SET THEORETIC DEFINING EQUATIONS OF THE VARIETY OF PRINCIPAL MINORS OF SYMMETRIC MATRICES

LUKE OEDING

Abstract. The variety of principal minors of $n \times n$ symmetric matrices, denoted $Z_n$, is invariant under the action of a group $G \subset GL(2^n)$ isomorphic to $(SL(2))^n \ltimes S_n$. We describe an irreducible $G$-module of degree 4 polynomials constructed from Cayley’s $2 \times 2 \times 2$ hyperdeterminant and show that it cuts out $Z_n$ set theoretically. This solves the set-theoretic version of a conjecture of Holtz and Sturmfels. Standard techniques from representation theory and geometry are explored and developed for the proof of the conjecture and may be of use for studying similar $G$-varieties.

1. Introduction

The problem of finding the relations among principal minors of a matrix of indeterminants dates back (at least) 1897 when Nanson [23] found relations among the principal minors of an arbitrary $4 \times 4$ matrix. In 1928 Stouffer [25] found an expression for the determinant of a matrix in terms of a subset of its principal minors. Griffin and Tsatsomeros [10] point out that the number of independent principal minors was essentially known to Stouffer in 1924, [27, 28]. In fact, Stouffer [28] claims that this result was already known to MacMahon in 1893 and later by Muir. Subsequently, interest in the subject seems to have diminished, however much more recently, there has been a renewed interest in the relations among principal minors and their application to matrix theory, probability, statistical physics and spectral graph theory.

In response to questions about principal minors of symmetric matrices, Holtz and Sturmfels [14] introduced the algebraic variety of principal minors of symmetric $n \times n$ matrices (denoted $Z_n$ herein – see section 4 for the precise definition) and asked for generators of its ideal. In the first nontrivial case, Holtz and Sturmfels showed that $Z_3$ is an irreducible hypersurface in $\mathbb{P}^7$ cut out by a special degree four polynomial, namely Cayley’s hyperdeterminant of format $2 \times 2 \times 2$. In the next case they showed (with the aid of a computer calculation) that the ideal of $Z_4$ is minimally generated by 20 degree four polynomials, but only 8 of these polynomials are copies of the hyperdeterminant constructed by natural substitutions. The other 12 polynomials were separated into classes based on their multidegrees. This was done in a first draft of [14], and at that point, the geometric meaning of the remaining polynomials and their connection to the hyperdeterminant was still somewhat mysterious. Because
of the symmetry of the hyperdeterminant, Landsberg suggested to Holtz and Sturmfels the following:

**Theorem 1.1** ([14] Theorem 12). The variety $Z_n$ is invariant under the action of

$$(SL(2)^n) \rtimes \mathfrak{S}_n.$$  

It should be noted that Borodin and Rains [3] found a similar result for two other cases; when the matrix is not necessarily symmetric and for a Pfaffian analog. In [24], we showed that $Z_n$ is a linear projection of the well known Lagrangian Grassmannian. We used this projection to give a geometric proof of Theorem 1.1.

Holtz and Sturmfels named the span of the $(SL(2)^n) \rtimes \mathfrak{S}_n$-orbit of the $2 \times 2 \times 2$ hyperdeterminant the hyperdeterminantal module (denoted $HD$ herein – see Section 3). It was then understood (and included in the final version of [14]) that the 20 degree four polynomials are a basis of the hyperdeterminantal module when $n = 4$. This interpretation led to the following:

**Conjecture 1.2** ([14] Conjecture 14). The prime ideal of the variety of principal minors of symmetric matrices, is generated in degree four by the hyperdeterminantal module for all $n \geq 3$.

While the first two cases of the conjecture ($n = 3, 4$) were proved using a computer, the dimension of the hyperdeterminantal module and the number of variables both grow exponentially with $n$ and this renders computational methods ineffective already in the next case $n = 5$, for which the hyperdeterminantal module has a basis of 250 degree 4 polynomials on 32 variables. Our point of departure is the use of the symmetry of $Z_n$ via tools from representation theory and the geometry of $G$-varieties.

The main purpose of this work is to solve the set-theoretic version of the Holtz and Sturmfels conjecture: (See Example 3.3 for the representation theoretic description of the hyperdeterminantal module in terms of Schur modules used in the following statement.)

**Theorem 1.3** (Main Theorem). The variety of principal minors of symmetric $n \times n$ matrices, $Z_n$, is cut out set theoretically by the hyperdeterminantal module, i.e. the irreducible $(SL(2)^n) \rtimes \mathfrak{S}_n$-module of degree 4 polynomials

$$HD = S_{(2,2)}S_{(2,2)}S_{(2,2)}S_{(4)}\ldots S_{(4)}.$$  

A second, unifying purpose of this work is to study $Z_n$ as a prototypical (non-homogeneous) $G$-variety, and in so doing, to show the use of standard constructions in representation theory and geometry, and to further develop general tools for studying geometric and algebraic properties of such varieties. We anticipate these techniques will be applicable to other $G$-varieties in spaces of tensors such as those that arise naturally in computational complexity [15], signal processing [4, 6, 20], and algebraic statistics [1, 25] (see also [19] for a unified presentation of the use of geometry and representation theory in these areas), and especially to the case of principal minors of arbitrary matrices studied by Lin and Sturmfels, [21] and Borodin and Rains [3].
1.1. Extended outline. First, we give the definitions of many of the objects and tools we will use in the rest of the paper. In Section 2 we recall basic notions concerning tensors, representations and $G$-varieties. In Section 3 we point out many standard facts from representation theory that we will use to study the ideal of $Z_n$ and the hyperdeterminantal module. In particular, we recall a method used by Landsberg and Manivel to study $G$-modules of polynomials via Schur modules. We also show how to use weights and lowering operators to make a two way transition between $G$-modules and polynomials. This is a tool which we use to prove Lemma 8.2. Lemma 8.2 is the key to Proposition 8.1 which is crucial to our proof of Theorem 1.3.

Next, we describe geometric aspects of the variety of principal minors of symmetric matrices and the zero set of the hyperdeterminantal module. In Section 4 we set up notation and give a precise definition of the variety. We also recall two useful facts; a symmetric matrix is determined up to the signs of its off-diagonal terms by its $1 \times 1$ and $2 \times 2$ principal minors, and the dimension of $Z_n$ is $\binom{n+1}{2}$. In Section 5 we describe the nested structure of $Z_n$. In particular, in Proposition 5.2 we show that $Z_n$ contains all possible Segre products of $Z_p$ and $Z_q$ where $p + q = n$. We use this interpretation in Proposition 8.1.

In Section 6 we study properties of the hyperdeterminantal module. In particular, we point out that it has dimension $\binom{n}{3} 5^{n-3}$. In Proposition 6.2 we show that it actually is an irreducible $(SL(2)^n) \ltimes \mathfrak{S}_n$-module of polynomials which occurs with multiplicity 1 in the space of degree 4 homogeneous polynomials. This is a consequence of a more general fact about modules with structure similar to that of the hyperdeterminantal module which we record in Lemma 6.3. In Proposition 6.5 we record the fact (originally proved in [14]) that the hyperdeterminantal module is in the ideal of $Z_n$. Then in Proposition 6.6 we generalize the idea to other varieties that have similar structure.

In Section 7 we extract a general property of the hyperdeterminantal module which we call augmentation. We explore properties of augmented modules via polarization of tensors, a technique used, for example, in the study of secant varieties. Of particular interest is the Augmentation Lemma 7.4 in which we give a geometric description of the zero set of a general augmented module. We apply the Augmentation Lemma 7.4 to give a geometric characterization of the zero set of the hyperdeterminantal module in Lemma 7.6. We use Lemma 7.6 in the proof of Theorem 1.3. Proposition 7.8 is another application of the Augmentation Lemma to polynomials which define Segre products of projective spaces. We use a slightly more complicated version of Lemma 7.8 in the proof of Lemma 8.2.

Then we pull together all of the ideas from the previous parts to prove Theorem 1.3. In particular, we show that any point in the zero set of the hyperdeterminantal module has a symmetric matrix that maps to it under the principal minor map.

In Section 8 we work to understand the case when all principal minors of a symmetric matrix agree with a given vector except possibly the determinant. Of particular importance is Proposition 8.1 which essentially says that for $n \geq 4$, if $z$ is a vector in the zero set of the hyperdeterminantal module, then a specific subset of the coordinates of $z$ determine the rest of its coordinates.

In order to prove Proposition 8.1 we use practically all of the tools from representation theory that we have introduced and developed earlier in the paper. With the aid of Proposition 8.1 the proof of Theorem 1.3 in Section 9 becomes relatively simple.
1.2. Applications of Theorem 1.3 We conclude this introduction by describing how Theorem 1.3 answers questions in other areas via three examples; in Statistics and the study of negatively correlated random variables, in Physics and the study of determinantal point processes, and in Spectral Graph Theory and the study of graph invariants. [2,10,12–14,22,29]

1.2.1. Application to covariance of random variables. Consider a non-singular real symmetric \( n \times n \) matrix \( A \). The principal minors of \( A \) can be interpreted as values of a function \( \omega : \mathcal{P}(\{1, \ldots, n\}) \to [0, \infty) \), where \( \mathcal{P} \) is the power set. This function \( \omega \), under various restrictions, is of interest to statisticians. In this setting, the off-diagonal entries of the matrix \( A^{-1} \) are associated to covariances of random variables. In D. Wagner’s [29] asked the following:

**Question 1.4.** When is it possible to prescribe the principal minors of the matrix \( A \) as well as the off-diagonal entries of \( A^{-1} \)?

Holtz and Sturmfels [14] (Theorem 6) gave an answer to this question using the hyperdeterminantal equations in degree 4, another set of degree 10 equations and the strict Hadamard-Fischer inequalities.

Our main result provides an answer to the first part of the question:

It is possible to prescribe the principal minors of a symmetric matrix if and only if the candidate principal minors satisfy all the relations given by the hyperdeterminantal module. For the second part of the question we can give a partial answer. It is not hard to see that the off-diagonal entries of \( A^{-1} \) are determined up to sign by the \( 0 \times 0 \), \( 1 \times 1 \) and \( 2 \times 2 \) principal minors, and the rest of the principal minors further restrict the freedom in the choices of signs.

Another useful fact is if \( A \) is invertible then

\[
A^{-1} = \frac{\text{adj}(A)}{\det(A)},
\]

where \( \text{adj}(A)_{i,j} = ((-1)^{i+j} \det(A^T_{i,j})) \) is the adjugate matrix.

This formula implies that up to scale, the vector of principal minors of \( A^{-1} \) is the vector of principal minors of \( A \) in reverse order. Therefore the determinant, \( n - 1 \times n - 1 \) and \( n - 2 \times n - 2 \) principal minors of \( A \) determine the off diagonal entries of \( A^{-1} \) up to \( \binom{n}{2} \) choices in combinations of signs, and the rest of the principal minors further restrict the choices of combinations of signs.

1.2.2. Application to determinantal point processes. Determinantal point processes were introduced by Macchi in 1975, and subsequently have received significant attention in many areas. A non zero point \( p_S \in \mathbb{C}^n \) is called determinantal if there is an integer \( m \) and an \( (n + m) \times (n + m) \) matrix \( K \) such that for \( S \subset \{1, 2, \ldots, n\} \)

\[
p_S = \det_{S,\bar{(n+1, \ldots, n+m)}}(K).
\]

Borodin and Rains were able to completely classify all such points for the case \( n = 4 \) (Theorem 4.6 [3]) by giving a nice geometric characterization. Lin and Sturmfels [21] studied the geometric and algebraic properties of the algebraic variety of determinantal points and independently arrived at the same result as Borodin and Rains. Moreover, Lin and Sturmfels
gave a complete proof of the claim of [3] that the ideal of the variety is generated in degree 12 by 718 polynomials.

Consider the case where we impose the restrictions that the matrix $K$ to be symmetric and the integer $m = 0$, and call these restricted determinantal points symmetric determinantal points.

**Restatement:** The variety of all symmetric determinantal points is cut out set theoretically by the hyperdeterminantal module.

This restatement is useful because it provides a complete list of necessary and sufficient conditions for determining which symmetric determinantal points can possibly exist.

1.2.3. **Application to spectral graph theory.** A standard construction in graph theory is the following. To a weighted directed graph $\Gamma$ one can assign an adjacency matrix $\Delta(\Gamma)$.

The eigenvalues of $\Delta(\Gamma)$ are invariants of the graph. The first example is with the standard graph Laplacian. Kirchoff’s well known Matrix–Tree theorem states that any $(n - 1) \times (n - 1)$ principal minor of $\Delta(\Gamma)$ counts the number of spanning trees of $\Gamma$.

There are many generalizations of the Matrix–Tree Theorem, such as the Matrix–Forest Theorem which states that $\Delta(\Gamma)|_S$, the principal minor of the graph Laplacian formed by omitting rows and columns indexed by the set $S \subset \{1, \ldots, n\}$, computes the number of spanning forests of $\Gamma$ rooted at vertices indexed by $S$.

The principal minors of the graph Laplacian are graph invariants. The relations among principal minors are then also relations among graph invariants. Relations among graph invariants are central in the study of the theory of unlabeled graphs. In fact, Mikkonen holds that “the most important problem in graph theory of unlabeled graphs is the problem of determining graphic values of arbitrary sets of graph invariants,” (see [22] p. 1).

Theorem [1.3] gives relations among the graph invariants that come from principal minors, and in particular, since a graph can be reconstructed from a symmetric matrix, Theorem [1.3] implies the following:

**Restatement:** There exists an undirected weighted graph $\Gamma$ with invariants $[v] \in \mathbb{P}^{2n-1}$ specified by the principal minors of a symmetric matrix $\Delta(\Gamma)$ if and only if $[v]$ is a zero of all the polynomials in the hyperdeterminantal module.

2. **Spaces of tensors, representations and $G$-varieties**

Here we recall basic definitions and facts coming from representation theory and geometry. For the sake of the reader not familiar with representation theory, we have tried to include many definitions and basic concepts that we might have skipped otherwise. For more background, see [7–9, 11, 19].

2.1. **Tensors and representations: basic notions.** Let $V_1, \ldots, V_d$ be vector spaces (herein we will only deal with finite dimensional complex vector spaces) and let $V_1 \otimes \cdots \otimes V_d$ denote their tensor product. The group $GL(V_1) \times \cdots \times GL(V_d)$ acts by change of coordinates in each factor. When $V_i$ are all isomorphic, here is also a natural action of the symmetric group $\mathfrak{S}_d$ on $V_1 \otimes \cdots \otimes V_d$ just by permuting the factors. With this convention one may define a left action of the semi-direct product $GL(V) \ltimes \mathfrak{S}_n$ on $V^\otimes n$. 

Let $V^*$ denote the dual vector space to $V$, which is the vector space of linear maps $V \to \mathbb{C}$. There is a natural action of $GL(V)$ on $V^*$ defined by $g.\omega(x) := \omega(g^{-1}.x)$ for every $x \in V$, $g \in GL(V)$ and $\omega \in V^*$. Let $S^dV^*$ denote the space of homogeneous degree $d$ polynomials on $V$.

A representation of a group $G$ on vector space $V$ is a group homomorphism $\rho : G \to GL(V)$. In this setting, $V$ becomes a $G$-module, i.e. a vector space with a compatible $G$-action. It is common to call $V$ a representation of $G$. A $G$-module said to be irreducible if it has no non-trivial $G$-invariant subspaces.

The groups we encounter in this study are reductive. By definition, a group $G$ acts on a vector space $V$ if every $G$-module splits into a unique direct sum of irreducible $G$-modules. Also, if $G$ is reductive, then $M$ is an irreducible $G$-module if and only if $M = \text{span}\{G.x\}$, in other words, $M$ is the linear span of the orbit of a single vector.

Because $G$ acts on $\mathbb{P}V$, we can consider its action on a subvariety $X \subset \mathbb{P}V$ which is said to be a $G$-variety (or $G$-invariant) if $g.x \in X$ for every $x \in X$ and $g \in G$. A variety $X$ is said to be a homogeneous variety if for some $x \in X$, $X = G.x = G/P$, where $P$ is the stabilizer of $x$. Homogenous varieties are often the first $G$-varieties that one encounters. They have rich geometric and algebraic properties and are well studied; for instance, see [16] for a modern treatment.

2.2. Examples of classical $G$-varieties. The following are two classic examples of $G$-varieties which happen to show up in the study of the variety of principal minors of symmetric matrices. These definitions can be found in many texts on algebraic geometry such as [11].

The space of all rank-one tensors (also called decomposable tensors) is the Segre variety, defined by the embedding,

$$\text{Seg} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n \longrightarrow \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \quad \begin{pmatrix} [v_1] \cdots [v_n] \end{pmatrix} \longmapsto \begin{pmatrix} v_1 \otimes \cdots \otimes v_n \end{pmatrix}.$$  

$\text{Seg} (\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$ is a $G$-variety for $G = GL(V_1) \times \cdots \times GL(V_n)$, moreover it is homogeneous since $\text{Seg} (\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n) = G.\{v_1 \otimes \cdots \otimes v_n\}$. If $X_1 \subset \mathbb{P}V_1, \ldots, X_n \subset \mathbb{P}V_n$ are varieties, let $\text{Seg}(X_1 \times \cdots \times X_n)$ denote their Segre product.

The $r^{th}$ secant variety to a variety $X \subset \mathbb{P}V$, denoted $\sigma_r(X)$, is the Zariski closure of all embedded secant $\mathbb{P}^{r-1}$’s to $X$, i.e.,

$$\sigma_r(X) = \bigcup_{x_1, \ldots, x_r \in X} \mathbb{P}(\text{span}\{x_1, \ldots, x_r\}) \subset \mathbb{P}V.$$

Secant varieties inherit the symmetry of the underlying variety. In particular, $\sigma_r(\text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n))$ is a $G$-variety for $G = GL(V_1) \times \cdots \times GL(V_n)$. However, homogeneity is not preserved in general.

3. Using representation theory to study the ideals of $G$-varieties

If $X \subset \mathbb{P}V$ is an algebraic variety we let $\mathcal{I}(X) \subset \mathbb{P}(\text{Sym}(V^*))$ denote the ideal of polynomials vanishing on $X$, and let $\hat{X} \subset V$ denote the cone over $X$. If $M$ is a set of polynomials, we let $\mathcal{V}(M)$ denote its zero set. Often algebraic varieties are given via an explicit parameterization by a rational map, but the vanishing ideal may be unknown. A basic question in
algebraic geometry is to find generators for the ideal of a given variety. Though there are
many known theoretical techniques, this remains a difficult practical problem.

**Fact:** $X$ is a $G$-variety if and only if $\mathcal{I}(X)$ is a $G$-module. This fact, which comes directly
from the definitions, is a key observation because it allows us to use the representation theory
of $G$-modules to study $\mathcal{I}(X)$.

By definition, all projective varieties have an action of $\mathbb{C}^*$ by rescaling which preserves
the projective variety. This action induces a grading by degree on the ideal. The $d^{th}$ graded piece
of $\mathcal{I}(X)$ will be denoted by $\mathcal{I}_d(X) := S^d(V^*) \cap \mathcal{I}(X)$. When a larger group $G$ acts on $X$, we
get more a finer decomposition of the ideal into irreducible $G$-modules. In particular, $S^dV^*$
and $\mathcal{I}_d(X)$ are $G$-modules, and we can decompose each of them as such. The irreducible
modules in $\mathcal{I}_d(X)$ are a subset of those in $S^dV^*$. This simple observation leads to a useful
ideal membership test, which is developed and discussed in [18,19].

We would like to be able to compare $G$-orbits of points and the zero sets of arbitrary sets
of polynomials (not necessarily $G$-modules). The following proposition is useful to that end,
and we will use it Section 8.

**Proposition 3.1.** Let $z \in \mathbb{P}V$, and let $B \subset \text{Sym}(V^*)$ be a collection of polynomials ($B$ is
not necessarily a $G$-module). Then

$$ G.z \subset \mathcal{V}(B) \text{ if and only if } z \in \mathcal{V}(\text{span}\{G.B\}) $$

**Proof.** $G.z \subset \mathcal{V}(B)$ if and only if $f(g.z) = 0$ for all $g \in G$ and for all $f \in B$. But from
the definition of the $G$-action on the dual space, $f(g.z) = (g^{-1}.f)(z)$, so $f(g.z) = 0$ for all
$g \in G$ and for every $f \in B$. This happens if and only if $(g.f)(z) = 0$ for all $g \in G$ and for
all $f \in B$, but, this is the condition that $z \in \mathcal{V}(\text{span}\{G.B\})$. \hfill \Box

The general linear group $GL(V)$ has nice representation theory. In particular Proposition
15.47 of [8] says that every $GL(V)$-module is isomorphic to a Schur module of the form $S_nV$,
where $\pi$ is a partition of some integer $d$. We refer the reader to [8,19] for general background
on Schur modules.

Two common representations (in this language) are the space of symmetric tensors $S^dV = S_{(d)}V$
and the space of skew-symmetric tensors $\bigwedge^dV = S_{(1,d)}V$, where $1^d$ denotes the partition
$(1,\ldots,1)$ with 1 repeated $d$ times.

We are interested in the case when $X \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$ is a variety in a space of tensors,
and $X$ is invariant under the action of $G = GL(V_1) \times \cdots \times GL(V_n)$. To study $\mathcal{I}_d(X)$ as
a $G$-module, we need to understand how to decompose the space of homogeneous degree $d$
polynomials $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ into a direct sum of irreducible $G$-modules. This is a standard
computation in representation theory, which was made explicit in [18]:

**Proposition 3.2** (Landsberg–Manivel [18] Proposition 4.1). Let $V_1, \ldots, V_n$ be vector spaces
and let $V = V_1 \otimes \cdots \otimes V_n$, and let $G = GL(V_1) \times \cdots \times GL(V_n)$. Then the following
decomposition as a direct sum of irreducible $G$-modules holds:

$$ S^d(V_1 \otimes \cdots \otimes V_n) = \bigoplus_{|\pi_1|=\cdots=|\pi_n|=d} ([\pi_1] \otimes \cdots \otimes [\pi_n]) \otimes S_{\pi_1}V_1 \otimes \cdots \otimes S_{\pi_n}V_n $$


where \([\pi_i]\) are representations of the symmetric group \(S_d\) indexed by partitions \(\pi_i\) of \(d\), and \(([\pi_1] \otimes \cdots \otimes [\pi_n])^{S_d}\) denotes the space of \(S_d\)-invariants (i.e., instances of the trivial representation) in the tensor product.

When \(V_i = V\), Proposition 3.2 specializes to give the following decomposition formula (as \(GL(V) \times \cdots \times GL(V)\)-modules) also found in [18]:

\[
S^d(V \otimes \cdots \otimes V) = \bigoplus_{|\pi_1| = \cdots = |\pi_n| = d} (S_{\pi_1} V \otimes \cdots \otimes S_{\pi_n} V)^{\otimes N_{\pi_1,\ldots,\pi_k}},
\]

where the multiplicity \(N_{\pi_1,\ldots,\pi_k}\) can be computed via characters. The modules \(S_{\pi_1} V \otimes \cdots \otimes S_{\pi_n} V\) are called isotypic components. The irreducible \(GL(V)^{\times n} \rtimes S_n\)-modules are constructed by taking a sum of modules of the form \(S_{\pi_1} V \otimes \cdots \otimes S_{\pi_n} V\) over all permutations in \(S_n\) which yield non-redundant modules. When the vector space is understood, we denote this compactly as

\[
S_{\pi_1} S_{\pi_2} \ldots S_{\pi_n} := \sum_{\sigma \in S_n} S_{\pi_{\sigma(1)}} V \otimes \cdots \otimes S_{\pi_{\sigma(n)}} V
\]

This decomposition formula is essential for understanding the structure of the ideals of \(G\)-varieties. This calculation is implemented in the computer program LiE, and we wrote an implementation in Maple.

This combinatorial description of Schur modules in terms of collections of partitions can be used to construct polynomials in spaces of tensors. We refer the reader to [18], [19] for a complete explanation. A copy of our implementation of these algorithms may be obtained by contacting the author.

3.1. Weight spaces. In this section and the next section we rephrase some of the ideas from the previous section in order to highlight how we use these ideas in practice.

The algebras \(\text{Sym}(V)\) and \(V^{\otimes}\) are graded by degree. We get a further decomposition of \(\text{Sym}(V_1 \otimes \cdots \otimes V_n)\) and \((V_1 \otimes \cdots \otimes V_n)^{\otimes}\) by weights as follows. We specialize to the case when \(V_i \simeq \mathbb{C}^2\) because this is notationally simpler, and it is all we need for this work.

Choose a basis \(\{x_0^i, x_1^i\}\) for each \(V_i\) and assign the weight \((0, \ldots, 0, -1, 0, \ldots, 0)\) to \(x_0^i\) and the weight \((0, \ldots, 0, +1, 0, \ldots, 0)\) to \(x_1^i\), where the non-zero entries occur in the \(i^{th}\) position.

Assign a weight to each monomial in \((V_1 \otimes \cdots \otimes V_n)^{\otimes}\) by requiring the weight to be additive. In particular, the monomial

\[
(((x_0^1)^{\otimes p_1} \otimes (x_1^1)^{\otimes q_1}) \otimes ((x_0^2)^{\otimes p_2} \otimes (x_1^2)^{\otimes q_2}) \otimes \cdots \otimes ((x_0^n)^{\otimes p_n} \otimes (x_1^n)^{\otimes q_n}))
\]

has weight

\[
(q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n).
\]

This is also known as grading by multi-degree. If we consider the lexicographic order on weights, we can make sense of the notion of highest weight, i.e. occurring soonest in the lexicographic order. As a caution, we point out that with this convention, smaller, or numbers that are more negative actually indicate a higher weight. A tensor is said to have weight zero if it has weight \((0, 0, \ldots, 0)\).
Each irreducible representation, $S_\pi V$ of $GL(V)$, has a highest weight vector, $v_\pi$ and since $GL(V)$ is reductive, we have the nice property that $S_\pi V = \text{span}\{GL(V).v_\pi\}$, i.e. each irreducible representation of $GL(V)$ is the span of the orbit of a highest weight vector.

3.2. Constructing new polynomials from old. Here we will recall a standard algorithm for constructing new vectors in a $G$-module from a known vector.

3.2.1. Producing $G$-modules from a polynomial. The following procedure is useful for finding more polynomials in an ideal of a $G$-variety when one polynomial is already known. In particular, we will use a variant of this procedure in the proof of Lemma 8.2 below.

Suppose we can write down a polynomial $h$ in some (unknown) $G$-module $B$. Since $B$ is a $G$-module, it is also a $g$-module, where $g$ is the Lie algebra associated to the Lie group $G$. The following algorithm is a standard idea in representation theory and can be used to find more polynomials in $B$, and in fact we will find submodules of $B$.

The essential fact we will use is that the Lie algebra $g$ associated to the Lie group $G$ acts on $G$-modules as a derivation and $B$ is a $G$-module if and only if $B$ is a $g$-module. We have a decomposition $g = g_- \oplus g_0 \oplus g_+$ into the lowering operators, the Cartan (abelian) subalgebra and the raising operators. The lowering operators can be thought of as lower triangular matrices when $G = SL_n$.

For example the lowering operator in $sl_2$ acts on $V = \{x_0, x_1\}$ by sending $x_0$ to $x_1$ and sending $x_1$ to 0. The Lie algebra associated to $SL(2)^\times n$ is $sl_2^\times n$ where each $sl_2$ acts on a single factor of the tensor product $V_1 \otimes \cdots \otimes V_n$.

By successively applying lowering operators, we will determine the lowest weight space that $h$ can live in. The lowest weight vector that we construct will generate a submodule of $B$.

**Input:** $h \in B$.

**Step 0.** Choose an ordered basis of lowering operators $g_- = \{\alpha_1, \ldots, \alpha_n\}$.

**Step 1.** Find the largest integer $k_1 \geq 0$ so that $\alpha_1^{k_1}.h \neq 0$, and let $h^{(1)} = \alpha_1^{k_1}.h$.

**Step 2.** Find the largest integer $k_2 \geq 0$ so that $\alpha_2^{k_2}.h^{(1)} \neq 0$, and let $h^{(2)} = \alpha_2^{k_2}.h^{(1)}$.

**Step n.** Find the largest integer $k_n \geq 0$ so that $\alpha_n^{k_n}.h^{(n-1)} \neq 0$, and let $h^{(n)} = \alpha_n^{k_n}.h^{(n-1)}$.

**Output:** The vector $h^{(n)}$ is a lowest weight vector in $B$ and $\text{span}\{G.h^{(n)}\}$ is a submodule of $B$ containing $h$.

3.2.2. Generating a weight basis of a $G$-module. Suppose we have successfully constructed a highest weight vector in some finite dimensional $G$-module $M$ and we want to know a basis of $M$. By modifying the above procedure, one also accomplishes this task, and moreover the vectors constructed will each have the property that all of their terms will have the same weight. We call such a basis a weight basis.

Successively apply all lowering operators to the highest weight vector to get all vectors of the next lowest weights. Keep the unique nonzero results. Since $M$ is a $g$-module, each new vector will be in $M$. Then apply all lowering operators to the result. Since $M$ is finite dimensional, the process will stop at the lowest (nonzero) weight vectors. The collection of weight vectors found in this way will be a weight basis of $M$, and will also be minimal set of generators of $\langle M \rangle$, the ideal generated by $M$.

Landsberg and Manivel [15] provide an algorithm for constructing highest weight vectors in Schur modules. By implementing this algorithm, we were able to carry out an ideal
membership test (for small degree), which is presented in [18]. The basic idea is that we can (in theory) write down a highest weight vector for each irreducible module of polynomials for a fixed small degree. Then we can test the vanishing of each highest weight vector on a general point of the variety \( X \). If the highest weight vector vanishes, then the entire module is in the ideal \( \mathcal{I}(X) \).

3.3. A simple transition between weights and modules. In the special case of \( GL(2) \) and \( SL(2) \), the representations are all of the form \( S_{(a,b)} \mathbb{C}^2 \), and the highest weight vectors in this representation have integer weight \( b - a \) (recall the convention is that the lower the number, the higher the weight).

We can also use this basic idea when studying the space of homogeneous polynomials \( S^d(V_1 \otimes \cdots \otimes V_n) \) in the case \( V_i \simeq \mathbb{C}^2 \). As mentioned before, up to isomorphism, every module in \( S^d(V_1 \otimes \cdots \otimes V_n) \) is of the form \( S_{\pi_1} V_1 \otimes \cdots \otimes S_{\pi_n} V_n \) with each \( \pi_i \) a partition of \( d \). Since \( V_i \simeq \mathbb{C}^2 \) for every \( 1 \leq i \leq n \), we may assume that each \( \pi_i \) is of the form \( (\pi_i^1, \pi_i^2) \) with \( \pi_i^1 + \pi_i^2 = d \). And the weight of a highest weight vector in \( S_{\pi_1} V_1 \otimes \cdots \otimes S_{\pi_n} V_n \) is \((\pi_1^2 - \pi_1^1, \pi_2^2 - \pi_2^1, \ldots, \pi_n^2 - \pi_n^1)\).

Moreover, given the data that \((w_1, w_2, \ldots, w_n)\) is the weight of a nonzero highest weight vector in \( S^d(V_1 \otimes \cdots \otimes V_n) \), one can immediately find (up to isomorphism) the module of the form \( S_{\pi_1} V_1 \otimes \cdots \otimes S_{\pi_n} V_n \) in which it occurs (we say that we know what isotypic component it lives in). To say this another way, the degree and weight of a polynomial plus the knowledge that it is a highest weight vector is necessary and sufficient information to find which module it lives in. In general, this information will not be sufficient to find how the module \( S_{\pi_1} V_1 \otimes \cdots \otimes S_{\pi_n} V_n \) is embedded in \( S^d(V_1 \otimes \cdots \otimes V_n) \) (i.e. which isotypic component the polynomial lives in). If one has more information, like for instance that the found module occurs with multiplicity one in \( S^d(V_1 \otimes \cdots \otimes V_n) \), then this will be sufficient.

Example 3.3. As a specific example, we consider the hyperdeterminant. The hyperdeterminant of format \( 2 \times 2 \times 2 \) is invariant under the action of \( SL(2) \times SL(2) \times SL(2) \), therefore it must have weight \([0, 0, 0]\). This, together with the knowledge that it is a degree 4 polynomial immediately tells us that it must be the module \( S_{(2,2)} \mathbb{C}^2 \otimes S_{(2,2)} \mathbb{C}^2 \otimes S_{(2,2)} \mathbb{C}^2 \) which occurs with multiplicity one in \( S^4(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \). Moreover, one can write the \( 2 \times 2 \times 2 \) hyperdeterminant on the variables \( X^{[i_1,i_2,i_3,0,\ldots,0]} \). The weight of this polynomial is \([0, 0, 0, -4, \ldots, -4]\) and it is a highest weight vector, therefore the span of its \((SL(2)^\times n) \ltimes \mathcal{S}_n\)-orbit is the hyperdeterminant module,

\[ HD = S_{(2,2)} S_{(2,2)} S_{(2,2)} S_{(4)} \cdots S_{(4)}. \]

4. Definition of the variety of principal minors of symmetric matrices

Let \( I = [i_1, \ldots, i_n] \) be a binary multi-index, with \( i_k \in \{0, 1\} \) for \( k = 1, \ldots, n \), and let \( |I| = \sum_{k=1}^n i_k \).

If \( A \) is an \( n \times n \) matrix, then let \( \Delta_I(A) \) denote the principal minor of \( A \) formed by taking the determinant of the principal submatrix of \( A \) indexed by \( I \) in the sense that the submatrix of \( A \) is formed by including the \( k^{th} \) row and column of \( A \) whenever \( i_k = 1 \) and striking the \( k^{th} \) row and column whenever \( i_k = 0 \). If one includes the \( 0 \times 0 \) minor, there are \( 2^n \) principal minors, therefore, a natural home for vectors of principal minors is \( \mathbb{C}^{2^n} \). Because of the
symmetry that will eventually become apparent, we will consider $\mathbb{C}^{2n}$ as a space of tensors as follows: For $1 \leq i \leq n$ let $V_i \simeq \mathbb{C}^2$ and consider $V_1 \otimes V_2 \otimes \cdots \otimes V_n \simeq \mathbb{C}^{2n}$. A choice of basis $\{x_i^0, x_i^1\}$ of $V_i$ for each $i$ determines a basis of $V_1 \otimes \cdots \otimes V_n$. We represent basis elements compactly by setting $X^I := x_i^1 \otimes x_j^1 \otimes \cdots \otimes x_n^1$. We use this basis to introduce coordinates on $\mathbb{PC}^{2n}$; if $P = [C_I X^I] \in \mathbb{PC}^{2n}$, the coefficients $C_I$ are the coordinates of the point $P$.

Let $S^2 \mathbb{C}^n$ denote the space of symmetric $n \times n$ matrices. The projective variety of principal minors of $n \times n$ symmetric matrices, $Z_n$, is defined by the following rational map,

$$\varphi : \mathbb{P}(S^2 \mathbb{C}^n \oplus \mathbb{C}) \longrightarrow \mathbb{P}\mathbb{C}^{2n}
\begin{align*}
[A, t] &\longmapsto \left[ t^{n-I} \Delta_I(A) X^I \right].
\end{align*}$$

The map $\varphi$ is defined on the open set where $t \neq 0$. Moreover, $\varphi$ is homogeneous of degree $n$, so it is well defined on projective space. The $1 \times 1$ principal minors of a matrix $A$ are the diagonal entries of $A = (a_{i,j})$, and if $A$ is a symmetric matrix, the $1 \times 1$ and $2 \times 2$ principal minors determine the off-diagonal entries of $A$ up to sign in light of the equation

$$a_{i,i}a_{j,j} - a_{i,j}^2 = \Delta_{[0,\ldots,0,1,0,\ldots,0]}(A),$$

where the 1’s in the index $[0,\ldots,0,1,0,\ldots,0,1,\ldots,0]$ occur in positions $i$ and $j$. So $\varphi$ is generically finite-to-one and $Z_n$ is a \((n+1)/2\)-dimensional variety. The affine map (on the set \( \{ t = 1 \} \)) defines a closed subset of $\mathbb{C}^2$, \([14]\).

5. The nested structure of $Z_n$ via Segre products

**Proposition 5.1.** The variety $\text{Seg}(Z_{(n-1)} \times \mathbb{P}V_n)$ is a subvariety of $Z_n$. Moreover, any point of $\text{Seg}(Z_{(n-1)} \times \mathbb{P}V_n)$ has an interpretation as the principal minors of a $(n-1) \times (n-1)$ matrix.

**Proof.** Notice that $\text{Seg}(Z_{(n-1)} \times \mathbb{P}\{x_n^0\}) \subset Z_n$, and the $SL(2)^{\times n}$-orbit of $\text{Seg}(Z_{(n-1)} \times \mathbb{P}\{x_n^0\})$ is $\text{Seg}(Z_{(n-1)} \times \mathbb{P}V_n)$. Since $Z_n$ is a $G$-variety for $G = (SL(2)^{\times n}) \ltimes \mathcal{S}_n$, it contains all of the $(SL(2)^{\times n}) \ltimes \mathcal{S}_n$-orbits of points within $Z_n$, so the first claim is proved.

Now we prove the “moreover” statement. Notice that we can parameterize $\text{Seg}(Z_{(n-1)} \times \mathbb{P}\{x_n^0\})$ as the image under $\varphi$ of matrices of the form

$$\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix},$$

where $P$ is an $(n-1) \times (n-1)$ matrix. Every point of $\text{Seg}(Z_{(n-1)} \times \mathbb{P}V_i)$ is in the $(SL(2)^{\times n}) \ltimes \mathcal{S}_n$-orbit of a point in $\text{Seg}(Z_{(n-1)} \times \mathbb{P}\{x_n^0\})$ which is the variety of principal minors of an $n-1 \times n-1$ block of an $n \times n$ matrix, so after a change of basis, we have the result.

With a little bit more work, one can show that a stronger result than Proposition 5.1 holds:

**Proposition 5.2.** Let $p + q = n$ and $Z_p \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_p)$ and $Z_q \subset \mathbb{P}(V_{p+1} \otimes \cdots \otimes V_n)$. Then $\text{Seg}(Z_p \times Z_q)$ is a subvariety of $Z_n$. 
Let \( U_0 = \{ [z] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \mid z = z_I X^I \in V_1 \otimes \cdots \otimes V_n, z_{[0, \ldots, 0]} \neq 0 \}. \) Then \( \varphi([A, t]) \in \text{Seg}(Z_p \times Z_q) \cap U_0, \) if and only if \( A \) is of the form

\[
\begin{pmatrix}
P & 0 \\
0 & Q
\end{pmatrix},
\]

where \( P \in S^2 \mathbb{C}^p \) and \( Q \in S^2 \mathbb{C}^q. \)

**Proof.** Let \( \varphi^i \) denote the principal minor map on \( i \times i \) matrices and let \( J \) and \( K \) be (respectively) multi-indices of length \( p \) and \( q. \) Let \( [x \otimes y] \in \text{Seg}(Z_p \times Z_q) \) be such that \( [x] = \varphi^p([P, r]) = [r^{p-|J|}\Delta_J(P)X^J] \) and \( [y] = \varphi^q([Q, s]) = [s^{q-|K|}\Delta_K(Q)X^K], \) with \( P \in S^2 \mathbb{C}^p \) and \( Q \in S^2 \mathbb{C}^q. \)

Notice that if \( r = 0, \) then \( [x] = [0, \ldots, 0, \det(P)] \in \text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_p) \), and similarly if \( s = 0, \) then \( [y] = [0, \ldots, 0, \det(Q)] \in \text{Seg}(\mathbb{P}V_{p+1} \times \cdots \times \mathbb{P}V_{p+q}). \) So the cases that \( r = 0 \) or \( s = 0 \) are covered by iterations of Proposition 5.1.

Now assume \( r \neq 0, s \neq 0 \) so we can set \( r = s = 1. \) Consider a blocked matrix of the form

\[
A = \begin{pmatrix}
P & 0 \\
0 & Q
\end{pmatrix},
\]

where \( P \in S^2 \mathbb{C}^p \) and \( Q \in S^2 \mathbb{C}^q. \) We claim that \( \varphi^{p+q}([A, 1]) = [x \otimes y]. \) The determinant of a block diagonal matrix is the product of the determinants of the blocks, and principal submatrices of block diagonal matrices are still block diagonal, so

\[
\varphi^n([A, 1]) = [\Delta_J(P)\Delta_K(Q) X^{J,K}].
\]

where \( X^{J,K} = X^J \otimes X^K. \) But we can reorder the terms in the product to find

\[
[\Delta_J(P)\Delta_K(Q) X^{J,K}] = [(\Delta_J(P)X^J) \otimes (\Delta_K(Q)X^K)] = [x \otimes y].
\]

For the second statement in the proposition, notice that for \( [x \otimes y] \in \text{Seg}(Z_p \times Z_q) \cap U_0, \) we have exhibited a matrix \( A \) as in (3) such that \( \varphi^n([A, 1]) = [x \otimes y]. \) But symmetric matrices are determined up to sign by their \( 1 \times 1 \) and \( 2 \times 2 \) principal minors. Any other matrix must have the same blocked form as the one in (3). \( \square \)

**Remark 5.3.** Proposition 5.2 gives a useful tool in finding candidate modules for \( I(Z_n): \) We are forced to consider

\[
I(Z_n) \subset \bigcap_{p+q = n} I(\text{Seg}(Z_p \times Z_q)).
\]

\[
p, q \geq 1
\]

6. **Properties of the hyperdeterminantal module**

As a consequence of Theorem 1.1, the defining ideal of \( Z_n, I(Z_n) \subset \text{Sym}(V_1^* \otimes \cdots \otimes V_n^*), \) is a \( G \)-module for \( G = (SL(2)^{\times n}) \ltimes \mathfrak{S}_n. \) As mentioned above, we will consider the \( G \)-module \( HD = S_{(2, 2)} S_{(2, 2)} S_{(2, 2)} S_{(4)} \ldots S_{(4)} \) (called the hyperdeterminantal module in [14]). In this section we compute the dimension of the hyperdeterminantal module and show that it occurs with multiplicity one in \( S^4(V_1^* \otimes \cdots \otimes V_n^*). \) Also, in the course of our observations, we arrive at a practical ideal membership test for a class of varieties that includes the variety of principal minors.
Observation 6.1. The dimension of the hyperdeterminantal module is
\[ \dim(S_{(2,2)}S_{(2,2)}S_{(2,2)}S_{(4)} \ldots S_{(4)}) = \binom{n}{3} 3^{n-3}. \]

Proof. The module \( S_{(2,2)} \mathbb{C}^2 \) is 1-dimensional and the module \( S_{(4)} \mathbb{C}^2 \) is 5-dimensional. \( \square \)

Proposition 6.2. The module \( HD = S_{(2,2)}S_{(2,2)}S_{(2,2)}S_{(4)} \ldots S_{(4)} \) occurs with multiplicity 1 in \( S^4(V_1^* \otimes \cdots \otimes V_n^*) \). Moreover, HD is an irreducible G-module for \( G = (SL(V_1) \times \cdots \times SL(V_n)) \ltimes \mathfrak{S}_n \approx (SL(2)^\times)^n \ltimes \mathfrak{S}_n \).

Remark 6.3. The fact that \( HD \) occurs with multiplicity 1 saves us a lot of work because we do not have to worry about which isomorphic copy of the module occurs in the ideal.

Proof. For the “moreover” part, notice that the module \( HD \) is the span of the \( G \)-orbit of a single polynomial (namely the hyperdeterminant of format \( 2 \times 2 \times 2 \) on the variables \( X_{[i_1,i_2,i_3,0,\ldots,0]} \) with \( 0 \leq i_1, i_2, i_3 \leq 1 \)) and therefore \( HD \) is an irreducible module.

We need to examine the \( SL(2)^\times \mathbb{C}^2 \)-module decomposition of \( S^4(V_1^* \otimes \cdots \otimes V_n^*) \). It suffices to prove for any fixed permutation \( \sigma \), that \( S_{(2,2)}V_{\sigma(1)}^* \otimes \cdots \otimes S_{(2,2)}V_{\sigma(3)}^* \otimes \cdots \otimes S_{(4)}V_{\sigma(n)}^* \) is an \( SL(2)^\times \mathbb{C}^2 \)-module which occurs with multiplicity 1 in the decomposition of \( S^4(V_1^* \otimes \cdots \otimes V_n^*) \).

We will follow the notation and calculations similar to [8]. For a representation \([\pi]\) of the symmetric group \( \mathfrak{S}_d \), let \( \chi_{\pi} \) denote its character. The number of occurrences of \( S_{\pi_1}V_1^* \otimes \cdots \otimes S_{\pi_n}V_n^* \) in the decomposition of \( S^d(V_1^* \otimes \cdots \otimes V_n^*) \) is computed by the dimension of the space of \( \mathfrak{S}_d \) invariants, \( \dim([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d} \). This may be computed by the formula

\[
\dim \left( ([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d} \right) = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \chi_{\pi_1}(\sigma) \cdots \chi_{\pi_n}(\sigma).
\]

In our case, we need to compute

\[
\dim \left( ([\pi_1] \otimes [(2,2)] \otimes [(2,2)] \otimes [(4)] \otimes \cdots \otimes [(4)])^{\mathfrak{S}_4} \right)
= \frac{1}{4!} \sum_{\sigma \in \mathfrak{S}_4} \chi_{\pi_1}(\sigma) \chi_{\pi_2}(\sigma) \chi_{\pi_2}(\sigma) \chi_{\pi_4}(\sigma) = 1,
\]

But, \( \chi_{\pi}(\sigma) = 1 \) for every \( \sigma \in \mathfrak{S}_4 \). So, our computation reduces to the following

\[
\dim \left( ([\pi_1] \otimes [(2,2)] \otimes [(2,2)] \otimes [(4)] \otimes \cdots \otimes [(4)])^{\mathfrak{S}_4} \right)
= \frac{1}{4!} \sum_{\sigma \in \mathfrak{S}_4} \chi_{\pi_1}(\sigma) \chi_{\pi_2}(\sigma) \chi_{\pi_2}(\sigma) \chi_{\pi_4}(\sigma) = 1,
\]

where the last equality is found by direct computation. The module \( S_{(2,2)}V_1^* \otimes S_{(2,2)}V_2^* \otimes S_{(2,2)}V_3^* \) occurs with multiplicity 1 in \( S^4(V_1^* \otimes V_2^* \otimes V_3^*) \). (The full decomposition of \( S^4(V_1^* \otimes V_2^* \otimes V_3^*) \) was computed in (prop 4.3 [8]).) Therefore the module \( S_{(2,2)}V_{\sigma(1)}^* \otimes S_{(2,2)}V_{\sigma(2)}^* \otimes S_{(2,2)}V_{\sigma(3)}^* \otimes S_{(4)}V_{\sigma(4)}^* \otimes \cdots \otimes S_{(4)}V_{\sigma(n)}^* \) occurs with multiplicity 1 in \( S^4(V_1^* \otimes \cdots \otimes V_n^*) \).
We have seen that each summand of $HD$ is an irreducible $SL(2) \times n$-module which occurs with multiplicity 1 in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$. Therefore $HD$ is an irreducible $G$-module, and it occurs with multiplicity 1 in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$. 

We remark that the above argument generalizes to:

**Lemma 6.4.** For every collection $\pi_1, \ldots, \pi_n$ of partitions of $d$,

$$\dim (\langle [\pi_1] \otimes \cdots \otimes [\pi_n]\rangle_{\mathfrak{S}_d}) = \dim (\langle [\pi_1] \otimes \cdots \otimes [\pi_n] \otimes [(d)]\rangle_{\mathfrak{S}_d}).$$

In particular, if $M$ is any irreducible $SL(V_1) \times \cdots \times SL(V_n)$-module which occurs with multiplicity $m$ in $S^d(V_1^* \otimes \cdots \otimes V_n^*)$, then $M \otimes S^dV_{n+1}$ is an irreducible $SL(V_1) \times \cdots \times SL(V_n) \times SL(V_{n+1})$-module which occurs with multiplicity $m$ in $S^d(V_1^* \otimes \cdots \otimes V_n^* \otimes V_{n+1}^*)$.

**Proof.** Use the formula

$$\dim (\langle [\pi_1] \otimes \cdots \otimes [\pi_n]\rangle_{\mathfrak{S}_d}) = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \chi_{\pi_1}(\sigma) \cdots \chi_{\pi_n}(\sigma).$$

and note that $\chi_{(d)}(\sigma) = 1$ for every $\sigma \in \mathfrak{S}_d$. 

**Proposition 6.5.** The hyperdeterminantal module is contained in the ideal of the variety of principal minors of symmetric matrices, i.e.

$$HD = S_{(2,2)}S_{(2,2)}S_{(2,2)}S_{(4)} \cdots S_{(4)} \subseteq \mathcal{I}(Z_n),$$

and in particular, $Z_n \subseteq V(HD)$.

**Proof.** Note, this statement is proved in [14]. The following is a slightly different proof that uses representation theory. Both $HD$ and $\mathcal{I}(Z_n)$ are $(SL(2) \times n) \rtimes \mathfrak{S}_n$-modules and $HD$ is an irreducible $(SL(2) \times n) \rtimes \mathfrak{S}_n$-module, so we only need to show that the highest weight vector of $HD$ vanishes on all points of $Z_n$. The highest weight vector of $HD$ is the hyperdeterminant of format $2 \times 2 \times 2$ on the variables $X^{i_1,i_2,i_3,0,\ldots,0}$. The set $Z_n \cap \text{span}\{X^{i_1,i_2,i_3,0,\ldots,0} | i_1, i_2, i_3 \in \{0,1\}\}$, is the set of principal minors of the upper $3 \times 3$ corner of $n \times n$ matrices. The hyperdeterminant vanishes on these principal minors because of the case $n = 3$, so there is nothing more to show. 

We can generalize this idea.

**Proposition 6.6.** Let $V,W$ be complex vector spaces with $\dim(V) \geq 2$. Suppose $Y \subset \mathbb{P}W$ and $X \subset \mathbb{P}(W \otimes V)$ are varieties such that $\text{Seg}(Y \times \mathbb{P}V) \subset X$. Suppose $B \subset S^dW^*$. Then $B \otimes S^dV^*$ is a space of polynomials. Moreover $B \otimes S^dV^* \subset \mathcal{I}_d(X)$ only if $B \subset \mathcal{I}_d(Y)$.

**Proof.** First notice that if $B \subset S^dW^*$, then $B \otimes S^dV^* \subset S^dW^* \otimes S^dV^*$. As $SL(W) \times SL(V)$-modules, $S^dW^* \otimes S^dV^* \subset S^d(W^* \otimes V^*)$. (This is a standard fact, and an easy consequence of Schur’s lemma or can be deduced from Proposition 3.2 see [8].) Therefore $B \otimes S^dV^* \subset S^d(W^* \otimes V^*)$ is a space of polynomials, establishing the first claim.

There exists a basis of $S^dV^*$ of vectors of the form $\alpha^d$, and moreover $S^dV^* = \text{span}\{SL(V), \alpha^d\}$ for any nonzero $\alpha \in V^*$. So $B \otimes S^dV^*$ has a basis of vectors of the form $f \otimes \alpha^d$ with $f \in B$ and $\alpha \in V^*$. It suffices to prove the proposition on this basis.

Suppose $f \otimes \alpha^d$ is a basis vector in $B \otimes S^d(V^*) \subset \mathcal{I}_d(X)$. Then since $\text{Seg}(Y \times \mathbb{P}V) \subset X$, $f \otimes \alpha^d \in \mathcal{I}_d(\text{Seg}(Y \times \mathbb{P}V)) \subset S^d(W^* \otimes V^*)$. This means that $f \otimes \alpha^d(y \otimes v) = 0$ for all $y \in Y$.
and for all \( v \in V \). It is a fact that \( \alpha^d(v) = \alpha(v)^d \) (this can be deduced from Lemma 7.3 below, for instance), so we can evaluate

\[
f \otimes \alpha^d(y \otimes v) = f(y)\alpha^d(v) = f(y)\alpha(w)^d.
\]

Since \( \dim(V) \geq 2 \), \( V(\alpha) \) is a hyperplane. It is no problem to choose a point that misses a hyperplane, so we can choose a particular \( v \in V \) so that \( \alpha(v) \neq 0 \).

This implies that \( f(y) = 0 \) for all \( y \in Y \) and hence \( f \in I_d(Y) \). This is true for any \( f \in B \) we choose, so we are done.

Proposition 6.6 fails to be an if and only if statement. Explicitly, we cannot say that every module in the ideal \( I_d(X) \) occurs as \( B \otimes S^dV^* \) for some \( B \in I_d(Y) \). In Section 7 we study the zero sets of modules of the form \( I_d(Y) \otimes S^dV^* \), and this sheds some light on the failure of the other direction of Proposition 6.6.

Remark 6.7. Proposition 5.1 says that \( \text{Seg}(Z_n \times \mathbb{P}V_{n+1}) \subset Z_{n+1} \). We can use this proposition to study the variety of principal minors in two ways. First, if \( M \) is a module in \( I_d(Z_n) \), then \( M \otimes S^dV_{n+1} \) is a module in \( I_d(Z_{n+1}) \). The second use is the contrapositive version. It gives an easy test for ideal membership for modules that have at least one \( S^dV_i^* \) factor. Suppose we know \( I_d(Z_n) \) for some \( n \). If we want to test whether \( M = S_{\pi_1} V_1^* \otimes \cdots \otimes S_{\pi_{n+1}} V_{n+1}^* \) is in \( I_d(Z_{n+1}) \) and we know that \( M \) has at least one \( \pi_i = (d) \), then we can remove \( S_{\pi_i} V_i^* \) and check whether the module we have left is in the ideal \( I_d(Z_n) \).

7. A geometric characterization of the zero set of the hyperdeterminantal module via augmentation

The hyperdeterminantal module has a useful inductive description that we would like to be able to exploit. In particular, for \( n \geq 3 \), the module is always of the form

\[
S_{(2,2)}S_{(2,2)}S_{(2,2)}S_{(4)} \cdots S_{(4)}
\]

where the number of \( S_{(4)} \) factors is \( n - 3 \). So to get from the module at stage \( n \) to the module at stage \( n + 1 \) we simply append another copy of \( S^4 \mathbb{C}^2 \). Here we notice a general structure. We will call a module of the form \( M \otimes S^dW^* \) an augmentation or augmented module.

In this section, we study augmented modules and we arrive at a geometric description of the zero set of an augmented module. By using applying this geometric description to the hyperdeterminantal module, we get a geometric description of its zero set. This description is essential in our proof of Theorem 1.3.

7.1. Polarization and its application to augmented modules. Augmentation is similar to prolongation, a concept found in the study of the ideals of secant varieties. A difference between the two is that augmentation does not change the degree of the polynomials, whereas prolongation increases the degree.

It is not a surprise that we can get inspiration from the techniques used to study secant varieties when studying augmented modules. In particular, polarization of a polynomial is a tool used to study ideals of secant varieties in [17] and in [26]. Let \( x_1, \ldots, x_n \) be a basis of \( V \). Given a homogeneous degree \( d \) polynomial \( f \) in the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \), its
polarization is a symmetric tensor $\overrightarrow{f} \in S^dV^*$ whose specific definition is given in the next lemma.

In the following, we use polarization to better understand the polynomials in an augmented module.

For this section only, we will follow the notation of [26] that is used and defined in the following useful lemma.

**Lemma 7.1** (Sidman–Sullivant [26] Lemma 2.5(1)). If $F$ is a homogeneous degree $d$ polynomial in $x_1, \ldots, x_n$, let $\overrightarrow{F}$ denote its polarization. Let $v = t_1x_1 + \cdots + t_kx_k$. Then

$$F(v) = \overrightarrow{F}(v, \ldots, v) = \sum_{\beta} \frac{1}{\beta!} t^{\beta} \overrightarrow{F}(x^{\beta}),$$

where $\beta = (\beta_1, \ldots, \beta_k)$, is a collection of non-negative integers whose sum is $\beta_1 + \cdots + \beta_k = |\beta| = d$, $\beta! = \beta_1! \cdots \beta_k!$, $t^{\beta} = t_1^{\beta_1} \cdots t_k^{\beta_k}$, and $\overrightarrow{F}(x^{\beta}) = \overrightarrow{F}(x_1^{\beta_1}, \ldots, x_k^{\beta_k})$, and $x_i^{\beta_i}$ is to be interpreted as $x_i$ repeated $i$ times.

The following is an example of the utility of this lemma that we will need later.

**Lemma 7.2.**

$L = \text{span}\{x_1, \ldots, x_k\} \subset V(f)$ if and only if $\overrightarrow{f}(x^{\beta}) = 0$ for every $\beta$ in Lemma 7.1.

**Proof.** A linear space $L = \text{span}\{x_1, \ldots, x_k\}$ is in the zero set of $f$ if and only if $f(t_1x_1 + \cdots + t_kx_k) = 0$ for all choices of $t_i \in \mathbb{C}$. Formula (7) says that

$$0 = f(t_1x_1 + \cdots + t_kx_k) = \sum_{\beta} \frac{1}{\beta!} t^{\beta} \overrightarrow{f}(x^{\beta}).$$

Now we consider a fixed element $\beta'$ and take the derivative $\frac{\partial}{\partial \overrightarrow{\gamma}}$ of the above expression to get

$$0 = \overrightarrow{f}(x^{\beta'}) + \sum_{\beta' > \beta} \frac{1}{(\beta - \beta')!} t^{\beta - \beta'} \overrightarrow{f}(x^{\beta}).$$

Then take limits, as $t_i \to 0$ to find that $0 = \overrightarrow{f}(x^{\beta'})$. We do this for each $\beta'$ to conclude the proof of the claim. □

In general, the polarization of the tensor product of two polynomials is not likely to be the product of the polarized polynomials; however, there is something we can say in the following special case:

**Lemma 7.3.** Let $F \in S^d(W^*)$ and let $\overrightarrow{F}$ denote its polarization. Then for $\gamma \in V^*$ we have

$$\overrightarrow{F} \otimes (\gamma)^d = \overrightarrow{F} \otimes (\gamma)^d = \overrightarrow{F} \otimes (\gamma)^d.$$

**Proof.** A standard fact about the polarization is that $\overrightarrow{F}$ is a symmetric multi-linear form. It is obvious that $(\gamma)^d = (\gamma)^d$, because $(\gamma)^d$ is already symmetric and multi-linear.
We will prove this by induction on the number of terms in $F$. Suppose $F$ is a monomial, $F = w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_n}$. Then use the isomorphism $W^\otimes d \otimes V^\otimes d \cong (W \otimes V)^{\otimes d}$, and write $w^\alpha \otimes \gamma^d = (w_1^{\alpha_1} \otimes \gamma^{\alpha_1}) \cdots (w_n^{\alpha_n} \otimes \gamma^{\alpha_n}) = (w_1 \otimes \gamma)^{\alpha_1} \cdots (w_n \otimes \gamma)^{\alpha_n} = (w \otimes \gamma)^\alpha$.

If $F$ is not a monomial, suppose $F = F_1 + F_2$ with $F_i$ nonzero polynomials for $i = 1, 2$. It is clear that $F_1 + F_2 = F_1^\gamma + F_2^\gamma$. Also, the operation $\otimes \gamma^d$ is distributive. So $F \otimes (\gamma^d) = F_1 \otimes (\gamma^d) + F_2 \otimes (\gamma^d)$ by induction on the number of monomials, we know that $F_i \otimes (\gamma^d) = F_i \otimes (\gamma^d)$ for $i = 1, 2$. We conclude that $F_1 \otimes (\gamma^d) + F_2 \otimes (\gamma^d) = (F_1 + F_2) \otimes (\gamma^d) = F \otimes (\gamma^d)$. □

The following lemma was inspired by methods found in [17]. It is a geometric description of the zero set of an augmented module.

**Lemma 7.4** (Augmentation Lemma). Let $W$ and $V$ be complex vector spaces with $\dim(V) \geq 2$. Let $X \subset \mathbb{P}W$ be a variety and suppose $\mathcal{I}(X) \subset S^{d}W^*$ is the ideal in degree $d$. Then

$$\mathcal{V}(\mathcal{I}(X) \otimes S^{d}V^*) = \text{Seg}(\mathcal{V}(\mathcal{I}(X)) \times \mathbb{P}V) \cup \bigcup_{L \subset \mathcal{V}(\mathcal{I}(X))} \mathbb{P}(L \otimes V),$$

where $L \subset \mathcal{V}(\mathcal{I}(X))$ are linear subspaces.

Note that since the linear spaces $L$ can be one dimensional, we do have

$$\text{Seg}(\mathcal{V}(\mathcal{I}(X)) \times \mathbb{P}V) \subset \bigcup_{L \subset \mathcal{V}(\mathcal{I}(X))} \mathbb{P}(L \otimes V),$$

and we will use Lemma [7.4] with the two terms on the right hand side of (8) combined, but we keep the two parts separate for emphasis here.

**Remark 7.5.** Note that if $X$ is generated in a single degree no larger than $d$, then one can replace $\mathcal{V}(\mathcal{I}(X))$ with $X$ in the statement of Lemma [7.4]. In particular, we will use the result of Lemma [7.4] with the induction hypothesis that $\mathcal{V}(HD) = Z_n$ and obtain a description of the zero set $\mathcal{V}(HD \otimes S^{d}V_{n+1})$ in terms of the geometry of $Z_n$.

**Proof of Proposition [7.4]**. First we prove “$\supseteq$”. Suppose $\dim(V) = n \geq 2$. Recall that we can choose a basis of $S^{d}V^*$ consisting of $d^{\text{th}}$ powers of linear forms, $\{(\gamma_1)^d, \ldots, (\gamma_r)^d\}$, where $r = \binom{n+d-1}{d}$ and the $\gamma_i$ are in general linear position. So one can construct a basis of the module $\mathcal{I}(X) \otimes S^{d}V^*$, consisting of polynomials of the form $f \otimes (\gamma^d)$, with $f \in \mathcal{I}(X)$ and $\gamma \in V^*$. We need to work on this basis.

Suppose $[x \otimes a] \in \text{Seg}(\mathcal{V}(\mathcal{I}(X)) \times \mathbb{P}V)$ and evaluate $(f \otimes (\gamma^d))(x \otimes a) = f(x)(\gamma^d)(a)$. But $x \in \mathcal{V}(\mathcal{I}(X))$, so $f(x) = 0$ for every $f \in \mathcal{I}(X)$, and in particular, $[x \otimes a] \in \mathcal{V}(\mathcal{I}(X) \otimes S^{d}V^*)$.

Now suppose $[v] \in \mathbb{P}(L \otimes V)$ for some linear subspace $L = \text{span}\{x_1, \ldots, x_l\} \subset \mathcal{V}(\mathcal{I}(X))$. By expanding an expression of $[v]$ in bases and collecting the coefficients of the $x_i$, we can write $[v] = [x_1 \otimes a_1 + \cdots + x_l \otimes a_l]$ for some $a_i \in V$ not all zero. Consider $f \otimes (\gamma^d) \in \mathcal{I}(X) \otimes S^{d}V$. By Lemma [7.3],

$$f \otimes (\gamma^d) = \overrightarrow{f} \otimes (\gamma^d)$$

and using the polarization formula [7], we write

$$(f \otimes (\gamma^d))(v) = \left(\overrightarrow{f} \otimes (\gamma^d)\right)(v, \ldots, v) = \sum_{\beta} \frac{1}{\beta!} \overrightarrow{f}(x^\beta)(\gamma^d)(a^\beta).$$

The choice of $L \subset \mathcal{V}(\mathcal{I}(X))$ means that $L \subset \mathcal{V}(f)$, so by Lemma [7.2], $\overrightarrow{f}(x^\beta) = 0$ for all $\beta$. Every term of $(f \otimes (\gamma^d))(v)$ vanishes so $(f \otimes (\gamma^d))(v) = 0$, and hence $[v] \in \mathcal{V}(\mathcal{I}(X) \otimes S^{d}V^*)$. 

---

**Remark 7.3**. Given a complex vector space $V$ and a homogeneous polynomial $f \in \mathbb{C}[V]^d$, the $d^{\text{th}}$ power $f^d$ is a polynomial of degree $d^2$. So we can choose a basis of $V^d$ consisting of $d^{\text{th}}$ powers of linear forms, $\{(\gamma_1)^d, \ldots, (\gamma_r)^d\}$, where $r = \binom{n+d-1}{d}$ and the $\gamma_i$ are in general linear position. Then $f \otimes (\gamma^d)$ is a linear combination of monomials $f \otimes (\gamma_1)^d \otimes \cdots \otimes (\gamma_r)^d$ in general linear position.
Now we prove \( \subseteq \). Consider any \([v] \in \mathbb{P}(W \otimes V)\). Choose a basis \( \{a_1, \ldots, a_k\} \) of \( V \) (by assumption \( k \geq 2 \)). Then expand the expression of \( v \) in bases and collect the coefficients of each \( a_i \) to find \([v] = [x_1 \otimes a_1 + \cdots + x_k \otimes a_k] \) with \( x_1, \ldots, x_k \in W \) and not all \( x_i \) zero. This is the statement that \( \sigma_k(\mathbb{P}W \times \mathbb{P}V) = \mathbb{P}(W \otimes V) \) when \( k = \dim(V) \).

Given \([v] \in \mathcal{V}(\mathcal{I}_d(X) \otimes S^dV^*) \subset \mathbb{P}(W \otimes V)\), we choose a basis of \( V \) (as above) so that \([v] = [x_1 \otimes a_1 + \cdots + x_k \otimes a_k] \in \mathcal{V}(\mathcal{I}_d(X) \otimes S^dV^*)\).

We need to show that \([v] \in \mathbb{P}(L \otimes V)\) for some linear space \( L \subset \mathcal{V}(\mathcal{I}_d(X))\). The natural linear space to consider is \( L = \text{span}\{x_1, \ldots, x_k\} \). Since we already have an expression \([v] = [x_1 \otimes a_1 + \cdots + x_k \otimes a_k]\), if we can show that \( L = \text{span}\{x_1, \ldots, x_k\} \subset \mathcal{V}(\mathcal{I}_d(X))\), we will be done.

For any \( f \otimes \gamma^d \in \mathcal{I}_d(X) \otimes S^dV^* \) we can write

\[
0 = (f \otimes \gamma^d)(v) = \sum_{\beta} \frac{1}{\beta!} \overrightarrow{f}(\mathbf{x}^\beta)\gamma^d(\mathbf{a}^\beta).
\]

Let \( \{a_1, \ldots, a_k\} \) be basis of \( V^* \) dual to \( \{a_1, \ldots, a_k\} \). Then we let \( \gamma \) vary continuously in \( V^* \) by writing it as

\[
\gamma = t_1a_1 + \cdots + t_ka_k
\]

where the parameters \( t_i \in \mathbb{C} \) vary. The polynomial \( \gamma^d \) is simple enough that we can expand it as follows:

\[
\gamma^d(\mathbf{a}^\beta) = \gamma^d(a_1^{\beta_1}, \ldots, a_k^{\beta_k}) = \gamma(a_1)^{\beta_1} \cdots \gamma(a_k)^{\beta_k}
\]

But our choices have made it so that \( \gamma(a_i) = t_i \), and therefore \( \gamma^d(\mathbf{a}^\beta) = \mathbf{t}^\beta \). So (9) becomes

\[
0 = (f \otimes \gamma^d)(v) = \sum_{\beta} \frac{1}{\beta!} \overrightarrow{f}(\mathbf{x}^\beta)\mathbf{t}^\beta = f(t_1x_1 + \cdots + t_kx_k),
\]

where we have used Lemma \( \ref{lemma3} \). So \( f(t_1x_1 + \cdots + t_kx_k) = 0 \) for all \( t_i \in \mathbb{C} \) and this is an equivalent condition that \( L = \text{span}\{x_1, \ldots, x_k\} \) is a subspace of \( \mathcal{V}(f) \). Since this was done for arbitrary \( f \in \mathcal{I}_d(X) \), we conclude that it holds for all such \( f \), so \( L \subset \mathcal{V}(\mathcal{I}_d(X)) \).

Now we can apply this geometric characterization of augmentation to the hyperdeterminantal module. To do this we need to set up some notation.

Assume \( n \geq 4 \). Let \( HD_i \) be the image of the hyperdeterminantal module at stage \( n - 1 \) under the following re-indexing isomorphism

\[
S^4(V_1^* \otimes \cdots \otimes V_{n-1}^*) \to S^4(V_1^* \otimes \cdots \otimes V_{i-1}^* \otimes V_i^* \otimes \cdots \otimes V_n^*),
\]

where we still have \( n - 1 \) vector spaces \( V_i \simeq \mathbb{C}^2 \), but we have shifted the index on the last \( n - i \) terms. Then the hyperdeterminantal module at stage \( n \) can be expressed as a sum of augmented modules as follows:

\[
HD = \sum_{i=1}^n (HD_i \otimes S^4V_i^*).
\]
Finally note that if dim(V) = k, then \( \sigma_s(\mathbb{P}W \times \mathbb{P}V) = \mathbb{P}(W \otimes V) \) for all \( s \geq k \). In the case \( V_i \cong \mathbb{C}^2 \), we have \( \mathbb{P}(L \otimes V_i) = \sigma_2(\mathbb{P}L \times \mathbb{P}V_i) \). Certainly

\[
Seg(\mathcal{V}(M_i) \times \mathbb{P}V_i) \subset \bigcup_{L \subset \mathcal{V}(M_i)} \mathbb{P}(L \otimes V_i),
\]

for any modules of polynomials \( M_i \). If \( L \subset \mathcal{V}(I_d(X)) \), then \( \sigma_s(\mathbb{P}L \times \mathbb{P}V) \subset \sigma_s(\mathcal{V}(I_d(X)) \times \mathbb{P}V) \). If \( A, B, C \) are vector spaces of polynomials such that \( C = A + B \) then \( \mathcal{V}(C) = \mathcal{V}(A) \cap \mathcal{V}(B) \). Collecting these ideas, we apply the Augmentation Lemma 7.4 to the hyperdeterminantal module to yield the following:

**Lemma 7.6 (Characterization Lemma).** Consider \( \sum_{i=1}^n HD_i \otimes S^d V_i^* \subset S^d(V_1^* \otimes \cdots \otimes V_n^*) \). Then

\[
\mathcal{V} \left( \sum_{i=1}^n HD_i \otimes S^d V_i^* \right) = \bigcap_{i=1}^n \left( \bigcup_{L \subset \mathcal{V}(HD_i)} \mathbb{P}(L \otimes V_i) \right) \subseteq \bigcap_{i=1}^n (\sigma_2(\mathcal{V}(HD_i) \times \mathbb{P}V_i)).
\]

**Remark 7.7.** A consequence of the characterization lemma is the following test for non-membership in the zero-set of \( HD \). Suppose \( [z] = [\zeta_1 \otimes x_1^2 + \zeta_2 \otimes x_1^2] \in \mathbb{P}^{2n-1} \). If either \( [\zeta_1] \) or \( [\zeta_2] \) is not a vector of principal minors of an \( (n - 1) \times (n - 1) \) symmetric matrix, then \( [z] \) is not a zero of the hyperdeterminantal module \( HD \) and hence not a vector of principal minors of a symmetric matrix since \( \mathcal{V}(HD) \supset Z_n \). This observation can be iterated, and each iteration cuts the size of the vector in question in half until one only need to check honest hyperdeterminants of format \( 2 \times 2 \times 2 \). This test, while relatively cheap and accessible, is necessary but not sufficient as Holtz and Sturmfels point out in [14].

It is well known that the ideal of the Segre product of projective spaces is generated in degree 2 by the \( 2 \times 2 \) minors of flattenings. In essence, this is saying that all of the polynomials in the ideal come from a Segre product of just two projective spaces. The following is a weaker, strictly set-theoretic result in the same spirit. It is another application of the Augmentation Lemma 7.4 and its proof is mimicked in the proof of Lemma 8.10 below.

**Proposition 7.8.** For \( 1 \leq i \leq n \), let \( V_i \) be complex vector spaces each with dimension \( \geq 2 \) and assume \( n \geq 2 \). If for each \( i \), \( B_i \subset S^d(V_1^* \otimes \cdots \otimes V_{i-1}^* \otimes V_{i+1}^* \otimes \cdots \otimes V_n^*) \) is a set of polynomials with the property

\[
\mathcal{V}(B^i) = Seg(\mathbb{P}V_1 \times \cdots \mathbb{P}V_{i-1} \times \mathbb{P}V_{i+1} \times \cdots \times \mathbb{P}V_n),
\]

then

\[
\mathcal{V} \left( \bigoplus_i (B_i \otimes S^d V_i^*) \right) = Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n).
\]

**Proof.** Work by induction and use the Augmentation Lemma 7.4. It is clear that \( \mathcal{V}(\bigoplus_i (B_i \otimes S^d V_i^*)) \supset Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n) \). All the linear spaces on \( Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n) \) are (up to permutation) of the form \( V_1 \otimes \widehat{a}_2 \otimes \cdots \otimes \widehat{a}_n \) where \( a_i \in V_i \) are nonzero and \( \widehat{a}_i \) denotes the line through \( a_i \). Then compute the intersection, \( \bigcup_{L^i} \bigcap_{i=1}^n \mathbb{P}(L^i \otimes V_i) \), and notice that in the intersection of just 3 factors, all of the resulting linear spaces must live in \( Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n) \). \( \square \)
8. Understanding the Case when All Principal Minors Agree with a Given Vector Except Possibly the Determinant

In the proof of Theorem 1.3 below we work to construct a matrix whose principal minors are a given point in the zero set of the hyperdeterminantal module. The main difficulty is the following. Suppose we have a point \( z \in V(HD) \) and a candidate matrix \( A \) which satisfies \( \Delta_I(A) = z_I \) for all \( I \neq [1, \ldots, 1] \). In other words, all of the principal minors of \( A \) except possibly for the determinant agree with the entries of \( z \). What can we say about \( z \)?

To answer this question, we must study the points in \( V(HD) \) which have all of their coordinates except one equal. Geometrically, we need to understand the points for which a line in the coordinate direction \( X^{[1, \ldots, 1]} \) above the point \( z \) intersects \( V(HD) \) in at least two points. We answer this question in Lemma 8.2 below. Using that lemma, we find the following

**Proposition 8.1.** Let \( n \geq 4 \). Suppose \( z = z_I X^I \) and \( w = w_I X^I \) are points in \( V(HD) \). If \( z_I = w_I \) for all \( I \neq [1, \ldots, 1] \) and \( z_{[0, \ldots, 0]} \neq 0 \), then \( z = w \).

To set up for the statement of the next lemma, we will use the following notation. If \( I = [i_1, \ldots, i_s] \) and \( 1 \leq i_j \leq n \) for all \( j \), then let \( V_I \simeq V_{i_1} \otimes \cdots \otimes V_{i_s} \). We assume \( V_i \simeq \mathbb{C}^2 \) for all \( i \), so that \( V_I \simeq (\mathbb{C}^2)^{\otimes s} \).

**Lemma 8.2.** Let \( n \geq 4 \). Suppose \( z = z_I X^I \) and \( w = w_I X^I \) are points in \( V(HD) \). If \( z_I = w_I \) for all \( I \neq [1, \ldots, 1] \) but \( z_{[1, \ldots, 1]} \neq w_{[1, \ldots, 1]} \), then

\[
[z], [w] \in \bigcup \limits_{I_p \cap I_q = \emptyset \text{ for } p \neq q, \cup_b I_b = \{1, \ldots, n\}, |I_s| \leq 2, 1 \leq s \leq m} \operatorname{Seg}(\mathbb{P}V_{i_1} \times \cdots \times \mathbb{P}V_{i_m}) \subset Z_n.
\]

Note that the notationally dense Segre product is just a product of \( \mathbb{P}^3 \)'s and \( \mathbb{P}^1 \)'s.

**Proof of Proposition 8.1.** Assume Lemma 8.2. Let \( z = z_I X^I \) and \( w = w_I X^I \) be points in \( V(HD) \cap \{ z \mid z_{[0, \ldots, 0]} \neq 0 \} \). Suppose that \( z_I = w_I \) for all \( I \neq [1, \ldots, 1] \), and suppose for contradiction that \( z_{[1, \ldots, 1]} \neq w_{[1, \ldots, 1]} \). Lemma 8.2 implies that \([z], [w]\) are in a Segre product of \( \mathbb{P}^1 \)'s and \( \mathbb{P}^3 \)'s.

Note that \( Z_1 \simeq \mathbb{P}^1 \) and \( Z_2 \simeq \mathbb{P}^3 \) and Proposition 5.2 implies that a point \([A,t]\) with \( t \neq 0 \) mapping to \( \operatorname{Seg}(\mathbb{P}V_{i_1} \times \cdots \times \mathbb{P}V_{i_m}) \) with \( |I_s| \leq 2 \) for each \( s \) and \( \cup_b I_b = \{1, \ldots, n\} \) is permutation equivalent to a block diagonal matrix consisting of \( 1 \times 1 \) and \( 2 \times 2 \) blocks. Moreover, such a block diagonal matrix is a special case of a symmetric tri-diagonal matrix, and therefore none of its principal minors depends on the sign of the off-diagonal terms. So fixing the \( 0 \times 0 \), \( 1 \times 1 \) and \( 2 \times 2 \) principal minors fixes the rest of the principal minors in such a matrix. If we take \( z_{[0, \ldots, 0]} = w_{[0, \ldots, 0]} = 1 \) and assume the \( 1 \times 1 \) and \( 2 \times 2 \) principal minors agree, then the rest of the principal minors must agree, including the determinants, thus the contradiction.

Note that the assumption \( z_{[0, \ldots, 0]} \neq 0 \) is necessary. If \( z_{[0, \ldots, 0]} = 0 \), then consider the image of any two matrices \( A, B \) with different nonzero determinants under the principal minor map with \( t = 0 \). Then \( \varphi([A,0]) = [0, \ldots, 0, \det(A)] \neq \varphi([B,0]) = [0, \ldots, 0, \det(B)] \). \( \square \)
Remark 8.3. A key point here is that we are not making the claim in Proposition 8.1 for $n = 3$. In this case, altering the sign of the off-diagonal terms of a $3 \times 3$ symmetric matrix can change the determinant without changing the other principal minors and without forcing the matrix to be blocked as a $2 \times 2$ block and a $1 \times 1$ block.

To see that the analog of Proposition 8.1 holds for $Z_n$ with $n \geq 4$ and $t \neq 0$ requires less work. We used Maple to construct a generic symmetric $4 \times 4$ matrix and computed its principal minors. Then we changed the signs of the off-diagonal terms in every possible combination and compared the number of principal minors that agreed with the principal minors of the original matrix. The result was that the two vectors of principal minors could agree in precisely 11, 13 or 16 entries, but not 15. We repeated the experiment in the $5 \times 5$ case and found that the two vectors could agree in precisely 16, 19, 20, 21, 23, 25 or 32 positions, but never 31 positions.

The general case follows from the $4 \times 4$ case as follows. Suppose $n \geq 4$ and $2^n - 1$ of the principal minors of an $n \times n$ symmetric matrix agree with the principal minors of another $n \times n$ symmetric matrix. Then we may assume that the $0 \times 0$, $1 \times 1$ and $2 \times 2$ principal minors of both matrices agree and hence the matrices must agree up to the signs of the off-diagonal terms. Then use the group to move the one position where the principal minors don’t agree to be a $4 \times 4$ determinant and use the $4 \times 4$ result for the contradiction.

To prove Lemma 8.2, we will show that if $w_I = z_I$ for all $I \neq [1, \ldots, 1]$ and $z_{[1, \ldots, 1]} \neq w_{[1, \ldots, 1]}$, then $z$ is a zero of an auxiliary set of polynomials denoted $B$. We will then show that the zero set $\mathcal{V}(B)$ is contained in the union of Segre varieties. Finally, Proposition 5.2 provides the inclusion into $Z_n$.

8.1. Reduction to one variable. Let $n \geq 4$. Suppose $z = z_I X^I$ and $w = w_I X^I$ are points in $\mathcal{V}(HD)$ are such that $z_I = w_I$ for all $I \neq [1, \ldots, 1]$. Both points are zeros of every polynomial in $HD$, but the only coordinate in which they can differ is $[1, \ldots, 1]$. Now consider the coordinates $z_I$ ($= w_I$) as fixed constants for all $I \neq [1, \ldots, 1]$, and for $f \in HD$ define $f_z$ by the substitution $f(X^{[0, \ldots, 0]}, \ldots, X^{[1, \ldots, 1]}) \mapsto f(z_{[0, \ldots, 0]}, \ldots, z_{[0, 1, \ldots, 1]}, X^{[1, \ldots, 1]}) =: f_z(X^{[1, \ldots, 1]})$. Let $HD_{[1, \ldots, 1]}(z) = \{f_z \mid f \in HD\}$ denote the resulting set of univariate polynomials. Then $z_{[1, \ldots, 1]}$ and $w_{[1, \ldots, 1]}$ are two (possibly different) roots of each univariate polynomial $f_z \in HD_{[1, \ldots, 1]}(z)$.

Lemma 8.4. If $f \in HD$, then the corresponding polynomial $f_z \in HD_{[1, \ldots, 1]}(z)$ is either degree 0, 1, or 2 in $X^{[1, \ldots, 1]}$.

Proof. It is sufficient to prove the lemma on a weight basis for $HD$. In particular, these polynomials have the property that all of their terms have the same weight. Recall in Section 3.1 we discussed how to compute weights of polynomials, and in Section 3.2 we discussed how to construct a weight basis of a module.

We recognize that $S_{(2,2)} S_{(2,2)} S_{(2,2)}$ is 1-dimensional, has weight zero, degree 4, and in particular, it is spanned by the hyperdeterminant, hyp. It is easy to see that hyp is a quadratic in $X^{[2,2,2]}$ just by looking at the expression in Corollary 10.1. We could also conclude this by the following argument. $(X^{[2,2,2]})^2$ has weight $[2, 2, 2]$, and the only way to raise this to $[0, 0, 0]$ is to multiply by $(X^{[1,1,1]})^2$. We cannot have anything of lower weight because we will not be able to raise its weight back up to $[0, 0, 0]$ and still be degree 4.
Consider the module \( S_{(2,2)} V_1^* \otimes S_{(2,2)} V_2^* \otimes S_{(2,2)} V_3^* \otimes S_{(4)} V_4^* \otimes \cdots \otimes S_{(4)} V_n^* \). A lowest weight vector in this module is constructed by taking the weight \([0,0,0]\) vector which spans \( S_{(2,2)} V_1^* \otimes S_{(2,2)} V_2^* \otimes S_{(2,2)} V_3^* \), \( i.e. \) hyp\(_{1,2,3}\)) and tensoring with \((x_4^4) \otimes \cdots \otimes (x_n^4)\) – the lowest weight vector for \( S_{(4)} V_4^* \otimes \cdots \otimes S_{(4)} V_n^* \). The leading term is
\[
(x_1^0 \otimes x_2^0 \otimes x_3^0)(x_1^1 \otimes x_2^1 \otimes x_3^1)^2 \otimes (x_4^4) \otimes \cdots \otimes (x_n^4) = (X^{[0,0,0,1]}\ldots 1)^2 (X^{[1,\ldots,1]}\ldots 1)^2.
\]

There cannot be any higher power of \( X^{[1,\ldots,1]} \) occurring in a polynomial in \( HD \), otherwise, the vector would have a weight that is lower than the lowest weight.

Now we know that \( w_{[1,...,1]} \) and \( z_{[1,...,1]} \) are both common zeros of univariate polynomials, all with degree 2 or less. The fact that \( w_{[1,...,1]} \) and \( z_{[1,...,1]} \) are both common zeros of more than one univariate polynomial comes from the fact that we have required \( n \geq 4 \) otherwise there is only one polynomial and what we are about to do would be trivial.

A quadratic (not identically zero) in one variable has at most two solutions, and a linear polynomial (not identically zero) has at most one solution. The only way then for us to have \( w \neq z \) and \([w], [z] \in \mathcal{V}(HD)\) is if all of the linear polynomials were identically zero and if all of the quadratics were scalar multiples of each other.

Therefore, we need to study the points \([z] \in \mathcal{V}(HD)\) for which \( HD_{[1,...,1]}(z) \) has dimension 1 or less. Define polynomials \( a_f, b_f, \) and \( c_f \) (which necessarily do not depend on \( X^{[1,\ldots,1]} \)) for each \( f_z \in HD_{[1,...,1]}(z) \) by
\[
f_z = a_f(z) (X^{[1,...,1]})^2 + b_f(z) (X^{[1,...,1]}) + c_f(z).
\]

The requirement that \( HD_{[1,...,1]}(z) \) have dimension 1 or less implies the weaker (but still sufficient condition) that the polynomials

\[
B' := \text{span}\{a_f(z)b_g(z) - a_g(z)b_f(z) \mid f, g \in HD\}
\]

be satisfied.

The polynomials in \( B' \) have the property that if \( h(z) \neq 0 \) for some nonzero \( h \in B' \), \( i.e. \) \([z] \notin \mathcal{V}(B')\), then there is a non-trivial pair of polynomials in \( HD_{[1,...,1]}(z) \) which are not scalar multiples of each other, and thus the zero set of \( HD_{[1,...,1]}(z) \) is a single point. In this case we must have \( w_{[1,...,1]} = z_{[1,...,1]} \). If, however \( h(z) = 0 \) for all \( h \in B' \) (\( i.e. \) \( z \in \mathcal{V}(B') \)), then it is possible that the polynomials in \( HD_{[1,...,1]}(z) \) have 2 common roots.

Notice that \( B' \) is not \( G \)-invariant for \( G = (SL(2)^{\times n}) \rtimes S_n \). Let \( B := \text{span}\{G.B'\} \) denote the corresponding \( G \)-module. If \( g.[z] \notin \mathcal{V}(B') \), then by our remarks above, \( g.[z] \in Z_n \), and in particular, \([z] \in Z_n \) (because \( Z_n \) is a \( G \)-variety). If \( G.[z] \subset \mathcal{V}(B') \), then \([z] \in \mathcal{V}(B) \) (cf. Proposition\([5,7]\)). So, we need to look at the variety \( \mathcal{V}(B) \). We conclude that our construction satisfies the property that if \([z] \in \mathcal{V}(HD) \) but \([z] \notin \mathcal{V}(B) \), then \([z] \in Z_n \).

We need to understand the types of points that can be in \( \mathcal{V}(B) \) and the following proposition gives sufficient information about \( \mathcal{V}(B) \).
Proposition 8.5. Let \( n \geq 4 \) and let \( B \) be the module of polynomials constructed above. Then

\[
\mathcal{V}(B) \subset \bigcup \text{Seg} \left( \mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m} \right) \subset Z_n.
\]

This proposition will be proved in the following sequence of lemmas.

1. \( S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)} \ldots S_{(5)} \subset B \).
2. \( \mathcal{V}(S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)} \ldots S_{(5)}) = \bigcup \text{Seg} \left( \mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m} \right) \) .
3. \( \text{Seg} \left( \mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m} \right) \subset Z_n \) is by Proposition 5.7.

Lemma 8.6. Suppose \( n \geq 4 \) and let \( B \) be constructed as above. Then

\[
S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)} \ldots S_{(5)} \subset B.
\]

Proof. Here we have some polynomials in \( B \) in an explicit form, and we would like to identify some \( (SL(2)^{\times n}) \times \mathfrak{S}_n \)-modules in \( B \) from this information. To do this we will carry out the steps in the algorithm given above in Section 3.2.

Suppose \( f_{[i_1,i_2,i_3]} \in S_{(2,2)}V_{i_1}^* \otimes S_{(2,2)}V_{i_2}^* \otimes S_{(4,4)}V_{i_3}^* \otimes \cdots \otimes S_{(4)}V_{i_n}^* \) is a lowest weight vector. Define \( a_{[i_1,i_2,i_3]} \), \( b_{[i_1,i_2,i_3]} \), \( c_{[i_1,i_2,i_3]} \) by the equation \( f_{[i_1,i_2,i_3]} = a_{[i_1,i_2,i_3]}(X^{[1,\ldots,1]})^2 + b_{[i_1,i_2,i_3]}(X^{[1,\ldots,1]}) + c_{[i_1,i_2,i_3]} \).

For this proof, we introduce some new notation. If \( i_1, i_2, i_3 \) are fixed, let \( X^{I_{p,q,r}} \) denote the coordinate vector with \( i_1 = p, i_2 = q, i_3 = r \) and \( i_k = 0 \) for \( k \geq 4 \).

Since \( f_{[i_1,i_2,i_3]} \) is a hyperdeterminant of format \( 2 \times 2 \times 2 \) we find

\[
a_{[i_1,i_2,i_3]} = (X^{I_{0,0,0}})^2
\]

\[
b_{[i_1,i_2,i_3]} = -2X^{I_{0,0,0}} \left( X^{I_{1,0,0}}X^{I_{0,1,1}} + X^{I_{0,1,0}}X^{I_{1,0,1}} + X^{I_{0,0,1}}X^{I_{0,1,1}} \right) + 4X^{I_{0,0,0}}X^{I_{1,0,0}}X^{I_{0,0,1}}.
\]

The weight of \( a_{[i_1,i_2,i_3]} \) is (up to permutation) \([-2, -2, -2, \ldots, 2] \), where the \(-2\)'s actually occur at \( \{i_1, i_2, i_3\} \). The weight of \( b_{[i_1,i_2,i_3]} \) is (up to permutation) \([-1, -1, -1, 3, \ldots, 3] \), where the \(-1\)'s actually occur at \( \{i_1, i_2, i_3\} \). Now consider

\[
h_{[i_1,i_2,i_3],[j_1,j_2,j_3]} = a_{[i_1,i_2,i_3]}b_{[j_1,j_2,j_3]} - a_{[j_1,j_2,j_3]}b_{[i_1,i_2,i_3]} \in B.
\]

We notice that \( h_{[i_1,i_2,i_3],[j_1,j_2,j_3]} \) can not have \([i_1,i_2,i_3] \) and \([j_1,j_2,j_3] \) all equal (this is the zero polynomial). So either two, one or zero pairs of \( i \)'s and \( j \)'s match in the indices \([i_1,i_2,i_3] \) and \([j_1,j_2,j_3] \). Therefore \( h_{[i_1,i_2,i_3],[j_1,j_2,j_3]} \) can have 3 different (up to permutation) weights, depending on how \([i_1,i_2,i_3] \) and \([j_1,j_2,j_3] \) match up. The three possible weights of \( h_{[i_1,i_2,i_3],[j_1,j_2,j_3]} \) are (up to permutation): \([-3, -3, 1, 1, 5, \ldots, 5] \), \([-3, 1, 1, 1, 1, 5, \ldots, 5] \), or \([1, 1, 1, 1, 1, 5, \ldots, 5] \).

In each case, apply the algorithm in Section 3.2 to lower \( h_{[i_1,i_2,i_3],[j_1,j_2,j_3]} \) to the lowest possible nonzero vector. We did this calculation in Maple. The output in each case is a
The space $V^*_1 \otimes V^*_2 \otimes V^*_3$ has a finite number of orbits under the action of $SL(2)^3$. This gives rise to a list of normal forms, which we record below together with the respective orbit closures to which they belong:

- $\text{Seg}(PV_1 \times PV_2 \times PV_3)$: Normal form $[x] = [a \otimes b \otimes c]$. 
- $\tau(\text{Seg}(PV_1 \times PV_2 \times PV_3))$ sing $= S_3 \Delta_2 V = \text{Seg}(PV_1 \times PV_2 \times PV_3)$: Normal form (up to permutation) $[x] = [a \otimes b \otimes c + a' \otimes b' \otimes c]$. This variety is called the singular orbit.
- $\sigma(\text{Seg}(PV_1 \times PV_2 \times PV_3))$: Normal form $[x] = [a \otimes b \otimes c + a' \otimes b' \otimes c + a \otimes b \otimes c']$.

The varieties containing each normal form are nested:

$$\text{Seg}(PV_1 \times PV_2 \times PV_3) \subset \tau(\text{Seg}(PV_1 \times PV_2 \times PV_3))_{\text{sing}} \subset \tau(\text{Seg}(PV_1 \times PV_2 \times PV_3)) \subset \sigma(\text{Seg}(PV_1 \times PV_2 \times PV_3))$$

The lowest weight vector for $S_{(4,1)} S_{(4,1)} S_{(4,1)}$ is

$$f_{(4,1),(4,1),(4,1)} = (X^{[1,1,1]})^2 (X^{[1,0,0]} (X^{[1,1,1]} X^{[1,0,0]} \times X^{[1,1,1]} X^{[1,0,0]} X^{[1,1,0]}) - X^{[1,1,1]} (X^{[0,0,0]} X^{[1,0,0]} + X^{[0,0,0]} X^{[0,0,0]} + X^{[1,1,0]}) X^{[1,1,0]} X^{[0,0,0]} + X^{[0,1,0]} X^{[0,0,0]} + X^{[1,1,0]}) X^{[1,1,0]} X^{[0,0,1]} )$$

By checking a generic point in the singular orbit, we find that $f_{(4,1),(4,1),(4,1)}(x) = 0$ for every $x \in \tau(\text{Seg}(PV_1 \times PV_2 \times PV_3))_{\text{sing}}$. So therefore $\tau(\text{Seg}(PV_1 \times PV_2 \times PV_3))_{\text{sing}} \subset \mathcal{V}(S_{(4,1)} S_{(4,1)} S_{(4,1)})$.

Next, we show that the other two varieties are not in $\mathcal{V}(S_{(4,1)} S_{(4,1)} S_{(4,1)})$. The varieties are nested, so consider the point $[x] = [X^{[1,1,1]} + X^{[0,1,1]} + X^{[1,0,1]} + X^{[0,1,0]}] \in \tau(\text{Seg}(PV_1 \times PV_2 \times PV_3))$. But $f_{(4,1),(4,1),(4,1)}(x) = 2 \neq 0$, so the other two varieties are not in $\mathcal{V}(S_{(4,1)} S_{(4,1)} S_{(4,1)})$.

Since we have considered all possible normal forms, we are done. □

**Notation 8.8.** Let $V_I = V_{i_1} \otimes \cdots \otimes V_{i_{|I|}}$ and let $\tilde{v}_I \in V_I$ denote the line through $v_{i_1} \otimes \cdots \otimes v_{i_{|I|}}$. If $\pi$ is a partition of $d$ for any integer $d$, let $S_{[\pi]} V_I = S_{\pi} V_{i_1} \otimes \cdots \otimes S_{\pi} V_{i_{|I|}}$. Note: This is not the same as $S_{[\pi]} (V_{i_1} \otimes \cdots \otimes V_{i_{|I|}})$.

**Observation 8.9.** All the linear spaces on $\text{Seg}(PV_{I_1} \times \cdots \times PV_{I_m})$ are (up to permutation) contained in one of the form $V_{I_1} \otimes \tilde{v}_{I_2} \otimes \cdots \otimes \tilde{v}_{I_m}$.

Let

$$\tilde{B} = \bigoplus_{|I|=n} \left( S_{[(4,1)]} V^*_I \otimes S_{[(5)]} ^* V^*_I \right)$$
and let

$$\tilde{B}_k = \bigoplus_{|I|=n-1, k \notin I} \left( S_{[(4,1)]} V_{\{i_1,i_2,i_3\}}^* \otimes S_{[(5)]} V_{\{i_1,i_2,i_3\}}^* \right).$$

Notice that $\tilde{B} = \sum_{k=1}^n \tilde{B}_k \otimes S_{(5)} V^*_k$.

We want to understand the zero set of this module $\tilde{B}$, and we do this in the next two lemmas by mimicking what we did for Lemma 7.8. We also point out that while notationally more complicated, the result is essentially the same as Lemma 7.8.

**Lemma 8.10.** Suppose

$$\mathcal{V}\left( \tilde{B}_k \right) = \bigcup_{|I_s| \leq 2, k \notin I_s, I_p \neq I_q \text{ for } p \neq q, \cup_s I_s = \{1, \ldots, n\} - \{k\}, 1 \leq s \leq m} \text{Seg} \left( \mathbb{P}V_{I_1} \times \mathbb{P}V_{I_2} \times \cdots \times \mathbb{P}V_{I_m} \right).$$

Then

$$\mathcal{V}\left( \tilde{B}_k \otimes S_{(5)} V^*_k \right) = \bigcup_{|I_s| \leq 2, k \notin I_s, I_p \neq I_q \text{ for } p \neq q, \cup_s I_s = \{1, \ldots, n\} - \{k\}, 1 \leq s \leq m} \text{Seg} \left( \mathbb{P}V_{I_1 \cup \{k\}} \times \mathbb{P}V_{I_2} \times \cdots \times \mathbb{P}V_{I_m} \right).$$

**Proof.** Apply the Augmentation Lemma 7.4 to the left hand side of (11). It remains to check that

$$\bigcup_{|I_s| \leq 2, k \notin I_s, I_p \neq I_q \text{ for } p \neq q, \cup_s I_s = \{1, \ldots, n\} - \{k\}, 1 \leq s \leq m} \text{Seg} \left( \mathbb{P}V_{I_1 \cup \{k\}} \times \mathbb{P}V_{I_2} \times \cdots \times \mathbb{P}V_{I_m} \right),$$

where $L \subset \mathcal{V}(\tilde{B}_k)$ are linear spaces. Because of symmetry, there is only one type of linear space to consider, $V_{I_1} \otimes \tilde{v}_{I_2} \otimes \cdots \otimes \tilde{v}_{I_m} \otimes V_k = V_{I_1 \cup \{k\}} \otimes \tilde{v}_{I_2} \otimes \cdots \otimes \tilde{v}_{I_m}$. It is clear that each of these linear spaces is on one of the Segre varieties on the right hand side of (11), and moreover every point on the right hand side of (11) is on one of these linear spaces. \hfill \Box

**Proposition 8.11.** Suppose $n \geq 4$.

$$\mathcal{V}\left( \tilde{B} \right) = \bigcup_{|I_s| \leq 2, I_p \neq I_q \text{ for } p \neq q, \cup_s I_s = \{1, \ldots, n\} \leq m} \text{Seg} \left( \mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m} \right).$$

**Proof.** Proof by induction. The base case is Lemma 8.7. For the induction step, use Lemma 8.10. We need to show that
Lemma 9.1. Choosing to work on this open set is no loss in generality because of the following hypothesis.

Proof. This can be done by writing a point \([p]\) in the first Segre variety in coordinates and then requiring the \(2 \times 2\) minors in the ideal of the second Segre variety to vanish on \([p]\).

We conclude this section by pointing out that we have established all of the ingredients for the proof of Lemma 8.2.

9. Proof of Theorem 1.3

The outline of the proof is the following. Proposition 6.5 says that \(Z_n \subseteq \mathcal{V}(HD)\). To show the opposite inclusion, we work by induction. In the cases of \(n = 3, 4\), the (stronger) ideal theoretic version of Theorem 1.3 was proved with the aid of a computer in [14]. Since the theorem is already proved for the cases \(n = 3, 4\) we will assume \(n \geq 5\). The induction hypothesis is that \(\mathcal{V}(HD_t) \simeq Z_{(n-1)}\). We need to show that given a point \([z] \in \mathcal{V}(HD)\), that \([z] \in Z_n\), i.e. that there exists a matrix \(A\) so that \(\varphi([A,t]) = [z]\). The key tools we use in this proof are Proposition 8.1 and Lemma 7.6.

We will work on a preferred open set \(U_0 = \{[z] = [z_I X^t] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \mid z_{[0,\ldots,0]} \neq 0\}\). Choosing to work on this open set is no loss in generality because of the following

Lemma 9.1. Let \(U_0 = \{[z] = [z_I X^t] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \mid z_{[0,\ldots,0]} \neq 0\}. Then \(\mathcal{V}(HD) \cap U_0 \subset Z_n\) implies that \(\mathcal{V}(HD) \subset Z_n\).

Proof. Since \(Z_n\) and \(\mathcal{V}(HD)\) are \(G\)-varieties, and \(G.U_0 = \mathbb{P}(V_1 \otimes \cdots \otimes V_n)\) the result follows.

Moreover, it suffices to work on the following section of the cone over projective space, \(\{z = z_I Z^t \in V_1 \otimes \cdots \otimes V_n \mid z_{[0,\ldots,0]} = 1\}\), because afterwards we can rescale everything to get the result on the whole open set \(U_0\).

Suppose we take a point in the zero set (as described by Lemma 7.6)

\[
[z] \in \mathcal{V}(HD) = \bigcap_{i=1}^n \bigcup_{L_i \subset \mathcal{V}(HD_t)} \mathbb{P}(L_i^t \otimes V_i).
\]
Since \([z]\) is fixed, we can also fix a single \(L^i\) for each \(i\) so that \([z] \in \bigcap_{i=1}^n \mathbb{P}(L^i \otimes V_i)\). Work in our preferred section of the cone over projective space and write \(n\) different expressions for the point \(z\) (one for each \(i\)):

\[
z = z_I X^I = \eta^i \otimes x^i_0 + \nu^i \otimes x^i_1,
\]

where \([\eta^i], [\nu^i] \in L^i \subset \mathcal{V}(HD_i)\). (These expressions are possible because each \(V_i\) is 2 dimensional.) Choosing \(z_{[0, \ldots, 0]} = 1\) also implies that \(\eta_{[0, \ldots, 0]} = 1\). The induction hypothesis says that \(Z_{n-1} \simeq \mathcal{V}(HD)\) for \(1 \leq i \leq n\). So each \(\eta^i\) satisfies \(\varphi([A^{(0)}, 1]) = \eta^i\) for some \(A(i) \in S^2 \mathbb{C}^{n-1}\). For each \(0 \leq j \leq n\) denote by \(A^j\) the following subset of matrices

\[
A^j = \{A \in S^2 \mathbb{C}^n | \Delta_I(A) = z_I \text{ for all } I = [i_1, \ldots, i_n] \text{ with } i_j = 0\}.
\]

Each matrix \(A \in A^j\) has the property that the principal submatrix of \(A\) formed by deleting the \(j^{\text{th}}\) row and column maps to \(\eta^j\) under the principal minor map. Thus each \(A \in A^j\) is a candidate matrix that might satisfy \(\varphi([A, 1]) = [z]\), however we don’t know if such a matrix will have a submatrix that maps to the other \(\eta^i\)’s. We claim that there is at least one matrix that satisfies all of these conditions.

**Lemma 9.2.** \(\cap_{i=1}^n A^i\) is non-empty.

**Proof.** By the induction hypothesis, each \(A^i\) is non-empty. Assume \(\cap_{i=2}^n A^i\) is no-empty. We show that if \(A \in \cap_{i=2}^n A^i\) then \(A \in A^1\). The same argument we use will also prove that if \(A \in \cap_{i=3}^n A^i\), then \(A \in A^1\), and so on, so it suffices to check the last, most restrictive case. Also because of the \(\mathfrak{S}_n\) action, we don’t have to repeat the proof for every permutation.  

If \(A \in \cap_{i=2}^n A^i\), then \(\Delta_I(A) = z_I\) for all \(I \neq [0, i_2, \ldots, i_n]\) with \(|I| \leq n-2\). The only possible exception we could have is for \(\Delta_{[0,1,\ldots,1]}\) might not be equal to \(z_{[0,1,\ldots,1]}\). Denote by \(A'\) the principal submatrix of \(A\) formed by deleting the 1st row and column of \(A\). Now since \(n \geq 5\), \(|I| \geq 3\), \(A'\) is at least as large as \(4 \times 4\), and we have determined that all of the principal minors of \(A'\) except possibly the determinant agree with a fixed point \(\eta^1 \in \mathcal{V}(HD_1)\) (in other words \(\Delta(A')_I = \eta^I\) for all \(I \neq [1, \ldots, 1]\), so we can apply Proposition 8.1 to conclude that the determinant of \(A'\) also agrees with \(\eta^1\) (i.e. \(\Delta_{[1,\ldots,1]}(A') = \eta^1_{[1,\ldots,1]}\)). Therefore any such \(A\) must have \(\Delta_{[0,1,\ldots,1]}(A) = z_{[0,1,\ldots,1]}\), and we have shown \(A \in A^1\).  

**Remark 9.3 (Building a matrix).** Note that when \(n \geq 4\), the proof we gave can be used also to construct a symmetric matrix whose principal minors are prescribed by a point \(z \in \mathcal{V}(HD) \cap \{z | z_{[0,\ldots,0]} \neq 0\}\). The entries of \(z\) corresponding to \(1 \times 1\) and \(2 \times 2\) principal minors determine a large finite set \(A\) of candidate matrices which could map to \(z\). Restrict the set \(A\) to only those matrices whose \(3 \times 3\) principal minors agree with the corresponding entries of \(z\), i.e. keep only the matrices \(A\) so that \(\Delta(A)_I = z_I\) for all \(|I| \leq 3\). We claim that the remaining set of matrices all map to \(z\) under the principal minor map. If \(A\) is such that all of the \(3 \times 3\) principal minors agree with \(z\), then Proposition 8.1 implies that each \(4 \times 4\) principal minor of \(A\) must agree with \(z\) also. Iterate this argument to imply that all of the principal minors of \(A\) must agree with \(z\).
10. Conclusion

The variety of principal minors of symmetric matrices is a prototypical $G$-variety in a space of tensors. We have studied it in the setting of $G$-varieties using a mix of representation theory and geometry with a secondary goal that the techniques used and presented here will be useful in the study of other $G$-varieties in spaces of tensors. Groebner basis techniques were used successfully by Holtz and Sturmfels to prove that the hyperdeterminantal module gives a set of minimal generators of the prime ideal $I(Z_n)$ only for the first two non-trivial cases \[14\]. Now the set theoretic result is established for all $n$, but there is still more work to be done for the full Holtz–Sturmfels Conjecture – i.e. the ideal theoretic case.

The set theoretic result is good enough for many applications related to principal minors of symmetric matrices. In particular, set theoretic defining equations of $Z_n$ are necessary and sufficient conditions for a given vector of length $2^n$ to be expressed as the principal minors of a symmetric matrix.

A practical membership test for the variety of principal minors of symmetric matrices is the following:

**Corollary 10.1.** Suppose $v = v_1 X^I \in \mathbb{C}^{2^n}$. Then $v$ represents the principal minors of a symmetric $n \times n$ matrix if and only if for any element $g = (a_{i_1,j_1}) \times \cdots \times (a_{i_n,j_n}) \in SL(2)^\times n$ and any $\sigma \in \Sigma = \{ \sigma \in \mathfrak{S}_n \mid \sigma(1) < \sigma(2) < \sigma(3), \text{ and } \sigma(4) < \sigma(5) < \cdots < \sigma(n) \}$, the transformed vector with coordinates $v_I$ defined by $g(\sigma,v) = a_{i_1,j_1} \cdots a_{i_n,j_n} v_{\sigma(1)} X^I = w_I X^I$ satisfies the $2 \times 2 \times 2$ hyperdeterminantal equation

\[
(w_{I[0,0,0]}^2 w_{I[0,0,1]}^2 + (w_{I[1,0,0]} w_{I[0,1,0]} + (w_{I[1,0,1]}^2 + (w_{I[0,0,1]}^2 w_{I[1,1,0]}^2 - 2 w_{I[0,0,0]} w_{I[0,0,1]} w_{I[1,1,0]} - 2 w_{I[0,0,0]} w_{I[0,1,0]} w_{I[1,1,0]} - 2 w_{I[1,0,0]} w_{I[0,0,1]} w_{I[1,1,0]} - 2 w_{I[0,0,0]} w_{I[0,1,0]} w_{I[1,1,0]} - 2 w_{I[1,0,0]} w_{I[1,0,1]} w_{I[1,1,0]} + 4 w_{I[0,0,0]} w_{I[0,1,0]} w_{I[1,1,0]} + 4 w_{I[0,0,1]} w_{I[0,1,0]} w_{I[1,1,0]} = 0,
\]

where $I_{[k,l,m]} = [k,l,m,0, \ldots, 0]$.

Acknowledgments

The author would like to thank J.M. Landsberg for suggesting this problem as a thesis topic and for his endless support and advice along the way. We thank the two anonymous reviewers who read the first draft of this paper and for their numerous useful suggestions for revision. We also thank Shaowei Lin, Linh Nguyen, Bernd Sturmfels, and Zach Teitler for useful conversations. Shaowei Lin pointed out the reference \[23\]. Bernd Sturmfels suggested the addition of Corollary \[10.1\].

References

1. E. Allman and J. Rhodes, Phylogenetic ideals and varieties for the general Markov model, Adv. in Appl. Math. 40 (2008), no. 2, 127–148. MR 2388607 (2008m:60145)
2. J. Borcea, P. Branden, and T. Liggett, Negative dependence and the geometry of polynomials, Journal of the American Mathematical Society 22 (2009), 521–567.
3. A. Borodin and E. Rains, Eynard-Mehta theorem, Schur process, and their Pfaffian analogs, J. Stat. Phys. 121 (2005), no. 3–4, 291–317. MR 2185331 (2006k:82039)
4. P. Bürgisser, M. Clausen, and M. Shokrollahi, Algebraic complexity theory, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 315, Berlin: Springer-Verlag, 1997. MR 1440179 (99c:68002)
5. P. Comon, G. Golub, L.-H. Lim, and B. Mourrain, Symmetric tensors and symmetric tensor rank, SIAM J. Matrix Anal. Appl. 30 (2008), no. 3, 1254–1279. MR 2447451
6. P. Comon and M. Rajih, Blind identification of under-determined mixtures based on the characteristic function, Signal Processing 86 (2006), no. 9, 2271–2281, http://dx.doi.org/10.1016/j.sigpro.2005.10.007.
7. D. Cox, J. Little, and D. O’Shea, Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra, third ed., Undergraduate Texts in Mathematics, New York: Springer, 2007. MR 2290010 (2007h:13036)
8. W. Fulton and J. Harris, Representation theory: A first course, Graduate Texts in Mathematics, vol. 129, New York: Springer-Verlag, 1991. MR 1153249 (93a:20069)
9. R. Goodman and N. Wallach, Representations and invariants of the classical groups, Encyclopedia of Mathematics and its Applications, vol. 68, Cambridge University Press, 1998. MR 1606831 (99b:20073)
10. K. Griffin and M. Tsatsomeros, Principal minors. II. The principal minor assignment problem, Linear Algebra Appl. 419 (2006), no. 1, 125–171. MR 2263115 (2008h:15015)
11. J. Harris, Algebraic geometry, Graduate Texts in Mathematics, vol. 133, New York: Springer-Verlag, 1992, A first course. MR 1182558 (93j:14001)
12. O. Holtz, Not all GKK $\tau$-matrices are stable, Linear Algebra Appl. 291 (1999), no. 1-3, 235–244. MR 1685605 (2000a:15011)
13. O. Holtz and H. Schneider, Open problems on GKK $\tau$-matrices, Linear Algebra Appl. 345 (2002), 263–267. MR 1883278
14. O. Holtz and B. Sturmfels, Hyperdeterminantal relations among symmetric principal minors, J. Algebra 316 (2007), no. 2, 634–648. MR 2358606 (2009c:15032)
15. J. M. Landsberg, Geometry and the complexity of matrix multiplication, Bull. Amer. Math. Soc. (N.S.) 45 (2008), no. 2, 247–284. MR 2383305 (2009b:68055)
16. J. M. Landsberg and L. Manivel, Construction and classification of complex simple Lie algebras via projective geometry, Selecta Math. (N.S.) 8 (2002), no. 1, 137–159. MR 1890196 (2002m:17006)
17. _____, On the projective geometry of rational homogeneous varieties, Comment. Math. Helv. 78 (2003), no. 1, 65–100. MR 1966752 (2004a:14050)
18. _____, On the ideals of secant varieties of Segre varieties, Found. Comput. Math. 4 (2004), no. 4, 397–422. MR 2097214 (2005m:14101)
19. J. M. Landsberg and J. Morton, The geometry of tensors: Applications to complexity, statistics and engineering, in preparation.
20. L. De Lathauwer and A. de Baynast, Blind deconvolution of DS-CDMA signals by means of decomposition in rank-(1, L, L) terms, IEEE Trans. Signal Processing 56 (2008), no. 4, 1562–1571.
21. S. Lin and B. Sturmfels, Polynomial relations among principal minors of a $4 \times 4$-matrix, arXiv:0812.0601v2.
22. T. Mikkonen, The ring of graph invariants - graphic values, 2007, preprint: arXiv:0712.0146.
23. E. J. Nanson, On the relations between the coaxial minors of a determinant, Philos. Magazine 5 (1897), 362–367.
24. L. Oeding, G-varieties and principal minors of symmetric matrices, Ph.D. Thesis.
25. L. Pachter and B. Sturmfels (eds.), Algebraic statistics for computational biology, New York: Cambridge University Press, 2005. MR 2205865 (2006i:92002)
26. J. Sidman and S. Sullivant, Prolongations and computational algebra, (2006), math/0611696, to appear in Canadian J. Math.
27. E. B. Stouffer, On the independence of principal minors of determinants, Trans. Amer. Math. Soc. 26 (1924), no. 3, 356–368. MR 1501282
28. ______, Expressions for the general determinant in terms of its principal minors, Amer. Math. Monthly 35 (1928), no. 1, 18–21. MR 1521341

29. D. Wagner, Negatively correlated random variables and Mason’s conjecture for independent sets in matroids, Ann. Comb. 12 (2008), no. 2, 211–239. MR 2428906

E-mail address: oeding@math.unifi.it