An extension of Fujita’s non extendability theorem for Grassmannians

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Abstract. In this paper we study smooth complex projective varieties $X$ containing a Grassmannian of lines $G(1, r)$ which appears as the zero locus of a section of a rank two nef vector bundle $E$. Among other things we prove that the bundle $E$ cannot be ample.

1. Introduction

A natural problem in algebraic geometry is to study to which extent the geometry of a smooth irreducible variety $X$ is determined by the geometry of its smooth subvarieties $Y \subset X$, under certain positivity conditions on the embedding $Y \subset X$. A typical result of this kind would characterize $X$ (possibly saying that it cannot exist) by containing a particular subvariety $Y \subset X$.

The classical setting in which the problem arose was the classification of smooth projective embedded varieties $X \subset \mathbb{P}^N$ in terms of their smooth linear sections $Y = X \cap \mathbb{P}^{N-k}$, for example the classification of low degree embedded varieties. Later on, it evolved in different settings. For instance one may impose positivity conditions on the normal bundle $N_{Y/X}$.

If $N_{Y/X}$ is (generically) globally generated then there exists a family of deformations of $Y$ sweeping out $X$. That is the case, for example, of the varieties swept out by linear subspaces of small codimension, see for instance [S] and [NO]. In a recent paper (cf. [MS]) the first and third author have dealt with embedded varieties $X \subset \mathbb{P}^N$ swept out by codimension two Grassmannians, that may be regarded as a projectively-embedded counterpart of this paper. The case of quadrics has been also studied, see [Fu] and [Bil].

If $N_{Y/X}$ is an ample line bundle, then it is well known that, up to a birational transformation, $Y$ can be considered as an ample divisor on $X$. See [H] for a foundational reference on ample subvarieties. With no assumption on the codimension, the hypothesis on $N_{Y/X}$ to be ample joint to some topological assumptions constitute the setup of [BdFL]. In that paper it is shown how some structural maps (RC-fibrations, nef-value morphism, Mori contractions) of $Y$ extend to $X$.

2000 Mathematics Subject Classification. Primary 14M15; Secondary 14E30, 14J45.

Key words and phrases. Ample subvarieties, extendability, uniform vector bundles, Grassmannians of lines, rational curves.

Partially supported by the Spanish government project MTM2006-04785 and by MIUR (PRIN project: Proprietà geometriche delle varietà reali e complesse).
In the context of complex geometry, Lefschetz Theorem shows us how the topology of a variety \( Y \subset \mathbb{P}^N \) containing \( Y \) as a linear section. Moreover an extension of this result, due to Sommese (cf. [So1] and [So2]), allows to work under weaker assumptions on the embedding \( Y \subset X \). In this way, Lefschetz-Sommese Theorem provides an important tool for the type of problems we are considering here.

In this paper we will consider varieties \( Y \) appearing as the zero locus of a regular section of an ample vector bundle \( E \) on \( X \). An interesting survey on this matter has been recently written by Beltrametti and Ionescu, see [BI2]. It deals mostly with the divisor case, but it also provides references for higher codimension. Among different results of this kind, let us recall a theorem by T. Fujita (cf. [F1] Thm. 5.2]). It states that, apart from the obvious cases, Grassmannians cannot appear as ample divisors on a smooth variety. Our goal here is to show how this result can be extended to codimension two:

**Theorem 1.1.** For \( r \geq 4 \) the Grassmannian of lines in \( \mathbb{P}^r \) cannot appear as the zero locus of a section of an ample vector bundle of rank two over a smooth complex projective variety.

Let us observe that the Grassmannian of lines in \( \mathbb{P}^r \), say \( G(1, r) \), is embedded naturally in \( G(1, r+1) \) as the zero locus of the universal quotient bundle \( Q \) which is not ample but globally generated. It is then natural to look for a broader positivity assumption on \( E \) in which this situation is included. Taking in account Lefschetz-Sommese Theorem, it makes sense to consider the notion of \( k \)-ampleness introduced by Sommese in [So2] Def. 1.3] (see also Definition 2.2 below). The main result of this paper, from which Theorem 1.1 is a straightforward corollary, is the following:

**Theorem 1.2.** Let \( X \) be a smooth complex projective variety of dimension \( 2r \) and \( G \subset X \) a subvariety isomorphic to the Grassmannian of lines in \( \mathbb{P}^r \), \( r \geq 4 \). We further assume that \( G \) equals the zero set of a section of a \((2r-4)\)-ample vector bundle \( E \) on \( X \) of rank two. Then \( X \) is isomorphic to the Grassmannian of lines in \( \mathbb{P}^{r+1} \) and \( E \) is the universal quotient bundle of this Grassmannian.

Our proof relies on proving that the normal bundle of \( G \) in \( X \) must be uniform, and on the classification of uniform vector bundles of low rank on Grassmannians.

The structure of the paper is the following. In Section 2 we recall some generalities on Grassmannians, positive vector bundles and vanishing results that we will use along the paper. In particular we find a lower bound on the degree of \( E \) in terms of the index of \( X \). Moreover one may show that this index is at most \( \dim X - 2 \), a fact that is crucial in our argumentation; this is the purpose of Section 3. Section 4 deals with the classification of uniform vector bundles on Grassmannians, and in Section 5 we determine the possible values of the restriction \( E|_G \). In Section 6 we present the proof of Theorem 1.2 and finally in Section 8 we use the results in [BdFL] in order to derive from Theorem 1.1 a non-extendability result for Grassmannian fibrations.

**Acknowledgements:** We would like to thank Tommaso de Fernex for his useful comments regarding Grassmannian fibrations.

**1.1. Conventions and definitions.** Along this paper \( X \) will denote a smooth complex projective variety of dimension \( 2r \) and \( G \subset X \) a subvariety isomorphic to the Grassmannian of lines in \( \mathbb{P}^r \), \( G(1, r) \), with \( r \geq 4 \). We further assume that \( G \)}
equals the zero set of a section of a vector bundle $E$ on $X$ of rank two which is $(2r - 4)$-ample in the sense of Sommese, see Definition 2.2. Denoting by $O(1)$ the ample generator of $\text{Pic}(G) \cong \mathbb{Z}$, the determinant of $E|_G$ is isomorphic to $O(c)$, for some $c \in \mathbb{Z}$. We call $c$ the degree of $E$. The notation $O(1)$ will be also used to denote the ample generator of a variety of Picard number one and the tautological line bundle on a projective bundle. Subscripts will be used if necessary.

Finally, on a Fano variety of Picard number one, a rational curve of degree one with respect to $O(1)$ will be called a line. By definition, the family of lines in $X$ is unsplit, i.e. the subscheme of Chow($X$) parametrizing them is proper. That amounts to say that a line is not algebraically equivalent to a reducible cycle. We will write $Q^n$ (or just $Q$ when its dimension is not relevant) for an $n$-dimensional smooth quadric.

2. Preliminaries

2.1. Generalities on Grassmannians. Let us recall some well-known facts on Grassmannians. We follow the conventions of [A]. As said before, the Grassmannian of lines in $P^r$ is denoted by $G(1, r)$. We will denote by $Q$ the rank two universal quotient bundle and by $S^r$ the rank $r - 1$ universal subbundle, related in the universal exact sequence:

$$0 \to S^r \to O^{r+1} \to Q \to 0.$$ 

The projectivization of $Q$ provides the universal family of lines in $P^r$:

$$\begin{array}{ccc}
P(E) & \xrightarrow{p_1} & G(1, r) \\
& p_2 & \xrightarrow{} \ P^r.
\end{array}$$

From right to left, this diagram may be thought of as the universal family of $P^{r-1}$'s in $G(1, r)$. These $P^{r-1}$'s have degree one with respect to the Plücker polarization and their normal bundles in $G(1, r)$ are isomorphic to $T_{P^{r-1}}(-1)$. Finally we recall that the Chow ring of $G(1, r)$ is generated by a well determined type of cycles, called Schubert cycles. The generators in dimension two are given by: the cycle parameterizing lines in a $P^3 \subset P^r$ passing by a point, and the cycle parameterizing lines in a $P^2 \subset P^r$ (we denote it by $G(1, 2)$). They are called a and b–planes, respectively.

**Remark 2.1.** In particular, the second Chern class of a vector bundle $E$ on $G(1, r)$ is given by two integers, corresponding to the second Chern classes of the restrictions of $E$ to the planes described above.

2.2. Positivity, topology and vanishing results. The hypotheses on $X$ in [11] impose severe restrictions on its topology. In order to describe them explicitly let us recall the definition of $k$-ampleness in the sense of Sommese, see [So2 Def. 1.3]:

**Definition 2.2.** Let $E$ be a semiample vector bundle over a projective variety, i.e. $O_{P(E)}(m)$ is free for $m$ big enough. The vector bundle $E$ is said $k$-ample if every fiber of the morphism $\phi : P(E) \to P(H^0(P(E), O_{P(E)}(m)))$ has dimension less than or equal to $k$. 

In particular any $k$-ample vector bundle is nef and is ample if it is 0-ample. Sommese’s extension of Lefschetz Hyperplane Section Theorem \[ \text{L, II, Thm. 7.1.1} \] admits an extension to $k$-ample vector bundles, see \[ \text{So2 Prop. 1.16} \] quoted in \[ \text{L, II, Rmk. 7.1.9} \], which applies to our case giving the following relations between the topologies of $X$ and $G$.

**Lemma 2.3.** Let $X$, $G$ and $E$ be as in \[ \text{L, I} \]. The restriction map $r : \text{Pic}(X) \to \text{Pic}(G)$ is an isomorphism.

**Proof.** Denote by $r_1 : H^i(X, \mathbb{Z}) \to H^i(G, \mathbb{Z})$ the corresponding restriction morphisms. By \[ \text{So2 Prop. 1.16} \] we get that $r_1$ is an isomorphism and $r_2$ is injective with torsion free cokernel. Furthermore we may compare the exponential sequences of $X$ and $G$ to get the following diagram:

$$
\begin{array}{cccccc}
H^1(X, \mathbb{Z}) & \longrightarrow & H^1(X, \mathcal{O}) & \longrightarrow & \text{Pic}(X) & \longrightarrow & H^2(X, \mathbb{Z}) & \longrightarrow & H^2(X, \mathcal{O}) \\
r_1 & & r_1,1 & & r & & r_2 & & r_2,0 \\
H^1(G, \mathbb{Z}) & \longrightarrow & 0 & \longrightarrow & \text{Pic}(G) & \cong & \mathbb{Z} & \longrightarrow & 0 \\
\end{array}
$$

Since $r_1$ is an isomorphism and is compatible with the Hodge decomposition then $r_{1,1}$ is an isomorphism. Since $r_2$ is injective and with torsion free cokernel then it is an isomorphism and moreover $r_{2,0}$ is an isomorphism. This implies that $r$ is an isomorphism.

We will denote by $\mathcal{O}_X(1)$ the ample generator of $\text{Pic}(X)$, whose restriction to $G$ is the Plücker line bundle. The degree of the canonical sheaf of $X$ equals $\deg(K_G) - \deg(E)$, that is $K_X = \mathcal{O}(-r - 1 - c)$ and $X$ is Fano. Hence Kobayashi-Ochiai Theorem \[ \text{KO} \] provides the bound

(1) $c \leq r$.

Along this paper we will make use several times of the following variant of a vanishing theorem due to Griffiths \[ \text{L, II, Variant 7.3.2} \]:

**Theorem 2.4.** Let $M$ be a smooth complex projective variety of dimension $n$, $L$ an ample line bundle on $M$ and $F$ a nef vector bundle of rank $k$ on $X$, then:

$$H^i(M, \omega_M \otimes S^m F \otimes \det F \otimes L) = 0 \text{ for all } i > 0, m \geq 0.$$

Applied to our setting, the previous theorem provides the following vanishing.

**Lemma 2.5.** Under the assumptions in \[ \text{L, I} \] and for every positive integer $l$, it follows that

$$H^i(X, S^m E(l-r-1)) = 0, \text{ for all } i \geq 1, m \geq 0.$$

Being $G$ the subscheme of zeroes of a section of the rank two vector bundle $E$, the ideal sheaf of $G$ in $X$ has the following locally free presentation:

(2) $0 \to \det(E^\vee) \cong \mathcal{O}(-c) \longrightarrow E^\vee \cong E(-c) \longrightarrow \mathcal{I}_{G/X} \to 0$.

Combining it with Lemma 2.3 we immediately obtain:

**Lemma 2.6.** With the assumptions of \[ \text{L, I} \] the restriction maps

$$H^0(X, \mathcal{O}(k)) \to H^0(G, \mathcal{O}(k))$$

are surjective for all $k > 0$. 

PROOF. In fact, it is enough to check that \( H^1(X, \mathcal{I}_{G/X}(k)) = 0 \). Taking cohomology on sequence (2), it suffices to show that \( H^1(X, E(k - c)) = H^2(X, \mathcal{O}(k - c)) = 0 \). By Lemma 2.5, the first vanishing holds whenever \( k - c + r + 1 \geq 1 \). Since \( c \leq r \), see (1), that inequality is fulfilled for every positive \( k \). For the second vanishing note that since \( \text{Pic}(X) \cong \mathbb{Z} \), Kodaira vanishing implies that line bundles on \( X \) have no intermediate cohomology.

Let us take a projective space of maximal dimension contained in \( G \), say \( \mathbb{P}^{r-1} \subset M \subset G \), and denote by \( E_M \) the restriction of \( E \) to \( M \). Later on we will need to apply Theorem 2.4 to \( E_M \):

**Lemma 2.7.** With the same assumptions as in (1) and for every positive integer \( l \) it follows that:

\[
H^i(M, S^m E_M(l + c - r)) = 0, \quad \text{for } i \geq 1, m \geq 0.
\]

3. High index Fano varieties containing codimension two Grassmannians

With the same assumptions as in (1) we will rule the cases \( c = r, r - 1 \) and \( r - 2 \) out, which correspond to projective spaces, quadrics and Del Pezzo varieties, respectively. In order to do that, it suffices to show that \( h^0(X, \mathcal{O}(1)) < r(r+1)/2 = h^0(G, \mathcal{O}(1)) \) contradicting Lemma 2.6.

In the case \( c = r \) we get \( h^0(X, \mathcal{O}(1)) = 2r + 1 \) and hence it is smaller than \( r(r+1)/2 \) whenever \( r \geq 4 \).

If \( c = r - 1 \), \( h^0(X, \mathcal{O}(1)) = 2r + 2 \), which is smaller than \( r(r+1)/2 \) if \( r \geq 5 \). The case \( r = 4 \) would correspond to a smooth quadric \( \mathbb{Q}^8 \subset \mathbb{P}^9 \) containing a Grassmannian \( G \cong \mathbb{G}(1,4) \), embedded in \( \mathbb{P}^9 \) via the Plücker map. But quadrics containing \( \mathbb{G}(1,4) \) are given by \( 4 \times 4 \) pfaffians, hence singular.

In order to rule out the case of Del Pezzo varieties, we will make use of Fubini’s classification (cf. [F3, 8.11, p. 72], see also [K, V.1.12]). Being \( \dim X = 2r \geq 8 \), the only possible values of \( h^0(X, \mathcal{O}(1)) \) are \( 2r, 2r+1, 2r+2 \) and \( 2r+3 \), which are smaller that \( r(r+1)/2 \) except in the following cases:

- \( X \) is a smooth cubic hypersurface in \( \mathbb{P}^9 \) containing a Plücker embedded Grassmannian \( G \cong \mathbb{G}(1,4) \). Recall the notation on Grassmannians established in (1) and note that the normal bundle of \( G \) in \( \mathbb{P}^9 \) is \( \mathcal{O}(1) \otimes \wedge^2 S \cong \mathcal{O}(2) \otimes S' \) where \( S' \) denotes the universal subbundle, see for instance [M, Prop. 4.5.1]. In particular, denoting by \( E_G \) the restriction of \( E \) to \( G \) we get the following exact sequence:

\[
0 \to E_G \to S'(2) \to \mathcal{O}(3) \to 0.
\]

Tensoring by \( \mathcal{O}(-3) \) we get \( H^1(G, E_G(-3)) \neq 0 \). Now use Serre duality to get \( h^1(G, E_G(-3)) = h^5(G, E_G'(3) \otimes \omega_G) = h^5(G, E_G(1) \otimes \omega_G) \). Since \( E_G(1) \) is ample we get a contradiction with Le Potier Vanishing Theorem [L, Thm. 7.3.5].

- \( X \) is a smooth complete intersection of two quadrics \( \mathbb{Q}_1 \) and \( \mathbb{Q}_2 \) in \( \mathbb{P}^{10} \) containing a Plücker embedded Grassmannian \( G \cong \mathbb{G}(1,4) \). We may argue as before: observe on one hand that as a consequence of Theorem 2.4 we get that \( h^1(G, E_G(-2)) = 0 \). But on the other hand taking cohomology on
the following exact sequences we get the contradiction \( h^1(G, E_G(-2)) \neq 0: \)

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & E_G & \longrightarrow & N_{G/P^{10}} & \longrightarrow & \mathcal{O}(2)^{\oplus 2} & \longrightarrow & 0, \\
0 & \longrightarrow & S^r(2) & \longrightarrow & N_{G/P^{10}} & \longrightarrow & \mathcal{O}(1) & \longrightarrow & 0.
\end{array}
\]

As a corollary of what we have proved and recalling that \( E \) is nef we get:

**Lemma 3.1.** Under the assumptions of [1.1] we get that \( 0 \leq c < r - 2 \).

4. **Uniform vector bundles on Grassmannians**

Uniform vector bundles of low rank on Grassmannians have been classified by Guyot, cf. [G]. For the sake of completeness we present here a proof for rank two vector bundles \( E \) on \( \mathbb{G}(1, r) \), using minimal sections of \( E \) over its lines. Although we need only the case \( r \geq 4 \) we include a proof working for any \( r \geq 2 \).

Let us recall that a rank \( k \) vector bundle \( E \) on \( \mathbb{G}(1, r) \) is *uniform of type* \((a_1, \ldots, a_k) (a_1 \leq \cdots \leq a_k)\) if for any line \( \ell \subset \mathbb{G}(1, r) \) the restriction of \( E \) to \( \ell \) splits as \( \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_k) \). The result is the following:

**Proposition 4.1.** Every uniform rank two vector bundle \( E \) on \( G := \mathbb{G}(1, r) \) of type \((0, 1)\) is isomorphic either to \( \mathcal{O} \oplus \mathcal{O}(1) \) or to the universal bundle \( \mathcal{O} \).

**Proof.** Note that for \( r = 2 \) the result is due to Van de Ven (cf. [VV, OSS Thm. 2.2.2]), and we may assume that \( r \geq 3 \).

First we show that there exists a family of linear subspaces of \( G \) of maximal dimension verifying that \( E|_{G} \cong \mathcal{O} \oplus \mathcal{O}(1) \). In fact, if \( r \geq 4 \), the restriction of \( E \) to a \( \mathbb{P}^{r-1} \) is isomorphic to \( \mathcal{O} \oplus \mathcal{O}(1) \) by the classification of uniform vector bundles on projective spaces (cf. [EHS, OSS Thm. 3.2.3]). For the case \( r = 3 \) recall that the Grassmannian \( \mathbb{G}(1, 3) \) contains two families of \( \mathbb{P}^2 \)'s that we call \( a \) and \( b \)-planes, see Section 2.1. Let us prove that the restriction of \( E \) could not be isomorphic to \( T_{\mathbb{P}^2}(-1) \) for both families. If this occurs then \( c_2(E) \) equals the union of two planes, one of each family, see Remark 2.1. Assume by contradiction that this is the case. Consider two \( a \)-planes \( a_1 \) and \( a_2 \) and denote by \( P \) their intersection and by \( r \) the corresponding line in \( \mathbb{P}^3 \). For every plane \( M \) containing \( r \) (determining a \( b \)-plane \( b_M \) containing \( P \)) we get two lines \( r_1(M) = b_M \cap a_1 \) and \( r_2(M) = b_M \cap a_2 \). For each line \( r_i (i = 1, 2) \) we get a lifting into \( \mathbb{P}(E) \) determined by the unique surjective map \( E|_{r_i(M)} \rightarrow \mathcal{O} \). Denote by \( R_1(M) \) and \( R_2(M) \) the intersections of these liftings with the fiber over \( P \). By hypothesis \( E|_{r_i} \cong T(-1) \), hence the maps sending \( M \rightarrow R_1(M) \) and \( M \rightarrow R_2(M) \) are isomorphisms from the set of planes containing \( r \) to the fiber over \( P \). In particular there exists \( M_0 \) such that \( R_1(M_0) = R_2(M_0) \). Now we consider the \( b \)-plane \( b_{M_0} \). It contains two lines whose distinguished liftings meet at one point. Then the restriction of \( E \) to \( b_{M_0} \) cannot be \( T(-1) \).

Recall that the family of \( \mathbb{P}^{r-1} \)'s of the previous paragraph is parameterized by a projective space \( \mathcal{M} \cong \mathbb{P}^r \). Each element of this family admits a lifting to \( \mathbb{P}(E) \) given by the unique surjective morphism \( E|_{\mathbb{P}^r} \rightarrow \mathcal{O} \) and we have the following diagram:

\[
\begin{array}{ccc}
\mathbb{P}^r & \overset{g}{\longrightarrow} & \mathbb{P}(E) \\
\downarrow_{p_1} & & \downarrow_{p_2} \\
\mathcal{M} & \cong & \mathbb{P}^r \\
& \overset{\pi}{\longrightarrow} & \mathbb{G}(1, r)
\end{array}
\]
where $Q$ stands for the universal quotient bundle on $G(1, r)$.

Now consider the restriction of $E$ to any $G(1, 2) \subset G$. The restriction $E|_{G(1, 2)}$ is either decomposable or isomorphic to $T(-1)$ by Van de Ven’s result. We claim that in the former case $E$ is decomposable. In fact, if this occurs, we push it down to $G$. By Lemma 2.7 we get $H^0(G(1, 2), T^2(-1)) = 0$ for any $G(1, 2) \subset G$. Arguing as in the previous paragraph, we may prove that in this case the map $g$ is surjective. Moreover there cannot be two liftings of $\mathbb{P}^{r-1}$’s passing by the same point of $\mathbb{P}(E)$. In fact, if this occurs, we push it down to $G$ and we find a $G(1, 2)$ meeting the two $\mathbb{P}^{r-1}$’s in two lines. But the (unique) lifting of this two lines to $\mathbb{P}(E)$ as curves of degree 0 with respect to $O(1)$ do not meet, since they lie in $G(1, 2)$ and $E|_{G(1, 2)} = O(1) \oplus O(1)$. In particular $g(\mathbb{P}(Q))$ meets the fiber $\pi^{-1}(x)$ in one point, hence $\pi : \mathbb{P}(E) \to G$ has a section and so $E$ splits as a sum of line bundles.

From now on we assume that $E|_{G(1, 2)} \cong T^2(-1)$ for any $G(1, 2) \subset G$. Arguing as in the previous paragraph, we may prove that in this case the map $g$ is surjective. Moreover there cannot be two liftings of $\mathbb{P}^{r-1}$’s passing by the same point of $\mathbb{P}(E)$. In fact, if this occurs, we push it down to $G$ and we find a $G(1, 2)$ meeting the two $\mathbb{P}^{r-1}$’s in two lines. But the (unique) lifting of this two lines to $\mathbb{P}(E)$ as curves of degree 0 with respect to $O(1)$ do not meet, since they lie in $G(1, 2)$ and $E|_{G(1, 2)} \cong T^2(-1)$, a contradiction.

Summing up, the morphism $g : \mathbb{P}(Q) \to \mathbb{P}(E)$ is bijective, and the proof is finished.

5. Determining $E$

In this section we will prove the following:

**Proposition 5.1.** Under the assumptions of 1.1 the vector bundle $E$ verifies that the restriction of $E$ to $G$ is isomorphic to $Q$, where $Q$ stands for the universal quotient bundle.

We begin by studying the restriction $E_M$ of $E$ to a projective space of dimension $r - 1$, $\mathbb{P}^{r-1} \cong M \subset G$. As a consequence of the upper bound on $c$ of 3.1 and of the numerical characterization of rank two Fano bundles onto projective spaces, see for example [APW], we get:

**Lemma 5.2.** Under the conditions above $E_M$ splits either as $E_M = O(1)^{\oplus 2}$ or as $E_M = O(1) \oplus O$.

**Proof.** Take the projective bundle $\pi : \mathbb{P}(E_M) \to M$. Since $-K_{\mathbb{P}(E_M)} = O(2) \otimes \pi^*O(r + 1 - c)$ then $\mathbb{P}(E_M)$ is a Fano variety, i.e. $E_M$ is a rank two Fano bundle. Hence we can use the classification of rank two Fano bundles, see [APW] Main Thm. and [SW] Thm. (2.1), to get that either $E_M$ splits as a sum of line bundles or $r = 4$, $c = 2$ and $E_M = N(1)$, being $N$ a null correlation bundle. This last possibility is excluded by the bound $c < r - 2$ of 3.1 so that $E_M$ splits as a sum of line bundles.

Moreover the Bend and Break lemma leads to the following vanishing:

$$H^0(M, E_M(-2)) = 0. \tag{3}$$

In fact, consider the exact sequence:

$$0 \to T_M(-3) \cong N_{M/G}(-2) \to N_{M/X}(-2) \to E_M(-2) \to 0.$$

By Lemma 2.7 we get $H^1(M, E_M(-2)) = 0$. Taking cohomology in the Euler sequence tensored with $O(-2)$ we get that $H^1(M, T_M(-3)) = 0$ and therefore $H^1(M, N_{M/X}(-2)) = 0$. In particular the subscheme $M_Q$ of the Hilbert scheme
parametrizing deformations of \( M \) in \( X \) containing a fixed smooth quadric \( Q \subset M \) is smooth at the point \([M]\) and its dimension equals \( H^0(M, N_{M/X}(-2))\).

But \( \mathcal{M}_Q \) must be zero dimensional, otherwise given two general points \( p, q \in Q \), for every deformation \( M_t \) of \( M \) we could consider the line \( t_t \subset M_t \) joining \( p \) and \( q \). Then a Bend and Break argument (cf. [De 3.2]) provides a reducible cycle \( C \) algebraically equivalent to \( t_t \), contradicting the fact that \( t_t \) has degree 1 with respect to \( \mathcal{O}(1) \). This implies that \( H^0(M, N_{M/X}(-2)) = 0 \), and so \( H^0(M, E_{M}(-2)) = 0 \), too.

Since \( E_M \) is nef then, by the splitting of \( E_M \) and (3), we get that \( E_M = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \) with \( 0 \leq a_1 \leq a_2 \leq 1 \) and \( a_1 + a_2 = c \). Hence \( c \leq 2 \), being \( E_M = \mathcal{O}(1)^{\oplus 2} \) if \( c = 2 \) and \( E_M = \mathcal{O}(1) \oplus \mathcal{O} \) if \( c = 1 \). If \( c = 0 \) then denote by \( E_G \) the restriction of \( E \) to \( G \). The rank two vector bundle is uniform with respect to the family of lines and in fact it is trivial, see [AW (1.2)]. This contradicts the fact that the Picard number of \( X \) is one, see [MS Lemma 3.6].

Now we can complete the proof of Proposition 5.1.

**Proof.** First we prove that the case \( c = 2 \) cannot occur. Recall that \( E_G \) stands for the restriction of \( E \) to \( G \). As \( E_M = \mathcal{O}(1)^{\oplus 2} \) we get that for any line \( \ell \subset G \) the restriction of \( E_G \) to \( \ell \) is \( \mathcal{O}(1)^{\oplus 2} \). This implies uniformity of \( E_G \) with respect to the family of lines and moreover \( E_G = \mathcal{O}(1)^{\oplus 2} \), [AW (1.2)]. Consider the exact sequence of (2)

\[
0 \to \mathcal{O}(1) \otimes \mathcal{I}_{G/X} \to E(1) = 0
\]

and tensor it by \( E(-1) \) to get:

\[
0 \to E(-3) \to E \otimes E(-3) \to E \otimes \mathcal{I}_{G/X}(-1) \to 0.
\]

By the usual decomposition \( E \otimes E \cong S^2 E \oplus \Lambda^2 E \) and the vanishing of Lemma 2.5 we get \( h^1(X, E \otimes \mathcal{I}_{G/X}(-1)) = 0 \). Now consider the exact sequence

\[
0 \to E \otimes \mathcal{I}_{G/X}(-1) \to E(-1) \to E_G(-1) = \mathcal{O}^{\oplus 2} \to 0
\]

to get that \( h^0(X, E(-1)) \geq 2 \) and that \( E(-1) \) is generically globally generated. Hence, see [MS Lemma 3.5], \( E(-1) = \mathcal{O}^{\oplus 2} \). Tensoring the exact sequence (3) by \( \mathcal{O}(1) \) we observe that \( \mathcal{I}_{G/X}(1) \) is globally generated. This implies, see [BS Cor. 1.7.5], that there exists a smooth element in the linear system \(|\mathcal{O}(1)|\) containing \( G \), which contradicts [FL Thm. 5.2].

If \( c = 1 \) we have shown in Proposition 4.1 that \( E_G \) is either as stated or splits as \( E_G = \mathcal{O} \oplus \mathcal{O}(1) \). If \( E_G \) splits, exactly as in the proof of the case \( c = 2 \), we get \( H^0(X, E(-1)) \neq 0 \). But this is a contradiction: in fact the exact sequence of (2)

\[
0 \to \mathcal{O}(-1) \to E(-1) \to \mathcal{I}_{G/X} \to 0
\]

gives \( H^0(X, E(-1)) = 0 \). This concludes that \( E_G = \mathcal{O} \).

\[\blacksquare\]

6. **Proof of the main Theorem**

Let us give the proof of Theorem 1.2.

**Proof.** Let us recall that as a consequence of what we proved in the Section 5 we can suppose that \( c = 1 \) and that \( E_G = \mathcal{O} \). Consider the projective bundle \( \pi : \mathbb{P}(E) \to X \), which is a Fano variety. Recall that \( E \) is nef by hypothesis and
not ample as \( c = 1 \). Hence we get that for \( m \) big enough the linear system \(|\mathcal{O}(m)|\) defines an extremal ray contraction \( \varphi \) leading to the following diagram:

\[
P(E) \xrightarrow{\varphi} Z, \\
\pi \\
X
\]

where \( Z \) is normal. For \( G \subset X \) we get that \( E_G = \mathcal{Q} \) so that, taking care of the Mori cone of \( \mathbb{P}(E_G) \), the following diagram appears:

\[
\begin{array}{ccc}
P(Q) & \xrightarrow{\varphi_1} & \mathbb{P}^r \\ \text{\( \pi_1 \)} & \downarrow & \downarrow \text{\( f \)} \\
\mathbb{P}(E) & \xrightarrow{\varphi} & Z, \\ \text{\( \pi \)} & \downarrow & \downarrow \\
G' & \xrightarrow{\pi} & X
\end{array}
\]

being \( \pi_1 \) and \( \varphi_1 \) the corresponding contractions of \( \mathbb{P}(Q) \) and \( f \) finite onto its image, which implies that \( \dim Z \geq r \).

Now we claim that the general fiber \( F \) of \( \varphi \) is isomorphic to \( \mathbb{P}^r \). In fact, \( F \) is irreducible and smooth by Bertini’s Theorem and, if it is not a single point, adjunction formula tells us that

\[-K_F = -K_{\mathbb{P}(E)|F} = \pi^*\mathcal{O}(1)^{\otimes(r+1)}.
\]

But \( \pi|_F \) is finite, hence \( \pi^*\mathcal{O}(1)|_F \) is ample and the above formula implies that, if not a point, \( F \) is a Fano manifold of index greater than or equal to \( r + 1 \). Recall that \( \dim F \leq r + 1 \), hence either \( F \) is a point or \( F \cong \mathbb{P}^r \) and \( \pi^*\mathcal{O}(1)|_F = \mathcal{O}(1) \) or \( F \) is a smooth quadric of dimension \( r + 1 \). In order to exclude the first and the last possibility let us introduce some notation. Since \( N_{G/X} = E_G = \mathcal{Q} \), which is globally generated, then there exists a \((r + 1)\)-dimensional irreducible variety \( G \) parameterizing deformations of \([G]\) which in fact contains the point corresponding to \( G \), say \([G] \in G \), as a smooth point. The family \( G \) dominates \( X \). By rigidity of Grassmannians, the general point \([G'] \in G \) is isomorphic to \( G(1, r) \). Moreover \( E_{G'} \) is nef and its Chern polynomial is that of \( E_G \). In particular it is uniform so that \( E_{G'} = \mathcal{Q} \), see Proposition 4.1. Thus, for the general point \( y \in \mathbb{P}(E) \) there exists \([G_y] \in G \) such that \( y \in \mathbb{P}(E_{G_y}) \) and provides a diagram as the one of (5). Therefore

\[
\varphi_1^{-1}f^{-1}(\varphi(y)) \supset \mathbb{P}^{r-1} \subset F
\]

and this inclusion excludes the possibility of \( F \) to be a point or a smooth quadric, being \( r \geq 4 \). Summing up we have shown that \( F \cong \mathbb{P}^r \) and \( \pi^*\mathcal{O}(1)|_F = \mathcal{O}(1) \) so that \( \pi(F) \cong \mathbb{P}^r \subset X \). Moreover, since the fibers of \( \varphi \) dominates \( X \) via \( \pi \), then the normal bundle \( N_{\pi(F)/X} \) is generically globally generated.

We claim that \( N_{\pi(F)/X} \cong T_{\mathbb{P}(E)}(-1) \). Consider the Euler sequence

\[
0 \to \mathcal{O} \to \pi^*(E^\vee) \otimes \mathcal{O}(1) \to T_{\mathbb{P}(E)/X} \to 0
\]

and restrict it to \( F \) to get that

\[
T_{\mathbb{P}(E)/X} \otimes \mathcal{O}_F = \mathcal{O}(1).
\]
Then, since \( \pi \) is an isomorphism, identifying isomorphic objects, we get the following diagram:

\[
\begin{array}{cccc}
0 & \rightarrow & T_{P(E)/X} \otimes O_F & \xrightarrow{\pi} & \mathcal{O}(-1) \\
0 & \rightarrow & T_P & \xrightarrow{\sim} & T_{P(E)} \otimes O_F \\
0 & \rightarrow & T_{\pi(F)} & \rightarrow & T_X \otimes \mathcal{O}_{\pi(F)} \\
0 & \rightarrow & N_{\pi(F)/X} & \rightarrow & 0.
\end{array}
\]

The last vertical sequence is that of Euler and \( N_{\pi(F)/X} = T_{\mathbb{P}^r}(-1) \) as claimed.

Now we claim that \( \varphi \) is equidimensional. Let us suppose the existence of a fiber \( F_0 \) such that \( \dim(\pi(F_0)) > r \). Recall that for the general point \( x \in X \) there passes the image by \( \pi \) of a general fiber \( F \) of \( \varphi \) and moreover \( \pi(F) \cong \mathbb{P}^r \), \( N_{\pi(F)/X} = T_{\mathbb{P}^r}(-1) \) and

\[(7)\]

\[c_r(N_{\pi(F)/X}) = 1.\]

Hence there exists a component \( \mathcal{M} \) of the Hilbert scheme of \( \mathbb{P}^r \)'s in \( X \) containing \([\pi(F)]\) as a smooth point and sweeping out \( X \). Through the general point \( x \in \pi(F_0) \) there exists \([M] \in \mathcal{M}\) such that \( x \in M \cong \mathbb{P}^r \subset X \). Since \( \dim(M \cap \pi(F_0)) \geq 1 \) and \( E|_M = O \oplus O(1) \) then the intersection \( M \cap \pi(F_0) \) admits a unique section into \( \mathbb{P}(E) \) contracted by \( \varphi \). It follows that \( F_0 \) intersects the only section \( M_0 \) over \( M \) contracted by \( \varphi \) so that it contains it, i.e. \( M_0 \subset F_0 \). Now consider a general \( y \in \mathbb{P}(E) \) Recall that \( F_y = \varphi^{-1}(\varphi(y)) \cong \mathbb{P}^r \) and \([\pi(F_y)] \in \mathcal{M}\). Moreover, since \( E|_{\pi(F_y)} = O \oplus O(1) \) then \( F_y \) is the unique section of \( E|_{\pi(F_y)} \) contracted by \( \varphi \). But now observe that as a consequence of the selfintersection formula and (7) any element in \( \mathcal{M} \) is meeting \( M \) and therefore \( \pi(F_y) \cap M \neq \emptyset \) which in particular gives \( \pi(F_y) \cap \pi(F_0) \neq \emptyset \). But this leads to the contradiction \( F_y \subset F_0 \).

From the fact that \( \varphi \) is equidimensional it follows that \( \varphi : \mathbb{P}(E) \rightarrow Z \) is a \( \mathbb{P}^r \)-bundle, that is all fibers are linear and \( \varphi \) is providing the structure of projective bundle, see [P2] 2.12 [quoted in [BS] Prop. 3.2.1]. In particular \( Z \) is smooth.

Recall that \( \varphi \) is defined by the system \([O(m)]\). We claim that we may assume \( m = 1 \). In fact, take \( x \in G \subset X \) and the fiber of \( \pi \) over it, that is \( \ell_x \cong \mathbb{P}^1 = \pi^{-1}(x) \). Consider \( y \in \ell_x \) and the fiber \( F_y \) of \( \varphi \) through \( y \). Now observe that \([\pi(F_y)] \in \mathcal{M}\) and that \( F_y \) corresponds to the only section of \( E|_{\pi(F_y)} \). Then \( F_y \cap \ell_x = \{y\} \) so that \( \varphi|_{\ell_x} \) is a one-to-one map from \( \mathbb{P}^1 \) onto its image in \( Z \). Hence the restriction of \( f \) to \( \varphi_1(\ell_x) \) is an isomorphism, for every \( x \in G \). Since \( G \) parametrizes all the lines of \( \mathbb{P}^r \), it follows that \( f \) itself is an isomorphism. Therefore we may consider \( \mathbb{P}^r \) as an effective divisor in the smooth variety \( Z \). Since \( \text{Pic}(Z) = \mathbb{Z} \), then \( \mathbb{P}^r \subset Z \) is ample and Kobayashi-Ochiai Theorem tells us that \( Z \cong \mathbb{P}^{r+1} \) and \( O_Z(\mathbb{P}^r) \cong O_{\mathbb{P}^{r+1}}(1) \). In particular, fibers of \( \pi \) map onto lines of \( Z \cong \mathbb{P}^{r+1} \).
The next step in the proof is to observe that through any two points \( x, y \in X \) there cannot pass two elements of \( \mathcal{M} \). In fact, by the self intersection formula and \( \mathcal{L} \) it holds that two possible different elements \( M_1, M_2 \) of \( \mathcal{M} \) through \( x \) and \( y \) must meet in a positive dimensional subvariety \( P = M_1 \cap M_2 \). But \( E_{M_i} = \mathcal{O}_{P} \oplus \mathcal{O}(1) \) for \( i = 1, 2 \) so that, exactly as in the proof of the equidimensionality of \( \varphi \), the corresponding unique sections \( \mathbb{P}^r \cong F_i \subset \mathbb{P}(E) \) such that \( \pi(F_i) = M_i \) are going to the same point by \( \varphi \), contradicting the fact that \( \varphi : \mathbb{P}(E) \to Z \) is a \( \mathbb{P}^r \)-bundle.

Recall that \( \ell_x := \varphi(\pi^{-1}(x)) \) is a line in \( Z \cong \mathbb{P}^{r+1} \) for all \( x \in X \). This provides a map \( g : X \to \mathbb{G}(1, r+1) \) sending \( x \) to \( \ell_x \). Since \( X \) and \( \mathbb{G}(1, r+1) \) are smooth of Picard number one then we conclude the proof of the theorem by showing that \( g \) is surjective and generically injective. It is then enough to prove that for the general \( x \in X \) there is no \( y \in X \) different from \( x \) such that \( \ell_x = \ell_y \). Suppose on the contrary the existence of such \( y \in X \setminus \{x\} \). For any point \( z \in \ell_x \) we get that \( \varphi^{-1}(z) = \mathbb{P}^r \) is meeting the lines \( \pi^{-1}(x) \) and \( \pi^{-1}(y) \). This implies that \( \pi(\varphi^{-1}(z)) \) is the only element \( \mathbb{P}^r = M \in \mathcal{M} \) through \( x \) and \( y \). This provides a one dimensional family of sections of \( E_{M}' \), which is a contradiction.

\begin{remark}
\textbf{Remark 6.1.} Let us remark that, as has been seen in the course of the proof, the hypothesis on the \((2r-4)\)-ampleness of \( E \) can be substituted by the hypothesis on the restriction map \( r : \text{Pic}(X) \to \text{Pic}(G) \) to be an isomorphism. Note that \( G \) appears as the zero set of a \((2r-2)\)-ample vector bundle on, for instance, the product \( X = G \times \mathbb{P}^2 \), but \( \text{Pic}(X) \neq \mathbb{Z} \). A similar situation appears by considering the desingularization of a cone over \( G \) with vertex a line. We do not know yet of any example in which \( E \) is \((2r-3)\)-ample and the restriction \( r \) is not an isomorphism.
\end{remark}

\section{7. Low values of \( r \)}

The case \( r = 3 \) can be seen as a particular case of the general problem of quadrics appearing as the zero locus of sections of positive rank two vector bundles. This is well understood in the case in which \( E \) is ample \cite{LM} (in fact in any codimension). Here we can prove the following:

\begin{proposition}
Let \( X \) be as smooth complex projective variety of dimension \( n \geq 6 \). Suppose the existence of a rank two nef vector bundle \( E \) on \( X \) and a section of \( E \) vanishing on a smooth quadric \( Q \subset X \). If the restriction map \( r : \text{Pic}(X) \to \text{Pic}(Q) \) is an isomorphism then \((X, E)\) is either
\begin{itemize}
  \item \((\mathbb{P}^n, \mathcal{O}(2) \oplus \mathcal{O}(1)), \) or
  \item \((Q, \mathcal{O}(1) \oplus \mathcal{O}(1)), \) or
  \item \((\mathbb{G}(1,4), \mathcal{Q}).
\end{itemize}
\end{proposition}

\begin{proof}
Denote as usual by \( \mathcal{O}(1) \) the ample generator of \( \text{Pic}(X) \) and by \( c \) the degree of the determinant of \( E \). Recall that since \( E \) is nef and has a section vanishing on \( E \) then \( c > 0 \). Now use adjunction formula to get that \( K_X = \mathcal{O}(-(n-2)-c) \). This implies that either \( c = 3 \) and \( X = \mathbb{P}^n \) or \( c = 2 \) and \( X = Q \) or \( c = 1 \). Hence we can suppose that \( c = 1 \) which means that \( X \) is a Del Pezzo Variety. Now we apply \cite{MS} Prop. 4.5 to get that \( X = \mathbb{G}(1,4) \). If \( X \cong \mathbb{G}(1,4) \) then \( E \) either splits as a sum of line bundles or \( E \cong Q \), see Proposition \cite{LM}. But in case \( E = \mathcal{O} \oplus \mathcal{O}(1) \) there are no sections vanishing on a codimension two variety and the result follows.
\end{proof}
The case $r = 2$ can be seen as a particular case of the general problem of linear spaces appearing as the zero locus of sections of positive rank two vector bundles. See [LM] for the case in which $E$ is ample. Here we can prove the following:

**Proposition 7.2.** Let $X$ be a smooth complex projective variety of dimension $n \geq 4$. Suppose the existence of a rank two nef vector bundle $E$ on $X$ and a section of $E$ vanishing on a linear space $\mathbb{P}^{n-2} \subset X$. If the restriction map $r : \text{Pic}(X) \to \text{Pic}(G)$ is an isomorphism then $(X, E)$ is either
- $(\mathbb{P}^n, \mathcal{O}(1) \oplus \mathcal{O}(1))$,
- $(\mathbb{G}(1, 3), \mathcal{Q})$.

**Proof.** With the same notation as before we get by adjunction that $K_X = \mathcal{O}(-(n-1) - c)$. Then either $c = 2$ and $X \cong \mathbb{P}^n$ or $c = 1$ and $X$ is a smooth quadric so that $\dim(X) \leq 4$, in fact equal by hypothesis. Since $E$ is uniform then either $E = \mathcal{O} \oplus \mathcal{O}(1)$ and no section vanishes in a codimension two subvariety or $E = \mathcal{Q}$ and we conclude.

**Remark 7.3.** For Propositions 7.1 and 7.2 let us remark that if we impose on $E$ to be $(n-4)$-ample then we get that the restriction morphism $r : \text{Pic}(X) \to \text{Pic}(G)$ is an isomorphism, exactly as in Lemma 2.3.

### 8. Fibrations in Grassmannians of lines

Inspired by [BdFL, Def. 5.1] we can give the following definition.

**Definition 8.1.** A surjective morphism $\pi : Y \to Z$ between a smooth projective variety $Y$ and a normal projective variety $Z$ is called a $\mathbb{G}(1, r)$-fibration if $\pi$ is an elementary Mori contraction and there is a line bundle $L$ on $Y$ such that the general fiber $G$ of $\pi$ is isomorphic to $\mathbb{G}(1, r)$ and $L|_G$ is the Plücker line bundle.

If $\dim(Z) \leq 2$ it suffices to check this hypothesis on a fiber:

**Lemma 8.2.** Let $\pi : Y \to Z$ be a morphism between a smooth projective variety $Y$ and a normal projective variety $Z$ such that $\dim(Z) \leq 2$. If there exists $z$ a smooth point of $Z$ and $L \in \text{Pic}(Y)$ such that $G := \pi^{-1}(z)$ is isomorphic to $\mathbb{G}(1, r)$ and $L|_G$ is the Plücker line bundle then $\pi : Y \to Z$ is a $\mathbb{G}(1, r)$-fibration. Moreover $Z$ is smooth and all smooth fibers of $\pi$ are isomorphic to $\mathbb{G}(1, r)$.

**Proof.** Since the normal bundle $N_{G/Y}$ is trivial then, in particular is generically globally generated and its determinant is also trivial. Up to replacing $L$ with $L \otimes \pi^* A$, with a suitable ample line bundle $A$ on $Z$ we may assume that $L$ is ample and we can apply [MS, Lemma 2.5] to get that $\pi$ is the contraction of an extremal ray. Moreover, by [AW, Cor. 1.4], $\pi$ is equidimensional and $Z$ smooth. By rigidity of Grassmannians, see for instance [HM], any smooth fiber is isomorphic to $\mathbb{G}(1, r)$ and the lemma follows.

**Proposition 8.3.** For $r \geq 4$ a $\mathbb{G}(1, r)$-fibration $Y$ cannot appear either as an ample divisor or as the zero locus of a section of a rank two ample vector bundle $E$ over a smooth projective variety $X$.

**Proof.** Suppose on the contrary that $Y \subset X$ appears as the zero locus of a section of $E$. Then, by Lefschetz-Sommessse Theorem, the restriction map from
Pic(X) to Pic(Y) is an isomorphism. Hence we may use [BdFL] Thm 4.1 to get a diagram:

\[
\begin{array}{c}
Y \xrightarrow{\phi} X \\
\pi \downarrow \quad \downarrow \\
Z \xrightarrow{\delta} S,
\end{array}
\]

where \( \phi \) is an elementary Mori contraction on \( X \) and \( \delta \) is a finite morphism.

Consider a general point \( s \in S \) and denote by \( F_s = \phi^{-1}(s) \) the fiber of \( \phi \) over \( s \), which is connected. Since \( \delta \) is finite then \( \delta^{-1}(s) = \{z_1, \ldots, z_d\} \). Denote by \( G_i = \pi^{-1}(z_i) \), \( 1 \leq i \leq d \), so that \( F_s \cap Y = G_1 \cup \cdots \cup G_d \), where \( G_i \cap G_j = \emptyset \) for \( i \neq j \). Recall that \( Y \) is defined as the zero locus of a section of an ample vector bundle and, since \( E|_{F_s} \) is ample then \( F_s \cap Y \) is also the zero locus of a section of an ample vector bundle. Then, by Lefschetz-Sommese Theorem, \( F_s \cap Y \) is connected so that \( d = 1 \), that is \( F_s \cap Y = G_1 \cong G(1,1) \). But \( G(1,1) \) cannot appear either as an ample divisor on \( F_s \) by [F1] Thm. 5.2 or as the zero locus of a section of \( E|_{F_s} \) by Theorem [BI]. This concludes the result.

**Remark 8.4.** Using [BdFL] Thm. 3.6, a similar statement holds under different hypotheses. We could have assumed that there exists an unsplit covering family \( V \) of rational curves in \( Y \) verifying the following: it restricts to a family \( V_Y \) covering \( Y \) and the general equivalence class in \( Y \) with respect to \( V_Y \) is isomorphic to \( G(1,1) \).

### References

[APW] Ancona, V., Peternell, T. and Wiśniewski, J. *Fano bundles and splitting theorems on projective spaces and quadrics*, Pacific J. Math. **163**, no. 1, 17-41 (1994).

[AW] Andreotti, A. and Wiśniewski, J. *On manifolds whose tangent bundle contains an ample locally free subsheaf*, Invent. Math. **146**, 209-217 (2001).

[A] Arrondo, E. *Subvarieties of Grassmannians*, Lecture Note Series Dipartimento di Matematica Univ. Trento, 10 (1996).

[BdFL] Beltrametti, M.C., de Fernex, T. and Lanteri, A. *Ample subvarieties and rationally connected fibrations*, Math. Ann. **341**, 897-926 (2008).

[BI1] Beltrametti, M.C., and Ionescu, P. *On manifolds swept out by high dimensional quadrics*. Math. Z. **260**, 229-234 (2008).

[BI2] Beltrametti, M.C., and Ionescu, P. *A view on extending morphisms from ample divisors*. To appear in Contemporary Mathematics.

[BS] Beltrametti, M.C. and Sommese, A. J. *The Adjunction Theory of Complex Projective Varieties*. De Gruyter Expositions in Mathematics 16, De Gruyter, Berlin-New York, 1995.

[BSW] Beltrametti, M. C., Sommese, A. J. and Wiśniewski, J. *Results on varieties with many lines and their applications to adjunction theory*. Complex Algebraic Varieties (Bayreuth, 1990), Lecture Notes in Mathematics 1507, Springer, Berlin, 1992, pp. 16-38.

[De] Debarre, O. *Higher-Dimensional Algebraic Geometry*. Springer-Verlag New York, 2001.

[EHS] Elencwajg, G., Hirschowitz, A., and Schneider, M. *Les fibres uniformes de rang au plus n sur \( \mathbb{P}_n(\mathbb{C}) \) sont ceux qu’on croit*. Vector bundles and differential equations (Proc. Conf., Nice, 1979), Progr. Math., **7**, 37-63 (1980).

[Fu] Fu, B. *Inductive characterizations of hyperquadrics*. Math. Ann. **340**, no. 1, 185-194 (2008).

[F1] Fujita, T. *Vector bundles on ample divisors*, J. Math. Soc. Japan **33**, no. 3, 405-414 (1981).

[F2] Fujita, T. *On Polarized manifolds whose adjoint bundles are not semipositive*. in *Algebraic Geometry, Sendai 1985*. Adv. Stud. Pure Math. **10**, 167-178 (1987).

[F3] Fujita, T. *Classification Theories of Polarized Varieties*. London Mathematical Society Lecture Note Series, no. 155. Cambridge University Press, Cambridge, 1990.

[G] Guyot, M. *Caractérisation par l’uniformité des fibrés universels sur la Grassmannienne*. Math. Ann. **270**, 47-62 (1985).
[H] Hartshorne, R. Ample subvarieties of algebraic varieties. Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin-New York 1970.

[HM] Hwang, J-M. and Mok, N. Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation, Invent. Math. 131, 393-418 (1998).

[KO] Kobayashi, S., Ochiai, T. Characterization of complex projective spaces and hyperquadrics, J. Math. Kyoto Univ. 13, 31-47 (1973).

[K] Kollár J. Rational curves on algebraic varieties, Berlin, Springer-Verlag, 1996.

[L] Lazarsfeld, R. Positivity in Algebraic Geometry II. Springer-Verlag, Berlin-Heidelberg, 2004.

[LM] Lanteri, A. and Maeda, H. Ample vector bundles with section vanishing on projective spaces or quadrics, Int. J. Math. 6, no. 4, 587-600 (1995).

[M] Manivel, L. Gaussian maps and plethysm. Algebraic geometry (Catania, 1993/Barcelona, 1994), Lecture Notes in Pure and Appl. Math., 200, Dekker, New York, 1998, 91-117.

[MS] Muñoz, R. and Solá-Conde, L. E. Varieties swept out by Grassmannians of lines to appear in Contemporary Mathematics.

[NO] Novelli, C., Occhetta, G. Projective manifolds containing a large linear subspace with nef normal bundle, preprint 2008 arxiv: 0712.3406v2.

[S] Sato, E. Projective manifolds swept out by large dimensional linear spaces, Tohoku Math. J. 49, 299-321 (1997).

[OSS] Okonek, C., Schneider, M. and Spindler, H. Vector Bundles on Complex Projective Spaces. Progress in Mathematics 3, Birkhäuser, Boston, 1980.

[S01] Sommese, A. J. On manifolds that cannot be ample divisors, Math. Ann. 221, no. 1, 55–72 (1976).

[S02] Sommese, A. J. Submanifolds of Abelian varieties, Math. Ann. 233, no. 4 229-256 (1978).

[4I] Szurek, M. and Wiśniewski, J. Fano bundles over $P^3$ and $Q^3$, Pacific J. Math. 141, no. 1 197-208 (1990).

[VV] Van de Ven, A. On uniform vector bundles. Math. Ann. 195, no. 4, 245-248 (1972).

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