Noether symmetry in the higher order gravity theory

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Abstract. Noether symmetry for higher order gravity theory has been explored, with the introduction of an auxiliary variable which gives the only correct quantum description of the theory, as shown in a series of earlier papers. The application of Noether theorem in higher order theory of gravity turned out to be a powerful tool to find the solution of the field equations. A few such physically reasonable solutions like power law inflation are presented.

PACS numbers: 98.80.Hw, 98.80.Bp,98.80.Jk,98.80.-k

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1. Introduction

The higher order gravity theory has important contribution in the early universe. The importance of fourth order gravity in the gravitational action was explored by several authors. Starobinsky [1] first presented a solution of the inflationary scenario without invoking phase transition in the early universe considering only geometric term in the field equations. In this direction, Hawking and Luttrell [2] shown the curvature square term in the action mimicks as a massive scalar field. Further, Starobinsky and Schmidt [3] have shown that the inflationary phase is an attractor in the neighbourhood of the solution of the fourth order gravity theory. Low energy effective action corresponding to Brane world cosmology also contains higher curvature invariant terms.

To elucidate the effect of the fourth order gravity theory in the early universe one needs more (exact) solutions of the field equations. The solutions of the classical fourth order gravity theory are very few due to non-linearity of the field equations. The field equations obtained from the action can be simplified by introducing an auxiliary variable following the prescription of Boulware et al [4].

In some recent papers [5], [6], [7] the minisuperspace quantization of fourth order gravity introducing auxiliary variable have been presented, whose canonical quantization yields Schrodinger like equation with a meaningful definition of quantum mechanical probability. Further, the extremization of the effective potential leads to the vacuum Einstein equation. The choice of auxiliary variable is the turning point in simplifying the field equations and it yields a transparent and simple quantum mechanical equation. With such a successful choice of auxiliary variable it is extremely necessary to study the solution of the classical field equations. The field equations with the auxiliary variable are quite complicated and require special attention for solution. We introduce Noether symmetry to the fourth order gravity theory in the FRW background as simplifying assumption instead of any adhoc assumption, or any equation of state for solution of the classical field equations.

we consider the Einstein-Hilbert action with a curvature squared term and a non-minimally coupled scalar field. Applying the Noether symmetry in the above action following the Noether symmetry approach of de-Rities et al [8] it is possible to extract a class of solutions. It is interesting to note that the introduction of Noether symmetry in the action of higher order gravity theory not only gives Noether symmetry but it also leads explicit time dependence of the scale factor (as well as the scalar field) as a consequence of Noether symmetry.

We now move on to extract Noether symmetry of the theory. It was earlier attempted by Capozziello [9], where he introduced the higher order term in the action as a constraint, through Lagrange multiplier. We shall try to compare the results also.
2. Classical field equations and the equations governing Noether symmetry

We consider the following action.

\[ S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G} (f(\phi)R + \frac{\beta}{6} R^2) - \frac{1}{2\pi^2} \frac{1}{2} \partial_{\mu} \phi_{\nu} \phi^{\mu} + V(\phi)) \right) \]  

(1)

In the Robertson-Walker metric

\[ ds^2 = e^{2\alpha}[-d\eta^2 + d\chi^2 + F(\chi)(d\theta^2 + \sin^2\theta d\phi^2)] \]

(2)

The action takes the form,

\[ S = \int \left[ \frac{3\pi}{2G} \{ f\alpha'' e^{2\alpha} + f(\alpha'^2 + k)e^{2\alpha} + \beta\alpha'' + \beta(\alpha^2 + k)^2 + 2\beta\alpha''(\alpha^2 + k) \} ight. \\
+ \frac{1}{2} \phi'^2 e^{2\alpha} - V(\phi)e^{4\alpha} d\eta. \] 

(3)

In the above, dash(') denotes derivative with respect to \( \eta \) and \( k = 0 \). Removing total derivative terms and choosing \( \frac{3\pi}{2G} = M \), the action can be expressed as,

\[ S = \int \left[ M \{ f(k - \alpha^2)e^{2\alpha} - f_{,\phi} \alpha' \phi' e^{2\alpha} + \beta(\alpha^2 + k)^2 + \beta\alpha'^2 \} ight. \\
+ \frac{1}{2} \phi'^2 e^{2\alpha} - V(\phi)e^{4\alpha} d\eta + \Sigma_1. \] 

(4)

Where \( \Sigma_1 = M[f\alpha'e^{2\alpha} + 2\beta(\frac{\alpha'^2}{3} + k\alpha')] \) is the surface term. Now we introduce the auxiliary variable \( Q \) as,

\[ MQ = \frac{\partial S}{\partial \phi''} = 2M\beta\alpha'', \ i.e., Q = 2\beta\alpha''. \]

(5)

Introducing the auxiliary variable in the action and writing the action in the canonical form we obtain,

\[ S = \int \left[ M \{ f(k - \alpha^2)e^{2\alpha} - f_{,\phi} \alpha' \phi' e^{2\alpha} + \beta(\alpha^2 + k)^2 + Q\alpha'' - \frac{Q^2}{4\beta} \} ight. \\
+ \frac{1}{2} \phi'^2 e^{2\alpha} - V(\phi)e^{4\alpha} d\eta + \Sigma_1. \] 

(6)

Finally removing total derivative term appearing in the auxiliary variable we obtain,

\[ S = \int \left[ M \{ f(k - \alpha^2)e^{2\alpha} - f_{,\phi} \alpha' \phi' e^{2\alpha} + \beta(\alpha^2 + k)^2 - Q\alpha' - \frac{Q^2}{4\beta} \} ight. \\
+ \frac{1}{2} \phi'^2 e^{2\alpha} - V(\phi)e^{4\alpha} d\eta + \Sigma, \] 

(7)

where, \( \Sigma = \Sigma_1 + MQ\alpha' \). It is not difficult to see that the action is canonical, since, Hessian determinant \( |\Sigma_{\alpha''\alpha''}| = -M^2 e^{2\alpha} \neq 0 \). Thus field equations are

\[ 4\beta(3\alpha^2 + k)\alpha'' - 2f(\alpha'' + \alpha'^2 + k)e^{2\alpha} - (\phi''f_{,\phi} + 2\alpha' \phi' f_{,\phi} + \phi'^2 f_{,\phi \phi})e^{2\alpha} \] \\
\[ - Q'' = \frac{1}{M}(\phi'^2 - 4V(\phi)e^{2\alpha})e^{2\alpha}. \] 

(8)

\[ Q = 2\beta\alpha'' \]

(9)

\[ f_{,\phi}(\alpha'' + \alpha'^2 + k) = \frac{1}{M}(\phi'' + 2\alpha' \phi' + V_{,\phi} e^{2\alpha}). \] 

(10)
Finally the Hamilton constraint equation is,

\[ f(\alpha'^2 + k) + f_\phi \alpha' \phi' e^{2\alpha} - \beta(\alpha'^2 + k)(3\alpha'^2 - k) + Q' \alpha' - \frac{Q^2}{4\beta} \]

\[ = \frac{1}{M} [f_{\phi} e^{2\alpha}] = \frac{1}{2} \frac{1}{M} \phi'^2 + V(\phi)e^{2\alpha} e^{2\alpha}. \quad (11) \]

In the above system the configuration space is three dimensional and its coordinates are \((\alpha, Q, \phi)\); whose tangent space is specified by the variables \((\alpha, Q, \phi, \alpha', Q', \phi')\). Hence we assume the infinitesimal generator of the Noether symmetry as

\[ X = A \frac{\partial}{\partial \alpha} + B \frac{\partial}{\partial Q} + C \frac{\partial}{\partial \phi} + A' \frac{\partial}{\partial \alpha'} + B' \frac{\partial}{\partial Q'} + C' \frac{\partial}{\partial \phi'}, \quad (12) \]

where \(A, B, C\) are functions of \(\alpha, Q, \phi\). The existence of Noether symmetry implies the existence of the vector field \(X\) such that the Lie derivative of the Lagrangian with respect to the vector field vanishes i.e.

\[ \mathcal{L}_X \mathcal{L} = 0. \quad (13) \]

The conserved quantity corresponding to the Noether symmetry is

\[ F = A \frac{\partial L}{\partial \alpha'} + B \frac{\partial L}{\partial Q'} + C \frac{\partial L}{\partial \phi'}. \quad (14) \]

equation (13) gives

\[ A[2M \{f(k - \alpha'^2) - f_{\phi} \alpha' \phi' \} e^{2\alpha} + (\phi'^2 - 4V e^{2\alpha}) e^{2\alpha}]
\]

\[ +(\frac{\partial A}{\partial \alpha} \alpha' + \frac{\partial A}{\partial Q} Q' + \frac{\partial A}{\partial \phi} \phi')M[(-2f \alpha' - f_{\phi} \phi') e^{2\alpha} + 4\beta \alpha'(\alpha'^2 + k) - Q']
\]

\[ +B(-\frac{MQ}{2\beta}) + (\frac{\partial B}{\partial \alpha} \alpha' + \frac{\partial B}{\partial Q} Q' + \frac{\partial B}{\partial \phi} \phi')(-M \alpha')
\]

\[ +C[M(f_{\phi} (k - \alpha'^2) - f_{\phi \phi} \alpha' \phi') e^{2\alpha} - V_{\phi} e^{4\alpha}]
\]

\[ +(\frac{\partial C}{\partial \alpha} \alpha' + \frac{\partial C}{\partial Q} Q' + \frac{\partial C}{\partial \phi} \phi')(-M f_{\phi} \alpha' e^{2\alpha} + \phi' e^{2\alpha}) = 0 \quad (15) \]

To satisfy equation (15) one has to satisfy the following equations: Collecting coefficients of \(\alpha'^4, Q' \alpha'^3, \) and \(\phi' \alpha'^3,\) we get

\[ A = A_0 \quad (16) \]

where, \(A_0\) is a constant. Co-efficient of \(Q' \phi'\) gives

\[ \frac{\partial C}{\partial Q} = 0, \quad (17) \]

ie., \(C\) is not a function of \(Q\). Further the co-efficients of \(\alpha' Q'\) gives,

\[ f_{\phi} \frac{\partial C}{\partial Q} + \frac{\partial B}{\partial Q} e^{-2\alpha} = 0, \quad (18) \]

ie., \(B\) also does not depend on \(Q\). Indeed it should be, since \(Q\) is an auxiliary variable only. Co-efficient of \(\phi'^2\) gives

\[ \frac{\partial C}{\partial \phi} + A = 0, \quad (19) \]
which implies
\[ C = -A_0\phi + g_1(\alpha). \] (20)
Finally, co-efficients of \( \alpha'' \), \( \phi'' \alpha' \) and \( Q'\phi' \) give,
\[ 2Af + f_{,\phi} (C + \frac{\partial C}{\partial \alpha}) + \frac{\partial B}{\partial \phi} e^{-2\alpha} = 0. \] (21)
\[ f_{,\phi} (2A + \frac{\partial C}{\partial \phi}) + C f_{,\phi\phi} - \frac{1}{M} \frac{\partial C}{\partial \alpha} + \frac{\partial B}{\partial \phi} e^{-2\alpha} = 0. \] (22)
and
\[ k(2Af + C f_{,\phi}) e^{2\alpha} - \frac{BQ}{2\beta} - \frac{1}{M}(CV_{,\phi} + 4AV) e^{4\alpha} = 0. \] (23)
The solution of \( A, B, C \) satisfying all these equations (16)-(23) yields Noether symmetry.

3. Solutions
Let us consider solution of equations (16)-(23) to find \( A, B, C \) and hence \( f(\phi), V(\phi) \) admitting Noether symmetry, further we shall consider solution of the field equations. Choosing \( B \) in the form \( B = B_1(\alpha)B_2(\phi) \), given by the separation of variables, equation (21) gives,
\[ A_0(2f - \phi f_{,\phi}) + f_{,\phi} (g_1 + \frac{dg_1}{d\alpha} + B_2 \frac{dB_1}{d\alpha} e^{-2\alpha}) = 0. \] (24)
Differentiating above equation (24) with respect to \( \phi \) we get,
\[ A_0 f_{,\phi} - A_0 \phi f_{,\phi\phi} + (g_1 + \frac{dg_1}{d\alpha}) f_{,\phi\phi} + B_2 \phi \frac{dB_1}{d\alpha} e^{-2\alpha} = 0. \] (25)
Eliminating, \( g_1 + \frac{dg_1}{d\alpha} \) between equations (23) and (24) we get,
\[ A_0 \frac{(2f - \phi f_{,\phi}) f_{,\phi} - (2f - \phi f_{,\phi}) f_{,\phi\phi}}{(f_{,\phi} B_{2,\phi} - B_2 f_{,\phi\phi})} = \frac{dB_1}{d\alpha} e^{-2\alpha} = N. \] (26)
Since, left hand side is a function of \( \phi \) and the right hand side is that of \( \alpha \), therefore both sides are equated to a constant \( N \). Hence,
\[ B_1 = \frac{N}{2} e^{2\alpha} + b_0, \] (27)
where \( b_0 \) is a constant, and
\[ 2f - \phi f_{,\phi} = N f_{,\phi} - \frac{N}{A_0} B_2, \] (28)
\( N_1 \) being yet another constant. In view of equation (28), equation (24) is,
\[ g_1 + \frac{dg_1}{d\alpha} + A_0 N_1 = 0, \] (29)
for \( f_{,\phi} \neq 0 \). Hence \( g_1 \) can be solved to find \( C \) as,
\[ C = \alpha_0 e^{-\alpha} - A_0(\phi + N_1). \] (30)
In view of which the equation (22) takes the following form,

\[
A_0[f_{,\phi} - (\phi + N_1)f_{,\phi\phi}] + \alpha_0(f_{,\phi\phi} + \frac{1}{M})e^{-\alpha} \\
+ \left(\frac{N}{2}e^{2\alpha} + b_0\right)e^{-2\alpha}B_{2,\phi} = 0. 
\]

(31)

This equation (31) is satisfied, provided \(\alpha_0 = 0\) and \(b_0\) or \(B_{2,\phi} = 0\). Note that, for \(f_{,\phi\phi} + \frac{1}{M} = 0\), \(f < 0\), which leads to negative Newton’s gravitational constant. Now, for the first choice, ie., \(\alpha_0 = b_0 = 0\), the above equation (31) reads,

\[
A_0[f_{,\phi} - (\phi + N_1)f_{,\phi\phi}] + \frac{N}{2}B_{2,\phi} = 0. 
\]

(32)

Comparing equation (32) with the relation between \(f\) and \(B_2\), given by equation (28) being differentiated with respect to \(\phi\) implies that these two equations are consistent either for \(N = 0\) or for \(B_2\) = a constant. The first choice leads to inconsistency. So finally we are left with only one option, ie., \(\alpha_0 = b_0 = 0\), \(B_2 = b_2\), a constant. For this choice equation (31) is

\[
f_{,\phi\phi}(\phi + N_1) - f_{,\phi} = 0. 
\]

(33)

Further, equation (28) gives,

\[
(\phi + N_1)f_{,\phi} = 2f + \frac{Nb_2}{2A_0}. 
\]

(34)

Equations (33) and (34) are thus consistent and yield the following solution,

\[
f = f_0(\phi + N_1)^2 - \frac{Nb_2}{2A_0},
\]

(35)

along with

\[
A = A_0; \quad B = b_2\left(\frac{N}{2}e^{2\alpha} + b_0\right); \quad C = -A_0(\phi + N_1). 
\]

(36)

In view of (35, 36) equation (23) reads,

\[
kN_b^2e^{2\alpha} + \frac{b_2}{2\beta}\left(\frac{N}{2}e^{2\alpha} + b_0\right)Q = \frac{A_0}{M}[(\phi + N_1)V_{,\phi} - 4V]e^{4\alpha}. 
\]

(37)

Now depending on values of the integration constants we consider the following different cases:

3.1. **Case 1**, \(b_0 = 0, b_2 \neq 0, (\phi + N_1)V_{,\phi} = 4V\).

Under this situation equation (37) yields,

\[
Q = -4k\beta, \text{ ie., } \alpha'' = -2k. \quad V(\phi) = V_0(\phi + N_1)^4 
\]

(38)

The scale factor \(e^{\alpha}\) can be obtained easily from equation (38) and it can be used in the Noether constant of motion (14) to find solution for \(\phi\). Equation (14) takes the form

\[
\frac{F}{A_0M} = 4\beta\alpha'(k + \alpha'^2) - Q' - \left[2f\alpha' + f_{,\phi}\phi' + \frac{b_2N}{2A_0}\alpha'ight. \\
\left.-(\phi + N_1)\alpha'f_{,\phi} + \frac{\phi + N_1}{M}\phi'\right]e^{2\alpha}.
\]

(39)
To find simple solution we choose $k = 0$, then
\[ e^\alpha = e^{g \eta}, \]  
where $g$ is a constant and integration of equation (39) yields
\[ (\phi + N_1)^2 = \phi_0^2 e^{-2g \eta} + \frac{C_1 + b_2 M Ng/2}{A_0(1 + 2M f_0)/2}, \]  
where $C_1$ is a constant and $\phi_0^2 = \frac{F/2g - 2A_0 M Ng^2}{A_0(1 + 2M f_0)/2}$. It is to be noted that the solution for $\alpha$ and $\phi$ presented here are obtained from Noether symmetry conditions and this solutions (40) and (41) satisfy the field equations (8)-(10) trivially under a simple restriction on the integration constants $C_1 = -b_2 M Ng/2$ and $V_0 = \frac{g^2(1 + 2M f_0)}{4\phi_0^2}$. This solution represents a power law inflation as the scale factor in proper time is $e^\alpha = a_0 t$.

3.2. Case 2. $b_0 = 0, b_2 \neq 0$, $(\phi + N_1)V_\phi - 4V = r = \text{constant}$

Equation (37) now takes the form
\[ Q = 2\beta \omega_0^2 e^{2\alpha} - 4k \beta, \]  
\[ V = V_0(\phi + N_1)^4 + r, \]  
where $\omega_0^2 = \frac{2A_0 r}{MN b_2}$. Now using (9) in (42) we get
\[ \alpha'^2 = \omega_0^2 e^{2\alpha} - 4k \alpha + q^2 \]  
whose integral gives
\[ e^{-\alpha} = \frac{\omega_0}{q} \sinh(q \eta) \]  
where $q$ is an integration constant. The solution (45) can be used in the Noether constant of motion to find $\phi$ and is given by
\[ (f_0 + \frac{1}{2M})(\phi + N_1)^2 = \frac{F \omega_0^2}{2M q^2 A_0}(\eta - \frac{\sinh(2q \eta)}{2q}) \]  
\[ - 2\beta \omega_0^2 \sinh(q \eta)^2 - \frac{Nb_2}{2A_0} \ln |\sinh(q \eta)| + C_2, \]  
where $C_2$ is a constant. It is important to note that the solutions (45) and (46) are obtained from the Noether symmetry only. To justify consistency of the solution above $e^\alpha$ and $\phi$ have to satisfy the field equations (8)-(10). Analytically this proof is too complicated so we leave this case.

Another simple solution of equation for $k = 0$ and $q = 0$ is
\[ e^{-\alpha} = \omega_0 \eta \]  
and as a consequence solution of $\phi$ from (14) is
\[ (f_0 + \frac{1}{2M})(\phi + N_1)^2 = C_2 - \frac{F \eta^3}{3A_0 M} - \frac{Nb_2}{2A_0 \omega_0^2} \ln \eta. \]  
the solutions (47) and (48) obtained from the Noether symmetry is not consistent with the field equations.
3.3. Case 3. $f=\text{constant}$

It is also possible to study a totally different case viz., $f = \text{constant} = 1$ (say).

Under this assumption, equation (21) gives

$$B = -A_0 e^{2\alpha} + B_2(\phi).$$

Equation (22) is,

$$MB_{2,\phi} \frac{dg_1}{d\alpha} e^{2\alpha} = N,$$

where $N$ is the separation constant. Equation (50) is solved to yield,

$$B_2 = \frac{N}{M} \phi + B_0; \quad g_1 = -\frac{N}{2} e^{-2\alpha} + C_0.$$

Equation (23) thus becomes,

$$A_0 e^{2\alpha} [2k + \frac{Q}{2\beta} + \frac{N}{2MA_0} V_{,\phi}] - \frac{N}{2M\beta} Q \phi$$
$$- \frac{B_0}{2\beta} Q + \frac{1}{M} [(A_0 \phi - C_0)V_{,\phi} - 4A_0 V] e^{4\alpha} = 0.$$

Now, the choice $Q = Q_0 e^{2\alpha}$ leads to inconsistency. The other choice is $B_0 = 0 = N$. Under this choice equation (50) gives,

$$Q = -4\beta k; \quad V = V_0 (A_0 \phi - C_0)^4.$$

together with

$$A = A_0; \quad B = -A_0 e^{2\alpha}; \quad C = C_0 - A_0 \phi.$$

The conserved current is,

$$\frac{F}{A_0 M} = 4\beta \alpha'(\alpha^2 + k) - Q' - \alpha' e^{2\alpha} + \frac{C_0}{A_0 M} \phi' e^{2\alpha} - \frac{1}{M} \phi \phi' e^{2\alpha}$$

Now from (53)

$$\alpha = -k \eta^2 + g \eta + h,$$

where $g, h$ are integration constant. This solution (56) can be used in equation (55) to find the scalar field and is given by

$$\phi^2 = \frac{F}{g A_0} e^{-2g\eta},$$

where we have assumed $k = 0, h = 0$. Further, one has to check the consistency of solutions (56) and (57) with the field equations. They are found to satisfy the field equations under restriction on the integration constants $g^2 = \frac{1}{4\beta}$, $V_0 = \frac{g^2}{4\beta^2}$, and $C_0 = 0$. This solution also leads to a power law inflation.
4. Concluding remarks

An excellent and remarkable feature of Noether symmetry has been explored in the context of higher order gravity theory. Not only the coupling parameters but also the solution of the field equations can directly be obtained by applying Noether theorem in such model. Earlier in a series of papers it has been shown that for a unique and correct quantum description of higher order gravity models, auxiliary variables should be chosen carefully in a unique manner. The technique of choosing such auxiliary variable now reveals new direction in the classical context also as Noether symmetry has been found to be a powerful tool to explore solutions to the field equations in highly nonlinear dynamics.

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