Rate Region of the Vector Gaussian CEO Problem with the Trace Distortion Constraint

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Abstract—We establish a new extremal inequality, which is further leveraged to give a complete characterization of the rate region of the vector Gaussian CEO problem with the trace distortion constraint. The proof of this extremal inequality hinges on a careful analysis of the Karush-Kuhn-Tucker necessary conditions for the non-convex optimization problem associated with the Berger-Tung scheme, which enables us to integrate the perturbation argument by Wang and Chen with the distortion projection method by Rahman and Wagner.

Index Terms—CEO problem, distributed source coding, extremal inequality, indirect source, lossy source coding, mean square error, rate region, vector Gaussian source.

I. INTRODUCTION

THE CEO problem, which is a special case of multi-terminal source coding, was first investigated by Berger, Zhang and Viswanathan [1]. Oohama [2] determined the asymptotic sum-rate-distortion function of the scalar Gaussian CEO problem via an ingenuous application of the entropy power inequality. A complete characterization of the rate region of the scalar Gaussian CEO was obtained in [3] and [4]. However, extending this result to the vector case is not straightforward due to the fact that the entropy power inequality is not necessarily tight in this setting. Tavilder and Viswanath [5] derived a lower bound on the sum rate of the vector Gaussian CEO problem by partially replacing the entropy power inequality with the worst additive noise lemma. An explicit lower bound on the weighted sum rate of the two-terminal vector Gaussian CEO problem can be found in [6]. Of particular relevance here is the work by Wang and Chen [7], [8], where they derived an outer bound on the rate region of the vector Gaussian CEO problem by establishing a certain extremal inequality; essentially the same result was obtained in [9]. The extremal inequality in [7], [8] is a variant of the Liu-Viswanath inequality [10], which is motivated by the seminal work of Weingarten, Steinberg and Shamai [11] on the characterization of the capacity region of the vector Gaussian broadcast channel.

However, the outer bound induced by the Wang-Chen extremal inequality is in general not tight. Our main result is a strengthened extremal inequality for the special case where the covariance distortion constraint is replaced with the trace distortion constraint. It turns out that this new extremal inequality yields a complete characterization of the rate region of the vector Gaussian CEO problem for this special case. The perturbation argument, which is widely used for establishing extremal inequalities, appears to be insufficient for our purpose. For this reason, we develop a spectral decomposition method, which can be effectively incorporated into the perturbation argument to obtain the desired inequality. It is worth mentioning that our spectral decomposition method is partly motivated by the distortion projection technique developed by Rahman and Wagner [12], [13] for the vector Gaussian one-help-one problem (see also [14] for a direct proof based on the perturbation method).

The rest of this paper is organized as follows. In Section II, we present the formulation of the vector Gaussian CEO problem under the trace distortion constraint and the corresponding Berger-Tung upper bound on the weighted sum rate. In Section III, we revisit some mathematical preliminaries which will be used frequently in our proof. In Section IV, we prove certain properties of the spectral decomposition of the mean squared error matrix of the Berger-Tung scheme based on a carefully analysis of the KKT conditions of an associated non-convex optimization problem. In Section V, we establish a new extremal inequality by considering projections into subspaces specified by the spectral decomposition result in the previous section, which is further leveraged to characterize the rate region of the vector Gaussian CEO problem with the trace distortion constraint. Finally, we conclude this paper in Section VI.

II. PROBLEM STATEMENT AND THE MAIN RESULT

The system model of the vector Gaussian CEO problem is depicted in Figure 1. Let \( \{X(t)\}_{t=1}^{\infty} \) be an \( n \times 1 \)-dimensional i.i.d. vector-valued sequence, where each \( X(t), t = 1, 2, \ldots, L \) is a Gaussian random vector with mean zero and covariance \( K > 0 \). For \( i = 1, 2, \ldots, L \), let

\[
Y_i(t) = X(t) + N_i(t), \quad i = 1, 2, \ldots, L
\]

where \( N_i(t), t = 1, 2, \ldots \) are i.i.d. Gaussian random \( m \times 1 \)-dimensional vectors independent of \( \{X(t)\}_{t=1}^{\infty} \) with mean zero and covariance \( \Sigma_i > 0 \). The noise processes \( \{N_i(t)\}_{t=1}^{\infty}, i = 1, 2, \ldots, L \) are mutually independent. For \( i = 1, 2, \ldots, L \), encoder \( i \) computes \( C_i = \phi_i^n(Y_i^n) \) based on its noisy observation \( Y_i^n = \{Y_i(1), \ldots, Y_i(n)\} \) using encoding function

\[
\phi_i^n : \mathcal{R}^{m \times n} \rightarrow \mathcal{M}_i^n = \{1, \ldots, 2^{nR_i}\}
\]

and sends \( C_i \) to the decoder. Upon receiving \( C_1, C_2, \ldots, C_L \), the decoder computes \( X^n = \{X(1), \ldots, X(n)\} = \varphi^n(C_1, \ldots, C_L) \), which is an estimate of the remove source \( X^n = \{X(1), \ldots, X(n)\} \), using decoding function

\[
\varphi^n : \mathcal{M}_1^n \times \cdots \times \mathcal{M}_L^n \rightarrow \mathcal{R}^{m \times n}.
\]
Throughout the paper, we adopt the trace distortion constraint. Specifically, a rate tuple \((\bar{R}_1, \ldots, \bar{R}_L, d)\) is said to be achievable subject to the trace distortion constraint \(d\) if there exist encoding functions \(\phi^n_1, \ldots, \phi^n_L\) and decoding function \(\varphi^n\) such that

\[
\frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \text{tr} (\text{cov}(X^n t - X^n)) \right] \leq d,
\]

where \(X^n t (t)\) and \(X^n t (i)\) represents the \(j\)-th component of \(X^n t \) and \(X^n t \) respectively. The rate region \(R(d)\) is the closure of all achievable rate tuples \((R_1, \ldots, R_L)\) subject to the trace distortion constraint \(d\).

Since the rate region is convex, it can be characterized by its supporting hyper-planes. As a consequence, it suffices to solve the following optimization problem

\[
R(d) = \inf_{(R_1, \ldots, R_L) \in R(d)} \sum_{i=1}^{L} \mu_i R_i
\]

for \(\mu_i \geq 0, i = 1, \ldots, L\); moreover, there is no loss of generality in assuming \(\mu_1 \geq \cdots \geq \mu_L \geq 0\). Note that if \(\mu_L = 0\), then one can reduce the \(L\)-terminal problem to the \((L-1)\)-terminal problem by providing \(Y^n L\) directly to the decoder and the first \(L-1\) encoders. For this reason, we shall focus on the case \(\mu_1 \geq \cdots \geq \mu_L \geq 0\) in the rest of this paper.

It is clear that \(R(d) = \infty\) when \(d \geq \text{tr}\{\Sigma^{-1} + \sum_{i=1}^{L} \Sigma_i\}^{-1}\), and \(R(d) = 0\) when \(d \geq \text{tr}\{K\}\). For this reason, we shall focus on the case \(\mu_1 \geq \cdots \geq \mu_L \geq 0\) in the rest of this paper.

The main result of this paper is the following theorem.

**Theorem 1**: For any \(\mu_1 \geq \cdots \geq \mu_L \geq 0\) and \(d \in (\text{tr}\{\Sigma^{-1} + \sum_{i=1}^{L} \Sigma_i\}^{-1}\}, \text{tr}\{K\})\),

\[
R(d) = R^{BT}(d).
\]

The rest of this paper is devoted to the proof of the converse part of the theorem, i.e.,

\[
R(d) \geq R^{BT}(d).
\]

## III. Mathematical Preliminaries

We first review some basic properties of conditional Fisher Information Matrix and MSE (mean square error).

**Definition 1**: Let \((X, U)\) be a pair of jointly distributed random vectors with differentiable conditional probability density function \(f(x|u)\). The vector-valued score function is defined as

\[
\nabla \log f(x|u) = \left[ \frac{\partial \log f(x|u)}{\partial x_1}, \ldots, \frac{\partial \log f(x|u)}{\partial x_m} \right]^T.
\]

The conditional Fisher Information of \(X\) with respect to \(U\) is given by

\[
J(X|U) = \mathbb{E} \left[ (\nabla \log f(x|u)) \cdot (\nabla \log f(x|u))^T \right].
\]

**Lemma 1 (Cramér–Rao Lower Bound)**: Let \((X, U)\) be a pair of jointly distributed random vectors. Assuming that the conditional covariance matrix \(\text{cov}(X|U) \succ 0\), then

\[
J(X|U)^{-1} \preceq \text{cov}(X|U).
\]

One can refer to the proof in [10] Appendix II.

**Lemma 2 (Complementary Identity)**: Let \((X, U)\) be a pair of jointly distributed random vectors. If \(N\) follows a Gaussian distribution \(N(0, \Sigma)\), and it is independent with \((X, U)\), then

\[
J(X + N|U) + \Sigma^{-1} \text{cov}(X|X + N, U) \Sigma^{-1} = \Sigma^{-1}
\]

The proof of this complementary identity can be found in [15] Corollary 1.

**Lemma 3 (de Bruijn’s Identity)**: Let \((X, U)\) be a pair of jointly distributed random vectors, and \(N \sim N(0, \Sigma)\) be a standard Gaussian random vector, which is independent of \((X, U)\), then

\[
\frac{d}{dU} h(X + \sqrt{U}N|U) = \frac{1}{2} \text{tr} \left\{ J(X + \sqrt{U}N|U) \Sigma \right\}.
\]

This lemma is the conditional version of [16] Theorem 14.
Replacing the variable $\gamma$ by $1/\gamma$ in de Bruijn’s identity and using the complementary identity in lemma 2, one can obtain the following result via simple algebraic manipulations.

**Corollary 1:**

$$
\frac{d}{d\gamma} \left( \sqrt{\gamma} X + N(U) \right) = \frac{1}{2} \text{tr} \left\{ \Sigma^{-1} \text{cov}(X|\sqrt{\gamma} X + N) \right\}.
$$

**Lemma 4 (Data Processing Inequality):** Let $(X, U, V)$ be a tuple of jointly distributed random vectors, and $U, V, X$ form a Markov chain, i.e. $U \to V \to X$, then

$$
J(X|U) \leq J(X|V).
$$

The proof follows easily by the chain rule of Fisher information matrix [17, Lemma 1].

**Lemma 5 (Fisher Information Inequality):** Let $(X, Y, U)$ be a tuple of jointly distributed random vectors, and $X$ and $Y$ be conditionally independent given $U$, then for any $\gamma \in (0, 1)$,

$$
J(\sqrt{1-\gamma} X + \sqrt{\gamma} Y|U) \leq (1-\gamma)J(X|U) + \gamma J(Y|U).
$$

This is an equivalent form of matrix Fisher information inequality. One can refer to [18, Proposition 3] for a detailed discussion.

**Lemma 6:** Let $(X, U)$ be a pair of jointly distributed random vectors, and $N \sim N(0, \Sigma)$ be a standard Gaussian random vector, which is independent of $(X, U)$, then for any $\gamma \in (0, 1)$, we have

$$
\text{cov}(X|\gamma X + N, U) \geq \gamma^2 \text{cov}(X|U) + (1-\gamma)^2 \Sigma.
$$

The proof is left in Appendix A.

**IV. PROPERTIES OF $R_{\text{BT}}(d)$**

In this section, we study the KKT (Karush–Kuhn–Tucker) conditions for the optimization problem $R_{\text{BT}}(d)$ and establish some basic properties of the subspaces induced by the eigen-decomposition of the MSE (mean square error) matrix. These properties play a key role in the proof of the converse theorem for the vector Gaussian CEO problem under the trace distortion constraint.

**A. KKT Conditions**

It is easy to observe that the objective function of the optimization problem $R_{\text{BT}}(d)$ goes to infinity as $|\Sigma_i - B_i| \to 0$ for any $i = 1, \ldots, L$. Hence the constraints $B_i \preceq \Sigma_i^{-1}, \ i = 1, \ldots, L$ are not active.

The Lagrangian of the optimization problem $R_{\text{BT}}(d)$ is given by

$$
\frac{\mu_1}{2} \log |K^{-1} + \sum_{j=1}^{L} B_j| - \sum_{i=1}^{L} \frac{\mu_i - \mu_{i+1}}{2} \log |\Sigma_i^{-1} - B_i| - \sum_{i=1}^{L} \frac{\mu_i}{2} \log |\Sigma_1^{-1} - B_1| + \sum_{i=1}^{L} \frac{\mu_i}{2} \log |\Sigma_i^{-1} - B_i| - \sum_{i=1}^{L} \text{tr}(B_i \Psi_i) + \lambda \left( \text{tr}\left( (K^{-1} + \sum_{j=1}^{L} B_j)^{-1} \right) - d \right),
$$

where matrices $\Psi_i, i = 1, \ldots, L$ and scalar $\lambda$ are Lagrange multipliers. Let $B_1^*, \ldots, B_L^*$ be the optimal solution of $R_{\text{BT}}(d)$. Define

$$
C_i = \left( K^{-1} + \sum_{j=1}^{L} B_j^* \right)^{-1}, \ i = 1, 2, \ldots, L.
$$

The KKT conditions for the optimization problem $R_{\text{BT}}(d)$ are given by

$$
\frac{\mu_1}{2} C_1 + \frac{\mu_1}{2} \left( \Sigma_1^{-1} - B_1^* \right)^{-1} - \Psi_1 - \lambda C_1^2 = 0; \quad \mu_1 C_1 + \frac{\mu_k}{2} \left( \Sigma_k^{-1} - B_k^* \right)^{-1} - \sum_{i=1}^{k-1} \mu_i - \mu_{i+1} C_{i+1} - \Psi_k - \lambda C_k^2 = 0; \quad k = 2, \ldots, L;
$$

$$
B_k^* \Psi_k = 0, \ k = 1, \ldots, L; \quad \lambda (\text{tr}(C_1) - d) = 0; \quad \Psi_k \succeq 0, \ k = 1, \ldots, L; \quad \lambda \geq 0.
$$

Notice that the optimization problem $R_{\text{BT}}(d)$ is not convex; therefore, the constraint qualifications need to be examined in order to show the existence of Lagrange multipliers $\Psi_i, i = 1, \ldots, L$ and $\lambda$ satisfying the KKT conditions. These technical details are relegated to Appendix B. Here we just point out the following implication of the KKT conditions.

**Corollary 2:** For $d \in (\text{tr}\{\Sigma^{-1} + \sum_{i=1}^{L} \Sigma_i\}, \text{tr}(K))$, we have

$$
\text{tr}(C_1) = d.
$$

**Proof:** According to the complementary slackness condition [15], for the purpose of proving [17], it suffices to show $\lambda \neq 0$. If $\lambda = 0$, then it follows by [12] that $\Psi_1 > 0$, which, together with the complementary slackness condition $B_1^* \Psi_1 = 0$ in [14], implies $B_1^* = 0$. Substituting $B_1^* = 0$ into the first equation in [13] gives $\Psi_2 > 0$. Along this way, we may inductively obtain $B_1^* = B_2^* = \cdots = B_L^* = 0$, which, in view of [15], implies $\text{tr}(K) \leq d$. This leads to a contradiction with the assumption $d < \text{tr}(K)$. Thus [17] is proved.

**B. Spectral-decomposition of MSE**

Since the mean square error matrix $C_1 = (K^{-1} + \sum_{i=1}^{L} B_j^*)^{-1}$ of the Berger-Tung scheme is positive definite, we can write its spectral representation as below:

$$
C_1 = \sum_{n=1}^{m} d_n e_n e_n^T,
$$

where the positive real numbers $d_n, n = 1, \ldots, m$ stand for the eigenvalues, and $e_1, e_2, \ldots, e_m \in \mathbb{R}^n$ are the corresponding normalized eigenvectors which form an orthogonal basis. It follows readily from (18) that

$$
C_1^2 = \sum_{n=1}^{m} d_n^2 e_n e_n^T.
$$
In what follows, we denote

\[
\Delta_1 \triangleq \frac{\mu_1}{2} (\Sigma_{i}^{-1} - B_i) - \Psi_i.
\]

(20)

By the matrix identity in KKT conditions (12), we see that

\[
\Delta_1 = \lambda C_1^2 - \frac{\mu_1}{2} C_1,
\]

Substituting (13) and (19) into the above equation leads to the following spectral representation of \( \Delta_1 \):

\[
\Delta_1 = \sum_{n=1}^{m} \left( \lambda_n C_n^2 - \frac{\mu_1}{2} d_n \right) e_n e_n^T.
\]

(21)

Now we divide the vector space \( \mathbb{R}^m \) into two orthogonal subspaces according to the sign of the eigenvalues

\[
\lambda_n C_n^2 - \frac{\mu_1}{2} d_n / n = 1, 2, \ldots, m.
\]

We may define \( m \times n_1 \) matrix \( U_1 \triangleq (e_1, e_2, \ldots, e_n) \) in which the eigenvectors \( e_n, n = 1, 2, \ldots, n_1 \), correspond to the positive eigenvalues. Similarly, we may define \( m \times (m - n_1) \) matrix \( V_1 \triangleq (e_{n+1}, e_2, \ldots, e_m) \), in which the eigenvectors \( e_{n+1}, n = n_1 + 1, n_1 + 2, \ldots, m \), correspond to non-positive eigenvalues. It can be verified that

\[
\begin{align*}
U_1^T \Delta_1 U_1 &> 0, \quad V_1^T \Delta_1 V_1 \leq 0, \quad U_1^T \Delta_1 V_1 = 0; \quad (22) \\
V_1^T C_1 U_1 &> 0, \quad V_1^T C_1 V_1 \geq 0, \quad U_1^T C_1 V_1 = 0. \quad (23)
\end{align*}
\]

At this stage we may rewrite the spectral decomposition of \( \Delta_1 \) and \( C_1 = (K^{-1} + \sum_{j=1}^{L} B_j) \) according to the positivity/non-positivity structure of eigenspaces as below:

\[
\begin{align*}
\Delta_1 &= U_1 U_1^T \Delta_1 U_1 U_1^T + V_1 V_1^T \Delta_1 V_1 V_1^T, \quad (24) \\
C_1 &= U_1 U_1^T C_1 U_1 U_1^T + V_1 V_1^T C_1 V_1 V_1^T. \quad (25)
\end{align*}
\]

Since \( V_1^T \Delta_1 V_1 \leq 0 \), we have

\[
V_1^T \Psi_1 V_1 \geq \frac{\mu_1}{2} V_1^T (\Sigma_{i}^{-1} - B_i) - V_1 > 0,
\]

which means that the subspace spanned by the column vectors of \( V_1 \) belongs to the image space of \( \Psi_1 \), i.e., \( V_1 \subseteq \text{Im}(\Psi_1) \). Thus by the complementary slackness conditions (13) in KKT conditions, we have \( B_i^* \Psi_1 = 0 \); as a consequence, the kernel space of \( B_i^* \) contains the image space of \( \Psi_1 \), i.e., \( \text{Ker}(B_i^*) \supseteq \text{Im}(\Psi_1) \), which implies

\[
B_i^* V_1 = 0. \quad (26)
\]

Henceforth, according to the definition of \( V_1 \), we have

\[
\begin{align*}
0 &= B_i^* V_1 \text{diag}(d_{n+1}, d_{n+2}, \ldots, d_m) \\
&= B_i^* (e_{n+1}, e_{n+2}, \ldots, e_m) \text{diag}(d_{n+1}, d_{n+2}, \ldots, d_m) \\
&= B_i^* (d_{n+1} e_{n+1}, d_{n+2} e_{n+2}, \ldots, d_m e_m) \\
&= B_i^* C_1 (e_{n+1}, e_{n+2}, \ldots, e_m) \\
&= B_i^* C_1 V_1.
\end{align*}
\]

(27)

Left-multiplying with \( C_2 = (K^{-1} + \sum_{j=2}^{L} B_j) \) at both sides of (27) yields

\[
0 = C_2 B_i^* C_1 V_1
\]

\[
= (K^{-1} + \sum_{j=2}^{L} B_j^*)^{-1} (K^{-1} + \sum_{j=1}^{L} B_j^*)^{-1} V_1
\]

\[
= (K^{-1} + \sum_{j=2}^{L} B_j^*)^{-1} V_1 - (K^{-1} + \sum_{j=1}^{L} B_j^*)^{-1} V_1
\]

\[
= C_2 V_1 - C_1 V_1,
\]

(28)

which implies that

\[
V_1^T C_1 V_1 = V_1^T C_2 V_1. \quad (29)
\]

In view of (29), \( e_{n_1+1}, e_{n_1+2}, \ldots, e_m \) are also the eigenvectors of matrix \( C_2 \equiv (K^{-1} + \sum_{j=1}^{L} B_j^*)^{-1} \) with the eigenvalues being \( d_{n_1+1}, d_{n_1+2}, \ldots, d_m \). On the other hand, we can conclude that

\[
U_1^T C_2 V_1 = U_1^T C_1 V_1 = 0. \quad (30)
\]

Subtracting (12) from the first equation in KKT conditions (13) and invoking (20) gives

\[
\Delta_2 = \frac{\mu_1 - \mu_2}{2} C_2 + \Delta_1. \quad (31)
\]

Combining equations (30) and (31) with \( U_1^T \Delta_1 V_1 = 0 \), we see

\[
U_1^T \Delta_2 V_1 = 0. \quad (32)
\]

Thus we may give matrix \( \Delta_2 \) the following spectral representation:

\[
\Delta_2 = U_1 U_1^T \Delta_2 U_1 U_1^T + V_1 V_1^T \Delta_2 V_1 V_1^T. \quad (33)
\]

From equation (31), we have \( \Delta_2 \succeq \Delta_1 \) and consequently

\[
U_1^T \Delta_2 U_1 \succ U_1^T \Delta_1 U_1 \succ 0.
\]

On the other hand,

\[
\begin{align*}
V_1 V_1^T \Delta_2 V_1 V_1^T &= \frac{\mu_1 - \mu_2}{2} V_1 V_1^T C_2 V_1 V_1^T + V_1 V_1^T \Delta_1 V_1 V_1^T \\
&= \sum_{n=n_1+1}^{m} \frac{\mu_1 - \mu_2}{2} d_n e_n e_n^T + \left( \lambda_n d_n^2 - \frac{\mu_1}{2} d_n \right) e_n e_n^T \\
&= \sum_{n=n_1+1}^{m} \left( \lambda_n d_n^2 - \mu_2 d_n \right) e_n e_n^T.
\end{align*}
\]

(34)

Now we are at the same situation as treating equation (21), and correspondingly the refined spectral representation of matrix \( \Delta_2 \) can be obtained through a procedure similar to that for \( \Delta_1 \). Here we may divide the subspace spanned by the column vector of \( V_1 \) into two orthogonal subspaces, according to the sign of \( \Delta_2 \)'s eigenvalues \( \lambda_n d_n^2 - \mu_2 d_n^2 / n = n_1 + 1, n_1 + 2, \ldots, m \). Specifically, we partition the matrix \( V_1 \) into a \( m \times (n_2 - n_1) \) matrix \( W_1 \triangleq (e_{n_1+1}, e_{n_1+2}, \ldots, e_m) \) and a \( m \times (m - n_2) \) matrix \( V_2 \triangleq (e_{n_2+1}, e_{n_2+2}, \ldots, e_m) \), in which \( n_2 \) represents the critical number such that

\[
\lambda_n d_n^2 - \mu_2 d_n^2 / n > 0, \quad n_1 < n \leq n_2
\]

\[
\lambda_n d_n^2 - \mu_2 d_n^2 / n \leq 0, \quad n > n_2.
\]

On the other hand, combining \( U_1 \) and \( W_1 \) will form a new \( m \times n_2 \) matrix \( U_2 \triangleq (e_1, e_2, \ldots, e_{n_2}) \). It is straightforward to verify that

\[
W_1^T \Delta_2 W_1 \succ 0; \quad W_1^T \Delta_2 V_2 \preceq 0; \quad W_1^T \Delta_2 V_2 = 0. \quad (35)
\]
\[ W_1^T C_2 W_1 \geq 0; \quad V_2^T C_2 V_2 \geq 0; \quad W_1^T C_2 V_2 = 0. \]  \hspace{1cm} (36)

We can further refine the spectral decomposition form of \( \Delta_2 \) and \( C_2 \):
\[ \Delta_2 = U_1 U_1^T \Delta_2 U_1 U_1^T + V_1 V_1^T \Delta_2 V_1 V_1^T \]
\[ = U_1 U_1^T \Delta_2 U_1 U_1^T + W_1 W_1^T \Delta_2 W_1 W_1^T + V_2 V_2^T \Delta_2 V_2 V_2^T \]  \hspace{1cm} (37)
\[ C_2 = U_2 U_2^T C_2 U_2 U_2^T + V_2 V_2^T C_2 V_2 V_2^T \]
\[ = U_1 U_1^T C_2 U_1 U_1^T + W_1 W_1^T C_2 W_1 W_1^T + V_2 V_2^T C_2 V_2 V_2^T \]  \hspace{1cm} (38)

Following the similar steps as in the derivation of (26), we obtain
\[ B_i^* V_i = 0. \]  \hspace{1cm} (39)
\[ V_i^T C_2 V_i = V_i^T C_2 V_i. \]  \hspace{1cm} (40)

Repeating this procedure \( L \) times yields the following theorem.

**Theorem 2:** In \( \mathbb{R}^m \), there exist three sets of column orthogonal matrices \( \{U_1, U_2, \ldots, U_L\}, \{V_1, V_2, \ldots, V_L\}, \{W_1, W_2, \ldots, W_{L-1}\} \), such that the following properties hold:

1. [Spectrum of \( C_i \)]  
\[ C_i = U_i U_i^T C_i U_i U_i^T + V_i V_i^T C_i V_i V_i^T, \quad i = 1, \ldots, L. \]  \hspace{1cm} (41)

2. [Spectrum of \( \Delta_i \)]  
\[ \Delta_i = U_i U_i^T \Delta_i U_i U_i^T + V_i V_i^T \Delta_i V_i V_i^T, \quad i = 1, \ldots, L. \]  \hspace{1cm} (42)

3. [Positive/Negative definiteness]  
\[ U_i^T \Delta_i U_i > 0, \quad i = 1, \ldots, L; \]
\[ W_i^T \Delta_i W_i > 0, \quad i = 1, \ldots, L - 1; \]
\[ V_i^T \Delta_i V_i < 0, \quad i = 1, \ldots, L. \]  \hspace{1cm} (44)

4. [Orthogonality] For any \( 1 \leq i \leq L \),
\[ B_i^* V_i = 0. \]  \hspace{1cm} (45)

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**V. Converse**

In this section we establish a new extremal inequality, which is further leveraged to give a complete characterization of the rate region of the vector Gaussian CEO problem with the trace distortion constraint. However, it appears difficult to give a direct proof of this extremal inequality using the perturbation method. To overcome this difficulty, we project the mean square error matrix of the Berger-Tung scheme into its eigenspaces, and estimate each term of the extremal inequality in its respective subspace. This approach is partly inspired by the work of Rahman and Wagner on the vector Gaussian one-help-one problem [13].

**A. Extremal Inequality**

**Theorem 3:** Let \( B_1^*, \ldots, B_L^* \) be the optimal solution of \( R^{BT}(d) \). For any random variables \( (M_1, \ldots, M_L, Q) \) jointly distributed with \( (X, Y_1, \ldots, Y_L) \) such that
\[ p(x, y_1, \ldots, y_L, m_1, \ldots, m_L, q) \]
\[ = p(x)p(y_1|x)p(m_1|y_1, q), \]  \hspace{1cm} (46)
and
\[ \sum_{n=1}^{m} \mathbb{E} [(x_n - E[x_n|\{M_1, \ldots, M_L\}])^2] \]
\[ = \text{tr} \{ \text{cov}(X|\{M_1, \ldots, M_L\}) \} \]
\[ \leq \Delta, \]  \hspace{1cm} (47)
we have
\[ \sum_{i=1}^{L-1} (\mu_i - \mu_{i+1}) h(X|M_i, \ldots, M_L) \]
\[ - \mu_1 h(X|M_1, \ldots, M_L) - \sum_{i=1}^{L} \mu_i h(Y_i|X, M_i, Q) \]
\[ \geq \sum_{i=1}^{L-1} \frac{\mu_i - \mu_{i+1}}{2} \log ((2\pi e)^{(2\pi e)C_{i+1}}) - \frac{\mu_1}{2} \log ((2\pi e)^{(2\pi e)C_1}) \]
\[ - \sum_{i=1}^{L} \frac{\mu_i}{2} \log ((2\pi e)^{(2\pi e)(\Sigma_i - \Sigma_i B_i^* B_i)})]. \]  \hspace{1cm} (48)

Note that
\[ h([U_i, (\Sigma_i^{-1} - B_i^*)^{-1} V_i]^T \Sigma_i^{-1} Y_i|X, M_i, Q) \leq h(U_i^T \Sigma_i^{-1} Y_i|X, M_i, Q) \]
\[ + h(V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} \Sigma_i^{-1} Y_i|X, M_i, Q). \]

On the other hand, following by the matrix equality,
\[ (2\pi e)^{U_i^T (\Sigma_i^{-1} - B_i^*)^{-1} V_i} \]
\[ \leq (2\pi e)^{U_i^T (\Sigma_i^{-1} - B_i^*)^{-1} V_i} \sum_i \]
\[ \left( (2\pi e)^{U_i^T (\Sigma_i^{-1} - B_i^*)^{-1} V_i} \right). \]
By Taking logarithm for the determinant of matrix to both sides, we have
\[
\frac{1}{2} \log |(2\pi e)(\Sigma_i - \Sigma_i B_i^T \Sigma_i)| + \log |\Sigma_i^{-1}|
\]
\[+ \log |U_i, (\Sigma_i^{-1} - B_i^T)^{-1} V_i|]
\[= \frac{1}{2} \log |(2\pi e)U_i^T (\Sigma_i^{-1} - B_i^T) U_i|
\]
\[+ \frac{1}{2} \log |(2\pi e) V_i^T (\Sigma_i^{-1} - B_i^T)^{-1} V_i|.
\]
Therefore, it suffices to prove
\[
\sum_{i=1}^{L-1} \{ (\mu_i - \mu_{i+1}) h(X|M_i, \ldots, M_L) - \mu_i h(X|M_1, \ldots, M_L) \}
\]
\[= \sum_{i=1}^{L} \mu_i h(U_i^T \Sigma_i Y_i | X, M_i, Q)
\]
\[= \sum_{i=1}^{L} h(V_i^T (\Sigma_i^{-1} - B_i^T)^{-1} \Sigma_i^{-1} Y_i | X, M_i, Q)
\]
\[\geq \frac{1}{2} \sum_{i=1}^{L} \mu_i - \mu_{i+1} + \frac{1}{2} \log |(2\pi e)C_{i+1}| - \frac{1}{2} \log |(2\pi e)C_i|
\]
\[- \sum_{i=1}^{L} \mu_i \log |(2\pi e)U_i^T (\Sigma_i^{-1} - B_i^T) U_i|
\]
\[- \sum_{i=1}^{L} \mu_i \log |(2\pi e) V_i^T (\Sigma_i^{-1} - B_i^T)^{-1} V_i| \] (49)

To the end of proving inequality (49), we define 2L mutually independent zero mean Gaussian distributed random vectors \(X_1^G, \ldots, X_L^G\) and \(N_1^G, \ldots, N_L^G\) which are independent of \((X, Y_1, \ldots, Y_L, M_1, \ldots, M_L, Q)\).

Following [10], [14], we use the covariance preserved transform proposed by Dembo et al. in [16]. Specifically, for any \(\gamma \in (0, 1)\), define

\[
X_{i, \gamma} = \sqrt{1 - \gamma} X_i + \sqrt{\gamma} X_{i^G}, \quad i = 1, \ldots, L;
\]
\[
Y_{i, \gamma} = \sqrt{1 - \gamma} Y_i + \sqrt{\gamma} N_{i^G}, \quad i = 1, \ldots, L.
\]

Consider the functional
\[
g(\gamma) = \sum_{i=1}^{L-1} \{ (\mu_i - \mu_{i+1}) h(X_{i+1, \gamma}|M_{i+1}, \ldots, M_L) - \mu_i h(X_{i, \gamma}|M_1, \ldots, M_L) \}
\]
\[- \sum_{i=1}^{L} \mu_i h(U_i^T \Sigma_i^{-1} Y_{i, \gamma}|X, M_i, Q)
\]
\[- \sum_{i=1}^{L} \mu_i h(V_i^T (\Sigma_i^{-1} - B_i^T)^{-1} \Sigma_i^{-1} Y_{i, \gamma}|X, M_i, Q).
\]

The following lemma is needed for evaluating the derivative of \(g(\gamma)\) with respect to \(\gamma\).

Lemma 7: For the afore-defined \(X_{i, \gamma}\) and \(Y_{i, \gamma}\), we have

1) \[2(1 - \gamma) \frac{d}{d\gamma} h(X_{i, \gamma}|M_1, \ldots, M_L) \]
\[= \text{tr} \{ C_i \{ J(X_{i, \gamma}|M_1, \ldots, M_L) - C_i^{-1} \} \} \] (51)

2) \[2(1 - \gamma) \frac{d}{d\gamma} h(U_i^T \Sigma_i^{-1} Y_{i, \gamma}|X, M_i, Q) \]
\[\geq \text{tr} \{ U_i Y_i^T (\Sigma_i^{-1} - B_i^T)^{-1} U_i U_i^T \Sigma_i^{-1} \text{cov}(Y_{i, \gamma}|X, M_i, Q) \Sigma_i^{-1} \} \] (52)

3) \[2(1 - \gamma) \frac{d}{d\gamma} h(V_i V_i^T (\Sigma_i^{-1} - B_i^T)^{-1} \Sigma_i^{-1} Y_{i, \gamma}|X, M_i, Q) \]
\[\geq \text{tr} \{ V_i V_i^T (\Sigma_i^{-1} - B_i^T)^{-1} V_i V_i^T (\Sigma_i^{-1} - B_i^T)^{-1} \} \] (53)

Proof:

1) Using de Bruijn’s identity [9] in Lemma 3 and taking \(\gamma' = \gamma/(1 - \gamma)\), we obtain
\[
\frac{d}{d\gamma} h(X_{i, \gamma}|M_1, \ldots, M_L) \]
\[= \frac{d}{d\gamma} \{ h(X + \sqrt{\frac{\gamma}{1 - \gamma}} X_{i^G}, \ldots, L) | M_1, \ldots, M_L \}
\]
\[+ n \log(1 - \gamma) \}
\[= \frac{1}{2} \text{tr} \{ \frac{1}{(1 - \gamma)^2} J(X + \sqrt{\frac{\gamma}{1 - \gamma}} X_{i^G}, \ldots, L) | M_1, \ldots, M_L \}
\]
\[\cdot C_i - \frac{1}{1 - \gamma} I \} \} \] (54)

Multiplying both sides with \(2(1 - \gamma)\) yields
\[2(1 - \gamma) \frac{d}{d\gamma} h(X_{i, \gamma}|M_1, \ldots, M_L) \]
\[= \text{tr} \{ J(\sqrt{1 - \gamma} X + \sqrt{\gamma} X_{i^G}, \ldots, L) | M_1, \ldots, M_L \} C_i - I \}
\[= \text{tr} \{ X_{i, \gamma} | M_1, \ldots, M_L \} C_i - I \}
\[= \text{tr} \{ C_i (X_{i, \gamma} | M_1, \ldots, M_L - C_i^{-1}) \} \} \} \] (55)

2) Using the alternative form of de Bruijn’s identity [7] in Corollary 1 and taking \(\gamma' = (1 - \gamma)/\gamma\), we obtain inequality [56] at the top of next page.

In [56], inequality (a) follows from Lemma 6. Multiplying both sides of [56] gives
\[2(1 - \gamma) \frac{d}{d\gamma} h(U_i^T \Sigma_i^{-1} Y_{i, \gamma}|X, M_i, Q) \]
\[\geq \text{tr} \{ \frac{1}{\sqrt{1 - \gamma}} U_i (\Sigma_i^{-1} - B_i^T)^{-1} U_i U_i^T \Sigma_i^{-1} \text{cov}(Y_{i, \gamma}|X, M_i, Q) \Sigma_i^{-1} U_i \}
\]
\[\geq \text{tr} \{ U_i U_i^T - U_i U_i^T (\Sigma_i^{-1} - B_i^T)^{-1} U_i U_i^T \Sigma_i^{-1} \text{cov}(Y_{i, \gamma}|X, M_i, Q) \Sigma_i^{-1} U_i \}
\]
\[= \text{tr} \{ U_i U_i^T - U_i U_i^T (\Sigma_i^{-1} - B_i^T)^{-1} U_i U_i^T \Sigma_i^{-1} \text{cov}(Y_{i, \gamma}|X, M_i, Q) \Sigma_i^{-1} U_i \}
\]
\[= \text{tr} \{ U_i U_i^T - U_i U_i^T (\Sigma_i^{-1} - B_i^T)^{-1} U_i U_i^T \Sigma_i^{-1} \text{cov}(Y_{i, \gamma}|X, M_i, Q) \Sigma_i^{-1} U_i \}
\] (56)
\[
\frac{d}{d\gamma} h(U_i^T \Sigma_i^{-1} Y_{i,\gamma} | X, M_i, Q) \\
= \frac{d}{d\gamma} \left\{ h\left( \frac{1 - \gamma}{\gamma} U_i^T \Sigma_i^{-1} Y_i + U_i^T \Sigma_i^{-1} N_i^G | X, M_i, Q \right) + n_i \log \gamma \right\} \\
= \frac{1}{2} \text{tr} \left\{ \frac{1}{\gamma} - \frac{1}{\gamma^2} \left( U_i^T (\Sigma_i^{-1} - B_i^*) U_i \right) \right\}^{-1} \text{cov}(U_i^T \Sigma_i^{-1} Y_i | X, M_i, Q) \\
= \frac{1}{2} \text{tr} \left\{ \frac{1}{\gamma} - \frac{1}{\gamma^2} \left( U_i^T (\Sigma_i^{-1} - B_i^*) U_i \right) \right\}^{-1} \text{cov}(\frac{1 - \gamma}{\gamma} U_i^T \Sigma_i^{-1} Y_i | X, M_i, Q) \\
= \frac{1}{2} \text{tr} \left\{ \frac{1}{\gamma} - \frac{1}{\gamma^2} \left( U_i^T (\Sigma_i^{-1} - B_i^*) U_i \right) \right\}^{-1} \left( \text{cov}(\frac{1 - \gamma}{\gamma} U_i^T \Sigma_i^{-1} Y_i | X, M_i, Q) + \gamma(1 - \gamma)^2 U_i^T (\Sigma_i^{-1} - B_i^*) U_i \right) \}.
\]

(56)

\[
\frac{d}{d\gamma} h(V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} \Sigma_i^{-1} Y_{i,\gamma} | X, M_i, Q) \\
= \frac{d}{d\gamma} \left\{ h(V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} \Sigma_i^{-1} Y_i + \sqrt{\frac{\gamma}{1 - \gamma}} V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} \Sigma_i^{-1} N_i^G | X, M_i, Q) + (n_i - n_i) \log \gamma \right\} \\
= \frac{1}{2} \text{tr} \left\{ \frac{1}{1 - \gamma^2} J(V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} \Sigma_i^{-1} Y_i + \sqrt{\frac{\gamma}{1 - \gamma}} V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} \Sigma_i^{-1} N_i^G | X, M_i, Q) V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} V_i - \frac{1}{1 - \gamma} I \right\} \\
= \frac{1}{2} \text{tr} \left\{ \frac{1}{1 - \gamma^2} J(V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} \Sigma_i^{-1} Y_i + \sqrt{\frac{\gamma}{1 - \gamma}} V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} \Sigma_i^{-1} N_i^G | X, M_i, Q) V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} V_i - \frac{1}{1 - \gamma} I \right\} \\
+ \frac{1}{2(1 - \gamma)} \text{tr} \left\{ V_i^T ((1 - \gamma)(\Sigma_i^{-1} - B_i^*) \Sigma_i (\Sigma_i^{-1} - B_i^*) + \gamma(\Sigma_i^{-1} - B_i^*)) V_i V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} V_i - I \right\} \\
= \frac{1}{2(1 - \gamma)} \text{tr} \left\{ V_i^T (\Sigma_i^{-1} - B_i^*) V_i V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} V_i - V_i^T V_i \right\}.
\]

(57)

\[
\Sigma_i^{-1} \text{cov}(Y_{i,\gamma} | X, M_i, Q) \Sigma_i^{-1},
\]

where (a) follows from the simple fact that for any positive definite matrix A and column orthogonal matrix P,

\[
(P^T A P)^{-1} \preceq P^T A^{-1} P.
\]

3) Again using de Bruijn’s identity [6] in Lemma [3] and taking \( \gamma' = \gamma/(1 - \gamma) \), we obtain inequality (57) at the top of next page.

In (57), (a) follows from the data processing inequality of Fisher information matrix in Lemma [3] (b) is due to the fact that \( (Y_i, X_i) \) and \( N_i^G \) are independently distributed Gaussians; (c) is due to \( B_i^* V_i = 0 \) (see Proposition [3] in Theorem [2]). By multiplying both sides of (57) with \( (2(1 - \gamma)) \), and switching the matrices in the trace operator, we obtain (53) as desired.

Since

\[
\text{tr} \left\{ V_i V_i^T (\Sigma_i^{-1} - B_i^*)^{-1} \Sigma_i^{-1} Y_i - V_i V_i^T \right\} = \text{tr} \left\{ U_i U_i^T (\Sigma_i^{-1} - B_i^*)^{-1} U_i U_i^T \right\},
\]

it follows by (58) and Lemma [7] that

\[
2(1 - \gamma) g'(\gamma) \leq \sum_{i=1}^{L-1} \text{tr} \left\{ (\mu_i - \mu_{i+1}) C_{i+1} \right\}
\]

(58)

Note that (60) implies the existence of a monotonically decreasing path from \( \gamma = 0 \) to \( \gamma = 1 \), from which the desired extremal inequality follows immediately.

B. Proof of Theorem [4]

To prove Theorem [4] we consider the right part of (59). Recall the KKT conditions (12) and (13):

\[
\frac{\mu_1}{2} C_1 = \lambda C_1^2 - \Delta_1.
\]
\[
I_1 = \sum_{i=1}^{L-1} \text{tr} \left\{ 2U_i U_i^T (\Delta_{i+1} - \Delta_i) U_i U_i^T \left( J(X_{i+1}, \gamma|M_{i+1}, \ldots, M_L) - C_{i+1}^{-1} \right) \right\} \\
+ \sum_{i=1}^{L-1} \text{tr} \left\{ 2U_i U_i^T \Delta_i U_i U_i^T \left( J(X_{i}, \gamma|M_{i}, \ldots, M_L) - C_{i}^{-1} \right) \right\}; \\
\]
(62a)

\[
I_2 = \sum_{i=1}^{L} \text{tr} \left\{ 2V_i V_i^T (\Delta_{i+1} - \Delta_i) V_i V_i^T \left( J(X_{i+1}, \gamma|M_{i+1}, \ldots, M_L) - C_{i+1}^{-1} \right) \right\} \\
+ \sum_{i=1}^{L} \text{tr} \left\{ 2V_i V_i^T \Delta_i V_i V_i^T \left( J(X_{i}, \gamma|M_{i}, \ldots, M_L) - C_{i}^{-1} \right) \right\}; \\
\]
(62b)

\[
I_3 = -\sum_{i=1}^{L} \text{tr} \left\{ \mu_i U_i U_i^T (\Sigma_i^{-1} - B_i^*)^{-1} U_i U_i^T \left( (\Sigma_i^{-1} - B_i^*) - \Sigma_i^{-1} \text{cov}(Y_i, \gamma|X_i, M, Q) \Sigma_i^{-1} \right) \right\}. \\
\]
(62c)

\[
I_4 = -2\lambda \text{tr} \left\{ C_i^2 \left( J(X_{i+1}, \gamma|M_{i+1}, \ldots, M_L) - C_{i+1}^{-1} \right) \right\}. \\
\]
(62d)

\[
\frac{\mu_i - \mu_{i+1}}{2} C_{i+1} = \Delta_{i+1} - \Delta_i, \quad i = 1, \ldots, L - 1.
\]

By using the spectral decomposition property \[ of \ C_i = (K^{-1} + \sum_{j=i}^{L} B_j^{-1})^{-1}, \quad i = 1, 2, \ldots, L, \] in Theorem 2, we obtain that
\[
2(1 - \gamma)g'(\gamma) \leq I_1 + I_2 + I_3 + I_4, \\
\]
(61)

where the terms in the r.h.s are defined at the top of this page.

In what follows, we estimate the above four terms respectively, starting with \[ I_2. \]

Lemma 8: The term \[ I_2 \] can be upper bounded by
\[
I_2 \leq I_5 + I_6, \\
\]
(63)

where
\[
I_5 = \sum_{i=1}^{L-1} \text{tr} \left\{ 2W_i W_i^T \Delta_{i+1} W_i W_i^T \left( J(X_{i+1}, \gamma|M_{i+1}, \ldots, M_L) - C_{i+1}^{-1} \right) \right\} \\
I_6 = \text{tr} \left\{ 2V_L V_L^T \Delta_L V_L V_L^T \left( J(X_L, \gamma|M_L) - C_L^{-1} \right) \right\}. \\
\]
(64a, 64b)

Proof: By Proposition \[ 2 \] in Theorem \[ 2 \]
\[
V_i V_i^T \Delta_{i+1} V_i V_i^T = W_i W_i^T \Delta_{i+1} W_i W_i^T + V_{i+1} V_{i+1}^T \Delta_{i+1} V_{i+1} V_{i+1}^T, \\
\]
\[ i = 1, \ldots, L. \]
we can rewrite \[ I_2 \] as follows:
\[
I_2 \leq \sum_{i=1}^{L-1} \text{tr} \left\{ 2W_i W_i^T \Delta_{i+1} W_i W_i^T \left( J(X_{i+1}, \gamma|M_{i+1}, \ldots, M_L) - C_{i+1}^{-1} \right) \right\} \\
+ \text{tr} \left\{ 2V_L V_L^T \Delta_L V_L V_L^T \left( J(X_L, \gamma|M_L) - C_L^{-1} \right) \right\}; \\
\]
(65a)

\[
\leq I_5 + I_6, \\
\]
where the last inequality is because \[ 65c \] is upper bounded by 0 as shown below.

By definition \[ 59, \]
\[
X_{i+1} \gamma \leftrightarrow X_i \gamma \leftrightarrow (M_i, M_{i+1}, \ldots, M_L) \leftrightarrow (M_{i+1}, \ldots, M_L), \\
\]
where the covariance matrices of \[ X_{i+1}^G \] and \[ X_i^G \] are \[ C_i = (K^{-1} + \sum_{j=i}^{L} B_j^{-1})^{-1}, \] and \[ C_{i+1} = (K^{-1} + \sum_{j=i+1}^{L} B_j^{-1})^{-1} \] respectively. In view of the positive semidefinite partial order
\[
C_i \preceq C_{i+1}, \\
\]
we can assume that
\[
X_{i+1} \gamma \leftrightarrow X_i \gamma \leftrightarrow (M_i, M_{i+1}, \ldots, M_L) \leftrightarrow (M_{i+1}, \ldots, M_L) \\
\]
form a Markov chain. Thus by the data processing inequality in Lemma \[ 4 \] we have
\[
J(X_{i+1}, \gamma|M_{i+1}, \ldots, M_L) \leq J(X_i, \gamma|M_i, M_{i+1}, \ldots, M_L). \\
\]
(66)

On the other hand, \[ B_i^* V_i = 0 \] (Proposition \[ 4 \] in Theorem \[ 2 \]) yields that
\[
V_i V_i^T C_i^{-1} V_i = V_i^T C_i^{-1} V_i, \\
\]
(67)

and Proposition \[ 3 \] in Theorem \[ 2 \] implies that
\[
V_i V_i^T \Delta_i V_i V_i^T \preceq 0. \\
\]
(68)

Finally, combining \[ 66, 67 \] and \[ 68 \] gives the upper bound \[ 63. \]

Substituting the upper bound \[ 63 \] into \[ 61 \] yields
\[
2(1 - \gamma)g'(\gamma) \leq I_1 + I_5 + I_6 + I_3 + I_4. \\
\]
(69)

We now upper bound the first two terms in r.h.s of \[ 69. \]

Lemma 9: For the terms \[ I_1 \] and \[ I_5, \]
\[
I_1 + I_5 \leq I_5, \\
\]
(70)
where
\[
I_7 = \sum_{i=1}^{L} \text{tr} \left\{ 2U_iU_i^T \Delta_i U_i U_i^T \right\} \\
\cdot \left( (\Sigma_i^{-1} - B_i^*) - \Sigma_i^{-1} \text{cov}(Y_{i,\gamma}|X, M_i, Q) \Sigma_i^{-1} \right). \tag{71}
\]

**Proof:** It follows from Proposition 3 in Theorem 2 that \(W_i^T \Delta_{i+1} W_i > 0, \quad i = 1, \ldots, L - 1.\) \(\tag{72}\)

On the other hand,
\[
J(X_{i+1, \gamma}|M_{i+1}, \ldots, M_L) - C_{i+1}^{(a)} \supseteq (1 - \gamma)J(X|M_{i+1}, \ldots, M_L) - (1 - \gamma)C_{i+1}^{(b)} \supseteq (1 - \gamma) \left( \sum_{j=1}^{L} (\Sigma_j^{-1} - \Sigma_j^{-1} \text{cov}(Y_j|X, M_j, Q) \Sigma_j^{-1}) \right) - (1 - \gamma) \left( K^{-1} + C_{i+1}^{-1} \right) \supseteq \sum_{j=1}^{L} (\Sigma_j^{-1} - B_j^*) - \Sigma_j^{-1} \text{cov}(Y_j|X, M_j, Q) \Sigma_j^{-1}, \tag{73}\]

where (a) follows from the definition of random vector \(\{X_{i+1, \gamma}\}\) and Fisher information inequality in Lemma 5; (b) can be proved by using the argument in [9, Section 6.2] for completeness, we rewrite the proof in [9] in Appendix 9, and (c) is due to the definition of random vector \(\{Y_{i,\gamma}\}\). Finally, we obtain the bound \(\sum_{j=1}^{L} (\Sigma_j^{-1} - B_j^*) - \Sigma_j^{-1} \text{cov}(Y_j|X, M_j, Q) \Sigma_j^{-1}\) in the following form: \(\tag{74}\)

**Lemma 11:** For the second term \(I_7\) and the third term \(I_3\) in \(\tag{72}\),
\[
I_7 + I_3 \leq 0. \tag{77}\]

**Proof:** By the definition of \(\Delta_i:\)
\[
\Delta_i = \frac{\mu_i}{2} (\Sigma_i^{-1} - B_i^*)^{-1} - \Psi_i, \tag{78}\]

we can write \(I_7 + I_3\) in the following form:
\[
I_7 + I_3 = - \sum_{i=1}^{L} \text{tr} \left\{ 2U_iU_i^T \Psi_i U_i U_i^T \right\} \cdot \left( (\Sigma_i^{-1} - B_i^*) - \Sigma_i^{-1} \text{cov}(Y_{i,\gamma}|X, M_i, Q) \Sigma_i^{-1} \right). \tag{79}\]

Considering that
\[
\text{cov}(Y_{i,\gamma}|X, M_i, Q) \supseteq (1 - \gamma) \text{cov}(Y_{i,\gamma}|X, M_i, Q) + \gamma (\Sigma_i - \Sigma_i B_i^* \Sigma_i) \supseteq (1 - \gamma) \text{cov}(Y_{i,\gamma}|X, M_i, Q) + \gamma (\Sigma_i - \Sigma_i B_i^* \Sigma_i) = \Sigma_i - \gamma \Sigma_i B_i^* \Sigma_i, \tag{80}\]

in which (a) is from the definition of random vector \(\{Y_{i,\gamma}\}\) in Section IV, we have
\[
I_7 + I_3 \leq - \text{tr} \left\{ 2U_iU_i^T \Psi_i U_i U_i^T \left( (\Sigma_i^{-1} - B_i^*) - (\Sigma_i^{-1} - \gamma B_i^*) \right) \right\} = \text{tr} \left\{ 2U_iU_i^T \Psi_i U_i U_i^T (1 - \gamma) B_i^* U_i \right\} \supseteq 2(1 - \gamma) \text{tr} \left\{ U_i^T \Psi_i U_i U_i^T B_i^* U_i \right\} + V_i^T \Psi_i V_i V_i^T B_i^* V_i \supseteq 2(1 - \gamma) \text{tr} \left\{ \Psi_i U_i U_i^T B_i^* \right\} = 2(1 - \gamma) \text{tr} \left\{ U_i^T B_i^* \Psi_i \right\}, \tag{81}\]

where (a) is from \(B_i V_i = 0\) of Proposition 3 in Theorem 2 (b) is from complementary slackness conditions in KKT conditions \(\tag{14}\): \(B_i^* \Psi_i = 0\). \(\tag{82}\)

**Lemma 12:** For the last term \(I_4\) in \(\tag{72}\), we have
\[
I_4 \leq 0. \tag{83}\]

**Proof:** Due to the spectral decomposition of \(C_1:\)
\[
C_1 = \sum_{n=1}^{m} d_n e_n e_n^T. \tag{84}\]
we see that

\[ -I_4/2\lambda = \sum_{n=1}^{m} d_n \text{tr}\{e_n^T J(X,\gamma|M_1,\ldots,M_L)e_n - d_n^{-1}\} \]

\[ \geq \sum_{n=1}^{m} d_n J(e_n^T X,\gamma|M_1,\ldots,M_L) - \sum_{n=1}^{m} d_n. \tag{82} \]

where the inequality in (82) is from [12] Corollary I-b: 

\( J(\Lambda N) \leq \Lambda^T J(N) \Lambda \) for any column orthogonal matrix \( \Lambda \).

Let

\[ c_n = \text{cov}(e_n^T X|M_1, M_2, \ldots, M_L). \]

By the definition of \{X,\gamma\} and the Cramér–Rao lower bound in Lemma 1

\[ J(e_n^T X,\gamma|M_1,\ldots,M_L)^{-1} \leq \text{cov}(e_n^T X,\gamma|M_1,\ldots,M_L) \]

\[ = (1-\gamma)c_n + \gamma d_n \]

To show that (82) is lower-bounded by 0 is equivalent to show:

\[ \sum_{n=1}^{m} d_n \frac{d_n}{(1-\gamma)c_n + \gamma d_n} \geq \sum_{n=1}^{m} d_n. \tag{83} \]

According to Corollary 1 we have

\[ \text{tr} (C_1) = \sum_{n=1}^{m} d_n = d. \]

Now consider the trace constraint

\[ \text{tr} \{\text{cov}(X|M_1, M_2, \ldots, M_L)\} = \text{tr} \{\text{cov}(e_1^T, e_2^T, \ldots, e_m^T)X|M_1, M_2, \ldots, M_L)\} \]

\[ = \sum_{n=1}^{m} \text{cov}(e_n^T X|M_1, M_2, \ldots, M_L) \]

\[ = \sum_{n=1}^{m} c_n \leq d. \]

Since \( f(x) = x^{-1} \) is convex, we have \( \sum_{n=1}^{m} \alpha_n f(x_n) \geq f(\sum_{n=1}^{m} \alpha_n x_n) \), where \( \sum_{n=1}^{m} \alpha_n = 1, \alpha_n \geq 0 \).

Let

\[ \alpha_n = \frac{d_n}{d}, \quad x_n = \frac{(1-\gamma)c_n + \gamma d_n}{d_n}. \]

It can be seen that

\[ \sum_{n=1}^{m} d_n \frac{d_n}{(1-\gamma)c_n + \gamma d_n} \geq \left( \sum_{n=1}^{m} \frac{d_n (1-\gamma)c_n + \gamma d_n}{d_n} \right)^{-1} \]

\[ = \left( 1-\gamma \right) \sum_{n=1}^{m} c_n + \gamma \sum_{n=1}^{m} d_n \geq 1, \tag{84} \]

which implies (83). Thus \( I_4 \) indeed upper-bounded by 0.

This completes the proof of Theorem 1 as well as the extremal inequality in Theorem 3.

### C. Rate Distortion Region

Now we proceed to prove Theorem 1 \textit{i.e.} \( R(d) \geq R^{BT}(d) \).

To this end we need the Wagner-Anantharam single-letter outer bound [19] on \( R(d) \).

\textit{Theorem 5:} [19] The rate region \( R(d) \) is contained in the union of rate tuples \( (R_1, R_2, \ldots, R_L) \) such that

\[ \sum_{i=1}^{L} R_i \]

\[ \geq \sum_{i=j}^{L} I(X; M_1, \ldots, M_i|M_{i+1}, \ldots, M_L) + \sum_{i=j}^{L} I(Y_j; M_j|X, Q) \]

where the union is over all joint distributions \( p(x, y_1, \ldots, y_L, m_1, \ldots, m_L, q) \), which can be factorized as follows:

\[ p(x, y_1, \ldots, y_L, m_1, \ldots, m_L, q) = p(x)p(q) \prod_{i=1}^{L} p(y_i|x)p(m_i|y_i, q), \]

and \( \text{tr}\{\text{cov}(X|M_1, \ldots, M_L)\} \leq d. \)

According to this single-letter outer bound, we have

\[ \sum_{i=1}^{L} \mu_i R_i \]

\[ \geq \sum_{i=1}^{L-1} (\mu_i - \mu_{i+1}) I(X; M_1, \ldots, M_i|M_{i+1}, \ldots, M_L) \]

\[ + \mu_L I(X; M_1, \ldots, M_L) + \sum_{i=1}^{L} I(Y_i; M_i|X, Q) \]

\[ = \sum_{i=1}^{L-1} (\mu_i - \mu_{i+1}) h(X|M_{i+1}, \ldots, M_L) \]

\[ - \mu_1 h(X|M_1, \ldots, M_L) - \sum_{i=1}^{L} \mu_i h(Y_i|X, M_i, Q) \]

\[ \quad + \mu_L h(X) + \sum_{i=1}^{L} h(Y_i|X). \]

Notice that the term (85) equals the l.h.s of extremal inequality 45 in Theorem 3 so that we have

\[ R(d) = \inf_{(R_1, \ldots, R_L) \in \mathcal{R}(d)} \sum_{i=1}^{L} \mu_i R_i \]

\[ \geq \inf_{(\text{tr}\{\text{cov}(X|M_1, \ldots, M_L)\} \leq d)} \sum_{i=1}^{L-1} (\mu_i - \mu_{i+1}) h(X|M_{i+1}, \ldots, M_L) \]

\[ - \mu_1 h(X|M_1, \ldots, M_L) - \sum_{i=1}^{L} \mu_i h(Y_i|X, M_i, Q) \]

\[ + \mu_L h(X) + \sum_{i=1}^{L} h(Y_i|X) \]

\[ \geq \sum_{i=1}^{L-1} \frac{\mu_i - \mu_{i+1}}{2} \log \left( \frac{2\pi e}{(K^{-1} + \sum_{j=i+1}^{L} B_j)} \right)^{-1}. \]
\[ -\frac{\mu_1}{2} \log [(2\pi e)(K^{-1} + \sum_{j=1}^{L} B_j^{-1})] \]
\[ -L \frac{\mu_i}{2} \log [(2\pi e)(\Sigma_i - \Sigma_i B_i^T \Sigma_i)] \]
\[ + \frac{\mu_i}{2} \log [(2\pi e)K] + L \frac{\mu_i}{2} \log [(2\pi e)\Sigma_i] \]
\[ = R^{BT}(d). \]

This completes the proof of Theorem and establishes the tightness of Berger-Tung inner bound for the vector Gaussian CEO problem with trace distortion constraint.

VI. CONCLUSION

This paper provides a complete characterization of the rate region of the vector Gaussian CEO problem with the trace distortion constraint. Our proof is based on, among other things, a careful analysis of the KKT conditions for the optimization problem associated with the Berger-Tung scheme. In particular, we exploit the special structure of the KKT conditions to bound the rate region by considering the projection into different subspaces, and the inherent symmetry of the CEO problem enables us to perform the projection procedure recursively.

It should be stressed that the approach in this work does not apply directly to the setting considered in [8], [9] where a procedure recursively. KKT conditions to bound the rate region by considering the optimization problem associated with the Berger-Tung trace distortion constraint. Our proof is based on, among others, a careful analysis of the KKT conditions for the optimization problem associated with the Berger-Tung scheme. In particular, we exploit the special structure of the KKT conditions to bound the rate region by considering the projection into different subspaces, and the inherent symmetry of the CEO problem enables us to perform the projection procedure recursively.

APPENDIX A

PROOF OF LEMMA

Note that
\[ \gamma^2 \text{cov}(X|U) + (1 - \gamma)^2 \Sigma \]
\[ \overset{(a)}{\geq} (\gamma \text{cov}(X|U))^{-1} + (1 - \gamma)((1 - \gamma) \Sigma)^{-1}^{-1} \]
\[ = (\text{cov}(X|U)^{-1} + \Sigma^{-1})^{-1}, \]
which (a) is because \( A^{-1} \) is matrix concave in \( A \). This together with the fact (see, e.g., [7] footnote 2)
\[ (\text{cov}(X|U)^{-1} + \Sigma^{-1})^{-1} \geq \text{cov}(X|X + N, U) \]
completes the proof of Lemma.

APPENDIX B

EXISTENCE OF KKT CONDITIONS FOR \( R^{BT}(d) \)

The proof is similar to those in [11] Appendix IV and [13] Appendix B. One can refer to [20] Sections 4-5 for the background materials. We first rewrite the optimization problem \( R^{BT}(d) \) in a general form:
\[
\min_b \quad f(b) \\
\text{subject to} \quad g(b) \leq 0, \\
b \in B \triangleq B_1 \cap B_2 \cap \ldots \cap B_L. \quad (86)
\]

The vector \( b \in \mathbb{R}^{Lm^2 \times 1} \) is constructed by concatenating the columns of \( m \times m \) matrices \( B_i \) through \( B_L \); moreover,
\[
f(b) \triangleq \sum_{i=1}^{L-1} \mu_i - \frac{\mu_i + \mu_{i+1}}{2} \log \frac{K^{-1} + \sum_{j=1}^{i} B_j}{K^{-1} + \sum_{j=1}^{i} B_j} \\
+ \sum_{i=1}^{L} \mu_i \log \frac{|\Sigma_i|}{|\Sigma_i - B_i|} + \mu_L \log \frac{|K^{-1} + \sum_{j=1}^{L} B_j|}{|K^{-1}|},
\]
\[
g(b) \triangleq \text{tr}\{(K^{-1} + \sum_{i=1}^{L} B_i) - d\},
\]
and
\[
B_i \triangleq \{ \text{column concatenation of} \ (B_1, B_2, \ldots, B_L) : B_i \geq 0 \}, \quad i = 1, 2, \ldots, L.
\]

Since \( f \) and \( g \) are continuously differentiable, the Fritz-John necessary conditions [20] Definition 5.2.1) hold: there exist \( \mu, \lambda \geq 0 \) for the local minima \( b^* \) such that
\[
- (\mu \nabla f(b^*) + \lambda \nabla g(b^*)) \in T_B(b^*), \quad (87)
\]
where \( T_B(b^*) \) is the tangent cone of \( B \) at \( b^* \) and \( T_B(b^*) \) is its polar cone.

As \( B_i, i = 1, 2, \ldots, L \) are nonempty convex sets such that \( \{a_i(b^*) \cap \cap \{a_i(b^*) \cap \cdots \cap \{a_i(b^*) \} \} \) is nonempty, it follows [20] Problem 4.23 and [20] Proposition 4.63) that
\[
T_B(b^*) = T_{B_1}(b^*) + T_{B_2}(b^*) + \cdots + T_{B_L}(b^*). \quad (88)
\]

As in [13] Section B), it can be verified that
\[
T_{B_i}(b^*) \cap A \subseteq \{ \text{column concatenation of} \ (O, \ldots, -\Psi_i, \ldots, O) : \Psi_i \geq 0, \text{tr} \{\Psi_i B_i^* \} = 0 \}
\]
in which \( A \) is the set of vectors constructed by concatenating the columns of \( L \) symmetric matrices.

Since l.h.s. of equation (87) is also in \( A \), to complete the proof of the existence of KKT conditions, we need to show \( \mu \neq 0 \). As in [11] Appendix IV, we will verify the constraint qualifications (CQ5a in [20] Section 5.4), i.e., there exists a vector
\[
d \in T_B(b^*) = T_{B_1}(b^*) \cap T_{B_2}(b^*) \cap \cdots \cap T_{B_L}(b^*),
\]
such that \( \nabla g(b^*)^T d < 0 \).
Given any \( \alpha > 1 \), let’s define a set of \( m^2 \times 1 \) vectors
\[
b_i = \text{vec}(B_i) \triangleq \text{vec} \left( \alpha B_i^* + \frac{\alpha - 1}{L} K^{-1} \right), i = 1, 2, \ldots, L. \quad (89)
\]
Here \( \text{vec}(\cdot) \) is the vectorization operator. It can be seen that \( b_i \in B_i \) since \( B_i \geq 0 \). We denote \( d_i = b_i - (b^*)_i \), where \( (b^*)_i \) denotes the \( i \)th \( L \)-components in \( b^* \). By [20] Definition 4.6.1) and [20] Proposition 4.6.2), we have \( d_i \in T_{B_i}(b^*) \). Now \( d \) can be constructed by
\[
d = \text{vec}(d_1, d_2, \ldots, d_L).
\]

In this way, the expression of \( \nabla g(b^*)^T d \) can be written as
\[
\sum_{i=1}^{L} \text{tr} \left\{ (K^{-1} + \sum_{j=1}^{L} B_j)^{-1} (B_i^* - B_i) \right\}
\]
\begin{equation}
\begin{align*}
&= \sum_{i=1}^{L} \text{tr} \left\{ (K^{-1} + \sum_{i=1}^{L} B_i^*)^{-2} \left( (1 - \alpha)B_i^* - \frac{\alpha}{L} K^{-1} \right) \right\} \\
&= (1 - \alpha) \text{tr} \left\{ (K^{-1} + \sum_{i=1}^{L} B_i^*)^{-1} \right\} < 0,
\end{align*}
\end{equation}

where the inequality is because \( 1 - \alpha < 0 \) and \((K^{-1} + \sum_{i=1}^{L} B_i^*)^{-1} > 0\). This completes the proof of the existence of KKT conditions for the non-convex optimization problem \( R_{BT}(d) \).

**APPENDIX C**

**PROOF OF INEQUALITY (b) IN (73)**

We shall show that

\begin{equation}
J(X|M_{i+1}, \ldots, M_L) \leq K^{-1} + \sum_{j=1}^{L} (\Sigma_j^{-1} - \Sigma_j^{-1} \text{cov}(Y_j|X, M_j, Q) \Sigma_j^{-1})
\end{equation}

Note that

\( X = \sum_{j=1}^{L} A_j Y_j + Z \triangleq \tilde{X} + Z \),

where \( Z \) is a Gaussian random vector, independent of \((Y_{i+1}, \ldots, Y_L)\), with mean zero and covariance matrix

\( K_Z \triangleq (K^{-1} + \sum_{j=i+1}^{L} \Sigma_j^{-1})^{-1} \), and \( A_j \triangleq K_Z \Sigma_j^{-1} \).

Using the complementary relationship between Fisher information and MSE in Lemma [2] we have

\begin{equation}
\begin{align*}
J(X|M_{i+1}, \ldots, M_L) \\
&\leq J(X|M_{i+1}, \ldots, M_L, Q) \\
&= J(\tilde{X} + Z|M_{i+1}, \ldots, M_L, Q) \\
&= K^{-1}_{Z} - K^{-1}_{Z} \text{cov}(\tilde{X}|X + Z, M_{i+1}, \ldots, M_L, Q) K^{-1}_{Z} \\
&= K^{-1}_{Z} - \sum_{j=i+1}^{L} \Sigma_j^{-1} \text{cov}(Y_j|X, M_j, Q) \Sigma_j^{-1} \\
&= K^{-1} - \sum_{j=i+1}^{L} (\Sigma_j^{-1} - \Sigma^{-1}_j \text{cov}(Y_j|X, M_j, Q) \Sigma^{-1}_j),
\end{align*}
\end{equation}

where (a) is from the data processing inequality in Lemma [2], and (b) is due to the fact that for any \( i \), the Markov chain \((Y_j, M_j) \leftrightarrow (X, Q) \leftrightarrow (Y_{(i)}^j, M_{(i)}^j)\) holds.

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