RANDOM POLYNOMIALS: THE CLOSEST ROOTS TO THE UNIT CIRCLE

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Abstract. Let \( f = \sum_{k=0}^{n} \varepsilon_k z^k \) be a random polynomial, where \( \varepsilon_0, \ldots, \varepsilon_n \) are iid standard Gaussian random variables, and let \( \zeta_1, \ldots, \zeta_n \) denote the roots of \( f \). We show that the point process determined by the magnitude of the roots \( \{1 - |\zeta_1|, \ldots, 1 - |\zeta_n|\} \) tends to a Poisson point process at the scale \( n^{-2} \) as \( n \to \infty \). One consequence of this result is that it determines the magnitude of the closest root to the unit circle. In particular, we show that

\[
\min_k ||\zeta_k| - 1|n^2 \to \text{Exp}(1/6),
\]

in distribution, where Exp(\( \lambda \)) denotes an exponential random variable of mean \( \lambda^{-1} \). This resolves a conjecture of Shepp and Vanderbei from 1995 that was later studied by Konyagin and Schlag.

1. INTRODUCTION

We consider the typical distribution of the zeros of the polynomial

\[
f(z) = \sum_{k=0}^{n} \varepsilon_k z^k,
\]

where \( \varepsilon_0, \ldots, \varepsilon_n \) are iid standard Gaussian random variables. This problem originates in the 1930s with seminal works of Bloch and Pólya [6], Littlewood and Offord, [18, 19, 20, 21] and Erdős and Offord [9] and, in the years since, many aspects of this problem have come to be well understood (see, for example, [12, 13, 14, 15, 23, 24, 25, 27]).

One immediately striking aspect of the zeros of a random polynomial is that they cluster tightly and uniformly around the unit circle. This phenomenon, now widely known, was first discovered by Šparo and Šur [28] in the 1960s and is now known to persist for a wide variety of coefficient distributions, thanks to the work of Arnold [2] and Ibragimov and Zaporozhets [11]. Finer aspects of this convergence are also known: the proportion of roots that are within \( \rho/n \) of the unit circle was determined by Shepp and Vanderbei [26] and the limiting distribution of the roots that have constant distance away from the unit circle, was determined in the celebrated work of Peres and Virág [23].

In this paper we determine the microscopic nature of the clustering around the unit circle by studying the point process determined by the roots at distance \( O(n^{-2}) \) from the unit circle, the

\[1\]Peres and Virág actually work in a slightly different setup: for them the \( \varepsilon_j \) are iid standard complex Gaussian random variables, a case which it is easy to adapt our results to.
scale at which the first zeros appear. To lay this out a little more carefully, let us order the roots of \( f \) according to their distance from the unit circle
\[
|1 - |\zeta_1|| \leq |1 - |\zeta_2|| \leq \cdots \leq |1 - |\zeta_n||
\]
and write \( 1 - |\zeta_i| = x_i/n^2 \). In their 1995 paper, Shepp and Vanderbei [26] conjectured that the closest root to the unit circle is at distance \( \Theta(n^{-2}) \) and that the point process determined by the roots at this distance \( \{x_1, x_2, x_3, \ldots \} \) is asymptotically a Poisson point process. In this paper we prove these conjectures. For this, let \( f \sim \mathcal{G}_n \) denote the probability space defined by (1).

**Theorem 1.** If \( f \sim \mathcal{G}_n \) then the set
\[
\{(||\zeta| - 1)n^2 : f(\zeta) = 0\}
\]
converges to a homogeneous Poisson process with intensity \( 1/12 \), as \( n \to \infty \), in the vague topology.

From this, we can immediately resolve the conjecture of Shepp and Vanderbei regarding the scale at which the first zeros appear. One direction of this conjecture was already proved by Konyagin and Schlag [17] who showed that the closest zero to the unit circle is at distance at least \( cn^{-2} \) with positive probability. Here we provide a matching upper bound and, in fact, determine the asymptotic law of this distribution. To state this result we write \( Z(f) \) for the set of complex zeros of \( f \), let \( S \) denote the unit circle in the complex plane, let \( d(\mathcal{A}, \mathcal{B}) = \inf\{|x - y| : x \in \mathcal{A}, y \in \mathcal{B}\} \) for sets \( \mathcal{A}, \mathcal{B} \subseteq \mathbb{C} \) and, for \( \lambda \geq 0 \), let \( \exp(\lambda) \) denote an exponential random variable with mean \( \lambda^{-1} \).

**Corollary 2.** Let \( f \sim \mathcal{G}_n \) then
\[
n^2d(Z(f), S) \to \exp(1/6),
\]
in distribution, as \( n \to \infty \).

Note that Corollary 2 is indeed a corollary of our Theorem 1 and requires no more information about polynomials: given a homogeneous Poisson point process \( x_1, x_2, \ldots \) on \( \mathbb{R} \), we have that \( \min\{|x_1|, |x_2|, \ldots \} \) is exponentially distributed.

It also appears that our techniques are strong enough to prove a slightly stronger and intuitive generalization of Theorem 1 the two-dimensional point process
\[
\{(||(\zeta) - 1)n^2, \text{arg}(\zeta)\) : f_n(\zeta) = 0, \text{arg}(\zeta) \in [0, \pi]\}
\]
converges to a homogeneous Poisson process on the strip \( \mathbb{R} \times [0, \pi] \) of intensity \( 1/(12 \pi) \). In this paper, however, we limit ourselves to providing a proof of Theorem 1.

An interesting point of contrast comes from another conjecture made by Shepp and Vanderbei [26] who also considered the closest real root to the unit circle. In this direction, they conjectured

\[\text{Note here that we restrict to roots } \zeta \text{ with } \text{arg}(\zeta) \in [0, \pi], \text{ since the roots with } \text{arg}(\zeta) \in [\pi, 2\pi] \text{ are simply a reflection of this set, due to the fact } f \text{ has real coefficients.}\]
that the closest real root is at distance $\Theta(n^{-1})$ and the point process determined by the real roots at distance $\Theta(n^{-1})$ converges to a Poisson process as $n \to \infty$. In contrast with Theorem 1, the first named author will show in a forthcoming paper [22] that the limiting point process converges to a point process which is not Poisson and will also affirm the conjecture of Shepp and Vanderbei, that the closest real roots to the unit circle appear at this scale. Indeed, in [22] it is shown that with probability $1 - \varepsilon$,

$$d(Z(f) \cap \mathbb{R}, S) = \Theta_\varepsilon(n^{-1})$$

for all $\varepsilon > 0$.

1.1. **A heuristic and proof sketch.** The main new idea behind the proof of Theorem 1 is conceptually simple and we hope it will inspire uses beyond the results of this paper. To get a feel for this idea, let us start by considering the zero set

$$\{z : \text{Re } f(z) = 0\}$$

in an annular neighbourhood of the unit circle. As we will see, this set appears as $\approx n$ curves in this annular region (which we will call *strings*, to borrow a phrase from combinatorial geometry) which begin somewhere inside the unit circle and then cross over to the outside of the unit circle. For the purposes of this discussion, it is also convenient to imagine these strings as coloured red. Likewise we imagine the zero set

$$\{z : \text{Im } f = 0\},$$

as $\approx n$ blue strings which behave in much the same fashion. Now while we will have essentially no crossings between strings of the same colour, crossings between strings of different colours correspond exactly to the zeros of $f$ in our annular neighbourhood. To find a root of $f$ near the unit circle, our strategy will be to find a pair of strings, one blue and one red, that are extremely close to each other on the unit circle. We will then see (with high probability) that these strings must cross near the unit circle, thereby giving us our zero of $f$.

This idea, assuming it can be made rigorous, reduces the problem to showing that there exist two strings, of different colours, that get quite “close” on the unit circle, meaning (as we’ll see), with distance $O(n^{-2})$. For this, consider the trigonometric polynomials

$$X(x) := \sum_{k=0}^{n} \varepsilon_k \cos kx = \text{Re } f(e^{ix}), \quad Y(x) := \sum_{k=0}^{n} \varepsilon_k \sin kx = \text{Im } f(e^{ix}),$$

and observe that we would like to show that there is a pair of roots $(x_0, y_0)$ of $X, Y$, respectively, with $|x_0 - y_0| = O(n^{-2})$. 

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To see that this is a reasonable goal, let $R$ be the set of (red) zeros of $X$ in $[0, \pi]$ and $B$ be the set of (blue) zeros of $Y$ in $[0, \pi]$. A classical result, due to Dunnage [8], tells us that $|R|, |B| \sim n/\sqrt{3}$. So if $R, B$ were each sets of $n/\sqrt{3}$ iid points in $[0, \pi]$, then we would have $d(R, B) \approx n^{-2}$, as desired.

While this heuristic appears promising, one aspect of the (true) distribution of the zeros of the real and imaginary parts of $f$ seem to point in a different direction: the roots of the real and imaginary parts of $f$ actually repulse each other (see Figure 1.1). Thus one may be lead to believe that the phenomena described above actually fails when applied to roots, as opposed to random sets of points. However, as we shall show, the behavior of the distribution of the real roots $x, y$ with $d(x, y) = \Theta(n^{-2})$ remains Poisson in the real case, albeit with a different parameter.

![Image](image1.png)

(a) The zeros of Re$f$ and Im$f$ on the unit circle for $n = 40$ in red and blue respectively.  
(b) The same number of red and blue points placed independently at random.

**Figure 1.**

Turning this heuristic into a proof is rather involved and consists of two main steps: first, we show (in Section 2) that to understand the point process corresponding to roots $\{\zeta\}$ of $f$ with $|\zeta| = 1 + \Theta(n^{-2})$, it is sufficient to study a different point process $\mu_f$ defined entirely in terms of the behavior of $f$ on the unit circle. Roughly speaking, $\mu_f(U)$ is the number of pairs of zeros of Re$(f)$, Im$(f)$ that have the correct position and velocity so that their associated strings will collide at some radius $r \in U$. The second step consists of showing that our new process $\mu_f$ converges to a Poisson point process. To do this, we use the method of moments along with a Kac-Rice formula which allows us to express the factorial moments of $\mu_f(U)$ as an integral of a certain kernel. We then work with this kernel: for tuples of zeros that are far apart we shall show that this kernel approximately factors, which roughly says that the behavior of far away roots is independent.

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3We ignore $[\pi, 2\pi]$ as the zeros here are simply a reflection of the zeros in $[0, \pi]$. 

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The main challenge lies in dealing with the case when the roots are clustered together (in various possible configurations) and thus are significantly dependent. Our main tool here is Lemma 19, which is our main technical contribution of this paper and consumes most of its length. We rephrase this lemma here in a slightly different way to give the reader a feel for the strength of the result.

**Lemma 3.** Let $I_1, \ldots, I_{2k}$ be disjoint arcs of the unit circle above the real line which satisfy $d(I_i, \{\pm 1\}) > n^{-1/2}$. For $f \sim \mathcal{G}_n$, let $A_i$ be the event that $\text{Re}(f)$ has a root in $I_i$ and $B_i$ be the event that $\text{Im}(f)$ has a root in $I_i + k$. Then

$$(2) \quad \mathbb{P}\left( \bigcap_{i=1}^{k} (A_i \cap B_i) \right) = O_k\left(n^{2k} \prod_{j=1}^{2k} |I_j| \right).$$

This result is sharp, up to constants, for all sizes of intervals $I_1, \ldots, I_k$, and thus gives very good control even when $|I_i|$ is much smaller than $n^{-1}$. Observe in the statement of Lemma 3 we specify that the intervals are above the real axis. This is because the roots of $f$ are identical above and below the real axis. We also specify that these intervals are not too close to the real axis. Indeed, some condition of this form is required as $f$ has a very different behavior (again due to its real coefficients) near $\pm 1$. For an extreme example, we point out that $\text{Im}(f)$ is always zero on the real line and thus $\text{Im}(f)$ always has a zero in any interval $I$ containing $\{\pm 1\}$, which would be detrimental to an estimate like (2).

We also point out that Lemma 19 (or, equivalently, Lemma 3) can be seen as extending a key lemma of Granville and Wigman [10, Proposition A.1] to its logical conclusion. Granville and Wigman, prove a variant of Lemma 19 for three zeros in a single interval of length $O(n^{-1})$. While there are some similarities in the approach, our generalization is not at all straightforward.

1.2. **Future research.** It appears that the notion of studying the zero sets $\{z : \text{Re} f = 0\}$ and $\{z : \text{Im} f = 0\}$ for a random polynomial is novel and many natural questions suggest themselves about the nature of these ensembles of “strings”. Most generally, one might ask if there is a natural probabilistic notion that models these ensembles of strings. It is also natural to ask if other phenomena, such as the fascinating results of Peres and Virág [23], have a pleasing explanation in terms of these trajectories.

It also appears natural to consider the microscopic structure of the zero set of a random polynomial about other circles $\{z : |z| = r\}$, where $r \geq 1$. For $r = 1 + O(n^{-1})$ we would expect a very similar behavior to what we see around $|z| = 1$, but with the “force” of the repulsion increasing with $r$. It seems particularly interesting to consider the distribution of roots about the circles with radius $r = 1 + \omega(n^{-1})$, where (we would imagine) the effects of the repulsion start to seriously warp the distribution.

Another direction would be to consider variants of Theorem 1 for different coefficient distributions. While we have not investigated this question here, we would imagine that the behavior seen
in Theorem 1 is “universal” in the sense that a similar result should remain true for a wide class of coefficient distributions. We offer the following as a target for future research.

Conjecture 4. Let \( \{ \varepsilon_j \} \) be iid real random variables with \( \mathbb{E} \varepsilon_1 = 0 \) and \( \mathbb{E} \varepsilon_1^2 = 1 \). Then the conclusion of Theorem 1 still holds.

Perhaps the most natural first step in this direction would be to extend Theorem 1 to the case of random Littlewood polynomials: polynomials where the \( \varepsilon_k \) are chosen in \( \{ \pm 1 \} \) independently and uniformly.

2. Reduction to the unit circle

The purpose of this section is to make rigorous a central piece of the heuristic outlined in the introduction. In particular, we show that to understand the zeros of \( f_n \) near the unit circle, it is sufficient to look at zeros of the real and imaginary part on the unit circle along with their derivatives at those points.

But before getting to this, we get an irritating matter out of the way: since \( f \) has real coefficients, we have \( |\zeta| = |\zeta| \) for all the roots \( \zeta \) of \( f \) and thus each distance in the sequence \( (|\zeta_1| - 1)n^2, \ldots, (|\zeta_n| - 1)n^2 \) occurs twice for roots \( \zeta \) with \( \zeta \in \mathbb{C} \setminus \mathbb{R} \). To sweep away this redundancy, we consider only the roots in the upper half plane; that is, with \( \text{Im} \zeta \geq 0 \). Now, for \( S \subseteq \mathbb{R} \) and \( n \in \mathbb{N} \), define the annulus in the upper-half plane

\[
A_n(S) = \{ z \in \mathbb{C} : (|z| - 1)n^2 \in S \text{ and } \text{Im} \zeta \geq 0 \}.
\]

For a polynomial \( f \) with \( \text{deg}(f) = n \), we define

\[
A_f(S) = \{ z \in A_n(S) : f(\zeta) = 0 \},
\]

and define the measure \( \nu_f \) on \( \mathbb{R} \) by

\[
\nu_f(S) = |A_f(S)|,
\]

for each Borel set \( S \subseteq \mathbb{R} \). The measure \( \nu_f \) is our main object of interest and in fact Theorem 1 is exactly the statement that the random counting measure \( \nu_f \) converges to a Poisson point process as \( n \to \infty \).

To study \( \nu_f \), we show in this section that it is sufficient to work with another measure \( \mu_f \), which is easier to work with and is defined solely in terms of the behavior of \( f \) on the unit circle. Throughout, we write \( \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}) \) but often just work in \( [0, \pi] \). Since \( f \) behaves differently near the real axis than elsewhere, it will be convenient for us to work only with points away from the real axis; with this in mind, define

\[
\mathbb{T}_0 := \{ x \in [0, \pi] : d(x, \{0, \pi\}) > n^{-1/2} \}.
\]
Now, to define the measure $\mu_f$, break $f$ into real and imaginary parts

$$f((1 + \rho)e^{ix}) = X(x, \rho) + iY(x, \rho),$$

where

$$X(x, \rho) = \sum_{j=1}^{n} \varepsilon_j (1 + \rho)^j \cos(jx), \quad \text{and} \quad Y(x, \rho) = \sum_{j=1}^{n} \varepsilon_j (1 + \rho)^j \sin(jx).$$

We also define $X(x) := X(x, 0)$ and similarly for $Y$. For a Borel set $S \subseteq \mathbb{R}$, we define $C_f(S)$ to be

$$\left\{ (x, y) \in \mathbb{T}_0^2 : X(x) = Y(y) = 0, \frac{(x - y)X'(x)Y'(y)n^2}{(X'(x))^2 + (Y'(y))^2} \in S, |x - y| \leq n^{-2}(\log n)^4 \right\}$$

and then set

$$\mu_f(S) := |C_f(S)|.$$ 

Roughly speaking we have designed the measure of $S$ to be the number of pairs of zeros, of $X$ and $Y$ respectively, that are at the right distance from each other and moving at the right “speed” (as $\rho$ changes) so that they will result in a zero $\zeta$ of $f$ with $(|\zeta| - 1)n^2 \in S$.

The following lemma, to which the remainder of this section is dedicated, makes the connection between $\nu_f$ and $\mu_f$ rigorous.

**Lemma 5.** Let $f \sim \mathcal{G}_n$ and let $I \subseteq \mathbb{R}$ be a bounded open interval. Then

$$\mathbb{P}(\nu_f(I) = \mu_f(I)) \to 1,$$

as $n$ tends to infinity.

The key idea behind the proof of Lemma 5 is that every zero $\zeta$ of $f$ that is close to the unit circle can be traced back to two nearby zeros on the unit circle of $\text{Re}(f)$ and $\text{Im}(f)$, respectively. This is established in the following two sister lemmas (Lemmas 8, 10), the proofs of which are quite similar yet different enough in a few important ways and so we have treated them separately. We now turn to state a few standard facts that we will make heavy use of.

**Fact 6.**

(1) **Salem-Zygmund Inequality:** For $f \sim \mathcal{G}_n$ there is a constant $C > 0$ so that we have

$$\max_{|z|=1} |f(z)| \leq C(n \log n)^{1/2},$$

with high probability.

(2) **Bernstein’s Inequality:** If $g$ is a polynomial of degree $n$ then

$$\max_{|z|=1} |g'(z)| \leq n \max_{|z|=1} |g(z)|.$$
(3) If $g$ is a polynomial of degree $n$ then for each $\rho \geq 1$ we have
\[
\max_{|z|=\rho} |g(z)| \leq \rho^n \max_{|z|=1} |g(z)|.
\]

We shall also make use of the following properties of Gaussian random variables.

**Fact 7.** For $\sigma > 0$, let $X \sim N(0, \sigma^2)$ be a centered Gaussian random variable.

1. For all $k \in \mathbb{N}$, there exists a constant $C_k$, independent of $\sigma$, for which $\mathbb{E}|X|^k = C_k \sigma^k$.  
2. If $(X, Y)$ is a bivariate Gaussian random variable and $y \in \mathbb{R}$ then $\text{Var}(X|Y = y) \leq \sigma^2$.  
3. If $U \subseteq \mathbb{R}$ is an open set then $\mathbb{P}(X \in U) = O(|U|/\sigma)$.

Let us say that $\zeta \in \mathbb{C}$ and $(x, y) \in \mathbb{T}^2$ are $\delta$-close if $|\zeta - e^{ix}|, |\zeta - e^{iy}| < \delta$. The following lemma allows us to associate a zero of $f$ to a $\delta$-close pair of zeros of $X$ and $Y$, assuming a regularity hypothesis is met.

**Lemma 8.** Put $\delta := n^{-2}(\log n)^3$, let $I \subseteq \mathbb{R}$ be an open, bounded interval. Then there exists $n_0 = n_0(I)$ for which the following holds for all $n > n_0(I)$. Let $f$ be a polynomial with
\[
(4) \quad \max_{x \in [0, 2\pi]} |f(e^{ix})| \leq n^{1/2} \log n.
\]

If $\zeta \in A_f(I)$ with $\arg(\zeta) = \theta \in \mathbb{T}_0$ and
\[
(5) \quad |X'(\theta)|, |Y'(\theta)| > n^{3/2} / \log n
\]
then $\zeta$ is $\delta$-close to a pair $(x_0, y_0) \in C_f(I)$.

**Proof.** We have that $\zeta$ is a root of $f$ where $\zeta = \rho_0 e^{i\theta} = (1 + \gamma/n^2) e^{i\theta}$ for some $\gamma \in I$ and $\theta \in \mathbb{T}_0$. We first select $x_0, y_0$ and then show that they satisfy the conclusions of Lemma 8.

We express $X, Y$ using the Taylor expansion in variables $\theta, x$, at $x = \theta, \rho = 0$,
\[
(6) \quad X(x, \rho) = X(\theta) + X'(\theta)(x - \theta) + Y'(\theta)\rho + O(n^{-3/2}(\log n)^3),
\]
\[
(7) \quad Y(y, \rho) = Y(\theta) + Y'(\theta)(y - \theta) - X'(\theta)\rho + O(n^{-3/2}(\log n)^3),
\]
where the error bound holds when $|x - \theta|, |y - \theta|, \rho = O(n^{-2} \log n)$. One can see this last point by bounding the second derivatives of $X, Y$:
\[
|X_{xx}(x, \rho)|, |Y_{xx}(x, \rho)| \leq \sup_{x \in [0, 2\pi]} |f^{(2)}((1 + \rho)e^{ix})| \leq 2n^2 \sup_{x \in [0, 2\pi]} |f(e^{ix})| = O(n^{5/2} \log n).
\]
Here we have used Bernstein’s inequality along with (3) in Fact 6 and our assumption (4).

Now, since $\zeta \in Z(f)$, we have $X(\theta, \rho_0) = Y(\theta, \rho_0) = 0$ which we write out as
\[
0 = X(\theta, \rho_0) = X(\theta) + X'(\theta)\rho_0 + O(n^{-3/2}(\log n)^3);
\]
0 = Y(\theta, \rho_0) = Y(\theta) - X'(\theta)\rho_0 + O(n^{-3/2}(\log n)^3).

We then choose \( x_0, y_0 \), so that \( X'(\theta)(x_0 - \theta) = Y'(\theta)\rho_0 \) and \( Y'(\theta)(y_0 - \theta) = -X'(\theta)\rho_0 \), up to lower-order terms. That is, there exist \( x_0, y_0 \) so that \( X(x_0) = Y(y_0) = 0 \) and

\[
(8) \quad x_0 = \theta + \frac{\gamma}{n^2} X'(\theta) + O(n^{-3}(\log n)^4);
\]

\[
(9) \quad y_0 = \theta - \frac{\gamma}{n^2} X'(\theta) + O(n^{-3}(\log n)^4).
\]

We now check that this choice of \( x_0, y_0 \) satisfies the conclusion of Lemma 8. Using \( (5) \), note that this choice of \( (x_0, y_0) \) is \( \delta \)-close to \( \zeta \), as desired. Rearranging \( (8) \) and \( (9) \), we may write

\[
(10) \quad \gamma = \frac{n^2(x_0 - y_0)X'(\theta)Y'(\theta)}{(X'(\theta))^2 + (Y'(\theta))^2} + o(1).
\]

We now need to replace \( X'(\theta), Y'(\theta) \), in \( (10) \), with \( X'(x_0), Y'(y_0) \) to fit the definition of \( \mathcal{C}_f(S) \). We achieve this with an easy application of the mean value theorem. Indeed, since \( x_0 \) is within \( O(n^{-2}(\log n)^2) \) of \( \theta \) there exists a \( \xi \) with \( |\theta - \xi| = O(n^{-2}(\log n)^2) \) so that

\[
X'(x_0) = X'(\theta) + X^{(2)}(\theta)(\theta - \xi).
\]

Since \( |X'(\theta)| > n^{3/2-o(1)} \), by \( (5) \) and \( |X^{(2)}(\theta)| < n^{5/2+o(1)} \) (again using \( (4) \) and Bernstein’s inequality) we see \( X'(x_0) = X'(\theta)(1 + n^{-1+o(1)}) \). Applying the same to \( Y \) yields

\[
\frac{n^2(x_0 - y_0)X'(\theta)Y'(\theta)}{(X'(\theta))^2 + (Y'(\theta))^2} = \gamma + o(1),
\]

and therefore the left-hand-side is in \( I \) for sufficiently large \( n \), since \( I \) is an open interval and \( \gamma \in I \). From \( (8), (9) \) and condition \( (5) \) we see that \( |x_0 - y_0| = O(n^{-2}(\log n)^2) \). Therefore \( (x_0, y_0) \in \mathcal{C}_f(I) \), for large enough \( n \). □

In a way very similar to the proof of Lemma 8 we may track how the roots move as \( \rho \) changes.

**Lemma 9.** Let \( f \) be a polynomial satisfying

\[
\max_{x \in [0, 2\pi]} |f(e^{ix})| \leq n^{1/2} \log n,
\]

and let \( x_0, y_0 \in \mathbb{T} \) be such that \( X(x_0) = 0 \) and \( Y(y_0) = 0 \), and

\[
|X'(x_0)|, |Y'(x_0)|, |X'(y_0)|, |Y'(y_0)| > n^{3/2} / \log n.
\]

Then, for each \( |\rho| \leq n^{-2} \log n \) there exists \( x_\rho, y_\rho \in \mathbb{T} \) so that

\[
X(x_\rho, \rho) = 0, \quad Y(y_\rho, \rho) = 0,
\]
where

\begin{align*}
x_\rho &= x_0 - \rho \frac{Y'(x_0)}{X'(x_0)} + O\left(n^{-3}(\log n)^4\right), \\
y_\rho &= y_0 + \rho \frac{X'(y_0)}{Y'(y_0)} + O\left(n^{-3}(\log n)^4\right).
\end{align*}

Note crucially, that if the \(x_0, y_0\) are very close to each other then \(Y'(x_0)/X'(x_0) \approx Y'(y_0)/X'(y_0)\) and thus Lemma 9 tells us that as we increase \(\rho\) the roots \(x_\rho, y_\rho\) are moving in opposite directions on the circle. This is, in fact, a direct consequence of the Cauchy-Riemann equations. The next lemma associates a nearby complex root of \(f\) to a pair of roots of \(X, Y\) on the unit circle.

**Lemma 10.** Put \(\delta = n^{-2}(\log n)^3\), let \(I \subseteq \mathbb{R}\) be an open interval. Then there exists \(n_0 = n_0(I)\) for which the following holds for all \(n > n_0(I)\). Let \(f\) be a polynomial with

\[
\max_{x \in [0, 2\pi]} |f(e^{ix})| \leq n^{1/2} \log n.
\]

If \((x_0, y_0) \in C_f(I)\) with

\[
|X'(x_0)|, |Y'(x_0)|, |X'(y_0)|, |Y'(y_0)| > n^{3/2}/\log n,
\]

then there exists \(\zeta \in A_f(I)\) which is \(\delta\)-close to \((x_0, y_0)\).

**Proof.** We apply Lemma 9 to see that \(X(x_\rho, \rho) = Y(x_\rho, \rho) = 0\) where

\begin{align*}
(11) & \quad x_\rho = x_0 - \rho \frac{Y'(x_0)}{X'(x_0)} + O(n^{-3}(\log n)^4), \\
(12) & \quad y_\rho = y_0 + \rho \frac{X'(y_0)}{Y'(y_0)} + O(n^{-3}(\log n)^4).
\end{align*}

To compare these two ratios, we apply the mean value theorem and use that \(|X'(y_0)| > n^{3/2}/\log n\) and Bernstein’s inequality to see that for some \(\xi\) with \(|\xi - y_0| \leq |y_0 - x_0|\), we have

\[
X'(x_0) = X'(y_0) + X^{(2)}(y_0)(\xi - y_0) = (1 + o(n^{-1/4}))X'(y_0),
\]

and likewise for \(Y'(x_0)\). As a result, we have that

\[
X'(y_0)/Y'(y_0) = (1 + o(1))X'(x_0)/Y'(x_0).
\]

Using this along with (11), (12) and

\[
(\log n)^2 > |X'(x_0)|/|Y'(x_0)| > 1/(\log n)^2
\]
we see that \( x_\rho, y_\rho \) are (up to lower order terms) traveling in opposite directions on the circle, and therefore we must have \( x_\rho = y_\rho \) for some \( \rho_0 = \gamma/n^2 \) where

\[
\gamma = \frac{n^2(x - y)X'(x_0)Y'(y_0)}{(X'(x_0))^2 + (Y'(y_0))^2} + o(1).
\]

Thus \( \zeta := (1 + \gamma/n^2)e^{i\theta} \in \mathcal{A}_f(I) \), for sufficiently large \( n \), since \( I \) is an open interval. Finally, we note that \((x_0, y_0)\) is \( \delta = n^{-2}(\log n)^{10} \) close to \( \zeta \), for large enough \( n \).

### 2.1. Dealing with pathological points.

Lemmas 8 and 10 showed that we could pair each zero \( \zeta \) of \( f \) with a nearby root-pair of \( X, Y \) (and vice versa), provided our function was not doing something atypical around our zero. Here we record a few lemmas that say these atypical behaviors will not be a problem for us. We postpone the fairly straightforward proofs of these lemmas to Appendix A, so we don’t distract from the main trajectory of our proof.

**Lemma 11.** Let \( f \sim G_n \). All \( \zeta \in Z(f) \) with \( ||\zeta| - 1| \leq n^{-2}(\log n)^{1/4} \) satisfy

\[
|X'(\arg(\zeta))|, |Y'(\arg(\zeta))| > n^{3/2}/\log n,
\]

with high probability.

**Lemma 12.** Let \( f \sim G_n \) and let \( I \) be a finite interval. Then all \((x_0, y_0) \in C_f(I)\) satisfy

\[
|X'(x_0)|, |Y'(x_0)|, |X'(y_0)|, |Y'(y_0)| > n^{3/2}/\log n,
\]

with high probability.

For generic \( z \in \mathbb{C} \), \( f \) is a non-degenerate two-dimensional Gaussian. However, near the real axis, the imaginary part of \( f \) is small, thus leading to a different, more “one-dimensional” behavior that we will have to treat in a slightly different manner. In particular, we shall show that \( f \) has no roots near the real axis that are within \( O(n^{-2}) \) of the unit circle.

**Lemma 13.** Let \( f \sim G_n \), \( M > 0 \) and \( \varepsilon > 0 \). Then \( f \) has no zeros in

\[
\{ z \in \mathbb{A}_n([-M, M]) : |\arg(z)| \leq n^{-\varepsilon} \text{ or } |\arg(-z)| \leq n^{-\varepsilon} \},
\]

with high probability.

The following lemma tells us that no two roots of \( f \) have distance \( \approx n^{-2+o(1)} \) in \( \mathbb{C} \).

**Lemma 14.** Let \( f \sim G_n \) and let \( I \) be a bounded interval. Then, with high probability, there are no pairs of distinct \( \zeta_1, \zeta_2 \in \mathcal{A}_f(I) \) so that

\[
|\zeta_1 - \zeta_2| \leq n^{-2}(\log n)^{10}.
\]

Similarly, the following says that on the unit circle, neither \( X \) nor \( Y \) has two roots that are very close together, another example of the repulsion of roots of random polynomials.
Lemma 15. Let $g$ be either $X$ or $Y$. Then, with high probability, there do not exist distinct points $x_1, x_2 \in T_0$ with $g(x_1) = g(x_2) = 0,$

$$|x_1 - x_2| \leq n^{-2}(\log n)^{10}.$$ 

2.2. Proof of Lemma 5

Proof of Lemma 5 As in Lemmas 8 and 10 we set $\delta = n^{-2}(\log n)^3$. We define a function

$$\alpha : A_f(I) \to C_f(I),$$

which is an injection with high probability. Say that $\zeta = (1 + \gamma/n^2)e^{i\theta} \in A_f(I)$ is good if

$$|X'(\theta)|, |Y'(\theta)| > n^{3/2}/\log n,$$

and bad otherwise. Given a good $\zeta \in A_f(I)$, we can apply Lemma 8 to see that there is a pair $(x_0, y_0) \in C_f(I)$, for large enough $n$, that is $\delta$-close to $\zeta$. Define $\alpha(\zeta) = (x_0, y_0)$. This defines $\alpha$ for all good $\zeta$. For bad $\zeta$, we define $\alpha$ to be an arbitrary point in $C_f(I)$.

We need to show that there are no bad $\zeta$ with high probability and that there is no pair $(x_0, y_0)$ with $\alpha(\zeta) = \alpha(\zeta') = (x_0, y_0)$, for distinct good $\zeta, \zeta'$. Starting with this latter point, assume that both of $\zeta, \zeta'$ are mapped to the $\delta$-close pair $(x_0, y_0)$. This implies that $|\zeta - \zeta'| < 2n^{-2}(\log n)^3$, which occurs with probability $o(1)$, by Lemma 14. We now turn to the former point and apply Lemma 11 to show that there are no bad $\zeta$ with high probability. Hence $\alpha$ is an injection with high probability and thus

$$\mathbb{P}(|A_f(I)| \leq |C_f(I)|) = 1 - o(1).$$

We now define a function $\beta : C_f(I) \to A_f(I)$, which will be an injection with high probability. For each $(x_0, y_0) \in C_f(I)$ we say that $(x_0, y_0)$ is good if

$$|X'(x_0)|, |Y'(x_0)|, |X'(y_0)|, |Y'(y_0)| > n^{3/2}/\log n,$$

and bad otherwise. Now if $(x_0, y_0) \in C_f(I)$ is good then we may apply Lemma 10 to find a $\zeta \in A_f(I)$ that is $\delta$-close to $(x_0, y_0)$ and define $\beta((x, y)) = \zeta$. We define $\beta(x_0, y_0)$ to be an arbitrary element of $A$ if $(x, y)$ is bad. Lemma 12 tells us that there are no bad pairs with high probability. To see that this is an injection with high probability, assume that $\beta((x_0, y_0)) = \beta((x_1, y_1)) = \zeta$. This means that both $(x_0, y_0), (x_1, y_1)$ are $\delta$ close to $\zeta$ and therefore $|x_0 - x_1| < n^{-2}(\log n)^3$. Lemma 15 tells us this happens with probability $o(1)$. Therefore $\beta$ is an injection with high probability and so

$$\mathbb{P}(|C_f(I)| \leq |A_f(I)|) = 1 - o(1),$$

thus completing the proof of Lemma 5. \qed
3. MOMENTS AND A KAC-RICE TYPE FORMULA

With the work of Section 2 in hand, it is enough to show that the related measure \( \mu_f \) converges to a Poisson point process. In this section we set up the remainder of the paper by expressing the moments of the random variables \( \{ \mu_f(U) \}_U \) in a convenient integral form, known as the Kac-Rice formula.

But first we note the following lemma which tells us that to prove Theorem 1, it is enough to study the factorial moments of the random variables \( \mu_f(U) \), for all appropriate \( U \). For a real number \( x \), we use the notation \((x)_k = x(x-1) \cdots (x-k+1)\).

**Lemma 16.** Let \( \{ m_n \} \) be a sequence of point processes on \( \mathbb{R} \). Then \( m_n \) converges in the vague topology to a Poisson point process of intensity \( \lambda \) if
\[
\mathbb{E}[(m_n(U))_k] \rightarrow (\lambda|U|)_k,
\]
for all \( U \subset \mathbb{R} \) and all \( k \in \mathbb{N} \), where \( |U| \) denotes is the Lebesgue measure of \( U \).

**Proof.** By a theorem of Kallenberg [16, Theorem 4.7], in order to show that \( m_n \) converges to the Poisson process of intensity \( \lambda \), it is sufficient to show that for each \( a < b \) and \( I \) of the form \((a,b]\) or \([a,b)\) we have that \( \mathbb{E} m_n(I) \rightarrow \lambda|I| \) and \( \mathbb{P}(m_n(I) = 0) \rightarrow e^{-\lambda|I|} \). By the method of moments [5, Theorems 30.1 and 30.2], both follow from showing convergence of the factorial moments; further, since \( m_n \) asymptotically assigns no mass to the endpoints, we may work with the interior of \( I \). Applying (13) then shows that \( m_n \) converges to the desired limit. \( \square \)

We now turn to our integral form for these moments. For this we define
\[
\varphi_U(x,y) := 1 \{(x,y) \in C_f(U)\},
\]
where \( f \) is a degree \( n \) polynomial, \( C_f(U) \) is as defined at (3), \( x,y \in T \) and \( S \subseteq \mathbb{R} \). We now make an important, admittedly somewhat jarring, definition, the utility of which will be apparent soon. For \( k \in \mathbb{N} \), let \( x \in T^k \), \( y \in T^k \) and define
\[
p_k(x,y,U) := \frac{\mathbb{E}\left[\prod_{j=1}^{k} \left|X'(x_j)\right| \cdot \left|Y'(y_j)\right| \cdot \varphi_U(x_j,y_j) \mid \{X(x_i) = Y(y_i) = 0\}_{i=1}^{k}\right]}{(2\pi)^k \det(\Sigma)^{1/2}},
\]
where \( \Sigma = \Sigma_k(x,y) = \text{Cov}(X(x_i),Y(x_i))_{i \in [k]} \) is the covariance matrix of the joint distribution \((X(x_i),Y(x_i))_{i=1}^{k}\). We now arrive at our Kac-Rice-type integral for the factorial moments of \( \mu_f(U) \).

**Lemma 17.** Let \( U \) be a bounded open set and \( k \in \mathbb{N} \). Then
\[
\mathbb{E}[(\mu_f(U))_k] = \int_{T^2} p_k(x,y,U) \, dx \, dy.
\]
We postpone the proof of Lemma 17 to Appendix B where we derive it from a very general form of the Kac-Rice integral.

On a technical note, we point out that if \( x_i = x_j \) for some \( i \neq j \), or similarly for \( y \), then \( \Sigma \) is singular and so \( p_k \) is undefined. In these cases, we just set \( p_k = 0 \), for completeness, although this is technically unnecessary, as we only care about the behavior of \( p_k \) up to sets of measure 0. As it turns out, this is “the correct” way of extending \( p_k \), since \( p_k \) tends to zero as \( |x_i - x_j| \to 0 \), for some \( i \neq j \), a fact which will be a consequence of our results.

Remark 18. When we write \( E[|X'(x)||X(x) = 0] \) we use the standard definition of a conditioned multivariate Gaussian. This may be thought of as \( \lim_{\varepsilon \to 0^+} E[|X'(x)|||X(x)| \leq \varepsilon] \), which is different from \( \lim_{\varepsilon \to 0^+} E[|X'(x)||\exists y \in [x-\varepsilon, x+\varepsilon] \text{ s.t. } X(y) = 0] \), which is another possible interpretation.

4. Using the integral form of the moments

In this section we set up the rest of the paper by introducing Lemma 19, which is our main technical lemma of the paper. After presenting this lemma, we will conclude the section by proving Theorem 1 assuming Lemma 19, thus providing motivation to our later sections.

In this direction, we introduce the density function

\[
(15) \quad p_k(x, y) := \frac{E \left[ \prod_{j=1}^{k} |X'(x_j)| \cdot |Y'(y_j)| \mid \{X(x_i) = Y(y_i) = 0\}_{i=1}^{k} \right]}{(2\pi)^k \det(\Sigma)^{1/2}},
\]

which is a natural upper bound on \( p_k(x, y, U) \), for every \( U \), and will be central to our discussion going forward. Our main technical lemma of this paper says that \( p_k \) never gets too large.

**Lemma 19.** For \( k \in \mathbb{N} \), let \( x, y \in T^k_0 \) and \( p_k \) be as above. Then

\[
p_k(x, y) \leq C_k n^{2k},
\]

where \( C_k > 0 \) is a constant depending only on \( k \).

In addition to being a natural upper bound for \( p_k(x, y, U) \), \( p_k(x, y) \) may be thought of as a density for \( k \)-tuples of zeros, a perspective we illustrate by deriving Lemma 3 from Lemma 19.

**Proof of Lemma 3.** Set \( S = \prod_{i=1}^{2k} I_i \) and bound

\[
P \left( \bigcap_{i=1}^{k} (A_i \cap B_i) \right) \leq \int_S p_k(x, y) \, dx \, dy \leq C_k n^{2k} |S| = C_k n^{2k} \prod_{j=1}^{2k} |I_j|.
\]

\[ \square \]

We now prove Theorem 1 assuming Lemma 19 by calculating the moments of \( \mu_f(U) \) using Lemma 19 for all open and bounded sets \( U \). Recall, \( \mu_f \) is the (random) measure defined in...
Section 2. As a first step in this direction, we lay out a few approximations that will be proved later.

**Lemma 20.** For \( d(\{x, y\}, \pi Z) \geq n^{-1/2} \) and \(|x - y| \geq n^{-1/2}\) we have

\[
\text{Cov}\left( \frac{X'(x)}{n^{3/2}}, \frac{Y(y)}{n^{1/2}}, \frac{Y'(y)}{n^{3/2}}, \frac{X(x)}{n^{1/2}} \right) = (1 + o(1)) \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{2} \end{bmatrix}.
\]

We also have

\[
\text{Cov}\left( \frac{X'(x)}{n^{3/2}}, \frac{Y'(y)}{n^{3/2}} \right| X(x) = Y(y) = 0) = (1 + o(1)) \begin{bmatrix} \frac{1}{24} & 0 \\ 0 & \frac{1}{24} \end{bmatrix}.
\]

Further, if \( F, G \in \{X, X', Y, Y'\} \) then

\[
\text{Corr}(F(x), G(y)) = O(n^{-1/2}).
\]

**Proof.** Lines (16) and (18) follow from Lemmas 36 and Fact 37 while (17) follows from (16) and the formula for the variance of a conditioned Gaussian. \( \square \)

We now consider our first moment \( \mathbb{E} \mu_f(U) \), where \( U \) is a finite union of intervals.

**Lemma 21.** Let \( U \) be an open set. We have that

\[
\mathbb{E} \mu_f(U) = \frac{|U|}{12} + o(1).
\]

**Proof.** We first apply Lemma 17 to express

\[
\mathbb{E} \mu_f(U) = \int_T \int_0 \left| X'(x) Y'(y) \right| \varphi_U(x, y) \ dxdy = \int_T \int_0 \int_{y: |x - y| \leq n^{-2}(\log n)^4} p_1(x, y, U) \ dxdy,
\]

where the second equality holds since pairs \((x, y)\) with \(|x - y| > n^{-2}(\log n)^4\) have \( p_1(x, y, U) = 0 \).

We now consider

\[
p_1(x, y, U) = \frac{\mathbb{E} \left[ |X'(x)Y'(y)| \cdot \varphi_U(x, y) \right| X(x) = Y(y) = 0]}{(2\pi) \det(\Sigma)^{1/2}},
\]

where \( \Sigma \) is the 2 by 2 covariance matrix \( \text{Cov}(X(x), Y(y)) \). We first directly compute the denominator of (20), by using (16) in Lemma 20

\[
\det(\Sigma) = (1 + o(1)) \det \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = (1 + o(1))n^2/4.
\]

Set \( \sigma^2 = n^3/24 \) and define the multivariate Gaussian

\[
(W_1, W_2) = (X'(x)/\sigma, Y'(y)/\sigma \mid X(x) = Y(y) = 0)
\]
and note that $E W_j^2 = 1 + o(1)$ by (17) and $\text{Cov}(W_1, W_2) = O(n^{-1/2})$ by (18). We may then write
\[
\mathbb{E} \left[ |X'(x)Y'(y)| \cdot \varphi_U(x, y) \mid X(x) = Y(y) = 0 \right] = \sigma^2 \mathbb{E} \left[ |W_1 W_2| \mathbf{1} \left( \frac{n^2 r W_1 W_2}{W_1^2 + W_2^2} \in U \right) \right]
\]
and note
\[
\mathbb{E} \left[ |W_1 W_2| \mathbf{1} \left( \frac{n^2 r W_1 W_2}{W_1^2 + W_2^2} \in U \right) \right] = (1 + o(1)) \mathbb{E} \left[ |Z_1 Z_2| \mathbf{1} \left( \frac{n^2 r Z_1 Z_2}{Z_1^2 + Z_2^2} \in U \right) \right]
\]
where $(Z_1, Z_2)$ is a standard 2-dimensional Gaussian. Putting the previous two lines together gives
\[
\text{(22)} \quad \mathbb{E} \left[ |X'(x)Y'(y)| \cdot \varphi_U(x, y) \mid X(x) = Y(y) = 0 \right] = (1 + o(1)) \frac{n^3}{24} \mathbb{E} \left[ |Z_1 Z_2| \mathbf{1} \left( \frac{n^2 r Z_1 Z_2}{Z_1^2 + Z_2^2} \in U \right) \right].
\]

Now, with (21) and (22) in hand, we return to (19). We set $\eta = n^{-2}(\log n)^4$ and, for each fixed $x$, we let $r = |x - \eta|$ and compute
\[
\int_{|y|:|y| \leq n^{-2}(\log n)^4} p_1(x, y, U) \, dy = (1 + o(1)) \frac{n^2}{24 \pi} \int_{-\eta}^\eta \mathbb{E} \left[ |Z_1| \cdot |Z_2| \mathbf{1} \left( \frac{n^2 r Z_1 Z_2}{Z_1^2 + Z_2^2} \in U \right) \right] \, dr.
\]
Anticipating an application of Fubini’s theorem, note that
\[
\int_{-\eta}^\eta \mathbf{1} \left( \frac{n^2 r Z_1 Z_2}{Z_1^2 + Z_2^2} \in U \right) \, dr = \int_{-\eta}^\eta \mathbf{1} \left( r \in \left( \frac{Z_1^2 + Z_2^2}{n^2 Z_1 Z_2} \right) U \right) \, dr = \left| [-\eta, \eta] \cap \left( \frac{Z_1^2 + Z_2^2}{n^2 Z_1 Z_2} \right) U \right|.
\]
We then have
\[
\int_{-\eta}^\eta \mathbb{E} \left[ |Z_1| \cdot |Z_2| \mathbf{1} \left( \frac{n^2 r Z_1 Z_2}{Z_1^2 + Z_2^2} \in U \right) \right] \, dr = (1 + o(1)) \mathbb{E} \left[ |U|^{n-2} \cdot |Z_1| \cdot |Z_2| \cdot \frac{Z_1^2 + Z_2^2}{|Z_1| \cdot |Z_2|} \right] = (1 + o(1)) |U|/n^2.
\]
Combining then gives
\[
\int_{|y|:|y| \leq n^{-2}(\log n)^4} p_1(x, y, U) \, dy = (1 + o(1)) \frac{n^2}{24 \pi} (2 \cdot |U|/n^2) = \frac{|U|}{12 \pi} + o(1).
\]
Thus, from (19), we have that
\[
\mathbb{E} \mu_f(U) = \int_{\mathbb{T}_0} \int_{|y|:|y| \leq n^{-2}(\log n)^4} p_1(x, y, U) \, dy \, dx = \frac{|U|}{12} + o(1),
\]
as desired. \qed

We now show that $p_k(x, y)$ approximately factors when the $x_j$ have pairwise distance at least $n^{-1/2}$. For this, we need an elementary fact about inverting matrices in a neighborhood of the identity. Here and throughout, we use the notation $\|B\|_{\text{op}}$ to denote the $\ell^2 \to \ell^2$ operator norm.

**Fact 22.** Let $M$ be a $d \times d$ matrix of the form $M = I + E$, where $\|E\|_{\text{op}} = \delta$. If $\delta < 1$ then
\[
M^{-1} = I + B,
\]
16
where $\|B\|_{\text{op}} \leq \delta/(1 - \delta)$.

For what follows, let $k \in \mathbb{N}$ and define

$$D_1 = D_{1,k,n} := \left\{ (x, y) \in \mathbb{T}_0^{2k} : |x_i - x_j| \geq n^{-1/2} \text{ for all } i \neq j \right\}.$$

**Lemma 23.** Let $U$ be an open set. For all $x \in D_{1,k}$ and all $y$, we have

$$p_k(x, y, U) = (1 + o(1)) \prod_{j=1}^k p_1(x_j, y_j, U) + O(e^{-c(\log n)^2}).$$

**Proof.** We consider the numerator and denominator of the density $p_k(x, y, U)$ separately. That is,

$$p_k(x, y, U) = \frac{\alpha_{n,k}(x, y)}{(2\pi)^k \det(\Sigma)^{1/2}},$$

where

$$\Sigma = \text{Cov}(X(x_j), Y(y_j))_{i,j \in [k]}.$$

Note that we may restrict ourselves to considering $y$ which satisfy $|y_j - x_j| \leq n^{-2}(\log n)^4$ for all $i \in [k]$, since $p_k = 0$ otherwise. Now, for $j \in [k]$, define $\Sigma_j$ to be the $2 \times 2$ covariance matrix $\text{Cov}((X(x_j), Y(x_j)))$. We may use Lemma 20 to see that

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 & \cdots & 0 \\ 0 & \Sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_k \end{pmatrix} + E,$$

where $E$ is a matrix with all entries $O(n^{1/2})$. Since the diagonal entries of each $\Sigma_j$ are $\Theta(n)$, it follows that

$$\det \Sigma = (1 + O(n^{-1/2})) \prod_{j=1}^k \det \Sigma_j.$$

We now turn to the numerator of $p_k(x, y, U)$,

$$\alpha_{n,k}(x, y) = \mathbb{E}\left[ \left( \prod_{j=1}^k |X'(x_j)| \cdot |Y'(y_j)| \right) \cdot \varphi \left| X(x_j) = Y(y_j) = 0 \text{ for all } j \right. \right].$$

Set

$$\psi := 1 \left( |X'(x_i)| \leq n^{3/2} \log n, |Y'(x_j)| \leq n^{3/2} \log n \text{ for all } i, j \in [k] \right).$$
and notice that we may introduce \( \psi \) into the expectation in \( \alpha_{n,k} \) and incur an additive error of at most \( O(e^{-c \log n^2}) \). That is,

\[
\alpha_{n,k}(x,y) = \mathbb{E} \left[ \left( \prod_{j=1}^{k} |X'(x_j)| \cdot |Y'(y_j)| \right) \cdot \varphi \cdot \psi \mid X(x_j) = Y(y_j) = 0 \text{ for all } j \right] + O(e^{-c \log n^2})
\]

\[
(25) = \int_{\mathbb{T}^{2k}} \frac{s_1 \cdots s_k t_1 \cdots t_k \cdot \varphi \cdot \psi}{(2\pi)^k \det(M)^{1/2}} \exp \left( - (s,t)^T M^{-1}(s,t) / 2 \right) ds \, dt + O(e^{-c \log n^2}),
\]

where we defined the covariance matrix

\[
M := \text{Cov} \left( (X'(x_j), Y'(y_j))_{j=1}^{k} \mid X(x_j) = Y(y_j) = 0 \right)
\]

and re-written \((s,t)\) in the permutation \((s_1, t_1, s_2, t_2, \ldots, s_k, t_k)\).

To finish the lemma, we need to show that we can “replace” the occurrences of \( M \) in \( (25) \) with \( M' \) where

\[
M' := \text{diag} \left( \text{Cov}((X'(x_j), Y'(y_j)) \mid X(x_j) = Y(y_j) = 0)_{j=1}^{k} \right).
\]

This is easily done as \( M, M' \) are both approximately multiples of the identity. Indeed, we see that

\[
M, M' = \frac{n^3}{24} I + O(n^{5/2}),
\]

by using \((17)\) for the diagonal entries and \((18)\) for the off-diagonal entries. It follows that \( \det(M) = (1 + O(n^{-1/2})) \det(M') \). To replace the occurrence of \( M^{-1} \) in \( (25) \), we apply Fact \( 22 \) to see that for all \((s,t)\) with \(|(s,t)|_{\infty} < n^{3/2} \log n\), we have

\[
|(s,t)^T M^{-1}(s,t) - (s,t)^T (M')^{-1}(s,t)| = O(n^{-1/2}(\log n)^2).
\]

This shows that on the support of the integrand of \((25)\), we may replace both instances of \( M \) with \( M' \) at the cost of a multiplicative error of at most \((1 + O(n^{-1/2}(\log n)^2))\) and therefore

\[
\alpha_{n,k}(x,y) = (1 + O(n^{-1/2}(\log n)^2)) \prod_{j=1}^{k} \mathbb{E} \left[ |X'(x_j)| \cdot |Y'(y_j)| \cdot \varphi \mid X(x_j) = Y(y_j) = 0 \right] + O(e^{-c \log n^2}).
\]

Combining this with \((24)\) completes the proof. \( \square \)

We now arrive at the main application of Lemma \( 19 \) in the proof of Theorem \( 1 \). This lemma deals the case when there are two root-pairs, \((x_0, y_0)\) and \((x_1, y_1)\) of \( X, Y \) that are close to each other. In particular, define

\[
D_0 = D_{0,k,n} := \{ (x,y) \in \mathbb{T}^{2k}_0 : |x_i - x_j| \leq n^{-1/2} \text{ for some } i \neq j \}.
\]

**Lemma 24.** For \( k \in \mathbb{N} \), let \( U \subseteq \mathbb{R} \) be open and bounded. We have

\[
\int_{D_0} p_k(x,y, U) \, dx \, dy = o(1),
\]

\[
\int_{D_0} p_k(x,y, U) \, dx \, dy = o(1),
\]
as \( n \) tends to infinity.

**Proof.** From the definition of \( p_k \), we see that \( p_k = 0 \) if \(|x_i - y_i| > n^{-2} (\log n)^4\) for some \( i \in [k] \). So it makes sense to consider the integral over the set

\[ D' := D_0 \cap \{ (x, y) : |x_j - y_j| \leq n^{-2} (\log n)^4 \text{ for all } j \} . \]

By Lemma 19 we have \( p_k(x, y, U) = O(n^{2k}) \) and so

\[ I := \int_{D_0} p_k(x, y, U) \, dx \, dy = \int_{D'} p_k(x, y, U) \, dx \, dy = O\left(|D'| \cdot n^{2k}\right) . \]

Since

\[ |D'| = O\left(\frac{(\log n)^{4k}}{n^{2k}} \cdot \frac{1}{n^{1/2}}\right) , \]

it follows that \( I = o(1) \), as desired. \( \square \)

We now prove Theorem 1, assuming Lemma 19.

**Proof of Theorem 1.** Let \( U \subseteq \mathbb{R} \) be open and bounded and let \( D_1, D_0 \) be as above. We have

\[ \mathbb{E}[(\mu_f(U))_k] = \int_{\mathbb{T}_0^{2k}} p_k(x, y, U) \, dx \, dy = \int_{D_0} p_k(x, y, U) \, dx \, dy + \int_{D_1} p_k(x, y, U) \, dx \, dy . \]

By Lemma 24, the first integral on the right-hand-side is \( o(1) \), while we may apply Lemma 23 to see

\[ \int_{D_1} p_k(x, y, U) \, dx \, dy = (1 + o(1)) \int_{\mathbb{T}_0^{2k}} p_k(x, y, U) \, dx \, dy = (1 + o(1)) \int_{D_0} \prod_{j=1}^{k} p_1(x_j, y_j, U) \, dx \, dy + O(e^{-(\log n)^2}) , \]

where we have applied Lemma 19 again to see that \( p_1(x_j, y_j, U) = O(n^2) \) and used that

\[ |D_0| = O((\log n)^{4k} n^{-(2k+1)/2}) . \]

We may now apply Lemma 21 to see that

\[ \prod_{j=1}^{k} \int p_1(x_j, y_j, U) \, dx \, dy = (1 + o(1)) \left(\frac{|U|}{12}\right)^k . \]

Thus for all bounded open sets \( U \) and all \( k \), we have

\[ \mathbb{E}[(\mu_f(U))_k] = (1 + o(1)) \left(\frac{|U|}{12}\right)^k . \]
and so we apply Lemma 16 to see that $\mu_f$ tends to a Poisson point processes in the vague topology.

In order to show that $\nu_n$ converges, it is enough to show that the finite dimensional distributions converge [16, Theorem 4.2(iii)]. For any finite set of intervals, $\mu_n$ and $\nu_n$ match on all of them with high probability. Since $\mu_n$ converges to the Poisson process of rate $1/12$, its marginal distributions converge to the corresponding marginals of the Poisson process and thus those of $\nu_n$ do as well. □

Proof of Corollary 2. Let $X_n = \min_{\zeta \in \zeta_k} |\zeta_k| - 1/n^2$. Then for each $t \geq 0$, Theorem 1 implies
$$\mathbb{P}(X_n \geq t) \to e^{-2t/12} = e^{-t/6},$$
which is sufficient to conclude that $X_n$ tends in distribution to an exponential random variable with mean 6. □

5. The numerator

We now turn to the first of several sections where we take the reader through a proof of Lemma 19. In this section, we tackle the numerator of $p_k(x, y) = p_{n,k}(x, y)$, as defined at (15). That is, write
$$p_k(x, y) = \frac{\alpha_k(x, y)}{(2\pi)^k |\Sigma|^{1/2}},$$
where we have defined $\alpha_k(x, y)$ to be
$$\Theta(26) \quad E\left[ |X'(x_1) \cdots X'(x_k) Y'(y_1) \cdots Y'(y_k)| \quad X(x_i) = Y(y_i) = 0 \text{ for all } i \in [k] \right].$$

The purpose of this section is to prove the following upper bound on $\alpha_k$.

Lemma 25. For $k \in \mathbb{N}$, there is a constant $C_k > 0$ so that for all $x = (x_1, \ldots, x_k) \in \mathbb{T}_0^k$ and $y = (y_1, \ldots, y_k) \in \mathbb{T}_0^k$ we have
$$\alpha_k(x, y) \leq C_k n^{2k^2 + k} \left( \prod_{i<j} \min\{|x_i - x_j|, n^{-1}\}^2 \cdot \min\{|y_i - y_j|, n^{-1}\}^2 \right).$$

Our first move in the direction of Lemma 25 is a basic property of Gaussian random variables.

Lemma 26. Let $(Z_1, \ldots, Z_d)$ be a mean-zero multivariate Gaussian random variable Then
$$E[|Z_1| \cdots |Z_d|] = \Theta_d \left( \sqrt{\text{Var}(Z_1) \cdots \text{Var}(Z_d)} \right).$$

Proof. Define the vector $(W_j)_{j \in [d]}$ by setting $W_j = Z_j / \sqrt{\text{Var}(Z_j)}$ and write
$$E[|Z_1| \cdots |Z_d|] = \sqrt{\text{Var}(Z_1) \cdots \text{Var}(Z_d)} \cdot E[|W_1| \cdots |W_d|].$$

We now show $E[|W_1| \cdots |W_d|] = \Theta(1)$. Let $M = \text{Cov}((W_j)_{j \in [j]})$ and note that $M_{jj} = 1$ for all $j$. The map $M \mapsto E[|W_1| \cdots |W_d|]$ is continuous and takes only positive values. Since the set of
covariance matrices with 1’s on the diagonal is compact, it follows that $\mathbb{E}[|W_1| \cdots |W_d|]$ is bounded above and away from 0, as desired.

With Lemma 26 in tow, we only need to understand the conditional variances

$$\text{Var}(X'(x_j) \mid X(x_i) = Y(y_i) = 0 \text{ for all } i)$$

and similarly for $Y'(y_j)$. To get a handle on the typical size of $X'(x_j)$ conditioned on $X(x_j) = Y(y_j) = 0$, we require a consequence of the mean value theorem.

**Lemma 27.** Let $x_1, \ldots, x_k$ be distinct points and let $f$ be a smooth function. If $f(x_j) = 0$ for all $j \in [d]$ then there exists a $\xi \in [\max_j \{x_j\}, \min_j \{x_j\}]$ so that

$$f'(x_1) = \frac{f^{(k)}(\xi)}{k!} \prod_{j=2}^{k} (x_1 - x_j).$$

Proof. Consider the polynomial $p(x) = c(x - x_1)(x - x_2) \cdots (x - x_k)$ with $c = \frac{f(x_1)}{(x_1 - x_2) \cdots (x_1 - x_k)}$. Note that $p(x_j) = 0$ for all $j \in [d]$ and $p'(x_1) = f'(x_1)$. Now, define $g(x) = f(x) - p(x)$ and note that $g(x_j) = 0$ for all $j \in [d]$ and $g'(x_1) = 0$. Let $y_1 < \cdots < y_k$ be the points $x_1, \ldots, x_k$ in order; by Rolle’s theorem, there are points $\xi_i \in (y_i, y_{i+1})$ for $i \in [d-1]$ with $g'(\xi_i) = 0$. Since $g'$ is zero on the $k$ distinct points $\{\xi_i\}_{i \in [d-1]}$ and $x_1$, there must exist a $\xi \in [\max_j \{x_j\}, \min_j \{x_j\}]$ so that $g^{(k)}(\xi) = 0$. Thus

$$0 = g^{(k)}(\xi) = f^{(k)}(\xi) - p^{(k)}(\xi) = f^{(k)}(\xi) - \frac{k! f'(x_1)}{(x_1 - x_2) \cdots (x_1 - x_k)}.$$

Solving for $f'(x_1)$ completes the proof.

We shall also need the following fact about Gaussian trigonometric polynomials. We state the next two lemmas for the function $X$, but the same is true of $Y$.

**Lemma 28.** For $k \in \mathbb{Z}_{\geq 0}$, let $I \subseteq \mathbb{T}$ be an interval with $|I| = An^{-1}$. Then

$$\mathbb{E} \sup_{y \in I} |X^{(k)}(y)| = O_{A,j}(n^{k+1/2})$$

and

$$\mathbb{E} \left[ \sup_{y \in I} |X^{(k)}(y)| \mid X(y) = 0 \right] = O_{A,j}(n^{k+1/2}).$$

Proof. For $x, x_0 \in I$, we may apply Taylor’s theorem about $x_0$ to obtain

$$|X^{(k)}(x)| \leq \sum_{j \geq 0} \frac{X^{(j+k)}(x_0)}{j!} (x - x_0)^j \leq \sum_{j \geq 0} \frac{|X^{(j+k)}(x_0)|}{j!} A^j n^{-j}.$$
To prove (27) we simply take expectations of both sides and use that
\[ \mathbb{E}|X^{(j+k)}(x_0)| \leq (\text{Var}(X^{(j+k)}(x_0)))^{1/2} = O(n^{k+j+1/2}). \]
The proof of (28) only requires one further twist: we take expectations of both sides of (29), while conditioning on on \( X(y) = 0 \). We then use the property of Gaussian random variables
\[ \mathbb{E}\left[|X^{(j+k)}(x_0)| \mid X(x_1) = 0\right] \leq \mathbb{E}|X^{(j+k)}(x_0)| \]
and then proceed as in the proof of (27).

**Lemma 29.** For \( k \in \mathbb{N} \), let \( x_1, \ldots, x_k \in \mathbb{T} \) be points with \( |x_i - x_j| < n^{-1} \) for all \( j \). Then
\[ \mathbb{E}[|X'(x_1)| \mid X(x_1) = X(x_2) = \cdots = X(x_k) = 0] \leq C_k n^{k+1/2} \prod_{j=2}^{k} |x_1 - x_j|, \]
where \( C_k > 0 \) is a constant depending only on \( k \).

**Proof.** Let \( a = \min_i x_i \) and \( b = \max_i x_i \). Given \( X(x_1) = X(x_2) = \cdots = X(x_k) = 0 \), apply Lemma 27 to obtain
\[ |X'(x_1)| \leq \max_{\xi \in [a,b]} |X^{(k)}(\xi)| \prod_{j=2}^{k} |x_j - x_1|. \]
Taking expectations, conditional on \( X(x_1) = X(x_2) = \cdots = X(x_k) = 0 \), and using Lemma 28 completes the proof.

**Proof of Lemma 25.** The multidimensional Gaussian random variable
\[ (X'(x_1), \ldots, X'(x_k), Y'(y_1), \ldots, Y'(y_k)) \]
is still Gaussian after we condition on \( X(x_i) = Y(y_i) = 0 \), for \( i \in [k] \). Thus, we may apply Lemma 26 to these conditioned random variables to learn
\[ \alpha_k(x,y) = \mathbb{E}\left[|X'(x_1) \cdots X'(x_k), Y'(y_1) \cdots Y'(y_k)| \mid X'(x_j) = Y'(y_j) = 0 \text{ for all } j\right] \]
is at most
\[ c_k \prod_{i=1}^{k} (\mathbb{E}[|X'(x_i)|^2 \mid X'(x_j) = Y'(y_j) = 0 \text{ for all } j] \mathbb{E}[|Y'(y_i)|^2 \mid X'(x_j) = Y'(y_j) = 0 \text{ for all } j])^{1/2}. \]
Define \( S_i = \{ j : |x_i - x_j| < n^{-1} \} \). We now use two further properties of Gaussian random variables, [2] and [1] from Fact 7 to bound
\[ (\mathbb{E}[|X'(x_i)|^2 \mid X'(x_j) = Y'(y_j) = 0 \text{ for all } j])^{1/2} \leq (\mathbb{E}[|X'(x_i)|^2 \mid X'(x_j) = 0 \text{ for all } j \in S_i])^{1/2} \leq CE[|X'(x_i)||X'(x_j) = 0 \text{ for all } j \in S_i] . \]
Thus we can use this together with Lemma 29 to see

\[(32) \quad (\mathbb{E} |X'(x_i)|^2 X'(x_j) = Y'(y_j) = 0 \text{ for all } j) \leq C_k n^{k+1/2} \prod_j \min\{|x_j - x_i|, n^{-1}\} \cdot \min\{|y_j - y_i|, n^{-1}\}^2,\]

Thus using (32) in (31), gives us

\[\alpha_k(x, y) \leq C_k n^{2k^2 + k} \prod_{i<j} \min\{|x_j - x_i|, n^{-1}\} \cdot \min\{|y_j - y_i|, n^{-1}\}^2,\]

as desired. \(\square\)

6. UNDERSTANDING THE COVARIANCE STRUCTURE

In this section we prepare for Section 7 where we will obtain a lower bound on the determinant of the covariance matrix \(\Sigma_k(x, y)\). In that section, we will relate such covariance matrices to covariance matrices involving \(X, Y\) and their derivatives at a “well separated” set of points. In this section, we study the structure of such covariance matrices, involving \(X, Y\) and their derivatives.

Our main goal here will be to prove Lemma 30. For this, we use the notation

\[(33) \quad X_0^{(j)}(x) := \frac{X^{(j)}(x)}{n^{j+1/2}}, \quad Y_0^{(j)}(x) := \frac{Y^{(j)}(x)}{n^{j+1/2}},\]

to denote the re-normalized versions of the derivatives of the trigonometric sums \(X, Y\) and, for a matrix \(A\), we let \(\lambda_{\min}(A)\) denote the smallest eigenvalue of \(A\). We will also abuse notation slightly and let \([s, t]\) denote the interval of integers \(\{s, s+1, \ldots, t\}\), when it is clear from context.

**Lemma 30.** For \(\varepsilon > 0\) and integers \(r \geq 1, s \geq 0\) there exists \(n(\varepsilon, r, s) \in \mathbb{N}\) so that the following holds. If \(z_1, \ldots, z_r \in T_0\) satisfy \(d(z_i, z_j) \geq \varepsilon/n\), for all \(i \neq j\) and

\[(34) \quad \Sigma = \text{Cov}\left(X_0^{(j)}(z_i), Y_0^{(j)}(z_i)\right)_{i \in [r], j \in [0, s]}\]

then

\[\lambda_{\min}(\Sigma) = \Omega_{\varepsilon, r, s}(1),\]

for all \(n \geq n(\varepsilon, r, s)\).

We approach Lemma 30 by first considering a limit object that captures the covariance structure of \(X, Y\), and associated derivatives, at the scale \(n^{-1}\). To this end, we define the Gaussian processes \((Z(t), W(t))\), where \(W(t)\) is defined to be the stationary mean-zero Gaussian process with covariance function

\[(35) \quad \text{Cov}(W(t), W(s)) = \frac{\sin(t - s)}{2(t - s)} = \frac{1}{2} \int_0^1 \cos((t - s)\theta) d\theta\]
and $Z(t)$ is defined to be the stationary mean-zero Gaussian process with the same covariance as $W(t)$ and

$$
E[Z(t)W(s)] = \frac{1 - \cos(t-s)}{2(t-s)} = \frac{1}{2} \int_0^1 \sin((t-s)\theta) \, d\theta.
$$

Our aim in this section will be to prove the following about the process $(Z(t), W(t))$.

**Lemma 31.** For $\varepsilon > 0$ and integers $r \geq 1$, $s \geq 0$ there exists $n(\varepsilon, r, s) \in \mathbb{N}$ so that the following holds. If $z_1, \ldots, z_r \in \mathbb{R}$ satisfy $|z_i - z_j| \geq \varepsilon$ and

$$
\Sigma_\infty := \text{Cov}\left(W^{(j)}(z_i), Z^{(j)}(z_i)\right)_{i \in [r], j \in [s]}
$$

then

$$
\lambda_{\min}(\Sigma_\infty) = \Omega_{\varepsilon, r, s}(1)
$$

for all $n \geq n(\varepsilon, r, s)$.

We shall then deduce Lemma 30 by showing that $\lambda_{\min}(\Sigma) \approx \lambda_{\min}(\Sigma_\infty)$, for sufficiently large $n$.

6.1. The process $(Z(t), W(t))$. We should remark that it is not actually clear, at this point, that the process $(W(t), Z(t))$ actually exists. While this is not strictly necessary for our work here, we do pause to take care of this point. Indeed, the question of existence is settled by the following lemma, along with some general machinery.

**Lemma 32.** Let $z_1, \ldots, z_r$ be distinct elements of $\mathbb{R}$. Then the covariance matrix

$$
\Sigma_\infty := \text{Cov}(Z(z_i), W(z_i))_{i=1}^r
$$

is positive definite.

We don’t prove Lemma 32 here as it will follow from our more general Lemma 33. But with Lemma 32 in hand, we may now apply Kolmogorov’s extension theorem [1, Section 1.2] to learn that $(Z(t), W(s))$ exists. Moreover, since the function $\sin(x)/x$ is smooth, we may assume that $(Z(t), W(s))$ has smooth paths [4, page 30].

We now turn to explore the covariance structure of the derivatives of this process. By differentiating under the integral we have

$$
\text{Cov}(W^{(a)}(t), W^{(b)}(s)) = \frac{(-1)^b}{2} \int_0^1 \theta^{a+b} \cos^{(a+b)}((t-s)\theta) \, d\theta.
$$

and

$$
\text{Cov}(W^{(a)}(t), Z^{(b)}(s)) = \frac{(-1)^b}{24} \int_0^1 \theta^{a+b} \sin^{(a+b)}((t-s)\theta) \, d\theta,
$$
where \( \cos^{(k)}, \sin^{(k)} \) denote the \( k \)th derivative of cosine and sine, respectively. In order to work with covariance matrices of \( Z, W \) and their derivatives, we prove the following lemma that allows us to express \( v^T \Sigma v \) in a convenient form. We remark that this useful form appears in [3, Lemma 5], but only for the process \( W \) and its derivative.

**Lemma 33.** For \( \varepsilon > 0 \) and integers \( r \geq 1, s \geq 0 \) and \( z_1, \ldots, z_r \in \mathbb{R} \), let

\[
\Sigma := \text{Cov}\left(W^{(i)}(z_i), Z^{(j)}(z_j)\right)_{i\in[r], j\in[0,s]}.
\]

Then

\[
v^T \Sigma v = \frac{1}{2} \int_0^1 |F_v(\theta)|^2 \, dt,
\]

where

\[
F_v(\theta) := \sum_{a,b} (x_{a,b} + iy_{a,b})(i\theta)^b e^{za i\theta},
\]

\( v = (x, y) \), \( x = (x_{i,j})_{i\in[r], j\in[0,s]} \), and \( y = (y_{i,j})_{i\in[r], j\in[0,s]} \).

**Proof.** To understand the indexing of this vector \( v \), we note that we can think of the rows and columns of \( \Sigma \) as indexed by \( Z^{(j)}(z_j) \), for all \( i \in [r], j \in [0,s] \) along with \( W^{(i)}(z_i) \) for all \( i \in [r], j \in [0,s] \). So we may write \( v = (x, y) \) where \( x = (x_{i,j})_{i\in[r], j\in[0,s]} \) and \( y = (y_{i,j})_{i\in[r], j\in[0,s]} \) and expand \( v^T \Sigma v \) as

\[
(40) \quad v^T \Sigma v = \sum_{a,b,c,d} x_{a,b} x_{c,d} \text{Cov}(W^{(b)}(z_a) W^{(d)}(z_c)) + \sum_{a,b,c,d} x_{a,b} y_{c,d} \text{Cov}(W^{(b)}(z_a) Z^{(d)}(z_c))
\]

\[
+ \sum_{a,b,c,d} y_{a,b} x_{c,d} \text{Cov}(Z^{(b)}(z_a) W^{(d)}(z_c)) + \sum_{a,b,c,d} y_{a,b} y_{c,d} \text{Cov}(Z^{(b)}(z_a) Z^{(d)}(z_c)).
\]

Working with the first of the terms on the right-hand-side, we have

\[
\sum_{a,b,c,d} x_{a,b} x_{c,d} \text{Cov}(W^{(b)}(z_a) W^{(d)}(z_c)) = \frac{1}{4} \sum_{a,b,c,d} x_{a,b} x_{c,d} \int_0^1 (i\theta)^b (-i\theta)^d e^{i(z_a - z_c)\theta} + (i\theta)^d (-i\theta)^b e^{i(z_c - z_a)\theta} \, d\theta
\]

\[
= \frac{1}{2} \sum_{a,b,c,d} x_{a,b} x_{c,d} \int_0^1 (i\theta)^b (-i\theta)^d e^{i(z_a - z_c)\theta} \, d\theta = \frac{1}{2} \int_0^1 \left| \sum_{a,b} x_{a,b} (i\theta)^b e^{i z_a \theta} \right|^2 \, d\theta
\]

where the second equality follows from swapping the roles of the pairs \( (a, b) \) and \( (c, d) \) and combining the two sums. Similarly, simplifying the other three terms in \((40)\) using \((38)\), \((39)\), allows us to rewrite \((40)\) to obtain

\[
v^T \Sigma v = \frac{1}{2} \int_0^1 |F_v(\theta)|^2 \, d\theta,
\]

as desired. \( \square \)
We now prove a general lower bound for the exponential-type polynomials that appear in Lemma 33.

**Lemma 34.** For \( r \geq 1, s \geq 0, \varepsilon > 0 \), let \( z_1, \ldots, z_r \in \mathbb{R} \) satisfy \( |z_i - z_j| \geq \varepsilon \) and let

\[
F(\theta) = \sum_{i=1}^{r} \sum_{j=0}^{s} \alpha_{i,j} \theta^j e^{i z_i \theta}.
\]

Then

\[
\int_0^1 |F(\theta)|^2 \, d\theta \geq c_{r,s,\varepsilon} \sum_{i=1}^{r} \sum_{j=0}^{s} |\alpha_{i,j}|^2.
\]

**Proof.** We apply induction on \( r \). The statement is clear for \( r = 1 \). For the induction step at \( r \), we define

\[ c_r^* = \min_{t \leq r} c_{t,s,\varepsilon}, \]

and we will choose \( M \) to be sufficiently large compared to \( 1/c_r^* \) \( r \) and \( s \). We consider two cases; (case 1) all of the \( z_i \) are contained in an interval of length \( rM \) or (case 2) there exists a partition into non-empty sets \( A \cup B = [r] \) so that \( |z_i - z_i'| > M \) for all \( i \in A \) and \( i' \in B \). We start by dealing with this latter case: let us write \( F = F_A + F_B \) where \( F_A := \sum_{(i,j): i \in A \alpha_{i,j} \theta^j e^{i z_i \theta} \) and \( F_B \) is defined similarly. We have

\[
\int_0^1 |F|^2 \, d\theta = \int_0^1 |F_A|^2 \, d\theta + \int_0^1 |F_B|^2 \, d\theta + \sum_{i \in A, i' \in B} 2\alpha_{i,j} \alpha_{i',j'} \int_0^1 \theta^{i+j'} \cos((z_i - z_j)\theta) \, d\theta,
\]

where this last sum is over all pairs \( (i, j), (i', j') \), where \( i \in A \) and \( i' \in B \). Now since \( |z_i - z_j| > M \) for all such pairs, we can integrate by parts to see

\[
\left| \int_0^1 \theta^{i+j'} \cos((z_i - z_j)\theta) \right| \leq 4r/M.
\]

Using this, along with \( 2|xy| \leq x^2 + y^2 \), we see

\[
\left| \sum_{i \in A, i' \in B} 2\alpha_{i,j} \alpha_{i',j'} \int_0^1 \theta^{i+j'} \cos((z_i - z_j)\theta) \right| \leq \frac{8r}{M} \sum_{i \in A, i' \in B} |\alpha_{i,j} \alpha_{i',j'}| \leq \frac{4r^2 s}{M} \sum_{i,j} |\alpha_{i,j}|^2.
\]

On the other hand, we may apply induction to the first two terms on the right hand side of (42) to obtain

\[
\int_0^1 |F|^2 \, d\theta \geq c_r^* \sum_{i,j} |\alpha_{i,j}|^2 - (4r^2 s/M) \sum_{i,j} |\alpha_{i,j}|^2.
\]

Thus we may choose \( M \) to be sufficiently large in terms of \( c_r^*, r \) and \( s \) so that the above sum is at least \( c_r^*/2 \sum_{i,j} |\alpha_{i,j}|^2 \), as desired.

In the case that \( z_1, \ldots, z_r \) lie in an interval of length \( rM \), we apply a compactness argument. First note that we may assume that \( z_1, \ldots, z_r \in [0, rM] \), as replacing \( \{z_1, \ldots, z_r\} \) with \( \{z_1 + T, \ldots, z_r + T\} \)
does not change the value of the integral. Also note that we may assume that $\sum_{i,j} |\alpha_{i,j}|^2 = 1$, by scaling both sides of (41). Now, define the function

$$I(z, \alpha) := \int_0^1 |F(\theta)|^2 d\theta,$$

where $z = (z_1, \ldots, z_r)$ and $\alpha = (\alpha_{i,j})_{i,j}$ and note that $I(z, \alpha)$ is a continuous function of its variables and that $I$ is always positive when $(\alpha_{i,j})_{i,j}$ is non-zero and $z_1, \ldots, z_r$ are distinct. Now define the compact set

$$S_{r,\varepsilon,M} := \{z \in [0,M]^r : \varepsilon \leq |z_i - z_j| \forall i \neq j\} \times \{\alpha \in \mathbb{C}^{r \times (s+1)} : |\alpha|_2 = 1\}.$$

Since $I$ is always positive on $S_{r,\varepsilon,M}$ it attains a positive minimum value on $S_{r,\varepsilon,M}$, this concludes the proof. \hfill \Box

We now are in a position to prove Lemma 31, our version of Lemma 30 for the limiting process $(W(t), Z(t))_t$.

**Proof of Lemma 31.** We use the variational definition of $\lambda_{\min}(\Sigma_\infty)$ along with Lemma 33 to write

$$\lambda_{\min}(\Sigma_\infty) = \min_{v : |v|_2 = 1} v^T \Sigma_\infty v = (1/2) \min_{v : |v|_2 = 1} \int_0^1 |F_v(\theta)|^2 d\theta,$$

where $F_v$ is finite sum of the form $F_v(\theta) := \sum_{i,j} \alpha_{i,j} \theta^i e^{iz_j}$ and the $\alpha_{i,j}$ are complex numbers satisfying $\sum_{i,j} |\alpha_{i,j}|^2 = 1$. We may now apply Lemma 34 to finish the proof of Lemma 31. \hfill \Box

6.2. **Proof of Lemma 30 from Lemma 31.** We now turn to prove our main lemma of this section by approximating covariance matrices in the statement of Lemma 30 with the covariance matrices of our limiting object $(Z(t), W(t))$.

**Lemma 35.** For $r, s \in \mathbb{Z}_{\geq 0}$, let $x_1, \ldots, x_r \in T_0$, let

$$\Sigma = \text{Cov} \left( X_0^{(j)}(z_i), Y_0^{(j)}(x_i) \right)_{i \in [r], j \in [0,s]}$$

and let

$$\Sigma_\infty := \text{Cov} \left( W^{(j)}(nx_i), Z^{(j)}(nx_i) \right)_{i \in [r], j \in [0,s]}.$$

Then

$$\|\Sigma - \Sigma_\infty\|_{op} = O_{r,s}(n^{-1/2}).$$

We show this by showing that $\Sigma - \Sigma_\infty$ tends to the zero matrix entry-wise.

**Lemma 36.** For $a, b \in \mathbb{Z}_{\geq 0}$, we have

$$\max_{x,y \in T_0} \left| \text{Cov} \left( X_0^{(a)}(x), X_0^{(b)}(y) \right) - \text{Cov} \left( W^{(a)}(nx), W^{(b)}(ny) \right) \right| = O(n^{-1/2});$$
\[
\max_{x,y \in \mathbb{T}_0} \left| \text{Cov} \left( X_0^{(a)}(x), Y_0^{(b)}(y) \right) - \text{Cov} \left( W^{(a)}(nx), Z^{(b)}(ny) \right) \right| = O(n^{-1/2}); \\
\max_{x,y \in \mathbb{T}_0} \left| \text{Cov} \left( Y_0^{(a)}(x), Y_0^{(b)}(y) \right) - \text{Cov} \left( Z^{(a)}(nx), Z^{(b)}(ny) \right) \right| = O(n^{-1/2}).
\]

The proof Lemma 36 relies on the following basic fact. Define

\[
D_{n,d}(x) := \sum_{k=0}^{n} k^d \cos \left( \frac{dk}{xn} \right) \quad S_n(x) := \sum_{k=0}^{n} k^d \sin \left( \frac{dk}{xn} \right)
\]

for each \( d \geq 0 \).

**Fact 37.** Let \( n, d \in \mathbb{Z}_{\geq 0} \). We have

\[
\max_{x \in \mathbb{T}} \left| n^{-(d+1)} D_{n,d}(x) - \int_{0}^{1} t^d \cos(xnt) \, dt \right| = O(n^{-1/2}); \\
\max_{x \in \mathbb{T}} \left| n^{-(d+1)} S_{n,d}(x) - \int_{0}^{1} t^d \sin(xnt) \, dt \right| = O(n^{-1/2}).
\]

**Proof.** We prove only the first part of the fact and note the other is almost identical. We write

\[
n^{-(d+1)} D_{n,d}(x) = \sum_{k=0}^{n} n^{-1} (k/n)^d \cos \left( \frac{dnk}{xn} \right) = \sum_{k=0}^{n} g(k/n),
\]

where we set \( g_{n,x,d}(t) := t^d \cos(xnt) \). Now note that

\[
|g'_{n,x,d}(t)| = |dt^{d-1} \cos(xnt) + xnt^d \sin(xnt)| \leq 2dxn,
\]

for \( t \in [0,1] \). Thus, using (43) and an effective form of convergence of the Riemann integral, we have

\[
\left| n^{-(d+1)} D_{n,d}(x) - \int_{0}^{1} g(t) \, dt \right| = \left| n^{-1} \sum_{k=0}^{n} g(k/n) - \int_{0}^{1} g(t) \, dt \right| \leq n^{-1} \int_{0}^{1} |g'(t)| \, dt = O(1/n).
\]

So this proves Fact 37 when \( x < n^{-1/2} \). For \( x > n^{-1/2} \) we show that both the sum and integral are small. Starting with the integral, integrate by parts to see

\[
\int_{0}^{1} t^d \cos(xnt) \, dt = \frac{\sin(xn)}{xn} - \frac{d}{xn} \int_{0}^{1} t^{d-1} \sin(xnt) \, dt = O(1/(xn)) = O(n^{-1/2}).
\]

On the other hand, apply Abel’s summation formula to express

\[
D_{n,d}(x) = n^d D_{n,1}(x) + d \int_{0}^{n} t^{d-1} D_{[x],1}(x) \, dt,
\]

and then use the bound \( |D_{m,1}(x)| \leq 1/x \) for all \( m \in \mathbb{N} \). □
Proof of Lemma 36. We treat the first case and note that the others are similar. We have
\[ \text{Cov} (X(x), X(y)) = \sum_{k=0}^{\infty} \cos(\lambda_k) \cos(y) = (1/2)D_n(x-y) + (1/2)D_n(x+y). \]

Thus applying the differential operator \( \left( \frac{d}{dx} \right)^a \left( \frac{d}{dy} \right)^b \) to both sides and multiplying by \( n^{-(a+b+1)} \) gives
\[ (44) \quad \text{Cov} \left( X_0(a), X_0(b) \right) = (1/2)(-1)^bn^{-(a+b+1)}D_n^{(a+b)}(x-y) + (1/2)n^{-(a+b+1)}D_n^{(a+b)}(x+y). \]

To deal with the second term on the right hand side of (44), we see that \( d(x+y, 2\pi) > n^{-1/2} \), since \( x, y \in S_0 \), and therefore we have \( (1/2)n^{-(a+b+1)}D_n^{(a+b)}(x+y) = O(n^{-1/2}) \), where the bound is uniform over all \( x, y \in S_0 \). We can then apply Fact 37 to (44) to conclude
\[ \text{Cov} \left( X_0(a), X_0(b) \right) = (-1)^b/2 \int_{0}^{1} \theta^{a+b} \cos((a+b)((t-s)\theta)) d\theta + O(n^{-1/2}), \]
which is \( \text{Cov} \left( W(a)(nx), W(b)(ny) \right) \), by definition. \( \square \)

Proof of Lemma 30. We compare the matrices
\[ \Sigma := \text{Cov} \left( X_0(j)(x_i), Y_0(j)(x_i) \right)_{i \in [k], j \in [0,s]} \quad \Sigma_{\infty} := \text{Cov} \left( W(j)(nx_i), Z(j)(ny_i) \right)_{i \in [k], j \in [0,s]}. \]

Using the variational definition of the least eigenvalue and the semidefinite property of \( \Sigma_{\infty}, \Sigma \), we write
\[ |\lambda_{\min}(\Sigma) - \lambda_{\min}(\Sigma_{\infty})| = \left| \min_{v:|v|=1} v^T\Sigma v - \min_{v:|v|=1} v^T\Sigma_{\infty} v \right| \leq \min_{v:|v|=1} \left\{ v^T\Sigma v - v^T\Sigma_{\infty} v \right\}, \]
which, by Lemma 35, is at most
\[ \|\Sigma - \Sigma_{\infty}\|_{op} = o(1). \]

Since \( \lambda_{\min}(\Sigma_{\infty}) = \Omega_{r,s,\varepsilon}(1) \), by Lemma 31 (for \( n \) large compared to \( \varepsilon, r, s \)) we conclude that
\[ \lambda_{\min}(\Sigma) = \Omega_{r,s,\varepsilon}(1), \]
for \( n \) sufficiently large, compared to \( r, s, \varepsilon \). \( \square \)

7. The Determinant of the Covariance Matrix

In this section we supply a complementary result to Lemma 25 by proving a lower-bound for the denominator in the expression (15). Here we write
\[ \Sigma_k(x, y) := \text{Cov} \left( X(x_i), Y(y_i) \right)_{i \in [k]}, \]
where \( x = (x_1, \ldots, x_k) \) and \( y = (y_1, \ldots, y_k) \).
Lemma 38. For \( k \in \mathbb{N} \) there exists \( c_k > 0 \) and \( n(k) \in \mathbb{N} \), so that for all \( x = (x_1, \ldots, x_k) \in \mathbb{T}_0^k \) and \( y = (y_1, \ldots, y_k) \in \mathbb{T}_0^k \) we have

\[
\det \Sigma_k(x, y) \geq c_k n^{2k^2} \prod_{i<j} \min\{|x_j - x_i|, 1/n\}^2 \cdot \min\{|y_j - y_i|, 1/n\}^2
\]

for all \( n > n(k) \).

To prove this, we will cluster the points \( \{x_i, y_j\}_{i,j} \) into groups so that points are “near” to points in their own group and sufficiently “far” from points in the other groups. In the following subsection, we make a few preparations for working within these clusters of “near” points.

7.1. Near points and derivatives. Assume that \( \{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_k\} \) are clustered tightly around points \( z_1, \ldots, z_k \), which, themselves, are reasonably well separated. In this subsection we show that we can compare the covariance matrix of \( \Sigma_k(x, y) \) (where \( x, y \) represent these “clustered” set of points) with a covariance matrix of \( X^{(j)}, Y^{(j)} \) evaluated at the points \( z_1, \ldots, z_r \). This will then allow us to use Lemma 30, the main result of Section 6, to prove Lemma 38.

A key technical device here will be a mean-value theorem for, so called, divided differences. Given \( x_0 < x_1 < \cdots < x_k \) set \( x = (x_0, x_1, \ldots, x_k) \) and define the divided differences with respect to \( x \), for a sequence \( y_0, \ldots, y_k \), inductively by

\[
[y_i, \ldots, y_{i+j}]_x := \frac{[y_{i+1}, \ldots, y_{i+j}]_x - [y_i, \ldots, y_{i+j-1}]_x}{x_{i+j} - x_i}
\]

(45)

For a function \( f \), we further extend this definition, by defining

\[
f[x_0, \ldots, x_k]_x := [f(x_0), \ldots, f(x_k)]_x.
\]

The reason for this definition becomes apparent with the following version of the mean value theorem, which is often attributed to Schwarz.

Lemma 39. Let \( x = (x_0, \ldots, x_k) \in \mathbb{R}^{k+1} \) satisfy \( x_0 < x_1 < \cdots < x_k \) and let \( f \) be a smooth function. Then there exists \( \xi \in [\min_j x_j, \max_j x_j] \) so that

\[
f[x_0, \ldots, x_k]_x = \frac{f^{(k)}(\xi)}{k!}.
\]

For our application of Lemma 39 we will need to use some basic properties of the linear map \( \Delta_x : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1} \) defined by

\[
\Delta_x y = ([y_0]_x, [y_0, y_1]_x, \ldots, [y_0, y_1, \ldots, y_{k+1}]_x)
\]

for all \( y = (y_0, \ldots, y_{k+1}) \in \mathbb{R}^{k+1} \).

\(^4\)For more information on divided differences and a proof of Lemma 39 see the survey [7].
Lemma 40. Let \( x = (x_1, \ldots, x_k) \in \mathbb{R}^k \) where \( x_1 < \cdots < x_k \) and let \( \Delta_x \) be as above. Then \( \Delta_x \) is a linear map with
\[
\det(\Delta_x) = \prod_{1 \leq i < j \leq n} (x_j - x_i)^{-1}.
\]

**Proof.** The linearity of \( \Delta_x \) is clear from the definition. To calculate the determinant of \( \Delta_x \), we apply induction on the dimension \( k \). The basis step holds by definition. Now, let \( e_k = (0, \ldots, 0, 1) \in \mathbb{R}^k \) denote the standard unit vector and note that
\[
\Delta_x e_k = [y_1, y_2, \ldots, y_k] e_k.
\]
To calculate \([y_1, \ldots, y_k]_x\) we use that \([0, \ldots, 0]_x = 0\) to see
\[
[y_1, \ldots, y_k]_x = (x_1 - x_k)^{-1}([y_1, \ldots, y_{k-1}]_x - [y_2, \ldots, y_k]_x) = (x_k - x_1)^{-1}[y_2, \ldots, y_k]_x.
\]
Iterating this gives \([y_1, \ldots, y_k]_x = \prod_{i<k} (x_k - x_i)^{-1}\). Now note that if \( y \perp e_k \) then \( \Delta_x y = \Delta_x y' \) where \( x', y' \in \mathbb{R}^{k-1} \) are the vectors \( x, y \) with the \( k \)th coordinates removed. Thus
\[
\det(\Delta_x) = [y_1, y_2, \ldots, y_k]_x \det(\Delta_{x'}) = \prod_{i<j} (x_j - x_i)^{-1},
\]
by induction. \( \square \)

Lemma 41. For \( \varepsilon > 0 \), let \( x_0, \ldots, x_k \in \mathbb{R} \) satisfy \( |x_0 - x_j| \leq \varepsilon/n \) for all \( 0 \leq j \leq k \). Then
\[
\Var \left( \frac{X[x_0, \ldots, x_k]}{n^{k+1/2}} - X^{(k)}(x_0)/k! \right) = O(\varepsilon^2).
\]

**Proof.** Since
\[
E := X[x_0, \ldots, x_k]n^{-k+1/2} - X^{(k)}(x_0)/k!
\]
is a linear combination of values of \( X, X^{(k)} \), which are jointly Gaussian, \( E \) itself is a Gaussian random variable. Thus it is sufficient to show that \( \mathbb{E} |E| = O(\varepsilon) \). For this, we apply Lemma 39 to obtain
\[
X[x_0, \ldots, x_k] = \frac{X^{(k)}(\xi)}{k!}
\]
for some \( \xi \in [x_1 - \varepsilon n^{-1}, x_1 + \varepsilon n^{-1}] =: I \) and so applying the (standard) mean value theorem, we have
\[
X^{(k)}(\xi) - X^{(k)}(x_1) = (x_1 - \xi)X^{(k+1)}(\xi'),
\]
for some \( \xi' \in I \). We now want to bound
\[
\mathbb{E} |E| = \mathbb{E} \left| X[x_0, x_1, \ldots, x_k] - X^{(k)}(x_0)/k! \right| n^{-(k+1/2)},
\]
from above. Using \((46)\) along with \((47)\) tells us that \((48)\) is at most
\[
(\varepsilon n^{-1})n^{-k+1/2} \sup_{y \in I} \left| X^{(k+1)}(\xi) \right| = O(\varepsilon),
\]
where the last equality follows form Lemma 28.

We apply Lemma 41 to arrive at the main result of this subsection.

Lemma 42. For \(\delta > 0\), let \(\{x_{i,j}\}_{i \in [r], j \in [k]}, \{y_{i,j}\}_{i \in [r], j \in [k]}, \{z_i\}_{i=1}^r \subset \mathbb{T}\) be such that
\[
|z_i - x_{i,j}| < \delta/n \quad \text{and} \quad |z_i - y_{i,j}| < \delta/n,
\]
for all \(i, j\). Let \(\Sigma\) be the covariance matrix of the joint distribution
\[
\left( \frac{X[x_{i,0}]}{n^{1/2}}, \ldots, \frac{X[x_{i,0}, \ldots, x_{i,k}]}{n^{k+1/2}}, \frac{Y[y_{i,0}]}{n^{1/2}}, \ldots, \frac{Y[y_{i,0}, \ldots, y_{i,k}]}{n^{k+1/2}} \right)_{i=1}^r
\]
and let \(\Sigma'\) be the covariance matrix of the joint distribution
\[
\left( \frac{X^{(0)}(z_i)}{0!}, \ldots, \frac{X^{(k)}(z_i)}{k!}, \frac{Y^{(0)}(z_i)}{0!}, \ldots, \frac{Y^{(k)}(z_i)}{k!} \right)_{i=1}^r.
\]

Then
\[
||\Sigma - \Sigma'||_{op} = O_{r,k}(\delta).
\]

Proof. We prove this by showing that all entries in the matrix \(\Sigma - \Sigma_0\) are at most \(3\delta\) in absolute value. An entry in the matrix \(\Sigma - \Sigma_0\) is of the form
\[
\mathbb{E} \frac{X[x_{0,a}, \ldots, x_{i,a}]}{n^{i+1/2}} \frac{Y[y_{0,b}, \ldots, y_{j,b}]}{n^{j+1/2}} - \mathbb{E} \left( i!^{-1} X^{(i)}_0(z_a) \cdot j!^{-1} Y^{(j)}_0(z_b) \right)
\]
or like this with the occurrences of \(X\) replaced with occurrences of \(Y\) or vice versa. However these cases are similar and so we ignore them. Momentarily suppressing the first subscript, we may express the first term in \((49)\) as
\[
\mathbb{E} \frac{X[x_{i}, \ldots, x_{j}]}{n^{i+1/2}} \frac{Y[y_{0}, \ldots, y_{j}]}{n^{j+1/2}} = \mathbb{E} \left( X^{(i)}_0(z_a)i!^{-1} - E_1 \right) \left( Y^{(j)}_0(z_b)j!^{-1} - E_2 \right),
\]
where \(E_1\), is defined as \(n^{-(i+1)/2} X[x_{0,a}, \ldots, x_{i,a}] - (i!)^{-1} X^{(i)}_0(z_b)\) and \(E_2\) is defined symmetrically. We now expand this out and obtain four terms, the first of which is
\[
\text{Cov} \left( i!^{-1} X^{(i)}_0(z_a), j!^{-1} Y^{(j)}_0(z_b) \right)
\]
and is the corresponding term in the matrix \(\Sigma'\). So to finish we need to bound the three “error” terms
\[
-\mathbb{E} E_1 X^{(j)}_0(x_0) j!^{-1} - \mathbb{E} E_2 X^{(i)}_0(x_0) i!^{-1} + \mathbb{E} E_1 E_2.
\]
We see that each of these are individually at most \(\delta\) by applying Cauchy-Schwarz and then Lemma 41.
As a result, we see that $\Sigma_0 - \Sigma'$ is a $2r(k+1) \times 2r(k+1)$ matrix with all entries at most $3\delta$. To finish, we use that the operator norm of a matrix is at most its Frobenius norm. So if we set $A = \Sigma_0 - \Sigma$, we have $\|A\|_{op} \leq \|A\|_F = \left(\sum_{i,j} A_{i,j}^2\right)^{1/2} = O_{k,r}(\delta)$, as desired. □

7.2. Proof of Lemma 38. We now turn to prove our main lemma of this section, Lemma 38. At the heart of the proof is Lemma 44, which allows us to deal with points that are grouped together in groups which are far apart from each other. As is common, we use the notation $B_0(z) \subseteq \mathbb{R}$ to denote the set of points within distance $\delta$ of $z$ and we recall that

$$T_0 = \{x \in (0, \pi) : d(x, \pi\mathbb{Z}) > n^{-1/2}\}.$$  

We also need the following elementary fact.

**Fact 43.** If the random vector $x$ has covariance matrix $\Sigma$ then the random vector $Ax$ has covariance matrix $A\Sigma AT$.

**Lemma 44.** For $k, r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\delta = \delta(k, \varepsilon) \in (0, \varepsilon)$, $c_{k,\varepsilon} > 0$ and $n(k, \varepsilon)$ so that the following holds. Let $z_1, \ldots, z_r \in T_0$ and let $x_1, \ldots, x_k, y_1, \ldots, y_k \in T_0$ satisfy

$$\{x_1, \ldots, x_k, y_1, \ldots, y_k\} \subseteq \bigcup_{i=1}^r \bar{B}_{\delta/n}(z_i)$$

and $|z_i - z_j| > \varepsilon/n$. If we set $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ then

$$\det \Sigma_k(x, y) \geq c_{k,\varepsilon} n^{2k^2} \prod_{i<j} \left( \min\{|x_j - x_i|, 1/n\}^2 \cdot \min\{|y_j - y_i|, 1/n\}^2 \right),$$

for all $n \geq n(k, \varepsilon)$.

**Proof.** We let $\delta \in (0, \varepsilon/2)$, which we shall specify later. Set $\Sigma := \Sigma_k(x, y)$ and define

$$S_x(z_i) := \{j : x_j \in B_{\delta/n}(z_i)\}, \quad S_y(z_i) := \{j : y_j \in B_{\delta/n}(z_i)\}.$$  

Note that, by choice of $\delta$, these sets are disjoint.

We now define the linear operator $\Delta$ on $\mathbb{R}^{2k}$ that “differences” points $x_i, x_{i'}$, in the sense (45), with $i, i' \in S_x(z_j)$, for each part of the partition (and likewise for the $y_i$). For this, split up the coordinates of $\mathbb{R}^{2k}$,

$$\mathbb{R}^{2k} = \mathbb{R}^{S_x(z_1)} \times \cdots \times \mathbb{R}^{S_x(z_k)} \times \mathbb{R}^{S_y(z_1)} \times \cdots \times \mathbb{R}^{S_y(z_k)},$$

according to the partition $\{S_y(z_i), S_x(z_i)\}$, and write $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_k)$. For a point

$$(u_1, \ldots, u_k, v_1, \ldots, v_k) \in \mathbb{R}^{2k} = \mathbb{R}^{S_x(z_1)} \times \cdots \times \mathbb{R}^{S_x(z_k)} \times \mathbb{R}^{S_y(z_1)} \times \cdots \times \mathbb{R}^{S_y(z_k)}$$


we define $\Delta$ by setting
\[
\Delta(u_1, \ldots, u_k, v_1, \ldots, v_k) = (\Delta_{x_1} u_1, \ldots, \Delta_{x_k} u_k, \Delta_{y_1} v_1, \ldots, \Delta_{y_k} v_k).
\]

Of course,
\[
(50) \quad \det \Sigma = (\det \Delta)^{-2} \det(\Delta \Sigma \Delta^T)
\]
and, by Fact 43, we see that $\Delta \Sigma \Delta^T$ is the covariance matrix of the random variable $\Delta(x, y)$. After renaming the points
\[
S_x(z_i) = \{x_{i,0}, \ldots, x_{i,k(i)}\} \quad S_y(z_i) = \{y_{i,0}, \ldots, y_{i,k'(i)}\},
\]
for each $i \in [r]$, we may use the definition of $\Delta$ to write
\[
\Delta \Sigma \Delta^T = \text{Cov}(X[x_{i,0}, \ldots, x_{i,k(i)}], Y[y_{i,0}, \ldots, y_{i,0}, y_{i,k'(i)})_{i=1}^r.
\]
We now rescale rows and columns corresponding to the terms $X[x_{i,0}, \ldots, x_{t,k(i)}]$ of “depth” $t$ by $n^{-(t+1)/2}$ to obtain
\[
(51) \quad \det \Delta \Sigma \Delta^T = n^T \det \text{Cov} \left( \frac{X[x_{i,0}]}{n^{1/2}}, \ldots, \frac{X[x_{i,0}, \ldots, x_{i,k(i)}]}{n^{(k(i)+1)/2}}, \frac{Y[y_{i,0}]}{n^{1/2}}, \ldots, \frac{Y[y_{i,0}, \ldots, y_{i,k'(i)}]}{n^{k'(i)+1/2}} \right)_{i=1}^r,
\]
where we have set
\[
T := \sum_{i=1}^r (k(i) + 1)^2 + \sum_{i=1}^r (k'(i) + 1)^2.
\]
Now let $\Sigma_0$ be the rescaled matrix in (51). We compare this renormalized matrix $\Sigma_0$ with the covariance matrix of the (re-scaled) derivatives of $X, Y$ at $\{z_i\}$. That is, the matrix
\[
\Sigma' := \text{Cov} \left( \frac{X_0^{(0)}(z_i)}{0!}, \ldots, \frac{X_0^{(k(i))}(z_i)}{(k(i))!}, \frac{Y_0^{(0)}(z_i)}{0!}, \ldots, \frac{Y_0^{(k'(i))}(z_i)}{(k'(i))!} \right)_{i=1}^r.
\]
We apply Lemma 30 (the main result of Section 6) to learn
\[
(52) \quad \lambda_{\min}(\Sigma') = \Omega_{k,\epsilon}(1),
\]
for all $n > n(\epsilon, k)$. Now, using the variational definition of the least eigenvalue of a symmetric matrix, we have
\[
(53) \quad \lambda_{\min}(\Sigma_0) \geq \lambda_{\min}(\Sigma') - \max_{v, ||v||_2=1} v'(\Sigma_0 - \Sigma')v = \lambda_{\min}(\Sigma') + O_k(\delta),
\]
where we have used Lemma 42 to bound
\[
\max_{v, ||v||_2=1} v'(\Sigma_0 - \Sigma')v = ||\Sigma_0 - \Sigma'||_{op} = O_k(\delta).
\]
Using (52) and (53) and choosing $\delta > 0$ to be sufficiently small compared to $k$ and $\epsilon$, gives
\begin{equation}
\det(\Sigma_0) \geq (\lambda_{\min}(\Sigma'))^{2k} = \Omega_{k,\epsilon}(1),
\end{equation}
for sufficiently small $\delta$ and $n > n(k, \epsilon)$.

Putting the pieces together, we use (50) along with (51) to express
\[ \det \Sigma = n^T (\det \Sigma_0)(\det \Delta)^{-2}. \]

We then apply Lemma 40 and (54) to write, for $c_{k,\epsilon} > 0$,
\begin{equation}
\det \Sigma \geq c_{k,\epsilon} n^T \prod_{i=1}^{r} \left\{ \prod_{0\leq a<b\leq k(i)} (x_{i,b} - x_{i,a})^2 \prod_{0\leq a<b\leq k'(i)} (y_{i,b} - y_{i,a})^2 \right\}.
\end{equation}

So, finally, using the clear fact $1 \geq n^2 \min\{|x - x'|^2, n^{-2}\}$, we rewrite (55) as
\[ \det \Sigma \geq c_{k,\epsilon} n^{2k} \prod_{i<j} \min\{|x_i - x_j, n^{-1}\}^2 \cdot \min\{|y_i - y_j, n^{-1}\}^2, \]
for all $n > n(\epsilon, k)$, thus completing the proof of Lemma 44.

To prove Lemma 38, all that remains is to apply Lemma 44 at an appropriate “scale”. That is, we need an appropriate distinction between “near” and “far” points. The following will provide us with just this.

**Lemma 45.** Let $\{\delta_k\}_k \subseteq \mathbb{R}^+$ be a decreasing sequence and let $S \subseteq \mathbb{R}$ have $|S| = s$. Then there exists $R \subseteq S$ with $|R| = r$ so that every $z \in S$ has $d(z, R) \leq \delta_r$ and any distinct $x, y \in R$ have $|x - y| > \delta_{r-1}$.

**Proof.** Let $r$ be the largest integer for which there exists a set $R := \{x_1, \ldots, x_r\} \subseteq S$ so that $|x_i - x_j| \geq \delta_{r-1}$, for all $i \neq j$. For a contradiction, assume there exists a point $z \in S \setminus R$ for which $|z - x_i| \geq \delta_r$ for all $i \in [r]$. But then if we take $x_{r+1} = z$ the set $\{x_1, \ldots, x_{r+1}\}$ contradicts the maximality of $r$, as $|x_i - x_j| \geq \delta_r$, for all $i \neq j$. Thus we conclude that all points of $S \setminus R$ are within $\delta_r$ of a point in $R$.

We now prove Lemma 38 the main result of this section.

**Proof of Lemma 38.** We define a decreasing sequence $\{\delta_m\}_m \subseteq \mathbb{R}_{>0}$ inductively by first setting $\delta_1 = \delta(k, 1)$, where $\delta(\cdot, \cdot)$ is the constant that appears in Lemma 44. Then, assuming that $\delta_t$ has been chosen, choose $\delta_{t+1} = \delta(k, \delta_t)/2$. We now apply Lemma 45 to find a value of $t$ and a set of points $z_1, \ldots, z_t$ so that $d(z_i, z_j) > \delta_t/n$ for $i \neq j$ and all elements of $\{x_1, \ldots, x_t, y_1, \ldots, y_k\}$ have distance $\leq \delta_{t+1}/n$ to one of the values $\{z_i\}$. We may now apply Lemma 44 with $\epsilon = \delta_t$ and $\delta = \delta_{t+1}$.
to obtain
\[ \det \Sigma_k(x, y) \geq c_k n^{2k^2} \prod_{i<j} \left( \min\{x_j - x_i, 1/n\} \cdot \min\{y_j - y_i, 1/n\} \right), \]
for all large enough \( n > n(k) \). \qed

8. PROOF OF LEMMA 19

We now have everything we need to prove Lemma 19 which, as we have seen in Section 4, implies Theorem 1, our main theorem.

Proof of Lemma 19. Recalling the definition of \( p_k(x, y) \) from (15), and applying Lemma 25 and Lemma 38 to the numerator and denominator, respectively, we have

\[
(56) \quad p_k(x, y) = \frac{\alpha_k(x, y)}{(2\pi)^k |\Sigma|^{1/2}} \leq C_k n^{k^2+k} \prod_{i<j} \min\{|x_i - x_j|, n^{-1}\} \cdot \min\{|y_i - y_j|, n^{-1}\} \leq C_k n^{k^2+k} \cdot n^{-k(k-1)} = C_k n^{2k}
\]
as claimed. \qed

Notice that in the proof of Lemma 19, we actually obtain a sharper bound on \( p_k \), at (56). It is our suspicion that this bound is actually the correct one.

Conjecture 46. For all \( x, y \in \mathbb{T}_0 \),

\[
p_k(x, y) = \Theta_k \left( n^{k^2+k} \prod_{i<j} \min\{|x_i - x_j|, n^{-1}\} \cdot \min\{|y_i - y_j|, n^{-1}\} \right).
\]

A resolution to this conjecture would then determine probabilities of events like those considered in Lemma 3 up to constant factors.

APPENDIX A. DETAILS FROM SECTION 2

In this appendix we will tie up a few loose ends from Section 2 by proving Lemmas 11, 12, 13, and 14. Perhaps unsurprisingly, we employ yet another Kac-Rice formula, which is actually much simpler than those which we have been working with. We won’t need to dive too deeply into the behavior of these integrals here; it will just allow us to get upper bounds on the number of roots in a region \( U \) when we have an upper bound on \( f' \) in \( U \).

Lemma 47 (Complex Kac-Rice). Let \( V \subset \mathbb{C} \) be an open set and let \( U \subset \mathbb{C} \) be a compact set. Then

\[
\mathbb{E} \left[ \left| \{ z \in U : f(z) = 0, f'(z) \in V \} \right| \right] = \int_U \frac{\mathbb{E}[|f'(z)|^2 \cdot 1[f'(z) \in V] \cdot f(z) = 0]}{2\pi |\det \text{Cov}(f(z))|^{1/2}} \, dz,
\]
where the integral is with respect to 2-dimensional Lebesgue measure.
Lemma 47 is derived from a more general Kac-Rice formula in Appendix B.

A.1. Roots close to the circle. For the logical flow of this appendix, it actually makes sense to start by proving Lemma 13, which will allow us to ignore the special behavior that $f$ has near the real axis. For this, we study the covariance of $f$ close to the real axis.

Lemma 48. For $M \geq 0$, $x \in [0, 1]$ and $|\rho| \leq M/n^2$ we have
\[
\det \Sigma := \det \text{Cov}(f((1 + \rho)e^{ix})) = \Omega \left( \min \{ n^4x^2, n^2 \} \right).
\]

Proof. For $\gamma > 0$ to be determined later, we consider three regions for $x$:
\[
x \leq \gamma/n, \quad x \in [\gamma/n, n^{-2/3}], \quad x > n^{-2/3}.
\]
Starting with the region $x \leq \gamma/n$, fix $\delta > 0$ and note that
\[
\Sigma_{1,1} = \sum_{k=0}^{n} (1 + \rho)^{2k} \cos^2(kx) \geq (1 - |\rho|)^n n \left( \int_{0}^{1} \cos^2(nxt) \, dt \right) \geq n(1 - \delta),
\]
where this last inequality holds provided $n$ is sufficiently large and $\gamma$ is small compared to $\delta$.

Similarly, bound
\[
\Sigma_{2,2} = \sum_{k=0}^{n} (1 + \rho)^{2k} \sin^2(kx) \geq (1 - \delta) \frac{n^3x^2}{6} \quad \text{and} \quad \Sigma_{1,2} = \sum_{k=0}^{n} (1 + \rho)^{2k} \sin(kx) \cos(kx) \leq (1 + \delta) \frac{n^2x}{4}.
\]

Putting these calculations together tells us that for $x \leq \gamma/n$ we have
\[
\det(\Sigma) \geq n^4x^2 \left( 1 + o_{\delta \to 0}(1) \right).
\]
Thus for $\gamma$ sufficiently small and $x < \gamma/n$ the Lemma is proven.

Turning to the region $x \leq n^{-2/3}$, we use the trapezoidal rule to see
\[
\Sigma_{1,1} = \sum_{k=0}^{n} (1 + \rho)^{2k} \cos^2(kx) = n \left( \int_{0}^{1} \cos^2(nxt) \, dt \right) + O(1).
\]
Evaluating the integral, setting $y = nx$ and doing the same for the other two entries of $\Sigma$ shows
\[
\Sigma = n \left[\begin{array}{cc}
\cos(y)\sin(y) + y & \sin^2(y) \\
\frac{2y}{\sin^2(y)} & \frac{2y}{y - \cos(y)\sin(y)} \\
\end{array}\right] + O(1).
\]
Thus for $nx = y \in [\gamma, n^{1/3}]$,
\[
\det(\Sigma) \sim \frac{n^2}{4} \left( \frac{\cos(y)^2 + y^2 - 1}{y^2} \right) = \Omega(n^2).
\]
For the remaining region $x > n^{-2/3}$, bound $\Sigma_{1,1} = \Omega(n)$, $\Sigma_{2,2} = \Omega(n)$ and
\[
\Sigma_{1,2} = \frac{1}{2} \left( \frac{\cos(x)(1 - \cos(x(n + 1))^2)}{\sin(x)} - \sin(x(n + 1)) \cos(x(n + 1)) \right) = O(x^{-1}) = O(n^{2/3}).
\]
This shows $\det(\Sigma) = \Omega(n^2)$ in this regime. $\square$

**Proof of Lemma 13.** First note that $f$ has real coefficients and that the law of $f(-z)$ is equal to that of $f(z)$. Therefore it is sufficient to show that the set
\[
S := \{ z \in \mathcal{A}_n ([-M, M]): 0 \leq \arg(z) \leq n^{-\varepsilon} \}
\]
is zero free, with high probability.

Set $z_k = \exp(ix_k)$, where $x_k = k/n^2$ for $k \in \{0, 1, \ldots, n^{-2\varepsilon}\}$. We will first show that $|f|$ is at least $n^{-1/2} \log n$ at each of each of the points $z_k$. We will then show that this implies that $f$ is non-zero in the balls $B_{2M/n^2}(z_k)$, which collectively cover $S$. We attack this former point first.

For this we show
\[
\sum_{k=0}^{n^{2-\varepsilon}} \mathbb{P}(|f(z_k)| \leq n^{-1/2} \log n) = o(1).
\]
To see this, note
\[
\mathbb{P}(|f(z_k)| \leq n^{-1/2} \log n) \leq \mathbb{P}(|\text{Re}(f(z_k))|, |\text{Im}(f(z_k))| \leq n^{-1/2} \log n) = O\left(n^{-1}(\log n)^2 \det(\Sigma)^{-1/2}\right)
\]
and then apply Lemma 48 to obtain
\[
\mathbb{P}(|f(z_k)| \leq n^{-1/2} \log n) = O\left(n^{-2}(\log n)^2 \max\{n x_k^{-1}, 1\}\right) = O\left(n^{-2}(\log n)^2 \max\{n/k, 1\}\right).
\]
We also have, by direct computation,
\[
\mathbb{P}(|f(z_0)| \leq n^{-1/2} \log n) = O\left(n^{-1} \log n\right).
\]
So summing this along with (58) over $k \in [0, n^{2-\varepsilon}]$, yields (57).

We now turn to show that if $|f(z_k)| \geq n^{-1/2} \log n$ then $f$ is zero-free in $B_{2M/n^2}(z_k)$. For this, note that, by Fact 6, we may condition on the event $|f'(z)| \leq (4M)^{-1}n^{3/2} \log n$ for all $|z| \leq 1 + 2M/n^2$ at only an additive loss in $o(1)$ in our probability calculations.

For any $z \in B_{2M/n^2}(z_k)$ the mean-value theorem for complex functions implies that there are points $\xi_1$ and $\xi_2$ on the line segment connecting $z$ and $z_k$ so that
\[
f'(\xi_1) = \text{Re}\left(\frac{f(z) - f(z_k)}{z - z_k}\right) \quad f'(\xi_2) = \text{Im}\left(\frac{f(z) - f(z_k)}{z - z_k}\right).
\]
This implies that
\[
\left|\frac{f(z) - f(z_k)}{z - z_k}\right| \leq \sqrt{2}(4M)^{-1}n^{3/2} \log n
\]
and so
\[|f(z) - f(z_k)| \leq (2n)^{-1/2} \log n.\]
Thus, if \(|f(z_k)| \geq n^{-1/2} \log n\), \(f(z)\) is non-zero. This, along with (58), implies that \(f\) has no zeros in \(\bigcup_k B_{2M/n^2}(z_k) \supseteq S\) with high probability. \(\square\)

A.2. Lemmas 11, 12 and 14

Proof of Lemma 11. Call a zero \(\zeta \in Z(f)\) bad if \(||\zeta| - 1| \leq n^{-2}(\log n)^{1/4}\) and
\[|X'(\arg(\zeta))|, |Y'(\arg(\zeta))| \leq n^{3/2} / \log n.\]
Note that by Lemma 13, we need only to show there are no bad zeros \(\zeta\) with \(\arg(\zeta) \in T_0\), with high probability. We start by counting the number of zeros with a slightly different property and then relate these to bad zeros.

Say that a zero \(\zeta \in Z(f)\) is nearly bad if \(||\zeta| - 1| \leq n^{-2}(\log n)^{1/4}\) and
\[|\text{Re}(\zeta f'(\zeta))| \leq 2n^{3/2} / \log n\text{ or }|\text{Im}(\zeta f'(\zeta))| \leq 2n^{3/2} / \log n\]
and let \(b\) be the number of nearly bad zeros. Put
\[U := \{x + iy \in \mathbb{C} : |x| < 2n^{3/2} / \log n \text{ or } |y| < 2n^{3/2} / \log n\},\]
set
\[V = \{z : ||z| - 1| \leq n^{-2}(\log n)^{1/4}, |\text{arg}(z)| \geq n^{-1/2} \text{ and } |\text{arg}(-z)| \geq n^{-1/2}\}\]
and apply Lemma 17 to the function \(g(z) = zf(z)\) to get an integral form for the number of \(\zeta \in V\) with \(g(\zeta) = 0\) and \(g'(\zeta) \in U\). Since \(\zeta \neq 0\) and \(g'(z) = f(z) + zf'(z)\) this is the same as counting \(\zeta \in V\) with \(f(\zeta) = 0\) and \(\zeta f'(\zeta) \in U\). That is,
\[(59) \quad b = \mathbb{E} \left[|\{z \in V : f(z) = 0, zf'(z) \in U\}|\right] = \int_V \frac{\mathbb{E}[|g'(z)|^2 \cdot 1_{g'(z) \in U} | g(z) = 0]}{2\pi \cdot \text{det Cov}(g(z))^{1/2}} \, dz.\]
Since we are working away from the real axis, Lemma 36 and Fact 37 apply and tell us that the denominator in this integrand satisfies
\[|\text{det Cov}(g(z))|^{1/2} = |z| |\text{det Cov}(f(z))|^{1/2} = \Theta(n).\]
To bound the numerator, we apply Cauchy-Schwarz
\[\mathbb{E}[|g'(z)|^2 \cdot 1_{g'(z) \in U} | g(z) = 0] \leq (\mathbb{E}[|g'(z)|^4 | g(z) = 0])^{1/2} (\mathbb{P}(g'(z) \in U | g(z) = 0))^{1/2} = O(n^3(\log n)^{-1/2}).\]
Where we have used
\[\mathbb{E}[|g'(z)|^4 | g(z) = 0] \leq C' (\mathbb{E}[|g'(z)|^2 | g(z) = 0])^2 \leq (\mathbb{E}[|g'(z)|^2])^2 = O(n^6),\]
where the first and second inequalities follow from properties of Gaussian Random variables (Fact 7).

We have also used

\[
(P(g'(z) \in U \mid g(z) = 0))^{1/2} \leq C|U| \left(\text{Var}(|g'(z)| \mid g(z) = 0)\right)^{-1/2} < (\log n)^{-1},
\]

where the first inequality is again due to a property of Gaussian random variables, the second inequality holds due to (17) in Lemma 20.

Thus, using (59), along with the fact that \(|V| = O(n^{-2}(\log n)^{1/4})\), we see

\[
b = O(n^3(\log n)^{-1/2} \cdot n^{-1}) \int_V |dz| = o(1)
\]

and so the probability there is a nearly bad zero \(\zeta\) is \(o(1)\).

We now show that the above implies that there are no bad zeros \(\zeta\) with high probability. So suppose that \(\zeta = e^{i\theta} \rho \in A_f(I)\) is bad and in particular: \(f(\zeta) = 0\) and \(|X'(\theta)| < n^{3/2}/\log n\) (the case with \(X\) replaced with \(Y\) is similar). This implies, by mean value theorem, that there is a \(\rho_0 \in [0, n^{-2} \max I]\) so that

\[
|X'(\theta, \rho)| \leq |X'(\theta, 0)| + \rho_0|X''(\theta, 0)| \leq (1 + o(1))n^{3/2}/\log n,
\]

with high probability. Since \(X'(\theta, \rho) = \text{Re}(\zeta f'(\zeta))\) we see that \(\zeta\) is a zero of \(f\) that satisfies \(|\text{Re}(\zeta f'(\zeta))| \leq 2n^{3/2}/\log n\) and therefore is a nearly bad zero. Since the probability there exists a nearly bad zero \(\zeta\) is \(o(1)\), the probability there is a bad zero is \(o(1)\).

\[\square\]

**Proof of Lemma 12** Say that \((x_0, y_0) \in C_f(I)\) is bad if \(\min\{|X'(x)|, |Y'(y)|\} < 2n^{3/2}(\log n)^{-1}\). We calculate the expected number of bad pairs

\[
b := \mathbb{E} \left[ \left\{ (x, y) \in C_f(I) : \min\{|X'(x)|, |Y'(y)|\} < 2n^{3/2}(\log n)^{-1} \right\} \right].
\]

Now use that fact that \(|x - y| \leq n^{-2}(\log n)^4\) and \(d(\{x, y\}, \pi Z) \geq n^{-1/2}\) for all \((x, y) \in C_f(I)\) and argue as in Lemma 21 to bound

\[
b \leq Cn^2 \int_{x \in \mathbb{T}_0} \int_{|r| \leq \eta} \mathbb{E} \left[ |Z_1| \cdot |Z_2| \mathbf{1}\left\{ \frac{n^2 \tau Z_1 Z_2}{Z_1^2 + Z_2^2} \in I \right\} \mathbf{1}\left\{ \min\{|Z_1|, |Z_2|\} \leq 2(\log n)^{-1} \right\} \right] dx dr
\]

where \(\eta = (\log n)^4\) and \((Z_1, Z_2)\) is a standard 2-dimensional Gaussian. Anticipating an application of Fubini’s theorem, bound

\[
\int_{|r| \leq \eta} \mathbf{1}\left\{ \frac{n^2 \tau Z_1 Z_2}{Z_1^2 + Z_2^2} \in I \right\} dr \leq \int_{-\infty}^{\infty} \mathbf{1}\left\{ \frac{n^2 \tau Z_1 Z_2}{Z_1^2 + Z_2^2} \in I \right\} dr = \frac{|I|}{n^2 |Z_1| \cdot |Z_2|} (Z_1^2 + Z_2^2).
\]

Combining with the bound on \(b\) then gives

\[
b \leq Cn^2 \int_{x \in \mathbb{T}_0} \mathbb{E} \left[ |Z_1| \cdot |Z_2| \frac{|I|}{n^2 |Z_1| \cdot |Z_2|} (Z_1^2 + Z_2^2) \mathbf{1}\left\{ \min\{|Z_1|, |Z_2|\} \leq 2(\log n)^{-1} \right\} \right] dx
\]
\[ \leq C\pi|I|\mathbb{E}[(Z_1^2 + Z_2^2)1\{\min\{|Z_1|, |Z_2|\} \leq 2(\log n)^{-1}\}] = o(1). \]

And so, for each pair \((x_0, y_0) \in \mathcal{C}_f(I)\), we have \(|X'(x_0)|, |Y'(y_0)| > 2n^{3/2}/\log n\), with high probability. Arguing as in the end of the proof of Lemma 11 with the mean value theorem shows \(|X'(y_0)|, |Y'(x_0)| > n^{3/2}/\log n\) with high probability. \(\square\)

**Proof of Lemma 14** We first apply Lemma 13 to see that there are no roots \(\zeta\) with \(\arg(\zeta) \in \mathbb{T} \setminus T_0\), with high probability. Thus we may restrict ourselves to the region \(\arg(z) \in T_0\). Say that \(\zeta_1, \zeta_2\) is a bad pair if

\[ \zeta_1, \zeta_2 \in \{z \in \mathbb{C} : ||z| - 1| \leq n^{-1} \text{ and } \arg(z) \in T_0\} =: V \]

are distinct roots of \(f\) with \(|\zeta_1 - \zeta_2| \leq n^{-2}(\log n)^{10}\).

To show that there are no bad pairs, we work with a slightly different notion: say that a \(\zeta\) is rotten if \(\zeta \in V\) is a zero of \(f\) and \(|f'(z)| \leq n^{1/2}(\log n)^{13}\). Let us first observe that there is no rotten root with high probability. Indeed, applying Lemma 47 and noting \(\det \text{Cov}(f(z)) = \Omega(n^2)\) for \(\arg(z) \in T_0\), we express the expected number of rotten points as

\[ \int_V \mathbb{E}[|f'(z)|^2 \cdot 1(\{|f'(z)| \leq n^{1/2}(\log n)^{13}\}) |f(z) = 0|] \frac{d\zeta}{2\pi(\det \text{Cov}(f(z)))^{1/2}} = O(n(\log n)^{26} \cdot n^{-1} \cdot |V|), \]

which is \(o(1)\) and thus there are no rotten points with high probability.

We now see that a bad pair implies a rotten point; suppose \(\zeta_1, \zeta_2\) is a bad pair. By the mean-value theorem for complex functions, there exists points \(u\) and \(v\) on the line segment connecting \(\zeta_1\) and \(\zeta_2\) so that

\[ \text{Re} f'(u) = \text{Re} \left( \frac{f(\zeta_1) - f(\zeta_2)}{\zeta_1 - \zeta_2} \right) = 0 \]
\[ \text{Im} f'(v) = \text{Im} \left( \frac{f(\zeta_1) - f(\zeta_2)}{\zeta_1 - \zeta_2} \right) = 0. \]

Recall that \(\max_{|z| \leq 1 + n^{-1}} |f^{(2)}(z)| = O(n^{5/2} \log n)\) with high probability and so we may apply mean value theorem again to see that for all \(z\) with \(|z - \zeta_1| < n^{-2}(\log n)^{11}\) we have

\[ |f'(z) - f'(\zeta_1)| = O(n^{-2}(\log n)^{11} \cdot n^{5/2} \log n) = O(n^{1/2}(\log n)^{12}), \]

with high probability. This shows that \(|f'(\zeta_1)| = O(n^{1/2} \log^{12} n)\) and thus \(\zeta_1\) is rotten. This shows that if \(\zeta_1, \zeta_2\) is a bad pair then one of \(\{\zeta_1, \zeta_2\}\) is a rotten point, up to a set of measure \(o(1)\). Thus there are no bad pairs with probability \(o(1)\). \(\square\)
Proof of Lemma 15. Suppose there exist distinct \( x_1, x_2 \in \mathbb{T}_0 \) with \( |x_1 - x_2| \leq n^2(\log n)^{10} \) and \( g(x_1) = g(x_2) = 0 \). Then there exists \( \xi \) with \( g'(\xi) = 0 \) and \( |\xi - x_1| \leq n^{-2}(\log n)^{10} \) by the mean-value theorem. Let \( E \) be the event that \( \max_{x \in [0, 2\pi]} |g''(x)| \leq Cn^{5/2}(\log n)^{1/2} \) and note that by the Salem-Zygmund and Bernstein inequalities, \( \mathbb{P}(E) = 1 - o(1) \). Now, conditioned on \( E \), the mean-value theorem implies that \( |g'(x_1)| \leq n^{1/2}(\log n)^{11} \). By the Kac-Rice formula (Lemma 50)

\[
\mathbb{E} \left\{ \left\{ x \in \mathbb{T}_0 : g(x) = 0, |g'(x)| \leq n^{1/2}(\log n)^{11} \right\} \right\} = \int_{\mathbb{T}_0} \frac{\mathbb{E}[|g'(x)|1(|g'(x)| \leq n^{1/2}(\log n)^{11} | g(x) = 0]}}{(2\pi \text{Var}(g(x)))^{1/2}} dx.
\]

We then bound the right hand side by

\[
n^{1/2}(\log n)^{11} \int_{\mathbb{T}_0} \frac{\mathbb{P}[|g'(x)| \leq n^{1/2}(\log n)^{11} | g(x) = 0]}{(2\pi \text{Var}(g(x)))^{1/2}} dx = O \left( n^{1/2}(\log n)^{11} \cdot (\log n)^{11} n^{-1} \cdot n^{-1/2} \right),
\]

which tends to 0 as \( n \to \infty \), as desired.

\[\square\]

Appendix B. Proof of Lemma 17. Our Kac-Rice-type formula

At first, formulas such as (14) can appear a bit unwieldy and their utility opaque. So we take a moment, before diving into the proof of Lemma 17 to say a few words about where these integrals come from. For this, we take a simplified version of the equation in Lemma 17

(60) \[
\mathbb{E}[\{x \in [0, 2\pi] : X(x) = 0\}] = \int_{[0, 2\pi]} \frac{\mathbb{E}[|X'(x)| | X(x) = 0]}{(2\pi)^{1/2} \text{Var}(X(x))^{1/2}} dx.
\]

Let \( p(x) \) denote the integrand on the right hand side of (60) and let us fix \( x \in [0, 2\pi] \) and take \( \varepsilon > 0 \) to be extremely small. We will now observe that the probability that there is a zero in the interval \((x - \varepsilon, x + \varepsilon)\) is roughly \( 2\varepsilon \cdot p(x) \). Of course, if we can show this to be true, linearity of expectation immediately gives us (60).

Now, when \( \varepsilon \) is sufficiently small, the function \( X(\cdot) \) is approximately linear on the interval \((x - \varepsilon, x + \varepsilon)\) with slope \( \approx X'(x) \). It is therefore easy to see when there is a zero in this interval: there is a zero in \((x - \varepsilon, x + \varepsilon)\) roughly when \( X(x - \varepsilon) \in (-2\varepsilon X'(x), 0) \), (assuming \( X'(x) > 0 \) without loss of generality). Using the properties of Gaussian random variables, one can calculate

\[
\mathbb{P}(X(x - \varepsilon) \in (-2\varepsilon X'(x), 0) | X'(x)) \approx \frac{2\varepsilon |X'(x)|}{\sqrt{2\pi \text{Var}(X(x))}}.
\]

We now want to average over all possible values of \( X'(x) \) to “eliminate” the conditional expectation. Here we note that \( \varepsilon \) can be taken small (compared to everything) and therefore we can restrict to considering the case when \( X(x) \) is small. Thus we condition on \( |X(x)| \leq \eta \), for some small \( \eta \) and take expectations on both sides. Finally, dividing this result by by \( 2\varepsilon \) gives exactly the Kac-Rice density, once we send \( \eta \) and \( \varepsilon \) to zero.
Of course, this is only a sketch and turning this into a genuine proof requires a bit of work. Instead of doing this here we derive our results from a much more general result on Gaussian processes.

B.1. Deriving Lemmas 17 and 47. Rather than prove our Kac-Rice formulas from scratch, we derive them from a general multivariate formula of Azaïs-Wschebor. For a smooth function $F : \mathbb{R}^n \to \mathbb{R}^n$, we write $F'$ for the Jacobian of $F$, given as an $n \times n$ matrix.

**Theorem 49** (Theorem 6.4 [4]). Let $U \subset \mathbb{R}^d$ be an open set and let $Z : U \to \mathbb{R}^d$ be Gaussian. Let $Z(t)$ be such that

1. the function $t \mapsto Z(t)$ is almost-surely of class $C^1$;
2. For each $t \in U$, the matrix $\text{Cov}(Z(t))$ is positive definite;
3. $\mathbb{P}(\exists t \in U, Z(t) = 0, \det(Z'(t)) = 0) = 0$.

Assume further that one has another random function $W : U \to \mathbb{R}^n$ satisfying:

1. The function $t \mapsto W(t)$ is continuous almost-surely.
2. For each fixed $t \in U$ the random process $(Z(s), W(t))_{s \in U}$ is Gaussian.

Then for every continuous bounded function $g : U \times \mathbb{R}^n \to \mathbb{R}$ and for every compact $I \subset U$ we have

$$
\mathbb{E} \left( \sum_{t \in I, Z(t) = 0} g(t, W(t)) \right) = \int_I \mathbb{E} \left( \frac{|\det(Z'(t))| g(t, W(t))}{(2\pi)^n/2 \det \text{Cov}(Z(t))^{1/2}} \right) dt.
$$

**Proof of Lemma 47.** Identify $\mathbb{C}$ with $\mathbb{R}^2$ and so we may view $f$ as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Seeking to apply Theorem 49 set $W(t) = f'(t)$ where $f'$ is the complex derivative of $f$ viewed as a function on $\mathbb{R}^2$. Let $g_\alpha$ be a sequence of continuous functions that increases monotonically to the indicator of $V$: $g_\alpha \uparrow 1_V$. For all $\alpha$ we have

$$
\mathbb{E} \left( \sum_{t \in U, f(t) = 0} g_\alpha(f'(t)) \right) = \int_U \mathbb{E} \left( \frac{|\det(J(t))| g_\alpha(f'(t))}{2\pi \det \text{Cov}(f(t))^{1/2}} \right) dt
$$

where we write $J(t)$ to be $2 \times 2$ Jacobian of $f$, where we view $f$ as a function in $\mathbb{R}^2$. Writing $f = u + iv$, the Cauchy-Riemann equations imply

$$
\det(J(t)) = u_x(t)v_y(t) - u_y(t)v_x(t) = (u_x(t))^2 + (v_x(t))^2 = |f'(t)|^2.
$$

Taking $\alpha \to 0$ sends $g_\alpha \uparrow 1_V$ and so applying dominated convergence theorem completes the proof. $\square$
Lemma 17 will follow from the Kac-Rice form below. Recall that for \( k \in \mathbb{N} \) and \( V \subset \mathbb{R}^{2k} \) we have
\[
(61) \quad p_k(x, y, V) := \frac{\mathbb{E} \left[ |X'(x_1) \cdots X'(x_k)Y'(y_1) \cdots Y'(y_k)| \varphi_V \mid X(x_i) = 0, Y(y_j) = 0 \text{ for all } i, j \right]}{(2\pi)^k |\Sigma|^{1/2}},
\]
where \( \Sigma \) is the \( 2k \times 2k \) covariance matrix \( \text{Cov}(X(x_i), Y(y_i)) \) and \( \varphi_V \) is the indicator for the event \((X'(x_1), \ldots, X'(x_k), Y'(y_1), \ldots, Y'(y_k)) \in V \). For any compact set \( T \subset \mathbb{T}^{2k} \), let \( N_T \) be the number of pairwise distinct tuples of pairs \(((x_1, y_1), \ldots, (x_k, y_k)) \in T \) where \( X(x_i) = 0, Y(y_j) = 0 \) for all \( i, j \) and \( \varphi_V = 1 \).

**Lemma 50.** We have
\[
\mathbb{E}[N_T] = \int_T p(x, y, V) \, dx \, dy.
\]

**Proof.** We adapt an argument from [1, Theorem 11.5.1]. Fix \( \delta > 0 \) and let
\[
Z(t_1, \ldots, t_{2k}) = (X(t_1), \ldots, X(t_k), Y(t_{k+1}), \ldots, Y(t_{2k})).
\]
Note that \( Z' \), the Jacobian of \( Z \), is a diagonal \( 2k \times 2k \) matrix with diagonal
\[
(X'(t_1), \ldots, X'(t_k), Y'(t_{k+1}), \ldots, Y'(t_{2k})).
\]
For a set \( T \subset [0, 2\pi]^{2k} \), define
\[
T_\delta := \{ t \in T : |t_i - t_j| \geq \delta \text{ for all } \{i, j\} \in [k]^{(2)} \cup [k + 1, 2k]^{(2)} \}.
\]
As in the proof of Lemma 47, let \( \{g_\alpha\}_\alpha \) be a sequence of continuous functions so that \( g_\alpha \uparrow 1_V \) as \( \alpha \to 0 \). By Theorem 49 we have
\[
(62) \quad \mathbb{E} \left( \sum_{t \in T_\delta, Z(t) = 0} g_\alpha(Z'(t)) \right) = \int_{T_\delta} \frac{\mathbb{E} \left( |\det(Z'(t))| g_\alpha(Z'(t)) \mid Z(t) = 0 \right)}{(2\pi)^k \det \text{Cov}(Z(t))^{1/2}} \, dt.
\]
Taking \( \alpha \to 0 \) implies \( g_\alpha \uparrow 1_V \). Monotone convergence theorem allows us to swap \( g_\alpha \) for \( 1_V \) on both sides of (62). Now, sending \( \delta \to 0 \), and again using monotone convergence theorem, allows us to replace \( T_\delta \) with \( T \) on both sides of (62). Noting that \( \det(Z'(t)) = \prod_{i=1}^{k} X(t_i)Y(t_{i+k}) \), completes the proof. \( \square \)

**Proof of Lemma 17**. Take \( T = \mathbb{T}^{2k} \) and \( V = C_f(U) \) in Lemma 50. Set \( R := \mu_f(U) = |C_f(U)| \) and note that \((R)_k \) counts the number of pairwise distinct tuples of pairs \(((x_1, y_1), \ldots, (x_k, y_k)) \in C_f(U) \). So
\[
\mathbb{E}(R)_k = N_T = \int_{T_0} p_k(x, y, U) \, dx \, dy,
\]
as desired. \( \square \)

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5If \( X \) is a set, we let \( X^{(k)} \) to be the set of \( k \) element subsets of \( X \).
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