Connection Between System Parameters and Localization Probability in Network of Randomly Distributed Nodes

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Abstract—This article deals with localization probability in a network of randomly distributed communication nodes contained in a bounded domain. A fraction of the nodes denoted as L-nodes are assumed to have localization information while the rest of the nodes denoted as NL-nodes do not. The basic model assumes each node has a certain radio coverage within which it can make relative distance measurements. We model both the case radio coverage is fixed and the case radio coverage is determined by signal strength measurements in a Log-Normal Shadowing environment. We apply the probabilistic method to determine the probability of NL-node localization as a function of the coverage area to domain area ratio and the density of L-nodes. We establish analytical expressions for this probability and the transition thresholds with respect to key parameters whereby marked change in the probability behavior is observed. The theoretical results presented in the article are supported by simulations.

Index Terms—Ad-hoc network, connectivity, GPS, iterative localization, LBS, localization, location based services, positioning, probabilistic method, random arrays, range-free localization, sensor networks.

I. INTRODUCTION

This article deals with a network of randomly distributed nodes some of which have a-priori knowledge about their position (these are the so called L-nodes) while others are supposed to localize themselves (these are the so called non-localized or NL-nodes).

This general framework can be recognized in many practical scenarios and in different types of communication networks, such as Distributed Sensor Networks (DSNs) and wireless networks. Generally speaking, we may classify the localization algorithms in at least two ways, centralized or distributed [1] and range-free or based on ranging techniques [2]. The most common techniques are based on measured range whereby the location of nodes are estimated through some standard methods such as triangulation. The ranging technique itself can be based on: 1) Received Signal Strength (RSS) measurements, 2) Time Of Arrival (TOA) measurements requiring clock synchronization among nodes, 3) Time Difference Of Arrival (TDOA) techniques requiring relative synchronicity of L-nodes, 4) Angle Of Arrival (AOA) measurements, and 4) combination of the above.

In range-free localization, connectivity between nodes is a binary event, either two nodes are within communication range of each other or they are not [3]. For simplicity, we may view this as a hard quantization of for instance RSS. If RSS is above a certain detection threshold, the nodes can communicate, otherwise they cannot. Of course, the nature of path loss and the terrain characteristics influences both the coverage radius and the deviation of the coverage zone from the idealistic circular geometry. Various range free algorithms have been proposed in the literature including the centroid algorithm [4], the DV-HOP algorithm [5], the Amorphous positioning algorithm [6], APIT [7], and ROCRSSI [2]. The fundamental characteristic of these algorithms is their ability to provide coarse (compared to ranging techniques) localization information with minimal per node communication and computation requirements.

Regardless of whether ranging based or range-free techniques are used for localization, several global issues of concern remain to be addressed in their application. In particular, this article deals with broader questions of whether a given NL-node can localize itself in a random array of nodes, a percentage of which have localization information. What are the requirements on the density of nodes over a terrain and coverage area of individual nodes that would allow node localization with very high probability?

These questions and others like them are at the heart of the localization problem and by their very nature, require a probabilistic setting [8]. To have a frame of reference, we present a communication scenario which is needed before any localization of NL-nodes can take place. In particular, we assume: 1) ranging techniques are employed and the coverage radius of each node is dependent on the path loss characteristic. While the coverage radius may vary from node to node, we assume that the coverage region is circular. The fact that the coverage geometry may not be perfectly circular has no significant bearing on the calculated probabilities. This is because of our assumption of uniform distribution of nodes on the terrain whereby what matters in so far as probability of finding other nodes in a given area around a reference node is concerned, is the area itself and not its geometric shape; 2) the nodes have calibrated power levels possibly achieved via an external beacon signal, 3) a CSMA/CA link layer protocol is employed for communication among nodes, hence, while collisions might occur, we can assume that the nodes within communication range can communicate without difficulty, and...
4) periodic broadcasts by L-nodes are used to inform NL-nodes that can hear them of the coordinate location of the L-nodes relative to some absolute reference frame.

A variety of localization techniques for both indoor and outdoor environments are currently available, offering various trade-offs between accuracy, cost and complexity. In this article we assume that each NL-node needs to communicate with at least three other L-nodes in order to be able to localize itself. A review of various localization techniques proposed in the literature may be found in [9]. In [10], the authors propose an approach based on connectivity information, to derive the locations of nodes in a network. In [11], the authors present some work in the field of source localization in sensor networks.

The rest of the article is organized as follows. In section-II we present the theoretical analysis for the localization networks. Section-III is devoted to presenting the simulation results in support of the theoretical analysis and section-IV to conclusions.

II. Analysis of Localization Probability Using the Probabilistic Method

The basic parameter set defining the problem is as follows.

- Total number of nodes distributed uniformly over a circular disk of radius $R$ (denoted as the domain) is $n$.
- $k$ of the $n$ nodes are assumed to be L-nodes (implying they have localization information relative to some coordinate frame. How this localization is established is irrelevant to problem formulation), the rest are denoted as non-localized nodes (NL-nodes) and need to localize themselves.
- The radio coverage radius is denoted as $d$ and is defined by $d = \sqrt{\frac{A_{cov}}{\pi}}$ (we assume $d << R$). In particular, the shape of the coverage area $A_{cov}$ is irrelevant in so far as the calculation of the probabilities is concerned due to the assumption of the uniform distribution of the nodes in the terrain.
- The localization problem is two dimensional and three distance measurements relative to nodes with known positions is sufficient to solve for the $(X,Y)$ coordinates of the NL-node unambiguously.
- We shall neglect the boundary problem in the sense that the nodes near the boundary of the domain can only see other network elements within the domain. Hence in effect, their radio coverage area within which they may identify other nodes is reduced. This assumption is validated to be reasonably good, even when the total number of nodes is not very large.

Various localization techniques have been reported in the literature. The basic strategy assumed in this article is as follows [12].

1) The graph corresponding to the set of nodes is created with bidirectional arcs based on range. In particular, any pair of nodes whose true distances are within a certain limit are connected by a bi-directional edge and can communicate with each other.

2) A given NL-node that has not localized itself, stores two sets of information generated locally: i) absolute coordinates of the L-nodes within its radio coverage (if any), ii) distance estimates to all other nodes within its radio coverage based on mean power measurements. We shall assume the measured distance information to be accurate.

The basic question that we wish to answer in this article is as follows; given that we are at a NL-node that is interior to our domain, what is the probability that the node can localize itself? To localize itself a NL-node needs at least three L-nodes within its radio coverage. Given the existence of degenerate cases, the formulas presented in the Theorems below really represent lower bounds to the node localization failure probability. However, since the degenerate cases are very rare, the calculated probabilities are very close to the true values (i.e., the bound is tight), and we have verified this experimentally.

**Theorem 2.1:** Under the assumption of a uniform distribution of $n$ nodes, $k < n$ of which are L-nodes while the rest are NL-nodes, over a circular domain of radius $R$, and per node radio coverage radius of $d < R$, the NL-node localization failure probability is tightly lower bounded by:

$$P_F \geq \left\{ \begin{array}{ll}
\frac{(1-a)^2}{2} + (1-a)\frac{a^2}{2} + 1 \times \{ \frac{b^2a + (1-b^2)}{n-1} \} \\
[1-(1-a)b^2]n^{-3} + 1 + b(1-a)(n-3) + b^3(1-a)^2(n-1)(n-2) \end{array} \right\}$$

where $a = \left(1 - \frac{d}{R} \right)$ (i.e., the fraction of the NL-nodes), $b = \left( \frac{d}{R} \right)$ and the operator notation of vector calculus is used to simplify the expression.

**Proof.** The assumption of uniform distribution of nodes over our domain implies that the probability of having a node within a circular disk (i.e., the radio coverage zone of a given node) of radius $d$ is given by $\left( \frac{A_{cov}}{\pi} \right) = \left( \frac{d^2}{\pi} \right)$. In $p$ independent trials, the probability of having $p$ nodes within the coverage zone is simply $\left( \frac{d}{R} \right)^{2p}$. Probability of a NL-node localization failure is bounded by:

$$P_F \geq P\{p \leq 2 \text{ or } (p \geq 3 \text{ and at most two nodes are L-nodes while the rest are NL-nodes)} \}$$

The probability that $p = 0$, is the probability that all other $(n-1)$ nodes are outside the coverage zone of the NL-node and is given by $[1 - b^2]^{n-1}$. Similarly, the probability that $p \geq 1$ nodes are within the coverage radius is:

$$\left( \begin{array}{c}
\frac{n-1}{p}
\end{array} \right) b^{2p}[1 - b^2]^{n-1-p}$$

If in addition, we require that of the $p$ nodes at most two be L-nodes while the other $(p-2)$ nodes are NL-nodes, then the probability of the event of interest is:

$$\left( \begin{array}{c}
\frac{n-1}{p}
\end{array} \right) b^{2p}[1 - b^2]^{n-1-p} \left[ a^p + p^a b^{p-1}(1-a)+ \frac{p(p-1)}{2} a^{p-2}(1-a)^2 \right]$$
A. Extension to Log-Normal Shadowing

Putting the pieces together, after some algebra the NL-node localization failure probability can be written as (assuming \( n \geq 4 \)):

\[
P_F \geq \sum_{p=0}^{(n-1)\cdot b^2} \left( \frac{n-1}{p} \right) b^p [1 - b^2]^{(n-1)-p} [a^p + \frac{p(a-1)}{2} a^{p-2}(1-a)^2]\]

(3)

This expression can further be written in the form:

\[
P_F \geq \frac{(1-a)^2}{2} \frac{\partial^2}{\partial a^2} \left\{ \left[ b^2 \cdot a + (1-b^2) \right]^{n-1} \right\} + (1-a) \frac{\partial}{\partial a} \left\{ \left[ b^2 \cdot a + (1-b^2) \right]^{n-1} \right\} + \left[ b^2 \cdot a + (1-b^2) \right]^{n-1}
\]

\[+ \left( 1-b \right)^{n-1} \left( 2 - b \right)^2 \]

\[= 1 \left[ (n-3) \cdot (1-a) \cdot b^2 \right]^2 \]

(5)

Fig. 1 depicts the localization probability \((1 - P_F)\) as a function of the fraction of the NL-nodes, \(a\), for exponentially increasing values of \(b\) defined via the expression \(b(j) = 2^{j/2} - 1\), \(j = 3, 4, ..., 20\) in a network containing \(n = 300\) nodes. It is evident from the figure that the localization probability increases as the fraction of the NL-nodes decreases for a fixed \(b\), and for a given fraction of NL-nodes, the localization probability increases rapidly as a function of increasing \(b\).

### A. Extension to Log-Normal Shadowing

Our analysis above on the node localization failure probability may really be taken to represent a probability conditioned on two things: 1) the total number of nodes \(n\) is known, and 2) the radio coverage area of the NL-node is known. In this subsection, we wish to demonstrate that it is relatively straightforward to remove these conditions. In particular, we shall take a particular propagation model and remove the conditioning on the knowledge of the coverage area. To this end, consider the problem of localizing a specific node by signal power measurements. Let us consider the received power \(P(d)\) at a distance \(d\) from a specific point \(P(d) = P_o - 10n_p \log_{10}(d) + X_s\), whereby \(P_o\) is the signal power at a reference distance \(d_0\) normalized to one for simplicity, \(n_p\) is the path loss exponent, and \(X_s\) is a Gaussian-distributed random variable taking into account the shadowing effect, i.e., \(X_s \sim N(\mu_s, \sigma^2)\) with \(\mu_s = 0\).

An estimator of \(d\) is given by \(\widehat{d} = 10^{\frac{P_o}{10n_p}}\), where \(P\) is the measured power. With simple mathematical calculation, it is possible to obtain:

\[
\widehat{d} = 10^{\frac{P_o}{10n_p}} = 10^{\frac{10b \log_{10}(d) - X_s}{10n_p}} = d \cdot 10^{\frac{X_s}{10n_p}}
\]

and then \(\frac{\widehat{d}}{d} = 10^{\frac{X_s}{10n_p}}\). The random variable \(X_1 = X_s/n_p\) is a gaussian random variable \(N(0, \sigma^2)\) with zero mean and variance \(\sigma^2 = \sigma^2/n_p^2\). We take \(\widehat{d}\) to be the true distance that would be measured if \(X_s = 0\). Hence, \(X_s\) behaves as a perturbation affecting the measured distance \(\widehat{d}\). Let the receiver have a detection threshold of \(\gamma\). Hence, if the received power is below \(\gamma\), the receiver does not detect the presence of a signal. This puts an upper limit on the estimated distance as \(\widehat{d}_{max} = 10^{\frac{\gamma}{10n_p}}\), leading to an upper limit of \(\frac{\widehat{d}}{d} \leq \frac{\widehat{b}}{b}\). We note that if the received power is below \(\gamma\), the transmit and receive nodes do not see each other. Hence the probability of finding a node in the coverage zone of the transmit node is zero. Since our probability values are proportional to the size of the coverage area, we interpret this as the event that \(\widehat{d} = 0\).

With this setup, the ratio between the estimated distance (when measurable) and actual one \(d\), i.e., \(Y = \frac{\widehat{d}}{d}\) is a random variable with log-normal probability density function \(f_Y(y)\) (initially we assume \(\gamma = -\infty\) dB). In fact, considering the following transformation applied to the random variable \(X_1 \sim N(0, \sigma^2)\): \(Y = g(X_1) = 10^{-\frac{X_1}{10n_p}}\) it is simple to see that \(\frac{\widehat{d}}{d}\) is distributed as follows:

\[
f_Y(y) = \frac{f_{X_1}(x_1)}{|g'(x_1)|} = \frac{\alpha}{\sqrt{2\pi\sigma^2}y} e^{-\frac{-(10\log_{10}(y))^2}{2\sigma^2}}
\]

(6)

\[\text{whereby } \alpha = \frac{10}{\ln(10)} f_{X_1}(x_1) \sim N(0, \sigma^2); \quad |g'(x_1)| = \frac{10}{\ln(10)} \frac{1}{10^{-\frac{x_1}{10n_p}}}.
\]

An important parameter in the analysis that follows is the coverage to domain radius ratio. Consider the random variable:

\[
\frac{\widehat{d}}{R} = \hat{b} = \left( \frac{\widehat{d}}{d} \right) \cdot \left( \frac{d}{R} \right) \text{ whereby } b_o = \frac{d}{R} \text{ is a fixed constant representing the true coverage to domain radius ratio. With this setup, we have } \frac{\widehat{d}}{R} = \hat{b} = b_0 \cdot Y.
\]

Define:

\[
f_1(x) = \frac{\alpha}{\sqrt{2\pi\sigma^2}x} e^{-\frac{-(10\log_{10}(x) - 10\log_{10}(b_0))^2}{2\sigma^2}}
\]

The probability density function (pdf) \(f_{\hat{b}}(\hat{b})\) of \(\hat{b}\) is:

\[
\begin{align*}
\int_{0}^{b_{max}} f_1(x) \, dx = \delta(\hat{b}) - b_{max} \\
\end{align*}
\]

Note that \(\hat{b}\) is a mixed random variable. The probability \(P(\hat{b} = 0) \neq 0\). This follows from the argument above that...
the event that the received signal is below the detection threshold is equivalent to the event that $\hat{d} = 0$ implying $\hat{b} = 0$. Again, considering the detection threshold, we have the upper limit $\hat{b} \leq \frac{d_{\text{max}}}{R}$. To proceed further, we make the basic assumption that $\frac{d_{\text{max}}}{R} < 1$ (this is almost always true) and that the coverage region is entirely contained in our domain. Note that the detection threshold $\gamma$ and path loss exponent have a major impact on $d_{\text{max}}$, but, we always assume $R$ is large enough that the above condition holds.

**Theorem 2.2:** Under the assumption of a uniform distribution of $n$ nodes, $k < n$ of which are L-nodes while the rest are NL-nodes, over a circular domain of radius $R$, and per node radio coverage governed by log-normal shadowing with $\frac{d_{\text{max}}}{R} < 1$, the NL-node localization failure probability $P_F$ is tightly lower bounded by:

$$\sum_{l=0}^{n-3} \binom{n-3}{l} (-k_1)^l \left\{ E \left[ \hat{b}^2 \right] + k_2 E \left[ \hat{b}^{2l+2} \right] + k_3 E \left[ \hat{b}^{2l+4} \right] \right\}$$

where $a = \left(1 - \frac{k_2}{k_3}\right)$ (i.e., the fraction of the NL-nodes), $k_1 = 1-a$, $k_2 = (1-a) \cdot (n-3)$, and $k_3 = (1-a)^2 \left( \frac{n^2-3n+2}{2} \right)$.

**Proof.** From Theorem 2.1 the NL-node localization failure probability conditioned on $\hat{b}$ can be written as follows:

$$P_{F|\hat{b}} \geq \left[ 1 - k_1 \cdot \hat{b}^2 \right]^{n-3} \cdot \left[ 1 + k_2 \hat{b}^2 + k_3 \hat{b}^4 \right]$$

Using the fact that:

$$\left[ 1 - k_1 \cdot \hat{b}^2 \right]^{n-3} = \sum_{l=0}^{n-3} \binom{n-3}{l} (-k_1)^l \left[ \hat{b}^2 \right]^{l}$$

it is possible to obtain the following relation:

$$P_{F|\hat{b}} \geq \sum_{l=0}^{n-3} \binom{n-3}{l} (-k_1)^l \cdot \left[ \hat{b}^{2l} + k_2 \hat{b}^{2l+2} + k_3 \hat{b}^{2l+4} \right]$$

With this setup, the NL-node localization probability failure $P_F$ can be lower bounded as follows:

$$P_F = \int_{0}^{+\infty} P_{F|\hat{b}} f_{\hat{b}}(\hat{b}) d\hat{b}.$$

By substituting (9) in the previous equation, it is possible to obtain:

$$P_F \geq \sum_{l=0}^{n-3} \binom{n-3}{l} (-k_1)^l \left\{ \int_{0}^{+\infty} \hat{b}^{2l} f_{\hat{b}}(\hat{b}) d\hat{b} + k_2 \int_{0}^{+\infty} \hat{b}^{2l+2} f_{\hat{b}}(\hat{b}) d\hat{b} + k_3 \int_{0}^{+\infty} \hat{b}^{2l+4} f_{\hat{b}}(\hat{b}) d\hat{b} \right\}$$

Note that the contribution of the delta function of $f_{\hat{b}}(\hat{b})$ in above integrals is zero since we are looking at non-central moments. This way a closed form bound for $P_F$ can be written as follows:

$$P_F \geq \sum_{l=0}^{n-3} \binom{n-3}{l} (-k_1)^l \left\{ E \left[ \hat{b}^{2l} \right] + k_2 E \left[ \hat{b}^{2l+2} \right] + k_3 E \left[ \hat{b}^{2l+4} \right] \right\}$$

Under certain conditions the expectations above are approximated by the moments of a log-normal random variable $\hat{b}$. The $k$-th order moment of a log-normal random variable is given by.

$$E \left[ y^k \right] = e^{\frac{k}{2} \mu + \frac{1}{2} \left( \frac{\sigma}{\mu} \right)^2 \sigma^2}$$

whereby $\mu$ is the mean of the random variable $10 \log_{10} X$. In particular, provided that the number of nodes $n$ is small (less than 10) and the variance of shadow variable $X_s$ is sufficiently small and recalling that $\alpha = \frac{10}{\ln 10}$ and $\mu = 10 \cdot \log_{10} (b_o)$, we have the approximation:

$$P_F \approx \sum_{l=0}^{n-3} \binom{n-3}{l} (-k_1)^l \left\{ e^{\frac{2l}{2} \mu + \frac{1}{2} \left( \frac{\sigma}{\mu} \right)^2 \sigma^2} + k_2 e^{\frac{2l+2}{2} \mu + \frac{1}{2} \left( \frac{\sigma}{\mu} \right)^2 \sigma^2} + k_3 e^{\frac{2l+4}{2} \mu + \frac{1}{2} \left( \frac{\sigma}{\mu} \right)^2 \sigma^2} \right\}$$

The above expression can be written in the alternate form:

$$P_F \approx \sum_{l=0}^{n-3} \binom{n-3}{l} (-k_1)^l \left\{ e^{\frac{2l(2l+1)}{2} \mu + \frac{1}{2} \left( \frac{\sigma}{\mu} \right)^2 \sigma^2} + k_2 e^{\frac{2l(2l+2)}{2} \mu + \frac{1}{2} \left( \frac{\sigma}{\mu} \right)^2 \sigma^2} + k_3 e^{\frac{2l(2l+4)}{2} \mu + \frac{1}{2} \left( \frac{\sigma}{\mu} \right)^2 \sigma^2} \right\}$$

Note that the limiting behavior is consistent in that if $X_s$ has variance of zero, $\sigma = 0$ and the above expression reduces to the already known result as expressed in (9).

For a numerical example, consider a scenario with the following parameters: $\gamma = -80$ dBm, $P_0 = 0$ dBm, $d_0 = 10$ cm, $n_p = 3.5$, $\sigma_s = 12$ dB, $\sigma_1 = 3.43$ dB, $R = 40$ m, $n = 50$ and $k = 10$. The computed $d_{\text{max}} = 19.3$ m and $b_{\text{max}} = 0.48$. Fig. 2 depicts the localization probability as a function of $b_o$ for $a = 0.2$ under both Log-Normal shadowing (solid line) and no shadowing (dashed line).

![Fig. 2. Localization probability as a function of $b_o$ for $a = 0.2$ under both Log-Normal shadowing (solid line) and no shadowing (dashed line).](image-url)
B. Threshold Conditions

Returning to our analysis where we assume the knowledge of the radio coverage area of a given NL-node, a common characteristic of many problems tackled using the probabilistic method is the existence of a transition threshold where the characteristic of interest exhibits a large variation. The transition threshold observable specially for large values of \( b \) can be obtained by taking the second partial derivative of \( P_F \) with respect to \( a \) and setting the result to zero. Given that \( P_F \) is defined through a differential equation, it is better to work directly with this form. This leads to the condition:

\[
\frac{(1-a)}{2a} \frac{\partial^2}{\partial a^2} \{[b^2 \cdot a + (1 - b^2)]^{n-1}\} = \frac{\partial^2}{\partial a^2} \{[b^2 \cdot a + (1 - b^2)]^{n-1}\} \tag{15}
\]

The resulting closed form expression for the transition threshold on \( a \) is given by:

\[
a^* = 1 - \frac{1}{b^2(0.5n - 1)} \tag{16}
\]

At the value \( a^* \), there is a marked change in the localization probability. Values of \( a \) below threshold lead to a high probability of localization of the NL-nodes. Values of \( a \) greater than \( a^* \) lead to very low probability of one shot localization. The transition threshold is plotted in Fig. 3 as a function of the coverage to domain radius ratio. The noticeable feature is the rapid increase in this threshold at a value of \( b \sim 0.15 \).

Fig. 4 depicts the localization probability \((1 - P_F)\) as a function of the coverage to domain radius ratio, \( b \), for exponentially increasing values of \( a \) defined via the expression \( a(j) = 2^{j/22} - 1 \), \( j = 1, 2, ..., 20 \) in a network containing \( n = 300 \) nodes.

The transition threshold in this scenario can be obtained by taking the second partial derivative of \( P_F \) with respect to \( b \) and setting the result to zero. This leads to the closed form expression:

\[
b^* = \left\{ \frac{4^{n^2-n-15}}{(1-a)(4^{n^2-n-6a}+10.5n-4.5)} \left[ 1 + \sqrt{1 + \frac{6(n-9)(2^{n^2-n-6a}+10.5n-4.5)}{(4n^2-n-15)^2}} \right] \right\}^{1/2} \tag{17}
\]

For large values of \( n \), the above expression simplifies to:

\[
b^* \simeq \left\{ \frac{1}{(1-a)n} \left[ 1 + \sqrt{1.75} \right] \right\}^{1/2} \tag{18}
\]

For a fixed \( a \), there is a transition threshold with respect to \( b \). At values of \( b \) below \( b^* \), the localization probability is small while for values of \( b \) greater than \( b^* \) the localization probability increases rapidly. The transition threshold is plotted in Fig. 5 as a function of the percentage of the NL-nodes. The noticeable feature is the relatively gradual increase in this threshold as a function of \( a \).

So far we have looked at what we may call a one-shot localization of the NL-nodes. What we mean is that our calculated localization probability looks at the event that a given NL-node finds at least three L-nodes within its radio coverage. Once a NL-node localizes itself, technically, it could become a reference localization node and change from the category of NL-node, to the category of L-node. The process of NL-node localization can then repeat itself with the potentially newly added L-nodes. The study of this iterative localization process is beyond the scope of the present work. The asymptotic behavior however in the limiting case of infinite iterations is relatively easy to deduce. In particular, there is always a non-zero probability that at least one node may not get localized even if all the other nodes are L nodes. This provides the lower bound:

\[
P_F(\infty) \geq \sum_{k=0}^{2} \binom{n-1}{k} b^{2k}[1-b^2]^{n-1-k}. \tag{18}
\]

### III. Simulation Results

We have simulated the node localization problem as outlined in the previous section. The details of our simulation setup is as follows:

- We consider uniformly distributing \( n \) nodes over a domain of radius \( R \) normalized to \( R = 1 \). To achieve this, we generate \( n \) pairs of uniformly distributed independent random variables \((r_i, \theta_i)\). Random variable \( r_i \) is uniform in the interval \([0,1]\) while \( \theta_i \) is uniform in the interval \([\pi, -\pi]\). To get a uniform distribution of nodes over the domain, the variable \( r_i \) is transformed to \( \sqrt{r_i} \) such that the pair \((\sqrt{r_i}, \theta_i)\) denotes the polar coordinates of the \( i \)-th node within our domain.
- \( k \) of the \( n \) nodes are identified as the L-nodes by selecting at random a set of \( k \) unique indices in the range \([1,2,\ldots,n]\).
- For a given NL-node, the set of distances to the \( k \) L-nodes are calculated and the number of L-nodes (if any) that fall within the coverage radius \( d \) is obtained. This is repeated for all the NL-nodes and the count of the total number of NL-nodes that can localize themselves is obtained.
- To get good statistical averages, we have generated 1000 realizations of the localization problem for a given set of parameters, \( n, k, a, b \). We have then obtained the empirical estimate of node localization failure probability by averaging the results over 1000 realizations.
As an example of sample simulation results while assessing the impact of the density of nodes over the domain, Fig. 6 depicts the NL-node localization probability versus $a$ for three different values of $n$ at a value of $b = 0.05$. Simulation points are marked by "*" while the analytical curves obtained using the formulas derived in the previous section are shown as solid lines. Note that at $n = 500$ the curve is convex, at $n = 1000$ it is almost linear and at $n = 3000$ it is almost concave. The other noticeable effect is the reduction in the localization probability for a fixed $a$ as the density of the nodes within the domain is reduced. The gap between theory and simulation can be attributed to the boundary points which as stated previously, have an effectively smaller coverage area than the interior points.

IV. CONCLUSIONS

In this article we have presented a probabilistic setup for NL-node localization in a randomly distributed network of mixed localized and non-localized nodes. The probabilistic method is then used to answer some fundamental questions regarding the feasibility of NL-node localization in such a network based on some basic parameters such as the density of the nodes and coverage to domain radius ratios. We have derived expressions for transition thresholds with respect to several key parameters whereby marked change in the localization probability is observed.

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