Next-to-Leading Order Shear Viscosity in $\lambda\phi^4$ Theory

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We show that the shear viscosity of $\lambda\phi^4$ theory is sensitive at next-to-leading order to soft physics, which gives rise to subleading corrections suppressed by only a half power of the coupling, $\eta = (3033.54 + 154.8) \sqrt{(N + 2)/N} T^3/\lambda^2$. The series appears to converge about as well (or badly) as the series for the pressure.

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I. INTRODUCTION

For some time there has been a growing interest in the theoretical prediction of long-time dynamics and equilibration in hot quantum field theories. One driving motivation has been understanding the early universe, particularly the possibility of electroweak baryogenesis (for an older but still quite pertinent review see [1]). In this case one is interested in dynamics of the electroweak sector, SU(2)+Higgs with a coupling $\alpha_w \sim 1/30$. Another motivation is attempting to describe high energy heavy ion collisions, which are being experimentally probed at RHIC and will soon be probed at even higher energies at the LHC. Since one of the original motivations for this program was to explore QCD in a regime where it is thermal and (relatively) weakly coupled, it makes sense to see how far one can go in its description using weak-coupling tools, even though the gauge coupling is not expected to be very small, $\alpha_s \sim 1/3$ at best.

Some of the most interesting and difficult problems involve understanding long-time dynamics and the approach to equilibrium. Transport coefficients are a theoretically well-defined and interesting subset of these problems. Recently there has been great interest in the shear viscosity, because RHIC results on elliptic and radial flow and spectra seem to indicate that the viscosity is surprisingly small [2–4]. On the theoretical side, great strides have been made in the theoretical evaluation of transport coefficients. In a pioneering paper, Jeon showed how to calculate the shear viscosity of $\lambda\phi^4$ theory at leading order in the coupling [5]. He showed that the evaluation reduces to a problem in kinetic theory. Since then, kinetic theory treatments of QED and QCD transport coefficients (electrical conductivity, conserved number diffusion, and shear and bulk viscosity) have appeared at leading-log [6] and full leading order [7, 8]. The kinetic treatment has also been justified diagrammatically at leading-log [9, 10] and leading order [11].

However, except for two unrealistic simplifying limits (large $N_c$ gauge theory [12] and large $N$ O(N) scalar field theory [13]), there are no results beyond leading order in weak coupling for any long-time dynamical quantities in any interacting relativistic 3+1 dimensional gauge theory [19]. We feel this is a big hole in the literature, because without any knowledge of subleading corrections, it is hard to know how quickly perturbation theory converges. On the other hand, a subleading result gives at least a hint of how fast the series converges; when the subleading correction becomes as large as the leading order result, perturbation theory has almost surely failed. With this in mind, this brief report will compute the shear viscosity of the simplest toy theory, $N$ component $\lambda\phi^4$ theory at general $N$, at next-to-leading order.

The reason this is possible, and a main reason it is interesting, is that the subleading correction emerges from soft physics. In this theory, the plasma of scalars induces an effective mass for all scalar excitations, $m^2_{\text{th}} = N^2/4 \lambda T^2$. But soft scalars play an important role as the "targets" in scattering processes. The scalars which are significantly influenced by this effective mass represent the target in $O(m_{\text{th}}/T)$ of scattering events. Therefore the scattering rate, and with it all long-timescale dynamical quantities, receive $O(\sqrt{N})$ corrections. Actually, both this thermal mass and the mass dependence of the shear viscosity were mentioned by Jeon [5], so his 1995 paper technically represented an NLO calculation, not just the leading-order one he claimed. But this was not mentioned (or recognized) in that paper, so we feel that a clear discussion is warranted.

II. LEADING-ORDER REVIEW

First we review Jeon’s leading order calculation, emphasizing where higher order corrections can arise. The shear viscosity is defined as a correction to the (local) equilibrium form for the stress tensor when a fluid undergoes nonuniform flow. For a flow velocity $v_i$ satisfying $\partial_i v_i = 0$, the stress tensor in the local rest frame is

$$T_{ij} = P\delta_{ij} - \eta \left[ \partial_i v_j + \partial_j v_i - \frac{2\delta_{ij}}{3} \partial_k v_k \right] + O(\partial^2), \quad (1)$$

with $\eta$ the viscosity. A fluctuation-dissipation relation (Kubo formula) relates this coefficient to a certain stress-stress correlation function [15]:

$$\eta = \frac{1}{2\omega} \lim_{\omega \rightarrow 0} \lim_{k \rightarrow 0} \int d^4x e^{i(k_i x_i - \omega x_0)} \frac{1}{\omega} \left[ \pi_{ij}, \pi_{ij} \right], \quad (2)$$

where $\pi_{ij} \equiv T_{ij} - \frac{\delta_{ij}}{3} T_{kk}$. This relation is the starting point for a perturbative analysis. One wants the correla-
In writing Bose stimulation functions we have assumed all bosonic species; we have also neglected higher order scattering processes, as justified by our expansion in $\lambda$.

Nonuniform flow velocity means that the equilibrium distribution function would be

$$f_0(x, p) = \left( \frac{\exp[(p^0 - v \cdot p)/\sqrt{T(1 - v^2)}] - 1}{T(1 - v^2)} \right)^{-1}$$

and the spatial derivative in Eq. (4) acts on this to give $(p_{\alpha}p_{\beta}\partial_{\alpha}v_{\beta})f_0(1+f_0)/\langle p^3T \rangle$. This term drives the system from equilibrium. The collision term vanishes if $f = f_0$ but is linear in departures $\delta f = f - f_0$; such a departure will arise and grow until the collision term and the gradient term cancel (remember that we want the vanishing frequency or long-time behavior). The departure from equilibrium will be of the same structural form as the spatial derivative term:

$$f(x, p) = f_0 + (\partial_i v_j)\frac{p_ip_j}{p^0}f_0(1+f_0)\chi(|p|),$$

with $\chi(|p|)$ a function of the magnitude of $p$ only, determined by solving Eqs. (4,5). This is most conveniently done by defining an inner product $\langle f|g \rangle \equiv \int \frac{d^3p}{(2\pi)^3}f(p)g(p)$ and a functional $Q[\chi] \equiv$

$$Q[\chi] = T\langle |\chi|^2/p^0 \rangle - \frac{1}{2}\chi\langle C_{\text{lin}}\chi \rangle,$$

where $C_{\text{lin}}$ is a linearization of the collision operator given below. The $\chi$ which extremizes $Q$ solves the Boltzmann equation, and the extremal value of $Q$ is $15\eta/2$.

### III. SUBLEADING CORRECTIONS

The advantage of the diagrams is that they help us see where the perturbative argument that further corrections are down by $O(\lambda)$ can break down. Specifically, for generic momenta an extra loop is $O(\lambda)$. The self-energy resummation is an $O(\lambda)$ correction unless the propagator momentum is soft or close to the light-cone. Near the light-cone, the width correction is essential, but it is already incorporated in the leading-order calculation and it naively receives only $O(\lambda)$ corrections from higher order diagrams. The one-loop self-energy leads to a mass correction mentioned in the introduction, $m_{th}^2 = \frac{\sqrt{\pi}\lambda T^2}{8\pi^2}$ for $N$ component scalar theory with $O(N)$ symmetry and interaction $\frac{1}{2}\langle \phi_n\phi_n \rangle^2$. This mass correction shifts the propagating pole by $O(\lambda)$ for generic momentum $p \sim T$.

However, for $p \sim m_{th} \sim T\sqrt{N}\lambda$, the shift to the propagating pole is an $O(1)$ correction. Therefore, in any diagram where one of the lines is soft, $p \sim m_{th}$, there will be $O(1)$ interaction corrections not taken into account in the leading-order analysis.

How important are soft momenta? It turns out that $\chi(p)$ vanishes as $\chi(p) \propto p$ at small momenta, meaning that particles at these momenta are very near equilibrium and do not contribute significantly to $T_{ij} - P_{ij}$, despite...
the \( f_0(1+f_0) \) enhancement factor. Therefore the first term in Eq. (7) is only sensitive to dispersion corrections at \( O(\lambda) \). However, such particles are of some importance as scatterers. To see this, first let us estimate what fraction of all particles are so soft. The total particle number density is \( n_{\text{tot}} = N \int (d^3p)/(2\pi)^3 f_0 \sim N T^3 \). The soft contribution is \( n_{\text{soft}} \sim N \int (d^3p)/(2\pi)^3 f_0 \Theta(m_\text{th} - p) \sim N m_{\text{th}}^2 T / (N T) \). (Phase space gives three factors of \( m_{\text{th}} \) but the distribution function is \( f_0 \sim T / m_{\text{th}} \) in the soft region.) Soft particles are therefore an \( O(N \lambda) \) fraction of all particles. In addition, scattering cross sections generically scale as \( \sigma \sim 1/s \) (the Mandelstam variable), and when \( k = O(T) \) particle scatters from a \( p = O(m_{\text{th}}) \) particle, \( s \sim -2p_\mu k^\mu \sim m_{\text{th}} T \) is smaller than \( s \sim T^2 \) valid for generic scattering. Therefore the cross-section is enhanced by a \( T/m_{\text{th}} \) factor, and the relative rate of scatterings from a soft target is actually \( O(m_{\text{th}}/T) = O(\sqrt{N \lambda}) \).

So soft scatterings are an \( O(\sqrt{N \lambda}) \) fraction of all scatterings. Do they matter to equilibration? The answer is yes; while the incoming particle is soft, most of the phase space for the scattering shares the final state energies roughly equally. This means that the hard particle retains its direction of flight but substantially changes its energy, and since the departure from equilibrium has a strong energy dependence, this is relevant. To calculate it, we need to perform the scattering calculation more carefully, incorporating the soft phase-space dispersion relation.

The collision term we need for the functional \( Q[\chi] \) is

\[
\langle \chi | C_{\text{lin}} | \chi \rangle \equiv \int \frac{d^3p_1 d^3k d^3p' d^3k'}{(2\pi)^{12} 2p_1^0 2k^0 2p'^0 2k'^0 T^5} \times (2\pi)^3 \delta^4(p+k-p'-k') \times \frac{1}{8} |\mathcal{M}|^2 f_0(p_1)f_0(k)[1+f_0(p')][1+f_0(k')] \times (\chi_{ij}(p) + \chi_{ij}(k) - \chi_{ij}(p') - \chi_{ij}(k'))^2, \tag{8}
\]

\[
\chi_{ij}(p) = \frac{p_ip_j - \delta_{ij} p^2}{p^2} \chi(p). \tag{9}
\]

The phase space is most conveniently dealt with in terms of the total energy \( \omega = p^0 + k^0 \) and momentum \( q = |p+k| \), two particle momenta, and an azimuthal angle. For massless dispersion relations, this “c-channel” parametrization reduces to [12]

\[
\int \frac{d^3p_1 d^3k d^3p' d^3k'}{(2\pi)^{12} 2p_1^0 2k^0 2p'^0 2k'^0 2\pi^6} (2\pi)^4 \delta^4(p+k-p'-k') = \frac{1}{2^8 \pi^5} \int_0^\infty d\omega \int_0^{\omega} dp \int_0^{\omega} dp' \int_0^{2\pi} d\phi / 2\pi. \tag{10}
\]

The relations between these variables and the relative angles of the particles are given in [12], and can be used to evaluate Eq. (8).

The particle energies are \( p, p', \omega - p, \omega - p' \). The \( \omega \sim m \) region is \( m^2 \) suppressed, as all four variables must be \( \sim m \). (There is a \( (T/m)^0 \) enhancement from the population functions, but a \( (m/T)^4 \) suppression from the small value of \( \chi_{ij}^2 \), so this region really is \( O(\lambda^2) \).) A region with one soft particle is \( m^2 \) suppressed; for instance, for soft \( p_i \), both \( p_i \) and \( \omega - q \) must be \( O(m) \). However, one of the statistical functions in Eq. (8) is \( O(T/m) \) in this case, and there is no suppression from the values of \( \chi_{ij} \). Therefore the contribution is \( O(m/T) \), as stated above. Note however that two soft particles (say, \( p, p' \sim m \)) is \( O(m^2) \) suppressed, compensated by two powers of population functions but further suppressed by cancellations in the \( \chi \) functions; so this region is \( O(\lambda) \) at most.

For the massive case it is more convenient to work in terms of particle energies rather than momenta. As just discussed, we need only consider the case where one particle is soft. In the small \( p^0 \) region, and taking \( \omega \gg m_{\text{th}} \), the \( p^0, q \) phase space is modified by

\[
\int dq \int dp = \int dp \int dq, \tag{11}
\]

with \( p \equiv \sqrt{(p^0)^2 - m_{\text{th}}^2} \). If we consider the difference between the collision integral with and without massive dispersion, we want the difference between these integrations. In this difference region, we can take \( p^0 \) small, allowing the substitution \( f_0(p^0) = \frac{T}{p^0} \) and \( f(k) = f(\omega) \). Furthermore, we may neglect \( \chi_{ij}(p) \) and set \( \chi_{ij}(k) = (k_i k_j / \omega) \chi(\omega) \). In this region, \( k \) and the final state momenta \( p', k' \) are collinear, so

\[
(\chi_{ij}(p) + \chi_{ij}(k) - \chi_{ij}(p') - \chi_{ij}(k'))^2 \sim \omega \chi(\omega) - p' \chi(p') - (\omega - p') \chi(\omega - p'). \tag{12}
\]

The \( q, p^0 \) and \( \omega, p' \) integrals then factorize, such that

\[
\langle \chi | C_{\text{lin, m=0}} | \chi \rangle - \langle \chi | C_{\text{lin, m}} | \chi \rangle = \left[ \int dp \int_{\omega - 2p^0}^{\omega} dq / p^0 - \int dp \int_{\omega - p^0 - p}^{\omega - p^0 + p} dq / p^0 \right] \times |\mathcal{M}|^2 \int_0^{\omega} dq \int dp f_0(\omega) e^{i p_0 f_0(p)} f_0(p), \tag{13}
\]

Here the factor of 4 counts the fact that there are 4 such “corners” of phase space where a particle becomes soft, and we have only considered the contribution of one of them. Note that the \( p' \) integral is insensitive to the edges \( p' \sim 0 \) and \( p' \sim \omega \) of its range, so there is no problem of overcounting. The bracketed integral gives \( \pi m_{\text{th}} T \).

We want to perturb in the small mass. Therefore it is fair to write the collision operator as

\[
C_{\text{lin}} = C_{\text{lin, m=0}} + \delta C \tag{14}
\]

with \( \delta C \) equal to minus the expression in Eq. (13). The correction to \( Q \) in Eq. (7) is therefore \(-\frac{1}{2} \langle \chi | \delta C | \chi \rangle \). The change in the extremal value at linear order in \( \delta C \) is just found by evaluating this using the unperturbed extremal value of \( \chi \).
What remains is to solve the variational problem with the unperturbed collision operator and to substitute the result for \( \chi \) into Eq. (13). We handle the variational problem using the multi-parameter Ansatz method developed in [6]. Since we are tuning the function \( \chi \) using the \( m = 0 \) collision operator, the evaluation of \( \langle \chi | \partial C | \chi \rangle \) only improves linearly with the quality of the variational Ansatz. Therefore we must use a large number of the variational functions presented there; going from 1 to 2 basis function expansion, is 0.8 basis function expansion, is 0.

We have shown that the first corrections to the leading-order calculation of the shear viscosity in scalar \( \phi^4 \) theory arise at order \( O(\sqrt{\lambda}) \). They originate because scattering processes are sensitive to soft (\( p \sim T \sqrt{\lambda} \)) particles at that order, and soft particles’ behavior is modified by plasma effects at the \( O(1) \) level. All of these remarks should apply to gauge theories including QCD. However, the details of the calculation in those cases will be more complex, since the dispersion relations are modified in a more complex manner in gauge theories, and in non-abelian gauge theories the mutual interactions between soft particles are also strongly modified [16].

One of the motivations for this work was to analyze the convergence of the perturbative treatment. What we have shown is that the first subleading correction is a small correction provided \( m_{th} \ll 2T \). This is satisfied if \( \lambda \ll \frac{288}{N^2} \). The thermal pressure converges in a comparable sized range [17]. So does vacuum perturbation theory; the ratio of the \( \beta \) function to the coupling \( \frac{\lambda}{N m_{th}} \) is small for \( \lambda \ll \frac{288^2}{N^4} \). Therefore the subleading corrections only become large when the theory is approaching its Landau pole. Since the origin of the leading-order correction is well understood, we might hope that the explicit inclusion of the thermal mass in the external state dispersion, and a resummation of the thermal mass via a gap equation, might improve the convergence of the series, as has been argued for the calculation of the pressure [18]. Aarts and Martinez-Resco have done this in the large \( N \) theory, where it is well justified at leading order in a \( 1/N \) expansion [13].

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