|Y) \right] - \frac{1}{n} \sum_{j=1}^{n} C(Y_j) \Gamma_D^{-1} C(Y_j) \right) \]

\[ + \left( \frac{1}{n} \sum_{j=1}^{n} C(Y_j) \Gamma_D^{-1} C(Y_j) - \frac{1}{n} \sum_{j=1}^{n} C(Y_j) \hat{\Gamma}_D^{-1} C(Y_j) \right) \]

\[ + \left( \frac{1}{n} \sum_{j=1}^{n} C(Y_j) \hat{\Gamma}_D^{-1} C(Y_j) - \frac{1}{n} \sum_{j=1}^{n} C_{e_n}(Y_j) \hat{\Gamma}_D^{-1} C_{e_n}(Y_j) \right) \]

\[ = A_1 + A_2 + A_3 + A_4. \]

From the central limit theorem we have \( A_1 = O_p \left( \frac{1}{\sqrt{n}} \right) \); then the required result is obtained by applying lemmas 6.2 to 6.5.

6.2.2 Proof of Theorem 4.2

Putting

\[ G = 2 \Gamma_e + \Psi - \Gamma, \quad G_D = 2 \Gamma_e + \mathbb{E}[\text{Var}(X|Y) \Gamma_D^{-1} \text{Var}(X|Y)] - \Gamma \]  

and

\[ \hat{G} = 2 \hat{\Gamma}_{e,n} + \frac{1}{n} \sum_{j=1}^{n} \hat{C}_{e_n}(Y_j) \hat{\Gamma}_D^{-1} \hat{C}_{e_n}(Y_j) - \Gamma_n = 2 \hat{\Gamma}_{e,n} + \hat{\Psi}_{e_n,D} - \Gamma_n, \]

we have:

\[ \| \hat{\Gamma}_{I,n} - \Gamma_I \|_{hs} \leq \| \Gamma^{-1} G - \Gamma^{-1} G_D \|_{hs} + \| \Gamma^{-1} G_D - \Gamma^{-1} G \|_{hs} + \| \Gamma^{-1} G_D - \hat{\Gamma}_D^{-1} \hat{G} \|_{hs} \]

\[ = K_{1n} + K_{2n} + K_{3n} + K_{4n}. \]

(7)
Then using (7) and the previous results we obtain:

\[ \hat{\beta} \]

and since \( K_{2n} = \| \Gamma^{-1} \Gamma_D - \Gamma_D^{-1} \|_{hs} \leq \| (\Gamma^{-1} - \Gamma_D^{-1}) G \|_{hs} \), we also have \( \lim_{D \to +\infty} K_{2n} = 0 \). Further,

\[
K_{3n} = \| \Gamma^{-1} G_D - \hat{\Gamma}^{-1} G_D \|_{hs} \leq \| (\Gamma^{-1} - \hat{\Gamma}^{-1}) G \|_{hs} \\
\leq \| \hat{\Gamma}^{-1} (\hat{\Gamma}_D - \Gamma_D) \Gamma_1^{-1} G \|_{hs} + \| \hat{\Gamma}^{-1} (\Gamma_n - \Gamma) \|_{hs}.
\]

then since \( \| \hat{\Gamma}_D^{-1} \|_{hs} = O_p(1/t) \) and \( \| \hat{\Gamma}_D - \Gamma_D \|_{hs} = o_p(t) \), we deduce that \( K_{3n} = o_p(1) \). On the other hand

\[
K_{4n} = \| \hat{\Gamma}^{-1} G_D - \hat{\Gamma}_D^{-1} G \|_{hs} \\
\leq \| \hat{\Gamma}^{-1} (\hat{\Gamma}_e - \Gamma_e) \|_{hs} + \| \hat{\Gamma}^{-1} \{ Var(X|Y) \Gamma^{-1} Var(X|Y) - \Psi_{e,n} \} \|_{hs} \\
+ \| \hat{\Gamma}^{-1} (\Gamma_n - \Gamma) \|_{hs}.
\]

Since \( \| \hat{\Gamma}_e - \Gamma_e \|_{hs} = O_p(1/\sqrt{n}) \) (see [8]), we deduce from the preceding inequality that

\[ K_{4n} = O_p \left( \frac{1}{tD\sqrt{n}} \right) + O_p \left( \frac{1}{tDn^2} \right) = o_p(1). \]

Then using (7) and the previous results we obtain: \( \hat{\Gamma}_{I,n} - \Gamma_I = o_p(1) \).

**6.2.3 Proof of Theorem 4.3**

Denoting by \((\hat{\beta}_k)_{1 \leq k \leq K}\) the orthonormal eigenvectors associated with the \( K \) largest eigenvalues \( \lambda_1 > \lambda_2 > \cdots > \lambda_K > 0 \) of \( \hat{\Gamma}_D^{-1} \hat{G} \) and by \((\hat{\beta}_k)_{1 \leq k \leq K}\) the orthonormal eigenvectors associated with the \( K \) largest eigenvalues \( \lambda_1 > \lambda_2 > \cdots > \lambda_K > 0 \) of \( \Gamma^{-1} G \), where \( G \) and \( \hat{G} \) are defined in (5) and (6), we will only show the convergence of the vector \( \hat{\beta}_1 \) as the proof for the others are the same. Clearly, \( \beta_1 = \lambda_1^{-1} \Gamma_2 \eta \) and \( \hat{\beta}_1 = \hat{\lambda}_1^{-1} \hat{\Gamma}_2 \hat{\eta} \) where \( \Gamma_2 = \Gamma^{-1} \{ 2 \Gamma_e + \Psi - \Gamma \} \Gamma^{-1/2}, \eta = \Gamma^{1/2} \beta_1 \) and \( \hat{\eta} = \hat{\Gamma}_D^{1/2} \hat{\beta}_1 \) with

\[
\hat{\Gamma}_2 \frac{1}{\hat{\lambda}_1} \hat{\Gamma}_D^{-1} \left[ 2 \hat{\Gamma}_e + \hat{\Psi} - \Gamma_n \right] \hat{\Gamma}_D^{-1/2}.
\]

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Hence
\[
\|\hat{\beta}_1 - \beta_1\|_H \leq \frac{1}{\lambda_1} \|(\hat{\Gamma}_2 - \Gamma_2)\hat{\eta}\|_H + \frac{1}{\lambda_1} \|\Gamma_2(\hat{\eta} - \eta)\|_H + \frac{(1 - \frac{1}{\lambda_1})\Gamma_2\hat{\eta}\|_H}
\leq \frac{\|\hat{\eta}\|_H}{|\lambda_1|} \|\hat{\Gamma}_2 - \Gamma_2\|_\infty + \frac{\|\Gamma_2\|_\infty}{|\lambda_1|} \|\hat{\eta} - \eta\|_H + \frac{\hat{\lambda}_1 - \lambda_1}{|\lambda_1\hat{\lambda}_1|} \|\Gamma_2\hat{\eta}\|_H.
\]

Then from Lemma 1 in [8] we obtain the inequalities
\[
|\hat{\lambda}_1 - \lambda_1| \leq \|\hat{\Gamma}^{1/2}_D \hat{\Gamma}_2 - \Gamma^{1/2}_2\|_\infty \quad \text{and} \quad \|\hat{\eta} - \eta\|_H \leq C_9 \|\hat{\Gamma}^{1/2}_D \hat{\Gamma}_2 - \Gamma^{1/2}_2\|_\infty, \quad (8)
\]
where \(C_9\) is an appropriate positive constant. Then, putting \(L_n = \|\hat{\Gamma}_2 - \Gamma_2\|_\infty\) and \(M_n = \|\Gamma^{1/2}_n \hat{\Gamma}_2 - \Gamma^{1/2}_2\|_\infty\), we have
\[
\|\hat{\beta}_1 - \beta_1\|_H \leq \frac{\|\hat{\eta}\|_H}{|\lambda_1|} L_n + \left( C_{10} + \frac{C_{11}\|\hat{\eta}\|_H}{|\lambda_1|} \right) M_n. \quad (9)
\]

Let us verify that \(L_n = o_p(1)\) and \(M_n = o_p(1)\). First,
\[
L_n = \|\Gamma^{-1}GT^{-1/2} - \Gamma_D^{-1} \hat{\Gamma}_D^{-1/2}\|_\infty \\
\leq \|\Gamma^{-1}GT^{-1/2} - \Gamma_D^{-1} \hat{\Gamma}_D^{-1/2}\|_\infty + \|\Gamma_D^{-1} \hat{\Gamma}_D^{-1/2} - \hat{\Gamma}_D^{-1} \hat{\Gamma}_D^{-1/2}\|_\infty + \|\hat{\Gamma}_D^{-1}(G - \hat{G}) \hat{\Gamma}_D^{-1/2}\|_\infty \\
= L_{1n} + L_{2n} + L_{3n}.
\]

We know that \(\Gamma^{-1}_D \hat{\Gamma}_D^{-1/2} = \Pi_D \Gamma^{-1} \Gamma^{-1/2}_D \Pi_D\) and putting \(\Pi_D = I - \Pi_D\) we have
\[
\Gamma^{-1}_D \hat{\Gamma}_D^{-1/2} - \Gamma^{-1}_D \Gamma^{-1/2}_D = \Pi_D \Gamma^{-1} \Gamma^{-1/2}_D \Pi_D - \Gamma^{-1}_D \Gamma^{-1/2}_D \\
= \Pi_D \Gamma^{-1} \Gamma^{-1/2}_D \Pi_D - \Pi_D \Gamma^{-1/2}_D \Pi_D - \Pi_D \Gamma^{-1} \Gamma^{-1/2}_D \Pi_D - \Pi_D \Gamma^{-1/2}_D \Pi_D \\
= -\Pi_D \Gamma^{-1} \Gamma^{-1/2}_D \Pi_D.
\]

Thus
\[
L_{1n} \leq \|\Pi_D \Gamma^{-1} \Gamma^{-1/2}_D \Pi_D\|_\infty + \|\Pi_D \Gamma^{-1} \Gamma^{-1/2}_D \Pi_D\|_\infty \\
\leq \|\Gamma^{-1} \Gamma^{-1/2}_D \Pi_D\|_\infty + \|\Pi_D \Gamma^{-1} \Gamma^{-1/2}_D \Pi_D\|_\infty.
\]
and, consequently, \( \lim_{D \to +\infty} L_{1n} = 0 \) because \( \lim_{D \to +\infty} \Pi_D^{1/2} = 0 \). On the other hand,

\[
L_{2n} \leq \|(\Gamma^{-1}_D - \hat{\Gamma}^{-1}_D)G\Gamma^{-1/2}_D\|_\infty + \|\Gamma^{-1}_D G(\Gamma^{-1/2}_D - \hat{\Gamma}^{-1/2}_D)\|_\infty + \|(\Gamma^{-1}_D - \hat{\Gamma}^{-1}_D)G(\Gamma^{-1/2}_D - \hat{\Gamma}^{-1/2}_D)\|_\infty
\]

and

\[
\|(\Gamma^{-1}_D - \hat{\Gamma}^{-1}_D)G\Gamma^{-1/2}_D\|_\infty = \|\hat{\Gamma}^{-1}_D (\hat{\Gamma}_D - \Gamma_D)\Gamma^{-1}_D G\Gamma^{-1/2}_D\|_\infty
\]

\[
\leq \|\hat{\Gamma}^{-1}_D (\hat{\Gamma}_D - \Gamma_D)\Gamma^{-1}_D G\Gamma^{-1/2}_D\|_\infty
\]

\[
\leq \|\hat{\Gamma}^{-1}_D\|_\infty \|\hat{\Gamma}_D - \Gamma_D\|_\infty \|\Gamma^{-1}_D G\Gamma^{-1/2}_D\|_\infty
\]

\[
= O_p \left( \frac{1}{t_D \sqrt{n}} \right)
\]

\[
= o_p(1).
\]

Using the following properties of operators (see, e.g., [5]):

\[
A^{-1/2} - B^{-1/2} = A^{-1/2}(B^{3/2} - A^{3/2})B^{-3/2} + (A - B)B^{-3/2}
\]

and \( \|A^{3/2} - B^{3/2}\|_\infty \leq C_{12}\|A - B\|_\infty \)

we obtain:

\[
\|\Gamma^{-1}_D (\Gamma_D^{-1/2} - \hat{\Gamma}_D^{-1/2})\|_\infty = O_p \left( \frac{1}{t_D^{3/2} \sqrt{n}} \right)
\]

and

\[
\|(\Gamma^{-1}_D - \hat{\Gamma}^{-1}_D)G(\Gamma_D^{-1/2} - \hat{\Gamma}_D^{-1/2})\|_\infty = O_p \left( \frac{1}{t_D^{5/2} n} \right).
\]
Therefore, $L_{2n} = o_p(1)$. For dealing with the last term $L_{3n}$ we consider the operator $G_D = 2\Gamma_e + \mathbb{E}[\text{Var}(X|Y)\Gamma_D^{-1}\text{Var}(X|Y)] - \Gamma$ and we have

$$
\|\hat{\Gamma}_D^{-1}(G - G_D)\hat{\Gamma}_D^{-1/2}\|_{hs} \leq \|\hat{\Gamma}_D^{-1} - \Gamma_D^{-1}\|_{hs} \|G - G_D\|_{hs} \|\hat{\Gamma}_D^{-1/2} - \Gamma_D^{-1/2}\|_{hs} \\
+ \|\hat{\Gamma}_D^{-1} - \Gamma_D^{-1}\|_{hs} \|G - G_D\|_{hs} \|\hat{\Gamma}_D^{-1/2} - \Gamma_D^{-1/2}\|_{hs} \\
+ \|\Gamma_D^{-1}(G - G_D)\|_{hs} \|\hat{\Gamma}_D^{-1/2} - \Gamma_D^{-1/2}\|_{hs} + \|\Gamma_D^{-1}(G - G_D)\Gamma^{-1/2}\|_{hs} \\
\leq \|\hat{\Gamma}_D^{-1} - \Gamma_D^{-1}\|_{hs} \|G - G_D\|_{hs} \|\hat{\Gamma}_D^{-1/2} - \Gamma_D^{-1/2}\|_{hs} \\
+ \|\Gamma_D^{-1} - \Gamma_D^{-1}\|_{hs} \|G - G_D\|_{hs} \|\hat{\Gamma}_D^{-1/2} - \Gamma_D^{-1/2}\|_{hs} + \|\Gamma_D^{-1}(G - G_D)\Gamma^{-1/2}\|_{hs} \\
+ \|\Gamma^{-1}G\|_{hs} \|\hat{\Gamma}_D^{-1/2} - \Gamma_D^{-1/2}\|_{hs} + \|\Gamma^{-1}(G - G_D)\Gamma^{-1/2}\|_{hs} \\
= O_p \left( \frac{1}{t_D \sqrt{n}} \right) + O_p \left( \frac{1}{t_D^{3/2} \sqrt{n}} \right) + O_p \left( \frac{1}{t_D^{5/2} n} \right) + o_p(1) \\
= o_p(1).
$$

Thus

$$
L_{3n} \leq \|\hat{\Gamma}_D^{-1}(G - G_D)\hat{\Gamma}_D^{-1/2}\|_{hs} + \|\hat{\Gamma}_D^{-1}(G_D - \hat{G})\hat{\Gamma}_D^{-1/2}\|_{hs} \\
\leq \|\hat{\Gamma}_D^{-1}(\Gamma - \Gamma_n)\hat{\Gamma}_D^{-1/2}\|_{\infty} + 2\|\hat{\Gamma}_D^{-1}(\Gamma_e - \hat{\Gamma}_e,n)\hat{\Gamma}_D^{-1/2}\|_{\infty} \\
+ \|\hat{\Gamma}_D^{-1} \left( \mathbb{E}[\text{Var}(X|Y)\Gamma_D^{-1}\text{Var}(X|Y)] - \frac{1}{n} \sum_{j=1}^{n} \hat{C}_e(Y_j)\hat{\Gamma}_D^{-1}\hat{C}_e(Y_j) \right) \hat{\Gamma}_D^{-1/2}\|_{\infty} + o_p(1) \\
= O_p \left( \frac{1}{t_D^{3/2} \sqrt{n}} \right) + O_p \left( \frac{1}{t_D^{5/2} n} \right) + o_p(1) \\
= o_p(1).
$$

From all what precedes we deduce that $L_n = o_p(1)$. From similar reasoning we also obtain $M_n = o_p(1)$. Then from (8) and what precedes, we deduce that $\hat{\lambda}_1 = O_p(1)$ and $\|\hat{\eta}\|_H = O_p(1)$. Therefore, (9) allows to conclude that $\|\hat{\beta}_1 - \beta_1\|_H = o_p(1)$. 

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References

[1] Bosq D. Linear processes in function spaces. Theory and Applications. Springer-Verlag; 2000.

[2] Cardot H, Ferrat F, Sarda P. Functional Linear Mode. Stat. Probab. Lett. 1999; 45:11–22.

[3] Cook RD. SAVE: a method for dimension reduction and graphics in regression. Comm. Statist. Theory Methods 2000; 29:2109-2121.

[4] Dauxois J, Ferré L, Yao AF. (2001). Un modèle semi-paramétrique pour variable aléatoire hilbertienne. C. R. Acad. Sci. Paris 2001; t.333, série I: 947–952.

[5] Fukumizu K, Bach FR, Gretton A (2007). Statistical consistency of kernel canonical correlation analysis. J. Mach. Learn. Res. 2007; 8:361-383.

[6] Ferraty F, Vieu P. The functional nonparametric model and application to spectrometric data. Comput. Stat. 2002; 17:545-564.

[7] Ferraty F, Vieu P. Nonparametric functional data analysis: theory and practice. Springer; 2016.

[8] Ferré L, Yao AF. Functional Sliced Inverse Regression analysis. Statistics 2003; 37:475–488.

[9] Ferré L, Yao AF. Smoothed functional inverse regression. Stat. Sin. 2005; 15: 665–683.

[10] Hall P, Horowitz JL. Methodology and convergence rates for functional linear regression. Ann. Statist. 2007; 35:70-91.

[11] Horvath L, Kokoszka P. Inference for functional data with applications. New York: Springer; 2012.

[12] Li KC. Sliced inverse regression for dimension reduction. J. Amer. Statist. Assoc. 1991; 86:316-342.

[13] Lian H, Li G. Series expansion for functional sufficient dimension reduction. J. Multivar. Anal. 2014; 124:150–165.
[14] Nkou EDD, Nkiet GM. Strong consistency of kernel estimator in a semi-parametric model. arXiv:1811.02663 [math.ST]; 2018.

[15] Prakasa Rao BLS. Nonparametric Functional Estimation. Orlando: Academic Press; 1983.

[16] Ramsay JO, Silverman BW. Functional data analysis. New York: Springer; 1997.

[17] Wanga G, Zhou Y, Feng XN, Zhang B. The hybrid method of FSIR and FSAVE for functional effective dimension reduction. Comput. Stat. Data Anal. 2015; 91: 64-77

[18] Yao F, Müller HG. Functional quadratic regression. Biometrika 2010; 97:49-64.

[19] Zhu L, Fang KT. Asymptotics for kernel estimate of sliced inverse regression. Ann. Stat. 1996; 24:1053–1068.

[20] Zhu LP, Zhu LX. On kernel method for sliced average variance estimation. J. Multivar. Anal. 2007; 98:970 - 991.