CAUCHY PROBLEM FOR STOCHASTIC NON-AUTONOMOUS
EVOLUTION EQUATIONS GOVERNED BY NONCOMPACT
EVOLUTION FAMILIES

PENGYU CHEN*, YONGXIANG LI AND XUPING ZHANG
Department of Mathematics, Northwest Normal University
Lanzhou 730070, China

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ABSTRACT. This paper investigates the Cauchy problem to a class of stochastic non-autonomous evolution equations of parabolic type governed by noncompact evolution families in Hilbert spaces. Combining the theory of evolution families, the fixed point theorem with respect to convex-power condensing operator and a new estimation technique of the measure of noncompactness, we established some new existence results of mild solutions under the situation that the nonlinear function satisfy some appropriate local growth condition and a noncompactness measure condition. Our results generalize and improve some previous results on this topic, since the strong restriction on the constants in the condition of noncompactness measure is completely deleted, and also the condition of uniformly continuity of the nonlinearity is not required. At last, as samples of applications, we consider the Cauchy problem to a class of stochastic non-autonomous partial differential equation of parabolic type.

1. Introduction. In this paper, we use a new estimation technique of the measure of noncompactness combined with the fixed point theorem with respect to convex-power condensing operator to discuss the existence of mild solutions for the Cauchy problem to a class of stochastic non-autonomous evolution equations of parabolic type governed by noncompact evolution families in the real separable Hilbert space $\mathbb{H}$

\[
\begin{cases}
    du(t) - A(t)u(t)dt = f(t, u(t))dW(t), & t \in [0, a], \\
    u(0) = u_0,
\end{cases}
\]

where $A(t)$ is a family of (possibly unbounded) linear operators depending on time and having the domains $D(A(t))$ for every $t \in [0, a]$, $a > 0$ is a constant, the state $u(\cdot)$ takes values in the real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let $\mathbb{K}$ be another separable Hilbert space with inner product $(\cdot, \cdot)_{\mathbb{K}}$ and norm $\|\cdot\|_{\mathbb{K}}$. Assume that $\{W(t) : t \geq 0\}$ is a cylindrical $\mathbb{K}$-valued Wiener process.
with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. We are also employing the same notation $\| \cdot \|$ for the norm of $\mathcal{L}(K, H)$, which denotes the space of all bounded linear operators from $K$ into $H$. We denote by $\mathcal{L}(H) = \mathcal{L}(H, H)$. $u_0 \in H$, the nonlinear function $f : [0, a] \times H \to \mathcal{L}(K, H)$ is Carathéodory continuous, which means that

1. for all $u \in H$, $f(\cdot, u)$ is strongly measurable,
2. for a.e. $t \in [0, a]$, $f(t, \cdot)$ is continuous.

In recent years, the stochastic evolution equations have attracted great interest because of their practical applications in many areas such as physics, chemistry, economics, social sciences, finance and other areas of science and engineering. For more details about stochastic evolution equations we refer to the books by Da Prato and Zabczyk [15], Grecksch and Tudor [19], Liu [24], Mao [28], Sobczyk [31] and the references therein. In 2002, Taniguchi, Liu and Truman [33] discussed the existence, uniqueness, $p$-th moment and almost sure Lyapunov exponents of mild solutions to a class of stochastic partial functional differential equations with finite delays by using semigroup methods. In 2008, Luo [27] discussed the exponential stability of mild solutions for stochastic partial differential equations by using the contraction mapping principle and stochastic integral technique. By introducing a suitable metric between the transition probability functions of mild solutions, Bao, Hou and Yuan [5] obtained the exponential stability of mild solutions to a class of stochastic partial differential equations in 2010. In 2011, Ren, Zhou and Chen [30] established the existence, uniqueness and stability of mild solutions for a class of time-dependent stochastic evolution equations with poission jumps and infinite delay under non-Lipschitz condition. Recently, Chen, Li and Zhang [7] discussed the existence of saturated mild solutions and global mild solutions, the continuous dependence of mild solutions on initial values and orders as well as the asymptotical stability in $p$-th moment for the initial value problem of a class of fractional stochastic evolution equations in real separable Hilbert spaces by using fractional resolvent operator theory, the Schauder fixed point theorem and piecewise extension method. In 2015, Wang [34] investigated the approximate mild solutions for a class of fractional stochastic autonomous evolution equations by constructing Picard type approximate sequences under the non-Lipschitz and linear growth conditions.

However, we observed that among the above-mentioned works, most of researchers focus on the case that the differential operators in the main parts are independent of time $t$, which means that the problems under consideration are autonomous. Nevertheless, when treating some parabolic evolution equations, it is usually assumed that the partial differential operators depend on time $t$ on account of this class of operators appears frequently in the applications, for the details please see [1], [2], [3], [8], [9], [10], [11], [12], [13], [17], [18], [23], [35] and [36]. Therefore, it is interesting and significant to investigate stochastic non-autonomous evolution equations, i.e., the differential operators in the main parts of the considered problems are dependent of time $t$. This is the first motivation of this article.

On the other hand, non-autonomous evolution equations have been extensively studied in recent years using various fixed point theorems when the corresponding evolution family is compact, see for example [3, 9, 11, 12, 13, 17, 18, 23, 35, 36], this is very convenient to the equations with compact resolvent. But for the case that the corresponding evolution family is noncompact, we have not seen the relevant papers to study non-autonomous stochastic evolution equations of parabolic
type with. Therefore, as the second motivation of this paper, we will investigate
the existence of mild solutions for Cauchy problem to stochastic non-autonomous
evolution equations of parabolic type (1) under the situation that the corresponding
evolution family is noncompact in this paper.

It is well known that the main difference between finite dimensional and infinite
dimensional space is that the unit ball in infinite dimensional space is not compact,
this fact means that Schauder’s fixed point theorem can not be applied to diffe-
rential equations in infinite dimensional Banach spaces. In order to overcome this
strong restriction, the condition of completely continuous operator in Schauder’s
fixed point theorem has been relaxed to condensing operator based on the concept
of condensing operator. In order to prove the corresponding solution operator for
the studied problem is condensing, people always add some noncompactness mea-
sure condition on nonlinear function, and also assume that the constants in the
noncompactness measure condition satisfy a very strong inequality. For example,
Lakshmikantham and Leela [22] studied the following initial value problem (IVP)
of ordinary differential equation in infinite dimensional Banach space $E$
\[
\begin{align*}
  x'(t) &= g(t, x(t)), \quad t \in [0, a], \\
  x(0) &= x_0.
\end{align*}
\]  
(2)

The authors proved that, if for any constant $R > 0$, $g$ is uniformly continuous on
$[0, a] \times B_R$ and satisfies the noncompactness measure condition
\[
\overline{\pi}(g(t, D)) \leq l \pi(D), \quad \forall t \in [0, a], \quad D \subset B_R,
\]  
(3)
where $B_R = \{ u \in E : \|x\| \leq R \}$, $\overline{\pi}(\cdot)$ present Kuratowski measure of noncompact-
ness, $l$ is a positive constant, then IVP (2) has a global solution provided that $l$
satisfies the condition
\[
al < 1.
\]  
(4)
One can easily to discover that the inequality (4) is a very strong restrictive con-
dition, and it is difficult to be satisfied in applications. In order to remove the
strong restriction on the constants in the conditions of noncompactness measure
like (3), Sun and Zhang [32] generalized the definition of condensing operator to
convex-power condensing operator. And based on the definition of this new kind of
operator, they established a new fixed point theorem with respect to convex-power
condensing operator which generalizes the famous Sadovskii’s fixed point theorem.
As an application, the authors investigated the existence of global mild solutions for
the initial value problem (IVP) of evolution equations in the real infinite dimensional
Banach space $E$
\[
\begin{align*}
  x'(t) + Ax(t) &= g(t, x(t)), \quad t \in [0, a], \\
  x(0) &= x_0,
\end{align*}
\]  
(5)
the authors assume the nonlinear term $g$ is uniformly continuous on $[0, a] \times B_R$ and
satisfies a suitable noncompactness measure condition similar to (3). We should
point out that the restriction condition similar to (4) has been deleted in [32].

Therefore, the third motivation of this paper is to get rid of the restriction
on the constants in the conditions of noncompactness measure assumed for the
nonlinear function $f$ of Cauchy problem to stochastic non-autonomous evolution
equations of parabolic type (1). As one can see, we successfully used the new kind
of fixed point theorem with respect to convex-power condensing operator given
by Lemma 2.12 established by Sun and Zhang in 2005 to study Cauchy problem to stochastic non-autonomous evolution equations of parabolic type (1) under the situation that the corresponding evolution family is noncompact, and completely deleted the strong restriction on the constants in the conditions of noncompactness measure. Furthermore, in this work we deleted the assumption that the nonlinear function \( f \) is uniformly continuous by using a new estimation technique of the measure of noncompactness given by Lemma 2.7 established by Chen and Li [6] in 2013. In fact, if \( f(t, u) \) is Lipschitz continuous on \([0, a] \times B_R\) with respect to \( u \), then the condition (3) is satisfied, but \( f \) may not necessarily uniformly continuous on \([0, a] \times B_R\).

The rest of this paper is organized as follows: in the sequel of Section 1, we give some general assumptions on the linear operator \( A(t) \), and also present the main results of this paper and some hypotheses. We provide in Section 2 some definitions, notations and necessary preliminaries on stochastic analysis, Kuratowski measure of noncompactness and fixed point theorem with respect to convex-power condensing operator. In particular, the definition of mild solution for the Cauchy problem to stochastic non-autonomous evolution equations of parabolic type (1) is also given. The proofs of the main theorems 1.1-1.4 are given in Section 3. In the final Section 4 we present an example of problem where the results of the previous sections apply.

Throughout the paper, we assume that \( \{A(t) : 0 \leq t \leq a \} \) is a family of closed and densely defined operator on Hilbert space \( \mathbb{H} \), which satisfies the following well-known “Acquistapace-Terreni condition”.

\((\text{AT}_1)\) There exist constants \( \lambda_0 \geq 0, \theta \in (\frac{\pi}{2}, \pi) \) and \( \lambda_1 \geq 0 \) such that \( \Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0) \) and for all \( \lambda \in \Sigma_\theta \cup \{0\} \) and \( t \in [0, a] \),

\[ \|R(\lambda, A(t) - \lambda_0)\|_{\mathcal{L}(\mathbb{H})} \leq \frac{M_1}{1 + |\lambda|}; \]

\((\text{AT}_2)\) There exist constants \( M_2 > 0 \) and \( \vartheta, \beta \in (0, 1] \) with \( \vartheta + \beta > 1 \) such that for all \( \lambda \in \Sigma_\theta \) and \( 0 \leq s \leq t \leq a \),

\[ \|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq \frac{M_2|t - s|^\vartheta}{|\lambda|^\beta}, \]

where

\[ \Sigma_\theta = \{ \lambda \in \mathbb{C} \setminus \{0\} : |\lambda| \leq \theta \}. \]

Conditions (\text{AT}_1) and (\text{AT}_2), which are initiated by Acquistapace and Terreni in [2] and Acquistapace in [1] for \( \lambda_0 = 0 \), are well understood and widely used in the literature. Under the above conditions (\text{AT}_1) and (\text{AT}_2), the family \( \{A(t) : 0 \leq t \leq a \} \) generates a unique linear evolution system, or called linear evolution family, \( \{U(t, s) : 0 \leq s \leq t \leq a \} \). Furthermore, by an obvious rescaling from [1, Theorem 2.3] and [2, Theorem 2.1] combined with the Acquistapace and Terreni conditions (\text{AT}_1) and (\text{AT}_2) one gets the following properties for the family of linear operator \( \{U(t, s) : 0 \leq s \leq t \leq a \} \):

(i) \( U(t, r)U(r, s) = U(t, s), U(t, t) = I \) for \( 0 \leq s \leq r \leq t \leq a \);
(ii) The map \( (t, s) \mapsto U(t, s)x \) is continuous for all \( x \in \mathbb{H} \) and \( 0 \leq s \leq t \leq a \);
(iii) \( U(t, s) \in C^1((s, \infty), \mathcal{L}(\mathbb{H})) \), \( \frac{\partial U(t, s)}{\partial t} = A(t)U(t, s) \) for \( t > s \), and \( \|A^k(t)U(t, s)\| \leq M|t - s|^{-k} \) for \( 0 < t - s \leq 1 \) and \( k = 0, 1 \);
(iv) \( \frac{\partial U(t, s)}{\partial s} = -U(t, s)A(s)x \) for \( t > s \) and \( x \in D(A(s)) \) with \( A(s)x \in \overline{D(A(s))} \).
From the property (iii) we know that
\[ \|U(t, s)\|_{L(E)} \leq M \quad \text{for} \quad 0 \leq s \leq t \leq a. \] (6)

In (6) and property (iii), \( M > 0 \) is a constant.

In order to show the existence of mild solutions to the Cauchy problem to stochastic non-autonomous evolution equations of parabolic type (1), it is sufficient to impose some natural growth conditions and noncompactness measure condition on the nonlinear function \( f \).

(C1) For some \( r > 0 \), there exist positive constant \( \rho \) and function \( \varphi_r \in L([0, a], \mathbb{R}_+) \) such that for all \( u \in \mathbb{H} \) satisfying \( \mathbb{E}\|u(t)\|^2 \leq r \) and a.e. \( t \in [0, a] \),
\[ \mathbb{E}\|f(t, u(t))\|^2 \leq \varphi_r(t) \quad \text{and} \quad \liminf_{r \to +\infty} \frac{\|\varphi_r\|_{L([0, a], \mathbb{R}^+)}}{r} : = \rho < +\infty, \]

(C1)* There exist a function \( \varphi \in L([0, a], \mathbb{R}_+) \) and a nondecreasing continuous function \( \Phi : \mathbb{R}^+ \to \mathbb{R}^+ \) such that for all \( u \in \mathbb{H} \) and a.e. \( t \in [0, a] \),
\[ \mathbb{E}\|f(t, u(t))\|^2 \leq \varphi(t)\Phi(\mathbb{E}\|u(t)\|^2), \]

(C2) There exists positive constant \( L \) such that for any bounded and countable sets \( D \subset \mathbb{H} \) and a.e. \( t \in [0, a] \),
\[ \alpha(f(t, D)) \leq L\alpha(D), \]
where \( \alpha(\cdot) \) denotes the Kuratowski measure of noncompactness on the bounded set of \( \mathbb{H} \).

By using the fixed point theorem with respect to convex-power condensing operator (see Lemma 2.12) which established by Sun and Zhang [32] in 2005, we can obtain the following two existence results, which do not add any restriction condition on the constant \( L \) in the condition (C2).

**Theorem 1.1.** Assume that the evolution family \( \{U(t, s) : 0 \leq s \leq t \leq a\} \) generated by \( \{A(t) : 0 \leq t \leq a\} \) is equicontinuous, the nonlinear function \( f : [0, a] \times \mathbb{H} \to \mathcal{L}(\mathbb{K}, \mathbb{H}) \) is Carathéodory continuous. If the conditions (C1) and (C2) are satisfied, then Cauchy problem (1) has at least one mild solution on \([0, a]\) provided that
\[ 2\text{Tr}(Q)M^2\rho < 1. \] (7)

**Theorem 1.2.** Assume that the evolution family \( \{U(t, s) : 0 \leq s \leq t \leq a\} \) generated by \( \{A(t) : 0 \leq t \leq a\} \) is equicontinuous, the nonlinear function \( f : [0, a] \times \mathbb{H} \to \mathcal{L}(\mathbb{K}, \mathbb{H}) \) is Carathéodory continuous. If the conditions (C1)* and (C2) are satisfied, then Cauchy problem (1) has at least one mild solution on \([0, a]\) provided that there exists a positive constant \( R \) such that
\[ 2M^2\mathbb{E}\|u_0\|^2 + 2\text{Tr}(Q)M^2\Phi(R)\|\varphi\|_{L([0, a], \mathbb{R}^+)} \leq R. \] (8)

If directly applying Sadovskii’s fixed-point theorem which given by Lemma 2.10, we can obtain the following results.

**Theorem 1.3.** Assume that the evolution family \( \{U(t, s) : 0 \leq s \leq t \leq a\} \) generated by \( \{A(t) : 0 \leq t \leq a\} \) is equicontinuous, the nonlinear function \( f : [0, a] \times \mathbb{H} \to \mathcal{L}(\mathbb{K}, \mathbb{H}) \) is Carathéodory continuous. If the conditions (C1) and (C2) are satisfied, then Cauchy problem (1) has at least one mild solution on \([0, a]\) provided that (7) and the following assumption
\[ 2ML\sqrt{2\alpha\text{Tr}(Q)} < 1 \] (9)
hold.
Theorem 1.4. Assume that the evolution family \( \{U(t, s) : 0 \leq s \leq t \leq a\} \) generated by \( \{A(t) : 0 \leq t \leq a\} \) is equicontinuous, the nonlinear function \( f : [0, a] \times \mathbb{H} \to \mathcal{L}(\mathbb{K}, \mathbb{H}) \) is Carathéodory continuous. If the conditions (C1)* and (C2) are satisfied, then Cauchy problem (1) has at least one mild solution on \([0, a]\) provided that there exists a positive constant \( R \) such that the assumption (8) and assumption (9) hold.

Remark 1. Compared Theorem 1.1 and Theorem 1.2 with Theorem 1.3 and Theorem 1.4 we know that one can delete the strong restriction condition (9) on the constant \( L \) in the condition (C2) by applying the fixed point theorem with respect to convex-power condensing operator which introduced by Sun and Zhang [32] given by Lemma 2.12, this is a huge improvement.

Remark 2. In this paper we only assume that the nonlinear function \( f \) is Carathéodory continuous, which is weaker than \( f \) is uniformly continuous required in [20], [22], [25], [26] and [32]. Furthermore, the results of Theorem 1.3 and Theorem 1.4 are also new.

Remark 3. The results obtained in this paper can be applied to the Cauchy problems for stochastic non-autonomous partial differential equations of parabolic type with noncompact evolution family. Therefore, Theorems 1.1-1.4 in this paper are supplement to the papers [5], [27], [30], [33], [34], [37] and [38]. This distinguishes the present paper from earlier works on stochastic evolution equations.

2. Preliminaries. We begin with this section by giving some notations. Through out this paper, we denote by \( \mathbb{H} \) and \( \mathbb{K} \) be two real separable Hilbert spaces, by \( (\cdot, \cdot) \) and \( (\cdot, \cdot)_K \) be their inner products, and by \( \| \cdot \| \) and \( \| \cdot \|_K \) be their vector norms, respectively. We use \( \theta \) to present the zero element in \( \mathbb{H} \). We denote by \( \mathcal{L}(\mathbb{H}) \) the Banach space of all linear and bounded operators on \( \mathbb{H} \) endowed with the topology defined by operator norm. Let \( L^1([0, a], \mathbb{H}) \) be the Banach space of all \( \mathbb{H} \)-value Bochner square integrable functions defined on \([0, a]\) with the norm

\[
\|u\|_{L^1} = \int_0^a \|u(t)\| \, dt \quad u \in L^1([0, a], \mathbb{H}).
\]

In this paper, we assume that \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) is a complete filtered probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets of \( \mathcal{F} \). Let \( \{e_k, k \in \mathbb{N}\} \) be a complete orthonormal basis of \( \mathbb{K} \). Suppose that \( \{\mathbb{W}(t) : t \geq 0\} \) is a cylindrical \( \mathbb{K} \)-valued Wiener process defined on the probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) with a finite trace nuclear covariance operator \( Q \geq 0 \), denote

\[
\text{Tr}(Q) = \sum_{k=1}^{\infty} \lambda_k = \lambda < \infty,
\]

which satisfies that \( Qe_k = \lambda_k e_k, \ k \in \mathbb{N} \). Let \( \{\mathbb{W}_k(t), k \in \mathbb{N}\} \) be a sequence of one-dimensional standard Wiener processes mutually independent on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) such that

\[
\mathbb{W}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \mathbb{W}_k(t) e_k.
\]

We further assume that \( \mathcal{F}_t = \sigma\{\mathbb{W}(s), 0 \leq s \leq t\} \) is the \( \sigma \)-algebra generated by \( \mathbb{W} \) and \( \mathcal{F}_b = \mathcal{F} \). For \( \varphi, \psi \in \mathcal{L}(\mathbb{K}, \mathbb{H}) \), we define \( \langle \varphi, \psi \rangle = \text{Tr}(\varphi^* Q \psi) \), where \( \psi^* \) is the
adjoint of the operator $\psi$. Clearly, for any bounded operator $\psi \in \mathcal{L}(K, \mathbb{H})$, 

$$
\|\psi\|_Q^2 = \text{Tr}(\psi Q \psi^*) = \sum_{k=1}^{\infty} \|\sqrt{\lambda_k} \psi e_k\|.
$$

If $\|\psi\|_Q^2 < \infty$, then $\psi$ is called a $Q$-Hilbert-Schmidt operator.

The collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables, denoted $L^2(\Omega, \mathbb{H})$, which is a Banach space equipped with the norm 

$$
\|u(\cdot)\|_{L^2} = (\mathbb{E}\|u(\cdot, \mathbb{W})\|^2)^{\frac{1}{2}},
$$

where the expectation $\mathbb{E}$ is defined by 

$$
\mathbb{E}u = \int_{\Omega} u(\mathbb{W})d\mathbb{P}.
$$

An important subspace of $L^2(\Omega, \mathbb{H})$ is given by 

$$
L^2_0(\Omega, \mathbb{H}) = \{u \in L^2(\Omega, \mathbb{H}) \mid u \text{ is } \mathcal{F}_0\text{-measurable}\}.
$$

We denote by $C([0, a], L^2(\Omega, \mathbb{H}))$ the space of all continuous $\mathcal{F}_t$-adapted measurable processes from $[0, a]$ to $L^2(\Omega, \mathbb{H})$ satisfying $\sup_{t \in [0, a]} \mathbb{E}\|u(t)\|^2 < \infty$. Then it is easy to see that $C([0, a], L^2(\Omega, \mathbb{H}))$ is a Banach space endowed with the supnorm 

$$
\|u\|_C = \left( \sup_{t \in [0, a]} \mathbb{E}\|u(t)\|^2 \right)^{\frac{1}{2}}.
$$

For any constant $r > 0$, let 

$$
B_r = \{u \in C([0, a], L^2(\Omega, \mathbb{H})) : \|u\|_C^2 \leq r\}.
$$

Clearly, $B_r$ is a bounded closed convex set in $C([0, a], L^2(\Omega, \mathbb{H}))$.

By [14, Proposition 2.8], we have the following result which will be used throughout this paper.

**Lemma 2.1.** If $f : [0, a] \times \mathbb{H} \to L(\mathcal{K}, \mathbb{H})$ is continuous and $u \in C([0, a], L^2(\Omega, \mathbb{H}))$, then 

$$
\mathbb{E}\left\| \int_0^a f(t, u(t))d\mathbb{W}(t) \right\|^2 \leq \text{Tr}(Q) \int_0^a \mathbb{E}\|f(t, u(t))\|^2 dt.
$$

**Definition 2.2.** ([29]) An evolution family $\{U(t, s) : 0 \leq s \leq t \leq a\}$ is said to be equicontinuous if the function $t \mapsto U(t, s)$ is continuous by operator norm for $t \in (s, +\infty)$.

**Definition 2.3.** An $\mathcal{F}_t$-adapted stochastic process $u : [0, a] \to \mathbb{H}$ is called a mild solution of Cauchy problem (1) if $u(t) \in \mathbb{H}$ has càdlàg paths on $t \in [0, a]$ almost surely and for each $t \in [0, a]$, $u(t)$ $\mathbb{P}$-almost surely satisfies the integral equation

$$
u(t) = U(t, 0)u_0 + \int_0^t U(t, s)f(s, u(s))d\mathbb{W}(s).
$$

Next, we introduce some basic definition and properties about Kuratowski measure of noncompactness that will be used in the proof of our main results.

**Definition 2.4.** ([4, 16]) The Kuratowski measure of noncoampactness $\overline{\alpha}(\cdot)$ defined on bounded set $S$ of Banach space $E$ is

$$
\overline{\alpha}(S) := \inf\{\delta > 0 : S = \bigcup_{i=1}^m S_i \text{ and } \text{diam}(S_i) \leq \delta \text{ for } i = 1, 2, \cdots, m\}.
$$
The following properties about the Kuratowski measure of noncompactness are well known.

**Lemma 2.5.** ([4, 16]) Let $E$ be a Banach space and $S, U \subset E$ be bounded. The following properties are satisfied:

(i) $\overline{\pi}(S) = 0$, if and only if $S$ is compact, where $\overline{S}$ means the closure hull of $S$;
(ii) $\overline{\pi}(S) = \overline{\pi}(\overline{S}) = \overline{\pi}(\text{conv } S)$, where conv $S$ means the convex hull of $S$;
(iii) $\overline{\pi}(\kappa S) = |\kappa|\overline{\pi}(S)$ for any $\kappa \in \mathbb{R}$;
(iv) $S \subset U$ implies $\overline{\pi}(S) \leq \overline{\pi}(U)$;
(v) $\overline{\pi}(S \cup U) = \max\{\overline{\pi}(S), \overline{\pi}(U)\}$;
(vi) $\overline{\pi}(S + U) \leq \overline{\pi}(S) + \overline{\pi}(U)$, where $S + U = \{x|y = z, y \in S, z \in U\}$;
(vii) If the map $Q : D(Q) \subset E \to X$ is Lipschitz continuous with constant $k$, then $\overline{\pi}(Q(V)) \leq k\overline{\pi}(V)$ for any bounded subset $V \subset D(Q)$, where $X$ is another Banach space.

**Remark 4.** If changes Banach space $E$ into Hilbert space $\mathbb{H}$, then Definition 2.4 and Lemma 2.5 are also valid.

In order to introduce the useful lemmas which will be used in our argument, we denote by $E$ be a Banach space and $C([0, a], E)$ be the Banach space of all continuous $E$-value functions on interval $[0, a]$. We use $\overline{\pi}(\cdot)$ and $\overline{\pi}_{C}(\cdot)$ to denote the Kuratowski measure of noncompactness on the bounded set of $E$ and $C([0, a], E)$, respectively. For any $D \subset C([0, a], E)$ and $t \in [0, a]$, set $D(t) = \{u(t) \mid u \in D\}$ then $D(t) \subset E$. If $D \subset C([0, a], E)$ is bounded, then $D(t)$ is bounded in $E$ and $\overline{\pi}(D(t)) \leq \overline{\pi}_{C}(D)$. For more details about the properties of the Kuratowski measure of noncompactness, we refer to the monographs [4] and [16].

**Lemma 2.6.** ([4]) Let $D \subset C([0, a], E)$ be bounded and equicontinuous. Then $\overline{\pi}(D(t))$ is continuous on $[0, a]$, and $\overline{\pi}_{C}(D) = \max_{t \in [0, a]} \overline{\pi}(D(t))$.

**Lemma 2.7.** ([6]) Let $E$ be a Banach space, and let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\overline{\pi}(D) \leq 2\overline{\pi}(D_0)$.

**Lemma 2.8.** ([21]) Let $E$ be a Banach space. If $D = \{u_n\}_{n=1}^{\infty} \subset C([0, a], E)$ is a countable set and there exists a function $m \in L^1([0, a], \mathbb{R}^+)$ such that for every $n \in \mathbb{N}$

$$\|u_n(t)\| \leq m(t), \quad \text{a.e. } t \in [0, a].$$

Then $\overline{\pi}(D(t))$ is Lebesgue integral on $[0, a]$, and

$$\overline{\pi}\left(\left\{\int_0^a u_n(t)dt \mid n \in \mathbb{N}\right\}\right) \leq 2\int_0^a \overline{\pi}(D(t))dt.$$

**Definition 2.9.** ([16]) Let $S$ be a nonempty subset of $E$. A continuous operator $Q : S \to E$ is called to be condensing if for every bounded set $D \subset S$,

$$\overline{\pi}(Q(D)) < \overline{\pi}(D).$$

**Lemma 2.10.** (Sadovskii’s fixed point theorem, [16]) Let $E$ be a Banach space. Assume that $D \subset E$ is a bounded closed and convex set on $E$ and $Q : D \to D$ is a condensing operator. Then $Q$ has at least one fixed point in $D$.

The following fixed point theorem with respect to convex-power condensing operator which introduced by Sun and Zhang [32] plays a key role in the proof of our main results.
Definition 2.11. Let $E$ be a real Banach space. If $Q : E \to E$ is a continuous and bounded operator, there exist $x_0 \in E$ and a positive integer $n_0$ such that for any bounded and nonprecompact subset $S \subset E$, 

$$\overline{\sigma}(Q^{(n_0,x_0)}(S)) < \overline{\sigma}(S),$$  

(10)

where

$$Q^{(1,x_0)}(S) \equiv Q(S), \quad Q^{(n,x_0)}(S) = Q(\overline{\sigma}(Q^{(n-1,x_0)}(S), x_0)), \quad n = 2, 3, \cdots.$$  

Then we call $Q$ a convex-power condensing operator about $x_0$ and $n_0$.

Lemma 2.12. (Fixed point theorem with respect to convex-power condensing operator, [32]) Let $E$ be a real Banach space, and let $D \subset E$ be a bounded, closed and convex set in $E$. If there exist $x_0 \in D$ and a positive integer $n_0$ such that $Q : D \to D$ be a convex-power condensing operator about $x_0$ and $n_0$, then the operator $Q$ exists at least one fixed point in $D$.

Remark 5. If $n_0 = 1$ in (10), then fixed point theorem with respect to convex-power condensing operator given by Lemma 2.12 will degrade into the famous Sadovskii’s fixed point theorem given Lemma 2.10. Noticed that Lemma 2.12 requires the operator $Q$ is neither condensing nor completely continuous. Therefore, fixed point theorem with respect to convex-power condensing operator is the generalization of Sadovskii’s fixed point theorem.

In this paper, we denote by $\alpha(\cdot)$ and $\alpha_C(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of $\mathbb{H}$ and $C([0,a], L^2(\Omega, \mathbb{H}))$, respectively.

3. Proof of the main results. In this section, we give the proofs of Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.1. Consider the operator $F : C([0,a], L^2(\Omega, \mathbb{H})) \to C([0,a], L^2(\Omega, \mathbb{H}))$ defined by

$$(Fu)(t) = U(t, 0)u_0 + \int_0^t U(t, s)f(s, u(s))d\mathbb{H}(s), \quad t \in [0, a].$$  

(11)

By direct calculation we know that the operator $F$ is well defined. From Definition 2.3, one can easily to verify that the mild solution of Cauchy problem (1) is equivalent to the fixed point of the operator $F$ defined by (11). In what follows, we will prove that the operator $F$ has at least one fixed point by applying Lemma 2.12.

Firstly, we prove that there exists a positive constant $R$ such that the operator $F$ defined by (11) maps the set $B_R$ to $B_R$. If this is not true, then there would exist $t_r \in [0, a]$ and $u_r \in B_r$ such that $\|(Fu_r)(t_r)\| > r$ for each $r > 0$. Combining with Lemma 2.1, the assumption (C1) and (6), we get that

$$r < \mathbb{E}\|(Fu_r)(t_r)\|^2 \leq 2\mathbb{E}\|U(t_r, 0)u_0\|^2 + 2\mathbb{E}\left(\int_0^{t_r} U(t_r, s)f(s, u_r(s))d\mathbb{H}(s)\right)^2$$

$$\leq 2M^2\mathbb{E}\|u_0\|^2 + 2\text{Tr}(Q)M^2\int_0^{t_r} \mathbb{E}\|f(s, u_r(s))\|^2ds$$

(12)

$$\leq 2M^2\mathbb{E}\|u_0\|^2 + 2\text{Tr}(Q)M^2\|f\|_{L([0, a], L^2)}. $$

Dividing both side of (12) by $r$ and taking the lower limit as $r \to +\infty$, combined with the assumption (7) we get that

$$1 \leq 2\text{Tr}(Q)M^2\rho < 1,$$  

(13)
which is a contradiction. Therefore, we have proved that \( F : B_R \to B_R \).

Secondly, we prove that the operator \( F : B_R \to B_R \) is continuous. To this end, let the sequence \( \{u_n\}_{n=1}^{\infty} \subset B_R \) such that \( \lim_{n \to \infty} u_n = u \) in \( B_R \). By the continuity of the nonlinear function \( f \), we know that when \( n \to +\infty \)
\[
\mathbb{E}\|f(s, u_n(s)) - f(s, u(s))\|^2 \to 0 \quad \text{a.e. } s \in [0, a].
\] (14)
Furthermore, from the assumption (C1), we get that for a.e. \( s \in [0, a] \),
\[
\mathbb{E}\|f(s, u_n(s)) - f(s, u(s))\|^2 \leq 2\mathbb{E}\|f(s, u_n(s))\|^2 + 2\mathbb{E}\|f(s, u(s))\|^2 \
\leq 4\varphi_R(s).
\] (15)
Using the fact that the functions \( s \to 4\varphi_R(s) \) is Lebesgue integrable for a.e. \( s \in [0, t] \) and every \( t \in [0, a] \), combined with Lemma 2.1, (6), (11), (14), (15) and the Lebesgue dominated convergence theorem, we know that
\[
\mathbb{E}\|(FQu_n) (t) - (Fu) (t)\|^2 \leq \mathbb{E}\left[\int_0^t U(t, s)[f(s, u_n(s)) - f(s, u(s))]dW(s)\right]^2 \\
\leq \text{Tr}(Q)M^2 \int_0^t \mathbb{E}\|f(s, u_n(s)) - f(s, u(s))\|^2 ds \\
\to 0 \quad \text{as } n \to \infty,
\]
which means that that
\[
\|(Fu_n) - (Fu)\|_C = \left( \sup_{t \in [0, a]} \mathbb{E}\|(Fu_n) (t) - (Fu) (t)\|^2 \right)^{\frac{1}{2}} \to 0 \quad \text{as } n \to \infty.
\]
Therefore, \( F : B_R \to B_R \) is a continuous operator.

Now, we are in the position to demonstrate that \( \{Fu : u \in B_R\} \) is a family of equicontinuous functions in \( C([0, a], L^2(\Omega, \mathbb{H})) \). For any \( u \in B_R \) and 0 \( \leq t_1 < t_2 \leq a \), by Lemma 2.1, (6), (11) and the condition (C1), we know that
\[
\mathbb{E}\|(Fu)(t_2) - (Fu)(t_1)\|^2 \leq 3\mathbb{E}\|U(t_2, 0)u_0 - U(t_1, 0)u_0\|^2 \\
+3\mathbb{E}\int_{t_1}^{t_2} U(t_2, s)f(s, u(s))dW(s)\|^2 \\
+3\mathbb{E}\int_0^{t_1} [U(t_2, s) - U(t_1, s)]f(s, u(s))dW(s)\|^2 \\
\leq I_1 + I_2 + I_3,
\]
where
\[
I_1 = 3\mathbb{E}\|U(t_2, 0) - U(t_1, 0)\|u_0\|^2,
I_2 = 3M^2\text{Tr}(Q) \int_{t_1}^{t_2} \varphi_R(s) ds,
I_3 = 3\text{Tr}(Q) \int_0^{t_1} \|U(t_2, s) - U(t_1, s)\|^2\varphi_R(s) ds.
\]
Therefore, in order to prove that \( \mathbb{E}\|(Fu)(t_2) - (Fu)(t_1)\|^2 \to 0 \) as \( t_2 - t_1 \to 0 \), we only need to check \( I_i \to 0 \) independently of \( u \in B_R \) when \( t_2 - t_1 \to 0 \) for \( i = 1, 2, 3 \).

For \( I_1 \), by the fact that the map \( (t, s) \mapsto U(t, s)x \) is continuous for all \( x \in \mathbb{H} \) and \( 0 \leq s \leq t \leq a \) given in Section 1 property (ii), we can easily get that \( I_1 \to 0 \) as \( t_2 - t_1 \to 0 \).
For $I_2$, from the fact that the function $\varphi_R(s)$ is Lebesgue integrable on $[0, a]$, we get that

$$I_2 = 3M^2 \text{Tr}(Q) \int_{t_1}^{t_2} \varphi_R(s) ds \to 0 \quad \text{as} \quad t_2 - t_1 \to 0.$$  

For $t_1 = 0$, $0 < t_2 \leq a$, it is easy to see that $I_3 = 0$. For $0 < t_1 < a$ and arbitrary $0 < \delta < t_1$, by Definition 2.2, (6), the assumption (C1) and the arbitrariness of $\delta$, we get that

$$I_3 \leq 3\text{Tr}(Q) \int_{0}^{t_1-\delta} \|U(t_2, s) - U(t_1, s)\|^2 \psi_R(s) ds$$
$$+ 3\text{Tr}(Q) \int_{t_1-\delta}^{t_1} \|U(t_2, s) - U(t_1, s)\|^2 \psi_R(s) ds$$
$$\leq \sup_{s \in [0, t_1-\delta]} \|U(t_2, s) - U(t_1, s)\|^2 \|\psi_R(s)\| \cdot 3\text{Tr}(Q) \int_{0}^{t_1-\delta} \psi_R(s) ds$$
$$+ 6\text{Tr}(Q)M^2 \int_{t_1-\delta}^{t_1} \psi_R(s) ds$$
$$\to 0 \quad \text{as} \quad t_2 - t_1 \to 0 \quad \text{and} \quad \delta \to 0.$$  

As a result, we have proved that $E\|(F_u)(t_2) - (F_u)(t_1)\|^2 \to 0$ independently of $u \in B_R$ as $t_2 - t_1 \to 0$, which means that the operator $F : B_R \to B_R$ is equicontinuous.

Denote

$$\Theta = \overline{\Theta},$$

where $\overline{\Theta}$ means the closure of convex hull. In what follows, we prove that $F : \Theta \to \Theta$ is a convex-power condensing operator. From above paragraph one can easily to verify that the operator $F$ maps $\Theta$ into itself and $\Theta \subset B_R$ is equicontinuous. Set $u_0 \in \Theta$. In the following, we will prove that there exists a positive integer $n_0$ such that for any bounded and nonprecompact subset $D \subset \Theta$

$$\alpha_C\left(F^{(n_0, u_0)}(D)\right) < \alpha_C(D).$$

(16)

For any $D \subset \Theta$ and $u_0 \in \Theta$, by the definition of operator $F^{(n, u_0)}$ and the equicontinuity of $\Theta$, we get that $F^{(n, u_0)}(D) \subset B_R$ is also equicontinuous. Therefore, we know from Lemma 2.6 that

$$\alpha_C\left(F^{(n, u_0)}(D)\right) = \max_{t \in [0, a]} \alpha\left(F^{(n, u_0)}(D)(t)\right), \quad n = 1, 2, \cdots.$$  

(17)

By Lemma 2.7, there exists a countable set $D_1 = \{u_n^1\} \subset D$, such that

$$\alpha(F(D)(t)) \leq 2\alpha(F(D_1)(t)).$$

(18)

For any $D', D'' \subset D_1$, we know from Lemma 2.1 and (6) that

$$E\left\|\int_0^t U(t, s)f(s, D'(s)) d\mathbb{W}(s) - \int_0^t U(t, s)f(s, D''(s)) d\mathbb{W}(s)\right\|^2$$
$$= E\left\|\int_0^t U(t, s)[f(s, D'(s)) - f(s, D''(s))] d\mathbb{W}(s)\right\|^2$$
$$\leq \text{Tr}(Q)M^2 \int_0^t E\|f(s, D'(s)) - f(s, D''(s))\|^2 ds.$$
Therefore, by (11), (18), Lemma 2.5 (vii) and Lemma 2.8 one gets that
\[
\alpha \left( \int_0^t U(t, s)f(s, D_1(s))d\mathcal{W}(s) \right) \leq M \left( 2\text{Tr}(Q) \int_0^t \left| \alpha(f(s, D_1(s))) \right|^2 ds \right)^{\frac{1}{2}}.
\]  
(19)

The above inequality combined with Lemma 2.5 (vii) and Lemma 2.8 and the assumption (C2), we get that
\[
\alpha(\mathcal{F}^{(1, u_0)}(D)(t)) = \alpha(\mathcal{F}(D)(t)) \leq 2\alpha(\mathcal{F}(D_1)(t))
\]
\[
\leq 2\alpha \left( \left\{ \int_0^t U(t, s)f(s, u_n^1(s))d\mathcal{W}(s) \right\} \right)
\]
\[
\leq 2M \left( 2\text{Tr}(Q) \int_0^t \left| \alpha(\{f(s, u_n^1(s))\}) \right|^2 ds \right)^{\frac{1}{2}}.
\]  
(20)

Again by Lemma 2.7, there exists a countable set \( D_2 = \{u_n^2\} \subset \mathcal{F}\{\mathcal{F}^{(1, u_0)}(D), u_0\} \), such that
\[
\alpha \left( \mathcal{F} \left( \mathcal{F}\{\mathcal{F}^{(1, u_0)}(D), u_0\} \right)(t) \right) \leq 2\alpha(\mathcal{F}(D_2)(t)).
\]  
(21)

Therefore, by (11), (19), (20), (21), Lemma 2.1, Lemma 2.8, Definition 2.11 and the assumption (C2), we get that
\[
\alpha(\mathcal{F}^{(2, u_0)}(D)(t)) = \alpha \left( \mathcal{F} \left( \mathcal{F}\{\mathcal{F}^{(1, u_0)}(D), u_0\} \right)(t) \right) \leq 2\alpha(\mathcal{F}(D_2)(t))
\]
\[
\leq 2\alpha \left( \left\{ \int_0^t U(t, s)f(s, u_n^2(s))d\mathcal{W}(s) \right\} \right)
\]
\[
\leq 2M \left( 2\text{Tr}(Q) \int_0^t \left| \alpha(\{f(s, u_n^2(s))\}) \right|^2 ds \right)^{\frac{1}{2}}.
\]  
(22)

If for \( \forall \, t \in [0, a] \), we assume that
\[
\alpha(\mathcal{F}^{(k, u_0)}(D)(t)) \leq \left( 2ML\sqrt{2\text{Tr}(Q)} \right)^k \sqrt{\frac{t^k}{k!}} \alpha_c(D).
\]  
(23)

Then by Lemma 2.6, there exists a countable set \( D_{k+1} = \{u_n^{k+1}\} \subset \mathcal{F}\{\mathcal{F}^{(k, u_0)}(D), u_0\} \), such that
\[
\alpha \left( \mathcal{F} \left( \mathcal{F}\{\mathcal{F}^{(k, u_0)}(D), u_0\} \right)(t) \right) \leq 2\alpha(\mathcal{F}(D_{k+1})(t)).
\]  
(24)
From (11), (19), (23), (24), Lemma 2.1, Lemma 2.8, Definition 2.11 and the assumption (C2), we get that
\[
\alpha \left( \mathbb{F}^{(k+1,u_0)}(D)(t) \right)
= \alpha \left( \mathbb{F}(\mathbb{F}^{(k,u_0)}(D), u_0)(t) \right) 
\leq 2\alpha(\mathbb{F}(D_{k+1})(t))
\leq 2\alpha \left( \int_0^t U(t,s)f(s,u_n^{k+1}(s))d\mathbb{W}(s) \right)
\leq 2M \left( 2\text{Tr}(Q) \int_0^t [\alpha(\{f(s,u_n^{k+1}(s))\})^2 ds \right)^{1/2}
\leq 2ML \left( 2\text{Tr}(Q) \int_0^t [\alpha(D_{k+1}(s))^2 ds \right)^{1/2}
\leq 2ML \left( 2\text{Tr}(Q) \int_0^t \left[ \left( 2M\sqrt{2\text{Tr}(Q)} \right)^k \frac{k!}{k!} \right] \frac{k!}{k!} ds \right)^{1/2} \alpha_C(D)
\leq \left( 2ML\sqrt{2\text{Tr}(Q)} \right)^{k+1} \sqrt{\frac{k+1}{(k+1)!}} \alpha_C(D).
\] (25)

Therefore, by the method of mathematical induction, we know that for any positive integer \( n \) and \( t \in [0,a] \)
\[
\alpha \left( \mathbb{F}^{(n,u_0)}(D)(t) \right) \leq \left( 2ML\sqrt{2\text{Tr}(Q)} \right)^n \sqrt{\frac{n}{n!}} \alpha_C(D).
\] (26)

Hence, by (17) and (26), we get that
\[
\alpha_C \left( \mathbb{F}^{(n,u_0)}(D) \right) = \max_{t \in [0,a]} \alpha \left( \mathbb{F}^{(n,u_0)}(D)(t) \right)
\leq \left( 2ML\sqrt{2\text{Tr}(Q)} \right)^n \sqrt{\frac{n}{n!}} \alpha_C(D).
\] (27)

Therefore, by the fact that
\[
\left( 2ML\sqrt{2\text{Tr}(Q)} \right)^n \sqrt{\frac{a^n}{n!}} \to 0 \quad \text{as} \quad n \to \infty,
\]
we know that there exists a large enough positive integer \( n_0 \) such that
\[
\left( 2ML\sqrt{2\text{Tr}(Q)} \right)^{n_0} \sqrt{\frac{a^{n_0}}{n_0!}} < 1.
\] (28)

Thus, from (27) and (28) one know that (16) is satisfied. By Definition 2.11 we get that \( \mathbb{F} : \Theta \to \Theta \) is a convex-power condensing operator. It follows from Lemma 2.12 that the operator \( \mathbb{F} \) defined by (11) has at least one fixed point \( u \in \Theta \), which is just a mild solution of the Cauchy problem (1). This completes the proof of Theorem 1.1.

\[\square\]

**Proof of Theorem 1.2.** From the proof of Theorem 1.1, we know that the mild solution of Cauchy problem (1) is equivalent to the fixed point of the operator \( \mathbb{F} \) defined by (11). In what follows, we prove that there exists a positive constant \( R \).
such that the operator $F$ maps the set $B_R$ to $B_R$. For any $u \in B_R$ and a.e. $t \in [0,a]$, by (11), (6), (8), Lemma 2.1 and the assumption $(C1)^*$, we know that

$$E\|F(u)(t)\|^2 \leq 2E\|U(t,0)u_0\|^2 + 2E\int_0^t U(t,s)f(s,u(s))d\mathcal{W}(s)\|^2$$

$$\leq 2M^2E\|u_0\|^2 + 2\text{Tr}(Q)M^2\int_0^t E\|f(s,u(s))\|^2 ds$$

$$\leq 2M^2E\|u_0\|^2 + 2\text{Tr}(Q)M^2\int_0^t \varphi(s)\Phi(E\|u(s)\|)^2 ds$$

$$\leq 2M^2E\|u_0\|^2 + 2\text{Tr}(Q)M^2\Phi(R)\|\varphi\|_{L([0,a],R^+)} \leq R,$$

which means that the operator $F : B_R \to B_R$. By adopting a completely similar method with which used in the proof of Theorem 1.1, we can prove that $F : B_R \to B_R$ is continuous and equicontinuous, and also $F : \overline{co}\mathcal{F}(B_R) \to \overline{co}\mathcal{F}(B_R)$ is a convex-power condensing operator. By Lemma 2.12, we know that the operator $F$ defined by (11) has at least one fixed point $u \in \overline{co}\mathcal{F}(B_R)$, which is just a mild solution of the Cauchy problem (1). This completes the proof of Theorem 1.2. \hfill \Box

**Proof of Theorem 1.3.** From the proof of Theorem 1.1, we know that the mild solution of Cauchy problem (1) is equivalent to the fixed point of the operator $F$ defined by (11) and $F : B_R \to B_R$ is a continuous operator. Furthermore, $\{F(u) : u \in B_R\}$ is a family of equicontinuous functions in $C([0,a], L^2(\Omega, \mathbb{H}))$ and for any $D \subset \overline{co}\mathcal{F}(B_R)$, $F(D) \subset B_R$ is equicontinuous. Therefore, by Lemma 2.6 we know that

$$\alpha_C(F(D)) = \max_{t \in [0,a]} \alpha(F(D)(t)). \quad (29)$$

By Lemma 2.7, there exists a countable set $D = \{u_n\} \subset D$, such that

$$\alpha(F(D)(t)) \leq 2\alpha(F(D))(t). \quad (30)$$

Hence, by Lemma 2.1, Lemma 2.8, (11), (19), (30) and the condition $(C2)$, we get that

$$\alpha(F(D)(t)) \leq 2\alpha(F(\overline{D}))(t)$$

$$\leq 2\alpha(U(t,0)u_0) + 2\alpha\left\{\int_0^t U(t,s)f(s,u_n(s))d\mathcal{W}(s)\right\}$$

$$\leq 2M\left(2\text{Tr}(Q)\int_0^t [\alpha\{\{f(s,u_n(s))\}\}]^2 ds\right)^{\frac{1}{2}} \quad (31)$$

$$\leq 2ML\left(2\text{Tr}(Q)\int_0^t [\alpha(\overline{D}(s))]^2 ds\right)^{\frac{1}{2}}$$

$$\leq 2ML\sqrt{2\text{Tr}(Q)}\alpha_C(D).$$

From (29), (31) and (9) we get that

$$\alpha_C(F(D)) = \max_{t \in [0,a]} \alpha(F(D)(t)) \leq 2ML\sqrt{2\text{Tr}(Q)}\alpha_C(D) < \alpha_C(D).$$

Therefore, it follows from Lemma 2.10 that the operator $F$ defined by (11) has at least one fixed point $u \in D$, which is just a mild solution of the Cauchy problem (1). This completes the proof of Theorem 1.3. \hfill \Box
Proof of Theorem 1.4. From the proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3, one can easily prove Theorem 1.4, we omit the details here. This completes the proof of Theorem 1.4. □

4. An example. In this section, we present an example, which do not aim at generality but indicate how our abstract result can be applied to concrete problem. Denote $O \subset \mathbb{R}^{N}$ $(N \geq 1)$ be a bounded domain with smooth boundary $\partial O$, $a > 0$ be a constant. As an application, we consider the Cauchy problem to the following stochastic non-autonomous partial differential equation of parabolic type

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} - A(x,t,D)u(x,t) = \frac{\sin(\pi t)}{(4 + |u(x,t)|)^\beta} dW(t), & x \in O, \quad t \in [0, a], \\
D^\alpha u(x,t) = 0, & (x,t) \in \partial O \times [0, a], \quad |\alpha| \leq n, \\
u(x,0) = \phi(x), & x \in O,
\end{cases}
\]

where $W(t)$ denotes a one-dimensional standard cylindrical Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$;

\[A(x,t,D)u = \sum_{|\alpha| \leq 2n} a_\alpha(x,t)D^\alpha u\]

are uniformly strongly elliptic operators in $O$ for $n \in \mathbb{Z}^+$, i.e., there exist a constant $C_1 > 0$ such that for every $x \in \overline{O}$, $t \in [0, a]$ and $\xi \in \mathbb{R}^N$,

\[(-1)^n \Re \sum_{|\alpha| = 2n} a_\alpha(x,t)\xi^\alpha \geq C_1 \xi^{2n};\]

the coefficients $a_\alpha(\cdot, t) \in C^{2n}(\overline{O})$ for $t \in [0, a]$ and $a_\alpha(\cdot, t) : [0, a] \to \mathbb{R}$ are uniformly Hölder continuous, i.e., there exist constants $C_2 > 0$ and $0 < \eta \leq 1$ such that for every $x \in \overline{O}$, $t, s \in [0, a]$ and $|\alpha| \leq 2n$,

\[|a_\alpha(x,t) - a_\alpha(x,s)| \leq C_2 |t - s|^{\eta};\]

$\phi \in L^2(O, \mathbb{R})$.

Let $H = L^2(O, \mathbb{R})$. Then $H$ is a Hilbert space with the norm $\| \cdot \|_2$ and inner product $(\cdot, \cdot)$. Consider the operator $A(t)$ on $H$ defined by

\[A(t)u(x) = A(x,t,D)u(x), \quad D(A(t)) = H^{2n}(O) \cap H^0_{0}(O).\]

It follows from [29, Lemma 6.1 in Chapter 7] that there are constants $\theta \in (\frac{\pi}{2}, \pi)$ and $M_1 \geq 0$ such that $A(t)$ satisfy the condition $\text{(AT}_1\text{)}$. Furthermore, by again [29, Lemma 6.1 in Chapter 7] together with Hölder continuity of coefficients $a_\alpha(x,t)$ one know that there exist constants $M_2 > 0$ and $\vartheta$, $\beta \in (0, 1]$ with $\vartheta + \beta > 1$ such that for all $\lambda \in \Sigma_T$ and $0 \leq s \leq t \leq a$, the condition $\text{(AT}_2\text{)}$ is satisfied. Therefore, the family $\{A(t) : 0 \leq t \leq a\}$ generates an equicontinuous evolution family $\{U(t,s) : 0 \leq s \leq t \leq a\}$. Denote

\[M := \sup_{0 \leq s \leq t \leq a} \|U(t,s)\|_{L(H)}.
\]

For every $t \in [0, a]$, denote

\[u(t) = u(\cdot, t), \quad f(t,u(t)) = \frac{\sin(\pi t)}{4 + |u(\cdot, t)|}, \quad u_0 = \phi(\cdot).
\]

Then the Cauchy problem of stochastic non-autonomous partial differential equation of parabolic type (32) can be rewritten into the abstract form of the Cauchy problem (1) in $L^2(O, \mathbb{R})$. 

Theorem 4.1. The Cauchy problem of stochastic non-autonomous partial differential equation of parabolic type (32) has at least one mild solution $u \in C(\mathcal{O} \times [0,a])$.

Proof. By the definition of nonlinear function $f$ and the norm $\| \cdot \|_2$, we can easily to verify that the condition (C1) hold with

$$\varphi_r(t) = \frac{|\mathcal{O}| \sin^2(\pi t)}{16}, \quad \rho = 0,$$

from which one can easily to verify that the assumption (7) is satisfied. Furthermore, from the definition of nonlinear function $f$ one know that $f(t,u)$ is Lipschitz continuous about the variable $u$ with Lipschitz constant $k = \frac{1}{16}$. Therefore, by Lemma 2.5 we know that the condition (C2) is satisfied with constant $L = \frac{1}{16}$. Therefore, all the conditions of Theorem 1.1 are satisfied. Hence, the Cauchy problem of stochastic non-autonomous partial differential equation of parabolic type (32) has at least one mild solution $u \in C(\mathcal{O} \times [0,a])$ due to Theorem 1.1. This completes the proof of Theorem 4.1.

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E-mail address: chpengyu123@163.com

E-mail address: liyxnwnu@163.com

E-mail address: lanyu9986@126.com