ON BOUNDS FOR RING-BASED CODING THEORY

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ABSTRACT. Coding Theory where the alphabet is identified with the elements of a ring or a module has become an important research topic over the last 30 years. Such codes over rings had important applications and many interesting mathematical problems are related to this line of research.

It has been well established, that with the generalization of the algebraic structure to rings there is a need to also generalize the underlying metric beyond the usual Hamming weight used in traditional coding theory over finite fields.

This paper introduces a new weight, called the overweight, which can be seen as a generalization of the Lee weight on the integers modulo 4. For this new weight we provide a number of well-known bounds, like a Plotkin bound, a sphere-packing bound, and a Gilbert-Varshamov bound. A further highlight is the proof of a Johnson bound for the homogeneous weight on a general finite Frobenius ring.

1. Introduction

Coding theoretic experience has shown that considering linear codes over finite fields often yields significant complexity advantages over the non-linear counterparts particularly, when it comes to complex tasks like encoding and decoding. On the other side, it was recognized early [6, 8] that the class of binary block codes contained excellent code families, which were not linear (Preparata, Kerdock codes, Goethals and Goethals-Delsarte codes). For a long time it could not be explained, why these families exhibit formal duality properties in terms of their distance enumerators that occur only on those among linear codes and their duals.

A true breakthrough in the understanding of this behavior came in the early 1990’s when after preceding work by Nechaev [7] the paper by Hammons et al. [5] discovered that these families allow a representation in terms of $\mathbb{Z}_4$-linear codes.

A crucial condition for this ring-theoretic representation was that $\mathbb{Z}_4$ was equipped with an alternative metric, the Lee weight, rather than with the traditional Hamming weight, which only distinguishes whether an element is zero or non-zero. The Lee weight is finer, assigning 2 a higher weight than the other non-zero elements of this ring.

The fact that the traditional settings of linear coding theory (finite fields with Hamming metric) are actually too narrow, suggests to expand the theory in at least two directions: on the algebraic part, the next more natural algebraic structure serving as alphabet for linear coding is that of finite rings (and modules). On the...
metrical part, the appropriateness of the Lee weight for $\mathbb{Z}_4$-linear coding suggests that the distance function for a generalized coding theory also requires generalization as well.

Since these ground-breaking observations, an entire discipline arose within algebraic coding theory. A considerable community of scholars have been developing results in various directions, among them code duality, weight-enumeration, code equivalence, weight functions, homogeneous weights, existence bounds, code optimality, decoding schemes, to mention only a few.

The paper at hand aims at providing a further contribution to this discipline, by introducing the overweight on a finite ring. To the authors best knowledge this concept appeared for the first time in the Master thesis of the first author [2] and has not been considered before. The overweight on a finite ring is extremal in the sense, that it is a positive definite function that satisfies the triangle inequality. For this overweight, we will develop a number of standard existence bounds, like a sphere-packing bound, a Plotkin bound, and a version of the (assertive) Gilbert-Varshamov bound.

In the final part of this article we derive a general Johnson bound for the homogeneous weight on a finite Frobenius ring. This result is important, as it is closely connected to list decoding capabilities.

2. Preliminaries

Throughout this paper we will consider $R$ to be a finite ring with identity, denoted by 1. If $R$ is a finite ring, we denote by $R^\times$ its group of invertible elements, also known as units.

Let us recall some preliminaries in coding theory, where we focus on ring-linear coding theory.

For $q$ a prime power, let us denote by $F_q$ the finite field with $q$ elements. In traditional coding theory we consider a linear code to be a subspace of a vector space over a finite field.

**Definition 1.** Let $q$ be a prime power, and let $k \leq n$ be non-negative integers. A linear subspace $C$ of $F_q^n$ of dimension $k$ is called an $[n, k]$-linear code.

In the paper at hand, we focus on a more general setting where the ambient space is a module over a finite ring.

**Definition 2.** Let $n \in \mathbb{N}$, and let $R$ be a finite ring. A submodule $C$ of $R^n$ of size $M = |C|$ is called a left $R$-linear $(n, M)$ code.

**Definition 3.** Let $R$ be a finite ring. A real-valued function $w$ on $R$ is called a weight, if it is non-negative, and if $w(0) = 0$. It is natural to identify $w$ with its additive extension to $R^n$, and so, we will always write $w(x) = \sum_{i=1}^n w(x_i)$ for all $x \in R^n$. Every weight $w : R \rightarrow \mathbb{R}$ induces what we define to be a distance $d : R \times R \rightarrow \mathbb{R}$ by $d(x, y) = w(x - y)$. Again, we will identify $d$ with its natural additive extension to $R^n \times R^n$.

The most prominent and best studied weight in traditional coding theory is the Hamming weight.
Definition 4. Let \( n \in \mathbb{N} \). The Hamming weight of a vector \( x \in R^n \) is defined as the size of its support
\[
w_H(x) = |\{i \in \{1, \ldots, n\} \mid x_i \neq 0\}|
\]
and the Hamming distance between \( x \) and \( y \in R^n \) is given by
\[
d_H(x, y) = |\{i \in \{1, \ldots, n\} \mid x_i \neq y_i\}| = w_H(x - y).
\]
The minimal Hamming distance of a linear code is then defined as the minimal distance between two different codewords
\[
d_H(C) = \min \{d_H(x, y) \mid x, y \in C, x \neq y\}.
\]
Note that the concept of minimal distance can be applied for any underlying weight \( w \).

Since we will establish a Plotkin bound, let us recall here the Plotkin bound over finite fields equipped with the Hamming metric.

Theorem 5 (Plotkin bound). Let \( C \) be an \((n, M)\) block code over \( \mathbb{F}_q \) with minimal Hamming distance \( d \). If \( d > \frac{q-1}{q}n \), then
\[
M \leq \frac{d}{d - \frac{q-1}{q}n}.
\]

Definition 6. A weight \( w : R \longrightarrow \mathbb{R} \) is called (left) homogeneous of average value \( \gamma > 0 \), if \( w(0) = 0 \) and the following conditions hold:

(i) For all \( x, y \) with \( Rx = Ry \) we have that \( w(x) = w(y) \).

(ii) For every non-zero ideal \( I \leq R \), it holds that
\[
\frac{1}{|I|} \sum_{x \in I} w(x) = \gamma.
\]
The homogeneous weight was first introduced by Constantinescu and Heise in [1] in the context of coding over integer residue rings. It was later generalised by Greferath and Schmidt [4] to arbitrary finite rings, where the ideal \( I \) in Definition 6 was assumed to be a principal ideal. In its original form, however the homogeneous weight only exists on finite Frobenius rings.

It can be shown that a left homogeneous weight is at the same time right homogeneous, and for this reason, we will omit the reference to any side for the sequel.

Theorem 7 (Plotkin bound for homogeneous weights, [3, Theorem 2.2]). Let \( w \) be a homogeneous weight of average value \( \gamma \) on \( R \), and let \( C \) be an \((n, M)\) block code over \( R \) with minimal homogeneous distance \( d \). If \( \gamma n < d \) then
\[
M \leq \frac{d}{d - \gamma n}.
\]

3. Bounds for the Overweight

In this section we introduce the overweight, a generalization of the Lee weight on \( \mathbb{Z}_4 \) to arbitrary finite rings. We will develop an analogue of the Plotkin bound for the overweight in this case.
Definition 8. Let $R$ be a finite ring. The overweight on $R$ is defined as

$$W : R \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in R^\times, \\ 2 & \text{otherwise.} \end{cases}$$

Clearly, the overweight function is a weight in the sense of our earlier definition. It is extremal in its property to still satisfy the triangle inequality. As agreed earlier, we will denote by $W$ also its additive expansion to $R^n$, given by $W(x) = \sum_{i=1}^n W(x_i)$.

Following from its definition, we get the following properties:

Lemma 9. Let $x, y \in R^n$. Then the overweight function satisfies:

i) $W(x) \geq 0$ and $W(x) = 0$ if and only if $x = 0$.
ii) If $Rx = Ry$ then $W(x) = W(y)$, in particular $W(x) = W(-x)$.
iii) $W(x + y) \leq W(x) + W(y)$.

Let us call the distance which is induced by the overweight the overweight distance, and denote it by $D$, i.e., $D(x, y) = W(x - y)$. We see that $D$ has the following properties:

Lemma 10. Let $R$ be a finite ring and $x, y, z \in R^n$. Then it holds that

i) $D(x, y) = D(x - y, 0)$,
ii) $D(x, y) \geq 0$ and $D(x, y) = 0$ if and only if $x = y$,
iii) $D(x, y) = D(y, x)$,
iv) $D(x, z) \leq D(x, y) + D(y, z)$.

3.1. A Sphere-Packing Bound. In this section we provide the sphere-packing bound and the Gilbert-Varshamov bound in the overweight distance. These are generic bounds and we are able to provide them in a simple form involving the volume of the balls in the underlying metric space.

We begin by defining balls with respect to the overweight distance.

Definition 11. For a given radius $r \geq 0$, the overweight ball $B_{r,D}(x)$ of radius $r$ centered in $x$ is defined as

$$B_{r,D}(x) := \{y \in R^n \mid D(x, y) \leq r\}.$$ 

Clearly, the volume of such a ball is invariant under translations, i.e.,

$$|B_{r,D}(x)| = |B_{r,D}(y)|,$$

for all $x, y \in R^n$.

Moreover, setting $u := |R^\times|$ and $v := |R| - 1 - u$, we have the generating function $f_W(z) = 1 + u z + v z^2$ for this weight function, so that the generating function for $W$ on $R^n$ takes the form

$$f_W^n(z) = (1 + uz + vz^2)^n = \sum_{k_0+k_u+k_v=n} \binom{n}{k_0, k_u, k_v} 1^{k_0} (uz)^{k_u} (vz^2)^{k_v} = \sum_{k=0}^n \sum_{\ell=0}^{n-k} \binom{n}{k} \binom{n-k}{\ell} u^k v^{\ell} z^{k+2\ell},$$
where we have set \( k = k_u \) and \( \ell = k_v \), and where the condition \( k_0 + k_u + k_v = n \) is transformed in \( 0 \leq k \leq n, 0 \leq \ell \leq n - k \). Now setting \( t = k + 2\ell \), we obtain the simplified expression for the generating function

\[
J_W^n(z) = \sum_{t=0}^{2n} \sum_{\ell=0}^{\lfloor t/2 \rfloor} \binom{n}{t - 2\ell} (n - t + 2\ell) u^{t-2\ell} v^\ell z^\ell.
\]

**Lemma 12.** The foregoing implies that the ball of radius \( e \) (centered in \( 0 \)) has volume exactly

\[
|B_{e,D}(0)| = \sum_{t=0}^{e} \sum_{\ell=0}^{\lfloor t/2 \rfloor} \binom{n}{t - 2\ell} (n - t + 2\ell) u^{t-2\ell} v^\ell.
\]

We thus provided an explicit formula for the cardinality of balls in \( \mathbb{R}^n \) with respect to the overweight distance.

We now obtain the sphere-packing bound for the overweight distance by combining the previous results. As before, \( R \) is a finite ring and \( u = |R^\times| \), whereas \( v = |R| - 1 - u \) represents the number of non-zero non-units.

**Corollary 13** (Sphere-Packing Bound). Let \( C \subseteq \mathbb{R}^n \) be a (not necessarily linear) non-zero code of length \( n \), and minimum distance \( d = 2e + 1 \). Then we have

\[
|C| \leq \frac{|R|^n}{|B_{e,D}(0)|},
\]

where the cardinality of \( |B_{e,D}(0)| \) is given in Equation (3.1).

### 3.2. A Gilbert-Varshamov Bound

With arguments similar to those for the sphere-packing bound, we can also get a lower bound to the maximal size of a code with fixed minimum distance.

**Proposition 14** (Gilbert-Varshamov bound). Let \( R \) be a finite ring, \( n \) a positive integer and \( d \in \{0, \ldots, 2n\} \). Then there exists a code \( C \subseteq \mathbb{R}^n \) of minimum overweight distance at least \( d \) satisfying

\[
|C| \geq \frac{|R|^n}{|B_{d-1,D}(0)|},
\]

where the volume is given in (3.1) for \( e = d - 1 \), i.e.,

\[
|B_{d-1,D}(0)| = \sum_{t=0}^{d-1} \sum_{\ell=0}^{\lfloor t/2 \rfloor} \binom{n}{t - 2\ell} (n - t + 2\ell) u^{t-2\ell} v^\ell.
\]

**Proof.** Assume \( C \subseteq \mathbb{R}^n \) of minimum distance at least \( d \) is the largest code of length \( n \) and minimum distance \( d \). Then the set of balls \( B_{d-1,D}(x) \) centered in the codewords \( x \in C \) must already cover the space \( \mathbb{R}^n \), because if they did not, one would find an element \( y \in \mathbb{R}^n \) that is not contained in the ball of radius \( d - 1 \) around any element of \( C \). This word \( y \) would have distance at least \( d \) to each of the words of \( C \), and thus \( C \cup \{y\} \) would be a code of properly larger size with distance at least \( d \), a contradiction to the choice of \( C \).

From the covering argument, we then see that

\[
|C| \geq \frac{|R|^n}{|B_{d-1,D}(0)|}.
\]
3.3. A Plotkin Bound. Over a local ring, we can use methods similar to the ones used for the classical Plotkin bound, to get an analogue of the Plotkin bound for (not necessarily linear) codes equipped with the overweight.

For the rest of this section, \( R \) is a finite local ring with maximal ideal \( J \). The notation stems from the Jacobson radical of the ring \( R \). Note that the factor ring \( R/J \) is a finite field, whose cardinality will be denoted by \( q \).

Similarly to the Hamming case, for a subset \( A \subseteq R \) we will denote by 
\[
W(A) = \sum_{a \in A} W(a)
\]
the average weight of the subset \( A \).

**Lemma 15.** Let \( I \subseteq R \) be a left or right ideal. Then
\[
\overline{W}(I) = \begin{cases} 
\frac{|R| + |J| - 2}{|R|} & \text{if } I = R, \\
2 \left( 1 - \frac{1}{|J|} \right) & \text{if } \{0\} \subsetneq I \subsetneq R, \\
0 & \text{else.}
\end{cases}
\]

**Proof.** Note that the last case is trivial as \( I = \{0\} \). If \( \{0\} \subsetneq I \subsetneq R \), then all non-zero elements of \( I \) have weight 2, so this case follows as well.

Finally, if \( I = R \), then there are \( |R \setminus J| = |R| - |J| \) elements of weight 1 and \( |J| - 1 \) elements of weight 2. Hence the total weight is \( |R| - |J| + 2(|J| - 1) \) and dividing by \( |R| \) yields the claim. \( \square \)

**Corollary 16.** Let \( R \) be a local ring with maximal ideal \( J \) and assume that \( |J| \geq 2 \). Then we have that \( \overline{W}(J) \geq \overline{W}(I) \) for all left or right ideals \( I \subseteq R \).

**Proof.** We immediately see that \( \overline{W}(J) \geq \overline{W}(I) \) for all \( I \subseteq J \). Now consider the case \( I = R \). We have that
\[
\overline{W}(R) = \frac{|R| - |J| - 2}{|R|} = \frac{|R \setminus J|}{|R|} + 2 \frac{|J| - 1}{|R|}
\]
\[
= \frac{|R \setminus J|}{|R|} + 2 \frac{|J| - 1}{|J|} \cdot \frac{|J|}{|R|}
\]
\[
\leq 2 \frac{|J| - 1}{|J|} \cdot \frac{|R \setminus J|}{|R|} + 2 \frac{|J| - 1}{|J|} \cdot \frac{|J|}{|R|}
\]
\[
= 2 \left( 1 - \frac{1}{|J|} \right) = \overline{W}(J),
\]
where we used that \( 2 \left( 1 - \frac{1}{|J|} \right) \geq 1 \). \( \square \)

To ease the notation, let us denote by \( \eta \) the following
\[
\eta = \overline{W}(J) = 2 \left( 1 - \frac{1}{|J|} \right).
\]

In what follows, we provide a Plotkin bound for the overweight over a local ring \( R \) with maximal ideal \( J \). The case \( |J| = 1 \) is already well studied, since in this case where \( R \) is a field and \( D \) is simply the Hamming distance. Hence, we will assume that \( |J| \geq 2 \).
We start with a lemma for the Hamming weight. The proof of it follows the idea of the classical Plotkin bound, which can be found in [9], and for the homogeneous weight in [3].

**Lemma 17.** Let $I \subseteq R$ be a subset and $P$ be a probability distribution on $I$. Then we have that

$$\sum_{x \in I} \sum_{y \in I} w_H(x - y)P(x)P(y) \leq 1 - \frac{1}{|I|}.$$  

**Proof.** We have that

$$\sum_{x \in I} \sum_{y \in I} w_H(x - y)P(x)P(y) = \sum_{x \in I} P(x)(1 - P(x)) = \sum_{x \in I} P(x) - \sum_{x \in I} P(x)^2.$$  

If we apply the Cauchy-Schwarz inequality to the latter sum, we obtain that

$$\sum_{x \in I} P(x) - \sum_{x \in I} P(x)^2 \leq 1 - \frac{1}{|I|} \left| \sum_{x \in I} P(x) \right|^2 = 1 - \frac{1}{|I|}.$$  

We are now ready for the most important step of the Plotkin bound. As before, $R$ is a local ring with non-zero maximal ideal $J$ and $\eta = W(J)$.

**Proposition 18.** Let $P$ be a probability distribution on $R$. Then it holds that

$$\sum_{x \in R} \sum_{y \in R} W(x - y)P(x)P(y) \leq \eta.$$  

**Proof.** Let $q = |R/J|$ and pick $x_1, \ldots, x_q$ such that $x_i + J \neq x_j + J$ if $i \neq j$. Then it follows that the cosets $\overline{x_i} := x_i + J$ form a partition of $R$. For all $k \in \{1, \ldots, q\}$, we denote by

$$P_k = \sum_{x \in \overline{x_k}} P(x).$$

It follows that $\sum_{k=1}^q P_k = 1$. By rewriting the initial sum as sum over all cosets we obtain that

$$\sum_{x \in R} \sum_{y \in R} W(x - y)P(x)P(y)$$

$$= \sum_{k=1}^q \sum_{x \in \overline{x_k}} \sum_{y \in \overline{x_k}} W(x - y)P(x)P(y)$$

$$= \sum_{k=1}^q \sum_{x \in \overline{x_k}} \left( \sum_{y \in \overline{x_k}} 2w_H(x - y)P(x)P(y) + \sum_{z \in R \setminus \overline{x_k}} w_H(x - z)P(x)P(z) \right)$$

$$= \sum_{k=1}^q \left( 2 \sum_{x \in \overline{x_k}} \sum_{y \in \overline{x_k}} w_H(x - y)P(x)P(y) + \sum_{x \in \overline{x_k}} \sum_{z \in R \setminus \overline{x_k}} P(x)P(z) \right)$$

$$= \sum_{k=1}^q \left( 2 \sum_{x \in \overline{x_k}} \sum_{y \in \overline{x_k}} w_H(x - y)P(x)P(y) + \sum_{x \in \overline{x_k}} P(x)(1 - P_k) \right).$$
If \( P_k \neq 0 \), then \( \tilde{P}(x) := P(x)/P_k \) defines a probability distribution on \( \mathbb{F}_q \). In this case we apply Lemma 17 to get that
\[
\sum_{x \in \mathbb{F}_q} \sum_{y \in \mathbb{F}_q} w_H(x - y) P(x) P(y) = P_k^2 \left( \sum_{x \in \mathbb{F}_q} \sum_{y \in \mathbb{F}_q} w_H(x - y) \frac{P(x) P(y)}{P_k^2} \right) \leq P_k^2 \left( 1 - \frac{1}{|J|} \right). \]

Note that the same inequality also trivially holds if \( P_k = 0 \). Applying this and using that \( \sum_{x \in \mathbb{F}_q} P(x) = P_k \), we obtain that
\[
\sum_{k=1}^{q} \left( 2 \sum_{x \in \mathbb{F}_q} \sum_{y \in \mathbb{F}_q} w_H(x - y) P(x) P(y) + \sum_{x \in \mathbb{F}_q} P(x)(1 - P_k) \right) \leq \sum_{k=1}^{q} \left( P_k^2 \cdot 2 \left( 1 - \frac{1}{|J|} \right) + P_k(1 - P_k) \right) \leq \sum_{k=1}^{q} P_k \cdot 2 \left( 1 - \frac{1}{|J|} \right) = 2 \left( 1 - \frac{1}{|J|} \right) = \eta,
\]
where we used that \( 2 \left( 1 - \frac{1}{|J|} \right) \geq 1 \) since \( |J| \geq 2 \) in the last inequality. \( \square \)

To complete the Plotkin bound for the overweight, we now follow the steps in [3]. Using Proposition 18 we get the following result:

**Proposition 19.** Let \( C \subseteq \mathbb{R}^n \) be a (not necessarily linear) code of minimum over-weight distance \( d \). Then
\[
|C|(|C| - 1)d \leq \sum_{x \in C} \sum_{y \in C} D(x, y) \leq |C|^2 n \eta.
\]

**Proof.** The first inequality follows since the distance between all distinct pairs of \( C \) is at least \( d \).

For the second inequality, let \( p_i : \mathbb{R}^n \to \mathbb{R} \) be the projection onto the \( i \)-th coordinate. Note that
\[
P_i(z) := \frac{|p_i^{-1}(z) \cap C|}{|C|}
\]
defines a probability distribution on \( R \) for all \( i \in \{1, \ldots, n\} \). Using Proposition 18, we get that
\[
\sum_{x \in C} \sum_{y \in C} D(x, y) = \sum_{i=1}^{n} \sum_{x \in C, y \in C} W(x_i - y_i)
\]
\[
= \sum_{i=1}^{n} \sum_{r \in R} \sum_{s \in R} W(r - s) P_i(r) P_i(s) |C|^2
\]
\[
\leq |C|^2 \sum_{i=1}^{n} \eta = |C|^2 n \eta.
\]
□

From this inequality, we obtain a Plotkin bound for the overweight distance. As before, \( R \) is a local ring with non-zero maximal ideal \( J \) and \( \eta = 2 \left( 1 - \frac{1}{|J|} \right) \).

**Theorem 20** (Plotkin bound for the overweight distance). Let \( C \subseteq R^n \) be a (not necessarily linear) code of minimum overweight distance \( d = D(C) \) and assume that \( d > n \eta \). Then
\[
|C| \leq \frac{d}{d - n \eta}.
\]

**Proof.** We divide both sides of the inequality in Proposition 19 by \( |C| \) to get that
\[
|C|(d - n \eta) \leq d.
\]
The result then follows from the assumption that \( d - n \eta > 0 \). □

By rearranging the same inequality, we also get the following version of the Plotkin bound, which does not require the assumption that \( d > n \eta \).

**Corollary 21.** Let \( C \subseteq R^n \) be a (not necessarily linear) code with \( |C| \geq 2 \) and let \( d = D(C) \). Then
\[
d \leq \frac{|C| n \eta}{|C| - 1}.
\]

**Proof.** We obtain this by dividing both sides of the inequality in Proposition 19 with \( |C|(|C| - 1) \), which is non-zero by assumption. □

**Remark 22.** Note that \( W \) is a homogeneous weight on \( \mathbb{Z}_4 \): the average weight on non-zero ideals is constant and if two elements generate the same ideal, they have the same weight. In this case, our bound coincides with the bound from [3] for the homogeneous weight on \( \mathbb{Z}_4 \).

4. **A Johnson Bound for the Homogeneous Weight**

In this section, we prove a Johnson bound for the homogeneous weight from Definition 6, denoted by \( wt \) and let \( \gamma \) be its average weight (on \( R \)). By abuse of notation we denote with \( wt \) also the extension of \( wt \) to \( R^n \), that is
\[
wt(x) = \sum_{i=1}^{n} wt(x_i).
\]
Note that $wt$ does not necessarily satisfy the triangle inequality. In [1, Theorem 2], it is shown that the homogeneous weight on $\mathbb{Z}_m$ satisfies the triangle inequality if and only if $m$ is not divisible by 6.

We define the ball of radius $r$ with respect to a homogeneous weight $wt$ to be the set of all elements having distance less than or equal to $r$.

**Definition 23.** Let $y \in \mathbb{R}^n$ and $r \in \mathbb{R}_{\geq 0}$. The ball $B_{r,wt}(y)$ of radius $r$ centered in $y$ is defined as

$$B_{r,wt}(y) := \{x \in \mathbb{R}^n \mid wt(x - y) \leq r\}.$$ 

Our aim is to provide a Johnson bound for the homogeneous weight over Frobenius rings. Thus, we begin by defining list-decodability.

**Definition 24.** Let $R$ be a finite ring. Given $\rho \in \mathbb{R}_{\geq 0}$, a code $C \subseteq R^n$ is called $(\rho, L)$-list decodable (with respect to $wt$) if for every $y \in \mathbb{R}^n$ it holds that

$$|B_{\rho n,wt}(y) \cap C| \leq L.$$ 

Over Frobenius rings, the following result holds, which will play an important role in the proof of the Johnson bound.

**Proposition 25** ([3, Corollary 3.3]). Let $R$ be a Frobenius ring, $C \subseteq R^n$ a (not necessarily linear) code of minimal distance $d$ and $\omega = \max\{wt(c) \mid c \in C\}$. If $\omega \leq \gamma n$, then

$$|C|(|C| - 1)d \leq \sum_{x,y \in C} wt(x - y) \leq 2|C|^2\omega - \frac{|C|^2\omega^2}{\gamma n}.$$ 

With this, we get an analogue of the Johnson bound for the homogeneous weight.

**Theorem 26.** Let $R$ be a Frobenius ring and $C \subseteq R^n$ be a (not necessarily linear) code of minimum distance $d$. Assume that $\rho \leq \gamma$. Then it holds that $C$ is $(\rho, d\gamma n)$ list-decodable if one of the following conditions is satisfied:

i) We have that $\gamma n(d - \gamma n) \geq 1$.

ii) It holds that $\rho \leq \gamma - \sqrt{(\gamma - \frac{d}{n})\gamma + \frac{1}{n^2}}$.

**Proof.** Assume that $\gamma n \geq \rho n$ and let $y \in \mathbb{R}^n$. We have to show that under the given conditions $|B_{e,wt}(y) \cap C| \leq d\gamma n$.

Note first that we may assume that $y = 0$, otherwise simply consider the translate

$$C' = \{c - y \mid c \in C\}.$$ 

Assume that $x_1, \ldots, x_N$ are in $B_{e,wt}(0) \cap C$. We have that $wt(x_i - x_j) \geq d$ for $i \neq j$, thus using Proposition 25 and $wt(x - y) = wt(y - x)$, we get that

$$N(N - 1)d \leq \sum_{i<j} wt(x_i - x_j) \leq N^2e - \frac{N^2e^2}{2\gamma n}.$$ 

Hence it follows that

$$N(d\gamma n - 2e\gamma n + e^2) \leq d\gamma n.$$ 

It holds that

$$(d\gamma n - 2e\gamma n + e^2) = (n\gamma - e)^2 - n\gamma(n\gamma - d).$$
If we assume that $n\gamma(n\gamma - d) \leq -1$, then we clearly have

$$(n\gamma - e)^2 - n\gamma(n\gamma - d) \geq 1.$$  

If this is not the case, we see that $\sqrt{(\gamma - d)n\gamma + \frac{1}{n^2}}$ is well-defined. So, if

$$\frac{e}{n} \leq \gamma - \sqrt{(\gamma - d)n\gamma + \frac{1}{n^2}},$$

then

$$(n\gamma - e) \geq \sqrt{(n\gamma - d)n\gamma + 1},$$

and hence

$$(n\gamma - e)^2 - n\gamma(n\gamma - d) \geq 1.$$  

It follows that $N \leq d\gamma n$. \hfill \Box 

**Remark 27.** Note that the second condition already forces $\rho \leq \gamma$.

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