CLUSTER ALGEBRAS AND MINIMAL AFFINIZATIONS OF REPRESENTATIONS OF THE QUANTUM GROUP OF TYPE $G_2$

LI QIAO AND JIAN-RONG LI

ABSTRACT. In this paper, we make a connection between cluster algebras and a class of modules of the quantum affine algebra $U_q\hat{g}$ of type $G_2$ called minimal affinizations. We introduce a system of equations satisfied by the $q$-characters of minimal affinizations of type $G_2$ which we called the M-system of type $G_2$. The M-system of type $G_2$ contains all minimal affinizations of type $G_2$ and only contains minimal affinizations. We show that the equations in the M-system of type $G_2$ correspond to mutations in some cluster algebra $\mathcal{A}$. Moreover, the minimal affinizations correspond to some cluster variables in $\mathcal{A}$.

Key words: cluster algebras; quantum affine algebras of type $G_2$; minimal affinizations; $q$-characters; M-systems

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1. Introduction

The theory of cluster algebras are invented by Fomin and Zelevinsky in [FZ02] as a combinatorial and algebraic framework for studying total positivity for semisimple algebraic groups developed by Lusztig [L94] and canonical bases of quantum groups introduced by Lusztig [L90] and Kashiwara [K91]. It has many applications including quiver representations, Teichmüller theory, tropical geometry, integrable systems, and Poisson geometry.

Let $g$ be the simple Lie algebra and $U_q\hat{g}$ the corresponding quantum affine algebras. In a remarkable paper [HL10], Hernandez and Leclerc make some connections between cluster algebras and finite dimensional representations of $U_q\hat{g}$. They show that the Grothendieck ring $\text{Rep}(U_q\hat{g})$ of some subcategory of the category of all finite dimensional representations of $U_q\hat{g}$ has a cluster algebra structure. In the paper [HL13], they apply the theory of cluster algebras to study the $q$-characters a family of $U_q\hat{g}$-modules called Kirillov-Reshetikhin modules and they give a new algorithm to compute the $q$-characters of these modules.

The family of minimal affinizations of quantum groups $U_qg$ is an important family of simple modules of $U_q\hat{g}$ which was introduced in [C95]. The family of minimal affinizations contains the celebrated Kirillov-Reshetikhin modules. Minimal affinizations are studied intensively in recent years, see for example, [CMY13], [CG11], [Her07], [LM13], [M10], [MP11], [MY12a], [MY12b], [MY14], [Nao13].

The aim of this paper is to make a connection between cluster algebras and minimal affinizations of the quantum affine algebras $U_q\hat{g}$ of type $G_2$. First we introduce a system of equations
which we call the M-system of type $G_2$ and prove that the equations in the M-system of type $G_2$ is satisfied by the $q$-characters of minimal affinizations of type $G_2$.

The extended T-system of type $G_2$ studied in [LM13] contains all minimal affinizations of type $G_2$ and some other types of modules. The M-system of type $G_2$ also contains all minimal affinizations of type $G_2$. But unlike the extended T-system of type $G_2$, the M-system of type $G_2$ contains only minimal affinizations of type $G_2$. Moreover, the M-system of type $G_2$ is much simpler than the extended T-system of type $G_2$.

We show that the equations in the M-system of type $G_2$ correspond to mutations in some cluster algebra $A$. The cluster algebra $A$ is the same as the cluster algebra of type $G_2$ introduced in [HL13]. The minimal affinizations correspond to some cluster variables in $A$.

We also show that the modules in the summands on the right hand side of each equation in the M-system is simple. In the last part of the paper, we study the dual M-system of type $G_2$.

The M-systems also exist for other Dynkin types of minimal affinizations. Since the method of proving that the $q$-characters of minimal affinizations satisfy the M-systems of other types are different from the method used in this paper and the M-systems of other types are much more complicated, we will publish them in other papers.

The paper is organized as follows. In Section 2, we give some background information about cluster algebras and representation theory of quantum affine algebras. In Section 3, we describe our main results in this paper. In Section 4 and 5, we prove two main theorems given in Section 3. In Section 6, we study the dual M-system of type $G_2$.

2. Background

2.1. Cluster algebras. Cluster algebras are invented by Fomin and Zelevinsky in [FZ02]. Let $\mathbb{Q}$ be the rational field and $F = \mathbb{Q}(x_1, x_2, \cdots, x_n)$ the field of rational functions. A seed in $F$ is a pair $\Sigma = (y, Q)$, where $y = (y_1, y_2, \cdots, y_n)$ is a free generating set of $F$, and $Q$ is a quiver with vertices labeled by $\{1, 2, \cdots, n\}$. Assume that $Q$ has neither loops nor 2-cycles. For $k = 1, 2, \cdots, n$, one defines a mutation $\mu_k$ by $\mu_k(y, Q) = (y', Q')$. Here $y' = (y'_1, \ldots, y'_n)$, $y'_i = y_i$, for $i \neq k$, and

$$y'_k = \prod_{i \rightarrow k} y_i + \prod_{k \rightarrow j} y_j,$$

where the first (resp. second) product in the right hand side is over all arrows of $Q$ with target (resp. source) $k$, and $Q'$ is obtained from $Q$ by

(i) adding a new arrow $i \rightarrow j$ for every existing pair of arrow $i \rightarrow k$ and $k \rightarrow j$;

(ii) reversing the orientation of every arrow with target or source equal to $k$;

(iii) erasing every pair of opposite arrows possible created by (i).

The mutation class $C(\Sigma)$ is the set of all seeds obtained from $\Sigma$ by a finite sequence of mutation $\mu_k$. If $\Sigma' = ((y'_1, y'_2, \cdots, y'_n), Q')$ is a seed in $C(\Sigma)$, then the subset $\{y'_1, y'_2, \cdots, y'_n\}$ is called a cluster, and its elements are called cluster variables. The cluster algebra $A_\Sigma$ as the subring of $F$ generated by all cluster variables. Cluster monomials are monomials in the cluster variables supported on a single cluster.

In this paper, the initial seed in the cluster algebra we use is of the form $\Sigma = (y, Q)$, where $y$ is an infinite set and $Q$ is an infinite quiver.
Definition 2.1 (Definition 3.1, [GG11]). Let Q be a quiver without loops or 2-cycles and with a countably infinite number of vertices labelled by all integers $i \in \mathbb{Z}$. Furthermore, for each vertex $i$ of Q let the number of arrows incident with $i$ be finite. Let $y = \{y_i \mid i \in \mathbb{Z}\}$. An infinite initial seed is the pair $(y, Q)$. By finite sequences of mutation at vertices of $Q$ and simultaneous mutation of the set $y$ using the exchange relation (2.1), one obtains a family of infinite seeds. The sets of variables in these seeds are called the infinite clusters and their elements are called the cluster variables. The cluster algebra of infinite rank of type $Q$ is the subalgebra of $\mathbb{Q}(y)$ generated by the cluster variables.

2.2. The quantum affine algebra of type $G_2$. In this paper, we take $g$ to be the complex simple Lie algebra of type $G_2$ and $\mathfrak{h}$ a Cartan subalgebra of $g$. Let $I = \{1, 2\}$. We choose simple roots $\alpha_1, \alpha_2$ and scalar product $(\cdot, \cdot)$ such that

$$(\alpha_1, \alpha_1) = 6, \ (\alpha_1, \alpha_2) = -3, \ (\alpha_2, \alpha_2) = 2.$$ 

Therefore $\alpha_1$ is the long simple root and $\alpha_2$ is the short simple root. Let $\{\alpha_i^\vee, \alpha_j^\vee\}$ and $\{\omega_1, \omega_2\}$ be the sets of simple coroots and fundamental weights respectively. Let $C = (C_{ij})_{i,j \in I}$ denote the Cartan matrix, where $C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. Let $r_1 = 3, r_2 = 1, D = \text{diag}(r_1, r_2)$ and $B = DC$. Then

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}.$$

Let $Q$ (resp. $Q^+$) and $P$ (resp. $P^+$) denote the $\mathbb{Z}$-span (resp. $\mathbb{Z}_{\geq 0}$-span) of the simple roots and fundamental weights respectively. Let $\leq$ be the partial order on $P$ in which $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in Q^+$. Let $\widehat{\mathfrak{g}}$ denote the untwisted affine algebra corresponding to $g$. Fix a $q \in \mathbb{C}^\times$, not a root of unity. Let $q_i = q^{r_i}, i = 1, 2$. Let $\mathcal{P}$ the free abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i \in I, a \in \mathbb{C}^\times}$.

The quantum affine algebra $U_q \widehat{\mathfrak{g}}$ in Drinfeld’s new realization, see [Dri88], is generated by $x_{i,n}^\pm (i \in I, n \in \mathbb{Z}), k_i^{\pm 1} (i \in I), h_{i,n} (i \in I, n \in \mathbb{Z}\setminus\{0\})$ and central elements $c^{\pm 1/2}$, subject certain relations.

The quantum affine algebra $U_q \widehat{\mathfrak{g}}$ contains two standard quantum affine algebras of type $A_1$. The first one is $U_q \widehat{\mathfrak{sl}_2}$ generated by $x_{1,n}^\pm (n \in \mathbb{Z}), k_1^{\pm 1}, h_{1,n} (n \in \mathbb{Z}\setminus\{0\})$ and central elements $c^{\pm 1/2}$. The second one is $U_q \widehat{\mathfrak{sl}_2}$ generated by $x_{2,n}^\pm (n \in \mathbb{Z}), k_2^{\pm 1}, h_{2,n} (n \in \mathbb{Z}\setminus\{0\})$ and central elements $c^{\pm 1/2}$.

The subalgebra of $U_q \widehat{\mathfrak{g}}$ generated by $(k_{i}^{\pm})_{i \in I}, (x_{i,0}^\pm)_{i \in I}$ is a Hopf subalgebra of $U_q \widehat{\mathfrak{g}}$ and is isomorphic as a Hopf algebra to $U_q \mathfrak{g}$, the quantized enveloping algebra of $g$. In this way, $U_q \widehat{\mathfrak{g}}$-modules restrict to $U_q \mathfrak{g}$-modules.

2.3. Finite-dimensional representations of $U_q \widehat{\mathfrak{g}}$ and $q$-characters. In this section, we recall the standard facts about finite-dimensional representations of $U_q \widehat{\mathfrak{g}}$ and $q$-characters of these representations, see [CP94], [CP95a], [FR98], [MY12a].
A representation $V$ of $U_q\mathfrak{g}$ is of type 1 if $c^{\pm 1/2}$ acts as the identity on $V$ and
\[ V = \bigoplus_{\lambda \in \mathcal{P}} V_\lambda, \quad V_\lambda = \{ v \in V : k_i v = q^{(\alpha_i, \lambda)} v \}. \quad (2.2) \]

In the following, all representations will be assumed to be finite-dimensional and of type 1 without further comment. The decomposition (2.2) of a finite-dimensional representation $V$ into its $U_q\mathfrak{g}$-weight spaces can be refined by decomposing it into the Jordan subspaces of the mutually commuting operators $\phi^{\pm}_{i, \pm r}$, see FR98:
\[ V = \bigoplus_{\gamma} V_\gamma, \quad \gamma = (\gamma^\pm_{i, \pm r})_{i \in I, r \in \mathbb{Z}_{\geq 0}}, \quad \gamma^\pm_{i, \pm r} \in \mathbb{C}, \quad (2.3) \]
where
\[ V_\gamma = \{ v \in V : \exists k \in \mathbb{N}, \forall i \in I, m \geq 0, (\phi^\pm_{i, \pm m} - \gamma^\pm_{i, \pm m})^k v = 0 \}. \]

Here $\phi^\pm_{i, n}$'s are determined by the formula
\[ \phi^\pm_{i, n}(u) := \sum_{n=0}^{\infty} \phi^{\pm}_{i, \pm n} u^{\pm n} = k^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{m=1}^{\infty} h_{i, \pm m} u^{\pm m} \right). \quad (2.4) \]
If $\dim(V_\gamma) > 0$, then $\gamma$ is called an $l$-weight of $V$. For every finite dimensional representation of $U_q\mathfrak{g}$, the $l$-weights are known, see FR98, to be of the form
\[ \gamma^\pm_{i, n}(u) := \sum_{r=0}^{\infty} \gamma^\pm_{i, \pm r} u^{\pm r} = q^{\deg Q_i - \deg R_i} \frac{Q_i(uq_i^{-1}) R_i(uq_i)}{Q_i(uq_i) R_i(uq_i^{-1})}. \quad (2.5) \]
where the right hand side is to be treated as a formal series in positive (resp. negative) integer powers of $u$, and $Q_i, R_i$ are polynomials of the form
\[ Q_i(u) = \prod_{a \in \mathbb{C}^\times} (1 - ua)^{w_{i,a}}, \quad R_i(u) = \prod_{a \in \mathbb{C}^\times} (1 - ua)^{x_{i,a}}, \quad (2.6) \]
for some $w_{i,a}, x_{i,a} \in \mathbb{Z}_{\geq 0}, i \in I, a \in \mathbb{C}^\times$. Let $\mathcal{P}$ denote the free abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i \in I, a \in \mathbb{C}^\times}$. There is a bijection $\gamma$ from $\mathcal{P}$ to the set of $l$-weights of finite-dimensional modules such that for the monomial $m = \prod_{i,a \in \mathbb{C}^\times} Y_{i,a}^{w_{i,a}}$, the $l$-weight $\gamma(m)$ is given by (2.5), (2.6).

Let $\mathbb{Z}\mathcal{P} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ be the group ring of $\mathcal{P}$. For $\chi \in \mathbb{Z}\mathcal{P}$, we write $m \in \mathcal{P}$ if the coefficient of $m$ in $\chi$ is non-zero.

The $q$-character of a $U_q\mathfrak{g}$-module $V$ is given by
\[ \chi_q(V) = \sum_{m \in \mathcal{P}} \dim(V_m)m \in \mathbb{Z}\mathcal{P}, \]
where $V_m = V_{\gamma(m)}$.

Let $\text{Rep}(U_q\mathfrak{g})$ be the Grothendieck ring of finite-dimensional representations of $U_q\mathfrak{g}$ and $[V] \in \text{Rep}(U_q\mathfrak{g})$ the class of a finite-dimensional $U_q\mathfrak{g}$-module $V$. The $q$-character map defines an
injective ring homomorphism, see [ER08],
\[ \chi_q : \text{Rep}(U_{q}\mathfrak{g}) \to \mathbb{Z}P. \]

For any finite-dimensional representation \( V \) of \( U_{q}\mathfrak{g} \), denote by \( \mathcal{M}(V) \) the set of all monomials in \( \chi_q(V) \). For each \( j \in I \), a monomial \( m = \prod_{i \in I, a \in \mathbb{C}^{\times}} Y_{i, a}^{u_{i, a}} \), where \( u_{i, a} \) are some integers, is said to be \( j \)-dominant (resp. \( j \)-anti-dominant) if and only if \( u_{j, a} \geq 0 \) (resp. \( u_{j, a} \leq 0 \)) for all \( a \in \mathbb{C}^{\times} \). A monomial is called \( j \)-dominant (resp. \( j \)-anti-dominant) if and only if it is \( j \)-dominant (resp. \( j \)-anti-dominant) for all \( j \in I \). Let \( \mathcal{P}^{+} \subset \mathcal{P} \) denote the set of all dominant monomials.

Let \( V \) be a representation of \( U_{q}\mathfrak{g} \) and \( m \in \mathcal{M}(V) \) a monomial. A non-zero vector \( v \in V_m \) is called a \text{highest l-weight vector} with \text{highest l-weight} \( \gamma(m) \) if
\[ x_{i, r}^{+} \cdot v = 0, \quad \phi_{i, r}^{\pm} \cdot v = \gamma(m)_{i, r}^{\pm} v, \quad \forall i \in I, r \in \mathbb{Z}, t \in \mathbb{Z}_{\geq 0}. \]
The module \( V \) is called a \text{highest l-weight representation} if \( V = U_{q}\mathfrak{g} \cdot v \) for some highest l-weight vector \( v \in V \).

It is known, see [CP94], [CP95], that for each \( m_{+} \in \mathcal{P}^{+} \) there is a unique finite-dimensional irreducible representation, denoted \( L(m_{+}) \), of \( U_{q}\mathfrak{g} \) that is highest l-weight with highest l-weight \( \gamma(m_{+}) \), and moreover every finite-dimensional irreducible \( U_{q}\mathfrak{g} \)-module is of this form for some \( m_{+} \in \mathcal{P}^{+} \). Also, if \( m_{+} \), \( m_{+}' \in \mathcal{P}^{+} \) and \( m_{+} \neq m_{+}' \), then \( L(m_{+}) \neq L(m_{+}') \). For \( m_{+} \in \mathcal{P}^{+} \), we use \( \chi_q(m_{+}) \) to denote \( \chi_q(L(m_{+})) \). If \( m \in \mathcal{M}(\chi_q(m_{+})) \), then we write \( m \in \chi_q(m_{+}) \).

The following lemma is well-known.

**Lemma 2.2.** Let \( m_{1}, m_{2} \) be two monomials. Then \( L(m_{1}m_{2}) \) is a sub-quotient of \( L(m_{1}) \otimes L(m_{2}) \). In particular, \( \mathcal{M}(L(m_{1}m_{2})) \subseteq \mathcal{M}(L(m_{1})). \mathcal{M}(L(m_{2})) \). □

For \( b \in \mathbb{C}^{\times} \), define the shift of spectral parameter map \( \tau_{b} : \mathbb{Z}P \to \mathbb{Z}P \) to be a homomorphism of rings sending \( Y_{i, a}^{\pm 1} \) to \( Y_{i, ab}^{\pm 1} \). Let \( m_{1}, m_{2} \in \mathcal{P}^{+} \). If \( \tau_{b}(m_{1}) = m_{2} \), then
\[ \tau_{b}\chi_q(m_{1}) = \chi_q(m_{2}). \quad (7.2) \]

A finite-dimensional \( U_{q}\mathfrak{g} \)-module \( V \) is said to be \text{special} if and only if \( \mathcal{M}(V) \) contains exactly one dominant monomial. It is called \text{anti-special} if and only if \( \mathcal{M}(V) \) contains exactly one anti-dominant monomial. It is said to be \text{prime} if and only if it is not isomorphic to a tensor product of two non-trivial \( U_{q}\mathfrak{g} \)-modules, see [CP97]. Clearly, if a module is special or anti-special, then it is simple.

Define \( A_{i, a} \in \mathcal{P}, i \in I, a \in \mathbb{C}^{\times} \), by
\[ A_{1, a} = Y_{1, a}Y_{1, a^{-1}}^{-1}Y_{2, a}^{-1}, \quad A_{2, a} = Y_{2, a}Y_{1, a}^{-1}Y_{2, a}^{-1}Y_{1, a}^{-1}Y_{1, a^{-1}}. \]
Let \( Q \) be the subgroup of \( \mathcal{P} \) generated by \( A_{i, a}, i \in I, a \in \mathbb{C}^{\times} \). Let \( Q^{\pm} \) be the monoids generated by \( A_{i, a}^{\pm 1}, i \in I, a \in \mathbb{C}^{\times} \). There is a partial order \( \leq \) on \( \mathcal{P} \) in which
\[ m \leq m' \text{ if and only if } m'm^{-1} \in Q^{+}. \quad (7.8) \]
For all \( m_{+} \in \mathcal{P}^{+}, \mathcal{M}(L(m_{+})) \subseteq m_{+}Q^{-} \), see [FM01].

Let \( m \) be a monomial. If for all \( a \in \mathbb{C}^{\times} \) and \( i \in I \), we have the property: if the power of \( Y_{i, a} \) in \( m \) is non-zero and the power of \( Y_{j, a}q^{k} \) in \( m \) is zero for all \( j \in I, k \in \mathbb{Z}_{>0} \), then the power of \( Y_{i, a} \) in \( m \) is negative, then the monomial \( m \) is called \text{right negative}. For \( i \in I, a \in \mathbb{C}^{\times}, A_{i, a}^{1} \)
right-negative. A product of right-negative monomials is right-negative. If \( m \) is right-negative and \( m' \leq m \), then \( m' \) is right-negative.

2.4. Minimal affinizations of \( U_q\mathfrak{g} \)-modules. Let \( \lambda = k\omega_1 + l\omega_2 \). A simple \( U_q\mathfrak{g} \)-module \( L(m_+) \) is called a minimal affinization of \( V(\lambda) \) if and only if \( m_+ \) is one of the following monomials

\[
\left( \prod_{i=0}^{k-1} Y_{1,aq^6i} \right) \left( \prod_{i=0}^{l-1} Y_{2,aq^{6k+2i+1}} \right), \quad \left( \prod_{i=0}^{l-1} Y_{2,aq^{2i}} \right) \left( \prod_{i=0}^{k-1} Y_{1,aq^{2i+6i+5}} \right),
\]

for some \( a \in \mathbb{C}^\times \), see [CP95b].

From now on, we fix an \( a \in \mathbb{C}^\times \) and denote \( i_s = Y_{i,aq^s}, i \in I, s \in \mathbb{Z} \). Without loss of generality, we may assume that a simple \( U_q\mathfrak{g} \)-module \( L(m_+) \) is a minimal affinization of \( V(\lambda) \) if and only if \( m_+ \) is one of the following monomials

\[
T^{(s)}_{k,l} = \left( \prod_{i=0}^{k-1} 1_{s+6i} \right) \left( \prod_{j=0}^{l-1} 2_{s+6k+2j+1} \right), \quad \bar{T}^{(s)}_{k,l} = \left( \prod_{j=0}^{l-1} 2_{-s-6k-2j-1} \right) \left( \prod_{j=0}^{k-1} 1_{-s-6j} \right).
\]

2.5. \( q \)-characters of \( U_q\mathfrak{sl}_2 \)-modules and the Frenkel-Mukhin algorithm. We recall the results of the \( q \)-characters of \( U_q\mathfrak{sl}_2 \)-modules which are well-understood, see [CP91], [FR98].

Let \( W_k^{(a)} \) be the irreducible representation \( U_q\mathfrak{sl}_2 \) with highest weight monomial

\[
X_k^{(a)} = \prod_{i=0}^{k-1} Y_{aq^{k-2i-1}},
\]

where \( Y_a = Y_{1,a} \). Then the \( q \)-character of \( W_k^{(a)} \) is given by

\[
\chi_q(W_k^{(a)}) = X_k^{(a)} \sum_{i=0}^{k} \prod_{j=0}^{i-1} A_{aq^{k-2j}}^{-1},
\]

where \( A_a = Y_{aq^{-1}} Y_{aq} \).

For \( a \in \mathbb{C}^\times, k \in \mathbb{Z}_{\geq 1} \), the set \( \Sigma_k^{(a)} = \{aq^{k-2i-1}\}_{i=0,\ldots,k-1} \) is called a string. Two strings \( \Sigma_k^{(a)} \) and \( \Sigma_{k'}^{(a')} \) are said to be in general position if the union \( \Sigma_k^{(a)} \cup \Sigma_{k'}^{(a')} \) is not a string or \( \Sigma_k^{(a)} \subset \Sigma_{k'}^{(a')} \) or \( \Sigma_{k'}^{(a')} \subset \Sigma_k^{(a)} \).

Denote by \( L(m_+) \) the irreducible \( U_q\mathfrak{sl}_2 \)-module with highest weight monomial \( m_+ \). Let \( m_+ \neq 1 \) and \( i \in \mathbb{Z}[Y_a]_{a \in \mathbb{C}^\times} \) be a dominant monomial. Then \( m_+ \) can be uniquely (up to permutation) written in the form

\[
m_+ = \prod_{i=1}^{s} \left( \prod_{b \in \Sigma_k^{(a)}} Y_b \right),
\]
where $s$ is an integer, $\sum_{k_i}^{(a_i)}, i = 1, \ldots, s$, are strings which are pairwise in general position and

$$L(m_+) = \bigotimes_{i=1}^{s} W_{k_i}^{(a_i)}, \quad \chi_q(L(m_+)) = \prod_{i=1}^{s} \chi_q(W_{k_i}^{(a_i)}).$$

For $j \in I$, let

$$\beta_j : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times} \rightarrow \mathbb{Z}[Y_{a}^{\pm 1}]_{a \in \mathbb{C}^\times}$$

be the ring homomorphism which sends, for all $a \in \mathbb{C}^\times$, $Y_{k,a} \mapsto 1$ for $k \neq j$ and $Y_{j,a} \mapsto Y_a$.

Let $V$ be a $U_q\hat{\mathfrak{g}}$-module. Then $\beta_i(\chi_q(V))$, $i = 1, 2$, is the $q$-character of $V$ considered as a $U_q\hat{\mathfrak{g}}$-module.

In some situation, we can use the $q$-characters of $U_q\hat{\mathfrak{g}}$-modules to compute the $q$-characters of $U_q\hat{\mathfrak{h}}$-modules for arbitrary $\mathfrak{g}$, see Section 5 in [FM01]. The corresponding algorithm is called the Frenkel-Mukhin algorithm. The Frenkel-Mukhin algorithm recursively computes the minimal possible $q$-character which contains $m_+$ and is consistent when restricted to $U_q\hat{\mathfrak{g}}$, $i = 1, 2$. For example, if a module $L(m_+)$ is special, then the Frenkel-Mukhin algorithm applied to $m_+$, see [FM01], produces the correct $q$-character $\chi_q(L(m_+))$.

### 3. Main results

In this section, we describe our main results.

#### 3.1. The $M$-system of type $G_2$

We use $\mathcal{T}_{k,l}^{(s)}$ to denote the irreducible finite-dimensional $U_q\hat{\mathfrak{g}}$-module with highest $l$-weight $T_{k,l}^{(s)}$. Here $T_{k,l}^{(s)}$ is defined in Section 2.4. Let $[\mathcal{T}]$ be the equivalence class of the $U_q\hat{\mathfrak{g}}$-module $\mathcal{T}$ in the Grothendieck ring $\text{Rep}(U_q\hat{\mathfrak{g}})$. Our first main result is the following system which we called $M$-system of type $G_2$.

**Theorem 3.1.** For $s \in \mathbb{Z}$, we have the following system of equations:

\[
\begin{align*}
[\mathcal{T}_{k,l}^{(s)}][\mathcal{T}_{k,0}^{(s+6)}] &= [\mathcal{T}_{k+1,l}^{(s)}][\mathcal{T}_{k-1,l}^{(s+6)}] + [\mathcal{T}_{0,3k+l}^{(s+6)}] \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\}), \quad (3.1) \\
[\mathcal{T}_{k,l+3}^{(s)}][\mathcal{T}_{k,l}^{(s+6)}] &= [\mathcal{T}_{k+1,l}^{(s)}][\mathcal{T}_{k-1,l+3}^{(s+6)}] + [\mathcal{T}_{0,l}^{(s+6k+6)}][\mathcal{T}_{0,3k+l+3}^{(s)}] \quad (k, l \in \mathbb{Z}_{\geq 1}). \quad (3.2)
\end{align*}
\]

Theorem 3.1 will be proved in Section 4. The equations in Theorem 3.1 are equivalent to the following equations.

\[
\begin{align*}
\chi_q(\mathcal{T}_{k,l}^{(s)})\chi_q(\mathcal{T}_{k,0}^{(s+6)}) &= \chi_q(\mathcal{T}_{k+1,l}^{(s)})\chi_q(\mathcal{T}_{k-1,l}^{(s+6)}) + \chi_q(\mathcal{T}_{0,3k+l}^{(s+6)}) \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\}), \\
\chi_q(\mathcal{T}_{k,l+3}^{(s)})\chi_q(\mathcal{T}_{k,l}^{(s+6)}) &= \chi_q(\mathcal{T}_{k+1,l}^{(s)})\chi_q(\mathcal{T}_{k-1,l+3}^{(s+6)}) + \chi_q(\mathcal{T}_{0,3k+l+3}^{(s+6k+6)})\chi_q(\mathcal{T}_{0,3k+l+3}^{(s)}) \quad (k, l \in \mathbb{Z}_{\geq 1}).
\end{align*}
\]
Example 3.2. The following are some equations in the M-system of type $G_2$.

\[
[1\cdot 72_0][1\cdot 1] = [1\cdot 71_1][2_0] + [2\cdot 62\cdot 42\cdot 2_0], \\
[1\cdot 92_0][1\cdot 3] = [1\cdot 91_3][2\cdot 2_0] + [2\cdot 82\cdot 62\cdot 42\cdot 2_0], \\
[1\cdot 112\cdot 42\cdot 2_0][1\cdot 5] = [1\cdot 111_5][2\cdot 42\cdot 2_0] + [2\cdot 102\cdot 82\cdot 62\cdot 42\cdot 2_0], \\
[1\cdot 132\cdot 62\cdot 42\cdot 2_0][1\cdot 72_0] = [1\cdot 131_7][2\cdot 62\cdot 42\cdot 2_0] + [2_0][2\cdot 122\cdot 10\cdots 2\cdot 2_0], \\
[1\cdot 331\cdot 272\cdot 20\cdots 2\cdot 2_0][1\cdot 271\cdot 212\cdot 14\cdots 2\cdot 2_0] \\
= [1\cdot 331\cdot 271\cdot 212\cdot 14\cdots 2\cdot 2_0][1\cdot 272\cdot 20\cdots 2\cdot 2_0] + [2\cdot 142\cdot 12\cdots 2\cdot 2_0][2\cdot 322\cdot 30\cdots 2\cdot 2_0].
\]

Moreover, we have the following theorem.

Theorem 3.3. For each relation in Theorem 3.1, all summands on the right hand side are irreducible.

Theorem 3.3 will be prove in Section 5

3.2. Definition of a cluster algebra $\mathcal{A}$. Let $S = \{-2n + 1 \mid n \in \mathbb{Z}_{\geq 1}\}$, $S' = \{-2n + 2 \mid n \in \mathbb{Z}_{\geq 1}\}$, and $V = \{(1) \times S\} \bigcup (2) \times S'$). We define a quiver $Q$ with vertex set $V$ as follows. The arrows of $Q$ are given by the following rules. For $s_1 \in S, s_2 \in S'$, there is an arrow from $(1,s_1)$ to $(2,s_2)$ if and only if $s_2 = s_1 - 5$ and there is an arrow from $(2,s_2)$ to $(1,s_1)$ if and only if $s_1 = s_2 - 1$. The quiver $Q$ is the same as the quiver $G$ of type $G_2$ in [HL13].

Let $t = \{t_{k,0}^{(s)}, t_{l,0}^{(s)} \mid s \in S, k, l \in \mathbb{Z}_{\geq 1}\}$. Let $\mathcal{A}$ be the cluster algebra defined by the initial seed $(t,Q)$. By Definition 2.4, $\mathcal{A}$ is the $\mathbb{Q}$-subalgebra of the field of rational functions $\mathbb{Q}(t)$ generated by all the elements obtained from some elements of $t$ via a finite sequence of seed mutations.

3.3. Mutation sequences. We use “$C_1$” to denote the column of vertices $(1,-1), (1,-7), \ldots, (1,-6n+5), \cdots$ in the quiver $Q$. We use “$C_2$” to denote the column of vertices $(1,-3), (1,-9), \ldots, (1,-6n+3), \cdots$ in $Q$. We use “$C_3$” to denote the column of vertices $(1,-5), (1,-11), \ldots, (1,-6n+1), \cdots$ in $Q$. We use “$C_4$” to denote the column of vertices $(2,0), (2,-2), \ldots, (2,1-2n+2), \cdots$ in $Q$. By saying that we mutate at the column $C_i$, $i \in \{1,2,3,4\}$, we mean that we mutate the vertices of $C_i$ as follows. First we mutate at the first vertex in this column, then the second vertex, an so on until the vertex at infinity. By saying that we mutate $(C_{i_1}, C_{i_2}, \ldots, C_{i_n})$, where $i_j \in \{1,2,3,4\}$, $j = 1,2,\ldots,n$, we mean that we first mutate the column $C_{i_1}$, then the column $C_{i_2}$, an so on up to the column $C_{i_n}$.

We define some variables $t_{k,l}^{(s)}$ ($k, l \in \mathbb{Z}_{\geq 0}, s \in S$) recursively as follows. The variables $t_{k,0}^{(s)}$, $t_{0,0}^{(s)}$, $s_1, s_2 \in S$, are already defined. They are cluster variables in the initial seed of $\mathcal{A}$ define in Section 3.2. For convenience, we write $t_{s_1}^{(-s_1/6,0)}$ at the vertex $(1,s_1)$ and write $t_{s_2}^{(-s_2+1)/2}$ at the vertex $(2,s_2)$ in the initial quiver $Q$, $s_1, s_2 \in S$. Then we obtain the quiver (a) in Figure 1.
Consider the mutation sequence \((C_1, C_1, \ldots, C_1)\) start from the initial seed, where the number of \(C_1\) is \(n\). First we mutate the first vertex in \(C_1\) and define \(t_{1,1}^{(-7)} = t_{1,0}^{(-1)}\). Therefore
\[
t_{1,1}^{(-7)} = t_{1,0}^{(-1)} = \frac{t_{2,0}^{(-7)} t_{0,1}^{(-1)} + t_{0,4}^{(-7)}}{t_{1,0}^{(-1)}}.
\] (3.3)

After this mutation, the quiver (a) in Figure 1 becomes the quiver (b) in Figure 1. Then we mutate the second vertex of \(C_1\) and define \(t_{2,1}^{(-13)} = t_{2,0}^{(-7)}\). Therefore
\[
t_{2,1}^{(-13)} = t_{2,0}^{(-7)} = \frac{t_{3,0}^{(-13)} t_{1,4}^{(-7)} + t_{0,7}^{(-13)}}{t_{2,0}^{(-7)}}.
\] (3.4)

After this mutation, the quiver (b) in Figure 1 becomes the quiver (c) in Figure 1. We continue this procedure and mutate the vertices of \(C_1\) in order and define \(t_{k,1}^{(-6k-1)} = t_{k,0}^{(-6k+5)}\) \((k = 3, 4, \ldots)\) recursively. Therefore
\[
t_{k,1}^{(-6k-1)} = t_{k,0}^{(-6k+5)} = \frac{t_{k+1,0}^{(-6k-1)} t_{k-1,1}^{(-6k+5)} + t_{0,3k+1}^{(-6k-1)}}{t_{k,0}^{(-6k+5)}};
\] (3.5)

Now we finish the mutation of the first \(C_1\) in \((C_1, C_1, \ldots, C_1)\). We start to mutate the second \(C_1\) in \((C_1, C_1, \ldots, C_1)\). First we mutate the first vertex in \(C_1\) and define \(t_{1,4}^{(-13)} = t_{1,1}^{(-7)}\). Therefore
\[
t_{1,4}^{(-13)} = t_{1,1}^{(-7)} = \frac{t_{2,1}^{(-13)} t_{0,4}^{(-7)} + t_{0,1}^{(-1)} t_{0,7}^{(-13)}}{t_{1,1}^{(-7)}}.
\] (3.6)

After this mutation, we obtain the quiver (e) in Figure 1. Then we mutate the second vertex of \(C_1\) and define \(t_{2,4}^{(-19)} = t_{2,1}^{(-13)}\). Therefore
\[
t_{2,4}^{(-19)} = t_{2,1}^{(-13)} = \frac{t_{3,1}^{(-19)} t_{1,4}^{(-13)} + t_{0,1}^{(-1)} t_{0,10}^{(-19)}}{t_{2,1}^{(-13)}}.
\] (3.7)

After this mutation, the quiver (e) in Figure 1 becomes the quiver (f) in Figure 1. We continue this procedure and mutate vertices of \(C_1\) in order and define \(t_{k,4}^{(-6k-7)} = t_{k,1}^{(-6k-1)}\) \((k = 3, 4, \ldots)\) recursively. Therefore
\[
t_{k,4}^{(-6k-7)} = t_{k,1}^{(-6k-1)} = \frac{t_{k+1,1}^{(-6k-7)} t_{k-1,4}^{(-6k-1)} + t_{0,1}^{(-1)} t_{0,3k+4}^{(-6k-7)}}{t_{k,1}^{(-6k-1)}};
\] (3.8)

Now we finish the mutation of the second \(C_1\) in the mutation sequence \((C_1, C_1, \ldots, C_1)\). We continue this procedure and mutate \(r\)-th \((r = 3, 4, \ldots, n)\) \(C_1\) in \((C_1, C_1, \ldots, C_1)\) in order. We define \(t_{k,3r-2}^{(-6k-6r+5)} = t_{k,3r-5}^{(-6k-6r+11)}\), where \((k, r) = (1, 3), (2, 3), (3, 3), (4, 3), \ldots; (1, 4), (2, 4), (3, 4), (4, 4), \ldots\).
\[ t_{k,3r-2}^{(-6k-6r+5)} = t_{k,3r-5}^{(-6k-6r+11)} = \frac{t_{k+1,3r-5}^{(-6k-6r+11)} + t_{k-1,3r-2}^{(-6k-6r+11)}}{t_{k,3r-5}^{(-6k-6r+11)}} \tag{3.9} \]

where \((k, r) = (1, 3), (2, 3), (3, 3), (4, 3), \ldots; (1, 4), (2, 4), (3, 4), (4, 4), \ldots; (1, 5), (2, 5), (3, 5), (4, 5) \ldots; (1, 6), (2, 6), (3, 6), (4, 6), \ldots\)

Similarly, we consider the mutation sequence \((C_2, C_2, \ldots, C_2)\) start from the initial seed, where the number of \(C_2\) is \(n\). First we mutate vertices in the first \(C_2\) in order and define \(t_{k,2}^{(-6k-3)} = t_{k,0}^{(-6k+3)} (k = 1, 2, \ldots)\) recursively. Therefore

\[ t_{k,2}^{(-6k-3)} = t_{k,0}^{(-6k+3)} = \frac{t_{k+1,0}^{(-6k-3)} - t_{k-1,2}^{(-6k-3)}}{t_{k,0}^{(-6k+3)}}, \quad k = 1, 2, \ldots \tag{3.10} \]

Then we mutate the second \(C_2\), the third \(C_2\) and so on in the mutation sequence \((C_2, C_2, \ldots, C_2)\).

We define \(t_{k,3r-1}^{(-6k-6r+9)} = t_{k,3r-4}^{(-6k-6r+9)}\), where \((k, r) = (1, 2), (2, 2), (3, 2), (4, 2), \ldots; (1, 3), (2, 3), (3, 3), (4, 3), \ldots; (1, 4), (2, 4), (3, 4), (4, 4) \ldots; (1, 5), (2, 5), (3, 5), (4, 5), \ldots\).

Similarly, we consider the mutation sequence \((C_3, C_3, \ldots, C_3)\) start from the initial seed, where the number of \(C_3\) is \(n\). First we mutate vertices in the first \(C_3\) in order and define \(t_{k,2}^{(-6k-5)} = t_{k,0}^{(-6k+1)} (k = 1, 2, \ldots)\) recursively. Therefore

\[ t_{k,3}^{(-6k-5)} = t_{k,0}^{(-6k+1)} = \frac{t_{k+1,0}^{(-6k-5)} - t_{k-1,3}^{(-6k+1)}}{t_{k,0}^{(-6k+1)}}, \quad k = 1, 2, \ldots \tag{3.12} \]

Then we mutate the second \(C_3\), the third \(C_3\) and so on in the mutation sequence \((C_3, C_3, \ldots, C_3)\). We define \(t_{k,3r-1}^{(-6k-6r+11)} = t_{k,3r-4}^{(-6k-6r+11)}\), where \((k, r) = (1, 2), (2, 2), (3, 2), (4, 2), \ldots; (1, 3), (2, 3), (3, 3), (4, 3), \ldots; (1, 4), (2, 4), (3, 4), (4, 4) \ldots; (1, 5), (2, 5), (3, 5), (4, 5), \ldots\), recursively.

Similarly, we consider the mutation sequence \((C_4, C_4, \ldots, C_4)\) start from the initial seed, where the number of \(C_4\) is \(n\). First we mutate vertices in the first \(C_4\) in order and define \(t_{k,2}^{(-6k-7)} = t_{k,0}^{(-6k+3)} (k = 1, 2, \ldots)\) recursively. Therefore

\[ t_{k,3}^{(-6k-7)} = t_{k,0}^{(-6k+3)} = \frac{t_{k+1,0}^{(-6k-7)} - t_{k-1,3}^{(-6k+3)}}{t_{k,0}^{(-6k+3)}}, \quad k = 1, 2, \ldots \tag{3.13} \]

where \((k, r) = (1, 2), (2, 2), (3, 2), (4, 2), \ldots; (1, 3), (2, 3), (3, 3), (4, 3), \ldots; (1, 4), (2, 4), (3, 4), (4, 4), \ldots; (1, 5), (2, 5), (3, 5), (4, 5), \ldots\)
The mutation sequence $(C_1, C_1, \ldots, C_1)$. 

Figure 1.
3.4. The equations in the $M$-system of type $G_2$ correspond to mutations in the cluster algebra $\mathcal{A}$. By (3.3), (3.4), (3.5), (3.10), (3.12), we have

$$t_{k,l}^{(s)} = t_{k,l}^{(s+6)} = \frac{t_{k+1,0}^{(s+6)}k_{k-1,l}^{(s)} + t_{0,3k+l}^{(s+6)}l_{k,0}^{(s+6)}}{(k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\}),} \quad (3.14)$$

where $s \in \{-2n - 5 | n \in \mathbb{Z}_{\geq 1}\}$. Equations (3.14) correspond to Equations (3.1) in the M-system. By (3.6), (3.7), (3.8), (3.9), (3.11), (3.13), we have

$$t_{k,l+3}^{(s)} = t_{k,l}^{(s+6)} = \frac{t_{k+1,1}^{(s)}k_{k-1,l+3}^{(s+6)} + t_{0,3k+l+3}^{(s+6)}l_{k,0}^{(s+6)}}{(k, l \in \mathbb{Z}_{\geq 1}),} \quad (3.15)$$

where $s \in \{-2n - 11 | n \in \mathbb{Z}_{\geq 1}\}$. Equations (3.15) correspond to Equations (3.2) in the M-system. Therefore we have the following theorem.

**Theorem 3.4.** Minimal affinizations of type $G_2$ correspond to cluster variables in $\mathcal{A}$ defined in Section 3.2.

3.5. The $m$-system of type $G_2$. For $k, l \in \mathbb{Z}_{\geq 0}$, let $m_{k,l} = \text{Res}(T_{k,l}^{(0)})$ be the restriction of $T_{k,l}^{(0)}$ to $U_q\mathfrak{g}$. Let $\chi(M)$ be the character of a $U_q\mathfrak{g}$-module $M$. By Theorem 3.1 we have the following theorem.

**Theorem 3.5.** We have

$$\chi(m_{k,l})\chi(m_{k,0}) = \chi(m_{k+1,0})\chi(m_{k-1,l}) + \chi(m_{0,3k+l}) \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\}), \quad (3.16)$$

$$\chi(m_{k,l+3})\chi(m_{k,l}) = \chi(m_{k+1,l})\chi(m_{k-1,l+3}) + \chi(m_{0,l})\chi(m_{0,3k+l+3}) \quad (k, l \in \mathbb{Z}_{\geq 1}). \quad (3.17)$$

We call the above system of equations the $m$-system of type $G_2$.

4. Proof of Theorem 3.1

In this section, we prove Theorem 3.1.

By the Frenkel-Mukhin algorithm, we have the following result.

**Lemma 4.1.** The fundamental $q$-characters for $U_q\mathfrak{g}$ of type $G_2$ are given by

$$\chi_q(1_0) = 1_0 + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11}$$

$$+ 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11}$$

$$+ 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11}$$

$$= 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11} + 2_0 2_1 2_3 2_5 2_7 2_9 2_{11}$$

$$4.1. \text{Classification of dominant monomials in the summands on both sides of the M-system.} \text{ By Theorem 3.8 in [Hir07] (see also Theorem 3.3 in [LM13]), the modules } T^{(s)}_{k,l} (s \in \mathbb{Z}, k, l \in \mathbb{Z}_{\geq 0}) \text{ are special. Therefore we can use the Frenkel-Mukhin algorithm to compute the } q\text{-characters of } T^{(s)}_{k,l} (s \in \mathbb{Z}, k, l \in \mathbb{Z}_{\geq 0}). \text{ Now we use the Frenkel-Mukhin algorithm to classify dominant monomials in the summands on both sides of the M-system.}

**Lemma 4.2.** We have the following cases.
(1) Let $M = T_{k,l}(s_1)T_{k,0}^{-1}(s+6)$, $(k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\})$. Then dominant monomials in $\chi_q(T_{k,l}(s))\chi_q(T_{k,0}^{(s+6)})$ $(k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\})$ are $M$ and

$$M_r = M \prod_{i=0}^{r-1} A^{-1}_{1,aq-2l-6i-2}, \quad r = 1, 2, \ldots, k.$$  

The dominant monomials in $\chi_q(T_{k-l+1,l}^{(s+6)})(k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\})$ are $M$, $M_1$, $\ldots$, $M_{k-1}$. The only dominant monomial in $T_{0,3k+4}^{(s)}$ is $M_k$. 

(2) Let $M = T_{k+l+3,k,l}^{(s+6)}$, $(k, l \in \mathbb{Z}_{\geq 1})$. Then dominant monomials in $\chi_q(T_{k,l}^{(s+6)})(k, l \in \mathbb{Z}_{\geq 1})$ are $M$ and 

$$M_r = M \prod_{i=0}^{r-1} A^{-1}_{1,aq+6k-6i-3}, \quad r = 1, 2, \ldots, k.$$  

The dominant monomials in $\chi_q(T_{k-l+3,l}^{(s+6)})(k, l \in \mathbb{Z}_{\geq 1})$ are $M$, $M_1$, $\ldots$, $M_{k-1}$. The only dominant monomial in $\chi_q(T_{0,3k+4}^{(s+6)})(k, l \in \mathbb{Z}_{\geq 1})$ is $M_k$. 

Proof. We prove the case of $\chi_q(T_{k,l}^{(s+6)})(k, l \in \mathbb{Z}_{\geq 1})$. The other cases are similar. Let $m'_1 = T_{k,l+3}^{(s)}, m'_2 = T_{k,l}^{(s+6)}$. Without loss of generality, we may assume that $s = 0$. Then 

$$m'_1 = (1_{0,1}1_{6k-6}6k+126k+3 \cdots 26k+2l+5),$$

$$m'_2 = (1_{6k}1_{6k}6k+726k+9 \cdots 26k+2l+5).$$

Let $m = m_1m_2$ be a dominant monomial, where $m_i \in \chi_q(m'_i), i = 1, 2$. We denote $m_3 = 2_{6k+1}26k+7 \cdots 26k+2l+5$, $m_4 = 2_{6k+7}26k+9 \cdots 26k+2l+5$. 

Suppose that $m_1 \in \chi_q(1_{0,1}6k-6)6k+126k+3 \cdots 26k+2l+5), m_2 \in \chi_q(1_{6k}1_{6k}6k+726k+9 \cdots 26k+2l+5)$. 

Similarly, if $m_2 \in \chi_q(1_{6k}1_{6k}6k+726k+9 \cdots 26k+2l+5)$, then $m = m_1m_2$ is right negative and hence $m$ is not dominant. This contradicts our assumption. Therefore $m_1 \in \chi_q(1_{0,1}6k-6)m_3$. 

Suppose that $m_2 \in \mathcal{M}(L(m'_2)) \cap \mathcal{M}(\chi_q(1_{6k}1_{6k}6k+726k+9 \cdots 26k+2l+5))(\chi_q(1_{6k})-1_{6k})m_4)$. By the Frenkel-Mukhin algorithm for $L(m'_2)$ and Lemma 4.1, $m_2$ must have the factor $1_{6k+6}^{-1}$. But by the Frenkel-Mukhin algorithm and the fact that $m_1 \in \chi_q(1_{0,1}6k-6)m_3$, $m_1$ does not have the factor $1_{6k+6}$. Therefore $m_1m_2$ is not dominant. Hence $m_2 \in \chi_q(1_{6k}1_{6k}6k+726k+9 \cdots 26k+2l+5)m_4$.

By the Frenkel-Mukhin algorithm and the fact that $m_1 \in \chi_q(1_{0,1}6k-6)m_3$, $m_1$ must be one of the following monomials,

$$n_1 = m_1^{-1}A^{-1}_{1,6k-3} = 1_{0,1}6k-121_{6k-12}^{-1}2_{6k-5}2_{6k-3} \cdots 2_{6k+2l+5},$$

$$n_2 = m_1^{-1}A^{-1}_{1,6k-3}A^{-1}_{1,6k-9} = 1_{0,1}6k-181_{6k-18}^{-1}2_{6k-11}2_{6k-9} \cdots 2_{6k+2l+5},$$

$$\cdots$$

$$n_k = m_1^{-1}A^{-1}_{1,6k-3}A^{-1}_{1,6k-9} \cdots A^{-1}_{1,3} = 1_{6k-1}^{-1}1_{6k-1}^{-1}2_{6k-6}2_{6k-9} \cdots 2_{6k+2l+5}.$$
It follows that the dominant monomials in $\chi_q(T_{k,l+3})\chi_q(T_{k,l}^{(s)})$ are

$$M = m'_1m'_2, \ M_1 = n_1m'_2 = MA_{1,6k-3}^{-1}, \ M_2 = n_2m'_2 = M \prod_{i=0}^{1} A_{1,6k-6i-3}^{-1}, \ \ldots,$$

$$M_{k-1} = n_{k-1}m'_2 = M \prod_{i=0}^{k-2} A_{1,6k-6i-3}^{-1}, \ M_k = n_km'_2 = M \prod_{i=0}^{k-1} A_{1,6k-6i-3}^{-1}.$$

\[\square\]

### 4.2. Proof of Theorem 3.1

By Lemma 4.2, the dominant monomials in the $q$-characters of the left hand side and of the right hand side of every equation in Theorem 3.1 are the same. Therefore the theorem is true.

### 5. Proof of Theorem 3.3

By Lemma 4.2, the modules in the second summand of every equation in Theorem 3.1 are special and hence they are irreducible. We only need to show that the modules in the first summand in every equation in Theorem 3.1 are irreducible. Let $S$ be a module in the first summand in an equation in Theorem 3.1. It suffices to prove that for each non-highest dominant monomial $M$ in $S$, we have $\chi_q(L(M)) \not\subseteq \chi_q(S)$, see [Her06, MY12a].

**Lemma 5.1.** We consider the same cases as in Lemma 4.2. In each case $M_i$ are the dominant monomials described by that Lemma 4.2.

1. For $k \in \mathbb{Z}_{\geq 1}$ and $l \in \{1, 2, 3\}$, let
   $$n_r = M_r A_{1,aoq-2l-6r+4}^{-1}, \quad r = 1, 2, \ldots, k - 1.$$  
   Then for $i = 1, 2, \ldots, k - 1, n_i \in \chi_q(M_i)$ and $n_i \not\in \chi_q(T_{k,l}^{(s)})\chi_q(T_{k,0}^{(s+6)})$.

2. For $k, l \in \mathbb{Z}_{\geq 1}$, let
   $$n_r = M_r A_{1,aoq+6k-6r+3}^{-1}, \quad r = 1, 2, \ldots, k - 1.$$  
   Then for $i = 1, 2, \ldots, k - 1, n_i \in \chi_q(M_i)$ and $n_i \not\in \chi_q(T_{k,l}^{(s)})\chi_q(T_{k,l}^{(s+6)})$.

**Proof.** We give a proof of the case of $\chi_q(T_{k,l+3}^{(s)})\chi_q(T_{k,l}^{(s+6)})$. The other cases are similar. By definition, we have

$$T_{k,l+3}^{(s)} = 1_s1_s+6 \cdots 1_s+6k-62_s+6k+12_s+6k+3 \cdots 2_s+6k+2l+5,$$

$$T_{k,l}^{(s+6)} = 1_s+61_s+12 \cdots 1_s+6k-61_s+6k2_s+6k+72_s+6k+9 \cdots 2_s+6k+2l+5,$$

$$M_1 = T_{k,l+3}^{(s)} T_{k,l}^{(s+6)} A_{1,aoq+6k-3}^{-1} = T_{k,l+3}^{(s)} T_{k,l}^{(s+6)} A_{1,aoq+6k-3}^{-1}.$$  

By $U_q(\tilde{sl}_2)$ argument, it is clear that $n_1 = M_1 A_{1,aoq+6k-3}^{-1}$ is in $\chi_q(M_1)$.
If \( n_1 \) is in \( \chi_q(T_{k,l}^{(s)})\chi_q(T_{k,l+3}^{(s+6)}) \), then \( T_{k,l}^{(s+6)}A_{1,aq+6k-3}^{-1} \) is in \( \chi_q(T_{k,l}^{(s+6)}) \) which is impossible by the Frenkel-Mukhin algorithm for \( T_{k,l}^{(s+6)} \). Similarly, \( n_i \in \chi_q(M_i), i = 1, 2, \cdots, k - 1 \), but \( n_2, n_3, \cdots, n_{k-1} \) are not in \( \chi_q(T_{k,l+3}^{(s)})\chi_q(T_{k,l}^{(s+6)}) \).

\[ \square \]

6. THE DUAL M-SYSTEM OF TYPE \( G_2 \)

In this section, we study the dual M-system of type \( G_2 \).

**Theorem 6.1** (Theorem 7.2, \[LM13\]). The module \( \tilde{T}_{k,l}^{(s)} \), \( s \in \mathbb{Z}, k, l \in \mathbb{Z}_{\geq 0} \) are anti-special.

**Lemma 6.2** (Lemma 7.3, \[LM13\]). Let \( \iota : \mathbb{Z}P \to \mathbb{Z}P \) be a homomorphism of rings such that \( Y_{1,aq}^s \mapsto Y_{1,aq}^{-1}, Y_{2,aq}^s \mapsto Y_{2,aq}^{-1} \) for all \( a \in \mathbb{C}^\times, s \in \mathbb{Z} \). Then

\[ \chi_q(\tilde{T}_{k,l}^{(s)}) = \iota(\chi_q(T_{k,l}^{(s)})). \]

**Theorem 6.3.** For \( s \in \mathbb{Z}, k, l \in \mathbb{Z}_{\geq 0} \), we have the following system of equations which we call the dual M-system of type \( G_2 \).

\[ \begin{align*}
[\tilde{T}_{k,l}^{(s)}][\tilde{T}_{k,l+3}^{(s+6)}] &= [\tilde{T}_{k+1,l}^{(s)}][\tilde{T}_{k+1,l+3}^{(s+6)}] + [\tilde{T}_{0,3k+l}^{(s)}] \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\}), \\
[\tilde{T}_{k,l+3}^{(s)}][\tilde{T}_{k,l}^{(s+6)}] &= [\tilde{T}_{k+1,l+3}^{(s)}][\tilde{T}_{k+1,l}^{(s+6)}] + [\tilde{T}_{0,l}^{(s+6k+6)}][\tilde{T}_{0,3k+l+3}^{(s)}] \quad (k, l \in \mathbb{Z}_{\geq 1}).
\end{align*} \]

Moreover, every module in the summands on the right hand side of every equation in the dual M-system is irreducible.

**Proof.** The lowest weight monomial of \( \chi_q(T_{k,l}^{(s)}) \) is obtained from the highest weight monomial of \( \chi_q(T_{k,l}^{(s)}) \) by the substitutions: \( 1_s \mapsto 1_{12+s}^{-1}, 2_s \mapsto 2_{12+s}^{-1} \). After we apply \( \iota \) to \( \chi_q(T_{k,l}^{(s)}) \), the lowest weight monomial of \( \chi_q(T_{k,l}^{(s)}) \) becomes the highest weight monomial of \( \iota(\chi_q(T_{k,l}^{(s)})) \). Therefore the highest weight monomial of \( \iota(\chi_q(T_{k,l}^{(s)})) \) is obtained from the lowest weight monomial of \( \chi_q(T_{k,l}^{(s)}) \) by the substitutions: \( 1_s \mapsto 1_{12-s}^{-1}, 2_s \mapsto 2_{12-s}^{-1} \). It follows that the highest weight monomial of \( \iota(\chi_q(T_{k,l}^{(s)})) \) is obtained from the highest weight monomial of \( \chi_q(T_{k,l}^{(s)}) \) by the substitutions: \( 1_s \mapsto 1_{-s}, 2_s \mapsto 2_{-s} \). Therefore the dual M-system is obtained applying \( \iota \) to both sides of every equation of the M-system.

The irreducibility of every module in the summands on the right hand side of every equation in the dual M-system follows from Theorem 6.3 and Lemma 6.2.

**Example 6.4.** The following are some equations in the dual M-system of type \( G_2 \).

\[ \begin{align*}
[20]1_7[1_1] &= [1_1]1_7[2_0] + [2_0]2_2[2_4]2_6], \\
[20]2_21_0[1_3] &= [1_3]1_9[2_0]2_2 + [2_0]2_2[2_4]2_6], \\
[20]2_22_1[1_1]1_1[1_5] &= [1_1]1_1[2_0]2_2[2_4] + [2_0]2_2[2_4]2_62_82_0], \\
[20]2_22_2[2_4][1_5][1_3][20]1_7 &= [2_0]1_71_3[2_0]2_2[2_4] + [2_0][2_0]2_2[2_4]2_6 + [2_0][2_0]2_2[2_4]2_62_0], \\
[2_0]2_2[2_0]2_0[1_7]1_3[2_0]1_7 &= [1_7]1_32_0[2_0]2_2[2_4]2_6 + [2_0][2_0]2_2[2_4]2_62_0], \\
[2_0]2_22_2[2_0]2_2[1_7]1_3[2_0]1_7 &= [2_0]2_2[2_0][1_7]1_3[2_0]2_2[2_4]2_6 + [2_0][2_0]2_2[2_4]2_62_0], \\
[2_0]2_22_2[2_0]2_2[1_7]1_3[2_0]1_7 &= [2_0]2_2[2_0]2_2[1_7]1_3[2_0]1_7 + [2_0][2_0]2_2[2_4]2_62_0].
\end{align*} \]
6.1. The dual \( m \)-system of type \( G_2 \). For \( k, l \in \mathbb{Z}_{\geq 0} \), let \( \tilde{m}_{k,l} = \text{Res}(\tilde{T}^{(0)}_{k,l}) \) be the restriction of \( \tilde{T}^{(0)}_{k,l} \) to \( U_q\mathfrak{g} \). By Theorem 6.3 we have the following theorem.

**Theorem 6.5.** We have

\[
\chi(\tilde{m}_{k,l})\chi(\tilde{m}_{k,0}) = \chi(\tilde{m}_{k+1,0})\chi(\tilde{m}_{k-1,l}) + \chi(\tilde{m}_{0,3k+l}) \quad (k \in \mathbb{Z}_{\geq 1}, l \in \{1, 2, 3\}),
\]

\[
\chi(\tilde{m}_{k,l+3})\chi(\tilde{m}_{k,l}) = \chi(\tilde{m}_{k+1,l})\chi(\tilde{m}_{k-1,l+3}) + \chi(\tilde{m}_{0,l})\chi(\tilde{m}_{0,3k+l+3}) \quad (k, l \in \mathbb{Z}_{\geq 1}).
\]

We call the above system of equations the dual \( m \)-system of type \( G_2 \).

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Li Qiao: Department of Mathematics, Lanzhou University, Lanzhou 730000, P. R. China. 
E-mail address: qiaol12@lzu.edu.cn

Jian-Rong Li: Department of Mathematics, Lanzhou University, Lanzhou 730000, P. R. China. 
E-mail address: lijr@lzu.edu.cn, lijr07@gmail.com