Primordial Power Spectrum reconstruction from CMB Weak Lensing Power Spectrum

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Abstract. We use the modified and improved Richardson-Lucy (IRL) deconvolution algorithm to reconstruct the Primordial Power Spectrum (PPS) from the Weak Lensing Power Spectrum $C_{\ell}^{\phi\phi}$ reconstructed from CMB anisotropies. This provides an independent window to observe and constrain the PPS $P_R(k)$ along different $k$ scales as compared to CMB Temperature Power Spectrum. The Weak Lensing Power Spectrum does not contain secondary variations in power and hence is cleaner, unlike the Temperature Power Spectrum which suffers from lensing which must be addressed during PPS reconstructions. We demonstrate that the physical behaviour of the weak lensing kernel is unique and reconstructs broad features over $k$. We provide an in-depth analysis of the error propagation using simulated data and Monte-Carlo sampling, using Planck best-fit cosmological parameters to simulate the data with cosmic variance limited error bars. The error and initial condition analyses provide a clear picture of the optimal reconstruction region for the estimator while providing a detailed statistical insight of the results. We also provide an algorithm for $P_R(k)$ sampling sparsity to be used based on the given data and errors, to optimize statistical significance. Eventually we plan to use this method on actual mission data and provide a cross reference to PPS reconstructed from other sectors and any possible features in them.

Keywords: CMBR theory, cosmological parameters from CMBR, gravitational lensing, inflation

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1 Introduction

The ΛCDM Standard Model of cosmology has been a mainstay in the field of cosmology for a long duration and is now in the realm of precision cosmology. Cosmic Microwave Background anisotropies have given us a powerful natural laboratory to infer the parametrizations used in ΛCDM. The Planck [30] and WMAP [27, 28] experiment and collaboration has provided a precise and consistent analysis of the data and parameter inference. A section of this analysis involves a parametrization of the Primordial Power Spectrum (PPS), which is usually modelled as a power law with the two parameters, amplitude $A_s$ and power law index $n_s$ [23, 24]. This is by no means the last word as many other parametrizations exist, from light adjustments like a $k$ dependent $n_s$ (running the power law), to specific model dependent features in the PPS like steps and valleys [25, 26]. Given the large and diverse range of inflation models, there is a well established framework that seeks to reconstruct the PPS using a model independent approach [5, 39–42]. Much work has been done on this approach, using the Richardson-Lucy deconvolution algorithm [1, 3, 4, 7, 10, 14]. The basic working premise of this algorithm is in the mathematical property that the power spectra $C_L, P_R(k)$ and transfer functions $G_{Lk}$ are positive definite, being squares of the underlying Gaussian random fields and transfer functions. It has been successfully implemented on the temperature and polarisation power spectra from CMB anisotropies. Recent precision reconstructions of the weak lensing power spectrum $C_{L\phi\phi}$ from Planck, SPT, ACT [33, 34, 36–38] and several other surveys, based on state of the art reconstruction formalisms, have opened up the possibility of reconstruction of the PPS from this data. Since the physical processes encoded in the transfer function of weak lensing is different from that of the CMB anisotropy transfer function, it is of great interest to study this approach to reconstruction of the PPS, as a way to cross validate PPS
reconstructions from other sectors. In recent years there has also been renewed interest in seeking alternatives to the established power law paradigm in order to find a resolution to interesting discrepancies in data, such as the $H_0$ tension [11, 13]. Also, further improvements are expected in the weak lensing power spectrum $C_L^{\phi\phi}$ observations from the S4 experiments and a future full-sky mission. In light of these, it is a worthwhile exercise to observe the behaviour of the error weighted Richardson-Lucy algorithm on the weak lensing transfer kernel and the statistical properties of a PPS reconstruction that can be obtained from $C_L^{\phi\phi}$ [43–45].

In this paper we will first give an overview of the weak lensing transfer functions and some numerical details of calculating it in section 2. Then we give a brief overview of the IRL algorithm and its applicability on $C_L^{\phi\phi}$ in section 3. After that we simulate the transfer kernel and provide a detailed view of its physical properties and transfer behaviour from PPS to $C_L^{\phi\phi}$ as well as reconstruction features, limitations and initial guess sensitivity in section 4. After that in section 5 we simulate $C_L^{\phi\phi}$ under specific conditions and error budgets and carry out a reconstruction of the PPS, showing the effect of our new $P_{R}(k)$ sampling algorithm that improves statistical significance of reconstruction. Finally we give a detailed overview of the statistical significance of our results and the overall robustness of the IRL algorithm in the $C_L^{\phi\phi}$ sector, for future reference for application to actual data.

2 Weak lensing angular power spectrum $C_L^{\phi\phi}$

The weak lensing power spectrum forms a distinct measure of the cosmological parameters given the baseline $\Lambda$CDM model with non-relativistic cold dark matter and a spatially flat FLRW cosmological model. Due to this, we have an independent probe of the cosmological model information by observing the $C_L^{\phi\phi}$ reconstructed from CMB observations. The convolution of the primordial power spectrum $P_R(k)$ with the radiative transfer function to give the $C_L^{\phi\phi}$ observable is given by

$$C_L^{\phi\phi} = 4\pi \int P_R(k) \left[ \int_0^{\chi_*} 2T_\phi(k; \eta_0 - \chi) \left( \frac{\chi_* - \chi}{\chi_* \chi} \right) j_L(k \chi) d\chi \right] \frac{dk}{k}$$

(2.1)

It can provide us with an independent probe to reconstruct the primordial power spectrum as the primordial curvature perturbations have evolved via a different sector of physical effects than CMB, resulting in an independent observable power spectra, encoded in the CMB via lensing. We provide a brief overview of the physics of weak lensing and its relation to the primordial comoving curvature perturbations. The weak lensing potential map and its power spectrum are reconstructed from CMB, which is observed as a lensed CMB anisotropy map, where the weak lensing effect is expressed by

$$\tilde{\Theta}(\hat{n}) = \Theta(\hat{n} + \tilde{\alpha})$$

(2.2)

This remapping angle $\tilde{\alpha}$ is in turn related to the weak lensing potential of the cosmological matter distribution and its effects integrated over all redshifts till the surface of last scattering

$$\tilde{\alpha} = -2 \int_0^{\chi_*} \nabla_\chi \Phi(\chi; \eta_0 - \lambda) \frac{f_K(\chi_* - \chi)}{f_K(\chi_*)f_K(\chi)} \frac{d\chi}{\chi}$$

(2.3)

Where the light from the source plane (last scattering surface) $\chi_*$ is being lensed by an instance of a lensing potential $\Phi$ at a comoving distance $\chi$ in the direction of the line of
We have used a cosmological model which assumes linear evolution of the primordial Gaussian fields when we model the transfer kernel which can be expressed as

$$\tilde{\alpha} = \nabla_\hat{n} \phi(\hat{n}) \quad \nabla_\hat{n}: \text{gradient on the sphere}$$

$$\phi(\hat{n}) = -2 \int_0^{\chi^*} \Phi(\chi\hat{n}; \eta_0 - \chi) \frac{fK(\chi_s - \chi)}{fK(\chi_s)fK(\chi)} d\chi$$

(2.4)

The lensing potential instance $\Phi(\chi\hat{n}; \eta_0 - \chi)$ at any given comoving distance can be expressed as the evolution of the primordial curvature perturbation $R(k)$ by a transfer function, which can be expressed as

$$\Phi(\chi\hat{n}; \eta_0 - \chi) = T_\Phi(k; \eta) R(k)$$

(2.5)

We have used a cosmological model which assumes linear evolution of the primordial Gaussian fields and hence we are discounting any non linear effects on the $C_L^{\phi\phi}$ which will have significant contribution from non linear effects at small scales of about $L \approx 2000$ and beyond when we model the transfer kernel $T_\Phi$ [20]. The transfer equation for the lensing potential power spectrum, in the linear regime is given by equation (2.1).

We now derive the numerical expression of equation (2.1) to be used for $P_R(k)$ reconstruction.

$$C^{\kappa\kappa}_L = \frac{[L(L + 1)]^2}{2\pi} C_L^{\phi\phi}$$

$$= \frac{[L(L + 1)]^2}{2\pi} 4\pi \int P_R(k) \left[ \int_0^{\chi^*} 2T_\phi(k; \eta_0 - \chi) \left( \frac{\chi_s - \chi}{\chi_s\chi} \right) j_L(k\chi) d\chi \right] \frac{d^2k}{k}$$

(2.6)

In the Limber approximation (Accurate at high L) case it is

$$C^{\kappa\kappa}_L = 2[L(L + 1)]^2 \int P_R(k) \left[ \sqrt{\frac{\pi}{2Lk}} 2T_\phi(k; \eta_0 - \chi_s) \left( \frac{\chi_s - \chi_s}{\chi_s\chi_s} \right) \right] \frac{d^2k}{k} \forall \chi_s = \frac{L}{k}$$

(2.7)

For our purposes we will simulate the transfer kernel of the power spectrum by the exact expression for $L \in [2 \rightarrow 100]$ and the Limber approximated expression for $L > 100$ to increase computation efficiency, while retaining numerical accuracy.

So our numerical expression for the kernel using the Trapezoidal rule, for the exact expression, we have

$$C^{\kappa\kappa}_L \approx 8[L(L + 1)]^2 \sum_{k = k_{\text{min}}}^{k_{\text{max}}} P_R(k) \left[ \sum_{j = 1}^{\chi^*} T_\phi^2(k; z(\chi)) W^2(\chi) j_L^2(k\chi) \Delta \chi \right] \frac{\Delta k}{k}$$

where

$$W(\chi) = \left( \frac{\chi_s - \chi}{\chi_s\chi} \right)$$

(2.8)

For the Limber approximated expression we have

$$C^{\kappa\kappa}_L \approx 4\pi \left[ \frac{L(L + 1)}{L} \right]^2 \sum_{k = k_s(L)}^{k_{\text{max}}} P_R(k) \left[ T_\phi^2(k; z(\chi_s)) \left( \frac{k}{L + 0.5} - \frac{1}{\chi_s} \right)^2 \right] \frac{\Delta k}{k^3}$$

$$\chi_s = \frac{L + 0.5}{k} \quad k_s(L) = \frac{L + 0.5}{\chi_s}$$

(2.9)
So we can put the net expression in the operational form of a discrete convolution

\[ C^{\kappa \kappa}_L = \sum_k G^{\kappa \kappa}_{Lk} P_R(k) \]  (2.10)

Where \( G^{\phi \phi}_{Lk} \) encodes the transfer function \( T_{\phi}(k; \eta) \). Here we use the following notation for the kernel in the discrete form to be used in the numerical R-L estimator

\[ G^{\kappa \kappa}_{Lk} = G^{\kappa \kappa}_{L}(k) \Delta k \]

\[ G^{\kappa \kappa}_{L}(k) = \begin{cases} \frac{8[L(L+1)]^2}{k} \left[ \sum_1^{N} T_{\phi}^2(k; z(\chi)) W^2(\chi) J^2_{L}(k \chi) \Delta \chi \right] & 2 \leq L \leq 100, \ k_{\min} \leq k \leq k_{\max} \\ \frac{4\pi}{L[L+1]} T_{\phi}^2(k; z(\chi)) \left( \frac{k}{L+0.5} - \frac{1}{\chi^{*}} \right)^2 & 100 < L, \ k_{\star}(L) \leq k \leq k_{\max} \end{cases} \]  (2.11)

This expression can be computed by obtaining the Transfer function using the public CAMB software for a specified chosen cosmological model [21].

Given this theoretical model it is evident that the observable power spectra is indeed an independent estimator of the primordial curvature perturbation power spectrum, which is unmodified by any significant higher order physical process that is also dependent on the \( P_R(k) \). We recall the weak lensing of the CMB temperature power spectrum where the lensed contribution is itself a function of \( P_R(k) \), making it a non linear dependency on the PPS and necessitating a template based ‘de-lensing’ approach before deconvolution [10, 19, 20]. In the following section we will explain the deconvolution process followed and the technical details involved. In section 4, Results Ia, we provide the details of the kernel simulation.

3 Method: improved Richardson-Lucy deconvolution

To proceed with the deconvolution of the primordial curvature perturbation power spectrum \( P_R(k) \) convolved by the radiative transfer function into the observable \( C^{\kappa \kappa}_L \) in equation (2.10), we use a deconvolution algorithm called the Richardson-Lucy deconvolution [15–18]. While the detailed derivation and its relation to other standard estimators may be looked up in previous literature [1, 3, 4, 7, 10], in principle it works by treating the transfer function \( G^{\kappa \kappa}_{Lk} \) as a probability distribution over the \( L \) index (Hence the normalisation to unity over \( L \)). The estimator iteratively adds the ‘remnant power’ weighted by the probability distribution \( \tilde{G}^{\kappa \kappa}_{Lk} \) to the previous iteration of \( P^{(i)}_{k} \), starting from some guess \( P^{(0)}_{k} \). The estimator is valid for positive definite functions being convolved, which is applicable here as we are working with the power spectra of Gaussian random fields in linear regimes.

We need a stable and accurate stopping criteria. While there are several options in the existing literature, such as minimizing the \( \chi^2 \) of the reconstructed \( C^{\kappa \kappa}_L \) with respect to the WMAP or Planck likelihood or comparing the change in the \( -\ln L \) with respect to number of iterations [1]. Other methods include simply checking for a 0.1% relative error cap between the final two iterations [4]. However for our work, we compare the relative difference between the final two iterations of the reconstructed \( C^{\kappa \kappa}_L \) across all multipoles and stopping iteration when at all multipoles are below 0.1% relative difference. The reconstructed \( P_R(k) \) does not necessarily converge to a stable reconstruction at multipoles where the propagated error bars are the largest. However seeing as it is a poorly constrained set of equations, this is to be expected because of degeneracy in solutions. However the reconstructed \( C^{\kappa \kappa}_L \) is much more stable with respect to iterations and hence we use it to identify the iteration stopping point.
The observed power spectrum will have a limitation in precision given by the error covariance matrix, the ideal case being limited to cosmic variance. Previous literature incorporates this error information by weighing each data multipole with its corresponding error bar and its correlations, given by the following equation [1, 3, 4, 7, 10].

\[
P_k^{(i+1)} = P_k^{(i)} \left[ 1 + \sum_L \tilde{G}_{Lk} \left( \frac{\hat{C}_L^{\kappa\kappa}}{C_L^{\kappa\kappa(i)}} - 1 \right) \tanh^2 \left( \frac{1}{2} \left[ (\hat{C}_L^{\kappa\kappa} - C_L^{\kappa\kappa(i)}) \Sigma^{-1}(\hat{C}_L^{\kappa\kappa} - C_L^{\kappa\kappa(i)})^T \right] \right) \right]
\]

where the \( \tanh^2 \) term accounts for the error information of the data and denotes the Improved reconstruction innovation in the Richardson Lucy estimator [1]. One should note that this estimator finds a free form estimate of the \( P_R(k) \). What this means in principle is that each \( P_R(k) \) is a variable being estimated. In application this means that the number of unknowns will depend on the number of gridpoints over \( k \) used to evaluate \( P_R(k) \). A large number of gridpoints increase the possibility of finding sharp features over the \( k \) range, but increase the correlation between different \( P_R(k) \) assuming the number of data points over \( L \) remain the same. In effect, one can only optimize what aspect of the data to infer best given a finite amount of data. In this regard it will be seen that the transfer kernel \( G_{Lk} \) is relatively smooth over \( k \) and damps out any sharp features in \( P_R(k) \) during convolution, leading to degeneracy between features that may contribute to the observable \( \hat{C}_L^{\kappa\kappa} \). The conclusions from this observation are that we can reduce the number of gridpoints over \( k \) at which to evaluate the different \( P_R(k_i) \). We expect this approach to reduce the correlation between different \( P_R(k_i) \) and reduce computation time, but also restrict the sensitivity of the estimator to sharp/high frequency features in \( P_R(k) \). We highlight that this restriction is a fundamental feature of the physics present in the transport kernel \( G_{Lk} \) and only broad features over \( P_R(k) \) can be resolved given this estimator. One simple way to mitigate this limitation is to use independent cross checks with other power spectra such as the \( \hat{C}_L^{TT} \) whose transport function \( G_{Lk}^{TT} \) have different sensitivities to different \( k \) ranges and can discover features in \( P_R(k) \) that \( G_{Lk}^{\kappa\kappa} \) will miss.

4 Results Ia: kernel simulation

4.1 Kernel behaviour

In this section we will discuss the behaviour of the simulated radiative transport kernel that we will use to deconvolve the primordial power spectrum from the observed angular power spectrum data realisation. We have used the Planck [30] inferred best fit background cosmological parameters given in table 1.

To simulate the \( G_{Lk}^{\kappa\kappa} \) kernel we used the CAMB [21] software. The numerical parameters used to simulate the kernel are given in table 2.
Table 1. The physical parameters on which the CAMB calculated ΛCDM cosmology radiative transfer kernel is based on [30].

| Parameter        | Planck Best-Fit |
|------------------|-----------------|
| $H_0$            | 67.36           |
| $\Omega_b h^2$  | 0.02237         |
| $\Omega_c h^2$  | 0.1200          |
| $m_\nu$          | 0.06 eV         |
| $\Omega_K$      | 0               |
| $\tau$           | 0.0544          |
| $A_s$            | 0.9             |
| $n_s$            | 0.9649          |
| $r$              | 0               |

Table 2. The simulation parameters with reference to CAMB. The $k$ range choice is explained in the text. The $L$ range choice is based on current observational data. The number of free variables $N_k$ is automatically assigned on the basis of numerical parameters such as $\Delta \log k$ (the number of $k$ intervals on the log $k$ scale) and a CAMB specific simulation accuracy parameter that controls $k$ sample resolution called $\text{SourcekAccuracyBoost}$.

| Simulation Parameters | Simulation Input |
|-----------------------|------------------|
| $k_{\text{min}}$     | $7 \times 10^{-6}$|
| $k_{\text{max}}$     | $1 \times 10^{1}$|
| $N_k$                | 3151             |
| $L_{\text{min}}$     | 2                |
| $L_{\text{max}}$     | 2500             |
| $\Delta \log k$      | 500              |
| Simulation Accuracy   | 8                |

Since the Limber approximation is accurate mainly at high $L$, we divide the transport kernel calculation into two parts. First we calculate the kernel for an $L$ range $L \in [2, 100]$ where we use the exact expression in equation (2.1). We calculate the transfer function $T_p(k; \eta_0 - \chi)$ from CAMB, then we integrate the function with the lensing window function and the spherical Bessel function of the first kind, over $\chi$. This section takes approximately an hour to compute. For the $L$ range $L \in [101, 2500]$ we use the Limber approximated kernel equation (2.9) to calculate the kernel and speed up the calculation over the intermediate steps by interpolating over $\chi$, since it is now a function of $L$. The accuracy of the reconstruction can be verified by comparing the $C_L^{\kappa \kappa}$ constructed from kernel transport $C_L^{\kappa \kappa} = \sum_k G_{Lk} P_R(k)$ to the CAMB simulated $C_L^{\kappa \kappa}$ for the same physical parameters. The entire calculation runs over a range of a few hours, using a system on the scale of a personal laptop/desktop with any new generation Intel i5 processor, 8 GB RAM or an equivalent configuration.

The results of this simulation are expressed in figure 1 and figure 2. We can clearly see that the kernel is extremely smooth throughout the $k$ range over the entire range of $L$, especially at the range where most of the transfer power is present. This can lead us to predict that any features in $P_R(k)$ that have a frequency higher than the radiative transfer kernel over the $k$ range will likely be degenerate with features that are smoother, yet transfer the
same power to the resultant $C_L$. This means that reconstruction of very sharp features in $P_R(k)$ such as spikes or oscillatory features with a frequency higher than the kernel, will not be optimal or may be missed altogether. Test example of such features and their effect on $C_L^{\nu\kappa}$ have been plotted in figure 3 which shows the lack of impact a high frequency feature PPS has on $C_L^{\nu\kappa}$. As a result, we can also make certain simplifications in the reconstruction process. We need not have a very fine grid over $k$ since the kernel is unable to reconstruct features with very high frequency in $k$. This reduces both our computation time as well as number of free parameters. For the sake of generality, in section 5.1 we keep a fairly fine $k$ grid, higher than the expected reconstruction fidelity over $k$, but low enough for it to be less computationally heavy. In the future if we sample over the cosmological parameter space to find the best fit parameters when using a free form $P_R(k)$, the freedom to reduce the grid density will aid in reducing the kernel computation time and making such an exercise computationally feasible. We also provide an algorithm to optimize the $P_R(k)$ $k$ grid sampling in order to improve the statistical properties of the reconstructed solutions. This is presented in section 5.2

We observe the transfer function’s properties along the $k$ ranges and their contribution across different $L$ multipoles. Figure 1 shows a top down heatmap of the transfer kernel $G_L^{\nu\kappa}$.

We can see that most of the kernel transfer power lies in the range of $k \in 10^{-3} \rightarrow 10^{-1}$. This means that the reconstruction support of the Richardson-Lucy estimator lies primarily in this range and smooth features in $P_R(k)$ within this region are more likely to be reconstructed with higher fidelity than in the regions beyond this range. Depending on the minimum $L$ range of the observed data, this will typically lead to the previously mentioned range of $k \in 10^{-3} \rightarrow 1$. Hence in short, this is the region where any $P_R(k)$ estimation from $C_L^{\nu\kappa}$ is likely to be meaningful. To verify this assumption, we plot the $C_L^{\nu\kappa}$ constructed by a power law $P_R(k)$ superimposed with some relatively low frequency feature, which we shall term Feature 1 and study how its contribution varies across the $k$ location of the feature. In figures 4 and 6 this is plotted with Feature 1 moving across the $k$ range over $P_R(k)$ and its contribution convolved with the transfer function is obtained as $C_L^{\nu\kappa}$ and compared to
Figure 2. Plot (a) shows the functional form of the $G_L^{\kappa}(k)$ kernel projected onto the $k$ vs $G_L^{\kappa}(k)$ plane where $L$ is parametrized as a color gradient within blocks of $L$ with corresponding $\Delta L$ step sizes showing the plotted frequency of $L$ blocks. The $L$ blocks are roughly segmented by the relative amount of power they transfer. Plot (b) shows the same kernel but with the numerical integration step size multiplied. $G_{Lk}^{\kappa} = G_L^{\kappa}(k)\Delta k$ used in equation (2.10).

The spectrum obtained with respect to a power law $P_R(k)$. As surmised earlier, the main contribution comes primarily when the feature moves across the $10^{-2}$ mark which coincides with the kernel peak in figure 2.

We now explain the numerical properties of the kernel and the role they play in optimal reconstruction of $P_R(k)$. As we can see from figure 2b, to make reconstruction efficient and optimal, we primarily need to assign an equal number of $P_R(k)$ samples with respect to the number of $C_{Lk}^{\kappa}$ data points. Refer to Fundamentally we are solving a set of linear equations and in order to have a well behaved non singular, invertible covariance matrix, we need the system to be an exactly determined and consistent set of linear equations. We can also work with an underdetermined and consistent set of equations where the $k$ sampling is much
Figure 3. This plot shows a high frequency feature in the form of a wavepacket (blue line) superimposed on a power law (orange dash) PPS, plotted on the top row. $C_{L_k}^{\kappa}$ from power law (orange dash) vs wavepacket superimposed PPS (blue line) is plotted in the mid row. Percent difference between the two $C_{L_k}^{\kappa}$ in blue in the bottom row, compared to cosmic variance in orange. The $k$ ranges from $10^{-5}$ to 10 and $L$ ranges from 2 to 2500. The feature moves across $k$ space from left to right.

Based on these discussions, we propose a mechanism where we establish a minimum power cutoff, $G^{\text{res}}_{L_k \text{cutoff}} = 0.001\% G^{\text{res}}_{L_k \text{max}}$. Then we can divide the kernel into several $k$ subsets and selectively choose $k$ samples out of those subsets, based on how many $L$ modes ‘enter’ the $G^{\text{res}}_{L_k \text{cutoff}}$ threshold within each subset. This will take care of both the constraint of having $N_k = N_L$ as well as ensuring that a given $k$ sample is associated with a corresponding and unique $L$ mode. This ensure that more ‘heterogeneous’ regions of the kernel are utilized for reconstruction, instead of the smooth regions, where it will be difficult to distinguish one $k$ contribution from a neighbouring one. In addition, based on the $G^{\text{res}}_{L_k \text{cutoff}}$ and with reference to the contour plot in figure 1, we will also cut off any $k$ and $L$ regions that fall outside...
Figure 4. This plot shows a low frequency Feature 1 (blue line), plotted on the top row superimposed with a power law (orange dash) PPS. Corresponding $C_{L}^{\text{ps}}$ from Feature 1 (blue line) based vs power law (orange dash) PPS in the mid row. Percent power difference in $C_{L}^{\text{ps}}$ (blue line) shown in the bottom row vs cosmic variance (orange line). The $k$ ranges from $5 \times 10^{-4}$ to 10 and $L$ ranges from 8 to 2500. Feature 1 moves across $k$ space from left to right.

| $k$ range          | $L$ range          |
|--------------------|--------------------|
| $k \in [2 \times 10^{-4} \rightarrow 2 \times 10^{-3}]$ | $L \in [2 \rightarrow 30]$ |
| $k \in [2 \times 10^{-3} \rightarrow 5 \times 10^{-2}]$ | $L \in [31 \rightarrow 700]$ |
| $k \in [5 \times 10^{-2} \rightarrow 2 \times 10^{-1}]$ | $L \in [701 \rightarrow 1350]$ |
| $k \in [2 \times 10^{-1} \rightarrow 1 \times 10^{0}]$ | $L \in [1350 \rightarrow 2500]$ |

Table 3. The reconstruction regimes ordered by data $C_{L}^{\text{ps}}$. $L$ range vs reconstruction $P_R(k)$ degrees of freedom over $k$ range.

the cutoff contour. We provide a general idea of the reconstruction regions in the table 3. A formal PCA based approach was also considered and implemented. However the sparse sampling from PCA is equivalent and gives similar results to our kernel informed sampling. In addition PCA also results in the new basis principal components being linear combinations of the $P_R(k)$ samples, which are difficult to interpret in the context of physical feature hunting. As a result we with the kernel informed sparse sampling.\(^1\)

\(^1\)We thank the anonymous referee for suggesting this comparison.
Another observation to note is that most of the kernel power lies within an $L$ range of $L \in 2 \rightarrow 700$. This means that any features in $P_R(k)$ will reflect in the $C_L^{\kappa \kappa}$ more significantly within this $L$ region. As a result, precise data in this $L$ range is likely to be more valuable in context of $P_R(k)$ deconvolution. Since a significant portion of the kernel support transfers power to the low $L$ multipoles, this calls for high precision full sky reconstructions of $C_L^{\kappa \kappa}$. To this end we shall examine how Cosmic Variance limited data would affect reconstruction and the precision in extracting features in $P_R(k)$ or lack thereof.

Lastly we highlight some key differences between the $C_L^{\kappa \kappa}$ and $C_L^{TT}$ kernels. With reference to figure 20 and figure 1 in [10] we note some key differences. The usable power range for $C_L^{TT}$ extends to a slightly larger domain, from approximately $k_{\text{min}} = 3 \times 10^{-4}$ to $k_{\text{max}} = 2 \times 10^{-1}$. The kernel for $C_L^{TT}$ is much more staggered and power contribution from low $k$ regions is comparable in magnitude to that of mid and high $k$ regions, unlike for $C_L^{\kappa \kappa}$, which is largely distributed around mid $k$ regions. We can also see that for each $L$ parametrized curve over $k$, the curves are more sharply peaked and straddle narrower $k$ ranges. As a result, the $C_L^{TT}$ kernel will be more sensitive to sharp features along $k$. We highlight that $C_L^{\kappa \kappa}$ is less sensitive at searching for narrow features. It can pick out the broad PPS and larger valleys of lensing power spectrum lies in that it can reconstruct the broad $P_R(k)$ independently to the temperature spectrum. Also $C_L^{TT}$ suffers from lensing smoothing. Since lensing itself is a function of the PPS, this means it suffers from a non-linear dependency on the PPS, which must be corrected before analysis [10]. $C_L^{\kappa \kappa}$ has no such ‘contamination’ and hence is a cleaner first probe into a free form PPS. This primary analysis can be used for the initial guess analysis for the lensing cleaned $C_L^{TT}$ later. We expect to perform further work along this line of enquiry, which is out of scope of this paper and reserved for a future study.

### 4.2 Initial guess behaviour

In this section we further expand upon the effects of the kernel on IRL reconstruction and the initial guess $P_R(K)^{(i=0)}$ used to begin the IRL iterations. Using the CAMB generated Limber approximated $C_L^{\kappa \kappa}$ data as before, we reconstruct the input Power-Law $P_R(K)$ with different initial guesses $P_R(K)^{(i=0)}$ and observe how the reconstruction varies. Ideally we should expect no change as the injected data does not change in either iteration. However we expect numerical and algorithm limitations to be visible in this exercise.

The figure 5 shows how a range of different initial guesses $P_R(K)^{(i=0)}$ obtained by varying the slope of the initial guess $P_R(K)^{(i=0)} = A_s (\frac{k}{k_s})^{(n_s-\frac{1}{2})}$ for integer $j \in [-20 \rightarrow 40]$, being used to recover the original $P_R(K)$. The oscillations due to different reconstructions show how the kernel is weak at high and low $k$ and hence high $L$ and variations are dominated by the initial guess for the PPS. More figures are given in the supplement section B. It is clear from the plot that slope of the initial guess has a large impact on the reconstruction at high and low $k$’s. For low $k$ this does not have much effect in the final reconstruction as the error bars in this region will too large anyway to affect feature hunting. But at high $k$, we impose a reconstruction bound of $k_{\text{max}} = 0.2$ when feature hunting and calculating the $\chi^2$ of the reconstructed $P_R(K)$. Usually we can use any initial guess with a slope similar to the theoretical power law. This in turn allows for a control over the reconstruction at edge $k$’s. But this is equivalent to imposing some prior information, in that we expect the overall slope of any reconstructed $P_R(K)$ to be close to the fiducial $n_s$ and only hunt for localized features ‘riding’ a global power law. Relaxing such limitations can be looked at in detail in future work.
Figure 5. Figure (a) shows the reconstructed $P_R(k)$ in blue dashed lines, given different initial guesses $P_R(k)^{(i=0)}$ in yellow lines varying by slope $n_s$. The red lines shows the original injected power spectrum $P_R(K)$. Figure (b) shows the relative % difference in the reconstructed $\hat{C}_\kappa^\kappa$ and the input data $C_\kappa^\kappa$.

We also provide some more initial guess variation plots in figure 21 and figure 22 to show how the reconstruction is highly dependent on the initial guess slope at high and low $k$’s.

5 Results Ib: simulated IRL reconstruction

In this section we perform a reconstruction of the PPS on simulated data with appropriate error bars, on the unbinned data, as well as analyze the error budget and covariance properties of the reconstruction. This seeks to provide a good foundation to implement the process on observed Planck data and predict the science output of future observations analyzed by this reconstruction process.
Figure 6. (a) Plots a low frequency feature, Feature 1, superimposed on the power law $P_R(k)$. (b) Plots the corresponding $C_{kL}^{\kappa}$ from $P_R(k)$ with and without Feature 1.

| Parameter           | Values                |
|---------------------|-----------------------|
| Input Data          | $C_{kL}^{\kappa}$     |
| Error Bars          | Cosmic Variance       |
| Input $P_R(k)$ Model | $A_s(k/k_*)^{(n_s-1)}$ |
| $k$ Range           | $[2 \times 10^{-4} \rightarrow 0.2]$ |
| $N_k$               | 1853                  |
| $L$ Range           | $[2 \rightarrow 2500]$ |
| $L$ Binning         | Unbinned              |
| Data Realisations   | $2 \times 10^6$       |

Table 4. The simulation parameters over which the IRL reconstruction is carried out.

5.1 Results i): high $k$ density

For the first reconstruction example, we demonstrate the application of the IRL reconstruction algorithm on the base $\Lambda$CDM cosmology model based on Planck best fit parameters and the Inflationary Primordial Power Spectrum $P_R(k)$ using a power law model $A_s(k/k_*)^{(n_s-1)}$. We list the simulation parameters and conditions in table 4.

The input data is limited by cosmic variance limited errorbars, to study the ideal case for reconstruction demonstration. The CAMB simulated $C_{kL}^{\kappa}$ power spectrum and its data realisation, are presented in figure 7.

Since we are working with an order of magnitude $10^3$ number of free form $P_R(k)$ variables, we expect a sample size of $10^6$ realisations of these measured variables to provide a statistically meaningful reconstruction of the IRL reconstructed $P_R(k)$ error covariance matrix $\Sigma_{kk'}$. Based on the input data realisation and errorbars in figure 7, we generate Monte-Carlo samples and carry out IRL reconstructions on them. We note that we are working with the error bars in
Figure 7. Plot of the CAMB simulated $C_{\kappa\kappa}^L$ in Orange and the data realisation by treating each $C_{\kappa\kappa}^L$ as a Gaussian random sample based on cosmic variance error bars, in blue.

Figure 8. (a) Plots the number of $\Sigma_{kk'}$ coefficients that exceed a 1% relative error change with respect to the previous realisation set, vs the realisation number set. We see that by $2 \times 10^6$ number of realisations, the accuracy saturates. Similarly the plot (b) plots the same but at 10% accuracy threshold, showing similar saturation.

$C_{\kappa\kappa}^L$ assuming that they are not correlated, meaning that $\Sigma_{\kappa\kappa'}$ is diagonal. Hence we will use the 2nd form of the IRL algorithm in equation (3.1). For real data usually this is not the case and the full $\Sigma_{LL'}$ may need to be incorporated. In figure 8 we plot the accuracy saturation of the reconstructed $\Sigma_{kk'}$ with respect to increasing realisation count and establish $2 \times 10^6$ as an acceptable number of realisations. These simulations are carried out in parallel on a computing cluster and take 40 hours per core, while utilizing 1000 Intel Skylake cores for $2 \times 10^6$ simulated data set realisations and their RL reconstruction.
Based on these reconstructed realisations we plot the reconstructed $P_R(k)$ and the $1\sigma, 2\sigma$ error bands from the diagonal part of $\Sigma_{kk'}$, in figure 9.

The reconstruction clearly has varying accuracy with respect to $k$. This follows from the nature of the input $C_{kL}^{\kappa\kappa}$ data. At low $L$, which correspond to power acquired from low $k$, (as can be noted from the kernel plots), the cosmic variance is high due to less number of azimuthal components, given by $N_m = 2L + 1$. This implies that the data realisation $C_{kL}^{\kappa\kappa}$ has a higher degree of fluctuations at low $L$, that deviate from the fiducial CAMB $C_{kL}^{\kappa\kappa}$, which are then transferred to the reconstructed $P_R(k)$ as fluctuations at low $k$. This behaviour makes it difficult to ascertain features in $P_R(k)$ at low $k$ due to the inherently noisy nature of the data. This can be addressed by aggressive binning at low $L$, which can clean the noise, but has the downside of reducing the $k$ sampling at those $L$s. The reason for this is addressed in the $\Sigma_{kk'}$ matrix later; an inordinately high $k$ sampling can lead to degenerate solutions for $P_R(k)$ and high degree of correlation in the free form $P_R(k)$ reconstruction, again making recovery of features uncertain as they may not be unique or independent of correlated features. Overall this leads to the conclusion that from low $L \in [2 \rightarrow 30]$, $k \in [10^{-4} \rightarrow 3 \times 10^{-3}]$, accurate, statistically uncorrelated sampling of $P_R(k)$ is sparse and not much information can be extracted over that range. What can be, gives a limited picture in terms of feature hunting. The reconstruction improves significantly at mid range $L \in [31 \rightarrow 1200]$ corresponding to $k \in [3 \times 10^{-3} \rightarrow 10^{-1}]$. See figure 2. In this range a major chunk of the $C_{kL}^{\kappa\kappa}$ data gets contribution from the said $k$ range and hence $P_R(k)$ can be sampled densely and will be largely uncorrelated, hence features detected in this range are relatively more statistically significant and unique. In addition the cosmic variance error bounds are significantly low here, so both the fluctuations in the data realisation are lesser, as well as propagated error bands in $P_R(k)$ are narrower, which means a higher precision reconstruction in this region makes it easier to narrow down potentially interesting deviations from the Power Law model. We propose that the estimator works well as a high precision reconstruction technique in this region of $k$ and is limited mainly by experimental bounds. At higher $k \in [10^{-1} \rightarrow 1]$ ranges, the reconstruction, though having a very low error bound due to low cosmic variance,
is unreliable due to a high degree of correlation between the $P_R(k)$ samples. The discussion about the kernel in section 4 explains that above $k = 1.5 \times 10^{-1}$, the kernel becomes uniform enough over $L$ for the IRL algorithm to be unable to distinctly reconstruct individual $P_R(k)$ points. Furthermore the kernel support itself drops rapidly and a full $k$ range reconstruction here is numerically unsound, as shown in section 4.2. It may be possible to study what happens when limited high $k$ ranges are reconstructed independently by analysing specific $C_L^{\kappa\kappa}$ bands over $L$ and high $L$, but that work is reserved for future updates.

We also plot the reconstructed $C_L^{\kappa\kappa}$ and compare it to the input data and fiducial CAMB simulated $C_L^{\kappa\kappa}$ in figure 10, along with the relative error plots between both reconstructed $C_L^{\kappa\kappa}$ and $P_R(k)$ in figure 11.
Figure 12. (a) is a plot of the $\Sigma_{kk'}$ from the reconstructed $P_R(k)$. Reds and Blues denote $\pm \log_{10} \Sigma_{kk'}$ respectively. The plot (b) plots the correlations matrix $\rho_{kk'}$ with the Reds, Blues being the $\pm 0 \rightarrow 1$ range respectively.

We demonstrate the discussion over the sparsity of $P_R(k)$ sampling at low $k$, and the highly correlated nature, numerical inaccuracy at high $k$, with the following plots on the covariance matrix $\Sigma_{kk'}$ and the Identity matrix expected from multiplying with its inverse, in figures 12a and 13.

In figure 12a, we have plotted the covariance matrix $\Sigma_{kk'}$ from the reconstructed $P_R(k)$. From the plot it is clear that there is correlation between the different free form $P_R(k)$ variables. This connects to our previous discussion on oversampling over $k$ with respect to the number of $L$ data points. To study this better we therefore also plot the correlation matrix in figure 12b, given by correlation coefficient (5.1). Again, as expected from our previous discussion of reconstruction over different $k$ ranges, the degree of correlation is very high at high $k$ and fairly strong at low $k$s as well. This opens up room for us to optimize the $k$ sampling comprehensively, and we will demonstrate this optimization using the inverse of $\Sigma_{kk'}$ and the corresponding Identity matrix.

$$\rho_{kk'} = \frac{\Sigma_{kk'}}{\sigma_k \sigma_{k'}}$$

We plot the visualisation used for optimizing the $P_R(k)$ sampling, in figure 13a. We use the $\text{numpy.linalg.inv}$ routine in the $\text{numpy}$ package to perform the covariance matrix inversion. The subroutine uses LU decomposition of a matrix to solve for it’s inverse. Ideally, $\Sigma_{kk'} \cdot \Sigma^{-1}_{kk'}$ should be an Identity matrix with a clear red diagonal of 1s and blue 0s in the off diagonals. However it is evident that the reconstruction in the low kernel support regions of $k < 7 \times 10^{-3}$ and $k > 1.5 \times 10^{-1}$, is highly correlated and the system of equations being solved by IRL is not well determined. As a result, the second plot in figure 13b, which should ideally by 0 throughout, also shows a lot of ‘noise’ in the regions of low information $k$ reconstruction, as well as the correlated regions. Hence using these figures as a guide and the kernel discussion, we can reduce the density of $k$ sampling in those regions and carry out reconstruction based
on the number of \( L \) data points per \( P_R(k) \) sample points, such that the system of equations being solved is reasonably well determined and the reconstruction \( \Sigma_{kk'} \) is invertible.

### 5.2 Results ii): sparse \( k \) density

In this section we propose an optimization method to reduce the degeneracy of solutions in \( P_R(k) \) space. In the previous example we are oversampling leading to both correlations over different \( k \) as well as degenerate solutions which can all fit the data used. This is expressed by the poorly reconstructed \( \Sigma_{kk'} \) matrix, which while numerically invertible, a dot product with its own inverse \( \Sigma^{-1}_{kk'} \) does not result in a clean Identity matrix \( I \). We therefore use a novel algorithm where we divide the \( k \) sampling grid into multiple subsets and ensure that the number of \( P_R(k) \) samples in a given subset is less than the number of new \( L \) modes that cross a minimum threshold of \( 0.001\% G_{\phi\phi}^L(k)_{\text{max}} \). This was described in section 4.1.

Other numerical parameters remain the same, including the input data based on the Power Law model with non correlated Cosmic-Variance limited errors bars given in figure 7.

Based on this algorithm we obtain a new \( P_R(k) \) sampling vector and generate \( 1 \times 10^6 \) data samples and IRL reconstruction samples. The cost of computation is lower given lesser number of \( P_R(k) \). The new simulation parameters are given in table 5. Figure 14 gives the numerical accuracy of the covariance matrix \( \Sigma_{kk'} \) with respect to accuracy.

We again plot the reconstructed \( P_R(k) \) with \( 1\sigma, 2\sigma \) error bands in figure 15, and we impose the constraint by the initial-guess sensitivity information and use the \( k \) range given in table 6. It is overplotted with the dense \( k \) sampling results from the previous section, in green. As visible, the error bars and features do not change in any statistically significant way, showing we have reduced a lot of degeneracy in the free form PPS, without affecting the amount of information gleaned from the analysis.

The reconstructed \( C_{L\kappa\kappa} \) and its comparison to the input fiducial CAMB simulated \( C_{L\kappa\kappa} \) is given in figure 16, and the relative error plots between both reconstructed \( C_{L\kappa\kappa} \) and \( P_R(k) \) with respect to theoretical CAMB \( C_{L\kappa\kappa} \) and Power Law model in figure 17.

We should note that we have naively plotted the \( 1\sigma, 2\sigma \) bands around the reconstructed \( P_R(k) \), but the standard deviation does not reflect the rest of the properties of the probability
\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{plot1.png}
\caption{}
\end{subfigure}
\hfill
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{plot2.png}
\caption{}
\end{subfigure}
\caption{(a) Plots the number of $\Sigma_{kk'}$ coefficients that exceed a 1\% relative error change with respect to the previous realisation set, vs the realisation number set. We see that by $2 \times 10^6$ number of realisations, the accuracy saturates. Similarly the plot (b) plots the same but at 10\% accuracy threshold, showing similar saturation.}
\end{figure}

\begin{table}[h]
\centering
\begin{tabular}{|l|l|}
\hline
Parameter & Values \\
\hline
Input Data & $C_L^{\nu\nu}$ \\
Error Bars & Cosmic Variance \\
Input $P_R(k)$ Model & $A_s(k/k_*)^{(n_s-1)}$ \\
k Range & $[2 \times 10^{-4} \rightarrow 0.2]$ \\
$N_k$ & 483 \\
$L$ Range & $[2 \rightarrow 2500]$ \\
$L$ Binning & Unbinned \\
Data Realisations & $1 \times 10^6$ \\
\hline
\end{tabular}
\caption{The simulation parameters over which the IRL reconstruction is carried out.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|l|l|}
\hline
Parameter & Values \\
\hline
Input Data & $C_L^{\nu\nu}$ \\
Error Bars & Cosmic Variance \\
Input $P_R(k)$ Model & $A_s(k/k_*)^{(n_s-1)}$ \\
k Range & $[10^{-4} \rightarrow 0.2]$ \\
$N_k$ & 483 \\
$L$ Range & $[2 \rightarrow 2500]$ \\
$L$ Binning & Unbinned \\
Data Realisations & $2 \times 10^6$ \\
\hline
\end{tabular}
\caption{The simulation parameters over which the IRL reconstruction is carried out.}
\end{table}
Figure 15. Plot of the reconstructed $P_R(k)$ (Blue) with $1\sigma$, $2\sigma$ bands from $2 \times 10^6$ data realisations with cosmic variance error, using the sparse $k$ sampling. Overplotted are the same reconstructions from dense $k$ sampling, from figure 9 in green for reference. The fiducial Power Law $P_R(k)$ (Orange) is added for reference.

Figure 16. Reconstructed $C_L^{\kappa\kappa}$ in Blue plotted over the fiducial CAMB $C_L^{\kappa\kappa}$ in Orange and the input data realisation in Green.

distribution of each $P_R(k)$ sample. For low $k$ we should also includes the skew statistic, since the $P_R(k)$ cannot be negative, being a power spectrum. However for the purpose of this analysis, we will ignore such details as the low $k$ regions have very high reconstruction error even for the ideal cosmic variance error limit in data, and as such are unlikely to ever be precise enough for feature hunting. We also plot the individual $P_R(k)$ sample points being evaluated, which are more sparse now, but at the advantage of being statistically more
Figure 17. The two figures show the relative % error between the reconstructed $C_{\ell}^m$ vs data realisation $C_{\ell}^m$, and the reconstructed $P_R(k)$ vs input Power Law $P_R(k)$. The same plots from figure 11 using dense $k$ sampling are overplotted for reference.

Figure 18. (a) is a plot of the $\Sigma_{kk'}$ from the reconstructed $P_R(k)$. Reds and Blues denote $\pm \log_{10} \Sigma_{kk'}$ respectively. Plot (b) plots the correlations matrix $\rho_{kk'}$ with the Reds, Blues being the $\pm 0 \rightarrow 1$ range respectively.

significant overall. This is expressed in the covariance/correlation matrix, its inverse and Identity matrix, $\Sigma_{kk'}, \Sigma_{kk'}^{-1}$ and $I$. These are plotted in figures 18 and 19.

We can see the drastic improvement in the covariance matrix for the new $P_R(k)$ reconstruction using the algorithm of $P_R(k)$ sampling following the kernel contributions per $L$ above a threshold. From this we can infer that our $P_R(k)$ sampling is such that chances of degenerate solutions for $P_R(k)$ are reduced. We obtain an invertible $\Sigma_{kk'}$ within numerical limitations. While correlations are still clearly present in figure 18b, we can better quantify our reconstruction statistically and make a statement if there are significant deviations from the null hypothesis of a Power-Law.
Figure 19. (a) Identity: plots the $\Sigma_{kk'} \times \Sigma^{-1}_{kk'}$ with Reds, Blues being $1 \pm 0.1$, $0 \pm 0.1$ respectively. (b) Residual: shows the same matrix but for the non $1 \pm 0.1$, $0 \pm 0.1$ terms. A clear Identity matrix is obtained within numerical bounds, showing that the $\Sigma_{kk'}$ is obtained from a unique $P_R(k)$ solution.

\[
\begin{array}{c|c}
\chi^2(\hat{P}_R(k) - P_R(k)_0) & \text{Values} \\
\chi^2_{\text{diag}} & 409.52 \\
\chi^2_{\text{full}} & 56.73 \\
\end{array}
\]

Table 7. The $\chi^2$ values for the reconstructed $\hat{P}_R(k)$ with respect to a Power-Law model $P_R(k)_0$ for $\Sigma_{kk'}$ and diagonal only components.

For our reconstruction we calculate the $\chi^2$ value using both the diagonal only error bars $\sigma_k$ as well as the full covariance matrix $\Sigma_{kk'}$.

\[
\chi^2_{\text{diag}} = \sum_{i=1}^{N_k} \frac{[\hat{P}_R(k_i) - P_R(k_i)_0]^2}{\sigma_{k_i}^2} \tag{5.2}
\]

\[
\chi^2_{\text{full}} = [\hat{P}_R(k) - P_R(k)_0]\Sigma^{-1}_{kk'}[\hat{P}_R(k') - P_R(k')_0]^T
\]

The results are given in table 7. It is observed that using the full covariance matrix $\Sigma_{kk'}$ gives us a drastic reduction in the $\chi^2$ value of the reconstruction with respect to power law. The reason for this is because of the correlation present between different $P_R(k)$ samples which are taken into account in the off diagonal terms and provide a better statistical estimate of the reconstruction. It is evident that a full reconstruction of the error covariance matrix carries a lot of statistical information of the recovered $P_R(k)$ when performed under the optimization we have defined earlier. This procedure helps validate any features being found in the reconstruction. The $p$-value for the full covariance matrix is close to 1 given the degrees of freedom and numerical limitations, which verifies that our null hypothesis of a power law PPS, is indistinguishable from the reconstructed PPS. This is expected from sample data generated using a power law PPS input.
6 Discussion

In this paper we analyse a hitherto unexplored data source for $P_R(k)$ reconstruction, using the $C^\kappa_L$ power spectrum.

We use the standard ΛCDM cosmology model with Planck best-fit parameters to carry out the IRL deconvolution algorithm and reconstruct a free-form $P_R(k)$ from simulated data under ideal observation conditions, namely limited by cosmic variance. We also establish reconstruction bounds from a detailed analysis of the transport kernel $G^\kappa_L(k)$ and its applicability for this reconstruction. We find that the reconstruction limits are from the $k$ range of $2 \times 10^{-4}$ to $2 \times 10^{-1}$ and the kernel is optimized when seeking broad features over $P_R(k)$ due to the smooth nature of the kernel $G^\kappa_L(k)$.

We also establish new paradigms of statistical precision for the reconstruction algorithm and provide a prescription for $P_R(k)$ sampling based on the kernel properties and verify the improvements using the full $P_R(k)$ reconstruction covariance matrix for $\Sigma_{kk'}$ and show that we can successfully avoid degenerate solutions by this method. These methods can potentially be applied to IRL reconstruction from other power spectra as well.

We also carry out a $\chi^2$ of reconstruction vs null-test estimation to demonstrate that the algorithm is well-behaved and reconstructs a $P_R(k)$, indistinguishable from the input power law model, without introducing statistically significant spurious features. We conclude that the algorithm is robust with respect to our reconstruction goals. We also show that accounting for the full covariance matrix significantly reduces the $\chi^2$-value, which is expected given the correlated nature of the reconstructed free form $P_R(k)$. One cannot ignore the complete covariance matrix in such a reconstruction while quantifying the statistical significance of the reconstructed PPS.

We should also mention that reconstruction from $C^\kappa_L$ has an added advantage in that the power spectrum does not contain secondary distortions such as the weak lensing damping of acoustic peaks of $C^{TT}_L$ at high $L$s. Reconstruction from temperate data currently requires a delensing template to be subtracted from $\tilde{C}^{TT}_L$. Hence $C^\kappa_L$ is a cleaner probe in this aspect and can be used as a starting point for $P_R(k)$ reconstructions.

We expect to carry out more work on this estimator and address several key concerns, from improving the statistical inference of the reconstructions, its behaviour on non power law input modelled data, as well as its applicability in improving $P_R(k)$ reconstruction jointly with existing CMB anisotropy power spectra and lensing corrections. We also expect to study the estimator behaviour on binned data as well as its application on actual experimental data from Planck and future full-sky CMB missions.

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Figure 20. The figure shows the functional form of the $G^{TT}_{Lk}$ kernel projected onto the $k$ vs $G^{TT}_{Lk}$ plane where $L$ is parametrized as a color gradient within blocks of $L$ with corresponding $\Delta L$ step sizes showing the plotted frequency of $L$ blocks. The $L$ blocks are roughly segmented by the relative amount of power they transfer. Here the numerical integration step size is multiplied.

A Kernel behaviour: supplementary figures

In this supplementary section we provide a kernel plot in Figure 20 of the $G^{TT}_{Lk}$ kernel to illustrate the different physics and consequent reconstruction differences as compared to $G^{\kappa\kappa}_{L}$.

B Initial guess: supplementary figures

In this supplementary section we add figures from section 4.2. Figure 21 and 22 plot the behaviour of the PPS reconstruction if the initial guess PPS is varied by a change in intercept $k^*$ for a given slope. As can be seen, the reconstructed PPS is largely insensitive to a phase shift and retains the same slope in the reconstructed PPS at high and low $k$, as was input in the initial guess. It is evident that the PPS initial guess in the IRL reconstruction algorithm is mainly sensitive to input slope at high and low $k$ regions.
Figure 21. Figure (a) shows the reconstructed $P_R(k)$ in blue dashed lines, given different initial guesses $P_R(k)^{(i=0)}$ in yellow lines varying by intercept $k_*$ for a given slope $n_s - 0.97$. The red line shows the original injected power spectrum $P_R(k)$. Figure (b) shows the relative % difference in the reconstructed $C_{L\kappa\kappa}$ and the input data $C_{L\kappa\kappa}$. 
Figure 22. Figure (a) shows the reconstructed $P_R(k)$ in blue dashed lines, given different initial guesses $P_R(k)^{(i=0)}$ in yellow lines varying by intercept $k^*$ for a given slope $n_s - 1.1$. The red line shows the original injected power spectrum $P_R(k)$. Figure (b) shows the relative % difference in the reconstructed $\hat{C}_L^{\kappa\kappa}$ and the input data $C_L^{\kappa\kappa}$. 
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