First-order Gradual Information Flow Types with Gradual Guarantees

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Gradual type systems seamlessly integrate statically-typed programs with dynamically-typed programs. The runtime for gradual type systems can be viewed as a monitor which refines and enforces constraints to ensure type-preservation. Gradual typing has been applied to information flow types, where information flow monitors are derived from gradual information flow types. However, existing work gives up the dynamic gradual guarantee — the property that loosening the policies of a program should not cause more runtime errors — in favor of noninterference — the key security property for information flow control systems. In this paper, we re-examine the connection between gradual information flow types and information flow monitors, and identify the root cause for the tension between satisfying gradual guarantees and noninterference. We develop a runtime semantics for a simple imperative language with gradual information flow types that provides both noninterference and the dynamic gradual guarantee. We leverage a proof technique developed for FlowML, which reduces noninterference proofs to preservation proofs, to prove the key security property.

1 INTRODUCTION

Static type systems can rule out problematic programs and check important properties at compile time. For instance, information flow type systems label each type with a security label (e.g., int\textsuperscript{H} to indicate a secret integer) and can ensure well-typed programs do not leak secrets to attackers [Volpano et al. 1996]. However, purely statically-typed languages face significant adoption challenges because average programmers are unfamiliar with and unwilling to use complex type systems and much of the legacy infrastructure was written in untyped or dynamically typed languages.

Gradual typing, which aims to seamlessly integrate statically-typed programs with dynamically-typed programs, is one promising technique to address these challenges [Siek and Taha 2006]. At a high-level, gradual type systems introduce a dynamic type, often written as ?, to accommodate untyped portions of the program. The type system allows any program to be typed under ?, and the runtime semantics of gradual type systems monitor the interactions between parts typed as ? and statically typed parts to ensure type preservation of the current execution.

Gradual typing has been applied to a number of type systems [Jafery and Dunfield 2017; Lehmann and Tantardini 2017; New and Ahmed 2018; Rastogi et al. 2012, 2015; Schwerter et al. 2014], including information flow types [Disney and Flanagan 2011; Fennell and Thiemann 2013, 2016; Toro et al. 2018], where an expression of type int\textsuperscript{H} has its information flow labels determined at runtime. As a result, information flow monitors are derived from the runtime semantics of gradual information flow types. Most of the prior works on gradual information flow typing have developed ad hoc approaches relying on using programmer-annotated upgrades and casts to ensure correctness [Disney and Flanagan 2011; Fennell and Thiemann 2013, 2016]. Formal properties related to gradual typing such as dynamic gradual guarantee [Siek et al. 2015], the property that loosening
the policies of a program should not cause more runtime errors, are not considered in those systems. Recent work aims to take a principled approach to generate runtime semantics from gradual typing theories, in hope to achieve gradual guarantees [Toro et al. 2018]. However, they have to give up dynamic gradual guarantee in favor of noninterference, the key security property for information flow control systems [Goguen and Meseguer 1982], in a higher-order setting.

In this paper, we re-examine the connection between gradual information flow types and information flow monitors (c.f. Austin and Flanagan [2009, 2010]; Russo and Sabelfeld [2010]). To identify the root cause of the tension between dynamic gradual guarantee and security, we focus on a simple imperative language with first-order functions and stores, which has been widely used in designing information flow control systems [Hunt and Sands 2006; Moore and Chong 2011; Russo and Sabelfeld 2010; Terauchi and Aiken 2005; Volpano et al. 1996; Volpano and Smith 1997]. We develop runtime semantics for this language with gradual information flow types that enjoy both noninterference and the dynamic gradual guarantee. We draw ideas from abstracting gradual typing, which advocates deriving runtime semantics for gradual types via preservation proof [Garcia et al. 2016].

We leverage a proof technique developed for FlowML, which reduces noninterference proofs to preservation proofs [Pottier and Simonet 2002]. The main idea here is to extend the language with pairs of expressions and commands, representing two executions with different secrets in one program. Noninterference follows from preservation.

The main insight is that to enforce noninterference and remove implicit flows, the runtime semantics need to take into consideration variable writes in the untaken branch for refining information about the imprecise labels. This is because the semantics that only gradually instantiate information flow labels resemble a flow-sensitive monitor and therefore inherit problems with implicit leaks of flow-sensitive monitors [Russo and Sabelfeld 2010]. On the other hand, too coarse-grained guessing without using information about the untaken branch yields rigid semantics that break gradual guarantees (More details in Section 3.4).

Our proof technique clearly illustrates the problem with flow-sensitive monitors and naturally suggests the “hybrid” approach [Moore and Chong 2011]. More concretely, paired execution pin-points that writing to memory inside a branch that depends on a secret results in two executions, which produces different attacker observations and therefore cause insecurity (Section 3.4).

While the language considered in this paper is simple (no higher-order functions or stores), it still manifests the problem that does not guarantee both noninterference and dynamic gradual guarantee. Further, much of the dynamic information flow monitoring literature uses this language, from which we draw inspirations for our monitoring semantics.

This paper is organized as follows. We briefly review information flow control systems in Section 2. Then, in Section 3, we give an overview of gradual information flow types and motivate our work. Next, we formally define the syntax, type system, and information flow monitor semantics of a simple while language (Section 4), prove that our monitor enforces noninterference (Section 5) and satisfies gradual guarantees (Section 6). We review related work in Section 7. Due to space constraints, the proofs of various lemmas and theorems have been included in the supplementary material included with the submission. Some of the uninteresting formalism has also been included in the supplementary material.

2 OVERVIEW OF INFORMATION FLOW CONTROL SYSTEMS

In information flow control systems, the variables are annotated with a security label that is an element of a security lattice, i.e., the elements have a partial-ordering and a well-defined join and meet operation. For instance, in a two-point lattice with elements \( \{L, H\} \) and \( L \preceq H \), \( L \) may
represent low or public, and $H$ may represent high or secret. $\preceq$ represents the partial-ordering between elements; $\ell_1 \preceq \ell_2$ indicates that information can flow from $\ell_1$ to $\ell_2$.

Information flows can be broadly classified as explicit and implicit [Denning and Denning 1977; Goguen and Meseguer 1982]. Explicit flows arise as a result of variable assignments. For instance, the statement $x = y + z$ causes an explicit flow of values in $y$ and $z$ to $x$. Implicit flows arise from control structures in the program. For example, in the program $l = false; if(h)\{l = true; \}$, there is an implicit flow of information from $h$ to the final value of $l$ (it is true iff $h$ is true). Implicit flows are handled by maintaining a $pc$ label (program-context label), which is an upper bound on the labels of all the predicates that have influenced the control flow thus far. In the previous example, the $pc$ label inside the branch is the label of $h$.

Information flow control systems aim to prevent leaks through these flows by either statically enforcing information flow typing rules and ruling out insecure programs at compile-time or dynamically monitoring programs and aborting the execution of insecure programs. In both systems, assignment to a variable is disallowed if either the $pc$ label or the join of the label of the operands is not less than the label of the variable being assigned [Austin and Flanagan 2009; Volpano et al. 1996]. Thus, in the above examples, if either the label of $y$ or $z$ is greater than the label of $x$ or the label of $h$ is greater than the label of $l$, the assignment does not type-check or the execution aborts at runtime. This guarantees a variant of noninterference, known as termination-insensitive noninterference [Volpano et al. 1996], which we prove for our gradual type system. We assume that an adversary cannot observe or gain any information if a program’s execution diverges or aborts, and can only observe “public” values output by the program.

3 GRADUAL SECURITY TYPING

Static information flow type systems do not scale up to scenarios where the security levels of some of the variables are not known at compile-time while dynamic approaches are not amenable to providing compile-time guarantees. Gradual typing extends the reach of type-system based analysis by adding an imprecise security label or a dynamic label, $\?$, for variables whose labels are not known at compile-time. The runtime semantics then ensures that no information is leaked due to the relaxation of the type-system for handling variables with $\?$ label.

3.1 Imprecise Security Label: Interpretations and Operations

The label $\?$ is not an actual element of the security lattice and its meaning is not universally agreed upon. Differences will manifest in the runtime monitoring semantics and proof of noninterference. For illustration, consider a variable $x$ of type $\text{int}^\?$. Semantically, in the literature $\?$ has meant one of the following. (1) $x$’s label is dynamic (flow-sensitive) and can change at runtime (e.g., from $\?$ representing $L$ to representing $H$ when $x$ is assigned a secret). (2) the set of possible labels for $x$ is refined at runtime and the set in a future state will be a subset of its the current state. Since we build on top of a flow-insensitive type system, we opt for the second meaning of $\?$. Our runtime monitor will keep track of the set of possible labels for $x$. Note that a typical flow-sensitive IFC monitor (e.g. Austin and Flanagan [2009]; Russo and Sabelfeld [2010]) takes an approach aligned with (1).

In the initial program state, $\?$ could be interpreted as: (1) $x$ could contain a secret or be observable to an adversary. Therefore, we should treat $x$ conservatively as if it is both secret and public. (2) $\?$ indicates indifference; the data $x$ contains in the initial state is of no security value; otherwise, $x$ should have been given the label $H$. We choose (2) again, as it is a cleaner interpretation. Note that this is only for the initial state. At runtime, the monitor maintains enough state to concretely know whether $x$ contains a secret or not.
A desirable formal property for gradual type systems is gradual guarantee, proposed by Siek et al. [2015]. Gradual guarantee relates programs that differ only in the precision of the type annotations. It states that changes that make the annotations of a gradually typed program less precise should not change the static or dynamic behavior of the program. In other words, if a program with more precise type annotations is well-typed in the static type system, and terminates in the runtime semantics, then the same program with less precise terms is also well-typed, and terminates, respectively.

For illustration, consider the previous example from Listing 1. Assume that the program is well-typed under a gradual type system with \( x : \text{bool}^H \) and \( y : \text{bool}^H \) as secret variables, and \( z : \text{bool}^L \) as a public variable. When \( x \) is \( \text{true} \), the branch on line 2 is taken and \( y \) is assigned the value \( \text{true} \) and has the label \( H \). When \( x \) is \( \text{false} \), \( y \) remains \( \text{false} \). This program is accepted by the security type system and dynamic information flow monitor, and runs to completion in all possible executions. As per the static gradual guarantee, the program should also be well-typed if, for instance, \( y \) had an imprecise \( ? \) security level as shown in Listing 2. As per the dynamic gradual guarantee, the program in Listing 2 should run to completion at runtime, even with the imprecise label for \( y \) in all executions of the program.
3.4 Implicit Flows vs. Dynamic Gradual Guarantee

The example in Listing 4 illustrates the tension between noninterference and dynamic gradual guarantee. Assume that $x : \text{bool}^H$ is a secret variable and $z : \text{bool}^L$ is a public variable. The security type of $y$ is unknown at compile-time, i.e., $y : \text{bool}^?$. Consider two runs of the program with different initial values of $x$. When $x$ is $\text{true}$, the branch on line 3 is taken and $y$ is assigned the value $\text{false}$ with some label. As $y$ is $\text{false}$, the branch on line 4 is not taken and $z$ remains $\text{true}$. When $x$ is $\text{false}$, the branch on line 3 is not taken and $y$ remains $\text{true}$ with the imprecise label $?$. As the $pc$ is $?$ and is possible to be $L$, the assignment on line 4 succeeds, and $z$ becomes $\text{false}$. Thus, in the two runs of the program, different values of $z$ are output for different values of $x$, thereby leaking the value of $x$ to the adversary. This is akin to the problem where dynamic information flow monitors are not able to reason about alternate branches, which results in aborting the program execution when the assignment on line 3 is being made (similar to the no-sensitive-upgrade check proposed by Austin and Flanagan [2009]).

This program can be rejected by deploying a special monitoring rule that preemptively aborts an assignment statement if there is possibility that the $pc$ is not lower than or equal to the variable’s label, as explained in Toro et al. [2018]. In the above example, the assignment on line 3 will be aborted, because the $pc$ is $H$, and $y$’s label could be $\bot$ or $L$, which might leak information. This ensures noninterference, but unfortunately, the extra check is not closed under policy weakening. That is, enlarging the set of possible labels for $y$ will cause the monitor to abort, which contradicts the dynamic gradual guarantee.

To tackle this problem, we leverage the static phase of the gradual type system to determine the set of variables being written to in different branches and loops, and refine their possible security labels to get a more permissive analysis that helps us recover the dynamic gradual guarantee. In other words, if the branch on line 2 is taken in Listing 2, we know that $y$ will be written to, therefore, we narrow the possibility of the labels for $y$ to $\{H, \top\}$ as the first step of executing the if statement, regardless of whether $x$ is true or false. This is very similar to how hybrid monitors stop implicit leaks [Moore and Chong 2011; Russo and Sabelfeld 2010]. We will discuss this further in Section 4.4.

4 A LANGUAGE WITH GRADUAL INFORMATION FLOW TYPES

In this section, we first present the syntax and typing rules for our language with gradual information flow types ($\text{WHILE}^G$). We then extend $\text{WHILE}^G$ to $\text{WHILE}^{\text{Evd}}$ by augmenting the syntax with evidences for gradual labels and define the translation from $\text{WHILE}^G$ to $\text{WHILE}^G_{\text{Evd}}$. Finally, we present the operational semantics for our monitor that can prevent implicit leaks.

4.1 Syntax and Typing Rules for $\text{WHILE}^G$

The syntactic constructs of $\text{WHILE}^G$ are summarized in Figure 1. Gradual types, $U$, consist of a type (bool, or int) and a gradual security label, $\ell$. This label is either a static security label, denoted $\ell$; or an imprecise dynamic label, denoted $?$. As is standard, $\ell$ is drawn from $\text{Labs}$, a set of labels, which are part of a security lattice $L = (\text{Labs}, \leq)$. Here $\leq$ is a partial order between labels in $\text{Labs}$. We commonly use the label $H$ to indicate secret, $L$ to indicate public data, and $L \leq H$.

As is common in gradual typing, the partial-ordering ($\leq$) and join operation ($\gamma$) on security labels ($\ell$) extends to consistent ordering ($\leq_c$) and consistent-join ($\gamma_c$) to account for $?$, as shown in Figure 2. The consistent subtyping relation is written as $\tau_? \leq_c \tau_\ell$.

A value is a raw value with a gradual label. Raw values are integer constants $n$, or boolean values. Expressions include values, variables, cast, and binary operations on expressions. Commands include skip, sequencing, assignments, branches, loops, and outputs. This language does not have
higher-order stores, so the left-hand side of the assignment is always a global variable. The output
command outputs a value at a fixed security label \( \ell \). We include this command mainly to have a

Fig. 1. Syntax for the language WHiLE\(^G\)

Fig. 2. Operations on gradual labels and types

Fig. 3. Typing rules for WHiLE\(^G\)
clear statement of the system’s observable behavior, so we do not allow output at the imprecise label \( \cdot \).

\( \text{WHILE}^G \) has two main typing judgments. The first types an expression \( e \) at type \( U \) given a typing context for variables: \( \Gamma \vdash e : U \). The second judgment types a command \( c \) given the typing context \( \Gamma \) and a \( pc \)-label: \( \Gamma ; g_{pc} \vdash c \). The \( pc \)-label \( g_{pc} \) is a gradual label. Typing rules for expressions and commands are summarized in Figure 3. \( \text{VAR} \) types the variables. Binary operations on expressions are typed by \( \text{Bop} \), which types the resulting expression with the join of the gradual labels. \( \text{CAST} \) casts an expression of type \( \tau^g \prime \) to the type \( \tau^g \) if the two labels satisfy consistent ordering. Note that the consistent subtyping relation is not transitive.

The rules for commands are pretty standard except that we check for the ordering of gradual labels instead of the actual labels. Thus, both the assignment and the output rule check for the labels of the variables to be higher than the current \( pc \), and the label of the operands and the channel, respectively.

4.2 \( \text{WHILE}^G_{\text{Evd}} \)

Recall that examples in Section 2 use a set of possible security labels for preventing information leaks. These are \( \text{evidences} \) attesting to the validity of gradual labels. We use an interval of labels, representing the least and the highest possible label, instead of tracking all possible labels as the refinement only narrows the interval of labels, similar to [Toro et al. 2018].

There are two types of evidences—one is a label-interval, \( \iota \), that justifies the imprecise dynamic label \( \cdot \); and a pair of intervals or the cast-evidence, \( E = (\iota_1, \iota_2) \) that justifies the consistent-subtyping relation between gradual types for casting an expression from one type to another. Intuitively, \( \iota \) represents the range of possible static labels that would allow the program to type-check. For static labels \( \ell \), the evidence is given as \( [\ell, \ell] \). An interval \( [\ell_1, \ell_r] \) is valid iff \( \ell_1 \preceq \ell_r \). In rest of the paper, we consider only those intervals that are valid. Any operation leading to invalid intervals is aborted or leads to divergence.

The syntax of \( \text{WHILE}^G_{\text{Evd}} \) is shown in Figure 4. A value in \( \text{WHILE}^G_{\text{Evd}} \) is a raw value with an interval of possible security labels with the gradual label. An explicit type cast is written \( E^g e \), where \( E \) is the evidence justifying the type cast and \( g \) is the label of the resulting type. The assignment command includes the label-interval of the expression to be written while the label-interval in the output command is that of the output expression. Both if and while commands include a label-interval, which is the initial evidence of their respective predicates. To prevent implicit leaks, we include a \( \text{write-set} \) of variables, \( X \), which takes into account variable writes in both conditional branches and the loop body.

**Label-interval operations.** We first define functions and operations on the label-intervals that are used by the typing rules and semantics (shown in Figure 5). The function \( \gamma(g) \) returns the
maximum possible label-interval for the gradual-label \( g \), assuming \( \perp \) and \( \top \) are the least and the greatest element in the lattice, respectively. Label-intervals form a lattice with the partial ordering defined as \( t_1 \subseteq t_2 \). Here, \( t_1 \) is said to be more precise than \( t_2 \). The label-intervals are refined throughout the execution of the program; i.e., getting more precise. For illustration, consider the example in Listing 3. Assume that the security lattice contains two elements \( L \) and \( H \) such that \( L \not\preceq H \). Initially, \( y \) has a label \( \ell \) with the evidence \([\ell, \ell]\) indicating that any of the two labels are possible. If \( x : \text{bool}^{12} \), then the only possible label of \( y \) that allows assignment on line 3 is \( H \). Thus, the evidence of the label of \( y \) is refined to \([H, H]\), which makes the program-context’s evidence on line 4 \([H, H]\) disallowing assignments to \( L \)-labeled variables.

We define \( t_1 \preceq t_2 \) to mean for every security label in \( t_1 \), there is a label in \( t_2 \), higher or in equal position in the security lattice; and for every security label in \( t_2 \), there is a label in \( t_1 \), lower or in equal position in the security lattice. Even though we do not use the relation directly in our typing rules or semantics, this is an invariant that must hold on the results of a set of the binary label-interval operations, necessitated by the noninterference proofs. The function \( \text{refine}(t_1, t_2) \) returns the largest sub-intervals of \( t_1 \) and \( t_2 \), \( t_1' \) and \( t_2' \), respectively, such that \( t_1' \preceq t_2' \). If the relation does not hold between \( t_1' \) and \( t_2' \), the function returns \( \text{undef} \) to abort the execution.

The join of label-intervals is defined as \( t_1 \sqcup t_2 \). Note that this is not to be confused with the join operation in the lattice that the intervals form. The join of the label-intervals computes the interval corresponding to all possible joins of security labels in those intervals.

### Evidence-based consistent subtyping

Next, we define consistent subtyping relations for both gradual labels and types as supported by label-intervals. The consistent subtyping relation between two gradual types is written as \( (t_1, t_2) \vdash \tau^{g_1} \preceq_c \tau^{g_2} \) (defined in Figure 5). In the relation, \( t_1 \), resp. \( t_2 \) represents the set of possible labels for \( g_1 \), resp. \( g_2 \), and \( g_1 \preceq_c g_2 \). The evidence \((t_1, t_2)\) is to justify the consistent security label partial ordering relation between the labels of the gradual types. Note that we do not have \( t_1 \preceq t_2 \) in the premise. The reason is that \((t_1, t_2) \vdash \tau^{g_1} \preceq_c \tau^{g_2} \) is used to type
### Typing rules for **WHILE**

Here are the rules for typing expressions and commands in **WHILE** with evidence insertion.

#### **WHILE**

| Rule | Description |
|------|-------------|
| $\Gamma \vdash e : U$ | $t \equiv y(g)$ |
| $\Gamma \vdash (t \ b)^g : \text{bool}^g$ | **G-BOOL** |
| $\Gamma \vdash (t \ n)^g : \text{int}^g$ | **G-INT** |
| $\Gamma \vdash x : \Gamma(x)$ | **G-VAR** |
| $\forall i \in \{1, 2\}, \Gamma \vdash e_i : \tau^{g_i}$ | $g = g_1 \gamma g_2$ |
| $\Gamma \vdash e_1 \ \text{bop} \ e_2 : \tau^g$ | **G-BOP** |
| $\Gamma ; t_{pc} \ g_{pc} + c$ | **G-SKIP** |
| $\Gamma ; t_{pc} \ g_{pc} + \text{skip}$ | **G-SEQ** |
| $\Gamma \vdash e : \tau^g$ | $g \preceq \ell$ |
| $\Gamma ; t_{pc} \ g_{pc} + \text{output}(\ell, e)$ | **G-OUT** |
| $\Gamma \vdash e : \text{bool}^g$ | $t_e = y(g)$ |
| $\Gamma ; t_{pc} \ X = \text{WtSet}(c)$ | **G-WHILE** |
| $\forall i \in \{1, 2\}, \Gamma ; t_{pc} \ Y \ t_e \ g_{pc} \gamma g + c_i$ | **G-IF** |

#### **WHILE** with evidence

| Rule | Description |
|------|-------------|
| $\Gamma \vdash e : \text{U}^g$ | $t \equiv y(g)$ |
| $\Gamma \vdash (t \ b)^g : \text{bool}^g$ | **G-BOOL** |
| $\Gamma \vdash (t \ n)^g : \text{int}^g$ | **G-INT** |
| $\Gamma \vdash x : \Gamma(x)$ | **G-VAR** |
| $\forall i \in \{1, 2\}, \Gamma \vdash e_i : \tau^{g_i}$ | $g = g_1 \gamma g_2$ |
| $\Gamma \vdash e_1 \ \text{bop} \ e_2 : \tau^g$ | **G-BOP** |
| $\Gamma ; t_{pc} \ g_{pc} + c$ | **G-SKIP** |
| $\Gamma ; t_{pc} \ g_{pc} + \text{skip}$ | **G-SEQ** |
| $\Gamma \vdash e : \tau^g$ | $g \preceq \ell$ |
| $\Gamma ; t_{pc} \ g_{pc} + \text{output}(\ell, e)$ | **G-OUT** |
| $\Gamma \vdash e : \text{bool}^g$ | $t_e = y(g)$ |
| $\Gamma ; t_{pc} \ X = \text{WtSet}(c)$ | **G-WHILE** |
| $\forall i \in \{1, 2\}, \Gamma ; t_{pc} \ Y \ t_e \ g_{pc} \gamma g + c_i$ | **G-IF** |

#### Typing rules for **WHILE**

Typing rules for expressions and commands with evidences are summarized in Figure 6. G-Cast allows assigning an expression of type $U_1$ the type $U_2$ if the cast-evidence $E$ shows that $U_1$ is a consistent sub-type of $U_2$.

We augment the command typing with an interval for the gradual $pc$ label; $t_{pc}$ is the range of possible static labels for the $g_{pc}$. The rules are similar to the ones in the original type-system except for the use of evidence for consistent ordering between the gradual labels. A notable exception is the use of $\text{WtSet}(c)$ that returns the set of variables being updated or assigned to in the command $c$. The definition for $\text{WtSet}$ is a straightforward inductive definition over the structure of $c$ and is shown in the supplementary material.

#### 4.3 From **WHILE** to **WHILE**

Figure 7 shows the rules for translating expressions and commands in **WHILE** to **WHILE** with evidence insertion. T-Bool and T-Int augment a raw value with an initial label-interval based on their gradual label. T-Cast inserts a cast-evidence that justifies the ordering between the gradual labels of the old and new type. T-Bool and T-Seq inductively define the translation on sub-expressions and sub-commands. The rules T-Assign and T-Out annotate the respective operations with a label-interval, which represents the set of possible labels based on the gradual
label of the expression to be assigned or output, while T-If and T-While augment the respective commands with the interval corresponding to the gradual labels of their respective predicates. Additionally, they insert a write-set X that includes the set of all variables that might be written to in both the branches and the loop body.

We prove the following theorem stating that any well-typed term in \textit{WHILE} can be translated to another well-typed term in \textit{WHILE}$_{\text{Evd}}$ The proof is included in the supplementary material.

**Theorem 1.** If $\Gamma; g_{pc} \vdash c$ and $\Gamma; g_{pc} \vdash c \leadsto c'$, then $\forall t_{pc} \subseteq \gamma(g_{pc}), \Gamma; t_{pc} g_{pc} \vdash c'$.

### 4.4 Operational Semantics

**Runtime constructs.** We first define additional runtime constructs for our semantics, shown below. The store, $\delta$, maps variables to values with their gradual labels and intervals. The gradual-labels of the variables are suffixed on the values for the purpose of evaluation.

\[
\begin{align*}
\text{Store} & : \delta :: = \cdot | \delta, x \mapsto \nu \\
\text{PC Stack} & : \kappa :: = \emptyset | (t_{pc} g_{pc}) | \kappa_1 \triangleright \kappa_2 \\
\text{Actions} & : \alpha :: = \cdot | (\ell, \nu) \\
\text{Commands} & : c :: = \cdots | \{c\} | \text{if } e \text{ then } c_1 \text{ else } c_2 \\
\text{Eval. Ctx.} & : C :: = [\cdot] | \ell, \text{output}(\ell, [\cdot]) | \text{if} [\cdot] \text{ then } c_1 \text{ else } c_2
\end{align*}
\]

![Fig. 7. Translation from \textit{WHILE} to \textit{WHILE}_{\text{Evd}}](image_url)

\[\begin{array}{ll}
\Gamma; \because e \leadsto e' : U & \\
\Gamma; e \vdash b^g \leadsto (b^g)^9 : \text{bool}^g & \text{T-BOOL} \\
\Gamma; e \vdash n^g \leadsto (n^g)^9 : \text{int}^g & \text{T-INT} \\
\Gamma; e \vdash x = : \tau^g & \text{T-VAR} \\
\forall i \in \{1, 2\}, \Gamma; \because e_i \leadsto e'_i : \tau^{g_i} & \\
\Gamma; e \vdash b \leadsto e'_1 \text{ bop } e'_2 : \tau^g & \text{T-BOP} \\
\Gamma; g_{pc} \vdash c \leadsto c' & \text{T-SKIP} \\
\Gamma; g_{pc} \vdash \text{skip} \leadsto \text{skip} & \text{T-SEQ} \\
\Gamma; g_{pc} \because x := e \leadsto x := e' & \text{T-ASSIGN} \\
\Gamma; g_{pc} \because x := e \leadsto \text{while } e \text{ do } c \leadsto \text{while }^X e' \text{ do } c' & \text{T-WHILE} \\
\Gamma; g_{pc} \because c \leadsto c' & \\
\Gamma; g_{pc} \because \gamma c \text{ bop } c \leadsto c' & \text{T-IF} \\
\end{array}\]
κ is a stack of pc labels, each of which is a gradual label, \( g_{pc} \), with the corresponding interval, \( t_{pc} \). The stack is used for evaluating nested if statements. The operation \( \kappa_1 \triangleright \kappa_2 \) indicates that \( \kappa_1 \) is on top of \( \kappa_2 \) in the stack. \( \alpha \) is an action, which may be silent or a labeled output. We add two runtime commands. \{c\} is used in evaluating if statements. The curly braces help the monitor to keep track of the scope of a branch. The if statement without the write set is used in an intermediate state for evaluating an if statement. We also define an evaluation context, which is standard.

**Additional label operations.** The function \( intvl(v) \) returns the label-interval of \( v \). Formally: \( intvl((i \ u)^g) = i \). Figure 8 defines operations on label-intervals used by the semantics. \( t_1 \triangleright \triangleright t_2 \) computes the intersection of the intervals \( t_1 \) and \( t_2 \). \( i' \triangleright (i \ u)^g \) merges the labels for a value. \( t \triangleright (t_1, t_2) \) refines \( t_2 \) based on the intersection of \( t \) and \( t_1 \).

**Expression monitoring semantics.** Our monitoring semantics for expressions is of the form \( \delta / e \downarrow e' \). Figure 9 presents key rules. As we mentioned before, when any of the interval computations fails to produce valid intervals, the monitor aborts the computation. Rules M-CONST, and M-VAR are standard. To perform a binary operation on two values, the operation is performed on the raw values, and the join of their associated intervals and gradual labels is assigned to the computed value. M-Cast refines a raw value’s interval according to the cast-evidence. If the refinement is not valid, the execution aborts (M-Cast-Err). Note that none of these operations modify the gradual label of the variable (the type of store locations remain the same); the operations only refine the intervals of the gradual label.
Fig. 10. Monitor semantics for commands.
Commands monitoring semantics. Our monitoring semantics for commands is summarized in Figure 10 and has the form $\kappa, \delta / c \xrightarrow{\alpha} \kappa', \delta' / c'$. Rules M-SEQ, and M-SEQ-ERR, and M-SKIP are standard. Rules M-Pc and M-Pop manage commands running in branches or loops. M-Pop pops the top-most pc-label from the stack, indicating the end of the branch or loop. We use braces around a command, \{c\} to indicate that c is executing in a branch or loop. Such a command is run taking into account only the specific branch’s pc-stack. When the command execution finishes, the braces are removed and the current pc-label is popped off the stack.

The assignment rule M-Assign updates the label-interval of the value being assigned according to the assignment’s context. The resulting interval is compared against the current pc, as well as the existing label-interval of the variable, to ensure that the assignment does not leak information. The interval operations can be found in Figure 11. The functions refineUB and refineLB refine a value’s label-interval to its greatest and least sub-interval based on the interval $\iota_c$. If either of these functions return an invalid interval, the execution aborts (M-Assign-Err).

We explain the assignment rule via examples. Consider a three-point lattice $L \sqsubseteq M \sqsubseteq H$, the following command $x := [H, H]^{5\gamma}$, and two stores $\delta_1 = x \mapsto [M, M]^{1\gamma}$ and $\delta_2 = x \mapsto [L, M]^{2\gamma}$. Assume the following current pc interval $\iota_{pc} = [L, H]$. Here, $\upsilon = [H, H]^{5\gamma}$. The second premise further refines the interval of $\upsilon$ to make sure that the pc context is lower than or equal to the interval of the value to be written. This is to prevent low assignment in a high context. For this example, refine([L, H], [H, H]) = ([L, H], [H, H]), so the intervals remain the same. Finally, we use the refined interval of the variable. This is to prevent storing a secret value to a low variable, but allow storing public data to a variable that holds secret. For $\delta_1$, $\upsilon'' = [H, H]^{5\gamma}$, so now $x$ stores a secret value with label $H$. For store $\delta_2$, refine([H, H], [L, M]) is not defined and the monitor aborts. Here, we prevent a secret from being written to a low variable. As we mentioned in Section 3.1, we do not taint the variable (allow $[H, H]^{5\gamma}$ to be written to $x$), as this is in contradiction with the fixed-label type system that we base our system on. M-OUT makes similar comparisons to ensure that the output is permitted.

Rule M-If is standard. The pc-label is determined by joining the current pc with the gradual label and interval of the branch-predicate’s value. Here, the pc-stack grows and the branch is placed in the scoping braces. The rule for while reduces it to if. Rule M-If-Refine is the key for preventing implicit leaks. We refine the intervals for variables in both branches according to the write set, $X,$
which contains the set of all variables being updated in either one of the two branches. Figure 11 includes the auxiliary definitions for refining the intervals of variables in a write-set. The function $rfl$ refines the label-interval of values in the store and is defined inductively while the function $\text{refineWSet}$ refine the intervals of the variables in the write-set to be at least as high as the lower label in the interval of the current $pc$. We show how this prevents implicit leaks using the classic example shown in Listing 4. When the functions $rfl$ or $\text{refineWSet}$ return under $\text{abort}$, the execution aborts. We omit the rules that transition to abort in the semantics defined further in the paper for clarity, and can be found in the supplementary material.

**Example** Below, we define two initial memories, $\delta_y$ maps $x$ to true and $\delta_f$ maps $x$ to false. Both $y$ and $z$ store true initially with $y$’s label being $?$, and $z$ being $L$.

\[
\begin{align*}
\delta_y &= y \mapsto [L, H]\text{true}^? \\
\delta_z &= z \mapsto [L, L]\text{true}^L \\
\delta_t &= x \mapsto [H, H]\text{true}^H \\
\delta_f &= x \mapsto [H, H]\text{false}^H \\
c_1 &= \text{if}(y) \text{ then } y := [L, H]\text{false}^? \text{ else skip} \\
c_2 &= \text{if}(z) \text{ then } z := [L, L]\text{false}^L \text{ else skip}
\end{align*}
\]

Below is the execution starting from the state where $x$ stores true.

\[
\begin{align*}
[L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{true}^?) \to c_1; c_2 \\
\to [L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{true}^?) \to \text{if } x \text{ then } y := [L, H]\text{false}^? \text{ else skip}; c_2 \\
\to [H, H] H \triangleright [L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{true}^?) \to \{y := [L, H]\text{false}^?\}; c_2 \\
\to [H, H] H \triangleright [L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{false}^?) \to \{\text{skip}\}; c_2 \\
\to [H, H] H \triangleright [L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{false}^?) \to \text{skip}; c_2 \\
\to [L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{false}^?) \to c_2 \\
\to \text{abort}
\end{align*}
\]

In the last step, $\text{refineWSet}$ fails, because the operation $\text{refine}([H, H], [L, L])$ produces an invalid label-interval.

Now let’s see the execution starting from the state where $x$ stores false.

\[
\begin{align*}
[L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{true}^?) \to c_1; c_2 \\
\to [L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{true}^?) \to \text{if } x \text{ then } y := [L, H]\text{false}^? \text{ else skip}; c_2 \\
\to [H, H] H \triangleright [L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{true}^?) \to \{\text{skip}\}; c_2 \\
\to [H, H] H \triangleright [L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{true}^?) \to \text{skip}; c_2 \\
\to [L, L] L, (\delta_t, \delta_z, y \mapsto [H, H]\text{true}^?) \to c_2 \\
\to \text{abort}
\end{align*}
\]

Notice that the label of $y$ is changed the same way as when we start the execution from $\delta_t$. Ultimately, the program aborts for the same reason.

## 5 NONINTERFERENCE

To prove noninterference, we extend our language to include pairs of values, expressions, and commands to simulate running two executions simultaneously which differ on secret values. This allows us to reduce our noninterference proof to a preservation proof [Pottier and Simonet 2002].

### 5.1 Paired Execution

**Syntax.** The augmented syntax with pairs is shown below. The store $\delta$ is extended to contain pairs of values. We also extend commands to be paired but do not allow pairs to be nested—
invariant maintained by our operational semantics. We only use pairs for values and commands
whose values and effects are not observable by the adversary (are “high” in other words).

\[
\begin{align*}
\text{Val} \quad v & := (t \ u) \mid \langle t_1 \ u_1 | t_2 \ u_2 \rangle \\
\text{Cmd} \quad c & := \cdots | \langle \kappa_1, t_1, c_1 | \kappa_2, t_2, c_2 \rangle
\end{align*}
\]

Pairs of commands are part of the runtime statement, generated as a result of evaluating a branching
statement. Each command represents an independent execution, capable of changing its own
\(pc\)-stack. As a result, we include local \(pc\)-stacks in the pair with each command. The rationale
behind additional \(pc\)-stacks in command pairs is explained with the semantics.

**Label-interval operations on pairs.** The interval of a paired value is a pair of intervals while
the intersection of an interval and a paired value is defined as \(I \bowtie (t_1, u_1 | t_2, u_2)\).

\[
\text{intvl}((t_1, u_1 | t_2, u_2)) = \langle t_1 | t_2 \rangle
\]

Other extensions to label-interval operations can be found in the supplementary material.

**Memory read and update operations.** As we allow the intervals of values to be refined, the
store read (\(\text{rd}_i\)) and update (\(\text{upd}_i\)) operations for paired values are a little involved, as shown in
Figure 12. \(\text{rd}_i \ v_1 \ v_n\) takes the value currently stored at a location as the first argument, the value
to be written to that location as the second argument and returns an updated value that will be
written to that location. The complication comes when \(i\) is 1 or 2, i.e., when only one execution is
updating the value. This function needs to make sure that the correct paired value is stored. The
\(\text{upd}_i\) operation does not take pairs as new-values, as these operations occur only in a specific run
after branching. We will give concrete examples when we explain the semantics for assignment.

**Operational semantics for pairs.** The operational semantics rules are augmented with an index,
\(i\). The judgments now are of the form \(\delta / i \parallel e'\) and \(\kappa, \delta / i \ c \rightarrow \kappa', \delta' / i \ c'\). The index \(i\) indicates
which branch of a pair is executing (when \(i \in \{1, 2\}\)) or if it is a top-level execution (when \(i\)
is omitted). Most of the rules can be directly obtained by adding the \(i\) to the monitor semantics
shown in Figures 9 and 10. Rules that deal with pairs, including read and write to the store need
to be modified. We explain important rule changes (shown in Figures 13 and 14).

Rule P-VAR uses the function \(\text{rd}_i \ v\) to retrieve the value indexed by \(i\) within \(v\). For example,
\(x \mapsto \langle v_1 | v_2 \rangle / i \ x \downarrow v_i\) when \(i \in \{1, 2\}\), and \(x \mapsto \langle v_1 | v_2 \rangle / x \downarrow \langle v_1 | v_2 \rangle\). To evaluate a cast over
a pair of values, we push the cast inside the pair (P-CAST).

Each command in the pair (P-C-PAIR) can make progress independently and the premise of the
rule is indexed by the corresponding \(i\). Here \(k_i\) is the \(pc\)-stack specific to \(c_i\). Consider a command
\(c = \langle c_1 | c_2 \rangle\), where both \(c_1\) and \(c_2\) have nested if statements. The execution of \(c\) will create different
\(k_1\) and \(k_2\) when executing \(c_1\) and \(c_2\). Next, \(i\) is the \(pc\)-label-interval demonstrating that \(c\) is supposed
to execute in a “high” context (unobservable by the adversary). The bottom \(pc\) in the stack is joined
with \(i\). We will come back to this point when explaining the typing rules.

Rule P-LIFT-IF lifts the pair that appears as branch conditions to generate a paired command. The
resulting commands on each side of the pair are determined by the value in the corresponding
side of the branch condition. The branching context \(i\) is the runtime interval of the branching
condition. Initial local \(pc\)-stack is empty.

Note that the individual branches do not contain pair of commands. To see how the semantics prevents nesting of command
pairs and how paired execution represents two runs with different
secrets, consider the program in Listing 5.

The only rule that generates paired command is P-LIFT-IF. Assume that \(a\) and \(b\) are variables containing paired-values such that

1. if \(a^H\) then
2. if \(b^H\) then
3. \(y^H := 1^H\)
4. else skip
5. else \(y^H := 2^H\)

Listing 5. Example program to explain branching on pairs
Value and interval projections:
\[ \langle i_1 \cdot u_1 \mid i_2 \cdot u_2 \rangle^g \]
\[ \langle i \cdot u \rangle^g \]
\[ \langle i_1 \mid i_2 \rangle_i = i_1 \mid i_2 \]
where \( i = \{1, 2\} \)

Read operations:
\[ \text{rd } v = v \]
\[ \text{rd}_1 v = [v]_1 \]
\[ \text{rd}_2 v = [v]_2 \]

Write operations:
\[ \text{intvl}(v_o) = \langle i_1 \mid i_2 \rangle \] or \( v_n = \langle v_1 \mid v_2 \rangle \)
\[ \forall i \in 1, 2, \quad [v_n]_i = (i'_i \cdot u_i)^g \]
\[ \text{refine}(i'_i, i) = (\_, i'_i)^g \]
\[ \text{upd} v_o v_n = \langle i''_1 u_1 \mid i''_2 u_2 \rangle^g \]
\[ \text{lab}(v_o) = g \]
\[ \text{refine}(i_n, i_o) = (\_, i'_o)^g \]
\[ \text{upd}_1 v_o (i_n u_n)^g = \langle i'_o u_n \mid i_2 u_2 \rangle^g \]
\[ \text{upd}_2 v_o (i_n u_n)^g = \langle i_1 u_1 \mid i'_o u_n \rangle^g \]

Binary operations:
\[ v_1 = \langle i_1 u_1 \mid i'_1 u'_1 \rangle^g \]
\[ v_2 = \langle i_2 u_2 \mid i'_2 u'_2 \rangle^g \]
\[ i = (i_1 \Rightarrow i_2) \]
\[ i' = (i'_1 \Rightarrow i'_2) \]
\[ u = (u_1 \text{ bop } u_2) \]
\[ u' = (u'_1 \text{ bop } u'_2) \]
\[ g = g_i \text{ \& g}_j \]
\[ v = \langle i u \mid i' u' \rangle^g \]
\[ v_1 \text{ bop } v_2 = v \]

Cast:
\[ v = \langle i u \rangle^g \]
\[ i' = i \Rightarrow E \]
\[ v = \langle i_1 u_1 \mid i_2 u_2 \rangle^g \]
\[ \forall i \in \{1, 2\}, \quad i'_i = i_i \Rightarrow E \]
\[ v' = \langle i u \rangle^g \]
\[ \text{upd} v''_o v = \langle v''_o \rangle^g \]

Fig. 12. Operations with pairs (the rules for error conditions are included in the supplementary material)

Expression Semantics:
\[ \delta / i e \Downarrow \! \! \! v \]
\[ \delta / i \langle i u \rangle^g \Downarrow \langle i u \rangle^g \quad \text{P-CONST} \]
\[ \delta / i x \Downarrow \! \! \! \text{rd}_1 \delta(x) \quad \text{P-VAR} \]
\[ \delta / i e \Downarrow \! \! \! v \]
\[ v' = \langle E, g' \rangle \Downarrow v \]
\[ \delta / i E^g e \Downarrow \! \! \! v' \quad \text{P-CAST} \]
\[ \delta / i e_1 \Downarrow \! \! \! v_1 \]
\[ \delta / i e_2 \Downarrow \! \! \! v_2 \quad \text{P-BOP} \]

Fig. 13. Operational semantics for expression evaluation in paired executions

\[ a = \langle u_1 \mid u_2 \rangle^H \] and \( b = \langle u_3 \mid u_4 \rangle^H \), meaning both \( a \) and \( b \) contain secrets and \( u_1 \) and \( u_3 \) are values for the first execution and \( u_2 \) and
Command Semantics: $\kappa, \delta / i \ c \xrightarrow{a} \kappa', \delta' / i \ c'$

$$
\begin{align*}
\kappa_i \triangleright t_{pc} \; \gamma \; t_1 \; g_{pc} \; \gamma_c \; g, \delta / i \; c_i \xrightarrow{a} \kappa'_i \triangleright t_{pc} \; \gamma \; t_1 \; g_{pc} \; \gamma_c \; g, \delta' / i \; c'_i \\
c_j = c'_j \quad \kappa_j = \kappa'_j \quad \{i, j\} = \{1, 2\} && \text{P-C-PAIR}
\end{align*}
$$

$$
\begin{align*}
t_{pc} \; g_{pc}, \delta / \langle \kappa_1, t_1, c_1 | \kappa_2, t_2, c_2 \rangle_g \xrightarrow{a} t_{pc} \; g_{pc}, \delta' / \langle \kappa'_1, t_1, c'_1 | \kappa'_2, t_2, c'_2 \rangle_g \\
i = \{1, 2\} \quad c_j = c_1 \text{ if } u_1 = \text{false} \quad c_k = c_1 \text{ if } u_2 = \text{true} \quad c_k = c_2 \text{ if } u_2 = \text{false} && \text{P-LIFT-IF}
\end{align*}
$$

$$
\begin{align*}
t_{pc} \; g_{pc}, \delta / \langle \theta, t_1, \text{skip} | \theta, t_2, \text{skip} \rangle_g \xrightarrow{} t_{pc} \; g_{pc}, \delta / \text{skip} && \text{P-SKIP-PAIR}
\end{align*}
$$

$$
\begin{align*}
\delta / i \; e \Downarrow \upsilon \quad \upsilon' = \text{refineUB}(t_{pc}, \upsilon) && \text{P-ASSIGN}
\end{align*}
$$

$$
\begin{align*}
\kappa, \delta / i \; c_1 \xrightarrow{a} \kappa', \delta' / i \; c'_1 && \text{P-SEQ}
\end{align*}
$$

$$
\begin{align*}
\kappa, \delta / i \; c_1; c_2 \xrightarrow{a} \kappa', \delta' / i \; c'_1; c'_2 && \text{P-Pop}
\end{align*}
$$

$$
\begin{align*}
\iota g \triangleright \kappa, \delta / i \{\text{skip}\} \xrightarrow{} \kappa, \delta / i \text{skip} && \text{P-Skip}
\end{align*}
$$

$$
\begin{align*}
\delta / i \; e \Downarrow \upsilon \quad \upsilon' = \text{refineWSet}_i(\delta, X, t_{pc}, \upsilon) && \text{P-IF-REFINE}
\end{align*}
$$

$$
\begin{align*}
t_{pc} \; g_{pc}, \delta / i \text{ if } \ell^X \text{ e then } c_1 \text{ else } c_2 \xrightarrow{} t_{pc} \; g_{pc}, \delta' / i \text{ if } \upsilon \text{ then } c_1 \text{ else } c_2
\end{align*}
$$

$$
\begin{align*}
\delta / i \; e \Downarrow \upsilon \quad \upsilon' = \text{refineLB}(\upsilon, [\ell, \ell]) \quad \upsilon_1 = \text{refineUB}(t_{pc}, \upsilon') \quad \text{P-OUT}
\end{align*}
$$

$$
\begin{align*}
t_{pc} \; g_{pc}, \delta / i \text{ output}(\ell, e) \xrightarrow{(i, \ell, \upsilon_1)} t_{pc} \; g_{pc}, \delta / i \text{ skip}
\end{align*}
$$

$$
\begin{align*}
\iota'_c = t_{pc} \; \gamma \; t \quad \ell_{pc}' = g_{pc} \; \gamma_c \; g \\
c_j = c_1 \text{ if } b = \text{true} \quad c_j = c_2 \text{ if } b = \text{false} && \text{P-IF}
\end{align*}
$$

$$
\begin{align*}
t_{pc} \; g_{pc}, \delta / i \text{ if } (i \ b)^g \text{ then } c_1 \text{ else } c_2 \xrightarrow{} t_{pc} \; g_{pc}, \delta' / i \{c_j\} \quad \text{P-WHILE}
\end{align*}
$$

Fig. 14. Operational semantics for paired executions (the rules that result in an “abort” or diverge are included in the supplementary material)

$u_4$ are for the second. We ignore the intervals in this example for simplicity of exposition. On line 1, we use the P-LIFT-IF rule since we branch on a pair of values to create paired commands. In the first execution, if $u_1 = \text{true}$, we take the then branch. When evaluating $b$ inside the branch we take the first part of the pair using the expression-evaluation rules and $\text{rd}_i$ operation (Figure 12)
for $i = 1$, i.e., $rd_1 b = [b]_1 = u_3$. Thus, the branch on line 2 becomes: if $u_3$ then . . . while the remaining parts remain the same. Similarly in the second execution, based on the value of $u_2$, either the then branch or the else branch is chosen. If the then branch is chosen, the branch on line 2 becomes if $u_4$ then . . . as we are in the second execution of the branch ($i = 2$) on line 1 and $rd_2 b = [b]_2 = u_4$. It is enough to generate two different runs of the program for reasoning about noninterference, which is what the projection semantics do.

Local $pc$ refinements in pairs are forgotten when both sides of the pair finish executing in P-Skip-Pair. This is similar to the P-Pop rule where the $pc$ for the branch or loop is forgotten.

Rule P-Assign deals with the complexity of pairs updating the store in one branch with the helper function $\text{upd}_j v_0 v_n$ (defined in Figure 12). The refinement of labels of store updates is the same as the monitor semantics. When the update comes from a specific branch of execution ($i \in \{1, 2\}$), the value for the other branch should be preserved. If the value in the store is already a pair, only the $i^{th}$ sub-expression is updated. Reconsider the example in Listing 5. The assignment on line 3 happens in either of the two branches, or both the branches depending on the values of $u_1$ to $u_4$. If it happens in only the first projection, the first part of the value-pair in $y$ is updated. Suppose that $y = \langle [\bot, T]_0 \mid [\bot, T]_2 \rangle^2$, initially, and $u_1 = u_3 = \text{true}$. Then, the value of $y$ after the assignment on line 3 becomes $y = \langle [H, T]_1 \mid [\bot, T]_2 \rangle^2$. If $u_2 = \text{false}$, then the else branch is taken, and at the end of the assignment on line 5 the value of $y$ is updated to $y = \langle [H, T]_1 \mid [H, T]_2 \rangle^2$. (Note that the first part of the pair already contains the value and the label-interval evaluated through the then branch as we evaluate the two runs one after the other when branching on a pair of values).

If the store value is not a pair, the value becomes a pair with the $i^{th}$ sub-expression the updated value, and the other sub-expression the old value. Considering the same example as above, if initially $y = \langle [\bot, T]_2 \rangle^2$, then at the end of then branch with $u_1 = \text{true}$, the updated value of $y$ is $y = \langle [H, T]_1 \mid [\bot, T]_2 \rangle^2$. When updates happen at the top-level, the entire value in the store should be updated. The first rule applies when either the old or the new value is a pair and the second rule applies when none of them are pairs. Note that this update rule is the reason why the intervals in a pair may differ.

The output rule is the same except that the event being output now includes the index to aid the statement and proof of noninterference. The P-If-Refine rule uses an augmented version of refineWSet, which only refines label-intervals for the $i^{th}$ branch as shown in Figure 15. These rules are included in the supplementary material along with the auxiliary functions they use.

5.2 Semantic Soundness and Completeness

To connect the semantics of the extended language with pairs to the monitor semantics, we prove soundness and completeness theorems. These theorems depend on projections of the store, expression- and command-configurations. Similar to the value projection seen in the previous section, the goal of these projections is to obtain one execution from a paired execution.

The projection of a paired value, a paired interval, a normal value and interval are defined in Figure 12. The projection of stores ($\delta$) and traces ($\top$) is inductively defined as shown in Figure 16. The projection function only keeps the output events produced by the execution of concern and ignores output performed by the other execution. The projection function for expression configurations is $[\delta \mathbin{/} e]_i = [\delta]_i \mathbin{/} e$ and for command configurations is defined in Figure 16. The interesting case is the projection of a command pair. We reassemble the $pc$-stack and wrap $c_i$ with curly braces to reflect the fact that these pairs only appear in an if branch.

The Soundness theorem ensures that if a configuration can transition to another configuration, then its projection can transition to the projection of the resulting configuration, generating the
same trace modulo projection. The Completeness theorem ensures that if both the projections of a configuration terminate, then the configuration terminates in an equivalent state. We write, $\vdash \kappa, \delta \mid c \text{ wf}$, to indicate that the configuration is well-formed (defined in the supplementary material). The formal soundness and completeness theorem statements are defined in Theorems 2 and 3. The proofs can be found in the supplementary material.

**Theorem 2 (Soundness).** If $\kappa, \delta \mid c \xrightarrow{T} \kappa', \delta' \mid c'$ where $\vdash \kappa, \delta \mid c \text{ wf}$, then $\forall i \in \{1, 2\}$, $[\kappa, \delta \mid c]_i \xrightarrow{T} [\kappa', \delta' \mid c']_i$
A gradual label is said to be "high" w.r.t an attacker, if the lower label in the interval is not lower than the attacker’s lower label. Before we explain the typing rules for the extended configuration, we define another label relation.

### 5.3 Preservation

Before we explain the typing rules for the extended configuration, we define another label relation. A gradual label is said to be “high” w.r.t an attacker, if the lower label in the interval is not lower than the attacker’s lower label.

#### Store projection:

\[
[-]_i = \cdot 
\]

\[
[\delta, x \mapsto v]_i = [\delta]_i, x \mapsto [v]_i 
\]

#### Trace projection:

\[
[-]_i = \cdot 
\]

\[
[T, (j, \ell, v)]_i = [T]_i, (\ell, [v]_i) \quad \text{if } i = j 
\]

\[
[T, (j, \ell, v)]_i = [T]_i \quad \text{if } i \neq j 
\]

#### Command-configuration projection:

\[
[t_{pc} g_{pc}, \delta / \text{skip}]_i = t_{pc} g_{pc}, [\delta]_i / \text{skip} 
\]

\[
[t_{pc} g_{pc}, \delta / x := e]_i = t_{pc} g_{pc}, [\delta]_i / x := e 
\]

\[
[k, \delta / c]_i = k', \delta' / c'_i 
\]

\[
[k \triangleright t_{pc} g_{pc}, \delta / \{c\}_i = k' \triangleright t_{pc} g_{pc}, \delta' / \{c'\} 
\]

\[
[t_{pc} g_{pc}, \delta / \text{output}(\ell, e)]_i = t_{pc} g_{pc}, [\delta]_i / \text{output}(\ell, e) 
\]

\[
\forall \{i, j\} \in \{1, 2\}, c'_i = \begin{cases} \text{skip} & \text{if } c_i = \text{skip and } c_j \neq \text{skip} \\ \{c_i\} & \text{else} \end{cases} 
\]

\[
[t_{pc} g_{pc}, \delta / \langle \kappa_1, t_1, c_1 \mid \kappa_2, t_2, c_2 \rangle_g]_i = k_i \triangleright (t_{pc} \gamma t_1) (g_{pc} \gamma c g) \triangleright t_{pc} g_{pc}, [\delta]_i / c'_i 
\]

\[
[t_{pc} g_{pc}, \delta / \langle \emptyset, t_1, \text{skip} \mid \emptyset, t_2, \text{skip} \rangle_g]_i = k_i \triangleright (t_{pc} \gamma t_1) (g_{pc} \gamma c g) \triangleright t_{pc} g_{pc}, [\delta]_i / \{\text{skip}\} 
\]

\[
[t_{pc} g_{pc}, \delta / \text{if } (t_1 u_1 \mid t_2 u_2)^g \text{ then } c_1 \text{ else } c_2]_i = t_{pc} g_{pc}, [\delta]_i / \text{if } (t_1 u_1)^g \text{ then } c_1 \text{ else } c_2 
\]

\[
[t_{pc} g_{pc}, \delta / \text{if } X e \text{ then } c_1 \text{ else } c_2]_i = t_{pc} g_{pc}, [\delta]_i / \text{if } X e \text{ then } c_1 \text{ else } c_2 
\]

\[
[t_{pc} g_{pc}, \delta / \text{while } X e \text{ do } c]_i = t_{pc} g_{pc}, [\delta]_i / \text{while } X e \text{ do } c 
\]

**Theorem 3 (Completeness).** If \(\forall i \in \{1, 2\}, [\kappa, \delta / c]_i \xrightarrow{T_i} \kappa_i, \delta_i / \text{skip and } t_{pc} \triangleright \kappa, \delta / c \text{ wf}, \) then \(\exists \kappa', \delta' / c \xrightarrow{T} \kappa', \delta' / \text{skip with } [\kappa', \delta' / \text{skip}]_i = \kappa_i, \delta_i / \text{skip and } T_i = [T]_i.\)

#### 5.3 Preservation

Before we explain the typing rules for the extended configuration, we define another label relation. A gradual label is said to be “high” w.r.t an attacker, if the lower label in the interval is not lower than the attacker’s lower label.
All the pair typing rules are parameterized over attacker’s label \( \ell_A \), which we omit from the rules for simplicity. The typing rule for value-pairs is shown below. The second premise checks that the interval is representative of the gradual type \( U \). The last premise checks that it is possible that \( U \)’s security label is high, meaning this pair of values is non-observable to the adversary.

\[
\forall i \in \{1, 2\}, \; \Gamma \vdash u_i : \tau^g \quad \forall i \in \{1, 2\}, \; \Gamma \vdash \gamma(g) \quad \forall i \vdash (g) \in H(\ell_A)
\]

R-V-PAIR

We type the runtime commands with pairs using judgement of the form \( \Gamma; \kappa \vdash_r c \). We list the rules for typing the commands in Figure 17.

Rule R-Pop types the inner command with pairs using judgement of the form \( \Gamma; \kappa \vdash_r c \). R-End directly uses command typing. For pairs, R-Seq first checks that each \( c_i \) is well-typing, using the \( pc \)-context assembled from the local \( pc \)-context. The second premise makes sure that these commands are typed (executed) in a high context. Here \( t_i \) is the witness for \( g \), which demonstrates that \( c_i \) are high commands. The sequencing statement types the second command using only the last \( pc \) on the stack because the execution order is from left to right. We can only encounter branches in the first part of a sequencing statement and not the second part before beginning the execution of the second command in the sequence. The typing rule for if statements without a write-set is straightforward.

We define store, trace and configuration typing in Figure 18. The store \( \delta \) types in the typing environment \( \Gamma \) if all variables in \( \delta \) are mapped to their respective type and gradual label in \( \Gamma \). We define top-level configuration typing as \( \vdash \kappa, \delta, c \). For the typing for traces and actions, the output value needs to be well-typed, and the label-interval of the value has to be lower than or equal to the channel label.

Using these definitions, we prove that our paired-execution semantics preserve the configuration typing and generate a well-typed trace. We write, \( \vdash \kappa, \delta, i, c \) for \( i \in \{\cdot, 1, 2\} \), to indicate that the configuration is safe. We say a configuration is safe if all of the following hold:

1. if \( i \in \{1, 2\} \), then \( \kappa \in H(\ell_A), \forall x \in WtSet(c), \intvl(\delta(x)) \in H(\ell_A) \)
We show that the gradual type system presented above satisfies termination-insensitive noninterference. The proofs are included in the supplementary material.

### 5.4 Noninterference

We show that the gradual type system presented above satisfies termination-insensitive noninterference. We start by defining equivalence for values and stores (Figure 19). Two values are said to be equivalent to an adversary at level $\ell_{A}$ if either they are both visible to the adversary and are the same, or both not observable by the adversary. The noninterference theorem states that given a program and two stores equivalent for an adversary at level $\ell_{A}$, if the program terminates in both the runs, then the $\ell_{A}$-observable actions on both the runs are same. We also define equivalence of traces w.r.t an adversary at level $\ell_{A}$ in Figure 19. We know that only when $\ell \not\equiv \ell_{A}$, the individual runs can produce $(i, \ell, v)$ because the two runs diverge only when branching on pairs. Similarly, $(\ell, \langle v_{1} \mid v_{2} \rangle)$ can only be produced if $\ell \not\equiv \ell_{A}$, because pairs can only be typed if each individual interval in the pair is high and rule P-OUT makes sure $\ell$ is lower than or equal to the pair’s interval. We prove a simple lemma that establishes that given a well-typed trace of actions $\Gamma \vdash [\ell_{A} \approx \ell \vdash [\ell_{A} \approx \ell \vdash c. \text{sf}]$, then $\vdash \ell \approx \ell_{A}$.

### Theorem 4 (Preservation).

If $\kappa, \delta / c \rightarrow^{+} \kappa', \delta' / c'$ with $\vdash \kappa, \delta, c$ and $\vdash \kappa, \delta / c \text{ sf}$, then $\vdash \kappa', \delta', c'$ and $\vdash T$

### Theorem 5 (Noninterference).

Given an adversary label $\ell_{A}$, a program $c$, and two stores $\delta_{1}, \delta_{2}$, s.t., $\vdash \delta_{1} \approx_{\ell_{A}} \delta_{2} : \Gamma$, and $\Gamma; [\bot, \bot] \perp \vdash c$, and $\forall i \in \{1, 2\}$, $[\bot, \bot] \perp \delta_{i} \vdash c \rightarrow^{+} \kappa_{i}, \delta'_{i}$, skip, then $\vdash T_{1} \approx_{\ell_{A}} T_{2}$. 

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5.5 Discussion

Implicit leaks manifested in noninterference proofs. Let’s revisit the example at the end of Section 4.4 to see how the implicit leak manifests in the paired execution and why it leads to our current design. We do not use the write sets in if statements and simply refine the label-intervals. Because x is H, it’s initialized with a paired value.

\[
\delta = \langle [H, H]true^{H} | [H, H]false^{H} \rangle, \ y \mapsto [L, H]true^{L}, \ z \mapsto [L, L]true^{L} \\
c_1 = \text{if } x \text{ then } y := [L, H]false^{L} \text{ else skip} \\
c_2 = \text{if } y \text{ then } z := [L, L]false^{L} \text{ else skip} \\
[L, L]L, \ \delta / c_1; c_2 \rightarrow \\
\ldots / \langle [H, H]true^{H} \rangle | [H, H]false^{H} \rangle \text{ then } y := [L, H]false^{L} \text{ else skip} ; c_2 \rightarrow \\
\ldots / \langle \ldots, y := [L, H]false^{L} ; c_2 | \ldots, \text{ skip} ; c_2 \rangle \rightarrow
\]

The variable y is updated only in the left branch. To prove soundness and completeness of the paired semantics, the two executions should be independent. Therefore, we try to update y in the store as \(\langle [H, H]false | [L, H]true^{L}\rangle\). However, this pair is not well-formed because pairs are only well-typed if both intervals are in H. Clearly, the right branch of y does not satisfy this requirement. Therefore, we cannot prove preservation for the assignment case. For preservation to succeed, we would need to refine the right branch to be \([H, H]true\) when assigning to y in the left execution. But then the two executions are no longer independent, which breaks the soundness (i.e., the projected execution is not guaranteed to make progress or stay in the same state).

With these constraints in place, we need to refine y before the if statement branches, which ultimately leads to our final design.

Generality of the write sets. The static write sets analysis in the typing rules is hard to scale to a language with first-order stores. For languages with first-class functions and references, the technique could be extended using regions and an effect type system to identify a sound overapproximation of the write sets statically [Foster et al. 2002; Nielson et al. 1999].

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Fig. 19. Equivalence definitions
Labels and intervals:

\[ \ell \subseteq \ell' \quad \ell \subseteq \ell \quad \ell_1 \subseteq \ell_2 \subseteq \ell_2' \quad t_1 \subseteq t_1' \quad t_2 \subseteq t_2' \]

Expressions:

\[ t_1 \subseteq t_2 \quad g_1 \subseteq g_2 \quad x \subseteq x \quad e_1 \subseteq e_1' \quad e_2 \subseteq e_2' \quad E \subseteq E' \quad g_1 \subseteq g_2 \quad e_1 \subseteq e_2 \]

Commands:

\[ \text{skip} \subseteq \text{skip} \quad c_1 \subseteq c_1' \quad c_2 \subseteq c_2' \quad e_1 \subseteq e_2 \]

\[ \text{if}^X e_1 \quad \text{then} \quad c_1 \quad \text{else} \quad c_2 \quad \text{while}^X e_1 \quad \text{do} \quad c_1 \quad \text{while}^X e_2 \quad \text{do} \quad c_2 \]

Store, Types and Typing-context:

\[ \forall x \in \delta, \delta(x) \subseteq \delta'(x) \quad g \subseteq g' \quad \tau \subseteq \tau' \quad \forall x \in \Gamma, \Gamma(x) \subseteq \Gamma'(x) \]

PC-stack and Configurations:

\[ \emptyset \subseteq \emptyset \quad t \subseteq t' \quad g \subseteq g' \quad \kappa_1 \subseteq \kappa_1' \quad \kappa_2 \subseteq \kappa_2' \quad \kappa \subseteq \kappa' \quad \delta \subseteq \delta' \quad c \subseteq c' \]

Fig. 20. Precision relations

6 GRADUAL GUARANTEES

The gradual guarantees state that if a program with more precise labels type-checks and is accepted by the runtime semantics of the gradual type system, then the same program with less precise labels is also accepted by the gradual type system. To establish these guarantees, we define precision relation between labels, expressions, and commands. The precision relations are shown in Figure 20.

Siek et al. [2015] proposed gradual guarantees for both the type-system and the runtime semantics. Our type system with gradual labels satisfies the static gradual guarantee. We also show dynamic gradual guarantee for our calculus, i.e., if a command takes a step under a store and pc-stack, then a less precise command can also take a step under a less precise store and pc-stack. The proofs can be found in the supplementary material included with the submission.

Theorem 6 (Static Guarantee). If \( \Gamma_1; g_1 \vdash c_1, \Gamma_1 \equiv \Gamma_2, g_1 \equiv g_2, \) and \( c_1 \equiv c_2, \) then \( \Gamma_2; g_2 \vdash c_2. \)

Theorem 7 (Dynamic Guarantee). If \( \kappa_1, \delta_1 / c_1 \xrightarrow{\alpha_1} \kappa_1', \delta_1' / c_1' \) and \( \kappa_1, \delta_1 / c_1 \equiv \kappa_2, \delta_2 / c_2, \) then \( \kappa_2, \delta_2 / c_2 \xrightarrow{\alpha_2} \kappa_2', \delta_2' / c_2' \) such that \( \kappa_1, \delta_1' / c_1' \equiv \kappa_2', \delta_2' / c_2' \) and \( \alpha_1 = \alpha_2. \)

7 RELATED WORK

Prior work on information flow control has focussed on static, dynamic, and hybrid approaches. Quite a few type-systems have been proposed to statically enforce noninterference by annotating variables with labels. Volpano et al. [1996] present the first type-system with information flow control. The type-system has labels with intervals, and the 

\[ \ell \subseteq \ell' \quad \ell \subseteq \ell \quad \ell_1 \subseteq \ell_2 \subseteq \ell_2' \quad t_1 \subseteq t_1' \quad t_2 \subseteq t_2' \]

Expressions:

\[ t_1 \subseteq t_2 \quad g_1 \subseteq g_2 \quad x \subseteq x \quad e_1 \subseteq e_1' \quad e_2 \subseteq e_2' \quad E \subseteq E' \quad g_1 \subseteq g_2 \quad e_1 \subseteq e_2 \]

Commands:

\[ \text{skip} \subseteq \text{skip} \quad c_1 \subseteq c_1' \quad c_2 \subseteq c_2' \quad e_1 \subseteq e_2 \]

\[ \text{if}^X e_1 \quad \text{then} \quad c_1 \quad \text{else} \quad c_2 \quad \text{while}^X e_1 \quad \text{do} \quad c_1 \quad \text{while}^X e_2 \quad \text{do} \quad c_2 \]

Store, Types and Typing-context:

\[ \forall x \in \delta, \delta(x) \subseteq \delta'(x) \quad g \subseteq g' \quad \tau \subseteq \tau' \quad \forall x \in \Gamma, \Gamma(x) \subseteq \Gamma'(x) \]

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\[ \emptyset \subseteq \emptyset \quad t \subseteq t' \quad g \subseteq g' \quad \kappa_1 \subseteq \kappa_1' \quad \kappa_2 \subseteq \kappa_2' \quad \kappa \subseteq \kappa' \quad \delta \subseteq \delta' \quad c \subseteq c' \]

Fig. 20. Precision relations

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Theorem 7 (Dynamic Guarantee). If \( \kappa_1, \delta_1 / c_1 \xrightarrow{\alpha_1} \kappa_1', \delta_1' / c_1' \) and \( \kappa_1, \delta_1 / c_1 \equiv \kappa_2, \delta_2 / c_2, \) then \( \kappa_2, \delta_2 / c_2 \xrightarrow{\alpha_2} \kappa_2', \delta_2' / c_2' \) such that \( \kappa_1, \delta_1' / c_1' \equiv \kappa_2', \delta_2' / c_2' \) and \( \alpha_1 = \alpha_2. \)

7 RELATED WORK

Prior work on information flow control has focussed on static, dynamic, and hybrid approaches. Quite a few type-systems have been proposed to statically enforce noninterference by annotating variables with labels. Volpano et al. [1996] present the first type-system with information flow control.
labels that satisfies a variant of noninterference, also known as termination-insensitive noninterference. Heintze and Riecke [1998] propose SLam that enforces information flow control in lambda calculus. Myers [1999] presents JFlow (and later JIF), enforcing information flow in Java. Our formalization borrows the proof-technique from FlowML, presented by Pottier and Simonet [2002], for enforcing noninterference using pairs. Hunt and Sands [2006] present a security type system for tracking information flow labels in a flow-sensitive manner.

Dynamic approaches use a runtime monitor to track the flow of information through the program. The labels are mostly flow-sensitive in nature. Austin and Flanagan [2009] present a purely-dynamic information flow monitoring approach that disallows assignments to public values in secret contexts. Our monitor semantics follows a similar approach to prevent information leaks at runtime. Subsequent work presents approaches to make the analysis more permissive and amenable to dynamic languages [Austin and Flanagan 2010, 2012; Hedin and Sabelfeld 2012]. Devriese and Pie [2010] present a different approach for enforcing noninterference dynamically by executing multiple copies of the program for different security levels. To leverage the benefits of static and dynamic approaches for precision and permissiveness, researchers have also proposed hybrid approaches to enforce noninterference [Buiras et al. 2015; Chandra and Franz 2007; Hedin et al. 2015; Just et al. 2011; Moore and Chong 2011; Russo and Sabelfeld 2010]. These approaches are not based on the precision of type annotations, which is the focus of gradual typing.

Siek and Taha [2006] pioneered gradual typing to type programs with unknown types and present an approach to combine static typing with dynamic typing. Subsequently, researchers have focussed on gradualizing type-systems with typestates [Wolff et al. 2011], ownership types [Sergey and Clarke 2012], effects [Schwerter et al. 2014], type annotations [Thiemann and Fennell 2014], and refinement types [Jafery and Dunfield 2017; Lehmann and Tanter 2017; Rastogi et al. 2015].

More closely related to our work are works on gradual security types. Disney and Flanagan [2011] study gradual security types for a pure lambda calculus, and Fennell and Thiemann [2013] present a gradual type system for a calculus with ML-style references. However, these works are based on adding explicit programmer-provided checks and casts to the code. Fennell and Thiemann [2016] extend their prior work to object-oriented programs in a flow-sensitive setting for a Java-like language. They use a hybrid approach to perform effect analysis that upgrades the labels of variables similar to the write-set used in our analysis but do not prove the gradual guarantees for their language. On the other hand, our approach has fixed gradual labels and refines only the label-intervals associated with the value to satisfy dynamic gradual guarantee. More recently, Toro et al. [2018] presented a type-driven gradual type system for a higher-order language with references based on abstract gradual typing [Garcia et al. 2016]. Their formalization satisfies the static gradual guarantee, but sacrifices the dynamic gradual guarantee for noninterference. They briefly discuss the idea of using hybrid approaches and faceted evaluation for regaining the dynamic gradual guarantee. The language presented in this paper is simpler than their language but has mutable global variables and hence, has a similar issue with proving noninterference while satisfying the dynamic gradual guarantee. It would be interesting to explore the possibility of extending the language with first-class references and functions, and performing an effect analysis to determine the write-set accurately for ensuring dynamic gradual guarantee alongside noninterference.

8 CONCLUSION AND FUTURE DIRECTIONS

We presented a gradual information flow type system for a simple imperative language that enforces termination-insensitive noninterference and enjoys the dynamic gradual guarantee at the same time. We demonstrated that our monitor can stop implicit flows by taking care of the write-sets of both branches, regardless of which branch is taken. The non-conventional proof technique of noninterference that we used helps us identify the conditions for ensuring gradual guarantees.
As a possible direction for future work, for languages with first-class functions and references, the technique can possibly be extended using techniques proposed by Foster et al. [2002]; Nielson et al. [1999], which use regions and side-effect analysis to determine aliases and points-to variables.

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A ADDITIONAL DEFINITIONS

Write-set
The function $WtSet$ is defined as:

$$WtSet(x) = \{x\}$$
$$WtSet(\text{skip}) = \emptyset$$
$$WtSet(E \ e) = WtSet(e)$$
$$WtSet(\text{while}^X \ e \ \text{do} \ c) = WtSet(c)$$
$$WtSet(x := e_2) = WtSet(x)$$
$$WtSet(\text{output}(\ell, e)) = \emptyset$$
$$WtSet({c}) = WtSet(c)$$
$$WtSet(c_1; c_2) = WtSet(c_1) \cup WtSet(c_2)$$
$$WtSet((k_1, t_1, c_1 | k_2, t_2, c_2)_g) = WtSet(c_1) \cup WtSet(c_2)$$
$$WtSet(\text{if}^X \ e \ \text{then} \ c_1 \ \text{else} \ c_2) = WtSet(c_1) \cup WtSet(c_2)$$

Evaluation-context
The evaluation contexts are defined as below:

$$\text{Eval. Ctx. } C ::= [\ ] | E[\ ] | [\ ] : bop e | v \ bop [\ ] | v ::= [\ ] | \text{output}(l, [\])$$

The semantics are shown in Figure 21.

B TRANSLATION FROM $\text{WHILE}^G$ TO $\text{WHILE}_{Evd}^G$

Lemmas 1. If $\iota = \gamma(g)$, then $\iota \sqsubseteq \gamma(g)$

Proof. Case: $g = ?$

\[ i = [\bot, \top] \text{ and } \gamma(g) = [\bot, \top]. \]
By the definition of $\preccurlyeq$, $\bot \preccurlyeq \bot$ and $\top \preccurlyeq \top$.
Thus by the definition of $\sqsubseteq$, $\iota \sqsubseteq \gamma(g)$

Case: $g = \ell$

\[ i = [\ell, \ell] \text{ and } \gamma(g) = [\ell, \ell]. \]
By the definition of $\preccurlyeq$, $\ell \preccurlyeq \ell$ and $\ell \preccurlyeq \ell$.
Thus by the definition of $\sqsubseteq$, $\iota \sqsubseteq \gamma(g)$

Box

Lemma 2. If $\Gamma \vdash e : U$ and $\Gamma \vdash e \leadsto e' : U$, then $\Gamma \vdash e' : U$

Proof. By induction on the expression typing derivation.

Case: BOOL, INT

By Lemma 1, $\gamma(g) \subseteq \gamma(g)$.
By T-BOOL, T-INT and G-BOOL, G-INT, the conclusion holds

Case: VAR

By definition of T-VAR and G-VAR

Case: BOP
∀i ∈ \{1, 2\}, Γ ⊢ e_i : τ^{g_i} \quad g = g_1 \sqsubseteq g_2
\[\Gamma ⊢ e_1 \textbf{bop} e_2 : τ^g\]

By IH,
(1) \(Γ ⊢ e'_i : τ^{g_i}\)

By (1), T-BOOL and G-BOOL, the conclusion holds

**Case:** CAST
\[Γ ⊢ e : U' \quad U' \leq_c U\]
\[\Gamma ⊢ e :: U : U\]

Assume \(U = τ^g\), By IH,
(1) \(Γ ⊢ e' : τ^{g'}\) such that \(τ^{g'} \leq_c τ^g\)
By (1), definition of evidence-based consistent subtyping and \(\sqsubseteq\), \(Γ ⊢ (e_1, e_2)^g e' : τ^g\)

**Lemma 3.** If \(Γ; g_{pc} ⊢ c\) and \(Γ; g_{pc} + c \sim c'\), then \(WtSet(c) = WtSet(c')\)

**Proof (sketch).** By induction on the command typing derivation. Follows from the definition of \(WtSet\) for SKIP, ASSIGN, OUT and additionally uses the IH for SEQ, IF, WHILE

**Theorem 1.** If \(Γ; g_{pc} ⊢ c\) and \(Γ; g_{pc} + c \sim c'\), then \(∀t_{pc} \subseteq γ(g_{pc}). Γ; t_{pc} g_{pc} ⊢ c'\)

**Proof.** By induction on the typing derivation - \(Γ; g_{pc} ⊢ c\).

**Case:** SKIP

By definition of T-SKIP, G-SKIP

**Case:** SEQ
\[Γ; g_{pc} ⊢ c_1 \quad Γ; g_{pc} ⊢ c_2\]
\[Γ; g_{pc} ⊢ c_1; c_2\]

By IH and inversion of T-SEQ,
(1) \(Γ; t_{pc} g_{pc} ⊢ c'_1\) and \(Γ; t_{pc} g_{pc} ⊢ c'_2\) such that \(t_{pc} \subseteq γ(g_{pc})\)
By (1), G-SEQ, the conclusion holds

**Case:** ASSIGN

By Lemma 2,
(1) \(Γ ⊢ x : τ^g\) and \(Γ ⊢ e' : τ^g\)
By assumption,
(2) \(t_{pc} \subseteq γ(g_{pc})\)
By inversion of ASSIGN,
(3) \(g_{pc} \preceq_c g\)
By (2), (3), the conclusion holds

**Case:** OUT

By Lemma 2,
(1) \(Γ ⊢ e' : τ^g\)
By inversion of OUT,
(2) \(g \preceq_c ℓ\) and \(g_{pc} \preceq_c g\)
By assumption, Lemma 1 and inversion of T-OUT,
(3) \(t_{pc} \subseteq γ(g_{pc}), t \subseteq γ(g)\) and \(|ℓ, ℓ| \subseteq γ(ℓ)\)
By (1),(2),(3), the conclusion holds

**Case:** IF (similar for WHILE)
C PAIRED EXECUTIONS - AUXILIARY DEFINITIONS AND ADDITIONAL RULES

Figures 22 define the auxiliary definitions used in the semantics of paired-executions and the additional rules for pairs.

D WELL-FORMEDNESS

(1) $\vdash v \text{wf}$ if
   (a) $v = \langle v_1 \mid v_2 \rangle$, then $\forall i \in \{1, 2\}. v_i = (u_i u_i)$
   (b) $v = (u u)^g$

(2) $\vdash \delta \text{wf}$, if $\forall x \in \delta. \vdash \delta(x) \text{wf}$

(3) $\vdash \delta \vdash / i e \text{wf}$ for $i \in \{1, 2\}$ if $\vdash \delta \text{wf}$

(4) $\vdash c \text{wf}$, when the following hold:
   (a) if $c = \langle \kappa_1, t_1, c_1 \mid \kappa_2, t_2, c_2 \rangle$, then $\vdash c_1 \text{wf}$, $\vdash c_2 \text{wf}$, and $c_1$ and $c_2$ do not contain pairs
   (b) if $c = \text{if } e \text{ then } c_1 \text{ else } c_2$, then $\vdash c_1 \text{wf}$, $\vdash c_2 \text{wf}$, and $c_1$ and $c_2$ do not contain pairs or braces
   (c) if $c = \text{while}^X e \text{ do } c$, then $\vdash c \text{wf}$ and $c$ does not contain pairs or braces
   (d) if $c = c_1; c_2$ then $\vdash c_1 \text{wf}$, $\vdash c_2 \text{wf}$ and $c_2$ does not contain pairs or braces
   (e) if $c = \{c_1\}$, then $\vdash c_1 \text{wf}$

(5) $\vdash \kappa, \delta \vdash / i \text{ c wf}$ for $i \in \{1, 2\}$ if all of the following hold
   (a) $\vdash c \text{wf}$ and $\vdash \delta \text{wf}$
   (b) if $i \in \{1, 2\}$, then $c$ does not contain pairs

Theorems and Proofs for Well-formedness

LEMMMA 4. If $\forall i \in \{1, 2\}, E :: \kappa, \delta \vdash / i c \rightarrow \kappa', \delta' / i c'$ and $\vdash \kappa, \delta \vdash / i \text{ c wf}$, then

(a). $c'$ does not contain pairs and

(b). $\text{WtSet}(c') \subseteq \text{WtSet}(c)$

PROOF. By induction on the structure of $E$.

$c'$ does not contain pairs: Follows from well-formedness definition for most rules. Pair-Seq, Pair-Pc use the IH additionally.

$\text{WtSet}(c') \subseteq \text{WtSet}(c)$:

PAIR-SEQ  By IH, $\text{WtSet}(c') \subseteq \text{WtSet}(c_1)$.
          Thus by definition of WtSet, $\text{WtSet}(c', c_2) \subseteq \text{WtSet}(c_1; c_2)$
PAIR-Pc   By IH, $\text{WtSet}(c') \subseteq \text{WtSet}(c)$
PAIR-POP   $\text{WtSet}(\text{skip}) = \text{WtSet}(\text{skip})$
PAIR-SKIP  $\text{WtSet}(\text{skip}) \subseteq \text{WtSet}(\text{skip}) \cup \text{WtSet}(c)$
PAIR-ASSIGN$\text{WtSet}(\text{skip}) \subseteq \text{WtSet}(x)$
\[
\begin{align*}
\text{refineUB}_i: \\
\text{refine}(t_c, i) &= (\_, i') \quad i_i = i' \\
\text{refine}(t_c, i) &= (\_, i'_j) \\
\text{refineUB}_i(t_c, (t \ u)^g) &= \langle i_1 \ u | t_2 \ u \rangle^g \\
\text{refineUB}_i(t_c, (t_1 \ u_1 | t_2 \ u_2)^g) &= \langle i'_1 \ u_1 | i'_2 \ u_2 \rangle^g
\end{align*}
\]

\[
\begin{align*}
\text{refineUB}_j: \\
\text{refine}(t_c, i) &= (\_, i') \\
\text{refine}(t_c, i) &= (\_, i'_j) \\
\text{refineUB}_j((t_{c1} | t_{c2}), (t \ u)^g) &= \langle i_1 \ u | t_2 \ u \rangle^g \\
\text{refineUB}_j((t_{c1} | t_{c2}), (t_1 \ u_1 | t_2 \ u_2)^g) &= \langle i'_1 \ u_1 | i'_2 \ u_2 \rangle^g
\end{align*}
\]

\[
\begin{align*}
\text{refineUB}: \\
\forall i \in \{1, 2\}. \text{refine}(t_c, i) &= (\_, i'_j) \\
\text{refineUB}(t_c, (t_1 \ u_1 | t_2 \ u_2)^g) &= \langle i'_1 \ u_1 | i'_2 \ u_2 \rangle^g \\
\forall i \in \{1, 2\}. \text{refine}(t_{c1}, i) &= (\_, i'_1) \\
\text{refineUB}(t_{c1} | t_{c2}, (t_1 \ u_1 | t_2 \ u_2)^g) &= \langle i'_1 \ u_1 | i'_2 \ u_2 \rangle^g
\end{align*}
\]

\[
\begin{align*}
\text{refineWSet}: \\
\delta' = \text{refL}(\delta, X, i) \\
\text{refL}(\delta, X, i) &= \delta \\
\forall i \in \{1, 2\}. \text{refine}(t_{c1}, i) &= (\_, i'_1) \\
\text{refineWSet}(\delta, X, t_{pc}(t \ u)) &= \delta' \\
\text{refL}_i(\delta, \cdot, \Pi) &= \delta \\
\text{refL}_i((\delta, X, i) \rightarrow v), (X, x, i) &= \delta', x \mapsto \text{refineUB}(i, v) \\
\text{refL}_i((\delta, X, i) \rightarrow v), (X, x, i) &= \delta', x \mapsto \text{refineUB}_i(\Pi, v) \\
\delta' &= \text{refL}_i(\delta, X, \Pi) \\
\text{refineWSet}(\delta, X, t_{pc}(t \ u)^g) &= \delta' \\
\forall i \in \{1, 2\}. \text{refine}(t_{c1}, i) &= (\_, i'_1) \\
\text{refineWSet}(\delta, X, t_{pc}, (t_1 \ u_1 | t_2 \ u_2)^g) &= \delta' \\
\text{refineLB}: \\
\text{refine}(t, t_c) &= (i', \_) \\
\text{refineLB}(t \ u)^g, t_c) &= (i' \ u)^g \\
\forall i \in \{1, 2\}. \text{refine}(t_1, i) &= (i'_1, \_) \\
\text{refineLB}(\langle t_1 \ u_1 | t_2 \ u_2 \rangle, i) &= \langle i'_1 \ u_1 | i'_2 \ u_2 \rangle
\end{align*}
\]

Fig. 22. Auxiliary definitions for paired executions
PAIR-If \( \text{WtSet}(c_1) \subseteq \text{WtSet}(c_1) \cup \text{WtSet}(c_2) \)
PAIR-If-Refine \( \text{WtSet}(c_1) \cup \text{WtSet}(c_2) = \text{WtSet}(c_1) \cup \text{WtSet}(c_2) \)
PAIR-While \( \text{WtSet}(c) \cup \text{WtSet}(c) \cup \text{WtSet}((\text{skip})) = \text{WtSet}(c) \)

LEMMA 5. \( \vdash v \text{ wf}, \vdash v' \text{ wf and } \vdash \delta \text{ wf}, \) then

1. \( \forall i \in \{1, 2, \ldots\}, \vdash \text{ refineUB}(i, v) \) \( \vdash \text{ refineLB}(v, i) \) \( \vdash \text{ upd}_i v \) \( v' \text{ wf and } \vdash \delta \text{ wf}, \)

2. \( \forall i \in \{1, 2, \ldots\}, X. \vdash \text{ refineWSet}(\delta, X, i, v) \)

PROOF. By examining the respective definitions and induction for \( \text{refineWSet}. \)

LEMMA 6 (WELL-FORMEDNESS PRESERVATION). \( \vdash \kappa, \delta / i c \xrightarrow{T} \kappa', \delta' / i c' \) where \( \vdash \kappa, \delta / i c \) \( \text{ wf}, \)
then \( \vdash \kappa', \delta' / i c' \) \( \text{ wf}. \)

PROOF. By induction of the number of steps in sequence. Base case follows by assumption.

Inductive Case: Holds for \( n \) steps; To show for \( n + 1 \) steps, i.e.,

if \( \kappa, \delta / i c \xrightarrow{T} \kappa', \delta' / i c' \)

By IH

(1) \( \vdash \kappa', \delta' / i c' \)

Then \( \vdash \kappa', \delta' / i c' \)

Induction over the derivation. The proof follows from Lemma 4(a) and 5 in most cases.
PAIR-SEQ, PAIR-Pc use the IH additionally and PAIR-C-Pair uses Lemma 4(b) additionally.

E. SOUNDNESS OF PAIRED-EXECUTION

LEMMA 7. \( \forall x \in \delta, i \in \{1, 2\}, [\delta(x)]_i = [\delta_i](x) \) and \( [\text{rd } \delta(x)]_i = [\text{rd } \delta_i](x) \)

PROOF. \( \delta(x) \) can be a pair or normal value

Case: \( \delta(x) = (i u)^g \)
By store-projection \( [\delta](x) = [(i u)^g] \) \( [i u]^g \)
By definition of \( \text{rd}, \text{rd } \delta(x) = [i u]^g \) and \( [\text{rd } \delta_i](x) = [i u]^g \)
By projection of values, the conclusion holds.

Case: \( \delta(x) = (i_1 u_1 | i_2 u_2)^g \)
By store-projection definition \( [\delta](x) = (i_1 u_1)^g \)
By definition of \( \text{rd}, \text{rd } \delta(x) = (i_1 u_1)^g \) and \( [\text{rd } \delta_i](x) = (i_1 u_1)^g \)
By projection of values, \( [\delta(x)]_i = (i_1 u_1)^g \) and \( [\text{rd } \delta_i](x)] = (i_1 u_1)^g \)

LEMMA 8. \( \forall i \in \{1, 2\}, [\text{upd } v_o v_n]_i = \text{upd } [v_o]_i [v_n]_i \)

PROOF. \( v_o \) and \( v_n \) can be a pair or normal value. We show when both are pairs and when \( v_o \) is pair. Other cases are similar.

Case: \( v_o = (i_o u_o | i_o' u_o'^g \) and \( v_n = (i_n u_n | i_n' u_n'^g \)
We show for \( i = 1 \).

By projection of values,

(1) \( [v_o]_1 = (i_1 u_o)^g \) and \( [v_n]_1 = (i_n u_n)^g \)

By (1), definition of upd,
(2) $\text{upd} \left[ v_0 \right]_1 [v_n]_1 = (t' u_n)^g$ where $\text{refine}(t_n, t_o) = (\_ , t')$
By definition of upd,

(3) $\text{upd} v_o v_n = \langle t_1 u_n \mid t'_1 u'_n \rangle^g$ where $\text{refine}(t_n, t_o) = (\_ , t_1)$ and $\text{refine}(t'_n, t'_o) = (\_ , t'_1)$
By (2), (3) and projection of values

$\text{upd} v_0 v_n \left[ v \right]_1 = (t_1 u_n)^g$ and $t_1 = t'$

Case: $v_o = \langle t_o u_o \mid t'_o u'_o \rangle^g$ and $v_n = (t_n u_n)^g$. We show for $i = 1$.

By projection of values,

(1) $[v_0]_1 = (t_o u_o)^g$ and $[v_n]_1 = (t_n u_n)^g$
By (1), definition of upd,

(2) $\text{upd} \left[ v_0 \right]_1 [v_n]_1 = (t' u_n)^g$ where $\text{refine}(t_n, t_o) = (\_ , t')$
By definition of upd,

(3) $\text{upd} v_o v_n = \langle t_1 u_n \mid t'_1 u'_n \rangle^g$ where $\text{refine}(t_n, t_o) = (\_ , t_1)$ and $\text{refine}(t'_n, t'_o) = (\_ , t'_1)$
By (2), (3) and projection of values

$\text{upd} v_0 v_n \left[ v \right]_1 = (t_1 u_n)^g$ and $t_1 = t'$

$\blacksquare$

**Lemma 9.** $\forall i \in \{1, 2\}, \left[ \text{refine}LB([v], i) \right]_i = \text{refine}LB([v]_i, i)$

**Proof.** $v$ can be a pair or normal value.

**Case:** $v = \langle t_1 u_1 \mid t'_2; u_2 \rangle^g$

By projection of values,

(1) $[v]_i = (t_1 u_i)^g$
By (1), definition of $\text{refine}LB$,

(2) $\text{refine}LB([v]_i, i) = (t'_1 u_i)^g$ where $\text{refine}(t_i, i) = (t'_i, \_)$
By definition of $\text{refine}LB$,

(3) $\text{refine}LB(v, i) = \langle t''_1 u_1 \mid t''_2 u_2 \rangle^g$ where $\text{refine}(t_i, i) = (t''_i, \_)$
By (2), (3) and projection of values, the conclusion holds

**Case:** $v = (t_1 u_1)^g$

By projection of values,

(1) $[v]_i = (t_1 u_i)^g$
By (1), definition of $\text{refine}LB$,

(2) $\text{refine}LB([v]_i, i) = (t'_1 u_i)^g$ where $\text{refine}(t_i, i) = (t'_i, \_)$
By definition of $\text{refine}LB$,

(3) $\text{refine}LB(v, i) = (t'_1 u_1)^g$ where $\text{refine}(t_i, i) = (t'_i, \_)$
By (2), (3) and projection of values, the conclusion holds

$\blacksquare$

**Lemma 10.** $\forall i \in \{1, 2\}, \left[ \text{refine}UB(\Pi, v) \right]_i = \text{refine}UB([\Pi]_i, [v]_i)$

**Proof.** $\Pi$ can be a pair of intervals or single interval and $v$ can be a pair or normal value.

**Case:** $\Pi = \langle t_1 \mid u_2 \rangle$, $v = \langle t'_1 u_1 \mid t'_2; u_2 \rangle^g$

By projection of values,

(1) $[v]_i = (t'_1 u_i)^g$ and $[\Pi]_i = t_i$
By (1), definition of $\text{refine}UB$,

(2) $\text{refine}UB([\Pi]_i, [v]_i) = (t''_1 u_i)^g$ where $\text{refine}(t_i, t'_i) = (\_ , t''_i)$
By definition of $\text{refine}UB$,

(3) $\text{refine}UB(\Pi, v) = \langle t''_1 u_1 \mid t''_2 u_2 \rangle^g$ where $\text{refine}(t_i, t'_i) = (\_ , t''_i)$
By (2), (3) and projection of values, the conclusion holds
Case: $\Pi = \langle t_1 | u_2 \rangle$, $\nu = (t \; u)^g$

By projection of values,

1. $[\nu]_i = (t \; u)^g$ and $[\Pi]_i = t$

By (1), definition of $\text{refineUB}$,

2. $\text{refineUB}(\Pi, [\nu]_i) = (t' \; u)^g$ where $\text{refine}(t, t_i) = (\_, t')$

By definition of $\text{refineUB}$,

3. $\text{refineUB}(\Pi, \nu) = \langle t'_1 \; u \mid t'_2 \; u \rangle^g$ where $\text{refine}(t, t_i) = (\_, t')$

By (2), (3) and projection of values, the conclusion holds

Case: $\Pi = t, \nu = \langle t_1 \; u_1 | t_2; u_2 \rangle^g$

By projection of values,

1. $[\nu]_i = (t_1 \; u_1)^g$ and $[\Pi]_i = t$

By (1), definition of $\text{refineUB}$,

2. $\text{refineUB}(\Pi, [\nu]_i) = (t' \; u_i)^g$ where $\text{refine}(t, t_i) = (\_, t')$

By definition of $\text{refineUB}$,

3. $\text{refineUB}(\Pi, \nu) = \langle t'_1 \; u_1 \mid t'_2 \; u_2 \rangle^g$ where $\text{refine}(t, t_i) = (\_, t')$

By (2), (3) and projection of values, the conclusion holds

Case: $\Pi = t_c, \nu = (t \; u)^g$

By projection of values,

1. $[\nu]_i = (t \; u)^g$ and $[\Pi]_i = t_c$

By (1), definition of $\text{refineUB}$,

2. $\text{refineUB}(\Pi, [\nu]_i) = (t' \; u)^g$ where $\text{refine}(t, t_i) = (\_, t')$

By definition of $\text{refineUB}$,

3. $\text{refineUB}(\Pi, \nu) = \langle t'_1 \; u \mid t'_2 \; u \rangle^g$ where $\text{refine}(t_c, t) = (\_, t')$

By (2), (3) and projection of values, the conclusion holds

\[ \square \]

Lemma 11. $\forall \{i, j\} \in \{1, 2\}, [\text{refineUB}_i(\Pi, \nu)]_i = \text{refineUB}_i([\Pi]_i, [\nu]_i) \land [\text{refineUB}_j(\Pi, \nu)]_j = [\nu]_j$

Proof. $\Pi$ can be a pair of intervals or single interval and $\nu$ can be a pair or normal value. We show for $i = 1, j = 2$.

Case: $\Pi = \langle t_{c1} | t_{c2} \rangle$, $\nu = \langle t_1 \; u_1 \mid t_2; u_2 \rangle^g$

By projection of values,

1. $[\Pi]_1 = t_{c1}$, $[\nu]_1 = (t_1 \; u_1)^g$ and $[\nu]_2 = (t_2 \; u_2)^g$

By (1), definition of $\text{refineUB}_1$,

2. $\text{refineUB}_1([\Pi]_1, [\nu]_1) = \langle t'_1 \; u_1 \mid t_1 \; u_1 \rangle^g$

By definition of $\text{refineUB}_1$,

3. $\text{refineUB}_1(\Pi, \nu) = \langle t'_1 \; u_1 \mid t_2 \; u_2 \rangle^g$

By (1), (2), (3) and projection of values,

4. $[\text{refineUB}_1(\Pi, \nu)]_1 = \text{refineUB}_1([\Pi]_1, [\nu]_1)$ and $[\text{refineUB}_1(\Pi, \nu)]_2 = [\nu]_2$

Case: $\Pi = \langle t_{c1} | t_{c2} \rangle$, $\nu = (t \; u)^g$

By projection of values,

1. $[\Pi]_1 = t_{c1}$, $[\nu]_1 = (t \; u)^g$ and $[\nu]_2 = (t \; u)^g$

By (1), definition of $\text{refineUB}_1$,

2. $\text{refineUB}_1([\Pi]_1, [\nu]_1) = \langle t' \; u \mid t \; u \rangle^g$

By definition of $\text{refineUB}_1$, 

\[ \square \]
(3) \( \text{refineUB}_1(\Pi, \nu) = \langle i' \mid u \rangle^9 \)
By (1), (2), (3) and projection of values,
\[
\text{refineUB}_1(\Pi, \nu) = \text{refineUB}_1([\Pi]_1, [\nu]_1) = \langle i' \mid u \rangle^9 \text{ and } \text{refineUB}_1(\Pi, \nu) = [\nu]_2 = \langle i \mid u \rangle^9
\]

**Case: \( \Pi = i, \nu = \langle t_1 \mid u_1 \mid t_2; u_2 \rangle^9 \)**

By projection of values,
\[(1) \ [\Pi]_1 = i, [\nu]_1 = \langle t_1 \mid u_1 \rangle^9 \text{ and } [\nu]_2 = \langle t_2 \mid u_2 \rangle^9
\]
(2) Let refine\((i, t_1) = (\_ , t_1')\)
By (1), definition of refineUB1,
\[(2) \text{refineUB}_1([\Pi]_1, [\nu]_1) = \langle t_1' \mid u_1 \mid i \rangle^9
\]
By definition of refineUB1,
(3) \( \text{refineUB}_1(\Pi, \nu) = \langle i' \mid u \rangle^9 \)
By (1), (2), (3) and projection of values,
(4) \( \text{refineUB}_1(\Pi, \nu) = \langle i' \mid u_1 \mid t_2; u_2 \rangle^9 \)

**Case: \( \Pi = i_c, \nu = \langle i \rangle^9 \)**

By projection of values,
(1) \( [\Pi]_1 = i_c, [\nu]_1 = \langle i \rangle^9 \text{ and } [\nu]_2 = \langle i \rangle^9 \)
(2) \( \text{Let refine}(i_c, i) = (\_ , i^{'}) \)
By (1), definition of refineUB1,
(2) \( \text{refineUB}_1([\Pi]_1, [\nu]_1) = \langle i^{' \_} \mid u \rangle^9 \)
By definition of refineUB1,
(3) \( \text{refineUB}_1(\Pi, \nu) = \langle i' \mid u \rangle^9 \)
By (1), (2), (3) and projection of values,
(4) \( \text{refineUB}_1(\Pi, \nu) = \text{refineUB}_1([\Pi]_1, [\nu]_1) = \langle i' \mid u_1 \rangle^9 \text{ and } \text{refineUB}_1(\Pi, \nu) = [\nu]_2 = \langle i \mid u \rangle^9 \)

**Lemma 12.** \( \forall \delta, X, x, x \in X \implies x \in \delta, \) we have \( \forall \{i, j\} \in \{1, 2\}, [\text{rfl}_{i}(\delta, X, \Pi)]_i = \text{rfl}_{i}([\delta]_i, X, [\Pi]_i) \) and \( [\text{rfl}_{i}(\delta, X, \Pi)]_j = [\delta]_j \)

**Proof.** By induction on the size of \( X \) and applying Lemma 11

**Lemma 13.** \( \forall \delta, X, X \subseteq \delta, \) we have \( \forall i \in \{1, 2\}, [\text{rfl}_{i}(\delta, X, \Pi)]_i = \text{rfl}_{i}([\delta]_i, X, [\Pi]_i) \)

**Proof.** By induction on the size of \( X \) and applying Lemma 10

**Lemma 14.** If \( \forall \delta, X, \nu, i_c. X \subseteq \delta \) and \( \nu \) is not a pair, then
\( \forall \{i, j\} \in \{1, 2\}, [\text{refineWSet}_{i}(\delta, X, i_c, \nu)]_i = \text{refineWSet}_{i}([\delta]_i, X, i_c, \nu) \) and \( [\text{refineWSet}_{i}(\delta, X, i_c, \nu)]_j = [\delta]_j \)

**Proof.** Follows from Lemma 12

**Lemma 15.** \( \forall \delta, X, \nu, i_c. X \subseteq \delta, \) we have \( \forall i \in \{1, 2\}, [\text{refineWSet}(\delta, X, i_c, \nu)]_i = \text{refineWSet}(\delta, X, i_c, \nu) \)

**Proof.** Follows from Lemma 13

**Lemma 16.** If \( \nu \) is not a pair, then
(1) \( \forall i, \text{refineUB}(i, \nu) \) is not a pair
(2) \( \forall i, \text{refineLB}(i, \nu) \) is not a pair

**Proof.** By examining the respective definitions.

**Lemma 17.** If \( \forall i \in \{1, 2\}, E :: \delta / i \Downarrow v_i \) and \( i \Downarrow i \) \text{ e wf, then } [\delta]_i / i \Downarrow v_i \) and \( v_i \) is not a pair

**Proof.** By induction on the structure of \( E. \)
Case: P-CONST
By P-CONST,
(1) \( \forall i \in \{1, 2\}, [\delta]_i / (i \ u)^g \downarrow (i \ u)^g \)

Case: P-VAR
By P-VAR
(1) \( \forall i \in \{1, 2\}, [\delta]_i / x \downarrow \text{rd}[\delta]_i(x) \)
\( \delta(x) \) is either a pair or normal value:

Subcase I: \( \delta(x) = (v_1 \mid v_2)^g \)
By definition of rd and value projection
(1) \( \text{rd}[\delta(x)] = v_i \)
By (1), definition of store projection and rd
(II) \( [\delta]_i(x) = v_i \) and \( \text{rd}[\delta]_i(x) = v_i \)
By (II) and (II), \( \text{rd}[\delta(x)] = \text{rd}[\delta]_i(x) \)
As \( \vdash \delta / i \ e \ \text{wf}, \delta(x) \) cannot have nested pairs. Hence, \( v_i \) is not a pair

Subcase II: \( \delta(x) = v = (i \ u)^g \)
By definition of rd and value projection
(II) \( \text{rd}[\delta(x)] = v; v \) is a normal value and not a pair
By (1), definition of store projection and rd
(II) \( [\delta]_i(x) = v \) and \( \text{rd}[\delta]_i(x) = v \)
By (II) and (II), \( \text{rd}[\delta(x)] = \text{rd}[\delta]_i(x) \)

Case: P-Bop
\[
\begin{array}{ccc}
\delta / i \ e_1 \downarrow v_1 & \delta / i \ e_2 \downarrow v_2 & v = v_1 \ \text{bop} \ v_2 \\
(1) & \delta / i \ e_1 \ \text{bop} \ e_2 \downarrow v \\
\end{array}
\]
By IH
(2) \( \forall i \in \{1, 2\}, [\delta]_i / e_1 \downarrow v_1 \) and \( [\delta]_i / e_2 \downarrow v_2 \) and \( v_1 \) and \( v_2 \) are not pairs
By (2), P-Bop and binary operation on values, the conclusion holds

Case: P-Cast
\[
\begin{array}{ccc}
\delta / i \ e \downarrow v & v' = (E, g) \triangleright v \\
(1) & \delta / i \ E^g e \downarrow v' \\
\end{array}
\]
By IH
(2) \( \forall i \in \{1, 2\}, [\delta]_i / e \downarrow v \) and \( v \) is not a pair
(3) Let \( v = (i \ u)^g \), then \( v' = (i' \ u)^g \) such that \( i' = i \Rightarrow E \)
By (2), (3) and P-Cast, the conclusion holds

\[ \square \]

Lemma 18. If \( \kappa, \delta / c \xrightarrow{\alpha} \kappa', \delta'/ / c' \), where \( \kappa = \vec{\kappa} \triangleright t_{pc} g_{pc} \), then \( \exists \vec{\kappa}' \ s.t. \kappa' = \vec{\kappa}' \triangleright t_{pc} g_{pc} \)

Proof. By induction on the structure of the derivation.

\[ \square \]

Lemma 19. If \( \forall \{i, j\} \in \{1, 2\}, \kappa, \delta / i \ c \xrightarrow{\alpha} \kappa', \delta'/ / i \ c' \), then \( \kappa, [\delta]_i / c \xrightarrow{[\alpha]_i} \kappa', [\delta']_i / c', [\delta]_j = [\delta']_j \) and \( [\alpha]_j = \).

Proof. By induction on the structure of command-evaluation derivation.

Case: P-Seq
\[
\begin{align*}
\kappa, \delta &\xrightarrow{\alpha} \kappa', \delta' \\
\kappa, \delta &\xrightarrow{\alpha} \kappa', \delta' \\
\end{align*}
\]

By IH

1. \(\forall \{i, j\} \in \{1, 2\}, \kappa, [\delta]_i / c_1 \xrightarrow{[\alpha]} \kappa', [\delta']_i / c'_1\) and \([\delta]_j = [\delta']_j\) and \([\alpha]_j = \cdot\),

2. \(\forall \{i, j\} \in \{1, 2\}, \kappa, [\delta]_j / c_1; c_2 \xrightarrow{[\alpha]} \kappa', [\delta']_j / c'_1; c_2\) and \([\delta]_j = [\delta']_j\) and \([\alpha]_j = \cdot\),

\textbf{Case: P-Pc}

\[
\kappa, \delta \xrightarrow{\alpha} \kappa', \delta' \\
\kappa, \delta \xrightarrow{\alpha} \kappa', \delta' \\
\]

By IH

1. \(\forall \{i, j\} \in \{1, 2\}, \kappa, [\delta]_j / e \xrightarrow{\cdot} \kappa', [\delta']_j / e'\) and \([\delta]_j = [\delta']_j\) and \([\alpha]_j = \cdot\),

\textbf{Case: P-Pop, P-Skip, P-If, P-While}

By definition of P-Pop, P-Skip, P-If, P-While

\textbf{Case: P-Assign}

By Lemma 17

1. \(\forall i \in \{1, 2\}, [\delta]_i / e \downarrow v_0\) such that \(v_0 = v\) and \(v\) is not a pair

By definition of P-Assign and Lemma 16,

1a) \(v' = v_0\) where \(v_0\) is obtained by operations on \(v_0\) and is not a pair

By assumption

2. \(\delta' = \delta[x \mapsto \text{upd}_i \delta(x) v']\),

By (1), and definition of P-Assign,

2a) \(\delta'' = [\delta]_j[x \mapsto \text{upd} \delta(j)(x) v']\).

T.S. \([\delta']_j = \delta''\) or \(\forall x \cdot [\delta'](x) j = \delta''(x)\)

We show for \(i = 1, j = 2\), the proof is similar for \(i = 2, j = 1\)

\textbf{Subcase I:} Suppose \(i = 1, j = 2\), \(v' = (i\ u)^9\) and \(\delta(x) = (\langle 1\ u_1 \uparrow 12\ u_2 \rangle)^9\)

By store projection

1. \([\delta]_1(x) = (1\ u_1)^9\) and \([\delta]_2(x) = (12\ u_2)^9\)

By definition of upd and value projection

2. \(\delta'(x) = (i'\ u \uparrow 12\ u_2)^9\) where \(\text{refine}(i, t_1) = (\_, i')\) and \([\delta']_2(x) = (12\ u_2)^9\)

By (1), definition of upd

3. \(\delta''(x) = (i''\ u)^9\) where \(\text{refine}(i, t_1) = (\_, i'')\)

By (2) and (3), \([\delta']_1 = \delta''(x), [\delta]_2 = [\delta']_2(x)\) and \(\forall y \cdot y \neq x \implies [\delta(y)]_1 = [\delta]_1(y)\)

\textbf{Subcase II:} Suppose \(i = 1, j = 2\), \(v' = (i\ u)^9\) and \(\delta(x) = (1\ u_1)^9\)

By store projection

1. \([\delta]_1(x) = (1\ u_1)^9\) and \([\delta]_2(x) = (1\ u_1)^9\)

By definition of upd and value projection

2. \(\delta'(x) = (i'\ u \uparrow 1\ u_1)^9\) where \(\text{refine}(i, t_1) = (\_, i')\) and \([\delta']_2(x) = (1\ u_1)^9\)

By (1), definition of upd

3. \(\delta''(x) = (i''\ u)^9\) where \(\text{refine}(i, t_1) = (\_, i'')\)

By (2) and (3), \([\delta']_1 = \delta''(x), [\delta]_2 = [\delta']_2(x)\) and \(\forall y \cdot y \neq x \implies [\delta(y)]_1 = [\delta]_1(y)\)

\textbf{Case: P-Out}

By Lemma 17,

1. \(\forall i \in \{1, 2\}, [\delta]_i / e \downarrow v_0\) such that \(v_0 = v\) and \(v\) is not a pair
By definition of P-OUT and Lemma 16,
(1a) \( v_i = v_i' \) where \( v_i' \) is obtained by operations on \( v_0 \) and is not a pair
By definition of trace-projection,
(2) \[ [i, \ell, v_1]_i = ([\ell, v_1]) \text{ and } [i, \ell, v_1]_j = \cdot \text{ where } i \neq j \]

**Case:** P-IF-REFINE

By Lemma 17
(1) \( \forall i \in \{1, 2\}, [\delta]_i / e \Downarrow v \) and \( v \) is not a pair,
By Lemma 15,
(2) \( \forall \{i, j\} \in \{1, 2\}, [\text{refineWSet}_i(\delta, X, t_{pc}, v)]_i = [\text{refineWSet}(\delta, X, t_{pc}, v)]_i \)
and \([\text{refineWSet}_j(\delta, X, t_{pc}, v)]_j = [\delta]_j\)
By (1), (2) and definition of P-IF-REFINE, the conclusion holds

**Lemma 20 (Expression soundness).** If \( E :: \delta / e \Downarrow v \) and \( \forall \delta / e \) wt then \( \forall i \in \{1, 2\}, [\delta / e]_i \Downarrow [v]_i \)

**Proof.** By induction on the structure of \( E \).

**Case:** P-CONST
By P-CONST and projection of values

**Case:** P-VAR
By assumption
(1) \( v = \text{rd } \delta(x) \)
T.S. \( \forall i \in \{1, 2\}, \text{rd } [\delta]_i(x) = [\text{rd } \delta(x)]_i \)
We show for \( i = 1 \), the proof is similar for \( i = 2 \)

**Subcase I:** \( \delta(x) = \langle i_1 u_1 | i_2 u_2 \rangle \)
By definition of rd
(11) \( [\text{rd } \delta(x)]_1 = \langle i_1 u_1 \rangle \)
By store-projection definition and definition of rd
(12) \( \text{rd } [\delta]_1(x) = \langle i_1 u_1 \rangle \)
By (11) and (12), the conclusion holds

**Subcase II:** \( \delta(x) = \langle i u \rangle \)
By definition of rd
(II1) \( [\text{rd } \delta(x)]_1 = \langle i u \rangle \)
By store-projection definition and definition of rd
(II2) \( \text{rd } [\delta]_1(x) = \langle i u \rangle \)
By (II1) and (II2), the conclusion holds

**Case:** P-BOP
\[
\frac{\delta / e_1 \Downarrow v_1 \quad \delta / e_2 \Downarrow v_2 \quad v = v_1 \text{ bop } v_2}{\delta / e_1 \text{ bop } e_2 \Downarrow v}
\]
(1)

By (1) and IH
(2) \( \forall i \in \{1, 2\}, [\delta]_i / e_1 \Downarrow [v_1]_i \) and \([\delta]_i / e_2 \Downarrow [v_2]_i \)
T.S. \( \forall i \in \{1, 2\}, [v_1]_i \text{ bop } [v_2]_i = [v_1 \text{ bop } v_2]_i \)
We show for \( i = 1 \), the proof is similar for \( i = 2 \)

**Subcase I:** \( v_1 = \langle i_1 u_1 | i_1' u_1' \rangle \), \( v_2 = \langle i_2 u_2 | i_2' u_2' \rangle \)
By definition of bop,
(11) \( v_1 \text{ bop } v_2 = \langle i u | _\rangle \) and \([v_1 \text{ bop } v_2]_1 = \langle i u \rangle \)
where \( i = i_1 \land i_2, u = u_1 \text{ bop } u_2 \) and \( g_1 \subseteq g_2 \)
By value-projection definition

(I2) \([v_1]_1 = (t_1 u_1)^g\) and \([v_2]_1 = (t_2 u_2)^g\)

By (I2) and definition of bop,

(I3) \([v_1]_1\) bop \([v_2]_1 = (t u)^g\) where \(t = t_1 \equiv t_2, u = u_1\) bop \(u_2\) and \(g = g_1 \equiv c \equiv g_2\)

By (I1) and (I3), the conclusion holds

Subcase II: \(v_1 = \langle t_1 u_1 | t'_1 u'_1 \rangle^g\), \(v_2 = (t_2 u_2)^g\), similar for \(v_1 = \langle t_1 u_1 \rangle^g\), \(v_2 = \langle t_2 u_2 | t'_2 u'_2 \rangle^g\).

By definition of bop,

(II1) \(v_1\) bop \(v_2 = \langle t u \_ \_ \rangle^g\) and \([v_1 \text{ bop } v_2]_1 = (t u)^g\)

where \(t = t_1 \equiv t_2, u = u_1\) bop \(u_2\) and \(g = g_1 \equiv c \equiv g_2\)

By value-projection definition

(II2) \([v_1]_1 = (t_1 u_1)^g\) and \([v_2]_1 = (t_2 u_2)^g\)

By (II2) and definition of bop,

(II3) \([v_1]_1\) bop \([v_2]_1 = (t u)^g\) where \(t = t_1 \equiv t_2, u = u_1\) bop \(u_2\) and \(g = g_1 \equiv c \equiv g_2\)

By (II1) and (II3), the conclusion holds

Subcase III: \(v_1 = (t_1 u_1)^g\), \(v_2 = (t_2 u_2)^g\)

By definition of bop,

(III1) \(v_1\) bop \(v_2 = (t u)^g\) and \([v_1 \text{ bop } v_2]_1 = (t u)^g\)

where \(t = t_1 \equiv t_2, u = u_1\) bop \(u_2\) and \(g = g_1 \equiv c \equiv g_2\)

By value-projection definition

(III2) \([v_1]_1 = (t_1 u_1)^g\) and \([v_2]_1 = (t_2 u_2)^g\)

By (III2) and definition of bop,

(III3) \([v_1]_1\) bop \([v_2]_1 = (t u)^g\) where \(t = t_1 \equiv t_2, u = u_1\) bop \(u_2\) and \(g = g_1 \equiv c \equiv g_2\)

By (III1) and (III3), the conclusion holds

Case: P-Cast

\[ \frac{\delta / e \Downarrow v \quad v' = (E, g) \triangleright v}{\delta / E^g e \Downarrow v'} \]

(1)

By IH and (1)

(2) \(\forall i \in \{1, 2\}, \left[\delta / e\right]_i \Downarrow [v]_i\)

We show for \(i = 1\), the proof is similar for \(i = 2\)

Subcase I: \(v = \langle t_1 u_1 | t_2 u_2 \rangle^g\),

By definition of \(\triangleright\) cast operation,

(I1) \(v' = \langle t'_1 u_1 | t'_2 u_2 \rangle^g\) and \([v']_1 = (t'_1 u_1)^g\) where \(t'_1 = t_1 \Rightarrow E\)

By value-projection definition and cast operation,

(II2) \([v']_1 = (t_1 u_1)^g\) and \([v'']_1 = (t'_1 u_1)^g\) where \(t'_1 = t_1 \Rightarrow E\)

By (I1) and (II2), the conclusion holds

Subcase II: \(v = (t u)^g\),

By definition of cast

(I1) \(v' = (t' u)^g\) and \([v']_1 = (t' u)^g\) where \(t' = t \Rightarrow E\)

By definition of value-projection and cast

(II2) \([v']_1 = (t u)^g\) and \([v''']_1 = (t'' u)^g\) where \(t'' = t \Rightarrow E\)

By (I1) and (II2), the conclusion holds

\(\square\)

Lemma 21 (Soundness). If \(\kappa, \delta / e \overset{\alpha}{\rightarrow} \kappa', \delta' / c'\) where \(\vdash \kappa, \delta / c\) wff,

then \(\forall i \in \{1, 2\}, \left[\kappa, \delta / c\right]_i \overset{\alpha}{{\rightarrow}} [\kappa', \delta' / c']_i\), or \([\kappa, \delta / c]_i = [\kappa', \delta' / c']_i\) and \([\alpha]_i = \cdot\)

Proof. By induction on the structure of the command derivation.
Case: P-SEQ
\[
\begin{align*}
\kappa, \delta / c_1 & \xrightarrow{a} \kappa', \delta' / c'_1 \\
\kappa, \delta / c_1; c_2 & \xrightarrow{a} \kappa', \delta' / c'_1; c'_2
\end{align*}
\]
By IH

(1) \(\forall i \in \{1, 2\}, [\kappa, \delta / c_i]_i \xrightarrow{[\alpha]_i} [\kappa', \delta' / c'_i]_i\), or \( [\kappa, \delta / c_1]_i = [\kappa', \delta' / c'_1]_i\)
By (1), projection of commands and definition of P-SEQ

(2) \(\forall i \in \{1, 2\}, [\kappa, \delta / c_1; c_2]_i \xrightarrow{[\alpha]_i} [\kappa', \delta' / c'_1; c'_2]_i\), or \( [\kappa, \delta / c_1; c_2]_i = [\kappa', \delta' / c'_1; c'_2]_i\)

Case: P-Pc
\[
\kappa, \delta / c \xrightarrow{a} \kappa', \delta' / c'
\]
By IH

(1) \(\forall i \in \{1, 2\}, [\kappa, \delta / c_i]_i \xrightarrow{[\alpha]_i} [\kappa', \delta' / c'_i]_i\), or \( [\kappa, \delta / c]_i = [\kappa', \delta' / c']_i\)
By (1), projection of commands and definition of P-Pc

(2) \(\forall i \in \{1, 2\}, [\kappa \triangleright t_{pc} g_{pc}, \delta / \{c\}]_i \xrightarrow{[\alpha]_i} [\kappa' \triangleright t_{pc} g_{pc}, \delta' / \{c'\}]_i\)
or \( [\kappa \triangleright t_{pc} g_{pc}, \delta / \{c\}]_i = [\kappa' \triangleright t_{pc} g_{pc}, \delta' / \{c'\}]_i\)

Case: P-Pop, P-Skip, P-If, P-While
By projection of commands, well-formedness definition and definition of P-Pop, P-Skip, P-If, P-While

Case: P-Assign
By Lemma 20

(1) \(\forall i \in \{1, 2\}, [\delta / e_i]_i \Downarrow v_i\) such that \(v_i = [v]_i\),
By projection of interval operations, Lemma 10,

(2) \(\forall i \in \{1, 2\}, v' = \text{refineUB}(t_{pc}, v)\) and \(v'_i = \text{refineUB}(t_{pc}, [v]_i)\) and \([v']_i = v'_i\),
By (2),

(3) \([\delta'(x)]_i = [\delta']_i(x)\)
By (1), projection of commands and definition of P-Assign,

(4) \([\delta'(x)]_i = \text{upd} \delta(x) v'\) and \(\delta'_i(x) = \text{upd} [\delta]_i(x) v'_i\),
By (3), (4) and Lemma 8, the conclusion follows

Case: P-Out
By Lemma 20

(1) \(\forall i \in \{1, 2\}, [\delta / e_i]_i \Downarrow [v]_i\),
By projection of interval operations, Lemmas 9 and 10,

(2) \(\forall i \in \{1, 2\}, [\text{refineLB}(v, [\ell, \ell])]_i = \text{refineLB}([v]_i, [\ell, \ell])\),
\(v'' = \text{refineUB}(t_{pc}, v')\) and \(v''_i = \text{refineUB}(t_{pc}, [v']_i)\) and \([v'']_i = v'_{i''}\)
By (1), (2), projection of commands and traces, and definition of P-Out,

(3) \(\forall i \in \{1, 2\}, ([\ell, \ell', v''])_i = ([\ell, \ell', v'']_i)\),

Case: P-If-Refine
By Lemma 20

(1) \(\forall i \in \{1, 2\}, [\delta / e_i]_i \Downarrow [v]_i\),
By Lemma 15,

(2) \(\forall i \in \{1, 2\}, [\text{refineWSet}(\delta, X, t_{pc}, v)]_i = \text{refineWSet}([\delta]_i, X, t_{pc}, [v]_i)\)
By (1), (2), projection of commands and definition of P-If-Refine, the conclusion holds

Case: P-C-Pair where \(c_1\) and \(c_2\) are not skip
We show for \( i = 1, j = 2 \). The proof is similar for \( i = 2, j = 1 \).

By assumption and Lemma 19

\[ (1) \quad \kappa_1 \triangleright (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}), [\delta]_1 / c_1 \overset{\alpha}{\rightarrow} \kappa'_1 \triangleright (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}), [\delta']_1 / c'_1, \]

and \( \kappa_2 = \kappa'_2, c_2 = c'_2, [\delta]_2 = [\delta']_2 \)

By projection of commands

\[ (2) \quad [t_{pc} g_{pc}, \delta / (k_1, t_1, c_1 | k_2, t_2, c_2)_g]_1 = \kappa_1 \triangleright (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta]_1 / \{c_1\} \]

\[ (3) \quad [t_{pc} g_{pc}, \delta' / (k'_1, t_1, c'_1 | k_2, t_2, c'_2)_g]_1 = \kappa'_1 \triangleright (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta']_1 / \{c'_1\} \]

\[ (4) \quad [t_{pc} g_{pc}, \delta / (k_1, t_1, c_1 | k_2, t_2, c_2)_g]_2 = \kappa_2 \triangleright (t_{pc} \gamma_1 t_2) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta]_2 / \{c_2\} \]

\[ (5) \quad [t_{pc} g_{pc}, \delta' / (k'_1, t_1, c'_1 | k_2, t_2, c'_2)_g]_2 = \kappa'_2 \triangleright (t_{pc} \gamma_1 t_2) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta']_2 / \{c'_2\} \]

By definition of P-Pc

\[ (6) \quad (\kappa, \delta / c \overset{\alpha}{\rightarrow} \kappa', \delta' / \{c'\}) \]

By (1), (2), (3) and (6)

\[ (7) \quad \kappa_1 \triangleright (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta]_1 / \{c_1\} \overset{\alpha}{\rightarrow} \kappa'_1 \triangleright (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta']_1 / \{c'_1\} \]

By (4), (5)

\[ (8) \quad \kappa_2 \triangleright (t_{pc} \gamma_1 t_2) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta]_2 / \{c_2\} = \kappa'_2 \triangleright (t_{pc} \gamma_1 t_2) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta']_2 / \{c'_2\} \]

By (7) and (8), the conclusion holds

**Case:** P-C-Pair where \( c_1 = \text{skip} \) and \( c_2 \) is not skip. Similar for the symmetric case.

We show for \( i = 2, j = 1 \).

By assumption and Lemma 19

\[ (1) \quad \kappa_2 \triangleright (t_{pc} \gamma_1 t_2) (g \gamma g_{pc}), [\delta]_2 / c_2 \overset{\alpha}{\rightarrow} \kappa'_2 \triangleright (t_{pc} \gamma_1 t_2) (g \gamma g_{pc}), [\delta']_2 / c'_2, \]

and \( \kappa_1 = \kappa'_1, [\delta]_1 = [\delta'_1] \)

By projection of commands

\[ (2) \quad [t_{pc} g_{pc}, \delta / (k_1, t_1, \text{skip} | k_2, t_2, c_2)_g]_1 = \kappa_1 \triangleright (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta]_1 / \text{skip} \]

\[ (3) \quad [t_{pc} g_{pc}, \delta' / (k'_1, t_1, \text{skip} | k_2, t_2, c'_2)_g]_1 = \kappa'_1 \triangleright (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta']_1 / \text{skip} \]

\[ (4) \quad [t_{pc} g_{pc}, \delta / (k_1, t_1, \text{skip} | k_2, t_2, c_2)_g]_2 = \kappa_2 \triangleright (t_{pc} \gamma_1 t_2) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta]_2 / \{c_2\} \]

\[ (5) \quad [t_{pc} g_{pc}, \delta' / (k'_1, t_1, \text{skip} | k_2, t_2, c'_2)_g]_2 = \kappa'_2 \triangleright (t_{pc} \gamma_1 t_2) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta']_2 / \{c'_2\} \]

By P-Pc

\[ \kappa, \delta / c \overset{\alpha}{\rightarrow} \kappa', \delta' / \{c'\} \]

By (1), (4), (5) and (6)

\[ (7) \quad \kappa_2 \triangleright (t_{pc} \gamma_1 t_2) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta]_2 / \{c_2\} \overset{\alpha}{\rightarrow} \kappa'_2 \triangleright (t_{pc} \gamma_1 t_2) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta']_2 / \{c'_2\} \]

By (1), (2), (3)

\[ (8) \quad \kappa_1 \triangleright (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta]_1 / \text{skip} = \kappa'_1 \triangleright (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta']_1 / \text{skip} \]

By (7) and (8), the conclusion holds

**Case:** P-Skip-Pair

By projection of commands

\[ (1) \quad [t_{pc} g_{pc}, \delta / (\emptyset, t_1, \text{skip} | \emptyset, t_2, \text{skip})_g]_1 = (t_{pc} \gamma_1 t_1) (g \gamma g_{pc}) \triangleright t_{pc} g_{pc}, [\delta]_1 / \{\text{skip}\} \]

By definition of P-Pop

\[ (2) \quad t_{pc} g_{pc} \triangleright \kappa, \delta / \{\text{skip}\} \rightarrow \kappa, \delta / \text{skip} \]

By (1), (2)

\[ (3) \quad [t_{pc} g_{pc}, \delta / (\emptyset, t_1, \text{skip} | \emptyset, t_2, \text{skip})_g]_1 \rightarrow t_{pc} g_{pc}, [\delta]_1 / \text{skip} \]

By (3) and projection of commands, the conclusion holds

**Case:** P-Lift-If

By P-Lift-If

\[ (1) \quad t_{pc} g_{pc}, \delta / (u_1 u_1 | t_1 u_2)_g \text{ then } c_1 \text{ else } c_2 \rightarrow t_{pc} g_{pc}, \delta / (\emptyset, t_1, c_1 | \emptyset, t_2, c_k)_g \]

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By projection of commands

(2) \[ \text{tpc } g_{pc}, \delta / (t_1 u_1 | t_2 u_2)^g \] then \( c_1 \) else \( c_2 \) \[ = \text{tpc } g_{pc}, \delta / (t_1 u_1)^g \] then \( c_1 \) else \( c_2 \)

(3) \[ \text{tpc } g_{pc}, \delta / (t_1, c_1) \] \[ = (\text{tpc } \gamma t_1) (g_{pc} \gamma g) \triangleright \text{tpc } g_{pc}, \delta / \{c_1\} \]

where \( c_j \) is \( c_1 \) if \( u_1 = \text{true} \) and \( c_j \) is \( c_2 \) if \( u_1 = \text{false} \)

(4) \[ \text{tpc } g_{pc}, \delta / (t_2 u_2) \] then \( c_1 \) else \( c_2 \) \[ = (\text{tpc } \gamma t_2) (g_{pc} \gamma g) \triangleright \text{tpc } g_{pc}, \delta / \{c_k\} \]

where \( c_k \) is \( c_1 \) if \( u_2 = \text{true} \) and \( c_k \) is \( c_2 \) if \( u_2 = \text{false} \)

By P-If

(5) \[ \text{tpc } g_{pc}, \delta / c_j \] then \( c_1 \) else \( c_2 \) \[ = (\text{tpc } \gamma t_1) (g_{pc} \gamma g) \triangleright \text{tpc } g_{pc}, \delta / \{c_j\} \]

where \( c_j \) is \( c_1 \) if \( u_1 = \text{true} \) and \( c_j \) is \( c_2 \) if \( u_1 = \text{false} \)

(6) \[ \text{tpc } g_{pc}, \delta / u_2 \] then \( c_1 \) else \( c_2 \) \[ = (\text{tpc } \gamma t_2) (g_{pc} \gamma g) \triangleright \text{tpc } g_{pc}, \delta / \{c_k\} \]

where \( c_k \) is \( c_1 \) if \( u_2 = \text{true} \) and \( c_k \) is \( c_2 \) if \( u_2 = \text{false} \)

By (2-6), the conclusion holds

\[ \square \]

**Theorem 2 (Soundness).** If \( \kappa, \delta / c \longrightarrow^* \kappa', \delta' / c' \) where \( + \kappa, \delta / c \text{ wf} \),

then \( \forall i \in \{1, 2\}, [\kappa, \delta / c]_i \longrightarrow^* [\kappa', \delta' / c']_i \).

**Proof.** By induction on the number of steps in the sequence. Base case follows from assumption.

**Inductive Case:** Holds for \( n \) steps; To show for \( n + 1 \) steps, i.e.,

if \( \kappa, \delta / c \longrightarrow^n \kappa', \delta' / c' \),

then \( \forall i \in \{1, 2\}, [\kappa, \delta / c]_i \longrightarrow^l \kappa_{0i}, \delta_{0i} / c_{0i} \xrightarrow{a_i} \kappa_{1i}, \delta_{1i} / c_{1i} \),

such that \( [\kappa', \delta' / c']_i = \kappa_{1i}, \delta_{1i} / c_{1i} \)

By induction hypothesis,

(1) \( \forall i \in \{1, 2\}, [\kappa_{0i}, \delta_{0i} / c_{0i}] = [\kappa', \delta' / c']_i \) and \( T_i = [T]_i \)

By Lemma 21, Lemma 6, (1) and projection of traces, the conclusion holds

\[ \square \]

**F COMPLETENESS OF PAIRED-EXECUTION**

**Lemma 22 (Expression Completeness).** If \( \forall i \in \{1, 2\}, \delta / e \Downarrow v_i \), then \( \exists v' \text{ s.t. } \delta / e \Downarrow v' \) with \( \{v'\}_i = v_i \)

**Proof.** By induction on the structure of the expression-evaluation derivation.

**Case:** P-Const

By P-Const and projection of values

**Case:** P-Var

By assumption

(1) \( v_i = \text{rd } [\delta]_i(x) \) and \( v' = \text{rd } \delta(x) \)

By (1) and Lemma 7,

(2) \( \text{rd } (\delta(x))_i = \text{rd } [\delta]_i(x) \) and \( \{v'\}_i = v_i \)

**Case:** P-Bop

\[ [\delta]_i / e_1 \Downarrow v_{1i}, [\delta]_i / e_2 \Downarrow v_{2i}, v_i = v_{1i} \oplus v_{2i} \]

(1) \( [\delta]_i / e_1 \text{ bop } e_2 \Downarrow v_i \)

By (1) and IH

(2) \( \delta / e_1 \Downarrow v_{1i}, \delta / e_2 \Downarrow v_{2i}, v_{1i} = [v_{1}]_i \) and \( v_{2i} = [v_{2}]_i \)

T.S. \( \forall i \in \{1, 2\}, [v_1 \text{ bop } v_{2i}]_i = [v_{1}]_i \text{ bop } [v_{2}]_i \)

We show for \( i = 1 \), the proof is similar for \( i = 2 \)
Subcase I: \( v_1 = (t_1 \ u_1 \mid t'_1 \ u'_1)^g, \ v_2 = (t_2 \ u_2 \mid t'_2 \ u'_2)^g \)

By definition of bop,

(I1) \( v_1 \text{ bop } v_2 = (t \ u \ | \ )^g \) and \( [v_1 \text{ bop } v_2]_1 = (t \ u)^g \)

where \( i = t_1 \ \succeq \ t_2, \ u = u_1 \text{ bop } u_2 \) and \( g = g_1 \ \chi \ c \ g_2 \)

By value-projection definition

(I2) \( [v_1]_1 = (t_1 \ u_1)^g \) and \( [v_2]_1 = (t_2 \ u_2)^g \)

By (I2) and definition of bop,

(I3) \( [v_1]_1 \text{ bop } [v_2]_1 = (t \ u)^g \) where \( i = t_1 \ \succeq \ t_2, \ u = u_1 \text{ bop } u_2 \) and \( g = g_1 \ \chi \ c \ g_2 \)

By (I1) and (I3), the conclusion holds

Subcase II: \( v_1 = (t_1 \ u_1 \mid t'_1 \ u'_1)^g, \ v_2 = (t_2 \ u_2)^g, \) similarly for \( v_1 = (t_1 \ u_1)^g, \ v_2 = (t_2 \ u_2 \mid t'_2 \ u'_2)^g \)

By definition of bop,

(II1) \( v_1 \text{ bop } v_2 = (t \ u \ | \ )^g \) and \( [v_1 \text{ bop } v_2]_1 = (t \ u)^g \)

where \( i = t_1 \ \succeq \ t_2, \ u = u_1 \text{ bop } u_2 \) and \( g = g_1 \ \chi \ c \ g_2 \)

By value-projection definition

(II2) \( [v_1]_1 = (t_1 \ u_1)^g \) and \( [v_2]_1 = (t_2 \ u_2)^g \)

By (II2) and definition of bop,

(II3) \( [v_1]_1 \text{ bop } [v_2]_1 = (t \ u)^g \) where \( i = t_1 \ \succeq \ t_2, \ u = u_1 \text{ bop } u_2 \) and \( g = g_1 \ \chi \ c \ g_2 \)

By (II1) and (II3), the conclusion holds

Subcase III: \( v_1 = (t_1 \ u_1)^g, \ v_2 = (t_2 \ u_2)^g \)

By definition of bop,

(III1) \( v_1 \text{ bop } v_2 = (t \ u)^g \) and \( [v_1 \text{ bop } v_2]_1 = (t \ u)^g \)

where \( i = t_1 \ \succeq \ t_2, \ u = u_1 \text{ bop } u_2 \) and \( g = g_1 \ \chi \ c \ g_2 \)

By value-projection definition

(III2) \( [v_1]_1 = (t_1 \ u_1)^g \) and \( [v_2]_1 = (t_2 \ u_2)^g \)

By (III2) and definition of bop,

(III3) \( [v_1]_1 \text{ bop } [v_2]_1 = (t \ u)^g \) where \( i = t_1 \ \succeq \ t_2, \ u = u_1 \text{ bop } u_2 \) and \( g = g_1 \ \chi \ c \ g_2 \)

By (III1) and (III3), the conclusion holds

Case: P-Cast

\[
\frac{[\delta]_i / \ e \Downarrow v_i \quad v'_i = (E, g) \triangleright v_i} {[\delta]_i / \ E^g \ e \Downarrow v'_i}
\]

(1)

By IH and (1)

(2) \( \forall i \in \{1, 2\}, \delta / \ e \Downarrow v \text{ and } [v]_i = v_i \)

We show for \( i = 1 \), the proof is similar for \( i = 2 \)

Subcase I: \( v = (t_1 \ u_1 \mid t_2 \ u_2)^g \),

By definition of \( \triangleright \) cast operation,

(I1) \( v' = (t'_1 \ u_1 \mid t'_2 \ u_2)^g \) and \( [v']_1 = (t'_1 \ u_1)^g \) where \( t'_1 = t_1 \ \models E \)

By value-projection definition and cast operation,

(I2) \( [v]_1 = (t_1 \ u_1)^g \) and \( v'_1 = (t'_1 \ u_1)^g \) where \( t'_1 = t_1 \ \models E \)

By (I1) and (I2), the conclusion holds

Subcase II: \( v = (t \ u)^g \),

By definition of cast

(II1) \( v' = (t' \ u)^g \) and \( [v']_1 = (t' \ u)^g \) where \( t' = t \ \models E \)

By definition of value-projection and cast

(II2) \( [v]_1 = (t \ u)^g \) and \( v''_1 = (t'' \ u)^g \) where \( t'' = t \ \models E \)

By (II1) and (II2), \( [v']_1 = v''_1 \)

\[\square\]
Lemma 23. If \( \kappa, \delta / c \) wf and \( c \) does not contain pairs or braces, then \( \forall i \in \{1, 2\}, [\kappa, \delta / c]_i = \kappa, [\delta]_i / c \)

Proof (sketch). By induction on the structure of \( c \). Most cases use the projection of commands and respective rules. \( c_1; c_2 \) use the IH additionally. \( \square \)

Lemma 24. If \( \forall i \in \{1, 2\}, [\kappa, \delta / c]_i = \kappa_0, \delta_0 / \) skip, and \( \kappa, \delta / c \) wf then \( \exists \kappa', \delta', c' \) s.t. \( \kappa, \delta / c \rightarrow^* \kappa', \delta' / c' \) with \( [\kappa', \delta' / c']_i = \kappa'_i, \delta'_i / c'_i \) and \( [\mathcal{T}]_i = \alpha_i \)

Proof. By induction on the structure of \( c \). If \( [\kappa, \delta / c]_1 = \kappa_0, \delta_0 / \) skip and \( [\kappa, \delta / c]_2 = \kappa_0, \delta_0 / \) skip, then \( c = \) skip (from the projection of command-configurations). If \( c = \) skip, then \( \kappa, \delta / c \) takes 0 steps and the conclusion follows from the assumption. If at least one of the projected runs steps:

Case: \( c = c_1; c_2 \)

By projection of commands, \( [\kappa, \delta / c_1; c_2]_i = \kappa_i, \delta_i / c_1; c_2 \)

By P-SEQ

(1) \( [\kappa, \delta / c_1; c_2]_i = \kappa_i, \delta_i / c_1; c_2 \)

By (2),

(3) \( [\kappa, \delta / c_1]_i = \kappa_i, \delta_i / c_1 \)

By (3) and IH,

(4) \( [\kappa, \delta / c_1]_i = \kappa_i, \delta_i / c_1 \)

By (1), (2), (4)

(5) \( \forall i \in \{1, 2\}, [\kappa', \delta' / c']_i = \kappa'_i, \delta'_i / c'_i \) and \( [\mathcal{T}]_i = \alpha_i \)

By (5), the conclusion holds

Case: \( c = \{c\} \)

By projection of commands, \( [\kappa, \delta / \{c\}]_i = \kappa_i, \delta_i / c_i \)

(1) \( [\kappa > t_{pc} g_{pc}, \delta / \{c\}]_i = \kappa_i > t_{pc} g_{pc}, \delta_i / c_i \)

By P-Pc,

(2) \( [\kappa > t_{pc} g_{pc}, \delta_i / c_i] \)

By (2), IH,

(3) \( [\kappa, \delta / \delta / \} / \{c\}]_i = \kappa'_i, \delta'_i / c'_i \) and \( [\mathcal{T}]_i = \alpha_i \),

By (1), (2) and (3), the conclusion holds

Case: \( c = \) skip; c

By projection of commands, \( [t_{pc} g_{pc}, \delta / \{c\}]_i = t_{pc} g_{pc}, [\delta]_i / \) skip

(1) \( [t_{pc} g_{pc}, \delta / \{c\}]_i = t_{pc} g_{pc}, [\delta]_i / \) skip; c

By P-Skip,
(2) \( t_{pc} g_{pc}, [\delta]_i \) / skip; c \( \rightarrow \) \( t_{pc} g_{pc}, [\delta]_i \) / c

(3) \( t_{pc} g_{pc}, \delta \) / skip; c \( \rightarrow \) \( t_{pc} g_{pc}, \delta \) / c

T.S. \( [t_{pc} g_{pc}, \delta / c]_i = t_{pc} g_{pc}, [\delta]_i / c \)

(4) From well-formedness definition, c does not contain pairs or braces

From Lemma 23 and (4), the conclusion holds

Case: \( c = \{ \text{skip} \} \)

By projection of commands,

\[ [t g, \delta / \text{skip}]_i = t g, [\delta]_i / \text{skip} \]

(1) \( [t g \triangleright t_{pc} g_{pc}, \delta / \{ \text{skip} \}]_i = t g \triangleright t_{pc} g_{pc}, [\delta]_i / \{ \text{skip} \} \)

The rule forces \( k = t g \)

By P-POp,

(2) \( t g \triangleright t_{pc} g_{pc}, \delta / \{ \text{skip} \} \rightarrow t_{pc} g_{pc}, \delta / \{ \text{skip} \} \)

(3) \( t g \triangleright t_{pc} g_{pc}, [\delta]_i / \{ \text{skip} \} \rightarrow t_{pc} g_{pc}, [\delta]_i / \{ \text{skip} \} \)

By (1), (2) and (3), the conclusion holds

Case: \( c = \text{if}(t b)^g \) then \( c_1 \) else \( c_2 \)

By projection of commands,

(1) \( [t_{pc} g_{pc}, \delta / (t b)^g \text{ then } c_1 \text{ else } c_2]_i = t_{pc} g_{pc}, [\delta]_i / (t b)^g \text{ then } c_1 \text{ else } c_2 \)

By P-If,

\[ t'_{pc} = t_{pc} \land t \quad g'_{pc} = g_{pc} \land g \quad c_j = c_1 \text{ if } b = \text{true} \quad c_j = c_2 \text{ if } b = \text{false} \]

(2) \( t_{pc} g_{pc}, [\delta]_i / (t b)^g \text{ then } c_1 \text{ else } c_2 \rightarrow t'_{pc} g'_{pc} \triangleright t_{pc} g_{pc}, [\delta]_i / \{ c_j \} \)

(3) \( t_{pc} g_{pc}, \delta / (t b)^g \text{ then } c_1 \text{ else } c_2 \rightarrow t'_{pc} g'_{pc} \triangleright t_{pc} g_{pc}, \delta / \{ c_j \} \)

Suppose \( b = \text{true}. \) Similar for \( b = \text{false} \)

T.S. \( [t'_{pc} g'_{pc} \triangleright t_{pc} g_{pc}, \delta / \{ c_1 \}]_i = t'_{pc} g'_{pc} \triangleright t_{pc} g_{pc}, [\delta]_i / \{ c_1 \} \)

By projection of commands,

\[ [t'_{pc} g'_{pc}, \delta / c_1]_i = t'_{pc} g'_{pc}, \delta' / c'_1 \]

(4) \( [t'_{pc} g'_{pc} \triangleright t_{pc} g_{pc}, \delta / \{ c_1 \}]_i = t'_{pc} g'_{pc} \triangleright t_{pc} g_{pc}, \delta' / \{ c'_1 \} \)

By well-formedness definition, \( c_1 \) does not contain pairs or braces.

By Lemma 23 and (5), the conclusion holds

Case: \( c = \text{while}^X e \text{ do } c \)

By projection of commands and P-WHILE

Case: \( c = x := e \)

By projection of commands,

T.S. \( [\delta'']_i = \delta' \) where \( \delta' = [\delta]_i[x \mapsto \text{upd} \ [\delta]_i(x)] \) and \( \delta'' = \delta[x \mapsto \text{upd} \delta(x)] \)

By Lemma 22

(1) \( \forall i \in \{1, 2\}, \) if \( [\delta]_i / e \downarrow v_i, \) then \( \delta / e \downarrow v \) such that \( [v]_i = v_i, \)

By (1), Lemma 10,

(2) \( [v''']_i = [\text{refineUB}(t_{pc}, v)]_i = \text{refineUB}(t_{pc}, [v]_i) = v' \)

By Lemma 8 and 7

(3) \( [v''''']_i = \text{upd} \delta(x) v'''_i = \text{upd} [\delta]_i(x) [v''']_i = v''''_i \)

(4) \( \delta''(x) = v''' \) and \( \delta''(x) = v'''' \)

By (3) and (4) the conclusion holds
Case: $c = \text{if}^X e$ then $c_1$ else $c_2$

By projection of commands,
T.S. $[\delta']_i = \delta'_i$ where $\delta'_i = \text{refineWSet}([\delta]_i, X, t_{pc}, v_i)$ and $\delta' = \text{refineWSet}(\delta, X, t_{pc}, v)$
By Lemma 22
(1) $[v]_i = v_i$
By (1), Lemma 15,
(2) $[\text{refineWSet}(\delta, X, t_{pc}, v)]_i = \text{refineWSet}([\delta]_i, X, t_{pc}, [v]_i)$
By (1), (2), the conclusion holds

Case: $c = \text{output}(\ell, e)$

By projection of commands, $[t_{pc} g_{pc}, \delta / \text{skip}]_i = t_{pc} g_{pc}, [\delta]_i / \text{skip}$
T.S. $[\ell, v'']_i = (\ell, v'')$

By Lemma 22
(1) $\forall i \in \{1, 2\}$, if $[\delta]_i / e \Downarrow v_i$, then $\delta / e \Downarrow v$ such that $[v]_i = v_i$
By (1), Lemma 9,
(2) $[v']_i = [\text{refineLB}(v, [\ell, \ell])]_i = \text{refineLB}([v]_i, [\ell, \ell]) = v'_i$
By (2), Lemma 10
(3) $[v'']_i = [\text{refineUB}(t_{pc}, v')]_i = \text{refineUB}(t_{pc}, [v']_i) = v''_i$
By (3) and projection of traces, the conclusion holds

Case: $c = (\kappa_1, t_1, c_1 | \kappa_2, t_2, c_2)$ where $c_1$ and $c_2$ are not skip

By projection of commands
(1) $[t_{pc} g_{pc}, \delta / \text{skip}]_i = (t_{pc} g_{pc}, [\delta]_i / \text{skip})$
By assumption, P-Pc and (1)
(2) $\kappa_1 \triangleright (t_{pc} \gamma t_1) g_{pc}, [\delta]_i / \{v_i\}$
(3) $\kappa_2 \triangleright (t_{pc} \gamma t_2) g_{pc}, [\delta]_i / \{v_i\}$
Suppose $i = 1, j = 2$ followed by $i = 2, j = 1$. Similar for the symmetric case

By (4), Lemma 19,
(5) $\kappa_1 \triangleright (t_{pc} \gamma t_1) g_{pc}, [\delta]_1 / \{v_i\}$
(6) $\kappa_2 \triangleright (t_{pc} \gamma t_2) g_{pc}, [\delta]_2 / \{v_i\}$
By (7), T.S. $\kappa_1 \triangleright (t_{pc} \gamma t_1) g_{pc}, [\delta]_1 / \{v_i\}$

Case: $c = (\kappa_1, t_1, c_1 | \kappa_2, t_2, c_2)$ where $c_1$ is skip and $c_2$ is not. Similar for $c_2$ is skip and $c_1$ is not skip

By projection of commands
(1) $[t_{pc} g_{pc}, \delta / \text{skip}]_i = (t_{pc} g_{pc}, [\delta]_i / \text{skip})$
(2) \[ t_{pc}(g_{pc}, \delta / \langle \kappa_1, t_1, \text{skip} | \kappa_2, t_2, c_2 \rangle) = \kappa_2 \triangleright (t_{pc} \triangleright t_2)(g_{pc} \triangleright g) \triangleright t_{pc}(g_{pc}, [\delta]'_2 / \{c_2\}) \]
(1) does not proceed as the projection of \( c \) gives skip

By assumption, P-Pc and (2)

(3) \[ \kappa_2 \triangleright (t_{pc} \triangleright t_2)(g_{pc} \triangleright g) \triangleright t_{pc}(g_{pc}, [\delta]'_2 / \{c_2\}) \]

(4) \[ \kappa_2 \triangleright (t_{pc} \triangleright t_2)(g_{pc} \triangleright g), [\delta]'_2 / c_2 \rightarrow \kappa''_2, [\delta''_2 / c''_2] \]

Suppose \( i = 2, j = 1 \).

(5) \[ t_{pc}(g_{pc}, \delta / \langle \kappa_1, t_1, \text{skip} | \kappa_2, t_2, c_2 \rangle) \rightarrow t_{pc}(g_{pc}, \delta' / \langle \kappa_1, t_1, \text{skip} | \kappa'_2, t'_2, c'_2 \rangle) \]

By Lemma 19,

(6) \[ \kappa_2 \triangleright (t_{pc} \triangleright t_2)(g_{pc} \triangleright g), [\delta]'_2 / c_2 \rightarrow [\alpha']_2 \rightarrow \kappa''_2 \triangleright (t_{pc} \triangleright t_2)(g_{pc} \triangleright g), [\delta'']'_2 / c''_2, \]

\[ [\delta]'_1 = [\delta]_1 \text{ and } [\alpha']_1 = . \]

T.S. \[ t_{pc}(g_{pc}, \delta' / \langle \kappa_1, t_1, \text{skip} | \kappa'_2, t'_2, c'_2 \rangle) \]

(7) \[ \kappa_2 \triangleright (t_{pc} \triangleright t_2)(g_{pc} \triangleright g), [\delta]'_2 / \{c_2\} \]

By projection of commands,

(8) \[ t_{pc}(g_{pc}, \delta' / \langle \kappa_1, t_1, \text{skip} | \kappa'_2, t'_2, c'_2 \rangle) \rightarrow \kappa'_2 \triangleright (t_{pc} \triangleright t_2)(g_{pc} \triangleright g), [\delta'']'_2 / \{c''_2\} \]

By (8), T.S. \[ \kappa''_2 \triangleright (t_{pc} \triangleright t_2)(g_{pc} \triangleright g) = \kappa''_2, [\delta'']'_2 = [\delta'']'_2, c''_2 = c''_2 \text{ and } [\alpha']_2 = \alpha_2 \]

By (4) and (6)

(9) \[ \kappa''_2 = \kappa''_2 \rightarrow (t_{pc} \triangleright t_2)(g_{pc} \triangleright g), \delta'' = [\delta'']'_2, c''_2 = c''_2, \alpha_2 = [\alpha']_2 \]

By (6), (7), (9) and projection of traces, the conclusion holds

Case: \( c = \langle \emptyset, t_1, \text{skip} | \emptyset, t_2, \text{skip} \rangle \)

By projection of commands

(1) \[ t_{pc}(g_{pc}, \delta / \langle \emptyset, t_1, \text{skip} | \emptyset, t_2, \text{skip} \rangle) = (t_{pc} \triangleright t_1)(g_{pc} \triangleright g) \triangleright t_{pc}(g_{pc}, [\delta]'_1 / \{\text{skip}\}) \]

(2) \[ t_{pc}(g_{pc}, \delta / \{\text{skip}\}) = t_{pc}(g_{pc}, [\delta]'_1 / \{\text{skip}\}) \]

By (1) and definition of P-Pop

(3) \[ (t_{pc} \triangleright t_1)(g_{pc} \triangleright g) \rightarrow t_{pc}(g_{pc}, [\delta]'_1 / \{\text{skip}\}) \rightarrow t_{pc}(g_{pc}, [\delta]'_1 / \{\text{skip}\}) \]

By (2) and (3), the conclusion holds

Case: \( c = \text{if } \langle t_1, u_1 | t_2, u_2 \rangle \) then \( c_1 \) else \( c_2 \)

By P-LIFT-IF

(1) \[ t_{pc}(g_{pc}, \delta / \langle t_1, u_1 | t_2, u_2 \rangle) \rightarrow t_{pc}(g_{pc}, \delta / \langle \emptyset, t_1, c_1 | \emptyset, t_2, c_2 \rangle) \]

By projection of commands

(2) \[ t_{pc}(g_{pc}, \delta / \langle t_1, u_1 | t_2, u_2 \rangle) \rightarrow t_{pc}(g_{pc}, [\delta]'_1 / \{c_2\}) \rightarrow t_{pc}(g_{pc}, [\delta]'_1 / \{c_2\}) \]

where \( c_2 = \text{if } u_1 = \text{true} \) and \( c_2 = \text{if } u_1 = \text{false} \)

(3) \[ t_{pc}(g_{pc}, \delta / \langle t_1, u_1, c_1 | t_2, c_2 \rangle) = (t_{pc} \triangleright t_1)(g_{pc} \triangleright g) \triangleright t_{pc}(g_{pc}, [\delta]'_1 / \{c_1\}) \]

where \( c_2 = \text{if } u_2 = \text{true} \) and \( c_2 = \text{if } u_2 = \text{false} \)

By P-IF

(5) \[ t_{pc}(g_{pc}, [\delta]'_2 / \{\text{skip}\}) \rightarrow t_{pc}(g_{pc}, [\delta]'_2 / \{\text{skip}\}) \rightarrow t_{pc}(g_{pc}, [\delta]'_2 / \{\text{skip}\}) \]

where \( c_2 = \text{if } c_2 = \text{if } u_2 = \text{true} \) and \( c_2 = \text{if } u_2 = \text{false} \)

By (2), (3), (4), (5), (6), the conclusion holds

\[ \square \]

Theorem 3 (Completeness). If \( \forall i \in \{1, 2\}, [\kappa, \delta / c]_i \rightarrow^* \kappa_i, \delta_i / \text{skip and } \triangleright^* \kappa, \delta / c \rightarrow^* \kappa', \delta' / \text{skip with } \triangleright^* \kappa', \delta' / \text{skip} \rightarrow^* \kappa_i, \delta_i / \text{skip and } [\triangleright^*]_i = \triangleright^* \]

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Proof. Suppose \( \forall i \in \{1, 2\}, [\kappa, \delta / c]_i \xrightarrow{\tau_i}^{n_i} \kappa_i, \delta_i / \text{skip} \). By induction over the number of steps \( n_1, n_2 \) in the two runs.

**Base Case:**

1. \( \forall i \in \{1, 2\}, [\kappa, \delta / c]_i = \kappa_i, \delta_i / \text{skip} \)
   
   By (1)
2. \( c = \text{skip} \) and \( [\delta]_i = \delta_i \)
   
   By assumption
3. \( \kappa, \delta / c = \kappa', \delta' / \text{skip} \)
   
   By (2), (3)
4. \( [\kappa', \delta' / \text{skip}]_i = \kappa_i, \delta_i / \text{skip} \)

**Inductive Case:**

By assumption

1. \( [\kappa, \delta / c]_i \xrightarrow{\alpha_i} \kappa'_i, \delta'_i / c'_i \) or \( [\kappa, \delta / c]_i \) ends in \( \text{skip} \) and doesn’t step
   
   By Lemma 24,
2. \( \exists \kappa_0, \delta_0, c_0, \kappa, \delta / c \xrightarrow{\tau'} \kappa_0, \delta_0 / c_0 \)
   
   By (1), (2), and Theorem 2,
3. \( \forall i \in \{1, 2\}, [\kappa_0, \delta_0 / c_0]_i = \kappa'_i, \delta'_i / c'_i \) and \( [\tau']_i = \alpha_i \)
   
   Conclusion follows by IH

\[ \square \]

G PRESERVATION

We define the following constraints on configurations to facilitate proofs related to paired values and commands. We start by defining \( i \in H(\ell_A), \Pi \in H(\ell_A) \) and \( \kappa \in H(\ell_A) \) for any observer at level \( \ell_A \).

\[
\begin{align*}
\frac{t = [\ell_t, \ell_I]}{i \in H(\ell_A)} & \quad \frac{\Pi = (t_1 \mid t_2)}{t_i \in H(\ell_A) \text{ where } i \in \{1, 2\}} & \quad \frac{\kappa \in H(\ell_A)}{\kappa \in H(\ell_A) \text{ where } i \in H(\ell_A)} \\
& \frac{\kappa = t \triangleright \Delta \triangleright \kappa'}{i \in H(\ell_A)} & \quad \frac{(\kappa' \in H(\ell_A) \lor \kappa' = \emptyset)}{\kappa \in H(\ell_A)}
\end{align*}
\]

We say a configuration is safe (written \( \vdash \kappa, \delta \vdash_i c \) sf for \( i \in \{\cdot, 1, 2\} \)) if all of the following hold

1. If \( i \in \{1, 2\} \), then \( \kappa \in H(\ell_A), \forall x \in \text{WtSet}(c), \text{intvl}(\delta(x)) \in H(\ell_A) \)
2. If \( c = \text{if} \langle \_ \_ \_ \rangle \) then \( c_1 \) else \( c_2 \), then \( \forall x \in \text{WtSet}(c), \text{intvl}(\delta(x)) \in H(\ell_A) \)
3. If \( c = \langle \kappa_1, t_1, c_1 \mid \kappa_2, t_2, c_2 \rangle_g \), then \( \forall i \in \{1, 2\}, t_i \vdash g \in H(\ell_A) \), and \( \forall x \in \text{WtSet}(c), \text{intvl}(\delta(x)) \in H(\ell_A) \)

**Lemma 25 (Interval Refine).** \( \forall i \in \{1, 2\}, \text{refine}(t_1, t_2) \subseteq t_i \)

Proof (Sketch). By examining the definitions of the operations.

**Lemma 26 (Join Refine).** \( \forall i \in \{1, 2\}, t_i \subseteq \gamma(g_i) \implies i \uparrow t_1 \subseteq \gamma(g_1) \cup \subseteq g_2 \).

Proof (Sketch). By examining the definitions of the operations.

**Lemma 27 (Evidence Subtyping).** \( E \vdash g_1 \leq g_2, t \subseteq \gamma(g_1) \implies t \Rightarrow E \subseteq \gamma(g_2) \).

Proof.

Let \( E = (t_1, t_2) \). By inversion, \( t_2 \subseteq \gamma(g_2) \).

By Lemma 25, \( t \Rightarrow E \subseteq t_2 \)
By $\subseteq$ is transitive, $t \subseteq \gamma(g_2)$.

\[\square\]

**Lemma 28 (Refinement maintains high label).** $t_1 \subseteq t_2$ and $t_2 \in H(\ell_A)$ imply $t_1 \in H(\ell_A)$

**Proof (sketch).** By examining the definitions of the operations.

\[\square\]

**Lemma 29 (Consistent subtyping maintains high).** $t \vdash g \in H(\ell_A)$, $E \vdash g \preceq \gamma g'$, and imply $t \gg E \vdash g' \in H(\ell_A)$

**Proof.**

By inversion, $t = [\ell_1, \ell_r], \ell_1 \not\approx \ell_A, t \subseteq \gamma(g)$

Assume $t \gg E = (\ell_x, \ell_y)$,

By the definition of $t \gg E, \ell_1 \preceq \ell_x$

Therefore, $\ell_x \not\approx \ell_A$

By Lemma 27, $t \gg E \sqsubseteq \gamma(g')$.

Therefore, $t \gg E \vdash g' \in H(\ell_A)$

\[\square\]

**Lemma 30 (Join maintains high).** $t \vdash g \in H(\ell_A), t' \sqsubseteq \gamma(g')$ imply $t \gamma t' \vdash (g \gamma c g') \in H(\ell_A)$

**Proof.**

By inversion, $t = [\ell_1, \ell_r], \ell_1 \not\approx \ell_A, t \subseteq \gamma(g)$

Assume $t \gamma t' = (\ell_x, \ell_y)$,

By the definition of $t \gamma t', \ell_1 \preceq \ell_x$

Therefore, $\ell_x \not\approx \ell_A$

By Lemma 26, $t \gamma t' \sqsubseteq \gamma(g \gamma c g')$.

Therefore, $t \gamma t' \vdash g \gamma c g' \in H(\ell_A)$

\[\square\]

**Lemma 31 (Preservation (cast)).** $\Gamma \vdash v: U_1$, $E \vdash U_1 \preceq c U_2$ and $U_2 = \tau g'$ imply $\Gamma \vdash (E, g') \triangleright v : U_2$.

**Proof.** By examining the definition of $(E, g') \triangleright v$.

**Case:** $v = (t u)^g$

By assumptions,

(1) $t' = t \gg E$ and $(E, g') \triangleright v = (t' u)^g$

By inversion of typing of $v$

(2) $U_1 = \tau g$, $t \subseteq \gamma(g)$

By inversion of subtyping

(3) $E \vdash g \leq c g'$

By Lemma 27 on (1), (3)

(4) $t' \sqsubseteq \gamma(g')$

By applying the same typing rule using (4), the conclusion holds

**Case:** $v = (t_1 u_1 | t_1 u_1)^g$

By assumptions,

(1) $t' = t \gg E (i \in \{1, 2\})$ and $(E, g') \triangleright v = (t' u_1 | t'_2 u_2)^g$

By inversion of typing of $v$,

(2) $U_1 = \tau g$, and for all $i \in \{1, 2\} t_i \sqsubseteq \gamma(g), u_i \vdash g \in H(\ell_A)$

By inversion of subtyping

(3) $E \vdash g \leq c g'$

\[\square\]
By Lemma 27 on (1), (3)

(4) \( t'_i \subseteq \gamma(g') \)

By Lemma 29

(5) \( t'_i \vdash g' \in H(\ell_A) \)

By applying the same typing rule using (4) and (5), the conclusion holds 

\[ \Box \]

**Lemma 32 (Preservation (bop)).** If \( \forall i \in \{1, 2\}, \Gamma \vdash v_1 : \tau_{g_1}, \text{then } \Gamma \vdash v_1 \text{ bop } v_2 : \tau_{g_1 \cap g_2} \).

**Proof (sketch).** By examining the definitions of \( v_1 \) bop \( v_2 \). Apply Lemma 26 and Lemma 30.

\[ \Box \]

**Lemma 33 (Preservation (expression)).** If \( E :: \delta / e \downarrow v, \vdash \delta : \Gamma, \text{and } \Gamma \vdash e, \text{then } \Gamma \vdash v \).

**Proof.** By induction on the structure of \( E \). The proof is straightforward when \( E \) ends in P-Const or P-Var.

**Case: \( E \) ends in P-Cast**

By assumptions,

\[ \delta / e \downarrow v \quad v' = (E, g') \triangleright v \]

(1) \( \delta / E' e \downarrow v' \)

(2) \( \vdash \delta : \Gamma, \text{and } \Gamma \vdash E' e : U_2 \)

By inversion of typing

(3) \( \Gamma \vdash e : U_1, E \vdash U_1 \leq c U_2 \text{ and } U_2 = \tau_{g'} \)

By I.H. on \( e \)

(4) \( \Gamma \vdash v : U_1 \)

By Lemma 31, \( \Gamma \vdash v' : U_2 \)

**Case: \( E \) ends in P-Bop**

By assumptions,

\[ \delta / i_1 \downarrow v_1 \quad \delta / i_2 \downarrow v_2 \quad v = v_1 \text{ bop } v_2 \]

(1) \( \delta / i_1 \text{ bop } i_2 \downarrow v \)

(2) \( \vdash \delta : \Gamma, \text{and } \Gamma \vdash i_1 \text{ bop } i_2 : U \)

By inversion of typing, \( \forall i \in \{1, 2\} \)

(3) \( \Gamma \vdash i_1 : \tau_{g_1}, \text{and } U = \tau_{g_1 \cap g_2} \)

By I.H. on \( i_1 \)

(4) \( \Gamma \vdash v_1 : \tau_{g_1} \)

By Lemma 32, \( \Gamma \vdash v : U \)

\[ \Box \]

**Lemma 34 (PC refinement).** If \( \Gamma; t_{pc} g_{pc} \vdash c, \text{and } t \subseteq t_{pc}, \text{then } \Gamma; t g_{pc} \vdash c. \)

**Proof (sketch).** By induction over the typing derivation of \( c. \)

\[ \Box \]

**Lemma 35.** If \( \vdash v : \tau^g, \text{refineUB}(\Pi, v) = v', \text{and when } i \in \{1, 2\} \text{ or } \Pi = \langle i | i' \rangle, \text{then } \vdash v' : \tau^g \text{ and intvl}(v') \in H(\ell_A) \text{ when } i \in 1, 2. \)

**Proof.** By examining all the rules. Use Lemma 25 to show that the resulting intervals are still \( H. \)

**Lemma 36.** If \( \vdash v : \tau^g, \text{refineLB}(v, i) = v', \text{then } \vdash v' : \tau^g \text{ and intvl}(v') \subseteq i. \)

**Proof (sketch).** By examining all the rules. Use Lemma 25.
LEMMA 37. If $\vdash \alpha : \Gamma$ and $\text{rfle}_{\delta, \Pi}(\delta, X, \Pi) = \delta'$ and when $i \in \{1, 2\}$ or $\Pi = \langle i | i' \rangle$, $\Pi \in H(\ell_A)$, then $\vdash \delta' : \Gamma$ and when $\Pi \in H(\ell_A)$, $\forall x \in X$, intvl($\delta'(x)$) $\in H(\ell_A)$.

PROOF. By induction over the size of $X$ and apply Lemma 35. \hfill \Box

LEMMA 38. If $\vdash \delta : \Gamma$, $\delta' = \text{refineWSet}_{\Pi}(\delta, X, t_{pc}, v)$, and when $i \in \{1, 2\}$, $t_{pc} \in H(\ell_A)$ then $\vdash \delta' : \Gamma$ and when $t_{pc} \in H(\ell_A)$ or intvl($v$) $\in H(\ell_A)$, $\forall x \in X$, intvl($\delta'(x)$) $\in H(\ell_A)$.

PROOF. Follows from Lemma 37. \hfill \Box

LEMMA 39. If $\vdash v_n : \tau^g$, $\vdash v_o : \tau^g$, and when $i \in \{1, 2\}$, intvl($v_o$) $\in H(\ell_A)$ and intvl($v_n$) $\in H(\ell_A)$, then $\vdash \text{upd}_i v_o v_n : \tau^g$.

PROOF. By examining the definition of $\text{upd}_i v_o v_n$

Case: $\text{upd}_i v_o v_n$ where either $v_o$ or $v_n$ is a pair

By inversion of typing rules, $\text{intvl}(v_o) \in H(\ell_A)$ or $\text{intvl}(v_n) \in H(\ell_A)$

By Lemma 25 and Lemma 28, $\text{intvl} \text{(\text{upd}_i v_o v_n)} \in H(\ell_A)$

Therefore V-PAIR applies and $\vdash \text{upd}_i v_o v_n : \tau^g$

Case: $\text{upd}_i v_o v_n$ where neither $v_o$ nor $v_n$ is a pair

We use Lemma 25 and transitivity of $\subseteq$ and apply value typing rule directly.

Case: $\text{upd}_i v_o v_n$, where $i \in \{1, 2\}$

By assumption, $\text{intvl}(v_o) \in H(\ell_A)$ and $\text{intvl}(v_n) \in H(\ell_A)$

By Lemma 25 and Lemma 28, the updated value’s interval is in $H(\ell_A)$

Therefore V-PAIR applies and $\vdash \text{upd}_i v_o v_n : \tau^g$ \hfill \Box

LEMMA 40 (Preservation one-step). If $\mathcal{E} :: \kappa, \delta /_i c \xrightarrow{\alpha} \kappa', \delta'/_i c'$, $\vdash \delta : \Gamma$ and $\mathcal{D} :: \Gamma; \kappa \vdash_r c$, and $\vdash \kappa, \delta /_i c$ sf then $\vdash \delta' : \Gamma; \kappa' \vdash_r c'$, $\vdash \alpha$, and $\vdash \kappa', \delta'/_i c'$ sf.

PROOF. By induction on the structure of $c$

Case: $c = \text{skip}$

There is no rule to step skip, so the conclusion holds trivially.

Case: $c = c_1; c_2$

By inversion of $\mathcal{D}$

(1) $\kappa = \kappa' ; t_{pc} g_{pc}, \Gamma; t_{pc} g_{pc} \vdash c_2$, and $\Gamma; \kappa \vdash_r c_1$ or $(\Gamma; t_{pc} g_{pc} \vdash c_1$ and $\kappa = t_{pc} g_{pc})$

By examining $\mathcal{E}$, there are two cases: P-SEQ or P-SKIP applies

Subcase I: $\mathcal{E}$ ends in P-SEQ

By assumption

$\kappa, \delta /_i c_1 \xrightarrow{\alpha} \kappa', \delta'/_i c'_1$

(12) $\kappa, \delta /_i c_1; c_2 \xrightarrow{\alpha} \kappa', \delta'/_i c'_1; c_2$

By I.H. on $c_1$

(13) $\vdash \delta' : \Gamma; \kappa' \vdash_r c'_1$ and $\vdash \alpha$

By Lemma 18, (13), (1), either R-END or R-C-SEQ applies

(14) $\Gamma; \kappa' \vdash_r c'_1; c_2$

Subcase II: $\mathcal{E}$ ends in P-SKIP

(III2) $\kappa, \delta /_i \text{skip}; c_2 \xrightarrow{\text{skip}} \kappa, \delta /_i c_2$

By (1) and $c_1 = \text{skip}$

(III3) $\kappa = t_{pc} g_{pc}$

By (1) and (III3) and R-END
\[(\Pi 4) \quad \Gamma; \kappa \vdash_{r} c_2\]

**Case:** \(c = x := e\)

By inversion of \(D\)

1. \(\kappa \vdash_{pc} g_{pc}, \Gamma \vdash x : \tau^g, \Gamma \vdash e : \tau^g, \text{ and } g_{pc} \preceq_c g\)

By examining \(E\), only P-ASSIGN applies

\[
\frac{\delta / i e \downarrow v'}{v'' = \text{refineUB}(t_{pc}, v')} \quad \delta' = \delta[x \mapsto \text{ upd}_i \delta(x) v'']
\]

By Lemma 33, \(\Gamma \vdash v' : \tau^g\).

By Lemma 35, \(\Gamma \vdash v'' : \tau^g\), and \(\text{intvl}(v'') \in H(\ell_A)\) when \(i \in \{1, 2\}\)

By assumption, \(\text{intvl}(\delta(x)) \in H(\ell_A)\)

By Lemma 39, \(\Gamma \vdash \text{ upd}_i \delta(x) v'' : \tau^g\)

By store typing \(\vdash \delta' : \Gamma\)

**Case:** \(c = \text{output}(\ell, e)\)

By inversion of \(D\)

1. \(\kappa \vdash_{pc} g_{pc}, \Gamma \vdash e : \tau^g, g \preceq_c \ell, g_{pc} \preceq_c \ell\)

By examining \(E\), only P-OUT applies

\[
\frac{\delta / i e \downarrow v'}{v'' = \text{refineLB}(v', [\ell, \ell])} \quad v_1 = \text{refineUB}(t_{pc}, v'')
\]

By Lemma 33

1. \(\Gamma \vdash v' : \tau^g\)

By Lemma 36

1. \(\Gamma \vdash v'' : \tau^g\) and \(\text{intvl}(v'') \subseteq [\ell, \ell]\)

By Lemma 35

1. \(\Gamma \vdash \text{ output}_i (\ell, e) \subseteq \text{intvl}(v_1)\)

By T-A-*, \(\vdash (i, \ell, v_1)\)

**Case:** \(c = \text{if}^X e \text{ then } c_1 \text{ else } c_2\)

By inversion of \(D\)

1. \(\kappa \vdash_{pc} g_{pc}, \Gamma \vdash e : \text{ bool}^g, t_c = \gamma(g_c), \)

2. \(\Gamma; t_{pc} \not\vdash_{pc} g_{pc} \preceq_c g_c \vdash c_1, \text{ where } i \in \{1, 2\}\) and

3. \(X = \text{ WtSet}(c_1) \cup \text{ WtSet}(c_2)\)

By examining \(E\), only P-IR-REFINE applies

\[
\frac{\delta / i e \downarrow v}{\delta' = \text{refineWSet}_i(\delta, X, t_{pc}, v)}
\]

By Lemma 33

1. \(\Gamma \vdash \text{ bool}^g\),

By (1) and R-C-IF

1. \(\Gamma; t_{pc} g_{pc} \vdash v \text{ if } v \text{ then } c_1 \text{ else } c_2\)

By Lemma 38, \(\vdash \delta' : \Gamma\)

By \(\vdash t_{pc} g_{pc}, \delta / i c : \text{ sf}\)

1. \(t_{pc} \in H(\ell_A)\) when \(i \in \{1, 2\}\)

By Lemma 38,

1. \(\text{ when } i \in \{1, 2\} \text{ or } \text{intvl}(v) \in H(\ell_A), \forall x \in X, \text{intvl}(\delta'(x)) \in H(\ell_A)\).

By sf definition, \(\vdash t_{pc} g_{pc}, \delta' / i \text{ if } v \text{ then } c_1 \text{ else } c_2 \text{ sf}\)
Case: $c = \text{if} \upsilon \text{then } c_1 \text{ else } c_2$

By inversion of $\mathcal{D}$

(1) $\kappa = t_{pc} g_{pc}, \Gamma \vdash e : \text{bool}^g, \ i_c = \gamma(g_c), \Gamma; t_{pc} \gamma \ i_c \ g_{pc} \ \gamma_c \ g_c \vdash c_i, \text{where } i \in \{1, 2\}$

By examining $\mathcal{E}$, there are three cases: P-Lift-If, P-If applies

Subcase I: $v = \langle t \text{ true} \rangle^g$

By assumption, $v = \langle t \text{ true} \rangle^g$

\[ t'_{pc} = t_{pc} \gamma \ i \quad g'_{pc} = g_{pc} \gamma_c \ g \]

By inversion of typing for $e$

(13) $\vdash g_c = g$

By Lemma 34 and (13), and (1)

(14) $\Gamma; t_{pc} \gamma \ i'_c \ g_{pc} \ \gamma_c \ g_c \vdash c_i$

By (14) and R-Pop, $\Gamma; t'_{pc} g_{pc} \triangleright t_{pc} g_{pc} \vdash \{c_i\}$.

Subcase II: $v = \langle t \text{ false} \rangle^g$, the proof is similar to the previous case.

Subcase III: $E$ ends in P-Lift-If,

By assumption, $v = \langle t_1 \ u_1 \mid t_2 \ u_2 \rangle^g$

\[ i = \{1, 2\} \quad c_j = c_1 \text{ if } u_1 = \text{true} \quad c_j = c_2 \text{ if } u_1 = \text{false} \]

\[ c_k = c_1 \text{ if } u_2 = \text{true} \quad c_k = c_2 \text{ if } u_2 = \text{false} \]

By inversion of typing for $v$

(III3) $g_c = g, \ i_1 = \gamma(g), \text{ and } t_i \vdash g \in \mathcal{H}(\ell_A)$

By similar arguments used in the previous subcase,

(III4) $\Gamma; t_{pc} \gamma \ i_c \ g_{pc} \ \gamma_c \ g_c \vdash c_i$

By (III3), (III4) and R-C-Pair, $\Gamma; t_{pc} g_{pc} \vdash \langle \emptyset, t_1, c_j \mid \emptyset, t_2, c_k \rangle_g$

By $\vdash t_{pc} g_{pc}, \delta / i \ c : \text{sf}$

(III5) $\forall x \in \text{WtSet}(c_i), \text{intvl}(\delta(x)) \in \mathcal{H}(\ell_A)$

By $t_i \in \mathcal{H}(\ell_A)$,

(III6) $t_i \in \mathcal{H}(\ell_A)$,

By (III5) and (III6), $\vdash t_{pc} g_{pc}, \delta / \langle \emptyset, t_1, c_j \mid \emptyset, t_2, c_k \rangle_g : \text{sf}$

Case: $c = \text{while X e do c'}$

By inversion of $\mathcal{D}$

(1) $\kappa = t_{pc} g_{pc}, \Gamma \vdash e_1 : \text{bool}^g, \ i_c = \gamma(g_c), \Gamma; t_{pc} \gamma \ i_c \ g_{pc} \ \gamma_c \ g \vdash c', \ X = \text{WtSet}(c')$

By examining $\mathcal{E}$, only P-While applies

(2) $t_{pc} g_{pc}, \delta / i \ \text{while X e do c'} \rightarrow t_{pc} g_{pc}, \delta / i \ \text{if X e then } (c') \text{ while X e do c'}$ else skip

To type the resulting if statement, we need to show the following:

$\Gamma; t_{pc} \gamma \ i_c \ g_{pc} \ \gamma_c \ g \vdash \text{while X e do c'}$

Because expression typing does not use pc context and

$\gamma (\gamma_c)$ the same interval (label) twice does not change the result, the conclusion holds

Case: $c = \{c'\}$

By inversion of $\mathcal{D}$

(1) $\kappa = \kappa_1 \triangleright t_{pc} g_{pc}, \Gamma; \kappa_1 \vdash c'$

By examining $\mathcal{E}$, there are two cases: P-Pc or P-Pop applies

Subcase I: $E$ ends in P-Pop

By $c' = \text{skip}$ and (1)

(12) $\kappa_1 = t \ g$

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By assumption

(I3) \( \vdash g \triangleright g_{pc}, \delta / i \{\text{skip}\} \rightarrow g_{pc}, \delta / i \text{ skip} \)

By G-Skip and R-END

(I4) \( \Gamma; g_{pc} \triangleright_r \text{skip} \)

**Subcase II:** \( E \) ends in P-Pc

By assumption

\[ \kappa, \delta / i \{c\} \xrightarrow{c} \kappa', \delta' / c' \]

(II2) \( \kappa \triangleright g_{pc}, \delta / i \{c\} \xrightarrow{\alpha} \kappa' \triangleright g_{pc}, \delta' / i \{c'\} \)

By I.H. on \( c \)

(II3) + \( \delta': \Gamma, \kappa'; \triangleright_r c' \) and + \( \alpha \)

By R-Pop

(II4) \( \Gamma; \kappa' \triangleright g_{pc} \triangleright_r \{c'\} \)

**Case:** \( c = \langle \kappa_1, i_1, c_1 | \kappa_2, i_2, c_2 \rangle_g \)

By inversion of \( D \)

(1) \( \kappa = g_{pc}, \Gamma; \kappa_i \triangleright (g_{pc} \triangleright y_i) (g_{pc} \triangleright c_i) \) for \( i \in \{1, 2\} \)

By examining \( E \), there are two cases: P-Skip-PAIR or P-C-PAIR applies

**Subcase I:** \( E \) ends in P-Skip-PAIR

(I2) \( \kappa_i = \emptyset \)

By assumption

(I3) \( g_{pc}, \delta / \langle \emptyset, i_1, \text{skip} | \emptyset, i_2, \text{skip} \rangle_g \rightarrow g_{pc}, \delta / \text{skip} \)

By G-Skip and R-END

(I4) \( \Gamma; g_{pc} \triangleright_r \text{skip} \)

**Subcase II:** \( E \) ends in P-C-PAIR

By assumption

\[ \kappa_i \triangleright g_{pc} \triangleright y_i g_{pc} \triangleright c_i \]

(II2) \( g_{pc}, \delta / \langle \kappa_i, i_1, c_1 | \kappa_2, i_2, c_2 \rangle_g \xrightarrow{c_i} g_{pc}, \delta' / \langle \kappa_i', i_1, c_1' | \kappa_2', i_2, c_2' \rangle_g \)

Assume \( c_1 \) takes a step. The other case when \( c_2 \) takes a step can be proven similarly.

By I.H. on \( c_1 \)

(II3) + \( \delta': \Gamma, \kappa'; \triangleright_r g_{pc} \triangleright y_i g_{pc} \triangleright c_1' \) and + \( \alpha \)

By R-C-PAIR, (II3), and (1)

(II4) \( \Gamma; g_{pc} \triangleright \langle \kappa_i', i_1, c_1' | \kappa_2', i_2, c_2' \rangle_g \)

\[ \Box \]

**Lemma 41.** If \( \kappa, \delta / c \xrightarrow{\alpha} \kappa', \delta' / c' \) with + \( \kappa, \delta, c \) and + \( \kappa, \delta / c \) sf, then + \( \kappa', \delta', c' \) and + \( \kappa', \delta' / c' \) sf and + \( \alpha \)

**Proof.** Follows from Lemma 40 \[ \Box \]

**Theorem 4 (Preservation).** If \( \kappa, \delta / c \xrightarrow{T} \kappa', \delta' / c' \) with + \( \kappa, \delta, c \) and + \( \kappa, \delta / c \) sf, then + \( \kappa', \delta', c' \) and + \( T \)

**Proof.** By induction on the number of steps in the sequence. Base case is trivial and follows from assumption.

**Inductive Case:** Holds for \( n \) steps; To show for \( n + 1 \) steps, i.e., if \( \kappa, \delta / c \xrightarrow{T^n} \kappa', \delta' / c' \xrightarrow{\alpha} \kappa'', \delta'' / c'' \), then + \( \kappa'', \delta'', c'' \) and + \( T, \alpha \)
We first define when a gradual label of an initial store location is not observable by the attacker. Formally:

\[ \text{H NONINTERFERENCE} \]

We define merging of two stores \( \delta \) as below:

\[ \Gamma \vdash \cdot \Rightarrow = \cdot \]

**Lemma 42.** If \( \vdash T \) then \( \vdash [T]_1 \approx_{\ell_A} [T]_2 \)

**Proof.** By induction on the structure of \( T \). The base case is trivial.

**Case:** \( T = \alpha, T' \)

By inversion of \( \vdash T \)

(1) \( \vdash \alpha \) and \( \vdash T' \)

By I.H. on \( T' \)

(2) \( \vdash [T']_1 \approx_{\ell_A} [T']_2 \)

By inversion of \( \vdash \alpha \), we have two cases

**Subcase I:** \( \alpha = (\ell, v) \)

By projection rules

(I3) \( [\alpha, T']_1 = \alpha, [T']_1 \), and \( [\alpha, T']_2 = \alpha, [T']_2 \)

By applying either EqT-L or EqT-H-L then EqT-H-r on (2), \( \vdash [T]_1 \approx_{\ell_A} [T]_2 \)

**Subcase II:** \( \alpha = (i, \ell, v) \), we show the case when \( i = 1 \), the other case when \( i = 2 \) is similar

By projection rules

(II3) \( [\alpha, T']_1 = (\ell, v), [T']_1 \), and \( [\alpha, T']_2 = [T']_2 \)

By typing of \( \alpha \)

(II4) \( \vdash \ell \not\in \ell_A \)

By (II4) and EqT-H-L and (2), \( \vdash [T]_1 \approx_{\ell_A} [T]_2 \)

**Lemma 43.** If \( \vdash \delta_1 \approx_{\ell_A} \delta_2 : \Gamma \) then \( \exists \delta \) s.t. \( \vdash \delta_1 \Rightarrow \delta_2 = \delta, \vdash \delta : \Gamma \) and \( [\delta]_i = \delta_i \)

**Proof.** By induction on the structure of the equivalence definition. Base case is trivial

**Inductive Case:** EqS-IND.

By assumption

\[ \vdash \delta_1 \approx_{\ell_A} \delta_2 : \Gamma \quad \vdash v_1 \approx_{\ell_A} v_2 : U \]

(1) \( \vdash \delta_1, x \mapsto v_1 \approx_{\ell_A} \delta_2, x \mapsto v_2 : \Gamma, x : U \)
By I.H. on $E_1$

(2) $\exists \delta$ s.t. $\Gamma \vdash \delta_1 \Rightarrow \delta_2 = \delta, \vdash \delta : \Gamma$ and $[\delta]_i = \delta_i$

By inversion of $E_2$, there are two subcases:

**Subcase I:** $E_2$ ends in EqV-L

By assumption

(I3) $v_i = (u u)_i^g, v_i \approx_c \ell_A, \vdash (u u)_i^g : U$

By casing on $g$ and (I3),

(I4) $g \notin H(\ell_A)$

By MgS-L on (I3), (I4), (2)

(I5) $\Gamma, x : U \vdash \delta_1, x \mapsto v_1 \Rightarrow \delta_2, x \mapsto v_1 = \delta, x \mapsto (u u)_i^g$,

By T-S-IND, (2), and (I3)

(I6) $\vdash \delta, x \mapsto (u u)_i^g : \Gamma, x : U$

**Subcase II:** $E_2$ ends in EqV-H

By assumption

(I13) $v_i = (u u)_i^g, v_i \approx g \in H(\ell_A)$, and $\vdash (u u)_i^g : U$ where $i \in \{1, 2\}$

By MgS-H on (I13), (2)

(I14) $\Gamma, x : U \vdash \delta_1, x \mapsto v_1 \Rightarrow \delta_2, x \mapsto v_1 = \delta, x \mapsto \langle u_2 u_2 \mid u_2 \rangle^g$,

By T-S-IND, (2), and (I13)

(I15) $\vdash \delta, x \mapsto \langle u_1 u_1 \mid u_2 u_2 \rangle^g : \Gamma, x : U$

\[\Box\]

**Theorem 5 (Noninterference).** Given an adversary label $\ell_A$, a program $c$, and two stores $\delta_1, \delta_2$, s.t., $\vdash \delta_1 \approx_{\ell_A} \delta_2 : \Gamma$, and $\Gamma; [\bot, \bot] \vdash c$, and $\forall i \in \{1, 2\}, [\bot, \bot] \vdash \delta_i / c \rightarrow^* \kappa_i, \delta'_i / \text{skip}$, then $\vdash T_1 \approx_{\ell_A} T_2$.

**Proof.**

By assumptions and Lemma 43

(1) $\exists \delta$ s.t. $\Gamma \vdash \delta_1 \Rightarrow \delta_2 = \delta, \vdash \delta : \Gamma$ and $[\delta]_i = \delta_i$

By Completeness Theorem (Theorem 3)

(2) $\exists \kappa', \delta', \tau$, s.t. $[\bot, \bot] \vdash \delta / c \rightarrow^* \kappa', \delta' / \text{skip}$

By Soundness Theorem (Theorem 2)

(3) $[\bot, \bot] \vdash \delta / c \rightarrow^* [\kappa, \delta' / \text{skip}]_i$

By the operational semantic rules are deterministic and (3)

(4) $T_i = [\hat{T}]_i$

By assumption and (1) and T-CONF

(5) $\vdash [\bot, \bot] \vdash \delta, c$

By Preservation Theorem (Theorem 4), (5), and (2)

(6) $\vdash \hat{T}$

By Lemma 42 and (6)

(7) $\vdash [\hat{T}]_1 \approx_{\ell_A} [\hat{T}]_2$

By (4) and (7), $\vdash T_1 \approx_{\ell_A} T_2$

\[\Box\]

## I GRADUAL GUARANTEES

### I.1 Auxiliary Lemmas and Proofs

**Lemma 44 (Operations are closed under refinement).**
We omit the definitions of $\varepsilon$. The precision of the cast operator is defined as shown above. As the precision operation is defined reflexive. When neither one is $\varepsilon$, the conclusion holds because $\subseteq$ is reflexive. When one of them is $\varepsilon$, $g_1 \subseteq g_2 \subseteq ?$, it is defined that $g_1 \subseteq g_2 \subseteq ?$.

proof

By assumptions

(1) $t_1 = [\ell_1l, \ell_1r], t'_1 = [\ell'_1l, \ell'_1r], \ell'_1l \ll \ell_1l$, and $\ell_1r \ll \ell'_1r$

(2) $t_2 = [\ell_2l, \ell_2r], t'_2 = [\ell'_2l, \ell'_2r], \ell'_2l \ll \ell_2l$, and $\ell_2r \ll \ell'_2r$

By (1) and (2)

(3) $\ell'_1l \ll \ell_1l \ll \ell_2l$ and $\ell_1r \ll \ell_2r \ll \ell'_2r$

(4) $\ell'_1r \ll \ell_1r \ll \ell_2r \ll \ell'_2r$

By definition of $\Rightarrow$

(5) $t_1 \Rightarrow t_2 = [\ell_1l \ll \ell_2l, \ell_1r \ll \ell_2r]$, and $t'_1 \Rightarrow t'_2 = [\ell'_1l \ll \ell'_2l, \ell'_1r \ll \ell'_2r]$

By (3) and (4), $t_1 \Rightarrow t_2 \subseteq t'_1 \Rightarrow t'_2$

By definition of $\Rightarrow$

(6) $t_1 \Rightarrow t_2 = [\ell_1l \ll \ell_2l, \ell_1r \ll \ell_2r]$ and $t'_1 \Rightarrow t'_2 = [\ell'_1l \ll \ell'_2l, \ell'_1r \ll \ell'_2r]$

By (3), and (4) $t_1 \Rightarrow t_2 \subseteq t'_1 \Rightarrow t'_2$

By definition of refine

(7) $\operatorname{refine}(t_1, t_2) = ([\ell_1l, \ell_1r \ll \ell_2r], [\ell_2l \ll \ell_1l, \ell_2r])$, and

(8) $\operatorname{refine}(t'_1, t'_2) = ([\ell'_1l, \ell'_1r \ll \ell'_2r], [\ell'_2l \ll \ell'_1l, \ell'_2r])$

T.S. $[\ell_1l, \ell_1r \ll \ell_2r] \subseteq [\ell'_1l, \ell'_1r \ll \ell'_2r]$ and $[\ell_2l \ll \ell_1l, \ell_2r] \subseteq [\ell'_2l \ll \ell'_1l, \ell'_2r]$

By (1) and (3), $[\ell_1l, \ell_1r \ll \ell_2r] \subseteq [\ell'_1l, \ell'_1r \ll \ell'_2r]$

By (2) and (4), $[\ell_2l \ll \ell_1l, \ell_2r] \subseteq [\ell'_2l \ll \ell'_1l, \ell'_2r]$

By assumptions

(9) $E = (t_a, t_b), E' = (t'_a, t'_b)$, and $t_a \subseteq t'_a, t_b \subseteq t'_b$

By $t_1 \subseteq t'_1$ and (9) and we have proven 1-3 of this lemma

(10) $t_a \Rightarrow t_1 \subseteq t'_a \Rightarrow t'_1$

By (10) and we have proven 1-3 of this lemma

(11) $\operatorname{refine}(t_a \Rightarrow t_1, t_b) \subseteq \operatorname{refine}(t'_a \Rightarrow t'_1, t'_b)$

By definition of $\Rightarrow E$ and (11), $t_1 \Rightarrow E \subseteq t'_1 \Rightarrow E'$

The proof of 5 cases on $g_1$ and $g_2$. When neither one is $\varepsilon$, the conclusion holds because $\subseteq$ is reflexive. When one of them is $\varepsilon$, $g_1 \subseteq g_2 = ?$, it is defined that $g_1 \subseteq g_2 \subseteq ?$.

1.2 Static Gradual Guarantee

We omit the definitions of $\varepsilon \subseteq \varepsilon'$ for \texttt{WHILEG}, which are inductively defined over the structure of $\varepsilon$. We only show the definitions for cast expression below, as it requires the types to be the same.

\[
\begin{align*}
e \subseteq \varepsilon' \\
e : U \subseteq \varepsilon' : U
\end{align*}
\]

The precision of the cast operator is defined as shown above. As the precision operation is defined over labels of values in stores, the expressions have to be cast to the same type $U$. If we cast $\varepsilon'$ to a different type $U'$ and try to show that $U \subseteq U'$, the proof results in cases that require a proof for $\ell_1 \subseteq \ell_2$ where $\ell_1 \neq \ell_2$, which does not hold.

lemma 45. If $g_1 \subseteq g_1', g_2 \subseteq g_2'$ and $g_1 \subseteq g_2$, then $g_1' \subseteq g_2'$.
PROOF. Assume \( g_1 = \ell_1 \) and \( g_2 = \ell_2 \) (L.H.S of \( \subseteq \) is a precise label).

By precision definition, \( g'_1 = \ell_1 \) or \( g'_1 =? \), and \( g'_2 = \ell_2 \) or \( g'_2 =? \).

If \( g'_1 = \ell_1 \) and \( g'_2 = \ell_2 \), then the conclusion holds by assumption.

If \( g'_1 =? \) and \( g'_2 = \ell_2 \), then \( \ll c \ell_2 \) holds by definition of \( \ll c \).

If \( g'_1 = \ell_1 \) and \( g'_2 =? \), then \( \ell_1 \ll c ? \) holds by definition of \( \ll c \).

If \( g'_1 =? \) and \( g'_2 =? \), then \( \ll c ? \) holds by definition of \( \ll c \).

Lemma 46 (Static Gradual Guarantee - Expressions). If \( \Gamma \vdash e : U, \Gamma \ll U', \) and \( e \ll e' \), then \( \Gamma' \vdash e' : U' \) and \( U \ll U' \).

PROOF. By induction on the structure of the typing derivation. Follows from assumption for Bool, Int, Var.

Case: Bop

By IH, \( \tau^{g_1} \subseteq \tau^{g'_1} \) and \( \tau^{g_2} \subseteq \tau^{g'_2} \).

By Lemma 44, \( g_1 \preceq g_2 \subseteq g'_1 \preceq g'_2 \).

By Bop, \( \tau^g \subseteq \tau^{g'} \).

Case: Sub

\[
\begin{array}{c}
\Gamma \vdash e : \tau^{g_1} \\
\Gamma \vdash e :: U : \tau^g \\
T.S. \Gamma' \vdash e' :: U : \tau^{g'}
\end{array}
\]

\( g_1 \ll_c g \) where \( \Gamma' \vdash e' : \tau^{g'_1} \).

\( e' \) is also cast to \( U \) instead of another type \( U' \) for reasons mentioned above.

By IH, \( \Gamma' \vdash e' : \tau^{g'_1}, \tau^{g_1} \subseteq \tau^{g'_1} \).

Thus, \( g_1 \subseteq g'_1 \).

By assumption, \( g_1 \ll_c g \).

\( g_1 = \ell_1 \), then \( g'_1 = \ell_1 \) or \( g'_1 =? \). In both cases, \( g'_1 \ll_c g \).

By SUB, the conclusion holds.

\[\Box\]

Theorem 6 (Static Gradual Guarantee). If \( \Gamma; g \vdash c, \Gamma \ll \Gamma', g \ll g', \) and \( c \ll c' \), then \( \Gamma'; g' \vdash c' \).

PROOF. By induction on command typing derivation. Most cases can be proven by using the induction hypothesis and the typing rule. When the derivation ends in ASSIGN, OUT, IF, WHILE, apply Lemma 46 on the premises and when the derivation ends in IF, WHILE, we additionally apply Lemma 44. We use the same typing rule to reach the conclusion.

\[\Box\]

1.3 Dynamic Gradual Guarantee

Lemma 47 (Dynamic Guarantee (Expressions)). If \( \delta_1 / e_1 \downarrow v_1, \delta_1 \subseteq \delta_2, \) and \( e_1 \subseteq e_2, \) then \( \delta_2 / e_2 \downarrow v_2 \) and \( v_1 \subseteq v_2 \).

PROOF. By induction on the structure of the expression evaluation. We apply the induction hypothesis directly for M-CTX-Err. The basecases M-CONST and M-VAR can be shown using assumptions directly.

Case: M-Bop

By assumption the evaluation ends in M-Bop rule:

\[
\begin{align*}
\delta / e_1 & \downarrow (t_1 u_1)^{g_1} \\
\delta / e_2 & \downarrow (t_2 u_2)^{g_2} \\
t & = (t_1 \gamma t_2) \\
g & = g_1 \gamma g_2 \\
u & = (u_1 \bop u_2)
\end{align*}
\]

\( \delta / e_1 \bop e_2 \downarrow (u)^{g} \)

(1) \( \delta / e_1 \bop e_2 \downarrow (u)^{g} \)

(2) \( e = e_1 \bop e_2, \delta \subseteq \delta', \) and \( e \subseteq e' \)

By inversion of (2)

(3) \( e' = e'_1 \bop e'_2, e_1 \subseteq e'_1, \) and \( e_2 \subseteq e'_2 \)

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(4) $\delta' \vdash e'_1 \downarrow (t'_1 u'_1)^\theta$, $t_1 \subseteq t'_1$, and $g_1 \subseteq g'_1$.

(5) $\delta' \vdash e'_2 \downarrow (t'_2 u'_2)^\theta$, $t_2 \subseteq t'_2$, and $g_2 \subseteq g'_2$.

By M-Bop

(6) $\delta' \vdash e'_1 \bop e'_2 \downarrow (t' u)^\theta$, where $t' = t'_1 \uplus t'_2$, $g' = g'_1 \uplus g'_2$.

By Lemma 44

(7) $(t u)^\theta \subseteq (t' u)^\theta$.

Case: M-Cast and Pair-Cast can be proven similarly by applying I.H. and Lemma 44.

\[ \square \]

Lemma 48.

(1) if $t_1 \subseteq t_2$ and $v_1 \subseteq v_2$, and $v'_1 = \text{refineLV}(t_1, v_1)$ and $v'_2 = \text{refineLV}(t_2, v_2)$, then $v'_1 \subseteq v'_2$.

(2) if $t_1 \subseteq t_2$ and $v_1 \subseteq v_2$, and $v'_1 = \text{refineVL}(v_1, t_1)$ and $v'_2 = \text{refineVL}(v_2, t_2)$, then $v'_1 \subseteq v'_2$.

(3) if $t_1 \subseteq t_2$ and $\delta_1 \subseteq \delta_2$, and $\delta'_1 = \text{rfl}(\delta_1, X, t_1)$, and $\delta'_2 = \text{rfl}(\delta_2, X, t_2)$, then $\delta'_1 \subseteq \delta'_2$.

(4) if $t_1 \subseteq t_2$ and $\delta_1 \subseteq \delta_2$, $t_{pc_1} \subseteq t_{pc_2}$, $t_{c_1} \subseteq t_{c_2}$, $g_1 \subseteq g_2$, and $\delta'_1 = \text{refineWSet}(\delta_1, X, t_{pc_1}, t_{c_1}, (t_1 u)^\theta)$ and $\delta'_2 = \text{refineWSet}(\delta_2, X, t_{pc_2}, t_{c_2}, (t_2 u)^\theta)$ then $\delta'_1 \subseteq \delta'_2$.

Proof. Proofs of (1) and (2) examine the definitions of the operations and apply Lemma 44 directly. Proof of (3) is by induction over the size of $X$ and (1). Proof of 4 invokes (3) directly. \[ \square \]

Theorem 7 (Dynamic Gradual Guarantee). If $\kappa_1, \delta_1 \vdash c_1 \xrightarrow{\alpha_1} \kappa'_1, \delta'_1 \vdash c'_1$ and $\kappa_1, \delta_1 \vdash c_1 \subseteq \kappa_2, \delta_2 \vdash c_2$, then $\kappa_2, \delta_2 \vdash c_2 \xrightarrow{\alpha_2} \kappa'_2, \delta'_2 \vdash c'_2$ such that $\kappa'_1, \delta'_1 \vdash c'_1 \subseteq \kappa'_2, \delta'_2 \vdash c'_2$ and $\alpha_1 = \alpha_2$.

Proof. By induction on the command semantics. Most cases can be proven by using the induction hypothesis and Lemma 44 directly.

When the derivation ends in M-ASSIGN, M-OUT, or M-IF-REFINE, apply Lemma 47 and Lemma 48 on the premises and use the same semantic rule to reach the conclusion. \[ \square \]