VARIETIES ASSOCIATED TO LINEAR OPERATORS

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Abstract. We introduce and study the notion of affine varieties associated to ordered bases and establish Galois connection between the power set of $\mathbb{A}^n_\mathbb{K}$ and the power set of $\mathbb{K}[x_1, ..., x_n]$, and then induce a Galois correspondence. We generalize the idea by defining affine varieties associated to linear operators. We produce Hilberts Nullstellensatz version for such varieties and show that there is a 1−1 correspondence between this kind of varieties in $\mathbb{A}^n_\mathbb{K}$ and the “usual” affine varieties in $\mathbb{A}^n_\mathbb{K}$. We prove that the “usual” affine varieties forms a skeleton for the category of all affine varieties associated to linear operators, and hence they are equivalent categories.

1. Introduction

The concept of affine varieties forms the base stone for the algebraic geometry theory. In this paper, we mix ordered bases and affine varieties (which we call here the “usual” affine varieties) as main ingredients to produce a new type of affine varieties (which we call affine varieties associated to bases). Then we generalize this concept by involving linear operators to obtain what we call affine varieties associated to linear operators.

To formulate our problem, we might need to recall some basic concepts. Let $\mathbb{K}$ be a field. The affine space of dimension $n$ over $\mathbb{K}$ is defined by

$$\mathbb{A}^n_\mathbb{K} := \mathbb{K}^n = \{ (a_1, ..., a_n) : a_i \in \mathbb{K}, \forall i \in \{1, ..., n\} \} \text{ [10] p. 2}.$$ 

The zero locus of a subset $S \subseteq \mathbb{K}[x_1, ..., x_n]$ is the set

$$V(S) = \{ a \in \mathbb{A}^n_\mathbb{K} : f(a) = 0 \text{ for all } f \in S \} \text{ [10] p. 2}.$$ 

Throughout this paper, we will ignore any kind of a structure on $\mathbb{A}^n_\mathbb{K}$, and we will simply think about $\mathbb{A}^n_\mathbb{K} = \mathbb{K}^n$ as a set. A subset $X \subseteq \mathbb{A}^n_\mathbb{K}$ is called an affine variety if $X = V(S)$ for some subset $S \subseteq \mathbb{K}[x_1, ..., x_n]$ [10] p. 20.

An affine variety $X$ is called irreducible if it is nonempty and there is no decomposition $X = X_1 \cup X_2$ ($X_1, X_2 \subseteq X$) into proper affine varieties subsets $X_1$ and $X_2$. Otherwise $X$ is called reducible [10] p. 74. Let $X$ be a subset of $\mathbb{A}^n_\mathbb{K}$. The set

$$I(X) = \{ f \in \mathbb{K}[x_1, ..., x_n] : f(a) = 0 \text{ for all } a \in X \}$$

is called the ideal of $X$ [15] p. 11.

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locally be given by the quotient of two polynomial functions, i.e. for any \( x \in X \), there exists an open neighborhood \( U_x \) of \( x \) in \( X \) and \( f, g \in \mathbb{K}[x_1, ..., x_n] \) such that for any \( u \in U_x \) we have

\[
g(u) \neq 0 \quad \text{and} \quad \varphi(u) = \frac{f(u)}{g(u)} \in \mathbb{K}[\mathbb{L}].
\]

We write \( \mathcal{O}(X) \) to be the set of regular functions on \( X \). (By convention, \( \mathcal{O}(\emptyset) = 0 \).) Let \( X \subseteq \mathbb{A}^n_{\mathbb{K}} \) and \( Y \subseteq \mathbb{A}^n_{\mathbb{K}} \) be affine varieties. A map \( f : X \to Y \) is a morphism (of varieties) if \( f_1, ..., f_n \in \mathcal{O}(X) \) \([14]\).

Our problem can be formulated as follows. Let \( V \) be a finite dimensional vector space over \( \mathbb{K} \) with \( \dim V = n \), and let \( B = (v_1, ..., v_n) \) is an ordered basis for \( V \) over \( \mathbb{K} \) (thus, we could think of \( B \) as an ordered basis for \( \mathbb{K}^n \) over \( \mathbb{K} \)). Let \( S \subseteq \mathbb{A}^n_{\mathbb{K}} \) and \( X \subseteq \mathbb{K}[x_1, ..., x_n] \). Consider the sets

\[
V_{B,j}(S) := \{a = (a_1, ..., a_n) \in \mathbb{A}^n_{\mathbb{K}} : f(\sum_{i=1}^n a_iv_i) = 0 \text{ for all } f \in S \}
\]

and

\[
I_{B,j}(X) := \{f \in \mathbb{K}[x_1, ..., x_n] : f(\sum_{i=1}^n a_iv_i) = 0 \text{ for all } (a_1, ..., a_n) \in X \}.
\]

We start with studying the pair \((V_{B,j}, I_{B,j})\) and show that it forms a Galois connection, induces a Galois correspondence and produces a bridge between algebra and geometry. Note that we have the following \( 1 - 1 \) correspondences

\[
[n] = \{1, ..., n\} \leftrightarrow \{V_{B,j} : 1 \leq j \leq n\}
\]

\[
\quad \quad \quad \quad \quad j \leftrightarrow V_{B,j}
\]

and

\[
[n] = \{1, ..., n\} \leftrightarrow \{I_{B,j} : 1 \leq j \leq n\}
\]

\[
\quad \quad \quad \quad \quad j \leftrightarrow I_{B,j}.
\]

It turns out that, one might see faces or simplices of \([n]\) as a collection of subsets of \( \{V_{B,j} : 1 \leq j \leq n\} \) (or a collection of subsets of \( \{I_{B,j} : 1 \leq j \leq n\} \)).

We can generalize the above idea by considering the pair \((V_L, I_L)\), where

\[
V_L(S) = \{a = (a_1, ..., a_n) \in \mathbb{A}^n_{\mathbb{K}} : f(L(a)) = 0 \text{ for all } f \in S \}
\]

and

\[
I_L(X) = \{f \in \mathbb{K}[x_1, ..., x_n] : f(L(a)) = 0 \text{ for all } a \in X \},
\]

for an arbitrary linear operator \( L : \mathbb{K}^n \to \mathbb{K}^n \). Recall that squarefree monomial ideals can be determined by simplicial complexes, and each subset \( \tau \subseteq [n] \) can be identified with its squarefree vector in \( \{0, 1\}^n \), which has entry 1 in the \( i \)-th position when \( i \in \tau \), and 0 in all other entries. By using this convention, we can write \( x^\tau = \prod_{i \in \tau} x_i \) \([13]\) p. 5).

Following \([13]\), one might give the following generalization for some basic notions of monomial algebra. An \( L\)-monomial in \( \mathbb{K}[x_1, ..., x_n] \) is a product \( x^{L,a} = \prod_{i=1}^n p_i(L(x_1, ..., x_n))^{a_i} \) for a vector \( a = (a_1, ..., a_n) \in \mathbb{N}^n \) of nonnegative integers, where \( p_i \) is the \( i \)-th projection map. An ideal \( J \subseteq \mathbb{K}[x_1, ..., x_n] \) is called an \( L\)-monomial ideal if it is generated by \( L\)-monomials. An \( L\)-monomial \( x^{L,a} \) is \( L\)-squarefree if every coordinate of \( a \) is 0 or 1. An ideal is \( L\)-squarefree if it is generated by \( L\)-squarefree monomials. Every \( \tau \subseteq [n] \) can be written as a complement \( \bar{\sigma} := [n] \setminus \sigma \) of some simplex \( \sigma \). We call the ideal \( Q^{\tau,L} := \langle p_j(L(x_1, ..., x_n)) : j \in \tau > as the monomial prime ideal associated to \( \tau \) and \( L \). The set

\[
V_{\Delta,L} := V_L(\prod_{\tau \in \Delta} Q^{\tau,L})
\]
(where $\bar{\tau} := [n] \setminus \tau$) is said to be the \textbf{Stanley-Reisner variety of the simplicial complex $\Delta$ associated to $L$}. If we write $x^{L, \tau} = \prod_{j \in \tau} p_j(L(x_1, ..., x_n))$, then the \textbf{Stanley-Reisner ideal of the simplicial complex $\Delta$ associated to $L$} is the ideal:

$$\mathcal{I}_{\Delta, L} := < x^{L, \tau} : \tau \notin \Delta > .$$

We will briefly study the Stanley-Reisner ideals of simplicial complexes associated to linear operators, and we will explicitly calculate them in terms of irreducible factors. In addition, we will show that the “usual” affine varieties forms a skeleton for the category of all affine varieties associated to linear operators, and this gives rise to an equivalence of categories between them.

Throughout this paper (unless otherwise specified), $\mathbb{K}$ is an algebraically closed field, $V$ is a finite dimensional vector space over $\mathbb{K}$ with $\dim V = n$, $L$ is a linear operator of $V$ (and hence of $\mathbb{K}^n$), $B = (v_1, ..., v_n)$ is an ordered basis for $V$ over $\mathbb{K}$ and $1 \leq j \leq n$.

2. Varieties Associated to Ordered Bases

An ordered basis for a finite dimensional vector space $V$ over $\mathbb{K}$ is an ordered $n$-tuple of vectors $B = (v_1, ..., v_n)$ such that the set $\{v_1, ..., v_n\}$ is a basis for $V$ [3 p. 292]. The order here is important; bases are written in a specific order. Thus, the ordered basis, for example, $(v_1, ..., v_n)$ is different from the ordered basis $(v_2, v_1, v_3, ..., v_n)$. For any $v_i \in B$, we write $v_i = (v_{i1}, ..., v_{in}) \in V \cong \mathbb{K}^n$. Note that $\sum_{i=1}^{j} a_i v_i = (\sum_{t=1}^{j} a_t v_{1t}, ..., \sum_{t=1}^{j} a_t v_{nt})$.

\textbf{Definition 2.1.} Let $\mathbb{K}$ be a field.

1. For a subset $S \subseteq \mathbb{K}[x_1, ..., x_n]$ and a positive integer $1 \leq j \leq n$, we call the set

$$V_{B,j}(S) = \{a = (a_1, ..., a_n) \in \mathbb{A}^n_\mathbb{K} : f(\sum_{i=1}^{j} a_i v_i) = 0 \text{ for all } f \in S\}$$

the $j$th zero locus of $S$ associated to the ordered basis $B$.

2. A subset $X \subseteq \mathbb{A}^n_\mathbb{K}$ is called a $j$th affine variety associated to the ordered basis $B$ if $X = V_{B,j}(S)$ for some subset $S \subseteq \mathbb{K}[x_1, ..., x_n]$.

\textbf{Remark 2.2.}

1. If $S = \{f_1, ..., f_t\}$ is a finite set, we will write $V_{B,j}(S) = V_{B,j}(\{f_1, ..., f_t\})$ also as $V_{B,j}(f_1, ..., f_t)$.

2. Affine $n$-space $\mathbb{A}^n_\mathbb{K}$ itself is an affine variety associated to $B$, since $\mathbb{A}^n_\mathbb{K} = V_{B,j}(0)$ for any positive integer $1 \leq j \leq n$. Similarly, for any positive integer $1 \leq j \leq n$, the empty set $\emptyset = V_{B,j}(1)$ is an affine variety associated to $B$.

3. Any single point in $\mathbb{A}^n_\mathbb{K}$ is an affine variety associated to $B$ since we have $(q_1, ..., q_n) = V_{B,j}(x_1 - \sum_{t=1}^{j} q_tv_{1t}, ..., x_n - \sum_{t=1}^{j} q_tv_{nt})$.

4. Let $B = (e_1, ..., e_n)$ be the standard ordered basis for $V$; that is,

$$e_1 = (1, 0, ..., 0), ..., e_n = (0, ..., 0, 1).$$
Then,

\[ V_{B,j}(S) = \{ a = (a_1, ..., a_n) \in \mathbb{A}^n_K : f(\sum_{i=1}^j a_i e_i) = 0 \text{ for all } f \in S \} \]

\[ = \{ (a_1, ..., a_n) \in \mathbb{A}^n_K : f(a_1, ..., a_j, 0, ..., 0) = 0 \text{ for all } f \in S \}. \]

In particular, if \( j = n \), then we have:

\[ V_{B,n}(S) = \{ (a_1, ..., a_n) \in \mathbb{A}^n_K : f(a_1, ..., a_n) = 0 \text{ for all } f \in S \} = V(S), \]

the “usual” affine variety of \( S \). Thus, the usual affine varieties in \( \mathbb{A}^n_K \) can be seen as a particular case of affine varieties associated to the standard ordered basis in \( \mathbb{A}^n_K \).

(5) Any polynomial ring \( K[x_1, ..., x_n] \) over a field \( K \) is Noetherian, and hence every ideal in \( K[x_1, ..., x_n] \) is finitely generated.

(6) As in the “usual” affine varieties (see [8] for example), if \( X = V_{B,j}(S) \) be an affine variety associated to \( B \), then \( (S) = (f_1, ..., f_m) \) for some \( f_1, ..., f_m \in S \), and hence \( X = V_{B,j}(S) = V_{B,j}(f_1, ..., f_m) \). Therefore, every affine variety associated \( B \) is a zero locus of finitely many polynomials.

(7) The affine varieties in \( \mathbb{A}^1_K \) associated to \( B \) has the same classification as the one of the usual affine varieties. Thus, the affine varieties in \( \mathbb{A}^1_K \) associated to ordered basis are exactly the finite sets and \( \mathbb{A}^1_K \) itself.

Example 2.3.

(1) Let \( B = ((1, 2), (1, 3)) \) and \( S = (x^2 - 5y^2 - 1) \subseteq \mathbb{R}[x, y] \) be the ideal generated by the polynomial \( x^2 - 5y^2 - 1 \). Then \( B \) is an ordered basis for \( \mathbb{R}^2 \) [17, p. 107], and the corresponding affine varieties associated to \( B \) are given by:

\[ V_{B,2}(S) = ((x + y)^2 - 5(2x + 3y)^2 - 1) \subseteq \mathbb{A}^2_{\mathbb{R}}, \]

\[ V_{B,1}(S) = (x^2 - 5(2x)^2 - 1) = \{1\} \subseteq \mathbb{A}^2_{\mathbb{R}}. \]

They can be shown in the following figure:

![Graphs of affine varieties](image1)

(2) Let \( B = ((1, 0, 0), (0, 1, 1), (0, 0, 1)) \) and \( S = (x^2 + y^2 + z^2 - 1) \subseteq \mathbb{R}[x, y, z] \) be the ideal generated by the polynomial \( x^2 + y^2 + z^2 - 1 \). Then \( B \) is an ordered basis for \( \mathbb{R}^3 \) and the corresponding affine varieties associated to \( B \) are given by:

![Graphs of affine varieties](image2)
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\[ V_{B,3}(S) = (x^2 + y^2 + z^2 - 1) \subseteq A_{\mathbb{R}}^3, \]
\[ V_{B,2}(S) = x^2 + y^2 - 1 \subseteq A_{\mathbb{R}}^3, \]
\[ V_{B,1}(S) = (x^2 - 1) \subseteq A_{\mathbb{R}}^3. \]

The following figure shows the corresponding affine varieties associated to \( B \).

3) Let \( B = ((1, 0, 0), (0, 1, 1), (0, 0, 1)) \) and \( S = (x^8 + y^3 + z^8 - 1) \subseteq \mathbb{R}[x, y, z] \) be the ideal generated by the polynomial \( x^8 + y^3 + z^8 - 1 \). Then \( B \) is an ordered basis for \( \mathbb{R}^3 \). As seen below, the corresponding affine varieties associated to \( B \) are given by:

\[ V_{B,3}(S) = (x^8 + y^3 + z^8 - 1) \subseteq A_{\mathbb{R}}^3, \]
\[ V_{B,2}(S) = (x^8 + y^3 - 1) \subseteq A_{\mathbb{R}}^3, \]
\[ V_{B,1}(S) = (x^8 - 1) \subseteq A_{\mathbb{R}}^3. \]
(4) Let \( B = ((1, 2, 1), (2, 3, 1), (-1, 2, -3)) \) and \( S = (x^2 - 2y^2 + 3z^2 - 1) \subseteq \mathbb{R}[x, y, z] \) be the ideal generated by the polynomial \( x^2 - 2y^2 + 3z^2 - 1 \). Then \( B \) is an ordered basis for \( \mathbb{R}^3 \), and as shown below, the corresponding affine varieties associated to \( B \) are given by:

\[
\begin{align*}
V_{B,3}(S) &= ((x + 2y - z)^2 - 2(2x + 3y + 2z)^2 + 3(x + y - 3z)^2 - 1) \subseteq \mathbb{A}^3_{\mathbb{R}}, \\
V_{B,2}(S) &= ((x + 2y)^2 - 2(2x + 3y)^2 + 3(x + y)^2 - 1) \subseteq \mathbb{A}^3_{\mathbb{R}}, \\
V_{B,1}(S) &= (x^2 - 2(2x)^2 + 3x^2 - 1) = \{ \} \subseteq \mathbb{A}^3_{\mathbb{R}}.
\end{align*}
\]

From the above example, one might note that every variety associated to \( B \) can be seen as a usual variety. This observation can be stated in the following lemma.

**Lemma 2.4.** Let \( J = (f_1, ..., f_n) \subseteq \mathbb{K}[x_1, ..., x_n] \). Then there are polynomials \( g_1, ..., g_n \in \mathbb{K}[x_1, ..., x_n] \) with \( V_{B,j}(J) = V(g_1, ..., g_n) \).

**Proof.** We have

\[
V_{B,j}(J) = \{ a = (a_1, ..., a_n) \in \mathbb{A}^n_{\mathbb{K}} : f_t(\sum_{i=1}^j a_i v_i) = 0 \text{ for all } t = 1, ..., n \}.
\]

For all \( t = 1, ..., n \), let \( g_t(x_1, ..., x_n) := f_t(\sum_{i=1}^j x_i v_i) \). Then \( g_t \in \mathbb{K}[x_1, ..., x_n] \) for every \( t = 1, ..., n \), and we obtain

\[
V_{B,j}(J) = \{ a \in \mathbb{A}^n_{\mathbb{K}} : g_t(a) = 0 \text{ for all } t = 1, ..., n \} = V(g_1, ..., g_n).
\]

**Remark 2.5.** Let \( B = (v_1, ..., v_n) \) be an ordered basis for \( \mathbb{V} \).

(1) Let \( \mathbb{L}_{B,j} : \mathbb{K}^n \to \mathbb{K}^n \) be a map defined by \( \mathbb{L}_{B,j}((a_1, ..., a_n)) = \sum_{i=1}^j a_i v_i \) for any \( (a_1, ..., a_n) \in \mathbb{K}^n \). Then \( \mathbb{L}_{B,j} \) is a linear operator which is isomorphism if \( j = n \). Nevertheless, \( \mathbb{L}_{B,j} \) needs not be one-to-one nor onto. For example, if \( \mathbb{K} = \mathbb{R} \) and \( B = (e_1, e_2, e_3) \) be the standard ordered basis for \( \mathbb{R}^3 \), then \( \mathbb{L}_{B,2} : \mathbb{K}^3 \to \mathbb{K}^3 \) is neither one-to-one nor onto.

The above discussion implies that the \( j \)th affine locus of \( S \) associated to the ordered
basis $B$ can be given as follows:

$$V_{B,j}(S) = \{a = (a_1, \ldots, a_n) \in \mathbb{A}^n_K : f(\mathbb{L}_{B,j}(a)) = 0 \text{ for all } f \in S\}.$$ 

Thus, every affine locus map $V_{B,j}$ gives rise to a linear operator $\mathbb{L}_{B,j} : \mathbb{K}^n \to \mathbb{K}^n$ and the converse is true as well. It turns out that we have the following 1–1 correspondence:

\[
\begin{array}{c}
\text{affine locus maps } V_{B,j} \text{ associated to } B \\
\longleftarrow 1-1 \longleftarrow \text{linear operators } \mathbb{L}_{B,j} : \mathbb{K}^n \to \mathbb{K}^n \\
\longleftarrow 1-1 \longleftarrow \text{matrices of the linear operators } \mathbb{L}_{B,j} \text{ w.r.t the ordered bases } B & \& B
\end{array}
\]

Lemma 2.6. (Properties of $V_{B,j}(-)$). Let $S, S_1, S_2 \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Then

1. If $S_1 \subseteq S_2$, then $V_{B,j}(S_1) \supseteq V_{B,j}(S_2)$.
2. $V_{B,j}(S_1) \bigcup V_{B,j}(S_2) = V_{B,j}(S_1 \cap S_2)$, where as usual we set $S_1 S_2 := \{fg : f \in S_1, \ g \in S_2\}$.
3. If $\mathcal{J}$ is any index set and $S_\alpha \subseteq \mathbb{K}[x_1, \ldots, x_n] \forall \alpha \in \mathcal{J}$, then $\bigcap_{\alpha \in \mathcal{J}} V_{B,j}(S_\alpha) = V_{B,j}(\bigcup_{\alpha \in \mathcal{J}} S_\alpha) = V_{B,j}(\sum_{\alpha \in \mathcal{J}} S_\alpha)$. In particular, finite unions and arbitrary intersections of affine varieties associated to $B$ are again affine varieties associated to $B$.
4. $V_{B,j}(\sqrt{S}) = V_{B,j}(S)$

Proof.

1. If $(a_1, \ldots, a_n) \in V_{B,j}(S_2)$, i.e., $f(\sum_{i=1}^n a_i v_i) = 0$ for all $f \in S_2$, then in particular $f(\sum_{i=1}^n a_i v_i) = 0$ for all $f \in S_1$.
2. “$\subseteq$” If $(a_1, \ldots, a_n) \in V_{B,j}(S_1) \bigcup V_{B,j}(S_2)$, then $f(\sum_{i=1}^n a_i v_i) = 0$ for all $f \in S_1$ or $g(\sum_{i=1}^n a_i v_i) = 0$ for all $g \in S_2$. In any case, this means that $(fg)(\sum_{i=1}^n a_i v_i) = 0$ for all $f \in S_1$ and $g \in S_2$, i.e. that $x \in V_{B,j}(S_1 S_2)$.
3. “$\supseteq$” If $(a_1, \ldots, a_n) \notin V_{B,j}(S_1) \bigcup V_{B,j}(S_2)$, i.e., $(a_1, \ldots, a_n) \notin V_{B,j}(S_1)$ and $(a_1, \ldots, a_n) \notin V_{B,j}(S_2)$, then there are $f \in S_1$ and $g \in S_2$ with $f(\sum_{i=1}^n a_i v_i) \neq 0$ and $g(\sum_{i=1}^n a_i v_i) \neq 0$. Then $(fg)(\sum_{i=1}^n a_i v_i) \neq 0$, and hence $(a_1, \ldots, a_n) \notin V_{B,j}(S_1 S_2)$.
4. We have $(a_1, \ldots, a_n) \in \bigcap_{\alpha \in \mathcal{J}} V_{B,j}(S_\alpha)$ if and only if $f(\sum_{i=1}^n a_i v_i) = 0$ for all $f \in S_\alpha$ and for all $\alpha \in \mathcal{J}$, which is the case if and only if $(a_1, \ldots, a_n) \in V_{B,j}(\bigcup_{\alpha \in \mathcal{J}} S_\alpha)$.
5. The inclusion “$\subseteq$” follows directly from part (1) since $S \subseteq \sqrt{S}$. For the other inclusion, let $(a_1, \ldots, a_n) \in V_{B,j}(S)$ and $f \in \sqrt{S}$. Then $f^t \in S$ for some $t \in \mathbb{N}$, so that $f^t(\sum_{i=1}^n a_i v_i) = 0$. Thus, $f(\sum_{i=1}^n a_i v_i) = 0$. This means that $(a_1, \ldots, a_n) \in V_{B,j}(\sqrt{S})$. 

\[ \Box \]

Example 2.7.

1. Let $B = ((1,0,0), (0,1,1), (0,0,1))$ and $S_1 = ((x - y)^2 - (z + 1)^2)$, $S_2 = ((x - y) + (z + 1)) \subseteq \mathbb{R}[x, y, z]$ be the ideal generated by the polynomial $(x - y)^2 - (z + 1)^2$ and $(x - y) + (z + 1)$ respectively. Then $B$ is an ordered basis for $\mathbb{K}^3$ and $S_1 \subseteq S_2$. While the corresponding affine varieties of $S_1$ associated to $B$ are given by:
\[ V_{B,3}(S_1) = ((x - y)^2 - (z + 1)^2) \subseteq \mathbb{A}^3_{\mathbb{R}}, \]
\[ V_{B,2}(S_1) = ((x - y)^2 - 1) \subseteq \mathbb{A}^3_{\mathbb{R}}, \]
\[ V_{B,1}(S_1) = (x^2 - 1) \subseteq \mathbb{A}^3_{\mathbb{R}}, \]
the corresponding affine varieties of \( S_2 \) associated to \( B \) are given by
\[ V_{B,3}(S_2) = ((x - y) + (z + 1)) \subseteq \mathbb{A}^3_{\mathbb{R}}, \]
\[ V_{B,2}(S_2) = ((x - y) + 1) \subseteq \mathbb{A}^3_{\mathbb{R}}, \]
\[ V_{B,1}(S_2) = (x + 1) \subseteq \mathbb{A}^3_{\mathbb{R}}. \]

Compatible to what the previous lemma implies, the following figure clearly shows that \( V_{B,j}(S_1) \supseteq V_{B,j}(S_2) \).

(2) Consider the ordered basis \( B = ((1, 0, 0), (1, -1, 1), (0, 1, 0)) \) for \( \mathbb{R}^3 \), and let \( S_1 = S_1 = ((x + y)^2z^2 - 1), \ S_2 = ((x + y)z - 1) \subseteq \mathbb{R}[x, y, z] \) be the ideals generated by the polynomials \((x + y)^2z^2 - 1\) and \((x + y)z - 1\) respectively. Again, it is obvious that \( S_1 \subseteq S_2 \). The corresponding affine varieties of \( S_1 \) associated to \( B \) are given by:
\[ V_{B,3}(S_1) = ((x + z)^2y^2 - 1) \subseteq \mathbb{A}^3_{\mathbb{R}}, \]
\[ V_{B,2}(S_1) = (x^2y^2 - 1) \subseteq \mathbb{A}^3_{\mathbb{R}}, \]
\[ V_{B,1}(S_1) = (x^2 - 1) \subseteq \mathbb{A}^3_{\mathbb{R}}, \]

On the other hand, the corresponding affine varieties of \( S_1 \) associated to \( B \) are given by
\[ V_{B,3}(S_1) = ((x + z)y - 1) \subseteq \mathbb{A}^3_{\mathbb{R}}, \]
\[ V_{B,2}(S_1) = (xy - 1) \subseteq \mathbb{A}^3_{\mathbb{R}}, \]
\[ V_{B,1}(S_1) = (x - 1) \subseteq A^3_\mathbb{R}. \]

We have the following figure (which shows that \( V_{B,j}(S_1) \supseteq V_{B,j}(S_2) \) for any \( 1 \leq j \leq n \)).

Definition 2.8. (Ideal of a subset of \( A^n_\mathbb{R} \)). Let \( X \subseteq A^n_\mathbb{R} \). Then

\[
I_{B,j}(X) := \{ f \in \mathbb{K}[x_1, \ldots, x_n] : f(\sum_{i=1}^{j} a_iv_i) = 0 \text{ for all } (a_1, \ldots, a_n) \in X \}
\]

is called the \( j \text{-th ideal of } X \text{ associated to } B. \)

Remark 2.9.

(1) Let \( B = (e_1, \ldots, e_n) \) be the standard ordered basis for \( V \); that is,

\[
e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1).
\]

Then,

\[
I_{B,j}(X) = \{ f \in \mathbb{K}[x_1, \ldots, x_n] : f(\sum_{i=1}^{j} a_i e_i) = 0 \text{ for all } (a_1, \ldots, a_n) \in X \}
= \{ f \in \mathbb{K}[x_1, \ldots, x_n] : f(a_1, \ldots, a_j, 0, \ldots, 0) = 0 \text{ for all } (a_1, \ldots, a_n) \in X \}.
\]

In particular, if \( j = n \), then we have:

\[
I_{B,j}(X) = \{ f \in \mathbb{K}[x_1, \ldots, x_n] : f((a_1, \ldots, a_n)) = 0 \text{ for all } (a_1, \ldots, a_n) \in X \},
\]
the usual ideal of $X$ in $\mathbb{K}[x_1, ..., x_n]$. Thus, the usual ideals of subsets in $\mathbb{K}[x_1, ..., x_n]$ can be seen as ideals of subsets in $\mathbb{K}[x_1, ..., x_n]$ associated to the standard ordered basis.

(2) Note that $I_{B,j}(X)$ is always a radical ideal: If $f^t \in I_{B,j}(X)$ for some $f \in \mathbb{K}[x_1, ..., x_n]$ and $t \in \mathbb{N}$, then $f^t(\sum_{i=1}^{j} a_i v_i) = 0$ for all $(a_1, ..., a_n) \in X$, and hence $f(\sum_{i=1}^{j} a_i v_i) = 0$ for all $(a_1, ..., a_n) \in X$ (since the ring $\mathbb{K}[x_1, ..., x_n]$ is reduced). Thus, $f \in I_{B,j}(X)$.

Investigating the basic properties of the maps $I_{B,j}$ implies the following:

**Proposition 2.10.** (Properties of $I_{B,j}(-)$). Let $X_1, X_2 \subseteq \mathbb{A}^n_{\mathbb{K}}$. Then

1. $I_{B,j}(\emptyset) = \mathbb{K}[x_1, ..., x_n]$.
2. For any point $(q_1, ..., q_n) \in \mathbb{A}^n_{\mathbb{K}}$, we have
   \[ I_{B,j}\{(q_1, ..., q_n)\} = (x_1 - \sum_{t=1}^{j} q_i v_1, ..., x_n - \sum_{t=1}^{j} q_i v_n). \]
3. If $X_1 \subseteq X_2$, then $I_{B,j}(X_1) \supseteq I_{B,j}(X_2)$.
4. $I_{B,j}(S_1 \cup S_2) = I_{B,j}(S_1) \cap I_{B,j}(S_2)$.
5. $I_{B,j}(\mathbb{A}^n_{\mathbb{K}}) = \{0\}$.

**Remark 2.11.** By using Remark (2.5), the $j$th affine ideal of $X \subseteq \mathbb{K}[x_1, ..., x_n]$ associated to the ordered basis $B$ can be given as follows:

\[ I_{B,j}(X) = \{ f \in \mathbb{K}[x_1, ..., x_n] : f(L_{B,j}(a)) = 0 \text{ for all } a = (a_1, ..., a_n) \in X \}. \]

Thus, every map $I_{B,j}$ corresponds to a linear operator $L_{B,j} : \mathbb{K}^n \rightarrow \mathbb{K}^n$ and vice versa. Consequently, we have the following 1–1 correspondence:

\[ \left\{ \begin{align*} \text{the maps } I_{B,j} \text{ associated to } B \end{align*} \right\} \xleftarrow{1-1} \left\{ \begin{align*} \text{the linear operators } L_{B,j} : \mathbb{K}^n \rightarrow \mathbb{K}^n \end{align*} \right\} \xrightarrow{1-1} \left\{ \begin{align*} \text{matrices of the linear operators } L_{B,j} \text{ w.r.t the ordered bases } B & \& B \end{align*} \right\}. \]

**Proposition 2.12.** Let $X \subseteq \mathbb{A}^n_{\mathbb{K}}$ and $S \subseteq \mathbb{K}[x_1, ..., x_n]$. Then

1. $X \subseteq V_{B,j}(I_{B,j}(X))$.
2. $S \subseteq I_{B,j}(V_{B,j}(S))$.
3. If $X$ is affine variety associated to $B$, then $V_{B,j}(I_{B,j}(X)) = X$.

**Proof.**

1. Let $(a_1, ..., a_n) \in X$. Then we have $f(\sum_{i=1}^{j} a_i v_i) = 0$ for all $f \in I_{B,j}(X)$. By definition, we have $(a_1, ..., a_n) \in V_{B,j}(I_{B,j}(X))$.
2. Let $f \in S$. Then $f(\sum_{i=1}^{j} a_i v_i) = 0$ for all $(a_1, ..., a_n) \in V_{B,j}(X)$. Again, By definition, we have $f \in I_{B,j}(V_{B,j}(S))$.
3. By $(i)$, it suffices to prove “$\subseteq$”. As $X$ is an affine variety associated to $B$, we can write $X = V_{B,j}(S)$ for some $S \subseteq \mathbb{K}[x_1, ..., x_n]$. Then $S \subseteq I_{B,j}(V_{B,j}(S))$ by $(i)$, and hence $V_{B,j}(S) \supseteq V_{B,j}(I_{B,j}(V_{B,j}(S)))$ by Lemma (2.6). Therefore, $V_{B,j}(I_{B,j}(X)) = X$. 

$\square$
Remark 2.13. As for the usual maps $V$ and $I$, $S \subseteq I_{B,j}(V_{B,j}(S))$ is a strict inclusion in general. For example, if $B = (e_1, ..., e_n)$ is the standard ordered basis for $V = \mathbb{R}^3$ and $S = (x_1^3, x_2^3, x_3^3)$. Then, $V_{B,3}(S) = V(x_1^3, x_2^3, x_3^3)$ and $I_{B,3}V_{B,3}(S) = IV(x_1^3, x_2^3, x_3^3) = (x_1, x_2, x_3)$ [10, p. 19].

Consider the maps

\[
\begin{align*}
&\{\text{subsets of } \mathbb{A}^n_\mathbb{K}\} \xleftarrow{I_{B,j}(-)} \quad \{\text{subsets of } \mathbb{K}[x_1, ..., x_n]\} \\
&\{\text{subsets of } \mathbb{A}^n_\mathbb{K}\} \xrightarrow{V_{B,j}(-)} \quad \{\text{subsets of } \mathbb{K}[x_1, ..., x_n]\}
\end{align*}
\]

The previous Proposition implies some quick nice consequences, but before that we might need to review some basic notions.

Definition 2.14. Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be preorders. Then, we say that a mapping $f : A \to B$ is antitone (or order-reversing) if

\[
(\forall x, y \in A) \quad x \preceq_A y \Rightarrow f(x) \succeq_B f(y) \quad [5, p. 5].
\]

Given preorders $(A, \preceq_A)$, $(B, \preceq_B)$ and antitone mappings $f : A \to B$ and $g : B \to A$, we say that the pair $(f, g)$ establishes an (antitone) Galois connection between $A$ and $B$ if $f(a) \succeq_B b$ and $a \preceq_A g(b)$, for all $a \in A$ and $b \in B$. If we further have $a = gf(a)$ and $b = fg(b)$, for all $a \in A$ and $b \in B$, then the pair $(f, g)$ is called an Galois correspondence between $A$ and $B$ [7, p. 145].

Proposition 2.15. Let $\mathcal{P}(\mathbb{A}^n_\mathbb{K})$ and $\mathcal{P}(\mathbb{K}[x_1, ..., x_n])$ be the power sets of $\mathbb{A}^n_\mathbb{K}$ and $\mathbb{K}[x_1, ..., x_n]$ respectively. Ordered by set inclusion, $\mathcal{P}(\mathbb{A}^n_\mathbb{K})$ and $\mathcal{P}(\mathbb{K}[x_1, ..., x_n])$ are preorders. Consider the maps

\[
\begin{align*}
&\mathcal{P}(\mathbb{A}^n_\mathbb{K}) \xleftarrow{I_{B,j}(-)} \mathcal{P}(\mathbb{K}[x_1, ..., x_n]) \\
&\mathcal{P}(\mathbb{A}^n_\mathbb{K}) \xrightarrow{V_{B,j}(-)} \mathcal{P}(\mathbb{K}[x_1, ..., x_n])
\end{align*}
\]

Then, the pair $(V_{B,j}, I_{B,j})$ forms an antitone Galois connection.

Any preorder set $(\mathcal{P}, \preceq_{\mathcal{P}})$ gives rise to a category in which the class of objects is simply $\mathcal{P}$ itself and for any two objects $a, b \in \mathcal{P}$, there is precisely one morphism $a \to b$ whenever $a \preceq_{\mathcal{P}} b$. It turns out that we could think of any map between preorder sets as a functor. Following [12, p. 95], we the following quick consequence:
Theorem 2.17. (Hilbert’s Nullstellensatz Theorem version for $V_{B,j}$ and $I_{B,j}$) Let $\mathbb{K}$ be an algebraically closed field. Then the following hold:

1. Every maximal ideal $m \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is of the form
   $$m \cong (x_1 - \sum_{t=1}^{j} a_t v_{t1}, \ldots, x_n - \sum_{t=1}^{j} a_t v_{tn}) = I_{B,j}(a)$$
   for some point $a = (a_1, \ldots, a_n) \in \mathbb{A}_\mathbb{K}^n$.

2. If $J \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a proper ideal, then $V_{B,j}(J) \neq 0$.

3. For every ideal $J \subseteq \mathbb{K}[x_1, \ldots, x_n]$, we have $I_{B,j}V_{B,j}(J) = \sqrt{J}$.

In particular, there is an inclusion-reversing bijection

$$\begin{align*}
\{ \text{jth affine varieties in } \mathbb{A}_\mathbb{K}^n \text{ associated to } B \} & \longleftrightarrow \{ \text{radical ideals in } \mathbb{K}[x_1, \ldots, x_n] \} \\
X \ & \longleftrightarrow \ I_{B,j}(X) \\
V_{B,j}(J) \ & \longleftrightarrow \ J.
\end{align*}$$

Proof. (Following [11] and [10])

1. By using Proposition (2.10), we have
   $$I_{B,j}(a) = I_{B,j}(\{(a_1, \ldots, a_n)\}) = (x_1 - \sum_{t=1}^{j} a_t v_{t1}, \ldots, x_n - \sum_{t=1}^{j} a_t v_{tn}) = I_{B,j}(a).$$

Let $\varphi : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}$, $f \mapsto f(\sum_{i=1}^{j} a_i v_i)$ be the evaluation map. Then $\varphi$ is obviously ring homomorphism with $\ker(\varphi) = I_{B,j}(a)$, and hence by Using the First Isomorphism Theorem, we have $\mathbb{K}[x_1, \ldots, x_n]/I_{B,j}(a) \cong \mathbb{K}$. Thus $I_{B,j}(a)$ is maximal.

For the other direction, let $m$ be a maximal ideal in $\mathbb{K}[x_1, \ldots, x_n]$.

For all $1 \leq i \leq n$, consider the $i$th inclusion ring homomorphism $\iota_i : \mathbb{K}[x_i] \to \mathbb{K}[x_1, \ldots, x_n]$. By Proposition 1.2, the intersection $\mathbb{K}[x_i] \cap m = \iota_i^{-1}(m)$ is a maximal
ideal in \( \mathbb{K}[x_i] \) for each \( i = 1, \ldots, n \). Since \( \mathbb{K}[x_i] \) is a principal ideal domain, \( \mathbb{K}[x_i] \cap m \) has the form \((p_i(x_i))\) with \( p_i(x_i) \) an irreducible polynomial. Since \( \mathbb{K} \) is algebraically closed, we obtain \((p_i(x_i)) = (x_i - b_i)\) with \( b_i \in \mathbb{K} \). Hence, there exist \( b_1, \ldots, b_n \in \mathbb{K} \) with \( x_i - b_i \in m \) for all \( i = 1, \ldots, n \). Since \( (x_i - b_i) \cong (x_i - \sum_{t=1}^{j} a_t v_t) \) (for all \( i = 1, \ldots, n \)), it follows that we have

\[
(x_1 - \sum_{t=1}^{j} a_t v_t, \ldots, x_n - \sum_{t=1}^{j} a_t a_{tn}) \cong (x_1 - b_1, \ldots, x_n - b_n) \subseteq m.
\]

Since \( m \) is maximal, we must have \( m \cong I_{B,j}(a) \) as desired.

(2) Suppose \( J \subsetneq \mathbb{K}[x_1, \ldots, x_n] \) is a proper ideal. Since \( \mathbb{K}[x_1, \ldots, x_n] \) is a Noetherian ring there is a maximal ideal \( m \) with \( J \subseteq m \). By using (1), \( m \cong I_{B,j}(a) \) for some point \( a = (a_1, \ldots, a_n) \in \mathbb{A}^n \), and hence \( \{a\} = V_{B,j}(I_{B,j}(a)) \subseteq V_{B,j}(J) \). Thus, \( V_{B,j}(J) \neq 0 \).

(3) As it might be expected, Rabinowitsch’s trick works here very similar to the case when we have the maps \( V \) and \( I \).

As a quick consequence to Theorem (2.17), we have the following correspondence:

\[
\left\{ \text{\#th affine varieties in } \mathbb{A}^n \text{ associated to } B \right\} \overset{1-1}{\longleftrightarrow} \left\{ \text{the usual affine varieties in } \mathbb{A}^n \right\}.
\]

Accordingly, for all \( 1 \leq i, j \leq n \), we have a 1 − 1 correspondence

\[
\left\{ \text{i\#th affine varieties in } \mathbb{A}^n \text{ associated to } B \right\} \overset{1-1}{\longleftrightarrow} \left\{ \text{j\#th affine varieties in } \mathbb{A}^n \text{ associated to } B \right\}.
\]

Another consequence is the following.

**Corollary 2.18.** Let \( \mathbb{K} \) be an algebraically closed field. The antitone Galois connection \((V_{B,j}, I_{B,j})\) in Proposition (2.15) induces the following Galois correspondence

\[
\left\{ \text{j\#th affine varieties in } \mathbb{A}^n \text{ associated to } B \right\} \overset{1-1}{\longleftrightarrow} \left\{ \text{radical ideals in } \mathbb{K}[x_1, \ldots, x_n] \right\}.
\]

3. Varieties Associated with Linear operators

3.1. Affine Varieties Associated to Linear Operators. As seen in Remark (2.5), the varieties associated to ordered bases can be given in terms of linear operators, and this
reasonably makes us think of a more general concept by replacing the linear operators $L_{B,j}$ by an arbitrary linear operators $L : \mathbb{K}^n \to \mathbb{K}^n$. Let $L : \mathbb{K}^n \to \mathbb{K}^n$ be any linear operator. We start this section with the following definition.

**Definition 3.1.** Let $S \subseteq \mathbb{K}[x_1, \ldots, x_n]$.

1. The set
   
   $$ V_L(S) = \{ a = (a_1, \ldots, a_n) \in \mathbb{A}^n_\mathbb{K} : f(L(a)) = 0 \text{ for all } f \in S \} $$

   is called the (affine) locus of $S$ associated to $L$ (or simply the $L$-(affine) locus of $S$).

   Subsets of $\mathbb{A}^n_\mathbb{K}$ of this form are called (affine) varieties associated to $L$ (or simply $L$-affine varieties).

2. The ideal (or simply the $L$-ideal of a set $X \subseteq \mathbb{A}^n_\mathbb{K}$ associated to $L$ is defined to be the set
   
   $$ I_L(X) = \{ f \in \mathbb{K}[x_1, \ldots, x_n] : f(L(a)) = 0 \text{ for all } a \in X \} $$

**Remark 3.2.**

1. Varieties associated to the ordered bases are a particular case of $L$-varieties when $L = L_{B,j}$.
2. If $L = id : \mathbb{K}^n \to \mathbb{K}^n$ is the identity linear operator, then the (affine) varieties associated to $L$ are precisely the (usual) varieties in $\mathbb{A}^n_\mathbb{K}$.
3. It might be noticeable that $V_L(S)$ needs not be an image of a variety mapped by $L$.
4. If $f \in \mathbb{K}[x_1, \ldots, x_n]$, then $fL \in \mathbb{K}[x_1, \ldots, x_n]$, where $fL(x) := f(L(x))$.
5. Clearly, we have the following $1 \leftrightarrow 1$ correspondence:

   $$ \begin{align*}
   \left\{ \text{L-affine locus maps } V_L \right\} & \leftrightarrow \left\{ \text{linear operators } L : \mathbb{K}^n \to \mathbb{K}^n \right\} \\
   \leftrightarrow \left\{ \text{standard matrices of the linear operators } L : \mathbb{K}^n \to \mathbb{K}^n \right\} 
   \end{align*} $$

**Example 3.3.** Let $L_1$ and $L_2$ be the linear operators corresponding to the following matrices

$$
A = \begin{bmatrix}
1 & 2 & 0 \\
-1 & 2 & -1 \\
2 & 1 & 3 
\end{bmatrix}, \quad B = \begin{bmatrix}
-1 & 2 & 1 \\
1 & -1 & 3 \\
-3 & 2 & -1 
\end{bmatrix}.
$$

1. Consider the ideal $S = (x^2 + y^2 - z^2 - 1) \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Then the varieties associated to $L_1$, $L_2$, $L_1 + L_2$ and $L_1L_2$ can be given in the following figure:
(2) Let $S = (x^2 + y^2 + z^2 - 1) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be the ideal generated by the polynomial $x^2 + y^2 + z^2 - 1$. We have the varieties associated to $L_1$, $L_2$, $L_1 + L_2$ and $L_1L_2$ can be visualized as follows:

Define $\mathfrak{V}_V := \{V_L : L \in Hom_K(V, V)\}$. We could define operations $\boxplus$ and $\odot$ on $\mathfrak{V}_V$ as follows:

$$V_L \boxplus V_{L'} = V_{L+L'} \quad \text{and} \quad V_L \odot V_{L'} = V_{LL'}.$$ 

Similarly, if $\mathcal{I}_V := \{I_L : L \in Hom_K(V, V)\}$, then we could define operations $\boxplus'$ and $\odot'$ on $\mathcal{I}_V$ as follows:

$$I_L \boxplus' I_{L'} = I_{L+L'} \quad \text{and} \quad I_L \odot' I_{L'} = I_{LL'}.$$ 

As a clear sequence, we have the following.

**Proposition 3.4.** $(\mathfrak{V}_V, \boxplus, \odot)$ and $(\mathcal{I}_V, \boxplus', \odot')$ are $K$-algebras which are isomorphic to the matrix algebra $M_n(\mathbb{K})$, and hence $\mathfrak{V}_V \cong \mathcal{V} \otimes \mathcal{V}^* \cong \mathcal{I}_V$. 

Proof. We have $\mathfrak{C}_V \cong \mathcal{J}_V \cong \text{Hom}_\mathbb{K}(V, V) \cong \text{Hom}_\mathbb{K}(\mathbb{K}^n, \mathbb{K}^n) \cong M_n(\mathbb{K})$. Since $V$ is a finite dimensional vector space, we also have $\mathfrak{C}_V \cong \mathcal{J}_V \cong \text{Hom}_\mathbb{K}(V, V) \cong V \otimes V^*$.

Let $H$ be a Hopf algebra over $\mathbb{K}$. If $V_1, V_2, V_3$ are finite-dimensional $\mathbb{K}$-vector spaces, then we have $V_1, V_2, V_3$ are rigid objects in the category $Vec_\mathbb{K}$ of vector spaces. Thus, we have

\begin{equation}
\varphi : Hom_{Vec_\mathbb{K}}(V_1 \otimes V_2, V_3) \rightarrow Hom_{Vec_\mathbb{K}}(V_1, V_3 \otimes V_2^*)
\end{equation}

In particular, if $V_1 = V_2 = V_3 = V$, we have

\begin{equation}
\varphi : Hom_{Vec_\mathbb{K}}(V \otimes V, V) \rightarrow Hom_{Vec_\mathbb{K}}(V, V \otimes V^*).
\end{equation}

Therefore, if $(V, \mu, u)$ is a finite dimensional algebra over $\mathbb{K}$ with a multiplication and unit maps $\mu$ and $u$ respectively, then $\varphi(\mu) : V \rightarrow V \otimes V^*$ is a morphism of algebras corresponding to the left regular representation of $V$, and hence $\mu$ uniquely corresponds to a morphism $V \rightarrow \mathfrak{C}_V$ of algebras. It turns out that we could have a system of cogenerators in terms of affine locus of maps associated to linear operators of finite dimensional vector spaces.

**Theorem 3.5.** [1, p. 9] Let $\mathcal{M}^H$ be the (monoidal category) of right $H$-comodules. Then the finite dimensional algebras of the form $\mathfrak{C}_V$ (hence also of the form $\mathcal{J}_V$) for finite dimensional $H$-comodules $V$, form a system of cogenerators in the category $\text{Alg}(\mathcal{M}^H)$ of algebras in $\mathcal{M}^H$.

A generalization of Lemma (2.4), we have the following proposition

**Proposition 3.6.** Let $J = (f_1, \ldots, f_m) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ and $L \in Hom_\mathbb{K}(V, V)$. Then $V_L(J) = V(J_L)$, where $J_L = (g_1, \ldots, g_m) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ and $g_t(x_1, \ldots, x_n) := f_t(L((x_1, \ldots, x_n)))$ for all $t = 1, \ldots, m$.

The proof of the following proposition is quite similar to the proof of Lemma (2.6), Proposition (2.10) and Proposition (2.12).

**Proposition 3.7.** Let $S, S_1, S_2 \subseteq \mathbb{K}[x_1, \ldots, x_n]$ and $X, X_1, X_2 \subseteq \mathbb{A}_\mathbb{K}^n$. Then

1. If $S_1 \subseteq S_2$, then $V_L(S_1) \supseteq V_L(S_2)$.
2. $V_L(S_1) \cup V_L(S_2) = V_L(S_1 \cap S_2)$, where as usual we set $S_1 \cap S_2 := \{fg : f \in S_1, g \in S_2\}$.
3. If $J$ is any index set and $S_\alpha \subseteq \mathbb{K}[x_1, \ldots, x_n] \forall \alpha \in J$, then $\bigcap_{\alpha \in J} V_L(S_\alpha) = V_L(\bigcup_{\alpha \in J} S_\alpha) = V_L(\sum_{\alpha \in J} S_\alpha)$. In particular, finite unions and arbitrary intersections of affine varieties associated to $B$ are again affine varieties associated to $B$.
4. $V_L(\sqrt{S}) = B_L(S)$
5. $I_L(\emptyset) = \mathbb{K}[x_1, \ldots, x_n]$.
6. For any point $a = (a_1, \ldots, a_n) \in \mathbb{A}_\mathbb{K}^n$ with $L(a) = (b_1, \ldots, b_n)$, we have
   $$I_L(\{(a_1, \ldots, a_n)\}) = (x_1 - b_1, \ldots, x_n - b_n).$$
7. If $X_1 \subseteq X_2$, then $I_L(X_1) \supseteq I_L(X_2)$.
8. $I_L(S_1 \cup S_2) = I_L(S_1) \cap I_L(S_2)$.
9. $I_L(\mathbb{A}_\mathbb{K}^n) = \{0\}$.
10. $X \subseteq V_L(I_L(X))$.
11. $S \subseteq V_L(V_L(S))$.
12. If $X$ is affine variety associated to $B$, then $V_L(I_L(X)) = X$. 
The following are more general consequences to (Proposition (2.15) and Proposition (2.16)) and Theorem (2.17) respectively.

**Proposition 3.8.** Let \( \mathcal{P}(\mathbb{A}^n_K) \) and \( \mathcal{P}(\mathbb{K}[x_1, \ldots, x_n]) \) be the power sets of \( \mathbb{A}^n_K \) and \( \mathbb{K}[x_1, \ldots, x_n] \) respectively. Ordered by set inclusion, \( \mathcal{P}(\mathbb{A}^n_K) \) and \( \mathcal{P}(\mathbb{K}[x_1, \ldots, x_n]) \) are preorders. Consider the maps

\[
\begin{align*}
\mathcal{P}(\mathbb{A}^n_K) & \xrightarrow{I_L(-)} \mathcal{P}(\mathbb{K}[x_1, \ldots, x_n]), \\
\mathcal{P}(\mathbb{K}[x_1, \ldots, x_n]) & \xrightarrow{V_L(-)} \mathcal{P}(\mathbb{A}^n_K).
\end{align*}
\]

Then, we have the following

1. The pair \( (V_L, I_L) \) forms an antitone Galois connection.
2. Regarded as a functor, the map \( V_L \) is a left adjoint to \( I_L \), and there is exactly one adjunction that makes \( V_L \) the left adjoint of \( I_L \). For all \( S \subseteq \mathbb{A}^n_K \) and \( X \subseteq \mathbb{K}[x_1, \ldots, x_n] \), \( S \subseteq I_L V_L(S) \) and \( V_L I_L(X) \supseteq X \). Thus,

\[
V_L I_L V_L(S) = V_L(S), \quad \text{and} \quad I_L V_L I_L(X) = I_L(X).
\]
3. Let \( \mathbb{K} \) be an algebraically closed field. Then the above antitone Galois connection \( (V_L, I_L) \) induces the following Galois correspondence

\[
\begin{align*}
\{ \text{L-affine varieties in } \mathbb{A}^n_K \} & \xrightarrow{I_L(-)} \{ \text{radical ideals in } \mathbb{K}[x_1, \ldots, x_n] \}, \\
\{ \text{radical ideals in } \mathbb{K}[x_1, \ldots, x_n] \} & \xleftarrow{V_L(-)} \{ \text{L-affine varieties in } \mathbb{A}^n_K \}.
\end{align*}
\]

**Theorem 3.9.** (Hilbert’s Nullstellensatz Theorem version for \( V_L \) and \( I_L \)) Let \( \mathbb{K} \) be an algebraically closed field. Then the following hold:

1. Every maximal ideal \( m \subseteq \mathbb{K}[x_1, \ldots, x_n] \) is of the form

\[
m \cong (x_1 - b_1, \ldots, x_n - b_n) = I_L(a)
\]

for some point \( a = (a_1, \ldots, a_n) \in \mathbb{A}^n_K \) with \( L(a) = (b_1, \ldots, b_n) \).
2. If \( J \subseteq \mathbb{K}[x_1, \ldots, x_n] \) is a proper ideal, then \( V_L(J) \neq 0 \).
3. For every ideal \( J \subseteq \mathbb{K}[x_1, \ldots, x_n] \), we have \( I_L V_L(J) = \sqrt{J} \).

In particular, there is an inclusion-reversing bijection

\[
\begin{align*}
\{ \text{L-affine varieties in } \mathbb{A}^n_K \} & \longleftrightarrow \{ \text{radical ideals in } \mathbb{K}[x_1, \ldots, x_n] \} \\
X & \mapsto I_L(X) \\
V_L(J) & \leftarrow J
\end{align*}
\]

**Corollary 3.10.** Let \( \mathbb{K} \) be an algebraically closed field. There is a \( 1 - 1 \) correspondence

\[
\{ \text{L-affine varieties in } \mathbb{A}^n_K \} \longleftrightarrow \{ \text{(usual) affine varieties in } \mathbb{A}^n_K \}.
\]
3.2. Varieties and Stanley-Reisner ideals associated Linear Operators. In this section, we follow [13] to generalize some monomial algebra concepts.

**Definition 3.11.** Let $\Delta$ be a simplicial complex and $\tau \subseteq [n]$.

1. For any $j \in \tau$, let $p_j : \mathbb{K}^n \to \mathbb{K}$ be the $j$th projection map. Then we could think of each $p_j(L(x_1, ..., x_n))$ as a linear polynomial.

2. An $L$-monomial in $\mathbb{K}[x_1, ..., x_n]$ is a product $x^{L,a} = \prod_{i=1}^{n} p_i(L(x_1, ..., x_n))^{a_i}$ for a vector $a = (a_1, ..., a_n) \in \mathbb{N}^n$ of nonnegative integers. An ideal $I \subseteq \mathbb{K}[x_1, ..., x_n]$ is called an $L$-monomial ideal if it is generated by $L$-monomials.

3. For notation, we write $x^{L,\tau} = \prod_{j\in\tau} p_j(L(x_1, ..., x_n))$.

4. An $L$-monomial $x^{L,a}$ is $L$-squarefree if every coordinate of $a$ is 0 or 1. An ideal is $L$-squarefree if it is generated by $L$-squarefree monomials.

5. Every $\tau \subseteq [n]$ can be written as a complement $\sigma := [n] \setminus \tau$ of some simplex $\sigma$.

6. The ideal $Q^{\tau,L} := \langle p_j(L(x_1, ..., x_n)) : j \in \tau \rangle$ is called the monomial prime ideal associated to $\tau$ and $L$.

7. The Stanley-Reisner variety of the simplicial complex $\Delta$ associated to $L$ is defined to be the set:

$$V_{\Delta,L} := V_L(\prod_{\tau \in \Delta} Q^{\tau,L}) = V(\prod_{\tau \in \Delta} Q^{\tau,L})_L \text{ (see Proposition 3.6).}$$

8. The Stanley-Reisner ideal of the simplicial complex $\Delta$ associated to $L$ is the ideal:

$$I_{\Delta,L} := \langle x^{L,\tau} : \tau \notin \Delta \rangle.$$

9. The Stanley-Reisner ring of $\Delta$ associated to $L$ is the quotient ring $\mathbb{K}[x_1, ..., x_n]/I_{\Delta,L}.$

The proof of the following theorem is straightforward.

**Theorem 3.12.** Fix a linear operator $L : \mathbb{K}^n \to \mathbb{K}^n$. The correspondence $\Delta \sim I_{\Delta,L}$ constitutes a bijection from simplicial complexes on vertices $[n] = \{1, ..., n\}$ to squarefree monomial ideals inside $\mathbb{K}[x_1, ..., x_n]$. Furthermore,

$$I_{\Delta,L} = \bigcap_{\tau \in \Delta} Q^{\tau,L}.$$

The following corollary is a quick consequence of and Theorem (3.12) and Proposition (3.7).

**Corollary 3.13.** For any linear operator $L : \mathbb{K}^n \to \mathbb{K}^n$, we have

1. $V_L(I_{\Delta,L}) = V_{\Delta,L}.$

2. If $\mathbb{K}$ is an algebraically closed field, then $I_{\Delta,L} = I_L V_{\Delta,L}$.

**Example 3.14.** Let $L$ be the linear operator corresponding to the matrix

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -1 \\ -2 & 1 & -2 \end{bmatrix}.$$

Let $\Delta$ be the simplicial complex on the vertex set $[5] = \{1, 2, 3, 4, 5\}$ consisting of all subsets of the sets $\{1, 2\}$ and $\{1, 3\}$. Then

$$I_{\Delta,L} = \langle -2x_1 + x_2 - 2x_3 \rangle \bigcap \langle x_1 + 3x_2 - x_3 \rangle = \langle (x_1 + 3x_2 - x_3)(-2x_1 + x_2 - 2x_3) \rangle.$$
The following figure shows how the Stanley-Reisner variety of the simplicial complex $\Delta$ associated to $L$ looks like.

$$V_{\Delta,L}$$

It might be worth mentioning that the above definitions allows us to explore such concepts in more detail. For instance, one might show that Stanley-Reisner rings of $\Delta$ associated to $L$ can be explicitly written in terms of $\Delta$. We could also investigate and establish minimal free resolutions of such interesting rings and adapt lots of consequences involved with them, such as, Alexander inversion formula, Reisner’s criterion, cohomology of links in the Stanley-Reisner complexes, etc. We refer the reader to [13] for further investigation.

3.3. The Category of all Affine Varieties Associated to Linear Operators.

Definition 3.15.

(1) An $L$-affine variety $X$ is called irreducible if it is nonempty and there is no decomposition $X = X_1 \cup X_2$ ($X_1, X_2 \subseteq X$) into proper $L$-affine varieties subsets $X_1$ and $X_2$. Otherwise $X$ is called reducible.

(2) We define the Zariski topology on $\mathbb{A}_K^n$ associated with $L$ (or simply the $L$-Zariski topology on $\mathbb{A}_K^n$) by taking the open subsets to be the complements of $L$-affine varieties. This is a topology due to Proposition (3.7).

(3) Let $X \subseteq \mathbb{A}_K^n$ be an $L$-affine variety and $f \in \mathbb{K}[x_1, \ldots, x_n]$. Define

$$D_{X,L}(f) = \{ x \in X : f(L(x)) \neq 0 \}.$$  

This is clearly an open subset of $X$, since $D_{X,L}(f) = X \setminus V_L(f)$. Note that

$$D_{X,L}(f) \cap D_{X,L}(g) = D_{X,L}(fg).$$

In fact, any open subset of $X$ is of the form: $X \setminus D_{X,L}(J)$ for some $J \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Since $J$ is finitely generated, we can write it as $J = \langle f_1, \ldots, f_m \rangle$ for some $f_1, \ldots, f_m \in J$, whence $X \setminus V_L(J) = D_{X,L}(f_1) \cup \ldots \cup D_{X,L}(f_m)$. Hence the sets $D_{X,L}(f)$ give a basis for the $L$-Zariski topology on $X$. We call them the principal $L$-affine open subsets.

(4) A regular function associated with $L$ (or simply an $L$-regular function) on $X$ is a map $\varphi : X \to \mathbb{K}$ which can locally be given by the quotient of two polynomial
functions, i.e. for any $x \in X$, there exists an open neighborhood $U_x$ of $x$ in $X$ and $f, g \in \mathbb{K}[x_1, \ldots, x_n]$ such that for any $u \in U_x$ we have
\[ g(L(u)) \neq 0 \text{ and } \varphi(u) = \frac{f(L(u))}{g(L(u))} \in \mathbb{K}. \]

We write $\mathcal{O}_L(X)$ to be the set of $L$-regular functions on $X$. (By convention, $\mathcal{O}_L(\emptyset) = 0$.)

(5) Let $X \subseteq \mathbb{A}^n_{\mathbb{K}}$ be an $L$-affine variety. Then the **coordinate ring of $X$ associated with $L$** (or simply the **$L$-coordinate ring**) is the quotient ring
\[ A_L(X) := \mathbb{K}[x_1, \ldots, x_n]/I_L(X). \]

**Remark 3.16.**

(1) As a consequence of Corollary (3.10), a subset $X \subseteq \mathbb{A}^n_{\mathbb{K}}$ is irreducible $L$-affine variety $\iff X$ is irreducible affine variety $\iff I_L(X) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a prime ideal $\iff I(X) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a prime ideal $\iff \mathbb{K}[x_1, \ldots, x_n]/I(X)$ is an integral domain $\iff \mathbb{K}[x_1, \ldots, x_n]/I_L(X)$ is an integral domain.

(2) Let $\varphi : X \to \mathbb{K}$ be an $L$-regular function. For any $x \in X$, there exists an open neighborhood $U_x$ of $x$ in $X$ and $f, g \in \mathbb{K}[x_1, \ldots, x_n]$ such that for any $u \in U_x$ we have
\[ g(L(u)) \neq 0 \text{ and } \varphi(u) = \frac{f(L(u))}{g(L(u))} \in \mathbb{K}. \]

This implies that for any $x \in X$, there exists an open neighborhood $U_x$ of $x$ in $X$ and $\alpha, \beta \in \mathbb{K}[x_1, \ldots, x_n]$ such that for any $u \in U_x$ we have
\[ \beta(u) \neq 0 \text{ and } \varphi(u) = \frac{\alpha(u)}{\beta(u)} \in \mathbb{K}, \]

where $\alpha = fL, \beta = gL \in \mathbb{K}[x_1, \ldots, x_n]$ by Remark (3.2). Thus, every $L$-regular function can be seen as a regular function. On the other hand, regular function can be viewed as an $L$-regular function with $L = \text{id}$ the identity linear operator.

(3) The $L$-Zariski topology on $\mathbb{A}^n_{\mathbb{K}}$ is a coarser topology than the (usual) Zariski topology on $\mathbb{A}^n_{\mathbb{K}}$ for any linear operator $L : \mathbb{K}^n \to \mathbb{K}^n$.

(4) If $L = 0$ is the zero linear operator, then for any $S \subseteq \mathbb{A}^n_{\mathbb{K}}$, we have
\[ V_0(S) = \{ a \in \mathbb{A}^n_{\mathbb{K}} : f((0, \ldots, 0)) = 0 \text{ for all } f \in S \} = \left\{ \begin{array}{ll} \emptyset & \text{if } (0, \ldots, 0) \notin V(S) \\ \mathbb{A}^n_{\mathbb{K}} & \text{if } (0, \ldots, 0) \in V(S) \end{array} \right. \]

Thus, the $0$-Zariski topology on $\mathbb{A}^n_{\mathbb{K}}$ is precisely the Indiscrete Topology on $\mathbb{A}^n_{\mathbb{K}}$.

(5) Clearly, we have
\[ \mathcal{O}_L(X) = \{ L\text{-regular functions on } X \} \subseteq_{\text{subalgebra}} \left\{ \text{functions } X \to \mathbb{K} \right\}. \]

Let $X \subseteq \mathbb{A}^n_{\mathbb{K}}$ be an $L$-affine variety and $U = D_{X,L}(g) = \{ x \in X : | g(L(x)) \neq 0 \}$ for some $g \in \mathbb{K}[x_1, \ldots, x_n]$. We have a $\mathbb{K}$-algebra homomorphism
\[ \Phi : \mathbb{K}[x_1, \ldots, x_n]_{gL} \to \mathcal{O}_L(U) \text{ defined by } \]
\[
\frac{f}{(gL)^n} \mapsto \left( a \mapsto \frac{f(L(a))}{(g(L(a)))^n} \right).
\]

Proposition 3.17. The induced \(\mathbb{K}\)-algebra homomorphism

\[\mathbb{K}[x_1, \ldots, x_n]_g/I_L(X)_g \to \mathcal{O}_L(U)\]

is an isomorphism. In particular, we have a \(\mathbb{K}\)-algebra isomorphism

\[A_L(X) \cong \mathcal{O}_L(X).\]

Proof. (Following [14, p. 13–14]) First, we have \(\ker(\Phi) = \{ \frac{f}{(gL)^n} \in \mathbb{K}[x_1, \ldots, x_n]_g : f(L(a)) = 0, \forall a \in U \}.\) Obviously, \(I_L(X)_g \subseteq \ker(\Phi).\) If \(\frac{f}{(gL)^n} \in \ker(\Phi),\) then \(f(g(L(a))) = f(L(a))g(L(a)) = 0\) for any \(a \in X,\) and hence \(fg \in I_L(X).\) Now \(\frac{f(L(a))}{(gL)^n} = \frac{fg(L(a))}{(gL)^n+1} = 0 \Rightarrow \frac{f}{(gL)^n} \in I_L(X)_g.\) Therefore, \(\ker(\Phi) = I_L(X)_g,\) and hence \(\Phi\) induces an injective homomorphism as stated above.

To prove surjectivity, let \(\varphi \in \mathcal{O}_L(U).\) Then there exist open \(W_1, \ldots, W_r\) such that \(U = W_1 \cup \ldots \cup W_r\) and \(f_i, g_i \in \mathbb{K}[x_1, \ldots, x_n]\) such that for any \(a \in W_i, g_i(L(a)) \neq 0\) and \(\varphi(a) = \frac{f_i(L(a))}{g_i(L(a))}.\)

Since the principal \(L\)-affine open subsets form a basis, we may assume \(W_i = D_{X,L}(h_i)\) for some polynomial \(h_i \in \mathbb{K}[x_1, \ldots, x_n],\) for any \(i.\) Since \(g_i(L(a)) \neq 0\) for any \(a \in D_{X,L}(h_i),\)

\[V_L(g_i) \subseteq V_L(h_i) \Rightarrow X \cap V_L(g_i) \subseteq V_L(h_i) \Rightarrow V_L(I_L(X) \cap V_L(g_i)) \subseteq V_L(h_i) \Rightarrow V_L(I_L(X) + V_L(g_i)) \subseteq V_L(h_i) \Rightarrow I_L(V_L(I_L(X) + V_L(g_i))) \supseteq I_L(V_L(h_i)) \Rightarrow (I_L(X) + V_L(g_i)) \supseteq (h_i) \Rightarrow h_i \in \sqrt{(I_L(X) + V_L(g_i))}.
\]

By Theorem [3.9], this shows that \(h_i \in \sqrt{I_L(X) + (g_i)}.\) Without changing \(W_i,\) we may replace \(h_i\) by a power or by an element with the same class modulo \(I_L(X).\) Hence we may assume without loss of generality that \(h_i \in (g_i),\) say \(h_i = g_i h'_i.\) After replacing \(f_i, g_i, h_i, g_i h'_i\) respectively, we may assume \(g_i = h_i\) for any \(i: \varphi(a) = f_i(L(a)) = f_i(L(a))h'_i(L(a)) = f_i(L(a))h'_i(L(a)) \Rightarrow \frac{f_i(L(a))}{g_i(L(a))} \Rightarrow \frac{f_i(L(a))}{g_i(L(a))} = \frac{f_i(L(a))}{g_i(L(a))}.\)

for all \(a \in W_i.\) Since \(\frac{f_i(L(a))}{g_i(L(a))} = \frac{f_i(L(a))}{g_i(L(a))} \Rightarrow \frac{f_i(L(a))}{g_i(L(a))} \Rightarrow \frac{f_i(L(a))}{g_i(L(a))} = \frac{f_i(L(a))}{g_i(L(a))}.\)

By Theorem [3.9], \(\sqrt{I_L(X) + (g)} = \sqrt{I_L(X)} + (g_1, \ldots, g_r).\)
Thus, there exists \( m \geq 1 \) and \( q_1, \ldots, q_r \in \mathbb{K}[x_1, \ldots, x_n] \) such that

\[
(3.4) \quad g^m - \sum_{i=1}^{r} q_i g_i \in I_L(X)
\]

We claim that \( \varphi \) is the image of \( \sum_{i=1}^{r} q_i L f_i L^{-m} \). To prove this, it suffices to show that for any \( a \in D_{X,L}(g_j) \), we have

\[
\sum_{i=1}^{r} q_i(L(a)) f_i(L(a)) g_i(L(a)) = f_i(L(a)) g_i(L(a)) \quad \text{for any } a \in U.
\]

Now

\[
\sum_{i=1}^{r} q_i(L(a)) f_i(L(a)) g_i(L(a)) = \sum_{i=1}^{r} q_i(L(a)) f_i(L(a)) g_i(L(a)) \quad \text{(by equation (3.3))}
\]

\[
= f_i(L(a)) g_i(L(a)) \quad \text{(by equation (3.4)).}
\]

Let \( X \subseteq \mathbb{A}_K^n \) and \( Y \subseteq \mathbb{A}_K^n \) be affine varieties associated to \( L \) and \( L' \) respectively. Let \( f : X \to Y \) be a map. Then we have the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\mathbb{A}_K^n & \xrightarrow{pr_i} & \mathbb{K}
\end{array}
\]

It turns out that we can write \( f = (f_1, \ldots, f_n) \) for \( f_i = pr_i f : X \to \mathbb{K} \) [14].

**Definition 3.18.** Let \( X \subseteq \mathbb{A}_K^n \) and \( Y \subseteq \mathbb{A}_K^n \) be affine varieties associated to \( L \) and \( L' \) respectively. A map \( f : X \to Y \) is a morphism \( \text{(of varieties associated to linear operators)} \) if \( f_1, \ldots, f_n \in \mathcal{O}_L(X) \).

As a quick consequence, we have the following

**Proposition 3.19.** Let \( X \subseteq \mathbb{A}_K^n \) and \( Y \subseteq \mathbb{A}_K^n \) be affine varieties associated to \( L \) and \( L' \) respectively. Then every morphism \( f : X \to Y \) is continuous \( \text{(with respect to the Zariski topologies associated to } L \text{ and } L' \text{ respectively).} \)

By Definition [3.18], the varieties associated to linear operators together with the morphisms defined in [3.18] form a category which we denote by \( \mathcal{S}_{lt} \).

Every affine variety \( X \subseteq \mathbb{A}_K^n \) associated to a linear operator \( L \) can be seen as a (usual) variety by Proposition [3.6]. Conversely, any variety can be identified as affine variety associated to an identity linear operator.

Furthermore, every morphism \( f : X \to Y \) in \( \mathcal{S} \) can be seen as morphism \( f : X \to Y \) from the \( id_{\mathbb{K}^m} \)-affine variety \( X \) to the \( id_{\mathbb{K}^n} \)-affine variety \( Y \), where \( id_{\mathbb{K}^m} \) and \( id_{\mathbb{K}^n} \) are the identity linear operators of \( \mathbb{K}^m \) and \( \mathbb{K}^n \) respectively. Thus, \( f : X \to Y \) is also a morphism in \( \mathcal{S}_{lt} \).

Using Proposition [3.6] and Remark [3.16] implies that every morphism \( f : X \to Y \) in \( \mathcal{S}_{lt} \) can be viewed as morphism from the (usual) affine variety \( X \) to the (usual) affine variety, and hence \( f : X \to Y \) is a morphism in \( \mathcal{S} \).

Accordingly, we have the following equivalence of categories.
Theorem 3.20. There is an equivalence of categories between the category \( S_{l.t.} \) of varieties associated to linear operators together with the morphisms and the category \( S \) of (usual) varieties.

Corollary 3.21. The category \( S_{l.t.} \) is equivalent to the category \( f.g.r.Alg_K \) of finitely generated reduced \( K \)-algebras.

A skeleton of a category \( B \) is any full subcategory \( B' \) such that each object of \( B \) is isomorphic (in \( B \)) to exactly one object of \( B' \) [12, p.93]. Let \( S_{id} \) be the category whose objects are the varieties associated to the identity linear operators and whose morphisms defined as in (3.18). Then \( S_{id} \) is a full subcategory of \( S_{l.t.} \) and isomorphic to the category \( S \). Indeed, we the following consequence:

Corollary 3.22. The category \( S_{id} \) is a skeleton of the category \( S_{l.t.} \). Thus, we have an equivalence of categories

\[ S_{id} \cong S_{l.t.}. \]

In other words, the category \( S \) of (usual) varieties can be viewed as a skeleton of the category \( S_{l.t.} \) of varieties associated to linear operators.

Proof. The first equivalence of categories is coming from the fact that any skeleton of a category \( B \) is equivalent to \( B \) [2, p. 51], [4, 183]. □

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