A variational formulation for steady surface water waves on a Beltrami flow

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This paper considers steady surface waves ‘riding’ a Beltrami flow (a three-dimensional flow with parallel velocity and vorticity fields). It is demonstrated that the hydrodynamic problem can be formulated as two equations for two scalar functions of the horizontal spatial coordinates, namely the elevation $\eta$ of the free surface and the potential $\Phi$ defining the gradient part (in the sense of the Hodge–Weyl decomposition) of the horizontal component of the tangential fluid velocity there. These equations are written in terms of a non-local operator $H(\eta)$ mapping $\Phi$ to the normal fluid velocity at the free surface, and are shown to arise from a variational principle. In the irrotational limit, the equations reduce to the Zakharov–Craig–Sulem formulation of the classical three-dimensional steady water-wave problem, while $H(\eta)$ reduces to the familiar Dirichlet–Neumann operator.

1. Introduction

(a) The main results

Consider an incompressible perfect fluid of unit density occupying a three-dimensional domain bounded below by a rigid horizontal plane and above by a free surface. A steady water wave is a fluid flow of this kind in which both the velocity field and free-surface profile are stationary with respect to a uniformly (horizontally) translating frame of reference. Working in this frame of reference, suppose that the fluid domain is $D_\eta = \{(x,y,z): -h < y < \eta(x,z)\}$ (so that the free surface is the graph $S_\eta$ of an unknown function $\eta$), and the flow is a (strong) Beltrami flow whose velocity and vorticity fields $\mathbf{u}$ and curl $\mathbf{u}$ are parallel, so that curl $\mathbf{u} = \alpha \mathbf{u}$ for some fixed constant $\alpha$. The hydrodynamic problem is to solve
where $n$ denotes the (outward-pointing) unit normal vector at $S_\eta$, $j = (0, 1, 0)$, $\nabla = (\partial_x, \partial_z)^T$, $\nabla^\perp = (-\partial_x, \partial_z)^T$ and the physical constants $g$, $\sigma$, $c = (c_1, c_3)^T$ are, respectively, the acceleration due to gravity, the coefficient of surface tension and the wave velocity; the pressure $p$ in the fluid is recovered using the formula $p(x, y, z) = -\frac{1}{2} |u(x, y, z)|^2 - gy$ (the variables $u$ and $p$ automatically solve the stationary Euler equation). Equations (1.4) and (1.5) are referred to as, respectively, the kinematic and dynamic boundary conditions at the free surface. It is natural to write $u$ as a perturbation of the trivial solution

$$\eta^* = 0, \quad u^* = c_1 \begin{pmatrix} \cos ay \\ 0 \\ \sin ay \end{pmatrix} + c_3 \begin{pmatrix} -\sin ay \\ 0 \\ \cos ay \end{pmatrix}$$

of (1.1)–(1.5), so that $v = u - u^*$ satisfies the equations

$$\begin{align*}
\text{div } v &= 0 \quad \text{in } D_\eta, \\
\text{curl } v &= \alpha v \quad \text{in } D_\eta, \\
v \cdot j &= 0 \quad \text{at } y = -h, \\
v \cdot n + u^* \cdot n &= 0 \quad \text{at } y = \eta, \\
\frac{1}{2} |v|^2 + v \cdot u^* + gn - \sigma \left( \frac{\eta_x}{(1 + |\nabla \eta|^2)^{1/2}} \right)_x - \sigma \left( \frac{\eta_z}{(1 + |\nabla \eta|^2)^{1/2}} \right)_z &= 0 \quad \text{at } y = \eta.
\end{align*}$$

This paper considers solutions $(\eta, v)$ of (1.7)–(1.11) which are evanescent as $|(x, z)| \to \infty$ and therefore represent localized waves ‘riding’ the trivial flow (1.6).

For $\alpha = 0$ and $u^* = (c_1, 0, c_3)^T$, equations (1.7)–(1.11) reduce to the classical three-dimensional irrotational steady water-wave problem, which is usually handled by writing $v = \text{grad } \phi$, where $\phi$ is a harmonic scalar potential, so that (1.7) and (1.8) are automatically satisfied. In fact, it is possible to formulate this problem in terms of the variables $\eta$ and $\xi = \phi|_{y=\eta}$ (see Zakharov [1] and Craig & Sulem [2]). Consider the variational principle

$$\delta L_0(\eta, \xi) = 0,$$

where

$$L_0(\eta, \xi) = \int_{D_\eta} \frac{1}{2} |\text{grad } \phi|^2 + \int_{\mathbb{R}^2} \left( -\eta \cdot \text{curl } \phi|_{y=\eta} + \frac{1}{2} g\eta^2 + \sigma (1 + |\nabla \eta|^2)^{1/2} - 1 \right)$$

and $\phi$ is the unique harmonic function with $\phi|_{y=-h} = 0$ and $\phi|_{y=\eta} = \xi$ (so that $v = \text{grad } \phi$ satisfies (1.7)–(1.9)); the Euler–Lagrange equations for $L_0(\eta, \xi)$ recover the boundary conditions at the free surface (see Luke [3]). In the Zakharov–Craig–Sulem formulation a Dirichlet–Neumann operator $G(\eta)$ defined by $G(\eta)\xi = \text{grad } \phi|_{y=\eta} \cdot N$ is introduced, where $N = (-\eta_x, 1, -\eta_z)^T$ (so that $n = N/|N|$). One finds that

$$L_0(\eta, \xi) = \int_{\mathbb{R}^2} \left( \frac{1}{2} \xi G(\eta) \xi - \eta \cdot \text{curl } \xi + \frac{1}{2} g\eta^2 + \sigma (1 + |\nabla \eta|^2)^{1/2} - 1 \right)$$
and that its Euler–Lagrange equations can be written as

\[
G(\eta)\xi + c \nabla \eta = 0,
\]

\[
\frac{1}{2}|\nabla \xi|^2 - \frac{(G(\eta)\xi + c \nabla \eta + \nabla \cdot \xi)^2}{2(1 + |\nabla \eta|^2)} - c \nabla \xi + g \eta - \sigma \left(\frac{\eta_x}{(1 + |\nabla \eta|^2)^{1/2}}\right)_x - \sigma \left(\frac{\eta_z}{(1 + |\nabla \eta|^2)^{1/2}}\right)_z = 0,
\]

which are readily confirmed to be equivalent to the boundary conditions at the free surface (with \(v = \text{grad} \ \phi\)).

This paper presents a generalization of the Zakharov–Craig–Sulem formulation to the case \(\alpha \neq 0\) (and includes the limit \(\alpha = 0\)). The velocity field \(v\) is represented by a solenoidal vector potential \(A\) with curl curl \(A = \alpha \text{curl} \ A\) and \(A \wedge j|_{y=-h} = 0\), so that \(v = \text{curl} \ A\) automatically satisfies (1.7)–(1.9); note that \(u^* = \text{curl} \ A^*\), where

\[
A^* = \frac{c_1}{\alpha} \begin{pmatrix} \cos ay - 1 \\ 0 \\ \sin ay \end{pmatrix} + \frac{c_3}{\alpha} \begin{pmatrix} -\sin ay \\ 0 \\ \cos ay - 1 \end{pmatrix}.
\]

Let \(F_\parallel\) denote the horizontal component of the tangential part of a vector field \(F\) at the free surface, so that \(F_\parallel = F_h + F_\eta \eta|_{y=\eta}\), where \(F_h = (F_1, F_2)^T\), and write, according to the Hodge–Weyl decomposition for vector fields in two-dimensional free space (see below),

\[
v_\parallel = \nabla \phi + \nabla \perp \psi,
\]

where \(\phi = \Delta^{-1}(\nabla \cdot v_\parallel)\), \(\psi = \Delta^{-1}(\nabla \perp \cdot v_\parallel) = -\Delta^{-1}(\nabla \cdot v_\perp)\) and \(\Delta^{-1}\) is the two-dimensional Newtonian potential. In §2, it is shown that the hydrodynamic problem can be formulated in terms of the variables \(\eta\) and \(\phi\). The Euler–Lagrange equations for the variational principle

\[
\delta \mathcal{L}(\eta, \Phi) = 0,
\]

where

\[
\mathcal{L}(\eta, \Phi) = \int_{D_\eta} \left( \frac{1}{2} |\text{curl} \ A|^2 - \frac{1}{2} \alpha A \cdot \text{curl} \ A \right) + \int_{\mathbb{R}^2} \left( - \frac{1}{2} \alpha \nabla \Delta^{-1}(\nabla \cdot A^\perp) \cdot A - \nabla \phi \cdot A^\perp \right)
\]

\[
+ \int_{\mathbb{R}^2} \left( \Gamma(\eta) + \frac{1}{2} g \eta^2 + \sigma \left(1 + |\nabla \eta|^2\right)^{1/2} - 1 \right),
\]

\[
\Gamma(\eta) = -\frac{1}{2} \alpha \nabla \Delta^{-1}(\nabla \cdot A^\perp) \cdot A + \frac{|c|^2}{2\alpha} (\sin ay - \alpha \eta)
\]

and \(A\) is the unique solution of the boundary-value problem

\[
\text{curl} \ \text{curl} \ A = \alpha \text{curl} \ A \quad \text{in} \ D_\eta,
\]

\[
\text{div} \ A = 0 \quad \text{in} \ D_\eta,
\]

\[
A \wedge j = 0 \quad \text{at} \ y = -h,
\]

\[
A \cdot n = 0 \quad \text{at} \ y = \eta,
\]

\[
(\text{curl} \ A)_\parallel = \nabla \phi - \alpha \nabla \Delta^{-1}(\nabla \cdot A^\perp) \quad \text{at} \ y = \eta,
\]

recover the boundary conditions at the free surface (with \(v = \text{curl} \ A\)); the existence and uniqueness of the solution to the above boundary-value problem for small values of \(|\alpha|\) is demonstrated by functional–analytic methods in §§4b and 4c. (Observe that

\[
\psi = \Delta^{-1}(\nabla \cdot v^\parallel) = -\Delta^{-1}(\text{curl} \ v \cdot N|_{y=\eta}) = -\alpha \Delta^{-1}(v \cdot N|_{y=\eta}) = -\alpha \Delta^{-1}(\nabla \cdot A^\parallel),
\]

in which the vector identity \(\text{curl} \ F \cdot N|_{y=\eta} = \nabla \cdot F^\parallel\) has been used, so that \(\psi\) is determined by (1.15)–(1.18) and (1.19) is equivalent to (1.12). The significance of the quantity \(v_\parallel\) has previously
been noted by Gavrilyuk et al. [4, §3.1] (a study of kinematic balance laws) and Castro & Lannes [5] (Hamiltonian formulations of water waves with general distributions of vorticity).

The appropriate generalization $H(\eta)$ of the Dirichlet–Neumann operator $G(\eta)$ is identified in §3: one defines $H(\eta)\Phi = \text{curl } A \cdot N|_{y=\eta}$. It is shown that

$$\mathcal{L}(\eta, \Phi) = \int_{\mathbb{R}^2} \left( \frac{1}{2} \Phi H(\eta) \Phi - \nabla \Phi \cdot A^*_{\eta} + \Gamma(\eta) + \frac{1}{2} \eta^2 + \sigma ((1 + |\nabla \eta|^2)^{1/2} - 1) \right)$$

and that its Euler–Lagrange equations can be written as

$$H(\eta)\Phi + u^* \cdot N|_{y=\eta} = 0,$$

$$\frac{1}{2} |K(\eta)|^2 - \frac{(H(\eta)\Phi + K(\eta)\Phi \cdot \nabla \eta)^2}{2(1 + |\nabla \eta|^2)} - \alpha \frac{H(\eta)\Phi (H(\eta)\Phi + K(\eta)\Phi \cdot \nabla \eta)}{1 + |\nabla \eta|^2}$$

$$+ K(\eta)\Phi \cdot u^*|_{y=\eta} + g \eta - \sigma \left( \frac{\eta_x}{(1 + |\nabla \eta|^2)^{1/2}} \right) x - \sigma \left( \frac{\eta_z}{(1 + |\nabla \eta|^2)^{1/2}} \right) z = 0,$$

where

$$K(\eta)\Phi = \nabla \Phi - \alpha \nabla^\perp \Delta^{-1}(H(\eta)\Phi).$$

(In the irrotational limit $\alpha = 0$ one finds that $\text{curl } A = \nabla \phi$, where $\phi$ is the unique harmonic function with $\phi|_{y=h} = 0$ and $\phi|_{y=\eta} = \Phi$, so that $H(\eta)\Phi = \nabla \phi N|_{y=\eta} = G(\eta)\Phi$, thus recovering the Craig–Sulem–Zakharov formulation.)

The treatment of the variational principle (1.13) in §2 consists in computing the formal first variation $\delta \mathcal{L}(\eta, \Phi)$ of the variational functional in terms of the infinitesimal variations $\eta$, $\Phi$; all variables are supposed to be as smooth as required for the relevant calculations. The mathematics can be made rigorous by ‘flattening’ the variable fluid domain $D_\eta$; that is, mapping it to the fixed reference domain $D_0$ by introducing the new vertical coordinate $\tilde{y} = h(y - \eta)/(y + \eta)$ and variable $\tilde{A}(\tilde{x}, \tilde{y}, \tilde{z}) = A(x, y, z)$. The variational functional is transformed into

$$\mathcal{L}(\eta, \Phi) = \int_{D_0} \left( \frac{1}{2} |\text{curl}^0 \tilde{A}|^2 - \frac{1}{2} \alpha \tilde{A} \cdot \text{curl}^0 \tilde{A} \right) \left( 1 + \frac{\eta}{R} \right)$$

$$+ \int_{\mathbb{R}^2} \left( - \frac{1}{2} \alpha \nabla \Delta^{-1}(\nabla \cdot \tilde{A}_{\eta}^\perp) \cdot \tilde{A}_{\eta} - \nabla \Phi \cdot A^*_{\eta} + \Gamma(\eta) + \frac{1}{2} g \eta^2 + \sigma ((1 + |\nabla \eta|^2)^{1/2} - 1) \right),$$

in which $\tilde{A}$ is the solution of the ‘flattened’ boundary-value problem

$$\text{curl}^0 \text{curl}^0 \tilde{A} = \alpha \text{curl}^0 \tilde{A} \quad \text{in } D_0,$$  

$$\text{div}^0 \tilde{A} = 0 \quad \text{in } D_0,$$  

$$\tilde{A} \land j = 0 \quad \text{at } \tilde{y} = -h,$$  

$$\tilde{A} \cdot N = 0 \quad \text{at } \tilde{y} = 0,$$  

$$\text{(curl}^0 \tilde{A})_{\|} = \nabla \Phi - \alpha \nabla^\perp \Delta^{-1}(\nabla \cdot \tilde{A}_{\eta}^\perp) \quad \text{at } \tilde{y} = 0$$

and the notation $\hat{F} = \hat{F}_h + \hat{F}_z \nabla \eta|_{y=0}$ for vector fields $\hat{F} : D_0 \to \mathbb{R}^3$ is used; explicit expressions for $\text{curl}^0 \tilde{A} := \text{curl } A$, $\text{curl}^0 \text{curl}^0 \tilde{A} := \text{curl } \text{curl } A$ and $\text{div}^0 \tilde{A} := \text{div } A$ are given below. This technique is used in §4c, where it is shown that the non-local operator $H(\eta)$ depends analytically upon $\eta$ in a sense made precise there.

The variational principle presented here is a combination of a classical result for Beltrami flows in fixed domains by Woltjer [6] and Laurence & Avellaneda [7] and a suggestion for an alternative variational framework for three-dimensional irrotational water waves by Benjamin [8, §6.6]. An alternative variational principle has been given by Lokharu & Wahlén [9], who use a vector potential $A$ within the flow as the principal variable and consider more general parametrizations of the free surface; in the present context their work shows that equations (1.7)–(1.11) (with
\( \mathbf{v} = \text{curl} \, \mathbf{A} \) follow from the variational principle

\[
\delta \left\{ \int_{D_\eta} \left( \frac{1}{2} |\text{curl} \, \mathbf{A}|^2 - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl} \, \mathbf{A} \right) \right\} - \int_{\mathbb{R}^2} \left( \frac{|f|^2}{2\alpha} (\sin \alpha \eta - \alpha \eta) + \frac{1}{2} g \eta^2 + \sigma ((1 + |\nabla \eta|^2)^{1/2} - 1) \right) \right\} = 0,
\]

where the variations are taken with respect to \( \eta \) and \( \mathbf{A} \) satisfying \( \text{div} \, \mathbf{A} = 0 \), \( \mathbf{A} \wedge |_{y=-h} = 0 \) and \( \mathbf{A} \wedge |_{y=\eta} = -\mathbf{A}^* \wedge |_{y=\eta} \).

(b) Notation and vector identities

In this article, vector fields \( D_\eta \rightarrow \mathbb{R}^3 \) and \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) are written in, respectively, bold upper and lower case, the horizontal component of \( \mathbf{F} = (F_1, F_2, F_3)^T \) is denoted by \( \mathbf{F}_h = (F_1, F_3)^T \) and the vector perpendicular to \( \mathbf{f} = (f_1, f_3) \) is denoted by \( \mathbf{f}^\perp = (-f_2, f_1)^T \). Evaluation at the free surface is indicated by an underscore, so that \( \mathbf{F} = (F_1, F_2, F_3)^T|_{y=\eta} \) and frequent use is made of the quantity \( \mathbf{F}_\parallel = \mathbf{F}_h + \mathbf{F}_2 \nabla \eta \) (the horizontal component of the tangential part of \( \mathbf{F} \) at the free surface). The usual three-dimensional vector operators are denoted by ‘grad’, ‘div’ and ‘curl’, while \( \nabla = (\partial_x, \partial_z)_T \), \( \nabla^\perp = (-\partial_x, \partial_z)_T \); the two- and three-dimensional Laplacians are both denoted by \( \Delta \) (the precise meaning being clear from the context).

In §§2 and 3, we proceed formally, assuming that all functions are as regular as required for the relevant calculations and making frequent use of the following identities (which are proved by explicit computation).

**Proposition 1.1.** The identities

\[
(i) \quad f^\perp \cdot g^\perp = f \cdot g, \quad f \cdot g^\perp = -f^\perp \cdot g, \quad f^\perp f = -f,
\]

\[
(ii) \quad (\text{grad} \, f)^\perp = \nabla^\perp f, \quad \nabla^\perp f = -\nabla f,
\]

\[
(iii) \quad f^\perp \cdot \nabla^\perp g = f \cdot \nabla g, \quad f \cdot \nabla^\perp g = -f^\perp \cdot \nabla g, \quad \nabla \cdot f = -\nabla^\perp \cdot f,
\]

\[
(iv) \quad (\mathbf{F} \wedge \mathbf{N}) \cdot \mathbf{G} = \mathbf{F}_\parallel \cdot \mathbf{G}_\parallel, \quad \text{curl} \, \mathbf{F} \cdot \mathbf{N} = \nabla \cdot \mathbf{F}_\parallel,
\]

\[
(v) \quad (\text{grad} \, f|_\parallel) = \nabla f \parallel, \quad (\text{grad} \, f|_\parallel) = \nabla \mathbf{F}_2 + (\text{curl} \, \mathbf{F}|_\parallel) \mathbf{F}_h - \text{curl} \, \mathbf{F} \parallel \mathbf{F}_2 - \text{curl} \, \mathbf{F} \parallel \mathbf{F}_h.
\]

\[
(vi) \quad \int_{\mathbb{R}^2} \nabla f \cdot \nabla^\perp g = 0, \quad \int_{\mathbb{R}^2} \nabla f \cdot \mathbf{G} = -\int_{\mathbb{R}^2} f \nabla \cdot \mathbf{G}, \quad \int_{\mathbb{R}^2} \nabla^\perp f \cdot \mathbf{G} = -\int_{\mathbb{R}^2} f \nabla \cdot \mathbf{G}
\]

are satisfied by all sufficiently regular vector fields \( \mathbf{F}, \mathbf{G} : D_\eta \rightarrow \mathbb{R}^3 \), \( f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and scalar fields \( f, g : \mathbb{R} \rightarrow \mathbb{R} \).

Each (sufficiently regular) vector field \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) admits a unique orthogonal decomposition

\[
f = \nabla \phi + \nabla^\perp \psi,
\]

where \( \phi = \Delta^{-1}(\nabla \cdot f) \) and \( \psi = \Delta^{-1}(\nabla^\perp \cdot f) = -\Delta^{-1}(\nabla \cdot f^\perp) \). Note that the projections \( f \mapsto \nabla \Delta^{-1}(\nabla \cdot f), f \mapsto \nabla^\perp \Delta^{-1}(\nabla^\perp \cdot f) \) onto the ‘gradient part’ and ‘orthogonal gradient part’ in the decomposition

\[
f = \nabla \Delta^{-1}(\nabla \cdot f) + \nabla^\perp \Delta^{-1}(\nabla^\perp \cdot f)
\]

are formally self-adjoint and have the property recorded in the following proposition.

**Proposition 1.2.** The identity

\[
\int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot g^\perp) \cdot f - \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot f^\perp) \cdot g = \int_{\mathbb{R}^2} f \cdot g^\perp
\]

holds for all sufficiently regular vector fields \( f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \).

A rigorous discussion of the Hodge–Weyl decomposition (which is used formally in §§2 and 3) is given in §4a.
It remains to record the expressions \( \text{curl}^n \tilde{\mathbf{F}}, \text{div}^n \tilde{\mathbf{F}}, \text{grad}^n f \) and \( \Delta^n f \) obtained from \( \text{curl} \mathbf{F}, \text{div} \mathbf{F}, \text{grad} \mathbf{f} \) and \( \Delta \mathbf{f} \) by the ‘flattening’ change of variables \( \tilde{y} = h(y - \eta)/(h + \eta) \), \( \tilde{\mathbf{F}}(x, \tilde{y}, z) = \mathbf{F}(x, y, z) \), \( f(x, \tilde{y}, z) = f(x, y, z) \); one finds that

\[
\text{curl}^n \tilde{\mathbf{F}} = \text{curl} \mathbf{F} - K^o_2(\tilde{F}_{3y}, 0, -\tilde{F}_1) + K^o_1(\eta_x \tilde{F}_{2y} + \eta_z \tilde{F}_{3y} - \eta_z \tilde{F}_{1y} - \eta_x \tilde{F}_{2y}),
\]

\[
\text{div}^n \tilde{\mathbf{F}} = \text{div} \mathbf{F} - K^o_1(\eta_x \tilde{F}_{1y} + \eta_z \tilde{F}_{3y}) - K^o_2 \tilde{F}_{2y},
\]

\[
\text{grad}^n \tilde{f} = \text{grad} \mathbf{f} - K^o_1(\eta_x \tilde{f}_{1y} + \eta_z \tilde{f}_{3y}) - K^o_2 \tilde{f}_{2y},
\]

\[
\Delta^n \tilde{f} = \Delta \mathbf{f} + K^o_1(\eta^2_x + \eta^2_z)(K^o_1 \tilde{f}_{1y} + 2K^o_2 \tilde{f}_{2y})
\]

\[
+ K^o_2(K^0_2 - 2) \tilde{f}_{2y} - K^o_1(\eta_{xx} + \eta_{zz}) \tilde{f}_y - 2K^o_1(\eta_x \tilde{f}_{xy} + \eta_z \tilde{f}_{yz}),
\]

where \( K^o_1 = (h + \tilde{y})/(h + \eta) \), \( K^o_2 = \eta/(h + \eta) \), \( K^o_3 = 1/(h + \eta) \).

\section{The variational principle}

In this section, we verify that equations (1.7)–(1.11) (with \( \mathbf{v} = \text{curl} \mathbf{A} \)) follow from the variational principle

\[
\delta \left\{ \int_{D_\eta} \left( \frac{1}{2} |\text{curl} \mathbf{A}|^2 - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl} \mathbf{A} \right) + \int_{\mathbb{R}^2} \left( -\frac{1}{2} \alpha \nabla \Delta^{-1}(\nabla \cdot \mathbf{A}^\perp) \cdot \mathbf{A}^\perp - \nabla \Phi \cdot \mathbf{A}^\perp \right) \right\} = 0,
\]

where

\[
\Gamma(\eta) = -\frac{1}{2} \alpha \nabla \Delta^{-1}(\nabla \cdot \mathbf{A}^\perp) \cdot \mathbf{A}^\perp + \frac{|\eta|^2}{2\alpha} (\sin \alpha \eta - \alpha \eta),
\]

\( \mathbf{A} \) is the unique solution of the boundary-value problem (1.15)–(1.19) and the variations are taken with respect to \( \eta \) and \( \Phi \); equations (1.7)–(1.19) are automatically satisfied, while (1.10) and (1.11) are recovered from the Euler–Lagrange equations for the variational functional \( \mathcal{L}(\eta, \Phi) \). To this end note that the rules

\[
\delta \mathbf{F} = \tilde{\mathbf{F}} + \mathbf{F}_y \tilde{\eta} \quad \text{and} \quad \delta \mathbf{F}^\parallel = \tilde{\mathbf{F}}^\parallel + (\mathbf{F}_y)^\parallel \tilde{\eta} + \nabla \tilde{\eta} \mathbf{F}_2,
\]

where, as is customary, \( \delta \mathbf{F} \) is abbreviated to \( \tilde{\mathbf{F}} \), imply that

\[
\text{curl} \text{curl} \mathbf{A} = \alpha \text{curl} \mathbf{A} \quad \text{in} D_\eta,
\]

\[
\mathbf{A} \wedge \mathbf{j} |_{y=-h} = 0,
\]

\[
(\text{curl} \mathbf{A}) \tilde{\eta} = -(\text{curl} \mathbf{A}_y) \tilde{\eta} - \nabla \tilde{\eta} (\text{curl} \mathbf{A})_2 + \nabla \Phi - \alpha \nabla \Delta^{-1}(\nabla \cdot (\mathbf{A}^\perp)^\perp).
\]

Observe that

\[
\delta \int_{D_\eta} \left( \frac{1}{2} |\text{curl} \mathbf{A}|^2 - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl} \mathbf{A} \right)
\]

\[
= \int_{D_\eta} \left( \text{curl} \mathbf{A} \cdot \text{curl} \mathbf{A} - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl} \mathbf{A} - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl} \mathbf{A} \right)
\]

\[
+ \int_{\mathbb{R}^2} \left( \frac{1}{2} |\text{curl} \mathbf{A}|^2 - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl} \mathbf{A} \right) \tilde{\eta}
\]

\[
= \int_{D_\eta} \left( \text{curl} \mathbf{A} \cdot \text{curl} \mathbf{A} - \alpha \text{curl} \mathbf{A} \cdot \mathbf{A} + \int_{\mathbb{R}^2} (\text{curl} \mathbf{A} \wedge \mathbf{N}) \cdot \mathbf{A} - \frac{1}{2} \alpha \int_{\mathbb{R}^2} (\mathbf{A} \wedge \mathbf{N}) \cdot \mathbf{A}
\]

\[
+ \int_{\mathbb{R}^2} \left( \frac{1}{2} |\text{curl} \mathbf{A}|^2 - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl} \mathbf{A} \right) \tilde{\eta}
\]

\[
= \int_{\mathbb{R}^2} (\text{curl} \mathbf{A}) \cdot \mathbf{A} - \frac{1}{2} \alpha \int_{\mathbb{R}^2} \mathbf{A} \cdot \mathbf{A} + \int_{\mathbb{R}^2} \left( \frac{1}{2} |\text{curl} \mathbf{A}|^2 - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl} \mathbf{A} \right) \tilde{\eta}
\] (2.1)
and
\[
\int_{R^2} (\text{curl } \mathbf{A})_h^+ \cdot \mathbf{A} = - \int_{R^2} (\text{curl } \mathbf{A})_\parallel \cdot \mathbf{A}^+ \\
= \int_{R^2} (\text{curl } \mathbf{A}^y) \eta + \nabla \eta (\text{curl } \mathbf{A})_2 \cdot \mathbf{A}^+ - \int_{R^2} \nabla \phi \cdot \mathbf{A}^+ \\
+ \alpha \int_{R^2} \nabla \Delta^{-1}(\nabla \cdot (\delta \mathbf{A}^+) \cdot \mathbf{A}^+) \\
= \int_{R^2} (\text{curl } \mathbf{A}^y) - \nabla (\text{curl } \mathbf{A})_2 \cdot \eta \mathbf{A}^+ - \int_{R^2} (\text{curl } \mathbf{A})_2 \eta \nabla \cdot \mathbf{A}^+ - \int_{R^2} \nabla \phi \cdot \mathbf{A}^+ \\
+ \alpha \int_{R^2} \nabla \Delta^{-1}(\nabla \cdot (\delta \mathbf{A}^+) \cdot \mathbf{A}^+) \\
= \alpha \int_{R^2} (\text{curl } \mathbf{A})_h^+ \eta \cdot \mathbf{A}^+ - \int_{R^2} (\text{curl } \mathbf{A})_2 \eta \nabla \cdot \mathbf{A}^+ - \int_{R^2} \nabla \phi \cdot \mathbf{A}^+ \\
+ \alpha \int_{R^2} \nabla \Delta^{-1}(\nabla \cdot (\delta \mathbf{A}^+) \cdot \mathbf{A}^+),
\] (2.2)
where an integration by parts and the fact that
\[
(\text{curl } \mathbf{A}^y)_\parallel - \nabla (\text{curl } \mathbf{A})_2 = (\text{curl } \text{curl } \mathbf{A})_h^+ = \alpha (\text{curl } \mathbf{A})_h^+
\]
has been used. Combining (2.1), (2.2) and the calculation
\[
\delta \int_{R^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{A}^+) \cdot \mathbf{A} = \int_{R^2} \nabla \Delta^{-1}(\nabla \cdot (\delta \mathbf{A}^+) \cdot \mathbf{A} + \int_{R^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{A}^+) \cdot \delta \mathbf{A}^+
\]
yields
\[
\delta \left\{ \int_{D_0} \left( \frac{1}{2} |\text{curl } \mathbf{A}|^2 - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl } \mathbf{A} \right) - \frac{1}{2} \alpha \int_{R^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{A}^+) \cdot \mathbf{A} \right\}
\]
\[
= \alpha \int_{R^2} (\text{curl } \mathbf{A})_h \eta \cdot \mathbf{A}^+ - \int_{R^2} (\text{curl } \mathbf{A})_2 \eta \nabla \cdot \mathbf{A}^+ - \int_{R^2} \nabla \phi \cdot \mathbf{A}^+ - \frac{1}{2} \alpha \int_{R^2} \mathbf{A}^+ \cdot \delta \mathbf{A}^+
\]
\[
+ \int_{R^2} \left( \frac{1}{2} |\text{curl } \mathbf{A}|^2 - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl } \mathbf{A} \right) \eta - \frac{1}{2} \alpha \int_{R^2} \mathbf{A}^+ \cdot \delta \mathbf{A}^+, (2.3)
\]
where proposition 1.2 has also been used. Repeating the argument leading to (2.2), one finds that
\[
\int_{R^2} \mathbf{A}^+ \cdot \delta \mathbf{A}^+ = \int_{R^2} \mathbf{A}^+ \cdot (\hat{\mathbf{A}}^+ + (\mathbf{A}^y) \eta + \nabla \eta \mathbf{A}_2)
\]
\[
= \int_{R^2} \mathbf{A}^+ \cdot \hat{\mathbf{A}}^+ + \int_{R^2} ((\mathbf{A}^y)_\parallel - \nabla (\mathbf{A}_2)) \eta \cdot \mathbf{A}^+ - \int_{R^2} \mathbf{A}_2 \eta \nabla \cdot \mathbf{A}^+
\]
\[
= - \int_{R^2} \mathbf{A}^+ \cdot \hat{\mathbf{A}}^+ + \int_{R^2} (\text{curl } \mathbf{A})_h \eta \cdot \mathbf{A}^+ - \int_{R^2} \mathbf{A}_2 \eta \nabla \cdot \mathbf{A}^+, (2.4)
\]
and it follows from (2.3), (2.4) and the calculation
\[
(\text{curl } \mathbf{A})_h \cdot \mathbf{A}^+ - \mathbf{A} \cdot \text{curl } \mathbf{A} = - \nabla \cdot \mathbf{A}^+ \mathbf{A}_2
\]
that
\[
\delta \left\{ \int_{D_0} \left( \frac{1}{2} |\text{curl } \mathbf{A}|^2 - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl } \mathbf{A} \right) - \frac{1}{2} \alpha \int_{R^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{A}^+) \cdot \mathbf{A} \right\}
\]
\[
= - \int_{R^2} (\text{curl } \mathbf{A})_2 \eta \nabla \cdot \mathbf{A}^+ + \int_{R^2} \phi \nabla \cdot \mathbf{A}^+ + \int_{R^2} \frac{1}{2} |\text{curl } \mathbf{A}|^2 \eta.
\]
Finally, note that

\[ \delta \int_{\mathbb{R}^2} \left( -\frac{1}{2} \alpha \nabla \Delta^{-1} (\nabla \cdot A^\perp \bigcdot A^\perp) \right) \]

\[ = -\frac{1}{2} \alpha \int_{\mathbb{R}^2} \left( \nabla \Delta^{-1} (\nabla \cdot A^\perp \bigcdot A^\perp) + \nabla \Delta^{-1} (\nabla \cdot A^\perp \bigcdot A^\perp) \right) \]

\[ = -\frac{1}{2} \alpha \int_{\mathbb{R}^2} \left( \nabla^\perp \Delta^{-1} (\nabla \cdot A^\perp) \bigcdot A^\perp + \nabla \Delta^{-1} (\nabla \cdot A^\perp) \bigcdot A^\perp \right) \]

\[ = \frac{1}{2} \alpha \int_{\mathbb{R}^2} \left( \nabla^\perp \Delta^{-1} (\nabla \cdot A^\perp) + \nabla \Delta^{-1} (\nabla \cdot A^\perp) \right) \cdot u^\perp \dot{\eta} \]

\[ = \frac{1}{2} \alpha \int_{\mathbb{R}^2} (A^\perp + 2 \nabla^\perp \Delta^{-1} (\nabla \cdot A^\perp)) \cdot u^\perp \dot{\eta}, \]

\[ \delta \int_{\mathbb{R}^2} \frac{|c|^2}{2 \alpha} (\sin \alpha \eta - \alpha \eta) = -|c|^2 \int_{\mathbb{R}^2} \sin^2 \left( \frac{1}{2} \alpha \eta \right) \dot{\eta} = -\frac{1}{2} \alpha \int_{\mathbb{R}^2} A^\perp \cdot u^\perp \dot{\eta} \]

and

\[ \delta \int_{\mathbb{R}^2} (-\nabla \cdot A^\perp) = \int_{\mathbb{R}^2} (\phi \nabla \cdot A^\perp - \nabla \phi \cdot A^\perp) \]

\[ = \int_{\mathbb{R}^2} (\phi \nabla \cdot A^\perp + \nabla \phi \cdot u^\perp \dot{\eta}) \]

\[ = \int_{\mathbb{R}^2} (\phi \nabla \cdot A^\perp + ((\text{curl } A)_\parallel + \alpha \nabla^\perp \Delta^{-1} (\nabla \cdot A^\perp)) \cdot u^\perp \dot{\eta}) \]

\[ = \int_{\mathbb{R}^2} (\phi \nabla \cdot A^\perp + ((\text{curl } A)_h + (\text{curl } A)_2 \nabla \eta + \alpha \nabla^\perp \Delta^{-1} (\nabla \cdot A^\perp)) \cdot u^\perp \dot{\eta}) \]

because \( A^\perp = -u^\perp \dot{\eta} \), and evidently

\[ \delta \int_{\mathbb{R}^2} \left( \frac{1}{2} g \eta^2 + \sigma ((1 + |\nabla \eta|^2)^{1/2} - 1) \right) \]

\[ = \int_{\mathbb{R}^2} \left( g \eta - \sigma \left( \frac{\eta_x}{(1 + |\nabla \eta|^2)^{1/2}} \right)_x - \sigma \left( \frac{\eta_z}{(1 + |\nabla \eta|^2)^{1/2}} \right)_z \right) \dot{\eta}. \]

The Euler–Lagrange equations for \( L(\eta, \phi) \) are therefore

\[ \nabla \cdot A^\perp + \nabla \cdot A^\perp = 0 \]

(2.5)

and

\[ \langle \text{curl } A \rangle_2 (-\nabla \cdot A^\perp + \nabla \eta \cdot u^\perp) + \alpha \nabla^\perp \Delta^{-1} (\nabla \cdot A^\perp) \bigcdot u^\perp \]

\[ + \frac{1}{2} |\text{curl } A|_2^2 + (\text{curl } A)_h \cdot u^\perp + g \eta - \sigma \left( \frac{\eta_x}{(1 + |\nabla \eta|^2)^{1/2}} \right)_x - \sigma \left( \frac{\eta_z}{(1 + |\nabla \eta|^2)^{1/2}} \right)_z = 0, \]

(2.6)

which are equivalent to equations (1.10) and (1.11) because \( \nabla \cdot A^\perp = -\nabla \eta \cdot u^\perp = u^\perp \cdot N \), \( \text{curl } A \cdot u^\perp = \text{curl } A \cdot u^\perp \) and \( \nabla \cdot A^\parallel = \text{curl } A \cdot N \).

### 3. A non-local operator

In this section, we express the variational functional \( L(\eta, \phi) \) and its Euler–Lagrange equations in terms of a non-local operator \( H(\eta) \) defined as follows: for fixed \( \phi \), let \( A \) denote the unique solution of (1.15)–(1.19) and define

\[ H(\eta) \phi = \nabla \cdot A^\perp. \]

(3.1)
Lemma 3.1. The formula
\[
\int_{\mathbb{R}^2} \Phi_1 H(\eta) \Phi_2 = \int_{\mathbb{R}^2} \left( \text{curl} \mathbf{B} \cdot \text{curl} \mathbf{C} - \frac{1}{2} \alpha \mathbf{B} \cdot \text{curl} \mathbf{C} - \frac{1}{2} \alpha \mathbf{C} \cdot \text{curl} \mathbf{B} \right)
\]
\[
- \frac{1}{2} \alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{B}^\perp) \cdot \mathbf{C}^\perp - \frac{1}{2} \alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{C}^\perp) \cdot \mathbf{B}^\perp
\]
holds for all \( \Phi_1, \Phi_2 \), where \( \mathbf{B} \) and \( \mathbf{C} \) denote the unique solutions of (1.15)–(1.19) with, respectively, \( \Phi = \Phi_1 \) and \( \Phi = \Phi_2 \) (so that \( H(\eta) \Phi_1 = \nabla \cdot \mathbf{B}^\perp, H(\eta) \Phi_2 = \nabla \cdot \mathbf{C}^\perp \)).

In particular, \( H(\eta) \) is formally self-adjoint; that is,
\[
\int_{\mathbb{R}^2} \Phi_1 H(\eta) \Phi_2 = \int_{\mathbb{R}^2} \Phi_2 H(\eta) \Phi_1
\]
for all \( \Phi_1, \Phi_2 \).

Proof. Note that
\[
\int_{\mathbb{R}^2} \left( \text{curl} \mathbf{B} \cdot \text{curl} \mathbf{C} - \frac{1}{2} \alpha \mathbf{B} \cdot \text{curl} \mathbf{C} - \frac{1}{2} \alpha \mathbf{C} \cdot \text{curl} \mathbf{B} \right)
\]
\[
- \frac{1}{2} \alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{B}^\perp) \cdot \mathbf{C}^\perp - \frac{1}{2} \alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{C}^\perp) \cdot \mathbf{B}^\perp
\]
and thus
\[
\int_{\mathbb{R}^2} \left( \text{curl} \mathbf{B} \cdot \text{curl} \mathbf{C} - \frac{1}{2} \alpha \mathbf{B} \cdot \text{curl} \mathbf{C} - \frac{1}{2} \alpha \mathbf{C} \cdot \text{curl} \mathbf{B} \right)
\]
\[
- \frac{1}{2} \alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{B}^\perp) \cdot \mathbf{C}^\perp - \frac{1}{2} \alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{C}^\perp) \cdot \mathbf{B}^\perp
\]
\[
= \int_{\mathbb{R}^2} \Phi_1 H(\eta) \Phi_2
\]
\[
+ \frac{1}{2} \alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{B}^\perp) \cdot \mathbf{C}^\perp - \frac{1}{2} \alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{C}^\perp) \cdot \mathbf{B}^\perp - \frac{1}{2} \alpha \int_{\mathbb{R}^2} \mathbf{B}^\perp \cdot \mathbf{C}^\perp
\]
where the last line follows by proposition 1.2. ■

Theorem 3.2. The variational functional \( \mathcal{L}(\eta, \Phi) \) and its Euler–Lagrange equations may be written as, respectively,
\[
\mathcal{L}(\eta, \Phi) = \int_{\mathbb{R}^2} \left( \frac{1}{2} \Phi H(\eta) \Phi - \nabla \Phi \cdot \mathbf{A}^\perp + \Gamma(\eta) + \frac{1}{2} \mathbf{g} \eta^2 + \sigma (1 + |\nabla \eta|^2)^{1/2} - 1 \right)
\]
and
\[
H(\eta) \Phi + \mathbf{u}^* \cdot \mathbf{N} = 0,
\]
\[ \frac{1}{2} |K(\eta)\Phi|^2 - \frac{(H(\eta)\Phi + K(\eta)\Phi \cdot \nabla \eta)^2}{2(1 + |\nabla \eta|^2)} - \alpha \frac{H(\eta)\Phi (H(\eta)\Phi + K(\eta)\Phi \cdot \nabla \eta)}{1 + |\nabla \eta|^2} \\
+ K(\eta)\Phi \cdot \mathbf{u}_h^\ast + g \eta - \sigma \left( \frac{\eta_x}{(1 + |\nabla \eta|^2)^{1/2}} \right)_x - \sigma \left( \frac{\eta_z}{(1 + |\nabla \eta|^2)^{1/2}} \right)_z = 0, \]

where \( K(\eta)\Phi = \nabla \Phi - \alpha \nabla \frac{1}{\Delta} \Delta^{-1}(H(\eta)\Phi) \).

**Proof.** Using lemma 3.1 with \( \Phi_1 = \Phi_2 = \Phi \), one finds that

\[
\int_{D_v} \left( \frac{1}{2} |\text{curl} \mathbf{A}|^2 - \frac{1}{2} \alpha \mathbf{A} \cdot \text{curl} \mathbf{A} \right) - \frac{1}{2} \alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot \mathbf{A}_\perp) \cdot \mathbf{A}_\parallel \Phi = \frac{1}{2} \int_{\mathbb{R}^2} \Phi H(\eta)\Phi, \]

where \( \mathbf{A} \) is the unique solution of (1.15)–(1.19), and the first result follows from this formula and the definition (1.14) of \( \mathcal{L}(\eta, \Phi) \). The result for the Euler–Lagrange equations is obtained from (2.5), (2.6) and the identities

\[ |\mathbf{v}|^2 = |\mathbf{v}|^2 + \frac{(|\mathbf{v}| \cdot \mathbf{N}|^2 - (|\mathbf{v}| \cdot \nabla \eta)^2)}{1 + |\nabla \eta|^2}, \quad \mathbf{v}_2 = \frac{\mathbf{v} \cdot \mathbf{N} + \mathbf{v} \cdot \nabla \eta}{1 + |\nabla \eta|^2} \]

with \( \mathbf{v} = \text{curl} \mathbf{A} \), so that

\[ \mathbf{v} \cdot \mathbf{N} = \nabla \cdot \mathbf{A}_\perp = H(\eta)\Phi, \quad \mathbf{v}_\parallel = \nabla \Phi - \alpha \nabla \frac{1}{\Delta} \Delta^{-1}(H(\eta)\Phi). \]  

**Remark 3.3.** Note that \( \Delta^{-1}(H(\eta)\Phi) \) is well defined because \( H(\eta)\Phi = \nabla \cdot \mathbf{A}_\parallel \).

4. Functional–analytic aspects

(a) Hodge–Weyl decomposition

Let us work in the Sobolev spaces

\[ H^s(\mathbb{R}^2) = \left\{ u \in S'(\mathbb{R}^2) : \|u\|_s^2 := \int_{\mathbb{R}^2} \left( 1 + |k|^2 \right)^s |\hat{u}(k)|^2 \, dk < \infty \}, \quad s \in \mathbb{R}, \]

where \( \hat{u} = \mathcal{F}[u] \) denotes the Fourier transform of \( u \), and the Beppo–Levi spaces

\[ H^s(\mathbb{R}^2) := (u \in L^2_{\text{loc}}(\mathbb{R}^2) : \|u\|_{H^s(\mathbb{R}^2)} := \|\nabla u\|_{H^{-s}(\mathbb{R}^2)} < \infty), \quad s \geq 0. \]

For each \( f \in L^2(\mathbb{R}^2)^2 \) there exist unique functions \( \Phi, \Psi \in H^1(\mathbb{R}^2) \) such that

\[ f = \nabla \Phi + \nabla \perp \Psi; \]

this (obviously orthogonal) decomposition of \( u \) is its **Hodge–Weyl decomposition**. The functions \( \Phi, \Psi \) are characterized as the weak solutions of the equations \( \Delta \Phi = \nabla \cdot f, \Delta \Psi = \nabla \perp \cdot f \); that is, \( \Phi \) and \( \Psi \) are the unique functions in \( H^1(\mathbb{R}) \) such that

\[ \int_{\mathbb{R}^2} \nabla \Phi \cdot \nabla \chi = \int_{\mathbb{R}^2} f \cdot \nabla \chi, \quad \int_{\mathbb{R}^2} \nabla \Psi \cdot \nabla \chi = \int_{\mathbb{R}^2} f \cdot \nabla \perp \chi \]

for all \( \chi \in H^1(\mathbb{R}^2) \), and one accordingly writes \( \Phi = \Delta^{-1}(\nabla \cdot f), \Psi = \Delta^{-1}(\nabla \perp \cdot f) \).

**Proposition 4.1.** For each \( s \geq 0 \) the formulae \( f \mapsto \Delta^{-1}(\nabla \cdot f) \) and \( f \mapsto \Delta^{-1}(\nabla \perp \cdot f) \) define continuous linear mappings \( H^s(\mathbb{R}^2)^2 \to H^{s+1}(\mathbb{R}^2) \).

(b) Well-posedness of the defining boundary-value problem

In this section, we suppose that \( \eta \) is a fixed function in \( W^{2,\infty}(\mathbb{R}^2) \) with \( \text{inf} \eta > -h \) and study the boundary-value problem (1.15)–(1.19) using the standard Sobolev spaces \( L^2(D_\eta)^3 \) and \( H^1(D_\eta)^3 \).
together with the closed subspaces

$$
\mathcal{X}_\eta = \{ \mathbf{F} \in H^1(D_\eta)^3 : \mathbf{F} \wedge \mathbf{j}|_{y=-h} = 0, \mathbf{F} \cdot \mathbf{n} = 0 \}
$$

and

$$
\mathcal{Y}_\eta = \{ \mathbf{F} \in H^1(D_\eta)^3 : \mathbf{F} \cdot \mathbf{j}|_{y=-h} = 0, \mathbf{F}^\perp = 0 \}
$$

of $H^1(D_\eta)^3$. The following proposition gives an alternative description of $\mathcal{X}_\eta$ and $\mathcal{Y}_\eta$ (see Castro & Lannes [5, lemma 3.3] for the result for $\mathcal{X}_\eta$; the result for $\mathcal{Y}_\eta$ is established in an analogous fashion).

**Proposition 4.2.** The spaces $\mathcal{X}_\eta$ and $\mathcal{Y}_\eta$ coincide, respectively,

$$
\{ \mathbf{F} \in L^2(D_\eta)^3 : \text{curl} \mathbf{F} \in L^2(D_\eta)^3, \text{div} \mathbf{F} \in L^2(D_\eta), \mathbf{F} \wedge \mathbf{j}|_{y=-h} = 0, \mathbf{F} \cdot \mathbf{n} = 0 \}
$$

and

$$
\{ \mathbf{F} \in L^2(D_\eta)^3 : \text{curl} \mathbf{F} \in L^2(D_\eta)^3, \text{div} \mathbf{F} \in L^2(D_\eta), \mathbf{F} \cdot \mathbf{j}|_{y=-h} = 0, \mathbf{F}^\perp = 0 \},
$$

and the function $\mathbf{F} \mapsto (\| \text{curl} \mathbf{F} \|^2_{L^2(D_\eta)^3} + \| \text{div} \mathbf{F} \|^2_{L^2(D_\eta)})^{1/2}$ is equivalent to their usual norm.

A weak solution of (1.15)–(1.19) is a function $\mathbf{A} \in \mathcal{X}_\eta$ such that

$$
\int_{D_\eta} (\text{curl} \mathbf{A} \cdot \text{curl} \mathbf{B} - \alpha \text{curl} \mathbf{A} \cdot \mathbf{B} + \text{div} \mathbf{A} \text{div} \mathbf{B}) = \alpha \int_{\mathbb{R}^2} \nabla^{-1}(\nabla \cdot \mathbf{A}^\perp) \cdot \mathbf{B}_\parallel = \int_{\mathbb{R}^2} \nabla^\perp \mathbf{\Phi} \cdot \mathbf{B}_\parallel \quad (4.1)
$$

for all $\mathbf{B} \in \mathcal{X}_\eta$, while a strong solution has the additional regularity requirement that $\mathbf{A} \in H^2(D_\eta)^3$ is solenoidal and satisfies (1.15) in $L^2(D_\eta)^3$ and (1.19) in $H^{1/2}(\mathbb{R}^2)^2$. The existence of weak and strong solutions is established in lemmas 4.5 and 4.7; both are preceded by auxiliary results (propositions 4.3, 4.4 and 4.6) necessary for their proof.

**Proposition 4.3.**

(i) The function $\mathbf{F} \mapsto \mathbf{F}$ is a continuous linear mapping $H^1(D_\eta)^3 \to H^{1/2}(\mathbb{R}^2)^3$ with continuous right inverse $H^{1/2}(\mathbb{R}^2)^3 \to \{ \mathbf{F} \in H^1(D_\eta)^3 : \mathbf{F}|_{y=-h} = 0 \}$.

(ii) The mapping $\mathbf{F} \mapsto \mathbf{F}^\perp$ defined on $\mathcal{D}(\overline{D_\eta})^3$ extends to a continuous linear mapping $\{ \mathbf{F} \in L^2(D_\eta)^3 : \text{curl} \mathbf{F} \in L^2(D_\eta)^3 \} \to H^{-1/2}(\mathbb{R}^2)^2$, where the former space is equipped with the norm $\mathbf{F} \mapsto (\| \text{curl} \mathbf{F} \|^2_{L^2(D_\eta)^3} + \| \text{curl} \mathbf{F} \|^2_{L^2(D_\eta)})^{1/2}$.

**Proof.** Assertion (i) follows from the corresponding result for $\mathbf{F} : D_0 \to \mathbb{R}^3$, the fact that $\mathbf{F} = \mathbf{F}|_{y=0}$ and the estimates $\| \mathbf{F} \|_{H^{1/2}(D_\eta)^3} \lesssim \| \mathbf{F} \|_{H^{1/2}(D_\eta)^3}$, $\| \mathbf{F} \|_{H^{1/2}(D_\eta)^3} \lesssim \| \mathbf{F} \|_{H^{1/2}(D_\eta)^3}$, while (ii) is obtained by a standard argument from the identity

$$
\int_{D_\eta} (\mathbf{F} \cdot \text{curl} \mathbf{G} - \text{curl} \mathbf{F} \cdot \mathbf{G}) = \int_{\mathbb{R}^2} \mathbf{F}^\perp \cdot \mathbf{G} \parallel, \quad \mathbf{F} \in \mathcal{D}(\overline{D_\eta}), \mathbf{G} \in H^1(D_\eta), \mathbf{G}|_{y=-h} = 0
$$

(see Dautray & Lions [10, (p. 207)]).

The proof of the next proposition has been given by Lannes [11, ch. 2].

**Proposition 4.4.** The boundary-value problem

$$
\Delta u = G \quad \text{in } D_0,
$$

$$
\partial_n u = 0 \quad \text{at } y = \eta,
$$

$$
u = 0 \quad \text{at } y = -h
$$

has a unique solution $u \in H^2(D_\eta)$ for each $G \in L^2(D_\eta)$.

**Lemma 4.5.** For all sufficiently small values of $|\alpha|$ the boundary-value problem (1.15)–(1.19) admits a unique weak solution for each $\mathbf{F} \in H^{1/2}(\mathbb{R}^2)^2$. The weak solution is solenoidal and satisfies (1.15) in the sense of distributions and (1.19) in $H^{-1/2}(\mathbb{R}^2)^2$. 
Proof. The estimates

\[
\left| \int_{D_\eta} \text{curl} \ A \cdot B \right| \lesssim \|A\|_{H^1(D_\eta)} \|B\|_{H^1(D_\eta)},
\]

\[
\left| \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot A_\eta) \cdot B \right| \lesssim \|A\|_0 \|B\|_0 \lesssim \|A\|_{1/2} \|B\|_{1/2} \lesssim \|A\|_{H^1(D_\eta)} \|B\|_{H^1(D_\eta)}
\]

and proposition 4.2 imply that for sufficiently small values of \(|\alpha|\) the left-hand side of (4.1) is a continuous, coercive, bilinear form \(X_\eta \times X_\eta \rightarrow \mathbb{R}\), while the estimate

\[
\left| \int_{\mathbb{R}^2} \nabla \cdot \Phi \cdot B \right| \lesssim \|\nabla \cdot \Phi\|_{-1/2} \|B\|_{1/2} \lesssim \|\nabla \Phi\|_{H^{1/2}(\mathbb{R}^2)} \|B\|_{H^1(D_\eta)}
\]

shows that its right-hand side is a continuous, bilinear form \(\dot{H}^{1/2}(\mathbb{R}^2)^2 \times X_\eta \rightarrow \mathbb{R}\) (note that proposition 4.3(i) has been used in both steps). The existence of a unique solution \(A \in X_\eta\) now follows from the Lax–Milgram lemma.

Let \(\Phi_A \in H^2(D_\eta)\) be the unique function satisfying \(\Delta \Phi_A = \text{div} \ A\) in \(D_\eta\) with boundary conditions \(\partial_\nu \Phi_A|_{y=0} = 0\), \(\Phi_A|_{y=-h} = 0\) (see proposition 4.4). It follows that \(B = \text{grad} \ \Phi_A \in X_\eta\) and hence

\[
\left( -\alpha \text{curl} \ A \cdot \text{grad} \ \Phi_A + (\text{div} \ A)^2 \right) - \alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot A_\eta^\perp) \cdot \nabla \Phi_A = 0
\]

(because \(B = \nabla (\Phi_A)\), which is orthogonal to \(\nabla \cdot \Phi\)). Since

\[
\alpha \int_{D_\eta} \text{curl} \ A \cdot \text{grad} \ \Phi_A = \alpha \int_{\mathbb{R}^2} \text{curl} \ A \cdot \nabla \Phi_A = \alpha \int_{\mathbb{R}^2} \nabla \cdot A_\eta^\perp \Phi_A = -\alpha \int_{\mathbb{R}^2} \nabla \Delta^{-1}(\nabla \cdot A_\eta^\perp) \cdot \nabla \Phi_A,
\]

one concludes that \(\text{div} \ A = 0\).

Choosing \(B \in D(D_\eta)^3\), one finds that \(A\) solves (1.15) in the sense of distributions and hence that \(\text{curl} \ \text{curl} \ A \in L^2(D_\eta)^3\). It follows that \((\text{curl} \ A)^\perp \in H^{-1/2}(\mathbb{R}^2)^2\) (proposition 4.3(ii)) and

\[
\int_{D_\eta} (\text{curl} \ \text{curl} \ A - \alpha \text{curl} \ A) \cdot B + \int_{\mathbb{R}^2} \left( ((\text{curl} \ A)^\perp - \nabla \cdot \Phi - \alpha \nabla \Delta^{-1}(\nabla \cdot A_\eta^\perp)) \right) \cdot B = 0.
\]

One concludes that (1.19) holds in \(H^{-1/2}(\mathbb{R}^2)^2\).

Proposition 4.6.

(i) The spaces

\[
\{F \in L^2(D_\eta)^3 : \text{curl} \ F \in L^2(D_\eta)^3, \ \text{div} \ F \in L^2(D_\eta), \ F \wedge |_{y=-h} \in H^{1/2}(\mathbb{R}^2)^3, \ F \cdot N \in H^{1/2}(\mathbb{R}^2) \}\]

and

\[
\{F \in L^2(D_\eta)^3 : \text{curl} \ F \in L^2(D_\eta)^3, \ \text{div} \ F \in L^2(D_\eta), \ F \cdot |_{y=-h} \in H^{1/2}(\mathbb{R}^2)^2, \ F^\perp_\parallel \in H^{1/2}(\mathbb{R}^2)^2 \}
\]

coincide with \(H^1(D_\eta)^3\).

(ii) The space

\[
\{F \in L^2(D_\eta)^3 : \text{curl} \ F \in H^1(D_\eta)^3, \ \text{div} \ F \in H^1(D_\eta), \ F \wedge |_{y=-h} = 0, \ F \cdot n = 0 \}
\]

coincides with \(\{F \in H^2(D_\eta)^3 : F \wedge |_{y=-h} = 0, \ F \cdot n = 0 \}\).

Proof. (i) Comparing the Sobolev–Slobodeckij norms for the two spaces (see Adams [12, §7.48]) shows that \(F, n \in H^{1/2}(S_\eta)\) if and only if \(F \cdot n \in H^{1/2}(\mathbb{R}^2)\) and \(F \wedge N \in H^{1/2}(S_\eta)^3\) if and only if \(F \wedge N \in H^{1/2}(\mathbb{R}^2)^3\); since \((F \wedge N)_h = F^\perp_\parallel\) and \((F \wedge N)_2 = F^\perp_\parallel \cdot \nabla \eta\) it follows that \(F \wedge n \in H^{1/2}(S_\eta)^3\) if and only if \((F^\perp_\parallel) \in H^{1/2}(\mathbb{R}^2)\).
The given spaces obviously contain $H^1(D_0)^3$; our task is to establish the reverse inclusions. Suppose that $F \in L^2(D_0)^3$ satisfies $\text{curl} \ F \in L^2(D_0)^3$, $F \wedge j|_{y=-h} \in H^{1/2}(\mathbb{R}^2)^2$ and $F\cdot n \in H^{1/2}(S_0)$. Letting $\phi \in H^2(D_0)$ be a function with

$$\partial_n \phi|_{y=-h} = F\cdot n$$

and $G \in H^1(D_0)^3$ be a function with

$$G = 0, \quad G|_{y=-h} = -(F \wedge j) \wedge j + (\text{grad} \ \phi \wedge j)|_{y=-h},$$

we find that $H = F - \text{grad} \ \phi - G$ satisfies $H \in L^2(D_0)^3$, $\text{curl} \ H \in L^2(D_0)^3$, $H \cdot n = 0$ and $H \wedge j|_{y=-h} = 0$. Proposition 4.2 asserts that $H \in \mathcal{X}_n$ and hence $F = H + \text{grad} \ \phi + G \in H^1(D_0)$.

Similarly, suppose that $F \in L^2(D_0)^3$ satisfies $\text{curl} \ F \in L^2(D_0)^3$, $F \in L^2(D_0)^3$, $F \cdot j|_{y=-h} \in H^{1/2}(\mathbb{R}^2)$ and $F_1 \in H^{1/2}(\mathbb{R}^2)^2$, so that $F \wedge n \in H^{1/2}(S_0)^3$. Letting $\phi \in H^2(D_0)$ be a function with

$$\partial_n \phi|_{y=-h} = F\cdot j|_{y=-h}$$

and $G \in H^1(D_0)^3$ be a function with

$$G|_{y=-h} = 0, \quad G = -(F \wedge n) \wedge n + (\text{grad} \ \phi \wedge n) \wedge n,$$

we find that $H = F - \text{grad} \ \phi - G$ satisfies $H \in L^2(D_0)^3$, $\text{curl} \ H \in L^2(D_0)^3$, $H \wedge n = 0$, and $H_1|_{y=-h} = 0$, so that $H_1 = 0$. Proposition 4.2 asserts that $H \in \mathcal{Y}_n$ and hence $F = H + \text{grad} \ \phi + G \in H^1(D_0)$.

(ii) This result follows by applying the results in part (i) to the derivatives of $F$.

**Lemma 4.7.** Suppose that $\Phi \in H^{3/2}(\mathbb{R}^2)$. Any weak solution $A$ of (1.15)–(1.19) is in fact a strong solution.

**Proof.** Recall that $\text{curl} \ \text{curl} \ A \in L^2(D_0)^3$ and

$$(\text{curl} \ A)|_{\|} = \nabla \Phi + \alpha \nabla \Delta^{-1}(\nabla \cdot A)|_{\|}$$

holds in $H^{-1/2}(\mathbb{R}^2)^2$; hence $(\text{curl} \ A)|_{\|} \in H^{1/2}(\mathbb{R}^2)^2$ (because the right-hand side of this equation belongs to $H^{1/2}(\mathbb{R}^2)^2$). Since $0 = \text{div} \ A \in L^2(D_0)$ and $\text{curl} \ A \cdot j|_{y=-h} = 0$ it follows that $\text{curl} \ A \in H^1(D_0)^3$ (proposition 4.6(i)), and furthermore $\text{curl} \ A \in H^1(D_0)^3$, $0 = \text{div} \ A \in H^1(D_0)$ with $A \wedge j|_{y=-h} = 0, A \cdot n = 0$ imply that $A \in H^2(D_0)^3$ (proposition 4.6(ii)). Finally, note that (1.15) holds in $L^2(D_0)$ because it holds in the sense of distributions and $A \in H^2(D_0)^3$.

We conclude this section with the following alternative characterization of a strong solution to (1.15)–(1.19).

**Proposition 4.8.** Suppose that $\Phi \in H^{3/2}(\mathbb{R}^2)$. The strong solutions of the boundary-value problems (1.15)–(1.19) and

$$-\Delta A = \alpha \text{curl} \ A \quad \text{in} \ D_0, \quad A \wedge j = 0 \quad \text{at} \ y = -h, \quad A_2\nu = 0 \quad \text{at} \ y = -h, \quad A \cdot n = 0 \quad \text{at} \ y = \eta, \quad (\text{curl} \ A)|_{\|} = \nabla \Phi - \alpha \nabla \Delta^{-1}(\nabla \cdot A)|_{\|} \quad \text{at} \ y = \eta$$

coincide (so that in particular (4.2)–(4.6) has a unique strong solution).

**Proof.** Suppose that $A \in H^2(D_0)^3$ is a strong solution of (1.15)–(1.19), so that $A$ satisfies (4.2) (owing to the identity $\text{curl} \ \text{curl} \ A = -\Delta A + \text{grad} \ \text{div} \ A$) and $\text{div} \ A|_{y=-h} = 0$. Since $A_1|_{y=-h} = 0$ and hence $\nabla \cdot A_1|_{y=-h} = 0$, we conclude that $A_2\nu|_{y=-h} = \text{div} \ A - \nabla \cdot A_1|_{y=-h} = 0$. 


The above argument shows that any strong solution \( A \in H^2(D_\eta)^3 \) of (4.2)–(4.6) satisfies \( \text{div} \ A|_{y=-h} = 0 \); it remains to show that in fact \( \text{div} \ A = 0 \) in \( D_\eta \). Writing

\[
-\Delta A = \text{curl} \ \text{curl} A - \text{grad} \ \text{div} A,
\]

taking the scalar product of (4.2) with a function \( B \in X_\eta \) and integrating by parts using the integral identities

\[
\int_{D_\eta} \text{curl} \ \text{curl} A \cdot B = \int_{D_\eta} \text{curl} A \cdot \text{curl} B - \int_{\mathbb{R}^2} (\text{curl} A)_\parallel \cdot B_\parallel
\]

and

\[
\int_{D_\eta} \text{grad} \ \text{div} A \cdot B = -\int_{D_\eta} \text{div} A \text{div} B
\]

(where we have used \( B \wedge j|_{y=-h} = 0, \ B \cdot n = 0 \) and \( \text{div} \ A|_{y=-h} = 0 \)), we find that \( A \) satisfies (4.1). It follows that \( A \) is a weak solution of (1.15)–(1.19), so that in particular \( \text{div} A = 0 \). \( \blacksquare \)

(c) Analyticity of the operator \( H(\eta) \)

In this section, we improve the result of lemmas 4.5 and 4.7 by quantifying the restriction that \( |\alpha| \) is small and showing that improved regularity of \( \eta \) and \( \Phi \) yields improved regularity of \( A \); we use these results to deduce that \( H(\eta) \) depends analytically upon \( \eta \) (see corollary 4.12 for a precise statement of this result). The starting point is the ‘flattened’ version (1.20)–(1.24) of the boundary-value problem (1.15)–(1.19), which according to proposition 4.8 is equivalent to the ‘flattened’ version of the boundary-value problem (4.2)–(4.6); that is,

\[
-\Delta \tilde{A} - \alpha \text{curl} \tilde{A} = H^0(\tilde{A}) \quad \text{in} \quad D_0,
\]

\[
\tilde{A} \wedge j = 0 \quad \text{at} \quad y = -h,
\]

\[
\tilde{A}_{2y} = 0 \quad \text{at} \quad y = -h,
\]

\[
\tilde{A} \cdot j = g^0(\tilde{A}) \quad \text{at} \quad y = 0,
\]

\[
(\text{curl} \tilde{A})_h + \alpha \nabla \perp \Delta^{-1} (\nabla \cdot \tilde{A}_h) = h^0(\tilde{A}) + \nabla \Phi \quad \text{at} \quad y = 0,
\]

where

\[
H^0(\tilde{A}) = \Delta^0 \tilde{A} + \alpha \text{curl}^0 \tilde{A} - \Delta \tilde{A} - \alpha \text{curl} \tilde{A},
\]

\[
g^0(\tilde{A}) = \nabla \eta \cdot \tilde{A}_h,
\]

\[
h^0(\tilde{A}) = -(\text{curl}^0 \tilde{A})_h + (\text{curl} \tilde{A})_h - \nabla \eta (\text{curl}^0 \tilde{A})_2 - \alpha \nabla \perp \Delta^{-1} (\nabla \cdot (\nabla \eta \tilde{A}_2));
\]

the definition (3.1) of \( H \) is accordingly replaced by

\[
H(\eta) \Phi = \nabla \cdot \tilde{A}_h + \nabla \cdot (\nabla \eta \tilde{A}_2).
\]

(With a slight abuse of notation \( \tilde{y} \) has been replaced by \( y \) for notational simplicity and the underscore now denotes evaluation at \( y = 0 \).)

The discussion of the boundary-value problem (4.7)–(4.11) begins with the corresponding inhomogeneous linear problem.

**Proposition 4.9.** Suppose that \( s \geq 2 \) and \( \alpha^* < (\pi/2h) \). The boundary-value problem

\[
-\Delta \tilde{A} - \alpha \text{curl} \tilde{A} = \mathbf{H} \quad \text{in} \quad D_0,
\]

\[
\tilde{A} \wedge j = 0 \quad \text{at} \quad y = -h,
\]

\[
\tilde{A}_{2y} = 0 \quad \text{at} \quad y = -h,
\]

\[
\tilde{A} \cdot j = g \quad \text{at} \quad y = 0,
\]

\[
(\text{curl} \tilde{A})_h + \alpha \nabla \perp \Delta^{-1} (\nabla \cdot \tilde{A}_h) = h \quad \text{at} \quad y = 0
\]
has a unique solution \( \hat{A} \in \mathcal{H}(D_0)^3 \) for each \( g \in \mathcal{H}^{-1/2}(\mathbb{R}^2) \) and \( H \in \mathcal{H}^{s-2}(D_0)^3 \), \( h \in \mathcal{H}^{s-3/2}(\mathbb{R}^2)^2 \). The solution operator defines a mapping \( \mathcal{H}^{-1/2}(\mathbb{R}^2) \times \mathcal{H}^{s-2}(D_0)^3 \times \mathcal{H}^{s-3/2}(\mathbb{R}^2)^2 \rightarrow \mathcal{H}(D_0)^3 \), which is bounded uniformly over \( |\alpha| \in [0, \alpha^*] \).

**Proof.** The solution to this boundary-value problem is

\[
\hat{A} = \mathcal{F}^{-1} \left[ \int_{-h}^{0} \frac{G(y, \xi) \hat{H}(\xi) d\xi}{\hat{h}_1 - i \hat{k}_1 \hat{g}} - G(y, 0) \begin{pmatrix} h_1 - i k_1 g \\ 0 \\ h_3 - i k_3 g \end{pmatrix} \right],
\]

where \( \mathcal{F}[u] \) denotes the Fourier transform of \( u \) with respect to \((x, z)\) (with independent variable \( k = (k_1, k_3) \)) and the Green’s matrix \( G \) is given by

\[
G(y, \xi) = \begin{cases}
U(y + h)C^TW(\xi)^T, & -h \leq y \leq \xi \leq 0, \\
W(y)CU(\xi + h)^T, & -h \leq \xi \leq y \leq 0;
\end{cases}
\]

explicit formulae for the matrices \( U(y), C \) and \( W(y) \) are stated in appendix A (note that the existence of \( G \) is subject to the non-resonance condition \( \alpha^2 - |k|^2 \equiv \frac{1}{2} \neq \pi n/2 \) for \( n \in \mathbb{N} \), which is guaranteed by choosing \( \alpha^* h < \pi/2 \)). We show by induction that the formulæ

\[
G_1(H) = \mathcal{F}^{-1} \left[ \int_{-h}^{0} G(y, \xi) \hat{h}(\xi) d\xi \right],
\]

\[
G_2(h) = \mathcal{F}^{-1} \left[ G(y, 0) \hat{h}(\xi) d\xi \right], \quad G_3(h) = \mathcal{F}^{-1} \left[ G_\xi(y, 0) \hat{h}(\xi) d\xi \right]
\]

define linear mappings \( G_1 : H_m^m(D_0)^3 \rightarrow H_{m+2}^m(D_0)^3 \), \( G_2 : H_{m+1/2}^m(\mathbb{R}^2)^3 \rightarrow H_{m+2}^m(D_0)^3 \), \( G_3 : H_{m+3/2}^m(\mathbb{R}^2)^3 \rightarrow H_{m+2}^m(D_0)^3 \) which are bounded uniformly over \( |\alpha| \in [0, \alpha^*] \) for \( m = 0, 1, 2, \ldots \) and hence for \( m \in [0, \infty) \) by interpolation.

The estimates given in appendix A ((A 1)–(A 5)) imply the existence of \( R > 0 \) such that

\[
|G(y, \xi)|, \quad |G_\xi(y, \xi)| \lesssim 1
\]

for \( |k| \leq R \) and

\[
|G(y, \xi)| \lesssim \frac{1}{|k|} e^{-|k||y-\xi|}, \quad |G_\xi(y, \xi)| \lesssim e^{-|k||y-\xi|}
\]

for \( |k| \geq R \), uniformly over \( y, \xi \in [-h, 0] \) and \( |\alpha| \in [0, \alpha^*] \). Using these inequalities, one finds by straightforward calculations that

\[
\|G_1(H)\|_{1,1}, \|\partial_\alpha G_1(H)\|_{1,1}, \|\partial_\xi G_1(H)\|_1 \lesssim \|H\|_0,
\]

and it follows from the equation

\[
\partial_\xi^2 G_1(H) = -\partial_\alpha^2 G_1(H) - \partial_\xi^2 G_1(H) - \alpha \text{curl } G_1(H) - H
\]

that

\[
\|\partial_\xi G_1(H)\|_0 \lesssim \|H\|_0.
\]

This argument establishes the result for \( G_1 \) for \( m = 0 \).

Suppose that \( \|G_1(H)\|_{m+2} \lesssim \|H\|_{m} \) for some \( m \in \mathbb{N}_0 \), so that

\[
\|G_1(H)\|_{m+2} \lesssim \|H\|_{m} \lesssim \|H\|_{m+1}
\]

and obviously

\[
\|\partial_\alpha G_1(H)\|_{m+2} \lesssim \|G_1(\partial_\alpha H)\|_{m+2} \lesssim \|\partial_\alpha H\|_{m} \lesssim \|H\|_{m+1}
\]

and

\[
\|\partial_\xi G_1(H)\|_{m+2} \lesssim \|G_1(\partial_\xi H)\|_{m+2} \lesssim \|\partial_\xi H\|_{m} \lesssim \|H\|_{m+1}.
\]
Furthermore, it follows from (4.13) that
\[
\| \partial_y^{m+3} G_1(H) \|_0 \lesssim \| D \partial_y^{m+1} G_1(D) \|_0 + \| \partial_y^{m+1} G_1(H) \|_0
\]
\[
\lesssim \| G_1(D) \|_{m+2} + \| G_1(H) \|_{m+2} + \| \partial_y^{m+1} H \|_0
\]
\[
\lesssim \| D \|_m + \| H \|_m + \| H \|_{m+1}
\]
\[
\lesssim \| H \|_{m+1},
\]
where \( |D| = \mathcal{F}^{-1}([k], \mathcal{F}[-]) \). One concludes that \( \| G_1(H) \|_{m+3} \lesssim \| H \|_{m+1} \).

The estimates for \( G_2 \) and \( G_3 \) are obtained in a similar fashion.

In view of the previous result, we henceforth fix \( \alpha^* < \pi/2 \) and assume that \( |\alpha| \in [0, \alpha^*] \); all results hold uniformly over these values of \( \alpha \).

**Theorem 4.10.** Suppose that \( s \geq 2 \). There exists an open neighbourhood \( V \) of the origin in \( H^{s+1/2}(\mathbb{R}^2) \) such that the boundary-value problem (4.7)–(4.11) has a unique solution \( \tilde{A} = \tilde{A}(\eta, \Phi) \) in \( H^s(D_0)^3 \) for each \( \eta \in V \) and \( \Phi \in H^{s-1/2}(\mathbb{R}) \). Furthermore, \( \tilde{A}(\eta, \Phi) \) depends analytically upon \( \eta \) and \( \Phi \) (and linearly upon \( \Phi \)).

**Proof.** Using the estimates
\[
\| f_1(\eta) \Delta \tilde{h} \|_{H^{s-2}(D_0)} \lesssim \| f(\eta) \|_s \| \Delta \eta \|_{s-3/2} \| \tilde{h} \|_{H^{s-1}(D_0)},
\]
\[
\| f_2(\eta, \nabla \eta) \tilde{h} \|_{H^{s-2}(D_0)} \lesssim \| f(\eta, \nabla \eta) \|_{s-1/2} \| \tilde{h} \|_{H^{s-1}(D_0)},
\]
\[
\| \nabla \tilde{h} \|_{s-1/2} \lesssim \| \nabla \eta \|_{s-1/2} \| \tilde{h} \|_{H^s(D_0)},
\]
\[
\| \nabla \eta \|_{s-3/2} \lesssim \| \nabla \eta \|_{s-1/2} \| \tilde{h} \|_{H^{s-1}(D_0)},
\]
where \( f_1 \) and \( f_2 \) are analytic mappings of a neighbourhood of the origin in, respectively, \( \mathbb{R} \) and \( \mathbb{R}^3 \) into \( \mathbb{R} \), one finds that the mappings \( (\eta, \tilde{A}) \mapsto g^\eta(\tilde{A}), (\eta, \tilde{A}) \mapsto H^\eta(\tilde{A}) \) are analytic in \( V \times H^s(D_0)^3 \), where \( V \) is an open neighbourhood of the origin in \( H^{s+1/2}(\mathbb{R}^2) \), their target spaces being, respectively, \( H^{s-1/2}(\mathbb{R}^2) \), \( H^{s-2}(D_0)^3 \) and \( H^{s-3/2}(\mathbb{R}^2)^2 \). (The above estimates are obtained from standard embedding theorems—see in particular Hörmander [13, theorem 8.3.1], noting that this theorem remains true when \( \mathbb{R}^3 \) is replaced by \( D_0 \)). It follows that the formula
\[
\mathcal{H}(\tilde{A}, \eta, \Phi) = \begin{pmatrix}
-\Delta \tilde{A} - \alpha \nabla \tilde{A} - H^\eta(\tilde{A}) \\
\tilde{A} \cdot j - g^\eta(\tilde{A}) \\
\text{curl} \tilde{A} h + \alpha \Delta^{-1}(\nabla \cdot \tilde{A}_h) - h^\eta(\tilde{A}) - \nabla \Phi
\end{pmatrix}
\]
defines an analytic mapping
\[
\mathcal{H} : S \times V \times H^{s-1/2}(\mathbb{R}) \to H^{s-2}(D_0)^3 \times H^{s-1/2}(\mathbb{R}^2)^2 \times H^{s-3/2}(\mathbb{R}^2)^2,
\]
where \( S = \{ \tilde{A} \in H^s(D_0)^3 : \tilde{A} \wedge j \}_{|y=-h} = 0, \tilde{A}_{2y} = 0 \} \).

Furthermore, \( \mathcal{H}(0,0,0) = (0,0,0) \), and the calculation
\[
d_1 \mathcal{H}[0,0,0](\tilde{A}) = \begin{pmatrix}
\text{curl} \text{ curl} \tilde{A} - \alpha \text{ curl} \tilde{A} \\
\tilde{A} \cdot j \\
\text{curl} \tilde{A}_h + \alpha \Delta^{-1}(\nabla \cdot \tilde{A}_h)
\end{pmatrix}
\]
and proposition 4.9 show that
\[
d_1 \mathcal{H}[0,0,0] : S \times H^{s+1/2}(\mathbb{R}^2)^2 \times H^{s-1/2}(\mathbb{R}) \to H^{s-2}(D_0)^3 \times H^{s-1/2}(\mathbb{R}^2)^2 \times H^{s-3/2}(\mathbb{R}^2)^2
\]
is an isomorphism. The analytic implicit-function theorem (see Buffoni & Toland [14, theorem 4.5.3]) asserts the existence of open neighbourhoods \( W \) and \( U \) of the origin in,
respectively, \( H^{s+1/2}(\mathbb{R}^2) \times H^s(\mathbb{R}) \) and \( H^s(D_0)^3 \) such that the equation
\[
\mathcal{H}(\vec{A}, \eta, \Phi) = (0, 0, 0)
\]
and hence the boundary-value problem (4.7)–(4.11) have a unique solution \( \vec{A}_0 = \vec{A}_0(\eta, \Phi) \) in \( U \) for each \((\eta, \Phi) \in W\); furthermore, \( \vec{A}_0(\eta, \Phi) \) depends analytically upon \( \eta \) and \( \Phi \). Since \( \vec{A}_0 \) depends linearly upon \( \Phi \) one can without loss of generality take \( W = V \times H^s(\mathbb{R}) \), and clearly \( U = H^s(D_0)^3 \) (with \( \Phi = 0 \) the construction yields a unique solution in a neighbourhood of the origin in \( H^s(D_0)^3 \), which is evidently the zero solution).

**Corollary 4.11.** Suppose that \( \eta \in H^{m+1/2}(\mathbb{R}^2) \cap W^{m,\infty}(\mathbb{R}^2) \) for some \( m \in \{2, 3, \ldots\} \). The strong solution \( A \) to (4.2)–(4.6) lies in \( H^m(D_0) \).

**Corollary 4.12.** Suppose that \( s \geq 2 \). There exists an open neighbourhood \( V \) of the origin in \( H^{s+1/2}(\mathbb{R}^2) \) such that formula (4.12) defines an analytic mapping \( \mathcal{H}(: V \rightarrow L(H^{s-1/2}(\mathbb{R}), H^{s-3/2}(\mathbb{R}))) \).

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**Appendix A. The Green’s matrix**

The entries of the matrices \( U(y) = (u_{mn}(y)) \) and \( W(y) = (w_{mn}(y)) \) are
\[
\begin{align*}
    u_{11}(y) &= \frac{ik_1}{|k|} \sinh |k|y, \\
    u_{12}(y) &= -\frac{k_1}{\alpha}[c(y) - \cosh |k|y] + k_3 \left[ s_1(y) - \frac{\sinh |k|y}{|k|} \right] + \frac{k_3}{|k|} \sinh |k|y, \\
    u_{13}(y) &= -\frac{ik_3}{\alpha}[c(y) - \cosh |k|y] - \frac{ik_1}{2|k|} \sinh |k|y + \frac{ik_1}{\alpha^2} \left[ s_2(y) - |k| \sinh |k|y + \frac{\alpha^2}{2} y \cosh |k|y \right], \\
    u_{21}(y) &= \cosh |k|y, \\
    u_{22}(y) &= \frac{|k|^2}{\alpha^2} \left[ s_1(y) - \frac{\sinh |k|y}{|k|} \right], \\
    u_{23}(y) &= \frac{|k|^2}{\alpha^2} \left[ c(y) - \cosh |k|y + \frac{\alpha^2}{2|k|} y \sinh |k|y \right], \\
    u_{31}(y) &= \frac{ik_3}{|k|} \sinh |k|y, \\
    u_{32}(y) &= -\frac{k_3}{\alpha}[c(y) - \cosh |k|y] - k_1 \left[ s_1(y) - \frac{\sinh |k|y}{|k|} \right] + \frac{k_1}{|k|} \sinh |k|y, \\
    u_{33}(y) &= \frac{ik_1}{\alpha}[c(y) - \cosh |k|y] - \frac{ik_3}{2|k|} \sinh |k|y + \frac{ik_3}{\alpha^2} \left[ s_2(y) - |k| \sinh |k|y + \frac{\alpha^2}{2} y \cosh |k|y \right]
\end{align*}
\]
and
\[
\begin{align*}
    w_{11}(y) &= \frac{ik_1}{|k|} \cosh |k|y, \\
    w_{12}(y) &= -\frac{ik_3}{|k|}[c(y) - \cosh |k|y] - \frac{ik_3}{|k|} \cosh |k|y + \frac{ik_1}{\alpha|k|} [s_2(y) - |k| \sinh |k|y],
\end{align*}
\]
\[ w_{13}(y) = \frac{k_1}{2|k|} \cosh |k|y + \frac{|k|k_3}{\alpha} \left[ s_1(y) - \frac{\sinh |k|y}{|k|} \right] - \frac{|k|k_1}{\alpha^2} \left[ c(y) - \cosh |k|y + \frac{\alpha^2}{2|k|^3} y \sinh |k|y \right], \]
\[ w_{21}(y) = \sinh |k|y, \]
\[ w_{22}(y) = \frac{|k|}{\alpha} [c(y) - \cosh |k|y], \]
\[ w_{23}(y) = \frac{i|k|^3}{\alpha^2} \left[ s_1(y) - \frac{\sinh |k|y}{|k|} + \frac{\alpha^2}{2|k|^2} y \cosh |k|y \right], \]
\[ w_{31}(y) = \frac{i k_3}{|k|} \cosh |k|y, \]
\[ w_{32}(y) = \frac{k_1}{|k|} [c(y) - \cosh |k|y] + \frac{i k_1}{|k|} \cosh |k|y + \frac{i k_3}{|k|} s_2(y) - |k| \sinh |k|y, \]
\[ w_{33}(y) = \frac{k_3}{2|k|} \cosh |k|y - \frac{|k|k_1}{\alpha} \left[ s_1(y) - \frac{\sinh |k|y}{|k|} \right] - \frac{|k|k_3}{\alpha^2} \left[ c(y) - \cosh |k|y + \frac{\alpha^2}{2|k|^3} y \sinh |k|y \right], \]

where
\[ c(y) = \begin{cases} \cos(\alpha^2 - |k|^2)^{1/2}y, & |k| \leq \alpha, \\ \cosh(|k|^2 - \alpha^2)^{1/2}y, & |k| \geq \alpha, \end{cases} \]

and
\[ s_1(y) = \begin{cases} \sin((\alpha^2 - |k|^2)^{1/2}y, & |k| \leq \alpha, \\ \sinh((|k|^2 - \alpha^2)^{1/2}y, & |k| \geq \alpha, \end{cases} \]

\[ s_2(y) = (|k|^2 - \alpha^2)s_1(y). \]

Fix \( \alpha^* > 0 \). Straightforward estimates show that
\[
|\partial_y^i (c(y) - \cosh |k|y)| \lesssim \alpha^3|k|^{-1} e^{\alpha |k| y},
\]
\[
|\partial_y^i \left( c(y) - \cosh |k|y + \frac{\alpha^2}{2|k|^3} y \sinh |k|y \right) | \lesssim \alpha^3|k|^{-1} e^{\alpha |k| y},
\]
\[
|\partial_y^i \left( s_1(y) - \frac{\sinh |k|y}{|k|} \right) | \lesssim \alpha^3|k|^{-1} e^{\alpha |k| y},
\]
\[
|\partial_y^i \left( s_1(y) - \frac{\sinh |k|y}{|k|} + \frac{\alpha^2}{2|k|^2} y \cosh |k|y \right) | \lesssim \alpha^3|k|^{-3} e^{\alpha |k| y},
\]
\[
|\partial_y^i (s_2(y) - |k| \sinh |k|y) | \lesssim \alpha^3|k|^{1} e^{\alpha |k| y},
\]
\[
|\partial_y^i \left( s_2(y) - |k| \sinh |k|y + \frac{\alpha^2}{2} y \cosh |k|y \right) | \lesssim \alpha^3|k|^{-1} e^{\alpha |k| y}, \quad i = 0, 1
\]

(uniformly over \( |y| \in [0, h] \) and \( |k| \geq \max(1, \sqrt{2}\alpha) \)); it follows that
\[
|\partial_y^i u_{mn}(y), |\partial_y^i w_{mn}(y) \lesssim |k| e^{\alpha |k| y}, \quad i = 0, 1 \quad (A1)
\]

(uniformly over \( |\alpha| \in [0, \alpha^*], |y| \in [0, h] \) and \( |k| \geq \max(1, \sqrt{2}\alpha) \)). Noting that
\[ c(y) \to \cos \alpha y, \quad s_1(y) \to \frac{\sin \alpha y}{\alpha}, \quad s_2(y) \to -\alpha \sin \alpha y \]

and
\[ \cosh |k|y \to 1, \quad \frac{\sinh |k|y}{|k|} \to y, \quad |k| \sinh |k|y \to 0 \]

as \( |k| \to 0 \) (uniformly over \( |y| \in [0, h] \)), we find that
\[
\frac{1}{|k|} |u_{mn}(y) \leq 1, \quad |w_{mn}(y) \leq 1, \quad (A2)
\]
as \(|k| \to 0\) (uniformly over \(|y| \in [0, h]\) and \(|\alpha| \in [0, \alpha^*]\)). Similarly, since \(c_y(y) = s_2(y)\), \(s_{1y}(y) = c(y)\) and \(s_{2y}(y) = (k^2 - \alpha^2)c(y)\), we find that

\[
|\partial_y u_{mn}(y)| \lesssim 1, \quad |\partial_y w_{mn}(y)| \lesssim 1
\]  

(A 3)

as \(|k| \to 0\) (uniformly over \(|y| \in [0, h]\) and \(|\alpha| \in [0, \alpha^*]\)).

The entries of the matrix \(C = (c_{mn})\) are

\[
c_{11} = -\frac{1}{2|k|} \text{sech}|k|/h + \frac{|k|}{\alpha^2} \left[ s - \text{sech}|k|/h + \frac{h\alpha^2}{2|k|} \text{sech}|k|/h \tan \alpha \right]
\]

\[
- \frac{2|k|}{\alpha^2} \tanh^2 |k|/h [s - \text{sech}|k|/h] + \frac{2}{\alpha^2} \text{sech}|k|/h \tan \alpha \left( t_2 - |k| \tanh |k|/h \right)
\]

\[
+ \text{sech}|k|/h \tan \alpha \left( t_1 - \frac{\tan \alpha}{|k|} \right) + \frac{1}{|k|} \text{sech}|k|/h \tanh^2 |k|/h,
\]

\[
c_{12} = -\frac{i}{\alpha} \tanh |k|/h [s - \text{sech}|k|/h] + \frac{i}{\alpha |k|} \text{sech}|k|/h [t_2 - |k| \tanh |k|/h],
\]

\[
c_{13} = -\frac{1}{|k|} \text{sech}|k|/h,
\]

\[
c_{21} = \frac{1}{\alpha} \tanh |k|/h [s - \text{sech}|k|/h] + \frac{|k|}{\alpha} \text{sech}|k|/h \left[ t_1 - \frac{\tan \alpha}{|k|} \right],
\]

\[
c_{22} = \frac{i}{|k|} [s - \text{sech}|k|/h] + \frac{i}{|k|} \text{sech}|k|/h,
\]

\[
c_{23} = 0,
\]

\[
c_{31} = \frac{i}{|k|} \text{sech}|k|/h,
\]

\[
c_{32} = 0,
\]

\[
c_{33} = 0,
\]

where

\[
s = \begin{cases} \text{sec}(\alpha^2 - |k|^2)^{1/2}/h, & |k| \leq \alpha, \\ \text{sech}[(|k|^2 - \alpha^2)^{1/2}/h, & |k| \geq \alpha, \end{cases}
\]

and

\[
t_1 = \begin{cases} \frac{\tan(\alpha^2 - |k|^2)^{1/2}/h}{(\alpha^2 - |k|^2)^{1/2}}, & |k| \leq \alpha, \\ \tan(|k|^2 - \alpha^2)^{1/2}/h}{(|k|^2 - \alpha^2)^{1/2}}, & |k| \geq \alpha, \end{cases}
\]

\[
t_2 = (|k|^2 - \alpha^2)t_1;
\]

the necessity that \(s\) is finite is met by choosing \(\alpha^*h < \pi/2\), so that \((\alpha^2 - |k|^2)^{1/2}/h < \pi/2\) for all \(|k| \leq \alpha\).

One finds that

\[
|s - \text{sech}|k|/h| \lesssim \frac{\alpha^2}{|k|} \text{sech}|k|/h,
\]

\[
|s - \text{sech}|k|/h + \frac{\alpha^2 h}{2|k|} \text{sech}|k|/h \tan \alpha \text{sech}|k|/h| \lesssim \frac{\alpha^4}{|k|^2} \text{sech}|k|/h,
\]

\[
|t_1 - \frac{\tan |k|/h}{|k|}| \lesssim \frac{\alpha^2}{|k|^2},
\]

\[
|t_2 - |k| \tanh |k|/h| \lesssim \alpha^2
\]
(uniformly over $|k| \geq \max(1, \sqrt{2\alpha})$), and hence

$$
|c_{nn}| \lesssim \frac{1}{|k|} \tanh |k|h 
$$

(A 4)

uniformly over $|\alpha| \in [0, \alpha^*]$ and $|k| \geq \max(1, \sqrt{2\alpha})$. Noting that

$$
s \to \sec ah, \quad t \to \frac{\tan ah}{\alpha}
$$

and

$$
\tanh |k|h \to 1, \quad \frac{\tanh |k|h}{|k|} \to h
$$

as $|k| \to 0$, we find that

$$
|k|C \to \begin{pmatrix} -\frac{1}{2} & -i \tan ah & -1 \\ 0 & i \sec ah & 0 \\ i & 0 & 0 \end{pmatrix}
$$

componentwise as $|k| \to 0$; it follows that

$$
|k|c_{nn} \lesssim 1 
$$

(A 5)

as $|k| \to 0$ (uniformly over $|\alpha| \in [0, \alpha^*]$).

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