Reproducing Kernels of Generalized Sobolev Spaces via a Green Function Approach with Differential Operators

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Abstract In this paper we introduce a generalization of the classical $L^2(\mathbb{R}^d)$-based Sobolev spaces with the help of a vector differential operator $P$ which consists of finitely or countably many differential operators $P_n$ which themselves are linear combinations of distributional derivatives. We find that certain proper full-space Green functions $G$ with respect to $L = P^*P$ are positive definite functions. Here we ensure that the vector distributional adjoint operator $P^*$ of $P$ is well-defined in the distributional sense. We then provide sufficient conditions under which our generalized Sobolev space will become a reproducing-kernel Hilbert space whose reproducing kernel can be computed via the associated Green function $G$. As an application of this theoretical framework we use $G$ to construct multivariate minimum-norm interpolants $s_{f,X}$ to data sampled from a generalized Sobolev function $f$ on $X$. Among other examples we show the reproducing-kernel Hilbert space of the Gaussian function is equivalent to a generalized Sobolev space.

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1 Introduction

There is a steadily increasing body of literature on radial basis function and other kernel-based approximation methods in such application areas as scattered data approximation, statistical or machine learning and numerical solutions of partial differential equations. Some recent books and survey papers on kernel methods are [2,3,5,9,11,12,16,17]. Given a set of data sites $X$ and associated values $Y$ sampled from a continuous function $f$, we use a positive definite function $\Phi$ to set up an interpolant $s_{f,X}$ to approximate $f$. It is well-known that within the reproducing-kernel Hilbert space framework one can also discuss the error analysis and optimal
recovery of the interpolation process whenever $f$ belongs to the related reproducing-kernel Hilbert space $\mathcal{H}_f(\mathbb{R}^d)$ (see Section 3 and [17]). However, there are still a number of difficulties and challenges associated with this method. Two important questions still in need of a satisfactory answer are: What kind of functions belong to a given reproducing-kernel Hilbert space? and Which kernel function should we utilize for a particular application?

In the survey paper [11] the authors give the reader some guidance for dealing with this problem. Others – especially statisticians – will attempt to find the “best” kernel function by selecting an “optimal” scaling parameter. Here such techniques as cross-validation and maximum likelihood estimation are often mentioned (see, e.g. [15, 16]). The recent paper [4] derives a new error bound of the approximation by the fundamental functions (Green functions) using scattered shifts.

In [10] the author noted that many kernels can be embedded into the classical Sobolev spaces, but that is was not so clear what the difference between the various kernels was. As a possible approach to this, he suggested that one might scale the classical Sobolev space with the help of a scaling parameter integrated into the kernel function. Once this is done, we can choose the “best” scaling parameter to set up the interpolant that minimizes the norm of the error functional for a given set of scattered data. Examples 5.1 and 5.2 will demonstrate that the classical Sobolev space can be reconstructed by different inner-products that allow us to balance the required derivatives by selecting various scaling parameters. The related Green functions are the Sobolev splines (Matérn functions) with appropriate scaling parameters (see [15]). Finally, Example 5.3 shows that the reproducing-kernel Hilbert space of the Gaussian function is also equivalent to a generalized Sobolev space. This generalized Sobolev space has been applied in the context of support vector machines and the study of motion coherence (see [13, 18]).

However, in this paper, we view this problem in a slightly different way. We hope to construct the reproducing kernel and the reproducing-kernel Hilbert space with the help of countably many differential operators \( \{P_n\}_{n=1}^\infty \) which are themselves linear combinations of distributional derivatives (defined in Section 4.1). Handling the vector differential operator \( \mathbf{P} := (P_1, \cdots, P_n, \cdots)^T \), we will generalize the concept of classical real-valued \( L_2(\mathbb{R}^d) \)-based Sobolev space \( H^n(\mathbb{R}^d) \) to be a generalized Sobolev space \( \mathcal{H}^n(\mathbb{R}^d) \) with a semi-inner product (see Definition 4.4).

In the following we use the notation \( \text{Re}(\mathcal{E}) \) to denote the collection of all real-valued functions of the function space \( \mathcal{E} \). For example, \( \text{Re}(C(\mathbb{R}^d)) \) expresses the collection of all real-valued continuous functions on \( \mathbb{R}^d \). The real classical Sobolev space is usually defined as

\[
\mathcal{H}^n(\mathbb{R}^d) := \left\{ f \in \text{Re}(L_2(\mathbb{R}^d)) : D^\alpha f \in L_2(\mathbb{R}^d) \text{ for all } |\alpha| \leq n, \alpha \in \mathbb{N}_0^d \right\}
\]

with inner product

\[
(f, g)_{\mathcal{H}^n(\mathbb{R}^d)} := \sum_{|\alpha| \leq n} (D^\alpha f, D^\alpha g), \quad f, g \in \mathcal{H}^n(\mathbb{R}^d),
\]
where $( \cdot, \cdot )$ is the standard $L^2(\mathbb{R}^d)$-inner product. Our real generalized Sobolev space will be of a very similar form, namely

$$H^p(\mathbb{R}^d) := \left\{ f \in \text{Re}(L^2(\mathbb{R}^d)) : \{ P_n f \}_{n=1}^{\infty} \subseteq L^2(\mathbb{R}^d) \text{ and } \sum_{n=1}^{\infty} \| P_n f \|_{L^2(\mathbb{R}^d)}^2 < \infty \right\}$$

with the semi-inner product

$$(f, g)_{H^p(\mathbb{R}^d)} := \sum_{n=1}^{\infty} (P_n f, P_n g), \quad f, g \in H^p(\mathbb{R}^d).$$

We further wish to connect the reproducing kernel to a (full-space) Green function $G$ with respect to some differential operator $L$, i.e., $LG = \delta_0$. Since the Dirac delta function $\delta_0$ at the origin is just a tempered distribution in the dual space of the Schwartz space, the Green function should be regarded as a tempered distribution as well. We find that $L$ can be computed by a vector differential operator $P := (P_1, \cdots, P_n)^T$ and its distributional adjoint operator $P^* := (P_1^*, \cdots, P_n^*)^T$, i.e., $L = P^* P = \sum_{j=1}^{n} P_j^* P_j$. The definition of the distributional adjoint operator is given in Section 4.1. Under some sufficient conditions, we can prove that the Green function $G$ is a positive definite function according to Theorem 4.1. In that case the reproducing-kernel Hilbert space $N_G(\mathbb{R}^d)$ related to $G$ is well-defined in Section 3 and [17], and its reproducing kernel $K$ on $\mathbb{R}^d$ is equal to $K(x, y) := G(x - y)$. Moreover, Theorems 4.2 shows that $N_G(\mathbb{R}^d)$ is equivalent to $H^P(\mathbb{R}^d)$ for some simple additional conditions. In the proof, we use techniques of distributional Fourier transforms defined in Section 4.2 and [14], as well as the classical Fourier transform.

2 Background

Given data sites $X = \{ x_1, \cdots, x_N \} \subset \mathbb{R}^d$ (which we also identify with the centers of our kernel functions below) and values $Y = \{ y_1, \cdots, y_N \} \subset \mathbb{R}$ of a real-valued continuous function $f$ on $X$, we wish to approximate this function $f$ by a linear combination of translates of a positive definite (see Section 3.1) function $\Phi$.

To this end we set up the interpolant in the form

$$s_{f, X}(x) := \sum_{j=1}^{N} c_j \Phi(x - x_j), \quad x \in \mathbb{R}^d,$$

and require it to satisfy the additional interpolation conditions

$$s_{f, X}(x_j) = y_j, \quad j = 1, \ldots, N.$$

The above system (2) is equivalent to a uniquely solvable linear system

$$A_{\Phi, X} c = y,$$
where $A_{\Phi X} := (\Phi(x_j - x_k))_{j,k=1}^{N \times N} \in \mathbb{R}^{N \times N}$, $c := (c_1, \ldots, c_N)^T$ and $Y := (y_1, \ldots, y_N)^T$. All of the above is discussed in detail in [17, Chapter 6.1].

**Example 2.1.** One of the best known examples that fits into this framework is the *univariate Sobolev spline (Matérn) interpolant*

$$s_{fX}(x) := \sum_{j=1}^{N} c_j \Phi(x - x_j), \quad \Phi(x) := \frac{1}{2\sigma} \exp(-\sigma |x|), \quad x \in \mathbb{R},$$

and $\sigma > 0$ is called the *Sobolev parameter*.

We can check that $\Phi$ is a positive definite function. Moreover, if we define $P = (P_1, P_2)^T := (d/dx, \sigma I)^T$ and $P^* = (P_1^*, P_2^*)^T := (-d/dx, \sigma I)^T$, then $\Phi$ is the Green function with respect to $L = P^T P = -d^2/dx^2 + \sigma^2 I$.

Furthermore, the interpolant $s_{fX}$ from (1)-(2) minimizes the norm $\|f\|_{\text{Sob}}$ of all $f \in C^1(\mathbb{R}) \cap \text{Re}(L_2(\mathbb{R}^d))$ with

$$\|f\|_{\text{Sob}}^2 := (Pf, Pf)_{\text{Sob}} = \int_{\mathbb{R}} \left| \frac{df}{dx}(x) \right|^2 dx + \int_{\mathbb{R}} \sigma^2 |f(x)|^2 dx < \infty$$

subject to the constraints $s_{fX}(x_j) = y_j, j = 1, \ldots, N$ (see [2]).

As we will show in Section 3.2, we can in general construct a reproducing-kernel Hilbert space $\mathcal{N}_\Phi(\mathbb{R}^d)$ from a positive definite function $\Phi$ such that the interpolant $s_{fX}$ is the best approximation of the function $f$ in $\mathcal{N}_\Phi(\mathbb{R}^d)$ fitting the values $Y$ on the data sites $X$.

### 3 Positive Definite Functions and Reproducing-Kernel Hilbert Spaces

Most of the material presented in this section can be found in the excellent monograph [17]. For the reader’s convenience we repeat here what is essential to our discussion later on.

#### 3.1 Positive Definite Functions

**Definition 3.1 ([17], Definition 6.1).** A continuous even function $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be a *positive definite function* if, for all $N \in \mathbb{N}$, all pairwise distinct centers $x_1, \ldots, x_N \in \mathbb{R}^d$, and all $c = (c_1, \ldots, c_N)^T \in \mathbb{R}^N \setminus \{0\}$, the quadratic form

$$\sum_{j,k=1}^{N} c_j c_k \Phi(x_j - x_k) > 0.$$
Theorem 3.1 ([17, Theorem 6.11]). Suppose that \( \Phi \in \text{Re}(C(\mathbb{R}^d)) \cap L_1(\mathbb{R}^d) \) is an even function and its \( L_1(\mathbb{R}^d) \)-Fourier transform is \( \hat{\phi} \). Then \( \Phi \) is positive definite if and only if \( \Phi \) is bounded and \( \hat{\phi} \) is nonnegative and nonvanishing.

3.2 Reproducing-Kernel Hilbert Spaces

Definition 3.2 ([17, Definition 10.1]). Let \( H(\mathbb{R}^d) \) be a real Hilbert space of functions \( f : \mathbb{R}^d \rightarrow \mathbb{R} \). A kernel \( K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) is called a reproducing kernel for \( H(\mathbb{R}^d) \) if

\[
(i) \quad K(\cdot, y) \in H(\mathbb{R}^d), \quad \text{for all } y \in \mathbb{R}^d,
(ii) \quad f(y) = (K(\cdot, y), f)_{H(\mathbb{R}^d)}, \quad \text{for all } f \in H(\mathbb{R}^d) \text{ and all } y \in \mathbb{R}^d.
\]

In this case \( H(\mathbb{R}^d) \) is called a reproducing-kernel Hilbert space.

Theorem 3.2 ([17, Theorem 10.12]). Suppose that \( \Phi \in C(\mathbb{R}^d) \cap \text{Re}(L_1(\mathbb{R}^d)) \) is an even function and its \( L_1(\mathbb{R}^d) \)-Fourier transform is \( \hat{\phi} \). If \( \Phi \) is a positive definite function, then

\[
\mathcal{N}_\Phi(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d) \cap \text{Re}(L_2(\mathbb{R}^d)) : \hat{\phi}^{-1/2} \hat{f} \in L_2(\mathbb{R}^d) \right\},
\]

is a reproducing-kernel Hilbert space with reproducing kernel

\[
K(x, y) = \Phi(x - y), \quad x, y \in \mathbb{R}^d,
\]

and its inner product satisfies

\[
(f, g)_{\mathcal{N}_\Phi(\mathbb{R}^d)} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\hat{f}(x)\hat{g}(x)}{\hat{\phi}(x)} \, dx, \quad f, g \in \mathcal{N}_\Phi(\mathbb{R}^d),
\]

where \( \hat{f} \) and \( \hat{g} \) are the \( L_2(\mathbb{R}^d) \)-Fourier transform of \( f \) and \( g \), respectively.

4 Connecting Green Functions and Generalized Sobolev Spaces to Positive Definite Functions and Reproducing-Kernel Hilbert Spaces

4.1 Differential Operators and Distributional Adjoint Operators

First, we define a metric \( \rho \) on the Schwartz space
such that

\[ f \in C^\infty(\mathbb{R}^d) : \forall \alpha, \beta \in \mathbb{N}_0^d, \exists C_{\alpha\beta} > 0 \text{ s.t. } \sup_{x \in \mathbb{R}^d} |x^\beta D^\alpha f(x)| \leq C_{\alpha\beta} \]  

so that it becomes a Fréchet space, where the metric \( \rho \) is given by

\[
\rho(\gamma_1, \gamma_2) := \sum_{\alpha, \beta \in \mathbb{N}_0^d} 2^{-|\alpha| - |\beta|} \frac{\rho_{\alpha\beta}(\gamma_1 - \gamma_2)}{1 + \rho_{\alpha\beta}(\gamma_1 - \gamma_2)}, \quad \rho_{\alpha\beta}(\gamma) := \sup_{x \in \mathbb{R}^d} |x^\beta D^\alpha \gamma(x)|,
\]

for each \( \gamma_1, \gamma_2, \gamma \in S \). This means that a sequence \( \{\gamma_n\}_{n=1}^\infty \) of \( S \) converges to an element \( \gamma \in S \) if and only if \( x^\beta D^\alpha \gamma_n(x) \) converges uniformly to \( x^\beta D^\alpha \gamma(x) \) on \( \mathbb{R}^d \) for each \( \alpha, \beta \in \mathbb{N}_0^d \). Together with its metric \( \rho \) the Schwartz space \( S \) is regarded as the classical test function space.

Let \( S' \) be the associated space of tempered distributions (dual space of \( S \) or space of continuous linear functionals on \( S \)). We introduce the notation

\[
\langle T, \gamma \rangle := T(\gamma), \quad \text{for each } T \in S' \text{ and } \gamma \in S.
\]

For any \( f, g \in L^1_{loc}(\mathbb{R}^d) \) whose product \( fg \) is integrable on \( \mathbb{R}^d \) we denote a bilinear form by

\[
(f, g) := \int_{\mathbb{R}^d} f(x)g(x)dx.
\]

If \( f, g \in L^2(\mathbb{R}^d) \) then \( f, g \) is equal to the standard \( L^2(\mathbb{R}^d) \)-inner product.

For each \( f \in L^1_{loc}(\mathbb{R}^d) \cap S \mathbb{J} \) there exists a unique tempered distribution \( T_f \in S' \) such that

\[
\langle T_f, \gamma \rangle = (f, \gamma), \quad \text{for each } \gamma \in S.
\]

So \( f \in L^1_{loc}(\mathbb{R}^d) \cap S \mathbb{J} \) can be viewed as an element of \( S' \) and we rewrite \( T_f := f \). This means that \( L^1_{loc}(\mathbb{R}^d) \cap S \mathbb{J} \) can be embedded into \( S' \), i.e., \( L^1_{loc}(\mathbb{R}^d) \cap S \mathbb{J} \subseteq S' \).

The Dirac delta function (Dirac distribution) \( \delta_0 \) concentrated at the origin is also an element of \( S' \), i.e., \( \langle \delta_0, \gamma \rangle = \gamma(0) \) for each \( \gamma \in S \) (see [14, Chapter 1] and [8, Chapter 11]).

Remark 4.1. \( S \mathbb{J} \) denotes the collection of slowly increasing functions which grow at most like any particular fixed polynomial, i.e.,

\[
S \mathbb{J} := \left\{ f \in \mathbb{R}^d \to \mathbb{C} : f(x) = \mathcal{O}(\|x\|^m_2) \text{ as } \|x\|_2 \to \infty \text{ for some } m \in \mathbb{N}_0 \right\},
\]

where the notation \( f = \mathcal{O}(g) \) means that there is a positive number \( M \) such that \( |f| \leq M|g| \).

First, we will define the distributional derivative \( P : S' \to S' \) by (strong) derivative

\[
D^\alpha := \prod_{k=1}^d \frac{\partial^\alpha_k}{\partial x_k^{\alpha_k}}, \quad \text{where } |\alpha| := \sum_{k=1}^d \alpha_k, \quad \alpha := (\alpha_1, \ldots, \alpha_d)^T \in \mathbb{N}_0^d.
\]

According to [14, Chapter 1], the derivative \( D^\alpha \) is a continuous linear operator from \( S \) into \( S \). Then \( P \) can be well-defined by
and the derivative

\[
(PT, \gamma) := (-1)^{|\alpha|}(T, D^\alpha \gamma), \quad \text{for each } T \in S' \text{ and } \gamma \in S.
\]

Since \((D^\alpha \gamma_1, \gamma_2) = (-1)^{|\alpha|}(\gamma_1, D^\alpha \gamma_2)\) for each \(\gamma_1, \gamma_2 \in S\), the restricted operator \(P|_S\) and the derivative \(D^\alpha\) coincide on \(S\). For convenience, we denote the distributional derivative as \(P := D^\alpha\) (similar as [1, Chapter 1]).

Next we wish to define the distributional adjoint operator of the distributional derivative. In the same way as before we can introduce the linear operator \(P^* : S' \to S'\) by using the derivative \((-1)^{|\alpha|}D^\alpha\), i.e.,

\[
(P^*T, \gamma) := (-1)^{|\alpha|}(T, (-1)^{|\alpha|}D^\alpha \gamma) = (T, D^\alpha \gamma), \quad \text{for each } T \in S' \text{ and } \gamma \in S.
\]

Moreover, the restricted operator \(P^*|_S\) and the derivative \((-1)^{|\alpha|}D^\alpha\) coincide on \(S\).

We call the linear operator \(P^*\) the distributional adjoint operator of the distributional derivative \(P = D^\alpha\). It can be written as \(P^* = (-1)^{|\alpha|}P = (-1)^{|\alpha|}D^\alpha\) also. Usually people call the derivative \(P^*|_S = (-1)^{|\alpha|}D^\alpha : S \to S\) as the classical adjoint operator of the distributional derivative \(P = D^\alpha : S' \to S'\). Here we can think of the classical adjoint operator \(P^*|_S\) extended to be the distributional adjoint operator \(P^*\).

Now we will define a more general differential operator and its distributional adjoint operator by using real linear combinations of distributional derivatives.

**Definition 4.1.** The differential operator (with real-valued constant coefficients) \(P : S' \to S'\) is defined as

\[
P := \sum_{|\alpha| \leq m} c_\alpha D^\alpha, \quad \text{where } c_\alpha \in \mathbb{R} \text{ and } \alpha \in \mathbb{N}_0^d, \ m \in \mathbb{N}_0.
\]

Its distributional adjoint operator \(P^* : S' \to S'\) is well-defined by

\[
P^* := \sum_{|\alpha| \leq m} (-1)^{|\alpha|}c_\alpha D^\alpha.
\]

To streamline terminology we will refer to differential operators (with real-valued constant coefficients) and distributional adjoint operators simply as differential operators and adjoint operators respectively in this article.

It is obvious that the adjoint operator \(P^*\) of the differential operator \(P\) is also a differential operator. We can further check that the differential operator \(P\) and its adjoint operator \(P^*\) have the following property

\[
\langle PT, \gamma \rangle = \langle T, P^* \gamma \rangle \quad \text{and} \quad \langle P^*T, \gamma \rangle = \langle T, P \gamma \rangle, \quad \text{for each } T \in S' \text{ and } \gamma \in S.
\]

Any differential operator \(P\) is also complex-adjoint invariant, i.e.,

\[
\overline{P \gamma} = P \overline{\gamma}, \quad \text{for each } \gamma \in S.
\]

**Remark 4.2.** Our distributional adjoint operator is different from the usual adjoint operators of bounded linear operators defined in Hilbert or Banach space. Our oper-
ator is formed in the dual space of the Schwartz space and it may be not a bounded operator if $S'$ is defined as a metric space. But it is continuous when $S'$ is given the weak-star topology as the dual of $S$. However, the idea of this construction is similar to the classical ones. Therefore we call it an adjoint as well.

Finally, we know that the classical Sobolev spaces are defined by weak derivatives. Here we will explain some relationships between the distributional derivative $P := D^\alpha$ and the $\alpha$th weak derivative. Fixing any $f \in L^{1}_{1}(\mathbb{R}^{d}) \cap SI$, if there is a generalized function $g \in L^{1}_{1}(\mathbb{R}^{d})$ such that

$$\langle g, \gamma \rangle = (-1)^{|\alpha|}\langle f, D^\alpha \gamma \rangle = \langle D^\alpha f, \gamma \rangle,$$

then $D^\alpha f = g$ is called the $\alpha$th weak derivative of $f$. This means that distributional derivatives and weak derivatives are the same on the classical Sobolev space.

**Remark 4.3.** In the book [1], the definition of the weak derivative has tiny differences from the one we use in this article. In particular, they use different test functions to derive the weak derivative. We now state their definition in order to compare it to the above mentioned weak derivative. Fixing any $f \in L^{1}_{1}(\mathbb{R}^{d})$, if there is a generalized function $g \in L^{1}_{1}(\mathbb{R}^{d})$ such that

$$\langle g, \gamma \rangle = (-1)^{|\alpha|}\langle f, D^\alpha \gamma \rangle,$$

then they call $D^\alpha f = g$ the $\alpha$th weak derivative of $f$. However, if $f \in L^{1}_{1}(\mathbb{R}^{d}) \cap SI$ and $D^\alpha f \in L^{1}_{1}(\mathbb{R}^{d}) \cap SI$, then the two definitions of weak derivatives are equivalent since $\mathcal{D}(\mathbb{R}^{d})$ is dense in $S$, where $\mathcal{D}(\mathbb{R}^{d}) := C_{0}^{\infty}(\mathbb{R}^{d})$ and its dual space $\mathcal{D}(\mathbb{R}^{d})'$ are defined in [1] Chapter 1. In this case, we can consider the two weak derivatives as being the same.

### 4.2 Distributional Fourier Transforms

Denote the **Fourier transform** and **inverse Fourier transform** of any $\gamma \in S$ by

$$\hat{\gamma}(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^{d}} \gamma(y) e^{-ix^{T}y} \, dy, \quad \check{\gamma}(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^{d}} \gamma(y) e^{ix^{T}y} \, dy, \quad i := \sqrt{-1}.$$

Since $\hat{\gamma}$ belongs to $S$ for each $\gamma \in S$ and the Fourier transform map is a homeomorphism of $S$ onto itself, the distributional Fourier transform $\hat{T} \in S'$ of the tempered distribution $T \in S'$ is well-defined by

$$\langle \hat{T}, \gamma \rangle := \langle T, \hat{\gamma} \rangle,$$

for each $\gamma \in S$.

Since $\check{\gamma} = \hat{\gamma}$ for each $\gamma \in S$, we have

$$\langle T, \check{\gamma} \rangle = \langle \hat{T}, \gamma \rangle,$$

for each $T \in S'$ and $\gamma \in S$. 


So the Fourier transform of $\gamma \in S$ is the same as its distributional transform. If $f \in L_1(\mathbb{R}^d)$ or $f \in L_2(\mathbb{R}^d)$ then its $L_1(\mathbb{R}^d)$-Fourier transform or $L_2(\mathbb{R}^d)$-Fourier transform is equal to its distributional Fourier transform. The distributional Fourier transform $\hat{\delta}_0$ of the Dirac delta function $\delta_0$ is equal to $(2\pi)^{-d/2}$. Moreover, we can also check that the distributional Fourier transform map is an isomorphism of the topological vector space $S'$ onto itself (see [14, Chapter 1] and [1, Chapter 7]).

Now we want to define the distributional Fourier transforms of differential operators. First, we will derive a linear operator $L : S' \to S'$ using any fixed complex-valued polynomial $\hat{p}$ on $\mathbb{R}^d$. Since the linear operator $\gamma \mapsto \hat{p} \gamma$ is a continuous operator from $S$ into $S$, the linear operator $L$ can be well-defined by $\langle L T, \gamma \rangle = \langle T, \hat{p} \gamma \rangle$, for each $T \in S'$ and $\gamma \in S$.

Since $L f = \hat{p} f \in L^1_{\text{loc}}(\mathbb{R}^d) \cap S$ when $f \in L^1_{\text{loc}}(\mathbb{R}^d) \in S$, we use the notation $L := \hat{p}$ for convenience.

Next we consider the case of distributional derivatives. According to [14, Chapter 1], we know that $\hat{D}^\alpha \gamma = \hat{p} \hat{\gamma}$ for each $\gamma \in S$, where $\hat{p}(x) := (ix)^\alpha$ is a complex-valued polynomial on $\mathbb{R}^d$. Hence we can verify that $\langle \hat{P} T, \gamma \rangle = \langle T, \hat{p} \hat{\gamma} \rangle$ for each $T \in S'$ and $\gamma \in S$, where $P := D^\alpha$ and $\hat{p}(x) := (ix)^\alpha$. This implies that $\hat{P} T = \hat{p} \hat{T}$ for each $T \in S'$. Therefore we can denote the distributional Fourier transform of a differential operator in the following way.

**Definition 4.2.** Let $P$ be a differential operator. If there is a complex-valued polynomial $\hat{p}$ on $\mathbb{R}^d$ such that $\langle \hat{P} T, \gamma \rangle = \langle T, \hat{p} \hat{\gamma} \rangle = \langle \hat{p} \hat{T}, \gamma \rangle$ for each $T \in S'$ and $\gamma \in S$, then $\hat{p}$ is said to be the distributional Fourier transform of $P$.

According to Definition 4.2, each differential operator $P$ possesses a distributional Fourier transform $\hat{p}$. The complex-valued polynomial $\hat{p}$ can be written explicitly as

$$\hat{p}(x) = \sum_{|\alpha| \leq m} c_\alpha (ix)^\alpha,$$

where $P = \sum_{|\alpha| \leq m} c_\alpha D^\alpha$, $c_\alpha \in \mathbb{R}$, $\alpha \in \mathbb{N}^d_0$, $m \in \mathbb{N}_0$.

Moreover, since $P$ is defined with the real-valued constant coefficients, we have $\hat{p}^* = \overline{p}$, where $\hat{p}^*$ is the distributional Fourier transform of the adjoint operator $P^*$.

### 4.3 Green Functions and Generalized Sobolev Spaces

**Definition 4.3.** $G$ is the (full-space) Green function with respect to the differential operator $L$ if $G \in S'$ satisfies the equation

$$L G = \delta_0,$$

(1)
Equation (10) is to be understood in the distributional sense which means that
\[ \langle G, L^\gamma \rangle = \langle LG, \gamma \rangle = \langle \delta_0, \gamma \rangle = \gamma(0) \quad \text{for each } \gamma \in S. \]

According to Theorem 3.1 we can obtain the following theorem.

**Theorem 4.1.** Let \( L \) be a differential operator and its Fourier transform \( \hat{L} \) be positive on \( \mathbb{R}^d \) so that \( \hat{L}^{-1} \in L_1(\mathbb{R}^d) \). If the Green function \( G \in C(\mathbb{R}^d) \cap \text{Re}(L_1(\mathbb{R}^d)) \) with respect to \( L \) is an even function, then \( G \) is a positive definite function and
\[
\hat{G}(x) := (2\pi)^{-d/2} \hat{f}(x)^{-1}, \quad x \in \mathbb{R}^d.
\]
is the \( L_1(\mathbb{R}^d) \)-Fourier transform of \( G \).

**Proof.** First we want to prove that \( \hat{G} \) is the \( L_1(\mathbb{R}^d) \)-Fourier transform of \( G \). The fact that \( \hat{L}^{-1} \in L_1(\mathbb{R}^d) \) implies that \( \hat{G} \in L_1(\mathbb{R}^d) \). If we can check that
\[
\langle \hat{G}, \gamma \rangle = \langle \hat{\gamma}, \gamma \rangle, \quad \text{for each } \gamma \in S,
\]
where \( \hat{G} \) is the distributional Fourier transform of \( G \), then we can conclude that \( \hat{G} \) is the \( L_1(\mathbb{R}^d) \)-Fourier transform of \( G \). Since \( \hat{L}^{-1} \in C^\infty(\mathbb{R}^d) \) and \( D^\alpha \varphi (\hat{L}^{-1}) \in S_\mathbb{R} \) for each \( \alpha \in \mathbb{N}^d_0, \hat{L}^{-1} \gamma \in S \) for each \( \gamma \in S \). Hence
\[
\langle \hat{G}, \gamma \rangle = \langle \hat{L}\hat{G}, \hat{L}^{-1}\gamma \rangle = \langle \hat{L}G, \hat{L}^{-1}\gamma \rangle = \langle \delta_0, \hat{L}^{-1}\gamma \rangle
\]
\[
= \langle (2\pi)^{-d/2} \hat{L}^{-1}\gamma \rangle = (2\pi)^{-d/2} \langle \hat{L}^{-1}\gamma \rangle = (\hat{\gamma}, \gamma).
\]

According to [17] Corollary 5.24, \( G \) can be recovered from its \( L_1(\mathbb{R}^d) \)-Fourier transform, i.e.,
\[
G(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{\gamma}(y)e^{ix'y}dy, \quad x \in \mathbb{R}^d.
\]

Then we have
\[
|G(x)| \leq (2\pi)^{-d} \int_{\mathbb{R}^d} \left| \hat{f}(y)^{-1}e^{ix'y}dy \right| \leq (2\pi)^{-d} \| L^{-1} \|_{L_1(\mathbb{R}^d)} < \infty,
\]
which shows that \( G \) is bounded. Since \( \hat{G} \) is positive on \( \mathbb{R}^d \), \( G \) is a positive definite function by Theorem 3.1.

**Definition 4.4.** Let the vector differential operator \( P = (P_1, \cdots, P_n)^T \) be set up by countably many differential operators \( \{P_n\}_{n=1}^\infty \). The real generalized Sobolev space induced by \( P \) is defined by
\[
H_P(\mathbb{R}^d) := \left\{ f \in \text{Re}(L_2(\mathbb{R}^d)) : \{P_n f\}_{n=1}^\infty \subseteq L_2(\mathbb{R}^d) \text{ and } \sum_{n=1}^\infty \| P_n f \|_{L_2(\mathbb{R}^d)}^2 < \infty \right\}
\]
and it is equipped with the semi-inner product
\[
\langle f, g \rangle_{H_P(\mathbb{R}^d)} := \sum_{n=1}^\infty \langle P_n f, P_n g \rangle, \quad f, g \in H_P(\mathbb{R}^d).
\]
What is the meaning of $H_p(\mathbb{R}^d)$? By the definition of the generalized Sobolev space, we know that $H_p(\mathbb{R}^d)$ is a real-valued subspace of $L_2(\mathbb{R}^d)$ and it is equipped with a semi-inner product induced by the vector differential operator $P$. On the other hand, $f \in \text{Re}(L_2(\mathbb{R}^d))$ belongs to $H_p(\mathbb{R}^d)$ if and only if there is a sequence $\{g_n\}_{n=1}^\infty \subset L_2(\mathbb{R}^d)$ such that $\sum_{n=1}^\infty \|g_n\|_{L_2(\mathbb{R}^d)} < \infty$ and

$$(g_n, \gamma) = (g_n, \gamma) = (P_n f, \gamma) = (f, P_n^* \gamma) = (f, P_n^* \gamma), \quad \text{for each } \gamma \in S, \ n \in \mathbb{N}.$$ 

In the following theorems of this section we only consider $P$ constructed by a finite number of differential operators $P_1, \ldots, P_n$. If $P := (P_1, \cdots, P_n)^T$, then the differential operator

$$L := P^T P = \sum_{j=1}^n P_j^* P_j$$

is well-defined, where $P_1 := (P_1^*, \cdots, P_n^*)^T$ is the adjoint operator of $P$. Furthermore, the distributional Fourier transform $\hat{f}$ of $f$ can be computed as the form

$$\hat{f}(x) = \sum_{j=1}^n \hat{p}_j(x) \hat{p}_j(x) = \sum_{j=1}^n \frac{\hat{p}_j(x) \hat{p}_j(x)}{2} = \|\hat{p}(x)\|^2_2, \quad x \in \mathbb{R}^d,$$

where $\hat{p} = (\hat{p}_1, \cdots, \hat{p}_n)$ is the distributional Fourier transform of $P$ and $\hat{p}^* = (\hat{p}_1^*, \cdots, \hat{p}_n^*)^T$ is the distributional Fourier transform of $P^*$. Now we can obtain the main theorem about the space $H_p(\mathbb{R}^d)$ induced by the vector differential operator $P := (P_1, \cdots, P_n)^T$.

**Theorem 4.2.** Let the vector differential operator be $P := (P_1, \cdots, P_n)^T$ and its distributional Fourier transform be $\hat{p} := (\hat{p}_1, \cdots, \hat{p}_n)^T$. Suppose that $\hat{p}$ is nonzero on $\mathbb{R}^d$ and $x \mapsto \|\hat{p}(x)\|^2_2 \in L_1(\mathbb{R}^d)$. If the Green function $G \in C(\mathbb{R}^d) \cap \text{Re}(L_1(\mathbb{R}^d))$ with respect to the differential operator $L = P^T P$ is chosen to be an even function, then $G$ is a positive definite function and its related reproducing-kernel Hilbert space $N_G(\mathbb{R}^d)$ is equivalent to the generalized Sobolev space $H_p(\mathbb{R}^d)$, i.e.,

$$(f, g)_{N_G(\mathbb{R}^d)} = (f, g)_{H_p(\mathbb{R}^d)}, \quad f, g \in N_G(\mathbb{R}^d) = H_p(\mathbb{R}^d).$$

**Proof.** By the above discussion, the distributional Fourier transform $\hat{l}$ of $L$ is equal to

$$\hat{l}(x) = \|\hat{p}(x)\|^2_2. \quad \text{Since } \hat{p} \text{ is nonzero on } \mathbb{R}^d \text{ and } x \mapsto \|\hat{p}(x)\|^2_2 \in L_1(\mathbb{R}^d), \hat{l} \text{ is positive on } \mathbb{R}^d \text{ and } \hat{l}^{-1} \in L_1(\mathbb{R}^d). \text{ According to Theorem 4.1, } \hat{l} \text{ is positive definite and its } L_1(\mathbb{R}^d) \text{-Fourier transform is given by}

$$\hat{g}(x) := (2\pi)^{-d/2} \hat{l}(x)^{-1} = (2\pi)^{-d/2} \|\hat{p}(x)\|^2_2, \quad x \in \mathbb{R}^d.$$ 

With the material developed thus far we are able to construct the reproducing-kernel Hilbert space $N_G(\mathbb{R}^d)$ related to $G$. We remark that since $\hat{l}$ is positive on $\mathbb{R}^d$, $Lf = 0$ for some $f \in L_2(\mathbb{R}^d)$ if and only if $f = 0$. This implies that $P f = 0$ if and only if $f = 0$. Hence we can conclude that $H_p(\mathbb{R}^d)$ is an inner-product space under this condition.
Next, we fix any \( f \in \mathcal{N}_G(\mathbb{R}^d) \). According to Theorem 3.2, \( f \in \text{Re}(C(\mathbb{R}^d)) \cap L^2(\mathbb{R}^d) \) possesses an \( L^2(\mathbb{R}^d) \)-Fourier transform \( \hat{f} \) and \( x \mapsto \hat{f}(x) \| \hat{p}(x) \|_2 \in L^2(\mathbb{R}^d) \). This means that the functions \( \hat{p}_j \hat{f} \in L^2(\mathbb{R}^d), \ j = 1, \ldots, n \). Therefore we can define
\[
 f_{P_j} := (\hat{p}_j \hat{f}) \in L^2(\mathbb{R}^d), \quad j = 1, \ldots, n,
\]
using the inverse \( L^2(\mathbb{R}^d) \)-Fourier transform.
Since \( \hat{p}_j \) is a polynomial for each \( j = 1, \ldots, n \), \( \hat{p}_j \mathcal{T} \mathcal{S} \gamma \in \mathbb{S} \) for each \( \gamma \in \mathbb{S} \). Moreover, \( \hat{p}_j \mathcal{T} \mathcal{S} = \hat{p}_j \mathcal{T} \mathcal{S} = \mathcal{T} \mathcal{S} \gamma \) implies that
\[
 (f_{P_j}, \mathcal{T}) = (\hat{p}_j \hat{f}, \mathcal{T}) = (\mathcal{T}, \hat{p}_j \hat{f}) = (\hat{f}, \mathcal{T}^* \mathcal{S} \gamma) = (\mathcal{T}^* \mathcal{S} \gamma, \hat{f}) = (P_j f, \mathcal{T}).
\]
This shows that \( P_j f = f_{P_j} \in L^2(\mathbb{R}^d) \). Therefore we know that \( f \in H_p(\mathbb{R}^d) \) and then \( \mathcal{N}_G(\mathbb{R}^d) \subseteq H_p(\mathbb{R}^d) \).

To establish equality of the inner products we let \( f, g \in \mathcal{N}_G(\mathbb{R}^d) \). Then the Plancherel theorem \([8]\) yields
\[
 (f, g)_{H_p(\mathbb{R}^d)} = \sum_{j=1}^{n} (f_{P_j}, g_{P_j}) = \sum_{j=1}^{n} \langle \hat{f}_{P_j}, \hat{g}_{P_j} \rangle = \sum_{j=1}^{n} \int_{\mathbb{R}^d} \hat{f}(x) \overline{\hat{g}(x)} \| \hat{p}(x) \|_2^2 \, dx
\]
\[
 = \int_{\mathbb{R}^d} \hat{f}(x) \overline{\hat{g}(x)} \| \hat{p}(x) \|_2^2 \, dx = \int_{\mathbb{R}^d} \hat{f}(x) \overline{\hat{g}(x)} \hat{p}(x) \, dx
\]
\[
 = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{f}(x) \overline{\hat{g}(x)} \| \mathcal{S}(x) \| \, dx = (f, g)_{\mathcal{N}_G(\mathbb{R}^d)}.
\]

Finally, we verify that \( \mathcal{N}_G(\mathbb{R}^d) = H_p(\mathbb{R}^d) \). We fix any \( f \in H_p(\mathbb{R}^d) \). Let \( \hat{f} \) and \( \overline{\mathcal{T} \mathcal{S}} \mathcal{T} \mathcal{S} \mathcal{f} \), respectively, be the \( L^2(\mathbb{R}^d) \)-Fourier transforms of \( f \) and \( P_j f, \ j = 1, \ldots, n \). Using the Plancherel theorem \([8]\) again we obtain
\[
 \int_{\mathbb{R}^d} \| \hat{f}(x) \|_{\mathcal{S}(x)}^2 \, dx = (\hat{p}_j \hat{f}, \overline{\hat{p}_j \hat{f}}) = (\mathcal{T} \mathcal{S} \mathcal{f}, \mathcal{T} \mathcal{S} \mathcal{f}) = (P_j f, \overline{P_j f}) < \infty.
\]
And therefore, with the help of the proof above, we have
\[
 \int_{\mathbb{R}^d} \frac{\| \hat{f}(x) \|^2}{\mathcal{S}(x)} \, dx = (2\pi)^{d/2} \int_{\mathbb{R}^d} \| \hat{f}(x) \|^2 \hat{p}(x) \, dx = (2\pi)^{d/2} \int_{\mathbb{R}^d} \| \hat{f}(x) \|^2 \| \hat{p}(x) \|^2 \, dx
\]
\[
 = (2\pi)^{d/2} \sum_{j=1}^{n} \int_{\mathbb{R}^d} \| \hat{f}(x) \|^2 \| \hat{p}(x) \|^2 \, dx < \infty
\]
showing that \( \mathcal{S}^{-1/2} \hat{f} \in L^2(\mathbb{R}^d) \). This means in particular that \( \hat{f} \in L^1(\mathbb{R}^d) \) because
\[
 \int_{\mathbb{R}^d} \| \hat{f}(x) \| \, dx \leq \left( \int_{\mathbb{R}^d} \frac{\| \hat{f}(x) \|^2}{\mathcal{S}(x)} \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} \mathcal{S}(x) \, dx \right)^{1/2} < \infty.
\]
Since the inverse $L_1(\mathbb{R}^d)$-Fourier transform of $f$ is continuous, $f \in C(\mathbb{R}^d)$. According to Theorem 3.1, $f \in \mathcal{N}_G(\mathbb{R}^d)$ which implies that $H_P(\mathbb{R}^d) \subseteq \mathcal{N}_G(\mathbb{R}^d)$.

5 Examples of Positive Definite Kernels Generated By Green Functions

5.1 One-Dimensional Cases

With the theory we developed in the preceding section in mind we again discuss the univariate Sobolev splines of Example 2.1.

Example 5.1 (Univariate Sobolev Splines). Let $\sigma > 0$ be a scalar parameter and consider $P := (d/dx, \sigma I)^T$ with $L := P^T P = \sigma^2 I - d^2/dx^2$. It is known that the Green function with respect to $L$ is $G(x) := \frac{1}{2\sigma} \exp(-\sigma |x|)$, $x \in \mathbb{R}$.

Since $P$ and $G$ satisfy the conditions of Theorem 4.2, we know $G$ is positive definite and that the interpolant formed by $G$ is given by

$$s_f, X(x) := \sum_{j=1}^{N} c_j G(x - x_j), \quad x \in \mathbb{R}. \quad (1)$$

Formula (1) denotes a Sobolev spline (or Matérn) interpolant. Since $f \in H_P(\mathbb{R})$ if and only if $f', f \in L_2(\mathbb{R}^d)$, $\mathcal{H}^1(\mathbb{R})$ is equivalent to $H_P(\mathbb{R})$. Applying Theorem 4.2 and [17, Theorem 13.2] we confirm that

$$\|s_f, X\|_{H_P(\mathbb{R})} = \min \left\{ \|f\|_{H_P(\mathbb{R})} : f \in \mathcal{H}^1(\mathbb{R}) \text{ and } f(x_j) = y_j, j = 1, \ldots, N \right\},$$

i.e., $s_f, X$ is the minimum $H_P(\mathbb{R})$-norm interpolant to the data from $\mathcal{H}^1(\mathbb{R})$.

5.2 d-Dimensional Cases

Example 5.2 (Sobolev Splines). This is a generalization of Example 5.1. Let $P := (Q_0^T, \ldots, Q_n^T)^T$ with a scalar parameter $\sigma > 0$, where

$$Q_j := \begin{cases} \frac{n! \sigma^{2n-2j}}{j!(n-j)!} \Delta^k & \text{when } j = 2k, \\ \frac{n! \sigma^{2n-2j}}{j!(n-j)!} \Delta^k \nabla & \text{when } j = 2k + 1, \end{cases} \quad k \in \mathbb{N}_0, \quad j = 0, 1, \ldots, n, \quad n > d/2.$$
Here we use $\Delta^0 := I$. With these definitions we get $L := P^T P = (\sigma^2 I - \Delta)^n$.

The Sobolev spline (or Matérn function) is known to be the Green function with respect to $L$ (see [2, Chapter 6.1.6]), i.e.,

$$G(x) := \frac{2^{1-n-d/2}}{\pi^{d/2} \Gamma(n) \sigma^{2n-d/2}} K_{d/2-n}(\sigma \|x\|_2), \quad x \in \mathbb{R}^d,$$

where $K_m(\cdot)$ is the modified Bessel function of the second kind of order $m$. Since $P$ and $G$ satisfy the conditions of Theorem 4.2, $G$ is positive definite and the associated interpolant $s_{f,X}$ is the same as the Sobolev spline (or Matérn) interpolant.

Since $f \in H_P(\mathbb{R}^d)$ if and only if $\Delta^{n/2} f \in L_2(\mathbb{R}^d)$, $H_P(\mathbb{R}^d)$ is equivalent to $H^n(\mathbb{R}^d)$ which implies that $N_G(\mathbb{R}^d)$ and $H^n(\mathbb{R}^d)$ are isomorphic by Theorem 4.2. It follows that the real classical Sobolev space $H^n(\mathbb{R}^d)$ becomes a reproducing-kernel Hilbert space with $H_P(\mathbb{R}^d)$-inner-product and its reproducing kernel is $K(x, y) := G(x - y)$.

In the following example we are not able to establish that the operator $P$ satisfies the conditions of Theorem 4.2 and so part of the connection to the theory developed in this paper is lost. We therefore use the symbol $\Phi$ to denote the kernel instead of $G$.

**Example 5.3 (Gaussians).** The Gaussian kernel $K(x, y) := \Phi(x - y)$ based on the Gaussian function $\Phi$ is very important and popular in current research fields such as scattered data approximation and machine learning. Many people would therefore like to better understand the reproducing-kernel Hilbert space associated with the Gaussian function. In this example, we will show that the reproducing-kernel Hilbert space of the Gaussian function is equivalent to a generalized Sobolev space.

We first consider the Gaussian function

$$\Phi(x) := \frac{\sigma^d}{\pi^{d/2}} \exp(-\sigma^2 \|x\|_2^2), \quad x \in \mathbb{R}^d, \quad \sigma > 0$$

We know that $\Phi$ is a positive definite function and its $L_1(\mathbb{R}^d)$-Fourier transform is given by (see [3, Chapter 4])

$$\hat{\Phi}(x) = (2\pi)^{-d/2} \exp \left(-\frac{\|x\|_2^2}{4\sigma^2}\right), \quad x \in \mathbb{R}^d.$$

According to Theorem 5.2, the reproducing-kernel Hilbert space of $\Phi$ is given by

$$N_\Phi(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d) \cap \text{Re} L_2(\mathbb{R}^d) : \tilde{\phi}^{-1/2} f \in L_2(\mathbb{R}^d) \right\},$$

and its inner-product is equal to

$$(f, g)_{N_\Phi(\mathbb{R}^d)} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\hat{f}(x)\hat{g}(x)}{\hat{\phi}(x)} dx, \quad f, g \in N_\Phi(\mathbb{R}^d).$$
where $\hat{f}, \hat{g}$ are the $L_2(\mathbb{R}^d)$-Fourier transforms of $f, g \in \mathcal{N}_\Phi(\mathbb{R}^d)$, respectively.

Let $P := (Q_0^T, \ldots, Q_n^T, \ldots)^T$, where

$$Q_n := \begin{cases} \left( \frac{(-1)^k}{n!^2 \pi^{d/2}} \right)^{1/2} \Delta^k & \text{when } n = 2k, \\ \left( \frac{1}{n!^2 \pi^{d/2}} \right)^{1/2} \Delta^k \nabla & \text{when } n = 2k + 1, \end{cases} \quad k \in \mathbb{N}_0. \ (2)$$

Here we again use $\Delta^0 := I$. Now we will verify that $\mathcal{N}_\Phi(\mathbb{R}^d)$ is equivalent to $\mathcal{H}_p(\mathbb{R}^d)$. Even though we find that $P$ does not satisfy the conditions of Theorem 4.2, we are still able to use other techniques in order to combine the results of this paper to complete the proof.

Let $P_n := (Q_0^T_n, \ldots, Q_n^T_n)^T$ and $L_n := P_n^T P_n$ for each $n \in \mathbb{N}$. We choose the Green function $G_n$ with respect to $L_n$, which is the inverse $L_2(\mathbb{R}^d)$-Fourier transform of $(2\pi)^{-d/2} f_n^{-1}$ when $n > d/2$. Therefore $P_n$ and $G_n$ satisfy the conditions of Theorem 4.2. This tells us that – as in Examples 5.1 and 5.2 – $H_{P_n}(\mathbb{R}^d)$ is equivalent to the classical Sobolev space $H^m(\mathbb{R}^d)$ for each $n \in \mathbb{N}$. Theorem 4.2 further tells us that

$\mathcal{N}_{G_n}(\mathbb{R}^d) \equiv H_{P_n}(\mathbb{R}^d), \quad \text{when } n > d/2.$

Furthermore, we can verify that

$$f \in H_p(\mathbb{R}^d) \iff f \in \cap_{n=1}^\infty H_{P_n}(\mathbb{R}^d) \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|f\|_{H_{P_n}(\mathbb{R}^d)} < \infty,$$

which implies that $\|f\|_{\mathcal{H}_p(\mathbb{R}^d)} \to \|f\|_{H_p(\mathbb{R}^d)}$ as $n \to \infty$.

Let $f \in \mathcal{N}_\Phi(\mathbb{R}^d)$ and $\hat{f}$ be the $L_2(\mathbb{R}^d)$-Fourier transform of $f$. We can check that $\|\hat{p}_1(x)\|_2^2 \leq \cdots \leq \|\hat{p}_n(x)\|_2^2 \leq \cdots \leq (2\pi)^{-d/2} \phi(x)^{-1}$ and $\|\hat{p}_n(x)\|_2^2 \to (2\pi)^{-d/2} \phi(x)^{-1}$ as $n \to \infty$. Hence, $\hat{G}_n^{-1/2} \hat{f} \in L_2(\mathbb{R}^d)$ which implies that $f \in \mathcal{N}_{G_n}(\mathbb{R}^d)$ by Theorem 4.2. According to the Lebesgue monotone convergence theorem [11] and Theorem 4.2, we have

$$\lim_{n \to \infty} \|f\|_{H_{P_n}(\mathbb{R}^d)}^2 = \lim_{n \to \infty} \|f\|_{\mathcal{N}_{G_n}(\mathbb{R}^d)}^2 = \lim_{n \to \infty} (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{|\hat{f}(x)|^2}{\hat{G}_n(x)} \, dx$$

$$= \lim_{n \to \infty} \int_{\mathbb{R}^d} |\hat{f}(x)|^2 \hat{G}_n(x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^d} |\hat{f}(x)|^2 \|\hat{p}_n(x)\|_2^2 \, dx$$

$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{|\hat{f}(x)|^2}{\phi(x)} \, dx = \|f\|_{\mathcal{N}_\Phi(\mathbb{R}^d)}^2 < \infty.$$

Therefore $f \in H_p(\mathbb{R}^d)$ and $\|f\|_{\mathcal{N}_\Phi(\mathbb{R}^d)} = \|f\|_{H_p(\mathbb{R}^d)}$.

On the other hand, we fix any $f \in H_p(\mathbb{R}^d)$. Then $f \in H_{P_n}(\mathbb{R}^d)$ for each $n \in \mathbb{N}$. We again use the Lebesgue monotone convergence theorem [11] and Theorem 4.2 to show that
and it is equipped with the semi-inner product

\[ \langle f, g \rangle_{H_\Phi^p(\mathbb{R}^d)} := \sum_{n=1}^{\infty} \langle P_n f, P_n g \rangle, \quad f, g \in H_\Phi^p(\mathbb{R}^d), \]

which establishes that \( \hat{\phi}^{-1/2} \hat{f} \in L_2(\mathbb{R}^d) \), and therefore \( f \in \mathcal{N}_\phi(\mathbb{R}^d) \).

Summarizing the above discussion, it follows that the reproducing-kernel Hilbert space of the Gaussian kernel is given by the generalized Sobolev space \( H_\Phi^p(\mathbb{R}^d) \), i.e.,

\[ \mathcal{N}_\phi(\mathbb{R}^d) \equiv H_\Phi^p(\mathbb{R}^d). \]

**Remark 5.1.** If \( f \in \mathcal{N}_\phi(\mathbb{R}^d) \equiv H_\Phi^p(\mathbb{R}^d) \), then \( f \in \mathcal{H}_\alpha(\mathbb{R}^d) \) for each \( n \in \mathbb{N} \). According to the Sobolev embedding theorem \([\Pi]\), we have \( \mathcal{N}_\phi(\mathbb{R}^d) \subseteq C_\alpha^p(\mathbb{R}^d) \). On the other hand, if a function \( f \in \text{Re}(C_\alpha^p(\mathbb{R}^d)) \) satisfies \( \|D^a f\|_{L_\infty(\mathbb{R}^d)} \leq C(\|f\|_{H_\Phi^p(\mathbb{R}^d)}) \) for some positive constant \( C \) and each \( \alpha \in \mathbb{N}_0^d \), then \( f \in H_\Phi^p(\mathbb{R}^d) \equiv \mathcal{N}_\phi(\mathbb{R}^d) \). Moreover, if we replace the test functions space to be \( \mathcal{D}(\mathbb{R}^d) \), then we can further think of the Gaussian function \( \Phi \) is a (full-space) Green function of \( L := \exp(-\frac{1}{4\sigma^2} \Delta) \), i.e.,

\[ L \Phi = \delta_0 \quad \text{and} \quad \Phi, \Phi_0 \in \mathcal{D}(\mathbb{R}^d). \]

### 6 Extensions and Future Works

In another paper \([6]\), we generalize the results of this paper in several ways.

Instead of being limited to differential operators we allow general distributional operators which, e.g., are allowed to be differential operators with non-constant coefficients. In that case the (full-space) Green functions and the generalized Sobolev spaces can be constructed by the vector distributional operators in a similar way.

In addition, we extend all the results from positive definite functions and their reproducing-kernel Hilbert space with respect to a vector differential operators to conditionally positive definite functions of some orders and their native space with respect to vector distributional operators. In this case the real generalized Sobolev space will be rewritten as the following from

\[ H_\Phi^p(\mathbb{R}^d) := \left\{ f \in \text{Re}(L_1^{loc}(\mathbb{R}^d)) \cap \mathcal{S} : \{P_n f\}_{n=1}^{\infty} \subseteq L_2(\mathbb{R}^d) \text{ and } \sum_{n=1}^{\infty} \|P_n f\|_{L_2(\mathbb{R}^d)}^2 < \infty \right\}, \]

and it is equipped with the semi-inner product

\[ \langle f, g \rangle_{H_\Phi^p(\mathbb{R}^d)} := \sum_{n=1}^{\infty} \langle P_n f, P_n g \rangle, \quad f, g \in H_\Phi^p(\mathbb{R}^d), \]
where \( P := (P_1, \cdots, P_n, \cdots)^T \) is a vector distributional operator (see [6]).

For example, if \( P := (\omega_\tau \partial^m / \partial x_1^m, \cdots, \omega_\tau D^\alpha, \cdots, \omega_\tau \partial^m / \partial x_d^m)^T \) which is set up by the differential operators (with the non-constant coefficients), then

\[
H_P(\mathbb{R}^d) := \left\{ f \in \text{Re}(L_1^{\text{loc}}(\mathbb{R}^d)) \cap C^1 : \omega_\tau D^\alpha f \in L_2(\mathbb{R}^d), |\alpha| = m, \alpha \in \mathbb{N}_0^d \right\},
\]

where \( \omega_\tau(x) := \|x\|_\tau^2 \) and \( 0 \leq \tau < 1 \).

In the work presented here and in [6] we do not specify any boundary conditions for the Green functions. Thus there may be many different choices for the Green function with respect to one and the same differential operator \( L \). In our future work we will apply a vector differential operator \( P := (P_1, \cdots, P_n)^T \) and a vector boundary operator \( B := (B_1, \cdots, B_s)^T \) on a bounded domain \( \Omega \) to construct a reproducing kernel and its related reproducing-kernel Hilbert space (see [7]). We further hope to use the distributional operator \( L \) to approximate the eigenvalues and eigenfunctions of the kernel function with the hope of obtaining fast numerical methods to solve the interpolating systems (1)-(2) similar as in [12] Chapter 15.

Finally, we only consider real-valued functions for the definition of our generalized Sobolev spaces and their Green functions in this paper. However, all conclusions and all theorems can be extended to complex-valued functions similar as was done in [17].

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