PROJECTIVE NORMALITY OF ARTIN-SCHREIER CURVES

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ABSTRACT. In this paper we study the projective normality of certain Artin-Schreier curves $Y_f$ defined over a field $\mathbb{F}$ of characteristic $p$ by the equations $y^q + y = f(x)$, $q$ being a power of $p$ and $f \in \mathbb{F}[x]$ being a polynomial in $x$ of degree $m$, with $(m,p) = 1$. Many $Y_f$ curves are singular and so, to be precise, here we study the projective normality of appropriate projective models of their normalizations.

1. INTRODUCTION

Let $\mathbb{P}^2$ denote the projective plane over an arbitrary field $\mathbb{F}$ of characteristic $p$, and let $q := p^k$ be a power of $p$ ($k > 0$). Denote by $Y_f \subseteq \mathbb{P}^2$ the curve defined over $\mathbb{F}$ by the equation $y^q + y = f(x)$, where $f(x) \in \mathbb{F}[x]$ is a polynomial of degree $m > 0$. Assume $(m,p) = 1$. The function field $\mathbb{F}(x,y)$ is deeply studied in [6], Proposition 6.4.1. In particular, the function $x$ is known to have only one pole $P_\infty$. Denote by $\pi : C_f \to Y_f$ the normalization of $Y_f$ (which is known to be a bijection) and set $Q_\infty := \pi^{-1}(P_\infty)$. For each $s \geq 0$ the (pull-backs of the) monomials $x^iy^j$ such that

$$i \geq 0, \quad 0 \leq j \leq q - 1, \quad qi + mj \leq s$$

form a basis of the vector space $L(sQ_\infty)$ (see [6], Proposition 6.4.1 again). The genus, $g$, of the curve $Y_f$ (which is by definition the genus of the normalization $C_f$) is known to be $g = (m-1)(q-1)/2$. In this paper we study the projective normality of certain embeddings $(X_f)$ of $C_f$ curves into suitable projective spaces. Let us briefly discuss the outline of the paper.

- Section 2 recalls a basic definition and contains a preliminary lemma.
- In Section 3 we take an arbitrary integer $m \geq 2$ which divides $q - 1$ and consider the curve $C_f$ embedded by $L(qQ_\infty)$ into the projective space $\mathbb{P}^r$, $r := (q - 1)/m + 1$. We show that this curve is in any case projectively normal and we compute the dimension of the space of quadric hypersurfaces containing it.

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In Section \[ \square \] we pick out an arbitrary integer \( m \geq 2 \) which divides \( tq + 1 \) (\( t \) being any positive integer) and consider the curve \( C_f \) embedded by \( L((tq + 1)Q_\infty) \). We show that if \( (tq - 1)/m \leq q - 1 \) and \( f(x) = x^m \) then the cited curve is in any case projectively normal.

Notice that the curve \( C_f \) and the line bundles \( \mathcal{L}(qQ_\infty) \), \( \mathcal{L}((tq + 1)Q_\infty) \) are defined over any field \( \mathbb{F} \supseteq \mathbb{F}_p \) containing the coefficients of the polynomial \( f(x) \). Hence when \( f(x) = x^m \) any field of characteristic \( p \) may be used.

2. Preliminaries

In this section we recall a basic definition and prove a general lemma. The result provides in fact sufficient conditions for the projective normality of a \( C_f \) curve as defined in the Introduction.

**Definition 1.** A smooth curve \( X \subseteq \mathbb{P}^r \) defined over a field \( \mathbb{F} \) is said to be projectively normal if for any integer \( d \geq 2 \) the restriction map

\[
\rho_{d,X} : S^d(H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))) \rightarrow H^0(X, \mathcal{O}_X(d))
\]

is surjective, \( S^d \) denoting the symmetric \( d \)-power of the tensor product.

**Lemma 2.** Consider a \( C_f \) curve as in Section \[ \square \] (\( q, m \) and \( f \) being as in the definitions). Set \( C := C_f \). Fix integers \( a, b, e \) such that \( a \geq 0, b \geq 0, a + b > 0 \) and \( e \geq aq + bm + (m - 1)(q - 1) - 1 \). The multiplication map

\[
\mu : L((aq + bm)Q_\infty) \otimes L(eQ_\infty) \rightarrow L((e + aq + bm)Q_\infty)
\]

is surjective.

**Proof.** As we will explain below, this is just a particular case of the base-point free pencil trick (\[ \square \], p. 126). Since the Weierstrass semigroup of non-gaps of \( Q_\infty \) contains \( m \) and \( q \), the line bundle \( \mathcal{O}_C((aq + bm)Q_\infty) \) is spanned by its global sections. Since \( (aq + bm) > 0 \), we have \( h^0(C, \mathcal{O}_C((aq + bm)Q_\infty)) \geq 2 \). Hence there is a two-dimensional linear subspace \( V \subseteq H^0(C, \mathcal{O}_C((aq + bm)Q_\infty)) \) (defined over \( \overline{\mathbb{F}} \) spanning \( \mathcal{O}_C((aq + bm)Q_\infty) \). Taking a basis, say \( \{w_1, w_2\} \), of \( V \), we get an exact sequence of line bundles on \( C \) (over \( \overline{\mathbb{F}} \)):

\[
0 \rightarrow \mathcal{O}_C((e - aq - bm)Q_\infty) \rightarrow \mathcal{O}_C(eQ_\infty) \rightarrow \mathcal{O}_C((e + aq + bm)Q_\infty) \rightarrow 0
\]

in which \( \phi \) is induced by the multiplication by the column vector \( (w_1, w_2) \). By assumption \( e - aq - bm > 2g - 2 \). Hence \( h^1(C, \mathcal{O}_C((e - aq - bm)Q_\infty)) = 0 \). It follows that the map

\[
\psi : H^0(C, \mathcal{O}_C(eQ_\infty)) \otimes \mathcal{O}_C((e + aq + bm)Q_\infty) \rightarrow H^0(C, \mathcal{O}_C((e + aq + bm)Q_\infty))
\]

induced in cohomology by the map \( \phi \) of the previous exact sequence is surjective. Since \( V \subseteq H^0(C, \mathcal{O}_C((aq + bm)Q_\infty)) \), \( \mu \) is surjective.

In the following sections the previous result will be applied to appropriate embeddings of \( C_f \) curves.

3. The case \( m|q - 1 \)

Assume that \( m \geq 2 \) is an integer which divides \( q - 1 \) and set \( c := (q - 1)/m \). If \( c = 1 \) then \( Y_f \) is a smooth plane curve and it is of course projectively normal. Hence we can focus on the case \( c \geq 2 \). Notice that the point \( P_\infty \in Y_f(\overline{\mathbb{F}}) \) defined in the Introduction is the only singular point of \( Y_f \), for any choice of \( f(x) \) as in the definitions. We have also an identity of vector spaces \( H^0(C_f, \pi^*(\mathcal{O}_{Y_f}(1))) = L(qQ_\infty) \) and by the results stated in the Introduction it can be easily seen
Lemma 4. This proves the theorem. □

**Remark 3.** Since $\pi$ is injective and has invertible differential at any point of $C_f \setminus \{Q_\infty\}$, then also $\varphi$ is injective with non-zero differential at any point of $C_f \setminus \{Q_\infty\}$. Moreover, the differential of $\varphi$ is non-zero even in $Q_\infty$. Indeed, since $L(qQ_\infty)$ has no base-points, in order to prove that $\varphi$ has non-zero differential at $Q_\infty$ it is sufficient to prove that

$$h^0(C_f, (q-2)Q_\infty) = h^0(C_f, qQ_\infty) - 2$$

([4], Chapter IV, proof of Proposition 3.1). To do this, we may notice that a basis of $L(qQ_\infty)$ is given by a basis of $L((q-2)Q_\infty)$ and the monomials $x$ and $y^c$ (see the Introduction). The result follows.

By the previous remark, $\varphi$ is in fact an embedding of $C_f$ into $\mathbb{P}^r$. Set $X_f := \varphi(C_f)$ and, for any integer $s \geq 1$, denote by

$$\mu_s : L(qQ_\infty) \otimes L(sqQ_\infty) \to L(q(s+1)Q_\infty)$$

the multiplication map.

**Lemma 4.** If $\mu_s$ is surjective for all $s \geq 1$ then $X_f$ is projectively normal.

**Proof.** Fix an integer $t \geq 2$ and assume that $\mu_s$ is surjective for all $s \in \{1, \ldots, t-1\}$. We need to prove the surjectivity of the linear map $\rho_t : S^t(L(qQ_\infty)) \to L(tqQ_\infty)$. Notice that in arbitrary characteristic $S^t(L(qQ_\infty))$ is defined as a suitable quotient of $L(qQ_\infty)^{\otimes t}$ ([2], §A2.3), i.e. $\tau_t = \rho_t \circ \eta_t$, where $\eta_t : L(qQ_\infty)^{\otimes t} \to L(tqQ_\infty)$ is the tensor power map and $\tau_t : L(qQ_\infty)^{\otimes t} \to S^t(L(qQ_\infty))$ is a surjection. Hence $\rho_t$ is surjective if and only if $\tau_t$ is surjective. Since $\tau_2 = \mu_1$, $\tau_2$ is surjective. So assume $t > 2$ and that $\tau_{t-1}$ is surjective. Since $\tau_{t-1}$ and $\mu_{t-2}$ are surjective, $\tau_t$ is surjective. □

**Proposition 5.** If $s \geq m$ then $\mu_s$ is surjective.

**Proof.** If $s \geq m$ then $sq \geq q + (m-1)(q-1) - 1$. Apply Lemma 2 by setting $e := sq$, $a := 1$ and $b := 0$. □

**Theorem 6.** The curve $X_f$ is projectively normal.

**Proof.** By Lemma 8 it is enough to prove that $\mu_s$ is surjective for all $s \geq 1$. The case $s \geq m$ is covered by Proposition 5. So let us assume $1 \leq s < m$. Let $i, j$ be integers such that $i \geq 0$, $0 \leq j \leq q - 1$ and $qi + mj \leq (s+1)q$.

- If $qi + mj \leq sq$ then $x^i y^j$ is in the image of $\mu_s$ because $1 \in L(qQ_\infty)$.
- If $sq < qi + mj \leq (s+1)q$ and $i > 0$ then $x^{i-1} y^j \in L(sqQ_\infty)$. Since $x \in L(qQ_\infty)$ then $x^i y^j$ is in the image of $\mu_s$.
- If $i = 0$ and $sq < mj < (s+1)q$ then $j > sq/m = s(c+1/m) > c$ and $mj \leq (s+1)q-1$. By the latter inequality we get $m(j-c) \leq (s+1)q-1-mc$. Observe that $(s+1)q-1-mc = sq$ and so $m(j-c) \leq sq$. This proves that $y^{j-c} \in L(sqQ_\infty)$. Finally, $\mu_s(y^c \otimes y^{j-c}) = y^j$.
- If $i = 0$ and $mj = (s+1)q$ then $j = (s+1)q/m = (s+1)(c+1/m) = (s+1)c + (s+1)/m$. Since $0 \leq j \leq q-1$ is a nonnegative integer, we must have $(s+1)/m \in \mathbb{N}$. Since $1 \leq s < m$ we get $s = m - 1$. It follows $mj = mq$ and $j = q$, a contradiction.

This proves the theorem. □
Corollary 7. Assume $m \geq 3$. The curve $X_f \subseteq \mathbb{P}^r$ is contained into $(c+3)/2 - 3c - 3$ linearly independent quadric hypersurfaces.

Proof. In the notations of Definition 1 set $X := X_f$ and $d := 2$. Define $r := c + 1$. By Theorem 6 the restriction map

$$
\rho_{2,X_f} : S^2(H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))) \to H^0(X_f, \mathcal{O}_{X_f}(2))
$$

is surjective. Hence, in particular, the restriction map

$$
\rho : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) \to H^0(X_f, \mathcal{O}_{X_f}(2))
$$

is surjective. Since $m \geq 3$ (by assumption) we can easily check that a basis of the vector space $L(2qQ_\infty)$ consists of the following monomials:

$$
\{1, y, \ldots, y^{2c}, x, xy, \ldots, xy^c, x^2\}.
$$

Hence $h^0(X_f, \mathcal{O}_{X_f}(2)) = \dim_{\mathbb{F}} L(2(q-1)Q_\infty) = 3c + 3$. The kernel of $\rho$ is exactly the space of the quadrics in $\mathbb{P}^r$ vanishing on $X_f$. By the surjectivity of $\rho$ we easily deduce its dimension:

$$
\dim_{\mathbb{F}} H^0(\mathbb{P}^r, \mathcal{I}_{X_f}(d)) = \left(\frac{r+2}{2}\right) - (3c+3) = \left(\frac{c+3}{2}\right) - 3c - 3.
$$

The result follows. □

4. The case $m|tq + 1$

Pick out any integer $t \geq 1$ and assume that $m \geq 2$ is an integer dividing $tq + 1$. Set $c := (tq + 1)/m$. As in Remark 3 it can be checked that $L((tq + 1)Q_\infty)$ defines an embedding, say $\varphi$, of $C_f$ into $\mathbb{P}^r$, $r := \dim L((tq + 1)Q_\infty) - 1$. Define $X_f := \varphi(C_f)$. For any integer $s \geq 1$ denote by

$$
\mu_s : L((tq + 1)Q_\infty) \otimes L(s(tq + 1)Q_\infty) \to L((s+1)(tq + 1)Q_\infty)
$$

the multiplication map. As in Section 3 the projective normality of $X_f$ is controlled by the $\mu_s$ maps.

Lemma 8. If $\mu_s$ is surjective for all $s \geq 1$ then $X_f$ is projectively normal.

Proof. Take the proof of Lemma 4 □

Proposition 9. If $s \geq m$ then $\mu_s$ is surjective.

Proof. Apply Lemma 2 by setting $a := 0$, $b := c$ and $e := s(tq + 1)$. □

In the following part of the section we focus on the case $f(x) = x^m$. In particular we are going to show that $X_f$ curves obtained with this choice of $f$ are projectively normal for any choice of $t \geq 1$, provided that $c \leq q - 1$.

Remark 10. The assumption $c \leq q - 1$ is not so restrictive from a geometric point of view. In fact, for any fixed $q$, the genus of $X_f$ is $g = (q-1)(m-1)/2$. Even if $c$ is small, here we study many curves of interesting genus.

Lemma 11. Set $f(x) := x^m$ and assume $c \leq q - 1$. Pick out an integer $b \geq 0$ such that $b \leq (s+1)c$. Then $y^b$ is in the image of $\mu_s$. □
Proof. Since \( c \leq q - 1 \) we get \( y^c \in L((tq + 1)Q_\infty) \). In particular, if \( b \leq c \) then we are done. Assume \( b > c \). Let us prove the lemma by induction on \( s \). If \( s = 1 \), then \( b \leq 2c \) and \( b - c \leq c \leq q - 1 \). Hence \( y^{b-c} \in L((tq + 1)Q_\infty) \) and so \( y^b = \mu(y^c \otimes y^{b-c}) \) is of course in the image of \( \mu_1 \). If \( s > 1 \), then write \( b = hc + \rho \) with \( h \leq s \) and \( 0 \leq \rho \leq c \). Since \( b - \rho = hc \leq sc \) we have that \( y^{b-\rho} \) is in the image of \( \mu_{s-1} \). In particular, it is in \( L(s(tq + 1)Q_\infty) \). Since \( y^\rho \in L((tq + 1)Q_\infty) \), we get \( y^b = \mu_s(y^\rho \otimes y^{b-\rho}) \). It follows that \( y^b \) is in the image of \( \mu_s \).

**Theorem 12.** Set \( f(x) = x^m \) and assume \( c \leq q - 1 \). Then \( X_f \) is projectively normal.

**Proof.** By Lemma \([8]\) it is enough to show that \( \mu_s \) is surjective for any \( s \geq 1 \). By Proposition \([9]\) we need only to prove that \( \mu_s \) is surjective for any \( 1 \leq s < m \). Let \( i, j \) be integers such that \( i \geq 0 \), \( 0 \leq j < q - 1 \) and \( qi + mj \leq (s+1)(tq + 1) \). We will examine separately the case \( 2 \leq s < m \) and the case \( s = 1 \).

To begin with, assume \( 2 \leq s < m \).

- If \( j \geq c \), then \( x^iy^{j-c} \in L(s(tq + 1)Q_\infty) \). Since \( y^c \in L((tq + 1)Q_\infty) \), we have \( x^iy^j = \mu_s(x^i \otimes x^j) \).
- If \( 0 \leq j < c \) and \( qi + mj \leq s(tq + 1) \), then \( x^iy^j \) is in the image of \( \mu_s \), because \( 1 \in L((tq + 1)Q_\infty) \).
- Assume \( 0 \leq j < c \) and \( s(tq + 1) \leq qi + mj \leq (s+1)(tq + 1) \). We have \( i \geq t \). Indeed, assume by contradiction that \( i < t \). Then

\[
qi + mj < tq + mj
\]

\[
< tq + mc
\]

\[
= tq + tq + 1
\]

\[
\leq stq + 1
\]

\[
< s(tq + 1),
\]

a contradiction (here we used \( s \geq 2 \)).

\( \text{(A) If } qi + mj < (s+1)(tq + 1) \text{ then } x^iy^{j} \in L(s(tq + 1)Q_\infty) \text{ and } x^iy^j = \mu_s(x^i \otimes x^j) \).

- Assume \( qi + mj = (s+1)(tq + 1) \). Since \( (m, p) = 1 \) we have \( i = am \) for an integer \( a > 0 \) and \( j = (s+1)c - aq \). Observe that \( x^iy^j = x^amy^{(s+1)c-aq} = (y^d + y)^n y^{(s+1)c-aq} \), which is a sum of monomials of the form \( y^b \) with \( b \leq (s+1)c \). Apply Lemma \([11]\) and the fact that \( \mu_s \) is linear to get that \( x^iy^j \) is in its image.

Now assume \( s = 1 \).

- Assume \( j \geq c \). Since \( qi + mj \leq 2(tq + 1) \) we get \( qi + m(j - c) \leq 2(tq + 1) - (tq + 1) = tq + 1 \). Hence \( x^iy^{j-c} \in L((tq + 1)Q_\infty) \). Finally, \( \mu_1(y^c \otimes x^iy^{j-c}) = x^iy^j \).
- Assume \( j < c \) and \( i \geq t \). Since \( qi + mj \leq 2(tq + 1) \) we get \( qi + mj = 2(tq + 1) - tq = tq + 2 \).

\( \text{(C) If } q(i - t) + mj \leq tq + 1 \text{ then } x^iy^j = \mu_1(x^i \otimes x^iy^j) \).

- Assume \( q(i - t) + mj = tq + 2 \), i.e. \( qi + mj = 2tq + 2 \). Repeat the proof of case (B) with \( s := 1 \).
- Assume \( j < c \) and \( i < t \). Then \( x^iy^j \in L((tq + 1)Q_\infty) \) and we easily get \( x^iy^j = \mu_1(x^i \otimes y^j) \).

The proof is concluded.

**Remark 13.** If \( t = 1 \) then the assumption \( c \leq q - 1 \) is trivially satisfied (we assumed \( m \neq q + 1 \)). In this case the curve \( y^d + y = x^m \) is covered by the Hermitian curve.
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