Optimal portfolio with unobservable market parameters and certainty equivalence principle

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Abstract

We consider a multi-stock continuous time incomplete market model with random coefficients. We study the investment problem in the class of strategies which do not use direct observations of the appreciation rates of the stocks, but rather use historical stock prices and an a priori given distribution of the appreciation rates. An explicit solution is found for case of power utilities and for a case when the problem can be embedded to a Markovian setting. Some new estimates and filters for the appreciation rates are given.

Key words: Optimal portfolio, continuous time market model, non-observable parameters, filters

JEL classification: D52, D81, D84, G11

Mathematical Subject Classification: 49K45, 60G15, 93E20

1 Introduction

The paper investigates an optimal investment problem for a market consisting of a locally risk free asset and a finite number, $n$, of risky stocks. It is assumed that the vector of stock prices $S(t)$ evolves according to an Itô stochastic differential equation with a vector of appreciation rates $a(t)$ and a volatility matrix $\sigma(t)$:

$$dS_i(t) = S_i(t)[a_i(t) \, dt + \sum_j \sigma_{ij}(t) \, dw_j(t)], \quad i = 1, \ldots, n.$$ 

The problem goes back to Merton (1969), who found strategies which solve the optimization problem in which $E(U(X(T)))$ is to be maximized, where $X(T)$ represents the wealth at the final time $T$ and where $U(\cdot)$ is a utility function. If the market parameters are observed, then the optimal strategies (i.e. current vector of stock holdings) are functions of the current
vector \( (a(t), \sigma(t), S(t), X(t)) \); see, e.g., survey in Hakansson (1997) and Karatzas and Shreve (1998). However, in practice, the process \( (a(t), \sigma(t)) \) is not given directly and has to be estimated from observations of prices given some prior hypothesis about the market dynamics. Some attempts have been made to construct winning strategies that are not using these parameters; see, e.g., Dokuchaev and Savkin (2002), Dokuchaev (2002, 2007). However, the mainstream approach is to consider models where \( a(t) \) and \( \sigma(t) \) have to be estimated from historical stock prices or some other observation process. There are many papers devoted to estimation of \( (a(t), \sigma(t)) \), mainly based on modifications of Kalman-Bucy filtering or the maximum likelihood principle; see e.g. Lo (1988), Chen and Scott (1993), Pearson and Sun (1994). Unfortunately, the process \( a(t) \) is usually hard to estimate in real-time markets, because the drift term, \( a(t) \), is usually overshadowed by the diffusion term, \( \sigma(t) \). On the other hand, \( \sigma(t) \) can, in principle, be found from stock prices; see (4) below. Thus, there remains the problem of optimal investment with unobservable \( a(t) \). A popular tool for this problem is the so-called “certainty equivalence principle”: agents who know the solution of the optimal investment problem for the case of directly observable \( a(t) \) can solve the problem with unobservable \( a(t) \) by substituting \( \mathbb{E}\{a(t)|S(\tau), \tau < t\} \) (see e.g. Gennotte (1986), Feldman (2007)). Unfortunately, this principle does not hold in the general case of non-log utilities (see Kuwana (1995)). Note that this principle is unrelated to the notion of “certainty equivalent value” to be found in the work of Frittelli (2000).

In fact, the problem is one of linear filtering. If \( R_i(t) \) is the return on the \( i \)th stock, then

\[
dR(t) = a(t)dt + \sigma(t)dw(t),
\]

so the estimation of \( a(t) \) given \( \{R(\tau), \tau < t\} \) (or \( \{S(\tau), \tau < t\} \)), is a linear filtering problem. If \( a(\cdot) \) is conditionally Gaussian, then the Kalman filter provides the estimate which minimizes the error in the mean square sense. Indeed, in this case the conditional mean and conditional variance of \( a(t) \) given the past prices completely describe the conditional distribution of \( a(t) \), \( \mathcal{P}_{a(t)}(\cdot|S(\tau), \tau < t) \). In this setting Williams (1977), Detemple (1986), Dothan and Feldman (1986), Gennotte (1986), Brennan (1998) solved the investment problem using the Kalman-Bucy filter. This solution is optimal in the class of admissible strategies which are functions of the current \( \left(X(t), S(t), \mathcal{P}_{a(t)}(\cdot|S(\tau), \tau < t)\right) \). But in general for non-Gaussian \( a \), the optimal strategy based on all historical prices does not lie in this set, hence their approach does not give the optimal strategy.

Karatzas (1997), Karatzas and Zhao (1998), Dokuchaev and Zhou (2000), Dokuchaev
and Teo (2000) have obtained optimal portfolio strategies in the class of strategies of the form \( \pi(t) = f(\{S(\tau) : \tau < t\}) \), where \( f(\cdot) \) is a deterministic function, when \( a(t) \) is random and unobservable, but under the crucial condition that \( a \) and \( \sigma \) are time independent. This assumption ensures that the optimal wealth has the form \( X(t) = H(S(t), t) \), where \( H(\cdot, \cdot) \) satisfies a deterministic parabolic backward equation of dimension \( n \), for the market with \( n \) stocks. Even if one accepts this restrictive condition, the solution of the problem is difficult to realise in practice for large \( n \) (say, \( n > 4 \)), since it is usually difficult to solve the parabolic equation. Karatzas (1997) gives the explicit solution of a goal achieving problem for the case of one stock which has conditionally normal growth rate. Karatzas and Zhao (2001) use linear filtering (via martingales) and dynamic programming to solve the problem for a general utility function with \( n > 1 \), \( \sigma \) diagonal and constant, and with random \( a \) with known (non-Gaussian) distribution. In Dokuchaev and Zhou (2000), additional constraints on the terminal wealth are added so that goal achieving problems are subsumed. Dokuchaev and Teo (2000) further generalize the constraints and utility functions allowed.

The restriction of the constant coefficients is relaxed in three seminal papers: Karatzas and Xue (1991) and Lakner (1995), (1998). Karatzas and Xue assumed that there are more Brownian motions than stocks. They assume that \( r \) and \( \sigma \) are adapted to the observable \( S \). After projecting onto an \( n \)-dimensional Brownian motion which generates the same filtration as \( S \), they obtain a reduced, completely observable model; existence of an optimal portfolio follows, but the optimal strategy is, as usual, defined only implicitly. Lakner (1995), (1998) assumes that \( S \) and \( w \) have equal dimension (as we do), and that \( r \) and \( \sigma \) are deterministic. This again guarantees that the filtration of \( S \) is Brownian. Results from filtering theory give a representation of the optimal portfolio, which is explicit in terms of a conditional expectation of a Malliavin derivative when the \( a_i \) are Ornstein-Uhlenbeck processes independent of \( w \). Zohar (2001) suggested an alternative approach based on a Cameron-Martin formula for a special single stock model when parameters are described by Ornstein-Uhlenbeck process.

We also consider the optimal investment problem with random, unobservable \( a(t) \), and we allow the random coefficients, \( r, \sigma \), to depend on time. Our approach, as usual, is to exhibit a claim which gives the optimal terminal wealth; the replication strategy for this claim will then be the optimal strategy. The replicating strategies for the very general models were obtained as a conditional expectation of a Malliavin derivative. In the present paper we are trying alternative approaches that can give a more explicit solution for some special cases.

We are targeting two special cases: (i) when \( U(x) = x^\delta \); (ii) when \( \hat{a}(t) \) can be presented
as a part of a diffusion Markov process, may be of a higher dimension.

First for the log utility and some power utilities, we can compute the hedging portfolio directly with few restrictions on \( r, a, \sigma \) (at least in the log case). For these utilities it is shown that the "certainty equivalence principle" can be reformulated with the following correction: the "equivalence filter" of \( a(t) \) must be derived (in place of \( \mathbb{E}\{a(t)|S(\tau), \tau < t\} \)). In general, it is neither \( \mathbb{E}\{a(t)|S(\tau), \tau < t\} \) nor any other function of \( \mathcal{P}_{a(t)}(\cdot|S(\tau), \tau < t) \). We show that for a general prior distribution of \( a(\cdot) \) and logarithmic utility, the equivalence filter of \( a(t) \) is in fact \( \hat{a}(t) \doteq \mathbb{E}\{a(t)|S(\tau), \tau < t\} \). If \( a(t) \) is Gaussian, then of course this is the Kalman-Bucy filter; this case was considered in Lakner (1995), (1998), Dokuchaev (2005).

Further, for the case of power utility the equivalence filter is not a function of \( \mathcal{P}_{a(t)}(\cdot|S(\tau), \tau < t) \) even under the Gaussian assumption. However, we show that this estimate can be written as a conditional expectation of \( a(t) \) under a new measure. Thus, under a Gaussian assumption on \( a(t) \), the equivalence filter can be obtained by a Kalman-Bucy filter, but with some correction to the parameters. In other words, our result gives new filters which (for Gaussian priors) are the classic Kalman-Bucy filters but with modified parameters. Cvitanić et al. (2002) presented the explicit optimal strategy for non-observable parameters for Gaussian priors for \( U(x) = \delta^{-1}x^\delta \), but only for the case when \( \delta < 0 \). Our approach also leads to the explicit solution but for positive \( \delta \); in fact, we cover only the case when \( \delta = (l-1)/l \), \( l = 2, 3, \ldots \).

The second class of considered problems includes problems which can be embedded to a Markovian setting. Dokuchaev and Zhou (2001) suggested to use linear parabolic equations to replicate the optimal claim when the coefficients are constant in time; in that case, the optimal claim can be presented as a function of the vector of stock prices at terminal time (see also Dokuchaev and Teo (2000)). We extended this approach to a more general model that covers the cases when \( \Theta \) is an Ornstein-Ulenbek process, or when \( \Theta \) is a finitely-valued Markov process; in the first case, the solution requires solving a linear parabolic equation of dimension \( n+2 \) and in the second case, solving a parabolic equation of dimension equal to the number of possible values of the Markov process \( \Theta \). Thus, we propose a simpler method than dynamic programming: the nonlinear parabolic Bellman equation is replaced for a linear parabolic equation. Note that Sass and Haussmann (2003) solved a more general problem for the case of parameters being driven by a finitely valued Markov chain, but their solution presents the replicating strategy as a conditional expectation of a Malliavin derivative.

In Section 2 we collect notation and definitions, and we set up the model. The problem
is stated in Section 3, and in Section 4 a formula for the optimal claim is presented in a very general setting. In Section 5 the solution is detailed for some power utilities. In Section 6 we consider the cases when the problem can be embedded to a Markovian setting. The Appendix contains most of the proofs.

2 The market model

On a given probability space \((\Omega, \mathcal{F}, P)\) satisfying the usual conditions, consider a market model consisting of a locally risk free asset or bank account with price \(B(t), t \geq 0,\) and \(n\) risky stocks with prices \(S_i(t), t \geq 0, i = 1, 2, ..., n,\) where \(n < +\infty\) is given. The prices of the stocks evolve according to the following equations:

\[
dS_i(t) = S_i(t) \left( a_i(t) dt + \sum_{j=1}^{d} \sigma_{ij}(t) dw_j(t) \right), \quad t > 0,
\]

where \((\top\) denoted transpose\) \(w(t) = (w_1(t), \ldots, w_d(t))\top\) is a standard \(d\)-dimensional Brownian motion, \(a(t) = (a_1(t), \ldots, a_n(t))\top\) is the vector of appreciation rates, and the \(\sigma_{ij}(t)\) are volatility coefficients. The initial prices \(S_i(0) > 0\) are given non-random constants. The price of the locally riskless asset evolves according to the following equation

\[
B(t) = B(0) \exp \left( \int_0^t r(t) dt \right),
\]

where \(B(0)\) is taken to be 1 without loss of generality, and \(r(t)\) is the progressively measurable random interest rate process. Write \(\sigma(t) = \{\sigma_{ij}(t)\}\) for the \(n \times d\) dimensional matrix process. Define the return to time \(t\) by \(dR_i(t) = dS_i(t)/S_i(t), \quad R_i(0) = 0,\) and introduce the vector of returns \(R(t) = (R_1(t), ..., R_n(t))\top\) and of excess returns \(\tilde{R}_i(t) = R_i(t) - \int_0^t r(\tau) d\tau.\)

Let \(\hat{r}(t) = r(t)(1, ..., 1)\top \in \mathbb{R}^n, \quad \tilde{a}(t) = a(t) - \hat{r}(t).\) Then

\[
dR(t) = a(t) dt + \sigma(t) dw(t), \quad d\tilde{R}(t) = \tilde{a}(t) dt + \sigma(t) dw(t).
\]

Remark 2.1. The first question is how to calibrate this model, i.e. what \(a_i\) and \(\sigma_{ij}\) to use. We can observe the prices, hence the returns, interest rate and excess return, but not the Brownian motion \(w.\) The volatility coefficients can in principle be estimated from \(R(\cdot)\) or \(\tilde{R}(\cdot).\) In fact,

\[
\int_0^t \sigma(\tau)\sigma(\tau)\top d\tau = <R>_t = <\tilde{R}>_t,
\]

the (observable) quadratic variation process of \(R\) or \(\tilde{R}.\) In fact we shall assume that \(\sigma\) is a non-random function of the excess return, \(\tilde{R}.\) It is more difficult to estimate the appreciation
the observation filtration and is also generated by \( \{R, r\} \). It is the observation filtration and is also generated by \( \{S, B\} \) or \( \{\tilde{R}, r\} \) (with augmentation). To describe the prior distribution of \( a(\cdot) \), we assume that there exist a separable linear normed space \( E \), a Borel measurable set \( \mathcal{T} \subseteq E \), and a random vector \( \Theta : \Omega \to \mathcal{T} \) with the distribution \( \nu \). Further we assume that we are given measurable functions \( A : [0, T] \times \mathcal{T} \times C([0, T]; \mathbb{R}^n) \to \mathbb{R}^n \), \( \alpha : [0, T] \times C([0, T]; \mathbb{R}^n) \to \mathbb{R}^{n \times n} \), and \( \rho : [0, T] \times C([0, T]; \mathbb{R}^n) \to \mathbb{R} \), such that
\[
\tilde{a}(t, \omega) \equiv A\left(t, \Theta(\omega), \tilde{R}(\cdot, \omega)|_{[0,t]}\right), \quad \sigma(t, \omega) \equiv \alpha(t, \tilde{R}(\cdot, \omega)|_{[0,t]}), \quad \tau(t, \omega) \equiv \rho\left(t, \Theta(\omega), \tilde{R}(\cdot, \omega)|_{[0,t]}\right).
\]
Here \( f(s, \omega)|_{[0,t]} = f(s \land t, \omega) \). Here \( f(s, \omega)|_{[0,t]} = f(s \land t, \omega) \).

**Assumption 2.1** \( \Theta \) and \( w(\cdot) \) are mutually independent;
\[
\sup_{t,f,\theta} (|\rho(t, \theta, f)| + |\alpha(t, f)|) < \infty \text{ a.s.; there exists a function } K(\cdot) \text{ and a constant } K_0 \text{ such that}
\]
\[
\sup_{t,f} |A(t, \theta, f)| \leq K(\theta) < \infty,
\]
\[
|A(t, \theta, f) - A(t, \theta, g)| \leq K(\theta) \sup_{\tau \in [0,t]} |f(\tau) - g(\tau)|;
\]
\[
|\alpha(t, f) - \alpha(t, g)| \leq K_0 \sup_{\tau \in [0,t]} |f(\tau) - g(\tau)|;
\]
\[
\alpha(t, f)\alpha(t, f)^\top \geq cI_n, \text{ where } c > 0 \text{ is a constant and } I_n \text{ is the identity matrix in } \mathbb{R}^{n \times n}.
\]

Let \( \Omega_w \triangleq C([0, T]; \mathbb{R}^n) \), let \( \mathcal{F}_w \) be the completion of the \( \sigma \)-algebra of subsets of \( \Omega_w \) generated by \( w(\cdot) \), and let \( \mathcal{F}_T \) be the completion of \( \sigma \)-algebra of subsets of \( \mathcal{T} \) generated by \( \Theta \). Further, let \( P_w \) be the probability measure on \( \mathcal{F}_w \) generated by \( w(\cdot) \). By the definitions, \( \nu \) is the probability measure on \( \mathcal{F}_T \).

Without loss of generality, we assume that the probability space \( (\Omega, \mathcal{F}, P) \) is such that \( \Omega = \mathcal{T} \times \Omega_w \), \( \mathcal{F} \) is the completion of \( \mathcal{F}_T \otimes \mathcal{F}_w \), and \( P \) is the completion of \( \nu \times P_w \).

**Remark 2.2.** (i) The conditions imply that the solutions of (1), (2) and (3) are well-defined.
(ii) The simplest models have \( \tilde{a}(\cdot) = \Theta(\cdot) \) for a process \( \Theta(t) \) independent of \( w(\cdot) \).

As usual it will be productive to work with an equivalent measure \( P_\pi \) under which the normalized wealth process (cf. next section) is a martingale. Set
\[
\mathcal{Z} \triangleq \exp\left(\int_0^T (\sigma(t)^{-1}\tilde{a}(t))^\top dw(t) + \frac{1}{2} \int_0^T |\sigma(t)^{-1}\tilde{a}(t)|^2 dt\right) .
\]
Clearly, there exists measurable function \( f : \mathcal{T} \times \Omega_w \rightarrow \mathbb{R} \) such that \( Z^{-1} = f(\Theta, w(\cdot)) \). Since \( \Theta \) and \( w \) are independent and \( |\sigma(\cdot)^{-1} a(\cdot)| \leq \sqrt{c} K(\Theta) \), then by Fubini’s Theorem it follows that

\[
E Z^{-1} = \int_{\mathcal{T}} \nu(d\theta) \int_{\Omega_w} P(d\omega) f(\theta, \omega_w) = \int_{\mathcal{T}} \nu(d\theta) 1 = 1,
\]

because the Lipschitz conditions on \( A \) and \( \alpha \) allow us to construct \( \tilde{R} \), hence \( Z \) for each value \( \theta \) of \( \Theta \). Define \( P_* \) by \( dP_* / dP = Z^{-1} \). Let \( E_* \) be the corresponding expectation. Note that \( dP / dP_* = Z \). By Girsanov’s Theorem, it follows that the process \((\tilde{R}(t), \mathcal{F}^{R,r})\) is a martingale with respect to \( P_* \).

We will require an expression for \( E_*(Z|\mathcal{F}^{R,r}_T) \). To prepare for this, define \( Q(t, \omega) \overset{\Delta}{=} (\sigma(t, \omega)\sigma(t, \omega)^\top)^{-1} \), and for each \( \theta \in \mathcal{T} \), introduce the process \( z(\theta, t) \) as a solution of the equations

\[
\begin{cases}
   dz(\theta, t) = z(\theta, t) A \left( t, \theta, \tilde{R}(\cdot)\right)_{[0, t]}^\top Q(t) d\tilde{R}(t), \\
   z(\theta, 0) = 1.
\end{cases}
\]

Now set

\[
\bar{Z} \overset{\Delta}{=} \int_{\mathcal{T}} d\nu(\theta) z(\theta, T).
\]

This is the required conditional density of \( P \) with respect to \( P_* \) given the observations.

**Proposition 2.1** \( E_*(Z|\mathcal{F}^{R,r}_T) = \bar{Z}. \)

**Remark 2.3.** Proposition 2.1 actually holds for more general \( \sigma \) (not of the form \( \alpha \)). It is only required that it be almost surely pathwise bounded, \( \sigma(t)\sigma(t)^\top \geq cI_n \) a.s., and that \( \Theta, \sigma \) and \( w \) be mutually independent. Under these conditions \( A \) can also depend additionally on \( r \) and \( \sigma\sigma^\top \). Now \( K(\theta) \) becomes \( K(\theta, q, \rho) \) where the last two arguments stand for the paths of \( \sigma\sigma^\top \) and \( r \) respectively. Proofs are given in the Appendix.

### 3 Problem statement

An investor holds a portfolio of the instruments; the pair \((\pi_0(t), \pi(t))\) describes the portfolio at time \( t \): the process \( \pi_0(t) \) is the investment in the bond, \( \pi_i(t) \) is the investment in the \( i \)th stock, \( \pi(t) = (\pi_1(t), \ldots, \pi_n(t))^\top, t \geq 0 \). Let \( X_0 > 0 \) be the wealth of the agent at time \( t = 0 \), i.e. the initial value of the portfolio, and let \( X(t) \) be the wealth at time \( t > 0 \), \( X(0) = X_0 \).

Then

\[
X(t) = \pi_0(t) + \sum_{i=1}^{n} \pi_i(t).
\]
The portfolio is said to be self-financing if \( dX(t) = \pi_0(t) \, dr(t) + \pi(t)^\top \, d\tilde{R}(t) \). For such portfolios
\[
dX(t) = r(t)X(t) \, dt + \pi(t)^\top \, d\tilde{R}(t),
\]
\( \pi_0(t) = X(t) - \sum_{i=1}^n \pi_i(t) \),
so \( \pi \) alone suffices to specify the portfolio; it is called a self-financing strategy. If we define \( \bar{X}(t) = B(t)^{-1}X(t) \), then
\[
\bar{X}(t) = X(0) + \int_0^t B(s)^{-1}\pi(s)^\top \, d\tilde{R}(s). \tag{9}
\]
For each \( \pi \) we denote the corresponding \( X \) or \( \bar{X} \) by \( X_{\pi}, \bar{X}_{\pi} \).

The investor’s problem is to choose \( \pi \) according to some criterion. First we note that the investor must base his decision at time \( t \) on his knowledge at time \( t \), which is \( \{S(s), r(s) : s \leq t\} \) or equivalently \( \{R(s), r(s) : s \leq t\} \). Hence to satisfy the agents observability requirement, \( \pi \) must be adapted to \( F_{R,r} \).

Let \( F_t \) be the filtration generated by \( R, r, a \) augmented by the null sets of \( F \).

**Definition 3.1** Let \( A \) (correspondingly \( A^a \)) be the class of all \( \{F_{R,r}\}_t \)-progressively measurable (correspondingly \( \{F_t\}_t \)-progressively measurable) processes \( \pi(\cdot) \) such that (i) \( \int_0^T |\pi(t)|^2 \, dt < \infty \) a.s. and (ii) there exists a constant \( q_\pi \) such that \( \mathbb{P}\{\bar{X}(t) - X_0 \geq q_\pi, \forall t \in [0, T]\} = 1 \).

A process \( \pi(\cdot) \in A \) is said to be an admissible strategy. For such \( \pi \) the integral in (9) is well defined. For each \( \pi \in A \), \( \bar{X}_{\pi}(t) \) is a \( \mathbb{P}_* \)-supermartingale with \( \mathbb{E}_*\bar{X}_{\pi}(t) \leq X_0 \) and \( \mathbb{E}_*|\bar{X}_{\pi}(t)| \leq |X_0| + 2|q_\pi| \). The following definition is standard.

**Definition 3.2** Let \( \xi \) be a given random variable. An admissible strategy \( \pi(\cdot) \) is said to replicate the claim \( \xi \) if \( X^\pi(T) = \xi \) a.s.

We observe that \( A^a \) denotes the class of admissible strategies when no observability requirement is imposed, i.e. the problem solved first by Merton.

Let \( T > 0 \), let \( \hat{D} \subset \mathbb{R} \) be convex and bounded below, and let \( X_0 \in \hat{D} \) be given. Let \( U(\cdot) : \hat{D} \to \mathbb{R} \cup \{-\infty\} \) be such that \( U(X_0) > -\infty \).

We may state our general problem as follows: Find an admissible self-financing strategy \( \pi(\cdot) \) which solves the following optimization problem:
\[
\text{Maximize } \mathbb{E}U(\bar{X}_\pi(T)) \text{ over } \pi(\cdot) \in A \tag{10}
\]
subject to \[
\begin{aligned}
\tilde{X}^\pi(0) &= X_0, \\
\tilde{X}^\pi(T) &= \tilde{D} \text{ a.s.}
\end{aligned}
\] (11)

The condition $\tilde{X}^\pi(T) \in \tilde{D}$ would represent a requirement for a minimal normalized terminal wealth if $\tilde{D} = [k, +\infty)$, $k > 0$.

Roughly speaking, the problem is solved as follows. Find the optimal terminal value by constrained maximization, then find the optimal $\pi$ by replicating this terminal value. The extra constraint to ensure possibility of replication is $E_\pi \tilde{X}(T) = X_0$. Hence we want to solve: \[
\max_{\xi} \{ EU(\xi) \mid \xi \in \tilde{D}, E_\pi \xi = X_0 \},
\]
or, using a Lagrange multiplier $\lambda$ and the fact that $\xi$ is $F_{r,T}$ measurable,
\[
E_\pi \max_{\xi \in \tilde{D}} \{ \tilde{Z}U(\xi) - \lambda \xi \} + \lambda X_0.
\] (12)

To make this program work, we assume that $U$, $X_0$ and $\tilde{D}$ satisfy the following three conditions.

Condition 3.1 There exists a measurable set $\Lambda \subseteq [0, \infty)$, and a measurable function $F(\cdot, \cdot) : (0, \infty) \times \Lambda \to \tilde{D}$ such that for each $z > 0$, $\tilde{x} = F(z, \lambda)$ is a solution of the optimization problem
\[
\text{Maximize } zU(x) - \lambda x \text{ over } x \in \tilde{D}.
\] (13)

This condition allows us to solve the maximization problem in (12). Of course the usual concavity hypotheses imply this condition, but more general utility functions are also covered.

Condition 3.2 There exists $\tilde{\lambda} \in \Lambda$ such that $E_\pi |F(\tilde{Z}, \tilde{\lambda})| < +\infty$ and $E_\pi F(\tilde{Z}, \tilde{\lambda}) = X_0$.

With this condition we now know that $\tilde{\lambda}$ is the correct multiplier to use. It will also be seen that the integrability implies that the optimal utility has well defined expectation.

The optimal solution of the problem (10)-(11) under Conditions 3.1-3.2 was obtained in the class $\mathcal{A}^{\alpha}$ in Dokuchaev and Haussmann (2001) under some additional conditions. By definition of $\mathcal{A}^{\alpha}$, this solution has the form $\pi(t) = \Gamma(t, S(\cdot)|[0,t], \tilde{a}(\cdot)|[0,t], r(\cdot)|[0,t])$, for some measurable function $\Gamma$.

In case $a(\cdot)$ is a Gaussian process and $U = \log$, it is known that the problem can be solved using Kalman filtering and the “certainty equivalence” principle, cf Gennotte, (1986). More precisely, solve the problem as if $a$ were known, to obtain the optimal strategy as $\pi(t, a)$ and find $m(t) \triangleq E\{a(t)|F_t\}$ (the Kalman filter). Then $\pi(t, m)$ is the optimal solution of the given problem. This result is incorrect for non-log utility functions, cf Kuwana, (1995); however we can resurrect it for some other utility functions if we allow proxies for $a$ other than $m$. 

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**Definition 3.3** Let \( \pi(t) = \Gamma(t, S(\cdot)|[0,t], a(\cdot)|[0,t], r(\cdot)|[0,t]) \) be an optimal solution of the problem (10)-(11) in the class \( \mathcal{A} \), where \( \Gamma \) is a measurable function. Further, let \( \tilde{\pi}(t) \) be an optimal solution of the problem (10)-(11) in the class \( \mathcal{A} \), and let there exists a \( n \)-dimensional \( \{\mathcal{F}_t^{R,r}\} \)-adapted random vector process \( \tilde{a}(\cdot) \) such that \( \tilde{\pi}(t) \equiv \Gamma(t, S(\cdot)|[0,t], \tilde{a}(\cdot)|[0,t], r(\cdot)|[0,t]) \). Then \( \tilde{a}(t) \) is said to be the equivalence filter of \( \tilde{a}(t) \) with respect to the problem (10)-(11).

Note that we do not assume that \( \tilde{a}(t) \) is a function of the current conditional distribution \( P_{\tilde{a}(t)}(\cdot|\mathcal{F}_t^{R,r}) \) of \( \tilde{a}(t) \).

### 4 Existence of the optimal claim and strategy

We solve our problem in two steps. First we show that \( E U(\tilde{\xi}) \) is an upper bound for the expected utility of normalized terminal wealth for \( \pi(\cdot) \in \mathcal{A} \). Then we show that a portfolio \( \tilde{\pi}(\cdot) \) which replicates the claim \( B(T)F(\tilde{\xi}) \) exists. This establishes the optimality of \( \tilde{\pi}(\cdot) \). We exhibit \( \tilde{\pi} \) for a couple of utility functions in the next section, and then treat the general case in the following one.

Let \( U^+(x) \overset{\Delta}{=} \max(0,U(x)), U^-(x) \overset{\Delta}{=} \max(0,-U(x)) \). Let \( F(\cdot) \) be as in Condition 3.1.

**Theorem 4.1** Under Assumption 2.1 and Conditions 3.1, 3.2, let

\[
\tilde{\xi} \overset{\Delta}{=} F(\tilde{Z}, \tilde{\lambda})
\]

with \( \tilde{\lambda} \) as in Condition 3.2. Then

(i) \( E U^-(\tilde{\xi}) < \infty, \tilde{\xi} \in \tilde{D} \text{ a.s.} \)

(ii) \( E U(\tilde{\xi}) \geq E U(\tilde{X}^\pi(T)), \forall \pi(\cdot) \in \mathcal{A} \).

(iii) The claim \( B(T)\tilde{\xi} \) is attainable in \( \mathcal{A} \), and there exists a replicating strategy in \( \mathcal{A} \). This strategy is optimal for the problem (10)-(11).

The proof is in the Appendix.

**Remark 4.1** It now follows that the optimal terminal wealth is \( B(T)F(\tilde{\xi}) \) and the optimal strategy is determined implicitly by replication. So we have a type of equivalence principle: proceed as for the completely observable problem, but replace \( Z \), the density of \( P \) with respect to \( P_* \), by \( \tilde{Z} \), the conditional expectation of \( Z \), cf Proposition 2.1.

The first two parts of the theorem hold under the weaker conditions mentioned in Remark 2.3, but then we cannot appeal to the martingale representation theorem to obtain the replication of part (iii). If we have another technique for establishing this replication, then the theorem holds under the weaker hypotheses. We pursue this idea in the next section.
5 Replication with myopic strategies and equivalence filters

We now consider two special utility functions, \( U(x) = \log(x + \delta) \), \( \delta \geq 0 \), and \( U(x) = x^\delta \) for some \( \delta \), but under the weaker assumptions of Remark 2.3. In these cases we can compute the replicating strategy directly and so solve the problem explicitly. We also find equivalence filters of the excess accumulation rates \( \bar{a}_i \).

**Lemma 5.1** Let \( U(x) \equiv \log(x + \delta), \delta \geq 0, X_0 > 0 \) and \( (0, +\infty) \subseteq \tilde{D} \). Then the optimal solution in the class \( \mathcal{A} \) of the problem (10)-(11) is

\[
\hat{\pi}(t)^\top = (X_0 + \delta)B(t) \int_T d\nu(\theta)z(\theta, t)A \left( t, \theta, \bar{R}(\cdot)\big|_{[0,t]}, \sigma(\cdot)\sigma(\cdot)^\top \big|_{[0,t]}, r(\cdot)\big|_{[0,t]} \right)^\top Q(t)
\]

\[
= (X\hat{\pi}(t) + \delta B(t)) \int_T d\nu(\theta)z(\theta, t)A \left( t, \theta, \bar{R}(\cdot)\big|_{[0,t]}, \sigma(\cdot)\sigma(\cdot)^\top \big|_{[0,t]}, r(\cdot)\big|_{[0,t]} \right)^\top Q(t)
\]

and

\[
X\hat{\pi}(t) = B(t) \left( (X_0 + \delta) \int_T d\nu(\theta)z(\theta, t) - \delta \right) \text{ for all } t.
\]

**Proof:** We must replicate the claim \( B(T)\hat{\xi} \). According to Condition 3.1, \( F(z, \lambda) = z/\lambda - \delta \), so Condition 3.2 gives \( \hat{\lambda} = \mathbb{E}_s \hat{Z}/(X_0 + \delta) = 1/(X_0 + \delta) \) since

\[
\mathbb{E}_s \hat{Z} = \mathbb{E}_s (Z|\mathcal{F}_T^R) = \mathbb{E}_s Z = \mathbb{E}Z^{-1}Z = 1.
\]

Write \( X_\delta \) for \( X_0 + \delta \). It follows that

\[
\hat{\xi} = F(\hat{Z}, \hat{\lambda}) = X_\delta \hat{Z} - \delta
\]

\[
= X_\delta \left\{ \int_T d\nu(\theta) \left[ 1 + \int_0^T z(\theta, t)A \left( t, \theta, \bar{R}(\cdot)\big|_{[0,t]}, \sigma(\cdot)\sigma(\cdot)^\top \big|_{[0,t]}, r(\cdot)\big|_{[0,t]} \right)^\top Q(t)d\bar{R}(t) \right] \right\} - \delta
\]

\[
= X_0 + \int_0^T B(t)^{-1}\hat{\pi}(t)^\top d\bar{R}(t) = \tilde{X}(T)
\]

if

\[
\hat{\pi}(t)^\top = B(t)X_\delta \int_T d\nu(\theta)z(\theta, t)A \left( t, \theta, \bar{R}(\cdot)\big|_{[0,t]}, \sigma(\cdot)\sigma(\cdot)^\top \big|_{[0,t]}, r(\cdot)\big|_{[0,t]} \right)^\top Q(t).
\]

Hence this strategy replicates \( B(T)\hat{\xi} \) and so is optimal.

Moreover

\[
X\hat{\pi}(t) = B(t)\tilde{X}\hat{\pi}(t) = B(t) \left( X_0 + \int_0^t B(s)^{-1}\hat{\pi}(s)^\top d\bar{R}(s) \right) = B(t) \left( X_\delta \int_T d\nu(\theta)z(\theta, t) - \delta \right),
\]

so in fact

\[
\hat{\pi}(t)^\top = (X(t) + \delta B(t)) \frac{\int_T d\nu(\theta)z(\theta, t)A \left( t, \theta, \bar{R}(\cdot)\big|_{[0,t]}, \sigma(\cdot)\sigma(\cdot)^\top \big|_{[0,t]}, r(\cdot)\big|_{[0,t]} \right)^\top Q(t)}{\int_T d\nu(\theta)z(\theta, t)}.
\]

\( \square \)
Corollary 5.1  (i) Under the conditions of Lemma 5.1, the equivalence filter of \( \tilde{a}(t) \) is

\[
\tilde{a}(t) = \frac{\int_T d\nu(\theta)z(\theta,t)A(t,\theta,\tilde{R}(\cdot)|\theta,t,\sigma(\cdot)\sigma(\cdot)^\top|\theta,t,\tau(\cdot)|\theta,t)}{\int_T d\nu(\theta)z(\theta,t)}. \tag{18}
\]

(ii) Assume that \( \mathbb{E}\left|K(\Theta,\sigma(\cdot),\sigma(\cdot)^\top,\tau(\cdot))\right|^2 < \infty \). The process \( \tilde{a}(t) \) defined by (18) is such that \( \tilde{a}(t) = \mathbb{E}\{\tilde{a}(t)|\mathcal{F}_t^{R,r}\} \), i.e. it is the minimum variance estimate in the class of estimates based on observations of \( (S,r) \) (or \( (R,r) \)) up to time \( t \) assuming the prior \( \nu \) for \( \Theta \). The optimal expected utility is

\[
\mathbb{E}\log(\tilde{X}(T) + \delta) = \frac{1}{2} \mathbb{E}\int_0^T \tilde{a}(t)^\top Q(t)\tilde{a}(t)dt + \log(X_0 + \delta). \tag{19}
\]

Part (i) is obvious if we recall that the optimal strategy in \( \mathcal{A}^a \) is \( \pi(t)^\top = (X(t) + \delta B(t)\tilde{a}(t))^\top Q(t) \), cf. Dokuchaev and Haussmann (2001) (or assume that \( \mathcal{T} \) is a singleton), and use (15). We give the proof of part (ii) in the Appendix. Observe that if \( \tilde{a}(t) \) is conditionally Gaussian, i.e.

\[
\tilde{d}(t) = \left(c_1(t,\tilde{R}(\cdot)|\theta,t,\sigma(\cdot)|\theta,t,\tau(\cdot)|\theta,t) - c_2(t,\tilde{R}(\cdot)|\theta,t,\sigma(\cdot)|\theta,t,\tau(\cdot)|\theta,t)\tilde{a}(t)\right)dt + c_3(t,\tilde{R}(\cdot)|\theta,t,\sigma(\cdot)|\theta,t,\tau(\cdot)|\theta,t)d\bar{w} \]

with \( \Theta = w'(\cdot) \) an independent Brownian motion, then the Kalman filter can be used to calculate \( \tilde{a} \) and hence \( \tilde{\pi}(t) = X(t)Q(t)\tilde{a}(t) \). This result extends Example 4.4 of Lakner (1998).

We can now characterize \( \tilde{Z} \); this will be helpful in Section 6. We add that the following corollary also delivers the result of Lakner (1998), Theorem 3.1, under our more general assumptions.

Corollary 5.2  Define \( \tilde{Z}(t) \triangleq \mathbb{E}_s(\tilde{Z}|\mathcal{F}_t^{R,r}) \), \( \tilde{a}(t) \triangleq \mathbb{E}(\tilde{a}(t)|\mathcal{F}_t^{R,r}) \). Then \( \tilde{Z} = \tilde{Z}(T) \) and

\[
\tilde{Z}(t) = \exp\left\{\int_0^t \tilde{a}(s)^\top Q(s)d\tilde{R}(s) - \frac{1}{2} \int_0^t \tilde{a}(s)^\top Q(s)\tilde{a}(s)ds\right\}. \tag{20}
\]

Proof: We take \( \delta = 0 \) and \( X_0 = 1 \). Then (17) implies that \( \tilde{Z} = X(t) \), so \( \tilde{Z}(t) = X(t) \) since the latter is a \( (\mathbb{P}_s,\mathcal{F}_t^{R,r}) \)-martingale; hence \( \log(\tilde{Z}(t)) = Y(t,\tilde{\pi}) \) where \( Y \) as in the proof of Corollary 5.1. The result follows from (46). \( \square \)

We can also carry out this program for certain power utility functions, those for which the function \( F \) has the form \( F(z,\lambda) = Cz^l \) with \( l > 1 \) an integer, i.e. \( U(x) = x^\delta/\delta \) with \( \delta = 1 - 1/l \), if we make further assumptions on the \( A \) and \( \sigma \). Specifically, \( A \) should be linear in \( \theta \) and \( \int_0^T |\sigma^{-1}A(t,\theta,\tilde{R},\sigma\sigma^\top,\tau)|dt \) must be deterministic, which realistically means that we take \( \Theta \) to be a process, \( \tilde{a}(t) = \Theta(t) \), and \( \sigma \) is non-random. Moreover we need some integrability, i.e.

\[
G \triangleq \int_T d\nu(\theta_1)\cdots d\nu(\theta_l)\gamma(\theta_1,\ldots,\theta_l) < \infty, \tag{21}
\]
where

\[ \gamma(\theta_1, \ldots, \theta_l) \triangleq \exp \left\{ \sum_{i,j=1}^{l} \int_0^T \theta_i(t)^\top Q(t) \theta_j(t) \, dt \right\}. \]

Here each \( \theta_i \) is an \( n \)-dimensional function, a sample path of \( \tilde{a} \).

We note that \( \gamma \) and \( G \) are non-random. It is convenient to introduce the notation \( \bar{T} \triangleq \{ \sum_{k=1}^l \theta_k : \theta_i \in T, \; i = 1, \ldots, l \} \), let \( \chi_D \) be the indicator of \( D \) and define a measure \( \bar{\nu} \) on \( \bar{T} \) by

\[ \bar{\nu}(D) \triangleq \frac{\int_T \chi_D(\sum_{k=1}^l \theta_k) \, d\nu(\theta_1) \cdots d\nu(\theta_l) \gamma(\theta_1, \ldots, \theta_l)}{\int_T d\nu(\theta_1) \cdots d\nu(\theta_l) \gamma(\theta_1, \ldots, \theta_l)}. \]

**Theorem 5.1** Assume that \( A(\cdot, \theta, f, q, \rho) = \theta(\cdot) \), \( \sigma \) deterministic, \( (0, +\infty) \subseteq \bar{D} \), \( X_0 > 0 \), \( U(x) \equiv x^\delta/\delta, \delta = (l - 1)/l \) for some integer \( l > 1 \), and \( G < \infty \). Then

(i) \( F(z, \lambda) \equiv z^\lambda - 1 \) and \( \hat{\lambda} = X_0^{-1/l}(E_* Z_l)^{1/l} \).

(ii) The optimal solution in the class \( A \) of the problem (10)-(11) is

\[ \hat{\pi}(t)^\top \triangleq X_0 B(t) \int_{\bar{T}} d\bar{\nu}(\theta) z(\theta, t) \theta(t)^\top Q(t) \]

\[ = X\hat{\pi}(t) \int_{\bar{T}} d\bar{\nu}(\theta) z(\theta, t) \theta(t)^\top Q(t), \quad (22) \]

and

\[ X\hat{\pi}(t) = X_0 B(t) \int_{\bar{T}} d\bar{\nu}(\theta) z(\theta, t). \quad (23) \]

Moreover

\[ E\{U(X\hat{\pi}(T))\} = X_0^\delta G^{1-\delta}/\delta. \quad (24) \]

**Remark.** We will see below that the equivalence filter \( \tilde{a}(t) \) under assumptions of Theorem 5.1 differs from \( E\{\tilde{a}(t) | \mathcal{F}_{t}^{R,r}\} \) and, in general, is not a function of the current conditional distribution \( \mathcal{P}_{\tilde{a}(t)}(\cdot | \mathcal{F}_{t}^{R,r}) \) of \( \tilde{a}(t) \). However, we can write \( \tilde{a} \) as a conditional expectation of \( \tilde{a} \) if we change measure. Let \( \tilde{P} \) be given by \( P \) when we replace \( \nu \) defined on \( T \) by \( \bar{\nu} \) defined on \( \bar{T} \).

**Corollary 5.3** Under the conditions of Theorem 5.1, the equivalence filter of \( \tilde{a}(t) \) is

\[ \tilde{a}(t) = \frac{Q(t)^{-1}\hat{\pi}(t)}{lX\hat{\pi}(t)} = l^{-1}E\{\tilde{a}(t) | \mathcal{F}_{t}^{R,r}\} \quad (25) \]

where \( X\hat{\pi}(t) \) is the wealth defined by (23).
In particular, if \( \tilde{a}(\cdot) = \Theta \) is time independent, Gaussian, with density function \( \varphi \), then \( G < \infty \) only if \( \int_0^T Q(t) \, dt \) is so small that \( \log \Phi(x_1, \ldots, x_l) \) is a negative definite quadratic form (plus an affine term), where

\[
\Phi(x_1, \ldots, x_l) \triangleq \varphi(x_1) \cdots \varphi(x_l) \exp \sum_{i,j=1 \atop i < j}^l x_i^T \left( \int_0^T Q(t) \, dt \right) x_j.
\]

It follows that if \( \tilde{\Theta} = (\Theta_1, \ldots, \Theta_l) \) is defined to have density function \( \Phi(x_1, \ldots, x_l)/G \), then \( \tilde{\Theta} \) is Gaussian, hence the distribution of \( \sum_{i=1}^l \Theta_i \), which is \( \tilde{\nu}(\cdot) \), is Gaussian. This means that Kalman filtering can be employed to calculate \( \hat{a}(t) \).

Cvitanić et al (2002) presented the explicit optimal strategy for non-observable parameters for the Gaussian prior for \( U(x) = \delta^{-1} x^\delta \), but only for the case when \( \delta < 0 \). Our approach is quite different and covers \( \delta > 0 \) but only for \( \delta = (l-1)/l, l = 2, 3, \ldots \).

The two special cases discussed above are of limited interest because of the special nature of the utility functions used even though we have generalized the market dynamics somewhat (\( r \) random in both cases and \( \sigma \) random in the first). Let us then find the optimal strategies for more general utility functions but under our more restrictive assumptions, cf. Assumption 2.1.

6 Embedding to a Markovian setting for the general utility

We can use a PDE-based approach to replication, hence to the solution of our problem, if the claim to be replicated, here a function of \( \bar{Z} \), is a functional of a Markov process. We mention three examples below. Suppose there exist an integer \( M > 0 \), a deterministic function \( \phi : \mathbb{R}^M \to \mathbb{R} \), and a \( M \)-dimensional Markov process \( y(\cdot) \) such that

\[
\bar{Z} = \phi(y(T)),
\]

and \( y(\cdot) \) is the solution of an Itô equation

\[
\begin{cases}
    dy(t) = f(y(t), t) \, dt + b(y(t), t) \, d\tilde{R}(t), \\
    y(0) = y_0 \in \mathbb{R}^M,
\end{cases}
\]

where \( f(\cdot) : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M, b(\cdot) : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^{M \times n} \) are measurable functions. We can always append the equation \( d\tilde{R} = d\tilde{R} \) so we may assume that \( \tilde{R} \) is included in \( y \) if needed. Then we assume that \( \alpha(t, \tilde{R}(\cdot)) = \alpha(t, y(t)) \). Write \( \tilde{b}(y, t) \) for \( b(y(t), t) \alpha(t, y) \).
We assume that the functions $\tilde{b}(y,t)$, $f(y,t)$ are Hölder and such that

$$|\tilde{b}(y,t)| + |f(y,t)| \leq \text{const} (|y| + 1).$$

Further, we assume that $\partial \tilde{b}(y,t)/\partial y$, $\partial^2 \tilde{b}(y,t)/\partial y^2$, $\partial f(y,t)/\partial y$ and $\partial^2 f(y,t)/\partial y^2$ are uniformly bounded and Hölder.

Let $y_*(\cdot)$ denote the solution of (26) with $\tilde{R}(\cdot)$ replaced by $\tilde{R}_*(\cdot) = \int_0^\alpha(t, \tilde{R}_*(t)) dw(t)$ and introduce the Banach space $\mathcal{Y}$ of functions $u : \mathbb{R}^M \times [0, T] \rightarrow \mathbb{R}$ with the condition

$$\mathcal{Y}^1 = \text{const} \times \mathbb{R}$$

where $\mathcal{Y}$ denotes the solution of (27) with $\mathbb{R}^M$ replaced by $\mathbb{R}^n$ and $\mathbb{R}^n = X_N$. Then there exists an admissible strategy $\pi(t) = (\pi_1(t), \ldots, \pi_n(t)) \in \mathcal{A}$ which replicates the claim $B(T)C(y(T))$. Furthermore,

$$\pi(t) = B(t)b(y(t), t)^\top \frac{\partial V}{\partial y}(y(t), t), \quad \tilde{X}^\pi(t) = V(y(t), t),$$

where $\frac{\partial V}{\partial y}$ denotes the gradient of $V$ with respect to its first argument and the function $V = V(y, t) : \mathbb{R}^M \times [0, T] \rightarrow \mathbb{R}$ is such that

$$\frac{\partial V}{\partial t}(y, t) + \frac{\partial V}{\partial y}^\top (y, t)f(y, t) + \frac{1}{2}\text{Tr}\left\{\frac{\partial^2 V}{\partial y^2}(y, t) \tilde{b}(y, t) \tilde{b}^\top(y, t)\right\} = 0, \quad (27)$$

$$V(y, T) = C(y). \quad (28)$$

The problem (27)--(28) admits a solution in the class $\mathcal{Y}^1$.

Let $V(x, t, \lambda) : \mathbb{R}^M \times [0, T] \times \Lambda \rightarrow \mathbb{R}$ be the solution of the partial differential equation (27) with the condition

$$V(y, T, \lambda) = F(\phi(y), \lambda). \quad (29)$$

The following result now is immediate.

**Theorem 6.1** Let the function $F(\cdot)$ be such that

$$\mathbb{E}_* F(\tilde{Z}, \tilde{\lambda})^2 < +\infty. \quad (30)$$

With $\tilde{\lambda}$ as in Condition 3.2, there exists an admissible self-financing strategy $\pi(\cdot) \in \mathcal{A}$ which replicates the claim $B(T)F(\tilde{Z}, \tilde{\lambda})$. This strategy is an optimal solution of the problem (10)--(11), and

$$\pi(t) = B(t)b(y(t), t)^\top \frac{\partial V}{\partial y}(y(t), t, \tilde{\lambda}), \quad \tilde{X}^\pi(t) = V(y(t), t, \tilde{\lambda}). \quad (31)$$
Example 1. Let us repeat briefly the solution from Dokuchaev (2005) for the problem solved first in Lakner (1998) by a different method. Both solutions involved the Kalman filter. Assume that we are given measurable deterministic processes \( \alpha(t), \beta(t), b(t) \) and \( \delta(t) \) such that
\[
d\tilde{\alpha}(t) = \alpha(t)[\delta(t) - \tilde{\alpha}(t)]dt + b(t)d\tilde{R}(t) + \beta(t)dW(t),
\]
where \( \alpha(t) \in \mathbb{R}^{n \times n}, \beta(t) \in \mathbb{R}^{n \times n}, b(t) \in \mathbb{R}^n, \delta(t) \in \mathbb{R}^n \), and where \( W \) is an \( n \)-dimensional Wiener process in \( (\Omega, \mathcal{F}, P) \), independent on \( w \) under \( P \). We assume that \( \alpha(t), \beta(t), b(t) \), and \( \delta(t) \) are Hölder in \( t \) and such that the matrix \( \beta(t) \) is invertible and \(|\beta(t)^{-1}| \leq c \), where \( c > 0 \) is a constant. Further, we assume that \( \tilde{\alpha}(0) \) follows an \( n \)-dimensional normal distribution with known mean vector \( m_0 \) and covariance matrix \( \gamma_0 \). We note that this setting covers the case when \( \tilde{\alpha} \) is an \( n \)-dimensional Ornstein-Uhlenbeck process with mean-reverting drift.

Let \( y(t) = (y_1(t), \ldots, y_{n+2}(t)) = (\tilde{y}(t), y_{n+1}(t), y_{n+2}(t)) \) be a process in \( \mathbb{R}^{n+2} \), where
\[
\tilde{y}(t) = \mathbb{E}\{\tilde{\alpha}(t)|\mathcal{F}_t^{R,y}\},
\]
\[
y_{n+1}(t) = \int_0^t \tilde{y}(s)^\top Q(s) d\tilde{R}(s),
\]
\[
y_{n+2}(t) = \exp\left(-\frac{1}{2} \int_0^t \tilde{y}(s)^\top Q\tilde{y}(s) ds\right).
\]

Clearly, \( y_{n+2}(t) \in (0,1] \), thus, \( \psi(y_{n+2}(t)) \equiv y_{n+2}(t) \). Theorem 10.3 from Liptser and Shiryaev (2000), p.396, gives the equation for \( \tilde{\alpha}(t) = \tilde{y}(t) \) such that the equation for \( y(t) \) is
\[
d\tilde{y}(t) = [A(t)\tilde{y}(t) + \alpha(t)\delta(t)]dt + [b(t)\sigma(t)^\top + \gamma(t)]Q(t)d\tilde{R}(t),
\]
\[
dy_{n+1}(t) = \tilde{y}(t)^\top Q(t)d\tilde{R}(t),
\]
\[
dy_{n+2}(t) = -\frac{1}{2}\psi(y_{n+2}(t))\tilde{y}(t)^\top Q(t)\tilde{y}(t)dt.
\]
Here \( \gamma(t) \) is \( n \times n \)-dimensional matrix defined from the Riccati’s equation
\[
\begin{cases}
\frac{d\gamma}{dt} = -[b(t)\sigma(t)^\top + \gamma(t)]Q(t)[b(t)\sigma(t)^\top + \gamma(t)]^\top - \tilde{\alpha}(t)\gamma(t) - \gamma(t)\tilde{\alpha}(t)^\top + \beta(t)\beta(t)^\top, \\
\gamma(0) = \gamma_0.
\end{cases}
\]
\( A(t) \triangleq -\tilde{\alpha}(t) - \gamma(t)Q(t) \). Note that the the corresponding \( f, b \) satisfy the required conditions.

Since \( \sigma \) is independent of \( \tilde{R} \) then \( \tilde{R} \) is not required as a component of \( y \).

Therefore, the equation for \( y(t) \) can be written as (26) and the corresponding \( f, b \) satisfy the required conditions. Since \( \sigma \) is independent of \( \tilde{R} \) then \( \tilde{R} \) is not required as a component of \( y \).

By Corollary 5.2, it follows that \( \bar{Z} = \phi(y(T)) \), where the function \( \phi(\cdot) : \mathbb{R}^{n+2} \rightarrow \mathbb{R} \) is such that \( \phi(y) = y_{n+2} \exp y_{n+1} \) for \( y = (y_1, \ldots, y_{n+1}, y_{n+2}) \). Thus, all assumptions of Theorem
6.1 are satisfied if (30) is satisfied. In particular, if \( F \) is bounded then (30) is satisfied; if \( F(\bar{Z}, \beta, \lambda) \) is polynomial with respect to \( \bar{Z} \), then (30) is satisfied if the variance of \( \bar{a}(t) \) is small enough.

Note that the solution in Lakner (1998) express the optimal strategy via a conditional expectation of an optimal claim; our solution borrowed from Dokuchaev (2005) is more constructive provided we can solve the Cauchy problem (27), (29).

For an Euclidean space \( E \) we shall denote by \( B([0, T]; E) \) the set of bounded measurable functions \( f(t) : [0, T] \to E \).

**Example 2.** Assume that the number of possible paths of \( \bar{a} \) is finite. Assume that \( \sigma \) is non-random, \( A(t, \theta, f) = \theta(t) \) and there exist an integer \( d > 1 \) and a set \( \{ \theta_i(\cdot) : i = 1, \ldots, d \} \subset B([0, T]; \mathbb{R}^n) \) such that \( \sum_{i=1}^d p_i = 1 \) where \( p_i = \mathbb{P}(\bar{a}(\cdot) = \theta_i(\cdot)) \). Set \( y(t) = (y_1(t), \ldots, y_d(t)) \top \), where \( y_i(t) = \Delta(\theta_i, t) \). Let \( \bar{F}(y) = F(y \top \lambda), D = (0, +\infty)^d \times [0, T) \) and let \( b(y, t) : D \to \mathbb{R}^{d \times n} \), be such that the \( i \)th row of \( b \) is \( y_i \theta_i(t) \top \). Then \( dy(t) = b(y(t), t)Q(t) \, d\bar{R} \) and (27),(29) becomes

\[
\begin{align*}
\frac{\partial V}{\partial t}(y, t) + \frac{1}{2} Tr[D_{yy}(y, t)b(y, t)Q(t)b(y, t)\top] = 0, \\
V(y, t) \to \bar{F}(y) \quad \text{as} \quad t \to T - 0 \quad \forall y.
\end{align*}
\]

Note that the equation is degenerate in general, so it may not be easy to solve. Nevertheless, the theorem gives the optimal strategy in terms of \( V \).

**Example 3.** Here \( \bar{a}(t) \) evolves as a function of a finitely-valued Markov process. For simplicity, let \( n = 1 \). Assume that \( \bar{a}(t) = A(t, \theta(t), R(t)) \) and \( \sigma(t) = \alpha(t, R(t)) \), where the process \( \theta(t) \) is a random Markov process such that \( \mathbb{P}(\theta(t) \in \Lambda) = 1 \), and \( \Lambda = \{ \theta_i : i = 1, \ldots, d \} \) is a given finite set, \( d > 1 \) is an integer. We assume that \( A(t, \cdot) : \Lambda \times \mathbb{R} \to \mathbb{R} \) and \( \alpha(t, \cdot) : \mathbb{R} \to \mathbb{R} \) are given measurable functions satisfying Assumption 2.1. We are given the initial distribution of \( \theta(0) \), i.e. we are given \( \bar{y}_i = \mathbb{P}(\theta(0) = \theta_i) \), and we are given bounded functions \( l_{ij}(\cdot) : [0, T] \to \mathbb{R} \) such that

\[
p_{ij}(t, s) = \delta_{ij} + \int_s^t \sum_{k=1}^d l_{ki}(\tau)p_{kj}(\tau, s) \, d\tau \quad \forall s \leq t,
\]

where \( p_{ij}(t, s) = \mathbb{P}(\theta(t) = \theta_i | \theta(s) = \theta_j) \), and where \( \delta_{ij} \) is the Kronecker delta, cf. Liptser and Shiryaev (2001), Lemma 9.1. This specifies \( \nu \).

Set \( M = d+2, y(t) = (y_1(t), \ldots, y_M(t)) \top \), where

\[
y_i(t) = \mathbb{P}(\theta(t) = \theta_i | \mathcal{F}^R_t), \quad i = 1, \ldots, d,
\]

\[
y_{d+1}(t) = \bar{R}(t); \quad y_{d+2}(t) = \bar{Z}(t).
\]
By Theorem 9.1 from Liptser and Shiryaev (2001), p.355, we have

\[
\begin{align*}
    dy_i(t) &= \sum_{k=1}^{d} l_{k,i}(t) y_k(t) dt + y_i(t) \alpha(t, \bar{R}(t))^{-2}\left[ A(t, \theta_i, \bar{R}(t)) \right. \\
    &\quad - \left. \sum_{k=1}^{d} A(t, \theta_k, \bar{R}(t)) y_k(t) \right] \left[ d\bar{R}(t) - \sum_{k=1}^{d} A(t, \theta_k, \bar{R}(t)) y_k(t) \right] dt, \\
    y_i(0) &= \bar{y}_i, \quad i = 1, \ldots, d.
\end{align*}
\]

(34)

To keep a linear bound we introduce a bounded smooth function \(\psi(\cdot) \in C^\infty(\mathbb{R})\) such that \(\psi(x) = x\) (\(\forall x \in [0,1]\)) (clearly, there exists such a function). As \(y_i \in [0,1], i = 1, \ldots, d\), we may replace such \(y_i\) by \(\psi(y_i)\) as needed. Then we have

\[
\tilde{a}(t) \triangleq \mathbb{E}\{\tilde{a}(t)|F_t^\mathbb{R}\} = \sum_{i=1}^{d} A(t, \theta_i, y_{d+1}(t)) y_i(t) = \sum_{i=1}^{d} A(t, \theta_i, y_{d+1}(t)) \psi(y_i(t)),
\]

so again (20) and (34) imply

\[
\begin{align*}
    dy_i(t) &= \sum_{k=1}^{d} l_{k,i}(t) y_k(t) dt + y_i(t) \alpha(t, \bar{R}(t))^{-2}\left[ A(t, \theta_i, \bar{R}(t)) \right. \\
    &\quad - \left. \sum_{k=1}^{d} A(t, \theta_k, \bar{R}(t)) \psi(y_k(t)) \right] \left[ d\bar{R}(t) - \sum_{k=1}^{d} A(t, \theta_k, \bar{R}(t)) \psi(y_k(t)) \right] dt, \\
    dy_{d+1}(t) &= d\bar{R}(t), \\
    dy_{d+2}(t) &= y_{d+2}(t) \sum_{i=1}^{d} A(t, \theta_i, y_{d+1}(t)) \psi(y_i(t)) \alpha(t, y_{d+1}(t))^{-2} d\bar{R}(t), \\
    y_i(0) &= \bar{y}_i, \quad i = 1, \ldots, d, \quad y_{d+1}(0) = 0, \ y_{d+2}(0) = 1.
\end{align*}
\]

(35)

Clearly, the system of equations (34)-(35) can be rewritten in the form of (26), the corresponding \(f, \tilde{b}\) satisfy the required conditions, and all assumptions of Theorem 6.1 are satisfied, and the optimal strategy can be found from the corresponding equation (27), (29).

7 Appendix: Proofs

First we prove Proposition 2.1. To this end define

\[
\begin{align*}
    \bar{R}_s(t) &\triangleq \int_{0}^{t} \alpha(\tau, \bar{R}_s(\cdot)) dw(\tau), \quad \bar{a}_s(t) \triangleq A(t, \Theta, \bar{R}_s(\cdot)|[0,t]), \\
    Z_s &\triangleq \exp \left( \int_{0}^{T} (\alpha(t, \bar{R}_s(\cdot))^{-1} \bar{a}_s(t)) \alpha(t, \bar{R}_s(\cdot))^{-1} \bar{a}_s(t)) \right. \\
    &\quad - \left. \left[ 1 \right. \int_{0}^{T} \left| \alpha(t, \bar{R}_s(\cdot))^{-1} \bar{a}_s(t) \right|^2 dt \right). \quad (36)
\end{align*}
\]

**Proposition 7.1** There exists a measurable function \(\psi: C([0,T];\mathbb{R}^n) \times B([0,T];\mathbb{R}^n) \to \mathbb{R}\) such that \(Z_s = \psi(\bar{R}_s(\cdot), \bar{a}_s(\cdot))\) and \(Z = \psi(\bar{R}(\cdot), \bar{a}(\cdot))\) a.s. Moreover, \(z(\theta, T) = \psi(\bar{R}(\cdot), A(\cdot, \theta, \bar{R}))\).
Proof. Define
\[ Q(t, f) \triangleq \alpha(t, f) \alpha(t, f)^\top. \] (37)
Then
\[ \log Z = \int_0^T \bar{a}(t)^\top Q(t, \bar{R}(\cdot)[0,t]) \left( \frac{1}{2} \bar{a}(t)dt \right), \] (38)
and
\[ \log Z_* = \int_0^T \tilde{a}_*(t)^\top Q(t, \bar{R}_*(\cdot)[0,t]) \left( \frac{1}{2} \tilde{a}_*(t)dt \right). \]
This defines \( \psi. \)

Since \( z(t, \theta) \) satisfies (38) with \( \bar{a}(\cdot) \) replaced by \( A(\cdot, \theta, \bar{R}) \), the last result follows.

\( \square \)

Let
\[ \tilde{Z}_* \triangleq \int_T d\nu(\theta) \psi(\bar{R}_*(\cdot), A(\cdot, \theta, \bar{R}_*(\cdot))) \triangleq \tilde{\psi}(\bar{R}_*(\cdot)). \] (39)
It follows from Proposition 7.1 that \( \tilde{Z} = \tilde{\psi}(\bar{R}(\cdot)). \) Finally, since \( \Theta \) is independent of \( w, r \), hence of \( \bar{R}_*, r \), it follows that
\[ \tilde{Z}_* = \mathbb{E}(Z_*|\bar{R}_*, r). \] (40)

Proposition 7.2 Let \( \phi : C([0, T]; \mathbb{R}^n) \times B([0, T]; \mathbb{R}^n) \times B([0, T]; \mathbb{R}) \rightarrow \mathbb{R} \) be a function such that \( \mathbb{E}\phi^{-1}(\bar{R}(\cdot), \bar{a}(\cdot), r(\cdot)) < +\infty \) and let \( \hat{\phi} \) be a similar function but with no dependence on \( \bar{a}. \)

Then
\[ \mathbb{E}\phi(\bar{R}(\cdot), \bar{a}(\cdot), r(\cdot)) = \mathbb{E}Z_*\phi(\bar{R}_*(\cdot), \tilde{a}_*(\cdot), r(\cdot)), \] (41)
\[ \mathbb{E}\hat{\phi}(\bar{R}(\cdot), r(\cdot)) = \mathbb{E}\tilde{Z}_*\hat{\phi}(\bar{R}_*(\cdot), r(\cdot)), \] (42)
\[ \mathbb{E}_*\hat{\phi}(\bar{R}(\cdot), r(\cdot)) = \mathbb{E}\hat{\phi}(\bar{R}_*(\cdot), r(\cdot)). \] (43)

Proof. By assumption \( \Theta \) is independent of \( w(\cdot) \). Define the probability measure \( \hat{\mathbb{P}} \) by
d\( \hat{\mathbb{P}} \) / \( \mathbb{P} = Z_* \). Then \( \mathbb{E}\{Z_*|\Theta\} = 1 \) and to prove (41) it suffices to prove
\[ \mathbb{E} \left\{ \phi(\bar{R}(\cdot), \bar{a}(\cdot), r(\cdot)) \bigg| \Theta \right\} = \mathbb{E} \left\{ Z_*\phi(\bar{R}_*(\cdot), \tilde{a}_*(\cdot), r(\cdot)) \bigg| \Theta \right\} \]
\[ = \hat{\mathbb{E}} \left\{ \phi(\bar{R}_*(\cdot), \tilde{a}_*(\cdot), r(\cdot)) \bigg| \Theta \right\} \quad \text{a.s.} \] (44)

Thus, for the next paragraph, without loss of generality, we will suppose that \( \Theta = \theta \) is deterministic, since for each value of \( \Theta \) we can construct \( \bar{R}, \bar{R}_*, \bar{a}, \tilde{a}_* \) and \( \hat{\mathbb{P}}. \)

By Girsanov’s Theorem, the process
\[ \hat{w}(t) \triangleq w(t) - \int_0^t \alpha(s, \bar{R}_*(\cdot))^{-1} \tilde{a}_*(s)ds \]
is a Wiener process under \( \hat{P} \). From this and (3) we obtain
\[
\begin{align*}
d\hat{R}(t) &= A(t, \Theta, \hat{R}(\cdot)|_{[0,t]})dt + \alpha(t, \hat{R}(\cdot)|_{[0,t]})dw(t), \\
d\hat{R}_*(t) &= A(t, \Theta, \hat{R}_*(\cdot)|_{[0,t]})dt + \alpha(t, \hat{R}_*(\cdot)|_{[0,t]})d\tilde{w}(t).
\end{align*}
\]

Then for each value of \( \Theta \) the processes \( \hat{R}(\cdot) \) and \( \hat{R}_*(\cdot) \) have the same distribution on the probability spaces defined by \( P \) and \( \hat{P} \) respectively, and (44), hence (41) follows.

Further, (42) follows by taking conditional expectation in (41). Finally, using Proposition 7.1 and (41),
\[
E_\ast \phi(\hat{R}(\cdot), r(\cdot)) = E_\ast \phi(\hat{R}_*(\cdot), r(\cdot)) = E_\ast \hat{Z}_\ast \phi(\hat{R}_*(\cdot), r(\cdot)) = E_\ast \phi(\hat{R}_*(\cdot), r(\cdot)).
\]

\[\square\]

**Proof of Proposition 2.1.** It suffices to show that \( E_\ast \phi Z = E_\ast \phi \tilde{Z} \) for all \( F^R_T \)-measurable functions \( \phi \). Such functions are of the form \( \tilde{\phi}(\hat{R}, r) \) above. But (42), (43) imply
\[
E_\ast \phi Z = E \phi = E \tilde{\phi}(\hat{R}, r) = E \tilde{Z}_\ast \phi(\hat{R}_*(\cdot), r(\cdot)) = E_\ast \tilde{Z}_\ast \phi(\hat{R}_*(\cdot), r(\cdot)) = E_\ast \phi \tilde{Z}.
\]

\[\square\]

Remark 2.3 can be verified by a similar technique. Just replace \( \alpha(t, \cdot) \) by \( \sigma(t, \omega) \) no matter what the argument in \( \alpha \). From (4) it follows that there exists a function \( Q(t, \omega) = Q(t, \hat{R}(\cdot, \omega)) = Q(t, \hat{R}_*(\cdot, \omega)) \). If \( A \) depends on \( r \) also then \( \tilde{\psi} \) depends additionally on \( r \). To obtain law uniqueness we now condition on \( \Theta, \sigma, r \) in the proof of Proposition 7.2.

We turn now to Theorem 4.1. Define \( \hat{\xi}_* \triangleq F(\hat{Z}_*, \hat{\lambda}) \). If we define \( \phi \) by \( \hat{\xi} = \phi(\hat{R}(\cdot)) \), then \( \hat{\xi}_* = \phi(\hat{R}_*(\cdot)) \).

**Proof of Theorem 4.1.** Let us show that \( EU^- (\hat{\xi}) < \infty \) so that \( EU(\hat{\xi}) \) is well defined. For \( k = 1, 2, ..., \) introduce the random events
\[
\Omega_*^{(k)} \triangleq \{ -k \leq U(\hat{\xi}_*) \leq 0 \}, \quad \Omega^{(k)} \triangleq \{ -k \leq U(\hat{\xi}) \leq 0 \},
\]
along with their indicator functions, \( \chi_*^{(k)} \) and \( \chi^{(k)} \), respectively. The number \( \hat{\xi}_* \) provides the unique maximum of the function \( \hat{Z}_* U(\xi_* \cdot - \hat{\lambda} \cdot) \) over \( \hat{D} \), and \( X_0 \in \hat{D} \). Hence by Proposition 7.2, we have, for all \( k = 1, 2, ..., \)
\[
E \chi_*^{(k)} U(\hat{\xi}) - E \chi_*^{(k)} \hat{\lambda} \hat{\xi}_* = E \chi_*^{(k)} \left( \hat{Z}_* U(\xi_* \cdot - \hat{\lambda} \cdot) \right) \geq E \chi_*^{(k)} \left( \hat{Z}_* U(X_0 - \hat{\lambda} X_0) \right) = E \chi^{(k)} U(X_0 - \hat{\lambda} X_0) P(\Omega_*^{(k)}) \geq -U(X_0) - |\hat{\lambda} X_0| > -\infty.
\]

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Furthermore, we have that $\mathbb{E}|\tilde{\xi}^*| = \mathbb{E}_*|\tilde{\xi}| < +\infty$. Hence $\mathbb{E}U^- (\tilde{\xi}) < \infty$.

Now observe that for any $\pi \in \mathcal{A}$ we can apply (42) and (43) to $U(\tilde{X}^\pi (T))$ (and use (39)) to obtain

$$\mathbb{E}U(\tilde{X}^\pi (T)) = \mathbb{E}_* (\tilde{Z}U(\tilde{X}^\pi (T))) \leq \mathbb{E}_* \{ \tilde{Z} (\tilde{X}^\pi (T)) - \tilde{\lambda} \tilde{X}^\pi (T) \} + \hat{\lambda} X_0 \leq \mathbb{E}_* \tilde{Z} U(\tilde{\xi}) - \tilde{\lambda} \tilde{\xi} + \hat{\lambda} X_0 = \mathbb{E}_* \tilde{Z} U(\tilde{\xi}) = \mathbb{E} U(\tilde{\xi}).$$

Thus (ii) is satisfied.

To show (iii), note that $\mathcal{F}_t^\pi = \mathcal{F}_t^\hat{R}$ so $\hat{\xi}_* = \phi(w(\cdot))$, where $\phi(\cdot): B([0, T]; \mathbb{R}^n) \rightarrow \mathbb{R}$ is a measurable function. By the martingale representation theorem,

$$\tilde{\xi}_* = \mathbb{E} \tilde{\xi}_* + \int_0^T f(t, w(\cdot) | [0, t]) \, dw(t),$$

where $f(t, \cdot) : B([0, t]; \mathbb{R}^n) \rightarrow \mathbb{R}$ is a measurable function such that $\int_0^T |f(t, w(\cdot) | [0, t])|^2 \, dt < +\infty$ a.s. There exists a unique measurable function $f_0(t, \cdot) : B([0, t]; \mathbb{R}^n) \rightarrow \mathbb{R}$ such that $f(t, w(\cdot) | [0, t]) \equiv f_0(t, \tilde{\tilde{R}}(\cdot) | [0, t])$. Thus,

$$\tilde{\xi}_* = \mathbb{E} \tilde{\xi}_* + \int_0^T f_0(t, \tilde{\tilde{R}}(\cdot) | [0, t]) \, dw(t) = \mathbb{E} \tilde{\xi}_* + \int_0^T f_0(t, \tilde{\tilde{R}}(\cdot) | [0, t]) \, \alpha(t, \tilde{\tilde{R}}^{-1} d\tilde{\tilde{R}}(t)).$$

Proposition 7.2 implies that $\mathbb{E} \tilde{\xi}_* = \mathbb{E} \hat{\xi}_* = X_0$, and

$$\hat{\xi} = X_0 + \int_0^T f_0(t, \tilde{\tilde{R}}(\cdot) | [0, t]) \, \sigma(t)^{-1} d\tilde{\tilde{R}}(t).$$

It follows that the strategy $\hat{\pi}(t)^\top = B(t) f_0(t, \tilde{\tilde{R}}(\cdot) | [0, t]) \, \sigma(t)^{-1}$ replicates $B(T) \bar{\xi}$. It belongs to $\mathcal{A}$; in particular, $\tilde{X}^\hat{\pi}(t) = X_0 + \int_0^T f_0(t, \tilde{\tilde{R}}(\cdot) | [0, t]) \, \sigma(t)^{-1} d\tilde{\tilde{R}}(t) = \mathbb{E}_*(\bar{\xi} | \mathcal{F}_t^\hat{R}) \in \tilde{D}$ since $\tilde{D}$ is convex. Hence $\tilde{X}^\hat{\pi}$ is bounded below. This completes the proof of Theorem 4.1. □

Proof of Corollary 5.1(ii). We shall employ the notation $Y(t, \pi) \triangleq \log \left( \frac{\tilde{X}^\pi(t) + \delta}{\lambda X_0 + \delta} \right)$. Let $\mathcal{B}_2$ be the set of all processes $\bar{a}(t) : [0, T] \rightarrow \mathbb{R}^n$ which are progressively measurable with respect to $\mathcal{F}_t^{\hat{R}, \pi}$ and such that $\mathbb{E} \int_0^T |\bar{a}(t)|^2 \, dt < +\infty$. For any $\bar{a}(\cdot) \in \mathcal{B}_2$, define $\bar{\pi}(t)^\top \triangleq (X^\pi(t) + \delta B(t) \bar{a}(t)^\top) Q(t)$ where $X^\pi(t) \triangleq B(t) \tilde{X}(t)$ and $\tilde{X}(\cdot)$ is found from (9) using $\pi^\top = B(\tilde{X} + \delta \bar{a}^\top) Q$. Then

$$Y(t, \bar{\pi}) = \int_0^t \left( \bar{a}(s)^\top Q(s) d\tilde{R}(s) - \frac{1}{2} \int_0^t \bar{a}(s)^\top Q(s) \bar{a}(s) ds \right),$$

and

$$\mathbb{E} Y(T, \bar{\pi}) = \frac{1}{2} \mathbb{E} \int_0^T \left( -|\sigma(t)^{-1}(\bar{a}(t) - \bar{a}(t))|^2 + \bar{a}(t)^\top Q(t) \bar{a}(t) \right) dt.$$

Set $a'(t) \triangleq \mathbb{E}\{\bar{a}(t) | \mathcal{F}_t^{\hat{R}, \pi}\}$. Since $\mathbb{E}|K(\Theta, \sigma(\cdot), \sigma(\cdot)^\top, r(\cdot))|^2 < \infty$, then Jensen’s inequality implies that $a'(\cdot) \in \mathcal{B}_2$. Consider the corresponding strategy

$$\pi'(t)^\top \triangleq (X^\pi'(t) + \delta B(t)) a'(t)^\top Q(t).$$

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It is well known that $\mathbf{E}Y(T, \pi^{'}) \geq \mathbf{E}Y(T, \hat{\pi})$, so the strategy (48) is optimal over all $\hat{\pi}(\cdot)$ which correspond to $\hat{a}(\cdot) \in \mathcal{B}_2$. Then (19) and the Corollary follow if $\hat{a}(\cdot) \in \mathcal{B}_2$.

Let us show this. For any $K > 0$, set

$$T_K \triangleq \inf\{t \in [0, T] : \int_0^t |\hat{a}(s)|^2 ds > \int_0^t |a'(s)|^2 ds + K\}.$$  

As usual we take $T_K = T$ if the set is empty. Note that

$$\mathbf{E}\log(\hat{X}^{\hat{\pi}}(T_K) + \delta) \geq \mathbf{E}\log(\hat{X}^{\pi'}(T_K) + \delta) \quad \forall K > 0,$$  

because if (49) fails, then $\mathbf{E}Y(T, \pi_K) > \mathbf{E}Y(T, \hat{\pi})$, where

$$\pi_K(t) \triangleq \begin{cases} \pi'(t) & t \leq T_K \\ \hat{\pi}(t) & t > T_K. \end{cases}$$

Further, let $\chi_K(t)$ denote the indicator function of the event $\{t < T_K\}$ and let $\tilde{a}_K(\cdot) \triangleq \chi_K(\cdot)\hat{a}(\cdot) \in \mathcal{B}_2$, $\tilde{a}_K(t) \triangleq \chi_K(t)\hat{a}(t)$. As in (47), we have

$$\mathbf{E}Y(T_K, \hat{\pi}) = \frac{1}{2} \mathbf{E} \int_0^{T_K} (-|t|^{-1}(\hat{a}(t) - \tilde{a}(t))^2 + \tilde{a}(t)^{\top} \mathbf{Q}(t)\tilde{a}(t)) dt$$

Then the process $\mathbf{E}\{\tilde{a}_K(t)|\mathcal{F}_t^{K,R}\} = \chi_K(t)\mathbf{E}\{\tilde{a}(t)|\mathcal{F}_t^{K,R}\} = \chi_K(t)a'(t)$ gives the maximum of $\mathbf{E}Y(T_K, \hat{\pi})$. It follows from (49) that $\chi_K(t)\hat{a}(t) = \chi_K(t)a'(t)$ for $t \in [0, T]$ and $K > 0$. Thus, $T_K = T$ a.s. for any $K > 0$, and $a'(\cdot) = \hat{a}(\cdot), \hat{a}(\cdot) \in \mathcal{B}_2$. Then (19) and (ii) follow. \(\square\)

**Proof of Theorem 5.1.** From Conditions 3.1 and 3.2 it follows that $F(z, \lambda) = \lambda^{-1}z^I$, so we want to replicate $B(T)(X_0/\mathbf{E}_*\hat{Z}^I)\hat{Z}^I$. Let us first find a representation for $\hat{Z}^I$.

$$\hat{Z}^I = \int_{T^I} d\nu(\theta_1) \cdots d\nu(\theta_t) \exp\left(\sum_{k=1}^l \int_0^T \theta_k(t)^{\top} Q(t) \mathbf{d}\hat{R}(t) - \frac{1}{2} \sum_{k=1}^l \int_0^T \theta_k(t)^{\top} Q(t) \theta_k(t) dt\right)$$

$$= \int_{T^I} d\nu(\theta_1) \cdots d\nu(\theta_t) \gamma(\theta_1, \ldots, \theta_t) z(\sum_{k=1}^l \theta_k, T)$$

$$= \int_T d\nu(\theta) z(\theta, T) G.$$ 

It follows that $\mathbf{E}_*\hat{Z}^I = G$ since $\mathbf{E}_*z(\theta, T) = \mathbf{E}_*\psi(\hat{R}, \theta) = \mathbf{E}\psi(\hat{R}_*, \theta) = 1$ because $\psi(\hat{R}_*, \theta)$ is a $\mathbf{P}$ martingale.

Now we must show that $\hat{X}^{\hat{\pi}}(T) = X_0 \hat{Z}^I/G$. But

$$\frac{X_0\hat{Z}^I}{G} = X_0 \int_T d\nu(\theta) z(\theta, T) = X_0 \left(1 + \int_T d\nu(\theta) z(\theta, T) \theta(t)^{\top} Q(t) \mathbf{d}\hat{R}(t)\right) = \hat{X}^{\hat{\pi}}(T)$$

with $\hat{\pi}$ defined by (22).
Now (23) and the equality in (22) follow from (9) and (6). Finally (24) follows from 
\[ \mathbb{E}\{\tilde{Z}^{l-1}\} = \mathbb{E}_s\{\tilde{Z}^l\} = G. \]

**Proof of Corollary 5.3.** If we take \( \mathcal{T} = \{\theta_o\} \) then \( \mathcal{T} = \{l\theta_o\} \), so (22) implies that the optimal strategy in case of complete observation is \( \pi(t) = lX^\pi(t)Q(t)\tilde{a}(t) \), hence the first equality in (25) follows. This and (22), (23) imply
\[
\lambda\tilde{a}(t)^\top = \frac{\int_{\mathcal{T}} d\bar{\nu}(\theta)z(\theta,t)\theta(t)^\top}{\int_{\mathcal{T}} d\bar{\nu}(\theta)z(\theta,t)}.
\]
Comparing this with Corollary 5.1(i) and (16), we see that \( \lambda\tilde{a}(t) \) is the equivalent filter for the problem with \( U(x) \equiv \log x \) and with the prior distribution of \( \Theta = \tilde{a}(\cdot) \) described by \( \bar{\nu} \) on \( \tilde{\mathcal{T}} \). By Corollary 5.1 (ii), \( \tilde{a}(t) = l^{-1}\mathbb{E}\{\tilde{a}(t)|\mathcal{F}^R_t\} \).

**Proof of Proposition 6.1.** It is required to show that the strategy defined in the Proposition does exist and is admissible. Assume that \( C(\cdot) \) has a finite support inside an open domain in \( \mathbb{R}^M \), and let the function \( C(\cdot) \) be smooth enough. Set \( V(x,s) \triangleq \mathbb{E}_sC(y^{x,s}(T)) \), where \( y^{x,s}(\cdot) \) is the solution of
\[
\begin{cases}
  dy(t) = f(y(t),t)dt + \tilde{b}(y(t),t)d\tilde{R}(t), \\
y(s) = x.
\end{cases}
\]
(51)
Then it can be shown that \( V(x,s) \) is the classical solution of the problem (27)-(28). Thus, \( V(x,t) \) is a classical solution of (27)-(28). Set \( \bar{X}_s(t) = V(y_s(t),t) \). From (27) and Itô’s Lemma, it follows that
\[
\bar{X}_s(T) = \bar{X}_s(t) + \int_t^T \frac{\partial V}{\partial y}(y_s(s),s)\tilde{b}(y_s(s),s)dw(s).
\]
It follows that \( \bar{X}_s(0) = V(y_s(0),0) = \mathbb{E}V(y_s(T),T) = X_0 \) and
\[
d\bar{X}_s(t) = \frac{\partial V}{\partial y}(y_s(t),t)^\top\tilde{b}(y_s(t),t)d\tilde{R}_s(t), \quad \bar{X}_s(T) = C(y_s(T)).
\]
(52)
Then \( \bar{X}_s(t) = \psi(t,\tilde{R}_s) \) for some measurable \( \psi \), and the result follows if we observe that \( \bar{X}^\pi(t) = \psi(t,\tilde{R}) \) for the given \( \pi \).

To continue, we require some a priori estimates. Let \( \zeta(t) \triangleq \alpha(t,\tilde{R}_s)^\top \pi(t) \). Define \( \pi_s \) in the obvious way. Consider the conditional probability space given \( r(\cdot) \). With respect to the conditional probability space, it follows from (52) that
\[
\begin{cases}
d\bar{X}_s(t) = B(t)^{-1}\zeta_s(t)^\top dw(t), \\
\bar{X}_s(T) = C(y_s(T)).
\end{cases}
\]
(53)
The solution \((Z_s(t),\bar{X}_s(t))\) of the stochastic backward equation (53) is a process in \( L_2([0,T],L^2(\Omega,\mathcal{F},P)) \times C([0,T],L^2(\Omega,\mathcal{F},P)) \) (see, e.g., El Karoui et al (1997), or Yong

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and Zhou (1999), Chapter 7, Theorem 2.2). Note that the equation (53) is linear. Thus, it can be shown by using Theorem 2.2 from Chapter 7 from Yong and Zhou (1999) again that there exists a constant $c_0$, independent of $C(\cdot)$, such that

$$\sup_t E \left\{ |\tilde{X}_s(t)|^2 | r(\cdot) \right\} + E \left\{ \int_0^T |\zeta_s(t)|^2 dt \right\} \leq c_0 E \left\{ C(y_s(T))^2 | r(\cdot) \right\} \quad \text{a.s.}$$

Hence

$$\sup_t E|\tilde{X}_s(t)|^2 + E \int_0^T |\zeta_s(t)|^2 dt \leq c_0 E C(y_s(T))^2.$$  \hspace{1cm} (54)

Let $C(\cdot)$ be a general measurable function satisfying the conditions specified in the proposition. Then, there exists a sequence $\{C^{(i)}(\cdot)\}$, where $C^{(i)}(\cdot)$ has a finite support inside the open domain $\mathbb{R}^M$ and is smooth enough, such that

$$E|C^{(i)}(y_s(T)) - C(y_s(T))|^2 \to 0 \quad \text{as} \quad i \to \infty.$$  \hspace{1cm}

Let $\tilde{X}^{(i)}(\cdot)$, $\pi^{(i)}(\cdot)$, $V^{(i)}(\cdot)$ be the corresponding processes and functions. By (54) and the linearity of (53), it follows that

$$\sup_t E|\tilde{X}_s^{(i)}(t)|^2 + E \int_0^T |\pi^{(i)}(t)|^2 dt \leq c_0 E|C^{(i)}(y_s(T)) - C^{(j)}(y_s(T))|^2 \to 0 \quad \text{as} \quad i \to \infty.$$  \hspace{1cm}

From (43) it follows that

$$\sup_t E_s|\tilde{X}^{(i)}(t) - \tilde{X}^{(j)}(t)|^2 + E_s \int_0^T |\pi^{(i)}(t) - \pi^{(j)}(t)|^2 dt \to 0 \quad \text{as} \quad i \to \infty.$$  \hspace{1cm}

Thus, $\{\tilde{X}^{(i)}(\cdot)\}$, $\{\pi^{(i)}(\cdot)\}$ are Cauchy sequences in the space the spaces $C([0,T], L^2(\Omega, \mathcal{F}, P_s))$ and $L^2([0,T], L^2(\Omega, \mathcal{F}, P_s))$ correspondingly, and hence, it can be shown that the corresponding limits $\tilde{X}(\cdot)$, $\pi(\cdot)$ exist, and belongs these spaces. Similarly Dokuchaev and Zhou (2001), it follows from the definition of $\mathcal{Y}^1$ that $V^{(i)}(\cdot)$ is a Cauchy sequence in and $\mathcal{Y}^1$. This completes the proof. \hspace{1cm} □

Proof of Theorem 6.1. As in the proof above, it can be shown that $\tilde{X}(t) = V(y(t), t, \tilde{\lambda})$ is the solution of some equation (52), i.e. it is the normalized wealth. Then the proof follows. \hspace{1cm} □

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