THE NUMBER OF ZEROS OF $\zeta'(s)$

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ABSTRACT. Assuming the Riemann Hypothesis, we prove that

$$N_1(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O\left(\frac{\log T}{\log \log T}\right),$$

where $N_1(T)$ is the number of zeros of $\zeta'(s)$ in the region $0 < \Im s \leq T$.

1. INTRODUCTION

The distribution of zeros of the first derivative of the Riemann zeta-function is interesting and important both in its own right and because of its connection with the distribution of zeros of zeta. As just one illustration of the latter we cite A. Speiser’s [5] theorem that the Riemann Hypothesis is equivalent to the nonexistence of non-real zeros of $\zeta'(s)$ in the half-plane $\Re s < 1/2$.

Let $\rho' = \beta' + i\gamma'$ denote a generic nonreal zero of $\zeta'(s)$, where $s = \sigma + it$ is a complex variable, and for $T \geq 2$ consider the zero counting function

$$N_1(T) := \sum_{\substack{0 < \gamma' \leq T \\ \beta' > 0}} 1.$$ 

In [2] B. C. Berndt proved that

$$N_1(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O(\log T) \quad (1)$$

as $T \to \infty$. This should be compared with the well-known formula (see [6])

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T), \quad (2)$$

which counts the zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the same region. That is,

$$N(T) = \sum_{\substack{0 < \gamma \leq T \\ \beta > 0}} 1.$$ 

By an old result of Littlewood [4], if the Riemann Hypothesis (RH) is true, the error term in (2) may be replaced by

$$O\left(\frac{\log T}{\log \log T}\right). \quad (3)$$

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Recently H. Akatsuka showed that if RH is true, the error term in (1) may also be reduced, namely to
\[ O \left( \frac{\log T}{\sqrt{\log \log T}} \right) \]  
(4)

Our purpose here is to show that if RH is true, the error term (3) holds in both (1) and (2).

**Theorem 1.** Assume RH. Then we have
\[ N_1(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O \left( \frac{\log T}{\log \log T} \right) \]
as \( T \to \infty \).

D. W. Farmer, S. M. Gonek and C. P. Hughes have conjectured that the error term in (2) is
\[ O \left( \sqrt{\log T \log \log T} \right) \]
This raises the question of what error term one should expect in (1) for a given error term in (2). By slightly modifying our proof of Theorem 1 we can show

**Theorem 2.** Assume RH and suppose that the error term in (2) is \( O(\Phi(T)) \) for some increasing function \( \log \log T \ll \Phi(T) \ll \log T \). Then we have
\[ N_1(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O \left( \max \left\{ \Phi(2T), \sqrt{\log T \log \log T} \right\} \right) \].  
(5)

Using this with the conjecture of Farmer et al., we obtain
\[ N_1(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O \left( \sqrt{\log T \log \log T} \right) \].

Of course, we really expect the error term in (5) to be \( O(\Phi(T)) \) in general.

2. **LEMMAS**

We first state three results of Akatsuka.

**Lemma 3.** Assume RH. For \( T \geq 2 \) satisfying \( \zeta(\sigma + iT) \neq 0 \) and \( G(\sigma + iT) \neq 0 \) for any \( \sigma \in \mathbb{R} \), we have
\[ N_1(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + \frac{1}{2\pi} \arg G(1/2 + iT) + \frac{1}{2\pi} \arg \zeta(1/2 + iT) + O(1), \]
where
\[ G(s) = -\frac{2^s}{\log 2} \zeta'(s), \]
and the argument is defined by continuous variation from \(+\infty\), with the argument at \(+\infty\) being 0.

This is Proposition 3.1 in [1].

**Lemma 4.** Assume RH. For \( 1/2 + (\log \log T)^2 / \log T \leq \sigma \leq 3/4 \), we have
\[ \arg G(\sigma + iT) \ll \frac{(\log T)^{2(1-\sigma)}}{\log \log T}. \]
See Remark 2.5 in [1].

**Lemma 5.** Assume RH. Then for $1/2 < \sigma < 20$, we have
\[
\arg \left( -\frac{2^{\sigma+iT}}{\log 2} \zeta'(\sigma + iT) \right) = O \left( \frac{\log \log T}{\sigma - \frac{1}{2}} \right).
\]

See Lemma 2.3 in [1].

We also require the following lemma.

**Lemma 6.** For all $t$ sufficiently large we have
\[
\frac{\zeta''}{\zeta'}(s) = \sum_{|s'-s|<5} \frac{1}{s - s'} + O(\log t),
\]
uniformly for $-1 \leq \sigma \leq 2$.

This can be proved in a standard way. See Theorem 9.6 (A) in [6] for example.

Let us define
\[
F_1(t) = \sum_{\beta' > 1/2} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2}.
\]

We require the following result of Y. Zhang [7].

**Lemma 7.** Assume RH and order the ordinates of the zeros of $\zeta(s)$ as $0 < \gamma_1 \leq \gamma_2 \leq \cdots$. Then
\[
\int_{\gamma_n}^{\gamma_{n+1}} F_1(t) \ll 1,
\]
where the implied constant is absolute.

This is a combination of Lemma 4 in [7] and equation (4.1) in [7].

To state our final lemma we write
\[
H = \frac{(\log \log T)^3}{\log T}
\]
and let $N_d$ denote the number of distinct zeros of $\zeta(s)$ on the vertical segment
\[
[1/2 + i(T - H), 1/2 + i(T + H)].
\]
We also let $N_1(\mathcal{R})$ be the number of zeros of $\zeta'(s)$ in the rectangular region $\mathcal{R}$ given by
\[
T - H \leq t \leq T + H, \quad 1/2 < \sigma \leq 1/2 + H.
\]

**Lemma 8.** Assume RH. We have
\[
N_1(\mathcal{R}) \ll N_d + 1.
\]
Proof. Consider the integral

\[ I := \int_{T-H}^{T+H} F_1(t) dt. \]

From Lemma 2 we see that

\[ I \ll N_d + 1. \]  \hspace{1cm} (6)

Thus, to prove the lemma, it suffices to show that

\[ N_1(\mathcal{R}) \ll I. \]

First observe that

\[ F_1(t) \geq \sum_{\rho' \in \mathcal{R}} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2}, \]

since each summand is positive. Hence,

\[ I \geq \int_{T-H}^{T+H} \sum_{\rho' \in \mathcal{R}} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2} dt \]

\[ = \sum_{\rho' \in \mathcal{R}} \int_{T-H}^{T+H} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2} dt. \]

Let \( \theta(\rho') \in (0, \pi) \) be the argument of the angle at \( \rho' \) with two rays through \( 1/2 + iT - H \) and \( 1/2 + iT + H \) respectively. It is easy to see that

\[ \theta(\rho') = \int_{T-H}^{T+H} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2} dt. \]

This gives us

\[ I \geq \sum_{\rho' \in \mathcal{R}} \theta(\rho'). \]

Now for \( \rho' \in \mathcal{R} \), we clearly have \( \theta(\rho') \geq c \) for some absolute positive constants \( c \). Therefore, we have

\[ I \geq cN_1(\mathcal{R}) \gg N_1(\mathcal{R}). \]

The result follows on combining this and (6). \( \square \)

3. PROOF OF THEOREM [1]

Since \( N_1(T) \) is right continuous with respect to \( T \), it suffices to consider \( T \)'s such that \( \zeta(\sigma + iT) \neq 0 \) and \( G(\sigma + iT) \neq 0 \) for any \( \sigma \in \mathbb{R} \). It is well known (see [6]) that RH implies

\[ \arg \zeta(1/2 + iT) \ll \frac{\log T}{\log \log T}, \]

(in fact, this is equivalent to (3)). Thus, in view of Lemma 3 to prove the theorem it suffices to prove that

\[ \arg G(1/2 + iT) \ll \frac{\log T}{\log \log T}. \]  \hspace{1cm} (7)

Recall that \( \arg G(1/2 + iT) \) is defined by continuous variation along the horizontal line from \( \infty + iT \) to \( 1/2 + iT \) starting with the value 0. Let \( \Delta_1 \) be the change in argument of \( G \) along the horizontal line from \( \infty + iT \) to \( 1/2 + (\log \log T)^2/ \log T + iT \), and let \( \Delta_2 \) be the change along the horizontal segment
from $1/2 + (\log \log T)^2/\log T + iT$ to $1/2 + iT$. We shall show that $\Delta_1$ and $\Delta_2$ are both bounded by $O(\log T/\log \log T)$.

The estimation of $\Delta_1$ is immediate, for by Lemma 4 with $\sigma = 1/2 + (\log \log T)^2/\log T$ we see that

$$\Delta_1 = \arg G(\sigma + iT) \ll \frac{(\log T)^{2(1-\sigma)}}{\log \log T} \ll \frac{\log T}{\log \log T}.$$ 

Next we bound $\Delta_2$. Using the notation

$$H = \frac{\log \log^3 T}{\log T},$$

we see that

$$\Delta_2 = \Im \int_{1/2}^{1/2 + H/\log \log T} \frac{G'(\sigma + iT)}{G}(\sigma + iT)d\sigma.$$ 

By the definition of $G(s)$ we have $(G'/G)(s) = (\zeta''/\zeta')(s) + O(1)$. Thus, it follows from Lemma 6 that

$$\frac{G'(s)}{G}(s) = \sum_{|\rho' - s| < 5} \frac{1}{s - \rho'} + O(\log T)$$

for $s$ on the segment $[1/2 + iT, 1/2 + H/\log \log T + iT]$. It is convenient to modify this formula slightly. Let $\mathcal{D}$ be the disk centered at $1/2 + H/2 \log \log T + iT$ with radius 5. We claim that

$$\frac{G'(s)}{G}(s) = \sum_{\rho' \in \mathcal{D}} \frac{1}{s - \rho'} + O(\log T)$$

for $s$ on the horizontal segment $[1/2 + iT, 1/2 + H/\log \log T + iT]$. Indeed, for such $s$ and for $\rho'$ not belonging to the intersection of $\mathcal{D}$ and the disk $|\rho' - s| < 5$, we see that $(s - \rho')^{-1} \ll 1$. Since the number of such $\rho'$ is $O(\log T)$, their contribution is also $O(\log T)$.

It now follows that

$$\Delta_2 = \Im \int_{1/2}^{1/2 + H/\log \log T + iT} \left( \sum_{\rho' \in \mathcal{D}} \frac{1}{s - \rho'} + O(\log T) \right) ds$$

$$= \sum_{\rho' \in \mathcal{D}} \left( \Im \int_{1/2}^{1/2 + H/\log \log T + iT} \frac{1}{s - \rho'} ds \right) + O((\log \log T)^2) \tag{8}$$

$$= \sum_{\rho' \in \mathcal{D}} \left[ \arg(1/2 + H/\log \log T + iT - \rho') - \arg(1/2 + iT - \rho') \right] + O((\log \log T)^2)$$

$$= \sum_{\rho' \in \mathcal{D}} f(\rho') + O((\log \log T)^2),$$

say. Notice that $f(\rho')$ is plus or minus the argument of the angle subtended by the segments from $1/2 + iT$ to $\rho'$ and from $1/2 + H/\log \log T + iT$ to $\rho'$. Thus, in particular, $f(\rho') \ll 1$.

It remains to prove that

$$\sum_{\rho' \in \mathcal{D}} f(\rho') \ll \frac{\log T}{\log \log T}.$$
We split the sum into three parts. We let $\sum_1$ denote the sum over the $\rho' \in \mathcal{R}$, that is, the $\rho'$ satisfying

$$T - H \leq \gamma' \leq T + H, \quad 1/2 < \beta' \leq 1/2 + H.$$  

We let $\sum_2$ be the sum over the zeros $\rho' = 1/2 + i\gamma'$, if any, with

$$T - H \leq \gamma' \leq T + H.$$  

Finally, we let $\sum_3$ denote the sum over the remaining $\rho'$ in $\mathcal{R}$.

By Lemma 8 and our observation above that $f(\rho') \ll 1$, we see that

$$\sum_1 \ll \max |f(\rho')| : N_1(\mathcal{R}) \ll N_1(\mathcal{R}) \ll N_d + 1.$$  

Recall that RH implies that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O\left(\frac{\log T}{\log \log T}\right).$$  

Thus $N_d \ll N(T + H) - N(T - H) \ll \log T / \log \log T$. We therefore obtain

$$\sum_1 \ll \frac{\log T}{\log \log T}. \quad (9)$$  

A similar argument (together with the well-known fact that $\zeta'(1/2 + it) = 0 \Rightarrow \zeta(1/2 + it) = 0$) shows that

$$\sum_2 \ll \frac{\log T}{\log \log T}. \quad (10)$$  

Now we consider $\sum_3$. Recall that RH implies that $\zeta'(s)$ has no non-real zero to the left of 1/2. Thus, it is easy to see that

$$f(\rho') = \arg(1/2 + H/\log \log T + iT - \rho') - \arg(1/2 + iT - \rho') \ll \frac{1}{\log \log T}$$  

for $\rho'$ in $\sum_3$. Since the number of $\rho'$ in $\mathcal{R}$ is at most $O(\log T)$, we see that

$$\sum_3 \ll \frac{\log T}{\log \log T}.$$  

Combining this with (9) and (10), we see that

$$\sum_{\rho' \in \mathcal{R}} f(\rho') \ll \frac{\log T}{\log \log T}.$$  

Thus, by (8) we obtain

$$\Delta_2 \ll \frac{\log T}{\log \log T}.$$  

This completes our proof.
4. Proof of Theorem 2

Let $X$, $g$ and $k$ be positive parameters with $X = o(1)$, $g \geq 2$ and $k \in \mathbb{Z}$, to be determined later. As before, we let $\Delta_1(X)$ be the change in argument of $G$ along the horizontal line from $\infty + iT$ to $1/2 + X + iT$, and let $\Delta_2(X)$ be the change along the horizontal segment from $1/2 + X + iT$ to $1/2 + iT$.

It is standard (see Theorem 14.14 (B) in [6]) to show that

$$\arg \zeta(\sigma + iT) \ll \Phi(2T)$$

for $\sigma \geq 1/2$. Using this with Lemma [5] we obtain

$$\Delta_1(X) \ll \frac{\log \log T}{X} + \Phi(2T). \quad (11)$$

Next we bound $\Delta_2$. For $0 \leq j \leq k$ let

$$Y_j = Xg^j,$$

and define $\mathcal{R}_j$ to be the rectangular region

$$\frac{1}{2} \leq \sigma \leq \frac{1}{2} + Y_j, \quad T - Y_j \leq t \leq T + Y_j.$$

Let $\mathcal{U}_1 = \mathcal{R}_1$, and for $2 \leq j \leq k$ let $\mathcal{U}_j = \mathcal{R}_j - \mathcal{R}_{j-1}$. We denote by $N(\mathcal{U}_j)$ the number of zeros of $\zeta'(s)$ in $\mathcal{U}_j$.

Let $\mathcal{D} = \mathcal{D}(X)$ be the disk centered at $1/2 + X/2 + iT$ with radius 5. As before we have

$$\frac{G'}{G}(s) = \sum_{\rho' \in \mathcal{D}} \frac{1}{s - \rho'} + O(\log T)$$

for $s$ on the horizontal segment $[1/2 + iT, 1/2 + X + iT]$. It follows that

$$\Delta_2(X) = \sum_{\rho' \in \mathcal{D}} f_X(\rho') + O(X \log T) \quad (12)$$

where $f_X(\rho')$ is plus or minus the argument of the angle subtended by the segments from $1/2 + iT$ to $\rho'$ and from $1/2 + X + iT$ to $\rho'$.

We split the sum into $k + 2$ parts. For $1 \leq j \leq k$ we let $\sum_j$ denote the sum over the $\rho' \in \mathcal{U}_j$. We let $\sum_{k+1}$ be the sum over the zeros $\rho' = 1/2 + i\gamma'$, if any, with

$$T - Y_k \leq \gamma' \leq T + Y_k.$$

Finally, we let $\sum_{k+2}$ denote the sum over the remaining $\rho'$ in $\mathcal{D}$.

Using the same argument as in Theorem 1, we can show that

$$\sum_j \ll g^{1-j} N(\mathcal{U}_j)$$
for $1 \leq j \leq k$. It follows from Lemma 8 that
\[ N(\varphi_j) \ll N(T + Y_j) - N(T - Y_j) + 1 \ll Y_j \log T + \Phi(T + Y_j). \]

This gives us
\[ \sum_j \ll g^{1-j}(Y_j \log T + \Phi(T + Y_j)) \leq Xg \log T + g^{1-j}\Phi(T + Y_k) \]
for $1 \leq j \leq k$. Thus, we have
\[ \sum_1 + \sum_2 + \cdots + \sum_k \ll Xgk \log T + \Phi(T + Y_k). \]

Similarly, we get
\[ \sum_{k+1} \ll Xgk \log T + \Phi(T + Y_k), \]
and
\[ \sum_{k+2} \ll g^{-k} \log T. \]

Combining the above estimates, we see that
\[ \sum_{\rho' \in \mathcal{D}} f_X(\rho') \ll Xgk \log T + \Phi(T + Y_k) + g^{-k} \log T. \]

Thus, by (12) we obtain
\[ \Delta_2 \ll Xgk \log T + \Phi(T + Y_k) + g^{-k} \log T. \]

This together with (11) gives us
\[ \arg G(1/2 + iT) \ll \frac{\log \log T}{X} + \Phi(2T) + Xgk \log T + \Phi(T + Y_k) + g^{-k} \log T. \tag{13} \]

Now we take $X = 1/\sqrt{\log T}$, $g = e$ and $k = \lfloor \log \log T/2 \rfloor + 1$. Note that in this case we have $1 \leq Y_k \leq e$.

From (13) it follows that
\[ \arg G(1/2 + iT) \ll \sqrt{\log T} \log \log T + \Phi(2T), \]
and this gives (5). \[ \Box \]

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