Deformation quantization of the $n$-tuple point

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Abstract

Contrary to the classical methods of quantum mechanics, the deformation quantization can be carried out on phase spaces which are not even topological manifolds. In particular, the Moyal star product gives rise to a canonical functor $F$ from the category of affine analytic spaces to the category of associative (in general, non-commutative) $\mathbb{C}$-algebras. Curiously, if $X$ is the $n$-tuple point, $x^n = 0$, then $F(X)$ is the algebra of $n \times n$ matrices.

1. Introduction. This short note, which is largely about an entertaining interpretation of the classical algebra of $n \times n$-matrices as a quantized $n$-tuple point, is almost a mathematical anecdote. This is also an attempt to understand what a quantum mechanical system may be on spaces like the “cross” $X_1 = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$, the “tick” $X_2 = \{(x, y) \in \mathbb{R}^2 \mid y^2 - x^3 = 0\}$ or the real line with one double point $X_3 = \{(x, y) \in \mathbb{R}^2 \mid xy = 0, y^2 = 0\}$ which either fail to be topological manifolds or/and have nilpotents in their structure sheaves. Contrary to the standard methods of quantum mechanics, the deformation quantization $\mathbb{P}$ (see also $\mathbb{P}$ for an up-to-date overview) easily sustains the introduction of this type of singularities and equips the (complexified) structure sheaves of the associated phase spaces with well-defined one-parameter non-commutative associative star products $\ast_{\hbar}$ which, however, depend meromorphically on the Planck constant $\hbar$. Their physical interpretation is left to the imagination of the reader.

2. The Moyal product $\ast_{\hbar}$ on affine spaces. Let $X$ be a subspace of $\mathbb{R}^n$ given by the equations

$$\phi_{\alpha}(x) = 0, \quad \alpha = 1, \ldots, k,$$

where $\phi_{\alpha}(x)$ are polynomial (or analytic or even smooth) functions on $\mathbb{R}^n$. The natural coordinates on $\mathbb{R}^n$ are denoted by $x^a$, $a = 1, \ldots, n$. We understand the affine space $X$ as a ringed space, i.e. as a pair $(X, \mathcal{O}_X)$ consisting of the subset of points (with the induced topology), $X \subset \mathbb{R}^n$, satisfying the above equations together with the structure sheaf $\mathcal{O}_X = \mathcal{O}_{\mathbb{R}^n}/J_X$, where $J_X$ is the ideal subsheaf of the sheaf $\mathcal{O}_{\mathbb{R}^n}$ of smooth functions on $\mathbb{R}^n$ generated by $\phi_{\alpha}(x)$. Remarkably, the deformation quantization on $X$ will not depend on the particular choice of generators $\phi_{\alpha}$ of $J_X$ giving therefore rise to a genuine functor on the category of affine spaces.

Let $M = \mathbb{R}^{2n}$ be the total space of the cotangent bundle to $\mathbb{R}^n$ with its canonical symplectic form $\omega = \sum_{a=1}^n dp_a \wedge dx^a$, where $p_a$ are the natural fibre coordinates. The
Moyal star product \([\mathbb{I}, \mathbb{J}]\) makes the sheaf \(\mathcal{O}_M\) of smooth functions on \(M\), or more precisely its extension \(\mathcal{O}_M[[\lambda]]\), \(\lambda\) being the formal deformation parameter, into the sheaf of non-commutative associative algebras with the product given by

\[
f \ast_\lambda g := e^{\sum_{n=1}^{\infty} \frac{\lambda}{n} \left( \frac{\partial^2}{\partial x a \partial x^a} - \frac{\partial^2}{\partial p_a \partial p_a} \right)} f(x^b, p_b) g(\bar{x}^c, \bar{p}_c) \bigg|_{x^a = \bar{x}^a \atop p_a = \bar{p}_a}.
\]

In the context of quantum mechanics the parameter \(\lambda\) is set to be \(\frac{i}{\hbar}\), \(\hbar\) being the Planck constant, and the Moyal product is denoted by \(\ast\).

With the affine subspace \((X, \mathcal{O}_X)\) of \(\mathbb{R}^n\) we associate

- two subsheaves of ideals of \((\mathcal{O}_M[[\lambda]], \ast_\lambda)\), the right ideal
  \[\mathcal{J}_r := \pi^*(J_X) \ast_\lambda \mathcal{O}_M[[\lambda]]\]
  and the left one
  \[\mathcal{J}_l := \mathcal{O}_M[[\lambda]] \ast_\lambda \pi^*(J_X),\]
  where \(\pi : M = \Omega^1 \mathbb{R}^n \to \mathbb{R}^n\) is the natural projection;
- two subsheaves of normalizers,
  \[\mathcal{N}_r := \{ f \in \mathcal{O}_M[[\lambda]] \mid f \ast_\lambda \pi^*(J_X) \subset \mathcal{J}_r \}\]
  and
  \[\mathcal{N}_l := \{ f \in \mathcal{O}_M[[\lambda]] \mid \pi^*(J_X) \ast_\lambda f \subset \mathcal{J}_l \},\]
  which are subsheaves of subrings of \((\mathcal{O}_M, \ast_\lambda)\);
- and, since \(\mathcal{J}_r \subset (\mathcal{N}_r, \ast_\lambda)\) and \(\mathcal{J}_l \subset (\mathcal{N}_l, \ast_\lambda)\) are subsheaves of two-sided ideals, the two quotient sheaves of (in general, non-commutative) associative algebras
  \[\mathcal{P}_X = \mathcal{N}_r / \mathcal{J}_r, \ast_\lambda\] and \(\mathcal{Q}_X = \mathcal{N}_l / \mathcal{J}_l, \ast_\lambda\).

The star products in the sheaves \(\mathcal{P}_X\) and \(\mathcal{Q}_X\) are naturally induced from the Moyal product and are thus denoted by the same symbol \(\ast_\lambda\). This could be a bit confusing because these new products may become singular when \(\lambda \to 0\).

Fixing the set of generators \(\phi_{\alpha}(x)\) of the ideal sheaf \(J_X\), one may equivalently define the above objects as follows

\[
\mathcal{J}_r = \{ f \in \mathcal{O}_M[[\lambda]] \mid f = \sum_{\alpha=1}^{n} \pi^*(\phi_{\alpha}) \ast_\lambda g_{\alpha} \text{ for some } g_{\alpha} \in \mathcal{O}_M[[\lambda]] \},
\]

\[
\mathcal{J}_l = \{ f \in \mathcal{O}_M[[\lambda]] \mid f = \sum_{\alpha=1}^{n} g_{\alpha} \ast_\lambda \pi^*(\phi_{\alpha}) \text{ for some } g_{\alpha} \in \mathcal{O}_M[[\lambda]] \},
\]

\[
\mathcal{N}_r = \{ f \in \mathcal{O}_M[[\lambda]] \mid f \ast_\lambda \pi^*(\phi_{\alpha}) \subset \mathcal{J}_r \text{ for all } \alpha = 1, \ldots, k \},
\]

\[
\mathcal{N}_l = \{ f \in \mathcal{O}_M[[\lambda]] \mid \pi^*(\phi_{\alpha}) \ast_\lambda f \subset \mathcal{J}_l \text{ for all } \alpha = 1, \ldots, k \},
\]

the equivalence (i.e. independence on the choice of generators) being due to the associativity of \(\ast_\lambda\) and the following elementary equality

\[
\pi^*(g(x)\phi_{\alpha}(x)) = \pi^*(g(x)) \ast_\lambda \pi^*(\phi_{\alpha}(x)) = \pi^*(\phi_{\alpha}(x)) \ast_\lambda \pi^*(g(x)), \quad \forall g(x) \in \mathcal{O}_{\mathbb{R}^n}.
\]
2.1. Lemma. The sheaves of \( \ast_\lambda \)-algebras \( P_X \) and \( Q_X \) are canonically isomorphic.

Proof. This statement follows almost immediately from an elementary observation that

\[ \pi^*(f) \ast_\lambda g = g \ast_{-\lambda} \pi^*(f) \]

for any \( f \in \mathcal{O}_{\mathbb{R}^n} \) and any \( g \in \mathcal{O}_M \). \( \square \)

The passage from \( P_X \) to \( Q_X \) (and vice versa) is essentially equivalent to the transformation \( \lambda \rightarrow -\lambda \).

Global sections of the sheaf of \( \ast_\lambda \)-algebras \( Q_X \) play the role of admissible observables for the affine space \( X \). The induced product \( \ast_\lambda \) allows us to define, at least in principle, a spectral theory of observables via the star-exponential and hence gives us the means to study quantum mechanical models on the background of a (singular) affine space \( X \). The physical interpretation of such models is very obscure — it is not even clear which observable would correspond to a “free particle” moving on \( X \)!

3. Quantization of the \( n \)-tuple point. The Moyal product on the cotangent bundle \( M = \Omega^1 \mathbb{R} \) to the real line is given explicitly by

\[ f \ast_\lambda g = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \sum_{k=0}^{i} \binom{i}{k} (-1)^k \frac{\partial^i f}{\partial x^{i-k} \partial p^k} \frac{\partial^i g}{\partial x^k \partial p^{i-k}} \]

where \( f, g \in \mathcal{O}_M \).

Let \( X \) be the \( n \)-tuple point in \( \mathbb{R} \) given by the equation

\[ x^n = 0. \]

Then the left ideal is

\[ \mathcal{J}_l = \mathcal{O}_M \ast_\lambda x^n = \{ f(x,p) \ast_\lambda x^n \text{ for some smooth function } f(x,p) \} \]

and the associated normalizer is

\[ \mathcal{N}_l = \{ f(x,p) \mid x^n \ast_\lambda f(x,p) = g(x,p) \ast_\lambda x^n \text{ for some smooth function } g(x,p) \}. \]

Any element \( h(x,p) \) of the quotient \( \mathcal{Q}_X = \mathcal{N}_l/\mathcal{J}_l \) can be uniquely represented as a sum

\[ h(x,p) = h_0 + h_1 \ast_\lambda x + \ldots + h_{n-1} \ast_\lambda x^{n-1}, \]

for some functions of one variable \( h_i = h_i(p), i = 0, 1, \ldots, n-1 \). Moreover, this sum must satisfy the equation

\[ x^n \ast_\lambda h = 0 \mod \mathcal{J}_l, \]

or equivalently

\[ \sum_{i=0}^{n-1} x^n \ast_\lambda h_i \ast_\lambda x^i = 0 \mod \mathcal{J}_l. \]
3.1. Lemma. For any smooth function \( g = g(x, p) \),

\[
x^i *_\lambda g = g *_\lambda x^i + \sum_{k=1}^{i} \binom{i}{k} (2\lambda)^k \frac{\partial^k g}{\partial p^k} *_\lambda x^{i-k}.
\]

Proof is straightforward.

3.2. Proposition. The ring of observables \( \mathcal{Q}_X \) consists of all possible sums \( \sum_{i=0}^{n} h_i *_\lambda x^i \), where the smooth functions \( h_i = h_i(p) \) are given explicitly by

\[
h_i = \sum_{k=0}^{n-1} a_{i,k} p^k + \frac{1}{(2\lambda)^k} \sum_{k=0}^{i-1} \frac{(2\lambda)^k}{(i-k)!} \left[ \sum_{j=0}^{i-k-1} (-1)^{j+1} \binom{i-k-1}{j} a_{k,n-i+k+j} p^{n+j} \right]
\]

with \( a_{i,k}, 0 \leq i, k \leq n-1 \), being arbitrary constants.

Proof. By Lemma 3.1,

\[
x^n *_\lambda h_i = \sum_{k=1}^{n} \binom{n}{k} (2\lambda)^k \frac{\partial^k h_i}{\partial p^k} *_\lambda x^{n-k} \mod \mathcal{J}_l.
\]

Hence, for any \( h = \sum_{i=0}^{n-1} h_i *_\lambda x^i \in \mathcal{Q}_X \),

\[
0 \mod \mathcal{J}_l = \sum_{i=0}^{n-1} x^n *_\lambda h_i *_\lambda x^i
\]

\[
= \sum_{i=0}^{n-1} \sum_{k=1}^{n} \binom{n}{k} (2\lambda)^k \frac{\partial^k h_i}{\partial p^k} *_\lambda x^{n-k+i}
\]

\[
= \sum_{i=0}^{n-1} \sum_{k=i+1}^{n} \binom{n}{k} (2\lambda)^k \frac{\partial^k h_i}{\partial p^k} *_\lambda x^{n-k+i}
\]

\[
= \sum_{i=0}^{n-1} \sum_{l=i}^{n-1} \binom{n}{l-i} (2\lambda)^{n-l+i} \frac{\partial^{n-l+i} h_i}{\partial p^{n-l+i}} *_\lambda x^l
\]

\[
= \sum_{l=0}^{n-1} \sum_{i=0}^{l} \binom{l}{l-i} (2\lambda)^{n-l+i} \frac{\partial^{n-l+i} h_i}{\partial p^{n-l+i}} *_\lambda x^l
\]

implying that the functions \( h_i = h_i(p) \) are solutions of the following system of differential equations

\[
\sum_{i=0}^{l} \binom{n}{i-l} (2\lambda)^{n-l+i} \frac{d^{n-l+i} h_i}{dx^{n-l+i}} = 0, \quad l = 0, 1, \ldots, n - 1.
\]

The general solution of this system can be found by induction and is precisely the one given in Proposition 3.2. \( \Box \)

If we assume that the deformation parameter \( \lambda \) takes values in a field \( \mathbb{K} \), which is either \( \mathbb{R} \) or \( \mathbb{C} \), then we have to work with functions on \( M \) with values in \( \mathbb{K} \) and hence to view the constants of integration \( a_{i,k} \) in Proposition 3.2 as elements of \( \mathbb{K} \).

3.3. Corollary. If \( X \) is the \( n \)-tuple point, then \( \dim_{\mathbb{K}} \mathcal{Q}_X = n^2 \).
3.4. Example. If $n = 2$, then a typical element $h$ of $\mathcal{Q}_X$ is of the form

$$h = a + bp + \left( c + dp - b \frac{p^2}{2\lambda} \right) \ast_\lambda x,$$

where $a, b, c$ and $d$ are constants, and the induced Moyal product is given by

$$h \ast_\lambda \tilde{h} = [a\tilde{a} + 2\lambda \tilde{c}\tilde{b}] + [ab + b\tilde{a} + 2\lambda \tilde{d}\tilde{b}]p$$

$$= \left( (ac + c\tilde{a} + 2\lambda \tilde{c}\tilde{d}) + (a\tilde{d} + d\tilde{a} + \tilde{b} - c\tilde{c} + 2\lambda d\tilde{d})p - (a\tilde{b} + \tilde{b} + 2\lambda d\tilde{b}) \frac{p^2}{2\lambda} \right) \ast_\lambda x.$$

Defining the map

$$\psi : \mathcal{Q}_X \longrightarrow \text{Mat}_\mathbb{K}(2, 2)$$

$$h \longrightarrow \left( \begin{array}{cc} a + 2\lambda d & b \\ 2\lambda c & 2\lambda a \end{array} \right)$$

one gets

$$\psi(h \ast_\lambda \tilde{h}) = \psi(h) \cdot \psi(\tilde{h})$$

where $\cdot$ stands for the usual matrix multiplication. Hence $\psi$ identifies $\mathcal{Q}_X$ together with the induced Moyal product $\ast_\lambda$ with the algebra of $2 \times 2$-matrices.

3.5. Theorem. If $X$ is the $n$-tuple point, then the quantum algebra $(\mathcal{Q}_X, \ast_\lambda)$ is canonically isomorphic, for any $\lambda \neq 0$, to the algebra of $n \times n$-matrices.

Proof. Let us define the $n$-dimensional vector space

$$V = \text{span}_\mathbb{K}(e_0 = x^{n-1}, e_1 = p \ast_\lambda x^{n-1}, \ldots, e_{n-1} = p^{n-1} \ast_\lambda x^{n-1}).$$

Since the induced Moyal product is associative, we have, for any $h \in \mathcal{Q}_X$ and any $k = 0, 1, \ldots, n - 1$,

$$h \ast_\lambda (p^k \ast_\lambda x^{n-1}) = \sum_{i=0}^{n-1} h_i \ast_\lambda (x^i \ast_\lambda p^k) \ast_\lambda x^{n-1}$$

$$= \sum_{i=0}^{n-1} \sum_{l=0}^{i} (2\lambda)^l \binom{i}{l} \left( h_i \ast_\lambda \frac{dp^k}{d^lp} \right) \ast_\lambda x^{n+i-l-1}$$

$$= \mod J_0 \sum_{i=0}^{k} (2\lambda)^i \binom{i}{l} \left( h_i \frac{dp^k}{d^lp} \right) \ast_\lambda x^{n-1}$$

$$= \mod J_0 \ g_k(p) \ast_\lambda x^{n-1},$$

where $g_k(p) = \sum_{i=0}^{k} (2\lambda)^i h_i \frac{dp^k}{d^lp}$. We claim that each function $g_k(p)$, $k = 0, 1, \ldots, n - 1$, is a polynomial in $p$ of order at most $n - 1$. Indeed,

$$\frac{d^n g_k(p)}{dp^n} = \sum_{i=0}^{k} \sum_{j=0}^{n} (2\lambda)^i \binom{n}{j} \frac{d^{n+j}p^k}{d^jp} \frac{d^{n-j}h_i}{d^{n-j}}$$

$$= \sum_{l=0}^{k} \frac{d^{p^k}}{d^p} \sum_{i=0}^{l} (2\lambda)^i \binom{n}{l-i} \frac{d^{n-l+i}h_i}{d^{n-l+i}}$$

$$= 0,$$
where we used the differential equations for $h_i(p)$ obtained in the proof of Proposition 3.2.

Thus the resulting equality

$$h \ast \lambda e_a = \sum_{b=0}^{n-1} A^h_{ab} e_b, \quad a = 0, 1, \ldots, n - 1,$$

with $A^h_{ab} \in \mathbb{K}$, defines a homomorphism from the algebra $(\mathcal{O}_X, \ast \lambda)$ to the algebra of $n \times n$ matrices,

$$\psi : \begin{array}{rcl} \mathcal{O}_X & \longrightarrow & \text{Mat}_{\mathbb{K}}(n, n) \\ h & \longrightarrow & A^h_{ab}. \end{array}$$

This homomorphism is injective. Indeed, if

$$h \ast \lambda p^k \ast \lambda x^{n-1} = 0, \quad \forall k \in \{0, 1, \ldots, n - 1\},$$

then

$$\sum_{i=0}^{k} (2\lambda)^i h_i \frac{d^i}{dp^i} = 0, \quad \forall k \in \{0, 1, \ldots, n - 1\},$$

implying $h_i = 0$.

Finally, the dimension counting implies that the map $\psi$ is an isomorphism. $\square$

4. Concluding remarks. (i) Theorem 3.5 implies that deformation quantization of the $k$th order infinitesimal neighbourhood of the embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ is equivalent to introducing matrix valued functions on the associated phase space $\mathbb{R}^{2n} = \Omega^1 \mathbb{R}^n$ with the quantum product being the tensor product of the $2n$-dimensional Moyal product and the matrix multiplication. Therefore, it is natural to expect the appearance of matrix algebras in any quantum theory where the background “space-time” is thickened into one extra dimension (cf. [1]).

(ii) The construction in subsection 3 can be easily generalized from the affine analytic subspaces, $X \hookrightarrow \mathbb{R}^n$, to analytic subspaces of arbitrary ambient manifolds, $X \hookrightarrow Y$, provided the cotangent bundles $\Omega^1 Y$ come equipped with the star products $\ast \lambda$ satisfying the equality

$$\pi^* (f(x)g(x)) = \pi^*(f(x)) \ast \lambda \pi^*(g(x)) = \pi^*(g(x)) \ast \lambda \pi^*(f(x)), \quad \forall f(x), g(x) \in \mathcal{O}_Y,$$

where $\pi : \Omega^1 Y \rightarrow Y$ is the natural projection. This will be the case, for example, if $\ast \lambda$ comes from the torsion-free affine connection on Y (lifted to $\Omega^1 Y$) via the Fedosov construction [3].

(iii) The computations of algebras of quantum observables for other examples mentioned in the introduction, say for the “tick” $X_2 = \{(x, y) \in \mathbb{R}^2 \mid y^2 - x^3 = 0\}$ or the real line with one double point $X_3 = \{(x, y) \in \mathbb{R}^2 \mid xy = 0, y^2 = 0\}$, are much easier than the one we did for the $n$-tuple point. We leave the details to the interested reader.
References

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