Some Results on the 
Second Gaussian Map for Curves

ELISABETTA COLOMBO & PAOLA FREDIANI

1. Introduction

The first Gaussian map for the canonical series has been intensively studied. It has been shown that, for a general curve of genus different from 9 and \( \leq 10 \), the first Gaussian map is injective, while for genus \( \geq 10 \) and different from 11 it is surjective [7; 9; 20]. In [21] it is proved that if a curve lies on a \( K3 \) surface then the first Gaussian map cannot be surjective, and it is known (see [18]) that the general curve of genus 11 lies on a \( K3 \) surface.

In this paper we study some properties of the second Gaussian map
\[ \mu_2 : I_2(K_X) \rightarrow H^0(X, 4K_X). \]

Our geometrical motivation comes from its relation with the curvature of the moduli space \( M_g \) of curves of genus \( g \) endowed with the Siegel metric induced by the period map \( j : M_g \rightarrow A_g \), which we started to analyze in [10]. There the curvature is computed using the formula for the associated second fundamental form given in [11]. In particular, in [11] it is proved that the second fundamental form lifts the second Gaussian map \( \mu_2 \), as stated in an unpublished paper of Green and Griffiths (cf. [15]).

In [10, Cor. (3.8)] we give a formula for the holomorphic sectional curvature of \( M_g \) along the a Schiffer variation \( \xi_P \), for \( P \) a point on the curve \( X \), in terms of the holomorphic sectional curvature of \( A_g \) and the second Gaussian map.

The relation of the second Gaussian map with curvature properties of \( M_g \) in \( A_g \) suggests that its rank could give information on the geometry of \( M_g \). Note that surjectivity can be expected for a general curve of genus at least 18. Recall that \( M_g \) is uniruled for \( g \leq 15 \), has Kodaira dimension at least 2 for \( g = 23 \), and is of general type for all other values of \( g \geq 22 \).

Along these lines, in this paper we exhibit infinitely many examples of curves lying on the product of two curves with surjective second Gaussian map. Other examples of curves whose second Gaussian map is surjective were given in [4] for complete intersections. Both classes of examples generalize constructions given by Wahl [21; 22] for the first Gaussian map.

Received May 27, 2008. Revision received July 23, 2008.
The present research took place in the framework of the PRIN 2005 of MIUR, “Spazi dei moduli e teoria di Lie”, and PRIN 2006 of MIUR, “Geometry on algebraic varieties”.

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We are also able to determine the rank of $\mu_2$ on the hyperelliptic and trigonal loci. More precisely: for any hyperelliptic curve of genus $g \geq 3$, we show that $\text{rk}(\mu_2) = 2g - 5$ and that its image has the Weierstrass points as base points. For any trigonal (non-hyperelliptic) curve of genus $g \geq 8$, we show that $\text{rk}(\mu_2) = 4g - 18$ and that its image has the ramification points of the $g_1^3$ as base points. Finally, we prove that for any non-hyperelliptic, non-trigonal curve of genus $g \geq 5$, the image of $\mu_2$ has no base points.

In [10] we apply these results to the holomorphic sectional curvature of $M_g$. In particular, along a Schiffer variation $\xi_P$, the holomorphic sectional curvature $H(\xi_P)$ of $M_g$ is strictly smaller than the holomorphic sectional curvature of $A_g$ for a non-trigonal, non-hyperelliptic curve [10, (4.4)]. Instead, if $P$ is either a Weierstrass point of a hyperelliptic curve or a ramification point of the $g_1^3$ on a trigonal curve, then the holomorphic sectional curvature $H(\xi_P)$ is equal to the holomorphic sectional curvature of $A_g$, which equals $-1$ [10, (4.4) and (5.3)].

The computations are based on the observation that, for a quadric $Q$ of rank at most $4$, $\mu_2(Q)$ can be written as the product of the first Gaussian maps associated to sections of the two adjoint line bundles $L$ and $K \otimes L^{-1}$, which define the quadric. As a first straightforward consequence we show that, for any curve, the rank of $\mu_2$ is greater or equal to $g - 3$.

In order to study the trigonal case, we use the related results of [6] and [13] on the first Gaussian map for trigonal curves. A crucial step in determining the rank of the first (and hence the second) Gaussian map on the trigonal locus is the observation that a trigonal curve lies on a rational normal scroll. A natural question is to understand whether restrictions on the rank of $\mu_2$ can be obtained if a curve lies on a special surface, as it happens for the first Gaussian map and $K3$ surfaces, or if it occurs in a nontrivial linear system of a surface (see [19] for related results). We intend to continue our investigations on the rank properties of $\mu_2$ for general curves and for curves on surfaces in the near future.

The paper is organized as follows. In Section 2 we study the second Gaussian map, on quadrics of rank $\leq 4$, in terms of the first Gaussian maps associated to sections of the two adjoint line bundles $L$ and $K \otimes L^{-1}$, which define the quadric. In Section 3 we show a class of infinitely many examples of curves with surjective second Gaussian map, and in Section 4 we determine the rank of $\mu_2$ for hyperelliptic and trigonal curves.

In Section 5 we prove injectivity of $\mu_2$ for general curves of genus at $\leq 6$ by specialization on trigonal curves and on smooth plane quintics. Finally, in Section 6 we study the global generation of the image of $\mu_2$.

ACKNOWLEDGMENTS. The authors thank Gilberto Bini and Pietro Pirola for several fruitful suggestions and discussions on the subject.

2. The Second Gaussian Map

We first recall the definition of the Gaussian maps (cf. [23]). Let $X$ be a smooth projective curve, let $S := X \times X$, and let $\Delta \subset S$ be the diagonal. Let $L$ be a line
bundle on $X$ and let $L_S := p_1^*(L) \otimes p_2^*(L)$, where $p_i : S \to X$ are the natural projections. Consider the restriction map

$$\mu_{n, L} : H^0(S, L_S(-n\Delta)) \to H^0(\Delta, L_S(-n\Delta)|_\Delta).$$

Notice that, since $O(\Delta)|_\Delta \cong T_X$,

$$H^0(\Delta, L_S(-n\Delta)|_\Delta) \cong H^0(X, 2L \otimes nK_X).$$

In the case $L = K_X$ we have $I_2(K_X) \subset H^0(S, K_S(-2\Delta))$, so we can define the second Gaussian map

$$\mu_2 : I_2(K_X) \to H^0(X, 4K_X)$$

as the restriction $\mu_{2, K}|_{I_2(K_X)}$.

As before, we fix a basis $\{\omega_i\}$ of $H^0(K_X)$. In local coordinates, $\omega_i = f_i(z)dz$. Let $Q \in I_2(K_X)$ and $Q = \sum_{i,j} a_{ij} \omega_i \otimes \omega_j$, and recall that $\sum_{i,j} a_{ij} f_i f_j \equiv 0$; since the $a_{ij}$ are symmetric, we also have $\sum_{i,j} a_{ij} f_i' f_j' \equiv 0$. The local expression of $\mu_2(Q)$ is

$$\mu_2(Q) = \sum_{i,j} a_{ij} f_i'' f_j' (dz)^{i,j} = -\sum_{i,j} a_{ij} f_i' f_j' (dz)^4.$$ (1)

We also recall the definition of the first Gaussian map (cf. [22])

$$\mu_{1, L} : \mathcal{H}^2H^0(L) \to H^0(2L \otimes K_X)$$

as the restriction of $\mu_{1, L}$ to $\mathcal{H}^2H^0(L) \subset H^0(S, L_S(-\Delta))$. In local coordinates, if $s_0, s_1 \in H^0(L)$ with $s_i = g_i l$, where $l$ is a local section of $L$, then

$$\mu_{1, L}(s_0 \wedge s_1) = (g_0 g_1' - g_1 g_0') l^2 dz.$$ (2)

Moreover, we have

$$\text{div}(\mu_{1, L}(s_0 \wedge s_1)) = 2F + R,$$

where $F$ is the base locus of $[[s_0, s_1]] \subset |H^0(L)|$ and $R$ is the ramification divisor of the induced morphism (see e.g. [9; 23]).

**Remark 2.1.** Recall that there is the following bijection:

$$\{\{Q\} \in \mathbb{P}(I_2(K_X)) \mid \text{rk}(Q) \leq 4\}
\longleftrightarrow \{\{L, K_X - L, V, W\} \mid V \subset H^0(L), \dim V = 2,
W \subset H^0(K_X - L), \dim W = 2\}.$$ (3)

Here $\text{rk}(Q) = 3$ if and only if $2L = K_X$ and $V = W$ (see e.g. [3, p. 261]).

**Lemma 2.2.** If a quadric $Q$ of rank at most 4 corresponds to $\{L, K_X - L, V, W\}$ and if $V = \langle s_0, s_1 \rangle$ and $W = \langle t_0, t_1 \rangle$, then

$$\mu_2(Q) = \mu_{1, L}(s_0 \wedge s_1) \mu_{1, K - L}(t_0 \wedge t_1).$$

In particular, $\mu_2(Q) \neq 0$.

**Proof.** By construction, $Q = (s_0 t_0) \otimes (s_1 t_1) - (s_0 t_1) \otimes (s_1 t_0) \in I_2(K_X)$. Locally we have $s_i = g_i l$, where $l$ is a local section of $L$, and $t_i = h_i l'^{-1}dz$, so

$$\mu_2(Q) = \sum_{i,j} a_{ij} f_i'' f_j' (dz)^{i,j} = -\sum_{i,j} a_{ij} f_i' f_j' (dz)^4.$$ (1)
We now make an easy linear algebra remark that will be useful in the sequel.

**Remark 2.3.** Recall that, by a theorem of M. Green ([14]; see also [3, p. 255]), for a non-hyperelliptic smooth curve of genus $g \geq 4$ it follows that $I_2$ is generated by quadrics of rank $\leq 4$.

We now make an easy linear algebra remark that will be useful in the sequel.

**Remark 2.4.** Let $X \subset \mathbb{P}^n = \mathbb{P}(V)$ be a projective variety. Let $f : V \to W$ be a linear map, $\dim(W) = m + 1$, and let $\tilde{f} : \mathbb{P}(V) = \mathbb{P}^n \dashrightarrow \mathbb{P}(W) = \mathbb{P}^m$ be the corresponding projection. Let $K$ be the kernel of $f$, and assume that $\mathbb{P}(K) \cap X = \emptyset$ (i.e., $\tilde{f}|_X$ is a morphism). This clearly implies that $\dim(X) + \dim(K) - 1 \leq n - 1$ or, equivalently, that $\text{rk}(f) \geq \dim(X) + 1$.

Consider the rational map

$$\tilde{\mu}_2 : \mathbb{P}(I_2(K_X)) \dashrightarrow \mathbb{P}(H^0(4K_X)), \quad [Q] \mapsto [\mu_2(Q)].$$

Let $\Gamma = \{[Q] \in \mathbb{P}(I_2(K_X)) \mid \text{rk}(Q) \leq 4\}$; then, by Lemma 2.2, the restriction of this map to $\Gamma$ is a morphism.

**Proposition 2.5.** For any curve of genus $g \geq 4$,

$$\text{rk}(\mu_2) \geq \dim \Gamma + 1 \geq g - 3.$$ 

**Proof.** Since $\tilde{\mu}_2|_\Gamma$ is a morphism, by Remark 2.4 we know that $\text{rk}(\mu_2) \geq \dim \Gamma + 1$. Denote by $\mathcal{W} \subset W^1_{g-1}$ the subset of line bundles $L \in W^1_{g-1}$ such that $h^0(L) = 2$; $\mathcal{W}$ is a nonempty open subset of $W^1_{g-1}$ of dimension $\geq g - 4$ (cf. e.g. [3]). If we set $\mathcal{Y} := \mathcal{W}/(\tau)$, where $\tau$ is the involution that maps $L$ to $K_X - L$, then we can identify $\mathcal{Y}$ with a subset of $\Gamma$. In fact, given a line bundle $L \in \mathcal{W}$, the set $\{L, K - L, H^0(L), H^0(K - L)\}$ determines a quadric of rank $\leq 4$, as we saw in Remark 2.1.

Therefore, $\dim(\Gamma) \geq \dim(\mathcal{W}) \geq g - 4$ and so $\text{rk}(\mu_2) \geq g - 3$. \qed

### 3. Surjectivity

In this section we give a class of examples of curves contained in the product of two curves for which the second Gaussian map is surjective, as Wahl does in [22, Thm. 4.11] for the first Gaussian map. Other examples of curves whose second Gaussian map is surjective have been obtained by Ballico and Fontanari [4] in the case of complete intersections, generalizing Wahl’s result [21] on the first Gaussian map for complete intersections.

Let $C_1, C_2$ be two smooth curves of respective genera $g_1, g_2$; denote by $K_i = K_{C_i}, i = 1, 2$; and choose $D_i$ divisors on $C_i$ of degree $d_i, i = 1, 2$. Let $X = C_1 \times C_2$, and let $C \in |p_1^*(D_1) \otimes p_2^*(D_2)|$ be a smooth curve, where $p_j$ is the projection from $C_1 \times C_2$ on $C_j$ and $K_X(C) = p_1^*(K_1(D_1)) \otimes p_2^*(K_2(D_2)).$
Theorem 3.1. If \( g_1, g_2 \geq 2 \) with \( d_i \geq 2g_i + 5 \) for \( i = 1, 2 \), or if \( g_1 \geq 2 \) and \( g_2 = 1 \) with \( d_1 \geq 2g_1 + 5 \) and \( d_2 \geq 7 \), or if \( g_2 = 0 \) with \( d_2 \geq 7 \) and \( d_2(g_1 - 1) > 2d_1 \geq 4g_1 + 10 \), then \( \mu_{2,K_X} \) is surjective for a smooth curve \( C \in |p_1^*D_1 \otimes p_2^*D_2| \).

Therefore, under these assumptions and for the general curve of genus \( g = 1 + (g_2 - 1)d_1 + (g_1 - 1)d_2 + d_1d_2 \), the second Gaussian map is surjective.

Proof. Denote by \( I_2(K_X(C)) \) the kernel of the multiplication map

\[
S^2H^0(K_X(C)) \rightarrow H^0(K_X^2(2C)).
\]

Let \( \mu^X_{2,K_X(C)} : I_2(K_X(C)) \rightarrow H^0(S^2\Omega_X^1 \otimes K_X^2(2C)) \) be the second Gaussian map of the line bundle \( K_X(C) \) on the surface \( X \). We have the following commutative diagram:

\[
\begin{array}{ccc}
I_2(K_X(C)) & \xrightarrow{\mu^X_{2,K_X(C)}} & H^0(S^2\Omega_X^1 \otimes K_X^2(2C)) \\
\downarrow g & & \downarrow p_1 \\
I_2(K_C) & \xrightarrow{\mu_2} & H^0(K_C^2)
\end{array}
\]

where \( p_1 \) is the restriction map and \( p_2 \) is the map that comes from the conormal extension.

We will prove that \( p_1, p_2 \) and \( \mu^X_{2,K_X(C)} \) are surjective. From this we clearly obtain the surjectivity of \( \mu_2 \).

We want to show that \( H^1(\Omega_X^1|_C \otimes K_C^2(-C)) = 0 \), from which the surjectivity of \( p_2 \) will follow. We have

\[
H^1(\Omega_X^1|_C \otimes K_C^2(-C)) = H^1(C, \mathcal{O}_C(p_1^*(K_1^2(D_1)) \otimes p_2^*(K_2^2(D_2))))
\]

\[
\oplus H^1(C, \mathcal{O}_C(p_1^*(K_1^2(D_1)) \otimes p_2^*(K_2^2(D_2)))),
\]

so it is sufficient to check that, under our assumptions,

\[
\mathcal{O}_C(p_1^*(K_1^2(D_1)) \otimes p_2^*(K_2^2(D_2))) \quad \text{and} \quad \mathcal{O}_C(p_1^*(K_1^2(D_1)) \otimes p_2^*(K_2^2(D_2)))
\]

both have degree greater than \( 2g(C) - 2 = d_1(2g_2 - 2 + d_2) + d_2(2g_1 - 2 + d_1) \).

Let us now consider the map \( p_1 \). We have

\[
S^2\Omega_X^1 \otimes K_X^2(C) = (p_1^*(K_1^2(D_1)) \otimes p_2^*(K_2^2(D_2)))
\]

\[
\oplus (p_1^*(K_1^2(D_1)) \otimes p_2^*(K_2^2(D_2)))
\]

\[
\oplus (p_1^*(K_1^2(D_1)) \otimes p_2^*(K_2^2(D_2)));
\]

thus, by Künneth, if \( g_i \geq 1 \) for \( i = 1, 2 \) or if \( g_2 = 0 \) with \( d_2 \geq 7 \) and \( g_1 \geq 1 \), then \( H^1(S^2\Omega_X^1 \otimes K_X^2(C)) = 0 \). Hence \( p_1 \) is surjective.
We want now to show that $\mu_{X,K_2(C)}^X$ is surjective. Observe that $$S^2H^0(K_X(C)) = (S^2H^0(K_1(D_1)) \otimes S^2H^0(K_2(D_2)))$$ $$\oplus (\Lambda^2H^0(K_1(D_1)) \otimes \Lambda^2H^0(K_2(D_2))),$$ so we have $$I_2(K_X(C)) = [(I_2(K_1(D_1)) \otimes S^2H^0(K_2(D_2)))$$ $$+ (S^2H^0(K_1(D_1)) \otimes I_2(K_2(D_2)))$$ $$\oplus (\Lambda^2H^0(K_1(D_1)) \otimes \Lambda^2H^0(K_2(D_2))).$$

Since $$H^0(S^2\Omega^1_X \otimes K^2_X(2C)) = (H^0(C_1, K^4_1(2D_1)) \otimes H^0(C_2, K^4_2(2D_2)))$$ $$\oplus (H^0(C_1, K^2_1(2D_1)) \otimes H^0(C_2, K^4_2(2D_2)))$$ $$\oplus (H^0(C_1, K^1_1(2D_1)) \otimes H^0(C_2, K^3_2(2D_2))),$$ one can easily check that $\mu_{X,K_2(C)}^X: I_2(K_X(C)) \rightarrow H^0(S^2\Omega^1_X \otimes K^2_X(2C))$ is the sum of the three following maps:

$$\mu_{2,K_1(D_1)} \otimes m_2: I_2(K_1(D_1)) \otimes S^2H^0(K_2(D_2))$$ $$\rightarrow H^0(K^4_1(2D_1)) \otimes H^0(K^2_2(2D_2)),$$

$$n_2 \otimes \mu_{2,K_2(D_2)}: S^2H^0(K_1(D_1)) \otimes I_2(K_2(D_2))$$ $$\rightarrow H^0(K^2_1(2D_1)) \otimes H^0(K^4_2(2D_2)),$$

$$\mu_{1,K_1(D_1)} \otimes \mu_{1,K_2(D_2)}: \Lambda^2(H^0(K_1(D_1)) \otimes \Lambda^2H^0(K_2(D_2))$$ $$\rightarrow H^0(K^4_1(2D_1)) \otimes H^0(K^2_2(2D_2)).$$

Here $m_2$ and $n_2$ are the multiplication maps. Now we apply [5, Thm. (1.7)] to the line bundles $L_i := K_i(D_i)$ on the curves $C_i, i = 1, 2$, to obtain that if $\deg(L_i) =: l_i$ satisfies $2l_i \geq 3(2g_i + 2) + 2g_i - 1$ then both $\mu_{2,L_i}$ and $\mu_{1,L_i}$ are surjective. Therefore, if $d_i \geq 2g_i + 5$, then $\mu_{2,L_i}$ and $\mu_{1,L_i}$ are surjective; hence $\mu_{X,K_2(C)}^X$ is surjective, and this concludes the proof.

**Remark 3.2.** The example of lowest genus of a smooth curve $C \in |p_1^*D_1 \otimes p_2^*D_2|$ with surjective second Gaussian map is 71, obtained by choosing $g_1 = 2, g_2 = 1, d_1 = 9$, and $d_2 = 7$.

### 4. Hyperelliptic and Trigonal Curves

Assume now that $X$ is either a hyperelliptic curve of genus $\geq 3$ or a trigonal curve of genus $g \geq 4$. Let $|F|$ denote the $g^1_2$ in the hyperelliptic case or the $g^1_1$ in the trigonal case. Let $\phi_F: X \rightarrow P^1$ be the induced morphism and let $\psi: P^1 \leftarrow P^{g-1}$ be the Veronese embedding, so that (in the hyperelliptic case) $\phi_K = \psi \circ \phi_F$, where $\phi_F$ is the canonical map. Observe that, in the hyperelliptic case, the hyperelliptic involution $\tau$ acts as $-\text{Id}$ on $H^0(K_X)$; we thus have an exact sequence

$$0 \rightarrow I_2(K_X) \rightarrow S^2(H^0(K_X)) \rightarrow H^0(2K_X)^+ \rightarrow 0.$$
where $H^0(2K_X)^+$ denotes the $\tau$-invariant part of $H^0(2K_X)$ whose dimension is $(2g - 1)$ and where $I_2(K_X)$ is the vector space of the quadrics containing the rational normal curve.

Set $L := K_X - F$. Fix a basis $\{x, y\}$ of $H^0(F)$ and a basis $\{t_1, \ldots, t_r\}$ of $H^0(L)$, both in the hyperelliptic and in the trigonal case. We have a linear map

$$
\psi: \Lambda^2(H^0(L)) \to I_2, \quad t_i \wedge t_j \mapsto Q_{ij} = xt_i \circ yt_j - xt_j \circ yt_i.
$$

We recall that, in both cases, the linear map $\psi: \Lambda^2(H^0(L)) \to I_2$ is an isomorphism, which can be easily checked or found in [1].

**Lemma 4.1.** Let $X$ be either a hyperelliptic curve of genus $\geq 3$ or a trigonal curve of genus $g \geq 4$, and let $q_1, \ldots, q_l$ be the ramification points of either the $g^2_1$ or the $g^3_1$. Then

$$
\mu_2(Q) = \mu_{1,F}(x \wedge y)\mu_{1,L}(\psi^{-1}(Q))
$$

where $Q$ is a quadric of rank 4.

**Proof.** The first statement is straightforward. So we have

$$
\text{div}(\mu_2(Q)) = \text{div}(\mu_{1,F}(x \wedge y)) + \text{div}(\mu_{1,L}(\psi^{-1}(Q)))
$$

$$
= q_1 + \cdots + q_l + \text{div}(\mu_{1,L}(\psi^{-1}(Q))).
$$

Therefore, $\mu_2(Q)(q_i) = 0$ for all $i = 1, \ldots, l$.

**Proposition 4.2.** Let $X$ be a hyperelliptic curve of genus $g \geq 3$. Then the rank of $\mu_2$ is $2g - 5$.

**Proof.** Given a hyperelliptic curve of genus $g$ with equation $y^2 = f(x)$, where $f$ has degree $2g + 2$ and only simple roots, a basis of $H^0(K_X)$ is $\{\omega_i = x^i \frac{dx}{\sqrt{f}} | 0 \leq i \leq g - 1\}$. Let $|F|$ be the $g_1^1$ on $X$, and assume that $F = \phi_F^{-1}(0) =: p_1 + p_2$.

Set $L = K_X - F = K_X - p_1 - p_2$, let $H^0(L) \subset H^0(K_X)$, and let

$$
\mu_{1,L}: \Lambda^2H^0(L) = \Lambda^2H^0(K_X - p_1 - p_2)
$$

$$
\to H^0(2L + K_X) = H^0(3K_X - 2p_1 - 2p_2)
$$

be the first Gaussian map of $L$; then

$$
\mu_{1,L} = \mu_{1,K} |_{\Lambda^2H^0(K_X - p_1 - p_2)}.
$$

By Lemma 4.1, the rank of $\mu_2$ equals the rank of $\mu_{1,L}$. As shown in [9],

$$
\mu_{1,K}(\omega_i \wedge \omega_j) = (i - j) x^{i+j-1} \frac{dx}{\sqrt{f}}, \quad 0 \leq i < j \leq g - 1,
$$

so there are exactly $2g - 3$ distinct powers of $x$.

Then clearly a basis of $H^0(K_X - p_1 - p_2)$ is given by $\{x^i \frac{dx}{\sqrt{f}} | i > 0\}$. We want to compute the dimension of the span of $\{\mu_{1,K}(\omega_i \wedge \omega_j) | 0 < i < j \leq g - 1\}$. Observe that $l := i + j - 1 = 0$ if and only if $i = 0$ and $j = 1$; $l = 1$ if and only if $i = 0$ and $j = 2$. But if $l \geq 2$, then $l = i + j - 1$ also for some $i, j > 0$.

Therefore, $\text{rk}(\mu_{1,L}) = \text{rk}(\mu_{1,K}) - 2 = 2g - 5$. 

\qed
Assume now that \( X \) is a non-hyperelliptic trigonal curve of genus \( g \geq 4 \). Let \( |F| \) be the \( g \) on \( X \) and assume that \( F = p_1 + p_2 + p_3, p_i \in X \). Let us denote by \( L = K_X - F = K_X - p_1 - p_2 - p_3, \deg(L) = 2g - 5 \), and \( h^0(L) = g - 2 \). So \( H^0(L) \subset H^0(K_X) \) and \( \mu_{1,L} = \mu_{1,K}|_{\Lambda^2 H^0(K_X-p_1-p_2-p_3)} \). In [9] it is proved that, for the general trigonal curve of genus \( g \geq 4 \), \( \dim(\text{coker}(\mu_{1,K})) = g + 5 \); moreover, specific examples of trigonal curves (whose genera are all equal to 1 modulo 3) such that the corank of \( \mu_{1,K} \) is \( g + 5 \) are exhibited. Using results of [13], in [6] Brawner proved that \( \dim(\text{coker}(\mu_{1,K})) = g + 5 \) for any trigonal curve of genus \( g \geq 4 \).

We shall now compute the rank of \( \mu_2 \) for trigonal curves. By Lemma 4.1 it suffices to compute \( \text{rk}(\mu_{1,L}) \), which we shall do following the computation used in [13] and [6] for \( \mu_{1,K} \).

Recall that a canonically embedded trigonal curve of genus \( g \) lies on a rational normal scroll \( S_{k,l} \) of genus \( g \) and \( k + l + k = g - 2 \); here \( k \) is the Maroni invariant, which is bounded by

\[
\frac{g - 4}{3} \leq k \leq \frac{g - 2}{2}
\]  

(cf. [17]).

The surface \( S_{k,l} \) is isomorphic to \( F_n \) with \( n = l - k \). Let us denote by \( H \) the hyperplane section and by \( R \) the fiber of the ruling; set \( B \equiv H - 1R \). We have

\[
H^2 = g - 2, \quad B^2 = -n, \\
C \equiv 3H - (g - 4)R, \\
K_S \equiv -2H + (g - 4)R;
\]

consequently,

\[
K_S + C - R \equiv H - R \equiv B + (l - 1)R
\]

and

\[
(K_S + C - R)|_C \equiv L.
\]

**Theorem 4.3.** For any trigonal curve \( C \) of genus \( g \geq 8 \), the rank of \( \mu_2 \) is \( 4g - 18 \). Hence, for the general curve of genus \( g \geq 8 \), \( \mu_2 \) has rank \( \geq 4g - 18 \).

**Proof.** As in [13, (2.1)], we have the following commutative diagram involving the first Gaussian map \( \mu_{1,H-R}^S \) for the scroll \( S := S_{k,l} \):

\[
\Lambda^2 H^0(S, \mathcal{O}_S(K_S + C - R)) \xrightarrow{\mu_{1,H-R}^S} H^0(S, \Omega_S^1(2K_S + 2C - 2R)) \xrightarrow{\text{Res}} \\
\Lambda^2 H^0(C, K_C - F)) \xrightarrow{\mu_{1,L}} H^0(C, 3K_C - 2F).
\]

We will prove that the map \( \mu_{1,H-R}^S \) is surjective, that \( \gamma' \) is injective, and that Res is surjective. This implies that \( \text{rk}(\mu_{1,L}) = h^0(S, \Omega_S^1(2K_S + 2C - 2R)) = h^0(S, \Omega_S^1(2B + 2(l - 1)R)). \)
Observe that, by the bound (3) of the Maroni invariant, \( k \geq 2 \) for \( g \geq 8 \); hence the hypotheses of [13, Cor. (3.3.2)] are satisfied, so \( h^0(S, \mathcal{O}_S^1(2B + 2(l - 1)R)) = 4g - 18 \). In fact, [13, Cor. (3.3.2)] asserts that \( h^0(S, \mathcal{O}_S^1(rB + sR)) = 2rs - nr^2 - 2 \) if \( r \geq 1 \) and \( s \geq nr + 2 \).

The surjectivity of \( \mu^S_{1,H-R} \) follows by [13, Thm. (4.5)], which states that \( \mu^S_{1,rB+sR} \) is surjective if \( r \geq 0 \) and \( s \geq nr + 1 \).

In [6, (3.4)] it is proved that the map \( \gamma : H^0(S, \mathcal{O}_S(2H)) \rightarrow H^0(C, 3KC) \) is injective. Since \( \gamma' \) is the restriction of \( \gamma \) to \( H^0(S, \mathcal{O}_S(-2H + (g - 5)R)) = 0 \), proving our assertion.

5. Injectivity for Low Genus

We now give some examples of computations of the rank of \( \mu_2 \) for genus \( \leq 7 \), from which will follow that \( \mu_2 \) is injective for the general curve of genus \( \leq 6 \). Note that if \( g(X) = 4 \), then \( I_2(K_X) \) has dimension 1 and so \( \mu_2 \) is injective.

**Proposition 5.1.** For any trigonal curve \( X \) of genus 5, \( \mu_2 \) is injective. Hence, for the general curve of genus 5, \( \mu_2 \) is injective.

**Proof.** For a curve of genus 5, the dimension of \( I_2(K_X) \) is 3. Let us assume that \( X \) is trigonal. Then there exists a line bundle \( L \) on \( X \) such that \( h^0(L) = 2 \) with \( \deg(L) = 3 \), so that \( h^0(K-L) = 3 \). Let \( \text{Gr}(2, H^0(K-L)) \) be the Grassmannian of the 2-dimensional subspaces in \( H^0(K-L) \). To any \( W \in \text{Gr}(2, H^0(K-L)) \) we associate the quadric \( Q_W \) of rank 4 corresponding to the set \( \{L, K-L, H^0(L), W\} \) as in Lemma 2.2. Thus we have a morphism

\[
\text{Gr}(2, H^0(K-L)) \rightarrow \mathbb{P}(H^0(4K)), \quad W \mapsto \tilde{\mu}_2(Q_W).
\]

Then, by Remark 2.4, we have

\[
\text{rk}(\mu_2) \geq \dim(\text{Gr}(2, H^0(K-L))) + 1 = 3.
\]

**Theorem 5.2.** Let \( X \) be a smooth plane quintic; then the map \( \mu_2 \) is injective and its image has no base points. Then, for the general curve of genus 6, \( \mu_2 \) is injective.
These three projections \( \pi_{ij} \) be distinct points of \( X \), in general position, such that the tangent line \( r_i \) of \( X \) at \( q_i \) is a simple tangent, \( i = 1, 2, 3 \). Assume also that \( P_{12} := r_1 \cap r_2 \), \( P_{13} := r_1 \cap r_3 \), and \( P_{23} := r_2 \cap r_3 \) are in general position and do not lie on \( X \). Denote by \( \pi_{12}, \pi_{13}, \pi_{23} \) the respective projections \( X \rightarrow \mathbb{P}^1 \) from the points \( P_{12}, P_{13}, P_{23} \). These three projections \( \pi_{ij} \) correspond to three pencils \( V_{ij} \subset H^0(L) \) with \( 2L \equiv K \). Let \( R_{ij} \) (for \( 1 \leq i < j \leq 3 \)) be the ramification divisor of \( \pi_{ij} \); then we have

\[
R_{ij} = q_i + q_j + A_{ij}.
\]

Observe that, by construction, \( q_3 \notin A_{12} \); otherwise, \( P_{12} \) would lie on \( r_3 \) and we would have \( P_{12} = P_{13} = P_{23} \). Analogously \( q_1 \notin A_{23} \) and \( q_2 \notin A_{13} \). Observe that, since \( r_2 \) and \( r_3 \) are simple tangents, it follows that \( q_2, q_3 \notin A_{23} \) and hence there must exist a point \( q_4 \in A_{23} \) that is different from \( q_2 \) and \( q_3 \). Notice that \( q_4 \notin R_{12} \cup R_{13} \). In fact, by construction \( P_{23} \in r_4 \) (\( r_4 \) is the tangent line of \( X \) at \( q_4 \)), hence \( P_{23} = r_2 \cap r_3 = r_4 \cap r_2 = r_4 \cap r_3 \). So if \( q_4 \in R_1 \) then \( P_{12} \in r_4 \); thus \( P_{12} = r_4 \cap r_2 = P_{23} \), a contradiction. Analogously, if \( q_4 \in R_3 \) then we get \( P_{12} = P_{23} = P_{13} \), which is impossible.

Define now the six quadrics in \( \Gamma \) by the following six sets as in Lemma 2.2:

\[
Q_{ij,kl} \leftrightarrow \{L, L = K - L, V_{ij}, V_{kl}\}
\]

for \( 1 \leq i < j \leq 3 \) and \( 1 \leq k < l \leq 3 \). Then, by Lemma 2.2, we have

\[
\text{div}(\mu_2(Q_{ij,12})) = 2R_{12} = 2q_1 + 2q_2 + 2A_{12},
\]

\[
\text{div}(\mu_2(Q_{ij,13})) = R_{12} + R_{13} = 2q_1 + q_2 + q_3 + A_{12} + A_{13},
\]

\[
\text{div}(\mu_2(Q_{ij,23})) = R_{12} + R_{23} = q_1 + 2q_2 + q_3 + A_{12} + A_{23},
\]

\[
\text{div}(\mu_2(Q_{ij,13})) = 2R_{13} = 2q_1 + 2q_3 + 2A_{13},
\]

\[
\text{div}(\mu_2(Q_{ij,23})) = R_{13} + R_{23} = q_1 + 2q_2 + 3q_3 + A_{13} + A_{23},
\]

\[
\text{div}(\mu_2(Q_{ij,23})) = 2R_{23} = 2q_2 + 2q_3 + 2A_{23}.
\]

Assume now that there exists a linear combination \( \sum \lambda_{ij,kl} \mu_2(Q_{ij,kl}) = 0 \). Then, evaluating in \( q_1 \) yields \( \lambda_{23,23} \mu_2(Q_{23,23})(q_1) = 0 \) and so \( \lambda_{23,23} = 0 \), since \( q_1 \notin R_{23} \). Evaluating in \( q_2 \) yields \( \lambda_{13,13} \mu_2(Q_{13,13})(q_2) = 0 \); thus \( \lambda_{13,13} = 0 \) since \( q_2 \notin R_{13} \). By evaluating in \( q_3 \) we obtain \( \lambda_{12,12} \mu_2(Q_{12,12})(q_3) = 0 \); thus \( \lambda_{12,12} = 0 \) since \( q_3 \notin R_{12} \). We now evaluate in \( q_4 \in A_{23} \) and find that \( \lambda_{12,13} \mu_2(Q_{12,13})(q_4) = 0 \). This implies \( \lambda_{12,13} = 0 \), since otherwise we would have \( q_4 \in A_{12} + A_{13} \), which is impossible. Finally, we must have \( \lambda_{12,23} = \lambda_{13,23} = 0 \), for otherwise we would have \( R_{12} = R_{13} \)—a contradiction. This proves the injectivity of \( \mu_2 \).

Notice that if \( P \) is a base point of the image of \( \mu_2 \), then the three projections \( \pi_{ij} \) must have a common ramification point, and by construction this is impossible.

\[\square\]

If \( X \) is a trigonal curve of genus 7, then the argument of Proposition 5.1 yields \( \text{rk}(\mu_2) \geq \dim \text{Gr}(2, H^0(K - L)) + 1 = 7 \). We will now exhibit an example of a
trigonal curve of genus 7 such that \( \text{rk}(\mu_2) = \text{rk}(\mu_{1,L}) = 9 \). Our example is the cyclic covering of \( \mathbb{P}^1 \), constructed in [9], whose affine equation is

\[ y^3 = x^9 - 1. \]

In [9] the map \( \mu_{1,K} \) is explicitly computed on the elements \( \sigma_{ij} = x^i y^j \frac{F}{x} \) for \( 0 \leq j \leq 1 \) and \( 0 \leq i \leq 3(2 - j) - 2 \), which form a basis of \( H^0(K_X) \). It is shown that the image of \( \mu_{1,K} \) (of dimension 18) is spanned by the following three types of elements:

\[ \mu_{1,K}(\sigma_{i0} \wedge \sigma_{i0}) = [(k - i)x^{i+k-1}y^{-3}] \cdot (dx)^3, \quad 0 \leq i < k \leq 4; \]
\[ \mu_{1,K}(\sigma_{i0} \wedge \sigma_{i1}) = [(k - i)x^{i+k-1}y^{-3} + 3x^{i+k+8}y^{-6}] \cdot (dx)^3, \quad 0 \leq i \leq 4, 0 \leq k \leq 1; \]
\[ \mu_{1,K}(\sigma_{i0} \wedge \sigma_{i1}) = y^{-2} \cdot (dx)^3. \]

The \( g_3 \) on our curve is the linear system \( |F| = |p_1 + p_2 + p_3| \), where \( p_i \) can be chosen to be the points \((0, y_i)\) with \( y_i^3 = -1 \). Therefore, if \( L = K_X - p_1 - p_2 - p_3 \) then we can identify \( H^0(L) \) with the subspace of \( \langle \sigma_{ij} \rangle \) generated by the elements \( \sigma_{ij} \), where \( i > 0 \). Since \( \mu_{1,L} = \mu_{1,K} |_{\text{Im}H^0(L)} \), one must compute the dimension of \( \langle \mu_{1,K}(\sigma_{ij} \wedge \sigma_{kl}) \mid i, k > 0 \rangle \), which turns out to be 9.

6. Base Points

We will now show global generation of the image of \( \mu_2 \) for curves that are neither hyperelliptic nor trigonal.

**Theorem 6.1.** Assume that \( X \) is a smooth curve, of genus \( g \geq 5 \), that is non-hyperelliptic and non-trigonal. Then, for any \( P \in X \), there exists a quadric \( Q \in I_2 \) such that \( \mu_2(Q)(P) \neq 0 \). Equivalently, \( \text{Im}(\mu_2) \cap H^0(4K_X - P) \neq \text{Im}(\mu_2) \) for all \( P \in X \).

**Proof.** We will show that, for any \( P \in X \), there exists a quadric \( Q \) of rank 4 such that \( \mu_2(Q)(P) \neq 0 \). We recall that any component of the space \( W^1_{g-1}(X) \) has dimension \( \geq g - 4 \), and in [12, Lemma (2.1.1)] it is proved that, if \( X \) is non-hyperelliptic, non-trigonal, and not isomorphic to a smooth plane quintic, then there exists a line bundle \( L \in W^1_{g-1}(X) \) such that both \( |L| \) and \( |K_X - L| \) are base point free. If \( X \) is a plane quintic, then by Theorem 5.2 we know that \( \mu_2 \) has no base points. So we can assume that there exists a nonempty irreducible open subset \( \mathcal{V} \) in \( W^1_{g-1}(X) \) consisting of line bundles \( L \) such that \( h^0(L) = 2, L \neq K_X - L \), and both \( |L| \) and \( |K_X - L| \) are base point free. So the condition \( \mu_2(Q)(P) = 0 \) for the quadric associated to \( |L| \) and \( |K_X - L| \) says that \( P \) is a ramification point either for the morphism \( \phi_{|L|}: X \to \mathbb{P}^1 \) or for the morphism \( \phi_{|K_X-L|}: X \to \mathbb{P}^1 \).

We claim that there exists an \( L \in \mathcal{V} \) such that \( P \) is at most a simple ramification point for both \( \phi_{|L|} \) and \( \phi_{|K_X-L|} \); that is, \( h^0(L - 3P) = 0 \) and \( h^0(K_X - L - 3P) = 0 \). In fact, assume for all \( L \in \mathcal{V} \) that either \( h^0(L - 3P) \geq 1 \) or \( h^0(K_X - L - 3P) \geq 1 \). Consider two maps: \( F_1: \text{Sym}^{\sigma-3}(X) \to \text{Pic}^{\sigma-1}(X) \) with \( F_1(D) = D + 3P \); and
$F_2 : \text{Sym}^{8-4}(X) \to \text{Pic}^{8-1}(X)$ with $F_2(D) = K_X - D - 3P$. Then $V$ is contained in $\text{Im}(F_1) \cup \text{Im}(F_2)$ and, since they have the same dimension and since $V$ is irreducible, it follows that either $\overline{V} = \text{Im}(F_1)$ or $\overline{V} = \text{Im}(F_2)$. This means that, for all $x_1, \ldots, x_{8-4} \in X, h^0(x_1 + \cdots + x_{8-4} + 3P) = h^0(K_X - x_1 - \cdots - x_{8-4} - 3P) \geq 2$; but this is absurd, since $h^0(K_X - 3P) = g - 3$ because $X$ is non-hyperelliptic and non-trigonal.

So assume that $P$ is a simple ramification point for $\phi := \phi_{|L|} : X \to \mathbb{P}^1$, and let $R$ be its ramification divisor. Consider the exact sequence

$$0 \longrightarrow TX \xrightarrow{\phi_*} \phi^*T_{\mathbb{P}^1} \longrightarrow N_\phi \longrightarrow 0.$$ Notice that $\phi^*T_{\mathbb{P}^1} = TX(R), N_\phi = TX(R)|_R$, and $\phi_*$ is the inclusion map of $TX$ in $TX(R)$; hence our exact sequence is

$$0 \longrightarrow TX \longrightarrow TX(R) \longrightarrow TX(R)|_R \longrightarrow 0.$$ The induced cohomology exact sequence is

$$0 \longrightarrow H^0(TX(R)) \xrightarrow{i} H^0(TX(R)|_R) \xrightarrow{\beta} H^1(TX) \longrightarrow 0.$$ We recall (see e.g. [2]) that $H^0(TX(R)) = H^0(\phi^*T_{\mathbb{P}^1})$ parameterizes the infinitesimal deformations of the morphism $\phi$ that do not move $X$, and it contains the 3-dimensional subspace $\phi^*(H^0(T_{\mathbb{P}^1}))$. Note that, since $H^1(TX(R)) = 0$, any first-order deformation extends.

The strategy of the proof is to exhibit an infinitesimal deformation $\rho \in H^0(TX(R))$ of $\phi$ such that $i(\rho) \notin \text{Im}(\phi_*(TX(R)|_R))$ and $\rho$ is not contained in $\phi^*(H^0(T_{\mathbb{P}^1}))$. The assumption that the ramification in $P$ is simple implies that $\rho$ extends to a deformation of $\phi$ such that the new map is no longer ramified in $P$.

Denote by $\xi_P \in H^1(TX)$ a Schiffer variation in $P$—that is, by definition a generator of the subspace $\text{Im}(H^0(TX(P))|_P) \subset H^1(TX)$. Since $P$ is a ramification point,

$$\xi_P \in \beta(H^0(TX(R)|_R)).$$ If we can prove that $\beta(H^0(TX(R) - P)|_{R - P}) \subset \beta(H^0(TX(R)|_R))$ generates $H^1(TX)$, then there exists an element $x \in H^0(TX(R - P)|_{R - P})$ such that $\xi_P = \beta(x)$. So if $c_P \in H^0(TX(R)|_P) \subset H^0(TX(R)|_R)$ is such that $\beta(c_P) = \xi_P$, then the element $\eta = c_P - x \in H^0(TX(R)|_R)$ maps to zero in $H^1(TX)$; hence there exists an element $\theta \in H^0(TX(R))$ such that $\eta = i(\theta)$ and $\eta \notin H^0(TX(R - P)|_{R - P})$, since the coefficient of $c_P$ in $\eta$ is nonzero.

Therefore, we seek to prove the existence of an $L \in V$ with simple ramification in $P$ such that $\beta(H^0(TX(R - P)|_{R - P}))$ generates $H^1(TX)$. Assume to the contrary that, for any $L \in V$, $\beta(H^0(TX(R - P)|_{R - P}))$ lies on a hyperplane in $H^1(TX)$—that is, there exists an element $\omega \in H^0(2K_X)$ such that $\omega(P_i) = 0$ for all $i \geq 2$ and $\text{ord}_{P_i} \omega = n_i$, where $\sum_{i \geq 2} n_i P_i = R - P$. Then we have

$$2K_X \equiv \text{div}(\omega) = P + \sum_{i \geq 2} n_i P_i - P + q \equiv K_X + 2L - P + q$$

for some $q \in X$. So for any $L$ there exists a point $q \in X$ such that
Since \( L \) varies in an open subset of \( W_{g-1}^1 \), which has dimension at least \( g - 4 \), while \( q \) varies in \( X \), it follows that if \( g \geq 6 \) then this cannot hold for all \( L \in \mathcal{V} \).

If \( g = 5 \), we still get a contradiction by noting that the multiplication by 2 in the Jacobian restricts to a connected topological covering \( \tilde{X} \) of the curve \( X \) of degree \( 2^{10} \) corresponding to the surjective homomorphism \( \pi_1(X) \to H_1(X, \mathbb{Z}/2\mathbb{Z}) \). Hence \( \tilde{X} \) cannot coincide with a component of \( W_4^1 \) that is a 2-to-1 covering of a quintic plane curve ramified along at most ten points (cf. [3, p. 270]).

Finally, we show that we can choose the deformation outside \( \phi^*(H^0(T_{\pi_1})) \). Set \( W = \pi_1(H^0(T_X(R))) \cap H^0(T_X(R - P)|_{R - P}) \). We have just proved that, for \( L \) general, \( \pi_1(H^0(T_X(R))) \) does not contain \( H^0(T_X(R - P)|_{R - P}) \); thus \( \dim(W) = h^0(T_X(R)) - 1 = g - 2 \geq 3 \) since \( g \geq 5 \). As a result, if \( \{e_1, e_2, e_3\} \) are three linearly independent elements in \( W \) then the four elements \( \{i(\mathfrak{d}) = \eta, \eta + e_1, \eta + e_2, \eta + e_3\} \) are linearly independent, since \( \eta \) is not contained in \( H^0(T_X(R - P)|_{R - P}) \). Hence there exists a deformation \( \rho \) with \( i(\rho) \in \{i(\mathfrak{d}) = \eta, \eta + e_1, \eta + e_2, \eta + e_3\} \). So \( \rho \) is the deformation we are looking for.

We have shown that if \( L \) does not belong to the curve \( \gamma \) given by the equation \( 2L \equiv K_X + P - q \) with \( q \in X \), then we can deform \( L \) in such a way that \( P \) is no longer a ramification point. Analogously, if \( L_1 := K_X - L \) does not belong to the curve \( \gamma \), then we can deform \( L_1 = K_X - L \) in such a way that \( P \) is no longer a ramification point of the corresponding morphism. So if we take \( L \in \mathcal{V} - \gamma \) then \( \iota : W_{g-1}^1 \to W_{g-1}^1 \) is the involution sending \( L \) to \( K_X - L \), we find deformations of \( L \) (then also of \( K_X - L \)) such that, for \( L' \) the deformed line bundle, \( P \) is not a ramification point of either \( \phi|_{L'} \) or \( \phi|_{K_X - L'} \).

\[ 2L \equiv K_X + P - q. \]

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E. Colombo
Dipartimento di Matematica
Università di Milano
I-20133 Milano
Italy
elisabetta.colombo@mat.unimi.it

P. Frediani
Dipartimento di Matematica
Università di Pavia
I-27100 Pavia
Italy
paola.frediani@unipv.it