A NOTE ON NEW TYPE DEGENERATE BERNOULLI NUMBERS

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ABSTRACT. Studying degenerate versions of various special polynomials have become an active area of research and yielded many interesting arithmetic and combinatorial results. Here we introduce a degenerate version of polylogarithm function, called the degenerate polylogarithm function. Then we construct new type degenerate Bernoulli polynomials and numbers, called degenerate poly-Bernoulli polynomials and numbers, by using the degenerate polylogarithm function and derive several properties on the degenerate poly-Bernoulli numbers.

1. INTRODUCTION

As is well known, for \( s \in \mathbb{C} \), the polylogarithm function is defined by a power series in \( z \), which is also a Dirichlet series in \( s \):

\[
\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots, \quad (\text{see } \{6, 9, 22\}).
\]

This definition is valid for arbitrary complex order \( s \) and for all complex arguments \( z \) with \( |z| < 1 \); it can be extended to \( |z| \geq 1 \) by analytic continuation.

From (1), we note that

\[
\text{Li}_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z).
\]

For any nonzero \( \lambda \in \mathbb{R} \) (or \( \mathbb{C} \)), the degenerate exponential function is defined by

\[
e_{\lambda}^x(t) = (1 + \lambda t)^\frac{x}{\lambda}, \quad e_{\lambda}^x(t) = (1 + \lambda t)^\frac{x}{\lambda} = e\lambda^z(t), \quad (\text{see } \{1, 14, 15, 17\}).
\]

By Taylor expansion, we get

\[
e\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{\lambda,n} \frac{t^n}{n!}, \quad (\text{see } \{13, 14, 15, 16, 17, 18\}),
\]

where \( (x)_{0,\lambda} = 1, \ (x)_{n,\lambda} = (x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \ (n \geq 1) \).

Note that

\[
\lim_{\lambda \to 0} e\lambda^x(t) = \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} = e^x.
\]

In [1,2], Carlitz considered the degenerate Bernoulli polynomials given by

\[
\frac{t}{e_{\lambda}^x(t) - 1} e_{\lambda}^x(t) = \frac{t}{(1 + \lambda t)^\frac{x}{\lambda} - 1} (1 + \lambda t)^\frac{x}{\lambda} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}.
\]

When \( x = 0 \), \( \beta_{n,\lambda} = \beta_{n,\lambda}(0) \) are called the degenerate Bernoulli numbers. Note that

\[
\lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x), \quad (n \geq 0),
\]

2010 Mathematics Subject Classification. 11B83; 05A19.

Key words and phrases. degenerate polylogarithm function; degenerate poly-Bernoulli polynomial; degenerate poly-Bernoulli number.
where \( B_n(x) \) are the ordinary Bernoulli polynomials given by

\[
\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see } [1-26]).
\]

The Stirling numbers of the second kind are defined as

\[
x^n = \sum_{l=0}^{n} S_2(n, l) (x)_l, \quad (n \geq 0), \quad (\text{see } [10, 12, 14, 20, 22]),
\]

where \((x)_0 = 1, (x)_n = x(x-1)(x-2)\cdots(x-n+1), (n \geq 1)\). Thus, we easily get

\[
\frac{1}{k!} (e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}, \quad (k \geq 0).
\]

In [11], Kim considered the degenerate Stirling numbers of the second kind which are defined as

\[
(x)_{n, \lambda} = \sum_{k=0}^{n} S_{2, \lambda}(n, k) (x)_k, \quad (n \geq 0).
\]

Note that

\[
\lim_{\lambda \to 0} S_{2, \lambda}(n, k) = S_2(n, k).
\]

The generating function of the degenerate Stirling numbers of the second kind is given by

\[
\frac{1}{k!} (e^{\lambda t} - 1)^k = \sum_{n=k}^{\infty} S_{2, \lambda}(n, k) \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see } [11]).
\]

In this paper, we will introduce the degenerate polylogarithm function as a degenerate version of the polylogarithm function for \( s = \frac{k}{\lambda} \in \mathbb{Z} \). Then we will construct new type degenerate Bernoulli polynomials and numbers, called degenerate poly-Bernoulli polynomials and numbers, by using the degenerate polylogarithm function and derive several properties on the degenerate poly-Bernoulli numbers.

2. NEW TYPE DEGENERATE BERNOULLI NUMBERS AND POLYNOMIALS

We define the degenerate logarithm function \( \log_{\lambda} (1 + t) \), which is the inverse of the degenerate exponential function \( e_\lambda(t) \) and the motivation for the definition of degenerate polylogarithm function, as:

\[
\log_{\lambda} (1 + t) = \sum_{n=1}^{\infty} \lambda^{n-1} (1)_{n, 1/\lambda} \frac{t^n}{n!}.
\]

From (4) and (11), we note that

\[
\log_{\lambda} (1 + t) = \sum_{n=1}^{\infty} \lambda^{n-1} \left( 1 - \frac{1}{\lambda} \right) \left( 1 - \frac{2}{\lambda} \right) \cdots \left( 1 - \frac{1}{\lambda} (n-1) \right) \frac{t^n}{n!}
\]

\[
= \frac{1}{\lambda} \sum_{n=1}^{\infty} (\lambda)_n \frac{t^n}{n!}
\]

\[
= \frac{1}{\lambda} ((1+t)^{\lambda} - 1).
\]

**Lemma 1.** For \( \lambda \in \mathbb{R} \), we have

\[
\log_{\lambda} (1 + t) = \frac{1}{\lambda} ((1+t)^{\lambda} - 1).
\]

In addition, \( e_\lambda \left( \log_{\lambda} (1 + t) \right) = 1 + t \).
It is easy to show that
\[
\lim_{\lambda \to 0} \log_\lambda (1 + t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} = \log(1 + t).
\]

For \( k \in \mathbb{Z} \), we define the degenerate polylogarithm function as
\[
I_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k} x^n, \quad (|x| < 1).
\]

Note that
\[
\lim_{\lambda \to 0} I_{k,\lambda}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \operatorname{Li}_k(x), \quad \text{(see [3, 22])}.
\]

From (12), we note that
\[
\frac{d}{dx} I_{k,\lambda}(x) = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k} x^n = \frac{1}{\lambda} I_{k-1,\lambda}(x).
\]

For \( k \geq 2 \), (13) can be written in the form of iterated integral which is given by
\[
I_{k,\lambda}(x) = \int_0^{\infty} \frac{1}{t} \frac{1}{t} \frac{1}{t} \cdots \frac{1}{t} I_{1,\lambda}(t) dt dt \cdots dt \quad \text{for } (k-2)\text{-times}
\]

By (11) and (12), we get
\[
I_{1,\lambda}(x) = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{n!} x^n = -\log_\lambda (1 - x).
\]

Thus, by (14) and (15), for \( k \geq 2 \) we get
\[
I_{k,\lambda}(x) = -\int_0^{\infty} \frac{1}{t} \frac{1}{t} \frac{1}{t} \cdots \frac{1}{t} \log_\lambda (1 - t) dt dt \cdots dt \quad \text{for } (k-2)\text{-times}
\]

For \( k \in \mathbb{Z} \), we define the new type degenerate Bernoulli numbers, which are called the degenerate poly-Bernoulli numbers, as
\[
\frac{1}{x} I_{k,\lambda}(x) \bigg|_{x=1 - e_\lambda(-t)} = \frac{1}{1 - e_\lambda(-t)} I_{k,\lambda}(1 - e_\lambda(-t)) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!}.
\]

Note that
\[
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(1)} \frac{t^n}{n!} = \frac{1}{1 - e_\lambda(-t)} I_{1,\lambda}(1 - e_\lambda(-t)) = \frac{-t}{e_\lambda(-t) - 1} = \sum_{n=0}^{\infty} \frac{(-1)^n \beta_{n,\lambda}^n}{n!}.
\]

Comparing the coefficients on both sides of (18), we have
\[
\beta_{n,\lambda}^{(1)} = (-1)^n \beta_{n,\lambda}, \quad (n \geq 0).
\]

Now, we consider the new type degenerate Bernoulli polynomials which are called the degenerate poly-Bernoulli polynomials and given by
\[
\frac{l_{k,\lambda}(1 - e_\lambda(-t))}{1 - e_\lambda(-t)} e_\lambda(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!}.
\]
Now, we observe that

\[
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} x^n/n! = \frac{k_{k,\lambda}(1-e_\lambda(-t))}{1-e_\lambda(-t)} e_\lambda^k(-t)
\]

(20)

\[
= \sum_{l=0}^{\infty} \beta_{l,\lambda}^{(k)} l! \sum_{m=0}^{\infty} \frac{(-1)^m(x)_{m,\lambda}}{m!} l^m
\]

\[
= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \left( \frac{n}{l} \right) \beta_{l,\lambda}^{(k)} (-1)^{n-l} (x)_{n-l,\lambda} l^n/n!
\]

Comparing the coefficients on both sides of (20), we have

\[
\beta_{n,\lambda}^{(k)} = \sum_{l=0}^{n} \left( \frac{n}{l} \right) \beta_{l,\lambda}^{(k)} (-1)^{n-l} (x)_{n-l,\lambda}, \quad (n \geq 0).
\]

Now, we observe that

\[
\frac{d}{dx} e_\lambda(-x) = \frac{d}{dx} \sum_{l=0}^{\infty} \frac{(-1)^l(1)_{l,\lambda}}{l!} x^l = \sum_{l=1}^{\infty} \frac{(-1)^l(1)_{l,\lambda}}{(l-1)!} x^{l-1}
\]

(22)

\[
= -\sum_{l=0}^{\infty} \frac{(-1)^l(1)_{l+1,\lambda}}{l!} x^l = -\sum_{l=0}^{\infty} \frac{(-1)^l(1)_{l,\lambda}}{l!} x^l (1-l\lambda)
\]

\[
= -e_\lambda(-x) + \lambda \sum_{l=1}^{\infty} \frac{(-1)^l(1)_{l,\lambda}}{(l-1)!} x^l = -e_\lambda(-x) + \lambda x \frac{d}{dx} e_\lambda(-x).
\]

Thus, by (22), we get

\[
(1-\lambda x) \frac{d}{dx} e_\lambda(-x) = -e_\lambda(-x).
\]

Therefore, by (23), we obtain the following lemma.

**Lemma 2.** For \( \lambda \in \mathbb{R} \), we have

\[
\frac{d}{dx} e_\lambda(-x) = -\frac{1}{1-\lambda x} e_\lambda(-x) = -e_\lambda^{1-\lambda}(-x).
\]

By Lemma 2, we easily get

\[
\frac{d}{dx} (1-e_\lambda(-x)) = \frac{1}{1-\lambda x} e_\lambda(-x) = e_\lambda^{1-\lambda}(-x)
\]

(24)

From (13), (15), (17), and (24), for \( k \geq 2 \) we have

\[
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} x^n/n! = \frac{1}{1-e_\lambda(-x)} \int_{0}^{\infty} e_\lambda^{1-\lambda}(-t) \int_{0}^{t} e_\lambda^{1-\lambda}(-t) \cdots \int_{0}^{t} e_\lambda^{1-\lambda}(-t) t dt \cdots dt.
\]

(25)

Therefore, by (25), we obtain the following theorem.

**Theorem 3.** For \( k \geq 2 \), we have

\[
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} x^n/n! = \frac{1}{1-e_\lambda(-x)} \int_{0}^{\infty} e_\lambda^{1-\lambda}(-t) \int_{0}^{t} e_\lambda^{1-\lambda}(-t) \cdots \int_{0}^{t} e_\lambda^{1-\lambda}(-t) t dt \cdots dt.
\]

(25)
From (17), we can derive the following equation:

\[
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} = \sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}(1)_{n,1/\lambda}}{(n-1)!n^k} (1 - e_{\lambda}(-t))^{n-1}
\]

\[
= \sum_{m=0}^{\infty} \frac{(-\lambda)^m(1)_{m+1,1/\lambda}}{m!(m+1)^k} (1 - e_{\lambda}(-t))^m
\]

\[
= \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{(m+1)^k} (1)_{m+1,1/\lambda} \sum_{n=m}^{\infty} (-1)^{m-n} S_{2,\lambda}(n,m) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( -1 \right)^n \sum_{m=0}^{n} \frac{\lambda^m(1)_{m+1,1/\lambda}}{(m+1)^k} S_{2,\lambda}(n,m) \frac{t^n}{n!}
\]

Therefore, by comparing the coefficients on both sides of (26), we obtain the following theorem.

**Theorem 4.** For \( n \geq 0 \), we have

\[
\beta_{n,\lambda}^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{\lambda^m(1)_{m+1,1/\lambda}}{(m+1)^k} S_{2,\lambda}(n,m).
\]

Note that

\[
(-1)^n B_n = \lim_{\lambda \to 0} \beta_{n,\lambda}^{(1)} = (-1)^n \sum_{m=0}^{n} \frac{m!}{n!} (-1)^{m} S_{2}(n,m)\quad (n \geq 0).
\]

For \( k = 2 \), by Theorem 3, we get

\[
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(2)} \frac{x^n}{n!} = \frac{1}{1 - e_{\lambda}(-x)} \int_{0}^{x} \frac{t}{1 - e_{\lambda}(-t)} e_{\lambda}^{1-\lambda}(-t) dt
\]

\[
= \frac{1}{1 - e_{\lambda}(-x)} \int_{0}^{\infty} \sum_{n=0}^{\infty} \beta_{n,\lambda} (1 - \lambda) (-1)^n \frac{n!}{n!} dt
\]

\[
= \frac{x}{1 - e_{\lambda}(-x)} \sum_{l=0}^{\infty} \frac{\beta_{l,\lambda} (1 - \lambda)}{l+1} \frac{x^l}{l!}
\]

\[
= \sum_{n=0}^{\infty} \left( -1 \right)^n \sum_{m=0}^{n} \frac{n!}{m!} \frac{\beta_{m,\lambda} (1 - \lambda)}{n-m+1} \frac{x^m}{m!}
\]

Therefore, by comparing the coefficients on both sides of (27), we obtain the following theorem.

**Theorem 5.** For \( n \geq 0 \), we have

\[
\beta_{n,\lambda}^{(2)} = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \beta_{m,\lambda} (1 - \lambda) \frac{1}{n-m+1} = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \beta_{m,\lambda} (1 - \lambda) \frac{1}{m+1}.
\]

In general, from (25), we note that

\[
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{x^n}{n!} = \frac{1}{1 - e_{\lambda}(-x)} \int_{0}^{x} \frac{e_{\lambda}^{1-\lambda}(-t)}{1 - e_{\lambda}(-t)} dt \int_{0}^{t} \frac{e_{\lambda}^{1-\lambda}(-t)}{1 - e_{\lambda}(-t)} dt \cdots \int_{0}^{t} \frac{e_{\lambda}^{1-\lambda}(-t)}{1 - e_{\lambda}(-t)} dt dt \cdots dt
\]

\[
= \sum_{n_1,n_2,\ldots,n_{k-1}=0}^{\infty} \frac{1}{n_1!n_2!\cdots n_{k-1}!} \frac{\beta_{n_1,\lambda} (1 - \lambda) \beta_{n_2,\lambda} (1 - \lambda) \cdots \beta_{n_{k-1},\lambda} (1 - \lambda)}{n_1 + n_2 + \cdots + n_{k-1} + 1} \frac{x}{1 - e_{\lambda}(-x)}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \sum_{n_1+n_2+\cdots+n_{k}=n} \binom{n}{n_1,n_2,\ldots,n_{k}} \frac{\beta_{n_1,\lambda} (1 - \lambda) \beta_{n_2,\lambda} (1 - \lambda) \cdots \beta_{n_{k},\lambda} (1 - \lambda)}{n_1 + n_2 + \cdots + n_{k} + 1} \frac{x^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \sum_{n_1,n_2,\ldots,n_{k}=n} \binom{n}{n_1,n_2,\ldots,n_{k}} \frac{\beta_{n_1,\lambda} (1 - \lambda) \beta_{n_2,\lambda} (1 - \lambda) \cdots \beta_{n_{k},\lambda} (1 - \lambda)}{n_1 + n_2 + \cdots + n_{k} + 1} \frac{x^n}{n!}
\]
Theorem 7. For \( n \in \mathbb{N} \) and \( n \geq 0 \), we have

\[
\beta_{n, \lambda}^{(k)} = (-1)^n \sum_{n_1 + \cdots + n_k = n} \binom{n}{n_1, n_2, \ldots, n_k} \beta_{n_1, \lambda} (1 - \lambda) \beta_{n_2, \lambda} (1 - \lambda) \cdots \beta_{n_k, \lambda} (1 - \lambda) \frac{n}{n_1 + n_2 + \cdots + n_k - 1 + \beta_{n, \lambda}}.
\]

From (17), we observe that

\[
l_{k, \lambda} (1 - e_{\lambda} (-t)) = (1 - e_{\lambda} (-t)) \sum_{l=0}^{\infty} \beta_{l, \lambda}^{(k)} \frac{t^l}{l!}
\]

\[
= \left(1 - \sum_{m=0}^{\infty} \frac{(-1)^m (1)_{m, \lambda}}{m!} t^m\right) \sum_{l=0}^{\infty} \beta_{l, \lambda}^{(k)} \frac{t^l}{l!}
\]

\[
= \sum_{n=0}^{\infty} \left(\beta_{n, \lambda} - \sum_{l=0}^{n} \binom{n}{l} \beta_{l, \lambda}^{(k)} (1)_{n-l, \lambda}\right) \frac{t^n}{n!}
\]

\[
= \sum_{n=1}^{\infty} \left(\beta_{n, \lambda} - \beta_{n}^{(k)} (1)\right) \frac{t^n}{n!}.
\]

On the other hand,

\[
l_{k, \lambda} (1 - e_{\lambda} (-t)) = \sum_{m=1}^{\infty} \binom{m}{m, \lambda} (-\lambda)^{m-1} \frac{1}{m!} (1 - e_{\lambda} (-t))^m
\]

\[
= \sum_{m=1}^{\infty} \binom{m}{m, \lambda} (-\lambda)^{m-1} \frac{1}{m!} (1 - e_{\lambda} (-t))^m
\]

\[
= \sum_{m=1}^{\infty} \binom{m}{m, \lambda} (-\lambda)^{m-1} \sum_{n=m}^{\infty} S_{2, \lambda} (n, m) (-1)^{n-m} \frac{t^n}{n!}
\]

\[
= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \binom{n}{m, \lambda} (-\lambda)^{m-1} \lambda^{n-m} S_{2, \lambda} (n, m)\right) \frac{t^n}{n!}.
\]

Therefore, by (29) and (30), we obtain the following theorem.

Theorem 7. For \( k \in \mathbb{Z} \), we have

\[
\beta_{n, \lambda}^{(k)} (1 - \beta_{n, \lambda}^{(k)}) = (-1)^n \sum_{m=1}^{\infty} \binom{m}{m, \lambda} (-\lambda)^{m-1} \lambda^{m-1} S_{2, \lambda} (n, m), \quad (n \geq 1).
\]

From (15), we note that

\[
t = l_{1, \lambda} (1 - e_{\lambda} (-t)) = \sum_{m=1}^{\infty} \binom{m}{m, \lambda} (-\lambda)^{m-1} \frac{1}{m!} (1 - e_{\lambda} (-t))^m
\]

\[
= \sum_{m=1}^{\infty} \binom{m}{m, \lambda} (-\lambda)^{m-1} \sum_{n=m}^{\infty} S_{2, \lambda} (n, m) (-1)^{n-m} \frac{t^n}{n!}
\]

\[
= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \binom{n}{m, \lambda} \lambda^{m-1} (-1)^{n-m} S_{2, \lambda} (n, m)\right) \frac{t^n}{n!}
\]

By comparing the coefficients on both sides of (31), we obtain the following theorem.

Theorem 8. For \( n \in \mathbb{N} \), we have

\[
(-1)^{n-1} \sum_{m=1}^{n} \binom{n}{m, 1} \lambda^{m-1} S_{2, \lambda} (n, m) = \delta_{n, 1},
\]

where \( \delta_{n, \lambda} \) is Kronecker’s symbol.
Remark. Note that
\[ \lim_{\lambda \to 0} \beta^{(1)}_{n,\lambda} = (-1)^n B_n, \quad \lim_{\lambda \to 0} \beta^{(1)}_{n,\lambda}(x) = (-1)^n B_n(x). \]

From Theorem 8 and Theorem 9, we note that
\[ B_0 = 1, \quad B_n(1) - B_n = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise}. \end{cases} \]

**Corollary 9.** For \( n \in \mathbb{N} \), we have
\[ \sum_{m=1}^{n} (-1)^{n-m} (m-1)! S_2(n,m) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise}. \end{cases} \]

It is well known that the Stirling numbers of the first kind are given by

\[ \frac{1}{k!} \left( \log(1+t) \right)^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}, \quad (k \geq 0). \]

From (33), we note that
\[ \sum_{k=0}^{\infty} \frac{(x)_{k,\lambda}}{k!} \left( \log_{\lambda}(1+t) \right)^k = \sum_{k=0}^{\infty} (x)_{k,\lambda} \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!} \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} S_1(n,k) (x)_{k,\lambda} \right) \frac{t^n}{n!}. \]

On the other hand,
\[ \sum_{k=0}^{\infty} \frac{(x)_{k,\lambda}}{k!} \left( \log_{\lambda}(1+t) \right)^k = e_{\lambda}^x \left( \log_{\lambda}(1+t) \right) = (1+t)^x \]
\[ = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!}. \]

By (34) and (35), we see that the degenerate Stirling numbers of the first kind are also given by

\[ (x)_n = \sum_{k=0}^{n} S_1(n,k)(x)_{k,\lambda}, \quad (n \geq 0). \]

Note that
\[ \lim_{\lambda \to 0} S_1(n,k) = S_1(n,k), \quad (n,k \geq 0). \]

Now, we observe that
\[ (x)_{n+1} = (x)_n (x-n) = \sum_{k=0}^{n} S_1(n,k)(x)_{k,\lambda} (x-k\lambda + k\lambda) - n \sum_{k=0}^{n} S_1(n,k)(x)_{k,\lambda} \]
\[ = \sum_{k=0}^{n} S_1(n,k)(x)_{k+1,\lambda} + \lambda \sum_{k=0}^{n} k S_1(n,k)(x)_{k,\lambda} - n \sum_{k=0}^{n} S_1(n,k)(x)_{k,\lambda} \]
\[ = \sum_{k=1}^{n+1} S_1(n,k-1)(x)_{k,\lambda} + \sum_{k=0}^{n} (\lambda k - n) S_1(n,k)(x)_{k,\lambda}. \]
On the other hand,

\[(x)_{n+1} = \sum_{k=0}^{n+1} S_{1,\lambda} (n+1, k)(x)_{k, \lambda}.\]

Therefore, by (37) and (38) and with the usual convention that \(S_{1,\lambda} (n, k) = 0\), for \(k > n\) or \(k < 0\), we obtain the following theorem.

**Theorem 10.** For \(\lambda \in \mathbb{N}\) and \(0 \leq k \leq n+1\), we have

\[S_{1,\lambda} (n+1, k) = S_{1,\lambda} (n, k-1) + (\lambda k - n)S_{1,\lambda} (n, k).\]

From (17), we note that

\[\sum_{n=0}^{\infty} B_{n,\lambda}^{(k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n (1)_{n+1,1/\lambda}}{n!(n+1)^k} (1 - e_{\lambda}(-t))^n.\]

Thus, by replacing \(t\) by \(-\log_{\lambda} (1 - t)\), we get

\[\sum_{m=0}^{\infty} \frac{B_{m,\lambda}^{(k)} (-1)^m}{m!} (\log_{\lambda} (1 - t))^m = \sum_{n=0}^{\infty} \frac{(-\lambda)^n (1)_{n+1,1/\lambda}}{n!(n+1)^k} t^n.\]

On the other hand,

\[\sum_{m=0}^{\infty} \frac{B_{m,\lambda}^{(k)} (-1)^m}{m!} (\log_{\lambda} (1 - t))^m = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{B_{m,\lambda}^{(k)} (-1)^m S_{1,\lambda} (n, m)}{(n-m)!} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} (-1)^{n-m} \frac{B_{m,\lambda}^{(k)}}{m!} S_{1,\lambda} (n, m) \right) \frac{t^n}{n!}.\]

Therefore, by (40) and (41), we obtain the following theorem.

**Theorem 11.** For \(k \in \mathbb{Z}\) and \(n \geq 0\), we have

\[\frac{1}{(n+1)^k} = \frac{1}{\lambda^n (1)_{n+1,1/\lambda}} \sum_{m=0}^{n} (-1)^m \frac{B_{m,\lambda}^{(k)}}{m!} S_{1,\lambda} (n, m).\]

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