Axiomatizing first-order consequences in inclusion logic

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Inclusion logic is a variant of dependence logic that was shown to have the same expressive power as positive greatest fixed-point logic. Inclusion logic is not axiomatizable in full, but its first-order consequences can be axiomatized. In this paper, we provide such an explicit partial axiomatization by introducing a system of natural deduction for inclusion logic that is sound and complete for first-order consequences in inclusion logic.

1 Introduction

In this paper, we axiomatize first-order consequences of inclusion logic. Inclusion logic was introduced by Galliani [6]. Together with independence logic, introduced by Grädel and Väänänen [11], inclusion logic is an important variant of dependence logic, which was introduced by Väänänen [29] as an extension of first-order logic and a new framework for characterizing dependency notions. Inclusion logic aims to characterize inclusion dependencies by extending first-order logic with inclusion atoms, which are strings of the form $x_1 \ldots x_n \subseteq y_1 \ldots y_n$, where $\langle x_1, \ldots, x_n \rangle = x$ and $\langle y_1, \ldots, y_n \rangle = y$ are sequences of variables of the same length. Inclusion logic adopts the team semantics of Hodges [21, 22], in which inclusion atoms and other formulas are evaluated in a model with respect to sets of assignments (called teams), in contrast to single assignments as in the usual first-order logic. Intuitively the inclusion atom $x \subseteq y$ specifies that all possible values for $x$ in a team $X$ are included in the values of $y$ in the same team $X$.

Galliani and Hella proved that inclusion logic is expressively equivalent to positive greatest fixed-point logic [8]. It then follows from the results of Immerman [23] and Vardi [31] that over finite ordered structures inclusion logic captures PTIME. Building on these results, Grädel defined model-checking games for inclusion logic [9], which then found applications in [10]. There also emerged some studies [13, 14, 16, 27] on the computational complexity and syntactical fragments of inclusion logic. Embedding the semantics of inclusion atoms into the semantics of the quantifiers, Rönnholm [28] introduced the interesting inclusion quantifiers that generalize the idea of the slashed quantifiers of independence-friendly logic [20] (a close relative to dependence logic). Inclusion atoms have also found natural applications in a recent formalization of Arrow’s Theorem in social choice in dependence and independence logic [26]. Motivated by the increasing interest in inclusion logic, we present in this paper a proof-theoretic investigation of inclusion logic, which is currently missing in the literature.

It is worth noting that inclusion atoms correspond exactly to the inclusion dependencies studied in database theory. The implication problem of inclusion dependencies, i.e., the problem of deciding whether $\Gamma \models \phi$ for a set $\Gamma \cup \{ \phi \}$ of inclusion dependencies (or inclusion atoms), is completely axiomatized in [4] by the following three rules/axioms:

- $x \subseteq x$ (identity)
- $x_1 \ldots x_n \subseteq y_1 \ldots y_n / x_{i_1} \ldots x_{i_k} \subseteq y_{i_1} \ldots y_{i_k}$ for $i_1, \ldots, i_k \in \{1, \ldots, n\}$ (projection and permutation)
- $x \subseteq y, y \subseteq z / x \subseteq z$ (transitivity)

The team semantics interpretation for inclusion atoms has recently been utilized to study the implication problems of inclusion atoms together with other dependency atoms [15, 18, 19]. In this paper, we study, instead, the

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We investigate the problem of finding a deduction system for which the completeness theorem

$$\Gamma \models \phi \iff \Gamma \vdash \phi$$

holds for $\Gamma \cup \{ \phi \}$ being a set of formulas of the logic.

It is known that dependence logic is not (effectively) axiomatizable, since the sentences of the logic are equi-expressive with sentences of existential second-order logic (ESO) [29]. Nevertheless, if one restricts the consequence $\phi$ in (1) to a first-order sentence and $\Gamma$ to a set of sentences in dependence logic, the axiomatization can be found. This is because, finding a model for such a set $\Gamma \cup \{ \neg \phi \}$ of sentences of dependence logic is the same as finding a model for a set of ESO sentences (i.e., sentences of the form $\exists f_1 \ldots f_n \alpha$ for some first-order $\alpha$), which is then reduced to finding a model for a set of first-order sentences (of the form $\alpha$). A concrete system of natural deduction for dependence logic admitting this type of completeness theorem was given in [25]. The proof of the completeness theorem uses a nontrivial technique based on the equivalence between a dependence logic sentence and its so-called game expression (an infinitary first-order sentence describing a semantic game) over countable models, and the fact that the game expression can be finitely approximated over recursively saturated models. Subsequently, using the similar method a system of natural deduction axiomatizing completely the first-order consequences in independence logic with respect to sentences was also introduced [12]. These partial axiomatizations for sentences were first generalized in [24] to cover the cases for formulas by expanding the language with a new predicate symbol to interpret the teams, and later generalized further in [32] to cover the case when the consequence $\phi$ in (1) is not necessarily first-order itself but has an essentially first-order translation by applying a trick that involves the weak classical negation $\sim$ and the addition of the RAA rule for $\sim$.

As we will demonstrate formally in this paper, inclusion logic is not (effectively) axiomatizable either. Since inclusion logic is less expressive than ESO, by the same argument as above, the first-order consequences of inclusion logic can also be axiomatized. In this paper, we give explicitly such an axiomatization. To be more precise, we introduce a system of natural deduction for inclusion logic for which the completeness theorem (1) holds for $\phi$ being a first-order formula and $\Gamma$ being a set of Inc-formulas. Our completeness proof uses the technique developed in [25] together with the trick in [32]. Our system of inclusion logic is a conservative extension of the system of first-order logic, in the sense that it has the same rules as that of first-order logic when restricted to first-order formulas only. The rules for inclusion atoms include some of those introduced in [12], and the rules characterizing the interactions between inclusion atoms and the connectives and quantifiers appear to be simpler than the corresponding ones in the systems of dependence and independence logic defined in [12,25]. The RAA rule for $\sim$, being a crucial (yet generally not effective) rule for applying the trick of [32], also behaves better in our system of inclusion logic than in the systems of dependence and independence logic. In particular, in the inclusion logic system, with respect to first-order formulas, the RAA rule for $\sim$ becomes effective and also derivable from other more basic rules.

The paper is organized as follows. In Section 2 we recall the basics of inclusion logic, and also give a proof that inclusion logic is not (effectively) axiomatizable. Section 3 discusses the normal form for inclusion logic. In Section 4, we define the game expressions and their finite approximations that are crucial for the proof of the completeness theorem of the system of natural deduction for inclusion logic. We introduce this system in Section 5, and also prove the soundness theorem as well as some useful derivable clauses in the section. The proof of the completeness theorem will be given in Section 6. We conclude in Section 7 by showing some applications of our system; in particular, we derive in our system the axioms for anonymity atoms proposed recently by Väänänen [30].

## 2 Preliminaries

In this section, we recall the basics of inclusion logic and prove formally that inclusion logic is not (effectively) axiomatizable. We consider first-order signatures $\mathcal{L}$ with a built-in equality symbol $\equiv$. Fix a set $\text{Var}$ of first-order variables, and denote its elements by $u, v, w, x, y, \ldots$ (with or without subscripts). First-order $\mathcal{L}$-terms $t$ are built recursively as usual. First-order $\mathcal{L}$-formulas $\alpha$ are defined by the grammar:

$$\alpha ::= \bot \mid t_1 = t_2 \mid R t_1 \ldots t_n \mid \neg \alpha \mid \alpha \land \alpha \mid \alpha \lor \alpha \mid \exists x \alpha \mid \forall x \alpha.$$
Throughout the paper, we reserve the first greek letters $\alpha, \beta, \gamma, \delta$ (with or without subscripts) for first-order formulas. As usual, we write $\alpha \rightarrow \beta := \neg \alpha \lor \beta$ and $\alpha \leftrightarrow \beta := (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)$ for first-order formulas $\alpha$ and $\beta$. Formulas $\phi$ of inclusion logic (Inc) are defined recursively as follows:

$$
\phi := \bot \mid \alpha \mid -\alpha \mid x_1 \ldots x_n \subseteq y_1 \ldots y_n \mid \phi \land \psi \mid \phi \lor \psi \mid \exists x \phi \mid \forall x \phi
$$

where $\alpha$ is an arbitrary first-order formula. The formula $x_1 \ldots x_n \subseteq y_1 \ldots y_n$ is called an inclusion atom. Note that in the literature on inclusion logic, Inc-formulas are usually assumed to be in negation normal form (i.e., negation occurs only in front of atomic formulas). We do not adopt this convention in this paper, but we do require that negation in Inc applies only to first-order formulas.

The set $\text{Fv}(\phi)$ of free variables of an Inc-formula $\phi$ is defined inductively as usual except that we now have the new case

$$
\text{Fv}(x_1 \ldots x_n \subseteq y_1 \ldots y_n) := \{x_1, \ldots, x_n, y_1, \ldots, y_n\}.
$$

We write $\phi(x_1, \ldots, x_k)$ to indicate that the free variables of $\phi$ are among $x_1, \ldots, x_k$. Inc-formulas with no free variable are called sentences. We write $\phi(t/x)$ for the formula obtained by substituting uniformly $t$ for $x$ in $\phi$, where we assume that $t$ is free for $x$.

We assume that the domain of a first-order model $M$ has at least two elements, and use the same letter $M$ to stand for both the model and its domain. An assignment of an $\mathcal{L}$-model $M$ for a set $\mathcal{V} \subseteq \text{Var}$ of variables is a function $s : \mathcal{V} \rightarrow M$. The interpretation of an $\mathcal{L}$-term $t$ under $M$ and $s$ (denoted $s(t^M)$) is defined as usual. For any sequence $x = (x_1, \ldots, x_k)$ of variables, we write $s(x_1, \ldots, x_k)$ or $s(x)$ for $s(x_1), \ldots, s(x_k))$. For any element $a \in M$, $s(a/x)$ is the assignment defined as

$$
s(a/x)(y) = \begin{cases} a, & \text{if } y = x; \\ s(y), & \text{otherwise}. \end{cases}
$$

A set $X$ of assignments of a model $M$ with the same domain $\text{dom}(X)$ is called a team (of $M$). In particular, the empty set $\emptyset$ is a team, and the singleton $\{\emptyset\}$ is a team with the empty domain.

**Definition 2.1** For any $\mathcal{L}$-formula $\phi$ of Inc, any $\mathcal{L}$-model $M$ and any team $X$ of $M$ with $\text{dom}(X) \supseteq \text{Fv}(\phi)$, we define the satisfaction relation $M \models_X \phi$ inductively as follows:

- $M \models_X \bot$ iff $X = \emptyset$.
- $M \models_X \alpha$ iff for all $s \in X$, $M \models_s \alpha$ in the usual sense.
- $M \models_X -\alpha$ iff for all $s \in X$, $M \not\models_s \alpha$ in the usual sense.
- $M \models_X x \subseteq y$ iff for all $s \in X$, there is $s' \in X$ such that $s(x) = s'(y)$.
- $M \models_X \phi \land \psi$ iff $M \models_X \phi$ and $M \models_X \psi$.
- $M \models_X \phi \lor \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ such that $M \models_Y \phi$ and $M \models_Z \psi$.
- $M \models_X \exists x \phi$ iff $M \models_{X(F/x)} \phi$ for some function $F : X \rightarrow \rho(M) \setminus \{\emptyset\}$, where
  $$X(F/x) = \{s(a/x) \mid s \in X \text{ and } a \in F(s)\}.$$
- $M \models_X \forall x \phi$ iff $M \models_{X(M/x)} \phi$, where $X(M/x) = \{s(a/x) \mid s \in X \text{ and } a \in M\}$.

For any set $\Gamma$ of Inc-formulas, we write $M \models_X \Gamma$ if $M \models_X \phi$ for all $\phi \in \Gamma$. We write $\Gamma \models \phi$ if $M \models_X \Gamma$ implies $M \models_X \phi$ for all models $M$ and teams $X$. We write simply $\models \phi$ for $\emptyset \models \phi$, and $\models \psi$ for $\{\emptyset\} \models \phi$. If both $\phi \models \psi$ and $\psi \models \phi$, we write $\models \psi$.

Our version of the team semantics for disjunction and existential quantifier is known in the literature as lax semantics; see [5] for further discussion. In some literature (e.g., [6]) inclusion terms are allowed to have arbitrary terms as arguments, namely strings of the form $t_1 \ldots t_n \subseteq t'_1 \ldots t'_m$ are considered well-formed formulas, and the semantics of these inclusion atoms are defined (naturally) as:
• \( M \models t_1 \cdots t_n \subseteq t'_1 \cdots t'_n \) iff for all \( s \in X \), there is \( s' \in X \) such that \( s(t^M_1, \ldots, t^M_n) = s'(t'_1)^M, \ldots, (t'_n)^M) \).

It is easy to verify that inclusion atoms of this type are definable in our version of inclusion logic, since \( t \subseteq t' \equiv \exists x(x = t \land y = t' \land x \subseteq y) \), where \( \exists v \) abbreviates \( \exists v_1 \cdots \exists v_k \) for some \( k \), and \( u = v \) is short for \( \wedge u_i = v_i \).

For any assignment \( s \) and any set \( V \subseteq \text{Var of variables} \), we write \( s \restriction V \) for the assignment \( s \) restricted to \( V \). For any team \( X \), define \( X \restriction V = \{ s \restriction V : s \in X \} \). We list the most important properties of \( \text{Inc-formulas} \) in the following lemma. The reader is referred to \([8]\) for other properties.

**Lemma 2.2** Let \( \phi \) be an \( \mathcal{L} \)-formula, \( M \) an \( \mathcal{L} \)-model, and \( X \), \( Y \), \( X_i \) (\( i \in I \)) arbitrary teams of \( M \) with \( \text{dom}(X), \text{dom}(Y), \text{dom}(X_i) \supseteq \text{Fv}(\phi) \).

**Locality:** If \( X \restriction V \models \phi \Rightarrow Y \restriction V \models \phi \), then \( M \models X \models \phi \iff M \models Y \models \phi \).

**Union Closure:** If \( M \models X_i \models \phi \) for all \( i \in I \), then \( M \models \bigcup_{i \in I} X_i \models \phi \).

**Flatness of First-order Formulas:** For any first-order \( \mathcal{L} \)-formula \( \alpha \),

\[
M \models X \alpha \iff M \models \{s\} \alpha \text{ for all } s \in X.
\]

Consequently, first-order formulas are also downwards closed, that is, \( M \models X \alpha \) and \( Y \subseteq X \) imply \( M \models Y \alpha \).

If \( \theta \) is a sentence, the locality property implies that \( M \models_{(\emptyset)} \theta \) iff \( M \models X \theta \) for all teams \( X \) of \( M \). We call \( M \) a model of \( \theta \), written \( M \models \theta \), if \( M \models_{(\emptyset)} \theta \).

By the result of \([6]\), Inc sentences can be translated into existential second-order logic (ESO), namely, for every Inc-sentence \( \theta \), there exists an ESO-sentence \( \tau(\theta) \) such that \( M \models \theta \iff M \models \tau(\theta) \). Since ESO is well-known to be compact, it follows that Inc is compact as well, that is, if every finite subset of a set \( \Gamma \) of Inc-sentences has a model, then the set \( \Gamma \) itself has a model. It was further proved in \([8]\) that Inc is expressively equivalent to positive greatest fixed point logic (posGFP) in the sense of the following theorem.

**Theorem 2.3** (\([3]\)) For any \( \mathcal{L} \)-formula \( \phi \) of Inc with \( \text{Fv}(\phi) = \{x_1, \ldots, x_n\} \), there exists an \( \mathcal{L}(R) \)-formula \( \psi(R) \) of posGFP with a fresh \( n \)-ary relation symbol \( R \) such that for all \( \mathcal{L} \)-models \( M \) and teams \( X \) of \( M \) with \( \text{dom}(X) = \{x_1, \ldots, x_n\} \),

\[
M \models X \phi \iff (M, \text{rel}(X)) \models_s \psi(R) \text{ for all } s \in X;
\]

and vice versa, where \( \text{rel}(X) = \{(s(x_1), \ldots, s(x_n)) : s \in X\} \) is an \( n \)-ary relation on \( M \) that serves as the interpretation for \( R \). In particular, Inc-sentences can be translated into posGFP and vice versa.

As a consequence of \([23]\), over finite models, Inc and least fixed point logic have the same expressive power. In particular, by \([23, 24]\), over ordered finite models, Inc captures PTIME.

Due to the strong expressive power, Inc is not (effectively) axiomatizable. We now give an explicit proof of this fact by following a similar argument to that in \([23]\).

Consider the signature \( \mathcal{L}_a = (+, \times, <, 0, 1) \) of arithmetic. We first show that the non-well-foundedness of \( < \) is definable in Inc.

**Proposition 2.4** For any model \( M \) in the signature \( \mathcal{L}_a \) of arithmetic, \( M \models \exists x \exists y (y \subseteq x \land y < x) \) iff \( <^M \) is not well-founded.

**Proof.** It is easy to prove that \( \exists x \exists y (y \subseteq x \land y < x) \) holds in \( M \) iff \( M \) contains an infinite \( <\)-descending chain \( \cdots < a_3 < a_2 < a_1 < a_0 \). We leave the proof details to the reader. \( \square \)

Now, put \( \phi = \exists x \exists y (y \subseteq x \land y < x) \), and let \( \mathcal{A}_\text{PA} \) be a (first-order) sentence stating that each of the (finitely many) axioms of Peano arithmetic except for the axiom schema of induction is true (or \( \mathcal{A}_\text{PA} \) is the conjunction of all axioms of Robinson arithmetic \( Q \)). For any \( \mathcal{L}_a \)-sentence \( \alpha \) of arithmetic, we have that

\[
\mathbb{N} \models \alpha \iff \alpha \lor \neg \mathcal{A}_\text{PA} \lor \phi,
\]  

\(1\) The author would like to thank Jouko Väänänen for suggesting this proof, and the formula used in Proposition 2.4 is taken essentially from \([8]\).
where \( \mathbb{N} \) is the standard model of Peano arithmetic. To see why, for the left to right direction, suppose that \( \mathbb{N} \models \alpha \) and that \( M \) is a model of \( \mathcal{PA} \) such that \( M \not\models \phi \). By Proposition 2.4, \( \phi \) is well-founded. Now, \( M \) is a model satisfying all axioms of Robinson arithmetic (including the axiom \( \forall x(x = 0 \lor \exists y(y + 1 = x)) \) and the axioms stating that \( < \) is a linear ordering), and the ordering \( \prec_M \) is a well-ordering. It is then easy to verify that \( M \) also satisfies the (second-order) induction axiom. Therefore \( M \) is (isomorphic to) the standard model \( \mathbb{N} \) of arithmetic, and \( M \models \alpha \). Conversely, suppose \( \models \alpha \lor \neg \mathcal{PA} \lor \phi \). The standard model \( \mathbb{N} \) of Peano arithmetic clearly satisfies \( \mathcal{PA} \), and by Proposition 2.4 the model \( \mathbb{N} \) falsifies \( \phi \). Thus we must have that \( \mathbb{N} \models \alpha \).

The equivalence (2) shows that truth in the standard model \( \mathbb{N} \) can be reduced to logical validity in inclusion logic. This means that validity in inclusion logic is not arithmetical, and therefore inclusion logic cannot have any (effective) complete axiomatization.

Nevertheless, there can be partial axiomatizations for the logic. The main objective of the present paper is to introduce a system of natural deduction for Inc that is complete for first-order consequences, in the sense that

\[
\Gamma \vdash \alpha \iff \Gamma \models \alpha
\]

holds whenever \( \Gamma \) is a set of Inc-formulas, and \( \alpha \) is a first-order formula. Our completeness proof will mainly follow the argument of [25], which roughly goes as follows: First, we show that any Inc-sentence is semantically equivalent to a formula \( \phi \) in certain normal form. Also, in the system to be introduced every Inc-formula implies its normal form. Then, we show that \( \phi \) is equivalent over countable models to a first-order sentence \( \Phi \) of infinite length (called its game expression). Next, we show that the game expression \( \Phi \) can be approximated in a certain sense (in the sense of Theorem 4.2) by some first-order sentences \( \Phi^\omega (n \in \omega) \) of finite length. Finally, making essential use of these approximations \( \Phi^\omega \) we will be able to prove the completeness theorem by certain model theoretic argument, together with a trick developed in [32] using the weak classical negation \( \sim \).

### 3 Normal form

In this section, we prove that every Inc-formula \( \phi(z) \) is (semantically) equivalent to a formula of the form

\[
Q^1 x_1 \ldots Q^n x_n \theta(x, z),
\]

where \( Q^i \in \{\exists, \forall\} \) and \( \theta \) is a quantifier-free formula.

**Theorem 3.1** ([71]) Every Inc-formula \( \phi(z) \) is semantically equivalent to a formula of the form

\[
Q^1 x_1 \ldots Q^n x_n \theta(x, z),
\]

where \( Q^i \in \{\exists, \forall\} \) and \( \theta \) is a quantifier-free formula.

Proof (sketch). The theorem follows from the fact that

- \( \neg \forall \alpha \equiv \exists \neg \alpha \) and \( \neg \exists \alpha \equiv \forall \neg \alpha \) for any first-order formula \( \alpha \),

and the fact that if \( x \notin \text{FV}(\psi) \), then

- \( \exists \phi \land \psi \equiv \exists \phi (\phi \land \psi) \),
- \( \exists \phi \lor \psi \equiv \exists \phi (\phi \lor \psi) \),
- \( \forall \phi \land \psi \equiv \forall \phi (\phi \land \psi) \),
- \( \forall \phi \lor \psi \equiv \exists \forall \forall \forall ((\phi \land y = z) \lor (\psi \land y \neq z)) \), where \( y, z \) are fresh variables.

**Theorem 3.2** ([13]) Every Inc-formula \( \phi(z) \) of the form (3) is semantically equivalent to a formula of the form

\[
\exists x \forall y \left( \bigwedge_{1 \leq j \leq n} z x_1 \ldots x_{j-1} y \subseteq z x_1 \ldots x_{j-1} x_j \land \theta(x, z) \right),
\]

where \( x = (x_1, \ldots, x_n) \), \( y \) is fresh and \( \theta \) is the quantifier-free formula in (3).
Proof (idea). This theorem is proved by exhaustively applying the equivalences

\[\forall v \phi(y, x, z) \equiv \exists v \phi(z, y \subseteq v \wedge \psi(v, x, z)),\]

and \(\forall y \forall z (z_1 y \subseteq z_1 v_1 \wedge z_2 y_2 \subseteq z_2 v_2 \wedge \chi(x, z_1, z_2, v_1, v_2)) \equiv \forall y (z_1 y_1 \subseteq z_1 v_1 \wedge z_2 y_2 \subseteq z_2 v_2 \wedge \chi(x, z_1, z_2, v_1, v_2)).\]

We show next that the quantifier-free formula \(\theta\) in the above two theorems can also be turned into an equivalent formula in some normal form.

**Lemma 3.3** Every quantifier-free \(\text{Inc}\)-formula \(\theta(z)\) is semantically equivalent to a formula of the form

\[\exists w \left( \bigwedge_{i \in I} u_i \subseteq v_i \wedge \alpha(w, z) \right),\]

where \(\alpha\) is a first-order quantifier-free formula, and each \(u_i\) and \(v_i\) are sequences of variables from \(w\).

**Proof.** We prove the lemma by induction on \(\theta\). The case when \(\theta\) is a first-order formula (including the case \(\theta = \neg \alpha\)) is trivial. If \(\theta = x \subseteq y\), clearly \(x \subseteq y \equiv \exists w (w \subseteq u \wedge w = x \wedge u = y)\).

Assume that \(\theta_0 = \exists w_0 (t_0(w_0) \wedge \alpha_0(w_0, x))\) and \(\theta_1 = \exists w_1 (t_1(w_1) \wedge \alpha_1(w_1, y))\), where \(\alpha_0, \alpha_1\) are first-order and quantifier-free, the sequences \(w_0\) and \(w_1\) do not have variables in common,

\[t_0(w_0) = \bigwedge_{i \in I} u_i \subseteq v_i \quad \text{and} \quad t_1(w_1) = \bigwedge_{j \in J} u_j \subseteq v_j.\]

If \(\theta = \theta_0 \wedge \theta_1\), then by (9) we have \(\exists w_0 \exists w_1 \exists p \exists p' \exists q \left( \bigwedge_{i \in I} (u_i \subseteq v_i) \wedge \bigwedge_{j \in J} (u_j \subseteq v_j) \wedge (\alpha_0 \wedge \alpha_1) \right).

If \(\theta = \theta_0 \vee \theta_1\), we show that \(\theta\) is equivalent to

\[\forall z \left( \bigwedge_{i \in I} u_i \subseteq v_i \wedge \alpha \right), \quad \left( \exists w \exists z \left( \bigwedge_{i \in I} u_i \subseteq v_i \wedge \bigwedge_{j \in J} (uj \subseteq v_j) \wedge (\alpha \wedge \psi) \right) \right),\]

where each \(u_i\) and \(v_i\) consist of variables from the sequence \(x = (x_1, \ldots, x_n)\). Then can prove \(\theta_0 \vee \theta_1 \equiv \psi\) by consecutively applying (10) as follows:

\[\exists w_0 \left( \bigwedge_{i \in I} u_i \subseteq v_i \wedge \alpha_0 \right) \wedge \theta_1\]

\[\equiv \exists w_0 \exists w_1 \exists p \exists p' \exists q \left( \bigwedge_{i \in I} u_i \subseteq v_i \wedge (\alpha_0 \leftrightarrow p = q) \wedge (\alpha_0 \wedge \exists w_1 (\bigwedge_{j \in J} u_j \subseteq v_j \wedge \alpha_1)) \right)\]

\[\equiv \exists w_0 \exists p \exists p' \exists q \left( \bigwedge_{i \in I} u_i \subseteq v_i \wedge (\alpha_0 \leftrightarrow p = q) \wedge \exists w_1 \exists p' \exists q' \left( \bigwedge_{j \in J} u_j \subseteq v_j \wedge (\alpha_1 \leftrightarrow p' = q') \wedge (\alpha_0 \wedge \exists w_1 \alpha_1) \right) \right)\]

\[\equiv \psi.\]

We now complete the proof by verifying claim (10). For the direction left to right, suppose \(M \models x \exists \chi \left( \bigwedge_{i \in I} u_i \subseteq v_i \wedge (\alpha \wedge \chi (x, z, y)) \right) \equiv \phi (y, z)\). Then there are teams \(Y, Z \subseteq X\) and suitable sequence of functions \(F = (F_1, \ldots, F_n)\) for \(\exists x\) such that \(X = Y \cup Z\), \(M \models \exists \chi (F(x)) \left( \bigwedge_{i \in I} u_i \subseteq v_i \wedge (\alpha \wedge \chi) \right)\) and \(M \models \phi (y, z)\). We now define suitable (sequence of) functions \(F' = (F_1', \ldots, F_n')\) for quantifications \(\exists x, \exists p, \exists q\) as follows: Pick two distinct elements \(a, b \in M\).
• Define $F'$ in such a way that the resulting team $(F'/x)$ satisfies

$$Y(F'/x) = Y(F/x)$$

and $(X \setminus Y)(F'/x) = (X \setminus Y)(a/x)$,

where $(X \setminus Y)(a/x) := \{s(a/x_1) \cdots (a/x_n) \mid s \in X \setminus Y\}$. We omit here the precise technical definition.

• Define $G : X(F'/x) \to \wp(M) \setminus \{\emptyset\}$ by taking $G(s) = \{a\}$.

• Define $H : X(F'/x)(G/p) \to \wp(M) \setminus \{\emptyset\}$ by taking

$$H(s) = \begin{cases} \{a\} & \text{if } M \models \alpha; \\ \{b\} & \text{otherwise}. \end{cases}$$

Put $W = X(F'/x)(G/p)(H/q)$. Clearly, $M \models \alpha \leftrightarrow p = q$. It remains to show that $M \models \alpha \lor \phi$ and $M \models \forall u_i pq \subseteq v_i pq$ for all $i \in I$.

For the former, define

$$U = Y(F'/x)(G/p)(H/q)$$

and $V = Z(F'/x)(G/p)(H/q)$.

Clearly $W = U \cup V$, as $X = Y \cup Z$. Since $M \models z \phi(y,z)$, $M \models Y(F'/x) \alpha(x,z)$ and $Y(F'/x) = Y(F'/x)$, we obtain $M \models \forall \phi(y,z)$ and $M \models \alpha(x,z)$ by locality.

For the latter, let $s \in W$ be arbitrary. If $s \in U$, since $M \models U u_i \subseteq v_i$, there exists $t_0 \in Y$ such that $t_0(v_i) = s(u_i)$. Now, since $M \models_U \alpha(x,z)$, by the definition of $H$ and $G$, we know that $s(q) = a = s(p)$. Thus, for $t = t_0(a/p)(a/q) \in W$, we have $t(v_i pq) = t_0(v_i,a,a) = s(u_i pq)$.

If $s \in W \setminus U$, then $s \models \text{dom}(X) \subseteq X \setminus Y$ and thereby $s(x) = (a, \ldots, a)$ by the definition of $F'$. Thus, $s(v_i pq) = \{a, \ldots, a, s(p), s(q)\} = s(u_i pq)$, namely that $s$ itself is the witness of $u_i pq \subseteq v_i pq$ for $s$.

For the direction right to left of the claim (10), suppose there are suitable (sequence of) functions $F = \langle F_1, \ldots, F_n \rangle, G, H$ for the quantifications $\exists x \exists y \exists z$ such that for $W = X(F'/x)(G/p)(H/q)$, we have that $M \models \forall \exists x \exists y \exists z \alpha(x,z) \land \phi(y,z)$.

Then there are teams $U, V \subseteq W$ such that $W = U \cup V$, $M \models_U \alpha$ and $M \models_V \phi$. Since $\alpha$ is flat, we may let $U \subseteq W$ be the maximal such team.

Consider $Y = U \cap \text{dom}(X)$ and $Z = V \cap \text{dom}(X)$. Clearly, $X = Y \cup Z$, and $M \models_Y \phi(y,z)$ by locality. It remains to show that $M \models \forall \exists x \exists y \exists z \alpha(x,z)$.

Define a suitable sequence of functions $F' = \langle F'_1, \ldots, F'_n \rangle$ for $\exists x$ in such a way that $Y(F'/x) = U \cap \text{dom}(X) \cup \{x_1, \ldots, x_n\}$. We omit the precise technical definition here. Now, since $M \models_U \alpha(x,z)$, we have $M \models F'(x,z)$ by locality. To show that $Y(F'/x)$ satisfies each $u_i \subseteq v_i$, take any $s \in Y(F'/x)$. Let $s \in W$ be an arbitrary extension of $s$. Since $M \models_U u_i pq \subseteq v_i pq$, there exists $t \in W$ such that $t(u_i pq) = t(v_i pq)$. Since $M \models_U \alpha(x,z)$ and $M \models_U \alpha \leftrightarrow p = q$, we have that $t(p) = t(q)$. It then follows that $t(p) = t(q)$, which in turn implies that $M \models \alpha$. Then $t \in U$, as $W$ was assumed to be the maximal subteam of $W$ that satisfies $\alpha(x,z)$. Hence, $t_0 = t \cap \text{dom}(X) \cup \{x_1, \ldots, x_n\} \subseteq Y(F'/x)$ and $t_0(v_i) = t(v_i) = t(u_i) = s(u_i)$.

Finally, by using the above normal form results we obtain the desired more refined normal form as follows.

**Theorem 3.4** Every Inc-formula $\phi(z)$ is semantically equivalent to a formula of the form

$$\exists w \exists x \forall y \left( \bigwedge_{i \in I} u_i \subseteq v_i \land \bigwedge_{j \in J} z_{x_1} \ldots x_{j-1} y \subseteq z_{x_1} \ldots x_{j-1} x_j \land \alpha(w,x,z) \right),$$

where $\alpha$ is a first-order quantifier-free formula, and each $u_i$ and $v_i$ are sequences of variables from $w$.

**Proof.** By Theorem 3.1 we may assume that $\phi$ is in prenex normal form 3. Furthermore, by Lemma 3.3 the quantifier free formula $\theta$ in 3 is equivalent to a formula of the form 7. Hence, $\phi(z)$ is equivalent to a formula of the form

$$Q^I x_1 \ldots Q^n x_n \exists w \left( \bigwedge_{i \in I} u_i \subseteq v_i \land \alpha(w,x,z) \right).$$

Finally, by applying Theorem 3.4 to the above formula (and rearranging the order of the existential quantifiers) we obtain an equivalent formula of the form 11. \[\square\]
To simplify notations in the normal form (11), we now introduce some conventions. For any permutation \( f : \{1, \ldots, n\} \to \{1, \ldots, n\} \) and \( k \leq n \), we define a function \( \sigma^{f,k}_{(j)} : \text{Var}^n \to \text{Var}^k \) by taking \( \sigma^{f,k}_{(j)} = x_{f(1)} \cdots x_{f(k)} \) for any sequence \( x = (x_1, \ldots, x_n) \). That is, \( \sigma^{f,k}_{(j)} \) is a sequence of variables from \( x \). When no confusion arises we drop the superscripts in \( \sigma^{f,k}_{(j)} \) and write simply \( \sigma_{(j)} \). We reserve the greek letters \( \pi, \rho, \sigma, \tau \) (with or without superscripts) for such functions. The normal form of an \text{Inc}-sentence (with no free variables) can then be written as

\[
\exists w \exists \gamma \exists y \left( \bigwedge_{i \in f} \rho^{f}_{i} \subseteq \sigma^{f}_{i} \land \bigwedge_{j \in I} \pi^{f}_{j} \subseteq \tau^{f}_{j} \land \alpha(w, x) \right).
\]

(12)

Observe that the formula in the above normal form has only one (explicit) universal quantifier (i.e., \( \forall y \)). Yet because of the inclusion atoms \( \pi^{f}_{j} \subseteq \tau^{f}_{j} \) in the formula, some existentially quantified variables from \( x \) are essentially universally quantified (cf. equivalence (6)).

4 Game expression and approximations

In this section, we define the game expression \( \Phi \) for every \text{Inc}-sentence \( \phi \) (with no free variables) in normal form. Intuitively the formula \( \Phi \) is a first-order sentence of infinite length that simulates all possible plays in the semantic game (in team semantics) of the formula \( \phi \). Over countable models \( \Phi \) and \( \phi \) are equivalent, as we will show in Theorem 4.1. For a game of finite length \( n \), we define a first-order formula \( \Phi^{n} \) of finite length, called the \( n \)-approximation of \( \Phi \). It follows from the model-theoretic argument in [25] that \( \Phi \) is equivalent to the (infinitary) conjunction of all its approximations \( \Phi^{n} \) over countable recursively saturated models. These game expressions and their finite approximations will be crucial for proving the completeness theorem for the system of \text{Inc} to be introduced in the next section.

Now, let \( \phi \) be an \text{Inc}-sentence (with no free variables). By Theorem 4.4 we may assume that \( \phi \) is in normal form (12). We now define the game expression of \( \phi \) as the following first-order sentence \( \Phi \) of infinite length:

\[
\Phi := \exists w \exists y_0 \forall y_0 \left( \alpha(w_0, x_0) \land \exists w^1 y_1 \forall y_1 \left( \alpha_1(w^1 x^1) \land \gamma_1(w_0 w^1) \land \delta_1(y_0, x_0 x^1) \land \ldots \right) \right),
\]

where

- \( w^n = (w_\xi \mid \xi \in E_n \cup U_n) \) and \( x^n = (x_\xi \mid \xi \in E_n \cup U_n) \) with
  - \( E_n \) being the set of indices \( \langle \xi, i \rangle \) of variables \( w_\xi, i \) introduced as witnesses for each \( \rho^{f}_{i} \subseteq \sigma^{f}_{i} \) with respect to the variables \( w_\xi \) from \( w^{n-1} \),
  - \( U_n \) being the set of indices \( \langle \eta, j \rangle \) of variables \( x_\eta, j \) introduced as witnesses for each \( \pi^{f}_{j} \subseteq \tau^{f}_{j} \) with respect to all new pairs \( x_\xi y_\eta \) with \( x_\xi \) from \( x_0 \cdots x^{n-1} \) and \( y_\eta \) from \( y_0 \cdots y_n \) (write \( A_n = \{ \langle \xi, \eta \rangle \mid \langle \xi, i \rangle, \langle \eta, j \rangle \in U_n \text{ for some } j \in J \} \),
  - and note that we are requiring that \( \xi, \eta \notin A_1 \cup \ldots \cup A_{n-1} \);

- \( \alpha_\xi(w^n x^n) := \bigwedge_{\xi \in E_n \cup U_n} \alpha(w_\xi, x_\xi) \);
- \( \gamma_\eta(w^{n-1} w^n) := \bigwedge_{\xi \in E_{n-1} \cup J} \rho^{f}_{w_\xi, i} = \sigma^{f}_{w_\xi, i} \);
- \( \delta_\eta(y_0 \ldots y_{n-1}, x_0 x^1 \ldots x^n) := \bigwedge_{\xi \eta \in A_n} \pi^{f}_{w_\xi, j} y_\eta = \tau^{f}_{w_\xi, j} \).
The formula $\Phi$ is defined in layers that correspond essentially to the plays in the semantic game of the formula $\phi$ (see e.g., [5] for the definition of the semantic game for Inc). Each layer of $\Phi$ consists of the subformula $\exists w^nx^\Phi \gamma_n(\alpha_0 \land \gamma_0 \land \delta_n \land \ldots)$ with $w^nx^\Phi = w_0 \delta_0$ and $\delta_0 = \alpha(w_0, x_0) \land T \land T \equiv \alpha(w_0, x_0)$. The intuitive reading of each layer is as follows: Each layer introduces new existentially quantified variables $w^n x^n$ and one universally quantified variable $y_n$, and specifies (in $\alpha_n$) that $\alpha$ holds for the existentially quantified variables $w^n x^n$. For each inclusion atom $p^x_n \subseteq \sigma^x_n$ in $\phi$, with respect to each sequence $w^x_n$ of existentially quantified variables introduced in layer $n - 1$, a witness sequence $w^x_{\xi,j}$ of variables (as specified in the formula $\gamma_n$), together with the accompanying sequence $w^x_{\eta,j}$, are introduced in layer $n$ as part of $w^n x^n$. Similarly, for each inclusion atom $\pi^y_n \subseteq \tau^y_n$ in $\phi$, with respect to each new combination $w^x_n y_n \in A_n$ of existentially quantified variables $w^x_n$ introduced up to layer $n - 1$ and universally quantified variables $y_n$ introduced up to layer $n$, a witness sequence $w^x_{\xi,n,j}$ of variables (as specified in the formula $\delta_n$) together with the accompanying sequence $w^x_{\eta,i}$ are introduced in layer $n$ as part of $w^n x^n$. Note that $E_{n+1} = \{ E_n \cup U_n, i \in I \}$ and $U_{n+1} = \{ E_n \cup U_n, i \in A_{n+1}, j \in J \}$.

We assume that the reader is familiar with the game-theoretic semantics of first-order and infinitary logic. Let us now recall the semantic game $G(M, \Phi)$ of the formula $\Phi$ over a model $M$, which is an infinite game played between two players $\forall$belard and $\exists$loise. At each round the players take turns to pick elements from $M$ for the quantified variables $w^n x^n$ and $y_n$, as illustrated in the following table:

| round | 0     | 1     | ... | n     | ... |
|-------|-------|-------|-----|-------|-----|
| $\forall$ | $a^0 b^n$ | $a^1 b^1$ | ... | $a^n b^n$ | ... |
| $\exists$ | $\pi^0_x$ | $\pi^1_x$ | ... | $\pi^n_x$ | ... |

The choices of the two players generate an assignment $s$ for the quantified variables $w^n x^n y_n$ defined as $s(w^n x^n) = a^n b^n$ and $s(y_n) = a_n$.

The player $\exists$loise wins the (infinite) game if for each natural number $n$, $M \models a_0 \land \gamma_n \land \delta_n$.

Finally, $M \models \Phi \iff \exists$loise has a winning strategy in the game $G(M, \Phi)$, where a winning strategy for $\exists$loise is a function that tells her what to choose at each round, and also guarantees her to win every play of the game. We now show that an Inc-sentence is semantically equivalent to its game expression over countable models by using the game-theoretic semantics.

**Theorem 4.1.** Let $\phi$ be an Inc-sentence, and $M$ a model. Then

(i) $M \models \phi \implies M \models \Phi$,

(ii) and $M \models \Phi \implies M \models \phi$, whenever $M$ is a countable model.

**Proof.** (i) Suppose $M \models \phi$. Then, there exists a suitable sequence $F$ of functions for $\exists w \exists x$ such that for $X = \{ \emptyset \}(F/wx)$,

$$M \models X(M/F) \wedge \rho^x_w \subseteq \sigma^x_w \land \bigwedge_{j \in J} \pi^y_j \subseteq \tau^y_j \land \alpha(w, x). \quad (13)$$

We prove $M \models \Phi$ by constructing a winning strategy for $\exists$loise in the semantic game $G(M, \Phi)$ as follows:

- In round 0, choose any assignment $s_0$ in $X$, and let $\exists$loise choose $a^0 = s_0(w)$ and $b^0 = s_0(x)$. Let $s_0$ be the assignment for $w_0 x_0$ generated by $\exists$loise’s choices so far. By (13), we have that $M \models \alpha_0(\alpha(w, x)$, which implies $M \models a_0 \land \alpha_0(w_0, x_0)$, thus the winning condition is maintained.

- Let $s_{n-1}$ be the assignment generated by the choices of the two players up to round $n - 1$. Assume that we have maintained that for each $w^x_{\xi,n} x^y_{\eta,j}$ in the domain of $s_{n-1}$, the assignment $x^y_{\xi}$ for $wx$ defined as $x^y_{\xi}(wx) = s_{n-1}(w^x_{\xi} x^y_{\eta})$ is in $X$, and assume that $\forall$belard has chosen $c_n$ in round $n$. 


For any $\xi \eta \in A_n$ with $a_\xi b_\xi c_\eta$ the corresponding choices by the two players in (at most) two earlier than $n$ rounds, the assignment $s_\xi (c_\eta / y)$ must be in $X(M/y)$. For each $j \in J$, since $M \models X(M/y)$, there exists $s' \in X(M/y)$ such that $s'(\tau_i) = (s_\xi (\pi_i), c_\eta)$. Let $\exists$ loise choose $b_\xi \eta_j = s'(x)$ and $a_\xi \eta_j = s'(w)$. Clearly, $\delta_n$ is satisfied by the assignment generated by the players’ choices so far, and $s_\xi \eta_j = s' \upharpoonright \text{dom}(X) \in X$.

Similarly, for any $\xi \in E_{n-1}$ and any $i \in I$, by using the fact that $s_\xi \in X$ and $M \models X(M/y)$, we can let $\exists$ loise choose $a_\xi b_\xi \eta_j$ so that $\gamma_n$ is satisfied by the assignment generated by the players’ choices so far, and $s_\xi \eta_j \in X$.

Moreover, since $M \models X(M/y) \alpha(w,x)$ and we have maintained that $s_\xi \in X$ for each $\xi \in E_n \cup U_n$, we conclude that each $\alpha(w, x)$ is satisfied by the assignment $s_n$ generated by the choices of the players till round $n$.

(ii) Suppose $M$ is a countable model of $\Phi$, and $\exists$ loise has a winning strategy in the game $G(M, \Phi)$. Let $(c_n)_{n<\omega}$ enumerate all elements of $M$, and let $\forall$ belard play $c_n$ at each round $n$. Suppose $s$ is the assignment generated by such choices of $\forall$ belard and the corresponding choices of $\exists$ loise given by her winning strategy. Let

$$X = \{s_\xi \mid \xi \in E_n \cup U_n, n < \omega\},$$

where recall that $s_\xi$ is the assignment for $w$ defined as $s_\xi (w) = s(w, x, \xi)$. Observe that $X = \{\emptyset\}(F/wx)$ for some suitable sequence $F$ of functions for $\exists w x$. To show $M \models \phi$, it suffices to verify that the team $X(M/y)$ satisfies [13].

To see that $M \models X(M/y) \alpha(w,x)$, for any $s_\xi (c_\eta / y) \in X(M/y)$, since $\exists$ loise wins the game, we know that $M \models_s \alpha(w, x)$, which implies $M \models s_\xi \alpha(w, x)$, as desired.

To see that $X(M/y)$ satisfies each $\pi_i^y \subseteq \tau_i^y$, take any $s_\xi (c_\eta / y) \in X(M/y)$ and assume $\xi \eta \in A_n$. Since $\exists$ loise wins the game, $s$ satisfies $\delta_n$, and in particular, $M \models s \pi_i^y \eta = \tau_i^y$. Thus, for any extension $s \in X(M/y)$ of $s_\xi \eta_j \in X$, we have that $s(\tau_i^y) = s(\pi_i^y \eta) = s(\pi_i^y)$. By a similar argument, we can also show that $X(M/y)$ satisfies each $\pi_i^y \subseteq \sigma_i^y$. This then finishes the proof.

For each natural number $n < \omega$, we define the $n$-approximation $\Phi_n$ of the infinitary sentence $\Phi$ as the finite first-order formula

$$\Phi_n := \exists w \cdot (x_0 \wedge \exists w \cdot x_1 \wedge y_1 \wedge \exists \alpha_1 \wedge \eta_1 \wedge \delta_1 \wedge \ldots \wedge \exists w \cdot (x_n \wedge y_n \wedge (\alpha_n \wedge \eta_n \wedge \delta_n) \ldots)) \quad \upharpoonright_{n+1}$$

The semantic game for $\Phi_n$ over a model $M$, denoted by $G(M, \Phi_n)$, is defined exactly as the infinite game $G(M, \Phi)$ except that $G(M, \Phi_n)$ has only $n + 1$ rounds. Using the game theoretic-semantics we show, as in [24], that the $\Phi_n$’s do approximate $\Phi$ over recursively saturated models, which (recall from, e.g., [11]) are models $M$ such that for any recursive set $\{\phi_n(x,y) \mid n < \omega\}$ of formulas,

$$M \models \forall x_n \left( \bigwedge_{n < \omega} \exists y \bigwedge_{m \leq n} \phi_m(x,y) \rightarrow \exists y \bigwedge_{n<\omega} \phi_n (x,y) \right).$$

**Theorem 4.2** If $M$ is a recursively saturated (or finite) model, then

$$M \models \Phi \iff M \models \Phi_n \text{ for all } n < \omega.$$

In particular, if $M$ is a recursively saturated countable (or finite) model, then

$$M \models \Phi \iff M \models \Phi_n \text{ for all } n < \omega.$$

**Proof.** The “in particular” part follows from Theorem 4.1. The direction “$\implies$” of the main claim follows from the observation that a winning strategy for $\exists$ loise in the infinite game $G(M, \Phi)$ is clearly also a winning strategy for $\exists$ loise in the finite game $G(M, \Phi_n)$ for every $n < \omega$. The other direction “$\impliedby$” follows from a similar argument to that of Proposition 15 in [25], which we omit here. ☐
### Table 1 Rules for equality, connectives and quantifiers

| Rule | Description |
|------|-------------|
| $t = t' = I$ | Equality rule |
| $t = t'/x = \text{Sub}$ | Substitution rule |
| $[\alpha] D \vdash \neg \alpha = \text{E}$ | Negation elimination |
| $\frac{\alpha}{\phi} \vdash \alpha = \phi$ | Explosion rule |
| $\frac{\phi \wedge \psi}{\phi \wedge \psi} \wedge I$ | Conjunction introduction |
| $\frac{\phi \wedge \psi}{\phi \wedge \psi} \wedge E$ | Conjunction elimination |
| $\frac{\phi(t/x)}{\exists x \phi} \exists I$ | Existential introduction |
| $\frac{D}{\forall x \phi} \forall I$ | Universal introduction |
| $\frac{\forall x \alpha \alpha(t/x)}{\forall x \phi(y)} \forall E$ | Universal elimination |
| $\frac{\forall x \phi(y)}{\forall x \phi} \forall E_0, (5)$ | Universal elimination (5) |
| $\frac{[\phi]}{\exists x \phi} \exists E(3)$ | Existential elimination |
| $\frac{\phi \vee \psi \vee \chi}{\phi \vee \psi} \vee I$ | Disjunction introduction (1) |
| $\frac{\phi \vee \psi \vee \chi}{\phi \vee \psi} \vee E(2)$ | Disjunction elimination (2) |
| $\frac{D_0 \ D_1}{[\psi]} \vee \text{RAA}$ | Reductio ad absurdum (1) |
| $\frac{D_0 \ D_1}{\forall y \psi} \forall \text{Sub}$ | Universal substitution (6) |
| $\frac{\forall x \phi(x,y) \forall y \psi}{\forall x \phi \forall y \psi} \forall \text{Exc}$ | Universal elimination (6) |
| $\frac{\forall x \phi \forall x \psi}{\forall x (\phi \wedge \psi)} \forall \text{Ext}$ | Universal elimination (5) |
| $\frac{\forall x \phi(x,v) \forall y \psi(v)}{\exists y \exists z \forall x ((\phi \wedge y = z) \vee (\psi \wedge y \neq z))} \forall \text{Ext}(7)$ | Universal elimination (7) |

(1) The undischarged assumptions in the derivation $D$ contain first-order formulas only.
(2) The undischarged assumptions in the derivations $D_0$ and $D_1$ contain first-order formulas only.
(3) $x$ does not occur freely in $\psi$ or in any formula in the undischarged assumptions of $D_1$.
(4) $x$ does not occur freely in any formula in the undischarged assumptions of $D$.
(5) $x$ is not in the sequence $y$ of free variables of $\phi$.
(6) $y$ does not occur freely in $\forall x \phi$ or in any formula in the undischarged assumptions of $D_1$.
(7) $x$ does not occur freely in $\psi(v)$, and $y,z$ are fresh variables.

### 5 A system of natural deduction for Inc

In this section, we introduce a system of natural deduction for inclusion logic and prove the soundness theorem of the system. We also prove in the system that every Inc-formula implies its normal form.

**Definition 5.1** The system of natural deduction for Inc consists of the rules for equality, connectives and quantifiers in Table 1 and the rules for inclusion atoms in Table 2, where $\alpha$ ranges over first-order formulas, and the letters $x,y,z,\ldots$ in serif font stand for arbitrary (possibly empty) sequences of variables. The rules with double horizontal bars are invertible, i.e., they can be applied in both directions.

We write $\Gamma \vdash_{\text{Inc}} \phi$ or simply $\Gamma \vdash \phi$ if $\phi$ is derivable from the set $\Gamma$ of formulas by applying the rules of the system of Inc. We write simply $\phi \vdash_{\text{Inc}} \psi$ for $\{\phi\} \vdash_{\text{Inc}} \psi$. Two formulas $\phi$ and $\psi$ are said to be *provably equivalent*, written $\phi \equiv_{\text{Inc}} \psi$, if both $\phi \vdash \psi$ and $\psi \vdash \phi$. 

∀ the scope of a existential quantifier as well as that of inclus ion atoms. These two rules are in a sense ad hoc to the present system. Simplifying these rules is left as futur e work.

The nonstandard features of the disjunction are also reflect ed in the rules

As shown in Table 1 restricted to first-order formulas only our system contains all rules of first-order logic (with equality). But classical rules are in general not sound for non-classical Inc-formulas, such as the rules for negation and ∀E. As a consequence, our system does not admit uniform substitution.

Recall that the usual disjunction elimination rule (∨E) is not sound for dependence and independence logic (see [12,25]). In our system of Inc the disjunction does admit the rule ∀E under the side condition that the undischarged assumptions in the sub-derivations contain classical formulas only. This side condition however, makes, among other things, the usual derivation of the distributive law φ ∧ (ψ ∨ χ) → (φ ∧ ψ) ∨ (φ ∧ χ) not applicable in the system. This distributive law actually fails in Inc in general, especially when φ is not closed downwards. The nonstandard features of the disjunction are also reflected in the rules ∀∧Ext and ∃⊆Ext. The invertible rule ∀∧Ext extends the scope of a universal quantifier over a disjunction. The rule ∃⊆Ext extends over a disjunction the scope of a existential quantifier as well as that of inclusion atoms. These two rules are in a sense ad hoc to the present system. Simplifying these rules is left as future work.

The universal quantifier of Inc turns out to be a peculiar connective, especially the usual elimination rule ∀xφ/φ(t/x) is not in general sound for arbitrary formulas. For instance, we have |= ∀x(y ⊆ x), whereas ⊭ y ⊆ z. The two weaker elimination rules ∀E and ∀E₀ we include in the system restrict the subformula φ in the premise either to a first-order formula or a formula in which x is not free. To compensate the weakness of the elimination rules we also add to our system a substitution rule ∀Sub, an exchange rule ∀Exc, and two rules ∀∧Ext and ∀∨Ext for extending the scope of universal quantifier over conjunction and disjunction. In this nonstandard setting, the derivations of some natural and simple rules for universal quantifier become not entirely trivial, as we will illustrate in the next proposition.

**Proposition 5.2** (i) ∀xφ ⊢ ∀yφ(y/x) if y /∈ Fv(∀xφ).

(ii) ∀x(φ ∧ ψ) ⊢ ∀xφ ∧ ∀xψ.

**Proof.** [3] Follows from ∀Sub, since y /∈ Fv(∀xφ).
The inclusion expansion rule \( \subseteq \text{Exc} \) and contraction rule \( \subseteq \text{Ctr} \) for inclusion atoms in our system, together with the rule \( xy \subseteq uv/xy \subseteq uvv \) that we will derive in the next proposition, are clearly equivalent to the projection rule \( x_1 \ldots x_n \subseteq y_1 \ldots y_n/x_1 \ldots x_n \subseteq y_1 \ldots y_n (i_1 \ldots i_k \subseteq \{1, \ldots n\}) \). As we mentioned in the introduction, the projection rule, the transitivity rule \( \equiv \text{Trs} \) and the reflexivity axiom \( x \subseteq x \) (that we will also derive in the proposition below) form a complete axiomatization of the implication problem of inclusion dependencies in database theory (\( \mathcal{E} \)). The inclusion compression rule \( \subseteq \text{Cmp} \) is a slight generalization of a similar rule introduced in \( \mathcal{E} \). The inclusion expansion rule \( \subseteq \text{Exp} \) characterizes the fact that

\[
\Gamma, y \subseteq x \models \alpha(y/z) \implies \Gamma \models \alpha(x/z)
\]

whenever variables in \( y \) are not free in \( \Gamma \) (observe that in this case \( \Gamma, y \subseteq x, \lnot \alpha(y/z) \models \bot \) iff \( \Gamma, y \subseteq x \models \alpha(y/z) \)). The weakening rule via existential quantifier \( \subseteq \text{W}_\exists \) was introduced in \( \mathcal{E} \), and the weakening rule via universal quantifier \( \subseteq \text{W}_G \) has a similar flavor. The invertiable simulation rule \( \subseteq \text{Sim} \) characterizes the fact that universal quantifiers can be simulated by existential quantifiers with the help of inclusion atoms.

**Proposition 5.3** 
(i) \( t \vdash t \subseteq x \).

(ii) If \( |x| = |y| = |z| \), then \( xy \subseteq zz \vdash t = y \).

(iii) \( xy \subseteq uv \vdash xy \subseteq uvv \).

**Proof.**

(i) By the soundness of the nontrivial rules \( \forall \text{E}, \forall \text{Ext}, \subseteq \text{Cmp}, \subseteq \text{Exp}, \subseteq \text{W}_\exists, \subseteq \text{W}_G \) and \( \forall \subseteq \text{Sim} \). The soundness of \( \exists \subseteq \text{Ext} \) follows from (10) in the proof of the disjunction case of Lemma 3.3.

\( \forall \text{E} \): It suffices to show that \( \Delta_0, \phi \models \chi \) and \( \Delta_1, \psi \models \chi \) imply \( \Delta_0, \Delta_1, \phi \lor \psi \models \chi \) for any two sets \( \Delta_0, \Delta_1 \) of first-order formulas. Suppose that \( M \models_X \Delta_0 \cup \Delta_1 \), and also that \( M \models_X \phi \lor \psi \). Then there exist \( Y, Z \subseteq X \) such that \( X = Y \cup Z, M \models_Y \phi \) and \( M \models_Z \psi \). Since formulas in \( \Delta_0 \cup \Delta_1 \) are closed downwards, we have that \( M \models_Y \Delta_0 \) and \( M \models_Z \Delta_1 \). Then it follows from the assumption that \( M \models_Y \chi \) and \( M \models_Z \chi \). Now, since \( \chi \) is closed under unions, we conclude that \( M \models_X \chi \), as required.

\( \forall \text{Ext} \): We first show that \( \forall \phi(x,v) \lor \psi(y) \models \exists y \exists z (\phi \land y = z) \lor (\psi \land y \neq z) \), where \( x \notin \text{Fv}(\psi) \) and \( y, z \notin \text{Fv}(\phi) \cup \text{Fv}(\psi) \). Suppose \( M \models_X \forall \phi \lor \psi \), where we may w.l.o.g. assume \( x, y, z \notin \text{dom}(X) \). Then there exist \( Y, Z \subseteq X \) such that \( X = Y \cup Z, M \models_{X|M/x} \phi \) and \( M \models_{Z} \psi \). Define functions \( F : X \to \phi(M) \setminus \emptyset \) and \( G : X/F/y \to \phi(M) \setminus \emptyset \) as follows: Let \( a, b \) be two distinct elements in \( M \).

\[
F(s) = \begin{cases} 
\{a\} & \text{if } s \in Y \setminus Z, \\
\{a, b\} & \text{if } s \in Y \cap Z, \\
\{b\} & \text{if } s \in Z \setminus Y,
\end{cases}
\]

and \( G(s) = \{a\} \).

Now, we split the team \( X' = X \setminus \{F/y\}(G/z)(M/x) \) into \( W = \{s \in X' \mid s(y) = a\} \) and \( U = \{s \in X' \mid s(y) = b\} \). Clearly, \( M \models_W y = a \) and \( M \models_U y \neq a \). Observe that \( W \cup \{\text{dom}(X) \cup \{x\}\} = Y \setminus F/M/x \) and \( U \cup \text{dom}(X) = Z \). Since \( M \models_{Y|M/x} \phi \) and \( M \models_Z \psi \), we conclude that \( M \models_W \phi \) and \( M \models_U \psi \).

\( |x| \) denotes the length of the sequence \( x \).
Conversely, suppose $M \models_{X} \exists y \exists z \forall x ((\phi(x, v) \land y = z) \lor (\psi(v) \land y \neq z))$. Then there are suitable functions $F, G$ and teams $Y, Z \subseteq X(F/\psi(G/z))(M/x)$ such that $X(F/\psi(G/z))(M/x) = Y \cup Z$, $M \models_{Y} \phi \land y = z$ and $M \models_{Z} \psi \land y \neq z$. Put $Y' = Y \setminus \text{dom}(X)$ and $Z' = Z \setminus \text{dom}(X)$. Clearly $X = Y' \cup Z'$. To show that $M \models_{X} \forall x \psi(v) \lor \phi$ and $M \models_{Z'} \psi$, it then suffices to verify $M \models_{Y'} \forall x \phi$ and $M \models_{Z'} \psi$. The latter is clear, since $M \models_{Z} \psi$ and $x, y, z \notin Fv(\psi)$. To see the former, first observe that for any $s \in Y$ and any $a \in M$, since $s(a/x)(y) = s(y) = s(z) = s(a/x)(z)$, we must have that $s(a/x) \notin Z$, or $s(a/x) \in Y$. This shows that $Y = Y'(M/x)$, thus $Y \setminus \text{dom}(X) \cup \{x\} = Y'$ \setminus \text{dom}(X)) = Y'(M/x)$. Now, since $Y \setminus \text{dom}(X) \cup \{x\}$ satisfies $\phi$, we conclude $M \models_{Y'} \phi$ and, thus $M \models_{Y'} \forall x \phi$ as required.

Cmp: Suppose $M \models_{X} y \subseteq x$ and $M \models_{x} \alpha(x/z)$, where the free variables of $\alpha(x/z)$ are among $x$. We show that $M \models_{X} \alpha(y/z)$. For any $s \in X$, since $M \models_{X} y \subseteq x$, there exists $s' \in X$ such that $s'(x) = s(y)$. Since $M \models_{X} \alpha(x/z)$ and $\alpha$ is first-order, we have that $M \models_{s'} \alpha(x/z)$, which implies $M \models_{s'} \alpha(y/z)$ by the locality property. Hence, we conclude that $M \models_{X} \alpha(y/z)$.

Exp: Assume $\Gamma, y \subseteq x, -\alpha(y/z) \subseteq \perp$, where the free variables of $\alpha(y/z)$ are among $y$, and the variables in $y$ do not occur freely in $\Gamma$. We show that $\Gamma \models \alpha(x/z)$. Suppose that $M \models_{X} \Gamma$ for some nonempty team $X$. It suffices to show that $M \models \alpha(x/z)$ for any $s \in X$. Consider the team $X(s(x)/y)$. Clearly, $M \models_{X(s(x)/y)} y \subseteq X$. On the other hand, since the variables in $y$ do not occur freely in $\Gamma$, by locality we obtain that $M \models_{X(s(x)/y)} \Gamma$. Now, since $X(s(x)/y) \neq \emptyset$, the assumption $\Gamma, y \subseteq x, -\alpha(y/z) \subseteq \perp$ gives that $M \not\models_{X(s(x)/y)} -\alpha(y/z)$, which by locality implies that $M \not\models \alpha(x/z)$, as required.

Wz: It suffices to show that $\Gamma \models x \subseteq y$ implies $\Gamma \models \exists w(xw \subseteq yz)$, where $w$ is not among $xyc$. Suppose $M \models_{X} \Gamma$. By the assumption, $M \models_{X} x \subseteq y$, meaning that for any $s \in X$, there exists $s' \in X$ such that $s'(y) = s(x)$. Now, to show that $M \models_{X} \exists w(xw \subseteq yz)$, we define a function $F : X \rightarrow \rho(M) \setminus \{\emptyset\}$ by taking $F(s) = \{s'(z)\}$.

To show that $M \models_{X(F/w)} xw \subseteq yz$, take any $s \in X(F/w)$. Consider the witness $s'_0 \in X$ for $x \subseteq y$ with respect to $s_0 = s \setminus \text{dom}(X)$. We have $s(xw) = s_0(x)w = s'_0(y)s'_0(z)$. Hence, any extension of $s'_0$ in $X(F/w)$ witnesses $xw \subseteq yz$.

WC: It suffices to show that $\Gamma \models x \subseteq y$ implies $\Gamma \models \forall w(xz \subseteq yw)$, where $w$ is not among $xyc$. Suppose $M \models_{X} \Gamma$, where we may assume w.l.o.g. that $w \notin \text{dom}(X)$ (if not, rename the bound variable $w$ in $\forall w(xz \subseteq yw)$). It then follows by locality that $M \models_{X(M/w)} \Gamma$ as well, and thus $M \models_{X(M/w)} x \subseteq y$ by assumption. To show $M \models_{X(M/w)} xz \subseteq yw$, take an arbitrary $s \in X(M/w)$. Since $M \models_{X(M/w)} x \subseteq y$, there exists $s' \in X(M/w)$ such that $s'(y) = s(x)$. Clearly, the assignment $s'' = s'(s(z)/w)$ belongs to the team $X(M/w)$, and $s''(yw) = s'(y)s'(z) = s(xz)$, as required.

Sim: For the top to bottom direction, suppose $M \models_{X} \forall x \phi(x, z)$. We show that $M \models_{X} \exists z \forall y (zy \subseteq xz \land \phi(x,z))$, where variables from $y$ are fresh. Let $x = \{x_1, \ldots, x_n\}$. Define a sequence $F = (F_1, \ldots, F_n)$ of functions for $\exists x$ by taking $F_i(s) = M$ for each $F_i$ from $F$, namely, we let $X(F/x) = X(M/x)$. It suffices to show that $M \models_{X(F/s_i)(M/y)} zy \subseteq xz \land \phi(x, z)$, or $M \models_{X(M/s_i)(M/y)} zy \subseteq xz \land \phi(x, z)$.

By assumption, $M \models_{X(M/s_i)(M/y)} \phi(x, z)$, which implies $M \models_{X(M/s_i)(M/y)} \phi(x, z)$. To show that $zy \subseteq xz$ is also satisfied by $X(M/s_i)(M/y)$, any $s \in X(M/s_i)(M/y)$. Observe that the function $s' = s(y)/s(z)$ belongs to the team $X(M/s_i)(M/y)$, and we have that $s'(zx) = s(z)s(y)$, as required.

For the bottom to top direction, suppose $M \models_{X} \exists z \forall y (zy \subseteq xz \land \phi(x, z))$, where no variable from $y$ are free in $\phi$, and we may assume w.l.o.g. that $\text{dom}(X)$ consists of all variables from $z$. Then there are suitable sequence $F$ of functions for $\exists x$ such that $M \models_{X(F/s_i)(M/y)} zy \subseteq xz \land \phi(x, z)$. We show that $M \models_{X} \forall x \phi(x, z)$, or $M \models_{X(M/s_i)} \phi(x, z)$, which, by locality, is further reduced to showing that $X(F/x) = X(M/x)$.

For any $s \in X(M/x)$, consider an arbitrary assignment $t \in X(F/x)(M/y)$ satisfying $t(z) = s(z)$ and $t(y) = s(x)$. Since $M \models_{X(F/s_i)(M/y)} zy \subseteq xz$, there exists $t' \in X(F/x)(M/y)$ such that $t'(zx) = t'(zy) = t(x)s(z)$, meaning that $s = t' \setminus \text{dom}(X) \cup \{x_1, \ldots, x_n\} \in X(F/x)$. Thus, $X(M/x) \subseteq X(F/x)$, thereby $X(M/x) = X(F/x)$.

We will prove the completeness theorem of our system in the next section. An important lemma for this proof is that every formula provably implies its normal form [12]. To prove this lemma we first prove a few useful propositions. The next three propositions concern the standard properties of quantifications as well as the monotonicity of the entailment relation in Inc. In the sequel, we will often apply Propositions [5.5] and [5.6] without explicit reference to them.

**Proposition 5.5** Let $\alpha$ be a first-order formula, and $x \notin Fv(\psi)$.
(i) \( \neg \forall x \alpha \vdash \exists x \neg \alpha \) and \( \neg \exists x \alpha \vdash \forall x \neg \alpha \).

(ii) \( \forall x \theta \wedge \psi \vdash \forall x (\theta \wedge \psi) \).

(iii) \( \exists x \theta \wedge \psi \vdash \exists x (\theta \wedge \psi) \).

(iv) \( \exists x \theta \vee \psi \vdash \exists x (\theta \vee \psi) \).

Proof. Since our system behaves exactly like first-order logic when restricted to first-order formulas only, item (i) can be proved as usual. Items (iii) and (iv) are proved also as usual. We only derive item (ii). For the right to left direction, we have by Proposition 5.2(ii) that \( \forall x (\theta \wedge \psi) \vdash \forall x \theta \wedge \forall x \psi \). Since \( x \not \in \text{Fv}(\psi) \), \( \forall x \psi \vdash \psi \) by \( \forall E_0 \). Thus \( \forall x (\theta \wedge \psi) \vdash \forall x \theta \wedge \forall x \psi \). For the other direction, since \( \theta, \psi \vdash \theta \wedge \psi \) and \( x \not \in \text{Fv}(\psi) \), we derive by applying \( \forall \text{Sub} \) that \( \forall x \phi, \psi \vdash \forall x (\phi \wedge \psi) \), thus \( \forall x \phi \wedge \forall x \psi \vdash \forall x (\phi \wedge \psi) \).

We write \( \phi(\theta) \) to indicate that \( \phi \) is a formula with an occurrence of \( \theta \) as a subformula, and write \( \phi[\theta'/\theta] \) for the formula obtained from \( \phi \) by replacing the occurrence of \( \theta \) by \( \theta' \).

**Proposition 5.6** If \( \theta \vdash \theta' \), then \( \phi(\theta) \vdash \phi(\theta'/\theta) \). Moreover, if the occurrence of \( \theta \) in \( \phi(\theta) \) is not in the scope of a negation, then \( \theta \vdash \theta' \) implies \( \phi(\theta) \vdash \phi(\theta'/\theta) \).

Proof. A routine inductive proof. Apply \( \forall \text{Sub} \) in the case \( \phi = \forall x \psi \).

**Proposition 5.7** Let \( \phi \) be a formula and \( Q \theta \) the semantically equivalent formula in prenex normal form as given in Lemma 3.3, where \( Qx = Q^1x_1 \cdots Q^nx_n (Q^i \in \{\forall, \exists\}) \) is a sequence of quantifiers and \( \theta \) is a quantifier free formula. Then \( \phi \vdash Q \theta \).

Proof. Repeatedly apply Propositions 5.5 and \( \forall, \text{Ext} \) (c.f. the proof of Theorem 3.1).

The next technical proposition shows, as a generalization of the rule \( \forall \subseteq \text{Sim} \), that universal quantifiers in a more general context can also be simulated using existential quantifiers and inclusion atoms.

**Proposition 5.8** \( \forall x Qu \phi(u, x, z) \vdash \exists x Qu \forall y (z y \subseteq z x \wedge \phi(u, x, z)) \).

Proof. We derive the proposition as follows:

\[
\begin{align*}
\forall x Qu \phi(u, x, z) & \vdash \exists x \forall y (z y \subseteq z x \wedge Qu \phi(u, x, z)) & \text{(\( \forall \subseteq \text{Sim} \))} \\
& \vdash \exists x (\forall y (z y \subseteq z x) \wedge Qu \phi(u, x, z)) \\
& \vdash \exists x Qu (\forall y (z y \subseteq z x) \wedge \phi(u, x, z)) \\
& \vdash \exists x Qu \forall y (z y \subseteq z x \wedge \phi(u, x, z)).
\end{align*}
\]

**Lemma 5.9** For any \( \text{inc-formula} \ \phi \), we have that \( \phi \vdash \phi' \), where \( \phi' \) is the semantically equivalent formula in normal form as given in Theorem 3.4.

Proof. We follow a similar argument to that of the semantic proof of Theorem 3.4. First, by Proposition 5.7 we obtain \( \phi \vdash Q \theta \), where \( Q \theta \) is the semantically equivalent formula of \( \phi \) as given in Theorem 3.1 with \( Qx = Q^1x_1 \cdots Q^nx_n (Q^i \in \{\forall, \exists\}) \) a sequence of quantifiers and \( \theta \) a quantifier free formula.

If we can show that \( \theta \vdash \exists x \forall y \theta' \) for some formula \( \theta' = \bigwedge_{i \leq j} u_i \subseteq v_i \wedge \alpha(w, x, z) \) as given in Lemma 3.3, we may obtain \( \phi \vdash Q \exists x \forall y \theta' \) by Proposition 5.6.

Next, we derive

\[
\begin{align*}
Qx \exists x \forall y \left( \bigwedge_{1 \leq j \leq n} \bigwedge_{Q^j \models \forall} x_1 \cdots x_{j-1} y_j \subseteq x_1 \cdots x_{j-1} x_j \wedge \theta'(w, x, z) \right) & \quad \text{(Proposition 5.8)} \\
\text{where } y = \langle y_j | Q^j = \forall, 1 \leq j \leq n \rangle \\
& \vdash \exists x \forall y \left( \bigwedge_{1 \leq j \leq n} \forall y_j (z_1 \cdots x_{j-1} y_j \subseteq x_1 \cdots x_{j-1} x_j) \wedge \theta'(w, x, z) \right) & \quad \text{(Proposition 5.4(ii))}
\end{align*}
\]
Putting all these together, we will complete the proof.

Now, we show that \( \vdash \exists w \theta' \) by induction on \( \theta \). The case \( \theta \) is a first-order formula (including the case \( \theta = \neg \alpha \)) is trivial. If \( \theta = x \subseteq y \), we have that \( x \subseteq y \vdash \exists w u (w \subseteq u \land w = x \land u = y) \). Indeed, we first derive that \( x = x \land y = y \vdash \exists w \exists u (w = x \land u = y) \). Then, by \( = \) we derive that \( x \subseteq y \vdash \exists w u (x \subseteq y \land w = x \land u = y) \).

Assume that \( \theta_0 \vdash \exists w_0 (t_0 (w_0) \land \alpha_0 (w_0, x)) \) and \( \theta_1 \vdash \exists w_1 (t_1 (w_1) \land \alpha_1 (w_1, y)) \), where \( \alpha_0, \alpha_1 \) are first-order and quantifier-free, the sequences \( w_0 \) and \( w_1 \) do not have variables in common, and \( t_0 \) and \( t_1 \) are as in \( \) in the proof of Lemma 3.3. If \( \theta = \theta_0 \land \theta_1 \), then we derive that \( \theta_0 \land \theta_1 \vdash \exists w_0 (t_0 \land \alpha_0) \land \exists w_1 (t_1 \land \alpha_1) \vdash \exists w_0 \exists w_1 (t_0 \land t_1 \land \alpha_0 \land \alpha_1) \) by Proposition 5.11(iii).

If \( \theta = \theta_0 \lor \theta_1 \), let \( \psi \) be the formula \( (9) \) as in the proof of the disjunction case of Lemma 3.3. We derive \( \theta \vdash \psi \) by following the semantic argument as in Lemma 3.3 in which we apply the rule \( \exists \subseteq \text{Ext} \) in the crucial steps.

We end this section by proving some facts concerning the weak classical negation \( \sim \) in the context of \( \text{Inc} \). This connective was introduced in [23], and a trick using \( \sim \) was developed in the paper to generalize the proof of the completeness theorem of dependence logic given in [25]. We will also apply this trick to prove the completeness theorem for our system in the next section. Recall that the team semantics of \( \sim \) is defined as

- \( M \models_X \sim \phi \) if \( X = \emptyset \) or \( M \not\models_X \phi \).

The weak classical negations \( \sim \phi \) of \text{Inc}-formulas \( \phi \) are not in general expressible in \( \text{Inc} \) (because positive greatest fixed point logic, being expressively equivalent to \( \text{Inc} \), is not closed under classical negation). Nevertheless, the weak classical negations \( \sim \alpha \) of first-order formulas \( \alpha \) are expressible (uniformly) in \( \text{Inc} \).

**Fact 5.10** If \( \alpha(x) \) is a first-order formula, then \( \sim \alpha(x) \equiv \exists y (y \subseteq x \land \neg \alpha(y/x)) \), where \( y \) is a sequence of fresh variables.

**Proof.** Since \( \alpha \) is flat, for any nonempty team \( X, M \not\models_X \alpha(x) \), iff \( M \models_s \neg \alpha(x) \) for some \( s \in X \), iff \( M \models_X \exists y (y \subseteq x \land \neg \alpha(y/x)) \).

Stipulating the string \( \sim \alpha(x) \) as a shorthand for the formula \( \exists y (y \subseteq x \land \neg \alpha(y)) \) of \( \text{Inc} \), we show next that the reductio ad absurdum (RAA) rule for \( \sim \) with respect to first-order formulas \( \alpha \), i.e., the rule

\[
\frac{[\sim \alpha]}{\alpha} \quad \text{RAA}_\sim
\]

is derivable in our system from the rule \( \subseteq \text{Exp} \).

**Lemma 5.11** If \( \Gamma, \sim \alpha \vdash \bot \), then \( \Gamma \vdash \alpha \).

**Proof.** Let \( \alpha = \alpha(x) \) and \( \sim \alpha(x) = \exists y (y \subseteq x \land \neg \alpha(y/x)) \), where \( y \) is a sequence of fresh variables. Suppose \( \Gamma, \sim \alpha \vdash \bot \). By \( \subseteq \text{Exp} \), it suffices to show that \( \Gamma, y \subseteq x, \neg \alpha(y/x) \vdash \bot \). But this follows easily from \( \exists y \) and the assumption \( \Gamma, \sim \alpha \vdash \bot \).
6 The completeness theorem

In this section, we prove the completeness theorem for our system of \(\text{Inc}\) with respect to first-order consequences. To be precise, we prove that

\[
\Gamma \vdash \alpha \iff \Gamma \models \alpha
\]

(14)

holds whenever \(\Gamma\) is a set of \(\text{Inc}\)-formulas, and \(\alpha\) is a first-order formula. As sketched in Section 2, our proof combines the technique introduced in [5] and a trick developed in [22] using the weak classical negation \(\sim\) and the RAA rule for \(\sim\). The former treats the case when the set \(\Gamma \cup \{\alpha\}\) of formulas in (14) are sentences (with no free variables) only, while the trick of the latter allows us to handle (open) formulas as well. Since the weak classical negation \(\sim\alpha\) of first-order formulas \(\alpha\) are definable uniformly in \(\text{Inc}\) (Fact 5.10), and the RAA rule for \(\sim\) is derivable in our system of \(\text{Inc}\) (Lemma 5.11), we will be able to apply the trick of [22] in a smoother manner than in the systems of dependence and independence logic [32] (in which the RAA rule for \(\sim\) was added in an ad hoc and non-effective manner).

We have prepared in the previous sections most relevant lemmas for the argument in [25] concerning the normal form of \(\text{Inc}\)-formulas (especially Lemma 5.9), the game expression and its approximations. Another important lemma for the completeness theorem is that any \(\text{Inc}\)-formula \(\phi\) implies every approximation \(\Phi_n\) of its game expression (as introduced in Section 4).

**Lemma 6.1** For any \(\text{Inc}\)-sentence \(\phi\), we have that \(\phi \vdash \Phi_n\) for every \(n < \omega\).

In order to prove the above lemma, we first need to prove a number of technical propositions and lemmas.

**Proposition 6.2** Let \(p : \text{Var}^n \rightarrow \text{Var}^k, \sigma : \text{Var}^n \rightarrow \text{Var}^m\) be functions. Then

\[
p_rz \subseteq \sigma_x, x_0y_0z_0 \subseteq xyz \vdash \exists x_1 y_1 (x_1 y_1 \subseteq x y \land p_{x_0} z_0 = \sigma_{x_1})
\]

where \(|x| = |x_0| \land |y| = |y_0| = |y_1|\) and \(|z| = |z_0|\). In particular, when \(z\) and \(z_0\) are the empty sequence we have \(p_rz \subseteq \sigma_x, x_0y_0 \subseteq x y \vdash \exists x_1 y_1 (x_1 y_1 \subseteq x y \land p_{x_0} = \sigma_{x_1})\).

**Proof.** Assume that \(p(x) = \sigma_x, r_z\) for some permutation \(p\) of the sequence \(x\). Then we have

\[
\begin{align*}
p_rz & \subseteq \sigma_{x_0}, x_0 y_0 z_0 \subseteq x y z \quad \text{(\(\subseteq \text{Ctr}, \subseteq \text{Exc}\))} \\
p_rz & \subseteq \sigma_x, \rho_{x_0} z_0 \subseteq p_rz \quad \text{((\(\subseteq \text{W}_{\exists}\), where \(|w| = |r_x|\)))} \\
p_{x_0} z_0 & \subseteq \sigma_x \quad \text{((\(\subseteq \text{Trs}\))} \\
\exists x_1 y_1 (p(x_1) = \rho_{x_0} z_0 w \land \rho_{x_0} z_0 y_1 \subseteq \sigma_x, r_y) & \vdash \exists x_1 y_1 (p(x_1) = \rho_{x_0} z_0 w \land \rho_{x_0} z_0 y_1 \subseteq \sigma_x, r_y) \quad \text{((= l, \exists l, |p_{x_0} z_0 w| = |x|))} \\
\exists x_1 y_1 (\sigma_{x_1}, r_y = \rho_{x_0} z_0 w \land p(x_1) y_1 \subseteq p(x) y) & \vdash \exists x_1 y_1 (\sigma_{x_1}, r_y = \rho_{x_0} z_0 w \land p(x_1) y_1 \subseteq p(x) y) \quad \text{((\text{Sub})} \\
\exists x_1 y_1 (\sigma_{x_1}, r_y = \rho_{x_0} z_0 w \land p(x_1) y_1 \subseteq p(x) y) & \vdash \exists x_1 y_1 (\sigma_{x_1}, r_y = \rho_{x_0} z_0 w \land x_1 y_1 \subseteq x y) \quad \text{((\text{Exc})}
\end{align*}
\]

We say that an occurrence of a subformula \(\theta\) in \(\phi(\theta)\) is not in the scope of a disjunction or negation if (1) \(\phi = \theta;\) or (2) \(\phi = \psi(\theta) \land \chi\) or \(\chi \land \psi(\theta),\) and \(\theta\) is not in the scope of a disjunction or negation in \(\psi(\theta);\) or (3) \(\phi = \bigvee Q \psi(\theta)\) \((Q \subseteq \{\forall, \exists\})\) and \(\theta\) is not in the scope of a disjunction or negation in \(\psi(\theta).\) For example, in the formula \(\phi(\theta) \land \psi(\bigvee \exists x \theta,\) the leftmost occurrence of \(\theta\) is in the scope of a disjunction, while the rightmost occurrence of \(\theta\) is not.

**Lemma 6.3** If the occurrence of the subformula \(\theta\) in \(\phi(\theta)\) is not in the scope of a disjunction or negation, then \(\phi(\theta), \psi \vdash \phi(\theta) \land \psi(\theta).\)

**Proof.** A routine inductive proof. Apply \forall \text{Sub}, \exists E, \exists l\) in the quantifier cases.

**Proposition 6.4** Suppose that \(\phi(x \subseteq y)\) is a formula in which the occurrence of \(x \subseteq y\) is not in the scope of a disjunction or negation, and the variables from \(y\) are free in \(\phi.\) If \(z\) does not have any common variable with \(x y,\) and \(w\) contains some variables occurring in \(\phi\) (either free or bound), then \(\forall z \phi(x \subseteq y) \vdash \forall z \phi[xw \subseteq yz/x \subseteq y].\)
Now, assuming \( \phi \models x \subseteq y \), since no variable from \( z \) occurs in \( x \subseteq y \), we derive by \( \forall z(x \subseteq y) \models x \subseteq y \). Next, we obtain by \( \subseteq \forall x \models \forall z(x \subseteq y) \). Thus, \( \forall z(x \subseteq y) \models \forall z(x \subseteq y) \).

If \( \phi = \psi(x \subseteq y) \land \chi \), then we have that

\[
\forall z(\psi(x \subseteq y) \land \forall z \chi) \quad \text{(Proposition 5.4 (ii))}
\]

\( \vdash \forall z \psi(x \subseteq y) \land \forall z \chi \) \quad \text{(induction hypothesis)}

\( \vdash \forall z(\psi(x \subseteq y) \land \chi) \). \quad \text{(Proposition 5.4 (ii))}

The case \( \phi = \chi \land \psi(x \subseteq y) \) is symmetric.

If \( \phi = \forall \psi(x \subseteq y) \), then we have that

\( \forall z \forall \psi(x \subseteq y) \vdash \forall \forall z \psi(x \subseteq y) \) \quad \text{(\forall \text{Exc})}

\( \vdash \forall \forall z \psi(x \subseteq y) \land \forall \forall z \chi \) \quad \text{(induction hypothesis)}

\( \vdash \forall z \forall \psi(x \subseteq y) \). \quad \text{(\forall \text{Exc})}

If \( \phi = \exists \psi(x \subseteq y) \), where \( \text{Fv}(\exists \psi) = u \) (note that all variables from \( y \) are among \( u \)), then we have that

\( \forall z \exists \psi(x \subseteq y) \vdash \exists z \exists \forall z_0(u_0 \subseteq u \land \psi(x \subseteq y)) \) \quad \text{(Proposition 5.8 where \( z_0 \) are fresh)}

\( \vdash \exists z \exists \forall z_0(u_0 \subseteq u \land \exists \forall z_0 \psi(x \subseteq y)) \) \quad \text{(Proposition 5.4 (ii))}

\( \vdash \exists z \exists \forall z_0(u_0 \subseteq u \land \forall \forall z_0 \psi(x \subseteq y)) \) \quad \text{(induction hypothesis)}

\( \vdash \exists z \exists \forall z_0(u_0 \subseteq u \land \forall \forall z_0 \psi(x \subseteq y)) \). \quad \text{(Proposition 5.8)}

Next, we obtain by \( \exists \psi(x \subseteq y) \) is not in the scope of a disjunction or negation)

\( \vdash \exists z \exists \forall z_0(u_0 \subseteq u \land \psi(x \subseteq y)) \) \quad \text{(Proposition 5.8)}

\( \vdash \exists z \exists \forall z_0(u_0 \subseteq u \land \psi(x \subseteq y)) \). \quad \text{(Proposition 5.8)}

Thus, \( x \subseteq y \) cannot occur in the scope of a negation)

\( \vdash \forall z \exists \psi(x \subseteq y) \). \quad \text{(Proposition 5.8)}

Now we are ready to give the proof of Lemma 6.1

Proof of Lemma 6.1. By Lemma 5.9 we may assume that \( \phi \) is in normal form (12). We prove the lemma by proving a stronger claim that \( \phi \models \Phi_n \) holds for every \( n < \omega \), where

\[
\Phi_n := \exists w_0 \exists x_0 \forall y_0 \left( \alpha_0 \land \mu_0 \land \exists w_1 x_1 \forall y_1 (\lambda_1 \land \mu_1 \land \cdots \land \exists w_n x_n \forall y_n (\lambda_n \land \mu_n) \cdots) \right).
\]

where each \( \lambda_n = \alpha_n \land \gamma_n \land \delta_n \).

\[
\mu_0 = \bigwedge_{i \in I} \rho_{i,0}^0 \subseteq \sigma_{i,0}^0 \land \bigwedge_{j \in J} \pi_{j,0}^0 \subseteq \tau_{j,0}^0 \quad \text{and} \quad \mu_n = \bigwedge_{i \in E_n \cup I_n} w_x x_y \subseteq w_0 x_0.
\]

If \( n = 0 \), then \( \Phi'_n := \exists w_0 x_0 \forall y_0 (\alpha(w_0, x_0) \land \mu_0) \), and \( \phi \models \Phi'_n \) can be derived by simply renaming the variables. Now, assuming \( \phi \models \Phi'_n \), we show that \( \phi \models \Phi_{n+1} \) by deriving \( \Phi'_n \models \Phi_{n+1}' \).

First, for each \( i \in E_n \) and each \( i \in I \), by Proposition 6.2 we derive that

\[
\rho_{i,0}^i \subseteq \sigma_{i,0}^0, w_x x_y \subseteq w_0 x_0 \vdash \exists w x (w x \subseteq w_0 x_0 \land \rho_{i,0}^i = \sigma_{i,0}^0).
\]
which, by $\subseteq \text{Cmp}$, yields
\[
\alpha(w_0, x_0), \rho_{w_0}^\ell \subseteq \sigma_{w_0}, w_0 x_0 \subseteq w_0 x_0 \vdash \exists w x (w x \subseteq w_0 x_0) \land \rho_{w_0}^\ell = \sigma_{w_0} \land \alpha(w, x)).
\]  
(15)

Similarly, for each $\xi \eta \in A_{n+1}$ and $j \in J$, we derive also by Proposition [6.2] and $\subseteq \text{Cmp}$ that
\[
\alpha(w_0, x_0), \pi_{x_0}^\ell y_0 \subseteq \tau_{x_0}^\ell y_0 \subseteq w_0 x_0 \vdash \exists w x (w x \subseteq w_0 x_0) \land \pi_{x_0}^\ell y_0 = \tau_{x_0}^\ell \land \alpha(w, x))
\]  
(16)

Next, we derive that
\[
\Phi''_n \vdash \exists w \exists x_0 \forall y_0 \left( \left( \alpha_0 \land \mu_0 \land \exists w^1 x_1 \forall y_1 (\cdots \land \exists w^n x^n y_n (\lambda_n \land \mu_n) \land \alpha(w_0, x_0) \land \left( \bigwedge_{i \in I} (\rho_{w_0}^\ell \subseteq \sigma_{w_0}) \land \left( \bigwedge_{\xi \in E_{n+1}} w_0 \xi_0 \subseteq w_0 x_0 \right) \right) \right) \right) \land \left( \bigwedge_{j \in J} (\tau_{x_0}^\ell \subseteq \tau_{x_0}^\ell) \land \left( \bigwedge_{\xi \eta \in A_{n+1}} w_0 \xi_0 \subseteq w_0 x_0 \right) \right)
\]  
(Proposition [5.3 ii iii])

\[
\vdash \exists w \exists x_0 \forall y_0 \left( \left( \alpha_0 \land \mu_0 \land \exists w^1 x_1 \forall y_1 (\cdots \land \exists w^n x^n y_n (\lambda_n \land \mu_n) \land \alpha(w_0, x_0) \land \left( \bigwedge_{i \in I} (\rho_{w_0}^\ell \subseteq \sigma_{w_0}) \land \left( \bigwedge_{\xi \in E_{n+1}} w_0 \xi_0 \subseteq w_0 x_0 \right) \right) \right) \right) \land \left( \bigwedge_{j \in J} (\tau_{x_0}^\ell \subseteq \tau_{x_0}^\ell) \land \left( \bigwedge_{\xi \eta \in A_{n+1}} w_0 \xi_0 \subseteq w_0 x_0 \right) \right)
\]  
(Proposition [6.4] applied to the subformula $\forall y_0 (\alpha_0 \land \cdots$ and each $w_0 \xi_0 \subseteq w_0 x_0$)

\[
\vdash \exists w \exists x_0 \forall y_0 \left( \left( \alpha_0 \land \mu_0 \land \exists w^1 x_1 \forall y_1 (\cdots \land \exists w^n x^n y_n (\lambda_n \land \mu_n) \land \left( \bigwedge_{i \in I} E_{n+1} \in I \right) \right) \right) \land \left( \bigwedge_{\xi \in E_{n+1}} w_0 \xi_0 \subseteq w_0 x_0 \right) \land \left( \bigwedge_{\xi \eta \in A_{n+1}} \exists w_0 \xi_0 \subseteq \exists w \xi_0 \right) \land \left( \bigwedge_{\xi \eta \in A_{n+1}} \exists w \xi_0 \subseteq \exists w \xi_0 \right)
\]  
(by (15) & (16))

\[
\vdash \exists w \exists x_0 \forall y_0 \left( \left( \alpha_0 \land \mu_0 \land \exists w^1 x_1 \forall y_1 (\cdots \land \exists w^n x^n y_n (\lambda_n \land \mu_n) \land \left( \bigwedge_{i \in I} \left( \exists w_0 \xi_0 \mid \xi \in E_{n+1} \right) \left( \alpha(w_0, x_0) \land \left( \bigwedge_{\xi \in E_{n+1}} \exists w_0 \xi_0 \subseteq w_0 x_0 \right) \right) \right) \right) \right) \land \left( \bigwedge_{\xi \eta \in A_{n+1}} \exists w_0 \xi_0 \subseteq \exists w \xi_0 \right) \land \left( \bigwedge_{\xi \eta \in A_{n+1}} \exists w \xi_0 \subseteq \exists w \xi_0 \right)
\]  
(Proposition [6.4] applied to the subformula $\forall y_0 (\alpha_0 \land \cdots$ and each $w_0 \xi_0 \subseteq w_0 x_0$)

This finishes the proof.

Finally, we are in a position to prove the completeness theorem of our system.

**Theorem 6.5 (Completeness)** Let $\Gamma$ be a set of Inc-formulas, and $\alpha$ a first-order formula. Then
\[
\Gamma \models \alpha \iff \Gamma \vdash_{\text{Inc}} \alpha.
\]

**Proof.** The direction "$\iff$" follows from the soundness theorem. For the direction "$\Rightarrow$", since Inc is compact, we may without loss of generality assume that $\Gamma$ is finite. Suppose now $\Gamma \models \alpha$ and $\Gamma \not\vdash_{\text{Inc}} \alpha$. Claim that $\exists \mathbf{z}(\bigwedge \Gamma \land \neg \alpha) \not\vdash_{\text{Inc}} \bot$, where $\mathbf{z}$ lists all free variables in $\Gamma$ and $\neg \alpha$. Indeed, if $\exists \mathbf{z}(\bigwedge \Gamma \land \neg \alpha) \vdash_{\text{Inc}} \bot$, then we derive $\Gamma, \neg \alpha \vdash_{\text{Inc}} \bot$ by $\exists \mathbf{z}$, and further $\Gamma \vdash_{\text{Inc}} \alpha$ by Lemma [5.11] a contradiction.
Now, let \( \Delta = \{ \Phi_n \mid \phi \equiv \exists z (\forall \Gamma \land \sim \alpha) \text{ and } n < \omega \} \). By Lemma 6.1, we must have that \( \Delta \vdash_{\text{Inc}} \bot \). It follows that \( \Delta \vdash_{\text{FO}} \bot \), since \( \Delta \cup \{ \bot \} \) is a set of first-order formulas, and the deduction system of \( \text{Inc} \) has the same rules as that of first-order logic when restricted to first-order formulas. By the completeness theorem of first-order logic, we know that the set \( \Delta \) of approximations of \( \phi \) has a model \( M \). By [1], every infinite model is elementary equivalent to a recursively saturated countable model. Thus, we may assume that \( M \) is a recursively saturated countable or finite model. By Theorem 6.2, \( M \) is also a model of \( \exists z (\forall \Gamma \land \sim \alpha) \), thereby \( M \models (\emptyset \cup z) \Gamma \) and \( M \not\models (\emptyset \cup z) \alpha \) for some suitable sequence \( F \) of functions for \( \exists z \). Hence \( \Gamma \not\models \alpha \).

7 Applications

In this final section of the paper, we illustrate the power of our system of \( \text{Inc} \) by discussing some applications.

Recall from Proposition 2.4 that the sentence \( \exists x \exists y (y \subseteq x \land y < x) \) defines the fact that \( < \) is not well-founded. By the completeness theorem (Theorem 6.3) we proved in the previous section, all first-order consequences of \( \exists x \exists y (y \subseteq x \land y < x) \) are derivable in our system. For instance, the property that there is a \( \exists \cdot \) chain of length \( n \) for any natural number \( n \), and the property that this \( \exists \cdot \) chain of length \( n \) descends from the greatest element (if exists). We now give explicit derivations of these properties in the example below.

**Example 7.1** Write \( x_1 < x_2 < \cdots < x_n \) for \( \bigwedge_{i=1}^{n-1} x_i < x_{i+1} \). For any \( n \in \mathbb{N} \),

(i) \( \exists x \exists y (y \subseteq x \land y < x) \vdash \exists x_1 \ldots \exists x_n (x_1 < x_2 < \cdots < x_n) \),

(ii) \( \exists x \exists y (y \subseteq x \land y < x), \forall y (y < x_0 \lor y = x_0) \vdash \exists x_1 \ldots \exists x_n (x_1 < \cdots < x_n < x_0) \).

**Proof.** (i) We only give an example of the proof for \( n = 3 \).

\[
\exists x \exists y (y \subseteq x \land y < x) \vdash \exists x_1 \exists x_2 \exists x_3 (x_1 < x_2 < x_3) \quad (\subseteq \text{W}_3)
\]

\[
\vdash \exists x_1 \exists x_2 \exists x_3 (x_1 < x_2 < x_3) \quad (\subseteq \text{Cmp})
\]

(ii) In view of item (i), it suffices to show \( \exists x_1 \ldots \exists x_n (x_1 < \cdots < x_n), \forall y (y < x_0 \lor y = x_0) \vdash \exists x_1 \ldots \exists x_n (x_1 < \cdots < x_n < x_0) \). But this is derivable in the system of first-order logic, and the same proof can also be performed in the system of \( \text{Inc} \).

In Proposition 5.3 in section 5 we have derived some interesting clauses in our system of \( \text{Inc} \). It is interesting to note that the formulas on the right side of the turnstile (\( \vdash \)) in items [1][11][13] of the proposition are not first-order formulas. While our completeness theorem (Theorem 6.3) does not apply to these cases, these clauses are indeed derivable. We now give some more examples in which our system can be successfully applied to derive non-first-order consequences in \( \text{Inc} \).

Consider the so-called anonymity atoms, introduced in [5] and studied recently by Väänänen [30] motivated by concerns in data safety. These atoms are strings of the form \( x_1 \ldots x_n y_{11} \ldots y_{mn} \) with the team semantics:

\[ M \models xYy \text{ iff } \text{ for all } s \in X, \text{ there exists } s' \in X \text{ such that } s(x) = s'(x) \text{ and } s(y) \neq s'(y). \]

Note that the anonymity atoms corresponds exactly to afunctional dependencies studied in database theory (see e.g., [2,8]). It was proved in [5] that first-order logic extended with anonymity atoms is expressively equivalent to inclusion logic, and in particular,

\[ xYy \equiv \exists v (xv \subseteq xy \vee v \neq y), \]

where \( v \neq y \) is short for \( \forall j, v_j \neq y_j \). We will then use \( xYy \) as a shorthand for the above equivalent \( \text{Inc} \)-formula. Write \( Tx \) for \( \{ \langle \rangle, T \} \), and stipulate \( xY = xY' := \bot \). The implication problem of anonymity atoms is shown in [30] to be completely axiomatized by the rules listed in the next example (read the clauses in the example as rules). We now illustrate that in our system of \( \text{Inc} \) all these rules are derivable.

**Example 7.2** (i) \( xyzYuuw \vdash yxzYuuw \land xyzYuuw \) (permutation).

(ii) \( xyzYz \vdash xYzu \) (monotonicity).
(iii) $xyTyz \vdash xyTz$ (weakening).

(iv) $x\Gamma \vdash \bot$.

**Proof.** Item (i) follows easily from $\subseteq \subseteq$ Exc, and item (iv) is trivial. We only prove the other two items. For item (ii), note that $xyTiz := \exists v(xyv \subseteq xyz \wedge v \neq z)$, and we have that

$$
\exists v(xyv \subseteq xyz \wedge v \neq z) \vdash \exists v(xyv \subseteq xzv \wedge v \neq z)
$$

$(\subseteq \text{Ctr})$

$$
\vdash \exists w(xvw \subseteq xzu \wedge v \neq z)
$$

$(\subseteq W_3)$

$$
\vdash \exists w(xvwv \subseteq xzu \wedge vwv vu \neq zu)
$$

$(\forall \text{I})$

$$
\vdash x\Gamma vu.
$$

For item (iii), note that $xyTzy := \exists u(xyuv \subseteq xyzy \wedge uv \neq zy)$, and we have

$$
\exists u(xyuv \subseteq xyzy \wedge uv \neq zy) \vdash \exists u(xyuv \subseteq xyzy \wedge uv \neq zy \wedge y = v)
$$

$(\text{Proposition } 5.3.4 \text{ (ii)})$

$$
\vdash \exists u(xyuv \subseteq xyzy \wedge u \neq z)
$$

$(\forall \text{E})$

$$
\vdash \exists u(xyuv \subseteq xyzy \wedge u \neq z)
$$

$(\subseteq \text{Ctr})$

$$
\vdash xyTz.
$$

The above example indicates that the actual strength of our deduction system of $\text{Inc}$ goes beyond the completeness theorem (Theorem 5.3) proved in this paper. How far can we actually go then? There are obviously barriers, as inclusion logic cannot be effectively axiomatized after all. For instance, in the context of anonymity atoms, the author was not able to derive a simple (sound) implication “$\text{Inc}$ and $x \subseteq y$ imply $\text{Inc}$” in the system of $\text{Inc}$. An easy solution for generating derivations of simple facts like this one would be to extend the current system with new rules. But then how many new rules or which new rules should we add to the current system in order to derive “sufficient” amount of sound consequences of $\text{Inc}$? One such candidate that is worth mentioning is the natural and handy rule $\phi \vee \neg \alpha \alpha \vee \psi / \phi \vee \psi$ (for $\alpha$ being first-order) that is sound and does not seem to be derivable in our system. Finding other such rules is left for future research.

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