Effect of Nucleon Dressing on Triton Binding Energy

Abstract The effect of nucleon dressing by pions, on the binding energy of three nucleons interacting via two-body forces, is calculated for the first time within a conventional nuclear physics approach. It is found that the dressing increases the binding energy of the triton by an amount approximately in the range from 0.3 MeV to 0.9 MeV, depending on the model used for dressing.

1 Introduction

We address a long-standing problem, dating back to the pre-Effective Field Theory (EFT) era of the 1960’s through to the early 1990’s, where most non-relativistic descriptions of the three-nucleon (3N) system, using as input realistic two-nucleon potentials, underestimated the triton binding energy by an amount ranging approximately from 0.5 to 1.0 MeV [1]. This was widely viewed as likely evidence for missing three-nucleon forces (3NFs), thereby spawning a huge effort into its investigation [2–8]. What appears to have been overlooked, however, was that the prime candidate for a 3NF, illustrated in Fig. 1(a), should be considered together with nucleon dressing effects as in Figs. 1(b) and (c), simply because a pion that is emitted by one nucleon, can be absorbed by any other nucleon, including the one that emitted it. One can only speculate as to why such effects of nucleon dressing were ignored while the effects of 3NF’s were so vigorously pursued, but it does seem likely that it was because the way to fully dress more than one nucleon at the same time in the nonrelativistic setting of time-ordered perturbation theory (TOPT), was not known at the time. Indeed, such was the observation of the authors of Ref. [9] who showed that the partial dressing of two nucleons, commonly used in the coupled $NN - \pi NN$ system, leads to inconsistent renormalisation [9]. However, in 1993 it was shown that the way to include nucleon dressing in multi-nucleon systems described by TOPT was through the use of convolution integrals [10]. This development led quickly to the formulation of few-body equations for the $NN - \pi NN$ system with consistent renormalisation [11,12], but it isn’t until recently that advances in computer power have made their solution feasible.

Here we apply the convolution approach to derive 3N bound-state equations where nucleons interact via two-body forces with all 3NF’s neglected, but otherwise where all nucleons are fully dressed. These equations take as input $NN$ t matrices whose off-shell behaviour is determined ideally through the use of realistic $NN$ potentials that use the same dressed nucleon propagators as the 3N equations. Since no such $NN$ potentials are as yet available, we adopt the strategy of solving the new 3N equations using a variety of input $NN$ t matrices constructed from various separable fits to two well-known realistic $NN$ potentials (Paris and Bonn) which
use undressed nucleon propagators. Not only does this strategy allow us to test the sensitivity of our results to the undetermined off-shell behaviour of the input \( N\bar{N} \) t matrices, it also provides an unambiguous way to determine the effect of nucleon dressing on the triton binding energy.

2 Bound-State Equations for Three Dressed Nucleons

We consider a non-relativistic TOPT of baryons and mesons described by a Hamiltonian \( H \). Green functions, defined as matrix elements of operator \( (E^+ - H)^{-1} \) taken between free-particle states, can then be expanded into a perturbation series whose terms are represented by time-ordered diagrams. To this end, we define Green function operators \( \hat{g}(E) \), \( \hat{D}(E) \), and \( \hat{G}(E) \), being identical to \( (E^+ - H)^{-1} \), but acting specifically in the space of 1, 2, and 3 nucleons, respectively. In this approach the formulation of the 3N bound state equations follows the same steps as for the corresponding Quantum Mechanical formulation, leading to the bound state equation for the wave function Faddeev components \( |\Psi_\alpha\rangle \) [13],

\[
|\Psi_\alpha\rangle = 2\hat{G}_0(E_b)\hat{w}_\alpha(E_b)|\Psi_\beta\rangle \tag{1}
\]

where \( \{\alpha, \beta, \gamma\} \) is any cyclic permutation of \( \{1, 2, 3\} \), \( E_b \) is the 3N bound state energy, and \( |\Psi\rangle = |\Psi_1\rangle + |\Psi_2\rangle + |\Psi_3\rangle \) is the fully antisymmetric 3N bound state wave function. However, in contrast to the Quantum Mechanical case, the 3N fully disconnected propagator \( \hat{G}_0(E) \) in Eq.(1) is that for three fully dressed nucleons, given by the convolution expressions [10]

\[
\hat{D}_0(E) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz \, \hat{g}(E - z)\hat{g}(z), \tag{2a}
\]

\[
\hat{G}_0(E) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz \, \hat{D}_0(E - z)\hat{g}(z), \tag{2b}
\]

where \( \hat{D}_0(E) \) is the fully disconnected part of the 2N Green function operator \( \hat{D}(E) \), and \( \hat{w}_\alpha(E_b) \) is the disconnected 3N t matrix operator (with nucleons \( \beta \) and \( \gamma \) interacting while nucleon \( \alpha \) is a spectator) where all possible dressings of all three nucleons is included, given by the convolution expression [10]

\[
\hat{G}_0(E)\hat{w}_\alpha(E)\hat{G}_0(E) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz \, \hat{D}_0(E - z)\hat{v}_\alpha(E - z)\hat{D}_0(E - z)\hat{g}_\alpha(z). \tag{3}
\]

In Eq.(3), \( \hat{g}_\alpha \) is the operator form of the dressed propagator of nucleon \( \alpha \), \( \hat{v}_\alpha \) is the operator 2N t matrix for nucleons \( \beta \) and \( \gamma \), and \( \hat{D}_0 \) the operator dressed 2N propagator of nucleons \( \beta \) and \( \gamma \). Equations Eq.(2) and Eq.(3) are illustrated in Fig. 2. To facilitate the calculation of the momentum matrix elements of the convolution integrals in Eqs.(2) and Eq.(3), we use the fact that our model dressed nucleon propagator \( g(E) \) (momentum matrix element of \( \hat{g} \)) is endowed with a simple pole at the physical nucleon mass \( m \), and a pion-nucleon (\( \pi N \)) unitarity cut starting at \( E = m + m_\pi \) where \( m_\pi \) is the pion mass. This analytic structure implies that \( g(E) \) satisfies the dispersion relation

\[
g(z) = \frac{Z}{z + m} - \frac{1}{\pi} \int_{m + m_\pi}^{\infty} d\omega \, \text{Im} \frac{g(\omega)}{z^+ - \omega}, \tag{4}
\]
where $Z$ is the nucleon wave function renormalisation constant.

To solve Eq.(1) numerically, we perform a partial wave decomposition using $J$-$J$ coupling (where $l_{a}, s_{a}$, $j_{a}$, $t_{a}$ are the relative orbital angular momentum (a.m.), spin, total a.m. and isospin of the ($\beta \gamma$) pair, $L_{a}$ is the relative orbital a.m. of the $e\gamma$ system, and $J$ ($T$) is the total a.m. (isospin)). Taking matrix elements of Eq. (3) between partial wave basis states and carrying out the convolution integral with the help of Eq.(4),

$$w_{\alpha} = \frac{\alpha}{\beta} \frac{\beta}{\gamma}$$

expressed in terms of an integral over variable $\omega$. For c.m. energies less than $3n + m_{\pi}$, this integral encounters no singularities and can be approximated using Gaussian quadratures. To simplify the solution of the 3N equations we make use of separable 2N potentials. For a rank-$M$ separable approximation, we write the partial wave 2N t matrix as

$$t_{l_{a}l_{b}}^{s_{a}l_{c}}(E, p_{a}, p_{b}) = h_{l_{a}l_{b}}(p_{a}) t_{l_{a}l_{b}}^{s_{a}l_{c}}(E) \overline{h}_{l_{a}l_{b}}(p_{b})$$  (5)

where $h_{l_{a}l_{b}}(p_{a})$ is an $1 \times M$ row matrix, $t_{l_{a}l_{b}}^{s_{a}l_{c}}(E)$ is an $M \times M$ square matrix, and $\overline{h}_{l_{a}l_{b}}(p_{b})$ is an $M \times 1$ column matrix. The resulting separable form for $w_{l_{a}l_{b}}^{s_{a}l_{c}}$ can be used in the bound state equation, Eq.(1), in an analogous way to that described in Ref. [14] for the 3N problem without dressing. In this way we are led to write the bound state equation in terms of a spectator wave function $\chi_{l_{a}l_{b}}^{N_{a}JT}(q_{a})$ satisfying the integral equation

$$\chi_{l_{a}l_{b}}^{N_{a}JT}(q_{a}) = 2 \sum_{l_{a}l_{b}l_{c}} \sum_{p_{a}p_{b}} t_{l_{a}l_{b}}^{s_{a}l_{c}}(E - \omega_{n} - \frac{3q_{a}^{2}}{4m})$$

$$\times \int_{0}^{\infty} dq_{b} \frac{2}{\omega_{n}} Z_{l_{a}l_{b}l_{c}}^{N_{a}N_{b}JT}(q_{a}, q_{b}, E) \chi_{l_{a}l_{b}l_{c}}^{N_{a}JT}(q_{b})$$  (6)

where $q_{a}$ is nucleon $a$'s momentum, $N_{a} = \{ l_{a}, t_{a}, L_{a}, J_{a} \}$, $\omega_{n} = m$, and $\omega_{n}$ ($n = 1, 2, \ldots$) is the $n$th quadrature point for the $\omega$ integral in Eq.(4). In Eq.(6)

$$Z_{l_{a}l_{b}l_{c}}^{N_{a}N_{b}JT}(q_{a}, q_{b}, E) = \frac{1}{2} W_{n} \sum_{L} \int_{-1}^{+1} dx \overline{h}_{l_{a}l_{b}}(p_{a}) D_{0} (E - \omega_{n} - E_{a\beta\gamma})$$

$$\times G_{0}^{-1}(E - E_{a\beta\gamma}) D_{0} (E - \omega_{n} - E_{a\beta\gamma}) h_{l_{a}l_{b}}^{\overline{m}_{a}}(p_{b}) P_{L}(x)$$

$$\times \left( \frac{q_{a}}{p_{a}} \right)^{l_{a}} \left( \frac{q_{b}}{p_{b}} \right)^{l_{b}} \sum_{a=0}^{L_{a}} \sum_{b=0}^{L_{b}} A_{\alpha, \beta, \gamma}^{L, a, b} \left( \frac{q_{a}}{q_{b}} \right)^{b-a}$$  (7)

where $W_{0} = Z$, $W_{n} = -\frac{1}{n} w_{n} \text{Im} \frac{\omega_{n}}{\omega}$ (for $n = 1, 2, \ldots$), $w_{n}$ being the $n$th quadrature weight for the $\omega$ integral, $E_{a\beta\gamma}$ is the total kinetic energy of the three nucleons, $P_{L}(x)$ (where $x = \hat{q}_{a} \cdot \hat{q}_{b}$) is the Legendre polynomial of order $L$, and $A_{\alpha, \beta, \gamma}^{L, a, b}$ is a numerical coefficient as specified in Ref. [14]. After nucleon wave function renormalisation, and discretisation, Eq.(6) becomes a matrix equation of the form $\chi = K(E)\chi$. The condition det($I - K(E_{b})$) = 0 then determines binding energy.
Table 1 Nucleon dressing model parameters as defined in Eq. (9), with \( m_0 \) being the bare nucleon mass and \( Z \) the nucleon wave function renormalisation constant

| Dressing model | \( \lambda \) (MeV) | \( \beta_1 \) (fm\(^{-1}\)) | \( \beta_2 \) (fm\(^{-1}\)) | \( C_0 \) | \( C_1 \) | \( C_2 \) | \( m_0 \) (fm\(^{-1}\)) | \( Z \) |
|----------------|---------------------|---------------------|---------------------|-----------|-----------|-----------|---------------------|-----|
| M8             | 537                 | 1.30764             | 1.60478             | 1.23727   | 0.304819  | 5.75485   | 5.3317              | 0.799532 |
| M7             | 800                 | 1.54233             | 1.60016             | 1.94223   | 0.42571   | 3.98739   | 5.71685             | 0.699705 |
| M6             | 2132                | 1.8706              | 1.5966              | 5.8692    | 0.627138  | 2.46274   | 6.56284             | 0.603483 |

Fig. 3 Real [(a)] and imaginary [(b)] parts of \( G_0^N(E)(E-3m) \) where \( G_0^N(E) = G_0(E)/Z^3 \) is the renormalised dressed 3N propagator. Curves labelled by the dressing models of Table 1

3 Numerical Results

To describe nucleon dressing, we use a formulation of pion-nucleon scattering that classifies diagrams of TOPT according to their multi-pion irreducibility [15]. In this scheme, the \( \pi N \) t matrix operator \( t_{\pi N} \) is expressed as

\[
t_{\pi N}(E) = \hat{\mathcal{f}}(E)\hat{\mathcal{g}}(E)\hat{\mathcal{f}}(E) + \hat{t}_{\pi N}(E)
\]

where \( \hat{\mathcal{f}}(E) \) (\( \hat{\mathcal{g}}(E) \)) is the \( N\to \pi N \) (\( \pi N\to N \)) dressed vertex operator, \( \hat{\mathcal{g}}(E) \) is the dressed nucleon operator that is to be used as input to the 3N binding energy calculation, and \( \hat{t}_{\pi N}(E) \) is the \( N \)-irreducible “background” part of the \( \pi N \) t matrix. The input to these equations consists of the “background” potential \( \hat{v}_{\pi N} \) and the “bare” \( \pi N N \) vertex \( f_0 \). Following Ref. [16], we choose energy-independent separable forms for the potential \( \hat{v}_{\pi N} \) in the \( P_{11} \) partial wave: \( v_{\pi N}(k', k) \equiv -h(k')h(k) \) with the form factors expressed as

\[
f_0(k) = \frac{k C_0}{\sqrt{\varepsilon(k)}} \left( k^2 + \lambda^2 \right), \quad h(k) = \frac{k C_1}{\sqrt{\varepsilon(k)}} \left( \frac{1}{k^2 + \beta_1^2} + \frac{C_2 k^4}{(k^2 + \beta_2^4)^3} \right)
\]

where \( \varepsilon(k) = \sqrt{k^2 + m_0^2} \). We likewise specify the \( \pi N \) propagator as \( G_{\pi N}(E, k) = (E^2 - k^2 + m_0 - \varepsilon(k))^{-1} \) and the bare nucleon propagator as \( g_0(E) = (E^2 - m_0)^{-1} \) where \( m_0 \) is the bare nucleon mass. To obtain a variety of models of nucleon dressing, we have carried out fits to the KH80 \( P_{11} \) \( \pi N \) phase shifts [17] (for pion laboratory energies up to 350 MeV) for a range of cutoff values \( \lambda \) for the bare \( \pi N N \) vertex function \( f_0(k) \). Each such fit was constrained to reproduce the \( \pi N N \) coupling constant \( f_{\pi N N}^2 = 0.079 \) in the way described in Ref. [16]. Results of three such fits are given in Table 1, and the corresponding graphs of the fully dressed 3N propagators \( G_0(E) \) in Fig. 3.

For numerical calculations of triton binding energies using Eq.(6), we limit the number of 3N partial wave channels to 5, one \( ^1S_0 \) and two coupled \( ^3S_1\rightarrow^3D_1 \) channels. For the input 2N potentials we use combinations of the so-called “PEST” separable approximations to the Paris potential [18,19], and “BEST” separable approximations to the Bonn [20] potential. In particular, we use four \( ^1S_0\rightarrow^3S_1\rightarrow^3D_1 \) combinations, denoted by P1/P1, P3/P4, B1/B1, and B3/B4, where, for example, P3/P4 denotes that the PEST3 potential is used in the \( ^1S_0 \) channel and PEST4 in the \( ^3S_1\rightarrow^3D_1 \) channels. Table 2 shows the resulting triton binding energy shifts, together with the binding energy when dressing is neglected. It is evident that for all models used, nucleon dressing results in an increase in the binding energy of the triton. Moreover, as might be expected, the nucleon dressing...
models that have the largest effect on the 3N propagator, as displayed in Fig. 3, also give the largest binding energy shifts. We are thus led to the conclusion that the triton binding energy shift due to the inclusion of nucleon dressing, is largely determined by the cutoff chosen for the bare $\pi NN$ vertex function used for dressing. For example, if the correct value of the cutoff $\lambda$ is 800 MeV, as suggested by QCD sum rules [21], then our calculations indicate that nucleon dressing will shift the undressed triton binding energy by an amount approximately in the range 0.5 to 0.7 MeV. However, if one takes into account the wide range of models in the literature, most of which propose cutoffs in the range 500 < $\lambda$ < 2200 MeV, the corresponding triton binding energy shifts would lie approximately in the range 0.3 to 0.9 MeV.

Although our results could be improved by using more accurate 2N forces and including more partial waves, it seems clear that the overall conclusion would be the same, namely, that nucleon dressing increases binding in the 3N system, and therefore that 3NF’s and nucleon dressing need to be considered together. Luckily modern EFT approaches do just that.

Funding We would like to thank R. J. McLeod and J. L. Wray for many stimulating discussions. A.N.K. was supported by the Georgian Shota Rustaveli National Science Foundation (Grant No. FR17-354).

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