SECTION I. Theoretical research in mathematics.

SOME PROPERTIES OF THE LATTICE OF F-CLOSED RIGHT IDEALS

Abstract: Throughout this paper $R$ is a unitary associative ring and $f$ is an injective ring endomorphism of $R$. In the present article, we introduce the notion of the lattice $\text{Lat}(R,f)$ of all $f$-closed right ideals of $R$ with some special operation instead of the intersection operation. The paper is devoted to the study of this lattice. In particular, we investigate the interrelationship between the lattice of all $f$-closed right ideals of $R$ and the lattice of right ideals of the Cohn-Jordan extension $A$. We obtained some results in this direction.

In Theorem 1 we give necessary and sufficient conditions, in terms of the lattice $\text{Lat}(R,f)$, for the Cohn-Jordan extension $A$ to be a right Artinian ring. This theorem implies in particular that $A$ is right Artinian provided that $R$ is right Artinian.

Theorem 2 is a structural theorem and states that a ring $R$ with a bounded length of chains of the right $f$-closed ideals is embeddable in a semisimple Artinian ring. The authors’ original proof is based on the Cohn-Jordan extension. The Cohn-Jordan extensions were first introduced in [8] for the study of skew polynomial rings constructed by means of a ring endomorphism. Five open questions are formulated.

Key words: lattice, composition length, right Artinian rings

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Introduction

Throughout this paper all rings are associative. In what follows let $R$ be a ring and $f$ be an injective ring endomorphism of $R$. Recall that David Alan Jordan introduced in [8] the construction of the smallest ring $A$ containing $R$ such that every endomorphism $f$ of $R$ to an automorphism of $A$ (see also [2, 10]). More precisely, let $\text{Aut}(R,f)$ be a ring, containing $R$ and $\tilde{f}$ be an automorphism of $A$ that extends the endomorphism $f$. Then the ring $A$ together with the automorphism $\tilde{f}$ is called the Cohn-Jordan extension of the ring $R$ with endomorphism $f$, if each element $a$ of $A$ can be presented as $a = f^{-n}(r)$, where $r \in R$ and $n$ is some positive integer.

Materials and Methods

Using the construction of a direct limit one can verify that this extension is unique. Let us consider a countable number of copies $R_i$ of the ring $R$ labeled by nonnegative integers $i$ and natural isomorphisms $\varepsilon_i: R \rightarrow R_i$. Given a pair of indexes $(m, n)$ with $m \leq n$, the mapping $f_{m,n}: R_m \rightarrow R_n$ is defined by $f_{m,n} = \varepsilon_n \circ f^{n-m} \circ \varepsilon_m^{-1}$. Then the equality $f_{m,n} = f_{m,k} \circ f_{k,n}$ holds for all $k$ such that $m \leq k \leq n$. Therefore, there is a direct limit $A(R,f) = \lim_{\rightarrow}(R_m f_{m,n}: m, n \geq 0)$.

One can check that the mapping defined by $\tilde{f}: \varepsilon_i(r) \rightarrow \varepsilon_{i+n}(r)$, where $i \geq 0, r \in R$, is a correctly defined automorphism of $A(R,f)$ and the
of all elements $x$ of closed right ideals $N$ of the form $x^n r x^{-n}$ is a ring containing $R$. Furthermore, the inner automorphism $f : x^n r x^{-n} \mapsto x^{-1} r x n^{-1}$ of $A$ is an extension of the endomorphism $f$. Moreover, $A = \bigcup_{n \geq 0} f^{-n}(R)$. Cohn-Jordan extensions are studied and used for different purposes in scientific papers [9, 11].

Throughout the sequel, let $A$ together with $\bar{f}$ denote the Cohn-Jordan extension of the ring $R$ and its injective endomorphism $f$.

**Definition 1.** A right ideal $I$ of $R$ is said to be $f$-closed (see [3, 7]), if

$$I = \bigcup_{n=1}^{\infty} f^{-n}(f^n(I) R).$$

One can check, that a right ideal $I$ of $R$ is $f$-closed if and only if $I = I A \cap R$. It implies that any $f$-closed right ideal $I$ of $R$ has the form $I = MA \cap R$ for some available right ideal $M$ of $A$. Conversely, all the right ideals of this kind are $f$-closed.

An ideal $N$ of $R$ is called an $f$-ideal if $f^{-1}(N) = N$ (see [1, 5]).

Let us consider the lattice $\text{Lat}(R, f)$ of all $f$-closed right ideals of $R$ supplied the following operations:

1. $B \wedge C = B \cap C$;
2. $B \vee C = \bigcup_{n \geq 0} f^{-n}(f^n(B) R + f^n(C) R)$.

The result of the first operation is the largest $f$-closed right ideal contained in the $f$-closed right ideals $B$ and $C$. The result of the second operation is the smallest $f$-closed right ideal containing both right ideals $B$ and $C$.

**Remark a.)** If $B$ and $C$ are $f$-closed right ideals of $R$, then the following two equalities hold:

- $f^{-n}(f^n(B) R \cap f^n(C) R) \subseteq f^{-n}(f^n(B) R) = B$
- $f^{-n}(f^n(B) R \cap f^n(C) R) \subseteq f^{-n}(f^n(C) R) = C$.

Therefore, we need not to describe the operation $B \wedge C$ in the same way as the operation $B \vee C$, because $\bigcup_{n \geq 0} f^{-n}(f^n(B) R \cap f^n(C) R) = B \wedge C$.

b. The following relation holds:

$$B \vee C = (B A + C A) \cap R.$$

Recall that, the submodules of some right module $M_R$ over a ring $R$, partially ordered by inclusion, form a modular lattice. In particular, the lattice of right ideals of some ring is a modular lattice. This means that the lattice satisfies the following condition called “Modular law”: if $B$, $C$ and $D$ are submodules of a module $M$ over a ring $R$ and $B \subseteq C$, then $(C \cap D) + B = C \cap (D + B)$.

**Proposition 1.** Let $B$, $C$ and $D$ are $f$-closed right ideals of $R$ with $B \subseteq C$. Then $B \vee (C \cap D) \subseteq C \cap (B \vee D)$.

**Proof.** First we show that “$B \vee (C \cap D) \subseteq C \cap (B \vee D)$”.

Let $r \in B \vee (C \cap D) = (B A + (C \cap D) A) \cap R$. Then $r = \sum b_i a_i + \sum x_j a_j$, where $b_i \in B$, $x_j \in C \cap D$, $a_i$, $a_j \in A$. Observe, that $\sum b_i a_i \in BA \subseteq CA$ and $\sum x_j a_j \in CA$. Hence $r \in CA \cap R = C$. Since $b_i \in B$, $x_j \in D$, we have $r \in (B A + D A) \cap R = B \vee D$. Therefore, $r \in C \cap (B \vee D)$.

Next we show that “$C \cap (B \vee D) \subseteq (B A + (C \cap D) A) \cap R$”. To prove this inclusion observe that $BA \subseteq CA$ and by modular law we obtain that

$c \in (B \vee D) \cap CA \cap (B A + D A) = BA + (C \cap D) A \subseteq CA$. QED.

**Corollary 1.** If $B \vee (C \cap D) = (B A + (C \cap D) A) \cap R$ for all $f$-closed right ideals of $R$, then the lattice $\text{Lat}(R, f)$ of all $f$-closed right ideals of $R$ is modular.

**Lemma 1.** If $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_d$ is a strictly ascending chain of right ideals of $A$ of the length $d$, then $R$ must have strictly ascending chain of $f$-closed right ideals of length $d$.

**Proof.** Choose elements $m_i \in M_i \setminus M_{i-1}$ ($i = 1, 2, \ldots, d$). By Definition 1 $m_i \in f^{-n}(R)$ for some non-negative integers $n_1, n_2, \ldots, n_d$. Let $n$ be the largest of these integer numbers. Then $b_i = f^{-n}(m_i) \in R$ for all $i = 1, 2, \ldots, d$ and right ideals $B_i = f^{-n}(M_i)$ form the strictly ascending chain $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_d$. Moreover, $b_i \in B_i \cap R$ and $b_i \notin B_{i-1} \cap R$ for all $i = 1, 2, \ldots, d$. Hence the chain of $f$-closed right ideals of $R$

$$B_0 \cap R \subseteq B_1 \cap R \subseteq B_2 \cap R \subseteq \cdots \subseteq B_d \cap R$$

is strictly ascending. But this chain has length $d$. QED.

**Lemma 2.** Let $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_d$ be a strictly ascending chain of $f$-closed right ideals of $R$ of length $d$. Then $B_0 A \subseteq B_1 A \subseteq B_2 A \subseteq \cdots \subseteq B_d A$ is a strictly ascending chain of right ideals of $A$ of the same length $d$.

**Proof.** If the relation $B_{i-1} A = B_i A$ were satisfied at some point in the second chain, then we would have

$$B_{i-1} A \subseteq B_i A \cap R = B_i A \cap R = B_i$$

by virtue of the $f$-closedness of the right ideals $B_{i-1}$ and $B_i$. But the last equality contradicts the condition of the lemma. QED.
Theorem 1. The following conditions (1) and (2) are equivalent:
(1) $A$ is right Artinian;
(2) there exists a non-negative integer $d$ such that all strictly ascending chains of $f$-closed right ideals of $R$ have length at most $d$.

Proof. "$(1) \Rightarrow (2)$". Let $A$ be right Artinian. Then by Hopkins–Levitki theorem $A$ is also right Noetherian and by Jordan-Holder theorem (see [6], Theorem 4.10, P. 44) $A$ has finite composition length $d$ (as a right module over itself). If $\text{Lat}(R, f)$ contained a strictly ascending chain of $f$-closed right ideals of length more than $d$, then by Lemma 2 the ring $A$ would contain a chain of right ideals of length more than $d$. This leads to a contradiction. Therefore, all strictly ascending chains of $f$-closed right ideals of $R$ have length at most $d$.

"$(2) \Rightarrow (1)$". Suppose that condition (2) holds. Then Lemma 1 shows that lengths of all strictly ascending chains of right ideals of $A$ do not exceed $d$. It follows that $A$ is right Artinian of length at most $d$. QED.

Proposition 2. Let be an endomorphism of $S$ and $N$ be an $F$-ideal of $S$. Suppose that $\text{Ker} F \subseteq N$. Then $F : S \rightarrow S$ induces the endomorphism $f : S/N \rightarrow S/N$ such that $f(s + N) = F(s) + N$ for all $s \in S$. In addition, the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{F^n} & S \\
\downarrow{\pi} & & \downarrow{\pi} \\
R & \xrightarrow{f^n} & R
\end{array}
\]

is commutative in the following sense:

a) $\pi \circ F^n(s) = f^n \circ \pi(s)$ for all positive integer $n$ and all $s \in S$;

b) if $Y$ is an ideal of $S$ and $N \subseteq Y$, then $\pi(F^{-n}(Y)) = f^{-n}(\pi(Y))$.

Proof. a). $f^n \circ \pi(s) = f^n(s + N) = F^n(s) + N = \pi \circ F^n(s)$.

Check equality b):

$\pi(\overline{f^{-n}(Y)}) = \{x + N \in R : f(x) \in Y\} =
= \{x + N \in R : f(x + N) \in \pi(Y)\} =
= \{x + N \in R : f(x + N) \in \pi(Y)\} = f^{-n}(\pi(Y))$.

Let $S$ be a ring and $N$ be a prime ideal of $S$. QED.

Lemma 3. Let $S$ be a ring satisfying ascending chain condition on right annihilators. Suppose that every nil-subring of $R$ is nilpotent. Let $F$ be an endomorphism of $S$ with $\text{Ker} F \subseteq N$. Then $f^{-1}(N) = N$.

For a proof we refer on [2; 4].

Theorem 2. Let $F$ be an endomorphism of $S$ and $\text{Ker} F \subseteq N$. Suppose that $d$ there exists a non-negative integer $d$ such that all strictly ascending chains of $F$-closed right ideals of $S$ have length at most $d$. Then the quotient-ring $R = S/N$ can be embedded in a product of finitely many matrix rings over division rings $D_i$.

Proof. The right annihilator of a set in the ring $S$ is the intersection of $S$ and the right annihilator of this set in the Cohn-Jordan extension $A = A(S, F)$, i.e.

\[ r_S(M) = S \cap r_{A(S, F)}(M). \]

It follows that all right annihilators in the ring $S$ are $F$-closed. Hence, by Theorem 1, the ring $S$ is a subring of the right Artinian ring, and every nil subring of an Artinian ring is nilpotent. Therefore, every nil subring of $S$ is nilpotent.

Let $P$ be a prime radical of $A(S, F)$ and $\overline{F}$ be an automorphism of $A(S, F)$ extending $F$. Then $P$ is a nilpotent ideal and, consequently, $P \cap \overline{F}^{-n}(S) \subseteq \text{rad}(\overline{F}^{-n}(S)) = \overline{F}^{-n}(N)$.

Thus $\bigcup_{n=0}^{\infty} \overline{F}^{-n}(N)$. It implies $N \subseteq P$.

Moreover, since $P$ is a nilpotent ideal of $S$, it follows that $P \cap S \subseteq N$. Therefore, $P \cap S = N$. By Proposition 2 the last equality shows that the map $s + N \rightarrow s + P$ is an embedding of the quotient-ring $R = S/N$ in the semisimple Artinian ring $A(S, F)/P$. To complete the proof of the theorem, it remains to note that the ring $A(S, F)/P$ is isomorphic to a finite direct product of complete matrix rings over the division rings by Weddeburn-Artin theorem (see [6], § 61, Theorem 5.16). QED.

Proposition 2. A right ideal $L$ of $A$ is essential in $A$ if and only if for each nonzero element $r \in R$ and for every nonnegative integer $n$, there is a nonnegative integer $m$ such that $f^n(r)R \cap \overline{f}^{n+m}(L) \neq 0$.

Proof. Let $L$ be an essential right ideal of the ring $A$, let $0 \neq r \in R$ and let $n$ be a nonnegative integer. Set $a = \overline{f}^{-n}(r)$. Since $aA(L) \neq 0$, there is a number $m \geq 0$ such that $a \overline{f}^{-n-m}(R) \cap L \neq 0$.

Applying the automorphism $\overline{f}^{n+m}$ to the last inequality, the demanded inequality $f^n(r)R \cap \overline{f}^{n+m}(L) \neq 0$ follows.

Suppose now that for each nonzero element $r \in R$ and for any nonnegative integer $n$ there is a nonnegative integer $m$ such that $f^n(r)R \cap \overline{f}^{n+m}(L) \neq 0$. Every element $a \in A$ can be represented in the form $a = \overline{f}^{-n}(r)$ where $r \in R$ and $n \geq 0$. Applying the automorphism $\overline{f}^{-n-m}$ to the inequality $f^n(r)R \cap \overline{f}^{n+m}(L) \neq 0$, we get that $a \overline{f}^{-n-m}(R) \cap L \neq 0$. QED.
Impact Factor:

| Country | Impact Factor |
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| ICV (Poland) | 6.630 |
| PIIF (India) | 1.940 |
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Conclusion

Here are some problems which will probably be useful for magistrates and graduate students.

Open problems:

1. Give necessary and sufficient conditions on $R$ and $f$ for the lattice $\text{Lat}(R, f)$ to be modular. Give some examples demonstrating that these conditions are essential.

2. If $\text{Lat}(R, f)$ satisfies the descending chain condition, then does $A$ need to be right Artinian?

3. Suppose that $\text{Lat}(R, f)$ contains some chain of length $d$, and all strictly ascending chains of $f$-closed right ideals of $R$ have length at most $d$. Is it true that all maximal strictly ascending chains of $f$-closed right ideals have length $d$?

4. What is the relationship between the essential elements of the lattice $\text{Lat}(R, f)$ and the essential right ideals of the Cohn-Jordan extension?

5. Find a necessary and sufficient condition for the Cohn-Jordan extension $A(R, f)$ to be a right Goldie ring.

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