On the Tail Problem in Cosmology

Valerio Faraoni

University of Victoria, Department of Physics and Astronomy,
P.O. Box 3055, Victoria, B.C., V8W 3P6 (Canada)

and

Sebastiano Sonego

Université Libre de Bruxelles, C.P. 231 Campus Plaine U.L.B.,
Boulevard du Triomphe, 1050 Bruxelles (Belgium)

Abstract

The tail problem for the propagation of a scalar field is considered in a cosmological background, taking a Robertson-Walker spacetime as a specific example. The explicit radial dependence of the general solution of the Klein-Gordon equation with nonminimal coupling is derived, and the inapplicability of the standard calculation of the reflection and transmission coefficients to the study of scattering of waves by the cosmological curvature is discussed.

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1 Introduction

The Klein-Gordon equation

\[ g^{ab} \nabla_a \nabla_b \Phi - \xi R \Phi - m^2 \Phi = 0, \]  
(1.1)

where \( \xi \) is a numerical constant, \( R \) is the curvature scalar, and \( m \) is the mass of the scalar field \( \Phi \), describes the simplest examples of relativistic wave propagation in curved spacetime and has consequently received much attention, especially in relation to the study of quantum processes \[2\]. Nevertheless, some interesting features have not yet been fully investigated: In particular, the physical consequences of the fact that the solutions of Eq. (1.1), even in the case in which \( \xi = m = 0 \), generally present tails \[3\]–\[10\] (i.e., propagation in the interior of the light cone), has been considered only in the context of radiation emitted by compact objects \[11\]–\[14\].

The formation of tails when \( \xi = m = 0 \) can be understood as an effect of the backscattering of curvature on the waves; this is schematically represented in Fig. 1, which represents a portion of spacetime in null coordinates \( u, v \). The curvature is nonzero only in the shaded region, and scatters waves analogously to what happens under the action of a potential. The ray 1, which does not enter the curved region, remains unperturbed, whereas the ray 2 can be decomposed into a transmitted (3) and a reflected (4) part. Nevertheless we emphasize that such a representation is purely pictorial and does not properly describe many features of the real phenomenon, which has a continuous, rather than localized character. A better, although still schematic, picture is given in Fig. 2. As in the usual scattering processes reflection is due to the inhomogeneities of the potential, so here it is caused by the curvature. In fact, a region of spacetime containing a homogeneous gravitational field is flat, and it is known that waves are not diffused by a flat four dimensional background \[3, 4, 6\].

As said above, this effect has been investigated for waves emitted by a compact source, by calculating explicitly the reflection and transmission coefficients, and has been found to be significant only in the induction zone and for long wavelengths, becoming negligible at large distances from the source and at late times, so that it is irrelevant for the problem of wave propagation in presence of astrophysical objects \[15\]. These general results can be qualitatively understood by remembering that scattering is non-negligible only in the regions where curvature is appreciable. Since the curvature of spacetimes associated to localized matter distributions drops off quickly as spatial infinity is approached, the region of scattering is localized near the source, as in Fig. 3. Moreover,

\footnote{We use units in which \( c = 1 \), and adopt the notations of Ref. \[1\]. The signature of the metric is +2.}
since the “potential barrier” turns out not to be very high, most of the radiation escapes in the region where the curvature is negligible, and where it is no more backscattered. Reflection is appreciable only for long wavelengths.

These considerations do not apply to spacetimes relevant as cosmological models, in which curvature is present over large scales and scattering could, in principle, take place everywhere (see Fig. 4). It is thus natural to ask whether in these cases the process can eventually “convey” a relevant part of radiation within the light cone; this constitutes the so-called cosmological tail problem [16]. Whereas qualitative criteria for the existence of tails have been already formulated and applied to special cases [3]–[10], no general quantitative characterization of the phenomenon, that could solve the problem above, seems to have been suggested up to now. A proposal of this kind, attempting to quantify the physical relevance of wave tails, will be described in a forthcoming article [17]. In this paper we restrict ourselves to present (Sec. 2) a solution of Eq. (1.1) in a form which can be useful for later references, and to point out (Sec. 3) the inapplicability of methods based on the calculation of reflection and transmission coefficients to investigate the occurrence of tails in a cosmological background. Sec. 4 contains some general remarks about the implications of this result.

All the calculations will be performed in a Friedmann-Lemaitre-Robertson-Walker (FLRW) universe, with metric

\[ ds^2 = a^2(\eta) \left[ -d\eta^2 + d\chi^2 + f^2(\chi) \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right], \quad (1.2) \]

where

\[ f(\chi) = \begin{cases} 
\sinh \chi & \chi \in (0, +\infty) \quad \text{if} \quad K = -1 \\
\chi & \chi \in (0, +\infty) \quad \text{if} \quad K = 0 \\
\sin \chi & \chi \in (0, \pi) \quad \text{if} \quad K = +1 
\end{cases}, \quad (1.3) \]

and \( a(\eta) \) is the scale factor. Although the general solution of Eq. (1.1) in a FLRW spacetime is well-known in an implicit form [18, 2], the explicit dependence on \( \chi \) has been worked out, to the authors’ knowledge, only in the case \( K = +1 \) [19] (the case \( K = 0 \) is trivial). As far as the study of tails is concerned, however, this is the most intricate case, because the phenomenon, if present, is complicated by the possibility that radiation travels more than once through closed spatial sections. For this reason, and since it represents the only one not yet considered in the literature, we shall work out explicitly only the case \( K = -1 \).

\(^2\eta\) is the so-called conformal time [2].
2 General solution

Equation (1.1) can be rewritten in the form
\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} \left( \sqrt{-g} g^{ab} \frac{\partial \Phi}{\partial x^b} \right) - \xi R \Phi - m^2 \Phi = 0. \tag{2.1}
\]

In the FLRW background defined by Eqs. (1.2) and (1.3), Eq. (2.1) becomes
\[
\frac{1}{f^2} \left[ \frac{\partial}{\partial \chi} \left( f^2 \frac{\partial \Phi}{\partial \chi} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} \right]
- \frac{1}{a^2} \frac{\partial}{\partial \eta} \left( a^2 \frac{\partial \Phi}{\partial \eta} \right) - a^2 \xi R \Phi - a^2 m^2 \Phi = 0, \tag{2.2}
\]
where now
\[
R = 6 \left( \frac{\dot{a}}{a^3} + \frac{K}{a^2} \right), \tag{2.3}
\]
a dot denoting derivative with respect to \( \eta \). Separation of time and space variables,
\[
\Phi(\eta, \chi, \theta, \varphi) = T(\eta)S(\chi, \theta, \varphi), \tag{2.4}
\]
leads to
\[
\frac{1}{a^4} \frac{d}{d\eta} \left( a^2 \frac{dT}{d\eta} \right) + \xi RT + m^2 T + \frac{k}{a^2} T = 0 \tag{2.5}
\]
and
\[
\frac{\partial}{\partial \chi} \left( f^2 \frac{\partial S}{\partial \chi} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 S}{\partial \varphi^2} + k S f^2 = 0, \tag{2.6}
\]
where \( k \) is a separation constant. Equation (2.3) is in general difficult to solve, but it becomes very simple when considering a static universe \[19, 2\], in which \( a = \text{const.} \). In this case the solutions are the usual exponentials, \( \exp(\pm i \omega \eta) \), where
\[
\omega^2 = k + a^2 m^2 + 6 \xi K. \tag{2.7}
\]
Further separation of radial and angular coordinates in Eq. (2.6),
\[
S(\chi, \theta, \varphi) = X(\chi) Y(\theta, \varphi), \tag{2.8}
\]
3The general solution of Eq. (2.5) is given in Ref. [20] also in the case \( a(\eta) \propto \eta^\alpha \) (\( \alpha \neq 1/2 \)).
leads to the usual equation for spherical harmonics \[ Y_{lm}(\theta, \phi) \] and to

\[
\frac{d}{d\chi} \left( f^2 \frac{dX_l}{d\chi} \right) + kf^2 X_l = l(l+1)X_l. \tag{2.9}
\]

Setting \( \Psi_l(\chi) \equiv f(\chi)X_l(\chi) \) and using Eq. (1.3), Eq. (2.9) takes the form of a one-dimensional Schrödinger equation:

\[
\frac{d^2 \Psi_l}{d\chi^2} + \left[ 2E - \frac{l(l+1)}{f^2(\chi)} \right] \Psi_l = 0, \tag{2.10}
\]

where

\[
2E \equiv k + K. \tag{2.11}
\]

The explicit solution of Eq. (2.10) in the case \( K = +1 \) has been already given by Ford [19] in terms of Gegenbauer functions. For \( K = 0 \), on the other hand, Eq. (2.10) becomes

\[
\frac{d^2 \Psi_l}{d\chi^2} + \left[ 2E - \frac{l(l+1)}{\chi^2} \right] \Psi_l = 0, \tag{2.12}
\]

which is familiar from the quantum mechanical description of a free particle in spherical coordinates (see, e.g., Ref. [22]). Its general solution is a linear combination of

\[
\Psi_l^{(1,2)}(\chi) \equiv \sqrt{\chi} H_{l+1/2}^{(1,2)}(\sqrt{2E} \chi), \tag{2.13}
\]

where \( E > 0 \) and \( H_{l+1/2}^{(1,2)} \) are the Hankel functions [21, 22].

In the case \( K = -1 \), Eq. (2.10) allows the eigenvalue \( E \) to assume any positive value. It is convenient to define

\[
2E \equiv p^2, \tag{2.14}
\]

with \( p > 0 \); then

\[
\frac{d^2 \Psi_l}{d\chi^2} + \left[ p^2 - \frac{l(l+1)}{\sinh^2 \chi} \right] \Psi_l = 0. \tag{2.15}
\]

Writing

\[
\Psi_l(\chi) \equiv e^{ip\chi} \phi_l(\chi), \tag{2.16}
\]

Eq. (2.15) transforms in the following equation for \( \phi_l \):

\[
\frac{d^2 \phi_l}{d\chi^2} + 2ip \frac{d\phi_l}{d\chi} - \frac{l(l+1)}{\sinh^2 \chi} \phi_l = 0. \tag{2.17}
\]
It is now useful to define the new variable $z \in (-\infty, 0)$ as
\[ z \equiv \frac{1}{2} (1 - \coth \chi) = \frac{1}{1 - e^{2\chi}}. \]  
(2.18)

In terms of $z$, Eq. (2.17) takes the form
\[ z (1 - z) \frac{d^2 \phi_l}{dz^2} + (1 - ip - 2z) \frac{d \phi_l}{dz} + l(l + 1) \phi_l = 0, \]  
(2.19)
which is immediately recognized as an hypergeometric equation [21, 22] with coefficients
\[ \alpha = -l \]
\[ \beta = l + 1 \]
\[ \gamma = 1 - ip. \]  
(2.20)

This equation admits two independent solutions [21, 22],
\[ \phi_l^{(1)} = F(\alpha, \beta, \gamma; z) = F(-l, l + 1, 1 - ip; z), \]  
(2.21)
and
\[ \phi_l^{(2)} = z^{1-\gamma}(1 - z)^{\gamma-\alpha-\beta} F(1 - \beta, 1 - \alpha, 2 - \gamma; z) = \]
\[ = (-1)^{ip} e^{-2ip\chi} F(-l, l + 1, 1 + ip; z), \]  
(2.22)
where $F$ denotes the hypergeometric function. Correspondingly, Eq. (2.16) gives the two independent solutions of Eq. (2.15) (an irrelevant factor $(-1)^{ip}$ has been dropped in getting Eq. (2.24) from Eq. (2.22)):
\[ \Psi_l^{(1)}(\chi) = e^{ip\chi} F(-l, l + 1, 1 - ip; z(\chi)); \]  
(2.23)
\[ \Psi_l^{(2)}(\chi) = e^{-ip\chi} F(-l, l + 1, 1 + ip; z(\chi)). \]  
(2.24)

The general solution of Eq. (2.15) is thus
\[ \Psi_l(\chi) = A_l(p) \Psi_l^{(1)}(\chi) + B_l(p) \Psi_l^{(2)}(\chi), \]  
(2.25)
with $A_l$ and $B_l$ arbitrary complex functions of $p$.

Notice that, since $\alpha$ is integer, the series defining $F$ terminates, and therefore $F$ reduces to a polynomial of degree $l$ in $z$:
\[ F(-l, l + 1, 1 \mp ip; z) = \sum_{n=0}^{l} \binom{l + n}{l} \frac{(-l)_n}{(1 \mp ip)_n} z^n, \]  
(2.26)
where \((\lambda)_n\) is defined \([21]\), for an arbitrary complex number \(\lambda\), by

\[
(\lambda)_n \equiv \lambda(\lambda + 1) \cdots (\lambda + n - 1) = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}, \tag{2.27}
\]

and \((\lambda)_0 \equiv 1\). This result is useful if one wants to compare the exact solution \((2.25)\) with the asymptotic one for \(\chi \rightarrow 0\).

### 3 Asymptotic character of the solution

In this section we argue that the concept of reflection and transmission coefficients is inapplicable to the study of tails in cosmological backgrounds. To this purpose, we shall restrict the discussion to the case \(K = -1\) and \(a = \text{const}\). For \(K = +1\), due to the existence of closed spatial sections of spacetime, the transmitted radiation might be allowed to pass more than once through a given point of space, superposing to the fraction of radiation which is possibly reflected. For non-constant \(a(\eta)\), the simple exponentials \(\exp(\pm ip\chi)\) might not correspond to outgoing and ingoing waves. In both cases the treatment would suffer from unnecessary complications.

The presence of tails is characterized by the fact that a pulse of radiation emitted at \(\chi = 0\) is partially backscattered. A stationary emission must therefore correspond to a solution of Eq. \((2.2)\) which for \(\chi \rightarrow 0\) contains incoming (reflected) as well as outgoing (emitted) radiation, whereas for \(\chi \rightarrow +\infty\) only outgoing (transmitted) waves are present. Therefore, we must impose to the general solution \((2.25)\) a boundary condition that corresponds to the absence of incoming waves at \(\chi \rightarrow +\infty\). This is easily accomplished by noticing that for \(\chi \rightarrow +\infty\) one has \(z \rightarrow 0\) and \(F \rightarrow 1\). Hence, the asymptotic form of \(\Psi_l^{(1)}\) and \(\Psi_l^{(2)}\) is

\[
\Psi_l^{(1)}(\chi \rightarrow +\infty) \approx e^{ip\chi}, \tag{3.1}
\]

\[
\Psi_l^{(2)}(\chi \rightarrow +\infty) \approx e^{-ip\chi}, \tag{3.2}
\]

which correspond, respectively, to outgoing and ingoing waves. The required boundary condition is therefore \(B_l = 0\), leading to

\[
\Psi_l(\chi) = A_l(p) e^{ip\chi} F(-l, l + 1, 1 - ip; z(\chi)) \tag{3.3}
\]

as the specific solution of the problem.
A calculation of the reflection and transmission coefficients along the same lines of that performed in Refs. [11]–[14] would now require to expand $\Psi_l(\chi)$ as

$$\Psi_l(\chi) = A_l^{(+)} \Psi_l^{(+)}(\chi) + A_l^{(-)} \Psi_l^{(-)}(\chi), \quad (3.4)$$

where $\Psi_l^{(\pm)}$ are independent solutions of Eq. (2.15) corresponding to waves which are purely outgoing ($+$) and ingoing ($-$) for $\chi \to 0$. Unfortunately, such solutions do not exist for $l \neq 0$ if we require purely ingoing/outgoing waves to have the form $\exp[-i(\omega \eta \pm p \chi)]$ as $\chi \to 0$. In fact, these simple exponentials are not solutions of the wave equation near the origin. This can be qualitatively understood by observing that in the neighbourhood of $\chi = 0$ the centrifugal potential $l(l+1)/\chi^2$ varies extremely rapidly, preventing the modes $\exp(\pm ip\chi)$ from occurring separately. As a consequence, one cannot select solutions of Eq. (2.15) which correspond to waves that near the origin have a purely ingoing or outgoing behaviour.

Formally, this can be realized by characterizing these solutions as asymptotic eigenfunctions of the operator $\hat{P}_\chi \equiv -id/d\chi$, i.e.,

$$-i \frac{d\Psi_l^{(\pm)}}{d\chi} = \pm \alpha \Psi_l^{(\pm)} + \text{(higher powers of } \chi \text{)}, \quad (3.5)$$

with $\alpha > 0$. Since for $l \neq 0$

$$[\hat{H}_l, \hat{P}_\chi] = O(\chi^{-3}) \neq 0, \quad (3.6)$$

where

$$\hat{H}_l \equiv -\frac{1}{2} \frac{d^2}{d\chi^2} + \frac{l(l+1)}{2 \sinh^2 \chi}, \quad (3.7)$$

it follows that there are no common eigenstates of the operators $\hat{H}_l$ and $\hat{P}_\chi$, i.e., there are no solutions of Eq. (2.13) which satisfy the asymptotic condition (3.5).

The following results are conditioned by the use of this characterization, which appears rather restrictive. However, other definitions do not seem sharp enough in selecting waves which can be considered as purely ingoing or outgoing. The formulation of a more general characterization would nevertheless constitute an important progress as far as the calculation of reflection and transmission coefficients is concerned.

It is easy to check that the same conclusion holds for a generic solution of the wave equation that contains components with $l \neq 0$. An analogy with the behaviour of particles can be drawn starting from the similarity of Eq. (2.13) with a Schrödinger equation in a central potential. The intuitive explanation is that particles with nonzero angular momentum cannot “enter in” or “exit from” the centre $\chi = 0$. 

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This conclusion can be also checked explicitly as follows. Looking for solutions of Eq. (2.15) of the form
\[ \Psi_l(\chi) = \chi^m + O(\chi^{m+1}) , \]  
(3.8)
one finds immediately that either \( m = -l \) or \( m = l+1 \). The general solution of Eq. (2.15) can therefore be written as
\[ \Psi_l(\chi) = C_l f_l(\chi) + D_l g_l(\chi) , \]  
(3.9)
with \( C_l \) and \( D_l \) arbitrary complex numbers and \( f_l, g_l \) two particular solutions which behave, for \( \chi \to 0 \), as
\[ f_l(\chi) = \frac{1}{\chi^l} + O(\chi^{1-l}) , \]  
(3.10)and
\[ g_l(\chi) = \chi^{l+1} + O(\chi^{l+2}) . \]  
(3.11)The coefficients \( C_l \) and \( D_l \) for any particular solution \( \Psi_l \) can be obtained by comparing Eqs. (3.9)–(3.11) with the expansion of \( \Psi_l(\chi) \) for \( \chi \to 0 \). In particular, for \( \Psi_l(\pm) \) one can write
\[ \Psi_l(\pm)(\chi) = C_l^{(\pm)} f_l(\chi) + D_l^{(\pm)} g_l(\chi) , \]  
(3.12)where the coefficients \( C_l^{(\pm)} \) and \( D_l^{(\pm)} \) should be chosen in such a way as to guarantee the prescribed character (purely outgoing or ingoing) of the solutions \( \Psi_l^{(\pm)} \) for \( \chi \to 0 \). Substituting Eq. (3.12) into (3.3) and requiring the coefficients of the leading powers of \( \chi \) to vanish, one obtains the trivial result \( C_l^{(\pm)} = D_l^{(\pm)} = 0 \).

The non-existence of solutions of Eq. (2.15) corresponding to waves which are purely ingoing or outgoing in the region \( \chi \to 0 \), prohibits one to define reflection and transmission coefficients as in the usual treatments of scattering problems. It seems that no method has been developed in the literature to deal with similar situations. The case considered here is quite different from the simpler ones arising when studying diffusion of waves by Schwarzschild black holes, where a coordinate transformation can be found for which the corresponding one dimensional Schrödinger problem involves a finite, localized potential barrier, and nontrivial asymptotic eigenfunctions of momentum can be found [11]–[14].

4 General remarks

The most appropriate treatment of the tail problem would be the explicit determination of the reflection coefficient describing backscattering by the cosmological curvature, and
characterizing quantitatively the fraction of radiation which does not propagate along the light cone. The impossibility of carrying on this approach leads one to look for alternative ways of studying the phenomenon. A straightforward idea could be to pursue the formal analogy between the tail problem and quantum scattering, by observing that Eq. (2.15) can be thought of as the radial part of a stationary Schrödinger equation describing a particle with Hamiltonian operator

\[ \hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{2} V(\chi) \hat{L}^2, \]  

(4.1)

where \( \hat{p} \) and \( \hat{L} \) are, respectively, the linear and angular momentum operators in a fictitious three-dimensional Euclidean space in which \( \chi \) plays the role of an ordinary radial coordinate, \( \Psi_l(\chi)/\chi \) is the radial part of the \( l \)-th component of the Schrödinger wave function, and

\[ V(\chi) \equiv \frac{1}{\sinh^2 \chi} - \frac{1}{\chi^2}. \]  

(4.2)

One possibility would then be to compute the cross section for this quantum mechanical system. Although such a calculation could be carried on, it cannot however be regarded as a satisfactory solution to our specific physical problem, which concerns waves emitted from \( \chi = 0 \) rather than incoming from infinity. A satisfactory study of the subject seems thus to require a radically different approach. In Ref. [17], a technique will be presented which characterizes tails in terms of the ratio between their energy content and the total energy of the field.

The impossibility of defining reflection and transmission coefficients is not the only reason (although obviously a compelling one) to look for a different characterization of tails. Even if such coefficients could be defined, in fact, they would not account in a reliable way for the fraction of radiation propagating inside the light cone. This can be realized by noticing that a crucial part of the information about the character of wave propagation is exclusively contained in Eq. (2.3), whereas the calculation of the reflection and transmission coefficients involves only Eq. (2.9). For example, as far as Eq. (2.9) (and, hence, the reflection and transmission coefficients) is concerned, all the values of \( \xi \) and \( m \) are equivalent, which is certainly not the case since, e.g., the choice \( m = 0, \xi = 1/6 \) leads to no tails in every conformally flat spacetime [3]. A way to take into account the time dependence as well is to perform the separation

\[ \Phi(\eta, \chi, \theta, \varphi) = \frac{\psi(\eta, \chi)}{f(\chi)} Y(\theta, \varphi) \]  

(4.3)
in Eq. (2.2), getting
\[
\frac{\partial^2 \psi_l}{\partial \chi^2} - \frac{\partial^2 \psi_l}{\partial \eta^2} - \left[ 1 - 6 \xi + a^2 m^2 \right] \psi_l = 0.
\]

(4.4)

Under the coordinate transformations
\[
\begin{align*}
\{ u &= \frac{1}{2} (\eta - \chi), \\
v &= \frac{1}{2} (\eta + \chi), \\
\tilde{u} &= \tanh u, \\
\tilde{v} &= \tanh v,
\end{align*}
\]

(4.5)

\[
\begin{align*}
\tilde{\eta} &= \tilde{v} + \tilde{u}, \\
\tilde{\chi} &= \tilde{v} - \tilde{u},
\end{align*}
\]

(4.6)

(4.7)

Eq. (4.4) becomes
\[
\frac{\partial^2 \psi_l}{\partial \tilde{\chi}^2} - \frac{\partial^2 \psi_l}{\partial \tilde{\eta}^2} - \left[ 1 - 6 \xi + a^2 m^2 \left( 1 - \frac{\tilde{u}^2}{1 - \tilde{v}^2} \right) \right] \psi_l = 0.
\]

(4.8)

If
\[
1 - 6 \xi + a^2 m^2 = 0,
\]

(4.9)

Eq. (4.8), and consequently Eq. (2.1), is clearly tail-free, since it corresponds formally to wave propagation in flat spacetime. This agrees with previous literature, and can be related to the known fact that \(\sinh^{-2} \chi\) is a so-called reflectionless potential. However, both this argument and previous works are not relevant when Eq. (4.9) does not hold – for example if \(m = \xi = 0\).

In spite of the deficiencies of the treatment, one feature of the phenomenon emerges nevertheless clearly: The tail problem for radiation in FLRW spacetimes regards in general only waves whose wavelengths are at least of order \(a(\eta)\). This can be seen explicitly for the case of a static universe with \(K = -1\), in which Eqs. (2.7), (2.11) and (2.14) give, for the case \(m = 0\),
\[
\omega^2 = p^2 + 1 - 6 \xi,
\]

(4.10)

that can be regarded as the dispersion relation for the waves. From Eq. (4.10) it follows that, except in the very special case \(\xi = 1/6\), only components with frequency and wave

\[\text{We thank an anonymous referee for pointing out this consideration.}\]
number smaller than (or comparable to) $a^{-1}$ are subject to diffusion. Since tails would then regard only a very small, extreme part of the spectrum of the radiation content of the universe, and since the variability of these waves on scales of order $a$ is too slow to be detected, it appears that the effect is hardly observable. However, no conclusion can be drawn without a careful investigation; in fact radiation of very long wavelengths can possibly lead to physically relevant effects (see, e.g., Ref. [25]).

Scalar fields satisfying a wave equation are considered in the inflationary models of the early universe (see, e.g., [26] and references therein). The tail problem for the inflaton regards length scales relevant for cosmology, so it could possibly have some importance in problems connected to the physics of the early universe. This is however beyond the scope of the present work.

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**References**

[1] R. M. Wald, *General Relativity* (Chicago: The University of Chicago Press, 1984).

[2] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press, 1982).

[3] J. Hadamard, *Lectures on Cauchy’s Problem in Linear Partial Differential Equations* (New York: Dover, 1952).

[4] H. P. Künzle, Proc. Camb. Phil. Soc. **64**, 779 (1968).

[5] W. Kundt and E. T. Newman, J. Math. Phys. **9**, 2193 (1968).

[6] F. G. Friedlander, *The Wave Equation on a Curved Spacetime* (Cambridge: Cambridge University Press, 1975).

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7The frequency and wave number measured by a fundamental observer differ from $\omega$ and $p$ by a factor $a^{-1}$.
[7] R. J. Torrence and W. E. Couch, Class. Quantum Grav. 2, 545 (1985).
[8] W. E. Couch and R. J. Torrence, Gen. Rel. Grav. 18, 767 (1986).
[9] R. J. Torrence and W. E. Couch, Gen. Rel. Grav. 20, 343 (1988).
[10] S. Sonego and V. Faraoni, J. Math. Phys. 33, 625 (1992).
[11] R. H. Price, Phys. Rev. D 5, 2419, 2439 (1972).
[12] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (San Francisco: Freeman, 1973).
[13] V. Moncrief, C. T. Cunningham and R. H. Price, in L. L. Smarr, ed., Sources of Gravitational Radiation (Cambridge: Cambridge University Press, 1979).
[14] I. D. Novikov and V. P. Frolov, Physics of Black Holes (Dordrecht: Kluwer Academic Publishers, 1989).
[15] K. S. Thorne, in N. Deruelle and T. Piran, eds., Gravitational Radiation (Amsterdam: North Holland, 1983).
[16] G. F. R. Ellis and D. W. Sciama, in L. O’Raifeartaigh, ed., General Relativity, Papers in Honour of J. L. Synge (Oxford: Clarendon Press, 1972).
[17] V. Faraoni and S. Sonego, in preparation.
[18] L. Parker and S. A. Fulling, Phys. Rev. D 9, 341 (1974).
[19] L. H. Ford, Phys. Rev. D 14, 3304 (1976).
[20] L. H. Ford and L. Parker, Phys. Rev. D 16, 1601 (1977).
[21] N. N. Lebedev, Special Functions and their Applications (New York: Dover, 1972).
[22] L. D. Landau and E. M. Lifshitz, Quantum Mechanics (Non-Relativistic Theory) (Oxford: Pergamon Press, 1977).
[23] V. Bargmann, Rev. Mod. Phys. 21, 488 (1949).
[24] I. Kay and H. E. Moses, J. Appl. Phys. 27, 1503 (1956).
[25] D. Hochberg and T. W. Kephart, Phys. Rev. Lett. 66, 2553; 67, 2403 (1991).
[26] E. W. Kolb and M. S. Turner, The Early Universe (New York: Addison Wesley, 1990).