Smart fuzzy graph

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Abstract: In recent times, the internet of things has gained more popularity. The internet of things connects people, objects and machines apart from distance through cloud based internet. In this paper, we have introduced the smart fuzzy graph along with some basic properties and related concepts. The aim of this paper is to solve some authentic real life problems using smart fuzzy graph.

Keyword: Smart fuzzy graph.

1. INTRODUCTION

Graph theory is useful in modelling fundamental attributes of given systems which consist of limited components. These can be utilized to constitute railway lines, communication network, traffic system, and so on. A given set of objects can be represented using a graph where each object is considered as vertex whereas the relation among these objects is entitled as an edge. Relationship among the items need not be absolutely characterized and hence the concept of an imprecise and the fuzziness arises.

The concept of uncertainty and vulnerability, in real life was presented by L.A. Zadeh in 1965, which he termed as fuzzy sets and fuzzy logics. The primary definition of a fuzzy graph was coined by Kaufman. In later period the establishment for fuzzy graph theory was proposed by Rosenfeld, Yeh and Bang. The relationship between the objects was given by A. Rosenfeld using fuzzy relation. Additionally he presented the analogues of several premise fuzzy graph-theoretic concepts, which include trees, forests, sub-graphs, connectedness, etc. Furthermore major contribution notably as connectivity on vertex and edge were presented by Yeh and Bang in fuzzy graphs based on Zadeh’s concept of fuzzy.

In recent time advances in connecting people around the globe through smart phones, computers, smart appliances, advanced network, etc., these smart things together has gained more popularity. This kind of connection between the people and the smart things is made possible through the internet of things, which acts as a platform for the smart appliances to work independently. The connected smart appliances speak among themselves to perform the scheduled or the un-scheduled task efficiently.

In this paper, the concept smart fuzzy graph has been introduced. The internet of things as mentioned above can be taken as a smart fuzzy graph and solved to find the connection among the elements in the network. This paper also provides some basic properties and related concepts. Also in this paper we have tried to solve some real life problems using smart fuzzy graph.
2. PRELIMINARIES:
This section contains some basic definition from [4] and [7].

2.1. Definition
A fuzzy graph \( G = (V, \sigma, \mu) \) is a triple consisting of a non empty set \( V \) together with a pair of functions \( \sigma : V \to [0,1] \) and \( \mu: E \to [0,1] \) such that for all \( x, y \in V, \mu(xy) \leq \sigma(x) \land \sigma(y) \). In the fuzzy set, the vertex set is denoted as \( \sigma \) and the edge set is denoted as \( \mu \). For convenience, we simply denote it as \( G \) or \( (\sigma, \mu) \) to represent the fuzzy graph \( G = (V, \sigma, \mu) \), where \( V \) is a finite set.

The fuzzy graph \( G = (\sigma, \mu) \) has a crisp graph that is denoted as \( G^* = (\sigma^*, \mu^*) \) where \( \sigma^* = \{u \in V | \sigma(u) > 0\} \), \( \mu^* = \{(u,v) \in E | \mu(u,v) > 0\} \).

2.2. Definition
A fuzzy graph \( H = (V, \tau, \nu) \) is a partial fuzzy subgraph of \( G \) if \( \tau \subseteq \sigma \) and \( \nu \subseteq \mu \). Similarly the fuzzy graph \( H = (V, \tau, \nu) \) is called a fuzzy subgraph of \( G \) induced by \( P \) if \( P \subseteq V \), \( \tau(x) = \sigma(x) \) for all \( x \in P \) and \( \nu(xy) = \mu(xy) \) for all \( x, y \in P \). The fuzzy sub graph induced by \( P \) is denoted as \( \langle P \rangle \).

2.3. Definition
Let \( G = (\sigma, \mu) \) be a fuzzy graph. Then a partial fuzzy subgraph \( (\tau, \nu) \) of \( G \) is said to span \( G \) if \( \sigma = \tau \). In this case, we call \( (\tau, \nu) \) a spanning fuzzy subgraph of \( (\sigma, \mu) \).

2.4. Definition
A fuzzy graph \( G = (\sigma, \mu) \) is said to be a complete fuzzy graph if \( \mu(x, y) = \sigma(x) \land \sigma(y) \) for all \( u, v \) in \( \sigma^* \).

2.5. Definition
An edge \((x, y)\) in \( \mu^* \) is said to be an effective edge if \( \mu(x, y) = \sigma(x) \land \sigma(y) \). A fuzzy graph \( G \) is said to be a strong fuzzy graph if \( \mu(x, y) = \sigma(x) \land \sigma(y) \) for all \((x, y)\) in \( \mu^* \).

2.6. Definition
If \( G = (\sigma, \mu) \) is denoted as a fuzzy graph, then the complement of the graph is defined as \( \bar{G} = (\sigma, \bar{\mu}) \) whereas \( \bar{\mu} = (x, y) = \sigma(x) \land \sigma(y) - \mu(x, y) \) for all \( x, y \in S \).

2.7. Definition
A node in the fuzzy graph \( G \) is said to be busy node if \( \sigma(x) \leq d(v) \) or else it is called as a free node.

2.8. Definition
The strength of connectedness between two vertices \( u \) and \( v \) is defined as the maximum strengths of all paths between \( u \) and \( v \) is denoted by \( \mu^*(u, v) \) or \( \text{CONN}_{\bar{G}(u,v)} \).

2.9 Definition
A path \( P \) in a fuzzy graph \( \bar{G} = (\sigma, \mu) \) is a sequence of distinct vertices \( v_0, v_1, ..., v_n \) such that \( \mu(v_{i-1}, v_i) > 0, 1 \leq i \leq n \).

Here \( n \geq 1 \) is called the length of the path \( P \). A single vertex \( v \) may also be considered as a path. In this case the path is of length 0. The consecutive pairs \((v_{i-1}, v_i)\) are called edges of the path. \( P \) is a cycle if \( v_n = v_1 \) and \( n \geq 3 \).

2.10 Definition
The strength of a path is defined to be the weight of the weakest arc of the path.
3. SMART FUZZY GRAPH

3.1. Definition of Smart Fuzzy Graph
Let V be a non-empty set. Then a smart fuzzy graph is defined as a pair of functions $G = (\sigma, \mu)$ where $\sigma$ is a fuzzy subset of V and $\mu$ is given as the symmetric fuzzy relation on $\sigma$.
(i.e.) $\sigma : V \rightarrow [0,1]$ and $\mu : [V \times V] \rightarrow [0,1] \ni \mu(x, y) \leq \sigma(x) \land \sigma(y)$ for all $u$ and $v$ in V with following criterion.

- If $i \neq j$, $\Sigma \mu_{ij} \leq \Sigma (\sigma_i \land \sigma_j) \leq 1$
- If $i = j$, $\Sigma \mu_{ij} = \Sigma (\sigma_i \land \sigma_j) = 0$

3.2. Scenario of usage of smart fuzzy graph
The system architecture’s supreme target is to build up a framework, which makes favourable beneficial interaction or communication among all the connected members in the framework of the system and utilize the system to the at-most. A list of feasible members of the system along with a short description of their possibilities of connections with others is presented below.

Let us consider the system architecture is a district which is connected with its subordinates through internet of things. The district administration is concerned in controlling the process and functioning of the district. Also the administration looks out for swift, rapid and legitimate ways for solving arguments and taking care of the problems. The district can equip and continue to maintain the smart city foundation through wireless network communication.

Let us assume the system consist of varies structures say, The Collectorate, police station, hospitals, banks, post office, schools, student hostels, colleges, restaurant, bus stand, railways, tourism, The Revenue Department and market. The system is interconnected among themselves through smart connectivity, through which the information is passed on to the other element in the system in real time. Whenever there arises an issue it is reported to the respective area and the problem is passed on to others on solving it, which can save time and make the system function efficiently.

![Figure 1 Smart Fuzzy Graph.](image-url)
From the above figure 1, we can see that each element is connected to every other element in the system. The elements are considered as the vertices and the connection between them is taken as the edges. The total connection between the systems is found using the smart fuzzy graph. The below table 1, displays the connection among the elements.

Table 1 \( \Sigma \mu_{ij} \leq \Sigma (\sigma_i \land \sigma_j) \leq 1. \)

|       | 0.91 | 0.89 | 0.95 | 0.96 | 0.9 | 0.88 | 0.91 | 0.94 | 0.97 | 0.99 | 0.92 | 0.9 | 0.89 | 0.93 | 0.95 |
|-------|------|------|------|------|-----|------|------|------|------|------|-----|----|-----|-----|-----|
| V1    | 0.91 | 0.1  | 0.99 | 0.05 | 0.02 | 0.01 | 0.02 | 0.01 | 0.03 | 0.01 | 0.11 | 0.11 | 0.1 | 0.1  | 0.02 |
| V2    | 0.89 | 0.1  | 0.05 | 0.04 | 0.02 | 0.02 | 0.02 | 0.01 | 0.02 | 0.01 | 0.03 | 0.02 | 0.02 | 0.02 | 0.09 |
| V3    | 0.95 | 0.09 | 0.05 | 0.02 | 0.02 | 0.1  | 0.05 | 0.09 | 0.06 | 0.01 | 0.09 | 0.08 | 0.07 | 0.04 | 0.12 |
| V4    | 0.96 | 0.06 | 0.04 | 0.02 | 0.02 | 0.1  | 0.06 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.01 | 0.09 | 0.02 |
| V5    | 0.9  | 0.02 | 0.02 | 0.02 | 0.02 | 0  | 0.04 | 0.01 | 0.02 | 0.02 | 0.03 | 0.04 | 0.06 | 0.1  | 0.12 |
| V6    | 0.88 | 0.01 | 0.05 | 0.1  | 0.04 | 0  | 0.06 | 0.01 | 0.06 | 0.02 | 0.11 | 0.01 | 0.04 | 0.07 | 0.04 |
| V7    | 0.91 | 0.02 | 0.02 | 0.05 | 0.06 | 0.01 | 0.06 | 0  | 0.02 | 0.1  | 0.07 | 0.02 | 0.03 | 0.06 | 0.02 |
| V8    | 0.94 | 0.01 | 0.01 | 0.09 | 0.01 | 0.02 | 0.01 | 0.02 | 0  | 0.1  | 0.09 | 0.05 | 0.1  | 0.03 | 0.06 |
| V9    | 0.97 | 0.03 | 0.02 | 0.06 | 0.02 | 0.02 | 0.06 | 0.1  | 0.1  | 0  | 0.03 | 0.03 | 0.05 | 0.12 | 0.02 |
| V10   | 0.99 | 0.01 | 0.01 | 0.01 | 0.03 | 0.03 | 0.02 | 0.1  | 0.09 | 0.05 | 0  | 0.15 | 0.06 | 0.07 | 0.13 |
| V11   | 0.92 | 0.12 | 0.03 | 0.09 | 0.04 | 0.04 | 0.11 | 0.07 | 0.05 | 0.05 | 0.15 | 0  | 0.02 | 0.03 | 0.07 |
| V12   | 0.9  | 0.11 | 0.02 | 0.08 | 0.05 | 0.06 | 0.01 | 0.02 | 0.1  | 0.05 | 0.06 | 0.02 | 0  | 0.13 | 0.04 |
| V13   | 0.89 | 0.01 | 0.02 | 0.07 | 0.01 | 0.1  | 0.04 | 0.03 | 0.03 | 0.12 | 0.07 | 0.07 | 0.13 | 0  | 0.01 |
| V14   | 0.93 | 0.1  | 0.02 | 0.04 | 0.09 | 0.12 | 0.07 | 0.06 | 0.06 | 0.02 | 0.13 | 0.07 | 0.04 | 0.01 | 0  |
| V15   | 0.95 | 0.02 | 0.09 | 0.12 | 0.02 | 0.02 | 0.04 | 0.02 | 0.01 | 0.05 | 0.04 | 0.01 | 0.12 | 0.02 | 0.05 |
| \( \sum_\mu_{ij} \) | 0.79 | 0.5  | 0.88 | 0.55 | 0.54 | 0.72 | 0.64 | 0.61 | 0.71 | 0.78 | 0.85 | 0.87 | 0.78 | 0.88 | 0.63 |

3.2. Example for smart fuzzy graph:

![Figure 2 A smart fuzzy graph.](image)

Table 2 \( \Sigma \mu_{ij} \leq \Sigma (\sigma_i \land \sigma_j) \leq 1. \)
|       | 0.8 | 0.6 | 0.5 | 0.9 | 0.5 | ∑μᵢⱼ |
|-------|-----|-----|-----|-----|-----|-------|
| V1    | 0.8 | 0   | 0.1 | 0.07| 0.13| 0.4   |
| V2    | 0.6 | 0.09| 0   | 0.2 | 0   | 0.39  |
| V3    | 0.5 | 0.11| 0.2 | 0   | 0.1 | 0.41  |
| V4    | 0.9 | 0.07| 0   | 0.1 | 0   | 0.27  |
| V5    | 0.5 | 0.13| 0.1 | 0   | 0   | 0.33  |
| ∑μᵢⱼ | 0.4 | 0.39| 0.41| 0.27| 0.33|       |

4. PROPERTIES OF SMART FUZZY GRAPH

4.1. Maximum degree of fuzzy graph:
Consider \( G = (\sigma, \mu) \) is a fuzzy graph. The maximum degree of \( G \) is denoted as
\[ \Delta(G) = \max \{ d(v_i) | v_i \in V \}. \]
Example: In the figure 2, \( d(v_1) \) has the maximum degree.

4.2. Order of fuzzy graph:
Let \( G = (\sigma, \mu) \) be a fuzzy graph, the order of fuzzy graph \( g \) is denoted as
\[ f_0(G) = \sum_{v_i \in V} \sigma(v_i) \]
Example: In the figure 2 the order of the smart fuzzy graph is \( O(G) = 5 \).

4.3. Size of fuzzy graph:
Let \( G = (\sigma, \mu) \) be a fuzzy graph, the size of fuzzy graph \( g \) is denoted as
\[ f_S(G) = \sum_{v_i, v_j \in V} \mu(v_i, v_j) \]
Example: In the figure 2 the order of the smart fuzzy graph is \( S(G) = 8 \).

4.4. Degree of a vertex:
Let \( G = (\sigma, \mu) \) be a fuzzy graph. The degree of a vertex \( u \) is
\[ d_G(u) = \sum_{v \in V} \mu(uv). \]
Since \( \mu(uv) > 0 \) for \( uv \in E \) and \( \mu(uv) = 0 \) for \( uv \notin E \), this is equivalent to
\[ d_G(u) = \sum_{v \in V} \mu(uv)dG(u). \]
The minimum degree of \( G \) is \( \delta(G) = \min \{ d(v) | v \in V \} \). The maximum degree of \( G \) is
\[ \Delta(G) = \max \{ d(v) | v \in V \} \].
Example: In the figure 2 the degree of the vertices are
\[ d(v_1) = 0.8, \quad d(v_2) = 0.6, \quad d(v_3) = 0.5, \quad d(v_4) = 0.9, \quad d(v_5) = 0.5. \]

4.5. Remarks:
1) Smart fuzzy graph cannot have cycles because smart fuzzy graph is an acyclic process.
2) Smart fuzzy graph is complete for \( K_3, K_5, K_9, K_{11} \).

5. Regular smart fuzzy graph and totally regular smart fuzzy wave

5.1. Definition of smart regular fuzzy graph
Let \( G = (\sigma, \mu) \) be taken as a fuzzy graph on the crisp graph \( G^*(V,E) \). If \( d_G(v) = k \) for all \( v \in V \), (i.e) when every vertex in the fuzzy graph has the same degree \( k \), then \( G \) is said to be a regular fuzzy graph of degree \( k \). It also known as \( k \)-regular. We can correlate this regular fuzzy graph definition in smart fuzzy graph theory.
Figure 3 A regular smart fuzzy graph.

Table 3 Regular smart fuzzy graph.

|       | 0.5 | 0.8 | 0.4 | \(\Sigma\mu_{ij}\) |
|-------|-----|-----|-----|-------------------|
| \(V1\) | 0   | 0.2 | 0.2 | 0.4               |
| \(V2\) | 0.2 | 0   | 0.2 | 0.4               |
| \(V3\) | 0.2 | 0.2 | 0   | 0.4               |
| \(\Sigma\mu_{ij}\) | 0.4 | 0.4 | 0.4 |                   |

For example consider the above figure 3, where \(d(v_1)=0.4\), \(d(v_2)=0.4\), \(d(v_3)=0.4\) and the graph satisfies the conditions of smart fuzzy graph. Hence the graph is regular.

5.2. Definition of a totally regular smart fuzzy graph

Let \(G = (\sigma, \mu)\) be a fuzzy graph. The total degree of a vertex \(u \in V\) is defined by \(\text{td}_G(u) = \sum_{E \ni uv} \mu(uv) + \sigma(u) = \sum_{E \ni uv} \mu(uv) + \sigma(u)\). If each vertex of \(G\) has the same total degree \(k\), then \(G\) is said to be a totally regular fuzzy graph of total degree \(k\) or also known as \(k\)-totally regular fuzzy graph. This definition may also apply to totally regular graphs in smart fuzzy graph.

Figure 4 A smart fuzzy graph that is totally regular.
Table 4 Totally regular smart fuzzy graph.

|       | 0.6 | 0.6 | 0.6 |  𝛹1 |  𝛹2 |  𝛹3 | Σ\(\mu_{ij}\) |
|-------|-----|-----|-----|-----|-----|-----|-------------|
|  0.6  |  𝛹1 | 0   | 0.2 | 0.2 | 0.4 |
|  0.6  |  𝛹2 | 0.2 | 0   | 0.2 | 0.4 |
|  0.6  |  𝛹3 | 0.2 | 0.2 | 0   | 0.4 |
| \(\Sigma\mu_{ij}\) | 0.4 | 0.4 | 0.4 | 0.4 |

For example consider the above figure 4, \(td(\nu_i) = 1\) for all \(i = 1, 2, 3\) and also the graph satisfies the condition of smart fuzzy graph. Therefore the smart fuzzy graph is totally regular.

5.3. THEOREM

A smart fuzzy graph is considered as regular and also totally regular, then \(\sigma\) is the given constant function in a smart fuzzy graph \(X\).

Proof:

Let the smart fuzzy graph \(X\) be a \(m\) regular and \(n\) totally regular smart fuzzy graph.

So \( (u) = m, \) where \( u \in V, \) \( td(u) = n, \) \( u \in V.\)

\[ \Rightarrow td(u) = n, \ u \in V \]

\[ \Rightarrow d(u) + \sigma(u) = n, \ u \in V \]

\[ \Rightarrow m + \sigma(u) = n, \ u, v \in V \]

\[ \Rightarrow \sigma(u) = n - m, \ u, v \in V \]

Therefore, \(\sigma\) is said to be a constant function.

Remark: The converse part need not be in accordance with the theorem.

5.4. Remarks

1) When there exist an edge between two vertices then the graph is said to be connected and it is considered as regular in smart fuzzy graph.
2) A smart fuzzy graph is said \(k\)-regular so if and only if \(\delta = \Delta = k.\)
3) In smart fuzzy graph not all complete graph is regular which is not the case in crisp graph.
4) In the smart fuzzy graph, regular and totally regular graphs have no relationship between them.

7. CONCLUSION

A new concept - smart fuzzy graph has been introduced in this paper along with some properties and calculations. The smart fuzzy graph can represent the wireless network communication (say, the internet of things) within a large structure that acts independently to solve the dispute or to pass the information to the required element in real time. This method by using smart fuzzy graph can be used to determine the connection present in the system. In the forthcoming papers we will extend this idea of smart fuzzy graph to other concepts.
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