Conjectures on
(0,2) Mirror Symmetry

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In this paper we conjecture a reformulation of the monomial-divisor mirror map for
(2,2) mirror symmetry, valid at a boundary of the moduli space, that is easily extended to
also include tangent bundle deformations – an important step towards understanding (0,2)
mirror symmetry. We check our conjecture in a few simple cases, and thereby illustrate
how to perform calculations using a description of sheaves recently published by Knutson,
Sharpe.

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1 Introduction

Historically mirror symmetry has provided a fascinating interplay between questions in string theory and algebraic geometry. To review, mirror symmetry is simply a symmetry under which two distinct Calabi-Yaus, say $X$ and $Y$, are both described by the same conformal field theory – string theory is unable to distinguish them. Mirror symmetry is to date not deeply understood, but a large number of empirical results have been obtained.

There exists a potential generalization of mirror symmetry known as $(0,2)$ mirror symmetry, which has been developed much less far. Whereas ordinary (so-called $(2,2)$) mirror symmetry relates Calabi-Yaus $X$ and $Y$, $(0,2)$ mirror symmetry relates pairs $(X, \mathcal{E})$ and $(Y, \mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ are torsion-free sheaves on $X$, $Y$, respectively. More precisely, there exists a single conformal field theory which describes both pairs. Ordinary mirror symmetry is recovered in the special case $\mathcal{E} = TX$, $\mathcal{F} = TY$.

Ordinary mirror symmetry maps complex and Kahler moduli into one another, so for three-folds

$$H^{(1,1)}(X) \cong H^{(2,1)}(Y)$$
$$H^{(2,1)}(X) \cong H^{(1,1)}(Y)$$

In at least the simplest possible examples of $(0,2)$ mirror symmetry, in which $\mathcal{E}$ and $\mathcal{F}$ are both deformations of the tangent bundles of $X$, $Y$,

$$\text{Ext}_X^1(\mathcal{O}, \mathcal{E}) \cong \text{Ext}_Y^1(\mathcal{F}, \mathcal{O})$$
$$\text{Ext}_X^1(\mathcal{E}, \mathcal{O}) \cong \text{Ext}_Y^1(\mathcal{O}, \mathcal{F})$$

Potentially $(0,2)$ mirror symmetry may relate the complex, Kahler, and sheaf moduli of $(X, \mathcal{E})$ to the complex, Kahler, and sheaf moduli of $(Y, \mathcal{F})$ in a highly intricate fashion, though as yet no one knows.

Although ordinary mirror symmetry is not deeply understood, it has been well-developed – there are well-known methods to construct mirror Calabi-Yaus, and precise relations between complex and Kahler moduli have been developed. No such statements are true of $(0,2)$ mirror symmetry\footnote{Except in the special limit in which $(0,2)$ mirror symmetry reduces to ordinary mirror symmetry, of course.}, although there has been a very limited amount of work on the subject\cite{1, 19, 20, 21}.

The purpose of this paper is to begin to rectify this situation, by describing steps towards a precise map between complex, Kahler, and sheaf moduli in the special case that the sheaf over either Calabi-Yau is a deformation of the tangent bundle.

More precisely, we make a precise conjecture for a reformulation of the existing monomial-divisor mirror map for ordinary mirror symmetry which easily generalizes to include sheaf
deformations. Our conjecture is only valid at large radius limits – we do not know how to improve its range of validity.

We make a few simple tests of our conjecture, which allow us to demonstrate explicitly how to perform calculations using a description of sheaves recently espoused in [1], which contained a review and significant extension of results in [2, 3, 4, 5].

2 The Monomial-Divisor Mirror Map

Let us briefly review the monomial-divisor mirror map as described in for example [10].

For any Calabi-Yau realized as a hypersurface in a Fano toric variety, one can associate two polytopes, call them $\mathcal{A}$ and $\mathcal{B}$.

One polytope, call it $\mathcal{A}$, is the Newton polytope of the complex structure – it is the polytope containing all points of the weight lattice $M$ which are associated to possible monomials appearing in the hypersurface equation. Put another way, $\mathcal{A}$ is the (convex polytope) image of the moment map associated to the $(S^1)^n \subset (\mathbb{C}^*)^n$ action on the toric variety, with symplectic form defined by the anticanonical divisor.

The polytope $\mathcal{B}$ can be constructed as the polyhedron generating the fan for the ambient toric variety, as cones over faces. (Note that such a polyhedron can only exist when the toric variety is projective.) This polyhedron is also the “polar polyhedron” of $\mathcal{A}$ [7, section 4.1]:

$$\mathcal{B} = \{ (x_1, \cdots, x_n) \in M_\mathbb{R} \mid \sum_i x_i y_i \geq -1 \forall (y_1, \cdots, y_n) \in \mathcal{A} \}$$

Mirror symmetry exchanges $\mathcal{A}$ and $\mathcal{B}$. By examining how vertices of the polyhedra are mapped to one another, we recover the monomial-divisor mirror map. In particular, vertices of $\mathcal{A}$ correspond to monomials and vertices of $\mathcal{B}$ correspond to toric divisors, so we can see explicitly which monomials are mapped to which divisors, and vice-versa.

Essentially the monomial-divisor mirror map acts by exchanging convex polytope moment map images. The vertices of such polytopes correspond to fixed points of the (Hamiltonian) torus action; by studying how fixed points are mapped to one another, we learn how monomials and divisors are exchanged.

In what follows we will present a formally distinct but (hopefully) equivalent formulation of the monomial-divisor mirror map, which directly exchanges specific components of sheaf cohomology groups.

\(^2\)Generalizations to complete intersections exist but are not relevant for this paper, so they are omitted.
3 Reformulation of Monomial-Divisor Mirror Map

How might we extend the monomial-divisor mirror map to include sheaf deformations? In order to do so, we shall work near large complex structure limit points. In particular, not just any large complex structure limit points, but those in which the tangent bundle of the Calabi-Yau is stably equivalent to the restriction of an “equivariant” sheaf on the ambient toric variety.

3.1 Equivariance

What does it mean for a sheaf on a toric variety to be equivariant? All toric varieties have a natural action of $(\mathbb{C}^\times)^n$. This action on the toric variety defines an action on any moduli space of sheaves: for any sheaf $\mathcal{E}$ and any $t \in (\mathbb{C}^\times)^n$, we can take $\mathcal{E} \mapsto t^* \mathcal{E}$. Typically $\mathcal{E} \neq t^* \mathcal{E}$, but in the special case that $\mathcal{E} = t^* \mathcal{E}$ for all $t \in (\mathbb{C}^\times)^n$, we say that $\mathcal{E}$ is equivariant (with respect to the algebraic torus).

Equivariant sheaves have a number of nice properties. For example, sheaf cohomology groups and global Ext groups of equivariant sheaves on toric varieties have a canonical decomposition, known as an isotypic decomposition, by elements of the weight lattice $M$ of the algebraic torus. More precisely, if $\mathcal{E}$ and $\mathcal{F}$ are equivariant, then

$$H^p(\mathcal{E}) = \bigoplus_{\chi} H^p(\mathcal{E})_{\chi}$$

$$\text{Ext}^p(\mathcal{E}, \mathcal{F}) = \bigoplus_{\chi} \text{Ext}^p(\mathcal{E}, \mathcal{F})_{\chi}$$

We should note that when authors speak of equivariant sheaves, they sometimes implicitly assume a specific choice of “equivariant structure” has been made. An equivariant structure is simply a precise choice of action of the algebraic torus on the sheaf; it is not sufficient to know the fact that the algebraic torus maps the sheaf back into itself, one must also know precisely how the algebraic torus maps the sheaf into itself.

Under what circumstances is the tangent bundle of a Calabi-Yau stably equivalent to the restriction of an equivariant sheaf? This will happen when the hypersurface defining the Calabi-Yau is a monomial. Such a Calabi-Yau is highly degenerate, and for technical reasons degenerate Calabi-Yaus are often excised from moduli spaces of complex structures. The degenerate Calabi-Yaus we shall consider, however, still exist on the moduli space. (For more information on complex structure moduli spaces, see appendix A.)

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\[3^3\text{In fact, it was noted in [1] that this definition of equivariant sheaves is not quite correct technically, but for the purposes of this article it will suffice.}\]
(degenerate) Calabi-Yaus whose tangent bundles are stably equivalent to the restriction of an equivariant sheaf as “equivariant Calabi-Yaus”.

If we describe the ambient toric variety in terms of homogeneous coordinates and $\mathbb{C}^\times$ actions, then the product of all homogeneous coordinates is a possible monomial appearing in a Calabi-Yau hypersurface equation, and by itself represents a degenerate Calabi-Yau that exists on the moduli space. Better, in fact: it is usually a large complex structure limit [6, 8].

Let us consider a specific example. Let $x_1, \cdots x_5$ represent homogeneous coordinates on $\mathbb{P}^4$ and $\mathbb{P}^4/(\mathbb{Z}_5)^3$, then in both ambient spaces the hypersurface

$$x_1x_2x_3x_4x_5 = 0$$

is a degenerate Calabi-Yau. Moreover, as is well-known, this hypersurface in $\mathbb{P}^4/(\mathbb{Z}_5)^3$ is precisely the large complex structure limit. Another example [11] is the family of degree 8 hypersurfaces in $\mathbb{P}^4_{1,1,2,2,2}/\mathbb{Z}_3^4$ given by

$$4\psi x_1x_2x_3x_4x_5 + \phi x_1^4x_2^4 = 0$$

where $x_1, \cdots x_5$ are the homogeneous coordinates. This family of Calabi-Yaus represents the large complex structure limits of the mirror to degree 8 hypersurfaces in $\mathbb{P}^4_{1,1,2,2,2}$. When either $\psi = 0$ or $\phi = 0$, we recover an equivariant Calabi-Yau.

As this point is quite important, we shall repeat it. In the “classical” limit of mirror symmetry, in which all worldsheet instanton corrections are suppressed in both mirror Calabi-Yaus, both Calabi-Yaus are degenerate and in limits have tangent bundles stably equivalent to the restriction of an equivariant sheaf. Thus, mirror Calabi-Yaus exist which are both equivariant, and such Calabi-Yaus correspond to a “classical” limit of mirror symmetry.

On an equivariant Calabi-Yau, sheaf cohomology groups and Ext groups inherit nontrivial properties from the ambient toric variety. In particular, in the case that a sheaf on an equivariant Calabi-Yau $X$ is the restriction of of an equivariant sheaf $\mathcal{E}$ on an ambient toric variety $Y$, the sheaf cohomology groups $H^p(X, \mathcal{E}|_X)$ inherit the isotypic decomposition of sheaf cohomology groups on the ambient space. In particular, recall that the restriction of $\mathcal{E}$ to $X$ is defined by

$$0 \to \mathcal{E} \otimes \mathcal{O}(K) \to \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_X \to 0$$

where $\mathcal{O}_X$ is a skyscraper sheaf with support on the Calabi-Yau $X$. Now, both $\mathcal{E} \otimes \mathcal{O}(K)$ and $\mathcal{E}$ are equivariant, and moreover the map $\mathcal{E} \otimes \mathcal{O}(K) \to \mathcal{E}$ is equivariant when $X$ is an equivariant Calabi-Yau, so clearly the sheaf cohomology groups $\tilde{H}^p(X, \mathcal{E}|_X)$ inherit the isotypic decomposition of sheaf cohomology groups on the ambient toric variety $Y$. Put another way, in the case of an equivariant Calabi-Yau, the skyscraper sheaf $\mathcal{O}_X$ is an equivariant torsionful sheaf, so $\mathcal{E} \otimes \mathcal{O}_X$ is equivariant and $\tilde{H}^p(Y, \mathcal{E} \otimes \mathcal{O}_X) = \tilde{H}^p(X, \mathcal{E}|_X)$ has an isotypic decomposition.
3.2 The Conjectured Reformulation

Now, let us re-examine the monomial-divisor mirror map in the limit of large complex structure and large radius, and more specifically, in a limit in which both Calabi-Yaus have tangent sheaves which are the restriction of an equivariant sheaf. In this limit, we conjecture the monomial-divisor mirror map exchanges isotypic components of sheaf cohomology. More precisely, if $X, Y$ are a mirror pair of Calabi-Yaus, then in this limit, for all $\chi$ in the weight lattice of the algebraic torus $M$, we claim

$$H^1(X, TX)_\chi \cong H^1(Y, T^*Y)_\chi$$
$$H^1(X, T^*X)_\chi \cong H^1(Y, TY)_\chi$$

We conjecture that this is precisely the statement of the monomial-divisor mirror map in the equivariant limit.

In fact, we have been slightly sloppy. The weight lattices of the algebraic tori underlying the ambient toric varieties in which $X$ and $Y$ are embedded are not canonically isomorphic. The more nearly correct statement of the conjecture is that there exists an isomorphism of the weight lattices such that components of sheaf cohomology groups are exchanged as above.

Not only have we been sloppy, we have also been somewhat naive. Equivariant Calabi-Yaus are highly singular, and in the limit of complex structure in which a Calabi-Yau becomes equivariant, the sheaf cohomology groups may jump— they may lose or gain elements. In particular, we shall see in specific examples that it is quite common for the sheaf cohomology groups of a smooth Calabi-Yau to not quite have the same dimensions as the sheaf cohomology groups of an equivariant Calabi-Yau. For example, we will find that whereas a smooth elliptic curve $E$ has $h^1(E, TE) = 1$, for a degenerate $E$ we compute $h^1(E, TE) = 0$. Unfortunately this jumping phenomenon makes our conjecture somewhat less palatable. In order for our conjecture to be genuinely useful, one would need to extend it away from equivariant limits.

3.3 (0,2) Generalization

How can we extend the monomial-divisor mirror map, as reformulated above, to include sheaf deformations? When phrased in the language above, it is straightforward. Deformations of a sheaf $\mathcal{E}$ are parametrized by elements of (global) $\text{Ext}^1(\mathcal{E}, \mathcal{E})$, so the answer should be relatively clear. In the special case that the gauge sheaves $\mathcal{E}, \mathcal{F}$ on $X, Y$, respectively are both deformations of the tangent sheaves, and also happen to be the restriction of equivariant sheaves, we can hypothesize that (0,2) mirror symmetry relates

$$\text{Ext}^1_X(\mathcal{E}, \mathcal{E})_\chi \cong \text{Ext}^1_Y(\mathcal{F}, \mathcal{F})_\chi$$
How can such a result be interpreted? In general, isotypic components of \( \text{Ext}^1(\mathcal{E}, \mathcal{E}) \) associated with character 0 describe deformations that preserve equivariance, whereas those with nonzero character destroy equivariance. If our conjecture is correct, then it implies that deformations preserving equivariance are mirror to deformations also preserving equivariance, and also relates specific non-equivariance-preserving deformations (at least in limits where worldsheet instanton effects are small).

So far we have given a conjecture for a reformulation of the monomial-divisor mirror map that works only in the limit in which both Calabi-Yaus are at large radius and large complex structure. How might one attempt to extend our ansatz away from this limit? Consider the case that one of the Calabi-Yaus is deformed to generic complex structure, but held at large radius (in fact at the same large radius limit). In this case, the algebraic torus defining the ambient toric variety no longer has a well-defined action on the sheaf cohomology groups – in fact, its action changes the complex structure of the Calabi-Yau. However, since the Calabi-Yau is at large radius, its mirror is at large complex structure (and in fact still equivariant) – thus, the algebraic torus defining the ambient toric variety of the mirror Calabi-Yau does have an action!

Put another way, if we deform one of our equivariant Calabi-Yaus to arbitrary complex structure but hold the Kahler moduli at the same large radius point, then its sheaf cohomology still has an isotypic decomposition – but under the algebraic torus defining the ambient toric variety of the mirror! Similarly, if we hold the complex structure fixed but vary the Kahler moduli, then the isotypic decomposition will still be well-defined, although cup and Yoneda products will receive worldsheet instanton corrections.

Unfortunately, because the sheaf cohomology groups jump in degenerations, there is no reason why the mirror isotypic decomposition of sheaf cohomology of a smooth Calabi-Yau described above should coincide with the isotypic decomposition of its degenerate mirror. Our calculations of sheaf cohomology of degenerations are somewhat naive – perhaps we have missed subtleties which give better-behaved results. In any event, we will not see the isotypic decomposition under a mirror algebraic torus explicitly in this paper. If it were possible to correct our ansatz for these cases, then one might wonder whether it can be extended farther – to the case when both the complex and Kahler moduli are generic. We do not have any comments on this case, except to speculate that some sort of derived categories argument may prove crucial.

In order to try to test our conjecture, we shall compute sheaf cohomology groups and Ext groups on degenerate elliptic curves. We shall not be able to check whether any matching suggested by the isotypic decomposition agrees with the usual monomial-divisor mirror map, we shall only be able to see whether (judging by dimensions of components) there exist matchings suggested by isotypic decompositions.
3.4 Discrete R-anomalies

There is a potential danger in our approach which we have glossed over so far. We shall be studying sheaves on toric varieties that restrict to a sheaf stably equivalent to the tangent bundle of the Calabi-Yau (for an elliptic curve $E$, sheaves $\mathcal{E}$ such that $\mathcal{E}|_E = TE \oplus \mathcal{O}$) and it was noted in [14] that linear sigma models describing such situations suffer from certain poorly understood effects.

However we feel this is not a problem in the present case. First, we are not trying to construct conformal field theories directly but rather are merely studying a mathematical construction designed to shed light on the monomial-divisor mirror map.

Second, we should point out the existence of discrete R-anomalies at Landau-Ginzburg points does not reflect instability in a gauge sheaf, despite claims in the existing literature. On a Calabi-Yau $X$, the sheaf $TX \oplus \mathcal{O}$ is not unstable but rather properly semistable\(^4\), and the $D$-term constraint in the low-energy supergravity allows not only stable sheaves but also split properly semistable sheaves\(^5\). Properly semistable sheaves are grouped\(^5\) in $S$-equivalence classes, and each $S$-equivalence class contains a unique split representative [15, p.23]. Thus, the discrete R-anomalies described in [14] do not have anything to do with stability per se, but rather quite possibly reflect the fact that the renormalization group flow is extremely subtle to understand because of the presence of a properly semistable sheaf.

4 Elliptic Curves

In this section we will perform consistency checks on our conjecture by explicitly calculating relevant sheaf cohomology groups on the degenerate elliptic curve in $\mathbb{P}^2$ and on the mirror curve in $\mathbb{P}^2/\mathbb{Z}_3$.

This particular case is extremely simple: there is only one complex, Kahler, and bundle deformation, so our conjecture automatically holds true in this case. In more complicated cases we can check that dimensions of isotypic components of sheaf cohomology groups match, but unfortunately nowhere in this paper (except in the trivial case of elliptic curves) will we be able to determine whether the monomial-divisor mirror map actually exchanges isotypic components. All we can do is check that dimensions of isotypic components match – a consistency test, no more.

\(^4\)More generally [14], the direct sum of two Mumford-Takemoto semistable bundles with the same slope is again Mumford-Takemoto semistable.

\(^5\)More precisely, points on a moduli space of sheaves that are properly semistable do not correspond to unique semistable sheaves, but rather to $S$-equivalence classes of properly semistable sheaves. Points that are stable do correspond to unique stable sheaves – $S$-equivalence classes are a phenomenon arising only for properly semistable objects.
As the techniques needed to perform our calculations are new to the physics literature, we shall work through the calculations in detail.

4.1 Elliptic Curve in $\mathbb{P}^2$

A fan describing $\mathbb{P}^2$ as a toric variety is shown in figure 1.

We will be interested in the degenerate elliptic curve $E$ defined by

$$xyz = 0$$

The tangent bundle of $E$ is stably equivalent to the restriction of an equivariant bundle, say $\mathcal{E}$, on $\mathbb{P}^2$. (In other words, $\mathcal{E}|_E = TE \oplus \mathcal{O}$.) In the notation of [1], the bundle $E$ on $\mathbb{P}^2$ is defined by the filtrations

$$(\mathcal{E})^\alpha(i) = \begin{cases} C^2 & i < 0 \\ \alpha & i = 0, 1 \\ 0 & i > 1 \end{cases}$$

(To check that this bundle restricts on $E$ to become stably equivalent to $TE$, i.e., that it is the same as the bundle on $\mathbb{P}^2$ defined by the short exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \bigoplus_{i=1}^3 \mathcal{O}(1) \rightarrow \mathcal{O}(3) \rightarrow 0$$

note that both bundles have $c_1 = 0$ and $c_2 = 3$.) In order to calculate the sheaf cohomology of $TE$, we shall calculate the sheaf cohomology of $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{O}(K)$ and use relations derived from the short exact sequence

$$0 \rightarrow \mathcal{E} \otimes \mathcal{O}(K) \rightarrow \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{O}_E \rightarrow 0$$
The modules associated to the toric divisors have the forms indicated below. The module associated to \( \{x = 0\} \), call it \((E)^x\), has the form

\[
\begin{array}{cccccc}
-3 & -2 & -1 & 0 & 1 & 2 \\
\vdots & & & & & \\
1 & 0 & (x) & (x) & C^2 & \\
0 & \cdots & 0 & (x) & (x) & C^2 & \cdots \\
-1 & 0 & (x) & (x) & C^2 & \\
\vdots & & & & & \\
\end{array}
\]

The complex one-dimensional subspace of \( \mathbb{C}^2 \) generated by \( x = (1,0) \) is denoted by \((x)\). Similarly, the module associated to \( \{y = 0\} \), call it \((E)^y\), has the form

\[
\begin{array}{cccccc}
-2 & -1 & 0 & 1 & 2 \\
\vdots & & & & & \\
1 & C^2 & C^2 & C^2 & \\
0 & \cdots & (y) & (y) & (y) & \cdots \\
-1 & (y) & (y) & (y) & \\
-2 & 0 & 0 & 0 & \\
\vdots & & & & & \\
\end{array}
\]

and finally the module \((E)^z\) associated to the divisor \( \{z = 0\} \) has the form

\[
\begin{array}{cccc}
-1 & 0 & 1 \\
\vdots & & & \\
3 & 0 & 0 & 0 \\
2 & (z) & 0 & 0 \\
1 & (z) & (z) & 0 \\
0 & \cdots & C^2 & (z) & (z) & \cdots \\
-1 & C^2 & C^2 & (z) & \\
-2 & C^2 & C^2 & C^2 & \\
\vdots & & & & & \\
\end{array}
\]

4.1.1 \( H^1(E, E) \)

Now, given the fact that

\[
H^0(E)_\chi = \bigcap_{\alpha \in |\Sigma|} (E)^\alpha(\chi)
\]
for reflexive sheaves $\mathcal{E}$, and that $(x) \cap (y) = 0$, we can easily show that

$$H^0(\mathbb{P}^2, \mathcal{E})_\chi = 0 \forall \chi$$

In general, for equivariant bundles $\mathcal{E}$ on a smooth toric surface

$$H^2(\mathcal{E})_\chi = \frac{V}{\sum_{\alpha \in |\Sigma|} (\mathcal{E})^\alpha(\chi)}$$

What is $V$? Over the open $T$-orbit any equivariant torsion-free sheaf is a trivial vector bundle; $V$ is the fiber of that bundle. (In the present case, $V = \mathbb{C}^2$.) For the case at hand we can quickly show

$$H^2(\mathbb{P}^2, \mathcal{E})_\chi = 0 \forall \chi$$

We can then use

$$\sum_i (-)^i \dim H^i(\mathcal{E})_\chi = \sum_{\sigma \in \Sigma} (-)^{\text{codim } \sigma} \dim (\mathcal{E})^\sigma(\chi)$$

(where the origin of the fan is included as a (zero-dimensional) cone, with associated module $V$) to show that, for the case at hand,

$$H^1(\mathbb{P}^2, \mathcal{E})_\chi = \begin{cases} \mathbb{C} & \chi = 0 \\ 0 & \text{otherwise} \end{cases}$$

The equivariant bundle $\mathcal{E} \otimes \mathcal{O}(K)$, for the “symmetric” equivariant structure on $K$ (namely, $-K = \sum_{\alpha \in |\Sigma|} D_\alpha$) is defined by the filtrations

$$(\mathcal{E} \otimes \mathcal{O}(K))^\alpha(i) = \begin{cases} \mathbb{C}^2 & i < -1 \\ \alpha & i = -1, 0 \\ 0 & i > 0 \end{cases}$$

and proceeding as before, one can quickly show

$$H^0(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}(K))_\chi = 0 \forall \chi$$
$$H^2(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}(K))_\chi = 0 \forall \chi$$
$$H^1(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}(K))_\chi = \begin{cases} \mathbb{C} & \chi = 0 \\ 0 & \text{otherwise} \end{cases}$$

Putting this together it is easy to see that

$$H^0(E, \mathcal{E})_\chi = H^1(E, \mathcal{E})_\chi = 0 \forall \chi$$

(For $\chi \neq 0$, the result is clear. The result for $\chi = 0$ must be obtained by a close examination of the exact sequence.)

This result is an example of an unfortunate phenomenon mentioned earlier – in the singular complex structure limits describing equivariant Calabi-Yaus, the sheaf cohomology groups may jump.
4.1.2 \( H^1(E, \mathcal{E}^\vee) \)

How does one calculate the dual \( \mathcal{E}^\vee \) of a sheaf \( \mathcal{E} \)?

More generally, given any two equivariant torsion-free sheaves \( \mathcal{E} \) and \( \mathcal{F} \), say, which over the open torus orbit look like vector bundles with fiber \( E, F \), respectively, then to any cone \( \sigma \in \Sigma \) we can define the module \( \text{Hom}(\mathcal{E}, \mathcal{F})^\sigma \) as follows. Each equivariant element \( \text{Hom}(\mathcal{E}, \mathcal{F})^\sigma(\chi) \) is precisely a subspace of \( \text{Hom}(E, F) \). Which subspace? Well, any element of \( \text{Hom}(E, F) \) defines a map between equivariant elements of the modules \( (\mathcal{E})^\sigma \) and \( (\mathcal{F})^\sigma \), and in particular \( \text{Hom}(\mathcal{E}, \mathcal{F})^\sigma(\chi) \) is the subspace of \( \text{Hom}(E, F) \) that maps \( (\mathcal{E})^\sigma(\mu) \) into \( (\mathcal{F})^\sigma(\mu + \chi) \) for all \( \mu \). (A general element of \( \text{Hom}(E, F) \) will map \( (\mathcal{E})^\sigma(\mu) \subseteq E \) into \( F \) for any \( \mu \), but not necessarily into the subspace \( (\mathcal{F})^\sigma(\mu + \chi) \subseteq F \).)

How can we calculate \( \mathcal{E}^\vee \) given \( \mathcal{E} \)? Well, \( \mathcal{E}^\vee = \text{Hom}(\mathcal{E}, \mathcal{O}) \), and in fact it is straightforward to show that for any equivariant torsion-free sheaf \( \mathcal{E} \), the equivariant reflexive sheaf \( \mathcal{E}^\vee \) is defined by the filtrations

\[
(\mathcal{E}^\vee)^\alpha(i) = (E/(\mathcal{E})^\alpha(1-i))^\ast
\]

In particular, for the case we are interested in,

\[
(\mathcal{E}^\vee)^\alpha(i) =
\begin{cases}
(C^2)^\ast \cong C^2 & \text{if } i < 0 \\
(C^2/(\alpha))^\ast = \{\omega \in (C^2)^\ast | \langle \omega, \alpha \rangle = 0 \} & \text{if } i = 0, 1 \\
0 & \text{if } i > 1
\end{cases}
\]

In order to derive the sheaf cohomology of \( \mathcal{E}^\vee \) and \( \mathcal{E}^\vee \otimes \mathcal{O}(K) \), we can either proceed directly as before, or we can use equivariant Serre duality, which says that for any equivariant bundle \( \mathcal{E} \) on an \( n \)-dimensional projective toric variety \( X \),

\[
H^i(X, \mathcal{E})_\chi \cong H^{n-i}(X, \mathcal{E}^\vee \otimes \mathcal{O}(K))_\chi
\]

where \( \mathcal{O}(K) \) is assumed to have the “symmetric” equivariant structure (namely, \( K = -\sum_{\alpha \in \Sigma} D_\alpha \)).

In either event, one quickly derives that

\[
H^0(E, \mathcal{E}^\vee)_\chi = H^1(E, \mathcal{E}^\vee)_\chi = 0 \forall \chi
\]

4.1.3 \( \text{Ext}^1_E(\mathcal{E}, \mathcal{E}) \)

Using the methods outlined earlier, it is straightforward to compute the filtrations defining the bundle \( \text{Hom}(\mathcal{E}, \mathcal{E}) \). Define

\[
\begin{align*}
M &= \text{Hom}(C^2, C^2) \\
P^\alpha &= \{\omega \in M | \omega(\alpha) \subseteq (\alpha)\} \\
Q^\alpha &= \{\omega \in M | \text{im } \omega \subseteq (\alpha) \text{ and } \omega(\alpha) = 0\}
\end{align*}
\]
then the filtrations defining $\text{Hom}(\mathcal{E}, \mathcal{E})$ are given by

$$\text{Hom}(\mathcal{E}, \mathcal{E})^\alpha(i) = \begin{cases} 
M & i < -1 \\
P^\alpha & i = -1, 0 \\
Q^\alpha & i = 1, 2 \\
0 & i > 2 
\end{cases}$$

For bundles $\mathcal{E}$, $\text{Ext}^k(\mathcal{E}, \mathcal{E}) = H^k(\text{Hom}(\mathcal{E}, \mathcal{E}))$; we shall use the notations interchangeably in the rest of this subsection.

Using the same techniques as earlier it is now straightforward to calculate

$$\text{Ext}^0_{\mathbb{P}^2}(\mathcal{E}, \mathcal{E})_\chi = \begin{cases} 
\mathbb{C} & \chi = 0 \\
0 & \text{otherwise} 
\end{cases}$$

$$\text{Ext}^2_{\mathbb{P}^2}(\mathcal{E}, \mathcal{E})_\chi = 0 \forall \chi$$

and $\text{Ext}^1_{\mathbb{P}^2}(\mathcal{E}, \mathcal{E})_\chi$ is given by

|     | -3 | -2 | -1 | 0  | 1  | 2  |
|-----|----|----|----|----|----|----|
| 2   | 0  | 0  | 0  | 0  | 0  | 0  |
| 1   | 0  | C  | C  | C  | 0  |    |
| 0   | 0  | 0  | C  | 0  | C  | 0  |
| -1  | 0  | 0  | 0  | C  | C  | 0  |
| -2  | 0  | 0  | 0  | 0  | C  | 0  |
| -3  | 0  | 0  | 0  | 0  | 0  | 0  |

As a check, it is well-known [13] that moduli spaces of rank 2 bundles on $\mathbb{P}^2$ of $c_1 = 0, c_2 = n$ have dimension $4n - 3$. In this case, $\mathcal{E}$ has $c_1 = 0, c_2 = 3$, so it has 9 complex deformations – and indeed, $\dim \text{Ext}^1_{\mathbb{P}^2}(\mathcal{E}, \mathcal{E}) = 9$.

This result for $\text{Ext}^1$ also implies that $\mathcal{E}$ has no “equivariant” deformations – that is, deformations that preserve equivariance. (In general, $\dim \text{Ext}^1(\mathcal{E}, \mathcal{E})_{\chi=0}$ is the number of sheaf moduli that preserve equivariance.) This also is relatively straightforward to check.

The groups $\text{Ext}^i_{\mathbb{P}^2}(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}(K))_\chi$ can either be worked out directly or calculated via equivariant Serre duality.

Putting those two sets of Ext groups together, we find that $\text{Ext}^0_{E}(\mathcal{E}, \mathcal{E})_\chi$ is given by
Our result for Ext$_1^E$ reveals some unfortunate pathologies that must be dealt with. It is well-known [12] that any moduli space of bundles on a smooth elliptic curve is isomorphic to the elliptic curve itself, so for smooth $E$, dim Ext$_1^E(\mathcal{E}, \mathcal{E}) = 1$. Yet here, by contrast, dim Ext$_1^E(\mathcal{E}, \mathcal{E}) > 1$. How is this possible? Strictly speaking Ext$_1^E(\mathcal{E}, \mathcal{E})$ is the Zariski tangent cone to a moduli space of sheaves at sheaf $\mathcal{E}$. If the moduli space is singular at the point represented by sheaf $\mathcal{E}$, for example, then Ext$_1^E(\mathcal{E}, \mathcal{E})$ need not have the same dimension as the moduli space. In particular, in the present case $E$ is extremely singular, and so there is no good reason why dim Ext$_1^E(\mathcal{E}, \mathcal{E}) = 1$. The “extra” generators of Ext$_1^E$ should be thought of as closely analogous to “fake” marginal operators in string theory – that is, operators with the right conformal dimension to be marginal, but which do not actually represent flat directions in the theory.

### 4.2 Elliptic Curve in $\mathbb{P}^2/\mathbb{Z}_3$

A fan describing $\mathbb{P}^2/\mathbb{Z}_3$ as a toric variety is shown in figure 4.

Strictly speaking the sheaf cohomology calculations presented in [1] should only be trusted on a smooth toric variety, however we shall apply them here.

The calculations in this case are extremely similar to those already presented, so we shall simply summarize the results.

Let $\mathcal{E}$ be a sheaf on $\mathbb{P}^2/\mathbb{Z}_3$ defined by the same type of filtrations used to define the $\mathcal{E}$ used in the previous section, then for an elliptic curve $\tilde{E}$ (the mirror to $E$) defined by $xyz = 0$

we have

$$H^0(\tilde{E}, \mathcal{E})_\chi = H^1(\tilde{E}, \mathcal{E})_\chi = 0 \forall \chi$$

$$H^0(\tilde{E}, \mathcal{E}^\vee)_\chi = H^1(\tilde{E}, \mathcal{E}^\vee)_\chi = 0 \forall \chi$$

and Ext$_1^E(\mathcal{E}, \mathcal{E})_\chi$ is given by

$$\text{Ext}_1^E(\mathcal{E}, \mathcal{E})_\chi = \text{Ext}_0^E(\mathcal{E}, \mathcal{E})_\chi$$
and

\[ \text{Ext}^0_E(\mathcal{E}, \mathcal{E})_\chi = \text{Ext}^1_E(\mathcal{E}, \mathcal{E})_{-\chi} \]

Here again we have the unfortunate phenomenon that there are more generators in \( \text{Ext}^1_E \) than there should be genuine moduli.

How do these results compare to those for the mirror \( E' \)? As only one of the groups \((\text{Ext}^1(\mathcal{E}, \mathcal{E}))\) is nonzero, and \( \dim \text{Ext}^1_E(\mathcal{E}, \mathcal{E}) = \dim \text{Ext}^1_{E'}(\mathcal{E}, \mathcal{E}) \), there exists an isomorphism of isotypic decompositions, and so this agrees – trivially – with the conjecture.

## 5 Higher Dimensional Calabi-Yaus

In principle one could work through the same calculations for higher dimensional Calabi-Yaus, though in practice these computations are more cumbersome. We shall not do so here, but instead will mention a few subtleties associated with such computations.

The tangent bundle of the equivariant elliptic curve was the restriction of a bundle on \( \mathbb{P}^2 \), but in higher dimensions the tangent sheaf of an equivariant Calabi-Yau will instead be
the restriction of an equivariant torsion-free sheaf on the ambient toric variety.

As a result, instead of calculating sheaf cohomology groups to count complex and Kahler moduli, one should consider Ext groups. More precisely, $H^1(X, T^*X)$ is replaced with $\text{Ext}^1_{X}(TX, O)$. The group $H^1(X, TX)$ is always isomorphic to $\text{Ext}^1_{X}(O, TX)$.

Global Ext groups of equivariant sheaves have an isotypic decomposition, just as the sheaf cohomology groups, and are relatively straightforward to calculate. As the methods involved may not be familiar to the reader, we review a simple example in appendix B.

6 Conclusions

In this paper we have presented a conjecture regarding how (0,2) mirror symmetry acts on sheaf deformations. More precisely, we have made a conjecture concerning how deformations of the tangent bundle of mirror symmetric Calabi-Yaus are mirror mapped into each other, in a limit that all worldsheet instanton corrections on both sides of the mirror are turned off.

We would like to make a more general conjecture concerning mirror symmetry. For Calabi-Yaus realized as hypersurfaces in toric varieties, an “inherently toric” description of mirror symmetry has been well-developed – Kahler moduli descending from the ambient space are exchanged with complex structure deformations to other hypersurfaces in the same ambient space. For Calabi-Yaus realized as special Lagrangian fibrations, mirror symmetry maps such Calabi-Yaus into other Calabi-Yaus also realized as special Lagrangian fibrations. More generally, it would appear that for Calabi-Yaus with any defining property “X”, there exists a description of mirror symmetry that is inherently “X”-ic. Perhaps such an observation can be made more precise – perhaps there exists some universal (in the sense of category theory) definition of mirror symmetry, which yields specific mirror conjectures for any given family of Calabi-Yaus.

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A Complex structure moduli spaces

Moduli spaces of complex structures can be constructed as GIT quotients [17, 18] (see the appendix of [1] for an introduction to GIT quotients for physicists). Typically smooth hypersurfaces are stable, and singular hypersurfaces are either semistable or (more commonly) unstable.

In this appendix we shall examine moduli spaces of complex structures explicitly, and check that the degenerate hypersurfaces used in the text really are on the moduli space (i.e., are semistable). Our presentation will be closely analogous to those in [17, section 4.2] and [18, section 4.4].

Let \( x_0, x_1, \ldots x_n \) be homogeneous coordinates on \( \mathbb{P}^n \). We shall study moduli spaces of degree \( n+1 \) hypersurfaces in \( \mathbb{P}^n \).

To set notation, let
\[
\sum a_{i_1i_2\cdots i_n} x_0^{n+1-i_1-i_2-\cdots-i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} = 0
\]
define a degree \( n+1 \) hypersurface in \( \mathbb{P}^n \).

Now, not every distinct set of \( a_{i_1i_2\cdots i_n} \) defines a distinct hypersurface – hypersurfaces related by an action of \( GL(n+1, \mathbb{C}) \) on the homogeneous coordinates are identical. As the overall breathing mode of \( GL(n+1, \mathbb{C}) \) is irrelevant, it suffices to consider the action of \( SL(n+1, \mathbb{C}) \).

The moduli space of complex structures is simply a GIT quotient of the space of \( a_{i_1i_2\cdots i_n} \) (namely, \( \mathbb{C}^k \) for some \( k \)) by \( SL(n+1, \mathbb{C}) \) (with ample line bundle \( \mathcal{O} \) on \( \mathbb{C}^k \)).

How can we determine whether a hypersurface is stable, semistable, or unstable under this GIT quotient? We shall use the numerical criterion for stability (see [17, section 2.1] or [18, section 4.2] or the appendix to [1]). This involves checking stability under all one-parameter subgroups of \( SL(n+1, \mathbb{C}) \).

Now, every one-parameter subgroup of \( SL(n+1, \mathbb{C}) \) is conjugate to one of the form
\[
\lambda(t) = \text{diag}(t^{r_0}, t^{r_1}, \ldots t^{r_n})
\]
with
\[
\sum r_i = 0
\]
and
\[
r_0 \geq r_1 \geq \cdots \geq r_n
\]
with not all \( r_i \) zero. Clearly, such a one-parameter subgroup will act on the \( a_{i_1i_2\cdots i_n} \) as
\[
\lambda(t) a_{i_1i_2\cdots i_n} = t^{-r_0(n+1-i_1-i_2-\cdots-i_n)-r_1i_1-\cdots-i_nr_n} a_{i_1i_2\cdots i_n}
\]
Now, define
\[ \mu(\{a_{i_1i_2\ldots i_n}\}, \lambda) = \text{unique integer } \mu \text{ such that } \lim_{t \to 0} t^\mu \lambda(t) a_{i_1i_2\ldots i_n} \text{ exists and is nonzero} \]
\[ = \max \left\{ r_0(n + 1 - i_1 - i_2 - \cdots - i_n) + r_1i_1 + \cdots + i_n r_n \mid a_{i_1i_2\ldots i_n} \neq 0 \right\} \]

According to the numerical criterion for stability, a point will be stable precisely when \( \mu > 0 \) for all one-parameter subgroups \( \lambda \), semistable when \( \mu \geq 0 \) for all one-parameter subgroups \( \lambda \), and unstable otherwise.

It is now easy to check that the equivariant Calabi-Yau defined by the hypersurface
\[ x_0x_1 \cdots x_n = 0 \]
in \( \mathbb{P}^n \) is always properly semistable. For this hypersurface, for any one-parameter subgroup \( \lambda(t) \),
\[ \mu = r_0(1) + r_1 + \cdots + r_n \]
\[ = 0 \]
Thus, equivariant Calabi-Yaus of this form are always semistable, and more to the point always exist on the moduli space.

Now we shall specialize to the case of elliptic curves in \( \mathbb{P}^2 \). Suppose the elliptic curve has a singular point, and that (without loss of generality) the point is located at
\[ (x_0, x_1, x_2) = (1, 0, 0) \]
In this case, we have constraints on the \( \{ a_{i_1i_2}\} \): \( a_{i_1i_2} = 0 \) if \( i_1 = i_2 = 0 \), or if precisely one of the \( i_j = 1 \) and the other is zero. We claim that in this case, there exists a one-parameter subgroup \( \lambda \) such that \( \mu(a_{i_1i_2}, \lambda) \leq 0 \). Let \( r_0 = 2, r_1 = r_2 = -1 \), then
\[ \mu(a_{i_1i_2}, \lambda) = 2(3 - i_1 - i_2) - i_1 - i_2 \]
\[ = 3(2 - i_1 - i_2) \]
\[ \leq 0 \text{ as } i_1 + i_2 \geq 2 \text{ for } a_{i_1i_2} \neq 0 \]
Thus, a singular elliptic curve in \( \mathbb{P}^2 \) can never be properly stable, but is either properly semistable or unstable.

B Ideal sheaf on \( \mathbb{P}^2 \)

In order to gain some basic experience with nontrivial global Ext calculations, we shall warm up by studying an ideal sheaf \( \mathcal{I} \) on \( \mathbb{P}^2 \), vanishing to order 1 at \( x = y = 0 \). (For notation,
consult figure [I].) In particular, we shall calculate the global Ext groups \( \text{Ext}^0(\mathcal{I}, \mathcal{O}) \) and \( \text{Ext}^1(\mathcal{I}, \mathcal{O}) \).

What modules do we associate to cones to describe this ideal sheaf \( \mathcal{I} \)? Over all cones \( \sigma \) except the dimension 2 cone spanned by \( x, y \), the module associated to \( \text{Spec} \ C[\sigma^\vee] \) is precisely \( C[\sigma^\vee] \). Over cone 1, spanned by \( x, y \), the module is the ideal \( I = (x, y) \subset C[x, y] \).

Now, let us calculate global Ext groups of this ideal sheaf, following the prescription in [I]. Let \( I^\sigma \) denote the module defining \( \mathcal{I} \) on cone \( \sigma \), \( \mathcal{O} \) the structure sheaf, and \( \mathcal{O}^\sigma \) the module defining \( \mathcal{O} \) on cone \( \sigma \) – namely, \( \mathcal{O}^\sigma = C[\sigma^\vee] \). In this notation, the global Ext group \( \text{Ext}^n(\mathcal{I}, \mathcal{O})_\chi \) is the limit of a spectral sequence with first-level terms

\[
E_1^{p,q} = \bigoplus_{\text{codim } \sigma = p} \text{Ext}^q_{C[\sigma^\vee]}(I^\sigma, \mathcal{O}^\sigma)_\chi
\]

where the Ext groups on the right side are the usual Ext groups of modules. It is straightforward to calculate these, with the results

\[
\begin{align*}
\text{Ext}^0(C[\sigma^\vee], C[\sigma^\vee]) &= C[\sigma^\vee] \\
\text{Ext}^n(C[\sigma^\vee], C[\sigma^\vee]) &= 0 \text{ for } n > 0 \\
\text{Ext}^0(I, C[x, y]) &= C[x, y] \\
\text{Ext}^1(I, C[x, y]) &= C[x, y]/I \\
\text{Ext}^n(I, C[x, y]) &= 0 \text{ for } n > 1
\end{align*}
\]

Now, let us calculate the second-level terms. Recall

\[
d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}
\]

so in particular

\[
d_1 : E_1^{p,q} \to E_1^{p+1,q}
\]

and this differential is precisely the Čech differential. Thus, \( E_2^{n,0} = H^n(\text{Hom}(\mathcal{I}, \mathcal{O})) \), and \( \text{Hom}(\mathcal{I}, \mathcal{O}) = \mathcal{O} \), so we can read off

\[
\begin{align*}
E_2^{0,0} &= \begin{cases} 
C & \chi = 0 \\
0 & \text{otherwise}
\end{cases} \\
E_2^{1,0} &= 0 \forall \chi \\
E_2^{2,0} &= 0 \forall \chi
\end{align*}
\]

In principle, identical methods give \( E_2^{n,1} = H^n(\text{Ext}^1(\mathcal{I}, \mathcal{O})) \). However, \( \text{Ext}^1(\mathcal{I}, \mathcal{O}) \) is not torsion-free, so we should go over the calculation in somewhat more detail.

Even for an equivariant torsion sheaf, we can still calculate sheaf cohomology (on the obvious Leray cover) as Čech cohomology of the complex

\[
0 \to \bigoplus_{\text{codim } \sigma = 0} E^\sigma(\chi) \to \bigoplus_{\text{codim } \sigma = 1} E^\sigma(\chi) \to E \to 0
\]
where $E^\sigma$ is the module defining the sheaf $\text{Ext}^1(I, O)$ over the open toric neighborhood associated to cone $\sigma$, namely

$$E^\sigma = \text{Ext}^1(I^\sigma, O^\sigma)$$

Over the open torus orbit, the equivariant sheaf $\text{Ext}^1(I, O)$ is a trivial rank 0 vector bundle, so $E = 0$. Also, from results above, $E^\sigma = 0$ for $\sigma$ a cone of codimension 1. Thus, the cohomology of this complex is trivial to compute, and we find

$$E^{0,1}_2 = \mathbb{C}[x, y]/I \cong \begin{cases} \mathbb{C} & \chi = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E^{1,1}_2 = 0 \forall \chi$$

$$E^{2,1}_2 = 0 \forall \chi$$

Now, it is easy to check that

$$E^{0,0}_\infty = E^{0,0}_2$$
$$E^{1,0}_\infty = E^{1,0}_2$$
$$E^{0,1}_\infty = E^{0,1}_2$$

so as a result we have

$$\text{Ext}^0(I, O)_\chi = E^{0,0}_2 = \begin{cases} \mathbb{C} & \chi = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Ext}^1(I, O)_\chi = E^{1,0}_2 \oplus E^{0,1}_2 = \begin{cases} \mathbb{C} & \chi = 0 \\ 0 & \text{otherwise} \end{cases}$$

which agrees precisely with the known (in fact, standard) result.

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