S-Expansion of Higher-Order Lie Algebras

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Abstract

By means of a generalization of the S-expansion method we construct a procedure to obtain expanded higher-order Lie algebras. It is shown that the direct product between an Abelian semigroup $S$ and a higher-order Lie algebra $(\mathcal{G},[\ldots,\ldots])$ is also a higher-order Lie algebra. From this S-expanded Lie algebra are obtained resonant submultialgebras and reduced multialgebras of a resonant submultialgebra.
I. INTRODUCTION

Higher-order (or multibracket) simple Lie algebras [1], [2], [3] are generalized ordinary Lie algebras. Their structure constants are given by Lie algebra cohomology cocycles which, by virtue of being such, satisfy a suitable generalization of the Jacobi identity.

As is noted in ref [1], [3] it could be interesting to find applications of these higher-order Lie algebras to know whether the cohomological restrictions which determine and condition their existence have a physical significance. Lie algebra cohomology arguments have already been very useful in various physical problems as in the description of anomalies or in the construction of the Wess-Zumino terms required in the action of extended supersymmetric objects. Other questions may be posed from a purely mathematical point of view. From the discussion in Sect.4 of ref. [1] we know that a representation of a simple Lie algebra may not
be a representation for the associated higher-order Lie algebras. Thus, the representation theory of higher-order algebras requires a separate analysis. A very interesting open problem from a structural point of view is the expansions of higher-order Lie algebras, which will take us outside the domain of the simple ones.

The purpose of this paper is to show that the S-expansion method developed in ref. [4] (see also [5], [6], [7]) can be generalized so that it permits obtaining expanded higher-order Lie algebras.

The paper is organized as follows: In section 2 we shall review some aspects of higher-order Lie algebras. The main point of this section is to display the differences between ordinary Lie algebras and higher-order Lie algebras and to generalize the definitions of higher-order Lie subalgebras and higher-order reduced Lie algebras. In section 3 we generalize the S-expansion method and we show that it is possible to obtain higher-order expanded Lie algebras. In section 4 is shown that, under determined conditions, relevant higher-order Lie subalgebras can be extracted from the S-expanded higher-order Lie algebras.

II. HIGHER-ORDER LIE ALGEBRAS

In this section we shall review some aspects of higher-order Lie algebras. The main point of this section is to display the differences between ordinary Lie algebras and higher-order Lie algebras and to generalize the concepts of subalgebra and reduced Lie algebra of ref. [4].

**Definition 1** An algebra is defined as a pair \((G, \circ)\) where \(G\) is a finite dimensional vector space, and \(\circ : G \times G \to G\) is a rule of composition defined over the vector space.

**Definition 2** A Lie algebra \(G\) is defined by the pair \((G, [\,])\) where \(G\) is a finite dimensional vector space, with basis \(\{T_A\}_{A=1}^{\dim G}\), over the field \(K\) of real or complex numbers; and \([\,]\) is a rule of composition \((T_A, T_B) \to [T_A, T_B] \in G\) which satisfies the following axioms:

- \([\alpha T_A + \beta T_B, T_C] = \alpha [T_A, T_C] + \beta [T_B, T_C]\) for \(\alpha, \beta \in K\) (linearity),
- \([T_A, T_B] = -[T_B, T_A]\) \(\forall T_A, T_B \in G\) (antisymmetry),
- \([[T_A, T_B], T_C] + [[T_B, T_C], T_A] + [[T_C, T_A], T_B] = 0,\)
  for all \(T_A, T_B, T_C \in G\) (Jacobi identity).
The Jacobi identity (JI) can be re-written
\[ \frac{1}{12} \sum_{\sigma \in S_3} (-1)^{\pi(\sigma)} \left[ [T_{A_\sigma(1)}, T_{A_\sigma(2)}], T_{A_\sigma(3)} \right] = 0. \] (1)
where \( S_3 \) is the permutation group of three elements and \( \pi(\sigma) \) is the parity of the permutation \( \sigma \).

**Definition 3** Let \( G \) be a Lie algebra. A \( n \)-bracket \([,\ldots,]\) or skew-symmetric Lie multibracket is a Lie algebra valued \( n \)-linear skew-symmetric mapping \([,\ldots,] : G \times \ldots \times G \rightarrow G \),
\[ (T_{A_1}, \ldots, T_{A_n}) \rightarrow [T_{A_1}, \ldots, T_{A_n}] = C_B^{A_1 \ldots A_n} T_B \] (2)
where the constants \( C_B^{A_1 \ldots A_n} \) are called higher-order structure constants which are completely antisymmetric in the indices \( A_1 \ldots A_n \). To define higher-order Lie algebras we need to find the generalization of the Jacobi identity. We postulate that the generalization of the left hand side of eq. (1) is given by
\[ \frac{1}{(n-1)!} \frac{1}{n!} \sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \left[ [T_{A_\sigma(1)}, \ldots, T_{A_\sigma(n)}], T_{A_\sigma(n+1)}, \ldots, T_{A_\sigma(2n-1)} \right] \] (3)
However we must find the conditions under which is possible the vanishing of the right hand side. Let \( T_A \) be the basis of the algebra in a representation of \( G \). Then is possible to realize the multibracket as
\[ [T_{A_1}, \ldots, T_{A_n}] = \varepsilon_{B_1 \ldots B_n} T_{B_1} \ldots T_{B_n} \]
\[ = \sum_{\sigma \in S_n} (-1)^{\pi(\sigma)} T_{A_{\sigma(1)}} \ldots T_{A_{\sigma(n)}} \]
where \( S_n \) is the permutation group of \( n \) element and \( \pi(\sigma) \) is the parity of the permutation \( \sigma \). In the appendix we will show that the realization (4) of the multibracket satisfy the identity
\[ \frac{1}{(n-1)!} \frac{1}{n!} \sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \left[ [T_{A_\sigma(1)}, \ldots, T_{A_\sigma(n)}], T_{A_\sigma(n+1)}, \ldots, T_{A_\sigma(2n-1)} \right] \]
\[ = \begin{cases} 0, & n \text{ even} \\ n [T_{A_1}, \ldots, T_{A_{2n-1}}], & n \text{ odd.} \end{cases} \]
This means that it is possible to obtain a generalization of the Jacobi identity for $n$ even. For $n$ odd we obtain an identity which contains a combination of multibrackets of different orders. Thus we can postulate that

$$\frac{1}{(n-1)! n!} \sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \left[ T_{A_{\sigma(1)}},..., T_{A_{\sigma(n)}} \right], T_{A_{\sigma(n+1)}},..., T_{A_{\sigma(2n-1)}} = 0, \quad (6)$$

is the appropriate generalization of the Jacobi Identity for $n$ even. This identity implies the following condition on the structure constants $C^{B}_{A_1...A_n}$:

$$\varepsilon^{B_1...B_{2n-1}}_{A_1...A_{2n-1}} C^{C}_{B_1...B_n} C^{D}_{C B_{n+1}...B_{2n-1}} = 0 \quad (7)$$

which is the generalization of the Jacobi condition [1].

By analogy with the standard Lie algebra, we may now give the following definition [1]:

**Definition 4** Let $\mathcal{G}$ be a Lie algebra and let $n$ be even. A higher-order Lie algebra or multialgebra on $\mathcal{G}$ is the algebra defined by the pair $(\mathcal{G}, [\,\,\,\,])$ where the multibracket $[\,\,\,\,\,\,\,\,]\quad (2)$ is multilinear, antisymmetric and satisfies the generalized Jacobi identity (6); and where the higher-order structure constants satisfy the generalized Jacobi condition (7).

The following definition generalizes the concept of Subalgebra:

**Definition 5** (Submultialgebra): Let $(\mathcal{G}, [\,\,\,\,\,\,\,\,])$ be a multialgebra, and consider the Lie algebra $\mathcal{G}$ of the form $\mathcal{G} = V_0 \oplus V_1$. The subspace $(V_0, [\,\,\,\,\,\,\,\,])$ will be called a submultialgebra of $(\mathcal{G}, [\,\,\,\,\,\,\,\,])$ if it satisfies

$$[V_0, V_0, ..., V_0] \subset V_0. \quad (8)$$

The existence of submultialgebras is reflected in certain definite restrictions on the structure constants. Let $C^{B}_{A_1...A_n}$ be the generalized structure constants of the multialgebra $(\mathcal{G}, [\,\,\,\,\,\,\,\,])$. If $\{T_{A_i}\}$, $\{T_{a^0_i}\}$ and $\{T_{a^1_i}\}$ denote the bases of $\mathcal{G}$, $V_0$ and $V_1$ respectively, where $A_i = 1, ..., \dim \mathcal{G}$, $a^0_i = 1, ..., \dim V_0$ and $a^1_i = \dim V_0 + 1, ..., \dim \mathcal{G}$, then the condition (8) can be expressed as

$$C^{b^1}_{a^1_1...a^0_n} = 0 \quad (9)$$

for $a^0_1...a^0_n \leq \dim V_0$ and $b^1 \geq \dim V_0 + 1$. In fact, If $V_0$ is a submultialgebra then $[V_0, V_0, ..., V_0] \subset V_0$. This mean that

$$\left[ T_{a^0_1}, ..., T_{a^0_n} \right] = C^{d^0}_{a^0_1...a^0_n} T_{d^0}. \quad (10)$$
i.e. for \( \dim V_0 < b^1 < \dim G \) we have \( C_{a_1^0 \ldots a_n^0}^{b_1^1} = 0 \).

The following theorem generalizes the concept of reduction of Lie algebras of ref. [4] to higher-order Lie algebras.

**Theorem 6** (Reduced Multialgebra): Let \(( \mathcal{G}, [\ldots, \ldots] )\) be a multialgebra, and consider the Lie algebra \( G \) of the form \( G = V_0 \oplus V_1 \), with \( \{ T_{A_i} \} \) being a basis for \( G \), \( \{ T_{a_i^0} \} \) a basis for \( V_0 \) and \( \{ T_{a_i^1} \} \) a basis for \( V_1 \). If the condition

\[
[V_1, V_0, \ldots, V_0] \subset V_1,
\]

is satisfied, then the structure constants \( C_{e^1 B_{n+1} \ldots B_{2n-1}}^{a_1^1 \ldots a_n^0} \) are zero, which lead to that the structure constants \( C_{a_1^0 \ldots a_n^0}^{b^1} \) satisfy the generalized Jacobi condition by themselves, and therefore

\[
\begin{bmatrix}
T_{a_i^0}, \ldots, T_{a_n^0}
\end{bmatrix} = C_{a_i^0 \ldots a_n^0}^{b^1} T_{b_i^1}
\]

corresponds by itself to a high-order Lie algebra. This algebra, with structure constants \( C_{a_1^0 \ldots a_n^0}^{b^1} \), is called a reduced multialgebra of \(( \mathcal{G}, [\ldots, \ldots] )\) and is symbolized as \([V_0, [\ldots, \ldots]]\).

**Proof.** If the condition

\[
[V_1, V_0, \ldots, V_0] \subset V_1
\]

is satisfied, we have

\[
\begin{aligned}
\begin{bmatrix}
T_{a_i^0}, \ldots, T_{a_n^0}
\end{bmatrix} &= C_{a_i^0 \ldots a_n^0}^{b^1} T_{b_i^1} + C_{a_i^0 \ldots a_n^0}^{b^1} T_{b_i^1} \\
\begin{bmatrix}
T_{b_i^1}, a_1^0, \ldots, T_{a_n^0}
\end{bmatrix} &= C_{b_i^1 a_1^0 \ldots a_n^0} C_{b_i^1} T_{c_i^1} \\
\begin{bmatrix}
T_{b_i^1}, \ldots, T_{b_h^1}
\end{bmatrix} &= C_{b_i^1 \ldots b_h^1} C_{b_i^1} T_{e_i^1} + C_{b_i^1 \ldots b_h^1} C_{b_i^1} T_{e_i^1}
\end{aligned}
\]

The structure constant of \( G \) satisfy the Jacobi identity

\[
\varepsilon_{A_1 \ldots A_{2n-1}}^{B_1 \ldots B_{2n-1}} C_{B_1 \ldots B_n}^{C} C_{C B_{n+1} \ldots B_{2n-1}}^{D} = 0.
\]

If \( \mathcal{G} = V_0 \oplus V_1 \) y \( \{ T_{A_i} \} \), \( \{ T_{a_i^0} \} \), y \( \{ T_{a_i^1} \} \) are the corresponding bases of \( \mathcal{G} \), \( V_0 \), y \( V_1 \) (where \( A_i = 1, \ldots, \dim \mathcal{G} \), \( a_i^0 = 1, \ldots, \dim V_0 \) and \( a_i^1 = \dim V_0 + 1, \ldots, \dim \mathcal{G} \)), then the generalized Jacobi condition on \( V_0 \) is given by

\[
\varepsilon_{a_1^0 \ldots a_{2n-1}^0}^{B_1 \ldots B_{2n-1}} C_{B_1 \ldots B_n}^{C} C_{C B_{n+1} \ldots B_{2n-1}}^{D} = 0
\]

which can be re-written as

\[
\begin{aligned}
\varepsilon_{a_1^0 \ldots a_{2n-1}^0}^{B_1 \ldots B_{2n-1}} C_{B_1 \ldots B_n}^{C} C_{C B_{n+1} \ldots B_{2n-1}}^{D}
+ \varepsilon_{a_1^0 \ldots a_{2n-1}^0}^{B_1 \ldots B_{2n-1}} C_{B_1 \ldots B_n}^{E} C_{E B_{n+1} \ldots B_{2n-1}}^{F} = 0.
\end{aligned}
\]
The Lie multialgebra obtained from the condition

\[ C_{e_{1}B_{n+1}...B_{2n-1}} = 0 \]

is called a reduced multialgebra of \( G \) and will be symbolized as \( | V_0 | \).

**Definition 7**

We consider now the indices \( B_1, ..., B_{2n-1} \). If one of these indices takes on a value in \( V_1 \), we have

\[
\begin{vmatrix}
\delta_{a_1}^{b_1} & \delta_{a_1}^{b_2} & \cdots & \delta_{a_1}^{b_{2n-1}} \\
\delta_{a_2}^{b_1} & \delta_{a_2}^{b_2} & \cdots & \delta_{a_2}^{b_{2n-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{a_{2n-1}}^{b_1} & \delta_{a_{2n-1}}^{b_2} & \cdots & \delta_{a_{2n-1}}^{b_{2n-1}} \\
\end{vmatrix}
= 0. \quad (17)
\]

From (17) we can see that a column of the determinant is zero and therefore \( b_1^1 b_2^1 ... b_{2n-1}^{2n-1} = 0 \).

Similarly, any permutation on the set \( (b_1^1 b_2^0 ... b_{2n-1}^0) \) in \( V_0 \) will be null. If two indices of the set \( (B_1, ..., B_{2n-1}) \) take on values in \( V_1 \), we have

\[
\begin{vmatrix}
\delta_{a_1}^{b_1} & \delta_{a_1}^{b_2} & \cdots & \delta_{a_1}^{b_{2n-1}} \\
\delta_{a_2}^{b_1} & \delta_{a_2}^{b_2} & \cdots & \delta_{a_2}^{b_{2n-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{a_{2n-1}}^{b_1} & \delta_{a_{2n-1}}^{b_2} & \cdots & \delta_{a_{2n-1}}^{b_{2n-1}} \\
\end{vmatrix}
= 0. \quad (18)
\]

From (18) we can see that a column of the determinant is zero and therefore \( b_1^1 b_2^1 b_3^1 ... b_{2n-1}^1 = 0 \).

In general the number of null columns increase with the number of indices of the set \( (B_1, ..., B_{2n-1}) \), which take on values in \( V_1 \). Thus, the equation (16) is then given by

\[
\begin{vmatrix}
\delta_{a_1}^{b_1} & \delta_{a_1}^{b_2} & \cdots & \delta_{a_1}^{b_{2n-1}} \\
\delta_{a_2}^{b_1} & \delta_{a_2}^{b_2} & \cdots & \delta_{a_2}^{b_{2n-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{a_{2n-1}}^{b_1} & \delta_{a_{2n-1}}^{b_2} & \cdots & \delta_{a_{2n-1}}^{b_{2n-1}} \\
\end{vmatrix}
= 0. \quad (19)
\]

From (19) we can see that the structure constant \( C_{a_1^0 ... a_n}^{d_0} \) satisfy the generalized Jacobi identity by themselves in two cases:

- When \( C_{b_1^1 ... b_0}^{a_1^0 ... a_n} = 0 \), i.e., when \( V_0 \) is a submultialgebra
- When \( C_{e_{1}B_{n+1}...B_{2n-1}} = 0 \), i.e., when \([V_1, V_0, ..., V_0] \subset V_1 \). This means that in this case the structure constant \( C_{a_1^0 ... a_n}^{d_0} \) satisfy the generalized Jacobi identity and

\[
\left[T_{a_1^0}, ..., T_{a_n^0}\right] = C_{a_1^0 ... a_n^0}^{b_0} T_{b_0} \quad (20)
\]

correspond by itself to a higher order Lie algebra. It is interesting to note that a reduced multialgebra \([V_0, [., . ]] \) does not correspond to a submultialgebra of \((\mathcal{G}, [., .])\).
III. S-EXPANSION OF HIGHER-ORDER LIE ALGEBRAS

In this section we shall review some aspects of the S-expansion procedure introduced in ref. [4]. The main point of this section and of this paper is to show that the generalization of the S-expansion method permits obtaining S-expanded higher-order Lie algebras.

A. S-Expansion of Lie Algebras

The S-expansion method is based on combining the structure constants of the Lie algebra \((\mathcal{G}, [],)\) with the inner law of a semigroup \(S\) to define the Lie bracket of a new, S-expanded algebra. Let \(S = \{\lambda_\alpha\}\) be a finite Abelian semigroup endowed with a commutative and associative composition law \(S \times S \rightarrow S, (\lambda_\alpha, \lambda_\beta) \mapsto \lambda_\alpha \lambda_\beta = K_{\alpha\beta}^\gamma \lambda_\gamma\). Let the pair \((\mathcal{G}, [],)\) a Lie algebra where \(G\) is a finite dimensional vector space, with basis \(\{T_A\}\) \(\dim G = 1\), over the field \(K\); and \([,]\) is a ruler of composition \(G \times G \rightarrow G, (T_A, T_A) \mapsto [T_A, T_A] = C_{A,B}^C T_C\). The direct product \(G = S \otimes G\) is defined as the Cartesian product set

\[
\mathfrak{G} = S \times G = \{T_{(A,\alpha)} : \lambda_\alpha \in S, T_A \in \mathcal{G}\}
\]

endowed with a composition law \([,]_S : G \times G \rightarrow G\) defined by

\[
[T_{(A,\alpha)}, T_{(B,\beta)}]_S =: \lambda_\alpha \lambda_\beta [T_A, T_B] = K_{\alpha\beta}^\gamma C_{AB}^C \lambda_\gamma T_C = C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} T_{(C,\gamma)}.
\]

where \(T_{(A,\gamma)} = \lambda_\gamma T_A\) is a basis of \(G\). The set (21) with the composition law (22) is called a S-expanded Lie algebra. This algebra is a Lie algebra structure defined over the vector space obtained by taking \(\text{ord} S\) copies of \(G\)

\[
\mathfrak{G} : \oplus_{\alpha \in S} W_\alpha \quad (W_\alpha \approx \mathcal{G}, \forall \alpha)
\]

\(\dim G = \text{ord} S \times \dim G\) by means of the structure constants

\[
C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = C_{AB}^C \delta_{\alpha\beta}^\gamma
\]

where \(\delta\) is the Kronecker symbol and the subindex \(\alpha, \beta \in S\) denotes the inner composition in \(S\) so that \(\delta_{\alpha\beta}^\gamma = 1\) when \(\alpha\beta = \gamma\) in \(S\) and zero otherwise. The constants \(C_{(A,\alpha)(B,\beta)}^{(C,\gamma)}\) defined by (23) inherit the symmetry properties of \(C_{AB}^C\) of \(G\) by virtue of the abelian character of the \(S\)-product, and satisfy the Jacobi identity.
In a nutshell, the S-expansion method can be seen as the natural generalization of the
Inönü-Wigner contraction, where instead of to multiply the generators by a numerical pa-
rameter, we multiply the generator by the elements of a Abelian semigroup.

**Theorem 8** The product $[,]_S$ defined in (22) is also a Lie product because it is linear,
antisymmetric and satisfies the Jacobi identity. This product defines a new Lie algebra
characterized by the pair $(\mathcal{G},[,]_S)$, and is called a S-expanded Lie algebra.

**Proof.** Since the S-product is abelian, the product $[,]_S$ defined by (22) inherits the sym-
petry properties of the product $[,]$ of $\mathcal{G}$, and satisfies the Jacobi identity. In fact,

\[
\begin{align*}
 &\left[[T_{(A_1,\alpha_1)}, T_{(A_2,\alpha_2)}]_S, T_{(A_3,\alpha_3)}\right]_S + \left[[T_{(A_2,\alpha_2)}, T_{(A_3,\alpha_3)}]_S, T_{(A_1,\alpha_1)}\right]_S \\
&+ \left[[T_{(A_3,\alpha_3)}, T_{(A_1,\alpha_1)}]_S, T_{(A_2,\alpha_2)}\right]_S \\
&= \frac{1}{1!} \sum_{\sigma \in S_3} (-1)^{\pi(\sigma)} \left[[T_{(A_{\sigma(1)},\alpha_{\sigma(1)})}, T_{(A_{\sigma(2)},\alpha_{\sigma(2)})}]_S, T_{(A_{\sigma(3)},\alpha_{\sigma(3)})}\right]_S \\
&= \frac{1}{1!} \sum_{\sigma \in S_3} (-1)^{\pi(\sigma)} \lambda_{\alpha_{\sigma(1)}} \lambda_{\alpha_{\sigma(2)}} \lambda_{\alpha_{\sigma(3)}} \left[[T_{A_{\sigma(1)}}, T_{A_{\sigma(2)}}]_S, T_{A_{\sigma(3)}}\right]_S \\
&= \frac{1}{1!} \sum_{\sigma \in S_3} (-1)^{\pi(\sigma)} K_{\alpha_{\sigma(1)}\alpha_{\sigma(2)}\alpha_{\sigma(3)}}^\gamma \lambda_\gamma \left[[T_{A_{\sigma(1)}}, T_{A_{\sigma(2)}}]_S, T_{A_{\sigma(3)}}\right]_S \\
&= K_{\alpha_1\alpha_2\alpha_3}^\gamma \lambda_\gamma \left(\frac{1}{1!} \sum_{\sigma \in S_3} (-1)^{\pi(\sigma)} \left[[T_{A_{\sigma(1)}}, T_{A_{\sigma(2)}}]_S, T_{A_{\sigma(3)}}\right]_S\right) = 0 (24)
\end{align*}
\]

where we have used the commutativity $K_{\alpha_{\sigma(1)}\alpha_{\sigma(2)}\alpha_{\sigma(3)}}^\gamma K_{\alpha_1\alpha_2\alpha_3}^\gamma = K_{\alpha_1\alpha_2\alpha_3}^\gamma$ and associativity of the
semigroup inner law, and the fact that the product $[,]$ satisfies the Jacobi identity. ■

From (24) we can see that the Jacobi identity of the S-expanded Lie algebra $(S \otimes \mathcal{G},[,]_S)$

\[
\begin{align*}
&\left[[T_{(A_1,\alpha_1)}, T_{(A_2,\alpha_2)}]_S, T_{(A_3,\alpha_3)}\right]_S + \left[[T_{(A_2,\alpha_2)}, T_{(A_3,\alpha_3)}]_S, T_{(A_1,\alpha_1)}\right]_S \\
&+ \left[[T_{(A_3,\alpha_3)}, T_{(A_1,\alpha_1)}]_S, T_{(A_2,\alpha_2)}\right]_S \\
&= 0 (25)
\end{align*}
\]

can be obtained if we multiply the Jacobi identity of the Lie algebra $(\mathcal{G},[,]_S)$ by $\lambda_{\alpha_1} \lambda_{\alpha_2} \lambda_{\alpha_3}$
or by the 3-selector $K_{\alpha_1\alpha_2\alpha_3}^\gamma$:

\[
\text{JI} (S \otimes \mathcal{G},[,]_S) = K_{\alpha_1\alpha_2\alpha_3}^\gamma (\text{JI} (\mathcal{G},[,]_S)). (26)
\]

Similarly, if multiply the Jacobi condition of the Lie algebra $(\mathcal{G},[,]_S)$

\[
\frac{1}{2} \varepsilon_{A_1A_2A_3} C_{B_1B_2}^C C_{B_3}^D = 0 (27)
\]
by \( K_{\alpha_1 \alpha_2 \alpha_3}^\beta = K_{\alpha_1 \alpha_3}^\gamma K_{\gamma \alpha_2}^\beta \), we obtain the Jacobi condition of the \( S \)-expanded Lie algebra \((S \otimes \mathcal{G}, [\cdot, \cdot]_S)\). In fact,

\[
K_{\alpha_1 \alpha_2 \alpha_3}^\beta \left( \frac{1}{2} \varepsilon_{A_1 A_2 A_3} C_{B_1 B_2 C}^{D} C_{B_1 B_2 C}^{E} \right) = \frac{1}{2} \varepsilon_{A_1 A_2 A_3} K_{\alpha_1 \alpha_2}^\gamma C_{B_1 B_2}^{C} K_{\gamma \alpha_3}^\beta C_{CB}^{D} = 0 \quad (28)
\]

\[
\frac{1}{2} \varepsilon_{A_1 A_2 A_3} C_{(B_1, \alpha_1)(B_2, \alpha_2)}^{(C, \gamma)} C_{(C, \gamma)(B_3, \alpha_3)}^{(D, \beta)} = 0. \quad (29)
\]

**B. \( S \)-Expansion of Lie Multialgebras**

The \( S \)-expansion method is based on combining the structure constants of \((\mathcal{G}, [\cdot, \cdot, \cdot]_S)\) with the inner law of a semigroup \( S \) to define the Lie bracket of a new, \( S \)-expanded multialgebra. Let \( S = \{\lambda_\alpha\} \) be a finite Abelian semigroup endowed with a commutative and associative composition law \( S \times S \rightarrow S, (\lambda_\alpha, \lambda_\beta) \mapsto \lambda_\alpha \lambda_\beta = K_{\alpha \beta}^\gamma \lambda_\gamma \). The direct product \( G = S \otimes G \) is defined as the cartesian product set

\[
\mathfrak{G} = S \times G = \{ T_{(A, \alpha)} = \lambda_\alpha T_A : \lambda_\alpha \in S, T_A \in \mathcal{G} \} \quad (30)
\]

with the composition law \([\cdot, \cdot, \cdot]_S : G \times \cdots \times G \rightarrow G\), defined by

\[
\left[ T_{(A_1, \alpha_1)}, \ldots, T_{(A_n, \alpha_n)} \right]_S = \lambda_{\alpha_1} \ldots \lambda_{\alpha_n} \left[ T_{A_1}, \ldots, T_{A_n} \right]
\]

\[
\left[ T_{(A_1, \alpha_1)}, \ldots, T_{(A_n, \alpha_n)} \right]_S = K_{\alpha_1 \ldots \alpha_n}^{\gamma} C_{A_1 \ldots A_n}^{\gamma} \lambda_{\gamma} T_C = C_{(A_1, \alpha_1) \ldots (A_n, \alpha_n)}^{(C, \gamma)} T_{(C, \gamma)} \quad (31)
\]

where \( T_{(A_i, \alpha_i)} \in G, \forall i = 1, \ldots, n, \) and \( C_{(A_1, \alpha_1) \ldots (A_n, \alpha_n)}^{(C, \gamma)} = K_{\alpha_1 \ldots \alpha_n}^{\gamma} C_{A_1 \ldots A_n}^{\gamma} \).

The set \( G = S \times G \) with the composition law \((31)\) define a new Lie multialgebra which will be called \( S \)-expanded Lie multialgebra. This algebra is a Lie algebra structure defined over the vector space obtained by taking \( S \) copies of \( G \) by means of the structure constant \( C_{(A_1, \alpha_1) \ldots (A_n, \alpha_n)}^{(C, \gamma)} = K_{\alpha_1 \ldots \alpha_n}^{\gamma} C_{A_1 \ldots A_n}^{\gamma} \) where \( K_{\alpha_1 \ldots \alpha_n}^{\gamma} = K_{\alpha_1 \ldots \alpha_{n-1}, \sigma_{\alpha_n}}^{\gamma} \). The structure constants \( C_{(A_1, \alpha_1) \ldots (A_n, \alpha_n)}^{(C, \gamma)} \) defined in \((31)\) inherit the symmetry properties of \( C_{A_1 \ldots A_n}^{\gamma} \) of \( G \) by virtue of the abelian character of the \( S \)-product.

**Theorem 9** The product \([\cdot, \cdot, \cdot]_S \) defined in \((31)\) is multilinear, antisymmetric and satisfies the generalized Jacobi identity (GJI).

\[
a \sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \left[ T_{(A_{\sigma(1)}, \alpha_{\sigma(1)})}, \ldots, T_{(A_{\sigma(n)}, \alpha_{\sigma(n)})} \right]_S \left[ T_{(A_{\sigma(n+1)}, \alpha_{\sigma(n+1)})}, \ldots, T_{(A_{\sigma(2n-1)}, \alpha_{\sigma(2n-1)})} \right]_S = 0 \quad (32)
\]
where

\[ a = \frac{1}{(n-1)! n!} \]

**Proof.** Since the \( S \)-product is abelian, the product \([, ..., S]\) defined by (31) inherits the symmetry properties of the product \([, ..., S]\) of \((G, [, ..., S])\), and satisfies the generalized Jacobi identity. In fact,

\[
\sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \left[ T(A_{\sigma(1)}, \alpha_{\sigma(1)}), ..., T(A_{\sigma(n)}, \alpha_{\sigma(n)}) \right]_{S}, T(A_{\sigma(n+1)}, \alpha_{\sigma(n+1)}), ..., T(A_{\sigma(2n-1)}, \alpha_{\sigma(2n-1)})_S \\
= \sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \lambda_{\alpha_{\sigma(1)}} \cdots \lambda_{\alpha_{\sigma(2n-1)}} \left[ T(A_{\sigma(1)}, ..., T(A_{\sigma(n)}), T(A_{\sigma(n+1)}, ..., T(A_{\sigma(2n-1)})_S \\
= \sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} K_{\alpha_{\sigma(1)}-\alpha_{\sigma(2n-1)}} \lambda_\gamma \left[ T(A_{\sigma(1)}, ..., T(A_{\sigma(n)}), T(A_{\sigma(n+1)}, ..., T(A_{\sigma(2n-1)}_S \\
= K_{\alpha_1...\alpha_{2n-1}}^\gamma \lambda_\gamma \left( \sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \left[ T(A_{\sigma(1)}, ..., T(A_{\sigma(n)}), T(A_{\sigma(n+1)}, ..., T(A_{\sigma(2n-1)}) \right) \right) = 0, \quad (33)
\]

where we have used the commutativity \( K_{\alpha_{\sigma(1)}-\alpha_{\sigma(2n-1)}} = K_{\alpha_1...\alpha_{2n-1}} \) and associativity of the semigroup inner law, and the fact that the product \([, ..., S]\) satisfies the generalized Jacobi identity.

From (33) we can see that the Jacobi identity of the \( S \)-expanded Lie multialgebra \((S \otimes G, [, ..., S])\) can be obtained if we multiply the generalized Jacobi identity of the Lie multialgebra \((G, [, ..., S])\) by \( K_{\alpha_1...\alpha_{2n-1}}^\gamma \).

Similarly, if we multiply the generalized Jacobi condition of the Lie algebra \((G, [, ..., S])\)

\[
\varepsilon_{B_1...B_{2n-1}} C_{C, C, C} B_1...B_n C_{C, C, C, C, C} B_{2n-1} = 0 \quad (34)
\]

by \( K_{\alpha_1...\alpha_{2n-1}}^\beta = K_{\alpha_1...\alpha_n}^\gamma K_{\gamma, \alpha_{n+1}...\alpha_{2n-1}}^\beta \), we obtain the generalized Jacobi condition of the \( S \)-expanded Lie multialgebra \((G, [, ..., S])\). In fact,

\[
K_{\alpha_1...\alpha_{2n-1}}^\beta \left( \varepsilon_{B_1...B_{2n-1}} C_{C, C, C} B_1...B_n C_{C, C, C, C, C} B_{2n-1} \right) = 0 \quad (35)
\]

\[
\varepsilon_{A_1...A_{2n-1}} K_{\alpha_1...\alpha_n} C_{C, C, C} B_1...B_n K_{\gamma, \alpha_{n+1}...\alpha_{2n-1}}^\beta C_{C, C, C, C, C} B_{2n-1} = 0 \quad (36)
\]

C. Multialgebra 0\(_S\)-Reduced

When the semigroup has a zero element 0\(_S\) \( \in S \), it plays a somewhat peculiar role in the \( S \)-expanded Lie multialgebra. Let us span \( S \) in nonzero elements \( \lambda_i, i = 0, \cdots, N \), and a zero
element $\lambda_{N+1} = 0_S$, i.e.,

$$S = \left\{ \lambda_0, \lambda_1, \ldots, \lambda_N, \lambda_{N+1} \right\}. \quad (37)$$

Then, the 2-selector satisfies

$$K_{N+1,i_2,\ldots,i_n}^j = K_{N+1,\ldots,N+1}^j = 0,$$

$$K_{N+1,i_2,\ldots,i_n}^{N+1} = K_{N+1,\ldots,N+1}^{N+1} = 1. \quad (38)$$

Therefore, the $S$-expanded multialgebra $(G, [\ldots, \ldots, \ldots_\downarrow S \downarrow, \ldots])$ can be split as

$$[T(A_1,i_1), \ldots, T(A_n,i_n)]_S = K_{i_1,\ldots,i_n}^k C_{A_1\ldots A_n} C T(C,k) + K_{i_1,\ldots,i_n}^{N+1} C_{A_1,\ldots,A_n} C T(C,N+1)$$

$$[T(A_1,N+1), T(A_2,i_2), \ldots, T(A_n,i_n)]_S = C_{A_1,\ldots,A_n} C T(C,N+1)$$

$$\vdots$$

$$[T(A_1,N+1), \ldots, T(A_r,N+1), T(A_{r+1},i_{r+1}), \ldots, T(A_n,i_n)]_S = C_{A_1,\ldots,A_n} C T(C,N+1)$$

$$\vdots$$

$$[T(A_1,N+1), \ldots, T(A_n,N+1)]_S = C_{A_1,\ldots,A_n} C T(C,N+1). \quad (39)$$

From (39) we can see that $(G, [\ldots, \ldots, \ldots_\downarrow S \downarrow, \ldots])$ can be written as $G = V_0 \oplus V_1$, with $V_0 = \{T(A,i)\}$, $V_1 = \{T(A,N+1)\}$. From (39) we also see that

$$[V_1, V_0, \ldots, V_0]_S \subset V_1 \quad (40)$$

$$\left[ V_1, \ldots, V_1, V_0, \ldots, V_0 \right]_S \subset V_1, \quad \text{con } r = 1, \ldots, n. \quad (41)$$

This means that the commutation relations

$$[T(A_1,i_1), \ldots, T(A_n,i_n)]_S = K_{i_1,\ldots,i_n}^k C_{A_1\ldots A_n} C T(C,k)$$

are those of a reduced Lie multialgebra $(G, [\ldots, \ldots, \ldots_\downarrow S \downarrow, \ldots])$. From (39) we see that the reduction procedure in this particular case is equivalent to imposing the condition

$$T_{(C,N+1)} = 0_S T_C = 0.$$

The above considerations motivate the following definition:

**Definition 10** Let $S$ be an Abelian semigroup with a zero element $0_S \in S$, and let $(S \otimes G, [\ldots, \ldots])$ be an $S$-expanded multialgebra. The multialgebra obtained by imposing the condition $0_S T_A = 0$ on $G$ is called a $0_S$-reduced multialgebra of $G$. 12
IV. S-EXPANSION OF SUBMULTIALGEBRAS

In this section is shown that there are at least two ways of extracting smaller multialgebras from \((S \otimes \mathcal{G}, [\ldots, \ldots])\). The first one gives rise to a “resonant submultialgebra” while the second produces reduced multialgebras of a resonant submultialgebra.

A. Resonant submultialgebras

The general problem of finding submultialgebras from an \(S\)-expanded multialgebra is a nontrivial one, which is met and solved in this section. In order to provide a solution, one must have some information about the subspace structure of \(\mathcal{G}, [\ldots, \ldots]\). This information is encoded in the following way:

Let \(\mathcal{G} = \bigoplus_{p \in I} V_p\) be a decomposition of \(\mathcal{G}\) in subspaces \(V_p\), where \(I\) is a set of indices. For each \((p_1, \ldots, p_n) \in I\) it is always possible to define \(i_{(p_1, \ldots, p_n)} \subset I\) such that

\[
[V_{p_1}, \ldots, V_{p_n}] \subset \bigoplus_{r \in i_{(p_1, \ldots, p_n)}} V_r. \tag{42}
\]

In this way, the subsets \(\{i_{(p_1, \ldots, p_n)}\}\) store the information on the subspace structure of \(\mathcal{G}\).

As for the Abelian semigroup \(S\), this can always be decomposed as \(S = \bigcup_{p \in I} S_p\), where \(S_p \subset S\). In principle, this decomposition is completely arbitrary; however, using the product from definition (2.2) of ref. [4], it is sometimes possible to pick out a very particular choice of subset decomposition. This choice is the subject of the following definition:

**Definition 11** Let \(\mathcal{G} = \bigoplus_{p \in I} V_p\) be a decomposition of \(\mathcal{G}\) in subspaces \(V_p\), with a structure described by the subsets \(i_{(p_1, \ldots, p_n)}\), as in Eq. (42). Let \(S = \bigcup_{p \in I} S_p\) be a resonant subset decomposition of the Abelian semigroup \(S\) such that

\[
S_{p_1} \times S_{p_2} \times \cdots \times S_{p_n} \subset \bigcap_{r \in i_{(p_1, \ldots, p_n)}} S_r. \tag{43}
\]

When such a subset decomposition \(S = \bigcup_{p \in I} S_p\) exists, then we say that this decomposition is in resonance with the subspace decomposition of \(\mathcal{G} = \bigoplus_{p \in I} V_p\).

**Theorem 12** Let \(\mathcal{G} = \bigoplus_{p \in I} V_p\) be a subspace decomposition of \(\mathcal{G}\), with a structure described by Eq. (42), and let \(S = \bigcup_{p \in I} S_p\) be a resonant subset decomposition of the Abelian semigroup
S, with the structure given in Eq. (13). Define the subspaces $W_p$ of $\mathfrak{S} = S \otimes \mathcal{G}$,

$$W_p = S_p \otimes V_p, \quad p \in I.$$  \hspace{1cm} (44)

Then,

$$\mathfrak{S}_R = \oplus_{p \in I} W_p$$

is called a resonant subalgebra of the S-expanded multialgebra $\mathfrak{S} = S \otimes \mathcal{G}$.

Proof. Using Eqs. (42) and (43) we have

$$[W_{p_1}, ..., W_{p_n}]_S = [S_{p_1} \otimes V_{p_1}, ..., S_{p_n} \otimes V_{p_n}]_S = (S_{p_1} \times \cdots \times S_{p_n}) \otimes [V_{p_1}, ..., V_{p_n}]$$

$$\subset \left( \bigcap_{s \in (p_1, \ldots, p_n)} S_s \right) \otimes \left( \biguplus_{r \in (p_1, \ldots, p_n)} V_r \right) = \biguplus_{r \in (p_1, \ldots, p_n)} \left( \bigcap_{s \in (p_1, \ldots, p_n)} S_s \right) \otimes V_r.$$  \hspace{1cm} (46)

But, it is clear that for each $r \in i(p_1, \ldots, p_n)$ one can write

$$\bigcap_{s \in (p_1, \ldots, p_n)} S_s \subset S_r.$$  \hspace{1cm} (47)

Then,

$$[W_{p_1}, ..., W_{p_n}]_S \subset \biguplus_{r \in (p_1, \ldots, p_n)} S_r \otimes V_r = \bigoplus_{r \in (p_1, \ldots, p_n)} W_r$$

$$[W_{p_1}, ..., W_{p_n}]_S \subset \biguplus_{r \in (p_1, \ldots, p_n)} S_r \otimes V_r = \bigoplus_{r \in (p_1, \ldots, p_n)} W_r$$

$$[W_{p_1}, ..., W_{p_n}]_S \subset \bigoplus_{r \in I} W_r = \mathfrak{S}_R$$  \hspace{1cm} (48)

Therefore, the algebra closes and $\mathfrak{S}_R$ is a submultialgebra of $\mathfrak{S}$.

This theorem translates the difficult problem of finding subalgebras from an $S$-expanded algebra $\mathfrak{S} = S \otimes \mathcal{G}$ into that of finding a resonant partition for the semigroup $S$.

Denoting the basis of $V_p$ by $\{T_{a_p}\}$, $\lambda_{a_p} \in S_p$, and $T_{(a_{p_1}, \ldots, a_{p_n})} = \lambda_{a_{p_1}} T_{a_{p_1}} \in W_{p_1}$ one can write

$$\left[ T_{(a_{p_1}, a_{p_2})}, \ldots, T_{(a_{p_n}, a_{p_1})} \right]_S = C_{(a_{p_1}, a_{p_2}) \ldots (a_{p_n}, a_{p_1})} (c_r, \gamma_r) T_{(c_r, \gamma_r)},$$

which means that the structure constants of the resonant submultialgebra are given by

$$C_{(a_{p_1}, a_{p_2}) \ldots (a_{p_n}, a_{p_1})} (c_r, \gamma_r) = K_{\alpha_{p_1} \ldots \alpha_{p_n}} \gamma_r C_{\alpha_{p_1} \ldots \alpha_{p_n}} c_r.$$  \hspace{1cm} (49)

An interesting fact is that the $S$-expanded multialgebra "subspace structure" encoded in $i(p_1, \ldots, p_n)$ is the same as in the original multialgebra, as can be observed from Eq. (48).
B. Reduced Multialgebras of a Resonant Submultialgebra

The following theorem provides necessary conditions under which a reduced multialgebra can be extracted from a resonant subalgebra:

**Theorem 13** Let $\mathcal{G}_R = \bigoplus_{p \in I} S_p \otimes V_p$ be a resonant submultialgebra $(\mathcal{G}, [\ldots, \ldots])$, i.e., let Eqs. (42) and (43) be satisfied. Let $S_p = \hat{S}_p \cup \check{S}_p$ be a partition of the subsets $S_p \subset S$ such that

\begin{equation}
\hat{S}_{p_1} \cap \check{S}_{p_1} = \phi
\end{equation}

\begin{equation}
\hat{S}_{p_1} \times \hat{S}_{p_2} \times \ldots \times \hat{S}_{p_n} \subset \bigcap_{r \in \{p_1, \ldots, p_n\}} \check{S}_r.
\end{equation}

The conditions (49) and (50) induce the decomposition $\mathcal{G}_R = \check{\mathcal{G}}_R \oplus \hat{\mathcal{G}}_R$ on the resonant subalgebra, where

\begin{equation}
\check{\mathcal{G}}_R = \bigoplus_{p \in I} \check{S}_p \otimes V_p
\end{equation}

\begin{equation}
\hat{\mathcal{G}}_R = \bigoplus_{p \in I} \hat{S}_p \otimes V_p.
\end{equation}

When conditions (49) and (50) hold, then

\begin{equation}
\left[ \check{\mathcal{G}}_R, \check{\mathcal{G}}_R, \ldots, \check{\mathcal{G}}_R \right]_S \subset \check{\mathcal{G}}_R
\end{equation}

and therefore $|\mathcal{G}_R|$ corresponds to a reduced algebra of $\mathcal{G}_R$.

**Proof.** $\check{W}_{p_1} = \hat{S}_{p_1} \otimes V_{p_1}$ and $\check{W}_{p_i} = \hat{S}_{p_i} \otimes V_{p_i}$. Then, using condition (50), we have:

\[
\left[ \check{W}_{p_1}, W_{p_2}, \ldots, W_{p_n} \right]_S = \left[ \hat{S}_{p_1} \otimes V_{p_1}, \hat{S}_{p_2} \otimes V_{p_2}, \ldots, \hat{S}_{p_n} \otimes V_{p_n} \right]_S
\]

\[
= \left( \hat{S}_{p_1} \times \hat{S}_{p_2} \times \ldots \times \hat{S}_{p_n} \right) \otimes \left[ V_{p_1}, V_{p_2}, \ldots, V_{p_n} \right]
\]

\[
\subset \bigg( \bigcap_{s \in \{p_1, \ldots, p_n\}} \hat{S}_s \bigg) \otimes \left( \bigoplus_{r \in \{p_1, \ldots, p_n\}} V_r \right)
\]

\[
= \bigoplus_{r \in \{p_1, \ldots, p_n\}} \left( \bigcap_{s \in \{p_1, \ldots, p_n\}} \hat{S}_s \right) \otimes V_r.
\]

For each $r \in \{p_1, \ldots, p_n\}$ we have

\[
\bigcap_{s \in \{p_1, \ldots, p_n\}} \hat{S}_s \subset \check{S}_r
\]
so that,

\[
\left[ \hat{W}_{p_1}, \hat{W}_{p_2}, \ldots, \hat{W}_{p_n} \right]_S \subset \bigoplus_{r \in I(p_1, \ldots, p_n)} \hat{S}_r \otimes V_r = \bigoplus_{r \in I(p_1, \ldots, p_n)} \hat{W}_r \\
\subset \bigoplus_{r \in I} \hat{W}_r = \hat{\mathfrak{g}}_R.
\]

Thus \( \left[ \hat{W}_{p_1}, \hat{W}_{p_2}, \ldots, \hat{W}_{p_n} \right]_S \subset \hat{\mathfrak{g}}_R \), i.e,

\[
\left[ \hat{\mathfrak{g}}_R, \hat{\mathfrak{g}}_R, \ldots, \hat{\mathfrak{g}}_R \right]_S \subset \hat{\mathfrak{g}}_R
\]

and therefore \(|\hat{\mathfrak{g}}_R|\) is a reduced algebra of \(\mathfrak{g}_R\).

The structure constants for the reduced algebra \(|\hat{\mathfrak{g}}_R|\) are given by,

\[
C_{(\alpha_{p_1}, \gamma_{r}) - (\alpha_{p_n}, \alpha_{p_n})}^{(c_r, \gamma_r)} = K_{\alpha_{p_1} \ldots \alpha_{p_n}}^{\gamma_r} C_{\alpha_{p_1} \ldots \alpha_{p_n}}^{c_r}
\]

with \(\alpha_{p_i}, \gamma_r\) such that \(\lambda_{\alpha_{p_i}} \in \hat{S}_{p_i} \) y \(\lambda_{\gamma_r} \in \hat{S}_r\).

C. \(S_E^{(N)}\)-Expansion of Multialgebras

Definition 14 Let us define \(S_E^{(N)}\) as the semigroup of elements

\[
S_E^{(N)} = \{\lambda_\alpha, \alpha = 0, \ldots, N + 1\}
\]

provided with a multiplication rule

\[
\lambda_\alpha \lambda_\beta = \lambda_{H_{N+1}(\alpha + \beta)} = \delta_{H_{N+1}(\alpha + \beta)}^{\gamma} \lambda_\gamma
\]

where \(H_{N+1}\) is defined as the function

\[
H_{n}(x) = \begin{cases} \\
\text{x, when } x < n, \\
\text{n, when } x \geq n.
\end{cases}
\]

The two-selectors for \(S_E^{(N)}\) read

\[
K_{\alpha \beta}^{\gamma} = \delta_{H_{N+1}(\alpha + \beta)}^{\gamma}
\]

where \(\delta_\alpha^{\beta}\) is the Kronecker delta.
The multiplication rule (55) can be directly generalized to
\[ \lambda_{\alpha_1} \ldots \lambda_{\alpha_n} = \lambda_{H^{N+1}(\alpha_1 + \ldots + \alpha_n)} = \delta_{H^{N+1}(\alpha_1 + \ldots + \alpha_n)}^\gamma \lambda_\gamma \] (57)
\[ K_{\alpha_1 \ldots \alpha_n} \gamma = \delta_{H^{N+1}(\alpha_1 + \ldots + \alpha_n)}^\gamma. \]

From Eq. (55), we have that \( \lambda_{N+1} \) is the zero element in \( S_E^{(N)} \), i.e., \( \lambda_{N+1} = 0 \).

The corresponding \( S \)-expanded multialgebra is given by the following commutation relation:
\[ [T(A_1, \alpha_1), \ldots, T(A_n, \alpha_n)]_S = \delta_{H^{N+1}(\alpha_1 + \ldots + \alpha_n)}^\gamma C_{A_1 \ldots A_n} T(C, \gamma), \] (58)
which implies that the structure constants for the \( S_E^{(N)} \)-expanded multialgebra can be written as
\[ C_{(A_1, \alpha_1) \ldots (A_n, \alpha_n)}^{(C, \gamma)} = \delta_{H^{N+1}(\alpha_1 + \ldots + \alpha_n)}^\gamma C_{A_1 \ldots A_n} \] (59)
with \( \gamma, \alpha_1, \ldots, \alpha_n = 0, \ldots, N+1 \). When the condition of \( 0_S \)-reduction is imposed, the Eq. (59) reduces to
\[ C_{(A_1, i_1) \ldots (A_n, i_n)}^{(C, k)} = \delta_{H^{N+1}(i_1 + \ldots + i_n)}^k C_{A_1 \ldots A_n}. \]

V. COMMENTS

We have shown that the successful \( S \)-expansion of the Lie algebras method, developed in ref. [4], can be generalized so as to obtain expanded higher-order Lie algebras.

The main results of this paper are: the generalizations of the definitions of Lie subalgebras and reduced Lie algebras to higher-order Lie subalgebras and higher-order reduced Lie algebras; to generalize the \( S \)-expansion method and to show that it is possible to obtain higher-order expanded Lie algebras, as well as to prove that under determined conditions can be extracted relevant higher-order Lie subalgebras from the \( S \)-expanded higher-order Lie algebras.

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VI. APPENDIX A

In this appendix we show that the realization (1) of the multibracket satisfies the identity

\[
\frac{1}{(n-1)!} \frac{1}{n!} \sum_{\sigma \in S_{2n-1}} (-1)^{\sigma(n)} \left[ T_{A_1}, \ldots, T_{A_{n(n+1)}}, \ldots, T_{A_{2n-1}} \right] \tag{60}
\]

\[
= \begin{cases} 
0, & n \text{ even} \\
 n \left[ T_{A_1}, \ldots, T_{A_{2n-1}} \right], & n \text{ odd.}
\end{cases}
\]

which can be re-written in the following way:

\[
\frac{1}{(n-1)!} \frac{1}{n!} \sum_{\sigma \in S_{2n-1}} (-1)^{\sigma(n)} \left[ T_{B_1}, \ldots, T_{B_n}, T_{B_{n+1}}, \ldots, T_{B_{2n-1}} \right] \tag{61}
\]

\[
= \begin{cases} 
0, & n \text{ even} \\
 nn! (n-1)! \left[ T_{A_1}, \ldots, T_{A_{2n-1}} \right], & n \text{ odd.}
\end{cases}
\]

In fact, if

\[
\varphi = \varepsilon_{B_1 \ldots B_{2n-1}} \left[ T_{B_1}, \ldots, T_{B_n}, T_{B_{n+1}}, \ldots, T_{B_{2n-1}} \right], \tag{62}
\]

then

\[
\varphi = \varepsilon_{C_1 \ldots C_n B_1 \ldots B_n} T_{C_1} \ldots T_{C_n}, T_{B_{n+1}}, \ldots, T_{B_{2n-1}} \tag{63}
\]

\[
= \varepsilon_{A_1 \ldots A_{2n-1}} C_1 \ldots C_n \varepsilon_{B_1 \ldots B_n} \left[ T_{C_1} \ldots T_{C_n}, T_{B_{n+1}}, \ldots, T_{B_{2n-1}} \right] \tag{64}
\]

\[
= n! \varepsilon_{C_1 \ldots C_n B_{n+1} \ldots B_{2n-1}} A_1 \ldots A_{2n-1} \left[ T_{C_1}, \ldots T_{C_n}, T_{B_{n+1}}, \ldots, T_{B_{2n-1}} \right]
\]

where we have used Eq. (4) and the property

\[
\varepsilon_{i_1 \ldots i_r} B^{h_1 \ldots h_r} = r! B^{i_1 \ldots i_r}. \tag{64}
\]

We consider now the multibracket \([T_{C_1} \ldots T_{C_n}, T_{B_{n+1}}, \ldots, T_{B_{2n-1}}]\). The expression \(T_{C_1} \ldots T_{C_n}\) is the matrix product of \(n\) elements, and therefore is a mapping onto another element of \(G\), which must be antisymmetrized together with \(T_{B_{n+1}}, \ldots, T_{B_{2n-1}}\). Thus, we can write

\[
[T_{C_1} \ldots T_{C_n}, T_{B_{n+1}}, \ldots, T_{B_{2n-1}}] \tag{65}
\]

\[
= \varepsilon_{B_{n+1} \ldots B_{2n-1}} \sum_{s=0}^{n-1} (-1)^s T_{C_{n+1}} \ldots T_{C_{n+s}} T_{C_1} \ldots T_{C_n} T_{C_{n+s+1}} \ldots T_{C_{2n-1}}
\]
where the \( n - 1 \) elements \( T_{B_{n+1}}, \ldots, T_{B_{2n-1}} \) are antisymmetrized with the contraction with \( \varepsilon_{B_{n+1} \ldots B_{2n-1}} \) and the element \( T_{C_1} \ldots T_{C_n} \) is antisymmetrized with \( \sum \). Introducing these results into (63) we have

\[
\varphi = n! \varepsilon_{A_1 \ldots A_{2n-1}} T_{C_{n+1}} \ldots T_{C_{n+s}} T_{C_1} \ldots T_{C_n} T_{C_{n+s+1}} \ldots T_{C_{2n-1}} \]

\[
\times \sum_{s=0}^{n-1} (-1)^s T_{C_{n+1}} \ldots T_{C_{n+s}} T_{C_1} \ldots T_{C_n} T_{C_{n+s+1}} \ldots T_{C_{2n-1}} \]

\[
= n! (n-1)! \varepsilon_{A_1 \ldots A_{2n-1}} \sum_{s=0}^{n-1} (-1)^s T_{C_{n+1}} \ldots T_{C_{n+s}} T_{C_1} \ldots T_{C_n} T_{C_{n+s+1}} \ldots T_{C_{2n-1}} \]

where we have used the identity (64). Since

\[
\varepsilon_{A_1 \ldots A_{2n-1}} T_{C_{n+1}} \ldots T_{C_{n+s}} T_{C_1} \ldots T_{C_n} T_{C_{n+s+1}} \ldots T_{C_{2n-1}} \]

\[
= (-1)^s \varepsilon_{A_1 \ldots A_{2n-1}} T_{C_1} T_{C_{n+1}} \ldots T_{C_{n+s}} T_{C_2} \ldots T_{C_n} T_{C_{n+s+1}} \ldots T_{C_{2n-1}} \]

\[
= (-1)^s (-1)^s \varepsilon_{A_1 \ldots A_{2n-1}} T_{C_1} T_{C_2} T_{C_{n+1}} \ldots T_{C_{n+s}} T_{C_3} \ldots T_{C_n} T_{C_{n+s+1}} \ldots T_{C_{2n-1}} \]

\[
\vdots
\]

\[
= (-1)^{ns} \varepsilon_{A_1 \ldots A_{2n-1}} T_{C_1} \ldots T_{C_n} T_{C_{n+1}} \ldots T_{C_{n+s}} T_{C_{n+s+1}} \ldots T_{C_{2n-1}} \]

\[
= (-1)^{ns} \varepsilon_{A_1 \ldots A_{2n-1}} T_{C_1} \ldots T_{C_{2n-1}}.
\]

we have that (66) takes the form

\[
\varphi = n! (n-1)! \sum_{s=0}^{n-1} (-1)^s (-1)^s \varepsilon_{A_1 \ldots A_{2n-1}} T_{C_1} \ldots T_{C_{2n-1}} \]

\[
= n! (n-1)! \varepsilon_{A_1 \ldots A_{2n-1}} T_{C_1} \ldots T_{C_{2n-1}} \sum_{s=0}^{n-1} (-1)^s (-1)^s \]

\[
= n! (n-1)! \left[T_{A_1}, \ldots, T_{A_{2n-1}}\right] \sum_{s=0}^{n-1} (-1)^s (n+1) .
\]

It is direct to check that

\[
\sum_{s=0}^{n-1} (-1)^s (n+1) = \begin{cases} 0, & \text{for } n \text{ even} \\ n, & \text{for } n \text{ odd} \end{cases}.
\]
Using (62) we find
\[
\frac{1}{n!} \frac{1}{(n-1)!} \varepsilon_{A_1...A_{2n-1}} B_1...B_{2n-1} \left[ [T_{B_1}, ..., T_{B_n}], T_{B_{n+1}}, ..., T_{B_{2n-1}} \right]
\]
\[
= \begin{cases} 
0, & \text{for } n \text{ even} \\
n \left[ T_{A_1}, ..., T_{A_{2n-1}} \right], & \text{for } n \text{ odd}
\end{cases}
\]
or
\[
\frac{1}{(n-1)!n!} \sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \left[ [T_{A_{\sigma(1)}}, ..., T_{A_{\sigma(n)}}], T_{A_{\sigma(n+1)}}, ..., T_{A_{\sigma(2n-1)}} \right]
\]
\[
= \begin{cases} 
0, & \text{for } n \text{ even} \\
n \left[ T_{A_1}, ..., T_{A_{2n-1}} \right], & \text{for } n \text{ odd}
\end{cases}
\].

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[8] where the order of the multialgebra is denoted by \( n \) and \( N \) denotes the number of elements of the semigroup \( S_E^{(N)} \).