The magnetic moment of $P_c(4312)$ as a $\bar{D}\Sigma_c$ molecular state

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(Dated:)

In this article, we tentatively assign the $P_c(4312)$ to be a $\bar{D}\Sigma_c$ molecular state with quantum number $J^P = \frac{1}{2}^-$, and calculate its magnetic moment using the QCD sum rule method in external weak electromagnetic field. Starting with the two-point correlation function in external electromagnetic field and expanding it in power of the electromagnetic interaction Hamiltonian, we extract the magnetic moment from the linear response to the external electromagnetic field. The numerical value is $\mu_{P_c} = 0.59^{+0.10}_{-0.20}$.

PACS numbers: 11.25.Hf, 11.55.Hx, 13.40.Gp.

I. INTRODUCTION

In Ref.[1], LHCb collaboration reported the discoveries of two pentaquark states $P_c(4380)$ and $P_c(4450)$ in the $J/\psi p$ invariant mass spectrum in the decay of $\Lambda_b \rightarrow J/\psi pK$. In 2019, they confirmed the $P_c(4450)$ state consisting of two narrow overlapping peaks $P_c(4440)$ and $P_c(4457)$, and observed a new narrow pentaquark state $P_c(4312)$ [2]. Following these experimental discoveries, there have been many theoretical studies concerning these pentaquark states through various models, such as the meson-baryon molecular scenario [3–29], the compact five quark states [30–46], kinematical triangle singularity [47] and so on.

In Ref.[29], we assumed the $P_c(4312)$ as a $\bar{D}\Sigma_c$ molecular state with quantum number $\frac{1}{2}^-$ and studied the decay of $P_c(4312)$ to $J/\psi p$ and to $\eta_c p$ with the QCD sum rule method. The QCD sum rule method [48] is a nonperturbative analytic formalism firmly entrenched in QCD with minimal modeling and has been successfully applied in almost every aspect of strong interaction physics. In Ref.[49–51], the QCD sum rule method was extended to calculate the magnetic moments of the nucleon and hyperon in the external field method. In this method, a statics electromagnetic field is introduced which couples to the quarks.

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and polarizes the QCD vacuum and magnetic moments can be extracted from the linear response to this field. Later, a more systematic studies was made for the magnetic moments of the octet baryons \[52–55\], the decuplet baryons \[56–59\] and the \(\rho\) meson \[60\]. In Ref.\[61, 62\], the authors calculated the magnetic moment of \(Z_c(3900)\) as an axialvector tetraquark state and an axialvector molecular state, respectively.

In this article, we extend this method to the investigation of the magnetic moment of the \(P_c(4312)\) state viewed as a \(\bar{D}\Sigma_c\) molecular state with quantum number \(J^P = \frac{1}{2}^-\). Electromagnetic and multipole moments are the major and meaningful parameters of hadrons. Analysis of the electromagnetic and multipole moments of the exotic states can help us get valuable knowledge about the electromagnetic properties of these states, the charge distributions inside them, their charge radius and geometric shapes and finally their internal substructures.

The rest of the paper is arranged as follows. In section II, we derive the sum rule for the magnetic moment of the \(P_c(4312)\) state. Section III is devoted to the numerical analysis and a short summary is given in section IV. In the Appendix B, the spectral densities are shown.

II. THE DERIVATION OF THE SUM RULES

The starting point of our calculation is the time-ordered correlation function in the QCD vacuum in the presence of a constant background electromagnetic field \(F_{\mu\nu}\),

\[
\Pi(p) = i \int dx^4 e^{ipx} \langle 0 \mid T[J^P_c(x)\bar{J}^{P_c}(0)] \mid 0 \rangle_F = \Pi^{(0)}(p) + \Pi^{(1)}(p)F_{\mu\nu} + \cdots ,
\]

where

\[
J^P_c(x) = [\bar{c}(x)i\gamma_5 d(x)][\bar{u}^a(x)C\gamma_\mu u_b(x)]\gamma^\mu \gamma_5 \Sigma_c(x),
\]

is the interpolating current of \(P_c(4312)\) considered as a \(\bar{D}\Sigma_c\) molecular state with \(J^P = \frac{1}{2}^-\) with \(T\) denoting the matrix transposition on the Dirac spinor indices, \(C\) meaning charge conjugation, and \(a, b, c\) being color indices. In the present work, we shall consider the linear response term, \(\Pi^{(1)}_{\mu\nu}(p)F_{\mu\nu}\), from which the magnetic moment will be extracted.

The external electromagnetic field can interact directly with the quarks inside the hadron and also polarize the QCD vacuum. As a consequence, the vacuum condensates involved in the operator product expansion of the correlation function in the external electromagnetic field \(F_{\mu\nu}\) are, dimension-2 operator

\[
F_{\mu\nu},
\]

dimension-3 operator

\[
\langle 0\mid \bar{q}\sigma_{\mu\nu}q\mid 0 \rangle_F,
\]

dimension-5 operators

\[
\langle 0\mid \bar{q}q\mid F_{\mu\nu} \rangle_F, \langle 0\mid \bar{q}g_\alpha G_{\mu\nu}q\mid 0 \rangle_F, \epsilon_{\mu\nu\alpha\beta} \langle 0\mid \bar{q}g_\alpha G^{\alpha\beta}q\mid 0 \rangle_F,
\]
where we make use of the following formulas,

\[ \langle 0 | \bar{q}q|0 \rangle \langle 0 | \bar{q}\sigma_{\mu\nu}q|0 \rangle_F, \langle 0 | g_s^2 G G|0 \rangle F_{\mu\nu}, \ldots, \]  

(6)

dimension-7 operators

\[ \langle 0 | g_s^2 G G|0 \rangle \langle 0 | \bar{q}\sigma_{\mu\nu}q|0 \rangle_F, \langle 0 | g_s q \sigma \cdot G q|0 \rangle F_{\mu\nu}, \ldots, \]  

(7)

dimension-8 operators

\[ \langle 0 | \bar{q}q|0 \rangle^2 F_{\mu\nu}, \langle 0 | g_s \bar{q}\sigma \cdot G q|0 \rangle \langle 0 | \bar{q}\sigma_{\mu\nu}q|0 \rangle_F, \langle 0 | \bar{q}q|0 \rangle \langle 0 | \bar{q}g_s G_{\mu\nu} q|0 \rangle_F, \]  

\[ \epsilon_{\mu\nu\alpha\beta} \langle 0 | \bar{q}q|0 \rangle \langle 0 | \bar{q}g_s G^{\alpha\beta} q|0 \rangle_F \ldots, \]  

(8)

and so on. The new vacuum condensates induced by the external electromagnetic field \( F_{\mu\nu} \) can be described by introducing new parameters, \( \chi, \kappa \) and \( \xi \), called vacuum susceptibilities as follows,

\[ \langle 0 | \bar{q}\sigma_{\mu\nu}q|0 \rangle_F = e\epsilon_0 \langle 0 | \bar{q}q|0 \rangle F_{\mu\nu}, \]  

\[ \langle 0 | \bar{q}g_s G_{\mu\nu} q|0 \rangle_F = e\epsilon_0 \langle 0 | \bar{q}q|0 \rangle F_{\mu\nu}, \]  

\[ \epsilon_{\mu\nu\alpha\beta} \langle 0 | \bar{q}q|0 \rangle \langle 0 | \bar{q}g_s G^{\alpha\beta} q|0 \rangle_F = i e \epsilon_0 \langle 0 | \bar{q}q|0 \rangle F_{\mu\nu}. \]  

(9)

In order to express the two-point correlation function \( \Pi \) physically, we expand it in powers of the electromagnetic interaction Hamiltonian \( H_{int} = -ie \int d^4 y j^e_{\mu}(y) A^\mu(y) \),

\[ \Pi(p) = i \int d^4 x e^{ipx} \langle 0 | T[J^{P_c}(x) J^{P_c}(0)] | 0 \rangle \]  

\[ + i \int d^4 x e^{ipx} \langle 0 | T[J^{P_c}(x)\{-ie \int d^4 y j^e_{\mu}(y) A^\mu(y)] J^{P_c}(0)\} | 0 \rangle + \cdots, \]  

(10)

where \( j^{e\mu}(y) \) is the electromagnetic current and \( A^\mu(y) \) is the electromagnetic four-vector.

Inserting complete sets of relevant states with the same quantum numbers as the current operator \( J^{P_c}(x) \) into the second term of (10) and carrying out involved integrals, one has

\[ \Pi^{(1)}_{\mu\nu}(p) F_{\mu\nu} = -\frac{\lambda_{P_c}^2}{4(p^2 - m_{P_c}^2)^2} [2m_{P_c}\mu_{P_c} \sigma^{\mu\nu} + \frac{\mu_{P_c} - 1}{m_{P_c}} (p^2 - m_{P_c}^2) \sigma^{\mu\nu} + \mu_{P_c} (p\sigma^{\mu\nu} + \sigma^{\mu\nu} \not{p}) + 2i \frac{\mu_{P_c} - 1}{m_{P_c}} (p^\mu \gamma^\nu - p^\nu \gamma^\mu) \not{p}] F_{\mu\nu} + \text{higher resonances}, \]  

(11)

where we make use of the following formulas,

\[ \langle 0 | J^{P_c}(0) | P_c(p, s) \rangle = \lambda_{P_c} u(p, s), \]  

(12)

and

\[ \langle P_c(k', s') | j^{e\mu}_{\mu}(0) | P_c(k, s) \rangle = \bar{u}(k', s') [F_1(Q^2) \gamma_{\mu} + F_2(Q^2) i\sigma_{\mu\nu} \frac{q^\nu}{2m_{P_c}}] u(k, s), \]  

(13)
the Lorentz structure of its better convergence. The spectral density \( \rho \)
where the constant \( a \) is introduced to parameterize the contributions of the pole-excited states transition. Subtracting the contributions of the excited-excited states transitions, 

\[
G_C(Q^2) = F_1(Q^2) - \frac{Q^2}{4m^2_{P_c}} F_2(Q^2),
\]

\[
G_M(Q^2) = F_1(Q^2) + F_2(Q^2).
\]

(14)

The magnetic moment \( \mu_{P_c} \) is given by \( G_M(0) \).

On the other hand, \( \Pi(p) \) can be calculated theoretically via OPE method at the quark-gluon level. To this end, one can insert the interpolating current \( J_{P_c}(x) \) into the correlation function \( \Pi \), contract the relevant quark fields by Wick’s theorem and find

\[
\Pi^{OPE}(p) = -2ie^{abc} \int d^4x e^{ipx} \left\{ \gamma^\mu \gamma_5 \gamma^\nu c(c) (x) \gamma^\nu c \right\}^\dagger \left\{ \gamma^\rho \gamma_5 \gamma^\sigma d(d) (-x) \right\} \left\{ \gamma^\rho \gamma_5 \gamma^\sigma u(u) (x) \gamma^\sigma c S^{(u)(T)} (x) C \right\} F,
\]

where \( S^{(u)}(x) \) and \( S^{(q)}(x) \), \( q = u, d \) are the full charm- and up (down)-quark propagators, whose expressions are given in the Appendix \( \text{A} \). Through dispersion relation, \( \Pi^{OPE}(p) \) can be written as

\[
\Pi^{OPE}(p) = \sigma^{\mu\nu} F_{\mu\nu} \int_{4m^2_{P_c}}^{\infty} ds \frac{\rho_1(s)}{s - p^2} + (p\sigma^{\mu\nu} + \sigma^{\mu\nu} p) F_{\mu\nu} \int_{4m^2_{P_c}}^{\infty} ds \frac{\rho_2(s)}{s - p^2}
\]

\[
+ i(p^\mu \gamma^\nu - p^\nu \gamma^\mu) F_{\mu\nu} \int_{4m^2_{P_c}}^{\infty} ds \frac{\rho_3(s)}{s - p^2} + \cdots,
\]

(16)

where \( \rho_i(s) = \frac{1}{\pi} \text{Im} \Pi^{OPE}(s), i = 1, 2, 3 \) are the spectral densities. We will choose the Lorentz structure \( i(p^\mu \gamma^\nu - p^\nu \gamma^\mu) F_{\mu\nu} \) to obtain our sum rule for the magnetic moment \( \mu_{P_c} \) because of its better convergence. The spectral density \( \rho_3(s) \) is given in the Appendix \( \text{B} \).

Finally, we match the phenomenological side \( \Pi(1) \) and the QCD representation \( \Pi^{OPE}(p) \) for the Lorentz structure \( i(p^\mu \gamma^\nu - p^\nu \gamma^\mu) F_{\mu\nu} \)

\[
- \frac{\lambda^2_{P_c}}{2(p^2 - m^2_{P_c})^2} \frac{\mu_{P_c} - 1}{m_{P_c}} + \text{higher resonances} = \int_{4m^2_{P_c}}^{\infty} ds \frac{\rho_3(s)}{s - p^2}.
\]

(17)

The higher resonances contain contributions from two parts, the pole-excited states transition and the excited-excited states transition induced by the external electromagnetic field. According to the quark-hadron duality, the later can be approximated by the QCD spectral density above some effective threshold \( s_0^{P_c} \), whose value will be determined in section \( \text{III} \)

\[
- \frac{\lambda^2_{P_c}}{2(p^2 - m^2_{P_c})^2} \frac{\mu_{P_c} - 1}{m_{P_c}} + \frac{a}{m^2_{P_c} - p^2} + \int_{s_0^{P_c}}^{\infty} ds \frac{\rho_3(s)}{s - p^2} + \text{subtractions} = \int_{4m^2_{P_c}}^{\infty} ds \frac{\rho_3(s)}{s - p^2},
\]

(18)

where the constant \( a \) is introduced to parameterize the contributions of the pole-excited states transition. Subtracting the contributions of the excited-excited states transitions,
one gets
\[-\frac{\lambda_{P_c}^2}{2(p^2 - m_{P_c}^2)^2} \frac{\mu_{P_c} - 1}{m_{P_c}} + \frac{a}{m_{P_c}^2 - p^2} + \text{subtractions} = \int_{4m_c^2}^{s_{P_c}} ds \rho_3(s) \frac{s}{s - p^2}. \quad (19)\]

In order to eliminate the subtractions, it is necessary to make a Borel transform which can also improve the convergence of the OPE series and suppress the contributions from the excited and continuum states. As a result, we have
\[(-\frac{\mu_{P_c} - 1}{2m_{P_c}M_B^2} + A)\lambda_{P_c}^2 e^{-m_{P_c}^2/M_B^2} = \int_{s_0}^{s_{P_c}} dse^{-s/M_B^2} \rho_3(s), \quad (20)\]
where \( A = \frac{a}{\lambda_{P_c}^2} \) and \( M_B^2 \) is the Borel parameter.

**III. NUMERICAL ANALYSIS AND THE PARTIAL DECAY WIDTHS**

The input parameters needed in numerical analysis are presented in Table I. For the vacuum susceptibilities \( \chi, \kappa \) and \( \xi \), we take the values \( \chi = -(3.15 \pm 0.30) \text{GeV}^{-2}, \kappa = -0.2 \) and \( \xi = 0.4 \) determined in the detailed QCD sum rules analysis of the photon light-cone distribution amplitudes \[63\]. Besides these parameters, we should determine the working intervals of the threshold parameter \( s_{0 P_c} \) and the Borel mass \( M_B^2 \) in which the magnetic moment is stable. The continuum threshold is related to the square of the first excited states having the same quantum number as the interpolating field and we use the value determined in Ref. \[29\], while the Borel parameter is determined by demanding that both the contributions of the higher states and continuum are sufficiently suppressed and the contributions coming from higher dimensional operators are small.

**TABLE I: Some input parameters needed in the calculations.**

| Parameter | Value |
|-----------|-------|
| \( \langle \bar{q}q \rangle \) | \(-0.24 \pm 0.01\) GeV³ |
| \( \langle g_s \bar{q}\sigma Gq \rangle \) | \((0.8 \pm 0.1)\langle \bar{q}q \rangle \text{GeV}^2 \) |
| \( \langle g_s^2 GG \rangle \) | \(0.88 \pm 0.25\) GeV⁴ |
| \( m_c \) | \(1.275^{+0.025}_{-0.035} \text{GeV}[64] \) |
| \( m_{P_c} \) | \(4311.9 \pm 0.74^{+0.8}_{-0.6} \text{MeV}[2] \) |
| \( \lambda_{P_c} \) | \(1.91^{+0.12}_{-0.13} \times 10^{-3} \text{GeV}^6[29] \) |

We define two quantities, the ratio of the pole contribution to the total contribution (RP) and the ratio of the highest dimensional term in the OPE series to the total OPE
series (RH), as followings,

\[
RP \equiv \frac{\int_{4m_c^2}^{s_0^{P_c}} ds \rho_3(s) e^{-s/M_B^2}}{\int_{4m_c^2}^{\infty} ds \rho_3(s) e^{-s/M_B^2}},
\]

\[
RH \equiv \frac{\int_{4m_c^2}^{s_0^{P_c}} ds \rho_3^{(d=9)}(s) e^{-s/M_B^2}}{\int_{4m_c^2}^{s_0^{P_c}} ds \rho_3(s) e^{-s/M_B^2}}. \tag{21}
\]

Firstly, we determine the working region of the \(M_B^2\). In Fig.1(a), we compare the various OPE contributions as functions of \(M_B^2\) with \(\sqrt{s_0^{P_c}} = 4.8\text{GeV}\). From it one can see that the OPE has good convergence. Fig.1(b) shows RP and RH varying with \(M_B^2\) at \(\sqrt{s_0^{P_c}} = 4.8\text{GeV}\). The figure shows that the requirement \(RP \geq 50\%\) \((RP \geq 40\%)\) gives \(M_B^2 \leq 4.2\text{GeV}^2\) \((M_B^2 \leq 4.8\text{GeV}^2)\).

![Graph](image1.png)

**FIG. 1:** (a) denotes the various OPE contributions as functions of \(M_B^2\) with \(\sqrt{s_0^{P_c}} = 4.8\text{GeV}\); (b) represents RP and RH varying with \(M_B^2\) at \(\sqrt{s_0^{P_c}} = 4.8\text{GeV}\).

Fig.2(a) shows the dependence of the magnetic moment \(\mu_{P_c}\) on the Borel mass \(M_B^2\) in the interval of \(2\text{GeV}^2 \leq M_B^2 \leq 7\text{GeV}^2\). From the figure we can see that \(\mu_{P_c}\) depends strongly on \(M_B^2\) as \(4\text{GeV}^2 \leq M_B^2\). In order to have a larger working interval of the Borel mass \(M_B^2\), we require \(RP \geq 40\%\). As a result, we limit \(M_B^2\) from \(4\text{GeV}^2\) to \(4.8\text{GeV}^2\). The result is shown in Fig.2(b), from which we can read reliably the value of the magnetic moment, \(\mu_{P_c} = 0.59^{+0.10}_{-0.20}\).

**IV. CONCLUSION**

In this article, we tentatively assign the \(P_c(4312)\) to be a \(\bar{D}\Sigma_c\) molecular state with quantum number \(J^P = \frac{1}{2}^-\), calculate its magnetic moment using the QCD sum rule method.
in the external weak electromagnetic field. Starting with the two-point correlation function in the external electromagnetic field and expanding it in power of the electromagnetic interaction Hamiltonian, we extract the magnetic moment from the linear response to the external electromagnetic field. The numerical value is $\mu_{P_c} = 0.59^{+0.10}_{-0.20}$. The prediction can be confronted to the experimental data in the future and give important information about the inner structure of the $P_c(4312)$ state.

**Acknowledgments**

This work was supported by the National Natural Science Foundation of China under Contract No.11675263.

**Appendix A: The quark propagators**

The full quark propagators are given as

$$S_{ij}^q(x) = \frac{i}{2\pi^2 x^4} \delta_{ij} - \frac{m_q}{4\pi^2 x^2} \delta_{ij} - \frac{(\bar{q}q)}{12} \delta_{ij} + \frac{i (\bar{q}q)}{48} m_q \not{x} \delta_{ij} - \frac{x^2}{192} (g_s \bar{q} \sigma G q) \delta_{ij}$$

$$+ \frac{i x^2}{152} m_q (g_s \bar{q} \sigma G q) \delta_{ij} - \frac{g_s m_q G^a_{\mu \nu} (\not{x} \sigma^{\mu \nu} + \sigma^{\mu \nu} \not{x})}{32\pi^2 x^2}$$

$$+ \frac{\delta_{ij} e_q F_{\mu \nu}}{288} (\sigma^{\mu \nu} - 2 \sigma^{\alpha \mu} x_\alpha x^\nu)$$

$$+ \frac{\delta_{ij} e_q (\bar{q}q) F_{\mu \nu}}{576} [\left(\kappa + \xi\right) \sigma^{\mu \nu} x^2 - \left(2\kappa - \xi\right) \sigma^{\alpha \mu} x_\alpha x^\nu] + \cdots$$

(A1)
for light quarks, and

\[ S_{ij}^Q(x) = i \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \left[ \frac{k + m_Q}{k^2 - m_Q^2} \delta_{ij} - \frac{g_s t_i^a G_{\mu\nu}^a}{4} \frac{(k + m_Q)\sigma^{\mu\nu}}{(k^2 - m_Q^2)^2} \right. \\
+ \left. \frac{\langle g_s^2 GG \rangle}{12} \delta_{ij} m_Q \frac{k^2 + m_Q}{(k^2 - m_Q^2)^4} + \frac{\delta_{ij} e_Q F_{\mu\nu} \sigma^{\mu\nu}(k + m_Q) + (k + m_Q)\sigma^{\mu\nu}}{(k^2 - m_Q^2)^2} \right] + \cdots \]  

(A2)

for heavy quarks. In these expressions \( t^a = \frac{\lambda^a}{2} \) and \( \lambda^a \) are the Gell-Mann matrix, \( g_s \) is the strong interaction coupling constant, and \( i, j \) are color indices, \( e_{Q(q)} \) is the charge of the heavy (light) quark and \( F_{\mu\nu} \) is the external electromagnetic field.

**Appendix B: The spectral densities**

In this appendix, we will give the explicit expression of the spectral density \( \rho_3(s) \).

\[ \rho_3(s) = \rho_3^{(d=2)}(s) + \rho_3^{(d=3)}(s) + \rho_3^{(d=5)}(s) + \rho_3^{(d=6)}(s) + \rho_3^{(d=7)}(s) + \rho_3^{(d=8)}(s) + \rho_3^{(d=9)}(s), \]  

with

\[ \rho_3^{(d=2)}(s) = \frac{m_c}{12288\pi^8} \int_{a_{\min}}^{a_{\max}} da \int_{b_{\min}}^{1-a} db \frac{1}{a^2 b^2} (1 - a - b)^4 (m_c^2 (a + b) - abs)^3 \]

(B3)

\[ \rho_3^{(d=5)}(s) = -\frac{m_c^2 \langle 0|\bar{q}q|0 \rangle}{384\pi^6} \int_{a_{\min}}^{a_{\max}} da \int_{b_{\min}}^{1-a} db \frac{1}{ab^2} (1 - a - b)^3 (m_c^2 (a + b) - abs) \]

(B4)

\[ \rho_3^{(d=6)}(s) = \frac{m_c^3 \langle 0|g_s^2 GG|0 \rangle}{147456\pi^8} \int_{a_{\min}}^{a_{\max}} da \int_{b_{\min}}^{1-a} db \frac{1}{ab^2} (1 - a - b)^4 \]

(B5)
\[
\rho_3^{(d=7)}(s) = -\frac{m_c^2}{512\pi^6} \langle 0 | g_s \bar{q} \gamma_\mu G \gamma^\mu q | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b} \frac{1}{(1-a-b)^2} \\
+ \frac{m_c^2}{768\pi^6} \langle 0 | g_s \bar{q} \gamma_\mu G \gamma^\mu q | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b} \frac{1}{ab} (1-a-b)^3 \\
+ \frac{m_c^2}{256\pi^6} \langle 0 | g_s \bar{q} \gamma_\mu G \gamma^\mu q | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b} (1-a-b) \\
- \frac{m_c^2}{256\pi^6} \langle 0 | g_s \bar{q} \gamma_\mu G \gamma^\mu q | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b} (1-a-b)^2,
\]

\[
\rho_3^{(d=8)}(s) = \frac{m_c^2}{576\pi^4} (2\kappa - \xi) \langle 0 | \bar{q} q | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b}
+ \frac{m_c^2}{144\pi^4} \langle 0 | \bar{q} q | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b},
\]

\[
\rho_3^{(d=9)}(s) = \frac{m_c^2}{27648\pi^6 M_B^2} \langle 0 | \bar{q} q | 0 \rangle \langle 0 | g_s^2GG | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b} \frac{(1-a-b_{\text{min}})^3}{(as-m_c^2)b_{\text{min}}^2} \\
- \frac{m_c^2}{9216\pi^6} \langle 0 | \bar{q} q | 0 \rangle \langle 0 | g_s^2GG | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b} \frac{(1-a-b_{\text{min}})^3}{(as-m_c^2)b_{\text{min}}^2} \\
- \frac{m_c^2}{9216\pi^6} \langle 0 | \bar{q} q | 0 \rangle \langle 0 | g_s^2GG | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b} \frac{a(1-a-b_{\text{min}})}{(as-m_c^2)} \\
- \frac{m_c^2}{4608\pi^6 M_B^2} \langle 0 | \bar{q} q | 0 \rangle \langle 0 | g_s^2GG | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b} \frac{(1-a-b_{\text{min}})^2}{(as-m_c^2)b_{\text{min}}^2} \\
+ \frac{m_c^2}{1536\pi^6} \langle 0 | \bar{q} q | 0 \rangle \langle 0 | g_s^2GG | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b} \frac{(1-a-b_{\text{min}})^2}{(as-m_c^2)b_{\text{min}}^2} \\
+ \frac{m_c^2}{3072\pi^6} \langle 0 | \bar{q} q | 0 \rangle \langle 0 | g_s^2GG | 0 \rangle \int_{a_{\text{min}}}^{a_{\text{max}}} \int_{b_{\text{min}}}^{1-a} \int_{b_{\text{min}}}^{1-b} \frac{da}{b} \frac{a(1-a-b_{\text{min}})}{(as-m_c^2)}. \tag{B8}
\]

In the above equations, \( a_{\text{max}} = \frac{1+\sqrt{1-\frac{4m_c^2}{s}}}{2}, \quad a_{\text{min}} = \frac{1-\sqrt{1-\frac{4m_c^2}{s}}}{2} \) and \( b_{\text{min}} = \frac{am_c^2}{as-m_c^2} \).

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