Abstract

We first derive the transverse Ward-Takahashi identities (WTI) of 3-dimensional quantum electrodynamics (QED) by means of the canonical quantization method and the path integration method, and then prove for the first time that QED is strictly solvable based on the transverse WTI and the longitudinal WTI, that is, the full vector and tensor vertices functions can be expressed in terms of the two-point functions, the Dyson-Schwinger equations (DSEs) which consists of an infinite set of coupled integral equations will form a closed system for the two-point functions. Among the many vertex approximations, the most famous one is the Ball-Chiu Ansatz [3] except for the bare vertex approximation. How to properly break through the bare vertex approximation and the Ball-Chiu Ansatz is a very challenging subject.

In what follows, we try to do some work in this area to see what kind of approximation can make the DSEs closed. One possible approach to this problem is to use the WTI to constrain the form of the vertex function. However, the normal WTI only contains the longitudinal part of the vertex functions, leaving its transverse part undetermined. In order to find further constraints on the vertex function, Takahashi derived so called transverse relations [4] relating Green's functions of different orders to complement the normal WTI, which have the potential to determine the full fermion-boson vertex in terms of the renormalization functions of the fermion propagator [5]. Some authors attempt to study this problem in various ways [6, 7, 8, 9]. He, Takahashi [8, 10] and Kondo [11], etc., find that the complete set of transverse WTI and longitudinal WTI for the vector, axial-vector and tensor vertex functions can form the complete solutions for these vertex functions in four-dimensions gauge theories. When it ignore the contribution of the three integral-term involving the Wilson line and chiral limit $m \to 0$, full vector vertex functions are expressed in terms of the two-point functions. Subsequently, Pennington and R. Williams [12, 13, 14] checked the transverse WT identity for the fermion-boson vertex to one-loop order. What we need to emphasize here is that although people have made important progress in constructing the fermion-boson vertex functions, it is still not possible to represent the full vector, axial-vector and tensor vertex functions in the four-dimensional gauge theory by two-point functions. That is to say, in four-dimensional quantum electrodynamics (QED), one cannot construct a completely closed DSEs by three-point Green’s functions and two-point Green’s functions. However, when the dimension of the gauge theory is reduced, it will change a lot. For example, in the case of two-dimensional gauge theory, Kondo first pointed out in Ref. [11] that “the transverse together with the usual (longitudinal) Ward-Takahashi identity are applied to specify the fermion-boson vertex function. ...... It is especially shown that in two dimensions, it becomes the exact and closed Schwinger-Dyson equation which can be exactly solved”. So a very natural question arises, can people get a completely Closed DSEs in three-dimensional QED (QED$_3$)? Since QED$_3$ can be regarded as an effective theory of high temperature superconductivity or as a toy model of quantum chromodynamics (QCD), this makes...

1. Introduction

The normal (longitudinal) Ward-Takahashi identities (WTI) [1] play an important role in various problems in quantum field theory, for example, it provides a consistency condition in the perturbative and non-perturbative approach of any quantum field theory and the proof of renormalizability of gauge theories [2]. In the Dyson-Schwinger equations (DSEs) approach, the fermion-boson vertex function is an important quantity to be specified. In order to use DSEs to do some actual calculations, we must artificially cut off the coupling of the $n$-point Green’s and the higher-order Green’s function to properly close the DSEs. If one can express the three-point vertices in terms of the two-point functions, the DSEs which consists of an infinite set of coupled integral equations will form a closed system for the two-point functions. Among the many vertex approximations, the most famous one is the Ball-Chiu Ansatz [3] except for the bare vertex approximation. How to properly break through the bare vertex approximation and the Ball-Chiu Ansatz is a very challenging subject.

In what follows, we try to do some work in this area to see what kind of approximation can make the DSEs closed. One possible approach to this problem is to use the WTI to constrain the form of the vertex function. However, the normal WTI only contains the longitudinal part of the vertex functions, leaving its transverse part undetermined. In order to find further constraints on the vertex function, Takahashi derived so called transverse relations [4] relating Green's functions of different orders to complement the normal WTI, which...
the research on QED, especially interesting (related references can be found in the literature in Ref. [15]). The main purpose of this paper is to try to construct a closed DSEs by three-point and two-point Green’s functions in the case of QED.

Due to the importance of Ward-Takahashi identities in the DSEs of QED, in this work, based on normal and transverse Ward-Takahashi Identities we discuss if the full vector and tensor vertices can be expressed in term of the fermion propagators in QED. In addition, given that there are two different expressions of the γ matrix in QED, we will also discuss whether the full vertices functions is equal in 4 × 4 and 2 × 2 representation. This work’s content is organized as follows: In Sect. 2, we present the generally applicable transverse WT relations for the vector and tensor vertex functions in N-dimensional space-time, and the detailed derivation is shown there; In Sect. 3, it is shown that the full vector and tensor vertices functions can be expressed in term of the fermion propagators in QED. And the transverse Ward-Takahashi identities and full vertices functions in different representation is also derived.

2. Identical Relations among Transverse Parts in N dimensions

The longitudinal (normal) WT identity determines its divergence, i.e., \( \partial \gamma \). The transverse WT identity [4] specifies the curl of the vertex function \( \partial^\gamma \gamma \) for QED, where \( \Gamma^\gamma \) is the fermion-boson (photon) vertex function. It was derived by Takahashi in 1986. The transverse Ward-Takahashi identity can be converted to

\[
\partial^\gamma \langle 0| T j(x) \gamma(y) \bar{\psi}(z) |0 \rangle = \partial^\gamma \langle 0| T j(x) \bar{\psi}(y) \gamma(z) |0 \rangle, \tag{1}
\]

where \( j(x) \) is the current operators. The above relation is valid for both QED and QCD.

Firstly let us introduce two bilinear covariant current operators,

\[
V^{\mu \nu \lambda}(x) = \frac{1}{4} \bar{\psi}(x) \left[ \gamma^\mu, \sigma^{\nu \lambda} \right] \phi(x) = g^{\mu \nu} f^{\lambda}(x) - g^{\nu \nu} f^{\lambda}(x),
\]

\[
V^{\nu \nu}(x) = -i \frac{1}{2} \bar{\psi}(x) \phi(x) = g^{\mu \nu} f^\nu(x) - g^{\nu \nu} f^\nu(x). \tag{2}
\]

One need to calculate the curl of the time-ordered products of the fermion’s three point functions involving the vector, axial-vector and tensor current operators, namely \( \mathcal{J}^\nu(x) = \bar{\psi}(x) \gamma^\nu \phi(x) \), \( \mathcal{J}^{\mu \nu}(x) = \bar{\psi}(x) \sigma^{\mu \nu} \phi(x) \) and \( \mathcal{F}^\nu(x) = \bar{\psi}(x) \gamma^\nu \gamma_5 \phi(x) \), respectively. Then the transverse WT identity for fermion’s vertex functions can be obtained by the curl of the \( T \) products of the corresponding fermion’ three-point function

\[
\partial^\nu \langle 0| T V^{\nu \nu \lambda}(x) \phi(y) \bar{\psi}(z) |0 \rangle = \partial^\nu \langle 0| T j^{\nu \nu}(x) \phi(y) \bar{\psi}(z) |0 \rangle - \partial^\nu \langle 0| T j^{\nu \nu}(x) \phi(y) \bar{\psi}(z) |0 \rangle. \tag{3}
\]

There are two ways to compute the curl of the time-ordered products of the above three-point functions. One is the canonical quantization method, another is the path integration method.

In the canonical quantization method, one notes here the general identity [2]

\[
\partial^\nu \langle 0| T V^{\nu \nu \lambda}(x) \phi(y_1) \phi(y_1) \ldots \phi(y_n) \bar{\psi}(y_n) |0 \rangle = \sum_{n=1}^\infty \delta_{00} \langle 0| T \left[ V^{\nu \nu \lambda}(x) \phi(y_2) \phi(y_3) \ldots \phi(y_n) \right] \delta(x_1 - y_1) \bar{\psi}(y_1) + \phi(y_1) [V^{\nu \nu \lambda}(x) \phi(y_2) \phi(y_3) \ldots \phi(y_n) \bar{\psi}(y_1)] \\
\times \bar{\psi}(y_1) \phi(y_1) \phi(y_1) \ldots \phi(y_1) \bar{\psi}(y_n) |0 \rangle + \langle 0| T \partial^\nu V^{\nu \nu \lambda}(x) \phi(y_1) \phi(y_1) \phi(y_1) \ldots \phi(y_n) \bar{\psi}(y_n) |0 \rangle. \tag{4}
\]

where the delimiter \( \nu \) above a term means its omission. The last term in above equation leads to a similar situation of \( \langle 0| T \bar{\psi}(y) N \left( \partial^\gamma + \partial^\gamma \right) \phi(x) \bar{\psi}(y) |0 \rangle \), normally, where \( N \) is matrix with an anti-communication relation. It means that the transverse WT identity exhibits different appearance depending on the dimensionality of space-time, because the anti-communication relation depends on the space-time dimension.

For the convenience of the discussion in \( N \)-dimensional gauge theory, we only uses the relations of gamma matrices that not depend on the spacetime dimensions and not introduce \( \gamma_5 \), as shown below

\[
[y^\nu, y^\mu] = 2g^{\mu \nu}, \quad \frac{1}{2}[y^\nu, y^\mu] = \sigma^{\nu \mu},
\]

\[
\frac{1}{2}[y^\nu, \sigma^{\nu \mu}] = ilg^{\nu \mu} y^\nu - g^{\nu \mu} y^\mu. \tag{5}
\]

Substituting the relations into Eqs. (2, 4), there are

\[
\partial_\mu \langle 0| T V^{\mu \nu \lambda}(x) \phi(y) \bar{\psi}(z) |0 \rangle
\]

\[
= \partial^\nu \langle 0| T j^\nu(x) \phi(y) \bar{\psi}(z) |0 \rangle - \partial^\nu \langle 0| T j^\nu(x) \phi(y) \bar{\psi}(z) |0 \rangle
\]

\[
= \langle 0| T \bar{\psi}(y) N \left( \partial^\gamma + \partial^\gamma \right) \phi(x) \bar{\psi}(y) |0 \rangle
\]

\[
+ \langle 0| T \partial_\mu V^{\mu \nu \lambda}(x) \phi(y) \psi(y) |0 \rangle.
\]

(6)

(7)

In order to relate the last term in the above equation to a definite Green’s function and to make the equations above more concise, here one need to consider two conditions. Firstly, the equation of motion for fermions with mass \( \bar{\psi}(i\partial + m) = 0 \), (i\( \partial - m \))\( \bar{\psi} = 0 \) are introduced to make the last term more concise. So the term \( \gamma^\mu \partial_\mu \psi(x) \) and \( \partial_\mu \bar{\psi}(x) \gamma^\mu \) need to be shown in the
equations as 
\[
\langle 0| T \partial_y V^{\mu \nu \lambda}(x) \psi(y) \bar{\psi}(z)| 0 \rangle = (0| i \bar{\psi}(x) \sigma^{\mu \nu} \gamma^\lambda \partial_y \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle \\
- (0| i \bar{\psi}(x) \sigma^{\nu \lambda} \gamma^\mu \partial_y \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle \\
+ \langle 0| \bar{\psi}(x) \frac{i}{2} [\gamma^\nu, \gamma^\lambda] (\gamma^\rho - \gamma^\sigma) \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle
\] 
(8)
and
\[
\langle 0| T \partial_y V^{\mu \nu \lambda}(x) \psi(y) \bar{\psi}(z)| 0 \rangle = \\
\frac{1}{4} \langle 0| \partial_{[\mu} \bar{\psi}(x) (\gamma^\nu \sigma^{\mu \nu} \gamma^\lambda + \sigma^{\mu \nu} \gamma^\lambda \gamma^\rho) \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle \\
+ \frac{1}{4} \langle 0| \partial_{\nu} \bar{\psi}(x) (\gamma^\lambda \sigma^{\mu \nu} + \gamma^\lambda \gamma^\rho \gamma^\nu) \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle \\
- \langle 0| \partial_{[\mu} \bar{\psi}(x) \sigma^{\mu \nu \lambda} \gamma^\rho \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle.
\] 
(9)
To further simplify the calculations, here one need to use the following relations to the first item of Eq. (9).
\[
\frac{1}{4} \langle 0| \partial_{[\mu} \bar{\psi}(x) (\gamma^\nu \sigma^{\mu \nu} \gamma^\lambda + \sigma^{\mu \nu} \gamma^\lambda \gamma^\rho) \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle = \\
\frac{1}{2} \langle 0| \partial_{\nu} \bar{\psi}(x) (\gamma^\lambda \sigma^{\mu \nu} + \gamma^\lambda \gamma^\rho \gamma^\nu) \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle \\
+ \frac{1}{2} \langle 0| \partial_{\nu} \bar{\psi}(x) (\gamma^\lambda \sigma^{\mu \nu} + \gamma^\lambda \gamma^\rho \gamma^\nu) \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle \\
- \langle 0| \partial_{[\mu} \bar{\psi}(x) \sigma^{\mu \nu \lambda} \gamma^\rho \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle \\
\] 
(10)
where one have defined $\frac{1}{4} [\gamma^\nu, \sigma^{\mu \nu} \gamma^\lambda] = A^{\mu \nu \lambda}$. With the similar procedure one derive the second item of Eq. (9), and defined $\frac{1}{2} [\gamma^\nu, \gamma^\lambda \sigma^{\mu \nu}] = B^{\mu \nu \lambda}$.

Secondly, one needs to move the derivative operators out of the T-product. For this purpose, one can write the form $\langle 0| T \bar{\psi}(x) N \psi(x) \bar{\psi}(z)| 0 \rangle$ as $\langle 0| T \bar{\psi}(x') N \psi(x) \bar{\psi}(y) \psi(y) \bar{\psi}(z)| 0 \rangle$ and then take $x' \rightarrow x$. The above expression including the nonlocal current is not gauge invariant. It needs to introduce a Wilson line $U(x,x') = \exp[-i \int_{x'}^x dy A_{\mu}(y)]$, joining the two space-time points $(x, x')$ to ensure that the current operators are locally gauge invariant. Comprehensive use of the Wilson line, the Eq. (4) and the equation of motion for fermions, there eventually are two relations
\[
(\partial_{\mu} - i \gamma_{\mu}) \langle 0| T \bar{\psi}(x') M^{\nu \lambda \mu}(x') \bar{\psi}(z)| 0 \rangle = \\
(0| T \bar{\psi}(x') M^{\nu \lambda \mu}(x') \gamma^\rho \bar{\psi}(z)| 0 \rangle - \partial^\nu (x - y) \gamma^\rho \delta^{\lambda \mu}(x - z) \\
+ (0| T \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle M^{\nu \lambda \mu}(x') \gamma^\rho \delta^{\lambda \mu}(x - z)
\] 
(11)
and
\[
(\partial_{\mu} - i \gamma_{\mu}) \langle 0| T \bar{\psi}(x') M^{\nu \lambda \mu}(x') \bar{\psi}(z)| 0 \rangle = \\
(0| T \bar{\psi}(x') M^{\nu \lambda \mu}(x') \gamma^\rho \bar{\psi}(z)| 0 \rangle - \partial^\nu (x - y) \gamma^\rho \delta^{\lambda \mu}(x - z) \\
- (0| T \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle M^{\nu \lambda \mu}(x') \gamma^\rho \delta^{\lambda \mu}(x - z)0 - 2i A_{\mu}(0) T \bar{\psi}(x') M^{\nu \lambda \mu}(x') \bar{\psi}(z)| 0 \rangle.
\] 
(12)
where $M^{\mu \nu \lambda}$ denotes a matrix.

Taking into account these equations on the above, submitting relations (10,11,12) to relations (6, 7, 8, 9) we arrive at the transverse WT relations for the fermions vertex functions in gauge theories in configuration space
\[
\partial^\nu \langle 0| T \bar{\psi}(x') \psi(y) \bar{\psi}(z)| 0 \rangle - \partial^\nu \langle 0| T \bar{\psi}(x') \psi(y) \bar{\psi}(z)| 0 \rangle = \\
\lim_{\epsilon \rightarrow 0} \frac{1}{2} (\partial_{\mu} - i \gamma_{\mu}) \langle 0| T \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle + i \mu \sigma^{\mu \nu} \delta^\nu (x - y) \langle 0| T \bar{\psi}(x') \psi(y) \bar{\psi}(z)| 0 \rangle
\] 
(13)
and
\[
\partial^\nu \langle 0| T \bar{\psi}(x') \psi(y) \bar{\psi}(z)| 0 \rangle - \partial^\nu \langle 0| T \bar{\psi}(x') \psi(y) \bar{\psi}(z)| 0 \rangle = \\
\frac{1}{2} (\sigma^{\mu \nu} \gamma^\lambda) \delta^\nu (x - y) \langle 0| T \bar{\psi}(x') \psi(y) \bar{\psi}(z)| 0 \rangle + \langle 0| T \bar{\psi}(y) \bar{\psi}(z)| 0 \rangle \sigma^{\mu \nu} \delta^\nu (x - z)
\] 
(14)
and
\[
\langle 0| \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle = \langle 0| \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle - \delta (x - y) \langle 0| T \bar{\psi}(x') \psi(y) \bar{\psi}(z)| 0 \rangle
\] 
(15)
So one need only to pay attention to the fermionic part $L_F = \bar{\psi} i \gamma^\nu (\partial_\nu - ie A_\nu) \psi - \bar{\psi} \gamma^\nu \partial_\nu \psi + \bar{\psi} \gamma^\nu \partial_\nu \psi$.

If one identify $A_\mu$ in $L_F$ as $A_\mu = A_\mu^{T^a} T^a$ with the generator $T^a$ of the gauge group $G$, the following relations also hold for the non-Abelian case irrespective of the gauge part. Then one can multiply Eq. (15) by the matrix $S$ from the left (right), where $S$ may be the spinor, flavor and color indices. Operating the differential operator $\frac{d}{dy} (\gamma^\nu \sigma^{\mu \nu})$ to the resulting equation, an then plus or minus, subsequently taking derivatives of both side with respect to $\frac{d}{dy}$ and $\frac{d}{dy}$ and setting all the sources equal to zero, one get the transverse WT identity
\[
\partial_{\mu} \langle 0| T \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle = \\
\langle 0| T \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle - \delta (x - y) \langle 0| T \bar{\psi}(x') \psi(y) \bar{\psi}(z)| 0 \rangle
\] 
(16)
and
\[
\partial_{\mu} \langle 0| T \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle = \\
\langle 0| T \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle - \delta (x - y) \langle 0| T \bar{\psi}(x') \psi(y) \bar{\psi}(z)| 0 \rangle
\] 
(17)
and
\[
\partial_{\mu} \langle 0| T \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle = \\
\langle 0| T \bar{\psi}(x') \bar{\psi}(z)| 0 \rangle - \delta (x - y) \langle 0| T \bar{\psi}(x') \psi(y) \bar{\psi}(z)| 0 \rangle
\] 
(18)
Let $S = S_A \otimes S_I \otimes S_r$ be a direct product of operators within the space of spinor, flavor and color. If choose $S = I_I \otimes I_I \otimes I_r$, one obtain the normal WT identity. It turns out that the transverse WT identity for vector current is obtained from Eq. (17) by choosing $S = \sigma_{\mu
u} \otimes I_I \otimes I_r$.

From the derivation of the above formula, the transverse WT identity exhibits different appearance depending on the dimensionality of spacetime. However, it is not easy to calculate the transverse WT identity for tensor current. In order to use the above relations (17, 18), we need to modify the bilinear covariant current operators (2) slightly:

$$V^{\mu\nu\rho\lambda}(x) = \frac{1}{4} \bar{\psi}(x) [\gamma^\rho, \sigma_{\mu\nu}], \gamma^4] \psi(x) = g^{\mu\nu} \rho^4(x) - g^{\rho\lambda} \nu^4(x)$$

$$= \bar{\psi}(x) \frac{1}{4} [\gamma^\rho, \{\sigma_{\mu\nu}, \gamma^4\}] \psi(x) - \bar{\psi}(x) g^{\rho\lambda} \gamma^{\mu\nu}\psi(x). \quad (19)$$

Through the above relations (17, 19), the transverse WT identity for fermion’s vertex functions can be obtained by

$$\partial_\mu (\bar{T} V^{\mu\nu\rho\lambda}(x) \psi(y) \bar{\psi}(z)) = 0$$

$$\partial_\mu (\bar{\psi}(x) [\gamma^\rho, \{\sigma_{\mu\nu}, \gamma^4\}] \psi(y) \bar{\psi}(z)) = \partial_\nu (\bar{\psi}(x) \gamma^{\mu\nu}\psi(y) \bar{\psi}(z)) - \partial_\lambda (\bar{\psi}(x) \gamma^{\rho\lambda}\psi(y) \bar{\psi}(z)). \quad (20)$$

where $S = \frac{1}{4} [\sigma_{\mu\nu}, \gamma^4]$. Then it can be verified that the transverse WT identity (13, 14) are obtained by the path integration method (17, 18).

As shown above, the transverse and longitudinal WT identities in the four-dimensional gauge theory do not specify the vertex function with a two-point Green’s function, thus forming a closed DSE. But in the case of low-dimension gauge theory, fundamental changes will occur, as we will show below. In QED${}_3$ theory, one can find a sets of transverse WT relations (for the vector and tensor vertex function) are coupled to each other, the transverse relations together with the longitudinal WT identities would lead to a complete set of WT-type constraint relations for the three-point functions. Then the complete expressions for three vertex functions can be deduced by solving this complete set of WT relations. These are the main content of the next section.

3. Anomaly, Dimension, Representation and Full Vertices Functions

The symmetry of the classical theory may be destroyed by quantum anomaly and there is the corresponding anomalous WT identity, as the Wilson line $U(x', x)$ introduced above (11, 12) are also useful for studying the Adler-Bell Jackiw anomaly contribution [16]. In four-dimensional gauge theories, by using perturbative method and Pauli-Villars regularization and dimensional regularization, Sun, et al., [17] find that there is no transverse anomaly term for both the axial-vector and vector current in the four-dimensional gauge theory. The absence of transverse anomalies for both axial-vector current in QED${}_2$ theory and vector (tensor) current in QED${}_3$ theory are also verified respectively [18]. So in transverse WT identity, one don’t need to discuss the problem of transverse quantum anomalies. However, the quantum anomaly of longitudinal WT identity needs to draw our attention.

In the previous section, we established the relationships of transverse WTI (13, 14) only by using matrix relations (5), which is suitable for $N$-dimensional time-space. As we will see shortly, the symmetric part $\{\gamma^\rho, \sigma_{\mu\nu}\}$ changes depending on the spacetime dimensions. In 3 + 1 dimensions time-space, submitting $\{\gamma^\rho, \sigma_{\mu\nu}\} = -2 \epsilon_{\mu
u\lambda\sigma} x^\lambda y_5$ into transverse WTI (13, 14), it is exactly the same as the previous results [8]. Now we turn to consider the 2 + 1 dimensional case, we choose the gamma matrices as

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i \sigma^1, \quad \gamma^2 = i \sigma^2, \quad \gamma^3 = \frac{1}{2} (\gamma^\rho, \sigma_{\mu\nu}) = \epsilon_{\mu\nu} \gamma_M, \quad \gamma_M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad [\gamma_M, \gamma^3] = 0. \quad (21)$$

where $\sigma^i$ denotes Pauli matrix. In this case, no other $2 \times 2$ matrix anti-commutes with all $\gamma^\rho$ and there is no $\gamma^3$ that anti-commute with $\gamma^\rho$. There is, therefore, nothing to generate a chiral symmetry that would be broken by a mass term $m \psi \bar{\psi}$.

Submit relations (21) to Eqs. (13, 14), the transverse Ward-Takahashi identity for the tensor and the vector vertex can be written in momentum space by introducing the standard definition for the three-point function,

$$q^0 T_V^0(p_1, p_2) - q^0 T_V^1(p_1, p_2) + q^0 T_V^0(p_1, p_2) = \epsilon_{\mu\nu} \epsilon^{\rho\lambda} S_F^\rho(q_1 - \epsilon_{\mu\nu} S_F^{\mu\nu} p_2). \quad (22)$$

and

$$q^0 T_\gamma^0(p_1, p_2) - q^0 T_\gamma^1(p_1, p_2) = -i S_F^{-1}(p_1) \sigma_{\mu\nu} - i \sigma_{\mu\nu} S_F^{-1}(p_2) - 2i \Gamma_\gamma^0(p_1, p_2)$$

$$+ i \epsilon_{\mu\nu} (p_1 + p_2) \bar{\gamma}_5 (p_1, p_2) - i \int \frac{d^3 k}{(2\pi)^3} 2k_\rho \epsilon_{\rho\nu} \Gamma_\gamma^3(p_1, p_2; k). \quad (23)$$

where $\Gamma_\gamma, \Gamma_\gamma^0, \Gamma_\gamma^3$ are the scalar, vector and tensor vertex functions, respectively, and $q = (p_1 - k) - (p_2 - k)$. The last term is called the integral-term involving the vertex function $\Gamma_\gamma^3(p_1, p_2; k)$ with the internal momentum $k$ of the gauge boson appearing in the Wilson line. The integral-term to one-loop order in four dimensions have been calculated in Ref. [13].

Noting that if we choose the basic fermion field to be a four component spinor, the three $4 \times 4$ matrices can be taken to be

$$\gamma^0 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}, \quad \gamma^1 = i \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix}, \quad \gamma^2 = i \begin{bmatrix} \sigma_2 \\ 0 \end{bmatrix}, \quad \gamma^3 = i \begin{bmatrix} 0 & 0 \\ 0 & \sigma_2 \end{bmatrix} \quad (24)$$

there will then be a $4 \times 4$ matrices $\gamma^5$ that anti-commute with all $\gamma^\rho$

$$\gamma^3 = i \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \gamma^5 = i \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \quad (25)$$

The massless theory will therefore be invariant under the chiral transformations. In this case, there are

$$\frac{1}{2} \{\gamma^\rho, \sigma_{\mu\nu}\} = \epsilon_{\mu\nu} \gamma_M, \quad \gamma_M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad [\gamma_M, \gamma^3] = 0. \quad (26)$$
So in this case, the above relation Eqs. (23, 22) will be modified as follows
\[ q^\mu \Gamma^{\nu}_{\mu}(p_1, p_2) - q^\nu \Gamma^{\mu}_{\nu}(p_1, p_2) = i e \epsilon^{\mu \nu \rho \sigma} (p_{2 \rho} + p_{2 \sigma}) \Gamma_M(p_1, p_2) - i \int \frac{d^3k}{(2\pi)^3} 2k_\rho e^{\mu \rho \sigma} \Gamma_M(p_1, p_2; k) \]
\[-i S F^- (p_1) + i S F^- (p_2) = 2 m \Gamma^{\mu}_{\nu}(p_1, p_2) \]
\[ (27) \]
and
\[ q^\nu \Gamma^{\mu}_{\nu}(p_1, p_2) - q^\mu \Gamma^{\nu}_{\nu}(p_1, p_2) + q^\nu \Gamma^{\mu}_{\nu}(p_1, p_2) = e^{\nu \alpha \beta} S^{-1}_{\alpha \beta} (p_1) \gamma_M = e^{\nu \alpha \beta} S^{-1}_{\alpha \beta} (p_2), \]
\[ (28) \]
where \( \Gamma_M \) denotes vertex function \((0)T \bar{\psi}(x) \gamma_M \psi(x) \bar{\psi}(y) \bar{\psi}(z) (0) \) in momentum space.

The above Eqs. (22, 23) show that the transverse part of the vertex function is related to the inverse of the fermion propagator and other vertex functions, namely the full vertex functions are coupled to each other and form a set of coupled equations. For instance, Eq. (23) shows that the transverse part of the vector vertex function is related to the fermion propagator, the tensor and scalar vertex functions. Noting that the transverse WT relation for the tensor vertex functions in four-dimensions have a mass term, which is different from the case of three-dimensions.

The well-known normal Ward-Takahashi identities
\[ q^\mu \Gamma^{\nu}_{\mu}(p_1, p_2) = S^{-1}_{\mu \nu} (p_1) - S^{-1}_{\mu \nu} (p_2) \]
\[ i q^\mu \Gamma^{\nu}_{\mu}(p_1, p_2) = S_{\mu \nu} (p_1) \gamma^\nu + \gamma^\nu S_{\mu \nu} (p_2) + 2 m \Gamma^{\mu}_{\nu}(p_1, p_2) + (p_{1 \nu} + p_{2 \nu}) \Gamma_S (p_1, p_2) \]
\[ (29) \]
denote the longitudinal part of the three-point vertex function, which together with the transverse WT relation form a complete set of WT-type relations for the fermion’s three-point vertex functions in QED theories. Then by this complete set of constraint relations one can obtain the complete solutions for these vector and tensor vertex functions.

Obviously, in four-dimensions space-time, it is extremely difficult to consider the full contributions of the above three integral-terms in Eqs. (13, 14). In order to get a set of closed DSEs, we should first ignore the contribution of integral-term involving the vertex functions and see what will happen. However, in three-dimensions space-time, we find that the full vertex function has a very simple expression, which can be expressed in terms of the fermion propagators. By using Eqs. (23, 22) and normal Ward-Takahashi identities Eq. (29), the complete expression in 2 × 2 representation for the vector vertex can be obtained as following
\[ \Gamma^{\mu}_{\nu}(p_1, p_2) = \frac{1}{(q^\mu - 4 m)} \left\{ q^\mu \left[ S^{-1}_{\mu \nu} (p_1) - S^{-1}_{\mu \nu} (p_2) \right] \right\} \]
\[ + i q^\mu \left[ S^{-1}_{\mu \nu} (p_1) \epsilon^{\nu \rho \sigma} + \epsilon^{\nu \rho \sigma} S^{-1}_{\nu \rho} (p_2) \right] + 2 m \left[ S^{-1}_{\mu \nu} (p_1) \gamma^\rho + \gamma^\rho S^{-1}_{\mu \nu} (p_2) \right] \]
\[ + 2 m (p_{1 \nu} + p_{2 \nu}) - i \epsilon^{\nu \rho \sigma} q_1 (p_{1 \rho} + p_{2 \rho}) \right\} \Gamma_S (p_1, p_2) \]
\[ + i \int \frac{d^3k}{(2\pi)^3} 2k_\rho q_\rho e^{\mu \rho \sigma} \Gamma_S (p_1, p_2; k) \right\}. \]
\[ (30) \]
And the tensor vertex function is
\[ \Gamma^{\mu}_{\nu}(p_1, p_2) = \frac{i}{q^\mu} \left\{ q^\mu \left[ S^{-1}_{\mu \nu} (p_1) (p_{1 \nu} - q^\nu - q^\nu - iq \epsilon^{\nu \rho \sigma}) \right] \right\} \]
\[ + (q^\mu q^\nu - q^\nu q^\mu + iq \epsilon^{\nu \rho \sigma}) S^{-1}_{\nu \rho} (p_2) \]
\[ + 2 m (q^\mu \Gamma^{\nu}_{\mu}(p_1, p_2) - q^\nu \Gamma^{\mu}_{\nu}(p_1, p_2)) \]
\[ + \left[i (q^\mu (p_{1 \nu} + p_{2 \nu}) - q^\nu (p_{1 \nu} + p_{2 \nu})) \right] \Gamma_S (p_1, p_2) \right\}, \]
\[ (31) \]
where
\[ \Gamma_S (p_1, p_2) = \Gamma_M (p_1, p_2) \]
\[ = \frac{-1}{q_\mu (p_{1 \mu} + p_{2 \mu}) \times \left[ \left[S^{-1}_{\mu \nu} (p_1) \gamma^\nu q_\nu + \gamma^\nu q_\nu S^{-1}_{\nu \mu} (p_2) \right] \right] - 2 m \left[ S^{-1}_{\nu \mu} (p_1) - S^{-1}_{\nu \mu} (p_2) \right] \}
\[ (32) \]
\[ \Gamma_{S(M)} (p_1, p_2) \]
\[ \] its derivation is as follows: Let \( q_\mu \) multiply the left side of the equation (23) and then use relations (29), one get
\[ q^\mu \Gamma^{\nu}_{\mu}(p_1, p_2) \]
\[ = q^\mu q_\mu q^\nu \Gamma^{\nu}_{\mu}(p_1, p_2) + i q^\mu q_\mu q^\nu \left[S^{-1}_{\mu \nu} (p_1) \epsilon^{\nu \rho \sigma} + \epsilon^{\nu \rho \sigma} S^{-1}_{\nu \mu} (p_2) \right] \]
\[ + 2 m q^\mu q_\mu q^\nu \left[ S^{-1}_{\nu \mu} (p_1) \gamma^\nu + \gamma^\nu S^{-1}_{\nu \mu} (p_2) \right] \]
\[ + 4 m^2 q_\mu q_\mu q_\nu \Gamma^{\nu}_{\mu}(p_1, p_2) + 2 m q_\nu q_\nu q^\mu (p_{1 \mu} + p_{2 \mu}) \Gamma_{S(M)} (p_1, p_2) \]
\[ - i \epsilon^{\nu \rho \sigma} q_\mu q_\rho q_{1 \nu} (p_{1 \nu} + p_{2 \nu}) \Gamma_{S(M)} (p_1, p_2) \]
\[ + i \int \frac{d^3k}{(2\pi)^3} 2k_\rho q_\rho e^{\mu \rho \sigma} \Gamma_{S(M)} (p_1, p_2; k). \]
\[ (33) \]
Exchanging the indicators \( \mu \) and \( \nu \), and adding the two equations thus obtained, due to the antisymmetry of \( \epsilon^{\mu \nu \rho \sigma} \) and \( \epsilon^{\nu \mu \rho \sigma} \), subsequently one get \( \Gamma_S (p_1, p_2) = \Gamma_M (p_1, p_2) \).

If we replace \( \Gamma_S \) with \( \Gamma_M \) in the above equation (30) we get the vector vertex function in 4 × 4 representation. Similarly, by replacing \( e^{\mu \nu \rho \sigma} \) with \( e^{\nu \mu \rho \sigma} \gamma_M \) in the equation (31) subsequently we get tensor vertex function in 4 × 4 representation.

In summary, we draw the following conclusions in 3-dimensional gauge theory
- \( \Gamma_S (p_1, p_2) = \Gamma_M (p_1, p_2) \);
- The full vector vertex function does not depend on the different \( \gamma \) matrix representation we use, i.e. it does not depend on whether we use 4 × 4 representation or 2 × 2 representation. But the tensor vertex function depends on the representation we use. Replacing \( e^{\mu \nu \rho \sigma} \) with \( e^{\nu \mu \rho \sigma} \gamma_M \), one can get tensor vertex function in 4 × 4 representation;
- In the chiral limit \( m \rightarrow 0 \), the vector and tensor vertex functions have a very simple form;
- The full vector and tensor vertex functions in three-dimensions space-time can be expressed in terms of fermion propagators only, which is different from vertex functions in the four-dimensional space-time (in four-dimensions gauge theory, only in the chiral limit, \( \Gamma^{\mu}_{\nu} \) and \( \Gamma^{\nu}_{\mu} \) at tree level are expressed in terms of the fermion propagators).
4. Summary and Conclusion

To summarize, we first derive the transverse WTI of 4-dimensional gauge theory by means of the canonical quantization method and the path integration method, and then using the characteristics of the \( \gamma \) matrix representation in three-dimensional gauge theory it is shown that the normal (longitudinal) WTI together with the transverse WTI form a complete set of Ward-Takahashi type constraint relations for the fermion-boson vertex functions in QED\(_3\) theory. By solving this complete set, the full vector and tensor vertex functions \( \Gamma^\nu_{\mu} \) and \( \Gamma^\mu_{\nu} \) can be expressed in terms of the fermion’s two-point functions, which is completely different from the situation in four-dimensions gauge theory (where only in the chiral limit, \( \Gamma^\nu_{\mu} \) and \( \Gamma^\mu_{\nu} \) at tree level are expressed in terms of the fermion propagators). Furthermore, it is found that the full tensor vertex function in \( 4 \times 4 \) representation is different from that in \( 2 \times 2 \) representation. But, the full vector vertex function does not depend on the representation we use. This means that when we study the dynamic behavior of three-dimensional gauge theory related to the tensor vertex function, we must specify the \( \gamma \) matrix representation in advance.

Finally, we need to emphasize that low-dimensional gauge theory has a very wide range of applications in condensed matter physics. In particular, QED\(_3\) has been suggested to be the effective low-energy field theory for the anomalous normal state of high-\( T_c \) cuprate superconductors [19, 20, 21]. It also provides a promising field-theoretic description for such exotic quantum many-body state as \( U(1) \) quantum spin liquid [22]. When massless Dirac fermions are coupled to \( U(1) \) gauge boson, they acquire a finite anomalous dimension due to the strong gauge interaction [19, 20, 21, 23]. This may lead to intriguing Luttinger-like behaviors, which has been used to understand the absence of well-defined quasiparticle peaks in the normal state of high-\( T_c \) cuprate superconductors [19, 20, 21, 23]. To reveal the nature of these Luttinger-like behaviors, one needs to compute certain types of Green’s function very carefully. The gauge invariance must be preserved during the analytical calculations [24, 25, 26, 27, 28]. In principle, these Green’s functions can be self-consistently obtained by solving a close set of DSEs. We expect that the generic WTI obtained in this work would be utilized to calculate the gauge invariant Green’s functions by means of DSEs.

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