Abstract

We study the (two-parameter) Segal–Bargmann transform $B^N_{s,t}$ on the unitary group $\mathbb{U}_N$, for large $N$. Acting on matrix valued functions that are equivariant under the adjoint action of the group, the transform has a meaningful limit $G_{s,t}$ as $N \to \infty$, which can be identified as an operator on the space of complex Laurent polynomials. We introduce the space of trace polynomials, and use it to give effective computational methods to determine the action of the heat operator, and thus the Segal–Bargmann transform. We prove several concentration of measure and limit theorems, giving a direct connection from the finite-dimensional transform $B^N_{s,t}$ to its limit $G_{s,t}$. We characterize the operator $G_{s,t}$ through its inverse action on the standard polynomial basis. Finally, we show that, in the case $s = t$, the limit transform $G_{t,t}$ is the “free Hall transform” $G^t$ introduced by Biane.

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The Segal–Bargmann transform (also known in the physics literature as the Bargmann transform or Coherent State transform) is a unitary isomorphism from $L^2$ to holomorphic $L^2$. It was originally introduced by Segal [30, 31, 32] and Bargmann [1, 2], as a map $S_t: L^2(\mathbb{R}^N, \gamma_t^N) \rightarrow \mathcal{H}L^2(\mathbb{C}^N, \gamma_t^{2N})$

where $\gamma_t^N$ is the standard Gaussian heat kernel measure $(\frac{1}{4\pi t})^{N/2} \exp(-\frac{1}{4t}|x|^2)\,dx$ on $\mathbb{R}^N$, and $\mathcal{H}L^2$ denotes the subspace of square-integrable holomorphic functions. The transform $S_t$ is given by convolution with the heat kernel, followed by analytic continuation.

In [17], the second author introduced an analog of the Segal–Bargmann transform for any compact Lie group $K$. Let $\Delta_K$ denote the Laplace operator over $K$ (determined, up to scale, by the $Ad$-invariant inner product on the Lie algebra $\mathfrak{k}$ of $K$), and denote by $e^{t\Delta_K}$ the corresponding heat operator. The generalized Segal–Bargmann transform $B_t$ maps functions on $K$ to holomorphic functions on the complexification $K_{\mathbb{C}}$ of $K$, by application of the heat operator and analytic continuation.

In this paper, we will work with the classical unitary groups $K = U_N$, and identify a limit as $N \rightarrow \infty$ of the Segal–Bargmann transform on $U_N$.

1.1 Main Definitions and Theorems

Denote by $M_N$ the algebra of $N \times N$ complex matrices, with unit $I_N$. Let $U_N$ denote the group of unitary matrices $U \in M_N: UU^* = I_N$, and let $GL_N$ denote the group of all invertible matrices in $M_N$; $GL_N$ is the complexification of $U_N$. The Lie algebra of $U_N$ is $u_N = \{X \in M_N: X^* = -X\}$, while the Lie algebra of $GL_N$ is $\mathfrak{gl}_N = M_N$. To describe the Laplace operator $\Delta_{U_N}$ explicitly, we fix the following notation.

**Notation 1.1.** For $Z \in M_N$ let

$$\text{Tr}_N(Z) \equiv \sum_{n=1}^{N} Z_{nn} \quad \text{and} \quad \text{tr}_N(Z) \equiv \frac{1}{N} \text{Tr}_N(Z) = \frac{1}{N} \sum_{n=1}^{N} Z_{nn}$$
denote the trace and normalized trace of $Z$, respectively. [We will usually drop the subscripts and write simply $\text{Tr}$ and $\text{tr}$, as the dimension will always be clear from context.] We also define (scaled) Hilbert-Schmidt norms on $u_N$ and on $M_N$ by

$$
\|X\|_{u_N}^2 \equiv N^2 \text{tr}(XX^*) = N\text{Tr}(XX^*) = N \sum_{j,k=1}^{N} |X_{j,k}|^2, \quad X \in u_N, \quad \text{and}
$$

$$
\|Z\|_{M_N}^2 \equiv \text{tr}(ZZ^*) = \frac{1}{N} \text{Tr}(ZZ^*) = \frac{1}{N} \sum_{j,k=1}^{N} |Z_{j,k}|^2, \quad Z \in M_N.
$$

**Definition 1.2.** For $\xi \in M_N$, let $\partial_\xi$ denote the left-invariant vector field on $GL_N$, whose action on smooth functions $f : GL_N \to \mathbb{C}$ is given by

$$
(\partial_\xi f)(Z) = \left. \frac{d}{dt} \right|_{t=0} f(Ze^{\xi t}), \quad Z \in GL_N.
$$

If $\xi = X \in u_N$ then $\partial_\xi$ is tangential to $U_N$ and so restricts to a left-invariant vector field on $U_N$ whose action on smooth functions $f : U_N \to \mathbb{C}$ is still given by (1.3).

**Definition 1.3.** The Laplace operator $\Delta_{U_N}$ is the second order elliptic operator on $U_N$ whose action on smooth functions $f : U_N \to \mathbb{C}$ is given by

$$
\Delta_{U_N} f = \sum_{X \in \beta_N} \partial_X^2 f
$$

where $\beta_N$ is an orthonormal basis for $u_N$ (with norm $\| \cdot \|_{u_N}$ given in (1.1)); the operator does not depend on which orthonormal basis is chosen.

Similarly, for $s, t > 0$ with $s > t/2$, let $A^{N}_{s,t}$ be the second order elliptic operator on $GL_N$ whose action on smooth functions $f : GL_N \to \mathbb{C}$ is given by

$$
A^{N}_{s,t} f = \left( s - \frac{t}{2} \right) \sum_{X \in \beta_N} \partial_X^2 f + \frac{t}{2} \sum_{X \in \beta_N} \partial^2_{t_X} f.
$$

Let $C^\infty_c(GL_N)$ denote the smooth compactly supported functions from $GL_N$ to $\mathbb{C}$. It is well known that the operators $\Delta_{U_N}$ and $A^{N}_{s,t}|_{C^\infty_c(GL_N)}$ are non-positive and essentially self-adjoint on $L^2(U_N)$ and $L^2(GL_N)$ respectively, where the measures on $U_N$ and $GL_N$ are taken to be any right invariant Haar measures. The self-adjoint closures of these operators induce (heat) semigroups $\left\{ e^{\tau \Delta_{U_N}} : \tau \geq 0 \right\}$ and $\left\{ e^{\tau A_{s,t}} : \tau \geq 0 \right\}$ on $L^2(U_N)$ and $L^2(GL_N)$ respectively. These semigroups then induce two (heat kernel) measures, $\rho^{N}_{t}$ and $\mu^{N}_{s,t}$, which satisfy

$$
\int_{U_N} f(U) \rho^{N}_{t}(dU) = \left( e^{\frac{t}{2} \Delta_{U_N} f} (I_N) \right), \quad f \in C(U_N),
$$

$$
\int_{GL_N} f(Z) \mu^{N}_{s,t}(dZ) = \left( e^{\frac{s}{2} A_{s,t} f} (I_N) \right), \quad f \in C_c(GL_N).
$$

We will sometimes write $E^{N}_{t}(f) = \int_{U_N} f(U) \rho^{N}_{t}(dU)$ and $E^{N}_{s,t}(f) = \int_{GL_N} f(Z) \mu^{N}_{s,t}(dZ)$.

**Remark 1.4.** The test functions $f$ on $GL_N$ we will use tend not to be compactly-supported (or bounded), but they do have sufficiently slow growth that (1.7) still holds true for such functions. This follows from Langland’s Theorem; cf. [26, Theorem 2.1 (p. 152)]. A gives a concise sketch of the heat kernel results we need in this paper.
Let \( \mathcal{H} L^2(\mathbb{G}_N, \mu_{s,t}^N) \) denote the Hilbert subspace of \( L^2(\mathbb{G}_N, \mu_{s,t}^N) \) consisting of those \( L^2 \) functions which possess a holomorphic representative. The following theorem with \( s = t \) is a special case of a Theorem 1.5 from page 2 of the two parameter form of this transform which we use here was introduced by the first and second authors in [11]; see also [8, 17, 18, 20].

**Theorem 1.5** (D, H, [11]). Fix \( s, t > 0 \) with \( s > t/2 \). For each \( f \in L^2(\mathbb{U}_N, \rho_s^N) \), the function \( e^{sA \Delta_{U_N}} f \) has a representative which has a unique analytic continuation to \( \mathbb{G}_N \); denote this analytic continuation by \( B_{s,t}^N f \). Then \( B_{s,t}^N f \in \mathcal{H} L^2(\mathbb{G}_N, \mu_{s,t}^N) \), and the resulting transform

\[
B_{s,t}^N : L^2(\mathbb{U}_N, \rho_s^N) \to \mathcal{H} L^2(\mathbb{G}_N, \mu_{s,t}^N)
\]

is a unitary isomorphism.

In this paper, we are interested in a slight extension of \( B_{s,t}^N \) to matrix-valued functions.

**Definition 1.6** (Boosted Segal-Bargmann Transform). Given a \( \mathbb{M}_N \)-valued function \( F \) on either \( \mathbb{U}_N \) or \( \mathbb{G}_N \), denote by \( \| F \|_{\mathbb{M}_N} \) the scalar-valued function \( Z \mapsto \| F(Z) \|_{\mathbb{M}_N} \). Fix \( s, t > 0 \) with \( s > t/2 \), and let

\[
\begin{align*}
L^2(\mathbb{U}_N, \rho_s^N; \mathbb{M}_N) &= \{ F : \mathbb{U}_N \to \mathbb{M}_N ; \| F \|_{\mathbb{M}_N} \in L^2(\mathbb{U}_N, \rho_s^N) \}, \\
L^2(\mathbb{G}_N, \mu_{s,t}^N; \mathbb{M}_N) &= \{ F : \mathbb{G}_N \to \mathbb{M}_N ; \| F \|_{\mathbb{M}_N} \in L^2(\mathbb{G}_N, \mu_{s,t}^N) \}.
\end{align*}
\]

Let \( \mathcal{H} L^2(\mathbb{G}_N, \mu_{s,t}^N; \mathbb{M}_N) \subset L^2(\mathbb{G}_N, \mu_{s,t}^N; \mathbb{M}_N) \) denote the subspace of (matrix-valued) holomorphic functions. These are Hilbert spaces in the norms

\[
\begin{align*}
\| F \|_{L^2(\mathbb{U}_N, \rho_s^N; \mathbb{M}_N)}^2 &= \int_{\mathbb{U}_N} \| F(U) \|_{\mathbb{M}_N}^2 \rho_s^N(U) \, dU, \\
\| H \|_{L^2(\mathbb{G}_N, \mu_{s,t}^N; \mathbb{M}_N)}^2 &= \int_{\mathbb{G}_N} \| H(Z) \|_{\mathbb{M}_N}^2 \mu_{s,t}^N(Z) \, dZ.
\end{align*}
\]

The **boosted Segal–Bargmann transform**

\[
B_{s,t}^N : L^2(\mathbb{U}_N, \rho_s^N; \mathbb{M}_N) \to \mathcal{H} L^2(\mathbb{G}_N, \mu_{s,t}^N; \mathbb{M}_N)
\]

is the unitary isomorphism determined by applying \( B_{s,t}^N \) componentwise; that is, it is determined by

\[
B_{s,t}^N(f \cdot V) = B_{s,t}^N f \cdot V \quad \text{for} \quad f \in L^2(\mathbb{U}_N, \rho_s^N) \quad \text{and} \quad V \in \mathbb{M}_N.
\]

The space \( L^2(\mathbb{U}_N, \rho_s^N; \mathbb{M}_N) \) can be naturally identified with the Hilbert space tensor product \( L^2(\mathbb{U}_N, \rho_s^N) \otimes \mathbb{C} \mathbb{M}_N \); under this identification, \( B_{s,t}^N \cong B_{s,t}^N \otimes \text{id}_{\mathbb{M}_N} \). To understand its action, consider the matrix-valued function \( F(U) = U^2 \) on \( \mathbb{U}_N \). Then, as calculated in Example 3.5

\[
(B_{s,t}^N f)(Z) = e^{-t} \cosh(t/N)Z^2 - Ne^{-t} \sinh(t/N)Z \cdot \text{tr}(Z).
\]

This highlights the fact that the Segal–Bargmann transform does not preserve the space of polynomial functions of a \( \mathbb{U}_N \)-variable; in general, it maps such functions to trace polynomials.

**Definition 1.7.** Let \( \mathbb{C}[u, u^{-1}] \) denote the algebra of Laurent polynomials in a single variable \( u \):

\[
\mathbb{C}[u, u^{-1}] = \left\{ \sum_{k \in \mathbb{Z}} a_k u^k : a_k \in \mathbb{C}, a_k = 0 \text{ for all but finitely-many } k \right\},
\]

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with the usual polynomial multiplication. The subalgebras \( \mathbb{C}[u] \) and \( \mathbb{C}[u^{-1}] \) denote polynomials in \( u \) and \( u^{-1} \) respectively.

We define the **Laurent polynomial functional calculus** as follows: for \( f \in \mathbb{C}[u, u^{-1}] \) as in (1.11), the function \( f_N : \mathbb{GL}_N \to \mathbb{M}_N \) is given by

\[
f_N(Z) = \sum_{k \in \mathbb{Z}} a_k Z^k,
\]

where the \( k = 0 \) term is interpreted as \( a_0 I_N \).

Let \( \mathbb{C}[\mathbf{v}] \) denote the algebra of complex polynomials in infinitely-many commuting variables \( \mathbf{v} = \{v_{\pm 1}, v_{\pm 2}, \ldots\} \), and let \( \mathbb{C}[u, u^{-1}; \mathbf{v}] \) denote the algebra of polynomials in the variables \( u, u^{-1}, v_{\pm 1}, v_{\pm 2}, \ldots \) (although we do not treat \( u \) and \( u^{-1} \) as independent in general). Thus

\[
\mathbb{C}[u, u^{-1}; \mathbf{v}] = \left\{ \sum_{k \in \mathbb{Z}} u^k Q_k(\mathbf{v}) : Q_k(\mathbf{v}) \in \mathbb{C}[\mathbf{v}], Q_k = 0 \text{ for all but finitely-many } k \right\}
\]  

(1.13)

In other words, we can realize \( \mathbb{C}[u, u^{-1}; \mathbf{v}] \) as the algebra \( (\mathbb{C}[\mathbf{v}])[u, u^{-1}] \) of Laurent polynomials in \( u \) with coefficients in the ring \( \mathbb{C}[\mathbf{v}] \); equivalently, \( \mathbb{C}[u, u^{-1}; \mathbf{v}] \cong \mathbb{C}[u, u^{-1}] \otimes \mathbb{C}[\mathbf{v}] \). We denote elements of \( \mathbb{C}[u, u^{-1}; \mathbf{v}] \) by \( P = P(u; \mathbf{v}) \).

Define the **trace polynomial functional calculus** as follows: for \( P \in \mathbb{C}[u, u^{-1}; \mathbf{v}] \), the function \( P_N : \mathbb{GL}_N \to \mathbb{M}_N \) is given by

\[
P_N(Z) = P(u; \mathbf{v})|_{u = Z, v_k = \text{tr}(Z^k), k \neq 0}.
\]

Functions of the form \( P_N \) for \( P \in \mathbb{C}[u, u^{-1}; \mathbf{v}] \) are called **trace polynomials**.

It might be more accurate to call such functions trace **Laurent** polynomials, but we will simply use trace polynomials as it should cause no confusion. For a concrete example: if \( P(u; \mathbf{v}) = v_2 v_{-4} u^5 + 8v_1^6 v_{-3} \) then

\[
P_N(Z) = \text{tr}(Z^2) \text{tr}(Z^{-1})^3 Z^5 + 8 \text{tr}(Z)^6 \text{tr}(Z^{-3}) I_N.
\]

**Remark 1.8.** It is important to note that, for any finite \( N \), there will be many distinct elements \( P \in \mathbb{C}[u, u^{-1}; \mathbf{v}] \) that induce the same trace polynomial, i.e. there will be \( P \neq Q \) with \( P_N = Q_N \). Nevertheless, it is true that if \( P_N = Q_N \) for all sufficiently large \( N \), then \( P = Q \); this is the statement of Theorem 2.10 below.

**Theorem 1.9.** Let \( P \in \mathbb{C}[u, u^{-1}; \mathbf{v}] \) as in Definition 1.7 and let \( N \in \mathbb{N} \) and \( s, t > 0 \) with \( s > t/2 \). There exists an element \( P_N \in \mathbb{C}[u, u^{-1}; \mathbf{v}] \) such that

\[
B_{s,t}^N P_N = [P_N]^N_N.
\]

(1.14)

The polynomial \( P_N \) can be computed as \( P_N = e^{\frac{\pi i}{2} \mathcal{D}_N} P \) where \( \mathcal{D}_N \) is a certain pseudodifferential operator on \( \mathbb{C}[u, u^{-1}; \mathbf{v}] \); cf. Theorem 1.18 and Definition 3.9 below.

The proof of Theorem 1.9 is on page 25.

**Remark 1.10.** We call \( \mathcal{D}_N \) a pseudodifferential operator because, if we identify \( u \) as variable in the unit circle \( \mathbb{U} \), then \( \mathcal{D}_N \) acts as a first order differential operator composed with a linear combination of the identity operator and the Hilbert transform on the circle. As explained in [24], the Hilbert transform on \( \mathbb{U} \) is a pseudodifferential operator. See Definition 3.6 and Remark 3.7 for more details.

For each \( N > 1 \), the boosted Segal–Bargmann transform’s range on Laurent polynomial calculus functions is contained in the larger space of trace polynomials. But as \( N \to \infty \), its image concentrates back on Laurent polynomials. This is our main theorem.
Theorem 1.11. Let \( s, t > 0 \) with \( s > \frac{t}{2} \). For each \( f \in \mathbb{C}[u, u^{-1}] \), there exist unique \( g_{s,t}, h_{s,t} \in \mathbb{C}[u, u^{-1}] \) such that

\[
\|B_{s,t}^N f_N - [g_{s,t}]_N\|_{L^2(\mathcal{G}_N; \mu_N^s)} = O\left(\frac{1}{N^2}\right), \quad \text{and} \quad (1.15)
\]

\[
\|B_{s,t}^N (g_{s,t})_{N}^{-1} f_N - [h_{s,t}]_N\|_{L^2(\mathcal{U}_N; \rho_N^s)} = O\left(\frac{1}{N^2}\right). \quad (1.16)
\]

We denote that map \( \mathcal{G}_{s,t} : \mathbb{C}[u, u^{-1}] \to \mathbb{C}[u, u^{-1}] \) given by \( f \mapsto g_{s,t} \) as the free unitary Segal–Bargmann transform, and we denote the map \( \mathcal{H}_{s,t} : \mathbb{C}[u, u^{-1}] \to \mathbb{C}[u, u^{-1}] \) given by \( f \mapsto h_{s,t} \) as the free unitary inverse Segal–Bargmann transform.

Remark 1.12. A concurrent paper by G. Cébron has recently proven a similar theorem; in particular, the \( s = t \) case of (1.15) is equivalent to [7, Theorem 4.7]. Cébron’s framework is somewhat different from ours, and should be consulted for a complementary approach. See Remark 1.22 for a detailed comparison.

Theorem 1.11 is proved on page 33. The “inverse” terminology is justified by the following, whose proof is on page 37.

Theorem 1.13. For \( s, t > 0 \) with \( s > \frac{t}{2} \), the maps \( \mathcal{G}_{s,t} \) and \( \mathcal{H}_{s,t} \) are invertible linear operators on \( \mathbb{C}[u, u^{-1}] \), and \( \mathcal{G}_{s,t}^{-1} = \mathcal{H}_{s,t} \).

To explain how the concentration phenomenon of Theorem 1.11 occurs, we recall the following theorem of Biane.

Theorem 1.14 (Biane, [4, Lemma 11]). For each \( s > 0 \) and \( k \in \mathbb{Z} \),

\[
\lim_{N \to \infty} \int_{U_N} \text{tr}(U^k) \rho_s^N (dU) = \nu_k(s),
\]

where \( \nu_0(s) = 1 \) and, for \( k \neq 0 \),

\[
\nu_k(s) = e^{-\frac{|k|}{2}s} \sum_{j=0}^{|k|-1} \frac{(-s)^j |k|-j}{j!} \left( \frac{|k|}{j+1} \right). \quad (1.17)
\]

From (1.17) it is clear that \( \nu_k = \nu_{-k} \) for all \( k \in \mathbb{N} \) and that each \( \nu_k(\cdot) \) has an analytic continuation to a holomorphic function on \( \mathbb{C} \) which we still denote by \( \nu_k \). For each \( s \in \mathbb{R} \), these constants are the moments of a probability measure \( \nu_s \), supported on either the unit circle \( \mathbb{U} \) (for \( s \geq 0 \)) or the positive real half-line \( (0, \infty) \) (for \( s \leq 0 \)). For \( s > 0 \), \( \nu_k(s) \) are the moments of the free unitary Brownian motion \( u_s \); see [4, Prop. 10]. These functions also encode the large-\( N \) limits of the moments of the measures \( \mu_s^N \).

Theorem 1.15. Let \( s, t > 0 \) with \( s > \frac{t}{2} \), and let \( k \in \mathbb{Z} \); then

\[
\lim_{N \to \infty} \int_{\mathbb{G}^N} \text{tr}(Z^k) \mu_s^N (dZ) = \nu_k(s - t). \quad \text{The proof of Theorem 1.15 is on page 33.}
\]

See, also, the third author’s concurrent paper [21] for several new convergence results for the empirical eigenvalues and singular values of random matrices sampled from \( \rho_s^N \) and \( \mu_s^N \).

Consider, again, the calculation of (1.10), which shows that, if \( f(u) = u^2 \), then the polynomial \( f_t^N \in \mathbb{C}[u, u^{-1}; v] \) of Theorem 1.9 can be identified as

\[
f_t^N(u; v) = e^{-t} \cosh(t/N)u^2 - Ne^{-t} \sinh(t/N)uv_1 = e^{-t}(u^2 - tuv_1) + O\left(\frac{1}{N^2}\right).
\]
The trace polynomial functional calculus evaluates $f_t^N(u; v)$ at $Z \in \mathcal{GL}_N$ by setting $u = Z$ and $v_1 = \text{tr}(Z)$; but as $N \to \infty$, $\text{tr}(Z) \to \nu_1(s-t) = e^{-(s-t)/2}$ by Theorem 1.15. This illustrates the fact that, in this case,

$$(\mathcal{G}_{s,t} f)(u) = e^{-t}(u^2 - te^{-(s-t)/2} u).$$

In general, this is how $g_{s,t}$ in (1.15) is produced: by evaluating the traces in the trace polynomial $P_t^N$ in Theorem 1.9 at the moments $\nu_k(s-t)$ of Theorem 1.15 and taking the large-$N$ limit of the resulting Laurent polynomial. To fully justify this, we prove the following concentration theorem, which shows, in a strong way, that the trace random variables $Z \mapsto \text{tr}(Z^k)$ over $\mathbb{U}_N$ and $\mathcal{GL}_N$ concentrate on their means as $N \to \infty$.

**Theorem 1.16.** For $s \in \mathbb{R}$, define the trace evaluation map $\pi_s : \mathbb{C}[u, u^{-1}; v] \to \mathbb{C}[u, u^{-1}]$ by

$$(\pi_s P)(u) = P(u; v)|_{\nu_k(s), k \neq 0}.$$  

Let $s, t > 0$, with $s > t/2$. For any $P \in \mathbb{C}[u, u^{-1}; v]$,

$$\|P_N - [\pi_s P]_N\|_{L^2(\mathbb{U}_N, \rho^N)} = O\left(\frac{1}{N^2}\right), \quad \text{and}$$

$$\|P_N - [\pi_{s-t} P]_N\|_{L^2(\mathbb{GL}_N, \mu^N)} = O\left(\frac{1}{N^2}\right).$$

The proof of Theorem 1.16 can be found on page 33. Combining it with Theorem 1.9, we see that the limit Segal–Bargmann transform $\mathcal{G}_{s,t} f$ in (1.13) is given by $\mathcal{G}_{s,t} f = \lim_{N \to \infty} \pi_{s-t}(f_N)$; see (1.25) below.

Finally, we explicitly describe the action of $\mathcal{H}_{s,t}$ via a generating function.

**Theorem 1.17.** Let $s, t > 0$ with $s > t/2$. For $k \geq 1$, let $f_k(u) \equiv u^k$ and $p_k^{s,t} \equiv \mathcal{H}_{s,t}(f_k)$. Then the generating function for $\{p_k^{s,t}\}$ is given by the power series

$$\Pi(s, t, u, z) = \sum_{k \geq 1} p_k^{s,t}(u) z^k,$$

which converges for all sufficiently small $u, z \in \mathbb{C}$. This generating function is determined by the implicit formula

$$\Pi(s, t, u, z e^{\frac{1}{2}(s-t)\frac{1+z}{1-z}}) = \left(1 - u z e^{\frac{1+z}{1-z}}\right)^{-1} - 1.$$  

In the special case $s = t$, this yields the generating function corresponding to the transform $\mathcal{G}^{t}$ of [4, Proposition 13], which Biane called the free Hall transform (after the second author of this paper). Thus, $\mathcal{G}_{t,t} = \mathcal{G}^{t}$, and the free unitary Segal–Bargmann transform is a generalization of the free Hall transform. The proof of Theorem 1.17 can be found on page 47.

### 1.2 Intertwining Operators and Partial Product Rule

The key ingredient needed to prove all the main theorems of this paper is the following intertwining formula, which shows that the Laplace operator $\Delta_{\mathbb{U}_N}$ factors through a pseudodifferential operator on $\mathbb{C}[u, u^{-1}; v]$.

**Theorem 1.18 (Intertwining Formulas).** Let $\mathbb{C}[u, u^{-1}; v]$ be the polynomial space of Definition 7.7, let $t \geq 0$, and let $N \in \mathbb{N}$. There exists a first order pseudodifferential operator $\mathcal{D}$ on $\mathbb{C}[u, u^{-1}; v]$ and a second order differential operator $\mathcal{L}$ on $\mathbb{C}[u, u^{-1}; v]$ (cf. (3.19) and (3.20) below) such that, setting

$$\mathcal{D}_N = \mathcal{D} - \frac{1}{N^2} \mathcal{L},$$

(1.22)
it follows that
\[ \Delta_{U_N} P_N = [D_N] P_N, \quad \text{for all } P \in \mathbb{C}[u, u^{-1}; v]. \] (1.23)

Moreover, the heat operator is given by
\[ e^{\frac{1}{2} \Delta_{U_N}} P_N = [e^{\frac{1}{2} D_N}] P_N, \quad \text{for all } P \in \mathbb{C}[u, u^{-1}; v]. \] (1.24)

A similar intertwining formula holds for the operator \( A_{s,t} \); cf. Theorem 3.26 on page 29.

The proof of Theorem 1.18 is on page 22.

Remark 1.19. (1) We will see in Section 3.3 below that the space \( \mathbb{C}[u, u^{-1}; v] \) is the union of a family \( \{ C_n[u, u^{-1}; v] \}_{n \in \mathbb{N}} \) of finite-dimensional subspaces, each of which is invariant under \( D_N \). Hence, the exponential \( e^{\frac{1}{2} D_N} \) makes sense as an operator on \( \mathbb{C}[u, u^{-1}; v] \), for all \( t \in \mathbb{R} \).

(2) Our intertwining formula (1.23) is closely related to results due to E. M. Rains [25] and A. N. Sengupta [33]. In both cases, the Laplacian \( \Delta_{U_N} \) was identified by a decomposition similar to (1.22) for some operators like our \( D \) and \( L \). We show that the component operators \( D \) and \( L \) can be realized as pseudodifferential operators on a polynomial intertwining space, which simplifies much of our analysis.

Since \( D_N = D + O(1/N^2) \), it follows that \( e^{\frac{1}{2} D_N} = e^{\frac{1}{2} D} + O(1/N^2) \); this is made precise in Lemma 4.1 below. As such, we will show in the proof of Theorem 1.11 that the free unitary Segal–Bargmann transform and its inverse are given by
\[ G_{s,t} = \pi_{s-t} \circ e^{\frac{1}{2} D}, \quad \text{and} \quad H_{s,t} = \pi_s \circ e^{-\frac{1}{2} D}. \] (1.25)

See Section 4.2 for details. The two operators \( e^{\frac{1}{2} D} \) and \( e^{-\frac{1}{2} D} \) are, of course, inverse to each other; Theorem 1.13 shows that this holds true even with the composed evaluations maps.

The operator \( D \) is a first order pseudodifferential operator, but it is not a differential operator: it does not satisfy the Leibnitz product rule. It does, however, satisfy the following partial product rule which is of both computational and conceptual importance.

**Theorem 1.20 (Partial Product Rule).** Let \( P \in \mathbb{C}[u, u^{-1}; v] \) and \( Q \in \mathbb{C}[v] \). Then
\[ D(PQ) = (DP)Q + P(DQ). \] (1.26)

Thus, for any \( t \in \mathbb{R} \),
\[ e^{\frac{1}{2} D}(PQ) = e^{\frac{1}{2} D} P \cdot e^{\frac{1}{2} D} Q. \] (1.27)

The proof of Theorem 1.20 can be found on page 25.

### 1.3 History and Discussion

Since the classical Segal–Bargmann transform \( S_t \) for Euclidean spaces admits an infinite dimensional version [32], it is natural to attempt to construct an infinite dimensional limit of the transform for compact Lie groups. One successful approach to such a limit is found in the paper [20] of the second author and A. N. Sengupta, in which they develop a version of the Segal–Bargmann transform for the path group with values in a compact Lie group \( K \). The paper [20] is an extension of the work of L. Gross and P. Malliavin [16] and reflects the origins of the generalized Segal–Bargmann transform for compact Lie groups in the work of Gross [15].

A different approach to an infinite dimensional limit is to consider the transform on a nested family of compact Lie groups, such as \( \mathbb{U}_N \) for \( N = 1, 2, 3, \ldots \). The most obvious approach to the \( N \to \infty \) limit would be to use on each \( u_N \) a fixed (i.e. \( N \)-independent) multiple of the Hilbert–Schmidt norm \( \| X \|_{\text{HS}}^2 = \text{Tr} (XX^*) \). Work of M.
Gordina [13, 14], however, showed that this approach does not work, because the target Hilbert space becomes undefined in the limit. Indeed, Gordina showed that, with the metrics normalized this way, in the large-\(N\) limit all nonconstant holomorphic functions on \(\mathbb{GL}_N\) have infinite norm with respect to the heat kernel measure \(\mu_{t,t}^N\).

In [4], Biane proposed scaling the Hilbert-Schmidt norm with \(N\) as in (1.1); he successfully carried out a large-\(N\) limit of the Lie algebra version of the transform. That is: taking the underlying space to be the Lie algebra \(u_N\) rather than the group \(U_N\), he considered a version of the classical Euclidean Segal–Bargmann transform, \(S_t^N\) acting on functions from \(u_N\) to \(\mathbb{M}_N\) given by polynomial functional calculus (cf. Definition 1.7). If \(f \in \mathbb{C}[u]\), the transformed functions \(S_t^N f_N\) have a limit (in a sense analogous to our Theorem 1.11) which can be thought of as a polynomial \(f_t \in \mathbb{C}[u]\). This defines a unitary transformation \(\mathcal{F}^t: f \mapsto f_t [4, \text{Theorem 3}]\) on the limiting \(L^2\) closure of polynomials with respect to the limit heat kernel measure—in this context Wigner’s semicircle law.

**Remark 1.21.** The results of [4, Section 1] are formulated in terms of the large-\(N\) limit of \(S_t^N\) on the space \(\mathcal{H}_N = iu_N\) of Hermitian \(N \times N\) matrices, which is of course equivalent to the formulation above. It also deals with a more general functional calculus on \(\mathcal{H}_N\); cf. Section 2.1 below. We have restricted our attention almost exclusively to the space of Laurent polynomial functions, for clarity of exposition. Section 2 also discusses equivariant functions: an extension of the space of functional calculus functions which forms a natural domain for the Segal–Bargmann transform, and subsumes all other function spaces discussed in this paper.

Biane proceeded in [4] to construct the free Hall transform transform \(\mathcal{G}^t\) as a kind of large-\(N\) limit \(U_N\) Segal–Bargmann, not by taking this limit directly as we have done, but instead developing a free probabilistic version of the Malliavin calculus techniques used by Gross and Malliavin [16] to derive the properties of \(B_t\) from an infinite dimensional version of \(S_t\). This laid the foundation for the modern theory of free Malliavin calculus and free stochastic differential equations, subsequently studied in [5, 6, 22] and many other papers, and was groundbreaking in many respects. Biane conjectured that his transform \(\mathcal{G}^t\) is the direct \(N \to \infty\) limit of the Segal–Bargmann transforms \(B_t^N\) on \(U_N\), and suggested that this could be proved using the methods of stochastic analysis, but left the details of such an argument out of [4] (see the Remark on page 263). One of the main motivations for the present paper is to prove (Theorems 1.11 and 1.17) that this connection indeed holds. Our methods and ideas are very different from those Biane suggested, however; they are analytic and geometric, rather than probabilistic. Moreover, we find the large-\(N\) limit of the two-parameter Segal–Bargmann transform \(B_{t,t}^N\), and this generalization is essential to our proof that \(\lim_{N \to \infty} B_{t,t}^N = \mathcal{G}^t\).

**Remark 1.22.** As noted above, the complementary paper [7] answers many of the same questions we do, using a somewhat different framework. Cébron’s paper uses the tools of free probability to construct a space of “formal trace polynomials” on which the limit Segal–Bargmann transform acts. He also realizes the Laplace operator \(\Delta_{U_N}\) via an intertwining formula, in his case formulated in terms of free conditional expectation, and finds a crucial \(O(1/N^2)\)-decomposition analogous to our (1.22). On the other hand, our method for connecting the large-\(N\) limit of the Segal–Bargmann transform to the work of Biane (Theorem 1.17) is completely different from that of [7], using PDE methods to derive the polynomial generating function for the limiting transform; moreover, our methods extend naturally to the two-parameter transform. A more complete understanding of the large-\(N\) limit of the Segal–Bargmann transform on \(U_N\) is likely achieved by considering both our approach and Cébron’s together.

### 2 Equivariant Functions and Trace Polynomials

In this section, we consider function spaces over \(U_N\) and \(\mathbb{GL}_N\) that are very natural domains for the Segal-Bargmann transform and its inverse.
Definition 2.1. Let $G \subset M_N$ be a matrix group. A function $F : G \to M_N$ is called equivariant if $F(BAB^{-1}) = BF(A)B^{-1}$ for all $A, B \in G$ (it is equivariant under the adjoint action of $G$).

The set of equivariant functions is a $C$-algebra. If $P \in \mathbb{C}[u, u^{-1}; v]$, then the trace polynomial $P_N$ is equivariant, as can be easily verified. This shows that the equivariant subspaces

$$L^2(U, \rho^N_s; M_N)_{eq} \quad \text{and} \quad \mathcal{H}L^2(GL_N, \mu^N_{s,t}; M_N)_{eq},$$

are non-trivial. The main results of this section, Theorem 2.3 and 2.7, show that $\mathbb{B}^N_{s,t}$ maps $L^2(\rho^N_s)_{eq}$ onto $\mathcal{H}L^2(\mu^N_{s,t})_{eq}$ (extending Theorem 1.9), and that trace polynomials are dense in these equivariant $L^2$-spaces. We conclude this section with Theorem 2.10, showing that the map $\mathbb{C}[u, u^{-1}; v] \to L^2(\rho^N_s)_{eq}$ given by $P \mapsto P_N$ is one-to-one when restricted to polynomials if a fixed maximal degree.

We begin with a brief discussion of functional calculus, which featured prominently in [4], and whose image is a (small) subspace of equivariant functions.

2.1 Functional Calculus

Definition 2.2. Let $U$ denote the unit circle in $C$. For every measurable function $f : U \to \mathbb{C}$, let $f_N$ be the unique function mapping $U_N$ into $M_N(\mathbb{C})$ with the property that

$$f_N \left( V \begin{pmatrix} \lambda_1 & \cdots & \lambda_N \\ \vdots & & \vdots \\ \lambda_N \end{pmatrix} V^{-1} \right) = V \begin{pmatrix} f(\lambda_1) & \cdots & \cdot \\ \cdot & & \cdot \\ \cdot & & f(\lambda_N) \end{pmatrix} V^{-1}$$

for all $V \in U_N$ and all $\lambda_1, \ldots, \lambda_N \in U$. The function $f_N$ is called the functional calculus function associated to the function $f$. The space of those functional calculus functions that are in $L^2(U, \rho^N_s; M_N(\mathbb{C}))$ is called the functional calculus subspace.

It is easy to check that $f_N(U)$ is well defined, independent of the choice of diagonalization. If, for example, $f$ is the function given by $f(\lambda) = e^{\lambda}$, then $f_N(U) = e^{\theta_U}$, computed by the usual power series. If $f \in \mathbb{C}[u, u^{-1}]$, then $f_N$ is the function given in (1.12); thus our notation $f_N$ for both is consistent. (By comparison: in [4], the functional calculus function $f_N$ is denoted $\theta^N_f$.) Trace polynomials are not, in general, functional calculus functions. For example, the function $F(U) = U \text{Tr}(U)$ is not a functional calculus function on $U_N$, except when $N = 1$. Indeed, if $N \geq 2$ and $U_N \ni U = \text{diag}(\lambda_1, \lambda_2)$, the $(1,1)$-entry of the diagonal matrix $U \text{Tr}(U)$ is $\frac{1}{2}(\lambda_1 + \lambda_2)\lambda_1$, which is not a function of $\lambda_1$ alone. This violates Definition 2.2. Functional calculus functions are, however, equivariant.

Since $\Lambda(f) \equiv \int_{U_N} \text{tr}(f_N(U)) \rho^N_s(dU)$ defines a positive linear functional on $C(U)$ with $\Lambda(1) = 1$, by the Riesz Representation Theorem [27, Theorem 2.14] there is a probability measure $\nu^N_s$ on $U$ such that

$$\int_{U_N} \text{tr}(f_N(U)) \rho^N_s(dU) = \Lambda(f) = \int_U f(\xi) \nu^N_s(d\xi), \quad f \in C(U). \quad (2.1)$$

(Theorem 1.16 shows, in particular, that $\nu^N_s$ converges weakly to $\nu_s$; cf. Theorem 1.14.) For any function $f$ on $U$, one can easily verify from Definition 2.2 that $||f||^2_{L^2(U)} = f_N(U)f_N(U)^*$; hence, by the density of $C(U)$ in $L^2(U, \nu^N_s)$, (2.1) shows that

$$||f_N||_{L^2(U, \rho^N_s; M_N(\mathbb{C}))} = ||f||_{L^2(U, \nu^N_s)}, \quad f \in L^2(U, \nu^N_s). \quad (2.2)$$

It follows that the functional calculus subspace is a closed subspace of $L^2(\rho^N_s)_{eq}$, and contains the functions $\{f_1 : f \in \mathbb{C}[u, u^{-1}]\}$ as a dense subspace. That this density result extends to trace polynomials in the full space $L^2(\rho^N_s)_{eq}$ is Theorem 2.7 below.
If \( F \) is a holomorphic function on \( \mathbb{C}^* \), there is a unique holomorphic function \( F_N \) from \( \mathbb{G}L_N \) to \( \mathbb{M}_N \) which satisfies
\[
F_N \left( A \begin{pmatrix} \lambda_1 & \cdots & \lambda_N \\ \vdots & \ddots & \vdots \\ \lambda_N & & \end{pmatrix} A^{-1} \right) = A \begin{pmatrix} F(\lambda_1) \\ \vdots \\ F(\lambda_N) \end{pmatrix} A^{-1}
\]
for every \( A \in \mathbb{G}L_N \) and all \( \lambda_1, \ldots, \lambda_N \in \mathbb{C}^* \); indeed, \( F_N \) is given by the same Laurent series expansion as \( F \), applied to the matrix variable. We call such a function a **holomorphic functional calculus function** on \( \mathbb{G}L_N \). As [4] shows, the boosted Segal–Bargmann transform \( \mathcal{B}_{s,t}^N \) does not, in general, map functional calculus functions on \( U_N \) to holomorphic functional calculus functions on \( \mathbb{G}L_N \). Nevertheless, [3] suggests that in the large-\( N \) limit, \( \mathcal{B}_{s,t}^N \) ought to map functional calculus functions to holomorphic functional calculus functions (at least in the \( s = t \) case). Since single-variable Laurent polynomial functions are dense in the functional calculus subspace, Theorem 1.11 can be interpreted as a rigorous version of this idea.

### 2.2 Results on Equivariant Functions

**Theorem 2.3.** Let \( s, t > 0 \) with \( s > t/2 \). The Segal–Bargmann transform \( \mathcal{B}_{s,t}^N \) maps the equivariant subspace \( L^2(U_N, \rho_s^N; M_N(\mathbb{C}))_{eq} \) isometrically onto \( \mathcal{H}L^2(\mathbb{G}L_N, \rho_s^N; M_N(\mathbb{C}))_{eq} \).

We begin with the following lemma.

**Lemma 2.4.** Let \( G \subset \mathbb{M}_N \) be a group. For any function \( F : G \to \mathbb{M}_N \), define
\[
C_V(F)(A) = V^{-1} F(VAV^{-1})V, \quad V, A \in G.
\]
Let \( s, t > 0 \) with \( s > t/2 \). Then for all \( F \in L^2(U_N, \rho_s^N; \mathbb{M}_N) \) and \( V \in U_N \),
\[
\mathcal{B}_{s,t}^N(C_V F) = C_V(\mathcal{B}_{s,t}^N F).
\]

**Proof.** Since \( \Delta_{U_N} \) is bi-invariant, it commutes with the left- and right-actions of the group; hence it, and therefore the semigroup \( e^{\frac{i}{2} \Delta_{U_N}} \), commutes with the adjoint action \( \text{Ad}_V(U) = VUV^{-1} \) on functions: for any \( V \in U_N \),
\[
e^{\frac{i}{2} \Delta_{U_N}} (F \circ (\text{Ad}_V)) = \left(e^{\frac{i}{2} \Delta_{U_N}} F\right) \circ \text{Ad}_V.
\]
Conjugating both sides of \( (2.5) \) by \( V^{-1} \) in the range of \( F \) (which commutes with the heat operator), it follows that
\[
C_V(e^{\frac{i}{2} \Delta_{U_N}} F) = e^{\frac{i}{2} \Delta_{U_N}} (C_V F), \quad V \in U_N.
\]
Uniqueness of analytic continuation now proves \( (2.4) \) from \( (2.6) \).

Theorem 2.3 now follows by analytically continuing \( (2.4) \) in the \( V \) variable.

**Proof of Theorem 2.3.** Let \( F \in L^2(U_N, \rho_s^N; \mathbb{M}_N) \) be equivariant; thus \( C_V F = F \) for all \( V \in U_N \). Then \( (2.4) \) shows that \( C_V(\mathcal{B}_{s,t}^N F) = \mathcal{B}_{s,t}^N F \equiv 0 \) for each \( V \in U_N \). Since \( \mathcal{B}_{s,t}^N F \) is holomorphic, it follows by uniqueness of analytic continuation that the function \( Z \mapsto C_Z(\mathcal{B}_{s,t}^N F) - \mathcal{B}_{s,t}^N F \equiv 0 \) for \( Z \in \mathbb{G}L_N \); thus, \( \mathcal{B}_{s,t}^N F \) is equivariant under \( \mathbb{G}L_N \), as required. An entirely analogous argument applies to the inverse transform, establishing the theorem. 

\[ \square \]
Let us remark here on an intuitive approach to the concentration of measure results in Section 4. If $U_t$ is a random matrix sampled from the distribution $\rho_t^N$ on $U_N$, its (random) eigenvalues converge to their (deterministic) mean as $N \to \infty$. To be precise: if $\lambda_1^N, \ldots, \lambda_N^N$ are the eigenvalues of $U_t$, the empirical eigenvalue measure

$$\tilde{\nu}_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j^N}$$

converges weakly almost surely to $\nu_t$. (The mean of the random measure $\tilde{\nu}_t^N$ is the measure $\nu_t^N$ of (2.1) which converges weakly to $\nu_t$; cf. Theorem 1.14. The stronger statement that the convergence is almost sure, not just in expectation, was first proved in [25]. See [21] for the strongest known convergence results.)

The conjugacy classes in the group $U_N$ are in one-to-one correspondence with the (symmetrized) list of eigenvalues. Each such list is, in turn, determined by its empirical measure $\tilde{\nu}_t$. The convergence of the random eigenvalues of $U_t$ to a deterministic limit therefore suggests that the heat kernel measure $\rho_t^N$ concentrates its mass on a single conjugacy class as $N \to \infty$. The following proposition therefore offers some insight into Theorem 1.16 (that trace polynomials concentrate on single-variable Laurent polynomials). Indeed, on a fixed conjugacy class, any equivariant function is given by a polynomial.

**Proposition 2.5.** Let $G \subseteq M_N$ be a group, and let $C$ be a conjugacy class in $G$. If $F: G \to M_N$ is equivariant, then there exists a single-variable polynomial $P_C$ such that $F(A) = P_C(A)$ for all $A \in C$.

**Proof.** Fix a point $A_0$ in $C$, and let $A_1$ commute with $A_0$. Then since $F$ is equivariant,

$$A_1^{-1}F(A_0)A_1 = F(A_1^{-1}A_0A_1) = F(A_0),$$

which shows that $F(A_0)$ commutes with any such $A_1$: that is, $F(A_0) \in \{A_0\}''$ is in the double commutant of $A_0$. A classical theorem in linear algebra (see, for example, [23] for a short proof) then asserts that there is a single-variable polynomial $P_{A_0}$ such that $F(A_0) = P(A_0)$. Every other point in the conjugacy class $C$ is of the form $A = B A_0 B^{-1}$ for some $B \in G$. Since applying a polynomial function to a matrix commutes with conjugation, we have

$$F(A) = F(BA_0 B^{-1}) = BF(A_0)B^{-1} = BP_{A_0}(A_0)B^{-1} = P_{A_0}(BA_0B^{-1}) = P_{A_0}(A)$$

which shows that the map $A_0 \mapsto P_{A_0}$ is constant for $A_0 \in C$, so relabel $P_{A_0} = P_C$. Thus, the identity $F(A) = P_C(A)$ holds for all $A \in C$. \hfill \square

**Remark 2.6.** Proposition 2.5 has the at-first-surprising consequence that the equivariant function $F(A) = A^{-1}$ is equal to a polynomial (not a Laurent polynomial) on any given conjugacy class. This can be seen as a consequence of the Cayley-Hamilton Theorem; cf. Section 2.4. Indeed, let $p_A(\lambda) = \det(\lambda I_N - A)$ be the characteristic polynomial of $A$; then $p_A(A) = 0$. This shows there are coefficients $c_k$ (determined by $A$) so that $\sum_{k=0}^N c_k A^k = 0$. Since $c_0 = (-1)^N \det(A)$, if $A$ is invertible we can therefore factor out $A$ from the $k \geq 1$ terms and solve for $A^{-1}$ as a polynomial in $A$. The above proof shows that this $A$-dependent polynomial is, in fact, uniform over the whole conjugacy class.

### 2.3 Density of Trace Polynomials

Conceptually, equivariant functions are a natural arena for the Segal–Bargmann transform in the large-$N$ limit. Computationally, it will be convenient to work on the subclass of trace polynomials. In fact, trace polynomials are dense in $L^2(U_N, \rho_s^N; M_N(\mathbb{C}))_{eq}$. Thus, understanding the action of $B_N^{eq}$ on this class tells the full story.

**Theorem 2.7.** For $s > 0$, the space of trace polynomials is dense in the equivariant space $L^2(U_N, \rho_s^N; M_N(\mathbb{C}))_{eq}.$
We begin by proving that equivariant functions whose entries are polynomials in $U$ and $U^*$ are dense.

**Lemma 2.8.** Every equivariant function $F \in L^2(\mathbb{U}_N, \rho^N_s; M_N(\mathbb{C}))_{\text{eq}}$ can be approximated by a sequence of equivariant matrix-valued functions $F_n$, where each entry of $F_n(U)$ is a polynomial in the entries of $U$ and their conjugates.

**Proof.** By the Stone–Weierstrass Theorem and the density of continuous functions in $L^2$, any $f \in L^2(\rho^N_s)$ can be approximated by scalar-valued polynomial functions of the entries of the $\mathbb{U}_N$ variable and their conjugates. Applying this result to the components of the matrix-valued function $F$, we see that there is a sequence $P_n$ of polynomials in the entries of $U$ and their conjugates such that

$$\lim_{n \to \infty} \|P_n - F\|_{L^2(\mathbb{U}_N, \rho^N_s; \mathbb{M}_N)} = 0. \tag{2.7}$$

Now, consider again the conjugation action $C_V$ of (2.3). It is easy to verify that this action preserves the space of homogeneous polynomials of degree $m$ in the entries $U_{jk}$ and their conjugates. Thus, the averaged function

$$F_n(U) = \int_{\mathbb{U}_N} C_V(P_n)(U) \, dV$$

is still a polynomial in the entries of $U$ and their conjugates; and $F_n$ is evidently equivariant. Therefore $C_V(F) = F$ for each $V \in \mathbb{U}_N$, and so

$$F_n(U) - F(U) = \int_{\mathbb{U}_N} C_V(P_n)(U) \, dV - F(U) = \int_{\mathbb{U}_N} [C_V(P_n) - C_V(F)](U) \, dV.$$

It follows from (2.7) (with an application of Minkowski’s inequality and the dominated convergence theorem) that $F_n$ approximates $F$ in $L^2(\mathbb{U}_N, \rho^N_s; \mathbb{M}_N)$ as claimed.

**Proof of Theorem 2.7.** We will show that each of the functions $F_n$ in Lemma 2.8 is actually a trace polynomial. Suppose, then, that $F$ is equivariant and that each entry of $F(U)$ is a polynomial in the entries of $U$ and their conjugates. Let $T(N) \subset \mathbb{U}_N$ denote the diagonal subgroup. By the spectral theorem, any $U \in \mathbb{U}_N$ has a unitary diagonalization $U = V \Lambda V^{-1}$ for some $\Lambda \in T(N)$. The equivariance of $F$ then gives that $F(U) = F(V \Lambda V^{-1}) = VF(\Lambda)V^{-1}$. In particular, any equivariant function $F$ is completely determined by its restriction $F|_{T(N)}$ to the diagonal subgroup.

Because $F$ is equivariant, by the same argument used in the proof of Proposition 2.5, $F(U) \in \{U\}''$ for each $U$. Let $U \in T(N)$ be in the dense subset of matrices with all eigenvalues distinct; then $\{U\}'$ is the set of all diagonal matrices, and so $F(U)$ commutes with all diagonal matrices, meaning that $F(U)$ is diagonal. By the initial assumption on $F$, all entries of $F(U)$ are polynomials in the entries and their conjugates; hence, since the off-diagonal entries are 0 on a dense set, $F(U)$ is diagonal for all $U \in T(N)$, and its diagonal entries are polynomials in the diagonal entries $\lambda_1, \ldots, \lambda_N$ of $U$ and their conjugates. Of course, for $U \in T(N)$, the diagonal entries of $U$ satisfy $\lambda_j = 1/\lambda_j$. Thus, each of the diagonal entries of $F|_{T(N)}(U)$ is a Laurent polynomial $q(\lambda_1, \ldots, \lambda_N)$ in the $\lambda_j$’s. The symmetric group $\Sigma_N$ is a subgroup of $\mathbb{U}_N$, so since $F|_{T(N)}$ is equivariant under $\mathbb{U}_N$, it is also equivariant under $\Sigma_N$. Hence each of the (matrix-valued) polynomials $q$ is equivariant under the action of $\Sigma_N$ on the diagonal entries.

Taking $k$ to be larger than the largest negative degree of any variable in $q$, and setting $r(\lambda_1, \ldots, \lambda_N) = (\lambda_1 \cdots \lambda_N)^k q(\lambda_1, \ldots, \lambda_N)$, $r$ is also equivariant under the action of $\Sigma_N$. We can then express

$$F|_{T(N)}(U) = (\lambda_1 \cdots \lambda_N)^{-k} r(\lambda_1, \ldots, \lambda_N) = \det(U^*)^k r(\lambda_1, \ldots, \lambda_N).$$

Since the diagonal entries of $r(\lambda_1, \ldots, \lambda_N)$ are equivariant under permutations, the first entry of $r$ must be invariant under permutations of the remaining $N - 1$ variables. This means that the first entry of $r$ is a linear
combination of terms of the form $\lambda_j^\ell s_\ell(\lambda_2, \ldots, \lambda_N)$, where $\ell$ ranges from 0 up to the degree $d$ of $r$ and $s_\ell$ is a symmetric polynomial in $N - 1$ variables. By equivariance under $\Sigma_N$, it now follows that, for $1 \leq j \leq N$, the $j$th diagonal component of $r$ itself must be a linear combination of terms of the form

$$\{ \lambda_j^\ell s_\ell(\lambda_1, \ldots, \hat{\lambda}_j, \ldots, \lambda_N) : 0 \leq \ell \leq d \}.$$ 

It is well-known that every symmetric polynomial in $N - 1$ variables $\lambda_1, \ldots, \lambda_{N-1}$ is a polynomial in power-sums $p_\ell(\lambda_1, \ldots, \lambda_{N-1})$ with $0 \leq \ell \leq N - 1$, where, for any integer $\ell$,

$$p_\ell(\lambda_1, \ldots, \lambda_{N-1}) = \lambda_1^\ell + \lambda_2^\ell + \cdots + \lambda_{N-1}^\ell. \tag{2.8}$$

(This result was known at least to Newton. For a proof, see [28, Theorem 4.3.7].) Furthermore, any power sum in $N - 1$ variables can be written as a linear combination of power sums of $N$ variables along with the monomials $\lambda_j^\ell$; for example

$$\sum_{j=2}^N \lambda_j^\ell = \left( \sum_{j=1}^N \lambda_j^\ell \right) - \lambda_1^\ell.$$

Thus, the first entry of $r$ is actually a polynomial in power-sums of all $N$ variables and in $\lambda_1$ with the remaining entries of $r$ then being determined by equivariance with respect to permutations.

Suppose now that $r$ is the permutation-equivariant polynomial whose $j$th entry is

$$\lambda_j^{\ell_0} \left( \lambda_1^{k_1} + \cdots + \lambda_N^{k_1} \right)^{\ell_1} \cdots \left( \lambda_1^{k_M} + \cdots + \lambda_N^{k_M} \right)^{\ell_M}.$$

Then $r$ is nothing but the restriction to $T(N)$ of the trace polynomial

$$R(U) = U^{\ell_0} \text{Tr}(U^{k_1})^{\ell_1} \cdots \text{Tr}(U^{k_M})^{\ell_M}.$$

Meanwhile, by the above-quoted result, the symmetric polynomial $(\lambda_1 \lambda_2 \cdots \lambda_N)^k$ can be expressed as a polynomial in the power-sums of the $\lambda_j$s. Taking the complex-conjugate of this result, we see that $\det(U^*)^k$ can be expressed as a scalar trace polynomial in $U^*$; thus $U \mapsto (\det U^*)^k R(U)$ is a trace polynomial. Hence $F|_{T(N)}$ is the restriction of the trace polynomial function $U \mapsto (\det U^*)^k R(U)$, and the result follows since $F$ is determined by $F|_{T(N)}$. \hfill \Box

### 2.4 Asymptotic Uniqueness of Trace Polynomial Representations

The Cayley–Hamilton theorem asserts that, for any matrix $A \in \mathbb{M}_N$, it follows that $p_A(A) = 0$ where $p_A(\lambda) = \det(\lambda I_N - A)$ is the characteristic polynomial of $A$. In fact, the coefficients of the characteristic polynomial $p_A$ are all scalar trace polynomial functions of $A$: this follows from the Newton identities. Using the operators $\mathcal{M}_{(\cdot)}$ and $\mathcal{A}_+$ of Definition [3,9] below, there is an explicit formula for $p_A$. Let

$$h_A(\lambda) = \exp \left( - \sum_{m=1}^\infty \frac{1}{m \lambda^m} \text{Tr}(A^m) \right).$$

Then for $A \in \mathbb{M}_N$, $p_A(\lambda) = (A_+ \mathcal{M}_{AN} h_A)(\lambda)$. (See the Wikipedia entry for the Cayley-Hamilton theorem.) Thus, the expression $p_A(A)$ is a (an $N$-dependent) trace polynomial in $A$, and the Cayley–Hamilton theorem asserts that this trace polynomial function vanishes identically on $\mathbb{M}_N$. We illustrate this result in the case $N = 2$. 

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For all $A \in M_2(\mathbb{C})$, the Cayley–Hamilton Theorem asserts that
\[ A^2 - \text{Tr}(A)A + \det(A)I_2 = 0. \] (2.9)

In the $2 \times 2$ case, however, it is easily seen that
\[ \det(A) = \frac{1}{2}((\text{Tr}(A))^2 - \text{Tr}(A^2)). \] (2.10)

Substituting (2.10) into (2.9) and expressing things in terms of the normalized trace gives
\[ A^2 - 2A\text{tr}(A) + 2\text{tr}(A)^2 I_2 - \text{tr}(A^2)I_2 = 0 \]
for all $A \in M_2(\mathbb{C})$. In particular, if $P \in \mathbb{C}[u, u^{-1}; v]$ denotes the nonzero polynomial $P(u; v) = u^2 - 2uv_1 + 2v_1^2 - v_2$, then $P_2: \mathbb{U}_2 \to \mathbb{M}_2$ is the zero function. Note, however, that $P_N$ is not the zero function on $\mathbb{U}_N$ for $N > 2$, since the minimal polynomial of a generic element of $\mathbb{U}_N$ has degree $N$. This demonstrates the following theorem.

**Theorem 2.10.** Let $P$ be a nonzero element of $\mathbb{C}[u, u^{-1}; v]$. Then, for all sufficiently large $N$, the trace polynomial function $P_N$ is not identically zero on $\mathbb{U}_N$. In particular, if $P, Q \in \mathbb{C}[u, u^{-1}; v]$ are such that $P_N = Q_N$ for all sufficiently large $N$, then $P = Q$.

In order to prove Theorem 2.10 the following lemma (from the theory of symmetric functions) is useful. The corresponding statement for symmetric polynomials (rather than Laurent polynomials) is a standard result. The Laurent polynomial case must be known, but is well hidden in the literature.

**Lemma 2.11.** If $N \geq 2n$, then the power sums $p_k(\lambda_1, \ldots, \lambda_N)$ (cf. (2.8)) with $0 < |k| \leq n$ are algebraically independent elements of the ring of rational function in $N$ variables.

**Proof.** Let $e_j$ denote the $j$th elementary symmetric polynomial in $N$ variables; that is, $e_j$ the sum of all products of exactly $j$ of the $N$ variables. Then the power sums $p_1, \ldots, p_n$ can be expressed as linear combinations of the functions $e_1, \ldots, e_n$. Thus, it suffices to prove the independence of the functions $e_j(\lambda_1, \ldots, \lambda_N)$ and $e_j(\lambda_1^{-1}, \ldots, \lambda_N^{-1})$ for $1 \leq j \leq n$. We may easily see, however, that
\[ e_j(\lambda_1^{-1}, \ldots, \lambda_N^{-1}) = \frac{e_{N-j}(\lambda_1, \ldots, \lambda_N)}{e_N(\lambda_1, \ldots, \lambda_N)}. \]

In the case $N = 2n$, we need to establish the independence of the functions $e_1, \ldots, e_{n/2}$ and $e_{n/2}/e_N, \ldots, e_{n-1}/e_N$, which follows easily from the known independence of $e_1, \ldots, e_n$; cf. [28 Theorem 4.3.7]. In the case $N > 2n$, if we had an algebraic relation among the functions $e_j(\lambda_1, \ldots, \lambda_N)$ and $e_j(\lambda_1^{-1}, \ldots, \lambda_N^{-1})$ for $1 \leq j \leq N$, we could clear $e_N$ from the denominator to obtain an algebraic relation among the functions $e_1, \ldots, e_n, e_{N-1}, \ldots, e_{N-n}$ and $e_N$, which is impossible.

We now proceed with the scalar version of Theorem 2.10.

**Lemma 2.12.** Let $Q \in \mathbb{C}[v]$. Let $N \geq 2n$. Then $Q_N$ is not identically zero on $\mathbb{U}_N$.

**Proof.** Since $Q_N$ is a trace polynomial, it also defines a holomorphic function on $\mathbb{GL}_N$. By uniqueness of analytic continuation, if $Q_N \equiv 0$ on $\mathbb{U}_N$, then $Q_N \equiv 0$ on $\mathbb{GL}_N$. To prove the lemma, it therefore suffices to find $A \in \mathbb{GL}_N$ with $Q_N(A) \neq 0$. Actually, we will find a diagonal matrix $A \in \mathbb{GL}_N$ with $Q_N(A) \neq 0$.

For clarity, we write out the polynomial $Q$ in terms of its coefficients:
\[ Q(v_1, v_{-1}, \ldots, v_n, v_{-n}) = \sum_{j_1, \ldots, j_n} \sum_{i_1, \ldots, i_n} a_{j_1, \ldots, j_n}^{i_1, \ldots, i_n} v_1^{i_1} v_{-1}^{j_1} \cdots v_n^{i_n} v_{-n}^{j_n}. \]
Consider any diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_N)$ in $\mathbb{GL}_N$; for convenience, denote $\lambda = (\lambda_1, \ldots, \lambda_N)$. Then $\text{tr}(A^k) = p_k(\lambda)$ (the power sum of (2.8)), and so

$$Q_N(\text{diag}(\lambda)) = \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n}^{j_1, \ldots, j_n} \cdot p_1(\lambda)^{i_1} p_{-1}(\lambda)^{j_1} \cdots p_n(\lambda)^{i_n} p_{-n}(\lambda)^{j_n}. \quad (2.11)$$

By Lemma 2.11 the power sums $p_1(\lambda), p_{-1}(\lambda), \ldots, p_n(\lambda), p_{-n}(\lambda)$ are algebraically independent since $\lambda = (\lambda_1, \ldots, \lambda_N)$ and $N \geq 2n$. Since $Q \neq 0$, some of the coefficients $a_{i_1, \ldots, i_n}^{j_1, \ldots, j_n}$ in (2.11) are $\neq 0$. It follows that $Q_N(\text{diag}(\lambda))$ is not identically 0, as desired.

This finally brings us to Theorem 2.10.

Proof of Theorem 2.10. Let $P(u; v) = \sum_{i} Q_i(v)u^i$ with $Q_i \in \mathbb{C}[v]$; then at least one $Q_\ell \neq 0$. Let us multiply $P_N(U)$ by $U^k$ for some large $k$, so that all the untraced powers of $U$ in $U^k P_N(U)$ are non-negative. Let $\ell$ be the highest untraced power of $U$ occurring in the expression for $U^k P_N(U)$. Choose $N$ large enough so that $N > \ell$ and so that (Lemma 2.12) the coefficient $Q_\ell$ of $U^\ell$ in $P_N(U)$ is not identically zero. Then $Q_\ell$ is nonzero on a nonempty open subset of $\mathbb{U}_N$. This set contains a matrix $U_0$ whose minimal polynomial has degree $N > \ell$. When we evaluate $P_N(U_0)$, the result will be a linear combination of powers of $U_0$ with the coefficient of $U_0^\ell$ being nonzero. Since the minimal polynomial of $U_0$ has degree $N > \ell$, the value of $P_N(U_0)$ is not zero. \qed

3 The Laplacian and Heat Operator on Trace Polynomials

This section is devoted to a complete description of the action of the Laplacian $\Delta_{\mathbb{U}_N}$ on trace polynomial functions, and its corresponding lift to $\mathcal{D}_N$ on the space $\mathbb{C}[u, u^{-1}; v]$; cf. Theorem 1.18. We begin by proving “magic formulas” expressing certain quadratic matrix sums in simple forms. We use these to give derivative formulas that allow for the routine computation of $\Delta_{\mathbb{U}_N} P_N$ for any $P \in \mathbb{C}[u, u^{-1}; v]$, and we then use these to prove the intertwining formula of Theorem 1.18. We conclude by proving a more general intertwining formula (Theorem 3.26) for the action of $A_{\mathbb{U}_N}^N$ on trace polynomial functions over $\mathbb{GL}_N$; in this latter case, we deal more generally with trace polynomials in $\mathcal{Z}$ and $\mathcal{Z}^*$ as this will be of use in Section 4.

3.1 Magic Formulas

We define an inner-product on $\mathbb{M}_N$ by

$$\langle X, Y \rangle = N \text{Tr} (Y^* X) = N^2 \text{tr}(Y^* X). \quad (3.1)$$

Restricted to the Lie algebra $\mathfrak{u}_N$ (consisting of all skew-Hermitian matrices in $\mathbb{M}_N$), $\langle \cdot, \cdot \rangle$ is real-valued; it is the polarized inner product corresponding to the norm $\| \cdot \|_{\mathbb{M}_N}$ of (1.1). (This is not to be confused with the polarized inner-product corresponding to the norm $\| \cdot \|_{\mathbb{M}_N}$ of (1.2).)

The main result of this section, which underlies all computations throughout this paper, is the following list of “magic formulas”.

Proposition 3.1. Let $\beta_N$ be any orthonormal basis for $\mathfrak{u}_N$ with respect to the inner-product in (3.1). Then we
have the following “magic” formulas: for any $A, B \in \mathbb{M}_N$,
\[ \sum_{X \in \beta_N} X^2 = -I_N, \quad (3.2) \]
\[ \sum_{X \in \beta_N} XAX = -\text{tr}(A)I_N, \quad (3.3) \]
\[ \sum_{X \in \beta_N} \text{tr}(XA)X = -\frac{1}{N^2}A, \quad (3.4) \]
\[ \sum_{X \in \beta_N} \text{tr}(XA)\text{tr}(XB) = -\frac{1}{N^2}\text{tr}(AB). \quad (3.5) \]

**Remark 3.2.** (1) Eq. (3.2) is the $A = I_N$ special-case of (3.3); similarly, (3.5) follows from (3.4) by multiplying by $B$ and taking $\text{tr}$. We separate them out as distinct formulas for convenience in repeated use below.

(2) These and related “magic” formulas appeared in [33, Lemma 4.1].

**Proof.** If $\beta_N$ is a basis for the real vector space $u_N$, it is also a basis for the complex vector space $\mathbb{M}_N = u_N \oplus iu_N$. Furthermore, if $\beta_N$ is (real) orthonormal in $u_N$ with respect to the (restricted real) inner product in (3.1), then $\beta_N$ is (complex) orthonormal in $\mathbb{M}_N$ with respect to the (complex) inner product in (3.1).

Thus, let $\tilde{\beta}_N$ be any orthonormal basis for $\mathbb{M}_N$ with respect to (3.1), and consider the linear map $\Phi: \mathbb{M}_N \to \mathbb{M}_N$ given by
\[ \Phi(A) = \sum_{X \in \beta_N} X^*AX. \]

A routine calculation shows that $\Phi$ is independent of the choice of orthonormal basis. We compute $\Phi$ by using the basis
\[ \tilde{\beta}_N \equiv \left\{ \frac{1}{\sqrt{N}}E_{jk} \right\}_{j,k=1}^N \quad (3.6) \]
where $E_{jk}$ is the $N \times N$ matrix with a 1 in the $(j, k)$-entry and zeros elsewhere. Writing things out in terms of indices shows that, for any $A \in \mathbb{M}_N$, we have
\[ N \cdot [\Phi(A)]_{\ell m} = \sum_{j,k=1}^N E_{kj}AE_{jk} \right\}_{\ell m} = \sum_{j,k,n,o=1}^N \delta_{kn}\delta_{j\ell}\delta_{j\ell}\delta_{km} = \sum_o A_{oo}\delta_{\ell m}, \]
which says that
\[ \Phi(A) = \frac{1}{N} \text{Tr}(A)I_N = \text{tr}(A)I_N. \]
The basis-independence of $\Phi$ allows us to replace (3.6) by any real orthonormal basis $\beta_N$ of $u_N$ (which, as noted above, is also a complex orthonormal basis for $\mathbb{M}_N$). The elements $X \in \beta_N$ are skew-Hermitian, and thus we obtain
\[ \sum_{X \in \beta_N} XAX = -\Phi(A) = -\text{tr}(A)I_N, \]
which is (3.3).

Meanwhile, if we multiply both sides of (3.4) by $-N^2$ and recall that each $X$ is skew, we see that (3.4) is equivalent to the assertion that
\[ A = \sum_{X \in \beta_N} N^2\text{tr}(X^*A)X = \sum_{X \in \beta_N} \langle A, X \rangle X. \]
3.2 Derivative Formulas

Theorem 3.3. Let \( m, n \in \mathbb{N} \). Let \( \beta_N \) denote an orthonormal basis for \( u_N \), and let \( X \in \beta_N \). The following hold true:

\[
\partial_X U^n = \sum_{j=1}^{n} U^j X U^{n-j}, \quad n \geq 0 \tag{3.7}
\]

\[
\partial_X U^n = -\sum_{j=n+1}^{0} U^j X U^{n-j}, \quad n < 0 \tag{3.8}
\]

\[
\partial_X \text{tr}(U^n) = n \cdot \text{tr}(X U^n), \quad n \in \mathbb{Z} \tag{3.9}
\]

\[
\Delta_{U_N} U^n = -nU^n - 2\mathbb{1}_{n \geq 2} \sum_{j=1}^{n-1} jU^j \text{tr}(U^{n-j}), \quad n \geq 0 \tag{3.10}
\]

\[
\Delta_{U_N} U^n = nU^n + 2\mathbb{1}_{n \leq -2} \sum_{j=n+1}^{-1} jU^j \text{tr}(U^{n-j}), \quad n < 0 \tag{3.11}
\]

\[
\Delta_{U_N} \text{tr}(U^n) = -n\text{tr}(U^n) - 2\mathbb{1}_{n \geq 2} \sum_{j=1}^{n-1} j\text{tr}(U^j) \text{tr}(U^{n-j}), \quad n \geq 0 \tag{3.12}
\]

\[
\Delta_{U_N} \text{tr}(U^n) = n\text{tr}(U^n) + 2\mathbb{1}_{n \leq -2} \sum_{j=n+1}^{-1} j\text{tr}(U^j) \text{tr}(U^{n-j}), \quad n < 0 \tag{3.13}
\]

\[
\sum_{X \in \beta_N} \partial_X U^m \cdot \partial_X \text{tr}(U^n) = \frac{mn}{N^2} U^{n+m}, \quad m, n \in \mathbb{Z} \tag{3.14}
\]

\[
\sum_{X \in \beta_N} \partial_X \text{tr}(U^m) \cdot \partial_X \text{tr}(U^n) = \frac{mn}{N^2} \text{tr}(U^{n+m}), \quad m, n \in \mathbb{Z}. \tag{3.15}
\]

These formulas are valid for all matrices \( U \in \mathbb{M}_N \); we will normally use them for \( U \in \mathbb{U}_N \).

Proof. By the product rule, for \( n \geq 0 \)

\[
\partial_X U^n = \left. \frac{d}{dt} \right|_{t=0} (Ue^{tX})^n = \sum_{j=1}^{n} U^j X U^{n-j}
\]

which proves (3.7). Similarly, for \( m > 0 \)

\[
\partial_X U^{-m} = \left. \frac{d}{dt} \right|_{t=0} (e^{-tX}U^{-1})^m = -\sum_{k=0}^{m-1} U^{-k}XU^{-(m-k)}
\]
and letting \( n = -m \) and reindexing \( j = -k \) proves (3.8). Taking traces of (3.7) and (3.8) then gives (3.9) after using \( \text{tr}(AB) = \text{tr}(BA) \) repeatedly. Making use of magic formulas (3.2) and (3.3), we then have, for \( n \geq 0 \),

\[
\Delta_{U_N} U^n = 2\mathbf{1}_{n \geq 2} \sum_{1 \leq j < k \leq n} \sum_{X \in \beta_N} U \ldots U X \ldots U X \ldots U + \sum_{j=1}^{n} \sum_{X \in \beta_N} U \ldots U X^2 \ldots U
\]

\[
= -2\mathbf{1}_{n \geq 2} \sum_{1 \leq j < k \leq n} U^{n-(k-j)} \text{tr}(U^{k-j}) - nU^n.
\]

A little index gymnastics then reduces this last expression to the result in (3.10). An entirely analogous computation proves (3.11). Equations (3.12) and (3.13) result from taking traces of (3.10) and (3.11), since the linear functional \( \text{tr} \) commutes with \( \Delta_{U_N} \). Finally, from (3.7) and (3.9), when \( m \geq 0 \),

\[
\sum_{X \in \beta_N} (\partial_X U^m) \text{tr}(\partial_X U^n) = n \sum_{X \in \beta_N} \sum_{j=1}^{m} U^j X X^m-j \text{tr}(XU^n)
\]

\[
= n \sum_{X \in \beta_N} \sum_{j=1}^{m} U^j \text{tr}(XU^n) X X^m-j
\]

\[
= -n \sum_{j=1}^{m} U^j U^m X^m-j = -\frac{mn}{N^2} U^{m+n}.
\]

An analogous computation for \( m < 0 \) yields the same result, proving (3.14), and taking the trace of this formula gives (3.15).

\( \square \)

Remark 3.4. Eq. (3.10) shows that the identity function \( \text{id}(U) = U \) on \( U_N \) satisfies \( \Delta_{U_N} \text{id} = -\text{id} \). It follows, for example, that all of the coordinate functions \( U \mapsto U_{jk} \) are eigenfunctions of \( \Delta_{U_N} \) with eigenvalue \(-1 \), independent of \( N \). This independence suggests that we are, in fact, using the “correct” scaling of the metric on \( U_N \), which in turn determines the scaling of \( \Delta_{U_N} \). If we used the unscaled Hilbert-Schmidt norm on \( u_N \), the function \( \text{id} \) would be an eigenvector for the Laplacian with eigenvalue \(-N \); that scaling would not bode well for an infinite dimensional limit of any quantities involving the Laplacian.

To illustrate how Theorem 3.3 may be used, we proceed to determine the action of the heat operator \( e^{\frac{t}{2} \Delta_{U_N}} \) on the polynomial \( P_N(U) = U^2 \).

Example 3.5. Eq. (3.10) shows that \( \Delta_{U_N} U^2 = -2U^2 - 2U \text{tr}U \). In order to calculate \( \Delta_{U_N}(U \text{tr}U) \), we use the definition (1.4) of \( \Delta_{U_N} \) and the product rule twice. For each \( X \in u_N \),

\[
\partial_X^2(U \text{tr}U) = \partial_X [(\partial_X U) \cdot \text{tr}U + U \cdot (\partial_X \text{tr}U)] = (\partial_X^2 U) \cdot \text{tr}U + 2(\partial_X U)(\partial_X \text{tr}U) + U \cdot \partial_X^2 \text{tr}U.
\]

Summing over \( X \in u_N \) and using (3.10), (3.12), and (3.14) then shows that

\[
\Delta_{U_N}(U \text{tr}U) = (-U) \cdot \text{tr}U - \frac{2}{N^2} U^2 + U \cdot (-\text{tr}U) = -\frac{2}{N^2} U^2 - 2U \text{tr}U.
\]

Thus, setting \( P_N(U) = U^2 \) and \( Q_N(U) = U \text{tr}U \), we have

\[
\Delta_{U_N} P_N = -2P_N - 2Q_N, \tag{3.16}
\]

\[
\Delta_{U_N} Q_N = -\frac{2}{N^2} P_N - 2Q_N. \tag{3.17}
\]

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When $N > 1$, the span of the two functions $P_N, Q_N$ is a 2-dimensional subspace of $C^\infty(U_N)$ (when $N = 1$, $P_N = Q_N$). Equations (3.16) – (3.17) show that this subspace is invariant under the action of $\Delta_{U_N}$, which is represented there by the matrix

$$D_N = \begin{bmatrix} -2 & -2/N^2 \\ -2 & -2 \end{bmatrix}.$$

The exponentiated matrix $e^{t D_N}$ is easily computed (cf. [19, Chapter 2, Exercises 6,7]) as

$$e^{t D_N} = e^{-t} \begin{bmatrix} \cosh(t/N) & -1/N \sinh(t/N) \\ -N \sinh(t/N) & \cosh(t/N) \end{bmatrix}.$$

It follows immediately (reading off from the first column of this matrix) that

$$e^{t \Delta_{U_N}} P_N = e^{-t} \cosh(t/N) P_N - e^{-t} N \sinh(t/N) Q_N$$

as claimed in (1.10).

Any trace polynomial function $P_N$ on $U_N$ is contained in a finite-dimensional subspace of matrix-valued functions that is invariant under $\Delta_{U_N}$; this follows from Theorem 1.18 and Corollary 3.19 below. Thus, the computation of $e^{t \Delta_{U_N}} P_N$ for any trace polynomial $P_N$ reduces to exponentiating a matrix of finite size.

### 3.3 Intertwining Formulas I

We now explore how operations on trace polynomials are reflected in their intertwining space $\mathbb{C}[u, u^{-1}; v]$. The derivative formulas of Theorem 3.3 show that $\Delta_{U_N}$ preserves the space of trace polynomials with only positive powers of $U$, and also preserves the space of trace polynomials with only negative powers of $U$. This motivates the following projection operators on $\mathbb{C}[u, u^{-1}; v]$.

**Definition 3.6.** Let $\mathcal{A}_\pm$ denote the positive and negative projection operators

$$\mathcal{A}_+: \mathbb{C}[u, u^{-1}; v] \to \mathbb{C}[u; v] \quad \text{and} \quad \mathcal{A}_-: \mathbb{C}[u, u^{-1}; v] \to \mathbb{C}[u^{-1}; v]$$

given by

$$\mathcal{A}_+ \left( \sum_{k=-\infty}^{\infty} u^k q_k(v) \right) = \sum_{k=0}^{\infty} u^k q_k(v), \quad \mathcal{A}_- \left( \sum_{k=-\infty}^{\infty} u^k q_k(v) \right) = \sum_{k=-\infty}^{-1} u^k q_k(v). \quad \text{(3.18)}$$

Note that $\mathcal{A}_+ + \mathcal{A}_- = \text{id}_{\mathbb{C}[u, u^{-1}; v]}$, while $\mathcal{A}_+ - \mathcal{A}_- = \text{sgn}$ is the signum operator, where $\text{sgn}(u^n) = \text{sgn}(n) u^n$, and $\text{sgn}(n) = n/|n|$ when $n \neq 0$ and $\text{sgn}(0) = 1$.

**Remark 3.7.** The Fourier transform conjugates the Hilbert transform with the signum multiplier; in this sense, the operators $\mathcal{A}_\pm$ are linear combinations of the identity and the Hilbert transform.

**Notation 3.8.** For any $k \in \mathbb{Z}$, let $\mathcal{M}_{u^k}$ denote the multiplication operator, $\mathcal{M}_{u^k} P(u; v) = u^k P(u; v)$. Let $\mathcal{L}$ be the second order linear differential operator on $\mathbb{C}[u, u^{-1}; v]$ defined by

$$\mathcal{L} = \sum_{|j|, |k| \geq 1} j k v_{k+j} \frac{\partial^2}{\partial v_j \partial v_k} + 2 \sum_{|k| \geq 1} k u^{k+1} \frac{\partial^2}{\partial v_k \partial u} \quad \text{(3.19)}$$
where, for convenience, \( v_0 = 1; \) and let \( \mathcal{D} \) be the first-order pseudodifferential operator on \( \mathbb{C}[u, u^{-1}; v] \) defined by
\[
\mathcal{D} = - \sum_{|k| \geq 1} |k| v_k \frac{\partial}{\partial v_k} - u \frac{\partial}{\partial u} (A_+ - A_-)
\]
\[
- 2 \sum_{k=2}^{\infty} \left[ \left( \sum_{j=1}^{k-1} j v_j v_{k-j} \right) \frac{\partial}{\partial v_k} + \left( \sum_{j=1}^{k-1} j v_{-j} v_{-k+j} \right) \frac{\partial}{\partial v_{-k}} \right]
\]
\[
- 2 \sum_{k=1}^{\infty} \left[ v_k u A_+ \frac{\partial}{\partial u} M_{u^{-k}} A_+ + v_{-k} u A_- \frac{\partial}{\partial u} M_{u^k} A_- \right].
\] (3.20)

It is also convenient to define
\[
\mathcal{D}_N = \mathcal{D} - \frac{1}{N^2} \mathcal{L}.
\] (3.21)

For the proof of Theorem 1.18 it is useful to decompose \( \mathcal{D} \) and \( \mathcal{D}_N \) as
\[
\mathcal{D} = -N - 2 \mathcal{Z} - 2 \mathcal{Y}
\] (3.22)
\[
\mathcal{D}_N = -N - 2 \mathcal{Z} - 2 \mathcal{Y} - \frac{1}{N^2} \mathcal{L} = \mathcal{D} - \frac{1}{N^2} \mathcal{L}
\] (3.23)
where \( N, \mathcal{Y}, \) and \( \mathcal{Z} \) are defined as follows.

**Definition 3.9.** Define the following operators on \( \mathbb{C}[u, u^{-1}; v] \).
\[
N_1 = u \frac{\partial}{\partial u} (A_+ - A_-), \quad N_0 = \sum_{|k| \geq 1} |k| v_k \frac{\partial}{\partial v_k}, \quad N = N_0 + N_1,
\] (3.24)
\[
\mathcal{Y} = \mathcal{Y}_+ - \mathcal{Y}_- = \sum_{k=1}^{\infty} v_k u A_+ \frac{\partial}{\partial u} M_{u^{-k}} A_+ - \sum_{k=-\infty}^{-1} v_k u A_- \frac{\partial}{\partial u} M_{u^k} A_-,
\] (3.25)
\[
\mathcal{Z} = \mathcal{Z}_+ - \mathcal{Z}_- = \sum_{k=2}^{\infty} \left( \sum_{j=1}^{k-1} j v_j v_{k-j} \right) \frac{\partial}{\partial v_k} - 2 \sum_{k=-\infty}^{-2} \left( \sum_{j=k+1}^{\infty} j v_j v_{k-j} \right) \frac{\partial}{\partial v_k}.
\] (3.26)

**Example 3.10.** The first order pseudodifferential operator \( \mathcal{Y} \) appears somewhat mysterious; we illustrate its action here.

- \( \mathcal{Y} \) annihilates \( \mathbb{C}[v] \); more generally, for \( P \in \mathbb{C}[u, u^{-1}; v] \) and \( Q \in \mathbb{C}[v] \), \( \mathcal{Y}(PQ) = \mathcal{Y}(P) \cdot Q \). It therefore suffices to understand the action of \( \mathcal{Y} \) on \( \mathbb{C}[u, u^{-1}] \).

- \( \mathcal{Y}_- \) annihilates \( \mathbb{C}[u] \) and \( \mathcal{Y}_+ \) annihilates \( \mathbb{C}[u^{-1}] \). The reader can calculate that
\[
\mathcal{Y}(u^n) = \mathcal{Y}_+(u^n) = \sum_{k=1}^{n-1} (n-k) v_k u^{n-k}, \quad n \geq 0
\]
\[
- \mathcal{Y}(u^n) = \mathcal{Y}_-(u^n) = \sum_{k=n+1}^{-1} (n-k) v_k u^{n-k}, \quad n < 0.
\]

**Notation 3.11.** For \( n \in \mathbb{Z} \) and \( Z \in \mathbb{M}_N \), let \( W_n(Z) = Z^n, V_n(Z) = \text{tr}(Z^n), \) and \( \mathcal{V}(Z) = \{ V_n(A) \}_{|n| \geq 1} \). (Technically we should write \( V_n^N \) for \( V_n \) and \( W_n^N \) for \( W_n \), but we omit this extra index since the meaning should be clear from the context.) With this notation we have \( P_N(U) = P(U; \mathcal{V}(U)) \) for \( P \in \mathbb{C}[u, u^{-1}; v] \).
Proof of Theorem 1.18  Given the notation introduced above our goal is to show that

\[ \Delta_{\mathcal{U}_N} P_N = [\mathcal{D}_N P]_N = \left[ \left( -N - 2Z - 2Y \cdot \frac{1}{N^2} \mathcal{L} \right) P \right]_N. \]  

(3.27)

Fix \( n \in \mathbb{Z} \setminus \{0\} \), and let \( P(u; v) = u^n q(v) \) where \( q \in \mathcal{C}[v]; \) thus \( P_N = W_n \cdot q(V) \). For \( X \in U_N \), by the product rule we have

\[ \partial_X P_N = \partial_X [W_n \cdot q(V)] = \partial_X W_n \cdot q(V) + W_n \cdot \partial_X q(V) \]

and therefore

\[ \Delta_{\mathcal{U}_N} P_N = \sum_{X \in \beta_N} \partial^2_X P_N \]

\[ = \sum_{X \in \beta_N} \left[ \partial^2_X W_n \cdot q(V) + 2\partial_X W_n \cdot \partial_X q(V) + W_n \cdot \partial^2_X q(V) \right] \]

\[ = (\Delta_{\mathcal{U}_N} W_n) \cdot q(V) + 2 \sum_{X \in \beta_N} \partial_X W_n \cdot \partial_X q(V) + W_n \cdot (\Delta_{\mathcal{U}_N} q(V)). \]

(3.28)

Using (5.14) and the chain rule, the middle term in (5.28) can be written as

\[ \sum_{X \in \beta_N} \partial_X W_n \cdot \partial_X q(V) = \sum_{X \in \beta_N} \partial_X W_n \cdot \sum_{|k| \geq 1} \left( \frac{\partial}{\partial v_k} q \right) (V) \cdot \partial_X V_k \]

\[ = \sum_{|k| \geq 1} \left( \sum_{X \in \beta_N} \partial_X W_n \cdot \partial_X V_k \right) \left( \frac{\partial}{\partial v_k} q \right) (V) \]

\[ = \sum_{|k| \geq 1} \left( -\frac{n k}{N^2} W_{n+k} \right) \left( \frac{\partial}{\partial v_k} q \right) (V) \]

\[ = -\frac{1}{N^2} \sum_{|k| \geq 1} n k W_{n+k} \left( \frac{\partial}{\partial v_k} q \right) (V). \]

(3.29)

Notice that \( n W_{n+k} = W_{k+1} \cdot n W_{n-1} = W_{k+1} [\frac{\partial}{\partial u} u^n]_N \), and so (3.29) may be written in the form

\[ \sum_{X \in \beta_N} \partial_X W_n \cdot \partial_X q(V) = -\frac{1}{N^2} \left[ \sum_{|k| \geq 1} k u^{k+1} \frac{\partial^2}{\partial u \partial v_k} P \right]_N. \]

(3.30)

For the last term in (5.28), we again use the chain and product rules repeatedly to find

\[ \partial^2_X q(V) = \partial_X \left( \sum_{|k| \geq 1} \left( \frac{\partial}{\partial v_k} q \right) (V) \cdot \partial_X V_k \right) \]

\[ = \sum_{|k| \geq 1} \left( \frac{\partial}{\partial v_k} q \right) (V) \cdot \partial^2_X V_k + \sum_{|j|, |k| \geq 1} \left( \frac{\partial^2}{\partial v_j \partial v_k} q \right) (V) \cdot (\partial_X V_j)(\partial_X V_k). \]

(3.31)

Summing this equation on \( X \in \beta_N \), (3.15) shows that the second sum in (3.31) simplifies to

\[ \sum_{X \in \beta_N} \sum_{|j|, |k| \geq 1} \left( \frac{\partial^2}{\partial v_j \partial v_k} q \right) (V) \cdot (\partial_X V_j)(\partial_X V_k) = -\frac{1}{N^2} \sum_{|j|, |k| \geq 1} j k V_{j+k} \cdot \left( \frac{\partial^2}{\partial v_j \partial v_k} q \right) (V). \]

(3.32)
For the first sum in (3.31), we break up the sum over positive and negative terms, and use (3.12) and (3.13) to see that
\[
\sum_{X \in \delta_N} \sum_{|k| \geq 1} \left( \frac{\partial}{\partial v_k} q \right) (V) \cdot \partial_X^2 V_k = \sum_{k=1}^\infty \left( \frac{\partial}{\partial v_k} q \right) (V) \left( -kV_k - \sum_{j=1}^{k-1} jV_j V_{k-j} \right)
\]
\[
+ \sum_{k=-\infty}^{-1} \left( \frac{\partial}{\partial v_k} q \right) (V) \left( kV_k + \sum_{j=k+1}^{1} jV_j V_{k-j} \right)
\]
which is equal to
\[
- \sum_{|k| \geq 1} |k|V_k \left( \frac{\partial}{\partial v_k} q \right) (V)
\]
\[
- 2 \sum_{k=2}^\infty \left( \sum_{j=1}^{k-1} jV_j V_{k-j} \right) \left( \frac{\partial}{\partial v_k} q \right) (V) + 2 \sum_{k=-\infty}^{-1} \left( \sum_{j=k+1}^{1} jV_j V_{k-j} \right) \left( \frac{\partial}{\partial v_k} q \right) (V).
\]
(3.33)
Combining (3.31) - (3.33) we see that the final term in (3.28) is
\[
W_n \cdot \Delta_{U_N} q(V) = -[N_0 P]_N - 2[\mathbb{Z} P]_N - \frac{1}{N^2} \left[ \sum_{|j|,|k| \geq 1} kv_j k^2 \partial^2 P \right]_N
\]
and combining this with (3.28) and (3.30) gives
\[
\Delta_{U_N} P_N = (\Delta_{U_N} W_n) \cdot q(V) - \left[ \left( N_0 + 2\mathbb{Z} + \frac{1}{N^2} \mathcal{L} \right) P \right]_N,
\]
(3.34)
where (3.30) and (3.32) are the terms responsible for \( \mathcal{L} \). To address the first term in (3.34), we treat the cases \( n \geq 0 \) and \( n < 0 \) separately. When \( n \geq 0 \), (3.10) gives
\[
(\Delta_{U_N} W_n) \cdot q(V) = -nW_n \cdot q(V) - \sum_{j=1}^{n-1} jW_j V_{n-j} q(V).
\]
The first term is \(- [u \frac{\partial}{\partial u} u^n q(v)]_N\), and the second is (reindexing \( k = n-j \))
\[
-2 \left[ \sum_{k=1}^{n-1} v_k u^{n-k} q(v) \right]_N = -2[\mathbb{Y}_+ P]_N
\]
from Example 3.10. An analogous computation in the case \( n < 0 \), using (3.11), shows that in this case
\[
\Delta_{U_N} W_n \cdot q(V) = \left[ u \frac{\partial}{\partial u} P \right]_N + 2[\mathbb{Y}_- P]_N.
\]
Combining these with (3.34) concludes the proof of (1.23); (1.24) follows immediately, with the help of Corollary 3.19.

\[\square\]

**Definition 3.12.** The **tracing map** \( T : \mathbb{C}[u, u^{-1}; v] \to \mathbb{C}[v] \) is the linear operator given as follows: if \( p \in \mathbb{C}[v] \) and \( k \in \mathbb{Z} \setminus \{0\} \), then
\[
T(u^k p(v)) = v_k p(v).
\]
(3.35)
Regarding \( \mathbb{C}[v] \) as a subalgebra of \( \mathbb{C}[u, u^{-1}; v] \), note that an element \( P \in \mathbb{C}[u, u^{-1}; v] \) is in \( \mathbb{C}[v] \) if and only if \( T(P) = P \).
The following intertwining formula is elementary to verify.

**Lemma 3.13.** For $P \in \mathbb{C}[u, u^{-1}; v]$ and $N \in \mathbb{N}$,

\[ [\mathcal{T}(P)]_N = \text{tr} \circ P_N. \quad (3.36) \]

In order to proceed further it is useful to know that $\mathbb{C}[u, u^{-1}; v]$ completely decomposes into the finite dimensional eigenspaces of the operator $N$. Indeed, the space $\mathbb{C}[u, u^{-1}; v]$ (Definition 1.7) is the span of monomials

\[ \mathbb{C}[u, u^{-1}; v] = \text{span}_\mathbb{C} \left\{ u^{k_0} v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} v_{-1}^{k_{-1}} : n \geq 0, \ k_0, k_j \in \mathbb{Z}, \ k_j \in \mathbb{N} \text{ for } j \in \mathbb{Z} \setminus \{0\} \right\} \]

where each monomial is an eigenvector of $N$ as the next example shows.

**Example 3.14.** The monomial, $P(u; v) = u^{k_0} v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} v_{-1}^{k_{-1}}$, is an eigenvector of $N$, with

\[ N(P) = \left( |k_0| + \sum_{1 \leq |j| \leq n} |j|k_j \right) P. \quad (3.37) \]

We will define this eigenvalue to be the trace degree of $P$.

**Definition 3.15.** The trace degree of a monomial in $\mathbb{C}[u, u^{-1}; v]$ is

\[ \deg \left( u^{k_0} v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} v_{-1}^{k_{-1}} \right) = |k_0| + \sum_{1 \leq |j| \leq n} |j|k_j. \quad (3.38) \]

More generally, the trace degree of any element of $\mathbb{C}[u, u^{-1}; v]$ is the maximum of the trace degrees of its monomial terms. For $n \geq 0$, denote by $\mathbb{C}_n[u, u^{-1}; v] \subset \mathbb{C}[u, u^{-1}; v]$ the subspace of polynomials of trace degree $\leq n$:

\[ \mathbb{C}_n[u, u^{-1}; v] = \{ P \in \mathbb{C}[u, u^{-1}; v] : \deg P \leq n \}. \quad (3.39) \]

Note that $\mathbb{C}_n[u, u^{-1}; v]$ is finite dimensional; indeed, it is contained in $\mathbb{C}[u, u^{-1}; v_{\pm 1}, \ldots, v_{\pm n}]$. Moreover, $\mathbb{C}[u, u^{-1}; v] = \bigcup_{n \geq 0} \mathbb{C}_n[u, u^{-1}; v]$. Define $\mathbb{C}_n[u; v]$, $\mathbb{C}_n[u^{-1}; v]$, $\mathbb{C}_n[v]$, $\mathbb{C}_n[u, u^{-1}]$, $\mathbb{C}_n[u]$, and $\mathbb{C}_n[u^{-1}]$ similarly.

**Remark 3.16.** The trace degree reflects the nature of the variables $v_{\pm 1}, v_{\pm 2}, \ldots$ in $\mathbb{C}[v]$ as stand-ins for traces of powers of a matrix variable. Informally, the trace degree of $P \in \mathbb{C}[u, u^{-1}; v]$ is the total degree of $P_N(Z)$, counting all instances of $Z$ inside and outside traces, where the degree of $Z^k$ is defined to be $|k|$.

**Lemma 3.17** (and $D_N$ commute with $\mathcal{T}$). Let $\mathcal{L}, D, D_N : \mathbb{C}[u, u^{-1}; v] \to \mathbb{C}[u, u^{-1}; v]$ be given as in Definition 3.9 and (3.22). The operators $D_N, D, \text{ and } \mathcal{L}$ preserve trace degree (3.38), and commute with the tracing map $\mathcal{T}$ (3.35).

**Proof.** Let $N, y_{\pm}, z_{\pm}$ be as in be given as in Definition 3.9. The reader may readily verify that $N, y_{\pm}, z_{\pm}$, and $\mathcal{L}$ all preserve trace degree. What’s more, it is elementary to calculate that $[\mathcal{T}, N] = 0$, while

\[ z_{\pm} \mathcal{T} = \mathcal{T} z_{\pm} + y_{\pm}, \quad y_{\pm} \mathcal{T} = 0. \]

Hence, it follows that $D = -N - 2(z_{\pm} + y_{\pm} = -N - 2(z_{\pm} + y_{\pm} + 2(z_{\pm} + y_{\pm})$ commutes with $\mathcal{T}$. Since $D_N = D - \frac{1}{N^2} \mathcal{L}$ (cf. (3.21)), we are left only to prove that $[\mathcal{T}, \mathcal{L}] = 0$. This is also straightforward to compute; instead, we offer an alternative proof. From (3.36), we see that, for any $P \in \mathbb{C}[u, u^{-1}; v]$,

\[ [\mathcal{T} D_N(P)]_N = \text{tr}(\Delta_U P_N) = \Delta_U \text{tr}(P_N) = [\mathcal{T} D_N](P)_N. \]

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That is: \(([\mathcal{J}, D_N]P)_N \equiv 0\). It follows, using the fact that \([\mathcal{J}, \mathcal{D}] = 0\), that

\[
([\mathcal{J}, \mathcal{L}]P)_N = ([\mathcal{J}, N^2(D_N - \mathcal{D})]P)_N = N^2 ([\mathcal{J}, D_N]P)_N \equiv 0, \quad \text{for all } N. \tag{3.40}
\]

Theorem 2.10 now proves that \([\mathcal{J}, \mathcal{L}]P = 0\). Since this holds true for any \(P \in \mathbb{C}[u, u^{-1}; v]\), the result is proved.

We now prove Theorem 1.20.

**Proof of Theorem 1.20.** For convenience, we restate (1.26): the desired property is

\[
\mathcal{D}(PQ) = (\mathcal{D}P)Q + P(\mathcal{D}Q), \quad P \in \mathbb{C}[u, u^{-1}; v], \quad Q \in \mathbb{C}[v].
\]

Recall from Definition 3.9 and (3.22) that \(\mathcal{D} = -N - 2y - 2z = -(N_0 + 2y) - (N_1 + 2z)\), where \(N_1\) and \(z\) are first order differential operators on \(\mathbb{C}[v]\), while \(N_0\) and \(y\) annihilate \(\mathbb{C}[v]\) and satisfy

\[
N_0(PQ) = (N_0P)Q, \quad y(PQ) = (yP)Q, \quad P \in \mathbb{C}[u, u^{-1}; v], \quad Q \in \mathbb{C}[v].
\]

Hence

\[
(N_0 + 2y)(PQ) = [(N_0 + 2y)P]Q = [(N_0 + 2y)P]Q + P[(N_0 + 2y)Q].
\]

Since \(N_1 + 2z\) satisfies the product rule on \(\mathbb{C}[u, u^{-1}; v]\) in general, this proves (1.26); (1.27) follows thence from the standard power series argument.

Remark 3.18. We could alternatively describe the intertwining operator \(\mathcal{D}\) as the unique operator on \(\mathbb{C}[u, u^{-1}; v]\) which satisfies the partial product rule (1.26), commutes with the tracing map \(\mathcal{J}\), and satisfies

\[
\mathcal{D}(u^k) = -|k|u^k - 2k \sum_{\ell = 1}^{k-1} (k - \ell) v_\ell u^{k-\ell} + 2 \sum_{\ell = k+1}^{-1} (k - \ell) v_\ell u^{k-\ell}.
\]

The next corollary follows immediately from the first statement of Lemma 3.17.

**Corollary 3.19.** For \(n, N \in \mathbb{N}\), the finite dimensional subspace \(\mathbb{C}_n[u, u^{-1}; v] \subset \mathbb{C}[u, u^{-1}; v]\) is invariant under \(\mathcal{D}_N\) and \(\mathcal{D}\). Hence, for \(t \in \mathbb{R}\), \(e^{t\mathcal{D}_N}\) and \(e^{t\mathcal{D}}\) are well-defined operators on \(\mathbb{C}[u, u^{-1}; v]\) that leave \(\mathbb{C}_n[u, u^{-1}; v]\) invariant.

This brings us to the proof of Theorem 1.9.

**Proof of Theorem 1.9.** For convenience, we restate the desired property (1.14): we will show that, for any \(P \in \mathbb{C}[u, u^{-1}; v]\), \(N \in \mathbb{N}\), and \(t > 0\), there exists \(P_t^N \in \mathbb{C}[u, u^{-1}; v]\) with

\[
\mathbf{B}_{s,t}^N P_N = [P_t^N]_N.
\]

Indeed, let \(\mathcal{D}_N\) be as in (3.22), and define \(P_t^N = e^{t\mathcal{D}_N}P \in \mathbb{C}[u, u^{-1}; v]\). By (1.24) of Theorem 1.18 we then have \([P_t^N]_N = e^{t\Delta U}P_N\). Since \([P_t^N]_N\) is a trace polynomial, the entries of \([P_t^N]_N(U)\) are (holomorphic) polynomials in the entries of \(U\). Thus, \([P_t^N]_N\) has an analytic continuation to an entire function on \(\mathbb{GL}_N\), whose entries are the very same polynomials. It follows that \([P_t^N]_N\), interpreted as a function on \(\mathbb{GL}_N\), is the analytic continuation of \(e^{t\Delta U}P_N\), which is, by Definition 1.6, equal to \(\mathbf{B}_{s,t}^N P_N\).

We conclude with the following Corollary to the proof of Theorem 1.9 characterizing the range of \(\mathbf{B}_{s,t}^N\) on trace polynomials.
Corollary 3.20. Let $s, t > 0$ with $s > t/2$, and let $N \in \mathbb{N}$. If $P \in \mathbb{C}[u, u^{-1}; v]$, there exists $Q \in \mathbb{C}[u, u^{-1}; v]$ such that $B_{s,t}^N Q_N = P_N$. Thus, $B_{s,t}^N$ maps the space $[\mathbb{C}[u, u^{-1}; v]]_N$ of trace polynomials onto itself.

Proof. Set $Q = e^{-\frac{1}{2}D_N} P$. Then the intertwining formula (1.24), combined with the above discussion, shows that
\[ B_{s,t}^N Q_N = [e^{-\frac{1}{2}D_N} Q]_N = P_N \]
as claimed. \qed

3.4 Intertwining Formulas II

This section is devoted to proving an intertwining formula for $\mathbb{GL}_N$ (cf. Theorem 3.26) which is analogous to the intertwining formula for $\mathbb{U}_N$ in Theorem 1.18. This result is only needed in order to prove concentration of measures on $\mathbb{GL}_N$ (Eq. (1.20) of Theorem 1.16) and hence we do not need as much detailed information about the operators involved. On the other hand, we will now have to consider scalar trace polynomials in both $Z$ and $Z^*$, which complicates the notation somewhat.

Notation 3.21. For $n \in \mathbb{N}$, let $\mathcal{E}_n$ denote the set of functions (words) $\varepsilon : \{1, \ldots, n\} \to \{\pm 1, \pm*\}$. For $\varepsilon \in \mathcal{E}_n$, we denote $|\varepsilon| = n$. Set $\mathcal{E} = \bigcup_n \mathcal{E}_n$. We define the word polynomial space $\mathcal{W}$ as
\[ \mathcal{W} = \mathbb{C} \left[ \{v_\varepsilon\}_{\varepsilon \in \mathcal{E}} \right] \]
the space of polynomials in the indeterminates $\{v_\varepsilon\}_{\varepsilon \in \mathcal{E}}$. Of frequent use will be the words
\[ \varepsilon(j,k) = (\pm1, \ldots, \pm1, \pm*, \ldots, \pm*) \in \mathcal{E}_{j+k}, \]
where we use $+1$ in the first slots if $j > 0$ and $-1$ if $j < 0$, and similarly we use $+*$ in the last slots if $k > 0$ and $-*$ if $k < 0$.

Notation 3.22. For $\varepsilon \in \mathcal{E}_n$ and $Z \in \mathbb{GL}_N$ we define $Z^\varepsilon = Z^{\varepsilon_1} Z^{\varepsilon_2} \cdots Z^{\varepsilon_n}$, where $Z^{*+} \equiv Z^*$ and $Z^{-*} \equiv (Z^*)^{-1} = (Z^{-1})^*$. Given $P \in \mathcal{W}$, we let $P_N : \mathbb{GL}_N \to \mathbb{C}$ be the function
\[ P_N(Z) = P(V(Z)) \]
where
\[ V(Z) = \{V_\varepsilon(Z) : \varepsilon \in \mathcal{E}\} \]
and
\[ V_\varepsilon(Z) = \text{tr}(Z^\varepsilon) = \text{tr}(Z^{\varepsilon_1} Z^{\varepsilon_2} \cdots Z^{\varepsilon_n}). \]
The notation $V$ here collides with Notation 3.11 but there should be no confusion as to which is being used. As in that case, we should technically write $V_\varepsilon = V^N_\varepsilon$ and $V = V^N$, but we suppress the $N$ throughout. Also, in terms of Notation 3.11 note that $V_{\varepsilon(k,0)}(Z) = \text{tr}(Z^k) = V_k(Z)$, while $V_{\varepsilon(0,k)}(Z) = \text{tr}((Z^*)^k) = V_k(Z^*)$. It is therefore natural to think of $\mathbb{C}[v]$ as included in $\mathcal{W}$, in the following way.

Notation 3.23. We can identify $\mathbb{C}[v]$ as a subalgebra of $\mathcal{W}$ in two ways: $\iota, \iota^* : \mathbb{C}[v] \hookrightarrow \mathcal{W}$, with $\iota$ linear and $\iota^*$ conjugate linear, are determined by
\[ \iota(v_k) = v_{\varepsilon(k,0)} \quad \iota^*(v_k) = v_{\varepsilon(0,k)}. \]
The inclusions $\iota$ and $\iota^*$ intertwine with the evaluation maps as follows: for $Q \in \mathbb{C}[v]$,
\[ [\iota(Q)]_N(Z) = Q_N(Z) \quad [\iota^*(Q)]_N(Z) = Q_N(Z^*). \]
The trace degree on \( \mathbb{C}[v] \) extends consistently to the larger space \( \mathcal{W} \).

**Definition 3.24.** The trace degree of a monomial \( \prod_{j=1}^{m} v_{\varepsilon}^{k_j} \in \mathcal{W} \) is given by
\[
\deg \left( \prod_{j=1}^{m} v_{\varepsilon}^{k_j} \right) = \sum_{j=1}^{m} |k_j| |\varepsilon_j|,
\]
and the trace degree of any element in \( \mathcal{W} \) is the highest trace degree of any of its monomial terms. Since \(|\varepsilon(k,0)| = |\varepsilon(0,k)| = k\), we have
\[
\deg t(Q) = \deg t^*(Q) = \deg Q
\]
for \( Q \in \mathbb{C}[v] \). Note, moreover, that \( \deg(RS) = \deg(R) + \deg(S) \) for \( R, S \in \mathcal{W} \) not identically 0. Finally, for \( n \in \mathbb{N} \) we set
\[
\mathcal{W}_n = \{ P \in \mathcal{W} : \deg(P) \leq n \}.
\]
Note that \( \mathcal{W}_n \) is finite dimensional, \( \mathcal{W}_n \subset \mathbb{C}[\{v_{\varepsilon}\}_{|\varepsilon| \leq n}] \), and \( \mathcal{W} = \bigcup_n \mathcal{W}_n \).

We now proceed to describe the action of \( A_{s,t}^N \) on functions on \( U_N \) or \( GL_N \) of the form \( R_N \) for some \( R \in \mathcal{W} \); recall from (3.4) that
\[
A_{s,t}^N \equiv \left( s - \frac{t}{2} \right) \sum_{X \in \beta_N} \partial_X^2 + \frac{t}{2} \sum_{X \in \beta_N} \partial_X^2,
\]
where \( \beta_N \) is an orthonormal basis for \( u_N \).

**Theorem 3.25.** Fix \( s, t \in \mathbb{R} \). There are collections \( \{ Q_{\varepsilon}^{s,t} : \varepsilon \in \mathcal{E} \} \) and \( \{ R_{\varepsilon,\delta}^{s,t} : \varepsilon, \delta \in \mathcal{E} \} \) in \( \mathcal{W} \) with the following properties:

1. For each \( \varepsilon \in \mathcal{E} \), \( Q_{\varepsilon}^{s,t} \) is a certain finite sum of monomials of trace degree \( |\varepsilon| \) such that
\[
A_{s,t}^N V_{\varepsilon} = [Q_{\varepsilon}^{s,t}]_N = Q_{\varepsilon}^{s,t}(V),
\]
(3.45)

2. For \( \varepsilon, \delta \in \mathcal{E} \), \( R_{\varepsilon,\delta}^{s,t} \) is a certain finite sum of monomials of trace degree \( |\varepsilon| + |\delta| \) such that
\[
\left( s - \frac{t}{2} \right) \sum_{X \in \beta_N} (\partial_X V_{\varepsilon}) (\partial_X V_{\delta}) + \frac{t}{2} \sum_{X \in \beta_N} (\partial_{ix} V_{\varepsilon}) (\partial_{ix} V_{\delta}) = \frac{1}{2} \frac{1}{N^2} [R_{\varepsilon,\delta}^{s,t}]_N = \frac{1}{N^2} R_{\varepsilon,\delta}^{s,t}(V).
\]
(3.46)

Please note that the polynomials \( Q_{\varepsilon}^{s,t} \) and \( R_{\varepsilon,\delta}^{s,t} \) do not depend on \( N \). The \( 1/N^2 \) in (3.46) comes from the magic formula (3.4), as we will see in the proof.

**Proof.** Fix \( \varepsilon \in \mathcal{E} \), and let \( n = |\varepsilon| \). Let \( \beta_N \) denote an orthonormal basis for \( u_N \), and let \( \beta_+ = \beta_N \) while \( \beta_- = i\beta_N \). For any \( \xi \in u_N \oplus iu_N = gl_N = M_N \) and \( Z \in GL_N \), we make the following conventions (for this proof only):
\[
(Z\xi)^1 \equiv Z\xi, \quad (Z\xi)^{-1} \equiv -Z^{-1}\xi, \quad (Z\xi)^* \equiv \xi^*Z^*, \quad (Z\xi)^{-*} \equiv -Z^*\xi^*.
\]
(3.47)

Note that, for \( \xi \in \beta_+, \xi^* = \mp\xi \). In the proof to follow, we do not precisely track all of the signs, and so \( \pm \) denotes a sign that may be different in different terms and on different sides of an equation. Thus, we have
\[
(\partial_{ix} V_{\varepsilon})(Z) = \sum_{j=1}^{n} \text{tr}(Z^{\varepsilon_1}Z^{\varepsilon_2} \cdots (Z\xi)^{\varepsilon_j} \cdots Z^{\varepsilon_n})
\]

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and so
\[
(\partial^2_\xi V_\varepsilon)(Z) = \sum_{j=1}^{n} \text{tr} (Z^{\varepsilon_1} Z^{\varepsilon_2} \cdots (Z \varepsilon_j)^{\varepsilon_j} \cdots Z^{\varepsilon_n}) + 2 \sum_{1 \leq j < k \leq n} \text{tr} (Z^{\varepsilon_1} Z^{\varepsilon_2} \cdots (Z \varepsilon_j)^{\varepsilon_j} \cdots (Z \varepsilon_k)^{\varepsilon_k} \cdots Z^{\varepsilon_n}).
\]
(3.48)

(3.49)

We must now sum over $\xi \in \beta_\pm$. It follows from magic formula (3.2) and convention (3.47) that each term in (3.48) simplifies to
\[
\sum_{\xi \in \beta_\pm} \text{tr} (Z^{\varepsilon_1} Z^{\varepsilon_2} \cdots (Z \varepsilon_j)^{\varepsilon_j} \cdots Z^{\varepsilon_n}) = \pm \text{tr} (Z^{\varepsilon_1} Z^{\varepsilon_2} \cdots Z^{\varepsilon_j} \cdots Z^{\varepsilon_n}) = \pm V_\varepsilon(Z).
\]
To be clear: the $\pm$ on the right varies with $j$ and whether the sum is over $\beta_+$ or $\beta_-$. Summing each of these terms over $1 \leq j \leq n$ shows that (3.48) summed over $\beta_\pm$ is
\[
\sum_{\xi \in \beta_\pm} \sum_{j=1}^{n} \text{tr} (Z^{\varepsilon_1} Z^{\varepsilon_2} \cdots (Z \varepsilon_j)^{\varepsilon_j} \cdots Z^{\varepsilon_n}) = n_\pm(\varepsilon) V_\varepsilon(Z)
\]
(3.50)

for some $n_\pm(\varepsilon) \in \mathbb{Z}$ with $|n_\pm(\varepsilon)| \leq |\varepsilon|$. For the terms in (3.49), applying (3.47) shows that
\[
\text{tr} (Z^{\varepsilon_1} Z^{\varepsilon_2} \cdots (Z \varepsilon_j)^{\varepsilon_j} \cdots (Z \varepsilon_k)^{\varepsilon_k} \cdots Z^{\varepsilon_n}) = \pm \text{tr}(Z^{\varepsilon_0}_{j,k} \varepsilon Z^{\varepsilon_1}_{j,k} \varepsilon Z^{\varepsilon_2}_{j,k})
\]
(3.51)

where $\{\varepsilon^\ell_{j,k}\}_{\ell=0,1,2}$ are certain substrings of $\varepsilon$, whose concatenation is all of $\varepsilon$: $\varepsilon^0_{j,k} \varepsilon^1_{j,k} \varepsilon^2_{j,k} = \varepsilon$. Applying magic formula (3.3) to (3.51) gives
\[
\sum_{\xi \in \beta_\pm} \text{tr}(Z^{\varepsilon_0}_{j,k} \varepsilon Z^{\varepsilon_1}_{j,k} \varepsilon Z^{\varepsilon_2}_{j,k}) = \pm \text{tr}(Z^{\varepsilon_0}_{j,k} \varepsilon Z^{\varepsilon_1}_{j,k} \varepsilon Z^{\varepsilon_2}_{j,k}) \text{tr}(Z^{\varepsilon_1}_{j,k} \varepsilon Z^{\varepsilon_2}_{j,k})
\]

where $\varepsilon_{j,k} = \varepsilon^0_{j,k} \varepsilon^2_{j,k}$. Note that $|\varepsilon_{j,k}| + |\varepsilon^1_{j,k}| = |\varepsilon|$. Hence, the sum in (3.49) summed over $\beta_\pm$ is equal to
\[
\sum_{1 \leq j < k \leq n} \pm \text{tr}(Z^{\varepsilon_{j,k}}) \text{tr}(Z^{\varepsilon_{j,k}}) = \sum_{1 \leq j < k \leq n} \pm V_{\varepsilon_{j,k}}(Z) V_{\varepsilon_{j,k}}(Z).
\]
(3.52)

Hence, if we define
\[
Q^\pm_\varepsilon = n_\varepsilon(\varepsilon) v_\varepsilon + 2 \sum_{1 \leq j < k \leq n} \pm v_{\varepsilon_{j,k}} v_{\varepsilon_{j,k}},
\]
(3.53)

which have homogeneous trace degree $|\varepsilon|$, then (3.48) – (3.52) show that
\[
Q^s,t_\varepsilon = \left( s - \frac{t}{2} \right) Q^+_\varepsilon + \frac{t}{2} Q^-_\varepsilon
\]
satisfies (3.48), proving item (1) of the theorem.

For item (2), fix $\delta \in \mathcal{E}$ and let $m = |\delta|$. We calculate for each $\xi \in \mathbb{M}_N$
\[
(\partial_\xi V_\varepsilon)(Z)(\partial_\delta V_\delta)(Z) = \sum_{j=1}^{n} \sum_{k=1}^{m} \text{tr}(Z^{\varepsilon_1} Z^{\varepsilon_2} \cdots (Z \varepsilon_j)^{\varepsilon_j} \cdots Z^{\varepsilon_n}) \cdot \text{tr}(Z^{\delta_1} Z^{\delta_2} \cdots (Z \varepsilon_k)^{\varepsilon_k} \cdots Z^{\delta_m}),
\]
again making use of convention (3.47). Using the cyclic property of the trace, we can write the terms in this sum in the form
\[
\pm \text{tr}(\xi Z^{(j)}) \text{tr}(\xi^{(k)})
\]

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where $\varepsilon^{(j)}$ is a certain cyclic permutation of $\varepsilon$, and $\delta^{(k)}$ is a certain cyclic permutation of $\delta$. Summing over $\xi \in \beta_{\pm}$ and using magic formula (3.5), we then have

$$
\sum_{\xi \in \beta_{\pm}} (\partial_\xi V_{\varepsilon})(Z)(\partial_\xi V_{\delta})(Z) = \frac{1}{N^2} \sum_{j=1}^{n} \sum_{k=1}^{m} \pm \text{tr}(Z^{\varepsilon^{(j)}} Z^{\delta^{(k)}}) = \frac{1}{N^2} \sum_{j=1}^{n} \sum_{k=1}^{m} \pm V_{\varepsilon^{(j)}} \delta^{(k)} (Z).
$$

(3.54)

Since $\varepsilon^{(j)} \delta^{(k)}$ has length $|\varepsilon| + |\delta|$, the $\mathcal{W}$ elements

$$
\mathcal{R}^{\pm}_{\varepsilon,\delta} = \sum_{j=1}^{n} \sum_{k=1}^{m} \pm v_{\varepsilon^{(j)}} \delta^{(k)}
$$

(3.55)

have homogeneous trace degree $|\varepsilon| + |\delta|$, and (3.54) therefore shows that

$$
i_s^{s,t}_{\varepsilon,\delta} = \left( s - \frac{t}{2} \right) R_{\varepsilon,\delta}^{+} + \frac{t}{2} R_{\varepsilon,\delta}^{-}
$$

(3.56)

satisfies (3.46), proving item (2) of the theorem.

---

**Theorem 3.26 (Intertwining Formula II).** Fix $s, t \in \mathbb{R}$. Let $\left\{ Q_{\varepsilon}^{s,t} : \varepsilon \in \mathcal{E} \right\}$ and $\left\{ R_{\varepsilon,\delta}^{s,t} : \varepsilon, \delta \in \mathcal{E} \right\}$ be the polynomials from Theorem 3.25 and define

$$
\tilde{D}_{s,t} = \frac{1}{2} \sum_{\varepsilon \in \mathcal{E}} Q_{\varepsilon}^{s,t} \frac{\partial}{\partial v_{\varepsilon}} \quad \text{and} \quad \tilde{L}_{s,t} = \frac{1}{2} \sum_{\varepsilon,\delta \in \mathcal{E}} R_{\varepsilon,\delta}^{s,t} \frac{\partial^2}{\partial v_{\varepsilon} \partial v_{\delta}}
$$

(3.57)

which are first and second order differential operators on $\mathcal{W}$ which preserve trace degree. Then, for all $N \in \mathbb{N}$ and $P \in \mathcal{W}$,

$$
\frac{1}{2} A_{s,t}^N P_N = \left[ \tilde{D}_{s,t} P + \frac{1}{N^2} \tilde{L}_{s,t} P \right]_N.
$$

(3.58)

**Remark 3.27.** Definition 1.3 of $A_{s,t}^N$ is stated for $s, t > 0$ and $s > t/2$; it is only in this regime that the operator $A_{s,t}^N$ is negative-definite and the tools of heat kernel analysis apply. The operator itself is well-defined-for any $s, t \in \mathbb{R}$, however, and it will be convenient to utilize this in some of what follows.

**Proof.** By the chain rule, if $\xi \in \mathcal{M}_N$ then

$$
\partial_\xi^2 P_N = \sum_{\varepsilon \in \mathcal{E}} \partial_\xi \left[ \frac{\partial P}{\partial v_{\varepsilon}} \right] (V) \cdot \partial_\xi V_{\varepsilon}
$$

$$
= \sum_{\varepsilon \in \mathcal{E}} \left( \frac{\partial P}{\partial v_{\varepsilon}} \right) (V) \cdot \partial_\xi^2 V_{\varepsilon} + \sum_{\varepsilon,\delta \in \mathcal{E}} \left( \frac{\partial^2 P}{\partial v_{\varepsilon} \partial v_{\delta}} \right) (V) \cdot (\partial_\xi V_{\varepsilon}) (\partial_\xi V_{\delta})
$$

from which it follows that

$$
A_{s,t}^N P_N = \sum_{\varepsilon \in \mathcal{E}} \left( \frac{\partial P}{\partial v_{\varepsilon}} \right) (V) \cdot A_{s,t}^N V_{\varepsilon}
$$

$$
+ \sum_{\varepsilon,\delta \in \mathcal{E}} \left( \frac{\partial^2 P}{\partial v_{\varepsilon} \partial v_{\delta}} \right) (V) \left[ \left( s - \frac{t}{2} \right) \sum_{\xi \in \beta} \partial_\xi V_{\varepsilon} \cdot \partial_\xi V_{\delta} + \frac{t}{2} \sum_{\xi \in i\beta} (\partial_\xi V_{\varepsilon}) (\partial_\xi V_{\delta}) \right].
$$

Combining this equation with the results of Theorem 3.25 completes the proof.
We record one further intertwining formula that will be useful in the proofs of Theorems 1.11 and 1.16.

**Lemma 3.28.** There exists a sesquilinear form (conjugate linear in the second variable)

\[ \mathcal{B} : \mathbb{C}[u, u^{-1}; v] \times \mathbb{C}[u, u^{-1}; v] \to \mathcal{W} \]

such that, for all \( P, Q \in \mathbb{C}[u, u^{-1}; v] \), we have \( \deg (\mathcal{B}(P, Q)) = \deg(P) + \deg(Q) \) and

\[ [\mathcal{B}(P, Q)]_N(Z) = \text{tr}[P_N(Z)Q_N(Z)^*] \quad \text{for all } Z \in \text{GL}_N. \]

**Proof.** By sesquilinearity, it suffices to define \( \mathcal{B} \) on \( P, Q \in \mathbb{C}[u, u^{-1}; v] \) of the form \( P(u; v) = u^k p(v) \) and \( Q(u; v) = u^\ell q(v) \) for \( k, \ell \in \mathbb{Z} \) and \( p, q \in \mathbb{C}[v] \). We compute, for \( Z \in \text{GL}_N \), that

\[ \text{tr}[P_N(Z)Q_N(Z)^*] = \text{tr}[Z^k p_N(Z)Z^\ell q_N(Z)^*] = \text{tr}(Z^k Z^\ell)p_N(Z)q_N(Z)^* \]

\[ = [v_{\varepsilon(k, \ell)}]_N(Z)[v(p)]_N(Z)[v^*(q)]_N(Z) \]

by (3.43), where \( \varepsilon(k, \ell) \) is defined in (3.41). Thus, we take \( \mathcal{B} : \mathbb{C}[u, u^{-1}; v] \times \mathbb{C}[u, u^{-1}; v] \to \mathcal{W} \) to be the unique sesquilinear form such that, for \( p, q \in \mathbb{C}[v] \),

\[ \mathcal{B}(u^k p, u^\ell q) = v_{\varepsilon(k, \ell)} v(p) v^*(q). \]

This is trace degree additive by (3.44). This concludes the proof. \( \square \)

## 4 Limit Theorems

In this section, we prove that the heat kernel measures \( \rho_s^N \) on \( U_N \) and \( \mu_{s, t}^N \) on \( \text{GL}_N \) each concentrate all their mass in such a way that the space of trace polynomials \( [\mathbb{C}[u, u^{-1}; v]]_N \) collapses onto the space of Laurent polynomials \( [\mathbb{C}[u, u^{-1}]]_N \) as \( N \to \infty \). To motivate this, consider the scalar-valued case: if \( Q \in \mathbb{C}[v] \), then Theorem 1.18 shows that

\[ e^{\frac{2}{N} \Delta_N} (Q_N) = \left[ e^{\frac{2}{N} \Delta_D} N^{-1} \right]_N Q_N = \left[ e^{\frac{2}{N} \Delta_D} Q \right]_N + O \left( \frac{1}{N^2} \right), \]

(4.1)

where the second equality will be made precise in Lemma 4.1 below. Evaluating (4.1) at \( I_N \) and using (1.6) shows that

\[ \mathbb{E} \rho_s^N (Q_N) = \left( e^{\frac{2}{N} \Delta_D} N^{-1} \right) (I_N) = \left( e^{\frac{2}{N} \Delta_D} Q \right) (1) + O \left( \frac{1}{N^2} \right), \]

(4.2)

where \( Q(1) = Q(v) \big|_{v=1} \) is the evaluation of \( Q \) at all variables \( v_k = 1 \). Theorem 1.20 shows that \( e^{\frac{2}{N} \Delta_D} \) is an algebra homomorphism on \( \mathbb{C}[v] \), and so

\[ \left[ e^{\frac{2}{N} \Delta_D} Q^2 \right]_N = \left[ \left( e^{\frac{2}{N} \Delta_D} Q \right)_N \right]^2. \]

(4.3)

If \( Q \) has real coefficients, then \( Q^2 = |Q|^2 \), and so (4.2) applied to \( Q^2 \) and (4.3) evaluated at 1 show that

\[ \text{Var} \rho_s^N (Q_N) = \int_{U_N} |Q_N(U)|^2 \rho_s^N (dU) - \left| \int_{U_N} Q_N(U) \rho_s^N (dU) \right|^2 = O \left( \frac{1}{N^2} \right). \]

Thus, the random variables \( Q_N \) concentrate on their limit mean (which is \( \pi_s Q \) by Theorem 1.14), summably fast. Section 4.11 fleshes out this argument in the general case (where \( Q \) need not have real coefficients, and is more generally in \( \mathbb{C}[u, u^{-1}; v] \)). Sections 4.2 and 4.3 then use these ideas to prove Theorems 1.11 and 1.13.
4.1 Concentration of Measures

We begin with an abstract result that will be the gist of all our concentration of measure theorems.

**Lemma 4.1.** Let $V$ be a finite dimensional normed $\mathbb{C}$-space and supposed that $D$ and $L$ are two operators on $V$. Then there exists a constant $C = C(D, L, \| \cdot \|_V) < \infty$ such that

$$\| e^{D+tL} - e^{D} \|_{\text{End}(V)} \leq C |t| \text{ for all } |t| \leq 1,$$  

(4.4)

where $\| \cdot \|_{\text{End}(V)}$ is the operator norm on $V$. It follows that, if $\varphi \in V^*$ is a linear functional, then

$$|\varphi(e^{D+tL}x) - \varphi(e^Dx)| \leq C\|\varphi\|_{V^*}\|x\|_V|t|, \quad x \in V, \quad |t| \leq 1,$$  

(4.5)

where $\| \cdot \|_{V^*}$ is the dual norm on $V^*$.

**Proof.** Using the well known differential of the exponential map (see for example [12] Theorem 1.5.3, p. 23), [19] Theorem 3.5, p. 70], or [29] Lemma 3.4, p. 35]),

$$\frac{d}{ds} e^{D+tsL} = e^{D+tsL} \int_0^1 e^{-t(D+sL)} Le^{t(D+sL)} dt$$  

$$= \int_0^1 e^{(1-t)(D+sL)} Le^{t(D+sL)} dt,$$

we may write

$$e^{D+tL} - e^{D} = \int_0^t e \frac{d}{ds} e^{D+tsL} ds = \int_0^t \left[ \int_0^1 e^{(1-t)(D+sL)} Le^{t(D+sL)} dt \right] ds.$$

Crude bounds now show

$$\| e^{D+tL} - e^{D} \|_{\text{End}(V)} \leq \int_0^{|t|} \left[ \int_0^1 \| e^{(1-t)(D+sL)} Le^{t(D+sL)} \|_{\text{End}(V)} dt \right] ds \leq C(D, L, \| \cdot \|_V)|t|,$$

proving (4.4), (4.5) follows immediately. \qed

Theorem 4.14 and Lemma 4.1 now allow us to give a useful alternate characterization of the evaluations maps $\pi_s$.

**Lemma 4.2.** For $P \in \mathbb{C}[u, u^{-1}; v]$ and $s \in \mathbb{R}$, the evaluation map $\pi_s$ can be written in the form

$$(\pi_sP)(u) = \left( e^{-\frac{s}{2} (N_0 + 2\zeta)} P \right)(u; 1)$$  

(4.6)

where, for $Q \in \mathbb{C}[u, u^{-1}; v]$, $Q(u; 1) = Q(u; v)|_{v_k=1,k\neq0}$.

**Proof.** First, note from Definition 3.9 that, for $P \in \mathbb{C}[v]$, $N_1p = \gamma P = 0$; thus, from (3.22), we have

$$\mathcal{D}|_{\mathbb{C}[v]} = (-N_0 - 2\zeta)|_{\mathbb{C}[v]}.$$

If $P(u; v) = \sum_k u^k p_k(v)$ with $p_k \in \mathbb{C}[v]$, then $((-N_0 - 2\zeta)P)(u; v) = \sum_k u^k (\mathcal{D}p_k)(v)$; hence, to prove (4.6), it suffices to show that

$$\pi_s(p) = \left( e^{\frac{s}{2} \mathcal{D}} p \right)(1), \quad p \in \mathbb{C}[v].$$  

(4.7)

By Theorem 4.20, $e^{\frac{s}{2} \mathcal{D}}$ is a homomorphism of $\mathbb{C}[v]$. Hence, to prove (4.7), it suffices to show that

$$\left( e^{\frac{s}{2} \mathcal{D}} v_k \right)(1) = \pi_s(v_k) = \nu_k(s), \quad k \in \mathbb{Z} \setminus \{0\}.$$  

(4.8)
Lemma 4.3. Let \( \nu_k(s) = \lim_{N \to \infty} \left( e^{s \Delta_{U_N}} \text{tr}[\cdot]^k \right) (I_N) = \lim_{N \to \infty} \left( e^{s \mathcal{D}_N} \nu_k \right) (1). \) (4.9)

On the other hand, \( \varphi(p) = p(1) \) is a linear functional on \( \mathbb{C}[v] \), and \( \nu_k \in \mathbb{C}_k[v] \) which is finite-dimensional. Since \( \mathcal{D}_N = \mathcal{D} - \frac{1}{N^2} \mathcal{L} \) and both \( \mathcal{D} \) and \( \mathcal{L} \) leave \( \mathbb{C}_k[v] \) invariant, Lemma 4.1 shows that

\[
\left| \left( e^{s \mathcal{D}_N} \nu_k \right) (1) - \left( e^{s \mathcal{D}} \nu_k \right) (1) \right| = O \left( \frac{1}{N^2} \right). \tag{4.10}
\]

Equations (4.9) and (4.10) imply (4.8), concluding the proof. \( \square \)

The next lemma relates \( \mathcal{D}_{s,t} \) to the evaluation map \( \pi_{s-t} \), which will lead to the proof of Theorem 1.15. Recall the inclusion maps \( t, t^* : \mathbb{C}[v] \hookrightarrow \mathcal{W} \) of Notation 3.23.

**Lemma 4.3.** Let \( s, t > 0 \) with \( s > t/2 \). Let \( \mathcal{D}_{s,t} \) be given as in (3.57). Then, for any \( Q \in \mathbb{C}[v] \),

\[
\left[ e^{\mathcal{D}_{s,t}}(Q) \right](1) = \pi_{s-t} Q. \tag{4.11}
\]

**Proof.** If \( f : \mathbb{GL}_N \to \mathbb{M}_N \) is holomorphic, then \( \partial_X f = i \partial_X f \) for all \( X \in u_N \), which then implies

\[
A^{N}_{s,t} f \big|_{\mathbb{U}_N} = \left( s - \frac{t}{2} \right) \sum_{X \in \beta_N} \partial_X f - \frac{t}{2} \sum_{X \in \beta_N} \partial_X^2 f = (s - t) \Delta_{\mathbb{U}_N} f.
\]

Since the scalar trace polynomial \( Q_N \) is holomorphic, it follows that

\[
e^{\frac{1}{2} A^{N}_{s,t} } Q_N = e^{\frac{1}{2} (s-t) \Delta_{U_N} } Q_N. \tag{4.12}
\]

(Note: when \( s < t \) the expression \( e^{\frac{1}{2} (s-t) \Delta_{U_N} } \) is not meaningful in general, but makes perfect sense as a power series when applied to a polynomial function such as \( Q_N \).) Using intertwining formulas (3.43) and (3.58) on the left-hand-side of (4.12) and intertwining formula (1.23) on the right-hand-side, we have

\[
\left[ e^{\mathcal{D}_{s,t} + \frac{1}{N^2} \bar{\mathcal{L}}_{s,t}}(Q) \right]_N = e^{\frac{1}{2} A^{N}_{s,t} } Q_N = e^{\frac{1}{2} (s-t) \Delta_{U_N} } Q_N = \left[ e^{\frac{1}{2} (s-t) \mathcal{D}_N } Q \right]_N,
\]

and evaluating both sides at \( I_N \) and using \( \mathcal{D}_N = \mathcal{D} - \frac{1}{N^2} \mathcal{L} \), we have

\[
\left( e^{\mathcal{D}_{s,t} + \frac{1}{N^2} \bar{\mathcal{L}}_{s,t}}(Q) \right) (1) = \left( e^{\frac{1}{2} (s-t) \mathcal{D} - \frac{1}{N^2} \mathcal{L} } Q \right) (1). \tag{4.13}
\]

Let \( n = \deg(Q) \). Using the linear functional \( \varphi(R) = R(1) \) on the finite-dimensional spaces \( \mathbb{C}_n[v] \) and \( \mathcal{W}_n \), Lemma 4.1 allows us to take the limit as \( N \to \infty \) in (4.13), yielding

\[
\left( e^{\mathcal{D}_{s,t}}(Q) \right) (1) = \left( e^{\frac{1}{2} (s-t) \mathcal{D} } Q \right) (1). \tag{4.14}
\]

Finally, since \( Q \in \mathbb{C}[v] \), Lemma 4.2 shows that the right-hand-side of (4.14) is \( \pi_{s-t} Q \). This concludes the proof. \( \square \)

**Remark 4.4.** A similar calculation shows that \( \left( e^{\mathcal{D}_{s,t}}(Q) \right) (1) = \pi_{s-t} Q. \)

Theorem 1.15 was really proved in the above proof.
Proof of Theorem 1.15. From (1.7) and Remark 1.4 together with intertwining formulas (3.43) and (3.58), we have
\[
\int_{GL_N} \text{tr}(Z^k) \mu_{s,t}^N(dZ) = \left( e^{\tilde{D}_{s,t} + \frac{1}{N^2} \tilde{Z}_{s,t}}(v_k) \right) (1).
\]
The result now follows as in the justification of (4.14) from (4.13).

We now proceed with the proof of Theorem 1.16.

Proof of Theorem 1.16. We begin with the proof of (1.20). By the triangle inequality, it suffices to prove the theorem for polynomials of the form \( P(u; v) = u^kQ(v) \) for \( k \in \mathbb{Z} \) and \( Q \in \mathbb{C}[v] \). Therefore
\[
P(u; v) - \pi_{s-t}P(u; v) = u^k[Q(v) - \pi_{s-t}Q] = u^kR_{s-t}(v)
\]
where \( R_{s-t} = Q - \pi_{s-t}Q \). Note that \( \pi_{s-t}R_{s-t} = 0 \). Now, for \( Z \in GL_N \),
\[
\|P_N(Z) - (\pi_{s-t}P)_N(Z)\|_{M_N}^2 = \text{tr}(Z^k[R_{s-t}]N(Z)[R_{s-t}]N(Z)^*Z^{*k})
\]
where, in the case \( k = 0 \), we interpret \( v_{\ell}(0,0) = 1 \). We calculate the \( L^2(\mu_{s,t}^N) \)-norm of the function \( [P - \pi_{s-t}P]_N = [u^kR_{s-t}]_N \) using (1.6). Thus, using the intertwining formula (3.58) together with (4.16), we have
\[
\|P_N - (\pi_{s-t}P)_N\|_{L^2(\mu_{s,t}^N)}^2 = e^{\frac{1}{2}A_{s,t}} (\|P_N - (\pi_{s-t}P)_N\|_{M_N}^2) (I_N)
\]
Thus
\[
\|P_N - (\pi_{s-t}P)_N\|_{L^2(\mu_{s,t}^N)}^2 = \left( e^{\tilde{D}_{s,t} + \frac{1}{N^2} \tilde{Z}_{s,t}}(v_{\ell(k,k)})(R_{s-t})(R_{s-t}) \right) (1). \tag{1.17}
\]
Now, let \( n = \deg Q = \deg R_{s-t} \). Using the linear functional \( \varphi(R) = R(1) \) on \( \mathcal{W}_{2n} \), Lemma 4.1 then yields
\[
\left| \left( e^{\tilde{D}_{s,t} + \frac{1}{N^2} \tilde{Z}_{s,t}}(v_{\ell(k,k)})(R_{s-t})(R_{s-t}) \right) \right| (1) - \left( e^{\tilde{D}_{s,t}}(v_{\ell(k,k)})(R_{s-t})(R_{s-t}) \right) (1) = O\left( \frac{1}{N^2} \right). \tag{1.18}
\]
But, since \( \tilde{D}_{s,t} \) is a first-order differential operator acting on \( \mathcal{W}_{2n} \), \( e^{\tilde{D}_{s,t}} \) is an algebra homomorphism, and we have
\[
e^{\tilde{D}_{s,t}}(v_{\ell(k,k)})(R_{s-t})(R_{s-t}) = e^{\tilde{D}_{s,t}}v_{\ell(k,k)} \cdot e^{\tilde{D}_{s,t}}(R_{s-t})(R_{s-t}) = 0 \tag{1.19}
\]
since \( e^{\tilde{D}_{s,t}}(R_{s-t}) = \pi_{s-t}R_{s-t} = 0 \) by Lemma 4.3. Thus, (1.17) - (1.19) prove (1.20).

Note that \( \frac{1}{2}A_{s,t}^N = \frac{1}{2}A_{s,0}^N \), thus taking \( t = 0 \) in (4.18) and restricting the function to \( U_N \) also proves (1.19), concluding the proof.

4.2 Proof of Main Limit Theorem 1.11

Proof of Theorem 1.11. We define \( \mathcal{H}_{s,t} \) and \( \mathcal{K}_{s,t} \) by (1.25); evidently, these are linear maps on \( \mathbb{C}[u, u^{-1}] \). Let \( f \in \mathbb{C}[u, u^{-1}] \); then by the intertwining formula (1.23),
\[
e^{\frac{1}{2}A_{s,t}^N} f_N = [e^{\frac{1}{2}D_{s,t}^N} f]_N,
\]
where \( D_{s,t}^N \) is defined in (3.22).
The function on the right is a trace polynomial function of $U \in \mathbb{U}_N$ (with no $U^*$s), and therefore its analytic continuation to $\mathbb{G}_Λ$ is given by the same trace polynomial function in $Z \in \mathbb{G}_Λ$. Thus

$$[B^N_{s,t}f_N](Z) = [e^{\frac{1}{2}DN} f]_N(Z), \quad Z \in \mathbb{G}_Λ.$$  

Hence

$$\|B^N_{s,t}f_N - [g_{s,t}f]_N\|_{L^2(\mu^N_{s,t})} = \|e^{\frac{1}{2}DN} f - [\pi_{s-t} \circ e^{\frac{1}{2}D} f]_N\|_{L^2(\mu^N_{s,t})}.$$  

By the triangle inequality, the last quantity is

$$\leq \|e^{\frac{1}{2}DN} f\|_N - \|e^{\frac{1}{2}D} f\|_N \|\pi_{s-t} \circ e^{\frac{1}{2}D} f\|_N \leq \|e^{\frac{1}{2}DN} f\|_N - \|e^{\frac{1}{2}D} f\|_N \|\pi_{s-t} \circ e^{\frac{1}{2}D} f\|_N.$$  

The second term in (4.20) is $O(1/N)$ by (1.20) (Theorem 1.16). Thus, to complete the (existence) proof of (1.15), it suffices to show that

$$\|e^{\frac{1}{2}DN} f - e^{\frac{1}{2}D} f\|_N^2 = O \left( \frac{1}{N^2} \right) \quad (4.21)$$

for each $f \in \mathbb{C}_N[u, u^{-1}]$. Let $n = \deg f$, let $\mathcal{B}$ be the sesquilinear form in Lemma 3.28 and let $R^{(N)} = e^{\frac{1}{2}DN} f - e^{\frac{1}{2}D} f$. Then by (1.7) and (3.58), the left side of (4.21) is given by

$$\|\|R^{(N)}\|_{L^2(\mu^N_{s,t})}^2 = \left( \left( [R^{(N)}]_N \right)_{M_N} + \left[ \left( \tilde{D}_{s,t} + \frac{1}{N^2} \tilde{D}_{s,t} \mathcal{B}(R^{(N)}, R^{(N)}) \right) (1) \right] \right) \leq \frac{C}{N^2} \|\mathcal{B}(R^{(N)}, R^{(N)})\|_{\mathcal{W}_{2n}}. \quad (4.22)$$

Let $\psi(P) = e^{\tilde{D}_{s,t} P}$ (1), another linear functional on $\mathcal{W}_{2n}$; then

$$\|\|\psi\|_{2n} \|\mathcal{B}(R^{(N)}, R^{(N)})\|_{\mathcal{W}_{2n}}.$$  

This, in conjunction with (4.22) and (4.23), shows that

$$\|\|R^{(N)}\|_{L^2(\mu^N_{s,t})}^2 \leq \left( \left( \left[ \psi \right]_{2n} + \frac{C}{N^2} \right) \|\mathcal{B}(R^{(N)}, R^{(N)})\|_{\mathcal{W}_{2n}}. \quad (4.24)$$

Since $\mathcal{B} : \mathbb{C}_n[u, u^{-1}; v] \times \mathbb{C}_n[u, u^{-1}; v] \to \mathcal{W}_{2n}$ is sesquilinear with finite dimensional domain and range, it is bounded with any choice of norms; in particular, given any norm $\| \cdot \|_{\mathbb{C}_n[u, u^{-1}; v]}$ on $\mathbb{C}_n[u, u^{-1}; v]$, there is a constant $C'$ (depending on $n$ but not on $N$) so that

$$\|\mathcal{B}(P, Q)\|_{\mathcal{W}_{2n}} \leq C' \|P\|_{\mathbb{C}_n[u, u^{-1}; v]} \|Q\|_{\mathbb{C}_n[u, u^{-1}; v]} \quad \text{for all } P, Q \in \mathbb{C}_n[u, u^{-1}; v].$$

Together with (4.24), this yields

$$\|\|R^{(N)}\|_{L^2(\mu^N_{s,t})}^2 \leq C' \left( \left[ \psi \right]_{2n} + \frac{C}{N^2} \right) \|R^{(N)}\|_{\mathbb{C}_n[u, u^{-1}; v]}^2. \quad (4.25)$$

Finally, Lemma 4.1 gives

$$\|R^{(N)}\|_{\mathbb{C}_n[u, u^{-1}; v]} = \|e^{\frac{1}{2}(D - \frac{1}{N^2} C)} f - e^{\frac{1}{2}D} f\|_{\mathbb{C}_n[u, u^{-1}; v]} = O \left( \frac{1}{N^2} \right)$$

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which proves (4.21). (In fact it shows this term is $O(1/N^4)$; however, since the square of the second term in (4.20) is $O(1/N^2)$, this faster convergence doesn’t improve matters.)

The proof of (1.16) is similar: the restriction of $(B_{jN}^N)^{-1}f_N$ to $\cup N$ is simply $e^{-\frac{i}{2}D_{N}}f_N$, and a similar triangle inequality argument now using (1.19) shows that it suffices to prove

$$\|\{e^{-\frac{i}{2}D_{N}}f\}_N - \{e^{-\frac{i}{2}D_{N}}f\}_N\|_{L^2(\rho_N)} = O\left(\frac{1}{N^2}\right).$$

(4.26)

The argument now proceeds identically to above, by redefining $R^{(N)}$ with the substitution $t \mapsto -t$, and taking all norms with the substitution $(s, t) \mapsto (s, 0)$ in all formulas from (4.22) onward.

Thus, we have shown that, with $G_{s, t}$ and $H_{s, t}$ defined as in (1.25), (1.15) and (1.16) hold. We reserve the proof of uniqueness until Corollary 4.9 below.

4.3 Limit Norms and the Proof of Theorem 1.13

We begin by proving that the transforms $G_{s, t}$ and $H_{s, t}$ are invertible on $C[u, u^{-1}]$. (This will be subsumed by Theorem 1.13 but it will be useful to have this fact in the proof.)

Lemma 4.5. $G_{s, t}$ and $H_{s, t}$ are invertible operators on $C_n[u, u^{-1}]$ for each $n > 0$, and hence on $C[u, u^{-1}]$.

Proof. Consider $e^{\pm \frac{i}{2}D}$ restricted to $C_n[u, u^{-1}; \nu]$. Expanding as power-series, a straightforward induction using the forms of the composite operators $N, Z$, and $Y$ shows that there exist $q_{k; \nu} \in C[\nu]$ with

$$e^{\pm \frac{i}{2}D}u^n = e^{\mp \frac{i}{2}t}u^n + \sum_{k=0}^{n-1} q_{k; \nu}u^k,$$

$$e^{\pm \frac{i}{2}D}u^{-n} = e^{\mp \frac{i}{2}t}u^{-n} + \sum_{k=-n+1}^{0} q_{k; \nu}u^k.$$

This shows that $e^{\pm \frac{i}{2}D}$ preserves $C_n[u]$ and $C_n[u^{-1}]$. Incorporating the evaluation maps $\pi_s$ or $\pi_{s-t}$, we find that

$$G_{s, t}(u^{\pm n}), H_{s, t}(u^{\pm n}) \in e^{\mp \frac{i}{2}t}u^{\pm n} + C_{n-1}[u, u^{-1}]$$

Consider, then, the standard basis $\{1, u^1, \ldots, u^n\}$ of $C_n[u]$; it follows that, in this basis, $G_{s, t}|C_n[u]$ and $H_{s, t}|C_n[u]$ are upper-triangular, with diagonal entries $e^{\mp \frac{i}{2}t}$ for $0 \leq k \leq n$. Thus the restrictions of $G_{s, t}$ and $H_{s, t}$ to $C_n[u]$ are invertible. A similar argument shows the invertibility on $C_n[u^{-1}]$, thus yielding the result on $C_n[u, u^{-1}]$. Since $C[u, u^{-1}] = \bigcup_n C_n[u, u^{-1}]$, the proof is complete.

We now introduce two seminorms on $C[u, u^{-1}; \nu]$.

Definition 4.6. Let $s, t > 0$ with $s > t/2$. For each $N$, define the seminorms $\| \cdot \|_{s, N}$ and $\| \cdot \|_{s, t, N}$ on $C[u, u^{-1}; \nu]$ by

$$\|P\|_{s, N} = \|P_N\|_{L^2(\cup_{N-\rho}, \rho_N; \mathcal{M}_N)}$$

$$\|P\|_{s, t, N} = \|P_N\|_{L^2(G_{s, t} \rho_N; \mathcal{M}_N)}.$$
In fact, for any \( n > 0 \) and sufficiently large \( N \), seminorms (4.27) and (4.28) are actually norms when restricted to \( \mathbb{C}_n[u, u^{-1}; \nu] \). Indeed, if \( \|P\|_{s,N} = 0 \) then \( P_N = 0 \) in \( L^2(\mathbb{U}_N, \rho_N^N; \mathbb{M}_N) \), and since \( P_N \) is a smooth function and \( \rho_N^N \) has a strictly positive density, this means \( P_N \) is identically 0. By Proposition 2.10, when \( N \) is sufficiently large (relative to \( n \)) it follows that \( P = 0 \).

For \( P \in \mathbb{C}[u, u^{-1}; \nu] \), define

\[
\|P\|_s = \lim_{N \to \infty} \|P\|_{s,N} \quad \text{(4.29)}
\]

\[
\|P\|^{s,t}_N = \lim_{N \to \infty} \|P\|^{s,t,N}_N. \quad \text{(4.30)}
\]

These are also seminorms on \( \mathbb{C}[u, u^{-1}; \nu] \), but they are not norms on all of \( \mathbb{C}[u, u^{-1}; \nu] \), or even on \( \mathbb{C}_n[u, u^{-1}; \nu] \) for any \( n > 1 \). However, restricted to \( \mathbb{C}[u, u^{-1}] \), they are in fact norms. To prove this, we look to the measure \( \nu_s \) described following Theorem 1.14 the law of the free unitary Brownian motion at time \( s > 0 \). The measure \( \nu_s \) is the weak limit of \( \nu_s^N \) of (2.1) (which exists by the Lévy continuity theorem). In [4, Proposition 10], it is shown that \( \nu_s \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{U} \), with a continuous density that is strictly positive in a neighborhood of \( 1 \in \mathbb{U} \); we will need this result (in particular the fact that \( \text{supp}(\nu_s) \) is not a finite set) in the following.

**Lemma 4.7.** The seminorms (4.29) and (4.30) are norms on \( \mathbb{C}[u, u^{-1}] \).

**Proof.** We begin with norm (4.29). Identify the Laurent polynomial \( P \in \mathbb{C}[u, u^{-1}] \) as a trigonometric polynomial function \( P_1 \) on the unit circle \( \mathbb{U} \). Then (2.2) shows that

\[
\|P\|_{s,N} = \|P_N\|_{L^2(\mathbb{U}_N, \rho_N^N; \mathbb{M}_N)} = \|P_1\|_{L^2(\mathbb{U}, \nu_s^N)}.
\]

Thus, since \( \nu_s^N \to \nu_s \),

\[
\|P\|_s = \lim_{N \to \infty} \|P_1\|_{L^2(\mathbb{U}, \nu_s^N)} = \|P_1\|_{L^2(\mathbb{U}, \nu_s)}.
\]

Since the support of \( \nu_s \) is infinite, (4.31) shows that seminorm (4.29) is indeed a norm on \( \mathbb{C}[u, u^{-1}] \).

For seminorm (4.30), we will utilize the isometry property of the finite dimensional Segal–Bargmann transform \( B_{s,t}^N \). Fix \( Q \in \mathbb{C}[u, u^{-1}] \), and let \( \deg Q = n \). By Lemma 4.5, there is a unique Laurent polynomial \( P \in \mathbb{C}_n[u, u^{-1}] \) so that \( \mathcal{G}_{s,t}(P) = Q \). Thus

\[
\|Q\|^{s,t} = \|\mathcal{G}_{s,t}P\|^{s,t} = \lim_{N \to \infty} \|\mathcal{G}_{s,t}P\|^{s,t,N}.
\]

By Theorem 1.11 and (4.28) we have

\[
\lim_{N \to \infty} \|\mathcal{G}_{s,t}P\|^{s,t,N} = \lim_{N \to \infty} \||B_{s,t}^N P||_{L^2(\mathbb{U}_N, \rho_N^N; \mathbb{M}_N)} = \lim_{N \to \infty} \||B_{s,t}^N P||_{L^2(\mathbb{U}, \nu_s^N; \mathbb{M}_N)}
\]

and by the isometry property of the Segal–Bargmann transform, we therefore have

\[
\|Q\|^{s,t} = \lim_{N \to \infty} \|P_N\|_{L^2(\mathbb{U}_N, \rho_N^N; \mathbb{M}_N)} = \|P\|_s.
\]

Thus, if \( \|Q\|^{s,t} = 0 \) then \( \|P\|_s = 0 \), so \( Q = \mathcal{G}_{s,t}(0) = 0 \). This concludes the proof.

**Remark 4.8.** Eq. (4.31) shows that norm (4.29) is just an \( L^2 \) norm, with respect to a well-understood measure. Norm (4.30) is, at present, much more mysterious. In [4], a great deal of work is spent trying to understand this norm in the case \( s = t \). It can, in that case, be identified as the norm of a certain reproducing kernel Hilbert space, built out of holomorphic functions on a bounded region \( \Sigma_t \subset \mathbb{C}^* \) which has few obvious symmetries, and which becomes non-simply-connected when \( t \geq 4 \). Understanding the norm (4.30) in general is a goal for future research of the present authors.
Corollary 4.9. For $s,t > 0$ with $s > t/2$ and $f \in \mathbb{C}[u, u^{-1}]$, the only Laurent polynomials $g_{s,t}$ and $h_{s,t}$ satisfying (1.15) and (1.16) are $g_{s,t} = \mathcal{G}_{s,t} f$ and $h_{s,t} = \mathcal{H}_{s,t} f$ as defined in (1.25).

Proof. Suppose that $g_{s,t}, g'_{s,t} \in \mathbb{C}[u, u^{-1}]$ both satisfy
\[
\|B^N_{s,t} f_N - [g_{s,t}]_N\|^2_{L^2(GL_N, \mu^N_{s,t}; \mathcal{M}_N)} = O \left( \frac{1}{N^2} \right) = \|B^N_{s,t} f_N - [g'_{s,t}]_N\|^2_{L^2(GL_N, \mu^N_{s,t}; \mathcal{M}_N)}.
\]
Then, by the triangle inequality, it follows that $\|g_{s,t} - g'_{s,t}\|^2_{L^2(GL_N, \mu^N_{s,t}; \mathcal{M}_N)} = O(1/N^2)$. Taking limits as $N \to \infty$, it follows that $\|g_{s,t} - g'_{s,t}\|_{s,t} = 0$, and it follows from Lemma 4.7 that $g_{s,t} = g'_{s,t}$. A similar argument shows uniqueness of $h_{s,t}$. The result now follows from the proof of Theorem 1.11 on page 33.

This leads us to the proof of Theorem 1.13.

Proof of Theorem 1.13 Fix $P \in \mathbb{C}[u, u^{-1}]$, and consider the Laurent polynomial $\mathcal{G}_{s,t} \mathcal{H}_{s,t} P \in \mathbb{C}[u, u^{-1}]$. By definition
\[
\|\mathcal{G}_{s,t} \mathcal{H}_{s,t} P - P\|^s,t = \lim_{N \to \infty} \|\mathcal{G}_{s,t} \mathcal{H}_{s,t} P - P\|^{s,t,N} = \lim_{N \to \infty} \|\mathcal{G}_{s,t} \mathcal{H}_{s,t} P\|_N - P_N\|_{L^2(\mu^N_{s,t})}.
\]
(4.32)

The triangle inequality yields
\[
\|\mathcal{G}_{s,t} \mathcal{H}_{s,t} P\|_N - P_N\|_{L^2(\mu^N_{s,t})} \\
\leq \|\mathcal{G}_{s,t} \mathcal{H}_{s,t} P\|_N - B^N_{s,t}[\mathcal{H}_{s,t} P]_N\|_{L^2(\mu^N_{s,t})} + \|B^N_{s,t}[\mathcal{H}_{s,t} P]_N - P_N\|_{L^2(\mu^N_{s,t})}.
\]
Applying (1.15) with $f = \mathcal{H}_{s,t} P$ shows that the first term is $O(1/N)$. For the second term, we use the isometry property of the Segal–Bargmann transform. The trace polynomial $P_N$ is in the range of $B^N_{s,t}$, by Corollary 3.20 and so
\[
\|B^N_{s,t}[\mathcal{H}_{s,t} P]_N - P_N\|_{L^2(\mu^N_{s,t})} = \|B^N_{s,t} (\mathcal{H}_{s,t} P)_N - (B^N_{s,t})^{-1} P\|_{L^2(\mu^N_{s,t})} \\
= \| (\mathcal{H}_{s,t} P)_N - (B^N_{s,t})^{-1} P\|_{L^2(\mu^N_{s,t})} = O \left( \frac{1}{N} \right),
\]
by (1.16). Hence, the quantity in the limit on the right-hand-side of (4.32) is $O(1/N)$, so its limit is 0. Therefore we have $\|\mathcal{G}_{s,t} \mathcal{H}_{s,t} P - P\|_{s,t} = 0$. Lemma 4.7 shows that $\|\cdot\|_{s,t}$ is a norm on $\mathbb{C}[u, u^{-1}]$, and so it follows that $\mathcal{G}_{s,t} \mathcal{H}_{s,t} P - P = 0$. Hence, since $\mathcal{G}_{s,t}$ and $\mathcal{H}_{s,t}$ are known to be invertible (Lemma 4.5), it follows that $\mathcal{H}_{s,t} = \mathcal{G}_{s,t}^{-1}$ as desired.

5 The Free Unitary Segal–Bargmann Transform

In this final section, we identify the limit Segal–Bargmann transform $\mathcal{G}_{s,t}$, which has been constructed as a linear operator on the space $\mathbb{C}[u, u^{-1}]$ of single-variable Laurent polynomials. We will characterize the Biane polynomials for $\mathcal{G}_{s,t}$:
\[
p^s,t_k = \mathcal{H}_{s,t}(\cdot)^k = \pi_s \circ e^{-\frac{1}{2}D}(\cdot)^k, \quad k \in \mathbb{Z}
\]
defined so that
\[
\mathcal{G}_{s,t}(p^s,t_k)(z) = z^k
\]
when $s,t > 0$ and $s > t/2$. We call them Biane polynomials since, as we will prove, they match the polynomials that Biane introduced in [4] Lemma 18] to characterize the free Hall transform $\mathcal{G}$, in the special case $s = t$. 37
There is classical motivation to understand these polynomials. Consider the classical Segal–Bargmann transform $S_t$ acting on $L^2(\mathbb{R}, \gamma^1_t)$. Since polynomials are dense in this Gaussian $L^2$-space, $S_t$ is completely determined by the polynomials $H_k(t, \cdot)$ satisfying $S_t(H_k(t, \cdot))(z) = z^k$. In this case, since the measure $\gamma^1_{t/2}$ is rotationally-invariant, the monomials $z \mapsto z^k$ are orthogonal, and since $S_1^1$ is an isometry, it follows that $H_k(t, \cdot)$ are the orthogonal polynomials of the Gaussian measure $\gamma^1_t$: the Hermite polynomials. We will determine the generating function II of these polynomials; cf. [1,21]. In the case $s = t$, this precisely matches the generating function in [4, Lemma 18], modulo a small correction; in this way, we verify that our limit Segal–Bargmann transform is the aforementioned free unitary Segal–Bargmann transform $G^s_t$.

Before proceeding, we make an observation. It is immediate from the form of the operators $D_{s,t}$ and $C_k$, acting on $C[u;\nu]$, and $C[u^{-1};\nu]$, and hence $p_k^{s,t}(u)$ is a polynomial in $u$ for $k \geq 0$, while $p_k^{s,t}(u)$ is a polynomial in $u^{-1}$ for $k < 0$. Hence, since $p_0^{s,t} = 1$, it will suffice to identify $p_k^{s,t}$ only for $k \geq 1$.

### 5.1 Biane Polynomials and Differential Recursion

It will be convenient to look at the related family of “unevaluated” polynomials.

**Definition 5.1.** For $t \in \mathbb{R}$ and $k \in \mathbb{N}$, define $B_k^t \in C[u, u^{-1}; v]$ and $C_k^t \in C[v]$ by

$$B_k^t(u; v) = e^{-\frac{t}{2}u}e^{-\frac{t}{2}D_1}u^k \quad \text{and} \quad C_k^t(v) = \mathcal{T}(B_k^t)(v),$$

where $\mathcal{T}$ is the tracing map of (3.35). For $s \in \mathbb{R}$, define $b_k(s, t, \cdot) \in C[u, u^{-1}]$ and $c_k(s, t) \in C$ by

$$b_k(s, t, u) = \pi_s(B_k^t)(u) \quad \text{and} \quad c_k(s, t) = \pi_s(C_k^t).$$

Note, by (5.7) and the linearity of $\pi_s$, that

$$b_k(s, t, u) = e^{-\frac{t}{2}u}p_k^{s,t}(u).$$

It is useful to note the following alternative expression for $c_k(s, t)$. From (5.3),

$$C_k^t(v) = e^{-\frac{t}{2}v}\mathcal{T}(e^{-\frac{t}{2}D_1}u^k) = e^{-\frac{t}{2}v}e^{-\frac{t}{2}D_1}v_k$$

since, by Lemma 3.17, $\mathcal{T}$ commutes with $D$. Thus, from Theorem 4.2, we have

$$c_k(s, t) = e^{-\frac{t}{2}v}\pi_s(e^{-\frac{t}{2}D_1}v_k) = e^{-\frac{t}{2}v}\left(e^{\frac{1}{2}(s-t)D_1}v_k\right)_{v=1} = e^{-\frac{t}{2}v}v_k(s - t).$$

The main computational tool that will lead to the identification of the Biane polynomials $p_k^{s,t}$ is the following recursion.
Proposition 5.2. Let \( s, t \in \mathbb{R}, u \in \mathbb{C} \), and \( k \geq 1 \). Let \( c_k(s, t) \) and \( b_k(s, t, u) \) be given as in Definition 5.7. Then
\[
c_k(s, t) = \nu_k(s) + \sum_{m=1}^{k-1} \int_0^t m c_{k-m}(s, \tau) c_m(s, \tau) \, d\tau, \quad k \geq 2
\] (5.8)
with \( c_1(s, t) = \nu_1(s) \); and
\[
b_k(s, t, u) = u^k + \sum_{m=1}^{k-1} \int_0^t m c_{k-m}(s, \tau) b_m(s, \tau, u) \, d\tau, \quad k \geq 2
\] (5.9)
with \( b_1(s, t, u) = u \).

Proof. First note that \( B^0_k(u; \nu) = u^k \) and \( C^0_k(\nu) = v_k \) by definition, and thus \( b_k(s, 0, u) = \pi_s(u^k) = u^k \), while \( c_k(s, 0) = \pi_s(v_k) = \nu_k(s) \). For \( k = 1 \), we have
\[
B^1_1(u) = e^{\frac{t}{2}} e^{-\frac{t}{2}D} u = u
\]
because \( D u = -u \). For \( k \geq 2 \),
\[
\frac{d}{dt} B_k^t = \frac{d}{dt} e^{\frac{t}{2} e^{-\frac{t}{2}D} u^k} = \frac{1}{2} e^{\frac{t}{2} e^{-\frac{t}{2}D} (k + D) u^k}.
\]
Recall (5.21) that \( D = -N - 2Z - 2y \). Eq. (5.24) shows that \( N(u^k) = k u^k \); (5.26) shows that \( Z \) annihilates \( u^k \); and Example 3.10 works out that \( y(u^k) = \sum_{j=1}^{k-1} (k-j) v_j u^{k-j} = \sum_{m=1}^{k-1} m u^m v_{k-m} \). Thus
\[
(k + D)(u^k) = k u^k - k u^k - 2 \sum_{m=1}^{k-1} m u^m v_{k-m} = -2 \sum_{m=1}^{k-1} m u^m v_{k-m}.
\]
Hence
\[
\frac{d}{dt} B_k^t = e^{-\frac{k}{2} t} e^{-\frac{t}{2}D} \left( \sum_{m=1}^{k-1} m u^m v_{k-m} \right) = e^{-\frac{k}{2} t} \sum_{m=1}^{k-1} m e^{-\frac{t}{2}D}(u^m v_{k-m}).
\]
(5.10)
We now use the partial homomorphism property of (1.27) at time \(-t\), which yields
\[
e^{-\frac{t}{2}D}(u^m v_{k-m}) = (e^{-\frac{t}{2}D} u^m)(e^{-\frac{t}{2}D} v_{k-m}).
\]
(5.11)
Now, \( v_{k-m} = \mathcal{J}(u^{k-m}) \), and, by Lemma 3.17 \( \mathcal{J} \) and \( D \) commute. We may therefore rewrite (5.11) as
\[
e^{-\frac{t}{2}D}(u^m v_{k-m}) = (e^{-\frac{t}{2}D} u^m) \mathcal{J}(e^{-\frac{t}{2}D} u^{k-m})
\]
(5.12)
Eq. (5.3) gives
\[
e^{-\frac{t}{2}D}(\cdot)^m = e^{\frac{m}{2}t} B_m^t \quad \text{and} \quad \mathcal{J}[e^{-\frac{t}{2}D}(\cdot)^{k-m}] = e^{\frac{k-m}{2}t} C_{k-m}^t.
\]
Thus, (5.10) and (5.12) combine to give
\[
\frac{d}{dt} B_k^t = e^{-\frac{k}{2} t} \sum_{m=1}^{k-1} m e^{\frac{m}{2}t} B_m^t e^{\frac{k-m}{2}t} C_{k-m}^t = \sum_{m=1}^{k-1} m C_{k-m}^t B_m^t.
\]
(5.13)
Integrating both sides of (5.13) from \( 0 \) to \( t \), and using the initial condition \( B_k^t(u; \nu) = u^k \), gives
\[
B_k^t = u^k + \sum_{m=1}^{k-1} m \int_0^t C_{k-m}^\tau B_m^\tau \, d\tau.
\]
(5.14)
The tracing map $\mathcal{T}$ is linear, and commutes with the integral (easily verified since all terms are polynomials); moreover, if $C \in \mathbb{C}[v]$, then $\mathcal{T}(CB) = C\mathcal{T}(B)$. Thus
\[
C'_k = \mathcal{T}(B'_k) = \mathcal{T}(u^k) + \sum_{m=1}^{k-1} m \int_0^t \mathcal{T}[C_{k-m}^\tau B^\tau_m] \, d\tau = v_k + \sum_{m=1}^{k-1} m \int_0^t C_{k-m}^\tau C^\tau_m \, d\tau. \tag{5.15}
\]
Finally, the evaluation map $\pi_s$ is an algebra homomorphism, and (as with $\mathcal{T}$) commutes with the integral; applying $\pi_s$ to (5.14) and (5.15) yields the desired equations (5.8) and (5.9), concluding the proof. 

Remark 5.3. By changing the index $m \mapsto k-m$ in (5.8) and averaging the results, we may alternatively state the recursion for $c_k$ as
\[
c_k(s, t) = \nu_k(s) + \frac{1}{2} \sum_{m=1}^{k-1} \int_0^t c_{k-m}(s, \tau) c_{m}(s, \tau) \, d\tau. \tag{5.16}
\]
A transformation of this form is not possible for the $b_k(s, t, u)$ recursion, however.

5.2 Exponential Growth Bounds
In Section 5.3, we will study the generating functions of the quantities $\nu_k(s)$, $c_k(s, t)$, and $b_k(s, t, u)$. As such, we will need a priori exponential growth bounds.

Lemma 5.4. For $s, t \in \mathbb{R}$ and $k \geq 2$,
\[
|\nu_k(t)| \leq C_{k-1}(1 + |t|)^{k-1}e^{-\frac{k}{2}t}, \quad \text{and} \quad |c_k(s, t)| \leq C_{k-1}(1 + |s-t|)^{k-1}e^{-\frac{k}{2}s}; \tag{5.17}
\]
where $C_k = \frac{1}{k+1}(\frac{2k}{k})$ are the Catalan numbers.

Remark 5.5. When $t > 0$, $\nu_k(t)$ is the $k$th moment of the probability measure $\nu_t$ on the unit circle $U$, and we therefore have the much better bound $|\nu_k(t)| \leq 1$; similarly, if $s \geq t$, $|c_k(s, t)| \leq e^{-\frac{k}{2}s}$. It is necessary to have a priori bounds for negative $t$ and $s-t$ as well, however. While (5.17) is by no means sharp, the known exact formula (1.17) for $\nu_k(t)$ shows that, when $t < 0$, $\nu_k(t)$ does grow exponentially with $k$ (at least for small $|t|$).

In the proof of Lemma 5.4, we will use the well-known fact that the Catalan numbers satisfy Segner’s recurrence relation
\[
C_k = \sum_{m=1}^{k} C_{m-1}C_{k-m}, \quad k \geq 1.
\]

Proof. Taking $s = 0$ in (5.16), and noting that $\nu_k(0) = 1$ for all $k$, we have
\[
c_k(0, t) = 1 + \frac{1}{2} \sum_{m=1}^{k-1} \int_0^t c_{m}(0, \tau) c_{k-m}(0, \tau) \, d\tau, \quad k \geq 2. \tag{5.19}
\]
We claim that
\[
|c_k(0, t)| \leq C_{k-1}(1 + |t|)^{k-1}, \quad k \geq 1. \tag{5.20}
\]
Since \( c_1(0, t) = 1 = C_1 \), we proceed by induction. Let \( k \geq 2 \), and assume that (5.20) holds below level \( k \); then (5.19) yields

\[
|c_k(0, t)| \leq 1 + \frac{k}{2} \int_0^{|t|} \sum_{m=1}^{k-1} C_{m-1} C_{k-m-1} (1 + \tau)^{k-2} d\tau
\]

\[
= 1 + \frac{k}{2(k-1)} \left( (1 + |t|)^{k-1} - 1 \right) \sum_{m=1}^{k-1} C_{m-1} C_{k-m-1}
\]

\[
= 1 - \frac{k}{2(k-1)} C_{k-1} + (1 + |t|)^{k-1} C_{k-1} \leq C_{k-1} (1 + |t|)^{k-1}
\]  

wherein we have used \( \frac{k}{2(k-1)} C_{k-1} \geq 1 \) for all \( k \geq 2 \). This completes the induction argument, proving (5.20) holds. Now, taking \( s = 0 \) in (5.7) yields

\[
c_k(0, t) = e^{-\frac{k}{2}t} \nu_k(-t)
\]  

meaning that \( \nu_k(t) = e^{-\frac{k}{2}t} C_k(0, -t) \), and this together with (5.20) proves (5.17). Then, using (5.7) once more, (5.17) implies that

\[
|c_k(s, t)| = e^{-\frac{k}{2}t} |\nu_k(s - t)| \leq e^{-\frac{k}{2}t} e^{-\frac{k}{2}(s-t)} \cdot C_{k-1}(1 + |s - t|)^{k-1}
\]

which prove (5.18).

**Remark 5.6.** Equations (5.19) and (5.22) together yield a recursion for the coefficients \( \theta_k(t) = e^{\frac{k}{2}t} \nu_k(t) = c_k(0, -t) \):

\[
\theta_k(t) = 1 - \frac{k}{2} \sum_{m=1}^{k-1} \int_0^t \theta_m(\tau) \theta_{k-m}(\tau) d\tau.
\]

This same recursion was derived in [4], Lemma 11, using free stochastic calculus, with \( \nu_k(s) \) being identified as the limit moments of the free unitary Brownian motion distribution. It is interesting that we can derive it directly from derivative formulas on the unitary group.

**Lemma 5.7.** Let \( s, t \in \mathbb{R} \) and \( u \in \mathbb{C} \). For \( k \geq 2 \), the \( b_k(s, t, u) \) of (5.9) satisfy

\[
|b_k(s, t, u)| \leq [5(1 + |s|)(1 + |t|)]^{k-1} |u|^k.
\]

**Proof.** Since \( b_1(s, t, u) = u \), (5.24) holds for \( k = 1 \). We proceed by induction, assuming (5.24) holds below level \( k \). Then (5.9) gives us

\[
|b_k(s, t, u)| \leq |u|^k + \sum_{m=1}^{k-1} \int_0^{|t|} m |c_{k-m}(s, \tau)||b_m(s, \tau, u)| d\tau.
\]

The Catalan number \( C_k \) is \( \leq 4^k \) (in fact it is \( \sim 4^k / k^{3/2} \sqrt{\pi} \)). Using the estimate \( 1 + |s - t| \leq (1 + |s|)(1 + |t|) \), (5.18) implies that \( |c_k(s, t)| \leq [4(1 + |s|)(1 + |t|)]^{k-1} \). Thus (5.25) and the inductive hypothesis give us, for \( k \geq 2 \),

\[
|b_k(s, t, u)| \leq |u|^k + \sum_{m=1}^{k-1} \int_0^{|t|} m [4(1 + |s|)(1 + \tau)]^{k-m-1} \cdot [5(1 + |s|)(1 + \tau)]^{m-1} |u|^k d\tau
\]

\[
= |u|^k + |u|^k \cdot (1 + |s|)^{k-2} \int_0^{|t|} (1 + \tau)^{k-2} d\tau \cdot \sum_{m=1}^{k-1} m 4^{k-m-1} 5^{m-1}.
\]
Summing the geometric series, we may estimate
\[5^{k-1} - 4^{k-1} \leq \sum_{m=1}^{k-1} m 4^{k-m-1} 5^{m-1} \leq (k-1) 5^{k-1}.\]
Substituting this into (5.26) we have
\[|b_k(s, t, u)| \leq |u|^k + |u|^k (1 + s)|k-2[(1 + |t|)^{k-1} - 1] \frac{1}{k-1} \sum_{m=1}^{k-1} m 4^{k-m-1} 5^{m-1}
\leq |u|^k \left(1 - (1 + s)|^{k-2} \frac{5^{k-1} - 4^{k-1}}{k-1}\right) + 5^{k-1} (1 + |s|)k-2(1 + |t|)^{k-1}|u|^k
\leq 5(1 + |s|)(1 + |t|)|^{k-1}|u|^k\]
where we have used that \(1 + |s| \geq 1\) and \(\frac{5^{k-1} - 4^{k-1}}{k-1} \geq 1\) for \(k \geq 2\). This concludes the inductive proof. \(\square\)

### 5.3 Holomorphic PDE

The double recursion of Proposition 5.2 can be written in the form of coupled holomorphic PDEs for the generating functions of \(c_k(s, t)\) and \(b_k(s, t, u)\).

**Definition 5.8.** Let \(s, t \in \mathbb{R}\). For \(z \in \mathbb{C}\), define
\[
\psi^s(t, z) = \sum_{k=1}^{\infty} c_k(s, t) z^k.
\]
Additionally, for \(u \in \mathbb{C}\) define
\[
\phi^{s,u}(t, z) = \sum_{k=1}^{\infty} b_k(s, t, u) z^k.
\]

By (5.18) and the Catalan bound \(C_k \leq 4^k\), the power series \(z \mapsto \psi^s(t, z)\) is convergent whenever \(|z| < e^{s/2}/(1 + |s - t|)\); similarly, by (5.24), the power series \(z \mapsto \phi^{s,u}(t, z)\) is convergent whenever \(|z| < 5(1 + |s|)(1 + |t|)|u|^{-1}\). Hence, \(\psi^s(t, \cdot)\) and \(\phi^{s,u}(t, \cdot)\) are holomorphic on a nontrivial disk with radius that depends continuously on \(s, t\). Note that, by (5.3),
\[
\Pi(s, t, u, z) = \sum_{k \geq 1} p_{k,t}^s(u) z^k = \sum_{k \geq 1} e^{-t} b_k(s, t, u) z^k = \phi^{s,u}(t, e^t z). \tag{5.27}
\]
So, identifying \(\phi^{s,u}(t, z)\) will also identify the sought-after generating function \(\Pi(s, t, u, z)\).

**Proposition 5.9.** For fixed \(s \in \mathbb{R}\), the functions \(\mathbb{R} \ni t \mapsto \psi^s(t, z)\) and \(\mathbb{R} \ni t \mapsto \phi^{s,u}(t, z)\) are differentiable for all sufficiently small \(|z|\) and \(|u|\). Their derivatives are given by
\[
\frac{\partial}{\partial t} \psi^s(t, z) = \sum_{k=1}^{\infty} \frac{\partial}{\partial t} c_k(s, t) z^k \quad \text{and} \quad \frac{\partial}{\partial t} \phi^{s,u}(t, z) = \sum_{k=1}^{\infty} \frac{\partial}{\partial t} b_k(s, t, u) z^k.
\]

**Proof.** From (5.8), \(\frac{\partial}{\partial t} c_1(s, t) = 0\), while for \(k \geq 2\) we have
\[
\frac{\partial}{\partial t} c_k(s, t) = k \sum_{m=1}^{k-1} \frac{c_k(s, t) c_m(s, t)}{m}.
\]
Thus, from Proposition 5.2, we have

\[ \left| \frac{\partial}{\partial t} c_k(s, t) \right| \leq \sum_{m=1}^{k-1} m|c_{k-m}(s, t)||c_m(s, t)| \leq (k-1)4^k e^{-\frac{k}{2}s}(1 + |s - t|)^k \]

for \( k \geq 2 \). It follows that \( \sum_{k=1}^{\infty} \frac{\partial}{\partial t} c_k(s, t) z^k \) converges to an analytic function of \( z \) on the domain \( |z| < e^{s^2/4(1 + |s - t|)} \). Integrating this series term-by-term over the interval \([0, t]\) shows that it is the derivative of \( \psi^s(t, z) \), as claimed. A completely analogous argument applies to \( \phi^{s,u}(t, z) \).

We will shortly write down coupled PDEs satisfies by \( \psi^s \) and \( \phi^{s,u} \). First, we remark on their initial conditions. From Proposition 5.2 we have

\[ c_k(s, 0) = \nu_k(s) \quad \text{and} \quad b_k(s, 0, u) = u^k. \]

Thus

\[ \psi^s(0, z) = \sum_{k \geq 1} \nu_k(s) z^k, \quad (5.28) \]

\[ \phi^{s,u}(0, z) = \sum_{k \geq 1} u^k z^k = \frac{uz}{1 - uz}. \quad (5.29) \]

It will be convenient to express \( \psi^s(0, z) \) in terms of the shifted coefficients \( g_k(s) = e^{kz} \nu_k(s) \) considered in Remark 5.6. Define

\[ g(s, z) = \sum_{k \geq 1} g_k(s) z^k = \psi^s(0, e^{-\hat{z}} z). \quad (5.30) \]

Note that, since \( \nu_k(0) = 1 \) for all \( k \), \( g(0, z) = \frac{z}{1 - z} \).

**Proposition 5.10.** For \( s, t \in \mathbb{R} \) and \( |z| \) and \( |u| \) sufficiently small, the functions \( g, \psi^s \), and \( \phi^{s,u} \) satisfy the following holomorphic PDEs:

\[ \frac{\partial g}{\partial s} = -z \frac{\partial g}{\partial z}, \quad g(0, z) = \frac{z}{1 - z}, \quad (5.31) \]

\[ \frac{\partial \psi^s}{\partial t} = z \psi^s \frac{\partial \psi^s}{\partial z}, \quad \psi^s(0, z) = g(s, e^{-\hat{z}} z), \quad (5.32) \]

\[ \frac{\partial \phi^{s,u}}{\partial t} = z \psi^s \frac{\partial \phi^{s,u}}{\partial z}, \quad \phi^{s,u}(0, z) = \frac{uz}{1 - uz}. \quad (5.33) \]

**Remark 5.11.** (1) PDE (5.31) was proved in [3, Lemma 1], using the recursion (5.23). We reprove it here, as a special case of (5.32).

(2) It is unusual that nonlinear PDEs with given “initial” conditions should have well-defined solutions for time flowing forwards or backwards. In fact, this is the case presently. In terms of (5.31), this is indicative of the fact that the measure \( \nu_s \) exists for all \( s \in \mathbb{R} \); although it becomes singular at \( s = 0 \), it is well-behaved for \( s > 0 \) and \( s < 0 \); see [21, Proposition 2.24] for a summary of known results about \( \nu_s \).

**Proof.** First, Remark 5.6 and (5.22) show that \( g_k(t) = c_k(0, -t) \), and hence \( g(t, z) = \phi^0(-t, z) \). Hence, (5.31) follows immediately from (5.32). Now, Proposition 5.9 yields that \( \psi^s(t, z) \) is differentiable in \( t \), and so by Proposition 5.2

\[ \frac{\partial}{\partial t} \psi^s(t, z) = \sum_{k=2}^{\infty} \frac{\partial}{\partial t} c_k(s, t) z^k = \sum_{k=2}^{\infty} \sum_{m=1}^{k-1} mc_m(s, t)c_{k-m}(s, t) z^k. \quad (5.34) \]
On the other hand, \( \psi^s(t, z) \) is analytic in \( z \), and
\[
z \frac{\partial}{\partial z} \psi^s(t, z) = \sum_{k=1}^{\infty} c_k(s, t) \cdot z \frac{\partial}{\partial z} z^k = \sum_{k=1}^{\infty} k c_k(s, t) z^k,
\]
and so
\[
z \psi^s(t, z) \frac{\partial}{\partial z} \psi^s(t, z) = \sum_{k_1=1}^{\infty} c_{k_1}(s, t) z^{k_1} \cdot \sum_{k_2=1}^{\infty} k_2 c_{k_2}(s, t) z^{k_2}
\]
\[
= \sum_{k=2}^{\infty} z^k \sum_{\substack{k_1 + k_2 = k \ k_1, k_2 \geq 1}} k_2 c_{k_1}(s, t) c_{k_2}(s, t).
\]
Reindexing the internal sum and comparing with (5.34) proves (5.32). The proof of (5.33) is entirely analogous. \( \square \)

### 5.4 Generating Function

We now proceed to prove the implicit formula (1.21), by solving the coupled PDEs (5.31) – (5.33). We do this essentially by the method of characteristics. These quasilinear PDEs have a fairly simple form; as a result, the characteristic curves are the same as the level curves in this case. As we will see, all three equations have the same level curves.

**Lemma 5.12.** Fix \( s_0 \geq 0 \) and \( w_0 \in \mathbb{C} \) with \( |w_0| < [4(1 + s_0)]^{-1} \). Consider the exponential curve
\[
w(s) = w_0 e^{\varrho(0, w_0) s}.
\]
Then \( s \mapsto \varrho(s, w(s)) \) is constant. In particular, \( \varrho(s, w(s)) = \varrho(0, w_0) \) for all \( s \in [0, s_0) \).

**Proof.** Lemma 5.4 shows that \( e^{\frac{s}{2} \nu_k(s)} \leq [4(1 + s)]^k \); thus
\[
\varrho(s, w) = \psi \nu^s(e^s w) = \sum_{k \geq 1} e^{\frac{k}{2} s} \nu_k(s) w^k
\]
converges to an analytic function of \( w \) for \( |w| < [4(1 + s)]^{-1} \). Thus, since \( s \mapsto [4(1 + s)]^{-1} \) is decreasing, \( \varrho(s, w) \) is differentiable in \( s \) and analytic in \( w \) for \( |w| < [4(1 + s_0)]^{-1} \) and \( 0 \leq s < s_0 \). Since \( 4(1 + s_0) > 1 \), the initial condition \( \varrho(0, w) = \frac{w}{1 - w} \) is also analytic on this domain. Thus, subject to these constraints, we can simply differentiate. To avoid confusion, we denote \( \dot{\varrho}(s, w) = \frac{\partial \varrho}{\partial s}(s, w) \) and \( \varrho'(s, w) = \frac{\partial \varrho}{\partial w}(s, w) \). Thus
\[
\frac{d}{ds} \varrho(s, w(s)) = \dot{\varrho}(s, w(s)) + \varrho'(s, w(s)) \dot{w}(s).
\] (5.35)
We now use (5.31), which asserts that \( \dot{\varrho}(s, w) = -w \varrho(s, w) \varrho'(s, w) \); hence
\[
\dot{\varrho}(s, w(s)) = -w(s) \varrho(s, w(s)) \varrho'(s, w(s)).
\]
Plugging this into (5.35) yields
\[
\frac{d}{ds} \varrho(s, w(s)) = \varrho'(s, w(s)) \left[-w(s) \varrho(s, w(s)) + \dot{w}(s) \right].
\] (5.36)
Note that \( w \) satisfies the ODE
\[
\dot{w}(s) = \frac{d}{ds} w_0 e^{\rho(0,w_0)s} = \rho(0,w_0) w_0 e^{\rho(0,w_0)s} = \rho(0,w_0) w(s).
\]
Substituting this into (5.35) yields
\[
\frac{d}{ds} \rho(s, w(s)) = \rho'(s, w(s)) w(s) \left[ \rho(0,w_0) - \rho(s, w(s)) \right],
\]
(5.37)
\[
\rho(s, w(s)) \big|_{s=0} = \rho(0,w_0).
\]
We now easily see that \( \rho(s, w(s)) \equiv \rho(0,w_0) = \frac{w_0}{1-w_0} \) is indeed the (unique) solution to this ODE. \( \Box \)

**Corollary 5.13.** Subject to the constraints on \( s, w \) in Lemma 5.12, the function \( \psi^s(0, w) = \rho(s, e^{-\frac{s}{2}} w) \) is constant along the curves \( s \mapsto e^{\frac{s}{2}} w(s) = w_0 e^{[\rho(0,w_0) + \frac{s}{2}] s} \). Note that
\[
\rho(0,w_0) + \frac{1}{2} = \frac{w_0}{1-w_0} + \frac{1}{2} = \frac{1+45}{2} w.
\]
Thus, for all sufficiently small \( \psi \) and \( s \),
\[
\psi^s(0, w e^{\frac{s}{2} \frac{1+w}{1-w}}) = \psi(0,w) = \rho(0,w) = \frac{w}{1-w}.
\]
Differentiation shows that the function \( w \mapsto w e^{\frac{s}{2} \frac{1+w}{1-w}} \) is strictly increasing for all \( w \in \mathbb{R} \) (provided \( s < 4 \)); and in general for all \( w > 0 \) for all \( s \); hence, (5.38) actually uniquely determines \( \psi^s(0, z) \) for \( z \) (by analytic continuation) when \( s < 4 \); moreover, by the inverse function theorem, it is analytic in \( z \).

Following the idea of Lemma 5.12 we now show that the level-curves of the functions \( \psi^s \) and \( \phi^{s,u} \) are also exponentials.

**Lemma 5.14.** For \( z_0 \in \mathbb{C} \), consider the exponential curve
\[
z(t) = z_0 e^{-\psi^s(0,z_0)t}.
\]
Then for \( z_0 \) and \( t \) sufficiently small, \( t \mapsto \psi^s(t, z(t)) \) and \( t \mapsto \phi^{s,u}(t, z(t)) \) are constant. In particular,
\[
\psi^s(t, z(t)) = \psi^s(0, z_0), \quad \text{and} \quad \phi^{s,u}(t, z(t)) = \phi^{s,u}(0, z_0).
\]
**Proof:** To improve readability, through this proof we suppress the parameters \( s, u \) and simply write \( \phi^{s,u}(t, z) = \phi(t, z) \) and \( \psi^s(t, z) = \psi(t, z) \). As per the discussion following Definition 5.8 these functions are differentiable in \( t \) and analytic in \( z \) for sufficiently small \( z \). As in the proof of Lemma 5.12 we set \( \dot{\psi}(t, z) = \frac{\partial}{\partial t} \psi(t, z) \), and \( \psi'(t, z) = \frac{\partial}{\partial z} \psi(t, z) \), and similarly with \( \dot{\phi} \) and \( \phi' \). Differentiating, we have
\[
\frac{d}{dt} \dot{\psi}(t, z(t)) = \dot{\psi}(t, z(t)) + \psi'(t, z(t)) \dot{z}(t),
\]
\[
\frac{d}{dt} \dot{\phi}(t, z(t)) = \dot{\phi}(t, z(t)) + \phi'(t, z(t)) \dot{z}(t).
\]
PDEs (5.32) and (5.33) say \( \dot{\psi}(t, z) = z \psi(t, z) \psi'(t, z) \) and \( \dot{\phi}(t, z) = z \psi(t, z) \psi'(t, z) \), and so
\[
\frac{d}{dt} \psi(t, z(t)) = [z(t) \psi(t, z(t)) + \dot{z}(t)] \psi'(t, z(t)),
\]
(5.39)
\[
\frac{d}{dt} \phi(t, z(t)) = [z(t) \psi(t, z(t)) + \dot{z}(t)] \phi'(t, z(t)).
\]
(5.40)
As in the proof of Lemma 5.12 we note that \( z \) satisfies the ODE
\[
\dot{z}(t) = -z_0 \psi(0, z_0) e^{-\psi(0, z_0) t} = -\psi(0, z_0) z(t).
\]
Substituting this into (5.39) and (5.40) yields
\[
\frac{d}{dt} \psi(t, z(t)) = \left[ \psi(t, z(t)) - \psi(0, z_0) \right] z(t) \psi'(t, z(t)), 
\]
(5.41)
\[
\frac{d}{dt} \phi(t, z(t)) = \left[ \psi(t, z(t)) - \psi(0, z_0) \right] z(t) \phi'(t, z(t)).
\]
(5.42)
The initial condition for (5.41) is \( \psi(t, z(t)) \big|_{t=0} = \psi(0, z_0) \), and it follows immediately that \( \psi(t, z(t)) = \psi(0, z_0) \) is the unique solution of this ODE. Hence, (5.42) reduces to the equation \( \frac{d}{dt} \phi(t, z(t)) = 0 \), and since its initial condition is \( \phi(t, z(t)) \big|_{t=0} = \phi(0, z_0) \), it follows that \( \phi(t, z(t)) = \phi(0, z_0) \) as well.

This brings us to the proof of (1.21). First, Lemma 5.14, together with the initial condition in (5.33), yields
\[
\phi^{s,u}(t, ze^{-\psi^{s}(0,z)t}) = \phi^{s,u}(0, z) = \frac{u z}{1 - u z} = \frac{1}{1 - u z} - 1.
\]
(5.43)
Next, Corollary 5.13 describes \( (s, z) \mapsto \psi^{s}(0, z) \) in terms of its level curves; (5.38) states that
\[
\psi^{s}(0, we^{\frac{s}{2}}) = \varrho(0, w) = \frac{w}{1 - w}.
\]
(5.44)
So set \( z = we^{\frac{s}{2}} \); then (5.43) and (5.44) say
\[
\phi^{s,u}(t, e^{-\varrho^{s}(0,z)t}we^{\frac{s}{2}}) = \phi^{s,u}(t, e^{-\psi^{s}(0,z)t}z) = \left( 1 - uwe^{\frac{s}{2}} \right)^{-1} - 1.
\]
(5.45)
Finally, note that
\[
-\frac{w}{1 - w} = -\frac{1}{2} \frac{1 + w}{1 - w} + \frac{1}{2}
\]
and so (5.45) may be written in the form
\[
\phi^{s,u}(t, e^{\frac{t}{2}we^{\frac{s}{2}}(s-t)}}{1 - w}) = \left( 1 - uwe^{\frac{s}{2}}(s-t)}{1 - w} \right)^{-1} - 1.
\]
(5.46)
Finally, recall (5.27), which says that
\[
\Pi(s, t, u, \zeta) = \phi^{s,u}(t, e^{\frac{t}{2}} \zeta).
\]
(5.47)
Setting \( \zeta = we^{\frac{t}{2}(s-t)}{1 - w} \), (5.46) and (5.47) combine to yield
\[
\left( 1 - uwe^{\frac{s}{2}}(s-t)}{1 - w} \right)^{-1} - 1 = \phi^{s,u}(t, e^{\frac{t}{2}we^{\frac{s}{2}}(s-t)}{1 - w}) = \phi^{s,u}(t, e^{\frac{t}{2}} \zeta) = \Pi(s, t, u, \zeta)
\]
which is precisely the statement of (1.21).

5.5 Proof of Theorem 1.17 (\( \mathcal{G}_{t,t} = \mathcal{G}^t \))

We are now in a position to complete the proof of Theorem 1.17 modulo a small error in [4].
Remark 5.15. In [4, Lemma 18], there is a typographical error that is propagated through the remainder of that paper. In the second line of the proof of that lemma, the function \( \imath(t, \cdot) \) should be the inverse of \( z \mapsto ze^{\frac{t}{1+z}} \) rather than the inverse of \( z \mapsto \frac{1}{1+z} e^{\frac{t}{1+2z}} \) as stated. That \( \imath(t, \cdot) \) has this different form follows from [4, Lemma 11], which defines the kernel function \( \kappa(t, z) \) (formula 4.2.2.a) implicitly by \( \frac{\kappa(t, z) - 1}{\kappa(t, z) + 1} e^{\frac{t}{2} \kappa(t, z)} = z \); then \( \imath(t, z) = \frac{\kappa(t, 1/z) + 1}{\kappa(t, 1/z) - 1} \) yields the result. Hence, the correct generating function for the Biane polynomials in [4] is the one in (1.21) above, in the special case \( s = t \). The third author of the present paper discovered this error as the result of the present work: early versions of the calculations in this section suggested the generating function \( \kappa(t, z) \) = \( \frac{1}{1+z} e^{\frac{t}{1+2z}} \) which defines the kernel function \( \kappa(t, z) \) = \( \frac{1}{1+z} e^{\frac{t}{1+z}} \) as stated. That \( \kappa(t, z) \) agrees with \( \kappa(t, z) \) = \( \frac{1}{1+z} e^{\frac{t}{1+2z}} \) for \( k \geq 1 \). Eq. (5.2) verifies that the Biane polynomials \( p_k^{t,t} \) for \( \mathcal{K}_{t,t} \) have the same generating function as the Biane polynomials of \( \mathcal{G}^t \) (cf. Remark 5.15), and this concludes the proof.

Proof of Theorem 1.17 By the density of trigonometric polynomials in \( L^2(U, \nu_t) \) for any measure \( \nu_t \), the transform \( \mathcal{G}^t \) is determined by its action on Laurent polynomial functions. Hence, to verify that \( \mathcal{G}_{t,t} = \mathcal{G}^t \), it suffices to verify that \((\mathcal{G}^t)^{-1} \) agrees with \( \mathcal{K}_{t,t} \) on monomials \( z \mapsto z^k \) for \( k \in \mathbb{Z} \). Eq. (5.2) is consistent with [4, Lemma 18], and so it suffices to prove this result for \( k \geq 1 \). Eq. (1.21) verifies that the Biane polynomials \( p_k^{t,t} \) for \( \mathcal{K}_{t,t} \) have the same generating function as the Biane polynomials of \( \mathcal{G}^t \) (cf. Remark 5.15), and this concludes the proof.

A Heat Kernel Measures on Lie Groups

Suppose that \( G \) is a connected Lie group and \( \beta \) is a basis for \( \text{Lie}(G) \). Then \( A = \sum_{X \in \beta} \partial_X^2 \) is a left-invariant non-positive elliptic differential operator which is essentially self adjoint on \( C_c^\infty(G) \) as an operator on \( L^2(G, dg) \) where \( dg \) is a right Haar measure on \( G \). Associated to the contraction semigroup \( \{e^{tA/2}\}_{t \geq 0} \) is a convolution semigroup of probability (heat kernel) densities \( \{h_t\}_{t \geq 0} \). In more detail, \( \mathbb{R}^+ \times G \ni (t, g) \mapsto h_t(g) = h_t(e) = h_t(e) \in \mathbb{R}_+ \) is a smooth function such that

\[
\partial_t h_t(g) = \frac{1}{2} Ah_t(g) \quad \text{for } t > 0
\]

and

\[
\lim_{t \downarrow 0} \int_G f(g) h_t(g) dg = f(e) \quad \text{for all } f \in C_c(G).
\]

(Throughout, \( e = 1_G \).) Basic properties of these heat kernels are summarized in [9, Proposition 3.1] and [10, Section 3]. For an exhaustive treatment of heat kernels on Lie groups see [26] and [34]. For our present purposes, we need to know that, if \( G = \mathbb{U}_N \) or \( G = \mathbb{G}_L \) (and so \( h_t \) is the density of \( \rho_{n,t} \) or \( \mu_{g,t} \), respectively), then

\[
\int_G f(g) h_t(g) dg = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{t}{2} \right)^n (A^n f)(I) \quad \text{for all } t \geq 0
\]

(A.1)

whenever \( f \) is a trace polynomial. This result can be seen as a consequence of Langland’s theorem; see, for example, [26, Theorem 2.1 (p. 152)]. As it is a bit heavy to get to Langland’s theorem in Robinson we will, for the reader’s convenience, sketch a proof of (A.1); see Theorem A.2 below. For the rest of this section let \( d \) denote the left-invariant metric on \( G \) such that \( \{\partial_X\}_{X \in \beta} \) is an orthonormal frame on \( G \) and set \( |g| = d(e, g) \). Also let us use the abbreviation \( h_t(f) \) for \( \int_G f(g) h_t(g) dg \).

Lemma A.1. Suppose \( f : [0, T] \times G \to C \) is a \( C^2 \) function such that \( |k(t, g)| \leq C e^{C|g|} \) for some \( C < \infty \), where \( k \) is any of the functions \( f, \partial_t f, \) or \( \partial_X f \) for any \( X \in \text{Lie}(A) \), or \( Af \). Then

\[
\partial_t h_t(f(t, \cdot)) = h_t \left( \partial_t f(t, \cdot) + \frac{1}{2} Af(t, \cdot) \right) \quad \text{for } t \in (0, T]
\]

(A.2)
and
\[
\lim_{t \downarrow 0} h_t (f(t, \cdot)) = f(0, \cdot). \tag{A.3}
\]

**Proof.** Let \( \{h_n\} \subset C^\infty (G, [0, 1]) \) be smooth cutoff functions as in [9, Lemma 3.6] and set \( f_n(t, g) \equiv h_n(g) f(t, g) \). Then it is easy to verify that it is now permissible to differentiate past the integrals and perform the required integration by parts in order to show that

\[
\frac{d}{dt} \left[ h_t (f_n(t, \cdot)) \right] = h_t \left( \partial_t f(t, \cdot) + \frac{1}{2} Af(t, \cdot) \right).
\]

Let \( F(t, \cdot) = \partial_t f(t, \cdot) + \frac{1}{2} Af(t, \cdot) \) and

\[
F_n(t, \cdot) = \partial_t f_n(t, \cdot) + \frac{1}{2} Af_n(t, \cdot)
\]

\[
= F(t, \cdot) h_n + \frac{1}{2} f(t, \cdot) h_n + \sum_{X \in \partial} \overline{X} f(t, \cdot) \partial_X h_n.
\]

From the properties of \( h_n \) and the assumed bounds on \( f \), given \( \epsilon \in (0, T) \) there exist \( C < \infty \) independent of \( n \) such that

\[
\sup_{\epsilon \leq t \leq T} |F_n(t, g) - F(t, g)| \leq 1_{|g| \geq n} C e^{C|g|}.
\]

It then follows by the standard heat kernel bounds (see for example [34] or [26] page 286) that

\[
\sup_{\epsilon \leq t \leq T} |h_t (F_n(t, \cdot)) - h_t (F(t, \cdot))| \to 0 \text{ as } n \to \infty.
\]

Hence we may conclude that \( \frac{d}{dt} [h_t (f(t, \cdot))] \) exists and

\[
\frac{d}{dt} [h_t (f(t, \cdot))] = \lim_{n \to \infty} \frac{d}{dt} [h_t (f_n(t, \cdot))]
\]

\[
= h_t \left( \partial_t f_n(t, \cdot) + \frac{1}{2} Af_n(t, \cdot) \right) \text{ for } \epsilon < t \leq T
\]

which proves (A.2). To prove (A.3) we start with the estimate

\[
|h_t (f(t, \cdot)) - f(0, \cdot)| \leq \left| \int_G [f(t, y) - f(0, \cdot)] h_t(y) \, dy \right|
\]

\[
\leq \int_G |f(t, y) - f(0, \cdot)| h_t(y) \, dy
\]

\[
\leq \delta(\epsilon, t) + C \int_{|y| > \epsilon} e^{C|y|} h_t(y) \, dy
\]

where

\[
\delta(\epsilon, t) = \int_{|y| \leq \epsilon} |f(t, y) - f(0, \cdot)| h_t(y) \, dy \leq \sup_{|y| \leq \epsilon} |f(t, y) - f(0, \cdot)|.
\]

From [9] Lemma 4.3] modified in a trivial way from its original form where \( \epsilon \) was take to be 1, we know that

\[
\limsup_{t \downarrow 0} \int_{|y| > \epsilon} e^{C|y|} h_t(y) \, dy = 0 \text{ for all } \epsilon > 0 \text{ and } c < \infty.
\]

Therefore, we conclude that

\[
\limsup_{t \downarrow 0} |h_t (f(t, \cdot)) - f(0, \cdot)| \leq \limsup_{t \downarrow 0} \delta(\epsilon, t) \to 0 \text{ as } \epsilon \downarrow 0
\]

as claimed. \( \square \)
Theorem A.2. Suppose now that $G = \mathbb{U}_N$ or $G = GL_N$ and $P_N$ is a trace polynomial function on $G$. Then for $T > 0$,

$$h_T(P_N) = \left( \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{T}{2} \right)^n A^n P_N \right) (I_N).$$  \hspace{1cm} (A.4)

Proof. Fix $T > 0$, and for $0 < t < T$ let

$$f(t, \cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{T - t}{2} \right)^n A^n P_N$$

where the sum is convergent as $A$ is a bounded operator on the finite dimensional subspace of trace polynomials of trace degree $\deg P$ or less. Moreover, $f(t, \cdot)$ is again a trace polynomial with time dependent coefficients and $f$ satisfies

$$\partial_t f(t, \cdot) + \frac{1}{2} A f(t, \cdot) = 0 \text{ with } f(T, \cdot) = P_N.$$  

From Lemma A.1 we may now conclude,

$$\frac{d}{dt} [h_t (f(t, \cdot))] = h_t \left( \partial_t f(t, \cdot) + \frac{1}{2} A f(t, \cdot) \right) = 0.$$  

Therefore $t \to h_t (f(t, \cdot))$ is constant for $t > 0$ and hence, using Lemma A.1 again,

$$h_T(P_N) = h_T (f(T, \cdot)) = \lim_{t \downarrow 0} h_t (f(t, \cdot)) = f(0, I_N) = \left( \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{T}{2} \right)^n A^n P_N \right) (I_N).$$

This concludes the proof.  \hspace{1cm} \square

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