Orbit Tomography of Binary Supermassive Black Holes with Very Long Baseline Interferometry

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Abstract

In this work, we study how to infer the orbit of a supermassive black hole binary (SMBHB) by time-dependent measurements with very long baseline interferometry, such as the Event Horizon Telescope (EHT). Assuming a pointlike luminosity image model, we show that with multiple years of observations by EHT, it is possible to recover the SMBHB orbital parameters—eccentricity, (rescaled) semimajor axis, orbital frequency, and orbital angles—from their time-varying visibilities even if the binaries’ orbital periods are a few times longer than the duration of observation. Together with the future gravitational wave detections of resolved sources of SMBHBs with the pulsar timing array, and/or the detections of optical-band light curves, we will be able to further measure the individual mass of the binary, and also determine the Hubble constant if the total mass of the binary is measured through the light curves of the two black holes or measured by alternative methods.

\textit{Unified Astronomy Thesaurus concepts: Radio astronomy (1338)}

1. Introduction

Most galaxies harbor supermassive black holes in their centers. Binaries of supermassive black holes may form as a consequence of mergers of galaxies (Kormendy & Richstone 1995; Kauffmann & Haehnelt 2000; Volonteri et al. 2003; Ferrarese & Ford 2005; Colpi & Dotti 2011; Kormendy & Ho 2013). Understanding the formation and evolution of supermassive black hole binaries (SMBHBs) is essential to revealing the evolutionary histories of galaxies. The evolution of SMBHBs may be classified into several stages (Begelman et al. 1980), depending on their separation and driving mechanisms. At their early stage with wide separations, the dynamical friction is capable of bringing the SMBHBs’ separations down to an order of parsecs within cosmological timescales (see, e.g., Callegari et al. 2011; Mayer 2013; Dosopoulou & Antonini 2017). At the separation of subparsec scales (i.e., $<0.01$ pc), gravitational wave (GW) emission is efficient to take away the energy and angular momentum of the SMBHB, so that they can merge within the Hubble timescale. As there is a gap between these two regimes for parsec-scale separations, it has been a long debate (the “final parsec problem”) whether and how SMBHBs migrate across the gap (Begelman et al. 1980; Colpi 2014). To overcome this final parsec problem, multiple mechanisms have been put forward to explain the efficient orbital damping, e.g., through the interactions of SMBHBs with environmental gas or stars in asymmetric nuclear potentials and on elongated orbits (e.g., see the review by Colpi 2014). To test these ideas, astrophysical observations over SMBHBs at subparsec separations are needed.

Currently, the ongoing observations on SMBHBs include telescopes targeting electromagnetic signals within multiple-frequency bands: radio, optical/infrared, X-ray, and also GW signals. In particular, the GWs emitted by SMBHBs at close separations are promising sources of pulsar timing arrays (PTAs; Hobbs 2013; McLaughlin 2013; Verbiest et al. 2016; Arzoumanian et al. 2020) at the frequency band of nano-Hz, or a period of order of one year. In the future, the space-borne GW detector Laser Interferometer Space Antenna (LISA; Amaro-Seoane et al. 2017) will be able to detect the merger signals of SMBHB coalescence, which are the loudest and most energetic GW events in our universe. Currently, the observational evidence for SMBHBs is all conducted by direct or indirect electromagnetic observations.

There are three known SMBHB systems found by direct electromagnetic imaging, which are identified as two distinct active galactic nuclei (AGNs) with projected separations of tens to thousands of parsec, in the radio, optical, and X-ray wavelengths (Komossa et al. 2003; Rodriguez et al. 2006; Fu et al. 2011). While nearly all the subparsec-SMBHB systems are spatially unresolved, their identification relies on indirect methods, such as the commonly used approach based on the semiperiodicity variation, including the emission-line dynamics (e.g., Bogdanović et al. 2009; Shen & Loeb 2010; Tsalmantza et al. 2011; Eracleous et al. 2012; Decarli et al. 2013; McKernan et al. 2013; Shen et al. 2013; Liu et al. 2014b, 2016), semiperiodic jet structures (e.g., Begelman et al. 1980; Conway & Wrobel 1995), semiperiodic light curves (e.g., D’Orazio et al. 2015; Graham et al. 2015; Kovačević et al. 2019; Saade et al. 2020; Komossa et al. 2021a, 2021b), tidal disruption event light curves (e.g., Liu et al. 2009, 2014a; Stone & Loeb 2011; Coughlin et al. 2017), and orbital motion of an unresolved radio core observed with very long baseline interferometry (VLBI; e.g., Sudou et al. 2003; D’Orazio & Loeb 2018; Breiding et al. 2021). Several SMBHB candidates have been selected with these indirect methods (Valtonen et al. 2008; Bogdanović et al. 2009; Liu et al. 2014a; Graham et al. 2015; Li et al. 2019).

In this work, we focus on the radio-band observation, which has the best chance of spatially resolving SMBHBs. We discuss the question of whether one can fully recover the orbit...
parameters of the binary based on the radio-interferometry measurement. This problem is nontrivial as the orbital parameters contain the eccentricity, semimajor axis, orbital frequency, and various Euler orbital angles—they may contribute to the visibility function with various degrees of degeneracy. For a given observation time, multiple sources with different periods may be simultaneously monitored. It is then interesting to find whether the orbit tomography for systems with periods longer than the observation duration can still be successful. We will study systems in different parameter regimes to answer these questions.

Some of the resolved SMBHBs may have close separations such that their GW emission are detectable by the PTAs. If these golden binary exist, we can combine the multi-messenger data to further determine the individual masses of the binary. Similarly, if the information from other electromagnetic frequency bands is available, e.g., optical light curves, we may further use these golden binaries to independently measure the Hubble constant.

This paper is organized as follows. In Section 2, we calculate the image and the visibility of SMBHB on the sky plane. In Section 3, in three representative examples of SMBHBs, we figure out the posterior distribution of their orbital parameters by doing the Markov-Chain Monte-Carlo simulations, and then we compare the ability of the constraining of SMBHB parameters for possible various-detection conditions. In Section 4, we discuss the multi-messenger applications when combining the image detections proposed in this paper with the future GW detections by PTA (in Section 4.1), as well as the multi-frequency applications when combining the radio image detections with the detections from optical-band light curves of the individuals (in Section 4.2). In the appendix, we give our mathematical proofs of the ways to break parameter degenerations.

In this paper, we use a natural unit with $c = G = 1$.

### 2. Imaging the Supermassive Black Hole Binary

As SMBHs move within a gas-rich environment, electromagnetic radiations in various frequency bands may be sourced from locations such as the circumbinary disk, circumsingle disks, possible jets, etc. It is a highly nontrivial task to model the emission in a given band as a function of the accretion disk conditions and SMBHB orbital parameters, which requires systematic numerical studies that are not currently available. As the first step to investigating the possibility of orbit tomography, we adopt a simple analytical model, assume the emission from a SMBHB is described by two individual pointlike luminosity functions given by

$$ I(r) = I_1 \delta(r - r_1) + I_2 \delta(r - r_2), $$

where $I_1$ and $I_2$ are the intensities of the individuals, $r$ is the sky position in radians, and $r_1$ and $r_2$ are the positions of the two components in the sky plane. This simple analytical model focuses on the emission in the vicinity of individual black holes and neglects emission from the extended regions in the circumbinary disk, as well as possible variation of the luminosity function within orbital timescales. Our simplified image model describing SMBHB as two-point emitters moving along the eccentric and oblique orbit is illustrated in Figure 1.

![Figure 1](image-url)

**Figure 1.** The image and orbit of the supermassive black hole binary. The orbital angels are defined in the reference frame $(X, Y, Z)$ where the X-axis, Y-axis are lying in the sky plane. Here, $\iota$ is the orbital inclination angle, $\omega$ is the periapsis, $\Omega$ is the angle of the longitude of ascending node, and $f$ is the phase of the individuals concerning the pericenter. The redshifted masses of the individuals in the binary are $m_1$ and $m_2$.

The complex visibility of the SMBHB is defined to be the Fourier transform of their sky image,

$$ V(u) = \int I(r) e^{-2\pi i u \cdot d^2 r}, $$

where $u$ is the projected baseline-vector orthogonal to the line of sight and measured in wavelengths. The polar coordinate components for $r$ and $u$ in the sky plane are $(r, \phi)$ and $(u, \varphi)$. After plugging Equation (1) back into Equation (2), the visibility in Equation (2) becomes

$$ V(u) = e^{-2\pi i u \cdot (r_1 - r_2)} (1 + e^{2\pi i u \cdot (r_1 - r_2)}). $$

We shall focus on the visibility-amplitude measurements in this study, which is

$$ |V(u)| = \sqrt{I_1^2 + I_2^2 + 2I_1I_2 \cos(2\pi u \cdot (r_1 - r_2))}. $$

We define $R$ as the projected vector of the binary separation vector $r_{12}'$ in the sky plane, and it is related to $r_1 - r_2$ by the angular diameter distance $L$ as $R = (r_1 - r_2)L$. The separation vector $r_{12}'$ is determined by the Keplerian orbit, with $r_{12}' = a(1-e^2)/(1+e \cos f(t))$, where $e$ and $a$ are the eccentricity and semimajor axis of the binary orbit. The components of $r_{12}'$ in the reference frame $(X, Y, Z)$ can be obtained through the transformation laws of the Euler angels $(\Omega, \iota, \omega)$ by

$$ r_{12}' = [r_{12}'] \\
\begin{pmatrix}
\cos \Omega \cos(f(t) + \omega) - \cos \iota \sin \Omega \sin(f(t) + \omega) \\
\cos \iota \cos \Omega \sin(f(t) + \omega) + \sin \Omega \cos(f(t) + \omega) \\
\sin \iota \sin(f(t) + \omega)
\end{pmatrix}, $$

where $r_{12}'$ is the separation vector in the sky plane.
therefore,
\[
R = \frac{a(1-e^2)}{1+e \cos f(t)} \times \left( \cos \Omega \cos (f(t) + \omega) - \cos \iota \sin \Omega \sin (f(t) + \omega) \right)
\times \left( \cos \iota \cos \Omega \sin (f(t) + \omega) + \sin \Omega \cos (f(t) + \omega) \right).
\]

(6)

With the coordinate expressions, we now rewrite the visibility in Equation (4) as
\[
|V(u)| = \sqrt{h^2 + 2hL \cos(2\pi u \cdot R/L)} = \sqrt{h^2 + 2hL \cos \Phi(t)},
\]
where the phase \(\Phi(t)\) in the cosine function is given by
\[
\Phi(t) = \frac{2\pi(1-e^2)}{1+e \cos f(t)} \left( \frac{u a}{L} \cos \iota \sin(\varphi - \Omega) \sin(f(t) + \omega) + \cos(\varphi - \Omega) \cos(f(t) + \omega) \right).
\]

(8)

The visibility amplitude is a function of time through \(f(t)\), according to the following relations that describe motion within an eccentric Keplerian orbit:
\[
\cos f = \cos g - e \cos g,
\]
\[
\tan \frac{f}{2} = \left( \frac{1 + e}{1 - e} \right)^{1/2} \tan \frac{g}{2},
\]
\[
g - e \sin g = \omega_0 t,
\]
where \(\omega_0 = (m_1 + m_2)^{1/2}(a(1+z))^{-3/2}\) is the orbital frequency of the binary seen by the observer. Notice that the function \(f(t)\) obtained from Equations (9)–(11) implicitly assumes a starting phase \(f(0) = 0\) at the starting time of observation. For generic initial conditions, we will need an initial phase \(f_0\) at the beginning of the observation. For the general case, the function \(f(t)\) in Equation (8) is replaced with \(f(t + f_0)\), where \(f(t_0) = f_0\).

3. Parameter Estimation of the SMBHB Orbit

To recover the orbital description for a SMBHB, there are nine unknown parameters to be determined by the visibility measurement, including the ratio of \(a/L\) (where \(a\) is the proper semimajor axis, and \(L\) is the angular diameter distance), the (redshifted) orbital frequency \(\omega_0\), the intensities \(I_1\) and \(I_2\), the eccentricity \(e\), the inclination angle \(i\), the periastron \(\omega\), the longitude of ascending node \(\Omega\), and the initial phase \(f_0\). In Appendix, we provide a mathematical procedure to show how to obtain the value of these orbital parameters assuming perfect measurement without errors. Although realistic data always comes with measurement uncertainties, this mathematical procedure shows that there is no intrinsic degeneracy between different parameters that prevent the exercise of orbit tomography.

Since the visibility is determined by the projected separation vector \(R\) of the binary, the projected orbital motion of the SMBHB on the celestial sphere can be traced by the measurements of the SMBHB visibility. If the visibilities observed by the two baselines \(u_1\) and \(u_2\) are \(V_1(t)\) and \(V_2(t)\), then, from such observations, in principle, we could obtain
\[
2\pi u R_1 / L = \Phi(\varphi \to \varphi_1),
\]
\[
2\pi u R_2 / L = \Phi(\varphi \to \varphi_2),
\]
with \(\varphi_{1,2}\) associated with the directions of the baselines, according to Equation (7). The information from two baselines is sufficient to determine the projected motion of the binary on the source plane, while additional baselines should provide better constraints on the orbit. In general, assuming independent observations at different observing times and from different baselines, the likelihood function can be written as follows:
\[
L = \prod_{m,n} \frac{1}{\sqrt{2\pi \sigma_{mn}^2}} e^{-\frac{(V_m - V_{mn}(t))^2}{2\sigma_{mn}^2}},
\]

(13)

where \(m\) is the index for baselines, \(n\) is the index for the observation time \(t_n\), \(V_m(t_n)\) is the observed visibility amplitude by the \(m\)th baseline at time \(t_n\), and \(\sigma_{mn}\) is the expected-measurement error bar of the corresponding data point. For the sake of illustration, we assume two orthogonal baselines given by \(\varphi_1 = 0\) and \(\varphi_2 = \pi/2\) for the examples presented below.

3.1. Parameter Estimation Assuming Constant Intensity Model

To illustrate the procedure to recover the orbital parameters, we consider the three examples listed in Table 1, indicating SMBHBs at 1 Gpc distance, with an observed orbital period of 10, 15, and 20 yr respectively. The second and the third examples represent SMBHBs with orbital periods larger than the observation period of the Event Horizon Telescope (EHT). The intensity of the individual black holes in the binary is set to be \(I_1 = 50\) mJy and \(I_2 = 30\) mJy, which are the sample values taken from the low-luminosity AGNs (D’Orazio & Loeb 2018), resulting in a total intensity of 80 mJy. With this total intensity, we set the error bars of the modeled visibility for these sources to be several mJy, as estimated according to the observations from (Breiding et al. 2021). The error bar \(\sigma\) in each observation is sampled from a [0, 8 mJy] uniform distribution (see Figures 2–4 for the details). The observation wavelength is assumed to be the millimeter. The two-dimensionless baselines \(u_1\) and \(u_2\) measured by the millimeter radio wavelength are taken to be \(u_{1,2} = 9.6\) G\(\lambda\). For SMBHBs with greater separations, the orbital period will be too large to generate a sufficient variation in the visibility function within the observation period. For more compact SMBHBs, i.e., the ones with angular separations smaller than 2 m\(\lambda\), the amplitude of variation of the visibility function will be too small to be resolved under the assumed condition of detection. The spatial resolution can be improved for visibility data with a higher signal-to-noise ratio.
or more arrays of baselines as discussed in the rest of this section.

Given the Likelihood function, we apply the Markov-Chain Monte-Carlo method to obtain the posterior distributions of various orbital parameters, assuming flat priors. For the examples listed in Table 1, the posterior distributions are shown in Figures 2–4, assuming a 10 yr observation time with VLBI. During this observation period, we assume a uniform sampling rate of 8 times per year—equivalently 80 times in the 10 yr observation period (so that $n$ ranges from 1 to 80 in Equation (13)). The true evolution of the visibility functions and the assumed error bars of the measurement data are shown in the upper-right corners of Figures 2–4. In these examples, the minimal value of $|V|$ does not reach $|I_1 - I_2|$, as the phase factor $\Phi$ never reaches $\pi$ with the binary separation and distance assumed in Table 1 (for all these cases, $a/L$ is less than $1/u$).

Based on the results shown the Figures 2–4, we find that the orbital parameters of the SMBHBs can be reasonably

\[ V(t) = \sum_{i=1}^{N} \frac{S_i(t)}{d_i^2} \cos(2\Phi(t)) + B(t) \]

where $S_i(t)$ is the visibility function, $d_i$ is the distance, $\Phi(t)$ is the phase, and $B(t)$ is the background. For the example in Figure 2, we assume $e = 0.3, i = \pi/3, \Omega = \pi/4, \omega = \pi/3, f_0 = \pi/2$. The upper-right plot shows the true values and the assumed error bars of the visibility function $|V|$ in this example: the curves represent the underlying value for $V$ (gray for $V_1$ and black for $V_2$), and the points represent the measurement data. The observation time is approximately equal to the orbital period of the SMBHB considered in this example.
constrained with the assumed radio-interferometry measurements. The fitting of the time-varying visibility is the key to recovering these parameters. By comparing the performance of orbit tomography for the three binaries, we conclude the following: (a) for a SMBHB in the example showed in Figure 2 that has an orbital period no more than the observation time, i.e., 10 yr, the orbital frequency $\omega_0$ and eccentricity $e$ that have an error bar within several percent are better constrained, while the flux density is less constrained due to the smaller variation of visibility amplitude; (b) for a moderate binary in Figure 3 that has an orbital period beyond 10 yr but within 20 yr, the ratio $a/L$ is better constrained to within several percent, while the flux densities $I_1$ and $I_2$ are still indistinguishable; (c) for a SMBHB shown in Figure 4 with a larger orbital period (of order 20 yr), the constraint over the orbital parameters are worse compared to the previous two cases, except for the flux densities, since the variation of visibility amplitude is larger; (d) both the inclination angles $\iota$ in these three examples are well constrained to be percent level.

Figure 3. The estimation of the orbital parameters for the SMBHB is considered in our second example in Table 1. The orbital parameters (except for the orbital separation) are assumed to be the same as in Figure 2. The upper-right plot shows the true values and the error bars of the visibility function $|V|$, where the curves represent the true values (gray for $V_1$ and black for $V_2$), and the points represent the measurement data. The observation time is about 2/3 of the orbital period of the SMBHB.
3.2. Parameter Estimation Assuming Time-varying Intensity Model

SMBHBs at subparsec separations are likely inspiraling with a highly eccentric orbit and circularize only briefly before the merger (see, e.g., Iwasawa et al. 2011). In an eccentric orbit, the accretion flows could vary considerably depending on the orbital phase (see, e.g., Sillanpaa et al. 1988). Therefore the constant intensity assumption made in Equation (1) has to be relaxed for eccentric binaries. However, the specific dependence of the intensity on the orbital phase is

\[ I(\phi) = \sum_n [a_n(\epsilon)\cos \phi + b_n(\epsilon)\sin \phi] \] (14)

where the expansion coefficients are unknown, which need to be explored and calibrated with magneto-hydrodynamics (MHD) simulations. In order to address the question of whether orbital tomography is still possible with variable intensities, we assume a fiducial model of time-varying...
The intensities described by

\[
I_1(t) = I_{01} \frac{1 - e^2}{1 - e \cos f(t)} ,
\]

\[
I_2(t) = I_{02} \frac{1 - e^2}{1 - e \cos f(t)} ,
\]

where \(I_{01}\) and \(I_{02}\) are the constant amplitude of the intensities \(I_1\) and \(I_2\), which are given by \(I_{01} = 50\) mJy and \(I_{02} = 30\) mJy.

The posterior distributions of the orbital parameters for a SMBHB similar to the second example in Figure 3 is shown in Figure 5. We can find from Figure 5 that the orbital parameters of the SMBHB are properly recovered when the intensities of the individual SMBHs are fitted with the proper template.

### 3.3. Detectabilities Under Various-detection and Parameter Conditions

We further explore the ability to constrain SMBHB parameters with a different number of baselines, error models, observation schedules, and orbital separations. The dependence of the posterior distributions of the binary parameters on these variations is presented in Figure 6. In the four-baselines case, we assume the directions of the baselines are at \(\phi = 0, \pi/4, \pi/2, 3\pi/4\) respectively. The corresponding distributions (see the third column with the label “eg2-4b,” which represents the second example in Table 1 with four baselines) are significantly narrower than those of the two-base case (“eg2”). The improvement in accuracy can be attributed to the additional information brought by the additional baselines. By comparing the first and the fourth column that we could see, the constraining ability of the parameters of SMBHB in the second example with two baselines and the 8 times observation frequency per year are almost the same as the results constrained by four baselines while with the 4 times observation frequency per year (“eg2-4b-N40”). In a separate example (the second column in Figure 6 with label “eg2-err”), we test the performance of a different visibility error prescription, with \(\sigma^2 = (0.5\sigma)^2 + (0.1|V|)^2\) (where \(\sigma\) is the error bar considered in Figures 2–4), so that the error also increases if the expected value of \(V\) is larger. As the noise fluctuation for the visibility data is generally greater in this prescription, the constraints on the orbital parameters become worse as expected. A similar trend is observed as we increase the waiting time between observations shown in “eg2-4b-N40.”

In the last two examples, we still assume a SMBH binary with a total mass of \(M = 10^7 M_\odot\) at distance 1 Gpc, four baselines, and the same error bar model as considered in Figures 2–4. The binary separations are chosen to be 1.2 \(\mu\) as (or 1.6 yr in period) and 8.4 \(\mu\) as (or 30 yr) respectively. In the lower-separation case, the binary period is much smaller than the observation period. The time-dependent visibility data contains multiple oscillation cycles, but the oscillation amplitude is small because of the smaller \(a/L\). In this case, the orbital frequency has the best relative precision among all the parameters. In the larger separation case, the binary period is three times that of the observation period. This means that the visibility measurement only lasts for a fraction of an oscillation cycle. The corresponding measurement accuracy of orbital frequency is much worse than the low-separation case. The orbital-angle-measurement uncertainties are both significantly larger than the “eg2-4b” case with modest orbital separation.

For much more compact binaries, i.e., the ones detectable by the successor of LISA, Advanced Millihertz Gravitational-wave Observatory (AMIGO; Baibhav et al. 2021), space-based VLBI is required to provide sufficient angular resolution to recover the binary orbit or even the final black hole itself. In this case, a multi-messenger test of General Relativity may be performed, as discussed in Yang (2021).

At this point, it is also worth thinking about how to claim a detection based on the visibility measurement. In principle, if we have multiple (more than two) baselines and the inferred binary parameters are consistent with each other assuming a different combination of baseline, this comparison provides a good indication that the underlying source is a SMBHB. Similar tests can also be performed at different wavelengths within the radio band. However, it will be difficult to distinguish the binary scenario from the case of the two arbitrarily moving blobs within the accretion disk, if the binary period is significantly longer than the observation period. It requires a more detailed study, along with better binary-disk emission models, to assess the appropriate binary-parameter range that allows successful model selection.

### 4. Multi-messenger/Band Detection

In addition to radio-interferometry observations, some of the SMBHBs may be observed in other frequency bands, such as optical/infrared and X-ray. They may also be resolved by PTAs for a suitable range of parameters. We shall discuss two examples to illustrate what further information can be obtained from multi-messenger/multi-band observations.

#### 4.1. Multi-messenger Observation with Gravitational Waves

Both extreme-mass-ratio inspirals and SMBHBs are promising sources for multi-messenger detection with GWs. Extreme-mass-ratio inspirals are mainly observed by LISA in their last stages of the inspiral/merger process, with the GW measurement providing the orbit information and radio signals coming from the common accretion disk and/or the jet (Pan & Yang 2021; Pan et al. 2021). On the other hand, SMBHBs resolvable by ground-based VLBI should have a much wider separation and a much lower frequency. The SMBHBs emitting GWs at the PTA band should have a have negligible frequency evolution in the timescale of years so that the GW alone is insufficient to infer the orbit. Without the chirp signal as commonly seen for ground-based detection, one can only measure an overall amplitude \(A = \mathcal{M}^{5/3}/D_L\) (where \(\mathcal{M}\) is the redshifted chirp mass, and \(D_L\) is the luminosity distance) coupled with inclination and polarization phase angles. Assuming a joint measurement with radio VLBI and PTA, the luminosity distance can be inferred from the redshift of the host galaxy of the SMBHB, and the orbital angles can be measured with VLBI through the varying visibility of the SMBHB images, as discussed in Section 3 (see Figures 2–4), which breaks the degeneracy between the chirp mass, the luminosity distance, and the orbital angles in the strain amplitude of GW. We may obtain the chirp mass of the binary up to measurement uncertainty. Supposing that we detect the GWs of a SMBHBs through PTA with a certain S/N, the uncertainty of the amplitude \(A\) of the system can be estimated.
directly through the Fisher matrix as follows:

\[ \Delta A^2 = \frac{1}{\langle h, h \rangle} = \frac{A^2}{A^2 \langle h, h \rangle} \sim \frac{A^2}{S/N^2}, \]  

(16)

where \( h \) is the waveform of SMBHB, and \( \langle , \rangle \) defines the inner product used in the Fisher matrix. Assuming the measurement uncertainty of the redshift and orbital angles are smaller than the relative uncertainty of the amplitude (i.e., Figures 2–4), the relative uncertainty of the chirp mass would be similar to that of the amplitude. For simplicity, the posterior of the chirp mass \( M_c \) is approximated by a Gaussian distribution

\[ P(M_c) = \frac{1}{\sigma_c \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{M_c - M_c^0}{\sigma_c} \right)^2}, \]  

(17)

where \( M_c^0 \) is the truth value of the chirp mass, and \( \sigma_c \approx M_c^0 / S/N \) is obtained from Equation (16).

On the other hand, through the measurement with EHT, the posterior distribution of the (redshifted) total mass \( M \) of the
binary could be extracted from the joint posterior distribution of $\omega_0$ and $a$ (see Figures 2-4), by the Kepler’s law: $M = \omega_0^2 a (1 + z)^3$, with the angular diameter distance inferred from the redshift.

Provided with the posterior distribution $P(M_c)$ and $P(\omega_0, a)$, and the fact that $M = m_1 + m_2$ and $M_c = m_1^{3/5} m_2^{3/5}/M^{1/3}$, we will try to recover the distributions of individual mass $m_1$ and $m_2$, with $m_1$ defined to be the less massive mass. First we sample the points in $a$, $\omega_0$ and use them to compute the total mass $M$. After that, we sample the points in the chirp mass according to its distribution $P(M_c)$, and then compute individual masses according to $m_1 = \frac{3}{2} M - \frac{1}{2} \sqrt{M^2 - 4M^{1/3} M_c^{5/3}}$ and $m_2 = \frac{1}{2} M + \frac{1}{2} \sqrt{M^2 - 4M^{1/3} M_c^{5/3}}$, from which we remove the samples giving complex numbers for $m_1$ and $m_2$. The statistical distributions of all sampling points give the probability densities $P(m_1)$ and $P(m_2)$. For the system discussed in the example of Figure 3, the results of the posterior distribution of $m_1$ and $m_2$ for a SMBHB that have $S/N = 10$ of PTA-GW detection are shown in Figures 7 and 8, assuming a different underlying mass ratio.

The underlying mass ratio assumed in Figures 7 and 8 are 9:1 and 1:1 respectively, with the same radio-measurement result shown in Figure 3. These assumptions may not be physical as one may expect the luminosity ratio to be correlated with the mass ratio. Nevertheless, for various mass ratios, the distributions for each component mass can be successfully constructed with the joint measurement, which agrees with the underlying injected values within 1$\sigma$–2$\sigma$. We also find that it is easier to separate out $m_1$ and $m_2$ in the first example, as expected.
4.2. Multi-band Observation to Determine the Hubble Constant

Since SMBHBs may be observed simultaneously in different frequency bands of electromagnetic waves, it is instructive to discuss multi-band measurements in this context. In particular, the periodic light curve in the optical band was found in the system PG 1302+102, which is explained as the relativistic Doppler boost modulation on the flux densities of the SMBHB individuals (D’Orazio et al. 2015; Graham et al. 2015). And the periodic variability arising from relativistic Doppler boost is found to be a promising electromagnetic signature to connect with GW detections (Charisi et al. 2022). In this section, we consider the scenario that the orbital velocities of individual black holes in the binary are measured using the modulation of flux densities, generated by the relativistic boost. If a black hole has a velocity of \( v \) and a rest-frame flux density of \( F^0_i \), then the variation of the observed flux density \( F_i \), assuming a general Keplerian orbital motion is

\[
\frac{\Delta F_i}{F_i} = (3 - \alpha) v \left[ e \cos \omega + \cos (f(t + t_0) + \omega) \right] \sin \iota, \tag{18}
\]

where \( v = m_i \mathcal{M}^{-1/2} (a(1 + z)(1 - e^2))^{-1/2} \) \((i = 1, 2) \) and \( \alpha \) is the exponent of the power law that best describes the spectrum in the frequency region of interest. It is usually assumed to be \( \alpha = 1.1 \) as a good proxy for the optical \( V \) band (D’Orazio et al. 2015; Dotti et al. 2022). Here, the eccentricity \( e \), inclination angle \( \iota \), periastron \( \omega \), and the phase angle \( f_0 \) may all be measured through the parameter estimation using the time-dependent visibility, as discussed in Section 3. In addition, measuring the optical light curves gives the instantaneous \( v \). The black hole that has a lower mass with higher velocity will have a larger relativistic boost of the flux density.

The measurement of \( v \) has two advantages when combined with the radio-interferometry measurements. First, based on the velocity data, we can compute the posterior distribution of \( m_1 \mathcal{M}^{-1/2} \), which further determines the probability density functions of individual masses \( m_1 \) and \( m_2 \). Second, as the angular separation of the SMBHB is directly measured through radio interferometry, one can determine the value of the angular diameter distance if the physical separation of the SMBHB is known. Since \( m_1, m_2 \) can be inferred from the optical light curves, and both the orbital frequency and the host galaxy redshift \( z \) are known, we can determine the physical separation \( a \) by using the relation \( \omega_0 = \mathcal{M}^{1/2}(a(1 + z))^{3/2} \). As a result, the angular diameter distance \( L \) can be determined. This can serve as an independent approach to measure the Hubble constant.

In addition to the optical light-curve measurement, there are also alternative ways to measure the mass of black holes within SMBHB, for example using the dynamical mass measurements or the relation between the SMBH mass and its host galaxy properties (see, e.g., Peterson 2014; Schutz & Ma 2016). Provided with mass measured from these alternative methods, we should also be able to determine the Hubble constant, similarly as mentioned above. One difference is that we can in principle detect the optical light curves at cosmological distances (say, about 1 Gpc; Valtonen et al. 2008; D’Orazio et al. 2015; Graham et al. 2015; Komossa et al. 2021a), while these alternative methods only resolve close sources (within about \( L \sim 100 \) Mpc; Schutz & Ma 2016).

5. Discussion and Conclusion

Assuming a point-emitter luminosity distribution, we have shown that the orbital parameters of a SMBHB can be recovered with time-dependent radio VLBI measurements. The orbit tomography is still possible if the observation period is a few times shorter than the period of the binary system. If additional measurement with GWs (using PTA) and/or electromagnetic signals in other frequency bands are available, the component masses of these golden binaries may be separately determined, and the joint multi-band observation may be used to measure the Hubble constant. Exploring the science potential of these golden SMBHBs is likely a fruitful direction for the next-generation EHT.

The source abundance of SMBHBs resolvable by millimeter VLBI has been studied in D’Orazio & Loeb (2018) considering the low-luminosity AGNs with periodic variations. They propose that the number of MBHBs out to redshift \( z \) over the entire sky with binary separation resolvable by VLBI is given by D’Orazio & Loeb (2018)

\[
N_{\text{EHT}} \approx 4\pi \int_0^z \frac{d^3V}{dzd\Omega} \int_{L_{\text{mm}} \text{min}}^{\infty} \frac{d^3N}{dL_{\text{mm}} dV} F(L_{\text{mm}}, z) dL_{\text{mm}} dz, \tag{19}
\]

where, from left to right, the terms refer to the cosmological volume element in a flat universe, a millimeter wavelength AGN-luminosity function (Yuan et al. 2017), and the binary probability distribution function that is proportional to the lifetime \( T \) of the SMBHB. The integration is over both the redshift and the millimeter luminosity \( L_{\text{mm}} \). Since the inspirals of SMBH binaries at separation \( \sim \mathcal{O}(10^4) \) Schwarzschild radii are driven by gravitational radiations (Loeb 2010), the binary lifetime is proportional to \( T \propto a^3 \). Therefore wider binaries are more abundant. In D’Orazio & Loeb (2018), it is assumed that SMBHBs are resolvable if the orbital periods are smaller than the observation period. With this assumption, it is predicted that there will be \( \mathcal{O}(1) - \mathcal{O}(10^2) \) SMBHBs resolvable by EHT with 10 yr observation, while the number increases up to several thousands for 20 yr observation. Interestingly, our MCMC simulations indicate that EHT with 10 yr of observation may resolve SMBHBs even if their orbital periods are of several decades. It means that the number of SMBHBs resolvable by EHT might be several thousands according to the model of D’Orazio & Loeb (2018).

In reality, the radio emission may come from not only the circumbinary disks around individual black holes but also the circumbinary-disk region or even possible jets. As the tidal steam feeding gas onto the circumbinary disks can vary based on the orbital phase, the emission from the vicinity of the individual black holes may also have nontrivial time dependence in orbital timescales. This may happen when the orbit is inclined with respect to the disk so that the gas accretion rate is maximized when black holes cross the disk and minimized when they move away from the disk. For in-disk binaries, if the orbit is highly eccentric, it is also possible to observe significant time-dependent luminosity variation as the black holes move through disk regions with different densities. To achieve a fully working model that applies to general cases for...
orbital tomography, it is necessary to properly describe the functional dependence of luminosity on an orbital phase, with various orbital inclinations, eccentricities, and disk conditions. Systematic calibration and characterization against MHD simulations with SMBHBs moving in accretion disks are needed for this purpose. There are relativistic MHD studies for emissions in other bands (see, e.g., Gold et al. 2014a; Gold 2019; Paschalidis et al. 2021), or binaries with smaller separations (see, e.g., Farris et al. 2012; Gold et al. 2014b) so that they fall into the detection band of LISA for multi-messenger observations. For radio emissions, these problems have not been explored. In a representative example, we also show that the orbit tomography of SMBHB is still possible even with time-varying intensities, and this needs a proper template of the individual intensities. Our work answers the question of whether orbital tomography is possible in the ideal setting, with much left to be studied for general and realistic cases.

A famous SMBHB measured in multi-bands is the blazar OJ 287 (other SMBHB candidates could be seen in Table 1 of Komossa et al. 2021a). The optical light curve of OJ 287 displays quasi-periodic bright flares with a period of 12 yr (Valtonen et al. 2006), which is explained by the collision of the secondary black hole with the disk of the primary black hole (Lehto & Valtonen 1996; Sundelius et al. 1997; Dey et al. 2018, 2021; Laine et al. 2020). The primary black hole in OJ 287 shows a bright radio jet, while the secondary is expected to have a radio jet only after it travels through the various impact sites (Dey et al. 2021; Gómez et al. 2022) since it is moving on an oblique orbit with respect to the primary disk. Therefore this system has a highly time-dependent $I_2$ function that violates the assumption of the simplified model in Equation (1). A better model should assign a larger luminosity at the disk crossings.

Although the uncertainty of the secondary radio luminosity for OJ 287-like SMBHB makes it a poor target to study the orbital tomography with the method developed in this paper, our MCMC simulations and the SMBHB event rate estimated in D’Orazio & Loeb (2018) indicate that the number of the SMBHB candidates in low-luminosity AGN, which are resolvable by EHT within the resolution limit, should be of the order of several hundreds (at least) in the universe. This large abundance enlarges the chance of finding in-disk SMBHB cases.

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Appendix

Recover the Orbital Parameters Assuming Perfect Detection

In this section, we outline a mathematical procedure of resolving the orbital parameters $I_1$, $I_2$, $\omega_0$, $e$, $i$, $\omega$, $\Omega$, and $a/L$, assuming a perfect observation. First, for a SMBHB that has a relatively large separation such that $\cos \Phi(t)$ could reach to the full range of $[-1, 1]$ (which is roughly equivalent to $au/L > 1$), we may compute the values of $I_1$ and $I_2$ through the maximal and minimal values of $|V|$.

Second, we could recover the phase $\Phi(t)$ in Equation (7) from the observed visibility $V(t)$ through

$$\Phi(t) = \arccos \left[ \frac{|V(t)| - I_1^2 - I_2^2}{2I_1I_2} \right]. \quad (20)$$

From Equations (8)–(11), we notice that $\Phi$ is a smooth function of $t$, so that we can obtain $\Phi(t)$ by solving Equation (20) with the continuity condition of the derivative, up to a sign of $\pm$. In Figure 9, we show an example of $\Phi(t)$ obtained using Equation (20) ($\Phi_{\text{observe}}$), the recovered $\Phi$ ($\Phi_{\text{recover1}} = - \Phi_{\text{recover2}}$), and the real $\Phi$ ($\Phi_{\text{real}}$).

Third, we rewrite $\Phi(t)$ in Equation (8) as

$$\Phi(t) = 2\pi u(1 - e^2) \frac{a}{L} \left[ \cos i \sin \omega \sin(\varphi - \Omega) + \cos \omega \cos(\varphi - \Omega) \left( \frac{\cos f(t)}{1 + e \cos f(t)} \right) + \left( \frac{\sin f(t)}{1 + e \cos f(t)} \right) \right], \quad (21)$$

and using the fact that (see, e.g., Equations (4.87), (4.88) in Maggiore 2007)

$$\frac{(1 - e^2) \cos f(t)}{1 + e \cos f(t)} = - \frac{3}{2} e + \sum_{n=1}^{\infty} \frac{1}{n} (J_{n-1}(ne) - J_{n+1}(ne)) \cos n\omega_0 t, \quad (22)$$

Figure 9. The case of SMBHB image visibility when the separation is large enough such that $\cos \Phi(t)$ could reach to the full range of $[-1, 1]$, and the reconstruction of the function $\Phi(t)$. 

we Fourier decompose \( \Phi(t) \) as follows,

\[
\Phi(t) = 2\pi u a (C_1 f_0 + C_2 f_0 + C_3 f_0 + C_4 f_0 + \ldots)
\]

where

\[
C_1 = \cos i \sin \sin(\varphi - \Omega) + \cos\omega \cos(\varphi - \Omega),
\]

\[
C_2 = \cos i \sin \sin(\varphi - \Omega) - \sin\omega \cos(\varphi - \Omega),
\]

\[
a_0 = -\frac{3}{2} e,
\]

\[
a_n = \frac{1}{n}(J_{n-1}(ne) - J_{n+1}(ne)), \quad n = 1, 2, \ldots
\]

\[
b_n = \frac{\sqrt{1 - e^2}}{n} (J_{n-1}(ne) + J_{n+1}(ne)), \quad n = 1, 2, \ldots
\]

(25)

Considering the observing time starting from an arbitrary initial phase \( f_0 = f(t_0) \), the phase function in Equation (24) is

\[
\Phi(t) = 2\pi u a \left[ (C_1 f_0 + C_2 f_0 + C_3 f_0 + C_4 f_0 + \ldots) \right]
\]

With the above preparation, we now recover the orbital parameters as follows:

The orbital frequency \( \omega_0 \). We obtain \( \omega_0 \) through the variation period of the phase \( \phi(t) \) of \( R \). Assuming \( \Phi_\parallel \) and \( \Phi_\perp \) are two phase functions of two visibilities observed from two orthogonal baselines \( u_\parallel \) and \( u_\perp \), then the phase \( \phi(t) \) of \( R \) could be obtained by

\[
\phi(t) = \arg \left[ \frac{\Phi_\parallel}{\Phi_\parallel^2 + \Phi_\perp^2 + i \Phi_\parallel^2 + \Phi_\perp^2} \right].
\]

(27)

The time render \( \phi \) to range a full circle is the orbital period \( 2\pi/\omega_0 \), which gives \( \omega_0 \).

Eccentricity \( e \) and initial phase \( f_0 \). We integrate \( \Phi(t) \) in Equation (26) by

\[
C_n = \int_0^{2\pi/\omega_0} \Phi(t) \cos n\omega_0 t dt = 2\pi u a \frac{\pi}{\omega_0} (C_1 a_n \cos n\omega_0 t_0
\]

\[
+ C_2 b_n \sin n\omega_0 t_0).
\]

(28)

and

\[
S_n = \int_0^{2\pi/\omega_0} \Phi(t) \sin n\omega_0 t dt = 2\pi u a \frac{\pi}{\omega_0} (C_2 b_n \cos n\omega_0 t_0
\]

\[
- C_1 a_n \sin n\omega_0 t_0).
\]

Combining Equations (28) and (29), we have

\[
C_1 = 2\pi u a \frac{C_m \cos m\omega_0 t_0 - S_n \sin m\omega_0 t_0}{a_n},
\]

\[
C_2 = 2\pi u a \frac{(C_n + S_m \cos m\omega_0 t_0) \sin m\omega_0 t_0}{b_n},
\]

where \( n \geq 1 \). Now replacing \( n \) with \( m(= n) \), we have

\[
C_1 = 2\pi u a \frac{C_m \cos m\omega_0 t_0 - S_m \sin m\omega_0 t_0}{a_m},
\]

\[
C_2 = 2\pi u a \frac{(C_m + S_m \cos m\omega_0 t_0) \sin m\omega_0 t_0}{b_m},
\]

By equaling Equations (30) to (32) and (31) to (33), we get two independent equations for \( e \) and \( t_0 \), which can be solved accordingly.

Orbital angles (inclination), \( \omega \) (periapsis), and \( \Omega \) (longitude of ascending node). Since we have obtained \( f_0 \) in the last step, to simplify the calculations, we adapt the starting time such that \( f(t) = 0 \), or \( t_0 = 0 \). We now define the coefficients \( C_1 \) and \( C_2 \) in Equation (24) for \( \Phi_\parallel \) and \( \Phi_\perp \) by

\[
C_1 = -\cos i \sin \sin \Omega + \cos \omega \cos \Omega,
\]

\[
C_2 = -\cos i \cos \sin \Omega - \sin \omega \cos \Omega,
\]

\[
C_3 = \cos i \sin \cos \Omega + \cos \sin \Omega,
\]

\[
C_4 = \cos i \cos \cos \Omega - \sin \omega \sin \Omega,
\]

(34)

and we integrate \( \Phi_\parallel \) and \( \Phi_\perp \) by

\[
\int_0^{2\pi/\omega_0} \Phi_\parallel \cos n\omega_0 t dt = 2\pi u a \frac{\pi}{\omega_0} (C_1 a_n \frac{\pi}{\omega_0}),
\]

\[
= 2\pi u a \frac{\pi}{\omega_0} a_n (\cos i \sin \sin \Omega + \cos \omega \cos \Omega),
\]

(35a)

\[
\int_0^{2\pi/\omega_0} \Phi_\parallel \sin n\omega_0 t dt = 2\pi u a \frac{\pi}{\omega_0} (C_2 b_n \frac{\pi}{\omega_0}),
\]

\[
= 2\pi u a \frac{\pi}{\omega_0} b_n (\cos i \cos \sin \Omega - \sin \omega \cos \Omega),
\]

(35b)

\[
\int_0^{2\pi/\omega_0} \Phi_\parallel \cos n\omega_0 t dt = 2\pi u a \frac{\pi}{\omega_0} (C_1 a_n \frac{\pi}{\omega_0}),
\]

\[
= 2\pi u a \frac{\pi}{\omega_0} a_n (\cos i \sin \sin \Omega + \cos \omega \cos \Omega),
\]

(35c)

\[
\int_0^{2\pi/\omega_0} \Phi_\parallel \sin n\omega_0 t dt = 2\pi u a \frac{\pi}{\omega_0} (C_2 b_n \frac{\pi}{\omega_0}),
\]

\[
= 2\pi u a \frac{\pi}{\omega_0} b_n (\cos i \cos \sin \Omega - \sin \omega \sin \Omega),
\]

(35d)
we define

\[ A_n = \int_{0}^{2\pi/\omega} \phi_n \sin n\omega_0 \, dt, \]
\[ B_n = \int_{0}^{2\pi/\omega} \phi_n \cos n\omega_0 \, dt, \]
\[ C_n = \int_{0}^{2\pi/\omega} \phi_n \sin n\omega_0 \, dt, \]
\[ D_n = \int_{0}^{2\pi/\omega} \phi_n \cos n\omega_0 \, dt. \]  

(36)

Therefore by combining Equations 35(a) and (b), 35(c) and (d), we have

\[ \cos \iota = \cot \Omega \frac{a_n C_n \cot \omega + b_n B_n}{a_n C_n - b_n B_n \cot \omega}, \]  

(37a)

\[ \cos \iota = \tan \Omega \frac{a_n A_n \cot \omega + b_n D_n}{a_n A_n - b_n D_n \cot \omega}. \]  

(37b)

Similarly, by combining Equations 35(a) and (c), 35(b) and (d), we have

\[ \cos \iota = \cot \omega \frac{a_n D_n - a_n B_n \tan \omega}{a_n B_n + a_n D_n \tan \omega}, \]  

(38a)

\[ \cos \iota = -\tan \omega \frac{b_n A_n - b_n C_n \tan \omega}{b_n C_n + b_n A_n \tan \omega}, \]  

(38b)

and by combining Equations 35(a) and (d), 35(b) and (c), we have

\[ \cos \iota = \frac{a_n A_n + b_n B_n \tan \omega \tan \Omega}{a_n A_n \tan \omega \tan \Omega + b_n B_n}, \]  

(39a)

\[ \cos \iota = -\frac{a_n C_n + b_n D_n \tan \omega \cot \omega}{a_n C_n \tan \omega \cot \omega + b_n D_n}. \]  

(39b)

At this point, by equaling Equations 37(a) and (b), 38(a) and (b), we arrive at

\[ \tan \Omega = \pm \sqrt{\frac{(a_n A_n \cot \omega + b_n B_n)(-a_n A_n + b_n D_n \cot \omega)}{(a_n C_n - b_n B_n \cot \omega)(a_n A_n \cot \omega + b_n D_n)}}, \]  

(40a)

\[ \tan \omega = \pm \sqrt{\frac{(-a_n D_n - a_n B_n \tan \omega)(b_n C_n + b_n A_n \tan \omega)}{(a_n B_n + a_n D_n \tan \omega)(b_n A_n - b_n D_n \tan \omega)}}. \]  

(40b)

we can obtain the values of \( \omega \) and \( \Omega \) by solving Equations 40(a) and (b) given values of \( a_n, b_n, A_n, B_n, C_n, \) and \( D_n \). There are constraints to these values. First, they are restricted to be real, and second, they must satisfy the equations from 37(a) to 39(b), and the range of \( \cos \iota \) is between \((-1, 1)\).

Ratio of \( a/L \). Finally, we can compute \( a/L \) by integrating \( \Phi \) with either \( \sin n\omega_0 t \) or \( \cos n\omega_0 t \), i.e., by solving

\[ \Phi_n \cos n\omega_0 t \, dt = \frac{2\pi a}{L} C_n a_n \frac{\pi}{\omega_0}. \]  

(41)
