OPTICS, MECHANICS AND QUANTIZATION OF
REPARAMETRIZATION INVARIANT SYSTEMS †

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March 28, 2022

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Abstract

In this paper we regard the dynamics obtained from the Fermat principle
as being the classical theory of light. We (first-)quantize the action and show
how close we can get to the Maxwell theory. We show that quantum Geometric
Optics is not a theory of fields in curved space. Considering Classical Mechanics
to be on the same footing, we show the parallelism between Quantum Mechanics
and Quantum Geometric Optics. We show that, due to the reparametrization
invariance of the classical theories, the dynamics of the quantum theories is given
by a Hamiltonian constraint. Some implications of the above analogy in the
quantization of true reparametrization-invariant theories are discussed.

† Work partially supported by the D.G.I.C.Y.T.
1 Introduction.

The theory that can be properly identified with a classical theory of Optics is Geometric Optics, which is obtained from the requirement for light paths to be of extremal optical length [1]. This theory was used to describe successfully the dynamics of light until Electrodynamics were formulated and the relationship between this theory and light discovered. Since then, Optics has been treated as a chapter of Electrodynamics and its quantization has been achieved as a byproduct of Quantum Electrodynamics. Hence, the history of Optics has been quite different from that of Mechanics, despite the close points of departure (a classical action in both cases) and the common point of arrival (Quantum Electrodynamics in both cases). This paper is meant to be a small step in filling this gap. In fact, the basics of this paper are similar to the ones that motivated the introduction of Quantum Mechanics on the grounds of its analogy to wave optics. Nevertheless, to our knowledge, no development similar to the present one has been made before.

In this paper we study firstly (Sec. 3) the theory obtained by (canonically) quantizing Geometric Optics as given by the Fermat principle†. We shall consider two distinct interpretations of Geometric Optics: it describes a particle with constant mass moving in a curved Euclidean 3+0-dimensional space(-time) or a particle moving in flat Euclidean 3+0-dimensional space(-time) but with a site-dependent mass. We show that, although both interpretations can be given to the same classical dynamics, they lead to different quantum theories. In fact, quantizing the classical theory à la Proca we find that the good interpretation, i.e. the one that approximates Maxwell theory, is the theory of a particle with site-dependent mass. Hence we find that, in contradiction with naive expectations, Quantum Geometric Optics cannot be identified with a theory of fields in curved space.

The above discussion leads us naturally to discuss the case of mechanical systems. In Sec. 4 we discuss the optical analogy of Classical Mechanics. We show that a complete parallelism can be established between Geometric Optics and “time-independent” Classical Mechanics, an analogy which can be carried throughout the quantization procedure, to the quantum theory. We show that Geometric Optics and the optical “image” of Classical Mechanics are described by reparametrization-invariant systems. Hence their quantum dynamics are tied to a Hamiltonian constraint. As a consequence, Classical Mechanics or Geometric Optics describes systems with Hamiltonian constraints whose good order is known, as well as the physical interpretation of the associated quantum theories. In Sec. 5 we discuss some implications of these results in the quantization of true

†In some literature on the topic, the procedure that we call “quantization” is referred to as “wavization” [see for instance Ref. [2] and references therein].
reparametrization-invariant systems, such as gravity theories.

Before pursuing the central aims of this paper we will describe briefly, in Sec. 2, the classical dynamics of (Geometric) Optics.

2 The classical dynamics of Optics.

The classical action of Optics, i.e., the action which gives the dynamics of the so-called Geometric Optics, is given by [1]:

\[ S = \int n(x) d s = \int n\sqrt{x^2}d \tau \]  

where \( \tau \) parametrizes the trajectory. In this action, \( n \), the refraction index, is physically identified with the quotient \( c/v \) where \( c \) (v) is the speed of light in the vacuum (medium). However, unlike the mechanical case, \( v \) (\( n \)) is not a degree of freedom but a datum which must be given in advance. Since \( v \) is the only reference to the physical time which appears in the action (1), it can be said that the action (1) corresponds to a frozen theory, i.e., a theory without time. The action (1) is invariant under reparametrization: \( \tau \rightarrow \tau(\xi) \), and thus, the study of its classical dynamics can proceed in two different ways: a) A non-reparametrization-invariant study which begins by fixing the parametrization of the trajectory and b) a reparametrization-invariant, or manifestly covariant, approach which does not require such a choice.

2.1 Non-explicitly covariant approach.

Let us choose as “time”, or preferred direction of motion, the co-ordinate \( z \). The action (1) takes the form

\[ S = \int L d z = \int d z n(x, y, z)\sqrt{1 + \dot{x}^2 + \dot{y}^2} \]  

The phase space is given by the co-ordinates \( x, y \) and the momenta

\[ P_x = \frac{\partial L}{\partial \dot{x}} = \frac{n^2 \dot{x}}{n\sqrt{1 + \dot{x}^2 + \dot{y}^2}} \]  

\[ P_y = \frac{\partial L}{\partial \dot{y}} = \frac{n^2 \dot{y}}{n\sqrt{1 + \dot{x}^2 + \dot{y}^2}} \]  

The Hamiltonian \( \mathcal{H} \) takes the form

\[ \mathcal{H} = P_x \dot{x} + P_y \dot{y} - L = -\sqrt{n^2(x, y, z) - P_x^2 - P_y^2} \]  

The equations of motion for any function \( F \) on the phase space are given, as usual, by
\[
\frac{dF}{dz} = \{F, \mathcal{H}\},
\]

where the Poisson bracket is determined by the symplectic form \( \omega \), which in the coordinates defined above has a Darboux form:

\[
\omega = d P_x \wedge dx + d P_y \wedge dy.
\]

### 2.2 Manifestly covariant description.

The action (1) has dimensions of length. In order to get an action with the adequate dimensions, it is convenient to multiply it by a factor \( \bar{h}/\lambda \) and take

\[
S = \frac{\bar{h}}{\lambda} \int n(x) ds = \frac{\bar{h}}{\lambda} \int n \sqrt{\dot{x}^2} d\tau
\]

Here \( \bar{h} \) is Plank’s constant and \( \lambda \) should be identified with the wavelength of light. The particular form of this factor clearly indicates that the theory we are studying describes only light with a definite wavelength or frequency. Hence, in the present theory, photons with different wavelengths will behave as different particles and will not interact at all with photons with a different wavelength. For the same reason, the interaction with the medium, as described by the present theory, will preserve the wavelength of light.

The action (3) essentially admits two different interpretations, which we will refer to in the sequel as first and second interpretations:

1) It can be interpreted as a theory of a particle with site-dependent mass \( \frac{\bar{h}}{\lambda} n \) evolving in a flat Euclidean 3+0-dimensional space(-time).

2) If we introduce in (8) the refraction index into the square root we get the action of a particle with mass \( \frac{\bar{h}}{\lambda} \) moving in an Euclidean 3+0 dimensional space(-time) with a conformally flat metric

\[
d s^2 = n^2 (d x^2 + d y^2 + d z^2).
\]

This analogy, together with the fact that in Electrodynamics in material media a tensor appears, the dielectric tensor \( \epsilon_{ij} \) having the form \( \epsilon_{ij} = n^2 \delta_{ij} \) for isotropic media, leads us to generalize and interpret Geometric Optics as the dynamics of a particle with constant mass \( \frac{\bar{h}}{\lambda} \) moving in a curved (Euclidean) 3+0 dimensional space(-time).

We shall study both interpretations at the same time by considering the action of a particle with site-dependent mass \( \frac{\bar{h}}{\lambda} m \) moving in a curved 3+0-dimensional space(-time) with metric \( N_{ij} \). Its action is

\[
S = \frac{\bar{h}}{\lambda} \int m ds = \frac{\bar{h}}{\lambda} \int d\tau m \sqrt{N_{ij} \dot{x}^i \dot{x}^j}
\]
The interpretation in 1) is recovered by putting $m = n$, $N_{ij} = \delta_{ij}$. The interpretation in 2) requires $m = 1$, $N_{ij} = \epsilon_{ij}^\prime$ or $N_{ij} = n^2 \delta_{ij}$ if the medium is isotropic.

Once the latter identifications have been made, everything proceeds as in the case of a massive particle in a $3+1$ dimensional curved space-time. The momenta are given by

$$p_i = \frac{\hbar}{\lambda} m \frac{N_{ij} \dot{x}^j}{\sqrt{N_{ij} \dot{x}^i \dot{x}^j}}.$$  \hspace{1cm} (11)

The invariance under reparametrization of the action (10) gives rise to a constraint:

$$p^2 = \frac{\hbar^2}{\lambda^2} m^2 \Leftrightarrow N^{ij} p_i p_j - \frac{\hbar^2}{\lambda^2} m^2 = 0$$ \hspace{1cm} (12)

It is interesting to point out that the constraint above completely describes the classical dynamics. In fact, following Landau [5] we can replace $p_i$ in (12) by $\frac{\partial S}{\partial x^i}$ and obtain a Hamilton-Jacobi-like equation

$$N^{ij} \partial_i S \partial_j S - \frac{\hbar^2}{\lambda^2} m^2 = 0,$$ \hspace{1cm} (13)

for which the general solution contains all the information about the classical dynamics of the system [4].

Obviously, equation (13) leads to the same dynamics irrespective of the interpretation one gives to the classical theory. In the next section we shall see that this is no longer the case in the quantum theory. Different interpretations of the same classical equations lead to different quantum theories. In the present case only the interpretation in 1) leads to the correct quantum theory.

3 Quantization.

In this section we quantize Geometric Optics in two different ways resorting to a massive Klein-Gordon field and a Proca field, respectively [the Dirac field is not considered here; the interested reader can find the relevant expressions in Ref. [6]]. The two different interpretations of the classical theory discussed above lead, as we shall see, to different quantum theories. We show that only one of these can be interpreted as the correct stationary Maxwell theory. As stated above we will deal with both interpretations at the same time by considering the more general case of a particle with site-dependent mass and moving in a curved space(-time). In this and next section, indices are raised and lowered with the metric $N_{ij}$.
3.1 Quantization as a scalar field.

As is well known, to quantize the system in (10) à la Klein-Gordon one introduces a complex scalar field \( \phi \) and makes use, in eq. (12), of the basic quantization rules: \( p_i \to -i \nabla_i \) to obtain for \( \phi \) the equation of motion

\[
-\hbar^2 \Box \phi - \frac{\hbar^2}{\lambda^2} m^2 \phi - \hbar^2 \alpha R \phi = 0
\]

or

\[
[\Box + \frac{m^2}{\lambda^2} + \alpha R] \phi = 0 .
\]

Here \( \alpha \) is an, in principle unknown, dimensionless constant; \( R \) is the scalar curvature, and \( \Box \) the Laplacian operator associated to the metric \( N_{ij} \),

\[
\Box \equiv \nabla^i \nabla_i = \frac{1}{\sqrt{N}} \partial_i \left( N^{ij} \sqrt{N} \partial_j \right) ,
\]

\( N \) being the determinant of the metric.

The equation of motion (15) can be obtained from the action

\[
S_{KG} = \int d^3 x \sqrt{N} \left[ -N^{ij} \partial_i \phi \partial_j \phi^* + \frac{m^2}{\lambda^2} \phi \phi^* + \alpha R \phi \phi^* \right] .
\]

Note that Plank’s constant has disappeared from the quantum equation of motion (15) for \( \phi \). Unlike the mechanical case, quantizing Geometric Optics does not require the introduction of the Plank constant but another parameter \( \lambda \) which apparently plays a quite different role. (In the following, we shall drop the parameter \( \hbar \) from the “mass” \( \frac{\hbar}{\lambda} m \)).

3.2 Quantization as a Proca field.

The action of a Proca field \( A^i \), with mass \( \frac{m}{\lambda} \), in a curved space with metric \( N_{ij} \) is given by (13):

\[
S_P = \int d^3 x \sqrt{N} \left[ -\frac{1}{4} F^{ij} F_{ij} + \frac{1}{2} \frac{m^2}{\lambda^2} A^i A_i + \frac{1}{2} \kappa R A^i A_i + \frac{1}{2} \gamma R_{ij} A^i A^j \right] .
\]

where

\[
F_{ij} = \partial_i A_j - \partial_j A_i = \nabla_i A_j - \nabla_j A_i ,
\]

and \( R_{ij}, R \) are respectively the Ricci tensor and the scalar curvature associated to the metric \( N_{ij} \). As in the scalar case, the dimensionless constants \( \kappa \) and \( \gamma \)
which appear in these terms are, in principle, undetermined. The equations of motion are easily obtained with the result

\[ \partial_i \sqrt{N} F^{ij} + \sqrt{N} \left( \frac{m^2}{\lambda^2} \delta^j_k + \kappa R \delta^j_k + \gamma R^j k \right) A^k = 0 \] (20)

or, what is equivalent,

\[ \nabla_i F^{ij} + \left( \frac{m^2}{\lambda^2} \delta^j_k + \kappa R \delta^j_k + \gamma R^j k \right) A^k = 0 \] (21)

Eq. (21) implies the following equations of motion for the basic field \( A^i \)

\[ \Box A^j - \nabla^j (\nabla_k A^k) - R^j k A^k + \left( \frac{m^2}{\lambda^2} \delta^j_k + \kappa R \delta^j_k + \gamma R^j k \right) A^k = 0 . \] (22)

Equation (20) implies in addition

\[ \partial_j (\sqrt{N} \left( \frac{m^2}{\lambda^2} \delta^j_k + \kappa R \delta^j_k + \gamma R^j k \right) A^k) = 0 \iff \nabla_j (\left( \frac{m^2}{\lambda^2} \delta^j_k + \kappa R \delta^j_k + \gamma R^j k \right) A^k) = 0 \] (23)

4 Physical interpretation.

In this section we shall compare the quantum theories constructed above with the Maxwell theory. We will see that only the first interpretation of the classical theory gives the correct quantum one, which coincides with the stationary Maxwell theory in a material medium. On the contrary, the interpretation of light rays as massive particles moving in a curved space does not lead to the correct wave theory.

Maxwell equations in material media without sources read [8]:

\[ \text{rot } E = -\frac{1}{c} \frac{\partial B}{\partial t} , \quad \text{div } B = 0 , \] (24)

\[ \text{div } D = 0 , \quad \text{rot } H = \frac{1}{c} \frac{\partial D}{\partial t} , \] (25)

which must be completed with the constitutive relations

\[ D^i = \sum_j \epsilon_{ij} E^j , \quad B^j = \sum_j \mu_{ij} H^j . \] (26)

In most material media the relevant constants are \( \epsilon_{ij} \) since \( \mu_{ij} \approx 1 \). This is the only case that we will consider in this paper.
The stationary equations are obtained by replacing \( \frac{\partial}{\partial t} \) everywhere with \( -\frac{i}{\lambda} \), and are given by:

\[
\frac{i}{\lambda} \mathbf{B} = \text{rot} \mathbf{E}, \quad \text{rot} \mathbf{H} = -\frac{i}{\lambda} \mathbf{D}
\]  

(27)

Let us consider the action

\[
S_{\text{Maxwell}} = \int \! d^4x \{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\},
\]

(28)

where indices are raised and lowered with the metric

\[
d s^2 = \frac{c^2}{n^2} d t^2 - d \mathbf{r}^2 = v^2 d t^2 - d \mathbf{r}^2.
\]

(29)

With the identifications

\[
F_{\mu\nu} = \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & -B_z & B_y \\
-E_y & B_z & 0 & -B_x \\
-E_z & -B_y & B_x & 0
\end{pmatrix},
\]

\[
F_{\mu\nu} = \begin{pmatrix}
0 & -D_x & -D_y & -D_z \\
D_x & 0 & -H_z & H_y \\
D_y & H_z & 0 & -H_x \\
D_z & -H_y & H_x & 0
\end{pmatrix},
\]

(30)

we get the correct Maxwell equations (24,25) (and constitutive relations (26)) in an isotropic medium with \( \epsilon = n^2 \) and \( \mu = 1 \).

The action in (28) differs from Maxwell action in a curved space-time in the volume element only. If we restrict the action in (28) to the stationary case, \( \frac{\partial^2}{\partial t^2} \equiv -\frac{1}{\lambda^2} \), and fix the gauge by putting \( A_0 = 0 \), we obtain the action in (25) with a flat metric \( N_{ij} = \delta_{ij} \), and a “mass” \( m = \frac{n}{\lambda} \). We see then that the first interpretation of the classical theory of light, when quantized, leads to the correct quantum theory which is the stationary Maxwell theory in a material medium.

The identities in (31) imply the identifications

\[
D_i = -\frac{i}{\lambda} n^2 A_i ,
\]

(31)

\[
E^i = -\frac{i}{\lambda} A_i ,
\]

(32)

\[
B^i = \frac{1}{2} \sum_{jk} \epsilon^{ijk} F_{jk} = (\text{rot} A)^i ,
\]

(33)

\[
H^i = \frac{1}{2} \sum_{jk} \epsilon^{ijk} F^{jk} ,
\]

(34)

which give, of course, the stationary Maxwell equations in an isotropic medium (27), together with the expected constitutive relations (\( \mu = 1 \)),
\[ D = n^2E, \quad B = H. \] (35)

Another proof of the equivalence above can be obtained by eliminating in (27) the magnetic field \( B \). We are led this way to a equation of motion for \( E \propto A \), that coincides with eq. (22) with a flat metric and mass \( m = n/\lambda \):

\[
\nabla^2 E^i - \partial_i (\partial_j E^j) + \frac{n^2}{\lambda^2} E^i = 0.\] (36)

4.1 Is Quantum Geometric Optics a theory of fields in curved space?

Let us now consider the second interpretation of Geometric Optics. Here \( m = 1 \) and \( N_{ij} \) is expected to be related to the dielectric tensor \( \epsilon_{ij} \) which, for isotropic media, is given by \( \epsilon_{ij} = n^2 \delta_{ij} \).

Let consider first the scalar case. The flat-metric interpretation gives for the scalar field \( \phi \) the equation of motion

\[
- \nabla^2 \phi - \frac{n^2}{\lambda^2} \phi = 0, \] (37)

equation which is known in the optical literature as Hemholtz’s equation. The curved-metric interpretation leads, in place, to the wave equation

\[
- \Box \phi - \frac{1}{\lambda^2} \phi - \alpha R \phi = 0. \] (38)

Let us restrict our attention to isotropic media, which is the only case that eq. (37) can take into account. The Ricci tensor and the scalar curvature for a metric of the form

\[
d s^2 = n^2 \delta_{ij} d x^i d x^j \] (39)

are given by

\[
R_{ij} = -\frac{1}{n} \partial_i \partial_j n + 2 \frac{\partial_i n \partial_j n}{n^2} - \delta_{ij} \frac{\partial_k \partial_k n}{n^2}, \] (40)

and

\[
R = \frac{1}{n^2} \left\{ -4 \frac{\partial_i \partial_i n}{n} + 2 \frac{\partial_i n \partial_i n}{n^2} \right\}. \] (41)

It it easy to check that if we make in (37) a change of field \( \phi = \sqrt{n} \phi \) we obtain for \( \phi \) the equation (38) with the specific value \(-\frac{1}{8}\) for \( \alpha \). Hence, for the scalar-field case, both interpretation of the classical theory lead to the same quantum theory. The interpretation with curved metric is more general since it can also take into account non-isotropic media. The value of \( \alpha \) is, of course, the one which
corresponds, for massless scalar fields, to a conformally invariant coupling with
the metric $\mathcal{L}$. This way of proceeding can serve as a guide to deal with the Proca
field.

The flat-metric equations of motion for the Proca field are (in the following
repeated indices are summed over except indicated otherwise):

$$
\partial_i F_{ij} + \frac{n^2}{\lambda^2} A_i = 0 \quad ; \quad \partial_i (n^2 A_i) = 0 .
$$
(42)

In the isotropic case, the equations of motion for the curved-space Proca field
are ($G_{ij} = \partial_i C_j - \partial_j C_i$):

$$
0 = \partial_i G_{ij} - \frac{\partial_i n}{n} G_{ij} + \frac{n^2}{\lambda^2} C_j
+ \kappa \left\{ -\frac{1}{n} \partial_i \partial_i n + 2 \frac{\partial_i n \partial_j n}{n^2} \right\} C_j
+ \gamma \left\{ -\frac{1}{n} \partial_i \partial_j n + 2 \frac{\partial_i n \partial_j n}{n^2} - \delta_{ij} \frac{\partial_k \partial_k n}{n} \right\} C_i .
$$
(43)

Let us make in (42) a change of fields $A_i = n^s C_i$. We obtain

$$
0 = \partial_i G_{ij} + s \frac{\partial_i n}{n} G_{ij} + \frac{n^2}{\lambda^2} C_j
+ s \left\{ \frac{\partial_i n}{n} C_j - \frac{\partial_j n}{n} C_i \right\}
+ s (s - 1) \frac{\partial_i n}{n} \left\{ \frac{\partial_j n}{n} C_j - \frac{\partial_j n}{n} C_i \right\} + s \left\{ \frac{\partial_i \partial_j n}{n} C_j - \frac{\partial_i \partial_j n}{n} C_i \right\} ,
$$
(44)

and

$$
0 = \partial_i C_i + (s + 2) \frac{\partial_i n}{n} C_i .
$$
(45)

It is easy to check that no choice of $\kappa$, $\gamma$ and $s$ can make the wave equations (K) and (44) to coincide.

In conclusion, we can state that Quantum Geometric Optics is not a theory
of fields in curved space.

We should point out here that the conclusion above is not in contradiction
with the result obtained by Plebanski [9]. He showed that Maxwell theory in a
curved space-time can be identified with the same theory in a flat space-time but
evolving in a material medium. The latter identification requires, nevertheless,
constitutive relations that are different from the ones we have been dealing with
in this paper: $D^i = \epsilon_i^j E^j$ and $H = B$. 

5 The optical analogy of Classical Mechanics.

As is well known [4], the trajectories with fixed energy $E$ for a time-independent mechanical system with kinetic energy $T = \frac{1}{2} M_{ij}(q) \dot{q}^i \dot{q}^j$ and potential $V$ can be obtained from the action

$$S = 2 \int \sqrt{E - V} \sqrt{T} \mathrm{d} \tau .$$  \hspace{1cm} (46)

The parameter $\tau$ has no physical relevance, since the action in (46) is reparametrization invariant. This principle, which we shall refer to as the Maupertuis principle, is the mechanical analogue of the Fermat principle. Here $\sqrt{E - V}$ plays the role of the refraction index.

Let us consider a mechanical system for which $M_{ij} = m \delta_{ij}$ with $m$ constant (the mass of the particle). We shall see that canonical quantization of the action in (46) gives the right time-independent Schrödinger equation.

The canonical momenta are given by

$$p_i = \frac{\sqrt{(E - V)}}{\sqrt{T}} m \dot{q}^i .$$  \hspace{1cm} (47)

The constraint associated with the reparametrization invariance of the action is

$$\sum_i p_i p_i = 2m(E - V) .$$  \hspace{1cm} (48)

If we apply the basic quantization rules $p_i \rightarrow -i\hbar \frac{\partial}{\partial x^i}$ to eq. (48) we find the stationary Schrödinger equation

$$-\hbar^2 \nabla^2 \Psi - 2m(E - V)\Psi = 0 .$$  \hspace{1cm} (49)

We see then that the exact analogue of the time-independent Schrödinger equation for Optics is eq. (13), the Hemboltz equation, which can be written in another way:

$$-\lambda^2 \nabla^2 \phi - n^2 \phi = 0 .$$  \hspace{1cm} (50)

Since $n^2$ is always positive, the solutions of eq. (50) belong to the continuous spectrum, that is to say, they are scattering solutions and hence not normalizable.

There is a simple procedure to go from the usual Hamilton Principle to the Maupertuis Principle and vice versa. Let us start with the action in the form that is required by the Hamilton Principle:

$$S = \int \mathrm{d} t \left\{ \frac{1}{2} M_{ij}(q) \dot{q}^i \dot{q}^j - (V - E) \right\} ,$$  \hspace{1cm} (51)
where we have added a convenient total derivative \( \frac{d}{dt}E t \) to the usual Lagrangian. Let us now introduce an arbitrary parameter \( \tau \) to describe the trajectories of the system. We have \( t = t(\tau) \), \( \mathrm{d}t = \mathrm{d}t' \mathrm{d}\tau \) (prime indicates derivative with respect to the parameter \( \tau \)). The action (51) takes the form

\[
S = \int \mathrm{d}\tau \left\{ \frac{1}{2} M_{ij} q'^i q'^j - t'(V - E) \right\} .
\tag{52}
\]

In order to obtain a reparametrization-invariant system we now hide the relationship between \( \tau \) and \( t \) by replacing \( t' \) with a new quantity, the vielbein or einvein \( e \). We obtain

\[
S = \int \mathrm{d}\tau \left\{ \frac{1}{2} M_{ij} q'^i q'^j e - e(V - E) \right\} .
\tag{53}
\]

The physical meaning of \( e \) is that of volume (length), or metric, along the trajectories: \( \mathrm{d}t^2 = e^2 \mathrm{d}\tau^2 \).

The action in (53) is, in fact, equivalent to the action that appears in the formulation of the Maupertuis principle. The quantity \( e \) plays the role of a Lagrangian multiplier, and can be eliminated by using its equation of motion:

\[
0 = \frac{\delta L}{\delta e} = -\frac{T}{e^2} - (V - E) \Rightarrow e = \sqrt{\frac{T}{E - V}}
\tag{54}
\]

If we replace in (53) \( e \) by its value in (54) we obtain the action in (46).

6 The case of true reparametrization-invariant systems.

The procedure sketched in the previous section to obtain a reparametrization-invariant system from a mechanical one can be applied the other way round. However, the procedure when applied in this direction, is not well defined: it does not lead to a unique result. The reason for this ambiguity is that the same reparametrization-invariant action can be obtained from different mechanical ones. This problem can be traced back to the fact that in the reparametrization-invariant action (44) there is no way of distinguishing between kinetic and potential energy.

In order to illustrate these points, let us consider, very briefly, 2-dimensional induced gravity which is a true reparametrization-invariant system. The action, after being restricted to the spatially homogeneous minisuperspace, reads (10):

\[
S = \int \mathrm{d}\tau \left[ \frac{1}{2} a \Phi'^2 + 2 \frac{\Lambda' \Phi'}{e} + \frac{\Lambda}{2} a e \right] .
\tag{55}
\]
Nevertheless, as stated above, this action is not unique: the same dynamics can be obtained from other actions related to (55) by a redefinition of \( e \). If we get rid of the Lagrange multiplier \( e \) we obtain

\[
S = 2 \int d\tau \sqrt{\frac{\Lambda}{2} a\sqrt{\frac{1}{2} a\Phi'^2 + 2a'\Phi'}}. \tag{56}
\]

The Hamiltonian constraint obtained from (55) can be written as

\[
-\frac{1}{8} ap_a^2 + \frac{1}{2} p_a p_\Phi - \frac{\Lambda}{2} a = 0. \tag{57}
\]

However, since there is no preferred choice of \( e \) here, the Hamiltonian constraint can be written in different forms, for instance,

\[
-\frac{1}{8} (ap_a)^2 + \frac{1}{2} p_ap_\Phi - \frac{\Lambda}{2} a^2 = 0. \tag{58}
\]

In fact, there are physical reasons which make this latter form of the constraint preferable \([10, 11]\).

For Geometric Optics there are also a series of actions involving a Lagrange multiplier (\( einvein \)) \( e \) which are equivalent to the Fermat action (1). Here, as it happens for mechanical systems, there is a preferred choice of parametrization, \( \tau = x^0 \), begin \( x^0 \) the physical time, which obeys

\[
\dot{x}^2 = \frac{1}{n^2}. \tag{59}
\]

This preferred choice of parametrization singularizes, among all possible actions, the one for which the interval \( ds^2 = e^2 d\tau^2 \) has the meaning of physical time:

\[
d s^2 \equiv d (x^0)^2. \tag{60}
\]

This preferred action is

\[
S = \frac{\hbar}{\lambda} \int d\tau \left\{ n^2 \frac{\dot{x}^2}{e} + e \right\}. \tag{61}
\]

However, the existence of this singularized action does not help us since it does not distinguish between the two interpretation of the classical theory studied above. In fact, it point to the wrong direction since it seems to indicate that the good interpretation is the curved-metric one.

To summarize, we can say that the existence of a true time enables us to distinguish kinetic energy from potential energy in the classical action. This

\*Note that in the literature of Quantum Cosmology what we call quantum Hamiltonian constraint is named Wheeler-DeWitt equation.
fact provides a preferred form of the classical Hamiltonian constraint which, in general, makes the ordering problem of its quantum counterpart, the Schrödinger equation, less poisonous, even though the problem is not completely solved.

7 Final remarks.

The structure of the present paper have been somewhat circular: we began by considering Geometric Optics from the point of view of Mechanics and found the analogy fruitful (Sections 2, 3 and 4). Then, in Sec. 5, we considered Mechanics from the point of view of Geometric Optics and again found the analogy fruitful. As a result, we have studied in depth the classical and quantum dynamics of Geometric Optics along with their relationship with Mechanics. We have widely shown that there is a close analogy between Optics and Mechanics and that between them, at a certain level, a complete isomorphism can be established. This explains the success of mechanical techniques when applied to Optics [2, 3].

The present paper illustrates both the power and the weakness of the quantum theory in its present form. For instance, we can obtain the stationary Maxwell theory from the Fermat principle. However, the Fermat principle, together with the canonical quantization procedure in its present form, does not indicate clearly which is the correct quantum theory. The same situation appears in connection with the Maupertuis principle and the time-independent Schrödinger equation. We need some information, a great amount of information indeed, that is provided neither by the classical theory nor the quantization procedure. These examples clearly illustrate the difficulties which appear when quantizing true reparametrization-invariant systems such as gravity theories [12]. In fact, the quantization procedure provides neither the correct equations of the quantum theory nor the physical interpretation we should give to it if we were able somehow to obtain these equations.

Acknowledgements. M. Navarro is grateful to the Spanish MEC for a postdoctoral FPU grant. J. Guerrero thanks the Spanish MEC for a FPU grant. M. Navarro is grateful to J. Navarro-Salas for very useful comments and suggestions.

References

[1] M. Born and E. Wolf, Principles of Optics, Pergamon, 1970.

[2] T. Sekiguchi and K.B. Wolf, Am.J.Phys. 55(9)1987. K.B. Wolf, J. Math. Phys. 33(7) 1992.
[3] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, 1991.

[4] H. Goldstein, *Classical Mechanics* 2nd edition, Addison-Wesley, Reading, Massachusetts, U.S.A.

[5] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press, Oxford, 1975.

[6] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, 1992.

[7] L.H. Ryder, *Quantum Field Theory*, Cambridge University Press, 1985.

[8] L.D. Landau and E.M. Lifshitz, *Electrodinámica de los Medios Continuos*, Reverté, Barcelona 1975.

[9] J. Plebanski, *Phys. Rev.* 118, 5 (1990)1396 (See also B. Mashhoon, *Phys. Rev. D*, 8, (1973)4297).

[10] J. Navarro-Salas, M. Navarro and V. Aldaya, *Phys. Lett.* B318 (1993).

[11] C. Teitelboim, in *Quantum Theory of Gravity*, ed. S. Christensen (Adam Hilger, Bristol, 1984) p. 327.

[12] C.J. Isham, *Conceptual and Geometrical Problems in Quantum Gravity*, Imperial-TP/90-91/14.