QUASISTATIC ADHESIVE CONTACT OF VISCO-ELASTIC BODIES AND ITS NUMERICAL TREATMENT FOR VERY SMALL VISCOSITY

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Abstract. An adhesive unilateral contact of elastic bodies with a small viscosity in the linear Kelvin-Voigt rheology at small strains is scrutinized. The flow-rule for debonding the adhesive is considered rate-independent and unidirectional, and inertia is neglected. The asymptotics for the viscosity approaching zero towards purely elastic material involves a certain defect-like measure recording in some sense natural additional energy dissipated in the bulk due to (vanishing) viscosity, which is demonstrated on particular 2-dimensional computational simulations based on a semi-implicit time discretisation and a spatial discretisation implemented by boundary-element method.

1. INTRODUCTION, QUASISTATIC DELAMINATION PROBLEM

Quasistatic inelastic processes on surfaces of (or interfaces between) solid elastic bodies like fracture or delamination (or debonding) of adhesive contacts have received intensive engineering and mathematical scrutiny during past decades. Often, the time scale of such processes is much faster than the external loading time scale, and such processes are then modelled as rate independent, which may bring theoretical and computational advantages. Yet, the above mentioned inelastic phenomena typically lead to sudden jumps during evolution, which is related with the attribute of nonconvexity of the governing stored energy (cf. here the non-convex term $\int_{\Gamma_C} \frac{1}{2} \mathcal{E}_C u \cdot u \, dS$ in (6d) below), and then it is not entirely clear which concept of solutions suits well for the desired specific application.

The “physically” safe way to cope with this problem is to reduce rate-independency on only such variables with respect to which the stored energy is convex, the resting ones being subjected to certain viscosity (or possibly also inertia). Here we neglect inertia from the beginning, which is addressed as a quasistatic problem; cf. [29, Sect.5] for the dynamical case. Moreover, such viscosities can be (and in most materials also are) very small, and in engineering literature are almost always completely neglected. However, although arbitrarily small, such viscosities are critically important to keep energetics valid.

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It therefore makes a sense to investigate the asymptotics towards purely elastic materials when these viscosities vanish. In the limit, we thus get some solutions of the underlying rate-independent system which, however, might (and, in specific applications, intentionally should) be different from solutions arising when viscosity are directly zero and global-energy-minimization principle is in play, cf. also Remark 2 below.

In this article, we will confine ourselves to visco-elastic bodies at small strains and we consider the viscosity in the Kelvin-Voigt rheology, which is the simplest rheology which makes the desired effect of natural prevention of the too-early delamination, cf. [28]. It should also be emphasized that our viscosity is in the bulk while the inelastic delamination itself is considered fully rate-independent, in contrast to a usual vanishing-viscosity approach as e.g. in [4, 8, 10, 15, 16, 21, 31]. A certain bulk viscosity but acting on displacement itself rather than on the strain was considered in [3].

For notational simplicity, we consider a single visco-elastic body occupying a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and the adhesive contact on a part $\Gamma_c$ of the boundary $\partial \Omega$, so that we consider $\partial \Omega = \Gamma_c \cup \Gamma_b \cup \Gamma_a \cup N$ with disjoint relatively open $\Gamma_c$, $\Gamma_b$, and $\Gamma_a$ subsets of $\partial \Omega$ and with $N$ having a zero $(d-1)$-dimensional measure. All results are, however, valid equally for delamination on boundaries inside $\Omega$, i.e. an adhesive contact between several visco-elastic bodies. For readers’ convenience, let us summarize the notation used below:

| $d$ dimension of the problem ($d = 2, 3$), $u$ displacement (defined on $\Omega$), $z$ delamination parameter (defined on $\Gamma_c$), $e(u) = \frac{1}{2} (Vu) + \frac{1}{2} Vu$ small-strain tensor, $C$ tensor of elastic moduli of the 4th-order, $\chi$ a “Kelvin-Voigt” relaxation time, $\chi C$ viscous-moduli tensor, $e = e(u_0 + u)$ $\mathbb{R}$ the matrix of elastic moduli of the adhesive. $E$ Young modulus, $\nu$ Poisson ratio, $\alpha$ fracture toughness, $\delta$ driving energy for delamination, $t$ traction stress vector (acting on $\Gamma_b \cup \Gamma_a$), $t_b$, $t_t$ normal or tangential component of $t$, $f$ bulk load (acting on $\Omega$), $g$ surface load (acting on $\Gamma_a$), $w_0$ surface displacement loading (on $\Gamma_a$). |  |

*Table 1. Summary of the basic notation used through the paper.*

We consider the standard model of a unilateral frictionless Signorini contact. The quasi-static boundary-value problem for the displacement $u$ on $\Omega$ and the so-called delamination parameter $z$ on $\Gamma_c$ valued in $[0, 1]$, representing Frémond’s concept [11] of delamination, considered in this paper is:

\[
\begin{align*}
& \text{div} \ C \epsilon + f = 0 \quad \text{with} \quad \epsilon = e(u, \dot{u}) = \chi e(\dot{u}) + e(u) \quad \text{on } \Omega, \quad (1a) \\
& u = w_0 \quad \text{on } \Gamma_b, \quad (1b) \\
& t(\epsilon) = g \quad \text{on } \Gamma_a, \quad (1c) \\
& t_b(\epsilon) + z(\mathbb{K} u - (\mathbb{K} u) \eta)n = 0, \\
& u n \geq 0, \quad t_n(\epsilon) + z(\mathbb{K} u) n \geq 0, \quad (t_b(\epsilon) + z(\mathbb{K} u) \eta)(u \eta) = 0, \quad (1d) \\
& \dot{z} \leq 0, \quad \delta \leq \alpha, \quad \dot{z}(\delta - \alpha) = 0, \quad \text{on } \Gamma_c, \\
& \delta \in \frac{1}{\chi} \mathbb{K} u + N_{\Gamma(1)}(z)
\end{align*}
\]

where we use the usual “dot-notation” for the time derivative, i.e. $(\cdot) = \frac{\partial}{\partial t}$, where the set-valued mapping $N_{\Gamma(1)} : \mathbb{R} \to \mathbb{R}$ assigns $z \in \mathbb{R}$ the normal cone $N_{\Gamma(1)}(z)$ to the convex set $[0, 1] \subset \mathbb{R}$ at $z \in \mathbb{R}$, and where the traction stress and its normal and tangential components are defined on $\Gamma_c \cup \Gamma_a$ respectively by the formulas

\[
\begin{align*}
t(\epsilon) &= (C \epsilon)[u]n, \\
t_b(\epsilon) &= n(C \epsilon)[u]n, \\
t_t(\epsilon) &= (C \epsilon)[u]n - t_n(\epsilon), \quad (1e)
\end{align*}
\]
where \( \vec{n} = \vec{n}(x) \) is the unit outward normal to \( \Gamma := \partial \Omega \). We further consider the initial-value problem for (1a-e) by prescribing the initial conditions
\[
    u(0) = u_0 \quad \text{and} \quad z(0) = z_0. \tag{1f}
\]
Of course, the loading \( f, g, \) and \( w_0 \) in (1a-c) depend on time \( t \). The parameter \( \alpha > 0 \) in (1d) is a given phenomenological number quantity (possibly as a function of \( x \in \Gamma \)) with a physical dimension \( \text{J/m}^2 \) with the meaning of a specific energy needed (and thus deposited in the newly created surface) to delaminate \( 1 \text{m}^2 \) of the surface under adhesion or, equally, the energy dissipated by this delamination process; in fact, (8) below reflects the latter interpretation. In engineering, \( \alpha \) is also called fracture toughness (or fracture energy).

As already mentioned, in conventional materials, the viscosity and thus relaxation time \( \chi > 0 \) is mostly very small in comparison with external force loading time-scale, and it is worth studying the asymptotics for \( \chi \rightarrow 0 \). Formally, the inviscid limit problem arising for \( \chi \rightarrow 0 \) is a quasistatic problem for purely elastic material which consists in replacing (1a) by
\[
    \text{div} \, C \epsilon(u) + f = 0 \quad \text{on} \ \Omega, \tag{2}
\]
and in replacing \( t(\epsilon)(u) \) by \( t(\epsilon(u)) \) in (1c) and similarly \( \tilde{t}_0(\epsilon) \) and \( \tilde{t}_u(\epsilon) \) by \( \tilde{t}_0(\epsilon(u)) \) and \( \tilde{t}_u(\epsilon(u)) \) in (1d) with \( t(\cdot), \tilde{t}_0(\cdot), \) and \( \tilde{t}_u(\cdot) \) again from (1c). This limit rate-independent problem itself, however, does not record any trace of energy dissipated by viscosity in the bulk during rupture of the delaminating surface, but there are explicit examples, cf. [29], showing that this energy is not negligible no matter how the viscosity coefficient \( \chi > 0 \) is small, which leads to a notion of Kelvin-Voigt approximable solution to this limit rate-independent problem involving a certain, so-called defect measure recording the “memory” of this dissipated energy which somehow remains even if viscosity coefficient \( \chi \) vanishes (i.e. is passed to 0).

The plan of the paper is as follows: First, in Section 2 we formulate the above initial-boundary-value problem (1) weakly and briefly present the main results about a-priori estimates and convergence for \( \chi \rightarrow 0 \) to the inviscid quasistatic rate-independent problem, leading to the above mentioned approximable solutions and defect measures, mainly taken from [29]. In Sect. 3 we perform time discretisation by a semi-implicit scheme and present some convergence results again from [29], and prove that the residual in the discrete energy balance converges to zero if the time step goes to 0. Merging Sections 2 and 3, this energy-residuum convergence serves as an important ingredient for controlling convergence of the discretisation with dependence on convergence of viscosity. This is eventually used in Section 4 where, making still a spacial discretisation by boundary-element method (BEM), we perform computational experiments both with a one-dimensional example from [29] with a known solution to tune parameters of the algorithm and eventually with a non-trivial two-dimensional example. In this last example, we (to our best knowledge historically for the first time) present numerical study of a nontrivial, spatially non-homogeneous defect measure.

2. INVISCID PROBLEM AS A VANISHING-VISCOSITY LIMIT

The weak formulation of the initial-boundary value problem (1) is a bit delicate due to the doubly-nonlinear structure of the flow rule for \( z \) on \( \Gamma \) without any compactness (i.e. without any gradient theory for \( z \)) and with both involved nonlinearities unbounded due to the constraints \( z \leq 0 \) and \( z \geq 0 \) (while the third constraint \( z \leq 1 \) is essentially redundant if the initial condition satisfies it). This would make serious difficulties in proving the existence of conventional weak solutions. Benefiting from rate-independency of the evolution rule for \( z \), we can cast a suitable definition by combining the conventional weak solution concept for \( u \) and the so-called energetic-solution concept [18, 22] of \( z \) as in [27].

Considering a fixed time horizon \( T > 0 \), we use the shorthand notation \( I = (0, T), \ I = [0, T), \ Q = I \times \Omega, \ Q_z = I \times \Omega \) with \( \Omega \) the closure of \( \Omega, \Sigma_w = I \times \Gamma_w, \) and \( \Sigma_\Gamma = I \times \Gamma_w \). We
will assume, without substantial restriction of generality of geometry of the problem, that
\[ \text{dist}(\Gamma_c, \Gamma_\delta) > 0, \quad \text{meas}_{d-1}(\Gamma_\delta) > 0, \quad \text{meas}_{d-1}(\Gamma_c) > 0. \] (3)

We first make a transformation of the problem to get time constant Dirichlet condition. To this goal, we first consider a suitable prolongation \( u_\delta \) of \( u_0 \) defined on \( Q \), i.e. \( u_\delta|_{\Sigma_0} = u_0 \). Then we shift \( u \) to \( u + u_\delta \), and rewrite (1) for such a shifted \( u \). Thanks to the first condition in (3), we can assume that \( u_\delta|_{\Sigma_0} = 0 \) so that (13) remains unchanged under this shift. The equations (11)-(13) transform in such a way that the original loading \( f, g \), and \( w_\delta \) as well as the original initial data \( u_0 \) are respectively modified as follows:
\[
\begin{align*}
\text{f replaced by } & f + \text{div} \, C \epsilon_\delta & \epsilon_\delta = \epsilon(\chi_\delta u_\delta + u_0), \quad (4a) \\
\text{g replaced by } & g + (C \epsilon_\delta)|_{\Gamma_\delta} \vec{n} & \quad (4b) \\
\text{w_\delta replaced by } & 0 & \quad (4c) \\
\text{u_0 replaced by } & u_0 - u_\delta(0) & \quad (4d)
\end{align*}
\]

We will use the standard notation \( W^{1,p}(\Omega) \) for the Sobolev space of functions having the gradient in \( L^p(\Omega; \mathbb{R}^d) \). If valued in \( \mathbb{R}^n \) with \( n \geq 2 \), we will write \( W^{1,p}(\Omega; \mathbb{R}^n) \), and furthermore we use the shorthand notation \( H^1(\Omega; \mathbb{R}^n) = W^{1,2}(\Omega; \mathbb{R}^n) \). We also use the notation of "\( \cdot \)" and "\( \langle \cdot, \cdot \rangle \)" for a scalar product of vectors and 2nd-order tensors, respectively. Let \( \text{Meas}(\bar{Q}) \equiv C(\bar{Q})' \) will denote the space of measures on the compact set \( \bar{Q} \). For a Banach space \( X, L^p(I; X) \) will denote the Bochner space of \( X \)-valued Bochner measurable functions \( u : I \rightarrow X \) with its norm \( ||u|| \) in \( L^p(I) \), here \( || \cdot || \) stands for the norm in \( X \). Furthermore, \( BV(\bar{Q}; I, X) \) will denote the space of mappings \( u : \bar{Q} \rightarrow X \) with a bounded variations, i.e. \( \sup_{0 \leq t_0 < t_1 \leq T} \sum_{i=1}^n ||u(t_i)-u(t_{i-1})|| < \infty \) where the supremum is taken over all finite partitions of the interval \([0, T]\). Also, we will use \( H^1(I; X) \) for the Sobolev space of \( X \)-valued functions with distributional derivatives in \( L^2(I; X) \). To accommodate the transformation (4) into the weak formulation, we introduce the functional \( \hat{f}(t) \in H^1(\Omega; \mathbb{R}^d)' \) by
\[
\langle \hat{f}(t), v \rangle := \int_\Omega f(t) \cdot v - C \epsilon(\chi_\delta u_\delta(t)+u_0(t)) : e(v) + \int_{\Gamma_\delta} g(t) \cdot v \, dS. \tag{5}
\]

**Definition 1.** The couple \((u_\delta, z_\delta)\) with \( u_\delta \in H^1(I; H^1(\Omega; \mathbb{R}^d)) \) and \( z_\delta \in BV(\bar{Q}; L^1(\Gamma_c)) \cap L^\infty(\Sigma_c) \) is called an energetic solution to the initial-boundary-value problem (1) under the transformation (4) if

(i) the momentum equilibrium (together with Signorini boundary conditions) in the weak form
\[
\int_Q C \epsilon(\chi_\delta u_\delta + u_0) : e(v-u_\delta) \, dx \, dt + \int_{\Sigma_c \cap \{v-u_\delta \geq 0\}} (v-u_\delta) \cdot \vec{n} \, dS \, dt \geq \int_0^T \langle \hat{f}(t), v-u_\delta \rangle \, dt \tag{6a}
\]

with \( \hat{f} \) defined in (5) holds for any \( v \in H^1(I; H^1(\Omega; \mathbb{R}^d)) \) with \( v|_{\Sigma_c} \cdot \vec{n} \geq 0 \) and \( v|_{\Sigma_0} = 0 \),

(ii) the so-called semi-stability of the delamination holds for any \( t \in [0, T] \):
\[
\| u_\delta(t, x) \|_{H^1} + z_\delta(t, x) \leq 2\alpha(x) \quad \text{or} \quad z_\delta(t, x) = 0 \quad \text{for a.a. } x \in \Gamma_c. \tag{6b}
\]

(iii) and the energy equality
\[
\mathcal{E}(t, u_\delta(t), z_\delta(t)) + \int_0^t \int_Q C \epsilon(\chi_\delta u_\delta) : e(u_\delta) \, dx \, dt + \int_{\Gamma_c} \alpha(z_\delta-\Delta x) \, dS = \mathcal{E}(0, u_0, z_0) + \int_0^t \langle \hat{f}, u_\delta \rangle \, dt \tag{6c}
\]

holds for any \( t \in [0, T] \) with \( \hat{f} \) defined again by (5), and with
\[
\mathcal{E}(t, u, z) := \left\{ \begin{array}{ll}
\frac{1}{2} \int_\Omega C \epsilon(u) : e(u) \, dx - \langle f(t), u \rangle & \text{if } u \cdot \vec{n} \geq 0, \ 0 \leq z \leq 1 \text{ on } \Gamma_c, \\
\frac{1}{2} \int_{\Gamma_c} C \epsilon(u) : e(u) \, dS + \int_\Omega C \epsilon(u) : e(u) \, dx + \int_{\Gamma_\delta} g(t) \cdot v \, dS & \text{if } u = 0 \text{ on } \Gamma_\delta, \\
\infty & \text{else},
\end{array} \right.
\tag{6d}
\]
the initial conditions \(u_0\) understood transformed as \(4d\) hold.

This definition is indeed well selective in the sense that any smooth energetic solution solves also \(1\) in the classical sense. Due to \(6c\), \(\mathcal{E}(t,u_0(t),z_0(t)) < \infty\) so that it holds \(u_t|_{\Sigma}\tilde{\gamma} \geq 0, u_{\epsilon}|_{\Sigma_0} = 0, \) and \(0 \leq z_{\epsilon} \leq 1\) if the initial conditions satisfies these constraints so that \(\mathcal{E}(0,u_0,z_0) < \infty\). Note also that \(1\) has an abstract structure of the initial-value problem for the trypical nonlinear system of two evolution inclusions:

\[
\begin{align*}
[\mathcal{R}_t]'u + \partial_u\mathcal{E}(t,u,z) &\ni 0, \quad u(0) = u_0, \\
\partial_t\mathcal{R}_t(\tilde{z}) + \partial_z\mathcal{E}(t,u,z) &\ni 0, \quad z(0) = z_0,
\end{align*}
\]

with \([\cdot]'\) denoting the (partial) Gâteaux differentials and \(\partial\) denoting the partial subdifferentials in the sense of convex analysis, with \(\mathcal{E}\) from \(6d\), and with the \(\chi\)-dependent (pseudo)potential of dissipative forces \(\mathcal{R}_t\) given by

\[
\mathcal{R}_t(\tilde{u}, \tilde{z}) = \begin{cases} \int_0^\infty \mathcal{E}(\tilde{u}) \chi(\tilde{u}) \, dx + \int_{\Gamma_c} \alpha|\tilde{z}| \, dS & \text{if } \tilde{z} \leq 0 \text{ a.e. on } \Gamma_c, \\
\text{otherwise.}
\end{cases}
\]

Also note that \(6b\) is equivalent to the integrated form of the abstract semistability \(\mathcal{E}(t,u_{\chi}(t),z_{\chi}(t)) \leq \mathcal{E}(t,u_0(t),\tilde{z}) + \mathcal{R}_t(\tilde{z} - z_{\chi}(t))\) holding for any \(\tilde{z} \geq 0\), where we wrote briefly \(\mathcal{R}_t(\tilde{z}, \tilde{z}) = \mathcal{R}_t(0, \tilde{z}) =: \mathcal{R}_t(\tilde{z})\). This means here:

\[
\forall \tilde{z} \in L^\infty(\Gamma_c), \quad 0 \leq \tilde{z} \leq z_{\chi}(t) : \int_{\Gamma_c} (z_{\chi}(t) - \tilde{z})(\mathcal{E}_0(t) - u_0(t) - 2\alpha) \, dS \leq 0. \tag{9}
\]

We will generally assume the following data qualification:

\[
\begin{align*}
C &> 0 \ (= \text{positive definiteness}), \tag{10a} \\
\mathcal{F} &\in W^{1,1}(I; H^1(\Omega; \mathbb{R}^d)), \quad u_0 \in H^1(\Omega; \mathbb{R}^d), \quad z_0 \in L^\infty(\Gamma_c), \tag{10b} \\
\|u_0\|_{H^1(\Omega; \mathbb{R}^d)} &\geq 0, \quad 0 \leq z_0 \leq 1 \text{ a.e. on } \Gamma_c, \quad \text{and} \tag{10c} \\
\|Ku_0(x) - u_0(x)\| &\leq 2\alpha \quad \text{or} \quad z_0(x) = 0 \quad \text{for a.a. } x \in \Gamma_c. \tag{10d}
\end{align*}
\]

Note that the qualification of \(\mathcal{F}\) in \(10b\) represents, in fact, assumptions on \(f, g, \) and \(w_0\) in the original problem \(1\), and that \(10d\) means semi-stability of the initial condition \((u_0, z_0)\). Under \(10\), existence of the solutions due to Definition 1 can, in fact, be proved by limiting the discrete solutions \(13\), cf. Lemma 1 and details in \(27, 29\).

**Proposition 1** (Vanishing viscosity limit, \(29\)). Let \(3\) and \(10\) hold, and let \(\chi > 0\). Then:

(i) Any solution \((u_\chi, z_\chi)\) according to Definition 1 satisfies the a-priori estimates:

\[
\begin{align*}
\|u_\chi\|_{L^1(I; H^1(\Omega; \mathbb{R}^d))} &\leq C/ \sqrt{\chi}, \tag{11a} \\
\|u_\chi\|_{L^\infty(I; H^1(\Omega; \mathbb{R}^d))} &\leq C, \tag{11b} \\
\|z_\chi\|_{L^\infty(I; \mathcal{L}(\Gamma_c))} &\leq C \tag{11c}
\end{align*}
\]

with \(C\) independent of \(\chi\).

(ii) There are \(u \in L^\infty(I; H^1(\Omega; \mathbb{R}^d)), z \in BV(\tilde{f}; L^1(\Gamma_c)), \) and \(\mu \in \mathcal{M}(\bar{Q})\), and a subsequence such that, for \(\chi \to 0\),

\[
\begin{align*}
u_\chi(t) &\to u(t) \quad \text{in } H^1(\Omega; \mathbb{R}^d) \text{ for a.a. } t \in [0, T], \tag{12a} \\
z_\chi(t) &\rightharpoonup z(t) \quad \text{in } L^\infty(\Gamma_c) \text{ for all } t \in [0, T], \tag{12b} \\
\chi|\mathcal{E}(u_\chi)\big|_{\chi} &\rightharpoonup \mu \quad \text{in } \mathcal{M}(\bar{Q}). \tag{12c}
\end{align*}
\]

(iii) Any triple \((u, z, \mu)\) obtained by this way fulfills, for a.a. \(t \in [0, T]\), the momentum equilibrium in the weak form, i.e.

\[
\int_{\Omega} \mathcal{C}(u(t)) \psi(t,u(t)) \, dx + \int_{\Gamma_c} z(t) \mathcal{K}(u(t) - v(t,u(t))) \, dS \geq \langle f(t), v - u(t) \rangle \quad \tag{13a}
\]
for all $v \in H^1(\Omega; \mathbb{R}^d)$ with $v|_{\Gamma_t}, \bar{n} \geq 0$, furthermore the semi-stability
\begin{equation}
\mathbb{K}u(t, x) \cdot u(t, x) \leq 2a(x) \quad \text{or} \quad z(t, x) = 0 \quad \text{for a.a. } x \in \Gamma_c \tag{13b}
\end{equation}
and eventually the energy equality
\begin{equation}
\mathcal{E}(t, u(t), z(t)) + \int_{\Gamma_c} \alpha(z_0 - z(t)) \, dS + \int_0^T \int_{\Omega} \mu(\text{d}x \, \text{d}t) = \mathcal{E}(0, u_0, z_0) + \int_0^T \langle f, u \rangle \, dt \tag{13c}
\end{equation}

The above assertion suggests the following:

**Definition 2.** A triple $(u, z, \mu)$ with $u \in L^\infty(I; H^1(\Omega; \mathbb{R}^d))$, $z \in BV(\tilde{f}; L^1(\Gamma_\tau))$, and $\mu \in \text{Meas}(\tilde{Q})$, $\mu \geq 0$, is called a Kelvin-Voigt-approximable solution to the quasistatic rate-independent delamination problem (1) with $\chi = 0$ transformed by (3) if (13) holds for a.a. $t \in I$, and $z(0) = z_0$, and if $(u, z, \mu)$ is attainable by a sequence of viscous solutions $((u_\xi, z_\xi))_{\xi \geq 0}$ in the sense (12).

The measure $\mu \in \text{Meas}(\tilde{Q})$, invented in [29], occurring in Proposition [11] represents a certain additional energy distributed over $\tilde{Q}$ specified rather implicitly by (12) but anyhow with a certain physical justification. Similar concept has been invented in various other problems in continuum mechanics (particularly of fluids) under the name of defect measures to reflect a possible additional energy dissipation of solutions lacking regularity and exhibiting various concentration effects in contrast to regular weak solutions where the defect measure vanishes, cf. [12, 23, 24]. Here, $\mu$ reflects the possible additional dissipated energy of Kelvin-Voigt-approximable solutions comparing to the so-called energetic solutions, cf. also Remark (2) below. A nonvanishing $\mu$ is vitally important and rather desirable in the context of fracture mechanics in contrast to the mentioned fluid-mechanical applications where the phenomenon of nonvanishing $\mu$ is related “only” to a possible lack of regularity of weak solutions and is not entirely clear whether it has some physical justification and supported experimental evidence.

### 3. TIME DISCRETISATION, CONVERGENCE

Some solutions to the initial-boundary value problem (1) in accord to Definition (1) can be obtained rather constructively by a *semi-implicit time discretisation*. To facilitate the a-priori estimates, we again consider the transformation (4). Using an equidistant partition of the time interval $[0, T]$ with a time step $\tau > 0$ such that $T/\tau \in \mathbb{N}$, we consider:

\begin{align*}
\text{div } C_\varepsilon^k + f_\tau^k &= 0 \quad \text{with } C_\varepsilon^k = \chi e \left( \frac{u_{\tau}^k - u_{\tau}^{k-1}}{\tau} \right) + e(u_{\tau}^k) \quad \text{on } \Omega, \tag{14a}
\end{align*}
\begin{align*}
u_{\tau}^k &\equiv 0 \quad \text{on } \Gamma_\tau, \tag{14b}
\end{align*}
\begin{align*}
t_0(e_{\tau}^k) + \varepsilon_{\tau}^{k-1} \left( [\varepsilon_{\tau}^k], n \right) + \mu_{\tau}^k \geq 0, \quad t_0(e_{\tau}^k) + \varepsilon_{\tau}^{k-1} \left( [\varepsilon_{\tau}^k], n \right) + \mu_{\tau}^k \geq 0, \quad (t_0(e_{\tau}^k) + \varepsilon_{\tau}^{k-1} \left( [\varepsilon_{\tau}^k], n \right) ) (n_{\tau}^k - \bar{n}) = 0, \tag{14c}
\end{align*}
\begin{align*}
\varepsilon_{\tau}^k &\leq \varepsilon_{\tau}^{k-1}, \quad \tau^e_{\tau} \leq \alpha, \quad (\varepsilon_{\tau}^k - \varepsilon_{\tau}^{k-1}) (\tau^e_{\tau} - \alpha) = 0, \quad \text{on } \Gamma_c, \tag{14d}
\end{align*}

with $t(\cdot)$, $t_0(\cdot)$, and $t_0(\cdot)$ from (16) and with $f_\tau^k = f(\tau k)$ and $g_\tau^k = g(\tau k)$ with $f$ and $g$ from (4), and proceeding recursively for $k = 1, ..., T/\tau$ with starting for $k = 1$ from
\begin{equation}
u_{\tau}^0 = u_0 \quad \text{and} \quad \varepsilon_{\tau}^0 = \tau_0. \tag{15}
\end{equation}

The adjective “semi-implicit” is related with usage of $\varepsilon_{\tau}^{k-1}$ in the first complementarity problem in (14a), instead of $e_{\tau}^k$ which would lead to a fully implicit formula. Such usage of $\varepsilon_{\tau}^{k-1}$ leads to the decoupling of the problem: first we can solve (14a-c) with the first
complementarity problem in (14d) for \( u^k_t \) and only after the rest of (14d) for \( z^k_t \); in fact, this can be understood as a fractional-step method, cf. also [28, Remark 8.25]. In addition, we can employ the variational structure of both decoupled problems. We thus obtain two convex minimization problems: first, we are to solve

\[
\begin{align*}
\text{minimize} & \quad E(k \tau, u^k_t, z^{k-1}_t) + \tau R_h\left(\frac{u^{k-1}_t - u^k_t}{\tau}, 0\right) \\
\text{subject to} & \quad u \in H^1(\Omega; \mathbb{R}^d), \ u|_{\Gamma_0} = 0, \ u|_{\Gamma_c} \bar{n} \geq 0
\end{align*}
\]

(16a)

and, denoting its unique solution by \( u^k_t \), then we solve

\[
\begin{align*}
\text{minimize} & \quad E(k \tau, u^k_t, z) + R_h(0, z - z^{k-1}_t) \\
\text{subject to} & \quad z \in L^\infty(\Gamma_c), \ 0 \leq z \leq z^{k-1}_t
\end{align*}
\]

(16b)

with the stored energy \( E \) and the dissipation (pseudo)potential \( R_h \) defined here by

\[
E(t, u, z) = \int_{\Omega} \frac{1}{2} \mathcal{C}(u):e(u)\,dx + \int_{\Gamma_c} \frac{1}{2} z^2 \chi_u \,dS - \langle f(t), u \rangle,
\]

(17a)

\[
R_h(\hat{u}, \hat{z}) = \int_{\Omega} \frac{1}{2} \mathcal{C}(\hat{u}):e(\hat{u}) \,dx - \int_{\Gamma_c} a \hat{z} \,dS.
\]

(17b)

Note that the constraints \( u|_{\Gamma_c} \bar{n} \geq 0, 0 \leq z \leq 1, \) and \( \hat{z} \leq 0, \) originally contained in \( E \) and \( R_h \) in (16) and (9), are now included in (15) so that we can equivalently use the smooth functionals \( E(t, \cdot, \cdot) \) and \( R_h \) in (17). Also note that \( R_h(\cdot, \cdot) \) is degree-1 homogeneous so that the factor \( \tau \) does not show up in the functional in (16b), in contrast to the degree-2 homogeneous functional \( R_h(\cdot, \cdot) \) in (16a).

The discrete analog of (6a) is by summation (for \( k = 1, \ldots, T/\tau \)) of the optimality conditions for (16a) written at \( u = u^k_t \), i.e.

\[
\int_{\Omega} \mathcal{C}^k e(v - u^k_t) \,dx + \int_{\Gamma_c} \frac{1}{2} \|v - u^k_t\|^2 \,dS \geq \langle f^0, v - u^k_t \rangle
\]

(18)

with \( C^k \) from (14a) and \( f^0 \) = \( f(\tau t) \), and tested by an arbitrary test-function \( v = v^k_t \). By comparison of values of (16b) at \( z^k_t \) and an arbitrary \( \hat{z} \), we get \( E(k \tau, u^k_t, z^k_t) + R_h(0, z^k_t - z^{k-1}_t) \leq E(k \tau, u^k_t, \hat{z}) + R_h(0, \hat{z} - z^{k-1}_t) \). By the degree-1 homogeneity and the convexity of \( R_h(\cdot, \cdot) \), we further get that the triangle inequality \( R_h(0, \hat{z} - z^{k-1}_t) \leq R_h(0, \hat{z} - z^{k-1}_t) + R_h(\hat{z} - z^{k-1}_t) \). Re-organizing the first estimate and merging it with the second one, we obtain the discrete analog of the semistability (5b), namely:

\[
E(k \tau, u^k_t, z^k_t) \leq E(k \tau, u^k_t, \hat{z}) + R_h(0, \hat{z} - z^{k-1}_t) \leq E(k \tau, u^k_t, \hat{z}) + R_h(0, \hat{z} - z^{k-1}_t).\]

(19)

A discrete analog of (6a) as an inequality “\( \leq \)” can be obtained by testing the optimality conditions for (16a) and (16b) respectively by \( u^k_t - u^{k-1}_t \) and \( z^k_t - z^{k-1}_t \) (which, in fact, means plugging \( v = u^{k-1}_t \) into (18) for the former test), and by adding it, benefitting from the cancellation of the terms \( \pm E(k \tau, u^k_t, z^{k-1}_t) \) and by the separate convexity of \( E(t, \cdot, \cdot) \). This gives the estimate

\[
E(k \tau, u^k_t, z^k_t) + \tau \sum_{i=1}^k R_h\left(\frac{u^{i-1}_t - u^i_t}{\tau}, \frac{z^{i-1}_t - z^i_t}{\tau}\right) \leq E(0, u_0, z_0) + \tau \sum_{i=1}^k \left\langle f^i, u^i_t - u^{i-1}_t \right\rangle.
\]

(20)

Let us by \( u_{k, \tau} \) denote the continuous piecewise affine interpolant of the values \( u^{i-1}_t \) at \( t = \tau i \), and by \( \tilde{u}_{k, \tau} \) the piecewise constant “backward” interpolant, while \( u^k_t \) the piecewise constant “forward” interpolant. Analogously, we introduce \( z_{k, \tau} \) and \( \tilde{z}_{k, \tau} \) interpolating values
Moreover, for a selected subsequence satisfying (22), the weak convergence (22a) is, in selection subsequences as in Lemma 1) that Proposition 2.

By a-priori estimates we have at disposal from (21c), using Banach’s and Helly’s selection principles, we have immediately:

**Lemma 1.** Assuming (10), \((u_{\kappa, \tau}, z_{\kappa, \tau})\) constructed recursively by (14) exists and, for \(\chi > 0\) fixed, there is a subsequence (indexed by \(\tau\)’s converging to 0) and \(u_{\kappa} \in H^1(I; H^1(\Omega; \mathbb{R}^d))\) and \(z_{\kappa} \in L^\infty(\Sigma) \cap \text{BV} (\tilde{I}; \text{Meas}(\Gamma_c))\) such that

\[
\begin{align*}
&u_{\kappa, \tau} \rightharpoonup u_{\kappa} \quad \text{in } H^1(I; H^1(\Omega; \mathbb{R}^d)), \\
&z_{\kappa, \tau} \rightharpoonup z_{\kappa} \quad \text{in } L^\infty(\Sigma) \cap \text{BV}(\tilde{I}; \text{Meas}(\Gamma_c)), \quad \text{and} \\
&z_{\kappa, \tau}(t) \rightharpoonup z_{\kappa}(t) \quad \text{in } L^\infty(\Gamma_c) \quad \text{for any } t \in [0, T].
\end{align*}
\]

Any \((u_{\kappa}, z_{\kappa})\) obtained by this way is an energetic solution to (1) due to Definition 1.

In fact, the last claim required a limit passage in (24) and then, a-posteriori, the proof of energy equality, which is rather technical and for details we refer to [27, 29].

Existence of \((\hat{\alpha}_{\kappa}, \hat{\tau}_{\kappa})\) solving (14) is simply by a direct method applied to the underlined variational problems [16]. Fixing \(\chi > 0\), we can investigate the convergence for \(\tau \to 0\).

By a-priori estimates we have at disposal from (21c), using Banach’s and Helly’s selection principles, we have immediately:

**Lemma 1.** Assuming (10), \((u_{\kappa, \tau}, z_{\kappa, \tau})\) constructed recursively by (14) exists and, for \(\chi > 0\) fixed, there is a subsequence (indexed by \(\tau\)’s converging to 0) and \(u_{\kappa} \in H^1(I; H^1(\Omega; \mathbb{R}^d))\) and \(z_{\kappa} \in L^\infty(\Sigma) \cap \text{BV}(\tilde{I}; \text{Meas}(\Gamma_c))\) such that

\[
\begin{align*}
&u_{\kappa, \tau} \rightharpoonup u_{\kappa} \quad \text{in } H^1(I; H^1(\Omega; \mathbb{R}^d)), \\
&z_{\kappa, \tau} \rightharpoonup z_{\kappa} \quad \text{in } L^\infty(\Sigma) \cap \text{BV}(\tilde{I}; \text{Meas}(\Gamma_c)), \quad \text{and} \\
&z_{\kappa, \tau}(t) \rightharpoonup z_{\kappa}(t) \quad \text{in } L^\infty(\Gamma_c) \quad \text{for any } t \in [0, T].
\end{align*}
\]

Any \((u_{\kappa}, z_{\kappa})\) obtained by this way is an energetic solution to (1) due to Definition 1.

In fact, the last claim required a limit passage in (24) and then, a-posteriori, the proof of energy equality, which is rather technical and for details we refer to [27, 29].

For further numerical study, cf. also Figures 4 and 5 below, the important feature is that the residuum \(\mathcal{E}_{\chi, \tau} \in L^2(I)\) in the discrete energy (im)balance (21c) can be controlled by making the time step \(\tau > 0\) sufficiently small:

**Proposition 2.** Assuming again (10) and \(\chi > 0\) fixed, it holds (even without any need of selection subsequences as in Lemma 1) that

\[
\lim_{\tau \to 0} \| \mathcal{E}_{\chi, \tau} \|_{L^2(I)} = 0 \quad \text{for any } 1 \leq p < \infty.
\]

Moreover, for a selected subsequence satisfying (22), the weak convergence (22a) is, in fact, strong, i.e. in particular

\[
e(\hat{u}_{\kappa, \tau}) \to e(\hat{u}_{\kappa}) \quad \text{in } L^2(Q; \mathbb{R}^{d \times d}).
\]

**Proof.** Essentially, for (23), the only important point is to prove (24). Using (21c) for \(t = T\), this can be seen from

\[
\int_Q \chi e(\hat{u}_{\kappa, \tau}) : e(\hat{u}_{\kappa, \tau}) \, dx \, dt \leq \lim \inf_{\tau \to 0} \int_Q \chi e(\hat{u}_{\kappa, \tau}) : e(\hat{u}_{\kappa, \tau}) \, dx \, dt \leq \lim \sup_{\tau \to 0} \int_Q \chi e(\hat{u}_{\kappa, \tau}) : e(\hat{u}_{\kappa, \tau}) \, dx \, dt
\]

\[
\leq \mathcal{E}(0, u_0, z_0) + \lim \sup_{\tau \to 0} \left( \int_0^T \langle \hat{T}, u_{\kappa, \tau} \rangle \, dt - \int_{\Gamma_c} \alpha(z_0 - z_{\kappa, \tau}(T)) \, dS - \mathcal{E}(T, u_{\kappa, \tau}(T), z_{\kappa, \tau}(T)) \right)
\]

\[
\leq \mathcal{E}(0, u_0, z_0) + \int_0^T \langle \hat{T}, u_{\kappa} \rangle \, dt - \int_{\Gamma_c} \alpha(z_0 - z_{\kappa}(T)) \, dS - \mathcal{E}(T, u_{\kappa}(T), z_{\kappa}(T))
\]

\[
= \int_Q \chi e(\hat{u}_{\kappa}) : e(\hat{u}_{\kappa}) \, dx \, dt.
\]
Note that, by (22a), we have at disposal \(u_{\chi}(T) \to u_k(T)\) in \(H^1(\Omega; \mathbb{R}^d)\) if \(\chi > 0\) is fixed, which we used together with (22c) for \(\tau = T\) to estimate \(\lim_{\tau \to 0} -\mathcal{E}(\tau, u_{\chi}(\tau), z_{\chi}(\tau)) \leq -\mathcal{E}(T, u_k(T), z_k(T))\) in (25). Eventually, the last equality in (25) can be proved by limiting a regularization of the Signorini condition, cf. [26, Step 4 in Sects. 8-9]. As a result, (25) proves
\[
\lim_{\tau \to 0} \int_0^T \chi C e(\dot{u}_{\chi,\tau}) : e(\dot{u}_{\chi,\tau}) \, dt dx = \int_0^T \chi C e(\dot{u}_{\chi}) : e(\dot{u}_{\chi}) \, dt dx. \tag{26}
\]
Using uniform convexity of the space \(L^2(\Omega; \mathbb{R}^{d \times d})\) equipped with the norm \(\|e\| := (\int_\Omega C e : e \, dt dx)^{1/2}\), (22a) allows for improvement of (22a) to the strong convergence
\[
u_{\chi,\tau} \to u_{\chi}
\]
in \(H^1(I; H^1(\Omega; \mathbb{R}^d))\), hence also (24) is proved.

Therefore, \(\int_0^T \int_\Omega \chi C e(\dot{u}_{\chi,\tau}) : e(\dot{u}_{\chi,\tau}) \, dt dx\) in (21c) converges to \(\int_0^T \int_\Omega \chi C e(\dot{u}_{\chi}) : e(\dot{u}_{\chi}) \, dt dx\) even uniformly in \(\tau\). From (27), we have certainly \(u_{\chi,\tau}(t) \to u_k(t)\) in \(H^1(\Omega; \mathbb{R}^d)\) for any \(t\), and using also (22b), we have \(\lim_{\tau \to 0} \mathcal{E}(t, u_{\chi,\tau}(t), z_{\chi,\tau}(t)) = \mathcal{E}(t, u_k(t), z_k(t))\) for any \(t \in [0, T]\). Thus we have the convergence in (21c) for any \(t \in [0, T]\). As the sequence \(\{\mathcal{E}_{\chi,\tau}\}_{\tau>0}\) does not alternate sign and is bounded in \(L^\infty(I)\), by Lebesgue theorem it converges to some \(\mathcal{E}_{\chi} \in L^\infty(I)\) strongly in \(L^p(I)\) for any \(1 \leq p < \infty\). Thus we showed that, in the limit for \(\tau \to 0\),
\[
\mathcal{E}(t, u_{\chi}(t), z_{\chi}(t)) + \int_0^t \int_\Omega \chi C e(\dot{u}_{\chi}) : e(\dot{u}_{\chi}) \, dx - \langle \dot{t}, u_{\chi} \rangle \, dt = \mathcal{E}(0, u_0, z_0) + \int_{\partial \chi} a(z_0 - z_{\chi}(t)) \, dS \equiv \mathcal{E}_{\chi}(t). \tag{28}
\]
We used already in (25) that the left-hand side of (25) is zero, i.e. here also \(\mathcal{E}_{\chi} = 0\), so that (23) is proved.

Eventually, we can realize that, in contrast to (22) and (24), the convergence (23) holds even for the whole sequence (indexed by a-priori chosen countable number of \(\tau's\)), which can be seen by a standard (contradiction) arguments based on uniqueness of the limit (here just 0).

It should be remarked that, however, we did not prove validity of (23) for \(p = \infty\). Anyhow, even a coarser mode of convergence (23) can ensure that the inequality (21c) yields eventually the energy equality (13c), as we will pursue in what follows.

Having the time-discrete viscous scheme, one may think about a convergence for both \(\chi \to 0\) and \(\tau \to 0\) simultaneously to obtain the Kelvin-Voigt-approximable solution to the quasistatic rate-independent problem. Here a certain circumspection has to be taken: obviously, \(\lim_{\tau \to 0} \chi C e(\dot{u}_{\chi}) : e(\dot{u}_{\chi}) = 0\) for any \(\tau > 0\) fixed, cf. also Remark 1 below; in fact, this convergence is even strong in \(W^{1,\infty}(I, L^1(\Omega))\). Therefore clearly,
\[
\lim_{\chi \to 0, \tau \to 0} \chi C e(\dot{u}_{\chi,\tau}) : e(\dot{u}_{\chi,\tau}) = 0
\]
and the energy balance (13c) would be obtained with \(\mu = 0\) as an inequality only; cf. also Figure 3 below. It is thus obvious that only some conditional convergence will lead to the desired \(\mu\) and the energy equality (13c) as before. Obviously, we must consider rather \(\lim_{\chi \to 0} \lim_{\tau \to 0}\). Linking (27) with (22c), we have
\[
w^{*}-\lim_{\tau \to 0} \chi C e(\dot{u}_{\chi,\tau}) : e(\dot{u}_{\chi,\tau}) = w^{*}-\lim_{\chi \to 0} \chi C e(\dot{u}_{\chi}) : e(\dot{u}_{\chi}) = \mu, \tag{29}
\]
meant in \(\text{Meas}(\Omega)\), and by (23) we have obviously also
\[
\lim_{\chi \to 0} \mathcal{E}_{\chi,\tau} = 0 \tag{30}
\]
meant in $L^p(I)$, $1 \leq p < \infty$. These two double limits can be merged under an implicit stability criterion $\mathcal{T} : \mathbb{R}^+ \to \mathbb{R}^+$ so that both

$$\lim_{\tau \to 0, \chi \to 0} w^* \lim_{\tau \to 0, \chi \to 0} \mathcal{C}e(\hat{u}_{\chi, \tau}) : e(\hat{u}_{\chi, \tau}) = \mu$$

and

$$\lim_{\tau \to 0, \chi \to 0} \mathcal{E}_{\chi, \tau} = 0; \quad (31)$$

cf. the arguments in the proof of [1, Cor. 4.8(ii)]. In this way, we obtain a Kelvin-Voigt approximable solution according Definition 2. However, the stability criterion $\tau \leq \mathcal{T}(\chi)$ is not explicit and thus not of a direct usage in general.

4. NUMERICAL IMPLEMENTATION AND COMPUTATIONAL EXPERIMENTS

A general observation is that (16) represents two recursive alternating linear-quadratic minimization problems which, after another spatial discretisation leads to linear-quadratic programming. On top of it, as no gradient of $z$ is involved in $\mathcal{E}$, (16b) has a local character and allows, after a suitable discretisation of $\Gamma_C$, decoupling on particular boundary elements. Therefore, conceptually the proposed scheme leads to a very efficient numerical strategy for fixed $\chi > 0$ and $\tau > 0$.

The essential difficulty is realization of the convergence (31). As the defect measure $\mu$ is typically not known (and, on top of it, is not unique), we can hardly control the former convergence in (31). Yet, we can at least control the latter one. To this goal, we devise the following conceptual algorithm relying on the convergence (23) considered with $p = 1$:

1. Set $\chi = \chi_0 > 0$ and $\tau = \tau_0 > 0$, and choose $\gamma > 0$ fixed.
2. Compute $(u_{\chi, \tau}, z_{\chi, \tau})$ and $\|\mathcal{E}_{\chi, \tau}\|_{L^1(I)}$.
3. If not $\|\mathcal{E}_{\chi, \tau}\|_{L^1(I)} \leq C \chi^\gamma$, then put $\tau := \tau/2$ and go to (2).
4. Put $\chi := \chi/2$ and $\tau := \tau/2$.
5. If not $\chi \leq \chi_{\text{final}}$, go to (2).
6. The end.

Table 2. Conceptual strategy to converge with the viscosity $\chi$ and the time step $\tau$ to approximate the correct energy balance.

In this way, we have at least the energetics in the limit under control if $\chi_{\text{final}}$ would be pushed to zero.

**Proposition 3.** The procedure from Table 2 is an algorithm in the sense that, for the parameters $\chi_0 > \chi_{\text{final}} > 0$, $\tau_0 > 0$, $C$, and $\gamma > 0$ given, it ends after a finite number of loops, giving a solution with a viscosity $\chi$ smaller than the a-priori chosen $\chi_{\text{final}}$.

**Proof.** The only notable point is that, after finite number of refinement of the time discretisation, the condition $\|\mathcal{E}_{\chi, \tau}\|_{L^1(I)} \leq C \chi^\gamma$ in Step (3) can be fulfilled. This follows from (23) and the fact, proved in Proposition 2, that this holds for the whole sequence of the time partitions.

4.1. Spatial discretisation by BEM

To launch computational experiments, one naturally needs to perform still a spatial discretisation. As $\mathcal{E}(t, \cdot, z)$ and $\mathcal{E}(\cdot, \cdot, z)$ are quadratic functionals, (16a) is a quadratic problem with the only constraint on $\Gamma_C$ which are linear. This allocation of all nonlinear effects exclusively on the boundary $\Gamma_C$ allows for using efficiently the boundary-element method (BEM) combined with linear-quadratic programming treating the variables on $\Gamma_C$.

BEM standardly uses so-called Poincaré-Steklov operators which are known in specific static cases, here in particular for the homogeneous isotropic elastic material which we
consider in what follows. Yet, we have to calculate the visco-elastic modification and here we benefit from choosing the ansatz of the tensor of viscous moduli as simply proportional to the elastic moduli, i.e. \( \chi \mathcal{C} \). Therefore we can use BEM with the same Poincaré-Steklov operators as in the static case only for a new variable \( v^\ast = u^\ast + \chi (u^\ast - u^{c-1})/\tau \); then, in terms of this new variable, one obviously has the Kelvin-Voigt strain \( \varepsilon^\ast = \varepsilon(v^\ast) \), the velocity \( (u^\ast - u^{c-1})/\tau \), and the displacement \( u^\ast = (\tau \varepsilon^\ast + \chi u^{c-1})/(\tau + \chi) \), which is to be used in (14), leading to the problem

\[
\text{div } \mathcal{C} \varepsilon(v^\ast) + \tau \dot{v}^\ast = 0 \quad \text{on } \Omega, \tag{32a}
\]

\[
v^\ast = 0 \quad \text{on } \Gamma_v, \tag{32b}
\]

\[
t(v^\ast) = \mathcal{E}^\ast \quad \text{on } \Gamma, \tag{32c}
\]

\[
\left. \begin{array}{l}
\int_{\Gamma} (\varepsilon(v^\ast) + \frac{\chi}{\tau} \frac{\varepsilon^\ast}{\tau + \chi}) : \mathcal{E}^\ast + \frac{\tau}{2} \mathcal{E}^\ast : \mathcal{E}^\ast + \frac{1}{2\tau} \mathcal{E}^\ast : \mathcal{E}^\ast - \frac{1}{2\tau} \mathcal{E}^\ast : \mathcal{E}^\ast = 0 \\
\end{array} \right\} \text{on } \Gamma_v, \tag{32d}
\]

with \( u^{c-1} = (\tau v^{c-1} + \chi u^{c-2})/(\tau + \chi) \) proceeding recursively for \( k = 1, \ldots, T/\tau \in \mathbb{N} \). Then, like (16), one constructs the corresponding minimization problems in terms of \( (v^\ast, \zeta^\ast) \). Also, evaluation of the energy balance (21c) in terms of \( v \) is possible at least approximately. More specifically, by using the Poincaré-Steklov operator for the auxiliary variable \( v^\ast \) which gives the equilibrium stress (in contrast to \( u^\ast \)), we calculate the test of the traction stress \( t(v^\ast) \) by velocity, i.e. the boundary integral

\[
\int_{\Gamma} \left( \varepsilon(v^\ast) : \left( \frac{u^\ast - u^{c-1}}{\tau} \right) \right) dS = \int_{\Omega} \mathcal{C} \varepsilon(v^\ast) : \mathcal{E}^\ast : \mathcal{E}^\ast + \frac{\tau}{2} \mathcal{E}^\ast : \mathcal{E}^\ast + \frac{1}{2\tau} \mathcal{E}^\ast : \mathcal{E}^\ast - \frac{1}{2\tau} \mathcal{E}^\ast : \mathcal{E}^\ast 
\]

where again \( \Gamma := \partial \Omega \), we obtain approximately the rate of stored energy and dissipation together, which can be used to express the overall energy balance as in (21) at least approximately by using boundary values only except the bulk contribution of \( f \) from (5), namely

\[
\int_{0}^{T} \left( \int_{\Gamma} \left( \varepsilon(v^\ast, t) \cdot u^\ast, t \right) dS - \left( \tilde{f}^\ast, u^\ast, t \right) \right) dt + \int_{\Omega} \left[ \frac{1}{2} \mathcal{C} \varepsilon(u^\ast, t) : \mathcal{E}^\ast : \mathcal{E}^\ast + \frac{\tau}{2} \mathcal{E}^\ast : \mathcal{E}^\ast + \frac{1}{2\tau} \mathcal{E}^\ast : \mathcal{E}^\ast - \frac{1}{2\tau} \mathcal{E}^\ast : \mathcal{E}^\ast \right] dV 
\]

\[
\int_{0}^{T} \left( \int_{\Gamma} \left( \varepsilon(u^\ast, t) \cdot u^\ast, t \right) dS + \left( \tilde{f}^\ast, u^\ast, t \right) \right) dt = \mathcal{C} \varepsilon(u^\ast, t) : \mathcal{E}^\ast : \mathcal{E}^\ast + \frac{\tau}{2} \mathcal{E}^\ast : \mathcal{E}^\ast + \frac{1}{2\tau} \mathcal{E}^\ast : \mathcal{E}^\ast - \frac{1}{2\tau} \mathcal{E}^\ast : \mathcal{E}^\ast 
\]

\[
\text{cf. (17a) and (21a). The coefficient } \chi + \tau/2 \text{ in (33) makes this expression only an estimate of the actual energetics (21a) which would need rather } \chi. \text{ This additional term } \frac{1}{2} \tau \mathcal{C} \varepsilon(u^\ast, t) : \mathcal{E}^\ast : \mathcal{E}^\ast \text{ will vanish if } \tau \to 0, \text{ being of the order } \mathcal{O} (\tau/\chi). \text{ Yet, it may not be entirely negligible for small } \chi, \text{ which is a certain drawback of the BEM implementation.}
\]

BEM also allows for avoiding transformation (4) of the Dirichlet condition if \( u \) (or here rather \( v \)) is considered only on \( \Gamma_v \) which is, due to (5), far from \( \Gamma_v \) and thus the partial derivative \( \partial v(t, u, \chi) \) have a good sense. In fact, the calculations presented below have been obtained by BEM implemented by a so-called collocation method.
In what follows, we use this implementation in a two-dimensional geometry and, as already mentioned, *isotropic material*. In this situation, the Poincaré-Steklov operator involved in BEM is well known; cf. [14, Sect.2.2]. As for the material, more specifically we use

\[ E = 70 \text{ GPa} \]  \hspace{1cm} \text{(Young modulus),} \tag{35a} \\
\[ \nu = \begin{cases} 
0 & \text{(in Sect. 4.2),} \\
0.35 & \text{(in Sect. 4.3).} 
\end{cases} \tag{35b} \]

thus, with \( \delta_{ij} \) standing for the Kronecker symbol, the elastic moduli tensor used in the previous sections takes the form

\[ C_{ijkl} = \frac{\nu E}{(1+\nu)(1-2\nu)} \delta_{ij}\delta_{kl} + \frac{E}{2(1+\nu)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \]

The viscosity of material describe by the relaxation time \( \chi \) will be varied and adjusted in particular cases below.

### 4.2. Computational experiments: a simple test geometry

In this section, we test the two-dimensional algorithm on a 0-dimensional example from [29] where the viscous solutions as well as the limit for \( \chi \to 0 \) are explicitly known, together with a resulting nontrivial defect measure \( \mu \). We choose a rectangular specimen glued on one side and pulled on the opposite one by gradually increasing Dirichlet load in the normal direction, cf. Figure 1. Choosing the Poisson ratio 0 makes the quasistatic problem essential 0-dimensional (i.e. the strain, stress, dissipation rates, and delamination \( z \) are spatially constant). Such sort of tests are common in building geophysical models where it is called a one-degree-of-freedom slider.

![Fig. 1. Essentially a 0-dimensional experiment (if the material is incompressible) with gradually increasing Dirichlet load and explicitly known solution.](image)

Considering still an isotropic adhesive \( K_{ij} = K \delta_{ij} \), the initial condition \( u_0 = 0 \) and \( z_0 = 1 \), and gradually increasing Dirichlet load \( w_D(t) = v_D t \), we know analytically the solution of the viscous problem as well as the Kelvin-Voigt approximable solution of the inviscid problem (which is probably even unique). More specifically, there is a time, let us denote it by \( t_{RUP,\chi} \) or (for the limit \( \chi \to 0 \)) by \( t_{RUP} \), when the spontaneous and complete rupture happens; \( t_{RUP,\chi} \) is determined only rather implicitly as a solution of the transcendental equation

\[ (a_0 t + b_\chi (1-e^{-t/	au}))^2 = 2\alpha/K \]

with the coefficients

\[ a_0 = \frac{E}{E+LK} v_D, \quad b_\chi = -\frac{LEK}{(E+LK)^2} v_D, \quad \text{and} \quad t_\chi = \frac{\chi E}{E+LK}, \tag{36} \]

and for \( \chi \to 0 \) it converges monotonically to some limit, let us denote it by \( t_{RUP} \); more specifically,

\[ t_{RUP,\chi} \to t_{RUP} = \frac{E+LK}{v_D E} \sqrt{\frac{2\alpha}{K}}. \tag{37} \]

For \( \chi > 0 \), the response is given by

\[ z_\chi(t,x) = \begin{cases} 
1 & \text{for } t < t_{RUP,\chi}, \\
0 & \text{for } t > t_{RUP,\chi}.
\end{cases} \tag{38a} \]
and the stress $\sigma_r(t, \cdot) = \sigma_r(t)$ and the viscous dissipation rate $\chi E e(\dot{u}_r) : e(\dot{u}_r)$ is constant in space (with values denoted by $\sigma_r(t)$ and $\rho_r(t)$, respectively), while the displacement $u_r(t, \cdot)$ is affine with $u_r(t, L) = w_r(t)$ and $u_r(t, 0) = w_r(t)$ with
\[
\bar{w}(t) = \begin{cases} 
\nu_0 - a_0 t - b_1 (1 - e^{-t/\tau}) & \text{for } t < t_{\text{rup}}, \vphantom{\frac{1}{2}} \\
(w_{\text{rup}} - \nu_0) e^{-\eta_{\text{rup}} t} & \text{for } t > t_{\text{rup}}, \vphantom{\frac{1}{2}} 
\end{cases}
\] (38b)

\[
\sigma_r(t) = K \frac{\mu t - w(t)}{L} = \chi E \frac{\nu_0 - w(t)}{L} + E \frac{\nu_0 t - w(t)}{L},
\] (38c)

\[
r_r(t) = \begin{cases} 
\chi E \frac{K^2 \chi^2}{E L^2} (1 - e^{-t/\tau})^2 & \text{for } t < t_{\text{rup}}, \vphantom{\frac{1}{2}} \\
1 - \chi E \frac{w_{\text{rup}} - \nu_0 t - \nu_0}{L} e^{-2(t - t_{\text{rup}})/\tau} & \text{for } t > t_{\text{rup}}, \vphantom{\frac{1}{2}} 
\end{cases}
\] (38d)

where $a_0, b_1, t_{\text{rup}}$ are from (36) and $w_{\text{rup}} := (\nu_0 - a_0) h_{\text{rup}} - b_1 (1 - e^{-t_{\text{rup}} / \tau})$. From (38d), one can see that the viscous dissipation rate $\chi E e(\dot{u}_r) : e(\dot{u}_r)$ concentrates in time when $\chi \to 0$ and, referring to (12c), the resulting defect measure $\mu$ takes the form
\[
\mu = \frac{\delta_{w_{\text{rup}}}}{\text{meas}(\Omega)} (\delta_{w_{\text{rup}}} \otimes 1) \quad \text{with} \quad \delta_{w_{\text{rup}}} = \alpha K E \] (39)

with $\delta_t \in \text{Meas}(\bar{I})$ denoting the Dirac measure supported at $t$ and $1 \in \text{Meas}(\Omega)$ is the spatial constant measure with density $1$ (i.e. the Lebesgue measure) on $\Omega$, and $\delta_{w_{\text{rup}}}$ is the energy stored in the bulk at the time of rupture $t_{\text{rup}}$ when also the driving force $\delta_t = \frac{4}{3} K w_t : w_t$ reaches the activation threshold $\alpha$; cf. (25) for details about this calculation. Although $\mu$ is known and thus we could design the strategy from Table 2 to control also the difference $\chi E e(\dot{u}_r) : e(\dot{u}_r) - \mu$ in some norm on $\text{Meas}(\bar{Q})$ which would be weakly* continuous, we intentionally do not want it because, in general (as also e.g. in Sect. 4.3 below), $\mu$ is not known.

In addition to (35), we consider $K = 150 \text{ GPa/m}$, $\alpha = 375 \text{ J/m}^2$, $\nu_0 = 267 \mu\text{m/s}$, and $T = 0, 375 \text{ s}$. The length of the specimen is $L = 0.1 \text{ m}$, as already depicted on Figure 1 while its cross-section is not important in this experiment. It is important that the implementation is able to hold the energetics with a good accuracy that can be efficiently controlled by making the time step small, as proved theoretically in Proposition 2 and shown on Figure 2 for a moderate selected viscosity $\chi$. The BEM spatial discretisation was coarse as all the quantities are either constant or affine in space in this “1-dimensional” example, so the coarseness of the spatial discretisation is irrelevant.

![Fig. 2](image-url) Illustration of the time-dependent residuum $-\xi_{w_{\text{rup}}}(\cdot)$ in the energy balance (21c) for $\tau$ gradually decreasing as depicted from up to down, while $\chi = 6.25 \times 10^{-3} \text{ s}$ is fixed. The numerical error occurs especially around sudden rupture but is shown to converge to 0 for $\tau \to 0$, as also proved in (45).

The interplay between $\chi$ and $\tau$ and its influence on the energy balance is depicted on Fig. 3 clearly showing a very slow (resp. no) convergence for small $\chi > 0$ (resp. for $\chi = 0$). The strategy from Table 2 chooses, in fact, a path decaying sufficiently slow from the left-upper corner towards the right-down corner in Fig. 3 (left):
the convergence of $L^1$-norm of $\mathcal{E}_{\chi,\tau}$ parametrized by $\chi$, documenting the theoretical result from Proposition 2 for $p = 1$.

For gradually vanishing viscosity $\chi$, Figure 4 displays respectively $w_\chi$ and $\sigma_\chi$ from (38c) and (38b) calculated numerically by a sufficiently small time step $\tau$.

Here, the defect measure $\mu$ is known from (19); now with $t_{RUP} = 0.322$ s and $\mathcal{E}_{RUP} = 803.75$ J/m$^3$; cf. (37) and (38). We can thus check the former convergence a-posteriori, which allowed us at least to tune the parameters for the algorithm from Table 2. Figure 5 left displays $r_\chi$ from (38d) calculated numerically by a sufficiently small time step $\tau$. To visualize the weak* convergence to a Dirac measure, we display rather the $L^\infty$-norm of $\mathcal{E}$ from Proposition 2 for $\chi$, cf. Figure 5 right; again realize that spatial dependence is not interesting here as all these quantities are one nonzero component.

Convergence of the viscous dissipation rate $r_\chi = \chi Ce(u_{\chi,r}) : e(u_{\chi,r})$ towards the defect measure $\mu$ from (19), i.e. here the Dirac at $t_{RUP} = 0.322$ s for $\chi = 0.025 \times 2^{-k}$ with $k = 0, 1, 2, 3$ and decreasing $\tau$ chosen according the strategy from Table 2, zoomed in and depicted on a selected time subinterval [0.3, 0.375].

Energy dissipated by viscosity over $[0, t]$, i.e. $\int_0^t \chi Ce(u_{\chi,r}) : e(u_{\chi,r}) \, dt$, converging to the jump at $t_{RUP} = 0.322$ s of the magnitude $\mathcal{E}_{RUP} = 803.75$ J. Also the convergence $t_{RUP,\chi} \neq t_{RUP}$ from (38) is well documented.
Remark 1. (Direct calculation of inviscid problem.) In principle, our semi-implicit time discretisation works for $\chi = 0$, too. Even, solving directly the inviscid problem is algorithmically much simpler. Yet, as pointed out at the end of Section 3, we cannot expect reasonable results if $\chi$ will converge to 0 too fast with respect to $\tau$, and in particular if straight $\chi = 0$ would be used. Here, we saw it already on Figure 3 where, for $\chi = 0$, the error in the energy balance practically remains constant no matter how the time discretisation refines. On Figure 5, $\chi = 0$ would cause all curves to degenerate simply to the $t$-axis, which shows fatal non-convergence of the overall viscous dissipation. Thus also the energy balance cannot hold. It is surprising that $u$, $\sigma$- and $z$-responses may still numerically converge to the correct solutions, as documented on Figure 6, which may seem to give a very efficient numerical strategy. We observed this phenomenon in all our calculations, in particular also on Figs. 10 and 14 below; cf. also the discussion in Sect. 5.

![Fig. 6.](image)

Fig. 6. A comparison of the strain (left) and stress (right) response of an energetically justified small-viscosity solution with an unphysical result without any viscosity obtained by a semi-implicit formula; strongly zoomed in and depicted on a selected short time subinterval around rupture [0.320, 0.324]: a surprisingly good match is achieved although energy does not match at all (since $\mu \equiv 0$ without viscosity), cf. also Fig. 3 for $\chi = 0$.

Remark 2. (Stress- versus energy-driven rupture.) The rupture of Kelvin-Voigt approximable solutions is essentially stress driven, while the energy-driven rupture (i.e. energy dissipated by delamination is compensated by the elastic energy got from the bulk and adhesive, being related to so-called Levitas’ maximum realizability principle and leading to so-called energetic solutions, cf. [22, 23]) occurs in general earlier, here in this simple example it would be at time $\sqrt{2\tau (L+KE)/(\nu KE)}$, as noted already in [29], i.e. already at time 0.292 s. The stress-driven rupture seems to be much more natural (especially if a large bulk would lead to extremely early delamination) and is also preferred in engineering (where mostly the existence of solution and the calculations are not analytically justified, however), cf. [17], or also the discussion about energy versus stress or global versus local minimization in mathematical literature [3, 15, 20, 30].

4.3. Computational experiments: a fully 2-D example

We now want to demonstrate applicability of the above developed methodology and algorithms to nontrivial situations where the defect measure $\mu$ is not known and typically is inhomogeneous, i.e. not distributed uniformly in space. Although we keep correct energetics via tracking numerically the latter convergence in (31), it should be emphasized that the calculations are not fully reliable because the former convergence in (31) cannot be checked. At this occasion, it should be however also emphasized that all the previous studies about defect measures have had been only purely theoretical and analytically motivated (and being related, like here, with possible lack of regularity of weak solutions of various continuum-mechanical problems, exhibiting various concentration effects in contrast to regular weak solutions where the defect measure vanishes, cf. [5, 12, 13, 9, 24]) and, except [29], existence of nontrivial defect measures had been rather only conjectured. In contrast to it, the simulations here represent, to our best knowledge, historically the very first attempt to see particular nontrivial spatially non-homogeneous defect measures.

For our computational experiment, we use a similar geometry as in Section 4.2 but, in contrast to Figure 1 with a delaminating surface $\Gamma_c$ on a different side and (in our 2nd experiment) also different direction of loading, both intentionally breaking the symmetry.
considered previously in Section 4.2. We also consider more realistic Poisson ratio, cf. (35). The speed of loading was taken the same in both experiments: \( w_0(t) = v_t \) with \( |v_t| = 333.3 \, \mu \text{m/s} \); the direction of \( v_t \) was varied: horizontal in the 1st experiment and vertical in the second experiment, cf. Figure 7. In addition to (35), we consider \( \mathbf{K} = \text{diag}(K_n, K_t) \) with \( K_n = 150 \, \text{GPa/m} \) and \( K_t = 75 \, \text{GPa/m} \), and the fracture toughness \( \alpha = 187.5 \, \text{J/m}^2 \). We use \( \chi = 0.01 \, \text{s} \) for all calculations.

Calculations with \( \tau = 5 \times 10^{-3} \text{s} \) and \( 9.33 \times 10^{-3} \text{s} \) have been performed 300 time steps, up to \( T = 1.5 \, \text{s} \) and \( T = 2.8 \, \text{s} \) for the 1st and the 2nd experiment, respectively. Such \( T \) was big enough to achieve a complete delamination of the whole contact surface.

### 4.3.1. Horizontal-loading experiment

In contrast to the example from Section 4.2, we have now the traction force on \( \Gamma_c \) nonhomogeneous, and it is interesting to see its evolution in time. This is depicted for 6 selected snapshots, starting from \( t = 0.21 \) s with an equidistant step \( 0.025 \, \text{s} \), on Fig. 8 (upper and middle rows) which shows the delamination gradually propagating on \( \Gamma_c \) from right to left. The displacement of the whole boundary has to be reconstructed by the Poincaré-Steklov operators which is a conventional procedure in BEM and is depicted on Fig. 8 (lower row).

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**Fig. 7** Geometry and boundary conditions of the 2-D problem considered.

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**Fig. 8** Upper row: distribution of the tangential traction force in the adhesive along \( \Gamma_c \). Middle row: distribution of the normal traction force in the adhesive along \( \Gamma_n \). Lower row: deformed configuration of gradually delaminating specimen under loading (1st experiment) from Fig. 7 the displacement depicted magnified 100× horizontally and 500× vertically to make the vertical deformation more visible.

---

To present spatial distribution of the defect measure, one must reconstruct the strain inside the domain \( \Omega \). This is a delicate (but anyhow executable) issue in BEM. To visualize the rate of viscous dissipation (which approximates the defect measure \( \mu \), cf. (12c), and may exhibit time oscillations which would make visualization difficult), we display rather the overall dissipation on the interval \([0, T]\), cf. Fig. 9 which approximates the total variation of the defect measure \( \int_0^T |\mu(t, x)| \, dt \) as a function of \( x \):
Fig. 9: The spatial distribution of the energy dissipated by (even very small) viscosity over the time interval $[0, t]$, i.e. $\int_0^t \chi C e (\dot{u}, \chi, \tau) \epsilon (\dot{u}, \chi, \tau) \, dt$ depicted in a gray scale at 6 selected time instances as also used on Fig. 8.

It is not surprising that the dissipated viscous energy is bigger in the right-hand part of the specimen which is particularly stretched during the delamination. Perhaps noteworthy phenomenon is that this energy is not localized along the delaminating surface; we saw this effect already in the example in Section 4.2 where it was distributed over the whole volume uniformly.

An analog of Figure 4 (right) displaying the force response $t \mapsto \int_{\Gamma} t(\epsilon(u, \dot{u}))(t, x) \, dS$ is on Figure 10 (left) together with a comparison with results obtained by the simplified inviscid algorithm from Remark 1:

Fig. 10: Vertical and horizontal components of the reaction force on the Dirichlet loading for small viscosity $\chi = 0.01$ s and energy well preserved (left), compared with the inviscid solution calculated by semi-implicit method but energy balance completely violated as in Remark 1 (right). As on Fig. 6 a surprisingly good match of this force response can be observed.

4.3.2. Vertical-loading experiment

Eventually, we briefly present most of the responses from Section 4.3.1 for another loading as indicated on Figure 7. Of course, the response is considerably different in some aspects, although the phenomena commented already for the horizontal loading are again observed. Now, the delamination propagates more slowly and we depict it with an equidistant step $0.45$ s (instead of $0.025$ s used in the 1st loading experiment) starting from $t = 0.05$ s:
The analog of Fig. 9 is on Fig. 12, showing again that the viscous energy (and also the defect measure) can be supported in the bulk far away from the delaminating surface $\Gamma_c$ and here even a tendency to surprising symmetry in spite of nonsymmetry of the boundary conditions:

Eventually, the force response corresponding to the previous Figures 11–12 is on Fig. 13(left). Like on Fig. 10, there is again a surprisingly good match if calculated by the simplified algorithm from Remark 1 as depicted Fig. 13(right), although there is no theoretical guaranty of such force-response match and obviously there is no match of energy.
5. CONCLUSION

We devised and tested a numerical strategy to approximate the natural notion of solution to delamination of purely elastic material (as usually considered in engineering applications). These solutions involve certain “defect measures” and follow the asymptotics arising from vanishing Kelvin-Voigt visco-elastic rheology which in the limit gives mere elastic material with the delamination driven naturally by stress rather than energy, as devised purely theoretically in [29] without any time discretisation.

We showed a delicate interaction between vanishing viscosity and time discretisation and difficulty to calculate physically relevant solutions. After testing the algorithm on a 0-dimensional example where exact solution is known, we calculated a couple of nontrivial 2-dimensional examples by using BEM. Beside, we also compared the results with those obtained by a simplified inviscid algorithm ignoring defect measures and thus violating the energy balance, and showed a very good match of stress-strain responses in all investigated particular cases, cf. Figs. 6, 10 and 13. Such an algorithm, called a Griffith model, was advocated already in [25] although the destruction of energy conservation was already pointed out there, and an investigation of some dynamical model leading to a correct limit in the quasistatic evolution advised, cf. [25, Sect. 3.2]. We conjecture that this simplified algorithm may converge to local solutions in the sense as introduced for a special crack problem in [31] and further generally investigated in [19], but the relation (and the phenomenon of good match) with the vanishing-viscosity approach remains still not justified.

The importance of the above presented methodology for calculation such defect measures would be pronounced in the full thermodynamical context like [6, 26], cf. also [7, Sect. 5.4], where the defect measure would naturally occur in the heat-transfer equation as a heat source and thus would influence temperature distribution inside the body and then backward the mechanics e.g. through thermal expansion or temperature dependence of mechanical properties of the adhesive. Interesting observation from Figures 9 and 12 is that the defect measure (and, the possible heat production) may occur even in spots which are quite distant from the surface undergoing inelastic dissipative process of delamination and it is certainly difficult (or rather impossible) to guess its distribution by intuition.

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