Wilson Line Inflation

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ABSTRACT: We present a general set-up for inflation in string theory where the inflaton field corresponds to Wilson lines in compact space in the presence of magnetic fluxes. T-dualities and limits on the value of the magnetic fluxes relate this system to the standard D-brane inflation scenarios, such as brane-antibrane inflation, D3/D7 brane inflation and different configurations of branes at angles. This can then be seen as a generalised approach to inflation from open string modes. Inflation ends when the Wilson lines achieve a critical value and an open string mode becomes tachyonic. Then hybrid-like inflation, including its cosmic string remnants, is realized in string theory beyond the brane annihilation picture. Our formalism can be incorporated within flux-induced moduli stabilisation mechanisms in type IIB strings. Also, contrary to the standard D-brane separation, Wilson lines can be considered in heterotic string models. We provide explicit examples to illustrate similarities and differences of our mechanism to D-brane inflation. In particular we present an example in which the η problem present in brane inflation models is absent in our case. We have examples with both blue and red tilted spectral index and remnant cosmic string tension \( G\mu \lesssim 10^{-7} \).

KEYWORDS: Strings, Cosmology, Inflation
1. Introduction

There are several classes of moduli fields in string compactifications. Closed string moduli include the geometric and dilaton fields, with the geometric ones separated into the deformations of the Kähler structure and deformations of the complex structure for Calabi-Yau and related compactifications. Open string moduli include the location of branes as well as Wilson lines which correspond to background values of gauge fields, usually coming from the open string sector of the type I/II string theory. In heterotic string however, Wilson lines are closed string modes.
Most of these fields have been proposed as inflatons in terms of explicit mechanisms to realize cosmological inflation in string theory. There are now concrete models for which the inflaton is a Kähler modulus [1, 3] and several scenarios in which the modulus is a brane separation [4, 5, 11, 6, 10, 13]. A general property of brane inflation models is that inflation terminates by means of a tachyon condensation mechanism similar to hybrid inflation. This open string tachyon itself has been also proposed to be the inflaton field in other scenarios [14]. Each of these scenarios has distinct physical implications which can be put to test by observations, regarding remnants of inflation such as cosmic strings, the value of the spectral index, etc.

Obtaining several realizations of inflation within string theory is a major achievement of the past five years. It would be highly desirable to have a general set-up that includes at least general classes of scenarios and try to extract as much model independent predictions. At the moment we are still at the stage of exploring the different possible realizations of inflation within string theory. For instance, there is not yet a concrete proposal for having the complex structure moduli as inflatons, something that could be desirable. Also Wilson lines have not been yet used as inflatons in an explicit string construction¹. The purpose of this paper is to explore this possibility.

Wilson lines correspond to background values of gauge fields that can be turned on in spaces of non-trivial homotopy. They provide most of the moduli in toroidal compactifications of the heterotic string parametrising the Narain lattices [17]. Discrete and continuous Wilson lines have also been used to break the gauge symmetries in phenomenological attempts to realistic models [18, 19, 20]. Furthermore, in the context of D-branes, Wilson lines have been playing a crucial role to understand dualities in type I/II strings [25]. A T-duality transforms branes at angles in such a way that the angles are mapped to magnetic fluxes and D-brane separations are mapped to Wilson lines in the dual model. Since brane separations have been successfully used as inflatons, it is natural to study the Wilson lines as inflatons. Furthermore, this formalism is the one that can naturally extend to the heterotic case in which the D-brane separation has no direct analogue.

We present our results as follows. Section 2 is dedicated to introduce the basic ideas and derive, in the case of toroidal compactifications, the string amplitude induced when magnetic fields and Wilson lines are turned on. We also comment on the fact that all potentials used in the literature as starting points for brane/antibrane inflation, D3/D7 inflation and D-branes at angles can be obtained from the amplitude we consider after performing T-dualities in different directions, as well as taking limits on the values of the magnetic fields. This means that all toroidal D-brane inflationary configurations can be obtained starting only with D9 branes with magnetic

¹ Notice however that the authors of [21] computed loop corrections to a variation of the KKLMMT model in terms of Wilson lines. Also, the possibility of using Wilson lines as inflatons was considered in [14] in a five dimensional effective field theory setup.
fluxes and Wilson lines in Type I string theory. The next section is dedicated to explore the Wilson line potential extracted from this amplitude for inflation and a concrete model of inflation is presented. Section 4 discusses the supergravity description of this set-up and compares with the KKLMMT approach to brane antibrane inflation in the context of moduli stabilization. We present our conclusions in section 5. We have an appendix explaining in detail the duality between Wilson lines and brane separation as well as the structure of Wilson lines in general. A further appendix discusses an explicit model in which the dilaton, complex structure and the location of D7 branes are fixed by RR and NS-NS fluxes but Wilson lines remain flat, leaving them as the right candidates for the inflaton field.

2. Wilson Lines and String Amplitudes

The aim of this section is to show how the vacuum energy generated in a supersymmetry breaking system of branes can have a dependence on Wilson line degrees of freedom generated in the world-volume of the branes. For the sake of clarity, let us stick to a simple configuration that illustrates the general idea. Consider IIB string theory compactified on a $T^6$, with a set of D7-branes filling the four dimensional space-time and wrapping the same toroidal four cycle in this $T^6$. The $T^6$ is constructed as $\mathbb{R}^6/\Gamma$, with $\Gamma$ being the lattice generated by six vectors $\{e_i\}$, $i = 1, ..., 6$. One can use affine parameters along these axes as $x^i, \in (-\pi, \pi)$, and we will set the metric so that the length of each lattice vector $e_i$ is $2\pi R_i$. To begin with we will consider the torus to be factorisable as a product of square tori, but later on we will generalise this setup.

In the absence of further orbifold/orientifold projections, this system preserves 16 supercharges and thus it leads to $\mathcal{N} = 4$ supersymmetry when reduced to four dimensions. The existence of non-trivial 1-cycles in the $T^4$ wrapped by the D-branes leads to the presence of Wilson line degrees of freedom, four for each brane, that show themselves as the lowest components of chiral multiplets in the reduced theory. These scalar fields have flat potentials and thus they can take an arbitrary vev consistently with supersymmetry, but with the consequence that these vevs can lead to rank-preserving adjoint gauge symmetry breaking, as explained in modern textbooks (see e.g. [25]).

\textsuperscript{2}There are other scalar degrees of freedom in the lower dimensional theory, apart from these Wilson lines and the position of the branes in the $T^2$ transverse to their worldvolume, corresponding to strings going from one brane to a different one. A vev for these scalar fields triggers a rank-reducing bifundamental gauge symmetry breaking, and is the field theory counterpart of the geometrical recombination of the system of branes. The only place where these fields will play a role in the following is in the symmetry breaking putting an end to inflation in some of the models, as it will be explained below, and thus one can consider their vevs to be set to zero during most of what follows.
This kind of flat potentials are known to be lifted in supersymmetry-breaking situations. This is indeed the case here. As we will see, the presence of some source of supersymmetry breaking, such as a non-zero magnetic field in the world-volume of one of the branes, is able to generate a potential for these previously shift-symmetric Wilson lines degrees of freedom$^3$.

For simplicity, let us restrict ourselves to a system with just two D7 branes, wrapping the same $T^4$ in the $T^6$. Let the coordinates of the $T^6$ be given by $\{x^i\}$, $i = 1, ..., 6$, and assume the pair of D7-branes to be wrapping the second and third torus ($i = 3, ..., 6$), being point-like (and one on top of each other) in the first one. Consider turning on a Wilson line along one of the directions of the $T^4$, say $x^5$, in only one of the branes$^4$ (the extension to a more general Wilson line is straightforward):

$$A = \frac{\lambda}{2\pi R_5} dx^5. \quad (2.1)$$

with $\lambda \in (-\pi, \pi)$. The presence of this Wilson line will break the gauge group $U(2) \rightarrow U(1)$. As stressed, in the absence of any source of supersymmetry breaking one has a flat potential for $\lambda$. A way of lifting this flat potential is to add a source of supersymmetry breaking, such as magnetic or B-field in the world-volume of the brane. The addition of magnetic field or the presence of B-field in the world-volume of the brane leads to a FI term in the four dimensional theory

$$\xi \sim \int_\Sigma J \wedge (B + 2\pi \alpha' F), \quad (2.2)$$

where $\Sigma$ is the four-cycle wrapped by the D-brane, $J$ is the Kähler form of the compactification space, $B$ is the pullback of the ambient B-field on $\Sigma$ and $F$ is the magnetic field in the world-volume of the D-brane. This FI term will generically break supersymmetry, and the system will try to evolve towards its restoration, generating non-trivial potentials for the fields involved.

$^3$This shift-symmetry is of course related to the gauge invariance in the world-volume of the D7 and consequently the potential will break this gauge invariance. The point here is that the potential between the first brane (call it $a$) and the second one (named $b$) will depend on the difference of Wilson lines, $\lambda_a - \lambda_b$, but not on the sum of them. When $\lambda_a - \lambda_b$ reaches some particular value (given in (2.12)), a tachyon with charge $(+1a, -1b)$ develops, leading the system to a final configuration in which the $U(1)$ gauge group with charge $Q_a - Q_b$ is broken. Since the potential is independent of $\lambda_a + \lambda_b$, the corresponding gauge symmetry associated to the group with charge $Q_a + Q_b$ is preserved all along the process.

$^4$So that $U_\gamma = e^{i\lambda}$. In the following we will use the term Wilson line to denote both $U_\gamma$ and $A_\gamma = (\lambda/L) d\sigma$, with $\sigma$ being the affine parameter parametrising the curve $\gamma$, $L$ the length of $\gamma$ and $\lambda \in (-\pi, \pi)$. Moreover, from the four dimensional point of view, $A_\gamma$ corresponds just to the zero mode of the field we will generically denote as Wilson line degree of freedom. We hope this abuse of language will not confuse the reader.
2.1 The interaction potential

Consider a magnetic field in the world volume of one of the D7-branes of the form

\[ F = \frac{2\pi m}{A_2} dx^3 \wedge dx^4 \]  

(2.3)

with \( m \in \mathbb{Z} \) and \( A_2 \) the area of the corresponding 2-cycle. In this situation, a Wilson line-dependent vacuum energy is generated, that can be computed via the Coleman-Weinberg formula, see [24, 25, 29]

\[ A = 2 \times V_{d+1} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \int_0^\infty \frac{dt}{2t} \text{Tr} e^{-2\pi\alpha'(k^2 + M^2)}, \]  

(2.4)

where \( d \) stands for the number of non-compact directions. The extra factor of 2 comes as usual from interchanging the two ends of the string. Note that the integral in the momenta \( k \) must only be computed for the external dimensions, since for the internal ones the momentum is a quantised quantity. We will then include the sum over the internal momentum in the trace of the mass operator.

For generality, we will start from a more general configuration of two Dp-branes, wrapping the same \( T^{(p-3)} \), with magnetic flux turned on in a \( T^2 \) submanifold wrapped by one of them. We allow for Wilson lines to be turned on along the \( n \) compact dimensions parallel to the branes in which there is no magnetic flux turned on, and we associate a brane separation degree of freedom \( x_i \) to each of the \( 9 - p = 4 - n \) transverse directions. The mass operator for the twisted string states lying between both branes is given by

\[ \alpha'M_{ab}^2 = \sum_{i=1}^{4-n} \frac{\Delta_i^2 \bar{R}_i^2}{4\pi^2\alpha'} + \sum_{i=1}^{n} \frac{P_i^2\alpha'}{4\pi^2R_i^2} + N_\nu + \nu(\theta_{ab} - 1), \]  

(2.5)

where the number operator \( N_\nu \) can be found in [35], and we have called \( R_i, i = 1, ..., n \) the compactification radii corresponding to the \( n \) dimensions in which Wilson lines can be switched on, \( \bar{R}_j, j = 1, ..., 4 - n \) the radii corresponding to the transverse directions. The number \( \theta \) is defined in analogy with the intersecting brane case (see appendix) as

\[ \tan \theta = \frac{(2\pi)^2\alpha'm}{A_{\text{flux}}}, \]  

(2.6)

with \( A_{\text{flux}} \) being the area of the torus and \( m \) the number of quanta of magnetic flux, and

\[ \Delta_i^2 = (x_i + 2\pi w_i)^2 \]  

(2.7)

\[ P_i^2 = (\lambda_i + 2\pi m_i)^2 \]  

(2.8)

\(^5\nu = 0, \frac{1}{2} \) for the R and NS sector, respectively.
with \( x_i, \lambda_i \in (-\pi, \pi) \). The amplitude (2.4) can be written as

\[
\mathcal{A} = \int_0^\infty \frac{dt}{t} \left( \int \frac{d^4k}{(2\pi)^4} e^{-2\pi\alpha' k^2} \right) \sum_{m_i,w_i} e^{\frac{x_i^2 R_i^2}{2\pi\alpha'}} e^{\frac{P_i^2 \alpha'}{2\pi R_i^2}} \text{Tr} e^{-2\pi\alpha' (N_i + \nu_i(\theta_{ab}))} 
\]

\[
= \frac{1}{(8\pi^2\alpha')^2} \int_0^\infty \frac{dt}{t^3} \sum_{m_i,w_i} \exp \left( -\sum_{i=1}^{4-n} \frac{\Delta_i^2 \tilde{R}_i^2}{2\pi\alpha'} \right) \exp \left( -\sum_{i=1}^{n} \frac{P_i^2 \alpha'}{2\pi R_i^2} \right) Z(\theta, t) 
\]

(2.9)

with

\[
Z(\theta, t) = \frac{\Theta_{11}^4(i|\theta|t/2\pi, it)}{i\Theta_{11}(i|\theta|t/\pi, it) \eta^6(it)}.
\]

The behaviour of this function \( Z(\theta, t) \) for \( \theta \neq 0 \) in the large and small \( t \) limit is

\[
Z(\theta, t) \to e^{\theta|t|}, \quad t \to \infty, \quad (2.10)
\]

\[
Z(\theta, t) \to 4t^3 \sin^2 \left( \frac{\theta}{2} \right) \tan \left( \frac{\theta}{2} \right), \quad t \to 0. \quad (2.11)
\]

Note that the mass of the lowest lying string state between both branes is given by

\[
M^2 = \sum_{i=1}^{4-n} \frac{x_i^2 R_i^2}{4\pi^2\alpha'} + \sum_{i=1}^{n} \frac{\lambda_i^2}{4\pi^2 R_i^2} - \frac{|\theta|}{2\pi\alpha'}. \quad (2.12)
\]

and this implies that, provided that the infinite sums converge, we are left with an infinite series of converging integrals, whenever the mass is not tachyonic. The presence of the exponentials allows us to substitute \( Z(\theta, t) \) by its low \( t \) limit and we get (assuming in the following \( \theta > 0 \))

\[
\mathcal{A} = \frac{\sin^2(\theta/2) \tan(\theta/2)}{2 \alpha'} \prod_{i=1}^{n} R_i \prod_{i=1}^{4-n} R_i \prod_{i=1}^{n} \exp(iw_ix_i) \prod_{i=1}^{n} \exp(im_i\lambda_i) 
\]

\[
\times \int_0^\infty \frac{dt}{t^2} \sum_{m_i,w_i} \exp \left( -\sum_{i=1}^{4-n} \frac{\pi\alpha'}{2R_i^2} w_i^2 - \sum_{i=1}^{n} \frac{\pi R_i^2}{2\alpha' m_i^2} \right) \quad (2.13)
\]

where we have used the Poisson resummation formula

\[
\sum_{n \in \mathbb{Z}} e^{-\pi an^2 + \pi bn} = \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}} e^{-\pi \left( n + \frac{b}{2a} \right)^2}. \quad (2.14)
\]

\footnote{There is also a spacetime volume factor \( V_4 \) which does not appear in the final expression for the potential, so we have not included it in the amplitude.}

\footnote{We are using the definitions of [25].}
The first term (with \( m_i = w_i = 0 \) \( \forall i \)) is divergent. This divergence is due to the sum over all images in a compact space, and, as explained in [7], it is an unphysical divergence. After discarding this infinity, we get

\[
A_{\text{reg}} = \frac{\sin^2(\theta/2) \tan(\theta/2)}{(8\pi^2)^2 \alpha^n} \prod_{i=1}^{n} R_i \sum_{(m_i, w_i) \neq (0, \ldots, 0)} \prod_{i=1}^{4-n} \exp(iw_i x_i) \prod_{i=1}^{n} \exp(im_i \lambda_i) \sum_{i=1}^{4-n} \frac{\pi \alpha'}{2R_i^2} w_i^2 + \sum_{i=1}^{n} \frac{\pi R_i^2}{2\alpha'} m_i^2.
\]

The interaction term is then

\[
V_{\text{int}} = -A_{\text{reg}}.
\]

Since we want to apply this to the D7 case, we set \( n = 2 \).

The potential (2.15) corresponds to the solution of Poisson’s equation with a \( \delta \)-function source, that is the Green’s function, on the torus. Due to the symmetry of the torus, the Green’s function is not rotationally symmetric at distances of order the compactification scale away from the source [6]. We now concentrate on the behaviour of the potential near the core \((x_i = \lambda_i = 0)\), where the rotational symmetry is restored. The infinite sum becomes

\[
\sum_{(m_i, w_i) \neq (0, \ldots, 0)} \prod_{i=1}^{4-n} \exp(iw_i x_i) \prod_{i=1}^{n} \exp(im_i \lambda_i) \sum_{i=1}^{4-n} \frac{\pi \alpha'}{2R_i^2} w_i^2 + \sum_{i=1}^{n} \frac{\pi R_i^2}{2\alpha'} m_i^2 \\approx \frac{2\pi^3 V}{||X, \Lambda||^2}.
\]

with

\[
||X, \Lambda||^2 = \sum_{i=1}^{2} \frac{x_i^2 R_i^2}{\alpha'} + \sum_{j=1}^{2} \frac{\lambda_j^2 \alpha'}{R_j^2} \equiv X^2 + \Lambda^2,
\]

and

\[
V = \frac{4}{\pi^2} \prod_{i=1}^{2} \frac{\tilde{R}_i}{R_i},
\]

so that the final amplitude reads

\[
A_{\text{reg}} = \frac{\sin^2(\theta/2) \tan(\theta/2)}{8\pi^4 \alpha'^2 ||X, \Lambda||^2}.
\]

This is, as expected, of the same form as the potential of the first reference of [7]. As we will see in the next section, the phenomenology in our setup can be quite different

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8The function we obtain after the regularisation is (apart from numerical factors) a solution to the Green’s equation in the torus, given by \( \nabla^2 G(x) = \delta(x) - 1/V \), with \( V \) the volume of the torus. The sum over images corresponds to an infinite sum over solutions of the non-compact Green’s equation and thus one cannot expect it to give the correct answer; this is the origin of this non-physical infinity.
from that of [3]. The inflaton in our case will be (up to a normalization) $\Lambda$ rather than $||X, \Lambda||$ and the fact that $\Lambda \propto 1/R_i$, $||X|| \propto \bar{R}_i$ implies that, for generic choices of brane separations $x_i$, we have $||X|| \gg \Lambda$. Then, expanding around $\Lambda/||X|| = 0$ yields a quadratic, rather than Coulombic, potential. In the limit of small brane separations such that $||X|| \ll \Lambda$ the potential is indeed Coulombic.

The computations we have done here are valid for square tori. In more general situations one might want to consider the effect of adding a real part to the complex structure of the torus. If the complex coordinates of a $T^{2n}$ are given by $dz^i = dx^i + \tau^i dy^i$, then the Green’s function is given by

$$G(z_i) = -\frac{1}{\prod_{i=1}^n(2\pi R_i)^2 \Im \tau_i} \sum_{(n_i, m_i) \neq (0,0)} \frac{\prod_{i=1}^n \exp \left( \frac{2\pi \Im (n_i z_i \tau_i + m_i z_i)}{\Im \tau_i} \right)}{\sum_{i=1}^n \frac{|m_i - n_i \tau_i|^2}{R_i^2 \Im \tau_i}},$$

and computing the corresponding inflationary potential is straightforward along the lines sketched here.

### 2.2 The Constant Term

The full inflationary potential will consist of a constant part, which gives the vacuum energy density responsible for inflation, plus an interaction, slowly decreasing, term. The interaction term was computed in the previous section. To compute the constant term, we will assume that the energy of the system after inflation has ended is of order the cosmological constant today (hence near zero); this means that all possible contributions to the vacuum energy included in the system, such as other branes, orientifolds and fluxes, sum up to zero.$^9$

The vacuum contribution can be found by considering the energy difference between the chosen configuration and the minimum energy configuration with the same charges. Here, the configuration of interest involves two D7-branes with magnetic flux$^{10}$ in a $T^2$ wrapped by one of them, but we can work in the T-dual picture of branes at angles and then T-dualise back. Consider for example two D1-branes wrapping $(n_1, m_1)$ and $(n_2, m_2)$ cycles respectively in a $T^2$. If $R_1, R_2$ are the radii of the torus then the energy of the configuration reads

$$E = TR_1 \left[ \sqrt{n_1^2 + m_1^2 R_2^2/R_1^2} + \sqrt{n_2^2 + m_2^2 R_2^2/R_1^2} \right]$$

$^9$Note that here we are assuming a local configuration, since we are mainly interested in the dynamics associated to the Wilson lines. The dynamics associated to the full system can be very complicated and its analysis is beyond the scope of this work. This, in particular, means that we are not explicitly cancelling RR tadpoles, an issue that should be addressed in a complete model.

$^{10}$Wilson lines do not contribute to the vacuum part of the potential at this level but, as we will see in section [4], moduli stabilization effects can induce such a dependence.
with $T$ the corresponding brane tension. The minimum energy state corresponds to a reconnected configuration of one brane wrapping a $(n_1 + n_2, m_1 + m_2)$ cycle

$$E_{\text{min}} = TR_1\sqrt{(n_1 + n_2)^2 + (m_1 + m_2)^2R_2^2/R_1^2}. \quad (2.23)$$

In the case of interest $n_1 = n_2 = 1, m_1 = 0, m_2 = m$ so that

$$\Delta E = TR_1 \left(1 + \sqrt{1 + m^2R_2^2/R_1^2} - 2\sqrt{1 + m^2R_2^2/(4R_1^2)}\right) \approx \frac{1}{4}TR_1^2m^2R_2^2. \quad (2.24)$$

Now, T-dualising in the $R_2$ direction, we have $R_2 \to \frac{\alpha'}{R_2}$ so $\Delta E$ becomes

$$\Delta E \approx \frac{1}{4}T\alpha' R_1 \frac{m^2\alpha'^2}{R_1^2 R_2^2}. \quad (2.25)$$

Similarly, for D7-branes wrapping a $T^4$ of volume $V_4$ as well as four spacetime dimensions, and with a magnetic flux switched on in a $T^2$ submanifold with radii $R_1$ and $R_2$, we have a vacuum energy density

$$\Delta E \approx \frac{1}{4}T^7V_4 \frac{m^2\alpha'^2}{R_1^2 R_2^2} \equiv \frac{1}{4}T^7V_4 \frac{(2\pi)^4 m^2 \alpha'^2}{V_{\text{flux}}^2}, \quad (2.26)$$

where $V_{\text{flux}} = (2\pi)^2R_1R_2$ is the volume of the $T^2$ in which the flux is turned on. $T^7$ is the D7-brane tension given (for $p = 7$) by

$$T_p = \frac{(2\pi)^{-p} \alpha'^{-p+1}}{g_s}. \quad (2.27)$$

This corresponds to a contribution to the potential

$$V_0 \simeq \frac{(2\pi)^{-7} \alpha'^{-4} V_4}{4g_s} \left(\frac{(2\pi)^2 \alpha' m}{V_{\text{flux}}}\right)^2 \equiv \frac{(2\pi)^{-7} \alpha'^{-4} V_4}{4g_s} \tan^2 \theta, \quad (2.28)$$

where we parametrised the magnetic flux in terms of an angle $\theta$, the same angle appearing in the interaction potential, cf. (2.20).

The full potential (in terms of $\lambda$) is the sum of the vacuum and interaction terms

$$V(\lambda) = V_0 + V_{\text{int}}(\lambda). \quad (2.29)$$

### 2.3 Unifying Models

Though straightforward, it is interesting to comment on the fact that most of the open string inflationary potentials (without the contributions of fluxes) may be regarded as coming from a common origin, namely Type I string theory with magnetic fluxes\textsuperscript{11}.

\textsuperscript{11}We refer the reader to the end of appendix A.2 for the conventions relevant to this section.
Consider Type I string theory on a $T^6$ with possibly magnetic flux in some of the cycles. Apart from the D9 branes included in Type I theory, all other possible branes can be seen as different configurations of magnetic flux on the world volume of these D9-branes. It follows that one can get any toroidal inflationary model previously considered in the literature just starting from different configurations of D9 branes with magnetic flux and Wilson lines in Type I theory.

- One of the possible ways of getting brane-antibrane inflation from this set-up is putting infinite magnetic flux in one of the 2-cycles wrapped by some brane and minus infinite magnetic flux in the same 2-cycle in some other brane. The inflaton will be a Wilson line (or a combination of them) along some 1-cycle.

- The model of inflation from branes at angles [6] can be obtained starting from Type I just considering a small amount of magnetic flux in some two-cycle wrapped by a D9 and making three T-dualities along three one-cycles, one of them belonging to the two cycle that supports the flux. The variation of this model considered in the last reference of [6] can be obtained considering magnetic flux in two different two-cycles. The inflaton(s) in both cases will be a (combination of) Wilson lines along one-cycles not supporting magnetic flux.

- The model of D3-D7 inflation [11] in a toroidal set-up can be obtained starting with magnetic flux along two 2-cycles in Type I and then performing six T-dualities. Again, the inflaton will be a Wilson line along some appropriate 1-cycle transverse to the two-cycles where magnetic flux has been turned on.

- One can try to generalise this picture considering more general compactification spaces that either have one-cycles or alternatively considering lower ($p < 9$) dimensional branes wrapping some cycles having non-trivial one-cycles on them, to be able to turn on Wilson lines on them. Magnetic flux along some two-cycle will break supersymmetry generating a non-trivial potential for the Wilson line.

3. Inflation from Wilson Lines

In this section we study the inflationary properties of the above potential, assuming that the volumes of all two-cycles in the compactification manifold have been fixed. In principle this is a possibility to consider, since there is no obstacle preventing this kind of mechanisms to occur in string theory, as commented in [9]. However, most known mechanisms to fix volumes proceed along the lines of superpotential stabilisation of Kähler moduli, and this situation will be analysed in section [11]. Along the present section we will simply assume that the volumes of the two-cycles have been stabilised by some unspecified mechanism.

We consider a specific set-up, in the spirit of the previous section, in which two D7-branes wrap a $T^4$ submanifold of a compactification manifold $T^6$. We write $T^6$
as a product of three 2-tori with radii \((R_4, R_5)\), \((R_6, R_7)\) and \((R_8, R_9)\) respectively, and arrange the branes to be pointlike in the first torus, with separations \(x_4, x_5\). In the world-volume of one of the branes, we turn on magnetic flux (parametrized by the angle \(\theta\)) in the second torus and a Wilson line along the 8-direction of the third torus. With this notation the potential becomes

\[
V(\lambda) = \frac{R_6 R_7 R_8 R_9 \tan^2(\theta)}{8\pi^3 \alpha'^4 g_s} - \frac{\sin^2(\theta/2) \tan(\theta/2)}{8\pi^3 \alpha^2 ||X, \Lambda||^2} \tag{3.1}
\]

with

\[
||X, \Lambda||^2 = \frac{x_4^2 R_4^2}{\alpha'} + \frac{x_5^2 R_5^2}{\alpha'} + \frac{\lambda^2 \alpha'}{R_8^2} \equiv X^2 + \Lambda^2. \tag{3.2}
\]

### 3.1 Normalization and Inflationary Parameters

We now move to analysing the inflationary properties of the potential (3.1). We assume that we have fixed the brane separations \(x_4, x_5\) by fluxes (see appendix \[B\]) but not the Wilson line \(\lambda\), which will play the role of the inflaton. In order to compute the slow-roll parameters we first need to identify the canonically normalised field. The kinetic term for the Wilson line comes precisely from the first term in the expansion of the DBI action, namely the Maxwell action. We have

\[
S = \int d^4 x \sqrt{-g} \left( \frac{1}{4g_{YM,A}^2} F_{\mu\nu} F^{\mu\nu} + \ldots \right) = - \int d^4 x \sqrt{-g} \left( \frac{1}{2g_{YM,A}^2} (\partial_{\mu} A_i)^2 + \ldots \right) \tag{3.3}
\]

with \(g_{YM,A}\) the (four dimensional) Yang-Mills gauge coupling constant, where we have used the fact that the Wilson lines have a constant profile in the internal dimensions. The relation between \(g_{YM,A}\) and other parameters in the model is

\[
\frac{1}{g_{YM,A}^2} = T_p (2\pi \alpha')^2 \text{Vol}_{p-3} \tag{3.4}
\]

with \(\text{Vol}_{p-3}\) the volume of the internal dimensions wrapped by the D-brane and \(T_p\) the D-brane tension:

\[
T_p = \frac{(2\pi)^{-p} \alpha'^{-\frac{p+1}{2}}}{g_s}. \tag{3.5}
\]

The relation between \(A_i\) and the numbers \(\lambda_i\) is given by

\[
A_i = \frac{\lambda_i}{2\pi R_i}. \tag{3.6}
\]

In this simplest case we have turned on only one Wilson line degree of freedom \(\lambda\), along the 8-direction of the third torus corresponding to radius \(R_8\). Then \(\lambda\), the
(non-canonically normalised) inflaton, will appear in the Lagrangian with a kinetic term

\[ S_{k,\phi} = - \int d^4x \sqrt{-g} \frac{1}{2} (T_p \alpha'^2 \text{Vol}_{p-3}) \left( \frac{\partial_{\mu} \lambda}{R_8} \right)^2. \] (3.7)

We have not pulled the $1/R_8$ out of the derivative for reasons that will become apparent afterwards. We can then define a canonically normalised field $\psi \equiv K^{1/2} \lambda$, with

\[ K \equiv T_\gamma \alpha'^2 V_4/R_i^2, \] (3.8)

where we have already substituted $p = 7$. If we assume that we can fix the volumes of the different two-cycles, then we already have all the ingredients to compute the inflationary parameters. These are given by

\[ \epsilon = \frac{M_p^2}{2} \left( \frac{V'}{V} \right)^2, \quad \eta = \frac{M_p^2}{V} \frac{V''}{V} \] (3.9)

where the prime means differentiation with respect to the canonically normalised field. It is convenient to recall the expression of the Planck mass in terms of stringy quantities, for unwarped compactifications. It is given by

\[ \frac{M_p^2}{2} = \frac{(2\pi)^{-7}}{g_s \alpha'^4 \text{Vol}_6}, \] (3.10)

where Vol$_6$ stands for the total six-dimensional volume. Then, with $\psi = K^{1/2} \lambda$, we get

\[ \epsilon = \frac{M_p^2}{2K} \frac{1}{V^2} \left( \frac{dV}{d\lambda} \right)^2 = \left( \frac{2\pi)^2 R_4 R_5 R_8}{g_s \alpha'^2} \right) \frac{1}{V^2} \left( \frac{dV}{d\lambda} \right)^2 \] (3.11)

\[ \eta = \frac{M_p^2}{K} \frac{1}{V^2} \frac{d^2V}{d\lambda^2} = \frac{2(2\pi)^2 R_4 R_5 R_8}{g_s \alpha'^2} \frac{1}{V} \frac{d^2V}{d\lambda^2}. \] (3.12)

### 3.2 Inflation Phenomenology

Before exploring whether the slow-roll conditions are satisfied in regions of the parameter space, let us pause and consider the qualitative features of the potential. First, we observe that $||X||$ scales with radius and $\Lambda$ with inverse radius, so for generic brane separations and similar compactification scales, we have $\Lambda \ll ||X||$. Expanding $||X, \Lambda||^{-2}$ around $\Lambda/||X|| = 0$ we find a quadratic term in $\lambda$ with positive mass. Inflation can then be realised by starting with $\lambda > 0$ and rolling towards the $\lambda = 0$ minimum. We can further arrange (see section 3.3) that a tachyonic instability appears when the field reaches a critical value $\lambda_{\text{crit}}$. Thus the inflationary phase comes to an end in a hybrid inflation fashion.
We now explore the conditions for slow-roll inflation. Using the potential (3.1) and the approximation $V \simeq V_0$ in the denominator\footnote{In (3.1) the presence of compactification radii, $g_s$, $||X,\Lambda||$ and $\theta$, all lead to a suppression of $V_{\text{int}}$ relative to $V_0$}, equation (\ref{eq:311}) reads

$$
\epsilon = 4(8\pi)^2 g_s \frac{R_4 R_5 \alpha' \sin^4(\theta/2) \tan^2(\theta/2)}{R_6^2 R_7^2 R_8^2 R_9^4} \frac{\lambda^2}{||X,\Lambda||^6}.
$$

(3.13)

Similarly, equation (\ref{eq:312}) for $\eta$ becomes

$$
\eta = (8\pi)^2 \frac{R_4 R_5 \alpha' \sin^2(\theta/2) \tan(\theta/2)}{R_6 R_7 R_8 R_9 \tan^2 \theta} \frac{1}{||X,\Lambda||^4} \left(1 - \frac{4\alpha' \lambda^2}{R_8^2 ||X,\Lambda||^2}\right).
$$

(3.14)

The slow-roll conditions $\epsilon \ll 1, \eta \ll 1$ can then be satisfied by choosing appropriately the relevant compactification volumes and, more importantly, by tuning the angle $\theta$ to be sufficiently small.

Note however that the choice of compactification volumes and angle needs to be consistent with the COBE normalization constraint. Indeed, if the inflaton $\lambda$ is responsible for generating the cosmological density perturbations, then the observed CMB anisotropies as measured by the COBE satellite, impose the normalization

$$
\delta_H \simeq \frac{1}{5\sqrt{3\pi} M_p^2} \sqrt{\frac{V}{\epsilon}} = 1.91 \times 10^{-5},
$$

(3.15)

or equivalently \footnote{Approximating $V \simeq V_0$ during slow-roll and using equation (3.10) we obtain a constraint for the average compactification radius}

$$
\left(\frac{V}{\epsilon}\right)^{1/4} = 0.027 M_p.
$$

(3.16)

Approximating $V \simeq V_0$ during slow-roll and using equation (3.10) we obtain a constraint for the average compactification radius

$$
\bar{R} \sim \left(\frac{g_s^3 \tan^2 \theta}{\epsilon}\right)^{1/8} \sqrt{\alpha'}.
$$

(3.17)

Then, equation (\ref{eq:311}) gives for the string scale

$$
\sqrt{\alpha'} M_p \sim \left(\frac{g_s}{\epsilon^2}\right)^{1/8} (\tan \theta)^{3/4}.
$$

(3.18)

For the example we will consider below with $\theta \sim 0.1$ and $g_s \sim 0.1$, we will find $\epsilon \sim 10^{-11}$ resulting in a (low) GUT string scale, as is typically the case in brane inflation models.

We highlight that the constraint (3.17) involves the angle $\theta$ as well as the compactification scale(s), so one does not have unlimited freedom in choosing these parameters to satisfy slow-roll. As we will now see, requiring that inflation ends in a hybrid-type exit further constrains the possible parameter choices.
3.3 Graceful Exit

An attractive feature of the model is the possession of a natural mechanism to exit inflation, via a tachyonic instability. The mass of the lowest string state stretching between the branes is (see equation (2.12))

\[
m^2 = \sum_{i=4}^{5} \frac{x_i^2 R_i^2}{4\pi^2 \alpha'^2} + \frac{\lambda^2}{4\pi^2 R_8^2} - \frac{|\theta|}{2\pi \alpha'}.
\]  

(3.19)

We observe that, if \( R_4, R_5 \) and \( \theta \) are chosen appropriately, then the mass squared of this mode becomes negative as the inflaton rolls towards \( \lambda = 0 \). Beyond this critical point, an instability appears and the field quickly rolls down in the tachyonic direction, violating slow-roll and thus ending the inflationary phase. This provides an elegant realization of hybrid inflation beyond the brane annihilation picture. Furthermore, since the tachyon is charged under the gauge groups on the branes, its condensation allows the formation of cosmic strings via the Kibble mechanism [5, 43].

It seems that the additional tuning of parameters required to guarantee a hybrid inflation type exit could be inconsistent with the COBE normalization or violate the slow-roll conditions, preventing inflation. Indeed, for a tachyon to develop, the first and last terms in equation (3.19) need to be comparable and this seems to require small compactification radii and large angles, which is the opposite of what is needed for slow-roll (equations (3.13-3.14)). However, equation (3.19) only involves \( R_4, R_5 \) and \( R_8 \) so the rest of the radii can still be much larger to guarantee that slow-roll is satisfied. Furthermore, the first term in (3.19) can be made small by putting the branes closer together (adjusting \( x_4, x_5 \) small) rather than reducing the compactification radii \( R_4, R_5 \). As a result, fine tuning can be achieved at the order of one part in 1000 to guarantee slow-roll, while satisfying the COBE normalisation and ensuring that tachyon condensation takes place after an appropriate number of e-folds. In the next section we present an explicit example of such a model.

3.4 An Explicit Example

We now consider a specific choice of parameters, which gives rise to slow-roll and tachyon condensation, while satisfying the COBE constraint. Most of the fine tuning comes from requiring that the tachyon condensation mechanism kicks in at the right value of \( \lambda \). Indeed, \( \lambda \) ranges from \(-\pi\) to \( \pi\) and to make sure that (3.19) changes sign for some critical value \( \lambda_{\text{crit}} \) in this range (in fact we need \( \lambda_{\text{crit}} < 1 \) to get enough e-folds) requires that the first and last terms in equation (3.19) are equal in magnitude to an accuracy of three significant figures. This corresponds to fine tuning of a few parts in a thousand. In our example we will choose \( R_4 = R_5 = R_8 = 5\sqrt{\alpha'} \) and magnetic flux such that \( \theta = 0.4 \). By choosing a configuration in which the branes are only separated in the 4-direction, setting \( x_4 = 0.317 \) and \( x_5 = 0 \), we obtain.
\( \lambda_{\text{crit}} = 0.161 \). Then, as long as we are able to satisfy the slow-roll conditions, there is enough room for inflation between \( \lambda \approx 1 \) and \( \lambda_{\text{crit}} \).

In order to satisfy slow-roll we can choose the rest of the compactification radii \( R_6, R_7 \) and \( R_9 \) to be large. As already mentioned however, we cannot make the slow-roll parameters arbitrarily small because the compactification scale is constrained by the COBE normalization. Fortunately, equation (3.17) allows for large enough radii to give rise to enough inflation. Small string coupling \( g_s \) leads to a smaller \( \epsilon \) but does not affect \( \eta \). Choosing for example \( R_6 = R_7 = R_9 = 20\sqrt{\alpha'} \) and \( g_s = 0.1 \) gives \( \epsilon(\lambda \sim 1) \approx 10^{-11}, \eta(\lambda \sim 1) \approx 10^{-3} \). Then, starting inflation at \( \lambda = 0.5 \), for example, gives rise to \( N \simeq 400 \) e-folds of inflation until the inflaton rolls down to \( \lambda_{\text{crit}} \), where tachyon condensation kicks in. Cosmological scales exit the horizon around 60 e-folds before the end of inflation (corresponding to \( \lambda \simeq 0.19 \)) at which point the slow-roll parameters are:

\[
\begin{align*}
\epsilon &\simeq 4.6 \times 10^{-13} \\
\eta &\simeq 2.8 \times 10^{-3}.
\end{align*}
\]

The scalar spectral index is then \( n_s = 1 - 6\epsilon + 2\eta \simeq 1.006 \). Note that although such a Harrison-Zel’dovich spectrum is disfavoured, it is still consistent with the combined WMAP3+SDSS data \([39]\). Also, if cosmic strings are indeed produced in the end of inflation as described in the previous section, then one should include the contribution of the string network when comparing to the CMB, in which case a flat inflationary spectrum plus strings is consistent, if not favoured, by the data \([45, 46]\). Due to the smallness of \( \epsilon \), the model predicts no significant running of the spectral index and no gravitational waves.

There are many other consistent choices of parameters with similar predictions. We can for example start with a different angle \( \theta \) and modify \( R_4, R_5 \) accordingly, or select the brane separations \( x_4, x_5 \) differently, to get the tachyonic instability at an appropriate value \( \lambda_{\text{crit}} \). We can also choose a smaller \( R_8 \) to make the effect of the second term in equation (3.19) more important, thus relaxing somewhat the fine tuning of brane separations. The rest of the radii are chosen so as to give rise to slow-roll, and simultaneously satisfy the COBE normalisation. Exploring the parameter space one can easily arrange \( \epsilon \lesssim 10^{-10} \), but the \( \eta \) parameter is typically much bigger \( \eta \gtrsim 10^{-3} \). Thus, a Harrison-Zel’dovich/slightly blue spectrum of scalar perturbations and no significant running or gravitational waves are robust predictions of the model in this limit.

The above picture is valid for relatively large angles, when the brane separations needed to arrange a satisfactory hybrid inflation model are such that \( ||X|| > \Lambda \). For small enough angles, successful tachyonic condensation can only happen for \( ||X|| < \Lambda \). In this case the potential is not of the \( 1 + \phi^2/\mu^2 \) form but instead goes like \( \phi^{-2} \), in close analogy to reference \([3]\). This leads to very different predictions than the
above situation, in particular it gives rise to negative $\eta$ and hence a slightly red scalar spectrum, in better agreement with WMAP3. For example, having $R_4 = R_5 = 5\sqrt{\alpha'}$, $R_8 = 2\sqrt{\alpha'}$ and $R_6 = R_7 = R_9 = 40\sqrt{\alpha'}$ with $\theta = 0.001$ and $g_s \approx 0.1$ yields $\epsilon(\lambda \approx 1) \approx 10^{-12}$, $\eta(\lambda \approx 1) \approx -3 \times 10^{-3}$. This gives rise to $N \approx 100$ e-folds of inflation from $\lambda \approx 0.7$ to $\lambda_{\text{crit}} \approx 0.16$. The slow roll parameters 60 e-folds before inflation are $\epsilon \approx 4 \times 10^{-11}$ and $\eta \approx -0.012$, corresponding to a spectral index $n_s \approx 0.976$. The smallness of $\theta$ follows from the requirement\footnote{For large angles, the third term in equation (3.19) cannot be balanced by the Wilson line term.} that the tachyonic regime appears at an acceptable value $\lambda_{\text{crit}} < 1$.

One comment is in order here. Apart from extra contributions to the $\eta$ parameter that will appear when one considers dynamical moduli stabilisation mechanism (see section\footnote{For large angles, the third term in equation (3.19) cannot be balanced by the Wilson line term.}), there is one extra contribution that has to be considered even in the present case. The interacting part of the inflationary potential is not given by (3.1), as already emphasized in the text, but rather by (2.15). Whereas both potentials are numerically close in the region of interest, it turns out that they are not equivalent from the point of view of the parameter $\eta$. As showed in\footnote{For large angles, the third term in equation (3.19) cannot be balanced by the Wilson line term.} the first of these potentials (3.1), roughly speaking, follows as a solution of a Green’s equation of the form $\nabla^2 V_1(x) = \delta(x)$, which implies that the $\eta$ parameter far from the source at $x = 0$ can be small, especially if there are more parameters to play with, like the fluxes in our case. However, a potential of the form (2.15) will rather satisfy an equation of the form $\nabla^2 V_2(x) = \delta(x) - 1/V$, giving typically a contribution to $\eta$ of order 1. This is certainly true case for brane-antibrane potentials. A detailed analysis\footnote{For large angles, the third term in equation (3.19) cannot be balanced by the Wilson line term.} showed that this is not the case for branes at angles, an analysis that applies also here. The easiest way of seeing this is that, in our case, the interaction potential fulfills $\nabla_x^2 V_{\text{int}}(y) = -\sin^2(\theta/2) \tan(\theta/2)/V$ whereas the constant part goes like $\tan^2(\theta)$, indicating that even when considering the complete (compact) potential the $\eta$ parameter will be suppressed. A full (necessary numerical) analysis of this issue is beyond the scope of this paper.

One may wonder how the above results would be affected by considering more complicated models, as for example having more than one Wilson lines. This situation is interesting since one of the Wilson lines can be used as the inflaton, while the other could play the role of a curvaton field, responsible for the generation of perturbations. Also, as a two-field model, it could allow for significant non-gaussianity, but the investigation of this would involve the study of non-linear perturbations (see for example\footnote{For large angles, the third term in equation (3.19) cannot be balanced by the Wilson line term.}). We leave this study for a further publication. One could also consider more realistic compactifications, or embedding a similar model in heterotic theory.

4. Supergravity Description and Moduli Stabilisation

The previous sections have been written in the spirit of the first articles on brane inflation. Even though we discussed a model in the appendix in which the dilaton,
complex structure moduli and D7 brane moduli are fixed, we have assumed that all Kähler moduli have been fixed by some unknown mechanism and only concentrated on the dynamics of the inflaton field. This is a strong assumption. Fortunately there has been much progress in fixing all geometric moduli by means of RR and NS fluxes combined with non-perturbative effects for the Kähler moduli. The prime example is that of [22]. Since [9], the standard approaches towards inflation now incorporate the dynamics of these moduli fixing that plays an important role in obtaining inflation.

4.1 Effective Action Generalities

In effective supergravity theories after Calabi-Yau compactifications we know that the D3 brane/antibrane system can be described in terms of an effective field theory. This has been discussed in detail starting from the work of [9]. In this case the anti D-brane is fixed at a location inside the Calabi-Yau, namely the tip of a Klebanov-Strassler throat in a deformed conifold geometry, and the D3 brane position is parametrised in terms of a scalar field $\varphi$. The supergravity theory includes the geometric moduli fields and $\varphi$. Within the KKLT scenario [22] of moduli stabilization, the effective field theory can be described in terms of only the Kähler moduli and $\varphi$ after fluxes fix the dilaton and complex structure moduli.

In the simplest one-modulus case, the Kähler potential takes the form

$$K = -3 \log (T + T^* - \varphi^* \varphi) \quad (4.1)$$

and the superpotential takes the KKLT form

$$W = W_0 + Ae^{-aT} \quad (4.2)$$

With $W_0$ a flux superpotential, taken constant after complex structure and dilaton stabilisation. To the supersymmetric Lagrangian constructed out of $K$ and $W$ we have to add the supersymmetry breaking interaction terms describing the tension of the anti D3 brane and the Coulomb interaction between the branes. This system has been analysed in some detail in the past few years in which $\varphi$ can be the inflaton field [9, 11, 44].

In this framework, moduli stabilization is fully considered when analyzing the prospects for inflation$^{14}$. With this set-up it has been found that it is possible to get inflation as long as some fine tuning of the parameters is performed. The fine-tuning is required because of the standard $\eta$ problem of $F$-term inflation. A conformally coupled scalar, such as $\varphi$ induces a contribution of order one to the slow-roll parameter $\eta$ and therefore it needs to be compensated by fine tuning the parameters of the model to obtain small $\eta$. More explicitly, for brane inflation the potential is of the form:

$^{14}$For related work regarding moduli stabilisation in D3/D7 inflation see [12].
\[ V = V_0(\mathcal{V}) + V_{\text{int}}(\varphi, \mathcal{V}) \quad (4.3) \]

where \( \varphi \) is the candidate inflaton and \( \mathcal{V} \) the volume modulus. For fixed volume, the first term \( V_0 > 0 \) is a constant that dominates the potential and gives rise to almost de Sitter expansion, while \( V_{\text{int}} \) is subdominant but by its dependence on \( \varphi \) provides the slow-roll conditions. The \( \eta \) problem appears in these string models because the Kähler modulus that is fixed by the KKLT mechanism is not just the volume \( \mathcal{V} \) but a combination of \( \mathcal{V} \) and \( \varphi \). This then induces a \( \varphi \) dependence in \( V_0 \) that gives rise to \( \eta \sim \mathcal{O}(1) \).

A fine tuning is required to have terms say in \( V_{\text{int}} \) that cancel the order \( \mathcal{O}(1) \) value of \( \eta \) to one part in at least 100. In general this fine tuning has been approached in several ways:

1. Taking into account the full potential as a function of \( T \) and \( \varphi \). The parameters \( W_0, A, a \) as well as the warp factor of the metric defining the throat can be fixed to find a point when both \( V' \) and \( V'' \) vanish (where the primes are derivatives with respect to the brane position field \( \varphi \)). Perturbing around this point there is a region in which both \( \eta \) and \( \epsilon \) satisfy the slow-roll conditions and give rise to at least 60 e-folds of inflation. In [10] it was found that this fine tuning is of order 1/1000, worse by one order of magnitude than the expected 1/100. But more general considerations in the second article of reference [10] reduced this tuning. In both cases a multi-brane configuration was needed to accommodate both, the scale of inflation (\( \sim 10^{15} \) GeV) and the standard model scale 1 TeV.

2. In [11] it is argued that inflation could appear naturally if there is a shift symmetry for the field \( \varphi \).

3. Non-perturbative corrections to the superpotential may depend on \( \varphi \) as first proposed in [9]. This means that the parameter \( A \) can be \( \varphi \)-dependent. Explicit string calculations for simple models have been performed in [21] where this \( \varphi \) dependence was found, providing an explicit way to fine tune.

4. Different configurations were proposed. In particular, having two throats with anti D3 branes at the tip of each throat (as in second article of [13]) provided a way to tune the potential such as to cancel the order one contribution to \( \eta \) by the competition between the two anti branes to attract the D3 brane.

5. In [13], the full DBI action was used to obtain ‘fast-roll’ inflation for the brane/anti-brane system:

\[ S_{\text{DBI}} = \int d^4x \ a^3(t) \left[ V(\varphi)\sqrt{1 - \ddot{\varphi}^2/V(\varphi)} + U(\varphi) \right] \quad (4.4) \]
With $T(\varphi)$ the space-dependent D3-brane tension. Its functional form comes from the warp-factor dependence of the metric and identifying the Calabi-Yau coordinate $r$ with the D3 brane location. The function $U(\varphi)$ includes the mass term induced from the conformal coupling as well as the interactions.

We may wonder if similar approaches can be used for Wilson lines. It so happens that the Kähler potential for Wilson lines in Calabi-Yau orientifold compactifications of type IIB string theory, has the same dependence on Wilson lines $\lambda$ as for the field $\varphi$ above [27]. Therefore we can use the Kähler potential above by substituting $\varphi \leftrightarrow \lambda$. Since the superpotential is the same as for KKLT and, as we have seen in the previous sections, the interactions have a similar dependence on the Wilson lines as for the Coulomb-like interaction among branes, we conclude that the physics is very much the same in both cases. An advantage of Wilson lines is that they provide more parameters to be tuned, for example the value of the fluxes of the magnetic fields.

Furthermore, in [21] the corrections to the non-perturbative superpotential above were found actually for the T-dual model in which the matter fields appearing in $W$ are precisely the Wilson lines. This dependence on Wilson lines permits the tuning to be made to reduce the value of $\eta$ as suggested originally in [8]. This applies directly to our case also.

It is clear that a similar dependence on the Wilson lines appears as in the original DBI action. It would be interesting to investigate how a 'potential' for the Wilson lines can be generated from the warp factor. But knowing the relationship between magnetised D7 branes and D3 branes, we expect a similar behaviour. Again, magnetised D7 branes will offer more parameters to play in order to compare with observations.

### 4.2 Wilson Line Inflation and Moduli Stabilisation

Let us now describe the model of the previous chapter in terms of the effective supergravity theory. As mentioned before, the known method to fix the Kähler parameters is the KKLT set-up which uses a combination of fluxes and non-perturbative superpotentials to fix the Kähler moduli. We can incorporate this to our type IIB toroidal model. Therefore we can have a proper treatment of inflation in which there is a potential for all the fields and we follow the evolution of the candidate inflaton field.

Previous experience [9, 10] shows that it is not possible to just assume that this mechanism is at work and just consider constant values of the Kähler moduli. As mentioned in the previous subsection, the main fact to be taken into account is that what is fixed by this mechanism is not the volumes but a combination of the volumes and the inflaton field. This is the source of the $\eta$ problem discussed in [1]. We see now how this problem is evaded in our example.
In a factorisable toroidal Type IIB set-up the supergravity fields are

\[ S \equiv -i \tau = e^{-\phi} + iC + \frac{1}{2} \sum_a \sum_{i=1}^3 |\zeta_{ai}^{i7i}|^2 \quad (4.5) \]

\[ T_i \equiv \frac{1}{2} e^{-\phi} A_j A_k - i \int C_A \wedge \omega_i + \frac{1}{2} \sum_a \sum_{j,k=1}^3 d_{ijk} |\phi_{ai}^{j,k}|^2 \quad i \neq j \neq k \quad (4.6) \]

\[ U_i \equiv \tau_i \quad (4.7) \]

with \( A_i \) the area of the \( i^{th} \) torus. \( d_{ijk} = 1 \) if \( i \neq j \neq k \) and zero otherwise. The \( \zeta_{ai}^{i7i} \) are the supergravity fields corresponding to the transverse position of a \( 7_i \) brane, whereas the \( \phi_{ai}^{j,k} \) are the ones corresponding to Wilson lines\(^{15}\). The Kähler potential is given by

\[
K = -\log \left( S + S^* - \sum_a \sum_{i=1}^3 |\zeta_{ai}^{i7i}|^2 \right) - \sum_{i=1}^3 \log \left( T_i + T_i^* - \sum_a \sum_{j,k=1}^3 d_{ijk} |\phi_{ai}^{j,k}|^2 \right) \\
- \sum_{i=1}^3 \log(U_i + U_i^*) \quad (4.8)
\]

The fact that the complex structure Kähler potential decouples is not completely clear but it is so at first order\(^{[38]}\). Let us assume this form in what follows. The outcome is that what we are fixing if we are applying some superpotential moduli fixing method are not the volumes but

\[
\text{Re} \, T_i = \frac{1}{2} e^{-\phi} A_j A_k + \frac{1}{2} \sum_a \sum_{j,k=1}^3 d_{ijk} |\phi_{ai}^{j,k}|^2. \quad (4.9)
\]

In our particular case, let us say that the D7 is pointlike in the first torus \((k = 1)\) and has flux in the second torus. Let us assume that we have been able to fix all the \( \text{Re} \, T_i \). We have fixed then

\[
\text{Re} \, T_1 = \frac{1}{2} e^{-\phi} A_2 A_3 \quad (4.10) \\
\text{Re} \, T_2 = \frac{1}{2} e^{-\phi} A_1 A_3 + \frac{1}{2} |\varphi|^2 \quad (4.11) \\
\text{Re} \, T_3 = \frac{1}{2} e^{-\phi} A_1 A_2 \quad (4.12)
\]

\(^{15}\)Here we are following the standard notation (see for instance \([34]\)) in which \( a \) labels the field and the index \( j \) refers to the complex dimension that has the Wilson line whereas \( k \) refers to the direction transverse to the D7 brane that hosts the Wilson line.
This basically means that the values of all areas are given in terms of $|\varphi|^2$ as

\begin{align}
A_1 &= \sqrt{\frac{2e^\varphi \text{Re}T_3}{\text{Re}T_1}} \left(\text{Re}T_2 - \frac{1}{2}|\varphi|^2\right) \\
A_2 &= \sqrt{\frac{2e^\varphi \text{Re}T_1 \text{Re}T_3}{\text{Re}T_2 - \frac{1}{2}|\varphi|^2}} \\
A_3 &= \sqrt{\frac{2e^\varphi \text{Re}T_1}{\text{Re}T_3}} \left(\text{Re}T_2 - \frac{1}{2}|\varphi|^2\right)
\end{align}

(4.13) \hspace{1cm} (4.14) \hspace{1cm} (4.15)

Assuming $|\varphi|^2 \ll 2(T_2 + T_2^*)$, we obtain for the canonically normalised inflaton

$$\psi = M_p \frac{\varphi}{\sqrt{T_2 + T_2^*}}.$$  \hspace{1cm} (4.16)

Now, in order to see if there is an $\eta$ problem in this model, we have to compute the value of $V_0$ in terms of the moduli $T_i$ and the candidate inflaton field $\psi$.

To express $V_0$ in the 4D Einstein frame we have to perform the standard Weyl transformation. This is achieved by the metric rescaling:

$$g_{\mu\nu} \rightarrow \Omega g_{\mu\nu}$$  \hspace{1cm} (4.17)

with $\Omega = g_s^2/(A_1 A_2 A_3)$. Therefore the constant term in the potential will be the product of three contributions: the D7 brane tension $T_7 V_4/g_s$ (with $V_4 = A_3 A_2$ the volume of the cycle wrapped by the D7 brane), the magnetic fluxes proportional to $m^2/A_2^2$ and $\Omega^2$ from the Weyl rescaling. This gives

$$V_0(\lambda) = \frac{T_7 A_3 A_2}{4A_1^2 A_2^2 A_3^2} \alpha'^2 g_s^4 m^2 = \frac{(2\pi)^{-3} m^2 \alpha'^{-2} g_s^3}{4A_1^2 A_2^2 A_3^2};$$  \hspace{1cm} (4.18)

expressed in terms of the fixed Kähler moduli this becomes

$$V_0 = \frac{(2\pi)^{-3} m^2 \alpha'^{-2}}{32 \text{Re}T_3^2 \text{Re}T_1},$$  \hspace{1cm} (4.19)

which does not depend on $\varphi$. We can then see that there is no contribution to a mass term for the inflaton field in $V_0$ that was the source of the $\eta$ problem in [9]. Essentially, the model of section 3 is unaffected by moduli stabilisation in this case, and we can just extract the same conclusions regarding slow-roll parameters, density fluctuations and the spectral index.

5. Conclusions

We have presented a general set-up for inflation in string theory with open string modes as inflatons. Wilson lines can play the role of inflaton fields in a way similar
to brane separations. In fact Wilson lines are T-dual to brane separations and therefore it is expected that they play a similar role. The physics of both systems is the same but in different regimes (large against small volume). Therefore if we are only interested on effective field theory descriptions then we would need only the large internal volume potential on each formalism and they are certainly not equivalent.

It is interesting that starting with a particular configuration of D-branes, fluxes and Wilson lines, all known proposals for open string inflation can be included by T-duality and limits on the value of the magnetic fluxes. Furthermore it is reassuring to know that the end of inflation, with corresponding reheating and topological remnants, such as cosmic strings, can go by the standard tachyon condensation mechanism.

The explicit models presented in section 3 provide examples of more possibilities than previously considered, such as the inclusion of the location of the D7 branes and the number of parameters involved, like magnetic fluxes, that can parametrise a fully realistic treatment of inflation. It is worth pointing out that both blue and red tilted values of the spectral index can be obtained and that the string scale tends to be of order the GUT scale leading to the remnant cosmic string tension to be of order $G\mu \lesssim 10^{-7}$.

For a discussion with moduli fixing a la KKLT, our formalism is in general very similar to the brane/antibrane case, including the need to confront the $\eta$ problem. But again there are more parameters that can be varied to contrast with experimental signatures such as the number of efoldings, the spectral index and the COBE normalized $\delta\rho/\rho$. Remarkably, we found that the $\varphi$ dependence in $V_0$ does cancel for the example of section 3 after including moduli stabilisation, as discussed in section 4, and therefore there is no $\eta$ problem, so the phenomenological results of section 3 hold after moduli stabilisation. Notice that even though there is no $\eta$ problem in these models, inflation is certainly not generic, as we needed to have the moduli, both geometric and D7 positions, as well as the magnetic fluxes, in particular ranges in order to satisfy the requirements of slow-roll, tachyon condensation and COBE normalisation. Still, it is encouraging that the fine-tuning required by the $\eta$ problem generic in D-brane inflation models is not present. It would be interesting to understand how general this cancellation is. For this it would be worth exploring more general Calabi-Yau compactifications with four cycles having non-trivial two-cycles carrying either magnetic fluxes or Wilson lines.

Notice that without a moduli fixing scheme, branes at angles provided a more flexible approach to inflation than the brane/antibrane system. However in the extension to include moduli fixing, only the brane/antibrane system has been considered so far. From the discussion above we can see that this omission is corrected here but in terms of the dual version with Wilson lines, instead of brane separation, as inflatons and magnetic fluxes representing the brane angles.
There are further open questions that may be worth exploring following our results. An explicit calculation on the heterotic string would be interesting, to have a concrete realization of inflation in the heterotic case similar to brane inflation in type II. For this nonsupersymmetric compactifications with Wilson lines would naturally provide both, the constant and interaction terms as in the case we discussed in section 3. Also, it would be interesting to check how the same cancellation we found to obviate the $\eta$ problem holds in T-dual configurations in terms of branes at angles. Finally, as stressed in the appendix, Wilson lines will be present whenever the homotopy of the submanifold wrapped by a given D-brane is not trivial, so that our mechanism is not restricted to the case of the torus. There are known examples of compactification manifolds richer than tori, which allow for cycles with non-trivial 1-homology. It would be interesting to see how our mechanism applies for these kind of manifolds. In particular, a complete discussion of our mechanism including warped geometries would be desirable.

Acknowledgements

We acknowledge useful conversations on the subject of this paper with S. Abdussalam, M. Berkooz, C.P. Burgess, P. G. Cámará, J. Conlon, M.P. García del Moral, F. Marchesano, R. Rabadán, E.P.S. Shellard, A. Sinha, K. Suruliz, and A. Uranga. D.C. thanks the Weizmann Institute for Science and the University of Zaragoza for hospitality during the completion of this work. A.A. is funded by the Cambridge European Trust and the Cambridge Newton Trust. The work of D.C. is supported by the University of Cambridge. F.Q. is partially funded by PPARC and a Royal Society Wolfson award.

A. General Aspects about Wilson Lines

A.1 Continuous and Discrete Wilson lines

Given a gauge field $A$ defined over a manifold $M$, a Wilson line along a given closed path $\gamma = \partial C$ is defined as

$$ U_\gamma = P \exp \oint_{\gamma} A. \tag{A.1} $$

In the Abelian case, the Wilson line is a gauge invariant quantity. We will restrict to this case henceforth. If $\gamma$ is a contractible loop, then $U = P \exp \int_C F$ and $U = 1$ whenever $F = 0$. The interesting case comes when $\gamma$ is not contractible, since then one can have $F = 0$ and $U \neq 1$ in a gauge invariant way.

In principle, Wilson lines are associated to the first homotopy group of the manifold $\pi_1(M)$ [15]. For each element of this homotopy group we have a non-contractible
1-cycle $\gamma$ and thus we can associate a Wilson line $U_\gamma$ to that cycle. However, most of the elements $U_\gamma$ are not independent and it is more useful to classify the Wilson lines in terms of homology. The Wilson lines in a given manifold are classified\(^{16}\) by group homomorphisms $H_1(M, \mathbb{Z}) \to U(1)$\(^{20}\). In general\(^ {27}\)

$$H_1(M, \mathbb{Z}) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \oplus \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_p}$$  (A.2)

for some $p$, where $b_1(M) = \dim H_1(M, \mathbb{R})$ is the first Betti number of $M$. The reason to use the homology group under the integers instead that under the real numbers in the classification is that the torsion 1-cycles (associated to discrete Wilson lines) are invisible under $H_1(M, \mathbb{R})$. These discrete Wilson lines correspond to cycles that are non trivial when going around them a given number $n-1$ of times but become trivial after going around $n$ times. All along this paper we will concentrate in continuous Wilson lines, that are elements of $H_1(M, \mathbb{R})$.

Upon compactification of string theory in a given manifold $M$ in the presence of D-branes, one will have continuous Wilson line degrees of freedom whenever $b_1(\Sigma) > 0$, where $\Sigma$ is some $p-3$ cycle wrapped by a D$p$-brane (for phenomenological reasons we will always consider D$p$ branes to be filling the four dimensional space-time and wrapping some $p-3$-cycle in the compact space $M$). This is, then, not a generic situation in a CY, where $h^{0,1} = h^{1,0} = 0$. It is possible, however, to find one-cycles inside higher dimensional cycles in a CY, see\(^ {28}\) for examples. Moreover, there are other spaces yielding $\mathcal{N} = 2$ supergravity in $D = 4$ upon compactification, like $SU(3)$ structure manifolds, where one can have $b_1(M) \neq 0$. Thus, Wilson lines are not generic in string theory but they are common enough for not to be considered as a pathology of the torus.

### A.2 On the Duality Between Angled and Magnetised Branes

Consider Type II string theory compactified on a $T^2$, and a pair of intersecting branes making angle $\theta$ in the directions $X_1, X_2$ that parametrise the torus. Without loss of generality, consider the first of these D-branes (call it brane $a$) to be parallel to the direction $X_1$. The boundary conditions for the string at $\sigma = 0$ (corresponding to the brane $a$) and $\sigma = \pi$ (brane $b$) are

$$\partial_\tau X_1 = \partial_\sigma X_2 = 0, \quad \sigma = 0,$$

$$\partial_\sigma (\cos \theta X_1 + \sin \theta X_2) = 0, \quad \sigma = 0,$$

$$\partial_\tau (- \sin \theta X_1 + \cos \theta X_2) = 0, \quad \sigma = \pi.$$  (A.3)

These boundary conditions can be trivially obtained starting from a standard Neumann-Dirichlet boundary condition in brane $b$ and performing a rotation of angle $\theta$.

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\(^{16}\)We thank F. Marchesano for a clear explanation of this point.
On the other hand, consider the boundary conditions for a string lying between a D-brane filling a torus (brane $a$), and another one (brane $b$) filling the same torus but with constant magnetic field turned on in its world-volume. The boundary conditions can be read from the worldsheet action (see e.g. [23]) and read

$$\begin{align*}
\partial_\tau X_1 &= \partial_\sigma X_2 = 0, \quad \sigma = 0, \\
\partial_\tau X_1 + \mathcal{F} \partial_\sigma X_2 &= 0 \\
-\mathcal{F} \partial_\tau X_1 + \partial_\sigma X_2 &= 0, \quad \sigma = \pi.
\end{align*}$$

(A.5)

with $\mathcal{F} = 2\pi \alpha' F_{12}$. We see then, comparing (A.4) with (A.5) that these descriptions are T-dual along the $X_2$ direction provided that

$$\tan \theta = 2\pi \alpha' F_{12}. \quad \text{(A.6)}$$

In practice we will have more involved configurations that the one described above, namely IIA configurations with even dimensional branes intersecting at points in the compact space (and filling the four dimensional space-time for obvious phenomenological reasons) or IIB constructions with odd-dimensional branes having non-trivial gauge bundles in their world-volume. The easiest of those constructions is given by $T^6$ compactifications or orbifolds/orientifolds thereof. In the IIA side, the setup consists of a group of D6-branes, each one wrapping an element of the 3-homology of the internal manifold; in the case of a $T^6 = \prod_i T^2_i$ each brane is completely characterised by 7 integers: six of them come in pairs as $\{(n^i, m^i)\}$, each one of them meaning the number of times the brane wraps each one of the directions of $T^2_i$ (say $n^i$ times wrapping the $x^i$ direction and $m^i$ times wrapping the $y^i$ direction). The other one, named $N$, corresponds to the number of branes that are on top of each other and gives the rank of the gauge group living in the world volume of the brane. We obtain the IIB description of this setup T-dualising three times along the direction $x^i$. The mapping of a given $\{(n^i, m^i)\}$ configuration is as follows:

- $N$ D6 branes specified by the vector $(1, 0)(1, 0)(1, 0)$ in the IIA picture is mapped to $N$ D3 branes in the IIB picture.

- $N$ D6 branes given by $(n, m)_i(1, 0)_j(1, 0)_k$, $m \neq 0$, $n$ and $m$ coprime, are mapped to a stack $Nm$ D5 branes filling the $i^{th}$ torus and point-like in the other two tori. $n \neq 0$ implies the presence of units magnetic field in the D5 in the $i^{th}$ torus that break the rank of the gauge group from $Nm$ down to $m$. This magnetic field is given by

$$F = \frac{\pi n}{m} 1_{N \times m} dx^i \wedge dy^i = \frac{\pi i}{\text{Im} \tau^i} \frac{n}{m} 1_{N \times m} dz^i \wedge dz^{\bar{i}} \quad \text{(A.7)}$$

where $dz^i = dx^i + \tau^i dy^i$.

---

17Remember that T-duality along $X_i$ exchanges the action of $\partial_\sigma$ and $\partial_\tau$ on $X_i$. 

---
• $N$ D6 branes characterised by a vector $(1,0)_i(n^j, m^j)_j(n^k, m^k)_k$, $m^j, m^k \neq 0$, $n^i$ and $m^i$ coprime for a given $i$, are mapped to a stack of $Nm^j m^k$ D7 branes that are point-like in the $i^{th}$ torus and whose magnetic field is given by

$$F = \sum_{a=1,2,3} \frac{\pi i}{\text{Im}\, \tau^a} \frac{n^a}{m^a m^j} 1_{N \times m^j m^k} \, dz^a \wedge d\bar{z}^a. \quad (A.8)$$

The rank of the gauge group is reduced from $Nm^j m^k$ down to $N$ because of the monopole background.

• Finally, a stack of $N$ D6 branes specified by a vector $(n^1, m^1)(n^2, m^2)(n^3, m^3)$ with all $m^i \neq 0$ and $n^i$ and $m^i$ coprime for a given $i$ is mapped to a stack of $Nm^1 m^2 m^3$ D9 branes whose magnetic field is

$$F = \sum_{a=1,2,3} \frac{\pi i}{\text{Im}\, \tau^a} \frac{n^a}{m^a m^1 m^2 m^3} 1_{N \times m^1 m^2 m^3} \, dz^a \wedge d\bar{z}^a. \quad (A.9)$$

Again, the rank of the gauge group is reduced from $Nm^1 m^2 m^3$.

It is well known [25] that under T-duality the adjoint fields corresponding to the position of a brane in the direction under which the T-duality is performed are mapped to Wilson line fields living in the world-volume of the brane, via the explicit mapping

$$\frac{\lambda}{R} \overset{\tau}{\longleftarrow} y$$

with $\lambda$ being the Wilson line, $y$ the position of the brane, and $2\pi R$ the length of the 1-cycle upon which we make the T-duality.

We must emphasize that we have performed the T-duality from Type IIA along the three $x^i$ axes. Alternatively, one can perform any other combination of dualities arriving to an equivalent situation with branes at angles and magnetised branes. One that is particularly interesting is starting from the Type IIA setup with branes at angles referred above and perform three T-dualities along the $y^i$-axes. One reaches in this way Type I theory (or an orbifold thereof). Again one can have all kinds of D$p$-branes with $p$ odd and fluxes. The dictionary to go from the Type IIB language quoted in this appendix and the Type I language[18] that is used in section 2.3 is as follows[19]. One must change all pairs of $(n^i, m^i)$ along this section by $(m^i, n^i)$. In this image are the $m$’s who play the role of magnetic flux quanta. Also, a D3 in one image is mapped to a D9 in the other image, and a D7 in one image goes to

---

[18] This is actually an imprecision we make in order to differentiate between both images, since both of them are Type IIB and are related by 6 T-dualities.

[19] One must stress that in order for this to be a real duality the complex structure parameters $\tau^i$ must be imaginary, since a real part for $\tau$ would complicate the duality.
a D5 in the other one. Finally, we would like to recall that one can generate lower
dimensional D-brane charges in the world-volume of higher dimensional ones when
turning on magnetic fluxes on them, or, alternatively, any Dp-brane can be seen as
a Dp′-brane, \( p < p' \), with magnetic flux on it. Thus, a D5-brane can be seen as a
D9-brane with infinite magnetic flux in one torus and minus infinite magnetic flux
in another torus. This implies, as emphasized in the main text, that, at least in the
absence of RR and NSNS fluxes, one can see every previously considered toroidal
inflationary model as coming just from Type I D9 branes with magnetic fluxes and
Wilson lines turned on.

B. A concrete model

As already mentioned, Wilson line inflation provides a T-dual image of angled-brane
inflation (studied in detail in [3]) and, as such, its predictions are rather similar to
this form of brane inflation in a large set of situations, albeit with some subtleties
like the role played in this situation by the momentum modes. However, we want to
stress that there are some situations in which Wilson lines and brane positions are
not entirely equivalent in a given physical situation (though, of course, T-duality will
change the roles of both kinds of fields). There are some situations, like compactifica-
tions with RR and NSNS fluxes, that are highly asymmetric with respect to Wilson
lines and brane positions: for example, brane positions get generically a (either soft
or supersymmetric) mass from the effect of these fluxes whereas the potential for
Wilson lines remains flat. We want to explore this fact to get inflation with Wilson
lines in a situation in which the relative positions between two stacks of branes are
stabilised.

In the following subsection we will review the mechanism (discovered in [32] and
reinterpreted in an elegant way in [31]; see also [33] for closely related work) by which
brane positions are stabilised in the presence of supersymmetric \( G_3 \) flux, and show
how and where brane positions are fixed. We present this model to show how our
model can be implemented in a fluxed Type IIB setup.

B.1 Fixing D7 brane positions with fluxes

Consider Type II string theory compactified on the \( T^6/Z_2 \) orientifold considered by
[30]. Consider a RR and NSNS flux of the form

\[
\frac{1}{4\pi^2\alpha'} F_3 = a_0\alpha_0 + a \sum_i \alpha_i + b \sum_i \beta_i + b_0\beta_0 \\
\frac{1}{4\pi^2\alpha'} H_3 = c_0\alpha_0 + c \sum_i \alpha_i + d \sum_i \beta_i + d_0\beta_0,
\]  

(B.1)  

(B.2)
where the $a, b, c, d$ and the $a_0$'s etc are integers and the $\alpha_a$'s and $\beta_a$'s form a basis of real 3-forms:

\[
\begin{align*}
\alpha_0 &= dx^1 \wedge dx^2 \wedge dx^3 \\
\beta_0 &= dy^1 \wedge dy^2 \wedge dy^3 \\
\alpha_i &= dy^i \wedge dx^j \wedge dx^k \\
\beta_i &= -dx^i \wedge dy^j \wedge dy^k,
\end{align*}
\] (B.3)

where the order of the indices in $\alpha_i$ and $\beta_i$ is defined to be the canonical one. We are going to use the mechanism described in [31] to fix the moduli of the branes at different positions in the torus. Consider a D7 brane wrapping the second and third torus, and being point-like in the first one. Given the $H_3$ flux (B.2), we can find a local description of $B_2$ in a patch that is convenient for us:

\[
B_2 = 4\pi^2 \alpha' \{(x^1 c_0 + y^1 c_0)dx^2 \wedge dx^3 + (-x^1 d + y^1 d_0)dy^2 \wedge dy^3 \\
+ (x^1 c - y^1 d)(dx^2 \wedge dy^3 + dy^2 \wedge dx^3)\}. \tag{B.4}
\]

Consider the following magnetic flux in the D7-brane

\[
F_2 = 2\pi \left( \alpha \ dx^2 \wedge dx^3 + \beta \ dy^2 \wedge dy^3 + \gamma \ (dx^2 \wedge dy^3 + dy^2 \wedge dx^3) \right) \tag{B.5}
\]

with $\alpha, \beta, \gamma$ integers. The supersymmetry condition for the D7 implies $B|_{D7} + 2\pi \alpha' F_2 = 0$, which is translated into the equations

\[
\begin{align*}
x_1 c_0 + y_1 c + \alpha &= 0 \\
x_1 d + y_1 d_0 + \beta &= 0 \\
x_1 c - y_1 d + \gamma &= 0.
\end{align*} \tag{B.6}
\]

Since the only variables here are $x_1, y_1$, the system is overdetermined. One must stress however that the flux integers $c, d$'s and the magnetic integers $\alpha, \beta, \gamma$ are not free parameters but are subject to some other conditions. At this point, it is better to take a precise example from where we extract the flux parameters and see what are the conditions on $F$. Before doing that, we can compute the mass corresponding to the open string field whose vev gives the position of the D7. It is given by [31, 32]

\[
m^2 = \frac{g_s}{2} |G_{123}|^2, \tag{B.7}
\]

where $G_3 = F_3 - \phi H_3$, with $\phi = C_0 + i/g_s$ the axio-dilaton. Now, from (B.1), (B.2) we get, using $dz^i = dx^i + \tau^i dy^i$,

\[
\begin{align*}
F_{123} &= K(\tau^i) \left( a_0 \tau^1 \bar{\tau}^2 \bar{\tau}^3 - a (\tau^1 \bar{\tau}^2 + \tau^1 \bar{\tau}^3 + \bar{\tau}^2 \bar{\tau}^3) - b(\tau^1 + \bar{\tau}^2 + \bar{\tau}^3) - b_0 \right) \tag{B.8}
\end{align*}
\]

\[
\begin{align*}
H_{123} &= K(\tau^i) \left( c_0 \tau^1 \bar{\tau}^2 \bar{\tau}^3 - c (\tau^1 \bar{\tau}^2 + \tau^1 \bar{\tau}^3 + \bar{\tau}^2 \bar{\tau}^3) - d(\tau^1 + \bar{\tau}^2 + \bar{\tau}^3) - d_0 \right) \tag{B.9}
\end{align*}
\]

\[\text{20}\text{We are using the symbols } F_2 \text{ for the magnetic field and } F_3 \text{ for the RR field. We hope this rather conventional choice will not confuse the reader.}\]

\[\text{21}\text{Note that satisfaction of these equations implies that no tadpole cancellation is required for the numbers } \alpha, \beta, \gamma.\]
with
\[ K(\tau^i) \equiv \frac{i\pi^2 \alpha'}{2 \prod_i \text{Im} \tau^i (2\pi)^3 R_1 R_2 R_3}. \quad (B.10) \]

Note that the prefactor $K$ encodes all the dependence of the masses in the Kähler moduli. Let us take for example the values\(^{22}\) $(a_0, a, b, b_0) = n_1(1, 0, 0, 1), (c_0, c, d, d_0) = n_2(1, -1, -1, -2)$ from \([30]\), with $n_i$ integers. These integers fix the complex structure moduli and dilaton to the values
\[ \tau^i = e^{\frac{2\pi i}{3}} \equiv \tau, \quad \phi = \frac{n_1}{n_2} e^{\frac{2\pi i}{3}} = \frac{n_1}{n_2} \tau. \quad (B.11) \]

This implies $g_s = (2n_2/\sqrt{3n_1})$. These fluxes produce a D3 tadpole
\[ N_{\text{flux}} = \frac{1}{(2\pi)^4 \alpha'^2} \int H_3 \wedge F_3 = 2n_1 n_2. \quad (B.12) \]

For these values, the equations (B.6) become
\[
\begin{align*}
x_1 - y_1 &= -\alpha/n_2 \\
x_1 - 2y_1 &= -\beta/n_2 \\
-x_1 + y_1 &= -\gamma/n_2.
\end{align*} \quad (B.13)
\]

One possible solution is to take $\alpha = -\gamma$. Then the position of the brane is fixed at
\[ x_1 = \frac{\beta - 2\alpha}{n_2}, \quad y_1 = \frac{\beta - \alpha}{n_2}. \quad (B.14) \]

We can also compute the mass of the open string excitation corresponding to the position of the brane. The relevant component of the $G_3$ flux is given by
\[ G_{123} = -n_1 K(\tau)(\tau + 2)^2 \quad (B.15) \]

So
\[ m^2 = \frac{n_1^2 g_s}{512\pi^2 (\text{Im} \tau)^2} (5 + 4 \text{Re} \tau)^2 \frac{\alpha'^2}{(R_1 R_2 R_3)^2} = \frac{n_1 n_2}{12\sqrt{3} \pi^2} \frac{\alpha'^2}{(R_1 R_2 R_3)^2}. \quad (B.16) \]

We must emphasize that this mass term is supersymmetric. On the other hand, there is no such constraint for Wilson lines. As emphasized in \([32]\), this is due to the fact that generic stabilisation of Wilson lines corresponding to a given D7 brane would be incompatible with gauge invariance.

We are computing the inflationary potential assuming that the positions are stabilised, but, in a given situation, we must check that this is indeed the case.

\(^{22}\)To avoid the kind of subtleties pointed out in \([30], [36]\), the values of $n_1, n_2$ can be restricted to be even.
To see this, consider a situation in which the Wilson lines are put to zero and the position field is expressed in terms of some canonically normalised field $\phi = y/2\pi\alpha'$, $y$ being the position of a brane [32]. The mass (B.16) is computed for this field. The potential for $y$ near the minimum $\phi_0 = 2\pi\alpha'y_0$, with $y_0$ given in (B.14) is roughly

$$V(\phi) \simeq m^2(\phi - \phi_0)^2 - \frac{k}{\phi_0^2}$$  \hspace{1cm} (B.17)

with $k$ arbitrarily small when $\theta \to 0$. The addition of the $\phi^{-2}$ part to the quadratic potential will only manifest itself in scales of order $\frac{k}{\phi_0^4}$ around $\phi_0$. That means that we can consider the field $\phi$ to be stabilised whenever

$$m^2 \gg \frac{k}{\phi_0^4},$$  \hspace{1cm} (B.18)

that can be satisfied easily just making the angle small. This basically means that any attempt of getting inflation with some field related to the position of a D7-brane in the presence of $G_3$ flux will necessarily face the issue that this field will unavoidably receive a non-negligible positive contribution to its mass coming from its backreaction to the flux.

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