Abstract

In $\mathcal{N} = 2$ Poincaré supersymmetry in four space-time dimensions, there exist off-shell supermultiplets with intrinsic central charge, including the important examples of the Fayet-Sohnius hypermultiplet, the linear and the nonlinear vector-tensor (VT) multiplets. One can also define similar supermultiplets in the context of $\mathcal{N} = 2$ anti-de Sitter (AdS) supersymmetry, although the origin of the central charge becomes somewhat obscure. In this paper we develop a general setting for $\mathcal{N} = 2$ AdS supersymmetric theories with central charge. We formulate a supersymmetric action principle in $\mathcal{N} = 2$ AdS superspace and then reformulate it in terms of $\mathcal{N} = 1$ superfields. We prove that $\mathcal{N} = 2$ AdS supersymmetry does not allow existence of a linear VT multiplet. For the nonlinear VT multiplet, we derive consistent superfield constraints in the presence of any number of $\mathcal{N} = 2$ Yang-Mills vector multiplets, give the supersymmetric action and elaborate on the $\mathcal{N} = 1$ superfield and component descriptions of the theory. Our description of the nonlinear VT multiplet in AdS is then lifted to $\mathcal{N} = 2$ supergravity. Moreover, we derive consistent superfield constraints and Lagrangian that describe the linear VT multiplet in $\mathcal{N} = 2$ supergravity in the presence of two vector multiplets, one of which gauges the central charge. These supergravity constructions thus provide the first superspace formulation for the component results derived in arXiv:hep-th/9710212. We also construct higher-derivative couplings of the VT multiplet to any number of $\mathcal{N} = 2$ tensor multiplets.
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1 Introduction

The vector-tensor (VT) supermultiplet is a dual version of the Abelian $\mathcal{N} = 2$ vector multiplet in four dimensions, obtained by dualizing one of the two physical scalars (belonging to the vector multiplet) into a gauge two-form. The auxiliary fields of these multiplets also differ, namely: a real isotriplet in the vector multiplet case, and a real scalar in the dual version. In contrast to the vector multiplet, the VT multiplet has an off-shell central charge, similar to the Fayet-Sohnius hypermultiplet $[1, 2]$.

The history of the VT multiplet is quite interesting. It was discovered by Sohnius, Stelle and West $[3]$ in 1980 (see $[4]$ for a review) as a spin-off of their attempts to construct an off-shell formulation for extended supersymmetric gauge theories. Soon after, it was shown by Milewski $[5]$ (see also $[6]$ for a review) that this multiplet has a simple structure from the point of view of $\mathcal{N} = 1$ supersymmetry. Specifically, its action functional in $\mathcal{N} = 1$ superspace is the sum of those describing $\mathcal{N} = 1$ vector and tensor multiplets

$$S_{\text{VT}} = \frac{1}{2} \int d^4x d^2\theta W^\alpha W_\alpha - \frac{1}{2} \int d^4x d^4\theta G^2 . \quad (1.1)$$

Here $W_\alpha$ is the chiral field strength of the $\mathcal{N} = 1$ vector multiplet,

$$\bar{D}_\alpha W_\alpha = 0 , \quad D^\alpha W_\alpha = \bar{D}^\alpha \bar{W}_{\dot{\alpha}} , \quad (1.2)$$

while $G$ is the real linear field strength of the $\mathcal{N} = 1$ tensor multiplet $[7],$

$$\bar{G} = G , \quad \bar{D}^2 G = 0 . \quad (1.3)$$

Then, the VT multiplet was completely forgotten for over a decade.

Research on the VT multiplet experienced a renaissance in the year 1995 when de Wit, Kaplunovsky, Louis and Lüst $[8]$ realized that this multiplet describes the dilaton-axion complex in heterotic $\mathcal{N} = 2$ four-dimensional supersymmetric string vacua $[3]$. This work triggered numerous studies of the VT multiplet and its Chern-Simons couplings in the component field approach (both in rigid supersymmetry and supergravity using the superconformal tensor calculus) $[9, 10, 11]$, as well as in the framework of conventional $\mathcal{N} = 2$ superspace $[12, 13, 14]$ and $\mathcal{N} = 2$ harmonic superspace $[15, 16, 17, 18]$ (see also $[19]$). In particular, it was found that besides the original ‘linear’ VT multiplet $[3]$,

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1 From a historical point of view, it is of interest to mention that Ref $[8]$ in fact announced the discovery of the VT multiplet, in spite of the existence of the original $[3, 5]$ and review $[4, 6]$ papers on the VT multiplet published in the early 1980s. It thus appears that this multiplet had been completely forgotten by mid-1990s.
there exists a ‘nonlinear’ VT multiplet [9, 10]. The difference between these inequivalent realizations is quite transparent in \( \mathcal{N} = 2 \) superspace, and here we would like to discuss this issue in some detail.

Let us consider \( \mathcal{N} = 2 \) central charge superspace [2]. The spinor derivatives with central charge, \( \Delta \), form the algebra

\[
\{ D^i, D^j \} = 2\varepsilon_{\alpha\beta}\varepsilon^{ij}\Delta ,
\]
\[
\{ \bar{D}_{\dot{i}}, \bar{D}_{\dot{j}} \} = -2\varepsilon^{\dot{i}\dot{j}}\varepsilon_{ij}\Delta ,
\]
\[
\{ D^i, \bar{D}_{\dot{j}} \} = -2i\delta^i_{\dot{j}}\partial_\alpha\delta_{\dot{\alpha}} .
\]

Following [13, 15, 14], the linear VT multiplet can be described by a real superfield, \( L \), constrained by

\[
D_{ij}L = 0 ,
\]
\[
D^i_{\dot{\alpha}}\bar{D}^j_{\dot{\beta}}L = 0 ,
\]

where we have denoted \( D_{ij} := D^{\alpha(i}D_{\alpha j)} \) and \( \bar{D}_{i\dot{j}} := \bar{D}_{\dot{\alpha}(i}\bar{D}^{\dot{\alpha}}_{\dot{j})} \). The multiplet is on-shell, \( \Box L = 0 \), if \( \Delta L = 0 \). Following [16], the nonlinear VT multiplet is described by a real superfield \( \mathbb{L} \) subject to the constraints

\[
D^{ij}\mathbb{L} = 2\kappa D^i\mathbb{L}D^j\mathbb{L} - \kappa \bar{D}_{ij}\mathbb{L} ,
\]
\[
D^i_{\dot{\alpha}}\bar{D}^j_{\dot{\beta}}\mathbb{L} = \kappa D^i_{\dot{\alpha}}\mathbb{L}\bar{D}^j_{\dot{\beta}}\mathbb{L} ,
\]

where \( \kappa \) is a real coupling constant of inverse mass dimension. These constraints can be written in an alternative form [17] using a new superfield \( L = \exp (-\kappa \mathbb{L}) \). One finds

\[
D^{ij}L = -\frac{1}{L}D^iLD^jL + \frac{1}{L}\bar{D}^i\bar{D}^jL ,
\]
\[
D^i_{\dot{\alpha}}\bar{D}^j_{\dot{\beta}}L = 0 .
\]

We can think of the constraints (1.6) as the unique consistent deformation of (1.5), see [16] for more details. The two VT multiplets have different Chern-Simons couplings to vector multiplets [9, 10, 18], including the one that gauges the central charge, and to \( \mathcal{N} = 2 \) supergravity [11].

It turns out the constraints for the VT multiplets have an interesting higher-dimensional origin. The linear VT multiplet constraints, eq. (1.5), can be interpreted as the equations obeyed by the gauge-invariant superfield strength of a free on-shell \( \mathcal{N} = 1 \) vector multiplet in five dimensions [20]. The same constraints also admit a six-dimensional origin [18] in
terms of the (1,0) self-dual tensor multiplet [21, 22]. The nonlinear VT multiplet constraints, eq. (1.7), coincide with the equations of motion for the five-dimensional $\mathcal{N} = 1$ supersymmetric U(1) Chern-Simons theory [20].

One of our goals in this paper is to study VT multiplets and their couplings in four-dimensional $\mathcal{N} = 2$ anti-de Sitter (AdS) supersymmetry. It is known that $\mathcal{N} = 1, 2$ rigid supersymmetric theories in AdS differ significantly from their counterparts defined in Minkowski space [23, 25, 26, 27, 28, 29]. We therefore may expect nontrivial restrictions to VT multiplet interactions in AdS. Indeed, our first observation is that a linear VT multiplet does not exist in AdS. The simplest way to prove this claim is by using a formulation in $\mathcal{N} = 1$ AdS superspace.

Let us assume that there exists an AdS extension of the linear VT multiplet. Then its dynamics can be formulated in $\mathcal{N} = 1$ AdS superspace where the linear VT multiplet decomposes into a vector and a tensor multiplet. The corresponding action should be a minimal AdS extension of (1.1), that is

$$S_{LVT} = \frac{1}{2} \int d^4x d^4\theta \frac{E}{\mu} W^\alpha W_\alpha - \frac{1}{2} \int d^4x d^4\theta \bar{E} G^2 ,$$

(1.8)

where $W_\alpha$ is the covariantly chiral field strength of the vector multiplet,

$$\bar{\nabla}_\alpha W_\alpha = 0 , \quad \nabla^\alpha W_\alpha = \bar{\nabla}_\dot{\alpha} W^{\dot{\alpha}} ,$$

(1.9)

and $G$ is the real linear field strength of the tensor multiplet,

$$\bar{G} = G , \quad (\nabla^2 - 4\mu)G = 0 .$$

(1.10)

The tensor multiplet sector of (1.8) can be dualized into a covariantly chiral superfield $\Phi$, $\bar{\nabla}_\alpha \Phi = 0$, and its conjugate $\bar{\Phi}$ [7]. Then the above action turns into

$$S = \frac{1}{2} \int d^4x d^4\theta \frac{E}{\mu} W^\alpha W_\alpha + \frac{1}{2} \int d^4x d^4\theta \bar{E} (\Phi + \bar{\Phi})^2 .$$

(1.11)

On the other hand, the linear VT multiplet should be dual to a free $\mathcal{N} = 2$ vector multiplet. The latter is described in $\mathcal{N} = 1$ AdS superspace by the action (see, e.g., [30])

$$S_{vector} = \frac{1}{2} \int d^4x d^4\theta \frac{E}{\mu} W^\alpha W_\alpha + \int d^4x d^4\theta \bar{E} \Phi \Phi .$$

(1.12)

By assumption, the dynamical systems (1.11) and (1.12) should be equivalent to each other. However the chiral sectors of (1.11) and (1.12) are different. This means that a
free linear VT multiplet does not exist in AdS.  

The above example provides enough rationale for studying the VT multiplet and its couplings in AdS. In $\mathcal{N} = 2$ supergravity, on the other hand, the VT multiplets and their couplings have been studied only in the component approach [11]. Developing a superspace formulation appears to be highly desirable.

This paper is organized as follows. In section 2 we describe a general setting for $\mathcal{N} = 2$ AdS supersymmetric theories with central charge and formulate a supersymmetric action principle in $\mathcal{N} = 2$ AdS superspace. In section 3 we derive consistent superfield constraints and a Lagrangian for the VT multiplet in $\mathcal{N} = 2$ AdS superspace, and then generalize them to include couplings to vector multiplets. The results of section 3 are then reformulated in $\mathcal{N} = 1$ AdS superspace in section 4. Section 5 is devoted to component results. Extension of our AdS constructions to supergravity is given in section 6. We sketch some interesting generalizations of our results and discuss open problems in section 7. The main body of the paper is accompanied by four technical appendices. Appendix A contains salient facts about $\mathcal{N} = 1$ AdS superspace. Appendix B is devoted to $\mathcal{N} = 2$ Killing vector fields. Some aspects of $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ AdS superspace reduction are discussed in Appendix C. Finally Appendix D contains a summary of the superspace formulation for $\mathcal{N} = 2$ conformal supergravity. Our notation and two-component spinor conventions follow [31].

2 \quad \mathbf{N} = 2 \text{ AdS supersymmetry and central charge}

The four-dimensional $\mathcal{N} = 2$ AdS superspace

$$\text{AdS}^{4|8} := \frac{\text{OSp}(2|4)}{\text{SO}(3,1) \times \text{SO}(2)}$$

can be realized as a maximally symmetric geometry that originates within the superspace formulation of $\mathcal{N} = 2$ conformal supergravity developed in [32]. Assuming the superspace is parametrized by local bosonic ($x$) and fermionic ($\theta, \bar{\theta}$) coordinates $z^M = (x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}})$ (where $m = 0, 1, \cdots, 3$, $\mu = 1, 2$, $\dot{\mu} = 1, 2$, and $\nu = 1, 2$), the corresponding covariant

In supergravity, it was shown in [11] that the linear VT multiplet can be consistently defined in the presence of an Abelian vector multiplet in addition to the central charge vector multiplet. In the rigid supersymmetric case, it was demonstrated in [10, 18] that in the case of the linear VT multiplet with gauged central charge one also needs at least two vector multiples (one of which is associated with the central charge) for ensuring the rigid scale and chiral symmetries of the action.
derivatives
\[ \mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}^i_a, \bar{\mathcal{D}}^\dot{a}) = E_A^M \partial_M + \frac{1}{2} \Omega_A^{bc} M_{bc} + \Phi_A S^{ij} J_{ij}, \quad i, j = 1, 2 \] (2.1)

obey the algebra \[30, 32\]
\[ \{\mathcal{D}^i_a, \mathcal{D}^j_\beta\} = 4 S^{ij} M_{a\beta} + 2 \varepsilon_{a\beta} \varepsilon^{ij} S^{kl} J_{kl}, \quad \{\mathcal{D}^i_a, \bar{\mathcal{D}}^j_\dot{\beta}\} = -2i \delta^i_j \mathcal{D}_a \dot{\beta}, \quad (2.2a) \]
\[ [\mathcal{D}_a, \mathcal{D}^j_\beta] = \frac{1}{2} (\sigma_a)_{\beta\gamma} S^{jk} \mathcal{D}^\gamma_k, \quad [\mathcal{D}_a, \mathcal{D}_b] = -S^2 M_{ab}, \quad (2.2b) \]

where \( S^{ij} \) is a covariantly constant and constant real isotriplet, \( S^{ij} = S^{ji}, \overline{S}^{ij} = S_{ij} = \varepsilon_{ik} \varepsilon_{jl} S^{kl}, \) and \( S^2 := \frac{1}{2} S^{ij} S_{ij} \). The SU(2) generators, \( J_{kl} \), act on the spinor covariant derivatives by the rule:
\[ [J_{kl}, \mathcal{D}^i_a] = -\frac{1}{2} (\delta^i_k \mathcal{D}_a l + \delta^i_l \mathcal{D}_a k). \] (2.3)

This superspace proves to be a conformally flat solution to the equations of motion for \( \mathcal{N} = 2 \) supergravity with a cosmological term [23].

Our goal in this section is to develop a general setting to formulate \( \mathcal{N} = 2 \) rigid supersymmetric theories with an off-shell central charge in the \( \mathcal{N} = 2 \) AdS superspace introduced. As compared with the super-Poincaré case, eq. (1.4), there appears to be a subtlety: the algebra of the AdS covariant derivatives (2.2) cannot be deformed to include a central charge. First of all, we address this issue by considering the AdS extension [28] of the Fayet-Sohnius hypermultiplet [2].

For further considerations it is useful to introduce the U(1) generator
\[ \mathcal{J} = S^{kl} J_{kl}, \] (2.4)
which appears in (2.1) and acts on the spinor covariant derivatives as follows
\[ [\mathcal{J}, \mathcal{D}^i_a] = S^{ij} \mathcal{D}^j_a, \quad [\mathcal{J}, \bar{\mathcal{D}}^\dot{a}_i] = -S^{ij} \bar{\mathcal{D}}^\dot{\beta}_j. \] (2.5)

As noted in [30], one can always choose
\[ S^{12} = 0 \] (2.6)
by applying a rigid SU(2) rotation. This choice is very useful for reduction to \( \mathcal{N} = 1 \) AdS superspace and is assumed in what follows. We denote the other components of \( S^{ij} \) as
\[ S^{11} = S_{22} = -\bar{\mu}, \quad S^{22} = S_{11} = -\mu. \] (2.7)
2.1 Fayet-Sohnius hypermultiplet

Our presentation in this subsection follows [28]. The Fayet-Sohnius hypermultiplet in AdS is described by a two-component superfield \( q_i \) and its conjugate \( \bar{q}^j := \bar{q}_i \) subject to the constraints

\[
\mathcal{D}^{(i} q^{j)} = \bar{\mathcal{D}}^{(i} q^{j)} = 0 .
\] (2.8)

Note that we do not assume a given action of \( \mathcal{J} \) on \( q_i \). Instead we rely on the constraints to determine its action. It follows from (2.2) that we may write

\[
\mathcal{J} = \frac{1}{4} \{ \mathcal{D}_{a\bar{a}}, \bar{\mathcal{D}}_{\bar{a}}^{\dot{a}} \} .
\] (2.9)

Using the constraints together with the covariant derivative algebra, one can show

\[
\begin{align*}
\mathcal{J} q_\perp &= -\frac{1}{4}(\mathcal{D}_\perp)^2 q_2 , & \mathcal{J} \bar{q}_\perp &= -\frac{1}{4}(\bar{\mathcal{D}}_\perp)^2 \bar{q}_2 , \\
\mathcal{J} q_\parallel &= \frac{1}{4}(\mathcal{D}_\parallel)^2 q_1 , & \mathcal{J} \bar{q}_\parallel &= \frac{1}{4}(\bar{\mathcal{D}}_\parallel)^2 \bar{q}_1 .
\end{align*}
\] (2.10a, b)

Motivated by the fact that \( q_\perp \) and \( \bar{q}_\perp \) are \( N = 1 \) chiral superfields,

\[
\begin{align*}
\mathcal{D}_\perp^{\dot{a}} q_\perp &= 0 , & \mathcal{D}_\perp^{\dot{a}} \bar{q}_\perp &= 0 ,
\end{align*}
\] (2.11)

we can rewrite (2.10a) as

\[
\begin{align*}
\mathcal{J} q_\perp + \mu q_\perp &= -\frac{1}{4}[(\mathcal{D}_\perp)^2 - 4\mu] q_2 , & \mathcal{J} \bar{q}_\perp + \mu \bar{q}_\perp &= -\frac{1}{4}[(\bar{\mathcal{D}}_\perp)^2 - 4\mu] \bar{q}_2 .
\end{align*}
\] (2.12)

Then introducing \( \mathcal{J} \), the U(1) operator transforming \( q_i \) as an isospinor,

\[
\begin{align*}
\mathcal{J} q_i := -S_{ij} q_j , & \quad \mathcal{J} \bar{q}_i := -S_{ij} \bar{q}_j ,
\end{align*}
\] (2.13)

we can write

\[
\begin{align*}
\Delta q_\perp &= -\frac{1}{4}[(\mathcal{D}_\perp)^2 - 4\mu] q_2 , & \Delta \bar{q}_\perp &= -\frac{1}{4}[(\bar{\mathcal{D}}_\perp)^2 - 4\mu] \bar{q}_2 .
\end{align*}
\] (2.14)

where we have introduced

\[
\Delta = \mathcal{J} - \mathcal{J} .
\] (2.15)

\(^3\)Isospinor indices are raised and lowered using antisymmetric tensors \( \varepsilon^{ij} \) and \( \varepsilon_{ij} \) normalized by \( \varepsilon^{12} = \varepsilon_{21} = 1 \). The rules are: \( q^i = \varepsilon^{ij} q_j \) and \( q_i = \varepsilon_{ij} q^j \). The conjugation property \( \overline{q_i} = q^i \) implies \( \overline{q^i} = -\bar{q}_i \).
Similarly we find
\[ \Delta q_2 = \frac{1}{4}[(D^1)^2 - 4\mu] q_1, \quad \Delta \bar{q}_2 = \frac{1}{4}[(\bar{D}^1)^2 - 4\bar{\mu}] \bar{q}_1. \] (2.16)

Here \( \Delta \) takes on the role of a central charge as it commutes with the covariant derivatives,
\[ [\Delta, D_a^i] = [\Delta, \bar{D}_{\dot{a}i}] = 0. \] (2.17)

Thus the constraints (2.8) allow us to specify the action of the generator \( J \) on the hypermultiplet as well as to separate a central charge, \( \Delta \). We will use this procedure in the next subsection and for the VT multiplet in AdS superspace.

### 2.2 Linear multiplet

Since there exist interesting \( \mathcal{N} = 2 \) AdS supermultiplets with central charge, we have to construct a supersymmetric action principle to describe their dynamics. This can be achieved by generalizing the famous construction due to Sohnius [2]. The idea is to make a linear multiplet in AdS\(^4/8\) take on the role of a superfield Lagrangian.

Following [2, 33], the linear multiplet is a real isotriplet superfield, \( L^{ij} = L^{ji} \) and \( \bar{L}^{ij} = L_{ij} \), subject to the constraints
\[ D_a^{(ij} L^{jk)} = \bar{D}_{\dot{a}}^{(i} L^{j)k} = 0. \] (2.18)

We define \( L^{ij} \) to transform under \( \text{OSp}(2|4) \), the isometry group of the \( \mathcal{N} = 2 \) AdS superspace, by the rule
\[ \delta L^{ij} = -\xi L^{ij} - 2\varepsilon J L^{ij}, \] (2.19)
where the first-order operator \( \xi \) and parameter \( \varepsilon \) are given by eqs. (B.1) and (B.3) respectively. The \( \text{U}(1) \) generator \( J = \mathbb{J} + \Delta \) acts on the linear multiplet as
\[ JL^{ij} = S_{i}^{k} L^{kj} + S_{j}^{k} L^{ki} + \Delta L^{ij}. \] (2.20)

In general, the linear multiplet has a non-zero central charge, \( \Delta L^{ij} \neq 0 \). Indeed, the constraints (2.18) imply that
\[ JL_{11} = -\frac{1}{2}(\bar{D}_{\dot{1}1})^2 L_{12}, \quad JL_{22} = \frac{1}{2}(D^1)^2 L_{12}. \] (2.21)

\( \Delta L^{ij} = 0 \) corresponds to the \( \mathcal{N} = 2 \) tensor multiplet.

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4The case \( \Delta L^{ij} = 0 \) corresponds to the \( \mathcal{N} = 2 \) tensor multiplet.
These results imply the action of the central charge on $L_1$ and $L_2$:

$$\Delta L_1 = -\frac{1}{2} [(D_1^2 - 4\mu) L_{12}] , \quad \Delta L_2 = \frac{1}{2} [(D_1^2 - 4\bar{\mu}) L_{12}] . \quad (2.22)$$

In accordance with (2.18), the superfield $L_1$ is $\mathcal{N} = 1$ chiral,

$$\bar{D}^i_1 L_1 = 0 . \quad (2.23)$$

The first equation in (2.22) shows that $\Delta L_1$ is also $\mathcal{N} = 1$ chiral, $\bar{D}^i_1 \Delta L_1 = 0$.

### 2.3 Supersymmetric action principle

When dealing with $\mathcal{N} = 2$ supersymmetric actions, it is convenient to use two types of reduction with respect to the Grassmann variables: (i) reduction to $\mathcal{N} = 1$ superspace; and (ii) complete reduction. Given an $\mathcal{N} = 2$ superfield $U(x, \theta, \bar{\theta})$, we define its $\mathcal{N} = 1$ projection as

$$U| := U(x, \theta, \bar{\theta})|_{\theta_2 = \bar{\theta}_2 = 0} , \quad (2.24)$$

while its component projection is defined by

$$U|| := U(x, \theta, \bar{\theta})|_{\theta_1 = \bar{\theta} = 0} . \quad (2.25)$$

Associated with the linear multiplet, $L_{ij}$, is the following functional

$$S = -\frac{1}{12} \int d^4 x \, e \left( D^{ij} + \bar{D}^{ij} + 36 S^{ij} \right) L_{ij} || , \quad e^{-1} = \det (e^a_m) , \quad (2.26)$$

where we have denoted $D^{ij} := D^{a(i} D_a^{j)}$ and $\bar{D}^{ij} := \bar{D}^{\dot{a}(i} \bar{D}_{\dot{a}}^{j)}$. It is assumed that $S$ is evaluated in a Wess-Zumino gauge of the form

$$D_a || = e_a + \omega_a , \quad e_a = e_a^m(x) \partial_m , \quad \omega_a = \frac{1}{2} \omega_{a}^{bc}(x) M_{bc} , \quad (2.27)$$

with no U(1) connection being present in $D_a ||$. The crucial property of the functional (2.26) is that it turns out to be invariant under arbitrary AdS isometry transformations (2.19). This means that (2.26) can be used as supersymmetric action for theories in $\mathcal{N} = 2$ AdS superspace.

To show that the action (2.26) is indeed $\mathcal{N} = 2$ supersymmetric, we first reformulate it in $\mathcal{N} = 1$ AdS superspace where one supersymmetry is manifestly realized. Making use of the constraints (2.18), we find

$$-\frac{1}{12} (D^{ij} + \bar{D}^{ij} + 36 S^{ij}) L_{ij} = -\frac{1}{4} [(D_1^2 - 12\bar{\mu}) L_{11} - \frac{1}{4} \left( (D_1^2 - 12 \bar{\mu}) L_{22} . \quad (2.28)$$
Projecting this to $\mathcal{N} = 1$ AdS superspace gives

$$-\frac{1}{12}(D^{ij} + \bar{D}^{ij} + 36S^{ij})|\mathcal{L}_{ij}| = -\frac{1}{4}(\nabla^2 - 12\bar{\mu})|\mathcal{L}_{11}| - \frac{1}{4}(\nabla^2 - 12\mu)|\mathcal{L}_{22}|. \quad (2.29)$$

The $\mathcal{N} = 1$ AdS integration rule (see, e.g., [35])

$$\int d^4x d^4\theta \frac{E}{\mu} \mathcal{L}_{\text{chiral}} = -\frac{1}{4} \int d^4x \epsilon (\nabla^2 - 12\bar{\mu})|\mathcal{L}_{\text{chiral}}|_{\theta = \bar{\theta} = 0}, \quad \bar{\nabla}_\alpha \mathcal{L}_{\text{chiral}} = 0 \quad (2.30)$$

can then be used to rewrite the action. One obtains

$$S = \int d^4x d^4\theta \frac{E}{\mu} |\mathcal{L}_{11}| + \int d^4x d^4\theta \frac{E}{\bar{\mu}} |\mathcal{L}_{22}|. \quad (2.31)$$

In accordance with (2.22), this functional is invariant under central charge transformations

$$\delta \mathcal{L}^{ij} = \zeta \Delta \mathcal{L}^{ij}, \quad \zeta = \text{const}. \quad (2.32)$$

Now we are finally prepared to prove invariance of the action (2.26) under arbitrary $\mathcal{N} = 2$ isometry transformations, eq. (2.19). As shown in [30] and reviewed in [28] and Appendix B any $\mathcal{N} = 2$ isometry transformation induces two different transformations in $\mathcal{N} = 1$ AdS superspace which are: (i) an isometry transformation of $\mathcal{N} = 1$ AdS superspace which is generated by a Killing vector superfield $\xi = \xi^a \nabla_a + \xi^\alpha \nabla_\alpha + \bar{\xi}_\dot{\alpha} \bar{\nabla}^\dot{\alpha}$; and (ii) an extended supersymmetry transformation described by a real superfield parameter $\epsilon$ constrained as in eq. (C.9). The $\mathcal{N} = 2$ transformation law (2.19) implies that $\mathcal{L}_{11}$ transforms as a scalar superfield under the $\mathcal{N} = 1$ AdS supergroup OSp(1|4),

$$\delta \xi |\mathcal{L}_{11}| = -\xi |\mathcal{L}_{11}|. \quad (2.33)$$

The action (2.31) is manifestly invariant under these transformations.

It remains to be shown that (2.31) is also invariant under the second supersymmetry. The second supersymmetry transformation of $|\mathcal{L}_{11}|$ (see Appendix B) is

$$\delta \epsilon |\mathcal{L}_{11}| = -\epsilon^\alpha \mathcal{D}_{\bar{\alpha}}^2 |\mathcal{L}_{11}| - \bar{\epsilon}_{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}}^2 |\mathcal{L}_{11}| - 2\epsilon \mathcal{J} |\mathcal{L}_{11}|. \quad (2.34)$$

Using the constraints on $\mathcal{L}_{ij}$ and (C.9a) we find

$$\delta \epsilon |\mathcal{L}_{11}| = (\nabla^2 - 4\mu)(\varepsilon \mathcal{L}_{12})|, \quad (2.35a)$$

and similarly

$$\delta \epsilon |\mathcal{L}_{22}| = -(\nabla^2 - 4\bar{\mu})(\varepsilon \mathcal{L}_{12})|. \quad (2.35b)$$

These results imply that the action is supersymmetric as required.
2.4 Hypermultiplet models

To illustrate the supersymmetric action principle, here we present examples involving the Fayet-Sohnius hypermultiplet. Given a linear multiplet, the corresponding action, (2.31) can be read directly from $L|_{11}$. For the examples that we shall consider it is necessary to first define the $N=1$ projection of the Fayet-Sohnius hypermultiplet.

It follows from the constraints (2.8) that

$$\Phi^+ := q_1| \quad , \quad \Phi^- = \bar{q}_1|$$

are covariantly chiral $N=1$ superfields,

$$\bar{\nabla}_\alpha \Phi^+ = 0 \quad , \quad \bar{\nabla}_\alpha \Phi^- = 0 .$$

Furthermore performing the $N = 1$ projection of (2.14) determines the action of the central charge on $\Phi^+$ and $\Phi^-$

$$\Delta \Phi^+ = -\frac{1}{4}(\bar{\nabla}^2 - 4\mu)\bar{\Phi}^- \quad , \quad \Delta \Phi^- = \frac{1}{4}(\nabla^2 - 4\bar{\mu})\Phi^+ .$$

These results imply that

$$(\Delta^2 + \Box_C)\Phi^\pm = 0 \quad , \quad \Box_C := \frac{1}{16}(\bar{\nabla}^2 - 4\mu)(\nabla^2 - 4\bar{\mu}) ,$$

with $\Box_C$ being the covariantly chiral d’Alembertian. It is seen that the hypermultiplet becomes on-shell if $\Delta$ is set to be a constant matrix.

A linear multiplet may be constructed using the hypermultiplet in a number of ways. Firstly, we consider the linear multiplets

$$(L_{\text{kin}})_{ij} = \frac{1}{2}(\bar{q}_i \Delta q_j - q_i \Delta \bar{q}_j) , \quad (2.40)$$

$$(L_{\text{der}})_{ij} = \frac{i}{2}(\bar{q}_i \Delta q_j + q_i \Delta \bar{q}_j) . \quad (2.41)$$

However only the former leads to a kinetic term while the latter leads to a total derivative in components. A straightforward evaluation of $(L_{\text{kin}})_{11}$ gives

$$\left|L_{\text{kin}}\right|_{11} = \frac{1}{2}(\Phi^- \Delta \Phi^+ - \Phi^+ \Delta \Phi^-) = -\frac{1}{8}(\bar{\nabla}^2 - 4\mu)(\Phi^+ \tilde{\Phi}^- + \Phi^- \tilde{\Phi}^+). \quad (2.42)$$

The corresponding action then reads

$$S_{\text{kin}} = \int d^4x d^4\theta \ E(\Phi^+ \tilde{\Phi}^- + \Phi^- \tilde{\Phi}^+). \quad (2.43)$$
Another possible linear multiplet that we can consider is a bilinear in $q_i$ and $\bar{q}_i$ of the form

$$ (\mathcal{L}_{\text{mass}})_{ij} = im\bar{q}_i q_j , \quad \bar{m} = m = \text{const} , \quad (2.44) $$

with corresponding action

$$ S_{\text{mass}} = im \int d^4x d^4\theta \frac{E}{\mu} \Phi_+ \Phi_- + c.c. \quad (2.45) $$

The specific feature of the actions generated by (2.40) and (2.44) is invariance under $U(1)$ transformations

$$ q_i \rightarrow e^{i\varphi} q_i , \quad \varphi \in \mathbb{R} . \quad (2.46) $$

This symmetry defines a charged hypermultiplet (when coupled to a Yang-Mills supermultiplet, such a hypermultiplet can transform in an arbitrary representation of the gauge group). Without demanding this symmetry it is possible to construct an additional linear multiplet

$$ (\tilde{\mathcal{L}}_{\text{mass}})_{ij} = \frac{1}{2} \mathcal{M} q_i q_j + \frac{1}{2} \bar{\mathcal{M}} \bar{q}_i \bar{q}_j , \quad (2.47) $$

with $\mathcal{M}$ a complex mass parameter. The corresponding action is found to be

$$ \tilde{S}_{\text{mass}} = \frac{1}{2} \int d^4x d^4\theta \frac{E}{\mu} (\mathcal{M} \Phi_+ \Phi_+ + \bar{\mathcal{M}} \Phi_- \Phi_-) + c.c. \quad (2.48) $$

3 Vector-tensor multiplet

In the case of Poincaré supersymmetry, $\mathcal{N} = 2$ superfield techniques proved fruitful to obtain consistent formulations for the linear and nonlinear VT multiplets and their Chern-Simons couplings to $\mathcal{N} = 2$ vector multiplets [13]–[18]. This section examines the possibility of formulating a VT multiplet in $\mathcal{N} = 2$ AdS superspace.

Here we derive consistent superfield constraints describing the nonlinear VT multiplet in AdS. As proved in Section 1 with the use of $\mathcal{N} = 1$ superfield techniques, a linear VT multiplet does not exist in AdS. We present an alternative and more direct proof of this result using $\mathcal{N} = 2$ superfields.
3.1 Consistent constraints

Given some real superfield $\mathbb{L}$, we make a general ansatz for the constraints

$$
\mathcal{D}^{ij} \mathbb{L} = S^{ij} f(\mathbb{L}) + a \mathcal{D}^i \mathbb{L} \mathcal{D}^j \mathbb{L} + b \mathcal{D}_{\alpha} \mathbb{L} \mathcal{D}^{ij} \mathbb{L} \mathcal{D}_{\alpha} \mathbb{L} , \tag{3.1a}
$$

$$
\mathcal{D}^{(i} \mathcal{D}^{j)} \mathbb{L} = c \mathcal{D}^i \mathbb{L} \mathcal{D}^{(i} \mathcal{D}^{j)} \mathbb{L} , \tag{3.1b}
$$

for some function $f(\mathbb{L})$ and parameters $a$, $b$ and $c$. The parameters are then fixed by consistency. There are two basic consistency requirements:

(i) $\mathcal{D}^{(i} \mathcal{D}^{jk)} \mathbb{L} = 0$;

(ii) expressions for $\mathcal{D}^{(i} \mathcal{D}^{jk)} \mathbb{L}$ derived using (3.1a) and (3.1b) respectively coincide.

It is worth noting that as a consequence of the consistency conditions we cannot include additional terms in the ansatz without also including terms with two covariant derivatives of $\mathbb{L}$. Imposing the consistency requirements yields two solutions:

$$
\mathcal{D}^{ij} \mathbb{L} = \frac{4}{\lambda} S^{ij} + \lambda \mathcal{D}^i \mathbb{L} \mathcal{D}^j \mathbb{L} , \tag{3.2a}
$$

$$
\mathcal{D}^{(i} \mathcal{D}^{j)} \mathbb{L} = 0 \tag{3.2b}
$$

and

$$
\mathcal{D}^{ij} \mathbb{L} = \frac{2}{\kappa} S^{ij} + 2 \kappa \mathcal{D}^i \mathbb{L} \mathcal{D}^j \mathbb{L} - \kappa \mathcal{D}^i \mathbb{L} \mathcal{D}^j \mathbb{L} , \tag{3.3a}
$$

$$
\mathcal{D}^{(i} \mathcal{D}^{j)} \mathbb{L} = \kappa \mathcal{D}^i \mathbb{L} \mathcal{D}^{(i} \mathcal{D}^{j)} \mathbb{L} , \tag{3.3b}
$$

where $\lambda$ is arbitrary and $\kappa$ is real. These solutions provide generalizations of the solutions found in [16, 17] and in an analogous way we can reject one of the solutions based on an additional consistency requirement, originating from the component structure of the multiplet. The superspace constraints give rise to differential constraints at the component level. We require that these component constraints can be solved for a gauge one-form and a gauge two-form. It turns out that only the second solution satisfies this deeper requirement and its component structure will be discussed in section 5. Furthermore, it is impossible to make the parameter $\kappa$ vanish and thus there exists no direct generalization of the linear VT multiplet in AdS, as expected.

It is useful to introduce a different superfield parameterization (compare with [17, 18])

$$
L = \exp (-\kappa \mathbb{L}) , \tag{3.4}
$$

13
which takes the solution to the simpler form

\[
\mathcal{D}^{ij} L = -2S^{ij} L - \frac{1}{L} \mathcal{D}^i L \mathcal{D}^j L + \frac{1}{L} \mathcal{D}^i L \mathcal{D}^j L ,
\]

(3.5a)

\[
\mathcal{D}^{(i} \mathcal{D}^{j)} L = 0 .
\]

(3.5b)

We shall use this parameterization in the rest of the paper.

### 3.2 Superfield Lagrangian

It remains to find the corresponding linear multiplet which takes on the role of a Lagrangian density for the VT multiplet. To find a Lagrangian density for the VT multiplet corresponding to the constraints (3.5), we try a general ansatz

\[
\mathcal{L}^{ij} = B(L) S^{ij} + A(L) \mathcal{D}^i L \mathcal{D}^j L + \tilde{A}(L) \mathcal{D}^i L \mathcal{D}^j L ,
\]

(3.6)

where \( B(L) \) is an arbitrary real function and \( A(L) \) is arbitrary. Imposing the constraints for a linear multiplet

\[
\mathcal{D}^{(i} \mathcal{L}^{jk)} = \mathcal{D}^{(i} \mathcal{L}^{jk)} = 0
\]

(3.7)

and using the constraints on \( L \) leads to the conditions

\[
B'(L) = -2A(L) L , \quad A'(L) = \frac{A(L)}{L} , \quad \tilde{A} = A .
\]

(3.8)

These are solved by \( A(L) = kL \) and \( B(L) = -\frac{2k}{3} L^3 \), for some real \( k \). Thus adopting a normalization for the Lagrangian density gives

\[
\mathcal{L}^{ij} = \frac{1}{4} L (\mathcal{D}^i L \mathcal{D}^j L + \mathcal{D}^i L \mathcal{D}^j L - \frac{2}{3} L^2 S^{ij})
\]

\[
= \frac{1}{12} (\mathcal{D}^{ij} + 4S^{ij}) L^3 = \frac{1}{12} (\mathcal{D}^{ij} + 4S^{ij}) L^3 ,
\]

(3.9)

which generalizes the result in [16, 17].

### 3.3 Coupling to vector multiplets

As an extension of the results in the previous two sections, here we consider couplings of the VT multiplet with \( \mathcal{N} = 2 \) super Yang-Mills fields.

The \( \mathcal{N} = 2 \) super Yang-Mills multiplet in AdS superspace is described by a chiral field, \( \mathcal{W} \) obeying the constraints

\[
\mathcal{D}_{\alpha i} \mathcal{W} = 0 ,
\]

(3.10)
We can choose the following consistent constraints in superspace, describing coupling of the VT multiplet to the Yang-Mills multiplet

\[
\mathcal{D}^{ij} L = -2 S^{ij} L - \frac{1}{L} \mathcal{D}^i L \mathcal{D}^j L + \frac{1}{L} \bar{\mathcal{D}}^i L \bar{\mathcal{D}}^j L \\
+ \frac{g}{2L} \text{tr}((\mathcal{D}^{ij} + 4S^{ij}) \mathcal{F}(\mathcal{W}) - (\bar{\mathcal{D}}^{ij} + 4S^{ij}) \bar{\mathcal{F}}(\bar{\mathcal{W}})),
\]

(3.11a)

\[
\mathcal{D}_\alpha^{(i} \bar{\mathcal{D}}^{j)} L = 0,
\]

(3.11b)

with \(g\) a real coupling constant and \(\mathcal{F}(\mathcal{W})\) some holomorphic function. However it turns out that the corresponding components do not allow for an appropriate gauge two-form to be defined in general and we must choose the simplest nontrivial case

\[
\mathcal{F}(\mathcal{W}) = \mathcal{W}^2,
\]

(3.12)

which generates Chern-Simons terms at the component level (see section 5).

A corresponding Lagrangian density can then be constructed

\[
\mathcal{L}^{ij} = \frac{1}{4} L(\mathcal{D}^i L \mathcal{D}^j L + \bar{\mathcal{D}}^i L \bar{\mathcal{D}}^j L - \frac{2}{3} L^2 S^{ij}) + \frac{g}{8} \text{tr} \left( L(\mathcal{D}^{ij} + 4S^{ij}) \mathcal{W}^2 + L(\bar{\mathcal{D}}^{ij} + 4S^{ij}) \bar{\mathcal{W}}^2 \\
- 2(\mathcal{D}^{ij} + 4S^{ij})(L\mathcal{W}^2) - 2(\bar{\mathcal{D}}^{ij} + 4S^{ij})(L\bar{\mathcal{W}}^2) \right)
\]

\[
= \frac{1}{12}(\mathcal{D}^{ij} + 4S^{ij})(L^3 - 3gL\text{tr}(\mathcal{W}^2)) \\
- \frac{g}{4} \text{tr} \left( (\bar{\mathcal{D}}^{ij} + 4S^{ij})(L\bar{\mathcal{W}}^2) - L(\bar{\mathcal{D}}^{ij} + 4S^{ij})\bar{\mathcal{W}}^2 \right),
\]

(3.13)

where reality of the last line follows from the constraints. This generalizes (3.9) to the case of Chern-Simons couplings.

4 Formulation in \(N = 1\) AdS superspace

Having derived the \(\mathcal{N} = 2\) constraints and Lagrangian density for the \(\mathcal{N} = 2\) VT multiplet in AdS superspace, it is natural to consider its formulation in terms of \(\mathcal{N} = 1\) superfields. This is especially apparent from the simplicity of the resulting action in \(\mathcal{N} = 1\) superspace. In this section we introduce \(\mathcal{N} = 1\) superfields in \(\mathcal{N} = 1\) AdS superspace describing the VT multiplet and its Chern-Simons coupling. We analyze the constraints obeyed by these superfields and formulate the corresponding action principle. In particular, we demonstrate that the action principle of the VT multiplet possesses a rather simple structure in terms of cubic interactions.
4.1 $\mathcal{N} = 1$ constraints

We begin by reformulating the constraints in terms of $\mathcal{N} = 1$ superfields. Starting with the $\mathcal{N} = 2$ super Yang-Mills multiplet we define the two independent $\mathcal{N} = 1$ projections (see Appendix C) as follows

$$\varphi := \mathcal{W}|, \quad \mathbb{W}_\alpha := \frac{i}{2}\bar{D}_2^2\mathcal{W}|.$$ (4.1)

A straightforward projection of the $\mathcal{N} = 2$ constraints of $\mathcal{W}$ give the following $\mathcal{N} = 1$ constraints

$$\bar{\nabla}_\dot{\alpha}\mathbb{W}_\alpha = 0,$$ (4.2a)

$$\bar{\nabla}_\dot{\alpha}\varphi = 0,$$ (4.2b)

$$\nabla^\alpha\mathbb{W}_\alpha = \bar{\nabla}_\dot{\alpha}\bar{\mathbb{W}}^\dot{\alpha}.$$ (4.2c)

Thus $\varphi$ is a chiral superfield and $\mathbb{W}_\alpha$ describes a vector multiplet in $\mathcal{N} = 1$ AdS superspace.

We are interested in the structure that the non-linear constraint s on the VT multiplet possess. Firstly we note that the $\mathcal{N} = 2$ VT superfield contains two independent $\mathcal{N} = 1$ projections defined as follows

$$G := L|, \quad W_\alpha := \frac{1}{2}\bar{D}_2^2L|.$$ (4.3)

The constraints for these $\mathcal{N} = 1$ superfields follow from those on $L$, which are equivalent to

$$(D_2^2) L = -2S^2 - \frac{1}{L}D_2^2L \bar{D}_2^2L + \frac{1}{L}\bar{D}_2^2L \bar{D}_2^2L$$

$$+ \frac{g}{2L} \text{tr}((D_2^2)^2 + 4S^2)\mathcal{W}^2) - \frac{g}{2L} \text{tr}((\bar{D}_2^2)^2 + 4S^2)\bar{\mathcal{W}}^2) ,$$ (4.4a)

$$(D_2^2) L = -2S^2 - \frac{1}{L}D_2^2L \bar{D}_2^2L + \frac{1}{L}\bar{D}_2^2L \bar{D}_2^2L$$

$$+ \frac{g}{2L} \text{tr}((D_2^2)^2 + 4S^2)\mathcal{W}^2) - \frac{g}{2L} \text{tr}((\bar{D}_2^2)^2 + 4S^2)\bar{\mathcal{W}}^2) ,$$ (4.4b)

$$D_2^2D_2^2 = -\frac{1}{L}D_2^2L \bar{D}_2^2L + \frac{1}{L}\bar{D}_2^2L \bar{D}_2^2L + \frac{9}{2L} \text{tr}(D_2^2\mathcal{W}^2) - \frac{9}{2L} \text{tr}(\bar{D}_2^2\bar{\mathcal{W}}^2) ,$$ (4.4c)

$$D_2^1\bar{D}_2^1 = 0 ,$$ (4.4d)

$$D_2^2\bar{D}_2^2 = 0 ,$$ (4.4e)

$$D_2^1\bar{D}_2^1 = -D_2^2\bar{D}_2^2L .$$ (4.4f)

From (4.4b) and (4.4f) we can see that $\varphi$, $\mathbb{W}_\alpha$, $G$ and $W_\alpha$ form a basis for independent $\mathcal{N} = 1$ projections. Namely, $\mathcal{N} = 1$ projections formed out of two covariant derivatives of $L$ can always be written in terms of the $\mathcal{N} = 1$ superfields defined.
Analyzing the projection of the constraints (4.4) lead to a number of $\mathcal{N} = 1$ conditions. (4.4a) and the constraint on $W$ gives a non-linear constraint for $G$

$$(\nabla^2 - 4\mu)G^2 = 8\bar{W}^2 + g\text{tr}((\nabla^2 - 4\mu)(\varphi - \bar{\varphi})^2 + 8\bar{\varphi}^2) .$$

(4.5)

Then using (4.4c) and its conjugate gives

$$\nabla^\alpha W_\alpha = \bar{\nabla}^\dot{\alpha} \bar{W}^{\dot{\alpha}} .$$

(4.6)

Finally, (4.4d) and (4.4e) imply chirality of $W_\alpha$

$$\nabla_\dot{\alpha} W_\alpha = 0 .$$

(4.7)

Thus we have the $\mathcal{N} = 1$ constraints for the VT multiplet with Chern-Simons terms

$$G^2\Delta G = \frac{1}{2} \nabla^\alpha (G^2 W_\alpha) + ig G \nabla^\alpha (\varphi \bar{W}_\alpha) + \text{c.c.} ,$$

(4.8a)

$$\Delta W_\alpha = \frac{1}{8}(\nabla^2 - 4\mu) \nabla_\alpha G ,$$

(4.8b)

(4.8c)

The constraints (4.4) also imply central charge transformations of the $\mathcal{N} = 1$ superfields. Using (4.4d) and (4.4f) one derives

$$G^2 \delta_\varepsilon G = -G^2 \nabla^\alpha \varepsilon W_\alpha - \nabla^\alpha (\varepsilon G^2 W_\alpha) - 2ig \varepsilon G \nabla^\alpha (\varphi \bar{W}_\alpha) + \text{c.c.} ,$$

(4.10a)

$$\delta_\varepsilon W_\alpha = -\frac{1}{4}(\nabla^2 - 4\mu) \nabla_\alpha (\varepsilon G) ,$$

(4.10b)

$$\delta_\varepsilon \varphi = -2\varepsilon^\alpha W_\alpha ,$$

(4.10c)

$$\delta_\varepsilon \bar{W}_\alpha = -\frac{1}{4}(\nabla^2 - 4\mu) \nabla_\alpha (\varepsilon (\varphi - \bar{\varphi})) .$$

(4.10d)

### 4.2 Supersymmetric action

In section 2.3 we presented the supersymmetric action associated with the linear multiplet. In particular we noted that the action can be written in terms of $\mathcal{N} = 1$ projections.
Here we make use of that result to derive the action rule for the VT multiplet in terms of its \(N = 1\) superfields.

Taking the \(N = 1\) projection of \(\mathcal{L}_{11}\) (and making use of the \(N = 1\) constraints (4.8)) gives

\[
\mathcal{L}_{11} = \frac{1}{12} (\nabla^2 - 4\mu) \left( G^3 + 3gG \text{tr}(\varphi^2 - \bar{\varphi}^2) \right) + 4i \text{tr}(W^\alpha \bar{\mathcal{W}}_\alpha \varphi) .
\]  

(4.11)

Putting this result in our action principle leads to

\[
S = -\frac{1}{3} \int d^4x d^4\theta \left( G^3 + 3gG \text{tr}(\varphi^2 - \bar{\varphi}^2) \right) + 4i \int d^4x d^4\theta \frac{E}{\mu} \text{tr}(W^\alpha \bar{\mathcal{W}}_\alpha \varphi) + \text{c.c.}
\]

\[
= -\frac{1}{3} \int d^4x d^4\theta E G^3 + 4i g \int d^4x d^4\theta \frac{E}{\mu} \text{tr}(W^\alpha \bar{\mathcal{W}}_\alpha \varphi) + \text{c.c.}
\]

\[
= -\frac{2}{3} \int d^4x d^4\theta E G^3 + 4i g \int d^4x d^4\theta E \left\{ \frac{1}{\mu} \text{tr}(W \bar{\mathcal{W}} \varphi) - \frac{1}{\bar{\mu}} \text{tr}(\bar{W} \mathcal{W} \bar{\varphi}) \right\} ,
\]  

(4.12)

where we have lifted part of the action from an integral over a chiral subspace to full superspace. As a check, one can show using (4.9) and (4.10) that the action is invariant under both central charge and supersymmetry transformations.

Turning off the Chern-Simons coupling reduces the action to

\[
S = -\frac{2}{3} \int d^4x d^4\theta E G^3 .
\]  

(4.13)

Both actions (4.12) and (4.13) are cubic. The reason for this is that both theories are related to five-dimensional \(N = 1\) supersymmetric Chern-Simons theories [20].

5 Component results

The superspace consistency conditions for the constraints of the VT multiplet do not guarantee the existence of a gauge one-form and a gauge two-form in its formulation. In order to verify their existence we must analyze the component fields.

We define the component fields of the external \(N = 2\) Yang-Mills multiplet, \(\mathcal{W}\), as

\[
w = \mathcal{W} \parallel, \quad \Sigma^i = D^i_\alpha \mathcal{W} \parallel, \quad \bar{\Sigma}^{\dot{i}} = \bar{D}^{\dot{i}}_\dot{\alpha} \bar{\mathcal{W}} \parallel,
\]

\[
F_{\alpha\beta} = -\frac{1}{8} D_{\alpha\beta} \mathcal{W} \parallel, \quad \bar{F}_{\dot{\alpha}\dot{\beta}} = \bar{D}_{\dot{\alpha}\dot{\beta}} \bar{\mathcal{W}} \parallel,
\]

\[
X^{ij} = (D^{ij} + 4S^{ij}) \mathcal{W} \parallel ,
\]  

(5.1)
and those of the VT multiplet as

$$l = L, \quad \lambda_a = \mathcal{D}_a L, \quad \tilde{\lambda}_{\dot{a}} = \bar{\mathcal{D}}_{\dot{a}} L, \quad U = \Delta L,$$

$$V_{a\dot{a}} = -\frac{1}{4} [\mathcal{D}_a, \bar{\mathcal{D}}_{\dot{a}}] L = -\frac{1}{2} \mathcal{D}_a \bar{\mathcal{D}}_{\dot{a}} L,$$

$$G_{a\beta} = -\frac{i}{8} [\mathcal{D}_a, \mathcal{D}_{\beta i}] L = -\frac{1}{4} \mathcal{D}_{a\beta} L, \quad \bar{G}_{\dot{a}\dot{\beta}} = \bar{G}_{a\beta}.$$  \hspace{1cm} (5.2)

We note that the central charge transformations of $G_{ab}$ and $V_a$ are

$$\Delta G_{ab} = -2 \mathcal{D}_{[a} V_{b]}, \hspace{1cm} \Delta V_a = -\frac{1}{l} V_a U - \frac{1}{4l} \varepsilon_{abcd} G^{bc} V^d - \frac{1}{l} \mathcal{D}^b G_{ab} - \mathcal{D}^b G_{ab}$$

$$+ \frac{g}{2l} \text{tr} (4i\mathcal{D}^b ((w - \bar{w}) F_{ab}) - 2\varepsilon_{abcd} \mathcal{D}^b (w + \bar{w}) F^{cd})$$

$$- i(\sigma_{ab})^{\dot{a}\dot{b}} \mathcal{D}^b (\Sigma_i^{\dot{a}} \Sigma_{\dot{b}i}) - i(\bar{\sigma}_{ab})^a\dot{b} \mathcal{D}^b (\Sigma_{\dot{a}i} \bar{\Sigma}_{\dot{b}i}))$$

$$+ \frac{1}{2l^2} (\mathcal{D}_a)^{\dot{a}} \mathcal{D}_b \lambda^i_{\dot{a}} \bar{\lambda}^{\dot{i}}_{\dot{b}} - \frac{1}{2l^2} (\mathcal{D}_a)^{\dot{a}} \mathcal{D}_b \lambda^i_{\dot{a}} \bar{\lambda}^{\dot{i}}_{\dot{b}}$$

$$+ \frac{1}{2l^2} (\mathcal{D}_a)^{\dot{a}} \mathcal{D}_b \lambda^i_{\dot{a}} \bar{\lambda}^{\dot{i}}_{\dot{b}} - \frac{1}{2l^2} (\mathcal{D}_a)^{\dot{a}} \mathcal{D}_b \lambda^i_{\dot{a}} \bar{\lambda}^{\dot{i}}_{\dot{b}}$$

$$+ \frac{1}{4l^3} (\mathcal{D}_a)^{\dot{a}} \mathcal{D}_b \lambda^i_{\dot{a}} \bar{\lambda}^{\dot{i}}_{\dot{b}} - \frac{1}{4l^3} (\mathcal{D}_a)^{\dot{a}} \mathcal{D}_b \lambda^i_{\dot{a}} \bar{\lambda}^{\dot{i}}_{\dot{b}}$$

$$- \frac{g}{8l^2} (\mathcal{D}_a)^{\dot{a}} \mathcal{D}_b \lambda^i_{\dot{a}} \bar{\lambda}^{\dot{i}}_{\dot{b}} (w - \bar{w}) - 4i\mathcal{D}_b \Sigma^{\dot{i}} \bar{\Sigma}_{\dot{b}i} w + X^{ij} \bar{\Sigma}_{\dot{a}j}$$

$$- 8 F_{a\dot{a}} \bar{\Sigma}^{\dot{b}i} - 4 S^{ij} \bar{\Sigma}_{\dot{a}j} w)$$

$$- \frac{g}{8l^2} (\mathcal{D}_a)^{\dot{a}} \mathcal{D}_b \lambda^i_{\dot{a}} \bar{\lambda}^{\dot{i}}_{\dot{b}} (w - \bar{w}) - 4i\mathcal{D}_b \Sigma^{\dot{i}} \bar{\Sigma}_{\dot{b}i} w + X^{ij} \Sigma_{\dot{a}j}$$

$$- 8 F_{a\dot{a}} \Sigma^{\dot{b}i} - 4 S^{ij} \Sigma_{\dot{a}j} w)$$

$$- \frac{g}{4l^3} (\mathcal{D}_a)^{\dot{a}} \mathcal{D}_b \lambda^i_{\dot{a}} \bar{\lambda}^{\dot{i}}_{\dot{b}} (w - \bar{w}) - 2S^{ij} (w^2 - \bar{w}^2) + \Sigma^{ijkl} \bar{\Sigma}_{ijkl} . \hspace{1cm} (5.3b)$$

Here $\mathcal{D}_a$ denotes the space-time covariant derivative.\(^5\) The superfield constraints lead to the following differential constraints on $F_{ab}$, $G_{ab}$ and $V_a$

$$\mathcal{D}_{[a} F_{bc]} = 0, \quad \mathcal{D}_{[a} G_{bc]} = 0, \hspace{1cm} \begin{equation} \tag{5.4a} \end{equation}$$

$$\mathcal{D}^a H_a = -\frac{1}{8} \varepsilon^{abcd} G_{ab} G_{cd} - \frac{g}{2} \varepsilon^{abcd} \text{tr} (F_{ab} F_{cd})$$

$$+ ig \text{tr} \mathcal{D}^a (D_a w w - D^a \bar{w} w + i(\sigma_a)_{\alpha\dot{a}} (\Sigma_{\alpha}^{\dot{a}} \bar{\Sigma}_i) - \frac{1}{2} D_a w^2 + \frac{1}{2} D_a \bar{w}^2) . \hspace{1cm} (5.4b)$$

\(^5\) Although this notation, $\mathcal{D}_a$, coincides with that used earlier for the vector covariant derivative in AdS\(^4\), we hope no misunderstanding will occur.
where we define
\[ H_a = lV_a + \frac{1}{2}(\sigma_a)_{\alpha\bar{\alpha}} \lambda^{\alpha\bar{\alpha}} \lambda_i^\alpha \bar{\lambda}_i^{\bar{\alpha}}. \] (5.5)

Now, we can solve the constraints in terms of gauge one-forms \( T_a \), \( A_a \) and a two-form \( B_{ab} \)
\[ F_{ab} = 2D_{[a}A_{b]} \quad \text{and} \quad G_{ab} = 2D_{[a}T_{b]} \] (5.6a)
\[ H^a = \frac{1}{2} e^{abcd} \left( D_b B_{cd} - \frac{1}{4} T_b D_c T_d - g \text{tr}(A_b D_c A_d) \right. \]
\[ \left. + 2ig \text{tr}(D_a w \bar{w} - D_a \bar{w} w + i(\sigma_a)_{\alpha\bar{\alpha}} \Sigma^{\alpha\bar{\alpha}}_{\alpha\bar{\alpha}} - \frac{1}{2} D_a w^2 + \frac{1}{2} D_a \bar{w}^2) \right) \]. (5.6b)

This confirms the claim that the superfield constraints lead to a one-form and a two-form at the component level.

As a final note, we give the component action in the case where, for simplicity, the Chern-Simons coupling is turned off
\[ S = \int d^4x e \left( - \frac{1}{4} l G_{ab} G^{ab} + \frac{1}{2l} V^a V_a - \frac{1}{2l} D_a l D^a l + \frac{1}{2} l U^2 + \frac{1}{4l^3} S^{ij} S_{ij} \right. \]
\[ + \frac{i}{2} G_{\alpha\bar{\beta}} \lambda^{\alpha\bar{\beta}} - \frac{i}{2} \bar{G}_{\bar{\alpha}\bar{\beta}} \bar{\lambda}^{\bar{\alpha}\bar{\beta}} - \frac{i}{2} \lambda^{\alpha\bar{\beta}} D_{\alpha\bar{\beta}} \bar{\lambda}^\alpha + \frac{i}{2} l D_{\alpha\bar{\beta}} \lambda^{\alpha\beta} \bar{\lambda}^{\alpha} \bar{\lambda}^\beta \]
\[ \left. + \frac{1}{16l} \lambda_i^\alpha \lambda_j^{\beta} \lambda_i^\alpha \lambda_j^{\beta} + \frac{1}{16l} \bar{\lambda}_i^{\bar{\alpha}} \bar{\lambda}_j^{\bar{\beta}} \bar{\lambda}_i^{\bar{\alpha}} \bar{\lambda}_j^{\bar{\beta}} - \frac{3}{8l} \lambda_i^\alpha \bar{\lambda}_i^{\bar{\alpha}} \lambda_j^{\beta} \bar{\lambda}_j^{\bar{\beta}} \right). \] (5.7)

6 Vector-tensor multiplet in supergravity

Having derived the appropriate constraints and Lagrangian density for the VT multiplet in AdS it is natural to look for an extension of our constructions to \( \mathcal{N} = 2 \) supergravity. We remind the reader that AdS\(^{4|8} \) is a maximally symmetric geometry that originates within the superspace formulation of \( \mathcal{N} = 2 \) conformal supergravity developed in [32] and reviewed in Appendix D. In the framework of supergravity, the central charge should be necessarily gauged. The \( \mathcal{N} = 2 \) vector supermultiplet, which gauges the central charge, should be part of the so-called minimal multiplet of \( \mathcal{N} = 2 \) supergravity [33]. The latter can be thought of as the \( \mathcal{N} = 2 \) Weyl multiplet [36, 37, 38] coupled to the central charge vector multiplet. Within the off-shell supergravity approach of [32], the action of any supergravity-matter system should be invariant under super-Weyl transformations, see Appendix D. In particular, the VT multiplet constraints in supergravity should respect super-Weyl invariance.

To describe the nonlinear VT multiplet, we introduce a real scalar superfield \( L \) chosen (by analogy with the component approach of [11] and the rigid superspace construction...
of [18] to be inert under the super-Weyl transformations,
\[ \delta_\sigma L = 0 \, . \] (6.1)

Making use of the central charge vector superfield, which is described by the covariantly chiral field strength \( \mathcal{Z} \) and its conjugate \( \bar{\mathcal{Z}} \), we find consistent super-Weyl invariant constraints
\[
\begin{align*}
\frac{1}{2} \mathcal{D}^{ij} L^2 &= \frac{\mathcal{Z}}{Z} \mathcal{D}^i L \mathcal{D}^j L - \frac{2}{Z} L \mathcal{D}^{(i} \mathcal{Z} \mathcal{D}^{j)} L - \frac{L^2}{2Z} (\mathcal{D}^{ij} + 4S^{ij}) \mathcal{Z} , \\
\mathcal{D}_\alpha^i \bar{\mathcal{D}}^j_\alpha L &= 0 ,
\end{align*}
\] (6.2a)

which generalize (3.5). Here the gauge-covariant derivatives \( \mathcal{D}_A \) are defined in eq. (D.7).

To derive a linear multiplet \( \mathcal{L}^{ij} \), which governs the dynamics of the VT multiplet in supergravity, we have two requirements. Firstly, we require that the constraints
\[
\mathcal{D}^{(i}_\alpha \mathcal{L}^{jk)} = \bar{\mathcal{D}}^{(i}_\alpha \mathcal{L}^{jk)} = 0 ,
\] (6.3)
be satisfied. Secondly, we require \( \mathcal{L}^{ij} \) to transform homogeneously under the super-Weyl transformations. Since the homogeneous super-Weyl transformation laws of covariant projective supermultiplets (to which \( \mathcal{L}^{ij} \) belongs) are uniquely fixed [32], the super-Weyl transformation of \( \mathcal{L}^{ij} \) should be
\[ \delta_\sigma \mathcal{L}^{ij} = (\sigma + \bar{\sigma}) \mathcal{L}^{ij} . \] (6.4)

The corresponding linear multiplet satisfying the conditions given can then be constructed as
\[ \mathcal{L}^{ij} = \frac{1}{12} (\mathcal{D}^{ij} + 4S^{ij})(\mathcal{Z} L^3) = \frac{1}{12} (\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij})(\bar{\mathcal{Z}} L^3) . \] (6.5)

These results generalize our formulation in AdS and provide the first superspace formulation of the nonlinear VT multiplet in \( \mathcal{N} = 2 \) supergravity.

In accordance with the component analysis of Claus et al. [11], in \( \mathcal{N} = 2 \) supergravity a linear VT multiplet can be consistently defined in the presence of a second vector multiplet in addition to the central charge vector multiplet. Within the superspace framework, such a supergravity-matter system can easily be constructed in conjunction with the rigid supersymmetric results of [18]. We make use of an additional vector multiplet, described
by the covariantly chiral field strength $\mathcal{Y}$ and its conjugate $\bar{\mathcal{Y}}$, to construct consistent super-Weyl invariant constraints

$$\mathcal{D}^{ij} L = \frac{2\bar{\mathcal{Y}}}{\bar{\mathcal{Y}} - \mathcal{Y}} (\mathcal{D}^{(i} \bar{\mathcal{Z}} \mathcal{D}^{j)} L + \bar{\mathcal{D}}^{(i} \mathcal{Z} \bar{\mathcal{D}}^{j)} L + \frac{1}{2} L (\mathcal{D}^{ij} + 4S^{ij}) \mathcal{Z})$$

$$- \frac{2\bar{Z}}{\bar{\mathcal{Y}} - \mathcal{Y}} (\mathcal{D}^{(i} \mathcal{Y} \mathcal{D}^{j)} L + \bar{\mathcal{D}}^{(i} \bar{\mathcal{Y}} \bar{\mathcal{D}}^{j)} L + \frac{1}{2} L (\mathcal{D}^{ij} + 4S^{ij}) \mathcal{Y}) \, , \quad (6.6a)$$

$$\mathcal{D}_\alpha^{(i} \bar{\mathcal{D}}^{j)} L = 0 \, . \quad (6.6b)$$

We note that although a pure linear VT multiplet does not exist in supergravity, the above is a consistent generalization of the constraints in the presence of the additional vector multiplet. In the flat superspace limit the constraints (6.6) reduce to those given in [18]. The corresponding Lagrangian density is given by

$$\mathcal{L}^{ij} = -\frac{i}{4} (\mathcal{Y} \mathcal{D}^i L \mathcal{D}^j L - \bar{\mathcal{Y}} \bar{\mathcal{D}}^i L \bar{\mathcal{D}}^j L) + \frac{i}{\bar{\mathcal{Y}} - \mathcal{Y}} \mathcal{Y} \mathcal{Z} + \frac{i}{\bar{\mathcal{Y}} - \mathcal{Y}} \bar{\mathcal{Y}} \mathcal{Z} L^2 (\mathcal{D}^{ij} + 4S^{ij}) \mathcal{Y}$$

$$- \frac{i}{2} \mathcal{Y} \mathcal{L} (\mathcal{D}^{(i} \mathcal{Z} \mathcal{D}^{j)} L + \bar{\mathcal{D}}^{(i} \bar{\mathcal{Z}} \bar{\mathcal{D}}^{j)} L + \frac{1}{2} L (\mathcal{D}^{ij} + 4S^{ij}) \mathcal{Z})$$

$$+ \frac{i}{2} \mathcal{L} (\bar{\mathcal{Y}} \mathcal{D}^{(i} \mathcal{Y} \mathcal{D}^{j)} L + \mathcal{Y} \mathcal{D}^{(i} \bar{\mathcal{Y}} \bar{\mathcal{D}}^{j)} L) \, . \quad (6.7)$$

Its flat superspace limit coincides with that derived in [18].

Although the constraints (6.2) and (6.6) satisfy the basic consistency requirements, it is possible to formulate another consistency condition. It was noticed in [18] that after casting the constraints in terms of harmonic variables $u^i_+$ and $u^i_-$ (normalized by $u^+ u^- = 1$), one must demand $L$ to be independent of the harmonics. This leads to a non-trivial consistency requirement. Making use of the harmonics we generalize the condition to supergravity. Independence of harmonics leads to the condition

$$\mathcal{D}^{--} L = 0 \, , \quad (6.8)$$

where $\mathcal{D}^{--} = u^- \partial / \partial u^+$ is one of the left-invariant vector fields on SU(2). Applying successive gauged central charge covariant derivatives,

$$\mathcal{D}_\alpha^\pm := u^\pm \mathcal{D}_\alpha \, , \quad \bar{\mathcal{D}}_\dot{\alpha}^\pm := u^\pm \bar{\mathcal{D}}_{\dot{\alpha}} \, , \quad (6.9)$$

to the above condition leads to a number of relations. In particular, using the (anti-) commutation relations for the covariant derivatives, one derives

$$0 = \mathcal{D}^+ \mathcal{D}^+ \bar{\mathcal{D}}^+ \bar{\mathcal{D}}^+ \mathcal{D}^{--} L$$

7The field strength $\mathcal{Y}$ obeys the constraints obtained from (D.10) and (D.11) by replacing $\mathcal{Z} \rightarrow \mathcal{Y}$. The super-Weyl transformation of $\mathcal{Y}$ is identical to that of $\mathcal{Z}$, eq. (D.12).

8We are grateful to Daniel Butter for assistance with the derivation of this consistency condition.
\[ -\mathcal{D}^- \mathcal{D}^+ \bar{\mathcal{D}}^+ \bar{\mathcal{D}}^+ L + 8i \mathcal{D}^{\alpha \dot{\alpha}} \mathcal{D}_\alpha \bar{\mathcal{D}}^\dot{\alpha} L - 2 \mathcal{D}^- \mathcal{D}^+ \bar{\mathcal{D}}^+ \mathcal{D}^+ L - 2 \mathcal{D}^- \bar{\mathcal{D}}^+ \mathcal{D}^+ \mathcal{D}^+ L \\
- 4\Delta \left( (\mathcal{D}^+)^2 (\mathcal{Z} L) + (\bar{\mathcal{D}}^+)^2 (\bar{\mathcal{Z}} L) + 2L(S^{++} \mathcal{Z} + \bar{S}^{++} \bar{\mathcal{Z}}) \\
- \frac{1}{2} L(\mathcal{D}^+)^2 \mathcal{Z} - \frac{1}{2} L(\bar{\mathcal{D}}^+)^2 \bar{\mathcal{Z}} \right). \] (6.10)

This consistency condition places restrictions on the possible constraints for \( L \). For instance, if we impose the constraint \( \mathcal{D}_a^{(i} \mathcal{D}^{j)}_\dot{\alpha} L = 0 \), we have the condition

\[ 0 = \Delta \left( (\mathcal{D}^{ij})^2 (\mathcal{Z} L) + (\bar{\mathcal{D}}^{ij})^2 (\bar{\mathcal{Z}} L) + 2L(S^{++} \mathcal{Z} + \bar{S}^{++} \bar{\mathcal{Z}}) \\
- \frac{1}{2} L(\mathcal{D}^{ij})^2 \mathcal{Z} - \frac{1}{2} L(\bar{\mathcal{D}}^{ij})^2 \bar{\mathcal{Z}} \right), \] (6.12)

which is equivalent to

\[ 0 = \Delta \left( (\mathcal{D}^{ij} + 4S^{ij})(\mathcal{Z} L) + (\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij})(\bar{\mathcal{Z}} L) - L(\mathcal{D}^{ij} + 4S^{ij}) \mathcal{Z} \right). \] (6.13)

We cannot impose the free constraint, \( \mathcal{D}^{ij} L = 0 \), without demanding annihilation of \( L \) by the central charge \( \Delta \), which would put \( L \) on-shell\(^{10} \). Furthermore consistency for our supergravity constraints are guaranteed by the general super-Weyl invariant identity

\[ 0 = (\mathcal{D}^{ij} + 4S^{ij})(\mathcal{Z} L) + (\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij})(\bar{\mathcal{Z}} L) - L(\mathcal{D}^{ij} + 4S^{ij}) \mathcal{Z} , \] (6.14)

which holds for both the linear and nonlinear cases.

We also note that the super-Weyl freedom can be completely fixed by imposing the gauge

\[ \mathcal{Z} = 1 . \] (6.15)

This is known to restrict the torsion superfield \( S^{ij} \) to be real,

\[ S^{ij} = \bar{S}^{ij} . \] (6.16)

Given the linear multiplet, \( \mathcal{L}^{ij} \), the corresponding locally supersymmetric action is constructed in terms of the components of \( \mathcal{L}^{ij} \) and the central charge vector multiplet

\(^9\text{One should keep in mind that the field strengths} \, \mathcal{Z} \, \text{and} \, \bar{\mathcal{Z}} \, \text{obey the Bianchi identity} \, (D.11)\).

\(^{10}\text{It should pointed out that a constraint of the form} \, (\mathcal{D}^{ij} + \mu S^{ij}) L = 0 , \text{with} \, \mu \, \text{a constant parameter, is not super-Weyl invariant and therefore it is not acceptable.}\)
The same action takes a compact form within the harmonic superspace approach to \( \mathcal{N} = 2 \) supergravity. As shown in [43], the harmonic superspace action is

\[
S = \int \mathrm{d}u \mathrm{d}\zeta (-4)^{4} V^{++} \hat{L}^{++}.
\]

(6.17)

Here \( V^{++} \) is the analytic gauge prepotential for the central charge vector multiplet, and \( \hat{L}^{++} \) is obtained from

\[
L^{++} := L_{i j}^{j} u_{i}^{+} u_{j}^{+}
\]

(6.18)

by performing a transformation to the so-called analytic frame. The integration in (6.17) is carried over the analytic subspace of harmonic superspace, see [42] for more details. The action (6.17) is a natural generalization of the rigid supersymmetric action principle given in [18].

7 Generalizations and further prospects

In this paper we have studied VT multiplets and their couplings to vector multiplets in AdS and, more generally, in \( \mathcal{N} = 2 \) supergravity within the superspace approach. In contrast to the super-Poincaré case, the striking feature of AdS supersymmetry is non-existence of a free linear VT multiplet.

Our results in section 6 provide the first superspace formulation of the nonlinear and the linear VT multiplets in \( \mathcal{N} = 2 \) supergravity. At the component level, a comprehensive study of the coupling of VT multiplets to \( \mathcal{N} = 2 \) supergravity was given in the past by Claus et al. [11]. Comparing our results in section 6 with those derived in [11], one can see that the superfield constraints and Lagrangian densities are more compact than their component counterparts. It was pointed out in [11] that “the complexity of our results clearly demonstrates the need for a suitable superspace formulation.” Such a formulation has been developed in our paper.

Using the locally supersymmetric constructions given in section 6, we can immediately derive new results in the case of AdS supersymmetry. It suffices to ‘freeze’ the background supergravity multiplet to a configuration describing the AdS geometry. This amounts to setting the torsion components \( Y_{\alpha\beta}, W_{\alpha\beta} \) and \( G_{\alpha\beta} \) to vanish,

\[
Y_{\alpha\beta} = 0, \quad W_{\alpha\beta} = 0, \quad G_{\alpha\beta} = 0
\]

(7.1)

and also choosing the remaining torsion \( S^{ij} \) to be real, eq. (6.16), and covariantly constant, \( \mathcal{D}_{A} S^{ij} = 0 \). Upon such a reduction, the constraints (6.2) describe the nonlinear VT
multiplet with gauged central charge in AdS. We can further freeze the central charge vector multiplet to that having a constant field strength

\[ Z = \bar{Z} = z = \bar{z} = \text{const}. \quad (7.2) \]

Due to the Bianchi identity (D.11) and the AdS condition (6.16), the parameter \( z \) must be real.\(^\text{11}\) In the limit \( Z \to z = \bar{z} = \text{const} \), the constraints (6.2) reduce to (3.5).

Furthermore, upon freezing the background supergravity multiplet to correspond to the AdS geometry, the constraints (6.6) describe the linear VT multiplet in AdS in the presence of two vector multiplets one of which gauges the central charge. This formulation can be further reduced to obtain two interesting special cases. First of all, in the AdS superspace we can freeze the \( \mathcal{Y} \) vector multiplet to that having a constant field strength,

\[ \mathcal{Y} = \bar{y} = \text{const}, \quad \bar{y} = y. \quad (7.3) \]

This leads to the linear VT multiplet with gauge central charge in AdS

\[ (\mathcal{D}^{ij} + 4S^{ij})L = \frac{2}{\bar{Z} - Z}\left( \mathcal{D}^{(i}Z\mathcal{D}^{j)}L + \bar{\mathcal{D}}^{(i}\bar{Z}\bar{\mathcal{D}}^{j)}L + \frac{1}{2}L\mathcal{D}^{ij}Z \right), \quad (7.4a) \]

\[ \mathcal{D}^{(i}\mathcal{D}^{j)}L = 0. \quad (7.4b) \]

Secondly, we can further freeze the central charge vector multiplet to that having a constant field strength, eq. (7.2). This leads to the linear VT coupled to a vector multiplet

\[ (\mathcal{D}^{ij} + 4S^{ij})L = \frac{2}{\mathcal{Y} - \bar{y}}\left( \mathcal{D}^{(i}\mathcal{Y}\mathcal{D}^{j)}L + \bar{\mathcal{D}}^{(i}\bar{y}\bar{\mathcal{D}}^{j)}L + \frac{1}{2}L\mathcal{D}^{ij}\mathcal{Y} \right), \quad (7.5a) \]

\[ \mathcal{D}^{(i}\mathcal{D}^{j)}L = 0. \quad (7.5b) \]

The constraints (7.4) and (7.5) look formally identical to each other, but it should be kept in mind that the first set of constraints correspond to the case of gauge central charge. It is not possible to freeze the remaining background vector multiplet in (7.4) or (7.5) to have a constant field strength since the corresponding expectation value should be real in AdS, as emphasized in eqs. (7.2) and (7.3), and hence the right hand side of (7.4) or (7.5) becomes singular when performing a limit \( Z \to z \) or \( \mathcal{Y} \to y \).

Chern-Simons couplings of the VT multiplet, such as those described by the relations (3.11) and (3.12), can be used as a tool to couple the VT multiplet to any number of \( \mathcal{N} = 2 \) tensor multiplets. This is achieved by making use of the techniques developed\(^\text{11}\)The existence of a frozen vector multiplet, eq. (7.2), in \( \mathcal{N} = 2 \) AdS superspace was proved in [30].

25
in [44] (see also [45]). It was shown in [44] how to generate a composite reduced chiral superfield \( \mathcal{W} \), from a system of \( n \) tensor multiplets described by their field strengths \( G^{ij}_I \), with \( I = 1, \ldots, n \).

\[
\mathcal{D}^{(i} G^{jk)}_I = \bar{\mathcal{D}}^{(i} C^{jk)}_I = 0 .
\] (7.6)

The construction is as follows:

\[
\mathcal{W} = \frac{1}{8\pi} \oint v^i dv_i \left( (\mathcal{D}^-)^2 + 4\bar{S}^- \right) \Omega(G^{++}_I) , \quad G^{++}_I := v_i v_j G^{ij}_I , \quad v^i \in \mathbb{C}^2 \setminus \{0\} ,
\] (7.7)

where \( \Omega(G^{++}_I) \) is a real homogeneous function of degree zero, \( \Omega(c G^{++}_I) = \Omega(G^{++}_I) \), when \( n > 1 \), and \( \Omega(G^{++}) \propto \ln G^{++} \) in the case of a single tensor multiplet. The integration in (7.7) is carried over a closed contour \( \gamma \) in \( \mathbb{C}^2 \setminus \{0\} \). The right hand side of (7.7) involves the second-order operator

\[
(\mathcal{D}^-)^2 + 4\bar{S}^- := \frac{u_i u_j}{(v, u)^2} \left( \mathcal{D}^{ij} + 4\bar{S}^{ij} \right) , \quad (v, u) := v^i u_i ,
\] (7.8)

which makes use of an isotwistor \( u_i \) constrained by \( (v, u) \neq 0 \) and fixed along the integration contour \( \gamma \); it can be shown that \( \mathcal{W} \) is independent of \( u_i \). As a simple example, we consider the case \( n = 1 \) and \( \Omega(G^{++}) = \ln G^{++} \) associated with the improved tensor multiplet [46, 47]. For this choice, eq. (7.7) leads to

\[
\mathcal{W} = -\frac{G}{8} \left( \mathcal{D}_{ij} + 4\bar{S}_{ij} \right) \left( \frac{G^{ij}}{G^2} \right) , \quad G^2 := \frac{1}{2} G^{ij} G_{ij} .
\] (7.9)

Replacing \( \mathcal{F}(\mathcal{W}) \rightarrow \mathcal{W}^2 \) in (3.11), with \( \mathcal{W} \) given by (7.7), yields a consistent higher-derivative coupling of the VT multiplet to \( N = 2 \) tensor multiplets.

Our analysis of the supergravity-matter systems in section 6 treated the cases of the linear and nonlinear VT multiplets separately. We also ignored Chern-Simons couplings to Yang-Mills vector multiplets. At the component level, Ref [11] provided a unified description of both the linear and nonlinear VT multiplets and their most general Chern-Simons couplings to vector multiplets. Using the results of our paper, it is possible to provide a superspace reformulation and generalization of the results in [11]. This will be discussed in a separate publication [48].

Ten years ago, Theis [49, 50] constructed a new nonlinear VT multiplet in Minkowski space. Interactions arise in this model as a consequence of gauging the central charge. The latter is achieved by using the gauge one-form belonging to the VT multiplet, unlike

\[12\] The chiral field strengths of Abelian vector multiplets are reduced chiral superfield.
the traditional approach of using a vector multiplet. It would be interesting to understand whether the construction of [49, 50] can be generalized to the case of AdS supersymmetry.

Recently, the supergravity results of [11] have been generalized in [59] to include $n_V$ vector and $n_T$ vector-tensor multiplets. As explained in [59], their constructions could be obtained from standard $\mathcal{N} = 2$ supergravity coupled to $n_V + n_T$ vector multiplets by dualizing $n_T$ imaginary components of the $n_V + n_T$ complex scalar fields parametrizing the special manifold. It would be interesting to understand how to obtain the results of [59] in an off-shell superconformal setting. Clearly, the case $n_T = 1$ is most interesting for string-theoretic applications [8].

In conclusion, let us summarize the main original results of this paper. We developed the general superspace setting for $\mathcal{N} = 2$ supersymmetric theories with central charge in AdS, including the supersymmetric action principle in $\mathcal{N} = 2$ AdS superspace. We proved that $\mathcal{N} = 2$ AdS supersymmetry does not allow existence of a linear VT multiplet. For the nonlinear VT multiplet in AdS, we derived consistent superfield constraints in the presence of any number of $\mathcal{N} = 2$ Yang-Mills vector multiplets, constructed the corresponding action and elaborated on the $\mathcal{N} = 1$ superfield and component descriptions of the theory. For the superfield constraints and Lagrangians of [18], which describe the linear and the nonlinear VT multiplets with gauged central charge, we provided the consistent extensions to $\mathcal{N} = 2$ supergravity. We also constructed higher-derivative couplings of the VT multiplet to any number of $\mathcal{N} = 2$ tensor multiplets.

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A $\mathcal{N} = 1$ AdS superspace

In this appendix we collect salient facts about the geometry of $\mathcal{N} = 1$ AdS superspace [51, 52, 53], AdS$^{4|4}$, and its isometries following [31].

The geometry of AdS$^{4|4}$ is determined by covariant derivatives,

$$\nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}^\dot{\alpha}) = E_A^M \partial_M + \frac{1}{2} \phi_A^{bc} M^{bc} \equiv E_A + \phi_A \quad (A.1)$$
obeying the following (anti-)commutation relations:

\[
\{\nabla_\alpha, \nabla_\beta\} = -4\bar{\mu}M_{\alpha\beta}, \quad \{\nabla_\alpha, \bar{\nabla}^\beta\} = -2i(\sigma^c)_\alpha^\beta \nabla_c \equiv -2i\nabla_\alpha^\beta, \quad (A.2a)
\]

\[
[\nabla_\alpha, \nabla_\beta] = -i\frac{\mu}{2}(\sigma_a)^{\beta\gamma} \bar{\nabla}_\gamma, \quad [\nabla_\alpha, \bar{\nabla}_\beta] = -|\mu|^2 M_{\alpha\beta}, \quad (A.2b)
\]

where \(\mu\) is a complex non-vanishing parameter. Here the Lorentz generators with vector indices \((M_{ab} = -M_{ba})\) are related to those with spinor indices \((M_{\alpha\beta} = M_{\beta\alpha})\) by the rules:

\[
M_{ab} = (\sigma_{ab})^{\alpha\beta} M_{\alpha\beta} - (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}}, \quad M_{\alpha\beta} = \frac{1}{2}(\sigma_{ab})^{\alpha\beta} M_{ab} - \frac{1}{2}(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \bar{M}_{ab}.
\]

The Lorentz generators act on the spinor covariant derivatives as

\[
[M_{\alpha\beta}, D_\gamma] = \varepsilon_{\gamma(\alpha} \nabla_{\beta)}, \quad [\bar{M}_{\dot{\alpha}\dot{\beta}}, \bar{\nabla}_\gamma] = \varepsilon_{\gamma(\dot{\alpha}} \bar{D}_{\dot{\beta})}, \quad (A.3)
\]

with \([M_{\alpha\beta}, \bar{D}_\gamma] = [\bar{M}_{\dot{\alpha}\dot{\beta}}, D_\gamma] = 0\).

A real vector field, \(\xi^4 = (\xi^a, \xi^\alpha, \xi^\dot{\alpha})\), on \(\text{AdS}^{4|4}\) is called a Killing vector field if

\[
[\xi + \frac{1}{2}\lambda^{cd} M_{cd}, \nabla_\alpha] = 0, \quad \xi := \xi^a \nabla_a + \xi^\alpha \nabla_\alpha + \bar{\xi}^\dot{\alpha} \bar{\nabla}^\dot{\alpha}, \quad (A.4)
\]

where \(\lambda^{cd}\) is uniquely determined in terms of \(\xi^4\) and corresponds to some local Lorentz transformation. The master equation \(\text{(A.4)}\) is equivalent to

\[
\lambda_{\alpha\beta} = \nabla_\alpha \xi_\beta, \quad \nabla^\alpha \xi_\alpha = 0, \quad \frac{i}{2}\mu \xi_{\alpha\dot{\alpha}} + \bar{\nabla}_{\dot{\alpha}} \xi_\alpha = 0, \quad (A.5)
\]

\[
\nabla_{(\alpha} \xi_{\beta)\dot{\gamma}} = 0, \quad \bar{\nabla}^{\dot{\alpha}} \xi_{\alpha\dot{\beta}} + 8i\xi_\alpha = 0, \quad (A.6)
\]

see [31] for a derivation. The AdS Killing vector fields generate the isometry group of the \(\mathcal{N} = 1\) AdS superspace, \(\text{OSp}(1|4)\). The infinitesimal isometry transformation associated with \(\xi^4\) acts on a tensor superfield \(U\) as follows

\[
\delta U = -\xi U - \frac{1}{2}\lambda^{cd} M_{cd} U. \quad (A.7)
\]

**B N = 2 Killing vector fields**

In this appendix we give a brief summary of the Killing vector fields of \(\mathcal{N} = 2\) AdS superspace, \(\text{AdS}^{4|8}\), which were used in section 2. These objects were originally introduced in [30] (see also [28]). A real vector field in \(\mathcal{N} = 2\) AdS superspace corresponding to the first-order operator

\[
\xi := \xi^a D_a + \xi_i^\alpha D_i^\alpha + \bar{\xi}_i^\dot{\alpha} \bar{D}_i^\dot{\alpha} \quad (B.1)
\]
is said to be a Killing vector field if it obeys the master equation

\[ [\xi + \frac{1}{2} \lambda^{cd} M_{cd} + 2 \varepsilon J, D_{a}] = 0 , \quad J := S^{kl} J_{kl} , \quad (B.2) \]

for uniquely determined parameters \( \lambda^{cd} \) and \( \varepsilon \) generating Lorentz and U(1) transformations respectively. The explicit expressions for these parameters are

\[ \lambda_{ab} = D_{\{a} \xi_{b\} , \quad \varepsilon = \frac{1}{8} S^{ij} D_{\alpha i} \xi_{j}^{\alpha} , \quad (B.3) \]

see [28, 30] for a derivation. The Killing vector fields generate the isometry group of the \( \mathcal{N} = 2 \) AdS superspace, OSp(2|4). If \( U \) is a tensor superfield in \( \mathcal{N} = 2 \) AdS superspace, its infinitesimal transformation associated with \( \xi \) is

\[ \delta U = -\xi U - \frac{1}{2} \lambda^{cd} M_{cd} U - 2 \varepsilon J U . \quad (B.4) \]

C  \( \mathcal{N} = 1 \) reduction

Any supersymmetric field theory in \( \mathcal{N} = 2 \) AdS superspace, AdS\(^{4|8}\), can be reformulated in terms of superfields on \( \mathcal{N} = 1 \) AdS superspace [30], AdS\(^{4|4}\). Such a reformulation proves to be useful for various applications. Here we give a summary of the \( \mathcal{N} = 2 \to \mathcal{N} = 1 \) reduction, more details can be found in [28, 30].

Given a tensor superfield \( U(x, \theta, \bar{\theta}) \) on AdS\(^{4|8}\), its \( \mathcal{N} = 1 \) projection is defined by

\[ U| := U(x, \theta, \bar{\theta})|_{\theta_2 = \bar{\theta}_2 = 0} \quad (C.1) \]

in a special coordinate system specified below. Given a gauge-covariant operator of the form \( D_{A_1} \ldots D_{A_n} \), its \( \mathcal{N} = 1 \) projection \( (D_{A_1} \ldots D_{A_n})| \) is defined as follows:

\[ \left( (D_{A_1} \ldots D_{A_n})| U \right) := (D_{A_1} \ldots D_{A_n} U) , \quad (C.2) \]

with \( U \) an arbitrary tensor superfield. The required coordinate system is specified by

\[ D_{\alpha}^{|} = \nabla_{\alpha} , \quad \bar{D}_{\dot{\alpha}}^{|} = \bar{\nabla}^{\dot{\alpha}} , \quad (C.3) \]

with \( \nabla_{A} = (\nabla_{a}, \nabla_{\alpha}, \bar{\nabla}^{\dot{\alpha}}) \) the covariant derivatives for AdS\(^{4|4}\) introduced in Appendix A. In such a coordinate system, the operators \( D_{\alpha}^{|} \) and \( \bar{D}_{\dot{\alpha}}^{|} \) do not involve any partial derivatives with respect to \( \theta_2 \) and \( \bar{\theta}_2 \), and therefore, for any positive integer \( k \), it holds that
\[ (D_{\hat{a}_1} \cdots D_{\hat{a}_k} U) = D_{\hat{a}_1} \cdots D_{\hat{a}_k} |U|, \text{ where } D_{\hat{a}} := (D_{\hat{a}}^1, D_{\hat{a}}^{\hat{d}}) \text{ and } U \text{ is a tensor superfield.} \]

We therefore obtain
\[ D_a \mid = \nabla_a. \tag{C.4} \]

The conceptual possibility to have a well-defined \( \mathcal{N}=2 \rightarrow \mathcal{N}=1 \) AdS superspace reduction follows from the fact that the operators \((D_a, D_{\hat{a}}^1, \bar{D}_{\hat{a}}^{\hat{d}})\) form a closed algebra
\[ \{D_{\hat{a}}^1, D_{\hat{b}}^{\hat{d}}\} = 4 S_{\hat{a}\hat{b}} M_{\alpha\beta}, \quad \{D_{\hat{a}}^1, \bar{D}_{\hat{b}}^{\hat{d}}\} = -2i D_{\hat{a}} \bar{D}_{\hat{b}}^{\hat{d}}, \tag{C.5a} \]
\[ [D_a, D_{\hat{b}}] = \frac{i}{2} (\sigma_a)_{\beta\gamma} s_{\hat{b}\hat{\gamma}} D_{\hat{\beta}}^1, \quad [D_a, \bar{D}_{\hat{b}}] = -S^2 M_{ab}, \tag{C.5b} \]

isomorphic to the covariant derivative algebra of \( \mathcal{N}=1 \) AdS superspace, eq. (A.2), with
\[ \mu = -S_{\hat{a}\hat{b}}. \tag{C.6} \]

The isometries of \( \text{AdS}^{4|8} \) are generated by the \( \mathcal{N}=2 \) Killing vector fields. Given such a Killing vector, \( \xi \), it induces two different transformations in \( \text{AdS}^{4|4} \) defined in terms of its \( \mathcal{N}=1 \) projection
\[ |\xi| = \xi + \xi_2^a D_{\alpha}^a + \bar{\xi}_2^{\dot{a}} \bar{D}_{\dot{a}}^{\dot{a}} \equiv \xi + \varepsilon^a D_{\alpha}^a + \bar{\varepsilon}^{\dot{a}} \bar{D}_{\dot{a}}^{\dot{a}}. \tag{C.7} \]

Here \( \xi = \varepsilon^a \nabla_a + \xi^\alpha \nabla_\alpha + \bar{\xi}_{\dot{a}} \bar{\nabla}_{\dot{a}} \) proves to be a Killing vector of the \( \mathcal{N}=1 \) AdS superspace. It can be shown that
\[ \varepsilon_\alpha = \nabla_\alpha \varepsilon, \quad \varepsilon := |\varepsilon|. \tag{C.8} \]

The real parameter \( \varepsilon \) satisfies the constraints \[54]\n\[ (\nabla^2 - 4\mu)\varepsilon = (\nabla^2 - 4\bar{\mu})\varepsilon = 0, \tag{C.9a} \]
\[ \nabla_\alpha \nabla_\dot{\alpha} \varepsilon = \nabla_{\dot{\alpha}} \nabla_\alpha \varepsilon = 0. \tag{C.9b} \]

The parameters \( \xi \) and \( \varepsilon \) generate two different transformations. The former generates an isometry transformation of \( \text{AdS}^{4|4} \) acting on \( |U| \) by
\[ \delta_\xi |U| = -\xi |U| - \frac{1}{2} \lambda^{ad} M_{cd} |U|. \tag{C.10} \]

The latter generates \( U(1) \) and second supersymmetry transformations,
\[ \delta_\varepsilon |U| = -\varepsilon^a (D_{\alpha}^a |U|) - \bar{\varepsilon}_{\dot{a}} (\bar{D}_{\dot{a}}^{\dot{a}} |U|) - 2\varepsilon J |U|. \tag{C.11} \]
\section*{D \ N = 2 conformal supergravity}

This appendix contains a summary of the superspace formulation for \( \mathcal{N} = 2 \) conformal supergravity developed in \cite{32}. The formulation is based on the curved superspace geometry introduced by Grimm \cite{55}. There exists a more general superspace formulation for \( \mathcal{N} = 2 \) conformal supergravity developed by Howe \cite{56}.\footnote{Howe’s formulation \cite{56} is a gauged fixed version of the superspace formulation for \( \mathcal{N} = 2 \) conformal supergravity developed by Butter \cite{57}.} The precise relationship between these two formulations is spelled out in \cite{58}. The results in this section are presented almost identically to \cite{32}.

Conformal supergravity can be realized in a curved 4D \( \mathcal{N} = 2 \) superspace, \( \mathcal{M}^{\mathfrak{g}8} \) parametrized by local coordinates \( z^M = (x^m, \theta^i, \bar{\theta}^\dot{i}) \), where \( m = 0, 1, ..., 3, \mu = 1, 2, \bar{\mu} = 1, 2 \) and \( i = 1, 2, 3, 4 \). The structure group is chosen to be \( \text{SL}(2, \mathbb{C}) \times \text{SU}(2) \), and the covariant derivatives \( D_A = (D_a, D^i, \bar{D}^\dot{i}) \) have the form

\[
D_A = E_A + \Phi_A^{kl} J_{kl} + \Omega_A^{bc} M_{bc} \nonumber \\
= E_A + \Phi_A^{kl} J_{kl} + \Omega_A^{\beta\gamma} M_{\beta\gamma} + \bar{\Omega}_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}}. 
\]

(D.1)

Here \( M_{cd} \) and \( J_{kl} \) are the generators of the Lorentz and \( \text{SU}(2) \) groups respectively, and \( \Omega_A^{bc} \) and \( \Phi_A^{kl}(z) \) the corresponding connections. The action of the generators on the covariant derivatives are defined as:

\[
[M_{\alpha\beta}, D^i] = \varepsilon_{\gamma(\alpha} D^i_{\beta)} \nonumber , \quad [M_{\dot{\alpha}\dot{\beta}}, \bar{D}^\dot{i}] = \varepsilon_{\dot{\gamma}(\dot{\alpha}} \bar{D}^\dot{i}_{\dot{\beta)}}, \quad (D.2)
\]

\[
[J_{kl}, D^i] = -\delta^i_{(k} D_{a)l} \nonumber , \quad [J_{kl}, \bar{D}^\dot{i}] = -\varepsilon_{(i(\dot{k} D_{\dot{l})a}}. \quad (D.3)
\]

The covariant derivatives satisfy the (anti-)commutation relations \cite{55}:

\[
\{D^i_{\alpha}, D^j_{\beta}\} = 4 S^{ij} M_{\alpha\beta} + 2 \varepsilon^{ij}_\alpha \varepsilon_{\alpha\beta} Y^{\gamma\delta} M_{\gamma\delta} + 2 \varepsilon^{ij}_\alpha \varepsilon_{\alpha\beta} W_{\gamma\delta} \bar{M}_{\gamma\delta} 
\]

\[
+ 2 \varepsilon^{ij}_\alpha \varepsilon_{\alpha\beta} S^{kl} J_{kl} + 4 Y_{\alpha\beta} J^{ij}, \quad (D.4a)
\]

\[
\{D^i_{\alpha}, \bar{D}^\dot{j}_{\beta}\} = -2 i \delta^j_{\dot{j}} D^i_{\alpha} \bar{D}^\dot{j}_{\beta} + 4 \delta^j_{\dot{j}} G^{\dot{\gamma}\dot{\beta}} M_{\alpha\beta} + 4 \delta^j_{\dot{j}} G^\dot{\gamma} \bar{M}_{\alpha\beta} + 8 G^{\dot{\gamma}} M_{\alpha\beta} J^{ij}, \quad (D.4b)
\]

\[
[D_{\alpha\dot{\alpha}}, D^i_{\beta}] = \pm 2 \varepsilon_{\alpha(\dot{\beta} G_{\gamma})\alpha} D^{\gamma j} - i (\varepsilon^{jk}_\alpha \varepsilon_{\alpha\beta} \gamma_{\dot{\alpha}\dot{\gamma}} Y_{\alpha\beta} + \varepsilon^{jk}_\alpha \varepsilon_{\alpha\beta} \bar{W}_{\alpha\dot{\beta}} + \varepsilon_{\alpha\beta} \bar{W}_{\alpha\dot{\gamma}} \gamma_{\dot{\alpha}\dot{\gamma}}) D^i_{\alpha} + \frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon^{\gamma j} D^{\gamma j} M^{\alpha\beta} \bar{M}^{\alpha\beta}
\]

\[
- \frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon^{\gamma j} Y_{\alpha\beta} + 2 \varepsilon_{\alpha\beta} D^i_{(\alpha} \bar{W}^{\gamma j)_{\beta}} + \frac{1}{3} \varepsilon_{\alpha\beta} \varepsilon_{\alpha\gamma} \bar{D}^i_{\gamma} S^{\gamma j} M^{\alpha\beta} \bar{M}^{\alpha\beta} - \frac{1}{2} (2 \varepsilon_{\alpha\beta} \varepsilon^{jk} D^i_{\alpha} \bar{W}^{\gamma j}_{\beta} + 2 \varepsilon_{\alpha\beta} \varepsilon^{jk} \bar{D}^i_{\alpha} \bar{W}^{\gamma j}_{\beta} + \varepsilon_{\alpha\beta} D^i_{\alpha} S^{\gamma j}) J_{kl}. \quad (D.4c)
\]
It was shown in [32] that the superspace geometry introduced is invariant under super-Weyl transformations:

\[
\delta_\sigma D^i_\alpha = \frac{1}{2} \bar{\sigma} D^i_\alpha + (D^\gamma i_\sigma) M_{\gamma \alpha} - (D_{\alpha k} \sigma) J^{ki}, \tag{D.5a}
\]

\[
\delta_\sigma D_a = \frac{1}{2} (\sigma + \bar{\sigma}) D_a + i \left( (\sigma_a)^\alpha_\gamma (D^k_\alpha \sigma) D_k^\gamma \right) + \frac{1}{4} (\sigma_a)^\alpha_\gamma (D_k^\gamma \bar{\sigma}) D_k^\alpha
\]

\[- \frac{1}{2} (D^b (\sigma - \bar{\sigma})) M_{ab}, \tag{D.5b}
\]

where \(\sigma\) is an arbitrary covariantly chiral superfield, \(\bar{D}_i^\alpha \sigma = 0\). The torsion components then transform as:

\[
\delta_\sigma S^{ij} = \bar{\sigma} S^{ij} - \frac{1}{4} (D^\gamma i_\sigma) \sigma, \tag{D.6a}
\]

\[
\delta_\sigma Y_{\alpha\beta} = \bar{\sigma} Y_{\alpha\beta} - \frac{1}{4} (\bar{D}_k^\alpha D_{\beta k}) \sigma, \tag{D.6b}
\]

\[
\delta_\sigma W_{\alpha\beta} = \sigma W_{\alpha\beta}, \tag{D.6c}
\]

\[
\delta_\sigma G_{\alpha\dot{\beta}} = \frac{1}{2} (\sigma + \bar{\sigma}) G_{\alpha\dot{\beta}} - \frac{i}{4} (D_\alpha \dot{\beta}) (\sigma - \bar{\sigma}). \tag{D.6d}
\]

Following [58], it is possible to incorporate a gauged central charge, \(\Delta\), into the supergeometry by modifying the covariant derivatives to contain a central charge gauge connection, \(V_A\):

\[
\mathcal{D}_A \rightarrow \mathcal{D}_A := \mathcal{D}_A + V_A \Delta, \quad \Delta V_A = 0. \tag{D.7}
\]

The corresponding algebra of gauge-covariant derivatives then becomes:

\[
[\mathcal{D}_A, \mathcal{D}_B] = T_{AB}^C \mathcal{D}_C + \frac{1}{2} R_{AB}^{cd} M_{cd} + R_{AB}^{kl} J_{kl} + F_{AB} \Delta, \tag{D.8}
\]

where the gauge-invariant field strength, \(F_{AB}\) is subject to covariant constraints and the torsion and curvature remain the same as the case without central charge. The components of \(F_{AB}\) are:

\[
F_{\alpha\beta}^{ij} = -2 \varepsilon_{\alpha\beta} \varepsilon^{ij} \bar{Z}, \quad F_{\dot{\alpha}\dot{\beta}}^{ij} = 2 \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon_{ij} \bar{Z}, \quad F_{\dot{\alpha} j} = 0, \tag{D.9a}
\]

\[
F_{\dot{\alpha} k} = \frac{i}{2} (\sigma_{\dot{\alpha}})^\dot{\gamma} \bar{D}_{k}^\gamma \bar{Z}, \quad F_{\dot{\alpha} j} = -\frac{i}{2} (\sigma_{\dot{\alpha}})^\dot{\gamma} D_{k}^\gamma \bar{Z}, \tag{D.9b}
\]

\[
F_{ab} = -\frac{1}{8} (\sigma_{ab})_{\alpha\beta} D^{\alpha k} D^\beta_k \bar{Z} + \frac{1}{8} (\bar{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} \bar{D}^{\dot{\alpha} k} \bar{D}^{\dot{\beta} k} \bar{Z}
+ \frac{i}{2} ((\sigma_{ab})_{\alpha\beta} W_{\alpha\beta} - (\bar{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} Y_{\dot{\alpha}\dot{\beta}}) \bar{Z} - \frac{1}{2} ((\sigma_{ab})_{\alpha\beta} W^{\alpha\beta} - (\bar{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} Y^{\dot{\alpha}\dot{\beta}}) \bar{Z}, \tag{D.9c}
\]

where \(\bar{Z}\) is a covariantly chiral superfield,

\[
\bar{D}_a^i \bar{Z} = 0. \tag{D.10}
\]
obeying the Bianchi identity

\[(D^\gamma (i \gamma D^j) + 4S^{ij}) \mathcal{Z} = (\bar{D}^i (\bar{\gamma}) \bar{\gamma} + 4 \bar{S}^{ij}) \bar{\mathcal{Z}}.\]  

(D.11)

To be consistent with the central charge interpretation, the field strength \(Z\) should be nowhere vanishing, \(Z \neq 0\). Super-Weyl transformations can then be seen to remain the same as in (D.5) and (D.6) with

\[\delta_\sigma \mathcal{Z} = \sigma \mathcal{Z}.\]  

(D.12)

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