Enhancing Scientific Inquiry by Mathematical Reasoning: 
Case of Applying Limits to Model Motion of a System of Objects

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Abstract. Mathematics and physics constitute parts of STEM that are widely recognized as a leading platform in contemporary education. While the effects of the methods of mathematics on discoveries in physics are unquestionable, using mathematical reasoning to enhance understanding of physics is still being researched. This study is posited to contribute to that research, and it investigates students’ skills to apply limiting the analysis to determine the acceleration of two objects connected by a massless string. Twenty-five (N=24) first-year students enrolled in the algebra-based college physics course took part in this case study. While the pretest showed that the students did not consider limits to answer questions that required limiting analysis, designed instructional unit whose goal was to bring forth the technique helped apply limits and made the solution process more explicit. Posttests results supported the study thesis that integrating more sophisticated mathematical apparatus into the analysis of physical systems opens a gate not only for advancing students’ scientific inquiry skills, but also their general STEM disposition.

1. Introduction and Theoretical Background

1.1 STEM as a Platform for Merging Scientific Contexts with Mathematical Reasoning

The acronym STEM has multiple definitions in educational research and practice. Moore [1] defined STEM as an effort to link some or all the four disciplines of science, technology, engineering and mathematics into one unit. McComas [2] termed it as an interdisciplinary approach to learning that intertwines academic concepts with real-world situations. Students’ successes in science and engineering depend on their skills of modeling where the ability to merge mathematics reasoning with scientific inquiry plays a significant part. Mathematics provides a computational system and helps conveniently encode a rule from natural phenomena and further hypothesize its behavior [3]. Hence, emphasizing modeling in physics that draws on algebraic concepts can benefit potential STEM students and ensure their college readiness. A broad range of learning objectives supported by STEM makes it prudent to create such learning experiences. Integrating science and mathematics is also in the center of interest in the mathematics education community. National Council of Teachers of Mathematics [4] has taken the position that observation, experiment, discovery, and conjecture are as much a part of the practice of teaching and learning mathematics as of any natural science and that school mathematics curriculum needs to be representative of that position. STEM activities that offer explorations and learning experiences during which students apply mathematical reasoning to
symbolically decode natural phenomena present a great platform for addressing not only STEM but also explicitly science research recommendations which in return can increase students’ motivation to undertake STEM-related fields of study. Honey [5] contended that science entails learning to express the behavior of natural systems as mathematical models, making this form of integration indispensable to learning science. This study is posited to be an example of such an undertaking. It focuses on enhancing students’ scientific inquiry skills by using algebraic functions and interpreting the limits of the functions in a specific context of motion of a system of two objects.

1.2 Venues of Enhancing Mathematical Reasoning in Physics
The idea of this research reflects recommendations of the physics education community that calls for enhancing the understanding of physical phenomena by mathematical reasoning [6]. In this regard, most of the current developments take the stance that students’ mathematical reasoning can be enriched by improving their conceptual understanding of physics formulas [7], [8]. While this line of methodology supports this recommendation, it initiates mathematical reasoning on algebraic representations, formulas that are not being considered as primary structures in mathematics curricula used to develop students’ algebraic thinking. The main algebraic structures are functions and the tools like taking limits, sketching or finding rates of change are performed on functions. It is hypothesized that functions will better be served to develop students’ mathematical reasoning and integrate them with a scientific inquiry to develop new knowledge. Thus, instead of using formulas to enhance scientific inquiry in this study functions will be applied. The theoretical framework is illustrated in figure 1.

![Figure 1](image-url)

**Figure 1.** Schematic representation of the theoretical framework.

This venue will require a change in perceiving formulas as functions by labeling physical quantities in the formula using a precise language of mathematics thus; outputs, inputs, variables, and constants and consequently analyzing the function using established criteria. It is assumed that this venue will better intertwine students’ algebraic skills with deriving scientific knowledge. Thus, the system’s acceleration will constitute the function output, and the inputs will be represented by the variable masses. The element of transitioning from formula to function will be discussed in details in the sections 2.4 and 2.5.

2. Research Methods
2.1. Research logistics, participants and questions
This undertaking can be classified as a pretest/posttest one group experimental case study [9]. The participants of the study consisted of a group of 25 college level algebra-based physics students. Within this group; 9 were female students and 16 male students. All participants possessed a prior calculus background, and they were familiar with sketching rational functions and evaluating limits of these type of functions for large values of the independent variable. They also possessed a physics background from high school on analyzing the motion of systems of objects. The pretest was assigned within the first week of the course. It was to assess students’ level of awareness that limits are to be considered to quantify the outputs of the formula. The results were not disclosed to students. The instructional unit was delivered within the section the dynamics about a month later. Students took posttest after two weeks from participating in the instructional unit. The posttests results were used to access students’ gain in handling algebraic structures (functions) and tasks to learn about the system
behavior. The pretest/posttest questions were similar except that on the posttest students were given a choice of using algebraic tools to answer the questions. The following emerged as research questions for the study:

(a) Are physics students comfortable with using limits to solve physics problems?
(b) Can limits enhance students’ understanding of natural phenomena?

The first research question was assessed using the pretest results. Posttest results were used to determine if converting formula to a function and introducing limits improves students’ handling the processes of applying limiting analysis and provide more insight into improving their mathematical reasoning.

2.2. Criteria for the study context selection
Analyzing motion of a system of two objects, with or without friction, is a common context in high school and undergraduate physics curricula. Traditional school laboratory exercises exploring such motion involve deriving expressions for the system acceleration, also known as an operational form of the Newton’s second law

\[ \ddot{a} = \frac{\sum F}{m} \quad (1) \]

Upon formulating the formula, students conclude that the acceleration depends directly upon the net force \( \sum F \), and inversely proportional upon the system’s total mass \( m \) [10]. By generating two separate graphs; acceleration versus the net force and acceleration versus the total system’s mass, these conclusions are visualized and confirmed. This approach, albeit commonly used in physics education, embraces two different types of algebraic proportionalities; direct and indirect that might inadvertently diminish the nature of the phenomenon’s cause and effect and perhaps confuse the learners. In fact, the hanging mass \( m_2 \) (see figure 2) contributes to the system mass as inertial and gravitational, and it seems that its effect cannot be fully understood without producing a graph when a high range of the values can be analyzed. Using the system to determine that the acceleration depends directly on the value of the pulling force produced by force of gravity acting on \( m_2 \) might also be misleading because, for a high range of the mass, the graph appears rational not linear. Such obstacles were reported [11]; for instance, students faced difficulties in identifying reasonable forces that affected the acceleration or could not correctly explain their effect on the acceleration. While this study is not to prove that using limits eliminates these obstacles, it is hoped that by using one algebraic structure students will be able to quantify better the motion of the system based on the effects of the limited values of the masses and consequently the forces acting on the system.

2.3 Discussion of applied algebraic tools
While applying limits is common for scientists, using limits in school practice and examining students' ability in applying limits to enhance scientific inquiry are not visible. In this study, I suggest conducting such analysis using a context of two objects connected by a massless string (see figure 1). While the study was conducted with college students, its context is suitable also for advanced high school physics courses. In practice, such transition provides opportunities for deriving valid conclusions especially in cases of extreme or limiting values for which direct laboratory measurements are not possible. Although at first the idea seemed to illustrate applications of limits in science rather than enhancing students’ scientific inquiry, developed instructional unit along with students’ responses has advanced the topic to a promising endeavor worthy of sharing it with physics colleagues. The necessary students’ math background was their familiarity with the techniques of evaluating limits of rational functions at infinity also called as evaluating function end behavior. The operational form of Newton’s second law of motion, \( \ddot{a} = \frac{\sum F}{m} \), represents a rational expression and the idea of taking limits should produce results that can be interpreted in the context of the phenomena. In general, using the
idea of limits should allow for extending both predictions and explanations of this type of motion that according to Redish [12] are essential goals of teaching physics. Quantifications resulting from applying limits is known as limiting case analysis [3]. Typically such analysis does not embrace in the formal processes of evaluating limits considering formulas as functions but assumes these structures [13]. This study is posited to bring forth precise tools of mathematics and apply the techniques of taking limits in a fashion that is consistent with what students learn in their mathematics classes. It is assumed that this approach will help students with transferring their skills and understand the outputs of the processes better.

2.4 Theoretical background of the instructional unit
While initially, the intent of this study was assessing students’ readiness in applying limits in physics, the pretest results prompted to move farther and propose a unit during which the underpinning of applying limits would be discussed with students with an attempt to improve their techniques. In their prior physics education, the participants studied how to construct a formula to find the acceleration of a half-Atwood machine. However, it appeared that enacting it from scratch and showing students that Newton’s laws can transition to more sophisticated algebraic embodiments appeared as a valuable element to be inserted into the unit. With these changes implied, the main purposes of the instructional unit were to (a) pinpoint the techniques of converting the formula to an algebraic function and (b) take the limits of the function and merge mathematical reasoning with the system behavior. The instructional unit was initiated from posting a problem about predicting the magnitude of the system’s acceleration when either of the masses had a zero or infinite magnitude. The problem context along with its both parts (Part 1 and Part 2) also constituted the pretest and posttest questions. The instructor presented Part 1 and discussed the tasks that lead to embracing the process of answering the problem questions using limits.

2.5. Problem Context, Part 1. Consider a system of two blocks (figure 2). The block $m_1$ of 2 kg is placed on a horizontal frictionless surface. The mass $m_2$, with an initial mass also of 2kg, is hanging over the side of the table. However, there are more masses continually being added to $m_2$, eventually increasing its amount to a vast and infinite value.

![Figure 2](image)

**Figure 2.** The system of two objects connected by a massless string.

a) Predict the value of the system acceleration when $m_2$ reaches an infinite value.
b) Formulate an expression for the acceleration of the system using Newton’s 2nd law and find its value when the mass $m_2$ reaches infinite amount. Use limits to support your answer.
2.5.1. Problem Solution. Students recorded their prediction on their notes, and the instructor uncovered the solution processes. The pulley is frictionless, so the motion of the system is driven only by the force of gravity acting on the mass, $m_2$, as shown in figure 2.

![Diagram of pulley system](image)

**Figure 3.** Primary forces acting on the system.

By considering the forward and downward directions of the masses as positive, equations for the net force acting on each object and thus the system was formulated.

$$
m_1a = T$$
$$m_2a = -T + m_2g$$

(2)

Adding the equations by sides and factoring out the symbol of acceleration, $a$, produces $(m_1 + m_2)a = m_2g$ and solving the equation for $a$ results in:

$$a = \frac{m_2g}{m_1 + m_2}$$

(3)

Equation (3) represents the earlier mentioned operational form of Newton’s 2nd law. It allows for computing system acceleration by substituting known mass values for $m_2$ and $m_1$. This iteration process might lead students to reach a correct conclusion. Alternatively, the students might conduct an experimental proof by attaching different masses on the hanger and justifying the rate of change of velocity. Both means, albeit valid, are limited, for example, to available equipment in the physics lab. Therefore, it is converting the formula to a function and approximating the function for a large value of $m_2$ will suit the purpose. The pretest disclosed that the students were unsure how to handle the evaluation of the formula for a large value of $m_2$. Therefore, more attention to this phase of the reasoning was dedicated. First, the independent and dependent variables in Equation (3) needed to be identified. Since the mass $m_2$ is changing, therefore this quantity will constitute the independent variable of the function. To highlight the variable nature of the acceleration and its dependence on the hanging mass $m_2$, we replace $a$ with $a(m_2)$. The magnitude of the mass $m_1$ is said to be constant and equal to 2kg; thus it can be replaced by 2. Formula (3) takes now the form of a rational function:

$$a(m_2) = \frac{m_2g}{2 + m_2}$$

(4)

It is to note that the mass $m_2$ contributes to the system acceleration as gravitational and inertial, thus from a physics point of view its effect on the system acceleration varies depending on its relative value with $m_1$ that can be better understood referring to the function graph (figure 3). Once the formula turns to a function, its graph can be sketched and the limits formally found. The processes will be discussed in 2.5.1.1 and 2.5.1.2.
2.5.1.1 Finding the acceleration from the graph. The function \( a(m_2) = \frac{m_2g}{2+m_2} \) can be sketched and its value for a large \( m_2 \) inferred (see figure. 3). Alternatively, one can find the limit of the function at infinity from its algebraic form. Using both representations should lead to the same conclusion. Referring to the graph, one will notice that when the mass \( m_2 \) is increasing, the acceleration is getting close to 9.8 m/s\(^2\). The limiting value also represents the equation of the horizontal asymptote of the rational function which is depicted by the dashed horizontal line. It is to note that the acceleration of the system will never reach the value of 9.8 m/s\(^2\) because the graph cannot cross the horizontal asymptote at infinity (assuming that the experiment is conducted on the Earth) and this conclusion corresponds with the system behavior.

\[ \lim_{m_2 \to \infty} \frac{m_2g}{2+m_2} = g \]  

**Figure. 3.** Graph of the acceleration function when the hanging mass changes.

The graph in figure 3 is arbitrarily generated for 0kg \( \leq m_1 \leq 35\) kg. The value of 0 kg was chosen to highlight the physical domain of the function \((m_2 \geq 0)\). Evaluating the function for \( m_2 = 0 \), produces zero acceleration which also represents the vertical intercept of the graph. It is to note that the graph provides an effective means to approximate the acceleration for various values of the hanging mass. For instance, when the mass is 5kg, the acceleration is about 7m/s\(^2\). By applying formal techniques of taking the limit of the function at an infinite value of the mass \( m_2 \) a concurrent conclusion can be reached that is discussed in the section that follows.

2.5.1.2 Finding the acceleration using the technique of evaluating the limit. By considering the dominant terms of the numerator and the denominator of the function \((m_2)\) and omitting 2kg as having insignificant value compared to an infinite value of \( m_2 \), one finds that the resulting function value will be close to 9.8 m/s\(^2\) that is illustrated in (5).

\[ a(m_2) = \lim_{m_2 \to \infty} \frac{m_2g}{m_2 + 2} = \lim_{m_2 \to \infty} \frac{m_2g}{m_2} = g \]  

The value of \( g = 9.8 \) m/s\(^2\) represents the equation of horizontal asymptote of the function simultaneously. The limit illustrates a boundary value of the system acceleration which cannot exceed 9.8 m/s\(^2\). It is also to note that the system will never reach that value because the mass of the cart will never be equal to zero. It is also interesting to note that when the mass \( m_2 \to \infty \), its gravitational and inertial effects on the system acceleration are equalized, and they cancel out (assuming that \( m_1 \) remains at 2kg). When \( m_2 \) is comparable to \( m_1 \), the effect of \( m_1 \) on the acceleration of the system cannot be ignored. Moreover, this operation is not allowed by the limiting analysis.

2.6. Problem Context, Part 2. Suppose that the mass \( m_2 \) remains constant (2kg) and the value of the mass \( m_1 \) increases to infinity.
a) Predict the value of the system acceleration
b) Formulate an expression for the acceleration of the system using Newton’s 2nd law and determine its value when the mass \( m_1 \) reaches a very high and infinite value. Use limits to support your answer.

2.6.1 Solution. The phases of the solution process are similar to the prior case due to the fact embracing the physical quantities in the variable notation also resulted in a rational function.

Considering \( a = \frac{m_1 g}{m_1 + m_2} \) and replacing \( m_2 \) by 2 kg produced \( a(m_1) = \frac{2g}{m_1 + 2} \). Suppose that we would like to take the limit of the function for \( m_1 \to \infty \) first. In this case, the mass of 2 kg can be removed from the expression as being insignificant which resulted in evaluating the quotient of \( 2g/m_1 \) for a large \( m_1 \). A formal limit evaluation proceeds in (6).

\[
a(m_1) = \lim_{m_1 \to \infty} \frac{2g}{m_1 + 2} = \lim_{m_1 \to \infty} \frac{2g}{m_1} = 0 \quad (6)
\]

A similar conclusion can also be reached from the graph of \( a(m_1) \) that is illustrated in figure 4.

![Graph of the acceleration function](image)

**Figure 4.** Graph of the acceleration function when the mass of the cart \( m_1 \) takes variable values.

The graph (figure 4) also provides a valid physical interpretation for its vertical intercept. It shows that when mass \( m_1 = 0 \), the acceleration of the system equals in magnitude to the acceleration due to gravity 9.8 m/s\(^2\). The graph interpretation assures the students about its precision in depicting the physical nature of the phenomenon. Likewise, in figure 3, the graph in figure 4 can also be used to find the system acceleration for various masses of the cart and help identify and interpret the horizontal asymptote of the graph that if the fact is \( y=0 \). The horizontal asymptote can be used to conclude that the value of the acceleration will never reach the value of zero, but it will be very close to zero. This conclusion also has a valid physical interpretation; since the surface is frictionless, even though \( m_1 \) is very large, the constant pulling force produced by the force of gravity acting on the 2kg mass will still move the mass. On the other hand, increasing \( m_1 \) constitutes only to increasing system’s inertia that decreases the rate of change of its velocity. As a medium of partial experimental verification and farther conceptualization, a real demonstration can be used.

3. Data Analysis

3.1 Analysis of the pre-test results

Even though the pretest contained more details regarding what process to take to answer the questions, the quality of student’s responses and the percent of correct answers were low.

In Part 1, 20\% of the students (N=5) concluded that the acceleration would be near 9.8 m/s\(^2\). Majority of the students 80\% (N=19) concluded that the acceleration of the system would increase, yet they did not explicate on the boundary. Some verbatim samples of student responses are as follows: The acceleration will be increased to infinity; the acceleration will be very large because there is more...
mass hanging, thereby making the acceleration larger; acceleration will increase because higher mass will increase the force of gravity; it will approach infinity because of the heavier weight pulled; acceleration is infinite; the acceleration would significantly increase; the acceleration will increase quickly. While the physics reasoning is excellent, the students did not attempt to use the algebraic properties of the formula to support their answers. They also did not notice, that the pulling mass has a dual contribution to the system acceleration; inertial and gravitational. The students derived the formula for the system acceleration, however, could not find adequate algebraic tools to evaluate the formula when the mass $m_2$ was increasing. Some students substituted large values for $m_2$ and evaluated the formula, but they did not foresee to use the idea of limits as a valid quantification tool. While all of these students did possess a background on limit evaluation, the background was not sufficient to realize that this problem called for applying the limiting analysis. This assumption prompted augmentation of the instructional unit from its original version [14] and placing more emphasis on highlighting its use when the quantity of interest does take a fixed value.

In Part 2, many of students concluded that the acceleration would be zero. However, the most accurate answer that the acceleration will be near the value zero but will never reach that value was concluded only by 25% (N=6) students. Verbatim samples of student responses are as follows: The acceleration will be positive but never close to zero; the acceleration will be very small because the object would require movement at the same acceleration; the object will have acceleration because of the gravitational field; the acceleration will be zero; the acceleration will decrease.

In both parts of problems, students faced difficulty in supporting their reasoning using the outputs of applied algebraic operations. Moreover, while their physics reasoning was excellent, it seemed that they realized that they were missing an algebraic tool that would confirm their physics reasoning. This was evident from their algebraic manipulations. A lack of paying more attention to the limit interpretation, even after the conduct of the instructional unit, could be accounted for a rare students’ exposure to limit interpretation in their mathematics and physics classes.

3.2 Analysis of the post-test results

On the post-test, the students were asked to conclude the magnitude of acceleration using any means. Such formulated task was to access their comfort of using limiting case analysis.

In Part 1, the results showed an improvement in handling the limit evaluation. The students’ work showed more consistency in identifying constant and variable quantities which lead to correct resulted 54% (N=13) answers. Some students still faced difficulties with identifying the independent variable in the mathematical sense. It was interesting that most students who renamed the independent variable to $x$ computed the limit correctly. These students who did not rename the variables usually stated that the limit is one instead of $g$. In supporting their answers, the majority (about 80%) analyzed the algebraic structure of the function, and the remaining 20% used simple physics reasoning. Some verbatim samples of student responses for Part 1 follow: The acceleration of the system will be $g$ because the two infinity values will cancel out; acceleration goes to infinity because increasing $m_2$ gives a high force of gravity and the system has the same inertia, if $m_2$ increases to infinity, then the acceleration would essentially just be $g$; the acceleration is infinity, and it will continue to increase when $m_2$ increases; the object cannot accelerate faster than $g$; therefore the answer $g$ from the limit makes sense.

In Part 2, the percent of correct answers was high 90% (N=22). Almost all students used the algebraic form and applied the limiting analysis to conclude and comment on their answers. Verbatim samples of student responses are as follows: As $m_1$ approaches $\infty$, the acceleration would be close to zero due to increased denominator, as $m_1 \to \infty$, $a \to \infty$, the larger the mass of $m_1$, the lower the rate of change of acceleration becomes, the acceleration would be very slow, but it would be there due to the presence of $m_2$, the acceleration of the system is $0 \text{m/s}^2$ because as in the limit, the denominator would be infinite where the numerator would become relatively insignificant, and the value would be approaching zero; inertia increases to infinity then the acceleration gets close to zero, $\frac{2g}{a_0}$ is going to be $0$; as $m_1$ increases, the acceleration will be smaller and smaller. A nuance about interpreting function limits emerged from analyzing the phenomenon; the function limit is typically denoted with
an equal sign not as approaching that value. For example, we write that $\lim_{m_2 \to \infty} \frac{m_2 g}{2 + m_2} = g$, not $\lim_{m_2 \to \infty} \frac{m_2 g}{2 + m_2} \rightarrow g$ which would change the verbal interpretation. This notation during the instructional unit was not augmented to assure its consistency with calculus notation, and instead, more emphasis was given to its interpretation. I realized that using arrow instead of the equal sign would more accurately describe the meaning of the limit in the context of the phenomenon.

A summary of pre-test/post-test results is shown in table 1.

Table 1. Summary of Pre-test/Post-test Results

|               | Pre-test | Post-test |
|---------------|----------|-----------|
| Part 1        | 20%      | 54%       |
| Part 2        | 25%      | 90%       |

4. Conclusions
The post-test results showed that the students improved their skills in applying the technique of taking limits to calculate physical quantities. The way they derived the solutions illustrated that they felt more comfortable to follow through the process of identifying variables and taking limits. While on the pretest most of these students did not realize that the idea of limits must be applied to answer the questions, upon being instructed, the students attempted to use the technique and were successful. Their verbal supports showed more consistency and clarity. Did such lesson enhance their general STEM disposition? While questions targeting this objective were not developed, the students were very eager to learn how to quantify physics problems using limits. Observing their genuine interest in knowing how limits can support physics understanding was encouraging to consider creating more contexts for such opportunities.

It seemed that such curriculum enhancement not only benefited their curiosity to learn physics but also equip them with tools that expand their algebraic reasoning skills and develop a more general view on physics. Upon applying limits, their processes of deriving solutions were more consistent with what they had learned in their calculus classes, and they seemed to appreciate the opportunity to integrate that knowledge.

While solving physics formulas with parameters is common in physics, using derived expressions to convert it to a function and learn more about system behavior is still rare which showed up on the pretest. The converted operational formula for acceleration turned into a reach algebraic function whose analysis generated many interesting inferences not possible to reach by viewing it as a pure formula.

Another exciting remark that emerged during the discussion after the posttest was if the system of two masses can be used to learn that the acceleration of the system increases proportionally to the force of gravity acting on the hanging mass. After analyzing the acceleration graph (see figure 3) and its algebraic function, the students, realized this is not possible. The graph appeared linear for small values of the hanging mass, but in fact, the relation for not linear and viewing the graph on its entire domain determined by a broader range of the hanging masses supported the claim.

While this paper presented an analysis of two motion cases, the experimental setup provided opportunities for creating more variations, where the idea of inducing algebraic function could be further explored. For example, having a system of three masses with two pulleys would not only diversify limiting case analysis but also allow for deepening the idea of modeling inertial mass.

Furthermore, by applying differential calculus and taking the first derivative of the acceleration function students could learn that the rate of change of acceleration concerning the hanging mass reaches zero for the substantial value of the mass. Consequently, by taking the second derivative, one could learn that the rate of change of the acceleration is decreasing on the entire function domain (that
can also be concluded by observing the concavity of the acceleration graph). The formulated algebraic function can return many more exciting facts about the behavior of the physical system as compared to its formula counterpart.

The element of further attention is to develop a general technique of converting formulas to functions due to isolated variables to provide more opportunities for the students to experience problem-solving with limiting analysis applied. It seemed that the term function, not often applied in undergraduate physics courses, showed as being pivotal in applying more sophisticated math apparatus. It also appeared that students have a strong association to limits applied to functions, not to formulas, therefore converting formulas to functions, provided means for limit taking. Using students’ knowledge from their math classes directly not by adjusting physics representations is aligned with general STEM objectives calling for unified methods and techniques across disciplines. Students perceive algebraic functions as enabling them to apply limits, whereas formulas do not enact that perception. The post-tests also revealed that students need to practice, perhaps more in their calculus classes the process of finding limits when parameters, not just function variables, are given.

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