We show, in particular, that, if a finite group $H$ is a retract of any finite group containing $H$ as a verbally closed subgroup, then the centre of $H$ is a direct factor of $H$.

1. Introduction

A subgroup $H$ of a group $G$ is called \textit{verbally closed} \cite{MR14} if any equation of the form

$$w(x, y, \ldots) = h,$$

where $w$ is an element of the free group $F(x, y, \ldots)$ and $h \in H$, having solutions in $G$ has a solution in $H$. If each finite system of equations with coefficients from $H$

$$\{w_1(x, y, \ldots) = 1, \ldots, w_m(x, y, \ldots) = 1\},$$

where $w_i \in H \ast F(x, y, \ldots)$ (and $\ast$ means the free product), having solutions in $G$ has a solution in $H$, then the subgroup $H$ is called \textit{algebraically closed} in $G$.

Algebraic closedness is a stronger property than verbal closedness; however these properties turn out to be equivalent in many cases (see \cite{Rom12}, \cite{RKh13}, \cite{MR14}, \cite{Mazh17}, \cite{RKhK17}, \cite{KM18}, \cite{KMM18}, \cite{Mazh18}, \cite{Bog18}, \cite{Bog19}, \cite{Mazh19}, \cite{RT19}, \cite{RT20}, \cite{Tim21}). A group $H$ is called \textit{strongly verbally closed} \cite{Mazh18} if it is algebraically closed in any group containing $H$ as a verbally closed subgroup. Thus, the verbal closedness is a subgroup property, while the strong verbal closedness is a property of an abstract group. The class of strongly verbally closed groups is fairly wide. For example, the following groups are strongly verbally closed:

- all abelian groups \cite{Mazh18},
- all free groups \cite{KM18},
- all virtually free groups containing no nonidentity finite normal subgroups \cite{KM18}, \cite{KMM18},
- all groups decomposing nontrivially into a free product \cite{Mazh19},
- fundamental groups of all connected surfaces except the Klein bottle \cite{Mazh18}, \cite{K21},

See also \cite{Bog18} and \cite{Bog19} for generalisations of some of these results. In particular, these results imply that

\textit{the infinite dihedral group is strongly verbally closed.}

This innocent looking special case is one of the most difficult to prove and requires a special argument. Actually, this particular proposition is the main result of \cite{KMM18} (used afterwards in \cite{Mazh19}, \cite{Bog18}, and \cite{Bog19}). In Section 5, we complement this fact with a description of finite dihedral strongly verbally closed groups.

Proving the strong verbal closedness of a group is not easy, but disproving this property is not easy either. The literature contains only the following examples of non-strongly-verbally-closed groups:

- the already mentioned fundamental group of the Klein bottle \cite{K21}
- and two nonabelian groups of order eight \cite{KM18}, \cite{RKhK17}.

We generalise the latter example substantially: in Section 3, we prove that

\textit{the centre of any finite strongly verbally closed group is its direct factor}

(and, more generally, strong verbal closedness of a finite group implies stringent constraints on its abelian normal subgroups, see the centre theorem in Section 3). In particular, this means that finite nilpotent nonabelian groups cannot be strongly verbally closed. This leads us to the following question.

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**Question 1.** Does there exist a finitely generated nilpotent nonabelian strongly verbally closed group?

We conjecture that the answer is negative; but so far, we can prove this only for the simplest infinite nilpotent nonabelian group (see Section 4):

\[
\text{the Heisenberg group } UT_3(\mathbb{Z}) = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\
0 & 1 & \mathbb{Z} \\
0 & 0 & 1 \end{pmatrix} \text{ is not strongly verbally closed.}
\]

The algebraic closedness can be characterised structurally if the group \(H\) is *equationally Noetherian* (i.e. any system of equations over \(H\) with finitely many unknowns has the same solutions in \(H\) as some its finite subsystem); namely, the algebraic closedness in this case is equivalent to the “local retractness” [KMM18]:

an equationally Noetherian subgroup \(H\) of a group \(G\) is algebraically closed in \(G\) if and only if \(H\) is a retract (i.e. the image of an endomorphism \(\rho\) of \(G\) such that \(\rho \circ \rho = \rho\) of each finitely generated over \(H\) subgroup of \(G\) (i.e. a subgroup of the form \(\langle H \cup X\rangle\), where the set \(X \subseteq G\) is finite).

Another well-known fact was proven in [RKh13] (Lemma 1.1):

if \(V(G)\) is a verbal subgroup of a group \(G\) and \(H\) is a verbally closed subgroup of \(G\), then \(H \cap V(G) = V(H)\) (i.e. the verbal subgroup of \(H\) corresponding to the same variety) and \(H/V(H) \subseteq G/V(G)\) is verbally closed in \(G/V(G)\).

These two facts imply that

\[
a \text{finite group } H \text{ is strongly verbally closed if and only if it is a retract of any finite group } G \text{ containing } H \text{ as a verbally closed subgroup and satisfying all identities of } H
\]

(because the variety generated by a finite group consists of locally finite groups [Neu69], Theorem 15.71). The following section contains a (short) proof that a finite group with nonabelian monolith (e.g., any finite simple group) is not only strongly verbally closed but also has a stronger property.

**Theorem 1.** Any finite group with nonabelian monolith is strongly verbally closed. Moreover, each such group \(H\) is a retract of any finite group containing \(H\) and satisfying all identities of \(H\).

So, the verbal-closedness requirement can be omitted — it holds automatically in this situation.)

Recall that the *monolith* of a group is the intersection of all nonidentity normal subgroups of this group. Many finite groups with abelian monoliths are also strongly verbally closed and even retracts in groups from the corresponding varieties (see Theorem 2 in the next section); in particular, semidirect products corresponding to irreducible faithful representations of finite \(p^t\)-groups over \(\mathbb{Z}_p\) enjoy this property.

We conclude this short introduction with the following simple observation (which, in particular, explains the words “finitely generated” in the open question above).

**Embedding theorem.** Any group \(H\) embeds into a strongly verbally closed group of cardinality \(|H| + \aleph_0\) that satisfies all identities of \(H\).

**Proof.** If a group \(H\) is algebraically closed in a variety \(M\) (i.e. algebraically closed in any group from \(M\) that contains \(H\)), then \(H\) is strongly verbally closed by the lemma from [RKh13] mentioned above. Thus, the embedding theorem follows from Scott’s theorem [Sco51]:

any group \(H\) from any variety \(M\) embeds into an algebraically closed in \(M\) group from \(M\) of cardinality \(|H| + \aleph_0\).

In [Sco51] (see also [LS80]) this theorem is stated for the variety of all groups, but it is easy to see that the (simple) proof works without any modifications for any variety (and even for any class of groups closed with respect to unions of ascending chains of groups).

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2. Monolithic groups

Let us recall some terminology (see [Neu69]):

- the variety generated by a class of groups \( \mathcal{K} \) is the class of all groups satisfying all identities that hold in all groups from \( \mathcal{K} \);
- a group \( X \) is called critical if it is not contained in the variety generated by the class consisting of all proper subgroups of \( X \) and all quotient groups of \( X \) by nontrivial normal subgroups.

**Proof of the nonabelian-monolith theorem.** Let \( M \) be the nonabelian monolith of a subgroup \( H \) of a finite group \( G \) satisfying all identities of \( H \). We should show that \( H \) is a retract of \( G \). Any nontrivial normal subgroup in a minimal hypothetical counterexample \( G \) nontrivially intersects \( H \) (and, hence, \( M \)). This implies that \( G \) has a unique minimal normal subgroup, and it contains \( M \). Both groups \( H \) and \( G \) are critical, because their monoliths are nonabelian ([KoN66], see also [Neu69], 53.44). Since they generate the same variety, they are isomorphic ([Neu69], 53.33), i.e. \( G = H \). This completes the proof.

For groups with abelian monoliths, the situation becomes more complicated.

**Lemma 1.** Let \( M \neq \{0\} \) be a finite module over a group ring \( \mathbb{Z}_p^m[G] \), where the prime \( p \) does not divide \( |G| \). Then

1) if the exponent of \( M \) is \( p^k \), and all proper submodules have smaller exponents, then
   a) \( M \) is homogeneous (i.e., a direct sum of cyclics of exponent \( p^k \)),
   b) all its layers \( p^i M/p^{i+1}M \), where \( i \in \{0, \ldots, k-1\} \), are isomorphic simple \( G \)-modules;
   c) elements of \( G \) acting identically on the lower layer act identically on the entire module \( M \);
2) if \( M \) does not decompose into a direct sum nontrivially, then all its proper submodules have smaller exponents.

**Proof.**

1) Put \( L = \{ x \in M \mid p^{k-1}x = 0 \} \). By Maschke’s theorem, we have a decomposition \( M/pM = L/pM \oplus N/pM \) for a \( G \)-module \( N \). The submodule \( N \) is of exponent \( p^k \), because \( L \) has smaller exponent. Therefore, \( L = pM \) by virtue of the minimality condition 1). This proves a).

The module \( M/pM \) is simple because of the same minimality condition: the preimage in \( M \) of any nonzero submodule of \( M/pM \) has exponent \( p^k \). By virtue of the homogeneity, the mapping

\[
M/pM \rightarrow M/p^{s+1}M, \quad a + pM \mapsto p^s a + p^{s+1}M
\]

is a well-defined isomorphism. This proves b). The property c) follows from b): an element acting identically on the lower layer acts identically on all layers; and an automorphisms of an abelian \( p \)-group acting identically on layers form a \( p \)-group; therefore, c) follows from the condition that \( |G| \) is not a multiple of \( p \).

2) Let us choose a minimal submodule \( N \subseteq M \) of exponent \( p^k \) equal to the exponent of \( M \). The homogeneity implies that \( N \) has a direct complement in the abelian group \( M \). Hence, it has a direct module complement (this is a simple generalisation of Maschke’s theorem, see, e.g., [Pas83], Lemma 1.1). Since \( M \) is indecomposable, we obtain that \( N = M \).

**Theorem 2.** Suppose that a finite group \( H \) contains a normal subgroup \( C \) such that \( C \) coincides with its centraliser, does not decomposes into a direct product of nontrivial subgroups normal in \( H \), and \( |C| \) is coprime to \( |H/C| \). Then \( H \) is a retract of any finite group \( G \supseteq H \) satisfying all identities of \( H \). In particular, \( H \) is strongly verbally closed.

**Proof.** Lemma 1 shows that \( H \) is monolithic, \( C \) is the centraliser of the monolith \( M = \{ c \in C \mid c^p = 1 \} \) and decomposes into a direct product of cyclics of equal order \( p^k \). Note also that the subgroup \( C \) is Sylow, because its order is coprime to the index.

We can assume that any nontrivial normal subgroup of \( G \) contains \( M \), and \( G \) is monolithic with a monolith \( L \supseteq M \) (because this is valid for any minimal hypothetical counterexample \( G \)).

Note that \( M \) is contained in the abelian group \( S = H^m \), where \( m = |G : C| \). Since \( G \in \text{var} \, N \), we have that \( M \subseteq G^m = S \), where \( S \) is a (normal) abelian Sylow subgroup of \( G \), and, hence, \( L \subseteq S \). The abelian Sylow subgroup \( S \) is a direct factor of its centraliser by the Schur–Zassenhaus theorem: \( C_G(S) = S \times U \). The
subgroup $U$ is characteristic (and even verbal) in $C_G(S)$, therefore, $U \unlhd G$. But $U \cap M = \{1\}$, i.e. $U = \{1\}$, since $G$ is monolithic. Thus, the Sylow $p$-subgroup $S$ of $G$ coincides with its centraliser. Therefore, Lemma 1 implies that $S$ is a homogeneous group of exponent $I$, and $l = k$, because $G \in \text{var } N$; moreover, $S = C_G(L)$ (by Assertion 1) c) of Lemma 1).

To prove that $G = H$, it remains to show that $|L| = |M|$ and $|G/C_G(L)| = |H/C_H(M)|$. Since the left-hand sides of these equalities do not exceed the right-hand sides (because $H$ is a subgroup of $G$), the equalities follow from Lemma 53.25 of [Neu69], which says, in particular, that

the monolith $L$ of any group $G$ from the variety generated by a finite group $H$ is isomorphic to the monolith $N$ of some factor $F$ (i.e. a quotient group of a subgroup) of $H$, and $G/C_G(L) \simeq F/C_F(N)$.

This means, that in the case under consideration, $L = M$ and $G/C_G(L) \simeq H/C_H(M)$, which completes the proof.

Theorems 1 and 2 suggest the following definition. We call a group $H$ a strong retract if it is a retract of any group $G \supseteq H$ from the variety var $H$. The property of being a strong retract is a property of an abstract group $H$ (stronger than the strong verbal closedness).

**Proposition.** A finite subgroup $H$ of a group $G$ is a retract if and only if $H$ is a retract of any finitely generated subgroup of $G$ containing $H$. In particular, the following groups are strong retracts:

- every finite group with nonabelian monolith
- and any finite group $H$ containing a normal subgroup $C$ such that $C$ coincides with its centraliser, does not decomposes into a direct product of nontrivial subgroups normal in $H$, and $|C|$ is coprime to $|H/C|$.

**Proof.** The assertion “In particular” follows immediately from the main assertion and Theorems 1 and 2. The main assertion is a corollary of Mal’cev’s local theorem (see [KaM82], Theorem 24.3.1):

if an object-universal formula $\Phi$ is true on subsystems which form

a local covering of an algebraic system $A$, then $\Phi$ is true on $A$.

It suffices to apply this theorem to the formula “there exists a retraction $G \to H$” on the algebraic system $(G, \cdot, -^1, h_1, \ldots, h_n)$, where $\{h_1, \ldots, h_n\} = H$; note that a retraction $\rho: G \to H$ corresponds to a binary predicate $\rho(x, y)$ (the graph of the retraction), and the condition “$\rho$ is a graph of a retraction” can be written as a universal formula (because $H$ is finite).

**Question 2.** What is an arbitrary finite strong retract?

3. The centres of finite strongly verbally closed groups

**Approximation lemma.** For any finite elementary abelian $p$-group $C$ (where $p$ is prime) and any positive integer $k$, there exists $t \geq k$ such that the direct product $P = \prod_{i=1}^{t} C_i$ of copies $C_i$ of $C$ contains a subgroup $R$ invariant with respect to the diagonal action on $P$ of the endomorphism algebra $\text{End } C \simeq M_d(\mathbb{Z}_p)$ (where $d = \log_p |C|$) of $C$ with the following properties:

a) $R$ is contained in the union of kernels $K_j$ of the natural retractions (projections) $P \to C_j$, 
b) but $R \cdot \prod_{j \notin J} C_j = P$ for any subset $J \subseteq \{1, \ldots, t\}$ of cardinality $k$; 
b') moreover, each such $J$ is contained in a set $J' \supseteq J$ such that $P = R \times \prod_{j \notin J'} C_j$; and there exist integers $n_{ij}$ such that the projection $\pi: P \to \prod_{j \notin J'} C_j$ with kernel $R$ acts as: $C_i \supseteq c_i \mapsto \prod_{j \notin J'} c_j^{n_{ij}}$, where $c_j \in C_j$ is the element corresponding to $c_i$ under the isomorphism $C_i \simeq C \simeq C_j$.

**Proof.** We regard the group $C$ (assumed to be nontrivial) as the additive group of the vector space $\mathbb{Z}_p^d$, and the group $P$ — as the group of all mappings from $X = V \setminus \{0\}$ to $\mathbb{Z}_p^d$, where $V$ is a finite vector space (of dimension $n$ chosen below) over $\mathbb{Z}_p$ (so, $C_i$ is the set of mappings vanishing at all but one vectors of $X$, i.e.
\( t = p^n - 1 \). As \( R \subseteq P \), we take the set of mappings defined by polynomials of degree at most \( r \) without free terms:

\[
R = R_r = \left\{ (\bar{x}_1, \ldots, \bar{x}_n) \mapsto (f_1(\bar{x}_1, \ldots, \bar{x}_n), \ldots, f_d(\bar{x}_1, \ldots, \bar{x}_n)) \mid f_i \in \mathbb{Z}_p[x_1, \ldots, x_n], \deg f_i \leq r, f_i(0, \ldots, 0) = 0 \right\}.
\]

Here \((\bar{x}_1, \ldots, \bar{x}_n)\) are coordinates of a vector \( v \in V \) with respect to some basis; but it is easy to see that the definition of \( R \) does not depend either on the choice of a basis in \( V \) or on the choice of a basis in \( \mathbb{Z}_p^d \) that ensures invariance of \( R \) with respect to the “diagonal” action of \( \text{GL}_d(\mathbb{Z}_p) \). By the same reason, \( R \) is \( M_d(\mathbb{Z}_p) \)-invariant (for any field \( F \), \( \text{GL}_d(F) \)-invariance implies \( M_d(F) \)-invariance, because any matrix decomposes into a sum of nonsingular matrices).

The Chevalley theorem ([Che35], see also, e.g., [Lan68], Chapter 5, Exercise 6) says that

polynomials \( f_1, \ldots, f_d \in F[x_1, \ldots, x_n] \) without free terms
over a finite field \( F \) has a common nonzero root if \( n > \sum \deg f_j \).

If we choose the space \( V \) (i.e. the integer \( t = p^\dim V - 1 = p^n - 1 \)) and the integer \( r \) such that \( n = \dim V > rd \), then property a) holds, because any mapping \( f \in R \) vanishes at a vector from \( X \), i.e. the projection of \( f \) to a factor \( C_i \) is zero.

To prove b), we should find, for any vectors \( v_1, \ldots, v_k \in X \) and \( w_1, \ldots, w_k \in \mathbb{Z}_p^d \), a mapping

\[
f = f_{v_1, w_1, \ldots, v_k, w_k} \in R
\]

such that \( f(v_i) = w_i \) for all \( i \) (because then, for any mapping \( g \in P \), we have \( g = f_{v_1, \ldots, g(v_1), \ldots, v_k, \ldots, g(v_k), h} \), where \( h(v_i) = 0 \), as required).

Since the definition of \( R \) does not depend on the choice of basis in \( V \), we can assume that, for all vectors \( v_i \), all coordinates, but the first \( k \) ones, are zero. Any mapping from a \( k \)-dimensional vector space over \( \mathbb{Z}_p \) to \( \mathbb{Z}_p \) is defined by a polynomial of degree at most \( p - 1 \) in each variable, i.e. the total degree of such polynomial is at most \( k(p - 1) \). Therefore, there exist polynomials \( f_1, \ldots, f_d \in \mathbb{Z}_p[x_1, \ldots, x_k] \) such that, for all \( i \in \{1, \ldots, k\} \) and \( j \in \{1, \ldots, d\} \),

\[
\text{the value of } f_j \text{ at the first } k \text{ coordinates of } v_i = (j\text{th coordinate of } w_i),
\]

\[
f_j(0, \ldots, 0) = 0, \quad \text{and } \deg f_j \leq k(p - 1).
\]

Therefore, if one choose \( r \geq k(p-1) \), then the mapping \( f: (\bar{x}_1, \ldots, \bar{x}_n) \mapsto (f_1(\bar{x}_1, \ldots, \bar{x}_k), \ldots, f_d(\bar{x}_1, \ldots, \bar{x}_k)) \) lies in \( R \) and maps \( v_i \) to \( w_i \), which completes the proof of b).

Condition b') follows from b), because the modules \( C_j \) are irreducible submodules of the semisimple \( M_d(\mathbb{Z}_p) \)-module \( P \); therefore, it remains to recall the following well known (and easy-to-prove) general fact about modules:

if \( R \) is a submodule of a semisimple module \( P = R + \bigoplus_{i \in I} C_i \), where the modules \( C_i \) are irreducible, then \( P = R \bigoplus \bigoplus_{i \in I'} C_i \) for some subset \( I' \subseteq I \).

To explain that the projection of \( P \) onto \( \bigotimes_{j \notin J'} C_j \) is defined by an integer matrix, it suffices to note that this projection is an endomorphism of the \( M_d(\mathbb{Z}_p) \)-module and apply the following well-known general fact on endomorphisms of semisimple modules:

any endomorphism of a module \( \bigoplus C_i \) over a semisimple algebra \( A \), where modules \( C_i \) are irreducible, is defined by a matrix with entries from the centre of \( A \),
i.e., for any endomorphism \( \varphi \), there exists a matrix \( M_\varphi = (m_{ij}) \) with elements from the centre of \( A \) such that, for each \( c_i \in C_i \), we have \( \varphi(c_i) = \sum_{j: C_j = C_i} m_{ij} c_j \), where \( c_j \in C_j \) are elements corresponding to \( c_i \) under a (fixed) isomorphism \( C_j \simeq C_i \). This completes the proof of the lemma.

A similar proposition holds for any finite fields. We state it for a future reference (but do not use it in this paper). The proof can be easily obtained from the argument above by obvious alterations.
Henceforth, the words “subgroup” and “centraliser” refer to the fibered product \( \bigoplus_{j \notin J} C_j \) of copies \( C_i \) of \( C \) with the following properties:

a) \( R \) is contained in the union of kernels \( K_j \) of the natural projections \( P \to C_j \);

b) but \( R + \bigoplus_{j \notin J} C_j = P \) for any subset \( J \subseteq \{1, \ldots, t\} \) of cardinality \( k \);

b') moreover, each such \( J \) is contained in a set \( J' \supseteq J \) such that \( P = R \oplus \bigoplus_{j \notin J'} C_j \); and there exist \( n_{ij} \in F \)

such that the projection \( \pi: P \to \bigoplus_{j \notin J'} C_j \) with kernel \( R \) acts as \( C_i \ni c_i \mapsto \sum_j n_{ij}c_j \), where \( c_j \in C_j \) are vectors corresponding to \( c_i \) under the isomorphism \( C_i \simeq C \simeq C_j \).

**Centre theorem.** The centre of any finite strongly verbally closed group \( H \) is a direct factor of \( H \). Moreover, for any normal subgroup \( N \) of a strongly verbally closed group \( H \), the centre \( Z\left(C\left(Z(N)\right)\right) \) of the centraliser \( C(Z(N)) \) of the centre \( Z(N) \) of \( N \) is a direct factor of this centraliser, and some complement is normal in \( H \):

\[
\frac{C(Z(N))}{\bigcup_{N} Z(N)} = \bigcup_{D \triangleleft H} \frac{Z(C(Z(N)))}{Z(N)} \times D
\]

**Proof.** The vertical inclusions are valid for any subgroup of any group (we added them to the statement for clarity).

Denote \( L = C(Z(N)) \). It suffices, for each prime \( p \), find a homomorphism \( \psi_p: L \to Z(L) \), commuting with the action of \( H \) on \( L \) by conjugations and injective on the \( p \)-component \( Z_p(L) \) of the centre \( Z(L) \) of \( L \), because then the homomorphism \( \psi: x \mapsto \prod_p \psi_p(\pi_p(x)) \) (where \( \pi_p: Z(L) \to Z_p(L) \) is the projection onto the \( p \)-component) is injective on \( Z(L) \) and, therefore, its kernel is the required complement \( D \).

Suppose that there is no such homomorphisms \( \psi_p \) for some \( p \), i.e. any homomorphism \( L \to Z(L) \) commuting with the action of \( H \) is not injective on \( Z_p(L) \) and, hence, on the maximal elementary subgroup \( C \subseteq Z_p(L) \). We should show that the group \( H \) is not strongly verbally closed.

Let us choose \( t \) by the approximation lemma applied to \( C \) (for some integer \( k \) to be specified later), and consider the fibered product

\[
Q = \left\{ (h_1, \ldots, h_t) \in H^t \mid h_1L = \ldots = h_tL \right\}
\]

of \( t \) copies of \( H \).

As an overgroup \( G \supset H \), we take the quotient group \( G = Q/R \), where the subgroup \( R \subseteq C^t \) is chosen by the approximation lemma (\( R \) is normal in \( Q \), because \( R \) is invariant with respect to the diagonal action of \( \text{Aut} C \); and the conjugation action of \( Q \) on \( P \) is diagonal, because \( L^t \) commutes with \( P = C^t \)).

Then \( H \) embeds into \( G \) diagonally: \( h \mapsto (h, \ldots, h) \). This homomorphism is an embedding not only into \( Q \) but also into \( G \) by the property a) of the approximation lemma, because all projections of a nontrivial diagonal element of \( Q \) are nontrivial.

This diagonal subgroup \( H \subseteq G \) is verbally closed in \( G \). Indeed, if an equation \( w(x_1, \ldots, x_n) = h \) is solvable in \( G \) and \( (\bar{x}_1, \ldots, \bar{x}_n) \in H^t \) is a preimage of a solution, then \( w(\bar{x}_1, \ldots, \bar{x}_n) = (hc_1, \ldots, hc_t) \), where \( (c_1, \ldots, c_t) \in R \) and, therefore, by the property a) \( c_i = 1 \) for some \( i \), i.e. the \( i \)th coordinates of the tuple \((\bar{x}_1, \ldots, \bar{x}_n)\) form a solution to the equation \( w(x_1, \ldots, x_n) = h \) in \( H \), as required.

It remains to show, that the diagonal subgroup \( H \subseteq G \) is not a retract. Let \( \rho: G \to H \) be a hypothetical retraction and let \( \rho: Q \to H \) be its composition with the natural epimorphism \( Q \to Q/R = G \).

Henceforth, the words “subgroup” and “centraliser” refer to the fibered product \( Q \) (which contains the diagonally embedded subgroup \( H \)); the centraliser of a set \( X \) in \( H \) is denoted by \( C_H(X) \), i.e. \( C_H(X) = C(X) \cap H \). If \( U \) is a subgroup of \( L \), then the symbol \( U_i \), where \( i = 1, \ldots, t \), denotes the corresponding subgroup \( \{(1, \ldots, 1, u, 1, \ldots, 1) \mid u \in U\} \) of \( Q \).
Let us verify that
\[ \hat{\rho}(L_i) \subseteq C_H(C_H(L)) = L \quad \text{for each } i. \] (*)

Indeed, if \( h \in C_H(L) \), then \( h \) commutes with each component of each element of \( L \), i.e. \( h \) commutes with \( L_i \).

Applying the retraction \( \hat{\rho} \) to this relation, we obtain that \( h = \hat{\rho}(h) \) commutes with \( \hat{\rho}(L_i) \), which proves the inclusion in (\( * \)). To prove the equality, note that \( L \) is a centraliser (of a centre of \( N \)), and the triple centraliser of any subgroup in any group coincides with the single centraliser.

On the other hand, the mutual commutator subgroup \([L_i, L_j]\) is trivial for \( i \neq j \). Therefore, we obtain \([\hat{\rho}(L_i), \hat{\rho}(L_j)] = \{1\}\). Hence, we have \( \prod_{j \neq i} \hat{\rho}(L_j) = \{1\} \).

If \( \hat{\rho}(L_i) = \hat{\rho}(L_i) \) for some different \( i \) and \( l \), then \( \hat{\rho}(L_i) \subseteq C_H(L) \), if \( \hat{\rho}(L_i) = \hat{\rho}(L_i) \) for some different \( i \) and \( l \).

Formulae (\( * \)) and (\( ** \)) implies immediately that
\[ \text{if } \hat{\rho}(L_i) = \hat{\rho}(L_i) \text{ for some different } i \text{ and } l, \text{ then } \hat{\rho}(L_i) \subseteq Z(L). \]

Let us take \( k \) in the approximation lemma to be the number of all subgroups of \( H \), and let \( J \) be the set of exclusive numbers \( i \), i.e. such that \( \hat{\rho}(L_i) \neq \hat{\rho}(L_i) \) for any \( l \neq i \). Then, according to (\( b' \)), we have a decomposition
\[ \bigotimes_{i=1}^t C_i = R \times \bigotimes_{i \in I} C_i, \quad \text{where all } i \in I \text{ are non-exclusive}, \]

and the projection \( \pi: \bigotimes_{i=1}^t C_i \rightarrow \bigotimes_{i \in I} C_i \) onto the second factor of this decomposition is defined by an integer matrix \((n_{ij})\) (i.e. \( C_i \ni c_i \mapsto \prod_j c_j^{n_{ij}} \)), where \( c_j \in C_j \) are elements corresponding to \( c_i \) under the isomorphism \( C_i \simeq C \simeq C_j \). This means that the restriction \( \pi: C \rightarrow \bigotimes_{i \in I} C_i \) of \( \pi \) to \( C \) is defined by the formula
\[ \pi: c \mapsto \prod_{j \in I} c_j^{m_j}, \quad \text{where } m_j = \sum_i n_{ij} \text{ and } c_j \in L_j \text{ are elements corresponding to } c \in C. \]

Therefore, the composition
\[ \Psi: C \rightarrow Z(L), \quad \Psi: c \mapsto \prod_{j \in I} c_j^{m_j} \overset{\hat{\rho}}{\mapsto} \prod_{j \in I} \hat{\rho}(c_j^{m_j}) \]

extends to a homomorphism \( \Phi: L \rightarrow Z(L) \) defined by the similar formula:
\[ L \ni g \mapsto \prod_{j \in I} \hat{\rho}(g_j^{m_j}), \quad \text{where } g_j \in L_j \text{ are elements corresponding to } g \in L. \]

(This is a homomorphism, because \( \hat{\rho}(L_j) \), for \( j \in I \), is contained in the abelian group \( Z(L) \).) The homomorphism \( \Phi \) commutes with the action of \( H \) and, therefore, its kernel intersects \( C \) nontrivially by the assumption. Thus, \( \Psi \), being the restriction of \( \Phi \) to \( C \), has a nontrivial kernel. On the other hand, \( \Psi \) is just the identity mapping, as can be seen from its definition (\( \hat{\rho} \circ \pi = \hat{\rho} \), because \( \hat{\rho}(R) = \{1\} \)). The obtained contradiction completes the proof.
4. Dihedral groups

We shall use the following simple facts:
- any group $G$ from the variety generated by a dihedral group of finite order $2n$ or $4n$, where $n$ is odd, decomposes into a semidirect product $C \ltimes Q$, where $C$ is an elementary abelian 2-group (Sylow 2-subgroup of $G$), and $Q$ is an abelian group of exponent $n$ (Hall 2'-subgroup of $G$ or, equivalently, the verbal subgroup generated by squares of all elements of $G$); thus, $Q$ is a $C$-module;
- any finite module $V$ of odd cardinality over any elementary abelian 2-group $D$ decomposes into the direct sum $V = \bigoplus_{\chi \in \hat{V}} V_{\chi}$, where $X$ is the set of all homomorphisms (characters) $C \to \{\pm1\}$, and $V_{\chi} = \{v \in V \mid cv = \chi(c)v \text{ for all } c \in C\}$. 

Semidirect-product lemma. In a semidirect product $G = C \ltimes Q$, where $C$ is an elementary abelian 2-group and $Q$ is a finite abelian group of odd exponent $n'$, the equality
\[ \{g^{2n'/d} \mid g \in G\} = \{g \in G \mid g^d = 1\} \tag{***} \]
holds for any divisor $d$ of $2n'$ such that either $d = 2$ or $d|n'$ and $\gcd(d, n'/d) = 1$.

**Proof.** Suppose that $g = cq$, where $c \in C$ and $q \in Q$. According to the facts stated above we obtain the decomposition $q = qa + q_{c,-}$, where $cq^{-1} = qa + q_{c,-}$, i.e. $q_{c,+} \in \prod_{\chi(c)=1} Q_{\chi}$ and $q_{c,-} \in \prod_{\chi(c)=-1} Q_{\chi}$. Then
\[ g^k = (cq)^k = (cq_{c,+} + q_{c,-})^k = c^k q_{c,+}^k q_{c,-}^{\frac{1}{2}(1+(-1)^k)}. \]
This means that, for $d = 2$, the sets in both sides of (***), consist of all possible products $cq_{c,-}$. If $d$ is odd, then these sets are $\{g^{2n'/d} \mid g \in Q\}$ and $\{q \in Q \mid q^d = 1\}$. In an abelian group $Q$ of exponent $n'$, these sets coincide (if $d$ and $n'/d$ are coprime): the both sets are the first direct factor in the decomposition $Q = \{q \in Q \mid q^d = 1\} \times \{q \in Q \mid q^{n'/d} = 1\}$. This completes the proof.

Dihedral-group theorem. The dihedral group $D_n$ of order $2n$ is strongly verbally closed if and only if $n$ is either infinite or not divisible by four.

Moreover, suppose that $H = D_n = \langle b \rangle_2 \rtimes \langle a \rangle_n$, where $n$ is not a multiple of four, is a dihedral subgroup of a finite group $G$ belonging to the variety generated by $H$. Then the following conditions are equivalent:
1) $H$ is verbally closed in $G$;
2) $H$ is algebraically closed in $G$;
3) $H$ is a retract of $G$;
4) in terms of the decompositions from the beginning of the section ($G = C \ltimes Q$ and $Q = \bigtimes_{\chi} Q_{\chi}$), the order of the $\chi$-component $(a^2)^{2} \chi$ of $a^2 \in H$ equals the order of $a^2$ for some character $\chi : C \to \{\pm1\}$.

**Proof.** The strong verbal closedness of infinite dihedral group is proven in [KMM18]. Finite dihedral groups $D_{4k}$ are not strongly verbally closed by the centre theorem.

It remains to prove the equivalence of Conditions 1) — 4) for finite $n$ not divisible by four. To prove the implication 4) $\implies$ 3), note that the cyclic (normal) subgroup $\langle (a^2)^{\chi} \rangle$ is a direct factor of $Q_{\chi}$ (because its order equals the period of the abelian group $Q_{\chi}$) and, therefore, it is a direct factor of $Q$, and the complement is normal in $G$ (because the action of $G$ on $Q_{\chi}$ is “scalar” and, hence, all subgroups contained in $Q_{\chi}$ are normal in $G$). Let $\pi$ be the composition $G \to \langle (a^2)^{\chi} \rangle \xrightarrow{\pi} \langle a^2 \rangle$ and let us verify that the mapping $\varphi : G = C \ltimes Q \to H$, $c \cdot q \mapsto b^{\frac{1}{2}(1-\chi(c))} \cdot \pi(q)$ is a homomorphism. A mapping from one semidirect product to another $X \ltimes Y \to Z \ltimes T$ of the form $xy \mapsto \alpha(x)\beta(y)$, where $\alpha : X \to Z$ and $\beta : Y \to T$ are homomorphisms, is a homomorphism if $\beta(xy^{-1}) = \alpha(x)\beta(y)\alpha(x)^{-1}$ for all $x \in X$ and $y \in Y$. In the case under consideration, this property holds:
\[ \pi(cqc^{-1}) = \pi(cq_{\chi}c^{-1}) = \pi(q_{\chi}^{x(c)}) = \pi(q_{\chi})^{x(c)} = b^{\frac{1}{2}(1-\chi(c))}\pi(q_{\chi})b^{-\frac{1}{2}(1-\chi(c))} = \alpha(c)\pi(q_{\chi})\alpha(c)^{-1}. \]
If \( n \) is odd, then \( \varphi \) is injective on \( H \), because any nontrivial normal subgroup of \( H \) intersects \( \langle a \rangle = \langle a^2 \rangle \) nontrivially, and the restriction of \( \varphi \) to \( \langle a^2 \rangle \) is injective by Condition 4.

If \( n \) is even, then take any homomorphism \( \gamma: G \to G/Q \to \langle a^{2} \rangle \) such that \( \gamma(a^{2}) = a^{2} \) and consider the map \( \varphi': G \to H = \langle a^{2}, b \rangle \times \langle a^{2} \rangle \), \( g \mapsto \varphi(g) \gamma(g) \), whose restriction to \( H \) is obviously injective. Thus, the composition of \( \varphi \) or \( \varphi' \) (depending on the parity of \( n \)) with an automorphism of \( H \) is the required retraction \( G = C \times Q \to H \).

The implications 3) \( \implies \) 2) \( \implies \) 1) are general facts valid for any groups (see Introduction).

It remains to prove the implication 1) \( \implies \) 4), i.e., to construct an equation of the form \( w(x, y, \ldots) = h \), where \( w \) is an element of the free group \( F(x, y, \ldots) \) and \( h \in H \), with the following properties:

a) it is solvable in \( G \);

b) the solvability of this equation in \( H \) implies Condition 4).

In an explicit form, such an equation looks fairly terrible. For simplification, we do three things:

1. First, we fix an element \( a' \in \langle a \rangle \) whose order is the product of all odd prime divisors of \( n \).

2. Secondly, we shall construct a multi-sort equation (or an equation with typed variables), i.e., an equation in which a type \([d]\) is assigned to each variable \( x[d] \), where \( d \in \{2\} \cup \{ \text{positive divisors of } n \text{ coprime to } n/d \} \); a solution of a multi-sort equation in a group is a substitution of elements of the group (instead of the variables), transforming the equation into a valid equality, where a variable of type \([d]\) is allowed to be substituted only by elements of order dividing \( d \). Let us verify that a multi-sort equation with properties a) and b) can be transformed to a usual equation with these properties: indeed, if we replace each typed variable \( x[d] \) with \( x^{2n'/d} \), where \( n' = n \) if \( n \) is odd, and \( n' = n/2 \) if \( n \) is even, then we obtain a usual equation whose solvability (in \( G \) or \( H \)) is equivalent to the solvability of the initial multi-sort equation in the same group, because \( \{ g^{2n'/d} \mid g \in G \} = \{ g \in G \mid g^{d} = 1 \} \) by the semidirect-product lemma.

3. Thirdly, we write the equation in the module language, i.e., the multiplication of elements from \( Q \) and \( H \) is a substitution of elements of the group (instead of the variables), transforming the equation \( \varphi \) nontrivially, and the restriction of \( \varphi \) to \( \langle a \rangle \) is injective.

For example, for \( n = 15 \), the multi-sort equation \( (x^{[2]} + 2y^{[2]}x^{[2]}) \cdot (3z^{15} + 4t^{[5]}) = a' \) is translated into the usual language as \( x^{15} (z^{6}t^{24})x^{-15} (yx)^{15} (z^{6}t^{24})^{2} (yx)^{-15} = a \). Certainly, the inverse translation is not always possible; but we do not need it: the only multi-sort equation we use will be written in the module language and it has the form similar to the example above: a sum of integer polynomials in variables of type \([2]\) multiplied by variables of odd types equals an element of \( Q \cap H \) (namely, a power of \( a' \)).

We need the following simple identity (almost copied from [KMM18]):

\[
\prod_{c \in C} \left(1 + \chi(c)c\right) \cdot q = 2^{\lvert C \rvert} q_{\chi} \quad \text{for any character } \chi \text{ and any } q \in Q.
\]

To prove it, note that the \( \chi \)-component of the element \( q \) in the left-hand side is multiplied by two \( \lvert C \rvert \) times; while the other components vanish, because, for each character \( \chi' \neq \chi \), there exists \( c \in C \) such that \( \chi(c) = -\chi'(c) \).

Let us decompose the elementary abelian group \( C \) into a direct product of cyclics \( C = \langle c_{1} \rangle_{2} \times \ldots \times \langle c_{m} \rangle_{2} \); for each \( c = \prod c_{i}^{e_{i}} \in C \) (where \( e_{i} \in \{0, 1\} \)), we put \( f_{c}(x_{1}^{[2]}, \ldots, x_{m}^{[2]}) = \prod (x_{i}^{[2]})^{e_{i}} \) and consider the multi-sort equation

\[
\sum_{\chi \in \chi_{C}} \left( \prod_{c \in C} \left(1 + \chi(c)f_{c}(x_{1}^{[2]}, \ldots, x_{m}^{[2]})\right) \right) \cdot y_{\chi}^{l \lvert \langle a'_{\chi} \rangle \rvert} = 2^{\lvert C \rvert} a',
\]

where \( l \) is the minimal positive integer such that \( l \lvert \langle a'_{\chi} \rangle \rvert \) is coprime with \( n/(l \lvert \langle a'_{\chi} \rangle \rvert) \). This equation is solvable in \( G \) just take \( x_{i}^{[2]} = c_{i} \) and \( y_{\chi}^{l \lvert \langle a'_{\chi} \rangle \rvert} = a'_{\chi} \), and Equation (2) becomes the sum over all characters \( \chi \) of Identities (1) with \( q = a'_{\chi} \). This proves Property a).
It remains to study Equation (2) in the dihedral group $H$. A substitution $x_i^{[2]} \to h_i \in H$ determines a homomorphism $\varphi: C \to H/\langle a \rangle$, $c \mapsto f_c(h_1, \ldots, h_m)\langle a \rangle$ and a character $\chi: C \xrightarrow{\varphi} H/\langle a \rangle \xrightarrow{\sim} \pm 1$. Any term in the left-hand side of (2) corresponding to any other character $\chi' \neq \chi$ vanishes, because there exists $c \in C$ such that $\chi'(c) \neq \chi(c)$ and, therefore, the element $1 + \chi'(c)f_c(h_1, \ldots, h_m) = 1 + \chi'(c)\varphi(c) \in \mathbb{Z}[H/\langle a \rangle]$ annihilate the $(H/\langle a \rangle)$-module $(a)$. Therefore, this substitution transforms (2) into

$$ky^t_{\chi}(\langle a_{\chi}^t \rangle) = 2^{\mid C \mid}a'$$

where $k = 2^{|C|}$, as it is easy to see.

If $|\langle a_{\chi}^t \rangle| < |\langle a^t \rangle|$, then

- an odd prime divisor $p$ of $n$ does not divide $|\langle a_{\chi}^t \rangle|$ (by the definition of $a^t$);
- hence, $p$ does not divide $l|\langle a_{\chi}^t \rangle|$ (by the definition of $l$),
- therefore, $y^t_{\chi}(\langle a_{\chi}^t \rangle)$ cannot be substituted by elements of orders divisible by $p$ (by the definition of the solvability of a multi-sort equation);
- so, the equation is not solvable in the dihedral group (because the order of the right-hand side is divisible by $p$).

Thus, if Equation (2) is solvable in the dihedral group, then $|\langle a_{\chi}^t \rangle| = |\langle a^t \rangle|$. But $\langle a^t \rangle$ is the socle (i.e. product of all minimal normal subgroups) of the group $\langle a^2 \rangle$, i.e. the injectivity of the restriction of the homomorphism $a^2 \mapsto (a^2)_{\chi}$ to $\langle a^t \rangle$ implies the injectivity of this homomorphism on the entire group $\langle a^2 \rangle$. Therefore, the solvability of Equation (2) in the dihedral group implies the equality $|\langle (a^2)^t \rangle| = |\langle a^2 \rangle|$, as required.

The following simplest nontrivial example illustrates the proof of the implication 1) $\implies$ 4).

**Example.** Suppose that $G = D_3 \times D_5 = \langle b_3, b_5 \rangle \times \langle b_5, a_5 \rangle$ and $H = D_{15} = \langle b_2 \rangle \times \langle a_{15} \rangle$ embeds into $G$ diagonally: $b = b_2 b_5$ and $a = a_3 a_5$. The subgroup $H$ is not a retract of $G$; so, there must exist an equation solvable in $G$, but not in $H$. Our argument allows us to construct such an equation explicitly.

In the case under consideration, $C = \{1, b_3, b_5, b_2 b_5\}$ (the Klein four-group), $Q = \langle a_{15} \rangle = \langle a_3 \rangle \times \langle a_5 \rangle$, and $a' = a$. There are four characters $C \to \{\pm 1\}$:

- two “important”: $\tau$ and $\pi$: $\tau(b_3) = -1 = -\tau(b_5) = \pi(b_5) = -\pi(b_3)$,
- and two “unimportant”: the trivial character $\varepsilon$ and the character $\delta = \tau \pi$: $\delta(b_3) = -1 = \delta(b_5)$.

Now, $Q_\tau = \langle a_3 \rangle$, $Q_\pi = \langle a_5 \rangle$, $a_\tau = a_3$, and $a_\pi = a_5$ (and $Q_\varepsilon = Q_\delta = \{1\}$). There are

- two variables of type $[2]}: x^{[2]}_3$ and $x^{[2]}_5$, whose translation into the usual language are $x^{15}_3$ and $x^{15}_5$,
- and two odd-type variables: $y^{[3]}_3$ and $y^{[5]}_5$, whose translations are $y^{10}_3$ and $y^{6}_5$.

The words $f_c$ (where $c \in C$) are $f_1 = 1$, $f_{b_3} = x^{15}_3$, $f_{b_5} = x^{15}_5$, and $f_{b_3 b_5} = x^{15}_3 x^{15}_5$. The expressions $1 + \chi(c)f_c \cdot q$ from (2) (where $q$ is an expression in variables taking value in $Q$) are translated from the module language to the group one as:

$$q f_c q^{\chi(c)} f_c^{-1} = \begin{cases} (q f_c)^2, & \text{if } \chi(c) = +1 \text{ (because } f_c^2 = 1 \text{ always)}; \\ [q, f_c], & \text{if } \chi(c) = -1. \end{cases}$$

Thus, Equation (2) takes the form

$$[[[[y^{10}_3 x^{15}_5]^2, x^{15}_3], x^{15}_3 x^{15}_5], [[[y^{12}_3 x^{15}_5]^2, x^{15}_3], x^{15}_3 x^{15}_5]] = a^{16} = a,$$

where the factors in the left-hand side correspond to the characters $\tau$ and $\pi$ (the factors corresponding to unimportant characters are omitted, because they are equal to 1 identically). Using the distributivity, we simplify this equation slightly:

$$[[[y^{20}_3 x^{15}_5]^2, x^{15}_3], [y^{12}_3 x^{15}_5]^2, x^{15}_5], x^{15}_3 x^{15}_5] = a.$$
In $G$, there is a solution: $x_3 = c_3$, $x_5 = c_5$, $y_3 = a_3^2$, $y_5 = a_5$ (i.e. $y_3^{[3]} = a_3$ and $y_5^{[5]} = a_5$). In the dihedral group $H$, there are no solutions, because
- if $x_3^{15} \in \langle a \rangle \neq x_3^{15}$, then the first inner commutator $[(\ldots)^2, x_3^{15}]$ is 1, and the second inner commutator $[(x_3^{15}y_3^{[3]}, x_3^{15})$ lies in $\langle a^3 \rangle$ (for any $y_i$); i.e. the left-hand side of the equation lies in $\langle a^3 \rangle$ and cannot be equal to the right-hand side;
- if $x_3^{15} \not\in \langle a \rangle \ni x_3^{15}$, then the left-hand side lies in $\langle a^5 \rangle$ (by similar reasons) and again cannot be equal to the right-hand side;
- if $x_3^{15} \in \langle a \rangle \ni x_3^{15}$, then the left-hand side is 1 and cannot be equal to the right-hand side.
- if $x_3^{15} \not\in \langle a \rangle \neq x_3^{15}$, then $x_3^{15}x_5^{15} \in \langle a \rangle$ and the left-hand side is 1 because of the outer commutator.

5. The Heisenberg group

**Affine-bilinear-function lemma.** Suppose that $U \supseteq U'$ and $V$ are finitely generated free abelian groups, $f: U \times V \to \mathbb{Z}$ is a bilinear function whose restriction to $U' \times V$ has rank at least two. Then, for any $u \in U$, the set $f(u + U', V) \subseteq \mathbb{Z}$ is a subgroup containing $f(U', V)$.

**Proof.** Since bases in $U'$ and $V$ can be chosen independently from each other, we can assume that the matrix of $f$ is diagonal, and each integer on the diagonal divides the next, because every integer matrix can be transformed to such a form (the Smith normal form) by integer invertible elementary transformations of rows and columns, see, e.g., [Vin99]. Thus, we can assume that the function $(u', v) \mapsto f(u + u', v)$ in coordinates takes the form $(u', v) \mapsto \sum n_i x'_i y'_i + \sum m_i y'_i$ (where $x'_i$ and $y'_i$ are coordinates of vectors $u'$ and $v$, respectively, $m_i, n_i \in \mathbb{Z}$, and $n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ linearly depend on the vector $u$).

By Dirichlet’s theorem, if $n_i \neq 0$, the arithmetic progression $\{n_i x'_i + m_i \mid x'_i \in \mathbb{Z}\}$ contains a number of the form $p_i \cdot \text{GCD}(n_i, m_i)$, where $p_i$ is an arbitrarily large prime. Thus, the primes $p_i$ can be chosen pairwise distinct. Therefore, for suitable $x'_i$, we have $\text{GCD}(n_1 x'_1 + m_1, n_2 x'_2 + m_2, \ldots) = \text{GCD}(n_1, m_1, m_2, \ldots)$ (because $n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ distinct).

Hence, choosing appropriate $y_i$, we obtain $(n_1 x'_1 + m_1 y_1 + n_2 x'_2 + m_2 y_2 + \ldots = \text{GCD}(n_1, m_1, m_2, \ldots).$ This shows that $f(u + U', V) = \text{GCD}(n_1, m_1, m_2, \ldots) \cdot \mathbb{Z} \supseteq n_1 \cdot \mathbb{Z} = f(U', V)$, as required.

**Heisenberg-group theorem.** The Heisenberg group $\text{UT}_3(\mathbb{Z})$ is not strongly verbally closed.

**Proof.** A verbal mapping in a group $H$ is a mapping $H^s \to H$ of the form

$$(h_1, \ldots, h_s) \mapsto w(h_1, \ldots, h_s),$$

where $w(t_1, \ldots, t_s)$ is an element of the free group $F_s = F(t_1, \ldots, t_s)$.

Let us verify that

in the Heisenberg group $H = \text{UT}_3(\mathbb{Z})$, the intersection of the image of each verbal mapping with the centre is a subgroup; and the image of this mapping is a union of some cosets of this subgroup. (3)

Consider a verbal mapping $\varphi: (h_1, \ldots, h_s) \mapsto w(h_1, \ldots, h_s)$ and put

$$h_i = T(x_i, y_i, z_i) \overset{def}{=} \begin{pmatrix} 1 & x_i & z_i \\ 0 & 1 & y_i \\ 0 & 0 & 1 \end{pmatrix}.$$ Then

$$\varphi(h_1, \ldots, h_s) = T(l(x_1, \ldots, x_s), l(y_1, \ldots, y_s), f(x_1, \ldots, x_s; y_1, \ldots, y_s) + l(z_1, \ldots, z_s)),$$

where $l: \mathbb{Z}^s \to \mathbb{Z}$ is a linear function and $f: \mathbb{Z}^s \times \mathbb{Z}^s \to \mathbb{Z}$ is a bilinear function. By an automorphism of the free group $F$, any word $w$ can be reduced to a normal form $w(t_1, \ldots, t_s) = t_1^m w'(t_1, \ldots, t_s)$, where $w'$ lies in the commutator subgroup of $F$. In the normal form, the function $l$ is just $l(x_1, \ldots, x_s) = mx_1$, and the restriction of $f$ to $U' \times V$ is skew-symmetric, i.e. $f(u', u') = 0$ for all $u' \in U'$; indeed, if $u' = (a_1, \ldots, a_s)$ (and, hence, $a_1 = 0$ or $m = 0$), then $f(u', u')$ is determined by the equality

$$w(T(a_1, a_1, 0), \ldots, T(a_s, a_s, 0)) = w'(T(a_1, a_1, 0), \ldots, T(a_s, a_s, 0)) = T(0, 0, f(u', u'))$$

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and, therefore, is zero, since matrices of the form $T(a, a, 0)$ commute and the word $w'$ is commutator.

To prove (3), we should show that

a) $\varphi(H^s) \cap Z(H)$ is a subgroup

b) and, for each element $h = T(a, b, c) = \varphi(\tilde{h}_1, \ldots, \tilde{h}_s) \in \varphi(H^s)$, the coset $h \cdot (\varphi(H^s) \cap Z(H))$ is contained in $\varphi(H^s)$.

The both facts follow from the affine-bilinear function lemma applied to the free abelianian groups

$$U = Z^s \supseteq U' = V = \{u \in Z^s \mid l(u) = 0\}.$$

The condition $\text{rk} f|_{U' \times V} \geq 2$ holds, because the rank of a skew-symmetric function is always even (if the rank is zero, we have nothing to prove).

Let us verify a). We have $\varphi(H^s) \cap Z(H) = \left\{ T(0, 0, f(u, v) + l(w)) \mid u, v, w \in Z^s, l(u) = l(v) = 0 \right\}$.

By the lemma, $\{f(U', V)\}$ is a subgroup of $Z$. Hence, $\{f(U', V)\} + l(Z^s)$ is a subgroup too (because $l(Z^s)$ is obviously a subgroup).

Let us verify b). We can assume that $b = 0$, because any element $h = T(a, b, c)$ can be reduced to such a form by an automorphism of $H$ (since $H$ is a free nilpotent of class two, and any pair of matrices $\{T(a_1, b_1, c_1), T(a_2, b_2, c_2)\}$, with $a_1 b_2 - a_2 b_1 = \pm 1$ forms a free basis). Now we obtain b) as follows. If $\tilde{h}_1 = T(\tilde{x}_1, \tilde{y}_1, \tilde{z}_1), u = (\tilde{x}_1, \ldots, \tilde{x}_s)$ and $v = (\tilde{y}_1, \ldots, \tilde{y}_s)$, then

$$h \cdot (\varphi(H^s) \cap Z(H)) = T(l(u), 0, f(u, v) + f(U', V) + l(Z^s)) \subseteq T(l(u), 0, f(u + U', V) + l(Z^s))$$

(where the inclusion follows from the lemma). For matrices $\tilde{h}_i = T(\tilde{x}_i + u'_i, v'_i, z'_i)$ we have

$$\varphi(\tilde{h}_1, \ldots, \tilde{h}_s) = T(l(u), 0, f(u + u', v') + l(q')),$$

where $u' = (u'_1, \ldots, u'_s) \in U', v' = (v'_1, \ldots, v'_s) \in U', v' = (v'_1, \ldots, v'_s) \in U'$ are vectors, which we can choose. Thus, $\varphi(H^s) \supseteq T(l(u), 0, f(u + U', V) + l(Z^s)) \supseteq h \cdot (\varphi(H^s) \cap Z(H))$. This completes the proof of (3).

Now, let us take the central product $G = H \times \bar{H} = (H \times \bar{H})/\langle \bar{c}c^{-1} \mid c \in Z(H) \rangle$ of $H$ and its copy $\bar{H}$.

The verbal closedness of $H$ in $G$ follows immediately from (3): if $w(\langle h_1, h'_1 \rangle, \ldots, \langle h_s, h'_s \rangle) = (h, 1)$ in $G$, then $w(h_1, \ldots, h_s) = hc^{-1}$ and $w(h'_1, \ldots, h'_s) = c \in Z(H)$ for some $c \in Z(H)$; and then, according to (3), $h$ lies also in the image of the corresponding verbal mapping. Thus, the verbal closedness is proven.

The group $H$ is equationally Noetherian, as well as any linear group [BMR99]; therefore, if $H$ is strongly verbally closed, then it is a retract of $G$ (see Introduction). But the retraction $G \to H$ is impossible, because $\bar{H}$ commutes with $H$, hence, $\bar{H}$ must be mapped in the centre by a hypothetical retraction, but then $Z(\bar{H}) = Z(H)$ is mapped to $\{1\}$. This contradiction completes the proof.

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