Anomalous Hall effect in superconductors with spin-orbit interaction

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We calculate the anomalous Hall conductance of superconductors with spin-orbit interaction and with either uniform or local magnetization. In the first case we consider a uniform ferromagnetic ordering in a spin triplet superconductor, while in the second case we consider a conventional s-wave spin singlet superconductor with a magnetic impurity (or a diluted set of magnetic impurities). In the latter case we show that the anomalous Hall conductance can be used to track the quantum phase transition, that occurs when the spin coupling between the impurity and electronic spin density exceeds a certain critical value. In both cases we find that for large spin-orbit coupling the superconductivity is destroyed and the Hall conductance oscillates strongly.

I. INTRODUCTION

Anomalous Hall effect (AHE) was observed in metallic ferromagnets long time ago as a Hall current generated by electric field in the absence of external magnetic field [11,2]. Since then several physical mechanisms of the AHE have been proposed, related to the side-jump and skew scattering from impurities [3,5], inhomogeneous internal magnetization [6,7], internal spin-orbit interaction [8], and topology of electron energy bands [9,19]. Theory of AHE attracted much attention recently [3,11–13] because it reveals some very unusual properties of solids such as the existence of monopoles in the momentum space or generation of topological gauge fields.

One of the most intriguing models of AHE is the one based on an intrinsic mechanism [9,10] related to the nontrivial topology of electron energy bands. In frame of this mechanism, the main contribution to AHE is due to electron states well below the Fermi energy [13]. The simplest model, in which this mechanism of AHE can be realized, is the model of a magnetized two-dimensional electron gas with Rashba spin-orbit (SO) interaction [15]. Unfortunately, it turns out that if the system is in the metallic state, i.e., if there is no gap at the Fermi surface, then the contribution of electron states at the Fermi surface can totally compensate the other contributions so that the resulting off-diagonal conductivity is zero [10]. In the opposite case, when the chemical potential lies in the gap, the anomalous Hall conductivity $\sigma_{xy}$ is nonzero and quantized in units of $e^2/h$. The theory of quantized AHE is quite similar to the theory of integer quantum Hall effect, where the gap is due to the Landau quantization in a strong magnetic field [17,18].

In this work we consider a 2D electron gas with nonzero magnetization and Rashba SO interaction. Such a model was used earlier for a description of intrinsic AHE [16]. However, we calculate the AHE in the case when the electron system is additionally superconducting. The superconductivity produces a gap at the Fermi level, suppressing the contribution to AHE from the Fermi surface. Thus, one can expect that only the filled electronic states below the gap contribute to the AHE. The possibility of AHE in superconductors has been already considered in the case of ferromagnet-superconductor double tunnel junctions [19], where side jump and/or skew scattering from impurities have been assumed as possible physical mechanisms responsible for the effect. This is, however, essentially different from our model, where we consider the intrinsic mechanism of AHE. Since in a superconductor charge is not conserved due to the particle-hole mixture, we do not expect any quantization of the anomalous Hall conductance. This was already shown for the usual Hall conductance in conventional superconductors in very high magnetic fields, where the Landau level description is appropriate [20].

Various materials are known to show the coexistence of ferromagnetism and superconductivity [21–28] and, in particular, the presence of spin-orbit interaction due to the lack of spatial inversion symmetry [29,32]. We note, that the possibility of magnetoelectric effects in non-centrosymmetric superconductors was predicted already long time ago [33], where it was shown that a supercurrent should induce a spin polarization and reversely a Zeeman-like term should induce a supercurrent [34] as a result of strong spin-orbit interaction. Other effects due to the interplay of ferromagnetism and superconductivity have also been considered [35,37]. Recently, the interplay between superconductivity, magnetism and spin-orbit interaction (or topological insulators [38,39]) has received additional attention due to the possibility of Majorana edge states in a finite system or inside superconducting vortices [10,42], with its possible applications in topological quantum computation. Moreover, the coexistence of magnetism and...
superconductivity turned out to be interesting also from the point of view of possible applications in spintronics [13, 14].

In this work we consider in section II a spin triplet superconductor while in section III a conventional superconductor with a magnetic impurity [15]. In both cases we analyze the influence of Rashba spin-orbit interaction. In the first case the magnetization is due to a ferromagnetic order, whereas in the second case the system is locally polarized by a magnetic impurity. The latter situation may also be achieved when considering a superconducting film with a magnetic dot juxtaposed. It has been shown before that if the coupling between the magnetic impurity and the spin density of conduction electrons is strong enough, the system becomes magnetized through a first order quantum phase transition [10, 17] that leads to discontinuities in various physical quantities [18]. In both cases we calculate the anomalous Hall conductance. We show that the Hall conductance of a superconductor with the magnetic impurity can be used to reveal the quantum phase transition. Finally, we conclude with section IV.

II. AHE IN A TRIPLET SUPERCONDUCTOR

We consider first a superconductor with a uniform magnetization. Since magnetism and superconductivity compete, a spin singlet superconductor is not stable due to Cooper pair breaking. Therefore, we consider a spin triplet superconductor, where magnetism and superconductivity can coexist. The system is described by the tight-binding model in two dimensions, to which we add a superconducting pairing term with the appropriate symmetry. Additionally, we also include the Rashba spin-orbit term [15], which is generally allowed in non-centrosymmetric materials. Due to the spin-orbit term, a spin singlet component, $\Delta_s$, is generally induced and therefore there is a pairing mixture in the system [19].

We write the electron operators, $\psi_{\vec{k},\sigma}$, in terms of the Bogoliubov operators, $\gamma_{n,\vec{k}}$, as

$$\psi_{\vec{k},\sigma} = \sum_n \left( u_n(\vec{k}, \sigma) \gamma_{n, \vec{k}} - \sigma v_n(\vec{k}, \sigma)^* \gamma_{n, -\vec{k}}^\dagger \right),$$

where $\vec{k}, n$ label the eigenstates of the system. The wave functions and energy eigenvalues satisfy the Bogoliubov – de Gennes equations [30], which can be written as

$$\begin{pmatrix} \epsilon_{\vec{k}} - h_z & \alpha(\sin k_y + i \sin k_x) & -d_x + id_y & d_z + \Delta_s \\ \alpha(\sin k_y - i \sin k_x) & \epsilon_{\vec{k}} + h_z & d_z - \Delta_s & -\epsilon_{\vec{k}} + h_z \\ -d_x - id_y & d_z - \Delta_s & \alpha(\sin k_y - i \sin k_x) & \epsilon_{\vec{k}} - h_z \\ d_z + \Delta_s & d_x - id_y & \alpha(\sin k_y + i \sin k_x) & -\epsilon_{\vec{k}} - h_z \end{pmatrix} \begin{pmatrix} u_n(\vec{k}, \uparrow) \\ u_n(\vec{k}, \downarrow) \\ v_n(-\vec{k}, \uparrow) \\ v_n(-\vec{k}, \downarrow) \end{pmatrix} = \epsilon_{\vec{k},n} \begin{pmatrix} u_n(\vec{k}, \uparrow) \\ u_n(\vec{k}, \downarrow) \\ v_n(-\vec{k}, \uparrow) \\ v_n(-\vec{k}, \downarrow) \end{pmatrix}.$$ (2)

Here, $\epsilon_{\vec{k}} = -2t(\cos k_x + \cos k_y) - \epsilon_F$ is the kinetic part, where $t$ denotes the hopping parameter set in the following as the energy scale, $t = 1$, $\epsilon_F$ is the chemical potential, chosen in the following as $\epsilon_F = -1$, $\vec{k}$ is a wave vector in the $xy$ plane, and we have taken the lattice constant to be unity, $a = 1$. Furthermore, $h_z$ in Eq. (2) is the magnetization, in energy units, along the $z$ direction, while the vector $\vec{d} = (d_x, d_y, d_z)$ is the vector representation of the superconducting pairing ($p$-wave). Finally, the Rashba spin-orbit term is written as $H_R = \vec{s} \cdot \vec{\alpha} = \alpha(\sin k_y \sigma_x - \sin k_x \sigma_y)$, where $\alpha$ is measured in the energy units, and $\sigma_x, \sigma_y$ are the Pauli matrices.

The pairing matrix can be written as

$$\Delta = \begin{pmatrix} \Delta_{\uparrow, \uparrow} & \Delta_{\uparrow, \downarrow} \\ \Delta_{\downarrow, \uparrow} & \Delta_{\downarrow, \downarrow} \end{pmatrix} = \begin{pmatrix} -d_x + id_y & d_z \\ d_z & -d_x - id_y \end{pmatrix}.$$ (3)

Thus, we can write $d_z = (\Delta_{\uparrow, \downarrow} - \Delta_{\downarrow, \uparrow})/2$, $d_y = -i(\Delta_{\uparrow, \downarrow} + \Delta_{\downarrow, \uparrow})/2$, and $d_z = \Delta_{\downarrow, \downarrow}$, while the vector $\vec{q} = i\vec{d} \times \vec{d}^*$ is given by $q_x = Real \left( (\Delta_{\downarrow, \downarrow} + \Delta_{\uparrow, \downarrow}) \Delta_{\uparrow, \downarrow}^* \right)$, $q_y = Im \left( (\Delta_{\uparrow, \downarrow} - \Delta_{\downarrow, \uparrow}) \Delta_{\uparrow, \downarrow}^* \right)$ and $q_z = 1/2 \left[ |\Delta_{\uparrow, \uparrow}|^2 - |\Delta_{\downarrow, \downarrow}|^2 \right]$. When this vector vanishes, the pairing is called unitary. We have verified that considering the s-wave component has generally a very small effect on our results, and therefore we assume $\Delta_s = 0$ in the following.

The energy eigenvalues of Eq. (2) can be written (for $\Delta_s = 0$) as

$$\epsilon_{\vec{k},\alpha_1,\alpha_2} = \alpha_1 \sqrt{z_1 + \alpha_2^2} \sqrt{z_2},$$ (4)

where

$$z_1 = \vec{d} \cdot \vec{s} + \vec{s} \cdot \vec{s} + \epsilon_{k}^2 + h_z^2,$$

$$z_2 = \left( \vec{d} \cdot \vec{s} \right)^2 + (\epsilon_{k}^2 + d_z^2)(\vec{s} \cdot \vec{s} + h_z^2),$$ (5)
and $\alpha_1, \alpha_2 = \pm$.

In the normal phase ($\vec{d} = 0$), the spin-orbit coupling lifts the spin degeneracy of the energy bands in the tight-binding model, except at $\vec{k} = (0, 0), (\pi, \pi)$ and $(0, \pi)$ (and equivalent points). These remaining degeneracies are lifted when including the magnetization. This is shown in Fig. 1, where the two energy bands are shown as a function of momentum for $\lambda_{so} = \alpha/2 = 2$ and various values of $h_z$. As can be seen from Eq. 4, the lowest band is gapless at the points where

$$\langle \vec{s} \cdot \vec{s} + h_z^2 \rangle + \epsilon_k^2 = 2\sqrt{(\vec{s} \cdot \vec{s} + h_z^2) \epsilon_k^2}. \quad (6)$$

In a general case ($\vec{d} \neq 0$), the lowest band has gapless points that are solutions of the equation $z_1 = 2\sqrt{z_2}$, which yields

$$\vec{d} \cdot \vec{d} + \langle \vec{s} \cdot \vec{s} + c_k^2 + h_z^2 \rangle = 2\sqrt{\left(\vec{d} \cdot \vec{s}\right)^2 + (c_k^2 + d_k^2)(\vec{s} \cdot \vec{s} + h_z^2)}. \quad (7)$$

Thus, in the superconducting phase the system is generally gapped. In particular, without the spin-orbit interaction the gapless points are obtained by $\vec{d} \cdot \vec{d} + c_k^2 = 0$ which implies particular values for the chemical potential.

The charge current along a link in the lattice can be obtained by adding a vector potential to the kinetic and spin-orbit terms and taking a functional derivative of the Hamiltonian with respect to the vector potential [50, 52], or through its definition in the charge continuity equation [53]. The zero-momentum charge current in the $\mu = x, y$ direction can be written as

$$j_\mu = \sum_{\vec{k}} \vec{\psi}_k^\dagger V^\mu_{\vec{k}} \vec{\psi}_k^\dagger, \quad (8)$$

where $\vec{\psi}_k = (\psi_{k,\uparrow} \psi_{k,\downarrow})^T$, and

$$V^x = \frac{2e}{\hbar} \left(-\eta_{k,\downarrow}^x I + \lambda_{so} \eta_{k,\uparrow}^x \sigma_y\right)$$
$$V^y = \frac{2e}{\hbar} \left(-\eta_{k,\downarrow}^y I - \lambda_{so} \eta_{k,\uparrow}^y \sigma_x\right) \quad (9)$$

Figure 1: Energy bands in units of the hopping, $t$, as a function of momenta $k_x, k_y$ in the normal phase for $\lambda_{so} = 2$ and various values of the magnetization: from left to right $h_z = 0$ and $h_z = 0.5$ (top); $h_z = 1$ and $h_z = 1.2$ (bottom).
is a velocity matrix operator \[\mathbf{V}_{\mu,\nu}\]. Here \(\eta_{k,+}^\mu = \cos(\vec{k} \cdot \delta_{\mu})\) and \(\eta_{k,-}^\mu = \sin(\vec{k} \cdot \delta_{\mu})\), where \(\delta_{\mu}\) is a vector displacement (in units of the lattice constant) between nearest-neighbors along the \(\mu\) direction. In turn, \(I\) is the \(2 \times 2\) unit matrix.

The Hall conductance can be now calculated using a Kubo like formula \[\text{[55]}\], which in the limit of uniform and stationary current, \(q \to 0\) and \(\omega \to 0\), is given by

\[
\text{Re}(\sigma_{xy}) = -i \frac{e}{h} \sum_{\vec{k}} \sum_{\alpha,\beta} \sum_{\gamma,\delta} \sum_{n,m} \frac{f_{n,\vec{k}} - f_{m,\vec{k}}}{E_{n,\vec{k}} - E_{m,\vec{k}} - i0^+} \left( V_{\vec{k},\alpha,\beta}^x V_{\vec{k},\gamma,\delta}^y u_n(\vec{k},\alpha) u_m(\vec{k},\gamma)* + V_{\vec{k},\alpha,\beta}^y V_{\vec{k},\gamma,\delta}^x u_n(\vec{k},\beta) u_m(\vec{k},\gamma)* \right).
\]

where \(N\) is the number of sites and \(f_{n,\vec{k}}\) is the Fermi function for the state described by \(n\) and \(\vec{k}\). In the normal phase the wave functions \(u\) and \(v\) are decoupled. The presence of superconducting pairing mixes the particle and hole character and, as already mentioned above, charge is no longer a good quantum number. The results for the Hall conductance depend then on the choice of the pairing matrix \[\text{[51, 56]}\] .

Let us now assume that the pairing amplitude is a free parameter. This describes the situations where superconductivity is induced by proximity and therefore no self-consistent solution is implied. This also applies to a situation where \(\sigma_{xy}\) is measured on a normal sample in which superconductivity pairing exists due to proximity effect in the presence of a nearby triplet superconductor. We consider both unitary and nonunitary cases. Then we consider the case, where the pairing amplitude is determined by solving the Bogoliubov – de Gennes equations self-consistently. In the latter case we consider a nonunitary situation, for which the amplitudes \(\Delta_{\uparrow,\downarrow}\) and \(\Delta_{\downarrow,\uparrow}\) are real, to simplify. This in turn implies that \(d_\mu\) is imaginary. In all cases we take \(\Delta_{\uparrow,\downarrow} = 0\) \((d_z = 0)\), which means that only the \(q_z\) component may be nonvanishing.

In Fig. 2 the anomalous Hall conductance in the normal phase (zero pairing amplitude) is plotted as a function of the magnetization \(h_z\) and spin-orbit coupling \(\lambda_{so}\). The Hall conductance vanishes if either the magnetization or the spin-orbit coupling vanishes. Then, the absolute value of the Hall conductance increases as either parameter increases. Dependence on \(h_z\) is more complex, as the conductance reaches a minimum around \(h_z = 1\) \(\sim -\epsilon_F\), which shifts if we change the chemical potential. The minimum in the Hall conductance as a function of the magnetization \(h_z\) (keeping the spin-orbit constant, for instance \(\lambda_{so} = 2\)) is associated with the gaplessness of the spectrum at the point \((0, \pi)\) and the equivalent points (see also Fig. 1).

Now, we consider the superconducting phase. Since the spin-orbit coupling renders the type of pairing undefined (with the mixture of spin triplet and spin singlet pairings), the strength of the triplet pairing is expected to be weakened in comparison to the same superconductor with a vanishing spin-orbit coupling. However, it has been shown before \[\text{[57]}\] that the amplitude of the triplet pairing is not affected by the spin-orbit term when the vector \(\vec{d}\) is parallel to the spin-orbit vector \(\vec{s}\). We have found that this pairing choice leads to results for the anomalous Hall conductance, that are very similar to those for the Hall conductance in the normal phase. This indicates that for this particular case, the superconducting order does not change significantly the Hall conductance, and therefore we do not show the corresponding results. We have also considered other choices of pairing, for which the vector \(\vec{d}\) is not parallel to the spin-orbit vector \(\vec{s}\). We have considered both unitary and non-unitary cases. It is already known for a unitary case \[\text{[57]}\], that even though the amplitude of the triplet coupling is somewhat weakened with respect to the case of vanishing spin-orbit term, it is still finite.

In Fig. 3 we show the anomalous Hall conductance in the superconducting phase for the two choices of the triplet
Figure 3: Anomalous Hall conductance for a spin triplet superconductor. Left panels present Hall conductance as a function of $h_z$ and $\lambda_{so}$ for $d = 1$, whereas right panels as a function of $d$ and $\lambda_{so}$ for $h_z = 0.5$. Top figures correspond to the unitary case, see Eq. [11] while bottom figures correspond to the nonunitary case, Eq. [12].

pairing. We consider a unitary choice given by

$$\Delta_{\uparrow,\uparrow} = d(-\sin k_y + i \sin k_x), \Delta_{\uparrow,\downarrow} = \Delta_{\downarrow,\uparrow} = 0, \Delta_{\downarrow,\downarrow} = d(\sin k_y + i \sin k_x)$$

$$q_x = 0, q_y = 0, q_z = 0.$$  

(11)

and a nonunitary choice given by

$$\Delta_{\uparrow,\uparrow} = d \sin k_x, \Delta_{\uparrow,\downarrow} = \Delta_{\downarrow,\uparrow} = 0, \Delta_{\downarrow,\downarrow} = 0$$

$$q_x = 0, q_y = 0, q_z = \frac{d^2}{2} \sin^2 k_x.$$  

(12)

In the case of unitary coupling (top panels of Fig. 3), $\sigma_{xy} = 0$ if either $\lambda_{so} = 0$ or $h_z = 0$. However, in the case of a non-unitary coupling (bottom panels of Fig. 3), $\sigma_{xy} = 0$ if $\lambda_{so} = 0$, but for a nonzero spin-orbit coupling there is a finite Hall conductance even if $h_z = 0$. In this nonunitary case there is a magnetization induced by the pairing, which leads to a finite $\sigma_{xy}$ in a similar way as in $^3$He.

In the unitary case the energy spectrum has a gap at the Fermi energy. This gap decreases as $\lambda_{so}$ increases. As $\lambda_{so}$ grows, the gap between the first and the second bands seems to decrease slightly and then it increases. In general, one can expects that small gaps between the bands will lead to large contributions to the Hall conductance. In the nonunitary case the energy spectrum also has a gap at the Fermi surface, which is small for small $\lambda_{so}$, increases for slightly larger spin-orbit coupling, but vanishes when $\lambda_{so}$ exceed $\lambda_{so} \sim 0.7$. As $\lambda_{so}$ grows further the gap between the first and second bands increases.

In the case when the superconductivity is intrinsic to the material, we have to solve the Bogoliubov – de Gennes equations self-consistently. We look for a situation of the type

$$\Delta_{\uparrow,\uparrow} = \tilde{d}(-\sin k_y + \sin k_x), \Delta_{\uparrow,\downarrow} = \Delta_{\downarrow,\uparrow} = 0, \Delta_{\downarrow,\downarrow} = \tilde{d}(\sin k_y + \sin k_x)$$

$$q_x = 0, q_y = 0, q_z = \frac{d^2}{2} (-4 \sin k_x \sin k_y).$$  

(13)

where the amplitude $\tilde{d}$ is determined self-consistently for a given magnetization, taking into account that

$$\tilde{d} = \frac{g}{N} \sum \limits_{\vec{k}} (-\sin k_x + \sin k_y) \langle \psi_{k_x} \psi_{-k_y} \rangle.$$  

(14)

where $g$ is the pairing interaction. The corresponding numerical results are shown in Fig. 4. As the left panel shows, the superconductivity is destroyed for large enough spin-orbit coupling. In the right panel we see that the Hall
conductance (as a function of $\lambda_{so}$) decreases with increasing $\lambda_{so}$, and as the transition to the normal phase appears, there are oscillations of the Hall conductance with relatively large amplitudes.

III. AHE IN A CONVENTIONAL SUPERCONDUCTOR WITH MAGNETIC IMPURITY

Consider now a classical spin immersed in a two-dimensional s-wave conventional superconductor. We use now a description of the system in the real space. In the center of the system, $\vec{r} = \vec{l}_c = (x_c, y_c)$, we place a classical spin along the $z$ direction. The kinetic energy part is described by a tight-binding model with hopping amplitude $t$, similarly as in the case of triplet superconductivity. The superconductor pairing is taken as $s$-wave, and the spin-orbit interaction is assumed as in the preceding section. The electron operator is written in terms of the Bogoliubov operators, 

$$\psi(\vec{r}, \sigma) = \sum_n \left( u_n(\vec{r}, \sigma) \gamma_n - \sigma v_n(\vec{r}, \sigma)^* \gamma_n^\dagger \right).$$

(15)

The zero momentum charge current in the $\mu = x, y$ direction can be written as $j_\mu = \sum_{\vec{r}} \bar{\psi}_\vec{r}^\dagger V^\mu \psi_{\vec{r}}$, where $\psi_{\vec{r}} = (\bar{\psi}_{\vec{r}, \uparrow} \quad \psi_{\vec{r}, \downarrow})^T$, and the velocity matrix operators are given by 

$$V^x = \frac{e}{\hbar} \left( it \eta_\uparrow \gamma_x + \lambda_{so} \eta_\uparrow \gamma_y \right)$$

and

$$V^y = \frac{e}{\hbar} \left( it \eta_\uparrow \gamma_y - \lambda_{so} \eta_\uparrow \gamma_x \right).$$

(16)

Here $f(\vec{r}) \eta_\uparrow \eta_\uparrow g(\vec{r}) = f(\vec{r} + \vec{\delta}_\mu) g(\vec{r}) + f(\vec{r}) g(\vec{r} + \vec{\delta}_\mu)$ and $f(\vec{r}) \eta_\uparrow \eta_\downarrow g(\vec{r}) = f(\vec{r} + \vec{\delta}_\mu) g(\vec{r}) - f(\vec{r}) g(\vec{r} + \vec{\delta}_\mu)$, where $\vec{\delta}_\mu$ is a displacement between nearest-neighbors along the $\mu$ direction, while $\sigma_x, \sigma_y$ are Pauli matrices, as above.

The real space wave functions obey the Bogoliubov – de Gennes equations for the energy excitations $\epsilon_n$

$$
\begin{pmatrix}
-h - \epsilon_F - J \delta_{\vec{r}, \vec{l}_c} & \Delta_F & \lambda_{so}(-\eta_x + i\eta_y) & 0 \\
\Delta_F^* & h + \epsilon_F - J \delta_{\vec{r}, \vec{l}_c} & 0 & \lambda_{so}(\eta_x - i\eta_y) \\
\lambda_{so}(\eta_x + i\eta_y) & 0 & -h - \epsilon_F + J \delta_{\vec{r}, \vec{l}_c} & \Delta_F \\
0 & \lambda_{so}(-\eta_x - i\eta_y) & \Delta_F^* & h + \epsilon_F + J \delta_{\vec{r}, \vec{l}_c}
\end{pmatrix}
\begin{pmatrix}
\bar{u}_n(\vec{r}, \uparrow) \\
v_n(\vec{r}, \downarrow) \\
v_n(\vec{r}, \uparrow) \\
v_n(\vec{r}, \downarrow)
\end{pmatrix}
= \epsilon_n
\begin{pmatrix}
\bar{u}_n(\vec{r}, \uparrow) \\
v_n(\vec{r}, \downarrow) \\
v_n(\vec{r}, \uparrow) \\
v_n(\vec{r}, \downarrow)
\end{pmatrix},
$$

(17)

where $h = t \delta_{\vec{r}, \vec{r}}$ with $\delta_{\vec{r}, \vec{r}} f(\vec{r}) = f(\vec{r} + \vec{\delta})$. Furthermore, $\eta_x = \pm 1$ if the neighbor along $x$ is $i_x + 1$ ($i_x - 1$) and $\eta_y = \pm 1$ if the neighbor along $y$ is $i_y + 1$ ($i_y - 1$). The parameter $J$ describes coupling between the impurity spin and the spin density of conduction electrons. Note that the solution of this problem requires diagonalization of a $4N \times 4N$ matrix, where $N$ is the number of lattice sites. This is in contrast to the problem of triplet superconductor described in the previous section, where a partial diagonalization was possible due to the translational invariance. Owing to this symmetry, the problem could be reduced to a simple diagonalization of a $4 \times 4$ matrix for each momentum value. Since the effect of the magnetic impurity is rather local, a system of $15 \times 15$ lattice sites is sufficient to have small...
finite size effects, as we have shown previously [47]. We solve the problem self-consistently, as in a previous study (see Ref. [47] for details).

As in the case of triplet superconductivity studied in section II, the Hall conductance can be obtained from a Kubo like formula, which now reads

$$Re \langle \sigma_{xy} \rangle = \frac{i \hbar}{V} \sum_{\vec{r}_1, \vec{r}_2} \sum_{\alpha, \beta} \sum_{\gamma, \delta} \sum_{n,m} \left( f_n - f_m \right) \left( \epsilon_n - \epsilon_m + i0^+ \right)^2 \left( \bar{V}^{x}_{\vec{r}_1; \alpha, \beta} V^{y}_{\vec{r}_2; \gamma, \delta} u_n(\vec{r}_1, \alpha)^* u_m(\vec{r}_2, \delta) u_m(\vec{r}_1, \beta) u_m(\vec{r}_2, \gamma)^* - V^{x}_{\vec{r}_1; \alpha, \beta} \bar{V}^{y}_{\vec{r}_2; \gamma, \delta} \gamma \delta u_n(\vec{r}_1, \alpha)^* u_n(\vec{r}_2, \gamma) u_m(\vec{r}_1, \beta) v_m(\vec{r}_2, \delta)^* \right)$$

(18)

In this expression $\bar{V}$ is the complex conjugate, and $V^\mu_{\vec{r}}$ means that the operator acts on the coordinate $\vec{r}$.

The corresponding numerical results are presented in Fig. 5 where we show the total magnetization, order parameter at the impurity site, and anomalous Hall conductance for the conventional superconductor with a magnetic impurity, calculated for a finite system including $15 \times 15$ lattice points as a function of $J$ and $\lambda_{so}$. In the lower panel we show some cuts of the Hall conductance for $J$ fixed and varying spin-orbit coupling.

In order to emphasize the connection between behavior of the Hall conductance and the quantum phase transition, we show in Fig. 6 (for different spin-orbit couplings) the Hall conductance, amplitude of the order parameter at the impurity location, and the total magnetization of the conduction electrons as a function of the coupling between the spin density of the conduction electrons and the impurity spin. At the quantum phase transition both the amplitude of the order parameter and the total magnetization have discontinuities. At this critical coupling the Hall conductance has a sharp minimum which therefore signals the phase transition.
Figure 6: Order parameter at the impurity site, total magnetization, and anomalous Hall conductance for the conventional superconductor as a function of coupling strength to the impurity spin, calculated for the spin-orbit coupling corresponding to $\lambda_{so} = 0.8$, $\lambda_{so} = 1.1$, and $\lambda_{so} = 1.4$, as indicated. The first two values cross the quantum phase transition and for the highest value the transition turns into a crossover.

IV. SUMMARY

We have analyzed the anomalous Hall effect in superconductors, considering only the intrinsic mechanism that results from the interplay of Rashba spin-orbit interaction and magnetization. In the normal phase, the effect appears when both spin-orbit term and magnetization are nonzero. In a conventional spin-singlet superconductor of $s$-wave symmetry, an extended magnetization destroys the superconductivity. As we have shown, to have a nonvanishing anomalous Hall conductance in the superconducting phase it is then sufficient to assume a single magnetic impurity in the presence of spin-orbit interaction. However, vanishing coupling between the conduction electrons and magnetic impurity, or vanishing spin-orbit coupling, lead to zero Hall conductance.

The case of a spin triplet superconductor is qualitatively different. An extended magnetization does not destroy the superconducting order. The magnetization, generally, either can be induced by an adjacent ferromagnet owing to the proximity effect (we may also consider the superconducting order as a proximity effect in heterostructures where some metal is coupled to a magnet and a superconductor), or it may be an intrinsic property of the material (described by a self-consistent solution for the pairing amplitude). In the first case, two pairing forms lead to different results. If the pairing is unitary, the results are similar to those for the normal phase, and both magnetization and spin-orbit coupling are required for a finite Hall conductance. The superconducting case is very similar to the normal phase also when $\vec{d}$ is parallel to $\vec{s}$. In the nonunitary case, however, there is a polarization associated with the pairing amplitude, and the Hall conductance is finite as long as the spin-orbit coupling is finite (a nonunitary pairing leads to a finite magnetization, as in the case of $^3$He).

Since the spin-orbit interaction generates spin flips, its moderate values destroy superconductivity in both the conventional and triplet superconductors. In the case of $s$-wave superconductors, critical values of the spin-coupling $J$ in the presence of spin-orbit coupling are larger than those for zero spin-orbit coupling, shifting thus the point at which the quantum phase transition appears. In the case of spin triplet superconductors with the pairing amplitude determined self-consistently, the spin-orbit coupling leads to suppression of the superconductivity through a continuous phase transition. Finally we have shown that the Hall conductance tracks the quantum phase transition induced by magnetic impurities in conventional superconductors. This provides transport measurement as a possible tool to detect the transition, related to earlier predictions that transport properties are affected by the presence of magnetic impurities in a superconductor [59]. We note that one of the interests of the AHE is that it can be easily measured.

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