BEYOND THE BORSUK–ULAM THEOREM:
THE TOPOLOGICAL TVERBERG STORY

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Dedicated to the memory of Jiří Matoušek

Abstract. Bárány’s “topological Tverberg conjecture” from 1976 states that any continuous map of an $N$-simplex $\Delta_N$ to $\mathbb{R}^d$, for $N \geq (d+1)(r-1)$, maps points from $r$ disjoint faces in $\Delta_N$ to the same point in $\mathbb{R}^d$. The proof of this result for the case when $r$ is a prime, as well as some colored version of the same result, using the results of Borsuk–Ulam and Dold on the non-existence of equivariant maps between spaces with a free group action, were main topics of Matoušek’s 2003 book “Using the Borsuk–Ulam theorem.”

In this paper we show how advanced equivariant topology methods allow one to go beyond the prime case of the topological Tverberg conjecture.

First, we explain in detail how equivariant cohomology tools (employing the Borel construction, comparison of Serre spectral sequences, Fadell–Husseini index, etc.) can be used to prove the topological Tverberg conjecture whenever $r$ is a prime power. Our presentation includes a number of improved proofs as well as new results, such as a complete determination of the Fadell–Husseini index of chessboard complexes in the prime case.

Secondly, we introduce the “constraint method,” which applied to suitable “unavoidable complexes” yields a great variety of variations and corollaries to the topological Tverberg theorem, such as the “colored” and the “dimension-restricted” (Van Kampen–Flores type) versions.

Both parts have provided crucial components to the recent spectacular counter-examples in high dimensions for the case when $r$ is not a prime power.

1. Introduction

Jiří Matoušek’s 2003 book “Using the Borsuk–Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry” [36] is an inspiring introduction to the use of equivariant methods in Discrete Geometry. Its main tool is the Borsuk–Ulam theorem, and its generalization by Albrecht Dold, which says that there is no equivariant map from an $n$-connected space to an $n$-dimensional finite complex that is equivariant with respect to a non-trivial finite group acting freely. One of the main applications of this technology in Matoušek’s book was a proof for Bárány’s “topological Tverberg conjecture” on $r$-fold intersections in the case when $r$ is a prime, originally due to Imre Bárány, Senya Shlosman and András Szűcs [8]. This conjecture claimed that for any continuous map $f: \Delta_N \to \mathbb{R}^d$, when $N \geq (d+1)(r-1)$, there are $r$ points in disjoint faces of the simplex $\Delta_N$ that $f$ maps to the same point in $\mathbb{R}^d$.

The topological Tverberg conjecture was extended to the case when $r$ is a prime power by Murad Özaydin in an unpublished paper from 1987 [38]. This cannot, however, be achieved via the Dold theorem, since in the prime power case the group actions one could use on the codomain are not free. So more advanced methods are needed, such as the Serre spectral sequence for the Borel construction and the Fadell–Husseini index. In this paper we present the area about and around the topological Tverberg conjecture, with complete proofs for all of the results, which include the prime power case of the topological Tverberg conjecture.

Özaydin in 1987 had not only proven the topological Tverberg theorem for prime power $r$, but he had also shown, using equivariant obstruction theory, that the approach fails when $r$ is not a prime power: In this case the equivariant map one looks for does exist.

Date: January 13, 2017.

To appear in “A Journey through Discrete Mathematics. A Tribute to Jiří Matoušek”, edited by Martin Loebl, Jaroslav Nešetřil and Robin Thomas, due to be published by Springer.

The research by Pavle V. M. Blagojević leading to these results has received funding from DFG via Collaborative Research Center TRR 109 “Discretization in Geometry and Dynamics.” Also supported by the grant ON 174008 of the Serbian Ministry of Education and Science.

The research by Günter M. Ziegler received funding from DFG via the Research Training Group “Methods for Discrete Structures” and the Collaborative Research Center TRR 109 “Discretization in Geometry and Dynamics.”
In a spectacular recent development, Isaac Mabillard and Uli Wagner [33] [34] have developed an \( r \)-fold version of the classical “Whitney trick” (cf. [50]), which yields the failure of the generalized Van Kampen–Flores theorem when \( r \geq 6 \) is not a prime power. Then Florian Frick observed that this indeed implies the existence of counterexamples to the topological Tverberg conjecture [26] [13] by a lemma of Gromov [27, p. 445] that is an instance of the constraint method of Blagojević, Frick, Ziegler [12, Lemma 4.1(iii) and Lemma 4.2]. (See [5] for a popular rendition of the story.)

The Tverberg theorem from 1966 [46] and its conjectured extension to continuous maps (the topological Tverberg conjecture) have seen a great number of variations and extensions, among them “colored” variants as well as versions with restricted dimensions (known as generalized Van Kampen–Flores theorems). Although many of these were first obtained as independent results, sometimes with very similar proof patterns, our presentation shows that there are many easy implications between these results, using in particular the “constraint method” applied to “unavoidable complexes,” as developed by the present authors with Florian Frick [12]. (Mikhail Gromov [27, p. 445] had sketched one particular instance: The topological Tverberg theorem for maps to \( \mathbb{R}^{n+1} \) implies a generalized Van Kampen–Flores theorem for maps to \( \mathbb{R}^n \).) Thus we can summarize the implications in the following scheme, which shows that all further main results follow from two sources, the topological Tverberg theorem for prime powers, and the optimal colored Tverberg theorem of the present authors with Benjamin Matschke [17], which up to now even for affine maps is available only for the prime case:

\[
\begin{align*}
\text{Topological Tverberg for } p^n & \quad \text{Optimal colored Tverberg for } p \\
\text{Generalized Van Kampen–Flores for } p^n & \quad \text{Topological Tverberg for } p \\
\text{Colored Van Kampen–Flores for } p^n & \quad \text{Bárány–Larman for } p - 1 \\
\text{Weak colored Tverberg for } p^n & \quad \text{Vrečica–Živaljević colored Tverberg for } p^n \text{ type B} \\
\text{Živaljević–Vrečica colored Tverberg for } p^n \text{ type A} &
\end{align*}
\]

Our journey in this paper starts with Radon’s 1921 theorem and its topological version, in Section 2. Here the Borsuk–Ulam theorem is all that’s needed. In Section 3 we state the topological Tverberg conjecture and first prove it in the prime case (with a proof that is close to the original argument by Bárány, Shlosman and Szűcs), and then for prime powers—and this is where we go “beyond the Borsuk–Ulam theorem.” Implications and corollaries of the topological Tverberg theorem are developed in Section 4—so that’s where we put constraints, and “add color.” In Section 5 we get to the counterexamples. And finally in Section 6 we discuss the “optimal colored Tverberg conjecture,” which is a considerable strengthening of Tverberg’s theorem, but up to now has been proven only in the prime case.

A summary of the main topological concepts and tools used in this paper is given at the end in the form of a dictionary, where a reference to the dictionary in the text is indicated by concept\textit{dict}.

\textbf{Acknowledgements.} We are grateful to Alexander Engström and Florian Frick for excellent observations on drafts of this paper and many useful comments. We want to express our gratitude to Peter Landweber for his continuous help and support in improving this manuscript.

\section{The Beginning}

\subsection{Radon’s theorem}

One of the first cornerstone results of convex geometry is a 1921 theorem of Johann Radon about overlapping convex hulls of points in a Euclidean space.

Let \( \mathbb{R}^d \) be a \( d \)-dimensional Euclidean space. Let \( x_1, \ldots, x_m \) be points in \( \mathbb{R}^d \) and let \( \alpha_1, \ldots, \alpha_m \) be non-negative real numbers that sum up to 1, that is, \( \alpha_1 \geq 0, \ldots, \alpha_m \geq 0 \) and \( \alpha_1 + \cdots + \alpha_m = 1 \). The \textit{convex combination} of the points \( x_1, \ldots, x_m \) determined by the scalars \( \alpha_1, \ldots, \alpha_m \) is the following point in \( \mathbb{R}^d \):

\[ x = \alpha_1 x_1 + \cdots + \alpha_m x_m. \]
For a subset $C$ of $\mathbb{R}^d$ we define the convex hull of $C$, denoted by $\text{conv}(C)$, to be the set of all convex combinations of finitely many points in $C$:

$$\text{conv}(C) := \{\alpha_1 x_1 + \cdots + \alpha_m x_m : m \in \mathbb{N}, x_i \in C, \alpha_i \in \mathbb{R}_{\geq 0}, \alpha_1 + \cdots + \alpha_m = 1\}.$$ 

Now Radon’s theorem can be stated as follows and proved using elementary linear algebra.

**Theorem 2.1** (Radon’s theorem, point configuration version [39]). Let $\mathbb{R}^d$ be a $d$-dimensional Euclidean space, and let $X \subseteq \mathbb{R}^d$ be a subset with (at least) $d + 2$ elements. Then there are disjoint subsets $P$ and $N$ of $X$ with the property that

$$\text{conv}(P) \cap \text{conv}(N) \neq \emptyset.$$ 

**Proof.** Let $X = \{x_1, \ldots, x_{d+2}\} \subset \mathbb{R}^d$. The homogeneous system of $d+2$ linear equations in $d+2$ variables

$$\alpha_1 x_1 + \cdots + \alpha_d x_d = 0, \quad \alpha_1 + \cdots + \alpha_d = 0$$

has a non-trivial solution, say $\alpha_1 = a_1, \ldots, \alpha_d = a_d$. Denote

$$P := \{i : a_i > 0\} \quad \text{and} \quad N := \{i : a_i \leq 0\}.$$ 

Then $P \cap N = \emptyset$ while $P \neq \emptyset$ and $N \neq \emptyset$, and

$$\sum_{i \in P} a_i x_i = \sum_{i \in N} -a_i x_i, \quad \sum_{i \in P} a_i = -\sum_{i \in N} a_i =: A,$$

where $A > 0$. Consequently, the following point is in the intersection of convex hulls of $P$ and $N$:

$$x := \sum_{i \in P} \frac{a_i}{A} x_i = \sum_{i \in N} -\frac{a_i}{A} x_i \in \text{conv}(\{x_i : i \in P\}) \cap \text{conv}(\{x_i : i \in N\}).$$

\(\square\)

In order to reformulate Radon’s theorem we recall the notion of an affine map. A map $f : D \to \mathbb{R}^d$ defined on a subset $D \subseteq \mathbb{R}^k$ is affine if for every $m \in \mathbb{N}, x_1, \ldots, x_m \in D$, and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ with $\alpha_1 + \cdots + \alpha_m = 1$ and $\alpha_1 x_1 + \cdots + \alpha_m x_m \in D$, we have

$$f(\alpha_1 x_1 + \cdots + \alpha_m x_m) = \alpha_1 f(x_1) + \cdots + \alpha_m f(x_m).$$

Here and in the following let $\Delta_k := \text{conv}\{e_1, \ldots, e_{k+1}\}$ be the standard $k$-dimensional simplex: This simplex given as the convex hull of the standard basis of $\mathbb{R}^{k+1}$ has the disadvantage of not being full-dimensional in $\mathbb{R}^k$, but it has the extra advantage of being obviously symmetric (with symmetry given by permutation of coordinates). With this, Radon’s theorem can be restated as follows.

**Theorem 2.2** (Radon’s theorem, affine map version). Let $f : \Delta_{d+1} \to \mathbb{R}^d$ be an affine map. Then there are disjoint faces $\sigma_1$ and $\sigma_2$ of the $(d+1)$-simplex $\Delta_{d+1} \subset \mathbb{R}^{d+2}$ with the property that

$$f(\sigma_1) \cap f(\sigma_2) \neq \emptyset.$$ 

With this version of Radon’s theorem at hand, it is natural to ask: Would Radon’s theorem still hold if instead of an affine map we consider an arbitrary continuous map $f : \Delta_{d+1} \to \mathbb{R}^d$?
2.2. The topological Radon theorem. The question we have just asked was answered in 1979 by Ervin Bajmóczy and Imre Bárány [4], using the Borsuk–Ulam theorem.

**Theorem 2.3** (Topological Radon theorem). Let \( f : \Delta_{d+1} \to \mathbb{R}^d \) be any continuous map. Then there are two disjoint faces \( \sigma_1 \) and \( \sigma_2 \) of \( \Delta_{d+1} \) whose images under \( f \) intersect,
\[
f(\sigma_1) \cap f(\sigma_2) \neq \emptyset.
\]

**Proof.** Let \( \Delta_{d+1} = \text{conv}\{e_1, \ldots, e_{d+2}\} \) be the standard simplex. Consider the subcomplex \( X \) of the polyhedral complex \( \Delta_{d+1} \times \Delta_{d+1} \) given by
\[
X := \{(x_1, x_2) \in \Delta_{d+1} \times \Delta_{d+1} : \text{there are faces } \sigma_1, \sigma_2 \subset \Delta_{d+1} \text{ such that } \sigma_1 \cap \sigma_2 = \emptyset, \ x_1 \in \sigma_1, \ x_2 \in \sigma_2\}.
\]
The group \( \mathbb{Z}/2 = \{e, \varepsilon\} \) acts freely on \( X \) by \( \varepsilon \cdot (x_1, x_2) = (x_2, x_1) \).

Let us assume that the theorem does not hold. Then there exists a continuous map \( f : \Delta_{d+1} \to \mathbb{R}^d \) such that \( f(x_1) \neq f(x_2) \) for all \((x_1, x_2) \in X\). Consequently the map \( g : X \to \mathbb{S}^{d-1} \) given by
\[
g(x_1, x_2) := \frac{f(x_1) - f(x_2)}{\|f(x_1) - f(x_2)\|},
\]
is continuous and \( \mathbb{Z}/2 \)-equivariant, where the action on \( \mathbb{S}^{d-1} = S(\mathbb{R}^d) \), the unit sphere in \( \mathbb{R}^d \), is the standard antipodal action.

Next we define a continuous \( \mathbb{Z}/2 \)-equivariant map from a \( d \)-sphere to \( X \). For this we do not use the standard \( d \)-sphere, but the unit sphere \( S(W_{d+2}) \) in the hyperplane \( W_{d+2} := \{(a_1, \ldots, a_{d+2}) \in \mathbb{R}^{d+2} : a_1 + \cdots + a_{d+2} = 0\} \subset \mathbb{R}^{d+2} \), that is,
\[
S(W_{d+2}) = \{(a_1, \ldots, a_{d+2}) \in \mathbb{R}^{d+2} : a_1 + \cdots + a_{d+2} = 0, a_1^2 + \cdots + a_{d+2}^2 = 1\}.
\]
This representation of the \( d \)-sphere also has the standard antipodal \( \mathbb{Z}/2 \)-action. The map \( h : S(W_{d+2}) \to X \) is defined by
\[
h(a_1, \ldots, a_{d+2}) := \left( \sum_{a_i \geq 0} \frac{a_i}{A} e_i, \sum_{a_i < 0} \frac{a_i}{A} e_i \right),
\]
where \( A := \sum_{a_i \geq 0} a_i = -\sum_{a_i < 0} a_i > 0 \). This is easily checked to be well-defined and continuous; the image point lies in the cell \( \text{conv}\{e_i : a_i > 0\} \times \text{conv}\{e_i : a_i < 0\} \) of the complex \( \Delta_{d+1} \times \Delta_{d+1} \).

The composition map \( g \circ h : S(W_{d+2}) \to \mathbb{S}^{d-1} \) yields a continuous \( \mathbb{Z}/2 \)-equivariant map from a free \( d \)-sphere to a free \((d-1)\)-sphere that contradicts the Borsuk–Ulam theorem \( \text{dict} \). Thus the theorem holds. \( \square \)

2.3. The Van Kampen–Flores theorem. The topological Radon theorem guarantees that for every continuous map \( \Delta_{d+1} \to \mathbb{R}^d \) there exist two pairwise disjoint faces whose \( f \)-images overlap. It is natural to ask: Is it possible to say something about the dimension of the disjoint faces whose \( f \)-images intersect? In the spirit of Poincaré’s classification of mathematical problems [2, Lec.1] this binary problem has a quick answer no, but if understood as an interesting problem it has an answer: If we are willing to spend an extra vertex/dimension, meaning, put the simplex \( \Delta_{d+2} \) in place of \( \Delta_{d+1} \), we get the following theorem from the 1930s of Egbert R. Van Kampen and Antonio Flores [24] [32].

**Theorem 2.4** (Van Kampen–Flores theorem). Let \( d \geq 2 \) be an even integer, and let \( f : \Delta_{d+2} \to \mathbb{R}^d \) be a continuous map. Then there are disjoint faces \( \sigma_1, \sigma_2 \subset \Delta_{d+2} \) of dimension at most \( d/2 \) whose images under \( f \) intersect,
\[
f(\sigma_1) \cap f(\sigma_2) \neq \emptyset.
\]

**Proof.** Let \( g : \Delta_{d+2} \to \mathbb{R}^{d+1} \) be a continuous map defined by
\[
g(x) := (f(x), \text{dist}(x, \text{sk}_{d/2}(\Delta_{d+2})))
\]
where \( \text{sk}_{d/2}(\Delta_{d+2}) \) denotes the \( d/2 \)-skeleton of the simplex \( \Delta_{d+2} \), and \( \text{dist}(x, \text{sk}_{d/2}(\Delta_{d+2})) \) is the distance of the point \( x \) from the subcomplex \( \text{sk}_{d/2}(\Delta_{d+2}) \). Observe that if \( x \in \text{relint } \sigma \) and \( \text{dist}(x, \text{sk}_{d/2}(\Delta_{d+2})) = 0 \), then the simplex \( \sigma \) belongs to the subcomplex \( \text{sk}_{d/2}(\Delta_{d+2}) \).

Now the topological Radon theorem can be applied to the continuous map \( g : \Delta_{d+2} \to \mathbb{R}^{d+1} \). It yields the existence of points \( x_1 \in \text{relint } \sigma_1 \) and \( x_2 \in \text{relint } \sigma_2 \), with \( \sigma_1 \cap \sigma_2 = \emptyset \), such that
\[
g(x_1) = g(x_2) \iff f(x_1) = f(x_2), \quad \text{dist}(x_1, \text{sk}_{d/2}(\Delta_{d+2})) = \text{dist}(x_2, \text{sk}_{d/2}(\Delta_{d+2})).
\]
If one of the simplices \( \sigma_1 \), or \( \sigma_2 \), would belong to \( \text{sk}_{d/2}(\Delta_{d+2}) \), then
\[
\text{dist}(x_1, \text{sk}_{d/2}(\Delta_{d+2})) = \text{dist}(x_2, \text{sk}_{d/2}(\Delta_{d+2})) = 0
\]
implicating that both \( \sigma_1 \) and \( \sigma_2 \) belong to \( \text{sk}_{d/2}(\Delta_{d+2}) \), which would concludes the proof of the theorem.
In order to prove that at least one of the faces $\sigma_1$ and $\sigma_2$ belongs to $\text{sk}_{d/2}(\Delta_{d+2})$, note that these are two disjoint faces of the simplex $\Delta_{d+2}$, which has $d+3$ vertices, so by the pigeonhole principle one of them has at most $[(d+3)/2] = d/2 + 1$ vertices.

The proof we have presented is an example of the constraint method developed in [12]. An important message of this proof is that the Van Kampen–Flores theorem is a corollary of the topological Radon theorem. It is clear that we could have considered a continuous map $f$ defined only on the $d/2$-skeleton.

All the results we presented so far have always claimed something about intersections of the images of two disjoint faces $\sigma_1, \sigma_2$, which we refer to as 2-fold overlap, or intersection. What about $r$-fold overlaps, for $r > 2$?

3. The topological Tverberg theorem

3.1. The topological Tverberg conjecture. In 1964, freezing in a hotel room in Manchester, the Norwegian mathematician Helge Tverberg proved the following $r$-fold generalization of Radon’s theorem [46]. It had been conjectured by Bryan Birch in 1954, who had established the result in the special case of dimension $d = 2$ [9]. The case $d = 1$ is easy, see below. (See [51] for some of the stories surrounding these discoveries.)

**Theorem 3.1** (Tverberg’s theorem). Let $d \geq 1$ and $r \geq 2$ be integers, $N = (d + 1)(r - 1)$, and let $f : \Delta_N \to \mathbb{R}^d$ be an affine map. Then there exist $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of the simplex $\Delta_N$ whose $f$-images overlap,

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset. \quad (1)$$

Any collection of $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of the simplex $\Delta_N$ having property (1) is called a Tverberg partition of the map $f$.

The dimension of the simplex in the theorem is optimal, it cannot be decreased. To see this consider the affine map $h : \Delta_{N-1} \to \mathbb{R}^d$ given on the vertices of $\Delta_{N-1} = \text{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ by

$$\mathbf{e}_i \mapsto u_{i(1)/(r-1)} \quad (2)$$

where $\{u_0, \ldots, u_d\}$ is an affinely independent set in $\mathbb{R}^d$, e.g., $(u_0, \ldots, u_d) = (0, \mathbf{e}_1, \ldots, \mathbf{e}_d)$. For each vertex of the simplex $\text{conv}\{u_0, \ldots, u_d\}$ the cardinality of its preimage is $r - 1$

$$|h^{-1}(\{u_0\})| = \cdots = |h^{-1}(\{u_d\})| = r - 1,$$

and so the map $h$ has no Tverberg partition. Even more is true: Any affine map $h : \Delta_{N-1} \to \mathbb{R}^d$ that is in general position cannot have a Tverberg partition.

As in the case of Radon’s theorem it is natural to ask: Would the Tverberg theorem still hold if instead of an affine map $f : \Delta_N \to \mathbb{R}^d$ we would consider an arbitrary continuous map? This was first asked by Bárany in a 1976 letter to Tverberg. In May of 1978 Tverberg posed the question in Oberwolfach, stating it for a general $N$-polytope in place of the $N$-simplex, see [28]. (The problem for a general $N$-polytope can be reduced to the case of the $N$-simplex by a theorem of Grünbaum: Every $N$-polytope as a cell complex is a refinement of the $N$-simplex [29, p. 200].) Thus, the topological Tverberg conjecture started its life in the late 1970s.

**Conjecture 3.2** (Topological Tverberg conjecture). Let $d \geq 1$ and $r \geq 2$ be integers, $N = (d + 1)(r - 1)$, and let $f : \Delta_N \to \mathbb{R}^d$ be a continuous map. Then there exist $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of the simplex $\Delta_N$ whose $f$-images overlap,

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset. \quad (3)$$

The case $r = 2$ of the topological Tverberg conjecture amounts to the topological Radon theorem, so it holds. The topological Tverberg conjecture is also easy to verify for $d = 1$, as follows.

**Theorem 3.3** (Topological Tverberg conjecture for $d = 1$). Let $r \geq 2$ be an integer, and let $f : \Delta_{2r-2} \to \mathbb{R}$ be a continuous map. Then there exist $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of the simplex $\Delta_{2r-2}$ whose $f$-images overlap,

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.$$ 

**Proof.** Let $f : \Delta_{2r-2} \to \mathbb{R}$ be continuous. Sort the vertices of the simplex $\Delta_{2r-2} = \text{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_{2r-1}\}$ such that $f(\mathbf{e}_{(1)}) \leq f(\mathbf{e}_{(2)}) \leq \cdots \leq f(\mathbf{e}_{(2r-2)}) \leq f(\mathbf{e}_{(2r-1)})$. Then the collection of $r - 1$ edges and one vertex of $\Delta_{2r-2}$

$$\sigma_1 = [\mathbf{e}_{(1)}, \mathbf{e}_{(2r-1)}], \quad \sigma_2 = [\mathbf{e}_{(2)}, \mathbf{e}_{(2r-2)}], \ldots, \quad \sigma_{r-1} = [\mathbf{e}_{(r-1)}, \mathbf{e}_{(r+1)}], \quad \sigma_r = \{\mathbf{e}_r\}$$

is a Tverberg partition for the map $f$. \□
At first glance the topological Tverberg conjecture is a *binary problem* in the Poincaré classification of mathematical problems. To our surprise it is safe to say, at this point in time, that the topological Tverberg conjecture was one of the most *interesting problems* that shaped interaction between Geometric Combinatorics on one hand and Algebraic and Geometric Topology on the other hand for almost four decades.

After settling the topological Tverberg conjecture for \( d = 1 \) and \( r = 2 \) we want to advance. How?

### 3.2. Equivariant topology steps in.

Let \( d \geq 1 \) and \( r \geq 2 \) be integers, and let \( N = (d + 1)(r - 1) \). Our effort to handle the topological Tverberg conjecture starts with an assumption that there is a counterexample to the conjecture with parameters \( d \) and \( r \). Thus there is a continuous map \( f : \Delta_N \to \mathbb{R}^d \) such that for every \( r \)-tuple \( \sigma_1, \ldots, \sigma_r \) of pairwise disjoint faces of the simplex \( \Delta_N \) their \( f \)-images do not intersect, that is,

\[
 f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset.
\]

In order to capture this feature of our counterexample \( f \) we parametrize all \( r \)-tuples of pairwise disjoint faces of the simplex \( \Delta_N \). This can be done in two similar, but different ways.

#### 3.2.1. The \( r \)-fold 2-wise deleted product.

The \( r \)-fold 2-wise deleted product \( K_{\Delta(2)}^{\times r} \) of a simplicial complex \( K \) is the cell complex

\[
 K_{\Delta(2)}^{\times r} := \{(x_1, \ldots, x_r) \in \sigma_1 \times \cdots \times \sigma_r \subset K^{\times r} : \sigma_i \cap \sigma_j = \emptyset \text{ for } i \neq j\},
\]

where \( \sigma_1, \ldots, \sigma_r \) are non-empty faces of \( K \). The symmetric group \( \mathfrak{S}_r \) acts (from the left) on \( K_{\Delta(2)}^{\times r} \) by

\[
 \pi \cdot (x_1, \ldots, x_r) := (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(r)}),
\]

for \( \pi \in \mathfrak{S}_r \) and \( (x_1, \ldots, x_r) \in K_{\Delta(2)}^{\times r} \). This action is free due to the fact that if \( (x_1, \ldots, x_r) \in K_{\Delta(2)}^{\times r} \) then \( x_i \neq x_j \) for all \( i \neq j \). We have seen a particular instance before: The complex \( X \) that we used in the proof of the topological Radon theorem 2.3 was \( (\Delta_{d+1})_{\Delta(2)}^{\times 2} \). For more details on the deleted product construction see for example [36, Sec. 6.3].

In the case when \( K \) is a simplex the topology of the deleted product \( K_{\Delta(2)}^{\times r} \) is known from the following result of Bárány, Shlosman and Szűcs [8, Lem. 1].

**Theorem 3.4.** Let \( N \) and \( r \) be positive integers with \( N \geq r - 1 \). Then \( (\Delta_N)^{\times r}_{\Delta(2)} \) is an \((N - r + 1)\)-dimensional and \((N - r)\)-connected CW complex.

**Proof.** A typical face of the CW complex \( (\Delta_N)^{\times r}_{\Delta(2)} \) is of the form \( \sigma_1 \times \cdots \times \sigma_r \), where \( \sigma_1, \ldots, \sigma_r \) are pairwise disjoint simplices. Consequently, the number of vertices of these simplices together cannot exceed \( N + 1 \), or in the language of dimension

\[
 \dim(\sigma_1) + 1 + \cdots + \dim(\sigma_r) + 1 \leq N + 1.
\]

The equality is attained when all the vertices are used, this is when \( \sigma_1, \ldots, \sigma_r \) is a maximal face of dimension

\[
 \dim(\sigma_1 \times \cdots \times \sigma_r) = \dim(\sigma_1) + \cdots + \dim(\sigma_r) = N - r + 1.
\]

Thus \( (\Delta_N)^{\times r}_{\Delta(2)} \) is an \((N - r + 1)\)-dimensional CW complex.

For \( N = r - 1 \) the deleted product \( (\Delta_N)^{\times r}_{\Delta(2)} \) is the 0-dimensional simplicial complex \([r]\) and the statement of the theorem holds. Thus, we can assume that \( N \geq r \).

For \( N \geq r \) we establish the connectivity of the deleted product of a simplex by induction on \( r \) making repeated use of the following classical 1957 theorem of Stephen Smale [43, Main Thm.]:

**Smale’s Theorem.** Let \( X \) and \( Y \) be connected, locally compact, separable metric spaces, and in addition let \( X \) be locally contractible. Let \( f : X \to Y \) be a continuous proper map, that is, any preimage of a compact set is compact. If for every \( y \in Y \) the preimage \( f^{-1}(\{y\}) \) is locally contractible and \( n \)-connected, then the induced homomorphism

\[
 f^* : \pi_i(X) \to \pi_i(Y)
\]

is an isomorphism for all \( 0 \leq i \leq n \), and is an epimorphism for \( i = n + 1 \).

Recall that \( \Delta_N \) denotes the standard simplex, whose vertices \( e_1, \ldots, e_{N+1} \) form the standard basis of \( \mathbb{R}^{N+1} \). The induction starts with \( r = 1 \) and the theorem claims that the simplex \( \Delta_N \), a contractible space, is \((N - 1)\)-connected, which is obviously true.
In the case $r = 2$ consider the surjection $p_1 : (\Delta_N)^{2(k+1)} \rightarrow sk_{N-1}(\Delta_N)$ given by the projection on the first factor. Any point $x_1$ of the $(N - 1)$-skeleton $sk_{N-1}(\Delta_N)$ of the simplex $\Delta_N$ lies in the relative interior of a face, 

$$x_1 \in \text{relint} \left( \text{conv} \{ e_i : i \in T \subseteq [N+1] \} \right)$$

where $1 \leq |T| \leq N$. Let us denote the complementary set of vertices by $S := \{ e_i : i \notin T \} \neq \emptyset$ and its convex hull by $\Delta_S := \text{conv}(S) \cong \Delta_{|S|-1}$. The fiber of the projection map $p_1$ over $x_1$ is given by

$$p_1^{-1}(\{ x_1 \}) = \{ (x_1, x_2) \in (\Delta_N)^{2(k+2)} : x_2 \in \Delta_S \} \cong \Delta_S,$$

and consequently it is contractible. By Smale’s theorem the projection $p_1$ induces an isomorphism between homotopy groups. Since we are working in the category of CW complexes the Whitehead theorem [19, Thm. 11.2] implies a homotopy equivalence of homotopy groups. Since we are working in the category of CW complexes the Whitehead theorem [19, Thm. 11.2] implies a homotopy equivalence of $(\Delta_N)^{2(k+2)}$ and $sk_{N-1}(\Delta_N)$. The $(N - 1)$-skeleton of a simplex is $(N - 2)$-connected and thus the theorem holds in the case $r = 2$.

For the induction hypothesis assume that $(\Delta_N)^{x(k+1)}$ is $(N - i)$-connected for all $i \leq k < r$. In the induction step we want to prove that $(\Delta_N)^{x(k+1)}$ is $(N - k - 1)$-connected.

Now consider the projection onto the first $k$ factors,

$$p_k : (\Delta_N)^{x(k+1)} \rightarrow sk_{N-k} \left( (\Delta_N)^{x(k)} \right).$$

Since $(\Delta_N)^{x(k)}$ is $(N - k)$-connected by induction hypothesis, its $(N - k)$-skeleton $sk_{N-k} \left( (\Delta_N)^{x(k)} \right)$ is $(N - k - 1)$-connected. For a typical point of the codomain we have that

$$(x_1, \ldots, x_k) \in \text{relint} \left( \text{conv} \{ e_i : i \in T_1 \subseteq [N+1] \} \right) \times \cdots \times \text{relint} \left( \text{conv} \{ e_i : i \in T_k \subseteq [N+1] \} \right),$$

where $T_i \cap T_j = \emptyset$ for all $1 \leq i < j \leq k$, and $|T_1| - 1 + \cdots + |T_k| - 1 \leq N - k$. As before, consider the complementary set of vertices $S := \{ e_i : i \notin \bigcup_{T} T_i \} \neq \emptyset$ and its convex hull $\Delta_S = \text{conv}(S) \cong \Delta_{|S|-1}$. The fiber of the projection map $p_k$ over $(x_1, \ldots, x_k)$ is given by

$$p_k^{-1}(\{ (x_1, \ldots, x_k) \}) = \{ (x_1, \ldots, x_k, x_{k+1}) \in (\Delta_N)^{x(k+1)} : x_{k+1} \in \Delta_S \} \cong \Delta_S,$$

so it is contractible. Again Smale’s theorem applied to the projection $p_k$ induces an isomorphism between homotopy groups of $(\Delta_N)^{x(k+1)}$ and $sk_{N-k} \left( (\Delta_N)^{x(k)} \right)$. Moreover, the Whitehead theorem implies that these spaces are homotopy equivalent. Since, $sk_{N-k} \left( (\Delta_N)^{x(k)} \right)$ is $(N - k - 1)$-connected we have concluded the induction step and the theorem is proved.

**Remark 3.5.** Our proof of Theorem 3.4 may be traced back to a proof in the lost preprint version of the paper [8]. Indeed, in the published version the first sentence of [8, proof of Lem. 1] says:

For this elementary proof we are indebted to the referee. Our original proof used the Leray spectral sequence.

Here we used Smale’s theorem in place of the Leray spectral sequence argument.

3.2.2. The $r$-fold $k$-wise deleted join. Let $K$ be a simplicial complex. The $r$-fold $k$-wise deleted join $K^{x(r)}$ of the simplicial complex $K$ is the simplicial complex

$$K^{x(r)}(k) := \{ \lambda_1 x_1 + \cdots + \lambda_r x_r \in \sigma_1 \ast \cdots \ast \sigma_r \subseteq K^{x(r)} : (\forall I \subseteq [n]) \text{card } I \geq k \Rightarrow \bigcap_{i \in I} \sigma_i = \emptyset \},$$

where $\sigma_1, \ldots, \sigma_n$ are faces of $K$, including the empty face. Thus in the case $k = 2$ we have

$$K^{x(r)}(2) := \{ \lambda_1 x_1 + \cdots + \lambda_r x_r \in \sigma_1 \ast \cdots \ast \sigma_r \subseteq K^{x(r)} : \sigma_i \cap \sigma_j = \emptyset \text{ for } i \neq j \}.$$

The symmetric group $\Sigma_r$ acts (from the left) on $K^{x(r)}(2)$ as follows

$$\pi : (\lambda_1 x_1 + \cdots + \lambda_r x_r) := \lambda_{\pi^{-1}(1)} x_{\pi^{-1}(1)} + \cdots + \lambda_{\pi^{-1}(r)} x_{\pi^{-1}(r)},$$

where $\pi \in \Sigma_n$ and $\lambda_1 x_1 + \cdots + \lambda_r x_r \in K^{x(r)}(2)$. This action is free only in the case when $k = 2$.

**Examples 3.6.** (Compare Figure 2.)

1. Let $K = \Delta_1$ be the 1-simplex. Then $K^{x2}(2) = S^0$ while $K^{x2}(2) \cong S^1$.
2. For $K = S^0$ we have that $K^{x2}(2) = S^0$, and $K^{x2}(2)$ is a disjoint union of two intervals.
3. If $K = [3]$ then $K^{x2}(2) \cong S^1$.
4. When $K = [k]$, the deleted join $K^{x(r)}(2)$ is the $k \times r$ chessboard complex $\text{chessboard complex}^{x(r)}$, which is denoted by $\Delta_{k,r}$. 


The following lemma establishes the commutativity of the join and the deleted join operations on simplicial complexes. We state it for \(k\)-wise deleted joins and prove it here only for 2-wise deleted joins. For more details and insight consult the sections “Deleted Products Good” and “...Deleted Joins Better” in Matoušek’s book, [36, Sections 5.4 and 5.5].

**Lemma 3.7.** Let \(K\) and \(L\) be simplicial complexes, and let \(n \geq 2\) and \(k \geq 2\) be integers. There exists an isomorphism of simplicial complexes:

\[
(K \ast L)^{\ast n}_{\Delta(k)} \cong K^{\ast n}_{\Delta(k)} \ast L^{\ast n}_{\Delta(k)}.
\]

**Proof.** We give a proof only for the case \(k = 2\). Let \(\sigma_1, \ldots, \sigma_n\) and \(\tau_1, \ldots, \tau_n\) be simplices in \(K\) and \(L\), respectively, such that \(\sigma_i \cap \tau_j = \emptyset\) and \(\tau_i \cap \tau_j = \emptyset\) for all \(i \neq j\). In addition, since the simplicial complexes \(K\) and \(L\) have disjoint vertex sets, we get that \(\sigma_i \cap \tau_j = \emptyset\) as well for all \(i\) and \(j\). Thus, for all \(i \neq j\) we obtain an equivalence:

\[
(\sigma_i \cup \tau_i) \cap (\sigma_j \cup \tau_j) = \emptyset \quad \text{if and only if} \quad \sigma_i \cap \sigma_j = \emptyset \quad \text{and} \quad \tau_i \cap \tau_j = \emptyset.
\]

It induces a bijection between the following simplices of \((K \ast L)^{\ast n}_{\Delta(2)}\) and \(K^{\ast n}_{\Delta(2)} \ast L^{\ast n}_{\Delta(2)}\) by:

\[
(\sigma_1 \ast \tau_1) \ast \cdots \ast (\sigma_n \ast \tau_n) \longleftrightarrow (\sigma_1 \ast (\Delta(2) \cdots \ast (\Delta(2) \ast (\tau_1 \ast \cdots \ast (\Delta(2) \tau_n)).
\]

□

A direct consequence of the previous lemma is the following useful fact.

**Lemma 3.8.** Let \(r \geq 2\) and \(2 \leq k \leq r\) be integers. Then

1. \((\Delta_N)^{r}_{\Delta(k)} \cong [r]^*(N+1)\),
2. \((\Delta_N)^{r}_{\Delta(k)} \cong (sk_{k-2}(\Delta_{r-1}))^{*(N+1)}\).

**Proof.** \((\Delta_N)^{r}_{\Delta(k)} \cong ([1]^*(N+1))^{r}_{\Delta(k)} \cong ([1]^r_{\Delta(k)})^{*(N+1)} \cong (sk_{k-2}(\Delta_{r-1}))^{*(N+1)}\). □

Thus the \(r\)-fold 2-wise deleted join of an \(N\)-simplex \((\Delta_N)^{r}_{\Delta(2)}\) is an \(N\)-dimensional and \((N-1)\)-connected simplicial complex.

### 3.2.3. Equivariant maps induced by \(f\). Recall that, at the beginning of Section 3.2, we have fixed integers \(d \geq 1\) and \(r \geq 2\), and in addition we assumed the existence of the continuous map \(f : \Delta_N \to \mathbb{R}^d\) that is a counterexample to the topological Tverberg theorem.

Define continuous maps induced by \(f\) in the following way:

- the product map is
  \[
P_f : (\Delta_N)^{\times r}_{\Delta(2)} \to (\mathbb{R}^d)^{\times r} \cong (\mathbb{R}^d)^{\oplus r}, \quad (x_1, \ldots, x_r) \mapsto (f(x_1), \ldots, f(x_r));
  \]

- the join map is
  \[
  J_f : (\Delta_N)^{\ast r}_{\Delta(2)} \to (\mathbb{R}^{d+1})^{\oplus r}, \quad \lambda_1 x_1 + \cdots + \lambda_r x_r \mapsto (\lambda_1, \lambda_1 f(x_1)) \oplus \cdots \oplus (\lambda_r, \lambda_r f(x_r)).
  \]

The codomains \((\mathbb{R}^d)^{\oplus r}\) and \((\mathbb{R}^{d+1})^{\oplus r}\) of the maps \(P_f\) and \(J_f\) are equipped with the action of the symmetric group \(\mathfrak{S}_r\) given by permutation of the corresponding \(r\) factors, that is

\[
\pi \cdot (y_1, \ldots, y_r) = (y_{\pi^{-1}(1)}, \ldots, y_{\pi^{-1}(r)})\quad \text{and}\quad \pi \cdot (z_1, \ldots, z_r) = (z_{\pi^{-1}(1)}, \ldots, z_{\pi^{-1}(r)}).
\]
for \((y_1, \ldots, y_r) \in (\mathbb{R}^d)^{\oplus r}\) and \((z_1, \ldots, z_r) \in (\mathbb{R}^{d+1})^{\oplus r}\). Then both maps \(P_f\) and \(J_f\) are \(\mathcal{G}_r\)-equivariant. Indeed, the following diagrams commute:

\[
\begin{array}{ccc}
(x_1, \ldots, x_r) & \xrightarrow{P_f} & (f(x_1), \ldots, f(x_r)) \\
\downarrow & & \downarrow \\
(x_{n-1(1)}, \ldots, x_{n-1(r)}) & \xrightarrow{P_f} & (f(x_{n-1(1)}), \ldots, f(x_{n-1(r)})),
\end{array}
\]

and

\[
\begin{array}{ccc}
\lambda_1 x_1 + \cdots + \lambda_r x_r & \xrightarrow{J_f} & (\lambda_1, \lambda_1 f(x_1)) \oplus \cdots \oplus (\lambda_r, \lambda_r f(x_r)) \\
\downarrow & & \downarrow \\
\lambda_{n-1(1)} x_{n-1(1)} + \cdots + \lambda_{n-1(r)} x_{n-1(r)} & \xrightarrow{J_f} & (\lambda_{n-1(1)}, \lambda_{n-1(1)} f(x_{n-1(1)})) \oplus \cdots \oplus (\lambda_{n-1(r)}, \lambda_{n-1(r)} f(x_{n-1(r)})).
\end{array}
\]

The \(\mathcal{G}_r\)-invariant subspaces

\[
D_P := \{(y_1, \ldots, y_r) \in (\mathbb{R}^d)^{\oplus r} : y_1 = \cdots = y_r\} \quad \text{and} \quad D_J := \{(z_1, \ldots, z_r) \in (\mathbb{R}^{d+1})^{\oplus r} : z_1 = \cdots = z_r\}
\]

of the codomains \((\mathbb{R}^d)^{\oplus r}\) and \((\mathbb{R}^{d+1})^{\oplus r}\), respectively, are called the thin diagonals. The crucial property of the maps \(P_f\) and \(J_f\), for a counterexample continuous map \(f : \Delta_N \to \mathbb{R}^d\), is that

\[
\text{im}(P_f) \cap D_P = \emptyset \quad \text{and} \quad \text{im}(J_f) \cap D_J = \emptyset.
\]

Indeed, the property (4) of the map \(f\) immediately implies that \(\text{im}(P_f)\) and \(D_P\) are disjoint. For the second relation of (5) assume that

\[
(\lambda_1, \lambda_1 f(x_1)) \oplus \cdots \oplus (\lambda_r, \lambda_r f(x_r)) \in \text{im}(J_f) \cap D_J \neq \emptyset
\]

for some \(\lambda_1 x_1 + \cdots + \lambda_r x_r \in (\Delta_N)^r(\lambda_{n-1})\). Then \(\lambda_1 = \cdots = \lambda_r = \frac{1}{r}\) and consequently \(f(x_1) = \cdots = f(x_r)\).

Therefore, the maps \(P_f\) and \(J_f\) induce \(\mathcal{G}_r\)-equivariant maps

\[
(\Delta_N)^r(\lambda_{n-1}) \to (\mathbb{R}^d)^{\oplus r} \setminus D_P \quad \text{and} \quad (\Delta_N)^r(\lambda_{n-1}) \to (\mathbb{R}^{d+1})^{\oplus r} \setminus D_J
\]

that, with an obvious abuse of notation, are again denoted by \(P_f\) and \(J_f\), respectively. Let us denote by

\[
R_P : (\mathbb{R}^d)^{\oplus r} \setminus D_P \to D_P^+ \setminus \{0\} \to S(D_P^-) \quad \text{and} \quad R_J : (\mathbb{R}^{d+1})^{\oplus r} \setminus D_J \to D_J^+ \setminus \{0\} \to S(D_J^-)
\]

the compositions of projections and deformation retracts. Here \(U^+\) denotes the orthogonal complement of the subspace \(U\) in the relevant ambient real vector space, while \(S(V)\) denotes the unit sphere in the real vector space \(V\). Both maps \(R_P\) and \(R_J\) are \(\mathcal{G}_r\)-equivariant maps with respect to the introduced actions.

Furthermore, let \(\mathbb{R}^r\) be a vectors space with the (left) action of the symmetric group \(\mathcal{G}_r\) given by the permutation of coordinates. Then the subspace \(W_r = \{(t_1, \ldots, t_r) \in \mathbb{R}^r : \sum_{i=1}^r t_i = 0\}\) is an \(\mathcal{G}_r\)-invariant subspace of dimension \(r - 1\). There is an isomorphism of real \(\mathcal{G}_r\)-representations

\[
D_P^- \cong W_r^{\oplus d} \quad \text{and} \quad D_J^- \cong W_r^{\oplus (d+1)}.
\]

Using this identification of \(\mathcal{G}_r\)-representations the \(\mathcal{G}_r\)-equivariant maps \(R_P\) and \(R_J\), defined in (7), can be presented by

\[
R_P : (\mathbb{R}^d)^{\oplus r} \setminus D_P \to S(W_r^{\oplus d}) \quad \text{and} \quad R_J : (\mathbb{R}^{d+1})^{\oplus r} \setminus D_J \to S(W_r^{\oplus (d+1)}).
\]

Finally we have the theorem we were looking for. It will give us a chance to employ methods of algebraic topology to attack the topological Tverberg conjecture.

**Theorem 3.9.** Let \(d \geq 1\) and \(r \geq 2\) be integers, and let \(N = (d+1)(r-1)\). If there exists a counterexample to the topological Tverberg conjecture, then there exist \(\mathcal{G}_r\)-equivariant maps

\[
(\Delta_N)^r(\lambda_{n-1}) \to S(W_r^{\oplus d}) \quad \text{and} \quad (\Delta_N)^r(\lambda_{n-1}) \to S(W_r^{\oplus (d+1)}).
\]

**Proof.** If \(f : \Delta_N \to \mathbb{R}^d\) is a counterexample to the topological Tverberg conjecture, then by composing maps from (6) and (8) we get \(\mathcal{G}_r\)-equivariant maps

\[
R_P \circ P_f : (\Delta_N)^r(\lambda_{n-1}) \to S(W_r^{\oplus d}) \quad \text{and} \quad R_J \circ J_f : (\Delta_N)^r(\lambda_{n-1}) \to S(W_r^{\oplus (d+1)}).
\]

□
Now we have constructed our equivariant maps. The aim is to find as many $r$'s as possible such that an $\mathcal{G}_r$-equivariant map
\[
(\Delta_N)_{\Delta(2)}^r \rightarrow S(W_r^{\oplus d}), \quad \text{or} \quad (\Delta_N)_{\Delta(2)}^r \rightarrow S(W_r^{\oplus (d+1)}),
\]
cannot exist. For this we keep in mind that the $\mathcal{G}_r$-action on $(\Delta_N)_{\Delta(2)}^r$ is free, while for $r \geq 3$ the $\mathcal{G}_r$-action on $(\Delta_N)_{\Delta(2)}^r$ is not free.

3.3. The topological Tverberg theorem. The story of the topological Tverberg conjecture continues with a 1981 breakthrough of Bárány, Shlosman and Szűcs [8]. They proved that in the case when $r$ is a prime, there is no $\mathbb{Z}/r$-equivariant map $(\Delta_N)_{\Delta(2)}^r \rightarrow S(W_r^{\oplus d})$, and consequently no $\mathcal{G}_r$-equivariant map can exist. Hence, Theorem 3.9 settles the topological Tverberg conjecture in the case when $r$ is a prime. We give a proof of this result relying on the following theorem of Dold [22] [36, Thm. 6.2.6]:

Dold’s theorem. Let $G$ be a non-trivial finite group. For an $n$-dimensional free $G$-CW complex $Y$ there cannot be any continuous $G$-equivariant map $X \rightarrow Y$.

**Theorem 3.10** (Topological Tverberg theorem for primes $r$). Let $d \geq 1$ be an integer, let $r \geq 2$ be a prime, $N \equiv (d+1)(r-1)$, and let $f : \Delta_N \rightarrow \mathbb{R}^d$ be a continuous map. Then there exist $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of the simplex $\Delta_N$ whose $f$-images overlap, that is
\[
f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.
\]

**Proof.** According to Theorem 3.9 it suffices to prove that there cannot be any $\mathcal{G}_r$-equivariant map $(\Delta_N)_{\Delta(2)}^r \rightarrow S(W_r^{\oplus d})$. Let $\mathbb{Z}/r$ be the subgroup of $\mathcal{G}_r$ generated by the cyclic permutation $(123 \ldots r)$. Then it is enough to prove that there is no $\mathbb{Z}/r$-equivariant map $(\Delta_N)_{\Delta(2)}^r \rightarrow S(W_r^{\oplus d})$. For that we are going to use Dold’s theorem.

The assumption that $r$ is a prime implies that the action of $\mathbb{Z}/r$ on the the sphere $S(W_r^{\oplus d})$ is free. Now, since

- $(\Delta_N)_{\Delta(2)}^r$ is an $(N-r)$-connected $\mathbb{Z}/r$-space, and
- $S(W_r^{\oplus d})$ is a free $(N-r)$-dimensional $\mathbb{Z}/r$-CW complex,

the theorem of Dold implies that a $\mathbb{Z}/r$-equivariant map $(\Delta_N)_{\Delta(2)}^r \rightarrow S(W_r^{\oplus d})$ cannot exist. \hfill \Box

The same argument yields that there cannot be any $\mathcal{G}_r$-equivariant map $(\Delta_N)_{\Delta(2)}^r \rightarrow S(W_r^{\oplus (d+1)})$, when $r$ is a prime. Observe that for an application of the theorem of Dold the nature of the group action on the domain is of no importance.

The next remarkable step followed a few years later. In 1987 in his landmark unpublished manuscript Özaydin [38] extended the result of Bárány, Shlosman and Szűcs and proved that the topological Tverberg conjecture holds for $r$ a prime power. He proved even more and left the topological Tverberg conjecture as a teaser for generations of mathematicians to come. But this story will come a bit later.

The first published proof of the topological Tverberg theorem for $r$ a prime power appeared in a paper of Aleksei Yu. Volovikov [47]; see Remark 3.12. Here we give a proof of the topological Tverberg theorem for prime powers based on a comparison of Serre spectral sequences which uses a consequence of the localization theorem for equivariant cohomology [30, Cor. 1, p. 45]. For background on spectral sequences we refer to the textbooks by McCleary [37] and by Fomenko and Fuchs [25].

**Theorem 3.11** (Topological Tverberg theorem for prime powers $r$). Let $d \geq 1$ be an integer, let $r \geq 2$ be a prime power, $N \equiv (d+1)(r-1)$, and let $f : \Delta_N \rightarrow \mathbb{R}^d$ be a continuous map. Then there exist $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of the simplex $\Delta_N$ whose $f$-images overlap, that is
\[
f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.
\]

**Proof.** Let $d \geq 1$ be an integer, and let $r = p^n$ for $p$ a prime. By Theorem 3.9 it suffices to prove that there cannot be any $\mathcal{G}_r$-equivariant map $(\Delta_N)_{\Delta(2)}^r \rightarrow S(W_r^{\oplus d})$.

Consider the elementary abelian group $(\mathbb{Z}/p)^n$ and the regular embedding $\text{reg} : (\mathbb{Z}/p)^n \rightarrow \mathcal{G}_r$, as explained in [1, Ex. 2.7, p. 100]. It is given by the left translation action of $(\mathbb{Z}/p)^n$ on itself: To each element $g \in (\mathbb{Z}/p)^n$ we associate the permutation $L_g : (\mathbb{Z}/p)^n \rightarrow (\mathbb{Z}/p)^n$ from $\text{Sym}((\mathbb{Z}/p)^n) \cong \mathcal{G}_r$ given by $L_g(x) = g + x$. We identify the elementary abelian group $(\mathbb{Z}/p)^n$ with the subgroup $\text{im}(\text{reg})$ of the symmetric group $\mathcal{G}_r$. Thus, in order to prove the non-existence of an $\mathcal{G}_r$-equivariant map it suffices to prove the non-existence of a $(\mathbb{Z}/p)^n$-equivariant map $(\Delta_N)_{\Delta(2)}^r \rightarrow S(W_r^{\oplus d})$. 


Our proof takes several steps; the crucial ingredient is a comparison of Serre spectral sequences. As it will be by contradiction, let us now assume that a \((\mathbb{Z}/p)^n\)-equivariant map \(\varphi : (\Delta_N)^{\times r}_{\Delta(2)} \to S(W_{r}^{\oplus d})\) exists.

(1) Let \(\lambda\) denote the Borel construction
\[\lambda : (\Delta_N)^{\times r}_{\Delta(2)} \to E(\mathbb{Z}/p)^n \times_{(\mathbb{Z}/p)^n} (\Delta_N)^{\times r}_{\Delta(2)} \to B(\mathbb{Z}/p)^n,\]
while \(\rho\) denotes the Borel construction fiber bundle
\[\rho : (S(W_{r}^{\oplus d})) \to E(\mathbb{Z}/p)^n \times_{(\mathbb{Z}/p)^n} S(W_{r}^{\oplus d}) \to B(\mathbb{Z}/p)^n.\]
Then the map \(\varphi\) would induce the following morphism between fiber bundles \(\lambda\) and \(\rho\):
\[
\begin{array}{ccc}
E(\mathbb{Z}/p)^n \times_{(\mathbb{Z}/p)^n} (\Delta_N)^{\times r}_{\Delta(2)} & \xrightarrow{\text{id} \times (\mathbb{Z}/p)^n \varphi} & E(\mathbb{Z}/p)^n \times_{(\mathbb{Z}/p)^n} S(W_{r}^{\oplus d}) \\
B(\mathbb{Z}/p)^n & \cong & B(\mathbb{Z}/p)^n \\
\end{array}
\]
This bundle morphism induces a morphism of associated cohomology Serre spectral sequences:
\[E^s_{t,j}(\lambda) := E^s_{t,j}(E(\mathbb{Z}/p)^n \times_{(\mathbb{Z}/p)^n} (\Delta_N)^{\times r}_{\Delta(2)}) \xrightarrow{\Phi^s_{t,j}} E^s_{t,j}(E(\mathbb{Z}/p)^n \times_{(\mathbb{Z}/p)^n} S(W_{r}^{\oplus d})) =: E^s_{t,j}(\rho)\]
such that on the zero row of the second term
\[E^2_{0,0}(\lambda) := E^2_{0,0}(E(\mathbb{Z}/p)^n \times_{(\mathbb{Z}/p)^n} (\Delta_N)^{\times r}_{\Delta(2)}) \xrightarrow{\Phi^2_{0,0}} E^2_{0,0}(E(\mathbb{Z}/p)^n \times_{(\mathbb{Z}/p)^n} S(W_{r}^{\oplus d})) =: E^2_{0,0}(\rho)\]
is the identity. Here for the morphisms we use the simplified notation \(\Phi^s_{t,j} := E^s_{t,j}(\text{id} \times (\mathbb{Z}/p)^n \varphi)\).

Before calculating both spectral sequences we recall the cohomology of \(B(\mathbb{Z}/p)^n\) with coefficients in the field \(\mathbb{F}_p\) and denote it as follows:
\[
p = 2 : \quad H^*(B((\mathbb{Z}/2)^n); \mathbb{F}_2) = H^*((\mathbb{Z}/2)^n; \mathbb{F}_2) \cong \mathbb{F}_2[t_1, \ldots, t_n], \quad \text{deg } t_i = 1
\]
\[
p > 2 : \quad H^*(B((\mathbb{Z}/p)^n); \mathbb{F}_p) = H^*((\mathbb{Z}/p)^n; \mathbb{F}_p) \cong \mathbb{F}_p[t_1, \ldots, t_n] \otimes \Lambda[e_1, \ldots, e_n], \quad \text{deg } t_i = 2, \text{deg } e_i = 1,
\]
where \(\Lambda[\cdot]\) denotes the exterior algebra.

(2) First we consider the Serre spectral sequence, with coefficients in the field \(\mathbb{F}_p\), associated to the fiber bundle \(\lambda\). The \(E_2\)-term of this spectral sequence can be computed as follows:
\[
E^2_{t,j}(\lambda) = \begin{cases} 
H^i((\mathbb{Z}/p)^n; \mathbb{F}_p), & j = 0, \\
H^i((\mathbb{Z}/p)^n; H^{N-r+1}((\Delta_N)^{\times r}_{\Delta(2)}; \mathbb{F}_p)), & j = N-r+1, \\
0, & \text{otherwise},
\end{cases}
\]
since by Theorem 3.4 the deleted product \((\Delta_N)^{\times r}_{\Delta(2)}\) is an \((N-r+1)\)-dimensional, \((N-r)\)-connected simplicial complex and consequently \(H^j((\Delta_N)^{\times r}_{\Delta(2)}; \mathbb{F}_p) = 0\) only for \(j = 0\) or \(j = N-r+1\). Thus, the only possibly non-zero differential of the spectral sequence is \(\partial_{N-r+2}\) and therefore \(E^2_{0,0}(\lambda) \cong E^\infty_{0,0}(\lambda)\) for \(i \leq N-r+1\).

(3) The second Serre spectral sequence, with coefficients in the field \(\mathbb{F}_p\), we consider is associated to the fiber bundle \(\rho\). In this case the fundamental group of the base space \(\pi_1(B((\mathbb{Z}/p)^n)) \cong (\mathbb{Z}/p)^n\) acts trivially on the cohomology \(H^*(S(W_{r}^{\oplus d}); \mathbb{F}_p)\). Indeed, when \(p = 2\) the group \((\mathbb{Z}/2)^n\) can only act trivially on \(H^0(S(W_{r}^{\oplus d}); \mathbb{F}_2) \cong \mathbb{F}_2\) and on \(H^{N-r}(S(W_{r}^{\oplus d}); \mathbb{F}_2) \cong \mathbb{F}_2\). For \(p\) an odd prime all elements of the group \((\mathbb{Z}/p)^n\) have odd order and therefore the action is trivial on the \(\mathbb{F}_p\) vector spaces \(H^j(S(W_{r}^{\oplus d}); \mathbb{F}_p) \cong \mathbb{F}_p\) and \(H^j(S(W_{r}^{\oplus d}); \mathbb{F}_p) \cong \mathbb{F}_p\). Thus the \(E_2\)-term of this spectral sequence is of the form
\[
E^2_{t,j}(\rho) = \begin{cases} 
H^i(B((\mathbb{Z}/p)^n); H^j(S(W_{r}^{\oplus d}); \mathbb{F}_p)), & j = 0 \text{ or } j = N-r,
\end{cases}
\]
\[
\cong \begin{cases} 
H^i((\mathbb{Z}/p)^n; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^j(S(W_{r}^{\oplus d}); \mathbb{F}_p) \cong \begin{cases} H^i((\mathbb{Z}/p)^n; \mathbb{F}_p), & j = 0 \text{ or } j = N-r, \\
0, \text{otherwise},
\end{cases}
\end{cases}
\]
Moreover, if \(\ell \in H^{N-r}(S(W_{r}^{\oplus d}); \mathbb{F}_p) \cong \mathbb{F}_p\) denotes a generator then the \((N-r)\)-row of the \(E_2\)-term is a free \(H^*(\mathbb{Z}/p)^n; \mathbb{F}_p\)-module generated by the element \(1 \otimes_{\mathbb{F}_p} \ell \in E^2_{0,N-r}(\rho) \cong H^{N-r}(S(W_{r}^{\oplus d}); \mathbb{F}_p)\). The only possible non-zero differential is
\[
\partial_{N-r+1} : E^N_{0,N-r}(\rho) \to E^N_{N-r+1,0}(\rho).
\]
Consequently we have that
Since the differentials in both spectral sequences are zero in all terms with a slight abuse of notation, we have that

\[ a \]

Finally, to reach a contradiction with the assumption that the map in cohomology

\[ \text{is not} \]

that the map in cohomology

\[ \text{is zero.} \]

Furthermore, if \( \partial_{N-r+1}(1 \otimes_{\mathbb{F}_p} \ell) = 0 \), then \( E_2^{0,r}(\rho) \cong E_2^{\bullet\bullet}(\rho) \). Hence, the projection map

\[ E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \rightarrow B(\mathbb{Z}/p)^n \]

induces a monomorphism in cohomology

\[ H^*(B(\mathbb{Z}/p)^n; \mathbb{F}_p) \rightarrow H^*(E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \rightarrow S((\mathbb{Z}/p)^d); \mathbb{F}_p) \].

Now the following consequence of the localization theorem [30, Cor. 1, p. 45], which in the case of finite groups holds only for elementary abelian groups, comes into play:

**Theorem.** Let \( p \) be a prime, \( G = (\mathbb{Z}/p)^n \) with \( n \geq 1 \), and let \( X \) be a finite \( G \)-CW complex. The fixed point set \( X^G \) of the space \( X \) is non-empty if and only if the map in cohomology

\[ H^*(BG; \mathbb{F}_p) \rightarrow H^*(EG \times_G X; \mathbb{F}_p) \],

induced by the projection \( EG \times_G X \rightarrow BG \), is a monomorphism.

Since the fixed point set \( S((\mathbb{Z}/p)^d)(\mathbb{Z}/p)^n = \emptyset \) of the sphere is empty, the theorem we just quoted implies that the map in cohomology

\[ H^*(B(\mathbb{Z}/p)^n; \mathbb{F}_p) \rightarrow H^*(E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \rightarrow S((\mathbb{Z}/p)^d); \mathbb{F}_p) \]

is not a monomorphism. Consequently, the element

\[ a := \partial_{N-r+1}(1 \otimes_{\mathbb{F}_p} \ell) \neq 0 \in E_{N-r+1}^{0,N-r+1} \]

is not zero.

(4) Finally, to reach a contradiction with the assumption that the \( (\mathbb{Z}/p)^n \)-equivariant map \( \varphi \) exists we track the element \( a := \partial_{N-r+1}(1 \otimes_{\mathbb{F}_p} \ell) \neq 0 \in E_{N-r+1}^{0,N-r+1} \) along the morphism of spectral sequences

\[ \Phi_s^{N-r+1.0} : E_s^{N-r+1.0} \rightarrow E_s^{N-r+1.0} \].

Since the differentials in both spectral sequences are zero in all terms \( E_s(\rho) \) and \( E_s(\lambda) \) for \( 2 \leq s \leq N-r \) we have that \( \Phi_s^{\bullet\bullet} \) is the identity for \( 2 \leq s' \leq N - r + 1 \). In particular, the morphism

\[ \Phi_s^{N-r+1.0} : E_{N-r+1}^{N-r+1.0} \rightarrow E_{N-r+1}^{N-r+1.0} \]

is still identity as it was in the second term, and so \( \Phi_{N-r+1}^{N-r+1.0}(a) = a \). Passing to the \( (N - r + 1) \)-term, with a slight abuse of notation, we have that

\[ \Phi_{N-r+2}^{N-r+1.0}(a) = [a] \].
where $[a]$ denotes the class induced by $a$ in the appropriate $(N - r + 2)$-term of the spectral sequences. Since $a := \partial_{N-r+1}(1 \otimes \mathbb{F}_p \ell) \in E^{N-r+1,0}_2(\rho)$ and $0 \neq a \in E^{N-r+1,0}_2(\lambda) \cong E^{\infty}_{N-r+1,0}(\lambda)$ passing to the next $E^{N-r+2,0}_2$-term we reach a contradiction:

$$\Phi^{N-r+1,0}_{N-r+2}(0) = [a] = a \neq 0,$$

because the class of the element $a$ in $E^{N-r+1,0}_2(\rho)$ vanishes (domain of $\Phi^{N-r+1,0}_{N-r+2}$) while in $E^{N-r+1,0}_2(\lambda)$ it does not vanish (codomain of $\Phi^{N-r+1,0}_{N-r+2}$). Hence, there cannot be any $(\mathbb{Z}/p)^n$-equivariant map $(\Delta_N)^{N-1}_{\Delta(2)} \to S(W^{r,d}_r)$, and the proof of the theorem is complete. \qed

In the language of the Fadell–Husseini index $\text{dict}$, as introduced in [23], we have computed that

$$\text{index}_{(\mathbb{Z}/p)^n}((\Delta_N)^{N-1}_{\Delta(2)}; \mathbb{F}_p) \subseteq H^{N-r+2}(B(\mathbb{Z}/p)^n; \mathbb{F}_p).$$

Furthermore, we showed the existence of an element $a \in H^{N-r+1}(B(\mathbb{Z}/p)^n; \mathbb{F}_p)$ that has the property

$$0 \neq a \in \text{index}_{(\mathbb{Z}/p)^n}(S(W^{r,d}_r); \mathbb{F}_p) \cap H^{N-r+1}(B(\mathbb{Z}/p)^n; \mathbb{F}_p). \tag{10}$$

Consequently $\text{index}_{(\mathbb{Z}/p)^n}(S(W^{r,d}_r); \mathbb{F}_p) \not\subseteq \text{index}_{(\mathbb{Z}/p)^n}((\Delta_N)^{N-1}_{\Delta(2)}; \mathbb{F}_p)$ and so the monotonicity property of the Fadell–Husseini index implies the non-existence of a $(\mathbb{Z}/p)^n$-equivariant map $(\Delta_N)^{N-1}_{\Delta(2)} \to S(W^{r,d}_r)$.

The element $a$ with the property (10) can be specified explicitly. It is the Euler class of the vector bundle

$$W^{r,d}_r \to E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n W^{r,d}_r \to B(\mathbb{Z}/p)^n.$$

From the work of Mann and Milgram [35] we get that for an odd prime $p$

$$a = \omega \cdot \left( \prod_{(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_p^n \setminus \{0\}} (\alpha_1 t_1 + \cdots + \alpha_n t_n)^{d/2} \right),$$

where $\omega \in \mathbb{F}_p \setminus \{0\}$, while for $p = 2$ we have that

$$a = \left( \prod_{(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_2^n \setminus \{0\}} (\alpha_1 t_1 + \cdots + \alpha_n t_n)^{d} \right).$$

The square root in $\mathbb{F}_p[t_1, \ldots, t_n]$ is not uniquely determined for an odd prime $p$ and $d$ odd: The factor $\omega$ accounts for an arbitrary square root being taken.

**Remark 3.12.** Volovikov, in his 1996 paper [47], proved the following extension of the topological Tverberg theorem for continuous maps to manifolds:

**Theorem.** Let $d \geq 1$ be an integer, let $r \geq 2$ be a prime power, and $N = (d + 1)(r - 1)$. For any topological $d$-manifold $M$ and any continuous map $f : \Delta_N \to M$, there exist $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of the simplex $\Delta_N$ whose $f$-images overlap, that is

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.$$

4. **Corollaries of the topological Tverberg theorem**

Over time many results were discovered that were believed to be substantial extensions or analogs of the topological Tverberg theorem, such as the generalized Van Kampen–Flores theorem of Karanbir Sarkaria [40] and Aleksei Volovikov [48], the colored Tverberg theorems of Rade Živaljević and Siniša Vrećica [53] [49] and Pablo Soberón’s result on Tverberg points with equal barycentric coordinates [44]. It turned out only recently that the elementary idea of constraint functions together with the concept of “unavoidable complexes” introduced in [12] transforms all these results into simple corollaries of the topological Tverberg theorem.

Well, if all these results are corollaries, is there any genuine extension of the topological Tverberg theorem? The answer to this question will bring us to the fundamental work of Bárany and Larman [7], and the optimal colored Tverberg theorem [17] from 2009. But this will be the story of the final section of this paper.
4.1. The generalized Van Kampen–Flores theorem. The first corollary we prove is the following generalized Van Kampen–Flores Theorem that was originally proved by Sarkaria [40] for primes and then by Volovikov [48] for prime powers. The fact that this result can be derived easily from the topological Tverberg theorem by adding an extra component to the map was first sketched by Gromov in [27, Sec. 2.9c]; this can be seen as a first instance of the constraint method [12, Thm. 6.3] “at work.”

**Theorem 4.1** (The generalized Van Kampen–Flores Theorem). Let \( d \geq 1 \) be an integer, let \( r \) be a prime power, let \( k \geq \left\lceil \frac{d+1}{2} \right\rceil \) and \( N = (d+2)(r-1) \), and let \( f : \Delta_N \to \mathbb{R}^d \) be a continuous map. Then there exist \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) in the \( k \)-skeleton \( \text{sk}_k(\Delta_N) \) of the simplex \( \Delta_N \) whose \( f \)-images overlap,

\[
\bigcap_{1 \leq i < j \leq r} f(\sigma_i) \cap \cdots \cap f(\sigma_r) \neq \emptyset.
\]

**Proof.** For the proof we use two ingredients, the topological Tverberg theorem and the pigeonhole principle. First, consider the continuous map \( g : \Delta_N \to \mathbb{R}^{d+1} \) defined by

\[
g(x) = (f(x), \text{dist}(x, \text{sk}_k(\Delta_N))).
\]

Since \( N = (d+2)(r-1) = ((d+1) + 1)(r-1) \) and \( r \) is a prime power we can apply the topological Tverberg theorem to the map \( g \). Consequently, there exist \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) with points \( x_1 \in \text{relint} \sigma_1, \ldots, x_r \in \text{relint} \sigma_r \) such that \( g(x_1) = \cdots = g(x_r) \), that is,

\[
f(x_1) = \cdots = f(x_r) \quad \text{and} \quad \text{dist}(x_1, \text{sk}_k(\Delta_N)) = \cdots = \text{dist}(x_r, \text{sk}_k(\Delta_N)).
\]

One of the faces \( \sigma_1, \ldots, \sigma_r \) has to belong to \( \text{sk}_k(\Delta_N) \). Indeed, if all the faces \( \sigma_1, \ldots, \sigma_r \) are disjoint, would not belong to \( \text{sk}_k(\Delta_N) \), then the simplex \( \Delta_N \) should have at least

\[
|\sigma_1| + \cdots + |\sigma_r| \geq r(k+2) \geq r\left(\left\lceil \frac{d+1}{2} \right\rceil + 2\right) \geq (r-1)(d+2) + 2 = N + 2
\]

vertices. Thus, since one of the faces is in the \( k \)-skeleton \( \text{dist}(x_1, \text{sk}_k(\Delta_N)) = \cdots = \text{dist}(x_r, \text{sk}_k(\Delta_N)) = 0 \), and consequently \( \sigma_1 \in \text{sk}_k(\Delta_N), \ldots, \sigma_r \in \text{sk}_k(\Delta_N) \), completing the proof of the theorem. \( \square \)

4.2. The colored Tverberg problem of Bárány and Larman. In their 1990 study on halving lines and halving planes, Bárány, Zoltan Füredi and László Lovász [6] realized a need for a colored version of the Tverberg theorem. The sentence from this paper

opened a new chapter in the study of extensions of the Tverberg theorem, both affine and topological. Soon after, in 1992, Bárány and David Larman in [7] formulated the colored Tverberg problem and brought to light a conjecture that motivated the progress in the area for decades to come.

Let \( N \geq 1 \) be an integer and let \( \mathcal{C} \) be the set of vertices of the simplex \( \Delta_N \). A coloring of the set of vertices \( \mathcal{C} \) by \( t \) colors is a partition \( (C_1, \ldots, C_t) \) of \( \mathcal{C} \), that is, \( \mathcal{C} = C_1 \cup \cdots \cup C_t \) and \( C_i \cap C_j = \emptyset \) for \( 1 \leq i < j \leq t \). The elements of the partition \( (C_1, \ldots, C_t) \) are called color classes. A face \( \sigma \) of the simplex \( \Delta_N \) is a rainbow face if \( |\sigma \cap C_i| \leq 1 \) for all \( 1 \leq i \leq t \). The subcomplex of all rainbow faces of the simplex \( \Delta_N \) induced by the coloring \( (C_1, \ldots, C_t) \) will be denoted by \( R((C_1, \ldots, C_t)) \) and will be called the rainbow subcomplex. There is an isomorphism of simplicial complexes \( R((C_1, \ldots, C_t)) \cong C_1 \star \cdots \star C_t \).

**Problem 4.2** (Bárány–Larman colored Tverberg problem). Let \( d \geq 1 \) and \( r \geq 2 \) be integers. Determine the smallest number \( n = n(d, r) \) such that for every affine map \( f : \Delta_{n-1} \to \mathbb{R}^d \), and every coloring \( (C_1, \ldots, C_{d+1}) \) of the vertex set \( \mathcal{C} \) of the simplex \( \Delta_{n-1} \) by \( d+1 \) colors with each color of size at least \( r \), there exist \( r \) pairwise disjoint rainbow faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_{n-1} \) whose \( f \)-images overlap,

\[
\bigcap_{1 \leq i < j \leq r} f(\sigma_i) \cap \cdots \cap f(\sigma_r) \neq \emptyset.
\]

A trivial lower bound for the function \( n(d, r) \) is \( (d+1)r \). Bárány and Larman proved that the trivial lower bound is tight in the cases \( n(r, 1) = 2r \) and \( n(r, 2) = 3r \), and presented a proof by Lovász for \( n(2, d) = 2(d+1) \). Furthermore, they conjectured the following equality.

**Conjecture 4.3** (Bárány–Larman conjecture). Let \( r \geq 2 \) and \( d \geq 1 \) be integers. Then \( n(d, r) = (d+1)r \).

Now we present the proof of Lovász for the Bárány–Larman conjecture in the case \( r = 2 \) from the paper of Bárány and Larman [7, Thm. (iii)].

**Theorem 4.4.** Let \( d \geq 1 \) be an integer. Then \( n(2, d) = 2(d+1) \).

**Proof.** Let \( n = 2(d+1) \), and let \( f : \Delta_{n-1} \to \mathbb{R}^d \) be an affine map. Furthermore, consider a coloring \( (C_1, \ldots, C_{d+1}) \) of the vertex set \( \mathcal{C} \) of the simplex \( \Delta_{n-1} \) by \( d+1 \) colors where \( |C_1| = \cdots = |C_{d+1}| = 2 \).

Denote \( C_i = \{v_i, -v_i\} \) for \( 1 \leq i \leq d+1 \). The subcomplex of all rainbow faces of the simplex \( \Delta_{n-1} \) is the join \( R := R((C_1, \ldots, C_{d+1})) \). In this case, the rainbow subcomplex \( R \) can be identified
with the boundary of the cross-polytope $[2]^{*+d+1} \cong S^d$. Here [2], as before, denotes the 0-dimensional simplicial complex with two vertices.

The restriction map $f|_R : [2]^{*+d+1} \to \mathbb{R}^d$ is a piecewise affine map, and therefore continuous. The Borsuk–Ulam theorem yields the existence of a point $x \in [2]^{*+d+1} \cong S^d$ on the sphere with the property that $f|_R(x) = f|_R(-x)$. The point $x \in [2]^{*+d+1}$ belongs to the relative interior of a unique simplex in the boundary of the cross-polytope $[2]^{*+d+1}$,

$$x \in \text{relint} \left( \text{conv} \{e_i v_i, \ldots, e_k v_k \} \right),$$

where $e_k \in \{-1,+1\}$ and $1 \leq k \leq d + 1$. Thus, $-x \in \text{relint} \left( \text{conv} \{-e_i v_i, \ldots, -e_k v_k \} \right)$. Since the rainbow faces $\text{conv} \{e_i v_i, \ldots, e_k v_k \}$ and $\text{conv} \{-e_i v_i, \ldots, -e_k v_k \}$ are disjoint, and $f|_R(x) = f|_R(-x) \in f|_R \left( \text{relint}(\text{conv} \{e_i v_i, \ldots, e_k v_k \}) \right) \cap f|_R \left( \text{relint}(\text{conv} \{-e_i v_i, \ldots, -e_k v_k \}) \right) \neq \emptyset$, we have proved the theorem.

\[ \square \]

4.3. The colored Tverberg problem of Živaljević and Vrećica. In response to the work of Bárány and Larman a modified colored Tverberg problem was presented by Živaljević and Vrećica in their influential paper [53] from 1992.

**Problem 4.5** (The Živaljević–Vrećica colored Tverberg problem). Let $d \geq 1$ and $r \geq 2$ be integers. Determine the smallest number $t = t(d, r)$ (or $t = tt(d, r)$) such that for every affine (or continuous) map $f : \Delta \to \mathbb{R}^d$, and every coloring $(C_1, \ldots, C_{d+1})$ of the vertex set $C$ of the simplex $\Delta$ by $d+1$ colors with each color of size at least $t$, there exist $r$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta$ whose $f$-images overlap, that is

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.$$

Observe that in the language of the function $t(d, r)$ the Bárány–Larman conjecture says that $t(d, r) = r$ for all $r \geq 2$ and $d \geq 1$. Furthermore, proving that $t(d, r) \leq +\infty$ does not imply $n(d, r) \leq +\infty$, while proving $t(d, r) = r$ would imply that $n(d, r) = r(d + 1)$.

In order to address the modified problem Živaljević and Vrećica needed to know the connectivity of chessboard complexes. For that they recalled the following result of Anders Björner, Lovász, Vrećica and Živaljević [11, Thm. 1.1]. Its connectivity lower bound is best possible according to [41].

**Theorem 4.6.** Let $m \geq 1$ and $n \geq 1$ be integers. The chessboard $\Delta_{m,n}$ is $\nu$-connected, where

$$\nu = \min \{m, n, \left\lfloor \frac{m+n+1}{3} \right\rfloor \} - 2.$$

**Proof.** Without loss of generality we can assume that $1 \leq m \leq n$. The proof proceeds by induction on $\min\{m,n\} = m$. In the case $m = 1$ the statement of the theorem is obviously true. For $m = 2$ we distinguish between two cases:

- If $n = 2$, then $\nu = \min \{2, 2, \left\lfloor \frac{4}{3} \right\rfloor \} - 2 = -1$ and $\Delta_{2,2}$ is just a disjoint union of two edges, and
- If $n \geq 3$, then $\nu = \min \{2, 2, \left\lfloor \frac{2n+1}{3} \right\rfloor \} - 2 = 0$ and $\Delta_{2,n}$ is path connected.

Let $m \geq 3$, and let us assume that the statement of the theorem holds for every chessboard $\Delta_{m',n'}$ where $1 \leq \min\{m',n'\} < m$. Now we prove the statement of the theorem for the chessboard $\Delta_{m,n}$.

Let $K_{\ell}$ for $1 \leq \ell \leq n$ be a subcomplex of $\Delta_{m,n}$ defined by

$$\{(i_0,j_0),\ldots,(i_k,j_k)\} \in K_{\ell} \iff \{(i_0,j_0),\ldots,(i_k,j_k),(1,\ell)\} \in \Delta_{m,n}.$$

The family of subcomplexes $\mathcal{K} := \{K_{\ell} : 1 \leq \ell \leq n\}$ covers the chessboard $\Delta_{m,n}$. Moreover, each subcomplex $K_{\ell}$ is a cone over the chessboard $\Delta_{m-1,n-1}$, and therefore contractible. Since, for $\sigma \subseteq [n]$ we have that

$$\bigcap\{K_{\ell} : \ell \in \sigma\} = \emptyset \iff \sigma = [n],$$

the nerve $\operatorname{dict} N_{\mathcal{K}}$ of the family $\mathcal{K}$ is homeomorphic to the boundary of an $(n-1)$-simplex $\partial \Delta_{n-1}$. Thus, $N_{\mathcal{K}} \cong S^{n-2}$ is $(n-3)$-connected. Furthermore, for $\sigma \subseteq [n]$ with the property that $2 \leq |\sigma| \leq n-1$ the intersection $\bigcap\{K_{\ell} : \ell \in \sigma\}$ is homeomorphism with the chessboard $\Delta_{m-1,n-|\sigma|}$. The induction hypothesis can be applied to each of these intersections. Therefore,

$$\operatorname{conn}(\bigcap\{K_{\ell} : \ell \in \sigma\}) = \operatorname{conn}(\Delta_{m-1,n-|\sigma|}) \geq \min\{m-1,n-|\sigma|,\left\lfloor \frac{m+n-|\sigma|}{3} \right\rfloor \} - 2.$$

Now we will apply the following connectivity version of the Nerve theorem $\operatorname{dict}$ due to Björner, see [10, Thm. 10.6].

**Theorem.** Let $K$ be a finite simplicial complex, or a regular CW-complex, and let $\mathcal{K} := \{K_i : i \in I\}$ be a cover of $K$ by a family of subcomplexes, $K = \bigcup\{K_i : i \in I\}$.
This concludes the induction step. (Colored Tverberg theorem of Živaljević and Vrećica) for primes, and for the following version of the proof for prime powers; see also the proof of Živaljević

Proof. \( \left( 1 \right) \) Let \( \sigma \) be a face of the nerve \( N_\mathcal{K} \) of \( \mathcal{K} \). The connectivity of \( \mathcal{K} \) ensures that \( \sigma \) is connected. Therefore, \( \mathcal{K} \) is \( \nu \)-connected, we have that

- for every face \( \sigma \) of the nerve \( N_\mathcal{K} \) the intersection \( \bigcap \{ K_i : i \in \sigma \} \) is contractible, then \( K \) and \( N_\mathcal{K} \) are homotopy equivalent, \( K \cong N_\mathcal{K} \).

(2) If for every face \( \sigma \) of the nerve \( N_\mathcal{K} \) the intersection \( \bigcap \{ K_i : i \in \sigma \} \) is \( (k - |\sigma| + 1) \)-connected, then the complex \( K \) is \( k \)-connected if and only if the nerve \( N_\mathcal{K} \) is \( k \)-connected.

In the case of the covering \( \mathcal{K} \) of the chessboard \( \Delta_{m,n} \), where \( 2 < m \leq n \), we have that

- for every face \( \sigma \) of the nerve \( N_\mathcal{K} \) the intersection \( \bigcap \{ K_i : i \in \sigma \} \) is contractible when \( |\sigma| = 1 \), and

\[
\text{conn} \left( \bigcap \{ K_i : i \in \sigma \} \right) \geq \min \left\{ m - 1, n - |\sigma|, \frac{m+n+|\sigma|-1}{3} \right\} - 2
\geq \min \left\{ m, n, \frac{m+n+1}{3} \right\} - 2 - |\sigma| + 1
\geq \nu - |\sigma| + 1,
\]

when \( 2 \leq |\sigma| \leq n - 1 \), while

- the nerve \( N_\mathcal{K} \) of the family \( \mathcal{K} \) is \( (n - 3) \)-connected with

\[
n - 3 \geq \min \left\{ m, n, \frac{m+n+1}{3} \right\} - 2 = \nu.
\]

Therefore, according to the Nerve theorem applied for the cover \( \mathcal{K} \) the chessboard \( \Delta_{m,n} \) is \( \nu \)-connected. This concludes the induction step. \( \square \)

The knowledge on the connectivity of the chessboard complexes was the decisive information both for the original proof of the Živaljević and Vrećica colored Tverberg theorem [53, Thm. 1], which worked only for primes, and for the following version of the proof for prime powers; see also the proof of Živaljević [52, Thm. 3.2 (2)].

**Theorem 4.7** (Colored Tverberg theorem of Živaljević and Vrećica). Let \( d \geq 1 \) be an integer, and let \( r \geq 2 \) be a prime power. For every continuous map \( f : \Delta \rightarrow \mathbb{R}^d \), and every coloring \( \left( C_1, \ldots, C_{d+1} \right) \) of the vertex set \( V \) of the simplex \( \Delta \) by \( d + 1 \) colors with each color of size at least \( 2r - 1 \), there exist \( r \) pairwise disjoint rainbow faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta \) whose \( f \)-images overlap, that is

\[
f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.
\]

In the language of the function \( tt(d, r) \) the previous theorem yields the upper bound \( tt(d, r) \leq 2r - 1 \) when \( r \) is a prime power. This bound implies the bound \( t(d, r) \leq tt(d, r) \leq 4r - 3 \) for arbitrary \( r \) via Bertrand's postulate.

**Proof.** Let \( r = p^n \) for \( p \) a prime and \( n \geq 1 \). Let \( f : \Delta \rightarrow \mathbb{R}^d \) be a continuous map from a simplex \( \Delta \) whose set of vertices \( V \) is colored by \( d + 1 \) colors \( \left( C_1, \ldots, C_{d+1} \right) \). Without loss of generality assume that \( |C_1| = \cdots = |C_{d+1}| = 2r - 1 \). In addition assume that the map \( f \) is a counterexample for the statement of the theorem. Set \( M := (d + 1)(2r - 1) - 1 \) and \( N := (d + 1)(r - 1) \), so \( \Delta \) is an \( M \)-dimensional simplex.

Now, the proof of the theorem will be presented in several steps.

(1) The rainbow subcomplex of the simplex \( \Delta \) induced by the coloring \( \left( C_1, \ldots, C_{d+1} \right) \) in this case is

\[
R(C_1, \ldots, C_{d+1}) \cong C_1 \ast \cdots \ast C_{d+1} \cong [2r - 1]^*(d+1).
\]

The \( r \)-fold 2-wise deleted join of the rainbow subcomplex \( R(C_1, \ldots, C_{d+1}) \) can be identified, with the help of Lemma 3.7 and Example 3.6, as follows

\[
(R(C_1, \ldots, C_{d+1}))^{\Delta(2)} \cong \left( [2r - 1]^*(d+1) \right)^{\Delta(2)} \cong \left( [2r - 1]^r \right)^{\Delta(2)} \cong (\Delta_{2r-1,r})^{\Delta(2)} \cong (\Delta_{2r-1,r})^{(d+1)}.
\]

The action of the symmetric group \( \mathfrak{S}_r \) on the chessboard \( \Delta_{2r-1,r} \) is assumed to be given by permutation of columns of the chessboard, that is

\[
\pi \cdot \{ (i_0, j_0), \ldots, (i_k, j_k) \} = \{ (i_0, \pi(j_0)), \ldots, (i_k, \pi(j_k)) \},
\]

for \( \pi \in \mathfrak{S}_r \) and \( \{ (i_0, j_0), \ldots, (i_k, j_k) \} \) a simplex in \( \Delta_{2r-1,r} \). Furthermore, the chessboard \( \Delta_{2r-1,r} \) is an \( (r - 1) \)-dimensional and according to Theorem 4.6 an \( (r - 2) \)-connected simplicial complex. Therefore

\[
\text{dim} \left( (\Delta_{2r-1,r})^{\Delta(2)} \right) = (d + 1)r - 1 = N + d,
\]

\[
\text{conn} \left( (\Delta_{2r-1,r})^{\Delta(2)} \right) = (d + 1)r - 2 = N + d - 1.
\]

(2) Now, along the lines of Section 3.2.3, the continuous map \( f : \Delta \rightarrow \mathbb{R}^d \) induces the join map

\[
J_f : (\Delta)^{\Delta(2)} \rightarrow (\mathbb{R}^{d+1})^{\Delta(2)}, \quad \lambda_1 x_1 + \cdots + \lambda_r x_r \mapsto (\lambda_1, \lambda_1 f(x_1)) \oplus \cdots \oplus (\lambda_r, \lambda_r f(x_r)).
\]
Both domain and codomain of the join map $J_f$ are equipped with the action of the symmetric group $S_n$ in such a way that $J_f$ is an $S_n$-equivariant map. The deleted join of the rainbow complex $\left(R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)}\right)$ is an $S_n$-invariant subcomplex of $\left(\Delta_r^{\Sigma(2)}\right)$. Thus, the restriction map

$$J'_f := J_f | (R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)}) : (R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)}) \to (\mathbb{R}^{d+1})^{\otimes r}$$

is also an $S_n$-equivariant map. Next consider the thin diagonal

$$D_J = \{(z_1, \ldots, z_r) \in (\mathbb{R}^{d+1})^{\otimes r} : z_1 = \cdots = z_r\}.$$ 

This is an $S_n$-invariant subspace of $(\mathbb{R}^{d+1})^{\otimes r}$. The key property of the map $J'_f$ we have constructed for any counterexample continuous map $f : \Delta \to \mathbb{R}^d$, is that $\text{im}(J'_f) \cap D_J = \emptyset$. Thus $J'_f$ induces an $S_n$-equivariant map

$$(R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)})^{\otimes r} \to (\mathbb{R}^{d+1})^{\otimes r} \backslash D_J$$

which we, with an obvious abuse of notation, again denote by $J'_f$. Furthermore, let

$$R_J : (\mathbb{R}^{d+1})^{\otimes r} \backslash D_J \to D_J \setminus \{0\} \to S(D^+_J)$$

be the composition of the appropriate projection and deformation retraction. The map $R_J$ is $S_n$-equivariant. Recall from Section 3.2.3 that there is an isomorphism of real $S_n$-representations $D_J^+ \cong W_r^{\otimes (d+1)}$. Here $W_r = \{(t_1, \ldots, t_r) \in \mathbb{R}^r : \sum_{i=1}^r t_i = 0\}$ and it is equipped with the (left) action of the symmetric group $S_n$ given by permutation of coordinates. After the identification of the $S_n$-representations the $S_n$-equivariant map $R_J$ defined in (13) has the form

$$R_J : (\mathbb{R}^{d+1})^{\otimes r} \backslash D_J \to S(W_r^{\otimes (d+1)}).$$

Thus we have proved that if there exists a counterexample map $f$ for the theorem, then there exists an $S_n$-equivariant map

$$(R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)})^{\otimes r} \to S(W_r^{\otimes (d+1)}).$$  

In the final step we reach a contradiction by proving that an $S_n$-equivariant map (15) cannot exist, concluding that a counterexample $f$ could not exist in the first place. The proof of the non-existence of an equivariant map is following the footsteps of the proof of Theorem 3.11.

(3) Consider the elementary abelian group $(\mathbb{Z}/p)^n$ and its regular embedding reg : $(\mathbb{Z}/p)^n \to S_n$. Now it suffices to prove the non-existence of a $(\mathbb{Z}/p)^n$-equivariant map $(R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)})^{\otimes r} \to S(W_r^{\otimes (d+1)})$. To prove the non-existence of such a map assume the opposite: let $\varphi : (R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)})^{\otimes r} \to S(W_r^{\otimes (d+1)})$ be a $(\mathbb{Z}/p)^n$-equivariant map.

Denote by $\lambda$ the Borel construction fiber bundle

$$\lambda : (R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)})^{\otimes r} \to E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \to (R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)})^{\otimes r} \to B(\mathbb{Z}/p)^n,$$

and by $\rho$ the following Borel construction fiber bundle

$$\rho : S(W_r^{\otimes (d+1)}) \to E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \to S(W_r^{\otimes (d+1)}) \to B(\mathbb{Z}/p)^n.$$ 

Then the map $\varphi$ induces the following morphism of fiber bundles

$$E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \to (R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)})^{\otimes r} \xrightarrow{\text{id} \times (\mathbb{Z}/p)^n \varphi} E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \to S(W_r^{\otimes (d+1)}) \xrightarrow{\varphi} B(\mathbb{Z}/p)^n.$$ 

In turn, this morphism induces a morphism of corresponding Serre spectral sequences

$$E_2^{s,j}(\lambda) := E_2^{s,j}(E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \to (R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)})^{\otimes r} \xrightarrow{\text{id} \times (\mathbb{Z}/p)^n \varphi} E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \to S(W_r^{\otimes (d+1)}) \xrightarrow{\varphi} B(\mathbb{Z}/p)^n).$$

with the property that on the zero row of the second term the induced map

$$E_2^{s,0}(\lambda) := E_2^{s,0}(E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \to (R_{(C_1,\ldots,C_{d+1})}^{\Sigma(2)})^{\otimes r} \xrightarrow{\text{id} \times (\mathbb{Z}/p)^n \varphi} E(\mathbb{Z}/p)^n \times (\mathbb{Z}/p)^n \to S(W_r^{\otimes (d+1)}) \xrightarrow{\varphi} B(\mathbb{Z}/p)^n)$$

is the identity. Again we use simplified notation by setting $\Phi_2^{s,j} := E_2^{s,j}(\text{id} \times (\mathbb{Z}/p)^n \varphi)$.

Depending on the parity of the prime $p$, we have that

$$p = 2 : H^r(B ((\mathbb{Z}/2)^n); \mathbb{F}_2) \cong H^r((\mathbb{Z}/2)^n) \otimes \mathbb{F}_2 \cong \mathbb{F}_2[t_1, \ldots, t_n], \quad \text{deg} \ t_i = 1$$

$$p > 2 : H^r(B ((\mathbb{Z}/p)^n); \mathbb{F}_p) \cong H^r((\mathbb{Z}/p)^n) \otimes \mathbb{F}_p \cong \mathbb{F}_p[t_1, \ldots, t_n] \otimes \Lambda[e_1, \ldots, e_n], \quad \text{deg} \ t_i = 2, \text{deg} e_i = 1,$$

where $\Lambda[\cdot]$ denotes the exterior algebra.
First we consider the Serre spectral sequence, with coefficients in the field $\mathbb{F}_p$, associated to the fiber bundle $\lambda$. Using the connectivity of $(R_{(C_1, \ldots, C_{d+1})})^s(\Delta(2))$, we get that the $E_2$-term of this spectral sequence is

$$E_2^{ij}(\lambda) = H^i(B((\mathbb{Z}/p)^n); H^j((R_{(C_1, \ldots, C_{d+1})})^s(\Delta(2)); \mathbb{F}_p)) = H^i((\mathbb{Z}/p)^n; H^j((R_{(C_1, \ldots, C_{d+1})})^s(\Delta(2)); \mathbb{F}_p))$$

$$\cong \begin{cases} H^i((\mathbb{Z}/p)^n; \mathbb{F}_p), & \text{for } j = 0, \\ H^i((\mathbb{Z}/p)^n; H^{N+d}((R_{(C_1, \ldots, C_{d+1})})^s(\Delta(2)); \mathbb{F}_p)), & \text{for } j = N + d, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, the only possibly non-zero differential of this spectral sequence is $\partial_{N+d+1}$ and therefore $E_2^{ij}(\lambda) \cong E_2^{ij}(\lambda)$ for $i \leq N + d$.

The Serre spectral sequence, with coefficients in the field $\mathbb{F}_p$, associated to the fiber bundle $\rho$, is similar to the one appearing in the proof of Theorem 3.11. Briefly, the $E_2$-term of this spectral sequence is

$$E_2^{ij}(\rho) = H^i((\mathbb{Z}/p)^n; H^j(S(W_r^\oplus(d+1)); \mathbb{F}_p)) \cong \begin{cases} H^i((\mathbb{Z}/p)^n; \mathbb{F}_p), & \text{for } j = 0 \text{ or } N - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Letting $\ell \in H^{N-1}(S(W_r^\oplus(d+1)); \mathbb{F}_p) \cong \mathbb{F}_p$ denote a generator, then the $(N-1)$-row of the $E_2$-term can be seen as a free $H^*(((\mathbb{Z}/p)^n; \mathbb{F}_p)$-module generated by $1 \otimes \mathbb{F}_p$, $\ell \in E_2^{N-1}(\rho) \cong H^{N-1}(S(W_r^\oplus(d+1)); \mathbb{F}_p)$. Thus, the only possible non-zero differential is $\partial_N : E_2^{N-1}(\rho) \to E_2^{N+0}(\rho)$ and it is completely determined by the image $\partial_N(1 \otimes \mathbb{F}_p)$. As in the proof of Theorem 3.11, a consequence of the localization theorem implies that $b := \partial_N(1 \otimes \mathbb{F}_p, \ell) \neq 0 \in E_2^{N,0}(\rho) \cong E_2^{N,0}(\rho)$ is not zero.

To reach the desired contradiction we track the element $b \in E_2^{N,0}(\rho) \cong E_2^{N,0}(\rho)$ along the morphism of spectral sequences

$$\Phi_s^{N,0} : E_s^{N,0}(\rho) \to E_s^{N,0}(\lambda).$$

The differentials in both spectral sequences are zero in all terms $E_s(\rho)$ and $E_s(\lambda)$ for $2 \leq s \leq N - 1$. Thus, $\Phi_s^{N,0}$ is an isomorphism for all $2 \leq s \leq N$. In particular, the morphism $\Phi_N^{N,0} : E_N^{N,0}(\rho) \to E_N^{N,0}(\lambda)$ is the identity, as it was in the second term, and so $\Phi_N^{N,0}(b) = b$. When passing to the $(N+1)$-term, with a slight abuse of notation, we get

$$\Phi_{N+1}^{N,0}(b) = [b],$$

where $[b]$ denotes the class induces by $b$ in the appropriate $(N+1)$-term of the spectral sequences. Since $b := \partial_N(1 \otimes \mathbb{F}_p, \ell) \in E_N^{N,0}(\rho)$ and $0 \neq b \in E_2^{N,0}(\lambda) \cong E_2^{N,0}(\lambda)$ we have reached a contradiction:

$$\Phi_{N+1}^{N,0}(0) = [b] = b \neq 0.$$

Therefore, there cannot be any $(\mathbb{Z}/p)^n$-equivariant map $(R_{(C_1, \ldots, C_{d+1})})^s(\Delta(2)) \to S(W_r^\oplus(d+1))$, and the proof of the theorem is complete.

As part of the proof of the previous theorem the following general criterion was derived.
Corollary 4.8. Let \((C_1, \ldots, C_m)\) be a coloring of the simplex \(\Delta\) by \(m\) colors. If there is no \(\mathcal{G}_r\)-equivariant map

\[ \Delta_{|C_1|,r} \ast \cdots \ast \Delta_{|C_m|,r} \cong (R_{C_1, \ldots, C_m})^{*r}_{\Delta(2)} \to S(W^\oplus_{r}^{(d+1)}), \]

then for every continuous map \(f : \Delta \to \mathbb{R}^d\) there exist \(r\) pairwise disjoint rainbow faces \(\sigma_1, \ldots, \sigma_r\) of \(\Delta\) whose \(f\)-images overlap,

\[ f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset. \]

The proof of Theorem 4.7 could have been written in the language of the Fadell–Husseini index [23]. The non-existence of an \((\mathbb{Z}/p)^n\)-equivariant map \((R_{C_1, \ldots, C_{d+1}})^{*r}_{\Delta(2)} \to S(W^\oplus_{r}^{(d+1)})\) would then follow from the observation that

\[ \text{index}_{(\mathbb{Z}/p)^n}((R_{C_1, \ldots, C_{d+1}})^{*r}_{\Delta(2)}; \mathbb{F}_p) \geq \text{index}_{(\mathbb{Z}/p)^n}(S(W^\oplus_{r}^{(d+1)}); \mathbb{F}_p). \]

More precisely, we have computed that

\[ \text{index}_{(\mathbb{Z}/p)^n}((R_{C_1, \ldots, C_{d+1}})^{*r}_{\Delta(2)}; \mathbb{F}_p) = \text{index}_{(\mathbb{Z}/p)^n}((\Delta_{2r-1,r})^{*(d+1)}; \mathbb{F}_p) \subseteq H_{\geq N+d+1}^N(B(\mathbb{Z}/p)^n; \mathbb{F}_p), \]

when \(|C_1| = \cdots = |C_{d+1}| = 2r - 1\). Actually we proved more:

\[ \text{index}_{(\mathbb{Z}/p)^n}((\Delta_{2r-1,r})^{*k}; \mathbb{F}_p) \subseteq H_{\geq k^r}^N(B(\mathbb{Z}/p)^n; \mathbb{F}_p). \]

Furthermore we have found an element \(b \in H^N(B(\mathbb{Z}/p)^n; \mathbb{F}_p)\) with the property that

\[ 0 \neq b \in \text{index}_{(\mathbb{Z}/p)^n}(S(W^\oplus_{r}^{(d+1)}); \mathbb{F}_p) \cap H^N(B(\mathbb{Z}/p)^n; \mathbb{F}_p), \]

and moreover

\[ \text{index}_{(\mathbb{Z}/p)^n}(S(W^\oplus_{r}^{(d+1)}); \mathbb{F}_p) = \langle b \rangle. \]

The element \(b\) with this property is the Euler class of the vector bundle

\[ W^\oplus_{r}^{(d+1)} \to E_{(\mathbb{Z}/p)^n} \times_{(\mathbb{Z}/p)^n} W^\oplus_{r}^{(d+1)} \to B(\mathbb{Z}/p)^n. \]

The work of Mann and Milgram [35] allows us to specify the element \(b\) completely: For \(p\) an odd prime it is

\[ b = \omega \cdot \left( \prod_{(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_p^* \setminus \{0\}} (\alpha_1 t_1 + \cdots + \alpha_n t_n)^{(d+1)/2} \right), \]

where \(\omega \in \mathbb{F}_p \setminus \{0\}\), while for \(p = 2\) it is

\[ b = \left( \prod_{(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_2^* \setminus \{0\}} (\alpha_1 t_1 + \cdots + \alpha_n t_n)^{d+1} \right). \]

The square root in \(\mathbb{F}_p[t_1, \ldots, t_n]\) is not uniquely determined for an odd prime \(p\) and \(d\) odd. Thus we consider an arbitrary square root.

Combining these index computations we have that

\[ 0 \neq b \in \text{index}_{(\mathbb{Z}/p)^n}(S(W^\oplus_{r}^{(d+1)}); \mathbb{F}_p) \cap H^N(B(\mathbb{Z}/p)^n; \mathbb{F}_p) \]

\[ \subseteq \text{index}_{(\mathbb{Z}/p)^n}((\Delta_{2r-1,r})^{*(d+1)}; \mathbb{F}_p) \subseteq H_{\geq N+d+1}^N(B(\mathbb{Z}/p)^n; \mathbb{F}_p), \]

which does not hold, as we just proved. Thus the \((\mathbb{Z}/p)^n\)-equivariant map in question does not exist.

Now observe the difference of dimensions in (18) and compare the dimension of the element \(b\) and the dimension of the group cohomology where the index of the join \((\Delta_{2r-1,r})^{*(d+1)}\) lives. We could have proved more. Indeed, using the index computation (16) we have that

\[ 0 \neq b \in \text{index}_{(\mathbb{Z}/p)^n}(S(W^\oplus_{r}^{(d+1)}); \mathbb{F}_p) \cap H^N(B(\mathbb{Z}/p)^n; \mathbb{F}_p) \]

\[ \subseteq \text{index}_{(\mathbb{Z}/p)^n}((\Delta_{2r-1,r})^{*k}; \mathbb{F}_p) \subseteq H_{\geq k^r}^N(B(\mathbb{Z}/p)^n; \mathbb{F}_p) \]

which is more than we can prove.
as long as $kr \geq N + 1$. We have just concluded that, if $kr \geq N + 1$, then there is no $(\mathbb{Z}/p)^n$-equivariant map

$$(\Delta_{2r-1,r}^*)^{k} \cong (R(C_1, \ldots, C_n))^{x_{r}}_{\Delta(2)} \to S(W_r^\oplus(d+1)).$$

Thus with Corollary 4.8 we have proved the following “colored Tverberg theorem of type B” [49, Thm. 4].

**Theorem 4.9** (The Colored Tverberg theorem of type B of Vrećica and Živaljević). Let $d \geq 1$ and $k \geq 1$ be integers, $N = (d + 1)(r - 1)$, and let $r \geq 2$ be a prime power. For every continuous map $f : \Delta \to \mathbb{R}^d$, and every coloring $(C_1, \ldots, C_k)$ of the vertex set $C$ of the simplex $\Delta$ by $k$ colors, with each color of size at least $2r - 1$ and $kr \geq N + 1$, there exist $r$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta$ whose $f$-images overlap, that is

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.$$

As we have just seen, the proof of the colored Tverberg theorem of Živaljević and Vrećica is in fact also a proof of a type B colored Tverberg theorem. Is it possible that this proof hides a way to prove, for example the Bárány–Larman conjecture? For this we would need to prove that for some all or $r$ and $|S| = \cdots = |C_{d+1}| = r$ there is no $\mathcal{G}_r$-equivariant map

$$\Delta_{*,r}^{(d+1)} \cong (R(C_1, \ldots, C_{d+1}))^{x_{r}}_{\Delta(2)} \to S(W_r^\oplus(d+1)).$$

The connectivity of the chessboard $\Delta_{*,r}$ is only $\left(\frac{2r+1}{2} - 2\right)$ and therefore the scheme of the proof of Theorem 4.7 cannot be used. Even worse, the complete approach fails, as the following theorem of Blagojević, Matschke and Ziegler [17, Prop. 4.1] shows that an $\mathcal{G}_r$-equivariant map (19) does exist.

**Theorem 4.10.** Let $r \geq 2$ and $d \geq 1$ be integers. There exists an $\mathcal{G}_r$-equivariant map

$$\Delta_{*,r}^{(d+1)} \to S(W_r^\oplus(d+1)).$$

**Proof.** For this we use equivariant obstruction theory, as presented by Tammo tom Dieck [21, Sec. II.3]. Let $N := (d + 1)(r - 1)$, $M := r(d + 1) - 1$, and let $(C_1, \ldots, C_{d+1})$ be a coloring of the vertex set of the simplex $\Delta_M$ by $d + 1$ colors of the same size $r$, that is $|C_1| = \cdots = |C_{d+1}| = r$. As we know the deleted join $(R(C_1, \ldots, C_{d+1}))^{x_{r}}_{\Delta(2)}$ of the rainbow complex is isomorphic to the join of chessboards $\Delta_{*,r}^{(d+1)}$. The action of the symmetric group $\mathcal{G}_r$ on the complex $\Delta_{*,r}^{(d+1)}$ is not free. The subcomplex $\Delta_{*,r}^{(d+1)}$ whose points have non-trivial stabilizers with respect to the action of $\mathcal{G}_r$ can be described as follows:

$$(\Delta_{*,r}^{(d+1)})^{>1} = ((R(C_1, \ldots, C_{d+1}))^{x_{r}}_{\Delta(2)})^{>1} = \{\lambda_1 x_1 + \cdots + \lambda_r x_r \in (R(C_1, \ldots, C_{d+1}))^{x_{r}}_{\Delta(2)} : \lambda_i = \lambda_j = 0 \text{ for some } i \neq j\}.$$

Here for a $G$-space (CW complex) $X$ we use notation $X^{>1}$ for the subspace (subcomplex) of all points (cells) with non-trivial stabilizer, meaning that $X/X^{>1}$ is a free $G$-space.

Let $f : \Delta_M \to \mathbb{R}^d$ be any continuous map. As explained in Section 3.2.3 the map $f$ induces the join map given by

$$J_f : (\Delta_M)^{x_{r}}_{\Delta(2)} \to (\mathbb{R}^{d+1})^{\oplus r}, \quad \lambda_1 x_1 + \cdots + \lambda_r x_r \mapsto (\lambda_1, f(x_1)) \oplus \cdots \oplus (\lambda_r, f(x_r)).$$

Since the rainbow complex $(R(C_1, \ldots, C_{d+1}))^{x_{r}}_{\Delta(2)}$ is an $\mathcal{G}_r$-invariant subcomplex of $(\Delta_M)^{x_{r}}_{\Delta(2)}$, the restriction

$$J_f : (R(C_1, \ldots, C_{d+1}))^{x_{r}}_{\Delta(2)} : (R(C_1, \ldots, C_{d+1}))^{x_{r}}_{\Delta(2)} \to (\mathbb{R}^{d+1})^{\oplus r}$$

is also an $\mathcal{G}_r$-equivariant map. Moreover $\text{im}(J_f) = \{(\lambda(x_1, \ldots, x_r) \in (\mathbb{R}^{d+1})^{\oplus r} : z_1 = \cdots = z_r\}$, where for $\Delta_{*,r}^{(d+1)} \cap D_f = \emptyset$ where, as before, $D_f = \{(x_1, \ldots, x_r) \in (\mathbb{R}^{d+1})^{\oplus r} : z_1 = \cdots = z_r\}$. Thus the map $J_f$ induces an $\mathcal{G}_r$-equivariant map

$$\Delta_{*,r}^{(d+1)} \rightarrow (R(C_1, \ldots, C_{d+1}))^{x_{r}}_{\Delta(2)} \rightarrow (\mathbb{R}^{d+1})^{\oplus r}.$$

Composing this map with the $\mathcal{G}_r$-equivariant retraction $R_f : (\mathbb{R}^{d+1})^{\oplus r} \setminus D_f \to S(D_f) \cong S(W_r^\oplus(d+1))$ introduced in (7), we get a continuous $\mathcal{G}_r$-equivariant map

$$(\Delta_{*,r}^{(d+1)})^{>1} = ((R(C_1, \ldots, C_{d+1}))^{x_{r}}_{\Delta(2)})^{>1} \rightarrow S(W_r^\oplus(d+1)).$$

The $(r - 1)$-dimensional chessboard complex $\Delta_{*,r}$ equivariantly retracts to a subcomplex of dimension $r - 2$. Indeed, for each facet of $\Delta_{*,r}$ there is an elementary collapse obtained by deleting all of its subfacets (faces of dimension $r - 2$) that contain the vertex in the $r$-th column. Performing these collapses to all facets of $\Delta_{*,r}$, we get that $\Delta_{*,r}$ collapses $\mathcal{G}_r$-equivariantly to an $(r - 2)$-dimensional subcomplex of $\Delta_{*,r}$. Consequently, the join $((\Delta_{*,r}^{(d+1)})^{>1}$ equivariantly retracts to a subcomplex $\Delta$ of dimension $(d+1)(r-1)-1$. Thus in order to prove the existence of an $\mathcal{G}_r$-equivariant map $\Delta_{*,r}^{(d+1)} \rightarrow S(W_r^\oplus(d+1))$ it suffices to construct an $\mathcal{G}_r$-equivariant map $\Delta \rightarrow S(W_r^\oplus(d+1))$. Since
\textbf{Proof.} disjoint rainbow faces colored by } 2 \text{ by [12, Thm. 5.3].}

4.4. \textbf{The weak colored Tverberg theorem.} How many colored Tverberg theorems can we get directly from the topological Tverberg theorem without major topological machinery? Here is an answer given by [12, Thm. 5.3].

**Theorem 4.11** (The weak colored Tverberg theorem). Let } d \geq 1 \text{ be an integer, let } r \text{ be a prime power, } N = (2d + 2)(r - 1), \text{ and let } f : \Delta_N \to \mathbb{R}^d \text{ be a continuous map. If the vertices of the simplex } \Delta_N \text{ are colored by } d + 1 \text{ colors, where each color class has cardinality at most } 2r - 1, \text{ then there are } r \text{ pairwise disjoint rainbow faces } \sigma_1, \ldots, \sigma_r \text{ of } \Delta_N \text{ whose } f\text{-images overlap, that is}

\[ f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset. \]

**Proof.** Let } C \text{ be the set of vertices of the simplex } \Delta_N \text{ and let } (C_1, \ldots, C_{d+1}) \text{ be a coloring of } C \text{ where } |C_i| \leq 2r - 1 \text{ for all } 1 \leq i \leq d + 1. \text{ To each color class } C_i \text{ we associate the subcomplex } \Sigma_i \text{ of } \Delta_N \text{ defined by}

\[ \Sigma_i := \{ \sigma \in \Delta_N : |\sigma \cap C_i| \leq 1 \}. \]

Observe that the intersection } \Sigma_1 \cap \cdots \cap \Sigma_{d+1} \text{ is the subcomplex of all rainbow faces of } \Delta_N \text{ with respect to the given coloring. Next consider the continuous map } g : \Delta_N \to \mathbb{R}^{2d+1} \text{ defined by}

\[ g(x) = (f(x), \text{dist}(x, \Sigma_1), \text{dist}(x, \Sigma_2), \ldots, \text{dist}(x, \Sigma_{d+1})). \]

Since } N = (2d + 2)(r - 1) = (2d + 1 + 1)(r - 1) \text{ and } r \text{ is a prime power, we can apply the topological Tverberg theorem to } g. \text{ Consequently there are } r \text{ pairwise disjoint faces } \sigma_1, \ldots, \sigma_r \text{ with points } x_1 \in \text{relint } \sigma_1, \ldots, x_r \in \text{relint } \sigma_r \text{ such that } g(x_1) = \cdots = g(x_r), \text{ that is,}

\[ f(x_1) = \cdots = f(x_r), \text{ dist}(x_1, \Sigma_1) = \cdots = \text{dist}(x_r, \Sigma_1), \ldots, \text{dist}(x_1, \Sigma_{d+1}) = \cdots = \text{dist}(x_r, \Sigma_{d+1}). \]

Now observe that for every subcomplex } \Sigma_i \text{ of the faces } \sigma_1, \ldots, \sigma_r \text{ which is contained in it. Indeed, if this would not hold then we would have } |\sigma_1 \cap C_i| \geq 2, \ldots, |\sigma_r \cap C_i| \geq 2, \text{ and consequently we would obtain the following contradiction:}

\[ 2r - 1 \geq |C_i| \geq |\sigma_1 \cap C_i| + \cdots + |\sigma_r \cap C_i| \geq 2r. \]

Hence the distances, which were previously known to be equal, have to vanish,

\[ \text{dist}(x_1, \Sigma_1) = \cdots = \text{dist}(x_r, \Sigma_1) = 0, \ldots, \text{dist}(x_1, \Sigma_{d+1}) = \cdots = \text{dist}(x_r, \Sigma_{d+1}) = 0, \]

implying that } x_1 \in \Sigma_1 \cap \cdots \cap \Sigma_{d+1} \text{ for every } 1 \leq i \leq r. \text{ Since } \Sigma_1, \ldots, \Sigma_{d+1} \text{ are subcomplexes of } \Delta_N \text{ and } x_i \in \text{relint } \sigma_1, \ldots, x_r \in \text{relint } \sigma_r \text{ it follows that the faces } \sigma_1, \ldots, \sigma_r \text{ belong to the subcomplex } \Sigma_1 \cap \cdots \cap \Sigma_{d+1}, \text{ that is, } \sigma_1, \ldots, \sigma_r \text{ are rainbow faces.} \]

A special case of the weak colored Tverberg theorem we just proved, namely } |C_1| = \cdots = |C_{d+1}| = 2r - 1, \text{ yields } t(d, r) \leq t(d, r) \leq 2r - 1 \text{ for } r \text{ a prime power. This is the colored Tverberg theorem of } \tilde{Z} \text{ivaljevi\v{c} and Vre\v{c}ica presented in Theorem 4.7.}

Along the lines of the previous theorem we can prove the following colored Van Kampen–Flores theorem, where the number of color classes is at most } d + 1.

**Theorem 4.12** (The colored Van Kampen–Flores theorem). Let } d \geq 1 \text{ be an integer, let } r \text{ be a prime power, let } k \geq \lceil d r^{-1} \rceil + 1 \text{ be an integer, and } N = (d + k + 1)(r - 1). \text{ Let } f : \Delta_N \to \mathbb{R}^d \text{ be a continuous map. If the vertices of the simplex } \Delta_N \text{ are colored by } k \text{ colors, where each color class has cardinality at most } 2r - 1, \text{ then there are } r \text{ pairwise disjoint rainbow faces } \sigma_1, \ldots, \sigma_r \text{ of } \Delta_N \text{ whose } f\text{-images overlap,}

\[ f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset. \]
Proof. Let $\mathcal{C}$ be the set of vertices of the simplex $\Delta_N$ and let $(C_1, \ldots, C_k)$ be a coloring where $|C_i| \leq 2r - 1$ for all $1 \leq i \leq k$. Such a coloring exists because $k(2r - 1) \geq (d + k + 1)(r - 1)$ is equivalent to our assumption $k \geq \lceil d \frac{r - 1}{d + k - 1} \rceil + 1$. To each color class $C_i$ we associate the subcomplex $\Sigma_i$ of $\Delta_N$ defined as before by

$$\Sigma_i := \{ \sigma \in \Delta_N : |\sigma \cap C_i| \leq 1 \}.$$ 

The subcomplex $\Sigma_1 \cap \cdots \cap \Sigma_k$ is a subcomplex of all rainbow faces of $\Delta_N$ with respect to the given coloring. Consider the continuous map $g : \Delta_N \to \mathbb{R}^{d+k}$ defined by

$$g(x) = (f(x), \text{dist}(x, \Sigma_1), \text{dist}(x, \Sigma_2), \ldots, \text{dist}(x, \Sigma_k)).$$ 

Since $N = (d+k+1)(r-1)$ and $r$ is a prime power the topological Tverberg theorem can be applied to the map $g$. Therefore, there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ with points $x_1 \in \text{relint} \sigma_1, \ldots, x_r \in \text{relint} \sigma_r$ such that $g(x_1) = \cdots = g(x_r)$, that is,

$$f(x_1) = \cdots = f(x_r), \text{dist}(x_1, \Sigma_1) = \cdots = \text{dist}(x_r, \Sigma_1), \ldots, \text{dist}(x_1, \Sigma_k) = \cdots = \text{dist}(x_r, \Sigma_k).$$ 

Now observe that every subcomplex $\Sigma_i$ contains one of the faces $\sigma_1, \ldots, \sigma_r$. Indeed, if this would not hold then $|\sigma_1 \cap C_i| \geq 2, \ldots, |\sigma_r \cap C_i| \geq 2$, and we would get the contradiction

$$2r - 1 \geq |C_i| \geq |\sigma_1 \cap C_i| + \cdots + |\sigma_r \cap C_i| \geq 2r.$$ 

Consequently the distances, which were previously known to be equal, have to vanish

$$\text{dist}(x_1, \Sigma_1) = \cdots = \text{dist}(x_r, \Sigma_1) = 0, \ldots, \text{dist}(x_1, \Sigma_k) = \cdots = \text{dist}(x_r, \Sigma_k) = 0,$$

implying that $x_i \in \Sigma_1 \cap \cdots \cap \Sigma_k$ for every $1 \leq i \leq r$. Since $\Sigma_1, \ldots, \Sigma_k$ are subcomplexes and $x_1 \in \text{relint} \sigma_1, \ldots, x_r \in \text{relint} \sigma_r$ it follows that

$$\sigma_1 \in \Sigma_1 \cap \cdots \cap \Sigma_k, \ldots, \sigma_r \in \Sigma_1 \cap \cdots \cap \Sigma_k,$$

that is, $\sigma_1, \ldots, \sigma_r$ are rainbow faces. \hfill $\square$

The “colored Tverberg theorem of type B” of Vrećica and Živaljević, Theorem 4.9, is a particular case of this theorem, when the color classes have the same size.

4.5. Tverberg points with equal barycentric coordinates. The last corollary of the Topological Tverberg theorem that we present here is the topological version [12, Thm. 8.1] of a recent result by Soberón [44, Thm. 1.1] [45, Thm. 1.1].

Let $N \geq 1$ be an integer, let $\mathcal{C}$ be the set of vertices of the simplex $\Delta_N$, and let $(C_1, \ldots, C_\ell)$ be a coloring of $\mathcal{C}$. Every point $x$ in the rainbow subcomplex $R(C_1, \ldots, C_\ell)$ has a unique presentation in barycentric coordinates as $x = \sum_{i=1}^\ell \lambda_i^x v_i^x$ where $0 \leq \lambda_i^x \leq 1$ and $v_i^x \in C_i$ for all $0 \leq i \leq \ell - 1$. Two points $x = \sum_{i=1}^\ell \lambda_i^x v_i^x$ and $y = \sum_{i=1}^\ell \lambda_i^y v_i^y$ in the rainbow subcomplex $R(C_1, \ldots, C_\ell)$ have equal barycentric coordinates if $\lambda_i^x = \lambda_i^y$ for all $1 \leq i \leq \ell$.

**Theorem 4.13.** Let $d \geq 1$ be an integer, let $r$ be a prime power, $N = r((r-1)d + 1) - 1 = (r-1)(rd+1)$, and let $f : \Delta_N \to \mathbb{R}^d$ be a continuous map. If the vertices of the simplex $\Delta_N$ are colored by $(r-1)d + 1$ colors where each colored class is of size $r$, then there are points $x_1, \ldots, x_r$ with equal barycentric coordinates that belong to $r$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ whose $f$-images coincide, that is

$$f(x_1) = \cdots = f(x_r).$$ 

**Proof.** Let $\ell = (r-1)d + 1$, and let $(C_1, \ldots, C_\ell)$ be a coloring of the vertex set $\mathcal{C} = \{v_0, \ldots, v_N\}$ of the simplex $\Delta_N$. Each point $x$ of the simplex $\Delta_N$ can be uniquely presented in the barycentric coordinates as $x = \sum_{j=0}^N \lambda_j^x v_j$. For every color class $C_i$, $1 \leq i \leq \ell$, we define the function $h_i : \Delta_N \to \mathbb{R}$ by $h_i(\sum_{j=0}^N \lambda_j^x v_j) = \sum_{v_j \in C_i} \lambda_j^x$. All functions $h_j$ are affine functions and $\sum_{i=1}^\ell h_i(x) = \sum_{j=0}^N \lambda_j^x = 1$ for every $x \in \Delta_N$.

Now consider the function $g : \Delta_N \to \mathbb{R}^\ell$ given by $g(x) = (f(x), h_1(x), \ldots, h_{\ell-1}(x))$. Since $N = (r-1)(rd+1)$, the topological Tverberg theorem applied to the function $g$ implies that there exist $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ and $r$ points $x_1 \in \text{relint} \sigma_1, \ldots, x_r \in \text{relint} \sigma_r$ such that $f(x_1) = \cdots = f(x_r)$ and $h_i(x_1) = \cdots = h_i(x_r)$ for $1 \leq i \leq \ell - 1$. In addition, the equality $\sum_{i=1}^\ell h_i(x) = 1$ implies that also $h_i(x_1) = \cdots = h_i(x_r)$.

Assume now that $|\sigma_j \cap C_i| \geq 1$ for some $1 \leq j \leq r$ and some $1 \leq i \leq \ell$. Then $h_i(x_j) > 0$ since $x_j \in \text{relint} \sigma_j$. Consequently, $h_i(x_1) = \cdots = h_i(x_r) > 0$ implying that $|\sigma_j \cap C_i| \geq 1$ for all $1 \leq j \leq r$. Since $|C_i| = r$ and $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint it follows that each $\sigma_j$ has precisely one vertex in the color class $C_i$. Thus, repeating the argument for each color class we conclude that all faces $\sigma_1, \ldots, \sigma_r$
are rainbow faces. The immediate consequence of this fact is that \( h_i(x_i), 1 \leq i \leq \ell \), are the barycentric coordinates of the point \( x_i \) and so all the points \( x_1, \ldots, x_\ell \) have equal barycentric coordinates. \( \square \)

5. **Counterexamples to the topological Tverberg conjecture**

Now we are going to get to a very recent piece of the topological Tverberg puzzle: We show how counterexamples to the topological Tverberg conjecture for any number of parts that is not a prime power were derived by Frick [26] [13] from the remarkable works of Özaydin [38] and of Mabillard and Wagner [33] [34], via a lemma of Gromov [27, p.445] that is an instance of the constraint method of Blagojević, Frick, Ziegler [12, Lemmas 4.1(iii) and 4.2].

5.1. **Existence of equivariant maps if \( r \) is not a prime power.** First we present the second main result of Özaydin’s landmark manuscript [38, Thm. 4.2].

**Theorem 5.1.** Let \( d \geq 1 \) be an integer, let \( r \geq 6 \) be an integer, and let \( N = (d + 1)(r - 1) \). If \( r \) is not a prime power, then there exists an \( \mathcal{S}_r \)-equivariant map

\[
(\Delta_N)^{\times r}_{\Delta(2)} \to S(W_r^{\oplus d}). \tag{21}
\]

**Proof.** In order to prove the existence of a continuous \( \mathcal{S}_r \)-equivariant map \((\Delta_N)^{\times r}_{\Delta(2)} \to S(W_r^{\oplus d})\) we again use the equivariant obstruction theory. Since

- \((\Delta_N)^{\times r}_{\Delta(2)}\) is an \((N - r + 1)\)-dimensional, \((N - r)\)-connected free \( \mathcal{S}_r \)-CW complex, and
- \(S(W_r^{\oplus d})\) is a path-connected \((N - r - 1)\)-connected, \((N - r)\)-simple \( \mathcal{S}_r \)-space,

we have that an \( \mathcal{S}_r \)-equivariant map \((\Delta_N)^{\times r}_{\Delta(2)} \to S(W_r^{\oplus d})\) exists if and only if the primary obstruction \( [\phi_{\mathcal{S}_r}^{N-r+1}(pt)] \in H^{N-r+1}_{\mathcal{S}_r}((\Delta_N)^{\times r}_{\Delta(2)}, S(W_r^{\oplus d})) \)

vanishes. The obstruction element \( [\phi_{\mathcal{S}_r}^{N-r+1}(pt)] = [\phi_{\mathcal{S}_r}^{N-r+1}(f)] \) does not depend on the particular \( \mathcal{S}_r \)-equivariant map \( f : \mathcal{S}_r \times (\Delta_N)^{\times r}_{\Delta(2)} \to S(W_r^{\oplus d}) \) used to define the obstruction cocycle \( \phi_{\mathcal{S}_r}^{N-r+1}(f) \). Thus, in order to prove the existence of an \( \mathcal{S}_r \)-equivariant map (21) it suffices to prove that the obstruction element \( [\phi_{\mathcal{S}_r}^{N-r+1}(f)] \) vanishes for some particular choice of \( f \).

Let \( p \) be a prime such that \( p \mid |\mathcal{S}_r| = r! \), and let \( \mathcal{S}_r^{(p)} \) denotes a \( p \)-Sylow subgroup of \( \mathcal{S}_r \). Since \( r \) is not a prime power each \( p \)-Sylow subgroup of \( \mathcal{S}_r \) does not act transitively on the set \([r]\), and hence the fixed point set \( S(W_r^{\oplus d})^{\mathcal{S}_r^{(p)}} \neq \emptyset \) is non-empty. Thus there exists a (constant) \( \mathcal{S}_r^{(p)} \)-equivariant map \((\Delta_N)^{\times r}_{\Delta(2)} \to S(W_r^{\oplus d})\), or equivalently the primary obstruction element with respect to \( \mathcal{S}_r^{(p)} \), vanishes, that is, \([\phi_{\mathcal{S}_r^{(p)}}^{N-r+1}(pt)] = [\phi_{\mathcal{S}_r^{(p)}}^{N-r+1}(f)] = 0 \). Here, the \( \mathcal{S}_r \)-equivariant map \( f \) is considered only as an \( \mathcal{S}_r^{(p)} \)-equivariant map.

The restriction \( \text{dict} \) homomorphism

\[
\text{res} : H^{N-r+1}_{\mathcal{S}_r}((\Delta_N)^{\times r}_{\Delta(2)}, S(W_r^{\oplus d})) \to H^{N-r+1}_{\mathcal{S}_r^{(p)}}((\Delta_N)^{\times r}_{\Delta(2)}, S(W_r^{\oplus d}))
\]

is defined on the cochain level in [14, Lem. 5.4]. According to the definition of the obstruction cochain (already on the cochain level) the restriction homomorphism sends the obstruction cochain \( \phi_{\mathcal{S}_r}^{N-r+1}(f) \) to the obstruction cochain \( \phi_{\mathcal{S}_r^{(p)}}^{N-r+1}(f) \). Consequently the same hold for obstruction elements

\[
\text{res}([\phi_{\mathcal{S}_r}^{N-r+1}(f)]) = [\phi_{\mathcal{S}_r^{(p)}}^{N-r+1}(f)] \tag{18}
\]

Now, composing the restriction homomorphism with the transfer \( \text{dict} \) homomorphism

\[
\text{tr} : H^{N-r+1}_{\mathcal{S}_r^{(p)}}((\Delta_N)^{\times r}_{\Delta(2)}, S(W_r^{\oplus d})) \to H^{N-r+1}_{\mathcal{S}_r}((\Delta_N)^{\times r}_{\Delta(2)}, S(W_r^{\oplus d}))
\]

also defined on the cochain level in [14, Lem. 5.4], we get

\[
[\mathcal{S}_r : \mathcal{S}_r^{(p)}] : [\phi_{\mathcal{S}_r^{(p)}}^{N-r+1}(f)] = \text{tr} \circ \text{res}([\phi_{\mathcal{S}_r}^{N-r+1}(f)]) = \text{tr}([\phi_{\mathcal{S}_r}^{N-r+1}(f)]) = \text{tr}(0) = 0.
\]

Finally, since \([\mathcal{S}_r : \mathcal{S}_r^{(p)}] : [\phi_{\mathcal{S}_r^{(p)}}^{N-r+1}(f)] = 0\) for every prime \( p \) that divides the order of the group \( \mathcal{S}_r \), it follows that the obstruction element \( [\phi_{\mathcal{S}_r^{(p)}}^{N-r+1}(f)] \) must vanish, and the existence of an \( \mathcal{S}_r \)-equivariant map (21) is established. \( \square \)
Corollary 5.2. Let $d \geq 1$ be an integer, let $r \geq 6$ be an integer that is not a prime power and let $N = (d + 1)(r - 1)$. For any free $\mathcal{S}_r$-CW complex $X$ of dimension at most $N - r + 1$ there exists an $\mathcal{S}_r$-equivariant map

$$X \to S(W_r^{\oplus d}).$$

Proof. The free $\mathcal{S}_r$-CW complex $X$ has dimension at most $N - r + 1$, and the deleted product $(\Delta_N)^{xr}_{\Delta(2)}$ is $(N - r)$-connected, therefore there are no obstructions for the existence of an $\mathcal{S}_r$-equivariant map $h : X \to (\Delta_N)^{xr}_{\Delta(2)}$. Next, let $f : (\Delta_N)^{xr}_{\Delta(2)} \to S(W_r^{\oplus d})$ be an $\mathcal{S}_r$-equivariant map whose existence was guaranteed by Theorem 5.1. The composition $f \circ h : X \to S(W_r^{\oplus d})$ yields the required $\mathcal{S}_r$-equivariant map.

With this theorem Özaydin only proved that the “deleted product approach” towards solving the topological Tverberg conjecture fails in the case that $r$ is not a prime power. What about the “deleted join approach”? This question was discussed in [16, Sec.3.4].

Theorem 5.3. Let $d \geq 1$ and $r$ be integers, and let $N = (d + 1)(r - 1)$. If $r$ is not a prime power, then there exists an $\mathcal{S}_r$-equivariant map

$$(\Delta_N)^{xr}_{\Delta(2)} \to S(W_r^{\oplus (d+1)}).$$

Proof. Since $r \geq 6$ is not a prime power Theorem 5.1 implies the existence of an $\mathcal{S}_r$-equivariant map

$$f : (\Delta_N)^{xr}_{\Delta(2)} \to S(W_r^{\oplus d}).$$

Now an $\mathcal{S}_r$-equivariant map

$$g : (\Delta_N)^{xr}_{\Delta(2)} \to S(W_r^{\oplus (d+1)}) \cong S(W_r \oplus W_r^{\oplus d})$$

can be defined by

$$g(\lambda_1 x_1 + \cdots + \lambda_r x_r) = \frac{1}{\nu} \left( (\lambda_1 - \frac{1}{r}, \ldots, \lambda_r - \frac{1}{r}) \oplus \prod_{i=1}^r \lambda_i \cdot f(x_1, \ldots, x_r) \right),$$

where $\nu := ||(\lambda_1 - \frac{1}{r}, \ldots, \lambda_r - \frac{1}{r}) \oplus \prod_{i=1}^r \lambda_i \cdot f(x_1, \ldots, x_r)||$. The function $g$ is well defined, continuous and $\mathcal{S}_r$-equivariant. Thus an $\mathcal{S}_r$-equivariant map (22) exists.

Now we see that not only the “deleted product approach” fails if $r$ is not a prime power, but the “deleted join approach” fails as well. Is this an indication that the topological Tverberg theorem fails if the number of parts is not a prime power?

5.2. The topological Tverberg conjecture does not hold if $r$ is not a prime power. It is time to show that the topological Tverberg conjecture fails in the case that $r$ is not a prime power. This will be done following the presentation given in [13].

Based on the work of Mabillard and Wagner [33] [34] we will prove that the generalized Van Kampen–Flores theorem for any $r$ that is not a prime power fails, as demonstrated by Frick [26] [13]. Since, by the constraint method, the generalized Van Kampen–Flores theorem for fixed number of overlaps $r$ is a consequence of the topological Tverberg theorem for the same number of overlaps $r$, failure of the generalized Van Kampen–Flores theorem implies the failure of the topological Tverberg theorem.

Theorem 5.4 (The generalized Van Kampen–Flores theorem fails when $r$ is not a prime power). Let $k \geq 3$ be an integer, and let $r \geq 6$ be an integer that is not a prime power. For any integer $N > 0$ there exists a continuous map $f : \Delta_N \to \mathbb{R}^k$ such that for any $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ from the $((r-1)k)$-skeleton $sk_{(r-1)k}(\Delta_N)$ of the simplex $\Delta_N$ the corresponding $f$-images do not overlap,

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset.$$

Proof. The deleted product $(sk_{(r-1)k}(\Delta_N))^{xr}_{\Delta(2)}$ is a free $\mathcal{S}_r$-space of dimension at most $d := (r-1)kr$. Since $r$ is not a power of a prime, according to Corollary 5.2, there exists an $\mathcal{S}_r$-equivariant map

$$h : (sk_{(r-1)k}(\Delta_N))^{xr}_{\Delta(2)} \to S(W_r^{\oplus d}).$$

Now we use the following result of Mabillard and Wagner [33, Thm.3] [34, Thm.7], for which an alternative proof is given in [3]. Skopenkov [42] gives a user’s guide.

**Theorem.** Let $r \geq 2$ and $k \geq 3$ be integers, and let $K$ be an $((r-1)k)$-dimensional simplicial complex. Then the following statements are equivalent:

(i) There exists a continuous $\mathcal{S}_r$-equivariant map $K^{xr}_{\Delta(2)} \to S(W_r^{\oplus kr}).$
(ii) There exists a continuous map \( f : K \to \mathbb{R}^r \) such that for any \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( K \) we have that \( f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset \).

If we apply this result to the \( \mathfrak{S}_r \)-equivariant map \( h \) in (23) we get a continuous map \( f : \text{sk}_{(r-1)k}(\Delta_N) \to \mathbb{R}^r \) with the property that for any collection of \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) in \( \text{sk}_{(r-1)k}(\Delta_N) \) the corresponding \( f \)-images do not overlap,

\[
f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset.
\]

\[ \square \]

Thus we have proved that in the case when \( r \) is not a prime power the generalized Van Kampen–Flores theorem fails. As we have pointed out this means that the corresponding topological Tverberg theorem also fails [13, Thm. 4.3].

**Theorem 5.5** (The topological Tverberg theorem fails for any \( r \) that is not a prime power). Let \( k \geq 3 \) and \( r \geq 6 \) be integers, and let \( N = (r-1)(rk+2) \). If \( r \) is not a prime power, then there exists a continuous map \( g : \Delta_N \to \mathbb{R}^{rk+1} \) such that for any \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_N \) the corresponding \( g \)-images do not overlap,

\[
g(\sigma_1) \cap \cdots \cap g(\sigma_r) = \emptyset.
\]

**Proof.** Since \( r \) is not a power of a prime, Theorem 5.4 yields a continuous map \( f : \Delta_N \to \mathbb{R}^r \) such that for any \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) in \( \text{sk}_{(r-1)k}(\Delta_N) \)

\[
f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset.
\]

Motivated by the proof of Theorem 4.1 we consider the function \( g : \Delta_N \to \mathbb{R}^{rk+1} \) defined by

\[
g(x) = (f(x), \text{dist}(x, \text{sk}_{(r-1)k}(\Delta_N))).
\]

We prove that the map \( g \) fails the topological Tverberg conjecture.

Assume, to the contrary, that there are \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) in \( \Delta_N \) and \( r \) points \( x_1 \in \text{relint} \sigma_1, \ldots, x_r \in \text{relint} \sigma_r \) such that \( g(x_1) = \cdots = g(x_r) \). Consequently,

\[
\text{dist}(x_1, \text{sk}_{(r-1)k}(\Delta_N)) = \cdots = \text{dist}(x_r, \text{sk}_{(r-1)k}(\Delta_N)).
\]

Next, at least one of the faces \( \sigma_1, \ldots, \sigma_r \) is in \( \text{sk}_{(r-1)k}(\Delta_N) \). Indeed, if all the faces \( \sigma_i \) would have dimension at least \( (r-1)k+1 \), then we would get the following contradiction:

\[
N + 1 = (r-1)(rk+2) + 1 = |\Delta_N| \geq |\sigma_1| + \cdots + |\sigma_r| \geq ((r-1)rk+2) = (r-1)(rk+2) + 2 > N + 1.
\]

Since one of the faces \( \sigma_1, \ldots, \sigma_r \) is in \( \text{sk}_{(r-1)k}(\Delta_N) \), all the distances vanish, meaning that

\[
\text{dist}(x_1, \text{sk}_{(r-1)k}(\Delta_N)) = \cdots = \text{dist}(x_r, \text{sk}_{(r-1)k}(\Delta_N)) = 0.
\]

Therefore, all the faces \( \sigma_1, \ldots, \sigma_r \) belong to \( \text{sk}_{(r-1)k}(\Delta_N) \) contradicting the choice of the map \( f \). Thus the map \( g \) is a counterexample to the topological Tverberg theorem. \[ \square \]

**Remark 5.6.** The smallest counterexample to the topological Tverberg theorem that can be obtained from Theorem 5.5 is a continuous map \( \Delta_{100} \to \mathbb{R}^{19} \) with the property that no six pairwise disjoint faces in \( \Delta_{100} \) have \( f \)-images that overlap. Recently, using additional ideas, Sergey Avvakumov, Isaac Mabillard, Arkadiy Skopenkov and Uli Wagner [3] have improved this to get counterexamples \( \Delta_{65} \to \mathbb{R}^{12} \).

6. The Bárány–Larman conjecture and the Optimal colored Tverberg theorem

Let us briefly recall the original colored Tverberg problem posed by Bárány and Larman in their 1992 paper [7], see Section 4.2.

**Problem 6.1** (Bárány–Larman colored Tverberg problem). Let \( d \geq 1 \) and \( r \geq 2 \) be integers. Determine the smallest number \( n = n(d, r) \) such that for every affine (continuous) map \( f : \Delta_{n-1} \to \mathbb{R}^d \), and every coloring \( (C_1, \ldots, C_{d+1}) \) of the vertex set \( C \) of the simplex \( \Delta_{n-1} \) by \( d+1 \) colors with each color of size at least \( r \), there exist \( r \) pairwise disjoint rainbow faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_{n-1} \) whose \( f \)-images overlap, that is

\[
f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset.
\]

A trivial lower bound for the function \( n(d, r) \) is \( (d+1)r \) and it is natural to conjecture the following.

**Conjecture 6.2** (Bárány–Larman Conjecture). Let \( r \geq 2 \) and \( d \geq 1 \) be integers. Then \( n(d, r) = (d+1)r \).
In Section 4.3 we tried to use the approach of Živaljević and Vrečica to solve the Bárány–Larman conjecture and we failed dramatically. We hoped to prove that an $S_r$-equivariant map

$$\Delta_r^{(d+1)} \to S(W_r^{(d+1)})$$

does not exist, but Theorem 4.10 gave us exactly the opposite, the existence of this map. What can we do now? We change the question, and prove the non-existence of an $S_{r+1}$-equivariant map

$$(R(C_1,\ldots,C_d+1))^r \to \Delta_r^{(d+1)} \to S(W_r^{(d+1)})$$

instead; here $|C_1| = \cdots = |C_{d+1}| = r$, and $|C_{d+2}| = 1$. Still, why should we be interested in such a result?

**Theorem 6.3.** Let $r \geq 2$ and $d \geq 1$ be integers. If there is no $S_{r+1}$-equivariant map

$$\Delta_r^{(d+1)} \to S(W_r^{(d+1)})$$

then $n(d,r) = (d+1)r$ and $tt(d,r) = r$.

**Proof.** Let $(C_1,\ldots,C_{d+1})$ be a coloring of the vertices of the simplex $\Delta$ with $|C_1| = \cdots = |C_{d+1}| = r$, and let $f: \Delta \to \mathbb{R}^8$ be a continuous map. Construct a simplex $\Delta'$ as a pyramid over $\Delta$, and let $C_{d+2}$ be the additional color class containing only the apex of the pyramid. Thus, $(C_1,\ldots,C_{d+2})$ is a coloring of the vertices of the simplex $\Delta'$.

Let us assume that an $S_{r+1}$-equivariant map $\Delta_r^{(d+1)} \to S(W_r^{(d+1)})$ does not exist. The non-existence of this map in combination with Corollary 4.8 implies that there exist $r+1$ pairwise disjoint rainbow faces $\sigma_1,\ldots,\sigma_{r+1}$ of the simplex $\Delta'$ whose $f$-images overlap, $f(\sigma_1) \cap \cdots \cap f(\sigma_{r+1}) \neq \emptyset$.

Without loss of generality we can assume that $\sigma_{r+1} \cap C_{d+2} \neq \emptyset$. Then the faces $\sigma_1,\ldots,\sigma_r$ are rainbow faces of the simplex $\Delta$ with respect to the coloring $(C_1,\ldots,C_{d+1})$ and $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.

Hence, $tt(d,r) = r$ and consequently $n(d,r) = (d+1)r$. \qed

The theorem we just proved tells us that in order to make an advance on the Bárány–Larman conjecture we should try to prove the non-existence of a continuous $S_{r+1}$-equivariant map

$$\Delta_r^{(d+1)} \to S(W_r^{(d+1)})$$

at least for some values of $r$.

Now in order to prove the non-existence of a continuous $S_{r+1}$-equivariant map (24), for $r+1 = p$ an odd prime, we will compute the Fadell–Husseini index of the join $\Delta_r^{(d+1)} \to S(W_r^{(d+1)})$ with respect to the cyclic group and compare the result with the index of the sphere $S(W_r^{(d+1)})$.

### 6.1. The Fadell–Husseini index of chessboards.

Let $p := r+1$ be an odd prime. In this section we compute the Fadell–Husseini index of chessboards

$$\text{index}_{Z/p}(\Delta_{k,p};F_p) \subseteq H^*(B(Z/p;F_p)) = H^*(Z/p;F_p), \quad k \geq 1,$$

and their joins with respect to the cyclic subgroup $Z/p$ of the symmetric group $S_p$. Recall that

$$H^*(B(Z/p;F_p)) = H^*(Z/p;F_p) = F_p[t] \otimes \Lambda[e],$$

where $deg t = 2$, $deg e = 1$, and $\Lambda[\cdot]$ denotes the exterior algebra. First, we collect some simple facts about the Fadell–Husseini index of chessboards.

**Lemma 6.4.** Let $k \geq 1$ be an integer, and let $p$ be an odd prime. Then

(i) $\text{index}_{Z/p}(\Delta_{1,p}) = H^{\geq 1}(B(Z/p;F_p))$,

(ii) $\text{index}_{Z/p}(\Delta_{2p-1,p}) = H^{\geq 1}(B(Z/p;F_p))$,

(iii) $\text{index}_{Z/p}(\Delta_{1,p}) \subseteq \text{index}_{Z/p}(\Delta_{2p-1,p}) \subseteq \cdots \subseteq \text{index}_{Z/p}(\Delta_{2p-1,p}) = \text{index}_{Z/p}(\Delta_{2p-1,p} = \cdots = H^{\geq p}(B(Z/p;F_p))$.

**Proof.** For the statement (i) observe that $\Delta_{1,p} = [p]$ and therefore $E(Z/p) \times Z/p \Delta_{1,p} \cong E(Z/p)$. Since $E(Z/p)$ is a contractible space,

$$\text{index}_{Z/p}(\Delta_{1,p}) = \ker(H^*(B(Z/p;F_p)) \to H^*(E(Z/p;F_p))) = H^{\geq 1}(B(Z/p;F_p)).$$
In order to prove (ii) recall that $\Delta_{2p-1,p}$ is a $(p-1)$-dimensional $(p-2)$-connected free $\mathbb{Z}/p$ simplicial complex, see Theorem 4.6. The Serre spectral sequence associated to the Borel construction fiber bundle $\Delta_{2p-1,p} \to \text{E}(\mathbb{Z}/p) \times_{\mathbb{Z}/p} \Delta_{2p-1,p} \to \text{B}(\mathbb{Z}/p)$ has the $E_2$-term given by

$$E_2^{i,j} = H^i(\text{E}(\mathbb{Z}/p); H^j(\Delta_{2p-1,p}; \mathbb{F}_p)) = H^i(\mathbb{Z}/p; H^j(\Delta_{2p-1,p}; \mathbb{F}_p))$$

for $j = 0$,  

$$= \begin{cases} 
H^i(\mathbb{Z}/p; \mathbb{F}_p), & \text{for } j = 0, \\
H^i(\mathbb{Z}/p; H^{p-1}(\Delta_{2p-1,p}; \mathbb{F}_p)), & \text{for } j = p-1, \\
0, & \text{otherwise}.
\end{cases}$$

Thus, $E_\infty^{i,0} \cong E_2^{i,0} \cong H^i(\mathbb{Z}/p; \mathbb{F}_p)$ for $0 \leq i \leq p-1$. Consequently $\text{index}_{\mathbb{Z}/p} \Delta_{2p-1,p} \subseteq H^{\geq p}(\text{B}(\mathbb{Z}/p); \mathbb{F}_p)$. Since $\Delta_{2p-1,p}$ is a free $\mathbb{Z}/p$ simplicial complex, we get $\text{E}(\mathbb{Z}/p) \times_{\mathbb{Z}/p} \Delta_{2p-1,p} \cong \Delta_{2p-1,p}/\mathbb{Z}/p$, implying that $H^i(\text{E}(\mathbb{Z}/p) \times_{\mathbb{Z}/p} \Delta_{2p-1,p}; \mathbb{F}_p) = 0$ for $i \geq p$. Since the spectral sequence $E_2^{*,*}$ converges to the cohomology of the Borel construction $H^*(\text{E}(\mathbb{Z}/p) \times_{\mathbb{Z}/p} \Delta_{2p-1,p}; \mathbb{F}_p)$, we have that $E_\infty^{i,j} = 0$ for $i + j \geq p$. In particular, $E_\infty^{i,0} = 0$ for $i \geq p$, implying that

$$\text{index}_{\mathbb{Z}/p} \Delta_{2p-1,p} = H^{\geq p}(\text{B}(\mathbb{Z}/p); \mathbb{F}_p).$$

For (iii) observe that there is a sequence of $\mathbb{Z}/p$-equivariant inclusions

$$\Delta_1 \hookrightarrow \Delta_2 \hookrightarrow \cdots \hookrightarrow \Delta_{k+1} \hookrightarrow \cdots$$

given by the inclusions of the corresponding vertex sets

$$[1] \times [p] \hookrightarrow [2] \times [p] \hookrightarrow \cdots \hookrightarrow [k] \times [p] \hookrightarrow [k+1] \times [p] \hookrightarrow \cdots.$$

Consequently the monotonicity property of the Fadell–Husseini index, combined with the fact that for $k \geq 2p-1$ all chessboards $\Delta_{k,p}$ are $(p-1)$-dimensional free $\mathbb{Z}/p$ simplicial complexes, implies that

$$\text{index}_{\mathbb{Z}/p} \Delta_{1,p} \subseteq \text{index}_{\mathbb{Z}/p} \Delta_{2,p} \subseteq \cdots \subseteq \text{index}_{\mathbb{Z}/p} \Delta_{2p-1,p} = \text{index}_{\mathbb{Z}/p} \Delta_{2p, p} = \cdots = H^{\geq p}(\text{B}(\mathbb{Z}/p); \mathbb{F}_p).$$

\[ \square \]

In the next step we compute the index of the chessboard $\Delta_{p-1,p}$. For that we need to establish the following fact.

**Lemma 6.5.** Let $p$ be an odd prime. There exists a $\mathbb{Z}/p$-equivariant map $f : \Delta_{p-1,p} \to S(W_p)$ such that the induced map in cohomology $f^* : H^{p-2}(S(W_p); \mathbb{F}_p) \to H^{p-2}(\Delta_{p-1,p}; \mathbb{F}_p)$ is an isomorphism.

**Proof.** Let $e_1, \ldots, e_p$ be a standard basis of $\mathbb{R}^p$, let $e := \frac{1}{p}(e_1 + \cdots + e_p)$, and let $v_i := e - e$ for $1 \leq i \leq p$. Denote now by $\Delta_{p-1} \subseteq W_p$ the simplex $\text{conv}\{v_1, \ldots, v_p\}$, which is invariant with respect to the action of the cyclic group $\mathbb{Z}/p$. Moreover, its boundary $\partial \Delta_{p-1}$ is equivariantly homeomorphic to the representation sphere $S(W_p)$.

We define a continuous map $f : \Delta_{p-1,p} \to \partial \Delta_{p-1} \cong S(W_p)$ to be the $\mathbb{Z}/p$-equivariant simplicial map given on the vertex set of $\Delta_{p-1,p}$ by $(i,j) \to v_j$, where $(i,j) \in [p-1] \times [p]$. It remains to be verified that $f^* : H^{p-2}(S(W_p); \mathbb{F}_p) \to H^{p-2}(\Delta_{p-1,p}; \mathbb{F}_p)$ is an isomorphism.

Since $p \geq 3$, the chessboard complex $\Delta_{p-1, p}$ is a connected, orientable pseudomanifold of dimension $p-2$, for this see [31, p. 145]. Thus $H_{p-2}(\Delta_{p-1,p}; \mathbb{Z}) = \mathbb{Z}$ and an orientation class is given by the chain

$$z_{p-1,p} = \sum_{\pi \in \Theta_p} (\text{sgn } \pi)(1, \pi(1), \ldots, (p-1), \pi(p-1)).$$

Then on the chain level we have that

$$f_#(z_{p-1,p}) = \sum_{\pi \in \Theta_p} (\text{sgn } \pi)(v_{\pi(1)}, \ldots, v_{\pi(p-1)}) = \sum_{\pi \in \Theta_p} (\text{sgn } \pi)(v_{\pi(1)}, \ldots, v_{\pi(p-1)}, \widehat{v}_{\pi(p)})$$

$$= \sum_{k=1}^{p} \sum_{\pi \in \Theta_p, \pi(p) = k} (-1)^{p+k}(\text{sgn } \pi)^2(v_1, \ldots, v_k, \ldots, v_p)$$

$$= \sum_{k=1}^{p} (-1)^{k-1} \sum_{\pi \in \Theta_p, \pi(p) = k} (v_1, \ldots, \widehat{v}_k, \ldots, v_p) = \sum_{k=1}^{p} (-1)^{k-1}(p-1)! (v_1, \ldots, \widehat{v}_k, \ldots, v_p)$$

$$= (p-1)! \sum_{k=1}^{p} (-1)^{k-1}(v_1, \ldots, \widehat{v}_k, \ldots, v_p).$$
For this calculation keep in mind that $p$ is an odd prime. The chain $\sum_{k=1}^{p}(-1)^{k-1}(v_1, \ldots, \widehat{v_k}, \ldots, v_p)$ is a generator of the top homology of the sphere $\partial \Delta_{p-1} \cong S(W_p)$. Therefore, the induced map in homology

$$f_* : H_{p-2}(\Delta_{p-1,p}; \mathbb{Z}) \to H_{p-2}(S(W_p); \mathbb{Z})$$

is just a multiplication by $(p-1)! \equiv -1 \pmod{p}$. Using the naturality of the universal coefficient isomorphism [19, Cor. 7.5] we have that the induced map in homology with $\mathbb{F}_p$ field coefficients

$$f_* : H_{p-2}(\Delta_{p-1,p}; \mathbb{F}_p) \to H_{p-2}(S(W_p); \mathbb{F}_p)$$

is again multiplication by $(p-1)!$. Since $(p-1)!$ and $p$ are relatively prime the multiplication by $(p-1)!$ is an isomorphism. Now using yet another universal coefficient isomorphism [19, Cor. 7.2] for the coefficients in a field we get that the induced map in cohomology with $\mathbb{F}_p$ coefficients

$$f^* : H^{p-2}(S(W_p); \mathbb{F}_p) \to H^{p-2}(\Delta_{p-1,p}; \mathbb{F}_p)$$

is an isomorphism. \hfill $\Box$

Now we have all ingredients needed to compute the index of the chessboard $\Delta_{p-1,p}$.

**Theorem 6.6.** \(\text{index}_{\mathbb{Z}/p} \Delta_{p-1,p} = \text{index}_{\mathbb{Z}/p} S(W_p) = H^{\geq p-1}(B(\mathbb{Z}/p); \mathbb{F}_p).\)

**Proof.** Let us denote by $\lambda$ the Borel construction fiber bundle

$$\lambda : \Delta_{p-1,p} \to E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} \Delta_{p-1,p} \to B(\mathbb{Z}/p),$$

and by $\rho$ the Borel construction fiber bundle

$$\rho : S(W_p) \to E(\mathbb{Z}/p)^n \times_{\mathbb{Z}/p} S(W_p) \to B(\mathbb{Z}/p).$$

The $\mathbb{Z}/p$-equivariant map $f : \Delta_{p-1,p} \to S(W_p)$ constructed in Lemma 6.5 induces a morphism of the Borel construction fiber bundles $\lambda$ and $\rho$:

$$E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} \Delta_{p-1,p} \xrightarrow{id \times_{\mathbb{Z}/p} f} E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} S(W_p) \xrightarrow{\rho} B(\mathbb{Z}/p).$$

This morphism induces a morphism of the corresponding Serre spectral sequences

$$E_\infty^{i,j}(\lambda) := E_\infty^{i,j}(E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} \Delta_{p-1,p}) \xrightarrow{f_\infty^{i,j}} E_\infty^{i,j}(E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} S(W_p)) =: E_\infty^{i,j}(\rho)$$

with the property that on the zero row of the second term the induced map

$$E_2^{i,0}(\lambda) = E_2^{i,0}(E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} \Delta_{p-1,p}) \xrightarrow{f_2^{i,0}} E_2^{i,0}(E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} S(W_p)) = E_2^{i,0}(\rho)$$

is the identity. Here we use simplified notation $f_\infty^{i,j} := E_\infty^{i,j}(\text{id} \times_{\mathbb{Z}/p} f)$. In the $E_2$-term, since the homomorphism $f^* : H^{p-2}(S(W_p); \mathbb{F}_p) \to H^{p-2}(\Delta_{p-1,p}; \mathbb{F}_p)$ induces an isomorphism on the $(p-2)$-cohomology, and $\mathbb{Z}/p$ acts trivially on both cohomologies $H^{p-2}(S(W_p); \mathbb{F}_p) \cong H^{p-2}(\Delta_{p-1,p}; \mathbb{F}_p) \cong \mathbb{F}_p$, the morphism of spectral sequences

$$f_2^{i,p-2} : E_2^{i,p-2}(\rho) \to E_2^{i,p-2}(\lambda)$$

(25)

is an isomorphism.

The $E_2$-term of the Serre spectral sequence associated to the fiber bundle $\rho$ is given by

$$E_2^{i,j}(\rho) = H^i(B(\mathbb{Z}/p); H^j(S(W_p); \mathbb{F}_p)) = H^i(B(\mathbb{Z}/p); H^j(S(W_p); \mathbb{F}_p)) \cong H^i(\mathbb{Z}/p; \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^j(S(W_p); \mathbb{F}_p),$$

because $\mathbb{Z}/p$ acts trivially on the cohomology $H^i(S(W_p); \mathbb{F}_p)$. Thus the only possible non-trivial differential is

$$\partial_{p-1} : E_2^{i,p-2}(\rho) \cong E_2^{i,p-2}(\rho) \to E_2^{i+p-1,0}(\rho) \cong E_2^{i+p-1,0}(\rho).$$

Let $\ell \in H^{p-2}(S(W_p); \mathbb{F}_p)$ denote a generator. Then the $(p-2)$-row of the $E_2$-term, as an $H^*(\mathbb{Z}/p; \mathbb{F}_p)$-module, is generated by $1 \otimes_{\mathbb{F}_p} \ell \in E_2^{0,p-2}(\rho)$. Since the differentials are $H^*(\mathbb{Z}/p; \mathbb{F}_p)$-module maps it follows that the differential $\partial_{p-1}$ is completely determined by its image $\partial_{p-1}(1 \otimes_{\mathbb{F}_p} \ell) \in E_2^{p-1,0}(\rho) \cong E_2^{p-1,0}(\rho)$. In order to find the image of the differential notice that $\mathbb{Z}/p$ acts freely on the sphere $S(W_p)$ and consequently $E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} S(W_p) \cong S(W_p)/\mathbb{Z}/p$. Since the spectral sequence $E_\infty^{i,j}$ converges to the cohomology $H^*(E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} S(W_p); \mathbb{F}_p)$ we have that $E_\infty^{i,j}(\rho) \cong E_{\infty}^{i,j}(\rho) = 0$ for $i + j \geq p - 1$. Thus

$$\partial_{p-1}(1 \otimes_{\mathbb{F}_p} \ell) = \omega \cdot \ell^{(p-1)/2} \neq 0$$
for some $\omega \in \mathbb{F}_p \backslash \{0\}$. Moreover,

$$\text{index}_{\mathbb{Z}/p} S(W_p) = \langle t^{(p-1)/2} \rangle = H^{\geq p-1}(B(\mathbb{Z}/p); \mathbb{F}_p).$$

The $E_2$-term of the Serre spectral sequence associated to the fiber bundle $\lambda$ is given by

$$E_2^{i,j}(\lambda) = H^i(B(\mathbb{Z}/p); H^j(\Delta_{p-1}; \mathbb{F}_p)) = H^j(\mathbb{Z}/p; H^i(\Delta_{p-1}; \mathbb{F}_p)).$$

In particular, $E_2^{0,0}(\lambda) \cong H^0(\mathbb{Z}/p; \mathbb{F}_p)$ and $E_2^{p-2,p-2}(\lambda) \cong H^p(\mathbb{Z}/p; \mathbb{F}_p)$, because $\mathbb{Z}/p$ acts trivially on the cohomology $H^{p-2}(\Delta_{p-1}; \mathbb{F}_p) \cong \mathbb{F}_p$. Let $z := f_s^{0,p-2}(1 \otimes \mathbb{F}_p)$. As we have seen in (25) the map $f_s^{0,p-2}$ is an isomorphism. Thus $z$ is a generator of $E_2^{0,p-2}(\lambda) \cong \mathbb{F}_p$, and moreover $z$ is a generator of the $(p-2)$-row of the $E_2$-term as an $H^*(\mathbb{Z}/p; \mathbb{F}_p)$-module. As in the case of the spectral sequence $E_2^{i,j}(\rho)$ the fact that $\mathbb{Z}/p$-acts freely on the chessboard $\Delta_{p-1}$ implies that $E_2^{i,j}(\lambda) \cong E_2^{i,j}(\lambda) = 0$ for $i + j > p - 1$.

Since $f_s^{i,j}$ is a morphism of spectral sequences it has to commute with the differentials. In particular, for $2 \leq s \leq p - 2$ we have

$$\partial_s (z) = \partial_s (f_s^{0,p-2}(1 \otimes \mathbb{F}_p)) = f_s^{s,p-s-1}(\partial_s (1 \otimes \mathbb{F}_p)) = 0.$$  

Now the fact that $z$ is a generator of the $(p-2)$-row of the $E_2$-term as an $H^*(\mathbb{Z}/p; \mathbb{F}_p)$-module yields

$$E_2^{p-2,p-2}(\lambda) \cong E_2^{p-2,p-2}(\lambda) \cong H^i(\mathbb{Z}/p; \mathbb{F}_p).$$

If in addition $\partial_{p-1}(z) = 0$, then for every $i \geq 0$

$$E_2^{i,p-2}(\lambda) \cong E_2^{i,p-2}(\lambda) \cong E_2^{i,p-2}(\lambda) \cong H^i(\mathbb{Z}/p; \mathbb{F}_p) \neq 0,$$

which contradicts the fact that $E_{2i}^{0,i}(\lambda) \cong E_{2i}^{0,i}(\lambda) = 0$ for $i + j > p - 1$. In summary we have that

$$\partial_{p-1}(z) = \partial_{p-1}(f_s^{0,p-2}(1 \otimes \mathbb{F}_p)) = f_{p-1}^{0,0}(\partial_{p-1}(1 \otimes \mathbb{F}_p)) = f_{p-1}^{0,0}(\omega \cdot t^{(p-1)/2}) = \omega \cdot f_{p-1}^{0,0}(t^{(p-1)/2}) \neq 0.$$  

Moreover, we have that

$$\partial_{p-1}: E_{p-1}^{i,p-2}(\lambda) \to E_{p-1}^{i+1,p-1}(\lambda)$$

must be an isomorphism for every $i \geq 0$. Hence, for $i \geq 0$ we have that

$$\omega \cdot f_{p-1}^{0,0}(t^{(p-1)/2}) = \omega \cdot f_{p-1}^{0,0}(t^{(p-1)/2}) \neq 0.$$  

By (26) we conclude that $f_{p-1}^{0,0}(t^{(p-1)/2}) = t^{(p-1)/2}$ and consequently,

$$\text{index}_{\mathbb{Z}/p} \Delta_{p-1} \preceq \langle t^{(p-1)/2} \rangle = H^{\geq p-1}(B(\mathbb{Z}/p); \mathbb{F}_p).$$

Finally we claim that no non-zero differential can arrive to the 0-row on $E_s$-term for $2 \leq s \leq p - 2$, implying that

$$\text{index}_{\mathbb{Z}/p} \Delta_{p-1} \subseteq \langle t^{(p-1)/2} \rangle = H^{\geq p-1}(B(\mathbb{Z}/p); \mathbb{F}_p),$$

and concluding the proof of the theorem. Indeed, if this is not true, then there exists a minimal $s$ such that $2 \leq s \leq p - 2$ and $0 \neq \partial_s (y) = t^i e^b \in E_2^{i,b}(\lambda)$ for some $y$ and $0 \leq i \leq p - 2$. Since differentials are $H^*(\mathbb{Z}/p; \mathbb{F}_p)$-module maps we have that $\partial_s (t^i e^b) = t^i \cdot \partial_s (e^b) = t^{i+1} e^b \in E_2^{i+1,0}(\lambda)$ for every $c \geq 0$. Consequently, $E_2^{i+1,0}(\lambda) = 0$ for every $c \geq 0$ contradicting the existence of the isomorphisms (26). Thus, no non-zero differential can arrive to the 0-row before the $E_{p-1}$-term. 

The proof of the previous theorem, combined together with the fact that a join of pseudomanifolds is a pseudomanifold, yields the following corollary [15, Cor. 2.6].

**Corollary 6.7.** Let $m \geq 1$ be an integer. Then

(i) $\text{index}_{\mathbb{Z}/p} \Delta_{p-1} = \text{index}_{\mathbb{Z}/p} (W_p^{\otimes m}) = H^{\geq m(p-1)}(B(\mathbb{Z}/p); \mathbb{F}_p)$,

(ii) $\text{index}_{\mathbb{Z}/p} (\Delta_{p-1} \ast [p]) = \text{index}_{\mathbb{Z}/p} (W_p^{\otimes m} \ast [p]) = H^{\geq m(p-1)+1}(B(\mathbb{Z}/p); \mathbb{F}_p)$,

(iii) $\text{index}_{\mathbb{Z}/p} (\Delta_{p-1} \ast \Delta_{p-2}) = \text{index}_{\mathbb{Z}/p} (W_p^{\otimes m} \ast [p]^{p-1}) = H^{\geq m(p-1)+p}(B(\mathbb{Z}/p); \mathbb{F}_p)$.

In the next step we compute the index of the chessboard $\Delta_{k,p}$ for $1 \leq k \leq p - 2$.

**Theorem 6.8.** $\text{index}_{\mathbb{Z}/p} \Delta_{k,p} = H^{\geq k}(B(\mathbb{Z}/p); \mathbb{F}_p)$, for $1 \leq k \leq p - 1$.

**Proof.** Let $1 \leq k \leq p - 2$ be an integer. The chessboard $\Delta_{k,p}$ is a $(k-1)$-dimensional free $\mathbb{Z}/p$ simplicial complex. Thus $E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} \Delta_{k,p} \cong \Delta_{k,p} / \mathbb{Z}/p$ and consequently $H^i(E(\mathbb{Z}/p) \times_{\mathbb{Z}/p} \Delta_{k,p}; \mathbb{F}_p) = 0$ for all $i \geq k$. Therefore, $\text{index}_{\mathbb{Z}/p} \Delta_{k,p} \supseteq H^{\geq k}(B(\mathbb{Z}/p); \mathbb{F}_p)$. For $k = 1$ the theorem follows from Lemma 6.4(i). Furthermore, for $k = p - 1$ the statement is the content of Theorem 6.6.

Now let us assume that $2 \leq k \leq p - 3$ is even. Then $p - 1 - k$ is also even. Now consider the $\mathbb{Z}/p$-equivariant inclusion map

$$\Delta_{p-1} \to \Delta_{k,p} \ast \Delta_{p-1-k,p}.$$
From the monotonicity and join properties of the Fadell–Husseini index we have that
\[
\text{index}_{\mathbb{Z}/p} \Delta_{k,p} \cdot \text{index}_{\mathbb{Z}/p} \Delta_{p-1-k,p} \subseteq \text{index}_{\mathbb{Z}/p}(\Delta_{k,p} \ast \Delta_{p-1-k,p}) \subseteq \text{index}_{\mathbb{Z}/p} \Delta_{p-1, p}.
\]
Since \(p - 1 - k\) is even and, as we have seen,
\[
\text{index}_{\mathbb{Z}/p} \Delta_{p-1-k,p} \supseteq H^{\geq p-1-k}(B(\mathbb{Z}/p); \mathbb{F}_p) = (t^{(p-1-k)/2})
\]
we have that \(t^{(p-1-k)/2} \in \text{index}_{\mathbb{Z}/p} \Delta_{p-1-k,p}\). On the other hand, assume that there is an element \(u \in \text{index}_{\mathbb{Z}/p} \Delta_{k,p}\) such that \(\deg(u) \leq k - 1\). Then we have reached a contradiction
\[
0 \neq u \cdot t^{(p-1-k)/2} \in \text{index}_{\mathbb{Z}/p} \Delta_{p-1-k,p} \subseteq \text{index}_{\mathbb{Z}/p} \Delta_{p-1-k,p} = H^{\geq p-1}(B(\mathbb{Z}/p); \mathbb{F}_p),
\]
because \(\deg(u \cdot t^{(p-1-k)/2}) = \deg(u) + \deg(t^{(p-1-k)/2}) = \deg(u) + p - 1 - k \leq p - 2\). Thus we have proved that for even \(k\)
\[
\text{index}_{\mathbb{Z}/p} \Delta_{k,p} = H^{\geq k}(B(\mathbb{Z}/p); \mathbb{F}_p).
\]
Next let us assume that \(3 \leq k \leq p - 2\) is odd. As we observed at the start of the proof
\[
\text{index}_{\mathbb{Z}/p} \Delta_{k,p} \supseteq H^{\geq k}(B(\mathbb{Z}/p); \mathbb{F}_p) = (t^{(k-1)/2}e, t^{(k+1)/2}).
\]
The \(\mathbb{Z}/p\)-equivariant inclusion map \(\Delta_{k-1,p} \subseteq \Delta_{k,p}\) together with the computation of the index for even integers implies that
\[
(t^{(k-1)/2}) = H^{\geq k-1}(B(\mathbb{Z}/p); \mathbb{F}_p) = \text{index}_{\mathbb{Z}/p} \Delta_{k-1,p} \supseteq \text{index}_{\mathbb{Z}/p} \Delta_{k,p} \supseteq H^{\geq k}(B(\mathbb{Z}/p); \mathbb{F}_p).
\]
In order to conclude the proof of the theorem it remains to prove that \(t^{(k-1)/2} \notin \text{index}_{\mathbb{Z}/p} \Delta_{k,p}\). This would yield the equality
\[
\text{index}_{\mathbb{Z}/p} \Delta_{k,p} = H^{\geq k}(B(\mathbb{Z}/p); \mathbb{F}_p)
\]
for all odd \(k\). Indeed, assume the opposite, that is, \(t^{(k-1)/2} \in \text{index}_{\mathbb{Z}/p} \Delta_{k,p}\). The \(\mathbb{Z}/p\)-equivariant inclusion \(\Delta_{k+1,p} \subseteq \Delta_{1,p} \ast \Delta_{k,p}\) combined with the monotonicity and join properties of the Fadell–Husseini index imply that
\[
\text{index}_{\mathbb{Z}/p} \Delta_{1,p} \cdot \text{index}_{\mathbb{Z}/p} \Delta_{k,p} \subseteq \text{index}_{\mathbb{Z}/p}(\Delta_{1,p} \ast \Delta_{k,p}) \subseteq \text{index}_{\mathbb{Z}/p} \Delta_{k+1,p}.
\]
Since \(e \in \text{index}_{\mathbb{Z}/p} \Delta_{1,p}\), and we have assumed that \(t^{(k-1)/2} \in \text{index}_{\mathbb{Z}/p} \Delta_{k,p}\), the previous relation implies that
\[
t^{(k-1)/2}e \in \text{index}_{\mathbb{Z}/p} \Delta_{k+1,p} = H^{\geq k+1}(B(\mathbb{Z}/p); \mathbb{F}_p) = (t^{(k+1)/2}),
\]
a contradiction. Hence \(t^{(k-1)/2} \notin \text{index}_{\mathbb{Z}/p} \Delta_{k,p}\), and the proof of the theorem is complete.

Let us review the results on the Fadell–Husseini index of chessboards we have obtained so far:
\[
\begin{array}{cccccccc}
\text{index}_{\mathbb{Z}/p} \Delta_{1,p} & \text{index}_{\mathbb{Z}/p} \Delta_{2,p} & \cdots & \text{index}_{\mathbb{Z}/p} \Delta_{p-1,p} & \text{index}_{\mathbb{Z}/p} \Delta_{p-2, p} & \cdots & \text{index}_{\mathbb{Z}/p} \Delta_{2, p}\n\hline
H^{\geq 1}(B(\mathbb{Z}/p); \mathbb{F}_p) & H^{\geq 2}(B(\mathbb{Z}/p); \mathbb{F}_p) & \cdots & H^{\geq p-1}(B(\mathbb{Z}/p); \mathbb{F}_p) & H^{\geq p}(B(\mathbb{Z}/p); \mathbb{F}_p) & \cdots & H^{\geq p}(B(\mathbb{Z}/p); \mathbb{F}_p) & \cdots
\end{array}
\]
The remaining question indicated by this diagram is: For which chessboard \(\Delta_{k,p}\) with \(p - 1 \leq k \leq 2p - 1\) does the first jump in the index \(H^{\geq p-1}(B(\mathbb{Z}/p); \mathbb{F}_p)\) to \(H^{\geq p}(B(\mathbb{Z}/p); \mathbb{F}_p)\) happen?

**Theorem 6.9.** \(\text{index}_{\mathbb{Z}/p} \Delta_{k,p} = H^{\geq p-1}(B(\mathbb{Z}/p); \mathbb{F}_p)\), for \(p - 1 \leq k \leq 2p - 2\).

**Proof.** It suffices to show that \(\text{index}_{\mathbb{Z}/p} \Delta_{2p-2, p} = H^{\geq p-1}(B(\mathbb{Z}/p); \mathbb{F}_p)\). For this we are going to prove that \(t^{(p-1)/2} \in \text{index}_{\mathbb{Z}/p} \Delta_{2p-2, p}\).

Consider the following composition of maps
\[
\Delta_{2p-2, p} \to \Delta_{p-1, p} \ast \Delta_{p-1, p} \xrightarrow{f \ast f} \partial \Delta_{p-1} \ast \Delta_{p-1} \to \{\lambda x + (1 - \lambda) y \in S^{p-2} \ast S^{p-2} : \lambda \neq \frac{1}{2} \text{ or } x \neq y\} \to S^{p-2} \cong S(W_p),
\]
where the first map is an inclusion, the second map is the 2-fold join of the map \(f : \Delta_{p-1, p} \to \partial \Delta_{p-1}\), \(\Delta_{p-1} \cong S(W_p)\) introduced in Lemma 6.5, the third map is again an inclusion, while the last map is a deformation retraction. All the maps in this composition are \(\mathbb{Z}/p\)-equivariant. The monotonicity property of the Fadell–Husseini index implies that
\[
\text{index}_{\mathbb{Z}/p} \Delta_{2p-2, p} \supseteq \text{index}_{\mathbb{Z}/p} S(W_p) = (t^{(p-1)/2}) = H^{\geq p-1}(B(\mathbb{Z}/p); \mathbb{F}_p),
\]
according to (17). Thus \(t^{(p-1)/2} \in \text{index}_{\mathbb{Z}/p} \Delta_{2p-2, p}\), and we have concluded the proof of the theorem. \(\square\)
Now we have the answer to our question. The jump happens in the last possible moment, that is for the index of $\Delta_{2p-1,p}$. The proof of this is due to Carsten Schultz.

We conclude the section with a very useful corollary [15, Cor. 2.6], which also hides a proof for the upcoming optimal colored Tverberg theorem 6.14.

**Corollary 6.10.** Let $1 \leq k_1, \ldots, k_n \leq p - 1$. Then
\[
\text{index}_{\mathbb{Z}/p}(\Delta_{k_1,p} \ast \cdots \ast \Delta_{k_n,p}) = H^{\geq k_1 + \cdots + k_n}(B(\mathbb{Z}/p); \mathbb{F}_p).
\]

**Proof.** Let $K := \Delta_{k_1,p} \ast \cdots \ast \Delta_{k_n,p}$, $K' := \Delta_{p-1-k_1,p} \ast \cdots \ast \Delta_{p-1-k_n,p}$, and $L := \Delta_{p-1,p}^n$. Then there is a $\mathbb{Z}/p$-equivariant inclusion $L \to K \ast K'$. Again the monotonicity and join properties of the Fadell–Husseini index imply that
\[
\text{index}_{\mathbb{Z}/p} L \supseteq \text{index}_{\mathbb{Z}/p}(K \ast K') \supseteq \text{index}_{\mathbb{Z}/p} K \cdot \text{index}_{\mathbb{Z}/p} K'.
\]
Furthermore $\dim L = \dim K + \dim K' + 1$. The complexes $K$ and $K'$ are free $\mathbb{Z}/p$-spaces and therefore, as previously observed, it follows that
\[
\text{index}_{\mathbb{Z}/p} K \supseteq H^{\geq \dim K + 1}(B(\mathbb{Z}/p); \mathbb{F}_p) \quad \text{and} \quad \text{index}_{\mathbb{Z}/p} K' \supseteq H^{\geq \dim K' + 1}(B(\mathbb{Z}/p); \mathbb{F}_p).
\]
Since, by Corollary 6.7, $\text{index}_{\mathbb{Z}/p} L = H^{\geq \dim L + 1}(B(\mathbb{Z}/p); \mathbb{F}_p)$ and $\dim L + 1$ is an even integer, the relation between the indexes
\[
\text{index}_{\mathbb{Z}/p} L \supseteq \text{index}_{\mathbb{Z}/p} K \cdot \text{index}_{\mathbb{Z}/p} K'
\]
implies that
\[
\text{index}_{\mathbb{Z}/p} K = H^{\geq \dim K + 1}(B(\mathbb{Z}/p); \mathbb{F}_p),
\]
as claimed. We have also proved that $\text{index}_{\mathbb{Z}/p} K' = H^{\geq \dim K' + 1}(B(\mathbb{Z}/p); \mathbb{F}_p)$. \hfill $\square$

6.2. **The Bárány–Larman conjecture and the optimal colored Tverberg Theorem.** Finally we will, motivated by Theorem 6.3, utilize the computation of the Fadell–Husseini index for the chessboards to prove the following result [17, Prop. 4.2].

**Theorem 6.11.** Let $d \geq 1$ be an integer, and let $p$ be an odd prime. There is no $\mathfrak{S}_p$-equivariant map
\[
\Delta_p^{\ast(d+1)} \ast [p] \to S(W_p^{\otimes(d+1)}).
\]

**Proof.** It suffices to prove that there is no $\mathbb{Z}/p$-equivariant map $\Delta_p^{\ast(d+1)} \ast [p] \to S(W_p^{\otimes(d+1)})$, where $\mathbb{Z}/p$ is a subgroup of the symmetric group $\mathfrak{S}_p$ generated by the cycle $(12 \ldots p)$. The proof uses the monotonicity property of the Fadell–Husseini index.

According to (17) and the join property for the spheres, the index of the sphere $S(W_p^{\otimes(d+1)})$ is
\[
\text{index}_{\mathbb{Z}/p} S(W_p^{\otimes(d+1)}) = \theta^{d+1}(p-1)/2 = H^{\geq (d+1)(p-1)/2}(B(\mathbb{Z}/p); \mathbb{F}_p).
\]
Using Corollary 6.7 we get that
\[
\text{index}_{\mathbb{Z}/p}(\Delta_p^{\ast(d+1)} \ast [p]) = H^{\geq (d+1)(p-1)/2}(E(\mathbb{Z}/p); \mathbb{F}_p),
\]
and consequently $\theta^{(d+1)(p-1)/2} \notin \text{index}_{\mathbb{Z}/p}(\Delta_p^{\ast(d+1)} \ast [p])$. Thus,
\[
\text{index}_{\mathbb{Z}/p} S(W_p^{\otimes(d+1)}) \not\subseteq \text{index}_{\mathbb{Z}/p}(\Delta_p^{\ast(d+1)} \ast [p]),
\]
implying that a $\mathbb{Z}/p$-equivariant map $\Delta_p^{\ast(d+1)} \ast [p] \to S(W_p^{\otimes(d+1)})$ cannot exist. \hfill $\square$

A direct corollary of Theorems 6.3 and 6.11 is that the Bárány–Larman conjecture holds for all integers $r$ such that $r + 1$ is a prime [17, Cor. 2.3].

**Corollary 6.12** (The Bárány–Larman conjecture for primes–1). Let $r \geq 2$ and $d \geq 1$ be integers such that $r + 1 = p$ is a prime. Then $n(d, r) = (d + 1)r$ and $lt(d, r) = r$.

Using the pigeonhole principle and the index computation for the chessboards we can in addition prove that in the case when $p$ is an odd prime the Bárány–Larman function $n(d, p)$ is finite.

**Theorem 6.13.** Let $p$ be an odd prime. Then $n(d, p) \leq (d + 1)(2p - 2) + 1$. 


Proof. Let $n = (d + 1)(2p - 2) + 1$, and let $(C_1, \ldots, C_{d+1})$ be a coloring of the vertex set of the simplex $\Delta_{n-1}$ by $d + 1$ colors with each color class of size at least $p$. Then by the pigeonhole principle at least one of the colors, let say $C_{d+1}$, has to be of the size at least $2p - 1$. According to Corollary 4.8: If we can prove that there is no $S_{p^r}$- or $\mathbb{Z}/p$-equivariant map

$$\Delta_{|C_1|, p} \ast \cdots \ast \Delta_{|C_{d+1}|, p} \cong (R_{(C_1, \ldots, C_{d+1})})_{p^r}^\Delta(2) \to S(W_{p}^{(d+1)}),$$

then for every continuous map $f : \Delta_{n-1} \to \mathbb{R}^d$ there are $p$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_{n-1}$ whose $f$-images overlap, that is $f(\sigma_1) \cap \cdots \cap f(\sigma_p) \neq \emptyset$. Thus we will now prove that there is no $\mathbb{Z}/p$-equivariant map $\Delta_{|C_0|, p} \ast \cdots \ast \Delta_{|C_{d+1}|, p} \to S(W_{p}^{(d+1)}).

Again, using (17) and the join property for the spheres, we have that

$$\text{index}_{\mathbb{Z}/p} S(W_{p}^{(d+1)}) = (t^{(d+1)}(p-1)/2) = H_{\geq (d+1)(p-1)+1}(\mathbb{BZ}/p; \mathbb{F}_p).$$

Since $|C_0| \geq p, \ldots, |C_d| \geq p$ and $|C_{d+1}| \geq 2p - 1$, there is a $\mathbb{Z}/p$-equivariant inclusion

$$\Delta_{p-1, p} \ast \cdots \ast \Delta_{d-1, p} \ast \Delta_{d-1, p} \rightarrow \Delta_{|C_0|, p} \ast \cdots \ast \Delta_{|C_d|, p} \ast \Delta_{|C_{d+1}|, p},$$

Thus the monotonicity property of the Fadell–Husseini index and Corollary 6.7 (iii) imply that

$$H_{\geq (d+1)(p-1)+1}(\mathbb{BZ}/p; \mathbb{F}_p) \supseteq \text{index}_{\mathbb{Z}/p} (\Delta_{|C_0|, p} \ast \cdots \ast \Delta_{|C_d|, p} \ast \Delta_{|C_{d+1}|, p}).$$

Therefore,

$$\text{index}_{\mathbb{Z}/p} S(W_{p}^{(d+1)}) \nsubseteq \text{index}_{\mathbb{Z}/p} (\Delta_{|C_0|, p} \ast \cdots \ast \Delta_{|C_d|, p} \ast \Delta_{|C_{d+1}|, p}),$$

and consequently there is no $\mathbb{Z}/p$-equivariant map $\Delta_{|C_0|, p} \ast \cdots \ast \Delta_{|C_{d+1}|, p} \to S(W_{p}^{(d+1)})$. This concludes the proof of the theorem. \qed

While focusing on the Bárán–Larman conjecture and the corresponding function $n(d, r)$, we almost overlooked that the index computations for the chessboards establish a considerable strengthening of the topological Tverberg theorem that is known as the optimal colored Tverberg theorem [17, Thm. 2.1].

**Theorem 6.14** (The optimal colored Tverberg theorem). Let $d \geq 1$ be an integer, let $p$ be a prime, $N \geq (d + 1)(p-1)$, and let $f : \Delta_N \rightarrow \mathbb{R}^d$ be a continuous map. If the vertices of the simplex $\Delta_N$ are colored by $m$ colors, where each color class has cardinality at most $p - 1$, then there are $p$ pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_p$ of $\Delta_N$ whose $f$-images overlap,

$$f(\sigma_1) \cap \cdots \cap f(\sigma_p) \neq \emptyset.$$

**Dictionary**

**Borel construction.** [1] [21] [30] Let $G$ be a finite group and let $X$ be a (left) $G$-space. The **Borel construction** of $X$ is the space given by $EG \times_G X := (EG \times X)/G$, where $EG$ is a contractible, right contractible $G$-space and $G$ acts on the product by $g \cdot (e, x) = (e \cdot g^{-1}, g \cdot x)$. The projection $EG \times X \rightarrow EG$ induces the following fiber bundle

$$X \rightarrow EG \times_G X \rightarrow BG.$$ This fiber bundle is called the **Borel construction fiber bundle**. The **Serre spectral sequence** associated to the Borel construction fiber bundle has the E₂-term given by

$$E_2^{r,s} = H^s(BG; H^r(X; R)) \cong H^s(G; H^r(X; R)),$$

where the coefficients are local and determined by the action of $\pi_1(BG) \cong G$ on the cohomology of $X$. Moreover, each row of the spectral sequence has the structure of an $H^* (BG; R)$-module, while all differentials are $H^* (BG; R)$-module morphisms.

The Borel construction and the associated fibration are natural with respect to equivariant maps, that is, any $G$-equivariant map $f : X \rightarrow Y$ between $G$-spaces induces the following morphism of fiber bundles

$$EG \times_G X \rightarrow EG \times_G Y$$

This morphism of fiber bundle induces a morphism of associated Serre spectral sequences

$$E_1^{r,s}(f) : E_1^{r,s}(EG \times_G Y) \rightarrow E_1^{r,s}(EG \times_G X),$$

such that

$$E_2^{r,0}(f) : E_2^{r,0}(EG \times_G Y) \rightarrow E_2^{r,0}(EG \times_G X)$$

is the identity.

**BG.** [1] [30] For a finite group $G$ the **classifying space** is the quotient space $BG = EG/G$. The projection $EG \rightarrow BG$ is the universal principal $G$-bundle, that is, the set of all homotopy classes of maps $[X, BG]$ is in bijection with the set of all isomorphism classes of principal $G$-bundles over $X$.

**Borsuk–Ulam theorem.** [36] Let $S^n$ and $S^m$ be free $\mathbb{Z}/2$-spaces. Then a continuous $\mathbb{Z}/2$-equivariant map $S^n \rightarrow S^m$ exists if and only if $m \leq n$. 


Cohomology of a group (algebraic definition). [1] [20]
Let $G$ be a finite group, and let $M$ be a (left) $G$-module. Consider a projective resolution $(P_n, d_n)_{n \geq 0}$ of the trivial (left) $G$ module $Z$, that is, an exact sequence

$$\cdots \rightarrow P_{n+1} \xrightarrow{d_n} P_n \xrightarrow{d_{n-1}} \cdots \rightarrow P_0 \xrightarrow{\pi} Z \rightarrow 0,$$

where each $P_n$ is a projective (left) $G$-module. The group cohomology of $G$ with coefficients in the module $M$ is the cohomology of the following cochain complex

$$\cdots \rightarrow \hom_G(P_{n+1}, M) \xrightarrow{d_{n+1}} \hom_G(P_n, M) \rightarrow 0,$$

Cohomology of group (topological definition). [1] [20]
Let $G$ be a finite group, and let $M$ be a (left) $G$-module. The group cohomology of $G$ with coefficients in the module $M$ is the cohomology of $BG$ with local coefficients in the $\pi_1(BG) \cong G$-module $M$, that is

$$H^*(G; M) := H^*(BG; M).$$

Connectedness. [19] [36] Let $n \geq -1$ be an integer. A topological space $X$ is $n$-connected if any continuous map $f : S^k \rightarrow X$, where $-1 \leq k \leq n$, can be continuously extended to a continuous map $g : B^{k+1} \rightarrow X$, that is $g|_{S^k} = f$. Here $B^{k+1}$ denotes a $(k+1)$-dimensional closed ball whose boundary is the sphere $S^k$. A topological space $X$ is $(1)$-connected if it is non-empty, it is 0-connected if and only if it is path-connected. If the space $X$ is $n$-connected and $Y$ is $m$-connected, then the join $X \ast Y$ is $(n+m+2)$-connected.

If the space $X$ is $n$-connected, but not $(n+1)$-connected, we write $\text{conn}(X) = n$. Then

$$\text{conn}(X \ast Y) \geq \text{conn}(X) + \text{conn}(Y) + 2.$$  

Chessboard complex. [31] [36] The $m \times n$ chessboard complex $\Delta_{m,n}$ is the simplicial complex whose vertex set is $[m] \times [n]$, and where the set of vertices $(i_0, j_0), \ldots, (i_k, j_k)$ spans a $k$-simplex if and only if $\prod_{0 \leq a < b \leq k}(i_a - i_b)(j_a - j_b) \neq 0$. For example, $\Delta_{2,3} \cong S^1 \ast S^3 \cong S^3$. The chessboard complex $\Delta_{m,n}$ is an $(\mathbb{S}^m \times \mathbb{S}^n)$-space by

$$(\sigma_1, \sigma_2) \cdot \{(i_0, j_0), \ldots, (i_k, j_k)\} = \{(\sigma_1(i_0), \sigma_2(j_0)), \ldots, (\sigma_1(i_k), \sigma_2(j_k))\},$$

where $(\sigma_1, \sigma_2) \in \mathbb{S}^m \times \mathbb{S}^n$, and $\{(i_0, j_0), \ldots, (i_k, j_k)\}$ is a simplex in $\Delta_{m,n}$. The connectivity of the chessboard complex $\Delta_{m,n}$ is

$$\text{conn}(\Delta_{m,n}) = \min \{m, n, \frac{m+n+1}{2} \} - 2.$$  

For $n \geq 3$, the chessboard complex $\Delta_{n-1,n}$ is a connected, orientable pseudomanifold of dimension $n-2$. Therefore, $\text{conn}(\Delta_{n-1,n}) = \frac{n(n-1)}{2}$ and an orientation homology class is given by the chain

$$z_{n-1,n} = \sum_{\pi \in \mathbb{S}_n} (\text{sgn } \pi)((1, \pi(1)), \ldots, (n-1, \pi(n-1))).$$

The symmetric group $\mathbb{S}_n \cong 1 \times \mathbb{S}_n \subseteq \mathbb{S}_{n-1} \times \mathbb{S}_n$ acts on $\Delta_{n-1,n}$ by the restriction action. Then $\pi \cdot z_{n-1,n} = (\text{sgn } \pi)z_{n-1,n} = 1$.

Deleted join. [36] Let $K$ be a simplicial complex, let $n \geq 2$, $k \geq 2$ be integers, and let $[n] := \{1, \ldots, n\}$. The $n$-fold $k$-wise deleted join of the simplicial complex $K$ is the simplicial complex

$$K^\times_{\Delta(k)} := \{\lambda_1 x_1 + \cdots + \lambda_n x_n \in \sigma_1 \ast \cdots \ast \sigma_n \subset K^\times : \forall I \subseteq [n] \text{ card } I \geq k \Rightarrow \bigcap_{i \in I} \sigma_i = \emptyset\},$$

where $\sigma_1, \ldots, \sigma_n$ are faces of $K$, including the empty face. The symmetric group $\mathbb{S}_n$ acts on $K^\times_{\Delta(k)}$ by

$$\pi \cdot (\lambda_1 x_1 + \cdots + \lambda_n x_n) := \lambda_{\pi^{-1}(1)} x_{\pi^{-1}(1)} + \cdots + \lambda_{\pi^{-1}(n)} x_{\pi^{-1}(n)},$$

for $\pi \in \mathbb{S}_n$ and $\lambda_1 x_1 + \cdots + \lambda_n x_n \in K^\times_{\Delta(k)}$.

Deleted product. [36] Let $K$ be a simplicial complex, let $n \geq k \geq 2$ be integers, and let $[n] := \{1, \ldots, n\}$. The $n$-fold $k$-wise deleted product of the simplicial complex $K$ is the cell complex

$$K^\times n_{\Delta(k)} := \{(x_1, \ldots, x_n) \in \sigma_1 \ast \cdots \ast \sigma_n \subset K^\times n : \forall I \subseteq [n] \text{ card } I \geq k \Rightarrow \bigcap_{i \in I} \sigma_i = \emptyset\},$$

where $\sigma_1, \ldots, \sigma_n$ are non-empty faces of $K$. The symmetric group $\mathbb{S}_n$ acts on $K^\times n_{\Delta(k)}$ by

$$\pi \cdot (x_1, \ldots, x_n) := (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}),$$

for $\pi \in \mathbb{S}_n$ and $(x_1, \ldots, x_n) \in K^\times n_{\Delta(k)}$.

Dold’s theorem. [36] Let $G$ be a non-trivial finite group. For an $n$-connected $G$-space $X$ and an at most $n$-dimensional free $G$-CW complex $Y$ there is no continuous $G$-equivariant map $X \rightarrow Y$.

EG. [1] [21] [30] For a finite group $G$ any contractible free $G$-CW complex equipped with the right $G$ cellular action is a model for an EG space. Milnor’s model is given by $EG = \text{colim}_{G \in \mathcal{G}} G^n$ where $G$ stands for a $0$-dimensional free $G$-simplicial complex whose vertices are indexed by the elements of the group $G$ and the action on $G$ is given by the right translation, and $G^n$ is an $n$-fold join of the $0$-dimensional simplicial complex with induced diagonal (right) action.

Equivariant cohomology (via the Borel construction). [1] [21] [30] Let $G$ be a finite group and let $X$ be a (left) $G$-space. The singular or Čech cohomology of the Borel construction $EG \times_G X$ of the space $X$ is called the equivariant cohomology of $X$ and is denoted by $H^*_G(X; R)$. Here $R$ denotes a group, or a ring of coefficients.

Equivariant cohomology of a relative $G$-CW complex. [21] Let $G$ be a finite group, let $(X, A)$ be a relative $G$-CW complex with a free action on $X \backslash A$, and let $C_*(X, A; Z)$ denote the integral cellular chain complex. The cellular free $G$-action on every skeleton of $X \backslash A$ induces a free $G$-action on the chain complex $C_*(X, A; Z)$. Thus $C_*(X, A; Z)$ is a chain complex of free $ZG$-modules.

For a $ZG$-module $M$ consider

- the $G$-equivariant chain complex

$$C^*_G(X, A; M) = C_*(X, A; Z) \otimes_{ZG} M,$$

and define the equivariant homology $H^*_G(X, A; M)$ of $(X, A)$ with coefficients in $M$ to be the homology of the chain complex $C^*_G(X, A; M)$;

- the $G$-equivariant cochain complex

$$C^*_G(X, A; M) = \text{Hom}_{ZG}(C_*(X, A; Z), M),$$

and define the equivariant cohomology $H^*_G(X, A; M)$ of $(X, A)$ with coefficients in $M$ to be the cohomology of the cochain complex $C^*_G(X, A; M)$. 
Exact obstruction sequence.  [21] Let $G$ be a finite group, let $n \geq 1$ be an integer and let $Y$ be a path-connected $n$-simple $G$-space. For every relative $G$-CW complex $(X,A)$ with a free action of $G$ on the complement $X\setminus A$, there exists the obstruction exact sequence

$[sk_{n+1}X,Y]_G \rightarrow \text{im}([sk_nX,Y]_G \rightarrow [sk_{n-1}X,Y]_G)
\rightarrow \mathcal{H}^{n+1}_G(X,A;\pi_nY),$

The sequence is natural in $X$ and $Y$. This should be understood as follows:

- A $G$-equivariant map $f : sk_{n-1}X \rightarrow Y$ that can be equivariantly extended to the $n$-skeleton $f' : sk_nX \rightarrow Y$, that is $f'|_{sk_{n-1}X} = f$, defines a unique element $\beta^{n+1}_G(f)$ living in $\mathcal{H}^{n+1}_G(X,A;\pi_nY)$, called the obstruction element associated to the map $f$.

- The exactness of the sequence means that the obstruction element $\beta^{n+1}_G(f)$ is zero if and only if there is a $G$-equivariant map $f' : sk_nX \rightarrow Y$ whose restriction is in the $G$ homotopy class of the restriction of $f$, that is $f'|_{sk_{n-1}X} \simeq_G f|_{sk_{n-1}X}$, which extends to the $(n+1)$-skeleton $sk_{n+1}X$.

The obstruction element $\beta^{n+1}_G(f)$ associated with the homotopy class $[f] \in [sk_nX,Y]_G$ can be introduced on the cochain level as well. Let $h : (D^{n+1},S^n) \rightarrow (sk_{n+1}X,sk_nX)$ be an attaching map and $e \in C_n(X,A;\mathbb{Z})$ the corresponding generator. The obstruction cochain $v^{n+1}_G(f) \in C^{n+1}_G(X,A;\pi_nY)$ of the map $f$ is defined on $e$ by

$v^{n+1}_G(f)(e) = [f \circ h] \in [S^n,Y].$

The cohomology class of the obstruction cocycle coincides with the obstruction element defined via the exact sequence.

Fadell–Husseini index. [23] Let $G$ be a finite group and $R$ be a commutative ring with unit. For a $G$-space $X$ and a ring $R$, the Fadell–Husseini index of $X$ is defined to be the kernel level of the map in equivariant cohomology induced by the $G$-equivariant map $px : X \rightarrow pt$:

$\text{index}_G(X,R) = \ker\left(H^*(BG;R) \rightarrow H^*(EG \times G;X,R)\right).$

Some basic properties of the index are:

- Monotonicity: If $X \rightarrow Y$ is a $G$-equivariant map then $\text{index}_G(X,R) \supseteq \text{index}_G(Y,R)$.

- Additivity: If $(X_1 \cup X_2, X_1 \times X_2)$ is an excisive triple of $G$-spaces, then $\text{index}_G(X_1,R) \cdot \text{index}_G(X_2,R) \subseteq \text{index}_G(X_1 \cup X_2, R)$.

- Joint: Let $X$ and $Y$ be $G$-spaces, then $\text{index}_G(X,R) \cdot \text{index}_G(Y;R) \subseteq \text{index}_G(X \times Y)$.

- Generalized Borsuk–Ulam–Bourgin–Yang theorem: Let $f : X \rightarrow Y$ be a $G$-equivariant map, and let $Z \subseteq Y$ be a closed $G$-invariant subspace. Then $\text{index}_G(f^{-1}(Z);R) \cdot \text{index}_G(Y,Z;R) \subseteq \text{index}_G(X;R)$. 

- Let $U$ and $V$ be finite dimensional real $G$-representations. If $H^*(SU(V);R)$ and $H^*(S(V);R)$ are trivial $G$-modules, $\text{index}_G(S(U);R) = \langle f \rangle$ and $\text{index}_G(S(V);R) = \langle g \rangle$, then $\text{index}_G(S(U \oplus V);R) = \langle f \cdot g \rangle \subseteq H^*(BG;R)$.

G-action. Let $G$ be a group and let $X$ be a non-empty set. A (left) $G$-action on $X$ is a function $G \times X \rightarrow X$, $(g,x) \mapsto g \cdot x$ with the property that:

$g \cdot (h \cdot x) = (gh) \cdot x$ and $1 \cdot x = x,$

for every $g,h \in G$ and $x \in X$. A set $X$ with a $G$-action is called a $G$-set. Let $G$ and $X$ in addition be topological spaces. Then a $G$-action is continuous if the function $G \times X \rightarrow X$ is continuous with respect to the product topology on $G \times X$. A topological space equipped with a continuous $G$-action is called a $G$-space.

G-equivariant map. Let $X$ and $Y$ be $G$-sets (spaces). A (continuous) $G$-equivariant map $f : X \rightarrow Y$ is a $G$-equivariant map if $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$ and all $g \in G$.

G-CW complex. [18] [21] Let $G$ be a finite group. A CW-complex $X$ is a $G$-CW complex if the group $G$ acts on $X$ by cellular maps and for every $g \in G$ the subspace $\{x \in X : g \cdot x = x\}$ is a CW-subcomplex of $X$. Let $X$ be a $G$-CW complex, and let $A$ be a subcomplex of $X$ that is invariant with respect to the action of the group $G$ and consequently a $G$-CW complex in its own right. The pair of $G$-CW complex $(X,A)$ is a relative $G$-CW complex.

Localization theorem. [21] [30] The following result is a consequence of the localization theorem for elementary abelian groups: Let $p$ be a prime, $G = (\mathbb{Z}/p^n)$ for $n \geq 1$, and let $X$ be a finite group $G$-CW complex. The fixed points set $X^p$ of the space $X$ is non-empty if and only the map in cohomology $H^*(BG;\mathbb{F}_p) \rightarrow H^*(EG \times G;X;\mathbb{F}_p)$ induced by the projection $EG \times G \rightarrow BG$, is a monomorphism.

n-simple. [19] A topological space $X$ is $n$-simple if the fundamental group $\pi_1(X,x_0)$ acts trivially on the $n$-th homotopy group $\pi_n(X,x_0)$ for every $x_0 \in X$.

Nerve of a family of subsets. Let $X$ be a set and let $\mathcal{I} := \{X_i : i \in I\}$ be a family of subsets of $X$. The nerve of the family $\mathcal{I}$ is the simplicial complex $N_{\mathcal{I}}$ with the vertex set $I$, and a finite subset $\sigma \subseteq I$ is a face of the complex if and only if $\bigcap_{(i \in \sigma)} X_i \neq \emptyset$.

Nerve theorem. [10] Let $K$ be a finite simplicial complex, or a regular CW-complex, and let $K := \{K_i : i \in I\}$ be a cover of $K$ by a family of subcomplexes, that is $K = \bigcup\{K_i : i \in I\}$.

1. If for every face $\sigma$ of the nerve $N_{\mathcal{K}}$ the intersection $\bigcap_{i \in \sigma} X_i$ is contractible, then $K$ and $N_{\mathcal{K}}$ are homotopy equivalent, that is $K \simeq N_{\mathcal{K}}$.

2. If for every face $\sigma$ of the nerve $N_{\mathcal{K}}$ the intersection $\bigcap_{i \in \sigma} X_i$ is $(k - |\sigma| + 1)$-connected, then the complex $K$ is $k$-connected if and only if the nerve $N_{\mathcal{K}}$ is $k$-connected.

Primary obstruction. [18] [21] Let $G$ be a finite group, let $n \geq 1$ be an integer and let $Y$ be an $(n+1)$-connected and $n$-simple $G$-space. Furthermore, let $(X,A)$ be a relative $G$-CW complex with the free $G$ action on $X\setminus A$, and let $f : A \rightarrow Y$ be a $G$-equivariant map. Then

- there exists a $G$-equivariant map $f' : sk_nX \rightarrow Y$ extending $f$, that is $f'|_A = f$,

- every two $G$-equivariant extensions $f', f'' : sk_nX \rightarrow Y$ of $f$ are $G$-homotopic, relative to $A$, on $sk_{n-1}X$, that is $\text{im}([sk_nX,Y]_G|_{sk_{n-1}X,Y}]_G) = \langle pt \rangle$,

- if $H : A \times I \rightarrow Y$ is a $G$-equivariant homotopy between $G$-equivariant maps $f : A \rightarrow Y$ and $f' : A \rightarrow Y$, and if $h : sk_nX \rightarrow Y$ and $h' : sk_nX \rightarrow Y$ are $G$-equivariant extensions of $f$ and $f'$, then there exists a $G$-equivariant homotopy $K : sk_{n-1}X \times I \rightarrow Y$ between $h|_{sk_{n-1}X}$ and $h'|_{sk_{n-1}X}$ that extends $H$.

In the case when $\text{im}([sk_nX,Y]_G|_{sk_{n-1}X,Y}]_G) = \langle pt \rangle$ the obstruction sequence becomes

$[sk_{n+1}X,Y]_G \rightarrow \langle pt \rangle \rightarrow \mathcal{H}^{n+1}_G(X,\pi_nY).$

The obstruction element $\beta^{n+1}_G(pt) \in \mathcal{H}^{n+1}_G(X,\pi_nY)$ is called the primary obstruction and does not depend on the choice of a $G$-equivariant map on the $n$-th skeleton of $X$. 
Restriction and transfer. [14] [20] Let $G$ be a finite group and let $H \subseteq G$ be its subgroup. Consider a $ZG$-chain complex $C_\ast = (C_\ast, d_\ast)$ and a $ZG$-module $M$. Denote by $res$ the restriction from $G$ to $H$. For every integer $n$ there exists a homomorphism

$$res : H^n(\hom_{ZG}(res C_\ast, res M)) \to H^n(\hom_{ZH}(C_\ast, M))$$

that we call the restriction from $G$ to $H$, and a homomorphism

$$tr : H^n(\hom_{ZH}(C_\ast, M)) \to H^n(\hom_{ZG}(res C_\ast, res M))$$

that is called the transfer from $H$ to $G$, with the property

$$tr \circ res = [G : H] \cdot id.$$

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