Exact particular solution for the blade-like surface configuration of a conducting liquid in an external electric field

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Abstract. Equilibrium surface configurations of the conducting liquid (molten metal) in an external electric field are studied with regard to the finiteness of the interelectrode distance. Equilibrium is achieved due to the balance between electrostatic and capillary forces. For the plane symmetric case, applying the conformal mapping method, we obtain an exact particular solution for the boundary shape: the liquid takes the shape of a blade. We find dependences of the curvature radius of the blade edge, of the distance from the edge to the flat electrode, and of the electric field strength on the applied potential difference and the surface tension coefficient.

1. Introduction

The free surface of the conducting liquid is deformed in an applied external electric field [1–9]. In the case of mutual compensation of capillary and electrostatic forces, a new equilibrium surface configuration differed from the original one can appear [10–15]. For the axial symmetry of the problem, it is known a particular stationary solution for the boundary shape, the so-called Taylor cone [16]. For the planar geometry, exact solutions were found for the shape of the charged jet [17], the shape of the neutral jet in a transverse external electric field [18–20], as well as for the periodic perturbations of the initially planar boundary of the liquid in a vertical electric field [21].

In the present work, we will consider the problem on the free surface shape of the liquid in an external electric field for the case of a finite interelectrode distance (a flat electrode is placed at some distance from the liquid surface). The electrode significantly affects the distribution of the electric field in the interelectrode gap: the electrostatic image forces arise.

We restrict ourselves to the plane symmetric case, when the conformal mapping method can be applied. It makes possible to simplify governing equations [17, 22]. As will be demonstrated, the problem admits an exact particular solution, for which the liquid takes the shape of a blade (the boundary inclination angle changes by $2\pi$ along the surface). The electric field strength is maximal at the edge of the blade and decreases to zero at the periphery; i.e., such a configuration ensures unlimited local amplification of the electric field. Note that, in the case of an infinite interelectrode distance, similar equilibrium configurations do not exist: if the
electric field strength exceeds a certain threshold value, drops break away from the bulk of the liquid [21, 23].

The obtained exact solution will enable us to determine the relation between the applied potential difference and the value of the field strength at the edge of the blade, and to find the curvature of the edge and the distance from the liquid to the flat electrode.

2. Initial equations

We will consider the situation when a flat infinite electrode is located at some distance from the free surface of a conducting liquid (denote this surface by \(S\)). There is a constant potential difference \(U\) between the liquid and the electrode. Let us assume that the problem has plane symmetry, i.e., all the quantities depend only on the pair of coordinates \(x\) and \(y\). The \(x\) axis of a rectangular system of coordinates coincides with the surface of the flat electrode. The \(y\) axis is perpendicular to the electrode and passes through the apex of the protrusion on the liquid surface; the position of the apex is given by the conditions \(x = 0\) and \(y = -D\), where \(D\) is the distance from the apex to the electrode.

The distribution of the electric field potential \(\varphi(x, y)\) is defined by the Laplace equation

\[
\varphi_{xx} + \varphi_{yy} = 0. \tag{1}
\]

It should be solved together with the condition of equipotentiality of the surface \(S\) of the conducting liquid, as well as of the electrode:

\[
\varphi|_{S} = 0, \quad \varphi|_{y=0} = -U.
\]

Suppose that the electric field strength tends to zero far away from the apex:

\[
|\nabla \varphi| \to 0 \quad \text{as} \quad |x| \to \infty.
\]

The equilibrium shape of the free surface of a conducting liquid is determined by the pressure balance (Laplace–Young) condition:

\[
\frac{\varepsilon_0}{2} (\varphi_x^2 + \varphi_y^2) + \sigma \kappa = 0.
\]

Here \(\kappa\) is the local curvature of the surface, \(\sigma\) is the surface tension coefficient, and \(\varepsilon_0\) is the dielectric constant. The first term corresponds to the electrostatic pressure, and the second, to the capillary pressure.

Let us pass to dimensionless variables by the substitutions \(x \to \lambda x\), \(y \to \lambda y\), and \(\varphi \to U \varphi\), where \(\lambda = \varepsilon_0 U^2 (2\sigma)^{-1}\) is the characteristic spatial scale. The quantity \(d = D/\lambda\) is a dimensionless analog of the distance \(D\).

In the new variables, the conditions on the surface of the liquid, on the surface of the electrode, and at infinity take the form

\[
\varphi_x^2 + \varphi_y^2 + \kappa = 0 \quad \text{on} \quad S, \tag{2}
\]

\[
\varphi = 0 \quad \text{on} \quad S, \tag{3}
\]

\[
\varphi = -1 \quad \text{at} \quad y = 0, \tag{4}
\]

\[
|\nabla \varphi| \to 0 \quad \text{as} \quad |x| \to \infty. \tag{5}
\]

The Laplace equation (1) in combination with the boundary conditions (2)–(5) represents the boundary value problem defining the shape of the liquid surface.
3. Complex variables

The boundary-value problem formulated in the previous section is convenient to solve by using the method of conformal mapping. We introduce the complex potential \( w = \phi - i\psi \) of the electric field, where the function \( \psi \) is the harmonic conjugate of the potential \( \phi \) (i.e., they are related by the Cauchy–Riemann conditions, \( \varphi_x = -\psi_y \) and \( \varphi_y = \psi_x \)). The potential \( w \) is an analytic function of the complex variable \( z = x + iy \). Then the complex electric field strength is

\[
F = \frac{dw}{dz} = \varphi_x - i\varphi_y. \tag{6}
\]

It can be represented in the polar form as

\[
F = -E \exp(-i\theta), \tag{7}
\]

where \( E = (\varphi_x^2 + \varphi_y^2)^{1/2} \) is the absolute value of the field strength and \( \theta = \arctan(\varphi_y/\varphi_x) \) is the angle of inclination of the field strength to the \( x \) axis.

The pressure balance condition (2) with use of the quantity \( E \) rewrites as

\[
E^2 + \kappa = 0 \quad \text{on} \quad S. \tag{8}
\]

The local curvature \( \kappa \) of the surface can be expressed in terms of \( \theta, \psi, \) and \( E \). As the vector of the strength is normal to the conducting liquid surface, then \( \theta|_S \) defines the angle of inclination of the outward normal to \( S \) to the \( x \) axis. Then, in accordance with definition, the surface curvature is \( \kappa = \partial\theta/\partial S \), where \( \partial S \) is an elementary arc of the surface. Using the Cauchy–Riemann equations for \( \phi \) and \( \psi \) functions, we find

\[
\kappa = \frac{\partial\theta}{\partial\psi} \frac{\partial\psi}{\partial S} = \frac{\partial\theta}{\partial\psi} \frac{\partial\phi}{\partial n}.
\]

As \( E \equiv -\partial\phi/\partial n \), where \( \partial/\partial n \) denotes the normal derivative, we finally obtain for the curvature:

\[
\kappa = -E \frac{\partial\theta}{\partial\psi}.
\]

Then condition (8) can be written in the simple form:

\[
\theta_\psi = E \quad \text{at} \quad \varphi = 0, \quad \tag{9}
\]

where we have taken into account (3). Hence it becomes clear that, in the considered problem, it is convenient to take the complex value \( F \) (or the pair of real functions \( E \) and \( \theta \)) as an unknown function and the complex potential (or the pair of real quantities \( \phi \) and \( \psi \)) as an independent variable. The latter corresponds to the conformal mapping the domain bounded by the electrode \( y = 0 \) and by the free surface \( S \) into the strip

\[
-1 \leq \varphi \leq 0, \quad -\infty < \psi < \infty. \tag{10}
\]

It is important that, after transition from \( z \) to the new variable \( w \), the complex strength of the electric field remains an analytic function, i.e., \( F = F(w) \) in the strip (10).

Finally, conditions on the flat electrode (4) and at infinity (5) can be rewritten as

\[
\theta = \pi/2 \quad \text{at} \quad \varphi = -1, \quad \text{as} \quad |\psi| \to \infty. \tag{11}
\]

Thus, the initial problem with unknown boundary is reduced to a much simpler problem on the strip. Its exact particular solution will be constructed in the following section.
4. Exact particular solution

Although equations (9), (11), (12) are much simpler than the original ones, there is no general method of constructing their solutions. Let us show that the particular solution of the problem can be found if we suppose that, on the surface of the liquid, the quantities $E$ and $\theta$ are related as follows:

$$E = E_{\text{max}} \sin \theta \quad \text{at} \quad \varphi = 0.$$  \hspace{1cm} (13)

Here $E_{\text{max}}$ is a constant which gives the maximum value of the field on the surface (its value will be defined later). Note that, in the recent work [24], the use of an analogous condition allowed us to find solutions for the equilibrium shape of a cylindrical column (a jet for applications) of a conducting liquid in a transverse electric field. A similar approach was also used to study the equilibrium configurations of a system of conducting liquid columns in a high-frequency magnetic field [25].

It is important that, when we apply the assumption (13), the boundary condition (9) takes the form of an ordinary differential equation (ODE):

$$\theta \psi' = E_{\text{max}} \sin \theta \quad \text{at} \quad \varphi = 0.$$  \hspace{1cm} (14)

This ODE can be easily solved analytically. For the angle $\theta$ on the liquid surface, we obtain

$$\theta = \frac{\pi}{2} + \arcsin \left( \tanh \left( E_{\text{max}} \psi \right) \right).$$  \hspace{1cm} (14)

Here the integration constant is chosen so that $\theta = \pi/2$ at $\psi = 0$ (i.e., the cusp is located at the point $w = 0$).

As can be seen from (14), $\theta \to \pi$ as $\psi \to \infty$ and $\theta \to 0$ as $\psi \to -\infty$. Then the inclination angle changes by $2\pi$ along the surface of a liquid. This corresponds to our a priori notion about the surface configuration (we suppose that the surface has the blade-like shape).

Substituting (13) and (14) into (7), we obtain the expression for the complex strength on the free surface:

$$F = -E_{\text{max}} \sin(\theta) \exp(-i\theta) = \frac{iE_{\text{max}}}{1 + i \sinh \left( E_{\text{max}} \psi \right)}.$$  \hspace{1cm} (15)

Since $F$ is an analytic function, it is possible to construct its analytic continuation from the free surface of the liquid to the strip (10). This corresponds to the substitution $\psi \to i w$ in expression (15). We get

$$F(w) = \frac{iE_{\text{max}}}{1 - \sin \left( E_{\text{max}} w \right)}. \hspace{1cm} (16)$$

Let us verify that expression (16) satisfies conditions (11) and (12). Remind that $w = -1 - i \psi$ on the electrode. Since the electrode surface is flat, the $x$-component of the field strength is absent, or $\Re F(-1 - i \psi) = 0$ [this expression is identical to condition (11)]. It is easy to see that this requirement is satisfied for

$$E_{\text{max}} = \pi/2,$$  \hspace{1cm} (17)

which uniquely determines the value of parameter $E_{\text{max}}$ from hypothesis (13). Condition (12) is naturally satisfied because the denominator becomes infinite at $\Im w \to \pm\infty$ in (16). Therefore, expression (16) together with (17) gives the solution of the considered problem.

For a known function $F(w)$, the distribution of the electric field potential over space, as well as the sought equilibrium shape of the free surface of the liquid, can be found by solving ODE $dw/dz = F(w)$, following from the definition (6). Substituting here (16) with $E_{\text{max}} = \pi/2$ and integrating, we obtain the following expression for the mapping function $z(w)$:

$$z(w) = -\frac{2i}{\pi} \left( 1 + w + \frac{2}{\pi} \cos \left( \frac{\pi w}{2} \right) \right). \hspace{1cm} (18)$$
Figure 1. The equilibrium shape of the surface of a conducting liquid corresponding to exact solution (21). The position of the electrode, \( y = 0 \), is also shown.

Here the integration constant is chosen so that the origin is located on the electrode surface, i.e., \( z(-1) = 0 \).

Dividing expression (18) into real and imaginary parts, we obtain parametric expressions for the equipotential surfaces (\( \varphi = \text{const} \)):

\[
x = -\frac{2\psi}{\pi} + \frac{4}{\pi^2} \sin\left(\frac{\pi \varphi}{2}\right) \sinh\left(\frac{\pi \psi}{2}\right),
\]

\[
y = -\frac{2}{\pi} - \frac{2\varphi}{\pi} - \frac{4}{\pi^2} \cos\left(\frac{\pi \varphi}{2}\right) \cosh\left(\frac{\pi \psi}{2}\right),
\]

where \( \psi \) plays the role of a parameter.

The equilibrium shape of the liquid can be found by substituting \( \varphi = 0 \) into (19) and (20) and excluding the parameter \( \psi \). As a result, we get

\[
y(x) = -\frac{2}{\pi} - \frac{4}{\pi^2} \cosh\left(\frac{\pi^2 x}{4}\right).
\]

Figure 1 shows the shape of the liquid surface, which is defined by the obtained exact solution (21). The thick straight line corresponds to the position of the electrode. As can be seen, the liquid takes the form of a blade. Formally, this blade is infinite in the direction of the \( y \) axis. However, it is clear that for a real system, the liquid blade will be bounded in space and attached to the electrode base on the left in the figure.

5. Analysis of the obtained solution

Let us analyze the exact particular solution (21) obtained for the equilibrium shape of the boundary of a conducting liquid. The apex of the liquid blade is located at \( x = 0 \), and, in this case, the distance (in dimensionless notation) between the boundary of the liquid and the electrode is determined by the expression

\[
d = -y(0) = 2(\pi + 2)/\pi^2 \approx 1.042.
\]
The curvature of the surface at the point of greatest deformation \((x = 0)\) can be determined from the formula

\[ \kappa_0 = \frac{d^2 y}{dx^2} \bigg|_{x=0}. \]

Then the dimensionless curvature radius of the edge is

\[ r = \frac{1}{\kappa_0} = \frac{4}{\pi^2} \approx 0.405. \] (23)

It is comparable to the distance to the electrode, \(d \approx 1.042\).

As for the electric field strength at the edge of the liquid blade, it is defined by the formula (17), i.e., \(E_{\text{max}} = \pi/2 \approx 1.571\).

Let us rewrite the formulas (22), (23), and (17) in the dimensional form. We obtain for the distance \(D\) between the edge of the liquid blade and the electrode, the radius \(R\) of the edge curvature, and the electric field strength \(E_{\text{max}}\) at the edge:

\[ D = \frac{\varepsilon_0 U^2 (\pi + 2)}{\pi^2 \sigma} \approx 0.521 \frac{\varepsilon_0 U^2}{\sigma}, \] (24)

\[ R = \frac{2 \varepsilon_0 U^2}{\pi^2 \sigma} \approx 0.203 \frac{\varepsilon_0 U^2}{\sigma}, \] (25)

\[ E_{\text{max}} = \frac{\pi \sigma}{\varepsilon_0 U} \approx 3.142 \frac{\sigma}{\varepsilon_0 U}. \] (26)

Thus, the values of \(D\), \(R\), and \(E_{\text{max}}\) are uniquely defined by the applied potential difference \(U\) and by the surface tension coefficient \(\sigma\).

Note that the values of \(D\) and \(R\) are comparable: \(D/R \approx 2.57\). It means that, for the obtained solution, the influence of the electrode on the field distribution is significant (the presence of an electrode leads to appearance of electrostatic image forces). It is noteworthy that, in the case of \(R/D \ll 1\) (the electrode is removed so far away from the liquid that it does not affect the field distribution near the cusp), equilibrium configurations providing the unlimited local amplification of the electric field do not exist.

In concluding this section, we estimate the characteristic values of \(D\), \(R\), and \(E_{\text{max}}\). Let us take for the potential difference \(U = 10\ \text{kV}\). For molten copper, the surface tension coefficient is \(\sigma \approx 1.37\ \text{N/m}\) [26]. Then formulas (24)–(26) give

\[ D \approx 0.34\ \text{mm}, \quad R \approx 0.13\ \text{mm}, \quad E_{\text{max}} \approx 4.9 \times 10^5\ \text{V/cm}. \]

For water (its electrostatic properties are similar to those of a conducting liquid) we can take \(\sigma \approx 0.073\ \text{N/m}\). Then we find

\[ D \approx 6.3\ \text{mm}, \quad R \approx 2.5\ \text{mm}, \quad E_{\text{max}} \approx 2.6 \times 10^4\ \text{V/cm}, \]

i.e., the characteristic distances and the field strength differ essentially from those for the copper.

When constructing the solution, the influence of gravity was neglected. For the free surface in the electric field, the characteristic scale for which the gravity pressure should be taken into account in the Laplace–Young condition (2) is given by the gravity-capillary length [2, 8]

\[ \lambda = 2\pi \sqrt{\sigma/(\rho g)} , \]

where \(\rho\) is the fluid density and \(g\) is the acceleration due to gravity. For liquid copper, we have \(\rho \approx 8\ \text{g/cm}^3\), \(\lambda \approx 26\ \text{mm}\), i.e., \(D \ll \lambda\) and \(R \ll \lambda\). It means that the influence of gravity is negligibly small. For water, we have \(\rho \approx 1\ \text{g/cm}^3\), \(\lambda \approx 17\ \text{mm}\), that exceeds several times \(D\) and \(R\). Nevertheless, it is possible to conclude that, for water, the gravity forces can influence the shape of the boundary.
Figure 2. The geometry of the rupture of the conducting liquid layer in an electric field. The family of the equipotential surfaces, \( \varphi = -1/3, -2/3, -1, -4/3, -5/3 \), is also shown [they are given by equations (19) and (20)]. Note that the liquid surface is also equipotential: \( \varphi = 0 \) to the left and \( \varphi = -2 \) to the right from the rupture.

6. Concluding remarks

Thus, we have found the particular exact solution for the equilibrium shape of the free surface of a conducting liquid in an external electric field with regard to the finiteness of the interelectrode distance. Equilibrium is achieved due to the mutual compensation of capillary and electrostatic forces on the free surface. For the obtained solution, the liquid takes the shape of a blade (see figure 1); the electric field strength is maximal at the edge of the blade [its value is given by formula (26)] and decreases to zero at infinity.

This solution can be formally continued into the domain \( y > 0 \). Then it will determine the geometry of the rupture of the free layer (sheet) of a conducting liquid in an electric field, which is illustrated in figure 2. The solution has a mirror symmetry with respect to the plane \( y = 0 \). For the problem considered above, this plane was the boundary of the electrode. Now it has a sense of the equipotential surface. The potential difference at the rupture equals \( 2U \), and the width of the rupture is \( 2D \).

If we decrease the rupture width with the unchanged geometry of the free surface and fixed potential difference, then the electrostatic pressure increases and, as a consequence, the pressure balance will be violated. This will result in a collapse of the rupture. On the contrary, for increasing rupture width, uncompensated capillary forces arise, and the rupture will expand. This indicates that the exact solution for the rupture is instable. It defines the threshold width of the rupture, which separates different regimes of evolution of the liquid layer: expansion or collapse of the rupture.

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