Simple considerations on the behaviour of bosonic modes with quantum group symmetry

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Abstract

While it is possible to introduce quantum group symmetry into the framework of quantum mechanics, the general problem of how to implement quantum group symmetry into (3 + 1) dimensional quantum field theory has not yet been solved. Here we try to estimate some features of the behaviour of bosonic modes.

1 Introduction

Quantum groups arose from the quantum inverse scattering method [1]. Mathematically they are deformations of groups [2, 3]. The underlying structure is that of noncommutative geometry: The (co-quasitriangular Hopf-) algebra of functions on the group becomes noncommutative when a new parameter, which is usually denoted 'q', deviates from 1, see e.g. [2, 3, 4, 5].

Quantum groups have found a wide field of applications from statistical mechanics to knot theory, see e.g. [6, 7] and are also extensively studied in their own right, see e.g. [8, 9, 10, 11, 12]. Particularly tempting is the idea to implement quantum groups as symmetries in quantum theory, which essentially means to make the commutation relations q-dependent. The usual results are then recovered as the special case q = 1.

The case q ≠ 1 may e.g. apply to the description of non-standard commutation relations of collective excitations or quasi-particles in solid state physics. But one can also consider q ≠ 1 as a check for the usual quantum theory. In this context, the observation of discretizing and regularizing effects of quantum group symmetry is of great interest [13, 14, 15, 16].

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2 Commutation relations with $SU_q(n)$- symmetry

There are already several approaches to the definition of quantum mechanical commutation relations with quantum group symmetry in the literature, see e.g. \[17, 18, 19, 20, 21, 22, 23\]. In particular Heisenberg algebras generated by adjoint pairs of operators $a^\dagger_i$ and $a^i$ were developed. The following $q$-dependent bosonic commutation relations are conserved under the action of the quantum group $SU_q(n)$, see e.g. \[17, 20\]:

\[
\begin{align*}
    a_i a_j - 1/qa_j a_i &= 0 \quad \text{for } i > j \\
    a^\dagger_i a^\dagger_j - 1/qa^\dagger_j a^\dagger_i &= 0 \quad \text{for } i < j \\
    a_i a^\dagger_j - q \ a^\dagger_j a_i &= 0 \quad \text{for } i \neq j \\
    a_i a_i - q^2 a^\dagger_i a^\dagger_i &= 1 + (q^2 - 1) \sum_{j<i} a^\dagger_j a_j
\end{align*}
\]

The ground state can be defined as usual:

\[
<0|0> = 1 \quad \text{and} \quad a_i|0> = 0 \quad \text{for } i = 1, ..., n
\]

One then obtains for the scalar product $<,>$:

\[
<0|(a_n)^{r_n} \cdots (a_1)^{r_1} (a^\dagger_1)^{r_1} \cdots (a^\dagger_n)^{r_n}|0> = \prod_{i=1}^{n} [r_i]_q! \tag{1}
\]

with

\[
[r]_q! := [1]_q \cdot [2]_q \cdot [3]_q \cdot ... \cdot [r]_q \quad \text{and} \quad [p]_q := \frac{q^{2p} - 1}{q^2 - 1} \tag{2}
\]

Although quantum groups do in general have more than one free parameter, no further parameters enter in these commutation relations [24]. The usual quantum mechanical programme, representation on a positive definit (Bargmann Fock-) Hilbert space of wave functions and definition of integral kernels like Green functions etc., could be performed [21], while preserving the hermiticity of the observables and the unitarity of time evolution.

3 q-bosonic modes

An extension of this formalism to relativistic quantum field theories proves to be difficult [25]. However, already by using simple nonrelativistic quantum mechanical techniques one can try to obtain some insight into the behaviour of modes of bosonic fields with quantum group symmetry. The reason for this is the estimation, that effects of $q \neq 1$ do not primarily occur as high energy effects, but instead as effects of high occupation numbers - at least on the basis of the quantum mechanical formalism.
mentionend above.
To see this, let us consider the simple quantum mechanical Hamiltonian of the $n$-dimensional isotropic harmonic oscillator:

$$ H := \hbar \omega \sum_{i=1}^{n} a_i^\dagger a_i \quad \text{with } \omega \in \mathbb{R}^+ $$

In the $SU_q(n)$ symmetric formalism it has the energy spectrum [20]:

$$ E_p = \hbar \omega (0 + 1 + q^2 + q^4 + \ldots + q^{2(p-1)}) = \hbar \omega \frac{q^{2p} - 1}{q^2 - 1} \quad \text{with } p = 0, 1, 2, \ldots $$

Thus its energy levels are no longer equidistant. The spacing of the levels increases or decreases depending on whether $q$ is larger or smaller than 1. In the latter case the series actually converges and the spectrum gets an upper bound $\frac{\hbar \omega}{1-q^2}$. In both cases it is true that the higher the energy level is, the more it deviates from the corresponding energy level for $q = 1$.

In quantum field theory there are harmonic oscillators at the points in 3-momentum space. Thus, if it is possible to extend this formalism to quantum field theory, it can be expected that quantum group effects do again increase with the level of excitation, i.e. then with the number of field quanta that are present.

Let us assume in the following, that we have a photon- or phonon- like bosonic quantum field and, for simplicity, that the momentum space is discretized. Apart from solids or from the situation in cavity, there are also hints, that some kind of discretization can occur naturally through quantum group symmetry [13, 14, 15, 16].

Picking out a mode with fixed wave vector $\vec{k}$, we do not need to specify the dispersion relation in order to write its free Hamiltonian in the form just given above:

$$ H_{\vec{k}} := \hbar \omega_{\vec{k}} \sum_{i=1}^{n} a_i^\dagger(\vec{k}) a_i(\vec{k}) $$

If we let $n = 2$, the two degrees of freedom can e.g. be interpreted as two polarizations. What follows will not depend on $n$ and its interpretation:

### 4 Induced emission of field quanta

Let us now consider a system with a source that is able to emit and to absorb field quanta of the $q$-mode. We follow the early treatment by Dirac [26]. The Hamiltonian, expanded in the $q$-mode’s variables reads:

$$ H = H_{\text{source}} + \sum_{i} (g_i^a a_i(\vec{k}) + a_i^\dagger(\vec{k}) g_i) $$

$$ + \sum_{i,j} (h_{ij} a_i(\vec{k}) a_j(\vec{k}) + f_{ij} a_i(\vec{k}) a_j(\vec{k}) + f_{ij}^\dagger a_i^\dagger(\vec{k}) a_j^\dagger(\vec{k})) + \ldots $$
This could e.g. be a laser-like system with an atom as the source coupled to a photon mode with a large wave length compared to the size of the atom. The above Hamiltonian is then obtained from the minimal coupling of the vector potential. In the Hamiltonian the part of the free source and the free field

\[ H_{\text{source}} + \hbar \omega_\kappa \hat{a}_i^\dagger(\kappa) \hat{a}_i^\dagger(\kappa) \]

are assumed to be dominant. Thus we can work in the direct product of the Hilbert spaces of states of the free source and of the free boson mode. Their interaction terms we then treat perturbatively. There are terms of one-boson emission, one boson absorption, two boson emission, etc. The coefficients \( f, g, h \) are operators that act nontrivially only on the source states.

Let us now calculate for example the probability \( p_m \) of the emission of one \( q \)-boson of the \( i \)’th degree of freedom, dependent on the number \( m \) of \( q \)-bosons already present. The probability amplitude \( \psi_m \) is proportional to the transition matrix element:

\[ \psi_m \propto \langle \text{source after} | \langle 0 \left| \left( a_i(\kappa) \right)^{m+1} \right| \frac{1}{\sqrt{|m+1,q!|}} \right| a_i^\dagger(\kappa) \right| | \text{source before} > \]  

where \( (|m,q!\rangle)^{-1/2}(a_i(\kappa))^m |0 > \) is a normalized state of \( m \) bosons (see Eq.2). Further evaluation yields:

\[ \psi_m \propto \langle \text{source after} | g_i | \text{source before} > | 0 \rangle \left( a_i(\kappa) \right)^{m+1} \frac{1}{\sqrt{|m+1,q!|}} \sqrt{|m+1,q!|} \frac{1}{\sqrt{|m,q!|}} (a_i^\dagger(\kappa))^m \]  

and finally

\[ \psi_m \propto \sqrt{|m+1,q!|} \quad \text{i.e.} \quad p_m \propto \sqrt{|m+1,q!|} \]  

Thus, while the probability of spontaneous emission does not depend on \( q \), the probability of induced emission does: It is no longer proportional to the number \( m+1 \), but to the \( q \)-number \( |m+1,q!| \). The deviation from the usual result increases with the occupation number of the boson mode.

The energy contained in the boson mode is proportional to \( |m,q!| \) (see Eq.3). Thus, while usually for the probability of boson emission holds

\[ p_m \propto I + \text{const} \]  

with \( I \) the intensity of the incident beam, we now obtain:

\[ p_m \propto I + q^{2m} \text{const} \quad \text{i.e.} \quad p_m/I \propto 1 + \frac{q^{2m}(q^2 - 1)}{q^{2m} - 1} \]  

The calculations for the absorption of bosons are of course analogous.
5 Conclusion

To summarize, from simple considerations, based on a $SU_q(n)$-symmetric quantum mechanical formalism, we expect that bosonic fields with quantum group symmetry should be recognizable from their particular behaviour for high occupation numbers $m$. Some of the characteristics should be:

1. The probability of induced emission into a fixed mode is no longer essentially proportional to its occupation number (Eq.[5]). For high occupation numbers the probability reaches a maximum if $q < 1$ or, if $q > 1$, it increases exponentially.

2. On the other hand, the probability of emission is still essentially proportional to the intensity of the beam (Eqs. [6, 7]). For $q < 1$ the intensity can only increase to a finite value.

3. Last but not least one can expect that laser-like systems of a $q$-bosonic mode with a source could come 'out of tune' for high occupation numbers. This is simply because the spacing of the energy levels of the mode changes with the occupation number, and the sources may only be prepared to provide quanta of energy within a certain fixed range.

6 References

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