Scaling, Self-similarity and Superposition

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Abstract

A novel procedure for the nonlinear superposition of two self-similar solutions of the heat conduction equation with power-law nonlinearity is introduced. It is shown how the boundary conditions of the superposed state conflicts with self-similarity, rendering the nonlinearly superposed state to be a non-exact solution. It is argued that the nonlinearity couples with the presence of the scale so that the superposition in the linear case can give an exact solution.

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I. INTRODUCTION

Phenomena exhibiting self-similarity at different time and/or spatial scales are ubiquitous in nature. Self-similarity techniques exploit such symmetries for reducing the number of variables for describing the system. This also implies the idea in dimensional analysis. Such techniques had much been explored in fluid phenomena and, more recently, in optics. Self-similar solutions can only be constructed in the absence of scales having the dimensions of independent variables. As a result they are always endowed with extreme boundary conditions (BCs) representing the intermediate asymptotic behavior of a system away from the initial conditions and boundaries.

A PDE may have more than one self-similar solution each corresponding to a specific BC. If the underlying equation is linear, its self-similar solutions can be superposed to obtain a solution satisfying a realistic boundary condition. Superposition of the solutions in the nonlinear case is not possible, but as is shown here, there is still an approximate symmetry to be exploited, leading to a procedure for nonlinear superposition of self-similar solutions.

II. HEAT EQUATION AND SELF-SIMILAR SOLUTIONS

As an example for illustrating the procedure, consider the nonlinear diffusion equation

\[
\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta^{n+1}}{\partial \xi^2},
\]

which is the dimensionless form of the heat conduction equation with power-law nonlinearity. This equation is also known as the porous medium equation describing the flow of an isentropic gas through a porous medium. The equation is nonlinear for \( n \neq 0 \) and we call \( n \) the nonlinearity parameter. This equation has two well-known self-similar solutions. The first solution,

\[
\theta(\xi, \tau) = \tau^{-\frac{1}{n+1}} \left[ \xi \tau^{-\frac{1}{2(n+1)}} \right]^{\frac{n}{n+1}} \left[ 1 - k_n \left( \xi \tau^{-\frac{1}{2(n+1)}} \right)^{\frac{n+2}{2(n+1)}} \right]^{1/n},
\]

satisfies the Dirichlet BC \( \theta(0, \tau) = 0 \), and the second solution,

\[
\theta(\xi, \tau) = \tau^{-\frac{1}{n+2}} \left[ 1 - k_n \left( \xi \tau^{-\frac{1}{n+2}} \right)^2 \right]^{1/n},
\]

satisfies the Neumann BC \( \theta'(0, \tau) = 0 \) where \( \theta' \equiv d\theta/d\xi \). In both these solutions

\[
k_n = \frac{n}{2(n+1)(n+2)}.
\]
The equation (1) can equivalently be replaced by two equations in conservative form. The first one is
\[
\frac{\partial \theta}{\partial \tau} + \frac{\partial \Phi}{\partial \xi} = 0 \tag{5}
\]
where
\[
\Phi(\xi, \tau) = -\frac{\partial \theta^{n+1}}{\partial \xi} \tag{6}
\]
is the Fick’s law in dimensionless form, and the second one is
\[
\frac{\partial (\theta \xi)}{\partial \tau} + \frac{\partial \Gamma}{\partial \xi} = 0 \tag{7}
\]
where
\[
\Gamma(\xi, \tau) = \Phi(\xi, \tau)\xi + \theta^{n+1} \tag{8}
\]
It is clear from the Eqns. (5) and (7) that, in the steady state, \( \Phi \) and \( \Gamma \) are the integration constants and solving equation (8) for \( \theta \) gives
\[
\theta(\xi) = \left( \Gamma - \Phi \xi \right)^{\frac{1}{n+1}}, \tag{9}
\]
which is the general solution in terms of \( \Phi \) and \( \Gamma \).

How do the time-dependent solutions given in Eqns. (2) and (3) look when written in terms of \( \Gamma(\tau) \equiv -\Gamma(0, \tau) \) and \( \Phi(\tau) \equiv -\Phi(0, \tau) \) corresponding to the time dependent case? Here the negative sign is to make the flux leaving the system from the left boundary positive. Using the divergence theorem for equation (5), it is possible to see from the first solution given in equation (2) that
\[
\Phi(\tau) = \frac{d}{d\tau} \int_{0}^{\infty} \theta d\xi \tag{10}
\]
decays as a power-law \( \Phi(\tau) = -\tau^{-\alpha} \) where
\[
\alpha = 1 + \frac{1}{2(n + 1)} \tag{11}
\]
while
\[
\Gamma(\tau) = \frac{d}{d\tau} \int_{0}^{\infty} \xi \theta d\xi \tag{12}
\]
vanishes (\( \Gamma(\tau) = 0 \)).

Similarly, it is possible to see from the second solution, given in equation (3), that \( \Gamma(\tau) = \tau^{-\beta} \) where
\[
\beta = 1 - \frac{1}{n + 2} \tag{13}
\]
while \( \Phi(\tau) = 0 \).
III. THE SOLUTIONS IN TERMS OF FLUXES

We write the first solution given in equation (2) in terms of \( \Phi(\tau) \) as

\[
\theta = [-\xi \Phi(\tau)]^{1/n} \left\{ 1 - k_n \frac{\xi^2}{\tau} [-\xi \Phi(\tau)]^{1/n - 1} \right\}^{1/n}.
\] (14)

It is possible to show that this satisfies equation (11) even if \( \Phi(\tau) \) is multiplied with a constant \( \Phi_0 \) so that

\[
\Phi(\tau) = -\Phi_0 \tau^{-\alpha}.
\] (15)

The second solution given in equation (3), can be written in terms of \( \Gamma(\tau) \) as

\[
\theta = [\Gamma(\tau)]^{1/n} \left\{ 1 - k_n \frac{\xi^2}{\tau} [\Gamma(\tau)]^{1/n - 1} \right\}^{1/n}
\] (16)

and it is possible to show that this satisfies equation (11) even if \( \Gamma(\tau) \) is defined as

\[
\Gamma(\tau) = \Gamma_0 \tau^{-\beta}
\] (17)

where \( \Gamma_0 \) is a constant. Note that there is a “duality” \( \Gamma \leftrightarrow -\xi \Phi \) between the two solutions (14) and (16).

The solutions for the linear case \( (n = 0) \), using \( \lim_{n \to 0} (1 + An)^{1/n} = e^A \), can be written, in terms of \( \Gamma \) and \( \Phi \), as

\[
\theta(\xi, \tau) = -\xi \Phi(\tau) e^{-\xi^2/4\tau}
\] (18)

where \( \Phi = -\Phi_0 \tau^{-3/2} \) and

\[
\theta(\xi, \tau) = \Gamma(\tau) e^{-\xi^2/4\tau}
\] (19)

where \( \Gamma = \Gamma_0 \tau^{-1/2} \), respectively. As the diffusion equation (11) is linear for \( n = 0 \), these two can be superposed to give

\[
\theta(\xi, \tau) = [\Gamma(\tau) - \xi \Phi(\tau)] e^{-\xi^2/4\tau},
\] (20)

and this satisfies the boundary condition \( \theta(0, \tau) = \Gamma(\tau) \). Note that the steady state solution given in Eqn.(9) for the linear case \( (n = 0) \) becomes \( \theta(\xi) = \Gamma - \xi \Phi \). The term in the square brackets in Eqn.(20) carries a form similar to the steady state solution but \( \Gamma \) and \( \Phi \) are time-dependent in the former case.

In the linear case multiplying a solution with any constant can be absorbed into the constants \( \Phi_0 \) or \( \Gamma_0 \). In the nonlinear case multiplying the solution with a constant does not
give a solution, but it is possible to gauge $\Phi_0$ or $\Gamma_0$ to obtain the similar effect. In other words, multiplying the flux with a constant in the general case reduces to multiplying the solution with a constant.

IV. NONLINEAR SUPERPOSITION

The self-similar solutions given by equations (14) and (16) correspond to extreme boundary conditions: According to the former, there is a sink at $\xi = 0$, such that all the heat carried to this boundary is totally absorbed. The latter solution describes the case in which there is a perfect insulator at $\xi = 0$ such that heat cannot leave the system from this boundary. In the steady state, the solution of which is given in equation (9), any ratio between the integration constants $\Gamma$ and $\Phi$ is possible allowing for intermediate BCs. This is not the case with the self-similar solutions which can only be constructed in the absence of a scale: If $\theta$ and $\theta'$ are both finite at the boundary, then $l \sim \theta/\theta'$ is a length scale associated with the system. It is the aim of this paper to find the time dependent version of the general steady state solution given in equation (9) even though it may not be an exact solution.

The self-similar solutions given in equations (14) and (16) appear to be the time-dependent versions of the steady state solution given in equation (9) with $\Gamma = 0$ and $\Phi = 0$, respectively. We have written the solutions given by equations (14) and (16) in a “dual” form suggesting a “nonlinear superposition” in which we add up not the solutions themselves but $\Gamma(\tau)$ and $-\xi \Phi(\tau)$ terms in separate corresponding parts of the solutions. This procedure gives

$$\theta(\xi, \tau) = [\Gamma(\tau) - \xi \Phi(\tau)]^{\frac{1}{n+1}} \left( 1 - k_n \frac{\xi^2}{\tau} [\Gamma(\tau) - \xi \Phi(\tau)]^{\frac{1}{n+1} - 1} \right)^{1/n} \quad (21)$$

which reduces to equations (14) and (16) for $\Gamma_0 = 0$ and $\Phi_0 = 0$, respectively. This expression is endowed with the BC $\theta(0, \tau) = [\Gamma(\tau)]^{1/(n+1)}$ which can describe cases in which a fraction of the heat is absorbed at the boundary.

The expression given in equation (21) cannot satisfy the diffusion equation (1) for a general value of $n$ but is always a good approximate solution. In Figure 1 we compare the self-similar solutions with the numerical solution of the diffusion equation (1) for two different BCs. In both cases we have taken $n = 7/3$ and initiated the numerical solution from a Gaussian temperature distribution. The analytical solutions are time shifted as $\tau \to \tau + 1$ exploiting the symmetry of the diffusion equation (1) under this transformation. This allows
FIG. 1: Evolution of a Gaussian heat distribution on a semi infinite bar described by the diffusion equation (11), with snapshots taken at $\tau = 0, \tau = 3, \tau = 9, \tau = 27$ and $\tau = 81$ where $\tau$ is time in units of diffusive time scale. Solid lines show the analytical solutions and dashed lines with data points (corresponding to the numerical grids) show the numerical solutions. Left panel corresponds to the case where we employed the Neumann boundary condition $\theta'(0, \tau) = 0$ and is compared with the exact solution given in equation (16) with $\tau \rightarrow \tau + 1$. Right panel corresponds to the case $\theta(0, \tau) = [\Gamma(\tau)]^{1/(n+1)}$ and is compared with the approximate solution given in equation (21).

us to evaluate the analytical expression at $t = 0$. The left panel shows the case with $\theta'(0, \tau) = 0$ where the exact solution given in equation (16) is compared with the numerical solution. It is seen that it takes about $\sim 10$ diffusive time scales for the numerical solution to forget its initial configuration and to settle onto the self-similar solution. The right panel shows a similar comparison for the approximate analytical solution given in equation (21) for $\Gamma_0 = 0.1, \Phi_0 = 1.0$. We see that the approximate solution is also remarkably accurate after $\sim 10$ diffusive time scales. The analytical approximate solution is even more successful if $n < 1$. 

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V. DISCUSSION

We have constructed an approximate solution of the nonlinear heat equation with power-law nonlinearity satisfying a more general BC, by using a “nonlinear superposition” of two self-similar solutions endowed with Dirichlet and Neumann BCs. We can not expect to find an exact solution for such a general BC introduces a length scale into the problem, thus, destroying the very condition for self-similarity. Requiring equation (21) be a solution of equation (1) reduces to the requirement that

$$
\left( \frac{\tau \Phi'}{\Phi} + \alpha \right) \xi^2 \Phi^2 + \left( \frac{\tau \Gamma'}{\Gamma} + \frac{\tau \Phi'}{\Phi} + 2 \right) \xi \Gamma \Phi + \left( \frac{\tau \Gamma'}{\Gamma} + \beta \right) \Gamma^2
$$

vanishes. The first and the third terms vanish by equations (15) and (17), respectively. The second term, noting that \(2 - \alpha - \beta = k_n\), does not vanish but simplify to \(k_n \xi \Gamma \Phi\). This vanishes for either \(\Gamma_0 = 0\) or \(\Phi_0 = 0\) corresponding to the well known self-similar solutions given in Eqns.(14) and (16). It also vanishes for the linear case, \(n = 0\), as \(k_0 = 0\). The expression given in equation (21) does not satisfy the diffusion equation (1) in general, but it is a very accurate approximate self-similar solution of it.

The procedure can be applied to other second order nonlinear PDEs as long as they have two self-similar solutions and the equation can be written in terms of two PDEs in conservative form. The application of this procedure to the viscous evolution of a thin accretion disk in the gravitational field of a central star, a ubiquitous phenomena in astrophysics [11], will be published elsewhere later.

Equation (22) vanishes when the nonlinearity parameter, \(n\), vanishes (note that \(k_0 = 0\)). As the “nonlinear superposition” technique we employed here reduces to ordinary summation for \(n = 0\), this gives a way of seeing why summing up two solutions in the linear case yields a solution even when a scale is present: the nonlinearity term \(n\) couples with the finite scale in the problem and vanishing of either yields a solution. It is interesting to see that superposing two solutions by addition is a special case of a more general procedure employed in reaching equation (21) from the solutions given by equations (14) and (16) although this general procedure does not necessarily give an exact solution.
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