ON ILLUMINATION OF THE BOUNDARY OF A CONVEX BODY IN $\mathbb{E}^n$,

$n = 4, 5, 6$

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Abstract. Let $H_n$ be the minimal number of smaller homothetic copies of an $n$-dimensional convex body required to cover the whole body. Equivalently, $H_n$ can be defined via illumination of the boundary of a convex body by external light sources. The best known upper bound in three-dimensional case is $H_3 \leq 16$ and is due to Papadoperakis. We use Papadoperakis’ approach to show that $H_4 \leq 96$, $H_5 \leq 1091$ and $H_6 \leq 15373$ which significantly improve the previously known upper bounds on $H_n$ in these dimensions.

1. Introduction and results

Let $\mathbb{E}^n$ denote the $n$-dimensional Euclidean space. A convex body in $\mathbb{E}^n$ is a convex compact set having non-empty interior. For two sets $A, B \subset \mathbb{E}^n$ we let $C(A, B)$ be the smallest number of translates of $B$ required to cover $A$, and let $\text{int}(A)$ denote the interior of $A$.

In [3] Hadwiger asked what is the smallest value $H_n$ of $C(K, \text{int}(K))$ for arbitrary convex body $K$ in $\mathbb{E}^n$, $n \geq 3$. This is equivalent to covering $K$ by smaller homothetic copies of $K$, or, as was shown by Boltyanski [2], to illuminating the boundary of the convex body by external light sources. Considering cube, one immediately gets $H_n \geq 2^n$. The related primary conjecture, which is commonly referred to as Levi-Hadwiger conjecture or as Gohberg-Markus covering conjecture, is that $H_n = 2^n$, but this is known (and is simple) only for $n = 2$. Below we give a brief overview of the known results about $H_n$. For a detailed history of the question and survey including many partial results for special classes of convex bodies see, e.g., [1].

The best known asymptotic upper bound on $H_n$ follows from the results [7,8] of Rogers and Shepard, see also [1] Section 2.2:

\[ H_n \leq \left(\frac{2n}{n}\right)^n (\ln n + \ln \ln n + 5), \]

2010 Mathematics Subject Classification. 52A20, 52A37, 52A40, 52C17.

Key words and phrases. Illumination problem, illumination number, covering number, covering by smaller homothetic copies, convex body.

The first author was supported by NSERC of Canada Discovery Grant RGPIN 04863-15.

The second author was partially supported by the PIMS Postdoctoral Fellowship.
where 5 can be replaced by 4 for sufficiently large \( n \). Lassak [5] showed that

\[
H_n \leq (n + 1)n^{n-1} - (n - 1)(n - 2)^{n-1},
\]

which is better than (1.1) for \( n \leq 5 \) and up to now was the best known bound for \( n = 4, 5 \). In [6] Papadoperakis showed that \( H_3 \leq 16 \), which is the best known bound in three dimensions.

The key idea of [6] is to reduce the illumination problem to that of covering specific sets of relatively simple structure by certain rectangular paralleloptopes. Namely, we have \( H_n \leq C_n \), where \( C_n \) is a related covering number which will be introduced in Section 2, see (2.1). In these terms, it was shown in [6] that \( C_3 \leq 16 \). In fact, it is not hard to prove the estimate in the other direction and establish that \( C_3 = 16 \), see Remark 4.4 so 16 is the best one can get with this method for three dimensions.

Our goal is to examine the behavior of \( C_n \) for \( n \geq 4 \). We begin with the bounds for \( n \geq 5 \).

**Theorem 1.1.** For any \( n \geq 5 \) we have

\[
4n^{n-2} + 2n \leq C_n \leq 2n(n - 1)(n - 2)^{n-2} + 2n + 1.
\]

We obtain much sharper estimates for \( n = 4 \).

**Theorem 1.2.** For the four-dimensional case we have \( 95 \leq C_4 \leq 96 \).

As an application, since \( H_n \leq C_n \), we obtain the following new upper bounds on \( H_n \) for \( n = 4, 5, 6 \).

**Theorem 1.3.** \( H_4 \leq 96 \), \( H_5 \leq 1091 \), \( H_6 \leq 15373 \).

Below we provide a table comparing various upper estimates on \( H_n \) stated above, which shows that our estimates for \( H_4 \), \( H_5 \), and \( H_6 \) are roughly one third of the previously known results.

| \( n \) | Theorem 1.3 | Lassak’s upper bound (1.2) | Asymptotic upper bound (1.1) |
|-------|-------------|---------------------------|-----------------------------|
| 4     | 96          | 296                       | 1879                        |
| 5     | 1091        | 3426                      | 8927                        |
| 6     | 15373       | 49312                     | 40886                       |

Theorem 1.2 means that in four dimensions the extension of the Papadoperakis’ approach allows to obtain the bound \( H_4 \leq 96 \) but one cannot do better than 95 using this method.
We tend do believe that in fact $C_4 = 95$, but proving this may require too much effort to justify such a small improvement (see Remark 6.7), recall that the conjectured value of $H_4$ is 16. Nevertheless, the proof of Theorem 1.2 implies the precise solution of two related covering problems (see Remark 6.6). Namely, the smallest number of rectangular parallelopopes with sides parallel to the coordinate axes and the sum of dimensions strictly less than 1 (or $\leq 1$) that are needed to cover the union of all two-dimensional faces of the four-dimensional unit cube is 89 (or 88, respectively).

Theorem 1.1 is a combination of Propositions 3.3 and 4.3 proved in Sections 3 and 4, respectively. Theorem 1.2 follows from Propositions 5.1 and 6.5 proved in Sections 5 and 6, respectively. Theorem 1.3 is an immediate corollary of Theorems 1.1 and 1.2 and Proposition 2.1.

2. Papadoperakis’ reduction to covering problem

We extend the notation $C(\cdot, \cdot)$ to the following two situations. For a set $A \subset \mathbb{E}^n$ and families $\mathcal{A}, \mathcal{B}$ of subsets of $\mathbb{E}^n$, we let $C(\mathcal{A}, \mathcal{B})$ be the smallest number of translates of elements of $\mathcal{B}$ required to cover $A$ and let $C(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}} C(A, \mathcal{B})$ be the smallest number of translates of elements of $\mathcal{B}$ needed to cover an arbitrary element of $\mathcal{A}$.

Let $B_{k,n}$ be the $k$-skeleton of the unit cube $[0,1]^n$, i.e., the union of all $k$-dimensional faces of $[0,1]^n$, or, in other words, the set of all points of the cube having at least $n-k$ coordinates equal to either 0 or 1.

Let $e_i$ denote the $i$-th basic unit vector in $\mathbb{E}^n$, $1 \leq i \leq n$. Note that if a point $x \in \mathbb{E}^n$ satisfies $x, x + e_i \in [0,1]^n$, then $x$ and $x + e_i$ are necessarily on some opposite $(n-1)$-dimensional faces of the cube $[0,1]^n$ (equivalently, the $i$-th coordinate of these points is 0 and 1, respectively). In the collection of sets

$$\mathcal{A}_n := \left\{ \bigcup_{i=1}^{n} \{x_i, x_i + e_i\} : x_i, x_i + e_i \in [0,1]^n \right\}$$

each set consists of at most $2n$ points on the boundary of $[0,1]^n$.

We will use the following two families of $n$-dimensional rectangular parallelopopes:

$$\mathcal{P}_n := \left\{ \prod_{i=1}^{n} [x_i, x_i + \delta_i] : x_i \in \mathbb{R}, \delta_i \geq 0, \sum_{i=1}^{n} \delta_i < 1 \right\} \quad \text{and}$$

$$\mathcal{P}_n^* := \left\{ \prod_{i=1}^{n} [x_i, x_i + \delta_i] : x_i \in \mathbb{R}, \delta_i \geq 0, \sum_{i=1}^{n} \delta_i \leq 1 \right\}.$$
Note that degeneration $\delta_i = 0$ in some coordinates is allowed. For simplicity, we will refer to rectangular parallelopetopes as boxes.

If $\mathcal{A}$ is a collection of subsets of $\mathbb{E}^n$ and $B \subset \mathbb{E}^n$, we let $\mathcal{A} \cup B := \{ A \cup B : A \in \mathcal{A} \}$. Finally, we are ready to define the needed covering number:

\begin{equation}
C_n := C(\mathcal{A}_n \cup B_{n-2,n}, P_n).
\end{equation}

In the above notations, it is established in [6, Lemmas 1-4] that $H_3 \leq C_3$. It turns out that the same arguments can be used in $\mathbb{E}^n$, $n \geq 3$, to prove the following.

**Proposition 2.1.** $H_n \leq C_n$ for any $n \geq 3$.

We only include an outline of the Papadoperakis’ approach since the exposition in [6] is very concise and the generalization to the higher dimensions is straightforward.

First, the shadow $s(u, X) = \{ tu + x : t > 0, x \in X \}$ of a set $X$ when the light comes from the direction $u$ is defined. Then illumination of the boundary of a convex body is equivalent to covering the boundary of the body by shadows of its interior ([6, Lemma 1(e)]). Next, the parallelopetope $P$ of smallest volume containing a given body $A$ is considered. Note that using affine transformations, one can assume that $P$ is a unit cube. Minimality of the volume of $P$ implies that the tangency points of the faces of $P$ with $A$ can be chosen in such a way that the pairs of points in the opposite faces are different by a unit vector ([6, Lemma 3]). (In our terminology this means that the set of tangency points belongs to $\mathcal{A}_3$.) In turn, this implies that the interior of $A$ contains a translate of any box from $P_3$. Finally, a combination of [6, Lemmas 2 and 4] yields that covering a one-dimensional skeleton of the unit cube together with the tangency points by $m$ boxes from $P_3$ implies that the whole $P$ and, in particular, the boundary of $A$ can be covered by $m$ shadows of $\text{int}(A)$, and hence $H_3 \leq C_3$. As was already mentioned, the same proof works for higher dimensions, and to arrive at our terminology one only needs to note that the union of the relative boundaries of the $(n-1)$-dimensional faces of $[0,1]^n$ is precisely $B_{n-2,n}$, the $(n-2)$-skeleton of the unit cube.

3. **Upper bound on $C_n$**

For $\varepsilon > 0$ and $A \subset \mathbb{E}^n$, the $\varepsilon$-neighborhood of $A$ is the set of all points $x \in \mathbb{E}^n$ such that for some $y \in A$ the distance between $x$ and $y$ is less than $\varepsilon$. By a neighborhood of $A$ we mean the $\varepsilon$-neighborhood of $A$ for some $\varepsilon > 0$. 


Lemma 3.1. If $A \subset \mathbb{E}^n$ is covered by a finite number of boxes from $P_n$, then each box can be modified so that a neighborhood of $A$ is covered by the resulting boxes while each new box is still from $P_n$.

Proof. We replace each box $\prod_{i=1}^{n}[x_i, x_i+\delta_i]$ with $\prod_{i=1}^{n}[x_i-\varepsilon, x_i+\delta_i+\varepsilon]$, where $\varepsilon = \frac{1}{3n}(1-\sum_{i=1}^{n}\lambda_i)$ and depends on the specific box.

Lemma 3.2. Suppose that the box $\prod_{i=1}^{n}[y_i, y_i+\gamma_i]$ is the union of a finite number of boxes from $P_n^*$. Then for any $k$ and for any $0 < \varepsilon < \gamma_k$ each box can be modified so that the union of the resulting boxes is

\[
\left(\prod_{i=1}^{k-1}[y_i, y_i+\gamma_i]\right) \times [y_k+\varepsilon, y_k+\gamma_k] \times \left(\prod_{i=k+1}^{n}[y_i, y_i+\gamma_i]\right)
\]

while each new box is from $P_n$.

Proof. The idea is to linearly compress the whole structure along the $k$-th coordinate. Let $l : \mathbb{R} \to \mathbb{R}$ be the linear function satisfying $l(y_k+\gamma_k) = y_k+\gamma_k$ and $l(y_k) = y_k+\varepsilon$ whose slope is clearly between 0 and 1. Now we simply replace each box $\prod_{i=1}^{n}[x_i, x_i+\delta_i]$ from the original union by

\[
\left(\prod_{i=1}^{k-1}[x_i, x_i+\delta_i]\right) \times [l(x_k), l(x_k+\delta_k)] \times \left(\prod_{i=k+1}^{n}[x_i, x_i+\delta_i]\right).
\]

Proposition 3.3. For every $n \geq 4$ we have $C_n \leq 2n(n-1)(n-2)^{n-2} + 2n + 1$.

Proof. Any $A \in \mathcal{A}_n$ can obviously be covered by $2n$ elements of $P_n$, so $C_n = C(\mathcal{A}_n \cup B_{n-2,n}, P_n) \leq 2n + C(B_{n-2,n}, P_n)$ and it remains to show that $C(B_{n-2,n}, P_n) \leq m + 1$, where $m = 2n(n-1)(n-2)^{n-2}$. Our strategy is to cover $B_{n-2,n}$ by $m$ elements of $P_n^*$ first, then add one more box and modify the cover so that all boxes belong to $P_n$.

Each $(n-2)$-dimensional face $F$ of $[0,1]^n$ is the unit $(n-2)$-dimensional cube which is the union of $(n-2)^{n-2}$ $(n-2)$-dimensional cubes with the side length $\frac{1}{n-2}$ that clearly belong to $P_n^*$. Since there are $\frac{n(n-1)}{2}$ faces of dimension $n-2$, we obtain $m$ boxes from $P_n^*$ that cover $B_{n-2,2}$. Let $\{P_i\}_{i=1}^{m}$ be the collection of all such boxes.
We will find one more box $P_0 \in \mathcal{P}_n$ and will describe a sequence of steps which modify the boxes from the collection $\{P_i\}_{i=0}^m$. At every step, we will have that $P_i \in \mathcal{P}_n^*$, $0 \leq i \leq m$.

\begin{equation}
B_{n-2,2} \subset \bigcup_{i=0}^m P_i \quad \text{and} \quad B \subset \bigcup_{0 \leq i \leq m, P_i \in \mathcal{P}_n} P_i
\end{equation}

for a certain set $B \subset E^n$.

We will be done when we can achieve the above for $B = B_{n-2,n}$. Whenever we apply Lemmas 3.1 and 3.2 below, we replace some boxes from $\{P_i\}_{i=0}^m$ with the boxes provided by the lemmas.

First we fix an $(n-2)$-dimensional face $F$ of $[0,1]^n$, say $F = [0,1]^{n-2} \times \{0\}^2$, and show how to get (3.1) with $B = F$. We set $P_0 := [0, \frac{1}{n-2}]^{n-3} \times \{0\}^3$. Invoking Lemma 3.1 for $A = P_0$, we get (3.1) for $B = [0, \frac{1}{n-2}]^{n-3} \times [0, \varepsilon_1] \times \{0\}^2$, where $0 < \varepsilon_1 < \frac{1}{n-2}$. With this $\varepsilon_1$ and $k = n - 2$, we can apply Lemma 3.2 to $[0, \frac{1}{n-2}]^{n-3} \times [0, 1] \times \{0\}^2$, which, by construction, is the union of $n-2$ boxes from $\{P_i\}_{i=1}^m$. This yields (3.1) for $B = [0, \frac{1}{n-2}]^{n-3} \times [0, 1] \times \{0\}^2$. Next we apply Lemma 3.1 to $A = [0, \frac{1}{n-2}]^{n-3} \times [0, 1] \times \{0\}^2$ and obtain (3.1) for $B = [0, \frac{1}{n-2}]^{n-4} \times [0, \frac{1}{n-2} + \varepsilon_2] \times [0, 1] \times \{0\}^2$ for some $\varepsilon_2 > 0$. Invoke Lemma 3.2 for this $\varepsilon_2$ and $k = n - 3$ to the box $[0, \frac{1}{n-2}]^{n-4} \times [\frac{1}{n-2}, 1] \times [0, 1] \times (0,0)$, which, by construction, is the union of $(n-3)(n-2)$ boxes from $\{P_i\}_{i=1}^m$ (none of these boxes were modified until now). This leads to (3.1) for $B = [0, \frac{1}{n-2}]^{n-4} \times [0, 1]^2 \times (0,0)$. Proceeding in this manner, subsequently applying Lemmas 3.1 and 3.2 decreasing $k$ by 1 at each step, we arrive at (3.1) for $B = F$.

Now suppose that we already established (3.1) with $B = B'$ being the union of some $(n-2)$-dimensional faces of $[0,1]^n$, but $B' \neq B_{n-2,2}$. Then we can pick an $(n-2)$-dimensional face $F'$ of $[0,1]^n$ which is not contained in $B'$ and has some $(n-3)$-dimensional face in common with some $(n-2)$-dimensional face inside $B'$. Without loss of generality, $F' = \{0\}^2 \times [0,1]^{n-2}$ and the common face is $\{0\}^3 \times [0,1]^{n-3} \subset B'$. We proceed similarly to the previous paragraph: first use Lemma 3.1 for $A = \{0\}^3 \times [0,1]^{n-3}$ which establishes (3.1) for $B = \{0\}^2 \times [0, \varepsilon] \times [0,1]^{n-3}$ and some $\varepsilon > 0$, and then invoke Lemma 3.2 for this $\varepsilon$, $k = 3$ and $F'$ to arrive at (3.1) for $B = B' \cup F'$.

Extending $B$ as above one $(n-2)$-dimensional face of $[0,1]^n$ at a time, in finitely many steps we get the desired (3.1) for $B = B_{n-2,n}$. \qed
4. Lower bound on $C_n$

In this section we assume that $n \geq 3$. For a set $A \subset \mathbb{E}^n$ let $\lambda_k(A)$ be the $k$-dimensional Lebesgue measure of its $k$-dimensional boundary.

**Lemma 4.1.** For any $P \in \mathcal{P}_n$ the inequality $\lambda_{n-2}(P \cap B_{n-2,n}) < \frac{n-1}{2n^{n-3}}$ holds.

**Proof.** Suppose $P = \prod_{i=1}^n [x_i, x_i + \delta_i]$. Then using [4, Th. 52, p. 52]

$$\lambda_{n-2}(P \cap B_{n-2,n}) \leq \sum_{1 \leq i < j \leq n, k \in \{1, \ldots, n\} \setminus \{i,j\}} \delta_k \leq \frac{n-1}{2n^{n-3}} \left( \sum_{i=1}^n \delta_i \right)^{n-2} < \frac{n-1}{2n^{n-3}}.$$  

\[\square\]

**Lemma 4.2.** If $P \in \mathcal{P}_n$ contains the center of an $(n-1)$-dimensional face of $[0,1]^n$, then $\lambda_{n-2}(P \cap B_{n-2,n}) < \frac{1}{(2n-4)^{n-2}}$.

**Proof.** If $(0, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in P = \prod_{i=1}^n [x_i, x_i + \delta_i]$, then since $\sum_{i=1}^n \delta_i < 1$ we have that $P$ can intersect only one $(n-2)$-dimensional face of $[0,1]^n$ which can be assumed to be $(0,0) \times [0,1]^{n-2}$.

So, if this intersection is non-empty, then $\delta_2 \geq \frac{1}{2}$ and

$$\lambda_{n-2}(P \cap B_{n-2,n}) \leq \prod_{i=3}^n \delta_i \leq \left( \frac{1}{n-2} \right)^{n-2} \left( \sum_{i=3}^n \delta_i \right)^{n-2} < \frac{1}{(2n-4)^{n-2}}.$$  

\[\square\]

**Proposition 4.3.** For $n \geq 5$ we have that $C_n \geq 4n^{n-2} + 2n$.

**Proof.** Let $A_n \in \mathcal{A}_n$ be the $2n$-element set of centers of all $(n-1)$-dimensional faces of $[0,1]^n$. Observe that any $P \in \mathcal{P}_n$ covers at most one point from $A_n$. Therefore, if $A_n \cup B_{n-2,n}$ is covered by $m$ boxes from $\mathcal{P}_n$, by Lemmas 4.1 and 4.2 we get

$$2n(n-1) = \lambda_{n-2}(B_{n-2,n}) < (m-2n) \frac{n-1}{2n^{n-3}} + 2n \frac{1}{(2n-4)^{n-2}},$$  

which leads to the claimed estimate using that $\frac{4n^{n-2}}{(n-1)(2n-4)^{n-2}} < 1$ for $n \geq 5$.  

\[\square\]

**Remark 4.4.** Note that we only used the condition $n \geq 5$ in the very end of our proof, and the inequality (4.1) still holds for $n = 3$ implying that in this case the corresponding $m > 15$, i.e. $C_3 \geq 16$.  


Remark 4.5. One can obtain sharper lower bounds on $C_n$ generalizing the forthcoming Lemmas 6.2 and 6.3. However, we feel that this will be too technical to be justified in the context of this work as the resulting bound will not be close to the available upper bound on $C_n$. In addition, even the presented lower bound on $C_n$ is very far from the conjectured value of $H_n$ for $n \geq 5$.

5. Upper bound on $C_4$

Proposition 5.1. For the four dimensional case we have $C_4 \leq 96$.

Proof. First we construct a specific cover to show that $C(B_{2,4}, \mathcal{P}_n^*) \leq 88$ and then modify that cover to establish $C(B_{2,4}, \mathcal{P}_n) \leq 89$ and $C_4 = C(A_4 \cup B_{2,4}, \mathcal{P}_n) \leq 96$. In this section, for simplicity, we will omit “of $[0,1]^4$” when referring to faces of various dimension of the 4-cube $[0,1]^4$. In particular, we simply say vertices, the term “edges” will be used for 1-dimensional faces, “faces” are reserved for 2-dimensional faces, and facets are 3-dimensional faces.

We begin with a cover satisfying $C(B_{2,4}, \mathcal{P}_n^*) \leq 88$. For each vertex, we take the box from $\mathcal{P}_4^*$ with all side lengths equal to $\frac{1}{4}$ containing the vertex and lying entirely in $[0,1]^4$. For each edge $E$, we take a box from $\mathcal{P}_4^*$ which is the image of $[\frac{1}{4}, \frac{3}{4}] \times [0, \frac{1}{4}]^2 \times \{0\}$ under a certain symmetry of the cube $[0,1]^4$ mapping the edge joining $(0,0,0,0)$ and $(1,0,0,0)$ to $E$. There are three choices of such a symmetry: of three faces containing $E$, two faces will intersect with the box by a $\frac{1}{2}$ by $\frac{1}{4}$ rectangle, and one face will intersect with the box only by the edge $E$. In such faces where the intersection is $E$, we will call the edge $E$ special. Otherwise, the edge will be referred to as normal.

We aim to obtain faces of two types, as illustrated in Figure 1.

![Figure 1. Decomposition of faces by boxes of the partition](image-url)
In type A face the horizontal edges are special and the vertical ones are normal, while in type B face all edges are normal. For each edge we want to pick exactly one of the three faces containing the edge in which this edge will be special (normal in the other two faces) so that the faces become either type A or type B. While there are many such choices, we will describe a specific one which will be convenient for the second part of the proof. For each face of type A, it suffices to indicate which direction is parallel to the two special edges. We use \( x_j, j = 1, \ldots, 4 \) as the coordinate axes in \( E^4 \). In each of the two facets \( x_4 = 0 \) and \( x_4 = 1 \) we assign all faces to be type A and directions (as vectors in \((x_1, x_2, x_3)\)) of special edges for each face to be as follows (here a single equation \( x_i = t, t \in \{0, 1\}, i = 1, 2, 3 \), describes a face as we already fixed \( x_4 \)):

\[
\begin{align*}
  x_1 &= 0 \quad \text{and} \quad x_1 = 1 : (0, 0, 1), \\
  x_2 &= 0 \quad \text{and} \quad x_2 = 1 : (1, 0, 0), \\
  x_3 &= 0 \quad \text{and} \quad x_3 = 1 : (0, 1, 0).
\end{align*}
\]

There are four more faces which will be assigned type A. Each such face is given by \( x_2 = t \) and \( x_3 = s \), where \( t, s \in \{0, 1\} \), the direction of the special edges is always \((0, 0, 0, 1)\). All faces that were not assigned type A are assigned type B. It is easy to verify that the described configuration satisfies our requirements, with 16 faces of type A and 8 faces of type B, and each edge being special in exactly one of the three containing faces. Overall, this forms a cover of \( B_{2, 4} \) with 16 boxes corresponding to the vertices, 32 boxes corresponding to the edges, 2 \cdot 16 additional boxes for faces of type A and 8 additional boxes for faces of type B, providing 88 boxes in total, as required for \( C(B_{2, 4}, \mathcal{P}_n^*) \leq 88 \). We remark that in this cover no two boxes have overlapping interior. We need to leave this cover alone for some time.

By \( [a, b] := \{ta + (1-t)b, t \in [0, 1]\} \) we denote the line segment joining two points \( a, b \in E^4 \). For \( k = 0, 1, 2 \) we refer to the image of \([\left(\frac{1}{4}, \frac{k}{4}, 0, 0\right), (\frac{2}{4}, \frac{k}{4}, 0, 0)\)] under some symmetry of \([0, 1]^4\) as side, semi-central and central segment, respectively. Our next goal is to show that for any \( A_4 \in \mathcal{A}_4 \) there exists a segment \( I \) which is either side, semi-central or central segment such that \( C(A_4 \cup I, \mathcal{P}_4) \leq 8 \). We have

\[
A_4 = \bigcup_{i=1}^{4} \{x_i, x_i + e_i\} : x_i, x_i + e_i \in [0, 1]^4,
\]
so $x_i$ has $i$-th coordinate equal to zero, and let us refer to the value of the remaining three coordinates as non-trivial coordinates. Suppose there is $i$ such that $x_i$ has all non-trivial coordinates in $[0, \frac{1}{3}) \cup \left(\frac{4}{3}, 1\right]$. By symmetry, we can assume $i = 4$, $x_4 = (q, s, t, 0)$ and $\frac{1}{3} > q \geq s \geq t \geq 0$, implying $s + t < \frac{1}{3} + q$. Then the box $[q, \frac{3}{4}] \times [0, s] \times [0, t] \times \{0\}$ is from $P_4$ and contains both $x_4$ and the side segment $I := \left[\left(\frac{1}{4}, 0, 0, 0\right), \left(\frac{2}{4}, 0, 0, 0\right)\right]$. We trivially cover the remaining 7 points from $A_4$ using a box per point and get $C(A_4 \cup I, P_4) \leq 8$.

So in what follows we can assume that for every $i$ at least one of the non-trivial coordinates of $x_i$ belongs to $[\frac{1}{4}, \frac{2}{3}]$. Suppose there is $i$ for which we have exactly one such coordinate. We can assume $i = 4$, $x_4 = (q, s, t, 0)$, $q \in \left[\frac{1}{4}, \frac{2}{3}\right]$ and $s, t \in \left[0, \frac{1}{2}\right)$. Then the box $[q, \frac{3}{4}] \times [0, s] \times [0, t] \times \{0\}$ is from $P_4$ and contains both $x_4$ and the side segment $I := \left[\left(\frac{1}{4}, 0, 0, 0\right), \left(\frac{2}{4}, 0, 0, 0\right)\right]$. We conclude as in the previous case. Next, if for some $i$ we have exactly two non-trivial coordinates of $x_i$ in $\left[\frac{1}{4}, \frac{2}{3}\right]$, then assuming $i = 4$, $x_4 = (q, s, t, 0)$, $q, s \in \left[\frac{1}{4}, \frac{2}{3}\right]$, and $t \in \left[0, \frac{1}{2}\right)$, the desired box will be $[\frac{1}{4}, \frac{2}{3}] \times [s, \frac{1}{2}] \times [0, t] \times \{0\}$ and the (central) segment is $\left[\left(\frac{1}{4}, \frac{1}{2}, 0, 0\right), \left(\frac{2}{4}, \frac{1}{2}, 0, 0\right)\right]$.

The above considerations allow us to assume that for every $i$ each non-trivial coordinate of $x_i$ is in $\left[\frac{1}{4}, \frac{2}{3}\right]$. If there exists $i$ such that at least one non-trivial coordinate of $x_i$ is in $\left[\frac{1}{4}, \frac{2}{3}\right] \cup \left(\frac{4}{3}, 1\right]$, then we can assume $i = 4$, $x_4 = (q, s, t, 0)$, $q \in \left[\frac{1}{4}, \frac{2}{3}\right]$ and $t \in \left[\frac{1}{4}, \frac{3}{4}\right)$. If $s < \frac{3}{8}$, we take $[\frac{1}{4}, \frac{2}{3}] \times [\frac{1}{4}, s] \times [0, t] \times \{0\}$ and the semi-central segment $\left[\left(\frac{1}{4}, \frac{1}{4}, 0, 0\right), \left(\frac{3}{4}, \frac{1}{4}, 0, 0\right)\right]$. If $s \geq \frac{3}{8}$, we take $[\frac{1}{4}, \frac{2}{3}] \times [s, \frac{1}{2}] \times [0, t] \times \{0\}$ and the central segment $\left[\left(\frac{1}{4}, \frac{1}{2}, 0, 0\right), \left(\frac{2}{4}, \frac{1}{2}, 0, 0\right)\right]$.

We will present two boxes from $P_4$ that cover $\{x_3, x_4\} \cup I$, $I := \left[\left(\frac{1}{4}, \frac{1}{2}, 0, 0\right), \left(\frac{2}{4}, \frac{1}{2}, 0, 0\right)\right]$, then the remaining six points form $A_4$ can be each covered by a box from $P_4$ leading to the desired $C(A_4 \cup I, P_4) \leq 8$. It is convenient to denote $J(u) := \min(\frac{1}{2}, u), \max(\frac{1}{2}, u)$, $u \in \left[\frac{3}{8}, \frac{5}{8}\right]$. Then the length of $J(u)$ is $|\frac{1}{2} - u| \leq \frac{1}{8}$. If $q' \leq \frac{1}{2}$, then we take $[q, \frac{3}{4}] \times J(s) \times [0, t] \times \{0\}$ and $\left[\frac{1}{4}, \frac{1}{2}\right] \times J(s') \times \{0\} \times [0, t']$. If $q' > \frac{1}{2}$, then we take $\left[\frac{1}{4}, \frac{1}{2}\right] \times J(s) \times [0, t] \times \{0\}$ and $\left[\frac{1}{2}, \frac{3}{4}\right] \times J(s') \times \{0\} \times [0, t']$. At last, we can assume that for every $i$ any non-trivial coordinate is either $\frac{3}{8}$ or $\frac{5}{8}$. In this case, we can assume $x_3 = (q', s', 0, \frac{3}{8})$, $x_4 = (q, s, \frac{3}{8}, 0)$, $q' \leq q$, and then simply take $\left[q', \frac{3}{4}\right] \times J(s') \times \{0\} \times [0, \frac{3}{8}]$ and $\left[\frac{1}{4}, q\right] \times J(s) \times [0, \frac{3}{8}] \times \{0\}$ that cover $\{x_3, x_4\} \cup I$. 
Now we can return to the cover fulfilling \( C(B_{2,4}, \mathcal{P}_4^*) \leq 88 \) that we constructed earlier. We also have \( C(A_4 \cup I, \mathcal{P}_4) \leq 8 \). Using a symmetry, if needed, we can make the following assumption. If \( I \) is a side segment, then \( I = [(\frac{1}{4}, 0, 0), (\frac{3}{4}, 0, 0)] \). If \( I \) is a semi-central segment, then \( I = [(0, 0, \frac{1}{4}), (0, 0, \frac{1}{2})] \). If \( I \) is a central segment, then \( I = [(\frac{1}{4}, 0, \frac{1}{2}), (\frac{1}{4}, 0, \frac{1}{2}, 0)] \). Let \( \{P_i\}_{i=1}^{96} \) be the boxes from \( \mathcal{P}_4^* \) we constructed above to cover \( B_{2,4} \cup A_4 \cup I \). Eight of them are already in \( \mathcal{P}_4 \), and similarly to what was done in Section 3 we will perform a sequence of modifications of the collection \( \{P_i\}_{i=1}^{96} \). At each step, we will have

\[
(5.1) \quad B_{2,4} \cup A_4 \subset \bigcup_{i=1}^{96} P_i \quad \text{and} \quad B \cup A_4 \subset \bigcup_{1 \leq i \leq 96, P_i \in \mathcal{P}_4} P_i,
\]

with the goal of reaching the above for \( B = B_{2,4} \cup A_4 \). Initially, we have (5.1) for \( B \) being a neighborhood of \( I \), after application of Lemma 3.1 to \( I \).

By \( B_{2,4}^* \) we denote the closure of the set obtained by removing from \( B_{2,4} \) the union of all boxes of the cover corresponding to the vertices. (This removes four “corner” squares with side length \( \frac{1}{4} \) from each face.) We will modify these boxes in the very end of the procedure and now we focus on covering \( B_{2,4}^* \). The idea is to obtain (5.1) for \( B = B_{2,4}^* \) by adding one “cornerless” face at a time.

Each box of the cover corresponding to an edge will be modified in a specific way, reducing the total sum of the dimensions of the box. Namely, when we are adding a face in which this edge is normal, say, without loss of generality, when the face is \( x_3 = x_4 = 0 \), the edge is \( x_2 = x_3 = x_4 = 0 \), and the original box is \([\frac{1}{4}, \frac{3}{4}] \times [0, \frac{1}{2}] \times \{0\} \times [0, \frac{1}{4}]\), then modified box will be \([\frac{1}{4}, \frac{3}{4}] \times [0, \frac{1}{2} - 3\epsilon] \times [0, \epsilon] \times [0, \frac{1}{4}]\) with sufficiently small \( \epsilon > 0 \). We will not make further changes to this box when other faces containing this edge are added to the union.

When a face of type A is added, there will be two possible situations. First one: a neighborhood of the corresponding central segment is already covered (parallel to the special edges). This can happen only when the first face is added, so the face is \( x_2 = x_4 = 0 \) and the central segment is \([\frac{1}{4}, 0, \frac{1}{2}, 0], (\frac{3}{4}, 0, \frac{1}{2}, 0)]\). Then we replace the two boxes corresponding to the face which are \([\frac{1}{4}, \frac{3}{4}] \times \{0\} \times [0, \frac{1}{2}] \times \{0\}\) and \([\frac{1}{4}, \frac{3}{4}] \times \{0\} \times [\frac{1}{2}, 1] \times \{0\}\) with \([\frac{1}{4} - \epsilon, \frac{3}{4} + \epsilon] \times \{0\} \times [0, \frac{1}{2} - 3\epsilon] \times \{0\}\) and \([\frac{1}{4} - \epsilon, \frac{3}{4} + \epsilon] \times \{0\} \times [\frac{1}{2} + 3\epsilon, 1] \times \{0\}\), for sufficiently small \( \epsilon > 0 \). The second (more typical) situation: a neighborhood of a side segment contained in a special edge is already covered. Let, without loss of generality, the face be \( x_2 = x_4 = 0 \) and the side segment be \([\frac{1}{4}, 0, 0, 0], (\frac{3}{4}, 0, 0, 0)]\). Then we replace \([\frac{1}{4}, \frac{3}{4}] \times \{0\} \times [0, \frac{1}{2}] \times \{0\}\) and \([\frac{1}{4}, \frac{3}{4}] \times \{0\} \times [\frac{1}{2}, 1] \times \{0\}\)
with $\left[ \frac{1}{4} - \varepsilon, \frac{3}{4} + \varepsilon \right] \times \{0\} \times \left[ 0, \frac{1}{2} + 3\varepsilon \right] \times \{0\}$ and $\left[ \frac{1}{4} - \varepsilon, \frac{3}{4} + \varepsilon \right] \times \{0\} \times \left[ \frac{1}{2} + 3\varepsilon, 1 \right] \times \{0\}$ for sufficiently small $\varepsilon > 0$. In either of the situations, we can modify any of the not yet modified boxes corresponding to the normal edges of this face and then add the face to the union.

When a face of type B is added, one of the boxes corresponding to the edges will be already modified or a neighborhood of a semi-central segment in this face will be covered. In either of situations, without loss of generality, we can assume that a neighborhood of $\left[ (0, 0, \frac{1}{4}, 1), (0, 0, \frac{3}{4}, 1) \right]$ is covered and the face is $x_1 = x_2 = 0$. We replace $\{0\}^2 \times \left[ \frac{1}{4}, \frac{3}{4} \right] \times \left[ \frac{1}{4}, \frac{3}{4} \right]$ with $\{0\}^2 \times \left[ \frac{1}{4} - \varepsilon, \frac{3}{4} + \varepsilon \right] \times \left[ \frac{1}{4} + 4\varepsilon, \frac{3}{4} + \varepsilon \right]$ for sufficiently small $\varepsilon > 0$ and modify the not yet modified boxes corresponding to the sides of this face.

Now we describe how to add all faces to the union using the above operations. If $I$ is a side or a central segment, we can begin with the face $x_2 = x_4 = 0$. Note that if the union already contains a face with a normal edge, then our operations allow to add another face in which this edge is special. So, we can add all faces from the facet $x_4 = 0$ for instance in the following order: $x_1 = 0, x_1 = 1, x_3 = 0, x_3 = 1, x_2 = 1$. Next we can add all faces having one side in $x_4 = 0$ and the other side in $x_4 = 1$, in arbitrary order. Such faces are either type B with the box corresponding to the edge belonging to $x_4 = 0$ already modified or type A with one of the special sides in $x_4 = 0$ which is already covered. We conclude by adding the faces from the facet $x_4 = 1$ in the same order as was done for $x_4 = 0$. Now if $I$ is a semi-central segment, we can begin with the face $x_1 = x_2 = 0$, proceed with adding $x_2 = x_4 = 0$ and then we can follow the order of adding faces as in the previous case when $I$ was a side or a central segment. The only difference will be that we simply skip the face $x_1 = x_2 = 0$.

So, we obtained (5.1) for $B = B_{2,4}^*$. Now we apply Lemma 3.1 for $A = B_{2,4}^*$ and then modify each of the boxes corresponding to the vertices so that the resulting boxes still contain the vertices and have all side lengths equal to $\frac{1}{4} - \varepsilon$, for sufficiently small $\varepsilon > 0$ yielding (5.1) for $B = B_{2,4}$. This completes the proof. \(\square\)

6. Lower bound on $C_4$

We will need several technical lemmas.

**Lemma 6.1.** If $P \in \mathcal{P}_4$ does not contain any vertices of $[0, 1]^4$, then $\lambda_2(P \cap B_{2,4}) < \frac{1}{4}$.

*Proof.* Suppose $P = \prod_{i=1}^4 [x_i, x_i + \delta_i]$. Let $m \geq 0$ be the number of indices $i$ such that $[x_i, x_i + \delta_i] \cap \{0, 1\} \neq \emptyset$. Without loss of generality, we may assume that these indices are $i = 1, \ldots, m$.
Lemma 6.2. If $0 \in P$, then $\lambda_2(P \cap B_{2,4}) \leq 1$.

Proof. If $P = \prod_{i=1}^4 [x_i, x_i + \delta_i]$, we have $x_i \leq 0 \leq x_i + \delta_i$ and can assume that $x_i = 0$, $i = 1, \ldots, 4$, which may only increase $\lambda_2(P \cap B_{2,4})$. We can also assume that $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4$ and let $m \geq 0$ be the largest index satisfying $\delta_m \geq \frac{1}{4}$. Since $\sum_{i=1}^4 \delta_i < 1$, we have $m \leq 3$. We compute

$$\lambda_2(P \cap (B_{2,4} \setminus [0, \frac{1}{4}]^4)) = \sum_{1 \leq i < j \leq 4} (\delta_i \delta_j - \min\{\delta_i, \frac{1}{4}\} \min\{\delta_j, \frac{1}{4}\})$$

$$= \sum_{1 \leq i < j \leq m} (\delta_i \delta_j - \frac{1}{16}) + \sum_{1 \leq i \leq m} (\delta_i - \frac{1}{4}) \sum_{m+1 \leq j \leq 4} \delta_j.$$

Therefore, if $m = 3$, we can use the proof of Lemma 4.1 for $n = 4$ to obtain

$$\lambda_2(P \cap (B_{2,4} \setminus [0, \frac{1}{4}]^4)) \leq \sum_{1 \leq i < j \leq 4} \delta_i \delta_j - \frac{3}{16} < \frac{3}{16}.$$

If $m = 2$, letting $s := \delta_1 + \delta_2 \in \left[\frac{1}{2}, 1\right)$, we have

$$\lambda_2(P \cap (B_{2,4} \setminus [0, \frac{1}{4}]^4)) = \delta_1 \delta_2 - \frac{1}{16} + \left(\delta_1 + \delta_2 - \frac{1}{2}\right) (\delta_3 + \delta_4)$$

$$< \left(\frac{s}{2}\right)^2 - \frac{1}{16} + \left(s - \frac{1}{2}\right) (1 - s) = \frac{3}{4} (s - 1)^2 + \frac{3}{16} < \frac{3}{16}.$$

If $m = 1$, then

$$\lambda_2(P \cap (B_{2,4} \setminus [0, \frac{1}{4}]^4)) = \left(\delta_1 - \frac{1}{4}\right) (\delta_2 + \delta_3 + \delta_4) \leq \left(\frac{\delta_1 + \delta_2 + \delta_3 + \delta_4 - \frac{1}{4}}{2}\right)^2 < \left(\frac{3}{8}\right)^2 < \frac{3}{16}.$$

Finally, if $m = 0$, then $\lambda_2(P \cap (B_{2,4} \setminus [0, \frac{1}{4}]^4)) = 0$. \hfill \Box

Lemma 6.3. Suppose $0 \in P_i$, $P_i \in \mathcal{P}_4$, $1 \leq i \leq k$, then

$$\lambda_2\left(\bigcup_{i=1}^k P_i \cap B_{2,4}\right) < \frac{1}{8} + \frac{k}{4}.$$
Proof. Let us begin with the easier cases.

If \( k = 1 \), the result follows directly from Lemma 4.1 If \( k \geq 4 \), then by Lemma 6.2

\[
\lambda_2 \left( \left( \bigcup_{i=1}^{k} P_i \right) \cap B_{2,4} \right) \leq \lambda_2 \left( \frac{[0, \frac{1}{4}]}{4} \cap B_{2,4} \right) + \sum_{i=1}^{k} \lambda_2 \left( \left( P_i \setminus \frac{[0, \frac{1}{4}]}{4} \right) \cap B_{2,4} \right)
\]

\[
< \frac{3}{8} + \frac{3}{16} k \leq \frac{1}{8} + \frac{k}{4}.
\]

Now let \( k = 2 \). Suppose \( P_1 = \prod_{i=1}^{4} [0, \gamma_i] \) and \( P_2 = \prod_{i=1}^{4} [0, \delta_i] \). Note that when any of the side lengths is increased, the quantity we need to estimate does not decrease. Therefore, it is sufficient to prove the desired inequality under the assumption \( \sum_{i=1}^{4} \gamma_i = \sum_{i=1}^{4} \delta_i = 1 \).

Let \( m \geq 0 \) be the number of indices \( i \) such that \( \gamma_i \geq \delta_i \). Without loss of generality, we can assume that these indices are \( i = 1, \ldots, m \) and that \( m \geq 2 \). The case \( m = 4 \) is obvious due to Lemma 4.1 If \( m = 3 \), then

\[
\lambda_2((P_2 \setminus P_1) \cap B_{2,4}) = (\delta_1 + \delta_2 + \delta_3)(\delta_4 - \gamma_4) < \frac{1}{4},
\]

so by Lemma 4.1

\[
\lambda_2((P_1 \cup P_2) \cap B_{2,4}) = \lambda_2(P_1 \cap B_{2,4}) + \lambda_2((P_2 \setminus P_1) \cap B_{2,4}) < \frac{3}{8} + \frac{1}{4} = \frac{5}{8}.
\]

If \( m = 2 \), let \( s := \gamma_1 + \gamma_2 \) and \( t := \delta_3 + \delta_4 \). From our assumption it follows that \( \delta_1 + \delta_2 = 1 - t \) and we obtain

\[
\lambda_2((P_1 \cup P_2) \cap B_{2,4}) \leq \sum_{1 \leq i < j \leq 4} (\gamma_i \gamma_j + \delta_i \delta_j - \min\{\gamma_i, \delta_i\} \min\{\gamma_j, \delta_j\})
\]

\[
= \gamma_1 \gamma_2 + \delta_3 \delta_4 + (\gamma_1 - \delta_1 + \gamma_2 - \delta_2)(\gamma_3 + \gamma_4) + (\delta_1 + \delta_2)(\delta_3 + \delta_4)
\]

\[
\leq \left( \frac{s}{2} \right)^2 + \left( \frac{t}{2} \right)^2 + (s + t - 1)(1 - s) + (1 - t)t
\]

\[
= \frac{3}{4}(s^2 + t^2) - st + 2s + 2t - 1 =: f(s, t).
\]

It is standard multivariate calculus to show that \( f \) has the absolute maximum value \( f(\frac{2}{3}, \frac{2}{3}) = \frac{3}{5} < \frac{5}{8} \).

It remains to consider the case \( k = 3 \), which is the most difficult one. Suppose \( P_1 = \prod_{i=1}^{4} [0, \beta_i] \), \( P_2 = \prod_{i=1}^{4} [0, \gamma_i] \), and \( P_3 = \prod_{i=1}^{4} [0, \delta_i] \). As in the previous case we can assume that

\[
\sum_{i=1}^{4} \beta_i = \sum_{i=1}^{4} \gamma_i = \sum_{i=1}^{4} \delta_i = 1
\]
Next, suppose that there exist three values of $i$ satisfying the inequality $\beta_i \geq \gamma_i$, in which case we say for the corresponding boxes that $P_1$ majorizes $P_2$. Then this inequality will be reversed for the remaining fourth value of $i$, and we can argue as in (6.1) to obtain $\lambda_2((P_1 \setminus P_2) \cap B_{2,4}) \leq \frac{1}{4}$.

So by the already considered case $k = 2$

$$\lambda_2((P_1 \cup P_2 \cup P_3) \cap B_{2,4}) \leq \lambda_2((P_2 \cup P_3) \cap B_{2,4}) + \lambda_2((P_1 \setminus P_2) \cap B_{2,4}) \leq \frac{5}{8} + \frac{1}{4} = \frac{7}{8}.$$ 

Therefore, we can assume that none of $P_1$, $P_2$ and $P_3$ majorizes any other. For each of the four values of the index $i$ one of the three boxes has the largest corresponding side length. Therefore, this happens at least twice for one of the boxes, so without loss of generality we can assume that either

$$\beta_1 \geq \gamma_1 \geq \delta_1 \quad \text{and} \quad \beta_2 \geq \gamma_2 \geq \delta_2,$$

or

$$\beta_1 \geq \gamma_1 \geq \delta_1 \quad \text{and} \quad \beta_2 \geq \delta_2 \geq \gamma_2.$$

No majorization assumption implies that in fact without loss of generality we have either

(6.3) \quad $$\beta_1 \geq \gamma_1 \geq \delta_1, \quad \beta_2 \geq \gamma_2 \geq \delta_2, \quad \delta_3 \geq \gamma_3 \geq \beta_3 \quad \text{and} \quad \delta_4 \geq \gamma_4 \geq \beta_4,$$

or

(6.4) \quad $$\beta_1 \geq \gamma_1 \geq \delta_1, \quad \beta_2 \geq \delta_2 \geq \gamma_2, \quad \gamma_3 \geq \delta_3 \geq \beta_3 \quad \text{and} \quad \delta_4 \geq \gamma_4 \geq \beta_4.$$

If (6.3) holds, then using the already considered case $k = 2$ we obtain

$$\lambda_2((P_1 \cup P_2 \cup P_3) \cap B_{2,4}) \leq \lambda_2((P_1 \cup P_3) \cap B_{2,4}) + \lambda_2((P_2 \setminus (P_1 \cup P_3)) \cap B_{2,4})$$

$$\leq \frac{5}{8} + \sum_{i=1,2} \sum_{j=3,4} (\gamma_i - \delta_i)(\gamma_j - \delta_j)$$

$$= \frac{5}{8} + (\gamma_1 + \gamma_2 - \delta_1 - \delta_2)(\gamma_3 + \gamma_4 - \beta_3 - \beta_4)$$

$$\leq \frac{5}{8} + (\gamma_1 + \gamma_2)(\gamma_3 + \gamma_4) \leq \frac{5}{8} + \frac{1}{4} = \frac{7}{8}.$$
If (6.4) holds, we introduce \( u := 1 - \gamma_1 - \delta_2 \) and compute using (6.2) that

\[
\lambda_2((P_1 \cup P_2 \cup P_3) \cap B_{2,4}) = \beta_1 \beta_2 + \beta_1 \beta_3 + \gamma_1(\gamma_3 - \beta_3) + \beta_2 \beta_4 + \delta_2(\delta_4 - \beta_4) + \gamma_3 \gamma_4 + \delta_3(\delta_4 - \gamma_4) + \beta_1 \beta_4 + \delta_1(\delta_4 - \beta_4) + (\gamma_1 - \delta_1)(\gamma_4 - \beta_4) + \beta_2 \beta_3 + \gamma_2(\gamma_3 - \beta_3) + (\delta_2 - \gamma_2)(\delta_3 - \beta_3)
\]

\[
= [\beta_1 \beta_2 + (\beta_1 + \beta_2 - 1 + u)(1 - \beta_1 - \beta_2)] + [\gamma_3(1 - \gamma_3) + \delta_4(1 - \delta_4) + \delta_3(\gamma_3 - \delta_4) + \gamma_4(\delta_4 - \gamma_4) + \delta_3 \gamma_4]
\]

\[
=: f(u, \beta_1, \beta_2) + g(u, \gamma_3, \gamma_4, \delta_3, \delta_4).
\]

We have \( f(u, \beta_1, \beta_2) \leq f(u, \beta, \beta) \) with \( \beta = (\beta_1 + \beta_2)/2 \in [0, \frac{1}{2}] \), and by standard calculus we obtain

\[
f(u, \beta_1, \beta_2) \leq f(u, \beta, \beta) \leq f(u, \frac{1}{2}, \frac{1}{2}) = \frac{1}{4} \quad \text{if} \quad u \leq \frac{1}{2}
\]

and

\[
f(u, \beta_1, \beta_2) \leq f(u, \beta, \beta) \leq f(u, \frac{2-u}{3}, \frac{2-u}{3}) = \frac{u^2 - u + 1}{3} \quad \text{if} \quad u > \frac{1}{2}.
\]

Now we turn to estimating \( g \). By (6.2) we have

\[
u = 1 - \gamma_1 - \delta_2 = -1 + \gamma_2 + \gamma_3 + \gamma_4 + \delta_1 + \delta_3 + \delta_4 \geq -1 + \gamma_3 + \gamma_4 + \delta_3 + \delta_4,
\]

so

\[
g(u, \gamma_3, \gamma_4, \delta_3, \delta_4) \leq g(-1 + \gamma_3 + \gamma_4 + \delta_3 + \delta_4, \gamma_3, \gamma_4, \delta_3, \delta_4)
\]

\[
= \gamma_3 + \gamma_4 + \delta_3 + \delta_4 - \gamma_3^2 - \gamma_4^2 - \delta_3^2 - \delta_4^2 - \gamma_3 \gamma_4 - \delta_3 \gamma_4 - \delta_3 \delta_4
\]

\[
=: \bar{g}(\gamma_3, \gamma_4, \delta_3, \delta_4).
\]

By standard multivariable calculus, \( \bar{g} \) attains the absolute maximum over \( \mathbb{R}^4 \) at \( (\frac{2}{5}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}) \) with the value equal to \( \frac{3}{5} \) which in combination with (6.5) provides the bound of \( \frac{17}{20} < \frac{7}{8} \) completing the proof of the case \( k = 3 \) if \( u \leq \frac{1}{2} \). If \( u > \frac{1}{2} \), due to (6.6), we need to estimate from above

\[
\bar{g}(u, \gamma_3, \gamma_4, \delta_3, \delta_4) := \frac{u^2 - u + 1}{3} + g(u, \gamma_3, \gamma_4, \delta_3, \delta_4),
\]

which is a quadratic function of \( u \) with positive leading coefficient. Therefore,

\[
\bar{g}(u, \gamma_3, \gamma_4, \delta_3, \delta_4) \leq \max\{\bar{g}(1, \gamma_3, \gamma_4, \delta_3, \delta_4), \bar{g}(\max\{\frac{1}{2}, -1 + \gamma_3 + \gamma_4 + \delta_3 + \delta_4\}, \gamma_3, \gamma_4, \delta_3, \delta_4)\}.
\]
We have
\[
\bar{g}(1, \gamma_3, \gamma_4, \delta_3, \delta_4) = \frac{1}{3} + \gamma_3(1 - \gamma_3) + \delta_4(1 - \delta_4) + \delta_3(\gamma_3 + \gamma_4 - 1) + \gamma_4(\delta_4 - 1)
\]
\[
\leq \frac{1}{3} + \gamma_3(1 - \gamma_3) + \delta_4(1 - \delta_4) \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{5}{6} < \frac{7}{8}.
\]
If \(-1 + \gamma_3 + \gamma_4 + \delta_3 + \delta_4 \leq \frac{1}{2}\), then
\[
(6.7) \quad \bar{g}(\frac{1}{2}, \gamma_3, \gamma_4, \delta_3, \delta_4) = \frac{1}{4} + g(\frac{1}{2}, \gamma_3, \gamma_4, \delta_3, \delta_4) \leq \frac{1}{4} + \bar{g}(\gamma_3, \gamma_4, \delta_3, \delta_4) \leq \frac{1}{4} + \frac{3}{5} = \frac{17}{20} < \frac{7}{8}
\]
by what was already done. Finally, assuming \(-1 + \gamma_3 + \gamma_4 + \delta_3 + \delta_4 > \frac{1}{2}\) we need to estimate
\[
\bar{g}(-1 + \gamma_3 + \gamma_4 + \delta_3 + \delta_4, \gamma_3, \gamma_4, \delta_3, \delta_4) = 1 - \frac{2}{3}(\gamma_3^2 + \gamma_4^2 + \delta_3^2 + \delta_4^2) - \frac{1}{3}(\gamma_3 \gamma_4 + \delta_3 \delta_4 + \delta_3 \gamma_4) + \frac{2}{3}(\gamma_3 \delta_3 + \gamma_4 \delta_4 + \gamma_3 \delta_4) =: 1 - h(\gamma_3, \gamma_4, \delta_3, \delta_4).
\]
We observe that \(h\) is a homogeneous function of degree 2 convex on \(\mathbb{R}^4\) with the absolute minimum at the origin implying that \(h\) is non-negative everywhere and
\[
h(\lambda \gamma_3, \lambda \gamma_4, \lambda \delta_3, \lambda \delta_4) = \lambda^2 h(\gamma_3, \gamma_4, \delta_3, \delta_4) \leq h(\gamma_3, \gamma_4, \delta_3, \delta_4)\]
for any \(\lambda \in (0, 1)\). We can choose \(\lambda \in (0, 1)\) so that \(-1 + \lambda(\gamma_3 + \gamma_4 + \delta_3 + \delta_4) = \frac{1}{2}\), and then
\[
\bar{g}(-1 + \gamma_3 + \gamma_4 + \delta_3 + \delta_4, \gamma_3, \gamma_4, \delta_3, \delta_4) \leq \bar{g}(\frac{1}{2}, \lambda \gamma_3, \lambda \gamma_4, \lambda \delta_3, \lambda \delta_4) < \frac{7}{8}
\]
arguing as in (6.7). This completes the proof. \(\square\)

Remark 6.4. While in the above lemma the proof of the case \(k = 3\) is quite involved, we were unable to deduce it using simpler arguments. This could be due to the fact that the estimate we need is almost sharp (for \(k = 2\) and \(k = 3\) the sharp bounds are \(\frac{3}{5}\) and \(\frac{17}{20}\) which are both just \(0.025\) smaller than the needed \(\frac{1}{8} + \frac{k}{4}\)) and the inequalities of the type \(\lambda_2((P_1 \setminus P_2) \cap B_{2,4}) \leq \frac{1}{4}\) turn out to be false in general.

Proposition 6.5. For the four-dimensional case we have \(C_4 \geq 95\).

Proof. Let \(A_4 \in A_4\) be the set consisting of 8 centers of the 3-dimensional faces of \([0, 1]^4\).
Suppose \(A_4 \cup B_{2,4} \subset \bigcup_{i=1}^m P_i, P_i \in \mathcal{P}_4\). Let \(k_j, 1 \leq j \leq 16,\) be the number of boxes from \(\{P_i\}_{i=1}^m\) covering the \(j\)-th vertex of \([0, 1]^4\) (enumerated arbitrarily), \(k := \sum_{j=1}^{16} k_j\). Let \(V\) be the set of vertices of \([0, 1]^4\). Observe that no element of \(\mathcal{P}_4\) can cover two points from \(A_4 \cup V\). By Lemmas 4.2, 6.1 and 6.3 we obtain
\[
24 = \lambda_2(B_{2,4}) < 8 \cdot \frac{1}{16} + (m - 8 - k) \cdot \frac{1}{4} + \sum_{j=1}^{16} \left( \frac{1}{8} + \frac{k_j}{4} \right) \leq \frac{1}{2} + \frac{m}{4},
\]
implying $m > 94$, as required.

\[
\square
\]

Remark 6.6. Following the proofs in Sections \textsuperscript{5} and \textsuperscript{6} one can readily see that $C(B_{2,4}, P_n^*) = 88$ and $C(B_{2,4}, P_n) = 89$.

Remark 6.7. In the proof of the upper bound $C_4 \leq 96$, we used at most two of the boxes covering arbitrary $A_4 \in A_4$ to cover a neighborhood of a side, semi-central or central segment in $B_{2,4}$ which was used to modify the 88 boxes from $P_4^*$ covering $B_{2,4}$ in a way that all of them become in $P_4$. From the lower bound above, we see that the total area of $B_{2,4}$ covered by such 88 boxes is strictly less than $24 = \lambda_2(B_{2,4})$, so in our proof of the upper bound, we used at most two from the boxes covering $A_4$ to cover some small strictly positive area in $B_{2,4}$ sufficient to maintain a cover of the remainder of $B_{2,4}$ by 88 boxes. If one wishes to prove $C_4 = 95$, this additional area of $B_{2,4}$ covered by boxes serving $A_4$ will need to be at least $\frac{1}{4}$ (replacing what one of the 88 boxes can cover). If points in $A_4$ are close to the centers of the facets, then (Lemma \textsuperscript{4.2}) one may need to use four or more boxes from those serving $A_4$ for this additional area which will probably need to be split between two faces leading to the need to consider more complicated covers of $B_{2,4}$. We also mention that there are many configurations $A_4$ for which an element of $P_4$ covers at most one point from $A_4$.

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