On inert properties of particles in classical theory

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Abstract

This is a critical review of inert properties of classical relativistic point objects. The objects are classified as Galilean and non-Galilean. Three types of non-Galilean objects are considered: spinning, rigid, and dressed particles. In the absence of external forces, such particles are capable of executing not only uniform motions along straight lines but also Zitterbewegungs, self-accelerations, self-decelerations, and uniformly accelerated motions. A free non-Galilean object possesses the four-velocity and the four-momentum which are in general not collinear, therefore, its inert properties are specified by two, rather than one, invariant quantities. It is shown that a spinning particle need not be a non-Galilean object. The necessity of a rigid mechanics for the construction of a consistent classical electrodynamics in spacetimes of dimension $D + 1$ is justified for $D + 1 > 4$. The problem of how much the form of fundamental laws of physics orders four dimensions of our world is revised together with its solution suggested by Ehrenfest. The present analysis made it apparent that the notion of the “back-reaction” does not reflect the heart of the matter in classical field theories with point-like sources, the notion of “dressed” particles proves more appropriate.

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1 Introduction

In modern textbooks on classical field theory (see, e.g., [1]–[4]) the concept of inertia of point-like objects has received not too much attention. Some authors are concerned with a parameter of the appropriate
dimension in the mechanical part of the Lagrangian identifying it with mass in the Newtonian sense, familiar from the school physics, while another authors derive relativistic concepts less formally, in an “inductive” way. Yet, whatever premises, the line of reasoning is basically aimed at the indoctrination of the universal significance (both on classical and quantum levels) of the quantity \( M \) defined by the relation \( p^2 = M^2 \). This quantity is called mass, with no adjectives. Many people think of it as the only quantity specifying inert properties of particles.

However, for a more rigorous treatment, particular emphasis should be placed upon the context. For example, if we are dealing with a classical picture, the quantity \( M \) alone is insufficient. Experts are well aware of this fact. However, they use to “feel too shy” to mention it in journals for the general physical audience and textbooks.

Put very simply, the essence of the problem is this. States of a relativistic point-like object may be characterized by the four-coordinate \( x^\mu \) in Minkowski space and the four-momentum \( p^\mu \). On the classical level, the four-velocity \( v^\mu = dx^\mu /ds \), \( s \) being the proper time, is also well defined. From the vectors \( p^\mu \) and \( v^\mu \), two invariants can be built:

\[
M^2 = p^2
\]

and

\[
m = p \cdot v,
\]

while the invariant \( v^2 = 1 \) is dynamically trivial, it manifests only the parametrization choice. \( M \) and \( m \) are called, respectively, the mass and the rest mass. For Galilean particles, these quantities are numerically equal. However, non-Galilean particles are also tolerable in classical theory. In the absence of external forces, such particles can execute not only uniform motion, but also trembling, self-accelerating, self-decelerating, and hyperbolic motions. A free non-Galilean object possesses the four-velocity \( v^\mu \) and the four-momentum \( p^\mu \) which in general are nonparallel, and hence inert properties of this object are characterized by two different quantities \( M \) and \( m \).

Three types of non-Galilean objects, spinning, rigid, and dressed particles, are a central preoccupation of this review. Schrödinger was the first to speak about a trembly regime, the visualization of solutions to the Dirac equation; since then this phenomenon bears the expressive German name “Zitterbewegung”.

More recently a classical realization of this phenomenon, a helical world line \([1,2]\), was found. An evolution mode for a free dressed charged particle with exponentially growing acceleration, see, e.g., \([3,4]\), was discovered (presumably by Lorentz) at the end of the 19th century. Although such solutions to the equations of motion are believed to be “unphysical”, their very existence changes the view on the Galilean motion as the exclusive regime for particles subjected to the “self-interaction”, actually for every real particle since any one possesses some charge (electric, Yang–Mills’, or gravitational), hence being “self-interacting”. The capability of a dressed colored particle for the self-decelerated motion in the absence of external forces was pointed out in \([8,9]\). The fact that rigid particles can execute Zitterbewegungs and runaways has also not gone unnoticed, see, e.g., \([1]\) and references therein.

Our discussion is restricted to the classical context, that is, any pure quantum problem is ruled out, and we do not touch on issues in curved spacetimes of general relativity. Quantum theory may be sporadically mentioned to make more prominent the classical character of the subject. The set of allowable world lines is taken to be composed of only smooth timelike or lightlike future-directed world lines, containing no spacelike curves or fragments of such curves (associated with superluminal motions), past-directed timelike curves (interpreted as the world lines of anti-particles), and piecewise smooth timelike curves made of adjacent future-directed and past-directed fragments. We leave aside any modification of the notion of mass related to extensions of spacetime symmetries, in particular the Schouten–Haantjes idea that mass behaves as a scalar density of the weight \(-1/2\) under conformal transformations, see, e.g., \([12]\). We dwell on elementary objects while the problem of mass of composite systems receives little attention. We adopt the retarded (rather than advanced or some else) boundary condition on classical fields generated by particles.

Among models of spinning particles, we address only J. Frenkel’s model \([13]\), the first description of a classical particle with spin, and the model with Grassmannian variables for spin degrees of freedom proposed by F. A. Berezin and M. S. Marinov \([14]\), and R. Casalbuoni \([15]\), and elaborated by C. A. P. Galvao and C. Teitelboim \([16]\). We discuss rigid particles with acceleration-dependent Lagrangians, even though models with higher derivatives may be found in the literature (for a complete list of references see \([17]\)). We are concerned with dressed charged and colored particles in four-dimensional Minkowski space \(E_{1,3}\), but we do not cover another dressed particles in \(E_{1,3}\), e. g., particles interacting with scalar or tensor fields \([18]\), dressed particles with spin interacting with electromagnetic \([13,20]\), scalar and tensor fields
dressed particles in curved manifolds and in flat spaces of other dimensions, for example, in $E_{1,5}$.

The purpose of these limitations is twofold. First, they reveal the presence of several quantities specifying inertia even in this restricted scope and separate the nonuniqueness problem in the given context from that in the general case. Second, a careful analysis of the notion of mass is requisite for other contexts. It is not improbable that this task may seem attractive to the reader of this paper.

The paper is organized as follows. Section 2 deals with the problem of mass for Galilean particles together with relevant issues omitted in the educational literature. It transpires that Newton’s second law is tailored for the relativistic mechanics being smoothly embedded into the four-dimensional geometry. We show how the embedding is accomplished. Two forms of the action of Galilean particles are considered. Their equivalence for a finite particle mass, and admissibility of only one of them for massless particles are established. Section 3 offers an account of inert properties of the Frenkel particle. In the modern formulation, this model is simple (at least in the absence of interactions) and instructive, but its description is scattered over research papers. Because of this, the model is discussed carefully and in the form convenient for the introductory learning. The fact that spinning particles do not necessarily behave as non-Galilean objects is demonstrated by the example of the model with Grassmannian variables in Sec. 4. Section 5 is devoted to the problem of inert properties of rigid particles. Motivations for studies of the higher derivative dynamics are given in Sec. 6. It is shown that the construction of a consistent classical electrodynamics in spacetimes of dimensions $D + 1 > 4$ leads inevitably to the notion of rigid particles. In this connection, we revise the problem of the four-dimensionality of our world together with its solution suggested by P. Ehrenfest. In Sec. 7, the notion of the “dressed” particle is shown to be more adequate for classical theories with point-like sources than the notion of the “self-interacting” particle. The problem of mass is illustrated by two comparatively simple examples of dressed particles. The summary of the discussion and points in favor of it are in Sec. 8.

The paper is intended mainly for readers with the basic knowledge of standard field theory. That is why major issues are self-contained whenever possible1 and, hopefully, their understanding will not require to consult original papers.

For the most part, we use standard notations. Gaussian units are adopted, the speed of light $c$ and elementary quantum $\hbar$ are set to 1. The metric is $\eta_{\mu\nu} = \text{diag}(+,-,-,-)$. Repeated Greek indices take the values 0, 1, 2, 3, while Latin indices run from 1 to 3. In some instances an obvious coordinate free geometric symbolism is applied to four-dimensional quantities, and three-dimensional vectors are denoted by boldface letters. World lines are parametrized either by an affine parameter $\lambda$ (derivative w. r. t. $\lambda$ is denoted by a prime) or with the aid of the proper time $s$ (derivative w. r. t. $s$ is denoted by a dot). The special symbols $v^\mu$ and $a^\mu$ stand, respectively, for the four-velocity $\dot{x}^\mu$ and four-acceleration $\ddot{x}^\mu$.

2 Galilean particle

Were we striving to embody the special relativity in a single phrase, this intention is best expressed as follows: “Spacetime of the physical world is described by pseudoeuclidean four-dimensional geometry of the signature $+,-,-,-$”. This means implicitly that all dynamical laws are represented as geometric statements.

An adherent of the deductive method of the “Course of Theoretical Physics” by Landau and Lifshitz, who normally views the principle of least action as Alpha and Omega of theoretical constructions, should meet such a geometric encoding with sympathy. Indeed, given a geometry, we can determine geometric invariants, write down Lagrangians as all possible invariant structures, and, varying the action, derive dynamical equations. We, therefore, can formally whittle things down to setting the geometry.

However, with closer inspection of the substantive aspect, it would transpire that an important issue was overlooked. It is impossible to verify experimentally the geometry by itself. The point, going back to Poincaré, is what to be verified is just the totality of geometry and physical laws, or, in symbolic form, $\Gamma + \Phi$. Changing $\Gamma$, one can modify $\Phi$ in such a way that theoretical predictions of phenomena are left intact. Therefore, it is insufficient to fix the spacetime description (patterned after the pseudoeuclidean

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1 Such a detailing may be justified by the fact that the analyzed problem still went unnoticed not only in the special monographic literature (the well-known book including), but also in essays for the general physical audience.
geometry), one should also clarify the way the operationally well defined physical notions (such as force, mass, energy and momentum) are incorporated into the theory.

Let us make clear the status of the Newtonian dynamics. A widespread misunderstanding is that the second Newton’s law in its primordial form

$$\frac{dp}{dt} = f$$

(3)
does not work at velocities comparable with the speed of light, it must be denounced in this domain, and the “true law of the relativistic mechanics”

$$\frac{d}{dt} m\gamma v = F, \quad \gamma = \frac{1}{\sqrt{1 - v^2}}$$

(4)
derived by Planck in 1906 [27] must be accepted. In actual fact, Eq. (3) need neither be rejected, nor modified, it should be only smoothly embedded it into the four-dimensional geometry of Minkowski space, which automatically yields Eq. (4).

The idea of the embedding is rested on the fact that Eq. (3) becomes an asymptotically exact law as \(v \to 0\). This means that Eq. (3) describes quite correctly the dynamics in an instantaneously accompanying inertial frame of reference where the velocity of the object is \(v = 0\), or, in the geometric language, the vector relation (3) is exact on the hyperplane \(\Sigma\) perpendicular to the world line. Meanwhile the hyperplane \(\Sigma\) rotates together with the normal vector \(v^\mu\) as one travels along the world line, Figure 1. Thus the algorithm for construction of a global relativistic picture is to jump in the instantaneously accompanying inertial frame, read and execute the local dynamical prescription (3); jump in the next instantaneously accompanying frame, etc. In other words, for the embedding, we need an operator \(\nu\) that would permanently project vectors of Minkowski space on the hyperplanes \(\Sigma\) perpendicular to the world line. This operator is

$$\nu_{\mu\nu} = \eta_{\mu\nu} - \frac{v^\mu v^\nu}{v^2},$$

(5)
and the projection of any vector \(X^\mu\) on the hyperplane \(\Sigma\) is

$$(\nu X)^\mu = X^\mu - X \cdot \frac{v}{v^2} v^\mu.$$  

(6)
The formulas (5) and (6) are valid also for arbitrarily parametrized world lines, one should only change \(v\) by \(x^\prime\). As to the parametrization by the proper time, we are dealing with even simpler formulas, because \(v^2 = 1\).

Consider how the projector (5) embeds the one-parameter family of the three-dimensional equations (3) in four dimensions. In the instantaneously accompanying frame, the time axis \(t\) is aligned with the tangent to the world line at the given instant, thus \(dt\) coincides with \(ds\) (this follows formally from the relation \(ds = \gamma^{-1}dt\) where \(\gamma \to 1\) as \(v \to 0\)), and the differentiation w. r. t. \(t\) can be replaced by the differentiation w. r. t. \(s\). From the Newtonian three-force \(f\), one can uniquely regain the Minkowski four-force \(f^\mu\). Indeed, components of \(f^\mu\) in an arbitrary frame of reference originate (through the Lorentz boost) from components of this vector in the rest frame where, by definition, they are

$$f^\mu = (0, f).$$

(7)
Define the four-momentum $p^\mu$ in such a way that the derivative of its spatial components w. r. t. $s$ coincide with components of the three-vector $dp/dt$ in Eq. (3) in the accompanying frame of reference. Then the required embedding is

$$v \perp (\dot{p} - f) = 0.$$  \hspace{1cm} (8)

We notice that the projector (3) is defined only on timelike tangent vectors and makes no sense for isotropic tangent vectors, thus solutions to Eq. (8) describe only smooth timelike world lines.

Mechanical objects of different types reveal different dependences of $p^\mu$ on kinematical variables. The simplest possibility provides an elementary Galilean object. This is a point-like object. Its states in the nonrelativistic limit are specified by the three-coordinate of its location $x$ and its three-momentum}

$$p = m v.$$  \hspace{1cm} (9)

Such objects are usually called particles, with no adjectives. We will follow this tradition, though one should bear in mind that we cover not the total set of point-like mechanical objects but only its Galilean subset.

The Newtonian mass $m$ is a fundamental characteristic of the particle, it remains constant no matter how great the influence on the particle (that is, under every possible force $f$):

$$\frac{d}{dt} m = 0.$$  \hspace{1cm} (10)

The “elementary” character of the object will be understood as the impossibility of its splitting which is formally controlled by the condition (10).

With the expression (9) for $p$, Eq. (3) reduces to

$$m a = f.$$  \hspace{1cm} (11)

For $f = 0$, Eq. (11) has a unique solution $v = \text{const}$. Thus free particles evolve in the Galilean regime.

From (3), it is clear that the particle four-momentum $p^\mu$ is

$$p^\mu = m v^\mu.$$  \hspace{1cm} (12)

In view of the relation $v \cdot a = 0$, the projector $v \perp$ in Eq. (8) acts as a unite operator, and this equation is reduced to

$$m a^\mu = f^\mu.$$  \hspace{1cm} (13)

Since the Minkowski four-force is orthogonal to the four-velocity, $f \cdot v = 0$, components of $f^\mu$ in an arbitrary Lorentz frame of reference are not independent, they are related by $f^0 = f^i v^i$. It is convenient to separate $\gamma$ as an overall factor of $f^\mu$:

$$f^\mu = \gamma (F \cdot v, F).$$  \hspace{1cm} (14)

Then $F$ is found to be the three-force in the Planck sense because the spatial component of Eq. (13) acquires the form (3). As to the time component,

$$\frac{d}{dt} m\gamma = F \cdot v,$$  \hspace{1cm} (15)

it may be interpreted as the equation of variation of energy $E = m\gamma$ due to the work performed by the force $F$ in a unite time.

So, the replacement of (3) by (11) does not imply that the Newtonian dynamics, as such, has been subjected to a revision or modification, it demonstrates only that Newton’s second law has been smoothly embedded into the four-dimensional pseudoeuclidean geometry. (Fixing such a kind of geometry, we

2 The particle may have another quantities specifying its individuality, for example, couplings with different fields. But, unlike mass specifying the particle “intrinsically”, these quantities characterize it relative to other objects.

3 The condition (14) is consistent with the assumption that $f^\mu$ in the rest frame takes the form (7). If we put $f^\mu = (k, f)$, rather than (7), then, apart from (3), the equation $\dot{m} = k$ would arise.
thus have maximally “loaded” Γ and left a minimum of the “load” for Φ – it is just the virtue of the pseudoeuclidean model of spacetime.

The mass $M$ and rest mass $m$ of a particle are defined by Eqs. (1) and (2). With the expression for $p^\mu$, Eq. (12), $M$ and $m$ are identical to one another and the Newtonian mass (denoted also by $m$). The latter remains constant not only when the particle is free, but also under the action of any force (this is called for by the convention of the elementary character of the particle).

The equality $M = m$ is crucial for the equivalence of mass and rest energy. Note in this connection that the concept of inertia is not replenished with a “relativistic” content. All the conceptual novelty of relativistic dynamics of Galilean particles, as opposed to the Newtonian dynamics, amounts to the mere geometrical fact that energy and momentum are temporal and spatial components of the timelike vector $p^\mu$ of length $M$, and, therefore, the particle energy $E$ is reckoned from $M$, rather than 0.

We now turn to the Lagrangian and Hamiltonian formulations. Let us see the way the projective structure of Eq. (8) is related to symmetries of the theory. The action $A$ depends on the world line configuration, rather than the parametrization, and hence $A$ remains unchanged under the reparametrization transformations

$$\lambda = \lambda(\xi), \quad x^\mu(\lambda) = x^\mu(\lambda(\xi))$$

where $\lambda(\xi)$ is an arbitrary continuous monotonic function of $\xi$. We represent the reparametrizations in the infinitesimal form:

$$\delta\lambda = \epsilon, \quad \delta x^\mu = \epsilon x'^\mu,$$

$\epsilon = \epsilon(\xi)$ is an arbitrary infinitesimal continuous positive function of $\xi$. From the invariance of $A$ under the transformations (16) follows the identity

$$\frac{\delta A}{\delta x'^\mu} x'^\mu = 0$$

which just implies that the Eulerian $\delta A/\delta x'^\mu$ involves the projective operator $\nabla$. Thus, given a reparametrization invariant action, this provides the embedding of the Newtonian dynamics into spacelike hyperplanes $\Sigma$ (which embody the ordinary three-space in instantaneously accompanying frames of reference).

The reparametrizations (16) is a kind of local gauge transformations; their analog in general relativity is general coordinate transformations $x^\mu = x^\mu(y)$, the so called diffeomorphisms of the pseudo-Riemannian space. Extending the dynamical framework, we may, along with $p^\mu$ of the form (12), consider any conceivable dependence of the momentum on kinematical variables, yet the reparametrization invariance requirement, or, equivalently, the presence of the projective structure remains therewith indisputable.

It might be well to point out that the projector $\nabla$ acts as a unite operator solely for $p^\mu = mv^\mu$. Thus, it would be erroneously to think, as is, alas, the case, that Eq. (13) is the equation of the relativistic dynamics in a broad sense. Such a role is assigned to Eq. (8). It is the equation that describes the evolution of any structureless mechanical object of finite mass.

The action for a relativistic Galilean particle proposed by Planck [27],

$$A = -\mu \int dt \sqrt{1 - v^2}$$

is readily rewritten in the reparametrization invariant form:

$$A = -\mu \int d\lambda \sqrt{x'^\mu x'^\mu}.$$  

The variation of (19) w. r. t. $v$ gives the canonical three-momentum

$$p^\mu = \mu \gamma v^\mu,$$

while the variation of (20) w. r. t. $x'^\alpha$ gives the canonical four-momentum

$$p^\alpha = \frac{\delta A}{\delta x'^\alpha} = \frac{\mu x'^\alpha}{\sqrt{x'^\mu x'^\mu}}.$$  

Note that the identity (18) is the simplest illustration of the second Noether theorem [28] in the case of the infinite-dimensional group of transformations (16) leaving the action $A$ invariant.
The expression (22) is coincident with the expression (12) when $\mu = m$. Therefore, the formal parameter $\mu$ should be identified with the quantities $m$ and $M$, and also the Newtonian mass. We will find in the following that such quantities specifying non-Galilean object are all distinct.

One further reparametrization invariant action for a Galilean particle proposed by L. Brink, P. Di Vecchia and P. Howe is

$$A = -\frac{1}{2} \int d\lambda \left( \frac{x'^2}{\eta} + \eta \mu^2 \right).$$

(23)

Here, $\eta(\lambda)$ is an auxiliary variable; the reader aware of elements of general relativity may interpret it as the square root of the determinant of the one-component metric world line tensor $\sqrt{\det g_{\lambda\lambda}} = \sqrt{g_{\lambda\lambda}}$ since its transformation law under the reparametrizations (16) is

$$\eta(\lambda) = \frac{d\xi}{d\lambda} \eta(\lambda(\xi)).$$

In the literature, the quantity $\eta^{-1}$ is referred to as “Einbein” or “monad”.

We now can define the massless Galilean particle as an object for which $\mu = 0$. The action (23) for such a particle is nonzero.

The variation of the action (23) w. r. t. $\eta$ gives the constraint equation

$$-\eta^{-2} x'^2 + \mu^2 = 0$$

(24)

from which in the case $\mu \neq 0$ we find

$$\eta^{-1} = \mu(x' \cdot x')^{-1/2}.$$  

(25)

When $\lambda$ is realized as the “laboratory time” $t$, we have $\eta^{-1} = \mu / \sqrt{1 - \nu^2}$, that is, the monad is identical to energy $E$ of the particle, while, for $\lambda = s$, we have $\eta^{-1} = \mu$.

Substituting (25) in (23), we revert to the Planck action (20).

Notice, for $\mu \neq 0$, the actions (24) and (23) are equivalent on the quantum level as well. This can readily be verify by means of the Feynman path integral. Indeed, when employing the action (23), the path integral involves an additional integration over $\eta(\lambda)$ that can be worked out through the use of the well-known result for the one-dimensional integration:

$$\int_0^\infty \frac{d\eta}{\sqrt{\eta}} \exp\left(-\frac{A}{\eta} - B\eta\right) = \sqrt{\frac{\pi}{B}} \exp(-2\sqrt{AB}), \quad A > 0, B > 0.$$  

From the action (23), we obtain the expression for the canonical momentum:

$$p^\alpha = \eta^{-1} x'^\alpha.$$  

(26)

With it, the constraint (24) is written in the form

$$-p^2 + \mu^2 = 0$$

(27)

which is identical to (1) when $M = \mu$.

The Hamiltonian corresponding to the action (23) is

$$H = p \cdot x' + L = \frac{1}{2} \eta (p^2 - \mu^2).$$

(28)

It is notable that $H = 0$ on the constraint (24). *Zero Hamiltonians are generally inherent in reparametrization invariant models.* [The Hamiltonian corresponding to the action (20) is identically zero.] The action (24) is then representable in the form quite clear from the canonical formalism viewpoint:

$$A = \int d\lambda \left( -p \cdot x' + H \right) = \int d\lambda \left[ -p \cdot x' + \frac{1}{2} \eta (p^2 - \mu^2) \right]$$

(29)

where $\eta$ may be interpreted as the Lagrangian multiplier of the problem with the constraint (27).

For $\mu = 0$, the actions (20) and (23) are *nonequivalent*. To describe a massless particle one should evidently proceed from the action (23). (Notice, there is a number of “desert islands” in this region, for one, the problem of the motion of a massless particle under the action of some simple force.)

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5In the subnuclear physics, this fact provokes occasionally the temptation to construe the quantities $\mu$ and $\eta^{-1}$ as, respectively, the *current* and *constituent* masses of quarks [30]. The reason for this is that light, $u$ and $d$, quarks confined in hadrons behave as ultra-relativistic objects for which $E \gg \mu$, hence a plausible explanation of great difference of values of the current and constituent masses of such quarks.
3 Pure gyroscope

As the first example of non-Galilean objects, we look at Frenkel’s spinning particle \[13\] (another name is pure gyroscope) following largely to \[31\]. For alternative treatments see, e. g., \[7, 4, 32\]. We consider a free particle, that is, the interaction with any field (in particular gravitational) is negligible. Because spacetime is homogeneous and isotropic, the four-momentum \( p^\mu \) and angular momentum tensor \( J_{\mu\nu} \) are conserved quantities. Write down explicitly the conservation laws:

\[
\dot{p}^\mu = 0, \tag{31}
\]

\[
\dot{J}_{\mu\nu} = 0. \tag{32}
\]

It is beyond reason to augment them by the addition of the spin conservation law \( \dot{\sigma}_{\mu\nu} = 0 \) since no extra symmetry is suggested. By (30)–(32), we have

\[
\dot{\sigma}_{\mu\nu} = p^\mu v_\nu - p^\nu v_\mu, \tag{33}
\]

thus the four-velocity and four-momentum of the spinning particle are in general not collinear.

The pure gyroscope is defined as such a particle that

\[
\sigma_{\mu\nu} v_\nu = 0. \tag{34}
\]

To understand the geometric meaning of this constraint, write a general 2-form \( \sigma \) as a combination of exterior products of 1-forms (covectors) \( \vec{v}, \vec{e}_1, \vec{e}_2, \vec{e}_3 \) which span a moving basis:

\[
\sigma = \sum_i D_i \vec{v} \wedge \vec{e}_i + \sum_{i<j} K_{ij} \vec{e}_i \wedge \vec{e}_j. \tag{35}
\]

Let this basis be orthonormal at any instant:

\[
\vec{v}^2 = 1, \quad \vec{v} \cdot \vec{e}_i = 0, \quad \vec{e}_i \cdot \vec{e}_j = -\delta_{ij}. \tag{36}
\]

Insertion of (35) in (34) gives

\[
D_i = 0. \tag{37}
\]

Now, only three terms in (37) are left:

\[
\sigma = K (\vec{e}_1 \wedge \vec{e}_2 + \alpha \vec{e}_1 \wedge \vec{e}_3 + \beta \vec{e}_2 \wedge \vec{e}_3). \tag{38}
\]

Using the bilinearity and skew-symmetry of exterior products, the expression in the parenthesis can be identically transformed to

\[
\vec{e}_1 \wedge (\vec{e}_2 + \alpha \vec{e}_3) - \beta \vec{e}_3 \wedge \vec{e}_2 = (\vec{e}_1 - \beta \vec{e}_3) \wedge (\vec{e}_2 + \alpha \vec{e}_3).
\]

Introducing two new base 1-forms \( \vec{f}_1 = \vec{e}_1 - \beta \vec{e}_3 \) and \( \vec{f}_2 = \vec{e}_2 + \alpha \vec{e}_3 \), we arrive at

\[
\sigma = K \vec{f}_1 \wedge \vec{f}_2. \tag{38}
\]

Equation (38) does not alter when the 1-form \( \vec{f}_1 \) is substituted by the 1-form

\[
\vec{g}_1 = \vec{f}_1 - \frac{\vec{f}_1 \cdot \vec{f}_2}{f_2} \vec{f}_2
\]

\[6\]If the spinning particle is electrically charged, the Lagrangian involves the so called Pauli term proportional to \( \sigma_{\mu\nu} F^{\mu\nu} \). In view of (33) and (34), this term takes the form \( d \cdot E + m \cdot B \) where the electric dipole moment \( d_i \) is proportional to \( D_i \), and the magnetic dipole moment \( m_i \) is proportional to \( \epsilon_{ijk} K_{jk} \). From (17) follows that the pure gyroscope corresponds to a spinning particle with a magnetic dipole moment, but with no electric dipole moment. Due to precession of \( m \) around magnetic lines of force, this object is referred to as the “gyroscope”.}
orthogonal to $\vec{f}_2$. Furthermore, we may normalize $\vec{g}_1$ and $\vec{f}_2$, and attribute their magnitudes to $K$. We single out the Planck constant $\hbar$ as an overall normalization in $K$ to yield

$$K\sqrt{\vec{g}_1^2 \vec{f}_2^2} = S\hbar.$$  

We now return to the initial notations of the base 1-forms, viz., $\vec{v}$, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, rather than $\vec{v}$, $\vec{g}_1$, $\vec{f}_2$, $\vec{e}_3$, stand hereafter for the resulted basis. This is quite legitimate, since only $\vec{v}$ is fixed (being cotangent to the world line) while the remainder of the basis is determined by the orthonormalization condition. The net result (recall that $\hbar = 1$) is

$$\sigma = S \vec{e}_1 \wedge \vec{e}_2$$  \hspace{1cm} (39)

where $S$ is the spin magnitude in the rest frame in which $v^\mu = (1, 0, 0, 0)$, $e_1^\mu = (0, 1, 0, 0)$, and $e_2^\mu = (0, 0, 1, 0)$. Equation (39) shows that $S$ can be defined in an invariant way:

$$\sigma_{\mu\nu}\sigma^{\mu\nu} = 2S^2.$$  \hspace{1cm} (40)

One further useful relation derivable from (39) is

$$\sigma_{\lambda\mu}\sigma^{\mu\nu}\sigma_{\lambda\rho} = -S^2\sigma_{\lambda\rho}.$$  \hspace{1cm} (41)

Let us return to the equation of the spin evolution (33). From (34), we obtain

$$\sigma_{\mu\nu}\frac{d}{ds}\sigma^{\mu\nu} = 1$$  \hspace{1cm} (44)

Contraction with $\zeta^\mu$ yields $\zeta^2 = \text{const}$. This means that only the direction of $\zeta^\mu$ varies in time, not the magnitude.

Equation (33) can be recast in the form

$$\ddot{\zeta}^\mu = -M^2 v^\mu + mp^\mu.$$  \hspace{1cm} (45)

Contraction with $\zeta_\mu$ yields $\zeta^2 = \text{const}$. This means that only the direction of $\zeta^\mu$ varies in time, not the magnitude.

Differentiation of (11) w. r. t. $s$, contraction with $v^\beta$, and making use of (33) leads to

$$m v^\lambda = p^\lambda + \frac{1}{S^2} \sigma^{\lambda\mu}\zeta_\mu,$$  \hspace{1cm} (46)

and further contraction with $p_\lambda$ results in

$$m^2 = M^2 - \frac{\zeta^2}{S^2}.$$  \hspace{1cm} (47)

We see that $m$ is a constant of motion because such are quantities in the right hand side of (47). Combining (47) and (44), we conclude that

$$m^2 > M^2.$$  \hspace{1cm} (48)
Why $M \neq m$? They differ since $v^\mu$ and $p^\mu$ are not collinear. To see this, differentiate (46) and take into account (31), (33), (43), (45), (34), and (42). The result is

$$S^2v^\mu = \zeta^\mu.$$  

Further differentiation leads to

$$S^2\dot{v}^\mu + M^2v^\mu = mp^\mu.$$  

One easily observes the similarity of (49) with the equation of harmonic oscillator under the action of an external constant force. Thus a solution is

$$v^\mu(s) = \frac{m}{M^2}p^\mu - \alpha^\mu \sin \omega s + \beta^\mu \cos \omega s$$

where $\alpha \cdot p = \beta \cdot p = \alpha \cdot \beta = 0$, $\alpha^2 = \beta^2$. Integration provides the world line:

$$x^\mu(s) = \frac{m}{M^2}p^\mu s + \frac{\alpha^\mu}{\omega} \cos \omega s + \frac{\beta^\mu}{\omega} \sin \omega s.$$  

This is a helical world line, a realization of the Zitterbewegung. The rotation with the frequency $\omega = M/S$ occurs on the plane spanned by two vectors $\alpha^\mu$ and $\beta^\mu$, perpendicular to the vector $p^\mu$. The amplitude of the rotation $\sqrt{\alpha^2} = \sqrt{\beta^2}$ being equal to the projection of the vector $p^\mu$ onto the plane spanned by two vectors $e_1^\mu$ and $e_2^\mu$ is arbitrary while the period of the rotation $T = 2\pi S/M$ is of order of the Compton wave length of the particle, $1/M$.

If we assume that $p^2 < 0$, then (51) is replaced by

$$x^\mu(s) = -\frac{m}{M^2}p^\mu s + \frac{\alpha^\mu}{\Omega} \cosh \Omega s + \frac{\beta^\mu}{\Omega} \sinh \Omega s.$$  

where $M^2 = -p^2$, $\Omega = M/S$, and $\alpha^\mu$ and $\beta^\mu$ meet the condition $\alpha^2 = -\beta^2$. This solution, describing a motion across the plane spanned by two vectors $p^\mu$ and $\alpha^\mu$, shows an enhancement of velocity. The solution (51) corresponds to a compactly supported motion, while the solution (52) corresponds to the motion in a noncompact region. If the momentum space is limited by the condition $p^2 \geq 0$, the configuration space contains the Zitterbewegung (51) but is free of the runaway (52).

Averaging (50) over $s$ gives

$$\langle v^\mu \rangle = \frac{m}{M^2}p^\mu.$$  

Let us trace the motion of the point with the coordinate

$$y^\mu = x^\mu + \frac{1}{M^2}\zeta^\mu.$$  

With (43) and (53), we have

$$\dot{y}^\mu = \frac{m}{M^2}p^\mu = \langle v^\mu \rangle.$$  

The point with the coordinate $y^\mu$ draws a straight world line with the guiding vector $p^\mu$. This point is interpreted as the center of mass. The conserved four-momentum $p^\mu$ must be assigned to an imagined carrier which is located at the center of mass and moves along the averaged world line.

The availability of two masses gives rise to two spins. Indeed, one may define spin as the internal angular momentum related to either kinematical rest frame where $v = 0$, i.e., $v^\mu = (1, 0, 0, 0)$, or dynamical rest frame where $p = 0$, i.e., $p^\mu = M(1, 0, 0, 0)$. So far we discussed the former possibility. In order to turn to the latter, we should, as is clear from (54) and (55), to use the notion of the center of mass. We express $x^\mu$ through $y^\mu$ and substitute the result in (30) to yield

$$J_{\mu\nu} = y_\mu p_\nu - y_\nu p_\mu + \Xi_{\mu\nu}$$  

where the tensor

$$\Xi_{\mu\nu} = \sigma_{\mu\nu} - (\zeta_\mu p_\nu - \zeta_\nu p_\mu)/M^2$$

plays now the same role as did $\sigma_{\mu\nu}$. In fact, the relation

$$\Xi_{\mu\nu}p^\nu = 0$$  

is an analog of the constraint (34), and hence relations analogous to (31)–(33) take place, in particular
\[ \Xi_{\mu \nu} \Xi^{\mu \nu} = 2 S^2, \]  
with \( S = \text{const}, \) and
\[ \dot{\Xi}_{\mu \nu} = 0 \]  
substitutes (33). Taking the square of both sides of (57), in view of (47), we find
\[ M^2 S^2 = m^2 S^2. \]  
It is clear that the difference between \( S \) and \( S \) is due to the difference between \( m \) and \( M \).

All these results could be derived in a more regular way starting from the Hamiltonian
\[ H = \frac{1}{2} \eta \left( p^2 - \mu^2 - \frac{\zeta^2}{S^2} \right) \]  
and the canonical Poisson brackets
\[ \{ x_\mu, x_\nu \} = \{ p_\mu, p_\nu \} = \{ x_\lambda, \sigma_{\mu \nu} \} = \{ p_\lambda, \sigma_{\mu \nu} \} = 0, \quad \{ x_\mu, p_\nu \} = \eta_{\mu \nu}, \]
\[ \{ \sigma_{\mu \nu}, \sigma_{\rho \sigma} \} = \sigma_{\mu \rho} \eta_{\nu \sigma} + \sigma_{\nu \rho} \eta_{\mu \sigma} - \sigma_{\mu \sigma} \eta_{\nu \rho} - \sigma_{\nu \sigma} \eta_{\mu \rho}, \]  
rather than from “heuristic” equations (31)–(33). The parametrization of the world line should be chosen such that the monad \( \eta^{-1} \) be fixed as \( \eta^{-1} = \mu \), and the parameter \( \mu \) be identified with \( m \). Note that the expression in the parenthesis is the constraint (47) whereby \( H = 0 \). The Hamiltonian (52) differs from that of Galilean particles (28) by the presence of the last term generating the evolution of spin degrees of freedom.

Thus the inertia of a pure gyroscope is specified by two invariants, \( M \) and \( m \). In the absence of interactions, they are constant, and \( m > M \) for all time. This poses the dilemma: Which quantity of these two is measured by experimenter? If only one of them is recorded in all cases, say, \( m \), then what is the reason for the prohibition from registration of another? Alternatively, if the result of the measurement is equipment-dependent, what is the peculiarity of the device that records, for example, only \( M \)?

While on the subject of a massless gyroscope, we encounter new troubles. What should be a criterion of the masslessness: \( M = 0 \) or \( m = 0 \)? If the masslessness is \( M = 0 \), then the role of the positive invariant conservative quantity \( m \) is obscure. If, on the other hand, the masslessness is \( m = 0 \), then \( p^2 < 0 \), i. e., dynamically, the object behaves as a tachyon (which, though, by no means suggests jumping through the light barrier!).

Upon quantization, only a single of these two quantities, \( M \) or \( m \), may survive. Which of them? Should the classical particle emerging in the limit \( \hbar \to 0 \) be massless (from some viewpoint) if the initial quantum particle is massless?

Finally, one further dilemma is to decide between \( M \) and \( m \) in the presence of gravitation; turning to the principle of equivalence of inert and gravitational masses, one of these two quantities, \( M \) or \( m \), should be set equal to the gravitational mass \( M_g \). Which of them?

Now, following the Ortega y Gasset lessons (8), we should reveal honesty and tell the truth: We are dealing with a “rebellion of the masses”.

Another model of a classical spinning particle with \( c \)-number spinor variables for the description of spin degrees of freedom was suggested by A. O. Barut and N. Zanghi (34). In the absence of external forces, such a particle behaves in a non-Galilean way, in many respects similar to the pure gyroscope. All the above problems related to the inequality \( M \neq m \) remain here. We do not pause on this model since its analysis would contribute little new to the present topic.

### 4 Model with Grassmannian variables

The issue of inert properties of a particle with spin degrees of freedom described by real-valued Grassmannian variables \( \theta^\mu \) and \( \theta_5 \) is another thing altogether. A refined version of this model (10) is specified by the reparametrization invariant action
\[ A = \int_{\lambda_1}^{\lambda_2} d\lambda \left[ -p \cdot x' + \frac{\eta}{2} (p^2 - \mu^2) - \frac{i}{2} (\theta' \cdot \theta + \theta' \cdot \theta_5 + i \chi(\theta \cdot p + \mu \theta_5)) - \frac{i}{2} [\theta(1) \cdot \theta(2) + \theta_5(1) \theta_5(2)] \right] \]

(64)
and the boundary variation conditions
\[ \delta x^\mu(1) = \delta x^\mu(2) = 0, \quad \delta \theta^\mu(1) + \delta \theta^\mu(2) = 0, \quad \delta \theta_5(1) + \delta \theta_5(2) = 0. \] (65)

The Grassmannian variable \( \chi(\lambda) \) plays the role of a Lagrange multiplier of the constraint.

From (64) and (65), one derives four dynamical equations
\[ \dot{p}^\mu = 0, \] (66)
\[ -\dot{x}^\mu + \eta p^\mu + i \chi \theta^\mu = 0, \] (67)
\[ -\dot{\theta}^\mu + \chi p^\mu = 0, \] (68)
\[ -\dot{\theta}_5 + \chi \mu = 0 \] (69)

(the proper time is chosen to be the parameter of evolution: \( \lambda = s \)) and two constraints
\[ p^2 - \mu^2 = 0, \] (70)
\[ \theta \cdot p + \mu \theta_5 = 0. \] (71)

A first glance, the dependence between the momentum and velocity, (67), is a direct analog of the dependence (46) responsible for the non-Galilean behavior of the pure gyroscope. But this resemblance is deceptive. Indeed, since \( \theta^0 \theta^0 = \theta^1 \theta^1 = \theta^2 \theta^2 = \theta^3 \theta^3 = 0 \), it follows from (67) that
\[ (\dot{x}^0 - \eta p^0)^2 = (\dot{x}^1 - \eta p^1)^2 = (\dot{x}^2 - \eta p^2)^2 = (\dot{x}^3 - \eta p^3)^2 = 0. \] (72)

We see that \( p^\mu \) is parallel to \( \dot{x}^\mu \). But \( \dot{x}^\mu \) is a timelike vector directed to the future, and hence, in view of (71), \( \eta = \mu \). Equation (67) is satisfied only for \( \chi = 0 \). From (68) and (69) follows that the Grassmannian variables do not vary in time, \( \dot{\theta}_5 = 0 \), while (66) and (67) imply \( \dot{x}^\mu = \text{const.} \).

Thus the spin and configuration variables evolve independently. The behavior of a free object with the Grassmannian variables proves to be strictly Galilean. The object is characterized by a single mass, \( \mu = m = M \).

It is obvious that the object capable of solely Galilean regime of evolution will not identified with a non-Galilean object. Thus the model of spinning particles with real-valued Grassmannian variables is not equivalent to the model of spinning particles with spinor \( c \)-number variables [34], contrary to the wrong assertion of Ref. [35].

5 Rigid particle

A point-like particle with the behavior governed by a Lagrangian dependent on higher derivatives is our next example of non-Galilean objects. Such a particle is called rigid. The velocity and momentum of the rigid particle are in general nonparallel, it can execute Zitterbewegung and runaway regimes. Thus the mass \( M \) and rest mass \( m \) of rigid particles are different quantities. Their dissimilarity is even greater than that of the pure gyroscope: \( M \) turns out to be a conserved quantity while \( m \) varies in time. Yet, it is not worth while to run ahead, it would be better to discuss the subject in succession.

Recall that, by our convention, allowable world lines are only timelike smooth curves. It would be sufficient for present purposes to consider a reparametrization invariant action dependent on velocities and accelerations,
\[ A = \int_{\lambda_1}^{\lambda_2} d\lambda L(x', x''). \] (73)

It immediately follows that the Lagrangian may be written as
\[ L(x', x'') = \gamma^{-1} \Phi(k), \] (74)
\[ \gamma^{-1} = \sqrt{x' \cdot x''}, \] (75)

where \( \Phi(k) \) is an arbitrary function of the world line curvature \( k \). It is well known that the curvature squared is equal and of opposite sign to the four-acceleration squared. We recall also that, in the general
In the case of a curve with an arbitrary parametrization \( x^{\mu}(\lambda) \), the four-acceleration \( a^{\mu} \) is calculated from the formula:

\[
a^{\mu} = \gamma \frac{d}{d\lambda} \left( \gamma \frac{dx^{\mu}}{d\lambda} \right). \tag{76}
\]

The Hamiltonian formalism of the rigid dynamics was originally developed in a fundamental M. V. Ostrogradskii memoir back in 1850 \[36\]. We will need only some findings of this formalism \[37\] (the derivation of them is left to the reader as a useful exercise). The infintesimal variation of the action can be represented in the form

\[
\delta A = \int_{\lambda_1}^{\lambda_2} d\lambda \left[ \frac{\partial L}{\partial x^{\mu}} - \frac{d}{d\lambda} \left( \frac{\partial L}{\partial x'^{\mu}} \right) \right] \bar{\delta}x^{\mu} + (H \delta \lambda - p \cdot \delta x - \pi \cdot \delta x') |_{\lambda_1}^{\lambda_2} \tag{77}
\]

where the symbol \( \bar{\delta} \) stands for the form variation of the world line, \( \bar{\delta}x^{\mu} = \delta x^{\mu} - x'^{\mu} \delta \lambda \),

\[
p^{\mu} = -\frac{\partial L}{\partial x'^{\mu}} + \frac{d}{d\lambda} \left( \frac{\partial L}{\partial x''^{\mu}} \right), \tag{78}
\]

\[
\pi^{\mu} = -\frac{\partial L}{\partial x''^{\mu}}, \tag{79}
\]

\[
H = p \cdot x' + \pi \cdot x'' + L. \tag{80}
\]

As far as the Lagrangian \( L \) is invariant under the four-coordinate translations

\[
x_{\mu} \to x_{\mu} + c_{\mu}, \tag{81}
\]

one infers (in line with the first Noether theorem) from (77) that \( p^{\mu} \) is a constant of motion. On the other hand, the Lagrangian \( L \) defies invariance under the four-velocity translations

\[
x'^{\mu} \to x'^{\mu} + d_{\mu}, \tag{82}
\]

since this would conflict with the reparametrization invariance which is assured by the presence in (74) of the quantity \( \gamma \) non-invariant under the transformation (82), hence the canonical momentum \( \pi^{\mu} \) is not conserved. Besides, using formulas (75), (76) and (78)–(80), one may check that the Hamiltonian \( H = 0 \) for any Lagrangians of the form (74); this is a consequence of the reparametrization invariance of the action (73). Thus \( \pi^{\mu} \) and \( H \) are unusable for the determination of inert properties of a rigid particle, from here on they will be of no interest.

Let the evolution parameter \( \lambda \) be the proper time \( s \). The Lagrangian (74) yields the equation of motion

\[
(\overset{\cdot}{v} \cdot p)^{\mu} = 0. \tag{83}
\]

This equation shows plainly that, in the absence of external forces, the canonical momentum \( p^{\mu} \) is a conserved quantity. Thus the invariants (1) and (2) built from \( p^{\mu} \) may characterize the inertia of a rigid particle.

The problem of integration of the equation of motion (83) in the generic case of arbitrary smooth functions \( \Phi(k) \) was investigated in \[38\] where the interested reader is referred to for detail. We will discuss here only a particular case

\[
\Phi(k) = -\mu + \nu k^2, \tag{84}
\]

where \( \mu \) and \( \nu \) are arbitrary real parameters. We choose \( \mu > 0 \) since, for \( \nu = 0 \), one regains the Planck Lagrangian \( L = -\mu \sqrt{x' \cdot x'} \) where \( \mu \) is taken to be the rest mass \( m \) of a Galilean particle. For the Lagrangian (84), one derives

\[
p^{\mu} = \mu v^{\mu} + \nu (2a^{\mu} + 3a^2 v^{\mu}). \tag{85}
\]

Let the particle be moving along z-axis. Then \( v^{\mu} \) may be represented as follows

\[
v^{\mu} = (\cosh \alpha, 0, 0, \sinh \alpha). \tag{86}
\]

Differentiation gives higher derivative expressions, specifically,

\[
a^{\mu} = \dot{\alpha} (\sinh \alpha, 0, 0, \cosh \alpha)\]
which implies $a^2 = -\dot{\alpha}^2$. Equation (83) reduces to

$$\mu\dot{\alpha} + \nu(2\ddot{\alpha} - \dot{\alpha}^3) = 0.$$ 

Denoting $\dot{\alpha} = q$ and $\mu/\nu = q_*^2$, rewrite it in the form

$$\ddot{q} + \frac{1}{2} q_*^2 q - \frac{1}{2} q^3 = 0. \quad (87)$$

The first integral of this equation is

$$\frac{1}{2} q^2 + U(q) = E, \quad (88)$$

$$U(q) = -\frac{1}{8}(q^2 - q_*^2)^2; \quad (89)$$

$E$ is an arbitrary integration constant.

Equations (87) and (88) may be viewed as the equations of motion of some fictitious particle of the unite mass in the potential field $U(q)$. For $\nu > 0$, the potential $U(q)$ has the shape schematically depicted in Figure 2, the left plot. For $-q_*^2/8 < E < 0$, the motion of the fictitious particle is compactly supported, and falls in the range $-q_* < q < q_*$. Thus, at not-too-large initial acceleration, $|a^2| < \mu/\nu$, the rigid particle executes a Zitterbewegung. For $E > 0$, or $E < -q_*^2/8$, the fictitious particle executes an infinite motion. In other words, if the initial acceleration of the rigid particle exceeds the critical value $(\mu/\nu)^{1/2}$, a runaway regime is certainly realized. For $E = 0$, the fictitious particle rests on either of two tops of the potential hill, that is, when $|a^2| = \mu/\nu$, the motion of the rigid particle proves to be uniformly accelerated. However, this regime is unstable, any small disturbance switches it to the runaway regime. For $E = -q_*^2/8$, the fictitious particle rests on the bottom of the potential pit, which corresponds to the Galilean regime of the rigid particle, $a^2 = 0$. It is clear also that, for $\mu = 0$, or $\nu < 0$, the rigid particle is capable of only a runaway regime, see the right plot in Figure 2. The Galilean regime of the rigid particle with such features is unstable, any small disturbance switches it to a runaway regime. Thus the instances with $\mu = 0$ and $\nu < 0$ are of no physical interest.

![Figure 2: The potential $U(q)$ of the fictitious particle](image)

With the aid of the Ansatz (86), (85) can be transformed to

$$p^\mu = (\mu - \nu\dot{\alpha}^2)(\cosh\alpha, 0, 0, \sinh\alpha) + 2\nu\ddot{\alpha}(\sinh\alpha, 0, 0, \cosh\alpha).$$

It follows

$$p^2 = (\mu - \nu\dot{\alpha}^2)^2 - 4\nu^2\ddot{\alpha}^2.$$ 

The comparison with (88) and (89) shows that $p^2 = -8\nu^2 E$. Thus the condition $p^2 < 0$ is tantamount to the condition $E > 0$ which is sufficient for the motion of the fictitious particle to be infinite, or, what is the same, sufficient for the rigid particle to be in a runaway motion.

Thus, if it is granted that the rigid particle is moving along a straight line, and the four-momentum space is limited by the condition $p^2 \geq 0$, the only non-Galilean regime, the Zitterbewegung, may occur.

The non-Galilean motions are feasible not only on straight lines but also on planes. Two regimes are realized here, the Zitterbewegung and Zitterbewegungs with amplitudes enhanced in time. (If the Lagrangian depends on velocities and accelerations, but independent of higher derivatives, the dimension of the subspace $d$ where the Zitterbewegung occurs is no more than $d = 2$.)
Expression (83) can be rewritten in a geometrically illuminating form:

\[ p^\mu = (\mu + \nu a^2) v^\mu + 2\nu (\perp \dot{a})^\mu \]  

(90)

where

\[ (\perp \dot{a})^\mu = \dot{a}_\mu + a^2 v^\mu. \]

It follows

\[ M^2 = p^2 = (\mu + \nu a^2)^2 + 4\nu^2 (\perp \dot{a})^2, \]

\[ m = p \cdot v = \mu + \nu a^2. \]  

(91)

(92)

Thus both \( M \) and \( m \) reveal nontrivial dependences on kinematic variables. Nevertheless, \( M \) is constant. As for \( m \), it varies in time both in Zitterbewegung and runaway regimes. It is time-independent only for uniformly accelerated motions. However, \( p^\mu = 0 \) for such motions. This is clear from (90) because the condition of relativistic uniformly accelerated motion is

\[ (\perp \dot{a})^\mu = 0. \]

Thus \( M \) is more fundamental than \( m \) for the rigid particle. The parameters \( \mu \) and \( \nu \) in the Lagrangian (84) should be taken positive, even though they have no direct physical meaning. If rigid particles are realized in nature, their inert properties are most likely specified by \( M \); just \( M \) is expected to be measured experimentally. In quantum picture, the inertia of rigid particles is represented by the sole \( M \) realized in nature, their inert properties are most likely specified by (84) should be taken positive, even though they have no direct physical meaning. If rigid particles are evaporation [39], the constancy of \( M \) remained to be found [40].

Strange as it may seem, the problem of the mass of the rigid particle is more simple than that of the field of local field theories in spaces of arbitrary dimensions. For example, let us express through the potential \( A \) only for uniformly accelerated motions. However, \( p^\mu = 0 \) for such motions. This is clear from (90) because the condition of relativistic uniformly accelerated motion is

\[ (\perp \dot{a})^\mu = 0. \]

Thus \( M \) is more fundamental than \( m \) for the rigid particle. The parameters \( \mu \) and \( \nu \) in the Lagrangian (84) should be taken positive, even though they have no direct physical meaning. If rigid particles are realized in nature, their inert properties are most likely specified by \( M \); just \( M \) is expected to be measured experimentally. In quantum picture, the inertia of rigid particles is represented by the sole quantity \( M \). Moreover, just \( M \) is natural to identify with the gravitational charge of the rigid particle \( M_g \) (provided that \( M_g \) is treated as a conserved quantity; with the advent of the idea of the black hole evaporation [39], the constancy of \( M_g \) became not so evident, however, the consensus on this subject still remain to be found [40]).

Strange as it may seem, the problem of the mass of the rigid particle is more simple than that of the pure gyroscope.

6 Why rigid particles?

Spinning and rigid particles are extravagant objects. While extravagances pertaining to spin are “Dei gratia”, as the saying goes, phenomenological justifications of peculiarities of the rigid dynamics still remain unknown. Is there a pure theoretical reason for recourse to the idea of higher derivatives? Newton imagined no such reason. What changed in the past three hundred years? What is the present-day role of the rigid dynamics? The following claims are quite common in the literature: The rigid particle is a toy model of rigid strings which in turn serve as a tool for effective description of phase transitions in quantum chromodynamics [41, 42]; properties of rigid particles are related to properties of hypothetical anyons [43]; the rigidity is a useful concept in the polymer chain physics [14], etc. It may well be that such arguments appear rather technical than fundamental.

To my mind, the dynamics with higher derivatives acquires its raison d’être in connection with the problem of the consistency of local field theories in spaces of arbitrary dimensions. For example, let us extend the four-dimensional classical electrodynamics of charged point-like particles to higher dimensions. Assume that the action is

\[ A = -\sum_{\nu=1}^{N} \mu_{i} \int ds_{i} \sqrt{v_{i} \cdot v_{i}} - \sum_{i=1}^{N} e_{i} \int dx_{i}^{\mu} A_{\mu}(x_{i}) - \frac{1}{4 \Omega_{D-1}} \int d^{D+1}x F_{\mu\nu} F^{\mu\nu} \]  

(93)

where \( \Omega_{D-1} \) is the area of a \( D - 1 \)-dimensional sphere of unite radius, and the field strength \( F_{\mu\nu} \) is expressed through the potential \( A_{\mu} \) in the usual fashion: \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \). Given the action (13), is it possible to build such a classical theory where all ultraviolet divergences are removed by some regularization-renormalization procedure?

Variation of the action w. r. t. \( A_{\mu} \) gives the \( D + 1 \)-dimensional Maxwell equations

\[ \partial_{\mu} F^{\mu\nu} = \Omega_{D-1} j_{\nu}. \]  

(94)

\[ j^{\mu}(x) = \sum_{i=1}^{N} e_{i} \int_{-\infty}^{\infty} ds_{i} v_{i}^{\nu}(s_{i}) \delta^{D+1}(x - x_{i}(s_{i})), \]  

(95)
The retarded solutions to these equations for the motion of the charged particles along arbitrary timelike world lines $x^\mu(s_i)$ are well known, see, e. g., [24]. We restrict our consideration to the simplest case of a single charge moving along a straight world line. Then the field is specified by a potential $\varphi(x)$, and Maxwell’s equations \( (94) – (95) \) are reduced to the Poisson equation

$$\Delta \varphi(x) = -\Omega_{D-1} \rho(x), \quad (96)$$

$$\rho(x) = e \delta^D(x). \quad (97)$$

The solution to \( (96) - (97) \) is

$$\varphi(x) = \begin{cases} |x|^{2-D}, & D \neq 2, \\ \log |x|, & D = 2. \end{cases} \quad (98)$$

The electrostatic energy of the rest particle with the $\delta$-shaped charge distribution \( (97) \), or, the self-energy, is

$$\delta m = \frac{1}{2} \int d^Dx \rho(x) \varphi(x) = \lim_{\epsilon \to 0} \frac{1}{2} e \varphi(\epsilon). \quad (99)$$

By \( (98) \), the self-energy $\delta m$ diverges linearly for $D = 3$, while the divergence is cubic for $D = 5$. These divergences are due to the singular behavior of the fields at short distances from the source, or, what is the same, slow decrease of the Fourier-transforms of the fields at high frequencies, hence the name “ultraviolet divergences”.

The standard approach to removal of these divergences is the infinite renormalization of parameters appearing in the Lagrangian. Specifically, we ascribe to the bare mass $\mu$ such a dependence on the regularization parameter $\epsilon$ as to render the sum

$$m = \lim_{\epsilon \to 0} \left( \mu(\epsilon) + \delta m(\epsilon) \right) \quad (100)$$

finite and positive. Then the renormalized mass $m$ is maintained to be the rest mass of the particle.

For small $\epsilon$, the self-energy $\delta m$ becomes large positive, thus the bare mass $\mu$ is large negative. However, this is not a particular problem since $\mu$ and $\delta m$ come to view only at intermediate stages and disappear once the passage to the limit \( (100) \) is performed. They are not observable quantities. This status is assigned only to the renormalized mass $m$.

The renormalizability is a necessary condition for consistency of local field theories \( [17, 46, 48] \). Since processes of creation and annihilation of particles are missing from the classical picture \( [1] \), the vacuum polarization responsible for the renormalization of the coupling constant $e$ is lacking. The problem is therefore reduced to the absorption of the self-energy divergences.

We now verify that $\mu$ and $\delta m$ have identical dimensions. The action is dimensionless in units $\hbar = 1$, $c = 1$. For the first term of \( (93) \), $|\mu| l [v] = 1$, and, with $[v] = 1$, we have $|\mu| = l^{-1}$. For the third term, $l^{D+1} [A^2] l^{-2} = 1$, and hence $[A] = l^{(1-D)/2}$. For the second term, $[\epsilon] l [A] = 1$, that is, $[\epsilon] = l^{(D-3)/2}$. In view of \( (100) \) and \( (34) \), $[\delta m] = [\epsilon^2] l^{2-D} = l^{-1}$. Thus the singularity of $\delta m(\epsilon)$ can be cancelled by the singularity of $\mu(\epsilon)$, and \( (100) \) becomes finite.

All troubles with divergences are then over for $D = 3$, and a consistent theory results from the action \( (34) \). However, the situation is more intricate for $D = 5$. For arbitrarily moving charged particle, the self-energy involves two divergent terms. The leading divergence is cubic. It occurs even in the static case, and is renormalized by $\mu$. However, there is one further, linear, divergence \( [24] \). It cannot be removed by the renormalization because the action \( (34) \) contains no term with a parameter $\nu$ of the appropriate dimension $[\nu] = [\nu^2] l^{-1} = l$. The corresponding contribution to the electromagnetic field momentum $P_\mu$ is proportional to

$$e^2 \epsilon^{-1} (2a_\mu + 3a^2 v_\mu).$$

When compared with \( (85) \), it becomes apparent that acceleration-dependent Lagrangians involve the parameter $\nu$ enabling the absorption of the linear divergence. Thus a consistent classical $D+1$-dimensional electrodynamics for $D + 1 > 4$ can be derived from the action with higher derivatives.

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7Indeed, the set of allowable world lines obviates timelike curves with abrupt breaks where a line going from the past to the future reverses its direction. Such world lines would correspond to processes of creation or annihilation of electron-positron pairs. World lines of this shape (“seagull” configurations) are forbidden in classical theory since the principle of last action does not apply to them.
However, what has \( D + 1 \geq 4 \) to do with us till we are in four dimensions and cannot escape to realms of higher dimensions? This raises the counter-question: Why do we think of our realm four-dimensional? Whether may four dimensions be illusory? However, given \( D + 1 = 4 \) as a plausible hypothesis, the question immediately arises: How stands out the case \( D = 3 \) against another dimensions physically? Ehrenfest [49] was the first to set and try to solve it. The essence of his solution is that. No stable composite particle system can exist in realms with \( D > 3 \), for example, a system similar to the hydrogen atom: It is imperative that the electron falls to the nucleus in it.

Greatly simplifying matters, we have to do with the solution of the relativistic Kepler problem. This is a two-particle problem which can be reduced to the problem of a single particle moving in the field of the potential \( U(r) \) and specified by the Hamiltonian (see, e. g., [1], Sec. 39)

\[
H = \sqrt{m^2 + \frac{p_\phi^2}{r^2} + \frac{p_r^2}{r^2} + U(r)},
\]

(101)

where \( p_\phi \) and \( p_r \) are the momenta canonically conjugate to the polar coordinates \( \phi \) and \( r \). Note that \( p_\phi \) is a conserved quantity, the orbital momentum \( J \). Switching off the dynamics, i. e., taking \( p_r = 0 \) in (101), we obtain the effective potential \( U(r) \) which is convenient for analyzing the particle behavior near the origin

\[
U(r) = \sqrt{m^2 + \frac{J^2}{r^2} + U(r)}.
\]

(102)

There are three alternatives. First, the attractive potential \( U(r) \) is more singular at the origin than the centrifugal term \( J/r \). The particle can in principle orbit in a circle of the radius corresponding to \( U_0 \), the local maximum of the potential \( U(r) \). But this orbiting is unstable, and the fall to the center is highly probable. If \( E > U_0 \), the fall to the center is unavoidable.

Second, \( U(r) \) is less singular than \( J/r \). In particular, for \( U(r) = -Ze^2/r \) this means that \( Ze^2 < J \). The particle executes a stable finite motion. The fall to the centre is impossible, except when \( J = 0 \).

Third, the singularities of \( U(r) \) and \( J/r \) are identical, i. e., \( U(r) = -Ze^2/r, \ Ze^2 = J \). The particle travels in a stable orbit that passes through the center.

The quantum-mechanical analysis essentially confirms these conclusions. It follows from the solutions of the Schrödinger equation [50] and relativistic wave equations for particles with spins 0 and 1/2 [51] that, in the case of sufficiently singular potentials \( U(r) \), bound states form a discrete spectrum extending from \( E = m \) to \( E = -\infty \). The system tends to more advantageous states associated with successively lower energy levels. As this take place, the dispersion of the wave function tends to zero as \( E_n \rightarrow -\infty \). The process resembles the fall to the center in its classical interpretation.

If the potential \( U(r) \) is less singular than the quantum-mechanical centrifugal term, the spectrum is bounded below. The only distinctive feature of the quantum-mechanical situation is that there exists a stable ground state with \( J = 0 \). However, this does not entail the fall to the centre, as the wave function behaves smoothly in the vicinity of the origin; there is balance between attraction and zero-point motion.

Since \( U(r) = c \varphi(r) \) where \( \varphi(r) \) is the solution of the \( D \)-dimensional Poisson equation [52], Ehrenfest inferred from this that the fall to the center is prevented for \( D = 3 \), but the fall is unavoidable for \( D > 4 \). The point \( D = 3 \) is critical, separating realms where stable bound states are feasible from those where such states are impossible.

The reason for prevention of the fall to the center is that the centrifugal manifestations of kinetic energy (the term \( J/r \) or zero-motions) dominate over attractive forces. However, the Hamiltonian [101] is essential for such a conclusion. It is derived from the action [18]. But this action is unsuited for a consistent description of electromagnetic interactions for \( D > 3 \). This action should be supplemented by terms with higher derivatives, for example, terms dependent of the world line curvatures are requisite for \( D = 5 \), terms dependent of curvatures and torsions are necessary for \( D = 7 \), etc. In the rigid dynamics, the two-particle problem is no longer Keplerian, it cannot be reduced to the problem of a single particle orbiting across a plane around the center of mass. The problem of two rigid particles in an exact setting is not solved. We can made only plausible conjecture of the behavior of such a system. The quantity responsible for the centrifugal effect is likely to be more singular than \( J/r \). For an acceleration-dependent Lagrangian, it is estimated to be \( \sim 1/r^3 \) [53], and hence the fall to the center can be prevented for \( D = 5 \) as well.

The presented reasonings are abundantly supplied with simplifications. For example, when disregarding relativistic effects of the retardation and radiation and restricting ourselves to the potential picture,
we miss the possibility of the fall to the center due to the dissipation of the particle energy (recall that the leading impetus to the invention of Bohr’s quantization rules was the problem of the fall of the radiating electron to the nucleus in the Rutherford model). However, a more complete analysis would take us away from the major theme. For a more full discussion of the suppressibility of collapse see [48].

A characteristic feature of the rigid dynamics is the Zitterbewegung. If the Zitterbewegung occurs in \(d\) dimensions, these dimensions may be considered to be frozen for rectilinear Galilean propagations. The last are feasible only in the remaining \(D - d\) dimensions. It can be shown [11] that the Zitterbewegung of a rigid particle with an acceleration-dependent Lagrangian is possible in two dimensions, and cannot stretch over larger number of dimensions. If we are dealing with \(D + 1 = 6\) where every point object executes a two-dimensional Zitterbewegung, these two dimensions are effectively compactified, and, from the point of view of a center-of-mass observer, the realm is four-dimensional. However, this effective compactification cannot be realized on the classical level. The reason is that accelerated motions of a charged particle in \(D + 1 = 6\) is attended with radiation [24]. The particle lost energy, thereby the amplitude of the Zitterbewegung is diminished, and the motion goes asymptotically to the Galilean regime.

The above construction of a consistent electrodynamics can with minor reservations be extended to the classical Yang–Mills theory: It is necessary to augment the mechanical part of the action of this theory by the addition of terms with higher derivatives [24].

The moral is that turning to the problem of four dimensions of our world inevitably leads to concepts of the rigid dynamics.

### 7 Dressed particles

The reader must be familiar with the notion of “dressed particles”, though, most probably, not from classical, but from quantum electrodynamics, where perturbation series of Feynman diagrams suggests the view of the electron wrapped up in the coat of electron-positron pairs. Owing to this coat, the renormalized mass \(m\) and charge \(e\) of the electron differ from the corresponding bare quantities \(m_0\) and \(e_0\) appearing in the initial Lagrangian. One may be under the impression that the notion of the “dressed particle” is essentially quantum, since processes of creation and annihilation of particles occur only in quantum picture. However, this impression is wrong.

The renormalization of mass takes place in any system with infinite degrees of freedom. For example, it has since midnineteenth century been known that a spherical body of mass \(m_0\) moving with velocity \(v\) through an ideal fluid behaves as an object with kinetic energy \((1/2)mv^2\) where \(m = m_0 + \delta m\), that is, its mass turns out to be augmented by the so called apparent additional mass \(\delta m\) equal to half the mass of the fluid displaced by the body. Dynamically, the dragged fluid train is integral part of this aggregate. The quantity \(m\) serves as a measure of its inertia, and the “bare” mass \(m_0\) no longer reveals itself.

The notion of the “dressed particle” is equally useful in the classical field theory with point-like sources. As is well known, historically, the idea of the electromagnetic mass precedes the quantum mechanics, originating from works by J. J. Thomson, who based himself on the analogy between the hydrodynamic medium and the aether (for more detail see, e. g., [53]). We turn to two comparatively simple models of classical dressed particles to proceed with the discussion of the problem of mass.

#### 7.1 Dressed charged particle

The Maxwell–Lorentz theory of \(N\) point-like charged particles is described by the action (93) with \(D = 4\). How does the “dressed particles” come there? Consider the source (95) composed of a single term. The generic solution to Maxwell’s equations (94) may be represented as \(F = F_{\text{ret}} + F_{\text{ex}}\) where \(F_{\text{ret}}\) is the retarded electromagnetic field generated by the source, and \(F_{\text{ex}}\) is an external field governed by the free Maxwell equations. The field \(F\) should be regularized (that is, the singularity of the function \(F_{\text{ret}}\) is smeared out in some relativistically invariant fashion) and inserted into the equation obtained by the variation of the action (93) w. r. t. \(x^\nu(s)\),

\[
\mu a^\lambda = e F^\lambda\mu \nu_{\mu},
\]
to yield, upon the renormalization of mass, Eq. (100), the Abraham–Lorentz–Dirac equation (see, e.g., [54]–[56]):

\[ m a^\lambda - \frac{2}{3} e^2 (\dot{a}^\lambda + v^\lambda a^2) = e F_{ex}^{\lambda\mu} v_\mu. \] (104)

Naively, one believes that Eq. (104) describes, as before, the evolution of mechanical degrees of freedom appearing in the action (93), but takes into account the individual actions on the particle of the external and self fields, \( F_{ex}^{\mu\nu} \) and \( F_{ret} \). The role of a finite “self-interaction” (or “back reaction”, or “radiation reaction force”, or “radiation damping force”, etc.) [1]–[4] is attributed to the higher derivative term in Eq. (104). Strange as it may seem, this interpretation exists happily for a good century, despite the fact that it is inconsistent and opens on numerous puzzles and paradoxes.

The self-interaction is, by definition, inherent in composite systems with reasonable autonomous constituents affecting each other. Such systems should possess sufficiently great number of degrees of freedom, at least \( \geq 6 \). As to Eq. (104), it is an ordinary differential equation describing the evolution of an object with the number of degrees of freedom certainly less than 6.

What is the object we are dealing with in actuality? Clearly, it is a synthetic object because it is characterized by the quantity \( m \) involving the mechanical \( \mu \) and field \( \delta m \) contributions. This object originates from the rearrangement of initial degrees of freedom in the action (13). It is natural to refer to it as a dressed particle. The dressed particle is a stretched object. It can be imagined as something like the de Broglie “pilot-wave” formed by the field train with a singularity at the point of the charge localization. Dynamical states of the dressed particle are specified by the four-coordinate of the singularity \( x^\mu \) and the attached to this point four-momentum

\[ p^\mu = m v^\mu - \frac{2}{3} e^2 a^\mu. \] (105)

The motion of the singularity is described by the equation

\[ v_\perp (\dot{p} - f) = 0 \] (106)

where \( f^\mu \) is an external four-force applied to the point \( x^\mu \). Indeed, the substitution of (105) in (106) results in the Abraham–Lorentz–Dirac equation (104) with \( f^\mu = e F_{ex}^{\mu\nu} v_\nu \). On the other hand, Eq. (106) is nothing but Newton’s second law in the invariant geometric representation. Equation (106) involves only the external force \( f^\mu \) but is deprived of an explicit “self-interaction” term occurrence. The dressed particle does not act on itself, it behaves as an elementary entity.

One further reason, advanced by C. Teitelboim [54], for the object with the four-momentum \( p^\mu \) of the form (105) to be singled out in its own right is that the Abraham–Lorentz–Dirac equation (104) stems from the energy-momentum balance at every point of the world line:

\[ \dot{p}^\mu + \mathcal{P}^\mu + P_{ex}^\mu = 0 \] (107)

where the four-momentum of the dressed particle \( p^\mu \) is defined by (105), the four-momentum of the radiation \( \mathcal{P}^\mu \) is derived from the Larmor formula

\[ \mathcal{P}^\mu = -\frac{2}{3} e^2 \int_{-\infty}^{s} d\tau \, v^\mu a^2, \] (108)

and the four-momentum \( P_{ex}^\mu \) relates to the integral of the external Lorentz four-force

\[ P_{ex}^\mu = -\int_{-\infty}^{s} d\tau \, f^\mu. \] (109)

Equation (107) reads: The four-momentum extracted from the external field \( -f^\mu ds \) is spent on the variation of the four-momentum of the dressed particle \( dp^\mu \) and the four-momentum \( \mathcal{P}^\mu ds \) carried away by the radiation.

The dressed particle can behave in a non-Galilean manner. With \( f^\mu = 0 \), Eq. (104) is satisfied by

\[ v^\mu(s) = \alpha^\mu \cosh(w_0 \tau_0 e^{s/\tau_0}) + \beta^\mu \sinh(w_0 \tau_0 e^{s/\tau_0}) \] (110)

where \( \alpha^\mu \) and \( \beta^\mu \) are constant four-vectors that meet the conditions

\[ \alpha \cdot \beta = 0, \quad \alpha^2 = -\beta^2 = 1, \] (111)
\( v_0 \) is an initial acceleration magnitude, \( \tau_0 = 2c^2/3m_0 \). The solution \( (10) \)–\( (11) \) describes a runaway motion, which degenerates to the Galilean regime when \( w_0 = 0 \).

It is often asserted that the solution \( (11) \) is “unphysical”, because it seems to contradict the energy conservation law: In the absence of external forces, the particle takes a run with the exponentially growing acceleration and radiates, that is, the energy of both the particle and electromagnetic field increases for no apparent reason. Using this line of reasoning, one keeps in mind either explicitly or implicitly that the mechanical object possesses the four-momentum \( p^\mu = mv^\mu \), with its time component \( \mathcal{E} = m\gamma \) being a positive definite quantity. However, it is beyond reason to insist on the existence of the object with such a four-momentum. A careful analysis with the use of different regularization procedures compatible with symmetries incorporated in the action \( (93) \) leads \[ 3 \] to the selection of an object possessing the four-momentum of the form \( (105) \) together with the balance equation \( (107) \). As shows this equation, there is no contradiction with the energy conservation law: The variation of energy of the dressed particle \( dp^0 \) is equal to the energy carried away by the radiation \( -P^0 ds \). A subtlety is that the object is characterized by the energy

\[
p^0 = m\gamma \left( 1 - \tau_0 \gamma^2 a \cdot v \right)
\]

which is not a positive definite quantity (this is scarcely surprising, if we recall the synthetic origin of the dressed particle). The indefiniteness of the expression \( (112) \) means that the increase of velocity may occasionally be accompanied by the decrease of energy. It would, therefore, make no sense to inquire: Where does the particle extract energy from to accelerate itself? The energy of the self-accelerated dressed particle is actually diminished.

Why did this problem not arise for the free gyroscope and rigid particle? As is shown in Sections 3 and 5, for such objects \( p^\mu = \text{const} \), and the invariability of \( p^\mu \) is due to a general reason, the translational invariance. Since the dependence of \( p^\mu \) on kinematical variables is intricate [see Eqs. \( (9) \) and \( (85) \)], the variation of velocity can be compensated by the variation of higher derivatives in such a way as to respect the condition \( p^\mu = \text{const} \). As to the dressed particle, in the absence of external forces, its four-momentum, in general, need not conserve. Now the constant of motion associated with the translational invariance is, by \( (107) \), the quantity \( p^0 + P^\mu \). To illustrate, for the motion in the regime \( (110) \), the quantity \( P^0 \) increases while the quantity \( p^0 \) decreases with the same rate.

When the result of the renormalization of mass, Eq. \( (100) \), is \( m = 0 \), the first term in Eq. \( (104) \) disappears, and, with \( f^\mu = 0 \), it reduces to

\[
(\dot{v}, \dot{\alpha})^\mu = 0
\]

which is the equation of a relativistic uniformly accelerated motion \[ 3 \]. The world line of the dressed particle with \( m = 0 \) in the absence of external forces is a hyperbola

\[
v^\mu(s) = \alpha^\mu \cosh w_0 s + \beta^\mu \sinh w_0 s, \quad \alpha \cdot \beta = 0, \quad \alpha^2 = -\beta^2 = 1.
\]

The curvature \( k = w_0 = \text{const} \) of such a world line may be arbitrary. The radiation goes with a constant intensity determined by the acceleration squared \( \alpha^2 = -w_0^2 \). As regards the energy of the dressed particle \( P^0 \), in view of \( (112) \), it is positive in the region of deceleration, \( s < 0 \), and negative in the region of acceleration, \( s > 0 \).

The reader can verify by a direct calculation that, when moving along the world lines \( (10) \) or \( (13) \) for a finite period of time \( \Delta s \), the increase of the radiation energy \( \Delta P^0 \) is exactly as the decrease of the energy of the dressed particle \( \Delta P^0 \).

From \( (105) \) follows that the invariant \( v \cdot p \) is a conserved quantity both in the absence and in the presence of interactions, because the renormalized mass \( m \) is taken to be constant. By contrast, \( M = \sqrt{p^2} \) depends on the form of the world line, that is, it is not conserved quantity,

\[
M^2 = m^2 (1 + \tau_0^2 a^2).
\]

(It is remarkable that, for a dressed rigid particle, \( v \cdot p, p^2 \), and any other invariant constructed from \( p^\mu \) and kinematical variables \( v^\nu, a^\mu \), etc., are not constants of motion \[ 24, 57 \], as distinct from the Abraham–Lorentz–Dirac particle which, fortunately, does have a conserved invariant quantity \( m = v \cdot p \). The problem of inert properties of a dressed particle in the general case is seen to be quite nontrivial.)

The expression \( (114) \) shows that, if \( \tau_0^2 a^2 < -1 \), the dressed particle turns to a tachyon state with \( M^2 < 0 \). Note, however, that the term \( \tau_0^2 a^2 \) is very small in the area of application of classical description.
Consider, for example, the Coulomb interaction of two electrons separated by a distance of order of Compton’s wave length of the electron $1/m$, the minimal allowable in the classical context separation. We then have the estimate
\[ \tau_0^2 |a^2| \sim e^8 \sim 10^{-8}. \]
It is clear that the critical acceleration $|a| = 1/\tau_0$ is inaccessible here. If it is granted that the class of acceptable world lines is comprised of smooth timelike curves with the curvature $k$ less than $\tau_0^{-1}$, the solution (110) falls outside the scope of this class, and the momentum space of the dressed particle contains no tachyon states, viz., states with $p^2 < 0$.

7.2 Dressed colored particle

The action of the $SU(N)$ Yang–Mills–Wong theory of $N$ colored particles is [13, 10]:
\[ A = -\sum_{i=1}^{N} \int ds_i \left( \mu_i \sqrt{v_i \cdot v_i} + \text{tr}(Z_i \xi_i^{-1} D_s \xi_i) \right) - \frac{1}{16\pi} \int d^4x \text{tr}(F_{\mu\nu} F^{\mu\nu}) \quad (115) \]
where $\xi_i = \xi_i(s_i)$ are time-dependent elements of the gauge group, $Z_i = e_i^a t_a$, $e_i^a$ are those constants whereby the colored charges of the particles are set $Q_i = \xi_i \xi_i^{-1}$, and $t_a$ are generators of the gauge group. The Yang–Mills field strength $F_{\mu\nu} = F_{\mu\nu}^a t_a$ is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]$ where $g$ is the Yang–Mills coupling constant. The covariant derivative $D_s$ is given by the formula $D_s = d/ds_i + v_i^a \lambda_i^a t_a$. Since $\xi_i$ transforms as $\xi_i \to \xi_i' = \Omega^{-1} \xi_i$ under local gauge transformation, the gauge invariance of the action (115) is evident.

Despite the similarity of the actions (113) and (117), they gives rise to quite distinct theories (the linear equations of electromagnetic field and the nonlinear Yang–Mills equations). This distinction reveals itself most sharply when the theories are “decoded”, that is, expressed in terms of exact solutions. Electrodynamics contains only two fundamental configurations, the plane wave and the Coulomb field. The former is peculiar to the situation without sources, while the latter is inherent in the situation with point-like sources. The set of extremals of the action (117) is much richer. Omitting the case in which there is no external sources, $F_{\mu\nu} = 0$, (for results of numerous investigations see, e. g., [59] and references therein), the situation with the source of the form
\[ j_\mu(x) = \sum_{i=1}^{N} \int ds_i Q_i(s_i) v_\mu^i(s_i) \delta^4(x - x_i(s_i)) \]
differs from the corresponding situation in electrodynamics in that there exist two classes of solutions [10] describing the Yang–Mills backgrounds of two vacuum phases, cold and hot. The solutions corresponding to the hot phase are fields of the Coulomb type constructed on the Cartan subgroup of the gauge group. For such solutions, all commutators disappear, and we return to the picture resembling that of electrodynamics. Every result obtained in Sec. 7.1 is reproduced here with minor change $e^2 \to \text{tr} Q^2$.

The solutions of the other class corresponding to the cold phase are non-Abelian. These solutions determine not only field configurations, but also the colored charges of the sources that generate such configurations [3]. When on the subject of a colored particle, we call it quark, and omit the particle label $i$. We now turn to the cold phase situation. The magnitude of the quark color charge takes a fixed value
\[ |\text{tr} Q^2| = \frac{4}{g^2} \left( 1 - \frac{1}{N} \right). \quad (116) \]
The equation of motion for a dressed quark in an external Yang–Mills field $F_{\mu\nu}$ is [8, 10]
\[ m \left[ a^\mu + \ell (a^\mu + v^\mu a^2) \right] = \text{tr}(QF_{\mu\nu}) v_\nu, \quad (117) \]
where $m$ is the renormalized mass, and
\[ \ell = \frac{2}{3m} |\text{tr} Q^2|. \quad (118) \]
\[ ^8 \text{We point out that the parameters } e_i \text{ in (113) and } e_i^a \text{ in (117) are in no way fixed a priori. There is no restrictions in choosing the number fields of their values. The solutions describing the cold phase fixes imaginary values of the colored charges, } e_i^a = 2i/g. \text{ In the hot phase, it is naturally to ascribe arbitrary real values to the quantities } e_i^a \text{ for stability reasons, for more details see [10].} \]
Equation (117) can be written in the form of Newton’s second law (8) in which
\[ p^\mu = m (v^\mu + \ell a^\mu) \]  
(119)
is the dressed quark four-momentum.

Equation (117) can be represented as the local energy-momentum balance
\[ \dot{p}^\mu + \dot{\mu}^\mu + \dot{P}_{\mu\nu} = 0 \]  
(120)
where the four-momentum of the dressed quark \( p^\mu \) is given by (119),
\[ \dot{\mu}^\mu = m \ell \int_{-\infty}^{s} d\tau v^\mu a^2, \]  
(121)
\[ P_{\mu\nu} = -\int_{-\infty}^{s} d\tau \text{tr}(QF^{\mu\nu}) v_\nu. \]  
(122)

The balance equations (107) and (120) differ only in their second terms. Based on the interpretation of \( P^\mu \) as the four-momentum carried away by a divergent wave from the source, \( \dot{\mu}^\mu \) should be taken as the four-momentum conveyed by a convergent wave to the source. While part of degrees of freedom of electromagnetic fields exists in the form of radiation, the pertinent degrees of freedom of the Yang–Mills field in the cold phase play the role of a “negative energy radiation”. The balance equation (120) reads: The four-momentum extracted from the external field \( -dP^\mu_{\text{ex}} \) is spent on the variation of the four-momentum of the dressed quark \( dp^\mu \) and the four-momentum \( \dot{\mu}^\mu ds \) carried away by the “negative energy radiation”.

Equation (117) with zero right hand side has a solution
\[ v^\mu(s) = \alpha^\mu \cosh(\omega_0 e^{-s/\ell}) + \beta^\mu \sinh(\omega_0 e^{-s/\ell}), \]  
(123)
\( \alpha^\mu \) and \( \beta^\mu \) meet the conditions (111). The solution (123) describes a self-decelerating motion. Although the energy of the dressed quark \( p_0^\mu \) increases, this increase exponentially weaken in time. As is seen from (120), the increase of \( p_0^\mu \) relates to the conveyance of energy of the Yang–Mills field attributed to the term \( \dot{\mu}^\mu \).

At first sight, the self-deceleration is an innocent phenomenon, because the motion becomes almost indistinguishable from Galilean in the short run. However, the presence of self-decelerations actually jeopardizes the consistency of the theory. Indeed, as we go to the past, the acceleration increases, and the intensity of the “negative energy radiation” grows along with it. Thus, the energy of the Yang–Mills field at any finite instant is divergent. This is clear from substituting the solution (123) in the integral (121).

Such “infrared” divergences cannot be removed from the theory by the renormalization of physical quantities. On may get ride of them only by a narrowing the class of acceptable world lines. Then the solution (123) would correspond to the world line that is ruled out in advance on the general grounds.

On the other hand, from (119) we have
\[ p^2 = m^2 (1 + \ell^2 a^2), \]  
(124)
Thus a dressed quark can turn to the tachyon state when \( |a| > \ell^{-1} \). By analogy with electrodynamics, we might require that the class of acceptable world lines be composed of curves with the curvature less than \( \ell^{-1} \). Then states with \( p^2 < 0 \) would be automatically excluded from the momentum space.

However, we have no longer phenomenological ground for such restrictions on the curvature. Equations (114) and (124) differ only in the change to \( t_0 \rightarrow t \). But this change radically alter the situation. From (116) and (118) follows that \( \ell \) depends on the coupling constant as \( g^{-2} \). If the coupling is strong, i. e., \( g \sim 1, \ell \) is of order of Compton’s wave length of the quark \( \Lambda_q = 1/m \), and if \( g \ll 1, \ell \) is even \( g^{-2} \) times greater than \( \Lambda_q \). As an illustration, let two quarks, interacting through the Coulomb-like colored force, be separated by a distance \( r \). As is easy to see, the critical acceleration whereby the quarks turn into

\[ \text{It is interesting that runaways do not play a similar role in electrodynamics with the retarded boundary condition. They entail no “infrared” divergences. Indeed, the insertion of the solution (110) in the integral (108) gives a finite result.} \]
tachyon states is attained at the separation \( r \approx |\text{tr}Q^2|/m \), which is more than Compton's wave length of the quark by a factor of \( g^{-2} \). Effects associated with great quark accelerations, the critical value \( |a| = \ell^{-1} \) included, fall within the area of application of classical theory.

Thus the conversion quarks to the tachyon state may provide some insight into subnuclear physics. A plausible assumption is that, crossing the point \( p^2 = 0 \) corresponds to the transition between the cold and hot phases rather than the would-be conversion of the quark to the tachyon state \([34]\).

8 Concluding remarks

We began with the assertion that \( M \) alone cannot provide an exhaustive account of inert properties of point objects. Two invariants, the mass \( M \) and the rest mass \( m \), played a key role in the following discussion. For a Galilean particle, \( M = m \), both of these quantities being identical to the operationally well defined Newtonian mass. On the other hand, for the Frenkel spinning particle, \( m > M \), and, therefore, the relation of \( m \) and \( M \) to experimentally measured quantities is an open question. The situation with rigid particles is somewhat simpler, because, in the absence of external forces, the only conserved quantity is \( M \). In the Lagrangian formalism, we encountered dimensionfull parameters \( \mu \) and \( \nu \), as well as the time-dependent monad \( \eta^{-1} \) and Lagrangian multiplier \( \chi \). While these and similar quantities are formally related to \( m \) and \( M \), they are of little physical concern. The reason for this is clarified by the example of the bare mass \( \mu \) which ceases from being a mere number and becomes a function of a regularization parameter \( \mu(\epsilon) \). The dependence on \( \epsilon \) is taken such that adding \( \mu(\epsilon) \) to the self-energy \( \delta m(\epsilon) \) results in the cancellation of their singularities rendering the renormalized mass \( m \) finite. (This seemingly awkward regularization-renormalization procedure is in fact an integral part of local field theories both on classical and quantum levels; albeit, mathematically, we are dealing with a quite respectable procedure of extraction of finite values based on the solution of the fundamental Riemann-Hilbert problem \([61]\).) In addition, the relation of \( m \) and \( M \) to the gravitation mass \( M_g \) was twice cursorily touched. A significant reduction of the number of quantities characterizing inert properties of non-Galilean objects can hardly be conceived.

'Gracious me! Why do we go into details of inertia of non-Galilean objects, even though no one observed Zitterbewegungs, runaways, and other extravagant regimes of free evolution?' the perplexed reader may interrupt at this point.

Surely anyone endeavored to observe them?

'Surely,' the sceptical reader may continue. 'But, is it really required a particular contrivance? Why are such regimes not immediately evident from fleeting glance?'

Surely anyone saw Galilean motions with the naked eye? Everyday observations convince us: In the absence of external forces, bodies are at rest. One day Aristotle arrived at this conclusion, and then, over 2000 years, none cast doubts on this subject. A lot of the credit must go to Galilei since he had the courage to make far-reaching extrapolations from everyday observations and verify the idea of the uniform motion in experiments specially adapted to the clarification of this issue.

Although the Zitterbewegung fails to be visible, there are circumstantial evidences that such a regime is yet feasible. Unfortunately, the frequency peculiar to the Zitterbewegung is of order of Compton's wave length of the object. This casts suspicion on the classical interpretation of this phenomenon. Nevertheless, the theoretical framework is large enough to expect that objects executing a Zitterbewegung with certainly classical value of frequency do exist.

As to processes with growing accelerations, they are inherent in unstable systems, specifically systems with two phases which are capable of a phase transition (e. g., the early Universe inflation \([22, 33]\), deconfinement \([24]\), etc.). It is not unlikely that such phenomena might be conveniently expressed in terms of self-accelerated dressed particles. Moreover, cosmological objects are every bit well suited for the role of self-accelerated particles. Indeed, great efforts are made to explain the recent discovery of the accelerated expansion of the Universe in models with the \( \Lambda \)-term \([33]\). An alternative explanation may be quite simple: Cosmological objects execute self-accelerated motions, analogous to the runaways of dressed charged particles, Eqs. \((110)\) and \((113)\).

Notice, we are dealing with objects (supernovas, galaxies, quasars, etc.) possessing internal angular momenta, thus their non-Galilean regimes may intricately combine the Zitterbewegung and motion with increasing velocity.

'And yet the condition of classicality is essential for this discussion altogether. However, fundamental laws of the Nature are quantum. The four-momentum \( p^\mu \) is the only well-defined dynamical variable in quantum theory, hence only \( M \) is relevant here (the four-velocity \( u^\mu \) is not a well-defined quantum
variable). Thus the problem $M \neq m$ is far-fetched. Although the notion of the non-Galilean particle exists (albeit under another names) in theoretical physics already over some 75 years, no tangible thing underlies it. What is the use of it? Should we ever trouble with the archaisms like the Abraham–Lorentz–Dirac equation, or Frenkel’s particle? Maybe, it is appropriate time to get ride of this theoretical rubbish,’ the irrepressible reader casts his further doubt.

It might be well to recall at this point that there are three radically different views of the nature of our world. One of them asserts that the most profound grasp of the physical reality is ensured by classical, deterministic laws. They form the fundamental level of cognition. One should establish the so called “hidden variable” theory to describe it. Quantum theory has a phenomenological status, it must be found by averaging over the hidden variables. In the late 1950s, this viewpoint was vigorously advocated and elaborated by L. de Broglie, J. Vigier [66], and especially D. Bohm [67]. In modern times it was revived by G. ’t Hooft [68], who maintains that deterministic, not quantum, states are the primary states in the sub-Planckian domain (with sizes $l < l_p = 1.6 \times 10^{-33}$ cm).

The opposite view is that our world is quantum. Macroscopic objects appear to be classical only effectively. Such classical manifestations are explained by the so called decoherence [69, 70]. This view is presently very popular. It is supported by results of experimental tests of the Bell inequalities [72, 73] (which, admittedly, do not lower the enthusiasm of adherents of the deterministic viewpoint, see, e. g., counter-arguments by ’t Hooft [68]). However, this view is difficult to accept when the human being or the Universe are concerned. With all the willingness to fall a victim to science, the present author would not dare to subscribe to the paper as “a superposition of alive and dead Kosyakov”. And you, the reader, are you really inspired with the role of a “decohered” homo sapiens?

At last, the third paradigm is based on the coexistence of the classical and quantum ontologies. In other words, we are dealing with two realms. In the classical realm, everything happens unambiguously, at least a given object certainly exists at the given place and at the given instant, and its individuality is preserved. In the quantum realm, every process (the being of objects included) is characterized by some probability amplitude. The individuality of a quantum object, say, a given electron, is not ensured since it is identical to any one of real or virtual electrons (that is, electrons that might not exist at the given instant certainly, but is ready to appear due to the electron-positron pair creation, muon decay, etc.). This paradigm is due to founders of the “Copenhagen interpretation” who repeatedly argued for the treatment of quantum objects on equal terms with macroscopic classical devices. A link between the classical and quantum realms is offered by the so called holographic principle. According to this principle (’t Hooft [74] and L. Susskind [75] were the fist to enunciate it in the context of quantum gravity), the physical reality may appear either classical or quantum, being imbedded in spacetimes of, respectively, $D + 1$ and $D$ dimensions.

Thus, to declare the supremacy of quantum notions over classical ones is to discriminate against the “Copenhagen” and “deterministic” minorities which involve not only scientific marginals.

One further comment on the troubles with old-fashioned concepts which are still not incarnated in observable objects is in order. It is interesting that the physical community fairly rich in lovers of these theoretical “relics”. The magnetic monopole was invented by Dirac 70 years ago, the ’t Hooft–Polyakov monopole is already over 25, and, although the experiment make no hint about the existence of these objects, whether have we not enough people who argue about the monopole, study its properties, and suggest resolutions of numerous problems, from subnuclear to cosmological, as if we deal with a real particle? However, could anybody bring himself/herself to call the monopole (together with a number of another somewhat obsolete things of the high energy physics props, e. g., Higgs bosons, axions, supersymmetric particles, etc.) the theoretical “rubbish”?

The all-powerful vogue can turn “the well forgotten out-of-date” to some up-to-date. Who remembered the Born–Infeld electrodynamics 15 years ago? One might confuse it with the Mie electrodynamics or, at best, elicit a vague recollection of something “maybe non-local, maybe nonlinear, or maybe gauge non-invariant”. However, this seemingly forgotten name is now flashy again in leading physical journals. The point is the Born–Infeld Lagrangian emerges in the low energy limit of the superstring theory. Peremptory decisions and “death sentences” especially those related to an authoritative scientist may be detrimental to his own reputation and studies of his colleagues. There is a great number of precedents. We turn to two of them.

One day, a young theorist A. Salam came to the formidable W. Pauli to submit to him a daring idea
of the two-component neutrino. Pauli responded with a note urging the visitor to “think of something better”. Discouraged Salam delayed his publication, and the credit for discovery of the parity violation fell to Lee and Yang.\footnote{Recall, this was just Pauli who derived the equation of the two-component massless spinor field 25 years prior to this event and who repudiated it at once, taking the violation of mirror symmetry to be absurdity. Giving up this equation for lost, Pauli turned down any proposal of its physical application.}

In the attempt to build a model of static Universe, Einstein introduced the cosmological constant (Λ-term) into the gravitation equations in 1917. Nobody felt the need or even naturalness for this step at that time. This was a likely reason why A. Friedmann concentrated on a nonstationary expanding model of Universe described by a solution to the gravitation equations with zero Λ-term. Einstein felt something “suspicious” in this solution, and he expressed his feeling in his comment of Friedmann’s paper. Later on, Einstein accepted both the very idea of nonstationary Universe and Friedmann’s solution, but went into another extreme and considered the Λ-term to be his greatest mistake. Following Einstein, most of theorists brought hastily the Λ-term in the category of regrettable \textit{ad hoc} constructions. This state of the art remained unchanged for about 40 years until Ya. B. Zel’dovich\footnote{observed that allowing for zero oscillatory modes makes the presence of the Λ-term in quantum gravity inevitable. Since then, the accounting for the Λ-term is a central (and challenging) problem in quantum gravity and cosmology.} observed that allowing for zero oscillatory modes makes the presence of the Λ-term in quantum gravity inevitable. Since then, the accounting for the Λ-term is a central (and challenging) problem in quantum gravity and cosmology.\footnote{\cite{79,80,81,82}}.

\section*{Acknowledgments}
I am indebted to V. G. Bagrov, A. O. Barut, T. Goldman, G. V. Efimov, I. B. Khriplovich, V. V. Nesterenko, F. Rohrlich, R. Woodard, and H. D. Zee for discussions of this subject at different time. This work is supported in part by the International Science and Technology Center, Project # 840.

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