NON-LINEAR APPROXIMATIONS TO GRAVITATIONAL INSTABILITY:
A COMPARISON IN THE QUASI-LINEAR REGIME

Dipak Munshi$^1$, Varun Sahni$^1$

and

Alexei A. Starobinsky$^{2,3}$

$^1$ Inter-University Centre for Astronomy and Astrophysics
   Post Bag 4, Ganeshkhind, Pune 411 007, India

$^2$ Yukawa Institute for Theoretical Physics, Kyoto University, Uji 611, Japan

$^3$ Landau Institute for Theoretical Physics, Russian Academy of Sciences,
   Moscow 117334, Russia (permanent address)
Abstract

We compare different nonlinear approximations to gravitational clustering in the weakly nonlinear regime, using as a comparative statistic the evolution of non-Gaussianity which can be characterised by a set of numbers $S_p$ describing connected moments of the density field at the lowest order in $\langle \delta^2 \rangle$: $\langle \delta^n \rangle_c \simeq S_n \langle \delta^2 \rangle^{n-1}$. Generalizing earlier work by Bernardeau (1992) we develop an ansatz to evaluate all $S_p$ in a given approximation by means of a generating function which can be shown to satisfy the equations of motion of a homogeneous spherical density enhancement in that approximation. On the basis of the values of we show that approximations formulated in Lagrangian space (such as the Zeldovich approximation and its extensions) are considerably more accurate than those formulated in Eulerian space such as the Frozen Flow and Linear Potential approximations. In particular we find that the $n$th order Lagrangian perturbation approximation correctly reproduces the first $n + 1$ parameters $S_n$. We also evaluate the density probability distribution function for the different approximations in the quasi-linear regime and compare our results with an exact analytic treatment in the case of the Zeldovich approximation.

Subject Headings: cosmology: theory – large scale structure of Universe
1. Introduction

Gravitational instability in the Universe can be characterised by two distinct epochs: during the first, density fluctuations $\delta \equiv (\rho - \rho_0)/\rho_0$ evolve self-similarly according to the tenets of linear theory $\delta^{(1)}(x, t) \propto D(t)\Delta(x)$ with the result that a density distribution that was initially Gaussian remains Gaussian at subsequent epochs as long as $|\delta| << 1$. The linear epoch clearly cannot continue indefinitely since a stage will arise when $\delta$, although small, becomes comparable to unity, so that weakly nonlinear effects can no longer be ignored. This is the quasi-linear regime; as long as $|\delta| < 1$ the evolution of the density field can be adequately described by means of a perturbative expansion: $\delta(x, t) = \sum_{n=1}^{\infty} \delta^{(n)}(x, t)$ where $\delta^{(n)}(x, t) = D_n(t)\Delta^{(n)}(x)$ if the Universe is spatially flat and matter dominated (Peebles 1980, Fry 1984). This epoch witnesses the growth of skewness and kurtosis and other higher moments of the one-point probability distribution function (PDF) of density perturbations characterising a non-Gaussian development of an initially Gaussian field $\delta^{(1)}$. During still later stages, $\delta$ becomes larger than unity, with the result that perturbative approximations break down and a fully nonlinear treatment of the problem is required. This later epoch is characterised by the formation of pancakes ($\delta \rightarrow \infty$ along two dimensional sheets) and the gradual development of cellular structure (Shandarin & Zeldovich 1989).

Although no exact treatment is available which encompasses the entire nonlinear epoch, some approximations have been suggested which attempt to mimic certain features of nonlinear gravitational instability. Our aim in this paper is to investigate the relative accuracy of five such approximations in the weakly nonlinear regime ($|\delta| < 1$) where they may be analytically compared to the exact solution in the form of a perturbative expansion in powers of $\delta$.

Our treatment overlaps with (and considerably extends) the recent work of Munshi & Starobinsky (1993) (hereafter MS) and Bernardeau et al. (1993), in which 3 approximations: the Zeldovich approximation (Zeldovich 1970), hereafter ZA, the frozen flow approximation (Matarrese et al. 1992), hereafter FF, and the linear potential approximation (Brainerd et al. 1993; Bagla & Padmanabhan 1994), hereafter LP, were compared to the exact solution in second and third order in perturbation theory. In this paper, we provide an ansatz whereby nonlinear approximations can be compared with the exact perturbative solution to any order in perturbation theory. We apply this ansatz to approximations formulated in Eulerian space such as FF and LP, as well as to approximations based on Lagrangian perturbation theory (of which ZA happens to be the lowest order solution). It appears that Lagrangian perturbative methods are, as a rule, much more accurate than either FF or LP to any given order in perturbation theory.

2. Nonlinear approximations

Gravitational instability in a spatially flat matter dominated FRW Universe before caustic formation can, in the Newtonian approximation, be described by the following system of equations (see, e.g., Zeldovich & Novikov 1983, Peebles 1980):

$$\Delta x \Phi = 4\pi Ga^2 \rho_0 \delta;$$  \hspace{1cm} (1a)
where \( \mathbf{u} = a\dot{x} \), \( x \) being the comoving coordinate \((a(t) \propto t^{2/3} \) and \( \rho_0 = 1/6\pi Gt^2 \)). If we assume that \( \mathbf{u} \) is irrotational then we can define a velocity potential such that \( \mathbf{u}(\mathbf{x},t) = \nabla_x v(\mathbf{x},t) \equiv \nabla U(\mathbf{x},t) \) where we have introduced the notation \( \nabla \equiv \frac{1}{aH} \nabla_x \), \( H = \dot{a}/a = 2/3t \) (the potential \( U \) is related to the potential \( V \) used in MS by the relation \( U = -HV \)). Moreover, \( \Delta U = \frac{1}{aH} \text{div}_x \mathbf{u} = \theta \) which is the dimensionless velocity divergence used in MS.

In this notation, we can rewrite the equations as follows:

\[
\begin{align*}
a \frac{\partial}{\partial a} \delta(\mathbf{x},a) + (1 + \delta(\mathbf{x},a))\theta(\mathbf{x},a) + \nabla \delta(\mathbf{x},a) \nabla U(\mathbf{x},a) &= 0; \\
\frac{1}{2} \frac{\partial}{\partial a} \theta(\mathbf{x},a) + \frac{1}{2} \theta(\mathbf{x},a) + \nabla U(\mathbf{x},a) \nabla \theta(\mathbf{x},a) + U_{ik} U_{ik} + \Delta \Phi(\mathbf{x},a) &= 0; \quad \text{(2c)}
\end{align*}
\]

where \( U_{ik} = \frac{1}{aH} \partial^2_{ik} U \) and summation over repeated indexes is assumed. The initial condition for the system (2) at \( t \to 0 \) is given by the linear approximation corresponding to a growing adiabatic mode:

\[
\Phi = \phi_0(\mathbf{x}); \quad U = -\frac{2}{3} \phi_0(\mathbf{x}); \quad \delta = \frac{2}{3} \Delta \phi_0(\mathbf{x}); \quad \theta = -\delta. \quad \text{(3)}
\]

Several of the nonlinear approximations which we will consider such as the Zeldovich approximation, the Frozen Flow approximation and the Linear Potential approximation arise as a result of the extrapolation of one of the linearised relations in (3) into the nonlinear regime. Thus in FF, the value of the velocity potential is kept fixed to its linearised value \( U = -\frac{2}{3} \phi_0 \), so that \( \mathbf{u} = -\frac{2}{3} \nabla \phi_0 \). The movement of a particle in FF is such that the particle upgrades its velocity with each time step to that prescribed by the local value of the linear velocity field. This results in a laminar flow for the velocity field since different particle trajectories can never intersect in principle.

The linear potential approximation on the other hand, is based upon the assumption that the gravitational potential does not evolve with time so that \( \Phi = \phi_0(\mathbf{x}) \). As a result particles effectively move along the lines of force of the primordial potential \( \phi_0 \). Similarly the Zeldovich approximation is based on extending \( U = -\frac{2}{3} \phi_0 \) into the nonlinear regime. In all three cases the constraint equations \( U = -\frac{2}{3} \Phi \) (ZA), \( U = -\frac{2}{3} \phi_0 \) (FF), \( \Phi = \phi_0 \) (LP) replace the Poisson equation (2c) which is not satisfied in these approximations beyond the linear regime.

In nonlinear approximations formulated in Lagrangian space, the main object of study is the particle trajectory. In these approximations the initial comoving (Lagrangian) coordinate \( \mathbf{q} \) and the Eulerian coordinate of a particle \( \mathbf{x}(t) \) are related by a displacement field \( \Psi(t, \mathbf{q}) \):

\[
\mathbf{x} = \mathbf{q} + \Psi(t, \mathbf{q}). \quad \text{(4)}
\]
If we introduce the matrix

$$M_{ik}(t, q) = \frac{\partial x_i}{\partial q_k} = \delta_{ik} + \frac{\partial \Psi_i}{\partial q_k}, \quad (5)$$

then $M_{ik}$ satisfies the equations

$$\frac{\partial}{\partial t} \left( a^2 \frac{\partial M_{ik}}{\partial t} \right) \cdot M^{-1}_{ki} + 2 \frac{a}{3} \left( J^{-1} - 1 \right) = 0, \quad (6a)$$

$$\epsilon_{ikl} M_{km} M^{-1}_{ml} = 0, \quad (6b)$$

where $J = |\det[M_{ik}]|$ is the Jacobian of the transformation (4), $\epsilon_{ikl}$ is the unit totally antisymmetric tensor, Eq. (6b) being the condition of potential motion in Eulerian space (see, e.g., Zeldovich & Novikov 1983). Density and velocity fields are determined from the relations

$$\rho \equiv \delta + 1 = J^{-1}, \quad u = a \left( \frac{\partial x}{\partial t} \right) = a \dot{\Psi}. \quad (7)$$

A solution of Eqs. (6a,b) in the quasi-linear regime may be obtained by expanding $\Psi$ in a power series $\Psi = \Psi^{(1)} + \Psi^{(2)} + \ldots$ where higher orders in $\Psi^{(n)} (n > 1)$ are related to lower orders via an iterative procedure (Moutarde et al. 1991, Buchert 1992, Lachieze-Rey 1993). It appears that for the matter dominated Universe all $\Psi^{(n)}$ factorize:

$$\Psi^{(n)}_i = D^n_+(t) \psi^{(n)}_i(q), \quad D_+(t) = \frac{3t^2}{2a^2} \propto a(t). \quad (8)$$

Subsequent Lagrangian approximations then arise if one truncates this series after a finite number of terms. As a result of this truncation Eq. (6a) becomes approximate and the Poisson equation (1a) ceases to be valid, but Eqs. (7) as well as the continuity equation (1b) continue to be exactly satisfied in these approximations.

The first-order Lagrangian approximation $\Psi = \Psi^{(1)}$ is simply the Zeldovich approximation:

$$x_i = q_i + D_+(t) \psi^{(1)}_i(q), \quad \psi^{(1)}_i = -\frac{\partial \phi_0(q)}{\partial q_i}. \quad (9)$$

The account of the next, second order terms results in what one may call the post-Zeldovich approximation (hereafter PZA). It is given by $\Psi = \Psi^{(1)} + \Psi^{(2)}$ where

$$\psi^{(2)}_{i,i} = \frac{3}{14} \left( (\psi^{(1)}_{i,i})^2 - \psi^{(1)}_{i,j} \psi^{(1)}_{j,i} \right), \quad (10a)$$

$$\psi^{(2)}_{i,j} = \psi^{(2)}_{j,i}, \quad (10b)$$

coma means partial derivative with respect to $q$ and summation over repeated indexes is assumed (Bouchet et al. 1992, Gramann 1993 and others). In the third order (post-post-Zeldovich approximation, hereafter PPZA), $\Psi = \sum_{n=1}^{3} \Psi^{(n)}$ with

$$\psi^{(3)}_{i,i} = -\frac{5}{9} \left( \psi^{(2)}_{i,i} \psi^{(1)}_{j,j} - \psi^{(2)}_{i,j} \psi^{(1)}_{j,i} \right) - \frac{1}{3} \text{det}[\psi^{(1)}_{i,j}], \quad (11a)$$
\[ \psi^{(3)}_{i,j} - \psi^{(3)}_{j,i} = \frac{1}{3} (\psi^{(2)}_{i,k} \psi^{(1)}_{k,j} - \psi^{(2)}_{j,k} \psi^{(1)}_{k,i}) \quad (11b) \]

(cf. Juszkiewicz et al. 1993, Bernardeau 1993) and so on. A distinguishing feature of PPZA is that motion becomes non-potential in Lagrangian space (but not in Eulerian space, of course) beginning from this approximation.

Our aim in the present paper will be to determine how well the five nonlinear approximations discussed above perform when compared within the framework of perturbation theory which is well defined in the quasi-linear regime when \(|\delta| < 1\). To investigate statistical behaviour in all orders of perturbation theory we follow Bernardeau (1992) who developed an elegant ansatz (which we summarise below) by means of which vertex weights characterising irreducible moments of \(\delta\) may be evaluated at every order by means of a generating function \(G_\delta\).

3. Generating function for the irreducible moments of \(\delta\).

Let us introduce a vertex generating function for any random field \(F(x, a)\) as

\[ G_F(\tau) = \sum_{n=1}^{\infty} \frac{\langle F^{(n)} \rangle_c}{n!} \tau^n \quad (12a) \]

where

\[ \langle F^{(n)} \rangle_c = \frac{\int \langle F(x, a) \delta^{(1)}(x_1, a) ... \delta^{(1)}(x_n, a) \rangle_c d^3x d^3x_1 ... d^3x_n}{(\int \langle \delta^{(1)}(x, a) \delta^{(1)}(x', a) \rangle d^3x d^3x')}^n, \quad (12b) \]

\(F^{(n)}\) is the n-th order of expansion of \(F\) in a power series with respect to \(\delta\), \(\delta^{(1)}\) is the linear approximation for \(\delta\) given in Eq. (3), and only connected diagrams are taken into account, which explains the subscript \(c\). Throughout the present analysis we assume that the initial density field \(\delta^{(1)}\) and the associated linear gravitational potential \(\phi_0(x)\) are Gaussian stochastic quantities though the results obtained below may be generalized to a non-Gaussian case as well. Higher vertices of \(\delta\) and the dimensionless velocity divergence \(\theta = \Delta U\) are denoted by \(\nu_n\) and \(\mu_n\) respectively so that

\[ G_\delta = \sum_{n=1}^{\infty} \frac{\nu_n}{n!} \tau^n, \quad G_\theta = \sum_{n=1}^{\infty} \frac{\mu_n}{n!} \tau^n \quad (13) \]

\((\nu_n \equiv \langle \delta^{(n)} \rangle_c, \quad \mu_n \equiv \langle \theta^{(n)} \rangle_c\) with \(\nu_1 = 1, \mu_1 = -1\). We define \(\tau\) with the opposite sign as compared to Bernardeau (1992) to simplify the appearance of some expressions.

It is well known that for small values of \(\sigma^2 = \langle \delta^{(1)}^2 \rangle\) the connected moments of the density field have the simple form (Fry 1984, Bernardeau 1992):

\[ \langle \delta^n \rangle_c \simeq S_n \langle \delta^2 \rangle^{n-1} \quad (14a) \]

where \(S_n\) are related to the vertex weights \(\nu_n\) introduced earlier by

\[ S_3 = 3 \nu_2 \]
\[ S_4 = 4 \nu_3 + 12 \nu_2^2 \]
\[ S_5 = 5\nu_4 + 60\nu_2\nu_2 + 60\nu_2^3 \]
\[ S_6 = 6\nu_5 + 120\nu_4\nu_2 + 90\nu_3^2 + 720\nu_2\nu_2^2 + 360\nu_2^4 \]  
(14b)

The equations relating \( T_n \) and \( \mu_n \) have the same form as (14b) with \( T_n \) replacing \( S_n \) and \( \nu_n \) being replaced by \((-1)^n\mu_n\). The additional \((-1)^n\) factor arises because \( \mu_n \) are defined with the use of \( \delta^{(1)} \) according to Eq. (12b). Formulas for the moments of \( \theta \) would be completely identical to those for \( \delta \) if we defined \( \mu_n \) using \( \theta^{(1)} = -\delta^{(1)} \). This just corresponds to the multiplication of \( \mu_n \) by \((-1)^n\).

Now we demonstrate how the vertex weights \( \nu_n \) (and hence \( S_n \)) can be determined in any arbitrary order for the nonlinear approximations described above. We do this by following Bernardeau (1992), who studied leading (tree level) diagrams in the limit when the density variance \( \sigma^2 \) was very small and showed that the generating functions (12a) defined for any arbitrary stochastic fields \( F \) and \( H \) possess the following properties:

\[ G_{a\partial b F} = \tau \frac{\partial}{\partial \tau} G_F, \]  
(15a)

\[ G_{FH} = G_F(\tau)G_H(\tau), \]  
(15b)

\[ G_{\nabla F, \nabla H} = 0, \]  
(15c)

\[ G_{F_{\alpha\beta} H_{\alpha\beta}}(\tau) = \frac{1}{3} G_{\Delta F \Delta G}. \]  
(15d)

Using (15a - d) we can rewrite equations (2a,b) in terms of vertex generating functions of the corresponding fields (as a result, they become equations for statistically averaged quantities):

\[ \tau \frac{d}{d\tau} G_\delta + (1 + G_\delta) G_\theta = 0, \]  
(16a)

\[ \tau \frac{d}{d\tau} G_\theta + \frac{1}{2} G_\theta + \frac{1}{3} G_\theta^2 + G_{\Delta \phi} = 0. \]  
(16b)

The Poisson equation (2c) is now replaced by:

1. \( G_{\Delta \phi} = \frac{1}{2} G_\delta \) for the exact solution,
2. \( G_{\Delta \phi} = -\frac{1}{2} G_{\Delta U} = -\frac{1}{2} G_\theta \) for ZA,
3. \( G_\theta = -\tau \) for FF,
4. \( G_{\Delta \phi} = G_{\Delta \phi_0} = \frac{1}{2} \tau \) for LP.

It is not straightforward to write the corresponding conditions replacing the Poisson equation for PZA, PPZA and higher Lagrangian approximations because they are formulated in Lagrangian space, and Eqs. (16a,b) originated from Eulerian equations. Remarkably, it appears unnecessary because as we shall show in the next section it is possible to circumvent this problem entirely.

4. Particle trajectories in the spherical model and values of the moments \( \nu_n, \mu_n \).

For the exact perturbative solution, the system of equations (16a,b) may be transformed into a second order differential equation for the variable \( y = 1 + G_\delta(\tau) \). For ZA, FF and
LP, Eq. (16b) may be solved directly after substituting the corresponding expressions for \( G_{\triangle\Phi} \). However, a much simpler and more physical method exists that is also applicable in the cases of PZA, PPZA and higher Lagrangian approximations.

Let us take a closer look at Eqs. (16a,b). It is easy to see that if we make the substitution \( G_\delta \to \delta, G_\theta \to \theta \) and \( \tau \to a \), then the statistical Eqs. (16a,b) become dynamical equations describing the isotropic and homogeneous expansion of the Universe (a) under the influence of gravity in the case of the exact solution, and (b) due to some other forces mimicking gravity for the different approximations (because the Poisson equation is not satisfied in this case). The initial conditions \( G_\delta = \tau, G_\theta = -\tau \) as \( \tau \to 0 \) can be satisfied simultaneously if a free scaling dimensional coefficient of proportionality in dynamical solutions is chosen so that \( \delta = a \) for \( a \to 0 \). Therefore, the dynamics in question is actually that of the spherical top-hat model. This agrees well with an intuitive picture that for \( \sigma \ll 1 \), large and rare fluctuations \( \sigma \ll \delta < 1 \) are approximately spherical. The same evidently refers to the cases of PZA, PPZA and so on, inspite of more complicated forms for the third equation replacing the Poisson equation, because it follows from Eqs. (15c,d) that to obtain statistical equations for generating functions one has to neglect inhomogeneities in dynamical equations and substitute all second-rank tensors by those proportional to a unit tensor.

Now, the easiest way to solve the top-hat model is to work with Lagrangian (comoving) coordinates \( q \). Then the problem reduces to one of finding appropriate particle trajectories \( x(q,a) \) associated with equations (16a,b) in each of the approximations and in the exact solution respectively.

The exact spherical top-hat solution:

Let \( r = |x|, r_0 = |q| = r(0) \). The gravitational potential inside a homogeneous sphere is \( \Phi = \frac{a^2 r^2 \delta}{9 t^2} \). It clearly follows from mass conservation that \( 1 + \delta = (r_0/r)^3 \). So the equation of motion of a particle (a spherical shell) is

\[
\frac{d}{dt}(a^2 \frac{d}{dt}) - \frac{d\Phi}{dr} = -\frac{2a^2 r}{9 t^2} \left( \left( \frac{r_0}{r} \right)^3 - 1 \right).
\]  

This is none other than the usual Newtonian equation \( \frac{d^2 R}{dt^2} = -\frac{2a^3}{9 t^2 R^2} \) for the reduced physical scale \( R = ar/r_0 \). Its first integral satisfying the initial condition \( \delta = a \) for \( a \to 0 \) is

\[
\left( \frac{dR}{da} \right)^2 = a \left( \frac{1}{R} - \frac{5}{3} \right)
\]  

(18)

(the independent variable is changed from \( t \) to \( a \)). The solution of Eq. (18) in a parametric form is

\[
R = \frac{3}{10} (1 - \cos \theta),
\]  

(19a)

\[
\delta = \frac{9}{2} \frac{(\theta - \sin \theta)^2}{(1 - \cos \theta)^2} - 1 = G_\delta,
\]  

(19b)

\[
a = \frac{3}{5} \left( \frac{3}{4} (\theta - \sin \theta) \right)^\frac{2}{3} = \tau
\]  

(19c)
Comparing (20a) with (13) we find: 

\[ G_\delta = \tau + 0.810\tau^2 + 0.601\tau^3 + 0.426\tau^4 + 0.293\tau^5 + \ldots, \quad (20a) \]

\[ G_\theta = - (\tau + 0.619\tau^2 + 0.376\tau^3 + 0.226\tau^4 + 0.135\tau^5 + \ldots). \quad (20b) \]

Comparing (20a) with (13) we find: \( \nu_2 = 34/21 \simeq 1.619, \ \nu_3 = 682/189 \simeq 3.608, \ \nu_4 \simeq 10.23. \)

Higher moments of the density distribution such as the skewness \( S_3 \), the kurtosis \( S_4 \) etc. can now be determined by substituting the values of \( \nu_n \) into Eq. (14b).

**The Zeldovich Approximation:**

In this case, particle trajectories are already given by Eq. (9). The initial gravitational potential \( \phi_0(r_0) \) satisfying the initial condition \( \delta = a \) for \( a \to 0 \) is \( \phi_0 = a^3r_0^2/9t^2 \) (it doesn’t depend on \( t \) since \( a(t) \propto t^{2/3} \)). As a result

\[ r(a) = r_0(1 - \frac{a}{3}), \quad (21a) \]

\[ \delta_{ZA} \equiv G_\delta = \left( \frac{r_0}{r(\tau)} \right)^{3} - 1 = \left( 1 - \frac{\tau}{3} \right)^{-3} - 1, \quad (21b) \]

\[ G_\theta = - \tau \left( 1 - \frac{\tau}{3} \right)^{-1}. \quad (21c) \]

Expanding \( G_\delta \) and \( G_\theta \) near \( \tau = 0 \), we get

\[ G_\delta = \tau + 0.667\tau^2 + 0.370\tau^3 + 0.185\tau^4 + 0.086\tau^5 + \ldots, \quad (22a) \]

\[ G_\theta = - (\tau + 0.333\tau^2 + 0.111\tau^3 + 0.037\tau^4 + 0.012\tau^5 + \ldots), \quad (22b) \]

from which we can readily obtain the values of \( \nu_n, \mu_n \) and \( S_n, T_n \). For \( n = 3, 4 \), they coincide with those found previously (Grinstein & Wise 1987, MS 1993, Bernardeau et al. 1993).

**PZA, PPZA and higher order Lagrangian approximations:**

Here the Lagrangian method shows its superiority over the Eulerian one because particle trajectories are explicitly given by Eqs. (4, 8-10) in the case of PZA, Eqs. (4, 8-11) for PPZA and so on. Just as in the previous paragraph, the initial gravitational potential for the top-hat spherical model is \( \phi_0(q) = a^3r_0^2/9t^2 \) where \( r_0 = |q| \). Then \( \psi^{(1)}_{i,i} = -2a^3/3t^2 = -a/D_{+}(t), \ D_{+}^2\psi^{(2)}_r = -a^2r_0/21 \) (the index \( r \) means a radial component). Therefore, a particle trajectory in PZA is

\[ r(a) = r_0 \left( 1 - \frac{a}{3} - \frac{a^2}{21} \right), \quad (23a) \]

and the generating functions for PZA are

\[ \delta_{PZA} \equiv G_\delta = \left( 1 - \frac{\tau}{3} - \frac{\tau^2}{21} \right)^{-3} - 1 = \tau + 0.810\tau^2 + 0.561\tau^3 + 0.358\tau^4 + 0.215\tau^5 + \ldots, \quad (23b) \]
\[ G_{\theta} = -\frac{\tau + \frac{2\tau^2}{3}}{1 - \frac{\tau^2}{3} - \frac{\tau^2}{21}} = -(\tau + 0.619\tau^2 + 0.254\tau^3 + 0.114\tau^4 + 0.050\tau^5 + \ldots). \quad (23c) \]

In this case, the first two coefficients of the expansion (20a,b) are reproduced exactly as a result of which PZA gives the correct value for the skewness \( S_3 \) in the lowest order in \( \sigma \).

Furthermore, \( D_{\psi}^3 \psi(3) = -23a^3r_0/1701 \) (see Eq. (11a)), so that a PPZA particle trajectory is given by

\[ r(a) = r_0 \left( 1 - \frac{a}{3} - \frac{a^2}{21} - \frac{23a^3}{1701} \right). \quad (24a) \]

Thus, the generating functions for PPZA have the forms:

\[ G_{\delta} = \left( 1 - \frac{\tau}{3} - \frac{\tau^2}{21} - \frac{23\tau^3}{1701} \right)^{-3} - 1 = \tau + 0.810\tau^2 + 0.601\tau^3 + 0.412\tau^4 + 0.268\tau^5 + \ldots, \quad (24b) \]

\[ G_{\theta} = -\frac{\tau + \frac{2\tau^2}{3} + \frac{23\tau^3}{189}}{1 - \frac{\tau^2}{3} - \frac{23\tau^3}{1701}} = -(\tau + 0.619\tau^2 + 0.376\tau^3 + 0.168\tau^4 + 0.082\tau^5 + \ldots). \quad (24c) \]

Now the first three terms in the expansion (20a,b) are correctly reproduced. Therefore, the PPZA values for both the skewness \( S_3 \) as well as the kurtosis \( S_4 \) are exact.

It is clear now that in the \( N \)-th order Lagrangian approximation defined by the condition \( \Psi = \sum_{n=1}^{N} \Psi^{(n)} \), a particle trajectory in the spherical top-hat model has the form \( r(a) = r_0 (1 - P_N(a)) \) where \( P_N \) is a \( N \)-th order polynomial with \( P_N(0) = 0 \), so the generating functions have the following structure:

\[ G_{\delta} = (1 - P_N(\tau))^{-3} - 1, \quad (25a) \]

\[ G_{\theta} = -3 \frac{dP_N(\tau)}{d\tau} (1 - P_N(\tau))^{-1}. \quad (25b) \]

The expansion of \( G_{\delta}, G_{\theta} \) around the point \( \tau = 0 \) correctly reproduces the first \( N \) terms of the series for the exact solution (20a,b) and consequently the first \( N + 1 \) values of \( S_n \).

**The Frozen Flow Approximation**

This is the only case when it is easier to solve the problem in Eulerian space because the velocity potential is equal to its initial, Gaussian distributed value. Therefore, \( \theta \) is Gaussian, too, and \( G_{\theta} = -\tau \). Consequently

\[ G_{\delta} = \exp(\tau) - 1. \quad (26) \]

However, the calculation using a particle trajectory method is not long either. The velocity potential in the top-hat model in FFA is equal to \( U = -2\phi_0(r)/3 = -2a^3r^2/27t^2 \). So, the equation of motion of a particle has the form

\[ u_r \equiv a \frac{dr}{dt} = -2a^2r/9t. \quad (27a) \]
Its solution in terms of $a$ is (Matarrese et al. 1992)

$$r(a) = r_0 \exp(-\frac{a}{3}) \quad (27b)$$

that just gives the expression (26) for the density generating function.

**The Linear Potential Approximation**

In LP, it is the gravitational potential $\Phi$ that stays equal to its initial linear value $\phi_0(r)$ which in our case is once more equal to $a^3 r^2 / 9 t^2$. The equation of motion of a particle is therefore

$$\frac{d}{dt} \left( a^2 \frac{dr}{dt} \right) = -\frac{2a^3}{9 t^2} r \quad (28a)$$

(cf. Eq. (17)) or, it terms of $a$,

$$\frac{d^2 r}{da^2} + \frac{3}{2a} \frac{dr}{da} + \frac{r}{2a} = 0. \quad (28b)$$

Solving Eq. (28b) we obtain (cf. Brainerd et al. 1993)

$$r = \frac{r_0}{\sqrt{2a}} \sin \sqrt{2a}. \quad (29a)$$

Therefore,

$$G_\delta = \left( \frac{\sqrt{2\tau}}{\sin \sqrt{2\tau}} \right)^3 - 1 = \tau + 0.567 \tau^2 + 0.242 \tau^3 + 0.087 \tau^4 + 0.028 \tau^5 + ..., \quad (29b)$$

$$G_\theta = \frac{3}{2} \left( \sqrt{2\tau cot \sqrt{2\tau}} - 1 \right) = -\left( \tau + 0.133 \tau^2 + 0.025 \tau^3 + 0.0051 \tau^4 + 0.0010 \tau^5 + ... \right) \quad (29c)$$

in LP.

In figure 1 we plot the density contrast in the spherical top-hat model obtained using PPZA, PZA, ZA, LP and FF against the exact solution. We find that for a positive density perturbation all models underestimate the density contrast, but PPZA, PZA and ZA are more accurate than LP and FF. This is quite surprising since it is well known that the Zeldovich approximations is the least accurate during spherical collapse when all eigenvalues of the deformation tensor $\partial^2 U / \partial q_i \partial q_j$ are equal (Shandarin & Zeldovich 1989). We also list the turnaround epoch and the recollapse epoch (in units of $\tau$) in each of the approximations and in the exact solution in Table 1. We find that both turnaround and recollapse occur later in the approximations than in the exact solution.

Expanding $G_\delta$ and $G_\theta$ in each of the approximations near $\tau = 0$ we get the reduced moments $\nu_n$, $\mu_n$ and, from (14b), the parameters $S_n$ and $T_n$. Our results are summarised in Table 2 for the first six moments of the distribution: $S_1 ... S_6$. Table 3 contains the values of $T_1 ... T_6$. The values which we obtain for $S_3$ and $S_4$ for ZA, FF and LP are identical to those obtained earlier by MS and Bernardeau et al. (1993). Interestingly we find that of all the
approximations considered by us, PPZA appears to be the most accurate in reproducing the results of the exact perturbative treatment. Next in accuracy comes PZA which reproduces the skewness only, then ZA followed by LP and FF. The results of our analysis lead us to conclude that nonlinear approximations formulated in Lagrangian space (ZA, PZA etc.) are significantly more accurate in the weakly nonlinear regime than those which are formulated in Eulerian space (FF and LP). It would be of great interest to extend this analysis to the strongly nonlinear regime where the different approximations should be compared with results of the adhesion model and N-body simulations. Some work in this direction is presently in progress (Sathyaprakash et al. 1994).

5. Density distribution functions

Having obtained the generating functions \(G_\delta\), it is fairly straightforward to calculate corresponding density PDFs \(\eta(\delta)\) for each of the approximations in the limit \(\sigma \ll 1\) and for a sufficiently small \(|\delta|\) (for the exact perturbative solution this was done by Bernardeau (1992)). A more detailed discussion of the region of applicability of the approach used in the paper will be given below.

Let \(\mathcal{P}(y)\) be the Laplace transform of \(\eta(\delta)\) (with \(\sigma\) displayed explicitly):

\[
\mathcal{P}(y) = \int_{-1}^{\infty} \eta(\delta) exp\left(-\frac{(1+\delta)y}{\sigma^2}\right) d\delta.
\]

Then the generating function of the moments \(S_n\) (see Eq. (14a)):

\[
\varphi(y) = \sum_{p=1}^{\infty} S_p \frac{(-1)^{p-1}}{p!} y^p, \quad S_1 = S_2 = 1,
\]

is connected to \(\mathcal{P}\) by the simple formula \(\mathcal{P}(y) = exp(-\varphi(y)/\sigma^2)\) (see, e.g., White 1979). On the other hand, as can be checked by a direct comparison of the coefficients of the series (13) and (31) using (14b) that \(\phi(y)\) can be expressed through the function \(\zeta(\tau) \equiv 1 + G_\delta(\tau)\) using the following relations:

\[
\varphi(y) = y\zeta(\tau(y)) + \frac{1}{2}\tau^2, \quad \tau(y) = -y\zeta'(\tau(y))
\]

where dot means derivative with respect to \(\tau\) (see, e.g., Bernardeau & Schaeffer 1992), so that

\[
\frac{d\varphi(y)}{dy} = \zeta(\tau(y)), \quad \frac{d^2\varphi(y)}{dy^2} = \zeta' \frac{d\tau}{dy} = -\zeta^2 \left(1 - \frac{\tau\zeta'}{\zeta}\right)^{-1}.
\]

Finally, the density PDF follows from the inverse Laplace transformation

\[
\eta(\delta) = \frac{1}{2\pi i\sigma^2} \int_{c-i\infty}^{c+i\infty} exp\left(\frac{(1+\delta)y - \varphi(y)}{\sigma^2}\right) dy
\]

where \(c = const\).
For example, for a purely Gaussian stochastic field, $\zeta(\tau) = 1 + \tau$, $y = -\tau$ and $\varphi(y) = y - y^2/2$ in accordance with the fact that all $S_p$ with $p > 2$ in Eq. (31) are equal to zero in this case. Then the inverse Laplace transform of this $\varphi(y)$ is just the Gaussian distribution $\eta(\delta) = (2\pi\sigma^2)^{-1/2}\exp\left(-\delta^2/(2\sigma^2)\right)$.

In the limit $\sigma \ll 1$, the main contribution to the integral in (33) is made by stationary points of the exponent in it, i.e. by roots of the equation

$$
\frac{d\varphi(y)}{dy} \equiv 1 + G_\delta(\tau) = 1 + \delta
$$

(complex roots of this equation should be considered, too). Thus, here we return to the prescription $\delta \to G_\delta$ used in the derivation of equations for the generating functions in Sec. 3. In this (steepest descent) approximation, Eq. (33) takes the form

$$
\eta(\delta) = \frac{1}{\sqrt{2\pi\sigma^2}} \left|\frac{1}{\tau' \zeta} \exp\left(-\frac{\tau^2}{2\sigma^2}\right)\right|
$$

where now $\tau(\delta)$ should be determined from Eq. (34) for any given generating function $G_\delta(\tau)$.

Let us now discuss the accuracy of the formulae (33, 35) for the density PDF. Since the generating functions $G_\delta(\tau)$ and $\varphi(y)$ have been calculated in the lowest order in $\sigma$ only, there exist small corrections to them beginning from terms proportional to $\sigma^2$ so that $\varphi(y) = \varphi_0(y) + \sigma^2 \varphi_1(y) + ...$ (here the zero-order term $\varphi_0(y)$ stands for $\varphi(y)$ used above) and similarly for $G_\delta$. The account of the $\sigma^2$ correction results in the multiplication of the integrand of Eq. (33) by the $\sigma$-independent term $\exp(-\varphi_1(y))$ having an unknown dependence on $y$. Similarly, the multiplicative factor $\exp(-\varphi_1(y(\tau)))$ will appear in the right-hand side of Eq. (35). In the limit $|\delta| \ll 1$ when $\delta \approx \tau \approx -y$, this factor has the form $(1 + O(\delta^2))$, the $\delta^2$ (or $\tau^2$) correction arising due to a $\sigma^4$ correction to the dispersion of density perturbations. Of course, we may renormalize $\sigma^2$ by defining it to be equal to the exact dispersion of density perturbations and not $\langle \delta^{(12)} \rangle$ as is used in the paper, but all other corrections beginning from a term $\propto \delta^3$ will remain.

Therefore, we have to conclude that if each term in the power series (13) for the generating function $G_\delta$ is computed in the lowest order in $\sigma$ as is done in Bernardeau 1992 and in the present paper, then the resulting density PDF has an exponential accuracy: the large exponential factor in it appears to be correct for not too large $\delta$, but the coefficient of the exponential should be taken in the limit $\delta \to 0$. One is allowed to keep terms linear in $\delta$ in the latter but we shall not do so except for retaining the coefficient $1/\sqrt{2\pi\sigma^2}$ in front of the large exponential (we make an exception for the Zeldovich approximation where it is instructive to keep the entire coefficient in Eq. (35) in order to compare it with an exact analytic solution). This is enough, however, to obtain very significant deviations from Gaussian behaviour in the quasi-linear regime.

On the other hand, to go beyond exponential accuracy and evaluate the coefficient of the exponential exactly, one has to calculate $\varphi_1(y)$. Generally, this quantity is spectrum-dependent although the ZA presents an important exception to this rule. But even then, the resulting expression may not be used for arbitrarily large $\delta$ in the limit $\sigma \ll 1$. In particular,
it is not possible to get the threshold value $\delta_c$ appearing in a Press-Schechter-like formula for the number of collapsed objects from such a treatment. This arises because other density configurations different from the spherical top-hat one may give a dominant contribution to the generating function $G_\delta$ in this regime. It results for instance, in that the value $\delta_c$ for ZA should be taken as $\sqrt{5}$ in the limit $\sigma \ll 1$ as shown in the Appendix, and not as 3 as seems from Eq. (38) below. Moreover, the case of the Zeldovich approximation shows that the value of $\delta$ for which such a new dominant configuration appears cannot be obtained from the spherical approximation for the vertex generating function.

Now we consider the concrete approximations.

The Zeldovich approximation:
This case is very important because we may compare our result with an exact analytic solution for the density PDF valid for all values of $\sigma$ and $\delta$ which was obtained by Kofman (1991). This permits us to check the range of validity of the method used. This solution and its limit for $\sigma \ll 1$ are given in the Appendix.

Using Eqs. (21b, 32a,b, 33) and changing the integration variable in (33) from $y$ to $\tau$, we obtain

$$ \zeta(\tau) = \left(1 - \frac{\tau}{3}\right)^{-3}, \quad y = -\tau \left(1 - \frac{\tau}{3}\right)^4, \quad \varphi(y) = -\tau + \frac{5}{6} \tau^2, \quad (36a) $$

$$ \eta(\delta) = -\frac{1}{2\pi i \sigma^2} \int_C \left(1 - \frac{\tau}{3}\right)^3 \left(1 - \frac{5\tau}{3}\right) \exp\left(\frac{\tau - \frac{5}{6} \tau^2 - (1 + \delta) \tau \left(1 - \frac{\tau}{3}\right)^4}{\sigma^2}\right) d\tau, \quad (36b) $$

where the integration contour $C$ corresponding to that in Eq. (33) is a continuous curve in the complex $\tau$ plane beginning in the sector $|\tau| \to \infty$, $0.7\pi < \arg(\tau) < 0.9\pi$ and ending in the sector $|\tau| \to \infty$, $-0.9\pi < \arg(\tau) < -0.7\pi$. The only irregular non-analytic point of the integrand is at $|\tau| = \infty$. So, for $\sigma \ll 1$, only stationary points of the exponent should be considered. There are four of them:

$$ \tau_n = 3 \left(1 - \frac{e^{i\beta_n}}{(1 + \delta)^{1/3}}\right), \quad n = 1, 2, 3, \quad \beta_n = \frac{2\pi(n - 1)}{3}, \quad \tau_4 = 0.6. \quad (37) $$

The last point does not make a contribution in the lowest order in $\sigma$ because of the coefficient in front of the exponential in Eq. (36b). The point $\tau_4$ gives the largest exponent for not too large $\delta$. The direction of the steepest descent for it is perpendicular to the real axis. Thus, we find using (35) that

$$ \eta(\delta) = \frac{1}{\sqrt{2\pi \sigma^2}} \left(1 - \frac{\tau}{3}\right)^{7/2} \left(1 - \frac{5\tau}{3}\right)^{1/2} e^{-\frac{\tau^2}{2\sigma^2}}, \tau(\delta) = 3 \left(1 - \frac{1}{(1 + \delta)^{1/3}}\right). \quad (38) $$

Comparing this expression with the rigorous result for ZA in the limit $\sigma \ll 1$ (Eq. (A.2)), we see that the large exponential is reproduced exactly up to a rather large value of $\delta$ but the coefficient of the exponential in Eq. (38) is correct in the limit of small $|\delta|$ only. In the latter limit, Eq. (38) just reduces to Eq. (A.3) if terms up to $O(\delta)$ inclusive are kept in the coefficient. $O(\delta^2)$ and higher terms in series expansions of the coefficients of the exponential in (38) and (A.2) are different. Moreover, the appearance of the square root
singularity in Eq. (38) at $\tau = 0.6$, $\delta \approx 0.95$ is an artifact of the approach used, this point is completely regular in the rigorous density PDF for ZA. All this is in complete agreement with the general discussion of limits of validity of the spherical approximation for the vertex generation function in the beginning of this section. (Note also that the exponent in Eq. (38) coincides with that found by Padmanabhan & Subramanian (1993) for the PDF of the final smoothed density field in ZA using a completely different approach.)

**PZA and PPZA:**
With the accuracy chosen above, we immediately get from Eqs. (23b, 24b, 34, 35) that

$$\eta(\delta) \approx \frac{1}{\sqrt{2\pi\sigma^2}} e^{\exp \left(-\frac{\tau^2(\delta)}{2\sigma^2}\right)} \quad (39)$$

where

$$\delta(\tau) = \left(1 - \frac{\tau}{3} - \frac{\tau^2}{21}\right)^{-3} - 1 \quad (39a)$$

for PZA and

$$\delta(\tau) = \left(1 - \frac{\tau}{3} - \frac{\tau^2}{21} - \frac{23\tau^3}{1701}\right)^{-3} - 1 \quad (39b)$$

for PPZA.

**The Frozen flow approximation:**
Now it follows from Eqs. (26, 34, 35) that $\tau = \ln(1 + \delta)$ and

$$\eta(\delta) \approx \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{1 + \delta} e^{\exp \left(-\frac{\ln^2(1 + \delta)}{2\sigma^2}\right)} \quad (40)$$

where we have introduced the coefficient $(1 + \delta)^{-1}$ (that is possible within our accuracy) to make the PDF being normalized to 1 exactly. PDF (40) is just the log-normal distribution which was independently proposed as a good statistical approximation for the density PDF in the quasi-linear regime by Hamilton (1988) and Coles & Jones (1991) (in contrast to the dynamical approximations which we are considering). The result (40) shows that there exists a close internal relationship between FFA and the log-normal approximation. In particular, we may expect that they have approximately the same accuracy in the quasi-linear regime.

**The Linear Potential approximation:**
Eqs. (29b, 34, 35) are now relevant, giving the result

$$\eta(\delta) \approx \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sin \sqrt{2\tau}} e^{\exp \left(-\frac{\tau^2(\delta)}{2\sigma^2}\right)} \quad \delta(\tau) = \left(\frac{\sqrt{2\tau}}{\sin \sqrt{2\tau}}\right)^3 - 1. \quad (41)$$

6. Conclusions

The nonlinear evolution of an initially Gaussian distribution leads to the following relationship between the connected density moments $\langle \delta^{(n)} \rangle_c$ at the lowest order in $\langle \delta^2 \rangle$: $\langle \delta^n \rangle_c \approx$
$S_n (\delta^2)^{n-1}$ where the parameters $S_p$ characterise the development of non-Gaussianity ($S_3$ describes the skewness of the density distribution and $S_4$ its kurtosis). Generalising earlier work by Bernardeau (1992) we have shown that for approximation methods attempting to mimic the effects of nonlinear gravity the values of $S_p$ can be derived by means of a generating function $G_\delta$. For a given nonlinear approximation $G_\delta \equiv \delta_{sph}$ is simply the overdensity in the spherical top hat model in that approximation. Thus knowing how particle trajectories evolve in the spherical top hat model in a given nonlinear approximation we can determine $G_\delta$ and consequently $S_p$ and the probability distribution function for the density.

Following this ansatz we determine the functional form of the generating function and the values of the first six parameters $S_1, \ldots S_6$ for five distinct nonlinear approximations, of which three are formulated in Lagrangian space (including the Zeldovich approximations and its extensions), and the remaining two in Eulerian space (frozen flow and linear potential). Comparing our results with those of an exact perturbative treatment (Bernardeau 1992) we find that an approximation formulated to $n^{th}$ order in Lagrangian space correctly reproduces the first $n + 1$ parameters $S_n$ (see Table 2). Our comparison leads us to conclude that nonlinear approximations which are formulated in Lagrangian space are considerably more accurate than those formulated in Eulerian space when tested in the weakly nonlinear regime.

Acknowledgements
The authors are grateful to F. Bernardeau, L. Kofman, T. Padmanabhan and B.S. Sathyaprakash for stimulating discussions. A.A.S. is grateful to Profs. Y. Nagaoka and J. Yokoyama for their hospitality at the Yukawa Institute for Theoretical Physics. The financial support for the research work of A.A.S. in Russia was provided by the Russian Foundation for Basic Research, Project Code 93-02-3631. D.M. was financially supported by the Council of Scientific and Industrial Research, India, under its JRF scheme.
APPENDIX

The rigorous formula for the density PDF in the Zeldovich approximation for a single-stream motion in Eulerian space with Gaussian initial conditions is (Kofman 1991, see also Kofman et al. 1994)

\[ \eta(\delta) = \frac{9 \cdot 5^{3/2}}{4\pi \sigma^4 (1 + \delta)^{3}} \int_{s_0(\delta)}^{s_0(\delta) + \infty} e^{-(s_0 - \delta)^2} \left( 1 + e^{-\frac{ds_0^2}{2\sigma^2}} \right) \left( e^{-\frac{\beta_s^2}{2\sigma^2}} + e^{-\frac{\beta_s^2}{2\sigma^2}} - e^{-\frac{\beta_s^2}{2\sigma^2}} \right) ds, \]  
(A.1a)

\[ s_0(\delta) = \frac{3}{(1 + \delta)^{1/3}}, \quad \beta_n(s, \delta) = s \cdot \sqrt{5} \left( \frac{1}{2} + \cos \left( \frac{2}{3}(n - 1)\pi + \frac{1}{3}\arccos \left( 2 \left( \frac{s_0}{s} \right)^3 - 1 \right) \right) \right). \]  
(A.1b)

For \( \sigma \ll 1 \) and not too large \( \delta \), the main contribution to the integral comes from the lower limit of integration \( s = s_0 \). If \( s = s_0(1 + x) \), \( x \ll 1 \), then \( \beta_1 = s_0 \cdot 3\sqrt{5}/2 + O(x) \), \( \beta_{2,3} = \mp s_0 \cdot \sqrt{5}x / (1 \mp \sqrt{5}/3 + O(x)) \). Then the terms with \( \beta_{2,3} \) are the leading ones (they almost cancel each other), and the formula (A.1a) takes the form

\[ \eta(\delta) = \frac{1}{\sqrt{2\pi \sigma^2}} \left( 1 - \frac{\tau}{3} \right)^{19/2} e^{-\frac{x^2}{2\sigma^2}} \tau = 3 \left( 1 - \frac{1}{(1 + \delta)^{1/3}} \right). \]  
(A.2)

If \( |\delta| \ll 1 \), too (but \( |\delta| \) may be much more than \( \sigma \), then (A.2) simplifies to

\[ \eta(\delta) = \frac{1 - 2\delta + O(\delta^2)}{\sqrt{2\pi \sigma^2}} \exp \left( -\frac{9}{2\sigma^2} \left( 1 - \frac{1}{(1 + \delta)^{1/3}} \right)^2 \right). \]  
(A.3)

The expression (A.2) clearly loses sense for \( \delta \geq (7/2)^3 - 1 \approx 42 \), but it is no more the dominant term in (A.1a) for this value and even for somewhat smaller values of \( \delta \). A numerical calculation shows that for \( \delta \geq 30.6 \), the main contribution to the PDF is produced by the maximum of the exponential \( \exp \left( -\left((s - 3)^2 + \beta_s^2\right)/2\sigma^2 \right) \) which is located at \( s \approx 4/3 \) for \( \delta \to \infty \), other terms in (A.1a) being exponentially smaller. As a result, \( \eta(\delta) \propto (1 + \delta)^{-3}\exp(-\delta_c^2/2\sigma^2) \) with \( \delta_c = \sqrt{5} \) for \( \delta \to \infty \) and \( \delta \ll 1 \). Thus, the Press-Schecheter-like density perturbation value \( \delta_c \) (a “threshold” for formation of compact objects) is equal to \( \sqrt{5} \) in the Zeldovich approximation, and not to 3 that would follow from a naive application of the formula (A.2) in the regime \( \delta \gg 1 \). Of course, even \( \sqrt{5} \) is a great overestimation of a real value of \( \delta_c \) in the exact solution (if it has sense at all) which is expected to be equal to, or a little bit less than 1.686.

Note also that the PDF (A.1a) is not normalized to unit total probability due to the appearance of multistreaming motion in the Zeldovich approximation (as well as in the exact solution) even for a small \( \sigma \). Actually, \( \int \eta(\delta) d\delta \) gives the mean number of streams \( N_s \) (see also Kofman et al. 1994). This is, however, an exponentially small effect, it is straightforward to show by direct integration of Eq. (A.1a) that

\[ N_s = 1 + \frac{16}{27\sqrt{10\pi}} \sigma \exp \left( -\frac{5}{2\sigma^2} \right) \]  
(A.4)

for \( \sigma \ll 1 \). Here, the threshold value \( \delta_c = \sqrt{5} \) appears once more. This formula is not bad even for \( \sigma = 1 \) where it gives \( N_s \approx 1.0087 \) instead of the exact value \( N_s(1) \approx 1.0137 \).
|       | Exact | PPZA | PZA | ZA  | LP  | FF  |
|-------|-------|------|-----|-----|-----|-----|
| $\tau_{ta}$ | 1.062 | 1.118| 1.194| 1.5 | 2.058| 3   |
| $\tau_{coll}$ | 1.686 | 2.050| 2.266| 3   | 4.935| $\infty$ |

Table 1: Spherical collapse

|       | Exact | PPZA | PZA | ZA  | LP  | FF  |
|-------|-------|------|-----|-----|-----|-----|
| $S_3$ | 4.857 | 4.857| 4.857| 4   | 3.4 | 3   |
| $S_4$ | 45.89 | 45.89| 44.92| 30.22| 21.22| 16  |
| $S_5$ | 656.3 | 654.6| 624.4| 342.2| 196.4| 125 |
| $S_6$ | 12,653| 12,568| 11,666| 5200| 2429 | 1296|

Table 2: Moments of $\delta$ field

|       | Exact | PPZA | PZA | ZA  | LP  | FF  |
|-------|-------|------|-----|-----|-----|-----|
| $T_3$ | -3.714| -3.714| -3.714| -2  | -0.8| 0.  |
| $T_4$ | 27.41 | 27.41| 24.49| 8   | 1.46| 0.  |
| $T_5$ | -308.4| -301.5| -240.8| -48.9| -4.19| 0.  |
| $T_6$ | 4694  | 4450 | 3180 | 404.4| 16.35| 0.  |

Table 3: Moments of $\theta$ field
References

[1] Bagla, J.S., & Padmanabhan, T., 1994. MNRAS, 266, 227.

[2] Bernardeau, F., 1992. ApJ, 392, 1.

[3] Bernardeau, F., 1993. CITA preprint 93/14; ApJ, in press.

[4] Bernardeau, F., & Schaeffer, R., 1992. A&A, 255, 1.

[5] Bernardeau, F., Singh, T.P., Banerjee B., & Chitre S.M., 1993. Preprint TIFR, astro-ph/9311053: MNRAS, submitted.

[6] Bouchet, F.R, Juszkiewicz, R., Colombi, S. & Pellat, R., 1992. ApJL, 394, L5.

[7] Brainerd, T.G., Scherer, R.J., & Villumsen, J.V., 1993. ApJ, 418, 570.

[8] Buchert, T., 1992. MNRAS, 254, 729.

[9] Coles, P., & Jones, B., 1991. MNRAS, 248, 1.

[10] Fry, J.N., 1984. ApJ, 279, 499.

[11] Gramman, M., 1993. ApJL, 405, L47.

[12] Grinstein, B., & Wise, M.B., 1987. ApJ, 320, 448.

[13] Hamilton, A., 1998. ApJL, 331, L59.

[14] Juszkiewicz, R., Weiberg, D.H., Amsterdamsky, P., Chodorovsky, M., & Bouchet, F.R., 1993. IAS preprint (IASSNS-AST 93/50).

[15] Kofman, L., 1991. In: Primordial Nucleosynthesis and Evolution of Early Universe, eds. K.Sato & J. Audouze Dordrecht: Kluwer, p. 495.
[16] Kofman, L., Bertschinger, E., Gelb, J.M., Nusser, A., & Dekel, A., 1994. ApJ, 420, 44.

[17] Lachieze-Rey, M., 1993. ApJ 408, 403.

[18] Matarrese, S., Lucchin, F., Moscardini, L., & Saez, D., 1992. MNRAS, 259, 437.

[19] Moutarde, F., Alimi, J.M., Bouchet, F.R., Pellat, R. & Ramani, A., 1991. ApJ, 382, 377.

[20] Munshi, D., & Starobinsky, A.A., 1993. Preprint IUCAA, astro-ph/9311056; ApJ, in press.

[21] Padmanabhan, T. & Subramanian, K., 1993. Ap.J, 410, 482.

[22] Peebles, P.J.E., 1980. The Large-Scale Structure of the Universe, Princeton, Princeton University Press.

[23] Sathyaprakash, B.S., Munshi, D., Sahni, V., Pogosyan, D. & Melott, A.L., 1994, in preparation.

[24] Shandarin, S.F., & Zeldovich, Ya. B., 1989. Rev. Mod. Phys., 61, 185.

[25] White, S.D.M., 1979. MNRAS, 186, 145.

[26] Zeldovich, Ya.B., 1970. Astron.Astroph., 5, 84.

[27] Zeldovich, Ya.B. and Novikov, I.D., 1983, The Structure and Evolution of the Universe (University of Chicago, Chicago/London).
Figure Captions

**Fig. 1**: The density in the top-hat spherical collapse model as estimated in the different approximations \(y(\tau) = 1 + \delta\) is plotted against the exact solution. The exact solution is labelled by 1; PPZA by 2; PZA by 3; ZA by 4; LP by 5; FF by 6; and the linear solution by 7.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/astro-ph/9402065v1