Exponentially Decreasing Critical Detection Efficiency for Any Bell Inequality

Nikolai Miklin,1, 2, Anubhav Chaturvedi,1, 3, Mohamed Bourennane,3,†
Marcin Pawłowski,1, 4, § and Adán Cabello5, 6, ¶
1Institute of Theoretical Physics and Astrophysics, National Quantum Information Center, Faculty of Mathematics,
Physics and Informatics, University of Gdansk, 80-952 Gdańsk, Poland
2Heinrich Heine University Düsseldorf, Universitätsstraße 1, 40225 Düsseldorf, Germany
3Department of Physics, Stockholm University, S-10691 Stockholm, Sweden
4International Centre for Theory of Quantum Technologies (ICTQT), University of Gdansk, 80-308 Gdańsk, Poland
5Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain
6Instituto Carlos I de Física Teórica y Computacional, Universidad de Sevilla, E-41012 Sevilla, Spain

We address the problem of closing the detection efficiency loophole in Bell experiments, which is crucial for real-world applications. Every Bell inequality has a critical detection efficiency \( \eta \) that must be surpassed to avoid the detection loophole. Here, we propose a general method for reducing the critical detection efficiency of any Bell inequality to arbitrary low values. This is accomplished by entangling two particles in \( N \) orthogonal subspaces (e.g., \( N \) degrees of freedom) and conducting \( N \) Bell tests in parallel. Furthermore, the proposed method is based on the introduction of penalized \( N \)-product (PNP) Bell inequalities, for which the so-called simultaneous measurement loophole is closed, and the maximum value for local hidden-variable theories is simply the \( N \)-th power of the one of the Bell inequality initially considered. We show that, for the PNP Bell inequalities, the critical detection efficiency decays exponentially with \( N \). The strength of our method is illustrated with a detailed study of the PNP Bell inequalities resulting from the Clauser-Horne-Shimony-Holt inequality.

**Introduction.**—Quantum correlations arising from local measurements on entangled particles [1] allow for multiple applications, including device-independent randomness expansion [2–5], quantum key distribution [6–10], secret sharing [11, 12], self-testing [13, 14], and certification of quantum measurements [15–17]. All these tasks require a loophole-free Bell test [18–21] as a necessary condition. The most challenging problem from the applications’ perspective is closing the detection loophole [22], since otherwise an adversary can simulate the behavior of entangled particles provided that a sufficient fraction of them remains undetected. Therefore, a fundamental problem is to identify quantum correlations that cannot be simulated with local hidden-variable (LHV) models even when the detection efficiency is relatively low.

The detection efficiency in a Bell inequality test is the ratio between the number of systems detected by the measuring devices and the number of systems emitted by the source. It depends not only on the properties of the detectors, but also on the losses in the channel. Closing the detection loophole requires surpassing a certain threshold detection efficiency, which depends on the quantum correlations chosen. For symmetric Bell tests (i.e., those in which all detectors have the same detection efficiency) and zero background noise, the necessary and sufficient threshold detection efficiency for entangled qubits can be as low as 2/3 for partially entangled states [23] and 0.828 for maximally entangled states [24]. Massar [25] showed that high-dimensional systems could tolerate a detection efficiency that decreases with the dimension \( d \) of the local quantum system. However, this result is of limited practical interest since an improvement over the qubit case occurs only for \( d > 1600 \). Vértesi, Pironio, and Brunner [26] identified a symmetric Bell inequality for which the efficiency can be lowered down to 0.618 for partially entangled states and 0.77 for maximally entangled states, using four-dimensional systems and assuming perfect visibility, which is still not sufficiently low for practical applications. Other proposals for loophole-free Bell tests with low detection efficiency either combine low-efficient detectors with nearly perfect ones [27–30] or use more than five spatially separated parties [31–33], which is impractical for real-world applications.

The critical detection efficiency \( \eta \) is not the only important parameter in a loophole-free Bell experiment. Another essential variable is the required visibility \( v \), which quantifies how much noise can be tolerated. The best combinations of parameters \((\eta, v)\) reported in photonic experiments in distances \( \lesssim 200 \text{ m} \) are: \((0.774, 0.99)\) [19], \((0.763, 0.99)\) [5], and \((0.8411, 0.9875)\) [4]. However, these values are very difficult to achieve in longer distances.

In this work, we propose a general method to reduce the detection efficiency requirement exponentially for any given Bell inequality. This is achieved by violating \( N \) Bell inequalities in parallel with a source of \( N \) entangled states carried by a single pair of particles. The value of the required detection efficiency then scales like \( \left( \frac{C}{\eta} \right)^N \), where \( C \) is the LHV bound, and \( Q \) is a quantum value, i.e., the decay is exponential. Moreover, our method reduces the
required detection efficiency for a given target visibility or a Bell inequality violation. We analyze in detail the case of parallel violation of N Clauser-Horne-Shimony-Holt (CHSH) Bell inequalities [34]. Another advantage of our approach is that the observed correlations can be directly used for practical applications, since the observed value of N CHSH inequalities can be connected to the violation of an individual CHSH inequality. Hence, there is no need to develop new protocols based on Bell inequalities with more settings [35].

Physical setup.—Consider a Bell experiment in which two spatially separated parties, Alice and Bob, have access to a source of high-dimensional entanglement carried by a single pair of particles. The key examples to keep in mind are hyperentangled states [36], in which two photons are entangled across multiple degrees of freedom, and photon pairs entangled in high-dimensional degrees of freedom [37]. Throughout the text, we consider photons as physical carriers of entanglement, however, similar reasoning can be applied to atoms, ions, etc.

Let us assume now that the carried high-dimensional entangled state is a product of N entangled states, as it is the case for hyperentanglement [36]. We also assume that Alice and Bob can perform joint measurements on their subsystems producing N outcomes each from a single click of their detectors. The main idea of the method is to use N outcomes from each run of the experiment to violate N Bell inequalities in parallel. In this way, the probability of detectors’ clicks for each of the N inequalities is of the order of the Nth root of the efficiency of the photon detection, i.e., it is effectively increased. We will provide a rigorous analysis that supports this claim.

To the best of our knowledge, the conjecture that the critical detection efficiency could be lowered by integrating several qubit-qubit entangled states in one pair of particles was first made in Ref. [38], without a proof. In Ref. [39], it was shown that the critical detection efficiency could be reduced for the so-called Einstein-Podolsky-Rosen-Bell inequalities that require perfect correlations [40]. Similar ideas have been developed in later works focused on quantum key distribution [41, 42] and the P value of a Bell test [43]. Very recently, the idea has been explored for the case of 2-qubit maximally entangled states [44]. In this paper, we introduce a much more powerful and practical tool: penalized N-product (PNP) Bell inequalities. This tool leads to smaller critical detection efficiencies than those obtained in Ref. [44] and applies to any quantum violation of any Bell inequality, thus opening a new path toward loophole-free Bell tests with longer distances and higher dimensions.

Product Bell inequalities.—Let us consider N Bell inequalities of the same type in parallel. Our first task is to identify a single parameter that quantifies the violation of local realism. One way to do it is to consider the product of the N parameters of all N Bell inequalities. Following this approach, let us start with a Bell inequality of the form

$$\sum_{a,b,x,y} p(a, b|x, y) c_{a,b}^{x,y} \leq C,$$  \hspace{2cm} (1)

where $p(a, b|x, y)$ denotes the conditional probability of Alice and Bob to observe outcomes $a$ and $b$ (with $a, b \in [m]$), respectively, given their choice of measurement settings $x$ and $y$ (with $x, y \in [n]$), and $C$ is the LHV bound. Throughout the text, $[n] = \{0, 1, \ldots, n-1\}$. An N-product Bell inequality based on Eq. (1) is defined as

$$\sum_{a,b,x,y} p(a, b|x, y) \prod_{i=1}^{N} c_{a_i,b_i}^{x_i,y_i} \leq C_N,$$  \hspace{2cm} (2)

where $a = (a_1, \ldots, a_N)$ is a tuple of Alice’s measurement outcomes, with $a_i \in [m]$ for all $i \in \{1, 2, \ldots, N\}$, and $b, x, y$ similarly defined. $C_N$ denotes the maximum value of the N-product Bell inequality attainable by LHV models.

One could expect that $C_N = C^N$. However, this is not the case for arbitrary Bell inequalities of the form given by Eq. (1), including the CHSH inequality [34]. Indeed, for the CHSH inequality, $C = \frac{3}{2}$ but $C_2 = \frac{10}{16}$ [38] and $C_3 = \frac{31}{48}$ [43]. This fact is also referred to as the simultaneous measurement loophole in Bell tests [38]. The problem of determining the closed form for $C_N$ (for the cases when $C_N > C^N$) is closely related to the so-called parallel repetition theorem in interactive proof systems [45]. This problem was tackled in Ref. [46–48], where only asymptotic upper-bounds on $C_N$ were reported. Moreover, the authors of Ref. [47] emphasized the difficulty of finding exact values of $C_N$.

In this work, we take a different approach to the problem. Instead of trying to find the values of $C_N$, we propose a method for modifying the Bell expression in Eq. (2) in a way that $C_N = C^N$ holds for all $N$. We achieve this by adding a nonlinear “penalty term” to the left-hand side of Eq. (2), which forces a product local strategy (i.e., one in which each outcome $a_i$ depends only on $x_i$, and similarly for Bob) to be optimal. Given a Bell expression specified by coefficients $c_{a,b}^{x,y}$ and the LHV bound $C$, we define a penalized N-product (PNP) Bell inequality as follows:

$$\sum_{a,b,x,y} p(a, b|x, y) \prod_{i=1}^{N} c_{a_i,b_i}^{x_i,y_i} - \kappa (A + B) \leq C^N,$$  \hspace{2cm} (3)

where $\kappa \in \mathbb{R}$ is some large positive number and

$$A = \sum_{i=1}^{N} \sum_{x_i=x'_i}^{x_i=m-2} \sum_{x_i \neq x'_i}^{x'_i=m-2} |p(a_i|x) - p(a_i|x')|,$$  \hspace{2cm} (4a)

$$B = \sum_{i=1}^{N} \sum_{y_i=y'_i}^{y_i=m-2} \sum_{y_i \neq y'_i}^{y'_i=m-2} |p(b_i|y) - p(b_i|y')|.$$  \hspace{2cm} (4b)

The sum over $x'$ is taken such that $x$ and $x'$ match on the $i$th element, but are not the same. The same holds for the
sum over \( y' \), \( p(a_i|x) \) denotes the marginal probability of outcome \( a_i \) of Alice’s measurement specified by \( x \). \( p(b_j|y) \) is analogously defined for Bob.

The general idea of the method is rather straightforward. By taking large enough \( \kappa \), we force both quantities \( A \) and \( B \) in Eq. (4) to be exactly 0. The condition \( A = 0 \) implies that Alice has to choose her local strategy among nonsignaling ones with respect to her local outcomes \( a_i \) and settings \( x_i \). \( B = 0 \) implies the same for Bob. Note that this set of strategies is larger than the set of product strategies for which \( p(a(x) = \prod_{i=1}^{N} p(a_i|x_i) \) holds. Nevertheless, it is not difficult to show that the nonsignaling constraints \( A = B = 0 \) enforce the bound to be \( C_N \) for the ideal case of infinite runs of the experiment (see Appendix C).

The remaining question is how large should one take \( \kappa \) to be. We answer this question below for the case of \( m = 2 \).

**Result 1.** Given a Bell inequality specified by the coefficients \( c_{a,b} \), with \( a, b \in \{0, 1\} \), \( x, y \in [n] \), it is sufficient to take \( \kappa = n^{-1}(\Sigma_N - C^N) \), such that the LHV bound of the corresponding PNP Bell inequality is \( C^N \), where \( \Sigma_N \) is the algebraic bound of the \( N \)-product Bell inequality without the penalty term.

Note that, instead of \( \Sigma_N \), any known, possibly tighter, bound on \( C_N \) can be used \([49]\).

**Proof.** For the proof, we use the terminology of probability vectors and local polytopes introduced in Ref. \([50]\).

For the Bell scenario with \( n \) settings per party and binary outcomes, the probability vector is defined as \( p = [p(0,0), \ldots, p(1,1)] = \{0,1\} - 1 \); i.e., it is a vector that uniquely specifies the behavior \( p(a,b|x,y) \). The local polytope \( P_{LHV} \) is the region in the space of \( p \), corresponding to LHV models. This polytope is convex and, by the Weyl-Minkowski theorem, it can be described either as a convex hull of its extremal points (in this case determined by local deterministic strategies) or as an intersection of half-spaces (which in this case are tight Bell inequalities and axioms of probabilities). The above concepts generalize straightforwardly to our scenario with multiple inputs and outputs, and we will use \( p \) and \( P_{LHV} \) to denote these concepts for our case.

For convex polytopes, the maximum of a linear function such as the one in Eq. (1) is attained at one of its extremal points. Although the expression in Eq. (3) is not linear on the whole \( P_{LHV} \), it is linear in each part of \( P_{LHV} \) for which every expression inside moduli in Eq. (4) has a definite sign. Hence, the global maximum has to be attained at either one of the extremal points of \( P_{LHV} \), or at a point resulting from the intersections of the facets of \( P_{LHV} \) by the hyperplanes \( p(a_i|x) - p(a_i|x') = 0 \) and \( p(b_j|y) - p(b_j|y') = 0 \), for some sets of \( i,j \) and some pairs \( x \neq x' \) and \( y \neq y' \). Let us denote the set of all of such points as \( E = \{p_{a_i}\}_i \).

Among all the points \( p_{a_i} \) in \( E \), there are some, let us call them \( E_0 \), for which \( A = B = 0 \) holds. For points in \( E_0 \), the minimal value of \( A + B \) is \( n^{-1}(N-1) \) (see Appendix B). On the other hand, the value of the expression in Eq. (3) on any of the points \( p_{a_i} \) without the penalty term, cannot exceed its algebraic maximum \( \Sigma_N \). Therefore, taking \( \kappa = n^{-1}(\Sigma_N - C^N) \) ensures that the LHV bound of Eq. (3) cannot exceed the one for strategies compatible with \( A = B = 0 \), i.e., the product bound \( C^N \) (see Appendix C).

The purpose of the upper bound on the sufficient value of \( \kappa \) is not only theoretical. In practice, even if we use a product quantum strategy, due to experimental errors both \( A \) and \( B \) will have small yet nonzero values. These errors will be multiplied by \( \kappa \) and could potentially result in large errors in the value of the violation.

**Lowering the critical detection efficiency.**—Here, we show that having a source of photon pairs carrying \( N \) entangled states each alongside with PNP Bell inequalities allows for a significant reduction in the critical detection efficiency requirements for the violation of local realism.

To avoid the fair sampling assumption \([51]\), the parties need to either treat “no-click” events as additional outcomes or employ a local assignment strategy \([52]\). The latter means that whenever one party’s detector does not click (when it should), the party draws an outcome according to some local (deterministic) strategy. This allows the parties to use the same Bell inequality without the need to find one with more outcomes.

In this work, we consider the local assignment strategy for mitigation of the “no-click” events. Let \( \otimes_{i=1}^{N} p_{AB} \) be a state carried by photon pair in out setup. Let \( \otimes_{i=1}^{N} A_{a_i}^{(i)} \) and \( \otimes_{i=1}^{N} B_{b_i}^{(i)} \) be the POVM (positive-operator valued measure) elements of Alice and Bob respectively, i.e., they are formed by the POVM elements \( A_{a_i}^{(i)} \) and \( B_{b_i}^{(i)} \), that are the same for all \( i \). Evidently, this leads to quantum behavior of the form \( p(a,b|x,y) = \prod_{i=1}^{N} \text{tr}(A_{a_i}^{(i)} \otimes B_{b_i}^{(i)} p_{AB}) \). Let \( \alpha : [n] \rightarrow \{0,1\} \) and \( \beta : [n] \rightarrow \{0,1\} \) be deterministic assignment strategies \( a_i = \alpha(x_i) \) and \( b_i = \beta(y_i) \), for all \( i \), employed by Alice and Bob respectively in case of a “no-click” event. If, for instance, Bob’s detector does not click but Alice’s does, the parties observed behavior is \( p(a,b|x,y) = \prod_{i=1}^{N} \text{tr}(A_{a_i}^{(i)} p_{A}) \delta_{b_i,\beta(y_i)} \), where \( p_{A} \) is Alice’s reduced state \( p_{AB} \) and \( \delta_{a,b} \) is the Kronecker delta. Similarly, the parties observe the behavior \( p(a,b|x,y) = \prod_{i=1}^{N} \text{tr}(B_{b_i}^{(i)} p_{B}) \delta_{a_i,\alpha(x_i)} \) whenever Alice’s detector does not click, but the one of Bob does. Finally, for the cases of no clicks on both detectors, the parties observe a local deterministic behavior \( p(a,b|x,y) = \prod_{i=1}^{N} \delta_{a_i,\alpha(x_i)} \delta_{b_i,\beta(y_i)} \).

Let us now take \( c_{a,b}^{xy} \geq 0 \) in the considered Bell inequality, which can always be achieved. Assuming the detection efficiency of Alice’s and Bob’s detectors to be \( \eta \), the value of the PNP Bell expression is the following:

\[
\eta^2 Q^N + \eta (1-\eta)(A^N + B^N) + (1-\eta)^2 C^N,
\]

(5)
with

\[ Q = \sum_{a,b,x,y} c_{a,b,x,y}^x \text{tr}(A_a^x \otimes B_b^y \rho_{AB}), \quad (6a) \]
\[ A = \sum_{a,b,x,y} c_{a,b,x,y}^x \text{tr}(A_a^x \rho_A) \delta_{b,\beta(y)}, \quad (6b) \]
\[ B = \sum_{a,b,x,y} c_{a,b,x,y}^x \text{tr}(B_b^y \rho_B) \delta_{a,\alpha(x)}, \quad (6c) \]

where we have assumed that the local strategies \( \alpha \) and \( \beta \) reproduce the LHV bound \( C \). Clearly, since all the aforementioned strategies are product, the penalty term is exactly 0. Notice that in Eq. (5), \( \eta \) appears only in its second power, precisely due to the fact that the \( N \)-qubit state \( \otimes_{i=1}^N \rho_{AB} \) is carried by a single pair of photons. This is what we meant when we said that the effective detection efficiency for each of the \( N \) Bell inequalities is of the order of \( \eta^2 \).

To observe a violation of local realism, one needs to ensure that the value of the expression in Eq. (5) is greater than the LHV bound \( C^N \). Solving this inequality with respect to \( \eta \), we obtain the following value of the required detection efficiency for given \( Q, A, \) and \( B \):

\[ \eta = \frac{2C^N - A^N - B^N}{Q^N + C^N - A^N - B^N}. \quad (7) \]

This equation has the following interesting implication.

**Remark 1.** For any given Bell inequality with binary outcomes and a quantum strategy with \( Q > C \), it follows from Eq. (7) that the detection efficiency requirement decays exponentially with \( N \).

Indeed, if we take \( A = B = \delta C \), then \( \eta = 2 \left( \frac{C}{Q} \right)^N (1 - \delta^N) + O \left( \frac{\delta^N}{2N} \right) \). For any Bell inequality, \( \delta < 1 \) whenever \( Q > C \). Hence, the decay of \( \eta \) with \( N \rightarrow \infty \) is at least exponential with the factor of \( \log \left( \frac{C}{Q} \right) \). The above remark is in parallel with the results of Massar [25]. It is worth noting that the critical detection efficiency in the Massar construction decreases exponentially with dimension, but necessitates an exponentially large number of measurement settings. Our method provides a polynomial in dimension, i.e., an exponential in the number of qubits, reduction in critical detection efficiency, but it only requires a polynomial number of measurement settings.

In order to find the critical detection efficiency \( \eta_{\text{crit}} \) for a given Bell inequality and its corresponding PNP Bell inequality, one needs to optimize \( \eta \) in Eq. (7) over all possible values of \((Q, A, B)\). In what follows, we solve this optimization problem for the \( N \)-product CHSH inequality.

**PNP inequality for the CHSH inequality.**—The coefficients of the CHSH inequality [34] in its nonlocal game formulation are \( c_{a,b}^{x,y} = \frac{1}{4} \delta_{a\oplus b, x\oplus y} \), where \( a, b, x, y \in \{0, 1\} \) and \( \oplus \) denotes addition modulo 2. For this form of the CHSH inequality, we have \( C = \frac{3}{4} \), the quantum bound \( Q_{\text{max}} = \frac{1}{2} + \frac{1}{2\sqrt{2}} \), and \( \Sigma = 1 \). In order to minimize the expression in Eq. (7) over all quantum states and measurements, first we determine the maximal values of \( A \) and \( B \), and then optimize Eq. (7) over \( Q \). In particular, due to the symmetry with respect to \( A \) and \( B \) in Eq. (7), we are interested in the situation \( A = B \). For this case, the optimal relation is the following:

\[ A = B = \frac{1}{2} + \frac{1}{4} \sqrt{(1 - q) \left( 1 + \frac{q}{\sqrt{1 + q^2}} \right)}, \quad (8) \]

where \( q = \sqrt{(4Q - 2)^2 - 1} \). As \( Q \) changes from \( \frac{3}{4} \) to \( \frac{1}{2} + \frac{1}{2\sqrt{2}} \), \( q \) increases from 0 to 1, and, hence, \( A \) and \( B \) decrease from \( \frac{3}{4} \) to \( \frac{1}{2} \). For the 2-qubit state \( \rho_{AB} \) and qubit measurements \( A_a^x \) and \( B_b^y \) that produce the relation in Eq. (8) see Appendix D. We used the Navascús-
Employing the relation in Eq. (8), we optimize $\eta$ in Eq. (7) over $Q$ in order to obtain the optimal value $\eta_{\text{crit}}$ for a given $N$. We plot the results in Fig. 1. In the same figure, we show the minimal visibility $v_{0.75}$ for which violation can still be observed with detectors of a given detection efficiency $\eta = 0.75$. As we can see, even though taking 2, 3, and 4-product CHSH inequalities does not decrease the value of $\eta_{\text{crit}}$, one can obtain a significant advantage in terms of visibility for $\eta > \eta_{\text{crit}}$.

In Fig. 2 we plot $\eta_v$, the required detection efficiency to observe a violation of the PNP Bell inequality with visibility as low as $v$. We also account for possible experimental imperfections by taking nonzero values of $A + B$. Note that the tolerance to the imperfections can be significantly increased if, instead of taking an algebraic maximum $\Sigma_N$ in Result 1, a known bound on the parallel repetition of the Bell inequality is used [49].

Summary and outlook.—In this work, we addressed the problem of reducing the detection efficiency requirements for loophole-free Bell experiments in order to achieve loophole-free Bell tests over longer distances. We presented a method that, when applied to any given Bell inequality, produces a new Bell inequality by taking a penalized product of $N$ copies of it, for which the critical detection efficiency decays exponentially with $N$. This implies that the critical detection efficiency can be drastically reduced in experiments using photon sources that allow for encoding multiple copies of a qubit-qubit (or qudit-qudit) entangled state on a single pair of particles. Examples of such sources are hyperentanglement sources and sources of high-dimensional entanglement.

We applied our method to several binary Bell inequalities and found that the lowest detection efficiencies occur for the PNP CHSH inequality. The advantage of the CHSH inequality is in terms of both critical detection efficiency and visibility of the violation. Our method can be applied to any Bell inequality with more outcomes, given that Result 1 can be extended to an arbitrary number of outcomes. A natural target for future work is to identify Bell inequalities for which the critical detection efficiencies are low enough for mid-distance photonic loophole-free Bell tests and related applications such as device-independent quantum key distribution.

Other important questions deserve separate investigation. For instance, we believe that the bound on the penalty coefficient $\kappa$ in Result 1 can be significantly lowered. Another relevant problem is the calculation of $P$ values for PNP Bell inequalities, which would depend on the value of the penalty term. Finally, it is interesting to see whether PNP Bell inequalities can be used for a single-shot Bell test [43].

ACKNOWLEDGMENTS

We would like to thank Konrad Banaszek, Costantino Budroni, Mateus Araújo, Jean-Marc Merolla, Miguel Navascués, and Marek Żukowski, for fruitful discussions and comments. This research was made possible by funding from QuantERA, an ERA-Net cofund in Quantum Technologies (www.quantera.eu), under projects SECRET (MINECO Project No. PCI2019-111885-2) and eDICT. We also acknowledge the financial support by First TEAM Grant No. 2016-1/5 and Project Qdisc (Project No. US-15097), with FEDER funds. This research was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), Project No. 441423094. M. P. acknowledges support by the Foundation for Polish Science (IRAP project, ICTQT, Contract No. 2018/MAB/5, cofinanced by EU within Smart Growth Operational Programme). A. Ch. acknowledges support by the NCN grant SHENG (Contract No. 2018/30/Q/ST2/00625). M. B. acknowledges support by the Knut and Alice Wallenberg Foundation and the Swedish Research Council. A. C. acknowledges support from the Knut and Alice Wallenberg Foundation through the Wallenberg Centre for Quantum Technology (WACQT). The numerical optimization was carried out using Ncpol2sdpa [54], YALMIP [55], MOSEK [56], and CVXOPT [57].
APPENDIX

Additional notation:

- In this Appendix we consider only the case of \(a, b \in \{0, 1\}^N\), i.e., \(a, b\) being binary strings of length \(N\), and \(x, y \in [n]^N\) tuples of integers in \([n]\) of length \(N\). For the sake of simplicity, the notation \(\forall a\), etc. is used instead of \(\forall a \in \{0, 1\}^N\). The same is used for the sums, e.g., \(\sum a\), unless the limits are specified.

- Probability vectors of marginal behaviors \(p_a = [p(a|x)]_{a,x}\) for Alice and \(p_b = [p(a|x)]_{a,x}\) for Bob (here the ordered sets \((\cdot,)\) are taken over all pairs \(a, x\) and \(b, y\), respectively).

- Polytopes of Alice’s marginal behaviors, \(P_A\), and Bob’s marginal behaviors, \(P_B\), both being intersections of \(n^N\) probability simplexes:

\[
P_A = \left\{ p_a \mid p(a|x) \geq 0, \forall a, x, \sum_a p(a|x) = 1, \forall x \right\}, \quad (9a)
\]

\[
P_B = \left\{ p_b \mid p(b|y) \geq 0, \forall b, y, \sum_b p(b|y) = 1, \forall y \right\}. \quad (9b)
\]

- The direct product \(P_A \times P_B\) of vectors \(p_a\) and \(p_b\) is a probability vector \(p\) with its entries given by products of the corresponding entries of \(p_a\) and \(p_b\), i.e., \(p(a_1, b_1) \cdots a_n, b_n) = p(a_1, b_1)p(a_2, b_2) \cdots p(a_n, b_n)\), \(\forall a, b, x, y\).

- The direct product of polytopes \(P_A\) and \(P_B\) is defined as

\[
P_A \times P_B = \left\{ p_a \times p_b \mid \forall p_a \in P_A, \forall p_b \in P_B \right\}. \quad (10)
\]

- The local polytope \(P_{LHV}\) corresponding to the considered Bell scenario is

\[
P_{LHV} = \text{conv}(P_A \times P_B), \quad (11)
\]

where \(\text{conv}(\cdot)\) denotes the operation of taking the convex hull.

- Sets of constraints \(L_A, L_B\), referred to below as the marginal nonsignaling constraints, resulting from the penalty terms being equal to zero, i.e., \(A = 0\) and \(B = 0\). More precisely,

\[
L_A = \left\{ p(a_i|x) - p(a_i|x') = 0 \mid \forall x \neq x', x_i = x'_i, a_i \in \{0, 1\}, \forall i \in \{1, 2, \ldots, N\} \right\}; \quad (12a)
\]

\[
L_B = \left\{ p(b_i|y) - p(b_i|y') = 0 \mid \forall y \neq y', y_i = y'_i, b_i \in \{0, 1\}, \forall i \in \{1, 2, \ldots, N\} \right\}. \quad (12b)
\]

In some parts of this Appendix, we use the same notation \(L_A\) and \(L_B\) for the polytopes defined by the corresponding constraints.

Appendix A: Extremal points of the intersection of the local polytope by the marginal nonsignaling constraints

Here, we show that the extremal points of the polytope resulting from the intersection of \(P_{LHV}\) by \(L_A\) and \(L_B\) are exactly the direct product of the points in \(P_A \cap L_A\) and \(P_B \cap L_B\). This fact is used in the subsequent sections of this Appendix. We state this fact more formally below as a lemma.

Lemma 1. Let \(P_A\) and \(P_B\) be the polytopes of Alice’s and Bob’s marginal behaviors, respectively, and let \(L_A\) and \(L_B\) be the polytopes defined by the marginal nonsignaling constraints. Then, the following two polytopes:

\[
P_1 = \text{conv}(P_A \times P_B) \cap L_A \cap L_B, \quad (A1a)
\]

\[
P_2 = \text{conv}((P_A \cap L_A) \times (P_B \cap L_B)) \quad (A1b)
\]

are equal.
Proof. It is clear that $\mathcal{P}_2 \subset \mathcal{P}_1$, since $\mathcal{P}_2 \subset \mathcal{P}_{\text{LHV}}, \mathcal{P}_2 \subset \mathcal{L}_A$, and $\mathcal{P}_2 \subset \mathcal{L}_B$. Inclusion in the other direction is less trivial. We need to show that there are no extremal points in $\mathcal{P}_1$ that do not belong to $\mathcal{P}_2$. For that purpose, we need to look more closely at the type of solutions to the intersection of the local polytope $\mathcal{P}_{\text{LHV}}$ with $\mathcal{L}_A$ and $\mathcal{L}_B$. First, every such distribution, specified by the probability vector $\mathbf{p}$, must be decomposable as a convex mixture of the direct product of extremal points of $\mathcal{P}_A$ and $\mathcal{P}_B$, i.e.,

$$\mathbf{p} = \sum_{\lambda_a \in \Lambda_A} \sum_{\lambda_B \in \Lambda_B} p(\lambda_a, \lambda_B) \mathbf{p}_a^{\lambda_a} \times \mathbf{p}_b^{\lambda_B}. \tag{A2}$$

Here, $\mathbf{p}_a^{\lambda_a}$ denotes an extremal point of $\mathcal{P}_A$ specified by $\lambda_a$ and, similarly, $\mathbf{p}_b^{\lambda_B}$ denotes an extremal point of $\mathcal{P}_B$ specified by $\lambda_B$. $\Lambda_A$ and $\Lambda_B$ are some subsets of all possible indices of the extremal points of $\mathcal{P}_A$ and $\mathcal{P}_B$, respectively. In the standard situation where no further constraints such as $\mathcal{L}_A$ and $\mathcal{L}_B$ apply, every extremal point in $\mathcal{P}_{\text{LHV}}$, written in the form in Eq. (A2), must have $|\Lambda_A| = |\Lambda_B| = 1$, i.e., there is only one term in the decomposition. This is, however, not true in our case, and we must consider the more general case of $|\Lambda_A| \geq 1$ and $|\Lambda_B| \geq 1$. At the same time, as shown below, for every choice of $\Lambda_A$ and $\Lambda_B$, in the decomposition in Eq. (A2) for an extremal point $\mathbf{p}$, it must be that $p(\lambda_a, \lambda_B) = p(\lambda_a)p(\lambda_B)$, with $p(\lambda_a)$ and $p(\lambda_B)$ being uniquely determined by the choice of sets $\Lambda_A$ and $\Lambda_B$.

Let us fix some $\Lambda_A$ and $\Lambda_B$. Among the vectors $\{\mathbf{p}_a^{\lambda_a}\}_{\lambda_a \in \Lambda_A}$, there could be some that belong to $\mathcal{L}_A$. Let us denote their subset by $\Lambda_A^{\text{NS}}$ (NS stands for nonsignaling) and its complement in $\Lambda_A$ by $\Lambda_A^\text{S}$. Similarly, let us define the sets $\Lambda_B^{\text{NS}}$ and $\Lambda_B^\text{S}$. Then, the decomposition in Eq. (A2) breaks down into four sums: $(\lambda_a, \lambda_B) \in \{\Lambda_A^\text{S} \times \Lambda_B^{\text{NS}}, \Lambda_A^\text{S} \times \Lambda_B^{\text{NS}}, \Lambda_A^{\text{NS}} \times \Lambda_B^\text{S}, \Lambda_A^{\text{NS}} \times \Lambda_B^{\text{NS}}\}$. We now show that, if $\mathbf{p}$ is an extremal point of $\mathcal{P}_1$, then it may contain only terms from the first sum, i.e., $(\lambda_a, \lambda_B) \in \Lambda_A^\text{S} \times \Lambda_B^\text{S}$.

Let us denote by $\mathbf{1}_* \cdot \mathbf{p}_a = 0$ a linear constraint from $\mathcal{L}_A$, i.e., one of the marginal nonsignaling constraints resulting from $\Lambda = 0$, where $\mathbf{1}_*$ stands for the transpose. Similarly, let $\mathbf{1}_*^T \cdot \mathbf{p}_a = 0 \in \mathcal{L}_B$. If $\mathbf{p} \in \mathcal{P}_1$, then necessarily $(\mathbf{1}_* \times \mathbf{1}_b)^T \cdot \mathbf{p} = 0$. For three out of four sums, $(\lambda_a, \lambda_B) \in \{\Lambda_A^\text{S} \times \Lambda_B^{\text{NS}}, \Lambda_A^{\text{NS}} \times \Lambda_B^\text{S}, \Lambda_A^{\text{NS}} \times \Lambda_B^{\text{NS}}\}$, the latter condition is trivially satisfied, since at least one of the parties’ strategies is nonsignaling. Hence, it must also be that the terms in Eq. (A2) corresponding to $(\lambda_a, \lambda_B) \in \Lambda_A^\text{S} \times \Lambda_B^{\text{NS}}$ and $(\lambda_a, \lambda_B) \in \Lambda_A^{\text{NS}} \times \Lambda_B^\text{S}$ must also individually result in a point in $\mathcal{P}_1$. Since we are looking for extremal points in $\mathcal{P}_1$ and they cannot be written as convex mixtures of points in $\mathcal{P}_1$, then we have four mutually exclusive possibilities: $(\Lambda_A, \Lambda_B) \in \{(\Lambda_A^\text{S} \times \Lambda_B^{\text{NS}}), (\Lambda_A^\text{S} \times \Lambda_B^{\text{NS}}), (\Lambda_A^{\text{NS}} \times \Lambda_B^\text{S}), (\Lambda_A^{\text{NS}} \times \Lambda_B^{\text{NS}})\}$. Below, we consider each of the cases separately.

The case $\Lambda_A = \Lambda_A^{\text{NS}}, \Lambda_B = \Lambda_B^{\text{NS}}$ corresponds directly to a point in $\mathcal{P}_2$ and, therefore, is trivial. A little less straightforward is the situation in the other three cases. We start by proving that the extremal points of $\mathcal{P}_1$ corresponding to the case $\Lambda_A = \Lambda_A^\text{S}, \Lambda_B = \Lambda_B^\text{S}$ must also belong to $\mathcal{P}_2$ and the other two possibilities ($\Lambda_A = \Lambda_A^{\text{NS}}, \Lambda_B = \Lambda_B^{\text{NS}}$ and $\Lambda_A = \Lambda_A^{\text{NS}}, \Lambda_B = \Lambda_B^\text{S}$) will follow as special cases. Below, for the sake of simplicity, we omit the superscript $S$ over $\Lambda_A$ and $\Lambda_B$. We also introduce the following notation for given $\mathbf{1}_*^T \cdot \mathbf{p}_a = 0 \in \mathcal{L}_A$ and $\mathbf{1}_*^T \cdot \mathbf{p}_b = 0 \in \mathcal{L}_B$,

$$s_a(\lambda_a) = \mathbf{1}_*^T \cdot \mathbf{p}_a^{\lambda_a}, \forall \lambda_a \in \Lambda_A, \tag{A3a}$$

$$s_b(\lambda_B) = \mathbf{1}_*^T \cdot \mathbf{p}_b^{\lambda_B}, \forall \lambda_B \in \Lambda_B. \tag{A3b}$$

Given $\Lambda_A$, let $p(\lambda_a) = \sum_{\lambda_B \in \Lambda_B} p(\lambda_a, \lambda_B)$ be the marginal distribution of $\lambda_a$ corresponding to extremal point $\mathbf{p}_a$ of the polytope $\mathcal{P}_A \cap \mathcal{L}_A$. Similarly, given $\Lambda_B$, let $p(\lambda_B)$ be the marginal distribution of $\lambda_B$ corresponding to extremal point $\mathbf{p}_b$ in $\mathcal{P}_B \cap \mathcal{L}_B$. For $p(\lambda_a)$ and $p(\lambda_B)$, we have that

$$\sum_{\lambda_a \in \Lambda_A} s_a(\lambda_a)p(\lambda_a) = 0, \tag{A4a}$$

$$\sum_{\lambda_B \in \Lambda_B} s_b(\lambda_B)p(\lambda_B) = 0, \tag{A4b}$$

which hold for every constraint in $\mathcal{L}_A$ and $\mathcal{L}_B$. We now show that there are no other distributions $\tilde{p}(\lambda_a)$ and $\tilde{p}(\lambda_B)$ over the same sets $\Lambda_A$ and $\Lambda_B$ for which the same conditions in Eq. (A4a) are satisfied. We prove it for Alice’s side; the proof for Bob’s side follows the same argumentation. Let us assume the opposite and define $\delta = \min_{\lambda_a \in \Lambda_A} \left| \frac{p(\lambda_a)}{\tilde{p}(\lambda_a)} \right|$, where $\delta \in (0, 1)$ since the distributions $p(\lambda_a)$ and $\tilde{p}(\lambda_a)$ are assumed to be different and it must be that $p(\lambda_a) > 0, \forall \lambda_a \in \Lambda_A$ (otherwise consider smaller $\Lambda_A$). Then, we can define another distribution $\check{p}(\lambda_a)$ as follows:

$$\check{p}(\lambda_a) = \frac{1}{1 - \delta} \left[ p(\lambda_a) - \delta \tilde{p}(\lambda_a) \right], \forall \lambda_a \in \Lambda_A. \tag{A5}$$
By construction, \( \hat{p}(\lambda_a) \geq 0, \forall \lambda_a \in \Lambda_A \), and, by linearity, \( \hat{p}(\lambda_a) \) also satisfies Eq. (A4a). However, in this case \( p(\lambda_a) \) can be written as a convex combination \( p(\lambda_a) = (1 - \delta)\hat{p}(\lambda_a) + \delta\bar{p}(\lambda_a) \) of points in \( \mathcal{P}_A \cap \mathcal{L}_A \), which contradicts our assumption that \( p(\lambda_a) \) corresponds to an extremal point.

Now we show that the only possibility for an extremal point in \( \mathcal{P}_1 \) corresponding to the fixed sets \( \Lambda_A \) and \( \Lambda_B \) is the situation in which \( p(\lambda_a, \lambda_b) = p(\lambda_a)p(\lambda_b) \). The condition for the extremal point corresponding to \( p(\lambda_a, \lambda_b) \) to belong to \( \mathcal{P}_1 \) are Eqs. (A4a) for its marginal distributions \( p(\lambda_a) \) and \( p(\lambda_b) \), together with the following set of constraints:

\[
\sum_{\lambda_a \in \Lambda_A} s_a(\lambda_a)p(\lambda_a) \sum_{\lambda_b \in \Lambda_B} s_b(\lambda_b)p(\lambda_b|\lambda_a) = 0, \quad \forall \mathbf{l}_a \cdot \mathbf{s}_a = 0 \in \mathcal{L}_A, \mathbf{l}_b \cdot \mathbf{s}_b = 0 \in \mathcal{L}_B. \tag{A6}
\]

We intentionally wrote down the above condition asymmetrically with respect to Alice’s and Bob’s distributions, to assist the proof. Now let us sum up the above condition with the constraint on \( p(\lambda_a) \) in Eq. (A4a) to obtain the following:

\[
\sum_{\lambda_a \in \Lambda_A} s_a(\lambda_a)p(\lambda_a) \sum_{\lambda_b \in \Lambda_B} [1 + s_b(\lambda_b)] p(\lambda_b|\lambda_a), \tag{A7}
\]

which must also hold for all the constraints in \( \mathcal{L}_A \) and \( \mathcal{L}_B \). At this moment, it is important to notice that we can define the constraint vectors \( \mathbf{l}_b \) such that \( s_b(\lambda_b) \in \{-1, 0, 1\} \) for all \( \lambda_b \in \Lambda_B \). Indeed, since vector \( \mathbf{l}_b \) represents a constraint on the equality of marginal distributions of \( \mathbf{p}_b \) for various settings, its entries can be taken equal to \(-1\) for one \( \mathbf{y} \), \( \mathbf{1} \) for some other \( \mathbf{y'} \) and the rest of the entries are \( 0 \). On the other hand, each of the extremal distributions \( \mathbf{p}_b^\mathbf{y}_b \) has exactly one entry equal to \( 1 \) for every \( \mathbf{y} \), which leads to \( s_b(\lambda_b) \in \{-1, 0, 1\}, \forall \lambda_b \in \Lambda_B^\mathbf{y}_b \).

From the above consideration, we immediately notice that the sums \( \sum_{\lambda_b \in \Lambda_B} [1 + s_b(\lambda_b)] p(\lambda_b|\lambda_a) \) is 0 for every constraint \( \mathbf{l}_b \). Now, there are two possibilities. First, these sums are different for different \( \lambda_a \). In that case, however, we will have a probability distribution

\[
\hat{p}(\lambda_a) \propto p(\lambda_a) \sum_{\lambda_b \in \Lambda_B} [1 + s_b(\lambda_b)] p(\lambda_b|\lambda_a) \tag{A8}
\]

different from \( p(\lambda_a) \) that satisfies the constraints in Eq. (A4a), which is already shown to be impossible. The other possibility is that the sums \( \sum_{\lambda_b \in \Lambda_B} [1 + s_b(\lambda_b)] p(\lambda_b|\lambda_a) \) are constant for each \( \lambda_a \), which is equivalent to saying that

\[
\sum_{\lambda_b \in \Lambda_B} s_b(\lambda_b)p(\lambda_b|\lambda_a) = \text{const}(\lambda_a). \tag{A9}
\]

However, in this case, multiplying by \( p(\lambda_a) \) and summing over \( \lambda_a \) leads to

\[
\text{const}(\lambda_a) = \sum_{\lambda_a \in \Lambda_A} p(\lambda_a) \sum_{\lambda_b \in \Lambda_B} s_b(\lambda_b)p(\lambda_b|\lambda_a) = \sum_{\lambda_b \in \Lambda_B} s_b(\lambda_b)p(\lambda_b) = 0, \quad \forall \mathbf{l}_b \cdot \mathbf{s}_b = 0 \in \mathcal{L}_B, \tag{A10}
\]

which means that every conditional distribution \( p(\lambda_b|\lambda_a) \) satisfies the constraints in Eq. (A4a) and hence must coincide with \( p(\lambda_b) \).

To sum up, we have shown that for every fixed minimal \( \Lambda_A^\mathbf{y}_a \) and \( \Lambda_B^\mathbf{y}_b \), for which there exists an extremal point \( \mathbf{p} \) in \( \mathcal{P}_1 \), this point \( \mathbf{p} \) must have \( p(\lambda_a, \lambda_b) = p(\lambda_a)p(\lambda_b) \) in its decomposition in Eq. (A2). This implies that this extremal point can be written as a direct product \( \mathbf{p} = \mathbf{p}_a \times \mathbf{p}_b \), with \( \mathbf{p}_a \in \mathcal{P}_A \cap \mathcal{L}_A \) and \( \mathbf{p}_b \in \mathcal{P}_B \cap \mathcal{L}_B \), and hence \( \mathbf{p} \in \mathcal{P}_2 \). The other two possibilities \( (\lambda_a, \lambda_b) \in \{\Lambda_A^\mathbf{y}_a \times \Lambda_B^\mathbf{y}_b, \Lambda_A^\mathbf{y}_a \times \Lambda_B^\mathbf{y}_b\} \) follow from the same proof as above with special cases of \( |\Lambda_A| = 1 \) and \( |\Lambda_B| = 1 \), respectively. This finishes the proof that \( \mathcal{P}_1 = \mathcal{P}_2 \).

**Appendix B: The smallest possible nonzero value of the penalty term**

Here, we provide an argument that supports the statement about the smallest nonzero value of the penalty term \( A + B \) for points \( \mathbf{p}_e \in \mathcal{E} \) in the proof of Result 1. First, we remind the reader the definition of \( \mathcal{E} \),

\[
\mathcal{E} = \bigcup_{\mathcal{L}_1 \subseteq \mathcal{L}_A, \mathcal{L}_2 \subseteq \mathcal{L}_B} \mathcal{E}_{\mathcal{L}_1, \mathcal{L}_2}, \tag{B1}
\]

where \( \mathcal{E}_{\mathcal{L}_1, \mathcal{L}_2} \) is the set of the extremal points of the polytope \( \mathcal{P}_{\mathcal{L}_{12}} \cap \mathcal{L}_1 \cap \mathcal{L}_2 \), i.e., the intersection of the local polytope by subsets \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) of the marginal nonsignaling constraints \( \mathcal{L}_A \) and \( \mathcal{L}_B \). The argument uses the following lemma.
Lemma 2. Let $\mathcal{P}_{LHV}$ be the local polytope and let $\mathcal{E}$ be the set of points resulting from the intersection of $\mathcal{P}_{LHV}$ by the hyperplanes of every possible subset of the marginal nonsignaling constraints $\mathcal{L}_A$ and $\mathcal{L}_B$. Then, the following holds:

$$
\min_{p_x \in \mathcal{E}, x \in [n]^N} \left\{ p(a_i | x) \left| p(a_i | x) > 0 \right\} \geq \frac{1}{n^{N-1}}, \right.
$$

(B2a)

$$
\min_{p_y \in \mathcal{E}, y \in [n]^N} \left\{ p(b_i | y) \left| p(b_i | y) > 0 \right\} \geq \frac{1}{n^{N-1}}, \right.
$$

(B2b)

where $p(a_i | x) = \sum_a \sum_b p(a, b | x, y)$ and $p(b_i | y) = \sum_a \sum_b p(a, b | x, y)$ are the marginal probabilities.

In the statement of the above Lemma, we used $a_i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$ to denote the tuple $a$ with the $i$-th element removed from it. Vector $b_i$ is defined similarly.

Proof. From Lemma 1, we know that the extremal points of the intersection $\mathcal{P}_{LHV} \cap \mathcal{L}_A \cap \mathcal{L}_B$ are of the product form, i.e., $p = p_a \times p_b$. Since Lemma 1 never used the fact that $\mathcal{L}_A$ and $\mathcal{L}_B$ must be all of the marginal nonsignaling constraints, its statement also holds for any subset of these constraints. Hence, it is also the case that the points in $\mathcal{E}$ are of the product form. Due to the symmetry of the problem with respect to two parties, it is sufficient to prove the statement of the current lemma for probabilities $p(a_i | x)$. Subsequently, since every $p \in \mathcal{E}$ is of the product form, it is sufficient to study the extremal points of $\mathcal{P}_A \cap \mathcal{L}_1$ for every $\mathcal{L}_1 \subset \mathcal{L}_A$, which we do below.

In this part of the proof, we turn to an alternative way of specifying the behaviors $p(a | x)$ by probability vectors $p_a$. Namely, instead of specifying probabilities $p(a | x)$ for every $a \in \{0, 1\}^N$ and $x \in [n]^N$, equivalently one can specify the probabilities of zero outcome events for every subset of $\{a_1 \in \{0, 1\} \} \times \{1, 2, \ldots, N\}$ for every setting $x$. In order to keep the notations simple, we adopt the convention that $p(a_1)$ is the probability of $a_1 = 0$, $p(a_i, a_j)$ is the joint probability of $a_1 = 0$ and $a_2 = 0$, etc. Probabilities of zero outcomes for every subset of variables $\{a_i \} \times \{1, 2, \ldots, N\}$ can then be joined together to form a probability vector as

$$
p_a = [p(a_1 | x), p(a_2 | x), \ldots, p(a_N | x), p(a_1, a_2 | x), \ldots, p(a | x)], \right.
$$

(B3)

where the tuples for each $x$ are concatenated together into a single vector. Importantly, the enumeration over $x$ is the same as before, i.e., $x \in [n]^N$. In the context of Bell inequalities, this form of probability vectors is sometimes referred to as the Collins-Gisin parametrization [35].

The condition for $p_a \in \mathcal{P}_A$ is a little more cumbersome to write down in terms of entries of $p_a$ in Eq. (B3), as compared to the standard way of composing a probability vector. For example, for $N = 2$, the condition reads

$$
p(a_1, a_2 | x) \geq 0, \right.
$$

(B4a)

$$
p(a_1 | x) - p(a_1, a_2 | x) \geq 0, \right.
$$

(B4b)

$$
p(a_2 | x) - p(a_1, a_2 | x) \geq 0, \right.
$$

(B4c)

$$
p(a_1 | x) + p(a_2 | x) - p(a_1, a_2 | x) \leq 1, \right.
$$

(B4d)

for all $x \in [n]^2$. At the same time, the conditions of $p_a$ to satisfy one of the constraints in $\mathcal{L}_A$ is very simple to write down in terms of the entries of $p_a$, e.g., $p(a_1 | x) = p(a_1 | x')$, where $x_1 = x'_1$.

The above convention of writing the probability vector is particularly convenient for us to use in the current proof. If we want to compute the extremal points of the polytope $\mathcal{P}_A \cap \mathcal{L}_1$ for some $\mathcal{L}_1 \subset \mathcal{L}_A$, we need to find intersections of all possible subsets of facets of this polytope. In particular, if the dimension of the polytope is $d$, we need to consider all subsets of size $d$. This dimension is easily computable to be $(2^N - 1)n^N - |\mathcal{L}_1|$, where $|\mathcal{L}_1|$ is the number of constraints in $\mathcal{L}_1$. However, considering all possible $(2^N - 1)n^N - |\mathcal{L}_1|$ subsets of facets of $\mathcal{P}_A \cap \mathcal{L}_1$ and finding their intersection is not an easy problem. Luckily, for our purposes it is sufficient to determine only the single-party marginal probabilities $p(a_1 | x)$ in every such solution, which we do below.

Finding an intersection of $d$ facets is equivalent to solving a system of $d$ linear equations for $d$ variables. This can be done, for example, by means of Gaussian elimination. The most common algorithm of Gaussian elimination is to bring the systems’ matrix to an upper-triangular form by row addition and multiplication. In that way, every variable, e.g., with index $j$, in a list $\{1, 2, \ldots, d\}$ is expressed in terms of values of variables from the right in that list, e.g., $\{j + 1, j + 2, \ldots, d\}$ and a constant term. The value of the last $d$-th variable is determined by the corresponding constant term only (in case the system has a solution). Importantly, there is freedom for choosing the order in which the variables are eliminated. In our case, we choose the last variables in the list to be the single-variable marginal probabilities $p(a_1 | x), p(a_2 | x)$, etc.

We know that, in case of no constraints added to $\mathcal{P}_A$, all possible solutions to a system of $(2^N - 1)n^N$ equations for every choice of subsets of facets satisfy $p(a_1 | x) \in \{0, 1\}, \forall x \in [n]^N$. This means that, when one performs Gaussian elimination, the coefficient in front of every single-variable marginal probability is at most 1. If the contrary were true,
then there would have existed extremal points in $\mathcal{P}_A$ with fractional probabilities. Now let us add one constraint from $\mathcal{L}_A$ to $\mathcal{P}_A$. This constraint reduces the total dimension of the system by 1 and substitutes one of the probabilities $p(a_j|x')$ by $p(a_j|x)$. Now, if we perform the Gaussian elimination, we can expect that the coefficient in front of $p(a_j|x)$ in some cases can be 2. Most importantly, it cannot be larger than 2 for any of the considered subsets of facets. This is a consequence of the fact that the process of Gaussian elimination is not affected by the constraint $p(a_j|x') = p(a_j|x)$ until the last point, when the rest of the probabilities are expressed in terms of $p(a_j|x')$ or $p(a_j|x)$. This argument is easy to extend to the case of two and more linear constraints from $\mathcal{L}_A$ that intersect $\mathcal{P}_A$, and one can see that the maximal possible coefficient in front of the marginal probabilities $p(a_j|x)$ grows linearly with the number of constraints considered. At the same time, the maximal number of constraints in $\mathcal{L}_A$ that includes a single probability $p(a_j|x)$ is $n^{N-1}$. By taking the constant term to be 1, we conclude that the smallest nonzero value the marginal probabilities $p(a_j|x)$ can take in extremal points of $\mathcal{P}_A \cap \mathcal{L}_1$ for $\mathcal{L}_1 \in \mathcal{L}_A$ is $n^{-(N-1)}$. 

If we want to determine the smallest possible nonzero value of $A+B$ in Eq. (3) for the intersection points, it is clear that we would take the ones with, e.g., $B = 0$, and with each term in the sum in $A$ except for one being zero. Since the smallest nonzero value of $p(a_i|x)$ is $n^{-(N-1)}$, this is also the smallest nonzero value of the difference $|p(a_i|x) - p(a_i|x')|$. 

Appendix C: No advantage from nonsignaling strategies

Here, we show that the LHV bound of Eq. (3) under constraints $A = 0$, $B = 0$ is indeed $C^N$. From Lemma 1 in Appendix A we know that the extremal points of the polytope $\mathcal{P}_{\text{LHV}} \cap \mathcal{L}_A \cap \mathcal{L}_B$, which is the local polytope intersected by the marginal nonsignaling constraints, are of the product form. At the same time, a Bell inequality must take its maximal LHV value at one of the extremal point of the polytope $\mathcal{P}_{\text{LHV}} \cap \mathcal{L}_A \cap \mathcal{L}_B$. Hence, we can write

$$
\sum_{a,b,x,y} p(a,b|x,y) \prod_{i=1}^N c_{a_i,b_i} = \sum_{a,b,x,y} p(a|x)p(b|y) \prod_{i=1}^N c_{a_i,b_i}
$$

\[= \sum_{a_1,b_1,x_1,y_1} c_{a_1,b_1} p(a_1|x)p(b_1|y) \sum_{a_1,b_1,x_1,y_1} p(a_1|x,a_1)p(b_1|y,b_1) \prod_{i=2}^N c_{a_i,b_i} \leq \sum_{a_1,b_1,x_1,y_1} c_{a_1,b_1} p(a_1|x)p(b_1|y) \max_{a_1,b_1,x_1,y_1} \left[ \sum_{a_1,b_1,x_1,y_1} p(a_1|x,a_1)p(b_1|y,b_1) \prod_{i=2}^N c_{a_i,b_i} \right],
\]

where we used the notation $a_1 = (a_2, \ldots, a_N)$, etc. Now we use the marginal nonsignaling constraints for Alice $p(a_1|x) = p(a_1|x')$, $\forall x' \neq x$, with $x_1 = x_1'$, which implies $p(a_1|x) = p(a_1|x_1)$ because inputs $\{x_i\}_{i \in \{1,2,\ldots,N\}}$ are statistically independent. Similarly, the marginal nonsignaling constraints for Bob imply $p(b_1|y) = p(b_1|y_1)$. It is clear that the maximum of the expression that comes after the sum $\sum_{a_1,b_1,x_1,y_1}$ cannot increase due to the conditioning on $a_1, x_1$ and $b_1, y_1$, which leads us to the original expression, but with $N - 1$ copies instead of $N$. Consequently, the total maximum for LHV theories under marginal nonsignaling condition is $C^N$.

Appendix D: Optimal states and measurements for the CHSH inequality

Here, we give exact form of the optimal states and measurements that reproduce the relation form Eq. (8) between $A$, $B$, and $Q$ [Eqs. (6)] for the CHSH inequality.

The state $\rho_{AB}$ is the following nonmaximally entangled pure state:

$$
\rho_{AB} = \frac{1 + \sqrt{1-q^2}}{2} |00\rangle\langle 00| + \frac{1 - \sqrt{1-q^2}}{2} |11\rangle\langle 11| + \frac{q}{2} \left( |00\rangle\langle 11| + |11\rangle\langle 00| \right).
$$

The observables $A_0^x - A_1^x$ and $B_0^y - B_1^y$ are sharp and given by

$$
A_0^x - A_1^x = \sqrt{\frac{1}{1+q} + \frac{(-1)^x q}{\sqrt{1+q^2}}} \sigma_x + (-1)^x \sqrt{\frac{q}{1+q} \left( 1 - \frac{(-1)^x}{\sqrt{1+q^2}} \right)} \sigma_x, \quad x \in \{0, 1\},
$$

$$
B_0^y - B_1^y = \sqrt{\frac{1}{1+q} + \frac{(-1)^y q}{\sqrt{1+q^2}}} \sigma_y - (-1)^y \sqrt{\frac{q}{1+q} \left( 1 - \frac{(-1)^y}{\sqrt{1+q^2}} \right)} \sigma_y, \quad y \in \{0, 1\},
$$
where $\sigma_z$ and $\sigma_x$ are the Pauli $z$ and $x$ matrices, respectively. The parameter $q$ is the one in Eq. (8) that gives the values of $A$ and $B$. The value $Q$ of the CHSH inequality is $Q = \frac{1}{2} + \frac{1}{4}\sqrt{1 + q^2}$.

[1] J. S. Bell, Physics Physique Fizika 1, 195 (1964).
[2] R. Colbeck, Quantum and relativistic protocols for secure multi-party computation, Ph.D. thesis, University of Cambridge (2009).
[3] S. Pironio, A. Acín, S. Massar, A. B. de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Nature (London) 464, 1021 (2010).
[4] W.-Z. Liu, M.-H. Li, S. Ragy, S.-R. Zhao, B. Bai, Y. Liu, P. J. Brown, J. Zhang, R. Colbeck, J. Fan, Q. Zhang, and J.-W. Pan, Nat. Phys. (2021).
[5] L. K. Shalm, Y. Zhang, J. C. Bienfang, C. Schlager, M. J. Stevens, M. D. Mazurek, C. Abellán, W. Amaya, M. W. Mitchell, M. A. Alhejji, H. Fu, J. Orstein, R. P. Mirin, S. W. Nam, and E. Knill, Nat. Phys. (2021).
[6] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[7] D. Mayers and A. Yao, in Proceedings 39th Annual Symposium on Foundations of Computer Science (Cat. No.98CB36280) (IEEE, Los Alamitos, CA, 1998) p. 503.
[8] J. Barrett, L. Hardy, and A. Kent, Phys. Rev. Lett. 95, 010503 (2005).
[9] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Phys. Rev. Lett. 98, 230501 (2007).
[10] S. Pironio, A. Acín, N. Brunner, N. Gisin, S. Massar, and V. Scarani, New J. Phys. 11, 045021 (2009).
[11] L. Aolita, R. Gallego, A. Cabello, and A. Acín, Phys. Rev. Lett. 108, 100401 (2012).
[12] M. G. M. Moreno, S. Brito, R. V. Nery, and R. Chaves, Phys. Rev. A 101, 052339 (2020).
[13] D. Mayers and A. Yao, Quantum Info. Comput. 4, 273 (2004).
[14] I. Šupić and J. Bowles, Quantum 4, 337 (2020).
[15] E. S. Gómez, S. Gómez, P. González, G. Cañas, J. F. Barra, A. Delgado, G. B. Xavier, A. Cabello, M. Kleinmann, T. Vértesi, and G. Lima, Phys. Rev. Lett. 117, 260401 (2016).
[16] M. Smania, P. Mironowicz, M. Nawareg, M. Pawłowski, A. Cabello, and M. Bourennane, Optica 7, 123 (2020).
[17] M. T. Quintino, C. Budroni, E. Woodhead, A. Cabello, and D. Cavalcanti, Phys. Rev. Lett. 123, 180401 (2019).
[18] B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau, and R. Hanson, Nature (London) 526, 682 (2015).
[19] M. Giustina, M. A. M. Versteegh, S. Wengerowsky, J. Handsteiner, A. Hochrainer, K. Phelan, F. Steinlechner, J. Kofler, J.-Å. Larsson, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, J. Beyer, T. Gerrits, A. E. Lita, L. K. Shalm, S. W. Nam, T. Scheidl, R. Ursin, B. Wittmann, and A. Zeilinger, Phys. Rev. Lett. 115, 250401 (2015).
[20] L. K. Shalm, E. Meyer-Scott, B. G. Christensen, P. Bierhorst, M. A. Wayne, M. J. Stevens, T. Gerrits, S. Glancy, D. R. Hamel, M. S. Allman, K. J. Coakley, S. D. Dyer, C. Hodge, A. E. Lita, V. B. Verma, C. Lambrocco, E. Tortorici, A. L. Migdall, Y. Zhang, D. R. Kurom, W. H. Farr, F. Marsili, M. D. Shaw, J. A. Stern, C. Abellán, W. Amaya, V. Pruneri, T. Jennewein, M. W. Mitchell, P. G. Kwiat, J. C. Bienfang, R. P. Mirin, E. Knill, and S. W. Nam, Phys. Rev. Lett. 115, 250402 (2015).
[21] W. Rosenfeld, D. Burchardt, R. Garthoff, K. Redeker, N. Ortegel, M. Rau, and H. Weinfurter, Phys. Rev. Lett. 119, 010402 (2017).
[22] P. M. Pearle, Phys. Rev. D 2, 1418 (1970).
[23] P. H. Eberhard, Phys. Rev. A 47, R747 (1993).
[24] A. Garg and N. D. Mermin, Phys. Rev. D 35, 3831 (1987).
[25] S. Massar, Phys. Rev. A 65, 032121 (2002).
[26] T. Vértesi, S. Pironio, and N. Brunner, Phys. Rev. Lett. 104, 060401 (2010).
[27] A. Cabello and J.-Å. Larsson, Phys. Rev. Lett. 98, 220402 (2007).
[28] N. Brunner, N. Gisin, V. Scarani, and C. Simon, Phys. Rev. Lett. 98, 220403 (2007).
[29] G. Garbarino, Phys. Rev. A 81, 032106 (2010).
[30] M. Araújo, M. T. Quintino, D. Cavalcanti, M. F. m. c. Santos, A. Cabello, and M. T. Cunha, Phys. Rev. A 86, 030101 (2012).
[31] J.-Å. Larsson and J. Semecolos, Phys. Rev. A 63, 022117 (2001).
[32] A. Cabello, D. Rodríguez, and I. Villanueva, Phys. Rev. Lett. 101, 120402 (2008).
[33] K. F. Pál, T. Vértesi, and N. Brunner, Phys. Rev. A 86, 062111 (2012).
[34] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[35] A. Cabello, D. Rodríguez, and I. Villanueva, Phys. Rev. Lett. 101, 120402 (2008).
[36] D. Collins and N. Gisin, J. Phys. A: Math. Gen. 37, 1775 (2004).
[37] P. G. Kwiat, J. Mod. Opt. 44, 2173 (1997).
[38] M. Erhard, M. Krenn, and A. Zeilinger, Nat. Rev. Phys. 2, 365 (2020).
[39] J. Barrett, D. Collins, L. Hardy, A. Kent, and S. Popescu, Phys. Rev. A 66, 042111 (2002).
[40] A. Cabello, Phys. Rev. Lett. 97, 140406 (2006).
[41] P. H. Eberhard and P. Rosselet, Found. Phys. 25, 91 (1995).
[42] R. Jain, C. A. Miller, and Y. Shi, IEEE Trans. Inf. Theory 66, 5567 (2020).
[43] M. Doda, M. Huber, G. Murta, M. Pivoluska, M. Plesch, and C. Vlatchou, Phys. Rev. Applied 15, 034003 (2021).
[44] M. Araújo, F. Hirsch, and M. T. Quintino, Quantum 4, 353 (2020).
[45] I. Márton, E. Bene, and T. Vértesi, (2021), arXiv:2103.10413 [quant-ph].
[46] R. Raz, SIAM J. Comput. 27, 763 (1998).
[47] T. Holenstein, in Proceedings of the Thirty-Ninth Annual ACM Symposium on Theory of Computing (Association for Computing Machinery, New York, NY, 2007) p. 411.
[48] K. W. O. Feige, G. Kindler, and R. O’Donnell, in Twenty-Second Annual IEEE Conference on Computational Complexity (CCC’07) (IEEE, Los Alamitos, CA, 2007).
[48] A. Rao, SIAM J. Comput. 40, 1871 (2011).
[49] I. Dinur and D. Steurer, in *Proceedings of the Forty-Sixth Annual ACM Symposium on Theory of Computing (STOC ’14)* (Association for Computing Machinery, New York, NY, 2014) p. 624.
[50] I. Pitowsky, Mathematical Programming 50, 395 (1991).
[51] C. Branciard, Phys. Rev. A 83, 032123 (2011).
[52] M. Czechlewski and M. Pawlowski, Phys. Rev. A 97, 062123 (2018).
[53] M. Navascués, S. Pironio, and A. Acín, Phys. Rev. Lett. 98, 010401 (2007).
[54] P. Wittek, ACM Transactions on Mathematical Software (TOMS) 41, 1 (2015).
[55] J. Löfberg, in *2004 IEEE International Conference on Robotics and Automation (IEEE Cat. No.04CH37508)* (IEEE, Los Alamitos, CA, 2004) p. 284.
[56] M. ApS, *The MOSEK optimization toolbox for MATLAB manual. Version 9.0.* (2019).
[57] M. S. Andersen, J. Dahl, and L. Vandenberghe, *CVXOPT: A Python package for convex optimization, version 1.1.6* (2013).