Furcation of resonance sets for one-point interactions

A.V. Zolotaryuk
Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine, Kyiv 03680, Ukraine

Abstract. Families of one-point interactions are derived from the system consisting of regularized two- and three-delta potentials using different paths of the convergence of corresponding transmission matrices in the squeezing limit. This limit is controlled by the relative rate of shrinking the width of delta-like functions and the distance between these functions using the power parameterization: width \( l = \varepsilon^{\mu-1} \), \( \mu \in [2, \infty] \) (for width) and \( r = \varepsilon^{\tau} \), \( \tau \in [1, \infty] \) (for distance). It is shown that at some values of real coefficients (intensities \( a_1 \), \( a_2 \) and \( a_3 \)) at the delta potentials, the transmission across the limit point interactions is non-zero, whereas outside these (resonance) values the one-point interactions are opaque splitting the system at the point of singularity into two independent subsystems. The resonance sets of intensities at which a non-zero transmission occurs are proved to be of four types depending on the way of squeezing the regularized system to one point. In its turn, on these sets the limit one-point interactions are observed to be either single- or multiple-resonant-tunnelling potentials also depending on the squeezing way. In the two-delta case the resonance sets are curves on the \((a_1, a_2)\)-plane and surfaces in the \((a_1, a_2, a_3)\)-space for the three-delta system. A new phenomenon of furcation of single-valued resonance sets to multi-valued ones is observed under approaching the parameter \( \mu > 2 \) to the value \( \mu = 2 \).

Keywords: one-point interactions, single- and multiple-resonant tunnelling, resonance curves and surfaces

PACS numbers: 03.65.-w, 03.65.Nk, 73.40.Gk

1. Introduction

The models described by the Schrödinger operators with singular zero-range potentials have widely been discussed in both the physical and mathematical literature (see books \cite{1, 2, 3, 4} for details and references). These models admit exact closed analytical solutions which describe realistic situations using different approximations via Hamiltonians describing point interactions \cite{5, 6, 7, 8, 9, 10}. Currently, because of the rapid progress in fabricating nanoscale quantum devices, of particular importance is the point modelling of different structures like quantum waveguides \cite{11, 12}, spectral filters \cite{13, 14} or infinitesimally thin sheets \cite{15, 16, 17}. A whole body of literature (see, e.g., \cite{18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30}, a few to mention), including the very recent studies \cite{31, 32, 33, 34, 35, 36, 37, 38} with references therein, has been
published where the one-dimensional Schrödinger operators with potentials given in the form of distributions are shown to exhibit a number of peculiar features with possible applications to quantum physics. A detailed list of references on this subject can also be found in the recent review [39]. On the other hand, using some particular regular approximations of the potential expressed in the form of the derivative of Dirac’s delta function, a number of interesting resonance properties of quantum particles tunnelling through this point potential has been observed [8, 40, 41, 42]. Particularly, it was found that at some values of the potential strength of the δ'-potential the transmission across this barrier is non-zero, whereas outside these values the barrier is fully opaque. In general terms, the existence of such resonance sets in the space of potential intensities has rigorously been established for a whole class of approximations of the derivative delta potential by Golovaty with coworkers [43, 44, 45, 46, 47, 48, 49]. This type of point interactions may be referred to as ‘resonant-tunnelling δ’-potentials’. These results differ from those obtained within Kurasov’s theory [21] which was developed for the distributions defined on the space of functions discontinuous at the point of singularity. Here the limit point interaction is also called a δ’-potential. The common feature of Kurasov’s point potential and a resonant-tunnelling δ’-potential is that the transmission matrices of both these interactions are of the diagonal form, but the elements of these matrices are different. It is of interest therefore to to find a way where it would be possible to describe both these types in a unique regularization scheme starting from the same initial regularized potential profile.

In the present work we address the problem on the relation between the point interactions realized within Kurasov’s theory and the resonant-tunnelling δ’-potentials studied in [8, 40, 41, 42, 43, 44, 46, 48, 49, 50]. Similarly to these papers, we explore the one-dimensional stationary Schrödinger equation

\[-d^2\psi(x)/dx^2 + V_\varepsilon(x)\psi(x) = E\psi(x)\]  

(1)

where \(\psi(x)\) is the wavefunction and \(E\) the energy of a particle. The potential \(V_\varepsilon(x)\) with a squeezing parameter \(\varepsilon > 0\) shrinks to one point, say \(x = 0\), as \(\varepsilon \to 0\). One of the ways to realize limit point interactions is to choose the potential \(V_\varepsilon(x)\) in the form of a sum of several Dirac’s delta functions as follows [31, 51, 52]

\[V_\varepsilon(x) = \sum_{j=1}^{N} c_j(\varepsilon)\delta(x - r_j(\varepsilon)), \quad r_j(\varepsilon) \in \mathbb{R},\]  

(2)

where all \(r_j(\varepsilon) \to 0\) and \(c_j(\varepsilon) \to \pm \infty\) as \(\varepsilon \to 0\). The particular case of the three-delta spatially symmetric potential [2], in the limit as the distances between the δ-functions tend to zero, has been studied by Cheon and Shigehara [51], and Albeverio and Nizhnik [52]. In this limit a whole four-parameter family of point interactions has been constructed, independently on whether or not potential (2) has a distributional limit. Here we follow the approach developed by Exner, Neidhardt and Zagrebnov [7], who have approximated the δ-potentials by regular functions and constructed a one-point limit interaction. In particular, they have proved that the limit takes place if the
distances between the ‘centers’ of regularized potentials tend to zero sufficiently slow relatively to shrinking the δ-like potentials. A similar research [6] concerns about the convergence of regularized δ-like structures to point potentials in higher dimensions.

In this paper we focus on the two cases when potential (2) consists of two (N = 2) and three (N = 3) δ-potentials separated equidistantly by a function r(ε) that tends to zero as ε → 0. All the coefficients at the δ-functions are specified as $c_j = -a_j/ε$ where $a_j$’s (‘intensities’, ‘charges’ or ‘amplitudes’) are non-zero constants. The sign ‘−’ has been chosen for convenience in the following notations, so that negative values of $a_j$ correspond to a δ-barrier and positive ones to a δ-well. Thus, in the case with $N = 3$ we have

$$V_{εr}(x) = -ε^{-1} [a_1 δ(x) + a_2 δ(x - r) + a_3 δ(x - 2r)], \quad (a_1, a_2, a_3) ∈ \mathbb{R}^3 \setminus \{0\}. \quad (3)$$

For the case of two δ-potentials, we just set in (3) $a_3 = 0$, so that $(a_1, a_2) ∈ \mathbb{R}^2 \setminus \{0\}$.

The transmission matrices for the two- and three-delta potentials are the products $Λ_{εr} = Λ_2 Λ_0 Λ_1$ and $Λ_{εr} = Λ_3 Λ_0 Λ_2 Λ_0 Λ_1$, respectively, where

$$Λ_0 = \begin{pmatrix} \cos(kl) & k^{-1} \sin(kl) \\ -k \sin(kl) & \cos(kl) \end{pmatrix}, \quad Λ_j = \begin{pmatrix} 1 & 0 \\ -a_j/ε & 1 \end{pmatrix}, \quad j = 1, 2, 3. \quad (4)$$

We restrict ourselves to the most simple approximation of the δ-potentials by piecewise constant functions resulting in a three (for $N = 2$) and a five (for $N = 3$) layered potential profile. In the limit as both the width of δ-like functions and the distance between them tends to zero simultaneously we obtain a family of one-point interactions. We observe that, starting from the same profile of the three- and five-layered structure that approximates potential (2), the limit point interactions crucially depend on the relative rate of tending the width of layers and the distance between them to zero. Within this approach one can realize both the point interactions obtained within Kurasov’s theory and the resonant-tunnelling potentials.

### 2. A piecewise constant approximation of the δ-potentials

Let us approximate the δ-potentials in (3) by piecewise constant functions. Then potential (3) is replaced by the rectangular function

$$V_{εlr}(x) = \begin{cases} 0 & \text{for } -∞ < x < 0, \quad l < x < l + r, \\ 2l + r < x < 2(l + r), \quad 3l + 2r < x < ∞, \\ -a_j/εl & \text{for } (j - 1)(l + r) < x < j(l + r) - r, \quad j = 1, 2, 3, \end{cases} \quad (5)$$

and, as a result, all the matrices $Λ_j, j = 1, 2, 3$, in the product for $Λ_{εr}$ are replaced by

$$Λ_{j,l} = \begin{pmatrix} \cos(k_jl) & k_j^{-1} \sin(k_jl) \\ -k_j \sin(k_jl) & \cos(k_jl) \end{pmatrix}, \quad (6)$$

where

$$k_j := \sqrt{k^2 + a_j/εl}, \quad k := \sqrt{E}, \quad j = 1, 2, 3. \quad (7)$$
In other words, the regularized transmission matrix \( \Lambda_{\text{str}} \) defined by the relations
\[
\begin{pmatrix}
\psi(x_2) \\
\psi'(x_2)
\end{pmatrix}
= \Lambda_{\text{str}}
\begin{pmatrix}
\psi(x_1) \\
\psi'(x_1)
\end{pmatrix},
\quad \Lambda_{\text{str}} = \Lambda_{3,l} \Lambda_0 \Lambda_{2,l} \Lambda_1 \Lambda_{t,l} =: \begin{pmatrix}
\tilde{\lambda}_{11} & \tilde{\lambda}_{12} \\
\tilde{\lambda}_{21} & \tilde{\lambda}_{22}
\end{pmatrix},
\tag{8}
\]
connects the boundary conditions for the wavefunction \( \psi(x) \) and its derivative \( \psi'(x) \) at
\( x = x_1 = 0 \) and \( x = x_2 = 3l + 2r \). For the case of the two-delta potential
\( (N = 2) \) we set in potential \( a_3 = 0 \), so that the boundary conditions are \( x_1 = 0 \) and \( x_2 = 2l + r \). The matrix elements in \( \Lambda \), denoted by overhead bars, depend on all the shrinking parameters \( \varepsilon, l, r \), whereas in the limit matrix elements, if they exist, the bars are omitted, i.e., we write \( \lim_{\varepsilon, l, r \to 0} \Lambda_{\text{str}} =: \Lambda = \begin{pmatrix}
\lambda_{11} & \lambda_{12} \\
\lambda_{21} & \lambda_{22}
\end{pmatrix} \). Having accomplished the limit procedure, we set \( x_1 = -0 \) and \( \lim_{l, r \to 0} x_2 = +0 \).

We follow the notations and the classification of one-point interactions given by Brasche and Nizhnik \[31\]. Thus, we denote
\[
\psi_s(0) := \psi(+0) - \psi(-0),
\psi'_s(0) := \psi'(+0) - \psi'(-0),
\psi_t(0) := \eta \psi(+0) + (1 - \eta) \psi(-0),
\psi'_t(0) := \eta \psi'(-0) + (1 - \eta) \psi'(+0),
\tag{9}
\]
where \( \eta \in \mathbb{R} \) is an arbitrary parameter (this is a generalization of the generally accepted case with \( \eta = 1/2 \), see, e.g., \[20, 21, 31, 33\]). Then the \( \delta' \)-interaction, or \( \delta' \)-potential, with intensity \( \alpha \) is defined by the boundary conditions \( \psi'_s(0) = 0 \) and \( \psi'_t(0) = \alpha \psi_t(0) \), so that the \( \Lambda \)-matrix in this case has the form
\[
\Lambda = \begin{pmatrix}
1 & 0 \\
\alpha & 1
\end{pmatrix}.
\tag{10}
\]
The dual interaction is termed a \( \delta' \)-interaction (the notation has been suggested in \[3, 19\] and adopted in the literature). This point interaction with intensity \( \beta \) defined by the boundary conditions \( \psi'_t(0) = 0 \) and \( \psi'_s(0) = \beta \psi'_t(0) \) has the \( \Lambda \)-matrix in the form
\[
\Lambda = \begin{pmatrix}
1 & \beta \\
0 & 1
\end{pmatrix}.
\tag{11}
\]
As follows from formulae \(10) \) and \(11) \), the usage of the parameter \( \eta \) for both the \( \delta \)- and \( \delta' \)-interactions does not play any role. However, for the \( \delta' \)-potential with intensity \( \gamma \) the potential part in equation \(11) \) is given by \( \gamma \delta'(x) \psi(x) \) where the wavefunction \( \psi(x) \) must be discontinuous at \( x = 0 \). Therefore, due to the ambiguity of the product \( \delta'(x) \psi(x) \), one can suppose the following generalized (asymmetric) averaging in the form
\[
\delta'(x) \psi(x) = [(1 - \eta) \psi(-0) + \eta \psi(+0)] \delta'(x) + [\eta \psi'(0) + (1 - \eta) \psi'(+0)] \delta(x).
\tag{12}
\]
This suggestion is also motivated by the studies \[53, 54, 55\] which demonstrate that the plausible averaging with \( \eta = 1/2 \) at the point of singularity in general does not work. The \( \delta' \)-potential with intensity \( \gamma \) is defined by the boundary conditions \( \psi_s(0) = \gamma \psi_t(0) \) and \( \psi'_s(0) = -\gamma \psi'_t(0) \) \[31\]. An equivalent form of these conditions is given by the \( \Lambda \)-matrix in the diagonal form
\[
\Lambda = \begin{pmatrix}
\theta & 0 \\
0 & \theta^{-1}
\end{pmatrix}.
\tag{13}
with
\[ \theta = \frac{1 + (1 - \eta)\gamma}{1 - \eta\gamma}. \]  

Finally, instead of the fourth type of point interactions defined in [31] as \( \delta \)-magnetic potentials, in this paper we shall be dealing with potentials which at some (resonant) values of intensities are fully transparent, whereas outside these values they are completely opaque satisfying the Dirichlet boundary conditions \( \psi(\pm 0) = 0 \). At the resonance sets the boundary conditions are given by the unit matrix \( \Lambda = I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Beside these, as a particular case, resonant-tunnelling \( \delta \)-potentials will also be shown to exist.

The convergence of the transmission matrix \( \Lambda_{\varepsilon,l,r} \) as \( \varepsilon, l, r \to 0 \) can be parameterized through the parameter \( \varepsilon \) using the powers \( \mu > 1 \) and \( \tau > 0 \) (keeping the same notation used in [56]) as follows
\[ l = \varepsilon^{\mu-1} \quad \text{and} \quad r = \varepsilon^{\tau}. \]  

Then, according to (7), we have the following asymptotic relations:
\[ k_j \to \sqrt{a_j/\varepsilon} = \sqrt{a_j} \varepsilon^{-\mu/2}, \quad k_j l \to \sqrt{a_j} \varepsilon^{\mu/2-1}, \quad k_j^2 l \to a_j \varepsilon^{-1}. \]  

Explicitly, using that \( k_j \to \infty, k_j l \) and \( k_i/k_j, i, j = 1, 2, 3 \), are finite and \( r \to 0 \) as \( \varepsilon \to 0 \), we find the asymptotic behaviour of the elements of the matrix \( \Lambda_{\varepsilon;r} = \Lambda_{2,l} \Lambda_0 \Lambda_{1,l} \) \((N = 2)\):
\[ \bar{\lambda}_{11} \to \cos(k_1 l) \cos(k_2 l) - (k_1/k_2) \sin(k_1 l) \sin(k_2 l) \]
\[ - k_1 r \sin(k_1 l) \cos(k_2 l), \]
\[ \bar{\lambda}_{12} \to 0, \]
\[ \bar{\lambda}_{21} \to - k_1 \sin(k_1 l) \cos(k_2 l) - k_2 \cos(k_1 l) \sin(k_2 l) \]
\[ + k_1 k_2 r \sin(k_1 l) \sin(k_2 l), \]
\[ \bar{\lambda}_{22} \to \cos(k_1 l) \cos(k_2 l) - (k_2/k_1) \sin(k_1 l) \sin(k_2 l) \]
\[ - k_2 r \cos(k_1 l) \sin(k_2 l). \]  

Similarly, for the three-delta potential the \( \bar{\lambda}_{ij} \)-asymptotes of the matrix product \( \Lambda_{\varepsilon;r} = \Lambda_{3,l} \Lambda_0 \Lambda_{2,l} \Lambda_0 \Lambda_{1,l} \) are as follows
\[ \bar{\lambda}_{11} \to \cos(k_1 l) \cos(k_2 l) \cos(k_3 l) - (k_1/k_2) \sin(k_1 l) \sin(k_2 l) \cos(k_3 l) \]
\[ - (k_1/k_3) \sin(k_1 l) \cos(k_2 l) \sin(k_3 l) - (k_2/k_3) \cos(k_1 l) \sin(k_2 l) \sin(k_3 l) \]
\[ - 2k_1 r \sin(k_1 l) \cos(k_2 l) \cos(k_3 l) - k_2 r \cos(k_1 l) \sin(k_2 l) \cos(k_3 l) \]
\[ + k_1 k_2 r^2 \sin(k_1 l) \sin(k_2 l) \cos(k_3 l) + (k_1 k_2 r/k_3) \sin(k_1 l) \sin(k_2 l) \sin(k_3 l), \]
\[ \bar{\lambda}_{12} \to - k_2 r^2 \cos(k_1 l) \sin(k_2 l) \cos(k_3 l), \]
\[ \bar{\lambda}_{21} \to - k_1 \sin(k_1 l) \cos(k_2 l) \cos(k_3 l) - k_2 \cos(k_1 l) \sin(k_2 l) \cos(k_3 l) \]
\[ + k_2 r \cos(k_1 l) \sin(k_2 l) \cos(k_3 l), \]
\[ \bar{\lambda}_{22} \to \cos(k_1 l) \cos(k_2 l) \cos(k_3 l) - (k_2/k_1) \sin(k_1 l) \sin(k_2 l) \cos(k_3 l) \]
\[ - k_2 r \cos(k_1 l) \sin(k_2 l) \cos(k_3 l) \]
\[ + k_1 k_2 r^2 \sin(k_1 l) \sin(k_2 l) \cos(k_3 l) + (k_1 k_2 r/k_3) \sin(k_1 l) \sin(k_2 l) \sin(k_3 l). \]
relations (17), (19)-(21), (23) and (24) are reduced to the analysis of the convergence of the corresponding transmission matrices as $\mu > 3$. Realizing point interactions under the convergence of the leads to quite different results.

In the limit as $\epsilon \to 0$ along the families of paths with $(\epsilon, l, r)$, therefore, as follows from asymptotes (25) and (27), the interval $0 \leq \tau < 11$ is not suitable for the existence of point interactions. Consequently, the interval $1 \leq \tau \leq \infty$ has to be considered for the further analysis of the convergence as $\epsilon \to 0$. Then, similarly to limit (18), from (22) we also have $\lambda_{12} \to -a_2 \epsilon^{2\tau-1} \to 0$ in the case with $N = 3$. However, the elements $\lambda_{21}$ given by asymptotes (26) and (28) are always divergent as $\epsilon \to 0$. The only possibility to make these terms finite is a cancellation of divergences in the shrinking limit. To accomplish such a cancellation procedure, in virtue of the form of formulae (25)-(28), we split the interval $2 \leq \mu \leq \infty$ into the four sets: $\{2\}$, $(2, 3)$, $\{3\}$, $(3, \infty]$ and the interval $1 \leq \tau \leq \infty$ into the sets: $\{1\}$, $(1, 2)$, $\{2\}$, $(2, \infty]$.

Next, for convenience we introduce a three-dimensional system of coordinates $(\epsilon, l, r)$ with the origin at $(\epsilon, l, r) = (0, 0, 0) =: \{0\}$ and consider the cube with the
vertices at \( \{0\} \), \((1,0,0),(1,1,0),(0,1,0) \) in the face \( r = 0 \) and \((0,0,1),(1,0,1),(1,1,1) \)
\((0,1,1) \) in the face \( r = 1 \), as shown in figure 1. Then the squeezing limit of potential \((5)\)
corresponds to a path (descent) for which \((\varepsilon, l, r) = (1,1,1) \) is a starting point and the
origin \((\varepsilon, l, r) = \{0\}\) a final point. For the whole interval \(1 \leq \tau \leq \infty \) we consider the four
families of paths parameterized by \(3 < \mu \leq \infty \) (paths 1), \(\mu = 3 \) (paths 2), \(2 < \mu < 3 \)
(paths 3) and \(\mu = 2 \) (paths 4). In its turn, for the \( j \)th \((j = 1, 4) \) family, we also single
out the same four subsets: \(ja \) (\(\tau = 1\)), \(jb \) (\(1 < \tau < 2\)), \(jc \) (\(\tau = 2\)) and \(jd \) (\(2 < \tau \leq \infty\)).
Some of these paths shown in the faces of the cube and along its edges are schematically
depicted in figure 1. As shown in this figure, in the limit case as \( \mu \to \infty \) (\( l \to 0 \)), all
the paths of family 1 follow first along the edge \((\varepsilon, r) = (1,1)\) and then each of these
paths descends in the face \( l = 0 \) approaching the cube origin \( \{0\} \) with different rates
depending on \( \tau \). Similarly, the case with \( \tau = \infty \) (\( r \to 0 \)) describes the situation when the
squeezing limit sequentially proceeds along the edge \((\varepsilon, l) = (1,1)\) and then along
the curve \( l = \varepsilon^{\mu-1} \) in the face \( r = 0 \) with rates depending on \( \mu \). Finally, note that the
limit paths when both \( \mu \) and \( \tau \) tend to infinity are different depending on the repeated
limit: first \( \mu \to \infty \), then \( \tau \to \infty \) or vice versa, first \( \tau \to \infty \) and then \( \mu \to \infty \). Below we
analyse both connected and separated point interactions which can be realized along all of
these paths starting at the point \((1,1,1)\) and ending at the origin \( \{0\} \).

3.1. Families of paths 1a, 2a and 3a \((1 < \mu \leq \infty, \tau = 1)\)

First we note that the \( \varepsilon \to 0 \) limit of asymptotes \((25)\) and \((27)\) with \( \mu > 2 \) and \( \tau = 1 \)
is finite and therefore this fact ensures the existence of point (connected or separated)
interactions. Next, as it can be seen from asymptotes \((26)\) and \((28)\), for all \( \mu > 1 \) and
\( \tau > 0 \) the \( \bar{\lambda}_{21} \)-terms are divergent as \( \varepsilon \to 0 \) in both the cases \( N = 2 \) and \( 3 \).
The necessary condition to make these terms finite in the \( \varepsilon \to 0 \) limit is to impose the equations

\[
K_2(a_1, a_2) := a_1 + a_2 - a_1 a_2 = 0 \quad \text{for} \quad N = 2,
K_3(a_1, a_2, a_3) := a_1 + a_2 + a_3 - a_1 a_2 - 2a_1 a_3 - a_2 a_3 + a_1 a_2 a_3 = 0 \quad \text{for} \quad N = 3.
\]

The first of these equations obtained earlier by Brasche and Nizhnik \((31)\) ensures the
finiteness of the limit term \( \lambda_{21} \) for all paths 1a, 2a and 3a, while for \( N = 3 \), because of
the presence of the term with \( \varepsilon^{\mu-3} \) in \((28)\), the existence of connected interactions
is impossible for path 3a (in virtue of the inequality \( \mu < 3)\). Using equations \((29)\) in
asymptotes \((25)\) and \((27)\), we obtain the diagonal elements of the limit matrix \( \Lambda \) (except
for paths 3a with \( N = 3)\):

\[
\lim_{\varepsilon \to 0} \bar{\lambda}_{11} =: \theta = \begin{cases}
1 - a_1 & \text{for} \quad N = 2, \\
1 - 2a_1 - a_2 + a_1 a_2 & \text{for} \quad N = 3,
\end{cases}
\]

\[
\lim_{\varepsilon \to 0} \bar{\lambda}_{22} =: \rho = \begin{cases}
1 - a_2 & \text{for} \quad N = 2, \\
1 - a_2 - 2a_3 + a_2 a_3 & \text{for} \quad N = 3.
\end{cases}
\]

In virtue of equations \((29)\), we have \( \rho = \theta^{-1} \) and therefore for paths 1a, 2a, 3a \((N = 2)\)
and 1a \((N = 3)\) the limit transmission matrix becomes of diagonal form \((13)\).
Figure 1. The $(\varepsilon, l, r)$-cube where the eight paths: 1a ($\mu \to \infty$, $\tau = 1$), 1c ($\mu \to \infty$, $\tau = 2$), 2d ($\mu = 3$, $\tau \to \infty$), 4a ($\mu = 2$, $\tau = 1$), 4c ($\mu = \tau = 2$), 4d ($\mu = 2$, $\tau \to \infty$) and both 1d (first $\mu \to \infty$, then $\tau \to \infty$ and first $\tau \to \infty$, then $\mu \to \infty$) are schematically shown by solid lines accompanied with arrows. The families of paths 1b ($\mu \to \infty$, $1 < \tau < 2$), 2d ($2 < \mu < 3$, $\tau \to \infty$) and 4b ($\mu = 2$, $1 < \tau < 2$) lie in sparsely shadowed regions, whereas the families of paths 1d ($\mu \to \infty$, $2 < \tau < \infty$), 4d ($\mu = 2$, $2 < \tau < \infty$) and 1d ($3 < \mu < \infty$, $\tau \to \infty$) are illustrated by densely shadowed areas.

This occurs at the following values of $(a_1, a_2)$ and $(a_1, a_2, a_3)$:

For $N = 2$ and

$$a_1 = \frac{\gamma}{1 - \eta \gamma}, \quad a_2 = \frac{\gamma}{1 + (1 - \eta) \gamma}$$

with arbitrary $a_2 \in \mathbb{R} \setminus \{2\}$, for $N = 3$. We call those intensities $(a_1, a_2) \in \mathbb{R}^2 \setminus \{0\}$ and $(a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\}$ which satisfy equations (29) the resonance sets $\mathcal{K}_2$ and $\mathcal{K}_3$, respectively. For $N = 2$ the first equation (29) describes a curve on the $(a_1, a_2)$-plane, whereas for $N = 3$ we have a surface in the $(a_1, a_2, a_3)$-space. Therefore the point interactions realized on the sets given by equations (29) along paths 1a, 2a, 3a $(N = 2)$ and 1a $(N = 3)$ may be called ‘single-resonant-tunnelling $\delta'$-potentials of the $\mathcal{K}$-type’.

There exists a particular subfamily of the intensities $(a_1, a_2) \in \mathbb{R}^2 \setminus \{0\}$ and $(a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\}$ from the $\mathcal{K}_{2,3}$-sets for which $\theta = \pm 1$ in (13), realizing the point
interactions with full transmission. Thus, for \( N = 2 \) these values are \( a_1 = a_2 = 2 \) resulting in the unit matrix \( \Lambda = -I \). In the three-delta case the two conditions \( a_1 = a_3 \) and \( 2a_1 + a_2 - a_1a_2 = 0 \) provide the unit matrix \( \Lambda = I \), whereas the other two conditions \( a_1 + a_3 = 2 \) and \( 2a_1 + a_2 - a_1a_2 = 2 \) (in general, an asymmetric structure) lead to the matrix \( \Lambda = -I \).

As regards path 2a in the case with \( N = 3 \), the cancellation of divergences in (28) leads in the limit as \( \varepsilon \to 0 \) to a non-zero constant. As a result, in virtue of (25) and (27), we have the same limit diagonal elements (30) and the limit transmission matrix of the form

\[
\Lambda = \begin{pmatrix}
\theta & 0 \\
\alpha & \theta^{-1}
\end{pmatrix}, \quad \alpha := \lim_{\varepsilon \to 0} \bar{\lambda}_{21},
\]

with \( \alpha = a_1a_3 \). Therefore, to be in agreement with the notation introduced above for paths 1a, 2a, 3a (\( N = 2 \)) and 1a (\( N = 3 \)), the point interaction realized along path 2a (\( \mu = 3, \tau = 1 \)) on the resonance \( \mathcal{K}_3 \)-set may be called a ‘single-resonant-tunnelling \( (\delta' + \delta) \)-potential of the \( \mathcal{K} \)-type’.

Finally, as follows from asymptote (28) for \( N = 3 \), the cancellation of divergences in the limit as \( \varepsilon \to 0 \) is impossible. Thus, everywhere beyond the \( \mathcal{K}_2 \)-set (paths 1a, 2a, 3a) and for all \( (a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\} \), the limit point interactions are separated. They are described by the boundary conditions of the Dirichlet type: \( \psi(\pm 0) = 0 \).

3.2. Families of paths jb, jc and jd, \( j = 1, 2, 3 \)

It follows from asymptotic relations (25) and (27) that for all paths jb, jc and jd, \( j = 1, 2, 3 \), we have the limits \( \bar{\lambda}_{11}, \bar{\lambda}_{22} \to 1 \), so that in these cases either connected or separated point interactions can be realized. The \( \varepsilon \to 0 \) analysis has to be carried out only for the \( \bar{\lambda}_{21} \)-terms given by asymptotes (26) and (28).

Families of paths 1b, 2b and 3b (\( 2 < \mu \leq \infty, 1 < \tau < 2 \)): Along these paths the \( \bar{\lambda}_{21} \)-terms are divergent for all non-zero \( a_1, a_2 \) and \( a_3 \). However, there exists a possibility to cancel the divergences in (28) for paths 1b and 2b (\( 3 \leq \mu \leq \infty \)) at \( \tau = 3/2 \) and for path 3b at \( \mu - 1 = \tau = 3/2 \). In the former case the last term in (28) is finite and the cancellation occurs if both the equations \( a_1 + a_2 + a_3 = 0 \) and \( a_1a_2 + 2a_1a_3 + a_2a_3 = 0 \) are fulfilled simultaneously. Similarly, in the latter case, instead of the last equation, we have \( a_1a_2 + 3a_1a_3 + a_2a_3 = 0 \). Excluding \( a_3 \) from these equations, we find the conditions \( a_1^2 + (a_1 + a_2)^2 = 0 \) and \( (3/4)a_1^2 + (3a_1/2 + a_2)^2 = 0 \), respectively, which are valid only if \( a_1 = a_2 = 0 \) and therefore \( a_3 = 0 \). Therefore the limit point interactions realized along the family of paths 1b, 1a and 1c are separated for all non-zero \( a_1, a_2 \) and \( a_3 \) with the boundary conditions \( \psi(\pm 0) = 0 \).

Families of paths 1c, 2c and 3c (\( 2 < \mu \leq \infty, \tau = 2 \)): Contrary to the previous case, for these paths the cancellation of divergences in the \( \bar{\lambda}_{21} \)-terms is possible, except for paths 3c with \( N = 3 \) because of the presence of the term with \( \varepsilon^{\mu-3} \) in (28). As a result,
a non-zero finite limit of the $\lambda_{21}$-terms takes place if the conditions
\begin{align*}
L_2(a_1, a_2) &:= a_1 + a_2 = 0 \quad \text{for } N = 2, \\
L_3(a_1, a_2, a_3) &:= a_1 + a_2 + a_3 = 0 \quad \text{for } N = 3
\end{align*}
hold true, being just a ‘linearized’ version of equations (29). In the following we refer
the intensities $(a_1, a_2) \in \mathbb{R}^2 \setminus \{0\}$ and $(a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\}$, which satisfy equations (34), to as $\mathcal{L}_2$- and $\mathcal{L}_3$-sets, respectively. On these sets the limit $\Lambda$-matrix describes the
$\delta$-potential with intensity $\alpha$. From asymptote (26) we obtain $\alpha = a_1 a_2$ for $N = 2$, while for $N = 3$ asymptote (28) results in
\begin{align*}
\alpha &= \begin{cases} 
  a_1 a_2 + 2a_1 a_3 + a_2 a_3 & \text{for paths } 1c, \\
  a_1 a_2 + 3a_1 a_3 + a_2 a_3 & \text{for path } 2c.
\end{cases}
\end{align*}
Everywhere beyond the $\mathcal{L}_2$-set for paths 1c, 2c, 3c and the $\mathcal{L}_3$-set for paths 1b, 2b as well as for all $(a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\}$ for paths 3b, the point interactions are separated satisfying the Dirichlet boundary conditions.

**Families of paths** 1d, 2d and 3d ($2 < \mu \leq \infty$, $2 < \tau \leq \infty$): The case with $\tau = \infty$ describes the situation when the squeezing limit sequentially follows the edge $r \to 0$ and then it goes along the curves $l = \varepsilon^{\mu-1}$ in the face $r = 0$ as shown in figure 1. The total cancellation of divergences takes place for paths 1d, 2d, 3d ($N = 2$, on the $\mathcal{L}_2$-set) and 1d ($N = 3$, on the $\mathcal{L}_3$-set), resulting in the existence of the resonant point interactions with full transmission ($\Lambda = I$). In virtue of the term with $\varepsilon^{\mu-3}$ in (28), the limit point interactions for paths 2d are of the $\delta$-potential type described by $\Lambda$-matrix (10) with $\alpha = a_1 a_3$. Finally, for paths 3d ($N = 3$) the cancellation of divergences in asymptotes (28) as $\varepsilon \to 0$ is impossible because of the presence of the term with $\varepsilon^{\mu-3}$. Consequently, for paths 1d, 2d, 3d outside the $\mathcal{L}_2$-set; 1d, 2d outside the $\mathcal{L}_3$-set and for all $(a_1, a_2, a_3) \in \mathbb{R} \setminus \{0\}$ for paths 3d, the limit interactions are separated satisfying the Dirichlet boundary conditions.

Thus, the $\Lambda$-matrices for paths 1b, 1c and 1d exhibit the transition of transmission that occurs on the $\mathcal{L}_{2,3}$-sets while varying the rate of the decrease of distance $r$ between the $\delta$-potentials. For sufficiently slow squeezing this distance ($1 < \tau < 2$, paths 1b), the limit point interactions are opaque, for intermediate shrinking ($\tau = 2$, paths 1c) the interactions become partially transparent ($\delta$-potential) and for fast shrinking ($2 < \tau < \infty$) the interactions appear to be fully transparent. One can check that these results are in agreement with those established by Šeba for $N = 2$ in the limit case $\mu \to \infty$ (see Theorem 3 in [18]).

4. Realizing point interactions under the convergence of the $\Lambda_{tllr}$-matrix along the families of paths with $\mu = 2$ and $1 \leq \tau \leq \infty$

Consider now the situation when $\mu = 2$ and $0 < \tau \leq \infty$. Then $l = \varepsilon$ and according to asymptotic relations (16), we have $k_j \to \sqrt{a_j}/\varepsilon$. In this case $r \to 0$ as $\varepsilon \to 0$ and
therefore asymptotes \((17), (19)\) and \((20)\) are reduced to

\[
\begin{align*}
\lambda_{11} & \rightarrow \cos \sqrt{a_1} \cos \sqrt{a_2} - \sqrt{a_1} a_2 \sin \sqrt{a_1} \sin \sqrt{a_2} - \sqrt{a_1} \sqrt{a_2} \cos \sqrt{a_2} \varepsilon^{\tau - 1}, \\
\lambda_{22} & \rightarrow \cos \sqrt{a_1} \cos \sqrt{a_2} - \sqrt{a_2} a_1 \sin \sqrt{a_1} \sin \sqrt{a_2} - \sqrt{a_2} \sqrt{a_1} \cos \sqrt{a_1} \varepsilon^{\tau - 1}, \\
\lambda_{21} & \rightarrow -(\sqrt{a_1} a_2 \cos \sqrt{a_2} a_2 \sin \sqrt{a_2} a_2 \varepsilon^{\tau - 1} + \sqrt{a_1} a_2 \sin \sqrt{a_1} \sin \sqrt{a_2} \varepsilon^{\tau - 2}) \quad (37)
\end{align*}
\]

for \(N = 2\). Similarly, in the case with \(N = 3\) asymptotes \((21), (23)\) and \((24)\) are transformed to

\[
\begin{align*}
\tilde{\lambda}_{11} & \rightarrow \cos \sqrt{a_1} \cos \sqrt{a_2} \cos \sqrt{a_3} - \sqrt{a_1} a_2 \sin \sqrt{a_1} \sin \sqrt{a_2} \cos \sqrt{a_3} \\
& \quad - \sqrt{a_1} a_2 \sqrt{a_1} \cos \sqrt{a_2} a_2 \sin \sqrt{a_2} a_2 \sin \sqrt{a_3} \varepsilon^{\tau - 1} + \sqrt{a_1} a_2 \sin \sqrt{a_1} \sin \sqrt{a_2} \cos \sqrt{a_3} \varepsilon^{2(\tau - 1)} \\
\tilde{\lambda}_{22} & \rightarrow \cos \sqrt{a_1} \cos \sqrt{a_2} \cos \sqrt{a_3} - \sqrt{a_2} a_1 \sin \sqrt{a_1} \sin \sqrt{a_2} \cos \sqrt{a_3} \\
& \quad - \sqrt{a_2} a_1 \sqrt{a_2} \cos \sqrt{a_1} a_1 \sin \sqrt{a_1} \sin \sqrt{a_2} a_2 \sin \sqrt{a_3} \varepsilon^{\tau - 1} + \sqrt{a_2} a_1 \cos \sqrt{a_2} \cos \sqrt{a_3} \varepsilon^{2(\tau - 1)} \\
\tilde{\lambda}_{21} & \rightarrow \left(\sqrt{a_1} a_2 / a_2 \sin \sqrt{a_1} \sin \sqrt{a_2} \cos \sqrt{a_3} - \sqrt{a_1} a_2 \sin \sqrt{a_1} \cos \sqrt{a_2} \cos \sqrt{a_3} \varepsilon^{-1} + \sqrt{a_1} a_2 \sin \sqrt{a_1} \sin \sqrt{a_2} \cos \sqrt{a_3} \varepsilon^{2(\tau - 3)}ight) \varepsilon^{2(\tau - 1)} \\
& \quad + k^2 \cos \sqrt{a_2} (\sqrt{a_1} a_2 \sin \sqrt{a_1} \cos \sqrt{a_3} + \sqrt{a_3} \cos \sqrt{a_1} \sin \sqrt{a_3}) \varepsilon^{2(\tau - 1)} \quad (39)
\end{align*}
\]

For the realization of (both connected and separated) interactions in the squeezing limit the elements \(\tilde{\lambda}_{11}\) and \(\tilde{\lambda}_{22}\) given by asymptotes \((36)\) and \((38)\) must be finite as \(\varepsilon \rightarrow 0\). Consequently, similarly to the case with \(\mu > 1\), the interval \(0 < \tau < 1\) is not suitable for realizing point interactions and therefore we have to consider the region \(1 \leq \tau \leq \infty\). Then limit \((22)\) becomes \(\tilde{\lambda}_{12} \rightarrow -\sqrt{a_2} \sin \sqrt{a_2} \varepsilon^{2\tau - 1} \rightarrow 0\).

All the paths of family 4 \((\mu = 2 \text{ and } 1 \leq \tau \leq \infty)\) are schematically shown in figure 1 starting at \((\varepsilon, l, r) = (1, 1, 1)\) and ending at the cube origin \((\varepsilon, l, r) = \{0\}\) within the diagonal plane: 4a \((\tau = 1)\), 4b \((1 < \tau < 2)\), 4c \((\tau = 2)\) and 4d \((2 < \tau \leq \infty)\) including the limit \(\tau \rightarrow \infty\) [first \(r \rightarrow 0\) along the edge \((\varepsilon, l) = (1, 1)\) and then along the diagonal \(l = \varepsilon \rightarrow 0\) in the face \(r = 0\)].

**Paths 4a and 4b \((\mu = 2, 1 \leq \tau < 2)\):** The cancellation of divergences in \((37)\) at \(\tau = 1\) leads to the resonance equation

\[
F_2(a_1, a_2) := \sqrt{a_1} \tan \sqrt{a_1} + \sqrt{a_2} \tan \sqrt{a_2} - \sqrt{a_1 a_2} \tan \sqrt{a_1} \tan \sqrt{a_2} = 0 \quad (40)
\]

for \(N = 2\). Using this equation in relations \((36)\) at \(\tau = 1\), we obtain the diagonal limit elements of the \(\Lambda\)-matrix in one of the following forms:

\[
\begin{align*}
\lambda_{11} & = (\cos \sqrt{a_1} - \sqrt{a_1} \sin \sqrt{a_1}) / \cos \sqrt{a_2} = -\sqrt{a_1} \sin \sqrt{a_1} \sqrt{a_2} \sin \sqrt{a_2}, \\
\lambda_{22} & = (\cos \sqrt{a_2} - \sqrt{a_2} \sin \sqrt{a_2}) / \cos \sqrt{a_1} = -\sqrt{a_2} \sin \sqrt{a_2} \sqrt{a_1} \sin \sqrt{a_1}.
\end{align*}
\]

\(\lambda_{12}\) becomes \(\tilde{\lambda}_{12} \rightarrow -\sqrt{a_2} \sin \sqrt{a_2} \varepsilon^{2\tau - 1} \rightarrow 0\).

For the realization of (both connected and separated) interactions in the squeezing limit the elements \(\lambda_{11}\) and \(\lambda_{22}\) given by asymptotes \((36)\) and \((38)\) must be finite as \(\varepsilon \rightarrow 0\). Consequently, similarly to the case with \(\mu > 1\), the interval \(0 < \tau < 1\) is not suitable for realizing point interactions and therefore we have to consider the region \(1 \leq \tau \leq \infty\). Then limit \((22)\) becomes \(\lambda_{12} \rightarrow -\sqrt{a_2} \sin \sqrt{a_2} \varepsilon^{2\tau - 1} \rightarrow 0\).
Using again equation (41), one can check that the equality $\lambda_{11}\lambda_{22} = 1$ holds true for matrix elements (41). Similarly, for the three-delta case the cancellation of divergences in (39) at $\tau = 1$ results in the resonance equation

$$F_3(a_1, a_2, a_3) := \sum_{j=1}^{3} \sqrt{a_j} \tan(a_j) - \sqrt{a_1 a_2} \tan(a_1) - 2 \sqrt{a_1 a_3} \tan(a_1) \tan(a_3)$$

$$- \sqrt{a_2 a_3} \tan(a_2) \tan(a_3) + (a_2 - 1) \sqrt{\frac{a_1 a_3}{a_2}} \prod_{j=1}^{3} \tan(a_j) = 0. \tag{42}$$

Using equation (42) in asymptotic relations (38) at $\tau = 1$, the expressions for the limit elements $\lambda_{11}$ and $\lambda_{22}$ can be simplified. As a result, we obtain the most simple representation of these elements:

$$\lambda_{11} = \left[ \cos(\sqrt{a_1} \cos(\sqrt{a_2} - 2 \sqrt{a_1} \sin(\sqrt{a_1} \cos(\sqrt{a_2} - \sqrt{a_2} \cos(\sqrt{a_1} \sin(\sqrt{a_2} - \sqrt{a_1} \sin(\sqrt{a_2})/\cos(a_3) = (\sqrt{a_1 a_2} \sin(\sqrt{a_1} \sin(\sqrt{a_2})) - \sqrt{a_1} \sin(\sqrt{a_1} \sin(\sqrt{a_2})/\cos(a_3) \right]$$

$$\lambda_{22} = \left[ \cos(\sqrt{a_2} \cos(\sqrt{a_3} - 2 \sqrt{a_2} \cos(\sqrt{a_3} - \sqrt{a_3} \cos(\sqrt{a_2} \sin(\sqrt{a_3} - \sqrt{a_3} \cos(\sqrt{a_2} \sin(\sqrt{a_3})) \right]$$

$$\lambda_{22} = \left[ \cos(\sqrt{a_2} \cos(\sqrt{a_3} - 2 \sqrt{a_2} \cos(\sqrt{a_3} - \sqrt{a_3} \cos(\sqrt{a_2} \sin(\sqrt{a_3} - \sqrt{a_3} \cos(\sqrt{a_2} \sin(\sqrt{a_3})) \right]$$

$$\lambda_{22} = \left[ \cos(\sqrt{a_2} \cos(\sqrt{a_3} - 2 \sqrt{a_2} \cos(\sqrt{a_3} - \sqrt{a_3} \cos(\sqrt{a_2} \sin(\sqrt{a_3} - \sqrt{a_3} \cos(\sqrt{a_2} \sin(\sqrt{a_3})) \right]$$

Using again equation (42), one can check that the formula $\lambda_{11}\lambda_{22} = 1$, in which the matrix elements are given by expressions (43), holds true.

The solutions to transcendental equations (40) and (42) determine countable sets of resonance curves on the $(a_1, a_2)$-plane and resonance surfaces in $(a_1, a_2, a_3)$-space. We refer these resonance curves and surfaces to as $\mathcal{F}_2$- and $\mathcal{F}_3$-sets, respectively. The limit transmission matrix on these sets is of diagonal form (13) with the element $\lambda_{22}$ given by (43), the values of which are determined by the solutions of equations (40) and (42). The point interactions of this countable family may be called ‘multiple-resonant-tunnelling $\delta$-potentials of the $\mathcal{F}$-type’.

Next, on the point subsets of $\mathcal{F}_{2,3}$ defined by

$$\mathcal{P}_2 := \{a_1, a_2 \mid \sin(\sqrt{a_1} = \sin(\sqrt{a_2} = 0)\}$$

$$\mathcal{P}_3 := \{a_1, a_2, a_3 \mid \sin(\sqrt{a_1} = \sin(\sqrt{a_2} = \sin(\sqrt{a_3} = 0)\} \tag{44}$$

we have $\Lambda = \pm I$. Note that no symmetry is required here, i.e., the reflectionless one-point potentials can be realized even if $a_1 \neq a_2$ $(N = 2)$ or $a_1 \neq a_2 \neq a_3$ $(N = 3)$.

Concerning paths 4b (both for $N = 2$ and 3), the cancellation of divergences in the $\lambda_{22}$-terms is impossible, except for the $\mathcal{P}_{2,3}$-subsets on which the divergences in (37) and (39) disappear as well. However, similarly to paths 1b and 2b, we have to analyse the case $\tau = 3/2$ in (39). Here the $\varepsilon \rightarrow 0$ limit of $\lambda_{21}$ will be finite if both the coefficients at $\varepsilon^{-1}$ and $\varepsilon^{-2}$ equal zero simultaneously resulting in two equations. Excluding from these equations the term $\sqrt{a_3} \tan(\sqrt{a_3})$, we find the condition $a_1 \tan^2(\sqrt{a_1} \cos^2(\sqrt{a_2}) + (\sqrt{a_1} \tan(\sqrt{a_1} + \sqrt{a_2} \tan(\sqrt{a_2})^2 = 0$ which cannot be satisfied for all $(a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\}$ and therefore the case with $\tau = 3/2$ does not produce connected point interactions. Thus, outside the $\mathcal{F}_{2,3}$-sets in the case of path 4a and
for all \((a_1, a_2) \in \mathbb{R}^2 \setminus \mathcal{P}_2\) and \((a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \mathcal{P}_3\) for paths 4b, the limit interactions are separated satisfying the boundary conditions \(\psi(\pm 0) = 0\).

Paths 4c and 4d \((\mu = 2, 2 \leq \tau \leq \infty)\): As follows from asymptotes (37) and (39) for these paths, the cancellation of divergences occurs if the equations

\[
G_2(a_1, a_2) := \sqrt{a_1} \tan \sqrt{a_1} + \sqrt{a_2} \tan \sqrt{a_2} = 0, \quad N = 2,
\]

\[
G_3(a_1, a_2, a_3) := \sum_{j=1}^{3} \sqrt{a_j} \tan \sqrt{a_j} - \sqrt{a_1a_3/a_2} \prod_{j=1}^{3} \tan \sqrt{a_j} = 0, \quad N = 3,
\]

are satisfied. Using the first of these equations in (36) at \(\tau = 2\), we find the following two representations for the diagonal elements of the \(\Lambda\)-matrix \((N = 2)\):

\[
\lambda_{11} = \lambda_{22}^{-1} = \cos \sqrt{a_1} / \cos \sqrt{a_2} = -\sqrt{a_1} \sin \sqrt{a_1} / \sqrt{a_2} \sin \sqrt{a_2}.
\] (46)

Similarly, using the second equation (45), we obtain from (38) the diagonal elements for \(N = 3\):

\[
\begin{align*}
\lambda_{11} &= (\cos \sqrt{a_1} \cos \sqrt{a_2} - \sqrt{a_1/a_2} \sin \sqrt{a_1} \sin \sqrt{a_2}) / \cos \sqrt{a_3} \\
&= -\sqrt{a_1} \sin \sqrt{a_1} \cos \sqrt{a_2} + \sqrt{a_2} \cos \sqrt{a_1} \sin \sqrt{a_2}) / \sqrt{a_3} \sin \sqrt{a_3}, \\
\lambda_{22} &= (\cos \sqrt{a_2} \cos \sqrt{a_3} - \sqrt{a_3/a_2} \sin \sqrt{a_2} \sin \sqrt{a_3}) / \cos \sqrt{a_1} \\
&= -\sqrt{a_2} \sin \sqrt{a_2} \cos \sqrt{a_3} + \sqrt{a_3} \cos \sqrt{a_2} \sin \sqrt{a_3}) / \sqrt{a_1} \sin \sqrt{a_1}.
\end{align*}
\] (47)

In virtue of the second equation (45), the equality \(\lambda_{11}\lambda_{22} = 1\), where the elements are given by (47), holds true. Next, as follows from asymptotes (37) and (39) at \(\mu = \tau = 2\) (path 4c), the off-diagonal elements \(\lambda_{21}\) on resonance sets (45) are in general non-zero. For this path they are given by

\[
\lambda_{21} = \begin{cases} \\
\sqrt{a_1a_2} \sin \sqrt{a_1} \sin \sqrt{a_2}, & N = 2, \\
\sqrt{a_1a_2} \sin \sqrt{a_1} \sin \sqrt{a_2} \cos \sqrt{a_3} + 2\sqrt{a_1a_3} \sin \sqrt{a_1} \cos \sqrt{a_2} \sin \sqrt{a_3} \\
& + \sqrt{a_2a_3} \cos \sqrt{a_1} \sin \sqrt{a_2} \sin \sqrt{a_3}, & N = 3.
\end{cases}
\] (48)

Setting \(\theta := \lambda_{11} = \lambda_{22}^{-1}\) and \(\alpha := \lambda_{21}\) where these elements are given by (46)-(48), we get the family of point interactions described by the \(\Lambda\)-matrix of form (38). The elements \(\theta\) and \(\alpha\) are determined by the countable sets of solutions to resonance equations (45). Therefore, similarly to the point interaction realized along path 2a on the single-resonance \(\mathcal{K}_3\)-set, the last family may be called ‘multiple-resonant-tunnelling \((\delta' + \delta)\)-potentials of the \(\mathcal{G}\)-type’.

Some particular cases of the potentials given by equations (46)-(48) should be singled out. First, we note that on the \(\mathcal{P}_{2,3}\)-sets, as follows from (46)-(48), the limit transmission matrix \(\Lambda = \pm I\). Similarly to (44), one can consider the following point subsets of the resonance sets \(\mathcal{G}_{2,3}\):

\[
Q_2 := \{a_1, a_2 | \cos \sqrt{a_1} = \cos \sqrt{a_2} = 0\},
\]

\[
Q_3^{(3)} := \{a_1, a_2, a_3 | \cos \sqrt{a_1} = \cos \sqrt{a_2} = \sin \sqrt{a_3} = 0\},
\]

\[
Q_3^{(2)} := \{a_1, a_2, a_3 | \cos \sqrt{a_1} = \sin \sqrt{a_2} = \cos \sqrt{a_3} = 0\},
\]

\[
Q_3^{(1)} := \{a_1, a_2, a_3 | \sin \sqrt{a_1} = \cos \sqrt{a_2} = \cos \sqrt{a_3} = 0\}.
\] (49)

On these subsets matrix elements (46)-(48) are simplified to

\[
\theta := \lambda_{11} = \pm \begin{cases} \\
\sqrt{a_1/a_2}, & N = 2, \\
\sqrt{a_1/a_2}, \quad \sqrt{a_1/a_3}, \quad \sqrt{a_2/a_3}, & N = 3
\end{cases}
\] (50)
and
\[ \alpha := \lambda_{21} = \mp \begin{cases} \sqrt{a_1a_2} / \sqrt{a_1a_2}, & \text{for } N = 2, \\ 2\sqrt{a_1a_3}, & \text{for } N = 3. \end{cases} \] (51)

In the particular case \( a_1 = a_2, a_1 = a_3 \) and \( a_2 = a_3 \), the multiple-resonant \((\delta' + \delta)\)-potentials are reduced to the multiple-resonant \(\delta\)-potentials given by \(\Lambda\)-matrix (10).

A more general case is the symmetric structure of the regularized potential for \( N = 3 \) if \( a_1 = a_3 \) and \( a_2 \) is arbitrary. Here the second resonance condition (45) is reduced to the following two equations:
\[ \sqrt{a_1} \tan \sqrt{a_1} = \sqrt{a_2}(\cos \sqrt{a_2} \mp 1) / \sin \sqrt{a_2}. \] (52)

Using these equations in (47) and (48), we get the limit transmission matrix that describes the two representations of the \(\delta\)-potential with
\[ \theta = \pm 1 \quad \text{and} \quad \alpha = \mp 2a_1 \sin^2 \sqrt{a_1} \] in \(\Lambda\)-matrix (33), where \( a_1 \) depends on \( a_2 \) through equation (52). As regards paths 4d, instead of equation (48) we have \( \lambda_{21} = 0 \), so that for this family of paths \( \Lambda = \pm I \).

Thus, similarly to the families of paths \( jb, jc \) and \( jd, j = 1, 2, 3 \), resulting in the one-point interactions with single resonances, the interactions realized along path 4c (\( \tau = 2 \)) describe an intermediate case with a partial multiple-resonant transmission, while for paths 4b (\( \tau < 2 \)) the limit interactions are opaque and for paths 4d (\( \tau > 2 \)) they are fully transparent being multiple-resonant as well. Everywhere beyond the \( G_{2,3} \)-sets we have the separated point interactions with the Dirichlet conditions \( \psi(\pm 0) = 0 \). Note that the results given by the first equation (45) and formulae (46) have been obtained earlier in [8, 43].

5. Concluding remarks

The main goal of this paper has been to approximate the system consisting of two and three \(\delta\)-potentials (with intensities \( a_j \neq 0, j = 1, 2 \) if \( N = 2 \) and \( j = 1, 2, 3 \) if \( N = 3 \)) by piecewise constant functions and then to investigate the convergence of the corresponding transmission matrices in the squeezing limit as both the width of \(\delta\)-like functions \( l \) and the distance between them \( r \) tend to zero. The admissible rates of shrinking the parameters \( l \) and \( r \) are controlled through the approximation given by equations (15), involving the two powers \( \mu \) and \( \tau \) as well as the parameter \( \varepsilon \to 0 \). For convenience of the presentation, the three-dimensional \((\varepsilon, l, r)\)-cube has been introduced and various paths inside it were considered. Starting from the same three-layer (for \( N = 2 \)) and five-layer (for \( N = 3 \)) potential profile described by (5), a whole family of limit one-point interactions with resonant-tunnelling behaviour has been realized. The resonance sets for these interactions are curves on the \((a_1, a_2)\)-plane \( (N = 2) \) and surfaces in the \((a_1, a_2, a_3)\)-space \( (N = 3) \). The number of resonances (one or infinite) depends on a path, along which the corresponding sequence of transmission matrices has a limit.
For both the cases with $N = 2$ and 3 we single out the four resonance sets named $\mathcal{K}_{2,3}$, $\mathcal{L}_{2,3}$, $\mathcal{F}_{2,3}$, $\mathcal{G}_{2,3}$ and defined by equations (29), (34), (40) and (42), (45), respectively. The first two of these sets describe single and the two others multiple resonances. Accordingly, the one-point interactions realized on these sets belong to $\mathcal{K}$-, $\mathcal{L}$-, $\mathcal{F}$- and $\mathcal{G}$-families and their $\Lambda$-matrices are given by (10), (13) and (33). The $\Lambda$-matrix elements for the interactions of the $\mathcal{K}$- and $\mathcal{L}$-families are single-valued, while for the families $\mathcal{F}$ and $\mathcal{G}$ these elements are multi-valued. All these interactions together with the paths along which they are realized, including the corresponding resonance sets and $\Lambda$-matrices are summarized in table 1. Here the following four subfamilies of one-point

### Table 1. Resonance sets and transmission matrices for resonant-tunnelling point interactions realized along all the possible paths in the $(\varepsilon, l, r)$-cube

| Resonant point interactions | Paths  | Resonance sets $(N = 2)$ | Resonance sets $(N = 3)$ | $\Lambda$-matrices |
|-----------------------------|--------|--------------------------|--------------------------|-------------------|
| $\delta'$-potentials        | 1a     | $\mathcal{K}_2$          | $\mathcal{K}_3$          | (13), (30)        |
|                             | 2a, 3a | $\mathcal{K}_2$          | -                        |                   |
|                             | 4a     | $\mathcal{F}_2$          | $\mathcal{F}_3$          | (13), (11), (43)  |
| $\delta$-potentials         | 1c, 2c | $\mathcal{L}_2$          | $\mathcal{L}_3$          | (10), $\alpha = a_1 a_2$, (35) |
|                             | 2d     | -                        | $\mathcal{L}_3$          | (10), $\alpha = a_1 a_2$ |
|                             | 3c     | $\mathcal{L}_2$          | -                        | (10), $\alpha = a_1 a_2$ |
|                             | 4c     | $\mathcal{Q}_2$, $a_1 = a_2$ | -                        | (33), (50), (51) |
|                             |        | $\mathcal{Q}_2^{[4]}$, $a_1 = a_2$ |                |                   |
|                             |        | $\mathcal{Q}_2^{[2]}$, $a_1 = a_3$ |                |                   |
|                             |        | $\mathcal{Q}_3^{[1]}$, $a_2 = a_3$ |                |                   |
|                             |        | $a_1 = a_3$, (52), (53) |                |                   |
| $(\delta' + \delta)$- potentials | 2a     | -                        | $\mathcal{K}_3$          | (30), (33), $\alpha = a_1 a_3$ |
|                             | 4c     | $\mathcal{G}_2$          | $\mathcal{G}_3$          | (33), (43), (48)  |
| Reflectionless potentials   | 1a     | -                        | $a_1 = a_3$, $2a_1 + a_2 - a_1 a_2 = 0$ | $\Lambda = I$ |
|                             |        | $a_1 = a_2 = 2$          | $a_1 + a_3 = 2$, $2a_1 + a_2 - a_1 a_2 = 2$ | $\Lambda = -I$ |
|                             | 1d     | $\mathcal{L}_2$          | $\mathcal{L}_3$          | $\Lambda = I$ |
|                             | 2d, 3d | $\mathcal{L}_2$          | -                        | $\Lambda = \pm I$ |
|                             | 4a, 4b | $\mathcal{P}_2$          | $\mathcal{P}_3$          | $\Lambda = \pm I$ |
|                             | 4c, 4d | $\mathcal{Q}_2$, $a_1 = a_2$ | -                        | (52), $a_1 = a_3$ |
|                             | 4d     | $\mathcal{Q}_2^{[3]}$, $a_1 = a_2$ |                |                   |
|                             |        | $\mathcal{Q}_2^{[2]}$, $a_1 = a_3$ |                |                   |
|                             |        | $\mathcal{Q}_3^{[1]}$, $a_2 = a_3$ |                |                   |
resonant-tunnelling interactions are singled out: (i) the $\delta'$-potentials (single-resonant of the $\mathcal{K}$- and multiple-resonant of the $\mathcal{F}$-, $\mathcal{G}$-types), (ii) the $\delta$-potentials (single-resonant of the $\mathcal{L}$-type, including multiple-resonant defined on the $\mathcal{Q}$-subsets), (iii) the $(\delta' + \delta)$-potentials (single-resonant of the $\mathcal{K}$- and multiple-resonant of the $\mathcal{G}$-types) and (iv) the reflectionless potentials (single-resonant of the $\mathcal{L}$-type, including multiple-resonant defined on the $\mathcal{P}$- and $\mathcal{Q}$-subsets).

Outside the resonance sets all the one-point interactions become separated with the Dirichlet boundary conditions $\psi(\pm 0) = 0$. The corresponding conditions for the existence of this type of interactions given on the $(a_1, a_2)$-plane and in the $(a_1, a_2, a_3)$-space depend on the paths and they are summarized in table 2.

It should be noticed that in this paper we have restricted ourselves to power parameterization (15). The admissible set of the powers $\mu$ and $\tau$ for realizing point interactions appears to be the set $Q := \{2 \leq \mu \leq \infty\} \times \{1 \leq \tau \leq \infty\}$. Using this parameterization as well as the piecewise constant approximation of the $\delta$-functions in potential (3), it is possible to get the explicit solutions for the corresponding $\Lambda$-matrices and to treat thus the reflection-transmission properties of the one-point interactions directly. It is of interest to note that inside the set $Q$ the resonance sets are single-valued and when approaching the boundary $\{\mu = 2, 1 \leq \tau \leq \infty\}$, the furcation of the resonance sets occurs. Qualitatively, all these results are the same for $N = 2$ and 3, except for the dimension of the resonance sets and the corresponding equations. In principle, a similar straightforward analysis could be carried out for higher $N$ resulting in the same types of one-point interactions with resonance sets $\mathcal{K}_N$, $\mathcal{L}_N$, $\mathcal{F}_N$ and $\mathcal{G}_N$ being $(N - 1)$-dimensional hypersurfaces, however, the corresponding formulae appear to be quite complicated. In the case, if we would like to deal with potentials (5) which admit distributional limits, for instance, the $\delta'(x)$ potential, constraint (34) has to be imposed in addition to sets (29), (40), (42) and (15). Therefore this constraint reduces the dimension of resonance sets by one, so that for $N = 2$, instead of the resonance curves, we have the corresponding set of discrete numbers and for $N = 3$ one-dimensional curves.

| Paths | Non-resonant conditions ($N = 2$) | Non-resonant conditions ($N = 3$) |
|-------|----------------------------------|----------------------------------|
| 1a, 2a | $(a_1, a_2) \notin \mathcal{K}_2 \cup \{0\}$ | $(a_1, a_2, a_3) \notin \mathcal{K}_3 \cup \{0\}$ |
| 1b, 2b, 3b | $(a_1, a_2) \in \mathbb{R}^2 \setminus \{0\}$ | $(a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{0\}$ |
| 1c, 1d, 2c, 2d | $(a_1, a_2) \notin \mathcal{L}_2 \cup \{0\}$ | $(a_1, a_2, a_3) \notin \mathcal{L}_3 \cup \{0\}$ |
| 3a | $(a_1, a_2) \notin \mathcal{G}_2 \cup \{0\}$ | $(a_1, a_2, a_3) \notin \mathcal{G}_3 \cup \{0\}$ |
| 3c, 3d | $(a_1, a_2) \notin \mathcal{F}_2 \cup \{0\}$ | $(a_1, a_2, a_3) \notin \mathcal{F}_3 \cup \{0\}$ |
| 4a | $(a_1, a_2) \notin \mathcal{P}_2 \cup \{0\}$ | $(a_1, a_2, a_3) \notin \mathcal{P}_3 \cup \{0\}$ |
| 4b | $(a_1, a_2) \notin \mathcal{G}_2 \cup \{0\}$ | $(a_1, a_2, a_3) \notin \mathcal{G}_3 \cup \{0\}$ |
Some of the particular cases for $N = 2$ have been treated in [8, 43, 50]. To conclude, it should be noticed that the approach developed in this paper can be a starting point for further studies on regular approximations of point interactions and understanding the resonant mechanism.

Acknowledgments

The financial support from the National Academy of Sciences of Ukraine under project No. 0112U000053 is acknowledged. The author would like to express gratitude to Yaroslav Zolotaryuk for stimulating discussions and valuable suggestions.

References

[1] Demkov Y N and Ostrovskii V N 1975 Zero-Range Potentials and Their Applications in Atomic Physics (Leningrad: Leningrad University Press)
[2] Demkov Y N and Ostrovskii V N 1988 Zero-Range Potentials and Their Applications in Atomic Physics (New York: Plenum)
[3] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 2005 Solvable Models in Quantum Mechanics (With an Appendix by Pavel Exner) 2nd revised edn (Providence: RI: American Mathematical Society: Chelsea Publishing)
[4] Albeverio S and Kurasov P 1999 Singular Perturbations of Differential Operators: Solvable Schrödinger-Type Operators (Cambridge: Cambridge University Press)
[5] Perez J F and Coutinho F A B 1991 Am. J. Phys. 59 52
[6] Brasche J F, Figari R and Teta A 1998 Potential Analysis 8 163
[7] Exner P, Neidhardt H and Zagrebnov V A 2001 Commun. Math. Phys. 224 593
[8] Christiansen P L, Arnbak N C, Zolotaryuk A V, Ermakov V N and Gaididei Y B 2003 J. Phys. A: Math. Gen. 36 7589
[9] Coutinho F A B and Amaku M 2009 Eur. J. Phys. 30 1015
[10] Exner P and Manko S S 2014 Lett. Math. Phys. 104 1079
[11] Albeverio S, Cacciapuoti C and Finco D 2007 J. Math. Phys. 48 032103
[12] Cacciapuoti C and Exner P 2007 J. Phys. A: Math. Theor. 40 F511
[13] Turek O and Cheon T 2012 Europhys. Lett. 98 50005
[14] Turek O and Cheon T 2013 Ann. Phys. (N.Y.) 330 104
[15] Zolotaryuk A V 2013 Phys. Rev. A 87 052121
[16] Zolotaryuk A V and Zolotaryuk Y 2015 Phys. Lett. A 379 511
[17] Zolotaryuk A V and Zolotaryuk Y 2015 J. Phys. A: Math. Theor. 48 035302
[18] Šeba P 1986 Rep. Math. Phys. 24 111
[19] Gesztesy F and Holden H 1987 J. Phys. A: Math. Gen. 20 5157
[20] Griffiths D J 1993 J. Phys. A: Math. Gen. 26 2265
[21] Kurasov P 1996 J. Math. Anal. Appl. 201 297
[22] Albeverio S, Dąbrowski L and Kurasov P 1998 Lett. Math. Phys. 45 33
[23] Coutinho F A B, Nogami Y and Perez J F 1997 J. Phys. A: Math. Gen. 30 3937
[24] Coutinho F A B, Nogami Y and Tomio L 1999 J. Phys. A: Math. Gen. 32 4931
[25] Albeverio S and Nizhnik L 2003 Lett. Math. Phys. 65 27
[26] Nizhnik L N 2003 J. Funct. Anal. Appl. 37 85
[27] Nizhnik L N 2006 J. Funct. Anal. Appl. 40 74
[28] Gadella M, Negro J and Nieto L M 2009 Phys. Lett. A 373 1310
[29] Arnbak H, Christiansen P L and Gaididei Y B 2011 Philos. Trans. R. Soc. A 369 1228
[30] Lange R-J 2012 J. High Energy Phys. JHEP11(2012), no. 32
[31] Brasche J F and Nizhnik L P 2013 Methods Funct. Anal. Topol. 19 4 [arXiv:1112.2545v1 [math.FA]]
Furcation of resonance sets for one-point interactions

[32] Gadella M, García-Ferrero M A, González-Martín S and Maldonado-Villamizar F H 2014 Int. J. Theor. Phys. 53 1614
[33] Lange R-J 2015 J. Math. Phys. 56 122105
[34] Zolotaryuk A V 2015 J. Phys. A: Math. Theor. 48 255304
[35] Kulinskii V L and Panchenko D Y 2015 Physica B: Physics of Condensed Matter 472 78
[36] Dias N C, Jorge C and Prata J N 2016 J. Differential Equations 260 6548
[37] Gadella M, Mateos-Guilarte J, Muñoz-Castañeda J M and Nieto L M 2016 J. Phys. A: Math. Theor. 49 015204
[38] Konno K, Nagasawa T and Takahashi R 2016 [arXiv:1605.05418v 1 [quant-ph]]
[39] Kostenko A and Malamud M 2013 Spectral Analysis, Differential Equations and Mathematical Physics - Proceedings of Symposia in Pure Mathematics eds H Holden et al. vol. 87 (Providence: RI: American Mathematical Society) p. 235
[40] Zolotaryuk A V, Christiansen P L and Iermakova S V 2006 J. Phys. A: Math. Gen. 39 9329
[41] Toyama F M and Nogami Y 2007 J. Phys. A: Math. Theor. 40 F685
[42] Zolotaryuk A V and Zolotaryuk Y 2014 Int. J. Mod. Phys. B 28 1350203
[43] Golovaty Y D and Man’ko S S 2009 Ukrainian Math. Bull. 6 169 (e-print [arXiv:0909.1034v 2 [math.SP]])
[44] Golovaty Y D and Hryniv R O 2010 J. Phys. A: Math. Theor. 43 155204
Golovaty Y D and Hryniv R O 2011 J. Phys. A: Math. Theor. 44 049802
[45] Man’ko S S 2010 J. Phys. A: Math. Theor. 43 445304
[46] Golovaty Y 2012 Methods Funct. Anal. Topol. 18 243
[47] Man’ko S S 2012 J. Math. Phys. 53 123521
[48] Golovaty Y D and Hryniv R O 2013 Proc. R. Soc. Edinb. A 143 791
[49] Golovaty Y 2013 Integr. Equ. Oper. Theor. 75 341
[50] Zolotaryuk A V 2010 Phys. Lett. A 374 1636
[51] Cheon T and Shigehara T 1998 Phys. Lett. A 243 111
[52] Albeverio S and Nizhnik L 2000 Ukr. Mat. Zh. 52 582, translation in 2001 Ukr. Math. J. 52 664.
[53] Griffiths D and Walborn S 1999 Am. J. Phys. 67 446
[54] De Vincenzo S and Sanchez C 2010 Can. J. Phys. 88 809
[55] Coutinho F A B, Nogami Y and Toyama F M 2012 Can. J. Phys. 90 383
[56] Zolotaryuk A V and Zolotaryuk Y 2011 J. Phys. A: Math. Theor. 44 375305