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Charged Particles and the Electro-Magnetic Field in Non-Inertial Frames of Minkowski Spacetime: II. Applications: Rotating Frames, Sagnac Effect, Faraday Rotation, Wrap-up Effect

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Abstract

We apply the theory of non-inertial frames in Minkowski space-time, developed in the previous paper, to various relevant physical systems. We give the 3+1 description without coordinate-singularities of the rotating disk and the Sagnac effect, with added comments on pulsar magnetosphere and on a relativistic extension of the Earth-fixed coordinate system. Then we study properties of Maxwell equations in non-inertial frames like the wrap-up effect and the Faraday rotation in astrophysics.
I. INTRODUCTION

In the first paper [1] (quoted as paper I) we developed the general theory of non-inertial frames in Minkowski space-time, whose starting point are its admissible 3+1 splittings defining the allowed conventions for clock synchronization, namely the allowed notions of instantaneous 3-spaces needed, for instance, for setting a well-posed Cauchy problem for Maxwell equations. In this way the coordinate singularities of the traditional 1+3 approach are avoided by construction. In particular it is shown that rigidly rotating frames are not admissible in special relativity.

Also the formulation of charged particles and of the electro-magnetic in non-inertial frames was given.

In this second paper we reformulate relevant physical system, usually described in the 1+3 framework, in the non-inertial frames based on the admissible 3+1 splittings.

In Section II there is a review of the rotating disk and of the Sagnac effect in the 1+3 point of view followed by their description in the framework of the 3+1 point of view (Subsection A) and by a discussion on the ITRS rotating 3-coordinates fixed on the Earth surface (Subsection B).

In Section III we give the 3+1 point of view in admissible nearly rigidly rotating frames of the Wrap Up effect, of the Sagnac effect and of the inertial Faraday rotation by studying electro-magnetic wave solutions of the non-inertial Maxwell equations.

In the Conclusions we give an overview of the results obtained in these two papers and we identify the still open problems about electro-magnetism in non-inertial frames.
II. THE ROTATING DISK AND THE SAGNAC EFFECT

In this Section we give the description of a rotating disk and of the Sagnac effect starting from an admissible 3+1 splitting of Minkowski space-time of the type of Eqs.(2.14) of I, i.e. whose embedding has the form $z^\mu(\tau, \sigma^u) = x^\mu(\tau) + \epsilon^\mu_r R^r_s(\tau, \sigma) \sigma^s$ with $x^\mu(\tau) = x^\mu_o + f^A(\tau) \epsilon^\mu_A$ describing the world-line of the observer origin of the 3-coordinates on the instantaneous 3-spaces $\Sigma_\tau$. This is the simplest non-inertial frame whose 3-spaces are space-like hyper-planes with admissible differentially rotating 3-coordinates. The rotation matrix $R^r_s(\tau, \sigma) = R^r_s(\alpha_i(\tau, \sigma)) = R^r_s(F(\sigma) \tilde{\alpha}_i(\tau))$ ($\sigma = |\vec{\sigma}|$) is admissible if the function $F(\sigma)$ satisfies the Møller conditions $0 < F(\sigma) < \frac{1}{A\sigma}$ and $\frac{dF(\sigma)}{d\sigma} \neq 0$.

An enlarged exposition of the material of this Section with a rich bibliography is given in Section I Subsection D and E and in Section VI Subsections B and C of the first paper in Ref.[2].

While at the non-relativistic level one can speak of a rigid (either geometrical or material) disk put in global rigid rotatory motion, the problem of the relativistic rotating disk is still under debate (see Refs.[3, 4]) after one century from the enunciation of the Ehrenfest paradox about the 3-geometry of the rotating disk. The problems arise when one tries to define measurements of length, in particular that of the circumference of the disk. Einstein [5] claims that while the rods along the radius $R_o$ are unchanged those along the rim of the disk are Lorentz contracted: as a consequence more of them are needed to measure the circumference, which turns out to be greater than $2\pi R_o$ (non-Euclidean 3-geometry even if Minkowski space-time is 4-flat) and not smaller. This was his reply to Ehrenfest [6], who had pointed an inconsistency in the accepted special relativistic description of the disk ¹ in which it is the circumference to be Lorentz contracted: as a consequence this fact was named the Ehrenfest paradox (see the historical paper of Grøn in Ref.[7]).

Since relativistic rigid bodies do not exist, at best we can speak of Born rigid motions [8] and Born reference frames ². However Grøn [7] has shown that the acceleration phase of a material disk is not compatible with Born rigid motions and, moreover, we do not have a well formulated and accepted relativistic framework to discuss a relativistic elastic material disk.

¹ If $R$ and $R_o$ denote the radius of the disk in the rotating and inertial frame respectively, then we have $R = R_o$ because the velocity is orthogonal to the radius. But the circumference of the rim of the disk is Lorentz contracted so that $2\pi R < 2\pi R_o$ inconsistently with Euclidean geometry.

² A reference frame or platform is Born-rigid [9] if the expansion $\Theta$ and the shear $\sigma_{\mu\nu}$ of the associated congruence of time-like observers vanish, i.e. if the spatial distance between neighboring world-lines remains constant.
As a consequence most of the authors treating the rotating disk (either explicitly or implicitly) consider it as a geometrical entity described by a congruence of time-like world-lines (helices in Ref.[10]) with non-zero vorticity, i.e. non-surface forming and therefore non-synchronizable (see for instance Ref.[11]). This means that there is no notion of instantaneous 3-space where to visualize the disk (see Ref.[3] for the attempts to define rods and clocks associated to this type of congruences): every observer on one of these time-like world-lines can only define the local rest frame and try to define a local accelerated reference frame as said in Section IIB of paper I.

In the 3+1 point of view the disk is considered to be a relativistic isolated system (either a relativistic material body or a relativistic fluid or a relativistic dust as a limit case 3) with compact support always contained in a finite time-like world-tube $W$, which in the Cartesian 4-coordinates of an inertial system is a time-like cylinder of radius R. Each admissible 3+1 splitting of Minkowski space-time, centered on an arbitrary time-like observer and with its two associated congruences of time-like observers (see Section IIB of paper I), gives a visualization of the disk in its instantaneous 3-spaces $\Sigma_{\tau}$: at each instant $\tau$ the points of the disk in $W \cap \Sigma_{\tau}$ are synchronized and through each one of them pass an Eulerian observer belonging to the surface forming congruence having as 4-velocity the unit normal to the instantaneous 3-spaces $\Sigma_{\tau}$. Instead the irrotational congruence of the disk is described by the second congruence (whose unit 4-velocity is $\sigma^{u}(\tau, \sigma^{u})/\sqrt{\epsilon g_{\tau\tau}(\tau, \sigma^{u})}$ and whose observers follow generalized helices $\sigma^{u} = \sigma_{u}^{u}$) associated to the admissible 3+1 splitting: each of the observers of this congruence, whose world-lines are inside $W$, has no intrinsic notion of synchronization.

As a consequence, each instantaneous 3-space $\Sigma_{\tau}$ of an admissible 3+1 splitting has a well defined (in general Riemannian) notion of 3-geometry and of spatial length: the radius and the circumference of the disk are defined in $W \cap \Sigma_{\tau}$, so that the disk 3-geometry is 3+1 splitting dependent. When the material disk can be described by means of a parametrized Minkowski theory, all these 3-geometry are gauge equivalent like the notions of clock synchronization.

The other important phenomenon connected with the rotating disk is the Sagnac effect (see the recent review in Ref.[13] for how many interpretations of it exist), namely the phase difference generated by the difference in the time needed for a round-trip by two light rays, emitted in the same point, one co-rotating and the other counter-rotating with the

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3 As an example of a congruence simulating a geometrical rotating disk we can consider the relativistic dust described by generalized Eulerian coordinates of Ref.[12] after the gauge fixing to a family of differentially rotating parallel hyper-planes.
disk\(^4\). This effect, which has been tested (see the bibliography of Refs.[13, 17]) for light, X rays and matter waves (Cooper pairs, neutrons, electrons and atoms), has important technological applications and must be taken into account for the relativistic corrections to space navigation, has again an enormous number of theoretical interpretations (both in special and general relativity) like for the solutions of the Ehrenfest paradox. Here the lack of a good notion of simultaneity leads to problems of time discontinuities or desynchronization effects when comparing clocks on the rim of the rotating disk.

Another area which is in a not well established form is electrodynamics in non-inertial systems either in vacuum or in material media (problem of the non-inertial constitutive equations). Its clarification is needed both to derive the Sagnac effect from Maxwell equations without gauge ambiguities [14] and to determine which types of experiments can be explained by using the locality hypothesis (see Section IIB of paper I) to evaluate the electro-magnetic fields in the comoving system (see the Wilson experiment and the associated controversy [18] on the validity of the locality principle) without the need of a more elaborate treatment like for the radiation of accelerated charges. It would also help in the tests of the validity of special relativity (for instance on the possible existence of a preferred frame) based on Michelson-Morley - type experiments [19, 20].

Instead (see also Ref.[14]) we remark that the Sagnac effect and the Foucault pendulum are experiments which signal the rotational non-inertiality of the frame. The same is true for neutron interferometry [21], where different settings of the apparatus are used to detect either rotational or translational non-inertiality of the laboratory. As a consequence a null result of these experiments can be used to give a definition of relativistic quasi-inertial system.

Let us remark that the disturbing aspects of rotations are rooted in the fact that there is a deep difference between translations and rotations at every level both in Newtonian

\[^4\text{For monochromatic light in vacuum with wavelength } \lambda \text{ the fringe shift is } \delta z = 4 \vec{\Omega} \cdot \vec{A}/\lambda c, \text{ where } \vec{\Omega} \text{ is the Galilean velocity of the rotating disk supporting the interferometer and } \vec{A} \text{ is the vector associated to the area } |\vec{A}| \text{ enclosed by the light path. The time difference is } \delta t = \lambda \delta z/c = 4 \vec{\Omega} \cdot \vec{A}/c^2, \text{ which agrees, at the lowest order, with the proper time difference } \delta \tau = (4 A \Omega/c^2) (1 - \Omega^2 R^2/c^2)^{-1/2}, \text{ where } A = \pi R^2, \text{ evaluated in an inertial system with the standard rotating disk coordinates. This proper time difference is twice the time lag due to the synchronization gap predicted for a clock on the rim of the rotating disk with a non-time orthogonal metric. See Refs.[13, 14, 15] for more details. See also Ref.[16] for the corrections included in the GPS protocol to allow the possibility of making the synchronization of the entire system of ground-based and orbiting atomic clocks in a reference local inertial system. Since usually, also in GPS, the rotating coordinate system has } t' = t \text{ (} t \text{ is the time of an inertial observer on the axis of the disk) the gap is a consequence of the impossibility to extend Einstein’s convention of the inertial system also to the non-inertial one rotating with the disk: after one period two nearby synchronized clocks on the rim are out of synchrony.}\]
mechanics and special relativity: the generators of translations satisfy an Abelian algebra, while the rotational ones a non-Abelian algebra. As shown in Refs.[22], at the Hamiltonian level we have that the translation generators are the three components of the momentum, while the generators of rotations are a pair of canonical variables ($L^3$ and $\text{arctg} \frac{L^2}{L^1}$) and an unpaired variable ($|\vec{L}|$). As a consequence we can separate globally the motion of the 3-center of mass of an isolated system from the relative variables, but we cannot separate in a global and unique way three Euler angles describing an overall rotation, because the residual vibrational degrees of freedom are not uniquely defined.

We will now give the 3+1 point of view on these topics (Subsection A), followed by a discussion on the rotating 3-coordinates fixed to the Earth surface (Subsection B).

A. The 3+1 Point of View on the Rotating Disk and the Sagnac Effect.

Let us describe an abstract geometrical disk with an admissible 3+1 splitting of the type (2.14) of I, in which the instantaneous 3-spaces are parallel space-like hyper-planes with normal $l^\mu$ centered on an inertial observer $x^\mu(\tau) = l^\mu \tau$

$$z^\mu(\tau, \vec{\sigma}) = l^\mu \tau + \ell^\mu R_r^s(\tau, \sigma) \sigma^s. \quad (2.1)$$

The rotation matrix $R_{(3)}$ describes a differential rotation around the fixed axis "3" (we take a constant $\omega$, but nothing changes with $\omega(\tau)$)

$$R_{(3)}^r_s(\tau, \sigma) = \begin{pmatrix} \cos \theta(\tau, \sigma) & -\sin \theta(\tau, \sigma) & 0 \\ \sin \theta(\tau, \sigma) & \cos \theta(\tau, \sigma) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\theta(\tau, \sigma) = F(\sigma) \omega \tau, \quad F(\sigma) < \frac{c}{\omega \sigma},$$

$$\Omega^r_s(\tau, \sigma) = \left( R_{(3)}^{-1} \frac{dR_{(3)}}{d\tau} \right)^r_s(\tau, \sigma) = \omega F(\sigma) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Omega(\tau, \sigma) = \Omega(\sigma) = \omega F(\sigma). \quad (2.2)$$

A simple choice for the gauge function $F(\sigma)$ is $F(\sigma) = \frac{1}{1+\frac{\omega^2 \sigma^2}{c^2}}$ (in the rest of the Section we put $c = 1$), so that at spatial infinity we get $\Omega(\tau, \sigma) = \frac{\omega}{1+\frac{\omega^2 \sigma^2}{c^2}} \rightarrow_{\sigma \rightarrow \infty} 0$. 

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By introducing cylindrical 3-coordinates $r, \varphi, h$ by means of the equations $\sigma^1 = r \cos \varphi$, $\sigma^2 = r \sin \varphi$, $\sigma^3 = h$, $\sigma = \sqrt{r^2 + h^2}$, we get the following form of the embedding and of its gradients

$$z^\mu(\tau, \vec{\sigma}) = l^\mu \tau + \epsilon_1^\mu [\cos \theta(\tau, \sigma) \sigma^1 - \sin \theta(\tau, \sigma) \sigma^2] +$$

$$+ \epsilon_2^\mu [\sin \theta(\tau, \sigma) \sigma^1 + \cos \theta(\tau, \sigma) \sigma^2] + \epsilon_3^\mu \sigma^3 =$$

$$= l^\mu \tau + \epsilon_1^\mu r \cos \theta(\tau, \sigma) + \epsilon_2^\mu r \sin \theta(\tau, \sigma) + \epsilon_3^\mu h,$$

\[ \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \tau} = z_r^\mu(\tau, \vec{\sigma}) = l^\mu - \omega r F(\sigma) \left( \epsilon_1^\mu \sin \theta(\tau, \sigma) + \epsilon_2^\mu \cos \theta(\tau, \sigma) \right), \]

\[ \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \varphi} = z_\varphi^\mu(\tau, \vec{\sigma}) = -\epsilon_1^\mu r \sin \theta(\tau, \sigma) + \epsilon_2^\mu r \cos \theta(\tau, \sigma) \]

\[ \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial r} = z_r^\mu(\tau, \vec{\sigma}) = -\epsilon_1^\mu \left( \cos \theta(\tau, \sigma) + \varphi \right) - \frac{r^2 \omega \tau}{\sqrt{r^2 + h^2}} \frac{dF(\sigma)}{d\sigma} \sin \theta(\tau, \sigma) + \varphi \]

\[ + \epsilon_2^\mu \left( \sin \theta(\tau, \sigma) + \varphi \right) + \frac{r^2 \omega \tau}{\sqrt{r^2 + h^2}} \cos \theta(\tau, \sigma) + \varphi \]

\[ \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial h} = z_h^\mu(\tau, \vec{\sigma}) = \epsilon_3^\mu - \epsilon_1^\mu \left( \frac{r h \omega \tau}{\sqrt{r^2 + h^2}} \frac{dF(\sigma)}{d\sigma} \sin \theta(\tau, \sigma) + \varphi \right) \]

\[ + \epsilon_2^\mu \left( \frac{r h \omega \tau}{\sqrt{r^2 + h^2}} \frac{dF(\sigma)}{d\sigma} \cos \theta(\tau, \sigma) + \varphi \right), \quad (2.3) \]

where we have used the notation $(r)$ to avoid confusion with the index $r$ used as 3-vector index (for example in $\sigma^r$).

In the cylindrical 4-coordinates $\tau, r, \varphi$ and $h$ the 4-metric is

\[ \epsilon g_{\tau\tau}(\tau, \vec{\sigma}) = 1 - \omega^2 r^2 F^2(\sigma), \quad \epsilon g_{\tau\varphi}(\tau, \vec{\sigma}) = -\omega r^2 F(\sigma), \quad \epsilon g_{\varphi\varphi}(\tau, \vec{\sigma}) = -r^2, \]

\[ \epsilon g_{r(r)}(\tau, \vec{\sigma}) = -\frac{\omega^2 r^3 \tau}{\sqrt{r^2 + h^2}} F(\sigma) \frac{dF(\sigma)}{d\sigma}, \quad \epsilon g_{\tau h}(\tau, \vec{\sigma}) = -\frac{\omega^2 r^2 h \tau}{\sqrt{r^2 + h^2}} F(\sigma) \frac{dF(\sigma)}{d\sigma}, \]
\[ \epsilon g_{(r)(r)}(\tau, \vec{\sigma}) = -1 - \frac{r^4 \omega^2 \tau^2 \left( \frac{dF(\sigma)}{d\sigma} \right)^2}{r^2 + h^2}, \]
\[ \epsilon g_{hh}(\tau, \vec{\sigma}) = -1 - \frac{r^2 h^2 \omega^2 \tau^2 \left( \frac{dF(\sigma)}{d\sigma} \right)^2}{r^2 + h^2}, \]
\[ \epsilon g_{(r)\varphi}(\tau, \vec{\sigma}) = -\frac{\omega^3 \tau}{\sqrt{r^2 + h^2}} \frac{dF(\sigma)}{d\sigma}, \quad \epsilon g_{h\varphi}(\tau, \vec{\sigma}) = -\frac{\omega r^2 h \tau}{\sqrt{r^2 + h^2}} \frac{dF(\sigma)}{d\sigma}. \]
\[ \epsilon g_{h(r)}(\tau, \vec{\sigma}) = -\frac{r^3 h \omega^2 \tau^2 \left( \frac{dF(\sigma)}{d\sigma} \right)^2}{r^2 + h^2}, \]

with inverse
\[ \epsilon g^{\tau\tau}(\tau, \vec{\sigma}) = 1, \quad \epsilon g^{\tau\varphi}(\tau, \vec{\sigma}) = -\omega \frac{F(\sigma)}{r}, \]
\[ \epsilon g^{(r)(r)}(\tau, \vec{\sigma}) = \epsilon g^{r\varphi}(\tau, \vec{\sigma}) = 0, \quad \epsilon g^{(r)(r)}(\tau, \vec{\sigma}) = \epsilon g^{hh}(\tau, \vec{\sigma}) = -1, \]
\[ \epsilon g^{\varphi\varphi}(\tau, \vec{\sigma}) = -\frac{1 + \omega^2 r^2 \left( \frac{r^2}{r^2 + h^2} \right)^2 - F^2(\sigma)}{r^2}, \]
\[ \epsilon g^{\varphi(r)}(\tau, \vec{\sigma}) = \frac{\omega r \tau}{\sqrt{r^2 + h^2}} \frac{dF(\sigma)}{d\sigma}, \quad \epsilon g^{r\varphi}(\tau, \vec{\sigma}) = \frac{\omega h \tau}{\sqrt{r^2 + h^2}} \frac{dF(\sigma)}{d\sigma}. \] (2.4)

It is easy to observe that the congruence of (non inertial) observers defined by the 4-velocity field
\[ z^\mu(\tau, \vec{\sigma}) = \frac{\mu - \omega r F(\sigma) \left( \epsilon_1^\mu \sin \left[ \phi(\tau, \sigma) + \varphi \right] + \epsilon_2^\mu \cos \left[ \phi(\tau, \sigma) + \varphi \right] \right)}{1 - \omega^2 r^2 F^2(\sigma)}, \] (2.5)
has the observers moving along the world-lines
\[ x^\mu_{\sigma_o}(\tau) = z^\mu(\tau, \vec{\sigma}_o) = \]
\[ = \mu \tau + r_o \left( \epsilon_1^\mu \cos \left[ \omega \tau F(\sigma_o) + \varphi_o \right] + \epsilon_2^\mu \sin \left[ \omega \tau F(\sigma_o) + \varphi_o \right] \right) + \epsilon_3^\mu h_o. \] (2.6)
The world-lines (2.6) are labeled by their initial value \( \vec{\sigma} = \vec{\sigma}_o = (\varphi_o, r_o, h_o) \) at \( \tau = 0 \).
In particular for $h_o = 0$ and $r_o = R$ these world-lines are helices on the cylinder in the Minkowski space

\[ \epsilon^\mu_3 z_\mu = 0, \quad (\epsilon^\mu_1 z_\mu)^2 + (\epsilon^\mu_2 z_\mu)^2 = R^2, \quad \text{or} \quad r = R, \quad h = 0. \quad (2.7) \]

These helices are defined the equations $\varphi = \varphi_o, r = R, h = 0$ if expressed in the embedding adapted coordinates $\varphi, r, h$. Then the congruence of observers (2.5), defined by the foliation (2.1), defines on the cylinder (2.7) the rotating observers usually assigned to the rim of a rotating disk, namely observers running along the helices $x^\mu_{\sigma_o}(\tau) = l^\mu \tau + R \left( \epsilon^\mu_1 \cos[\Omega(R) \tau + \varphi_o] + \epsilon^\mu_2 \sin[\Omega(R) \tau + \varphi_o] \right)$ after having put $\Omega(R) \equiv \omega F(R)$.

On the cylinder (2.7) the line element is obtained from the line element $ds^2 = g_{AB} d\sigma^A d\sigma^B$ for the metric (2.4) by putting $dh = dr = 0$ and $r = R, h = 0$. Therefore the cylinder line element is

\[ \epsilon (ds_{cyl})^2 = \left[ 1 - \omega^2 R^2 F^2(R) \right] (d\tau)^2 - 2 \omega R^2 F(R) d\tau d\varphi - R^2 (d\varphi)^2. \quad (2.8) \]

We can define the light rays on the cylinder, i.e. the null curves on it, by solving the equation

\[ \epsilon (ds_{cyl})^2 = (1 - R^2 \Omega^2(R)) d\tau^2 - 2 R^2 \Omega(R) d\tau d\varphi - R^2 d\varphi^2 = 0, \quad (2.9) \]

which implies

\[ R^2 \left( \frac{d\varphi(\tau)}{d\tau} \right)^2 + 2 R^2 \Omega(R) \left( \frac{d\varphi(\tau)}{d\tau} \right) - (1 - R^2 \Omega(R)) = 0. \quad (2.10) \]

The two solutions

\[ \frac{d\varphi(\tau)}{d\tau} = \pm \frac{1}{R} - \Omega(R), \quad (2.11) \]

define the world-lines on the cylinder for clockwise or anti-clockwise rays of light.

\[ \Gamma_1 : \quad \varphi(\tau) - \varphi_o = \left( + \frac{1}{R} - \Omega(R) \right) \tau, \]
\[ \Gamma_2 : \quad \varphi(\tau) - \varphi_o = \left( - \frac{1}{R} - \Omega(R) \right) \tau. \quad (2.12) \]

This is the geometric origin of the Sagnac effect. Since $\Gamma_1$ describes the world-line of the ray of light emitted at $\tau = 0$ by the rotating observer $\varphi = \varphi_o$ in the increasing sense of $\varphi$ (anti-clockwise), while $\Gamma_2$ describes that of the ray of light emitted at $\tau = 0$ by the
same observer in the decreasing sense of \( \varphi \) (clockwise), then the two rays of light will be re-absorbed by the same observer at different \( \tau \)-times \( \tau(\pm 2\pi) \), whose value, determined by the two conditions \( \varphi(\tau(\pm 2\pi)) - \varphi_o = \pm 2\pi \), is

\[
\Gamma_1 : \quad \tau(+2\pi) = \frac{2\pi R}{1 - \Omega(R) R}, \quad \Gamma_2 : \quad \tau(-2\pi) = \frac{2\pi R}{1 + \Omega(R) R}.
\]  

(2.13)

The time difference between the re-absorption of the two light rays is

\[
\Delta \tau = \tau(+2\pi) - \tau(-2\pi) = \frac{4\pi R^2 \Omega(R)}{1 - \Omega^2(R) R^2} = \frac{4\pi R^2 \omega F(R)}{1 - \omega^2 F^2(R) R^2},
\]

(2.14)

and it corresponds to the phase difference named the Sagnac effect (see footnote 4)

\[
\Delta \Phi = \Omega \Delta \tau, \quad \Omega = \Omega(R) = \omega F(R).
\]  

(2.15)

We see that we recover the standard result if we take a function \( F(\sigma) \) such that \( F(R) = 1 \). In the non-relativistic applications, where \( F(\sigma) \rightarrow 1 \), the correction implied by admissible relativistic coordinates is totally irrelevant.

With an admissible notion of simultaneity, all the clocks on the rim of the rotating disk lying on a hyper-surface \( \Sigma_\tau \) are automatically synchronized. Instead for rotating observers of the irrotational congruence there is a desynchronization effect or synchronization gap because they cannot make a global synchronization of their clocks: usually a discontinuity in the synchronization of clocks is accepted and taken into account (see Ref.[16] for the GPS).

To clarify this point and see the emergence of this gap, let us consider a reference observer \( (\varphi_o = \text{const}., \tau) \) and another one \( (\varphi = \text{const}., \neq \varphi_o, \tau) \). If \( \varphi > \varphi_o \) we use the notation \( (\varphi_R, \tau) \), while for \( \varphi < \varphi_o \) the notation \( (\varphi_L, \tau) \) with \( \varphi_R - \varphi_o = -(\varphi_L - \varphi_o) \).

Let us consider the two rays of light \( \Gamma_{R-} \) and \( \Gamma_{L-} \), with world-lines given by Eqs.(2.12), emitted in the right and left directions at the event \( (\varphi_o, \tau_-) \) on the rim of the disk and received at \( \tau \) at the events \( (\varphi_R, \tau) \) and \( (\varphi_L, \tau) \) respectively. Both of them are reflected towards the reference observer, so that we have two rays of light \( \Gamma_{R+} \) and \( \Gamma_{L+} \) which will be absorbed at the event \( (\varphi_o, \tau_+) \). By using Eq.(2.13) for the light propagation, we get

\[
\text{5} \text{ Sometimes the proper time of the rotating observer is used: } d\mathcal{T}_o = d\tau \sqrt{1 - \Omega^2(R) R^2}.
\]
\[ \Gamma_R^- : (\varphi - \varphi_o) = \frac{1 - R\Omega(R)}{R} (\tau - \tau_-), \quad \Gamma_R^+ : (\varphi - \varphi_o) = \frac{1 + R\Omega(R)}{R} (\tau_+ - \tau), \]

\[ \Gamma_L^- : (\varphi - \varphi_o) = -\frac{1 + R\Omega(R)}{R} (\tau - \tau_-), \quad \Gamma_L^+ : (\varphi - \varphi_o) = -\frac{1 - R\Omega(R)}{R} (\tau_+ - \tau). \]

\[ (2.16) \]

As shown in Section II, Eqs.(2.17) and (2.18), of the first paper in Ref.[2], in a neighborhood of the observer \((\varphi_o, \tau)\) \([\varphi, \tau] \text{ is an observer in the neighborhood}\) we can only define the following local synchronization \(^6\)

\[ c \Delta \tilde{T} = \sqrt{1 - R^2 \Omega^2(R)} \Delta \tau_E = \sqrt{1 - R^2 \Omega^2(R)} \Delta \tau - \frac{R^2 \Omega^2(R)}{\sqrt{1 - R^2 \Omega^2(R)}} \Delta \varphi. \]

(2.17)

If we try to extend this local synchronization to a global one for two distant observers \((\varphi_o, \tau)\) and \((\varphi, \tau)\) in the form of an Einstein convention (the result is the same both for \(\varphi = \varphi_R\) and \(\varphi = \varphi_L\))

\[ \tau_E = \frac{1}{2} (\tau_+ + \tau_-) = \tau - \frac{R^2 \Omega(R)}{1 - R^2 \Omega^2(R)} (\varphi - \varphi_o), \]

we arrive at a contradiction, because the curves defined by \(\tau_E = \text{constant}\) are not closed, since they are helices that assign the same time \(\tau_E\) to different events on the world-line of an observer \(\varphi_o = \text{constant}\). For example \((\varphi_o, \tau)\) and \((\varphi_o, \tau + 2\pi \frac{R^2 \Omega(R)}{1 - R^2 \Omega^2(R)})\) are on the same helix \(\tau_E = \text{constant}\). As a consequence we get the synchronization gap.

As shown in both papers of Ref.[2], by using the global synchronization on the instantaneous 3-spaces \(\Sigma_\tau\) we can define a generalization of Einstein’s convention for clock synchronization by using the radar time \(\tau\). If an accelerated observer A emits a light signal at \(\tau_-\), which is reflected at a point P of the world-line of a second observer B and then reabsorbed at \(\tau_+\), then the B clock at P has to be synchronized with the following instant of the A clock \([n = + \text{ for } \varphi = \varphi_R, n = - \text{ for } \varphi = \varphi_L]\)

\[ \tau(\tau_-, n, \tau_+) = \frac{1}{2} (\tau_+ + \tau_-) - \frac{n R \Omega(R)}{2} (\tau_+ - \tau_-) = \tau_+ + \mathcal{E}(\tau_-, n, \tau_+) (\tau_+ - \tau_-), \]

with \(\mathcal{E}(\tau_-, n, \tau_+) = \frac{1 - n R \Omega(R)}{2}, \quad \Omega(R) = \omega F(R). \)

\[ (2.19) \]

\(^6\) See Ref.[15] for a derivation of the Sagnac effect in an inertial system by using Einstein’s synchronization in the locally comoving inertial frames on the rim of the disk and by asking for the equality of the one-way velocities in opposite directions.
Finally in the first paper of Ref.[2] [see Eqs.(6.37)-(6.47) of Section VI] there is the evaluation of the radius and the circumference of the rotating disk. If we choose the spatial length of the instantaneous 3-space $\Sigma_\tau$ of the admissible embedding (2.1), we get an Euclidean 3-geometry, i.e. a circumference $2\pi R$ and a radius $R$ at each instant $\tau$ independently from the choice of the gauge function $F(\sigma)$. With other admissible 3+1 splittings we would get non-Euclidean results: as said they are gauge equivalent when the disk can be described with a parametrized Minkowski theory. Instead the use of a local notion of synchronization from the observers of the irrotational congruence located on the rim of the rotating disk implies a local definition of spatial distance based on the 3-metric $^3\gamma_{uv} = -e \left( g_{uv} - \frac{\partial x^{\sigma}}{\partial \sigma^{\nu}} \right)$, i.e. a non-Euclidean 3-geometry. In this case the radius is $R$, but the circumference is $2\pi R/\sqrt{1-R^2 \Omega^2(R)}$. However this result holds only in the local rest frame of the observer with the tangent plane orthogonal to the observer 4-velocity (also called the abstract relative space) identified with a 3-space (see Section IIB of paper I).

See Subsection D of Section III for a derivation of the Sagnac effect in nearly rigid rotating frames.

**B. The Rotating ITRS 3-Coordinates fixed on the Earth Surface.**

The embedding (2.14) of I, describing admissible differential rotations in an Euclidean 3-space, could be used to improve the conventions IERS2003 (International Earth Rotation and Reference System Service) [23] on the non-relativistic transformation from the 4-coordinates of the Geocentric Celestial Reference System (GCRS) to the International Terrestrial Reference System (ITRS), the Earth-fixed geodetic system of the new theory of Earth rotation replacing the old precession-nutation theory. It would be a special relativistic improvement to be considered as an intermediate step till to a future development leading to a post-Newtonian (PN) general relativistic approach unifying the existing non-relativistic theory of the geo-potential below the Earth surface with the GCRS PN description of the geo-potential outside the Earth given by the conventions IAU2000 (International Astronomical Union) [23] for Astrometry, Celestial Mechanics and Metrology in the relativistic framework.

In the IAU 2000 Conventions the Solar System is described in the Barycentric Celestial Reference System (BCRS) as a quasi-inertial frame, centered on the barycenter of the Solar System, with respect to the Galaxy. BCRS is parametrized with harmonic PN 4-coordinates $x_{BCRS}^\mu = (x_{BCRS}^0 = c t_{BCRS}; x_{BCRS}^i)$, where $t_{BCRS}$ is the barycentric coordinate time and
the mutually orthogonal spatial axes are *kinematically non-rotating* with respect to fixed radio sources. This a nearly Cartesian 4-coordinate system in a PN Einstein space-time and there is an assigned 4-metric, determined modulo $O(c^{-4})$ terms and containing the gravitational potentials of the Sun and of the planets, PN solution of Einstein equations in harmonic gauges: in practice it is considered as a special relativistic inertial frame with nearly Euclidean instantaneous 3-spaces $t_{BCRS} = const.$ (modulo $O(c^{-2})$ deviations) and with Cartesian 3-coordinates $x^i_{BCRS}$. This frame is used for space navigation in the Solar System. The geo-center (a fictitious observer at the center of the earth geoid) has a world-line $y^\mu_{BCRS}(x^0_{BCRS}) = \left(x^0_{BCRS}; y^i_{BCRS}(x^0_{BCRS})\right)$, which is approximately a straight line.

For space navigation near the Earth (for the Space Station and near Earth satellites using NASA coordinates) and for the studies from spaces of the geo-potential one uses the GCRS, which is defined outside the Earth surface as a local reference system centered on the geo-center. Due to the earth rotation of the Earth around the Sun, it deviates from a nearly inertial special relativistic frame on time scales of the order of the revolution time. Its harmonic 4-coordinates $x^\mu_{GCRS} = \left(x^0_{GCRS} = c t_{GCRS}; x^i_{GCRS}\right)$, where $t_{GCRS}$ is the geocentric coordinate time, are obtained from the BCRS ones by means of a PN coordinate transformation which may be described as a special relativistic pure Lorentz boost without rotations (the parameter is the 3-velocity of the geo-center considered constant on small time scales) plus $O(c^{-4})$ corrections taking into account the gravitational acceleration of the geo-center induced by the Sun and the planets. As a consequence the GCRS spatial axes are kinematically non-rotating in BCRS and the relativistic inertial forces (for instance the Coriolis ones) are hidden in the geodetic precession; the same holds for the aberration effects and the dependence on angular variables. A PN 4-metric, determined modulo $O(c^{-4})$ terms, is given in IAU2000: it also contains the GCRS form of the geo-potential and the inertial and tidal effects of the Sun and of the planets. Again the instantaneous 3-spaces are considered nearly Euclidean (modulo $O(c^{-2})$ deviations) 3-spaces $t_{GCRS} = const.$.

In IAU2000 the coordinate times $t_{BCRS}$ and $t_{GCRS}$ are then connected with the time scales used on Earth: SI Atomic Second, TAI (International Atomic Time), TT (Terrestrial Time), $T_{EPH}$ (Ephemerides Time), UT and UT1 and UTC (Universal Times for civil use), GPS (Mastr Time), ST (Station Time).

Finally we need a 4-coordinate system fixed on the Earth crust. It is the ITRS with 4-coordinates $x^\mu_{ITRS} = \left(x^0_{ITRS} \overset{\text{def}}{=} c t_{GCRS}; x^i_{ITRS}\right)$, which uses the same coordinate time as GCRS. It is obtained from GCRS by making a suitable set of non-relativistic time-dependent *rigid* rotations on the nearly Euclidean 3-spaces $t_{GCRS} = const.$. The geocentric rectangular 3-coordinates $x^i_{ITRS}$ match the reference ellipsoid WGS-84 (basis
of the terrestrial coordinates (latitude, longitude, height) obtainable from GPS) used in geodesy. As shown in IERS2003, we have \( x_{\text{ITRS}}^i = \left( W^T(t_{\text{GCRS}}) R_3^T(-\theta) C \right)^i_j x_{\text{GCRS}}^j \), where \( C = R_3^T(s) R_3^T(E) R_2^T(-d) R_3^T(-E) \) and \( W(t_{\text{GCRS}}) = R_3(-s') R_2(x_p) R_1(y_p) \) are rotation matrices. This convention is based on the new definition of the Earth rotation axis (\( \theta \) is the angle of rotation about this axis): it is the line through the geo-center in direction of the Celestial Intermediate Pole (CIP) at date \( t_{\text{GCRS}} \), whose position in GCRS is \( n_{\text{GCRS}}^i = \left( \sin d \cos E, \sin d \sin E, \cos d \right) \). The new non-rotating origin (NLO) of the rotation angle \( \theta \) on the Earth equator (orthogonal to the rotation axis) is a point named the Celestial Intermediate Origin (CIO), whose position in CGRS requires the angle \( s \), called the CIO locator. Finally in the rotation matrix \( W^T(t_{\text{GCRS}}) \) (named the polar motion or wobble matrix) the angles \( x_p \) and \( y_p \) are the angular coordinates of CIP in ITRS, while the angle \( s' \) is connected with the re-orientation of the pole from the ITRS z-axis to the CIP plus a motion of the origin of longitude from the ITRS x-axis to the so-called Terrestrial Intermediate Origin (TIO), used as origin of the azimuthal angle.

Let us now consider the embedding \( z^\mu(\tau, \sigma^u) = x^\mu(\tau) + \epsilon^\mu_r R^r_s(\tau, \sigma) \sigma^s \) of Eq.(2.14) of I. Let us identify \( x^\mu = z^\mu(\tau, \sigma^u) \) with the GCRS 4-coordinates \( x_{\text{GCRS}}^\mu \) centered on the world-line of the geo-center assumed to move along a straight line. Then, if we identify the space-like vectors \( \epsilon^\mu_r \) with the GCRS non-rotating spatial axes, we have \( x^\mu(\tau) = \epsilon^\mu_r \tau = l^\mu \), where \( l^\mu \) is orthogonal to the nearly Euclidean 3-spaces \( t_{\text{GCRS}} = \text{const.} \). The proper time \( \tau \) of the geo-center coincides with \( c t_{\text{GCRS}} \) modulo \( O(c^{-2}) \) corrections from the GCRS PN 4-metric.

Then a special relativistic definition of ITRS can be done by replacing the rigidly rotating 3-coordinates \( x_{\text{ITRS}}^i \) with the differentially rotating 3-coordinates \( \sigma^r \). The rotation matrix \( R(\tau, \sigma) \), with the choice \( F(\sigma) = \frac{1}{1 + \frac{\omega^2}{2}} \) for the gauge function (\( \omega \) can be taken equal to the mean angular velocity for the Earth rotation), will contain three Euler angles determined by putting \( R(\tau, \sigma)|_{F(\sigma)=1} = C^T R_3(-\theta) W(t_{\text{GCRS}}) \).

In this way a special relativistic version of ITRS could be given as a preliminary step towards a PN general relativistic formulation of the geo-potential inside the Earth to be joined consistently with GCRS outside the Earth. Even if this is irrelevant for geodesy inside the geoid, it could lead to a refined treatment of effects like geodesic precession taking into account a model of geo-potential interpolating smoothly between inside and outside the geoid and the future theory of heights over the reference ellipsoid under development in a formulation of relativistic geodesy based on the use of the new generation of microwave and optical atomic clocks both on the Earth surface and in space.
III. NON-INERTIAL MAXWELL EQUATIONS IN NEARLY RIGID ROTATING FRAMES

In the 3+1 point of view the Maxwell equations (4.17) of I in an arbitrary inertial frame are identical to the Maxwell equations in general relativity, but now the 4-metric is describing only the inertial effects present in the given frame. Therefore we can adapt the techniques used in general relativity to non-inertial frames, for instance the definition of electric and magnetic fields done in Ref.[24] (see Appendix A of paper I) or the geometrical optic approximation to light rays of Ref.[25].

For the 1+3 point of view on this topic see for instance Ref.[26] and its bibliography. In particular, for the treatment of electromagnetic wave in rotating frame by means of Fermi coordinates [27] and for the determination of the helicity-rotation coupling, as a special case of spin-rotation coupling [28, 29]. In all these calculations the locality hypothesis (see Section IIB of paper I) is used.

In the case of linear acceleration an analysis of the inertial effects has been done in Ref.[20]. The same non-inertial 4-metric has been used in Ref.[30] to study the optical position meters constituents of the laser interferometers on ground used for the detection of gravitational waves. However the 4-metric used has a bad behavior at spatial infinity, so that the conclusions on the electro-magnetic waves in these frames (even if supposed to hold at distances smaller than those where there are coordinate singularities) are questionable because the Cauchy problem for Maxwell equations is not well posed.

In this Section we study some properties of electro-magnetic waves and of geometrical optic approximation to light rays in the radiation gauge in the admissible rotating non-inertial frame defined by the embedding (2.14) of I, ensuring a well-posed Cauchy problem, at small distances from the rotation axis where the $O(c^{-1})$ deviations from rigid rotations is governed by Eqs.(2.15) and (2.16) of I. Even if we will ignore these deviations, doing the calculations in the radiation gauge in locally rigidly rotating frames, they could be taken into account in a more refined version of the subsequent calculations base on the 3+1 point of view, which is free from coordinate singularities. This would also allow to verify the validity of the locality hypothesis. In particular we consider the Phase Wrap Up effect [16, 27], the Sagnac effect [14, 31] and the Faraday Rotation [32].

A. The 3+1 Point of View on Electro-Magnetic Waves and Light Rays in Nearly Rigidly Rotating Non-Inertial Frames.

Let us consider a non-inertial frame of the type (2.14) of I with vanishing linear acceleration and $\tau$-independent angular velocity and centered on an inertial observer. In the
notation of Eqs.(2.15), (2.16) and (4.47) of I, we have \( x^\mu(\tau) = e^\mu_\tau \tau \), i.e. \( \ddot{v}(\tau) = \ddot{w}(\tau) = 0 \), and \( \Omega(\tau) = \tilde{\Omega} = \text{const.} \) (whose components are \( \tilde{\Omega}^r = \text{const.} \)). We will ignore the higher order terms, so that locally we have a rigidly rotating frame, but with more effort small deviations from rigid rotation could be taken into account.

In this case the Hamiltonian (4.35), or (4.51), of I gives the following Hamilton equations for the transverse electro-magnetic field \( \vec{A}_\perp = \{ A^\perp_\tau = \tilde{A}^\perp_\tau \neq A_\perp^\tau \} + O(c^{-2}) \)

\[
\frac{\partial \tilde{A}^r_\perp(\tau, \vec{\sigma})}{\partial \tau} = \pi^r_\perp(\tau, \vec{\sigma}) - \frac{1}{c} \int d^3 \sigma' \left[ -\tilde{\Omega} \cdot \vec{\sigma}' \times \vec{\partial} \tilde{A}^r_\perp(\tau, \vec{\sigma}') + \tilde{\Omega} \times \tilde{A}^r_\perp(\tau, \vec{\sigma}') \right] s \mathbf{P}^{sr}(\vec{\sigma}', \vec{\sigma}),
\]

\[
\frac{\partial \pi^r_\perp(\tau, \vec{\sigma})}{\partial \tau} = \Delta \tilde{A}^r_\perp(\tau, \vec{\sigma}) - \frac{1}{c} \int d^3 \sigma' \left[ -\tilde{\Omega} \cdot \vec{\sigma}' \times \vec{\partial} \pi^r_\perp(\tau, \vec{\sigma}') + \tilde{\Omega} \times \tilde{\pi}^r_\perp(\tau, \vec{\sigma}') \right] s \mathbf{P}^{sr}(\vec{\sigma}', \vec{\sigma}) +
\]

\[
+ \sum_i Q_i \left( \tilde{n}_i(\tau) + \tilde{\Omega} \times \tilde{n}_i(\tau) \right) s \mathbf{P}^{sr}(\vec{n}_i, \vec{\sigma}). \tag{3.1}
\]

For the study of homogeneous solutions of these equations, i.e. for incoming electromagnetic waves propagating in regions where there are no charged particles, these equations can be replaced with the following ones (we use the vector notation of Section IVC of paper I)

\[
\frac{\partial \tilde{A}_\perp(\tau, \vec{\sigma})}{\partial \tau} = \tilde{\pi}_\perp(\tau, \vec{\sigma}) - \frac{1}{c} \left[ -\tilde{\Omega} \cdot \vec{\partial} \tilde{A}_\perp(\tau, \vec{\sigma}) + \tilde{\Omega} \times \tilde{A}_\perp(\tau, \vec{\sigma}) \right],
\]

\[
\frac{\partial \tilde{\pi}_\perp(\tau, \vec{\sigma})}{\partial \tau} = \Delta \tilde{A}_\perp(\tau, \vec{\sigma}) - \frac{1}{c} \left[ -\tilde{\Omega} \cdot \vec{\partial} \tilde{\pi}_\perp(\tau, \vec{\sigma}) + \tilde{\Omega} \times \tilde{\pi}_\perp(\tau, \vec{\sigma}) \right]. \tag{3.2}
\]

As shown in Appendix A of I, this result allows to recover the form given by Schiff in Appendix A of ref.[24] for the Landau-Lifschitz non-inertial electro-magnetic fields [33].

Let us look at solutions of Eqs.(3.2) in the following two ways.

1. **Going back to an Inertial Frame**

Let us look at solution by reverting to an inertial frame.

By introducing the 3-coordinates

\[
X^a(\tau) = R^a_\tau(\tau) \sigma^r, \tag{3.3}
\]

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at each value of $\tau$ by means of a $\tau$-dependent rotation (it would become also point-dependent if we go beyond rigid rotations) we can go from the rigidly rotating non-inertial frame with radar 4-coordinates $(\tau; \sigma^u)$ to an instantaneously comoving inertial frame, centered on the same inertial observer, with 4-coordinates $(\tau; X^a)$.

Let us assume that the non-inertial transverse electromagnetic potential $A_{\perp r}(\tau, \sigma^u)$ can be obtained from the instantaneously comoving inertial transverse potential $A_{\perp a}^{(\text{com})}(\tau, X^a(\tau))$ by using the rotation matrix $R(\tau)$

$$A_{\perp r}(\tau, \sigma^u) = A_{\perp a}^{(\text{com})}(\tau, X^a(\tau) = R_a^s(\tau) \sigma^s) R_r^a(\tau). \quad (3.4)$$

By definition $A_{\perp a}^{(\text{com})}(\tau, X^a(\tau))$ satisfies the inertial Maxwell equations in the radiation gauge (obtainable by putting Eqs.(3.4) into Eqs.(3.2))

$$\frac{\partial^2 A_{\perp a}^{(\text{com})}(\tau, X^b)}{\partial \tau^2} - \Delta_X A_{\perp a}^{(\text{com})}(\tau, X^b) = 0, \quad \sum_a \frac{\partial}{\partial X^a} A_{\perp a}^{(\text{com})}(\tau, X^b) = 0. \quad (3.5)$$

This result is in accord with the general covariance of non-inertial Maxwell equations and is also consistent with the locality hypothesis (see Section IIB of paper I) of the the 1+3 approach.

If we consider the following plane wave solution with constant $F_a$ and $\hat{K}_a$ and $\sum_a \hat{K}_a F_a = 0$ (transversality condition)

$$A_{\perp a}^{(\text{com})}(\tau, X^b) = \frac{1}{\omega} F_a e^{i \frac{\omega}{c} (\tau - \sum_a \hat{K}_a X^a)}, \quad (3.6)$$

we get the following expression for the non-inertial solution

$$A_{\perp r}(\tau, \sigma^u) = F_a R_r^a(\tau) e^{i \frac{\omega}{c} \Phi(\tau, \sigma^u)},$$

$$\Phi(\tau, \sigma^u) = \tau - \hat{K}_a R^a_r(\tau) \sigma^r \approx |\Omega|_{\text{const.}} \tau \left(1 + \frac{\tilde{\Omega}}{c} \cdot \tilde{\sigma} \times \hat{K}\right) - \hat{K} \cdot \tilde{\sigma} + O(\Omega^2/c^2). \quad (3.7)$$

2. Eikonal Approximation

Let us now look at solutions by making the following eikonal approximation (without any commitment with the locality hypothesis)

$$A_{\perp r}(\tau, \sigma^u) = \frac{1}{\omega} \alpha_r(\tau, \sigma^u) e^{i \frac{\omega}{c} \Phi(\tau, \sigma^u)} + O(1/\omega^2). \quad (3.8)$$
and by putting this expression in Eqs.(3.2).

Let us consider the case in which we have \( \omega/c \gg 1 \) e \( \Omega/c \ll 1 \), so that Eqs.(3.2) become a power series in \( \omega/c \). By neglecting terms in \( \Omega^2/c^2 \) and terms in \( (c/\omega)^{-k} \) for \( k \geq 0 \), the dominant terms are:

a) at the order \( \omega/c \) the equation for the phase \( \Phi \), named \textit{eikonal equation};

b) at the order \( (\omega/c)^o = 1 \) the equation for the amplitude \( a_r \), named \textit{first-order transport equation}.

These equations have the following form \( (\bar{a} = \{a_r\}) \)

\[
\left[ \left( \frac{\partial \Phi}{\partial \tau} \right)^2 - 2 \frac{\bar{\Omega}}{c} \cdot \bar{\sigma} \times \bar{\partial} \Phi - \left( \bar{\partial} \Phi \right)^2 \right] (\tau, \sigma^u) + O(\Omega^2/c^2) = 0
\]

\[
\left[ \frac{\partial \Phi}{\partial \tau} \left( \frac{\partial \bar{a}}{\partial \tau} + \frac{\bar{\Omega}}{c} \times \bar{a} - \frac{\bar{\Omega}}{c} \cdot \bar{\sigma} \times \bar{\partial} \bar{a} \right) - \frac{\partial \bar{a}}{\partial \tau} \frac{\bar{\Omega}}{c} \cdot \bar{\sigma} \times \bar{\partial} \Phi - \left( \bar{\partial} \Phi \cdot \bar{\sigma} \right) \bar{a} \right] (\tau, \sigma^u) =
\]

\[
= -\frac{1}{2} \left( \frac{\partial^2 \Phi}{\partial \tau^2} - 2 (\bar{\Omega} \times \bar{\sigma} \cdot \bar{\partial}) \frac{\partial \Phi}{\partial \tau} - \Delta \Phi \right) (\tau, \sigma^u) + O(\Omega^2/c^2)
\]

\[
\left[ \bar{a} \cdot \bar{\partial} \Phi \right] (\tau, \sigma^u) = 0 \quad (\text{transversality condition}). \tag{3.9}
\]

Let us look for solutions of the \textit{eikonal equation} for \( \Phi \) of the form

\[
\Phi(\tau, \sigma^u) = \tau + F(\sigma^u), \tag{3.10}
\]

where we have chosen the boundary condition

\[
\frac{\partial \Phi}{\partial \tau} = 1. \tag{3.11}
\]

This condition implies that the solution of Eq.(3.8) describes a ray emitted from a source having a characteristic frequency \( \omega \) when it is at rest in the non-inertial frame. Let us remark that in more general cases this type of boundary conditions are possible only if the 3-metric \( h_{rs} \) and the lapse \( (n) \) and shift \( (n^r) \) functions are stationary in the non-inertial frame.

An expansion in powers of \( \Omega/c \) of \( F(\sigma^u) \), namely \( F(\sigma^u) = F_o(\sigma^u) + \frac{\Omega}{c} F_1(\sigma^u) + O\left(\frac{\Omega^2}{c^2}\right) \),

gives the following form of the eikonal equation

\[
\left[ 1 - \left( \bar{\partial} F_o(\sigma^u) \right)^2 \right] - \frac{2\Omega}{c} \left[ \bar{\Omega} \cdot \bar{\sigma} \times \bar{\partial} F_o(\sigma^u) + \bar{\partial} F_o(\sigma^u) \cdot \bar{\partial} F_1(\sigma^u) \right] + O\left(\frac{\Omega^2}{c^2}\right) = 0, \tag{3.12}
\]
implying:
a) the equation 
\[ 1 - \left( \hat{\partial} F_o(\sigma^u) \right)^2 = 0 \]
at the order zero in \( \Omega \). If \( \hat{k} \) is an arbitrary unit vector (the propagation direction of the plane wave in the inertial limit \( \Omega \rightarrow 0 \)), its solution is

\[ F_o(\sigma^u) = -\hat{k} \cdot \hat{\sigma}. \tag{3.13} \]

b) the equation 
\[ \hat{k} \cdot \hat{\partial} F_1(\sigma^u) = -\hat{\Omega} \cdot \hat{\sigma} \times \hat{k} \]
for \( F_1(\sigma^u) \), after having used Eq.(3.13), at the order one in \( \Omega \). Since we have \( (\hat{k} \cdot \hat{\partial}) (\hat{\Omega} \cdot \hat{\sigma} \times \hat{k}) = 0 \) and \( (\hat{k} \cdot \hat{\partial}) (\hat{k} \cdot \hat{\sigma}) = 1 \), the solution for \( F_1(\sigma^u) \) is

\[ F_1(\sigma^u) = - \left( \hat{\Omega} \cdot \hat{\sigma} \times \hat{k} \right) (\hat{k} \cdot \hat{\sigma}). \tag{3.14} \]

Therefore the solution for \( \Phi \) is

\[ \Phi(\tau, \sigma^u) = \tau - \hat{k} \cdot \hat{\sigma} \left( 1 + \hat{\Omega} \cdot \hat{\sigma} \times \hat{k} \right). \tag{3.15} \]

The phases in the solutions (3.7) and (3.15) of Eqs.(3.2) are different since the solutions have different boundary conditions. The solution (3.7) satisfies also the eikonal equation but not the boundary condition (3.11), since we have \[ \frac{\partial \Phi}{\partial \tau} = 1 - \hat{K}_a \epsilon_{rur} \hat{\Omega}^u \sigma^v \neq 1. \]

Let us remark that both the solutions (3.7) and (3.15) have the following structure

\[ \tilde{A}_1^r(\tau, \sigma^u) \sim A^r(\tau, \sigma^u) e^{i\varphi(\tau, \sigma^u)}, \tag{3.16} \]
where \( A^r(\tau, \sigma^u) \sim O(1/\omega) \) is the amplitude and \( \varphi(\tau, \sigma^u) \sim O(\omega) \) is the phase. The only difference is that the solution (3.7) holds for every value of \( \omega \) (also for the small values corresponding to the radio waves of the GPS system), while the solution (3.15) for the phase of the eikonal approximation (3.8) holds only for higher values of \( \omega \), corresponding to visible light.

### 3. Light Rays

Given the phase of Eq.(3.16), the trajectories of the light rays are defined as the lines orthogonal (with respect to the 4-metric \( g_{AB} \) of the 3+1 splitting) to the hyper-surfaces \( \varphi(\tau, \sigma^u) = \text{const.} \). Therefore the trajectories \( \sigma^A(s) \) (\( s \) is an affine parameter) satisfy the equation

\[ \frac{d\sigma^A(s)}{ds} = g^{AB}(\sigma(s)) \frac{\partial \varphi}{\partial \sigma^B}(\sigma(s)). \tag{3.17} \]
For instance in the case of our rigidly rotating foliation, for which Eqs. (2.14)-(2.16) of I imply \( g^{rr} = 1, g^{rr} = -(\vec{\Omega} \times \vec{\sigma})^r, g^{rs} = -\delta^r s + O(\Omega^2/c^2) \), Eqs. (3.17) take the form

\[
\frac{d\tau(s)}{ds} = \omega + \vec{k} \cdot \left( \frac{\vec{\Omega}}{c} \times \vec{\sigma} \right) + O(\Omega^2/c^2),
\]

\[
\frac{d\sigma^r(s)}{ds} = \omega \left( \frac{\vec{\Omega}}{c} \times \vec{\sigma} \right)^r + k^r \left( 1 + \frac{\vec{\Omega}}{c} \times \vec{\sigma} \cdot \vec{k} \right) - \left( \frac{\vec{\Omega}}{c} \times \vec{k} \right)^r (\vec{k} \cdot \vec{\sigma}) + O(\Omega^2/c^2),
\]

whose solution has the form

\[
\vec{\sigma}(\tau) - \vec{\sigma}(0) = \vec{k} \tau + \left( \frac{\vec{\Omega}}{c} \times \vec{k} \right) \tau^2 + O(\Omega^2/c^2).
\]

This equation shows that in the rotating frame the ray of light appears to deviate from the inertial trajectory \( \vec{\sigma}(\tau) = \vec{k} \tau \) due to the centrifugal correction \( \vec{c}(\tau) = \left( \frac{\vec{\Omega}}{c} \times \vec{k} \right) \tau^2 + O(\Omega^2/c^2) \) implying \( \vec{k} \cdot \vec{c}(\tau) = 0 + O(\Omega^2/c^2) \).

\[\text{B. Sources and Detectors}\]

To connect the previous solutions to the interpretation of observed data we need a schematic description of sources and detectors.

In many applications sources and detectors are described as point-like objects, which follow a prescribed world-line \( \zeta^A(\tau) = (\tau, \eta^A(\tau)) \) with unit 4-velocity \( v^A(\tau) = \frac{d\zeta^A(\tau)}{d\tau} \left( g^{CD}(\zeta(\tau)) \frac{d\zeta^C(\tau)}{d\tau} \frac{d\zeta^D(\tau)}{d\tau} \right)^{-1/2} \).

This description is enough for studying the influence of the relative motion between source and detector on the frequency emitted from the source and that observed by the detector (it works equally well for the Doppler effect and for the gravitational redshift in presence of gravity). With solutions like Eq. (3.16) the frequency emitted by a source located in \( \zeta^s_A \) and moving with 4-velocity \( v^s_A \) and that observed by a detector in \( \zeta^r_A \) and moving with 4-velocity \( v^r_A \) are \( \omega_s = v^s_A \partial_A \varphi(\zeta^s) \) and \( \omega_r = v^r_A \partial_A \varphi(\zeta^r) \), respectively.

This justifies the boundary condition (3.11), because sources at rest in the rotating frame with coordinates \( (\tau, \sigma^r) \) have 4-velocity \( v^A = (1, 0) \).

However, to measure the electro-magnetic field in assigned (spatial) polarization direction we must assume that the detector is endowed with a tetrad orthonormal with respect to the 4-metric of the 3+1 splitting, such that the time-like 4-vector is the unit 4-velocity of the detector: in 4-coordinates adapted to the 3+1 splitting they are \( \mathcal{E}^A_{(\alpha)}(\tau) = (\mathcal{E}^A_{(\alpha)})(\tau) = \ldots \).
Let us consider the following two cases.

1. Detectors at Rest in an Inertial Frame

A detector at rest in the instantaneous inertial frame with coordinates $(\tau; X^a(\tau))$ follows the straight world-line $z^\mu(\tau, \sigma^u) = e^\mu_\tau \tau + e^\mu_\sigma R^\tau_s(\tau) \sigma^s + O(\Omega^2/c^2)$, so that $z^\mu(\tau, \sigma^u) = e^\mu_\tau \tau + e^\mu_\sigma R^\tau_s(\tau) \sigma^s + O(\Omega^2/c^2)$ and $z^\mu(\tau, \sigma^u) = e^\mu_\sigma R^\sigma_r(\tau) + O(\Omega^2/c^2)$.

The world-line of these objects will have the form $\zeta^\mu(\tau) = e^\mu_\tau \tau + e^\mu_\sigma R^\tau_s(\tau) \eta^s + O(\Omega^2/c^2) = e^\mu_A \zeta^A(\tau)$ with $\eta^s = \text{const}$. We have $\zeta^\tau(\tau) = \tau$ and $\zeta^\sigma(\tau) = R^\tau_s(\tau) \eta^s + O(\Omega^2/c^2)$. Therefore these objects coincide with some of the observers belonging at the non-surface forming congruence generated by the evolution vector field as said in Section IIB of paper I. Since the world-lines of the Eulerian observers of the other congruence are not explicitly known, it is not possible to study the behavior of objects coinciding with some of these observers.

Therefore the unit 4-velocity $u^\mu(\tau) = e^\mu_A v^A(\tau)$ will have the components $v^A(\tau)$ proportional to $\dot{\zeta}^A(\tau) = \left(1; \dot{R}^\tau_s(\tau) \eta^s + O(\Omega^2/c^2)\right) \approx \left|_{\eta^s = \text{const.}} \left(1; \dot{R}^\tau_s(\tau) (\eta^s \times \vec{\Omega}/c)^s + O(\Omega^2/c^2)\right)\right.$, where the definitions after Eq.(2.14) of I have been used.

We can also write $u^\mu(\tau) = \tilde{u}^A(\tau) z^\mu_A(\tau, \eta^s)$ by using the non-orthonormal tetrads $z^\mu_A(\tau, \sigma^u)$. Then we get $v^\tau(\tau) = \tilde{u}^\tau(\tau) + O(\Omega^2/c^2)$ and $v^\sigma(\tau) = \tilde{u}^\sigma(\tau) \dot{R}^\tau_s(\tau) \eta^s + R^\tau_s(\tau) \tilde{u}^\tau(\tau) + O(\Omega^2/c^2)$. While the quantities $v^A(\tau)$ give the description of the 4-velocity with respect to the asymptotic non-rotating inertial observers, the quantities $\tilde{u}^A(\tau)$ explicitly show the effect of the rotation at the position $\eta^s$ of the object. Therefore it should be $\tilde{u}^A(\tau) = (1; 0)$ at the lowest or-
der: indeed we get $\tilde{u}^s(\tau) = 1 + O(\Omega^2/c^2)$ and $\tilde{u}^r(\tau) = v^s(\tau) R_s^r(\tau) - \tilde{u}^r \left( R^{-1}(\tau) \tilde{R}(\tau) \right)^r_s \eta_o^s = 0 + O(\Omega^2/c^2)$.

For the constant unit normal to the instantaneous 3-spaces we get $l^\mu = e^\mu_x = \tilde{l}^A(\tau, \eta_o^r) z^\mu(\tau, \eta_o^r)$ with $\tilde{l}^r(\tau, \eta_o^r) = 1 + O(\Omega^2/c^2)$ and $\tilde{l}^r(\tau, \eta_o^u) = -\tilde{l}^r(\tau, \eta_o^u) \left( R^{-1}(\tau) \tilde{R}(\tau) \right)^r_s \eta_o^s = -\left( \eta_o \times \frac{\hat{\Omega}}{c} \right)^r + O(\Omega^2/c^2)$.

Let us introduce an orthonormal tetrad $W_{(\alpha)}^\mu(\tau, \eta_o^r)$, $\eta_{\mu\nu} W_{(\alpha)}^\mu W_{(\beta)}^\nu = \eta_{(\alpha)(\beta)}$, whose time-like 4-vector is $l^\mu$, i.e. We have $W_{(\alpha)}^\mu = l^\mu = e^\mu_x = W_{(\alpha)}^A e^A_x = \tilde{W}_{(\alpha)}^A z^A(\tau, \eta_o^r)$ with $W_{(\alpha)}^A = (1; 0)$ and $\tilde{W}_{(\alpha)}^r = \tilde{l}^r(\tau, \eta_o^r) z^r(\tau, \eta_o^r)$ with $l^r, W_{(i)}^r = \tilde{l}^r(\tau, \eta_o^r) \tilde{W}_{(i)}^r = 0$ must be non-rotating with respect to the observer with 4-velocity proportional to $z^r(\tau, \eta_o^r)$. Therefore we must have $\tilde{W}_{(i)}^A = \tilde{l}^A(\tau, \eta_o^r)$ with $\tilde{W}_{(i)}^r = constant$. As a consequence we have $W_{(i)}^A(\tau) = 0; R^s(\tau) \tilde{W}_{(i)}^s + O(\Omega^2/c^2)$.

The polarization axes of sources and detectors will be defined by a tetrad $E_{(\alpha)}^\mu(\tau, \eta_o^r) = E_A^\mu(\tau, \eta_o^r) e^A_x = \tilde{E}_A^\mu(\tau, \eta_o^r) z^A(\tau, \eta_o^r)$, $\eta_{\mu\nu} E_{(\alpha)}^\mu E_{(\beta)}^\nu = \eta_{(\alpha)(\beta)}$ with the following properties:

a) the time-like 4-vector $E_{(\alpha)}^\mu(\tau, \eta_o^r)$ is such that its components $\tilde{E}_{(\alpha)}^\mu(\tau, \eta_o^r)$ coincide with the components $\tilde{u}^A(\tau) = (1; 0) + O(\Omega^2/c^2)$ of the 4-velocity $u^\mu(\tau)$ of the object located at $z^\mu(\tau, \eta_o^r)$: as a consequence we have $E_{(\alpha)}^\mu(\tau, \eta_o^r) = z^\mu(\tau, \eta_o^r) + O(\Omega^2/c^2) = u^\mu(\tau)$;

b) the spatial axes $E_{(i)}^\mu(\tau, \eta_o^r)$, orthogonal to the 4-velocity $u^\mu(\tau)$, must be at rest in the rotating frame: we have to identify their components $\tilde{E}_{(i)}^\mu(\tau, \eta_o^r)$.

If at the observer position we consider the Lorentz transformation sending $l^\mu$ to $u^\mu(\tau)$, i.e. $L^\mu_\nu(l \mapsto u(\tau))$, its projection $L^A_B(\tilde{\beta}) \rightarrow elf = e^A_B L^\nu_\nu(l \mapsto u(\tau)) e^\nu_B$ is a Wigner boost, see Eq.(2.8) of I, with parameter $\tilde{\beta} = \{ \beta^r = R^s(\tau) \left( \eta_o \times \frac{\hat{\Omega}}{c} \right)^s \}$ (so that $\gamma = \sqrt{1 - \beta^2} = 1 + O(\Omega^2/c^2)$).

Therefore the transformation sending the components $\tilde{l}^A(\tau, \eta_o^r)$ of the unit normal into the components $\tilde{u}^A(\tau)$ of the 4-velocity modulo terms of order $O(\Omega^2/c^2)$ is

$$
\tilde{E}_{(\alpha)}^A(\tau, \eta_o^r) = \tilde{u}^A(\tau) = (1; 0) + O(\Omega^2/c^2) = \\
= \left( z^A \mu e^\mu C D(\tilde{\beta}) e^D e^\nu B(\tilde{l}_{(\alpha)}^r) \right) (\tau, \eta_o^r) + O(\Omega^2/c^2) = \\
= \left( z^A \mu e^\mu C D(\tilde{\beta}) e^D e^\nu B \tilde{W}_{(\alpha)}^B \right) (\tau, \eta_o^r) + O(\Omega^2/c^2),
$$

$$
\tilde{E}_{(i)}^A(\tau, \eta_o^r) = \left( z^A \mu e^\mu C D(\tilde{\beta}) e^D e^\nu B \tilde{W}_{(i)}^B \right) (\tau, \eta_o^r) \tilde{W}_{(i)}^B. \tag{3.20}
$$

This complete the construction of the non-rotating tetrads $E_{(\alpha)}^\mu(\tau, \eta_o^r)$ for the objects at rest at $\eta_o^r$. 

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A detector endowed of such a non-rotating tetrad will measure the following projections of the electro-magnetic field strength on its polarization directions

$$\hat{E}_i = F_{AB} \tilde{u}^A \tilde{E}^B_{(i)}, \quad \hat{B}_i = \frac{1}{2} \epsilon_{(i)(j)(k)} F_{AB} \tilde{E}^A_{(j)} \tilde{E}^B_{(k)}. \quad (3.21)$$

These quantities have to be confronted with the non-inertial electric and magnetic fields $E_r$ and $B_r$, whose projections on the non-rotating spatial axes $\tilde{W}^A_{(i)} = (0; \tilde{W}^r_{(i)})$ inside the instantaneous 3-space are

$$E_{(i)} = E_r \tilde{W}^r_{(i)}, \quad B_{(i)} = B_r \tilde{W}^r_{(i)}. \quad (3.22)$$

Eqs.(3.20) imply the following connection among these quantities

$$\hat{E}_{(i)} = E_{(i)} + \mathcal{O}(\Omega^2/c^2),$$

$$\hat{B}_{(i)} = B_{(i)} - \epsilon_{ijk} \tilde{W}^r_{(j)} \delta_{rs} \left( \eta_o \times \frac{\tilde{\Omega}}{c} \right)^s E_{(k)} + \mathcal{O}(\Omega^2/c^2). \quad (3.23)$$

For radio wave (like in the case of GPS) the directions $G^a_{(i)}$ or $\tilde{E}^r_{(i)}$ are realized by means of antennas attached to both emitters and receivers. In the optical range the antennas are replaced by components of the macroscopic devices used for the emission and the detection.

C. The Phase Wrap Up Effect

The phase wrap up is a modification of the phase when a receiver in rotational motion analyzes the circularly polarized radiation emitted by a source at rest in an inertial frame. Till now the effect has been explained by using the 1+3 point of view and the locality hypothesis in Refs.[27], where it showed that it is a particular case of helicity-rotation coupling (the spin-rotation coupling for photons). It has been verified experimentally, in particular in GPS [16], where the receiving antenna on the Earth surface is rotating with Earth.

We will explain the effect by using the non-inertial solution (3.7) and an observer at rest in an inertial frame endowed of the tetrad $G^\mu_{(A)}$ defined in Subsubsection 1 of Subsection B. We rewrite the spatial axes in the form $\tilde{G}^a_{(i)} = \left( \tilde{I}_1^a, \tilde{I}_2^a, \tilde{K}^a \right)$ with the vectors satisfying $\tilde{I}_1 \cdot \tilde{I}_2 = 0, \tilde{I}_1 \cdot \tilde{K} = 0 (\lambda = 1, 2), \tilde{I}_2^2 (\lambda) = 1$. Then we pass to a circular basis by introducing the vectors $\tilde{I}_{(\pm)} = \frac{\tilde{I}_1 + \tilde{I}_2}{\sqrt{2}}$, which satisfy $\tilde{K} \cdot \tilde{I}_{(\pm)} = 0, \tilde{I}_{(\pm)}^2 = 0$ and $\tilde{I}_{(+)} \cdot \tilde{I}_{(-)} = 1$.

In the rotating non-inertial frame a right-circularly polarized wave, emitted in the inertial frame, will have the form (3.7) ($\tilde{K} \cdot \tilde{I}_{(+)} = 0$ is the transversality condition)
Let us remark that in the circular basis we have \( \vec{A}_\perp = A_n \hat{n} + A_\perp \vec{I}_\perp = A_\perp \vec{I}_\perp \), but the components \( A_n, A_\pm \), coincide with either linearly or circularly polarized states of the electro-magnetic field only for \( \hat{n} = \hat{k} \), since \( \vec{K} = \frac{\omega}{c} \hat{k} \ (\vec{K}^2 = \frac{\omega^2}{c^2}) \) is the wave vector.

From Eqs (3.24) we obtain the following non-inertial magnetic and electric fields (2.19) of I

\[
B_r = -\frac{F}{c} I_{(+)} a R^a_r(\tau) e^{i \frac{\omega}{c} \Phi(\tau, \vec{s})},
\]

\[
E_r = -i \frac{F}{c} I_{(+)} a R^a_r(\tau) e^{i \frac{\omega}{c} \Phi(\tau, \vec{s})} + \frac{1}{c} (\vec{\Omega} \times \vec{s}) \times \vec{B} = E_o I_{(+)} a R^a_r(\tau) e^{i \frac{\omega}{c} \Phi(\tau, \vec{s})} + E_\ell \vec{K}_a R^a_r(\tau) e^{i \frac{\omega}{c} \Phi(\tau, \vec{s})},
\]

\[
B_o = -\frac{F}{c}, \quad E_o = -i \frac{F}{c} + \frac{1}{c} (\vec{\Omega} \times \vec{s}) \times \vec{B} \cdot \vec{I}_\perp, \quad E_\ell = \frac{1}{c} (\vec{\Omega} \times \vec{s}) \times \vec{B} \cdot \vec{K}. \quad (3.25)
\]

Let us now consider a receiver at rest in the rotating frame. Since its 4-velocity is \( \vec{u}^A = (1; 0) \), it can be endowed with the non-rotating tetrad \( \vec{W}^A_{(o)} \) of Subsubsection 2 of Subsection B. If \( \hat{n} \) is the unit vector in the direction of the rotation axis, i.e. if \( \vec{\Omega} = \Omega \hat{n} \), we can choose the spatial axes \( \vec{W}^A_{(i)} = (\vec{e}_{(1)}, \vec{e}_{(2)}, \hat{n}^\tau) \) with \( \vec{e}_{(1)} \cdot \vec{e}_{(2)} = 0, \vec{e}_{(A)} \cdot \vec{K} = 0, \vec{e}_{(A)}^2 = 1 \).

If we introduce the circular basis \( \vec{e}_{(\pm)} = \frac{\vec{e}_{(1)} \pm i \vec{e}_{(2)}}{\sqrt{2}} \), we have \( \hat{n} \cdot \vec{e}_{(\pm)} = 0, \vec{e}_{(\pm)}^2 = 0, \vec{e}_{(+)} \cdot \vec{e}_{(-)} = 1 \) and \( R^a_r(\tau) \epsilon_{(\pm)} = \epsilon_{(\pm)}^a \).

The receiver will measure the following magnetic and electric fields

\[
B_n = B_r \hat{n}^\tau = B_o (\vec{I}_{(+)} a \hat{n}^a) \exp \left[ i \frac{\omega}{c} \Phi \right],
\]

\[
B_{(\pm)} = B_r \vec{e}_{(\pm)}^\tau = B_o (\vec{I}_{(+)} a \vec{e}_{(\pm)}^a) \exp \left[ i \frac{\omega}{c} (\mp \Omega \tau \mp \omega \Phi(\tau, \vec{s})) \right],
\]

\[
E_n = E_r \hat{n}^\tau = \left[ E_o (\vec{I}_{(+)} a \hat{n}^a) + E_\ell \vec{K}_a \hat{n}^a \right] \exp \left[ i \frac{\omega}{c} \Phi \right],
\]

\[
E_{(\pm)} = E_r \vec{e}_{(\pm)}^\tau = \left[ E_o (\vec{I}_{(+)} a \vec{e}_{(\pm)}^a) + E_\ell \vec{K}_a \vec{e}_{(\pm)}^a \right] \exp \left[ i \frac{\omega}{c} (\mp \Omega \tau \mp \omega \Phi(\tau, \vec{s})) \right]. \quad (3.26)
\]
In the case $\hat{n}^a = \hat{K}^a$ we find

$$B_n = B_{(-)} = 0$$

$$B_{(+)} = B_o e^{i\left[(\omega-\Omega)\tau+\hat{K} \cdot \vec{\sigma}\right]} ,$$

$$E_n = E_\ell e^{i\omega_{\tau+\hat{K} \cdot \vec{\sigma}}}, \quad E_{(+)} = 0$$

$$E_{(+)} = E_o e^{i\left[(\omega-\Omega)\tau+\hat{K} \cdot \vec{\sigma}\right]}.$$  \quad (3.27)

Therefore the components $B_{(+)}$, $E_{(+)}$ have the frequency modified to $\omega \mapsto \omega - \Omega$: this is the phase wrap up effect. These are same results as in Ref.[27] at the lowest order in $\Omega/c$. The only new fact is the presence of the component $E_n \neq 0$.

It would be interesting to make the calculation of the deviations of order $O(\Omega^2/c^2)$ from rigid rotation, to see whether the result $\omega \mapsto \gamma (\omega \pm \Omega)$ ($\gamma$ is a Lorentz factor), found in Ref.[27] by using the locality hypothesis and supporting the interpretation with the helicity-rotation coupling, is confirmed.

D. The Sagnac Effect

Following a suggestion of Ref.[14] let us consider the solution (3.8) in the eikonal approximation, which describes the propagation of the radiation along a ray of light whose trajectory is given in Eq.(3.19). This solution allows to get a derivation of the Sagnac effect (described in Section II) along the lines of Ref.[31].

Let us consider two receivers $A$ and $B$ at rest in the rotating frame and characterized by the 3-coordinates $\eta_A^r$ and $\eta_B^r$ respectively. Let us assume that $A$ and $B$ lie in the same 2-plane containing the origin $\sigma^r = 0$ and orthogonal to $\vec{\Omega}$. Therefore we have $\vec{\Omega} \cdot \vec{\eta}_A = \vec{\Omega} \cdot \vec{\eta}_B = 0$. Let us assume that $A$ and $B$ are both on the trajectory of a ray of light, so that Eq.(3.19) implies the existence of a time $\tau_{AB}$ such that we have

$$\vec{\eta}_B - \vec{\eta}_A = \hat{k} \tau_{AB} + \left(\frac{\vec{\Omega}}{c} \times \hat{k}\right) \tau_{AB}^2 + O(\Omega^2/c^2).$$  \quad (3.28)

The phase difference between $A$ and $B$ at the same instant $\tau$ is
\[ \Delta \varphi_{AB} = \frac{\omega}{c} \left[ \Phi(\tau, \vec{n}_B) - \Phi(\tau, \vec{n}_A) \right] = \]

\[ = -\frac{\omega}{c} \left[ \hat{k} \cdot (\vec{n}_B - \vec{n}_A) + \hat{k} \cdot \vec{n}_B \left( \frac{\vec{\Omega}}{c} \cdot \vec{n}_B \times \hat{k} \right) - (\hat{k} \cdot \vec{n}_A) \left( \frac{\vec{\Omega}}{c} \cdot \vec{n}_A \times \hat{k} \right) \right] + \]

\[ + O(\Omega^2/c^2). \quad (3.29) \]

Eq. (3.28) implies

\[ \vec{n}_B = \vec{n}_A + \hat{k} \tau_{AB} + O(\Omega/c) \Rightarrow \frac{\vec{\Omega}}{c} \cdot \vec{n}_B \times \hat{k} = \frac{\vec{\Omega}}{c} \cdot \vec{n}_A \times \hat{k} + O(\Omega^2/c^2), \quad (3.30) \]

so that we get

\[ \Delta \varphi_{AB} = -\frac{\omega}{c} \left[ \hat{k} \cdot (\vec{n}_B - \vec{n}_A) + \hat{k} \cdot \vec{n}_B \left( \frac{\vec{\Omega}}{c} \cdot \vec{n}_B \times \hat{k} \right) + O(\Omega^2/c^2) \right]. \quad (3.31) \]

Since Eq. (3.28) also implies \( \vec{n}_B - \vec{n}_A = | \vec{n}_B - \vec{n}_A | \hat{k} + O(\Omega^2/c^2) \), we arrive at the result

\[ \Delta \varphi_{AB} = -\frac{\omega}{c} \left[ | \vec{n}_B - \vec{n}_A | + \frac{\vec{\Omega}}{c} \cdot \vec{n}_A \times (\vec{n}_B - \vec{n}_A) \right] + O(\Omega^2/c^2). \quad (3.32) \]

If \( A_{BAO} \) is the area of the triangle BAO in the 2-plane orthogonal to \( \vec{\Omega} \), we have \( \frac{\vec{\Omega}}{c} \cdot \vec{n}_A \times (\vec{n}_B - \vec{n}_A) = \pm 2 \frac{\Omega}{c} A_{BAO} \) (the choice of \( \pm \) depends on the direction of motion of the ray). As a consequence, the phase difference is the sum of the following two terms

\[ \Delta \varphi_{AB} = -\frac{\omega}{c} | \vec{n}_B - \vec{n}_A | + \delta \varphi_{AB} + O(\Omega^2/c^2). \quad (3.33) \]

While the first term, \( -\frac{\omega}{c} | \vec{n}_B - \vec{n}_A | \), is present also in the inertial frames, the second term

\[ \delta \varphi_{AB} = \pm \frac{2 \omega \Omega}{c^2} A_{BAO}, \quad (3.34) \]

is the extra phase variation due to the rotation of the frame. This is the Sagnac effect.

**E. The Inertial Faraday Rotation**

Let us give the derivation of the rotation of the polarization of an electro-magnetic wave in a rotating frame, named inertial Faraday rotation, which is important in astrophysics
[32], were it is induced by the gravitational field (due to the equivalence principle only non-inertial frames are allowed in general relativity). Our approach is analogous to the one of Ref.[25] in the case of Post-Newtonian gravity.

Let us consider the amplitude \( \vec{a} \) of the solution (3.8) in the eikonal approximation: it carries the information about the polarization of a ray of light. To study the first-order transport equation for it, the second of Eqs.(3.9), let us make the series expansion

\[
\vec{a}(\tau, \vec{\sigma}) = \vec{a}_o(\tau, \vec{\sigma}) + \Omega c \vec{a}_1(\tau, \vec{\sigma}) + O\left(\frac{\Omega^2}{c^2}\right),
\]

(3.35)

and let us make the ansatz (in an inertial frame it corresponds to a plane wave)

\[
\vec{a}_o(\tau, \vec{\sigma}) = \vec{a}_o = \text{const.}, \quad \Rightarrow \quad \frac{\partial \vec{a}_o}{\partial \tau} = 0, \quad \partial_r \vec{a}_o = 0.
\]

(3.36)

This ansatz implies the following form of the second and third equation in Eqs.(3.9)

\[
\frac{\Omega}{c} \left[ \left( \frac{\partial \vec{a}_1}{\partial \tau} + \hat{\Omega} \times \vec{a}_o \right) - (\hat{k} \cdot \hat{\sigma}) \vec{a}_1 \right] + O\left(\frac{\Omega^2}{c^2}\right) = 0,
\]

\[
\vec{a}_o \cdot \hat{k} + \frac{\Omega}{c} \left[ \vec{a}_o \cdot \left( \hat{k} (\hat{\Omega} \cdot \hat{\sigma} \times \hat{k}) - (\hat{k} \cdot \hat{\sigma}) (\hat{\Omega} \times \hat{k}) \right) + \vec{a}_1 \cdot \hat{k} \right] + O\left(\frac{\Omega^2}{c^2}\right) = 0.
\]

(3.37)

To study these equations, let us assume that each rotating receiver is endowed with a tetrad of the type given in Eq.(3.20): the spatial axes \( \vec{W}^r_{(i)} = (R^r_1(k), R^r_2(k), \hat{k}^r) \) with \( \vec{R}_\lambda(k) \cdot \vec{R}_{\lambda'}(k) = \delta_{\lambda\lambda'} \), \( \vec{R}_\lambda(k) \cdot \hat{k} = 0 \).

The second of Eqs.(3.37) for the unknown \( \vec{a}_o, \vec{a}_1 \) is the transversality condition and it implies

\[
\text{order 0 in } \Omega \rightarrow \vec{a}_o \cdot \hat{k} = 0 \Rightarrow \vec{a}_o = a_o^\lambda \vec{R}_\lambda(k),
\]

\[
\text{order 1 in } \Omega
\]

\[
\vec{a}_1 \cdot \hat{k} = -\vec{a}_o \cdot \left( \hat{k} (\hat{\Omega} \cdot \hat{\sigma} \times \hat{k}) + (\hat{k} \cdot \hat{\sigma}) (\hat{\Omega} \times \hat{k}) \right) =
\]

\[
= -a_o^\lambda \vec{R}_\lambda(k) \cdot (\hat{\Omega} \times \hat{k}) (\hat{k} \cdot \hat{\sigma}).
\]

(3.38)

Due to the ansatz (3.36) the first of Eqs.(3.37) is of order 1 in \( \Omega \) and gives the following condition on \( \vec{a}_1 \)
\[ \frac{\partial \vec{a}_1}{\partial \tau} - \hat{\Omega} \times \vec{a}_o + (\hat{k} \cdot \vec{\partial}) \vec{a}_1 = 0. \]  

(3.39)

If we project this equation on the directions \( \hat{k}, \vec{R}_\lambda(k) \), we get

\[ \frac{\partial}{\partial \tau} (\vec{a}_1 \cdot \hat{k}) - \hat{\Omega} \times \vec{a}_o \cdot \hat{k} + (\hat{k} \cdot \vec{\partial}) (\vec{a}_1 \cdot \hat{k}) = 0, \]

\[ \frac{\partial \vec{a}_1^\lambda}{\partial \tau} - \hat{\Omega} \times \vec{R}_\lambda'(k) \cdot \vec{R}_\lambda(k) a_\alpha^\nu + (\hat{k} \cdot \vec{\partial}) a_1^\lambda = 0. \]  

(3.40)

While the first of Eqs.(3.40) is automatically satisfied, the second one is an equation for the components \( a_1^\lambda \). The simplest solutions are obtained with the following ansatz

\[ \frac{\partial \vec{a}_1^\lambda}{\partial \tau} = 0, \implies \vec{a}_1^\lambda(\tau) = [\hat{\Omega} \times \vec{R}_\lambda'(k) \cdot \vec{R}_\lambda(k)] \hat{k} \cdot \vec{\sigma} a_\alpha^\nu. \]  

(3.41)

The final solution for the transverse electro-magnetic potential is

\[ \vec{A}_\perp = \frac{a_o}{\omega} \left[ \vec{R}_1 + \theta(\vec{\sigma}) \vec{R}_2(k) - \frac{\hat{k} \cdot \vec{\sigma}}{c} (\vec{\Omega} \cdot \vec{R}_2(k)) \hat{k} \right] e^{i \frac{\vec{\sigma}}{c} \Phi} + \]

\[ + \frac{a_o^2}{\omega} \left[ \vec{R}_2(k) - \theta(\vec{\sigma}) \vec{R}_1(k) + \frac{\hat{k} \cdot \vec{\sigma}}{c} (\vec{\Omega} \cdot \vec{R}_1(k)) \hat{k} \right] e^{i \frac{\vec{\sigma}}{c} \Phi} + O(1/\omega^2), \]

with

\[ \theta(\vec{\sigma}) = \frac{1}{c} (\hat{k} \cdot \vec{\sigma}) (\vec{\Omega} \cdot \hat{k}). \]  

(3.42)

The resulting non-inertial magnetic and electric fields are \( \vec{B} = \{B_r\}, \vec{E} = \{E_r\} \)
\[ \vec{B} = -i a_o \frac{1}{c} \left[ \vec{R}_2(k) - \theta(\vec{\sigma}) \vec{R}_1(k) \right] e^{(i \vec{\omega} \cdot \vec{\phi})} - \]
\[ - i a_o \frac{1}{c} \left[ \left( \frac{\vec{\Omega}}{c} \times \vec{\sigma} \cdot \hat{k} \right) \vec{R}_2(k) - \frac{\hat{k} \cdot \vec{\sigma}}{c} (\vec{\Omega} \cdot \vec{R}_1(k)) \hat{k} \right] e^{(i \vec{\omega} \cdot \vec{\phi})} + \]
\[ + i a_o \frac{2}{c} \left[ \vec{R}_1(k) + \theta(\vec{\sigma}) \vec{R}_2(k) \right] e^{(i \vec{\omega} \cdot \vec{\phi})} + \]
\[ + i a_o \frac{2}{c} \left[ \left( \frac{\vec{\Omega}}{c} \times \vec{\sigma} \cdot \hat{k} \right) \vec{R}_1(k) + \frac{\hat{k} \cdot \vec{\sigma}}{c} (\vec{\Omega} \cdot \vec{R}_2(k)) \hat{k} \right] e^{(i \vec{\omega} \cdot \vec{\phi})} + \]
\[ + O(1/\omega) + O(\Omega^2/c^2) = \]
\[ \overset{\text{def}}{=} b(\vec{\sigma}) e^{(i \vec{\omega} \cdot \vec{\phi})} + O(1/\omega) + O(\Omega^2/c^2), \]
\[ \vec{E} = -i a_o \frac{1}{c} \left[ \vec{R}_1(k) + \theta(\vec{\sigma}) \vec{R}_2(k) - \frac{\hat{k} \cdot \vec{\sigma}}{c} (\vec{\Omega} \cdot \vec{R}_2(k)) \hat{k} \right] e^{(i \vec{\omega} \cdot \vec{\phi})} - \]
\[ - i a_o \frac{1}{c} \left[ \vec{R}_2(k) - \theta(\vec{\sigma}) \vec{R}_1(k) + \frac{\hat{k} \cdot \vec{\sigma}}{c} (\vec{\Omega} \cdot \vec{R}_1(k)) \hat{k} \right] e^{(i \vec{\omega} \cdot \vec{\phi})} + \]
\[ + O(1/\omega) + O(\Omega^2/c^2) = \]
\[ \overset{\text{def}}{=} f(\vec{\sigma}) e^{(i \vec{\omega} \cdot \vec{\phi})} + O(1/\omega) + O(\Omega^2/c^2). \quad (3.43) \]

As in the case of the Sagnac effect let us consider two receivers A and B at the endpoints of the same light ray described by Eqs, (3.19) and (3.28). The magnetic field observed by A, \( \vec{B}(\tau, \vec{\eta}_A) \), differs from the one observed by B, \( \vec{B}(\tau, \vec{\eta}_B) \). Since the phase changes have been already analyzed for the Sagnac effect, let us concentrate on the amplitudes \( \vec{b}(\vec{\eta}_A) \) and \( \vec{b}(\vec{\eta}_B) \). Since Eq.(3.28) gives \( \vec{\eta}_B - \vec{\eta}_A = \hat{k} \tau_{AB} + O(\Omega/c) \), we find
\[ \vec{b}(\vec{\eta}_B) - \vec{b}(\vec{\eta}_A) = \frac{ia_o^1}{c} \delta \theta_{BA} \vec{R}_1(k) + \frac{ia_o^2}{c} \delta \theta_{BA} \vec{R}_2(k) + \]

\[ + \frac{ia_o^1}{c} \left[ \frac{1}{c} | \vec{\eta}_B - \vec{\eta}_A | (\vec{\Omega} \cdot \vec{R}_1(k)) \hat{k} \right] + \]

\[ + \frac{ia_o^2}{c} \left[ \frac{1}{c} | \vec{\eta}_B - \vec{\eta}_A | (\vec{\Omega} \cdot \vec{R}_2(k)) \hat{k} \right] + O(\Omega^2/c^2), \]

with

\[ \delta \theta_{BA} = | \vec{\eta}_B - \vec{\eta}_A | (\vec{\Omega} \cdot \hat{k}) + O(\Omega^2/c^2). \quad (3.44) \]

\( \delta \theta_{BA} \) is the angle of the inertial Faraday rotation (in this case it is small, \( \delta \theta_{AB} \sim \Omega/c \)). It agrees with Eq.(4) of Ref.[32], where it has the form \( \delta \theta_{AB} = -\frac{1}{2} \int_A^B \sqrt{g_{\tau \tau}} (\nabla \times \vec{n}) \cdot d\vec{\sigma} \) as a line integral along the spatial trajectory of the light ray. This formula agrees with our result, because, due to the approximations we have done, we have \( g_{\tau \tau} = 1, (\nabla \times \vec{n}) = -\frac{2\vec{\Omega}}{c} \) and our ray trajectory is \( \vec{\sigma}(\tau) = \hat{k} \tau + \vec{\sigma}_o + O(\Omega^2/c^2) \).

To make the rotation explicit, let us write the components along the two polarization directions: \( b(\lambda)(\vec{\eta}_A) = \vec{b}(\vec{\eta}_A) \cdot \vec{R}_\lambda(k) \) and \( b(\lambda)(\vec{\eta}_B) = \vec{b}(\vec{\eta}_B) \cdot \vec{R}_\lambda(k) \). In this way we get

\[ b(1)(\vec{\eta}_B) = b(1)(\vec{\eta}_A) + \delta \theta_{AB} \frac{ia_o^1}{c} + O(\Omega^2/c^2) = b(1)(\vec{\eta}_A) + \delta \theta_{AB} b(1)(\vec{\eta}_A) + O(\Omega^2/c^2), \]

\[ b(2)(\vec{\eta}_B) = b(2)(\vec{\eta}_A) + \delta \theta_{AB} \frac{ia_o^2}{c} + O(\Omega^2/c^2) = b(2)(\vec{\eta}_A) + \delta \theta_{AB} b(2)(\vec{\eta}_A) + O(\Omega^2/c^2). \quad (3.45) \]

This is just a small angle rotation with \( b(\lambda)(\vec{\eta}_B) = R_{\lambda \lambda'}(k) (\delta \theta_{AB}) b(\lambda')(\vec{\eta}_A) \).

The electric field may be treated in the same way.
IV. CONCLUSIONS

The theory of non-inertial frames developed in these two papers is free by construction from the coordinate singularities of all the approaches to accelerated frames based on the 1+3 point of view, in which the instantaneous 3-spaces are identified with the local rest frames of the observer. The pathologies of this approach are either the horizon problem of the rotating disk (rotational velocities higher than $c$), which is still present in all the calculations of pulsar magnetosphere in the form of the light cylinder, or the intersection of the local rest 3-spaces. The main difference between the 3+1 and 1+3 points of view is that the Møller conditions forbid rigid rotations in relativistic theories.

In this paper we have given the simplest example of 3+1 splitting with differential rotations and we have revisited the rotating disk and the Sagnac effect following the 3+1 point of view. This splitting is also used to give a special relativistic generalization of the non-relativistic non-inertial International Terrestrial Reference System (ITRS) used to describe fixed coordinates on the surface of the rotating Earth in the conventions IERS2003 [23].

Then we re-examined some properties of the electro-magnetic wave solutions of non-inertial Maxwell equations, which till now were described only by means of the 1+3 point of view, in the 3+1 framework, where there is a well-posed Cauchy problem due to the absence of coordinate singularities. By considering admissible nearly rigid rotating frames we recover the results of the 1+3 approach and open the possibility to make these calculations in presence of deviations from rigid rotations.

A still open problem are the constitutive equations for electrodynamics in material media in non-inertial systems. For linear isotropic media see the Wilson-Wilson experiment in Refs.[18] and Refs.[14, 34], while for an attempt towards a general theory in arbitrary media (including the premetric extension of electro-magnetism) see Refs.[35]

In conclusion we have now a good understanding of particles and electro-magnetism in non-inertial frames in Minkowski space-time, where the 4-metric induced by the admissible 3+1 splitting describes all the inertial effects. Going to canonical gravity, in asymptotically Minkowskian space-times without super-translations and in the York canonical basis of Refs.[36, 37], it is possible to see which components remain inertial effects and which become dynamical tidal effects (the physical degrees of freedom of the gravitational field). Moreover the inertial 3-volume element and some inertial components of the extrinsic curvature of the instantaneous 3-spaces become complicated functions of both general relativistic inertial and tidal effects, because they are determined by the solution of the super-Hamiltonian constraint (the Lichnerowicz equation) and of the super-momentum constraints. Finally,
in accord with the equivalence principle, the instantaneous 3-spaces are only partially de-
termined by the freedom in choosing the convention for clock synchronization: after such a
convention the final instantaneous 3-spaces associated to each solution of Einstein’s equations
are dynamically determined, because in general relativity the metric structure of space-time
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