Effect of viscous dissipation on hydromagnetic fluid flow and heat transfer of nanofluid over an exponentially stretching sheet with fluid-particle suspension

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Abstract: This paper considers the problem of steady, boundary layer flow and heat transfer of a nanofluid with fluid-particle suspension over an exponentially stretching surface in the presence of transverse magnetic field and viscous dissipation. The stretching velocity and wall temperature are assumed to vary according to specific exponential form. The governing equations in partial forms are reduced to a system of coupled non-linear ordinary differential equations using suitable similarity transformations. An effective Runge–Kutta–Fehlberg (RKF-45) is used to solve the obtained differential equations with the help of a symbolic software MAPLE. The effects of flow parameters—such as nanofluid interaction parameter, magnetic parameter, solid volume fraction of nanoparticle parameter, Prandtl number and Eckert number—on the flow field and heat-transfer characteristics were obtained and are tabulated. Useful discussions were carried out with the help of plotted graphs and tables. Under the limiting cases, comparison with the existing results was made and found to be in good agreement.

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PUBLIC INTEREST STATEMENT

This paper considers the problem of steady, boundary layer flow and heat transfer of a nanofluid with fluid-particle suspension over an exponentially stretching surface in the presence of transverse magnetic field and viscous dissipation. The governing equations in partial forms are reduced to a system of coupled non-linear ordinary differential equations using suitable similarity transformations. The heat-transfer analysis is carried for two heating process, namely (1) prescribed exponential-order surface temperature and (2) prescribed exponential-order heat flux. Useful discussions were carried out with the help of plotted graphs. Under the limiting cases, comparison with the existing results was found to be in good agreement.
agreement. The results demonstrate that the skin friction coefficient increases for both magnetic and solid volume fraction nanoparticle parameters. However, dusty fluid with copper (Cu) nanoparticles has the appreciable cooling performance than other fluids.

Subjects: Applied Mathematics; Mathematical Modeling; Mathematics & Statistics; Non-Linear Systems; Science; Thermodynamics

Keywords: boundary layer flow; dusty fluid; nanofluid; viscous dissipation; heat transfer; exponentially stretching surface

1. Introduction
During the past few decades, the study of boundary layer flow and heat transfer over a stretching surface has achieved a lot of success because of its large number of applications in industry and technology. Few of these applications are materials manufactured by polymer extrusion, drawing of copper wires, continuous stretching of plastic films, artificial fibres, hot rolling, wire drawing, glass fibre, metal extrusion and metal spinning etc. After the pioneering work by Sakiadis (1961), a large amount of literature is available on boundary layer flow of Newtonian and non-Newtonian fluids over linear and non-linear stretching surfaces. Crane (1970) investigated the flow caused by the stretching of a sheet. Many researchers such as Gupta and Gupta (1977), Dutta, Roy, and Gupta (1985), Chen and Char (1988), Andersson (2002) extended the work of Crane (1970) to study the effect of heat- and mass-transfer analysis under different physical situations. On the other hand, Gupta and Gupta (1977) stressed that realistically, stretching surface is not necessarily continuous. Most of the available literature deals with the study of boundary layer flow over a stretching surface where the velocity of the stretching surface is assumed to be linearly proportional to the distance from the fixed origin. However, it is often argued that (Gupta & Gupta, 1977) stretching of plastic sheet realistically may not necessarily be linear. This situation was effectively dealt by Kumaran and Ramanaiah (1996) in their work on boundary layer flow of fluid where a general quadratic stretching sheet has been assumed for the first time.

In determining the particle accumulation and impingement on the surface, the study on boundary layer flow of fluid-particle suspension flow finds its importance. In view of this, Saffman (1962) formulated governing equations for the flow of dusty fluid and has discussed the stability of the laminar flow of a dusty gas in which dust particles are uniformly distributed. Based on this model, Chakraborti (1974) analysed the boundary layer flow for a dusty gas. Datta and Mishra (1982) investigated the boundary layer flow of a dusty fluid over a semi-infinite flat plate. Numerical investigations carried out by Vajravelu and Nayfeh (1992) to analyse the hydromagnetic flow of a dusty fluid over a stretching sheet. Recently, Gireesha, Ramesh, and Bagewadi (2012) have critically analysed the magnetohydrodynamics (MHDs) flow and heat transfer of a dusty fluid over a stretching sheet using numerical technique.

Ali (1995) has investigated the thermal boundary layer flow by considering the non-linear stretching surface. Later, Magyari and Keller (1999) focused on heat- and mass-transfer analysis on boundary layer flow due to an exponentially continuous stretching sheet. Portha, Murthy, and Rajasekhar (2005) investigated the effect of viscous dissipation on the mixed convection heat transfer from an exponentially stretching surface. Sajid and Hayat (2008) studied the influence of thermal radiation on the boundary layer flow due to an exponentially stretching sheet using homotopy analysis method. The study of MHD has important applications, and may be used to deal with problems such as cooling of nuclear reactors by liquid sodium and induction flow meter, which depends on the potential difference in the fluid in the direction perpendicular to the motion and to the magnetic field. Elbashbeshy (2001) added new dimension to the study on exponentially continuous stretching surface. Khan and Pop (2010) and Sanjayan and Khan (2006) have studied the viscoelastic boundary layer flow and heat transfer due to an exponentially stretching sheet. Recently, Bidin and Nazar (2009) numerically analysed the effect of thermal radiation on the steady laminar two-dimensional boundary layer flow and heat transfer over an exponentially stretching sheet, which was earlier solved analytically by Sajid and Hayat (2008). Pal (2010) reported the mixed convection flow past an exponentially stretching surface in the presence of a magnetic field. Nadeem, Zaheer, and Fang (2011) addressed the flow of Jeffrey
fluid and heat transfer past an exponentially stretching sheet. Ishak (2011) studied the effect of radiation on MHD boundary layer flow of a viscous fluid over an exponentially stretching sheet. It was found that the local heat-transfer rate at the surface decreases with increasing values of the magnetic and radiation parameters. Sahoo and Poncet (2011) addressed the flow of third-grade fluid past an exponentially stretching sheet with slip condition.

Nanofluids, a new class of nano-engineered liquid solutions of colloidal particles with a diameter of 1–100 nm, have been shown great energy saving potentials and attractive properties for applications such as energy, bio and pharmaceutical industry, and chemical, electronic, environmental, material, medical and thermal engineering, among others. The concept of nanofluid was first introduced by Choi (1995) in the article enhancing thermal conductivity of fluids with nanoparticles. Based on this pioneer work, Mabood, Khan, and Ismail (2015a, 2015b) focused on the study of combined heat and mass transfer of electrically conducting nanofluid over a non-linear stretching surface in the presence of a first-order chemical reaction and viscous dissipation. Further, they (Mabood & Khan, 2014) have obtained the series solution for MHD stagnation point flow in porous medium for different values of Prandtl number and suction/injection parameter. Abbasi, Shehzad, Hayat, Alsaedi, and Obid (2015) presented an analysis to address the MHD two-dimensional boundary layer flow of Jeffrey nanofluid over a stretching sheet with thermal radiation. Hussain, Hayat, Shehzad, Alsaedi, and Chen (2015) examined the flow problem resulting from the stretching of a surface with convective conditions in a MHD third-grade nanofluid in the presence of thermal radiation. Hayat, Hussain, Shehzad, and Alsaedi (2015) investigated the Brownian motion and thermophoresis effects on two-dimensional boundary layer flow of an Oldroyd-B nanofluid in the presence of thermal radiation and heat generation.

Recently, Pavithra and Gireesha (2013) discussed the effect of internal heat generation/absorption on dusty fluid flow over an exponentially stretching sheet with viscous dissipation. Mukhopadhyay and Gorla (2012) analysed the effects of partial slip on flow past an exponentially stretching sheet. Rudraswamy and Gireesha (2014) investigated the influence of chemical reaction and thermal radiation on MHD boundary layer flow and heat transfer of a nanofluid over an exponentially stretching sheet. To the best of author’s knowledge, studies on hydromagnetic fluid flow and heat transfer of nanofluid over an exponentially stretching sheet with fluid particle suspension in the presence of viscous dissipation have not been considered so far. The governing boundary layer equations of the flow problem are transformed into ordinary differential equations, using similarity transformations. The resulting ordinary differential equations are then solved numerically, using Runge–Kutta–Fehlberg (RKF-45) method. The effects of flow pertinent parameters on the flow- and heat-transfer characteristics are discussed and are presented in detail. Obtained results are compared with the known results available in the literature and the comparison shows good agreement. It is believed that the results of this study can be used in the design of an effective cooling system for electronic machinery to help ensure effective and safe operational conditions, which includes nuclear power plants, gas turbines, aircraft, missiles, satellites, space vehicles, etc.

2. Mathematical formulation and solution of the problem

Consider a steady, two-dimensional, boundary layer flow and heat transfer of an incompressible nanofluid with fluid-particle suspension over an exponentially stretching surface. Cartesian coordinate system is considered in such a way that x-axis is taken along the stretching surface in the direction of motion and y-axis is normal to it. The plate is stretched in the x-direction with a velocity \( U(x) = U_0 e^x \), defined at \( y = 0 \). The flow is generated as a consequence of exponential stretching of the sheet caused by simultaneous application of equal and opposite forces along x-axis keeping the origin fixed. A uniform magnetic field \( B_0 \) is assumed to be applied in the y-direction. The geometry of the flow configuration is as shown in Figure 1.

The flow- and heat-transfer characteristics under the boundary layer approximations are governed by the following equations,
where $x$ and $y$, respectively, represents coordinate axes along the continuous surface in the direction of motion and perpendicular to it. $(u, v)$ and $(u_p, v_p)$ denotes the velocity components of the nanofluid and dust phases along $x$ and $y$ directions respectively, $N$ denotes number density of the dust particles, $m$ is the mass concentration of dust particles and $\sigma$ symbolises electrical conductivity.

The associated boundary conditions are,

\begin{align}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{u \partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 u}{\partial y^2} + \frac{KN}{\rho_{nf}} (u_p - u) - \frac{\sigma B_0^2}{\rho_{nf}} u \\
\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y} &= 0 \\
\frac{u_p \partial u_p}{\partial x} + v_p \frac{\partial u_p}{\partial y} &= \frac{K}{m} (u - u_p)
\end{align}

where $x$ and $y$, respectively, represents coordinate axes along the continuous surface in the direction of motion and perpendicular to it. $(u, v)$ and $(u_p, v_p)$ denotes the velocity components of the nanofluid and dust phases along $x$ and $y$ directions respectively, $N$ denotes number density of the dust particles, $m$ is the mass concentration of dust particles and $\sigma$ symbolises electrical conductivity.

The associated boundary conditions are,

\begin{align}
u &= U_w(x), \quad v = 0, \quad T = T_w \text{ at } y = 0 \\
u &= 0, \quad u_p \rightarrow 0, \quad v_p \rightarrow v, \text{ as } y \rightarrow \infty
\end{align}

where $U_w(x) = U_0 e^x$ is the sheet velocity, $U_0$ is reference velocity and $L$ is the reference length. The effective density of the nanofluid is given by,

$$\rho_{nf} = (1 - \phi) \rho_f + \phi \rho_s$$

where $\phi$ is the solid volume fraction of nanoparticles and the heat capacitance of the nanofluid is obtained as,

$$(\rho C_p)_{nf} = (1 - \phi)(\rho C_p)_f + \phi(\rho C_p)_s$$

and the thermal conductivity of the nanofluid $k_{nf}$ for spherical nanoparticles can be written as Maxwell (1904),

$$\frac{k_{nf}}{k_f} = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f - 2\phi(k_f - k_s)}$$
Also the effective dynamic viscosity of the nanofluid given by Brinkman (1952) as,

\[ \mu_{nf} = \frac{\mu_f}{(1 - \phi)^{2.5}} \]  

(2.9)

Here the subscripts \( nf, f, s \) represent the thermophysical properties of the nanofluids, base fluid and the nano-solid particles, respectively.

We are interested in similarity solution of the above boundary value problem; therefore, introduce the following similarity transformations;

\[ u = U_0 e^{\frac{\eta}{L}} f'(\eta), \quad v = -\sqrt{\frac{\nu_f}{2L}} e^{\frac{\eta}{L}} [f(\eta) + nf'(\eta)] \]

\[ u_p = U_0 e^{\frac{\eta}{L}} F'(\eta), \quad v_p = -\sqrt{\frac{\nu_f}{2L}} e^{\frac{\eta}{L}} [F(\eta) + nF'(\eta)] \]

\[ \eta = \sqrt{\frac{U_0}{2\nu_f L}} e^{\frac{\eta}{L}} y \]

(2.10)

Making use of the transformations (2.10), Equation 2.1 and 2.3 are identically satisfied and Equations 2.2 and 2.4 will take the following form,

\[ f'''' + (1 - \phi)^{2.5} \left[ (1 - \phi) + \phi \frac{\rho_f}{\rho_f} \right] (ff'' - 2f'^2) + (1 - \phi)^{2.5} [2\beta(f' - F') - Mf] = 0 \]

(2.11)

\[ FF'' - 2[F'^2] + 2\beta[f' - F'] = 0 \]

(2.12)

where a prime denotes differentiation with respect to \( \eta, I = \frac{mN}{\nu_f} \) is the mass concentration, \( \nu_f \) is the kinematic viscosity of the fluid, \( \tau_v = m/K \) is the relaxation time of the particle phase, \( \beta = \frac{L}{rU_0} \) is the fluid particle interaction parameter for velocity and \( M = \frac{2\nu_f L}{\rho_s u_s} \) is the magnetic parameter.

The boundary conditions defined as in Equation 2.5 will become;

\[ f'(\eta) = 1, \quad f(\eta) = 0 \] at \( \eta = 0 \)

\[ f'(\eta) = 0, \quad F'(\eta) = 0 \] \( F(\eta) = f(\eta) \) as \( \eta \rightarrow \infty \)

(2.13)

3. Heat-transfer analysis

The governing steady, boundary layer, heat transport equations for both fluid and dust phases with viscous dissipation are given by,
\[ (\rho C_p)_{nf} \left[ \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = k_{nf} \frac{\partial^2 T}{\partial y^2} + \frac{NC_{pf}}{\tau_f} (T_p - T) + \frac{N}{\tau_c} (u_p - u)^2 + \mu_f \left( \frac{\partial u}{\partial y} \right)^2 \]  

(3.1)

\[ u_p \frac{\partial T_p}{\partial x} + v \frac{\partial T_p}{\partial y} = \frac{C_{pf}}{C_{nf}\tau_f} (T - T_p) \]  

(3.2)

where \( T \) and \( T_p \) are the temperatures of the fluid and dust particle inside the boundary layer, \( c_p \) and \( c_{pf} \) are the specific heat of fluid and dust particles, \( \tau_f \) is the thermal equilibrium time i.e. the time required by a dust cloud to adjust its temperature to the fluid, \( k_{nf} \) is the thermal conductivity, \( \tau_c \) is the relaxation time of the dust particle, that is, the time required by a dust particle to adjust its velocity relative to the fluid.

Heat-transfer phenomenon are solved for two types of heating process, namely,

- Prescribed exponential-order surface temperature (PEST) and
- Prescribed exponential-order heat flux (PEHF).

### 3.1. Case-1: Prescribed exponential-order surface temperature (PEST)

For this heating process, the following boundary conditions are employed;

\[ T = T_w(x) \text{ at } y = 0 \]

\[ T \rightarrow T_\infty, \quad T_p \rightarrow T_\infty \text{ as } y \rightarrow \infty \]  

(3.3)

where \( T_w = T_\infty + T_0 e^{\frac{x}{\tau}} \) is the temperature distribution in the stretching surface, \( T_0 \) is the reference temperature and \( c_1 \) is a constant.

Defining the non-dimensional variables for fluid-phase temperature \( \theta(\eta) \) and dust-phase temperature \( \theta_p(\eta) \) as,

\[ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \theta_p(\eta) = \frac{T_p - T_\infty}{T_w - T_\infty} \]  

(3.4)

where \( T = T_\infty = T_0 e^{\frac{x}{\tau}} \theta(\eta) \). Using the similarity variable \( \eta \) and Equation 3.4 into Equations 3.1-3.2 and on equating the coefficient of \( \left( \frac{x}{\tau} \right)^0 \) on both sides, one can arrive at the following system of equations:

\[ \frac{k_{nf}}{k_f} \theta'' + Pr \left( (1 - \phi) + \phi \left( \frac{\rho C_p}{\rho_{nf}} \right) \right) \left( \theta' - c_1 f' \theta \right) \]

\[ + \frac{2NPf \beta_f}{\rho_f} \left[ \theta_p - \theta \right] + \frac{2NPf Ec \beta_f}{\rho_f} [F' - f']^2 + EcPr[f'']^2 = 0 \]  

(3.5)

\[ F\theta_p' - c_1 F' \theta_p + 2\gamma \beta_f \left[ \theta - \theta_p \right] = 0 \]  

(3.6)

where \( Pr = \frac{(\rho C_p)}{\eta} \) is the Prandtl number, \( Ec = \frac{\nu_f}{\tau_c c_{pf}} \) is the Eckert number, \( \beta_f = \frac{1}{\nu_f T_c} \) is the fluid particle interaction parameter for temperature and \( \gamma = \frac{c_p}{c_{pf}} \) is the ratio of specific heat.

The corresponding thermal boundary conditions becomes,

\[ \theta(\eta) = 1 \text{ at } \eta = 0 \]

\[ \theta(\eta) \rightarrow 0, \quad \theta_p(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty \]  

(3.7)
3.2. Case-2: Prescribed exponential-order heat flux (PEHF)

For this heating process, consider the boundary conditions as follows,

\[
\frac{\partial T}{\partial y} = -\frac{q_w}{k_{nf}} \text{ at } y = 0
\]

\[T \to T_\infty, \quad T_p \to T_\infty \text{ as } y \to \infty \quad (3.8)
\]

where \(q_w(x) = T_1 e^{\frac{c_2 x^2}{L}}\), \(T_\infty + T_1 k_{nf} e^{\frac{c_2 x^2}{L}} = \frac{T_1}{\nu_1^2} \sqrt{\frac{2k_{rf}}{\nu_1^2}}\), \(T_1\) is reference temperature, \(c_2\) is constant and \(T - T_\infty = T_1 k_{nf} e^{\frac{c_2 x^2}{L}} = \frac{T_1}{\nu_1^2} \sqrt{\frac{2k_{rf}}{\nu_1^2}}\). Using the similarity variable \(\eta\) and Equation 3.4 into Equation 3.1–3.2 and on equating the coefficient of \(e^{\frac{c_2 x^2}{L}}\) on both sides, one can arrive the following system of equations

\[
\frac{k_{nf}}{k_f} \psi'' + Pr \left[ (1 - \phi) + \frac{\phi (\rho C_p)_s}{(\rho C_p)_f} \right] (f'\theta' - c_1 f'' \theta) + \frac{2NPr \beta_f}{\rho_f} \left[ \theta_p - \theta \right] + \frac{2NPr Ec \beta_f}{\rho_f^2} \left[ f'' \theta' \right]^2 = 0
\]

\(F\theta_p' - c_1 F' \theta_p + 2 \gamma \beta_f \left[ \theta - \theta_p \right] = 0 \quad (3.9)

where \(Ec = \frac{k_{nf} U_0}{T_\infty} \sqrt{\frac{\nu_1}{\nu_1^2}}\) is the Eckert number and the corresponding thermal boundary conditions are written as

\(\theta' (\eta) = -1 \text{ at } \eta = 0\)

\(\theta(\eta) \to 0, \quad \theta_p(\eta) \to 0 \text{ as } \eta \to \infty \quad (3.11)\)

The important physical parameters for the boundary layer flow are the skin friction coefficient and heat-transfer coefficient which are, respectively, defined as,

\[C_f = \frac{\tau_w}{\rho_f U_w^2}, \quad \text{Nu}_x = \frac{\text{Nu}_s}{k_{nf} T_w - T_\infty} \quad (3.12)\]

where the skin friction \(\tau_w\) and the heat transfer from the sheet \(q_w\) are given by,

\[\tau_w = \mu_f \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k_{nf} \left( \frac{\partial T}{\partial y} \right)_{y=0} \quad (3.13)\]

Using the non-dimensional variables, one obtains,

\[\sqrt{2Re} C_f = f''(0)\]

\[\frac{\text{Nu}_x}{\sqrt{2Re}} = -\frac{\chi}{2L} \phi'(0) \text{ (PEST Case) and } \frac{\text{Nu}_s}{\sqrt{2Re}} = \frac{\chi}{2L} \frac{1}{\theta(0)} \text{ (PEHF Case)} \quad (3.14)\]

where \(Re = \frac{U_0 L}{\nu_1}\) is the Reynolds number.

4. Numerical solution

A two-dimensional hydromagnetic fluid flow and heat transfer of nanofluid at an exponentially stretching sheet with fluid-particle suspension in presence of viscous dissipation is considered. The non-linear differential Equation 2.11 and 2.12, 3.5 and 3.6 or 3.9 and 3.10 together with the
boundary conditions 2.13, 3.7 and 3.11 for both PEST and PEHF cases are solved numerically using Runge–Kutta–Fehlberg (RKF-45) scheme with the help of Maple software. We have chosen suitable finite values of \( \eta \rightarrow \infty \) say \( \eta = 5 \).

The results for \( f''(0) \) obtained in the present work and those by Vajravelu and Nayfeh (1992) and Das (2012) are recorded in Table 2, for different values of \( \phi \). Further, Table 3 shows the comparative values of \( \theta'(0) \) with Magyari and Keller (1999), Bidin and Nazar (2009), Ishak (2011) and Abd El-Aziz (2009).

### Table 2. Comparison of the results of \( f''(0) \) for Cu-water for various values of solid volume fraction of nanoparticles \( \phi \) when \( \beta = M = 0 \)

| \( \phi \) | \( f''(0) \) |
|---------|-------------|
|         | Vajravelu and Nayfeh (1992) | Das (2012) | Present result |
| 0.0     | −1.001411   | −1.001411   | −1.001396   |
| 0.1     | −1.175209   | −1.175251   | −1.175394   |
| 0.2     | −1.218301   | −1.218315   | −1.218580   |

### Table 3. Values of \( \theta'(0) \) for several values of Prandtl number and in the absence of \( \beta, \beta_T, Ec, M \) and \( \phi \)

| \( Pr \) | \( \theta'(0) \) |
|---------|----------------|
|         | Magyari and Keller (1999) | Bidin and Nazar (2009) | Ishak (2011) | El-Aziz (2009) | Present result |
| 1       | −0.9548 | −0.9547 | −0.9548 | −0.9548 | −0.95764 |
| 2       | −1.4714 | −1.4715 | −1.4715 | −1.4715 | −1.47084 |
| 3       | −1.8691 | −1.8691 | −1.8691 | −1.8691 | −1.86854 |
| 4       | −2.5001 | −2.5001 | −2.5001 | −2.5001 | −2.49972 |
| 5       | −3.6604 | −3.6604 | −3.6604 | −3.6604 | −3.66005 |

### Table 4. Values of wall temperature gradient \( [-\theta'(0)] \) in PEST case and wall temperature \( \theta(0) \) in PEHF case for different values of the parameters \( Pr, Ec, M, N, \phi \) and \( \beta \)

| \( \beta \) | \( M \) | \( Pr \) | \( Ec \) | \( N \) | \( \phi \) | \( -\theta'(0) \) (PEST) | \( \theta(0) \) (PEHF) |
|---------|-----|-----|-----|-----|-----|--------------|-----------------|
| 0.2     | 0.5 | 6.2 | 0.5 | 0.5 | 0.2 | 1.15302      | 0.92894         |
| 0.6     | 1   | 0   | 6.2 | 0.5 | 0.5 | 0.68437      | 1.28081         |
| 1       | 0.5 | 2.2 | 0.5 | 0.5 | 0.2 | 0.92198      | 1.04621         |
| 0.6     | 0.5 | 6.2 | 0   | 0.5 | 0.2 | 2.13308      | 0.46880         |
| 0.6     | 0.5 | 6.2 | 0   | 0.5 | 0.2 | 1.07472      | 0.96496         |
| 0.6     | 0.5 | 6.2 | 0   | 0.5 | 0.2 | 1.42719      | 0.82803         |
| 0.6     | 0.5 | 6.2 | 0   | 0.5 | 0.2 | 2.00800      | 0.67278         |

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for various values of Pr in the absence of magnetic field. From these tables, one can notice that there is a close agreement between these approaches and thus verifies the accuracy of the method used. The results of thermal characteristics at the wall are examined for the values of the temperature gradient function \( \theta'(0) \) in PEST case and \( \theta(0) \) in PEHF case, and are tabulated in Table 4. The effects of various physical parameters such as nanofluid interaction parameter \( (\beta) \), magnetic parameter \( (M) \), Prandtl number \( (Pr) \) Eckert number \( (Ec) \) number density parameter \( (N) \) and solid volume fraction parameter \( (\phi) \) are examined and discussed in detail.

5. Results and discussion

The hydromagnetic boundary layer flow and heat transfer of a dusty nanofluid over an exponentially stretching sheet are investigated in the presence of viscous dissipation. Similarity transformations are used to convert the governing time-independent non-linear boundary layer equations into a system of non-linear ordinary differential equations. The obtained highly non-linear ordinary differential equations are then solved numerically by means of most efficient numerical technique, fourth–fifth-order Runge–Kutta–Fehlberg scheme with the help of Maple software. The temperature profile \( \theta(\eta) \) and \( \theta_p(\eta) \) for both PEST and PEHF cases are depicted graphically. A parameter of interest for the present study is the nanofluid interaction parameter \( (\beta) \), magnetic parameter \( (M) \) Prandtl number \( (Pr) \) Eckert number \( (Ec) \) number density parameter \( (N) \) and solid volume fraction of nanoparticle parameter \( (\phi) \).

The variation of velocity profiles with \( \eta \) for different values of the nanofluid interaction parameter for both nanofluid and dust phases are illustrated in Figure 2. It is noticed from this figure that the velocity profiles decrease with increasing values of \( \beta \) for nanofluid phase and increase for dust phase in the boundary layer. The increasing values of \( \beta \) reduce the velocity \( f'(\eta) \) and thereby increase the boundary layer thickness.

The effects of magnetic parameter on velocity profiles for both nanofluid and dust phases are illustrated graphically through Figure 3. It is interesting to note that as strength of the magnetic field increases, velocity profiles for both the phases decrease. This is due to the fact that, the introduction of transverse magnetic field has a tendency to create a drag, known as the Lorentz force which tends to resist the flow. This behaviour is even true in the case of increasing values of nanofluid interaction parameter for fluid phase.

Figure 4 depicts the temperature profiles \( \theta(\eta) \) and \( \theta_p(\eta) \) vs. \( \eta \) for different values of nanofluid interaction parameter \( \beta \). We infer from these figures that the temperature increases with increase in
nanofluid interaction parameter for both PEST and PEHF cases. Also one can observed that nanofluid phase temperature is higher than that of dust phase.

The effects of magnetic field parameter ($M$) on temperature profiles $\theta(\eta)$ and $\theta_p(\eta)$ for both PEST and PEHF cases are depicted in Figure 5. We infer from this figure that the temperature profiles increases with increase in magnetic parameter and it also indicates that both the nanofluid and the dust particle temperature are parallel to each other. This is true for both PEST and PEHF cases.

The temperature profile for various values of Prandtl number ($Pr$) is represented for both PEST and PEHF cases in Figure 6. From this figure, it reveals that the temperature decreases with increase in the value of $Pr$. Hence Prandtl number can be used to increase the rate of cooling.

Figure 7 explains the effect of viscous dissipation in terms of Eckert number on temperature profiles. Viscous dissipation can changes the temperature distribution by playing a role like an energy source, which leads to affect heat transfer rates. Here the temperature of both nanofluid and dust phase increase with increase in the value of $Ec$. It is because heat energy is stored in the liquid due to frictional heating and this is true in both the cases.
Figure 5. Effect of $M$ on temperature profiles for both PEST and PEHF cases.

Figure 6. Effect of $Pr$ on temperature profiles for both PEST and PEHF cases.

Figure 7. Effect of $Ec$ on temperature profiles for both PEST and PEHF cases.

Figure 8 shows the temperature distribution $\theta(\eta)$ and $\theta_r(\eta)$ v. $\eta$ for different values of number density ($N$). We infer from this figure that the temperature decreases with increase in $N$ for both PEST and PEHF cases.
The effect of solid volume fraction of nanoparticle parameter $\phi$ on velocity profiles for both fluid and dust phases are illustrated graphically through Figure 9. From this figure, it is observed that as the volume fraction of nanoparticles increases from 0 to 0.2, velocity profile for both fluid and dust phases decrease inside the boundary layer, while they increase outside.

Figure 10 is plotted over temperature profiles of different values of nanoparticles volume fraction $\phi$ for both PEST and PEHF cases. It is observed that, by increasing nanoparticles volume fraction, the thermal boundary layer increases for both fluid and dust phases.

Figure 11 shows the velocity profile for different type of fluids in the region of uniform magnetic field. It is observed that the velocity profile for regular fluid is much higher than that of nanofluid, dusty fluid and dusty nanofluid, respectively, in their order. This result is well evident to say that dusty nanofluid has high thermal conductivity than dusty fluid, nanofluid and regular fluid.

The effect of nanoparticles (Ag, Cu, CuO and Al₂O₃) on velocity profiles for both fluid and dust phases are depicted graphically in Figure 12. It is interesting to note that, velocity profile increases in the order of silver (Ag), copper (Cu), copper oxide (CuO) and aluminium oxide (Al₂O₃). From this, it is concluded that velocity profile for aluminium oxide (Al₂O₃) is much higher than that of others (see Table 1).

Figure 9. Effect of $\phi$ on velocity profile.
Figure 10. Effect of $\phi$ on temperature profiles for both PEST and PEHF cases.

Figure 11. Effect of all fluids on velocity profiles.

Figure 12. Effect of nanoparticles on velocity profiles.
Figure 13 exhibit the temperature profile for few nanoparticles in both PEST and PEHF cases. It is observed that temperature profile for silver (Ag) is much higher than that of copper (Cu), copper oxide (CuO) and Aluminium oxide (Al₂O₃).

Figure 14 displays the variation of skin friction (−f′′(0)) for increasing values of nanofluid particle interaction parameter (β), magnetic parameter (M) as well as solid volume fraction nanoparticle parameter (ϕ). It can be noticed that the nanofluid particle interaction parameter increases the skin friction. The same effect can also be found with the magnetic parameter and solid volume fraction nanoparticle parameter.
Figures 15 and 16 depict the nature of heat-transfer coefficient ($\theta'(0)$) in PEST case and $\theta(0)$ in PEHF case with nanofluid particle interaction parameter ($\beta$) for different values of magnetic parameter ($M$) and solid volume fraction nanoparticle parameter ($\phi$), respectively. It is very clear that the heat transfer increases with nanofluid particle interaction parameter ($\beta$) where as it decreases with magnetic parameter ($M$) and solid volume fraction nanoparticle parameter ($\phi$) in PEST case but it increases with both ($\beta$) and ($\phi$) in PEHF case. The temperature gradient function $\theta'(0)$ in PEST case is negative, by this we mean that the heat is transferred from fluid to the stretching surface. Further, the temperature function $\theta(0)$ in PEHF case at the sheet is positive means, heat is absorbing from the fluid.

6. Conclusions

A mathematical analysis has been carried out to study the effect of viscous dissipation on hydromagnetic flow and heat transfer of nanofluid over an exponentially stretching sheet with fluid particle suspension. Some of the important observations of our analysis obtained by the graphical representation are reported as follows.

The effect of magnetic parameter is to increase the temperature distribution in the flow region for both the cases of PEST and PEHF for both nanofluid and dust particle phases. The nanofluid phase temperature is higher than that of the dust phase. The temperature of nanofluid and dust phase increases with increasing values of $Ec$ in both PEST and PEHF cases. The velocity of nanofluid and dust phases decreases while the temperature of nanofluid and dust phases increases as solid volume fraction of nanoparticles ($\phi$) increases. For the aluminium oxide nanoparticles, the velocity is higher than that of other nanoparticles, and for silver (Ag) nanoparticles, the thermal conductivity is higher than other particles such as copper (Cu), copper oxide (CuO) and aluminium oxide (Al$_2$O$_3$). It is found that the dusty fluid with copper (Cu) nanoparticles have the appreciable cooling performance. The dusty nanofluids are found to have higher thermal conductivity, when compared with dusty fluid and nanofluid.

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