ZEROS OF FUNCTIONS IN BERGMAN–TYPE HILBERT SPACES OF DIRICHLET SERIES

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Abstract. For a real number \( \alpha \) the Hilbert spaces \( D_\alpha \) consists of those Dirichlet series \( \sum_{n=1}^{\infty} a_n/n^s \) for which \( \sum_{n=1}^{\infty} |a_n|^2/(d(n))^\alpha < \infty \), where \( d(n) \) denotes the number of divisors of \( n \). We extend a theorem of Seip on the bounded zero sequences of functions in \( D_\alpha \) to the case \( \alpha > 0 \). Generalizations to other weighted spaces of Dirichlet series are also discussed, as are partial results on the zeros of functions in the Hardy spaces of Dirichlet series \( H^p \), for \( 1 \leq p < 2 \).

1. Introduction

Let \( d(n) \) denote the divisor function let \( \alpha \) be a real number. We are interested in the following Hilbert spaces of Dirichlet series:

\[
D_\alpha = \left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} : \|f\|_{D_\alpha}^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{[d(n)]^\alpha} < \infty \right\}.
\]

The functions of \( D_\alpha \) are holomorphic in \( \mathbb{C}_{1/2}^+ = \{ s = \sigma + it : \sigma > 1/2 \} \). Bounded Dirichlet series are almost periodic, and this implies that they have either no zeros or infinitely many zeros, as observed by Olsen and Seip in [10]. This leads us to restrict our investigations to bounded zero sequences for spaces of Dirichlet series. In [13], Seip studied bounded zero sequences for \( D_\alpha \), when \( \alpha \leq 0 \). This includes the Hardy–type (\( \alpha = 0 \)) and Dirichlet–type (\( \alpha < 0 \)) spaces. The topic of the present work is the Bergman–type spaces (\( \alpha > 0 \)).

Let us therefore introduce the weighted Bergman spaces in the half-plane, \( A_\beta \). For \( \beta > 0 \), these spaces consists of functions \( F \) which are holomorphic in \( \mathbb{C}_{1/2}^+ \) and satisfy

\[
\|F\|_{A_\beta} = \left( \int_{\mathbb{C}_{1/2}^+} |F(s)|^2 \left( \sigma - \frac{1}{2} \right)^{\beta-1} \ dm(s) \right)^{\frac{1}{2}} < \infty.
\]

It was shown by Olsen in [9] that the local behavior of the spaces \( D_\alpha \) are similar to the spaces \( A_\beta \), where \( \beta = 2^\alpha - 1 \). This relationship between \( \alpha \) and \( \beta \) will be retained throughout this paper.

For any space of holomorphic functions \( \mathcal{C} \) on some domain, we let \( Z(\mathcal{C}) \) denote the set of sequences in the domain such that there is some \( F \in \mathcal{C} \) vanishing on \( S \), taking into account multiplicities.

A result proved by Horowitz in [6] shows that if \( \mathcal{C} = A_\beta \) we may assume that \( F \) vanishes precisely on \( S \in Z(A_\beta) \), i.e. \( F \) has no extraneous zeros in \( \mathbb{C}_{1/2}^+ \). We will exploit this fact to prove our main result.

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Theorem 1. Suppose \( S = (\sigma_j + it_j) \) is a bounded sequence of points in \( \mathbb{C}_{1/2}^+ \) and that \( \alpha > 0 \). Then there is a non-trivial function in \( \mathcal{D}_\alpha \) vanishing on \( S \) if and only if \( S \in Z(A_\beta) \).

The “only if” part follows from the local embedding of \( \mathcal{D}_\alpha \) into \( A_\beta \) of Theorem 1 and Example 4 from [9]. To prove the “if” part, we will adapt the methods of [13], where an analogous result for \( \alpha \leq 0 \) was obtained.

The “if” part can essentially be split into two steps. The first step is a discretization lemma, which depends on the properties of \( \mathcal{D}_\alpha \)—or rather the weights \([d(n)]^\alpha\).

2. Proof of Theorem 1

We begin with the Paley–Wiener representation of functions \( F \in A_\beta \), and seek to construct a Dirichlet series \( f \in D_\alpha \) which approximates \( F \).

Lemma 2 (Paley–Wiener Representation). \( A_\beta \) is isometrically isomorphic to

\[
L^2_\beta = \left\{ \phi \text{ measurable on } [0, \infty) : \|\phi\|_{L^2_\beta}^2 = \frac{2\pi \Gamma(\beta)}{2^\beta} \int_0^\infty |\phi(\xi)|^2 \frac{d\xi}{\xi^{\beta}} < \infty \right\},
\]

under the Laplace transformation

\[
F(s) = \int_0^\infty \phi(\xi)e^{-(s-1/2)\xi} d\xi.
\]

Proof. A proof can be found in [2]. \qed

The other ingredient needed for the discretization lemma is estimates on the growth of \([d(n)]^\alpha\). We will partition the integers into blocks and use an average order type estimate.

Lemma 3. Let \( \alpha \) be a real number and \( 0 < \gamma < 1 \). Then

\[
\sum_{j^{\gamma} \leq \log n \leq (j+1)^{\gamma}} \frac{|d(n)|^\alpha}{n} \asymp j^{2^\alpha - 1},
\]
as \( J \to \infty \). The implied constants may depend on \( \alpha \) and \( \gamma \).

Proof. We will first assume that \( 2^\alpha \) is not an integer. To obtain the required estimate, we apply the precise form of a formula by Ramanujan [11] and Wilson [14]: For any real number \( \alpha \) and any integer \( \nu > 2^\alpha - 2 \), we have

\[
D_\alpha(x) = \sum_{n \leq x} [d(n)]^\alpha = x(\log x)^{2^\alpha - 1} \left( \sum_{\lambda=0}^{\nu} \frac{A_\lambda}{(\log x)^{\lambda}} + O\left( \frac{1}{(\log x)^{\nu+1}} \right) \right).
\]

Fix \( \nu \) such that \( \nu > 2^\alpha - 1 \) and \( \nu > 1/\gamma - 1 \). We use Abel summation to rewrite

\[
\sum_{y < n \leq x} \frac{|d(n)|^\alpha}{n} = \frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} + \int_y^x \frac{D_\alpha(z)}{z^2} dz.
\]
By using (2) and the fact that $2^\alpha - 1 - \nu < 0$ we perform some standard calculations to estimate

$$\frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} = \sum_{\lambda=0}^{\nu} A_\lambda \left( (\log x)^{2^\alpha - 1 - \lambda} - (\log y)^{2^\alpha - 1 - \lambda} \right) + O \left( (\log y)^{2^\alpha - 1 - \nu} \right),$$

$$\int_y^x \frac{D_\alpha(z)}{z^2} \, dz = \sum_{\lambda=0}^{\nu} A_\lambda \left( (\log x)^{2^\alpha - \lambda} - (\log y)^{2^\alpha - \lambda} \right) + O \left( (\log y)^{2^\alpha - 1 - \nu} \right).$$

Let us now take $x = \exp \left( (j + 1)^\gamma \right)$ and $y = \exp (j^\gamma)$. For any exponent $\eta$ it is clear that

$$(\log x)^{\eta} - (\log y)^{\eta} = \gamma \eta j^{\gamma - 1} \left( 1 + O \left( \frac{1}{j} \right) \right).$$

Hence we have

$$\frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} \asymp \sum_{\lambda=0}^{\nu} A_\lambda (\gamma (2^\alpha - 1 - \lambda)) j^{\gamma (2^\alpha - 1 - \lambda) - 1} + O \left( j^{\gamma (2^\alpha - 2 - \nu)} \right),$$

$$\int_y^x \frac{D_\alpha(z)}{z^2} \, dz \asymp \sum_{\lambda=0}^{\nu} A_\lambda j^{\gamma (2^\alpha - \lambda) - 1} + O \left( j^{\gamma (2^\alpha - 1 - \nu)} \right).$$

We combine these estimates with (3) to obtain

$$\sum_{\gamma \leq \log n \leq (j + 1)^\gamma} \left[ \frac{d(n)}{n} \right]^{\alpha_\gamma} \asymp j^{\gamma 2^{\alpha - 1} - 1} \left( 1 + \sum_{\lambda=1}^{\nu} B_\lambda \frac{1}{j^{\gamma \lambda}} + O \left( \frac{1}{j^{\gamma 2^{\alpha - 1} - \nu (2^{\alpha - 1} - \nu)} \right) \right),$$

which proves (1) since $\nu > 1 / \gamma - 1$. By continuity on both sides of (1), the assumption that $2^\alpha$ is not an integer may be dropped.

The parameter $0 < \gamma < 1$ will be used to control the “block size” of our partition of the integers. It will become apparent that as $\alpha$ grows to infinity, we must be able to let $\gamma$ tend to 0. In [13] it was sufficient to have a similar estimate only for $1/2 < \gamma < 1$.

**Lemma 4** (Discretization Lemma). Let $\alpha > 0$ and let $N$ be a sufficiently large positive integer. Then there exists positive constants $A$ and $B$ (depending on $\alpha$, but not $N$) such that the following holds: For every function $f \in L^2_{\beta}$ supported on $[\log N, \infty)$, there is a function of the form

$$f(s) = \sum_{n=0}^{\infty} a_n s^n$$

in $\mathcal{D}_\alpha$ such that $\|f\|_{\mathcal{D}_\alpha} \leq A \|\phi\|_{L^2_{\beta}}$. Moreover, $f$ may be chosen so that

$$\Phi(s) = \int_{\log N}^{\infty} \phi(\xi) e^{-(s - 1/2)\xi} \, d\xi - f(s)$$

enjoys the estimate

$$|\Phi(s)| \leq B |s - 1/2| N^{-\alpha + 1/2} (\log N)^{-1} \|\phi\|_{L^2_{\beta}},$$

in $\mathbb{C}^+_{1/2}$. 


Proof. Let \( \gamma = 2/(4+2^\alpha) \) and let \( J \) be the largest integer smaller than \((\log(N))^{1/\gamma}\).
For \( j \geq J \), let \( n_j \) be the smallest integer \( n \) such that \( e^j \leq n \). When \( \gamma \) is small it is possible that \( n_j = n_{j+1} \). This can be avoided by taking \( N \) sufficiently large. Set \( \xi_{n_j} = j^\gamma \) and for \( n_j < n \leq n_{j+1} \) iteratively choose \( \xi_n \) such that

\[
\frac{\xi_{n+1}^{\alpha+1} - \xi_n^{\alpha+1}}{\alpha+1} = A_j \frac{[d(n)]^\alpha}{n},
\]

where \( A_j \) is chosen so that \( \xi_{n_{j+1}} = (j+1)^\gamma \). Clearly, Lemma \[\text{[1]}\] implies that \( A_j \) is bounded as \( j \to \infty \). Let us set

\[
a_n = \sqrt{n} \int_{\xi_{n}}^{\xi_{n+1}} \phi(\xi) \, d\xi.
\]

A simple computation using the Cauchy–Schwarz inequality shows that

\[
|a_n|^2 = n \left| \int_{\xi_n}^{\xi_{n+1}} \phi(\xi) \, d\xi \right|^2 \leq n \cdot \frac{\xi_{n+1}^{\alpha+1} - \xi_n^{\alpha+1}}{\alpha+1} \int_{\xi_n}^{\xi_{n+1}} |\phi(\xi)|^2 \, d\xi.
\]

In view of \[\text{(5)}\] it is clear that \( \|f\|_{L^2} \leq A \|\phi\|_{L^2} \). Now, if \( n_j < n \leq n_{j+1} \) and \( \xi \in [\xi_{n_j}, \xi_{n_{j+1}}] \) we see that

\[
|e^{-(s-1/2)} - n^{-(s-1/2)}| \leq N^{-\sigma+1/2}|s - 1/2|j^{-1}.
\]

Then, by \[\text{(6)}\] and the Cauchy–Schwarz inequality

\[
|\Phi(s)| \leq N^{-\sigma+1/2}|s - 1/2| \sum_{j=J}^{\infty} j^{-1} \sum_{n=n_j}^{n_{j+1}-1} \left( \frac{\xi_{n+1}^{\alpha+1} - \xi_n^{\alpha+1}}{\alpha+1} \right)^{1/2} \left( \int_{\xi_n}^{\xi_{n+1}} |\phi(\xi)|^2 \, d\xi \right)^{1/2}.
\]

By using the Cauchy–Schwarz inequality again with \[\text{(5)}\] we get

\[
|\Phi(s)| \ll N^{-\sigma+1/2}|s - 1/2| \sum_{j=J}^{\infty} j^{-1} \left( \sum_{n=n_j}^{n_{j+1}-1} \frac{[d(n)]^\alpha}{n} \right)^{1/2} \left( \int_{\xi_n}^{\xi_{n+1}} |\phi(\xi)|^2 \, d\xi \right)^{1/2}.
\]

Now Lemma \[\text{[3]}\] and the Cauchy–Schwarz inequality yield

\[
|\Phi(s)| \ll N^{-\sigma+1/2}|s - 1/2| \left( \sum_{j=J}^{\infty} j^{(2+2^\alpha)\gamma - 3} \frac{[d(n)]^\alpha}{n} \right)^{1/2} \left( \int_{\log N}^{\infty} \left| \phi(\xi) \right|^2 \, d\xi \right)^{1/2}.
\]

The series converges since \( \gamma < 2/(2 + 2^\alpha) \). The proof is completed by a standard estimate of the convergent series,

\[
\left( \sum_{j=J}^{\infty} j^{(2+2^\alpha)\gamma - 3} \right)^{1/2} \ll (\log N)^{(2+2^\alpha)\gamma - 2}/(2\gamma) = (\log N)^{-1},
\]

where we used that \( J \approx (\log N)^{1/\gamma} \).

The final result needed for the iterative scheme is the following simple lemma on the \( \partial \)-equation. We omit the proof, which is obvious.
Lemma 5. Suppose $g$ is a continuous function on $\mathbb{C}_1^{+}$, supported on 
\[ \Omega(R, \tau) = \{ s = \sigma + it : 1/2 \leq \sigma \leq 1/2 + \tau, -R \leq t \leq R \}. \]

Then 
\[ u(s) = \frac{1}{\pi} \int_{\Omega} \frac{g(w)}{s - w} dm(w) \]
solves $\overline{\partial}u = g$ in $\mathbb{C}_1^{+}$ and satisfies $\|u\|_{\infty} \leq C_{\Omega}\|g\|_{\infty}$.

We have now collected all our preliminary results and are ready to begin the proof of Theorem 1. For any positive integer $N$ we set $E_N(s) = N^{-s+1/2}$ and consider the space $E_N A_{\beta}$. By a substitution it is evident that any $F \in E_N A_{\beta}$ can be represented 
\[ F(s) = \int_{\log N}^{\infty} \phi(\xi)e^{-(s-1)/2} d\xi \]
for some $\phi \in L^2_{\beta}[\log N, \infty)$, in view of Lemma 2.

Final step in the proof of Theorem 1. Let us fix $\alpha > 0$ and a bounded sequence $S = (\sigma_j + it_j) \in Z(A_{\beta})$. From this point all constants may depend on $\alpha$ and $S$. Since $S$ is bounded we may assume $S \subset \Omega(R-2, \tau-2)$ for some $R, \tau > 2$. Let $\Theta$ be some smooth function defined on $\mathbb{C}_1^{+}$ with the following properties:

- $\Theta$ is supported on $\Omega(R, \tau)$,
- $\Theta(s) = 1$ for $s \in \Omega(R-1, \tau-1)$,
- $|\partial \Theta(s)| \leq 2$.

Let $G \in A_{\beta}$ vanish precisely on $S$ and assume furthermore that $\|G\|_{A_{\beta}} = 1$. Now, suppose that $F \in E_N A_{\beta}$, and let $f \in \mathcal{D}_{\alpha}$ be the function obtained by applying Lemma 4 to $F$, and $\Phi = F - f$. Moreover, let $u$ denote the solution to the equation
\[ \overline{\partial}u = \frac{\partial(\Theta\Phi)}{GE_N}. \]

The right hand side of (7) is a smooth function compactly supported on $\Omega(R, \tau)$ since $|G(s)|$ is bounded from below where $\partial \Theta(s) \neq 0$. We can use Lemma 5 and Lemma 2 to estimate
\[ \|u\|_{\infty} \ll \left\| \frac{\partial(\Theta\Phi)}{GE_N} \right\|_{\infty} \ll (\log N)^{-1}\|\phi\|_{L^2_{\beta}} = (\log N)^{-1}\|F\|_{A_{\beta}}. \]

We set $T_N F = \Theta\Phi - GE_N u$. The function $T_N F$ has the following properties:

- $T_N F(s) = \Phi(s)$ for $s \in S$,
- $T_N F$ is holomorphic in $\mathbb{C}_1^{+}$ since $\partial T_N F(s) = 0$ for $s \in \mathbb{C}_1^{+}$,
- $T_N F \in E_N A_{\beta}$, by the compact support of $\Theta$ and the estimate (8).

Hence $T_N$ defines an operator on $E_N A_{\beta}$. By the triangle inequality, Lemma 4 and the fact that $\Theta$ has compact support, it is clear that 
\[ \|T_N F\|_{A_{\beta}} \leq \|\Theta\Phi\|_{A_{\beta}} + \|GE_N u\|_{A_{\beta}} \ll (\log N)^{-1}\|\phi\|_{L^2_{\beta}} + \|u\|_{\infty}\|G\|_{A_{\beta}}. \]

Since $\|G\|_{A_{\beta}} = 1$ and $\|\phi\|_{L^2_{\beta}} = \|F\|_{A_{\beta}}$ we have $\|T_N F\| \ll (\log N)^{-1}$ in view of (8).

Let $N$ be large, but arbitrary, and define $F_0(s) = E_N(s)G(s)$. Then $F_0 \in E_N A_{\beta}$ and its norm in this space is $\leq 1$.

\[ F_j = T_N^j F_0. \]
Let $f_j$ be the Dirichlet series of Lemma 4 obtained from $F_j$. Then $f_0 + F_1$ vanishes on $S$, since

$$f_0(s) + F_1(s) = f_0(s) + T_N F_0(s) = f_0(s) + F_0(s) - f_0(s) = F_0(s) = 0,$$

for $s \in S$, by the fact that $T_N F(s) = \Phi(s)$ for $s \in S$. Iteratively, the function $f_0 + f_1 + \cdots + f_j + F_{j+1}$ also vanishes on $S$. Define

$$f(s) = \sum_{j=0}^{\infty} f_j(s)$$

and choose $N$ so large that $\|T_N\| < 1$ so that $\|F_j\|_{A_\beta} \to 0$ and, say $|f(1)| > \sum_{j=1}^{\infty} |f_j(1)|,$

so that $f$ is non-trivial in $\mathscr{D}_\alpha$ and vanishing on $S$. □

By again following [13], we can modify the iterative scheme in the following way: Let $F \in A_\beta$ be arbitrary, and set $F_0 = F$. Using the algorithm in the same manner as above, we see that $F_1(s) + f_0(s) = F_0(s)$ for $s \in S$. Moreover,

$$F_{j+1}(s) + f_j(s) + f_{j-1}(s) + \cdots + f_0(s) = F(s),$$

for $s \in S$. Continuing as above, we obtain the following result:

**Corollary 6.** Suppose $S = (\sigma_j + it_j) \in Z(A_\beta)$ is bounded. For every function $F \in A_\beta$ there is some $f \in \mathscr{D}_\alpha$ such that $f(s) = F(s)$ on $S$.

We can extend Theorem 1 and Corollary 6 by considering different weights. Let $w = (w_1, w_2, \ldots)$ be a non-negative weight. Define the Hilbert space of Dirichlet series $\mathscr{D}_w$ in the same manner as above, with the added convention that the basis vector $n^{-s}$ is excluded if $w_n = 0$. Recall from [9] that $\mathscr{D}_w$ embeds locally into $A_\beta$ if and only if

$$\sum_{n \leq x} w_n \ll x (\log x)^\beta,$$

where $\beta > 0$. By modifying the proof of Theorem 1 we can obtain a similar result for $\mathscr{D}_w$ with respect to $A_\beta$ provided we have

$$\sum_{j^\gamma \leq \log n \leq (j+1)^\gamma} \frac{w_n}{n} \asymp j^{\gamma (\beta - 1)} - 1,$$

as $j \to \infty$, for some $0 < \gamma < 2/(3 + \beta)$. Several of the weights considered in [9] are possible, but we only mention the case $w_n = (\log n)^\beta$ for $\beta > 0$. These spaces were introduced by McCarthy in [8]. It is easy to show that these weights satisfy (9) and (10) for any $0 < \gamma < 1$, and similar results with respect to $A_\beta$ are obtained.

**Remark.** The embeddings of [9] extend to any $\beta \leq 0$, in view of (9), and we get the Hardy space ($\beta = 0$) and Dirichlet spaces ($\beta < 0$) in the half-plane. We can extend the results in [13] in a similar manner as above. However, this is only possible for $-1 \leq \beta < 0$. The method of [13] breaks down for $\beta < -1$ due to the fact that the norms of the corresponding Dirichlet spaces in the half-plane uses higher order derivatives and different estimates are needed.
3. Zeros of function in Hardy spaces of Dirichlet series

The Hardy spaces of Dirichlet series $\mathcal{H}^p$, $1 \leq p \leq \infty$, can be defined as the closure of the set of all Dirichlet polynomials with respect to the norms

$$
\left\| \sum_{n=1}^{N} \frac{a_n}{n^s} \right\|_{\mathcal{H}^p} = \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} \frac{a_n}{n^u} \right|^p dt \right)^{\frac{1}{p}}.
$$

For the basic properties of these spaces, we refer to [4] and [1]. We immediately observe that $\mathcal{H}^2 = \mathcal{D}_0$. In [13], the bounded zero sequences of the spaces $\mathcal{H}^p$, $2 \leq p \leq \infty$, are studied. In particular, for $\mathcal{H}^2$ the Blaschke condition

$$
\sum_{j} (\sigma_j - 1/2) < \infty
$$

is shown to be both necessary and sufficient. Results for $2 < p < \infty$ are obtained through embeddings $\mathcal{D}_\alpha \subset \mathcal{H}^p \subset \mathcal{H}^2$, where $\alpha < 0$ depends on $p$. The embedding of $\mathcal{H}^p$ into $\mathcal{H}^2$ implies that the Blaschke condition (11) is necessary for $\mathcal{H}^p$.

The sufficient conditions are obtained through a similar result as Theorem 1: For $\alpha < 0$, the spaces $\mathcal{D}_\alpha$ have the same bounded zero sequences as certain weighted Dirichlet spaces in $\mathbb{C}_{1/2}^+$. In particular, for $2 < p < \infty$ there is some $0 < \beta < 1$ such that a sufficient condition for bounded zero sequences of $\mathcal{H}^p$ is

$$
\sum_{j} (\sigma_j - 1/2)^{1-\beta} < \infty,
$$

and moreover $\beta \to 0$ as $p \to 2^-$. We omit the details, which can be found in [13], where also a complete description for the bounded zero sequences of $\mathcal{H}^\infty$ is contained.

We will consider the case $1 \leq p < 2$, where we have the embeddings

$$
\mathcal{H}^2 \subset \mathcal{H}^p \subset \mathcal{D}_\alpha,
$$

and $\alpha = 2/p - 1$. That $\mathcal{H}^2 \subset \mathcal{H}^p$ for $1 \leq p < 2$ is trivial, and this shows that (11) is a sufficient condition for bounded zero sequences of $\mathcal{H}^p$. The other embedding is more interesting: In [5], Helson proved the beautiful inequality

$$
\|f\|_{\mathcal{D}_{1/p}} = \left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{H}^1}.
$$

In [9] it is observed that by interpolating between (14) and $\mathcal{H}^2 = \mathcal{D}_0$ one obtains the inequality $\|f\|_{\mathcal{D}_{1/p}} \leq \|f\|_{\mathcal{H}^p}$, which provides the required embeddings.

This leads us to consider necessary conditions for zero sequences of the spaces $A_\beta$. The zero sequences of Bergman spaces in the unit disc $\mathbb{D}$ have attracted considerable attention. For $\beta > 0$, these are the spaces

$$
A_\beta(\mathbb{D}) = \left\{ F \in H(\mathbb{D}) : \|F\| = \int_{\mathbb{D}} |F(z)|^2 (1-|z|)^{\beta-1} dm(z) < \infty \right\}.
$$

Results pertaining to zero sequences of $A_\beta(\mathbb{D})$ are relevant to our case since

$$
\phi(s) = \frac{s - 3/2}{s + 1/2}
$$
is a conformal mapping from $\mathbb{C}_1^{+}/2$ to $\mathbb{D}$. Moreover,

$$F \mapsto (s + 1/2)^{-2(\beta + 1)}F \left( \frac{s - 3/2}{s + 1/2} \right)$$

defines an isometric isomorphism from $A_\beta(\mathbb{D})$ to $A_\beta$. This implies that $S \in Z(A_\beta)$ if and only if $\phi(S) \in Z(A_\beta(\mathbb{D}))$. Theorem 4.1 of [3] shows that the Blaschke condition (11) is both necessary and sufficient provided $S$ is contained in any cone $|t - t_0| \leq c(\sigma - 1/2)$. Unfortunately, the situation becomes more complicated in the general setting and we do not have a Blaschke type condition. The statement analogous to (12) for Bergman spaces is: For every $\epsilon > 0$ and every $A_\beta$ a necessary condition for zero sequences is

$$\sum_j (\sigma_j - 1/2)^{1+\epsilon} < \infty.$$ 

Clearly, this condition does not offer any insight into what happens as $p \to 2^+$. However, using the notion of density introduced by Korenblum in [7] we can provide a generalized condition describing the geometrical information of the zero sequences of $A_\beta(\mathbb{D})$. The most precise results on Korenblum’s density are obtained by Seip in [12]. We omit the details, and only mention that this condition in a certain sense tends to (11) when $\beta \to 0^+$, that is when $p \to 2^+$.

**Remark.** The Blaschke condition (11) is well-known to be necessary and sufficient for zero sequences of the Hardy spaces $H^p(\mathbb{C}_1^{+}/2)$. By a theorem in [4], $H^2$ embeds locally into $H^2(\mathbb{C}_1^{+}/2)$. This trivially extends to even integers $p$. Whether the local embedding extends to every $p \geq 1$ is an open question. Observe that if (11) is not the optimal necessary condition for zero sequences of $H^p$, $1 \leq p < 2$, then the local embedding would be impossible for these $p$. Since (12) is a sufficient condition for $p \geq 2$, this does not contradict the local embedding when $p > 2$.

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