The partial derivative of Okamoto’s functions with respect to the parameter

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Abstract

The differentiability of the one parameter family of Okamoto’s functions as functions of $x$ has been analyzed extensively since their introduction in 2005. As an analogue to a similar investigation, in this paper, we consider the partial derivative of Okamoto’s functions with respect to the parameter $a$. We place a significant focus on $a = 1/3$ to describe the properties of a nowhere differentiable function $K(x)$ for which the set of points of infinite derivative produces an example of a measure zero set with Hausdorff dimension 1.

1 Introduction

In 1957, De Rham [6] studied a unique continuous solution $L_a(x)$ of the following functional equation.

$$L_a(x) = \begin{cases} aL_a(2x), & 0 \leq x \leq \frac{1}{2}; \\ (1 - a)L_a(2x - 1) + a, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

(1.1)

where $0 < a < 1, a \neq 1/2$. Also known as Lebesgue’s singular function, $L_a(x)$ is strictly increasing and has a derivative equal to zero almost everywhere. From (1.1), it is clear that $L_a$ is self-affine: The portions of the graph above the intervals $[0, 1/2]$ and $[1/2, 1]$ are affine images of the whole graph, contracted horizontally by $1/2$ and vertically by $a$ and $1 - a$, respectively.

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Another famous continuous nowhere differentiable function, namely that of Takagi \[17\] was introduced in 1903. It is defined by

\[
T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \phi(2^n x),
\]  

(1.2)

where \(\phi(x) = \text{dist}(x, \mathbb{Z})\), the distance from \(x\) to the nearest integer. Functions, like Takagi’s function, that are not sufficiently smooth or regular have often been ignored as ‘pathological’ and not worthy of study. Prior to 1984, there was no known relationship between the functional equation, \(L_a(x)\), and Takagi’s function, \(T(x)\). However, that year Hata and Yamaguti \[20\] showed the following beautiful relationship between \(L_a(x)\) and \(T(x)\):

\[
\left. \frac{\partial L_a(x)}{\partial a} \right|_{a=1/2} = 2T(x).
\]  

(1.3)

A generalization of this result of Hata and Yamaguti was considered by Sekiguchi and Shiota \[16\]. They computed the \(k\)-th partial derivative of \(L_a(x)\) with respect to the real parameter \(a\) and applied it to digital sums problems \[12\]. Also, Tasaki, Antoniou and Suchanecki pointed out applications of Hata and Yamaguti’s results in physics \[18\]. Later, Kawamura expanded Hata and Yamaguti’s results to classify all self-similar sets determined by two contractions in the plane \[9\]. Note that both Lebesgue’s singular function and Takagi’s nowhere differentiable function are expressed by binary expansions of \(x\) on the interval \([0, 1]\).
In 2005, H. Okamoto [13] introduced another one-parameter family of self-affine functions \( \{ F_a \} \), which are expressed by ternary expansions of \( x \) on the interval \([0, 1]\). He defined \( F_a \) as follows. Let \( f_0(x) = x \), and inductively, for \( n = 0, 1, 2, \ldots \), let \( f_{n+1} \) be the unique continuous function which is linear on each interval \([j/3^{n+1}, (j+1)/3^{n+1}]\) with \( j \in \mathbb{Z} \) and satisfies, for \( k = 0, 1, \ldots, 3^n - 1 \), the equations

\[
\begin{align*}
    f_{n+1}(k/3^n) &= f_n(k/3^n), \\
    f_{n+1}((k+1)/3^n) &= f_n((k+1)/3^n), \\
    f_{n+1}((3k+1)/3^{n+1}) &= f_n(k/3^n) + a \left[ f_n((k+1)/3^n) - f_n(k/3^n) \right], \\
    f_{n+1}((3k+2)/3^{n+1}) &= f_n(k/3^n) + (1-a) \left[ f_n((k+1)/3^n) - f_n(k/3^n) \right].
\end{align*}
\]

The sequence \( (f_n) \) defined above converges uniformly on \([0, 1]\). Let \( F_a := \lim_{n \to \infty} f_n \), then \( F_a \) is a continuous function from the unit interval \([0, 1]\) onto itself. By changing the parameter \( a \), one can produce some interesting examples: Perkins’ nowhere differentiable function [15] (when \( a = 5/6 \)), Bourbaki-Katsuura’s function [5, p. 35, Problem 1-2], [8] (when \( a = 2/3 \)) and Cantor’s Devil’s staircase function (when \( a = 1/2 \)). Okamoto and Wunsch [14] proved that \( F_a \) is singular when \( 0 < a < 1/2 \) and \( a \neq 1/3 \).

Let \( a_0 \approx 0.5592 \) be the unique real root of \( 54a^3 - 27a^2 = 1 \). Okamoto [13] showed that

1. \( F_a \) is nowhere differentiable if \( 2/3 \leq a < 1 \);
2. \( F_a \) is nondifferentiable at almost every \( x \in [0, 1] \) but differentiable at uncountably many points if \( a_0 < a < 2/3 \); and

Figure 2: The first two steps in the construction of \( F_a \)
3. $F_a$ is differentiable almost everywhere but nondifferentiable at uncountably many points if $0 < a < a_0$.

Okamoto left the case $a = a_0$ open, but Kobayashi \[10\] later showed, using the law of the iterated logarithm, that $F_{a_0}$ is nondifferentiable almost everywhere. Later, Allaart \[1\] investigated the set of points at which $F_a$ has an infinite derivative and showed that this set has an interesting connection with $\beta$-expansions.

The graph of $F_a$ is self-affine: The portions of the graph above the intervals $[0,1/3]$, $[1/3,2/3]$ and $[2/3,1]$ are affine images of the whole graph, contracted horizontally by $1/3$ and vertically by $a$, $|2a - 1|$ and $a$, respectively. The middle one is also reflected vertically when $a > 1/2$. In fact, $F_a(x)$ satisfies the following functional equation:

$$F_a(x) = \begin{cases} aF_a(3x), & 0 \leq x \leq 1/3, \\ (1-2a)F_a(3x-1) + a, & 1/3 \leq x \leq 2/3, \\ aF_a(3x-2) + (1-a), & 2/3 \leq x \leq 1, \end{cases}$$

\hspace{1cm} (1.4)

where $0 < a < 1$. As a result, the box-counting dimension of the graph of $F_a$ follows from Example 11.4 in \[7\]. It is 1 if $a \leq 1/2$, and $1 + \log_3(4a - 1)$ if $a > 1/2$.

Notice that $F_{1/3}(x) = x$ and $L_{1/2}(x) = x$. The main purpose of this article is to investigate the partial derivative of $F_a(x)$ as an analogy of Hata and Yamaguti’s result. Let

$$K(x) := \left. \frac{\partial F_a(x)}{\partial a} \right|_{a=1/3}.$$ 

In Section 2, we derive a simpler expression for $K(x)$:

$$K(x) = \sum_{n=0}^{\infty} \frac{1}{3^n} \Phi(3^n x),$$

\hspace{1cm} (1.5)

where

$$\Phi(x) := \begin{cases} 3x, & 0 \leq x \leq 1/3, \\ 3(1-2x), & 1/3 \leq x \leq 2/3, \\ 3(x-1), & 2/3 \leq x \leq 1. \end{cases}$$

This particular expression helps us prove that $K(x)$ is continuous but nowhere differentiable, just like Takagi’s function.
In Section 3, we give a complete description of the set of points at which \( K(x) \) has an infinite derivative in terms of the ternary expansion of \( x \). That is, for \( x \in [0,1] \), the ternary expansion of \( x \) is the sequence \( \varepsilon_1, \varepsilon_2, \ldots \) defined by
\[
x = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{3^k}, \text{ where } \varepsilon_k \in \{0, 1, 2\}
\]
for all \( k \). If \( x \) has two ternary expansions we take the one ending in all 0’s, except when \( x = 1 \), in which case we take the expansion ending in all 2’s. Then, for \( n \in \mathbb{N} \), let \( I_1(n) \) be the number of 1’s occurring in the first \( n \) ternary digits of \( x \). That is,
\[
I_1(n) := \#\{j \leq n : \varepsilon_j = 1\}.
\]
The definition of which allows us to state our main Theorem.

**Theorem 1.1.** \( K'(x) = \pm \infty \) if and only if \( x \) satisfies the following condition:
\[
n - 3I_1(n) \rightarrow \pm \infty \quad \text{as } n \rightarrow \infty.
\]
(1.6)
The infinite derivatives of Takagi’s function were first discussed by Begle and Ayres [3] in 1936, and were finally characterized fully by Allaart and Kawamura [2] and Krüppel [11] in 2010. Compared with this result, our characterization of the infinite derivatives of \( K(x) \) is surprisingly simple. As a Corollary of this Theorem, we also obtain yet another example of a measure zero set with Hausdorff dimension 1.

## 2 Nowhere differentiability of \( K(x) \)

Kobayashi mentioned that \( F_a(x) \) has the following representation
\[
F_a(x) = \sum_{n=1}^{\infty} \prod_{i=1}^{n-1} p(\varepsilon_i(x))q(\varepsilon_n(x)),
\]
(2.1)
where
\[
p(0) = a, \quad p(1) = 1 - 2a, \quad p(2) = a,
\]
\[
q(0) = 0, \quad q(1) = a, \quad q(2) = 1 - a.
\]
Although Kobayashi did not give a proof, it is easy to see that (2.1) satisfies the functional equation (1.4). Therefore, it is clear that \( F_a(x) \) is an analytic function with respect to \( a \in (0,1) \).
Define the partial derivative of \( F_a(x) \) with respect to \( a \) as follows:

\[
\frac{\partial F_a(x)}{\partial a} := \lim_{h \to 0} \frac{F_{a+h}(x) - F_a(x)}{h},
\]

provided the limit exists. A simple calculation gives

\[
\frac{\partial F_a(x)}{\partial a} = s(\varepsilon_1(x)) + \sum_{n=1}^{\infty} a^{n-I_1(1,n)}(1-2a)^{I_1(1,n)}q(\varepsilon_{n+1}(x)),
\]

where \( s(0) = 0, s(1) = 1 \) and \( s(2) = -1 \). In particular, we have

\[
K(x) = \frac{\partial F_a(x)}{\partial a} \bigg|_{a=1/3} = \sum_{n=0}^{\infty} \frac{1}{3^n} \left\{ s(\varepsilon_{n+1}(x)) + (n - 3I_1(1,n))\varepsilon_{n+1}(x) \right\}.
\]

The graph of \( K(x) \) is shown in Figure 3. Note that \( F_{1/3}(x) = x \).

Another expression of \( K(x) \) can be derived from (1.4). It is easy to check that \( K(x) \) satisfies the following functional equation.

\[
K(x) = \begin{cases} 
K(3x)/3 + 3x, & 0 \leq x \leq 1/3, \\
K(3x-1)/3 + 3(1-2x), & 1/3 \leq x \leq 2/3, \\
K(3x-2)/3 + 3(x-1), & 2/3 \leq x \leq 1,
\end{cases}
\]

where \( K(0) = 0 = K(1) \). Here, we recall the following theorem of Yamaguti and Hata [19]:

\[\text{(2.2)}\]
Theorem 2.1 (Yamaguti-Hata, 1983). Consider the functional equation:

\[ F(t, x) - tF(t, \psi(x)) = g(x), \quad (2.4) \]

where \( \psi(x) \) is a given one-dimensional dynamical system on \([0,1] \to [0,1]\) and \( g(x) \) is a bounded measurable function on \([0,1]\). Then, there exist a unique bounded solution of (2.4) such that

\[ F(t, x) = \sum_{n=0}^{\infty} t^n g(\psi^n(x)), \quad (2.5) \]

where \( \psi^n(x) \) is the \(n\)-fold iteration of \( \psi \).

Notice that the functional equation (2.3) is a special case of (2.4). More precisely, let \( t = 1/3 \), and let

\[ \psi(x) := \begin{cases} 
3x, & 0 \leq x \leq 1/3, \\
3x - 1, & 1/3 \leq x \leq 2/3, \\
3x - 2, & 2/3 \leq x \leq 1, 
\end{cases} \]

and

\[ g(x) := \Phi(x) := \begin{cases} 
3x, & 0 \leq x \leq 1/3, \\
3(1 - 2x), & 1/3 \leq x \leq 2/3, \\
3(x - 1), & 2/3 \leq x \leq 1. 
\end{cases} \]

Then Theorem 2.1 gives another expression for \( K(x) \):

\[ K(x) = \sum_{n=0}^{\infty} \frac{1}{3^n} \Phi(3^n x), \]

where we extend \( \Phi \) to all of \( \mathbb{R} \) by \( \Phi(x+1) = \Phi(x) \). Notice that this expression is different from (2.2).

Proposition 2.2. \( K(x) \) is an odd function.

Proof. Notice that \( \Phi(x) = -\Phi(1 - x) = \Phi(x + 1) \). Thus,

\[
K(x) = \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n \Phi(3^n x) = -\sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n \Phi(1 - 3^n x) \\
= -\sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n \Phi(3^n (1 - x)) = -K(1 - x).
\]

\[\square\]
Remark 2.3. A unique bounded solution of another special case of (2.5) is Takagi’s function. Comparing (1.2) with (1.5), $K(x)$ is somewhat similar to Takagi’s function $T(x)$. Notice that $T(x)$ is based on the binary expansion of $x$ while $K(x)$ is based on the ternary expansion of $x$. Furthermore, $T(x)$ is even, but $K(x)$ is an odd function.

Theorem 2.4. The function $K$ is continuous, but it does not possess a finite derivative at any point.

Proof. First, define

$$K_n(x) := \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k \Phi(3^k x).$$

For each $n \in \mathbb{N}$, it is clear that $K_n$ is continuous on $[0, 1]$ and the sequence $(K_n)$ converges uniformly to $K$ since $|\Phi(3^k x)| \leq 1$. Thus, $K$ is continuous.

Next, we modify the nowhere differentiability proof for Takagi’s function by Billingsley [4]. Put $\Phi_k(x) := 3^{-k} \Phi(3^k x)$ for $k = 0, 1, \ldots$. Fix a point $x$, and, for each $n \in \mathbb{N}$, let $u_n$ and $v_n$ be ternary rationals of order $n$ with $v_n - u_n = 3^{-n}$ and $u_n \leq x < v_n$. Then

$$\frac{K(v_n) - K(u_n)}{v_n - u_n} = \sum_{k=0}^{n-1} \frac{\Phi_k(v_n) - \Phi_k(u_n)}{v_n - u_n},$$

since $\Phi_k(u_n) = \Phi_k(v_n) = 0$ for all $k \geq n$. But for $k < n$, $\Phi_k$ is linear on $[u_n, v_n]$ with slope $\Phi_k^+(x)$, the right-hand derivative of $\Phi_k$ at $x$. Thus,

$$\frac{K(v_n) - K(u_n)}{v_n - u_n} = \sum_{k=0}^{n-1} \Phi_k^+(x).$$

Since $\Phi_k^+(x) = 3$ or $-6$ for each $k$, this last sum cannot converge to a finite limit. Hence, $K$ does not have a finite derivative at $x$. \qed

3 Improper infinite derivatives of $K(x)$

Since $K(x)$ has no finite derivative at any point, it is natural to ask at which points $x \in [0, 1]$ does $K(x)$ have an infinite derivative. These points are completely characterized by Theorem 1.1. In order to prove our main
Theorem, we use (1.5) as an expression for $K(x)$, and we define the right-hand and left-hand derivative of $K(x)$ by

$$K'_+(x) := \lim_{h \to 0^+} \frac{K(x + h) - K(x)}{h},$$

$$K'_-(x) := \lim_{h \to 0^-} \frac{K(x + h) - K(x)}{h},$$

provided the limits exist.

**Proof of Theorem 1.1.** First we consider the right-hand derivative of $K(x)$. Let $x$ and $h$ be real numbers such that $0 \leq x < x + h < 1$ and write

$$x = \sum_{k=1}^{\infty} \varepsilon_k/3^k, \quad x + h = \sum_{k=1}^{\infty} \varepsilon'_k/3^k,$$

where $\varepsilon_k, \varepsilon'_k \in \{0, 1, 2\}$. As noted earlier, when $x$ is a triadic rational, there are two ternary expansions, but we choose the one which is eventually all zeros, except when $x = 1$, in which case we take the expansion ending in all 2’s. We adopt the same convention for $x + h$.

Let $p := p(h) \in \mathbb{N}$ such that $3^{-p} \leq h < 3^{-p+1}$. In other words, $p$ is the position of the first nonzero ternary digit of $h$. Let

$$k_0 := \max\{k \in \mathbb{N} : \varepsilon_1 = \varepsilon'_1, \varepsilon_2 = \varepsilon'_2, \ldots \varepsilon_k = \varepsilon'_k\}.$$

Clearly, $0 \leq k_0 \leq p - 1$. Observe that by the assumption for the expression of $x$, $k_0 \to \infty$ as $h \to 0^+$. Set

$$D_n(x, h) := \frac{\Phi(3^n(x + h)) - \Phi(3^n x)}{3^n h}.$$

Notice that

$$3^n x = \sum_{k=1}^{\infty} \varepsilon_{n+k}/3^k, \quad 3^n(x + h) = \sum_{k=1}^{\infty} \varepsilon'_{n+k}/3^k,$$

and $D_n(x, h)$ is the slope of the secant line between the two points $(3^n x, \Phi(3^n x))$ and $(3^n(x + h), \Phi(3^n(x + h)))$.

Then there are three cases to consider: (i) $k_0 \leq p - 3$, (ii) $k_0 = p - 2$ and (iii) $k_0 = p - 1$.

(i) Assume first that $k_0 \leq p - 3$. Then

$$\frac{K(x + h) - K(x)}{h} = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4,$$
where

\[ \Sigma_1 = \sum_{n=0}^{k_0-1} D_n(x, h), \quad \Sigma_2 := D_{k_0}(x, h), \]
\[ \Sigma_3 := \sum_{n=k_0+1}^{p-2} D_n(x, h), \quad \Sigma_4 := \sum_{n=p-1}^{\infty} D_n(x, h). \]

- For \( \Sigma_1 \), since \( \varepsilon_n = \varepsilon'_n \) for \( 1 \leq n \leq k_0 \),
  \[ D_n(x, h) = 3(-2)^{\varepsilon_{n+1} \mod 2} \] for \( 0 \leq n \leq k_0 - 1 \).

Define

\[ I_i(a, b) := \#\{j : a \leq j \leq b, \varepsilon_j = i\}, \]
\[ f(a, b) := f(x)(a, b) = 3I_0(a, b) - 6I_1(a, b) + 3I_2(a, b). \]

Then we have

\[ \Sigma_1 = \sum_{n=0}^{k_0-1} D_n(x, h) = \sum_{n=1}^{k_0} 3(-2)^{\varepsilon_n \mod 2} = f(1, k_0). \]

- For \( \Sigma_2 := D_{k_0}(x, h) \), since \( \varepsilon_{k_0+1} \neq \varepsilon'_{k_0+1} \) and \( D_{k_0}(x, h) \) is the slope of the secant line between the two points \((3^{k_0}x, \Phi(3^{k_0}x))\) and \((3^{k_0}(x+h), \Phi(3^{k_0}(x+h)))\), we have
  \[ -6 \leq \Sigma_2 \leq 3. \]

- For \( \Sigma_3 \), the assumption \( k_0 \leq p - 3 \) implies that
  \[ \varepsilon_n = 2 \quad \text{and} \quad \varepsilon'_n = 0 \quad \text{for} \quad k_0 + 2 \leq n \leq p - 1. \]

For each \( n \) with \( k_0 + 2 \leq n \leq p - 1 \), there is an integer \( m_n \) such that
\[ 3^n x \in [m_n - 1/3, m_n) \quad \text{and} \quad 3^n(x+h) \in [m_n, m_n + 1/3). \]

Since \( \Phi(x + 1) = \Phi(x) \) and the \( \Phi \) has the same slope of 3 on both \([0, 1/3]\) and \([2/3, 1]\), we have

\[ \Sigma_3 = \sum_{n=k_0+1}^{p-2} D_n(x, h) = \sum_{n=k_0+2}^{p-1} 3 = 3(p-k_0-2) = f(k_0+2, p-1). \]
• For $\Sigma_4$, since $3^{-p} \leq h < 3^{-p+1}$ and $-1 \leq \Phi(x) \leq 1$, we may bound $\Sigma_4$ with a geometric series:

$$|\Sigma_4| \leq (2)3^p \sum_{n=p-1}^{\infty} \frac{1}{3^n} = 9.$$ 

Therefore, if $k_0 \leq p - 3$, we have

$$f(1, k_0) + f(k_0 + 2, p - 1) - 15 \leq \frac{K(x + h) - K(x)}{h} \leq f(1, k_0) + f(k_0 + 2, p - 1) + 12.$$ 

Since $-6 \leq f(k_0 + 1, k_0 + 1) \leq 3$,

$$f(1, p - 1) - 18 \leq \frac{K(x + h) - K(x)}{h} \leq f(1, p - 1) + 18.$$ 

The other cases are much simpler.

(ii) If $k_0 = p - 2$, notice that we may split them the sum in nearly the same way as in case (i), except we exclude $\Sigma_3$. So,

$$f(1, p - 1) - 18 \leq \frac{K(x + h) - K(x)}{h} \leq f(1, p - 1) + 18.$$ 

(iii) If $k_0 = p - 1$, we exclude $\Sigma_2$ and $\Sigma_3$. So,

$$f(1, p - 1) - 9 \leq \frac{K(x + h) - K(x)}{h} \leq f(1, p - 1) + 9.$$ 

This completes the preliminary analysis of the cases (i), (ii), and (iii). In any case,

$$f(1, p - 1) - 18 \leq \frac{K(x + h) - K(x)}{h} \leq f(1, p - 1) + 18.$$ 

Next notice that as $h \to 0^+$, $p \to \infty$. Therefore, it follows that

$$K'_+(x) = \pm \infty \text{ iff } f(1, n) \to \pm \infty \text{ as } n \to \infty.$$ 

From Proposition 2.2, we have

$$K'_-(x) = K'_+(1 - x).$$ 

Notice that $f_x(1, n) \to \pm \infty \iff f_{1-x}(1, n) \to \pm \infty \text{ as } n \to \infty$. Therefore, the left-side derivative follows

$$K'_-(x) = \pm \infty \text{ iff } f(1, n) \to \pm \infty \text{ as } n \to \infty.$$ 

This completes the proof. \qed
Define
\[ p_1(x) := \lim_{n \to \infty} \frac{I_1(n)}{n}, \]
assuming the limit exists, where \( I_1(n) = \# \{ j \leq n : \varepsilon_j = 1 \} \). Note that \( p_1(x) \) is the frequency of the digit 1 in the ternary expansion of \( x \). We obtain the following immediate consequence of the main theorem.

**Corollary 3.1.** Assume \( p_1(x) \) exists. Then
\[
\begin{cases} 
K'(x) = +\infty, & \text{if } p_1(x) < 1/3, \\
K'(x) = -\infty, & \text{if } p_1(x) > 1/3.
\end{cases}
\]

It is well-known that almost all numbers (in the sense of Lebesgue measure) are normal to all bases; that is, they have base-\( m \) expansions containing equal proportions of the digits 0, 1, \ldots, \( m-1 \) for all \( m \). Notice that \( p_1(x) = 1/3 \) is the boundary case in Corollary 3.1.

**Corollary 3.2.** Let \( F := \{ x \in [0,1] : K'(x) = \pm \infty \} \). The Lebesgue measure of \( F \) is 0 while the Hausdorff dimension of \( F \) is 1.

**Proof.** Let \( I_1(n; x) := \# \{ j \leq n : \varepsilon_j(x) = 1 \} \). We may view \( I_1(n; x) \) as a random variable on the probability space \([0,1]\) equipped with Borel sets and Lebesgue measure. It is easy to see that the random sequence \((n - 3I_1(n; x))_{n \in \mathbb{N}}\) is a mean zero random walk, so it oscillates infinitely often between positive and negative values with probability one. Thus, by Theorem 1.1, the Lebesgue measure of \( F \) is 0.

Let \( p_0, p_1, p_2 \) be real numbers of summing to 1, so that \( 0 < p_i < 1 \) and \( \sum_{i=0}^2 p_i = 1 \). Let \( F(p_0, p_1, p_2) \) be the set of numbers \( x \in [0,1] \) with ternary expansions containing the digits 0, 1, 2 in proportions \( p_0, p_1, p_2 \) respectively. Let \( (\alpha_n) \) be an increasing sequence converging to 1/3 as \( n \to \infty \). By Proposition 10.1 in Falconer’s book [7], we have
\[
\dim_H F \left( \frac{1 - \alpha_n}{2}, \alpha_n, \frac{1 - \alpha_n}{2} \right) = -\frac{(1 - \alpha_n) \ln((1 - \alpha_n)/2) - \alpha_n \ln \alpha_n}{\ln 3},
\]
which converges to 1 as \( n \to \infty \).

Since \( \bigcup_n F \left( \frac{1-\alpha_n}{2}, \alpha_n, \frac{1-\alpha_n}{2} \right) \subset F \), the Hausdorff dimension of \( F \) is 1.

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