EMBEDDINGS OF NON-SIMPLY-CONNECTED 4-MANIFOLDS IN 7-SPACE. I. CLASSIFICATION MODULO KNOTS

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Abstract. We work in the smooth category. Let N be a closed connected orientable 4-manifold with torsion free $H_1$, where $H_q := H_q(N;\mathbb{Z})$. The main result is a complete readily calculable classification of embeddings $N \to \mathbb{R}^7$, up to equivalence generated by isotopies and embedded connected sums with embeddings $S^4 \to \mathbb{R}^7$. Such a classification was earlier known only for $H_1 = 0$ by Boéchat-Haefliger-Hudson 1970. Our classification involves the Boéchat-Haefliger invariant $\kappa(f) \in H_2$, Seifert bilinear form $\lambda(f) : H_3 \times H_3 \to \mathbb{Z}$ and $\beta$-invariant assuming values in the quotient of $H_3$ defined by values of $\kappa(f)$ and $\lambda(f)$. In particular, for $N = S^4 \times S^3$ we define geometrically a 1–1 correspondence between the set of equivalence classes of embeddings and an explicitly defined quotient of $\mathbb{Z} \oplus \mathbb{Z}$.

Our proof is based on development of Kreck modified surgery approach, involving some simpler reformulations, and also uses parametric connected sum.

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1. **INTRODUCTION AND MAIN RESULTS**

In §1.1 we provide a broader context for the results in this paper. The main results are described in §1.1 and formally stated in §1.2. Our ideas are described in §1.3 and formally realized in §§2-4. Except for the definitions in bold, ‘some notation’ in bold and Lemma 1.5, the material of §1.1 and §1.3 is not formally used in the rest of this paper.

1.1. **The Knotting Problem.** Let us start with a citation of [MAE].

‘Three important classical problems in topology are the following, cf. [Ze93, p. 3].

*The Manifold Problem:* Classify $n$-manifolds.

*The Embedding Problem:* Find the least dimension $m$ such that a given manifold admits an embedding into $m$-dimensional Euclidean space $\mathbb{R}^m$.

*The Knotting Problem:* Classify embeddings of a given manifold into another given manifold up to isotopy.

The Embedding and Knotting Problems have played an outstanding role in the development of topology. Various methods for the investigation of these problems were created by such classical figures as G. Alexander, H. Hopf, L. S. Pontryagin, R. Thom, H. Whitney, M. Atiyah, F. Hirzebruch, R. Penrose, J. H. C. Whitehead, C. Zeeman, W. Browder, J. Levine, S. P. Novikov, A. Haefliger, M. Hirsch, J. F. P. Hudson, M. Irwin and others.’

The Knotting Problem is related to other branches of mathematics, most importantly, to algebraic topology. For recent surveys see [RS99, Sk08, MAE]; whenever possible we refer to these surveys not to original papers.

This paper is on the Knotting Problem for 4-manifolds. We consider smooth manifolds, embeddings and isotopies. By a classification we mean a complete, readily calculable classification.\(^1\) For $m \geq n + 2$ the classifications of embeddings of compact $n$-manifolds into $S^m$ and into $\mathbb{R}^m$ are the same, see details in [MAE, Remark 1.3]. In this subsection $P$ is a closed connected $n$-manifold.

**Definition of $E^m(P)$.** Let $E^m(P)$ be the set of isotopy classes of embeddings $f: P \to S^m$, where $S^m \subset \mathbb{R}^{m+1}$ is the unit $m$-sphere.

The Knotting Problem is more accessible for

$$2m \geq 3n + 4,$$

\(^1\)For a discussion of the terms ‘smooth’ and ‘readily calculable’ see Remark 2.19, [MAE, Remark 1.2].
where there are some classical classifications of embeddings, which are surveyed in [Sk08, §2, §3], [MAE]. When \( n = 4 \), classification was obtained for \( m \geq 9 \) by Whitney–Wu, for \( m = 8 \) by Haefliger–Hirsch, and for \( N = S^4 \) and \( m = 7 \) by Haefliger, giving
\[
|E^m(P)| = 1 \quad \text{for} \quad m \geq 9, \quad E^8(P) = H_1(P; \mathbb{Z}_2), \quad E^7(S^4) \cong \mathbb{Z}_{12}.
\]

Here the equality sign between sets denotes the existence of a ‘geometrically defined’ bijection, and the isomorphism is a group isomorphism for the group structure defined below. Any orientable 4-manifold embeds into \( \mathbb{R}^7 \) (this follows easily from results of Donaldson and Boechat-Haefliger, see detailed references in [MAM]).

The Knotting Problem is much harder for \( 2m < 3n+4 \). If \( P \) is a closed manifold that is not a disjoint union of homology spheres, then until recently no classification of embeddings \( P \to \mathbb{R}^m \) was known. This is in spite of the existence of many interesting approaches including methods of Haefliger-Weber, Browder-Wall and Goodwillie-Weiss [Sk08, Sk08', Sk10, CS11].

In this paper we develop the approach which uses the modified surgery of M. Kreck, see [CRS04], see [MAE, Remark 1.2].

Recent classifications for \( 2m < 3n+4 \) concern
- embeddings of 3- and 4-dimensional manifolds [Sk08’, Sk10, CS11],
- embeddings of \( d \)-connected \( n \)-manifolds for \( 2m \geq 3n + 3 - d \) [Sk02], and
- embeddings \( S^p \times S^d \to S^m \) [CRS07, CRS12, CFS14, Sk15].

These results are based on three productive approaches. One of them involves almost embeddings and the \( \beta \)-invariant of [Sk02, Sk07, Sk14, CRS07, CRS12] (which, though related to, is different from the \( \beta \)-invariant in this paper), another is based on relations between different sets of embeddings [Sk11, Sk15]. However, these and other approaches are not sufficient to classify the embeddings of \( m \)-manifolds \( N \) into \( S^7 \), even in the case of \( N = S^3 \times S^3 \).

In this paper we develop the approach which uses the modified surgery of M. Kreck, see §1.3 and [Sk08’, Sk10, CS11].

This paper is the first and most important one in the program for the classification of (smooth and piecewise linear) embeddings into \( S^m \) of non-simply connected 4-manifolds (under the ‘torsion free’ condition and up to an indeterminacy in certain cases). See [II, III]; some parts of those results easily follow from this paper, but we state those parts in [II, III] not here. In order to explain what is done in the present paper, let us recall some more definitions.

Denote by \([f]\) the isotopy class of an embedding \( f : P \to S^m \).

**Definitions of the group** \( E^m(S^n) \) for \( m \geq n + 3 \), of its action on \( E^m(P) \) and of \( g\# \), \( E^m_\#(P) \). Represent elements of \( E^m(P) \) and of \( E^m(S^n) \) by embeddings \( f : P \to S^m \) and \( g : S^n \to S^m \) whose images are contained in disjoint balls. Join the images of \( f, g \) by an arc whose interior misses the images. Let \([f]\#[g]\) be the isotopy class of the embedded connected sum of \( f \) and \( g \) along this arc, cf. [Ha66, Theorem 1.7], [Av16, §1]. The isotopy class of the embedded connected sum depends only on the isotopy classes \([f]\) and \([g]\) [MAE, §5]. Hence we can define the operation
\[
\#: E^m(P) \times E^m(S^n) \to E^m(P) \quad \text{by} \quad ([f],[g]) \mapsto [f]\#[g].
\]

For \( P = S^n \) and \( m \geq n + 3 \) this defines a group structure on \( E^m(S^n) \) [Ha66]. Clearly \( \# \) is an action of \( E^m(S^n) \) on the set \( E^m(P) \). We define
\[
E^m_\#(P) := E^m(P)/E^m(S^n).
\]

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2For more information and references see [MAM]. For classifications of embeddings of \( n \)-manifolds into \( \mathbb{R}^{2n-1} \) when \( n \neq 4 \) see [Ya84, Sa99, Sk10’, To10].
to be the quotient of this action, i.e. the set of ‘embeddings modulo knots’. Let $q_\# : E^m(P) \to E^m_\#(P)$ be the quotient map.

A simpler version of the knotting problem is to classify the set $E^m_\#(P)$. For $n = 4$ Boéchat and Haefliger classified $E^7_\#(P)$ when $H_1(P; \mathbb{Z}) = 0$ [BH70]. The action of the knots was investigated in [Sk10] and determined when $H_1(P; \mathbb{Z}) = 0$ in [CS11], which also classified $E^7(P)$ in this case. In general, even the simpler version is hard: If $2m < 3n + 4$ and $P$ is a closed manifold that is not (homologically) $[(n-2)/2]$-connected, then until recently no classification of $E^m_\#(P)$ was known.

The main result of this paper is a classification of $E^7_\#(P)$ when $P$ is a closed connected orientable 4-manifold with torsion free $H_1(P; \mathbb{Z})$; see Theorems 1.1 and 1.3 below. This requires finding a complete set of invariants and constructing embeddings realizing particular values of these invariants. Although the invariants come from modified surgery theory (see §1.3), we work with them using basic algebraic topology. Thus we make this paper and its relation to other work (independent of surgery) accessible to non-specialists: e.g. we emphasize relations between our $\lambda$-invariant and [Sa99, To10]. Lemma 1.4 and §2.2, §2.3 describe the invariants we use. In §1.3 we explain how the invariants appear in our approach to classification and give an overview of the proof of their completeness. The beginning of §1.2 gives explicit construction of embeddings $S^1 \times S^3 \to S^7$. For our general $P$, we use a parametric connected sum operation on embeddings which is described in §2.4. We create new embeddings $P \to S^7$ from a fixed embedding $f_0 : P \to S^7$ using parametric connected sum with embeddings $S^1 \times S^3 \to S^7$. So embeddings of $S^1 \times S^3$ play a key role in the classification of embeddings for general $P$.

1.2. Main results. We first define a family of embeddings $\tau_\alpha : S^1 \times S^3 \to S^7$ and a corresponding map

$$\tau : \mathbb{Z}^2 \to E^7(S^1 \times S^3).$$

Let $V_{4,2}$ denote the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^4$. Take a smooth map $\alpha : S^3 \to V_{4,2}$. Regarding $V_{4,2} \subset (\mathbb{R}^4)^2$, write $\alpha(x) = (\alpha_1(x), \alpha_2(x))$. Define the adjunction map $\mathbb{R}^2 \times S^3 \to \mathbb{R}^4$ by $((s,t), x) \mapsto \alpha_1(x)s + \alpha_2(x)t$. (Regarding $V_{4,2} \subset (\mathbb{R}^4)^2$, this map is obtained from $\alpha$ by the exponential law $Z^X \times Y = (Z^X)^Y$.) Denote by $\tau_\alpha : S^1 \times S^3 \to S^7$ the restriction of the adjunction map. We define the embedding $\tau_\alpha$ to be the composition

$$S^1 \times S^3 \xrightarrow{\pi \times \text{pr}} S^3 \times S^3 \xrightarrow{i} S^7,$$

where $i(x,y) := (y,x)/\sqrt{2}$ and $\text{pr}_2(x,y) = y$.

We define the map $\tau$ by $\tau(l,b) := [\tau_\alpha]$, where $\alpha : S^3 \to V_{4,2}$ represents $(l, b) \in \pi_3(V_{4,2})$ (for the standard identification $\pi_3(V_{4,2}) = \mathbb{Z}^2$ described in §2.1).

We define $\tau_\# := q_\# \tau$.

Theorem 1.1. The map

$$\tau_\# : \mathbb{Z}^2 \to E^7_\#(S^1 \times S^3)$$

is a surjection such that

$$\tau_\#(l,b) = \tau_\#(l',b') \iff (l = l' \text{ and } b \equiv b' \text{ mod } 2l).$$

Remark 1.2. (a) There is a map $\tau : \pi_q(V_{m-q,p+1}) \to E^m(S^p \times S^q)$ which is defined analogously to above. For $m \geq 2p+q+3$ the sets $E^m(S^p \times S^q)$ and $E^m_\#(S^p \times S^q)$ admit a group structure such that $\tau$ and $q_\#$ are homomorphisms [Sk15]. For $p \leq q$ and $2m \geq 2p+3q+4$ (conjecturally for $2m \geq p+3q+4$) the map $\tau_\# := q_\# \tau$ is an isomorphism. Theorem 1.1 shows that the case of embeddings $S^1 \times S^3 \to S^7$ is different: there are no group structures on
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$E^7(S^1 \times S^3)$ or on $E^7_\#(S^1 \times S^3)$ such that $\tau$ or $\tau_\#$ is a homomorphism (because by Theorem 1.1 $\tau^{-1}_\#(0,0)$ is infinite while $\tau^{-1}_\#(1,0)$ is finite; analogous argument works for $\tau$ since $E^7(S^4)$ is finite).

(b) The isotopy classes $\tau(1,0)$ and $\tau(0,1)$ are represented by embeddings

$$S^1 \times S^3 \xrightarrow{\text{for } k} S^3 \times S^3 \xrightarrow{1} S^7,$$

where the maps $T^k : S^1 \times S^3 \to S^3$ are defined as follows:

- $T^1(s, y) := sy$, where $S^3$ is identified with the set of unit length quaternions and $S^1 \subset S^3$ with the set of unit length complex numbers;
- $T^2(e^{i\theta}, y) := \eta(y) \cos \theta + \sin \theta$, where $\eta : S^3 \to S^2$ is the Hopf map, and $S^2$ here is identified with the 2-sphere formed by unit length quaternions of the form $ai + bj + ck$.

For other constructions see [MAM, Examples of knotted tori].

Before stating our main results for the general case, we establish some conventions, notation and definitions.

**Convention on coefficients.** Unless otherwise stated, we omit $\mathbb{Z}$-coefficients from the notation of (co)homology groups. We identify the coefficient group $\mathbb{Z}_d$ with $H_0(X; \mathbb{Z}_d)$, the zero-dimensional homology group of a connected oriented manifold $X$.

**Notation for $N$, $f$, characteristic classes and intersections in manifolds.** Throughout this paper $N$ is a closed connected oriented 4-manifold and $f : N \to S^7$ an embedding. Let $H_q := H_q(N)$. We denote the Poincaré dual of a characteristic class by adding a superscript ‘∗’, so for example $w_2^\ast(N) \in H_2(N; \mathbb{Z}_2)$ is the Poincaré dual of the second Stiefel-Whitney class. The homology intersection product in an oriented $n$-manifold $M$ is denoted by

$$\cap_M : H_i(M) \times H_j(M) \to H_{n-i-j}(M).$$

The well-known definition of such product is recalled in [MAI]. For the intersection powers we omit subscripts indicating the manifold $M$, so, for example, $x^2$ denotes $x \cap_M x$. Let $\rho_n$ be the reduction modulo $n$. The intersection $x \cap_M y$ of a $\mathbb{Z}$-homology class $x$ and a $\mathbb{Z}_n$-homology class $y$ is defined as the $\mathbb{Z}_n$-homology class $\rho_n x \cap_M y$. Let $\sigma(N)$ be the signature of the intersection form $H_2 \times H_2 \to \mathbb{Z}$.

If $H_1 = 0$, then the map

$$\kappa_\# : E^7_\#(N) \to H^2_\text{DIFF} := \{ u \in H_2 \mid \rho_2 u = w_2^\ast(N), \ u^2 = \sigma(N) \} \subset H_2$$

(which is the Bočchat-Haefliger invariant defined below) is 1–1. This statement is easily deduced from known results in Remark 2.20.e. Our second main result is a generalization of this statement to non-simply-connected 4-manifolds.

**Definition of div, $B(H_3)$, 7 and a symmetric pair.** For an element $u$ of a free abelian group denote by $\text{div } u$ the divisibility of $u$, i.e. $\text{div } 0 = 0$ and $\text{div } u$ is the largest integer which divides $u$ for $u \neq 0$. For an element $u$ of an abelian group $G$ denote by $\text{div } u$ the divisibility of $[u] \in G / \text{Tors}(G)$.

Denote by $B(H_3)$ the space of bilinear forms $H_3 \times H_3 \to \mathbb{Z}$. For $l \in B(H_3)$ denote by $\eta : H_3 \to H_1$ the adjoint homomorphism uniquely defined by the property $l(x, y) = x \cap_N \eta y$. A pair $(u, l) \in H_2 \times B(H_3)$ is called symmetric if

$$l(y, x) = l(x, y) + u \cap_N x \cap_N y \quad \text{for all } \quad x, y \in H_3.$$

The maps

$$\kappa : E^7(N) \to H^2_\text{DIFF}, \quad \lambda : E^7(N) \to B(H_3) \quad \text{and}$$
a statement involving \( \kappa \) because of the additivity (Lemmas 2.3 and 2.9 below). But not for \( \kappa \) of Theorem 1.3.

Let \( \kappa \), we write \( \kappa \) required for Theorem 1.3 below are well-defined by the product of \( \kappa \) of Theorem 1.3 below are defined in §2.2 and §2.3. Then the maps \( \kappa \#: E^7_\#(N) \to H^2_{\text{DIFF}} \), \( \lambda_\#: E^7_\#(N) \to B(H_3) \) and \( \beta_{u,l,\#} : (\kappa \# \times \lambda_\#)^{-1}(u,l) \to C_{u,l} \) of Theorem 1.3 below are well-defined by \( \kappa = \kappa \# q_\#, \lambda = \lambda \# q_\# \) and \( \beta_{u,l} = \beta_{u,l,\#}(q_\# \times q_\#) \) because of the additivity (Lemmas 2.3 and 2.9 below).

In order to avoid double statements of similar properties, we use the following convention: a statement involving \( \kappa \) holds for both \( \kappa \) and \( \kappa \# \). If a statement holds for \( \kappa \# \) but not for \( \kappa \), we write \( \kappa \# \) in the formulation. In this paper there are no statements which hold for \( \kappa \) but not for \( \kappa \# \). Analogous remark holds for \( \lambda \) vs \( \lambda_\# \), \( \beta_{u,l} \) vs \( \beta_{u,l,\#} \) etc.

**Theorem 1.3.** Let \( N \) be a closed connected orientable 4-manifold with torsion free \( H_1 \). Then the product

\[ \kappa \# \times \lambda_\#: E^7_\#(N) \to H^2_{\text{DIFF}} \times B(H_3) \]

has non-empty image consisting of all symmetric pairs, and for every \( (u,l) \in \text{im}(\kappa \# \times \lambda_\#) \) each map \( \beta_{u,l,\#} \) is 1–1 (see the remark immediately below).

We call geometrically defined maps invariants. In particular, the maps \( \lambda \) and \( \kappa \) are invariants.

**Remark on relative invariants.** The map \( \beta_{u,l} \) is a relative invariant. By this we mean that for \( [f_0], [f_1] \in (\lambda \times \kappa)^{-1}(u,l) \) there is an invariant \( ([f_0], [f_1]) \mapsto \beta(f_0, f_1) \) (defined in §2.3) and that \( \beta_{u,l}(f) := \beta(f, f') \) for a fixed choice of \( [f'] \in (\lambda \times \kappa)^{-1}(u,l) \). We suppress the choice of \( [f'] \) from the notation.

The Seifert bilinear form \( \lambda(f) : H_3 \times H_3 \to \mathbb{Z} \) (defined in §2.2) measures the linking of 3-cycles in \( N \) under \( f \). For \( N = S^1 \times S^3 \) identify \( B(H_3) \) with \( \mathbb{Z} \).

**Lemma 1.4** (Calculation for \( \lambda \); proved in §2.2). (a) For an embedding \( f : S^1 \times S^3 \to S^7 \) we have \( \lambda(f) = \text{lk}_{S^7}(f|_{(1,0) \times S^3}, f|_{(-1,0) \times S^3}) \in \mathbb{Z} \).

(b) \( \lambda(\tau(l,b)) = l \).

(c) We have \( \lambda(f)(x, y) = \text{lk}_{S^7}(f|_X, f|_Y) \) if classes \( x \) and \( y \) are represented by disjoint closed oriented 3-submanifolds (or integer 3-cycles) \( X \) and \( Y \).

See Remark 2.20.

1.3. **An approach to the Knotting Problem.** The proofs of our main results are based on the ideas we explain below. These ideas are useful in a wider range of dimensions [Sk08'] and for problems other than classification of embeddings [Kr99]. In this paper we do not assume the reader is familiar with surgery. Hence we describe the application of modified surgery in non-specialist terms and make parenthetical remarks for specialists.

**Some notation.** Take the standard orientation on \( \mathbb{R}^m \). For an oriented manifold with boundary we use the orientation on the boundary whose completion by ‘the first vector pointing outside’ gives the orientation on the manifold. So an orientation of \( S^{m-1} = \partial D^m \) is defined. Denote by

- \( N \) a closed connected oriented 4-manifold and \( f : N \to S^7 \) an embedding;
- \( C = C_f \) the closure of the complement in \( S^7 \) to a sufficiently small tubular neighborhood of \( f(N) \); the orientation on \( C \) is inherited from the orientation of \( S^7 \);
• \( \nu = \nu_f : \partial C \to N \) the sphere subbundle of the normal vector bundle of \( f \): the total space of \( \nu \) is identified with \( \partial C \).

In this paper a bundle isomorphism is an oriented vector bundle isomorphism identical on the base, or the restriction to the sphere bundle of such. In this and other notation we sometimes omit the subscript \( f \) on the base, or the restriction to the sphere bundle of such. In this and other notation we sometimes omit the subscript \( f \). We shall also change the subscript ‘\( f \)’ to ‘\( k \)’.

**Lemma 1.5.** For a closed connected 4-manifold \( N \) two embeddings \( f_0, f_1 : N \to S^7 \) are isotopic if and only if there is an orientation preserving diffeomorphism \( C_0 \to C_1 \) whose restriction to the boundary \( \partial C_0 \to \partial C_1 \) is a bundle isomorphism.

Lemma 1.5 is well-known to the experts (for a proof see, e.g., [Sk10, Lemma 1.3]) and holds in more general situations.

**Remark 1.6.** We shall not only decide if there is a diffeomorphism \( C_0 \to C_1 \) as in Lemma 1.5 but we also prove a general ‘relative diffeomorphism criterion’ for certain 7-manifolds with boundary. This is the Almost Diffeomorphism Theorem 4.5. It generalizes [CS11, Almost Diffeomorphism Theorem 2.8, Diffeomorphism Theorem 4.7]. It is a new non-trivial analogue of [KS91, Theorem 3.1] and of [Kr99, Theorem 6] for 7-manifolds \( M \) with non-empty boundary and infinite \( H_4(M) \).

Lemma 1.5 reduces the classification of embeddings to a two-step classification problem for their complements. Firstly, we classify the complements relative to fixed identifications of their boundaries, and secondly we determine the action of the bundle automorphisms on the relative diffeomorphism classes of the complements. This is the starting point of both the classical and modified surgery approaches.

To find out if there exists a diffeomorphism \( C_0 \to C_1 \) as in Lemma 1.5 using classical surgery (see [Wa70]), we would first need decide if the complements \( C_0 \) and \( C_1 \) have the same homotopy type. If they do, then we take homotopy equivalence \( h : C_0 \to C_1 \) and apply surgery relative to the boundary to the Poincaré pairs \((C_1, h(\partial C_0))\) and \((C_1, \partial C_1)\).

In this paper to determine if there is a diffeomorphism \( C_0 \to C_1 \) as in Lemma 1.5 we use modified surgery [Kr99]; cf. [CS11, Remark 2.2] and the text after it. For this we fix for \( k = 0, 1 \)

- the spin structures on \( C_k \) which they inherit from \( S^7 \) and also
- the Seifert classes which are algebraic analogues of Seifert surfaces; the Seifert classes are the relative homology classes \( A_k[N] \in H_5(C_k, \partial C_k) \cong \mathbb{Z} \), which are the images of the fundamental class of \( N \) under homology Alexander duality (defined in §3.1).

(This data on \( C_k \) defines a normal \( 2 \)-smoothing of \( C_k \) [Kr99, §2], i.e. a normal 3-equivalence \( C_k \to BSpin \times \mathbb{C}P^\infty \). We use a particular case of the modified surgery approach which corresponds to spin surgery over the homotopy 2-type of the complement.)

The modified surgery approach to the embedding problem requires that we find a bundle isomorphism \( \varphi : \partial C_0 \to \partial C_1 \) preserving both the spin structures and the homology classes \( \partial A_k[N] \in H_5(\partial C_k) \). We prove that there is always a bundle isomorphism \( \varphi : \partial C_0 \to \partial C_1 \) preserving spin structure (Lemma 3.6, cf. [CS11, Lemma 2.4]).

The first obstruction we encounter to the existence of a diffeomorphism as in Lemma 1.5 is the difference \( \varphi_* \partial A_0[N] - \partial A_1[N] \in H_3(\partial C_1) \). The analysis of this obstruction leads to the definition of \( \kappa \)-invariant (see §2.2 and Remark 2.21)

\[ \kappa : E^7(N) \to H_2. \]

Assume further that \( \kappa(f_0) = \kappa(f_1) \). We prove that \( \varphi_* \partial A_0[N] = \partial A_1[N] \) for every bundle isomorphism \( \varphi : \partial C_0 \to \partial C_1 \) (Lemma 3.5.a for \( q = 4 \), [CS11, Agreement Lemma 2.5]). We then identify the spin boundaries of \((C_0, A_0[N])\) and \((C_1, A_1[N])\) via a bundle isomorphism
φ which preserves the spin structures. For any such identification there is a spin bordism (W, z) between (C₀, A₀[N]) and (C₁, A₁[N]) relative to the boundaries (because the complete obstruction to the existence of such a bordism assumes values in $\Omega_2^{Spin}(\mathbb{C}P^\infty) = 0$ [KS91, Lemma 6.1]). It remains to determine whether we can replace the bordism (W, z) by an h-cobordism. This problem is addressed in [Kr99, Theorem 3], where a complete algebraic obstruction is defined. Analysis of the obstruction for the bordism (W, z) to have the homology of an h-cobordism ‘outside $H_4(W)$’ leads to the definition of λ-invariant (see §2.2 and Remark 2.21)

$$\lambda: E^7(N) \to B(H_3).$$

Assume further that $\lambda(f₀) = \lambda(f₁)$. From the surgery point of view, the β-invariant (see definition in §2.3) arises as the obstruction for the bordism (W, z) to have the homology of an h-cobordism ‘in the summand of $H_4(W)$ coming from $H_4(\partial W)$’ (i.e. in the singular part of the intersection form on $H_4(W)$). This invariant assumes values in a quotient of $H_1$ defined by $\kappa(f₀)$ and $\lambda(f₀)$.

Assume further that $\beta(f₀) = \beta(f₁)$. We may now assume that the bordism (W, z) ‘has the homology of an h-cobordism’ away from the unimodular part of $H_4(W)$. Extending arguments from [CS11], we prove that we can modify $f₀$ by connected sum with a knot $g: S^4 \to S^7$ so that for some corresponding

- spin bundle isomorphism $\varphi' : \partial C_{f₀#g} \to \partial C₁$,
- identification of the spin boundaries of the pairs $(C_{f₀#g}, A_{f₀#g}[N])$ and $(C₁, A₁[N])$,
- spin null bordism ($W', z'$) between the above pairs, relative to the boundaries,
- the pair ($W', z'$) is bordant to an h-cobordism. Then by the h-cobordism theorem [Mi65] and Lemma 1.5, $f₀#g$ and $f₁$ are isotopic.

The above discussion outlines the proof that the $\kappa$-, $\lambda$- and $\beta$-invariants combine to give a complete systems of invariants for embeddings modulo knots. This is stated in the MK Isotopy Classification Theorem 2.8 and the behaviour of these invariants under connected sum with knots is described in the Additivity Lemmas 2.3, 2.9.

Plan of the paper. We introduce further notation in §2.1 and §3.1. In §2 we present the important constructions and lemmas used in the proof of our main results. The lemmas from §2 are proven in §3 and §4. The subsection titles in §3 indicate the most important lemmas proven in that subsection. A reader who wants to check a particular lemma from §2 does not need to read all of §3 and §4.

2. Definitions of the invariants and proofs modulo lemmas

2.1. Main notation. Recall that some notation was introduced in §1.

Throughout this paper ‘(sub)manifold’ is shorthand for ‘compact oriented (sub)manifold, possibly with non-empty boundary’.

Some identifications.

Identify $\pi_n(S^n)$ and $\mathbb{Z}$ by the degree isomorphism.

Identify $S^2$ and $\mathbb{C}P^1$. Represent $S^3 = \{(z₁, z₂) \in \mathbb{C}^2 : |z₁|^2 + |z₂|^2 = 1\}$. The Hopf map $\eta : S^3 \to S^2$ is defined by $\eta(z₁, z₂) = [z₁ : z₂]$. Identify $\pi_3(S^2)$ and $\mathbb{Z}$ by the Hopf isomorphism (that sends the homotopy class of $\eta$ to 1).

Identify $\mathbb{R}^4$ and the algebra $\mathbb{H}$ of the quaternions. Identify $\pi_3(V_{4,2})$ and $\pi_3(S^3) \oplus \pi_3(S^2) = \mathbb{Z}^2$ by the standard isomorphism, which is defined using the projection $V_{4,2} \to S^3$ given by $(x, y) \mapsto x$ and the section $S^3 \to V_{4,2}$ given by $x \mapsto (x, xi)$. Identify $\pi_3(SO_3)$ and $\pi_3(S^2) = \mathbb{Z}$ by the map induced by the action of $SO_3$ on $S^2$. 
General notation.
Denote by
- \( N \) a closed connected orientable 4-manifold with torsion free \( H_1 \);
- \( f, f_0, f_1 : N \to S^7 \) embeddings;
- \( \equiv \) a congruence modulo \( n \);
- \( \text{pr}_k \) the projection of a Cartesian product onto the \( k \)-th factor;
- \( \text{id} X \) the identity map of the set \( X \);
- \( 1_m := (1,0,\ldots,0) \in S^m \);
- \( \text{Cl} X \) the closure of a subset \( X \) in the ambient space, which is clear from the context;
- \( N_0 := \text{Cl}(N - B^4) \), where \( B^4 \) is an embedded closed 4-ball in \( N \).

For every \( q \leq m \) identify the space \( \mathbb{R}^q \) with the subspace of \( \mathbb{R}^m \) given by the equations \( x_{q+1} = x_{q+2} = \cdots = x_m = 0 \). Analogously identify \( D^q, S^{q-1} \) with the corresponding subspaces of \( D^m, S^{m-1} \).

Define \( \mathbb{R}_+, \mathbb{R}_- \subset \mathbb{R}^m \) and \( D_+, D_- \subset S^m \) by the equations \( x_1 \geq 0 \) and \( x_1 \leq 0 \), respectively. Then
\[
S^m = D_+^m \bigcup_{\partial D_+^m = \partial D_-^m} D_-^m \quad \text{and} \quad \partial D_+^m = \partial D_-^m = D_+^m \cap D_-^m = 0 \times S^{m-1} \neq S^{m-1}.
\]

We denote the union of oriented manifolds in the same way as set-theoretic union. So both formulas \( S^4 = D_+^4 \cup (-D_-^4) \) and \( S^4 = D_+^4 \cup D_-^4 \) are correct, the sign “\( \cup \)” means union of oriented manifolds in the first formula and union of manifolds in the second one.

Homological notation.
Denote by \([x]\) the homology class of \( x \) or the equivalence class of \( x \) which is an element of a quotient group.

We denote the maps induced in homology by the same letters as the inducing maps. Thus if \( g : X \to Y \) is a map of spaces, \( g : H_*(X) \to H_*(Y) \) denotes the induced map on homology.

Homomorphisms between homology groups with \( \mathbb{Z}_d \)-coefficients are denoted in the same way as those for \( \mathbb{Z} \)-coefficients. So the coefficients are to be understood from the context. When this could lead to confusion, we specify coefficients by indicating the domain and the range of the homomorphism, e.g. \( i : H_3(C_6; \mathbb{Z}_d) \to H_3(M; \mathbb{Z}_d) \).

We denote by \( i_{A,X}, j_{A,X}, \partial_{A,X} \) or shortly by \( i_A, j_A, \partial_A \) or shortly by \( i, j, \partial \), the homomorphisms from the exact sequence of the pair \((X,A)\). If \( A = C_k \) or \( A = C_f \), then we shorten the subscript \( C_k \) or \( C_f \) to just \( k \) or just \( f \), respectively. Denote by \( \text{ex} : H_q(X,A) \to H_q(X - B, A - B) \) the excision isomorphism, where \( B \) is a subset of \( A \).

For a \( p \)-manifold \( P \) denote \( H_q(P, \partial) := H_q(P, \partial P) \).

Let \( P \) and \( Q \) be \( p \)- and \( q \)-manifolds. Denote by
\[
PD : H^n(P) \to H_{p-n}(P, \partial) \quad \text{and} \quad \text{PD} : H^n(P, \partial) \to H_{p-n}(P),
\]
the Poincaré duality isomorphisms. We sometimes identify homology and cohomology groups by Poincaré duality. We choose to work mostly with homology classes, since this has technical advantages for our arguments, see [CS11, Remark 2.3].

For a map \( \xi : P \to Q \) denote the ‘preimage’ homomorphism by
\[
\xi^! := \text{PD} \circ \xi \circ \text{PD}^{-1} : H_n(Q, \partial) \to H_{p-q+n}(P, \partial),
\]
where \( \xi \) is the homomorphism induced in cohomology.

We now consider simplicial cycles or cycles represented by maps of manifolds, the intersection of transverse cycles and linking number of disjoint cycles [MAI].
For set-theoretic intersection we write $X \cap Y$. (This notation is also used for restriction, see §3.1.) For the algebraic intersection of chains or integer cycles or oriented manifolds in an ambient manifold $M$ we write $X \cap_M Y$. Recall that $X \cap_M Y = (-1)^{\text{codim}X \text{codim}Y} Y \cap_M X$, and that if $X, Y$ are cycles, then $X \cap_M Y$ depends only on the homology classes represented by $X$ and $Y$.

Let $A$ and $B$ be integer $a$- and $b$-cycles in $\mathbb{R}^m$ having disjoint supports with $a+b+1 = m$. Define the linking number of $A$ and $B$ by $\text{lk}(A, B) := A \cap_{\mathbb{R}^m} B$, where $\beta$ is a $(b+1)$-cycle in $\mathbb{R}^m$ with $\partial \beta = B$. It is easy to check that $\text{lk}(A, B) = \alpha \cap_{\mathbb{R}^m} B$, where $\alpha$ is an $(a+1)$-cycle in $\mathbb{R}^m$ with $\partial \alpha = A$. Recall that $\text{lk}(A, B) = (-1)^{(m-a)(m-b)} \text{lk}(B, A)$.

2.2. Definitions of the $\kappa$ and $\lambda$-invariants. Let $\zeta : N_0 \to \nu^{-1}N_0$ be a ‘partial’ section of $\nu : \partial C \to N$. Consider the following diagram:

$$
\mathbb{Z} \cong H_4(N_0, \partial) \xrightarrow{\zeta} H_4(\nu^{-1}N_0, \partial) \xrightarrow{\text{ex}} H_4(\partial C, \nu^{-1}B^4) \xrightarrow{j_{\partial C}} H_4(\partial C) \xrightarrow{i_c} H_4(C).
$$

Here $j_{\partial C}$ and $\text{ex}$ are isomorphisms. The composition $N_0 \xrightarrow{\zeta} \nu^{-1}N_0 \xrightarrow{\text{ex}} \partial C$ of $\zeta$ and the inclusion is called a weakly unlinked section if $i_c j_{\partial C}^{-1} \text{ex}^{-1} \zeta = 0 \in H_4(C)$.

We remark that

- a section $\zeta : N_0 \to \nu^{-1}N_0$ exists because the Euler class of $\nu$ is zero, since vector bundle associated to $\nu$ is 3-dimensional, and $N_0$ retracts to a 3-polyhedron;
- any section $\zeta : N_0 \to \nu^{-1}N_0$ is weakly unlinked for $N = S^1 \times S^3$ because there is an isomorphism $H_4(S^7 - f(S^1 \times S^3)) \cong H_2(S^1 \times S^3) = 0$. Cf. Lemma 3.3.a.

**Lemma 2.1.** A weakly unlinked section exists and is unique up to vertical homotopy over the 2-skeleton of any triangulation of $N$.

**Proof.** This holds by [BH70, Proposition 1.3] because by [Sk10, Remark 2.4 and footnote 14] our definition of a weakly unlinked section is equivalent to the original definition [BH70]. Cf. proof of Lemma 3.3.b and [Sk08’, the Unlinked Section Lemma (a)].

**Definition of the Boéchat-Haefliger invariant** $\kappa : E^7(N) \to H_2$. Take a weakly unlinked section $\xi : N_0 \to \partial C$. Recall that $H_2$ is torsion-free. By Poincaré duality $\cap_N : H_2 \times H_2 \to \mathbb{Z}$ is unimodular. Hence $\kappa(f) \in H_2$ can be defined by the equation

$$
\kappa(f) \cap_N [X] = \text{lk}_{S^7}(fN, \xi X),
$$

for any 2-cycle $X \subset N_0$.

This is well-defined, i.e. independent of the choice of $\xi$, by Lemma 3.2.$\kappa'$. This definition is equivalent to those of [BH70, Sk10, CS11] by Lemma 3.2.$\kappa'e$. Clearly, the map $\kappa : E^7(N) \to H_2$ is well-defined by $\kappa([f]) := \kappa(f)$.

**Definition of the Seifert form** $\lambda : E^7(N) \to B(H_3)$. Represent classes $x, y \in H_3$ by closed oriented 3-submanifolds (or integer 3-cycles) $X, Y \subset N_0$. Take a weakly unlinked section $\xi : N_0 \to \partial C$. Define

$$
\lambda(f)(x, y) := \text{lk}_{S^7}(fX, \xi Y) \in \mathbb{Z}.
$$

This is well-defined; i.e. independent of the choice of $\xi$, by Lemma 3.2.$\lambda'$. Clearly, the pairing $\lambda(f) : H_3 \times H_3 \to \mathbb{Z}$ is indeed a bilinear form. Clearly, the map $\lambda : E^7(N) \to B(H_3)$ is well-defined by $\lambda([f]) := \lambda(f)$. Cf. [Sa99, To10].

---

3Those definitions do not require the assumption that $H_1$ is torsion free. In those papers the invariant was denoted by $w_f$ [BH70], or $BH(f)$ [Sk10], or $\kappa(f)$ [CS11], instead of $\kappa(f)$.
Proof of Lemma 1.4. Part (c) follows because $\xi Y$ and $fY$ are homologous in $S^7 - fX$. Part (a) follows by (c). Part (b) follows by (a).

We give equivalent definitions of $\kappa$ and $\lambda$ in Lemma 3.2.

**Lemma 2.2** ($\kappa$-symmetry; proved in §3.2). We have $\lambda(f)(y, x) = \lambda(f)(x, y) - \kappa(f)\cap_N x\cap_N y$.

Cf. [Sa99, Lemma 2.2], [To10, Theorem 1.5(2) and Lemmas 1.6 and 2.10].

**Lemma 2.3** (Additivity of $\kappa$). For every pair of embeddings $g : S^4 \to S^7$ and $f : N \to S^7$

$$\kappa(f \# g) = \kappa(f) \quad \text{and} \quad \lambda(f \# g) = \lambda(f).$$

Proof. We may assume that $g(S^4) \cap C_f = \emptyset$ and $\nu_f = \nu_{f \# g}$ over $N_0$. Then additivity for $\lambda$ and $\kappa$ follows because a weakly unlinked section for $f$ is also a weakly unlinked section for $f \# g$.

2.3. **Definition of the $\beta$-invariant and the map $\beta_{u,1}$.** Take a small oriented disk $D^3_f \subset \mathbb{R}^7$ whose intersection with $f(N)$ consists of exactly one point of sign $+1$ and such that $\partial D^3_f \subset \partial C_f$. Define the **meridian of $f$** by

$$S^2_f := [\partial D^3_f] \in H_2(C_f).$$

Then $S^2_f$ is a generator of $H_2(C_f)$ (by homology Alexander duality, see §3.1).

Recall that $f_0, f_1 : N \to S^7$ are embeddings. For a bundle isomorphism $\varphi : \partial C_0 \to \partial C_1$ define the **closed oriented 7-manifold**

$$M = M_\varphi := C_0 \cup_\varphi (-C_1).$$

A **joint Seifert class for a bundle isomorphism** $\varphi : \partial C_0 \to \partial C_1$ is a class

$$Y \in H_5(M_\varphi) \quad \text{such that} \quad Y \cap_{M_\varphi} [\partial D^3_f] = 1.$$ We shall omit the phrase ‘for a bundle isomorphism $\varphi$’ if its choice is clear from the context.

**Lemma 2.4** (proved in §3.4). If $\kappa(f_0) = \kappa(f_1)$ and $\varphi : \partial C_0 \to \partial C_1$ is a bundle isomorphism, then there is a joint Seifert class $Y \in H_5(M_\varphi)$.

We call a bundle isomorphism $\varphi : \partial C_0 \to \partial C_1$ a **$\pi$-isomorphism** if $M_\varphi$ is parallelizable.

**Lemma 2.5** (proved in §3.6). If $\kappa(f_0) = \kappa(f_1)$ and $\lambda(f_0) = \lambda(f_1)$, then there is a $\pi$-isomorphism $\varphi : \partial C_0 \to \partial C_1$. A $\pi$-isomorphism is unique (up to vertical homotopy through linear isomorphisms) over $N_0$.

**Definitions of $\beta(f_0, f_1)$.** For $u \in H_2$ and $l \in B(H_3)$ recall that $\tilde{l} : H_3 \to H_1$ is the adjoint of $l$ and that $\rho_d : H_1 \to H_1(N; \mathbb{Z}_d)$ is reduction modulo $d$. Assume that $\kappa(f_0) = \kappa(f_1)$ and that $\lambda(f_0) = \lambda(f_1)$. Denote $d := \text{div}(\kappa(f_0))$. By Lemmas 2.4 and 2.5 there is a joint Seifert class $Y \in H_5(M_\varphi)$ and a $\pi$-isomorphism $\varphi : \partial C_0 \to \partial C_1$. Define

$$\beta(f_0, f_1) := [(i_{\partial C_0, M_\varphi} \nu_l)^{-1} \rho_d Y^2] \in C_{\kappa(f_0), \lambda(f_0)}$$

using the composition $H_1(N; \mathbb{Z}_d) \xrightarrow{i_{\partial C_0, M_\varphi}} H_3(\partial C_0; \mathbb{Z}_d) \xrightarrow{i_{\partial C_0, M_\varphi}} H_3(M_\varphi; \mathbb{Z}_d)$.

**Lemma 2.6** (proved in §3.7). The class $\beta(f_0, f_1)$ is well-defined; i.e.

- for every joint Seifert class $Y$ and $\pi$-isomorphism $\varphi$ there is a unique element $b_{\varphi,Y} \in H_1(N; \mathbb{Z}_d)$ such that $i_{\partial C_0, M_\varphi} \nu_l b_{\varphi,Y} = \rho_d Y^2 \in H_3(M_\varphi; \mathbb{Z}_d)$,
- $[b_{\varphi,Y}] \in C_{\kappa(f_0), \lambda(f_0)}$ is independent of the choice of joint Seifert class $Y$ and $\pi$-isomorphism $\varphi$. 


Lemma 2.7 (Calculation of $\beta$; proved in §3.8). (a) $\beta(\tau(0,0), \tau(0,b)) = b[S^1 \times 1_3] \in H_1(S^1 \times S^3)$.
(b) $\beta(\tau(l,b), \tau(l,b)) = \rho_{2l}(b-b')[S^1 \times 1_3] \in H_1(S^1 \times S^3; \mathbb{Z}_{2l})$ (cf. Remark before Lemma 3.15).

Theorem 2.8 (Isotopy Classification Modulo Knots; proved in §4.2). If we have $\lambda(f_0) = \lambda(f_1)$, $\kappa(f_0) = \kappa(f_1)$ and $\beta(f_0, f_1) = 0$, then there is an embedding $g : S^4 \to S^7$ such that $f_0$ is isotopic to $f_1 \# g$.

Lemma 2.9 (Additivity of $\beta$; proved in §3.7). For every pair of embeddings $g : S^4 \to S^7$ and $f : N \to S^7$ we have $\beta(f \# g, f) = 0$.

Lemma 2.10 (Transitivity of $\beta$; proved in §3.7). For every triple of embeddings $f_0, f_1, f_2 : N \to S^7$ having the same values of $\kappa$- and $\lambda$-invariants we have $\beta(f_2, f_0) = \beta(f_2, f_1) + \beta(f_1, f_0)$.

Definitions of the maps $\beta : [(\kappa \times \lambda)^{-1}(u,l)]^2 \to C_{u,l}$, $\beta_{\#} : [((\kappa_{\#} \times \lambda_{\#})^{-1}(u,l)]^2 \to C_{u,l}$, $\beta_{u,l} : (\kappa \times \lambda)^{-1}(u,l) \to C_{u,l}$.

Clearly, the map $\beta$ is well-defined by $\beta([f], [g]) := \beta(f, g)$.

The map $\beta_{\#}$ is well-defined by $\beta_{\#} = \beta_{\#}(q_{\#} \times q_{\#})$, according to the additivity and the transitivity of $\beta$ (Lemmas 2.9 and 2.10).

Take an embedding $f' : N \to S^7$ representing an isotopy class in $(\kappa \times \lambda)^{-1}(u,l)$. Let $\beta_{u,l}(f) := \beta(f, f')$. The map $\beta_{u,l}$ depends on $f'$ but we do not indicate this in the notation.

2.4. Parametric connected sum and parametric additivity. In our proof of the realizability of the invariants we extensively use the so-called parametric connected sum operation defined below. We first recall, with minor modifications, some definitions and results of [Sk07, §2], [Sk15, §2.1], [MAP].

Definition of a standardized map. The base point $\ast$ of $V_{4,2}$ is the standard inclusion $\mathbb{R}^2 \to \mathbb{R}^4$. Take the embedding $\tau_0 : S^1 \times S^3 \to S^7$ for the constant map $\alpha_0 : S^3 \to V_{4,2}$ (as defined in §1.2). Clearly, $\tau_0(S^1 \times D^3_{\bar{3}}) \subset D^7_+$. For an embedding $s : S^1 \times D^3_{\bar{3}} \to N$, a map $h : N \to S^7$ is called s-standardized if $h(N - \text{im } s) \subset \text{Int } D^7_+$ and $h \circ s = \tau_0|_{s \times D^3_{\bar{3}}}$.

Lemma 2.11. For every embedding $s : S^1 \times D^3_{\bar{3}} \to N$, any $f$ is isotopic to an s-standardized embedding $\tilde{f} : N \to S^7$.

This is a smooth version of [Sk07, Standardization Lemma] which is proved analogously in Remark 2.23.a, cf. [Sk15, Standardization Lemma 2.1.a].

Definition of parametric connected sum $[f] +_s \tau(l,b)$. Take a map $\alpha : S^3 \to V_{4,2}$ representing an element $(l,b) \in \mathbb{Z}^2 = \pi_3(V_{4,2})$ such that $\alpha(D^3_{\bar{3}}) = \ast$. Then the embedding $\tau_\alpha$ is $i$-standardized for the inclusion $i : S^1 \times D^3_{\bar{3}} \to S^1 \times S^3$. Take an embedding $s : S^1 \times D^3_{\bar{3}} \to N$ and an $s$-standardized embedding $\tilde{f} : N \to S^7$ isotopic to $f$. Let $R : \mathbb{R}^m \to \mathbb{R}^m$ be the symmetry of $\mathbb{R}^m$ with respect to the hyperplane given by equations $x_1 = x_2 = 0$, i.e., $R$ is defined by $R(x_1, x_2, x_3, \ldots, x_m) := (-x_1, -x_2, x_3, \ldots, x_m)$. Define the embedding

$$h : N \to S^7 \text{ by } h(a) := \begin{cases} \tilde{f}(a) & a \notin \text{im } s, \\ R\tau_\alpha(x, R\gamma) & a = s(x, y). \end{cases}$$
The two formulas agree on $\partial \text{im} s$ because $\tau_0(x, y) = R\tau_0(x, Ry)$. Clearly, $h$ is a smooth embedding, i.e. it is injective, differentiable and has non-degenerate derivative.

Let $[f] + s \tau(l, b) \subset E^7(N)$ be the set of isotopy classes of the embeddings $h$, for all choices of $\tilde{f}$ and $\alpha$ as above. (In fact, $[h]$ clearly does not depend on the choice of $\alpha$, for fixed $l, b, \tilde{f}, s$. Still, $[h]$ may depend on $\tilde{f}$, i.e. $+s$ is not defined at the level of isotopy for embeddings of 4-manifolds into $S^7$, as opposed to other situations [Sk07, Sk15, MAP]. Cf. Corollary 2.13.c,d,e.)

**Lemma 2.12** (Parametric additivity; proved in §3.3). For any embedding $s : S^1 \times D^3 \to N$ let $[s] := [s|_{S^1 \times \{0\}}] \in H_1$. Then for any embedding $h \in f + s \tau(l, b)$ and $x, y \in H_3$ we have

$$\tau(h) = \tau(f), \quad \lambda(h)(x, y) = \lambda(f)(x, y) + l([s] \cap N)\tau([s] \cap N)$$

and

$$\tau(h) = \tau(f), \quad \beta(h) = \beta(f, h) \in C_{\tau(f), \lambda(f)}. $$

**Corollary 2.13.** (a) For every $u \in H_2^{\text{DIFF}}$ every isotopy class in $\tau^{-1}(u)$ can be obtained from a single isotopy class in $\tau^{-1}(u)$ by a finite sequence of parametric connected sum operations.

(b) Denote by $B_0(H_3)$ the group of symmetric bilinear forms $H_3 \times H_3 \to \mathbb{Z}$. Then there is a surjection

$$\tau_\# : H_1 \times H_2^{\text{DIFF}} \times B_0(H_3) \to E^7(N)$$

such that

$$\tau_\#(b, u, l) = \tau_\#(b', u', l') \iff u = u', \quad l = l' \quad \text{and} \quad b - b' \in C_{u, l + \lambda_\# \tau_\#(0, u, 0)}. $$

(c) The set $q_\#([f] + s \tau(l, b))$ is independent of $s$ for fixed $l, b, \tilde{f}$ and $[s|_{S^1 \times \{0\}}] \in H_1$ (cf. [Sk14]).

(d) Both sets $[f] + s \tau(l, b)$ and $q_\#([f] + s \tau(l, b))$ may consist of more than one element, i.e. $[h]$ from the definition of $[f] + s \tau(l, b)$ can depend on the choice of $\tilde{f}$, even for $N = S^1 \times S^3$.

(e) The set $q_\#(\tau(0, b')) + \tau(0, b)$ consists of one element, i.e. $[h]$ from the definition of $\tau(0, b') + \tau(0, b)$ does not depend on the choice of $\tilde{f}$.

**Proof.** Parts (a,c) follow from Theorem 1.3 and the parametric additivity (Lemma 2.12). Part (b) follows from (a,c) and Remark 2.20.c. Part (d) follows from Remark 2.23.b and the parametric additivity (Lemma 2.12). Part (e) follows from Theorem 1.1 and the parametric additivity (Lemma 2.12).

2.5. **Proof of Theorems 1.1 and 1.3 assuming lemmas.**

**Proof of Theorem 1.1: the surjectivity of $\tau_\#$.** Take any embedding $f : S^1 \times S^3 \to S^7$. Identify $B(H_3(S^1 \times S^3))$ with $\mathbb{Z}$. Denote $l := \lambda(f) \in \mathbb{Z}$. Take a representative $\alpha : S^3 \to V_{4,2}$ of $(l, 0) \in \pi_3(V_{4,2})$. Then $[\tau_\alpha] = \tau(l, 0)$. By the calculation of $\lambda$ (Lemma 1.4.b) we have $\lambda(\tau_\alpha) = l$.

We have $\text{im } \overline{\lambda(f)} = l\mathbb{Z}[S^1 \times 1_3]$ and $\tau(f) = \tau_\#(\tau_\alpha) = 0$. Hence $C_{\tau(f), \lambda(f)} \cong H_1(S^1 \times S^3; \mathbb{Z}[2]) \cong \mathbb{Z}[2]$ and the class $\beta(f, \tau_\alpha) \in C_{\tau(f), \lambda(f)}$ is defined. Take an integer $b$ such that $\beta(f, \tau_\alpha) = -\rho_2[b[S^1 \times 1_3]$. By the transitivity of $\beta$ (Lemma 2.10) and the calculation of $\beta$ (Lemma 2.7.b)

$$\beta([f], \tau(l, b)) = \beta(f, \tau_\alpha) + \beta(\tau(l, 0), \tau(l, b)) \rho_2(-b + b)[S^1 \times 1_3] = 0.$$ 

Hence by the MK Isotopy Classification Theorem 2.8 $q_\#[f] = \tau_\#(l, b)$. □
Proof of Theorem 1.1: description of preimages of $\tau_\#$. Denote $\tau := \tau_\#(l, b)$ and $\tau' = \tau_\#(l', b')$. By the calculation of $\lambda$ (Lemma 1.4.b) we have $\lambda(\tau) = l$ and $\lambda(\tau') = l'$. So for $l = l'$ by the calculation of $\beta$ (Lemma 2.7.b) we have $\beta(\tau', \tau) = \rho_{2l}(b - b')[S^1 \times 1_3]$. Hence

$$\tau = \tau' \iff l = l' \quad \text{and} \quad \beta(\tau', \tau) = 0 \iff l = l' \quad \text{and} \quad b \equiv b' \mod 2l$$

by the MK Isotopy Classification Theorem 2.8.

Proof of Theorem 1.3. The map $\beta_{u,l,\#}$ is surjective by the parametric additivity (Lemma 2.12) and is injective by the MK Isotopy Classification Theorem 2.8.

In this proof denote $\cdot = \cap_N$. By Lemma 3.2, the definition of $\kappa$ is equivalent to that of [BH70], cf. [Sk10, CS11]. Hence by [BH70] $\im \kappa_\# = \im \kappa = H_2^{DIFF}$. So it remains to prove that for every $u \in \im \kappa$

$$\lambda(\kappa^{-1}(u)) = \{l \in B(H_3) : (l(y, x) = l(x, y) + u \cdot x \cdot y \text{ for every } x, y \in H_3)\}.$$

By the $\kappa$-symmetry (Lemma 2.2) and Remark 2.20.c this is implied by the following claim.

Claim. For every embedding $f : N \to S^7$ and every symmetric bilinear form $m : H_3 \times H_3 \to \mathbb{Z}$ there is an embedding $g = g(f, m) : N \to S^7$ such that $\kappa(g) = \kappa(f)$ and $\lambda(g) = \lambda(f) + m$.

Proof of claim. We can set $g(f, m_1 + m_2) := g(g(f, m_1), m_2)$. Thus it suffices to construct $g(f, m)$ only for basic forms

$$m_p(x, y) = (p \cdot x)(p \cdot y) \quad \text{and} \quad m_{p,q}(x, y) = (p \cdot x)(q \cdot y) + (p \cdot y)(q \cdot x),$$

where $p, q \in H_1$.

Take embeddings $s, u, v : S^1 \times D_2 \to N$ whose restrictions to $S^1 \times 0$ represent elements $p, q, p + q \in H_1$, respectively. By the parametric additivity (Lemma 2.12) we can take as $g(f, m_p)$ and $g(f, m_{p,q})$ any elements of

$$[f] +_s \tau(1, 0) \quad \text{and} \quad ([f] +_v \tau(1, 0)) +_s \tau(-1, 0) +_u \tau(-1, 0),$$

respectively, where the latter set is the set $h_1 +_u \tau(-1, 0)$ for some $h_1 \in h_2 +_s \tau(-1, 0)$ and for some $h_2 \in [f] +_v \tau(1, 0)$. \hfill \square

2.6. Appendix: on regular homotopy and the Compression problem.

Proposition 2.14 (Regular homotopy classification). If $f_0, f_1 : N \to S^7$ are embeddings and $(\lambda(f_0) - \lambda(f_1))(x, x) \equiv 0$ for all $x \in H_3$, then $f_0$ and $f_1$ are regular homotopic.

Define the map $W : E_7(N) \to H_1(N; \mathbb{Z}_2)$ by $\rho_2 \lambda(f)(x, x) = W(f) \cap_N x$ for all $x \in H_3(N; \mathbb{Z}_2)$. By Proposition 2.14 $W$ induces an injection on the set of regular homotopy classes of embeddings. By Theorem 1.3 $\im W$ consists of those $y \in H_1(N; \mathbb{Z}_2)$ for which there is $u \in H_2^{DIFF}$ and a $u$-symmetric bilinear form $l \in B(H_3)$ such that $\rho_{2l}(x, x) = y \cap_N x$ for every $x \in H_3(N; \mathbb{Z}_2)$. It would be interesting to obtain a more direct description of $\im W$.

Definition of the Whitney invariant $W'_0$. (See [Sk10', §1].) The singular set of a smooth map $H : X \to Y$ between manifolds is $S(H) := \{x \in X : d_x H \text{ is degenerate}\}$.

Let $f_0, f_1 : P \to S^7$ be immersions of a 4-manifold $P$. Take a general position homotopy $H : P \times I \to S^7 \times I$ between $f_0$ and $f_1$. By general position, $\text{Cl} S(H)$ is a closed 1-submanifold. Define $W'_0(f_0, f_1) := [\text{Cl} S(H)] \in H_1(P, \partial; \mathbb{Z}_2)$.

(It is well-known that $W'_0(f_0, f_1)$ is indeed independent of $H$ for fixed $f_0$ and $f_1$.)

Lemma 2.15. Let $f_0, f_1 : P \to \mathbb{R}^7$ be embeddings of a 4-manifold $P$ and $X \subset P$ a closed connected 3-submanifold. Take the normal vector field of $X$ in $P$ defined by the orientations of $X$ and $P$. Let $X'$ be the shift of $X$ along this vector field. Then

$$W'_0(f_0, f_1) \cap_P \rho_{2}[X] = \rho_{2}[\text{lk}_{\mathbb{R}^7}(f_0X, f_0X') - \text{lk}_{\mathbb{R}^7}(f_1X, f_1X')]$$.
Proof. It suffices to prove this equality for \( P = X \times I, X = X \times 0 \) and \( X' = X \times 1 \). By the strong Whitney Isotopy Theorem [Sk08, Theorem 2.2.b] \( f_0|_X \) and \( f_1|_X \) are isotopic. Since both sides of the required equality do not change under isotopy of \( \text{id} \mathbb{R}^7 \), we may assume that \( f_0 = f_1 \) on \( X \). Take a general position homotopy \( H : X \times I \times I \rightarrow \mathbb{R}^7 \times I \) between \( f_0 \) and \( f_1 \) that is fixed on \( X \). The homotopy \( H \) gives a homotopy \( G : X \times I \rightarrow \mathbb{R}^7 \) of a normal vector field on \( f_0(X) \subset \mathbb{R}^7 \) (through normal vector fields which are not assumed to be non-zero). Since \( H|_{X \times t} \) is an embedding, for every \( x \in X \) and \( t \in I \) the differential \( d_{(x,t)}H \) is degenerate if and only if \( G(x, t) = 0 \). By general position, \( G^{-1}(0) = H(X' \times I) \cap f_0(X) \) is a finite number of points. Then

\[
W_0'(f_0, f_1) \cap P \rho_2[X] = \rho_2|H(X' \times I) \cap f_0(X)| = \rho_2[\text{lk}_{\mathbb{R}^7}(f_0 X, f_0 X') - \text{lk}_{\mathbb{R}^7}(f_1 X, f_1 X')].
\]

\( \square \)

Proof of Proposition 2.14. The following statements are equivalent:

(i) \( f_0 \) and \( f_1 \) are regular homotopic;
(ii) \( f_0|_{N_0} \) and \( f_1|_{N_0} \) are regular homotopic;
(iii) \( W_0'(f_1|_{N_0}, f_0|_{N_0}) = 0 \);
(iv) \( \lambda(f_0)(x, x) \equiv \lambda(f_1)(x, x) \mod 2 \) for every \( x \in H_3 \).

Indeed,

- (i)\iff(ii) because by the Smale-Hirsch classification of immersions [Hi60] the complete obstruction to extension of a regular homotopy from \( N_0 \) to \( N \) assumes values in \( \pi_4(V_7,4) = 0 \) [Pa56].
- (ii)\iff(iii) because by the Smale-Hirsch classification of immersions [Hi60] the first obstruction to regular homotopy between \( f_0|_{N_0} \) and \( f_1|_{N_0} \) assumes values in \( H_1(N_0, \partial; \pi_3(V_7,4)) \) and is complete, and because this obstruction clearly coincides with \( W_0'(f_1|_{N_0}, f_0|_{N_0}) \).
- (iii)\iff(iv) by Lemma 2.15 because by the calculation of \( \lambda \) (Lemma 1.4.c) \( \lambda(f_k)([X],[X]) = \text{lk}(f_k|_X, f_k|_{X'}) \), so \( W_0'(f_0, f_1) \cap N x = \rho_2(\lambda(f_0) - \lambda(f_1))(x, x) \) for all \( x \in H_3(N; \mathbb{Z}_2) \).

\( \square \)

Problem 2.16 (Compression problem). For an integer \( j \in \{1, 2\} \) describe those embeddings \( N \rightarrow S^7 \) which are isotopic to embeddings with image in \( S^{7-j} \subset S^7 \).

Clearly, \( \lambda(f) = \varepsilon(f) = 0 \) for an embedding \( f : N \rightarrow \mathbb{R}^7 \) such that \( f(N) \subset \mathbb{R}^6 \).

Proposition 2.17. There are embeddings \( f_0, f_1 : N \rightarrow S^7 \) such that \( f_0(N) \cup f_1(N) \subset S^6 \) and \( \beta(f_0, f_1) \neq 0 \).

Proof. This follows by Lemma 2.18 below because \( \beta(\tau(0,0), \tau(0,1)) \neq 0 \) by the calculation of \( \beta \) (Lemma 2.7.a).

\( \square \)

Lemma 2.18. There is a representative of \( \tau(0,1) \) whose image is in \( S^6 \subset S^7 \).

Proof of Lemma 2.18. We use the construction of \( \tau(0,1) \) from Remark 1.2. Denote by \( n : S^2 \rightarrow TS^3 \) a non-zero vector field normal to \( S^2 \subset S^3 \) and looking to the northern hemisphere of \( S^3 \). Then

- for every \( x \in S^3 \) the image \( T^2(S^1 \times x) \) is the round circle in \( S^3 \) passing through \( x \) in the direction \( n(\eta(x)) \), and
- \( T^2 \) is uniform on this circle.

Consider the normal bundle of \( \text{id} S^3 \times \eta : S^3 \rightarrow S^3 \times S^2 \). The obstructions to the existence of a non-zero section of this bundle are in \( H^{i+1}(S^3, \pi_i(S^1)) = 0 \). Hence there is such a section \( v(x) \in T_{(x)}S^2, x \in S^3 \). Define a map \( T^3 : S^1 \times S^3 \rightarrow S^3 \) by setting

- for every \( x \in S^3 \) the image \( T^3(S^1 \times x) \) to be the round circle in \( S^3 \) passing through \( x \) in the direction \( v(x), \) and
• $T^3$ to be ‘linear’ uniform on this circle.
We have $T^3(S^3) \subset S^2$, hence $i \circ (pr_2 \times T^3)(S^1 \times S^3) \subset i(S^3 \times S^2) \subset S^6 \subset S^7$.

Take a linear homotopy $v_t(x) := \frac{tn(\eta(x)) + (1-t)v(x)}{|tn(\eta(x)) + (1-t)v(x)|} \in T_{i(x)}S^3$ between non-zero vector fields $n(\eta(x))$ and $v(x)$ on $S^2 \subset S^3$. This homotopy defines a homotopy between $T^2$ and $T^3$ which keeps the image of $S^1 \times x$ embedded. The latter homotopy defines an isotopy from a representative $i \circ (pr_2 \times T^2)$ of $\tau(0,1)$ to the embedding $i \circ (pr_2 \times T^3)$ whose image is in $S^6 \subset S^7$. $\square$

2.7. Appendix: some remarks to §1 and §2.

Remark 2.19 (to §1.1). In this paper ‘smooth’ means ‘$C^1$-smooth’. Recall that a smooth embedding is ‘orthogonal to the boundary’. For each $C^\infty$-manifold $N$ the forgetful map from the set of $C^\infty$-isotopy classes of $C^\infty$-embeddings $N \to \mathbb{R}^m$ to $E^m(N)$ is a 1–1 correspondence. For a (possibly folklore) proof of this result see [Zh16].

Remark 2.20 (to §1.2). (a) Theorem 1.1 is not an immediate corollary of Theorem 1.3, see Remark 2.24.
(b) For every $u \in H^D_{2\text{DIFF}}$ the set $\lambda(\varkappa^{-1}(u))$ consists of those $l \in B(H_3)$ for which $(u,l)$ is symmetric.
(c) For fixed $u$, the set of symmetric pairs $(u,l)$ is in bijection with the group $B_0(H_3)$. Indeed $B_0(H_3)$ acts freely and transitively on this set, for if $(u,l)$ and $(u,l')$ are symmetric pairs, then $l - l'$ is a symmetric form.
(d) Theorem 1.3 has a restatement similar to Theorem 1.1, see Corollary 2.13.b.
(e) Deduction of the italicized statement in p. 5. Suppose that $H_1 = 0$. The forgetful map from $E^m_\#(N)$ to the set of PL isotopy classes is injective for $H_1 = 0$ [Bo71, p. 141], [Ha68]. Boéchat and Haefliger classified PL embeddings $f : N \to S^7$ up to PL isotopy [BH70, Theorem 1.6]. They also characterized smoothable PL embeddings [BH70, Theorem 2.1]. All this implies the above result. An alternative proof of the injectivity of $\varkappa_\#$ is given in [CS11].

Remark 2.21 (to §1.3). Note that $\varkappa$-invariant and $\lambda$-invariants can alternatively be defined using the intersection in the homology of the complements $C_0, C_1$. However, that definition corresponds to the classical surgery not modified surgery approach.

Remark 2.22 (to §2.3). The property of $Y$ identified in Lemma 3.13.a below provides an equivalent definition of a joint Seifert class which explains the name and which was used in [Sk10, CS11], together with the name ‘joint homology Seifert surface’.

Remark 2.23 (to §2.4). (a) Proof of Lemma 2.11. Define

$$i : \sqrt{2}D^2 \times D^4 \to S^7 \quad \text{by} \quad i(x,y) := (y\sqrt{2 - |x|^2}, 0, 0, x)/\sqrt{2}. $$

(No confusion with the map $i$ defined in §1.2 will appear.) Then $i = \tau_0$ on $S^1 \times D^3$. For $\gamma \leq \sqrt{2}$ denote $\Delta_\gamma := i(\gamma D^2 \times \{-1\}) \subset \text{Int } D_\gamma^7$.

In this proof we omit the sign $\circ$ for composition.

Any two embeddings $S^1 \times D^3 \to S^7$ are isotopic. So we can make an isotopy and assume that $f s = i$ on $S^1 \times D^3_\gamma$.

Since $7 > 2 \cdot 1 + 3 + 1$, by general position we may assume that $f(N) \cap \Delta_1 = \partial \Delta_1$. Then there is $\gamma$ slightly greater than 1 such that $f(N) \cap \Delta_\gamma = \partial \Delta_1$. Take the standard 3-framing on $\Delta_\gamma$ tangent to $i(\gamma D^2 \times S^3)$ whose restriction to $\partial \Delta_1$ is the standard normal 3-framing of
\[\partial \Delta_1 \text{ in } f(N). \text{ Then the standard 2-framing normal to } i(\gamma D^2 \times S^3) \text{ is a 2-framing on } \partial \Delta_1 \text{ normal to } f(N). \text{ Using these framings we construct}
\]
- an orientation-preserving embedding \( H : D^7_\gamma \to D^7_\gamma \) onto a sufficiently small neighborhood of \( \Delta_1 \) in \( D^7_\gamma \), and
- an isotopy \( h_t \) of \( \text{id}(S^1 \times S^3) \) shrinking \( S^1 \times D^3_\gamma \) to a sufficiently small neighborhood of \( S^1 \times \{-1\} \) in \( S^1 \times D^3_\gamma \) such that

\[ H(\Delta_{\sqrt{2}}) = \Delta_\gamma, \quad H i(S^1 \times D^3_\gamma) = H(D^7_\gamma) \cap f(N) \quad \text{and} \quad H i = i h_1 \quad \text{on} \quad S^1 \times D^3_\gamma. \]

The embedding \( H \) is isotopic to \( \text{id} D^7_\gamma \) by [Hi76, Theorem 3.2]. This isotopy extends to an isotopy \( H_t \) of \( \text{id} S^7 \) by the Isotopy Extension Theorem [Hi76, Theorem 1.3]. Then \( H_t^{-1} f h_t \) is an isotopy of \( f \). Let us prove that the embedding \( H_t^{-1} f h_t \) is standardized.

We have \( H_t^{-1} f h_t = H_t^{-1} i h_1 = i \) on \( S^1 \times D^2_\gamma \). Also if \( H_t^{-1} f h_t N - \text{im} s \not\subset \text{Int} D^m_+ \), then there is \( x \in N - \text{im} s \) such that \( f h_t x \in H(D^7_\gamma) \). Then \( f h_t x = H_t(y) = i h_1(y) = f h_1(y) \) for some \( y \in S^1 \times D^3_\gamma \). This contradicts the fact that \( f h_1 \) is an embedding.

(b) Note that \( [f] \in [f] + s \tau(0, 0) \) for every \( s \). Also \( \tau(l + l', b + b') \in \tau(l, b) + \tau([S^1 \times 1], \tau(l', b')) \), and in particular, \( \tau(l, b') \in \tau(l, b) + \tau([S^1 \times 1], \tau(0, b' - b)). \)

(c) The calculation of \( \lambda \) and \( \beta \) (Lemma 1.4.b and 2.7.b) follows by (b) and the parametric additivity (Lemma 2.12). However, Lemma 2.12 for \( \lambda \) and \( \beta \) is proved using Lemma 1.4.b and the particular case Lemma 2.7.a of Lemma 2.7.b, respectively. For this reason, as well as for applications, it is convenient to state Lemma 1.4.b and Lemmas 2.7.a,b separately.

(d) In Corollary 2.13.b it would be interesting to canonically construct at least part of the map \( \tau_\# \), and to give an algebraic (possibly non-canonical) construction of an \( u \)-symmetric form instead of \( \lambda_{\tau_\#}(0, u, 0) \).

(e) Using parametric connected sum one can define a map \( E^7(S^1 \times S^3)^2 \to 2 E^7(S^4 \times S^3) \), and the same statement holds with \( E^7 \) replaced by \( E^7_\# \), cf. [Sk15, §2.1], [MAP]. Corollary 2.13.de means that this map
- is not single-valued for either \( E^7 \) or \( E^7_\# \), this is unlike the situation in other dimensions [Sk15, §2.1], [MAP],
- is single-valued on \( \lambda_\#^{-1}(0) \subset E^7_\#(S^1 \times S^3) \); then it defines a group structure on \( \lambda_\#^{-1}(0) \) (an unpublished direct proof was sketched by S. Avvakumov).

Cf. [II, III] for smooth and PL analogues.

Remark 2.24 (to §2.5). Theorem 1.1 also follows from the construction of the map \( \tau \), the calculation of \( \lambda \) and \( \beta \) (Lemmas 1.4.b and 2.7.b), together with a version of Theorem 1.3 stating that \( \beta_{u,l,\#} \) is a 1–1 map for every representative \( f' \) of an isotopy class in \( (\lambda_\# \times \lambda_\#)^{-1}(u, l) \) (such a version is essentially proved in the proof of Theorem 1.3); we take \( f' = \tau(0, l) \).

3. Proofs of lemmas

3.1. More notation. Recall that some notation was introduced in §§1.2, 1.3 and 2.1.

Denote by \( D
\nu = D
\nu_f : S^7 \to \text{Int } C_f \to N \) the oriented normal disk bundle of \( f \) (the orientation of \( D
\nu \) is inherited from the orientation of \( S^7 \) and \( N \)).
**Definition of homology Alexander duality.** Consider the following diagram:

\[
\begin{array}{cccc}
H_{q-2}(N) & \xrightarrow{PD} & H^{6-q}(N) \\
\downarrow{\nu'} & & \downarrow{\hat{A}} \\
H_{q+1}(C, \partial) & \xrightarrow{\partial C} & H_q(\partial C) & \xrightarrow{i_C} H_q(C) \\
\downarrow{PD} & & \downarrow{AD} \\
H^{6-q}(C) & \xleftarrow{\hat{A}} & H_q(N) \\
\end{array}
\]

Here \(AD\) is Alexander duality and \(A = A_f, \hat{A} = \hat{A}_f\) are homology Alexander duality isomorphisms.\(^4\) The lines are exact and the triangles are commutative by the well-known Alexander Duality Lemmas of [Sk08', Sk10).

Fix an orientation on \(N\) and denote by \([N] \in H_4\) and \([N_0] \in H_4(N_0, \partial)\) the corresponding fundamental classes. We often use the class \(A[N] \in H_5(C, \partial)\) which may be called the homology Seifert surface of \(f\).

**Lemma 3.1** (Intersection Alexander duality). For every \(y \in H_q\) and for every \(z \in H_{4-q}\) we have \(y \cap_N z = Ay \cap_C \hat{A}z\).

**Proof.** For every \(x \in H_1(\partial C)\) we have \(\nu(x \cap_{\partial C} \nu'z) = \nu x \cap_N z\). Take \(x = \partial Ay\). Since \(\nu x = y\) and \(y \cap_N z \in \mathbb{Z}\), we obtain \(y \cap_N z = \partial Ay \cap_{\partial C} \nu'z = Ay \cap_C \hat{A}z\). \(\square\)

**Definition of the restriction homomorphism** \(r\). If \(P\) is a (compact oriented) codimension \(c\) submanifold of a manifold \(Q\) and either \(y \in H_k(Q)\) or \(y \in H_k(Q, \partial)\), denote

\[
r_{P,Q}(y) = r_{P}(y) = y \cap P := PD((PDy)|_P) \in H_{k-c}(P, \partial).
\]

If \(y\) is represented by a closed submanifold \(Y \subset Q\) transverse to \(P\), then \(y \cap P\) is represented by \(Y \cap P\). Clearly, \(y \cap_Q [P] = i_{P,Q}(y \cap P)\).

**Definition of the difference class** \(d(\xi, \xi')\). (This definition is not used until §3.3.) Let \(Q\) be a \(q\)-manifold, and \(\xi, \xi'\) non-zero sections of a \(k\)-dimensional vector bundle over \(Q\). We define the difference class

\[
d(\xi, \xi') \in H_{q-k+1}(Q, \partial) = H^{k-1}(Q) = H^{k-1}(Q; \mathbb{Z})
\]

of \(\xi\) and \(\xi'\) to be the class of the preimage of the zero section under a general position homotopy from \(\xi\) to \(\xi'\). This class is the homology primary obstruction to a vertical homotopy from \(\xi\) to \(\xi'\), and is equal to the Poincaré dual of the cohomology primary obstruction to a vertical homotopy from \(\xi\) to \(\xi'\), which is defined in [Wh78, Theorem 6.4 Ch. VI].

Difference classes between other structures, e.g. spin structures or framings, on (a part of) a manifold are defined analogously. (In fact, such structures can be represented as sections of certain bundles. Then one can use the homological or cohomological definition of the primary obstruction to vertical homotopy, the two definitions being related by Poincaré duality.)

---

\(^4\) We use this name because they are compositions of the ‘ordinary Alexander duality’ and canonical (Poincaré or Poincaré-Lefschetz) isomorphisms. If we identify canonically isomorphic groups, then the ‘homological Alexander duality’ is the same as the ‘ordinary Alexander duality’. Our definition of \(A, \hat{A}\) reveals that the geometric meanings of the ‘ordinary Alexander duality’ isomorphisms \(H_q(X) \to H^{n-q}(S^n - X)\) and \(H^q(X) \to H_{n-q}(S^n - X)\) are different, so the isomorphism are not as analogous to each other as it seems from the notation.
Definition of cobordism of homology classes together with supporting manifolds. (This definition is not used until §3.8.) Assume that \( P \) and \( Q \) are closed manifolds and \( x_j \in H_{k_j}(P) \), \( y_j \in H_{k_j}(Q) \) for \( j = 1, \ldots, n \). The tuples \((P, x_1, \ldots, x_n)\) and \((Q, y_1, \ldots, y_n)\) are called cobordant if there is a manifold \( V \) and classes \( v_j \in H_{k_{j+1}}(V, \partial) \) such that

\[
\partial V = P \sqcup (-Q), \quad \partial v_j \cap P = x_j \quad \text{and} \quad \partial v_j \cap Q = y_j \quad \text{for every} \quad j = 1, \ldots, n.
\]

The following definitions of a spin structure on a manifold \( Q \) and of the spin characteristic class \( p_*^Q \) will not be used until §3.6. Take a manifold \( Q \) and its triangulation. We write ‘skeleta of \( Q \)’ for ‘skeleta of the triangulation’.

Definition of a spin structure. A stable tangent spin structure on \( Q \) is a stable equivalence class of framing of the tangent bundle of \( Q \) over the 1-skeleton of \( Q \), which extends to the 2-skeleton of \( Q \). Two such framings are stably equivalent if they are homotopic, perhaps after addition of the trivial bundle with trivial framing to the tangent bundle of \( Q \). For brevity we omit ‘stable tangent’. (This will not lead to a confusion because our manifolds \( Q \) will have dimensions at least 3, so stable tangent spin structures are in 1–1 correspondence with tangent spin structures in the ordinary sense. See also Lemma 4.2.)

The trivial spin structure on \( S^7 \) is the one induced from the spin structure on \( D^8 \) compatible with the orientation. (Note that this is the only spin structure on \( S^7 \).)

If \( P \subset Q \) is a compatibly triangulated codimension zero submanifold, then a spin structure on \( Q \) induces a spin structure on \( P \) by restricting the framing over the 1-skeleton of \( Q \) to a framing over the 1-skeleton of \( P \). If \( Q \) has boundary \( \partial Q \), then a spin structure on \( Q \) induces a spin structure on \( \partial Q \).

If \( F : Q \to P \) is a diffeomorphism with differential \( dF \) and \( s \) is a spin structure on \( Q \), then the induced spin structure \( F_*s \) on \( P \) is obtained by applying \( dF \) to the vector fields over the 1-skeleton of \( Q \) which define \( s \).

Remark on spin structures via maps to \( BSpin \). Take a manifold \( Q \). A stable oriented vector bundle over \( Q \) is a sequence \( \xi_j, j \geq q, \) of \( j \)-dimensional oriented vector bundles over \( Q \), together with isomorphisms \( \xi_j \oplus n \to \xi_{n+j} \) of \((n+j)\)-dimensional oriented vector bundles for any \( n \) (here \( q \)’s can be different for different bundles, and \( n \) is the trivial bundle of dimension \( n \) [KL05, §18.10]). Analogously one defines a stable spin vector bundle. Let \( BSO \) and \( BSpin \) be the classifying spaces for stable oriented and spin vector bundles respectively. Recall that \( BSpin = BSO(4) \) is the (unique up to homotopy) 3-connected space for which there exists a fibration \( \gamma : BSpin \to BSO \) inducing an isomorphism on \( \pi_i \) for every \( i \geq 4 \).

Let \( \tau'_Q \) be the oriented tangent bundle of \( Q \). Let \( \tau_Q \) be the stable oriented vector bundle which is the sequence \( \tau'_Q \oplus n, n \geq 0 \), with the identical isomorphisms. We use the same notation \( \tau_Q \) for the classifying map \( Q \to BSO \) of \( \tau_Q \), which is the composition of the canonical map \( BSO_q \to BSO \) with the classifying map of \( \tau_Q \). A spin lift of \( \tau_Q \) is a map \( \pi_Q : Q \to BSpin \) such that \( \tau_Q = \gamma \circ \pi_Q \). Obstruction theory shows that a spin structure on \( Q \) may be regarded as a homotopy class of a spin lift \( \pi_Q \) of the map \( \tau_Q \), up to homotopy through spin lifts of \( \tau_Q \).

Definition of \( p^Q \) for a spin \( q \)-manifold \( Q \). It is well-known that there is a generator \( p \in H^4(BSpin) \cong \mathbb{Z} \) such that \( 2p \) is the pull back in \( H^4(BSpin) \) of the universal first Pontryagin class \( p_1 \in H^4(BSO) \) [CS11, §3, proof of Lemma 2.11]. Take a map \( \pi_Q : Q \to \)
Let by $\lambda$ (for simply-connected $Q$), we define $\lambda^* := \text{PD}_\mathbb{F}_p \in H_{q-4}(Q, \partial)$.

We remark that $\lambda^*$ does not depend of the choice of spin structure on $Q$ [CCV08, page 170] (for simply-connected $Q$ this is obvious).

3.2. Lemmas on the $\varkappa$- and $\lambda$-invariants (3.2 and 2.2). In this subsection
- the larger intersection symbol $\cap$ denotes the intersection of homology classes in $\nu^{-1}N_0$
- we identify $H_q(N_0, \partial)$ with $H_q$ by the isomorphism $\iota_{N_0}$ for each $q \in \{1, 2, 3\}$
- for a section $\xi : N_0 \rightarrow \nu^{-1}N_0$ we use without mention that $\xi[N_0] \cap \nu'y = \xi y$ for each $q \in \{1, 2, 3\}$ and $y \in H_q$.

We shorten $\lambda(f)$, $\lambda(f)$ and $\varkappa(f)$ to $\lambda$, $\lambda$ and $\varkappa$ respectively.

We define $\varkappa : H_2 \rightarrow \mathbb{Z}$ by $\varkappa(y) := \varkappa \cap N y$.

We prove that $\lambda$ and $\varkappa$ are independent of $\xi$ (Lemma 3.2, $\lambda'$, $\varkappa'$) we denote them by $\lambda_\xi$ and $\varkappa_\xi$ respectively.

**Lemma 3.2.** Let $\xi : N_0 \rightarrow \nu^{-1}N_0$ be a section such that $i_C\xi$ is weakly unlinked.

(a) $i_C(\xi[N_0] \cap x) = A[N] \cap i_C x$ for each $q \in \{1, 2, 3\}$ and $x \in H_q(\nu^{-1}N_0)$.

(b) $\lambda_\xi \lambda_\xi = \hat{A}^{-1}i_C\xi$ on $H_3$.

(c) $\varkappa_\xi \varkappa_\xi = \hat{A}^{-1}i_C\xi$ on $H_2$.

(d) $\lambda_\xi(y) = \hat{A}^{-1}(A[N] \cap C \hat{A} y)$ for every $y \in H_3$.

We use without mention that $i_C$, $\lambda_\xi$, $\varkappa_\xi$, and $\varkappa_\xi$ are independent of $\xi$.

**Proof of (a).** By [Sk10, Section Lemma 2.5.a] $A[N] \cap \nu^{-1}N_0 = \xi[N_0]$. Hence we have the equalities $A[N] \cap C i_C x = i_C((A[N] \cap \nu^{-1}N_0) \cap x) = i_C(\xi[N_0] \cap x)$.

**Proof of (b) and (c).** The formulas follow because for every closed oriented $q$-submanifold $X \subset N$, $q \in \{3, 4\}$,

$$\text{lk}_{S^7}(fX, \xi Y) \overset{(1)}{=} \text{lk}_{S^7}(\partial AX, \xi Y) \overset{(2)}{=} A[X] \cap C i_C\xi y \overset{(3)}{=} [X] \cap N \hat{A}^{-1}i_C\xi y$$

where
- $AX$ is any $(q+1)$-chain in $C$ whose boundary is in $\partial C$ and represents $\partial A[X]$ there;
- $\xi Y$ is a small shift of $\xi Y$ into the interior of $C$;
- (1) holds because $\nu\partial A[X] = [X]$, so $fX$ is homologous to $\partial AX$ in $S^7 - \text{Int} C$, and then because $\xi Y$ is homologous to $\xi Y'$ in $C$;
- (2) holds by definition of the linking coefficient;
- (3) holds by intersection Alexander duality (Lemma 3.1).

**Proof of (d) and (e).** By (a) for $x = \nu'y$ we have $A[N] \cap C \hat{A} y = i_C(\xi[N_0] \cap \nu'y) = i_C\xi y \in H_q(C)$ for each $q \in \{2, 3\}$. So (d) implies (d'). Also (e) implies that for every $y \in H_2$ we have

$$\varkappa\cap_N y = \hat{A}^{-1}(A[N] \cap C \hat{A} y) = A[N] \cap C (A[N] \cap C \hat{A} y) = A^{-1}(A[N] \cap C A[N]) \cap_N y.$$

Here the second and the third equalities follow by intersection Alexander duality (Lemma 3.1). This proves (e).
Proof of (e). Part (e) follows because for every $y \in H_2$

$$\nu \cap y \overset{1}{=} A[N] \cap_{C} i_C \xi y \overset{2}{=} [\xi[N_0]] \bigcap \xi y = [\xi[N_0]] \bigcap [\xi[N_0]] \bigcap \nu^1 y = e^*(\xi) \cap y.$$ 

Here (1) holds by (x) and intersection Alexander duality (Lemma 3.1), (2) holds by (a) and the other two equalities are obvious. □

Proof of $\kappa$-symmetry (Lemma 2.2). Let $\xi: N_0 \to \partial C$ be a weakly unlinked section obtained from a section $\zeta: N_0 \to \nu^{-1}N_0$ by composing with the inclusion $\nu^{-1}N_0 \to \partial C$. Let $-\xi: N_0 \to \partial C$ be the weakly unlinked section obtained by composing $-\zeta$ with the inclusion $N_0 \to \partial C$. If the homology classes $x, y \in H_3$ are represented by closed oriented 3-submanifolds (or integer 3-cycles) $X, Y \subset N_0$, then

$$\lambda(y, x) = \text{lk}_{S^7}(fY, \xi X) = \text{lk}_{S^7}(\xi X, fY) = \text{lk}_{S^7}(fX, -\xi Y).$$

Hence

$$\lambda(x, y) - \lambda(y, x) \overset{(1)}{=} fX \cap S^7 Y \overset{(2)}{=} \xi Y \cap S^7 Y \overset{(3)}{=} \xi x \cap_{\partial C} \xi y \overset{(4)}{=} \xi x \cap_{\partial C} \xi y = \text{lk}_{S^7}(fX, -\xi Y).$$

where

- $Y_{\xi} \subset S^7$ is the 4-submanifold (with boundary) that is the union over $a \in Y$ of segments joining $\xi a$ to $(-\xi)a$ (or $Y_{\xi}$ is the corresponding integer 4-chain); we have $Y_{\xi} \cong Y \times I$;
- the algebraic intersection of submanifolds (or the cycle and the chain) in $S^7$ is defined because the first one does not intersect the boundary of the second one;
- (1) holds by definition of the linking coefficient and the above formula for $\lambda(y, x)$;
- $Y'_{\xi}$ is obtained from $Y_{\xi}$ by a small shift along $\xi - f$ considered as vector field on $fN$;
- (2), (4), (6) are clear;
- (3) holds because $Y'_{\xi} \cap \partial C = Y$;
- (5) holds because $\dim \nu^{-1}N_0 - \dim \xi[N_0]$ is even, so we can exchange the order of terms in the cap product without changing the sign, and because $\nu'x \cap \nu'y = \nu'(x \cap y)$;
- (7) holds by Lemma 3.2.a;
- (8) holds by Lemma 3.2.$\kappa$ and intersection Alexander duality (Lemma 3.1). □

3.3. Parametric additivity of $\kappa$ and $\lambda$ (Lemma 2.12). Let $V$ be a 4-manifold with non-empty boundary. Recall that an embedding $v: V \to D_+^7$ is called proper, if $f^{-1}\partial D_+^7 = \partial V$. Denote by

- $C = C_v$ the closure of the complement in $D_+^7$ to a tubular neighborhood of $v(V)$;
- $v = v_v: \text{Cl}(\partial C_v - \partial D_+^7) \to V$ the restriction of the oriented normal vector bundle of $v$.

A section $\zeta: V \to \nu^{-1}V = \text{Cl}(\partial C_v - \partial D_+^7)$ of $v$ is called weakly unlinked if

$$i_{\partial C_v, C_v}[V] = 0 \in H_4(C, \partial D_+^7 \cap \partial C).$$

We remark that if we would take $\partial V = \emptyset$ in this definition, we would not obtain the definition of a weakly unlinked section for a closed manifold.

Lemma 3.3. (a) Any section is weakly unlinked for any proper embedding $v: D^1 \times S^3 \to D_+^7$.

(b) For any proper embedding $v: V \to D_+^7$ of a connected 4-manifold $V$ with non-empty boundary and torsion free $H_1(V, \partial)$, a weakly unlinked section exists and is unique up to vertical homotopy over any 2-skeleton of $V$. 
(c) Let \( g: N \to S^7 \) and \( s: D^1 \times S^3 \to N \) be embeddings such that \( g|_{g^{-1}(D^7_+)} \) is a proper embedding into \( D^7_+ \) and \( g^{-1}(D^7_-) = \text{im} \, s \). Then any weakly unlinked section for the abbreviation of \( g \), \( N - \text{Int}(\text{im} \, s) \to D^7_+ \), extends over \( N_0 := N - \text{Int} (D^1 \times D^3_+) \) to a weakly unlinked section for \( g \).

**Proof of (a).** Part (a) follows because

\[
H_4(C_v, \partial D^7_+ \cap \partial C_v) \cong H_4(D^7 - i(D^1 \times S^3), \partial D^7_+ - i(S^0 \times S^3)) \cong H_2(D^1 \times S^3, \partial) = 0
\]

by Alexander duality, the homology exact sequence of a pair and the 5-lemma. (Cf. the proof of additivity of \( \beta \), Lemma 2.9, in §3.7.)

**Proof of (b).** (The proof is analogous to the ‘absolute’ case \([BH70, \text{Proposition 1.3}] \).) For sections \( \xi \) and \( \xi' \) of the normal bundle of \( v \) the difference class \( d(\xi, \xi') \in H_2(V, \partial) \) is defined in §3.1. Alexander duality \( \tilde{A}: H_q(V, \partial) \to H_{q+2}(C_v, \partial D^7_+ \cap \partial C) \) is defined analogously to the absolute case. Then \( d(\xi, \xi') = \pm \tilde{A}^{-1}(\xi - \xi')[V, \partial V] \) analogously to \([BH70, \text{Lemme 1.2}] \). This implies the uniqueness of a weakly unlinked section for \( v \).

Let us prove the existence of a weakly unlinked section for \( v \). The normal Euler class \( \overline{\nu}(v) \) assumes values in \( H^3(V) \cong H_1(V, \partial) \). Since the normal bundle of \( v \) is odd-dimensional, \( 2\overline{\nu}(v) = 0 \).

Since \( H_1(V, \partial) \) is torsion free, \( \overline{\nu}(v) = 0 \). Since \( V \) is connected and has non-empty boundary, it retracts to a 3-dimensional subpolyhedron. Hence there is a section \( \zeta: V \to \nu^{-1}V \) of \( v \). Denote by \( U \) a closed neighbourhood of a 2-skeleton in \( V \). Construct a section \( \zeta': U \to \nu^{-1}V \) such that \( d(\zeta', \zeta) = \mp \tilde{A}^{-1}\zeta[V] \cap U \in H_2(U, \partial) \). By \([St99, \text{Theorem 37.4}] \) there is an extension \( \xi \) of \( \zeta' \) to \( V \) such that \( d(\xi, \zeta) = \pm \tilde{A}^{-1}\xi[V] \). Since \( d(\xi, \zeta) = \pm \tilde{A}^{-1}(\xi - \zeta)[V], \) the extension \( \xi \) is weakly unlinked. \( \Box \)

**Proof of (c).** Since \( H^q(D^1 \times D^3_+, \partial D^1 \times D^3_+; \pi_{q-1}(S^2)) = 0 \) for every \( q \), obstruction theory entails that there is a section \( \zeta: N_0 \to \partial C_g \) extending a given weakly unlinked section for the abbreviation of \( g \). Define \( x \in H_4(C_g) \) by \( x := i_{C_g} j^{-1}_{\partial C_g} \text{ex}^{-1} \xi[N_0, \partial] \), as in the definition of a weakly unlinked section for closed manifolds (§2.2). We have

\[
x \cap D^7_+ = \xi[g^{-1}(D^7_+)] = 0 \in H_4(C_g \cap D^7_+, C_g \cap \partial D^7_+).
\]

Consider the following part of the homology exact sequence of the pair \( (C_f, C_f \cap \partial D^7_-) \):

\[
\begin{array}{ccc}
H_4(C_g \cap D^7_+) & \xrightarrow{j} & H_4(C_g, C_g \cap D^7_-) \\
\| & & \|_{\text{ex}_+} \\
H_2(D^1 \times S^3) & = 0 & H_4(C_g \cap D^7_+, C_g \cap \partial D^7_+)
\end{array}
\]

Since \( 0 = x \cap D^7_+ = \text{ex}_+ jx \), we have \( x = 0 \), i.e. \( \xi \) is weakly unlinked for \( g \). \( \Box \)

Some notation for the proof of parametric additivity. Recall that \( s: S^1 \times D^3 \to N \) is an embedding realizing \( [s] \in H_1 \) and \( R \) is the symmetry of \( S^m \) with respect to the subspace defined by \( x_1 = x_2 = 0 \). Let \( N_+ := \text{Cl}(N - \text{im} \, s) \) and \( N_- := \text{im} \, s \subset N \). For a cycle \( X \) representing \( [X] \in H_1 \) and in general position to \( N_+ \cap N_- \) denote \( X_+ := X \cap N_+ \) a relative cycle representing \( [X] \cap N_+ \in H_1(N_+, \partial) \).

---

\(^6\)For a map \( f : X \to Y \) and \( A \subset X, f(A) \subset B \subset Y \), the abbreviation \( g : A \to B \) of \( f \) is defined by \( g(x) := f(x) \).

\(^7\)Alternatively, since \( \overline{\nu} = 0 \) for embeddings of closed manifolds, \( 2\overline{\nu}(v) = \overline{\nu}(2v) = 0 \) for the ‘double’ \( 2v \) of \( v \).
By Lemma 2.11 we may assume that \( f \) and an embedding \( \tau_\alpha \) representing \( \tau(l,b) \) are \( s \)-standardized and \( i \)-standardized, respectively. Take the embedding \( h \) given in the definition of parametric connected sum in §2.4. Then both \( f \) and \( h \) satisfy the assumptions of Lemma 3.3.c. We have \( f = h \) on \( N_+ \). Using Lemma 3.3.abc we can form a weakly unlinked section \( \xi_h \) for \( h \) as follows: we take the union of

- a weakly unlinked section for the abbreviation \( N_+ \to D^7_+ \) of \( h \) with
- the restriction to \( s(D^1_+ \times D^3) \) of a weakly unlinked section for the abbreviation \( N_- \to D^7_- \) of \( h \).

A weakly unlinked section \( \xi_f \) for \( f \) can be constructed analogously: simply replace \( h \) by \( f \).

Completion of the proof of parametric additivity for \( \kappa \). For every \( x \in H_2 \) take an integer 2-cycle (or closed oriented 2-submanifold) \( X \subset N \) representing \( x \). By general position we may assume that \( X \subset N_+ \). There is an integer 3-chain \( X' \) in \( D^7_+ \) such that \( \partial X' = \xi_h X \). So parametric additivity for \( \kappa \) holds because

\[
\kappa(h) \cap_N x = \text{lk}_{S^7}(hN, \xi_h X) = hN \cap_{S^7} X' \overset{(3)}{=} fN \cap_{S^7} X' \overset{(4)}{=} \kappa(f) \cap_N x,
\]

where

- the equality (3) follows because \( h = f \) on \( N_+ \) and \( hN_-, fN_- \subset D^7_- \);
- the equality (4) is proved in the same ways as the first two equalities, with \( h \) replaced by \( f \).

Completion of the proof of parametric additivity for \( \lambda \). For every \( x, y \in H_3 \) take integer 3-cycles (or closed oriented 3-submanifolds) \( X, Y \subset N \) representing \( x, y \). There are integer 4-chains \( Y'_+ \) in \( D^7_+ \) such that \( \partial(Y'_+ + Y'_-) = \xi_h Y \) and \( \partial Y'_+ \cap hN = \emptyset \). We have

\[
\lambda(h)(x,y) \overset{(a)}{=} \text{lk}_{S^7}(hN, \xi_h Y) = hX_+ \cap_{S^7} Y'_+ + hX_- \cap_{S^7} Y'_-.
\]

So parametric additivity for \( \lambda \) follows because

\[
hX_+ \cap_{S^7} Y'_+ \overset{(s)}{=} fX_+ \cap_{S^7} Y'_+ \overset{(**)}{=} \lambda(f)(x,y) \quad \text{and}
\]

\[
hX_- \cap_{S^7} Y'_- \overset{(1)}{=} \tau_\alpha s^{-1}X_- \cap_{S^7} RY'_- \overset{(2)}{=} (\lambda \tau_\alpha)(x,y) \overset{(3)}{=} l([s] \cap_N x) ([s] \cap_N y).
\]

Here

- equality (**) holds because \( h = f \) on \( N_+ \) and \( hN_-, fN_- \subset D^7_- \);
- equality (***) holds by equality (a) for \( h \) replaced by \( f \), because \( fs = \tau_0 | S^1 \times D^3 \), so \( fX_- \cap_{S^7} Y'_- = 0 \) analogously to the calculation of \( \lambda \) (Lemma 1.4.c);
- equality (1) holds because \( R \) preserves the orientation;
- \( x_s := ([s] \cap_N x)[1_1 \times S^3] \in H_3(S^1 \times S^3) \) and analogously define \( y_s \);
- equality (2) is proved below;
- equality (3) holds by the calculation of \( \lambda \) (Lemma 1.4.b).

To prove equality (2), we first apply the analogue of equality (**) for \( f \) replaced by \( \tau_\alpha \). Observe that \( \tau_\alpha \) is a weakly unlinked section for \( \tau_\alpha \). This shows that the left hand side of equality (2) equals to the value of \( \lambda \tau_\alpha \) on certain homology classes in \( H_3(S^1 \times S^3) \). We have the equality \([s^{-1}X_-] = ([s] \cap_N x)[1_1 \times D^3_+] = x_s \cap (1_1 \times D^3_+) \) and the same with \( X, x \) replaced by \( Y, y \). Hence these homology classes are \( x_s \) and \( y_s \). \( \square \)
3.4. Agreement of Seifert classes (Lemmas 2.4 and 3.5).

**Lemma 3.4.** If \( \varphi(f_0) = \varphi(f_1) \) and \( \lambda(f_0) = \lambda(f_1) \), \( \varphi : \partial C_0 \to \partial C_1 \) is a bundle isomorphism and \( \xi : N_0 \to \partial C_0 \) a weakly unlinked section for \( f_0 \), then \( \varphi \xi \) is a weakly unlinked section for \( f_1 \).

**Proof.** The proof we give follows the same line of reasoning as [CS11, §3, end of proof of Lemma 2.5]. By Lemma 2.1 there exists a weakly unlinked section \( \xi_1 \) for \( f_1 \). By Lemma 3.2.e

\[
e^*((\varphi \xi)^{\perp}) = e^*(\xi^\perp) = \varphi(f_0) = \varphi(f_1) = e^*(\xi_1^\perp).
\]

For any pair of sections \( \zeta, \eta : N_0 \to \partial C_1 \) we have

\[
e^*(\zeta^\perp) - e^*(\eta^\perp) = \pm 2d(\zeta, \eta) = \pm 2icj_0^{-1}\text{ex}^{-1}(\zeta - \eta),
\]

where \( d(\zeta, \eta) \in H_2(N_0; \pi_2(S^2)) \) is the difference class [BH70, Lemma 1.7], defined in §3.1 above. We apply this for \( \zeta = \xi_1 \) and \( \eta = \varphi \xi \). Since \( H_2(N) \) has no 2-torsion, we obtain the equation \( icj_0^{-1}\text{ex}^{-1} \varphi \xi = icj_0^{-1}\text{ex}^{-1} \xi_1 = 0 \); i.e. \( \varphi \xi \) is weakly unlinked. \( \square \)

**Definition of a joint Seifert class.** A joint Seifert class for \( x \in H_q \) and a bundle isomorphism \( \varphi : \partial C_0 \to \partial C_1 \) is an element

\[
X \in H_{q+1}(M_\varphi) \quad \text{such that} \quad X \cap C_k = A_kx \in H_{q+1}(C_k, \partial) \quad \text{for each} \quad k = 0, 1.
\]

When the bundle isomorphism \( \varphi \) is clear from the context, we shall simply call \( X \) a joint Seifert class for \( x \in H_q \). Note that a joint Seifert class, as defined in §2.3, is a joint Seifert class for \( [N] \in H_4 \) by Lemma 3.13.a below.

**Lemma 3.5** (Agreement of Seifert classes). Assume that \( \varphi(f_0) = \varphi(f_1) \), that \( \lambda(f_0) = \lambda(f_1) \), and that \( \varphi : \partial C_0 \to \partial C_1 \) a bundle isomorphism. Assume that the coefficients are \( \mathbb{Z} \) or \( \mathbb{Z}_d \) for some \( d \); they are omitted from the notation. Let \( \partial \kappa := \partial_{\partial C_k, \mathbb{C}_k} \).

(a) \( \varphi \partial_0 A_0 = \partial_1 A_1 : H_q \to H_q(\partial C_1) \).

(b) \( \partial : H_{q+1}(M_\varphi, C_0) \to H_q(\partial C_1) \) is zero for each \( q \).

(c) For every \( x \in H_q \) there is a joint Seifert class for \( x \).

**Proof.** Part (c) follows by (a) and the Mayer-Vietoris sequence for \( M_\varphi \):

\[
\begin{align*}
H_5(M_\varphi) & \xrightarrow{\partial_0 \oplus \partial_1} H_5(C_0, \partial) \oplus H_5(C_1, \partial) \xrightarrow{\partial_0 - \partial_1} H_4(\partial C_0) .
\end{align*}
\]

For \( q \geq 5 \) part (a) is trivial and part (b) follows because \( H_{q+1}(M_\varphi, C_0) \cong H_q = 0 \).

For \( q = 1 \) part (b) is trivial and part (a) follows because \( \varphi \partial_0 A_0 = \partial_1 A_1 = \nu_0^{-1} \).

Now assume that \( q \in \{2, 3, 4\} \). Let \( \xi_0 : N_0 \to \partial C_0 \) be a weakly unlinked section for \( f_0 \).

Since \( \varphi(f_0) = \varphi(f_1) \), \( \lambda(f_0) = \lambda(f_1) \) and \( H_2 \) has no 2-torsion, by Lemma 3.4 \( \xi_1 := \varphi \xi_0 \) is a weakly unlinked section for \( f_1 \). In this proof \( k \in \{0, 1\} \). The map

\[
\xi_k \oplus \nu_k^1 : H_q \oplus H_{q-2} \to H_q(\partial C_k)
\]

is an isomorphism for each \( q \in \{2, 3\} \). The map

\[
(j_k^{-1}\text{ex}^{-1} \xi_k) \oplus \nu_k^1 : H_4(N_0, \partial) \oplus H_2 \to H_4(\partial C_k)
\]

is an isomorphism. Let \( i_k := i_{\partial C_k, C_k} \). We have \( i_k \nu_k^1 = \widehat{A}_k \). By Lemma 3.2, \( \lambda, \varphi \) and [Sk10, Lemma 2.5.b] we have

\[i_k \xi_k = \widehat{A}_k \varphi(f_k) \quad \text{on} \quad H_2, \quad i_k \xi_k = \widehat{A}_k \lambda(f_k) \quad \text{on} \quad H_3 \quad \text{and} \quad i_k j_k^{-1}\text{ex}^{-1} \xi_k[N_0] = i_k \partial \kappa A_k[N] = 0.\]

\[8\]Of these assumptions we need none for (a,b) and \( q \notin \{2, 3, 4\} \), and only \( \varphi(f_0) = \varphi(f_1) \) for (a,b) and \( q \in \{2, 4\} \).
Hence \( \varphi \ker i_0 = \ker i_1 = \im \partial_1 \). Then the following commutative diagram

\[
\begin{array}{ccc}
H_{q+1}(M_\varphi, C_0) & \xrightarrow{\partial} & H_q(C_0) \\
\text{ex} & & \text{id} \\
H_q & \xrightarrow{A_1} & H_{q+1}(C_1, \partial) \xrightarrow{\partial_1} H_q(\partial C_1) \xrightarrow{i_1} H_q(C_1),
\end{array}
\]

shows that \( i_0 \varphi^{-1} \partial_1 = 0 \), which implies (b).

Since \( \nu_1 \varphi \partial_0 A_0 = \nu_1 \partial_1 A_1 \), we have \( \varphi \partial_0 A_0 - \partial_1 A_1 = \nu_1 y \) for some map \( y : H_q \to H_{q-2} \). Applying \( i_1 \) to both sides and using that \( \varphi \partial_0 A_0 x \in \varphi \ker i_0 = \ker i_1 \) we obtain \( 0 = A_1 y \). Hence \( y = 0 \), i.e. (a) holds. \qed

**Proof of Lemma 2.4.** By Lemma 3.5.b, \( i_{C_0, M_\varphi} \) is injective. Since \( H_2(M_\varphi, C_0) \cong H_2(C_1, \partial) \cong H_1 \) is torsion free, \( i_{C_0, M_\varphi} \) is split injective. As \( S^2_{j_0} \in H_2(C_0) \cong \mathbb{Z} \) is primitive, \( i_{C_0, M_\varphi} S^2_{j_0} \in H_2(M_\varphi) \) is primitive. So by Poincaré duality there is a joint Seifert class \( Y \in H_5(M_\varphi) \). \qed

**3.5. Spin bundle isomorphisms (Lemma 3.6).** For \( k = 0, 1 \) let \( \mathbf{sp}_k \) be the (stable tangent) spin structure on \( \partial C_k \) induced from the trivial spin structure on \( S^7 \).

A bundle isomorphism \( \varphi : \partial C_0 \to \partial C_1 \) is called **spin** if it carries \( \mathbf{sp}_0 \) to \( \mathbf{sp}_1 \).

**Lemma 3.6 (Spin Lemma).** (a) For a bundle isomorphism \( \varphi : \partial C_0 \to \partial C_1 \) the manifold \( M_\varphi \) is spin if and only if \( \varphi \) is spin. Moreover, for every spin bundle isomorphism \( \varphi : \partial C_0 \to \partial C_1 \), there is a unique spin structure on \( M_\varphi \) whose restrictions to \( C_0, C_1 \) are induced from \( S^7 \).

(b) A spin bundle isomorphism exists and is unique (up to homotopy) over the 2-skeleton of any triangulation of \( N \).

**Remark.** (a) The ‘if’ part of the Spin Lemma 3.6.a can be proved as follows: If \( \varphi \) is spin, then the spin structures on \( C_0, C_1 \) coming from \( S^7 \) agree up to homotopy on the boundaries. Hence they can be glued together to give a spin structure on \( M_\varphi \).

Below we give another proof together with the proof of the ‘only if’ and the ‘moreover’ parts.

(b) In order to illustrate the main idea of the Spin Lemma 3.6.b let us sketch of a proof for \( N = S^1 \times S^3 \). (The sketch is not formally used in the proof.) Take a smooth map \( \alpha : S^1 \to SO_3 \) representing the generator of \( \pi_1(SO_3) \). Identify \( \partial C_f \) and \( S^1 \times S^3 \times S^2 \) by the formula \( f_\alpha(x, y, z) = (x, y, \alpha(x)z) \).

The manifold \( \partial C_f \) has precisely two stable tangent spin structures and the self-bundle-isomorphism \( f_\alpha \) acts by exchanging these. This implies the existence. The uniqueness follows from the fact that every spin bundle isomorphism is isotopic to \( f_\alpha \) or the identity.

**Definition of the difference class** \( d(\mathbf{sp}, \mathbf{sp}') \). Let \( Q \) be a \( q \)-manifold. For spin structures \( \mathbf{sp} \) and \( \mathbf{sp}' \) on \( Q \) their difference is

\[
H_{q-1}(Q, \partial; \mathbb{Z}_2) = H^1(Q; \mathbb{Z}_2) = H^1(Q; \pi_1(SO))
\]

is the primary obstruction to homotopy from \( \mathbf{sp} \) to \( \mathbf{sp}' \), cf. §3.1. (This is the homology class represented by the degeneracy set of a general position homotopy, through ordered \((q-1)\)-sets of vectors, from \( \mathbf{sp} \) to \( \mathbf{sp}' \).)

The following facts about spin structures are well known, follow by elementary obstruction theory, and will be used without mention:

- if the difference of \( \mathbf{sp} \) and \( \mathbf{sp}' \) is zero, then \( \mathbf{sp} \) and \( \mathbf{sp}' \) are equivalent, and
- for a \( q \)-manifold \( Q \) the difference with a fixed spin structure is a 1–1 correspondence between \( H_{q-1}(Q, \partial; \mathbb{Z}_2) = H^1(Q; \mathbb{Z}_2) \) and spin structures on \( Q \) up to equivalence.
For spin structures $sp$ and $sp'$ on $\partial C_1$ let

$$d(sp, sp') \in H_3(N; \mathbb{Z}_2)$$

be the preimage of the difference class in $H_3(\partial C_1; \mathbb{Z}_2)$ under the isomorphism $\nu^1$.

**Proof of the Spin Lemma 3.6.a.** We have

$$\varphi \text{ is spin } \iff d(\varphi_* sp_0, sp_1) = 0 \iff w_2^*(M_\varphi) = 0 \iff M_\varphi \text{ is spin}.$$ 

Here the second equivalence holds because

- $w_2^*(M_\varphi) = i_{C_0, M_\varphi} \hat{A}_0 d(\varphi_* sp_0, sp_1)$ by the naturality of the primary obstruction (the details are analogous to Lemma 3.10 below), and
- $H_6(M_\varphi, C_0) \cong H_5 = 0$, so $i_{C_0, M_\varphi}$ is injective (cf. the Agreement Lemma 3.5.b for $s = 5$).

Let us prove the ‘moreover’ part.

Existence follows by the proof of the ‘if’ part above.

Let us prove uniqueness. The Mayer-Vietoris sequence for $H_6(N, S)$, the class $w_2^*(M_\varphi)$ is represented by an oriented 3-submanifold $\nu$ of $N$. Since $w_2^*(M_\varphi)$ is spin, it is stably equivalent to the stable normal bundle of a parallelizable manifold. Hence $\nu$ is the primary obstruction to homotopy of bundle isomorphisms between $\varphi$ and $\psi$.

**Lemma 3.7.** (a) For every bundle isomorphism $\varphi: \partial C_0 \to \partial C_1$ and $v \in H_3(N; \mathbb{Z}_2)$ there is a bundle isomorphism $\varphi_v: \partial C_0 \to \partial C_1$ such that $d(\varphi_v, \varphi) = v$.

(b) For every pair of bundle isomorphisms $\varphi, \psi: D_0 \to D_1$ and every spin structure $sp$ on $\partial C_0$ we have $d(\psi_* sp, \varphi_* sp) = d(\psi, \varphi)$.

**Proof of (a).** Since $H_3$ has no torsion, $v = \rho_2 \pi$ for some $\pi \in H_3$. Since $H_3 \cong H^1(N) \cong [N, S^1]$, the class $\pi$ is represented by an oriented 3-submanifold $P \subset N$ that is the preimage of a regular value of a map $N \to S^1$ representing $\pi$. Denote by $V \times D^1$ a tubular neighborhood of $V$ in $N$. We have that $e(\nu_0|_V) = e(\nu_0) \cap V = 0$ and that $\nu_0|_V$ is stably equivalent to the stable normal bundle of a parallelizable manifold. Hence $\nu_0|_V$ is trivial. Take a trivialization of $\nu_0$ over $V \times D^1$, i.e. identify $\nu_0^{-1}(V \times D^1)$ and $V \times D^1 \times S^2$. Take a smooth map $\alpha: D_1 \to SO_3$ which maps a neighbourhood of the boundary to the identity and which, modulo the boundary, represents the generator of $\pi_1(SO_3) \cong \mathbb{Z}_2$. Then define

$$\varphi_1(x) := \begin{cases} 
\varphi(x) & x \notin \nu_0^{-1}(V \times D^1), \\
\varphi(a, t, \alpha(t)z) & x = (a, t, z) \in V \times D^1 \times S^2 = \nu_0^{-1}(V \times D^1).
\end{cases}$$

By construction $d(\varphi_1, \varphi) = v$. So part (a) follows by taking $\psi := \varphi_1$.\end{proof}

**Proof of (b).** Carry out the construction of (a) for $\varphi$ and $v := d(\psi, \varphi)$. Now (b) follows because

$$d(\psi_* sp, \varphi_* sp) = d(\varphi_1* sp, \varphi_* sp) = vd = v,$$

where

- the first equality follows because $\psi$ is equivalent to $\varphi_1$ over the 1-skeleton;

\footnote{By (b) the equivalence class of $\varphi_1$ depends only on $v$ not on $V$ and on the trivialization.}
• $a \in V$, $b \in S^2$ are any points and $d \in H^1(a \times D^1 \times b, \partial; \mathbb{Z}_2) \cong \mathbb{Z}_2$ is the relative difference class of the spin structures $\varphi_1 \circ \text{sp}|_{a \times D^1 \times b}$, $\varphi_2 \circ \text{sp}|_{a \times D^1 \times b}$ which coincide on the boundary;
• the second equality follows by the naturality of the primary obstruction;
• the last equality is proved as follows. Take a smooth map $\alpha : D^1 \to SO_3$ which maps a neighbourhood of the boundary to the identity and which, modulo the boundary, represents the generator of $\pi_1(SO_3) \cong \mathbb{Z}_2$. Since the standard inclusion $SO_2 \to SO_3$ induces an epimorphism $\pi_1(SO_2) \to \pi_1(SO_3)$,\(^{10}\) we may assume that $\alpha(t) \in SO_2$ for all $t \in D^1$; i.e. that $\alpha(t)b = b$ for all $t \in D^1$. So $\varphi = \varphi$ over $V \times D^1 \times b$, thus $d = 1$ by definition of a spin structure.

Proof of the Spin Lemma 3.6.b. The result [CS11, Lemma 2.4] asserts that there is a bundle isomorphism $\varphi' : \partial C_0 \to \partial C_1$. Hence using Lemma 3.7 we can modify $\varphi'$ over the 1-skeleton of $N$ to obtain a spin bundle isomorphism $\varphi$. Applying Lemma 3.7 again, and using the fact that $\pi_1(SO_3) = 0$, we obtain that $\varphi$ is unique up to vertical homotopy over the 2-skeleton of $N$. \hfill \square

3.6. String bundle isomorphisms (Lemmas 2.5 and 3.8). For $k = 0, 1$ let $D_k := S^7 - \text{Int} C_k$ and let $st_k$ be the homotopy class of the restriction to $D_k$ of the stable tangent framing on $S^7$.

The following String Lemma 3.8 can be regarded as a ‘complex’ version of the Spin Lemma 3.6.

Lemma 3.8 (String Lemma). Assume that $\varphi(f_0) = \varphi(f_1)$ and $\lambda(f_0) = \lambda(f_1)$. A bundle isomorphism $\varphi : \partial C_0 \to \partial C_1$ is a $\pi$-isomorphism if and only if its extension $\Phi : D_0 \to D_1$ carries $st_0$ to $st_1$.

Remarks. (a) The ‘if’ direction of the String Lemma 3.8 is simple: When $\Phi_* st_0 = st_1$, the stable tangent framings on $C_0, C_1$ coming from $S^7$ agree up to homotopy after identifying $\partial C_0$ and $\partial C_1$ by $\varphi$. So these framings can be glued together to give a stable tangent framing on $M_\varphi$.

(b) For a $\pi$-isomorphism $\varphi : \partial C_0 \to \partial C_1$, a stable tangent framing on $M_\varphi$ whose restrictions to $C_0, C_1$ are induced from $S^7$ is not unique.

For the proof of Lemmas 3.8 and 2.5 we need the following definitions and lemmas.

Lemma 3.9. For every map $\alpha : S^3 \to SO_3$ denote by $\xi(\alpha)$ the oriented 3-dimensional vector bundle over $S^4$

• whose total space is $(\mathbb{R}^3 \times D^4_+) \cup_{\phi(\alpha)} (\mathbb{R}^3 \times (-D^4))$, where $\phi(\alpha)(v, x) = (\alpha(x)v, x)$,

• and whose projection maps $(v, x)$ to $x$.

For a map $\alpha_1 : S^3 \to SO_3$ representing $1 \in \pi_3(SO_3) = \mathbb{Z}$ we have $p_1^*(\xi(\alpha_1)) = 4 \in H_0(S^4) = \mathbb{Z}$.

Proof. The lemma is well-known; we present the proof for completeness. We start with the identity $p_1^* := p_1^*(\xi(\alpha_1)) = \pm 4 \in H_0(S^4) = \mathbb{Z}$ [Mi56]. To determine the sign in this equation let $S\xi(\alpha_1)$ be the total space of the oriented $S^2$-bundle associated to $\xi(\alpha_1)$ and $z \in H_4(S\xi(\alpha_1)) \cong \mathbb{Z}$ a generator. By [Wa66, Theorem 5] $p_1^*(S\xi(\alpha_1)) \cap z \equiv 4z^3 \mod 24$. This and $p_1^* = \pm 4$ imply that $p_1^*(S\xi(\alpha_1)) = 4z^2$, consequently $p_1^* = +4$.

For stable tangent framings $st$ and $st'$ on $D_1$ which are homotopic on the 2-skeleton of $D_1$ their difference

\[ d(st, st') \in H^3(D_1; \pi_3(SO)) = H^3(N) = H_1, \]

\(^{10}\)There is an alternative proof not using this fact, cf. proof of Lemma 3.11 below.
is the primary obstruction to vertical homotopy between them, cf. §3.1. Here we use the zero section \( N \to D^1 \) to identify the cohomology groups.

A bundle isomorphism \( \Phi: D_0 \to D_1 \) is called \textit{spin} if its restriction to the boundary is spin. Since the restriction induces an isomorphism \( H^1(D_k; \mathbb{Z}_2) \to H^1(C_k; \mathbb{Z}_2) \), this is equivalent to carrying the spin structure on \( D_0 \) induced from \( S^7 \) to the spin structure on \( D_1 \) induced from \( S^7 \).

Thus if \( \Phi: D_0 \to D_1 \) is a spin bundle isomorphism, then the difference class \( d(\Phi_*, \text{st}_0, \text{st}_1) \in H_1 \) is well-defined by the uniqueness statements in the Spin Lemma 3.6.a,b.

**Lemma 3.10.** For a spin bundle isomorphism \( \Phi: D_0 \to D_1 \) we have \( p^*_M \phi = i_{C_0, M_\phi} \widetilde{A}d(\Phi_*, \text{st}_0, \text{st}_1) \).

**Proof.** Let

- \( \delta_\varphi \in H^4(D_0 \times I, D_0 \times \partial I) \) be the primary obstruction to the extension of \( \text{st}_0|_{D_0 \times \partial} \cup \text{st}_1|_{D_0 \times 1} \) to a stable tangent framing of \( D_0 \times I \), and
- \( \gamma_\varphi \in H^4(\partial C_0 \times I, \partial) \) be the primary obstruction to the extension of \( \text{st}_0|_{\partial C_0 = \partial C_0 \times 0} \cup \text{st}_1|_{\partial C_0 = \partial C_0 \times 1} \) to a stable tangent framing of \( \partial C_0 \times I \).

We consider the following commutative diagram:

\[
\begin{array}{c}
\delta_\varphi \in H^4(D_0 \times I, D_0 \times \partial I) \xrightarrow{\text{restriction}} H^4(\Sigma(D_0 \times I)) \xrightarrow{\nu^*} H^3(D_0) \xrightarrow{\iota_{C_0, M_\phi}} H_1 \\
\gamma_\varphi \in H^4(\partial C_0 \times I, \partial) \xrightarrow{=} H_3(\partial C_0 \times I) \xrightarrow{=} H_3(M_\phi) \xrightarrow{\phi} H_1
\end{array}
\]

The lemma follows because by the naturality of the primary obstruction

- the image of \( d(\Phi_*, \text{st}_0, \text{st}_1) \) under the first line of isomorphisms is \( \delta_\varphi \);
- the restriction of \( \delta_\varphi \) is \( \gamma_\varphi \);
- the image of \( \gamma_\varphi \) under the second line homomorphisms is \( p^*_M \phi \).

The latter statement follows because

\[
M_\varphi \cong C_0 \bigcup_{\partial C_0 = \partial C_0 \times 0} \partial C_0 \times I \bigcup_{\varphi: \partial C_0 \times 1 \to \partial C_1} (-C_1)
\]

and \( p^*_M \phi \) is the primary obstruction to extending the spin structure on \( M_\varphi \) to a stable tangent framing on \( M_\varphi \). (To see the latter, observe that class \( p^*_M \phi \) is the primary obstruction to lifting \( \overline{\gamma}_M : M_\varphi \to BSpin \) to \( \gamma^* EO \), where \( \gamma^* EO \to BSpin \) is the pullback to \( BSpin \) of the universal \( O \)-bundle along the canonical map \( \gamma: BSpin \to BO \).)

**Proof of the String Lemma 3.8.** By the Spin Lemma 3.6.a, it suffices to prove the result for spin \( \varphi \). For every spin bundle isomorphism \( \varphi \) with extension \( \Phi: D_0 \to D_1 \) we have

\[
\Phi_*, \text{st}_0 = \text{st}_1 \iff d(\Phi_*, \text{st}_0, \text{st}_1) = 0 \iff p^*_M = 0 \iff M_\varphi \text{ is parallelizable.}
\]

Here

- the first equivalence is the completeness of the obstruction;
- the second equivalence holds by Lemma 3.10 because \( \tau(f_0) = \tau(f_1) \), \( \lambda(f_0) = \lambda(f_1) \) and \( H_2 \) has no torsion, so \( i_{C_0, M_\phi} \) is injective by the Agreement Lemma 3.5.b for \( s = 3 \);
- the last equivalence holds because \( \pi_l(SO) = 0 \) for \( l = 4, 5, 6 \). \( \square \)

By the Spin Lemma 3.6.b every two spin bundle isomorphisms \( \Phi, \Psi: D_0 \to D_1 \) are homotopic over a neighborhood of the 2-skeleton of some triangulation of \( N \). Hence the primary (and only) obstruction,

\[
d(\Phi, \Psi) \in H^3(N; \pi_3(SO)) = H_1,
\]

to homotopy of spin bundle isomorphisms from \( \Phi \) to \( \Psi \) is well-defined.
Lemma 3.11. (a) For every \( v \in H_3 \) and spin bundle isomorphism \( \Phi: D_0 \to D_1 \), there is a spin bundle isomorphism \( \Phi_v: D_0 \to D_1 \) such that \( d(\Phi_v, \Phi) = v \).

(b) For every pair of spin bundle isomorphisms \( \Phi, \Psi: D_0 \to D_1 \) and for every stable tangent framing \( st \) on \( D_0 \), we have \( d(\Psi_{s, st}, \Phi_{s, st}) = 2d(\Psi, \Phi) \).

Proof of (a). Take an oriented circle \( V \subset N \) representing \( v \). Denote by \( V \times D^3 \) a tubular neighborhood of \( V \) in \( N \). The restriction \( D\nu_0|_V \) is an oriented bundle over a circle and so is trivial. Take a trivialization of \( D\nu_0 \) and represents (modulo the boundary) the generator \( 1 \in \pi_3(SO_3) \cong \mathbb{Z} \). We define \( \Phi_1 \) by

\[
\Phi_1(x) := \begin{cases} 
\Phi(x) & x \notin (D\nu_0)^{-1}(V \times D^3), \\
\Phi((a, t, \alpha(t) z)) & x = (a, t, z) \in V \times D^3 \times B^3 = (D\nu_0)^{-1}(V \times D^3).
\end{cases}
\]

By construction \( d(\Phi_1, \Phi) = v \). \( \square \)

Proof of (b). Make the construction of (a) for \( v := d(\Psi, \Phi) \). Now (b) follows because

\[
d(\Psi_{s, st}, \Phi_{s, st}) = d(\Phi_{s, st}, \Phi_{s, st}) = vd = 2v,
\]

where

- the first equation follows because \( \Psi \) is equivalent over \( N_0 \) to \( \Phi_1 \);
- \( a \in V \) is any point and \( d \in H^3(a \times D^3 \times 0, \partial) \cong \mathbb{Z} \) is the relative difference class of stable tangent framings \( \Phi_{1, st}|_{a \times D^3 \times 0}, \Phi_{s, st}|_{a \times D^3 \times 0} \) which coincide on the boundary;
- the second equation follows by the naturality of the primary obstruction;
- the last equality is proved as follows. The relative difference class of tangent framings \( \Phi_{1, st}|_{a \times D^3 \times 0}, \Phi_{s, st}|_{a \times D^3 \times 0} \) coinciding on the boundary is \( 1 \). Since the stabilization homomorphism \( \pi_3(SO_3) \to \pi_3(SO) \) is identified with multiplication by \( 2 \), we have \( d = 2 \). \( \square \)

Lemma 3.12. If \( M \) is a closed spin 7-manifold, then \( p^*_M \) is divisible by \( 2 \).

Proof. We have \( \rho_2(p_M) = w_4(M) = v_4(M) = 0 \). Here

- the first equality is proved in [CS11, §3, Proof of Lemma 2.11.b];
- the second equality holds because \( M \) is spin;
- the third equality holds because \( Sq^4: H^3(M; \mathbb{Z}_2) \to H^7(M; \mathbb{Z}_2) \) is trivial. \( \square \)

Proof of Lemma 2.5. We use the String Lemma 3.8. Since \( H_1 \) has no \( 2 \)-torsion, the uniqueness follows by Lemma 3.11.b. Let us prove the existence.

By the Spin Lemma 3.6.b there is a spin bundle isomorphism \( \varphi: \partial C_0 \to \partial C_1 \). Let \( \Phi \) be the extension of \( \varphi \). By Lemmas 3.10 and 3.12, \( i_{C_0\varphi} \hat{A}_0 d(\Phi_{s, st}, st) = p^*_M \varphi \) is even. Since \( \varphi(f_0) = \varphi(f_1), \lambda(f_0) = \lambda(f_1) \) and \( H_2 \) has no torsion, \( i_{C_0\varphi} \) is injective by the Agreement Lemma 3.5.b for \( s = 3 \). Since we assume that \( H_3(M, C_0) \cong H_3(C_1, \partial) \cong H_2 \) is torsion free, it follows that \( d(\Phi_{s, st}, st) \) is also even; i.e., \( d(\Phi_{s, st}, st) = 2v \) for some \( v \in H_1 \). By Lemma 3.11.a there is a spin bundle isomorphism \( \Psi: D_0 \to D_1 \) such that \( d(\Phi, \Psi) = -v \). Then by Lemma 3.11.b

\[
d(\Psi_{s, st}, st) = d(\Phi_{s, st}, st) + d(\Phi_{s, st}, \Psi_{s, st}) = 2v - 2v = 0.
\]

Then by the String Lemma 3.8 \( \Psi \) is a \( \pi \)-isomorphism. \( \square \)
3.7. Joint Seifert classes (Lemmas 2.6, 2.9, 2.10 and 3.13).

**Lemma 3.13** (Description of joint Seifert classes). Let \( \varphi: \partial C_0 \to \partial C_1 \) be a bundle isomorphism and \( i := i_{C_0, M_\varphi} \).

(a) A class \( Y \in H_5(M_\varphi) \) is a joint Seifert class if and only if \( Y \cap C_k = A_k[N] \) for each \( k = 0, 1 \).

(b) Let \( Y \in H_5(M_\varphi) \) be a joint Seifert class. A class \( Y' \in H_5(M_\varphi) \) is a joint Seifert class if and only if

\[
Y' = Y_y := Y + i\widehat{A}_0y \quad \text{for some} \quad y \in H_3.
\]

(c) Let \( Y \in H_5(M_\varphi) \) be a joint Seifert class. Then \( Y_y^2 - Y^2 = 2i\widehat{A}_0\lambda(f_0)(y) \) for every \( y \in H_3 \).

(d) If \( p \in H_4(M_\varphi) \) and \( q \in H_3(C_0) \), then \( p \cap_{M_\varphi} iq = A_0^{-1}(p \cap C_0) \cap_N \widehat{A}_0^{-1}q \).

**Proof of (a).** The ‘if’ part follows because \( Y \cap M_\varphi iS^2_{f_0} = (Y \cap C_0) \cap_{C_0} S^2_{f_0} = A_0[N] = 1 \).

Let us prove the ‘only if’ part. Since \( H_1(C_k) = 0 \) and \( H_2(C_k) \cong \mathbb{Z} \), we have \( H_5(C_k, \partial) \cong \mathbb{Z} \). Since \( (Y \cap C_k) \cap S^2_{f_k} = 1 \), the class \( Y \cap C_k \) equals the generator \( A_k[N] \) of \( H_5(C_k, \partial) \). \( \square \)

**Proof of (b).** Look at the segment of (the Poincaré-Lefschetz dual to) the Mayer-Vietoris sequence:

\[
H_5(\partial C_0) \xrightarrow{i_{\partial C_0, M_\varphi}} H_5(M_\varphi) \xrightarrow{r_0 \oplus r_1} H_5(C_0, \partial) \oplus H_5(C_1, \partial) \xrightarrow{\partial - \partial_1} H_4(\partial C_0).
\]

The ‘only if’ part follows because \( (Y' - Y) \cap S^2_{f_k} = 0 \) and \( \nu_1^1: H_3(N) \to H_5(\partial C_0) \) is an isomorphism, so \( Y' - Y \in \ker(r_0 + r_1) = \text{im } i_{\partial C_0, M_\varphi} = \text{im } (i\widehat{A}_0) \).

The ‘if’ part follows analogously because \( iS^2_{f_k} \cap \text{im } i_{\partial C_0, M_\varphi} = 0 \). \( \square \)

**Proof of (c).** We have

\[
Y_y^2 - Y^2 = 2Y \cap_{M_\varphi} i\widehat{A}_0y = 2i((Y \cap C_0) \cap_{C_0} \widehat{A}_0y) \overset{(3)}{=} 2i(A_0[N] \cap_{C_0} \widehat{A}_0y) = 2i\widehat{A}_0\lambda(f_0)(y).
\]

Here (3) holds by (a) and (4) holds by Lemma 3.2.\( \lambda' \).

**Proof of (d).** We have \( p \cap_{M_\varphi} iq = (p \cap C_0) \cap_{C_0} q = A_0^{-1}(p \cap C_0) \cap_N \widehat{A}_0^{-1}q \). Here the last equality holds by intersection Alexander duality (Lemma 3.1). \( \square \)

**Proof of Lemma 2.6.** Consider the following diagram:

\[
H_4(M_\varphi, C_0; \mathbb{Z}_d) \xrightarrow{i} H_3(M_\varphi; \mathbb{Z}_d) \xrightarrow{j} H_3(C_0; \mathbb{Z}_d) \xrightarrow{\text{ex}} H_3(C_1, \partial; \mathbb{Z}_d).
\]

By Lemmas 3.13.a and 3.2.\( \lambda' \) we have \( \text{ex } j_{C_0, M_\varphi} Y^2 = Y^2 \cap C_1 = (A_1[N])^2 = A_1u \) with \( \mathbb{Z} \)-coefficients (these maps \( \text{ex} \) and \( j_{C_0, M_\varphi} \) are not to be confused with the above \( \mathbb{Z}_d \)-homomorphisms \( \text{ex} \) and \( j \) which are used elsewhere in this proof). Hence \( j\rho_d Y^2 = 0 \). By the Agreement Lemma 3.5.b for \( s = 3, \partial = 0 \). So \( i \) is an isomorphism onto \( \ker j \). Now the existence and the uniqueness of \( b_{\varphi,y} \) follow because \( \widehat{A}_0 = i_{\partial C_0, C_0} \nu_1^1 \) is an isomorphism.

The independence of \( \beta(f_0, f_1) \) from \( Y \) for fixed \( \varphi \) follows by Lemma 3.13.b,c because a change of \( Y \) by \( Y_y \) leads to a change of \( b_{\varphi,y} = \widehat{A}_0^{-1}i_{M}^{-1}\rho_d Y^2 \) by \( 2\rho_d \lambda(f_0)(y) \).

The independence of \( \beta(f_0, f_1) \) from \( \varphi \) is implied by the uniqueness of Lemma 2.5, the independence of \( Y \) for fixed \( \varphi \) and the following Lemma 3.14 applied to \( f_0 = f_1 \) and \( d = \text{div}(\varphi(f_0)) \) (then \( C_0 = C_1 \) but \( \varphi_0 \neq \varphi_1 \) is possible). \( \square \)
Lemma 3.14. Assume that \( C_1 \supset C_0, H_5(C_1, C_0) = 0 \), and \( \varphi_k: \partial C_f \to \partial C_k \), \( k = 0, 1 \), are bundle isomorphisms coinciding over \( N_0 \). Then for every joint Seifert class \( Y_0 \in H_5(M_{\varphi_0}) \) there is a joint Seifert class \( Y_1 \in H_5(M_{\varphi_1}) \) such that \( i_{C_f, M_{\varphi_1}}^{-1} \rho_d Y_1^2 \subset i_{C_f, M_{\varphi_0}}^{-1} \rho_d Y_0^2 \subset H_3(C_f; \mathbb{Z}_d) \) for every \( d \).

Proof. Denote

\[
M_k := M_{\varphi_k} \quad \text{and} \quad \overline{M}_k := C_0 \cup_{\varphi_k|_{N_0}} (-C_1)
\]

so that \( M_k = \overline{M}_k \cup S^2 \times \partial B^5 \). Since \( C_1 \supset C_0 \), we have \( \overline{M}_1 \supset \overline{M}_0 \). Consider the following diagram, where \( r = r_{\overline{M}_0, \overline{M}_1} \) and the coefficients are \( \mathbb{Z} \) or \( \mathbb{Z}_d \):

\[
\begin{array}{c}
\xymatrix{ H_q(M_1) \ar[r]^-{r} \ar[dr]_-{\overline{i}} & H_q(\overline{M}_1, \partial) \ar[r]^-{r} \ar[d]^-{\overline{j}} & H_q(\overline{M}_0, \partial) \ar[r]^-{r} \ar[d]^-{\overline{j}} & H_q(M_0) \ar[d]^-{\overline{i}} \\
H_q(C_f) \ar[ur]_-{i_{C_f, M_1}} & H_q(\overline{M}_1) \ar[ur]_-{\overline{i}_{C_f, M_0}} & H_q(\overline{M}_0) \ar[ur]_-{\overline{i}} & H_q(C_f) \ar[ur]_-{i_{C_f, M_0}} }
\end{array}
\]

From the (Poincaré dual of the cohomology) exact sequence of the pair \((M_1, \overline{M}_1)\) we obtain that \( r_{M_1} \) is an epimorphism for \( q = 5 \). Analogously \( r_{M_0} \) is a monomorphism for \( q = 3 \). By excision \( H_5(\overline{M}_1, \overline{M}_0) \cong H_5(C_1, C_0) = 0 \). Hence from the homology and the cohomology exact sequences of the pair \((\overline{M}_1, \overline{M}_0)\) we obtain that \( r \) is an epimorphism for \( q = 5 \). So we can take \( Y_1 \in r_{\overline{M}_1}^{-1} r r_{M_0} Y_0 \in H_5(M_1) \). Clearly, \( Y_1 \cap S^2 = 1 \), i.e., \( Y_1 \) is a joint Seifert class for \( \varphi_1 \). The required inclusion follows from \( \rho_d Y_1^2 \subset r_{\overline{M}_1}^{-1} r_{M_0} \rho_d Y_0^2 \subset H_3(\overline{M}_1, \partial) \) and the commutativity of the diagram because \( r_{M_0} \) is a monomorphism for \( q = 3 \).

\[\square\]

Proof of the additivity of \( \beta \) (Lemma 2.9). Denote \( h := f \# g \). We have

\[
\beta(h, f) = \beta(f, f) = 0.
\]

Let us prove the second equality. (It also follows by the transitivity of \( \beta \), Lemma 2.10.) The manifold \( M_{id \partial C_f} = \partial(C_f \times I) \) is parallelizable. Take \( Y := \partial(A_f[N] \times I) \in H_5(M_{id \partial C_f}) \).

Clearly, \( Y \) is a joint Seifert class for \( id \partial C_f \). By Lemma 3.2 \( \varphi \) we have \( A_f[N]^2 = A_f \varphi(f) \in H_3(C_f, \partial) \). Then \( Y^2 \in d(\varphi(f)) H_3(M_{id \partial C_f}) \). Hence \( \beta(f, f) = 0 \).

Let us prove the first equality. We may assume that \( g(S^4) \cap C_f = \emptyset \), \( \nu_f = \nu_h \) over \( N_0 \) and \( C_h \supset C_f \). Since \( \pi_4(V_{S^4}) = 0 \) [Pa56], all embeddings \( S^4 \to S^7 \) are regular homotopic [Sm59]. Hence \( f \) and \( h \) are regular homotopic identically on \( N_0 \). A regular homotopy between them extends to a regular homotopy of a tubular neighborhood of \( f N \) in \( S^7 \), identical over \( \nu_f^{-1} N_0 \). The bundle isomorphism \( \varphi: \partial C_f \to \partial C_h \) defined by this regular homotopy is identical over \( N_0 \). Extend \( \varphi \) to a bundle isomorphism \( S^4 \to S^7 \) \( - \int \partial C_f \to \int S^7 \to S^4 \) identical over \( N_0 \). Then by the String Lemma 3.8 \( \varphi \) is a \( \pi \)-isomorphism. Now the first equality holds by Lemma 2.6 and Lemma 3.14 for \( f_0 = f, \varphi_0 = id \partial C_f, Y_0 \) any Seifert class, \( f_1 = h, \varphi_1 = \varphi \) and \( d = d(\varphi(f)) \).

The assumptions of Lemma 3.14 are fulfilled because

\[
H_5(C_h, C_f) \cong H_5(B^4 \times D^3 - h(B^4), B^4 \times \partial D^3) \cong H_5(B^4 \times D^3 - B^4 \times 0, B^4 \times \partial D^3) = 0.
\]

Here the second isomorphism holds by the 5-lemma because of the Alexander duality isomorphism \( H_q(B^4 \times D^3 - h(B^4)) \cong H_{q-6}(B^4, \partial) \cong H_q(B^4 \times D^3 - B^4 \times 0) \). \(\square\)
Proof of the transitivity of \( \beta \) (Lemma 2.10). By Lemma 2.5 we have that there are \( \pi \)-isomorphisms \( \varphi_{01} : \partial C_0 \to \partial C_1 \) and \( \varphi_{21} : \partial C_2 \to \partial C_1 \). Let

\[
V := M_{\varphi_{01}} \times [-1,0] \cup \ldots \cup M_{\varphi_{21}} \times [0,1].
\]

Denote \( \varphi_{kl} := \varphi_{lk}^{-1} \). Observe that \( M_{\varphi_{kl}} = -M_{\varphi_{lk}} \). Then \( \partial V = M_{\varphi_{10}} \cup M_{\varphi_{21}} \cup M_{\varphi_{02}} \).

In this paragraph \( k \in\{10,21\} \). Take any \( x \in H_3 \). By Lemmas 2.4 and 3.5.c there are joint Seifert classes \( Y_{4,k} \in H_5(M_{\varphi_k}) \) and \( Y_{3,k} \in H_4(M_{\varphi_k}), Y_{3,k} \) for \( x \). Denote \( I_{10} = [-1,0] \) and \( I_{21} = [0,1] \). Applying the Mayer-Vietoris sequences for \( V \) we see that there is a class \( \overline{Y}_q \in H_{q+2}(V,\partial) \) such that

\[
\overline{Y}_q \cap (M_{\varphi_k} \times I_k) = Y_{q,k} \times I_k \in H_{q+2}(M_{\varphi_k} \times I_k,\partial) \quad \text{for each} \quad k \in\{10,21\}, \quad q \in\{3,4\}.
\]

Then for each \( q \in\{3,4\} \) the class \( Y_{q,20} := \partial \overline{Y}_q \cap M_{\varphi_{20}} \in H_{q+1}(M_{\varphi_{20}}) \) is a joint Seifert class for \( f_2, f_0 \) and \( \varphi_{20} \), where \( Y_{3,20} \) corresponds to \( x \). So the triple \((V,\overline{Y}_4,\overline{Y}_3)\) is a cobordism between

\[
(M_{\varphi_{20}},Y_{4,20},Y_{3,20}) \quad \text{and} \quad (M_{\varphi_{10}},Y_{4,10},Y_{3,10}) \cup (M_{\varphi_{21}}, Y_{4,21}, Y_{4,21}).
\]

Since \( Y_{4,rl} \cap Y_{3,rl} \) is a characteristic number of such triples,

\[
Y_{3,20} \cap M_{\varphi_{20}} Y_{4,20}^2 = Y_{3,10} \cap M_{\varphi_{10}} Y_{4,10}^2 + Y_{3,21} \cap M_{\varphi_{21}} Y_{4,21}^2 \in \mathbb{Z}.
\]

Denote \( d := \text{div}(\varphi(f_0)) \). By Lemma 2.6 there are \( b_{rl} := \tilde{A}_t^{-1} \tilde{I}_t^{-1} \tilde{M}_{rl} \rho_d Y_{4,rl}^2 \in H_1(N;\mathbb{Z}_d) \). Then by Lemma 3.13.d \( Y_{3,rl} \cap M_{\varphi_{rl}}, Y_{4,rl}^2 = x \cap N b_{rl} \in \mathbb{Z}_d \). Hence \( x \cap N (b_{20} - b_{10} - b_{21}) = 0 \in \mathbb{Z}_d \).

Since this holds for every \( x \in H_3 \) and \( H_1 \) is torsion free, \( b_{20} = b_{10} + b_{21} \in H_1(N;\mathbb{Z}_d) \). By the String Lemma 3.8 the composition \( \varphi_{20} := \varphi_{01}^{-1} \varphi_{21} \) is a \( \pi \)-isomorphism. Hence taking the quotients modulo \( 2\lambda(f_0)(H_3) \) of both sides we obtain \( \beta(f_2,f_0) = \beta(f_1,f_0) + \beta(f_2,f_1) \).

3.8. Calculations of the \( \beta \)-invariant (Lemmas 2.7 and 2.12 for \( \beta \)).

Lemma 3.15 (proved below in §3.8). Let \( f_0, f_1 : S^1 \times S^3 \to S^7 \) be embeddings such that \( \lambda(f_0) = \lambda(f_1) = 0 \). Then

\[
i_{C_0,M_\varphi}\tilde{A}_0^{-1} \tilde{I}_t^{-1} \tilde{M}_{rl} \rho_d Y_{4,rl}^2 = Y^2 = -\frac{1}{4} p_1^0(M_\varphi) \in H_5(M_\varphi)
\]

for any bundle isomorphism \( \varphi : \partial C_0 \to \partial C_1 \) such that \( M_\varphi \) is spin and any joint Seifert class \( Y \in H_5(M_\varphi) \).

Lemma 3.16 (proved below in §3.8). Let \( f_0, f_1 : S^1 \times S^3 \to S^7 \) be embeddings and suppose that \( \varphi, \varphi' : \partial C_0 \to \partial C_1 \) are bundle isomorphisms such that \( \varphi = \varphi' \) over \( S^1 \times D_3^3 \), and over \( S^1 \times D_3^3 \) the bundle isomorphism \( \varphi \) is obtained from \( \varphi' \) by twisting with the \(+1 \in \pi_3(SO_3) = \mathbb{Z} \) (i.e. for the extension \( \Phi : D_0 \to D_1, N = S^1 \times S^3 \) and \( V := S^1 \times 1_3 \) we have \( \varphi' = \Phi_1|_{\partial C_0} \) in the notation of the proof of Lemma 3.11.a.). Then the triple

\[
(M_{\varphi'}, Y_{4}', Y_{3}') \quad \text{is cobordant to} \quad (M_{\varphi}, Y_{4}, Y_{3}) \cup ([CP^3 \times S^1], [C\mathbb{P}^2 \times S^1], [C\mathbb{P}^2 \times 1_3])
\]

for some joint Seifert classes \( Y_q \in H_{q+1}(M_{\varphi}) \) and \( Y_q' \in H_{q+1}(M_{\varphi'}) \), \( q = 3,4 \), where \( Y_3 \) and \( Y_3' \) are those for \( 1 \times S^3 \).

Lemmas 3.16, 4.6 and [Sk08, Cobordism Lemma] are analogous.

\[\text{11} \]There is a unique \( x \in H_3(M_\varphi) \) such that \( 4x = p_1^0(M_\varphi) \). This follows by the proof of the lemma (or because \( p_1^0(M_\varphi) \) is divisible by \( 4 \) by Lemma 3.12 and \( H_3(M_\varphi) \cong \mathbb{Z} \); one proves the latter using the agreement of Seifert classes, Lemma 3.5.b, and the exact sequence of pair \((M_\varphi, C_0)\).
Proof of Lemma 3.15. By Lemma 2.5 there is a $\pi$-isomorphism $\partial C_0 \to \partial C_1$. If $\varphi : \partial C_0 \to \partial C_1$ is a $\pi$-isomorphism, then $M_\varphi$ is parallelizable, so $p_1(M_\varphi) = 0$, hence the required equality holds by definition of $\beta(f_0,f_1)$.

The proof of Lemma 2.5 shows that every spin bundle isomorphism $\varphi : \partial C_0 \to \partial C_1$ can be modified by a sequence of twistings with $\pm 1 \in \pi_3(SO_3) = \mathbb{Z}$ to obtain a $\pi$-isomorphism $\varphi'$. Hence it suffices to prove that if $\varphi, \varphi'$ are spin bundle isomorphisms and $\varphi$ is obtained from $\varphi'$ by twisting with $\pm 1 \in \pi_3(SO_3) = \mathbb{Z}$, then $i^{-1}_{C_0,M_\varphi}(Y') = i^{-1}_{C_0,M_{\varphi'}}((Y')^2 - \frac{1}{i}p^*_{i}(M_{\varphi'}))$ for some joint Seifert classes $Y \in H_5(M_\varphi)$ and $Y' \in H_5(M_{\varphi'})$.

Let us prove this assertion. Take any joint Seifert classes $Y_3 \in H_4(M_\varphi)$ and $Y'_3 \in H_4(M_{\varphi'})$ for $1 \times S^3$. The map $(i_{C_0,M_\varphi}A_0)^{-1} : H_3(M_\varphi) \to H_1(S^1 \times S^3) = \mathbb{Z}$ is an isomorphism coinciding with $x \mapsto x \cap M_\varphi Y_3$. Analogous assertion holds with $\varphi, Y, Y_3$ replaced by $\varphi', Y', Y'_3$. Now the assertion for `+1-modification' follows because by Lemma 3.16

$$(Y'_3)^2 \cap M_{\varphi'} Y'_3 - Y^2 \cap M_\varphi Y_3 = [CP^2 \times S^1]^2 \cap [CP^2 \times S^1] = [CP^1 \times S^1]^2 \cap [CP^2 \times S^1] = 1$$

and

$$p^*_{i}(M_{\varphi'}) \cap M_{\varphi'} Y'_3 - p^*_{i}(M_\varphi) \cap M_\varphi Y_3 = p^*_{i}(CP^3 \times S^1) \cap [CP^2 \times S^1] = 4.$$

The assertion for `-1-modification' is analogous. □

Proof of Lemma 3.16. Take a map $p_3 : S^3 \to SO_3$ representing $+1 \in \pi_3(SO_3)$ and such that $\alpha|_{D^3_+} = id S^2$. Identify $\partial C_0$ with $S^2 \times S^1 \times S^3$ by any bundle isomorphism. Identify $\partial C_1$ with $\partial C_0 = S^2 \times S^1 \times S^3$ by $\varphi$. Define a self-diffeomorphism $\pi$ of $(S^2 \times S^1 \times S^3 \times I) - \left(S^2 \times S^1 \times Int D^3_+ \times \left[\frac{1}{3}, \frac{2}{3}\right]\right)$ by $\pi(x) := \begin{cases} (a(b)a,z,b,t) & x = (a,z,b,t) \in S^2 \times S^1 \times D^3_+ \times \left[\frac{2}{3}, 1\right] \\ x & \text{otherwise.} \end{cases}$

Let $V := (C_0 \times I) \bigcup_{\pi} (C_1 \times I)$ and $\Sigma := S^2 \times D^3 \times \left[\frac{1}{3}, \frac{2}{3}\right]$.

Then $V$ is a cobordism between $M_{\varphi'}$ and $M_\varphi \sqcup E_\alpha \times S^1$, where

$$\hat{\alpha} : \partial \Sigma \to \partial \Sigma \text{ maps } (a,b,t) \text{ to } \begin{cases} (a,b,t) & t < 2 \\ (\alpha(b)a, b, 2) & t = 2 \end{cases}$$

and $E_\alpha := \Sigma \cup \hat{\alpha} \Sigma \cong (S^2 \times D^4 \setminus \{(a,b) \sim (\alpha(b)a, R(b))\} |_{(a,b) \in S^2 \times D^3_+}) \cong CP^3$.

Here $R : D^3_+ \to D^3_+$ is the reflection with respect to $\partial D^3_+ = \partial D^3$. The diffeomorphism (2) is well-known and is proved using a retraction to the dual $(CP^1)^* \subset CP^3$ of the complement to a tubular neighborhood of $CP^1 \subset CP^3$.

By the Agreement Lemma 3.5, $\varphi \partial A_0 = \partial A_1$ and $\varphi' \partial A_0 = \partial A_1$. Hence using the (Poincaré dual of the cohomology) Mayer-Vietoris sequence for $V$ we see that for each $q \in \{3, 4\}$ there are $\nabla_q \in H_q + 2(V, \partial)$ such that $\nabla_q \cap (C_k \times I) = A_k[S^1 \times S^3] \times I \in H_6(C_k \times I, \partial)$ and $\nabla_q \cap (C_k \times I) = A_4[S^1 \times S^3] \times I \in H_5(C_k \times I, \partial)$ for each $k = 0, 1$.\footnote{Alternatively, (2) holds because $E_\alpha$ is an $S^2$-bundle over $S^4$ with characteristic map representing $+1 \in \pi_3(SO_3)$.}
Then $Y_q := \partial \mathcal{Y}_q \cap M_\varphi$ and $Y_q' := \partial \mathcal{Y}_q \cap M_{\varphi'}$ are joint Seifert classes; $Y_4$ is for $[S^1 \times S^3]$ and $Y_3$ is for $[1_1 \times S^3]$. Hence $(V, \mathcal{Y}_4, \mathcal{Y}_3)$ is a cobordism between

$$(M_\varphi, Y_4', Y_3') \quad \text{and} \quad (M_\varphi, Y_4, Y_3) \cup (S^1 \times E_\alpha, \partial \mathcal{Y}_4 \cap (S^1 \times E_\alpha), \partial \mathcal{Y}_3 \cap (S^1 \times E_\alpha)).$$

We have $\partial \mathcal{Y}_4 \cap (S^1 \times E_\alpha) = [S^1] \otimes y_4$ for a certain $y_4 \in H_4(E_\alpha)$. Since

$$\mathcal{Y}_4 \cap (C_0 \times I) = A_0[S^1 \times S^3] \times I,$$

we have

$$y_4 \cap \Sigma = \left[ * \times D^3_\times \left[ \frac{1}{3}, \frac{2}{3} \right] \right] \in H_4(\Sigma, \partial).$$

Hence under (1) $y_4$ goes to a class whose intersection with $[S^2 \times 0]$ in the quotient manifold is $+1$. Therefore under the composition of (1) and (2) $y_4$ goes to a class whose intersection with $[CP^1]$ in $CP^3$ is $+1$, i.e. to $[CP^2]$.

We have $\partial \mathcal{Y}_3 \cap (S^1 \times E_\alpha) = * \otimes y_3$ for a certain $y_3 \in H_4(E_\alpha)$. Analogously to the above under the composition of (1) and (2) $y_3$ goes to $[CP^2]$. □

Take a map $\alpha : S^3 \to \pi_3(V_{4,2})$ representing the element $(0, b)$. In this subsection we abbreviate the subscript $\tau_k$ to $k$, e.g. $\nu_\alpha = \nu_{\tau_0}$.

**Proof of the calculation of $\beta$ (Lemma 2.7.a).** Take a normal vector field $e_1$ on $S^3 \subset S^7$ tangent to $D^4 \supset S^3$ and pointing outside $D^4$. Take the standard framing $S^3 \times D^3 \to S^7$ of the orthogonal complement to $e_1$ in the normal bundle of $S^3 \subset S^7$. Take a normal vector field $e_2$ on $S^3 \subset S^7$ orthogonal to $e_1$ and representing an element $b \in \pi_2(S^3)$ w.r.t. the standard framing. Since the ‘action’ map $\pi_3(SO_3) \to \pi_3(S^3)$ is an isomorphism, $e_2$ can be completed to a framing $e_2, e_3, e_4$ of the orthogonal complement to $e_1$, and this framing is unique up to homotopy. Recall that in $\tau_0$ is formed by ends of $\varepsilon$-length vectors normal to $S^3 \subset S^7$ in the subbundle spanned by $e_1$ and $e_2$ of the normal bundle to $S^3$ in $S^7$ (for some small $\varepsilon$).

Recall that $\nu_\alpha = \nu_{\tau_0}$ is the normal bundle of $\tau_0$. Take a section $\xi_0$ of $\nu_\alpha$ in the 2-plane subbundle, spanned by $e_1$ and $e_2$, of the normal bundle to $S^3 \subset S^7$, so that for every point $(x, y) \in S^1 \times S^3$ the vector $\xi_0(x, y)$ looks into the 2-disk bounded by $\tau_0(S^1 \times y)$ but not outside. Then $\xi_0(x, y)$ is a framing of $\nu_\alpha$.

For every $x \in S^3$ the circle $\xi_0(S^1 \times x) \subset S^3 \times x$ bounds a 2-disk in $x \times D^4$. The union of such 2-disks along $x \in S^3$ is a 5-manifold $W_\alpha \cong D^2 \times S^3$ with boundary $\xi_0(S^1 \times S^3)$. Choose the orientation on $W_\alpha$ so that $\partial W_\alpha = \xi_0(S^1 \times S^3)$. Since $A_\alpha^{-1} = \nu_\alpha \partial$ and $\nu_\alpha \xi_0 = id N$, the manifold $W_\alpha$ represents the relative homology class $[W_\alpha] = A_\alpha[S^1 \times S^3] \in H_5(C_\alpha, \partial)$.

Let

$$D^4_\alpha := (1 - \varepsilon)D^4 \subset S^7 \quad \text{and} \quad S^3_\alpha := 1_1 \times S^3.$$ We have $\partial D^4_\alpha = \xi_0(S^3_\alpha)$. Hence $[D^4_\alpha] = A_\alpha[S^3_\alpha] \in H_4(C_\alpha, \partial)$ for a certain orientation on $D^4_\alpha$.

Make analogous construction for $\alpha$ replaced by the constant map $\alpha_0$. We obtain the standard embedding $\tau_0 = \tau_{\alpha_0}$, a section $\xi_0$, a framing,

$$[W_0] = A_0[S^1 \times S^3] \in H_5(C_0, \partial) \quad \text{and} \quad [D^4_0] = A_0[S^3_\alpha] \in H_4(C_0, \partial).$$

Take a bundle isomorphism $\varphi : \partial C_\alpha \to \partial C_\alpha$ defined by the constructed framings. We may assume that $f_0 = f_\alpha$ over a neighborhood of $S^1 \times 1_3$. Hence $\varphi$ carries the spin structure $sp_\alpha$ on $\partial C_\alpha$ coming from $S^7$ to the spin structure $sp_0$ on $\partial C_0$ coming from $S^7$. Therefore $\varphi$ is spin. Since $\varphi \xi_0 = \xi_0$, we have

$$\varphi \partial W_0 = \varphi \xi_0(S^1 \times S^3) = \xi_0(S^1 \times S^3) = \partial W_\alpha \quad \text{and} \quad \varphi \partial D^4_\alpha = \varphi \xi_0(S^3_\alpha) = \xi_0(S^3_\alpha) = \partial D^4_\alpha.$$ Hence $W_0 \cup_\partial (-W_\alpha)$ and $\Sigma^4 := D^4_0 \cup_\partial (-D^4_\alpha)$ with their natural orientations represent joint Seifert classes $Y_4 \in H_5(M_\varphi)$ and $[\Sigma^4] \in H_4(M_\varphi)$, where $[\Sigma^4]$ corresponds to $[S^7]$. We have $W_0 \cap D^4_\alpha = W_\alpha \cap D^4_\alpha = \emptyset \quad \Rightarrow \quad (W_0 \cup_\partial (-W_\alpha)) \cap \Sigma^4 = \emptyset \quad \Rightarrow \quad Y_4 \cap M_\varphi [\Sigma^4] = 0 \quad \Rightarrow \quad Y^2_4 \cap M_\varphi [\Sigma^4] = 0,$
and \( p_1^*(M_\varphi \cap M_\varphi) [\Sigma^4] = p_1^*(\tau_{M_\varphi}) = p_1^*(\Sigma^4) + p_1^*(\mu) = -4b \),

where

- \( \mu \) is the normal bundle of \( \Sigma^4 \) in \( M_\varphi \);
- the last equality follows because \( S^4 \) is stably parallelizable, the map \( p_1: \pi_3(SO_3) \to \mathbb{Z} \) is multiplication by 4 [DNF12], and \( \mu \) corresponds to the preimage of \(-b\) under the `action’ map \( \pi_3(SO_3) \to \pi_3(S^2) \) because the obstruction to existence of a non-zero section of \( \mu \) is \(-b\).

Let us prove the latter statement. Take the normal vector field \( e_2^0 \) on \( S^3 \subset S^7 \) orthogonal to \( e_1 \) and representing the constant map \( S^3 \to S^2 \) w.r.t. the standard framing. The corresponding section of the normal bundle of \( \partial D_0^4 \subset S^7 \) is tangent to \( \partial C_0 \). That section is mapped under \( d\varphi \) to the section of the normal bundle of \( \partial D_0^4 \subset S^7 \) corresponding to \( e_2 \) (here \( \varphi: \partial C_0 \to \partial C_0 \) is thought of as a diffeomorphism rather than a bundle isomorphism). Clearly, \( e_2^0 \) extends to a non-zero section of the normal bundle of \( D_0^4 \subset S^7 \). The obstruction to extension of \( e_2 \) to a non-zero section of the normal bundle of \( D_0^4 \subset S^7 \) is \( b \). Since \( \mu \) is the bundle over \( D_0^4 \subset S^7 \), the obstruction to the existence of a non-zero section of \( \mu \) is \(-b\).

Thus \( \beta(\tau_0, \tau_\alpha) \in S_\alpha \times S_\alpha [S^3_1] = (Y^4_2 - \frac{1}{4} p_1(M_\varphi)) \cap M_\varphi \Sigma^4 = b \) by Lemmas 3.15 and 3.13.d. Hence \( \beta(\tau_0, \tau_\alpha) = b[S^1 \times S^3] \).

**Proof of the parametric additivity of \( \beta \) (Lemma 2.12).** We use the notation from §3.3. Analogously to Lemma 2.5 one proves that there is a \( \pi \)-isomorphism \( \psi: \partial C_\alpha \to \partial C_0 \) such that \( \psi(\partial C_\alpha \cap D_\pm^7) = (\partial C_0 \cap D_\pm^7) \).

For an embedding \( g \) denote \( C_{g\pm} := C_g \cap D_\pm^7 \). Observe that

\[
C_{f-} = C_{a-} = C_{0-} \cong C_{0+}, \quad C_{h+} = C_{f+} \quad \text{and} \quad C_{h-} = C_{a+}.
\]

Then \( \text{id} C_{f+} + R \) gives diffeomorphisms

\[
C_h \cong C_{f+} \bigcup_{C_{f+} \cap \partial D_+^7 = C_{a+} \cap \partial D_+^7} C_{a+} \quad \text{and} \quad \partial C_h \cong \partial C_{f+} \cap \partial D_+^7 \bigcup_{\partial C_{f+} \cap \partial D_+^7 = \partial C_{a+} \cap \partial D_+^7} \partial C_{a+} \cap \partial D_+^7
\]

(\( C_{a+} \) comes with the positive orientation because \( R \) preserves the orientation). Identify \( \partial C_h \) with the right-hand side of the latter equality. Let \( \varphi: \partial C_h \to \partial C_f \) be \( \text{id}(\partial C_{f+} \cap D_+^7) \cup \psi|_{\partial C_{a+} \cap D_+^7} \). Then by the String Lemma 3.8 \( \varphi \) is a \( \pi \)-isomorphism. (This is proved by considering the restrictions of stable tangent framings coming from \( S^7 \) to \( \partial C_f \cap D_+^7 \), \( \partial C_\alpha \cap D_+^7 \) and \( \partial C_0 \cap D_+^7 \).)

Let

\[
V := M_{\text{id} \partial C_f} \times [-1, 0] \bigcup_{R_3} (-M_\psi) \times [0, 1],
\]

where \( R_3: (C_{f-} \cup (-C_{f-})) \times 0 \to ((-C_{f-}) \cup C_{a-}) \times 0 \) is the union of two copies of the reflection with respect to the hyperplane \( x_3 = 0 \). Then \( \partial V = M_\varphi \cup (-M_\psi) \cup (-M_{\text{id} \partial C_f}) \). Now the proof is completed analogously to the second paragraph of the proof of \( \beta \)-transitivity (Lemma 2.10) using the calculation of \( \beta \) (Lemma 2.7.a).

**3.9. Appendix: some remarks to §3.**

**Remark 3.17** (to §3.2). (a) The class \( \varpi \in H_2 \) measures the linking of 2-cycles in \( N \) and the ‘top cell’ of \( N \) under \( f: f = f’ \) on \( N_0 \), then \( \langle \varpi(f’ - \varpi(f)) \cap N_0 = 2A^{-1}_{f|_{N_0}} [f(B^4) \cup f’(B^4)] \rangle \in H_2(N_0; \mathbb{Z}) \approx H_2 \) (this is proved analogously to [Sk08’, §2, The Boéchat-Haefliger invariant Lemma]).
(b) Weakly unlinked sections may differ on a 3-skeleton of \( N_0 \) even up to vertical homotopy, and \textit{a priori} changing \( \xi \) on a 3-skeleton could change the integer \( \text{lk}_{S^7}(fX, \xi Y) \) in the definition of \( \lambda \). However different choices of \( \xi \) do not change \( \text{lk}_{S^7}(fX, \xi Y) \). The formal explanation for this is given in Lemma 3.2. Informally, the change is trivial because it ‘factors through’ \( H_3(S^2) = 0 \).

(c) If in the definitions of \( \varpi \) and \( \lambda \) we would take an arbitrary (i.e. not weakly unlinked) section \( \xi \), we would obtain different values. Note that \( 2\lambda(x, y) = \text{lk}_{S^7}(f(x, \xi y) + \text{lk}_{S^7}(f(x, -\xi y) \mod 2 \text{ for every } x \text{.} \) \( \text{We actually only need the regular homotopy over } N \). Proposition 2.14. homotopy classification of embeddings (Proposition 2.14) \( f_0 \) and \( f_1 \) are regular homotopic. (We only need the regular homotopy over \( N \).

(d) Although a weakly unlinked section is only defined over \( N_0 \), its construction involves all of the embedding \( f \) via the inclusions \( \nu^{-1}N_0 \to \partial C \to C \), and not only \( f|_{N_0} \). For embeddings \( N_0 \to S^7 \) an analogue of \( \varpi \) is not defined and only ‘a part’ of \( \lambda \) is defined (D. Tonkonog, unpublished, cf. [To10]).

(e) Let \( \xi : N_0 \to \nu^{-1}N_0 \) be a section such that \( i_C \xi \) is weakly unlinked. Then \( \nu \varpi = \lambda - \partial A \) on \( H_3 \) and \( \nu \varpi = \lambda - \partial A \) on \( H_2 \).

\textbf{Proof.} Let \( q \in \{2, 3\} \). Since \( \nu \partial A = \text{id} H_q = \nu \xi \), for every \( y \in H_q \) there is a class \( x(y) \in H_{q-2} \) such that \( \partial Ay - \xi y = \nu'x(y) \). Then \( 0 = i_C \partial Ay = i_C \xi y + i_C \nu'x(y) = i_C \xi y + \bar{A}x(y) \). Now the required equalities follow by Lemma 3.2.λ, \( \varpi \).

\textbf{Remark 3.18} (to \S 3.4). (a) Lemma 2.4 also follows from Lemmas 3.5.c and 3.13.a.

(b) An alternative proof of the Agreement Lemma 3.5.a,b for \( q \in \{2, 3, 4\} \). We generalize the proof of [CS11, Agreement Lemma 2.5] which is part (a) for \( q = 4 \).

Let \( \xi_0 : N_0 \to \partial C_0 \) be a weakly unlinked section for \( f_0 \). Since \( \varpi(f_0) = \varpi(f_1) \), \( \lambda(f_0) = \lambda(f_1) \) and \( H_2 \) has no 2-torsion, by Lemma 3.4 \( \xi_1 = \partial \xi_0 \) is a weakly unlinked section for \( f_1 \).

We have \( \nu_1 \varphi \partial_0 A_0 = \nu_0 \partial_0 A_0 = \text{id} H_q = \nu_1 \partial_1 A_1 \). Also

\[
\xi_1 \varphi \partial_0 A_0 = \xi_0 \partial_0 A_0 \overset{(2)}{=} \left\{ \begin{array}{ll}
\varpi(f_0) = \xi_0 \nu \varpi = \xi_1 \xi_0 & q = 4 \\
\xi_1 \xi_0 - \xi_1 \nu \varpi = \xi_0 \nu \varpi - \varpi(f_0) & q = 3 \\
\xi_1 \xi_0 - \xi_1 \nu \varpi = \xi_0 \nu \varpi - \varpi(f_0) & q = 2
\end{array} \right. \overset{(3)}{=} \xi_1 \partial_1 A_1.
\]

Here

- (2) holds because \( \xi_0 \partial_0 A_0 [N] = \varpi \) [Sk10, Lemma 2.5.b] and by Remark 3.17.e.
- (3) holds because \( \xi_1 \xi_0 = \xi_0 \varphi^{-1} \varphi \xi_0 = \xi_1 \xi_1 \).

Now (a) follows because the map \( \varphi_1 : H_3(\partial C_1) \to H_3 \oplus H_1 \) is an isomorphism.

Let \( i_0 : = i_{\partial C_1, C_1} \) and consider the diagram from §3.4. Part (b) follows from (a) because

\[
(a) \quad \Rightarrow \quad \varphi \im(\partial_0 A_0) = \im(\partial_1 A_1) \quad \Leftrightarrow \quad \varphi \im(\partial_0 A_0) = \im(\partial_1) \quad \Leftrightarrow \quad \varphi \ker i_0 = \ker i_1 \quad \Rightarrow \\
\Rightarrow \quad \varphi \ker i_0 \supset \im(\partial_1) \quad \Leftrightarrow \quad \ker(\varphi^{-1} \partial_1) = 0 \quad \Leftrightarrow \quad (b).
\]

(In fact, (a) \( \Leftrightarrow \) \( \varphi \ker i_0 = \ker i_1 \), see the proof of §3.4.)

\textbf{Remark 3.19} (to §3.6). The existence part of Lemma 2.5 (under weaker assumption \( \lambda(f_0)(x, x) \equiv \lambda(f_1)(x, x) \mod 2 \text{ for every } x \)) can be proved as follows. By the regular homotopy classification of embeddings (Proposition 2.14) \( f_0 \) and \( f_1 \) are regular homotopic.

(We actually only need the regular homotopy over \( N_0 \) which is easier to prove.) A regular homotopy between \( f_0 \) and \( f_1 \) extends to a regular homotopy of a tubular neighborhood of \( f_0(N) \) in \( S^7 \). By the String Lemma 3.8 the bundle isomorphism defined by this regular homotopy is a \( \pi \)-isomorphism. (In §3.6 we give another proof together with the proof of the uniqueness part of Lemma 2.5.)
Remark 3.20 (to §3.7). (a) In Lemma 3.13 we do not assume that \( \lambda(f_0) = \lambda(f_1) \) or \( \chi(f_0) = \chi(f_1) \). However, we apply this lemma when \( \lambda(f_0) = \lambda(f_1) \), because the existence of a joint Seifert class (Lemma 2.4) requires this assumption. If \( \lambda(f_0) \neq \lambda(f_1) \), then \( \lambda(f_0)(H_3) \neq \lambda(f_1)(H_3) \) is possible, (but \( i_{C_0,M_\varphi}\lambda(f_0)(H_3) = i_{C_1,M_\varphi}\lambda(f_1)(H_3) \) by (b) and (c)). This is possible because \( i_{C_0,M_\varphi} \) or \( i_{C_1,M_\varphi} \) need not be injective when \( \lambda(f_0) \neq \lambda(f_1) \).

(b) In applications of Lemma 3.14 both subsets of \( H_3(C_f;\mathbb{Z}_3) \) consist of one element \( \hat{\lambda}f_0,\nu_0 = \hat{\lambda}f_1,\nu_1 \), where \( \nu_\varphi,Y \) is defined in Lemma 2.6.

Remark 3.21 (to §3.8). The following result (not used in this paper) is proved analogously to Lemma 3.15. If \( \lambda(f_0) = \lambda(f_1), \chi(f_0) = \chi(f_1) \), then \( \varphi: \partial C_0 \to \partial C_1 \) is a bundle isomorphism such that \( M_{\varphi} \) is spin and \( Y \in H_5(M_{\varphi}) \) is a joint Seifert class, then \( \beta(f_0,f_1) = [\hat{A}_0^{-1}i_{C_0,M_{\varphi}}\rho_{\text{div}(\chi(f_0))}(Y^2 - \frac{1}{2}p_1(M_{\varphi}))] \in \mathcal{C}_{\lambda(f_0),\chi(f_0)} \).

4. PROOF OF THE MK ISOTOPY CLASSIFICATION THEOREM 2.8

4.1. The obstruction \( \eta(\varphi,Y) \) and its properties. Recall that some notation was introduced in §§1.2, 1.3, 2.1 and 3.1.

Denote \( d_0 := \text{div}(\chi(f_0)) \). In what follows, a statement involving \( k \) holds for both \( k = 0,1 \).

A joint Seifert class \( Y \in H_5(M_{\varphi}) \) is called a \( d \)-class for an integer \( d \) if \( \rho_dY^2 = 0 \) (or, equivalently, \( Y^2 \in dH_3(M_{\varphi}) \)).

Lemma 4.1. If \( \lambda(f_0) = \lambda(f_1), \chi(f_0) = \chi(f_1), \beta(f_0,f_1) = 0 \) and \( \varphi: \partial C_0 \to \partial C_1 \) is a \( \pi \)-isomorphism, then there is a \( d_0 \)-class for \( \varphi \).

Lemma 4.1 follows from Lemmas 2.4 and 3.13 bc together with the definition of \( \beta(f_0,f_1) \).

For a 8-manifold \( W \) we consider the intersection products

\[
\cap_\theta: H_4(W) \times H_4(W,\partial) \to \mathbb{Z} \quad \text{and} \quad \cap_{\partial_0}: H_6(W,\partial) \times H_6(W,\partial) \to H_4(W,\partial).
\]

As before, for the corresponding squares \( H_6(W,\partial) \to H_4(W,\partial) \) we do not put any subscript.

The following lemma is well-known. Let \( \nu_Q \) be the stable equivalence class of the normal bundle of a manifold \( Q \). A stable normal spin structure on \( Q \) is a framing \( \xi \) of \( \nu_Q \) over the 2-skeleton of \( Q \) compatible with the orientation of \( Q \), considered up to the homotopy of the restriction of \( \xi \) to the 1-skeleton. For brevity we omit ‘stable’ (analogously to stable tangent spin structures).

Lemma 4.2. If a manifold has a spin structure, then it has a normal spin structure.

Definition of \( \eta(\varphi,Y) \) for a \( \pi \)-isomorphism \( \varphi: \partial C_0 \to \partial C_1 \) and a \( d \)-class \( Y \in H_5(M_{\varphi}) \).

Since \( \varphi: \partial C_0 \to \partial C_1 \) is a \( \pi \)-isomorphism, \( M_{\varphi} \) is spin. Take any normal spin structure on \( M \) given by Lemma 4.2. Since \( M_{\varphi} \) is simply-connected, a normal spin structure on \( M_{\varphi} \) is unique. Since \( \Omega_7^{\text{Spin}}(CP\infty) = 0 \) [KS91, Lemma 6.1] there is a 8-manifold \( W \) with a normal spin structure and \( z \in H_6(W,\partial) \) such that \( \partial W = M_{\varphi} \) and \( \partial z = Y \). Consider the following fragment of the exact sequence of the pair \( (W,\partial W) \):

\[
H_4(\partial W;\mathbb{Z}_d) \xrightarrow{jw} H_4(W;\mathbb{Z}_d) \xrightarrow{jw} H_4(W,\partial;\mathbb{Z}_d) \xrightarrow{\partial w} H_5(\partial W;\mathbb{Z}_d).
\]

Since \( \partial W \rho_dz^2 = \rho_dY^2 = 0 \), there is a class \( \overline{z^2} \in H_4(W;\mathbb{Z}_d) \) such that \( jw \overline{z^2} = \rho_dz^2 \). Define

\[
\eta(\varphi,Y) = \eta(f_0,f_1,d,\varphi,Y) := \rho_d(\overline{z^2} \cap_\theta (z^2 - p_1^*)) \in \mathbb{Z}_{\tilde{d}}, \quad \text{where} \quad \tilde{d} := \gcd(d,24).
\]
Proof that $\eta(\varphi, Y)$ is well-defined, i.e. independent of the choice of $W, z$ and $\overline{z^2}$. The proof is analogous to [CS11, 2.3 and footnote (q)]. For the independence from $\overline{z^2}$ instead of $[CS11, \text{Lemma } 2.7]$ we use $\partial_W p_W = p_{M_\varphi}^* = 0$. For the independence from $W, z$ instead of the uniqueness of $\partial_W z$ of [CS11, \text{Lemma } 2.6] we use that $\partial_W z = Y$ is fixed. □

Lemma 4.3 (proved in §4.3). Let $\varphi: \partial C_0 \to \partial C_1$ be a $\pi$-isomorphism and $Y \in H_5(M_\varphi)$ a $d_0$-class.

(a) (Divisibility of $\eta$ by 2) The residue $\eta(\varphi, Y) \in \mathbb{Z}_{d_0}$ is even.

(b) (Change of $\eta$) There is an embedding $g : S^4 \to S^7$, a $\pi$-isomorphism $\varphi' : \partial C_0 \to \partial C_{f_1 \# g}$ and a $d_0$-class $Y' \in H_5(M_{\varphi'})$ such that $\eta(\varphi', Y', f_0, f_1, \# g, d_0) = \eta(\varphi, Y, f_0, f_1, d_0) + 2$.

(c) (Change of $\varphi$) For every $\pi$-isomorphism $\varphi' : \partial C_0 \to \partial C_1$ there is a $d_0$-class $Y' \in H_5(M_{\varphi'})$ such that $\eta(\varphi', Y') = \eta(\varphi, Y)$.

Note that Lemma 4.3.a is trivial for $\widehat{d}_0$ odd.

Other properties of $\eta$ which are not used in this paper will be discussed in [II].

4.2. Proof of Theorem 2.8 using Theorem 4.5 and Lemmas 4.3, 4.4. In the definition of $\eta(\varphi, Y)$ in §4.1 instead of $C_0, C_1, \varphi, Y$ we can take any pair of simply-connected parallelizable 7-manifolds $M_0, M_1$, a diffeomorphism $\varphi : \partial M_0 \to \partial M_1$ such that the manifold $M := M_0 \cup_\varphi (-M_1)$ is parallelizable and any class $Y \in H_6(M)$ such that $\rho_4 Y^2 = 0$. Denote by $\eta(\varphi, Y) = \eta(M_0, M_1, d, \varphi, Y) \in \mathbb{Z}_{\overline{d}}$ the obtained residue. Also, in this situation for $d$ even we can define $\eta'(\varphi) = \eta(M_0, M_1, d, \varphi) := \rho_2(\overline{z^2} \cap_0 \overline{z^2}) \in \mathbb{Z}_2$.

The proof that $\eta'(\varphi)$ is independent of the choice of $W, z$ and $\overline{z^2}$ is analogous to the case of $\eta(\varphi, Y)$.

Lemma 4.4. Assume that $d_0$ is even and $\varphi : \partial C_0 \to \partial C_1$ is a $\pi$-isomorphism.

(a) (proved in §4.4) The residue $\eta'(\varphi)$ is well-defined, i.e. is independent of the choice of $Y$.

(b) (Change of $\eta'$; proved in §4.3) There is a $\pi$-isomorphism $\varphi' : \partial C_0 \to \partial C_1$ such that $\eta'(\varphi') = \eta'(\varphi) + 1$.

Theorem 4.5 (Almost Diffeomorphism Theorem; proved in §4.5). Let

- $M_0, M_1$ be oriented simply-connected 7-manifolds whose homology groups are free abelian and such that $H_5(M_k, \partial) \cong \mathbb{Z}$;
- $\varphi : \partial M_0 \to \partial M_1$ be a diffeomorphism such that $M := M_0 \cup_\varphi (-M_1)$ is a parallelizable oriented manifold for which $H_2(M), H_3(M)$ are free abelian and $j_k := j_{M_k, M} : H_4(M) \to H_4(M, M_k), k = 0, 1$, are epimorphisms having the same kernel;
- $Y \in H_5(M)$ be a class such that $Y \cap M_k$ is a generator $\alpha_k \in H_5(M_k, \partial)$, $\text{div}(Y^2) = \text{div}(\alpha_0^2) = \text{div}(\alpha_1^2) =: d$ and there is a class $Q \in H_4(\partial M_0)$ such that $i_M Q \cap_0 Y^2 = d$.

For some homotopy 7-sphere $\Sigma$ there is an orientation-preserving diffeomorphism $\overline{\varphi} : M_0 \to M_1 \# \Sigma$ extending $\varphi$ and such that $\overline{\varphi} \alpha_0 = \alpha_1 \# 0$ if\footnote{We conjecture that the ‘only if’ statement is also true.}

$$
\eta(\varphi, Y) = 0 \quad \text{and, for } d \text{ even}, \quad \eta'(\varphi) = 0.
$$

Proof of Theorem 2.8 using the Almost Diffeomorphism Theorem 4.5. Since $\lambda(f_0) = \lambda(f_1)$ and $\pi(f_0) = \pi(f_1)$, by Lemma 2.5 there is a $\pi$-isomorphism $\varphi : \partial C_0 \to \partial C_1$. Since $\beta(f_0, f_1) = 0$, by Lemma 4.1 there is a $d_0$-class $Y \in H_5(M_\varphi)$.\footnote{We conjecture that the ‘only if’ statement is also true.}
By the divisibility of $\eta(\varphi, Y)$ by 2 (Lemma 4.3.a) and by a change of $\eta$ (Lemma 4.3.b) we can change $f_1$ (by connected sum with a knot), $\varphi$ and $Y$, and assume that $\eta(\varphi, Y) = 0$.

In this paragraph assume that $d_0$ is even. Then by a change of $\eta'$ (Lemma 4.4.b) we obtain a new $\pi$-isomorphism $\varphi$ such that $\eta'(\varphi) = 0$. By a change of $\varphi$ (Lemma 4.3.c) we obtain a new $d_0$-class $Y$ such that $\eta(\varphi, Y) = 0$.

Since $7 = 4 + 3$, by general position $C_k$ are simply-connected. The groups $H_3(C_k) \cong H_1$ and $H_4(C_k, \partial) \cong H_2$ are free abelian. Hence by Lefschetz duality $H_2(C_k)$ is free abelian. So all homology groups of $C_k$ are free abelian. Since $\lambda(f_0) = \lambda(f_1)$, the group $H_2(M_\varphi)$ is free abelian;

- by the exact sequence of the pair $(M_\varphi, C_k)$ and the Agreement Lemma 3.5.b for $q = 2$ the group $H_2(M_\varphi)$ is free abelian;

- by the Agreement Lemma 3.5.b for $q = 3$ the map $j_k: H_4(M_\varphi) \to H_4(M_\varphi, C_k)$ is onto.

Since $\tilde{A}_k: H_2 \to H_4(C_k)$ is an isomorphism and $i_{C_1, M_\varphi} \tilde{A}_1 = i_{\partial C_1, M_\varphi} \varphi \nu_0 = i_{C_0, M_\varphi} \tilde{A}_0$, we have $im i_{C_0, M_\varphi} = im i_{C_1, M_\varphi}$. Hence $ker j_{C_0, M_\varphi} = ker j_{C_1, M_\varphi}$. Then from the exact sequence of the pair $(M_\varphi, C_0)$ we obtain that $H_3(M_\varphi)$ and $H_2(M_\varphi)$ are free abelian.

By Alexander duality, $\alpha_k := A_k[N]$ is a generator of $H_5(C_k, \partial)$. We have $d_0 = div(\lambda(f_0)) = div(\alpha_k^2)$. By Lemmas 3.13.a and 3.5.c there is a class $Y \in H_5(M)$ such that $\partial C_1 = \alpha_k$. Since $Y^2$ is divisible by $d_0$ and $A_0 \lambda(f_0) = A_0 (Y \cap C_0)^2$, we have $div(Y^2) = d_0$.

Take $Q' \in H_2$ such that $Q' \cap N \lambda(f_0) = d_0$. Let $Q := \nu_0 Q' \in H_4(\partial C_0)$. Then

$$i_{C_0, M_\varphi} Q \cap_{M_\varphi} Y^2 = i_{C_0, M_\varphi} \tilde{A}_0 Q' \cap_{M_\varphi} Y^2 = \tilde{A}_0 Q' \cap_{C_0} (Y \cap C_0)^2 = Q' \cap N \lambda(f_0) = d_0.$$ 

Hence by the Almost Diffeomorphism Theorem 4.5 for $M_0 = C_0$, $M_1 = C_1$ and $M = M_\varphi$ there is an orientation preserving diffeomorphism $\varphi: C_0 \to C_1 \# \Sigma$ extending the bundle isomorphism $\varphi$. The bundle isomorphism $\varphi$ also extends to an orientation-preserving diffeomorphism $S^7 - Int C_0 \to S^7 - Int C_1$. So $S^7 \cong S^7 \# \Sigma \cong \Sigma$. Then $f_0$ and $f_1$ are isotopic by Lemma 1.5.

### 4.3. Proofs of Lemmas 4.3 and 4.4.b.

**Proof of Lemma 4.3.a.** Take any pair $(W, z)$ from the definition of $\eta(\varphi, Y)$ and take some map $Z: W \to \mathbb{CP}^\infty$ corresponding to $z \in H_6(W, \partial) \cong H^2(W) \cong [W, \mathbb{CP}^\infty]$. By spin surgery of $Z$ relative to $\partial W$ we may assume that $Z$ is 3-connected. The residue $\tilde{z}^2 \cap_\partial (z^2 - p_W)$ does not change throughout this surgery because it is ‘spin $\mathbb{CP}^\infty$-characteristic residue modulo $d$ relative to the boundary’. Since $Z$ is 3-connected, by the Hurewicz Theorem for the mapping cylinder of $Z$ we have $H_3(W) = \pi_3(W) = \pi_3(\mathbb{CP}^\infty) = 0$. Hence $Tors H_4(W) \cong Tors H_5(W) = 0$. So there is a class $\tilde{Z}^2 \in H_4(W)$ such that $\rho_{d_0} \tilde{Z}^2 = \tilde{Z}^2$. Then

$$\tilde{Z}^2 \cap_\partial (z^2 - p_W) = \rho_{d_0} (z^2 \cap_\partial z^2 - z^2 \cap_\partial p_W) = \rho_{d_0} (\tilde{Z}^2 \cap_\partial \tilde{Z}^2 - \tilde{Z}^2 \cap_\partial p_W).$$

The latter residue is divisible by 2 by [CS11, Lemma 2.11].

**Lemma 4.3.b** is proved analogously to [CS11, §3, the second equality of Addendum 1.3].

For the proofs of Lemmas 4.3.c and 4.4.b we need the following result.

**Lemma 4.6** (proved below in §4.3). Assume that $\pi$-isomorphisms $\varphi', \varphi: \partial C_0 \to \partial C_1$ coincide over $N_0$ and that over $Cl(N - N_0)$ they differ by the generator of $\pi_4(SO_3) \cong \mathbb{Z}_2$. Then for every integer $d$ and $d$-class $Y \in H_5(\varphi')$ there is a $d$-class $Y' \in H_5(\varphi')$ such that the pair

$$(M_{\varphi'}, Y')$$

is cobordant to

$$(M_{\varphi}, Y) \sqcup (S^2 \times S^5, A),$$
where $S^2 \tilde{\times} S^5$ is the total space of the non-trivial $S^2$-bundle over $S^5$ (i.e. the bundle corresponding to the non-trivial element of $\pi_4(\text{SO}_3) \cong \mathbb{Z}_2$) and $A \in H_5(S^2 \tilde{\times} S^5) \cong \mathbb{Z}$ is a generator.

**Proof of Lemma 4.3.c.** By Lemma 2.5 we may assume that $\varphi' = \varphi$ over $N_0$. We may also assume that over $\text{Cl}(N - N_0)$ isomorphism $\varphi'$ obtained from $\varphi$ by twisting with $d(\varphi', \varphi) \in \pi_4(\text{SO}_3) \cong \mathbb{Z}_2$. If $d(\varphi', \varphi) = 0$, then we may assume that $\varphi' = \varphi$ and take $Y' = Y$. If $d(\varphi', \varphi) \neq 0$, then by Lemma 4.6 and a calculation in [CS11, Proof of the Framing Theorem 2.9] $\eta(\varphi', Y') = \eta(\varphi, Y)$.

**Proof of Lemma 4.4.b.** We do not assume Lemma 4.4.a. and so we write $\eta'(\varphi, Y)$ instead of $\eta'(\varphi)$ and prove the lemma in the following form.

For every $\pi$-isomorphism $\varphi: \partial C_0 \to \partial C_1$, even integer $d$ and $d$-class $Y \in H_5(M_\varphi)$ there is a $\pi$-isomorphism $\varphi': \partial C_0 \to \partial C_1$ and a $d$-class $Y' \in H_5(M_{\varphi'})$ such that $\eta(\varphi, Y') = \eta'(\varphi, Y) + 1$.

Take a bundle isomorphism $\varphi': \partial C_0 \to \partial C_1$ coinciding with $\varphi$ over $N_0$ and over $\text{Cl}(N - N_0)$ obtained from $\varphi$ by twisting with the non-trivial element of $\pi_4(\text{SO}_3) \cong \mathbb{Z}_2$. Then by the String Lemma 3.8 $\varphi'$ is a $\pi$-isomorphism. By Lemma 4.6 and a calculation in [CS11, Proof of the Framing Theorem 2.9] $\eta'(\varphi', Y') = \eta'(\varphi, Y) + 1$.

**Proof of Lemma 4.6.** Take a smooth map $\alpha: S^4 \to \text{SO}_3$ representing the non-trivial element of $\pi_4(\text{SO}_3) \cong \mathbb{Z}_2$ and such that $\alpha|_{D^+_4} = \text{id} S^2$. For $k = 0, 1$ identify

$$\nu_k^{-1} \text{Cl}(N - N_0) \times \left[ \frac{1}{3}, \frac{2}{3} \right] \text{ with } \Sigma_k := S^2 \times D^4 \times \left[ \frac{1}{3}, \frac{2}{3} \right] \text{ (so } \Sigma_0 = \Sigma_1).$$

Let $U_k := \partial C_k \times I - \text{Int} \Sigma_k$. Define

$$\overline{\alpha}: U_0 \to U_1 \text{ by } \overline{\alpha}(s, t): = \begin{cases} (\varphi(s), t) & s \in \nu_0^{-1}(N - N_0), \ t \in \left[ \frac{2}{3}, 1 \right] \\ (\varphi'(s), t) & \text{otherwise} \end{cases}$$

and $V := C_0 \times I \bigcup_{\pi} C_1 \times I$.

Hence $V$ is a cobordism between $M_{\varphi'}$ and $M_{\varphi} \sqcup E_{\alpha}$, where

$$\widehat{\alpha}: \partial \Sigma_0 \to \partial \Sigma_1 \text{ maps } (a, b, t) \text{ to } \begin{cases} (a, b, t) & t < \frac{2}{3} \\ (\alpha(b)a, b, \frac{2}{3}) & t = \frac{2}{3} \end{cases}$$

and $E_{\alpha} := \Sigma_0 \cup_{\partial} \Sigma_1 \cong S^2 \times D^5 / \{(s, b) \sim (\alpha(b)s, R(b)) \} (s, b) \in S^2 \times D^+_4$.

Here $R: D^4_+ \to D^4_+$ is the reflection with respect to $0 \times \mathbb{R}^4$.

Consider the following commutative diagram:

$$
\begin{array}{ccc}
H^q(V, C_0 \times I) & \xrightarrow{\text{ex}} & H^q(C_1 \times I, U_1) \\
\downarrow \text{i} & & \downarrow \text{i} \\
H^q(M_{\varphi}, C_0) & \xrightarrow{\text{ex}} & H^q(C_1, \partial) \\
& \xrightarrow{\text{i}_{C_1 \times I}} & H^q(C_1 \times I, \partial C_1 \times I) \\
\end{array}
$$

We have

$$H^q(\partial C_1 \times I, U_1) \cong H^q(\Sigma_1, \partial) \cong H_{7-q}(\Sigma_1) = 0 \quad \text{for} \quad q = 1, 2, 3, 4.$$
Hence from the exact sequence of the triple \( U_1 \subset \partial C_1 \times I \subset C_1 \times I \) we see that \( i_{C_1 \times I} \) is injective for \( q = 2, 3, 4, 5 \). Hence \( i \) is an isomorphism for \( q = 2, 3, 4, 5 \). Look at the inclusion-induced mapping of the exact sequences of pairs \((M_\varphi, C_0)\) and \((V, C_0 \times I)\). By the 5-lemma we see that the inclusion \( M_\varphi \rightarrow V \) induces an isomorphism in \( H^q(\cdot) \) for \( q = 2, 4 \). Or, in Poincaré dual form, \( r_{M_\varphi} \partial_V : H_q(V, \partial) \rightarrow H_{q-1}(M_\varphi) \) is an isomorphism for \( q = 4, 6 \). The same holds for \( \varphi \) replaced by \( \varphi' \).

Let \( \overline{Y} := (r_{M_\varphi} \partial_V)^{-1} Y \in H_0(V, \partial) \). Then by Lemma 3.13.a

\[
\overline{Y} \cap (C_k \times I) = A_k[N] \times I \in H_0(C_k \times I, \partial) \quad \text{for each } k = 0, 1.
\]

So by Lemma 3.13.a \( Y' := r_{M_\varphi} \partial_V \overline{Y} \in H_5(M_\varphi) \) is a joint Seifert class. We have the equation

\[
\rho_d(Y')^2 = \rho_d r_{M_\varphi} \partial_V (r_{M_\varphi} \partial_V)^{-1} Y'^2 = 0, \quad \text{i.e. } Y' \text{ is a } d\text{-class}.
\]

Let \( Y_\alpha := \partial \overline{Y} \cap E_\alpha \in H_5(E_\alpha) \).

Since

\[
\overline{Y} \cap (C_0 \times I) = A_0[N] \times I, \quad \text{we have } \ Y_\alpha \cap \Sigma_1 = \left[ \ast \times D^4 \times \left[ \frac{1}{3}, \frac{2}{3} \right] \right] \in H_5(\Sigma_1, \partial).
\]

Hence under (1) \( Y_\alpha \) goes to a class whose intersection with \([S^2 \times 0]\) is +1. Therefore under the composition of (1) and (2) \( Y_\alpha \) goes to a class whose intersection with the fiber \( S^2 \) is +1, i.e. to \( A \).

Therefore \((V, \overline{Y})\) is the required cobordism. \(\square\)

### 4.4. Proof of Lemma 4.4.a. Definition of \( M_f \) and \( Y_{f,y} \).

Identify \( C_f \) and \( C_f \times 0 \). Denote \( M_f := \partial(C_f \times I) = M_{id} \partial C_f \) and, for \( y \in H_3 \), \( Y_{f,y} := \partial(A_f[N] \times I) + i_{C_f, M_f} \tilde{A}_{f,y} \in H_5(M_f) \).

**Lemma 4.7** (Description of \( d \)-classes for \( M_f \)). A class \( Y \in H_5(M_f) \) is a \( d \)-class if and only if \( Y = Y_{f,y} \) for some \( y \in \ker(2\rho_{d_\infty(x(\partial)f)})\lambda(f)) \).

This follows by Lemma 3.13.b,c.

**Lemma 4.8** (proved below in §4.4). For every \( y \in H_3 \) there is a spin null-bordism \((W, z)\) of \((M_f, Y_{f,y})\) such that \( p_{1W} \) is even.\(^\text{14}\)

**Proof of Lemma 4.4.a.** Before we prove that \( \eta'(\varphi) \) is independent of \( Y \) we denote it by \( \eta'(\varphi, Y) \). Take any pair of \( d_0 \)-classes \( Y', Y'' \in H_5(M_\varphi) \). We have

\[
\eta'(\varphi, Y') - \eta'(\varphi, Y'') = (1) \eta'(\text{id} \partial C_{f_0}, Y) = (2) \eta'(\text{id} \partial C_{f_0}, Y_{f_0,y}) = (3) 0 \in \mathbb{Z}_2,
\]

where

- equality (2) holds for some \( y \in \ker(2\rho_{d_\infty(x(\partial)f)}\lambda(f_0)) \) by the description of \( d \)-classes for \( M_f \) (Lemma 4.7);
- equality (3) holds by Lemmas 4.3.a and 4.8;
- equality (1) holds for some \( d_0 \)-class \( Y \in H_5(M_{f_0}) \) by the following result.

Let \( f_0, f_1, f_2 : N \rightarrow S^7 \) be embeddings, \( \varphi_{01} : \partial C_0 \rightarrow \partial C_1 \) and \( \varphi_{12} : \partial C_1 \rightarrow \partial C_2 \) \( \pi \)-isomorphisms, \( Y_{01} \in H_5(M_{\varphi_{01}}) \) and \( Y_{12} \in H_5(M_{\varphi_{12}}) \) \( d \)-classes. Then \( \varphi_{02} := \varphi_{12} \varphi_{01} \) is a \( \pi \)-isomorphism and there is a \( d \)-class \( Y_{02} \in H_5(M_{\varphi_{02}}) \) such that \( \eta'(\varphi_{02}, Y_{02}) = \eta'(\varphi_{01}, Y_{01}) + \eta'(\varphi_{02}, Y_{12}) \).

This result is proved analogously to [CS11, Lemma 2.10], cf. [Sk08’, §2, Additivity Lemma] (the property that \( Y_{02} \) is a \( d \)-class is achieved analogously to the proof of Lemma 4.6). \(\square\)

\(^{14}\)We cannot take \( W = C_f \times I \) because \( \partial H_5(C_f \times I, \partial) \neq Y_{f,y} \). So we note that the following equality holds \( \partial(A_f[N] \times I + \tilde{A}_{f,y} \times I) = Y_{f,y} + \tilde{A}_{f,y} \times 1 \neq Y_{f,y} \) and ‘surger out’ \( \tilde{A}_{f,y} \times 1 \) shifted into the interior.
Definition of a simplifying 6-bordism $V$ and maps $v = v_0, v_1, v_2, v_3$. A simplifying 6-bordism for $f$ and an oriented 3-submanifold $P \subset N$ is a 6-manifold $V \subset C_f$ with boundary $\partial V = \nu^{-1}P \cup v(S^2 \times S^3)$ for some embedding $v = v_0 : S^2 \times S^3 \to \text{Int} C_f$ such that $V \cap \partial C_f = \nu^{-1}P$ and $v(S^2 \times 1_4)$ is homologous to $S_2^5$ in $C_f$. (Then $[\text{im } v] = [\hat{A}_f(P)] \in H_5(C_f)$.)

E.g. for $N = S^3 \times S^3$ and $P = 1_1 \times S^3$ we can take a simplifying 6-bordism $S^2 \times S^3 \times I \cong V \subset C_f$.

Let $v_1 : S^2 \times S^3 \times D^1 \to \text{Int} C_f$ be an embedding such that $v_1|_{S^2 \times S^3 \times 1} = v$, $\text{im} v_1 \cap V = \text{im} v$, and $v_1(c \times D^1)$ is tangent to $V$ for every $c \in S^2 \times S^3$.

Extend $v_1$ to an orientation-preserving embedding $v_2 : S^2 \times S^3 \times D^2 \to \text{Int} C_f = \text{Int} C_f \times \frac{1}{2}$ transversal to $V$ and such that $\text{im} v_2 \cap V = \text{im} v$.

Extend $v_2$ to an orientation-preserving embedding $v_3 : S^2 \times S^3 \times D^3 \to \text{Int}(C_f \times I)$.

**Lemma 4.9.** For every oriented 3-submanifold $P \subset N$ there is a simplifying 6-bordism.

**Proof.** Equip $\nu^{-1}P$ with the spin structure induced from $C_f$. (This spin structure is compatible with the orientation of $\nu^{-1}P$.) Since $C_f$ is simply connected, we can perform spin surgeries on 1-spheres in the 5-manifold $\nu^{-1}P$ to obtain a spin bordism between the inclusion $\nu^{-1}P \to C_f$ and a map $\mu : X \to C_f$ of some closed simply connected 5-manifold $X$. Since the induced map $i_{C_f} : H_2(\nu^{-1}P) \to H_2(C_f) \cong \mathbb{Z}$ is surjective, $\mu : H_2(X) \to H_2(C_f)$ is surjective. By Smale’s classification of simply connected spin 5-manifolds [Sm62, Theorem A] (see also [Cr11, Theorem 4.1]), there is a closed simply connected spin 5-manifold $X'$ with $H_2(X') = \ker \mu$. Choose any isomorphism $H_2(X) \to \ker \mu \oplus \mathbb{Z}$. Then by Barden’s classification of simply connected 5-manifolds [Ba65, Theorem 2.2] (see also [Cr11, Theorem 5.1]), we may identify $X$ with $(S^2 \times S^3)\#X'$ so that $\mu H_2(X') = \{0\} \subset H_2(C_f)$. Also by Smale’s classification [Sm62, Theorem 1.1], $X'$ is spin diffeomorphic to the boundary of a handlebody obtained by attaching 3-handles to $D^6$ (for some spin structures on these manifolds). So the co-cores of these handles give framed embeddings of 2-spheres such that spin surgery on these 2-spheres gives $S^5$. Applying this to 2-spheres in $X'$ and using $\mu H_2(X') = 0 \in H_2(C_f)$ we obtain a spin bordism over $C_f$, $g : V \to C_f$, between $\mu$ and a map $S^2 \times S^3 \to C_f$ inducing an isomorphism on $H_2$. Then

- $V$ is a spin 6-manifold obtained from $\nu^{-1}P \times I$ by attaching 2-handles $D^2 \times D^4$ and 3-handles $D^3 \times D^3$ to $\nu^{-1}P \times 1$;
- $\partial V \cong spin \nu^{-1}P \times 0 \sqcup S^2 \times S^3$;
- $g|_{\nu^{-1}P \times 0}$ is the identity and $g|_{S^2 \times S^3}$ induces an isomorphism on $H_2$.

Now the lemma follows by (the second part of) the following Semiproper Embedding Theorem 4.10.b for $\partial_+ V := \nu^{-1}P$ and $\partial_- V := S^2 \times S^3$.

**Theorem 4.10** (Semiproper Embedding). Let $V$ and $X$ be $v$- and $x$-manifolds such that $\partial V = \partial_+ V \sqcup \partial_- V$. Then every map $g : V \to X$ such that $g|_{\partial_+ V}$ is an embedding into $\partial_- V$ is homotopic rel $\partial_+ V$ to an embedding $V \to X$, provided either

(a) $x < 2v$ and $(V, \partial_- V)$ is $(2v - x - 1)$-connected, or
(b) $x = 7 = v + 1$ and $(V, \partial_- V)$ is 2-connected, $X$ and $V$ have spin structures $s_X$ and $s_V$ such that $s_V|_{\partial_+ V} = g^* s_X|_{\partial X}$.

**Proof.** First assume that $(V, \partial_- V)$ is $(2v - x - 1)$-connected. Then there is a handle decomposition of $V$ relative to $\partial_+ V$ without handles of index more than $v - (2v - x - 1) - 1 = x - v$. In particular, $V$ is a regular neighborhood (in itself) of an $(x - v)$-polyhedron.
Use induction on the number of handles. The base case is \( V = \partial_+ V \times I \) (when there are no handles). Then define an embedding \( V \to X \) as \( \partial_+ V \times I \xrightarrow{g|_{\partial_+ V \times I}} \partial X \times I \xrightarrow{i_X} X \), where \( i_X \) is the collar inclusion.

Let us prove the inductive step for (a). We may assume that \( V' := V \cup D^k \times D^{v-k} \), \( k \leq x - v \), \( g: V' \to X \) is a map such that \( g|_{\partial_+ V} \) is an embedding into \( \partial X \) and \( g|_V \) is an embedding and \( g^* s_X|_V = s_V \). Since \( x < 2v \), we have \( x \geq 2(x - v) + 1 \geq 2k + 1 \). Since \( V \) is a regular neighborhood (in itself) of an \( (x-v) \)-framing of \( \partial X \), we may assume that \( g|_{D^k \times 0} \) is an embedding and \( g(D^k \times I) \cap g(V) = g(\partial D^k \times 0) \). Since \( k \leq x - v \), we have \( \pi_{k-1}(V_{x-k,v-k}) = 0 \). Hence the normal \((v-k)\)-framing of \( g(\partial D^k \times 0) \) in \( g(\partial V) \) extends to a normal \( v - k \) framing of \( g(D^k) \) in \( X \). Thus \( g|_{D^k \times 0} \) extends to an embedding \( D^k \times D^{v-k} \to X \) whose image intersects \( g(V) \) at \( g(\partial D^k \times D^{v-k}) \). This extension defines an embedding \( V' \to X \) extending \( g|_V \) and homotopic to \( g \).

Now we prove (b). By hypothesis, there is a handle decomposition of \( V \) relative to \( \partial_+ V \) without handles of index more than \( 6 - 2 - 1 = 3 \). The proof is the same as the proof of (a) above except that \( x \geq 2k + 1 \) is verified directly and \( k > x - v = 1 \) is possible. The required extension of the \((v-k)\)-framing exists

- for \( k = 3 \) because \( \pi_{k-1}(V_{x-k,v-k}) = \pi_2(SO_4) = 0 \);
- for \( k = 2 \) because \( g^* s_X|_V = s_V \) (in spite of \( \pi_{k-1}(V_{x-k,v-k}) = \pi_1(SO_5) \neq 0 \)).

**Proof of Lemma 4.8.** Take any \( y \in H_3 \). Since \( H_3 \cong H^1(N) \cong [N,S^1] \), the class \( y \) is represented by an oriented 3-submanifold \( P \subset N \) that is the preimage of a regular value of a map \( N \to S^1 \) representing \( y \); orientations on \( N \) and \( S^1 \) give an orientation on the preimage. Take a simplifying 6-bordism \( V \subset C_f \) given by Lemma 4.9. Take the corresponding maps \( v, v_1, v_2, v_3 \). Let

\[
W_+ := (C_f \times I) - \text{Int im } v_3 \quad \text{and} \quad W := W_+ \cup_{v_3} (S^2 \times S^4 \times S^2) (S^2 \times D^4 \times S^2).
\]

(The manifold \( W \) may be called the result of an \( S^2 \)-parametric surgery along \( v_3 \).)

Denote

\[
t := v_3(S^2 \times 0 \times S^2) \quad \text{and} \quad \Delta := 1_2 \times D^4 \times 1_1.
\]

Identify \( S^2 \times D^4 \times S^2 \) with \( t \times \Delta \).

Consider the cohomology exact sequence of the pair \( (W,W_-) \) in the following Poincaré dual form:

\[
\begin{array}{ccccccccc}
\vdots & H_0(t \times \Delta) & \xrightarrow{r_1} & H_0(W,-) & \xrightarrow{r_2} & H_0(W_-,\partial) & \xrightarrow{r_3} & H_0(W_-,\partial) & \xrightarrow{r_4} & H_0(t \times \Delta) \\
& PD_{\text{iso}} & \cong & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
& H^2(W,W_-) & \xrightarrow{PD_{\text{iso}}} & \cong & H^3(W,W_-)
\end{array}
\]

Since \( H_5(t \times \Delta) = 0 \), the map \( r_{W_-} \) is an epimorphism. Take any

\[
Z \in r_{W_-}^{-1}(A_f[N] \times I \cap W_-) \subset H_6(W,\partial).
\]

Denote

\[
\hat{V} := V \cup (S^2 \times D^4 \times 1) \subset W \quad \text{and} \quad z := Z + [\hat{V}] \in H_6(W,\partial).
\]

Objects constructed above depend upon \( y, f \) and the choices in the construction. We do not indicate this in their notation.

Since \( H_6(t \times \Delta) = 0 \), the spin structure on \( W_- \) coming from \( S^7 \times I \) extends to \( W \). Clearly, \( \partial W = \partial(C_f \times I) = M_f \) (for the ‘boundary’ spin structure on \( M_f \) coming from \( C_f \times I \)).
Since
\[ \partial V Z = \partial C_f \times I (A_f [N] \times I) = Y_f, \quad \text{and} \quad \partial V [\hat{V}] = \left[ \nu_f^{-1} P \times \frac{1}{2} \right] = \hat{M}_{f} \hat{A}_f y, \]
we have \( \partial V z = Y_{f,y} \).

Consider the first line of diagram (*) with subscripts 6,5 changed to 4,3, respectively. Since \( p^*_W \cap W^- = 0 \), by exactness \( p^*_W = n[t] \) for some \( n \in \mathbb{Z} \).

Denote
\[ W'_+ := (S^7 - \text{Int } v_2) \cup_{v_2 \mid s_2 \times s_3 \times S^1} S^2 \times D^4 \times S^1. \]

Then
\[ n = n[t] \cap t \times [\Delta] = (p^*_W \cap t \times \Delta) \cap t \times [\Delta] \equiv 0 \mod 2. \]

Here
- the homology classes \([t]\) and \([\Delta]\) are taken in the space indicated under `\(\cap\)` (so \([\Delta]\) has different meanings in different parts of the formula);
- the equality (3) holds because \( r_{s_2 \times D^4 \times S^1}: H_4(t \times \Delta, \partial) \to H_3(S^2 \times D^4 \times S^1, \partial) \) is an isomorphism;
- the congruence holds because \( H_5(S^2 \times D^4 \times S^1) = 0 \), so the spin structure on \( S^7 - \text{Int } v_2 \) coming from \( S^7 \) extends to \( W'_+ \), hence by Lemma 3.12 \( p^*_{W'_+} \) is even. \( \square \)

4.5. Proof of Theorem 4.5 using Lemmas 4.12 and 4.13. Definition of an elementary pair. Suppose that \( U, V_0 \) and \( V_1 \) are abelian groups and that \( \cap_{01}: V_0 \times V_1 \to \mathbb{Z} \) a unimodular pairing. (Then \( V_k \) has to be free abelian.) An elementary pair is a pair \( v_k: U \to V_k, k = 0, 1 \), of monomorphisms such that \( v_0 U \cap_{01} v_1 U \neq 0 \) and \( v_k U \) is a half-rank direct summand in \( V_k \) for each \( k = 0, 1 \). (Then \( \text{rk } V_k \) has to be even.)

The following theorem is an easy corollary of a theorem of Kreck. In it and in §4.7 we consider the intersection product
\[ \cap_{01}: H_4(W, M_0) \times H_4(W, M_1) \to \mathbb{Z}. \]

**Theorem 4.11** (Modified surgery theorem). For \( l \geq 2 \)
- \( M_0, M_1 \subset \mathbb{R}^n \) be \((4l-1)\)-manifolds with common boundary;
- \( p: B \to BO \) be a fibration such that \( \pi_i(B) = 0 \) and \( \pi_i(p) = 0 \) for every \( i \geq 2l \);
- \( \overline{Sv_k}: M_k \to B, k = 0, 1 \), be \((2l-1)\)-connected maps coinciding on the boundary and such that \( p \overline{Sv_k} \) is the classifying map of the normal bundle of \( M_k \).

A diffeomorphism \( M_0 \to M_1 \) commuting with \( \overline{Sv_k} \) and identical on \( \partial M_0 \) exists if there is
- a \( 4l \)-manifold \( W \) such that \( \partial W = M_0 \cup (-M_1) \),
- a \( 2l \)-connected map \( \overline{Sv}: W \to B \) extending \( \overline{Sv_0} \cup \overline{Sv_1} \),
- a subgroup \( U \subset \ker \overline{Sv} \subset H_{2l}(W) \) such that the pair \( j_{M_k, \overline{Sv}} \mid U, k = 0, 1 \), is elementary.

**Proof.** For an elementary pair \( v_k: U \to V_k, k = 0, 1 \), the quotient \( v_0 U \times V_1 / v_1 U \to \mathbb{Z} \) of \( \cap_{01} \) is unimodular. So by [CS11, the Kreck Theorem 4.1], cf. [Kr99, Theorem 4], \( \overline{Sv} \) is bordant (relative to the boundary) to an \( h \)-cobordism. Hence the theorem holds by the relative \( h \)-cobordism theorem [Mi65]. \( \square \)

**Definitions of \( i_W, j_W, \partial W \), convenient manifold and pre-elementary class.** Let \( W \) be an 8-manifold.

Denote by \( i_W, j_W, \partial W \) the homomorphisms from the exact sequence of the pair \((W, \partial W)\).

The manifold \( W \) is called convenient if \( H_3(\partial W) \) is free abelian, \( H_5(W, \partial) = H_5(W) = 0 \) and \( \partial W \) is parallelizable.

A class \( z \in H_6(W, \partial) \) is called pre-elementary if there is a homomorphism \( s: H_4(W, \partial) \to H_4(W) \) such that
(1) \( H_4(W) = \text{im} \ i_W \oplus \text{im} \ s, \)
(2) \( su \cap u, su = su \cap_\partial v \) for every \( u, v \in H_4(W, \partial), \)
(3) \( \sigma(W) = sp_W \cap_\partial p_W = sz^2 \cap_\partial p_W = 0. \)

**Lemma 4.12** (Pre-elementary class; proved in §4.6). Let \( W \) be a convenient 8-manifold and \( z \in H_6(W, \partial) \) a class such that for \( d := \text{div}(\partial_W z^2) \) and some \( \overline{z^2} \in H_6(W, \mathbb{Z}_d) \)

\[ j_W \overline{z^2} = \rho_d z^2, \quad \overline{z^2} \cap_\partial (z^2 - p_W) \equiv 0 \mod \overline{d} \quad \text{and, if } d \text{ is even, } \overline{z^2} \cap_\partial z^2 \equiv 0 \mod 2. \]

Then there is a spin 8-manifold \( W' \) such that \( \partial W' \) is a homotopy 7-sphere and \( z \preceq 0 \in H_6(W, z W', \partial) \) is pre-elementary.

**Lemma 4.13** (Elementary pair; proved in §4.7). Let \( W \) be a convenient 8-manifold such that

\[(*) \quad \partial W = M_0 \cup_{\partial M_0 = \partial M_1} (-M_1) \quad \text{for some 7-manifolds } M_0, M_1 \quad \text{without torsion in their homology and having a common boundary, and}
\]
\n\[(**) \quad j_{M_k, \partial W} : H_4(\partial W) \rightarrow H_4(\partial W, M_k), \quad k = 0, 1, \quad \text{are epimorphisms having the same kernel.} \]

Let \( z \in H_6(W, \partial) \) be a pre-elementary class for which there is a class \( q \in \ker j_{M_0, \partial W} \) such that \( q \cap_\partial \partial_W z^2 = \text{div}(\partial_W z^2) \).

Then there is a subgroup \( U \subset H_4(W) \) such that \( U \cap_\partial z^2 = U \cap_\partial p_W = 0 \) and the pair of homomorphisms \( j_{M_k, W|U}, k = 0, 1 \), is elementary.

We remark that the proofs of the Elementary pair Lemma 4.13 modulo Lemma 4.15 (found in §4.7) and of the Pre-elementary class Lemma 4.12 modulo Lemma 4.14 (found in §4.6) are similar to [CS11, Proof of Bordism Theorem 4.3 and of Lemma 4.5]. However, these proofs are different in details from those in [CS11], even when \( H_1 = 0. \)

**Proof of the Almost Diffeomorphism Theorem 4.5 modulo Lemmas 4.12 and 4.13.** Take the spin structure on \( M \) corresponding to a tangent framing on \( M \). Take any normal spin structure on \( M \) given by Lemma 4.2. Since \( \Omega_7^{Spin}(\mathbb{C}P^\infty) = 0 \) [KS91, Lemma 6.1] there is an 8-manifold \( W \) with a normal spin structure and \( z \in H_6(W, \partial) \) such that \( \partial W = M \) and \( \partial z = Y. \)

Recall that \( BSpin = BO(4) \) is the (unique up to homotopy) 3-connected space for which there exists a fibration \( \gamma : BSpin \rightarrow BO \) inducing an isomorphism on \( \pi_i \) for every \( i \geq 4 \). Let \( B := BSpin \times \mathbb{C}P^\infty, p := \gamma \text{pr}_2 \) and \( \overline{\nu} : W \rightarrow B \) be the map corresponding to the given normal spin structure on \( W \) and to \( z \in H_6(W, \partial) \approx H^3(W) \approx [W, \mathbb{C}P^\infty]. \)

For each \( k = 0, 1 \) since \( M_k \) is torsion free, \( H_2(M_k) \cong H_5(M_k, \partial) \cong \mathbb{Z} \). Then the homomorphism \( \overline{\nu}|_{M_k} : H_2(M_k) \rightarrow H_2(\mathbb{C}P^\infty) \) is an isomorphism. This and the fact that \( \pi_1(M_k) = 0 \) imply that the map \( \overline{\nu}|_{M_k} \) is 3-connected.

Performing \( B \)-surgery below the middle dimension we can change \( \overline{\nu} \) relative to the boundary and assume that \( \overline{\nu} \) is 4-connected [Kr99, Proposition 4]. Then

\[ H_5(W, \partial) \cong H^3(W) \cong H^3(B) = 0, \quad H_3(W) \cong H_3(B) = 0 \quad \text{and} \quad H_2(W) \cong H_2(B) \cong \mathbb{Z}. \]

From Poincaré-Lefschetz duality it follows that \( \text{im} j_W \) is a direct summand in \( H_4(W, \partial) \). The manifold \( W \) is now convenient. By the Pre-elementary class Lemma 4.12 we can change \( W \) to obtain a new manifold, again denoted \( W \), with \( \partial W = M_0 \# \Sigma \) and \( z \in H_6(W, \partial) \) elementary.

Let \( q := i_M Q \). Take a subgroup \( U \) given by the Elementary pair Lemma 4.13. Recall that there is an isomorphism \( H_4(B) \rightarrow \mathbb{Z} \oplus \mathbb{Z} \) mapping \( \overline{\nu}(x) \) to \( (x \cap_\partial z^2, x \cap_\partial p_W) \) for every \( x \in H_4(W) \). Then \( U \subset \ker \overline{\nu} \). Apply the Modified Surgery Theorem 4.11 for \( l = 2 \) and \( \overline{\nu}_k := \overline{\nu}|_{M_k} \). The obtained diffeomorphism commutes with \( \overline{\nu}_k \) and so is orientation-preserving. \( \square \)
4.6. **Proof of the Pre-elementary class Lemma 4.12.** We first construct a homomorphism $s$ satisfying (1) from the definition of a pre-elementary class (and some additional properties). Then we show how to achieve (2) keeping (1), and finally we show how to achieve (3) keeping (1) and (2).

**Lemma 4.14.** Let $V, V'$ be free abelian groups, $\cdot : V \times V' \rightarrow \mathbb{Z}$ a unimodular form, $j : V \rightarrow V'$ a homomorphism whose image is a direct summand and $(\text{im } j)^\perp = \ker j$.

A homomorphism $s : V' \rightarrow V$ is called a 1-homomorphism if

$$V = \ker j \oplus \text{im } s, \quad jsj = j \quad \text{and} \quad sjs = s.$$  

A homomorphism $s : V' \rightarrow V$ is called a 2-homomorphism if

$$V = \ker j \oplus s \quad \text{and} \quad x \cdot jsv = x \cdot v \quad \text{for every} \quad x \in \text{im } s, v \in V'.$$

(a) There is a 1-homomorphism.

(b) For a 1-homomorphism $t : V' \rightarrow V$ define a homomorphism $t^* : \text{im } t \rightarrow V$ by the property

$$t^* x \cdot u = x \cdot jtu \quad \text{for every} \quad u \in V'.$$

Then $t^* t$ is a 2-homomorphism.

(c) If $s$ is a 2-homomorphism, then

(c1) $jsj = j$;

(c2) $V' = \text{im } j \oplus \ker s$;

(c3) $sjs = s$;

(c4) $t$ is a 1-homomorphism for every homomorphism $t : V' \rightarrow \text{im } s$ such that $tj = sj$.

**Proof of (a).** Since $V$ and $V'$ are free abelian, there is a subgroup $T \subset V$ such that $V = \ker j \oplus T$. Then $j|_T$ is injective and $j(T) = \text{im } j$. Since $j(T) = \text{im } j$ is a direct summand in $V'$, the inverse of the abbreviation $j : T \rightarrow j(T)$ extends to an epimorphism $t : V' \rightarrow T$. We have $tjt = t$ and $jtj = j$. \hfill $\square$

**Proof of (b).** Denote $s := t^* t$. Take any $x \in \text{im } t$. Since $jtj = j$, we have $t^* x \cdot jtu = x \cdot jtu = x \cdot jtu$ for every $u \in V'$. Hence $t^* x - x \perp \text{im } (jt)$. Since $V = \ker j \oplus \text{im } t$, we have $\text{im } (jt) = \text{im } j$. Then $t^* x - x \in \ker j$, i.e. $jt^* x = jx$. Since $tjt = t$, we have $tjt^* x = tjt = x$.

Since $t^* x - x \in \ker j$, we have $V = \ker j + \text{im } t^*$. If $j^* t^* x = 0$, then $jx = 0$, and consequently $x \in \ker j \cap \text{im } t = \{0\}$. Hence $V = \ker j \oplus \text{im } t^*$. Since $\text{im } s = t^* \text{im } t = \text{im } t^*$, we obtain $V = \ker j \oplus \text{im } s$.

Since $t^* x \cdot j^* y = x \cdot jtu = x \cdot jy$ for all $x, y \in \text{im } t$,

we have $su \cdot jsv = t^* tu \cdot j^* yv = tu \cdot jtv = t^* tu \cdot v = su \cdot v$ for all $u, v \in V'$. \hfill $\square$

**Proof of (c1).** Take any $y \in V$. Since $\ker j \perp \ker j$, we have $x_1 \cdot jy = 0 = x_1 \cdot jsjy$ for every $x_1 \in \ker j$. Also $x_2 \cdot jy = x_2 \cdot jsjy$ for every $x_2 \in \text{im } s$. Since $V = \ker j \oplus \text{im } s$, we have $x \cdot jy = x \cdot jsjy$ for every $x \in V$. Then by the unimodularity of $\cdot : V \times V' \rightarrow \mathbb{Z}$ we have $j = jsj$. \hfill $\square$

**Proof of (c2).** If $sjx = 0$, then $jsjx = 0$. So by (c1) $jx = 0$. Therefore $\text{im } j \cap \ker s = 0$. Since $\text{im } j$ is a direct summand, by rank considerations $V' = \text{im } j \oplus \ker s$. \hfill $\square$
Proof of (c3). By (c2) \( \text{im } s = s \text{ im } j \). Also \( \text{im } j = j \text{ im } s \). So the abbreviations \( j \colon \text{im } s \to \text{im } j \) and \( s \colon \text{im } j \to \text{im } s \) are surjective. Hence
\[
jsj = j \iff js|\text{im } j = \text{id}(\text{im } j) \iff sj|\text{im } s = \text{id}(\text{im } s) \iff sjs = s.
\]
\(\square\)

Proof of (c4). We have \( j \in j = j \text{ by (c1)} \). We have \( \text{im } t \supset j\text{ im } j(V) = sj(V) = \text{im } s \) by (c2). Hence \( V = \ker j \oplus \text{im } t \) and \( t|\text{im } s = s|\text{im } s = \text{id}(\text{im } s) \) by (c3).

Proof of the Pre-elementary class Lemma 4.12. Since \( H_3(\partial W) \) is free abelian, \( \text{im } j \) is a direct summand in \( H_4(W, \partial) \). Apply Lemma 4.14.ab to \( V = H_4(W), V' = H_4(W, \partial), \cdot \equiv \cap_0 \) and \( j = j_w \). We obtain a 2-homomorphism \( s \). Let us show how to modify \( (W, z, s) \) to achieve property (3) from the definition of a pre-elementary class.

Since \( \rho_d \partial W \equiv 0 \), \( \rho_d \partial W \equiv 0 \). Hence there are \( a \in H_4(W) \) and \( b \in H_4(W, \partial) \) such that \( z^2 = j_w a + db \). So \( \rho_d j_w s \equiv 2 = \rho_d j_w a \equiv \rho_d j_w a = \rho_d z^2 \). Here (2) holds by Lemma 4.14.c1. Since the residues in the Pre-elementary class Lemma 4.12 are independent of the choice of \( \equiv j_w \equiv b \), we may take \( z^2 := \rho_d s z^2 \).

Below we prove that
\[
\begin{align*}
(a) & \text{ we can change } \eta(z, s) := s z^2 \cap_0 (z^2 - p_W) \text{ by } 2d \text{ without changing } s z^2 \cap_0 z^2; \\
(b) & \text{ we can simultaneously change } \eta(W, z, s) \text{ by } d^2 - d \text{ and } s z^2 \cap_0 z^2 \text{ by } 2d^2.
\end{align*}
\]
If \( d \) is odd, applying (b) we make \( s z^2 \cap_0 z^2 \) even keeping \( \eta(z, s) \) divisible by \( d = \gcd(d, 3) \).
Then applying (a) we can change \( \eta(z, s) \) by \( 2d \) keeping \( s z^2 \cap_0 z^2 \) even.
If \( d \) is even, \( s z^2 \cap_0 z^2 \) is even by the hypothesis. Applying (a,b) we can change \( \eta(z, s) \) by \( \gcd(2d, \partial W - d) = d \) keeping \( s z^2 \cap_0 z^2 \) even.

Take \( (S^2)^4 \) and the class \( z_S \) which is the sum of four summands, each represented by a product of three 2-spheres and a point. Then \( z_S^4 = 24 \). Since \( (S^2)^4 \) is almost parallelizable, we have \( p_{(S^2)^4} = 0 \). Taking connected sums with copies of \( (S^2)^4, z_S \) we can change \( \eta(z, s) \) by any multiple of 24 while keeping \( s z^2 \cap_0 z^2 \) even.

So we obtain that \( \eta(z, s) = 0 \) and \( s z^2 \cap_0 z^2 \) is even.

By [KS91, spin case of (2.4) and Proposition 2.5] there is a closed spin 8-manifold \( W_0 \) and \( z_0 \in H_0(W_0) \) such that \( z_0^2 \cap W_0 z_0^2 - p_{W_0} = 0 \) and \( z_0^2 \cap W_0 z_0^2 = 2 \). Taking connected sums with copies of \( (W_0, z_0) \) we can change \( s z^2 \cap_0 z^2 \) by any multiple of 2 without changing \( \eta(W, z, s) \). So we can obtain \( s z^2 \cap_0 z^2 = 0 \) while keeping \( \eta(z, s) = 0 \).

Let \( \mathbb{H} P^2 \) be quaternionic projective space oriented so that its signature is given by \( \sigma(\mathbb{H} P^2) = 1 \). Recall that \( \mathbb{H} P^2 \) is 3-connected and \( (p^*_{\mathbb{H} P^2})^2 = 1 \) [Mi56, Lemmas 3 and 4]. There is a 3-connected parallelizable 8-manifold \( E_8 \) whose boundary is a homotopy sphere and whose signature is 8. Then \( p_{E_8}^* = 0 \).

Since \( \partial W \) is parallelizable, \( \partial W p_{W}^* = 0 \). By [CS11, Lemma 2.11.b] \( p_{W}^* \) is a characteristic element for \( \cap W|\text{im } s \). Hence by Lemma 4.15.a \( \sigma(W) = \sigma(\cap W|\text{im } s) \equiv \mod 8 sp^*_{W} \cap W sp^*_{W} \). Therefore taking connected sums with copies of \( \mathbb{H} P^2 \) and \( E_8 \) we can achieve \( \sigma(W) = sp^*_{W} \cap_0 p_{W}^* = 0 \) while keeping \( \eta(z, s) = s z^2 \cap_0 z^2 = 0 \).\(\square\)

Proof of (b). Denote \( W_1 := W \# \mathbb{H} P^2 \# (-\mathbb{H} P^2) \). We have \( H_4(\mathbb{H} P^2 \# (-\mathbb{H} P^2)) \cong \mathbb{Z}^2 \) with evident basis. In this basis the intersection form is \( \text{diag}(1, -1) \) and \( p_{\mathbb{H} P^2 \# (-\mathbb{H} P^2)} = (1, 1) \). Let \( z_1 \) be the preimage of \( z \) under the ‘connected sum’ isomorphism \( H_0(W_1, \partial) \to H_0(W, \partial) \). In

\footnote{This covers a minor gap in [CS11, §4]: there we needed additionally to take connected sums with the \( E_8 \)-manifold to kill \( \alpha_W \), and so \( \partial W \) will in general be changed by connected sum with a homotopy sphere.}
order to construct the new $s$ (this is $t^*t$ not $s_1$, both defined below) let us define the lower two lines of the following diagram:

\[
\begin{array}{cccccc}
(z^2_{p_{W,1}}) & \xrightarrow{c_{\varnothing}} & (z^2_{p_{W,1}},0,0) & \xrightarrow{s':=(s\oplus \varnothing)\oplus \text{id}} & (sz^2,\partial_W z^2,0,0) & \xrightarrow{t':=i\oplus(t''\oplus \text{id})} (sz^2,0,d) \\
H_4(W_1,\partial) & \xrightarrow{c_{\varnothing}} & H_4(W,\partial) \oplus \mathbb{Z}^2 & \xrightarrow{\text{im } s \oplus H_3(\partial W) \oplus \mathbb{Z}^2} & H_4(W) \oplus \mathbb{Z}^2 \\
& & & & \xrightarrow{t'} & H_4(W_1) \\
\end{array}
\]

Let $c_{\varnothing}$ and $c$ be the ‘connected sum’ orthogonal isomorphisms (for the form $\text{diag}(1,-1)$ on $\mathbb{Z}^2$). Let $\text{id} := \text{id } \mathbb{Z}^2$. Let $s'(u,a,b) := s(u) \oplus \partial u \oplus (a,b)$. Since $H_3(W) = 0$, by Lemma 4.14.e2 $s \oplus \partial W : H_4(W,\partial) \to \text{im } s \oplus H_3(\partial W)$ is an isomorphism. Hence $s'$ is an isomorphism.

Since $H_3(\partial W)$ is free abelian and $d = \text{div}(\partial_W z^2)$, there is a map

\[
t'' : H_3(\partial W) \to \mathbb{Z}^2 \quad \text{such that} \quad t''(\partial_W z^2) = (0,d).
\]

Let $t'(u,v,a,b) := u \oplus (t''(v) + (a,b))$. Let $V := H_4(W_1), \quad V' := H_4(W_1,\partial), \quad j := j_{W_1}, \quad s_1 := c^{-1}(s \oplus \text{id})c_{\varnothing}, \quad t := c^{-1}t's'c_{\varnothing},$

so that the undashed lines of the diagram commute. Then $\text{im } s_1 = \text{im } s \oplus \mathbb{Z}^2 = \text{im } t' = c \text{ im } t$. Clearly, $s_1$ is a 2-homomorphism. Also

\[
tj = c^{-1}t's'c_{\varnothing} = c^{-1}t's'(j_{W} \oplus \text{id})c = c^{-1}(t(j_{W} \oplus 0)) \text{id} c = (s(j_{W} \oplus 0)) \text{id} c = c^{-1}((s \oplus \text{id})c_{\varnothing})j = s_1 j.
\]

Hence by Lemma 4.14.b,c for $s_1$ we obtain that $t^*t$ is a 2-homomorphism.

For every $u_1, u_2 \in H_4(W_1,\partial)$ by definition of $t^*$ we have

\[
t^*tu_1 \cap_{W_1} u_2 = tu_1 \cap W v_2 + a_1a_2 - b_1b_2, \quad \text{where} \quad (v_k,a_k,b_k) = ctu_k.
\]

Clearly, the images of $z_1^2$ are as shown in the first line of the diagram. Since $\partial W$ is parallelizable, the images of $p_{v_1}$ are as shown in the first line of the diagram. Hence

\[
t^*t z_1^2 \cap_{W_1} z_1^2 = sz^2 \cap W sz^2 - d^2 \quad \text{and} \quad \eta(z_1,t^*t) - \eta(z,s) = 0 \cdot (0-1) - d \cdot (d-1) = -d^2 + d.
\]

\[\square\]

\textbf{Proof of (a).} By [Mi56] there is a $D^4$-bundle over $S^4$ whose Euler class is 0 and whose first Pontryagin class is 4. The double of this bundle is an $S^4$-bundle $S^4 \times S^4$ over $S^4$ whose first Pontryagin class is 4. We have $H_4(S^4 \times S^4) \cong \mathbb{Z}^2$ with evident basis. In this basis $p_{S^4 \times S^4} = (2,0)$ and the intersection form is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Analogously to the proof of (b) with $\mathbb{H}P^2\#(-\mathbb{H}P^2)$ replaced by $S^4 \times S^4$ we construct $W_1, z_1$ and $t^*t$. Then for every $u_1,u_2 \in H_4(W_1,\partial)\text{ we have}$

\[
t^*tu_1 \cap_{W_1} u_2 = tu_1 \cap W v_2 + a_1b_2 + a_2b_1, \quad \text{where} \quad (v_k,a_k,b_k) = ctu_k.
\]

Also $ct \begin{pmatrix} z_1^2 \\ p_{W_1}^* \end{pmatrix} = \begin{pmatrix} sz^2,0,d \\ sp_{W_1},2,0 \end{pmatrix}$. Then $t^*t z_1^2 \cap_{W_1} z_1^2 = sz^2 \cap W sz^2$ and $\eta(z_1,t^*t) = \eta(z,s) - 2d$. \[\square\]
4.7. Proof of the Elementary pair Lemma 4.13.

**Lemma 4.15.** Let $W$ be an 8-manifold satisfying the assumptions (*) and (**) of the Elementary pair Lemma 4.13. Let $s: H_4(W, \partial) \to H_4(W)$ be a homomorphism such that $H_4(W) = \text{im } i_W + \text{im } s$ (additively which implies orthogonally w.r.t. $\cap_W$). Denote $j_k := j_{M_k, W}$ and $S := \text{im } s$. Denote by the superscript $\perp$ the orthogonal complement with respect to $\cap_W$, unless another intersection product is indicated as subscript. Then

(a) $S$ is free abelian and the form $\cap_W|_S$ is unimodular;

(b) $j_0|_S, j_1|_S$ are injective, $H_4(W, M_k) = (j_{1, k} S) \perp \oplus j_k S$, $k = 0, 1$, and the restrictions of $\cap_W$ both to $j_0 S \times j_1 S$ and to $(j_1 S) \perp \times (j_0 S) \perp$ are unimodular;

(c) $j_k \text{im } i_W$ is a half-rank direct summand in the free abelian group $(j_{1, k} S) \perp$;

(d) if

\[ a \in H_4(W, \partial), \quad q \in \ker j_{M_0, \partial W} \subset H_4(\partial W) \quad \text{and} \quad q \cap \partial W = \text{div}(\partial W a), \]

then there is a subgroup $U \subset \text{im } i_W$ such that $U \cap_W a = 0$ and the pair $j_k|_U: U \to (j_{1, k} S) \perp$, $k = 0, 1$, is elementary.

**Proof of (a).** Since the torsion of $H_4(W)$ is contained in $\text{im } i_W = H_4(W)_{\cap_W}$, the group $S$ is free abelian. Since $\cap_W: H_4(W) \times H_4(W, \partial) \to \mathbb{Z}$ is unimodular, $x \cap_W y = x \cap_W j_W y$ for all $x, y \in H_4(W)$ and $H_3(\partial W)$ is free abelian, it follows that the form $\cap_W|_S$ is unimodular. \[ \square \]

**Proof of (b).** Since $j_0 x \cap_1 y = x \cap_W y$ for all $x, y \in H_4(W)$, it follows that $\cap_W|_{j_0 S \times j_1 S}$ is unimodular. Then $j_0|_S$ and $j_1|_S$ are injective. So if $x \in S$ and $j_k x \cap_1 j_{1, k} S = 0$, then $x = 0$. Also, for every $y \in H_4(W, M_k)$ the $\cap_W$-intersection with $y$ defines a linear map $j_{1, k} S \to \mathbb{Z}$. Hence there is a class $y_S \in j_k S$ such that $y \cap_1 x = y_S \cap_1 x$ for every $x \in j_{1, k} S$. Then $y = y_S + (y - y_S) \cap_1 j_{1, k} S = 0$. Thus $H_4(W, M_k) = (j_{1, k} S) \perp \oplus j_k S$, $k = 0, 1$. Since $\cap_W$ and $\cap_W|_{j_0 S \times j_1 S}$ are unimodular, $\cap_W|_{(j_1 S) \perp \times (j_0 S) \perp}$ is unimodular. \[ \square \]

Some notation for the proofs of Lemmas 4.15.c,d and 4.13. Denote by $i_{W, k}, \partial_{W, k}$ homomorphisms from the exact sequence of the triple $(W, \partial W, M_k)$. Denote by $i_k, j_k, \partial_k$ and $\tilde{i}_k, \tilde{j}_k, \tilde{\partial}_k$ the homomorphisms from the exact sequences of the pairs $(W, M_k)$ and $(\partial W, M_k)$, respectively. Recall that $H_4(W)_{\cap_W} = \text{im } i_W$. Consider the following diagram:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {$H_5(W, \partial)$};
\node (B) at (3,0) {$H_4(W)$};
\node (C) at (6,0) {$H_3(W)$};
\node (D) at (0,-1) {$H_4(M_k)$};
\node (E) at (3,-1) {$H_4(\partial W)$};
\node (F) at (6,-1) {$H_3(M_k)$};
\node (G) at (0,-2) {$\text{im } i_W \oplus S$};
\node (H) at (3,-2) {$(j_{1, k} S) \perp \oplus j_k S$};
\draw[->] (A) -- (B) node[midway,above] {$\partial_{W, k}$};
\draw[->] (B) -- (C) node[midway,above] {$\partial_k = 0$};
\draw[->] (B) -- (E) node[midway,above] {$j_k$};
\draw[->] (E) -- (F) node[midway,above] {$i_{W, k}$};
\draw[->] (B) -- (H) node[midway,above] {$i_k$};
\draw[->] (A) -- (D) node[midway,above] {$\tilde{i}_k$};
\draw[->] (D) -- (E) node[midway,above] {$\tilde{j}_k$};
\end{tikzpicture}
\end{array}
\]

**Proof of (c).** We have

\[
\frac{(j_{1, k} S) \perp}{j_k \text{im } i_W} \overset{(1)}{=} \text{coker } j_k \overset{(2)}{=} H_3(M_k) \overset{(3)}{=} H_4(\partial W, M_k) \overset{(4)}{=} \text{im } i_{W, k} \overset{(5)}{=} j_k \text{im } i_W.
\]

Here

- (1) is obtained by adding $j_k S$ both to nominator and denominator and using (b);
- (2) holds because $H_3(W) = 0$, hence $\partial_k$ is an epimorphism;
\[ (3) \text{ holds by Poincaré-Lefschetz duality because both } H_3(M_k) \text{ and } H_4(\partial W, M_k) \cong_{\text{ex}} H_4(M_{1-k}, \partial) \text{ are free abelian;} \]
\[ (4) \text{ holds because } H_5(W, \partial) = 0, \text{ hence } i_{W, k} \text{ is injective;} \]
\[ (5) \text{ holds because } j_k \text{ is surjective.} \]

Since \( H_3(M_k) \) is free abelian, \( j_k \im i_W \cong \frac{(j_{1-k} S)_{\perp}}{j_k \im i_W} \) is free abelian. This implies (c). \( \square \)

**Proof of (d).** Since \( M_k \) is torsion free, by Poincaré-Lefschetz duality \( H_4(\partial W, M_0) \cong_{\text{ex}} H_4(M_1, \partial) \) is free abelian. Since \( j_0 \) is surjective, it follows that there is a subgroup
\[
U'' \subset H_4(\partial W) \quad \text{such that } \quad j_{0|U''}: U'' \rightarrow H_4(\partial W, M_0) \quad \text{is an isomorphism.}
\]

Since \( H_3(\partial W) \) is free abelian, there is a class \( a_0 \in H_3(\partial W) \) such that \( \partial_W a = a_0 \div (\partial_W a) \).

Define \( U' := \{ u - (a_0 \cap_{\partial_W} u) q \mid u \in U'' \} \) and \( U := i_W U' \).

Since \( q \cap_{\partial W} \partial_W a = \div(\partial_W a) \), we have \( \partial_W a \cap_{\partial W} U' = 0 \). Thus \( U \cap_{\partial W} a = 0 \). So by (c) it remains to prove that \( j_k |_U \) is an isomorphism onto \( j_k \im i_W \).

Since \( \ker j_0 = \ker j_1 \), the map \( j_1 |_{U''} \) is injective. Then \( U'' \cap_{\partial W} \im j_k = 0 \). This and the fact that \( q \in \im i_0 = \im i_1 = \ker j_0 = \ker j_1 \) imply that \( U' \cap_{\partial W} \im j_0 = 0 \). Since \( \ker j_0 = \ker j_1 \), we have \( \im j_0 = \im i_1 \). Therefore \( U' \cap_{\partial W} \im j_1 = 0 \). Thus \( j_k |_{U''} \) is injective. Since \( H_5(W, \partial) = 0 \), the map \( i_W \) is injective. Hence \( U \cap_{\partial W} i_W \ker j_k = 0 \). We have \( i_W \ker j_k = i_W \im i_k = \im j_k \). Thus \( j_k |_{U''} \) is injective.

Since \( \ker j_0 = \ker j_1 \), we have \( H_4(\partial W) = U'' + \ker j_0 = U'' + \ker j_1 \). Since \( q \in \ker j_0 = \ker j_1 \), it follows that \( H_4(\partial W) = U' + \ker j_0 = U' + \ker j_1 \). So \( j_1 U' = j_1 H_4(\partial W) \). Therefore we have \( j_k U = j_k i_W U' = i_{W,k} j_k H_4(\partial W) = j_k \im i_W \).

**Proof of the Elementary pair Lemma 4.13.** The group \( H_4(W, M_k) \) is torsion free for \( k = 0, 1 \).

(Indeed, consider the Poincaré dual of the exact sequence of the pair \((W, M_{3-k})\):
\[
H_5(W, \partial) \rightarrow H_4(M_{3-k}, \partial) \rightarrow H_4(W, M_k) \rightarrow H_4(W, \partial).
\]

By the assumptions
\[
H_5(W, \partial) = 0, \quad \text{Tors } H_4(M_{3-k}, \partial) = \text{Tors } H_2(M_{3-k}) = 0 \quad \text{and} \quad \text{Tors } H_4(W, \partial) = \text{Tors } H_3(W) = 0.
\]

Hence \( H_4(W, M_k) \) is torsion free.)

Since \( z \) is pre-elementary, there is a homomorphism \( s \) from the definition of a pre-elementary class. Denote \( S := \im s \).

Let
\[
\tilde{U} := \{ u \in S \mid lu = msz^2 + nsp^2 \text{ for some integers } l, m, n \}.
\]

Since \( z \) is pre-elementary, \( \tilde{U} \cap_{\partial W} \tilde{U} = 0 \). By Lemma 4.15.a \( S \) is free abelian and the form \( \cap_{\partial W}|_S \) is unimodular. Then there is a subgroup
\[
T \subset S \quad \text{such that } \quad \tilde{U} \subset T, \quad \rk T = 2 \rk \tilde{U} \quad \text{and} \quad \cap_{\partial W}|_T \text{ is unimodular.}
\]

Hence \( \sigma(T) = 0 \). Since both \( \cap_{\partial W}|_S \) and \( \cap_{\partial W}|_T \) are unimodular, \( T \cap T_{\partial W}^\perp = 0 \) and \( \rk T_{\partial W}^\perp = \rk S - \rk T \), we have \( S = T \oplus T_{\partial W}^\perp \). So \( \sigma(T_{\partial W}^\perp) = \sigma(W) - \sigma(T) = 0 \). Hence there is a half-rank direct summand
\[
\tilde{U} \subset T_{\partial W}^\perp \quad \text{such that} \quad \tilde{U} \cap_{\partial W} \tilde{U} = 0.
\]

Let \( U_S := \tilde{U} \oplus \tilde{U} \).
We have \( U_S \cap_\partial z^2 = U_S \cap_\partial p^*_W = 0 \) and the pair \( j_k|_{U_S}: U_S \rightarrow j_kS \), \( k = 0, 1 \), is elementary.

(Indeed, since \( z \) is pre-elementary, \( \tilde{U} \cap_\partial z^2 = \tilde{U} \cap_\partial p^*_W = 0 \). Also \( \tilde{U} \cap_\partial \tilde{U} = 0 \). Hence by the properties (2) and (3) of \( s \) we obtain \( U_S \cap_\partial z^2 = U_S \cap_\partial p^*_W = 0 \). Since \( \tilde{U} \cap_\partial \tilde{U} = 0 = \tilde{U} \cap_\partial \tilde{U} \), we have \( j_0U_S \cap_0j_1U_S = 0 \). By Lemma 4.15.b \( j_k|_{S} \) is injective. Since \( \tilde{U} \) and \( \tilde{U} \) are half-rank direct summands in \( T \) and in \( T^\perp_\partial \), respectively, \( U_S \) is a half-rank direct summand in \( S \). So \( j_kU_S \) is a half-rank direct summand in \( j_kS \).

Applying Lemma 4.15.d to \( a = z^2 \) we obtain a subgroup \( U_\partial \subset \text{im } i_W \). Since \( \partial W \) is parallelizable, \( p_1(\partial W) = 0 \). Hence \( \text{im } i_W \cap_\partial p^*_W = 0 \). Therefore \( U := U_S \oplus U_\partial \) is as required. 

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