Characteristic foliation of twisted Jacobi manifolds

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Abstract

We study the characteristic foliation of a twisted Jacobi manifold. We show that a twisted Jacobi manifold is foliated into leaves that are, according to the parity of the dimension, endowed with a twisted contact or a twisted locally conformal symplectic structure.

Keywords : Twisted Jacobi manifold, characteristic foliation.

A.M.S. classification (2000): 53C12, 53D10, 53D17.

1 Introduction

The notion of twisted Jacobi manifold was introduced by the authors in [7] and several of its properties and relations with other geometric structures, such as twisted Dirac-Jacobi bundles and quasi-Jacobi bialgebroids, were studied in [8]. Twisted Jacobi manifolds appeared as a natural generalization of the twisted Poisson manifolds, which were introduced by Severa and Weinstein in [10], motivated by works on topological field theory [9] and on string theory [5]. In [8], some examples of twisted Jacobi structures on manifolds were presented, including twisted locally conformal symplectic structures. In this Note, we show that twisted contact structures also provide examples of twisted Jacobi structures. Twisted contact and twisted locally conformal symplectic structures are two important types of twisted Jacobi structures on a manifold. In fact, we prove that, according to the parity of its dimension, a transitive twisted Jacobi manifold is either a twisted contact manifold, or a twisted locally conformal symplectic manifold. The characteristic foliation of a twisted Jacobi manifold is also discussed in this Note and we show that each characteristic leaf of a twisted Jacobi manifold is endowed with a transitive twisted Jacobi structure.
The paper starts with a very brief review, in section 2, of the main properties of twisted Jacobi manifolds. Section 3 is devoted to the study of the characteristic foliation of a twisted Jacobi manifold.

2 Twisted Jacobi manifolds

A twisted Jacobi manifold \((7, 8)\) is a differentiable manifold \(M\) equipped with a bivector field \(\Lambda\), a vector field \(E\) and a 2-form \(\omega\) such that

\[
\frac{1}{2}[(\Lambda, E), (\Lambda, E)]^{(0,1)} = (\Lambda, E)^\#(d\omega, \omega). \tag{1}
\]

In \((11)\), \([\cdot, \cdot]^{(0,1)}\) denotes the Schouten bracket of the Lie algebroid \((TM \times \mathbb{R}, [, ], \pi)\) over \(M\), modified by the 1-cocycle \((0,1)\) of the Lie algebroid cohomology complex with trivial coefficients \([3]\), and \((\Lambda, E)^\#\) is the natural extension of \((\Lambda, E)^\# : \Gamma(TM \times \mathbb{R}) \to \Gamma(TM \times \mathbb{R})\), given, for all \((\alpha, f) \in \Gamma(TM \times \mathbb{R})\), by \((\Lambda, E)^\#(\alpha, f) = (\Lambda^\#(\alpha) + fE, -\langle \alpha, E \rangle)\), to a homomorphism from \(\Gamma(\wedge^k(TM \times \mathbb{R}))\) to \(\Gamma(\wedge^k(TM \times \mathbb{R}))\), \(k \in \mathbb{N}\), \((k = 3, \text{ in } (1))\) defined, for all \((\eta, \xi) \in \Gamma(\wedge^3(TM \times \mathbb{R}))\) and \((\alpha_1, f_1), \ldots, (\alpha_k, f_k) \in \Gamma(TM \times \mathbb{R})\), by

\[
(\Lambda, E)^\#((\eta, \xi)((\alpha_1, f_1), \ldots, (\alpha_k, f_k)) = (-1)^k(\eta)(\langle \Lambda^\#(\alpha_1), \ldots, \Lambda^\#(\alpha_k) \rangle).
\]

For a bivector field \(\Lambda\) on \(M\), we consider the usual homomorphism \(\Lambda^\# : \Gamma(TM^* M) \to \Gamma(TM)\) associated to \(\Lambda\) and we define its natural extension \(\Lambda^\# : \Gamma(\wedge^kTM^* M) \to \Gamma(\wedge^kTM)\), \(k \in \mathbb{N}\), by setting, for all \(\eta \in \Gamma(\wedge^kTM)\) and \(\alpha_1, \ldots, \alpha_k \in \Gamma(TM^*)\),

\[
\Lambda^\#(\eta)(\alpha_1, \ldots, \alpha_k) = (-1)^k(\eta)(\langle \Lambda^\#(\alpha_1), \ldots, \Lambda^\#(\alpha_k) \rangle).
\]

Also, following \([10]\), we denote by \((\Lambda^\# \otimes 1)(\eta)\) the section of \(\wedge^{k-1}TM \otimes TM\) that acts on multivector fields by contraction with the factor in \(TM^*\): for all \(X \in \Gamma(TM)\) and \(\alpha_1, \ldots, \alpha_{k-1} \in \Gamma(TM^*)\),

\[
(\Lambda^\# \otimes 1)(\eta)(\alpha_1, \ldots, \alpha_{k-1})(X) = (-1)^k(\eta)(\langle \Lambda^\#(\alpha_1), \ldots, \Lambda^\#(\alpha_{k-1}), X \rangle).
\]

The next Proposition gives an equivalent expression of \((1)\) in terms of the usual Schouten bracket.

**Proposition 2.1** \([3]\) The pair \((\Lambda, E), \omega\), with \((\Lambda, E) \in \Gamma(\wedge^2(TM \times \mathbb{R}))\) and \(\omega \in \Gamma(\wedge^2TM^*)\), defines a twisted Jacobi structure on \(M\) if and only if

\[
[\Lambda, \Lambda] + 2E \wedge \Lambda = 2\Lambda^\#(d\omega) + 2(\Lambda^\#(\omega) \wedge E) \tag{2}
\]

and

\[
[E, \Lambda] = (\Lambda^\# \otimes 1)(d\omega)(E) - ((\Lambda^\# \otimes 1)(\omega)(E)) \wedge E. \tag{3}
\]

As in the case of a Jacobi manifold \([3]\), given a twisted Jacobi structure \(((\Lambda, E), \omega)\) on \(M\), \((\Lambda, E)\) defines on \(C^\infty(M, \mathbb{R})\) the internal composition law

\[
\{f, g\} = \Lambda(df, dg) + \langle fdg - gdf, E \rangle, \quad f, g \in C^\infty(M, \mathbb{R}), \tag{4}
\]
that is bilinear, skew-symmetric but it does not, in general, satisfy the Jacobi identity. We have \( \{8\} \), for all \( f, g, h \in C^\infty(M, \mathbb{R}) \),

\[
\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = (\Lambda, E)^\#(d\omega, \omega)((df, f), (dg, g), (dh, h)).
\]

Therefore, \( (C^\infty(M, \mathbb{R}), \{\cdot, \cdot\}) \) is not a local Lie algebra. However, \( ((\Lambda, E), \omega) \) defines a Lie algebroid structure \( \{\cdot, \cdot\} \omega, \pi \circ (\Lambda, E)^\# \) on the vector bundle \( T^*M \times \mathbb{R} \rightarrow M \). \( \[8\] \). The bracket on the space \( \Gamma(T^*M \times \mathbb{R}) \) of smooth sections is given, for all \( (\alpha, f), (\beta, g) \in \Gamma(T^*M \times \mathbb{R}) \), by

\[
\{\alpha, f\}, \{\beta, g\}\} \omega = \{\alpha, f\}, \{\beta, g\}\} + (d\omega, \omega)((\Lambda, E)^\#(\alpha, f), (\Lambda, E)^\#(\beta, g)), \cdot),
\]

where \( \{\cdot, \cdot\} \) denotes the Kerbrat-Souici-Benhammadi bracket \( \[4\] \) and the anchor map is \( \pi \circ (\Lambda, E)^\# \), where \( \pi : TM \times \mathbb{R} \rightarrow TM \) denotes the projection on the first factor.

Next, we present two important examples of twisted Jacobi structures on a manifold.

**Examples 2.2**

1. **Twisted locally conformal symplectic manifolds:** A twisted locally conformal symplectic manifold \( \[8\] \) is a manifold \( M \) of even dimension \( 2n \) equipped with a non-degenerate 2-form \( \Theta \), a closed 1-form \( \vartheta \), and a 2-form \( \omega \) such that

\[
d(\Theta - \omega) + \vartheta \wedge (\Theta - \omega) = 0.
\]

Let \( E \) be the unique vector field and \( \Lambda \) the unique bivector field on \( M \) which are defined by

\[
i(E)\Theta = -\vartheta \quad \text{and} \quad i(\Lambda^\#(\alpha))\Theta = -\alpha, \quad \text{for all} \ \alpha \in \Gamma(T^*M). \quad (5)
\]

Then, we have

\[
E = \Lambda^\#(\vartheta) \quad \text{and} \quad \Lambda = \Lambda^\#(\Theta).
\]

By a simple, but very long computation, we prove that the pair \( ((\Lambda, E), \omega) \) satisfies the relations \( \[2\] \) and \( \[3\] \). Whence, \( ((\Lambda, E), \omega) \) endows \( M \) with a twisted Jacobi structure.

2. **Twisted contact manifolds:** A twisted contact manifold is a manifold \( M \) of odd dimension \( 2n + 1 \) equipped with a 1-form \( \vartheta \) and a 2-form \( \omega \) such that \( \vartheta \wedge (d\vartheta + \omega)^n \neq 0 \), everywhere in \( M \). Let us consider on \( M \) the vector field \( E \) defined by

\[
i(E)\vartheta = 1 \quad \text{and} \quad i(E)(d\vartheta + \omega) = 0,
\]

and the bivector field \( \Lambda \) whose associated morphism \( \Lambda^\# \) is given, for all \( \alpha \in \Gamma(T^*M) \), by

\[
\Lambda^\#(\vartheta) = 0 \quad \text{and} \quad i(\Lambda^\#(\alpha))(d\vartheta + \omega) = -(\alpha - \langle \alpha, E \rangle \vartheta).
\]

Then, by a simple, but very long computation, we prove that \( ((\Lambda, E), \omega) \) satisfies \( \[2\] \) and \( \[3\] \). Thus, \( ((\Lambda, E), \omega) \) endows \( M \) with a twisted Jacobi structure.
3 The characteristic foliation of a twisted Jacobi manifold

It is well known [2] that any Jacobi manifold is decomposed into leaves equipped with transitive Jacobi structures that are, according to the parity of the dimension of the leaves, contact or locally conformal symplectic structures. In this section, we will prove a similar result for twisted Jacobi manifolds.

Let \((M, (\Lambda, E), \omega)\) be a twisted Jacobi manifold and consider its associated Lie algebroid over \(M\), \((T^*M \times \mathbb{R}, \{\cdot, \cdot\}, \pi \circ (\Lambda, E)^\#)\). The image \(\text{Im}(\pi \circ (\Lambda, E)^\#)\) of the anchor map defines a completely integrable distribution on \(M\), called the characteristic distribution of \((M, (\Lambda, E), \omega)\), that determines a foliation of \(M\) into leaves, which are called the characteristic leaves of \((\Lambda, E), \omega\) [1]. If, at every point of \(M\), the dimension of the characteristic leaf of \((\Lambda, E), \omega)\) through this point is equal to the dimension of \(M\), the twisted Jacobi manifold \((M, (\Lambda, E), \omega)\) is said to be transitive. According to the parity of the dimension of \(M\), there are two kinds of transitive twisted Jacobi manifolds.

**Proposition 3.1** Let \((M, (\Lambda, E), \omega)\) be a transitive twisted Jacobi manifold.

1) If \(M\) is of even dimension, then \((\Lambda, E), \omega)\) comes from a twisted locally conformal symplectic structure.

2) If \(M\) is of odd dimension, then \((\Lambda, E), \omega)\) comes from a twisted contact structure.

**Proof.**

1) Let \(\text{dim} \ M = 2n\). Since \((\Lambda, E), \omega)\) is transitive, \(\text{rank} \Lambda^\# = 2n\), everywhere on \(M\), and \(E\) is a section of \(\text{Im} \Lambda^\#\), i.e. there exists a 1-form \(\vartheta\) on \(M\) such that \(E = \Lambda^\# (\vartheta)\). Let \(\Theta\) be the 2-form on \(M\) obtained by the inversion of \(\Lambda\), i.e., for any \(\alpha \in \Gamma (T^*M)\), \(i(\Lambda^\#(\alpha))\Theta = -\alpha\). A simple computation shows that equations (2) and (3) give, respectively, \(d(\Theta - \omega) + \vartheta \wedge (\Theta - \omega) = 0\) and \(d\vartheta = 0\). Whence, we conclude that \((\Lambda, E), \omega)\) is provided by the twisted locally conformal symplectic structure \((\vartheta, \Theta, \omega)\) on \(M\).

2) Let \(\text{dim} \ M = 2n + 1\). Since \((\Lambda, E), \omega)\) is transitive, \(\text{rank} \Lambda^\# = 2n\), everywhere on \(M\), and \(E\) is not a section of \(\text{Im} \Lambda^\#\). Let \(\vartheta\) be the 1-form on \(M\) defined by \(i(E)\vartheta = 1\) and \(\Lambda^\#(\vartheta) = 0\) and let \(\Theta\) be the 2-form on \(M\) obtained by the inversion of \(\Lambda\), i.e., for any \(\alpha \in \Gamma (T^*M)\), \(i(\Lambda^\#(\alpha))\Theta = - (\alpha - \langle \alpha, E \rangle \vartheta)\) and \(i(E)\Theta = 0\). Clearly, \(\vartheta \wedge \Theta^n \neq 0\), everywhere on \(M\), and \(\Lambda = \Lambda^\#(\Theta)\). So, we have \([\Lambda, \Lambda] = 2\Lambda^\#(d\Theta) - 2E \wedge \Lambda^\#(d\vartheta)\) and, by a simple argumentation, we prove that (2) and (3) give \(\Theta = d\vartheta + \omega\). Thus, \((\Lambda, E), \omega)\) comes from the twisted locally conformal symplectic structure \((\vartheta, \Theta, \omega)\) on \(M\).

**Theorem 3.2** Let \((M, (\Lambda, E), \omega)\) is a twisted Jacobi manifold. Then, the bracket (4) induces a transitive twisted Jacobi structure on each characteristic leaf of \(M\).

**Proof.** Let \(S\) be a characteristic leaf of \((M, (\Lambda, E), \omega)\) through a point \(p\), with \(\text{dim} \ S = k\), and \((x_1, \ldots, x_k, y_1, \ldots, y_{n-k})\), \(n = \text{dim} \ M\), a system of
adapted local coordinates of $M$. Given two functions $\tilde{f}, \tilde{g} \in C^\infty(S, \mathbb{R})$, we can extend them locally to functions $f, g \in C^\infty(M, \mathbb{R})$, i.e. $f(x, 0) = \tilde{f}(x)$ and $g(x, 0) = \tilde{g}(x)$. On $C^\infty(S, \mathbb{R})$ we define the bracket $\{ , \}_S$ by setting
\[
\{ \tilde{f}, \tilde{g} \}_S(x) = \{ f, g \}(x, 0), \quad \text{for all } \tilde{f}, \tilde{g} \in C^\infty(S, \mathbb{R}). \tag{6}
\]
We have,
\[
\{ \tilde{f}, \tilde{g} \}_S(x) = \{ f, g \}(x, 0) = (\Lambda^\#(df) + fE)|_{(x,0)}g - (df, E)|_{(x,0)}g
= -(\Lambda^\#(dg) + gE)|_{(x,0)}f + (dg, E)|_{(x,0)}f
\]
and we realize that the bracket (6) only depends on $\tilde{f}$ and $\tilde{g}$ because it is computed along the integral curves of the vector fields $\Lambda^\#(df) + fE$, $\Lambda^\#(dg) + gE$ and $E$ through $(x, 0)$, which lie on $S$. Clearly, (6) yields a transitive twisted Jacobi structure on $S$. □

From Proposition 3.1 and Theorem 3.2, we conclude that a twisted Jacobi manifold is foliated into leaves that are endowed, according to the parity of the dimension, with a twisted locally conformal symplectic structure or a twisted contact structure.

Acknowledgments. The work of Joana M. Nunes da Costa has been partially supported by POCI/MAT/58452.

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