Sheaf theoretic characterization of étale groupoids

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Abstract

The study of Haefligr [2, 3] suggests that it is natural to regard a pseudogroup as an étale groupoid. We show that any étale groupoid corresponds to a pseudogroup sheaf, a new generalization of a pseudogroup. This correspondence is an analog of the equivalence of the two definitions of a sheaf: as an étale space and as a contravariant functor.

0 Introduction

Let $X$ be a topological space, and let $X_{\text{top}}$ be the set of the open sets in $X$. $X_{\text{top}}$ is an ordered set, therefore, $X_{\text{top}}$ is a small category. A presheaf on $X$ is a functor $X_{\text{top}} \to \text{Set}$, where $\text{Set}$ is the category of the (small) sets and the maps. A presheaf $\mathcal{F}$ is a sheaf if, for any open set $U \subset X$ and any open covering $\{U_\lambda\}$ of $U$, the following diagram is an equalizer:

$$\mathcal{F}(U) \to \prod_\lambda \mathcal{F}(U_\lambda) \Rightarrow \prod_{\lambda, \mu} \mathcal{F}(U_{\lambda \mu}),$$

where $U_{\lambda \mu} = U_\lambda \cap U_\mu$. The category of the sheaves on $X$ is equivalent to the category of the étale spaces over $X$. An étale space over $X$ is a local homeomorphism $E \to X$ from a topological space $E$ to the given topological space $X$. (cf. [1] or Section 1.)

For two open sets $U$ and $V$ in $X$, let $\text{Homeo}_X(U, V)$ be the set of the homeomorphisms from $U$ to $V$. The correction of the sets $\text{Homeo}_X(U, V)$ determines a groupoid $\mathcal{H}$ of $\text{Homeo}_X$. A pseudogroup is a subgroupoid $\mathcal{H}$ of $\text{Homeo}_X$ satisfying the sheaf property:

- For any homeomorphism $f \in \text{Homeo}_X(U, V)$, $f$ belongs to $\mathcal{H}(U, V)$ if and only if there exists an open covering $\{U_\alpha\}$ of $U$ such that the restrictions $f|_{U_\alpha}$ belong to $\mathcal{H}(U_\alpha, f(U_\alpha))$. 

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According to Haefliger [2, 3], it is natural to regard pseudogroups as étale groupoids. A topological groupoid (over $X$) is a groupoid object in the category of the topological spaces (such that the object space is $X$). A topological groupoid is étale if the source map is a local homeomorphism (cf. [5] or Section 1.1). Resende [6] introduced an abstract pseudogroup as a complete and infinitely distributive inverse semigroup, and he gave equivalent correspondence among the étale groupoids, the abstract pseudogroups and the quantales, by locale theory.

We define a pseudogroup sheaf (cf. Definition 3.2), a new generalization of a pseudogroup, and prove the equivalence of the étale groupoids over $X$ and the pseudogroup sheaves on $X$, as an analogue of the equivalence of the étale spaces over $X$ and the sheaves on $X$. The main result in this paper is the following theorem.

**Theorem 4.3.** Let $X$ be a $T_1$ space. Then, there exists an equivalence $\text{Pse}(X) \simeq \text{ÉtGrpd}_X$.

$\text{Pse}(X)$ is the category of the pseudogroup sheaves on $X$, and $\text{ÉtGrpd}_X$ is the category of the étale groupoids over $X$. The requirement that the base space is a $T_1$ space in Theorem 4.3 is contrasted with the requirement that the base space is a sober space in Resende’s result.

We review sheaves and étale spaces in Section 1. We see the correspondence from étale groupoids to pseudogroup sheaves in Section 2. We define a pseudogroup sheaf, and see the correspondence from pseudogroup sheaves to étale groupoids in Section 3. The main result is obtained by summarizing the contents of the previous sections in categorical viewpoints in Section 4. As an added note, the interpretation of a pseudogroup in classical sense as a pseudogroup sheaf is given in Section 5.

In this paper, we write the operation of a given category in three different manners.

$(-) \circ (-)$ is the composition of two maps.

$(-) \cdot (-)$ is the operation of a given étale groupoid.

$(-) \bullet (-)$ is the operation of a given pre-pseudogroupoid.

We often call a morphism “an arrow” in a given étale groupoid or a given pre-pseudogroupoid. The purpose of these manners is to eliminate confusion.

1 \hspace{1cm} Sheaves

Let $X$ be a topological space. A presheaf on $X$ is a contravariant functor on the ordered set of the open sets in $X$ with valued in the category of
the (small) sets and the maps. A sheaf on \(X\) is a presheaf satisfying the sheaf condition (cf. Definition 1.3). The category of the sheaves on \(X\) is equivalent to the category of the étale spaces over \(X\) (cf. Section 1.5).

1.1 Preliminaries

Let \(X\) be a topological space. First, we mention some notations for topological spaces in this paper.

**Definition 1.1.**

- \(X_{\text{top}}\) is the set of the open sets in \(X\).
- \(\mathcal{N}(x)\) is the set of the open neighborhoods of a point \(x \in X\).
- For two points \(x, y \in X\), we denote \(x \to y\) if \(\mathcal{N}(y) \subset \mathcal{N}(x)\). (In other words, we denote \(x \to y\) if a sequence \(\{x\}\) converges to \(y\).)
- A topological space \(X\) is a \(T_1\) space if a relation \(x \to y\) implies the equality \(x = y\) for any two points \(x, y \in X\).

\(X_{\text{top}}\) is an ordered set, therefore, \(X_{\text{top}}\) is a small category. We also mention some notations for categories.

**Definition 1.2.**

- \(\text{Ob}(\mathcal{C})\) is the class of the objects of a given category \(\mathcal{C}\).
- \(\mathcal{C}^{\text{op}}\) is the opposite category of a category \(\mathcal{C}\).
- A set is small if it is in the given universe (cf. [iii]). A category is small if the set of the objects and the set of the morphisms are small.
- \(\text{Set}\) is the category of the (small) sets and the maps.
- A filtered set \(\Lambda\) is an ordered set \(\Lambda\) such that, for any two element \(\lambda, \mu \in \Lambda\), there exists an element \(\nu \in \Lambda\) such that it satisfies the relations \(\lambda, \mu \leq \nu\).
- A direct system (resp. a inverse system) indexed by a filtered set \(\Lambda\) is a functor \(\Lambda \to \text{Set}\) (resp. \(\Lambda^{\text{op}} \to \text{Set}\)).
- A direct limit \(\varinjlim F\) (resp. a inverse limit \(\varprojlim G\)) of a direct system \(F\) (resp. a inverse system \(G\)) indexed by a filtered set \(\Lambda\) is a set defined by the follows:
\[ \lim_{\to} F = \prod_{\lambda} F_{\lambda}/\sim, \]
where \( \sim \) is an equivalent relation defined by the follows:
\[ x \sim y \ (x \in F_{\lambda}, y \in F_{\mu}) \iff \exists \nu \geq \lambda, \mu \text{ s.t. } F_{\lambda \leq \nu}(x) = F_{\mu \leq \nu}(y). \]

\[ \lim_{\leftarrow} G = \{ (x_{\lambda}) \in \prod_{\lambda} G_{\lambda} \mid G_{\mu \geq \lambda}(x_{\lambda}) = x_{\mu} \}. \]

1.2 Sheaves

We define sheaves on a topological space.

**Definition 1.3** (cf. [1]). A presheaf on a topological space \( X \) is a functor \( X^{\text{op}} \to \text{Set} \). A presheaf \( F \) is a sheaf (or a sheaf of sections) if, for any open set \( U \subset X \) and any open covering \( \{U_{\lambda}\} \) of \( U \), the following diagram is an equalizer:
\[ F(U) \to \prod_{\lambda} F(U_{\lambda}) \rightrightarrows \prod_{\lambda, \mu} F(U_{\lambda \mu}), \]
where \( U_{\lambda \mu} = U_{\lambda} \cap U_{\mu} \). A section of a (pre)sheaf \( F \) on an open set \( U \) is an element \( s \in F(U) \).

A morphism between (pre)sheaves is a natural transformation between functors.

1.3 Étale spaces

The concept of étale spaces is equivalent to sheaves.

**Definition 1.4** (cf. [1]). An étale space (or a sheaf of germs) over a topological space \( X \) is a local homeomorphism \( E \to X \) from a topological space \( E \) to \( X \).

Let \( p : E \to X \) and \( q : F \to X \) be étale spaces over \( X \). A morphism \( f : p \to q \) over \( X \) is a continuous map \( f : E \to F \) such that the following diagram is commutative:
\[
\begin{array}{ccc}
E & \xrightarrow{f} & F \\
\downarrow p & & \downarrow q \\
X & \xrightarrow{q} & X
\end{array}
\]

Let \( p : E \to X \) be a étale space over \( X \). The sheaf \( \Gamma(-; p) \) associated with \( p \) is defined by the follows:
\[ \Gamma(U; p) = \{ s : U \to E \mid p(s(x)) = x \ (\forall x \in U) \}. \]
1.4 Étale spaces associated with presheaves

We will consider the inverse construction of the previous subsection. Let $\mathcal{F}$ be a presheaf on $X$. Let $x \in X$ be a point. The set $\mathcal{N}(x)$ of open neighborhoods of $x$ is a preordered set, therefore, $\mathcal{N}(x)$ is a small category. The opposite category $\mathcal{N}(x)^{\text{op}}$ of $\mathcal{N}(x)$ is a filtered set.

**Definition 1.5** (cf. [1]). The stalk $F_x$ of $F$ at $x$ is the direct limit of the direct system $F|_{\mathcal{N}(x)^{\text{op}}}$. For a section $s \in \mathcal{F}(U)$ and a point $x \in U$, the germ $s_x$ of $s$ at $x$ is the image of $s$ for the natural map $\mathcal{F}(U) \to F_x$.

We define a set $E_F$ and a map $p_F : E_F \to X$ as the follows:

$$E_F = \coprod_x F_x, \quad p_F(s_x) = x (s_x \in F_x).$$

For an open set $U \subset X$ and a section $s \in \mathcal{F}(U)$, a subset $[s, U] \subset E_F$ is defined as $[s, U] = \{s_x | x \in U\}$.

**Proposition 1.6** (cf. [1]). The correction of subsets $\{[s, U] | s, U\}$ determines a unique topology on $E_F$ such that $\{[s, U] | s, U\}$ is a basis for the topology. Moreover, the map $p_F : E_F \to X$ is a local homeomorphism.

$p_F : E_F \to X$ is the étale space associated with the presheaf $\mathcal{F}$.

1.5 Categorical viewpoints

Some concrete categories are defined as follows.

- $\mathbf{PSh}(X)$ the category of the presheaves on $X$.
- $\mathbf{Sh}(X)$ the category of the sheaves on $X$.
- $\mathbf{Et}_X$ the category of the étale spaces over $X$.

The construction $p \mapsto \Gamma(-; p)$ defines a functor $\Gamma : \mathbf{Et}_X \to \mathbf{Sh}(X) \subset \mathbf{PSh}(X)$. The construction $\mathcal{F} \mapsto p_F$ defines a functor $p : \mathbf{PSh}(X) \to \mathbf{Et}_X$. Then, there exists an adjunction $p \dashv \Gamma : \mathbf{PSh}(X) \to \mathbf{Et}_X$. The unit map $\mu : Id \to \Gamma \circ p$ of the adjunction $p \dashv \Gamma$ is called the sheafification. For a presheaf $\mathcal{F}$, the component $\mu_\mathcal{F} : \mathcal{F} \to \Gamma(-, p_F)$ of $\mu$ is defined as follows:

$$\mu_\mathcal{F} : \mathcal{F}(U) \to \Gamma(U, p_F); s \mapsto [x \mapsto s_x]$$

The counit map $p \circ \Gamma \to Id$ of the adjunction $p \dashv \Gamma$ is isomorphic. A component of the unit map $\mu_\mathcal{F} : \mathcal{F} \to \Gamma(-, p_F)$ is an isomorphism if and only if the presheaf $\mathcal{F}$ is a sheaf. Therefore, the restriction of the adjunction $p \dashv \Gamma$ induces an equivalence $\mathbf{Sh}(X) \simeq \mathbf{Et}_X$. 


2 Étale groupoids

A topological groupoid (over $X$) is a groupoid object in the category of the topological spaces (such that the object space is $X$). A topological groupoid is étale if the source map is a local homeomorphism (cf. Definition 2.1). Any étale groupoid has a sheaf associated with the étale groupoid (cf. Section 2.2).

2.1 Étale groupoids

Recall, a (small) groupoid $G$ is a (small) category $G$ such that any morphism in $G$ is an isomorphism. (In this paper, we assume that a groupoid is small unless it is confusing.) A topological groupoid is a groupoid $G = (G_0, G_1, s, t, i, \text{inv}, \text{comp})$ such that the set of objects $G_0$ and the set of arrows $G_1$ are topological spaces, and that the structure maps (i.e. source map $s$, target map $t$, identities map $i$, inversion map $\text{inv}$ and composition map $\text{comp}$) are continuous.

Definition 2.1. A topological groupoid is étale if the source map is a local homeomorphism.

Remark 2.2. A topological groupoid is étale if and only if the all structure maps are local homeomorphisms.

In this paper, we fix the object space $G_0$.

Definition 2.3. Let $X$ be a topological space. A topological groupoid over $X$ is a topological groupoid such that the object space is $X$.

Morphisms between topological groupoids are continuous functors.

Definition 2.4. Let $G$ and $\mathcal{H}$ be topological groupoids over a topological space $X$. A morphism $\phi : G \to \mathcal{H}$ over $X$ is a functor $\phi : G \to \mathcal{H}$ such that the object map of $\phi$ is the identity map of $X$ and that the arrow map of $\phi$ is continuous.

2.2 Sheaves associated with étale groupoids

Let $G$ be an étale groupoid over $X$. We define $\mathcal{G}(U, V)$ as the set $\{f : U \to t^{-1}(V) \mid s \circ f = \text{id}\}$ for each open sets $U, V \in X_{\text{top}}$. The correction of these sets $\mathcal{G}(U, V)$ define a small category $\mathcal{G}$ such that the objects are the open sets in $X$. (The composition $(-) \bullet (-)$ of the category $\mathcal{G}$ is defined as $(g \bullet f)(x) = g(t(f(x))) \cdot f(x)$ for $x \in U$, where the operation $(-) \cdot (-)$ is the
composition of the groupoid $\mathcal{G}$.) The category $\widehat{\mathcal{G}}$ includes the category $X_{\text{top}}$ as a subcategory.

For each open set $V \in X_{\text{top}}$ and each point $x \in X$, let $\widehat{\mathcal{G}}(V) = \varprojlim_{U \ni x} \mathcal{G}(U, V)$. For each points $x, y \in X$, let $\widehat{\mathcal{G}}_{xy}(V) = \varprojlim_{V \ni y} \mathcal{G}_{xy}(V).

\textbf{Proposition 2.5.} The above $\widehat{\mathcal{G}}$ satisfies the following:

1. $\text{Ob}(X_{\text{top}}) = \text{Ob}(\widehat{\mathcal{G}})$.

2.1 The natural projections $\widehat{\mathcal{G}}_{xy}(V) \to \mathcal{G}_{xy}(V)$ are injections. (This identifies the set $\mathcal{G}_{xy}(V)$ as the set $\text{Im}[\mathcal{G}_{xy}(V) \to \mathcal{G}_{xy}(V)]$.)

2. The map $\prod_{y \in V} \mathcal{G}_{xy}(V)$ is surjection.

2.3 For three points $x, y, z \in X$, if we have a relation $y \to z$ (cf. Definition 1.1), then we obtain the inclusion $\mathcal{G}_{xy}(V) \subset \mathcal{G}_{xz}(V)$.

2.4 For a germ $f_x \in \mathcal{G}_{xy}(V)$, $f_x$ belongs to $\mathcal{G}_{xz}(V)$ if and only if we have the relation $f(x) \to y$.

3. The presheaf $\widehat{\mathcal{G}}(-, V)$ is a sheaf for each $V \in X_{\text{top}}$.

\textbf{Proof.} The claims (1) and (3) is obvious by definition.

We will prove the claim (2.1). The map $\widehat{\mathcal{G}}(U, V_1) \to \widehat{\mathcal{G}}(U, V_2)$ is an injection by definition for any open sets $U$, $V_1$ and $V_2$ ($V_1 \subset V_2$). The map $\widehat{\mathcal{G}}(V_1) \to \widehat{\mathcal{G}}(V_2)$ is an injection for any point $x$ and any open sets $V_1$ and $V_2$ ($V_1 \subset V_2$). Therefore, the set $\mathcal{G}_{xy}(V) = \varprojlim_{V \ni y} \mathcal{G}_{xy}(V)$ is identified with the set $\bigcap_{V \ni y} \mathcal{G}_{xy}(V)$. In other words, the natural projections $\mathcal{G}_{xy}(V) \to \mathcal{G}_{xy}(V)$ are injections.

We identify the set $\mathcal{G}_{xy}(V)$ as the set $\text{Im}[\mathcal{G}_{xy}(V) \to \mathcal{G}_{xy}(V)]$ by the claim (2.1). For a germ $f_x \in \mathcal{G}_{xy}(V)$, $f_x$ belongs to $\mathcal{G}_{xy}(V)$ if and only if, for any open neighborhood $V'$ of $y$, there exist an open neighborhood $U$ of $x$ and a section $f \in \mathcal{G}(U, V')$ such that the germ of $f$ at $x$ coincides with $f_x$. We obtain the claim (2.2) because $f_x$ belongs to the set $\mathcal{G}_{xy}(V)$ for any $f_x \in \mathcal{G}_{xy}(V)$.

We will prove the claim (2.3). Suppose that we have the relation $y \to z$. Take any germ $f_x \in \mathcal{G}_{xy}(V)$. For any open neighborhood $V$ of $y$, there exist an open neighborhood $U$ of $x$ and a section $f \in \mathcal{G}(U, V)$ such that the germ of $f$ at $x$ coincides with $f_x$. Any open neighborhood of $z$ is an open neighborhood
of $y$ because of the relation $y \rightarrow z$. Therefore, the germ $f_x$ belongs to the set $\hat{G}^y_x$.

We will prove the claim (2.4). A proof of the implication $f(x) \rightarrow y \Rightarrow f_x \in \hat{G}^y_x$ is the same as the proof of the claim (2.3). We will prove the inverse. Suppose that a germ $f_x \in \hat{G}_x(V)$ belongs to $\hat{G}^y_x$. Take any open neighborhood $V'$ of $y$. There exist an open neighborhood $U$ of $x$ and a section $f \in \hat{G}(U,V')$ such that the germ of $f$ at $x$ coincides with $f_x$. The point $f(x)$ belongs to the open set $V'$. Then, $V'$ is an open neighborhood of $f(x)$. Therefore, we obtain the relation $f(x) \rightarrow y$. \hfill \Box

Remark 2.6. Suppose that $X$ is a $T_1$ space (cf. Definition 1.1). If there exists a germ $f_x$ belonging to $\hat{G}^y_x \cap \hat{G}^z_x$, then we obtain the equality $y = f(x) = z$ because of the relation $y \leftarrow f(x) \rightarrow z$. Therefore, if $X$ is a $T_1$ space, all together the above conditions (2.1), (2.2), (2.3) and (2.4) equivalents the following:

(2) The cannonical map $\prod_{y \in V} \hat{G}^y_x \rightarrow \hat{G}_x(V)$ induced by the natural projections $\hat{G}^y_x \rightarrow \hat{G}_x(V)$ is an isomorphism for each open set $V \in X_{top}$ and each point $x \in X$.

3 Pseudogroup sheaves

A pseudogroup is a subgroupoid $\mathcal{H}$ of $\text{Homeo}_X$ satisfying the sheaf property. We define a pseudogroup sheaf (cf. Definition 3.2), a new generalization of a pseudogroup. The category of the pseudogroup sheaves on $X$ is equivalent to the category of the étale groupoids over $X$ (cf. Section 4). The interpretation of a pseudogroup in classical sense as a pseudogroup sheaf is given in Section 5.

3.1 Pseudogroup sheaves

Let $X$ be a topological space. Suppose that $\mathcal{C}$ is a small category with including $X_{top}$ as a subcategory, and $\text{Ob}(X_{top}) = \text{Ob}(\mathcal{C})$. Then, the functor $\mathcal{C}(-,V) : X_{top}^{op} \subset \mathcal{C}^{op} \rightarrow \textbf{Set}$ is a presheaf on $X$ for each $V \in X_{top}$.

For each open set $V \in X_{top}$ and each point $x \in X$, let $\mathcal{C}_x(V) = \lim_{\longrightarrow \#x} \mathcal{C}(-,V)$. For each points $x, y \in X$, let $\mathcal{C}^y_x = \lim_{\longleftarrow \#y} \mathcal{C}_x(V)$.

Suppose that the canonical map $\prod_{y \in V} \mathcal{C}^y_x \rightarrow \mathcal{C}_x(V)$ induced by the natural projections $\mathcal{C}^y_x \rightarrow \mathcal{C}_x(V)$ is an isomorphism for each open set $V \in X_{top}$ and
each point \( x \in X \). We identify the set \( \hat{G}^y_x \) as the set \( \text{Im}[\hat{G}^y_x \to \hat{G}_x(V)] \). Then, the correction of these sets \( C^y_x \) define a small category \( \hat{G} \) such that the sets of the objects is \( X \) and that the sets of the arrows from \( x \) to \( y \) is \( C^y_x \). (The composition \((-) \cdot (-)\) of the category \( C^\star \) is defined as \((g_y \cdot f_x) = (g \cdot f)_x\) for \( x \in U \), where the operation \((-) \cdot (-)\) is the composition of the category \( C \).)

**Lemma 3.1.** The above composition \((-) \cdot (-)\) of the category \( C^\star \) is well-defined.

**Proof.** Take a composable pair \((g_y, f_x) \in C^y_z \times C^y_x\). For any open neighborhood \( W \) of \( z \), there exist an open neighborhood \( V \) of \( y \) and a section \( g \in C(V,W) \) such that the germ of \( g \) at \( y \) coincides with \( g_y \). There exist an open neighborhood \( U \) of \( x \) and a section \( f \in C(U,V) \) such that the germ of \( f \) at \( x \) coincides with \( f_x \). The composition \( g_y \cdot f_x \) is defined as the germ \((g \cdot f)_x\).

Take any sections \( g' \in C(V',W) \) and \( f' \in C(U',V') \) such that the germ of \( g' \) at \( y \) coincides with \( g_y \) and that the germ of \( f' \) at \( x \) coincides with \( f_x \). There exist an open neighborhood \( V'' \subset V \cap V' \) of \( y \) and a section \( g'' \in C(V'',W) \) such that the restrictions \( g|_{V''} \) and \( g'|_{V''} \) coincide with \( g'' \).

There exist an open neighborhood \( U'' \) of \( x \) and a section \( f'' \in C(U'',V'') \) such that the restrictions \( f|_{U''} \) and \( f'|_{U''} \) coincide with \( f'' \). Then, we obtain the equality \((g \cdot f)_x = (g'' \cdot f'')_x = (g' \cdot f')_x\). Therefore, the operation \((-) \cdot (-)\) is well-defined.

**Definition 3.2.** Let \( X \) be a \( T_1 \) space. A *pre-pseudogroup* on \( X \) is a small category \( C \) with an embedding functor \( X_{\text{top}} \subset C \) satisfying the following:

1. \( \text{Ob}(X_{\text{top}}) = \text{Ob}(C) \).
2. The canonical map \( \coprod_{y \in V} C^y_x \to C_x(V) \) induced by the natural projections \( C^y_x \to C_x(V) \) is an isomorphism for each open set \( V \in X_{\text{top}} \) and each point \( x \in X \).
3. The category \( C^\star \) is a groupoid.

A *pseudogroup sheaf* on \( X \) is a pre-pseudogroup on \( X \) satisfying the following:

1. The presheaf \( C(-, V) \) is a sheaf for any \( V \in X_{\text{top}} \).

**Remark 3.3.** When \( X \) is not a \( T_1 \) space, this definition is possible, but could not be reasonable (cf. Proposition 2.5).
The first example is the pseudogroup sheaf associated with an étale groupoid.

**Example 3.4.** Let \( G \) be an étale groupoid, and let \( \widehat{G} \) be the category defined in Section 2.2. If \( X \) is a \( T_1 \) space, then the category \( \widehat{G} \) satisfies the conditions (1), (2) and (4) in Definition 3.2. In fact, the category \( \widehat{G} \) is a pseudogroup sheaf on \( X \) because of Proposition 3.5.

**Proposition 3.5.** The groupoid \( G \) is isomorphic to the category \( \widehat{G}^* \). In particular, the category \( \widehat{G}^* \) is a groupoid.

**Proof.** Define a functor \( \phi: \widehat{G}^* \to G \) as the follows:

\[
\phi(f_x) = f(x).
\]

In fact, the map \( \phi \) is compatible with the operations by the follows:

\[
\phi(g_y \cdot f_x) = \phi(g \cdot f)_x = (g \cdot f)(x) = g(y) \cdot f(x) \quad \text{(cf. Section 2.2)}
\]

( Remark that the pair \((g_y, f_x)\) is a composable pair, therefore, we obtain the equality \( y = t(f(x)) \).) The functor \( \phi \) is a bijection because the source map of the groupoid \( G \) is a local homeomorphism. Therefore, we obtain an isomorphism \( \phi: \widehat{G}^* \to G \). \( \square \)

We give other examples.

**Example 3.6.** The category \( X_{\text{top}} \) is a pseudogroup sheaf on \( X \).

**Example 3.7.** Let \( \text{LoHomeo}_X(U, V) = \{ f: U \to V : \text{local homeomorphisms} \} \) for each open sets \( U, V \in X_{\text{top}} \). The correction of these sets \( \text{LoHomeo}_X(U, V) \) define a small category \( \text{LoHomeo}_X \) such that the objects are the open sets in \( X \). The category \( \text{LoHomeo}_X \) satisfies the conditions (1), (3) and (4) in Definition 3.2. If \( X \) is a \( T_1 \) space, then \( \text{LoHomeo}_X \) is a pseudogroup sheaf on \( X \).

**Example 3.8.** Let \( \mathcal{F} \) be a sheaf of group on \( X \). We define \( \mathcal{C} \) as

\[
\mathcal{C}(U, V) = \begin{cases} 
\mathcal{F}(U) & (U \subset V) \\
\emptyset & (U \not\subset V).
\end{cases}
\]

Then \( \mathcal{C} \) is a pseudogroup sheaf on \( X \).
3.2 Étale groupoids associated with pre-pseudogroups

Let $\mathcal{C}$ be a pre-pseudogroup on a $T_1$ space $X$. Let $s : E_\mathcal{C} \to X$ be an étale space associated with the presheaf $\mathcal{C}(-, X)$. Then, $E_\mathcal{C} = \prod_{x \in X} \mathcal{C}_x(X) \cong \prod_{x \neq y} \mathcal{C}_x(X) \cong \prod_{x \neq y} \mathcal{C}_y(X)$ and $s(f_x) = x$ for $f_x \in \mathcal{C}_y(X)$ (cf. Section 1.4). In other words, $E_\mathcal{C}$ is the set of arrows of the groupoid $\mathcal{C}^*$, and $s$ is the source map of $\mathcal{C}^*$.

Proposition 3.9. $\mathcal{C}^*$ is an étale groupoid.

Proof. We have to prove that the identities map $i : X \to E_\mathcal{C}$, the inversion map $\text{inv} : E_\mathcal{C} \to E_\mathcal{C}$ and the composition map $\text{comp} : E_\mathcal{C} \times X E_\mathcal{C} \to E_\mathcal{C}$ are continuous. (Then, the target map $t : E_\mathcal{C} \to X$ is continuous because of the equality $t = s \circ \text{inv}$.)

For an open set $U \subset X$ and a section $f \in \mathcal{C}(U, X)$, a subset $[s, U] \subset E_\mathcal{C}$ is defined as $[s, U] = \{f_x | x \in U\}$. Recall that the collection of these $[f, U]$ generates the open sets of $E_\mathcal{C}$ (cf. Section 1.4).

Claim 1. $i$ is continuous.

The map $i$ coincides with the inverse of the homeomorphism $s|_{[id_X, X]} : [id_X, X] \to X$. Therefore, $i$ is continuous.

Claim 2. $\text{inv}$ is continuous.

Take any point $g_y \in \text{inv}^{-1}([f, U])$. Denote $\text{inv}(g_y) = f_x$. This satisfies the equality $f_x \cdot g_y = id_y$. $U$ is an open neighborhood of $x(= \overline{f}(y))$. There exist an open neighborhood $V$ of $y$ and a section $g \in \mathcal{C}(V, U)$ such that the germ of $g$ at $y$ coincides with $g_x$. We obtain the equality $(f \cdot g)_y = id_y$. There exists an open neighborhood $V' \subset V$ of $y$ such that $f \cdot g|_{V'} = id|_{V'}$. We obtain the inclusion $g_y \in [g, V'] \subset \text{inv}^{-1}([f, U])$. Therefore, $\text{inv}$ is continuous.

Claim 3. $\text{comp}$ is continuous.

Take any point $(g_y, f_x) \in \text{comp}^{-1}([h, U])$. Denote $g_y \cdot f_x = h_x$. There exist an open neighborhood $V$ of $y$ and a section $g \in \mathcal{C}(V, X)$ such that the germ of $g$ at $y$ coincides with $g_y$. There exist an open neighborhood $U' \subset U$ of $x$ and a section $f \in \mathcal{C}(U', V)$ such that the germ of $f$ at $x$ coincides with $f_x$. Because of the equality $g_y \cdot f_x = h_x$, there exists an open neighborhood $U'' \subset U'$ of $x$ such that $g \cdot f|_{U''} = h|_{U''}$. We obtain the inclusion $(g_y, f_x) \in [g, V] \times [f, U'] \subset \text{comp}^{-1}([h, U])$. Therefore, $\text{comp}$ is continuous. 

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4 Main result

Let $X$ be a $T_1$ space. Some concrete categories are defined as follows.

- $\text{PPse}(X)$ the category of the pre-pseudogroups on $X$.
- $\text{Pse}(X)$ the category of the pseudogroup sheaves on $X$.
- $\text{ÉtGrpd}_X$ the category of the étale groupoids over $X$.

The construction $\mathcal{G} \mapsto \widehat{\mathcal{G}}$ (cf. Section 2.2) defines a functor $G : \text{ÉtGrpd}_X \to \text{Pse}(X) \subset \text{PPse}(X)$. The construction $\mathcal{C} \mapsto \mathcal{C}^*$ (cf. Section 3.2) defines a functor $F : \text{PPse}(X) \to \text{ÉtGrpd}_X$. We will prove that there exists an adjunction $F \dashv G$.

We will construct the unit map $\mu : \text{Id} \to GF$ of the adjunction $F \dashv G$.

Let $\mathcal{C}$ be a pre-pseudogroup on $X$. Define a morphism $\mu_\mathcal{C} : \mathcal{C} \to \widehat{\mathcal{C}}^*$ over $X$ as follows:

$\mu_\mathcal{C}(f)(x) = f_x$ (for $f \in \mathcal{C}(U, V)$ and $x \in U$).

Remark that the morphism $\mathcal{C}(-, V) \to \widehat{\mathcal{C}}^*(-, V)$ between presheaves induced by $\mu_\mathcal{C}$ is the sheafification (cf. Section 1.5).

**Proposition 4.1.** Let $\mathcal{C}$ and $\mu_\mathcal{C}$ be as above. For any étale groupoid $\mathcal{G}$ on $X$ and any morphism $\phi : \mathcal{C} \to \widehat{\mathcal{G}}$ between pre-pseudogroups, there exists a unique morphism $\hat{\phi} : \mathcal{C}^* \to \widehat{\mathcal{G}}$ between étale groupoids such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mu_\mathcal{C}} & \mathcal{C}^* \\
\downarrow{\phi} & & \downarrow{G(\hat{\phi})} \\
\widehat{\mathcal{G}} & & \\
\end{array}
$$

**Proof.** The morphism $\mathcal{C}(-, X) \to \widehat{\mathcal{C}}^*(-, X)$ between presheaves induced by $\mu_\mathcal{C}$ is the sheafification. $s_{\mathcal{C}^*} : \mathcal{C}_1^* \to X$ is the étale space associated with the presheaf $\mathcal{C}(-, X)$. The sheaf $\widehat{\mathcal{C}}^*(-, X)$ (resp. $\widehat{\mathcal{G}}(-, X)$) is the sheaf associated with the étale space $s_{\mathcal{C}^*}$ (resp. $s_{\mathcal{G}} : \mathcal{G}_1 \to X$, where $s_{\mathcal{G}}$ is the source map of the groupoid $\mathcal{G}$). The morphism $\phi : \mathcal{C}(-, X) \to \widehat{\mathcal{G}}(-, X)$ between presheaves induces a unique morphism $\hat{\phi}_1 : s_{\mathcal{C}^*} \to s_{\mathcal{G}}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{C}(-, X) & \xrightarrow{\mu_\mathcal{C}} & \widehat{\mathcal{C}}^*(-, X) \\
\downarrow{\phi} & & \downarrow{G(\hat{\phi}_1)} \\
\widehat{\mathcal{G}}(-, X) & & \\
\end{array}
$$
(where the functor $\Gamma$ is the same as in Section 1.5.) Then, the map $\tilde{\phi}_1 : \mathcal{C}_1^* \to \mathcal{G}_1$ can be written concretely as follows:

$$\tilde{\phi}_1(f_x) = \phi(f)(x).$$

Write the target maps as $t$, and the following equality holds:

$$t(\tilde{\phi}_1(f_x)) = t(\phi(f)(x)) = t(\phi(f)_x) = t(f_x).$$

For any composable pair $(g_y, f_x)$, the following equality holds:

$$\tilde{\phi}_1(g_y \cdot f_x) = \tilde{\phi}_1((g \bullet f)_x) = \phi(g \bullet f)(x) = (\phi(g) \bullet \phi(f))(x) = \phi(g)(y) \cdot \phi(f)(x) = \phi_1(g_y) \cdot \phi_1(f_x).$$

Therefore, the map $\tilde{\phi}_1$ determines a morphism $\tilde{\phi} : \mathcal{C}^* \to \mathcal{G}$ between étale groupoids. Then, the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mu^c} & \mathcal{C}^* \\
\phi \downarrow & & \downarrow G(\tilde{\phi}) \\
\tilde{\mathcal{G}} & & \\
\end{array}$$

Conversely, suppose there exists a morphism $\psi : \mathcal{C}^* \to \mathcal{G}$ that makes the above diagram commutative. Then, the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{C}(-, X) & \xrightarrow{\mu^c} & \mathcal{C}^*(-, X) \\
\phi \downarrow & & \downarrow \Gamma(\psi) \\
\tilde{\mathcal{G}}(-, X) & & \\
\end{array}$$

We obtain the equality $\psi_1 = \tilde{\phi}_1$ because of the universality of the sheafification. This implies the equality $\psi = \tilde{\phi}$. Therefore, the morphism $\tilde{\phi}$ is unique.

**Proposition 4.2.** The counit map $\epsilon : FG \to Id$ of the adjunction $F \dashv G$ is isomorphic.
Proof. For an étale groupoid $\mathcal{G}$, the morphism $\epsilon_\mathcal{G} : \hat{\mathcal{G}}^* \to \mathcal{G}$ is the unique morphism such that the following diagram is commutative:

\[
\begin{array}{ccc}
\hat{\mathcal{G}} & \xrightarrow{\hat{\mu}_\mathcal{G}} & \hat{\mathcal{G}}^* \\
\downarrow{id} & & \downarrow{\mathcal{G}(\epsilon_\mathcal{G})} \\
\hat{\mathcal{G}} & \to & \mathcal{G}.
\end{array}
\]

Then, the following diagram is commutative:

\[
\begin{array}{ccc}
\hat{\mathcal{G}}(-,V) & \xrightarrow{\hat{\mu}_\mathcal{G}} & \hat{\mathcal{G}}^*(-,V) \\
\downarrow{id} & & \downarrow{\Gamma(\epsilon')} \\
\hat{\mathcal{G}}(-,V) & \to & \mathcal{G}(-,V),
\end{array}
\]

where $\epsilon'$ is the morphism between étale spaces determined by the arrow map $(\epsilon_\mathcal{G})_1 : \hat{\mathcal{G}}_1^* \to \mathcal{G}_1$. These morphisms $\epsilon'$ coincide with the components of the counit map of the adjunction $p \dashv \Gamma : \text{PSh}(X) \to \text{Et}_X$ (cf. Section 1.5) because of the universality of the sheafification. These are isomorphisms. Therefore, $\epsilon$ is isomorphic.

A component of the unit map $\mu_C : C \to \hat{C}^*$ is an isomorphism if and only if the pre-pseudogroup $C$ is a pseudogroup sheaf. Therefore, the restriction of the adjunction $F \dashv G$ induces an equivalence $\text{Pse}(X) \simeq \text{EtGrpd}_X$. We obtain the following theorem.

**Theorem 4.3.** Let $X$ be a $T_1$ space. Then, there exists an equivalence $\text{Pse}(X) \simeq \text{EtGrpd}_X$.

## 5 Classical pseudogroups

Let $X$ be a $T_1$ space, and let $C$ be a pre-pseudogroup on $X$. Any section $f \in C(U,V)$ has the underlying map $\overline{f} : U \to V$ in the following way: Take any point $x \in U$. The germ $f_x$ of $f$ at $x$ belongs to $C_x(V) \cong \bigsqcup_{y \in V} C_x^y$. Then, there exists exactly one point $y \in V$ such that the germ $f_x$ belongs to $C_x^y$. We define $\overline{f}(x) = y$.

The map $\overline{f} : U \to V$ is a local homeomorphism because of the equality $\overline{f} = t \circ (s|_{[f,U]})^{-1}$, where $t$ is the target map and $s$ is the source map. Moreover, the correspondence $f \mapsto \overline{f}$ define a morphism over $X$ from the pre-pseudogroup $C$ to the pseudogroup sheaf $\text{LoHomeo}_X$ (cf. Example 3.7).
Lemma 5.1.

- \( \overline{id} = id \).
- \( \overline{g \circ f} = \overline{g} \circ \overline{f} \).

Proof. We will prove the equality \( \overline{id} = id \). Take any identity \( id \in \mathcal{C}(U, U) \) and any point \( x \in U \). For any open neighborhood \( V \subset U \) of \( x \), the restriction \( id|_V \) of \( id \) belongs to the set \( \mathcal{C}(V, V) \). Therefore, the germ \( id_x \) belongs to the set \( \mathcal{C}_x^x \). We obtain the equality \( \overline{id}(x) = x \).

We will prove the equality \( \overline{g \circ f} = \overline{g} \circ \overline{f} \). Take any composable pair \( (g, f) \in \mathcal{C}(V, W) \times \mathcal{C}(U, V) \) and any point \( x \in U \). Let \( y = \overline{f}(x) \). The germ \( g_x \) belongs to the set \( \mathcal{C}_{y}^{g(y)} \). For any open neighborhood \( W' \subset W \) of \( g(y) \), there exist an open neighborhood \( V' \subset V \) of \( y \) and a section \( g \in \mathcal{C}(V', W') \) such that the germ of \( g \) at \( y \) coincides with \( g_y \). There exist an open neighborhood \( U' \subset U \) of \( x \) and a section \( f \in \mathcal{C}(U', V') \) such that the germ of \( f \) at \( x \) coincides with \( f_x \). Therefore, the germ \( (g \circ f)_x \) belongs to the set \( \mathcal{C}_{x}^{g(y)} \). We obtain the equality \( \overline{(g \circ f)(x)} = \overline{g(y)} = \overline{(g \circ f)(x)} \). \( \square \)

A pseudogroup in classical sense corresponds with a concrete pseudogroup sheaf.

Definition 5.2. A pseudogroup sheaf \( \mathcal{C} \) on \( X \) is concrete if the functor \( \mathcal{C} \to \text{LoHomeo}_X; f \mapsto \overline{f} \) is faithful.

Let \( \mathcal{C} \) be a concrete pseudogroup on \( X \). Then the subcategory of invertible morphisms of \( \mathcal{C} \) is a pseudogroup in a classical sense.

Conversely, let \( \mathcal{Q} \) be a pseudogroup in a classical sense. For two open sets, define a set \( \mathcal{C}'(U, V) \) by the follows:

\[
\mathcal{C}'(U, V) = \{ i \circ f \mid f \in \mathcal{Q}(U, V'), i \text{ is the inclusion map } V' \hookrightarrow V \}.
\]

The correction of these sets \( \mathcal{C}'(U, V) \) determines a pre-pseudogroup \( \mathcal{C}' \). We can obtain a concrete pseudogroup sheaf \( \mathcal{C} \) by the sheafification of the pre-pseudogroup \( \mathcal{C}' \).

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